Chern-Weil map for principal bundles over groupoids

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Abstract

The theory of principal $G$-bundles over a Lie groupoid is an important one unifying various types of principal $G$-bundles, including those over manifolds, those over orbifolds, as well as equivariant principal $G$-bundles. In this paper, we study differential geometry of these objects, including connections and holonomy maps. We also introduce a Chern-Weil map for these principal bundles and prove that the characteristic classes obtained coincide with the universal characteristic classes.

As an application, we recover the equivariant Chern-Weil map of Bott-Tu. We also obtain an explicit chain map between the Weil model and the simplicial model of equivariant cohomology which reduces to the Bott-Shulman map $S(g^*)^G \to H^*(BG)$ when the manifold is a point.

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1. Introduction

A remarkable, and well-known, theory based on principal $G$-bundles over a manifold is the theory of Chern-Weil \[11, 33\] which constructs characteristic classes geometrically in terms of differential forms. Recently, there has been an increasing interest in other types of principal bundles, for instance, principal bundles over orbifolds or equivariant principal bundles. It is therefore important to develop a theory of characteristic classes for these more general settings. For equivariant principal bundles, this has been done by many authors \[5, 9, 19\], who introduced a Chern-Weil map in which the de Rham cohomology groups in the usual Chern-Weil theory were replaced by equivariant cohomology (using the Weil model or the Cartan model). More recently, Alekseev-Meinrenken gave a construction of Chern-Weil map for non-commutative $g$-differential algebras \[2\].

In this paper, we develop Chern-Weil theory for principal $G$-bundles over Lie groupoids. The notion of a groupoid is a standard generalization of the concepts of spaces and groups, in which spaces and groups are treated on an equal footing. Simplifying somewhat, one could say that a groupoid is a mixture of a space and a group, with properties of both interacting in a delicate way. In a certain sense, groupoids provide a uniform framework for many different geometric objects. For instance, when a Lie group $G$ acts on a manifold $M$ the transformation groupoid $G \times M \rightrightarrows M$ can be used to replace the quotient space. On the other hand, an orbifold can be represented by a proper étale groupoid \[24\]. Thus the notion of principal bundles over groupoids unifies various existing notions of principal bundles.

A Lie groupoid $\Gamma \rightrightarrows \Gamma_0$ gives rise to a simplicial manifold $\Gamma_\ast$ in a natural fashion. Thus one can use the double complex $\Omega(\Gamma_\ast)$ associated to the simplicial manifold $\Gamma_\ast$ as a model for differential forms. Its cohomology $H^*_{dR}(\Gamma_\ast)$ is isomorphic to the cohomology of
the classifying space \( B\Gamma \). It is well known that \( H^*_{dR}(\Gamma) \) becomes a ring under the cup product [13]. For a Lie group, this is the model studied by Bott-Shulman in connection with characteristic classes [17, 18, 29]. Under this model, for a principal \( G \)-bundle \( P \to \Gamma_0 \) over a Lie groupoid \( \Gamma \Rightarrow \Gamma_0 \), the relevant notion involved is a pseudo-connection, i.e. a connection 1-form \( \theta \in \Omega^1(P) \otimes g \) for the \( G \)-bundle \( P \to \Gamma_0 \) (forgetting the groupoid structure). The total pseudo-curvature is a degree 2 element in \( \Omega(\Gamma_0) \otimes g \) consisting of two parts. The first part is the usual curvature 2-form \( \Omega = d\theta + \frac{1}{2}[\theta, \theta] \in \Omega^2(P) \otimes g \), while the second part \( \partial \theta \in \Omega^1(P) \otimes g \to \Omega^1(Q) \otimes g \) measures the failure of \( \theta \) to be \( \Gamma \)-basic (Proposition 3.6). Here \( Q \Rightarrow P \) denotes the transformation groupoid associated to the \( \Gamma \)-action on \( P \), i.e. \( Q = \Gamma \times_{\Gamma_0} P \), and \( \partial : \Omega^1(P) \otimes g \to \Omega^1(Q) \otimes g \) is the boundary map with respect to this transformation groupoid. Comparing this with the usual picture, it is useful to note that the differential stack corresponding to the groupoid \( Q \Rightarrow P \) is indeed a principal \( G \)-bundle over the differential stack of \( \Gamma \). We prove

**Theorem A.**

1. Associated to any pseudo-connection \( \theta \in \Omega^1(P) \otimes g \), on the cochain level there is a canonical map
   \[
   z_\theta : S(g^*)^G \to Z^*_{dR}(\Gamma_\star),
   \]
called the Chern-Weil map, where \( Z^*_{dR}(\Gamma_\star) \) is the space of closed forms. On the level of cohomology, \( z_\theta \) induces an algebra homomorphism
   \[
   w_\theta : S(g^*)^G \to H^*_{dR}(\Gamma_\star),
   \]
   which is independent of the choice of the pseudo-connection and thus may be denoted by \( w_P \). Moreover, \( z_\theta \) is completely determined by the total pseudo-curvature (Proposition 6.3).

2. If \( \varphi \) is a strict homomorphism from \( \Gamma' \Rightarrow \Gamma_0 \) to \( \Gamma \Rightarrow \Gamma_0 \), then the following diagrams
   \[
   S(g^*)^G \xrightarrow{z_\theta} Z^*_{dR}(\Gamma_\star) \quad \quad S(g^*)^G \xrightarrow{w_P} H^*_{dR}(\Gamma_\star)
   \]
   commute, where \( P' = \varphi^* P \) is the pull-back of \( P \) via \( \varphi \).

As an application, we show that this construction reduces to various existing constructions in the literature in specific cases.

**Theorem B.**

1. If \( \Gamma \) is a manifold \( M \), then \( w_P : S(g^*)^G \to H^*_{dR}(M) \) reduces to the usual Chern-Weil map [13].

2. If \( \Gamma \) is a Lie group \( G \), \( P \) the \( G \)-bundle \( G \to \cdot \), and \( \theta \) the left Maurer-Cartan form, then \( z_\theta : S(g^*)^G \to Z^*_{dR}(G_\star) \) and \( w_P : S(g^*)^G \to H^*_{dR}(G_\star) \) coincide with the Bott-Shulman maps [7, 8].
Another interesting application is the case of equivariant principal $G$-bundles. Let $P \to M$ be an $H$-equivariant principal $G$-bundle. In [9], Bott-Tu introduced a Chern-Weil map, with values in the Weil model, associated to any $H$-invariant connection $\theta$ on $P \to M$. More precisely, they constructed an $H$-basic connection $\Xi$ on the $G$-differential algebra $W(\mathfrak{h}) \otimes \Omega(P)$, which induces a map $z_{BT} : S(g^*)^G \to (W(\mathfrak{h}) \otimes \Omega(M))^H$-basic. On the other hand, $P$ can be considered as a $G$-bundle over the transformation groupoid $H \times M \Rightarrow M$, and $\theta \in \Omega(P) \otimes g$ as a pseudo-connection. Our construction in Section 6 induces a map $z_\theta : S(g^*)^G \to Z((H \times M)_\bullet)$. We have

**Theorem C.** The following diagram commutes

$$
\begin{array}{ccc}
S(g^*)^G & \xrightarrow{z_{BT}} & Z^*(W(\mathfrak{h}) \otimes \Omega(M))^H \text{-basic} \\
\downarrow z_\theta & & \downarrow K \\
Z((H \times M)_\bullet)
\end{array}
$$

where $K : W(\mathfrak{h}) \otimes \Omega(M) \to \Omega((H \times M)_\bullet)$ is the natural chain map between the Weil model and the simplicial model as described in Proposition 6.11.

There is a conceptually simpler way to think of characteristic classes, namely via universal characteristic classes. A principal $G$-bundle $P$ over $M$ induces a map from $M$ to $BG$ (the classifying space of $G$), which in turn induces a homomorphism of cohomology groups $H^*(BG) \to H^*(M)$. Composing the Bott-Shulman map with this map, one gets a map $S(g^*)^G \to H^*(M)$. It was shown in [29] that this coincides with the usual Chern-Weil map $w_P$.

We show that this picture can be generalized to our more general context. While it is possible to make sense of the above argument by replacing the manifold $M$ everywhere with a groupoid $\Gamma$, it is conceptually even simpler to broaden the concept of “maps” between groupoids by allowing generalized homomorphisms, which can be thought of as a groupoid version of smooth maps between differential stacks. Under this framework, a principal $G$-bundle $P$ over a groupoid $\Gamma \Rightarrow \Gamma_0$ is then equivalent to a generalized homomorphism from $\Gamma$ to $G$. Hence it induces a homomorphism on the level of cohomology $H^*_{\text{dR}}(G_\bullet) \to H^*_{\text{dR}}(\Gamma_\bullet)$. Composing the Bott-Shulman map with this map, one obtains a map $w : S(g^*)^G \to H^*_{\text{dR}}(\Gamma_\bullet)$. This is called the universal Chern-Weil map. We have

**Theorem D.** The universal Chern-Weil map and the Chern-Weil map are equal.

A connection on a principal $G$-bundle $P$ over a groupoid $\Gamma \Rightarrow \Gamma_0$ is a pseudo-connection $\theta \in \Omega^1(P) \otimes g$ such that $\partial \theta = 0$. In contrast to pseudo-connections, the notion of connections is well-defined for differential stacks in the sense that they may pass to Morita equivalent groupoids in a natural fashion (Corollary 3.15). Unlike the case of manifolds, connections do not always exist for every principal bundle over a groupoid $\Gamma \Rightarrow \Gamma_0$. However, we prove that

**Theorem E.** Any principal $G$-bundle over an orbifold always admits a connection.

Whenever a connection exists, the Chern-Weil map admits a much simpler form. It essentially results from applying a polynomial function to the curvature of the connection exactly as in the manifold case. More precisely, consider the subcomplex $(\Omega(\Gamma_0)^\Gamma, d)$ of
the de Rham complex \((\Omega(\Gamma_0), d)\), where \(\Omega(\Gamma_0)^\Gamma = \{ \omega \in \Omega(\Gamma_0) \mid \partial \omega = 0 \}\). Let \(H^*_\text{dR}(\Gamma_0)^\Gamma\) be its cohomology. The inclusion \(i : \Omega(\Gamma_0)^\Gamma \to \Omega(\Gamma_\ast)\) is a chain map and therefore induces a morphism \(i : H^*_\text{dR}(\Gamma_0)^\Gamma \to H^*_\text{dR}(\Gamma_\ast)\).

**Theorem F.** Assume that the principal \(G\)-bundle \(P \xrightarrow{\pi} \Gamma_0\) over the groupoid \(\Gamma \Rightarrow \Gamma_0\) admits a connection \(\theta \in \Omega^1(P) \otimes \mathfrak{g}\). Then the following diagrams commute

\[
\begin{align*}
S(\mathfrak{g}^*)^G & \xrightarrow{z} Z^*(\Gamma_0)^\Gamma \\
\downarrow \circ \alpha & \quad \downarrow i \\
Z^*(\Gamma_\ast) & \quad Z^*(\Gamma_\ast)
\end{align*}
\]

and

\[
\begin{align*}
S(\mathfrak{g}^*)^G & \xrightarrow{w} H^*_\text{dR}(\Gamma_0)^\Gamma \\
\downarrow \circ \beta & \quad \downarrow i \\
H^*_\text{dR}(\Gamma_\ast) & \quad H^*_\text{dR}(\Gamma_\ast)
\end{align*}
\]

where \(z : S(\mathfrak{g}^*)^G \to Z^*(\Gamma_0)^\Gamma\), and \(w : S(\mathfrak{g}^*)^G \to H^*_\text{dR}(\Gamma_0)^\Gamma\) denote the usual Chern-Weil maps obtained by forgetting the groupoid action. I.e., \(z(f) = f(\Omega) \in Z^*(P)\text{basic} \cong Z^*(\Gamma_0)\), \(\forall f \in S(\mathfrak{g}^*)^G\), where \(\Omega \in \Omega^2(P) \otimes \mathfrak{g}\) is the curvature form.

Therefore, as a consequence, when a flat connection exists, i.e. a pseudo-connection whose total curvature vanishes, the Chern-Weil map vanishes (except in degree 0). Indeed, a flat connection resembles in many ways a flat connection on usual principal bundles. In particular we prove

**Theorem G.** A flat connection on a principal \(G\)-bundle \(P \xrightarrow{\pi} \Gamma_0\) over a groupoid \(\Gamma \Rightarrow \Gamma_0\) induces a group homomorphism

\[
\pi_1(\Gamma_\ast, x) \to G,
\]

where \(\pi_1(\Gamma_\ast, x)\) denotes the fundamental group of the groupoid \(\Gamma\).

Since our construction of the Chern-Weil map can be done purely algebraically, following the idea of Cartan [10], we carry out this construction for any \(G\)-differential simplicial algebra, which we believe is of interest in its own right. This is the content of Section 4. We present two equivariant constructions. One is through fat realizations and the other is a functorial approach in terms of simplicial algebras. The Chern-Weil map for principal \(G\)-bundles over a groupoid is discussed in Section 5. Section 2 is preliminary on principal \(G\)-bundles. We give several equivalent pictures of principal \(G\)-bundles over a groupoid which should be of independent interest. Section 3 is devoted to the study of differential geometry of these principal bundles including connections, curvatures and holonomy maps.

Finally, a remark about notation. In this paper, \(G\) always denotes a Lie group and \(\mathfrak{g}\) its Lie algebra. By a “groupoid”, we always mean a “Lie groupoid” unless it is specified otherwise; the source and target maps are denoted by \(\beta\) and \(\alpha\), respectively.

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2 Principal G-bundles over groupoids

2.1 Simplicial G-bundles

In this subsection, we recall some useful results concerning simplicial G-bundles.

Recall that a simplicial set $M$ is a sequence of sets $(M_n)_{n \in \mathbb{N}}$ together with face maps $\epsilon^n_i : M_n \to M_{n-1}, i = 0, \ldots, n$ and degeneracy maps $\eta^n_i : M_n \to M_{n+1}, i = 0, \ldots, n$, which satisfy the simplicial identities: $\epsilon^{n-1}_i \epsilon^n_j = \epsilon^{n-1}_j \epsilon^n_i$ if $i < j$, $\eta^{n+1}_i \eta^n_j = \eta^{n+1}_j \eta^n_i$ if $i \leq j$, $\epsilon^{n+1}_i \eta^n_j = \eta^{n+1}_j \epsilon^n_i$ if $i < j$, $\epsilon^{n+1}_i \eta^n_j = \eta^{n+1}_j \epsilon^n_i$ if $i > j + 1$ and $\epsilon^{n+1}_i \eta^n_j = \epsilon^{n+1}_j \eta^n_i = \text{id}_{M_n}$.

A simplicial manifold is a simplicial set $\Gamma = (M_n)_{n \in \mathbb{N}}$, where, for every $n \in \mathbb{N}$, $M_n$ is a smooth manifold and all the degeneracy and face maps are smooth maps.

Associated to any Lie groupoid $\Gamma \rightrightarrows \Gamma_0$, there is a canonical simplicial manifold $\Gamma_\bullet$, constructed as follows. Set $\Gamma_n = \{(\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Gamma^n | \beta(\gamma_i) = \alpha(\gamma_{i+1}), i = 1, \ldots, n - 1\}$, the manifold consisting of all composable $n$-tuples, and define the face maps $\epsilon^n_i : \Gamma_n \to \Gamma_{n-1}$ by, for $n > 1$

\[ \epsilon^n_i(\gamma_1, \gamma_2, \ldots, \gamma_n) = (\gamma_1, \ldots, \gamma_i \gamma_{i+1}, \ldots, \gamma_n), \quad 1 \leq i \leq n - 1, \]

and for $n = 1$ by, $\epsilon^n_0(\gamma) = \beta(\gamma)$, $\epsilon^n_1(\gamma) = \alpha(\gamma)$. Also define the degeneracy maps by $\eta^n_0 = \epsilon : \Gamma_0 \to \Gamma_1$ (being the unit map of the groupoid) and $\eta^n_i : \Gamma_n \to \Gamma_{n+1}$ by:

\[ \eta^n_0(\gamma_1, \ldots, \gamma_n) = (\alpha(\gamma_1), \gamma_1, \ldots, \gamma_n) \]

\[ \eta^n_0(\gamma_1, \ldots, \gamma_n) = (\gamma_1, \ldots, \gamma_i, \beta(\gamma_i), \gamma_{i+1}, \ldots, \gamma_n), \quad 1 \leq i \leq n. \]

We now recall the notion of simplicial G-bundles over a simplicial manifold.

**Definition 2.1** A principal G-bundle over a simplicial manifold $\Gamma_\bullet := (M_n)_{n \in \mathbb{N}}$ is a simplicial manifold $P_\bullet := (P_n)_{n \in \mathbb{N}}$ such that

- for every $n \in \mathbb{N}$, $P_n$ is a principal G-bundle over $M_n$, and
- the degeneracy and face maps are morphisms of principal G-bundles.

2.2 Principal G-bundles over a groupoid

We recall in this subsection the definition of a principal G-bundle over a groupoid, and give some basic examples.

**Definition 2.2** A principal G-bundle over a groupoid $\Gamma \rightrightarrows \Gamma_0$ consists of a principal right G-bundle $P \xrightarrow{\pi} \Gamma_0$ over the manifold $\Gamma_0$ such that

(i) there is a map $\sigma : Q \to P$, where $Q$ is the fibered product $Q = \Gamma \times_{\beta, \Gamma_0, \pi} P$. We write $\sigma(\gamma, p) = \gamma \cdot p$. This map is subject to the constraints

(ii) for all $p \in P$ we have $\pi(p) \cdot p = p$;

(iii) for all $p \in P$ and all $\gamma_1, \gamma_2 \in \Gamma$ such that $\pi(p) = \beta(\gamma_2)$ and $\alpha(\gamma_2) = \beta(\gamma_1)$ we have $\gamma_1 \cdot (\gamma_2 \cdot p) = (\gamma_1 \gamma_2) \cdot p$;

(iv) for all $p \in P$, $g \in G$ and $\gamma \in \Gamma$, such that $\beta(\gamma) = \pi(p)$, we have $(\gamma \cdot p) \cdot g = \gamma \cdot (p \cdot g)$;

Axioms (i)-(iii) simply mean that the groupoid $\Gamma \rightrightarrows \Gamma_0$ acts on $P \xrightarrow{\pi} \Gamma_0$. Axiom (iv) means that this action commutes with the G-action.
Example 2.3 Let $\Gamma$ be a transformation groupoid $H \times M \rightrightarrows M$, where $H$ acts on $M$ from the left. Then a principal $G$-bundle over $\Gamma$ corresponds exactly to an $H$-equivariant principal (right) $G$-bundle over $M$.

The following proposition clarifies the relation between principal $G$-bundles over a groupoid $\Gamma \rightrightarrows \Gamma_0$ and principal $G$-bundles over its corresponding simplicial manifold $\Gamma_\bullet$.

**Proposition 2.4** Let $\Gamma \rightrightarrows \Gamma_0$ be a Lie groupoid. There is an equivalence of categories between the category of $G$-bundles over the groupoid $\Gamma \rightrightarrows \Gamma_0$ and the category of simplicial $G$-bundles over the simplicial manifold $\Gamma_\bullet$.

This proposition is an immediate consequence of Lemma 2.5 and Lemma 2.7 below.

First, let us introduce some terminology. For a given Lie groupoid $\Gamma \rightrightarrows \Gamma_0$, by a principal $G$-groupoid over $\Gamma \rightrightarrows \Gamma_0$, we mean a Lie groupoid $Q \rightrightarrows P$ together with a groupoid morphism from $Q$ to $\Gamma$:

\[
\begin{array}{ccc}
G & \to & Q \\
\downarrow & & \downarrow \\
G & \to & P
\end{array}
\]

such that both $G \to Q \xrightarrow{\pi} \Gamma$ and $G \to P \xrightarrow{\pi} \Gamma_0$ are principal $G$-bundles and the source and target maps are morphisms of principal $G$-bundles. There is an obvious notion of morphisms for principal $G$-groupoids over a given groupoid $\Gamma \rightrightarrows \Gamma_0$ so that one obtains a category.

**Lemma 2.5** The category of principal $G$-bundles over the groupoid $\Gamma \rightrightarrows \Gamma_0$ is equivalent to the category of principal $G$-groupoids over $\Gamma \rightrightarrows \Gamma_0$.

**Proof.** Associated to any principal $G$-bundle $P \xrightarrow{\pi} \Gamma_0$ over $\Gamma \rightrightarrows \Gamma_0$, there is a groupoid $Q \rightrightarrows P$, namely, the transformation groupoid, which is defined as follows. Let $Q = \Gamma \times_{\beta,\Gamma_0,\pi} P$, the source and target maps are, respectively, $\beta(\gamma, p) = p$, $\alpha(\gamma, p) = \gamma \cdot p$, and the multiplication is

\[
(\gamma_1, q) \cdot (\gamma_2, p) = (\gamma_1 \gamma_2, p), \quad \text{where } q = \gamma_2 \cdot p.
\]

Let $\pi : Q \to \Gamma$ be the projection map. It is simple to see that $\pi$ is a groupoid homomorphism and one obtains a commutative diagram (6). Hence $Q$ is a principal $G$-groupoid over $\Gamma$.

Conversely, assume that $Q$ is a principal $G$-groupoid over $\Gamma$. Define a $\Gamma$-action on $P \xrightarrow{\pi} \Gamma_0$ by $\gamma \cdot p = \alpha(q)$, where $q \in Q$ is the unique element on the fiber $\pi^{-1}(\gamma)$ satisfying $\beta(q) = p$. This endows $P \xrightarrow{\pi} \Gamma_0$ with a structure of principal $G$-bundle over $\Gamma \rightrightarrows \Gamma_0$ such that $Q \rightrightarrows P$ is its corresponding transformation groupoid.

We leave the reader to check the functoriality of the above construction. □

**Lemma 2.6** Let $\Gamma \rightrightarrows \Gamma_0$ be a Lie groupoid. Assume that $Q$ and $P$ are principal $G$-bundles over $\Gamma$ and $\Gamma_0$, respectively, which admit two $G$-bundle maps $\beta, \alpha : Q \to P$ such that diagram (6) commutes. Then $Q$ admits a structure of principal $G$-groupoid over $\Gamma \rightrightarrows \Gamma_0$. 

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Proof. We take $\beta : Q \to P$ and $\alpha : Q \to P$ as the source and target maps. The product $q_1 \cdot q_2$ of two elements $q_1$ and $q_2$ with $\beta(q_1) = \alpha(q_2)$ is defined to be the unique element on the $\pi$-fiber over $\pi(q_1)\pi(q_2) \in \Gamma$ whose source equals to $\beta(q_2)$. For any $p \in P$, define the unit $\epsilon(p) \in Q$ to be the unique element in $\pi^{-1}(\epsilon(\pi(p)))$ such that $\beta(\epsilon(p)) = p$. For any $q \in Q$, the inverse $q^{-1}$ is defined to be the unique element on the $\pi$-fiber over $\pi(q)^{-1} \in \Gamma$ such that $\beta(q^{-1}) = \alpha(q)$.

Since for any composable triple $(q_1, q_2, q_3)$, both $q_1 \cdot (q_2 \cdot q_3)$ and $(q_1 \cdot q_2) \cdot q_3$ are mapped to $\pi(q_1)\pi(q_2)\pi(q_3) \in \Gamma$ under $\pi$, and to $\beta(q_3)$ under $\beta$, hence $q_1 \cdot (q_2 \cdot q_3) = (q_1 \cdot q_2) \cdot q_3$. Therefore the multiplication is associative. The rest of the axioms can be verified in a similar fashion. □

Lemma 2.7 The category of principal $G$-groupoids over $\Gamma \rightrightarrows \Gamma_0$ is equivalent to the category of simplicial $G$-bundles over the simplicial manifold $\Gamma_\bullet$.

Proof. Given a principal $G$-groupoid $Q \rightrightarrows P$ over $\Gamma \rightrightarrows \Gamma_0$, $Q_\bullet$ is clearly a principal $G$-bundle over the simplicial manifold $\Gamma_\bullet$.

Conversely, assume that $R_\bullet := (R_n)_{n \in \mathbb{N}}$ is a simplicial $G$-bundle over $\Gamma_\bullet$. Let $Q = R_1$ and $P = R_0$. By Definition 2.1, we have the commutative diagram as in Eq. (6). Therefore, according to Lemma 2.6, $Q \rightrightarrows P$ is endowed with a groupoid structure, which is a principal $G$-groupoid over $\Gamma \rightrightarrows \Gamma_0$.

To complete the proof, it suffices to show that the associated simplicial manifold $Q_\bullet$ is isomorphic to $R_\bullet$ as simplicial $G$-bundles over $\Gamma_\bullet$. By abuse of notation, we use $\epsilon^p_t$ to denote the face maps of both $R_\bullet$ and $Q_\bullet$. We denote by $p_n$ the maps from $Q_n$ or $R_n$ to $P$ defined by $p_n = \epsilon^1_0 \cdots \epsilon^n_0$. Define $\varphi_n : R_n \to Q_n$ by $\varphi_n(r) = q$, where $q \in Q_n$ is the unique element in the fiber of $\pi : Q_n \to \Gamma_n$ over $\pi(r) \in \Gamma_n$ with $p_n(q) = p_n(r)$. It is simple to see that $(\varphi_n)_{n \in \mathbb{N}}$ is a simplicial map and is therefore an isomorphism of principal $G$-bundle over $\Gamma_\bullet$. □

2.3 Generalized homomorphisms

In this subsection, we will give another definition of $G$-bundles over a groupoid using the notion of generalized homomorphisms [17, 18, 32]. Let us recall its definition below.

Definition 2.8 A generalized groupoid homomorphism $\varphi := (Z, \sigma, \tau)$ from $\Gamma \rightrightarrows \Gamma_0$ to $H \rightrightarrows H_0$ is given by a manifold $Z$, two smooth maps $\Gamma_0 \xrightarrow{\tau} Z \xrightarrow{\sigma} H_0$, a left action of $\Gamma$ with respect to $\tau$, a right action of $H$ with respect to $\sigma$, such that the two actions commute, and $Z$ is an $H$-principal bundle over $\Gamma_0$.

In the sequel, we will use both notations $(Z, \sigma, \tau)$ and $\Gamma_0 \xrightarrow{\tau} Z \xrightarrow{\sigma} H_0$ interchangeably to denote a generalized homomorphism.

Recall that given a Lie groupoid $H \rightrightarrows H_0$, by an (right) $H$-principal bundle over $M$, we mean an (right) $H$-space $P \to H_0$ and a surjective submersion $\pi : P \to M$ such that for all $p, p' \in P$, such that $\pi(p) = \pi(p')$, there exists a unique $\gamma \in H$, such that $p \cdot \gamma$ is defined and $p \cdot \gamma = p'$.

The following proposition lists several useful equivalent definitions of (right) $H$-principal bundles, or (right) $H$-torsors over a manifold $M$.

Proposition 2.9 ([17, §2], [18, Def. 1.1]) Let $H \rightrightarrows H_0$ be a Lie groupoid. The following statements are equivalent.
1. $Z \xrightarrow{\sigma} H_0$ is an (right) $H$-principal bundle over $M$;

2. $Z \xrightarrow{\sigma} H_0$ is an right $H$-space, where the $H$-action is free and proper so that $M \cong Z/H$ (note that the projection map $Z \to M$ is always a surjective submersion).

3. $Z \xrightarrow{\sigma} H_0$ is an right $H$-space with a smooth map $\pi : Z \to M$ such that at every point $x \in M$ there is a local section $s : U \to Z$ of $\pi$ satisfying $\pi^{-1}(U) \cong U \times_{\sigma s,H_0,0} H$.

If $f : \Gamma \to H$ is a strict homomorphism (i.e. a smooth map satisfying the property that $f(\gamma_1 \gamma_2) = f(\gamma_1) f(\gamma_2)$, then $Z_f = \Gamma_0 \times_{f,H_0,0} H$, with $\tau(x,h) = x$, $\sigma(x,h) = \beta(h)$, and the actions $\gamma \cdot (x,h) = (\alpha(\gamma), f(\gamma) h)$ and $(x,h) \cdot h' = (x, h'h)$, defines a generalized homomorphism from $\Gamma \rightrightarrows \Gamma_0$ to $H \rightrightarrows H_0$.

Two generalized homomorphisms $\varphi_1 := (Z, \sigma, \tau)$ and $\varphi_2 := (Z', \sigma', \tau')$ from $\Gamma$ to $H$ are said to be equivalent if there is a $\Gamma$-$H$-equivariant diffeomorphism $\varphi : Z \to Z'$.

Generalized homomorphisms can be composed as follows. If $\varphi : \Gamma_0 \xrightarrow{\ell} Z \xrightarrow{\tau} \Gamma_0$ and $\varphi' : \Gamma_0 \xrightarrow{\ell'} Z' \xrightarrow{\tau'} \Gamma_0$ are generalized homomorphisms from $\Gamma \rightrightarrows \Gamma_0$ to $H \rightrightarrows H_0$, and from $H \rightrightarrows H_0$ to $R \rightrightarrows R_0$, respectively, then the composition $\varphi \varphi' : \Gamma_0 \xleftarrow{\ell} Z \xrightarrow{\tau} \Gamma_0 \xrightarrow{\ell'} Z' \xrightarrow{\tau'} \Gamma_0 \rightrightarrows R_0$ defined by

$$Z'' = Z \times_H Z' := (Z \times_{\sigma,H_0,\tau'} Z')/(z,z') \sim (zh,h^{-1}z')$$

is a generalized homomorphism from $\Gamma \rightrightarrows \Gamma_0$ to $R \rightrightarrows R_0$. Moreover, the composition of generalized homomorphisms is associative. Thus one obtains a category $\mathcal{G}$ whose objects are Lie groupoids and morphisms are generalized homomorphisms $[18, 32]$. There is a functor $\mathcal{G}_a \to \mathcal{G}$, where $\mathcal{G}_a$ is the category of Lie groupoids with strict homomorphisms, given by $f \mapsto Z_f$ as described above. The isomorphisms in the category $\mathcal{G}$ are called Morita equivalences $[25, 34]$. Indeed it is simple to see that any generalized homomorphism can be decomposed as the composition of a Morita equivalence with a strict homomorphism $[32]$. Let us briefly recall this construction below.

Let $\varphi := (Z, \sigma, \tau)$ be a generalized homomorphism from $\Gamma' \rightrightarrows \Gamma'_0$ to $\Gamma \rightrightarrows \Gamma_0$. We denote by $\Gamma'[Z] \rightrightarrows Z$ the pull-back of $\Gamma' \rightrightarrows \Gamma'_0$ via the surjective submersion $Z \xrightarrow{\tau} \Gamma_0$, i.e. the groupoid $Z \times_{\tau,0} \Gamma' \times_{\beta,\tau} Z$ with the multiplication law $(z_1, \gamma_1', z_2)(z_2, \gamma_2', z_3) = (z_1, \gamma_1' \gamma_2', z_3)$. Then the projection map from $\Gamma'[Z] \rightrightarrows Z$ to $\Gamma' \rightrightarrows \Gamma'_0$, denoted by $\tau$, by abuse of notation, $\tau(z_1, \gamma', z_2) = \gamma'$

is a strict homomorphism, which is indeed a Morita equivalence $[3]$. On the other hand the map from $\Gamma'[Z] \rightrightarrows Z$ to $\Gamma \rightrightarrows \Gamma_0$,

$$\sigma(z_1, \gamma', z_2) = \gamma,$$

where $\gamma \in \Gamma$ is the unique element such that $\gamma' \cdot z_2 = z_1 \cdot \gamma$, is a strict homomorphism. Thus we have proved the following

**Lemma 2.10** Let $\varphi := (Z, \sigma, \tau)$ be a generalized homomorphism from $\Gamma' \rightrightarrows \Gamma'_0$ to $\Gamma \rightrightarrows \Gamma_0$. Then $\varphi$ is the composition of the Morita equivalence $\Gamma' \rightrightarrows \Gamma' \rightrightarrows \Gamma'[Z]$ with the strict homomorphism $\sigma : \Gamma'[Z] \to \Gamma$.

The following proposition follows immediately from the definition of generalized homomorphisms.

**Proposition 2.11** The category of $G$-principal bundles over a groupoid $\Gamma \rightrightarrows \Gamma_0$ is equivalent to the category of generalized homomorphisms from $\Gamma \rightrightarrows \Gamma_0$ to $G \rightrightarrows$.
As a consequence, we have the following

**Corollary 2.12** If $\Gamma'$ and $\Gamma$ are Morita equivalent groupoids, there is an equivalence of categories of principal $G$-bundles over $\Gamma'$ and $\Gamma$.

As another consequence, principal $G$-bundles over groupoids can be pulled back via generalized homomorphisms. We describe this pull-back construction explicitly in the following:

**Proposition 2.13** Let $P$ be a principal $G$-bundle over the groupoid $\Gamma$ and $\varphi : \Gamma_0 \leftarrow Z \rightarrow \Gamma_0$ a generalized homomorphism from $\Gamma'$ to $\Gamma$. Then the pull-back $G$-bundle $P' \xrightarrow{\pi'} \Gamma_0'$ can be described as follows.

- $P' = (Z \times_{\Gamma_0} P)/\Gamma$. We denote by $(z, p) \in P'$ the class in $P'$ corresponding to $(z, p) \in Z \times_{\Gamma_0} P$;
- $G$ acts on $P'$ by $(z, p) \cdot g = (z, p \cdot g)$, $\forall g \in G$;
- $\pi' : P' \rightarrow \Gamma_0'$ is given by $\pi'(z, p) = \tau(z)$;
- the $\Gamma'$-action on $P' \rightarrow \Gamma_0'$ is given by $\gamma' \cdot (z, p) = (\gamma' \cdot z, p)$, $\forall \gamma' \in \Gamma'$ with $\beta(\gamma') = \tau(z)$.

Sometimes it is useful to consider a generalized homomorphism $\varphi := (P, \sigma, \pi)$ defined by a principal $G$-bundle $P \xrightarrow{\pi} \Gamma_0$ over a groupoid $\Gamma \rightrightarrows \Gamma_0$ as the composition of a Morita equivalence with a strict homomorphism. Below we present two equivalent pictures of such compositions.

First consider the semi-direct product groupoid $Q': Q \times G \rightrightarrows P$ (using the right $G$-action on the groupoid $Q \rightrightarrows P$ as introduced in the proof of Lemma 2.5). Here the source and target maps, and the multiplication are

$$\beta(q, g) = \beta(q)g, \quad \alpha(q, g) = \alpha(q), \quad (q_1, g_1) \cdot (q_2, g_2) = (q_1 \cdot (q_2 g_1^{-1}) , g_1g_2). \quad (10)$$

It is simple to check that this groupoid is Morita equivalent to $\Gamma \rightrightarrows \Gamma_0$. In fact the map $(\gamma, p, g) \mapsto (\gamma p, \gamma, pg)$ is a groupoid isomorphism from $Q' \rightrightarrows P$ onto the pull-back groupoid of $\Gamma \rightrightarrows \Gamma_0$ via the surjective submersion $P \xrightarrow{\pi} \Gamma_0$. Moreover the natural projection on the second factor

$$
\begin{array}{ccc}
Q' & \xrightarrow{p'} & G \\
\downarrow & & \downarrow \\
P & \rightarrow & \\
\end{array}
$$

defines a strict groupoid homomorphism.

Alternatively, consider the gauge groupoid $P \times P \xrightarrow{\beta} \Gamma_0$. We denote by $(p_1, p_2)$ the class in $P \times P$ corresponding to $(p_1, p_2) \in P \times P$. A strict homomorphism $\rho : \Gamma \rightarrow P \times P$ is defined by $\gamma \mapsto (\gamma p, p)$, where $p \in P$ is any element satisfying $\pi(p) = \beta(\gamma)$. Thus we obtain the following groupoid homomorphism:

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\rho} & P \times P \\
\downarrow & & \downarrow \\
\Gamma_0 & \rightarrow & \Gamma_0
\end{array}
$$

10
Since any transitive groupoid is Morita equivalent to its isotropy group, \( P \times_G M \) is Morita equivalent to \( G \Rightarrow . \)

**Proposition 2.14** Let \( P \overset{\pi}{\rightarrow} \Gamma \) be a principal \( G \)-bundle over a groupoid \( \Gamma \Rightarrow \Gamma_0 \). Then the generalized homomorphism \( \Gamma \sim \Gamma_0 \Rightarrow \cdot \) from \( \Gamma \Rightarrow \Gamma_0 \) to \( G \Rightarrow \cdot \) is equal to the \( \pi : \Gamma \rightarrow P \times_G P \) and \( \pi \times \pi : P \times G \Rightarrow \sim \) Morita \( G \).

**Proof.** This follows from a direct verification, and is left to the reader. \( \square \)

We end this section by summarizing Proposition 2.4 and Proposition 2.11 in the following:

**Theorem 2.15** Let \( \Gamma \Rightarrow \Gamma_0 \) be a Lie groupoid. The following categories are equivalent:

- the category of principal \( G \)-bundles over \( \Gamma \Rightarrow \Gamma_0 \);
- the category of principal \( G \)-bundles over the simplicial manifold \( \Gamma \); and
- the category of generalized homomorphisms from \( \Gamma \Rightarrow \Gamma_0 \) to \( G \Rightarrow \cdot \).

3 **Pseudo-connections and connections**

The purpose of this section is to study differential geometry of a principal \( G \)-bundle over a Lie groupoid and in particular connections and curvatures.

3.1 **De Rham cohomology**

Considering a Lie groupoid \( \Gamma \Rightarrow \Gamma_0 \) as a generalization of a manifold (indeed it defines a differential stack \( \mathbb{H} \)), one can also speak about de Rham cohomology, which is given by the following double complex \( \Omega^\bullet(\Gamma) \):

\[
\begin{array}{ccc}
\ldots & \ldots & \ldots \\
\Omega^1(\Gamma_0) & \Omega^1(\Gamma_1) & \Omega^1(\Gamma_2) & \ldots \\
\downarrow d & \downarrow d & \downarrow d & & \\
\Omega^0(\Gamma_0) & \Omega^0(\Gamma_1) & \Omega^0(\Gamma_2) & \ldots \\
\downarrow d & \downarrow d & \downarrow d & & \\
\downarrow d & \downarrow d & \downarrow d & & \\
\end{array}
\]

(13)

Its boundary maps are \( d : \Omega^k(\Gamma_p) \rightarrow \Omega^{k+1}(\Gamma_p) \), the usual exterior differential, and \( \partial : \Omega^k(\Gamma_p) \rightarrow \Omega^k(\Gamma_{p+1}) \), the alternating sum of the pull-back of the face maps of the corresponding simplicial manifolds \( \Gamma_p \). We denote the total differential by

\[
\delta = (-1)^p d + \partial.
\]

The cohomology groups of the total complex \( H^*_{dR}(\Gamma) = H^k(\Omega^\bullet(\Gamma)) \) are called the de Rham cohomology groups of \( \Gamma \Rightarrow \Gamma_0 \). Note that \( H^*_{dR}(\Gamma) \) is a super-commutative algebra with respect to the cup-product \( \Box \) defined as follows. For any \( \omega \in \Omega^k(\Gamma_m) \) and \( \omega_2 \in \Omega^l(\Gamma_n) \), let \( \omega_1 \wedge \omega_2 \in \Omega^{k+l}(\Gamma_{m+n}) \) be the differential form given by

\[
(\omega_1 \wedge \omega_2) (\gamma_1, \ldots, \gamma_{m+n}) = (-1)^{kn} p_1^* \omega_1 \wedge p_2^* \omega_2,
\]

(14)
where \( p_1 : \Gamma_{m+n} \to \Gamma_m \) is defined by
\[
\begin{align*}
p_1(\gamma_1, \ldots, \gamma_{m+n}) &= (\gamma_1, \ldots, \gamma_m) \quad \text{if } m \geq 1 \\
p_1(\gamma_1, \ldots, \gamma_n) &= \alpha(\gamma_1) \quad \text{if } m = 0 \text{ and } n \geq 1 \\
p_1 &= \text{Id} \quad \text{if } m = n = 0
\end{align*}
\]
and \( p_2 : \Gamma_{m+n} \to \Gamma_n \) is the map
\[
\begin{align*}
p_2(\gamma_1, \ldots, \gamma_{m+n}) &= (\gamma_{m+1}, \ldots, \gamma_{m+n}) \quad \text{if } n \geq 1 \\
p_2(\gamma_1, \ldots, \gamma_m) &= \beta(\gamma_m) \quad \text{if } n = 0 \\
p_2 &= \text{Id} \quad \text{if } m = n = 0
\end{align*}
\]
Eq. (14) can be rewritten in terms of face maps into a more compact form:
\[
\omega_1 \vee \omega_2 = (-1)^{kn} (\epsilon_{m+n})^* \cdots (\epsilon_{n+1})^* (\omega_1) \land (\epsilon_{m+n})^* \cdots (\epsilon_{n+1})^* (\omega_2).
\]

One can prove that \( \vee \) induces a graded ring structure on the de Rham cohomology groups \( H^*_dR(\Gamma_\approx) \). An important property of the de Rham cohomology is that it is functorial with respect to generalized homomorphisms. More precisely, a generalized homomorphism \( \varphi \) from \( \Gamma' \Rightarrow \Gamma_0 \) to \( \Gamma \Rightarrow \Gamma_0 \) induces a homomorphism \( \varphi^* : H^*_dR(\Gamma_0) \to H^*_dR(\Gamma'_0) \). We recall its construction below.

First, note that a strict homomorphism \( \psi \) of Lie groupoids induces a natural chain map between their de Rham double complexes, and therefore a morphism \( \psi^* \) of de Rham cohomology groups. Now given a generalized homomorphism \( \varphi := (Z, \sigma, \tau) \) from \( \Gamma' \Rightarrow \Gamma_0 \) to \( \Gamma \Rightarrow \Gamma_0 \), let \( \Gamma'[Z] \Rightarrow Z \) be the pull-back groupoid of \( \Gamma' \Rightarrow \Gamma_0 \) under the surjective submersion \( \tau : Z \to \Gamma_0' \). Denote again by \( \tau \) the strict homomorphism from \( \Gamma'[Z] \Rightarrow Z \) to \( \Gamma' \Rightarrow \Gamma_0 \), and the strict homomorphism from \( \Gamma'[Z] \Rightarrow Z \) to \( \Gamma \Rightarrow \Gamma_0 \) as in Eq. (8) and (9), respectively. Then

**Lemma 3.1** \( \Delta \)[21] The homomorphism \( \tau^* : H^*_dR(\Gamma'_0) \to H^*_dR(\Gamma'_0) \) is an isomorphism.

Define \( \varphi^* : H^*_dR(\Gamma_0) \to H^*_dR(\Gamma'_0) \) by
\[
\varphi^* = (\tau^*)^{-1} \circ \sigma^*.
\]

It is simple to check that two equivalent generalized homomorphisms induce the same morphism for the de Rham cohomology groups. Hence \( \varphi \) is well-defined. In particular, if \( \Gamma' \Rightarrow \Gamma_0 \) and \( \Gamma \Rightarrow \Gamma_0 \) are Morita equivalent groupoids, then \( H^*_dR(\Gamma_0) \Rightarrow H^*_dR(\Gamma'_0) \).

### 3.2 Pseudo-connections and pseudo-curvatures

We are now ready to introduce the following

**Definition 3.2** Let \( P \to \Gamma_0 \) be a principal \( G \)-bundle over a Lie groupoid \( \Gamma \Rightarrow \Gamma_0 \).

- A pseudo-connection is a connection one-form \( \theta \in \Omega^1(P) \otimes \mathfrak{g} \) of the \( G \)-bundle \( P \to \Gamma_0 \) (ignoring the groupoid action);
- the total pseudo-curvature \( \Omega_{\text{total}} \in \Omega^2(Q_\approx) \otimes \mathfrak{g} \) is defined by
\[
\Omega_{\text{total}} = \delta \theta + \frac{1}{2}[\theta, \theta] = \partial \theta + \Omega,
\]
where \( \Omega = d\theta + \frac{1}{2}[\theta, \theta] \in \Omega^2(P) \otimes \mathfrak{g} \) is the curvature form corresponding to \( \theta \).
The total pseudo-curvature $\Omega_{\text{total}}$ consists of two terms: the term $\partial \theta := \beta^* \theta - \alpha^* \theta$ is a $\mathfrak{g}$-valued $1$-form on $Q$, and the term $\Omega$ is a $\mathfrak{g}$-valued $2$-form on $P$. Both terms have a total degree 2 in the double complex $\Omega^*(Q_\bullet) \otimes \mathfrak{g}$.

Introduce a bracket of degree $-1$ on $\Omega^*(Q_\bullet) \otimes \mathfrak{g}$ by

\[ [\omega_1 \otimes X_1, \omega_2 \otimes X_2] = (\omega_1 \lor \omega_2) \otimes [X_1, X_2], \tag{18} \]

where $\omega_1, \omega_2 \in \Omega^*(Q_\bullet)$, and $X_1, X_2 \in \mathfrak{g}$. Note that this bracket does not satisfy the graded-Jacobi identity, and is neither graded skew-symmetric.

**Proposition 3.3** The total pseudo-curvature satisfies the Bianchi identity:

\[ \delta \Omega_{\text{total}} = \frac{1}{2}([\Omega_{\text{total}}, \theta] - [\theta, \Omega_{\text{total}}]). \tag{19} \]

**Proof.** Applying the total differential $\delta$ to Eq. (17), we obtain

\[
\delta \Omega_{\text{total}} = \frac{1}{2} \delta ([\theta, \theta]) \\
= \frac{1}{2} (\delta \theta, \theta) - [\theta, \delta \theta] \\
= \frac{1}{2} ([\Omega_{\text{total}}, \theta] - [\theta, \Omega_{\text{total}}]) + \frac{1}{4} ([[\theta, \theta], \theta] - [\theta, [\theta, \theta]]).
\]

Note that $\theta$ is an element in $\Omega^1(P) \otimes \mathfrak{g}$, and the restriction of the cup-product to $\Omega(P)$ is simply the wedge product. It follows from the Jacobi identity that $[[\theta, \theta], \theta] = [\theta, [\theta, \theta]] = 0$ $[20]$. Therefore, Eq. (19) follows. □

**Remark 3.4** We end this subsection by presenting a geometric interpretation of $\partial \theta$.

Note that any smooth path $p(t)$ in $P$ induces a smooth path $g_p(t)$ in $G$ as follows. By $\tilde{p}(t) \in P$, we denote the horizontal lift of the projected path $\pi(p(t))$ starting at $p(0)$. Then the path $g_p(t) \in G$ is defined by the relation $p(t) = \tilde{p}(t) \cdot g_p(t)$. It is clear that $g_p(t)$ satisfies the relation $\dot{g}_p(0) = \dot{p}(0) \mathbb{J} \theta$. Let $\delta_q \in T_qQ$ be any tangent vector and $q(t)$ a path in $Q$ with $\dot{q}(0) = \delta_q$. Set $a(t) = \alpha(q(t))$ and $b(t) = \beta(q(t))$. Define $g(t) \in G$ by $g(t) = g_a^{-1}(t)g_b(t)$. Then we have

\[ g(0) = 1, \quad \dot{g}(0) = \delta_q \mathbb{J} \partial \theta. \tag{20} \]

### 3.3 Connections

**Definition 3.5** Let $P \to \Gamma_0$ be a principal $G$-bundle over a Lie groupoid $\Gamma \rightrightarrows \Gamma_0$.

- A pseudo-connection $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ is called a connection if $\partial \theta = 0$;
- if $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ is a connection, then $\Omega_{\text{total}} = \Omega \in \Omega^2(P) \otimes \mathfrak{g}$ is called the curvature;
- a connection is said to be flat if its curvature vanishes.

In what follows, we investigate the criteria for a connection to exist. As we see below, this imposes a strong assumption on the $G$-bundles.

By $U_{\text{loc}}(\Gamma)$ we denote the pseudo-group of local bisections of the groupoid $\Gamma \rightrightarrows \Gamma_0$. It is simple to see that there is a local action of $U_{\text{loc}}(\Gamma)$ on $P$ preserving the $G$-bundle structure. In other words, there is a group homomorphism

\[ \varphi : U_{\text{loc}}(\Gamma) \to \text{Diff}_{\text{loc}}(P), \]
where Diff\textsubscript{loc}(P) denotes the pseudo-group of local automorphisms of the principal \(G\)-bundle \(P\). The map \(\varphi\) can be defined as follows. For any \(\mathcal{L} \in \mathcal{U}_{\text{loc}}(\Gamma)\), \(\varphi(\mathcal{L}) \in \text{Diff}_{\text{loc}}(P)\) is defined for any \(p \in P\) such that \(\pi(p)\) is in the support of \(\varphi \in U_{\text{loc}}(\Gamma)\) by

\[
\varphi_{\mathcal{L}}(p) = \mathcal{L} \cdot p, \tag{21}
\]

where \(\cdot\) denotes the \(\Gamma\)-action. Note that \(\mathcal{L} \cdot p\) is uniquely determined by assumptions.

**Proposition 3.6** Assume that \(P \to \Gamma_0\) is a principal \(G\)-bundle over a Lie groupoid \(\Gamma \Rightarrow \Gamma_0\) and \(\theta \in \Omega^1(P) \otimes \mathfrak{g}\) is a pseudo-connection. The following statements are equivalent.

1. \(\theta\) is a connection;
2. The \(\mathfrak{g}\)-valued one-form \(\theta\) is basic with respect to the \(U_{\text{loc}}(\Gamma)\)-action;
3. For each \(p \in P\), we have the inclusion \(\hat{A}_{\pi(p)} \subset H_p\), and moreover the distribution \(\{H_p \subset T_pP, \forall p \in P\}\) is preserved under the action of \(U_{\text{loc}}(\Gamma)\). Here \(H_p \subset T_pP\) is the horizontal subspace defined by the connection 1-form \(\theta\), and \(\hat{A}_{\pi(p)}\) denotes the \(\text{Lie algebra of } \mathcal{U}_{\text{loc}}(\Gamma)\).

The following technical lemma and Corollary 3.8 will be useful.

**Lemma 3.7** Let \(\Gamma \Rightarrow \Gamma_0\) be a Lie groupoid and \(\omega \in \Omega^k(\Gamma_0)\) a \(k\)-form on \(\Gamma_0\). Then the following statements are equivalent

1. \(\partial \omega = 0\);
2. \(\omega\) is basic with respect to the \(U_{\text{loc}}(\Gamma)\)-action.

**Proof.** Note that the Lie algebra of \(U_{\text{loc}}(\Gamma)\) is the Lie algebra \(\mathfrak{g}_{\text{loc}}(A)\) of local sections of \(A\).

1) \(\Rightarrow\) 2). \forall \gamma \in \Gamma\), let \(\mathcal{L}\) be a local bisection through \(r\) and \(m = \beta(r)\). For all \(\delta^1_m, \ldots, \delta^k_m \in T_m\Gamma_0\), there are unique \(\delta^1_m, \ldots, \delta^k_m \in T_r\Gamma\) tangent to the submanifold \(\mathcal{L}\) satisfying \(\beta_*(\delta^i_m) = \delta^i_m\) for all \(i \in \{1, \ldots, k\}\). It is simple to see that \(\varphi_{\mathcal{L}}(\delta^i_m) = \alpha_*\delta^i_m\) for all \(i \in \{1, \ldots, k\}\). Since \(\partial \omega = 0\), we have

\[
0 = (\partial \omega)(\delta^1_m, \ldots, \delta^k_m) = \omega(\delta^1_m, \ldots, \delta^k_m) - \omega(\varphi_{\mathcal{L}}(\delta^1_m), \ldots, \varphi_{\mathcal{L}}(\delta^k_m)). \tag{22}
\]

Therefore \(\varphi_{\mathcal{L}}^*\omega = \omega\).

\(\forall X \in \Gamma(A)\), let \(\hat{X}\) be its corresponding right invariant vector field on \(\Gamma\), and \(\hat{X}\) the vector field on \(\Gamma_0\) generated by the infinitesimal action of \(X\) (this is also denoted by \(a(X)\), where \(a : A \to TT\Gamma_0\) is the anchor map). It is known that these vector fields are related by the relation \(\alpha_*\hat{X}(r) = \hat{X}(m)\). Thus it follows that

\[
\hat{X} \cdot \partial \omega = \beta^*((\beta_*(\hat{X}(r)) \cdot \partial \omega) - \alpha^*((\alpha_*\hat{X}(r)) \cdot \partial \omega) = -\alpha^*((\alpha_*\hat{X}(r)) \cdot \partial \omega), \tag{23}
\]

since \(\beta_*\hat{X}(r) = 0\). Since \(\alpha\) is a surjective submersion, it follows that

\[
\hat{X}(m) \cdot \partial \omega = \alpha_*\hat{X}(r) \cdot \partial \omega = 0. \tag{24}
\]

By Eq. (22) and (24), \(\omega\) is a basic form with respect to the \(U_{\text{loc}}(\Gamma)\)-action.
2) \(\Rightarrow\) 1). By working backwards using Eq. (23), one obtains that \(\mathbf{X}_\gamma \partial \omega = 0\). I.e., \(\delta_r \partial \omega = 0\) if \(\beta, \delta_r = 0\). Thus it follows that \((\partial \omega)(\delta^1_r, \ldots, \delta^k_r) = 0\) whenever \(\beta, \delta^1_r \wedge \cdots \wedge \beta, \delta^k_r = 0\). Similarly, we have \((\partial \omega)(\alpha^1_r, \ldots, \alpha^k_r) = 0\) whenever \(\alpha, \delta^1_r \wedge \cdots \wedge \alpha, \delta^k_r = 0\).

Assume now that \(\beta, \delta^1_r \wedge \cdots \wedge \beta, \delta^k_r \neq 0\) and \(\alpha, \delta^1_r \wedge \cdots \wedge \alpha, \delta^k_r \neq 0\). Then there exists a local bisection \(\mathcal{L}\) through \(r\) such that the vectors \(\delta^1_r, \ldots, \delta^k_r\) are all tangent to the submanifold \(\mathcal{L}\). Thus, \(\forall i \in \{1, \ldots, k\}\), we have \(\varphi_{\mathcal{L}, \beta, \delta^i_r} = \alpha, \delta^i_r\). It thus follows that

\[
(\partial \omega)(\delta^1_r, \ldots, \delta^k_r) = \omega(\varphi_{\mathcal{L}, \beta, \delta^1_r}, \ldots, \varphi_{\mathcal{L}, \beta, \delta^k_r}) - \omega(\beta, \delta^1_r, \ldots, \beta, \delta^k_r) = 0.
\]

Therefore \(\partial \omega = 0\). This completes the proof of the lemma. \(\square\)

**Corollary 3.8** Let \(X \to \Gamma_0\) be a \(\Gamma\)-space and \(\omega \in \Omega^k(X)\). Then the following statements are equivalent

1. \(\partial \omega = 0\), where \(\partial\) is with respect to the transformation groupoid \(Q := \Gamma \times_{\beta, \Gamma_0, \pi} X \rightrightarrows X\);

2. \(\omega\) is basic with respect to the \(U_{\text{loc}}(\Gamma)\)-action.

**Proof.** Note that \(\omega \in \Omega^k(X)\) is basic with respect to the action of \(U_{\text{loc}}(\Gamma)\) if and only if it is basic with respect to the action of \(U_{\text{loc}}(Q)\). Then the conclusion follows immediately from Lemma 3.7 \(\square\)

Now we are ready to prove Proposition 3.6

**Proof.** 1) \(\iff\) 2) follows from Corollary 3.8

2) \(\Rightarrow\) 3) is trivial.

3) \(\Rightarrow\) 1) First, let us fix some notations. We denote by \(\Psi\) the natural map from (an open subset of) \(U_{\text{loc}}(\Gamma) \times P\) to \(Q\) given by

\[
\Psi(\mathcal{L}, p) = (\mathcal{L}(\pi(p)), p) \in \Gamma \times_{p, \Gamma_0, \pi} P \quad (= Q).
\]

Let \(\delta_q \in T_qQ\) be any tangent vector in \(Q\) and let \(q(t) = (\gamma(t), p(t)) \in Q\) be a path through \(q\) such that \(\dot{q}(0) = \delta_q\). Let \(\mathcal{L}(t)\) be a family of local bisections in \(U_{\text{loc}}(\Gamma)\) through \(\gamma(t) \in \Gamma\). Thus \(\Psi(\mathcal{L}(t), p(t)) = q(t)\). Hence we have \(\beta, \delta_q = \dot{p}(0)\) and \(\alpha, \delta_q = \frac{d}{dt}|_{t=0}(\mathcal{L}(t) \cdot p(t)) = \varphi_{\mathcal{L}, \dot{p}(0)} + \frac{d}{dt}|_{t=0}(\mathcal{L}(t) \cdot p(t))\), where \(\mathcal{L} = \mathcal{L}(0)\) and \(p(0) = p\). Since \(\frac{d}{dt}|_{t=0}(\mathcal{L}(t) \cdot p) \in \dot{A}_{\pi(\mathcal{L}, p)}\), by assumption we have

\[
\delta_q \mathcal{L} \partial \theta = \dot{p}(0) \mathcal{L} \partial \theta - \varphi_{\mathcal{L}, \dot{p}(0)} \mathcal{L} \partial \theta.
\]

From the decomposition \(T_pP = V_p \oplus H_p\), we write \(\dot{p}(0) = \dot{X}(p) + v_{\text{hor}}\), where \(v_{\text{hor}} \in H_p\) is a horizontal vector and \(\dot{X}\) is the vector field on \(P\) corresponding to the infinitesimal action of \(X \in \mathfrak{g}\). Therefore

\[
\delta_q \mathcal{L} \partial \theta = v_{\text{hor}} \mathcal{L} \partial \theta - \varphi_{\mathcal{L}, v_{\text{hor}}} \mathcal{L} \partial \theta + \dot{X}(p) \mathcal{L} \partial \theta - (\varphi_{\mathcal{L}, \dot{X}(p)} \mathcal{L} \partial \theta) = (\varphi_{\mathcal{L}, \dot{X}(p)}) \mathcal{L} \partial \theta.
\]

Since \(v_{\text{hor}}\) is horizontal and the distribution of horizontal subspaces is preserved under the \(U_{\text{loc}}(\Gamma)\)-action, we have \(v_{\text{hor}} \mathcal{L} \partial \theta = \varphi_{\mathcal{L}, v_{\text{hor}}} \mathcal{L} \partial \theta = 0\). Since the \(U_{\text{loc}}(\Gamma)\)-action commutes with the \(G\)-action, we have \(\varphi_{\mathcal{L}, \dot{X}(p)} = \dot{X}(\mathcal{L} \cdot p)\) and therefore

\[
\dot{X}(p) \mathcal{L} \partial \theta - \varphi_{\mathcal{L}, \dot{X}(p)} \mathcal{L} \partial \theta = X - X = 0.
\]

From Eq. (26), it follows that \(\partial \theta = 0\). \(\square\)
As a special case, let us consider an $H$-equivariant principal (right) $G$-bundle over $M$ as in Example 2.8, we have the following

**Corollary 3.9** Let $\Gamma$ be the transformation groupoid $H \times M \rightrightarrows M$, where $H$ acts on $M$ from the left. And let $P \to M$ be an $H$-equivariant principal (right) $G$-bundle over $M$ considered as a $G$-bundle over $\Gamma$. A one-form $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ defines a connection in the sense of Definition 3.5 if and only if $\theta$ is a connection 1-form for the $G$-bundle $P \to M$ and is basic with respect to the $H$-action.

In particular, if $H$ acts on $M$ freely and properly, a connection on the principal $G$-bundle $P \to M$ over $\Gamma$ is equivalent to a connection on the (ordinary) principal $G$-bundle $P/H \to M/H$.

**Proof.** Let $Q$ be the transformation groupoid $\Gamma \times_M P \rightrightarrows P$. It is simple to see that $Q$ is isomorphic to $H \times P \rightrightarrows P$. By $\beta$ and $\alpha$ we denote the source and target maps. $\forall X \in \mathfrak{h}$, by $\dot{X}$ we denote its corresponding right-invariant vector field on $H$ and by $\dot{X}$ the vector field on $P$ induced by the infinitesimal action of $X$. Thus $\forall p \in P, h \in H$ and $v \in T_p P$, we have

$$\theta(v) = (\beta^* \theta)(\dot{X}(h), v) = (\alpha^* \theta)(\dot{X}(h), v) = \theta(\dot{X}_p) + \theta(\varphi_{hv} v),$$

where $\varphi_{hv}$ denotes the $H$-action on $P$. Therefore $\partial \theta = 0$ if and only if $\theta(\varphi_{hv} v) = \theta(v)$, $\forall h \in H$, $v \in T_p P$ and $\theta(\dot{X}_p) = 0$, $\forall X \in \mathfrak{h}$, i.e. $\theta$ is $H$-basic. $\square$

Next we study some obvious obstructions to the existence of a connection. Recall that a principal $G$-bundle $P$ over $\Gamma \rightrightarrows \Gamma_0$ induces a Lie groupoid homomorphism $\rho : \Gamma \to \frac{P \times P}{G}$ given by Eq. (12). For any $m \in \Gamma_0$, we denote by $I_m = \Gamma_m^0$ the isotropy group at $m \in \Gamma_0$ and $(I_m)_0$ its connected component of the identity. A groupoid is said to be a foliation groupoid if all its isotropy groups are finite groups.

**Proposition 3.10** If the principal $G$-bundle $P$ over $\Gamma \rightrightarrows \Gamma_0$ admits a connection, then

1. $(I_m)_0$ acts on $P$ trivially,
2. $(I_m)_0 \subseteq \ker \rho$, and
3. if $\Gamma$ is proper, the image $\rho(\Gamma)$ is an immersed foliation subgroupoid of the gauge groupoid $\frac{P \times P}{G}$.

**Remark 3.11** We note that $\rho(\Gamma) \rightrightarrows \Gamma_0$ is neither a submanifold of $\frac{P \times P}{G}$ nor a Lie groupoid in general.

**Proof.** 1) For any $p \in P$, the infinitesimal action of the Lie algebra $\mathfrak{i}_m$ of $I_m$, where $m = \pi(p)$, defines a subspace $\mathfrak{i}_m \subset T_p P$. On the one hand, since $\mathfrak{i}_m \subset \mathfrak{a}_m$, by Proposition 3.6.3), we have $\mathfrak{i}_m \subset H_p$, where $H_p$ is the horizontal tangent space at $p$. On the other hand, since the action of $I_m$ maps $\pi^{-1}(m)$ to itself, $\mathfrak{i}_m$ must be a subspace of $T_p(\pi^{-1}(m))$ or $V_p$, the vertical tangent space at $p$, i.e., $\mathfrak{i}_m \subseteq V_p$. Since $H_p \cap V_p = 0$, it thus follows that $\mathfrak{i}_m = 0$. Hence, the Lie group $(I_m)_0$ acts on $P$ trivially.

2) is just a rephrase of 1).

3) Since $\rho$ is the identity map when being restricted to the unit space, the isotropy group $J_m$ of $\rho(\Gamma)$ over $m \in \Gamma_0$ is given by $J_m = \rho(I_m)$. Since $(I_m)_0 \subseteq \ker(\rho)$, it thus follows that $\rho$ passes to the quotient $J_m = \rho((I_m)_0)$. The groupoid $\Gamma$ is proper, so $I_m$ must be compact and therefore $\frac{I_m}{(I_m)_0}$ is discrete. It thus follows that the group $J_m$ is a finite group. Hence the groupoid $\rho(\Gamma)$ is a foliation groupoid. $\square$
Example 3.12 Let $G$ be a Lie group. Then $G \to \cdot$ can be considered as a principal $G$-bundle over the groupoid $G \rightrightarrows \cdot$, where the groupoid $G \rightrightarrows \cdot$ acts on $G$ by left translation. It is clear that connection does not exist unless $G$ is discrete.

3.4 Pull-back connections

In this subsection, we show that there is a natural notion of pull-back connections under generalized homomorphisms which generalizes the usual pull-back connections of principal bundles over manifolds.

Let $P$ be a principal $G$-bundle over $\Gamma \rightrightarrows \Gamma_0$, and $\varphi: \Gamma'_0 \xrightarrow{\tau} Z \xrightarrow{\sigma} \Gamma_0$ a generalized homomorphism from $\Gamma' \rightrightarrows \Gamma_0$ to $\Gamma \rightrightarrows \Gamma_0$. By $P'$, we denote the pull-back principal $G$-bundle over $\Gamma' \rightrightarrows \Gamma'_0$ constructed as in Proposition 2.13.

We need to introduce some notations. Let $\tilde{\tau}: Z \times_{\Gamma_0} P \to P'$ ($= (Z \times_{\Gamma_0} P)/\Gamma$) be the natural projection and $\tilde{\sigma}: Z \times_{\Gamma_0} \to P'$ the projection map to the second component.

**Proposition 3.13** If $\theta \in \Omega^1(P) \otimes g$ is a connection for the principal $G$-bundle $P$ over $\Gamma \rightrightarrows \Gamma_0$, then

1. there is a unique one-form $\theta' \in \Omega^1(P') \otimes g$ satisfying the condition:
   \[
   \tilde{\tau}^*\theta' = \tilde{\sigma}^*\theta; \tag{28}\n   \]
2. $\theta'$ defines a connection on the pull-back principal $G$-bundle $P'$ over $\Gamma' \rightrightarrows \Gamma'_0$, which is called the pull-back connection and is denoted by $\varphi^*\theta$;
3. the curvature $\Omega'$ of $\theta'$ and the curvature $\Omega$ of $\theta$ are related by
   \[
   \tilde{\tau}^*\Omega' = \tilde{\sigma}^*\Omega; \tag{29}\n   \]
4. $\varphi^*\theta$ is flat if $\theta$ is flat;
5. if $\psi$ and $\varphi$ are generalized homomorphisms from $\Gamma'' \rightrightarrows \Gamma'_0$ to $\Gamma' \rightrightarrows \Gamma'_0$ and from $\Gamma' \rightrightarrows \Gamma'_0$ to $\Gamma \rightrightarrows \Gamma_0$ respectively, then
   \[
   (\varphi \psi)^*\theta = \psi^*(\varphi^*\theta).\n   \]

**Proof.** 1) Note that $Z \times_{\Gamma_0} P \to \Gamma_0$ is a $\Gamma$-space and hence admits a $U_{\text{loc}}(\Gamma)$-action. Moreover we have
   \[
   (Z \times_{\Gamma_0} P)/U_{\text{loc}}(\Gamma) \simeq (Z \times_{\Gamma_0} P)/\Gamma = P'. \tag{30}\n   \]

Let $\theta_Z = \tilde{\sigma}^*\theta \in \Omega^1(Z \times_{\Gamma_0} P) \otimes g$. Since the projection $\tilde{\sigma}: Z \times_{\Gamma_0} P \to P$ commutes with the $U_{\text{loc}}(\Gamma)$-action, and $\theta$ is basic with respect to the $U_{\text{loc}}(\Gamma)$, the 1-form $\theta_Z$ is also basic. Hence there exists an unique $\theta' \in \Omega^1(P') \otimes g$ such that $\tilde{\tau}^*\theta' = \theta_Z$.

2) The triple $(Z \times_{\Gamma_0} P, \tilde{\sigma}, \tilde{\tau})$ defines a generalized homomorphism from $Q' \rightrightarrows P'$ to $Q \rightrightarrows P$, where $Q' \rightrightarrows P'$ is the transformation groupoid of the principal $G$-bundle $P' \to \Gamma'_0$ over $\Gamma' \rightrightarrows \Gamma'_0$ and $Q \rightrightarrows P$ is the pull-back groupoid of $Q' \rightrightarrows P'$ via $\tilde{\tau}: Z \times_{\Gamma_0} P \to P'$. By abuse of notation, we denote by $\tilde{\tau}$ and $\tilde{\sigma}$ the homomorphisms from $Q'[Z \times_{\Gamma_0} P] \rightrightarrows Z \times_{\Gamma_0} P$ to $Q' \rightrightarrows P'$ and from $Q'[Z \times_{\Gamma_0} P] \rightrightarrows Z \times_{\Gamma_0} P$ to $Q \rightrightarrows P$, respectively, defined as in Eqs. (8) and (9):

\[
\begin{array}{ccc}
Q' & \xrightarrow{\tilde{\tau}} & Q'\rightrightarrows Z \times_{\Gamma_0} P \\ \\
\downarrow & \Downarrow \tilde{\sigma} & \downarrow \\
Q & \xrightarrow{\tilde{\sigma}} & Q \rightrightarrows P \\
\end{array}
\]

\[
\begin{array}{ccc}
P' & \xrightarrow{\tilde{\tau}} & Z \times_{\Gamma_0} P \\ \\
\downarrow & \downarrow & \downarrow \\
P & \tilde{\sigma} & P \rightrightarrows P \\
\end{array}
\]
We can now prove 2) easily. Since \( \tilde{\tau} \) and \( \tilde{\sigma} \) are strict groupoid homomorphisms, we have,

\[ \tilde{\tau}^* \partial \theta' = \partial \tilde{\tau}^* \theta' = \partial \theta_Z, \tag{31} \]

and on the other hand,

\[ 0 = \tilde{\sigma}^* \partial \theta = \partial \theta_Z. \tag{32} \]

From Eqs. (31-32), it follows that \( \tilde{\tau}^* \partial \theta' = 0 \). Since \( \tilde{\tau} : Q'[Z \times \Gamma_0 P] \to Q' \) is a surjective submersion, we must have \( \partial \theta' = 0 \). This proves 2).

3) follows from Eq. (28). 4) is straightforward. 5) is a direct verification and is left to the reader. □

**Remark 3.14** If \( \varphi : \Gamma' \to \Gamma \) is a groupoid strict homomorphism, then the pull-back can be described more explicitly. In this case \( P' = \varphi^* P = \Gamma' \times_{\varphi, \Gamma_0, \pi} P \) and \( \theta' = \text{pr}^* \theta \), where \( \text{pr} : \Gamma' \times_{\varphi, \Gamma_0, \pi} P \to P \) is the projection on the second component.

**Corollary 3.15** If \( \Gamma' \) and \( \Gamma \) are Morita equivalent groupoids, there is an equivalence of categories of principal \( G \)-bundles with connections over \( \Gamma' \) and \( \Gamma \). There is also an equivalence of categories of principal \( G \)-bundles with flat connections over \( \Gamma' \) and \( \Gamma \).

Therefore, this allows us to speak about connections and flat connections of principal \( G \)-bundles over a differential stack. As we see from Example 3.12, connections may not always exist. However, for orbifolds, we will show that they always exist.

**Theorem 3.16** Any principal \( G \)-bundle over an orbifold admits a connection.

**Proof.** It is well-known [24] that an orbifold can be represented by a proper étale groupoid \( \Gamma \rightrightarrows \Gamma_0 \).

Let \( \theta \in \Omega^1(P) \otimes g \) be a pseudo-connection of the principal \( G \)-bundle \( P \xrightarrow{\pi} \Gamma_0 \). For any \( \gamma \in \Gamma \), there is always a unique local bisection through \( \gamma \) since \( \Gamma \) is an étale groupoid. We denote by \( \varphi_{\gamma} \) the local diffeomorphism on \( P \) induced by this local bisection. Note that \( \varphi_{\gamma} \) is defined in an open neighborhood of any \( p \in \pi^{-1}(\beta(\gamma)) \).

Since \( \Gamma \rightrightarrows \Gamma_0 \) is a Lie groupoid, there exists a (right) Haar system, denoted by \( \lambda = (\lambda_x)_{x \in \Gamma_0} \), where \( \lambda_x \) is a measure with support on \( \Gamma_x = \beta^{-1}(x) \) such that

- for all \( f \in C^\infty_c(\Gamma) \), \( x \mapsto \int_{\gamma \in \Gamma_x} f(\gamma) \lambda_x(d\gamma) \) is smooth, and
- the right translation by \( R_{\gamma'} : \Gamma_y \to \Gamma_x \) (where \( y = \alpha(\gamma') \) and \( x = \beta(\gamma') \) for all \( \gamma' \in \Gamma \)) preserves the measure, i.e.

\[ \int_{\gamma \in \Gamma_y} f(\gamma') \lambda_y(d\gamma) = \int_{\gamma \in \Gamma_x} f(\gamma) \lambda_x(d\gamma). \tag{33} \]

Let us recall that a smooth function \( c : \Gamma_0 \to \mathbb{R}_+ \) is called a cutoff function if

(i) for any \( x \in \Gamma_0 \), \( \int_{\gamma \in \Gamma_x} c(\alpha(\gamma)) \lambda_x(d\gamma) = 1 \);

(ii) for any \( K \subset \Gamma_0 \) compact, the support of \((c \circ \alpha)|_{\Gamma_K}\) is compact.
It is known [30, Proposition 6.7] that a cutoff function exists if and only if $\Gamma \Rightarrow \Gamma_0$ is proper. We can now define a connection $\tilde{\theta}$ on $P \overset{\pi}{\to} \Gamma_0$ by

$$\tilde{\theta}_p = \int_{\gamma \in \Gamma_\pi(p)} c(\alpha(\gamma)) \varphi_\gamma^* \theta_{\gamma.p} \lambda_{\pi(p)}(d\gamma).$$  \hfill (34)

Since $c(\alpha(\gamma))$ has a compact support in $\Gamma_\pi(p)$, the integral in Eq. (34) is well-defined. Let us check that $\tilde{\theta}$ is indeed a connection. It is easy to see, by Eq. (34), that $\tilde{\theta}$ is $G$-invariant. For any $X \in g$,

$$\tilde{\theta}(p) \mathcal{J} \tilde{\theta}_p = \int_{\gamma \in \Gamma_\pi(p)} c(\alpha(\gamma)) \left( \varphi_{\gamma^*} \tilde{X}(p) \right) \mathcal{J} \theta_{\gamma.p} \lambda_{\pi(p)}(d\gamma).$$  \hfill (35)

Since $\varphi_{\gamma^*} \tilde{X}(p) = \tilde{X}(\gamma \cdot p)$ and $\tilde{X}(\gamma \cdot p) \mathcal{J} \theta_{\gamma.p} = X$, Eq. (35) implies that $\tilde{X}(p) \mathcal{J} \tilde{\theta}_p = X$. We check that $\tilde{\theta}$ is basic with respect to the $U_{\text{loc}}(\Gamma)$-action. Since $\Gamma$ is étale, we only need to check that $\tilde{\theta}$ is invariant under the $U_{\text{loc}}(\Gamma)$-action, or $\varphi_{\gamma^*} \tilde{\theta} = \tilde{\theta}$ for any $\gamma \in \Gamma$.

By Eq. (34), we have, for any $\gamma'$ with $\alpha(\gamma') = \pi(p)$,

$$\varphi_{\gamma'}^* \tilde{\theta}_p = \int_{\gamma \in \Gamma_\pi(p)} c(\alpha(\gamma)) \varphi_{\gamma'}^* \varphi_\gamma^* \theta_{\gamma.p} \lambda_{\pi(p)}(d\gamma).$$  \hfill (36)

Now, using the relations $\varphi_{\gamma'}^* \varphi_\gamma^* = \varphi_{\gamma^*}^*$, $c(\alpha(\gamma)) = c(\alpha(\gamma'))$, $\theta_{\gamma.p} = \theta_{\gamma' \cdot p'}$, where $p' = \gamma'^{-1} \cdot p$, we have

$$\varphi_{\gamma'}^* \tilde{\theta}_p = \int_{\gamma \in \Gamma_\pi(p)} c(\alpha(\gamma')) \varphi_{\gamma'^*}^* \theta_{\gamma'^* \cdot p'} \lambda_{\pi(p)}(d\gamma).$$  \hfill (37)

Since the Haar measure $\lambda_{\pi(p)}(d\gamma)$ is invariant under the right translation, Eqs. (37) and (33) imply that

$$\varphi_{\gamma'}^* \tilde{\theta}_p = \int_{\gamma \in \Gamma_\pi(p')} c(\alpha(\gamma)) \varphi(\gamma)^* \theta_{\gamma' \cdot p'} \lambda_{\pi(p')}(d\gamma).$$  \hfill (38)

The right hand side of Eq. (38) is $\tilde{\theta}_{p'}$ by definition. Therefore we have proved that $\tilde{\theta}_{p'} = \varphi_{\gamma'}^* \tilde{\theta}_p$. By Corollary 3.8 we have $\partial \tilde{\theta} = 0$. This completes the proof. □

3.5 Holonomy map

As usual, a flat connection on a principal $G$-bundle over a groupoid $\Gamma$ induces a representation of the fundamental group of $\Gamma$ to $G$ via the holonomy map.

Let us recall the definition of the fundamental group of a topological groupoid. We need a few preliminaries.

Proposition 3.17 Let $\Gamma$ be a topological groupoid. There is a bijection between

(a) equivalence classes of generalized homomorphisms from $\{\text{pt}\}$ to $\Gamma$, and

(b) elements in the quotient space $\Gamma_0/\Gamma$.  

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Proof. A generalized homomorphism from \( \{pt\} \) to \( \Gamma \) is given by a space \( Z \), a continuous map \( p: Z \to \Gamma_0 \) and a right action of \( \Gamma \) on \( Z \) whose momentum map is \( p \), such that \( Z/\Gamma \simeq \{pt\} \). The element of \( \Gamma_0/\Gamma \) corresponding to that generalized homomorphism is the class of \( p(z) \) for any \( z \in Z \).

Conversely, if \( \bar{x} \in \Gamma_0/\Gamma \), choose a representative \( x \in \Gamma_0 \). Then \( Z = \alpha^{-1}(x) \) together with \( p = \beta : Z \to \Gamma_0 \) and the multiplication on the right as the right action on \( Z \), is a generalized homomorphism from \( \{pt\} \) to \( \Gamma \). It is clear that if \( x, y \in \Gamma_0 \) are in the same groupoid orbit their corresponding generalized homomorphisms are equivalent. \( \square \)

We call a generalized homomorphism \( \{pt\} \xrightarrow{} X \to \Gamma_0 \) a point in \( \Gamma \). A pointed groupoid is a pair \((\Gamma, X)\), where \( \Gamma \) is a groupoid, and \( \{pt\} \xrightarrow{} X \to \Gamma_0 \) a point in \( \Gamma \). If \( \Gamma_0 \hookrightarrow Z \xrightarrow{\phi} \Gamma_0 \) is a generalized homomorphism from a groupoid \( \Gamma' \) to a groupoid \( \Gamma \), and \( X' \) is a point in \( \Gamma' \), then one can define the image \( Z \xrightarrow{} X' \), a point in \( \Gamma' \) by using the composition of generalized homomorphisms

\[
\{pt\} \xrightarrow{\phi} \Gamma' \xrightarrow{\gamma} \Gamma.
\]

Suppose that \((\Gamma, X)\) and \((\Gamma', X')\) are two pointed groupoids. Then a pointed generalized homomorphism from \((\Gamma', X')\) to \((\Gamma, X)\) is a pair \((Z, \varphi)\), where \( Z \) is a generalized homomorphism from \( \Gamma' \) to \( \Gamma \) and \( \varphi \) is an equivalence from \( Z \xrightarrow{} X' \) to \( X \). It is clear that pointed groupoids with pointed generalized homomorphisms form a category.

**Remark 3.18** Consider the special case when \( \Gamma' \) is a topological space \( M \), and \( X' \) is a basepoint \( m_0 \in M \). For any \( x_0 \in \Gamma_0 \), let \( \{pt\} \xrightarrow{} \alpha^{-1}(x_0) \to \Gamma_0 \) be the point in \( \Gamma \) as in the proof of Proposition 3.17, which is denoted by \( x_0 \) by abuse of notation. Then a pointed generalized homomorphism from \((M, m_0)\) to \((\Gamma, x_0)\) is equivalent to giving a generalized homomorphism \( \overline{\epsilon} : Z \xrightarrow{\tau} \Gamma_0 \) and a point \( z_0 \in Z \) such that \( \tau(z_0) = m_0 \) and \( \sigma(z_0) = x_0 \). Here \( z_0 \in Z \) is the unique element corresponding to the inverse image of \( \epsilon(x_0) \) under the isomorphism \( \varphi : \tau^{-1}(m_0) \to \alpha^{-1}(x_0) \) (where \( \epsilon : \Gamma_0 \to \Gamma \) is the unit map).

We say that two pointed generalized homomorphisms \((Z_0, \varphi_0)\) and \((Z_1, \varphi_1)\) from \((M, m_0)\) to \((\Gamma, X)\) are homotopic if there exists a pointed generalized homomorphism \((Z, \varphi)\) from \(((M \times [0, 1])/(\{m_0\} \times [0, 1]), m_0)\) to \((\Gamma, X)\) such that

- there exist equivalences \( i_0 : Z_0 \to Z|_{M \times \{0\}} \) and \( i_1 : Z_1 \to Z|_{M \times \{1\}} \);
- via the identifications \( i_0 \) and \( i_1 \) above, the restriction of \( \varphi \) to \( Z_0 \xrightarrow{} X' \) and \( Z_1 \xrightarrow{} X' \) is equal to \( \varphi_0 \) and \( \varphi_1 \), respectively.

We denote by \([[(M, m_0), (\Gamma, X)]\) the set of homotopy classes of pointed generalized homomorphisms from \((M, m_0)\) to \((\Gamma, X)\).

**Remark 3.19** In the non-pointed situation, the set of homotopy classes of generalized homomorphisms from a space \( M \) to a groupoid \( \Gamma \) is also called the set of concordance classes of principal \( \Gamma \)-bundles over \( M \), or \( \Gamma \)-structures on \( M \).

Let \((M', m_0')\) be another topological space. By \((M, m_0) \vee (M', m_0')\), we denote the pointed space \((M \cup M')/(m_0 \sim m_0')\) with the basepoint \( m_0 = m_0' \). It is clear that if \((Z, \varphi)\) (resp. \((Z', \varphi')\)) is a pointed generalized homomorphism from \((M, m_0)\) (resp. \((M', m_0')\)) to \((\Gamma, X)\) then one can form a generalized homomorphism from \((M, m_0) \vee (M', m_0')\) to \((\Gamma, X)\). For \( M = M' = S^n \), using the usual map \((M, \ast) \to (M, \ast) \vee (M, \ast)\) we get a group structure on \( \pi_n(\Gamma \ast X) := [(S^n, \ast), (\Gamma, X)]\). It is clear that if \( X \) and \( Y \) are equivalent
points in $\Gamma$, then $\pi_n(\Gamma, X)$ is isomorphic to $\pi_n(\Gamma, Y)$. The group $\pi_1(\Gamma, X)$ is called the fundamental group of $\Gamma$. For any $x \in \Gamma_0$, consider the point $\{pt\} \mapsto x^{-1}(x) \rightarrow \Gamma_0$ in $\Gamma$ as in the proof of Proposition 3.17. We denote by $\pi_1(\Gamma, x)$ its corresponding fundamental group. Since the isomorphism class of $\pi_1(\Gamma, x)$ only depends on the class $\bar{x} \in \Gamma_0/\Gamma$, we also denote it by $\pi_1(\Gamma, \bar{x})$ as well.

**Remark 3.20** We recently learned that the fundamental group of a stack is also being introduced by Behrang Noohi [27]. It will become clear below (Proposition 3.21) that if $x$ and $y$ are in the same path-connected component of $\Gamma_0$ then $\pi_1(\Gamma, x)$ and $\pi_1(\Gamma, y)$ are isomorphic (but the isomorphism may be not canonical). As a consequence, we have

**Proposition 3.21** Let $\Gamma$ be a topological groupoid. Suppose that $\Gamma_0/\Gamma$ is path connected, then $\pi_1(\Gamma, x)$ is independent of the choice of $x$ up to isomorphism.

**Remark 3.22** It is clear by construction that if $(\Gamma, X)$ and $(\Gamma', X')$ are two Morita-equivalent pointed groupoids then $\pi_n(\Gamma, X)$ and $\pi_n(\Gamma', X')$ are isomorphic.

**Remark 3.23** By a generalized path on $\Gamma$, we mean a generalized homomorphism from an interval $[a, b]$ to $\Gamma$. If $P$ and $P'$ are generalized paths from $[a, b]$ and $[b, c]$ to $\Gamma$ respectively, and if an equivalence $\psi$ from $P \circ b$ to $P' \circ b$ is given, then the composition of $P$ and $P'$ is the generalized path $P''$ from $[a, c]$ to $\Gamma$ defined by $(P \cup P')/(p \sim \psi(p))$.

We will see below that the homotopy groups of $\Gamma$ are in fact isomorphic to the homotopy groups of the classifying space $B\Gamma$, which is the fat geometric realization of the simplicial space $\Gamma$.

Recall that the space $B\Gamma$ is defined by $\lim_{\longrightarrow} (B\Gamma)_n$, where

$$(B\Gamma)_n = \left( \coprod_{0 \leq k \leq n} \Gamma_k \times \Delta_k \right) / \sim,$$

and the equivalence relation $\sim$ is generated by

$$(\gamma, \tilde{e}^k_i(t)) \sim (e^k_i(\gamma), t) \quad \text{and} \quad (\gamma, \tilde{\eta}^k_i(t')) \sim (\eta^k_i(\gamma), t') \quad \forall \gamma \in \Gamma_k, \forall t \in \Delta_{k-1}, \forall t' \in \Delta_{k+1}.$$

Here $e^k_i$ and $\eta^k_i$ are the face and degeneracy maps of $\Gamma$, defined by Eqs. (7.11), while $\tilde{e}^k_i$ and $\tilde{\eta}^k_i$ are the face and degeneracy maps of $\Delta$, defined by Eqs. (7.14, 15). It is known that $B\Gamma$ is naturally endowed with a topology [16, 28].

**Proposition 3.24** Let $(M, m_0)$ be a locally compact $\sigma$-compact space, $\Gamma$ a topological groupoid and $x_0 \in \Gamma_0$. Then there is a canonical isomorphism

$$[(M, m_0), (\Gamma, x_0)] \cong [(M, m_0), (B\Gamma, x_0)].$$

(In the theorem above, $\Gamma_0$ is identified to a subspace of $B\Gamma$).

**Proof.** The proof is almost identical to that of [216, Theorem 1.7] or [31], where the proposition is proved in the unpointed case.□
Corollary 3.25 For all $n \in \mathbb{N}$ and $x \in \Gamma_0$, $\pi_n(\Gamma, x) \cong \pi_n(B\Gamma, x)$.

Before we introduce the holonomy map, we need to characterize geometrically flat $G$-principal bundles over a groupoid $\Gamma \rightrightarrows \Gamma_0$ (i.e. $G$-principal bundles which admit a flat connection).

Proposition 3.26 Let $P$ be a $G$-principal bundle over $\Gamma \rightrightarrows \Gamma_0$. The following are equivalent:

(i) $P$ is flat;

(ii) there exist an open cover $(U_i)$ of $\Gamma_0$ and a local trivialization $\varphi_i: U_i \times G \rightarrow P|_{U_i}$ such that the transition function $\psi_{ij}: \Gamma_{U_i}^{U_j} \rightarrow G$ defined by the equation

$$
\gamma \cdot \varphi_j(\beta(\gamma), g) = \varphi_i(\alpha(\gamma), \psi_{ij}(\gamma)g)
$$

is a constant map for any $\Gamma_{U_i}^{U_j}$;

(iii) As a $G$-principal bundle over $\Gamma$, $P$ is induced from a $G^d$-principal bundle $P'$ under the natural group homomorphism $G^d \rightarrow G$, where $G^d$ is the group $G$ endowed with the discrete topology.

Proof. (i) $\implies$ (ii): Since $P$ is flat as a $G$-principal bundle over $\Gamma_0$, there exists a local trivialization $\varphi_i: U_i \times G \rightarrow P|_{U_i}$ such that $\varphi_i^*\theta$ is the trivial connection on $U_i \times G$ for all $i$ (i.e. $\varphi_i(U_i \times \{g\})$ is horizontal for all $g \in G$). Since $\partial\theta = 0$, horizontal sections are invariant under the $U(\Gamma)$-action according to Lemma 3.7. It thus follows that $\psi_{ij}(\gamma)$ must be independent of $\gamma$.

(ii) $\implies$ (iii): Let $P'$ be the $G^d$-principal bundle over $\Gamma$ defined by the cocycle $\{\psi_{ij}\}$. More explicitly, let $\Gamma[U_i] : \coprod U_i$ be the pull back groupoid of $\Gamma \rightrightarrows \Gamma_0$ via the open covering $\coprod U_i \rightarrow \Gamma_0$. Then $\Gamma[U_i] = \coprod \Gamma_{U_i}^{U_j}$ and $\coprod \psi_{ij}: \coprod \Gamma_{U_i}^{U_j} \rightarrow G$ is a groupoid homomorphism, which is constant on each $\Gamma_{U_i}^{U_j}$. Therefore, it defines a continuous groupoid homomorphism from $\Gamma[U_i]$ to $G^d$. Composing the Morita equivalence from $\Gamma$ to $\Gamma[U_i]$ with this homomorphism, one obtains a generalized homomorphism from $\Gamma$ to $G^d$, which corresponds to a $G^d$-principal bundle $P'$ over $\Gamma$.

(iii) $\implies$ (i) By assumption, there is a local trivialization $\varphi_i: U_i \times G \rightarrow P|_{U_i}$ such that the transition functions $\psi_{ij}: \Gamma_{U_i}^{U_j} \rightarrow G$ defined by Eq. (39) are constant on each $\Gamma_{U_i}^{U_j}$. Therefore $\varphi_i(U_i \times \{g\})$, for all $g \in G$, induces a well-defined horizontal distribution on $P$, which is clearly a flat connection according to Proposition 3.6.

We are now ready to prove the main theorem of this subsection:

Theorem 3.27 Let $P \xrightarrow{\pi} \Gamma_0$ be a flat $G$-principal bundle over a Lie groupoid $\Gamma \rightrightarrows \Gamma_0$, $x \in \Gamma_0$ and $p \in P$ such that $\pi(p) = x$. There is a group homomorphism

$$
\text{Hol}_{\Gamma,x,p}: \pi_1(\Gamma_*, x) \rightarrow G,
$$

called the holonomy map, defined as follows. For every generalized pointed loop $(Z, z_0)$ from $(S^1, *)$ to $(\Gamma, x)$, let $P_{S^1} = Z \times_{\Gamma} \Gamma$ be the pull back $G^d$-principal bundle over $S^1$,.
where \( P' \) is the corresponding \( G^d \)-principal bundle over \( \Gamma \) as in Proposition 3.26. Let \( p' = (z_0, p) \). Then

\[
\text{Hol}_{\Gamma, x, p}(Z, z_0) := \text{Hol}_{S^1, x, p'}(\text{Id}_{S^1}).
\]

In other words, if \( f(t) \) is the horizontal lift (with respect to the pull back connection) of the path \( \text{Id}_{S^1} \) on \( P_{S^1} \) starting at \( p' \), then \( \text{Hol}_{\Gamma, x, p}(Z, z_0) \) is the element \( g \in G \) satisfying \( f(1) = f(0)g \).

**Proof.** It remains to prove that the holonomy map only depends on the homotopy class of the generalized path. This clearly follows from Proposition 3.26(iii) and the continuity of the holonomy map. \( \square \)

**Remark 3.28** As usual, if \( x \) and \( y \) are in the same path-connected component, then \( \pi_1(\Gamma, x) \) and \( \pi_1(\Gamma, y) \) are isomorphic, the isomorphism being well-defined up to an inner automorphism. Via this identification, there exists \( g \in G \) such that \( \text{Hol}_{\Gamma, y} = \text{Ad}_g \circ \text{Hol}_{\Gamma, x} \).

### 4 Chern-Weil map for \( G \)-differential simplicial algebras

In this section, we present an algebraic construction of Chern-Weil map for \( G \)-differential simplicial algebras. By an algebra, we always mean an \( \mathbb{N} \)-graded super-commutative algebra.

#### 4.1 Simplicial algebras

We start with recalling the definition of \( G \)-differential algebras [15].

**Definition 4.1** A \( G \)-differential algebra is an \( \mathbb{N} \)-graded differential algebra \((A, d)\) (\( d \) is of degree 1) equipped with a \( G \)-action and a linear map \( i \), called the contraction, from \( g \) to the space of derivations of degree \(-1\) satisfying

- the action of \( G \) preserves the \( \mathbb{N} \)-graded differential algebra structure;
- the following identities hold: \( \forall X, Y \in g \) and \( \forall g \in G \)

\[
\begin{align*}
  i_X i_Y + i_Y i_X &= 0, \\
  g \circ i_X g^{-1} &= i_{\text{Ad}_g(X)}, \\
  i_X d + di_X &= L_X,
\end{align*}
\]

where \( L_X \) denotes the infinitesimal \( g \)-action.

An element \( a \in A \) is called basic if \( L_X a = 0 \) and \( i_X a = 0 \) for any \( X \in g \).

A standard example of \( G \)-differential algebra is the Weil algebra.

**Example 4.2** (Weil algebra) Let \( g \) be the Lie algebra of a Lie group \( G \), and let \( W(g) = S(g^*) \otimes \Lambda g^* \) be the tensor product of \( S(g^*) \) and \( \Lambda g^* \).

- The degree of an element in \( S^k(g^*) \otimes \Lambda^l g^* \) is \( 2k + l \). With respect to this grading, \( W(g) \) is clearly super-commutative.
• On generators, the differential $d$ is defined by
\[
d(1 \otimes \xi^i) = \xi^i \otimes 1 - \frac{1}{2} \sum_{j,k} f^i_{jk} (1 \otimes (\xi^j \wedge \xi^k))
\]
\[
d(\xi^i \otimes 1) = \sum_{j,k} f^i_{jk} (\xi^j \otimes \xi^k),
\]
where $(\xi^i)_{i=1}^{\dim(\mathfrak{g})}$ is a basis of $\mathfrak{g}^*$ and $(f^i_{jk})_{i,j,k=1}^{\dim(\mathfrak{g})}$ are the structure constants of the Lie algebra $\mathfrak{g}$ with respect to the dual basis of $(\xi^i)_{i=1}^{\dim(\mathfrak{g})}$.

• The Lie group $G$ acts on $W(\mathfrak{g})$ by coadjoint action.

• For any $X \in \mathfrak{g}$, the contraction $i_X$ is defined to be an odd derivation, which, on the generators, is given by
\[
i_X(\xi \otimes 1) = 0 \quad \text{and} \quad i_X(1 \otimes \xi) = \xi(X).
\]

In this case, the space of basic elements can be identified with $S(\mathfrak{g}^*)^G$.

Another example is the following:

**Example 4.3** Let $P$ be a $G$-space. Then the algebra $\Omega(P)$ of differential forms on $P$ with the de Rham differential $d$, the natural $G$-action and the contraction: $i_X \omega = \tilde{X} \underline{\omega}$, where $\tilde{X}$ is the infinitesimal vector field on $P$ corresponding to $X \in \mathfrak{g}$, is a $G$-differential algebra. In this case, the space of basic elements can be identified with $\Omega(M)$.

A homomorphism between two $G$-differential algebras $A$ and $B$ is an $\mathbb{N}$-graded differential algebra homomorphism which commutes with the $G$-actions and the contractions. It is easy to see that the $G$-differential algebras form a category.

**Definition 4.4**
1. A $G$-differential simplicial algebra $A_\bullet$ is a sequence of differential $\mathbb{N}$-graded algebras $(A_n)_{n \in \mathbb{N}}$ together with a sequence of morphisms of $G$-differential algebras, called face maps $\epsilon^n_i : A_{n-1} \to A_n$, $i = 0, \ldots, n$ and degeneracy maps $\eta^n_i : A_{n+1} \to A_n$, $i = 0, \ldots, n$, which satisfy the co-simplicial relations: $\epsilon^n_i \epsilon^{n-1}_j = \epsilon^n_{i-1} \epsilon^n_{j+1}$ if $i < j$, $\eta^n_j \eta^n_{j+1} = \eta^n_{j+1} \eta^n_{j+1}$ if $i = j$ or $\eta^n_j \epsilon^{n+1}_i = \epsilon^n_i \eta^n_{i+1}$ if $i < j$, $\eta^n_j \epsilon^{n+1}_i = \epsilon^n_i \eta^n_{i-1}$ if $i > j$ and $\eta^n_j \epsilon^{n+1}_i = \eta^n_i \epsilon^{n+1}_i = \text{id}_{A_n}$.

2. A differential simplicial algebra is a $G$-differential simplicial algebra where $G$ is the trivial group.

**Example 4.5** Given a ($G$-)simplicial manifold $M_\bullet = (M_n)_{n \in \mathbb{N}}$, let $A_n = \Omega(M_n)$ be the differential algebra of differential forms on $M_n$ equipped with the de Rham differential. It is simple to see that $(A_n)_{n \in \mathbb{N}}$ is a ($G$-)differential simplicial algebra with the face and degeneracy maps being the pull-back of the face and degeneracy maps of the simplicial manifold.

**Example 4.6** In particular, given a simplicial $G$-bundle $Q_\bullet$ over $\Gamma_\bullet$, according to Example 4.5 $(\Omega(Q_n))_{n \in \mathbb{N}}$ is a $G$-differential simplicial algebra.

Equivalently (see Proposition 2.1), if $P$ is a simplicial $G$-bundle over the groupoid $\Gamma \rightrightarrows \Gamma_\ast$, and $Q \rightrightarrows P$ is the transformation groupoid, then $Q_\bullet = (Q_n)_{n \in \mathbb{N}}$ is a principal $G$-bundle over the simplicial manifold $\Gamma_\bullet$. Therefore one obtains a $G$-differential simplicial algebra $(\Omega(Q_n))_{n \in \mathbb{N}}$. 

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The following construction gives us a useful way of constructing $G$-differential simplicial algebras.

Let $A$ be a $G$-differential unital algebra. Set $A_n = A^\otimes (n+1), \forall n \in \mathbb{N}$ be the tensor algebra of graded algebras constructed according to the Quillen rule [14]. The $G$-action on $A$ extends naturally to an action of $G$ on $A_n$ by the diagonal action. Similarly, the contraction operation extends naturally to $A_n$ as well using the derivation rule. Define $\epsilon^n_i : A^\otimes n \to A^\otimes n+1$ and $\eta^n_i : A^\otimes n+2 \to A^\otimes n+1$ by

$$\epsilon^n_i(x_0 \otimes \cdots \otimes x_{n-1}) = x_0 \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_{n-1},$$

and

$$\eta^n_i(x_0 \otimes \cdots \otimes x_{n+1}) = x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_{n+1}.$$ 

It is simple to check that $\epsilon^n_i$ and $\eta^n_i$ satisfy all the compatibility conditions of face maps and degeneracy maps. Indeed we have the following:

**Proposition 4.7** If $A$ is a $G$-differential unital algebra, then the sequence $A_\bullet := (A^\otimes (n+1))_{n \in \mathbb{N}}$ with the structures described above is a $G$-differential simplicial algebra.

**Proof.** The proof is a direct verification and is left for the reader. $\square$

In particular, when $A$ is the Weil algebra $W(g)$, one obtains the simplicial Weil algebra $W(g)_\bullet$ (see [21]):

**Definition 4.8** The simplicial Weil algebra $W(g)_\bullet := (W(g)^\otimes (n+1))_{n \in \mathbb{N}}$ is the $G$-differential simplicial algebra constructed as in Proposition 4.7 when $A$ is taken to be the Weil algebra $W(g)$.

One can also define homomorphisms between (resp. G-differential) simplicial algebras so that one obtains a category.

**Definition 4.9** Let $A_\bullet$ and $B_\bullet$ be differential simplicial algebras (resp. G-differential simplicial algebras). A homomorphism is a sequence of differential graded algebra homomorphisms (resp. G-differential algebra homomorphisms) $\varphi_n : A_n \to B_n$ that commute with the degeneracy and face maps.

### 4.2 Cohomology and fat realization of differential simplicial algebras

Recall that the cohomology of an $\mathbb{N}$-graded differential algebra $(A,d)$ is

$$H^*(A) = \text{Ker } d/\text{Im } d,$$

and that $H(\cdot)$ is a covariant functor from the category of $\mathbb{N}$-graded differential algebras to the category of graded algebras.

Similarly, one can introduce cohomology for simplicial differential algebras. Let $A_\bullet = (A_n)_{n \in \mathbb{N}}$ be a differential simplicial algebra. By $A^k_n$, we denote the space of elements of degree $k$ in $A_n$. The collection $\{A^k_n\mid (k,n) \in \mathbb{N}^2\}$ becomes a double complex with respect to the differential $d_n : A^k_n \to A^{k+1}_n$ of $A_n$ and the map $\delta : A^k_n \to A^{k+1}_{n+1}$, which is defined to be the alternate sum of the face maps. We denote by $H^*(A_\bullet)$, the cohomology group with
More precisely, since \( \Omega(\Delta^n_M) \) is a realization of a differential algebra and then the cohomology functor, or we can apply the cohomology functor, namely via the fat realization. First, let us introduce the fat realization of differential forms on a simplicial manifold \( M \) completion) the space of differential forms on the geometric fat realization of \( M \) is a differential algebra, called the de Rham cohomology of the simplicial manifold as in Section 3.1.

**Proposition 4.10** \( H^*(\cdot) \) is a covariant functor from the category of simplicial differential algebras to the category of \( \mathbb{N} \)-graded algebras.

There is an alternative way to talk about the cohomology of a simplicial differential algebra, namely via the fat realization. First, let us introduce the fat realization of differential forms on a simplicial manifold \( M = (M_n)_{n \in \mathbb{N}} \), \( H^*(A^*) \) reduces to the de Rham cohomology of the simplicial manifold as in Section 3.1.

**Example 4.11** When \( A_* \) is the simplicial differential algebra of differential forms on a simplicial manifold \( M_* = (M_n)_{n \in \mathbb{N}} \), the fat realization \( \| A_* \| \) is (up to some topological completion) the space of differential forms on the geometric fat realization of \( M_* \). More precisely, since \( \Omega(\Delta_n) \otimes \Omega(M_n) \) is a dense subset of \( \Omega(\Delta_n \times M_n) \), the fat realization \( \| A_* \| \) is a dense subset of the space of simplicial differential forms on the geometric fat realization of \( M_* \).

Given a simplicial differential algebra, we can apply the fat realization functor to get a differential algebra and then the cohomology functor, or we can apply the cohomology functor.
functor directly. To compare these two approaches, we need the following integration map following \[13\]:

\[ I : \parallel A \parallel \to \bigoplus_n A_n \quad (48) \]

\[ (k_n)_{n \in \mathbb{N}} \to \sum_n \int \Delta_n k_n. \quad (49) \]

Note that the sum on the right-hand side of Eq. (48) is always finite. Hence \( I \) is well-defined.

**Proposition 4.12**

1. The integration map induces an isomorphism

\[ I : H^*(\parallel A \parallel) \simeq H^*(A). \]

2. If \( \varphi : A \to B \) is a homomorphism of simplicial differential algebras, then the following diagram commutes

\[
\begin{array}{ccc}
\parallel A \parallel & \xrightarrow{\parallel \varphi \parallel} & \parallel B \parallel \\
I \downarrow & & \downarrow I \\
A & \xrightarrow{\varphi} & B
\end{array}
\]

3. Therefore one obtains a commutative diagram

\[
\begin{array}{ccc}
H^*(\parallel A \parallel) & \xrightarrow{H(\parallel \varphi \parallel)} & H^*(\parallel B \parallel) \\
I \downarrow & & \downarrow I \\
H^*(A) & \xrightarrow{H(\varphi)} & H^*(B)
\end{array}
\]

**Proof.** The proof is similar to that of Proposition 6.1 in \[13\] and we omit it here. \( \square \)

We summarize some important functorial properties of the above constructions in the following

**Proposition 4.13**

1. The fat realization \( \parallel \cdot \parallel \) is a covariant functor from the category of differential simplicial algebras (resp. the category of \( G \)-differential simplicial algebras) to the category of differential algebras (resp. the category of \( G \)-differential algebras). Both functors will be denoted by \( R \).

2. Taking basic elements is a covariant functor from the category of \( G \)-differential simplicial algebras (resp. the category of \( G \)-differential algebras) to the category of differential simplicial algebras (resp. the category of differential algebras). Both functors will be denoted by \( B \).

3. The two functors \( R \circ B \) and \( B \circ R \) are isomorphic as functors from the category of \( G \)-differential simplicial algebras to the category of differential algebras.

From now on, we will denote by \( A_{\text{basic}} \parallel \cdot \parallel \) and \( H^*(A) \) the image of a \( G \)-differential simplicial algebra \( A \) under the functors \( B \), \( R \) and \( H(\cdot) \) respectively. Accordingly, we will denote by \( \varphi_{\text{basic}} \parallel \varphi \parallel \) and \( H(\varphi) \) the image of a homomorphism \( \varphi \) of \( G \)-differential simplicial algebras under these functors.
5 Chern-Weil map

5.1 Chern-Weil map for $G$-differential simplicial algebras: first construction

Following [1], a connection on a $G$-differential algebra $(A,d)$ is an element $\theta \in (A^1 \otimes g)^G$ such that for any $X \in g$,

$$i_X \theta = 1 \otimes X.$$  \hspace{1cm} (50)

Note that if $\varphi : A \rightarrow B$ is a homomorphism of $G$-differential algebras and $\theta$ is a connection on $A$, then $\varphi(\theta) \in (B^1 \otimes g)^G$ is a connection on $B$.

A connection induces a homomorphism of $G$-differential algebras

$$c_\theta : W(g) \rightarrow A,$$  \hspace{1cm} (51)

which is defined on generators as follows. For any $\xi \in g^*$,

$$c_\theta(1 \otimes \xi) = \langle 1 \otimes \xi, \theta \rangle,$$

$$c_\theta(\xi \otimes 1) = \langle 1 \otimes \xi, d\theta + \frac{1}{2}[\theta, \theta] \rangle.$$  

Applying the basic functor, we get a morphism $z_\theta : S(g^*)^G \rightarrow A^{\text{basic}}$, where the differential on $S(g^*)^G$ is the zero differential. The map $z_\theta$ takes values in the subspace $Z(A^{\text{basic}})$ of $A^{\text{basic}}$. More precisely, we have

$$z_\theta(f) = f(\Omega), \quad \forall f \in S(g^*)^G$$  \hspace{1cm} (52)

where $\Omega = d\theta + \frac{1}{2}[\theta, \theta] \in (A^2 \otimes g)^G$ is the curvature of $\theta$. We call $z_\theta$ the Chern-Weil map on the cochain level [15].

Applying the cohomology functor, one obtains a morphism $w_\theta : S(g^*)^G \rightarrow H^*(A^{\text{basic}})$ called the Chern-Weil map. The following result is well-known [15].

**Proposition 5.1**

1. The Chern-Weil map $w_\theta : S(g^*)^G \rightarrow H^*(A^{\text{basic}})$ does not depend on the connection. In the sequel, we denote $w_\theta$ by $w$.

2. If $\varphi : A \rightarrow B$ is a homomorphism of $G$-differential algebras and $\theta$ is a connection on $A$, then

$$c_{\varphi(\theta)} = \varphi \circ c_\theta, \quad z_{\varphi(\theta)} = \varphi \circ z_\theta, \quad \text{and} \quad w_{\varphi(\theta)} = \varphi \circ w_\theta.$$

**Definition 5.2** Let $A_* := (A_n)_{n \in \mathbb{N}}$ be a $G$-differential simplicial algebra. We call a connection on $A_0$ a pseudo-connection on $A_*$. Starting with a pseudo-connection on $A_*$, we will construct a connection on its fat realization $\|A_*\|$. Let us define $(n+1)$ maps from $A_0$ to $A_n$ by

$$p_0^n = e_0^n \cdots e_i \cdots e_1,$$

$$p_i^n = e_0^n \cdots e_0^{n-i+1} \cdots e_{n-i} \cdots e_1, \quad 1 \leq i < n,$$

$$p_n^n = e_0^n \cdots e_0 \cdots e_0.$$  

Now define, for any $\alpha \in A_0$,

$$\alpha_n = \sum_{i=0}^n t_i \otimes p_i^n(\alpha) \in \Omega^0(\Delta_n) \otimes A_n.$$  \hspace{1cm} (53)
The sequence $\tilde{\alpha} := (\alpha_n)_{n \in \mathbb{N}}$ satisfies the compatibility conditions of Eqs. [46, 47]. Therefore $\tilde{\alpha} \in \| A_* \|$. In particular, if $\theta \in (A_n^1 \otimes \mathfrak{g})^G$ is a pseudo-connection on $A_*$, then

$$\tilde{\theta} = \left( \sum_{i=0}^{n} t_i \otimes p_i^* (\theta) \right)_{n \in \mathbb{N}} \in \| A_* \| \otimes \mathfrak{g}.$$  \hspace{1cm} (54)

**Proposition 5.3** $\tilde{\theta} \in \| A_* \| \otimes \mathfrak{g}$ defines a connection on the fat realization $\| A_* \|$.

**Proof.** Since all the maps $p_i^*, i = 0, \ldots, n$, are homomorphisms of $G$-differential algebras, $p_i^* (\theta) \in (A_n^1 \otimes \mathfrak{g})^G, i = 0, \ldots, n$, is a connection on $A_n$. Since $\sum_{i=0}^{n} t_i = 1$, hence $\theta_n = \sum_{i=0}^{n} t_i p_i^* (\theta)$ is also a connection on $A_n$. In particular, for any $n \in \mathbb{N}, \theta_n$ is $G$-invariant and therefore $\tilde{\theta}$ is $G$-invariant as well. To show that $\tilde{\theta}$ is a connection, it suffices to check Eq. [50] for any $X \in \mathfrak{g}$. Now

$$\hat{X} \cdot \tilde{\theta} = \left( \hat{X} \cdot \tilde{\theta}_n \right)_{n \in \mathbb{N}}$$

$$= \left( \Omega(\Delta_n) \otimes A_n \otimes X \right)_{n \in \mathbb{N}}$$

$$= \left( \Omega(\Delta_n) \otimes A_n ight)_{n \in \mathbb{N}} \otimes X$$

$$= 1 \| A_* \| \otimes X.$$

This completes the proof. $\square$

According to Eq. [51], one obtains a homomorphism of $G$-differential algebras

$$c_{\tilde{\theta}}^* : W(\mathfrak{g}) \rightarrow \| A_* \|.$$

Applying the basic functor, one obtains a homomorphism of differential algebras

$$c_{\tilde{\theta}}^{\text{basic}} : S(\mathfrak{g}^*)^G \rightarrow \| A_* \|^{\text{basic}}.$$

Using Proposition [4.13(3)], we have $\| A_* \|^{\text{basic}} \simeq \| A_*^{\text{basic}} \|$. Therefore we obtain a homomorphism, denoted again by $c_{\tilde{\theta}}^{\text{basic}}$:

$$c_{\tilde{\theta}}^{\text{basic}} : S(\mathfrak{g}^*)^G \rightarrow \| A_*^{\text{basic}} \|.$$  \hspace{1cm} (55)

Composing $c_{\tilde{\theta}}^{\text{basic}}$ with the cohomology functor and using the isomorphism $H^* (S(\mathfrak{g}^*)^G) \cong S(\mathfrak{g}^*)^G$, one obtains a homomorphism of (graded) algebras

$$H(c_{\tilde{\theta}}^{\text{basic}}) : S(\mathfrak{g}^*)^G \rightarrow H^*(\| A_*^{\text{basic}} \|).$$

Finally composing $H(c_{\tilde{\theta}}^{\text{basic}})$ with the integration map $I$, we obtain a map

$$w_{\theta} : S(\mathfrak{g}^*)^G \rightarrow H^*(A_*^{\text{basic}}),$$  \hspace{1cm} (56)

which is called the Chern-Weil map.

Note that $I : H^*(\| A_*^{\text{basic}} \|) \rightarrow H^*(A_*^{\text{basic}})$ is a homomorphism of (graded) algebras. Eq. [56] implies that $w_{\theta} : S(\mathfrak{g}^*)^G \rightarrow H^*(A_*^{\text{basic}})$ is a homomorphism of algebras, whose image lies in $H^{\text{even}}(A_*^{\text{basic}})$.

On the cochain level, composing $c_{\tilde{\theta}}^{\text{basic}}$ with the integration map $I : \Omega(\| A_*^{\text{basic}} \|) \rightarrow \Omega(A_*^{\text{basic}})$, we obtain a map $z_{\theta}$, called the Chern-Weil map on the cochain level:

$$z_{\theta} : S(\mathfrak{g}^*)^G \rightarrow Z(A_*^{\text{basic}}),$$  \hspace{1cm} (57)

where $Z(A_*^{\text{basic}})$ denotes the space of cocycles in $A_*^{\text{basic}}$. 29
Example 5.4 Consider the simplicial Weil algebra $W(g)$. It is simple to check that $\eta = \sum (1 \otimes \xi^i) \otimes X_i \in W(g)^1 \otimes g$ is a pseudo-connection, where $(\xi^i)_{i=1}^{\dim(g)}$ and $(X_i)_{i=1}^{\dim(g)}$ are dual basis of $g^*$ and $g$. The Chern-Weil map on the cochain level induces a map

$$Z : S(g^*)^G \rightarrow Z^*(W(g)) \quad (58)$$

and the Chern-Weil map induces a morphism

$$T : S(g^*)^G \rightarrow H^*(W(g))^{\text{basic}}. \quad (59)$$

Indeed $T$ is an isomorphism according to Theorem 5.5 [21].

Theorem 5.5 Let $A_\bullet$ be a $G$-differential simplicial algebra. Then

1. a pseudo-connection $\theta \in (A_1^1 \otimes g)^G$ induces a canonical map

$$z_\theta : S(g^*)^G \rightarrow Z(A_\bullet^{\text{basic}}), \quad (60)$$

where $Z(A_\bullet^{\text{basic}})$ denotes the space of cocycles in $A_\bullet^{\text{basic}}$. On the cohomology level, $z_\theta$ induces a morphism

$$w_\theta : S(g^*)^G \rightarrow H^*(A_\bullet^{\text{basic}}), \quad (61)$$

which is called the Chern-Weil map.

2. The Chern-Weil map $w_\theta$ does not depend on the choice of pseudo-connections on $A_\bullet$. In the sequel, we will denote $w_\theta$ by $w$.

Proof. 1) is already proved. 2) follows from Proposition 5.1 □

Remark 5.6 It would be interesting to compare our construction above with the noncommutative Chern-Weil map introduced by Alekseev-Meinrenken [2].

5.2 Second construction

We now describe our second construction of the Chern-Weil map. Assume that $A_\bullet := (A_n)_{n \in \mathbb{N}}$ is a $G$-differential simplicial algebra and $\theta \in (A_1^1 \otimes g)^G$ is a pseudo-connection on $A_\bullet$.

For any $0 \leq i \leq n$, $\theta^i = p_i^\theta(\theta) \in (A^1_n \otimes g)^G$ is a connection on $A_n$. Thus we have $(n+1)$-homomorphisms of $G$-differential algebras $W(g) \rightarrow^{c_{g_0}^a} A_n$, $i = 0, \ldots, n$. Define $c : W(g) \rightarrow A_\bullet$ by

$$c(x_0 \otimes \cdots \otimes x_n) = c_{g_0}^a(x_0) \cdots c_{g_n}^a(x_n), \quad \forall x_i \in W(g), \quad i = 0, \ldots, n. \quad (62)$$

Lemma 5.7 $c : W(g) \rightarrow A_\bullet$ is a homomorphism of $G$-differential simplicial algebras.

Proof. The proof is a direct verification and is left to the reader. □

Now applying the basic functor to Eq. (62) yields a homomorphism of differential simplicial algebras:

$$c^{\text{basic}} : W(g)_\bullet^{\text{basic}} \rightarrow A_\bullet^{\text{basic}}.$$

Hence

$$c^{\text{basic}} : Z^*(W(g)_\bullet^{\text{basic}}) \rightarrow Z^*(A_\bullet^{\text{basic}}). \quad (63)$$
By composing with the map \( Z : S(g)^* G \rightarrow Z^*(W(g)^{\text{basic}}) \) defined as in Eq. (58), we obtain an homomorphism
\[
z^\theta : S(g)^* G \rightarrow Z^*(A^\text{basic}). \tag{64}
\]
Applying the cohomology functor, we obtain a morphism of algebras
\[
w^\theta : S(g)^* G \rightarrow H^*(A^\text{basic}). \tag{65}
\]
We see in Corollary 5.9 that indeed \( z^\theta = z^\theta \) and \( w^\theta = w \).

### 5.3 Properties of the Chern-Weil map

**Proposition 5.8** Let \( \varphi : A_* \rightarrow B_* \) be a homomorphism of \( G \)-differential simplicial algebras, and \( \theta \in (A_0^1 \otimes g)^G \) a pseudo-connection on \( A_* \). Then the following diagrams commute
\[
S(g)^* G \xrightarrow{z^\theta} Z^*(A^\text{basic}) \quad \text{and} \quad S(g)^* G \xrightarrow{w} H^*(A^\text{basic}). \tag{66}
\]

**Proof.** From Eq. (54), it follows that
\[
\| \varphi \| (\tilde{\theta}) = (\varphi(\sum_{i=0}^n t_i \otimes p^n_i(\theta)))_{n \in \mathbb{N}} = (\sum_{i=0}^n t_i \otimes p^n_i(\varphi(\theta)))_{n \in \mathbb{N}} = \varphi(\theta).
\]
According to Proposition 5.1, we have the commutative diagram
\[
W(g) \xrightarrow{c^\theta} \| A_* \| \xrightarrow{c^\theta} \| B_* \| \tag{67}
\]
Applying the basic functor \( B \), one obtains the commutative diagram:
\[
S(g)^* G \xrightarrow{\varphi^\text{basic}} \| A_* \|^\text{basic} \xrightarrow{\varphi^\text{basic}} \| B_* \|^\text{basic} \tag{68}
\]
By Proposition 4.13 the realization and basic functors commute. Therefore, we have
\[
S(g)^* G \xrightarrow{\varphi^\text{basic}} \| A_* \|^\text{basic} \xrightarrow{\varphi^\text{basic}} \| B_* \|^\text{basic} \tag{68}
\]
According to Proposition 4.12 \( I \circ \| \varphi^\text{basic} \| = \varphi^\text{basic} \circ I \). Composing with the integration map, we have
\[
S(g)^* G \xrightarrow{\varphi^\text{basic}} \| A_* \|^\text{basic} \xrightarrow{\varphi^\text{basic}} \| B_* \|^\text{basic} \tag{68}
\]
Since \( I \circ \varphi^\text{basic} \) and \( I \circ \varphi^\text{basic} \) take values in \( Z(A_*^\text{basic}) \) and \( Z(B_*^\text{basic}) \) respectively, one obtains the commutative diagrams (66). □
Corollary 5.9 For any $G$-differential simplicial algebra $A_\bullet$ and any pseudo-connection $\theta$ on $A_\bullet$, we have $z_\theta = z_0^\theta$ and $w = w^\theta$. Thus the first and the second Chern-Weil constructions coincide.

**Proof.** Let $\eta \in W(\mathfrak{g})^1 \otimes \mathfrak{g}$ be the pseudo-connection constructed as in Example 5.4. It is simple to check that $c(\eta) = \theta$, where $c: W(\mathfrak{g}) \to A_\bullet$ is the map defined by Eq. (62). By Proposition 5.8 we have $c^{\text{basic}}_\theta z_\theta = z_\theta$. Passing to the cohomology, we have $w_\theta = w$. □

Remark 5.10 From Eq. (67), it follows that for any homomorphism of $G$-differential simplicial algebras $\phi: A'_\bullet \to A_\bullet$, the equality $c_\phi(\theta) = \parallel \phi \parallel c_\theta$ holds. Composing with $I$ we obtain

$$I \circ c_\phi(\theta) = I \circ \parallel \phi \parallel c_\theta.$$

Therefore, by Proposition 4.12(2), we obtain

$$I \circ c_\phi(\theta) = \phi \circ I \circ c_\theta.$$

In particular, if $A'_\bullet = W(\mathfrak{g})$ is endowed with the pseudo-connection $\eta \in W(\mathfrak{g})^1 \otimes \mathfrak{g}$ as in Example 5.4, $\zeta$ is a pseudo-connection on $A_\bullet$ and $c: W(\mathfrak{g}) \to A_\bullet$ is the homomorphism of $G$-differential simplicial algebras constructed as in Eq. (62), then $c(\eta) = \zeta$. Thus Eq. (69) implies that

$$I \circ c_\zeta = c_\phi I \circ c_\eta.$$

6 Chern-Weil map for principal $G$-bundles over groupoids

6.1 Main theorem

In this subsection, we apply the results of Section 5 to the case of a principal $G$-bundle over a groupoid. Let $P \xrightarrow{\pi} \Gamma_0$ be a principal $G$-bundle over $\Gamma \xrightarrow{\pi} \Gamma_0$. Then, according to Example 4.4, $\Omega(Q_\bullet)$ is a $G$-differential simplicial algebra and a pseudo-connection $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ defines a pseudo-connection on the $G$-differential simplicial algebra $\Omega(Q_\bullet)$, where $Q \xrightarrow{\pi} P$ is the transformation groupoid. Note that $\Omega(Q_\bullet)^{\text{basic}} = \Omega(\Gamma_\bullet)$. Therefore, one obtains a Chern-Weil map $w_P: S(\mathfrak{g}^*)^G \to H^*_{dR}(\Gamma_\bullet)$ and a Chern-Weil map $z_\theta: S(\mathfrak{g}^*)^G \to Z^*_{dR}(\Gamma_\bullet)$ on the cochain level.

**Theorem 6.1**

1. Associated to any pseudo-connection $\theta \in \Omega^1(P) \otimes \mathfrak{g}$, there is a canonical map

$$z_\theta: S(\mathfrak{g}^*)^G \to Z^*_{dR}(\Gamma_\bullet),$$

called the Chern-Weil map on the cochain level, where $Z^*_{dR}(\Gamma_\bullet)$ is the space of closed forms. On the cohomology level, $z_\theta$ induces an algebra homomorphism

$$w_\theta: S(\mathfrak{g}^*)^G \to H^*_{dR}(\Gamma_\bullet),$$

which is independent of the choice of pseudo-connections and is denoted by $w_P$. Moreover, $z_\theta$ is completely determined by the total pseudo-curvature (Proposition 6.3).
2. If \( \varphi \) is a strict homomorphism from \( \Gamma' \cong \Gamma'_0 \) to \( \Gamma \Rightarrow \Gamma_0 \), then the following diagrams

\[
S(\mathfrak{g}^*)^G \xrightarrow{\varphi'^*} Z_{dR}^*(\Gamma'_0) \xrightarrow{\varphi^*} S(\mathfrak{g}^*)^G \xrightarrow{\varphi'^*} H_{dR}^*(\Gamma'_0)
\]

commute, where \( P' = \varphi^* P \) is the pull-back of \( P \) via \( \varphi \).

The following proposition lists some important properties of this Chern-Weil map.

**Theorem 6.2**

1. If \( \Gamma \) is a manifold \( M \), then \( w_P : S(\mathfrak{g}^*)^G \to H_{dR}^*(M) \) reduces to the usual Chern-Weil map \([73]\).

2. If \( \Gamma \) is a Lie group \( G \), \( P \) is the \( G \)-bundle \( G \to \cdot \) and \( \theta \) is the left Maurer-Cartan form, then \( z_\theta : S(\mathfrak{g}^*)^G \to Z_{dR}^*(G) \) and \( w_P : S(\mathfrak{g}^*)^G \to H_{dR}^*(G) \) coincide with the Bott-Shulman maps \([7, 8]\).

**Proof.** 1) When \( \Gamma = M \), the one-forms \( \alpha_n \) defined by Eq. (53) are all equal. Therefore, \( \tilde{\theta} = (1_{\Omega(D_n)} \otimes \theta)_{n \in \mathbb{N}} \) and

\[
c^\text{basic}_\tilde{\theta} = (1_{\Omega(D_n)})_{n \in \mathbb{N}} \otimes c^\text{basic}_\theta. \tag{71}
\]

The integration map \( I \) is equal to zero on \( \Omega^0(\Delta_n) \otimes \Omega^*(M) \) unless \( n = 0 \). Composing Eq. (71) with \( I \), we obtain \( z_\theta = I \circ c^\text{basic}_\tilde{\theta} = I \circ c^\text{basic}_\theta \). The conclusion follows by passing to the cohomology.

2) Let \( \theta \in \Omega^1(G) \otimes \mathfrak{g} \) be the left Maurer-Cartan form of \( G \). Then our construction reduces to that as in \([13\, \text{Chapter 6}]\). Hence, the conclusion follows from \([8]\). \( \square \)

Now we turn to the study of the relation between the Chern-Weil map and the total pseudo-curvature.

As above, let \( \theta \in \Omega^1(P) \otimes \mathfrak{g} \) be a pseudo-connection and \( \Omega_{\text{total}} = \partial \theta + \Omega \in \Omega^1(Q) \otimes \mathfrak{g} \oplus \Omega^2(P) \otimes \mathfrak{g} \) its pseudo-curvature. Let \( \mathcal{C} \) be the subalgebra of \( \Omega(Q, \mathfrak{g}) \) generated (under the cup-product) by the images of both maps \( \Lambda \mathfrak{g}^* \to \Omega^1(Q) \) and \( S(\mathfrak{g}^*) \to \Omega^2(P) \) induced by \( \partial \theta \) and \( \Omega \) respectively. Let \( \mathcal{D} \subset \Omega(\Gamma_*) \) be the subalgebra of \( \Omega(\Gamma_*) \) consisting of basic elements of \( \mathcal{C} \).

**Proposition 6.3** For any \( f \in S(\mathfrak{g}^*)^G \), \( z_\theta(f) \) is a non-commutative polynomial \( P_f(\partial \theta, \Omega) \) in \( \partial \theta \) and \( \Omega \). In particular, \( z_\theta \) belongs to \( \mathcal{D} \).

In other words the Chern-Weil map is completely determined by the total pseudo-curvature.

**Proof.** The curvature of \( \tilde{\theta} \) on \( \| Q, \| \) is the simplicial \( \mathfrak{g} \)-valued 2-form \( \tilde{\Omega} = d\tilde{\theta} + \frac{1}{2}[\tilde{\theta}, \tilde{\theta}] \). More precisely

\[
\tilde{\Omega}_n = \sum_{i=0}^n dt_i \wedge \theta_i + \sum_{i=0}^n t_i \Omega_i - \frac{1}{2} \sum_{i=0}^n t_i[\theta_i, \theta_i] + \frac{1}{2} \sum_{i=0}^n \sum_{j=0}^n t_i t_j[\theta_i, \theta_j]. \tag{72}
\]

where \( \theta_i = (p_i^n)^* \theta \in \Omega^1(Q_n) \otimes \mathfrak{g} \) and \( \Omega_i = (p_i^n)^* \Omega \in \Omega^2(Q_n) \otimes \mathfrak{g} \).
Introducing new variables $\eta_i = \theta_{i+1} - \theta_i$, $i = 1, \ldots, n$ we have

$$
\sum_{i=0}^{n} t_i[\theta_i, \theta_i] - \sum_{i=0}^{n} \sum_{j=0}^{n} t_i t_j[\theta_i, \theta_i] = 
\sum_{i=0}^{n} t_i[\theta_0 + \sum_{k<i} \eta_k, \theta_0 + \sum_{k<i} \eta_k] - \sum_{i,j} t_i t_j[\theta_0 + \sum_{k<i} \eta_k, \theta_0 + \sum_{k<i} \eta_k] = 
\sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \sum_{i>\text{max}(k,l)} (\sum_{i=0}^{n-1} t_i)[\eta_k, \eta_l] - \sum_{i,j} t_i t_j[\eta_k, \eta_l]
$$

Moreover the identity \( \sum_{i=0}^{n} dt_i \wedge \theta_i = -\sum_{i=0}^{n-1} ds_i \wedge \varphi_i \) holds where \( s_i = \sum_{k=0}^{i} t_k \) for \( i = 0, \ldots, n - 1 \). Therefore Eq. (73) can be rewritten as

$$
\tilde{\Omega}_n = -\sum_{i=0}^{n} ds_i \wedge (\theta_{i+1} - \theta_i) + \sum_{i=0}^{n} t_i \Omega_i - \frac{1}{2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} ((\sum_{i>\text{max}(k,l)} t_i) + (\sum_{i>k,j>l} t_i t_j)) [\theta_{k+1} - \theta_k, \theta_{l+1} - \theta_l].
$$

For any \( f \in S(\mathfrak{g}^*)^G \), we have \( p^*(z_\theta(f)) = I_\bullet f(\tilde{\Omega}) \), where \( p^* : \Omega(\Gamma_\bullet) \to \Omega(Q_\bullet) \) is the pull-back of the projection \( Q \to \Gamma \). Now, by Eq. (73), it is simple to see that \( f(\tilde{\Omega}_n) \) is a linear combination of terms of the form:

$$
\omega \wedge f_0(\Omega_0) \wedge g_0(\theta_1 - \theta_0) \wedge \cdots \wedge f_1(\Omega_1) \wedge g_1(\theta_{i+1} - \theta_i) \wedge \cdots \wedge f_n-1(\Omega_n-1) \wedge g_n-1(\theta_n - \theta_{n-1}) \wedge f_n(\Omega_n),
$$

where \( \omega \in \Omega(\Delta_n) \), \( f_i \in S(\mathfrak{g}^*) \), \( i = 0, \ldots, n \) and \( g_i \in \mathfrak{g}^* \), \( i = 0, \ldots, n - 1 \).

Note that by applying the integration map to the above forms, \( \omega \) is integrated and the remaining terms are linear combinations of the form \( f_0(\Omega_0) \wedge g_0(\theta_1 - \theta_0) \wedge \cdots \wedge f_i(\Omega_i) \wedge g_i(\theta_{i+1} - \theta_i) \wedge \cdots \wedge f_n-1(\Omega_n-1) \wedge g_n-1(\theta_n - \theta_{n-1}) \wedge f_n(\Omega_n) \). Now by the definition of cup-product, one can rewrite the latter as

$$
f_0(\Omega_0) \wedge g_0(\theta_1 - \theta_0) \wedge \cdots \wedge f_i(\Omega_i) \wedge g_i(\theta_{i+1} - \theta_i) \wedge \cdots \wedge f_n-1(\Omega_n-1) \wedge g_n-1(\theta_n - \theta_{n-1}) \wedge f_n(\Omega_n) = f_0(\Omega) \vee g_0(\partial \theta) \vee \cdots \vee f_i(\Omega) \vee g_i(\partial \theta) \vee \cdots \vee f_n-1(\Omega) \vee g_n-1(\partial \theta) \vee f_n(\Omega)
$$

This completes the proof. \( \square \)

Consider the subcomplex \( \Omega^\bullet(\Gamma_0)^{\Gamma} \) of the de Rham complex \( (\Omega^\bullet(\Gamma_0), d) \), where \( \Omega^\bullet(\Gamma_0)^{\Gamma} = \{ \omega \in \Omega^\bullet(\Gamma_0) \mid d \omega = 0 \} \). Let \( H^\bullet(\Gamma_0)^{\Gamma} \) be its cohomology group. The natural inclusion \( i : \Omega^\bullet(\Gamma_0)^{\Gamma} \to \Omega^\bullet(\Gamma_\bullet) \) is a chain map and induces a homomorphism \( H^\bullet(\Gamma_0)^{\Gamma} \to H_{\text{dR}}^\bullet(\Gamma_\bullet) \).

When a connection exists, the Chern-Weil map admits an explicit simple form.

**Theorem 6.4** Assume that the principal \( G \)-bundle \( \xrightarrow{\pi} \Gamma_0 \) over the groupoid \( \Gamma \Rightarrow \Gamma_0 \) admits a connection \( \theta \in \Omega^1(P) \otimes \mathfrak{g} \). Then the following diagrams commute

$$
\begin{align*}
S(\mathfrak{g}^*)^G & \xrightarrow{z} Z^\bullet(\Gamma_0)^{\Gamma} \\
{\uparrow}_{\pi} & \quad \downarrow i \\
\mathfrak{g} & \xrightarrow{\iota} Z^\bullet(\Gamma_\bullet)
\end{align*}
$$

and

$$
\begin{align*}
S(\mathfrak{g}^*)^G & \xrightarrow{w} H^\bullet_{\text{dR}}(\Gamma_0)^{\Gamma} \\
{\uparrow}_{\pi} & \quad \downarrow i \\
\mathfrak{g} & \xrightarrow{\iota} H^\bullet_{\text{dR}}(\Gamma_\bullet)
\end{align*}
$$

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where \( z : S(\mathfrak{g}^*)^G \to \mathcal{Z}^*(\Gamma_0)^\Gamma \), and \( w : S(\mathfrak{g}^*)^G \to H_{dR}^*(\Gamma_0)^\Gamma \) denote the usual Chern-Weil maps by forgetting about the groupoid action i.e., \( z(f) = f(\Omega) \in Z^*(P)^{\text{basic}} \cong \mathcal{Z}^*(\Gamma_0) \) and \( w(f) = [z(f)] \in H^*(\Gamma_0), \forall f \in S(\mathfrak{g}^*)^G \), where \( \Omega \in \Omega^2(P) \otimes \mathfrak{g} \) is the curvature form.

**Proof.** We have
\[
\partial \Omega = \partial(d\theta + \frac{1}{2}[\theta, \theta]) = d\partial \theta + \frac{1}{2}([\partial \theta, \theta] + [\theta, \partial \theta]) = 0.
\]
Hence the image of \( z : S(\mathfrak{g}^*)^G \to \mathcal{Z}^*(\Gamma_0) \) lies in \( \mathcal{Z}^*(\Gamma_0)^\Gamma \). Therefore, the image of \( w : S(\mathfrak{g}^*)^G \to H_{dR}^*(\Gamma_0) \) lies in \( H_{dR}^*(\Gamma_0)^\Gamma \).

Since \( \partial \theta = 0 \), it follows that all the terms \( \theta_i - \theta_{i-1} \) in Eq. (73) vanish for \( i \in \{0, \ldots, n-1\} \). Thus we have \( \bar{\Omega} = \sum_{i=0}^{n} t_i \Omega_i \). This implies that for any \( f \in S(\mathfrak{g}^*)^G \), \( I(f(\bar{\Omega}_n)) = 0 \) except for \( n = 0 \). Therefore the identities \( z_0(f) = I(f(\bar{\Omega}_0)) = i(f(\Omega)) \) hold. \( \square \)

**Remark 6.5** In the case of an étale groupoid or the holonomy groupoid of a foliation, the Chern-Weil map has been studied by Crainic and Moerdijk in [12].

**Corollary 6.6** If a principal \( G \)-bundle \( P \xrightarrow{\pi} \Gamma_0 \) over a groupoid \( \Gamma \Rightarrow \Gamma_0 \) admits a flat connection, then the Chern-Weil map vanishes (except for degree 0).

**Example 6.7** The Chern-Weil map associated to a \( G \)-bundle over a finite group is always zero in strictly positive degree. This is due to the fact that this bundle admits a flat connection.

### 6.2 Universal Chern-Weil map

According to Proposition 2.11, a principal \( G \)-bundle \( P \to \Gamma_0 \) over \( \Gamma \Rightarrow \Gamma_0 \) corresponds to a generalized homomorphism \( \varphi_P \) from \( \Gamma \Rightarrow \Gamma_0 \) to \( G \Rightarrow \cdot \). According to Eq. (16), such a generalized homomorphism induces a map \( \varphi^* : H_{dR}^*(G_*) \to H_{dR}^*(\Gamma_*) \). Let \( S(\mathfrak{g}^*)^G \to H_{dR}^*(G_*) \) be the Bott-Shulman map. Composing these two morphisms, we obtain a morphism
\[
w^u_P : S(\mathfrak{g}^*)^G \to H_{dR}^*(\Gamma_*)
\]
called the **universal Chern-Weil map**. It is immediate from the definition that if \( \varphi : \Gamma' \to \Gamma \) is a generalized homomorphism then
\[
w^u_{P'} = \varphi^* \circ w^u_P.
\]
(74)
where \( P' = \varphi^* P \) is the pull-back of \( P \xrightarrow{\pi} \Gamma_0 \) by \( \varphi \) as constructed in Proposition 2.13 and \( \varphi^* : H_{dR}^*(\Gamma_*) \to H_{dR}^*(\Gamma_*) \) is the map constructed as in Eq. (16).

**Theorem 6.8** The universal Chern-Weil map and the Chern-Weil map are equal.

**Proof.** According to Theorem 6.2 (2), we know that the claim holds for the principal \( G \)-bundle \( G \to \cdot \) over \( G \Rightarrow \cdot \).

Let \( Q' \) be the groupoid \( Q \times G \Rightarrow P \) as in Eq. (10). And let \( p r : Q' \to G \) and \( p : Q' \to \Gamma \) be the strict homomorphisms given by the natural projections on \( G \) and \( \Gamma \), respectively. It is simple to see that the pull-back of the principal \( G \)-bundle \( G \to \cdot \) by \( p r \) is isomorphic to \( P = P \times G \to P \). According to Eq. (74) and Theorem 6.1 (2), we have
\[
w^u_P = w^u_\pi : S(\mathfrak{g}^*)^G \to H_{dR}^*(Q'_*).
\]
(75)
On the other hand, the pull-back of $P \to \Gamma_0$ via $p$ is also isomorphic to $\tilde{P} \to P$. Hence Eq. (74) implies that $w_\rho^\mu = p^* w_P^\mu$, while Theorem 6.12 implies that $w_\rho = p^* w_P$. Therefore

$$p^* w_P^\mu = p^* w_P.$$

(76)

It is easy to see that $Q' \rightrightarrows P$ is indeed the pull-back of $\Gamma_0 \rightrightarrows \Gamma_0$ via $P \xrightarrow{\pi} \Gamma_0$. So these two groupoids are Morita equivalent, and therefore, according to Lemma 3.1, $p^* : H_{dR}^*(\Gamma_\bullet) \to H_{dR}^*(Q'_\bullet)$ is an isomorphism. Therefore, Eq. (76) implies that $w_P^\mu = w_P$. □

The following corollary follows immediately from Proposition 6.8 and Eq. (74).

**Corollary 6.9** Let $\varphi$ be a generalized homomorphism from $\Gamma_0 \rightrightarrows \Gamma_0'$ to $\Gamma_0 \rightrightarrows \Gamma_0$, then the following diagram commutes

$$
\begin{array}{ccc}
S(g^*)^G & \xrightarrow{w_\varphi^*} & H_{dR}^*(\Gamma_\bullet) \\
\uparrow w_P & & \uparrow \varphi^* \\
H_{dR}^*(\Gamma_\bullet) & & H_{dR}^*(\Gamma_\bullet')
\end{array}
$$

where $P' = \varphi^* P$ is the pull-back of $P$ via $\varphi$ and $\varphi^* : H_{dR}^*(\Gamma_\bullet) \to H_{dR}^*(\Gamma_\bullet')$ is the morphism induced by $\varphi$ as given by Eq. (10).

**Remark 6.10** From Corollary 6.9, it follows that the universal Chern-Weil map is defined for $G$-bundles over differential stacks. Note that generalized homomorphisms correspond to smooth maps between their stacks. Theorem 6.8 means that, when a presentation of a differential stack, i.e. a Lie groupoid, is chosen, the universal Chern-Weil map can be computed using a pseudo-connection, as in the manifold case.

### 6.3 Equivariant principal $G$-bundles

For a compact Lie group $H$ and a smooth manifold $M$ on which $H$ acts, there are many different ways to define the equivariant cohomology $H^*_H(M)$. One definition, called the simplicial model, is to define $H^*_H(M)$ as $H^*_dR(\Gamma_\bullet)$, where $\Gamma$ is the transformation groupoid $H \times M \rightrightarrows M$. Another definition, called the Weil model, is to define $H^*_H(M)$ as the cohomology of the complex $(W(\mathfrak{h}) \otimes \Omega(M))^{H-\text{basic}}$, where $\mathfrak{h}$ is the Lie algebra of $H$. These two models are known to be equivalent (see Section 4.4 [15]).

As an application of the tools developed in the previous sections, in what follows, we describe an explicit chain map that establishes a quasi-isomorphism from the Weil model to the simplicial model. Consider the right principal $H$-bundle $H \to \cdot$ over the groupoid $H \rightrightarrows \cdot$, together with the pseudo-connection $\zeta$ equal to the Maurer-Cartan form, i.e., $\zeta = \sum_{i=1}^{\dim(\mathfrak{h})} \zeta^i \otimes X_i$ where $\{X_1, \ldots, X_{\dim(\mathfrak{h})}\}$ is a basis of $\mathfrak{h}$ and $\zeta^1, \ldots, \zeta^{\dim(\mathfrak{h})} \in \Omega^1(H)$ be the dual basis of the left invariant vector fields associated to $X_1, \ldots, X_{\dim(\mathfrak{h})}$.

As in Example 5.3 let $\eta = \sum_{i=1}^{\dim(\mathfrak{h})} (1 \otimes \zeta^i) \otimes X_i$ be a connection on the algebra $W(\mathfrak{h})$, where $\{\xi^1, \ldots, \xi^{\dim(\mathfrak{h})}\}$ is the dual basis of $\{X_1, \ldots, X_{\dim(\mathfrak{h})}\}$.

Note that we can consider $\eta$ as a pseudo-connection on $W(\mathfrak{h})$. Similarly, let $H \times H \rightrightarrows H$ be the pair groupoid. We can consider $\zeta$ as a pseudo-connection on $\Omega((H \times H)_\bullet)$. Let $c : W(\mathfrak{h}) \to \Omega((H \times H)_\bullet)$ be the homomorphism of $H$-differential simplicial algebras defined as in Eq. (72) using the pseudo-connection $\zeta$. It is clear that $c(\eta) = \zeta$. Hence, Eq. (70), applied to the case where $A_\bullet = \Omega((H \times H)_\bullet)$ and $\theta = \eta$ implies that the following two sequences of compositions of chain maps are equal

$$W(\mathfrak{h}) \xrightarrow{\zeta} \| \Omega((H \times H)_\bullet) \| \xrightarrow{I} \Omega((H \times H)_\bullet)$$
and
\[ W(h) \xRightarrow{c_h} W(h) \xrightarrow{l} W(h) \xleftarrow{c} \Omega((H \times H)_*) . \]

In other words, we have a commutative diagram
\[
\begin{array}{ccc}
\| \Omega((H \times H)_*) \| & \xrightarrow{l} & \Omega((H \times H)_*) \\
W(h) \downarrow & & \downarrow c \\
\| W(h)_* \| & \xrightarrow{l} & W(h)_* \\
\end{array}
\]

Taking the tensor product with \( \Omega(M) \), we obtain
\[
\begin{array}{ccc}
\| \Omega((H \times H)_*) \| \otimes \Omega(M) & \xrightarrow{l} & \Omega((H \times H)_*) \otimes \Omega(M) \\
W(h) \otimes \Omega(M) \downarrow & & \downarrow c \otimes \text{id} \\
\| W(h)_* \| \otimes \Omega(M) & \xrightarrow{l} & W(h)_* \otimes \Omega(M) \\
\end{array}
\]

where \( F = c_h \otimes \text{id} \). Let \( R \Rightarrow H \times M \) be the product of the pair groupoid \( H \times H \Rightarrow H \) with the manifold \( M \Rightarrow M \). Since we have the inclusion \( \Omega((H \times H)_*) \otimes \Omega(M) \subset \Omega(R_*) \) and \( \| \Omega((H \times H)_*) \| \otimes \Omega(M) \subset \| \Omega(R_*) \| \), we have
\[
\begin{array}{ccc}
\| \Omega(R_*) \| & \xrightarrow{l} & \Omega(R_*) \\
W(h) \otimes \Omega(M) \downarrow & & \downarrow c \otimes \text{id} \\
\| W(h)_* \| \otimes \Omega(M) & \xrightarrow{l} & W(h)_* \otimes \Omega(M) \\
\end{array}
\]

There is a natural projection from \( R \) onto \( \Gamma \) given by \( \pi(h_1, h_2, m) = (h_1^{-1} h_2, h_2^{-1} \cdot m) \). This endows \( R \Rightarrow H \times M \) with a structure of principal \( H \)-groupoid over \( \Gamma \Rightarrow M \), where \( H \) acts on \( R \) by \( h \cdot (h_1, h_2, m) = (h h_1, h h_2, h \cdot m) \).

Therefore \( \Omega(R_*)^{H-\text{basic}} \simeq \Omega(\Gamma_*) \). Taking \( H \)-basic elements in the previous diagram, we obtain
\[
\begin{array}{ccc}
\| \Omega(\Gamma_*) \| & \xrightarrow{l} & \Omega(\Gamma_*) \\
(W(h) \otimes \Omega(M))^{H-\text{basic}} \downarrow & & \downarrow (c \otimes \text{id})^{H-\text{basic}} \\
\| W(h)_* \| \otimes \Omega(M) \xrightarrow{H-\text{basic}} & & \xrightarrow{H-\text{basic}} (W(h)_* \otimes \Omega(M))^{H-\text{basic}} \\
\end{array}
\]
Therefore we obtain two equivalent descriptions of the natural chain map
\[ K : (W(h) \otimes \Omega(M))^{H-\text{basic}} \to \Omega(\Gamma_\ast). \]

**Proposition 6.11** If \( H \) is a compact Lie group, then \( K \) is a quasi-isomorphism.

To prove this result, we need some preliminaries. Recall that a \( W(h) \)-module \( \Omega \) is an \( H \)-differential algebra endowed with an action \( W(h) \otimes \Omega \to \Omega \) which is a homomorphism of \( H \)-differential algebras. A \( W(h) \)-algebra is a \( H \)-differential algebra with a structure of \( W(h) \)-module. Note that an \( H \)-differential algebra can be endowed with a structure of \( W(h) \)-module if and only if it admits a connection. A \( W(h) \)-module is said to be acyclic if its cohomology vanishes in strictly positive degree and is \( \mathbb{R} \) in degree 0 (see [15]).

According to (Section 4.4 [15]), for any acyclic \( W(h) \)-algebras \( A \) and \( B \), the inclusion \( B \otimes \Omega \hookrightarrow A \otimes B \otimes \Omega \) induces an isomorphism
\[
H^*((B \otimes \Omega)^{H-\text{basic}}) \simeq H^*((A \otimes B \otimes \Omega)^{H-\text{basic}}). \tag{80}
\]

Hence for any acyclic \( W(h) \)-algebras \( A \) and \( B \), an isomorphism \( K(A, B) \) from \( H^*((A \otimes \Omega)^{H-\text{basic}}) \) to \( H^*((B \otimes \Omega)^{H-\text{basic}}) \) can be canonically constructed by

\[
K(A, B) = H(j^{H-\text{basic}})^{-1}H(\tau^{H-\text{basic}})H(i^{H-\text{basic}}). \tag{81}
\]

Here,

- \( \tau : A \otimes B \otimes \Omega \to B \otimes A \otimes \Omega \) is obtained by flipping \( A \) and \( B \) according to the Quillen rule, i.e.,
  \[
  \tau(a \otimes b \otimes \omega) = (-1)^{kl}b \otimes a \otimes \omega \quad \forall a \in A^k, b \in B^l, \omega \in \Omega
  \]
- \( i : A \otimes \Omega \to B \otimes A \otimes \Omega \) is the canonical inclusion \( i(a \otimes \omega) = (1_B \otimes a \otimes \omega) \),
- \( j : B \otimes \Omega \to A \otimes B \otimes \Omega \) is the canonical inclusion \( j(b \otimes \omega) = (1_A \otimes b \otimes \omega) \).

**Lemma 6.12** Let \( \Omega \) be an \( H \)-differential algebra and \( A, B \) acyclic \( W(h) \)-algebras. Assume that \( \varphi : A \to B \) is a homomorphism of \( W(h) \)-algebras. Then the homomorphism

\[
H((\varphi \otimes \text{id})^{H-\text{basic}}) : H^*((A \otimes \Omega)^{H-\text{basic}}) \to H^*((B \otimes \Omega)^{H-\text{basic}})
\]

is equal to \( K(A, B) \) and is therefore an isomorphism.

**Proof.** Let \( j_A \) be the homomorphism of \( G \)-differential algebra \( j_A : W(h) \to A \) defined by \( w \to w \cdot 1_A \). As a first step, we show that

\[
H((j_A \otimes \text{id})^{H-\text{basic}}) : H^*((W(h) \otimes \Omega)^{H-\text{basic}}) \to H^*((A \otimes \Omega)^{H-\text{basic}})
\]

is an isomorphism.

It is simple to check that the following diagram is commutative,

\[
\begin{array}{ccc}
W(h) \otimes \Omega & \xrightarrow{j_A \otimes \text{id}} & A \otimes \Omega \\
\downarrow{i_1} & & \downarrow{i_2} \\
A \otimes W(h) \otimes \Omega & \xrightarrow{i_3} & A \otimes A \otimes \Omega \\
\downarrow{i_4} & & \\
A \otimes \Omega
\end{array}
\]

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where $i_1, i_2, i_3, i_4$ are defined by, $\forall w \in W(h), a \in A, \omega \in \Omega$

\[
i_1(w \otimes \omega) = (1_A \otimes w \otimes \omega) \quad i_2(a \otimes \omega) = (1_A \otimes a \otimes \omega) \quad i_3(a \otimes \omega) = (a \otimes 1_W(h) \otimes \omega) \quad i_4(a \otimes \omega) = (a \otimes 1_A \otimes \omega).
\]

By Eq. (80), $i_1, i_2, i_3, i_4$, when being restricted to the $H$-basic elements, induce isomorphism of cohomology. Therefore

\[
H((1_A \otimes \text{id})^{H_{\text{basic}}}) = H(i_2^{H_{\text{basic}}})^{-1} \circ H(i_4^{H_{\text{basic}}}) \circ H(i_3^{H_{\text{basic}}})^{-1} \circ H(i_1^{H_{\text{basic}}})
\]

is indeed an isomorphism.

The commutativity of the diagram

\[
\begin{array}{ccc}
W(h) & \xrightarrow{J_A} & A \\
J_B \downarrow & & \downarrow \varphi \\
B & & \\
\end{array}
\]

implies the commutativity of the diagram

\[
H^*((W(h) \otimes \Omega)^{H_{\text{basic}}}) \xrightarrow{H((1_A \otimes \text{id})^{H_{\text{basic}}})} H^*((A \otimes \Omega)^{H_{\text{basic}}})
\]

Since $H((1_A \otimes \text{id})^{H_{\text{basic}}})$ and $H((1_B \otimes \text{id})^{H_{\text{basic}}})$ are isomorphisms, the commutativity of diagram (82) implies that $H((\varphi \otimes \text{id})^{H_{\text{basic}}})$ must be an isomorphism. Moreover, if $\varphi$ and $\psi$ are two such homomorphisms, then

\[
H((\varphi \otimes \text{id})^{H_{\text{basic}}}) = H((\psi \otimes \text{id})^{H_{\text{basic}}}).
\]

Now consider the following two homomorphisms of acyclic $W(h)$-algebras from $A$ to $A \otimes B$ given by $i_1(a) = a \otimes 1_B$ and $j_1(a) = 1_A \otimes \varphi(a), \forall a \in A$. Then we have $i = i_1 \otimes \text{id}$ and $\tau \circ j_1(\varphi \otimes \text{id}) = j_1 \otimes \text{id}$. By Eq. (83), we have

\[
\begin{align*}
H(i^{H_{\text{basic}}}) &= H((i_1 \otimes \text{id})^{H_{\text{basic}}}) = H((j_1 \otimes \text{id})^{H_{\text{basic}}}) \\
&= H(\tau^{H_{\text{basic}}}) \circ H(j^{H_{\text{basic}}}) \circ H((\varphi \otimes \text{id})^{H_{\text{basic}}}).
\end{align*}
\]

The conclusion thus follows from Eq. (81) immediately. □

Now we can prove Proposition 6.11.

PROOF. According to [15], $\|\Omega((H \times H)_\bullet)\|$ (denoted by $\Omega(\mathcal{E})$ in [15]) is an acyclic $W(h)$-algebra. Lemma 6.12 applied to the case that $\Omega = \Omega(M), A = W(h)$ and $B = \|\Omega((H \times H)_\bullet)\|$, implies that

\[
c_\zeta \otimes \text{id} : W(h) \otimes \Omega(M) \to \|\Omega((H \times H)_\bullet) \otimes \Omega(M)\|
\]

induces an isomorphism in cohomology when restricted to basic elements.

Now the cohomology of $\|\Omega((H \times H)_\bullet) \otimes \Omega(M)\|^{H_{\text{basic}}}$ is equal to the cohomology of $\Omega(\Gamma_\bullet)$. Proposition 6.11 then follows immediately from the fact that $I : \|\Omega(\Gamma_\bullet)\| \to \Omega(\Gamma_\bullet)$ induces an isomorphism in cohomology according to Proposition 4.12. □
Remark 6.13 An explicit quasi-isomorphism between the Cartan model and the simplicial model of an equivariant cohomology was also constructed by Meinrenken [23]. It would be interesting to compare these two constructions.

Let $P \to M$ be an $H$-equivariant principal $G$-bundle. In [9], Bott-Tu introduced a Chern-Weil map with values in the Weil model associated to any $H$-invariant connection $\theta$ on $P \to M$. More precisely, they constructed an $H$-basic connection on $W(h) \otimes \Omega(P)$ by

$$\Xi = \dim(h) \sum_{i=1}^{\dim(h)} (1 \otimes \xi_i) \otimes L_i + 1 \otimes \theta \in W(h) \otimes \Omega(P) \otimes g, \quad (84)$$

where $L_i = - < \theta, \hat{X}_i >$ is a $g$-valued function on $P$. This connection, according to Proposition 5.10, induces a map $S^*(g^*)^G \to (W(h) \otimes \Omega(M))^{H\text{-basic}}$ that we denote by $z_{BT}$. On the other hand, $\theta \in \Omega(P) \otimes g$ can be considered as a pseudo-connection of $(W(h) \otimes \Omega(P))^{H\text{-basic}}$. The construction of Section 6 induces a map $z_\theta : S^*(g^*)^G \to Z^*(\Gamma_\bullet)$.

Theorem 6.14 The following diagram commutes

$$\begin{array}{ccc}
S^*(g^*)^G & \xrightarrow{z_{BT}} & (W(h) \otimes \Omega(M))^{H\text{-basic}} \\
\downarrow_{z_\theta} & & \downarrow_{K} \\
Z^*(\Gamma_\bullet) & & 
\end{array}$$

We will need the following lemma first.

Lemma 6.15 Assume that $\theta \in \Omega^1(P) \otimes g$ is an $H$-invariant connection on the principal $G$-bundle $P \to M$. Let $\pi : H \times P \to P$ be the map defined by $\pi(h, p) = h^{-1} \cdot p$. We have

$$\pi^* \theta = \sum_{i=1}^{\dim(h)} \zeta_i \otimes L_i + 1 \otimes \theta \in \Omega(H) \otimes \Omega(P) \otimes g \subset \Omega(H \times P) \otimes g.$$

Proof. The proof is a direct verification and is left to the reader. □

Now we are ready to prove Theorem 6.14

Proof. The 1-form $\Xi$, as defined in Eq. (84), can be considered as a pseudo-connection on the $G$-differential simplicial algebra $W(h) \otimes \Omega(P)$. By definition of $\psi_\eta$, we have

$$(\psi_\eta \otimes \text{id})(\Xi) = \sum_{i=1}^{\dim(h)} \psi_\eta(1 \otimes \xi_i) \otimes L_i + 1 \otimes \theta = \sum_{i=1}^{\dim(h)} \zeta_i \otimes L_i + 1 \otimes \theta.$$  

By Lemma 6.15, we have $(\psi_\eta \otimes \text{id})(\Xi) = \pi^* \theta \in \Omega(R_\bullet)$, where $R$ denotes the direct product of the pair groupoid $H \times H \rightrightarrows H$ with the manifold $P \rightrightarrows P$.

By the functoriality properties of the Chern-Weil map, we obtain the commutative diagram

$$\begin{array}{ccc}
\| \Omega(R_\bullet) \| & \xrightarrow{I} & \Omega(R_\bullet) \\
\| W(h) \| & \xrightarrow{c_{\pi^* \theta}} & \Omega(R_\bullet) \\
\| W(h) \| & \xrightarrow{c_\Xi} & \| W(h) \| \otimes \Omega(P) \xrightarrow{I} W(h) \otimes \Omega(P) \\
\| W(h) \| & \xrightarrow{c_\Xi} & \| W(h) \| \otimes \Omega(P) \xrightarrow{I} W(h) \otimes \Omega(P) \\
\end{array}$$
Since the image of $\Xi$ under $F$ is $\widetilde{\Xi}$, which is a connection for the $G$-differential algebra $\| W(\mathfrak{h}) \| \otimes \Omega(P)$, we also have a commutative diagram

$$
\begin{array}{ccc}
W(\mathfrak{g}) & \xrightarrow{c_\mathfrak{g}} & W(\mathfrak{h}) \otimes \Omega(P) \\
\downarrow{c_\Xi} & & \downarrow{T} \\
\| W(\mathfrak{h}) \| \otimes \Omega(P)
\end{array}
$$

(86)

Combining Eq. (85) and Eq. (86), we obtain the commutative diagram

$$
\begin{array}{ccc}
\| \Omega(R_\ast) \| & \xrightarrow{I} & \Omega(R_\ast) \\
\downarrow{c_{\mathfrak{g} + \theta}} & & \downarrow{\psi_{\eta} \otimes \text{id}} \\
W(\mathfrak{g}) & \xrightarrow{c_\mathfrak{g}} & W(\mathfrak{h}) \otimes \Omega(P) \\
\downarrow{\Xi} & & \downarrow{\psi_{\eta} \otimes \text{id}} \\
\| W(\mathfrak{h}) \| \otimes \Omega(P) & \xrightarrow{I} & W(\mathfrak{h}) \otimes \Omega(P)
\end{array}
$$

(87)

Since all the arrows of this diagram are $G$-$H$-bimodule maps ($W(\mathfrak{g})$ is considered as a trivial $H$-module), we can restrict ourself to elements which are both $G$- and $H$-basic. We then obtain the following commutative diagram:

$$
\begin{array}{ccc}
\| \Omega(\Gamma_\ast) \| & \xrightarrow{I} & \Omega(\Gamma_\ast) \\
\downarrow{c_{\mathfrak{g}}} & & \downarrow{(\psi_{\eta} \otimes \text{id})^H_{\text{basic}}} \\
S(\mathfrak{g}^*)^G & \xrightarrow{z_{BT}} & (W(\mathfrak{h}) \otimes \Omega(P))^{H_{\text{basic}}} \\
\downarrow{T_{H_{\text{basic}}}} & & \downarrow{(\psi_{\eta} \otimes \text{id})^H_{\text{basic}}} \\
\left( \| W(\mathfrak{h}) \| \otimes \Omega(P) \right)^{H_{\text{basic}}} & \xrightarrow{I} & \left( W(\mathfrak{h}) \otimes \Omega(P) \right)^{H_{\text{basic}}}
\end{array}
$$

(88)

Now the composition of $c_\mathfrak{g}$ and $I$ is $z_{\theta}$, the Chern-Weil map on the cochain level, while $K$ is the composition $(\psi_{\eta} \otimes \text{id})^{H_{\text{basic}}}I_{T^{H_{\text{basic}}}}$. Therefore we have $z_{\theta} = K \circ z_{BT}$.

□

**Remark 6.16** For equivariant Chern-Weil map in the Cartan model, we refer the reader to [5][19].

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