ON CERTAIN GEOMETRIC AND HOMOTOPY PROPERTIES OF CLOSED SYMPLECTIC MANIFOLDS

Raul Ibanez, Yuli Rudyak, Aleksy Tralle and Luis Ugarte

1. Introduction

Homotopy properties of closed symplectic manifolds attract the attention of geometers since the classical papers of Sullivan [S] and Thurston [Th]. On one hand, "soft" homotopy techniques help in the solution of many "hard" problems in symplectic geometry, cf. [G1, McD, RT1, TO]. On the other hand, it is still unknown if there are specific homotopy properties of closed manifolds dependent on the existence of symplectic structures on them. It turns out that symplectic manifolds violate many specific homotopy conditions shared by the K"ahler manifolds (which form a subclass of symplectic manifolds). In particular, if $M$ is a closed K"ahler manifold then the following holds:

1. all the odd-degree Betti numbers $b_{2i+1}(M)$ are even;
2. $M$ has the Hard Lefschetz property;
3. all Massey products (of all orders) in $M$ vanish.

It is well known (and we shall see it below) that closed symplectic manifolds violate all the homotopy properties (1) – (3). However, it is not clear whether properties (1) – (3) are independent or not, in case of closed symplectic manifolds or certain classes of these ones. In other words, can a combination of the type

(1) – (2) – non-(3)

be realized by a closed symplectic manifold (possibly, with prescribed properties). The knowledge of an answer to this question might shed a new light on the whole understanding of closed symplectic manifolds.

In Theorem 3.1 we have summarized our knowledge by writing down the corresponding tables. We have considered two classes of symplectic manifolds: the class of symplectically aspherical symplectic manifolds and the class of simply-connected symplectic manifolds. Recall that a symplectically aspherical manifold is a symplectic manifold $(M, \omega)$ such that $\omega|_{\pi_2(M)} = 0$, i.e.

$$\int_{S^2} f^\# \omega = 0$$

for every map $f : S^2 \to M$. In view of the Hurewicz Theorem, a closed symplectically aspherical manifold always has a non-trivial fundamental group. It is well known that symplectically aspherical manifolds play an important role in geometry and topology of symplectic manifolds, [F, G2, H, RO, RT2].
The next topic of the paper is about symplectically harmonic forms on closed symplectic manifolds. Brylinski [B] and Libermann (Thesis, see [LM]) have introduced the concept of a symplectic star operator * on a symplectic manifold. In a sense, it is a symplectic analog of the Hodge star operator which is defined in terms of the given symplectic structure \( \omega \). Using this operator, one defines a symplectic codifferential \( \delta := (-1)^{k+1}(*d*) \), \( \deg \delta = -1 \). Now we define symplectically harmonic differential forms \( \alpha \) by the condition

\[
\delta \alpha = 0, \quad d \alpha = 0.
\]

Let \( \Omega^*_{hr}(M, \omega) \) denote the space of all symplectically harmonic forms on \( M \). Clearly, the space \( H^k_{hr}(M) := \Omega^k_{hr}/(\Omega^k_{hr} \cap \text{Im} \ d) \) is a subspace of the de Rham cohomology space \( H^k(M) \).

Here we also have an interesting relation between geometry and homotopy theory. For example, Mathieu [M] proved that \( H^k_{hr}(M, \omega) = H^k(M) \) if and only if \( M \) has the Hard Lefschetz property. We will also see that the Lefschetz map

\[
L^k : H^{m-k}(M) \to H^{m+k}(M), \quad \dim M = 2m
\]

(multiplication by \([\omega]^k\)) plays an important role in studying of \( H^k_{hr}(M, \omega) \).

We set \( h_k(M, \omega) = \dim H^k_{hr}(M, \omega) \). According to Yan [Y], the following question was posed by Boris Khesin and Dusa McDuff.

**Question:** Are there closed manifolds endowed with a continuous family \( \omega_t \) of symplectic structures such that \( h_k(M, \omega_t) \) varies with respect to \( t \)?

Yan [Y] constructed a closed 4-dimensional manifold \( M \) with varying \( h_3(M) \). So, he answered affirmatively the above question.

(Actually, Proposition 4.1 from [Y] is wrong, the Kodaira–Thurston manifold is a counterexample, but its Corollary 4.2 from [Y] is correct because it follows from our Lemma 4.4. Hence, the whole construction holds.)

However, the Yan’s proof was essentially 4-dimensional. Indeed, Yan [Y] wrote:

“For higher dimensional closed symplectic manifolds, it is not clear how to answer the question in the beginning of this section”,
i.e. the above stated question.

In this note we prove the following result (Theorem 4.6): There exists at least one 6-dimensional indecomposable closed symplectic manifold \( N \) with varying \( h_5(N) \).

Moreover, Yan remarked that there is no 4-dimensional closed symplectic nilmanifolds \( M \) with varying \( \dim H^*_{hr}(M) \). On the contrary, our example is a certain 6-dimensional nilmanifold.

2. Preliminaries and notation

Given a topological space \( X \), let \( (M_X, d) \) be the Sullivan model of \( X \), that is, a certain natural commutative DGA algebra over the field of rational numbers \( \mathbb{Q} \) which is a homotopy invariant of \( X \), see [DGMS, TO, S] for details. Furthermore, if \( X \) is a nilpotent \( CW \)-space of finite type then \((M_X, d)\) completely determines the rational homotopy type of \( X \).
A space $X$ is called formal if there exists a DGA-morphism 

$$\rho : (\mathcal{M}_X, d) \to (H^*(X; \mathbb{Q}), 0)$$

inducing isomorphism on the cohomology level. Recall that every closed Kähler manifold is formal [DGMS].

We refer the reader to [K, Ma, RT1] for the definition of Massey products. It is well known and easy to see that Massey products yield an obstruction to formality [DGMS, RT1, TO]. In other words, if the space is formal then all Massey products must be trivial. Thus, all the Massey products in every Kähler manifold vanish.

We need also the following result of Miller [Mi]:

2.1. Theorem. Every closed simply-connected manifold $M$ of dimension $\leq 6$ is formal. In particular, all Massey products in $M$ vanish. □

The next homotopy property related to symplectic (in particular, Kähler) structures is the Hard Lefschetz property. Given a symplectic manifold $(M^{2m}, \omega)$, we denote by $[\omega] \in H^2(M)$ the de Rham cohomology class of $\omega$. Furthermore, we denote by $L_\omega : \Omega^k(M) \to \Omega^{k+2}(M)$ the multiplication by $\omega$ and by $L[k] : H^k(M) \to H^{k+2}(M)$ the induced homomorphism in the de Rham cohomology $H^*(M)$. As usual we write $L$ instead of $L_\omega$ or $L[k]$ if there is no danger of confusion. We say that a symplectic manifold $(M^{2m}, \omega)$ has the Hard Lefschetz property if, for every $k$, the homomorphism

$$L^k : H^{m-k}(M) \to H^{m+k}(M)$$

is surjective. In view of the Poincaré duality, for closed manifolds $M$ it means that every $L^k$ is an isomorphism. We need also the following result of Gompf [G1, Theorem 7.1].

2.2. Theorem. For any even dimension $n \geq 6$, finitely presented group $G$ and integer $b$ there is a closed symplectic $n$-manifold $M$ with $\pi_1(M) \cong G$ and $b_i(M) \geq b$ for $2 \leq i \leq n-2$, such that $M$ does not satisfy the Hard Lefschetz condition. Furthermore, if $b_1(G)$ is even then all degree-odd Betti numbers of $M$ are even. □

We denote such manifold $M$ by $M(n, G, b)$.

2.3. Remark. Theorem 7.1 in [G1] is formulated in a slightly different way, but the proof is based on constructing of $M$ by some "symplectic summation" in a way to violate the Hard Lefschetz property.

In our explicit constructions we will need some particular classes of manifolds, namely, nilmanifolds, resp. solvmanifolds. These are homogeneous spaces of the form $G/\Gamma$, were $G$ is a simply connected nilpotent, resp. solvable Lie group and $\Gamma$ is a co-compact discrete subgroup (i.e. a lattice). The most important information for us is the following (see e.g. [TO] for the proofs):

2.4. Recollection. (i) Let $g$ be a nilpotent Lie algebra with structural constants $c_{ij}^k$ with respect to some basis, and let $\{\alpha_1, \ldots, \alpha_n\}$ be the dual basis of $g^*$. Then the differential in the Chevalley–Eilenberg complex $(\Lambda^*g^*, d)$ is given by the formula

$$d\alpha_k = - \sum_{1 \leq i < j < k} c_{ij}^k \alpha_i \wedge \alpha_j.$$
(ii) Let $g$ be the Lie algebra of a simply connected nilpotent Lie group $G$. Then, by Malcev’s theorem, $G$ admits a lattice if and only if $g$ admits a basis such that all the structural constants are rational. Moreover, this lattice is unique up to an automorphism of $G$.

(iii) Let $G$ and $g$ be as in (ii), and suppose that $G$ admits a lattice $\Gamma$. By Nomizu’s theorem, the Chevalley–Eilenberg complex $(\Lambda^*g^*, d)$ is quasi-isomorphic to the de Rham complex of $G/\Gamma$. Moreover, $(\Lambda^*g^*, d)$ is a minimal differential algebra, and hence it is isomorphic to the minimal model of $G/\Gamma$:

$$(\Lambda^*g^*, d) \cong (M_{G/\Gamma}, d).$$

Also, any cohomology class $[a] \in H^k(G/\Gamma)$ contains a homogeneous representative $\alpha$. Here we call the form $\alpha$ homogeneous if the pullback of $\alpha$ to $G$ is left invariant.

Let $\omega_0$ be the standard symplectic form on $\mathbb{C}P^m$. Recall that every closed symplectic manifold $(M^{2n}, \omega)$ with integral form $\omega$ can be symplectically embedded into $\mathbb{C}P^m$ for $m$ large enough, with the (known) smallest possible value of $m$ equal to $n(n+1)$ [Gr, Ti]. We will use the blow-up construction with respect to such embedding [McD, RT1]. We need the following result.

2.5. Theorem. Let $(M^{2n}, \omega)$ be a closed connected symplectic manifold, let $i : (M, \omega) \rightarrow (\mathbb{C}P^m, \omega_0)$ be a symplectic embedding, and let $\overline{\mathbb{C}P}^m$ be the blow-up along $i$. Then the following holds:

(i) $\overline{\mathbb{C}P}^m$ is a simply-connected symplectic manifold;
(ii) if there exists $i$ such that $b_{2i+1}(M)$ is odd, then there exists $k$ such that $b_{2k+1}(\overline{\mathbb{C}P}^m)$ is odd;
(iii) if $M$ possesses a non-trivial Massey triple product and $m - n \geq 4$, then $\overline{\mathbb{C}P}^m$ possesses a non-trivial Massey triple product. Moreover, if there is a non-trivial Massey product $\langle \alpha, \beta, [\omega] \rangle \in H^*(M)$, $\alpha, \beta \in H^*(M)$, then $\overline{\mathbb{C}P}^m$ possesses a non-trivial Massey triple product even for $m - n = 3$.

Proof. (i) and (ii) are proved in [McD], (i) and (iii) are proved in [RT1]. □

3. Relation between homotopy properties of closed symplectic manifolds

3.1. Theorem. The relations between the Hard Lefschetz property, evenness of odd-degree Betti numbers and vanishing of the Massey products for closed symplectic manifolds are given by the following tables:

TABLE 1: symplectically aspherical case;
TABLE 2: simply-connected case.

The word Impossible in the table means that there is no closed symplectic manifold (aspherical or simply connected) that realizes the combination in the corresponding line.

The sign ? means that we (the authors) do not know whether a manifold with corresponding properties exists.
Table 1: Symplectically Aspherical Symplectic Manifolds

| Triviality of Massey Products | Hard Lefschetz Property | Evenness of $b_{2i+1}$ | Property                             |
|-------------------------------|-------------------------|-------------------------|-------------------------------------|
| yes                           | yes                     | yes                     | Kähler ($T^{2n}$)                   |
| yes                           | yes                     | no                      | Impossible                           |
| yes                           | no                      | yes                     | ?                                   |
| yes                           | no                      | no                      | ?                                   |
| no                            | yes                     | yes                     | ?                                   |
| no                            | yes                     | no                      | Impossible                           |
| no                            | no                      | yes                     | $K \times K$                        |
| no                            | no                      | no                      | $K$                                 |

Table 2: Simply-Connected Symplectic Manifolds

| Triviality of Massey Products | Hard Lefschetz Property | Evenness of $b_{2i+1}$ | Property                           |
|-------------------------------|-------------------------|-------------------------|------------------------------------|
| yes                           | yes                     | yes                     | Kähler ($CP^n$)                    |
| yes                           | yes                     | no                      | Impossible                          |
| yes                           | no                      | yes                     | $M(6,\{e\},0)$                     |
| yes                           | no                      | no                      | ?                                  |
| no                            | yes                     | yes                     | ?                                  |
| no                            | yes                     | no                      | Impossible                          |
| no                            | no                      | yes                     | $\mathbb{C}P^5 \times \mathbb{C}P^5$|
| no                            | no                      | no                      | $\mathbb{C}P^5$                    |
Proof. We prove the theorem via line-by-line analysis of Tables 1 and 2.

**Line 1 in Tables 1 and 2.** For closed Kähler manifolds, the Hard Lefschetz property is proved in [GH], the evenness of \(b_{2i+1}\) follows from the Hodge theory [W], the triviality of Massey products follows from the formality of any closed Kähler manifold [DGMS].

One can ask if there are non-Kähler manifolds having the properties from line 1. In the symplectically aspherical case the answer is affirmative. Let \(G = \mathbb{R} \times \varphi \mathbb{R}^2\) be the semidirect product determined by the one-parameter subgroup \(\varphi(t) = \text{diag}(e^{kt}, e^{-kt}), t \in \mathbb{R}, e^k + e^{-k} \neq 2.\) One can check that \(G\) contains a lattice, say \(\Gamma.\) Then the compact solvmanifold \(M = G/\Gamma \times S^1\)

is symplectic and has the same minimal model as the Kähler manifold \(S^2 \times T^2.\) Hence such manifold fits into line 1. It cannot be Kähler, since it admits no complex structure. The latter follows from the Kodaira–Yau classification of compact complex surfaces (see [TO] for details).

**Line 2 in Tables 1 and 2.** Any manifold satisfying the Hard Lefschetz property must have even \(b_{2i+1}.\) Indeed, consider the usual non-singular pairing \(p : H^{2k+1}(M) \otimes H^{2m-2k-1}(M) \to \mathbb{R}\) of the form

\[
p([\alpha], [\beta]) = \int_M \alpha \wedge \beta.
\]

Define a skew-symmetric bilinear form \(\langle - , - \rangle : H^{2k+1}(M) \otimes H^{2k+1}(M) \to \mathbb{R}\) via the formula

\[
\langle [\alpha], [\gamma] \rangle = p ([\alpha], L^{m-2k-1}[\gamma]),
\]

for \([\alpha], [\gamma] \in H^{2k+1}(M).\) Since this form is non-degenerate and skew-symmetric, its domain \(H^{2k+1}(M)\) must be even-dimensional, i.e. \(b_{2k+1}\) is even.

**Line 3 in Table 1.** We do not know any non-simply-connected (and, in particular, symplectically aspherical) examples to fill in this line.

**Line 3 in Table 2.** We use Theorem 2.2 with \(n = 6\) and \(G = \{e\}.\) Then, for every \(b\), the corresponding manifold \(M(6, \{e\}, b)\) has even odd-degree Betti numbers and does not have the Hard Lefschetz property. Furthermore, all the Massey products in \(M\) vanish by 2.1.

**Line 4 and 5 in Tables 1 and 2.** We do not know any examples to fill in these lines.

**Line 6 in Tables 1 and 2.** This is impossible, see the argument concerning line 2.

**Lines 7 and 8 in Table 1.** Consider the Kodaira-Thurston manifold \(K\) [Th]. Recall that this manifold is defined as a nilmanifold

\[
K = N_3/\Gamma \times S^1,
\]
where $N_3$ denotes the 3-dimensional nilpotent Lie group of triangular unipotent matrices and $\Gamma$ denotes the lattice of such matrices with integer entries. One can check that the Chevalley–Eilenberg complex of the Lie algebra $n_3$ is of the form

$$(\Lambda(e_1, e_2, e_3), d), \quad de_1 = de_2 = 0, \quad de_3 = e_1e_2.$$ 

with $|e_i| = 1$. We have already mentioned that the minimal model of any nilmanifold $N/\Gamma$ is isomorphic to the Chevalley–Eilenberg complex of the Lie algebra $n$. In particular, one can get the minimal model of the Kodaira–Thurston manifold in the form

$$(\Lambda(x, e_1, e_2, e_3), d), \quad dx = de_1 = de_2 = 0, \quad de_3 = e_1e_2$$ 

with degrees of all generators equal 1. One can check that the vector space $H^1(K)$ has the basis $\{[x], [e_1], [e_2]\}$. Hence, $b_1(K) = 3$, which also shows that $K$ does not have the Hard Lefschetz property. Furthermore, $K$ possesses a symplectic form $\omega$ with $[\omega] = [e_1e_3 + e_2x]$, and one can prove that the Massey triple product $\langle [e_1], [e_1], [\omega] \rangle$ is non-trivial. Thus, $K$ realizes Line 8 of Table 1.

Finally, $K \times K$ realizes Line 7 of Table 1.

**Lines 7 and 8 in Table 2.** We use Theorem 2.5. Consider a symplectic embedding $i : K \to \mathbb{C}P^m, m \geq 5$, and perform the blow-up along $i$. Then, by 2.5(i), $\tilde{\mathbb{C}P}^m$ is simply-connected. Furthermore, it realizes Line 8 of Table 2 by 2.5(ii) and 2.5(iii).

Finally, $\tilde{\mathbb{C}P}^m \times \tilde{\mathbb{C}P}^m$ realizes Line 7 of Table 2. □

**3.2. Remark.** The result of Lupton [L] shows that the problem of constructing a non-formal manifold with the Hard Lefschetz property turns out to be very delicate. In [L] there is an example of a DGA, whose cohomology has the Hard Lefschetz property, but which is not intrinsically formal. This means that there is also a non-formal minimal algebra with the same cohomology ring. Sometimes, using Browder–Novikov theory, one can construct a smooth closed manifold $M$ with such non-formal Sullivan minimal model. However, there is no way in sight to get a symplectic structure on $M$.

**4. Flexible symplectic manifolds**

Let $(M^{2m}, \omega)$ be a symplectic manifold. It is known that there exists a unique non-degenerate Poisson structure $\Pi$ associated with the symplectic structure (see, for example [LM, TO]). Recall that $\Pi$ is a skew symmetric tensor field of order 2 such that $[\Pi, \Pi] = 0$, where $[\cdot, \cdot]$ is the Schouten-Nijenhuis bracket.

The Koszul differential $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined for Poisson, in particular symplectic, manifolds as

$$\delta = [i(\Pi), d].$$

Brylinski has proved in [B] that the Koszul differential is a symplectic codifferential of the exterior differential with respect to the symplectic star operator. We choose the volume form associated to the symplectic form, say $v_M = \omega^m/m!$. Then we define the symplectic star operator

$$* : \Omega^k(M) \to \Omega^{2m-k}(M)$$
by the condition \( \beta \wedge (\ast \alpha) = \Lambda^k(\Pi)(\beta, \alpha)v_M \), for all \( \alpha, \beta \in \Omega^k(M) \). It turns out to be that
\[
\delta = (-1)^{k+1}(\ast \circ d \circ \ast).
\]

4.1. Definition. A \( k \)-form \( \alpha \) on the symplectic manifold \( M \) is called symplectically harmonic, if \( d\alpha = 0 = \delta\alpha \).

We denote by \( \Omega^k_{\text{hr}}(M) \) the space of symplectically harmonic \( k \)-forms on \( M \). We set
\[
H^k_{\text{hr}}(M, \omega) = \Omega^k_{\text{hr}}(M) / (\text{Im } d \cap \Omega^k_{\text{hr}}(M)), \quad h^k(M, \omega) = \dim H^k_{\text{hr}}(M, \omega).
\]

We say that a de Rham cohomology class is symplectically harmonic if it contains a symplectically harmonic representative, i.e. if it belongs to the subgroup \( H^*_{\text{hr}}(M) \) of \( H^*(M) \).

4.2. Definition. We say that a closed smooth manifold \( M \) is flexible, if \( M \) possesses a continuous family of symplectic forms \( \omega_t, t \in [a, b] \), such that \( h^k(M, \omega_a) \neq h^k(M, \omega_b) \) for some \( k \).

So, the McDuff–Khesin Question (see the introduction) asks about existence of flexible manifolds.

In order to prove our result on the existence of flexible 6-dimensional nilmanifolds, we need some preliminaries. The following lemma is proved in [IRTU] and generalizes an observation of Yan [Y].

4.3. Lemma. For any symplectic manifold \( (M^{2m}, \omega) \) and \( k = 0, 1, 2 \) we have
\[
H^{2m-k}_{\text{hr}}(M) = \text{Im } \{ L^{m-k} : H^k(M) \to H^{2m-k}(M) \} \subset H^{2m-k}(M).
\]

In other words,
\[
h_{2m-k}(M, \omega) = \dim \text{Im } \{ L^{m-k} : H^k(M) \to H^{2m-k}(M) \}. \quad \square
\]

The following fact can be deduced from 4.3 using standard arguments from linear algebra, see [IRTU].

4.4. Lemma. Let \( \omega_1 \) and \( \omega_2 \) be two symplectic forms on a closed manifold \( M^{2m} \). Suppose that, for \( k = 1 \) or \( k = 2 \), we have
\[
h_{2m-k}(M, \omega_1) \neq h_{2m-k}(M, \omega_2).
\]

Then \( M \) is flexible. \( \square \)

4.5. Proposition. Let \( G \) be a simply connected 6-dimensional nilpotent Lie group such that its Lie algebra \( \mathfrak{g} \) has the basis \( \{X_i\}^6_{i=1} \) and the following structure relations:
\[
[X_1, X_2] = -X_4, \quad [X_1, X_4] = -X_5, \quad [X_1, X_5] = [X_2, X_3] = [X_2, X_4] = -X_6
\]
(all the other brackets \( [X_i, X_j] \) are assumed to be zero). Then \( G \) admits a lattice \( \Gamma \), and the corresponding compact nilmanifold \( N := G/\Gamma \) admits two symplectic forms \( \omega_1 \) and \( \omega_2 \) such that
\[
\dim \text{Im } L^2_{[\omega_1]} = 0, \quad \dim \text{Im } L^2_{[\omega_2]} = 2.
\]
Proof. First, $G$ has a lattice by 2.4(ii). Furthermore, by 2.4(iii), in the Chevalley–Eilenberg complex $(\Lambda^* g^*, d)$ we have

\[
\begin{align*}
    d\alpha_1 &= d\alpha_2 = d\alpha_3 = 0, \\
    d\alpha_4 &= \alpha_1 \alpha_2, \\
    d\alpha_5 &= \alpha_1 \alpha_4, \\
    d\alpha_6 &= \alpha_1 \alpha_5 + \alpha_2 \alpha_3 + \alpha_2 \alpha_4, 
\end{align*}
\]

where we write $\alpha_i \alpha_j$ instead of $\alpha_i \wedge \alpha_j$. One can check that the following elements represent closed homogeneous 2-forms on $N$:

\[
\begin{align*}
    \omega_1 &= \alpha_1 \alpha_6 + \alpha_2 \alpha_5 - \alpha_3 \alpha_4, \\
    \omega_2 &= \alpha_1 \alpha_3 + \alpha_2 \alpha_6 - \alpha_4 \alpha_5. 
\end{align*}
\]

Since $[\omega_1^2] \neq 0 \neq [\omega_2^2]$, these homogeneous forms are symplectic. Indeed, by 2.4(iii) the cohomology classes $[\omega_1]$ and $[\omega_2]$ have homogeneous representatives whose third powers are non-zero. Then the same is valid for their pull-backs to invariant 2-forms on the Lie group $G$. But for invariant 2-forms this condition implies non-degeneracy. Since $G \to N$ is a covering, the homogeneous forms $\omega_1$ and $\omega_2$ on $N$ are also non-degenerate.

Obviously, the $\mathbb{R}$-vector space $H^1(N)$ has the basis $\{[\alpha_1], [\alpha_2], [\alpha_3]\}$. One can check by direct calculation that

\[
[\omega_1]^2[\alpha_i] = 0, \quad i = 1, 2, 3
\]

and that

\[
[\omega_2]^2[\alpha_1] = -2[\alpha_1 \alpha_2 \alpha_4 \alpha_5 \alpha_6], \quad [\omega_2]^2[\alpha_2] = 0, \quad [\omega_2]^2[\alpha_3] = 2[\alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6].
\]

Finally, it is straightforward that the above cohomology classes span 2-dimensional subspace in $H^5(N)$. □

4.6. Theorem. There exists a flexible 6-dimensional nilmanifold.

Proof. Consider the nilmanifold $N$ as in 4.5. Because of 4.3 and 4.5, we conclude that $h_5(N, \omega_1) = 0 \neq 2 = h_5(N, \omega_2)$,

and the result follows from 4.4. □

Acknowledgment. The first and the fourth authors were partially supported by the project UPV 127.310-EA147/98. This work was partially done in Oberwolfach and financed by Volkswagen-Stiftung. The second and third authors were also partially supported by Max-Planck Institut für Mathematik, Bonn.

REFERENCES

[B] J.-L. Brylinski, A differential complex for Poisson manifolds, J. Diff. Geom. 28 (1988), 93-114.

[FG] M. Fernández and A. Gray, Compact symplectic solvmanifold not admitting complex structures, Geom. Dedic. 34 (1990), 295-299.
[DGMS] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, Real homotopy theory of Kähler manifolds, Invent. Math. 29 (1975), 245-274.

[F] A. Floer, Symplectic fixed points and holomorphic spheres, Commun. Math. Phys 120 (1989), 575–611.

[G1] R. Gompf, A new construction of symplectic manifolds, Ann. Math. 142 (1995), 527-597.

[G2] R. Gompf, On symplectically aspherical manifolds with nontrivial π2, Math. Res. Letters 5 (1999), 599-603.

[GH] P. Griffiths and J. Harris, Principles of Algebraic Geometry, Wiley, New York, 1978.

[Gr] M. Gromov, Partial Differential Relations, Springer, Berlin, 1986.

[H] H. Hofer, Lusternik–Schnirelmann theory for Lagrangian intersections, Annales de l’inst. Henri Poincaré—analyse nonlineare 5 (1988), 465–499.

[IRTU] R. Ibáñez, Yu. Rudyak, A. Tralle and L. Ugarte, Symplectically harmonic forms on 6-dimensional nilmanifolds, to appear.

[K] D. Kraines, Massey higher products, Trans. Amer. Math. Soc. 124 (1966), 431-449.

[LM] P. Libermann and C. Marle, Symplectic Geometry and Analytical Mechanics, Kluwer, Dordrecht, 1987.

[L] G. Lupton, Intrinsic formality and certain types of algebras, Trans. Amer. Math. Soc. 319 (1990), 257-283.

[M] O. Mathieu, Harmonic cohomology classes of symplectic manifolds, Comment. Math. Helvetici 70 (1995), 1-9.

[Ma] J. P. May, Matric Massey products, J. Algebra 12 (1969), 533-568.

[McD] D. McDuff, Examples of symplectic simply connected manifolds with no Kähler structure, J. Diff. Geom. 20 (1984), 267-277.

[Mi] T. J. Miller, On the formality of (k – 1)-connected compact manifolds of dimension less than or equal to 4k – 2, Illinois J. Math. 23 (1979), 253-258.

[RO] Yu. Rudyak and J. Oprea, On the Lusternik–Schnirelmann category of symplectic manifolds and the Arnold conjecture, Math. Z. 230 (1999), 673-678.

[RT1] Yu. Rudyak and A. Tralle, On Thom spaces, Massey products and non-formal symplectic manifolds, available as DG9907035, MPI Preprint Series 71 (1999).

[RT2] Yu. Rudyak and A. Tralle, On symplectic manifolds with aspherical symplectic form, available as DG9908001, MPI Preprint Series 88 (1999).

[S] D. Sullivan, Infinitesimal computations in topology, Publ. Math. IHES 47 (1978), 269-331.

[Th] W. P. Thurston, Some simple examples of compact symplectic manifolds, Proc. Amer. Math. Soc. 55 (1976), 467-468.

[Ti] D. Tischler, Closed 2-forms and an embedding theorem for symplectic manifolds, J. Diff. Geom. 12 (1977), 229-245.

[TO] A. Tralle and J. Oprea, Symplectic manifolds with no Kähler structure, Springer, LNM 1661, 1997.

[W] R.O. Wells, Differential Analysis on Complex Manifolds, Prentice Hall, 1978.

[Y] D. Yan, Hodge structure on symplectic manifolds, Adv. Math. 120 (1996), 143-154.
E-mail address: mtpibtor@lg.ehu.es

FB6/Mathematik, Universität Siegen, 57068 Siegen, Germany
E-mail address: rudyak@mathematik.uni-siegen.de, july@mathi.uni-heidelberg.de

University of Warmia and Masuria, 10561 Olsztyn, Poland
E-mail address: tralle@tufi.wsp.olsztyn.pl

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain
E-mail address: ugarte@posta.unizar.es