Effectiveness of Variable Distance Quantum Error Correcting Codes

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Abstract

Quantum error correction is capable of digitizing quantum noise and increasing the robustness of qubits. Typically, error correction is designed with the target of eliminating all errors - making an error so unlikely it can be assumed that none occur. In this work, we use statistical quantum fault injection on the quantum phase estimation algorithm to test the sensitivity to quantum noise events. Our work suggests that quantum programs can tolerate non-trivial errors and still produce usable output. We show that it may be possible to reduce error correction overhead by relaxing tolerable error rate requirements. In addition, we propose using variable strength (distance) error correction, where overhead can be reduced by only protecting more sensitive parts of the quantum program with high distance codes.

1. Introduction

Physical quantum noise is hard to model accurately as it is highly complex and can have counter-intuitive impacts [4]. Additionally, the noise present in a physical system can change over time [16], making it difficult to properly characterize it via benchmarking procedures [17].

Quantum error correction (QEC) groups collections of physical qubits together to represent one logical qubit. The logical qubit is much more resilient to noise than the individual physical qubits it is made of. An additional useful property of QEC is that it restricts the quantum noise, forcing it to take on a restricted set of noise events, eg. X and Z errors. This allows us to accurately model the quantum noise acting on logical qubits with just X and Z errors [15].

For example, physical quantum noise can manifest as slight over- or under-rotations of individual qubits [1]. The angle of over- or under- rotation can be any amount. In some sense, the noise is analog in nature. QEC involves the measurement of ancillary qubits, which introduces non-unitary transformations of the quantum state (while leaving the computational quantum state intact). The measurement forces a decision on the error. Either the qubit snaps back to desired state, removing the noise, or it causes a complete noise event, flipping the state of the qubit. This is effectively a digitization of the noise into a set of errors. This digital nature of the errors allows them to be detected and corrected.

QEC can only detect and correct errors on physical qubits if the physical noise rates are sufficiently low. If a sufficient number of noise events occur on multiple physical qubits simultaneously, this may result in an error which is either undetectable or uncorrectable - a logical error. QEC is typically designed with the target of removing logical noise events entirely. If a QEC code can tolerate a large number of simultaneous physical noise events, it becomes very unlikely that sufficient number of simultaneous noise events will occur. The minimum number of simultaneous error a QEC code tolerate is called the code distance.

Unfortunately, the cost of creating large distance QEC codes is very high. The number of physical qubits required for each logical qubit grows quadratically with the code distance, potentially reaching thousands of physical qubits for each logical qubits [5], even for reasonable noise rates. The resources quickly grow beyond what can realistically be provided by the available hardware.

In this work we suggest reducing the code distance and risk allowing some logical errors to occur. This can significantly reduce the resources required to build a (nearly) fault-tolerant computer. If the quantum application can tolerate some amount of error (still produces the correct output with high probability), such a strategy will prove effective. There are two methods to attempt

1. Lowering the code distance overall (on all logical qubits at all times)
2. Selectively lowering the code distance for a subset of the qubits at specific times

In both cases, performing statistical fault injection with quantum applications will allow us to accurately estimate the probability of success, despite the existence of errors. Method 2 is enabled because Surface Codes [5] allow us to change the number of physical qubits which are dedicated to each logical qubit. Hence, statistical fault injection can allow us to detect more and less sensitive regions of the application, and dedicate the resources more efficiently based on need.

2. Background

Attempting to improve the reliability of quantum programs at the physical level is a well studied problem. Numerous works analyzed the effectiveness of scheduling quantum programs on more reliable physical qubits [16] and others have focused on swap minimization [3, 9]. Additionally, statistical fault injection was suggested for studying the sensitivity of programs at the physical level to enable more optimal gate scheduling and qubit placement [12]. An important distinction of this work is that it focuses on the logical level and is effectively quantum
software sensitivity analysis. This significantly changes the considerations.

First, noise can be modelled more accurately at the logical level as QEC forces noise to manifest as X and Z errors. This is in contrast to modelling noise as the physical level, where there are many different models, each noise model can result in different outcomes [11], and the noise seen in physical experiment changes over time [17].²

Second, the noise mitigation strategy is different. Work at the physical level will use gate scheduling, qubit mapping, and swap minimization to reduce noise. At the logical level, the level of QEC can be optimized. We can find the minimal amount of overall QEC for a given application, or we can fine tune the QEC for each qubit over time.

2.1. Surface Codes

Surface Codes are a promising form of QEC as they require only local interactions between physical qubits [5, 10]. Such operations can be performed easily on quantum computers which only allow nearest neighbor interactions, such as superconducting quantum computers. Surface codes can be conceptually visualized as “patched” of physical qubits, where the patches can be moved and interacted with each other [10]. Initially, qubit braiding was considered the ideal method of performing logic operations on qubits [5], however a new strategy called lattice surgery has become the state of the art [10, 6].

A surface code which provides a code distance of $d$ requires $d^2$ physical qubits per logical qubit. Logical gates on logical qubits require roughly $d$ time cycles [10]. Additionally, changing the $d$ for each qubit will take roughly $d$ cycles.

2.2. Gate Decomposition

When operating modern quantum computers without QEC, we have access to a wide variety of quantum gates (operations). The specific gate types available will be dependent on the specific machine. However, precise rotations around at least two axes are typically available. For example, IBM machines [2] offer $R_x$, $R_y$, and $R_z$ gates, which are rotations around the $x$, $y$, and $z$ axes of the Bloch sphere. Unfortunately, this gate set is not compatible with QEC. Hence, it cannot be used in fault-tolerant quantum computers.

In this work we use the Clifford+T gate set, which is the most widely used universal set which is also compatible with QEC. $R_x$, $R_y$, and $R_z$ gates are still required for many fault-tolerant quantum algorithms. In order to perform them, we will need to approximate them with sequences of gates within our Clifford+T set. This can be done with the gridsynth algorithm [14, 13], which converts $R_z(\theta)$ into sequences of $X$, $H$, $S$, and $T$ gates. The length of the sequence depends on the angle of rotation, $\theta$, and the precision to which we need to approximate it. For example, a rotation by $\pi/3$ can be done with

$$R_z(\pi/3) \approx H S T H S T H S T$$  (1)

Longer sequences can achieve better approximations. However, this also significantly increases the length of the program and increases the resources required (T gates are challenging to perform on the surface code [5]). We can use shorter sequences to reduce overhead, however this will introduce error into the program (even in the absence of noise). For example, using approximation rotations with a bit precision of 4 (average sequence length of 33.8) results in the quantum phase estimation algorithm producing the correct output only 50% of the time. Using a bit precision of 5 (average sequence length of 44) increases this to 98.5%.

It is sufficient to approximate $R_z$ gates, as any unitary operation $U$ can be implemented with its Euler angles

$$U = R_z(\beta) R_y(\gamma) R_z(\delta) = R_z(\beta) H R_z(\gamma) H R_z(\delta)$$  (2)

In this work we use the standard Clifford+T gates. However, further optimizations can reduce the circuit to operations more efficiently performed by Surfaces codes. Any circuit can be represented by a set of $T$ gates, followed by $S$ gates, and then finally by Pauli Measurements [10].

3. Trade-Off

Reducing the QEC code distance will reduce overhead and the latency it takes to complete the program. It will also increase the probability of failure, so it should be shown that the benefit it worth the cost.

Most quantum algorithms, even in the absence of noise, produce the correct result with some probability. This means the algorithm will have to be run multiple times before the correct answer is produced. How many times it will need to run depends on the probability of success. If an algorithm has a probability $P_{success}$ of success, it is expected that $\frac{1}{P_{success}}$ runs will be required to get the correct result, on average. Hence, the mean time until success is the metric of interest. If the algorithm has a latency of $L$, the mean time to success will be $L \times \frac{1}{P_{success}}$.

Hence, we can determine the success of reducing the QEC distance by comparing the mean time to success. The latency of the reduced QEC version, $L_{reduced}$ and the probability of success $P_{success_{reduced}}$ will both be lower than the originals

$$L_{reduced} < L$$

$$P_{success_{reduced}} < P_{success}$$  (3)

The reduction in the QEC overhead will be a sucess if

$$\frac{L_{reduced}}{P_{success_{reduced}}} < \frac{L}{P_{success}}$$  (4)

²In fact, statistical fault injection at the physical level produced contradicting conclusions depending on the noise model used [12].
Figure 1: Sensitivity heatmap of a 6-qubit QPE algorithm. Thin green lines represent insensitive regions and thick red lines indicate sensitive regions. Vertical lines are single qubit gates and arrows between qubits are two qubit gates. Lines terminate when the last operation on the qubits have been completed, and they can be measured early. Qubit 5 has no operations on it after state preparation.

4. Quantum Sensitivity Analysis

If we allow logical errors to occur, it is of interest how detrimental they will be to the success of the program. The impact of a logical error depends on the time (cycle) and space (qubit) on which it occurs [12]. We can gather this information with the classical computer architecture strategy of statistical fault injection, where errors are inserted at different locations and we view the impact on the output of the program. Since we are working with QEC, we can assume that errors are restricted to X and Z error (Y errors are a combination of X and Z errors). As a case study, we look at the quantum phase estimation (QPE) algorithm, which consists of the inverse quantum fourier transform (QFT). QPE is representative of algorithms a fault-tolerant quantum computer is likely to run, and it can be used to solve important problems such as quantum chemistry [8].

The input to QPE is a quantum state which has gone through a noiseless state preparation, where a sequence of controlled-phase gates are performed from qubits 0-4 to qubit 5. This state is chosen so that a noiseless QPE implementation will produce a single output with high probability (> 98%). This allows easy validation of the classical measurement values.

Statistical fault injection is performed by running many individual simulations, where in each an error is inserted on a single gate. We use Qiskit’s [2] depolarizing noise model with the error rate set to 100%, where X, Z, and Y (both X and Z) are all equally likely. We evaluate the quality of the output by comparing the probability of successful trial (PST) relative to that of the noiseless output. Given we set the input so that there is one correct answer, this metric can be written as:

\[
\text{Relative PST} = \frac{P_{\text{correct, noisy}}}{P_{\text{correct, ideal}}}
\]

A heatmap of the relative PST is shown in Figure 1, where the significance of error at each location can be visually inspected. Thick red lines indicate sensitive regions. Notably, qubits become more sensitive after they have been the control qubit of a two-qubit gate. Once entangled in this manner, they remain sensitive to the end of the program.

5. Variable Strength QEC

Given knowledge of the underlying sensitivity of a quantum program, it should be possible to more appropriately apply QEC. We consider here the idea of applying variable distance QEC, where different logical qubits are given different levels (different code distances \(d\)) of QEC at different times during the program. The core idea is: If a logical qubit is less susceptible to logical-level noise, we can risk applying lower error correction to it.

We start by acknowledging a fundamental limitation to this idea. The logical noise rate decreases exponentially with \(d\). The physical qubit count increases with \(d^2\). The time overhead increases linearly with \(d\). Hence, we can save quadratic space and linear time, but risk exponential increases in noise. This suggests this approach can easily backfire if applied too aggressively.

5.1. Success Rate as a Function of Code Distance

We estimate the probability of logical error on a qubit, \(P_L\), from the physical error rate, \(p\), for code distance \(d\) with the analytical formula provided by Fowler et. al. for surface codes [5]:

\[
P_L \approx 0.03 \times (p/0.0057)^{(d+1)/2}
\]

The probability of successful trial (PST) is the probability that we will measure the correct result at the end of the quantum program. With sufficiently high \(d\), no logical errors will occur, and PST will be 1 (for quantum programs that have a single, correct output). As \(d\) decreases, there will be an exponential decrease of PST until it hits nearly 0. Where this decay occurs is a function of the physical noise rate. Figure 2 shows the PST of the quantum phase estimation (QPE) algorithm for different \(d\) over a wide range of physical noise rates. The higher \(d\) is, the higher physical noise than can be tolerated.

Using information from Section 4, we know which logical qubits are more sensitive on QPE. This allows us to define
QEC which uses variable distance, depending on the underlying sensitivities. We define QEC which is combinations of two different distances. For example, 3, 5 is a QEC which uses \( d = 3 \) on less susceptible qubits and \( d = 5 \) on more susceptible qubits. The dashed lines in Figure 2 are variable distance codes, and it can be seen that they have a resilience in-between their constituent code distances.

6. Time to Solution

We can combine the sensitivity information from Section 4 along with the associated latency for each logic operation for a given code distance to estimate the time to solution. As noted in Section 3, the time to solution will be \( L_{\text{P\_success}} \).

We estimate \( P_{\text{success}} \) by finding the probability of two possibilities:
1. No logical error occurs
2. A single logical error occurs, but the output is the same as in the case of no error
These two probabilities combined provide a lower bound on \( P_{\text{success}} \). It is also possible that two or more logical errors occur and the output remains the same. However, counting these possibilities quickly becomes intractable because a total of \((N_{\text{gates}})^N_{\text{errors}}\) must be considered, where \( N_{\text{gates}} \) is the number of quantum gates performed in the algorithm and \( N_{\text{errors}} \) is the number of possible errors considered. Hence, we consider two or more errors to be a failure.

A gate on a logical qubit with distance \( d \) takes \( d \) cycles to complete. Since it is known what the distance is of each logical qubit throughout the program, we can easily find the latency \( L \). When we combine this latency with \( P_{\text{success}} \), this provides the time to solution, which is shown in Figure 3. It is noteworthy that the optimal code distance depends on the error rate. At low error rates lower code distances are preferable due to lower overhead. However, as the error rate increases the codes begin to fail. Since error suppression is exponential, once the codes begin to fail they dramatically drop in reliability; \( P_{\text{success}} \) drops quickly. Hence, a large number of trials are going to be required to obtain the correct solution, leading to a large latency.

Each variable distance code is optimal within a range of error rates. For example, the code distance \( d = 3, 5 \) is optimal at error rates where \( d = 3 \) begins to fail. \( d = 3, 5 \) can provide a faster solution until it breaks down and \( d = 5 \) will be necessary. However, interestingly, variable distance codes tend to provide the best solution for most error rates. When \( d = 3, 5 \) fails, \( d = 5, 7 \) is already preferable to \( d = 5 \), as shown in Figure 3.

7. Practicality

We have shown that variable distance QEC can provide a benefit at a specific range of error rates. However, we should note variable QEC can be challenging to practically implement. In expected superconducting quantum architectures, for which surface codes are most appropriate, logical qubits are arranged next to each other in a two-dimensional lattice [7]. Non-uniform layouts, which would be incurred by variable QEC, would result in wasted physical qubits. Additionally, state-of-the-art quantum computer architecture is limited by magic state distillation to perform T-gates [10], hence improvements on the gate speed of logical qubits may not be significantly
8. Conclusion

In this work we suggested both the decrease in the QEC code distance and the application of variable distance codes. We showed with statistical fault injection that quantum programs can be relatively insensitive to isolated logical errors. Additionally, we demonstrated that variable distance QEC codes can reduce the time to solution - by using profiling information from the fault injection process. However, we note that such strategies are fundamentally limited by the exponentially increasing risk of error with decreasing code distances.

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