GEOMETRIC INVARIANT THEORY
FOR PRINCIPAL THREE-DIMENSIONAL SUBGROUPS
ACTING ON FLAG VARIETIES

HENRIK SEPPÄNEN AND VALDEMAR V. TSANOV

ABSTRACT. Let $G$ be a semisimple complex Lie group. In this article, we study Geometric Invariant Theory on a flag variety $G/B$ with respect to the action of a principal 3-dimensional simple subgroup $S \subset G$. We determine explicitly the GIT-equivalence classes of $S$-ample line bundles on $G/B$. We show that, under mild assumptions, among the GIT-classes there are chambers, in the sense of Dolgachev-Hu. The Hilbert quotients $Y = X//S$ with respect to various chambers form a family of Mori dream spaces, canonically associated with $G$. We are able to determine the three important cones in the Picard group of any of these quotients: the pseudo-effective, the movable and the nef cones.

CONTENTS

Introduction 1
1. Setting 4
  1.1. The flag variety $G/B$ and GIT for subgroups 4
  1.2. The principal subgroup $S \subset G$ 6
  1.3. Representations, coadjoint orbits and momentum maps 7
2. The $S$-action on $G/B$ and GIT 8
  2.1. The orbit structure 8
  2.2. The Hilbert-Mumford criterion 9
  2.3. The Kirwan stratification and the $S$-ample cone 10
  2.4. GIT classes of ample line bundles 11
  2.5. Unstable divisors 12
3. Quotients and their Picard groups 13
References 14

INTRODUCTION

Let $G$ be a semisimple complex Lie group. We explore the interaction of two remarkable objects from the theory of semisimple groups - flag varieties and principal 3-dimensional simple subgroups. One context in which these two objects interact is Geometric Invariant Theory (GIT), quotients, and their variations (VGIT). As a result, we obtain a nontrivial example, whose strong properties allow for various explicit calculations. This example stems from the structure of $G$, which makes it interesting in its own right, supplies the computational tools, and presents a potential field for applications. In particular, we find a family of varieties, obtained as Hilbert quotients of the flag variety with respect to a principal subgroup. These varieties are Mori dream spaces, canonically associated to $G$. What we present in this article are some initial results. It becomes evident from these results, that this example could be

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developed further to illustrate further elements of the theoretical framework of GIT in the sense of [Kir84], [T96], [DH98], [KKV89], [S14]. Our main source for the properties of principal subgroups is the classical work of Kostant, [Kos59].

We summarize some of our results in the following theorem, which is a compilation of Proposition 2.2, Theorem 2.5, Theorem 2.9, Theorem 3.1, Theorem 3.2. To state the theorem, we briefly introduce some standard notation. We recall some definitions and basic properties in the next section. Assume that $G$ is connected and simply connected. Let $H \subset B \subset G$ be nested Cartan and Borel subgroups. Let $W = N_G(H)/H$ be the Weyl group and $w_0$ be the longest element with respect to $B$. Let $\Lambda$ be the weight lattice of $H$, let $\Lambda^+$ and $\Lambda^{++}$ denote the monoids of dominant and strictly dominant weight with respect to $B$. Let $X = G/B$ be the flag variety. We have $\text{Pic}(X) \cong \Lambda$, with $\mathcal{L}_\lambda = G \times_B \mathbb{C}_\lambda$. The sets $\Lambda^+$ and $\Lambda^{++}$ represent respectively the sets of effective and ample line bundles. Furthermore, all ample bundles are very ample and $G$-equivariant. By the Cartan-Weyl classification of irreducible modules and the Borel-Weil theorem, the space of global section of any effective line bundle is an irreducible $G$-module and all irreducible $G$-modules are obtained this way, $H^0(X, \mathcal{L}_\lambda) = V_\lambda^*$ for $\lambda \in \Lambda^+$. Let $S \subset G$ be a principal 3-dimensional simple subgroup with Cartan and Borel subgroups $H_S \subset B_S \subset S$ satisfying $H_S = S \cap H$ and $B_S = S \cap B$. The line bundle $\mathcal{L}_\lambda$ on $X$ being $G$-equivariant, is also $S$-equivariant, hence there are well-defined notions of stability, instability and semistability on $X$ with respect to $S$ and $\mathcal{L}_\lambda$. There is also a Hilbert quotient $Y = X//S$ for $S$-ample line bundles.

**Theorem:** Assume that every simple factor of $G$ has at least 5 positive roots. The following hold:

(i) The $S$-orbits of $X$ of dimension less than 3 are exactly the orbits through $H$-fixed points, $X^H = \{x_w = wB : w \in W\}$. There is an unique 1-dimensional orbit $Sx_1 = Sx_{w_0} \cong S/B_S \cong \mathbb{P}^1$. There are $\frac{1}{2}|W| - 1$ two-dimensional orbits $Sx_w = Sx_{w_0w} \cong S/H_S \cong (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \text{diag}(\mathbb{P}^1)$. The rest of the orbits are three dimensional with trivial or finite abelian isotropy groups.

(ii) All ample line bundles on $X$ are $S$-ample, i.e., some power admits $S$-invariant sections. The $S$-unstable locus of any ample line bundle on $X$ has codimension at least 2. The GIT-equivalence classes of $S$-ample line bundles on $X$ are defined by the subdivision of the dominant Weyl chamber $\Lambda^+_R$ by the system of hyperplanes $\mathcal{H}_w, w \in W$ given by

$$\mathcal{H}_w = \{\lambda \in \Lambda_R : \lambda(wh) = 0\},$$

where $h$ is an arbitrary fixed nonzero element in the Lie algebra of $H_S$.

(iii) The GIT-equivalence classes given by the connected components of $\Lambda^+_R \setminus (\cup_w \mathcal{H}_w)$ are chambers, in the sense of Dolgachev-Hu, which in our case means that the semistable locus consists only of 3-dimensional orbits. The hyperplanes $\mathcal{H}_w$ are walls, in the sense of Dolgachev-Hu.

(iv) The Hilbert quotient $Y = X//S$ with respect to any fixed chamber $\mathcal{C} \subset \Lambda^+_R \setminus (\cup_w \mathcal{H}_w)$ is a geometric quotient. The variety $Y$ is a Mori dream space, whose Picard group is a lattice of the same rank as $\Lambda$. There is an isomorphism of $\mathbb{Q}$-Picard groups $\text{Pic}(X)_\mathbb{Q} \cong \text{Pic}(Y)_\mathbb{Q}$ induced by descent in one direction, and pullback followed by extension in the other. There are the following isomorphisms involving the pseudo-effective, movable, and nef cones in the Picard group of $Y$:

$$\text{Eff}(Y) = \text{Mov}(Y) \cong \Lambda^+_R, \quad \text{Nef}(Y) \cong \mathcal{C}.$$
Moreover, every nef line bundle on $Y$ is semi-ample, i.e., admits a base-point-free power.

(v) Fix $Y$ as in (iv). For every $\lambda \in \Lambda^+$, there exists $k \in \mathbb{N}$ such that $(V_{k\lambda})^S \neq 0$ and a line bundle $\mathcal{L}$ on $Y$ such that

$$H^0(Y, \mathcal{L}^j) \cong (V_{k\lambda})^S$$

for all $j \in \mathbb{N}$.

Remark: The assumption made in the above theorem allows us to reduce the technicality of the statement. The excluded cases are those $G$ admitting simple factors of type $A_1$, $A_2$, $B_2$. They are accessible with our methods and indeed many of the propositions in the main text are proven without this assumption. Recall that $\dim X$ is equal to the number of positive roots of $G$. The particularities of the excluded cases are due to the low dimension of the respective factors of $X$, which results in low codimension of the unstable locus for some line bundles. Let us note here the following, in order to give an idea of the occurring phenomena. The presence of simple factors of $G$ of rank 1 implies the existence of ample line bundles on $X$, which are not $S$-ample, and with respect to which the whole $X$ is unstable. The presence of simple factors of type $A_2$ or $B_2$ implies the existence of $S$-ample line bundles whose unstable loci contain divisors. This interferes in the relations between the Picard groups of $X$ and the Hilbert quotient.

Geometric Invariant Theory finds one of its applications in the theory of branching laws for reductive groups, a.k.a. eigenvalue problem. This relates to part (v) of the above theorem. Let us outline the general ideas or order to see how our example fits in, what phenomena it exhibits, reductive groups, a.k.a. eigenvalue problem. This relates to part (v) of the above theorem. Let $S$ be a finitely generated submonoid of $\mathbb{R}$, the nulleigencone, cf. [R10]. In the particular case of a principal subgroup $S \subset G$, the nulleigencone was computed as an example by Berenstein and Sjamaar, [BS00]. The global description of the multiplicities still presents an open problem in the general situation. There are results concerning specific weights, e.g. Kostant’s multiplicity formula, cf. [V78]. There are also methods for specific types of subgroups, e.g. Littlemann’s path method, [L95], the method of Berenstein-Zelevinski, [BZ01]. Recently, the first author has constructed a global Okounkov body, $\Delta_Y$ (we do not use this notation elsewhere in the text), a strongly convex cone with a surjective map $p : \Delta_Y \to \text{Cone}(\mathcal{E})$, such that the fibre $\Delta_Y(\hat{\lambda}, \lambda) = p^{-1}(\hat{\lambda}, \lambda)$ is in turn Okounkov bodies, whose volume varies along the ray $\mathbb{R}_+(\hat{\lambda}, \lambda)$ asymptotically as the dimension of $(\hat{V}_\lambda^* \otimes V_\lambda)^G$, cf. [S14]. This result is proven under the assumption that there are

$$\mathcal{E}(\hat{G} \subset G) = \{(\hat{\lambda}, \lambda) \in \hat{\Lambda}^+ \times \Lambda^+ : \text{Hom}_G(\hat{V}_\lambda, V_\lambda) \neq 0\}$$

$$m(\lambda) = \dim \text{Hom}_G(\hat{V}_\lambda, V_\lambda).$$

The relation to invariant theory comes via the isomorphisms

$$\text{Hom}(\hat{V}_\lambda, V_\lambda) \cong (V_\lambda^* \otimes V_\lambda)^\hat{G} \cong H^0(\hat{G}/\hat{B} \times G/B, \hat{\mathcal{L}}_{-w_0\lambda} \otimes L_\lambda)^\hat{G}.$$
chambers among the GIT-classes of line bundles on $\hat{G}/\hat{B} \times G/B$. This existence of chambers is not guaranteed for any action of a reductive subgroup $\hat{G} \subset G$ on $G/B$. There are also problems related to thick walls. The results of [SL14] do not admit an immediate generalization encompassing the nulleigencone $\text{Cone}(\mathcal{E}_0)$ for arbitrary subgroups $\hat{G} \subset G$. In the case of a principal subgroup $S \subset G$, however, our results show that there are no thick walls, and the dimension of $V_\mathcal{S}$ could be measured, asymptotically, by volumes of slices of a convex cone. This convex cone is a global Okounkov body for one fixed quotient $Y = X//S$, and carries the information for all dominant weights $\lambda$. Thus the body is not uniquely determined, it depends on a choice of the quotient $Y$ and a flag of subvarieties in $Y$. One open question, relevant for combinatorial interpretations of the results, is whether the Okounkov body can be made polyhedral. This gives a motivation for further study of the geometry of the quotients $Y = X//S$ and the Hilbert functions of their line bundles.

1. Setting

1.1. The flag variety $G/B$ and GIT for subgroups. Let $G$ be a connected, simply connected semisimple complex Lie group. Let $B \subset G$ be a Borel subgroup and $X = G/B$ be the flag variety of $G$. Let $H \subset B$ be Cartan subgroup and $\Delta = \Delta^+ \sqcup \Delta^-$ be the root systems of $G$ with respect to $H$, split into positive and negative part with respect to $B$. Let $\Pi$ be the set of simple roots. Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the associated triangular decomposition of the Lie algebra of $G$, where $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is the nilradical of $\mathfrak{b}$ and $\mathfrak{n}$ is the nilradical of the opposite Borel $\mathfrak{b}$. The Weyl group $W = N_G(H)/H$ acts simply transitively on the Borel subgroups of $G$ containing $H$, and thus on the set of $H$-fixed points $X^H = \{x_B = wB, w \in W\}$. The $B$-orbits in $X$ define the Schubert cell decomposition

$$X = \bigcup_{w \in W} Bx_w.$$  

The unipotent radical $N \subset B$ acts transitively on each Schubert cell and, if $N_{x_w}$ denotes the stabilizer of $x_w$, the set of positive roots is partitioned roots as $\Delta^+ = \Delta(N_{x_w}) \cup \Delta(N/N_{x_w})$, with $\Delta(N/N_{x_w})$ being the set of weights for the $T$-action on $T_{x_w}Bx_w$. We have $\Delta(N/N_{x_w}) = \Delta^+ \cap w\Delta^-$, this set is called the inversion set of $w^{-1}$, if we adhere to the popular notation $\Phi_w = \Delta^+ \setminus w^{-1}\Delta^-$. The two sets $\Phi_w$ and $\Phi_{w^{-1}}$ have the same number of elements, the length $l(w)$ of $w$, also equal to the number of simple reflections in a reduced expression for $w$. Thus $\dim Bx_w = l(w)$. Inversion sets are closed and co-closed under addition in $\Delta^+$, we have $\Delta^+ \setminus \Phi_w = \Phi_{w^{-1}}$, where $w_0$ is the longest element of $w$. Thus $\Delta(N_{x_w}) = \Phi_{w_0w^{-1}}$ and $\Delta(N/N_{x_w}) = \Delta(N_{x_{w_0w}}) = \Phi_{w^{-1}}$.

Let $\Lambda \in \mathfrak{h}^*$ denote the weight lattice of $H$, $\Lambda^+$ the set of dominant weights with respect to $B$ and $\Lambda^{++}$ the set of strictly dominant weights, i.e., those belonging to the interior of the Weyl chamber. For $\lambda \in \Lambda^+$, let $V_{\lambda}$ denote an irreducible $G$-module with highest weight $\lambda$ and let $v^\lambda$ be the highest weight vector of $G$, the unique $B$-eigenvector. We have an equivariant orbit-map $X \to G[v^\lambda] \subset \mathbb{P}(V_{\lambda})$, and this is the unique closed orbit of $G$ in $\mathbb{P}(V_{\lambda})$. We have $\varphi(x_w) = [v^{\lambda}]$. The map $\varphi$ is an embedding if and only if $\lambda \in \Lambda^{++}$. For $\lambda \in \Lambda^+ \setminus \Lambda^{++}$, the orbit $G[v^\lambda]$ is a partial flag variety $G/P$, where $P \supset B$ is a parabolic subgroup of $G$. We shall focus mostly on the complete flag variety $G/B$ and the interior of the Weyl chamber.

Every line bundle on $X$ admits a unique $G$-linearization. The Picard group of $X$ is identified with the weight lattice. For $\lambda \in \Lambda$, we denote by $L_{\lambda} = G \times_B \mathbb{C}_{-\lambda}$ the associated homogeneous line bundle on $X$. For dominant $\lambda$, $L_{\lambda}$ is the pullback of $\mathcal{O}_{\mathbb{P}(V_{\lambda})}(1)$. The Borel-Weil theorem asserts that

$$H^0(X, L_{\lambda}) \cong V_{\lambda} \quad \text{for} \quad \lambda \in \Lambda^+ \quad \text{and} \quad H^0(X, L_{\lambda}) = 0 \quad \text{for} \quad \lambda \notin \Lambda \setminus \Lambda^+.$$
In invariant theory one considers a subgroup $\hat{G} \subset G$ and the space of $\hat{G}$-invariant vectors $V^G_{\lambda}$. We restrict ourselves to the case when $\hat{G}$ is semisimple, and our results concern the very special case of a principal simple subgroup of rank 1, but now we outline the general scheme. The first natural questions one may ask are: What is the dimension of the space of invariants and when is it nonempty? How does it vary with $\lambda$? Two central objects associated with these questions are the nilleigenmonoid, or null-Littlewood-Richardson monoid (which is indeed finitely generated submonoid of $\Lambda^+$ by a theorem of Brion and Knop), and the multiplicity

$$E_0(\hat{G} \subset G) = \{ \lambda \in \Lambda : V^G_{\lambda} \neq 0 \} \quad m_{\lambda} = \dim V^G_{\lambda}.$$ 

Note that $V^G_{\lambda}$ has a canonical nondegenerate pairing with $(V^G_{\lambda})^*$ and recall that $V^G_{\lambda} \cong V_{-w_0\lambda}$, whence $E^*_0 = -w_0E_0 = E_0$ and $m_{\lambda} = m_{-w_0\lambda}$.

The Borel-Weil theorem allows to rephrase these questions in terms of invariant sections of line bundles, and $E_0^{++} = E_0 \cap \Lambda^{++}$ corresponds to the cone of $\hat{G}$-linearized ample line bundles in the Picard group of $X$. Of particular interest is the variation of the multiplicity as the weight varies along a ray, $m_{k\lambda}$, $k \in \mathbb{N}$. Since $\mathcal{L}_k^\lambda = \mathcal{L}_{k\lambda}$, this relates to the ring of $\hat{G}$-invariants in the homogeneous coordinate ring of $X \subset \mathbb{P}(V_{\lambda})$, which is given by

$$R(\lambda) = \bigoplus_{k \in \mathbb{N}} H^0(X, \mathcal{L}_\lambda) = \mathbb{C}[V_{\lambda}] / I(X), \quad R_k(\lambda) = V^*_k(\lambda).$$

In this setting, there are the notions of instability, semistability and stability on $X$, with respect to $\mathcal{L}_{\lambda}$, defined by

$$X_{us}(\lambda) = X_{us,\hat{G}}(\mathcal{L}_\lambda) = \{ x \in X : f(x) = 0, \forall f \in R(\lambda)^G \setminus \mathbb{C} \}$$
$$X_{ss}(\lambda) = X_{ss,\hat{G}}(\mathcal{L}_\lambda) = \{ x \in X : \exists f \in R(\lambda)^G \setminus \mathbb{C}, f(x) \neq 0 \}$$
$$X_s(\lambda) = X_{s,\hat{G}}(\mathcal{L}_\lambda) = \{ x \in X : \hat{G}x \text{ is finite and } \hat{G}x \subset X_{ss}(\lambda) \text{ is closed} \}.$$

With these definitions, we have, for $\lambda \in \Lambda^+$.

$$\lambda \in \text{Cone}(E_0) \iff X_{ss}(\lambda) \neq \emptyset.$$

This relation has been the basis for the descriptions of eigencone initiated with Heckman’s thesis and culminating with Ressayre’s minimal list of inequalities defining Cone($E_0$). We are going to consider subgroups for which Cone($E_0$) has a fairly simple structure. In fact, in many cases we will have Cone($E_0$) = $\Lambda^+_R$. We are rather interested in the structure of the unstable, semistable and stable loci. Let us note, however, that the cases when Cone($E_0$) $\neq$ $\Lambda^+_R$ are indeed of specific interest for the study of branching laws, and we shall see a manifestation of this later on.

**Remark 1.1.** The group $\hat{G}$ necessarily has closed orbits in $X$. These closed orbits are flag varieties of $\hat{G}$. In fact, they are all complete flag varieties, i.e., have the form $\hat{G} / \hat{B}$, because the all isotropy groups in $X$ are solvable. Thus the closed $\hat{G}$-orbits are parametrized by the Borel subgroups of $G$ containing a fixed Borel subgroup $\hat{B} \subset \hat{G}$. Note that

$$\hat{G}x \subset X \text{ closed } \implies \hat{G}x \subset X_{us}(\lambda) \text{ for all } \lambda \in \Lambda^{++}.$$ 

We end this section by recalling the notions of GIT-equivalence classes and chambers, following Dolgachev and Hu, [DH98]. We have $\Lambda_R \cong \text{Pic}(X)_R$. The dominant Weyl chamber $\Lambda^+_R$ is identified with the pseudo-effective cone $\overline{\text{Eff}}(X)$. Then Cone($E_0$) $\cong C^\hat{G}(X)$ is the $\hat{G}$-ample cone on $X$.

**Definition 1.1.** Two $\hat{G}$-ample line bundles $\mathcal{L}_{\lambda_1}$ and $\mathcal{L}_{\lambda_1}$ on $X$ are called GIT-equivalent, if they have the same semistable loci, i.e., $X_{ss}(\lambda_1) = X_{ss}(\lambda_2)$. If $D$ is an equivalence class of line bundles, we denote by $X_{ss}(D)$ the corresponding semistable locus.
Recall that the equivalence relation on line bundles in $C^\hat{G}(X)$ is extended to an equivalence relation on $C^\hat{G}(X)$, by a natural extension of the notion of stability and semistability to $\mathbb{R}$-divisors, cf. [DH98]. The following definition singles out a specific type of equivalence classes, chambers, the existence of which has remarkable consequences, as shown by Dolgachev and Hu. The definition we adopt here differs from the original one in [DH98], but is shown therein to be an equivalent characterization.

**Definition 1.2.** An equivalence class $C \subset C^\hat{G}(X)$ is called a chamber, if $X_{ss}(C) = X_s(C)$.

In what follows, unless otherwise specified, we apply the notation $E_0, X_{us}$ etc. for the case $\hat{G} = S$, where $S$ is the principal subgroup of $G$ defined in the next section.

**1.2. The principal subgroup $S \subset G$.** Among the conjugacy classes of three dimensional simple subgroups of a semisimple complex Lie group $G$, there is a distinguished one - the class of principal subgroups, cf. [Kos59]. They admit several characterizations. Since we focus on the flag variety $X = G/B$, we define a principal subgroup $S \subset G$ to be a three dimensional simple subgroup with a unique closed orbit in $X$. Such an orbit is a rational curve, which we call the principal curve $C \subset X$. In the next proposition we recall other characterizations of principal subgroups.

**Proposition 1.1.** Let $s \subset g$ be a three dimensional simple subalgebra, i.e., $s \cong \mathfrak{sl}_2 \mathbb{C}$, and let $S \subset G$ be the corresponding subgroup. Let $\{e_+, h_0, e_-\} \subset s$ be a standard $\mathfrak{sl}_2$-triple. Then the following are equivalent:

(i) $S$ has a unique closed orbit in $X$.

(ii) Every Borel subgroup of $S$ is contained in a unique Borel subgroup of $G$.

(iii) These exists a unique triangular decomposition $g = n \oplus h \oplus \bar{n}$ compatible with $s = \mathbb{C}e_+ \oplus \mathbb{C}h_0 \oplus \mathbb{C}e_-$.

(iv) $e_+$ is contained in a unique maximal subalgebra $n \subset g$ of nilpotent elements. (The elements with this property are called principal nilpotent elements. They form a single conjugacy class.)

(v) If $n \subset g$ is any maximal subalgebra of nilpotent elements containing $e_+$, $h \subset g$ is a Cartan subalgebra normalizing $n$ and $e_\alpha \in n, \alpha \in \Delta^+$ are the root vectors, upon writing $e_+ = \sum_{\alpha \in \Delta^+} c_\alpha e_\alpha$,

we have $c_\alpha \neq 0$ for all simple roots $\alpha$.

Furthermore, all subalgebras (respectively subgroups) of $g$ (respectively $G$) with the above properties form a single conjugacy class.

From now on we fix a principal subgroup $S \subset G$ and a triple $\{e_+, h_0, e_-\} \subset s$ and the associated triangular decomposition of $g$. We may further take the nilpotent element to be the sum of the simple root vectors:

$$e_+ = \sum_{\alpha \in \Pi} e_\alpha.$$

Then we necessarily have

$$\alpha(h_0) = 2 \quad \text{for all} \quad \alpha \in \Pi,$$

which determines $h_0$ uniquely in $h$. We denote by $\mathfrak{h}_S = \mathbb{C}h_0$ and $\mathfrak{b}_S = \mathbb{C}h_0 \oplus \mathbb{C}e_+$ the Cartan and Borel subalgebras of $s$ associated with the given triple, respectively, and by $H_S$ and $B_S$ the corresponding subgroups of $S$. For the attributes of $G$ we use the notation introduced earlier in the text, with reference to the given triangular decomposition.
Remark 1.2. For any finite dimensional $G$-module $V$, we have

\[ V^G = V^H \cap V^S. \]

In what follows, we shall make extensive use of restrictions of weights from $H$ to $H_S$. We denote the inclusion map by $i : S \subset G$, and keep the same notation for the restriction of $i$ to subgroups; we denote by $i^*$ the resulting restrictions and pullbacks. In particular, for weights we have

\[ i^* : \Lambda \to \Lambda_S \cong \mathbb{Z}, \quad \nu \mapsto \nu(h_0). \]

This map determined by the values of the simple roots \([\Pi]\). However, weights, especially dominant weights, are often given in terms of the fundamental weights $\omega_\alpha$. The values of fundamental weights on the principal element $h_0$ can be computed using the classification and structure of root systems. We record in the next proposition some inequalities, which we need for our estimates on codimension of unstable loci.

**Proposition 1.2.** In the bases of fundamental weights for $\Lambda$ and $\Lambda_S$ the restriction $i^*$ is given by

\[ i^* = (2 \ 2 \ldots 2) A^{-1} : \mathbb{Z}^\ell \to \mathbb{Z}, \]

where $A$ and $\ell$ are the Cartan matrix and the rank of $G$, respectively.

Denote $m = m(\mathfrak{g}) = \min \{\omega_\alpha(h_0) : \alpha \in \Pi\}$. The value of a fundamental weight on $h_0$ depends only on the simple ideal to which the fundamental weight belongs. For the types of simple Lie algebras we have:

1) $m(A_\ell) = \ell$ for $\ell \geq 1$.
2) $m(B_\ell) = 2\ell$ for $\ell \geq 3$.
3) $m(C_\ell) = 2\ell - 1$ for $\ell \geq 2$.
4) $m(D_\ell) = 2\ell - 2$ for $\ell \geq 4$.
5) $m(E_6) = 16$.
6) $m(E_7) = 27$.
7) $m(E_8) = 58$.
8) $m(F_4) = 16$.
9) $m(G_2) = 6$.

In particular, if $\mathfrak{g}$ has no simple factors of rank 1, we have $\omega(h_0) \geq 2$ for all fundamental weights.

**Proof.** The first statement follows from \([\Pi]\) and the characterization of the Cartan matrix as the matrix for change of basis between the fundamental weights and the simple roots. The second statement is obtained by direct calculation using for instance the tables at the end of [Bou68]. \[\square\]

**Lemma 1.3.** Let $W_S = \{1, \sigma = -1\}$ be the Weyl group of $S$ and recall that $w_0$ denotes the longest element of $W$. We have

\[ i^*(w_0\lambda) = \sigma i^*(\lambda) \quad \text{for all} \quad \lambda \in \Lambda. \]

**Proof.** The statement follows immediately from the definition of $h_0$ and the fact that $w_0$ sends the set of simple roots $\Pi$ to $-\Pi$. \[\square\]

1.3. **Representations, coadjoint orbits and momentum maps.** Let $T_S \subset H_S$ be the maximal compact subgroup of $H_S$ and $K_S \subset S$ be a maximal compact subgroup containing $T_S$. Let $K \subset G$ be a maximal compact subgroup containing $K_S$. Then $K$ necessarily contains the maximal compact subgroup $T \subset H$. We fix a $K$-invariant positive definite Hermitian form $\langle \cdot, \cdot \rangle$ on $V_\lambda$, which induces the Fubini-Study form on $\mathbb{P} = \mathbb{P}(V_\lambda)$. We take $\langle \cdot, \cdot \rangle$ to be $\mathbb{C}$-linear on the first argument. We shall consider momentum maps for this action. The classical target
space of momentum maps is $\mathfrak{t}^*$. For our calculations it is suitable to replace $\mathfrak{t}^*$, by $i\mathfrak{k}$, which is harmless, since the representations are isomorphic. The Killing form is positive definite on $i\mathfrak{k}$; we denote it by $(.,.)$ and use the same notation for the induced forms on subspaces and dual spaces. This allows us to embed the weight lattice as $\Lambda \in \Lambda$ harmless, since the representations are isomorphic. The Killing form is positive definite on

$$\mu_K : \mathbb{P} \rightarrow i\mathfrak{k}^*, \quad \mu[v](\xi) = \frac{\langle \xi v, v \rangle}{\langle v, v \rangle}, \quad [v] \in \mathbb{P}, \xi \in i\mathfrak{k}.$$  

We have $\mu_K(X) = K\lambda$. The momentum map for the $K_S$-action is given by restriction

$$\mu = \mu_{K_S} = i^* \circ \mu_K : \mathbb{P} \rightarrow i\mathfrak{k}_S^*.$$  

For $\xi \in i\mathfrak{k}_S$ we denote the $\xi$-component of $\mu$ by

$$\mu^\xi : \mathbb{P} \rightarrow \mathbb{R}, \quad \mu^\xi[v] = \mu[v](\xi).$$  

In the following we shall make particular use of the restrictions of $\mu$, $||\mu||^2$ and $\mu^{h_0}$ to the smooth subvariety $X \subset \mathbb{P}$.

2. The $S$-action on $G/B$ and GIT

2.1. The orbit structure. Here we discuss orbit structure for the action of the principal subgroup $S \subset G$ on the flag variety $X = G/B$. Assume $\dim X \geq 3$.

Lemma 2.1. The set of $H_S$ fixed points in $X$ is $X^{H_S} = X^H = \{x_w = wB; w \in W\}$. The Weyl group $W_S = \{1, \sigma\}$ acts on $X^H$ by

$$\sigma : X^H \rightarrow X^H, \quad x_w \mapsto x_{w_0w}.$$  

Proof. Since $H_S$ is a regular one-parameter subgroup in $G$, i.e., it is contained in a unique Cartan subgroup, we have $X^{H_S} = X^H$. The Weyl group $W_S$ acts on $X^{H_S}$. We shall use a projective embedding to show that $\sigma$ acts in $X^H$ as $w_0$. Let $\lambda^{++}$ and let $\varphi X \rightarrow \mathbb{P}(V_\lambda)$ be the corresponding embedding. Then we have $\varphi(x_w) = [v^w_{w\lambda}]$, which defines a $W$-equivariant embedding $X^H \rightarrow \Lambda$, $x_w \mapsto w\lambda$. We may further restrict weights by $i^*\Lambda \rightarrow \Lambda$ and the composition

$$\mu^{h_0} = i^* \circ \varphi : X^H \rightarrow \Lambda_S \cong \mathbb{Z}, \quad x_w \mapsto w\lambda(h_0)$$  

is $W_S$ equivariant. We may further choose $\lambda$ avoiding the hyperplanes defined by $w\lambda(h_0) = 0$ and $w\lambda(h_0) = w'\lambda(h_0)$ for all $w, w' \in W$, so that the above map $\mu^{h_0}$ is injective. It follows that $\sigma$ has no fixed points in $X^H$, so it is uniquely determined and by Lemma 1.3 we must have $\sigma(x_w) = x_{w_0w}$. \hfill $\Box$

Proposition 2.2. (i) The orbits of $S$ in $X$ have dimensions 1, 2 and 3.

(ii) The orbits of dimension 1 and 2 are exactly the orbits through the fixed point set of the maximal torus $H \subset G$, $X^H = \{x_w = wB; w \in W\}$. There is a unique 1-dimensional orbit

$$C = S[x_1] = S[x_{w_0}] \cong S/B_S \cong \mathbb{P}^1.$$  

There are $\frac{|W|}{2} - 1$ two-dimensional orbits

$$S[x_w] = S[x_{w_0w}] \cong S/H_S \cong (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \{(\text{diagonal})\} \quad \text{for} \quad w \in W \setminus \{1, w_0\}.$$  

(iii) The isotropy subgroup of any 3-dimensional orbit is either trivial or finite abelian.

(iv) Every $S$-orbit in $X$ has a finite number of orbits in its closure.
Proof. The 1-dimensional orbit $S$-orbit in $C \subset X$ is unique by definition. A 2-dimensional orbit has a 1-dimensional isotropy group. In $S \cong SL_2 \mathbb{C}$, the 1-dimensional subgroups are conjugate to the unipotent $N_S = N \cap S$, the Cartan subgroup $H_S$, or its normalized $N_S(H_S)$. The unipotent subgroup $N_S$ has a unique fixed point in $X$, $x_1$, because $N$ is the only maximal unipotent subgroup of $G$ containing $N_S$ (see Prop. (iv)). Since $Sx_1 = B_S$, there are no orbits of the form $S/N_S$ in $X$. It follows that any 2-dimensional orbit must contain an $H_S$-fixed point $x_w$. Since $H_S$ is a regular one-parameter subgroup in $G$, i.e., it is contained in a unique Cartan subgroup, we have $X^H_S = X^H$. We know from Lemma 3 that $\sigma \in W_S$ has no fixed points in $X^H$ and acts by $\sigma(x_w) = x_{w_0w}$. Hence $N_S(H_S)$ has no fixed points in $X$. We can conclude that the 2-dimensional orbits are $S[x_w] = S[x_{w_0w}] \cong S/H_S$ for $w \in W \setminus \{1, w_0\}$. In particular they are finitely many. This implies (ii) and also (i), with the assumption $\dim X \geq 3$. Part (iv) follows immediately, since we have an algebraic action, so the closure of any orbit contains only orbits of smaller dimension. For part (iii) we use again the fact that the action is algebraic, so a 0-dimensional isotropy subgroup $\Gamma$ must be finite, As such it is compact, and hence contained in a maximal compact subgroup $K \subset G$. The group $K$ acts transitively on $X$ and $X \cong K/T$, where $T$ is a Cartan subgroup of $K$. Hence $\Gamma$ must be abelian.

Remark 2.1. Nonabelian finite subgroups of $S$ can be obtained as isotropy subgroups for actions on partial flag varieties $G/P$. For instance the symmetry group of a tetrahedron appears as the isotropy subgroup of the unique 3-dimensional orbit in $\mathbb{P}^3$ of the principal subgroup $S \subset SL_3 \mathbb{C}$, given by the 4-dimensional irreducible representation of $SL_2$.

The above proposition leads us to define an equivalence relation with finitely many equivalence classes on the set of 3-dimensional $S$-orbits as follows.

Definition 2.1. We denote by $X^1_S$, $X^2_S$, $X^3_S$ the subsets of $X$, where the $S$-orbits have dimension 1, 2, 3, respectively. Two points $x, y \in X^3_S$ are said to have the same $S$-orbit type, if $\overline{Sx} \setminus Sx = \overline{Sy} \setminus Sy$.

Let $x \in X^3_S$. Then the above proposition implies that the type of the orbit $Sx$ is determined by the set $\overline{Sx}^{H_S}$.

2.2. The Hilbert-Mumford criterion. We shall use the Hilbert-Mumford criterion to detect instability. Here we present its specific form in the case of the principal subgroup. The criterion, in its general formulation, reduces verification of the instability for a reductive group to verification of instability for dominant $\mathbb{C}^*$-subgroups. Since $S$ has rank 1, all its $\mathbb{C}^*$-subgroups are Cartan subgroups and are conjugate. Thus the detection of $S$-instability is reduced to $H_S$-instability. We formulate this in the following lemma, which is a direct application of the Hilbert-Mumford criterion for our case, and which is essential for our calculations.

Lemma 2.3. Assume $\dim X \geq 3$. Let $\lambda \in \Lambda^{++}$. Let $x \in X^3_S$, i.e., $\dim Sx = 3$. Then $H_S$ has two fixed points in $\overline{Sx}$, say $x_{w_1}$ and $x_{w_2}$. The following are equivalent:

(i) $x$ is $S$-unstable with respect to $L_\lambda$.
(ii) $x$ is $H_S$-unstable with respect to $L_\lambda$.
(iii) $w_1 \lambda(h_0)$ and $w_2 \lambda(h_0)$ are both nonzero and have the same sign, i.e., $w_1 \lambda(h_0)w_2 \lambda(h_0) > 0$.

In particular, $X^3_S \cap X_{us,S}(\lambda) = X^3_S \cap X_{us,H_S}(\lambda)$.

Remark 2.2. Let $x \in X$. Consider the set $H_Sx$. The set of $H_S$-fixed points $\overline{Sx}^{H_S}$ has two elements. The nontrivial element $\sigma \in W_S$ acts on both $\overline{Sx}^{H_S}$ and $N_S(H_S)^{H_S}$. The set $N_S(H_S)^{H_S}$ has either 2 or 4 elements; its image under $\mu^{w_0}$ belongs to $\mathbb{Z}$ and is stable under $\sigma$ which acts here by -1.

1Apologies for the notation!
2.3. The Kirwan stratification and the Sample cone. For this subsection, we fix $$\lambda \in \Lambda^{++}$$, the associated ample line bundle $$L_\lambda$$ on $$X$$, the projective embedding $$\varphi : X \subset \mathbb{P} = \mathbb{P}(V_\lambda)$$ and the resulting momentum map $$\mu$$ defined in (\ref{momentum_map}). We have a $$K_S$$ equivariant function

$$||\mu||^2 : X \to \mathbb{R},$$

This function defines a Morse-type stratification of $$X$$, described in the symplectic setting by Kirwan and in the projective setting by Ness, [Kir84], [N84].

Lemma 2.4. Let $$w \in W$$. Then the Schubert variety $$Bx_w$$ is contained in $$X_{us}(\lambda)$$ if and only if $$w\lambda(h_0) > 0$$. Furthermore, $$x_w$$ is a critical point of $$||\mu_K||^2$$ and, if $$w\lambda(h_0) > 0$$, then the associated Kirwan stratum is $$S_{w\lambda} = SBx_w$$, with the Schubert cell $$Bx_w$$ being the prestratum.

Proof. This follows directly from the definitions of Kirwan and the description of the tangential representations given in the section on the flag variety.

Theorem 2.5. The Kirwan stratification of the S-unstable locus in $$X$$ with respect to $$L_\lambda$$ is given by

$$X_{us}(\lambda) = \bigcup_{w \in W: w\lambda(h_0) > 0} S_{w\lambda}, \quad S_{w\lambda} = SBx_w. \quad (5)$$

The dimension of the strata is given by $$\dim S_{w\lambda} = l(w) + 1$$, and consequently

$$\dim X_{us}(\lambda) = 1 + \max \{l(w) : w\lambda(h_0) > 0\} , \quad \text{codim}_X X_{us}(\lambda) = 1 + \min \{l(w) : w\lambda(h_0) < 0\} .$$

Proof. The stratification (5) is obtained directly from Kirwan’s formula and Lemma 2.4. The dimension formulae follow from $$\dim Bx_w = l(w)$$, $$S \cap B = B_S$$ and the fact that the tangent line $$\mathbb{C}e_\cdot x_w \cong s/b_S$$ is transversal to $$T_{x_w}(Bx_w)$$. \hfill \Box

The above theorem allows us, in particular, to detect the cases when the semistable locus is nonempty. Thus we can determine the Sample cone $$C^S(X)$$, which, as we already mentioned is identified with the the nulleigencone Cone($$E_0(S \subset G)$$) $$\subset \Lambda^+_R$$. This description is already known from the work of Berenstein and Sjamaar, [BS00]. We formulate it below.

Corollary 2.6. If $$G$$ has no simple factors of rank 1, then the nulleigencone (a.k.a. the Sample cone on $$X$$) consists of the entire Weyl chamber:

$$C^S(X) \cong \text{Cone}(E_0) = \Lambda^+_R .$$

If $$G$$ has simple factors of rank 1, then each such factor has a simple root $$\alpha$$ orthogonal to all other simple roots. The nulleigencone is defined by the following inequalities:

$$C^S(X) \cong \text{Cone}(E_0) = \{ \lambda \in \Lambda^+_R : \lambda(s_\alpha h_0) \geq 0 \text{ for all } \alpha \in \Pi \text{ with } \alpha \perp \Pi \setminus \{\alpha\} \} .$$

More explicitly, upon writing a dominant weight $$\lambda$$ as $$\lambda = \sum_{\beta \in \Pi} \lambda_\beta \omega_\beta$$ in the basis of fundamental weights, we have, for $$\alpha$$ as above,

$$\lambda(s_\alpha h_0) = -\lambda_\alpha + \sum_{\beta \in \Pi \setminus \{\alpha\}} \lambda_\beta \omega_\beta(h_0) .$$

Proof. Theorem 2.5 implies that $$X = X_{us}(\lambda)$$ if and only if there exists a simple root $$\alpha \in \Pi$$ such that $$s_\alpha \lambda(h_0) = \lambda(s_\alpha h_0) < 0$$. For $$\gamma \in \Delta$$ and $$\nu \in \Lambda$$, we denote $$n_{\gamma,\nu} = 2 \langle \gamma, \nu \rangle$$, so that $$s_\gamma \nu = \nu - n_{\gamma,\nu} \gamma$$. For any simple root $$\alpha \in \Pi$$, we have $$n_{\alpha, \omega_\alpha} = 1$$ and $$n_{\alpha, \omega_\beta} = 0$$ for $$\beta \in \Pi \setminus \{\alpha\}$$. Hence

$$s_\alpha \lambda = \lambda - \lambda_\alpha \alpha , \quad s_\alpha \lambda(h_0) = \lambda(h_0) - 2\lambda_\alpha = \lambda_\alpha(\omega_\alpha(h_0) - 2) + \sum_{\beta \in \Pi \setminus \{\alpha\}} \lambda_\beta \omega_\beta(h_0) .$$

Now the claims of the corollary follow from Proposition 1.2. \hfill \Box
We record the following geometric formulation of the above corollary.

**Corollary 2.7.** If $L_\lambda$ is an ample and $S$-ample line bundle on $X$, i.e., $\lambda \in \Lambda^{++} \cap \text{Cone}(E_0)$, then all Schubert divisors intersect the $S$-semistable locus $X_{ss}(\lambda)$. If furthermore $\lambda$ belongs to the interior of the $S$-ample cone, i.e., $\lambda \in \Lambda^{++} \cap \text{Int Cone}(E_0)$, then all Schubert divisors intersect the $S$-stable locus.

2.4. GIT classes of ample line bundles. The description of the unstable loci of $S$-ample line bundles on $X$ given in Theorem 2.5 allows us to determine the GIT-equivalence classes of line bundles according to the definitions given at the end of Section 1.1. In particular, we determine the chambers, in the sense of Definition 1.2.

We are lead by Theorem 2.5 to consider the following partition of $W$, defined for any dominant weight $\lambda$

$$ W = W^+(\lambda) \sqcup W^-(\lambda) \sqcup W^0(\lambda) , $$

where

$$ W^\pm(\lambda) = \{ w \in W : w\lambda(h_0) \in \mathbb{Z}_\pm \} , \ W^0(\lambda) = \{ w \in W : w\lambda(h_0) = 0 \} . $$

**Lemma 2.8.** The following hold:

(i) $w_0W^\pm(\lambda) = W^\pm(\lambda)$, $w_0W^0(\lambda) = W^0(\lambda)$.

(ii) The partition (6) depends only on the GIT-equivalence class of $\lambda$, say $C$. Different GIT-equivalence classes correspond to different partitions. We denote $W^0\pm(\lambda) = W^0\pm(C)$.

(iii) For every GIT-equivalence class $C$ and every $\nu \in \text{Cone}(E_0)$, the following are equivalent:

(a) $\nu \in C$ ;

(b) $X_{ss}(C) \subset X_{ss}(\nu)$ ;

(c) $W^+(\nu) \subset W^+(C)$ .

(iv) A GIT-equivalence class $C$ is a chamber if and only if $W^0(C) = \emptyset$.

**Proof.** Part (i) follows directly from Lemma 1.3. Part (ii) follows from the definition of GIT-equivalence classes and Theorem 2.5. For part (iii), the equivalence between (a) and (c) follows from part (i); the equivalence between (b) and (c) follows from Theorem 2.5. For part (iv), notice first that, since our group $S$ is 3-dimensional, we have that $C$ is a chamber if and only if $X_{ss}(C)$ consists only of 3-dimensional $S$-orbits. According to Proposition 2.2 the orbits of dimension 1 and 2 are exactly $Sx_w$ for $w \in W$. Let $\lambda \in C$. We consider the projective embedding $\varphi : X \subset \mathbb{P}(V_\lambda)$ given by the ample line bundle $L_\lambda$. The point $x_w$ is sent to a weight vector $\varphi(x_w) = [v_w^\lambda]$. A weight vector is semistable if and only if its weight is 0. Hence the orbit $S[x_w]$ is semistable if and only if $w\lambda(h_0) = \lambda(w^{-1}h_0) = 0$. Hence $C$ is a chamber if and only if $W^0 = \emptyset$. $\square$

Summing up the preceding results we obtain the following.

**Theorem 2.9.** The partition of the $S$-ample cone $\text{Cone}(E_0)$ into GIT-equivalence classes is given by the system of hyperplanes

$$ \mathcal{H}_w = \{ \lambda \in \Lambda_\mathbb{R} : \lambda(wh_0) = 0 \} , \ w \in W . $$

The chambers are the connected components of $\text{Cone}(E_0) \setminus \cup_w \mathcal{H}_w$. 
2.5. **Unstable divisors.** Assume $\lambda \in \Lambda^+ \cap \text{Int Cone}(\mathcal{E}_0)$, so that $X_{us}(\lambda) \neq X$. According to Theorem 2.5, we have $\text{codim}_XX_{us}(\lambda) = -1 + \min\{l(w) : w\lambda(h_0) < 0\}$. Corollary 2.7 tells us that $X_{us}(\lambda)$ cannot contain Schubert divisors. In this section, we use explicit calculations with Weyl groups to identify the cases when the unstable locus contains divisors. The dimension formula implies:
\[
\text{codim}_XX_{us}(\lambda) = 1 \text{ if and only if there exists } w \in W \text{ with } l(w) = 2 \text{ and } w\lambda(h_0) < 0.
\]

The Weyl group elements of length 2 have the form $w = s_\beta s_\alpha$, with two distinct simple roots $\alpha, \beta \in \Pi$. Next we compute the action of $s_\beta s_\alpha$ on a dominant weight $\lambda$ in coordinates given by the fundamental weights $\omega_\gamma, \gamma \in \Pi$, and then we evaluate $s_\beta s_\alpha \lambda$ at $h_0$ using the fact that $\gamma(h_0) = 2$ for all $\gamma \in \Pi$. Let
\[
\lambda = \sum_{\gamma \in \Pi} \lambda_\gamma \omega_\gamma,
\]
We proceed as in Corollary 2.6, where we computed the action of one simple reflection. Applying the second simple reflection, we get
\[
s_\beta s_\alpha \lambda = s_\beta (\lambda - n_\alpha \alpha) = \lambda - \lambda_\alpha \alpha - \lambda_\beta \beta + \lambda_\alpha n_\beta \alpha \beta.
\]
Hence
\[
s_\beta s_\alpha \lambda(h_0) = \lambda(h_0) - 2(\lambda_\alpha + \lambda_\beta) + 2n_\beta \alpha \lambda_\alpha = \lambda_\alpha (\omega_\alpha(h_0) - 2(1 - n_\beta \alpha)) + \lambda_\beta (\omega_\beta(h_0) - 2) + \sum_{\gamma \in \Pi \setminus \{\alpha, \beta\}} \lambda_\gamma \omega_\gamma(h_0).
\]

Since $\alpha, \beta$ are simple roots, we have $n_\beta \alpha = \leq 0$, We need to estimate the numbers $(\omega_\beta(h_0) - 2)$ and $(\omega_\alpha(h_0) - 2(1 - n_\beta \alpha))$. This may be done using Proposition 1.2.

**Lemma 2.10.**

(i) If $\beta$ is not orthogonal to all other simple roots, then $\omega_\beta(h_0) - 2 \geq 0$.

(ii) The inequality
\[
\omega_\alpha(h_0) \geq 2(1 - n_\alpha \beta)
\]
holds for all $\alpha, \beta \in \Pi$, except when $\alpha$ is orthogonal to all other simple roots, or when $\alpha$ and $\beta$ are the simple roots of a simple ideal $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ of type $A_2$ or $C_2$ with $\alpha$ being the long simple root.

**Proof.** The first statement is already contained in Proposition 1.2. For the second statement, recall that $n_{\beta, \alpha} \in \{0, -1, -2, -3\}$, whence $2(1 - n_{\beta, \alpha}) \in \{2, 4, 6, 8\}$. Thus the inequality $(\omega_\alpha(h_0) - 2(1 - n_{\beta, \alpha})) < 0$ puts, via Proposition 1.2 a restriction on the simple ideal $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ to which $\alpha$ belongs. We consider the possible values of $n_{\beta, \alpha}$.

If $n_{\beta, \alpha} = 0$, i.e., $\alpha$ and $\beta$ are orthogonal, then $(\omega_\alpha(h_0) - 2(1 - n_{\beta, \alpha})) < 0$ if and only if if $\alpha$ is orthogonal to all other simple roots.

If $n_{\beta, \alpha} = -3$, then $\alpha$ and $\beta$ are the long and short simple roots of $\hat{\mathfrak{g}} = \mathfrak{g}_2$, respectively. In this case we have $\omega_\alpha(h_0) = 10 > 8$ (and $\omega_\beta(h_0) = 6$), so $(\omega_\alpha(h_0) - 2(1 - n_{\beta, \alpha})) < 0$.

If $n_{\beta, \alpha} = -2$, then $\hat{\mathfrak{g}}$ has two root lengths, $\alpha$ is long and $\beta$ is short. According to Proposition 1.2, if $\omega_\alpha(h_0) < 6$, then $\hat{\mathfrak{g}}$ must have type $C_2, C_3$. For the long simple root of $C_\ell$ we have $\omega_\alpha(h_0) = \ell^2$. Hence $C_3$ is excluded, and we are left with $C_2$, where we have indeed $\omega_2(h_0) - 2(1 - n_{\alpha_1, \alpha_2}) = 4 - 6 = -2$.

If $n_{\beta, \alpha} = -1$, then we have the inequality $\omega_\alpha(h_0) < 4$. Proposition 1.2 implies that this occurs exactly when $\hat{\mathfrak{g}}$ has type $A_1$ or $A_2$.

Using the above lemma, we deduce the following.

**Theorem 2.11.** Suppose that $G$ has no simple factors with of type $A_1, A_2$ or $C_2$. Then for every $\lambda \in \Lambda^+$ the $S$-unstable locus $X_{us}(\lambda)$ has codimension at least 2.
3. Quotients and their Picard groups

In this section, we use the language of divisors rather than line bundles, as it is more suitable for the context. This is unproblematic, since $X$ is smooth. Thus $C^S(X)$ is interpreted as a cone of $\mathbb{R}$-divisors on $X$ (recall from Section 1.1 that $C^S(X)$ is the $S$-ample cone on $X$, identified with the eigencone $\text{Cone}(E_0) \subset \Lambda^+_S$).

We assume from now on that all simple factors of $G$ have at least 5 positive roots. This is equivalent to the hypothesis of Theorem 2.11, whence we get $\text{codim}_X(X_{us}(\lambda)) \geq 2$ for all $\lambda \in \Lambda^{++}$.

Let $C$ be a fixed chamber in $C^S(X)$, let $Y := Y(C) := X_{ss}(C)//S$ be the corresponding quotient, and let $\pi : X_{ss}(C) \to Y$ be the quotient morphism. Since the unstable locus of $C$ is of codimension at least two, the results of [S14] apply and we obtain the following.

**Theorem 3.1.** The quotient $Y$ is a Mori dream space and the following hold:

(i) there is an isomorphism of $\mathbb{Q}$-Picard groups

$$\text{Pic}(Y)_{\mathbb{Q}} \cong \text{Pic}(X)_{\mathbb{Q}};$$

(ii) there is an isomorphism of cones

$$\overline{\text{Eff}}(Y) \cong C^S(X).$$

Moreover, for any line bundle $L$ on $X$, there exists $k \in \mathbb{N}$ and a line bundle $L$ on $Y$ such that

$$H^0(X, L^{jk})^S \cong H^0(Y, L^j), \quad j \in \mathbb{N}.$$

In fact, in this particular case we can say more about the convex geometry of the divisors on $Y$. We first recall that the stable base locus, $\text{Bs}(D)$, of a Cartier divisor $D$ on a variety $Z$ is the intersection of the base loci of all positive multiples of $D$;

$$\text{Bs}(D) = \bigcap_{m \geq 1} \text{Base}(mD).$$

An effective divisor $D$ is said to be movable if $\text{codim}(\text{Bs}(D), Z) \geq 2$. In the case when $Z$ has a finitely generated Picard group, we define the movable cone $\text{Mov}(Z)$ to be the closed convex cone in $\text{Pic}(Z)_{\mathbb{R}}$ generated by all movable divisors.

We now have the following result about cones of divisors on the quotient $Y$.

**Theorem 3.2.** Let $Y = X_{ss}(C)//S$ be the quotient above. Then there is an equality of cones

$$\overline{\text{Eff}}(Y) = \text{Mov}(Y),$$

and an isomorphism of cones

$$\text{Nef}(Y) \cong \overline{\mathcal{C}}.$$ 

Moreover, every nef $\mathbb{Q}$-divisor $D$ on $Y$ is semi-ample, i.e., some positive multiple of $D$ is basepoint-free.

**Proof.** In order to prove the first identity, let $D$ be an effective divisor on $Y$. The divisor $\pi^*D$ on $X_{ss}(C)$ then extends uniquely to a divisor on $X$, which we also denote by $\pi^*D$. By the isomorphism (9) the stable base locus of $D$ is given by

$$\text{Bs}(D) = \pi(X_{us}(\pi^*D) \cap X_{ss}(C)).$$

Since the unstable locus of any divisor in $C^S(X)$ is of codimension at least two, and since the fibres of $\pi$ all have the same dimension, (10) shows that $D$ is movable. Hence, $\overline{\text{Eff}}(Y) \subseteq \text{Mov}(Y)$, and this proves the first identity.
For the second identity, we first note that every ample divisor is identified with a divisor in $\overline{C}$ by the isomorphism \( \mathfrak{S} \). Indeed, if $D$ is a divisor on $Y$ such that $\pi^*D \not\subseteq \overline{C}$, then, by part (iii) of Lemma 2.8, the unstable locus $X_{us}(\pi^*D)$ contains an irreducible component $F$ which intersects $X_{ss}(C)$, so that $\pi(F \cap X_{ss}(C))$ is an irreducible component of the stable base locus $Bs(D)$. Hence, $D$ cannot be ample. Hence, $\text{Ample}(Y) \subseteq \overline{C}$, so that we also have $\text{Nef}(Y) \subseteq \overline{C}$.

On the other hand, if $A \in \overline{C}$ is any $\mathbb{R}$-divisor on the boundary of $\overline{C}$, the inclusion of semi-stable loci $X_{ss}(C) \subseteq X_{ss}(A)$ holds \[DH98\], so that $X_{us}(A) \subseteq X_{us}(C)$. If $A$ is a $\mathbb{Q}$-divisor, the identity \[10\] applied to $mA$, for $m \in \mathbb{N}$ such that $mA$ is an effective integral divisor, then shows that $mA = \pi^*D$ for a semi-ample divisor $D$. This proves the second identity as well as the final claim. \hfill \square

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References

- [BS00] A. Berenstein, R. Sjamaar, Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion, J. of AMS 13 (2000), 433-466.
- [BZ01] A. Berenstein, A. Zelevinski, Tensor product multiplicities, canonical bases and totally positive varieties, Invent. math. 143 (2001), 77-128.
- [Bour68] N. Bourbaki, Groupes et algèbres de Lie. Chapitre VI: Systèmes de racines, Actualités Scientifiques et Industrielles, no. 1337, Herman, Paris, 1968.
- [DH98] I. V. Dolgachev, Y. Hu, Variation of geometric invariant theory quotients, Pub. IHES 78 (1998), 5–56.
- [E92] A. G. Elashvili, Invariant algebras. In: Lie groups, their discrete subgroups, and invariant theory, Adv. Soviet Math. 8 (1992), 57-64.
- [H95] Y. Hu, (W-R)-matroids and thin Schubert-type cells attached to algebraic torus actions, Proc. AMS 123 (1995), 2607–2617.
- [Kac80] V. Kac, Some remarks on nilpotent orbits, J. of Algebra 64 (1980), 190–213.
- [Kir84] F. C. Kirwan, Cohomology of Quotients in Symplectic and Algebraic Geometry, Mathematical Notes, Vol. 31, Princeton Univ. Press, 1984.
- [KKV89] F. Knop, H. Kraft, T. Vust, The Picard group of a G-variety, Algebraische Transformationsgruppen und Invariantentheorie, DMV Sem. 13, 77-87, Birkhäuser, Basel, 1989.
- [Kos59] B. Kostant, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, Amer. J. Math. 81 (1959), 973–1032.
- [L95] P. Littelmann, Paths and root operators in representation theory, Ann. of Math. 142 (1995), 499–525.
- [N84] L. Ness, A Stratification of the null cone via the moment map, Amer. J. of Math. 106 (1984), 1281–1329.
- [R98] N. Ressayre, An example of a thick wall. Appendix to “Variation of geometric invariant theory quotients” by Dolgachev and Hu, Pub. IHES 87 (1998), 53–56.
- [R10] N. Ressayre, Geometric invariant theory and the generalized eigenvalue problem, Invent math 180 (2010), 389-441.
- [Sepp14] H. Seppänen, Global branching laws by global Okounkov bodies, arXiv:1409.2025, 2014.
- [Thad96] M. Thaddeus, Geometric invariant theory and flips. J. Amer. Math. Soc. 9 (1996), 691-723.
- [Vogan78] D. A. Vogan, Jr., Lie algebra cohomology and a multiplicity formula of Kostant J. of Algebra 51 (1978), 69–75.

Address:
Mathematisches Institut, Georg-August-Universität Göttingen, Bunsenstraße 3-5, D-37073 Göttingen, Deutschland.

Emails:
Henrik.Seppaenen@mathematik.uni-goettingen.de
Valdemar.Tsanov@mathematik.uni-goettingen.de