A Nash type result for Divergence Parabolic Equation related to Hörmander’s vector fields

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Abstract. In this paper we consider the divergence parabolic equation with bounded and measurable coefficients related to Hörmander’s vector fields and establish a Nash type result, i.e., the local Hölder regularity for weak solutions. After deriving the parabolic Sobolev inequality, (1,1) type Poincaré inequality of Hörmander’s vector fields and a De Giorgi type Lemma, the Hölder regularity of weak solutions to the equation is proved based on the estimates of oscillations of solutions and the isomorphism between parabolic Campanato space and parabolic Hölder space. As a consequence, we give the Harnack inequality of weak solutions by showing an extension property of positivity for functions in the De Giorgi class.

Keywords: Hörmander’s Vector Fields; Divergence Parabolic Equation; Weak Solution; Hölder Regularity; Harnack Inequality.

1 Introduction

Schauder theory for the solutions to linear elliptic and parabolic equations with $C^\alpha$ coefficients or VMO coefficients has been completed. De Girogi has followed the local Hölder continuity for the solutions to the divergence elliptic equation with bounded and measurable coefficients

$$-\sum_{i,j=1}^n D_i \left( a^{ij}(x) D_j u \right) = 0, x \in \mathbb{R}^n$$

and given the a priori estimate of Hölder norm (see [8]). Nash in [31] derived independently the similar result for the solutions to the parabolic equation with a different approach from [8]. Hereafter Moser in [29] developed a new method (nowadays it is called the Moser iteration method) and proved again results above-mentioned to elliptic and parabolic equations. These important works break a new path for the study of regularity for weak solutions to partial differential equations.

In [11] Fabes and Stroock proved the Harnack inequality for linear parabolic equations by going back to Nash’s original technique in [31]. A very interesting approach has been raised by De Benedetto (see [9]) for proving a Harnack inequality of functions belonging to parabolic De Giorgi classes. The approach was used to derive the Hölder continuity of solutions to linear second order parabolic equations with

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bounded and measurable coefficients (see [25]). Giusti [17] applied the approach to give a proof of the Harnack inequality in the elliptic setting.

Square sum operators constructed by vector fields satisfying the finite rank condition were introduced by Hörmander (see [20]), who deduced that such operators are hypoelliptic. Many authors carried on researches to such operators and acquired numerous important results ([14],[15],[21],[24],[32]). Nagel, Stein and Wainger ([30]) concluded the deep properties of balls and metrics defined by vector fields of this type. Many other authors obtained very appreciable results related to Hörmander’s vector fields. Schauder estimates to degenerate elliptic and parabolic equations related to noncommutative vector fields have been handled in [6],[18],[34], etc. Bramanti and Brandolini in [3] investigated Schauder estimates to the following Hörmander type nondivergence parabolic operator

\[ H = \partial_t - \sum_{ij} a_{ij}(t,x)X_i X_j - \sum_{i=1}^q b_i(t,x)X_i - c(t,x), \]

where coefficients \( a_{ij}(t,x), b_i(t,x) \) and \( c(t,x) \) are \( C^n \).

In this paper, we are concerned with the divergence parabolic equation with bounded and measurable coefficients related to Hörmander’s vector fields and try to establish a Nash type result for weak solutions which will play a crucial role for corresponding nonlinear problems.

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, \( Q_T = \Omega \times (0,T) \subset \mathbb{R}^n \times \mathbb{R}, T > 0 \). Consider the following divergence parabolic equation

\[ u_t + X_j^* (a^{ij}(x,t)X_i u) + b_i(x,t)X_i u + c(x,t)u = f(x,t) - X_j^* f^i(x,t), \quad (x,t) \in Q_T \quad (1.1) \]

where \( X_i = \sum_{k=1}^n b_{ik}(x) \frac{\partial}{\partial x_k} \) \( b_{ik}(x) \in C^\infty(\Omega), i = 1,...,q, q \leq n \) is the smooth vector field, \( X_j^* = -\sum_{k=1}^n \frac{\partial}{\partial x_k} (b_{jk}(x) \cdot) \) is the adjoint of \( X_j, j = 1,...,q \). The summation coonventions in (1.1) are omitted. Denote

\[ Lu = X_j^* (a^{ij}(x,t)X_i u) + b_i(x,t)X_i u + c(x,t)u, \]

then (1.1) is written by

\[ u_t + Lu = f(x,t) - X_j^* f^i(x,t). \]

Throughout this paper we make the following assumptions:

(C1) \( a^{ij}(x,t) \in L^\infty(Q_T) (i,j = 1,2,...q) \) and there exists \( \Lambda > 0 \) such that

\[ \Lambda^{-1}|\xi|^2 \leq a^{ij}(x,t)\xi_i \xi_j \leq \Lambda|\xi|^2, \quad (x,t) \in Q_T; \xi \in \mathbb{R}^q; \]

(C2) for \( m > (Q+2)/2, Q \) is the local homogeneous dimension relative to \( \Omega, b_i(x,t)(i=1,2,...,q) \) and \( c(x,t) \) satisfy

\[ \sum_i \|b_i^2\|_{L^\infty(Q_T)} + \|c\|_{L^\infty(Q_T)} \leq \Lambda; \]

(C3) \( f \in L^{\frac{p(Q+2)}{2(Q+2-p)}}(Q_T), f^i \in L^{p}(Q_T), p > Q + 2, i = 1,...,q \).

Let

\[ (Lu, \varphi) = \int_\Omega [(a^{ij} X_i u) X_j \varphi + (b_i X_i u + cu) \varphi] dx, \quad \text{for } \varphi \in W^{1,1}_{Q,0}(Q_T); \]
we call that \( u \) is a weak sub-solution (super-solution) to the equation (1.1), if \( u \in V_2(Q_T) \) satisfies
\[
\int_0^t (u, \phi) d\tau + \int_0^t (Lu, \phi) d\tau \lesssim (\geq) \int_0^t [(f, \phi) + (f') \cdot X(\phi)] d\tau, t \in (0, T),
\]
for any \( \phi \in W^{1,1}_{2,0}(Q_T), \phi \geq 0 \). If \( u \) is not only a sub-solution but also a super-solution to (1.1), we say that \( u \) is a weak solution to (1.1); at this time, (1.2) becomes an integral equality and the restriction \( \varphi \geq 0 \) is eliminated. Spaces \( W^{1,1}_{2,0}(Q_T), V_2(Q_T) \) here and \( V_2^{1,0}(Q_T) \) appeared in the following paragraph will be described in Section 2 in detail.

We call \( u \in V_2^{1,0}(Q_T) \) belongs to the De Giorgi class
\[
DG(Q_T) := DG(Q_T; \lambda_0, \eta, M, F_0, \gamma(\cdot), \delta),
\]
if \( \|u\|_{L^\infty(Q_T)} \leq M \) and for \( 0 < \text{ess sup} u \leq M, \delta \in (0, 1], \)
\[
\max \left\{ \sup_{0 < t < t_0 < t_0 + \tau} \| \xi(u - k)_{\pm}(\cdot, t) \|_{L^2(B_\rho)}, \lambda_0 \|X(u - k)_{\pm}\|_{L^2(Q_{\rho, \tau})} \right\}
\leq (1 + \epsilon)\| \xi(u - k)_{\pm}(\cdot, t_0) \|_{L^2(B_\rho)}
+ \gamma(\epsilon) \left\{ \|X\|_{L^{\infty}(Q_{\rho, \tau})} + \| \xi \|_{L^{\infty}(Q_{\rho, \tau})} + \| \xi \|_{L^{\infty}((Q_{\rho, \tau}))} \| (u - k)_{\pm} \|_{L^2(Q_{\rho, \tau})}
+ (k^2 + F_0\|\)_{Q_{\rho, \tau}} [(u - k)_{\pm} > 0]]^{1-2/\eta} \right\}, \tag{1.3}
\]
where \( Q_{\rho, \tau} = B_\rho \times (t_0, t_0 + \tau) \subset Q_T, 0 < \rho, \tau < 1, \zeta \in W^{1,1}_{2,0}(Q_{\rho, \tau}), 0 < \zeta(x, t) \leq 1, \epsilon \in (0, 1], \eta \geq Q + 2, F_0 \geq 0, \lambda_0 > 0 \) is parameter, \( \gamma(\cdot) \) is a non-negative decreasing functions. If (1.3) is holds for \((u - k) \pm ((1.3) \text{ is holds for } (u - k) \_ \_), \) we denote \( u \in DG^+(Q_T) \) \((u \in DG^-(Q_T). \) Clearly,
\[
DG(Q_T) := DG^+(Q_T) \cap DG^-(Q_T).
\]

Now we state the main results of this paper.

**Theorem 1.1** (Hölder Regularity). Let \( u \in V_2^{1,0}(Q_T) \) with \( \|u\|_{L^\infty(Q_T)} \leq M \) be the weak solution to (1.1) with (C1), (C2) and (C3), then for \( Q \subset Q_T, \) there exist \( C = C(Q, \eta, \Lambda, \delta, d_p) \geq 1, \) and \( \beta, 0 < \beta \leq 1 - \frac{Q + 2}{\eta}, \) such that
\[
[u]_{\beta, Q} \leq C\overline{\eta}^{-\beta} \left(M + F_0\|\right)_{Q_T}[\frac{1}{2}], \tag{1.4}
\]
where \( \overline{\eta} = \min\{1, d_p(Q, \partial_P Q_T)\}, d_p \text{ is the parabolic metric (see (2.6) below ) and } F_0 = \sum_i \|f_i\|_{L^p(Q_T)} + \|f\|_{L^p(Q_T)} \}
\]

**Theorem 1.2** (Harnack Inequality). Let \( u \in V_2^{1,0}(Q_T) \) with \( \|u\|_{L^\infty(Q_T)} \leq M \) be the weak solution to (1.1) with (C1), (C2) and (C3), and \( u \neq 0 \) on \( Q^0_R = B_R(x_0) \times (t_0, t_0 + a'R^2) \subset Q_T, a' > 1, \) then
\[
\inf_{B_{R/2}(x_0, t_0 + a'R^2)} u \geq C^{-1} u(x_0, t_0 + R^2) - CF_0\|\right)_{Q_T}[\frac{1}{2}], \tag{1.5}
\]
where \( C \) depends on \( Q, \Lambda, \eta, \) and \((a' - 1)^{-1}.\)

The proofs of Theorems are based on the readjustment of De Giorgi’s approach, and some new ingredients applying to our setting are replenished. It is worth emphasizing that we do not impose any artificial condition to the measure of metric ball induced by Hörmander’s vector fields.
We note that (1.1) involves the special case
\[ u_t + X^* x (a^{ij}(x,t)X_j u) = 0. \]

The rest of the paper is organized as follows: Section 2 explains Hörmander’s vector fields, the parabolic Campanato space and parabolic Hölder space; several new preliminary results including the parabolic Sobolev inequality, (1,1) type Poincaré inequality of Hörmander’s vector fields and a De Giorgi type Lemma are inferred. In section 3 we prove that weak solutions to (1.1) are actually in the De Giorgi class $DG(Q_T)$ and derive some properties of functions in $DG(Q_T)$. Section 4 is devoted to proofs of main results. Theorem 1.1 is proved based on the estimates of oscillations of solutions and the isomorphism between the parabolic Campanato space and parabolic Hölder space. Theorem 1.2 is followed by using Theorem 1.1 and an extension property of positivity for functions in the De Giorgi class.

2 Preliminaries

Let $X_1, ..., X_q (q \leq n)$ are $C^\infty$ vector fields in $\Omega \subset \mathbb{R}^N$. Throughout this paper, we always suppose that these vector fields satisfy the finite rank condition [20], i.e., there exists a positive integer $s$ such that $\{X^\beta(x_0)\}_{|\beta| \leq s}$ spans $\mathbb{R}^N$ at every point $x_0 \in \Omega \subset \mathbb{R}^N$.

**Definition 2.1** (Carnot-Carathéodory distance,[12]). An absolutely continuous curve $\gamma: [0, T] \rightarrow \Omega$ is said sub-unitary, if it is Lipschitz continuous and satisfies that for every $\xi \in \mathbb{R}^N$ and $a \in (0, T]$, 
\[ < \gamma'(t), \xi >^2 \leq \sum_{j=1}^q < X_j(\gamma(t)), \xi >^2. \]

Let $\Phi(x,y)$ be the class of sub-unitary curves connected $x$ and $y$, we define the Carnot-Carathéodory distance (C-C distance) by 
\[ d(x,y) = \inf \{T \geq 0 : \gamma \in \Phi(x,y)\}. \]

The C-C metric ball is defined by 
\[ B_R(x_0) = B(x_0, R) = \{x \in \Omega : d(x_0, x) < R\} \]
and the Lebesgue measure of metric ball $B_R(x_0)$ by $|B_R(x_0)|$ . A fundamental doubling property with respect to the metric balls was showed in [30], namely, there are positive constants $C_1 > 1$ and $R_0$, such that for $x_0 \in \Omega$ and $0 < R < R_0$,
\[ |B(x_0,2R)| \leq C_1|B(x_0, R)|, \]
(2.1)
where $Q = \log_2 C_1$, $Q$ acts as a dimension and is called the local homogeneous dimension relative to $\Omega$. It is easy to see from (2.1) that for any $0 < R \leq R_0$ and $\theta \in (0, 1)$ ,
\[ |B_{\theta R}| \geq C_1^{-1}\theta^Q |B_R|, \]
(2.2)
where $C_1$ and $R_0$ are constants in (2.1).

The gradient of $u \in C^1(\Omega)$ is denoted by $Xu = (X_1 u, ..., X_q u)$, and the norm of $Xu$ is of the form 
\[ |Xu| = \left( \sum_{j=1}^q (X_j u)^2 \right)^{1/2}. \]
The Sobolev space $S^p_{0}(\Omega)(1 \leq p < \infty)$ related to vector fields $X_1, ..., X_q$ is
Definition 2.3. The parabolic Sobolev space $V_2(Q_T)$ on vector fields $X_1, ..., X_q$ is the set of all functions $u$ satisfying $Xu \in L^2(Q_T)$ and \( \sup_{t \in (0,T)} \int_{Q_t} |u|^2 dx < \infty \). The norm on $V_2(Q_T)$ is
\[
\|u\|_{V_2} := \left( \int_{Q_T} |u|^2 dx dt + \sup_{t \in (0,T)} \int_{Q_t} |u|^2 dx \right)^{\frac{1}{2}} < \infty.
\] (2.4)

In the sequel, we also need spaces $V^{1,0}_2(Q_T)$ and $W^{1,1}_2(Q_T)$, where $V^{1,0}_2(Q_T)$ is the set of functions in $V_2(Q_T)$ satisfying
\[
\lim_{h \to 0} \|u(\cdot, t + h) - u(\cdot, t)\|_{L^2(Q)} = 0, t, t + h \in [0,T];
\] $W^{1,1}_2(Q_T)$ contains functions satisfying $u \in L^2((Q)_T), Xu \in L^2(Q_T)$ and $u_t \in L^2(Q_T)$. The norm on $W^{1,1}_2(Q_T)$ is
\[
\|u\|_{W^{1,1}_2} := \left( \int_{Q_T} |u|^2 dx dt + \int_{Q_T} |Xu|^2 dx dt + \int_{Q_T} u_t^2 dx dt \right)^{\frac{1}{2}}.
\] (2.5)

Obviously, we have
\[W^{1,1}_2(Q_T) \subset V^{1,0}_2(Q_T) \subset V_2(Q_T).\]
Moreover, $V_2(Q_T)$ and $W^{1,1}_2(Q_T)$ are collections of functions in $V_2(Q_T)$ and $W^{1,1}_2(Q_T)$ satisfying
\[u(\cdot, t)|_{\partial \Omega} = 0, \text{a.e. } t \in (0,T)\]
respectively, where $\partial \Omega$ is the boundary of $\Omega$.

Let $Q \subset \subset Q_T$, we define the parabolic metric $d_P$:
\[
d_P((x, t), (y, s)) = (d(x, y)^2 + |t - s|)^{1/2}, \text{ for } (x, t), (y, s) \in Q.
\] (2.6)

Definition 2.3 (Hölder space). For $\alpha \in (0,1]$, let $C^\alpha(Q)$ be the set of functions $u : Q \to \mathbb{R}$ satisfying
\[|u|_{\alpha,Q} := \sup \left\{ \frac{|u(x,t) - u(y,s)|}{d_P((x,t),(y,s))} \mid (x, t), (y, s) \in Q, (x, t) \neq (y, s) \right\} < \infty;
\]
its norm is
\[\|u\|_{\alpha,Q} = |u|_{\alpha,Q} + \|u\|_{L^\infty(Q)}.
\]

Definition 2.4 (Campanato space). For $1 \leq p < +\infty$ and $\lambda \geq 0$, if $u \in L^p(Q)$ satisfies
\[|u|_{p,\lambda} := \left\{ \sup_{Z \in Q, 0 \leq R \leq d} \left( \frac{R^{-\lambda}}{|Q_R|} \int_{Q_R} |u(Z) - u_{Q_R}|^p dZ \right)^{\frac{1}{p}} \right\} < \infty,
\]
where $d = \text{diam}Q, Z = (x, t), Q_R = (B_R(x) \times (t, t + R^2)) \cap Q, u_{Q_R} = \frac{1}{|Q_R|} \int_{Q_R} u(Y) dY, Y = (y, s)$, then we say that $u$ belongs to the Campanato space $L^{p,\lambda}(Q)$ with the norm
\[\|u\|_{p,\lambda} = |u|_{p,\lambda} + \|u\|_{L^p}.
\]
Let us state two useful cut-off functions \( \xi(x) \) and \( \eta(t) \) ([10],[16]) which satisfy \( 0 \leq \xi \leq 1, \xi = 1 \) on \( B_\rho(B_\rho \subset B_R \subset \Omega, \xi = 0 \) outside \( B_R \) and \( |X\xi| \leq \frac{1}{R^p} \).

\[
\eta(t) = \begin{cases} \frac{t-(t_0-R^2)}{R^2-t_0^2}, & t \in (t_0-R^2, t_0-R^2), \\ 1, & t \in [t_0-R^2, t_0+R^2]. \end{cases}
\]

The following result is well known.

**Lemma 2.5** (Sobolev inequality,[4, 5, 27, 28]). For \( 1 \leq p < Q \), there exist \( C > 0 \) and \( R_0 > 0 \) such that for any \( x \in \Omega \) and \( 0 < R \leq R_0 \), we have for any \( u \in S^1_p(B_R), B_R = B_R(x), \)

\[
\left( \frac{1}{|B_R|} \int_{B_R} |u|^p dx \right)^{\frac{1}{p}} \leq CR \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{1}{p}}, \tag{2.7}
\]

where \( 1 \leq k \leq Q/(Q-p) \). In particular, let \( k = \frac{Q}{Q-p} \), then

\[
\left( \int_{B_R} |u|^{pQ/(Q-p)} dx \right)^{(Q-p)/pQ} \leq CR |B_R|^{-\frac{Q-p}{Q}} \left( \int_{B_R} |Xu|^p dx \right)^{\frac{1}{p}}. \tag{2.8}
\]

In the light of (2.8) we can prove

**Theorem 2.6** (Parabolic Sobolev inequality). For \( u \in \hat{V}_2(Q_T) \), it follows \( u \in L^{2(Q+2)/Q}(Q_T) \) and

\[
\int_{Q_T} |u|^{2(Q+2)/Q} dx dt \leq CR^2 |B_R|^{-\frac{Q}{Q-1}} \left( \max_{t \in (0,T)} \left( \int_{B_R} |u(x,t)|^2 dx \right)^{2(Q-1)/Q} \int_{Q_T} |Xu|^2 dx dt, \tag{2.9}
\]

where \( Q_R = B_R(x_0) \times (t_0 - R^2, t_0] \subset Q_T, B_R = B_R(x_0) \) and

\[
\|u\|_{L^{2(Q+2)/Q}(Q_T)} \leq CR |Q_R|^{-\frac{Q}{Q-1}} \|u\|_{\hat{V}_2(Q_T)}. \tag{2.10}
\]

**Proof.** Using the Hölder inequality and (2.8) with \( p = 1, \) it sees

\[
\int_{B_R} |u|^{2(Q+2)/Q} dx = \int_{B_R} |u|^{2/Q} |u|^{2(1+Q)/Q} dx \\
\leq \left( \int_{B_R} |u|^2 dx \right)^{1/Q} \left( \int_{B_R} \left( |u|^{2(1+Q)/Q} \right)^{(Q-1)/Q} dx \right)^{(Q-1)/Q} \\
\leq CR |B_R|^{-\frac{Q}{Q-1}} \left( \int_{B_R} |u|^2 dx \right)^{1/Q} \int_{B_R} X \left( |u|^{2(1+Q)/Q} \right) dx \\
\leq CR |B_R|^{-\frac{Q}{Q-1}} \left( \int_{B_R} |u|^2 dx \right)^{1/Q} \int_{B_R} |u|^{(2+Q)/Q} |Xu| dx \\
\leq CR |B_R|^{-\frac{Q}{Q-1}} \left( \int_{B_R} |u|^2 dx \right)^{1/Q} \left( \int_{B_R} |u|^{2(2+Q)/Q} dx \right)^{1/2} \int_{B_R} |Xu|^2 dx \right)^{1/2}
\]

and then

\[
\int_{B_R} |u|^{2(Q+2)/Q} dx \leq CR |B_R|^{-\frac{Q}{Q-1}} \left( \int_{B_R} |u|^2 dx \right)^{2/Q} \int_{B_R} |Xu|^2 dx \\
\leq CR |B_R|^{-\frac{Q}{Q-1}} \left( \max_{s} \int_{B_R} |u(x,s)|^2 dx \right)^{2/Q} \int_{B_R} |Xu|^2 dx.
\]

Integrating it with respect to \( t, \) (2.9) is derived.
We arrive at by (2.9) and the Young inequality that
\[
\|u\|_{L^2(Q_{1/2}/Q_R)} \leq CR^{\frac{\alpha}{\kappa}}|B_R|^{-\frac{\alpha}{\kappa}} \sup_{0<\xi<T} \|u(\cdot,\xi)\|_{L^\infty(B_R)} \|Xu\|_{L^2(Q_R)}^{\frac{\alpha}{\kappa}}
\]
\[
\leq CR(R^2|B_R|)^{-\frac{\alpha}{\kappa}} \left(\sup_{0<\xi<T} \|u(\cdot,\xi)\|_{L^2(Q_T)} + \|Xu\|_{L^2(Q_T)}\right)
\]
\[
\leq CR|Q_R|^{-\frac{\alpha}{\kappa}} \left(\sup_{0<\xi<T} \|u(\cdot,\xi)\|_{L^2(Q_T)}^2 + \|Xu\|_{L^2(Q_T)}^2\right)^{\frac{1}{2}}
\]
\[
= CR|Q_R|^{-\frac{\alpha}{\kappa}} \|u\|_{V^{2/(\kappa)}(Q_T)}
\]
and (2.10) is proved.

**Lemma 2.7.** For \(0 < \theta < 1\) and \(\kappa > 1\), there exists a positive constant \(C_\kappa\) such that
\[
(1 - \theta) \leq C_\kappa (1 - \theta)^{1 - \frac{1}{\kappa}}.
\]

**Proof.** Denote
\[
f(\theta) = \frac{1 - \theta}{1 - \theta^{1 - \frac{1}{\kappa}}},
\]
then
\[
\lim_{\theta \to 0^+} f(\theta) = 1, \quad \lim_{\theta \to 1^{-}} f(\theta) = \frac{\kappa}{\kappa - 1}.
\]
Let
\[
F(\theta) = \begin{cases} 
1, & \theta = 0, \\
f(\theta), & \theta \in (0, 1), \\
\frac{\kappa - 1}{\kappa}, & \theta = 1,
\end{cases}
\]
we have that \(F(\theta)\) is continuous and uniformly bounded on \([0, 1]\), which implies \(f(\theta) \leq C_\kappa\) for some \(C_\kappa > 0\).

**Theorem 2.8** ((1,1) type Poincar inequality). Let \(u \in W^{1,1}(B_R)\) and
\[
E_0 = \{x \in B_R|u(x) = 0\}.
\]
If \(|E_0| > 0\), then
\[
\int_{B_R} |u|dx \leq \frac{CR|B_R|}{|E_0|} \int_{B_R} |Xu|dx,
\]
where \(C > 0\) relies only on \(Q\).

**Proof.** Consider first \(u \in \text{Lips}(B(x, R))\), then by the result of Jerison [22],
\[
\int_{B_R} |u(x) - u_{B_R}|dx \leq CR \int_{B_R} |Xu|dx,
\]
where \(u_{B_R} = \frac{1}{|B_R|} \int_{B_R} udx\). Since the above inequality has the self-improvement property (see [19]), i.e. there exists \(\kappa > 1\), such that
\[
\left(\frac{1}{|B_R|} \int_{B_R} |u(x) - u_{B_R}|^\kappa dx\right)^{\frac{1}{\kappa}} \leq CR|B_R|^\frac{1}{\kappa - 1} \int_{B_R} |Xu|dx,
\]
it follows
\[
\left(\int_{B_R} |u(x) - u_{B_R}|^\kappa dx\right)^{\frac{1}{\kappa}} \leq CR|B_R|^\frac{1}{\kappa - 1} \int_{B_R} |Xu|dx.
\]
Applying
\[ |u_{B_R}| = \frac{1}{|B_R|} \left| \int_{B_R} u \, dx \right| \leq \frac{1}{|B_R|} \int_{B_R \setminus E_0} |u| \, dx \leq \frac{1}{|B_R|} \left( \int_{B_R} |u|^{\frac{\sigma}{2\sigma - 1}} \right)^{\frac{2\sigma - 1}{\sigma}} |B_R \setminus E_0|^{1 - \frac{1}{\sigma}}, \]

it arrives at
\[ \left( \int_{B_R} |u|^{\sigma} \, dx \right)^{\frac{1}{\sigma}} \leq \left( \frac{1}{|B_R|} \int_{B_R} |u(x) - u_{B_R}|^{\sigma} \, dx \right)^{\frac{1}{\sigma}} + |u_{B-R}| |B_R|^{\frac{1}{\sigma}}, \]

and then
\[ \left(1 - \frac{(B_R) \setminus E_0}{|B_R|} \right)^{\frac{1}{\sigma - 1}} \left( \int_{B_R} |u|^{\sigma} \, dx \right)^{\frac{1}{\sigma}} \leq C_R |B_R|^{\frac{1}{\sigma - 1}} \int_{B_R} |Xu| \, dx. \]

Noting
\[ 1 - \frac{(B_R) \setminus E_0}{|B_R|} \leq C_{\kappa} \left(1 - \frac{(B_R) \setminus E_0}{|B_R|} \right)^{\frac{1}{\sigma - 1}} \int_{B_R} |Xu| \, dx. \]

we have from (2.7) that
\[ C_{\kappa}^{-1} \left(1 - \frac{(B_R) \setminus E_0}{|B_R|} \right)^{\frac{1}{\sigma - 1}} \left( \int_{B_R} |u|^{\sigma} \, dx \right)^{\frac{1}{\sigma}} \leq C_R |B_R|^{\frac{1}{\sigma - 1}} \int_{B_R} |Xu| \, dx. \]

Using it and the Hölder inequality, it follows
\[ \frac{|E_0|}{|B_R|} \int_{B_R} |u| \, dx \leq \frac{|E_0|}{|B_R|} |B_R|^{\frac{1}{\sigma - 1}} \left( \int_{B_R} |u|^{\sigma} \, dx \right)^{\frac{1}{\sigma}} \]
\[ \leq C_R |B_R|^{\frac{1}{\sigma - 1}} |B_R|^{\frac{1}{\sigma - 1}} \int_{B_R} |Xu| \, dx \]
\[ = C_R \int_{B_R} |Xu| \, dx. \]

Now (2.11) is obtained by combining it and the density of Lip($B_R$) in $W^{1,1}(B_R)$.

**Theorem 2.9** (De Giorgi type lemma). Let $u \in W^{1,1}(B_R)$ and
\[ A(k) = \{x \in B_R | u(x) > k\}, \quad \text{for} \ l > k, \]
we have
\[ (l - k) |A(l)| \leq \frac{C_R |B_R|}{|B_R \setminus A(k)|} \int_{A(k) \setminus A(l)} |Xu| \, dx, \quad (2.12) \]
where $C > 0$ relies only on $Q$.

**Proof.** Denoting the function
\[ \hat{u}(x) = \begin{cases} 
  l - k, & x \in A(l), \\
  u(x) - k, & x \in A(k) - A(l), \\
  0, & x \in B_R - A(k), 
\end{cases} \]

\[ (l - k) |A(l)| \leq \frac{C_R |B_R|}{|B_R \setminus A(k)|} \int_{A(k) \setminus A(l)} |Xu| \, dx, \quad (2.12) \]
and noting $\hat{u} \in W^{1,1}(B_R)$, it obtains by using (2.11) that

$$(l - k)|A(l)| \leq \int_{B_R} |\hat{u}(x)|dx \leq \frac{C R |B_R|}{|B_R \setminus A(k)|} \int_{B_R \setminus A(k)} |X \hat{u}|dx = C \frac{C R |B_R|}{|B_R \setminus A(k)|} \int_{A(k) \setminus A(l)} |X u|dx.$$ 

This proves (2.12).

### 3 Several auxiliary lemmas

For the weak sub-solution (super-solution) to (1.1), we have

**Lemma 3.1.** Let $u \in V_{2}^{1,0}(Q_T)$ be the bounded weak sub-solution (or super-solution) to (1.1) with (C1), (C2) and (C3) then

$$u \in DG^+ \text{ (or } u \in DG^-)$$

where $p > Q + 2, \eta = \min \{p, 2m\}, \gamma(\cdot)$ relies only on $Q, \Lambda$ and $p$, 

$$F_0 = \sum \|f^i\|_{L^p(Q_T)} + \|f^i\|_{L^p(Q_T)} < \infty. \tag{3.1}$$

**Proof.** We only prove the conclusion for the weak sub-solution and the proof for the weak supersolution is similar. Multiplying the test function $\zeta^2(u - k)_+$ to (1.1) and integrating on $B_{\rho} \times (t_0, t)$, it yields

$$\int_{t_0}^{t} (u_t, \zeta^2(u - k)_+) dt = \int_{t_0}^{t} \int_{B_{\rho}} u_t \zeta^2(u - k)_+ dx dt$$

$$= \frac{1}{2} \int_{B_{\rho}} \zeta^2(u - k)_+ (\cdot, t) dx |_{t_0}^{t} - \frac{1}{2} \int_{t_0}^{t} \int_{B_{\rho}} \zeta \zeta_t(u - k)^2_+ dx dt$$

$$= \frac{1}{2} \int_{B_{\rho}} \zeta^2(u - k)_+ (x, t) dx - \frac{1}{2} \int_{B_{\rho}} \zeta^2(u - k)_+ (x, t_0) dx$$

$$- \int_{t_0}^{t} \int_{B_{\rho}} \zeta \zeta_t(u - k)^2_+ dx dt$$

and so

$$\frac{1}{2} \int_{B_{\rho}} \zeta^2(u - k)_+ (x, t) dx - \frac{1}{2} \int_{B_{\rho}} \zeta^2(u - k)_+ (x, t_0) dx$$

$$- \int_{t_0}^{t} \int_{B_{\rho}} \zeta \zeta_t(u - k)^2_+ dx dt + \int_{t_0}^{t} \int_{B_{\rho}} a^{ij} X_i u X_j (\zeta^2(u - k)_+) dx dt$$

$$+ \int_{t_0}^{t} \int_{B_{\rho}} (b^i X_i u + cu) (\zeta^2(u - k)_+) dx dt$$

$$\leq \int_{t_0}^{t} \int_{B_{\rho}} [f^i X_i (\zeta^2(u - k)_+) + f^j \hat{\zeta}^2(u - k)_+] dx dt, \tag{3.2}$$

where $t_0 < t \leq t_0 + \tau, 0 < \tau < 1$. Noticing $X_t u = X_i (u - k)_+$ and

$$X_i u X_j (\zeta^2(u - k)_+) = X_i u X_j (\zeta \cdot (u - k)_+)$$

$$= \zeta X_i u [(u - k)_+ X_j \zeta + X_j (\zeta (u - k)_+)]$$

$$= [X_i \zeta (u - k)_+] - (u - k)_+ X_i \zeta [((u - k)_+ X_j \zeta + X_j (\zeta (u - k)_+)$$

$$= X_i (\zeta (u - k)_+) X_j (\zeta (u - k)_+) - (u - k)_+ X_i \zeta X_j \zeta,$$
we obtain from (3.2) that

\[
\frac{1}{2} \int_{B_\rho} \zeta^2 (u - k)_+(x, t) dx + \int_{t_0}^t \int_{B_\rho} a^{ij} X_i u X_j (\zeta^2 (u - k)_+) dx dt \\
\leq \frac{1}{2} \int_{B_\rho} \zeta^2 (u - k)_+(x, t_0) dx + \int_{t_0}^t \int_{B_\rho} \zeta_t (u - k)_+^2 dx dt \\
+ \int_{t_0}^t \int_{B_\rho} a^{ij} (u - k)_+^2 X_i X_j \zeta dx dt - \int_{t_0}^t \int_{B_\rho} (b^i X_i u + cu) \zeta^2 (u - k)_+ dx dt \\
+ \int_{t_0}^t \int_{B_\rho} [f^i X_i (\zeta^2 (u - k)_+) + f \zeta^2 (u - k)_+] dx dt.
\]  

(3.3)

Denote the third, fourth and fifth term in the right hand side of (3.3) respectively by

\[I_1 := \int_{t_0}^t \int_{B_\rho} a^{ij} (u - k)_+^2 X_i X_j \zeta dx dt,\]
\[I_2 := - \int_{t_0}^t \int_{B_\rho} (b^i X_i u + cu) \zeta^2 (u - k)_+ dx dt,\]
\[I_3 := \int_{t_0}^t \int_{B_\rho} [f^i X_i (\zeta^2 (u - k)_+) + f \zeta^2 (u - k)_+] dx dt.\]

Applying (C1) to \(I_1\), it has

\[I_1 \leq \Lambda \int_{t_0}^t \int_{B_\rho} (u - k)_+^2 |X \zeta|^2 dx dt.\]

In virtue of the Cauchy inequality, \([u > k] = \{(x, t) \mid u > k\}\) and

\[\zeta X_i u = X_i (\zeta (u - k)_+) - (u - k)_+ X_i \zeta,\]

we see

\[I_2 \leq \int_{t_0}^t \int_{B_\rho} |b^i \zeta (u - k)_+ + \zeta X_i u| dx dt + \int_{t_0}^t \int_{B_\rho} \left| \sqrt{c} \zeta (u - k)_+ + \sqrt{c} u \zeta \right| dx dt\]
\[\leq \gamma(c) \int_{Q_{\rho, \tau \cap [u > k]}} \zeta^2 (u - k)_+ + \sum (b^i)^2 dx dt + \epsilon \int_{Q_{\rho, \tau \cap [u > k]}} |X_i (\zeta (u - k)_+) - (u - k)_+ X_i \zeta|^2 dx dt\]
\[+ \gamma(c) \int_{Q_{\rho, \tau \cap [u > k]}} |c \zeta^2 (u - k)_+|^2 dx dt + \epsilon \int_{Q_{\rho, \tau \cap [u > k]}} |c|((u - k)_+)^2 \zeta^2 dx dt\]
\[\leq \gamma(c) \int_{Q_{\rho, \tau \cap [u > k]}} \zeta^2 (u - k)_+ + \sum (b^i)^2 + |c| dx dt\]
\[+ \epsilon \int_{Q_{\rho, \tau \cap [u > k]}} |X_i (\zeta (u - k)_+) - (u - k)_+ X_i \zeta|^2 dx dt + 2 \epsilon \int_{Q_{\rho, \tau \cap [u > k]}} |c|((u - k)_+)^2 + k^2 \zeta^2 dx dt\]
\[\leq \gamma(c) \int_{Q_{\rho, \tau \cap [u > k]}} \left\{ \zeta^2 (u - k)_+ + \sum (b^i)^2 + |c| \right\} + |c| k^2 \zeta^2 dx dt\]
\[+ 2 \epsilon \int_{Q_{\rho, \tau \cap [u > k]}} |X_i (\zeta (u - k)_+)|^2 + (u - k)_+^2 |X \zeta|^2 dx dt\]
and

\[ I_3 \leq \gamma(e) \int_{Q_{p, r} \cap [u > k]} \left\{ \sum (f^i)^2 + |f| \zeta^2(u - k)_+ \right\} \, dx \, dt + \epsilon \int_{Q_{p, r} \cap [u > k]} [(u - k)_+ X_j \zeta + X_j (\zeta(u - k)_+)]^2 \, dx \, dt \]

\[ \leq \gamma(e) \int_{Q_{p, r} \cap [u > k]} \left\{ \sum (f^i)^2 + |f| \zeta^2(u - k)_+ \right\} \, dx \, dt + 2\epsilon \int_{Q_{p, r} \cap [u > k]} (u - k)_+^2 |X\zeta|^2 + |X_i (\zeta(u - k)_+)|^2 \, dx \, dt. \]

Substituting these estimates into (3.3) and using (C1) to the second term in the left hand side of (3.3), it is not difficult to derive

\[
\frac{1}{2} \int_{B_p} \zeta^2(u - k)_+(x, t) \, dx + \Lambda^{-1} \int_{t_0}^t \int_{B_p} |X(\zeta(u - k)_+)|^2 \, dx \, dt \\
\leq \frac{1}{2} \int_{B_p} \zeta^2(u - k)_+(x, t_0) \, dx + \int_{t_0}^t \int_{B_p} \zeta^2(u - k)_+ \, dx \, dt + \Lambda \int_{t_0}^t \int_{B_p} (u - k)_+^2 |X\zeta|^2 \, dx \, dt \\
+ \gamma(e) \int_{Q_{p, r} \cap [u > k]} \left\{ \zeta^2(u - k)_+ \left[ \sum (b^i)^2 + |c| \right] + |c| k^2 \zeta^2 \right\} \, dx \, dt \\
+ \gamma(e) \int_{Q_{p, r} \cap [u > k]} \left\{ \sum (f^i)^2 + |f| \zeta^2(u - k)_+ \right\} \, dx \, dt \\
+ 4\epsilon \int_{Q_{p, r} \cap [u > k]} (u - k)_+^2 |X\zeta|^2 + |X_i (\zeta(u - k)_+)|^2 \, dx \, dt. \]

Choosing \( \epsilon = \frac{\Lambda^{-1}}{16} \), we have

\[
\frac{1}{2} \int_{B_p} \zeta^2(u - k)_+(x, t) \, dx + \frac{3\Lambda^{-1}}{4} \int_{t_0}^t \int_{B_p} |X(\zeta(u - k)_+)|^2 \, dx \, dt \\
\leq \frac{1}{2} \int_{B_p} \zeta^2(u - k)_+(x, t_0) \, dx + C \left\{ \int_{t_0}^t \int_{B_p} (u - k)_+^2 (|X\zeta|^2 + |\zeta|) \, dx \, dt \\
+ \int_{Q_{p, r} \cap [u > k]} \zeta^2(u - k)_+^2 \left[ \sum (b^i)^2 + |c| \right] + |c| k^2 \zeta^2 \right\} \, dx \, dt \\
+ \int_{Q_{p, r} \cap [u > k]} \left\{ \sum (f^i)^2 + |f| \zeta^2(u - k)_+ \right\} \, dx \, dt. \tag{3.4} \]

Denote the third and fourth term in the right hand side of (3.4) by

\[ I_{11} := \int_{Q_{p, r} \cap [u > k]} \zeta^2(u - k)_+ \left[ \sum (b^i)^2 + |c| \right] + |c| k^2 \zeta^2 \right\} \, dx \, dt, \]

\[ I_{12} := \int_{Q_{p, r} \cap [u > k]} \sum (f^i)^2 + |f| \zeta^2(u - k)_+ \right\} \, dx \, dt. \]

Employing (C2), (3.1) and the Hölder inequality, it shows

\[ I_{11} \leq \left( \int_{Q_{p, r} \cap [u > k]} \left[ \sum (b^i)^2 + |c| \right] \, dx \, dt \right)^{\frac{m}{m+1}} \left( \int_{Q_{p, r} \cap [u > k]} \zeta^2(u - k)_+^2 \, dx \, dt \right)^{\frac{m}{m+1}} \\
+ k^2 \left( \int_{Q_{p, r} \cap [u > k]} |c|^m \, dx \, dt \right)^{\frac{1}{m}} \left( |Q_{p, r} \cap [u > k]|^{1 - \frac{1}{m}} \right) \\
\leq \Lambda \|\zeta(u - k)_+\|_{2m/(m-1)}^2 + k^2 \Lambda \|Q_{p, r} \cap [u > k]\|^{1 - \frac{1}{m}}, \]
where $m > \frac{Q+2}{2}$. As $2 < \frac{2m}{m-1} < \frac{2(Q+2)}{Q}$, it nows by the Interpolation inequality and the Young inequality that for $0 < \theta < 1$,

$$
\|\zeta(u-k)\|_{2m/(m-1)}^2 \leq \|\zeta(u-k)\|_{2(Q+2)/2(Q+2)}^{2(1-\theta)} \|\zeta(u-k)\|_{2}^{2\theta} \\
\leq \frac{e}{\Lambda} \|\zeta(u-k)\|_{2(Q+2)/2(Q+2)}^2 + \gamma(e) \|\zeta(u-k)\|_{2}^2.
$$

It implies from it and (2.10) that

$$
II_1 \leq \epsilon \|\zeta(u-k)\|_{\mathcal{L}_t^2(\rho,\tau)}^2 + \gamma(e) \|\zeta(u-k)\|_{\mathcal{L}_t^2(\rho,\tau)}^2 + k^2\Lambda|\rho,\tau \cap [u > k]|^{1-\frac{1}{p}}.
$$

Using (2.10) to $II_2$, it gets

$$
II_2 \leq \left(\int_{\rho,\tau \cap [u > k]} \left( \sum (f')^2 \right)^{\frac{2}{3}} \right)^{\frac{3}{2}} |\rho,\tau \cap [u > k]|^{1-\frac{2}{p}} \\
+ \left(\int_{\rho,\tau \cap [u > k]} |\zeta f|^2 + |\zeta(\frac{Q+2}{Q+2}) dx \right) \left( \int_{\rho,\tau \cap [u > k]} |\zeta(\frac{Q+2}{Q+2})|^2 dx \right)^{\frac{2(Q+2)}{Q+2}} |\rho,\tau \cap [u > k]|^{1-\frac{1}{p} - \frac{1}{p}} \\
\leq F^2_0 |\rho,\tau \cap [u > k]|^{1-\frac{1}{p} + \epsilon \|\zeta(u-k)\|_{\mathcal{L}_t^2(\rho,\tau)}^2 + \gamma(e) F^2_0 |\rho,\tau \cap [u > k]|^{1-\frac{1}{p}}.
$$

Putting estimates for $II_1$ and $II_2$ into (3.4) and choosing $\eta = \min \{2m, p \} (m > \frac{Q+2}{2}, p > Q + 2)$, it leads to

$$
\frac{1}{2} \int_{B_t} \zeta^2(u-k)^+ (x,t) dx + \frac{3\Lambda^{-1}}{4} \int_{t_0}^t \int_{B_t} |X(\zeta(u-k))^2| dx dt \\
\leq \frac{1}{2} \int_{B_t} \zeta^2(u-k)^+ (x,t_0) dx + C \left\{ \int_{t_0}^t \int_{B_t} (u-k)^2 (|X\zeta|^2 + |\zeta|^2) dx dt \\
+ 2\epsilon \|\zeta(u-k)\|_{\mathcal{L}_t^2(\rho,\tau)}^2 + \gamma(e) \|\zeta(u-k)\|_{\mathcal{L}_t^2(\rho,\tau)}^2 + k^2\Lambda|\rho,\tau \cap [u > k]|^{1-\frac{1}{p}} \\
+ F^2_0 |\rho,\tau \cap [u > k]|^{1-\frac{1}{p} + \gamma(e) F^2_0 |\rho,\tau \cap [u > k]|^{1-\frac{1}{p}}.\right\}
$$

Taking $\epsilon$ small enough and $\lambda_0 = \Lambda^{-1}$, we conclude

$$
\sup_{0 < t_0 < t < t_0 + \tau} \|\zeta(u-k)^+(\cdot,t)\|_{\mathcal{L}_t^2(B_t)}^2 + \lambda_0 \|X(\zeta(u-k))\|_{\mathcal{L}_t^2(\rho,\tau)}^2 \\
\leq \frac{1}{2} \int_{B_t} \zeta^2(u-k)^+ (x,t_0) dx \\
\quad + \gamma(e) \left\{ \|X\zeta\|_{\mathcal{L}_t^\infty(\rho,\tau)}^2 + \|\zeta\|_{L^\infty(\rho,\tau)}^2 + \|\zeta\|_{L^\infty(\rho,\tau)}^2 \right\} \|(u-k)\|_{\mathcal{L}_t^2(\rho,\tau)}^2 \\
\quad + (k^2 + F^2_0 |\rho,\tau \cap [u > k]|^{1-2/\eta}) \right\}
$$

and the proof is completed.

**Remark 3.1** Lemma 3.1 indicates that the weak solution to (1.1) belongs to the De Giorgi class.

Next we give several useful properties for functions in the De Giorgi class.
Lemma 3.2. Let \( u \in DG^+(\mathcal{Q}_T) \),

\[
\mathcal{Q}_R = B_2R(x_0) \times (t_0, t_0 + aR^2) \subset \mathcal{Q}_T, \text{ for } 0 < R \leq \frac{1}{2}, \]

then for any positive integer \( s \), either

\[
|B_R(x_0) \cap [u(\cdot, t) > k]| \leq (1 - \sigma)|B_R|, \text{ for } 0 < \sigma < 1, \ t \in (t_0, t_0 + aR^2) \subset (0, T), \tag{3.5}
\]

then for any positive integer \( s \), either

\[
H \leq 2^s(M + F_0)R|\mathcal{Q}_R|^{-\frac{1}{2}}, \tag{3.6}
\]

or

\[
\left| \mathcal{Q}_R \cap [u > \mu - \frac{H}{2^s}] \right| \leq \frac{C}{\sigma^2 \mu}Q_R^*|\mathcal{Q}_R|, \tag{3.7}
\]

where \( C \) relies only on \( Q, \lambda_0, \eta, \delta \) and \( \gamma(\cdot), \mathcal{Q}_R^* = B_\rho(x_0) \times (t_0, t_0 + a\rho^2) \), for \( 0 < \rho < 2R \).

Proof. Denote

\[
A_R(k, t) = B_2R(x_0) \times [\{u(\cdot, t) > k\}],
\]

\[
A_R(k) = \mathcal{Q}_R \cap [u > k],
\]

\[
k_l = \mu - \frac{H}{2^l}, \ l = 0, 1, ...,
\]

then \( k_l \) is increasing, \( A_R(k_l) \) is decreasing, and \( |A_R(k, t)| \leq (1 - \sigma)|B_R| \) by (3.5). Applying Theorem 2.9 and (3.5), we have for \( t_0 \leq t \leq t_0 + aR^2 \),

\[
(k_{l+1} - k_l)^2|A_R(k_{l+1}, t)| \leq \frac{CR^2|B_R|^2}{|B_R \setminus A_R(k_l, t)|^2} \left( \int_{A_R(k_l, t) \setminus A_R(k_{l+1}, t)} |Xu| dx \right)^2
\]

\[
\leq \frac{CR^2|B_R|^2}{|B_R - (1 - \sigma)|B_R|^2} \left( \int_{A_R(k_l, t) \setminus A_R(k_{l+1}, t)} |Xu| dx \right)^2
\]

\[
\leq \frac{CR^2}{\sigma^2} |A_R(k_l, t) \setminus A_R(k_{l+1}, t)| \int_{A_R(k_l, t)} |Xu|^2 dx.
\]

Integrating it in \( t \) over \( (t_0, t_0 + aR^2) \) and noting \( k_{l+1} - k_l = \mu - \frac{H}{2^{l+1}} - (\mu - \frac{H}{2^l}) = \frac{H}{2^{l+1}} \) and

\[
\int_{t_0}^{t_0 + aR^2} |A_R(k_l, t) \setminus A_R(k_{l+1}, t)| dt
\]

\[
= \int_{t_0}^{t_0 + aR^2} |B_R(x_0) \cap [k_l < u(\cdot, t) < k_{l+1}]| dt
\]

\[
= |\mathcal{Q}_R \cap [k_l < u < k_{l+1}]| = |A_R(k_l) \setminus A_R(k_{l+1})|
\]

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Letting $\xi$ be the cut-off function between $B_R(x_0)$ and $B_{2R}(x_0)$ and using

$$|X\xi|^2 + |\xi| + |\xi|^2 \leq \frac{C}{R^2}, \quad |u - k_l| \leq \mu - k_l = \frac{H}{2^l},$$

and (2.2), we obtain from (1.3) (k is changed to $k_l$) that

$$\int_{Q_{2n}^R} |X(u - k_l)_+|^2 dxdt$$

$$\leq (1 + \epsilon) \|\xi(u - k_l)+(\cdot, t_0)\|_{L^2(B_R)}^2 + (k^2 + F_0^2)\|\xi\|_{L^2(Q_{2n}^R)}^2$$

$$\leq C \left( \left( \frac{H^2B_R}{4^l} + \frac{H^2|Q_{R}|}{4^lR^2} \right) \right).$$

(3.9)

If (3.6) is invalid, i.e., there exists $l, 0 \leq l \leq s - 1$, such that $(M + F_0) \leq 2^{-l}HR^{-1}|Q_R|^{1/2}$, then we note $|Q_R^a| \leq |Q_R| = R^2|B_R|$ and so $|Q_R^a|^{1-2/\eta} \leq |Q_R|^{1-2/\eta}$ to arrive at from the previous estimate (3.9) that

$$\int_{Q_{2n}^R} |X(u - k_l)_+|^2 dxdt$$

$$\leq C \left( \left( \frac{H^2B_R}{4^l} + \frac{H^2|Q_{R}|}{4^lR^2} + \frac{H^2|Q_{R}|^{2/\eta}}{4^lR^2} \right) \right).$$

(3.10)

Taking it into (3.8), it obtains

$$|A_R(k_{l+1})| = |Q_R^a \cap [u > k_{l+1}]| = \int_{t_0}^{t_0+aR^2} |B_R \cap [u(\cdot, t) > k_{l+1}]| dt$$

$$= \int_{t_0}^{t_0+aR^2} |A_R(k_{l+1}, t)| dt \leq \frac{CR|B_R|^{1/2}}{\sigma}|A_R(k_l) \setminus A_R(k_{l+1})|^2.$$
Squaring both sides and summing with respect to $l$ from 0 to $s - 1$, we have by using $|A_R(k_s)| \leq |A_R(k_{s+1})|$, $0 \leq l \leq s - 1$, $|A_R(k_0)| \leq |Q_R^s|$ and $aR^2|B_R| = |Q_R^s|$ that

$$s|A_R(k_s)|^2 \leq \sum_{l=0}^{s-1} |A_R(k_{l+1})|^2$$

$$\leq \sum_{l=0}^{s-1} \frac{C}{\sigma^2} R^2 |B_R| |A_R(k_l) \setminus A_R(k_{l+1})|$$

$$\leq \frac{C}{\sigma^2} R^2 |B_R| |A_R(k_0)| \leq \frac{CaR^2|B_R|}{\alpha\sigma^2} |Q_R^s|$$

$$= \frac{C}{\alpha\sigma^2} |Q_R^s|^2,$$

which implies (3.7).

**Lemma 3.3.** Let $u \in DG^+(Q_T)$,

$$\bar{Q}_{2R} = B_{2R}(x_0) \times (t_0, t_0 + R^2) \subset Q_T,$$ for $0 < R \leq \frac{1}{2},$

$\mu \geq \text{ess sup } u, \bar{Q}_{2R} = B_{2R}(x_0) \times (t_0, t_0 + aR^2), 0 < a \leq 1.$ For $0 < H := \mu - k \leq \delta M, 0 < \sigma < 1$, if $u$ satisfies

$$|B_R(x_0) \cap [u(\cdot, t_0) > k]| \leq (1 - \sigma)|B_R|,$$ (3.10)

then there exists a positive integer $s_0 = s_0(\sigma) \geq 1$ relying on $Q, \lambda_0, \eta, \delta, \sigma$ and $\gamma(\cdot)$, such that either

$$H \leq 2^{s_0}(M + F_0)R|Q_R|^{-\frac{1}{\sigma}},$$ (3.11)

or

$$\sup_{t_0 < t < t_0 + R^2} \left| B_R(x_0) \cap \left[ u(\cap, t) > \mu - \frac{H}{2^{s_0}} \right] \right| \leq \left[ 1 - \sigma + \frac{1}{2} \min \left\{ \sigma, 1 - \sigma \right\} \right] |B_R|. $$ (3.12)

**Proof.** Suppose that $\xi(x)$ is a cut-off function between $B_{\beta R}(x_0)(0 < \beta < 1)$ and $B_R(x_0)$, such that $|X\xi|^2 + |\xi|^2 \leq \frac{C}{(1 - \beta)^2 R^2}$, and denote

$$Q_R^s = B_R(x_0) \times (t_0, t_0 + aR^2), 0 < a \leq 1,$$

$$A_R^s(k) = Q_R^s \cap [u > k].$$

We observe $u - k \leq \mu - k = H$ and apply (1.3) on $Q_R^s$ to see

$$\sup_{t_0 < t \leq t_0 + aR^2} \|\xi(u - k) + (\cdot, t)\|_{L^2(B_R)}^2 \leq (1 + \epsilon) \|\xi(u - k) + (\cdot, t_0)\|_{L^2(B_R)}^2 + \gamma(\epsilon) \left\{ \frac{CH^2}{(1 - \beta)^2 R^2} |A_R^s(k)| + (M + F_0)^2 \frac{C}{(1 - \beta)^2 R^2} \right\}^{1 - 2/\alpha}. $$ (3.13)

On the other hand, for any integer $s_1 \geq 1,$

$$u - k > \mu - \frac{H}{2^{s_1}} - k = H - \frac{H}{2^{s_1}} = (1 - 2^{-s_1})H, \text{ on } B_{\beta R} \cap [u(\cdot, t) > \mu - \frac{H}{2^{s_1}}],$$

it follows

$$\|\xi(u - k) + (\cdot, t)\|_{L^2(B_R)}^2 \geq \int_{B_{\beta R}} |u(\cdot, t) - k|^2 \, dx$$

$$\geq \int_{B_{\beta R} \cap [u(\cdot, t) > \mu - \frac{H}{2^{s_1}}]} |u(\cdot, t) - k|^2 \, dx$$

$$\geq (1 - 2^{-s_1})^2 H^2 \left| B_{\beta R} \cap [u(\cdot, t) > \mu - \frac{H}{2^{s_1}}] \right|. $$ (3.14)
If (3.11) is invalid, i.e. \( (M + F_0) \leq 2^{-s_1} H R^{-1} |Q_R|^\frac{2}{s_2} \), we note (3.10) and

\[
\| \xi(u - k)_+ (\cdot, t_0) \|_{L^2(B_R)}^2 \leq H^2 \left| B_R(x_0) \right| | u(\cdot, t_0) > k |,
\]

and obtain from (3.13) and (3.14) that

\[
\begin{aligned}
sup_{t_0 < t \leq t_0 + aR^2} \left| B_{\beta R} \right| | u(\cdot, t) > \mu - \frac{H}{2^{s_1}} | \\
\leq \frac{1}{(1 - 2^{-s_1})^2 H^2} \sup_{t_0 < t \leq t_0 + aR^2} \| \xi(u - k)_+ (\cdot, t) \|_{L^2(B_R)}^2 \\
\leq \frac{1}{(1 - 2^{-s_1})^2 H^2} \left\{ (1 + \epsilon) | B_R(x_0) \cap | u(\cdot, t_0) > k | \\
+ \gamma(\epsilon) \left[ \frac{CH^2}{(1 - \beta)^2 |Q_R|} | A_R^\beta(k) | + (M + F_0)^2 | A_R^\beta(k) |^{1 - 2/\eta} \right] \right\} \\
\leq \left( 1 + \epsilon \right) \left( 1 - \sigma \right) \left( 1 - 2^{-s_1} \right)^2 | B_R | + C \gamma(\epsilon) \left[ \frac{| A_R^\beta(k) |}{(1 - \beta)^2 |Q_R|} + \left( \frac{| A_R^\beta(k) |}{|Q_R|} \right)^{1 - 2/\eta} \right],
\end{aligned}
\]

(3.15)

where the fact \( R^2 | B_R | = |Q_R| \) is used. Obviously, \( \frac{(1 - \sigma)}{(1 - 2^{-s_1})^2} \leq 4\epsilon \) from \( 1 - \sigma \leq 1 \) and \( (1 - 2^{-s_1})^2 \geq \frac{1}{4} \).

\[
\frac{1}{(1 - 2^{-s_1})^2} > 1; | B_{\beta R} | \geq C_{\beta}^{-1} \beta^Q | B_R | \quad \text{and} \quad | B_R | \| B_{\beta R} \| \leq (1 - C_{\beta}^{-1} \beta^Q) | B_R | \quad \text{by (2.2)} \quad \text{and} \quad \frac{| A_R^\beta(k) |}{|Q_R|} \leq \left( \frac{| A_R^\beta(k) |}{|Q_R|} \right)^{1 - \frac{2}{\eta}}.
\]

Combining these and (3.15), it derives that for \( t_0 < t \leq t_0 + aR^2 \),

\[
\begin{aligned}
\left| B_R(x_0) \cap | u(\cdot, t) > \mu - \frac{H}{2^{s_1}} | \right| \\
\leq | B_R | \| B_{\beta R} | + \left| B_{\beta R}(x_0) \cap | u(\cdot, t) > \mu - \frac{H}{2^{s_1}} | \right| \\
\leq (1 - C_{\beta}^{-1} \beta^Q) | B_R | + \left( 1 + \epsilon \right) \left( 1 - \sigma \right) \left( 1 - 2^{-s_1} \right)^2 | B_R | + C \gamma(\epsilon) | B_R | \left[ \frac{1}{(1 - \beta)^2 |Q_R|} + \left( \frac{| A_R^\beta(k) |}{|Q_R|} \right)^{1 - 2/\eta} \right] \\
\leq | B_R | \left[ 1 - C_{\beta}^{-1} \beta^Q + \left( 1 - \sigma \right) \left( 1 - 2^{-s_1} \right)^2 + 4\epsilon + C \gamma(\epsilon) \left( \frac{| A_R^\beta(k) |}{|Q_R|} \right)^{1 - 2/\eta} \right].
\end{aligned}
\]

(3.16)

Choosing \( \beta \in (0, 1) \) such that

\[
1 - \beta = \left( \frac{| A_R^\beta(k) |}{|Q_R|} \right)^{(1/3)(1 - 2/\eta)}
\]

and watching \( 1 - C_{\beta}^{-1} \beta^Q \leq C_Q (1 - \beta) \) for a positive constant \( C_Q \), it obtains from (3.16) that

\[
\begin{aligned}
\left| B_R(x_0) \cap | u(\cdot, t) > \mu - \frac{H}{2^{s_1}} | \right| \\
\leq | B_R | \left[ C_Q \left( \frac{| A_R^\beta(k) |}{|Q_R|} \right)^{(1/3)(1 - 2/\eta)} + \left( 1 - \sigma \right) \left( 1 - 2^{-s_1} \right)^2 + 4\epsilon + C \gamma(\epsilon) \left( \frac{| A_R^\beta(k) |}{|Q_R|} \right)^{(1/3)(1 - 2/\eta)} \right] \\
\leq | B_R | \left[ \frac{(1 - \sigma)}{(1 - 2^{-s_1})^2} + 4\epsilon + (C_Q + C \gamma(\epsilon)) \left( \frac{| A_R^\beta(k) |}{|Q_R|} \right)^{(1/3)(1 - 2/\eta)} \right].
\end{aligned}
\]

(3.17)

Since \( \gamma(\epsilon) \) is decreasing, we know that \( q = \epsilon^{-1} \gamma^{-1}(\epsilon) \in \mathbb{R}^+ \), is strictly increasing and its inverse function \( \epsilon = \varphi(q) \) satisfies \( \varphi(q) \to 0 \) as \( q \to 0 \). Using \( \gamma(\epsilon) = \frac{3}{2} \) and choosing

\[
\epsilon = \varphi \left( \left( \frac{| A_R^\beta(k) |}{|Q_R|} \right)^{(1/3)(1 - 2/\eta)} \right),
\]

16
it follows
\[

g(\epsilon) = \epsilon \left( \frac{|A^\theta_R(k)|}{|Q_R|} \right)^{-(1/3)(1-2/\eta)} = \varphi \left( \left( \frac{|A^\theta_R(k)|}{|Q_R|} \right)^{(1/3)(1-2/\eta)} \right) \left( \frac{|A^\theta_R(k)|}{|Q_R|} \right)^{-(1/3)(1-2/\eta)}
\]
and so (3.17) becomes
\[
|B_R(x_0) \cap [u(\cdot, t) > \mu - \frac{H}{2^{s_1}}]| \\
\leq |B_R| \left\{ \frac{(1 - \sigma)}{(1 - 2^{-s_1})^2} + C\varphi \left( \left( \frac{|A^\theta_R(k)|}{|Q_R|} \right)^{(1/3)(1-2/\eta)} \right) \right\}.
\]
(3.18)

As \( \epsilon = \varphi(q) \) is increasing and \( |A^\theta_R(k)| \leq |Q^R_R| = a|Q_R| \), it implies
\[
C\varphi \left( \left( \frac{|A^\theta_R(k)|}{|Q_R|} \right)^{(1/3)(1-2/\eta)} \right) \leq C\varphi(a^{(1/3)(1-2/\eta)}).
\]
Picking \( a = a(\sigma) > 0 \) satisfying
\[
C\varphi(a^{(1/3)(1-2/\eta)}) \leq \frac{1}{4} \min\{1 - \sigma, \sigma\}
\]
and using \( \lim_{s_1 \to \infty} \frac{(1 - \sigma)}{(1 - 2^{-s_1})^2} = 1 - \sigma \), we can take \( s_1 = s_1(\sigma) \) large enough, such that
\[
\frac{(1 - \sigma)}{(1 - 2^{-s_1})^2} \leq 1 - \sigma + \frac{1}{4} \min\{1 - \sigma, \sigma\}.
\]

With it and (3.18), it derives
\[
\sup_{t_0 \leq t \leq t_0 + aR^2} |B_R(x_0) \cap [u(\cdot, t) > \mu - \frac{H}{2^{s_1}}]| \leq \left\{ 1 - \sigma + \frac{1}{2} \min\{1 - \sigma, \sigma\} \right\} |B_R|.
\]
(3.19)

Similarly, denoting
\[
Q^{1-a}_R = B_R(x_0) \times (t_0 + aR^2, t_0 + R^2],
\]
\[
A^{1-a}_R(k) = Q^{1-a}_R \cap [u > k],
\]
and repeating the process above, we have for a large \( s_2 \geq 1 \),
\[
\sup_{t_0 + aR^2 \leq t \leq t_0 + R^2} |B_R(x_0) \cap [u(\cdot, t) > \mu - \frac{H}{2^{s_2}}]| \leq \left\{ 1 - \sigma + \frac{1}{2} \min\{1 - \sigma, \sigma\} \right\} |B_R|.
\]
(3.20)

Letting \( s_0 = \max\{s_1, s_2\} \), it shows by combining (3.19) and (3.20) that
\[
\sup_{t_0 \leq t \leq t_0 + R^2} |B_R(x_0) \cap [u(\cdot, t) > \mu - \frac{H}{2^{s_1}}]| \leq \left\{ 1 - \sigma + \frac{1}{2} \min\{1 - \sigma, \sigma\} \right\} |B_R|,
\]
which is (3.12).

**Lemma 3.4.** Let \( u \in DG^+(Q_T) \), \( Q_R = B_R(x_0) \times (t_0 - R^2, t_0] \subset Q_T, 0 < R \leq 1 \), and \( \mu \geq \text{ess sup}_{Q_R} u. \) there exists \( \theta \in (0, 1) \) depending on \( Q, \lambda_0, \eta, \delta, \sigma \) and \( \gamma(\cdot) \), such that for \( k < \mu \), if
\[
|Q_R \cap [u > k]| \leq \theta |Q_R|,
\]
(3.21)

\[
\delta M \geq H := \mu - k > (M + F_0)R|Q_R|^{-\frac{1}{\gamma}},
\]
(3.22)

then
\[
\text{ess sup}_{Q_{R/2}} u \leq \mu - \frac{H}{2}.
\]
(3.23)
To prove Lemma 3.4, we recall a known result.

**Lemma 3.5 ([7]).** Given a non-negative sequence \( \{ y_h \} (h = 0, 1, 2, \ldots) \) satisfying the recursion relation
\[
y_{h+1} \leq C b^h y_h^{1+\epsilon},
\]
where \( b > 1 \) and \( \epsilon > 0 \), if
\[
y_0 \leq \theta = C^{-1/\epsilon} b^{-1/\epsilon^2},
\]
then
\[
\lim_{h \to \infty} y_h = 0.
\]

**Proof of Lemma 3.4.**

Denote \( R_m = \frac{R}{2} + \frac{R}{2m+1}, k_m = \mu - \frac{H}{2} - \frac{H}{2m+1} \), and \( Q_m = Q_{R_m} \), \( \tilde{m} = 0, 1, 2, \ldots \),
then \( R_m \) is decreasing, \( Q_m \) is increasing and
\[
R_m - R_{m+1} = \left( \frac{R^2}{4} + \frac{R^2}{2m+1} + \frac{R^2}{22m+2} \right) - \left( \frac{R^2}{4} + \frac{R^2}{2m+2} + \frac{R^2}{22m+4} \right) \geq \frac{R^2}{2m+2}.
\]

Take a cut-off function \( \zeta_m(x, t) \) between \( Q_m \) and \( Q_{m+1} \), then
\[
|X \zeta_m|^2 \leq \left( \frac{C}{R_m - R_{m+1}} \right)^2 \leq \frac{C2^{2m+4}}{R^2}, \zeta_m \leq \frac{C}{R_m - R_{m+1}} \leq \frac{C2^{2m+2}}{R^2}, |\zeta_m| \leq 1,
\]
and
\[
|X \zeta_m|^2 + |\zeta_m| + |\zeta_m|^2 \leq C\frac{2^{4m}}{R^2}.
\]

Using
\[
\| \zeta_m(u - k_m) + (\cdot, t_0) \|^2_{L^2(B_{R_m})} = \frac{1}{R^2} \int_{t_0 - R^2}^{t_0} \int_{B_{R_m}} |(u - k_m) + (\cdot, t)|^2 \, dx \, dt
\leq \| (u - k_m) + \|^2_{L^2(Q_m)},
\]
and replacing \( Q_{\rho, \tau, \zeta} \) and \( k \) in (1.3) by \( Q_m, \zeta_m \) and \( k_m \), we obtain
\[
\| \zeta_m(u - k_m) + \|^2_{L^2(Q_m)} \leq \sup_{t_0 < t < t_0 + T} \| \zeta_m(u - k_m) + (\cdot, t) \|^2_{L^2(B_{R_m})} + \| X(\zeta_m(u - k_m) + ) \|^2_{L^2(Q_m)}
\leq C \left[ \frac{2^{4m}}{R^2} \right] \| (u - k_m) + \|^2_{L^2(Q_m)} + (M + F_0)^2 |Q_m \cap \{ u > k_m \} |^{-1/\eta}. \quad (3.25)
\]

Denoting \( A_m = Q_m \cap \{ u > k_m \} \) it follows \( \| (u - k_m) + \|^2_{L^2(Q_m)} \leq H^2 |A_m| \) from \( u - k_m \leq \mu - k_m \leq H \).
Applying it into (3.25), we have by (2.10),
\[
\| \zeta_m(u - k_m) + \|^2_{L^2(Q_m)}
\leq C R^2 |Q_R|^{-\frac{\eta}{2 + \eta}} \| \zeta_m(u - k_m) + \|^2_{L^2(Q_m)}
\leq C |Q_R|^{-\frac{\eta}{2 + \eta}} \left[ \frac{2^{4m} H^2 |A_m| + R^2 (M + F_0)^2 |A_m| |^{-1/\eta} \right]. \quad (3.26)
\]
On the other hand, $u > k_{m+1}$ on $A_{m+1}$ and
\[
\|\zeta_m(u - k_m)\| \|_{2(Q+2)}^{(Q+2)} A_m = \left( \int_{Q_m} (\zeta_m(u - k_m))^{\frac{2(Q+2)}{Q+2}} dx \right)^{\frac{Q}{Q+2}} \\
\geq \left( \int_{A_{m+1}} (u - k_m)^{\frac{2(Q+2)}{Q+2}} dx \right)^{\frac{Q}{Q+2}} \\
\geq (k_{m+1} - k_m)^2 |A_{m+1}|^{\frac{Q}{Q+2}},
\]
it shows
\[
(k_{m+1} - k_m)^2 |A_{m+1}| \leq \|\zeta_m(u - k_m)\| \|_{2(Q+2)}^{(Q+2)} (Q_m) |A_{m+1}|^{\frac{Q}{Q+2}}. \quad (3.27)
\]
Substituting $k_{m+1} - k_m = \frac{H}{m+1}$ into (3.27), we derive from (3.26) that
\[
|A_{m+1}| \leq \frac{2^{m+4}}{H^2} C |Q_R|^{-\frac{Q}{Q+2}} \left[ 2^{4m} H^2 |A_m| + R^2 (M + F_0)^2 |A_m|^{1-\frac{2}{\eta}} \right] |A_{m+1}|^{\frac{Q}{Q+2}} \\
\leq 2^{4m} C |Q_R|^{-\frac{Q}{Q+2}} \left[ 2^{4m} |A_m| + \frac{R^2 (M + F_0)^2}{H^2} |A_m|^{1-\frac{2}{\eta}} \right] |A_{m+1}|^{\frac{Q}{Q+2}} \\
\leq C 2^{4m} |Q_R|^{-\frac{Q}{Q+2}} \left[ |A_m| + \frac{R^2 (M + F_0)^2}{H^2} |A_m|^{1-\frac{2}{\eta}} \right] |A_{m+1}|^{\frac{Q}{Q+2}}. \quad (3.28)
\]
Due to (3.22) and $|A_m| \leq |Q_R|$, it yields from (3.28) that
\[
|A_{m+1}| \leq C 2^{4m} |Q_R|^{-\frac{Q}{Q+2}} \left( |A_m| + |A_m|^{1-\frac{2}{\eta}} |Q_R|^{\frac{2}{\eta}} \right) |A_{m+1}|^{\frac{Q}{Q+2}} \\
= C 2^{4m} |Q_R|^{-\frac{Q}{Q+2}} |A_m|^{1-\frac{2}{\eta}} \left( |A_m|^{\frac{2}{\eta}} + |Q_R|^{\frac{2}{\eta}} \right) |A_{m+1}|^{\frac{Q}{Q+2}} \\
\leq C 2^{4m} |A_m|^{1-\frac{2}{\eta} + \frac{2}{Q+2}} |Q_R|^{\frac{2}{\eta} - \frac{Q}{Q+2}},
\]

hence
\[
\frac{|A_{m+1}|}{|Q_R|} \leq C 2^{4m} \left( \frac{|A_m|}{|Q_R|} \right)^{1-\frac{2}{\eta} + \frac{2}{Q+2}}. \quad (3.29)
\]

Let
\[
y_m = \frac{|A_m|}{|Q_R|},
\]
then (3.29) becomes
\[
y_{m+1} \leq C 2^{4m} y_m^{1-\frac{2}{\eta} + \frac{2}{Q+2}}.
\]

Observing $1 - \frac{2}{\eta} + \frac{2}{Q+2} > 1$ (as $\eta > Q + 2$ ) and (3.21), we have
\[
y_0 = \frac{|A_0|}{|Q_R|} = \frac{|Q_R \cap [u > k]|}{|Q_R|} \leq \theta, \ \theta \in (0, 1), \quad (3.30)
\]
then $\lim_{m \to \infty} y_m = 0$ by Lemma 3.5, therefore $\lim_{m \to \infty} |A_m| = 0$.

Since
\[
R_m \to \frac{R}{2}, k_m \to \mu - \frac{H}{2} \text{ as } m \to \infty
\]
we have
\[
0 = \lim_{m \to \infty} |A_m| = \lim_{m \to \infty} |Q_{R_m} \cap [u > k_m]| = |Q_{R/2} \cap [u > \mu - \frac{H}{2}]|,
\]
which gives (3.23).
Lemma 3.6. Let $u \in DG^+(Q_T)(u \in DG^-(Q_T))$,
\[ \hat{Q}_{2R} = B_{2R}(x_0) \times (t_0 - R^2, t_0) \subset Q_T, \text{ for } 0 < R \leq 1/2, \]
\[ \mu \geq \text{ess sup} u (\tilde{\mu} \leq \text{ess inf} u). \]
If for $0 < \mu - k \leq \delta M$ ($0 < k - \tilde{\mu} \leq \delta M$) and $0 < \sigma < 1$, $u$ satisfies
\[ |B_R(x_0) \cap [u(\cdot, t_0 - R^2) > k]| \leq (1 - \sigma)|B_R| \quad (|B_R(x_0) \cap [u(\cdot, t_0 - R^2) \leq k]| \leq (1 - \sigma)|B_R|), \]
then there exists $s = s(\sigma) \geq 1$ depending on $Q, \lambda_0, \eta, \delta, \sigma$ and $\gamma(\cdot)$ such that either
\[ H := \mu - k \leq 2^s(M + F_0)R|\hat{Q}_R|^{-\frac{1}{n}} \quad (H := k - \tilde{\mu} \leq 2^s(M + F_0)R|\hat{Q}_R|^{-\frac{1}{n}}), \]
or
\[ \text{ess sup} u \leq \frac{\mu - H}{2} \quad \text{ess inf} u \geq \frac{\mu + H}{2}, \]
where $\hat{Q}_R = B_R(x_0) \times (t_0 - \rho^2, t_0]$.

Proof. Let $s_0$ be the constant in Lemma 3.3. If (3.32) is invalid for $s \geq s_0$, then from Lemma 3.3 and (3.31),
\[ \sup_{t_0 - R^2 \leq t \leq t_0} |B_R(x_0) \cap [u(\cdot, t) > \mu - \frac{H}{2^{s_0}}]| \leq \left[ 1 - \sigma + \frac{1}{2} \min\{\sigma, 1 - \sigma\} \right] |B_R|. \]
Employing Lemma 3.2 ($s$ and $H$ are changed into $s - s_0 - 1$ and $\frac{H}{2^{s_0}}$, respectively), it follows
\[ \left| \hat{Q}_R \cap [u > \mu - \frac{H}{2^{s_0 - 1}}] \right| \leq \frac{C}{\sigma \sqrt{s - s_0 - 1}} |\hat{Q}_R|, \]
where $C$ relies on $Q, \lambda_0, \eta$, and $\gamma(\cdot)$. Let $\theta$ be the constant in Lemma 3.4 and choose $s$ large enough satisfying
\[ \frac{C}{\sigma \sqrt{s - s_0 - 1}} \leq \theta. \]
It implies (3.33) from Lemma 3.4.

4 Proofs of main results

We first prove an oscillation estimate for the weak solution to (1.1).

Lemma 4.1. Let $u \in V^{1,0}_2(Q_T)$ be the weak solution to (1.1) with (C1), (C2) and (C3),
\[ Q_{R_0} = B_{R_0}(x_0) \times (t_0 - R^2, t_0) \subset Q_T, \quad 0 < R_0 \leq 1, \]
then for any $R \in (0, R_0]$, there exists $\beta, 0 < \beta \leq 1 - \frac{\alpha + 2}{n}$, such that
\[ \text{osc} u \leq C \left( \frac{R}{R_0} \right)^\beta \text{osc} u + (M + F_0)R_0|\hat{Q}_{R_0}|^{-\frac{1}{n}}, \]
where $\text{osc} u = \text{ess sup} u - \text{ess inf} u, C \geq 1$ relies on $Q, \eta, \Lambda, \delta$ and
\[ F_0 = \sum_i \|f_i\|_{L^p(Q_T)} + \|f\|_{L^{\frac{n(\alpha + 2)}{n - \alpha + 2}}(Q_T)}. \]
Proof. Denote $\nu(R) = \operatorname{ess sup}_Q u$, $\tilde{\nu}(R) = \operatorname{ess inf}_Q u$ and $\omega(R) = \nu(R) - \tilde{\nu}(R)$, then
\[
\operatorname{osc}_Q u = \omega(R)
\]
and one of the following two inequalities holds:
\[
\begin{align*}
|B_{R/2}(x_0) \cap \{u \left(\cdot, t_0 - \left(\frac{R}{2}\right)^2\right) < \tilde{\nu}(R) + \frac{1}{2} \omega(R)\}| & \leq \frac{1}{2} |B_{R/2}|, \\
|B_{R/2}(x_0) \cap \{u \left(\cdot, t_0 - \left(\frac{R}{2}\right)^2\right) > \nu(R) - \frac{1}{2} \omega(R)\}| & \leq \frac{1}{2} |B_{R/2}|.
\end{align*}
\tag{4.2}
\]

If (4.2) is valid, then Lemma 3.6 implies that for $\omega(R/2) \leq H := \frac{\mu - \tilde{\mu}}{2} \leq \delta M$, there exists $s_1 = s_1(1/2) \geq 1$, such that one of the following two inequalities holds:
\[
\begin{align*}
\frac{\omega(R)}{2} & \leq 2^{s_1}(M + F_0)R|Q_R|^{-\frac{1}{4}}, \\
\operatorname{ess inf}_Q u & \geq \tilde{\nu}(R) + \frac{\omega(R)}{2^{s_1+2}}.
\end{align*}
\tag{4.4}
\]

It sees that by (4.4),
\[
\omega(R/4) \leq \omega(R) \leq 2^{s_1+1}(M + F_0)R|Q_R|^{-\frac{1}{4}},
\tag{4.6}
\]
and by (4.5),
\[
\omega(R/4) = \operatorname{ess sup}_Q u - \operatorname{ess inf}_Q u
\leq \operatorname{ess sup}_Q u - \tilde{\nu}(R) - \frac{\omega(R)}{2^{s_1+2}}
= \omega(R) - \frac{\omega(R)}{2^{s_1+2}}
= \omega(R)(1 - 2^{s_1+2}).
\tag{4.7}
\]

If (4.3) is valid, then by Lemma 3.6 there exists $s_2 = s_2(1/2) \geq 1$, such that one of the following two inequalities holds:
\[
\begin{align*}
\frac{\omega(R)}{2} & \leq 2^{s_2}(M + F_0)R|Q_R|^{-\frac{1}{4}}, \\
\operatorname{ess sup}_Q u & \leq \tilde{\nu}(R) - \frac{\omega(R)}{2^{s_2+2}}.
\end{align*}
\tag{4.8}
\]

It shows that from (4.8),
\[
\omega(R/4) \leq \omega(R) \leq 2^{s_2+1}(M + F_0)R|Q_R|^{-\frac{1}{4}},
\tag{4.10}
\]
and from (4.9),
\[
\omega(R/4) = \operatorname{ess sup}_Q u - \operatorname{ess inf}_Q u
\leq \nu(R) - \frac{\omega(R)}{2^{s_2+2}} - \operatorname{ess inf}_Q u
= \nu(R) - \tilde{\nu}(R) - \frac{\omega(R)}{2^{s_2+2}}
= \nu(R) - \frac{\omega(R)}{2^{s_2+2}}
= \omega(R)(1 - 2^{s_1+2}).
\tag{4.11}
\]
Let us take \( s_0 = \max\{s_1, s_2\} \), we derive by (4.6) and (4.10) that

\[
\omega(R/4) \leq \omega(R) \leq 2^{s_0+1}(M + F_0)R|\mathcal{Q}_R|^{-\frac{4}{n}},
\]

and by (4.7) and (4.11) that

\[
\omega(R/4) = \text{ess sup}_{Q_{R/4}} u - \text{ess inf}_{Q_{R/4}} u \leq \omega(R)(1 - 2^{s_0+2}).
\]

Combining (4.12) and (4.13), it follows that for \( R \in (0, R_0), 0 < R_0 \leq 1, \)

\[
\omega(R/4) = \text{osc}_{Q_{R/4}} u \leq \omega(R)(1 - 2^{s_0+2} + 2^{s_0+1}(M + F_0)R|\mathcal{Q}_R|^{-\frac{4}{n}}).
\]

In terms of \( |B_{R_0/4}| \geq c_1^{-1} (\frac{1}{4})^Q |B_{R_0}| \), we find

\[
\frac{R_0}{4} |Q_{R_0/4}|^{-\frac{4}{n}} = \frac{R_0}{4} \left[ \frac{(R_0/4)^2}{|B_{R_0/4}|} \right]^{-\frac{4}{n}} \leq C_1 \left( \frac{1}{4} \right)^{1 - \frac{4\beta_
u}{n}} R_0 |Q_{R_0}|^{-\frac{4}{n}}.
\]

Replacing \( R \) in (4.14) by \( R_0 \) and denoting \( \vartheta = 1 - 2^{-(s_0+1)} \) and \( v = (\frac{1}{4})^{1 - \frac{4\beta_
u}{n}} \), it implies \( 0 < \vartheta, \psi < 1 \), and from (4.14) and the above estimate that

\[
\omega(R_0/4^2) \leq \vartheta \omega(R_0/4) + 2^{s_0+1}(M + F_0)\frac{R_0}{4} |Q_{R_0/4}|^{-\frac{4}{n}}
\]

\[
\leq \vartheta \left( \vartheta \omega(R_0) + 2^{s_0+1}(M + F_0)R_0 |Q_{R_0}|^{-\frac{4}{n}} \right) + 2^{s_0+1}(M + F_0)C_1^\frac{1}{\vartheta} R_0 |Q_{R_0}|^{-\frac{4}{n}}
\]

\[
\leq \vartheta^2 \omega(R_0) + (\vartheta + \psi)2^{s_0+1}(M + F_0)C_1^\frac{1}{\psi} R_0 |Q_{R_0}|^{-\frac{4}{n}}.
\]

Generally, for \( t = \log_4 \frac{R_0}{R} \), we have

\[
\omega(R_0/4^t) \leq \vartheta^t \omega(R_0) + \sum_{t=0}^{t-1} \vartheta^t \psi^{t-1} - 1 \times 2^{s_0+1} C_1^\frac{1}{\psi} (M + F_0)R_0 |Q_{R_0}|^{-\frac{4}{n}}.
\]

Let us discuss two cases: \( \vartheta \geq \psi \) and \( \vartheta < \psi \).

1. If \( \vartheta \geq \psi \), it gives \( \frac{1}{\psi} \leq \frac{1}{\vartheta} = 4^{1 - \frac{4\beta_
u}{n}} \) and then \( \log_4 \frac{1}{\vartheta} \leq 1 - \frac{Q+2}{\eta} \). Choosing \( \beta_1 = \log_4 \frac{1}{\vartheta} \), we derive from (4.15) that

\[
\omega(R) \leq \vartheta^t \omega(R_0) + \frac{1 - \vartheta^t}{1 - \vartheta} 2^{s_0+1} C_1^\frac{1}{\psi} (M + F_0)R_0 |Q_{R_0}|^{-\frac{4}{n}}
\]

\[
\leq \vartheta^t \left( \omega(R_0) + \frac{(\vartheta - t - 1)2^{s_0+1} C_1^\frac{1}{\psi}}{1 - \vartheta} (M + F_0)R_0 |Q_{R_0}|^{-\frac{4}{n}} \right)
\]

\[
\leq C \left( \frac{R}{R_0} \right)^{\beta_1} \left[ \omega(R_0) + (M + F_0)R_0 |Q_{R_0}|^{-\frac{4}{n}} \right],
\]

where we used \( \vartheta^t = \left( \frac{R}{R_0} \right)^{\beta_1} \) and \( C = \max \left\{ 1, \frac{(\vartheta - t - 1)2^{s_0+1} C_1^\frac{1}{\psi}}{1 - \vartheta} \right\} \).

2. If \( \vartheta < \psi \), then \( \frac{1}{\vartheta} > \frac{1}{\psi} = 4^{1 - \frac{4\beta_
u}{n}} \) and \( \log_4 \frac{1}{\psi} > 1 - \frac{Q+2}{\eta} \). Choosing \( \beta_2 = 1 - \frac{Q+2}{\eta} \), it leads to from
Lemma 4.2. Let $\alpha$ where

\[
\omega(R) \leq v^t \omega(R_0) + \frac{v^t - 1}{v - 1} 2^{\alpha + 1} C_1^\alpha (M + F_0) R_0 |Q_{R_0}|^{-\frac{1}{\alpha}}
\]

\[
\leq v^t \left( \omega(R_0) + \frac{1 - v^{-t}}{v - 1} 2^{\alpha + 1} C_1^\alpha (M + F_0) R_0 |Q_{R_0}|^{-\frac{1}{\alpha}} \right)
\]

\[
\leq C \left( \frac{R}{R_0} \right)^{\beta_2} \left[ \omega(R_0) + (M + F_0) R_0 |Q_{R_0}|^{-\frac{1}{\alpha}} \right],
\]

where $v^t = \left( \frac{R}{R_0} \right)^{-\frac{1}{1 - \beta_1}}$ and $C = \max \left\{ 1, \frac{1 - v^{-t}}{v - 1} 2^{\alpha + 1} C_1^\alpha \right\}$.

Select $\beta = \min\{\beta_1, \beta_2\}$, we prove (4.1) by combining estimates in cases (1) and (2).

The following is an isomorphism between $L^{p', \lambda}(Q)$ and $C^\alpha(Q)$.

\[ L^{p', \lambda}(Q) \cong C^\alpha(Q), \]

where $\alpha = \frac{1}{p'}$.

\[ \text{Proof.} \quad \text{Suppose } u \in C^\alpha(Q). \text{ For any } Z = (x, t) \in Q, \text{ denote } \]

\[ \hat{Q}_R(Z) = (B_R(x_0) \times (t, t + R^2)) \cap Q, \text{ for } 0 < R \leq d = \text{diam} Q, \]

we have for $Y_1 = (y_1, s_1)$ and $Y_2 = (y_2, s_2) \in \hat{Q}_R(Z)$,

\[
\left| u(Y_1) - u(\hat{Q}_n(Z)) \right| \leq \frac{1}{|Q_R(Z)|} \int_{\hat{Q}_n(Z)} |u(Y_1) - u(Y_2)| dY_2
\]

\[
\leq \frac{|u|_\alpha}{|Q_R(Z)|} \int_{\hat{Q}_n(Z)} |d^p Y_1, Y_2| dy_2
\]

\[
\leq C |u|_\alpha R^\alpha
\]

and

\[
\frac{R^{-\lambda}}{|Q_R(Z)|} \int_{\hat{Q}_n(Z)} |u(Y_1) - u(\hat{Q}_n(Z))|^{p'} dY_1 \leq CR^{\alpha p'} |u|_{\alpha}'.
\]

Noting $\alpha = \frac{1}{p'}$, it yields

\[
\sup_{Z \in Q, 0 < R \leq d} \frac{R^{-\lambda}}{|Q_R(Z)|} \int_{\hat{Q}_n(Z)} |u(Y_1) - u(\hat{Q}_n(Z))|^{p'} dY_1 \leq CR^{\alpha p'} |u|_{\alpha} = C |u|_{\alpha}
\]

and

\[
C^\alpha(Q) \subset L^{p', \lambda}(Q). \quad (4.16)
\]

On the contrary, if $u \in L^{p', \lambda}(Q)$, we have for any $Y \in \hat{Q}_R(Z), 0 < \rho < R \leq d$,

\[
\left| u(\hat{Q}_\rho(Z)) - u(\hat{Q}_n(Z)) \right|^{p'} \leq 2^{p'-1} \left[ \left| u(\hat{Q}_\rho(Z)) - u(Y) \right|^{p'} + \left| u(Y) - u(\hat{Q}_n(Z)) \right|^{p'} \right].
\]

(4.17)

Integrating it over $\hat{Q}_\rho(Z)$, it follows

\[
\left| u(\hat{Q}_\rho(Z)) - u(\hat{Q}_n(Z)) \right|^{p'} \left| \hat{Q}_\rho(Z) \right| u^{p'}_{p', \lambda}
\]

\[
\leq 2^{p'-1} \int_{\hat{Q}_n(Z)} u(\hat{Q}_\rho(Z)) - u(Y) dY + \int_{\hat{Q}_n(Z)} u(Y) - u(\hat{Q}_n(Z)) dY
\]

\[
\leq 2^{p'} R^\lambda \left| \hat{Q}_\rho(Z) \right| u^{p'}_{p', \lambda}
\]

\[
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\]
and

\[ |u_{\hat{Q}_n}(Z) - u_{\hat{Q}_m}(Z)| \leq C R^p \left\| \frac{\hat{Q}_n(Z)}{\hat{Q}_m(Z)} \right\|^p |u|_{p',\lambda}. \]

Using \( \left| \frac{\hat{Q}_n(Z)}{\hat{Q}_m(Z)} \right| \leq \left( \frac{R}{r} \right)^{Q+2} \) by (2.2), it views that for any \( k, m, k \leq m \),

\[
\left| u_{\hat{Q}_{2^{-m}R}}(Z) - u_{\hat{Q}_{2^{-m}R}}(Z) \right| \leq \sum_{j=k+1}^{m} \left| u_{\hat{Q}_{2^{-j}R}}(Z) - u_{\hat{Q}_{2^{-j}R}}(Z) \right|
\]

\[
\leq \sum_{j=k+1}^{m} C |2^{-j+1}R|^{\alpha} 2^{-j\alpha} [u]_{p',\lambda}
\]

\[
= C 2^{\alpha+2} \frac{\hat{Q}_m}{\hat{Q}_n} \sum_{j=k+1}^{m} 2^{-j\alpha} [u]_{p',\lambda}
\]

\[
\leq C R^\alpha [2^{-k\alpha} - 2^{-m\alpha}] |u|_{p',\lambda}, \tag{4.18}
\]

which implies that \( \{ u_{\hat{Q}_{2^{-m}R}}(Z) \} \) is a Cauchy sequence, and its limit is denoted by \( \tilde{u}(Z) \). Letting \( m \to \infty \) in (4.18), we have

\[
\left| u_{\hat{Q}_{2^{-k}R}}(Z) - \tilde{u}(Z) \right| \leq C R^\alpha 2^{-k\alpha} [u]_{p',\lambda}. \tag{4.19}
\]

Since \( u_{\hat{Q}_{2^{-k}R}}(Z) \) (for any \( k \)) is continuous on \( Z \), it knows by (4.19) that \( \tilde{u}(Z) \) is continuous in \( Q \). In addition, Lebesgue’s Theorem assures

\[
\lim_{m \to \infty} u_{\hat{Q}_{2^{-m}R}}(Z) = u(Z), \text{ a.e. } Z \in Q,
\]

hence \( u(Z) = \tilde{u}(Z) \) a.e. \( Z \in Q \) and \( \tilde{u}(Z) \) is independent of \( R \). Now we see from (4.19) that

\[
\left| u_{\hat{Q}_n}(Z) - \tilde{u}(Z) \right| \leq C R^\alpha [u]_{p',\lambda}. \tag{4.20}
\]

Let us prove further that \( \tilde{u}(Z) \) is Hölder continuous. For any \( Y_1, Y_2 \in Q \), we take \( R = d_P(Y_1, Y_2) \) and obtain from (4.20) that

\[
|\tilde{u}(Y_1) - \tilde{u}(Y_2)| \leq |u_{\hat{Q}_n(Y_1)} - \tilde{u}(Y_1)| + |u_{\hat{Q}_n(Y_2)} - \tilde{u}(Y_2)| + |u_{\hat{Q}_n(Y_1)} - u_{\hat{Q}_n(Y_2)}|
\]

\[
\leq C R^\alpha [u]_{p',\lambda} + |u_{\hat{Q}_n(Y_1)} - u_{\hat{Q}_n(Y_2)}|. \tag{4.21}
\]

To estimate \( \left| u_{\hat{Q}_n(Y_1)} - u_{\hat{Q}_n(Y_2)} \right| \) in (4.21), we note

\[
\left| u_{\hat{Q}_n(Y_1)} - u_{\hat{Q}_n(Y_2)} \right|^{p'} \leq 2^{p'-1} \left[ \left| u_{\hat{Q}_n(Y_1)} - \tilde{u}(Z') \right|^{p'} + \left| u_{\hat{Q}_n(Y_2)} - \tilde{u}(Z') \right|^{p'} \right], \text{ for any } Z' \in Q,
\]

and integrate it over \( \hat{Q}_R(Y_1) \cap \hat{Q}_R(Y_2) \) to gain

\[
\left| \hat{Q}_R(Y_1) \cap \hat{Q}_R(Y_2) \right| \left| u_{\hat{Q}_n(Y_1)} - u_{\hat{Q}_n(Y_2)} \right|^{p'} \leq 2^{p'-1} \int_{\hat{Q}_R(Y_1) \cap \hat{Q}_R(Y_2)} \left| u_{\hat{Q}_n(Y_1)} - \tilde{u}(Z') \right|^{p'} dZ' + \int_{\hat{Q}_R(Y_1) \cap \hat{Q}_R(Y_2)} \left| u_{\hat{Q}_n(Y_2)} - \tilde{u}(Z') \right|^{p'} dZ'
\]

\[
\leq C R^\alpha \left| \hat{Q}_R(Y_1) \cap \hat{Q}_R(Y_2) \right| |u|_{p',\lambda},
\]

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which shows

$$|u_{Q_n}(y_1) - u_{Q_n}(y_2)| \leq CR^\alpha |u|_{p', \lambda}.$$

Substituting it into (4.21), it happens

$$|\tilde{u}(Y_1) - \tilde{u}(Y_2)| \leq CR^\alpha |u|_{p', \lambda}$$

and noting $R = d_{P}(Y_1, Y_2)$, we arrive at

$$\mathcal{L}^{p', \lambda}(Q) \subset C^{\alpha}(Q).$$

**Proof of Theorem 1.1.** For any $Z_0 = (x_0, t_0) \in Q$, denote

$$Q_R(Z_0) = B(R) \times (t_0 - R^2, t_0],$$

By employing Lemma 4.1, we derive for $R \in (0, d]$, $d = \min\{1, d_{P}(Q, \partial Q)_T\}$,

$$\text{osc}_{Q_R(Z_0)} u \leq C \left(\frac{R}{d}\right)^{\beta} \left[\text{osc}_{Q_T} u + (M + F_0)d|Q_T|^{-\frac{1}{p}}\right]$$

$$\leq C \left(\frac{R}{d}\right)^{\beta} \left[M \left(1 + d|Q_T|^{-\frac{1}{p}}\right) + F_0d|Q_T|^{-\frac{1}{p}}\right]$$

$$\leq C \left(\frac{R}{d}\right)^{\beta} \left[M + F_0d|Q_T|^{-\frac{1}{p}}\right],$$

(4.22)

where $1 + d|Q_T|^{-\frac{1}{p}}$ is a constant, and then

$$\frac{R^{-\beta}}{|Q_R(Z_0)|} \int_{Q_R(Z_0)} |u(Z) - u_{Q_R(Z_0)}| dZ \leq R^{-\beta} \text{osc}_{Q_R(Z_0)} u \leq Cd^\beta \left(M + F_0d|Q_T|^{-\frac{1}{p}}\right),$$

which implies $u \in L_{loc}^{1, \beta}(Q_T; d_{P})$ and so $u \in C_{loc}^{\beta}(Q_T; d_{P})$ by Lemma 4.2. Now let us estimate $[u]_{\beta, Q}$.

For any $Z_0 = (x_0, t_0), Z_1 = (x_1, t_1) \in Q$, without losing of generality, suppose $t_0 \geq t_1$. If $d_{P}(Z_0, Z_1) \leq d$, then $Z_1 \in Q_R(Z_0) \subset Q_T, R = d_{P}(Z_0, Z_1)$, and from (4.22),

$$|u(Z_0) - u(Z_1)| \leq \text{osc}_{Q_R(Z_0)} u \leq C \left(\frac{d_{P}(Z_0, Z_1)}{d}\right)^{\beta} \left(M + F_0d|Q_T|^{-\frac{1}{p}}\right);$$

(4.23)

if $d_{P}(Z_0, Z_1) > d$, then

$$|u(Z_0) - u(Z_1)| \leq \frac{2M}{d^\beta} \left[d_{P}(Z_0, Z_1)\right]^\beta.$$  

(4.24)

It follows (1.4) by combining (4.23) and (4.24).

To prove Theorem 1.2, we need an extension property of positivity for functions in the De Giorgi class.

**Lemma 4.3.** Let $u \in DG^{-}(Q_T)$ and $u \geq 0$ in

$$Q_R = B(x_0) \times (t_0, t_0 + aR^2) \subset Q_T, 0 < R \leq 1.$$

For $\epsilon \in (0, 1)$, if

$$\text{ess inf}_{R, a} u(x, t_0) \geq k \geq 0,$$

(4.25)

then there exist $R_0 > 0$ and a positive integer $s \geq 1$ relying on $Q, \lambda_0, \eta$ and $\gamma(\cdot)$, such that for $B_R = B(x_0), 0 < R \leq R_0$, either

$$k \leq 2^{s+2} \frac{F_0}{R} |Q_R|^{-\frac{1}{p}}.$$  

(4.26)
or

\[
\text{ess inf } B(x, t + aR^2) \geq c^s (k - 2^{s+2} F_0 R |Q_R|^{-\frac{1}{s}}). \tag{4.27}
\]

**Proof.** Let \( \epsilon \leq 1/8 \), we have from (4.25) and (2.2) that

\[
\left| B_{4R} \cap \{ u(\cdot, t_0) > k \} \right| \geq |B_{4R}|
\]

and then

\[
\left| B_{4R} \cap \{ u(\cdot, t_0) < k \} \right| \leq |B_{4R}| - \left| B_{4R} \cap \{ u(\cdot, t_0) > k \} \right| \leq |B_{4R}| - |B_{4R}| \leq (1 - C_1^{-1} 4^{-Q}) |B_{4R}|.
\]

As a result of Lemma 3.6, there exists \( s > 1 \), such that either

\[
|k| \leq 2^s (k + F_0) (4 \epsilon R) |Q_{4R}|^{-\frac{1}{s}}, \tag{4.28}
\]

(\( M \) in (3.32) is changed to \( k \); actually, it is suitable from the proofs of lemmas in Section 3), or

\[
\text{ess inf } u \geq \frac{k}{2^s}, \tag{4.29}
\]

where we used \( H = k \) which is followed from \( H := k - \text{inf } u \) and \( u \geq 0 \).

If (4.28) holds, we choose \( R_0 \) satisfying

\[
2^{s+1} R_0 |Q_{R_0}|^{-\frac{1}{s}} = \frac{1}{2}
\]

and for \( 0 < R \leq R_0 \), it yields by (2.2) that \( |Q_{4R}|^{-\frac{1}{s}} \leq C_1^\frac{1}{s} R_0 |Q_{R_0}|^{-\frac{1}{s}} \) and \( |Q_{4R}|^{-\frac{1}{s}} \leq C_1^\frac{1}{s} R |Q_{R_0}|^{-\frac{1}{s}} \).

Noting \( C_1^\frac{1}{s} = 2^\frac{1}{s} < 2 \), it implies from (4.28) that

\[
\begin{align*}
|k| & \leq 2^s k (4 \epsilon R) |Q_{4R}|^{-\frac{1}{s}} + 2^s F_0 (4 \epsilon R) |Q_{4R}|^{-\frac{1}{s}} \\
& \leq 2^s k C_1^\frac{1}{s} R |Q_{R_0}|^{-\frac{1}{s}} + 2^s C_1^\frac{1}{s} F_0 R |Q_{R_0}|^{-\frac{1}{s}} \\
& \leq \frac{1}{2} R_0 + 2^{s+1} F_0 R |Q_{R_0}|^{-\frac{1}{s}}.
\end{align*}
\]

This proves (4.26).

If (4.29) holds, choose a suitable \( \epsilon > 0 \) such that \( N = \frac{\log \epsilon}{\log 2} - 2 \) is an integer, and arrive at by employing Lemma 3.6 with \( N \) times that

\[
\begin{align*}
\text{ess inf } u(x, t_0 + aR^2) &= \text{ess inf } u(x, t_0 + aR^2) \\
& \geq \text{ess inf } u(x, t_0 + aR^2) \\
& \geq \epsilon \left( k - 2^{s+2} F_0 R |Q_R|^{-\frac{1}{s}} \right)
\end{align*}
\]

which is (4.27).

**Proof of Theorem 1.2.** For simplicity, we assume \( a' = 2 \) and denote

\[
Q_r = B_r (x_0) \times (t_0 + R^2 - r^2, t_0 + R^2 + r^2).
\]

Consider two functions

\[
m(r) = u(x_0, t_0 + R^2) \left( 1 - \frac{r}{R} \right)^{-\alpha} \quad \text{and} \quad \mu(r) = \max_{Q_r} u(x, t),
\]

where we used \( H = k \) which is followed from \( H := k - \text{inf } u \) and \( u \geq 0 \).
where \( s \) is determined in Lemma 4.3, and let \( r_0(r_0 < R) \) be the largest root to the equation \( m(r) = \mu(r) \). Since \( m(r) \to \infty \) as \( r \to R - 0 \), and \( u \) is continuous and bounded in \( \mathcal{Q}_R \), it sees that \( r_0 \) is well defined, \( m(r) > \mu(r) \) for \( r_0 < r \leq R \), and there exists \((x_1, t_1) \in \mathcal{Q}_{r_0} = B_{r_0}(x_0) \times (t_0 + R^2 - r_0^2, t_0 + R^2 + r_0^2)\), such that

\[
 u(x_1, t_1) = m(r_0) = \mu(r_0).
\]

Now introduce

\[
 \mathcal{Q} = \left\{(x, t) \mid d(x, x_1) \leq \frac{R - r_0}{2}, t_1 - \frac{(R - r_0)^2}{4} < t \leq t_1 \right\},
\]

i.e. \( \mathcal{Q} = B_{(R-r_0)/2}(x_1) \times \left(t_1 - \frac{(R-r_0)^2}{4}, t_1\right) \). For any \((x, t) \in \mathcal{Q}\),

\[
 d(x, x_0) = d(x, x_1) + d(x_0, x_1) \leq \frac{R - r_0}{2} + r_0 = \frac{R + r_0}{2}
\]

and

\[
 t_0 + R^2 = \frac{(R + r_0)^2}{4} < t_0 + R^2 - r_0^2 = \frac{(R - r_0)^2}{4} < t_1 < \frac{(R - r_0)^2}{4} < t_1
\]

\[
 < t_0 + R^2 + r_0^2 < t_0 + R^2 + \frac{(R + r_0)^2}{4}, \tag{4.30}
\]

where \( r_0^2 < r_0^2 + \frac{(R-r_0)^2}{4} < \frac{(R+r_0)^2}{4} \), then we have \( \mathcal{Q} \subset \mathcal{Q}_{(R+r_0)/2} \). Noting the meaning of \( r_0 \), it follows

\[
 \sup_{\mathcal{Q}} u(x, t) \leq \mu \left( \frac{R + r_0}{2} \right) \leq m \left( \frac{R + r_0}{2} \right)
\]

\[
 = u(x_0, t_0 + R^2) \cdot 2^s \left( 1 - \frac{r_0}{R} \right)^{-s}
\]

\[
 = 2^s m(r_0)
\]

\[
 = 2^s \mu(r_0). \tag{4.31}
\]

From Lemma 4.1, (2.2), (4.31) and \( \frac{R-r_0}{2} < R \), we obtain

\[
 |u(x_1, t_1) - \inf_{d(x, x_1) \leq \epsilon(R-r_0)} u(x, t_1)| \leq B_{\epsilon(R-r_0)} u(x, t_1)
\]

\[
 \leq C_2^2 2^\beta \left[ \sup_{B_{\epsilon(R-r_0)/2}} u(x, t_1) + F_0 \frac{R-r_0}{2} |Q_{R-r_0}|^{-\frac{1}{4}} \right]
\]

\[
 \leq C \epsilon^\beta \left[ \sup_{\mathcal{Q}} u + F_0 |Q_{R-r_0}|^{-\frac{1}{4}} \right]
\]

\[
 \leq C \epsilon^\beta \left[ 2^s \mu(r_0) + F_0 R |Q_R|^{-\frac{1}{4}} \right]
\]

and

\[
 \inf_{d(x, x_1) \leq \epsilon(R-r_0)} u(x, t_1) \geq u(x_1, t_1) - C \epsilon^\beta \left[ 2^s \mu(r_0) + F_0 R |Q_R|^{-\frac{1}{4}} \right]
\]

\[
 = \mu(r_0) - C \epsilon^\beta \left[ 2^s \mu(r_0) + F_0 R |Q_R|^{-\frac{1}{4}} \right].
\]

Choosing \( \epsilon \) so small that \( C \epsilon^\beta < 1 \) and \( C \epsilon^\beta 2^s = \frac{1}{2} \), it yields

\[
 \inf_{d(x, x_1) \leq \epsilon(R-r_0)} u(x, t_1) \geq \frac{1}{2} \mu(r_0) - F_0 R |Q_R|^{-\frac{1}{4}}. \tag{4.32}
\]
For $x \in B_{R/2}(x_0)$, we have
\[ d(x, x_1) \leq d(x, x_0) + d(x_0, x_1) \leq \frac{R}{2} + r_0 \leq \frac{3R}{2} \]
and so
\[ B_{R/2}(x_0) \subset B_{3R/2}(x_1). \]

Employing it and (4.25) in Lemma 4.3 on the domain $B_6 = (x_1) \times (t_1, t_0 + 2R^2]$, it shows
\[
\begin{align*}
\inf_{d(x, x_0) \leq R/2} u(x, t_0 + 2R^2) &\geq \inf_{d(x, x_1) \leq 3R/2} u(x, t_0 + 2R^2) \\
&\geq \left( \frac{\epsilon(R - r_0)}{6R} \right)^s \left( \frac{1}{2} \mu(r_0) - (2^{s+2} + 1)F_0R|Q_0|^{-\frac{4}{s}} \right) \\
&\geq \frac{1}{2} \left( \frac{\epsilon(R - r_0)}{6R} \right)^s \cdot u(x, t_0 + R^2) \left( 1 - \frac{r}{R} \right)^{-s} - \left( \frac{\epsilon(R - r_0)}{6R} \right)^s \cdot \frac{2 \cdot 6^s F_0 R |Q_0|^{-\frac{4}{s}}}{s} \\
&\geq \frac{1}{2} \left( \frac{\epsilon}{6} \right)^s u(x, t_0 + R^2) - 2 \left[ \frac{\epsilon(R - r_0)}{R} \right]^s \cdot F_0 R |Q_0|^{-\frac{4}{s}} \\
&= \frac{1}{2} \left( \frac{\epsilon}{6} \right)^s u(x, t_0 + R^2) - 2 \left( \frac{6}{\epsilon} \right)^s \cdot F_0 R |Q_0|^{-\frac{4}{s}},
\end{align*}
\]
where we used $2^{s+2} + 1 < 2 \cdot 6^s$ and $\frac{\epsilon(R - r_0)}{R} < \frac{\epsilon}{6}$. This proves (1.5).

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