Characterization of \((R, p, q)\)-deformed Rogers–Szegő polynomials: associated quantum algebras, deformed Hermite polynomials and relevant properties

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Abstract
This paper addresses a new characterization of \((R, p, q)\)-deformed Rogers–Szegő polynomials by providing their three-term recurrence relation and the associated quantum algebra built with corresponding creation and annihilation operators. The whole construction is performed in a unified way, generalizing all known relevant results which are straightforwardly derived as particular cases. Continuous \((R, p, q)\)-deformed Hermite polynomials and their recurrence relation are also deduced. Novel relations are provided and discussed.

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1. Introduction
Deformed quantum algebras, namely the \(q\)-deformed algebras \([14, 17, 18]\) and their extensions to \((p, q)\)-deformed algebras \([1, 2]\), continue to attract much attention. One of the main reasons is that these topics represent a meeting point of nowadays fast developing areas in mathematics and physics like the theory of quantum orthogonal polynomials and special functions, quantum groups, integrable systems and quantum and conformal field theories and statistics. Indeed, since the work of Jimbo \([14]\), these fields have known profound interesting developments which can be partially found, for instance, in the books by Chari and Pressley \([3]\), Klimyk and Schumudgen \([15]\), Ismail Moudard \([11]\) and references therein.

The two-parameter quantum algebra, \(U_{p,q} (gl(2))\), was first introduced in \([2]\) in view to generalize or/and unify a series of \(q\)-oscillator algebra variants, known in the earlier physics and mathematics literature on the representation theory of single-parameter quantum algebras. Then there follows a series of works in the same direction, among which the work of Burban and Klimyk \([1]\) on representations of two-parameter quantum groups and models of two-parameter quantum algebra \(U_{p,q} (su_{1,1})\) and \((p, q)\)-deformed oscillator algebra. Almost
simultaneously, Gelfand et al [7] introduced the \((r, s)\)-hypergeometric series satisfying a two-parameter difference equation, including \(r\)- and \(s\)-shift operators. This new series reproduces the Burban and Klimyk \(P, Q\)-hypergeometric functions. The \((p, q)\)-deformation rapidly found applications in physics and mathematical physics, as described for instance in [4, 9, 10].

Upon recalling a technique of constructing explicit realizations of raising and lowering operators that satisfy an algebra akin to the usual harmonic oscillator algebra, through the use of the three-term recurrence relation and the differentiation expression of Hermite polynomials, Galetti [5] has shown that a similar procedure can be carried out in the case of the three-term recurrence relation for Rogers–Szegő and Stieltjes–Wigert polynomials and the Jackson \(q\)-derivative. This technique furnished new realizations of the \(q\)-deformed algebra associated with the \(q\)-deformed harmonic oscillator, which obey commutation relations, well known and spread in the literature.

In the same vein, after recalling the connection between the Rogers–Szegő polynomials and the \(q\)-oscillator, Jagannathan and Sridhar [13] have defined \((p, q)\)-Rogers–Szegő polynomials, shown that they are connected with the \((p, q)\)-deformed oscillator associated with the Jagannathan–Srinavasa \((p, q)\)-numbers [12] and proposed a new realization of this algebra. In a previous paper [8], we proposed a theoretical framework for the \((p, q)\)-deformed state generalization and provided a generalized deformed quantum algebra, based on a work by Odzijewicz [17] on a generalization of \(q\)-deformed states in which the realizations of creation and annihilation operators are given by multiplication by \(z\) and the action of the deformed derivative \(\partial_{R, p, q}\) on the space of analytic functions defined on the disc.

The present investigation aims at giving a new realization of the previous generalized deformed quantum algebras and an explicit definition of the \((R, p, q)\)-Rogers–Szegő polynomials, together with their three-term recurrence relation and the deformed difference equation giving rise to the creation and annihilation operators.

The paper is organized as follows. As a matter of clarity, we present in section 2 a brief review of known results on \((R, p, q)\)-deformed numbers, binomial coefficients and quantum algebra. In section 3, we perform the realization of \((R, p, q)\)-deformed quantum algebras using the \((R, p, q)\)-difference equation and the three-term recurrence relation satisfied by \((R, p, q)\)-Rogers–Szegő polynomials. The key result of this section is theorem 3.1 giving the method of computation of relevant quantities. Section 4 is devoted to the study of the continuous \((R, p, q)\)-Hermite polynomials. We then give their definition and recurrence relation. In section 5, relevant examples and their properties are provided and demonstrated. Finally, section 6 ends with the concluding remarks.

2. \((R, p, q)\)-numbers and associated \((R, p, q)\)-deformed quantum algebras

This section addresses the general theoretical framework as well as a brief review of known results on deformed numbers and deformed binomial coefficients. The calculus methodology leading to the definition and the computation of the three-term recurrence relation of polynomials is also exposed.

In [8], we derived the \((R, p, q)\)-numbers which are a generalization of the Heine \(q\)-number

\[
[R]_q = \frac{1 - q^n}{1 - q}, \quad n = 0, 1, 2, \ldots
\]  

and Jagannathan–Srinivasa \((p, q)\)-numbers [12]

\[
[R]_{p, q} = \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \ldots
\]
Consider \( p \) and \( q \), two positive real numbers such that \( 0 < q < p \), and a given meromorphic function \( \mathcal{R} \), defined on \( \mathbb{C} \times \mathbb{C} \) by

\[
\mathcal{R}(x, y) = \sum_{k, l=-\infty}^{\infty} r_{kl} x^k y^l
\]

(3)

with an eventual isolated singularity at zero, where \( r_{kl} \) are complex numbers, \( L \in \mathbb{N} \cup \{0\} \), \( \mathcal{R}(p^n, q^n) > 0 \) \( \forall n \in \mathbb{N} \) and \( \mathcal{R}(1, 1) = 0 \). Denote by \( \mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\} \) a complex disc and by \( \mathcal{O}(\mathbb{D}_R) \) the set of holomorphic functions defined on \( \mathbb{D}_R \). Then, the \((\mathcal{R}, p, q)\)-number is given by [8]

\[
[n]_{\mathcal{R}, p, q} = \mathcal{R}(p^n, q^n), \quad n = 0, 1, 2, \ldots
\]

(4)

leading to define \((\mathcal{R}, p, q)\)-factorials

\[
[n]!_{\mathcal{R}, p, q} = \begin{cases} 1 & \text{for } n = 0 \\ \mathcal{R}(p, q) \cdots \mathcal{R}(p^n, q^n) & \text{for } n \geq 1, \end{cases}
\]

(5)

and the \((\mathcal{R}, p, q)\)-binomial coefficients

\[
\begin{bmatrix} m \\ n \end{bmatrix}_{\mathcal{R}, p, q} = \frac{[m]!_{\mathcal{R}, p, q}}{[n]!_{\mathcal{R}, p, q} [m-n]!_{\mathcal{R}, p, q}}, \quad m, n = 0, 1, 2, \ldots; \quad m \geq n
\]

(6)

that satisfy the relation

\[
\begin{bmatrix} m \\ n \end{bmatrix}_{\mathcal{R}, p, q} = \begin{bmatrix} m \\ m-n \end{bmatrix}_{\mathcal{R}, p, q}, \quad m, n = 0, 1, 2, \ldots; \quad m \geq n.
\]

(7)

There also result the following linear operators defined on \( \mathcal{O}(\mathbb{D}_R) \) by (see [8] and references therein for more details)

\[
Q : \varphi \mapsto \mathcal{Q}\varphi(z) = \varphi(qz)
\]

\[
P : \varphi \mapsto \mathcal{P}\varphi(z) = \varphi(pz)
\]

\[
\partial_{p, q} : \varphi \mapsto \partial_{p, q} \varphi(z) = \frac{\varphi(pz) - \varphi(qz)}{z(p - q)}
\]

(8)

and the \((\mathcal{R}, p, q)\)-derivative given by

\[
\partial_{\mathcal{R}, p, q} := \partial_{p, q} \mathcal{R}(P, Q) = \frac{p - q}{pP - qQ} \mathcal{R}(pP, qQ) \partial_{p, q}.
\]

(9)

The quantum algebra associated with the \((\mathcal{R}, p, q)\)-deformation, denoted by \( \mathcal{A}_{\mathcal{R}, p, q} \), is generated by the set of operators \([1, A, A^\dagger, N] \) satisfying

\[
AA^\dagger = [N + 1]_{\mathcal{R}, p, q}, \quad A^\dagger A = [N]_{\mathcal{R}, p, q};
\]

\[
[N, A] = -A, \quad [N, A^\dagger] = A^\dagger.
\]

(10)

with the realization on \( \mathcal{O}(\mathbb{D}_R) \) given by [8]

\[
A^\dagger \equiv z, \quad A \equiv \partial_{\mathcal{R}, p, q}, \quad N \equiv \frac{d}{dz} \text{ is the usual derivative on } \mathbb{C}.
\]

(11)

where \( \partial_z \equiv \frac{d}{dz} \) is the usual derivative on \( \mathbb{C} \).

3. \((\mathcal{R}, p, q)\)-Rogers–Szegö polynomials and their related quantum algebras

This section aims at providing the realizations of \((\mathcal{R}, p, q)\)-deformed quantum algebras induced by \((\mathcal{R}, p, q)\)-Rogers–Szegö polynomials. We first define the latter and their three-term recurrence relation, and then following the procedure elaborated in [5, 13], we prove that every sequence of these polynomials forms a basis for the corresponding deformed quantum algebra.
Indeed, Galetti in [5], upon recalling the technique of construction of raising and lowering operators which satisfy an algebra akin to the usual harmonic oscillator algebra, by using the three-term recurrence relation and the differentiation expression of Hermite polynomials, has shown that a similar procedure can be carried out to construct a $q$-deformed harmonic oscillator algebra, with the help of relations controlling the Rogers–Szegő polynomials. Following this author, Jagannathan and Sridhar in [13] adapted the same approach to construct a Bargman–Fock realization of the harmonic oscillator as well as the realizations of $q$- and $(p, q)$-deformed harmonic oscillators based on Rogers–Szegő polynomials.

As a matter of clarity, this section is stratified as follows. We first develop the synoptic schemes of known different generalizations and then display the formalism of $(R, p, q)$-Rogers–Szegő polynomials.

### 3.1. Hermite polynomials and harmonic oscillator approach

The Hermite polynomials are defined as orthogonal polynomials satisfying the three-term recurrence relation

$$H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z)$$  \hspace{1cm} (12)

and the differentiation relation

$$\frac{d}{dz}H_n(z) = 2nH_{n-1}(z).$$  \hspace{1cm} (13)

Inserting equation (13) in equation (12), one gets

$$H_{n+1}(z) = \left(2z - \frac{d}{dz}\right)H_n(z),$$  \hspace{1cm} (14)

which includes the introduction of a raising operator (see [5] and references therein), defined as

$$\hat{a}_+ = 2z - \frac{d}{dz}$$

such that the set of Hermite polynomials can be generated by the application of this operator to the first polynomial $H_0(z) = 1$, i.e.

$$H_n(z) = \hat{a}_+^n H_0(z).$$  \hspace{1cm} (16)

From equation (13), one defines the lowering operator $\hat{a}_-$ as

$$\hat{a}_- H_n(z) = \frac{1}{2} d\frac{d}{dz} H_n(z) = nH_{n-1}(z).$$  \hspace{1cm} (17)

Furthermore, one constructs a number operator in the form

$$\hat{n} = \hat{a}_+ \hat{a}_-.$$  \hspace{1cm} (18)

One can readily check that these operators satisfy the canonical commutation relations

$$[\hat{a}_-, \hat{a}_+] = 1, \quad [\hat{n}, \hat{a}_-] = -\hat{a}_-, \quad [\hat{n}, \hat{a}_+] = \hat{a}_+,$$

although the operators $\hat{a}_-$ and $\hat{a}_+$ are not the usual creation and annihilation operators associated with the quantum mechanics harmonic oscillator. Thus, we see that one can obtain raising, lowering and number operators from the two basic relations satisfied by the Hermite polynomials, i.e. the three-term recurrence relation and the differentiation relation, respectively, so that they satisfy the well-known commutation relations.
On the other hand, if one considers the usual Hilbert space spanned by the vectors $|n\rangle$, generated from the vacuum $|0\rangle$ by the raising operator $\hat{a}_+$, then together with the lowering operator $\hat{a}_-$, the following relations hold,

$$\hat{a}_- \hat{a}_+ - \hat{a}_+ \hat{a}_- = 1,$$
$$\langle 0 | 0 \rangle = 1,$$
$$|n\rangle = \hat{a}_n^\dagger |0\rangle,$$
$$\hat{a}_- |0\rangle = 0.$$  \hspace{1cm} (20)

In particular, the next expressions, established using the previous equations, are in order:

$$\hat{a}_+ |n\rangle = |n + 1\rangle,$$
$$\hat{a}_- |n\rangle = |n - 1\rangle,$$
$$\langle m | n \rangle = n! \delta_{mn}. \hspace{1cm} (21)$$

Now, on the other hand, examining the procedure given in [13], the authors considered the sequence of polynomials

$$\psi_n(z) = \frac{1}{\sqrt{n!}} h_n(z), \hspace{1cm} (22)$$

where

$$h_n(z) = (1 + z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k, \hspace{1cm} (23)$$

obeying the relations

$$\frac{d}{dz} \psi_n(z) = \sqrt{n} \psi_{n-1}(z), \hspace{1cm} (24)$$
$$\frac{d}{dz} \psi_n(z) = \sqrt{n+1} \psi_{n+1}(z), \hspace{1cm} (25)$$
$$\frac{d}{dz} \psi_n(z) = n \psi_n(z), \hspace{1cm} (26)$$
$$\frac{d}{dz} ((1 + z) \psi_n(z)) = (n + 1) \psi_n(z). \hspace{1cm} (27)$$

Here equations (25) and (26) are the recurrence relation and the differential equation for polynomials $\psi_n(z)$, respectively. By analogy to the work done by Galleti, Jagannathan and Sridhar proposed the following relations:

$$\hat{a}_+ = (1 + z), \hspace{1cm} \hat{a}_- = \frac{d}{dz}, \hspace{1cm} \hat{n} = (1 + z) \frac{d}{dz}, \hspace{1cm} (28)$$

for creation (or raising), annihilation (or lowering) and number operators, respectively, and found that the set $\{\psi_n(z) | n = 0, 1, 2, \ldots\}$ forms a basis for the Bargman–Fock realization of the harmonic oscillator (19).

### 3.2. Rogers–Szegő polynomials and $q$-deformed harmonic oscillator

Here in an analogous way to Jagannathan and Sridhar [13], we perform a construction of the creation, annihilation and number operators from the three-term recurrence relation and the $q$-difference equation finding the Rogers–Szegő polynomials. This procedure differs a little from that used by Galetti [5] to obtain raising, lowering and number operators.
The Rogers–Szegö polynomials are defined as
\[ H_n(z; q) = \sum_{k=0}^{n} \binom{n}{k}_q z^k, \quad n = 0, 1, 2, \ldots \] (29)
and satisfy a three-term recurrence relation
\[ H_{n+1}(z; q) = (1 + z)H_n(z; q) - z(1 - q^n)H_{n-1}(z; q) \] (30)
as well as the \(q\)-difference equation
\[ \partial_q H_n(z; q) = [n]_q H_{n-1}(z; q). \] (31)
In the limit case \( q \to 1 \), the Rogers–Szegö polynomial of degree \( n \) \( n = 0, 1, 2, \ldots \) well converges to
\[ h_n(z) = \sum_{k=0}^{n} \binom{n}{k} z^k \]
as required. Defining
\[ \psi_n(z; q) = \frac{1}{\sqrt{|n|}_q} H_n(z) = \frac{1}{\sqrt{|n|}_q} \sum_{k=0}^{n} \binom{n}{k}_q z^k, \quad n = 0, 1, 2, \ldots, \] (32)
one can straightforwardly infer that
\[ \partial_q \psi_n(z; q) = \sqrt{|n|}_q \psi_{n-1}(z; q) \] (33)
with the property that for \( n = 0, 1, 2, \ldots \)
\[ \partial_q^{n+1} \psi_n(z; q) = 0 \quad \text{and} \quad \partial_q^m \psi_n(z; q) \neq 0 \quad \text{for any} \quad m < n + 1. \] (34)
It follows from equations (30) and (32) that the polynomials \{\psi_n(z; q) | n = 0, 1, 2, \ldots\} satisfy the following three-term recurrence relation,
\[ \sqrt{|n+1|}_q \psi_{n+1}(z; q) = (1 + z)\psi_n(z; q) - z(1 - q^n)\sqrt{|n|}_q \psi_{n-1}(z; q) \] (35)
and the \(q\)-difference equation
\[ ((1 + z) - (1 - q^n)z \partial_q) \psi_n(z; q) = \sqrt{|n+1|}_q \psi_{n+1}(z; q) \] (36)
obtained from equation (33). Hence, it is natural to formally define the number operator \( N \) as
\[ N \psi_n(z; q) = n \psi_n(z; q) \] (37)
determined for the creation and annihilation operators expressed as
\[ A^+ = 1 + z - (1 - q^n)z \partial_q \quad \text{and} \quad A = \partial_q, \] (38)
respectively. Indeed, the proofs of the following relations are immediate:
\[ N \psi_n(z; q) = n \psi_n(z; q), \] (39)
\[ A^+ \psi_n(z; q) = \sqrt{|n+1|}_q \psi_{n+1}(z; q), \] (40)
\[ A \psi_n(z; q) = \sqrt{|n|}_q \psi_{n-1}(z; q), \] (41)
\[ A^+ A \psi_n(z; q) = [n]_q \psi_n(z; q) = [N]_q \psi_n(z; q), \] (42)
\[ AA^+ \psi_n(z; q) = [n + 1]_q \psi_n(z; q) = [N + 1]_q \psi_n(z; q). \] (43)
Therefore, one concludes that the set of polynomials \{\psi_n(z; q) | n = 0, 1, 2, \ldots\} provides a basis for a realization of the \(q\)-deformed harmonic oscillator algebra given by
\[ AA^+ - qA^+ A = 1, \quad [N, A] = -A, \quad [N, A^+] = A^+. \] (44)
We can now supply the general procedure for constructing the recurrence relation for the \((R, p, q)\)-generalized Rogers–Szegö polynomials and quantum algebras. This is summarized as follows.

**Theorem 3.1.** If \(\phi_i(x, y) (i = 1, 2, 3)\) are functions satisfying
\[
\phi_i(p, q) \neq 0 \quad \text{for } i = 1, 2, 3, \quad (45)
\]
and if, moreover, the following relation between \((R, p, q)\)-binomial coefficients holds,
\[
\binom{n+1}{k}_{R, p, q} = \phi_1^k(p, q) \binom{n}{k}_{R, p, q} + \phi_2^{n+1-k}(p, q) \binom{n}{k-1}_{R, p, q} - \phi_3(p, q) [n]_{R, p, q} \binom{n-1}{k-1}_{R, p, q} \quad (47)
\]
for \(1 \leq k \leq n\), then the \((R, p, q)\)-Rogers–Szegö polynomials defined as
\[
H_n(z; R, p, q) = \sum_{k=0}^{n} \binom{n}{k}_{R, p, q} z^k, \quad n = 0, 1, 2, \ldots \quad (48)
\]
satisfy the three-term recurrence relation
\[
H_{n+1}(z; R, p, q) = H_n(z; R, p, q) - z\phi_2^n(p, q)H_n(z\phi_2^{-1}(p, q); R, p, q) - z\phi_3(p, q)[n]_{R, p, q}H_{n-1}(z; R, p, q) \quad (49)
\]
and the \((R, p, q)\)-difference equation
\[
\partial_{R, p, q} H_n(z; R, p, q) = [n]_{R, p, q} H_{n-1}(z; R, p, q). \quad (50)
\]

**Proof.** Multiplying the two sides of relation (47) by \(z^k\) and adding for \(k = 1\) to \(n\), we get
\[
\sum_{k=1}^{n} \binom{n+1}{k}_{R, p, q} z^k = \sum_{k=1}^{n} \phi_1^k(p, q) \binom{n}{k}_{R, p, q} z^k + \sum_{k=1}^{n} \phi_2^{n+1-k}(p, q) \binom{n}{k-1}_{R, p, q} z^k - \phi_3(p, q) [n]_{R, p, q} \sum_{k=1}^{n} \binom{n-1}{k-1}_{R, p, q} z^k. \quad (51)
\]
After a short computation and using condition (47) we get equation (49). Then there immediately results the proof of equation (50). \(\square\)

Setting
\[
\psi_n(z; R, p, q) = \frac{1}{[n]_{R, p, q}} H_n(z; R, p, q), \quad (52)
\]
and using equations (49) and (50) yield the three-term recurrence relation
\[
(\phi_1(p, Q) + z\phi_2^n(p, q)\phi_2^{-1}(p, Q) - z\phi_3(p, q)\partial_{R, p, q}) \psi_n(z; R, p, q) = \sqrt{n+1}[n]_{R, p, q} \psi_{n+1}(z; R, p, q) \quad (53)
\]
and the \((R, p, q)\)-difference equation
\[
\partial_{R, p, q} \psi_n(z; R, p, q) = \sqrt{n!} \partial_{R, p, q} \psi_{n-1}(z; R, p, q)
\]
for the polynomials \(\psi_n(z; R, p, q)\) with the virtue that for \(n = 0, 1, 2, \ldots\)
\[
\partial_{R, p, q}^{n+1} \psi_n(z; R, p, q) = 0 \quad \text{and} \quad \partial_{R, p, q}^n \psi_n(z; R, p, q) \neq 0 \quad \text{for} \ m < n + 1.
\]
Now, formally defining the number operator \(N\) as
\[
N \psi_n(z; R, p, q) = n \psi_n(z; R, p, q),
\]
and the raising and lowering operators by
\[
A^\dagger = \left( \phi_1(P, Q) + z \phi_2^N(p, q) \phi_2^{-1}(P, Q) - z \phi_3(p, q) \partial_{R, p, q} \right)
\]
and
\[
A = \partial_{R, p, q},
\]
respectively, the set of polynomials \(\{\psi_n(z; R, p, q) \mid n = 0, 1, 2, \ldots\}\) provides a basis for a realization of \((R, p, q)\)-deformed quantum algebra \(A_{R, p, q}\) satisfying the commutation relations (10). Provided the above-formulated theorem, we can now show how the realizations in terms of Rogers–Szegő polynomials can be derived for different known deformations simply by determining the functions \(\phi_i (i = 1, 2, 3)\) that satisfy relations (45)–(47).

4. Continuous \((R, p, q)\)-Hermite polynomials

We exploit here the peculiar relation established in the theory of \(q\)-deformation between Rogers–Szegő polynomials and Hermite polynomials [11, 12, 15, 16], and given by
\[
\mathbb{H}_n(\cos \theta; q) = e^{i \theta} H_n(e^{-2i \theta}; q) = \sum_{k=0}^{n} \binom{n}{k} q^k e^{i(n-2k)\theta}, \quad n = 0, 1, 2, \ldots,
\]
where \(\mathbb{H}_n\) and \(H_n\) stand for the Hermite and Rogers–Szegő polynomials, respectively. The property that all the \(q\)-Hermite polynomials can be explicitly recovered from the initial one \(\mathbb{H}_0(\cos \theta; q) = 1\), using the three-term recurrence relation
\[
\mathbb{H}_{n+1}(\cos \theta; q) = 2 \cos \theta \mathbb{H}_n(\cos \theta; q) - (1 - q^n) \mathbb{H}_{n-1}(\cos \theta; q)
\]
with \(\mathbb{H}_{-1}(\cos \theta; q) = 0\), is also of interest.

In the same way, we define the \((R, p, q)\)-Hermite polynomials through the \((R, p, q)\)-Rogers–Szegő polynomials as
\[
\mathbb{H}_n(\cos \theta; R, p, q) = e^{i \theta} H_n(e^{-2i \theta}; R, p, q), \quad n = 0, 1, 2, \ldots.
\]
Then the next statement is true.

**Proposition 4.1.** Under the hypotheses of theorem 3.1, the continuous \((R, p, q)\)-Hermite polynomials satisfy the following three-term recurrence relation:
\[
\mathbb{H}_{n+1}(\cos \theta; R, p, q) = e^{i \theta} \phi_1^\dagger(p, q) \phi_1(P, Q) \mathbb{H}_n(\cos \theta; R, p, q) \\
+ e^{-i \theta} \phi_2^\dagger(p, q) \phi_2^{-1}(P, Q) \mathbb{H}_n(\cos \theta; R, p, q) \\
- \phi_3(p, q) [n]_{R, p, q} \mathbb{H}_{n-1}(\cos \theta; R, p, q).
\]
Proposition 5.1. In this case, the
\[ e^{i(n+1)\theta} H_{n+1}(e^{-2i\theta}; \mathcal{R}, p, q) = e^{i(n+1)\theta} H_n(\phi_1(p, q) e^{-2i\theta}; \mathcal{R}, p, q) + e^{i(n-1)\theta} \phi_2^2(p, q) H_n(\phi_2^{-1}(p, q) e^{-2i\theta}; \mathcal{R}, p, q) \]
\[ - e^{i(n-1)\theta} \phi_3(p, q)[n]_{\mathcal{R}, p, q} H_{n-1}(e^{-2i\theta}; \mathcal{R}, p, q) \]
\[ = e^{i\theta} e^{i(n+1)\theta} \phi_1(P, Q) H_n(e^{-2i\theta}; \mathcal{R}, p, q) + e^{i\theta} \phi_2(P, Q) H_n(e^{-2i\theta}; \mathcal{R}, p, q) \]
\[ - \phi_3(p, q)[n]_{\mathcal{R}, p, q} e^{i(n-1)\theta} H_{n-1}(e^{-2i\theta}; \mathcal{R}, p, q). \]

Proof Multiplying the two sides of the three-term recurrence relation (49) by \( e^{i(n+1)\theta} \), we obtain, for \( z = e^{-2i\theta} \),
\[ e^{i(n+1)\theta} H_{n+1}(e^{-2i\theta}; \mathcal{R}, p, q) = e^{i(n+1)\theta} H_n(\phi_1(p, q) e^{-2i\theta}; \mathcal{R}, p, q) \]
\[ + e^{i(n-1)\theta} \phi_2^2(p, q) H_n(\phi_2^{-1}(p, q) e^{-2i\theta}; \mathcal{R}, p, q) \]
\[ - e^{i(n-1)\theta} \phi_3(p, q)[n]_{\mathcal{R}, p, q} H_{n-1}(e^{-2i\theta}; \mathcal{R}, p, q) \]
\[ = e^{i\theta} e^{i(n+1)\theta} \phi_1(P, Q) H_n(e^{-2i\theta}; \mathcal{R}, p, q) \]
\[ + e^{i\theta} \phi_2(P, Q) H_n(e^{-2i\theta}; \mathcal{R}, p, q) \]
\[ - \phi_3(p, q)[n]_{\mathcal{R}, p, q} e^{i(n-1)\theta} H_{n-1}(e^{-2i\theta}; \mathcal{R}, p, q). \]

The required result follows from the use of the equalities
\[ e^{i\theta} \phi_1(P, Q) H_n(e^{-2i\theta}; \mathcal{R}, p, q) = \phi_1^2(p, q) \phi_1(P, Q) e^{i\theta} H_n(e^{-2i\theta}; \mathcal{R}, p, q), \]
\[ e^{i\theta} \phi_2^{-1}(P, Q) H_n(e^{-2i\theta}; \mathcal{R}, p, q) = \phi_2^{-1}(P, Q) \phi_2^2(P, Q) e^{i\theta} H_n(e^{-2i\theta}; \mathcal{R}, p, q) \]
with
\[ \phi_j(P, Q) e^{-2i\theta} = \phi_j(p, q) e^{-2i\theta}, \quad j = 1, 2, k = 0, 1, 2, \ldots \]
□

5. Relevant particular cases

The following pertinent cases deserve to be raised, as their derivation from the previous general theory appeals concrete expressions for the deformed function \( \mathcal{R}(p, q) \).

5.1. \( \mathcal{R}(x, y) = \frac{x+y}{p-q} \)

In this case, the \( \mathcal{R}(x, y) \)-numbers are simply given by
\[ [n]_{p, q} = \mathcal{R}(p^n, q^n) = \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \ldots \]
with the \( \mathcal{R}(p, q) \)-factorials defined by
\[ [n]!_{p, q} = \begin{cases} 1 & \text{for } n = 0 \\ \prod_{k=1}^{n} \frac{p^k - q^k}{p - q} = \frac{(p, q); (p, q)_n}{(p - q)^n} & \text{for } n \geq 1. \end{cases} \]
(64)

They correspond to the Jagannathan–Srinivasa \( (p, q) \)-numbers and \( (p, q) \)-factorials [12, 13].

There result the following relevant properties.

Proposition 5.1. If \( n \) and \( m \) are non-negative integers, then
\[ [n]_{p, q} = \sum_{k=0}^{n-1} p^{n-1-k} q^k, \]
\[ [n + m]_{p, q} = q^m [n]_{p, q} + p^m [n]_{p, q} = p^m [n]_{p, q} + q^m [m]_{p, q}, \]
\[ [-m]_{p, q} = -q^{-m} p^{-m} [m]_{p, q}, \]
\[ [n - m]_{p, q} = q^{-m} [n]_{p, q} - q^{-m} p^{-m} [m]_{p, q} = p^{-m} [n]_{p, q} - q^{-m} p^{-m} [m]_{p, q}, \]
\[ [n]_{p, q} = [2]_{p, q} [n - 1]_{p, q} - pq [n - 2]_{p, q}. \]
Proposition 5.2. The \((p, q)\)-binomial coefficients

\[
\binom{n}{k}_{p,q} = \frac{[n]!_{p,q}}{[k]!_{p,q}[n-k]!_{p,q}} = ((p, q); (p, q))_n \tilde{(p^n - q^n)}_{n-k},
\]

where \(0 \leq k \leq n\), \(n \in \mathbb{N}\), and \(((p, q); (p, q))_m = (p - q)(p^2 - q^2) \cdots (p^m - q^m), m \in \mathbb{N}\), satisfy the following identities:

\[
\binom{n}{k}_{p,q} = \binom{n}{n-k}_{p,q} = p^{k(n-k)} \binom{n}{k}_{q/p},
\]

\[
\binom{n+1}{k}_{p,q} = p^k \binom{n}{k}_{p,q} + q^{n+1-k} \binom{n}{k-1}_{p,q},
\]

\[
\binom{n+1}{k}_{p,q} = p^k \binom{n}{k}_{p,q} + p^{n+1-k} \binom{n}{k-1}_{p,q} - (p^n - q^n) \binom{n-1}{k-1}_{p,q},
\]

with

\[
\binom{n}{k}_{q/p} = \frac{(q/p; q/p)_n}{(q/p; q/p)_{n-k}}.
\]

The algebra \(A_{p,q}\), generated by \([1, A, A^\dagger, N]\), associated with \((p, q)\)-Janagathan–Srinivasa deformation, satisfies the following commutation relations [12, 13]:

\[
[\mathcal{A}, A^\dagger] = A^\dagger, \quad [\mathcal{A}, A] = -A.
\]

The \((p, q)\)-Rogers–Szegö polynomials studied in [13] appear as a particular case obtained by choosing \(\phi_1(x, y) = \phi_2(x, y) = \phi(x, y) = x\) and \(\phi_3(x, y) = x - y\). Indeed, \(\phi(p, q) = q \neq 0, \phi(P, Q) = P - Q \neq 0, \phi(P, Q) = \phi(P, Q)\), and equation (69) shows that

\[
\binom{n+1}{k}_{p,q} = p^k \binom{n}{k}_{p,q} + p^{n+1-k} \binom{n}{k-1}_{p,q} - (p^n - q^n) \binom{n-1}{k-1}_{p,q}.
\]

Hence, the hypotheses of the above theorem are satisfied and, therefore, the \((p, q)\)-Rogers–Szegö polynomials

\[
H_n(z; p, q) = \sum_{k=0}^{n} \binom{n}{k}_{p,q} z^k, \quad n = 0, 1, 2, \ldots
\]

satisfy the three-term recurrence relation

\[
H_{n+1}(z; p, q) = H_n(z; p, q) + z p^n H_n(p^{-1} z; p, q) - z(p^n - q^n) H_{n-1}(z; p, q)
\]

and \((p, q)\)-difference equation

\[
\partial_{p,q} H_n(z; p, q) = [n]_{p,q} H_{n-1}(z; p, q).
\]

Finally, the set of polynomials

\[
\psi_n(z; p, q) = \frac{1}{\sqrt{[n]_{p,q}}} H_n(z; p, q), \quad n = 0, 1, 2, \ldots
\]
forms a basis for a realization of the \((p, q)\)-deformed harmonic oscillator and quantum algebra \(A_{p,q}\) satisfying the commutation relations (71) with the number operator \(N\) defined as

\[
N\psi_n(z; p, q) = n\psi_n(z; p, q),
\]
relating the annihilation and creation operators given by

\[
A = \partial_{p,q} \quad \text{and} \quad A^\dagger = P + zp^Np^{-1} - z(p - q)\partial_{p,q},
\]
respectively. Naturally, setting \(p = 1\) one recovers the results of subsection 3.2.

The continuous \((p, q)\)-Hermite polynomials have been already suggested in [12] without any further details. In the above-achieved generalization, these polynomials are given by

\[
\mathbb{H}_n(\cos \theta; p, q) = e^{i\theta}H_n(e^{-2i\theta}; p, q) = \sum_{k=0}^{n} \binom{n}{k}_{p,q} e^{i(2n-2k)\theta}, \quad n = 0, 1, 2, \ldots
\]

Since for the \((p, q)\)-deformation \(\phi_1(x, y) = \phi_2(x, y) = x\) and \(\phi_3(x, y) = x - y\), from proposition 4.1 we deduce that the corresponding sequence of continuous \((p, q)\)-polynomials satisfies the three-term recurrence relation

\[
\mathbb{H}_{n+1}(\cos \theta; p, q) = p^2(e^{i\theta}P + e^{-i\theta}P^{-1})\mathbb{H}_n(\cos \theta; p, q) - (p^q - q^p)\mathbb{H}_{n-1}(\cos \theta; p, q),
\]

with \(Pe^{i\theta} = p^{-1/2}e^{i\theta}\). This relation turns to be the well-known three-term recurrence relation (59) for continuous \(q\)-Hermite polynomials in the limit \(p \to 1\). As a matter of illustration, let us explicitly compute the first three polynomials using relation (79), with \(\mathbb{H}_{-1}(\cos \theta; p, q) = 0\) and \(\mathbb{H}_0(\cos \theta; p, q) = 1:\)

\[
\mathbb{H}_1(\cos \theta; p, q) = p^1(e^{i\theta}P + e^{-i\theta}P^{-1})1 - (p^q - q^p)0 = e^{i\theta} + e^{-i\theta} = 2\cos \theta
\]

\[
= \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{p,q} e^{i\theta} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{p,q} e^{-i\theta}.
\]

\[
\mathbb{H}_2(\cos \theta; p, q) = p^2(e^{i\theta}P + e^{-i\theta}P^{-1})(e^{i\theta} + e^{-i\theta}) - (p^q - q^p)1
\]

\[
= e^{2i\theta} + e^{-2i\theta} + p + q = 2\cos 2\theta + p + q
\]

\[
= \begin{bmatrix} 2 \\ 0 \end{bmatrix}_{p,q} e^{2i\theta} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}_{p,q} e^{-2i\theta} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}_{p,q} e^{-i\theta}.
\]

\[
\mathbb{H}_3(\cos \theta; p, q) = p(e^{i\theta}P + e^{-i\theta}P^{-1})(e^{2i\theta} + e^{-2i\theta} + p + q) - (p^q - q^p)(e^{i\theta} + e^{-i\theta})
\]

\[
= e^{3i\theta} + e^{-3i\theta} + (p^2 + pq + q^2)(e^{i\theta} + e^{-i\theta})
\]

\[
= 2\cos 3\theta + 2(p^2 + pq + q^2)\cos \theta
\]

\[
= \begin{bmatrix} 3 \\ 0 \end{bmatrix}_{p,q} e^{3i\theta} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}_{p,q} e^{-3i\theta} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{p,q} e^{i\theta} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{p,q} e^{-i\theta}.
\]

5.2. \(\mathcal{R}(x, y) = \frac{1 - xy}{(p^{-1} - q^n)}\)

The \((\mathcal{R}, p, q)\)-numbers and \((\mathcal{R}, p, q)\)-factorials are reduced to \((p^{-1}, q)\)-numbers and \((p^{-1}, q)\)-factorials, namely

\[
[n]_{p^{-1},q} = \frac{p^{-n} - q^n}{p^{-1} - q}
\]

and

\[
[n]_{p^{-1},q} = \begin{cases} 1 & \text{for } n = 0 \\ \frac{((p^{-1}, q); (p^{-1}, q))_n}{(p^{-1} - q)^n} & \text{for } n \geq 1. \end{cases}
\]
respectively, which exactly reproduce the \((p, q)\)-numbers and \((p, q)\)-factorials introduced by Chakrabarty and Jagannathan [2].

The other properties can be recovered similarly to those of section 5.1 replacing the parameter \(p\) by \(p^{-1}\).

The \((R, p, q)\)-derivative is also reduced to \((p^{-1}, q)\)-derivative. Indeed,

\[
\partial_{R,p,q} = \frac{p - q}{P - Q} \frac{1 - PQ}{(p - q)P} \equiv \partial_{p^{-1},q},
\]

obtained by a simple replacement of the dilatation operator \(P\) by \(p^{-1}\).

The algebra \(\mathcal{A}_{p^{-1},q}\) generated by \([1, A, A^\dagger, N]\), associated with \((p, q)\)-Chakrabarty and Jagannathan deformation satisfies the following commutation relations:

\[
\begin{align*}
A A^\dagger - p^{-1} A^\dagger A &= q^N, & A A^\dagger - q A^\dagger A &= p^{-N} \\
[N, A^\dagger] &= A^\dagger & [N, A] &= -A.
\end{align*}
\]

Hence, the \((p^{-1}, q)\)-Rogers–Szegö polynomials

\[
H_n(z; p^{-1}, q) = \sum_{k=0}^{n} \binom{n}{k}_{p^{-1},q} z^k \quad n = 0, 1, 2, \ldots
\]

obey the three-term recurrence relation

\[
H_{n+1}(z; p^{-1}, q) = H_n(p^{-1}z; p^{-1}, q) + z p^{-n} H_n(pz; p^{-1}, q) - z(p^{-n} - q^n) H_{n-1}(z; p^{-1}, q)
\]

and the \((p^{-1}, q)\)-difference equation

\[
\partial_{p^{-1},q} H_n(z; p, q) = [n]_{p^{-1},q} H_{n-1}(z; p, q).
\]

Finally, the set of polynomials

\[
\psi_n(z; p^{-1}, q) = \frac{1}{\sqrt{[n]_{p^{-1},q}}} H_n(z; p^{-1}, q), \quad n = 0, 1, 2, \ldots
\]

forms a basis for a realization of the \((p^{-1}, q)\)-deformed harmonic oscillator and quantum algebra \(\mathcal{A}_{p^{-1},q}\) generating the commutation relations (82) with the number operator \(N\) formally defined as

\[
N \psi_n(z; p^{-1}, q) = n \psi_n(z; p^{-1}, q),
\]

and the annihilation and creation operators given by

\[
A = \partial_{p^{-1},q} \quad \text{and} \quad A^\dagger = P^{-1} + z P^{-N} P - z(p^{-1} - q) \partial_{p^{-1},q},
\]

respectively. Naturally, setting \(p = 1\) permits us to recover the results of subsection 3.2.

5.3. \(R(x, y) = \frac{xy - 1}{q - p^{-1} y}\)

In this case, the \((R, p, q)\)-numbers and \((R, p, q)\)-factorials are reduced to

\[
[n]_{p,q}^R = \frac{p^n - q^{-n}}{q - p^{-1}},
\]

and

\[
[n]_{p,q}!^R = \begin{cases} 1 & \text{for } n = 0 \\ ((p, q^{-1})); (p, q^{-1})_n & \text{for } n \geq 1 \end{cases} \frac{(q - p^{-1})^n}{(q - p^{-1})^n}
\]

respectively, introduced in our previous work [9], generalizing the \(q\)-Quesne algebra [18].

Then follow some remarkable properties.
Proposition 5.3. If $n$ and $m$ are non-negative integers, then

$$[-m]_{p,q}^Q = -pq^{-m}q^m[m]_{p,q}^0,$$  \hfill (90)

$$[n + m]_{p,q}^Q = q^{-m}[n]_{p,q}^Q + pq^m[m]_{p,q}^0 = p^n[n]_{p,q}^Q + q^{-m}[m]_{p,q}^Q,$$  \hfill (91)

$$[n - m]_{p,q}^Q = q^{-m}[n]_{p,q}^Q - pq^{-m}q^m[m]_{p,q}^0 = p^{-n}[n]_{p,q}^Q + p^{-m}q^{m-n}[m]_{p,q}^Q,$$  \hfill (92)

$$[n]_{p,q}^Q = \frac{q - p^{-1}}{p - q^{-1}}[2]_{p,q}^Q[n - 1]_{p,q}^Q - pq^{-1}[n - 2]_{p,q}^Q.$$  \hfill (93)

**Proof.** Equations (90) and (91) are immediate by the application of the relations $p^{-m} - q^m = -pq^{-m}q^m(p^n - q^m)$ and $p^{n+m} - q^{-m} = q^{-m}(p^n - q^m) + p^m(p^n - q^{-m}) = p^m(p^n - q^{-m}) + q^{-m}(p^{n+m} - q^{-m})$, respectively, while equation (92) results from the combination of equations (90) and (91). Finally, the relation

$$[n]_{p,q^{-1}} = \frac{p^n - q^m}{p - q^{-1}} = \frac{q - p^{-1}}{p - q^{-1}} = \frac{q - p^{-1}}{p - q^{-1}}[n]_{p,q}, \quad n = 1, 2, \ldots$$  \hfill (94)

cumulatively taken with the identity

$$[n]_{p,q^{-1}} = [2]_{p,q^{-1}}[n - 1]_{p,q^{-1}} - pq^{-1}[n - 2]_{p,q^{-1}},$$

gives equation (93). \hfill \Box

**Proposition 5.4.** The $(p, q)$-Quesne binomial coefficients

$$\begin{bmatrix} n \end{bmatrix}_{p,q}^Q = \frac{((p, q^{-1}); (p, q^{-1}))_n}{((p, q^{-1}); (p, q^{-1}))_{n-k}((p, q^{-1}); (p, q^{-1}))_{n-k}},$$  \hfill (95)

where $0 \leq k \leq n, \quad n \in \mathbb{N}$, satisfy the following properties:

$$\begin{bmatrix} n \end{bmatrix}_{p,q}^Q = \begin{bmatrix} n \end{bmatrix}_{p,q}^Q - k^{(n-k)}\begin{bmatrix} n \end{bmatrix}_{1/q_p}^{1/q_p} - k^{(n-k)}\begin{bmatrix} n \end{bmatrix}_{1/q_p}^{1/q_p},$$  \hfill (96)

$$\begin{bmatrix} n + 1 \end{bmatrix}_{p,q}^Q = p^k\begin{bmatrix} n \end{bmatrix}_{p,q}^Q + q^{-n-1+k}\begin{bmatrix} n \end{bmatrix}_{p,q}^Q,$$  \hfill (97)

$$\begin{bmatrix} n + 1 \end{bmatrix}_{p,q}^Q = p^k\begin{bmatrix} n \end{bmatrix}_{p,q}^Q + p^{n+1-k}\begin{bmatrix} n \end{bmatrix}_{p,q}^Q - (p^n - q^{-n})\begin{bmatrix} n - 1 \end{bmatrix}_{p,q}^Q.$$  \hfill (98)

**Proof.** It is straightforward, using proposition 5.1 and

$$\begin{bmatrix} n \end{bmatrix}_{p,q}^Q = \begin{bmatrix} n \end{bmatrix}_{p,q^{-1}}.$$  \hfill (99)

Finally, the algebra $A_{p,q}^Q$, generated by $\{1, A, A^+, N\}$, associated with $(p, q)$-Quesne deformation satisfies the following commutation relations:

$$p^{-1}A A^+ - A^+A = q^{-N-1}, \quad qA A^+ - A^+A = p^{N+1}$$

$$[N, A^+] = A^+, \quad [N, A] = -A.$$  \hfill (100)
The \((p, q)\)-Rogers–Szegő polynomials corresponding to the Quesne deformation [9] are deduced from our generalization by choosing \(\phi_1(x, y) = \phi_2(x, y) = \phi(x, y) = x\) and \(\phi_3(x, y) = y - x^{-1}\). Indeed, it is worthy of attention that we get in this case \(\phi(p, q) = p \neq 0\), \(\phi_3(p, q) = q - p^{-1} \neq 0\), \(\phi(P, Q)\), and from equation (98)

\[
\left[\frac{n + 1}{k}\right]_p^q = p^k \left[\frac{n}{k}\right]_p^q + p^{q+1-k} \left[\frac{n}{k-1}\right]_p^q - (q-p^{-1})[n]_p^q \left[\frac{n-1}{k-1}\right]_p^q.
\]

Hence, the hypotheses of the theorem are satisfied and, therefore, the \((p, q)\)-Rogers–Szegő polynomials

\[
H_n^Q(z; p, q) = \sum_{k=0}^{n} \left[\frac{n}{k}\right]_p^q z^k, \quad n = 0, 1, 2, \ldots
\]

satisfy the three-term recurrence relation

\[
H_{n+1}^Q(z; p, q) = H_n^Q(pz; p, q) + zp^n H_n^Q(p^{-1}z; p, q) - z(p^n - q^{1-n}) H_n^Q(z; p, q)
\]

and the \((p, q)\)-difference equation

\[
\partial_{p,q}^Q H_n^Q(z; p, q) = [n]_p^q [n]_{p,q}^Q H_{n-1}^Q(z; p, q).
\]

Thus, the set of polynomials

\[
\psi_n^Q(z; p, q) = \frac{1}{\sqrt|[n]_p^q}} H_n^Q(z; p, q), \quad n = 0, 1, 2, \ldots
\]

forms a basis for a realization of the \((p, q)\)-Quesne deformed harmonic oscillator and quantum algebra \(A_{p,q}^Q\) engendering the commutation relations (100) with the number operator \(N\) formally defined as

\[
N \psi_n^Q(z; p, q) = n \psi_n^Q(z; p, q),
\]

and the annihilation and creation operators given by

\[
A = \partial_{p,q}^Q \quad \text{and} \quad A^+ = P + zp^N p^{-1} - z(q-p^{-1}) \partial_{p,q},
\]

respectively. Naturally, setting \(p = 1\) gives the Rogers–Szegő polynomials associated with the \(q\)-Quesne deformation [18].

The continuous \((p, q)\)-Hermite polynomials corresponding to the \((p, q)\)-generalization of Quesne deformation [9] can be defined as follows:

\[
\mathcal{H}^Q_n(\cos \theta; p, q) = e^{i\theta^p} H_n^Q(e^{-2i\theta}; p, q)
\]  

\[
= \sum_{k=0}^{n} \left[\frac{n}{k}\right]_p^q e^{i(n-2k)\theta}, \quad n = 0, 1, 2, \ldots
\]

Since for the \((p, q)\)-generalization of Quesne deformation [9] \(\phi_1(x, y) = \phi_2(x, y) = x\) and \(\phi_3(x, y) = y - x^{-1}\), from proposition 4.1 we deduce that the corresponding sequence of continuous \((p, q)\)-Hermite polynomials satisfies the three-term recurrence relation

\[
\mathcal{H}^Q_{n+1}(\cos \theta; p, q) = p^x (e^{i\theta^p} P + e^{-2i\theta} P^{-1}) \mathcal{H}^Q_n(\cos \theta; p, q) - (p^n - q^{1-n}) \mathcal{H}^Q_{n-1}(\cos \theta; p, q).
\]

(108)
5.4. $R(x, y) = h(p, q)y^\nu /x^\mu$

Here $0 < pq < 1$, $p^\mu < q^{-\nu}$, $p > 1$, and $h$ is a well-behaved real and non-negative function of deformation parameters $p$ and $q$ such that $h(p, q) \to 1$ as $(p, q) \to (1, 1)$.

The $(R, p, q)$-numbers become $(p, q; \mu, \nu, h)$-numbers introduced in our previous work [10] and defined by

$$[n]_{p,q,h}^{\mu,\nu} = h(p, q) q^{n\mu} p^n - q^{-n \nu}/p^\mu q^{-1}. \quad (109)$$

**Proposition 5.5.** The $(p, q; \mu, \nu, h)$-numbers verify the following properties, for $m, n \in \mathbb{N}$:

$$[-m]_{p,q,h}^{\mu,\nu} = -q^{2m-n-m} [m]_{p,q,h}^{\mu,\nu}, \quad (110)$$

$$[n + m]_{p,q,h}^{\mu,\nu} = q^{n+m} [n]_{p,q,h}^{\mu,\nu} + q^{n-m} [m]_{p,q,h}^{\mu,\nu}, \quad (111)$$

$$[n - m]_{p,q,h}^{\mu,\nu} = q^{-n-m} [n]_{p,q,h}^{\mu,\nu} + q^{n+m} [m]_{p,q,h}^{\mu,\nu}, \quad (112)$$

$$[n]_{p,q,h}^{\mu,\nu} = \frac{q - q^{-1}}{p - q^{-1}} \frac{1}{p^n h(p, q)} q^{n+1} [2]_{p,q,h}^{\mu,\nu} - \frac{q^{2n}}{p^{n+1}} [n - 2]_{p,q,h}^{\mu,\nu}. \quad (113)$$

**Proof.** It is straightforward using proposition 5.3 and the fact that

$$[n]_{p,q,h}^{\mu,\nu} = h(p, q) q^{n\mu} [n]_{p,q}^Q. \quad (114)$$

□

**Proposition 5.6.** The $(p, q; \mu, \nu, h)$-binomial coefficients

$$\binom{n}{k}_{p,q,h}^{\mu,\nu} := \frac{[n]_{p,q,h}^{\mu,\nu}}{[k]_{p,q,h}^{\mu,\nu} [n-k]_{p,q,h}^{\mu,\nu}} = q^{k(n-k)} \binom{n}{k}_{p,q}^Q, \quad (115)$$

where $0 \leq k \leq n$; $n \in \mathbb{N}$, satisfy the following properties:

$$\binom{n}{k}_{p,q,h}^{\mu,\nu} = \binom{n}{n-k}_{p,q,h}^{\mu,\nu}, \quad (116)$$

$$\binom{n+1}{k}_{p,q,h}^{\mu,\nu} = q^{k} \frac{[n]_{p,q,h}^{\mu,\nu}}{p^k [n]_{p,q,h}^{\mu,\nu}} + q^{(v-1)(n+1-k)} \binom{n}{k-1}_{p,q,h}^{\mu,\nu}, \quad (117)$$

$$\binom{n+1}{k}_{p,q,h}^{\mu,\nu} = q^{k} \frac{[n]_{p,q,h}^{\mu,\nu}}{p^k [n]_{p,q,h}^{\mu,\nu}} + q^{(\mu-1)(n+1-k)} \binom{n}{k-1}_{p,q,h}^{\mu,\nu}, \quad (118)$$
Proof. The proof follows from proposition 5.4 and the fact that
\[ [n]_{p,q,h}^{\mu,v} = H_n(p, q) \frac{q^{(v+1)/2}}{p^{(v+1)/2}} [n]_{p,q}^{Q}, \]  
where use of equation (114) has been made.

The algebra $A_{p,q,h}$ generated by $[1, A, A^\dagger, N]$, associated with $(p, q, \mu, v, h)$-deformation, satisfies the following commutation relations:
\[ p^{-1}A A^\dagger = \frac{q^v}{p^{v}} A^\dagger A = h(p, q) \left( \frac{q^{v-1}}{p^{v-1}} \right)^{N+1}, \]
\[ qA A^\dagger = \frac{q^v}{p^{v}} A^\dagger A = h(p, q) \left( \frac{q^v}{p^{v-1}} \right)^{N+1}, \]
\[ [N, A] = A^\dagger, \quad [N, A^\dagger] = -A. \]

The $(p, q, \mu, v, h)$-Rogers–Szegö [10] polynomials are deduced from the above general construction by setting $\phi_1(x, y) = x^{1+\nu}y^v$, $\phi_2(x, y) = x^{-\mu}y^{-1}$ and $\phi_3(x, y) = \frac{x}{h(p, q)}$. Indeed, $\phi_i(p, q) \neq 0$ for $i = 1, 2, 3$; $\phi(P, Q)z^\mu = \phi(p, q)z^\mu$ for $i = 1, 2$ and the property (118) furnishes
\[ \binom{n+1}{k}^{\mu,v}_{p,q,h} = \frac{q^n}{p^{(n+1)\mu}} \sum_{k=0}^{n} \binom{n}{k}^{\mu,v}_{p,q,h} + \frac{q^{(n+1-k)}}{p^{(n+1-k+1)\mu}} \binom{n}{k-1}^{\mu,v}_{p,q,h} - \frac{q-p^{-1}}{h(p, q)} \binom{n}{k}^{\mu,v}_{p,q,h}. \]

Therefore, the $(p, q, \mu, v, h)$-Rogers–Szegö polynomials are defined as follows:
\[ H_n(z; p, q, \mu, v, h) = \sum_{k=0}^{n} \binom{n}{k}^{\mu,v}_{p,q,h} z^k, \quad n = 0, 1, 2, \ldots \]

with the three-term recurrence relation
\[ H_{n+1}(z; p, q, \mu, v, h) = H_n \left( \frac{q^{v}}{p^{v-1}} z; p, q, \mu, v, h \right) + \frac{q^{(v-1)n}}{p^{v-n}} H_n \left( \frac{p^{v}}{q^{v-1}} z; p, q, \mu, v, h \right) - \frac{q^{-n}}{p^{n}} (p^n - q^{-n}) H_{n-1}(z; p, q, \mu, v, h) \]
\[ \text{and the } (p, q, \mu, v, h)-\text{difference equation} \]
\[ \partial_{p,q,h}^{\mu,v} H_n(z; p, q, \mu, v, h) = [n]^{\mu,v}_{p,q,h} H_{n-1}(z; p, q, \mu, v, h). \]

Hence, the set of polynomials
\[ \psi_n(z; p, q, \mu, v, h) = \frac{1}{\sqrt{[n]^{\mu,v}_{p,q,h}}} H_n(z; p, q, \mu, v, h), \quad n = 0, 1, 2, \ldots \]
forms a basis for a realization of the $(p, q, \mu, v, h)$-deformed algebra $A_{p,q,h}$ satisfying the commutation relations (120) with the number operator $N$ formally defined as
\[ N \psi_n^Q(z; p, q, \mu, v, h) = n \psi_n(z; p, q, \mu, v, h), \]

together with the annihilation and the creation operators given by
\[ A = \partial_{p,q,h}^{\mu,v} \text{ and } A^\dagger = \frac{q^v}{p^{v-1}} + z \left( \frac{q^{v-1}}{p^{v}} \right)^N \frac{p^{v}}{Q^{v-1}} = z \frac{(q-p^{-1})}{h(p, q)} \partial_{p,q,h}^{\mu,v}, \] respectively.
The continuous \((p, q, \mu, v, h)\)-Hermite polynomials \([10]\) can now be deduced as
\[
\mathbb{H}_n(\cos \theta; p, q, \mu, v, h) = \frac{q^{n+\frac{1}{2}}}{p^{n+\frac{1}{2}}} Q^n \mathbb{H}_n(\cos \theta; p, q, \mu, v, h) \\
+ \frac{q^{n-\frac{1}{2}}}{p^{n+\frac{1}{2}}} P^n \mathbb{H}_n(\cos \theta; p, q, \mu, v, h) \\
- (p^n - q^{-n}) \frac{q^{2n}}{p^{2n}} \mathbb{H}_{n-1}(\cos \theta; p, q, \mu, v, h). \tag{128}
\]

Since for the \((p, q, \mu, v, h)\)-deformation \(\phi_1(x, y) = x^{1-\frac{1}{2}\mu} y^\mu\), \(\phi_2(x, y) = x^{\frac{-1}{2}} y^{\mu - 1}\) and \(\phi_3(x, y) = \frac{y^\mu - x^{\mu - 1}}{(p^p q^q)^\mu - 1}\), from proposition 4.1 the corresponding sequence of continuous \((p, q, \mu, v, h)\)-Hermite polynomials satisfies the three-term recurrence relation
\[
\mathbb{H}_{n+1}(\cos \theta; p, q, \mu, v, h) = \frac{q^{n+\frac{1}{2}}}{p^{n+\frac{1}{2}}} Q^n \mathbb{H}_n(\cos \theta; p, q, \mu, v, h) \\
+ \frac{q^{n-\frac{1}{2}}}{p^{n+\frac{1}{2}}} P^n \mathbb{H}_n(\cos \theta; p, q, \mu, v, h) \\
- (p^n - q^{-n}) \frac{q^{2n}}{p^{2n}} \mathbb{H}_{n-1}(\cos \theta; p, q, \mu, v, h). \tag{127}
\]

6. Concluding remarks

In this paper, we have defined and discussed a general formalism for constructing \((R, p, q)\)-deformed Rogers–Szegö polynomials. The displayed approach not only provides novel relations, but also generalizes the well-known standard and deformed Rogers–Szegö polynomials. A full characterization of the latter, including the data on the three-term recurrence relations and difference equations, has been provided. We have succeeded in elaborating a new realization of the \((R, p, q)\)-deformed quantum algebra generalizing the construction of \(q\)-deformed harmonic oscillator creation and annihilation operators performed in \([5, 13]\). The continuous \((R, p, q)\)-Hermite polynomials have also been investigated in detail.

Finally, relevant particular cases and examples have been exhibited.

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