Hyperentangled States

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Abstract

We investigate a new class of entangled states, which we call hyperentangled, that have EPR correlations identical to those in the vacuum state of a relativistic quantum field. We show that whenever hyperentangled states exist in any quantum theory, they are dense in its state space. We also give prescriptions for constructing hyperentangled states that involve an arbitrarily large collection of systems.

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1 EPR Correlations in the Vacuum

It is well-known that the Bell inequality is generically violated in the states of free relativistic quantum fields (see [1] for a recent review). This reinforces the point, usually driven home in the context of nonrelativistic quantum mechanics, that quantum correlations cannot be explained locally in terms of common causes or pre-existing elements of reality. However, it is important to remember that Bell’s original derivation of his inequality [2] did not simply assume local realism from the outset. Rather, Bell (pp. 14-5) launched his derivation from a version of the Einstein-Podolsky-Rosen (EPR) argument [3] in order to motivate his assumption that elements of reality exist for all the observables that figure in the inequality he derives. Granting that EPR had successfully established the dilemma that either locality must fail or quantum mechanics fails to account for certain elements of reality, Bell showed that even if we grasp the second horn of EPR’s dilemma, locality must fail in any case on pain of failing to account for the correlations actually obtained. This raises the question of whether one can reproduce Bell’s own EPR-inspired argument for nonlocality in relativistic quantum field theory by employing, for example, vacuum correlations.

Recall that to apply their sufficient condition for identifying an element of reality, EPR needed to exploit correlations with the feature that if a local measurement of a particle’s position (or momentum) were performed, a distant particle’s position (or momentum) could be predicted with certainty. So EPR’s argument relies essentially on the availability of maximal correlations. Are such correlations present in the vacuum? Indeed they are, all over the place. Redhead [4, 5] has recently shown that any field state with bounded energy—the vacuum state in particular—dictates a host of maximal correlations between the values of observables associated with any two spacelike-separated regions. Thus vacuum correlations supply plenty of resources to run Bell’s EPR-inspired derivation of his inequality.

On the other hand, even if the Bell inequality were not violated in this context, so allowing the possibility of a common cause explanation of the correlations by events in the overlapping backwards lightcones of the correlated events, this would still not give an acceptable local explanation of the correlations. This is because the vacuum is time-translation invariant, so any common cause events employed to explain correlations at a later time are inevitably involved in correlations that themselves need explaining. It seems we must either embark on an infinite regress of explanation, or accept
the correlations as ‘brute facts’.

Of course the situation is not so hopeless as that. Even if no satisfactory causal explanation of maximal vacuum correlations is available, this does not preclude pursuing a deeper understanding of those correlations in purely quantum-mechanical terms. The present paper aims to do precisely this, following on from the initial investigations undertaken in [4, 5, 6]. In particular, we uncover necessary and sufficient conditions for maximal correlations like those in the vacuum to obtain in the states posited by any quantum theory (not necessarily relativistic or field-theoretic), and investigate how generic such states are. But before we can get started, we need to delve into Redhead’s argument [4] for the existence of maximal vacuum correlations.

2 Maximal Vacuum Correlations

The formal proof of the pervasiveness of maximal vacuum correlations proceeds within the algebraic approach to relativistic quantum field theory [7, 8]. On that approach, one associates with each bounded open region \( O \) in Minkowski spacetime \( M \) a von Neumann algebra \( \mathcal{U}(O) \) of bounded operators whose self-adjoint elements represent observables measurable in region \( O \) (constructed from field operators smeared with test functions having support in \( O \)). Each algebra in the collection \( \{\mathcal{U}(O)\}_{O \subseteq M} \) acts on the same Hilbert space \( H \) consisting of states of the entire field on \( M \).

If these algebras satisfy certain plausible general conditions—notably that algebras associated with spacelike-separated regions must commute and that the energy-momentum spectrum of the field must be confined to the forward lightcone (see [4] for the complete list of conditions)—then it becomes possible to prove the Reeh-Schlieder theorem [9]. It is this theorem that is pivotal for establishing maximal correlations in the vacuum. If we call a field state \( \Psi \) cyclic for \( \mathcal{U}(O) \) whenever the set \( \{A \Psi : A \in \mathcal{U}(O)\} \) is dense in \( H \), then the Reeh-Schlieder theorem asserts that for any bounded open region \( O \), the vacuum state \( \Omega \) (or any other field state \( \Psi \) with bounded energy) is cyclic for \( \mathcal{U}(O) \).

In addition to the Reeh-Schlieder theorem, Redhead’s argument uses a simple result about commuting von Neumann algebras \( \mathcal{U}(O) \) and \( \mathcal{U}(O') \), viz. that if a vector \( \Psi \) is cyclic for one of the algebras \( \mathcal{U}(O) \) or \( \mathcal{U}(O') \) it must separating for the other, where \( \Psi \) is called a separating vector for an algebra if no nonzero operator in the algebra maps \( \Psi \) to 0. Since the proof of
this result is elementary, and we will revisit it a number of times below, we
give its proof. If, say, \( \Psi \) is cyclic for \( \mathcal{U}(O) \) then there is a sequence of operators 
\( \{A_m\} \subseteq \mathcal{U}(O) \) that commute with any \( A' \in \mathcal{U}(O') \) and can be applied to \( \Psi \)
to approximate any vector \( \Phi \in H \) as closely as desired. Therefore, if \( A'\Psi = 0 \) we have 
\[
A'\Phi = A'\lim A_m\Psi = \lim A_mA'\Psi = 0
\]  
for any \( \Phi \in H \), which implies \( A' = 0 \).

Now call a von Neumann algebra \( \mathcal{U}(O') \) maximally correlated with 
a von Neumann algebra \( \mathcal{U}(O) \) in the state \( \Psi \) if for any nonzero projection 
operator \( P' \in \mathcal{U}(O') \) and any \( \epsilon > 0 \), there is a nonzero projection operator 
\( P \in \mathcal{U}(O) \) such that 
\[
\text{Prob}_{\Psi}(P' = 1/P = 1) = 1 - \epsilon.
\]  
(Note that this relation between algebras given a state \( \Psi \) need not be symmetric.) What Redhead’s argument establishes is that if any state \( \Psi \) is cyclic for \( \mathcal{U}(O) \) and \( O' \) spacelike-related to \( O \)—so that \( \mathcal{U}(O') \) commutes with \( \mathcal{U}(O) \)—then \( \mathcal{U}(O') \) is maximally correlated with \( \mathcal{U}(O) \) in state \( \Psi \). Without pursuing 
the details of the argument, the basic idea is intuitive. Since \( \Psi \) is assumed 
cyclic for \( \mathcal{U}(O) \) it is separating for \( \mathcal{U}(O') \), so that for any nonzero projection 
\( P' \in \mathcal{U}(O') \) the state \( \Phi = P'\Psi/\|P'\Psi\| \) is well-defined. Moreover, by construction of \( \Phi \), 
\( \text{Prob}_{\Psi}(P' = 1/P = 1) = 1 \). One then uses the possibility of getting 
arbitrarily close to \( \Phi \) by acting on \( \Psi \) with operators in \( \mathcal{U}(O) \) to infer (via the 
spectral theorem) the existence of a sequence of projections \( \{P_m\} \subseteq \mathcal{U}(O) \) such that 
\[
\lim \text{Prob}_{\Psi}(P' = 1/P_m = 1) = 1.
\]  
Of course, what makes Redhead’s theorem relevant for the physics of the vacuum is the Reeh-Schlieder theorem. For together these theorems imply that the vacuum state \( \Omega \) is actually filled with maximal correlations: for any two spacelike-separated (bounded, open) regions \( O \) and \( O' \), \( \mathcal{U}(O') \) is maximally correlated with \( \mathcal{U}(O) \) in the state \( \Omega \).

Having made precise the sense in which there are maximal vacuum correlations, it may now appear that there are some obstacles to running the EPR argument in this setting. In the first place, EPR’s sufficient condition for 
identifying an element of reality only permits the inference to an element of
reality in situations where the outcome of measuring it can be predicted with absolute certainty. For maximal vacuum correlations, all we have is prediction with arbitrarily high certainty. Is it possible to remove this restriction? No. Suppose that for some nonzero \( P' \in \mathcal{U}(O') \) there is a nonzero \( P \in \mathcal{U}(O) \)
such that Prob$_{\Omega}(P' = 1/P = 1) = 1$. Then Prob$_{\Omega}(P' = 0/P = 1) = 0$ which requires $(I - P')P\Omega = 0$. But $(I - P')P$ is an element of the algebra generated by $U(O) \cup U(O')$, which is the local algebra associated with the bounded open region $O \cup O'$ (cf. [7], p. 107). Since $\Omega$ is separating for this algebra, it follows that $(I - P')P = 0$. And this contradicts a well-known consequence of the axioms of algebraic relativistic quantum field theory ([8], p. 76): that nonzero operators (such as $(I - P')$ and $P$) associated with (generic) spacelike separated commuting algebras must have a nonzero product. The upshot is that vacuum correlations cannot supply conditional predictions with absolute certainty. However, we see no reason why the ability to predict the outcome of measuring some local observable with arbitrarily high certainty should not give one just as strong grounds to infer the existence of an element of reality corresponding to that observable as one would get if its value were predictable with absolute certainty.

There also appears to be a second worry. The correlations EPR originally exploited had the additional feature that whatever the result of the local measurement of position (or momentum), the distant particle’s position (or momentum) could be predicted with certainty. By contrast, even if the probability for $P' = 1$ given $P = 1$ is close to 1 as per Eqn. 2, it does not follow that either $P' = 1$ or $P' = 0$ must be assigned a probability near to 1 given that $P = 0$ (though this may be true for certain special choices of $P'$). Again, however, we do not think this undermines the validity of EPR’s argument in the vacuum. It was important to their argument that if without in any way disturbing a system we can predict with certainty the outcome of measuring one of its observables, then there exists an element of reality corresponding to the observable. Behind this was the idea that one should not be able to make sharp the value of a local observable by performing measurements at a distance. But to the extent that this locality assumption is plausible, it should also be plausible to assume that one cannot make sharp the value of a local observable by performing a distant measurement and getting some particular outcome. Indeed, since Redhead’s theorem tells us that for any local observable and any possible outcome of measuring it, it is always possible to perform a distant measurement on a second observable and get an outcome that dictates with (virtual) certainty the outcome of measuring the first observable, EPR-type reasoning leads immediately to the conclusion that there are pre-existing elements of reality in the vacuum state for all local observables! Of course, while valid we are not saying EPR’s reasoning is sound, since it is based upon locality assumptions which, in turn,
lead to the Bell inequality.

One last point before we probe deeper into the origin of maximal vacuum correlations. These correlations are of exactly the sort that troubled Schrödinger when he wrote of the ‘sinister’ possibility in quantum mechanics of steering a distant system into any desired state by a suitable local measurement. But one must not be misled into thinking that the correlations lead to any empirically detectable nonlocality, in violation of spacelike commutativity. In the vacuum, the outcome of measuring the projection $P$ in Eqn. 2 is generally going to be probabilistic and cannot itself be controlled, which is what would be needed to truly ‘steer’ a distant system close to an eigenstate of $P'$. In fact, it turns out that the well-known Fredenhagen bound on correlations in the vacuum entails that for any given $P' \in U(O')$, the maximally correlated $P \in U(O)$ in the vacuum state must have a probability of occurring that falls off exponentially with the minimum Lorentz distance between $O'$ and $O$ (see [4], Sec. 3). Moreover, the outcome of measuring the projection $P$ in Eqn. 2 will be certain to be 1 in the vacuum only in the uninteresting case when $P'$ is the identity operator. For if $\text{Prob}_\Omega(P' = 1 | P = 1) = 1$ when $(\Omega, P\Omega) = 1$, then $I - P$ annihilates $\Omega$ which forces $P = I$ because $\Omega$ is a separating vector for all the local algebras. This, in turn, means we must have $\text{Prob}_\Omega(P' = 1) = 1$, which by the same reasoning forces $P' = I$ as well.

3 Hyperentanglement

Since the pervasiveness of maximal vacuum correlations rests on the Reeh-Schlieder theorem, this is the natural place to look for an explanation for their presence. The theorem is remarkable because if $O$ is, for example, the neighbourhood of some particular point in $M$, how could acting on $\Omega$ with operators in $U(O)$ approximate an arbitrary state of the field, in particular one which looks quite unlike the vacuum in some remote region spacelike-separated from $O$? The short answer, given by Haag ([7], p. 102), is that requiring the energy-momentum spectrum of the field to be confined to the forward lightcone forces the vacuum to be a highly correlated state, and it is these correlations which are ‘judiciously exploited’ to prove the Reeh-Schlieder theorem. However, the analyticity arguments that go into the proof (cf. [8], pp. 25-6) shed little light on this, nor do they give any sense of exactly what structure a state needs to have to be saturated with maximal
correlations between spacelike-separated observables.

Our approach to this issue will be to abstract away from the context of quantum field theory and analyze the matter in terms of the more familiar concept of entanglement. We shall see that the sorts of states that give rise to pervasive maximal correlations can occur in any quantum theory. Furthermore, contrary to what one might expect, these states need not differ from other entangled states in their degree of entanglement (according to the measure recently proposed by Shimony [12]), but rather involve a completely new kind of entanglement.

In order to characterize vacuum correlations in terms of entanglement, consider the implications the Reeh-Schlieder theorem has for only a finite number of mutually spacelike-separated regions \( \{O_i\}_{i=1}^n \) \((n > 1)\). Each of the algebras \( \mathcal{U}(O_i) \) has a representation on a separable Hilbert space \( H_i \) of \( \text{dim} > 1 \), and since these algebras mutually commute the von Neumann algebra generated by \( \bigcup_{i=1}^n \mathcal{U}(O_i) \) can be represented on the tensor product Hilbert space

\[
H = H_1 \otimes H_2 \otimes \cdots \otimes H_n. \tag{3}
\]

Note that for any (proper) subset \( S \) of the indices \( \{1, \ldots, n\} \), \( H \) can be factored as

\[
H = H_S \otimes H_{S'}, \tag{4}
\]

where

\[
H_S = \bigotimes_{i \in S} H_i \quad \text{and} \quad H_{S'} = \bigotimes_{i \in S} H_i, \tag{5}
\]

and \( S' \) is the complement in \( \{1, \ldots, n\} \) of \( S \). We shall also use the letter \( S \) to refer to the algebra represented on \( H_S \). In that case, \( S' \) will refer to the commutant of \( S \), i.e. the set of all bounded operators on \( H \) that commute with those in \( S \), which are exactly the operators represented on \( H_{S'} \). (Whether \( S \) denotes a subset of indices labelling the factors in \( H \) or the corresponding operator algebra will always be clear from context.) By analogy with the previous section, call a state vector \( v \in H \) \( S \)-cyclic if the set

\[
\{(A \otimes I)v : A = \text{a bounded operator on } H_S, I = \text{the identity on } H_{S'}\} \tag{6}
\]

is dense in \( H \). Since operators in the algebra \( S \) are associated with the region of spacetime \( \bigcup_{i \in S} O_i \) (itself bounded and open), the Reeh-Schlieder theorem asserts that \( \Omega \) is \( S \)-cyclic for all \( S \).
Now forget about the details of relativistic quantum field theory. State spaces having the tensor product form of $H$ above occur in all quantum theories and are used to describe compound systems with $n$ components, such as $n$ spinless particles. (So sometimes we shall refer to a subset of the indices $\{1, \ldots, n\}$ as a subsystem.) In this more general context, there no longer need be a vacuum state, and we no longer have any Reeh-Schlieder theorem. However, Redhead’s theorem is still valid in the form:

**Theorem 1** If a state $v \in H$ is $S$-cyclic, then $S'$ is maximally correlated with $S$ in state $v$.

By analogy with the vacuum, we want to investigate the entanglement properties of states with the following feature:

*For any two nonoverlapping subsystems $S$ and $T$, the algebra $S$ is maximally correlated with the algebra $T$.*

Anticipating the results of our investigation, it will be appropriate to call such states **hyperentangled**.

The first thing to note is that hyperentangled states, so defined, are indeed entangled—indeed, they must be entangled with respect to any of the possible factorizations of $H$ as $H = H_S \otimes H_{S'}$. For suppose that some state $v$ is both hyperentangled and a product state. Then, on the one hand, for any nonzero projection $P' \in S'$ and $\epsilon > 0$, there must exist a nonzero projection $P \in S$ such that

$$ (v, (P \otimes P')v) > (1 - \epsilon)(v, (P \otimes I)v), \quad (7) $$

while, on the other,

$$ (v, (P \otimes P')v) = (v, (P \otimes I)v)(v, (I \otimes P')v) \quad (8) $$

since $v$ is a product state with respect to $H = H_S \otimes H_{S'}$. Taken together, Eqns. 7 and 8 entail $(v, (I \otimes P')v) > 1 - \epsilon$. But since $\epsilon$ can be chosen arbitrarily small, $(v, (I \otimes P')v) = 1$ for any nonzero projection $P' \in S'$, which of course is absurd.

### 4 Tests for Hyperentanglement

In order to be able to say more about hyperentanglement, we turn now to establishing the equivalence of $S$-cyclicity and maximal correlation of algebras.
with two further conditions (3. and 4. below) which serve as simple tests of a state’s hyperentanglement.

**Theorem 2** Let \( v \) be any state vector in \( H = H_S \otimes H_{S'} \), \( D^v_S \), be the reduced density operator for the subsystem \( S' \) as determined by \( v \), and \( \{ b_j \} \) be any orthonormal basis for \( H_{S'} \). Then the following are equivalent:

1. \( v \) is \( S \)-cyclic.
2. \( S' \) is maximally correlated with \( S \) in state \( v \).
3. \( D^v_{S'} \) does not have 0 as an eigenvalue.
4. \( v \) may be expanded as \( v = \sum_j v_j \otimes b_j \) where the set of vectors \( \{ v_j \} \subseteq H_S \) is linearly independent.

**Proof.** 1. \( \Rightarrow \) 2. This is just Redhead’s theorem (Thm. 1 above).

2. \( \Rightarrow \) 3. Let \( w \) be any nonzero vector in \( H_{S'} \) and \( P'_w \) be the projection onto the subspace generated by \( w \). Clearly a necessary condition for \( S' \) to be maximally correlated with \( S \) in state \( v \) is that \( \text{Prob}_v(P'_w = 1) > 0 \). It follows that \( \text{Tr}(D^v_{S'}P'_w) \neq 0 \), and hence that \( D^v_{S'}w \neq 0 \). Since \( w \) was arbitrary, \( D^v_{S'} \) cannot have 0 as an eigenvalue.

\(~4. \Rightarrow ~3. \) Since we can pick an orthonormal basis \( \{ a_i \} \subseteq H_S \) and expand any vector in \( H \), in particular \( v \), as

\[ v = \sum_{ij} c_{ij} a_i \otimes b_j = \sum_j (\sum_i c_{ij} a_i) \otimes b_j, \tag{9} \]

there exist vectors \( \{ v_j \} \subseteq H_S \) such that \( v = \sum_j v_j \otimes b_j \). Assuming \(~4. \) is false, (i.e. 4. is false), at least one element of \( \{ v_j \} \) must be a finite linear combination of other elements in \( \{ v_j \} \). (To avoid possible confusion, note that we are using the general notion of linear independence, applicable to any vector space (topological or not), according to which a set of vectors is linearly independent exactly when all its finite subsets are.) Without loss of generality, we may assume that \( v_1 \) is a linear combination of \( \{ v_2, \ldots, v_m \} \)—for if this were not the case, we could always renumber the basis vectors \( \{ b_j \} \) so that it is. Thus, for some coefficients \( \{ c_j \}^m_{j=2} \) we have

\[ v_1 = \sum_{j=2}^m c_j v_j. \tag{10} \]
This gives

\[ v = v_1 \otimes b_1 + \sum_{j>1} v_j \otimes b_j \] (11)

\[ = (\sum_{j=2}^{m} c_j v_j) \otimes b_1 + \sum_{j>1} v_j \otimes b_j \] (12)

\[ = \sum_{j=2}^{m} v_j \otimes (c_j b_1 + b_j) + \sum_{j>m} v_j \otimes b_j. \] (13)

Now consider the subspace \( T \) in \( H_{S'} \) spanned by the orthonormal vectors \( \{b_j\}_{j=1}^{m} \subseteq \{b_j\} \). Clearly \( T \) has dimension \( m \), and the \( m - 1 \) vectors \( \{c_j b_1 + b_j\}_{j=2}^{m} \) span a proper subspace of \( T \). Therefore there is at least one nonzero vector \( w \in T \) orthogonal to all the vectors \( \{c_j b_1 + b_j\}_{j=2}^{m} \). Moreover, since \( w \in T \), \( w \) is orthogonal to all the remaining basis vectors \( \{b_j\}_{j>m} \) as well. So if we let \( P'_w \) be the projection operator whose range is the subspace generated by \( w \), then the action of \( I \otimes P'_w \) on \( v \) —considering the expansion of \( v \) given in Eqn. 13 —produces the 0 vector. It follows that \( Tr(D_{S'}^v P'_w) = 0 \), and hence that \( w \) is an eigenvector of \( D_{S'}^v \) corresponding to eigenvalue 0, so that condition 3. fails.

4. \( \Rightarrow 1 \). As we saw at the beginning of the previous argument, any vector \( w \in H \) can be expanded as \( w = \sum j w_j \otimes b_j \) where \( \{w_j\} \) is some set of vectors in \( H_S \). To establish \( S \)-cyclicity of \( v \), all we need to do is construct a sequence of bounded operators \( \{A_m \otimes I\} \) which act on \( v = \sum j v_j \otimes b_j \) to bring it arbitrarily close in norm to \( w = \sum j w_j \otimes b_j \). So define the mapping \( A_m \) \( (m = 1, 2, \ldots) \) by:

\[ A_m v_j = w_j \text{ if } j \leq m, \] (14)

\[ = 0 \text{ if } j > m. \] (15)

Since the set \( \{v_j\} \) is linearly independent by hypothesis, this definition extends to a linear operator \( A_m \) from the closed subspace generated by \( \{v_j\} \) to \( H_S \), an operator which is bounded, since (for finite \( m \)) \( A_m \) annihilates all but a finite number of the \( \{v_j\} \). If we further extend the definition of \( A_m \) to the whole of \( H_S \) by stipulating that \( A_m u = 0 \) for all \( u \) orthogonal to the closed subspace generated by \( \{v_j\} \), we get a bounded linear operator \( A_m \) acting on \( H_S \) for each \( m \). Now just note that, by construction of the \( \{A_m\} \), \( \lim(A_m \otimes I)v = w \). QED.
Looking at Bohm’s version of the EPR state using two spin-1/2 particles
\[ \frac{1}{\sqrt{2}}(|\sigma_{1z} = +1\rangle|\sigma_{2z} = -1\rangle + |\sigma_{1z} = -1\rangle|\sigma_{2z} = +1\rangle), \] (16)
we see straightaway, using condition Thm. 2.4 (i.e. condition 4 of Thm. 2), that 1 is maximally correlated with 2 and vice-versa. In fact, every entangled state of two spin-1/2 particles is hyperentangled. For it is easy to verify that a state \( v \in H_1^2 \otimes H_2^2 \) (here, the superscripts indicate dimension) is entangled if and only if neither \( D_1^v \) nor \( D_2^v \) have 0 as an eigenvalue which, in view of Thm. 2.3, is exactly the condition for hyperentanglement. Thus the entangled spin-1/2 Hardy state
\[ \frac{1}{\sqrt{3}}[2|\sigma_{1x} = +1\rangle|\sigma_{2x} = +1\rangle - |\sigma_{1y} = +1\rangle|\sigma_{2z} = +1\rangle], \] (17)
is hyperentangled too.
This conclusion that every entangled state is hyperentangled is actually an artifact of \( H \) having two-dimensional factors. For the same reason that Bohm’s singlet state is hyperentangled, the spin-1 singlet state
\[ \frac{1}{\sqrt{3}}[|S_{1y} = 0\rangle|S_{2y} = 0\rangle - |S_{1x} = 0\rangle|S_{2x} = 0\rangle - |S_{1z} = 0\rangle|S_{2z} = 0\rangle] \] (18)
employed by Heywood and Redhead in their algebraic proof of nonlocality is hyperentangled. But the closely related entangled state
\[ |v\rangle = \frac{1}{\sqrt{2}}[|S_{1y} = 0\rangle|S_{2y} = 0\rangle - |S_{1x} = 0\rangle|S_{2x} = 0\rangle] \] (19)
is not, since relative to the orthonormal basis
\[ b_1 = |S_{2y} = 0\rangle, \ b_2 = |S_{2x} = 0\rangle, \ b_3 = |S_{2z} = 0\rangle, \] (20)
for \( H_2^3 \), the corresponding vectors
\[ v_1 = \frac{1}{\sqrt{2}}|S_{1y} = 0\rangle, \ v_2 = -\frac{1}{\sqrt{2}}|S_{1x} = 0\rangle, \ v_3 = |0\rangle, \] (21)
are obviously not linearly independent, in violation of Thm. 2.4 for \( S = 1 \). (A similar violation occurs for \( S = 2 \), but of course the \( S = 1 \) violation is enough to defeat the hyperentanglement.)
When $H$ has more than two factors, the possibility of hyperentanglement looks even less likely. The three spin-1/2 Greenberger-Horne-Zeilinger state
\[
\frac{1}{\sqrt{2}}[(|\sigma_1 z = +1\rangle|\sigma_2 z = +1\rangle|\sigma_3 z = +1\rangle + |\sigma_1 z = -1\rangle|\sigma_2 z = -1\rangle|\sigma_3 z = -1\rangle]
\]
fails the test of Thm. 2.3 in the case $S = 1$, because all vectors orthogonal to both $|\sigma_2 z = +1\rangle|\sigma_3 z = +1\rangle$ and $|\sigma_2 z = -1\rangle|\sigma_3 z = -1\rangle$ in $H_2^2 \otimes H_3^2$ are eigenvectors of the density operator for $\{2, 3\}$ with eigenvalue 0 (similar remarks applying for the density operators of $\{1, 2\}$ and $\{1, 3\}$). We leave the reader the task of checking that the three spin-1/2 Hardy state
\[
\frac{1}{\sqrt{7}}[2^{3/2}(|\sigma_1 x = +1\rangle|\sigma_2 x = +1\rangle|\sigma_3 x = +1\rangle - |\sigma_1 z = +1\rangle|\sigma_2 z = +1\rangle|\sigma_3 z = +1\rangle],
\]
also fails to pass muster.
These last two nonexamples suggest that hyperentangled states may not be all that common in quantum theories after all. On the contrary, we shall now show that when a compound system actually has a state space that permits the existence of hyperentangled states, they must be norm dense in the unit sphere of that space.

\section{The Existence and Density of Hyperentangled States}

We start by noting that the test of hyperentanglement supplied by the conditions of Thm. 2 can be simplified so that it is only necessary to check satisfaction of (any one of) those conditions for the ‘atomic’ subsystems represented by the individual factors $H_1, H_2, \ldots, H_n$ of $H$.

\textbf{Theorem 3} A state $v \in H$ is hyperentangled if and only if any (and therefore all) of Thm. 2’s equivalent conditions on $v$ hold in all the cases $S = 1, 2, \ldots, n$.

\textbf{Proof.} ‘Only if’. If $v$ is hyperentangled, then (by definition) for any two nonoverlapping subsystems $S$ and $T$, the algebra $S$ is maximally correlated to the algebra $T$ in state $v$. In particular, for any index $i$ the algebra $i'$
(corresponding to the set of indices in \{1, \ldots, n\} unequal to \(i\)) is maximally correlated to the algebra \(i\), so that Thm. 2.2 holds for \(v\) in the case \(S = i\).

‘If’. Consider any two nonoverlapping subsystems \(S\) and \(T\). If \(j\) is any index in the set \(T\), then since by hypothesis Thm. 2 holds for \(v\) when \(S = j\), \(v\) is \(j\)-cyclic which \textit{ipso facto} means \(v\) must be \(T\)-cyclic. Using 1. \(\Rightarrow\) 2. of Thm. 2, this entails that \(T'\) is maximally correlated with \(T\) in state \(v\), so that in particular \(S\) is maximally correlated with \(T\) in state \(v\), because \(S \subseteq T'\).

\textit{QED.}

For our next result below, we shall be employing Thm. 3 and in particular the equivalence between hyperentanglement, Thm. 2.3 for \(S = 1\) to \(n\) (the condition on the atomic system’s density operators), and Thm. 2.4 for \(S = 1\) to \(n\) (the linear independence condition).

\textbf{Theorem 4} Let the state space be given by \(H = H_1 \otimes H_2 \otimes \cdots \otimes H_n\) where \(n > 1\) and each factor space is separable and has nontrivial (> 1) dimension. Then the following are equivalent:

1. There exists a hyperentangled state in \(H\).

2. All the factors of \(H\) have the same dimension, and if \(n > 2\) their (common) dimension is infinite.

3. The set of hyperentangled states is norm dense in the unit sphere of \(H\).

\textit{Proof.} 1. \(\Rightarrow\) 2. We begin by establishing:

\textbf{Lemma}: For \(i = 1\) to \(n\), \(\dim H_i = \dim H_i'\).

(As before, we write \(i\) for \(\{i\}\) and \(i'\) for \(\{i\}' = \{j : j \neq i\}\).)

By hypothesis, some state \(v \in H\) is hyperentangled. Let

\[ v = \sum_k c_k (a_k \otimes b_k) \tag{24} \]

be a Schmidt decomposition of \(v\) with respect to the factorization \(H = H_i \otimes H_{i'}\), where \(\{a_k\}\) and \(\{b_k\}\) are orthonormal sets in \(H_i\) and \(H_{i'}\) respectively. If \(\dim H_i < \dim H_{i'}\), then the set \(\{b_k : c_k \neq 0\}\) cannot form a basis for \(H_{i'}\). Therefore there must be a nonzero vector \(w \in H_{i'}\) orthogonal to all the vectors in \(\{b_k : c_k \neq 0\}\). If \(P'_w\) is the projection onto the subspace \(w\) generates, then by Eqn. 24 \((I \otimes P'_w)v = 0\), and \(w\) must be an eigenvalue
of $D_i'$ corresponding to eigenvalue 0. But since $v$ is hyperentangled, this contradicts Thm. 2.3 for $S = i$.

All that remains is to show that $\dim H_i > \dim H_{i'}$ leads to a similar contradiction, and then the Lemma follows. To this end, let $H_j$ be any one of the Hilbert space factors of $H$ that occurs in $H_{i'}$ and re-express $H$ as $H = H_j \otimes H_{i'}$. Since

$$\dim H_j \leq \dim H_{i'} < \dim H_i \leq \dim H_{j'},$$

(25)

$\dim H_j < \dim H_{j'}$. So we are in exactly in the same situation as we were before; that is, by our hypothesis that there is a hyperentangled $v \in H$, $v$ must (in particular) satisfy Thm. 2.3 for $S = j$, and we can run through the argument of the previous paragraph, with $j$ in place of $i$, to get a contradiction with $\dim H_j < \dim H_{j'}$.

With Lemma in hand, the proof that $1 \Rightarrow 2$ is now straightforward. If $n = 2$, then the Lemma immediately yields that both factors of $H$ must have the same dimension. If $n > 2$, write $H$ as $H = H_i \otimes H_j \otimes H_{\{i,j\}'}$ isolating the $i$th and $j$th Hilbert space factors in $H$ and denoting the tensor product of the rest of the factors by $H_{\{i,j\}'}$. Again using the Lemma:

$$\dim H_i = \dim[H_j \otimes H_{\{i,j\}'}] = \dim H_j \dim H_{\{i,j\}'},$$

(26)

$$\dim H_j = \dim[H_i \otimes H_{\{i,j\}'}] = \dim H_i \dim H_{\{i,j\}'},$$

(27)

Since $\dim H_{\{i,j\}'} > 1$, there is no solution to Eqns. 26 and 27 when either $\dim H_i$ or $\dim H_j$ is finite. Since $i$ and $j$ were arbitrary, this shows that all of $H$’s factors must be infinite-dimensional.

2. $\Rightarrow$ 3. Observe first that since the Hilbert space $H$ is a complete metric space (by definition), its unit sphere is closed and defines a complete metric subspace of $H$. Now suppose we could establish the following:

Claim: For any $i = 1$ to $n$, the set of states satisfying Thm. 2.4 for $S = i$ is a countable intersection of dense open sets in the unit sphere of $H$.

If so, then since hyperentanglement amounts to satisfying Thm. 2.4 for all $S = 1$ to $n$ (cf. Thm. 3), the set of all hyperentangled states would also have to be a countable intersection of dense open sets in the unit sphere of $H$. But the Baire category theorem [18] asserts that in a complete metric space—such as the unit sphere of $H$—a countable intersection of dense open
sets must itself be dense! So we would have the desired conclusion if we could establish the **Claim**, which we now proceed to do.

Fix, once and for all, an \( i \) such that \( 1 \leq i \leq n \). Note that, by hypothesis, all of \( H \)'s factors have the same dimension, and infinite dimension when \( n > 2 \). So in either case we have \( \dim H_i = \dim H_i' \). Also fix (once and for all) an orthonormal basis \( \{ b_j \} \) for \( H_i' \). Let \( T \) be any finite subset of the indices that enumerate the vectors \( \{ b_j \} \). Remembering that each \( v \in H_i \otimes H_i' \) can be expanded (indeed uniquely, given \( \{ b_j \} \)) as \( v = \sum_j v_j \otimes b_j \), define the set of states:

\[
F(T) = \{ v \in H_i \otimes H_i' : ||v|| = 1 \text{ and } \{ v_j \}_{j \in T} \text{ is linearly independent} \}.
\]

(28)

Clearly a state \( v \in H_i \otimes H_i' \) satisfies Thm. 2.4 for \( S = i \) if and only if, for every finite subset \( T \) of the indices enumerating \( \{ b_j \} \), \( v \in F(T) \). Since there are at most countably many such finite subsets (even if the basis \( \{ b_j \} \) is infinite), it suffices for the **Claim** to show that for any finite \( T \), \( F(T) \) is both dense and open in the unit sphere of \( H \).

1) \( F(T) \) is norm dense in the unit sphere of \( H \). Choose any state \( w \in H_i \otimes H_i' \) and let

\[
w = \sum_{k \in K} c_k (x_k \otimes y_k)
\]

be a Schmidt decomposition for \( w \). Here the \( \{ c_k \} \) are coefficients (not necessarily all nonzero), \( \{ x_k \} \) and \( \{ y_k \} \) are orthonormal bases in \( H_i \) and \( H_i' \) respectively, and \( K \) is either a finite or countably infinite index set (depending on whether \( H_i \) and \( H_i' \) are both finite or infinite-dimensional). For density, we need to show that we can always find a state in \( F(T) \) arbitrarily close to \( w \). There are two cases.

(Case 1): \( c_k \) is nonzero for all \( k \in K \). In this case, \( w \) is itself in \( F(T) \). For the expansion of \( w \) in Eqn. 29 satisfies Thm. 2.4 for \( S = i \), taking the orthonormal basis of \( H_i' \) in that theorem to be \( \{ y_k \} \). However, the equivalences in Thm. 2 hold no matter what orthonormal basis for \( H_i' \) we choose, so that satisfaction of Thm. 2.4 for \( S = i \) relative to one such basis entails satisfaction relative to them all. Thus relative to the specific orthonormal basis \( \{ b_j \} \) in the definition of \( F(T) \), \( w = \sum_j w_j \otimes b_j \) must be such that the vectors \( \{ w_j \} \subseteq H_i \) are linearly independent, and in particular \( w \in F(T) \).
(Case 2): $c_k$ is zero for at least one $k \in K$. If so, consider the family of states of the form:

$$u = \frac{\sum_{k \in K} d_k (x_k \otimes y_k)}{\sum_{k \in K} |d_k|^2}$$

(30)

where $d_k = c_k$ if $c_k \neq 0$, $d_k \neq 0$ if $c_k = 0$, and the sequence $\{d_k : c_k = 0\}_{k \in K}$ is square-summable. (Note that since $\dim H_i = \dim H_i'$, there are indeed states in $H$ of form $u$.) By the argument of (Case 1), every state of form $u$ lies in $F(T)$. Moreover, we can make $u$ as close to $w$ as we like by making the coefficients $d_k$ corresponding to $c_k = 0$ arbitrarily small.

2) $F(T)$ is open in the unit sphere of $H$. Let $B$ denote the span in $H_\nu$ of $\{b_j\}_{j \in T}$. If $W$ is any subspace of $H_i$ of (finite) dimension $m = \dim(B) = |T|$, the projection $P_W \otimes P_B$ maps $H$ onto $W \otimes B$. Applied to $v = \sum_j v_j \otimes b_j \in H$, this projection yields $\sum_{j \in T} P_Wv_j \otimes b_j$. Let $F(W,T)$ denote the set of those vectors $v = \sum_j v_j \otimes b_j \in H$ for which $\{P_Wv_j\}_{j \in T}$ is linearly independent. This last implies that $\{v_j\}_{j \in T}$ is linearly independent as well, so we have $F(W,T) \cap X \subseteq F(T)$ where $X$ is the unit sphere of $H$. On the other hand, if $v \in F(T)$, then taking $V = \text{span}\{v_j\}_{j \in T}$, plainly $v \in F(V,T)$. Thus, $F(T) = \bigcup_{W} F(W,T) \cap X$ where the union is taken over all $m$-dimensional subspaces of $H_i$.

To show that $F(T)$ is open, it now suffices to show that each $F(W,T)$ is open. Let $U$ denote the collection of vectors $u \in W \otimes B$ of the form $\sum_{j \in T} u_j \otimes b_j$ with $\{u_j\}_{j \in T}$ linearly independent. Since $F(W,T) = (P_W \otimes P_B)^{-1}(U)$ and projections are continuous, our task further reduces to that of showing that $U$ is open in $W \otimes B$.

Choosing a basis $\{w_i\}_{i \in T}$ for $W$, each vector $u \in W \otimes B$ has a unique expansion as $u = \sum_{i,j \in T} c_{ij} w_i \otimes b_j$. Let $[u] = [c_{ij}]$ denote the $m \times m$ matrix consisting of the coefficients $c_{ij}$, and define $\text{det} : W \otimes B \to C$ by $\text{det}(u) := \text{det}[u]$. Note that $\text{det}$ is continuous (and in fact independent of the choice of $\{w_i\}_{i \in T}$). Notice also that if we express $u \in W \otimes B$ as $\sum_{j \in T} u_j \otimes b_j$, where $u_j = \sum_{i \in T} c_{ij} w_i$, then since $\text{det}(u) = 0$ just in case the columns of $[c_{ij}]$ are linearly dependent, $\text{det}(u) = 0$ exactly when the vectors $\{u_j\}_{j \in T}$ are linearly dependent. It follows that $U = \text{det}^{-1}(C \setminus \{0\})$, and since $\text{det}$ is continuous, $U$ is open in $W \otimes B$ as claimed. $QED.$

The above theorem lays bare the fundamental obstacle to states of three or more spin-1/2 particles being hyperentangled: hyperentangled states cannot live in finite-dimensional state spaces when there are more than two particles! (In fact, this was anticipated by Wagner [6], but his arguments do
not establish it in full generality.

Moreover, the density of hyperentangled states (when they exist) entails that such states need not differ from other entangled states simply by their degree of entanglement. Recently Shimony [12] has proposed the following definition for the degree of entanglement of a state \( v \in H \):

\[
E(v) = \frac{1}{2} \min \|v - w\|^2
\]

(31)

where \( w \) is a product state in \( H \) and the minimum is taken over the set of all product states. Shimony appears to propose this only for the case \( n = 2 \) but there is no reason not to adopt his definition in general, so that the minimum is taken over all \( n \)-fold product states. Shimony shows that if \( H \) is finite-dimensional \( E(v) \in (0, 1] \) (\( E(v) = 0 \) corresponding to no entanglement), whereas in the infinite-dimensional case no state actually attains an entanglement degree of 1. In any case, since hyperentangled states are dense in \( H \) (when its factors have the appropriate dimension), it follows that their degrees of entanglement lie dense in the interval \((0, 1)\), and in particular, that there are hyperentangled states with degrees of entanglement arbitrarily close to 0! This shows that hyperentanglement, despite the pervasive maximal correlations involved, should be viewed as a new kind of entanglement, and certainly not as a case of maximal entanglement.

6 Constructing Hyperentangled States

An unusual feature of our proof that hyperentangled states are dense is that in the most interesting case of \( n > 2 \), when the factor spaces of \( H \) must be infinite-dimensional, our proof is not constructive because it relies essentially on the Baire category theorem. Popescu has conjectured ([6], p. 32), by analogy with the vacuum state, that the ground state of \( n \) quantized coupled harmonic oscillators should provide an explicit example of (what we have been calling) a hyperentangled state. But while Wagner ([6], pp. 32-4) has confirmed this for \( n = 2 \), our own efforts to find suitable couplings in the case \( n > 2 \) have not been successful. This in turn raises the more general question of what conditions on the Hamiltonian governing a collection of \( n > 2 \) systems will guarantee that it will spend most of its time in a hyperentangled state.

On the other hand, with a little ingenuity it is perfectly possible to write down a state that is hyperentangled when \( n > 2 \). Expand a general state
$w \in H$ in terms of a product basis for $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$ as

$$w = \sum_{a,b,\ldots,z \in N} v_{a,b,\ldots,z} y_1^a \otimes y_2^b \otimes \cdots \otimes y_n^z$$

(32)

where $N$ is the natural numbers (including 0) and $[v_{a,b,\ldots,z}]$ are the elements of $w$’s square-summable and countable by countable by countable… ($n$ times) coefficient matrix. Using Thm. 2.4 for $S = 1, \ldots, n$ it is easy to see that $w$ will be hyperentangled exactly when the rows of $[v_{a,b,\ldots,z}]$ are linearly independent and the columns of $[v_{a,b,\ldots,z}]$ are linearly independent and the ‘files’ of $[v_{a,b,\ldots,z}]$ are linearly independent, etc.

Of course, it is trivial to find a square-summable coefficient matrix of this sort when $n = 2$. For $n > 2$, things get more complicated. We shall give two methods for constructing a suitable coefficient matrix in this case in the hope that one of them might be parlayed into an actual physical example of a hyperentangled state.

For the first method, start with the case $n = 3$. Fix an injection $j : N \times N \to N$ such that $j(a,b) \geq \max(a,b)$. (For example, we could choose $j(a,b) = 2^a 3^b$.) Consider those functions $h : N \times N \times N \to C$ such that

$$h(a,b,c) \neq 0 \iff \text{either } a = j(b,c) \text{ or } b = j(a,c) \text{ or } c = j(a,b).$$

(33)

We shall need to employ the following preliminary result.

**Lemma:** $c > j(a,b)$ (resp. $a > j(b,c)$, resp. $b > j(a,c)$) implies $h(a,b,c) = 0$.

**Proof.** Suppose that $c > j(a,b)$ and $h(a,b,c) \neq 0$. Then $c \neq j(a,b)$, so either $a = j(b,c)$ or $b = j(a,c)$. But if $a = j(b,c)$, then $c > j(j(b,c),b) \geq \max(j(b,c),b) \geq \max(b,c)$, a contradiction since obviously $c \leq \max(b,c)$. Similarly, if $b = j(a,c)$, then $c > j(a,j(a,c)) \geq \max(a,j(a,c)) \geq \max(a,\max(a,c)) = \max(a,c)$ again a contradiction. So $h(a,b,c) = 0$, as desired. (By symmetry, the same conclusion follows if $a > j(b,c)$ or $b > j(a,c)$.) QED.

We now write $v_{a,b,\cdot}$ for the row vector $(h(a,b,c))_{c \in N}$ (and similarly $v_{\cdot,b,c}$ for the column vector $(h(a,b,c))_{a \in N}$ and $v_{a,\cdot,c}$ for the ‘file’ vector $(h(a,b,c))_{b \in N}$).

**Theorem 5** \{v_{a,b,\cdot}\}_{a,b \in N} (resp. \{v_{\cdot,b,c}\}_{b,c \in N}, resp. \{v_{a,\cdot,c}\}_{a,c \in N}) constitutes a linearly independent set.
Proof. Suppose (contrary to hypothesis) that \( \sum_{i=1}^{m} s_i v_{a_i, b_i, c_i} = 0 \) with all coefficients \( s_i \neq 0 \) and all pairs \((a_i, b_i)\) distinct. Then for all \( k \in \mathbb{N}, \)
\[ \sum_{i=1}^{m} s_i h(a_i, b_i, k) = 0. \]
In particular, setting \( k' = \max_{i=1}^{m} j(a_i, b_i) \) we have
\[ \sum_{i=1}^{m} s_i h(a_i, b_i, k') = 0. \] (34)
But by the distinctness of the \((a_i, b_i)\), the injectivity of the \(j\) function and the Lemma above, the sum on the left-hand side of Eqn. 34 involves exactly one non-zero term, a contradiction which proves the linear independence of \( \{v_{a,b}\}_{a,b \in \mathbb{N}} \). (The linear independence of each of the sets \( \{v_{.,b,c}\}_{b,c \in \mathbb{N}} \) and \( \{v_{a,.,c}\}_{a,c \in \mathbb{N}} \) follows by symmetry.). QED.

This little theorem supplies a recipe for building infinite 3-dimensional matrices with linearly independent rows, linearly independent columns, and linearly independent files. Such matrices must have non-zero values at position \((a, b, c)\) exactly when either \( a = j(b, c), b = j(a, c) \) or \( c = j(a, b) \), but then we have the freedom to choose as we please any non-zero value for \( h(a, b, c) \). In particular, we can easily arrange for the square-summability of the entries of the matrix \([v_{a,b,c}]\).

(For a bijection \( j : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) such that \( j(a, b) \geq \max(a, b) \), \( \{v_{a,b}\}_{a,b \in \mathbb{N}} \) (resp. \( \{v_{.,b,c}\}_{b,c \in \mathbb{N}} \), resp. \( \{v_{a,.,c}\}_{a,c \in \mathbb{N}} \) actually constitutes a Hilbert space basis. Indeed the set of vectors with finitely many non-zero entries lies dense in Hilbert space and we can express any such vector as a linear combination of vectors from \( \{v_{a,b}\}_{a,b \in \mathbb{N}} \) (resp. \( \{v_{.,b,c}\}_{b,c \in \mathbb{N}} \), resp. \( \{v_{a,.,c}\}_{a,c \in \mathbb{N}} \). Thus, consider any vector \((q_i)\) with last non-zero entry \( q_I \). Then the vector \((q_i) - (q_I/h(j^{-1}(I), I))v_{j^{-1}(I)} \) has its last non-zero entry before position \( I \), and we may proceed inductively. For an explicit example of a bijective \( j \) function, define \( j(a, b) \) so it has binary expansion \( \beta_n \alpha_n \ldots \beta_1 \alpha_1 \) when \( a \) has binary expansion \( \alpha_n \ldots \alpha_1 \) and \( b \) has binary expansion \( \beta_n \ldots \beta_1 \).

The arguments above extend \textit{mutatis mutandi} to provide higher dimensional matrices with the analogous property. To get a 4-dimensional matrix, for example, we should consider functions \( h \) where \( h(a, b, c, d) \neq 0 \) if and only if either \( a = j(b, j(c, d)), b = j(a, j(c, d)), c = j(a, j(b, d)) \) or \( d = j(a, j(b, c)) \). We may even construct analogous infinite-dimensional matrices of the required sort, with entries indexed by finitely non-zero natural number sequences.

We turn now to describing a second independent method of constructing coefficient matrices for hyperentangled states. Let \( v \) denote any \( p \times p \times p \)-
matrix with complex entries, viewing \(v\) as a function \(v : \{1, \ldots, p\}^3 \rightarrow \mathbb{C}\). Write \(v_{a,b,c}\) for the vector \((v(a,b,c))^p_{c=1}\) (and similarly \(v_{b,c}\) for the vector \((v(a,b,c))^p_{a=1}\) and \(v_{a,c}\) for the vector \((v(a,b,c))^p_{b=1}\)). Ultimately we shall use such a matrix \(v\) as the ‘seed’ from which to a ‘grow’ a countable by countable by countable matrix with the required properties. Again, we need a preliminary result to make this construction go through.

**Theorem 6** Let \(v\) denote a \(p \times p \times p\) matrix with complex entries. Fix \(m \leq p\) and assume the linear independence of the vector sets \(\{v_{a,b,c}\}_{1 \leq a,b \leq m}\), \(\{v_{a,c}\}_{1 \leq a \leq m}\) and \(\{v_{b,c}\}_{1 \leq b,c \leq m}\). Now set \(p' = p^2 + p - m^2\). Then the matrix \(v\) has an extension to a \(p' \times p' \times p'\)-matrix \(v'\) with

1. linearly independent vector sets \(\{v'_{a,b}\}_{1 \leq a,b \leq p}\), \(\{v'_{a,c}\}_{1 \leq a,c \leq p}\) and \(\{v'_{b,c}\}_{1 \leq b,c \leq p}\);
2. \(v'(a,b,c) = 0\) whenever \(a,b \leq m\) and \(c > m\), \(a,c \leq m\) and \(b > m\) or \(b,c \leq m\) and \(a > m\).

Moreover we can arrange to have the sum of the squares of the absolute values of the new entries equal to any \(\epsilon > 0\).

**Proof.** First we arrange for the vectors in \(\{v'_{a,b,c}\}_{1 \leq a,b,c \leq p}\) to extend the corresponding vectors in \(\{v_{a,b,c}\}_{1 \leq a,b \leq m}\). For the sake of condition 2. we must certainly stipulate that \(v'(a,b,c) = 0\) whenever \(a,b \leq m\) and \(c > m\). This leaves us with \(p^2 - m^2\) length \(p\) vectors to extend, namely

\[
\{v_{a,b,c} : 1 \leq a,b \leq p\} \quad \text{and} \quad a,b \text{ are not both } \leq m, \tag{35}
\]

and we need to extend each of these to a length \(p + p^2 - m^2\) vector. Thus we must append \(p^2 - m^2\) entries to each vector. Do this by simply appending the \(p^2 - m^2\) standard basis vectors in \(\mathbb{C}^{p^2 - m^2}\) multiplied by \(\sqrt{\epsilon/(3(p^2 - m^2))}\) (in any order). This makes the vector set \(\{v'_{a,b,c}\}_{1 \leq a,b \leq p}\) linearly independent because we could project a linear dependence to a linear dependence either among the standard basis vectors in \(\mathbb{C}^{p^2 - m^2}\) or among the vectors \(\{v_{a,b,c}\}_{1 \leq a,b \leq m}\).

To finish, we repeat the procedure so as to extend likewise the vectors \(\{v_{a,c}\}_{1 \leq a,c \leq p}\) and \(\{v_{b,c}\}_{1 \leq b,c \leq p}\). Finally, we may set the values of the remaining entries of \(v'\) to 0. \(QED.\)

Using this theorem, we can now build up the coefficient matrix for an \(n = 3\) hyperentangled state as follows. Begin, say, with \(p = 2\), \(m = 1\) and \(v\)
any $2 \times 2 \times 2$-matrix with $v(1,1,1) \neq 0$. Now iterate the process described in Thm. 6. (Each time the new $p$ equals the old $p'$ and the new $m$ equals the old $p$.) Observe that each row, column and file has only finitely many nonzero entries, since it stabilizes at some finite stage of the process. Thus, we obtain an infinite 3-dimensional matrix with linearly independent row sets, column sets and file sets; any potential dependence involves just finitely many vectors, all of these stable at some finite stage of the construction, so a dependence would contradict the theorem.

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