Triangleland: I. Classical dynamics with exchange of relative angular momentum

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Abstract
In Euclidean relational particle mechanics, only relative times, relative angles
and relative separations are meaningful. Barbour–Bertotti (1982 Proc. R. Soc.
Lond. A 382 295) theory is of this form and can be viewed as a recovery of (a
portion of) Newtonian mechanics from relational premises. This is of interest
in the absolute versus relative motion debate and also shares a number of
features with the geometrodynamical formulation of general relativity, making
it suitable for some modelling of the problem of time in quantum gravity. I
also study similarity relational particle mechanics (‘dynamics of pure shape’),
in which only relative times, relative angles and ratios of relative separations
are meaningful. This I consider first as it is simpler, particularly in 1 and 2D,
for which the configuration space geometry turns out to be well known, e.g.
$S^2$ for the ‘triangleland’ (3-particle) case that I consider in detail. Second,
the similarity model occurs as a sub-model within the Euclidean model: that
admits a shape-scale split. For harmonic oscillator like potentials, similarity
triangleland model turns out to have the same mathematics as a family of rigid
rotor problems, while the Euclidean case turns out to have parallels with the
Kepler–Coulomb problem in spherical and parabolic coordinates. Previous
work on relational mechanics covered cases where the constituent subsystems
do not exchange relative angular momentum, which is a simplifying (but in
some ways undesirable) feature paralleling centrality in ordinary mechanics.
In this paper I lift this restriction. In each case I reduce the relational problem to
a standard one, thus obtain various exact, asymptotic and numerical solutions,
and then recast these into the original mechanical variables for physical
interpretation.

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1. Introduction

The traditional formulation of mechanics [1] has rested on absolute space and absolute time. However, Leibniz [2] and Mach [3] raised philosophically well-motivated relational objections to these foundations. (See, e.g. [4–7] for further discussion of this ‘absolute versus relative motion debate’.) It is reasonable to consider whether relational principles apply to physics as a whole. However, for many years no means were known by which physical theories could be built along these lines. Then Barbour and Bertotti [8] and Barbour [9] found some relational particle mechanics (Reissner’s earlier theory—subsequently rediscovered by Schrödinger and by Barbour–Bertotti—[6, 10] is incompatible with mass-anisotropy experiments.)

Note that the present paper uses the word ‘relational’ in Barbour’s sense. This is worth some discussion because Rovelli [11] uses the same word, but each (Rovelli and Barbour) take it to mean something different. In outline, Rovelli’s classical relationalism involves objects not being located in spacetime but being located with respect to each other. Rovelli also has a quantum relationalism that quantum states for a particular subsystem only make sense with respect to another subsystem. He then speculates that these two relationalisms of his might be related (p 157 of the online version of [11]: ‘Is there a connection . . . . This is of course very vague, and might lead nowhere, but I find the idea intriguing.’ Barbour on the other hand, has specific spatial and temporal relationalism postulates that embody particular ideas of Mach (that time is to be abstracted from change) and Leibniz (the identity of indiscernibles), each of which is sharply implemented by particular mathematics at the classical level, as follows.

A physical theory is temporally relational if there is no meaningful primary notion of time for the whole system thereby described (e.g. the universe) [8, 12, 13]. This is implemented by using actions that are manifestly reparametrization invariant while also being free of extraneous time-related variables (such as Newtonian time or general relativity (GR)’s lapse). This reparametrization invariance then directly produces primary constraints quadratic in the momenta (such as the energy constraint of mechanics or the Hamiltonian constraint of GR).

A physical theory is configurationally relational if a certain group $G$ of transformations that act on the theory’s configuration space $Q$ are physically meaningless [8, 12–14]. This can be implemented by such as using arbitrary-$G$-frame-corrected quantities rather than ‘bare’ $Q$-configurations. For, despite this augmenting $Q$ to the principal bundle $P(Q, G)$, variation with respect to each adjoined independent auxiliary $G$-variable produces a secondary constraint linear in the momenta (e.g. the GR momentum constraint arises as a vectorial collection of three such constraints) which removes one $G$ degree of freedom and one redundant degree of freedom among the $Q$ variables. Thus one ends up dealing with the desired reduced configuration space—the quotient space $Q/G$. Configurational relationalism includes as subcases both spatial relationalism and internal relationalism (in the sense of gauge theory). Configurational relationalism can also be implemented, at least in some cases [13, 17], by working directly on reduced configuration space. (Reissner’s theory was formulated along these direct lines, while I also found that the theories [8, 9] can be arrived at 1- and 2D from a direct implementation [13].)

One difference between Barbour and Rovelli’s approaches is as follows (another is mentioned in subsection 2.1). In section 2.4.4 of [11], Rovelli discusses ‘meanings of time’ and identifies time in Newtonian physics as a metric line that exists alongside the configuration space of the system, nowhere reflecting Barbour’s starting point that time is derived from change alongside consideration of the configuration spaces alone, on which

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2 This is my own passive, mathematician’s implementation, whereas Barbour thinks about this in active terms that physicists sometimes use. This difference in thinking does not, however, lead to any tangible discrepancies in the material of this paper.
Jacobi-type variational principles are defined. This then takes Barbour straight to the situation in which no variable is distinguished as time at the kinematic level; while Rovelli characterizes this as an essential difference between non-relativistic and relativistic mechanics in his approach, in Barbour’s approach this distinction has dissolved. The recovery of Newtonian dynamics from relational particle models also has features by which such models are closer in objective structure to GR than Rovelli generally holds nonrelativistic mechanics models to be in [11]. Thereby, relational particle models provide tractable models outside the scheme in [11].

As far as I know, Barbour and collaborators have not as yet reached a specific notion of quantum relationalism. Rovelli’s own idea of relationalism at the quantum level above mentioned does share a number of features with record-theoretic positions that Barbour and I have previously advocated [18–20]. It is not as yet known whether such records-theoretic positions really have any substantial conceptual or technical ties to Barbour’s notion of classical relationalism, any more than Rovelli knows whether his similar quantum position to be tied to his own classical relationalism (as above quoted).

While Barbour’s relationalism has the virtue of specifically being in line with Leibniz and Mach (whereby alongside having attained a concrete mathematical implementation of these ideas it is definitely of interest to the foundations of physics and theoretical physics and so deserves full investigation), I would not dismiss the possibility that Rovelli’s relationalism is also in line with other (interpretations of) Leibniz, Mach or other such historical figures, and, in any case, has original value and is useful in a major quantum gravity program (loop quantum gravity).

The current paper and its sequel [21] concern technical advances with examples specifically of Barbour’s relational program—I subsequently use ‘relational’ in this sense except where I specifically say otherwise. Section 2 presents Euclidean relational particle mechanics [5, 8, 13, 17, 22–26] (also referred to as scaled models) and similarity relational particle mechanics [9, 13, 17, 25, 27] (also referred to as scalefree models) and further motivates the relational scheme by arguing that some (conformal)geometrodynamical formulations of GR can be regarded as arising therein too. Section 3 explains the configuration space structure of relational particle mechanics which has further parallels with GR. All these parallels are eventually relevant as regards relational particle mechanics furnishing useful toy models [18–20, 26, 27, 29–32] for the problem of time in quantum gravity [29, 33] and other issues in the foundations of quantum cosmology [19, 34]. Use of relational particle mechanics in both the absolute versus relative motion debate and the study of conceptual strategies suggested towards resolving the problem of time in quantum gravity would benefit from having a good working understanding of explicit examples of relational particle mechanics, which is the subject of this paper at the classical level.

Relational particle models concern \( N \) particles in dimension \( d \). As the general configuration for \( d = 3 \) is an \( N \)-haedron, I term the 3D \( N \)-particle relational particle model \( N \)-haedronland. Likewise, I term the 2D \( N \)-particle relational particle model \( N \)-a-gonland, and the 1D one \( N \)-stop metroland (as in urban public transport maps). The special 3-a-gonland considered as the principal example in this paper I term triangleland.

Noting that scalefree \( N \)-stop metroland and \( N \)-a-gonland have fairly standard geometry (\( \mathbb{S}^{N-2} \) and \( \mathbb{C}P^{N-2} \), respectively [13]) permits explicit reductions [13, 17] and subsequent availability of useful coordinate systems and methods of mathematical physics. For scalefree triangleland, \( \mathbb{C}P^{1} = \mathbb{S}^{2} \), so one has ‘twice as many techniques’, so in this paper I choose to study this case. Scaled triangleland also permits explicit reduction and its configuration space in shape-scale variables takes the form \( C(\mathbb{S}^{2}) \), which also makes for tractable and interesting
explicit examples, where the cone $C(X)$ over a space $X$ is $\mathbb{R}_+ \times X \cup \emptyset$, the special cone-point or ‘apex’.

In section 4 I give Euler–Lagrange equations for scalefree $N$-stop metroland and $N$-agonland and give further specific forms for the exceptional triangleland case. I then consider examples with potentials that are independent of the relative angle $\Phi^3$ as well as harder ones that depend on this. (The former is a substantial simplification [17]—in close analogy with centrality in ordinary mechanics—but it is the harder latter case that is relevant to various problem of time schemes—semiclassical emergent time and possibly the semblance of dynamics in timeless records schemes discussed in paper II [21].) In particular, I consider the general case of harmonic oscillator like potentials between all particles, looking at the ‘special’ ($\Phi$-independent) and ‘very special’ (constant) subcases within as well as the general small and large asymptotic behaviour, and recast this in terms of the problem’s ‘original variables’—mass weighted relative Jacobi variables $\iota_1, \iota_2$ that still contain an absolute orientation. Identifying the ‘very special’ problem’s mathematics as corresponding to the linear rigid rotor, the ‘special’ problem’s as that with additionally a background homogeneous electric field in the axial direction and the general problem’s likewise but now with a general direction, is valuable in this paper and in subsequent work at the QM level in paper II. In section 5 I use that a rotation (or, equivalently, a normal modes construction) maps the general case to the special case for this particular problem. Thus I can get as far in solving for the general case as I can with the ‘special’ case.

In section 6, I consider scaled triangleland’s Euler–Lagrange equations in terms of the straightforwardly relational variables $(\iota_1, \iota_2, \Phi)$, the useful $(I_1, I_2, \Phi)$ coordinates (that turn out to be parabolic coordinates), the $C(\mathbb{CP}^1)$ presentation shape-scale coordinates $(I, R, \Phi)$ and the $C(S^2)$ presentation shape-scale coordinates $(I, \Theta, \Phi)$. I then exactly solve the ‘special’ case of multi-harmonic oscillator like potential for scaled triangleland, by mapping it in $(I_1, I_2, \Phi)$ coordinates to a close analogue of the Kepler–Coulomb problem (which move remains useful at the quantum level [37]). I then give Euclidean relational particle mechanics’ own rotation/normal modes construction, whereby my having obtained the special solution enables me to also obtain the general solution, albeit this paper only has room for presenting the special solution and its physical interpretation.

I conclude in section 7, including an outline of further promising relational particle mechanics examples. Paper II considers this paper’s similarity models at the quantum level and [37] likewise for this paper’s Euclidean models. Interesting problem of time in quantum gravity applications of these models will be developed in yet further papers [31].

2. Examples of relational theories

2.1. Euclidean relational particle mechanics

In Euclidean, or scaled, relational particle mechanics [5, 8, 19, 23], only relative times, relative angles and relative separations are meaningful. For example for 3-particles in dimension $d > 1$, Euclidean relational particle mechanics is a dynamics of the triangle that the 3-particles form. Euclidean relational particle mechanics was originally [8] conceived for $\mathbb{Q} = \mathbb{Q}(N, d) = \mathbb{R}^{Nd}$ the positions $q_{i\alpha}$ of $N$ particles in $d$-dimensional space with $G$ the $d$-dimensional Euclidean

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3 The notation for this and the following paragraph is as follows. $R_i, i = 1, 2$, are relative Jacobi coordinates [36]. $\Phi$ is the angle between these. By $\iota_i$ being mass-weighted, I mean that $\iota_i = \sqrt{\mu_i R_i}$, where $\mu_i$ are the particle (cluster) masses associated with $R_i$ [36]. $\iota_i = \iota_i I$, the magnitude of $\iota_i$, and $I_i = \iota_i^2 = \mu_i R_i^2$, the $i$th Jacobi (barycentric) partial moment of inertia. $R$ is the simple ratio variable $\iota_1/\iota_2$ and $\Theta = 2 \arctan \mathcal{R}$, which turns out to geometrically be the azimuthal spherical angle; see also figure 1 and sections 2–3 for further interpretation and depictions of these.
group $\text{Eucl}(d)$ of translations, $\text{Tr}(d)$, and rotations, $\text{Rot}(d)$. However, eliminating $\text{Tr}(d)$ is trivial and produces a theory of essentially the same form as the original if relative Jacobi coordinates $R_{\alpha \alpha}$ are employed [26], so I take that as my starting position. Relative Jacobi coordinates are inter-particle (cluster) separations chosen such that the kinetic term is diagonal [36]; thus I take my $Q$ to be $\mathbb{R}(N, d)$, the space of relative separations (this is $\mathbb{R}^{nd}$), and $G$ to be $\text{Rot}(d)$.

Then configurational relationalism takes the form that one is to construct one’s action using the arbitrary $\text{Rot}(d)$ frame expressions $\circ R_{\alpha \alpha} \equiv R_{\alpha \alpha} - \epsilon_{\alpha \beta \gamma} B_\beta R_{\gamma \gamma}$ rather than ‘bare’ $R_{\alpha \alpha}$.

The action that one builds is of Jacobi type [38] so as to implement temporal relationalism,

$$S[R_{\alpha \alpha}, R_{\alpha \alpha}, B_\alpha] = 2 \int d\lambda \sqrt{T[U + E]}.$$  \hspace{1cm} (1)

Here the kinetic term $T(R_{\alpha \alpha}, R_{\alpha \alpha}, B_\alpha) = M^{ij\alpha\beta} \circ R_{\alpha \alpha} \circ R_{ij\beta} / 2$ is homogeneous quadratic in the velocities. $U$ is minus the potential energy $V$ which is a function of $\sqrt{R \cdot R}$ alone, and $E$ is the total energy of the closed system/universe; hitherto this has been taken to have a fixed value (but should it: no observer would know it exactly?) (Note that each such action $S_j = 2 \int d\lambda \sqrt{T[E + U]}$ is indeed equivalent to the more well-known Euler–Lagrange actions $S = \int dt[T - V]$).

From (1), the conjugate momenta are then

$$p^{\alpha \beta} = M^{ij\alpha\beta} \circ R_{ij\beta} \quad \text{for} \quad * R_{\alpha \alpha} \equiv R_{\alpha \alpha} - \epsilon_{\alpha \beta \gamma} B_\beta R_{\gamma \gamma}$$  \hspace{1cm} (3)

and $* \equiv \sqrt{(U + E) / T} = d/dt$ for $t$ the emergent ‘Leibniz–Mach–Barbour’ time that coincides here with Newtonian time. This object introduced, another difference between Barbour’s scheme and Rovelli’s can be pointed out. In Rovelli’s scheme, one clock/timesstandard is much as good as another (at the conceptual level rather than what is convenient or accurate), while in Barbour’s scheme there emerges a Leibniz–Mach–Barbour timesstandard to which everything in the universe contributes (an ‘ephemeris’ timesstandard [23, 30, 40]). This choice of time is distinguished by how it substantially simplifies both the above momentum–velocity relations and the Euler–Lagrange equations, and it amounts to an emergent recovery of other notions of time like Newtonian time here in mechanics, or proper time or cosmic time in section 2.3, and it has some parallels with the actual ephemeris time in official use in the first half of the 20th century [39], which is based on the totality of the motions of the objects in the solar system.

Reparametrization invariance implies [41] that these must obey at least one primary constraint; here there is one, which takes the form of an energy constraint

$$H \equiv N_{ij\alpha\beta} p^{\alpha \beta} p^{ij\gamma} / 2 + V = E,$$  \hspace{1cm} (4)

4 Lower-case Greek letters are spatial indices. Capital indices are used for the $N$ particle position coordinates, lower-case indices for $n = N - 1$ coordinates describing relative particle (cluster) separations, barred and tilded lower-case indices to take values 1 to $n - 1$ values and hatted lower-case indices to take values 1 to $n - 2$. The dot denotes $d/d\lambda$ for $\lambda$ a label-time parameter that has no physical meaning since (1) is reparametrization-invariant. $B_\alpha$ generates the rotations. As well as the obvious 3D case, the 1 and 2D cases are incorporable into this form as follows. Take $B_\alpha = (0, 0, B)$ in 2D. Then $L^\alpha$ has just one component that is nontrivially zero. Take furthermore $B = 0$ in 1D. Then there is no $L^\alpha$ constraint at all. $M^{\mu\nu\alpha\beta}$ is the kinetic metric $\mu_1 \mu_2 \mu_3 \mu_4 / m_1 m_2 m_3 m_4$, where $\mu_1$ are the Jacobi masses [36]. For example for this paper’s triangleland example, these are $\mu_1 = m_2 m_3 / (m_2 + m_3)$ and $\mu_2 = m_1 (m_2 + m_3) / (m_1 + m_2 + m_3)$ for $m_1$ the particle masses.) $N_{ij\alpha\beta}$ is the inverse of this array.
to which the momenta contribute quadratically but not linearly. Variation with respect to $B_\alpha$ yields as a secondary constraint the zero total angular momentum constraint

$$L^\alpha \equiv \epsilon^\alpha_\beta_\gamma R_\beta P_\gamma = 0,$$

which is linear in the momenta and is interpretable as the physical content of the theory being in relative separations and relative angles and not in absolute angles.

Euclidean relational particle mechanics is Leibnizian/Machian in form, and yet is in agreement with a subset of Newtonian mechanics—the zero total angular momentum universes. The recovery from relationalism of many of the results previously obtained by standard absolutist physics and the finding of similar mathematical structures along both routes (at least in the simpler cases) is an interesting reconciliation as regards the extent to which we have not been prejudiced by studying absolutist physics. This paper and [21] show that this trend extends further, albeit the mathematics arising from similarity relational particle mechanics is distinct from that in the ordinary central force problem in having restricted, unusual potentials inherited from the scale invariance). Sections 2.3 and 2.4 recollect that a similar trend is also present in GR.

2.2. Similarity relational particle mechanics

In similarity, or scalefree, relational particle mechanics [9, 17, 27], only relative times, relative angles and ratios of relative separations are meaningful. That is, it is a dynamics of shape excluding size: a dynamics of pure shape. In the case of 3-particles in dimension $d > 1$, similarity relational particle mechanics is the dynamics of the shape of the triangle that the 3-particles form. $G$ is now augmented by the dilations $\text{Dil}(d) = \text{Dil}$ to form the Similarity group $\text{Sim}(d)$, so that $\circ R_{i\alpha}$ now takes the form $\circ R_{i\alpha} \equiv R_{i\alpha} = \epsilon_\alpha_\rho_\gamma B_\rho P_\gamma + C R_{i\alpha}$, where $C$ generates the dilations. Noting that the ‘banal conformal transformation’, $^5$

$$T \rightarrow T = \Omega^2 T, \quad E + U \rightarrow E + U = \{E - V\}/\Omega^2,$$

leaves the Jacobi action invariant, the most natural presentation [17] for the action is

$$S[R_{i\alpha}, R_{i\alpha}, B_\alpha, C] = 2 \int \sqrt{T[U + E]}$$

with $T(R_{i\alpha}, R_{i\alpha}, B_\alpha, C) = M^{ijkl} \circ R_{i\alpha} \circ R_{j\beta}/2I$ for $I$ the barycentric moment of inertia of the system, whereupon $V$ is a function only of manifestly scale-invariant ratios of $\sqrt{E_\alpha - E_\beta}$ (i.e. homogeneous of degree zero) and $E$ comes unweighted. (Barbour’s original presentation [9], in which the potential is homogeneous of degree $-2$ and an energy $E$ cannot be added on to this (but $E/I$ can), is related to this one by use of $\Omega^2 = I$, i.e. by not dividing through by the moment of inertia in the first place.) Then (in the conformally natural presentation I use above) the conjugate momenta are given by (3) but divided by $I$ and containing the $*$ corresponding to the new $\circ$. There are again primary and secondary constraints respectively of form (4) with the first term multiplied by $I$ and (5), as well as a secondary constraint from variation with respect to $C$

$$D \equiv R_{i\alpha} P^\alpha = 0,$$

$^5$ Performing this transformation clearly leaves invariant product-type actions and so of Jacobi-type actions (1) and its similarity relational particle mechanics counterpart and of the reduced actions that follow from these in section 3, and of their GR counterparts such as the Baierlein–Sharp–Wheeler action [42], (9) and (13) and its variants. $T$ and $E + U$ scale compensatingly and then deduce from $^* \equiv \sqrt{(E + U)/T}$ that $*$ scales as $* \rightarrow \epsilon_\alpha_\alpha = \Omega^{-2}$. (Euler–Lagrange or Arnowitt–Deser–Misner type actions have this invariance too but its presence therein is less obvious to spot [43].) I term each particular choice of $\Omega$ a ‘banal conformal representation’. Note: classically Euler–Lagrange equations are invariant [43] so applying a banal conformal transformation makes no difference and it is just a means of computational convenience in some cases. But retaining this lack of difference at the quantum level has implications [21, 37, 43].
which is also linear in the momenta. This is the dilational (or Euler) constraint—it says that the dilational momentum of the whole system is zero so that the physical content of the theory is not in relative separations but in ratios of relative separations (relative angles already being functions of ratios, they are unaffected).

2.3. Relational formulation of geometrodynamics

Important further motivation for the study of relationalism and relational particle mechanics is that general relativity (GR) can also be formulated as a relational theory. Begin by recollecting that as well as being a spacetime theory, GR can be studied as a dynamics obtained by splitting spacetime with respect to a family of spatial hypersurfaces [45]. However, answering a question of Wheeler [46], this dynamics can be taken to follow from first principles of its own [12, 15, 47]. And one of the two known such sets of first principles are relational first principles, with $Q = \text{Riem}(\Sigma, 3)$—the space of positive-definite 3-metrics on some fixed topology $\Sigma$ which I take to be a compact without boundary one for simplicity—and $G = \text{Diff}(\Sigma, 3)$ the diffeomorphisms on $\Sigma$ [12, 14, 15]. The arbitrary Diff($\Sigma, 3$) frame expressions are $\epsilon h_{\mu\nu} = \dot{h}_{\mu\nu} - \xi \epsilon h_{\mu\nu}$ rather than ‘bare’ $\dot{h}_{\mu\nu}$.6

The action that one builds so as to explicitly implement temporal relationalism is [12, 14, 15, 49]

$$S_{\text{GR}}[h_{\alpha\beta}, \dot{h}_{\alpha\beta}, \dot{F}_\mu] = 2 \int d\lambda \int d^3x \sqrt{h} \left( \frac{1}{4} M^{\mu\nu\rho\sigma} \circ h_{\mu\nu} \circ h_{\rho\sigma} \right)$$

for $\Sigma_{\text{GR}}[x^\alpha, h_{\mu\nu}, \dot{h}_{\mu\nu}; \dot{F}_\mu] = \frac{1}{4} M^{\mu\nu\rho\sigma} \circ h_{\mu\nu} \circ h_{\rho\sigma}$, (9)

which bears many similarities to the better-known Baierlein–Sharp–Wheeler [42] action.7 Also note that one does not need to assume the GR form of the kinetic metric or potential; relational postulates plus a few simplicities give this since the Dirac procedure [41] prevents most other likewise simple choices of kinetic term $T$ from working [12, 14, 15, 50, 51]. Yet further motivation is that (1) configurational relationalism is closely related [14, 44] to certain formulations of gauge theory. (2) The above relational formulation of GR is furthermore robust to the inclusion of a sufficiently full set of fundamental matter sources so as to describe nature [14–16, 52, 53].

Then the conjugate momenta are

$$\pi^{\mu\nu} = \sqrt{h} M^{\nu\rho\sigma} \ast h_{\rho\sigma}$$

for $\ast \equiv \sqrt{\left[ \text{Ric}(h) - 2\Lambda \right]} / T_{\text{GR}}$. Equation (9) being reparametrization invariant, there must likewise be at least one primary constraint, which is in this case the GR Hamiltonian constraint

$$\mathcal{H} \equiv \frac{1}{\sqrt{h}} N_{\mu\nu\rho\sigma} \pi^{\mu\nu} \pi^{\rho\sigma} - \sqrt{h} [\text{Ric}(h) - 2\Lambda] = 0$$

6 The dot now denotes $\partial / \partial \lambda$. $F_\mu$ generates Diff($\Sigma, 3$). $\xi$ is the Lie derivative with respect to the vector field $F_\mu$. $h$, $D_\mu$ and $\text{Ric}(h)$ are the determinant, covariant derivative and Ricci scalar associated with $h_{\mu\nu}$. $\Lambda$ is the cosmological constant. $M^{\mu\nu\rho\sigma} = h^{\mu\sigma} h^{\nu\rho} - h^{\mu\rho} h^{\nu\sigma}$ is the kinetic supermetric of GR, with determinant $M$ and inverse $N_{\mu\nu\rho\sigma} = h_{\mu\rho} h_{\nu\sigma} - h_{\mu\sigma} h_{\nu\rho} / 2$ (which is the undensitized version of the DeWitt supermetric [48]).

7 The difference is that the Baierlein–Sharp–Wheeler action contains the shift, $\beta_\mu$, while the action (9) contains the velocity associated with the frame variable $F_\mu$, which is such that $F_\mu = \beta_\mu$. Both of these actions are free of extraneous time-related variables (unlike the Arnowitt–Deser–Misner action which contains a such—the lapse—multiplier elimination of which produces the Baierlein–Sharp–Wheeler action). The above-mentioned difference does not affect the outcome of the variational procedure [44]; however, it does make (9) homogeneous quadratic in $\partial / \partial \lambda$, so that $\lambda$’s cancel out of $d\lambda \sqrt{T}$ so that it and not the Baierlein–Sharp–Wheeler action is manifestly reparametrization invariant as well as free of extraneous time-related variables, i.e. temporally relational as defined on page 1.
to which the momenta contribute quadratically but not linearly. Variation with respect to $F_\mu$ yields as a secondary constraint the GR momentum constraint

$$H_\mu \equiv -2D_\nu \pi^{\nu\mu} = 0,$$

(12)

which is linear in the gravitational momenta and interpretable as GR being more than just a theory of dynamical 3-metrics: the physical information is in the underlying geometry and not in the allocation of points to that geometry. While, the purely quadratic nature of $H$ leads to $\dot{H} = E$ and not $i\hbar \partial \Psi / \partial T$ for some notion of time $T$. The $H$ of relational particle mechanics also has this feature and so itself manifests the problem of time (section 2.1); it is a reasonable model for this in that a number of the conceptual strategies subsequently suggested for GR have nontrivial counterparts for relational particle mechanics.

The presence of square roots in the above actions so as to implement manifest temporal relationalism has caused some concern with referees due to square roots causing substantial difficulties at the quantum level. In this respect, I note that the above square roots occur at the classical level in the Lagrangian formulation. However, there are no square roots in the Hamiltonians resulting from such actions, and it is these that I promote to quantum equations in [21], so that there are no square roots in the quantum equations, and so this work encounters none of the problems associated with handling square roots at the quantum level.

2.4. Relational formulation of conformogeometrodynamics

This has a number of ties with the work of Lichnerowicz [54] and York [55] on maximal and constant mean curvature spatial slices respectively, which is important in numerical relativity [56]. A preliminary action is

$$S^{GR} = \int d\lambda \int d^3 x \sqrt{h} \phi^6 \sqrt{\mathcal{T}^{GR} \{ \phi - 4 \{ \text{Ric}(h) - 8D^2 \phi / \phi \} \}},$$

(13)

where

$$\mathcal{T}^{GR}(x^\alpha, h_{\mu\nu}, \phi, h_{\mu\nu}, \dot{h}_{\mu\nu}; \dot{F}_\mu) = \{ \phi^{-4} h^{\mu\nu} \phi^{-4} h^{\nu\sigma} - \phi^{-4} h^{\mu\nu} \phi^{-4} h^{\nu\sigma} \} \circ \{ \phi^{4} h_{\mu\nu} \} \circ \{ \phi^{4} h_{\rho\sigma} \}$$

$$= \{ h^{\mu\nu} h^{\nu\sigma} - h^{\mu\nu} h^{\rho\sigma} \} \circ \{ h_{\mu\nu} + 4h_{\mu\nu} \phi / \phi \} \circ \{ h_{\rho\sigma} + 4h_{\rho\sigma} \phi / \phi \};$$

(14)

$\phi$ is a conformal factor. This action then gives as a primary constraint

$$H_{\phi} \equiv \phi^{-8} \frac{1}{\sqrt{h}} \pi_{\mu\nu} \pi^{\mu\nu} - \sqrt{h} \left\{ \text{Ric}(h) - \frac{8D^2 \phi}{\phi} \right\} = 0$$

(15)

(Lichnerowicz equation), $H_\mu$ as a secondary constraint from $F_\mu$ variation, and

$$\pi = 0$$

(16)

(condition for a maximal slice) as one part of the free end hypersurface [44, 59] variation of $\phi$, but the other part of this variation entails frozenness (the well-known non-propagability of maximal slicing for spatially compact without boundary GR.

Anderson et al, Barbour and Ó Murchadha [49, 57] got round this frozenness by considering a new action with division by $\text{Vol}^{2/3}$ which amounts to fully using $G = \text{Diff}(\Sigma, 3) \times \text{Conf}(\Sigma, 3)$ where $\text{Conf}(\Sigma, 3)$ is the group of conformal transformations on $\Sigma$. But the subsequent variational principle no longer gives GR and is furthermore questionable as an alternative theory [14, 49, 51]).

I, like [57], present this for simplicity with no $\Lambda$ term; see [49] for inclusion of this. Also, Barbour and Ó Murchadha [57] first considered this with two separate multipliers instead of a single more general auxiliary whose velocity also features in the action and has to be free end hypersurface varied; the way this is presented here is that of [58, 59]. See also [44, 59] for justification of the type of variation in use.
In [58] it was circumvented rather by using $G = \text{Diff}(\Sigma, 3) \times \text{VPConf}(\Sigma, 3)$, where VPConf$(\Sigma, 3)$ are the (global) volume-preserving conformal transformations on $\Sigma$ as implemented by using

$$\hat{\phi} = \phi / \left\{ \int d^3 x \sqrt{h} \phi^6 \right\}^{1/6}. \quad (17)$$

Subsequently, the primary constraint (now denoted by $\mathcal{H}_\phi$) picks up more terms (making it a Lichnerowicz–York equation rather than a Lichnerowicz equation). While, $\phi$ variation now gives

$$\pi / \sqrt{h} = C, \quad (18)$$

(condition for a constant mean curvature slice) alongside an equation that successfully maintains this. The addition of matter to the present paragraph’s scheme has not to date been extensively studied, but no significant hindrances are known to date [60], and the addition of matter to the preceding paragraph’s scheme has been extensively studied [49, 61].

3. Geometry of the configuration spaces

The following configuration space structure issues play an important underlying role in papers I and II.

3.1. Euclidean relational particle mechanics configuration spaces and reduction

For Euclidean relational particle mechanics, such a study was carried out in [24] in a ‘rigged’ fashion (auxiliary variables included), and in [24, 26, 62] in a reduced fashion (auxiliary variables eliminated). [63] exemplifies related work. In [27] I noted that this elimination has a simpler nature in 2D than in 3D, and used this to pass to completely relational variables in [13, 17] for both Euclidean and similarity relational particle mechanics. Relative space $R(N, d)$ is the quotient space $Q(N, d)/\text{Tr}(d)$. This is flat space, so the kinetic metric is just $T(R_i) = \sum_i \mu_i R_i^2 / 2$. At this level we have Jacobi coordinates and constraints as in section 2.1.

I use mass-weighted Jacobi variables as the most succinct ‘original variables of the problem’. Relational space is $R(N, d) = Q(N, d) / \text{Eucl}(d)$, which is what one has on eliminating the angular momentum constraint. These coincide in 1D. In 2D one can do elimination explicitly [13, 17], obtaining, e.g. in the 3-particle case the $T_{\triangle}(i_1, i_2, \Phi)$ corresponding to the line element

$$ds^2_{\triangle} = di_1^2 + di_2^2 + \frac{i_1^2 i_2^2 d\Phi^2}{i_1^2 + i_2^2}. \quad (19)$$

(To obtain a configuration space kinetic term $T$ from the line element of a metric $ds^2 = M_{AB} dQ^A dQ^B$, use (A.2).) This elimination is done by: step (1) passing to mass-weighted Jacobi coordinates. Step (2) eliminating the rotational auxiliary from the Lagrangian form of the constraint (5), casting the subsequent expression in mass-weighted Jacobi bipolar coordinates, which I denote by $(i_1, \theta_1, i_2, \theta_2)$ (figure 1(iv); these are still with respect to fixed axes.) Step (3) one can then pass to fully Euclideanly relational coordinates (figure 1(v)) ($(i_1, i_2, \Phi)$) (by an additional ‘purely absolute’ angle dropping out of the working [17, 26]). In 3D, the situation is considerably harder.

3.2. Similarity relational particle mechanics configuration spaces and reduction

For similarity relational particle mechanics, such a study was carried out in [13] in a reduced fashion (both by reduction and in an already-reduced form based on [64]).
space is $P(N, d) = Q(N, d)/\text{Tr}(d) \times \text{Dil}$ and $\text{Shape space}$ is $S(N, d) = Q(N, d)/\text{Sim}(d)$. Furthermore, $P(N, d) = S^{n-1}$ and $S(N, 1) = S^{n-1}$ where in both cases ‘=’ here means equal both topologically and metrically, with the standard (hyper)spherical metric. Thus one has $T^{N\text{-stop}}(R_p, R_{\hat{p}})$ or $T^{N\text{-stop}}(\Theta_p, \Theta_{\hat{p}})$ corresponding to the line element

$$\text{d}s^2_{N\text{-stop}} = \frac{\left\{1 + \sum_{p=1}^{n-1} R_p^2\right\} \sum_{\hat{p}=1}^{n-1} \text{d}R_{\hat{p}}^2 - \left\{\sum_{p=1}^{n-1} R_p \text{d}R_p\right\}^2}{\left\{1 + \sum_{p=1}^{n-1} R_p^2\right\}^2} = \sum_{r=1}^{n-1} \prod_{p=1}^{r-1} \sin^2 \Theta_p \text{d}\Theta_p^2 \quad (20)$$

in terms of simple ratio coordinates (21) and (ultra)spherical coordinates (22), respectively, where $\prod_{p=1}^{n-1} \text{ terms are defined to be } 1$. It is obtained from the similarity relational particle mechanics action in Jacobi coordinates by: Step (1) passing to mass-weighted coordinates. Step (2) eliminates dilational auxiliaries from the Lagrangian form of the constraint (8) and re-express in mass-weighted Jacobi bipolar coordinates. Step (3) pass to fully relational simple ratio variables

$$R_p \equiv t_p/t_{n-1} \quad (21)$$

(in the present context these are, geometrically, Beltrami coordinates); these are related to ultraspherical coordinates by

$$\Theta_p = \arctan\left(\frac{\sum_{j=1}^{p} R_j^2}{R_{p+1}}\right) \quad (22)$$

Figure 1. Coordinate systems for scaled triangleland. (i) Absolute particle position coordinates $(q_1, q_2, q_3)$ with respect to fixed axes and a fixed origin $O$; the corresponding particle masses are $m_I, I = 1–3$. (ii) Relative particle position coordinates, any two of which form a basis. (iii) Relative Jacobi coordinates $(R_1, R_2)$; the corresponding Jacobi masses are $\mu_i, i = 1, 2$. The mass-scaled relative Jacobi coordinates are related to these by $\iota_i = \sqrt{\mu_i R_i}$. (iv) Bipolar relative Jacobi coordinates $(\rho_1, \theta_1, \rho_2, \theta_2)$. The mass-scaled radial Jacobi coordinates are $\iota_i = \sqrt{\mu_i}$. These coordinates still refer to fixed axes. (v) Fully relational coordinates $(\rho_1, \rho_2, \Phi)$, for $\Phi = \arccos(\frac{R_1 \cdot R_2}{\|R_1\|\|R_2\|})$ the relational ‘Swiss army knife angle’ between the two relative Jacobi vectors. The coordinate ranges are $0 \leq \rho_i < \infty, 0 \leq \Phi < 2\pi$. 

for \( R_{n-1} \equiv \eta_{n-1}/\eta_{n-1} = 1 \). The coordinate ranges for these are \( \Theta_p \in (0, \pi), \Theta_{nd-1} \in [0, 2\pi) \).

For example for 4-stop metroland \( \Theta_1 \) is the azimuthal angle \( \Theta \) and \( \Theta_2 \) is the polar angle \( \Phi \). Then one has \( T^{\text{4-stop}}(R_1, R_2, R_3) \) or \( T^{\text{4-stop}}(\Theta, \Phi) \) corresponding to the line element

\[
d s^2_{\text{4-stop}} = \left\{ 1 + \frac{1}{2} R_2^2 \right\} \frac{d R_1^2}{1 + \frac{1}{2} R_2^2} - 2 R_1 R_2 d R_1 d R_2 = d\Theta^2 + \sin^2 \Theta d\Phi^2
\]

(23)

for ‘simple’ coordinates \( (R_1, R_2) \) (4-stop metroland subcase of (21)) and spherical coordinates \( (\Theta, \Phi) = (\Theta_1, \Theta_2) \), given by the 4-stop metroland subcase of (22)).

While, for \( N \) particles in 2D \( S(N, 2) = \mathbb{CP}^{n-1} \) both topologically and metrically, with the standard Fubini–Study metric corresponding to the line element

\[
d s^2_{\text{\mathbb{CP}^{n-1}}} = \frac{1}{1 + \sum_{p} |Z_p|^2} \sum_{q} |dZ_q|^2 - \frac{1}{1 + \sum_{i} |Z_i|^2} \sum_{p} |Z_p| dZ_p^2
\]

(24)

for \( Z_i \) inhomogeneous coordinates on \( \mathbb{CP}^{n-1} \), from which the kinetic term \( T^{\text{\mathbb{CP}^{n-1}}}(Z_p, \dot{Z}_p) \) is constructed. This is obtained from the similarity relational particle mechanics action in Jacobi coordinates by step (1) passing to mass-weighted coordinates. Step (2) eliminate both a rotational auxiliary and a dilational auxiliary from the Lagrangian forms of (5), (8).

Step (3) write this action in terms of ratio variables. Furthermore, one can use the polar form \( \dot{Z} = R_s \exp(i\Theta) \); indexing moduli (real ratio coordinates) as \( R_p \) and arguments (relative-angle coordinates) as \( \Theta_p \) (for both \( \dot{p} \) and \( \dot{p} \) taking \( 1 \) to \( N - 2 \)). Then the configuration space metric can be written in two blocks \( (p, q) = 0) \)

\[
M_{pq} \equiv \{1 + \sum_{p} |Z_p|^2\}^{-2} d_\delta pq - |1 + \sum_{p} |Z_p|^2\}^{-2} R_p R_q,
\]

(25)

where for a given \( p \) \( R_p \) is the \( R_p \) that forms a complex coordinate pair with \( \Theta_p \).

As a particular example, \( S(3, 2) = \mathbb{CP}^{1} = \mathbb{S}^2 \), making this triangled land case particularly amenable to study due to the availability of both projective and spherical techniques (while, as we shall see, this example meets many of the nontrivialities required by problem of time strategies); this paper principally considers this case. In this case the kinetic term collapses to \( T^{\text{\mathbb{CP}^{1}}}(Z, \dot{Z}), T^{\text{\mathbb{CP}^{1}}}(R, \dot{R}, \Phi) \) or \( T^{\text{\mathbb{CP}^{1}}}(\Theta, \dot{\Theta}, \Phi) \) as constructed from the line element

\[
d s^2_{\text{\mathbb{CP}^{1}}} = \frac{|dZ|^2}{1 + |Z|^2} = \frac{dR^2 + R^2 d\Phi^2}{1 + R^2} = \frac{d\Theta^2 + \sin^2 \Theta d\Phi^2}{4}.
\]

(26)

where \( R \) is the simple ratio variable \( R \equiv \frac{t_1}{t_2} \) This is physically the square root of the ratio of partial moments of inertia, \( \sqrt{T_1/T_2} \), and mathematically a choice of inhomogeneous coordinate’s modulus on \( \mathbb{CP}^{1} \). This is related to the azimuthal angle \( \Theta \) on \( \mathbb{S}^2 = \mathbb{CP}^{1} \) by

\[
\Theta = 2 \arctan R \Leftrightarrow R = \tan \frac{\Theta}{2}
\]

(27)

for \( \Theta \) the spherical azimuthal angle (see also figure 2). (Thus \( R \) is also geometrically interpretable as a radial stereographic coordinate on \( \mathbb{S}^2 \).) \( \Phi \) is the relative angle between the two Jacobi coordinate vectors. Useful relations between the two representations include that \( R = 1 \) is the equator, \( R \) small (compared to 1) corresponds to near the North Pole (\( \Theta \) a small angle) and \( R \) large corresponds to near the South Pole (the complement angle \( \Xi = \pi - \Theta \) is small). Also, one can use the barred banal conformal representation \( T \rightarrow 4T, U + E \rightarrow 4[U + E] \), whereupon this sphere of radius 1/2 becomes the unit sphere, and I write \( T_{\mathbb{S}^2}^{\text{triangle}} \).

For use in paper II, this line element in terms of \( \Phi \) and \( I_i, i = 1 \) or 2 is (no sum)

\[
d s^2 = dI_1^2/4I_1[I - I_1] + (I - I_1) I_1 d\Phi^2/I_1^2.
\]

(28)
Figure 2. Interrelation between $R, U, \Theta$ and $\Xi$ coordinates. $N$ is the North Pole, $S$ is the South Pole, $O$ is the centre. Then point $P$ in the spherical representation corresponds to point $P'$ in the stereographic tangent plane at $N$ with radial coordinate $R$, and to point $P''$ in the stereographic tangent plane at $S$ with radial coordinate $U$.

One can also use $T = \tilde{T}_{\text{triangle}}(R, \dot{R}, \Phi)$ constructed from the line element
\[
\text{d}s_{\text{triangle}}^2 = \frac{1}{4} \left\{ \text{d}I^2 + 4I^2 \frac{|\text{d}Z|^2}{[1 + |Z|^2]^2} \right\} = \frac{1}{4I} \left( \text{d}I^2 + 4I^2 \frac{dR^2 + R^2 d\Phi^2}{[1 + R^2]^2} \right)
\]
by performing a banal conformal transformation with conformal factor $\Omega^2 = (1 + R^2)^2$. This is geometrically trivial, while the other above forms are both geometrically natural and mechanically natural (equivalent to $E$ appearing as an eigenvalue free of weight function); using $(1 + R^2)$ alone would be the conformally natural choice.

In 3D, the situation is, again, harder [64] though at least similarity relational particle mechanics is free of collisions that are maximal (i.e. between all the particles at once).

3.3. Euclidean relational particle mechanics in scale-shape variables

Finally, these ‘shapes’ are also relevant within the corresponding Euclidean relational particle mechanics, as these admit conceptually interesting formulations in terms of scale-shape split variables. For scaled $N$-stop metroland, the configuration space is the generalized cone $C(S^{n-1})$ while for $N$-a-gonland it is $C(S^{p-1})$, scaled triangleland also being $C(S^2)$. The special cone-point or ‘apex’ 0 here corresponds physically to the maximal collision. Now use the new coordinate
\[
I = t_1^2 + t_2^2,
\]
which is physically the moment of inertia and mathematically a radius, and the same $R$ coordinate as in section 3.2. This coordinate transformation then inverts to
\[
t_1 = \sqrt{I/[1 + R^2]} R, \quad t_2 = \frac{\sqrt{I}}{1 + R^2}.
\]
Also, $R$ can be supplanted by $\Theta = 2 \arctan R$.

The kinetic term is $T_{\text{triangle}}(Z, \dot{Z}, \dot{I}, \dot{\Theta}, \dot{\Phi})$, $T_{\text{triangle}}(\dot{R}, \dot{\Phi}, \dot{\Theta}, \dot{\Phi})$ or $T_{\text{triangle}}(I, \dot{I}, \dot{\Theta}, \dot{\Phi})$ as constructed from the line element
\[
\text{d}s_{\text{triangle}}^2 = \frac{1}{4I} \left\{ \text{d}I^2 + 4I^2 \frac{|\text{d}Z|^2}{[1 + |Z|^2]^2} \right\} = \frac{1}{4I} \left( \text{d}I^2 + 4I^2 \frac{dR^2 + R^2 d\Phi^2}{[1 + R^2]^2} \right)
\]
Moreover, one can use instead the banal-conformally related kinetic term $\tilde{T}_{\text{Flat}}(I, \dot{I}, \dot{\Theta}, \dot{\Phi})$ constructed from
\[
\text{d}s_{\text{Flat}}^2 = \text{d}I^2 + I^2 [d\Theta^2 + \sin^2 \Theta d\Phi^2].
\]
(the corresponding conformal factor being $4I$, so that also
\[ \tilde{U} + \tilde{E} = \{U + E\}/4I, \]
away from $I = 0$ in which place this conformal transformation is invalid). The form of (33) clearly makes this a flat (i.e. geometrically trivial) representation in spherical polar coordinates, with $I$ as the radius. While, the other forms above are mechanically natural.

3.4. Geometrodynamical configuration spaces

While, GR is a geometrodynamics [46, 48] on the quotient configuration space Superspace $(\Sigma, 3) \equiv \text{Riem}(\Sigma, 3)/\text{Diff}(\Sigma, 3)$ [46, 48], which is studied topologically and geometrically in, e.g. [46, 48, 65]. Superspace is an infinite-dimensional complicatedly stratified manifold; explicit reduction is not in general possible here.

3.5. Conformogeometrodynamical configuration spaces

One can obtain relational theories or formulations using $G = \text{Diff}(\Sigma, 3) \times \text{Conf}(\Sigma, 3)$ that reproduces the maximal condition (however this formulation freezes unless one alters one’s theory from GR [49]) and using $G = \text{Diff}(\Sigma, 3) \times \text{VPConf}(\Sigma, 3)$ (the volume-preserving conformal transformations) that does reproduce GR in the York formulation from an action principle [58].

Here the associated configuration spaces are conformal superspace $\text{CS}(\Sigma, 3) \equiv \text{Riem}(\Sigma, 3)/\text{Diff}(\Sigma, 3) \times \text{Conf}(\Sigma, 3)$, which is studied geometrically in [66, 67], and $[\text{CS} + \text{V})(\Sigma, 3) = \text{Riem}(\Sigma, 3)/\text{Diff}(\Sigma, 3) \times \text{VPConf}(\Sigma, 3)$, which previously featured in, e.g. [55], but has not to my knowledge been studied from a geometrical perspective (the $\text{CRiem}(\Sigma, 3) \equiv \text{Riem}(\Sigma, 3)/\text{Conf}(\Sigma, 3)$ analogue of preshape space has been studied geometrically in, e.g. [48, 67]).

Parallels between this and Euclidean relational particle mechanics in shape-scale split variables include homogeneous degrees of freedom playing a similar role to scale, York time [68] having an ‘Euler time’ analogue [27], similarity relational particle mechanics corresponding to maximal slicing in having this time frozen, and in the analogy between $[\text{CS} + \text{V})(\Sigma, 3)$ and the Euclidean relational particle mechanics configuration spaces’ cone structure. In the configuration space of GR, the analogous ‘special point’ at zero scale is the Big Bang.

4. Similarity relational particle mechanics at the classical level

4.1. 1 and 2D cases

In appendix A, I provide how the general curved space mechanics unfolds. In the case of scalefree $N$-stop metroland, the Jacobi action is
\[ S^{N-\text{stop}}(\Theta_p, \dot{\Theta}_p) = 2 \int d\lambda \sqrt{g^{00}}[E - V]. \]
The Euler–Lagrange equations following from this are given in [13] in Beltrami coordinates (the notation there uses $s_p$ in place of the present paper’s $R_p$), but in this paper I use, rather, (ultra)spherical coordinates, so I provide the Euler–Lagrange equations afresh in these
\[ \left\{ \prod_{p=1}^{n-1} \sin^2 \Theta_p \dot{\Theta}_p \right\}^{*} - \left\{ \sum_{r=q+1}^{n-1} \prod_{p=1, p \neq q}^{r-1} \sin^2 \Theta_p \right\} \sin \Theta_q \cos \Theta_q \Theta_q^{*2} = - \frac{\partial V}{\partial \Theta_q}. \]
There is also a first energy integral,
\[
\sum_{r=1}^{n-1} \prod_{\beta=1}^{r-1} \sin^2 \Theta_r \Theta_\beta^2 / 2 + V(\Theta_r) = E. \tag{37}
\]

For scalefree $N$-a-gonland, the Jacobi action
\[
S = 2 \int d\lambda \sqrt{T_{FS}[E - \nabla]} \tag{38}
\]
gives the Euler–Lagrange equations as presented implicitly in [13].

In the exceptional triangleland case, there are spherical flat (conformal-to-stereographic) presentations
\[
S = 2 \int d\lambda \sqrt{T_{flat}[\tilde{E} - \tilde{V}]} \tag{39}
\]
Then, in the spherical presentation, the Euler–Lagrange equations simplify to
\[
\Theta^2 - \sin \Theta \cos \Phi \Phi^2 = -\frac{\partial \nabla}{\partial \Theta}, \quad [\sin^2 \Theta \Phi^2]^{\tilde{r}} = -\frac{\partial \nabla}{\partial \Phi} \tag{40}
\]
with an accompanying energy integral
\[
\Theta^2 / 2 + \sin^2 \Theta \Phi^2 / 2 + \nabla = \tilde{E}. \tag{41}
\]
(Above,
\[
\tilde{r} \equiv \sqrt{[E + U]/T} = \sqrt{|E + U|/T}/4 = \ast/4. \tag{42}
\]
If $\nabla$ is independent of $\Phi$, then (40ii) becomes another first integral
\[
\sin^2 \Theta \Phi^{\tilde{r}} = \mathcal{J}, \quad \text{constant.} \tag{43}
\]
This is a relative angular momentum quantity; see appendix C for various interpretations of it.

Then (41) becomes
\[
\Theta^2 / 2 + \mathcal{J}^2 / 2 \sin^2 \Theta + \tilde{V}(\Theta) = \tilde{E}. \tag{44}
\]
Or, in terms of the $R$ coordinate, the important results (43) and (44) take the form
\[
R^2 \Phi^{\tilde{r}} = \mathcal{J}, \quad \frac{1}{2} \left\{ \frac{dR}{d\tilde{r}} \right\}^2 + \frac{\mathcal{J}^2}{R^2} = \tilde{E} + \tilde{U}, \tag{45}
\]
where $\tilde{\ast} \equiv \sqrt{|E + U|/T}$. This is illuminating though it is direct parallel with the usual flat planar presentation of the ordinary mechanics of a test particle moving in a central potential
\[
\begin{pmatrix}
\text{ratio of square roots of the two subsystems' Jacobi partial moments of inertia} \\
\text{(relative angle between Jacobi coordinates)} = \Phi \leftrightarrow \theta \\
\text{(relative angular momentum of the two subsystems)} = \mathcal{J} \leftrightarrow L (= L_z)
\end{pmatrix}
\]
where $\tilde{\ast} \equiv \sqrt{|E + U|/T}$. This is illuminating though it is direct parallel with the usual flat planar presentation of the ordinary mechanics of a test particle moving in a central potential
\[
\begin{pmatrix}
\text{radial coordinate of test particle}, \\
\text{(polar coordinate of test particle),} \\
\text{angular momentum component perpendicular to the plane}
\end{pmatrix}
\]
This analogy is furthermore a pointer to parallel the well known $u = 1/r$ substitution by the $\mathcal{U} = 1/R$ one, which turns out to be exceedingly useful in my study below. (In the spherical presentation the counterpart of this inversion map $\mathcal{R} \longrightarrow \mathcal{U} = 1/R$ is the supplementary map $\Theta \longrightarrow \pi - \Theta = \theta$ the supplementary angle, $\theta$. In $i_i$ or $I_i$ variables, it takes the form of interchanging the 1-indices and the 2-indices. I term the underlying operation that takes these forms in these presentations as the duality map.)

Next, $\tilde{\mathcal{V}}_e \equiv \tilde{V} + \tilde{J}^2/R^2 - \tilde{E}$ is the potential quantity that is significant for motion in time, and combining (44) and (43), $\tilde{U}_{\text{orb}} \equiv -R^2\tilde{V}_e$ is the potential quantity that is significant as regards the shapes of the classical orbits. Translating into the spherical language, the corresponding quantities are $\tilde{V}_e = \tilde{V} + J^2 / \sin^2 \Theta - \tilde{E}$ and $\tilde{U}_{\text{orb}} = -\sin^4 \Theta \tilde{V}_e$.

Finally, combining (44) and (43) or their $R$-analogues gives as quadratures for the shapes of the orbits

$$\Phi - \Phi_0 = \int \mathcal{J} d\Theta / \sin \Theta \sqrt{2[\tilde{E} - \tilde{V}_e]} \sin^2 \Theta - \mathcal{J}^2 = \int \mathcal{J} d\mathcal{R} / \mathcal{R} \sqrt{2[\tilde{E} + \tilde{U}]/\mathcal{R}^2 - \mathcal{J}^2}.$$  

(50)

While, (44) and its $R$-analogue give as quadratures for the time traversals

$$\tilde{t} - \tilde{t}_0 = \int \sin \Theta d\Theta / \sqrt{2[\tilde{E} + \tilde{U}]} \sin^2 \Theta - \mathcal{J}^2, \quad \text{or} \quad \tilde{t} - \tilde{t}_0 = \int \mathcal{R} d\mathcal{R} / \sqrt{2[\tilde{E} + \tilde{U}]/\mathcal{R}^2 - \mathcal{J}^2}.$$  

(51)

4.2. A class of separable potentials for scalefree triangleland

As $\theta$-independent (central) potentials considerably simplify classical and quantum mechanics, by the above analogy it is clear that $\Phi$-independent (relative angle independent) potentials will also considerably simplify classical and quantum scalefree triangleland. In particular, these simplifications include separability. Also, in each case, there is a conserved quantity: angular momentum $L$ in the case of ordinary mechanics with central potentials, and relative angular momentum $\mathcal{J}$ of the two constituent subsystems in the present context (see appendix C for details of the interpretation of $\mathcal{J}$). A general class of separable potentials consists of linear combinations of

$$V_{(a,\beta)} \propto \left\{ \begin{array}{l} \ell_1 \sqrt{\mathcal{J}} \\ \ell_2 \sqrt{\mathcal{J}} \end{array} \right\}^{\alpha - \beta} \left\{ \begin{array}{l} \mathcal{R} \\ \sqrt{1 + \mathcal{R}^2} \end{array} \right\}^{\beta - \beta} \propto \left\{ \begin{array}{l} 1 \\ \sqrt{1 + \mathcal{R}^2} \end{array} \right\}^{\beta} \propto \sin^{\alpha - \beta} \Theta \cos^{\beta} \Theta \frac{1}{2}$$

so that

$$V_{(a,\beta)} \propto \mathcal{R}^{\alpha - \beta} / \sqrt{1 + \mathcal{R}^2}$$

and

$$V_{(a,\beta)} \propto \sin^{\alpha - \beta} \Theta \cos^{\beta} \Theta \frac{1}{2}.$$  

(52)

The most physically relevant subcase therein are the power-law-mimicking potentials $\beta = 0$ (remember that $I$ turns out to be constant in similarity relational particle mechanics after variation is done). These correspond to potential contributions solely between particles 2, 3. The case $\beta = \alpha$ are potential contributions solely between particle 1 and the centre of mass of particles 2, 3 (which is less widely physically meaningful). It also turns out that the duality map sends $V_{(a,\beta)}$ to $V_{(a,\alpha - \beta)}$.

Simple examples therein of quantum-mechanical interest are the constant potential ($\alpha = 0$) and similarity relational particle mechanics’ mimicker of harmonic oscillators ($\alpha = 2$). I choose these for explicit study due to the ubiquity of constant and harmonic oscillator potentials in theoretical physics; moreover harmonic oscillator potentials confer
nice boundedness features at the quantum level. I use the following notation for the constants of proportionality for the single harmonic oscillator between particles 2, 3:

\[ 2V_{(2,0)} = h_23(q_2 - q_3)^2 = H_1^{(1)} \rho_1^2 \] for \( h_23 \) the ordinary position space Hooke’s coefficient and \( H_1 \) the relative configuration space’s ‘Jacobi–Hooke’ coefficient. I use the obvious cyclic permutations of this for harmonic oscillators between particles 3, 1, and particles 1, 2. Moreover, the linear combination of these last two such that \( m_2h_{13} = m_3h_{12} \) (resultant force of the second and third ‘springs’ pointing along the line joining the centre of mass of particles 2, 3 and the position of particle 1) is

\[ 2\left[ V_{(2,0)}^{(2)} + V_{(2,2)}^{(2)} \right] = H_1^{(2)} \rho_1^2 + H_2^{(2)} \rho_2^2, \]

where\(^{16}\)

Additionally, one can consider \( V = V^{(1)} + V^{(2)} \), whereupon one has \( H_1 = H_1^{(1)} + H_1^{(2)} \) and \( H_2 = H_2^{(2)} \); then define \( K_i \equiv H_i/\mu_i \) so that

\[ 2V = H_1 \rho_1^2 + H_2 \rho_2^2 = K_1 \rho_1^2 + K_2 \rho_2^2. \] (54)

I refer to this case as the ‘special’ case of the 3-harmonic oscillator problem, the general case having angle-dependent potentials (see the following subsection) due to not having the resultant force of the second and third ‘springs’ point along \( \xi_2 \). Three notes of importance below and paper II are as follows.

(1) In \( (R, \Phi) \) coordinates this leads to the form

\[ 2\tilde{V} = \{K_1 R^2 + K_2 \}/[1 + R^2]^3, \] (55)

or, in \( (\Theta, \Phi) \) coordinates, by the simplifications \( \cos^2 \Theta/2 = [1 + \cos \Theta]/2 \) and \( \sin^2 \Theta/2 = [1 - \cos \Theta]/2 \), to the form

\[ \tilde{V} = A + B \cos \Theta \]

for \( A = [K_1 + K_2]/16, \quad B = [K_2 - K_1]/16 \) (56)

(2) This situation is the linear combination of a \( V_{(2,0)} \) and a \( V_{(2,2)} \) and as such is self-dual provided that the values of \( K_1 \) and \( K_2 \) are also interchanged (or, equivalently, the sign of \( B \) is reversed).

(3) if \( K_1 = K_2 \) (corresponding to the highly symmetric balance \( m_1h_{23} = m_2h_{31} = m_3h_{12} \) in position space), then

\[ \tilde{V} = A, \] (57)

to which I refer as the ‘very special case’; this is unconditionally self-dual.

4.3. Scalefree triangleland with harmonic oscillator type potential

In all other multiple harmonic oscillator cases, there is a \( \Phi \)-dependent cross-term, rendering the potential nonseparable in these coordinates. The general triple harmonic oscillator like dipotential \( 2V = h_{123}[q_2 - q_3]^2 \) cycles maps to

\[ 2\tilde{V} = \{K_1 R^2 + LR \cos \Phi + K_2 \}/[1 + R^2]^3 \]

for \( L \equiv [\mu_1 \mu_2]^{-1/2} [h_{13}m_2 - h_{12}m_3]/[m_2 + m_3] \). (58)

Or, alternatively, in the spherical presentation, the potential is

\[ \tilde{V} = A + B \cos \Theta + C \sin \Theta \cos \Phi \]

for \( C = L/16 \). (59)

This clearly includes the previous subsection’s special case by setting \( C = 0 \) (and so \( L = 0 \), corresponding to \( m_2h_{13} = m_3h_{12} \)).
4.4. Analogy with some linear rigid rotor set-ups

Some useful mathematical analogies for scalefree triangleland with multiple harmonic oscillator like potentials are as follows:

very special harmonic oscillator $\longleftrightarrow$ linear rigid rotor,  \hspace{1cm} (60)

special harmonic oscillator $\longleftrightarrow$ linear rigid rotor in a background homogeneous electric field in the axial ('z')-direction,  \hspace{1cm} (61)

general harmonic oscillator $\longleftrightarrow$ linear rigid rotor in a background homogeneous electric field in an arbitrary direction.  \hspace{1cm} (62)

In particular, this classical problem has

$$T_{\text{rotor}} = I_{\text{rotor}} (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)/2, \hspace{1cm} V_{\text{rotor}} = -\mathcal{M} E \cos \theta,$$  \hspace{1cm} (63)

where $I_{\text{rotor}}$ is the single nontrivial value of the moment of inertia of the linear rigid rotor, $\mathcal{E}$ is a constant external electric field in the axial 'z' direction and $\mathcal{M}$ is the dipole moment component in that direction. Thus the correspondence is $\Theta \leftrightarrow \theta, \Phi \leftrightarrow \phi,$  \hspace{1cm} (energy) \hspace{1cm} (64)

$$1 \leftrightarrow I_{\text{rotor}},$$  \hspace{1cm} (65)

$$E/4 - A = \mathcal{E} - A \leftrightarrow E = \text{(energy)},$$  \hspace{1cm} (66)

These all being well studied at the quantum level [69, 70], this identification is of considerable value in solving the relational problem in hand, by the string of techniques in figure 3.

4.5. A brief study of the potential

Working in spherical coordinates, set $0 = \dot{\mathcal{V}}/\dot{\Theta} = -B \sin \Theta + C \cos \Theta \cos \Phi, 0 = \dot{\mathcal{V}}/\dot{\Phi} = -C \sin \Theta \sin \Phi$ to find the critical points. These are at $(\Theta, \Phi) = (\arctan(C/B), 0), (\pi - \arctan(C/B), \pi)$ which are antipodal (see figure 4); in fact the potential
Figure 4. The preferred axis that is picked at by the potential.

is axisymmetric about the axis these lie on. The critical points are, respectively, a maximum and a minimum (the very special case $B = C = 0$ is also critical, for all angles—this case ceases to have a preferred axis.)

$\mathcal{R}$ small corresponds to $\Theta$ small, for which

$$
\mathcal{U} + \mathcal{E} = [\mathcal{E} - A - B] - C\Theta \cos \Phi + B\Theta^2/2 + O(\Theta^3),
$$

(67)

or

$$
2[\tilde{\mathcal{U}} + \tilde{\mathcal{E}}] = 2\mathcal{E} - K_2 - L\mathcal{R} \cos \Phi - [4\mathcal{E} + K_1 - 3K_2]\mathcal{R}^2 + O(\mathcal{R}^3)
$$

$$
\equiv Q_0 - L\mathcal{R} \cos \Phi - \{4\mathcal{R}^2\} + O(\mathcal{R}^3).
$$

(68)

Thus the leading term is a constant, unless $Q_0 = 2\mathcal{E} - K_2 (\propto \mathcal{E} - A - B) = 0$, in which case it is linear in $\Theta$ or $\mathcal{R}$ and with a $\cos \Phi$ factor, unless also $L (\propto C) = 0$ (which is also the condition for the ‘special’ case), in which case it is quadratic in $\Theta$ or $\mathcal{R}$, unless $B = 0$ (given previous conditions, this is equivalent to $Q_2 = 4\mathcal{E} + K_1 - 3K_2 = 0$), which means that one is in the $K_2 = 2\mathcal{E}$ subcase of the ‘very special’ case, for which $\mathcal{U} + \mathcal{E}$ has no terms at all.

$\mathcal{R}$ large corresponds to the supplementary angle $\Xi \equiv \pi - \Theta$ being small, so

$$
\mathcal{U} + \mathcal{E} = [\mathcal{E} - A - B] + C\Xi \cos \Phi + B\Xi^2/2 + O(\Xi^3),
$$

(69)

or

$$
2[\tilde{\mathcal{U}} + \tilde{\mathcal{E}}] = [2\mathcal{E} - K_1]\mathcal{R}^{-4} - L\mathcal{R}^{-5} \cos \Phi - [4\mathcal{E} + K_2 - 3K_1]\mathcal{R}^{-6} + O(\mathcal{R}^{-7})
$$

$$
\equiv Q_4\mathcal{R}^{-4} - L\mathcal{R}^{-5} \cos \Phi - \{4\mathcal{R}^2\} + O(\mathcal{R}^{-7}).
$$

(70)

Thus the leading term goes as a constant in $\Xi$ or as $\mathcal{R}^{-4}$, unless $Q_4 = 2\mathcal{E} - K_1 (\propto \mathcal{E} - A + B) = 0$, in which case it goes linearly in $\Xi$ or as $\mathcal{R}^{-5}$ in each case also with a $\cos \Phi$ factor, unless also $L (\propto C) = 0$ (‘special’ case), in which case it goes quadratically in $\Xi$ or as $\mathcal{R}^{-6}$, unless $B = 0$ (given previous conditions, this is equivalent to $Q_6 = 4\mathcal{E} + K_2 - 3K_1 = 0$), which means that one is in the $K_1 = 2\mathcal{E}$ subcase of the ‘very special’ case, for which $\mathcal{E} + \mathcal{U}$ has no terms at all. Note that $B = 0$ implies $K_1 = K_2$, so this very special subcase indeed coincides with the previous paragraph’s. Finally note that the large and small asymptotics are dual to each other (the difference of four powers is accounted for by how the kinetic energy scales under the duality map), so that one need only analyse the parameter space for one of the two regimes and then obtain everything about the other regime by simple transcription.
4.6. Classical equations of motion

The Jacobi-type action for this problem is, in spherical coordinates and using the barred banal conformal representation

$$S = \int d\lambda \sqrt{\dot{\Theta}^2 + \sin^2 \Theta \dot{\Phi}^2 / 2 \left\{ E - A - B \cos \Theta - C \sin \Theta \cos \Phi \right\}},$$

(71)

Then the Euler–Lagrange equations are

$$\Theta^{\prime\prime} - \sin \Theta \cos \Theta \left\{ \Phi^{\prime 2} \right\} = B \sin \Theta - C \cos \Theta \cos \Phi,$$

(72)

$$\sin^2 \Theta \Phi^{\prime 2} = C \sin \Theta \sin \Phi.$$  

(73)

On account of the action being independent of $t$, one of these can be replaced by the energy integral

$$\frac{\left\{ \Theta^{\prime 2} + \sin^2 \Theta \Phi^{\prime 2} \right\}}{2} + A + B \cos \Theta + C \sin \Theta \cos \Phi = E.$$  

(74)

A dynamical systems and phase space analysis of these equations and their interpretation in terms of $I_1, I_2$ and $\Phi$ will be presented elsewhere [71]. Also note that the notions of $V_{\text{eff}}$ and $V_{\text{orb}}$ are inapplicable in the nonseparable case.

4.7. ‘Special case’

For $C = 0$, (74) reads

$$\frac{1}{2} \left\{ \Theta^{\prime 2} + \frac{\mathcal{J}^2}{\sin^2 \Theta} \right\} + A + B \cos \Theta = E,$$

(75)

which can be integrated (using (43) to eliminate $t^{\prime\prime}$ in favour of $\Phi$)

$$\Phi - \Phi_0 = \int \mathcal{J} \, d\Theta / \sin \Theta \sqrt{2(E - A - B \cos \Theta) \sin^2 \Theta - \mathcal{J}^2}. $$

(76)

For the very special case $B = C = 0$, this reduces to the well-known integral for the computation of geodesics for the sphere—cf [17], whose exact solution I now cast below in terms of $\mathcal{R}, \Phi$ and then in terms of the ‘original’ $I_1, I_2$ variables. I have also solved the $B \neq 0$ case exactly by, e.g. $\tan^{\frac{1}{2}} \mathcal{R} = \mathcal{R} \equiv \sqrt{x - 1}$, obtaining then via Maple a composition of polynomials, roots and elliptic functions in $x$ that I consider to be too complicated to present here.

Anderson [17] sketches $\tilde{V}_{\text{e}}$ and $\tilde{U}_{\text{orb}}$ for the single harmonic oscillator and their large and small asymptotic behaviours. $\mathcal{J} = 0$ is rather simpler: 1D motion, i.e. the orbits are straight lines. Assume $\mathcal{J} \neq 0$ from now on in looking for orbits with less trivial shapes. The universal large $\mathcal{R}$ asymptotic solution’s $\tilde{V}_{\text{e}}$ exhibits a finite potential barrier and the orbits are bounded from above. The small $\mathcal{R}$ asymptotic solution for a fixed $E > 0$ harmonic oscillator problem is the usual radial/isotropic harmonic oscillator problem. Here, $V_{\text{eff}}$ is an infinite well formed by the harmonic oscillator’s parabolic potential on the outside and the infinite centrifugal barrier on the inside, and the orbits are bounded from below. For a fixed $E > 0$ harmonic oscillator scalefree triangleland problem, in contrast with the usual radial harmonic oscillator problem, the potential tends to 0 rather than $+\infty$ as $\mathcal{R} \rightarrow \infty$, and the orbits are bounded from both above and below.

10 This was done using Maple 10 to 12.
4.8. Exact solution for the ‘very special case’

This effectively constant-V case is equivalent to the geodesic problem on the sphere. Thus it is solved in \( \Theta, \Phi \) variables by great circles (first equality)

\[
F \cos(\Phi - \Phi_0) = 2 \cot \Theta = \{1 - R^2\}/R = \{t_1^2 - t_1^2\}/t_1 t_2
\]  

(77)

(following through with variable transformations so as to cast it in terms of straightforward relational variables, and using \( JF = 2\sqrt{2(E - A) - J^2} = \sqrt{2E - K_2 - 4J^2} = \sqrt{Q_0 - 4J^2} \). The \( R \) form above can be rearranged to the equation of a generally off-centre circle (figure 5(a)).

While, in terms of the original quantities of the problem,

\[
F^2(t_1 \cdot t_2)^2 + \|t_1\|^4 + \|t_2\|^4 + 2F\{\|t_1\|^2 - \|t_2\|^2\}t_1 \cdot t_2 \cos \Phi_0 = \{2 + F^2 \sin^2 \Phi_0\}\|t_1\|^2\|t_2\|^2.
\]  

(78)

Note

1. This solution is homogeneous in \( \|t_1\|, \|t_2\| \) and \( \sqrt{t_1 \cdot t_2} \) (in fact it is a fourth-order homogeneous polynomial in these quantities). These quantities are all invariant under rotation, so the zero total angular momentum constraint is manifestly satisfied, while the zero dilational momentum constraint is manifestly satisfied due to the homogeneity. (‘Homogeneous equation’ is another way of expressing ‘equation in terms of ratios alone’.)

2. It has simpler subcases: \( \sin \Phi_0 = 0 \) gives the merely second order

\[
F_{t_1 \cdot t_2} + \|t_1\|^2 = \|t_2\|^2,
\]  

(79)

which I sketch in figure 5, while \( \cos \Phi_0 = 0 \) gives

\[
F^2(t_1 \cdot t_2)^2 + \|t_1\|^4 + \|t_2\|^4 = \{2 + F^2\}\|t_1\|^2\|t_2\|^2.
\]  

(80)

3. The solution is invariant under \( t_1 \leftrightarrow t_2 \) provided that also \( \Phi_0 \) is shifted by a multiple of \( \pi \).

4.9. Small relative scale asymptotic behaviour

The first approximation for \( R \) is the generic asymptotics \((Q_0 = 2E - K_2 \propto E - A + B \neq 0, \) so \( \hat{U} + \hat{E} \) goes as \( Q_0/2 \). (Note however that not all dynamical orbits enter such a regime—sometimes the quadrature’s integral goes complex before the small \( R \) regime is attained (\( R \) large ‘classically forbidden’) when \( \hat{U}(R)R^2 - J^2 < 0 \).) Then integrating the \( R \) quadrature (50) gives the orbits

\[
\pm \sqrt{2Q_0} \sec(\Phi - \Phi) = R = \tan \frac{\Theta}{2} = t_1/t_2
\]  

(81)

(following through with variable transformations so as to cast it in terms of straightforward relational variables, and using the \( J \)-absorbing constant \( q_0 = Q_0/2J^2 = E - K_2/2J^2 = 4(E - A - B)/J^2 \). Note that the \( R \) form of the orbits are parallel straight lines (vertical for \( \Phi_0 = 0, \pi \) (figure 6(a)).

Moreover, these are known not to be very good approximands in that they totally neglect the non-constant part of the potential and thus are precisely rectilinear motions. Thus at least here (and one may suspect likewise in the following section), it is often necessary to use the second approximation in studies. The other equalities in (81) convert that result into what form the asymptotic orbits take in straightforward relational variables; I sketch these in figures 6(a) and 7(a).

11 This result was first given in its original context in chapter 10 of book I of [1].
While, in terms of the problem’s original quantities

\[ \|\xi_2\|^4 + 2q_0\|\xi_1\|^2\|\xi_2\|^2 \sin^2 \Phi_0 + 2\sqrt{2q_0}\|\xi_3\|^2\|\xi_1 \cdot \xi_2\|^2 \cos \Phi_0, \]  

(82)

which is again a fourth-order homogeneous polynomial in \(\|\xi_1\|, \|\xi_2\|\) and \(\sqrt{\frac{1}{\|\xi_1\|} - \frac{1}{\|\xi_2\|}}\). It admits the simpler cases (1) \(\sin \Phi_0 = 0\) (which is merely second order)

\[ \|\xi_3\|^2 = \sqrt{2q_0}\|\xi_1 \cdot \xi_2\|, \]

(83)

and (2) \(\cos \Phi_0 = 0\)

\[ \|\xi_3\|^4 = 2q_0\|\xi_1\|^2\|\xi_2\|^2 - \|\xi_1 \cdot \xi_2\|^2. \]

(84)
The second approximation for $R$ small in which $L = 0$ but the $Q_2 R^2/2$ term is also kept turns out to also often be necessary. If $Q_0 = 0$ but $L \neq 0$ one has non-generic asymptotics not directly covered in this subsection, but one can use the technique of section 5 to reduce this case to one of those covered in the present subsection in a new set of variables. If $Q_0, L = 0$, one resides within the special case, and the next leading term is $Q_2 R^2/2$, which case is included in my working below; indeed now the ‘second’ approximation is always necessary. If $Q_2 = 4E + K_1 - 3K_2 = 16(E - A - 2B) = 0$ also, one is within the very special case, and so one does not need any asymptotic calculations as one has the exact solution of subsection 4.8.

For the second approximation integrating the $R$ quadrature gives the orbits

$$\pm 1/\sqrt{q_0 + q_0^2 - q_2 \cos(2(\Phi - \Phi_1))} = R = \tan \frac{\Theta}{2} = \frac{\ell_1}{\ell_2}$$

(85)

(following through with variable transformations so as to cast it in terms of straightforward relational variables, and using the $J$-absorbing constant $q_2 = Q_2/J^2$). This is straightforwardly rearrangeable into quite a standard form (e.g. [72, 73])

$$R^2 = \frac{1}{q_0 + q_0^2 - q_2 \cos(2(\Phi - \Phi_0))},$$

(86)

the case-by-case analysis of which is provided in figure 6(b).

While, in terms of the problem’s original quantities, and using $g$ for $\sqrt{q_0^2 - q_2}$,

$$\|\xi\|^8 + \left\{q_2 + g^2 \cos^2(2\Phi_0)\right\}\|\xi_1\|^4\|\xi_2\|^4 - 2q_0\|\xi_2\|^6\|\xi_1\|^2 + 4g^2\{\ell_1 \cdot \ell_2\}^4 + 2g\|\xi_2\|^2\{q_0\|\xi_1\|^2 - \|\xi_2\|^2\}^2 \cos(2\Phi_0) = 4g^2\{\ell_1 \cdot \ell_2\}^2\|\xi_1\|^2\|\xi_2\|^2.$$
which is an eighth-order homogeneous polynomial in $\|\iota_1\|$, $\|\iota_2\|$ and $\sqrt{\iota_1 \cdot \iota_2}$. It admits the simpler cases (1) $\sin(2\Phi_0) = 0$ (which is merely fourth order)

$$\|\iota_2\|^4 = (q_0 - g)\|\iota_1\|\|\iota_2\|^2 + 2g(\iota_1 \cdot \iota_2)^2$$

(88)

and (2) $\cos(2\Phi_0) = 0$

$$\|\iota_2\|^8 + q_0^2\|\iota_1\|^4\|\iota_2\|^4 - 2q_0\|\iota_2\|^6\|\iota_1\|^2 = 4g^2(\iota_1 \cdot \iota_2)^2(\|\iota_1\|^2\|\iota_2\|^2 - (\iota_1 \cdot \iota_2)^2).$$

(89)
Taking \( q_0 = 0 \) further simplifies the simple subcases. While, taking \( g = 0 \) collapses the general solution to
\[
\|\mathbf{\iota}_2\| = 0 \quad (2, 1, 3 \text{ collinearity with } 1 \text{ at the centre of mass of } 2, 3) \text{ or } \|\mathbf{\iota}_2\| = \pm \sqrt{q_0}\|\mathbf{\iota}_1\| \quad \text{(rectilinear motion: } \Phi \text{ fixed).} \quad (90)
\]

4.10. Large relative scale asymptotic behaviour

For analogous notions of first and second large approximations, now the quadrature in \( U = 1/R \) takes the same form as the \( R \)-quadrature in the above workings with
\[
Q_0 \rightarrow Q_4, \quad Q_2 \rightarrow Q_6
\]
(91) (\( q_4, q_6 \) and \( f \) below are then the obvious analogues of \( q_0, q_2 \) and \( g \)). Hence the solutions are dual to those of subsection 4.9’s. Thus all of subsection 4.9’s results apply again under the duality substitutions (and the subsequently induced language changes ‘small’ \( \rightarrow \) ‘large’, and ‘2, 1, 3 collinearity with particle 1 at the centre of mass of particles 2, 3’ \( \rightarrow \) ‘collision between particles 2, 3, also interpretable as particle 1 escaping to infinity’). In particular, the \( I_1, I_2 \) plots of figure 7(b) and which parts of the parameter space the various cases hold in and where the now small asymptotics is valid can all just be read off the existing figures under these substitutions. Then the region in which small asymptotics applied before (figure 7(c)) is now that in which large asymptotics apply.

New \( R, \Phi \) plots are, however, required (figure 8). Some particular comments are: (1) the first approximation is then
\[
\pm \sqrt{2q_4} \cos(\Phi - \Phi_0) = R = \tan \frac{\Theta}{2} = t_{11/t_2}. \quad (92)
\]
In the \( (R, \Phi) \) plane and for \( \Phi_0 = 0 \), this takes the form of a family of circles of radius \( \sqrt{q_4/2} \) and centre \( (\sqrt{q_4/2}, 0) \), so that they are all tangent to the vertical axis through the origin ([17] and figure 6(a)). (2) The second approximation gives
\[
\pm \sqrt{q_4 + \sqrt{q_4^2 - q_6} \cos(2(\Phi - \Phi))} = R = \tan \frac{\Theta}{2} = t_{11/t_2}. \quad (93)
\]
In the \( R, \Phi \) plane and for \( \Phi_0 = 0 \), this take the forms in figure 8(b) in the parameter regions delineated by figure 6(c).

Note: for general \( \tilde{V}_{(a,0)} \) with constant of proportionality \( \Lambda_a \), one gets exactly the same large-asymptotics analysis as here, with \( q_0 = 2E - \Lambda_a, q_2 = 4E - (4 + a)\Lambda_a \). Thus generally this duality to the isotropic harmonic oscillator of the universal large-scale asymptotics of scalefree triangleland is a useful and important result for this Machian mechanics, for the classical and quantum mechanics of isotropic harmonic oscillators is rather well studied and thus a ready source of classical and quantum methods, results and insights.

4.11. Investigation of the intermediate-\( R \) region

Note that only two of \( Q_0, Q_2, Q_4 \) and \( Q_6 \) are independent. Thus prescribing a particular small asymptotics entails prescribing a particular large asymptotics too. There are however no compatibility restrictions: each small asymptotic behaviour is capable of connecting to each large asymptotic behaviour.

Also note from the twin circles, tear drops and peanuts that more than one region in which large asymptotics applies is possible. Then numerical integration using Maple’s (see footnote 9). rkf45 solver reveals that precession can occur in the intermediate-\( R \) region, so that orbits can have arbitrarily many large (and small) asymptotic regions. However, the
Figure 7. Using a form valid for both first and second small approximations, the partial moments of inertia are, for \( \Phi_0 = 0 \), \( I_1 = \frac{I}{1 + q_0 + g \cos(2\Phi)} \) and \( I_2 = \frac{I(q_0 + g \cos(2\Phi))}{1 + q_0 + g \cos(2\Phi)} \). (a) Then the three types of behaviour of the first large approximation are: ‘\( I_2 \) is surrounded by \( I_1 \)’, ‘\( I_1 \) and \( I_2 \) touching’, and ‘\( I_1 \) and \( I_2 \) cross-over’ (of which I provide two representatives). The touching case is the limiting case between the other two cases. The only instance in which the small approximation is self-consistent is in the last picture for \( \Phi \) within wedges around five angular tick-marks wide either side of 0 and of \( \pi \). (b) Next, here are sketches of the nine further behaviours exhibited by the second approximation. The first three are limiting behaviours on the \( q_2 = q_2^0 \) parabola, of which the second is itself the limiting behaviour between the other two a single point (corresponding again to the \( R = 1 \) solution). The next three are the unfolded version of ‘cross-over’, the reverse case of ‘touching’ (again a limiting behaviour) and the reverse case of ‘is surrounded’. The last four are all cases in which there occurs 2, 1, 3 collinearity with particle 1 at the centre of mass of particles 2, 3: ‘is surrounded’, a ‘touching’ limiting case, and two instances of ‘cross-over’. The outwards-lying arrows indicate for which \( \Phi \) in each case the small approximation is self-consistent. (c) Here I provide a sketch (not to scale) of which regions of the parameter space these various cases reside in. The dashed curve touches the parabola, thus trisecting the classically allowed region into: a region above and to the left of the dashed curve where the small asymptotics is everywhere valid, a region to the right of the dashed curve where it is valid in some wedges, and a region below and to the left of the dashed curve where it is nowhere valid. (d) A few examples of reading off what is happening in the particle position picture. The third and sixth cases in 7(b) correspond to particle 1 describing a closed curve round particles 2, 3. The fifth case describes particle 1 coming in from infinity (= particles 2, 3 colliding) and the approximation breaking down for some \( \Phi_0 \). The tenth case describes particle 1 coming in from infinity and reaching the centre of mass of particles 2, 3 at some angle \( \Phi_0 \).
the input $R$ initial datum (and checks that the evolution does not take one away from $E = 0$),
thus amending the error.

5. Normal coordinates for scalefree triangleland with multi-harmonic oscillator type potentials

5.1. A rotation sending the general case to the special case in new coordinates

One can avoid having a $C$-term by performing a rotation or normal coordinates construction,
which preserves the form of the kinetic term while sending the potential to a $C$-free form in
the new coordinates. One can get to these by inserting such a rotation between steps 1 and 2
in section 3.2, or by rotating coordinates at the level of the equations of motion. I denote
the new coordinates with N-subscripts (N for normal). Due to the general case in N-subscripted
coordinates taking the same form as the special case in the original coordinates, we can uplift
section 4 to an exact solution of general multiple harmonic oscillator case by inserting N-
subscripts and, at the end of the calculation, rotating back from normal Jacobi coordinates to
a more complicated form in terms of the original Jacobi coordinates.

What is the requisite rotation angle? From the matrix equation $R(\text{through angle } \alpha \text{ about y axis}) \ n = n_N$,

\[
\begin{pmatrix}
\cos \alpha & 0 & -\sin \alpha \\
0 & 1 & 0 \\
\sin \alpha & 0 & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
C \\
0 \\
B
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
B_N
\end{pmatrix},
\]
where $n, n_N$ are unit vectors considered to be in unit-radius spherical coordinate form about
the original and normal-coordinate axes, the requisite rotation is by

\[
\alpha = \arctan(C/B):
\]
this sends $B \cos \Theta + C \sin \Theta \cos \Phi$ to $B_N \cos \Theta_N$. Then

\[
\sin \alpha = C/\sqrt{B^2 + C^2}, \quad \cos \alpha = B/\sqrt{B^2 + C^2}, \quad B_N = \sqrt{B^2 + C^2},
\]
Figure 8. (a) The first large approximation’s family of tangent circles in the \((R, \Phi)\) plane. (b) Additional behaviours of the second large approximation: bulging, rectangular and thin tear drops, ellipse-like and peanut-like curves and circles centred on the origin. The partition of \(q_4, q_6\) parameter space here is the same as that of \(q_0, q_1\) parameter space in figure 6(c); indeed figure 6(c) is an important clarification of the behaviour exhibited by the large approximation.

\[
\cos \Theta_N = \frac{B \cos \Theta + C \sin \Theta \cos \Phi}{\sqrt{B^2 + C^2}} \quad \text{and} \quad \Phi_N = \arctan(\sqrt{B^2 + C^2} \sin \Theta \sin \Phi / \{B \sin \Theta \cos \Phi - C \cos \Theta\}).
\]  

(97)

The potential is now

\[
\left\{ K_1^N \rho^2_1 + K_2^N \rho^2_2 \right\} / 8I = A_N + B_N \cos \Theta_N.
\]  

(98)
It is also useful to note for later use the following coefficient interconversions:

\[ A_N = A, \quad K_N^1 = 8(A - \sqrt{B^2 + C^2}), \quad K_N^2 = 8(A + \sqrt{B^2 + C^2}), \]  
\[ K_1^N = K_1 + K_2 - \sqrt{[K_1 - K_2]^2 + L^2}, \quad K_2^N = K_1 + K_2 + \sqrt{[K_1 - K_2]^2 + L^2}. \]  

From the spherical perspective, the normal coordinates solution has the same form as the special solution in the original coordinates, but now one is to project onto the general tangent plane rather than the tangent plane at the North Pole, interpreting the general stereographic coordinate thereat as the ratio of the square roots of the barycentric partial moments of inertia. This permits graphical sketches of the qualitative behaviour rather than fairly lengthy analytical expressions constructed by passing from \((\Theta_N, \Phi_N)\) coordinates to \((\Theta, \Phi)\) coordinates and then on to mechanically significant variables.

5.2. Examples: preamble

Note that the very special case’s solution is invariant under this rotation: that has no preferred axis. So one needs to look slightly further to obtain nontrivial examples. First I give the simplest nontrivial example of the first small asymptotics from both an analytical and a graphical presentation. See figure 9 for this section’s different meaning of ‘small’ and ‘large’ regimes. For this paper to be of manageable length I then only provide a graphical perspective for the first large asymptotics, and brief comments on further cases.

5.3. First small asymptotics solution for general case

From (81) with \(N\)-subscripts, \(\tan \frac{\Theta_N}{2} = \sin \Theta / \{1 + \cos \Theta_N\}\) (97), (99) and elementary cancellations, for \(\Phi_N^0 = 0\) the analytic solution for this takes the form

\[ B_N + B \cos \Theta + C \sin \Theta \cos \Phi = \sqrt{2q_N^0} \{B \sin \Theta \cos \Phi - C \cos \Theta\}, \]  
where \(q_N^0\) bears the same relation to \(K_2^N\) as \(q_0\) bears to \(K_2\). Then in terms of \((R, \Phi)\),

\[ \{B_N - B - \sqrt{2q_N^0 C}\} R^2 + 2\{C - \sqrt{2q_N^0 B}\} R \cos \Phi + B_N + B + \sqrt{2q_N^0 C} = 0. \]  
Or, in terms of straightforward relational variables \(I_1, I_2, \Phi)\,

\[ \{B_N - B - \sqrt{2q_N^0 C}\} I_1 + \{B_N + B + \sqrt{2q_N^0 C}\} I_2 + 2\{C - \sqrt{2q_N^0 B}\} \sqrt{I_1 I_2} \cos \Phi = 0. \]  
Finally in terms of the original variables for the problem,

\[ \{B_N - B - \sqrt{2q_N^0 C}\} \|\parallel I_1\parallel^2 + \{B_N + B + \sqrt{2q_N^0 C}\} \|\parallel I_2\parallel^2 + 2\{C - \sqrt{2q_N^0 B}\} \parallel I_1\parallel \cdot \parallel I_2\parallel = 0. \]  
Like for the \(C = 0\) case (83), this is a second-order homogeneous polynomial in \(\parallel I_1\parallel, \parallel I_2\parallel\) and \(\sqrt{I_1 \cdot I_2}\). See also figure 10 for interpretation of the above formulae.

5.4. First large asymptotics solution for general case

See figure 11.

5.5. Further examples

One can go on from here to provide in rotated coordinates general solutions to the second small and large asymptotics, and of the general numerical behaviour in the region in between. Here are some brief comments.
Figure 9. \(N\)-large and \(N\)-small domains of applicability map to the original \((R, \Phi)\) plane as indicated.

(1) Similarly to the \(C = 0\) approximation having turn-around ellipses rather than having to follow straight lines, the second small approximation allows for turn-around behaviour rather than having to complete the first small approximation’s circles: approximate circular arc, turn-around, approximate circular arc in opposite direction to the original.

(2) One can now have asymmetric bulges where one had symmetric ones before (e.g. in the first large approximation or in the peanut case of the second large approximation); indeed one bulge can non-generically become infinitely big (like the straight line in the first large approximation) and even form another contribution ‘beyond infinity’ which shows up on the other side of the opposite bulge.

6. Scaled triangleland at the classical level

6.1. Scale-shape coordinates \((I, R, \Phi)\) or \((I, \Theta, \Phi)\)

The general scaled triangleland with multi-harmonic oscillator like potential’s Jacobi-type action is

\[
S[I, \Theta, \Phi, I, \dot{\Theta}, \dot{\Phi}] = 2 \int d\lambda \sqrt{T[E + U]}
\]

\[
= 2 \int d\lambda \sqrt{I^2 + I^2(\Theta^2 + \sin^2 \Theta \Phi^2)} \left\{ \frac{E + U(I, \Theta, \Phi)}{4I} \right\}.
\]

(105)
Figure 10. (a) Sketch of how the first small approximation’s parallel straight lines in the \((R, \Theta)\) plane project onto the \((\Theta, \Phi)\) sphere. (b) \((R, \Phi)\) sketches for \(C \neq 0\), either from plotting the analytical function or from projecting the sketch on the sphere onto the tangent plane of the appropriately rotated North Pole. The family of parallel straight lines for \(C = 0\) is now a family of circles tangent to a single point. Within the (unshaded) region where this solution is valid, one therefore obtains circular arcs. (c) Subsequent sketches of \(I_1\) and \(I_2\) as functions of \(\Phi\) showing some of the distortions which occur when a \(C\)-term is switched on in the ‘cross-over’ case of figure 7(a). All subfigures have \(A = 1 = B\) and \(E = 9\). The first picture is for \(C = 0.1\). The second picture is for \(C = 0.5\), by which stage \(I_2\) encloses the origin. The third picture is for \(C \approx 2.9\), for which \(I_1\) and \(I_2\) touch, after which \(I_2\) ‘is surrounded’ by \(I_1\) (fourth picture).

The Euler–Lagrange equations are

\[
I_{\dot{\Theta}} - I_1 \dot{\Theta} \{\dot{\Theta}^2 + \sin^2 \Theta \dot{\Phi}^2\} + \frac{1}{4I} \left\{ \frac{\partial V}{\partial I} + E + U \right\} = 0,
\]

(106)

\[
\{I_2 \dot{\Phi}\}^2 - I_2 \Phi \dot{\Theta} \sin \Theta \cos \Theta + \frac{1}{4I} \frac{\partial V}{\partial \Theta} = 0,
\]

(107)

\[
\{I_2 \sin^2 \Theta \dot{\Phi}\}^2 + \frac{1}{4I} \frac{\partial V}{\partial \Phi} = 0,
\]

(107)

and there is an accompanying energy integral

\[
\{I_{\dot{\Theta}}^2 + I_1 \{\dot{\Theta}^2 + \sin^2 \Theta \dot{\Phi}^2\}\}/2 + V/4I = E/4I.
\]

(108)

(Above,

\[
* = \sqrt{\frac{E + U}{T}} = \frac{1}{4I} \sqrt{\frac{E + U}{T}} = \frac{1}{4I} \ast
\]

(109)
Figure 11. (a) Sketch of how the first large approximation’s family of tangent circles in the 
($\mathcal{N}, \Theta$) plane project onto the ($\Theta, \Phi$) sphere. (b) ($\mathcal{N}, \Phi$) sketches for $C \neq 0$ either from plotting the
analytical function or from projecting the sketch on the sphere onto the tangent plane of the
appropriately rotated North Pole. One still has arcs that touch at a point, but this point is no longer
at the origin but is rather shifted away from it along the $\Phi = 0$ axis. This does not cause any
notable changes to $I_1$ and $I_2$ as functions of $\Phi$, so I do not provide sketches of these.

In the case in which $V$ is independent of $\Phi$, (107) simplifies to the first integral
\[ I^2 \sin^2 \Theta \Phi^2 = J. \]  
(110)
See appendix C for physical interpretations of the relative angular momentum quantity $J$.

Then the other Euler–Lagrange equations and the energy integral take the forms
\[
\begin{align*}
I^{\dot{\dot{\phantom{\Psi}}} - I^{\dot{\Phi}}} & = -\frac{J^2}{I^3 \sin^2 \Theta} + \frac{1}{4I} \left\{ \frac{\partial V}{\partial I} + \frac{E - V}{I} \right\} = 0, \\
\{I^2 \dot{\Theta}^2 - \frac{J^2 \cos \Theta}{I^2 \sin^3 \Theta} + \frac{1}{4I} \frac{\partial V}{\partial \Theta} &= 0, \\
\frac{I^{\dot{\Phi}}}{2} + \frac{J^2 \dot{\Theta}^2}{2} + \frac{J^2}{2I^2 \sin^2 \Theta} + \frac{V}{4I} &= \frac{E}{4I}.
\end{align*}
\]  
(111)
(112)

6.2. Scaled triangleland with general multi-harmonic oscillator potential

The usual general multi-harmonic oscillator potential maps to
\[
\frac{1}{4I} \left\{ \left\{ K_1 + K_2 \right\}/4 + \left\{ K_2 - K_1 \right\}/4 \cos \Theta + \left\{ L/4 \right\} \sin \Theta \cos \Phi \right\}
= A + B \cos \Theta + C \sin \Theta \cos \Theta.
\]  
(113)

6.3. Sketch of the potential

The sketch of potential is as for the corresponding similarity problem. The sketch of $\tilde{V} - \tilde{E}$
is slightly different from that of $V - E$ for the similarity problem in that one has a further
looking for critical points,
\[
\frac{\partial (\tilde{V} - \tilde{E})}{\partial I} = E/4I^2,
\]  
(114)
so one also needs $E = 0$ in order to have these. After that the analysis is as before (including
the picking out of a preferred axis except in the $B = C = 0$ case), except that the Hessian has
an extra row and column of zeros.
6.4. Classical equations of motion

The Jacobi-type action for this problem is then
\[ S = 2 \int \sqrt{\frac{I^2 + I^2(\Theta^2 + \sin^2 \Phi^2)}{2}} \left\{ \frac{E}{4I} - A - B \cos \Theta - C \sin \Theta \cos \Phi \right\}. \]  
(115)

The Euler–Lagrange equations are
\[ I \dot{\Theta} + I \dot{\Phi} \dot{\Theta} + \frac{E}{I} = 0, \]  
(116)
\[ \{I^2 \dot{\Theta}^2 - I^2 \dot{\Phi}^2 \sin \Theta \cos \Theta + C \cos \Theta \cos \Phi - B \sin \Theta \} = 0, \]  
(117)
\[ \{I^2 \sin^2 \Theta \dot{\Phi} \}^2 - C \sin \Theta \sin \Phi = 0. \]  
(118)

The energy integral is
\[ \frac{I^{42} + I^2 \{\Theta^2 + \sin^2 \Phi^2\}}{2} + A + B \cos \Theta + C \sin \Theta \cos \Phi = \frac{E}{I}. \]  
(119)

For small \( \mathcal{R} \) or \( \Theta \) and for large \( \mathcal{R} \) or small \( \Xi = \pi - \Theta \), section 4’s results for approximations to the potential hold again.

6.5. Special case

In the special case of \( C = 0 \), (107) applies and the remaining Euler–Lagrange equations and energy integral become
\[ \frac{I^{42}}{2} + I^2 \{\Theta^2 + \sin^2 \Phi^2\} + A + B \cos \Theta + C \sin \Theta \cos \Phi = \frac{E}{I}. \]  
(120)
\[ \frac{I^{42}}{2} + I^2 \{\Theta^2 + \sin^2 \Phi^2\} + A + B \cos \Theta + C \sin \Theta \cos \Phi = \frac{E}{I}. \]  
(121)

6.6. Analogy between very special case and the Kepler–Coulomb problem

The very special Euclidean relational particle mechanics harmonic oscillator banal-conformally maps to the Kepler problem with
\begin{align*}
\text{(radius)} &= r \leftrightarrow I \text{ (total moment of inertia)}, \\
\text{(test mass)} &= m \leftrightarrow 1, \\
\text{(angular momentum)} &= L \leftrightarrow J \text{ (relative angular momentum—see appendix C)}, \\
\text{(total energy)} &= E \leftrightarrow -A = -(\text{sum of mass-weighted Jacobi–Hooke coefficients})/16
\end{align*}
(122) to (125)

and
\begin{align*}
\text{(Newton’s gravitational constant)(massive mass)(test mass)} &= GMm \leftrightarrow \Xi \text{ (total energy)/4} \\
\text{(or to the 1-electron atom Coulomb problem with this last analogy replaced by)}
\end{align*}
(126)
\begin{align*}
\text{(nuclear charge) (test charge of electron)/4\pi} \text{ (permittivity of free space)} &= (Ze)e/4\pi\varepsilon_0 \leftrightarrow \bar{E}(\text{total energy)/4}).
\end{align*}
(127)
Also note that the positivity of the Hooke’s coefficients translates to the requirement that the gravitational or atomic energy be negative, i.e. to bound states. While, the positivity of $E$ required for classical consistency corresponds to attractive problems like the Kepler problem or the atomic problem being picked out, as opposed to repulsive Coulomb problems.

Also, the special case corresponds to the same ‘background electric field’ that the rotor was subjected to in section 4, which, moreover, is proportional to $\cos \Theta$ which is analogous to $\cos \theta$, which is in the axial (‘$z$’) direction but is not the well-known mathematics of the axial (‘$z$’) direction Stark effect for the atom, which involves, rather, $r \cos \theta$. But, nevertheless, the situation in hand is both closely related to the rotor situation in section 4 and to the mathematics of the atom in parabolic coordinates (see, e.g. [70, 74]). The general case is then the same situation but with the ‘electric field’ pointing in an arbitrary direction. The idea then is to use the obvious analogue of the scheme in figure 3 to solve Euclidean relational particle mechanics problems in straightforward relational, relative and absolute terms.

6.7. Exact solution for the very special case

Now

$$I^{1/2}/2 + J^{1/2}/2 I \sin^2 \Theta_0 + A = E/I$$  \hspace{1cm} (128)

(usually one would set $\Theta_0 = \pi/2$ without loss of generality, however the present physical interpretation has the value of $\Theta_0$ be meaningful, as $\Theta_0 = 2 \arctan(i_1/i_2)$). Thus the solutions are conic sections

$$I = l/\{1 + e \cos(\Phi - \Phi_0)\},$$  \hspace{1cm} (129)

where the semi-latus rectum and the eccentricity are given by

$$l = J^2/E \sin^2 \Theta_0, \quad e = \sqrt{1 - 2AJ^2/E^2 \sin^2 \Theta_0}.$$  \hspace{1cm} (130)

So, in terms of straightforward relational variables,

$$I_1 + I_2 = l/\{1 + e \cos(\Phi - \Phi_0)\},$$  \hspace{1cm} (131)

and in terms of the original variables of the problem,

$$\{l_1^2 + l_2^2\}/\{l_1^2 l_2^2\} + e\{l_1 \cdot l_2 \cos \Phi_0 + \sqrt{l_1^2 l_2^2 - (l_1 \cdot l_2)^2} \sin \Phi_0\} = l.$$  \hspace{1cm} (132)

Note that this is non-homogeneous (it rearranges to a non-homogeneous eighth-order polynomial); this is OK as solutions of Euclidean relational particle mechanics do not have to be scale invariant.

In moment of inertia-relative angle space, for $2A = \{\mathcal{E} \sin \Theta_0/J\}^2$ one has circles, for $0 < 2A < \{\mathcal{E} \sin \Theta_0/J\}^2$ one has ellipses, for $A = 0$ one has parabolas (corresponding to the case with no springs). The hyperbolic solutions ($A < 0$) are not physically relevant here because this could only be attained with negative Hooke’s coefficient springs. The circle’s radius is $I = l = \mathcal{E}/2A$ while for the ellipses $I$ is bounded to lie between $J/\sqrt{2A \sin \Theta_0}$ and $\mathcal{E}/2A$. The smallest $I$ attained in the parabolic case is $J^2/2\mathcal{E} \sin^2 \Theta_0$. The period of motion for the circular and elliptic cases is $\pi \mathcal{E}/\sqrt{2A^3}$.

As regards the individual subsystems, combining the fixed plane equation and the $I(\Phi)$ relation,

$$I_1 = l \sin^2 \frac{\Theta}{2}/\{1 + e \cos(\Phi - \Phi_0)\}, \quad I_2 = l \cos^2 \frac{\Theta}{2}/\{1 + e \cos(\Phi - \Phi_0)\}$$  \hspace{1cm} (133)

so each of these behave individually similarly to the total $I$. In the $\Theta_0 = \pi/2$ plane, they are both equal (and so equal to $I/2$). Circle and ellipse cases have $I_1$ and $I_2$ as closed bounded curves which sit inside the curve that $I$ traces. The parabolic case has $I_1$, $I_2$ curves to the ‘inside’ of the parabola that $I$ traces.

33
6.8. Special case solved

Use $I_1$, $I_2$, $\Phi$ coordinates given by

$$I_1 \equiv (I - Z)/2 \equiv I(1 - \cos \Theta)/2 \quad \text{and} \quad I_2 \equiv (I + Z)/2 \equiv I(1 + \cos \Theta)/2,$$  \hfill (134)

which invert to

$$I = I_1 + I_2, \quad \Theta = \arccos((I_2 - I_1)/(I_1 + I_2))$$  \hfill (135)

and are mathematically parabolic coordinates scaled by $1/2$, which moreover in the present relational context have the physical interpretation of partial moments of inertia of the two subsystems. Then

$$S = 2 \int d\lambda \sqrt{T[E + U]} = 2 \int d\lambda \sqrt{\frac{1}{2} \left\{ \frac{I_1^2}{I_1} + \frac{I_2^2}{I_2} + \frac{4I_1I_2\Phi^2}{I_1 + I_2} \right\} \left\{ \frac{E - K_1I_1 + K_2I_2}{8} \right\}},$$  \hfill (136)

for $T, U, E$ as before. Then the $\Phi$-Euler–Lagrange equation is

$$\frac{4I_1I_2\Phi^2}{I_1 + I_2} = J,$$  \hfill (137)

and the energy integral is, subsequently,

$$\frac{I_1^2}{2I_1} + \frac{I_2^2}{2I_2} + \frac{J^2}{8} \left\{ \frac{1}{I_1} + \frac{1}{I_2} \right\} + \frac{K_1I_1 + K_2I_2}{8} = \frac{E}{4},$$  \hfill (138)

which separates into

$$4I_1^2 + J^2 + K_1I_1^2 = 2EiI_i$$  \hfill (139)

for $E_1 + E_2 = E$. This is solved by

$$\hat{t} - t_0 = \left\{ \frac{2}{\sqrt{K_1}} \right\} \arccos\left( \frac{I_iK_i - E_i}{\sqrt{E_i^2 - K_iJ_i^2}} \right)$$  \hfill (140)

(in agreement with [17], once differences in convention are taken into account). Thus, synchronizing, one part of the equation for the orbits is

$$\sqrt{K_2} \arccos\left( \left\{ \frac{I_1K_1 - E_1}{\sqrt{E_1^2 - K_1J_1^2}} \right\} \right) = \sqrt{K_1} \arccos\left( \left\{ \frac{I_2K_2 - E_2}{\sqrt{E_2^2 - K_2J_2^2}} \right\} \right).$$  \hfill (141)

(One can see how the arccosines cancel in the very special case. Then $E_1 = E_2 = E/2$ gives $t_1 = t_2$, i.e. $\Theta = 2\arctan(t_1/t_2) = 2\arctan(1) = \pi/2$, so are confined to the plane perpendicular to the chosen $Z$-axis.) Then the $\Phi$-Euler–Lagrange equation implies, in the synchronized case

$$\Phi - \Phi_0 = J \int \frac{d\tau}{\sqrt{F_i \cos \tau_i + E_i}} = \frac{2}{\sqrt{K_1}} \sum_{i=1}^{2} \arctan\left( \frac{\sqrt{[E_i - F_i][F_i - A_i]}}{[E_i + F_i][F_i + A_i]} \right)$$  \hfill (142)

(for $\tau_i = 2\sqrt{K_i}(t - t_0)$, $F_i = \sqrt{E_i^2 - K_iJ_i^2}$ and $A_i = K_iI_i - E_i$), which simplifies to

$$\Phi - \Phi_0 = \frac{2}{\sqrt{K_1}} \sum_{i=1}^{2} \arctan\left( \frac{\sqrt{[E_i^2 - K_iJ_i^2 - E_i][I_i + J_i^2]}}{[E_i^2 - K_iJ_i^2 + E_i][I_i^2 - J_i^2]} \right)$$  \hfill (143)

in the straightforward relational variables. While, in the original variables of the problem,

$$\sqrt{K_2} \arccos\left( \left\{ \frac{E_1}{\sqrt{E_1^2 - K_1J_1^2}} \right\} \right) = \sqrt{K_1} \arccos\left( \left\{ \frac{E_2}{\sqrt{E_2^2 - K_2J_2^2}} \right\} \right).$$  \hfill (144)
\[
\arccos\left(\frac{\iota_1 \cdot \iota_2}{\|\iota_1\| \|\iota_2\|}\right) = \Phi_0 + \sum_{i=1}^{2} \arctan
\times \left(\sqrt{\left[\left\{\sqrt{E_1^2 - K_1 J^2 - E_1}\right\} \|\iota_1\|^2 + J^2\right] / \left[\left\{\sqrt{E_1^2 - K_1 J^2 + E_1}\right\} \|\iota_1\|^2 - J^2\right]}\right). \tag{144}
\]

6.9. The single harmonic oscillator case requires a separate working

For \(K_1 = K_2 = 0\), the trajectories are given by, after some manipulation,
\[
\sqrt{E_2/E_1} I_2 = I_1 = \sec(E_1(\Phi - \Phi_0)/E)/\sqrt{2E_1}, \tag{145}
\]
which is obviously the expected straight line in the absence of forces.

For \(K_2 = 0, K_1 \neq 0\), the trajectories are given by, in straightforward relational variables, and synchronizing,
\[
\{I_1 K_1 - E_1\}/\sqrt{E_1^2 - K_1 J^2} = \cos\left(\sqrt{2K_1/E_2\sqrt{2E_2 I_2} - J^2}\right) \tag{146}
\]
and
\[
\Phi - \Phi_0 = \arctan\left(\sqrt{\left[\left\{\sqrt{E_1^2 - K_1 J^2 - E_1}\right\} I_1 + J^2\right] / \left[\left\{\sqrt{E_1^2 - K_1 J^2 + E_1}\right\} I_1 - J^2\right]}\right)
\]
\[+ \arctan\left(\sqrt{2E_2 I_2/J^2 - 1}\right). \tag{147}\]

While, in terms of the original variables of the problem,
\[
\{K_1 \|\iota_1\|^2 - E_1\}/\sqrt{E_1^2 - K_1 J^2} = \cos\left(\sqrt{2K_1/E_2\sqrt{2E_2 \|\iota_2\|^2} - J^2}\right). \tag{148}\]

6.10. A brief interpretation of the previous two subsections’ examples

In subsection 6.9’s example, \(I_2\) (or \(I_2\)) makes a good timestandard as the absolute space intuition of it ‘moving in a straight line’ survives well enough to confer monotonicity. It is convenient then to rewrite (146), (147) as a curve in parametric form with \(I_2\) playing the role of parameter, leading to the plots in figure 12.

In subsection 6.8’s example, \(I_1\) and \(I_2\) oscillate boundedly, so neither of these (or \(I_1\) or \(I_2\) is a good clock parameter from the point of view of monotonicity. There is again some scope for variation in relative angle \(\Phi\), including ‘sporadic’ amplitude variations.

6.11. Normal coordinates for scaled triangleland with multi-harmonic oscillator potential

The working of section 5 holds again (using now \(x, x_N\) in place of \(n\) and \(n_N\) but radii are unaffected by rotations and so cancel out giving the same rotation and \(\Theta, \Phi\) to \(\Theta_N, \Phi_N\)
Figure 12. (a) 3D plot showing oscillatory behaviour including some changes in the size of the relative angle that occur at regular intervals but can involve ‘sporadic’ changes in how much the relative angle changes in each interval. (The particular plot given is for the 1 harmonic oscillator case, with $K_1 = E_1 = E_2 = 1$ and $J = 0.1$, with $\Phi$ plotted vertically, $I_2$ out of the page and $I_1$ into the page.) This is because although 1 is ‘moving in a straight line’ in absolute space, the position with respect to which its separation is being measured from in relational space (the centre of mass of particles 2, 3) is then also moving around due to the oscillations of the ‘spring’ between these particles. (b) Polar plots of $I_1$ and $I_2$ as functions of $\Phi$ for the first oscillation. Further oscillations correspond to similar angular variations at larger radius. Note well that $I_1$ and $I_2$ are independent in Euclidean relational particle mechanics, as opposed to summing to a constant $I$ in similarity relational particle mechanics. In the solution exhibited, the particles expand away from triple collision while relative angle varying oscillations occur, involving almost-isosceles to almost-collinear changes in shape.

coordinate change as before). For Euclidean relational particle mechanics, one can uplift from the preceding parts of section 6 by inserting the above extra steps into the triangleland case of section 3.1.

The potential is now

$$\left\{ K_N^1 \rho_{1N}^2 + K_N^2 \rho_{2N}^2 \right\}/8 = A_N + B_N \cos \Theta_N. \quad (150)$$

Then the Jacobi-type action for scaled triangleland with general multi-harmonic oscillator potential in shape-scale variables is

$$S = 2 \int d\lambda \sqrt{\frac{\dot{I}_1^2 + \dot{I}_2^2 \Theta_N^2 + \sin^2 \Theta_N \Phi_N^2}{2}} \left\{ \frac{E}{I} - A_N - B_N \cos \Theta_N \right\}. \quad (151)$$

The Euler–Lagrange equations and energy integral that follow from this are, respectively and after discovering the conserved quantity $J$ and eliminating it, (120), (121) with N-subscripts appended. The momenta are (117) with N’s appended in the last two and the Hamiltonian and the energy constraint are (118) treated likewise.

I have then rotated the above two exact solutions for the special case to obtain solutions to the general case, but these are too lengthy to include in this paper.
7. Conclusion

7.1. Results summary

Relational particle models are useful as regards the long-standing absolute versus relative motion debate, and also due to structural similarities with the geometrodynamical formulation of general relativity, for problem of time in quantum gravity. 1- and 2D relational particle mechanics are tractable due to the simple nature of their configuration space geometries: these are, respectively, $S^k$ and $CP^k$ for 1- and 2D similarity relational particle mechanics. Additionally, for the 3-particle case of the latter (‘scalefree triangleland’), $CP^3 = S^3$ holds, making this case even more tractable. This and its Euclidean relational particle mechanics counterpart (‘scaled triangleland’) furish this paper’s particular examples.

I consider models with general multiple harmonic oscillator type potentials which, as compared with the earlier study [17], include the new feature of relative angular momentum exchange between the two constituent subsystems. I get there by first considering the ‘special’ subcase (which has no relative angle $\Phi$ dependence in its potential and involves no relative angular momentum exchange) and its ‘very special’ sub-subcase (which has constant potential in one presentation). I then identify scalefree triangleland’s very special case’s mathematics with that of the linear rigid rotor; the special case is then analogous to that with a background electric field aligned with its axis. The Euclidean relational particle mechanics special and very special cases’ mathematics reduces to some of the mathematics that arises in the Kepler–Coulomb problem. Finally, I use a rotation or normal coordinates construction to cast the general ($\Phi$-dependent, relative angular momentum exchanging) case as the special case in the transformed coordinates. This has the mathematics corresponding to the analogue background electric field being unaligned with the axis. In each case I use the standard spherical or planar mathematics that I have been able to cast the problem into so as to obtain solutions, and then map back to provide physical interpretation in terms of various mechanically significant quantities that are more intuitively associated with the original relational problems: (barycentric partial moment of inertia, $\Phi$) variables, mass-weighted Jacobi inter-particle (cluster) coordinates, and by sketches, what the particles themselves are doing. In returning to these various levels, the standard spherical, planar and flat space mathematics becomes unusual and nontrivial.

Highlights of my results are as follows.

1. The very special multiple harmonic oscillator like potential scalefree triangleland problem is straightforwardly soluble on the sphere and retains a manageable form in terms of the underlying mechanical variables.

2. While the special multiple harmonic oscillator like potential scalefree triangleland problem is also classically exactly soluble on the sphere, its form is very complicated, so I just provide more manageable large and small asymptotic solutions.

3. In the stereographic plane, the small asymptotics solution has the mathematics of the 2D isotropic harmonic oscillator $((k, \text{constant}) \times \text{radius})^2$ potential, including the upside-down ($k < 0$) and degenerate ($k = 0$) cases, whereby I obtain complete control of it and then characterize it in terms of the underlying mechanical variables.

4. The large asymptotics solution maps to the small asymptotics solution again, under inversion of the radius, so I also obtain complete control of it and can characterize it in terms of the underlying mechanical variables. Moreover, this is important beyond the case with harmonic oscillator like potentials, as scalefree triangleland exhibits universal large-scale behaviour, various cases of which I can now understand through their being
related by the inversion to the various cases of conic sections that occur for \( k \times (\text{radius})^2 \) potentials.

(5) The very special and special multiple harmonic oscillator Euclidean relational particle mechanics problems are also classically exactly soluble in the conformally related flat space in which they respectively take the forms of the Kepler–Coulomb problem and a nonstandard composition of parabolic coordinate subproblems thereof.

(6) Then by a rotation or normal coordinates construction, the general relative-angle dependent similarity and Euclidean relational particle mechanics multiple HO (type) potential problems are exactly soluble because they map to their special cases in the new coordinates. However, these solutions are very complicated in terms of the underlying mechanical variables, and as such I mostly only provide sketches of some aspects of their behaviour.

7.2. Further tractable cases: 4-stop and N-stop metrolands

Because for scalefree 4-stop metroland (relational particle mechanics of 4-particles in 1D) the \( R_j \) are different and related to \( \Theta, \Phi \) in a different way, physically interesting potentials in this case generally map to different functions of these coordinates for the 4-stop metroland interpretation and for the triangleland interpretation. Thus one needs new calculations for 4-stop metroland rather than straightforward deduction from this paper’s triangleland results. For example the harmonic oscillator type potential maps to

\[
V = \frac{K_3 \cos^2 \Theta + K_1 \sin^2 \Theta \cos^2 \Phi + K_2 \sin^2 \Theta \sin^2 \Phi}{2} = \tilde{A} \cos(2\Theta) + \tilde{B} \sin^2 \Theta \cos(2\Phi)
\]

(152)

for \( \tilde{A} = \{K_1 + K_2 + 2K_3\}/8, \tilde{B} = \{-K_1 - K_2 + 2K_3\}/8, \tilde{C} = \{K_1 - K_2\}/4 \). Thus this model admits direct analogues of this paper’s ‘special’ (\( C = 0 \)) and ‘very special’ (\( B = C = 0 \)) subcases.

While, generalizing to scalefree N-stop metroland, the multiple harmonic oscillator type potential maps to

\[
V = \frac{1}{2} \sum_{p=1}^{n-1} K_p R_p^2 + K_n}{\sum_{p=1}^{n-1} R_p^2 + 1} = \frac{1}{2} \sum_{p=1}^{n} K_p n_p^2
\]

(153)

for \( n_p \) the unit vector in the Euclidean configuration space \( \mathbb{R}^n \). Scaled N-stop metroland is also of interest [35].

7.3. Further work and applications

This paper’s models remain to be studied from the perspective of dynamical systems [71]. Its principal application at the moment is that many aspects of this work carry over to QM in paper II (similarity case) and [37] (Euclidean case), and on towards the study of many problem of time strategies (in particular emergent semiclassical time and records theory [19, 20, 31] but also conceivably internal time approaches and histories theory, as well as investigation of various quantum gravity and quantum cosmology applications such as the problem of observables, operator ordering, closed-universe effects, finite-universe effects and the study of small inhomogeneities/clumps). Some of these applications would benefit from study of further models: with other potentials (for which universal large asymptotics results in this paper will be useful), the above 4-stop and N-stop metrolands of comparable tractability to this paper, as well as somewhat harder relational particle mechanics in 2D of \( N > 3 \) particles.
(requiring $C\mathbb{P}^{N-2}$ geometry based methods) and 3D models that are much harder [64] even for modest values of $N$.

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Appendix A. Mechanics on an in general curved configuration space

For a general finite theory of the quantities $Q_A$ with a curved configuration space, consider the Jacobi action

$$S = 2 \int \mu \sqrt{\mathcal{T} \{U + E\}},$$

(A.1)

where $U(Q_A)$ is minus the potential energy $V(Q_A)$, $E$ is the total energy and $T$ is the kinetic energy

$$T = \mathcal{M}_{AB} Q^A Q^B / 2,$$

(A.2)

for $\mathcal{M}_{AB}$ the curved configuration space metric.

This action works as follows. For $* \equiv \sqrt{\mathcal{T} \{U + E\} / T}$, the Euler–Lagrange equations are, in geometrical form,

$$Q^{A*} + \Gamma^A_{BC} Q^{B*} Q^{C*} = - \frac{\partial V}{\partial Q_A}$$

(A.3)

and there is an energy first integral

$$\mathcal{M}_{AB} Q^{A*} Q^{B*} / 2 + V(Q^C) = E.$$  

(A.4)

Appendix B. Momenta, Hamiltonians and energy constraints

These are important as regards the passage to quantum theory in paper II and [37].

B.1. General curved configuration space mechanics

The conjugate momenta are

$$P_A = \mathcal{M}_{AB} Q^B,$$

(B.1)

and there is then as a primary constraint the quadratic energy constraint

$$H \equiv N^{AB} P_A P_B / 2 + V = E$$

(B.2)

for $N^{AB}$ the inverse of $\mathcal{M}_{AB}$ and $H$ the Hamiltonian for the system. This is propagated by (A.3). Such energy constraints that have quadratic but not linear dependence on the momenta are analogous to the GR Hamiltonian constraint and carries associated with its form the frozen formalism aspect of the problem of time [29, 33].
B.2. Scaled triangleland in \((i_1, i_2, \Phi)\) coordinates

The conjugate momenta are

\[
P_\rho^i = i_\rho^i, \quad P_\Phi = \frac{i_1^2 i_2^2 \Phi}{\sqrt{i_1^2 + i_2^2}}. \tag{B.3}
\]

The classical Hamiltonian and energy constraint are then

\[
H = \frac{1}{2} \sum_{i=1}^{2} \left\{ P_\rho^i \frac{d}{dt} P_\rho^i + P_\Phi^2 \right\} + V = E. \tag{B.4}
\]

In the \(\Phi\)-independent case, \(P_\Phi = J\), constant, so one has furthermore

\[
H = \frac{1}{2} \sum_{i=1}^{2} \left\{ P_\rho^i \frac{d}{dt} P_\rho^i + J^2 \right\} + V = E. \tag{B.5}
\]

B.3. Scaled triangleland in \((I_1, I_2, \Phi)\) coordinates

These are useful in the context of a \(\Phi\)-independent potential energy in the special and very special cases, for which the conjugate momenta are

\[
P_i = I_i^x / I_1, \quad P_\Phi = 4I_1 I_2 \Phi^x / I = J, \text{ constant}. \tag{B.6}
\]

The Hamiltonian and the energy constraint are then

\[
\mathcal{H} = \frac{I_1 P_1^2}{2} + \frac{I_2 P_2^2}{2} + \frac{J^2}{8} \left( \frac{1}{I_1} + \frac{1}{I_2} \right) + \frac{K_1 I_1 + K_2 I_2}{8} = \frac{E}{4}. \tag{B.7}
\]

B.4. N-particle d-dimensional preshape space theory and scalefree N-stop metroland

The conjugate momenta are

\[
P_q = \left\{ \prod_{p=1}^{q-1} \sin^2 \Theta_p \right\} \Theta_q^*, \tag{B.8}
\]

The Hamiltonian and the energy constraint are then

\[
H = \frac{1}{2} \sum_{q=1}^{n-1} \frac{P_q^2}{\prod_{p=1}^{q-1} \sin^2 \Theta_p} + V(\Theta_p) = E. \tag{B.9}
\]

B.5. Scalefree N-a-gonland and the exceptional case of triangleland

The conjugate momenta are

\[
\mathcal{P}_{R_p} = \left\{ \frac{\delta_{pq}}{1 + \| \mathcal{R} \|^2} - \frac{\mathcal{R}_p \mathcal{R}_q}{(1 + \| \mathcal{R} \|^2)^2} \right\} \mathcal{R}_q^*, \tag{B.10}
\]

\[
\mathcal{P}_{\Theta_p} = \left\{ \frac{\delta_{pq}}{1 + \| \mathcal{R} \|^2} - \frac{\mathcal{R}_p \mathcal{R}_q}{(1 + \| \mathcal{R} \|^2)^2} \right\} \mathcal{R}_q \Theta_q^*. \tag{B.10}
\]

The Hamiltonian and the energy constraint are then

\[
H = \frac{1}{2(1 + \| \mathcal{R} \|^2)} \left\{ \delta_{pq} + \mathcal{R}_p \mathcal{R}_q \mathcal{P}_{R_p} \mathcal{P}_{R_q} + \left\{ \frac{\delta_{pq}}{\mathcal{R}_p^2} + 1 \right\} \mathcal{P}_{\Theta_p} \mathcal{P}_{\Theta_q} \right\} + V(\mathcal{R}_p, \Theta_p) = E. \tag{B.11}
\]
For the specific example of scalefree triangleland with harmonic oscillator like potentials in stereographic coordinates, the conjugate momenta are

\[ p_R = \mathcal{R}^2, \quad p_\theta = \mathcal{R}^2 \phi^2 \]  

and the Hamiltonian and the energy constraint are given by

\[ \mathcal{H} = \frac{1}{2} \left( p_R^2 + p_\phi^2 \mathcal{R}^2 \right) + \frac{K_1 \mathcal{R}^2 + L \mathcal{R} \cos \Phi + K_2}{2(1 + \mathcal{R}^2)^3} = \frac{E}{(1 + \mathcal{R}^2)^2}. \]  

While, in spherical coordinates, one has

\[ p_\phi = \tilde{\phi}, \quad p_\phi = \sin^2 \phi \tilde{\phi} \]  

and

\[ \tilde{\mathcal{H}} = \frac{1}{2} \left( p_\phi^2 + \frac{p_\phi^2}{\sin^2 \phi} \right) + A + B \cos \phi + C \sin \phi \cos \phi = \tilde{E}. \]  

\[ B.6. \text{ Scale-shape formulation of scaled triangleland} \]

The conjugate momenta are

\[ p_1 = I^2, \quad p_\phi = I^2 \phi^2, \quad p_\phi = I^2 \sin^2 \phi \Phi^2. \]  

The Hamiltonian and the quadratic energy constraint are then

\[ \tilde{\mathcal{H}} = \frac{p_1^2}{2} + \frac{p_\phi^2}{2I^2} + \frac{p_\phi^2}{2I^2 \sin^2 \phi} + \frac{V(I, \phi)}{4I} = \tilde{E}. \]  

In the special case, \( p_\phi = J \). The Hamiltonian and the energy constraint are then

\[ \mathcal{H} = \frac{p_1^2}{2} + \frac{p_\phi^2}{2I^2} + \frac{J^2}{2I^2 \sin^2 \phi} + \frac{V(I, \phi)}{4I} = \tilde{E}. \]  

In particular, for scaled triangleland with multi-harmonic oscillator potential, this is

\[ \mathcal{H} = \frac{p_1^2}{2} + \frac{p_\phi^2}{2I^2} + \frac{p_\phi^2}{2I^2 \sin^2 \phi} + A + B \cos \phi + C \sin \phi \cos \phi = \tilde{E} = \frac{E}{4I}, \]  

which is close to but not exactly the same as the classical Hamiltonian for an atom in a background homogeneous electric field pointing in an arbitrary direction. And in normal coordinates, the Hamiltonian and the energy constraint are

\[ \mathcal{H} = \frac{p_1^2}{2} + \frac{p_\phi^2}{2I^2} + \frac{p_\phi^2}{2I^2 \sin^2 \phi} + A_N + B_N \cos \phi = \frac{E}{4I}, \]  

which is close to but not exactly the same as the classical Hamiltonian for an atom in a background homogeneous electric field pointing in the axial 'z' direction.

\[ \text{Appendix C. Physical interpretation of this paper’s relative angular momentum quantities} \]

The relative angular momentum quantity in scaled triangleland in simple relational variables is

\[ J = I_1 I_2 \Phi^*/I. \]  

This is equivalent to the scale-shape spherical polar form

\[ J = I^2 \sin^2 \phi \Phi^*. \]  

\[ \text{E Anderson} \]
by (134) and (109). It has the following interpretation:

\[ J_1 = I_1 I_2 \Phi^* = I_1 I_2 [\theta_2^* - \theta_1^*] = I_1 L_2 - I_2 L_1 = (I_1 + I_2) L_2 = -(I_1 + I_2) L_1, \]  

where the fourth equality uses the zero angular momentum constraint, and so, as \( I_1 + I_2 = I \),

\[ J = L_2 = -L_1 = (L_2 - L_1)/2. \]  

So it is interpretable as the angular momentum of one of the two constituent subsystems, minus the angular momentum of the other or half of the difference between the two subsystems’ angular momenta, which is a relative angular momentum presentation.

Using spherical coordinates, scalefree triangleland’s relative angular momentum quantity is

\[ J = \sin^2 \Theta \Phi^* = \frac{I^2 \sin^2 \Theta}{I} \Phi^* = \frac{I^2 \sin^2 \Theta \Phi^*}{I} = \frac{J}{I} \]  

by (42), (109) and (C.2). Thus, by (C.4),

\[ J = J/I = L_2/I = -(L_2 - L_1)/2I. \]  

but \( J \) is constant in similarity relational particle mechanics, so this is still, up to a constant of proportion, the angular momentum of one of the two constituent subsystems, minus the angular momentum of the other or half of the difference between the two subsystems’ angular momenta. Additionally, it has the dimensions of rate of change of angle, which makes sense since on the sphere only angles are meaningful.

In either case, \( \Phi^-dependence \) in the potential corresponds to there being no means for angular momentum to be exchanged between the subsystem composed of particles 2, 3 and that composed of particle 1.

For scaled triangleland, one can consider the configuration space vector

\[ \mathbf{l} = \begin{pmatrix} I \sin \Theta \cos \Phi \\ I \sin \Theta \sin \Phi \\ I \cos \Theta \end{pmatrix} = \begin{pmatrix} 2l_1 l_2 \cos \Phi \\ 2l_1 l_2 \sin \Phi \\ i^2 - i_1^2 \end{pmatrix}. \]  

Then from this and its conjugate momentum \( \mathbf{p} \), the vector

\[ \mathbf{l} = \mathbf{l} \times \mathbf{p} \]  

can be formed, which is conserved in the very special case; \( J \) is the axial ‘\( Z \)’ component of this, \( J_Z \), while the new components are

\[ J_X = \frac{2}{i_1^2 + i_2^2} \{ i_1^2 + i_2^2 \} \{ i_1^2 i_2^2 - i_2 i_1^2 \} \sin \Phi + i_1^2 - i_2^2 \} i_1 l_2 \cos \Phi \Phi^* \]  

and

\[ J_Y = \frac{2}{i_1^2 + i_2^2} \{ -i_1^2 + i_2^2 \} \{ i_1 l_2^2 - i_2 l_1^2 \} \cos \Phi + i_1^2 - i_2^2 \} i_1 l_2 \sin \Phi \Phi^*. \]  

Additionally, the configuration space Laplace–Runge–Lenz type vector

\[ \mathbf{q} = \mathbf{p} \times \mathbf{l} = -\mathbf{e}_1/I \]  

\[ = \left\{ \frac{4}{i_1^2 + i_2^2} \{ i_1^2 + i_2^2 \} \{ i_1^2 i_2^2 + i_2^2 i_1^2 \} + i_1^2 i_2^2 \Phi^* \right\} \frac{2l_1 l_2 \cos \Phi}{i_1^2 + i_2^2} - \mathbf{e}_1 \]  

\[ - 4 \left\{ i_1^2 + i_2^2 \} \{ i_1^2 i_2^2 + i_2^2 i_1^2 \} \left( i_1 l_2^2 + i_2 l_1^2 \right) \cos \Phi - i_1 l_2 \sin \Phi \Phi^* \right\} \left( i_1 l_2^2 - i_1 ^2 \right) \]  

\[ + 4 \left\{ i_1^2 + i_2^2 \} \{ i_1^2 i_2^2 + i_2^2 i_1^2 \} \left( i_1 l_2^2 + i_2 l_1^2 \right) \sin \Phi + i_1 l_2 \sin \Phi \Phi^* \right\} \left( i_1 l_2^2 - i_1 ^2 \right) \]  

(111)
which is also conserved. However, this only furnishes one further independent conserved quantity due to the interdependences
\[ \mathbf{J} \cdot \mathbf{Q} = 0 \quad \text{and} \quad Q^2 = E^2 - 2AJ^4/E^2. \] (C.13)

The very special multiple harmonic oscillator like potential case of scalefree triangleland also has not just a conserved quantity \( \mathbf{J} \) but a conserved vector \( \mathbf{J} \) of which \( \mathbf{J} \) is the axial ‘Z’ component (so that \( \mathbf{J} = J/1 \)). \( \Phi \)-independent scalefree triangleland can still have yet more conserved quantities, but these are of a rather more complicated nature along the lines described in, e.g. [75, 76].

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