ON EXISTENCE OF WAVEFRONT SOLUTIONS IN MIXED MONOTONE REACTION-DIFFUSION SYSTEMS

WEI FENG
Department of Mathematics and Statistics
University of North Carolina Wilmington
Wilmington, NC 28403, USA

WEIHUA RUAN
Department of Mathematics, Computer Science, and Statistics
Purdue University Calumet
Hammond, IN 46323, USA

XIN LU
Department of Mathematics and Statistics
University of North Carolina Wilmington
Wilmington, NC 28403, USA

(Communicated by Yuan Lou)

Abstract. In this article, we give an existence-comparison theorem for wavefront solutions in a general class of reaction-diffusion systems. With mixed quasi-monotonicity and Lipschitz condition on the set bounded by coupled upper-lower solutions, the existence of wavefront solution is proven by applying the Schauder Fixed Point Theorem on a compact invariant set. Our main result is then applied to well-known examples: a ratio-dependent predator-prey model, a three-species food chain model of Lotka-Volterra type and a three-species competition model of Lotka-Volterra type. For each model, we establish conditions on the ecological parameters for the presence of wavefront solutions flowing towards the coexistent states through suitably constructed upper and lower solutions. Numerical simulations on those models are also demonstrated to illustrate our theoretical results.

1. Introduction. The existence, propagation, and asymptotic behavior of wavefront solutions of reaction-diffusion systems have been attracting the attention of researchers on mathematical models in biology, ecology, chemistry and physics. Among various approaches to the existence of wavefront solutions, the method of upper-lower solutions was widely applied to both single-equation and multi-equation systems. For the case of multi-equation systems, the existence of wavefront solutions were established for various two-dimensional competition models and three-dimensional cooperative-competitive models, which can be transformed into a cooperative system through a simple substitution [6, 9, 10, 19, 21]. For those models

2010 Mathematics Subject Classification. Primary: 35K57, 35C07; Secondary: 74H20, 74H15, 93A30.

Key words and phrases. Reaction-diffusion systems, mixed monotone functions, existence of wavefront solutions, coexistence in ecological models, numerical simulations.

1Contacting Author.

2Weihua Ruan is partially supported by Simons Foundation Grant 245488.
with all reaction functions being quasimonotone nondecreasing, an ordered pair of upper and lower solutions were constructed through wavefront solutions of the classical K.P.P. equation (Kolmogorov-Petrovskii-Piskunov \cite{16}), and the wavefront solutions were generated by monotone iterations starting from the upper solution. The convergence of the iterative sequence also helped to obtain the monotonicity and asymptotic rates of the wavefront solutions in those models. However, it is not an easy task to extend this method to general reaction-diffusion systems with mixed quasi-monotone properties, including the two-species predator-prey models and three-dimensional competition models. “Note that if the dimension of the Lotka-Volterra competition system is greater than or equal to three, then one is not able to converse it into a cooperative system any more” \cite{21}. Actually, it is impossible to transform most of the real-life models into quasi-monotone non-increasing systems.

Comparing with other methods for establishing the existence of wavefront solutions (for example, search for the generalized upper and lower solution pairs\cite{3, 24, 30, 31}, geometric singular perturbation \cite{2, 8} and phase plane analysis \cite{1, 4, 11, 12, 13, 14, 15, 16, 27, 28, 29, 32}) construction of upper-lower solution pairs in the classical sense gives more straightforward results on the existence of wavefront solutions in terms of the the natural parameters, and also provides valuable information on the ultimate bounds and asymptotic rates of the solutions. However, there is a gap on the proof for compactness of the operator $F$ (in Lemma 3.5) because the application of Ascoli-Arzela Theorem is restricted to functional spaces on a compact domain but not for $C(R, R^n)$. In addition, the definition of upper and lower solutions in (P1) (on page 398) requires that they have the same limits at both $-\infty$ and $+\infty$, which creates difficulty on construction of upper-lower solutions in complex models, therefore the general definition \cite{25} should allow inequalities to hold on boundary conditions. Furthermore, the assumption on wavefront solutions having limit as 0 at $-\infty$ is also unnecessarily restrictive since for many models there are traveling wavefront solutions connecting various equilibrium states.

In this article, we establish an existence-comparison theorem for wavefront solutions connecting two equilibrium states of a general reaction-diffusion system with mixed quasi-monotone properties. In Section 2, the upper-lower solutions are defined to satisfy mixed differential inequalities and their limits at $\mp\infty$ also satisfy corresponding inequalities related to the equilibrium states. A complete proof of the existence-comparison theorem (Theorem 2.6) is given by applying Schauder’s Fixed Point Theorem to a compact operator. We also apply our main theoretical result to three reaction-diffusion models with mixed quasi-monotone properties and give conditions for the existence of wavefront solutions through suitably constructed upper-lower solutions. In Section 3, we show the existence of wavefront solutions for a ratio-dependent predator-prey model flowing from the prey-only state $(1, 0)$ to the coexistence state $(u^*, v^*)$. In Sections 4 and 5, we show the existence of wavefront solutions for two 3-species Lotka-Volterra models (food chain and competition) flowing from the trivial state $(0, 0, 0)$ to the coexistence state $(u^*, v^*, w^*)$. In our derivation of the wavefront solutions, the construction of the upper-lower solution pairs is based on the classical results on K.P.P. equation and relevant conditions on ecological parameters. For all models, we assume different diffusion rates for the species.
different population density functions, and find the minimum wave speed for wavefront solutions in terms of those diffusion rates and other ecological parameters. Finally, in Section 6, we demonstrate numerical simulations of wavefront solutions for the ratio-dependent predator-prey model and the food chain model of Lotka-Volterra type. Two numerical examples are illustrated in accordance with derived conditions for the existence of specific wavefront solutions. Through the numerical simulations, we observe that unlike the 2-dimensional competition models (transformable to a quasi-monotone nondecreasing system), the mixed quasi-monotone models no longer warrant the monotonicity of wavefront solutions on $(-\infty, +\infty)$.

2. Existence-comparison theorem for wavefront solutions. We start with a general $n$-dimensional reaction-diffusion system

$$\frac{\partial u}{\partial t} = D\Delta u + f(u) \quad -\infty < x < \infty, \quad t > 0,$$

where $u = (u_1, \ldots, u_n)^T$, $f = (f_1, \ldots, f_n)^T$, $D = \text{diag}(d_1, \ldots, d_n)$ with $d_i > 0$ for $i = 1, \ldots, n$. A wavefront solution for (2.1) is a solution in the form $u(x, t) = w(x + ct)$ for some $c > 0$, where $w$ satisfies

$$\begin{cases}
Dw''(s) - cw'(s) + f(w(s)) = 0 & \quad -\infty < s < \infty, \\
\lim_{s \to -\infty} w(s) = u^-, \quad \lim_{s \to \infty} w(s) = u^+,
\end{cases}$$

for some $u^-, u^+ \in \mathbb{R}^n$. We first define a pair of coupled upper and lower solutions for the wavefront system (2.2):

**Definition 2.1. Upper and lower solutions.**

A pair of bounded functions $\tilde{w} \equiv (\tilde{w}_1, \ldots, \tilde{w}_n)$ and $\hat{w} \equiv (\hat{w}_1, \ldots, \hat{w}_n)$ are coupled upper and lower solutions for (2.2) if $\tilde{w} \geq \hat{w}$,

$$\begin{align*}
d_i \tilde{w}_i'' - c\tilde{u}_i' + f_i\left(\tilde{w}_i, [\hat{w}]_{k_i}, [\tilde{w}]_{\hat{k}_i}\right) & \leq 0, \\
d_i \hat{w}_i'' - c\hat{u}_i' + f_i\left(\hat{w}_i, [\hat{w}]_{k_i}, [\tilde{w}]_{\hat{k}_i}\right) & \geq 0,
\end{align*}$$

where $k_i$ and $\hat{k}_i$ are subsets of positive integers such that

$$k_i \cup \hat{k}_i = \{1, \ldots, n\} \setminus \{i\},$$

for each $i = 1, \ldots, n$ and $\lim_{t \to -\infty} \tilde{w}(t)$, and $\lim_{t \to \infty} \hat{w}(t)$ both exist with

$$\lim_{t \to -\infty} \tilde{w}(t) \geq u^- \geq \lim_{t \to \infty} \hat{w}(t), \quad \lim_{t \to -\infty} \hat{w}(t) \geq u^+ \geq \lim_{t \to \infty} \tilde{w}(t).$$

In this section, we show that under the following hypotheses for the reaction functions, there exists a wavefront solution (between the upper and lower solutions) for system (2.1).

**H:**

1. (Equilibrium States) $f(u^-) = f(u^+) = 0$. Furthermore, $u^-$ is the only zero of $f$ between $\lim_{t \to -\infty} \tilde{w}(t)$ and $\lim_{t \to -\infty} \hat{w}(t)$ and $u^+$ is the only zero of $f$ between $\lim_{t \to \infty} \hat{w}(t)$ and $\lim_{t \to \infty} \tilde{w}(t)$.

2. (Mixed quasi-monotonicity) For each $i$, there are subsets of positive integers, $k_i$ and $\hat{k}_i$, such that (2.4) holds and the function

$$f_i(u) = f_i\left(u_i, [u]_{k_i}, [u]_{\hat{k}_i}\right)$$
is monotone nondecreasing in \([u]_{k_i}\) and is monotone nonincreasing in \([u]_{k_i}\).
Also, there is \(\beta \geq 0\) such that \(f_i(u) + \beta u_i\) is nondecreasing in \(u_i\) for each \(i\).
Without loss of generality, we assume that \(\beta > 0\).

3. (Lipschitz condition) \(f\) satisfies the Lipschitz condition in any bounded set \(S\) of \(\mathbb{R}^n\). That is, there is \(L > 0\) such that \(|f(u) - f(v)| \leq L|u - v|\) for all \(u, v \in S\).

For the convenience of establishing the existence-comparison theorem, we look into a fixed point problem by writing the equation in (2.2) in the form

\[
\begin{cases}
    d_i w''_i - cw'_i - \beta w_i + F_i(w) = 0, \\
    \lim_{t \to -\infty} w_i(t) = u^-_i, \\
    \lim_{t \to \infty} w_i(t) = u^+_i,
\end{cases}
\]

(2.6)

where

\[F_i(w) = f_i(w) + \beta w_i.\]

Let

\[
\lambda^-_i = \frac{c - \sqrt{c^2 + 4\beta d_i}}{2d_i}, \quad \lambda^+_i = \frac{c + \sqrt{c^2 + 4\beta d_i}}{2d_i}
\]

and let \(\mu > 0\) satisfy

\[
\mu < \min_{1 \leq i \leq n} \{ |\lambda^-_i|, |\lambda^+_i| \}.
\]

(2.7)

We denote the Banach space

\[X = \left\{ \phi \in C(\mathbb{R}; \mathbb{R}^n) : \sup_{t \in \mathbb{R}} |\phi(t)| e^{-\mu|t|} < \infty \right\}\]

with the norm

\[|\phi|_X = \sup_{t \in \mathbb{R}} |\phi(t)| e^{-\mu|t|},\]

and define the operator \(K\) on \(X\) by \(K\phi = \psi \equiv (\psi_1, \ldots, \psi_n)\) where

\[
\psi_i(t) = \frac{1}{d_i (\lambda^+_i - \lambda^-_i)} \left[ \int_{-\infty}^t e^{\lambda^-_i (t-s)} F_i(\phi(s)) \, ds + \int_t^\infty e^{\lambda^+_i (t-s)} F_i(\phi(s)) \, ds \right].
\]

(2.8)

It can be verified that \(\psi_i\) satisfies

\[d_i \psi''_i - cw'_i - \beta \psi_i + F_i(\phi) = 0.\]

Further more, by l’Hôpital’s rule

\[
\begin{align*}
\lim_{t \to -\infty} \psi_i(t) &= \frac{1}{d_i (\lambda^+_i - \lambda^-_i)} \lim_{t \to -\infty} \left[ \frac{1}{e^{\lambda^-_i t}} \int_{-\infty}^t e^{\lambda^-_i s} F_i(\phi(s)) \, ds \\
&\quad + \frac{1}{e^{\lambda^+_i t}} \int_t^\infty e^{\lambda^+_i s} F_i(\phi(s)) \, ds \right] \\
&= \frac{1}{d_i (\lambda^+_i - \lambda^-_i)} \lim_{t \to -\infty} \left[ F_i(\phi(t)) - \frac{F_i(\phi(t))}{-\lambda^-_i} \right] \\
&= -\frac{1}{d_i \lambda^+_i \lambda^-_i} \lim_{t \to -\infty} F_i(\phi(t)).
\end{align*}
\]
Let

Then for any \( \phi \)

Thus a fixed point of Lemma 2.2.

The following lemma shows that \( S \)

Proof. Hence by (2.8)

provided that \( \lim_{t \to -\infty} \phi(t) \) and \( \lim_{t \to -\infty} \phi(t) \) both exist. Since

it follows that

it follows that

Thus a fixed point of \( K \) between \( \tilde{w} \) and \( \hat{w} \) is a solution to (2.6). Now let

Then for any \( \phi \in S \) the function \( \psi = K \phi \) satisfies

The following lemma shows that \( S \) is an invariant set for \( K \).

Lemma 2.2. Let (H) hold. Then \( KS \subset S \).

Proof. It suffices to show that \( \hat{w}_1 \leq \psi_i \leq \tilde{w}_i \) for \( i = 1, \ldots, n \). Since \( \hat{w}_i \leq \phi_i \leq \tilde{w}_i \) in \( \mathbb{R} \) for all \( i = 1, \ldots, n \), by the quasi-monotonicity of \( F_i \), it follows that

Hence by (2.8)

\[
\begin{align*}
\psi_i(t) &\geq \frac{1}{d_i(\lambda_i^+ - \lambda_i^-)} \left[ \int_{-\infty}^{t} e^{\lambda_i^- (t-s)} F_i \left( \tilde{w}_i, [\tilde{w}]_{k_i}, [\hat{w}]_{k_i} \right) ds \right. \\
&\quad + \left. \int_{t}^{\infty} e^{\lambda_i^- (t-s)} F_i \left( \tilde{w}_i, [\tilde{w}]_{k_i}, [\hat{w}]_{k_i} \right) ds \right], \\
\psi_i(t) &\leq \frac{1}{d_i(\lambda_i^+ - \lambda_i^-)} \left[ \int_{-\infty}^{t} e^{\lambda_i^+ (t-s)} F_i \left( \hat{w}_i, [\hat{w}]_{k_i}, [\tilde{w}]_{k_i} \right) ds \right. \\
&\quad + \left. \int_{t}^{\infty} e^{\lambda_i^+ (t-s)} F_i \left( \hat{w}_i, [\hat{w}]_{k_i}, [\tilde{w}]_{k_i} \right) ds \right].
\end{align*}
\] (2.10)

Let

\[
\hat{F}_i(t) = -d_i \hat{w}_i'' + c \hat{w}_i' + \beta \hat{w}_i, \quad \tilde{F}_i(t) = -d_i \tilde{w}_i'' + c \tilde{w}_i' + \beta \tilde{w}_i \quad \text{for} \quad i = 1, \ldots, n.
\]
Then
\[
\hat{w}_i(t) = \frac{1}{d_i(\lambda_i^+ - \lambda_i^-)} \left[ \int_{-\infty}^t e^{\lambda_i^-(t-s)} \hat{F}_i(s) \, ds + \int_t^\infty e^{\lambda_i^+(t-s)} \hat{F}_i(s) \, ds \right],
\]
\[
\hat{w}_i(t) = \frac{1}{d_i(\lambda_i^+ - \lambda_i^-)} \left[ \int_{-\infty}^t e^{\lambda_i^-(t-s)} \dot{F}_i(s) \, ds + \int_t^\infty e^{\lambda_i^+(t-s)} \dot{F}_i(s) \, ds \right],
\]
and by (2.3),
\[
F_i\left(\hat{w}_i(t), [\hat{\psi}(t)]_{k_i}, [\hat{w}(t)]_{k_i}\right) \geq \hat{F}_i(t),
\]
\[
F_i\left(\tilde{w}_i(t), [\tilde{\psi}(t)]_{k_i}, [\tilde{w}(t)]_{k_i}\right) \leq \tilde{F}_i(t)
\]
for each \(i\). Now, by (2.10), we can see that
\[
\psi_i(t) \geq \tilde{w}_i(t), \quad \psi_i(t) \leq \hat{w}_i(t) \quad \text{for } t \in \mathbb{R}, \quad i = 1, \ldots, n.
\]
This completes the proof. \(\square\)

The next lemma gives the continuity of \(K\) on \(S\).

**Lemma 2.3.** Let \((H)\) hold. Then there is an \(M > 0\) such that
\[
|K\xi - K\eta|_X < M |\xi - \eta|_X \quad \text{for all } \xi, \eta \in S.
\]

**Proof.** Since \(f\) satisfies the Lipschitz condition given in \((H)\), there is a constant \(L > 0\) such that
\[
|f(y) - f(z)| < L |y - z| \quad \text{for any } y, z \in S.
\]
It follows that
\[
|F(y) - F(z)| \leq (L + \beta) |y - z| \quad \text{for any } y, z \in S.
\]
Let \(\zeta \equiv (\zeta_1, \ldots, \zeta_n)\) and \(\chi \equiv (\chi_1, \ldots, \chi_n)\) denote \(K\xi\) and \(K\eta\), respectively. By (2.8) and the above inequality,
\[
|\zeta_i(t) - \chi_i(t)| \leq \frac{1}{d_i(\lambda_i^+ - \lambda_i^-)} \int_{-\infty}^t e^{\lambda_i^-(t-s)} |F_i(\xi(s)) - F_i(\eta(s))| \, ds
\]
\[
+ \int_t^\infty e^{\lambda_i^+(t-s)} |F_i(\xi(s)) - F_i(\eta(s))| \, ds \leq \frac{L + \beta}{d_i(\lambda_i^+ - \lambda_i^-)} \int_{-\infty}^t e^{\lambda_i^-(t-s)} |\xi(s) - \eta(s)| \, ds + \int_t^\infty e^{\lambda_i^+(t-s)} |\xi(s) - \eta(s)| \, ds.
\]
Since
\[
|\xi(s) - \eta(s)| = |\xi(s) - \eta(s)| e^{-\mu|s|} e^{\mu|s|} \leq |\xi - \eta|_X e^{\mu|s|},
\]
it follows that
\[
|\zeta_i(t) - \chi_i(t)| \leq \frac{L + \beta}{d_i(\lambda_i^+ - \lambda_i^-)} \int_{-\infty}^{t} e^{\lambda_i^-(t-s)+\mu|s|} ds + \int_t^\infty e^{\lambda_i^+(t-s)+\mu|s|} ds.
\]
From (2.7) and the fact that \( \lambda_i^- < 0 < \lambda_i^+ \), we see that \( \lambda_i^- + \mu < 0 \) and \( \lambda_i^+ - \mu > 0 \). Thus
\[
\int_{-\infty}^{t} e^{\lambda_i^- (t-s) + \mu s} ds + \int_{t}^{\infty} e^{\lambda_i^+ (t-s) + \mu s} ds
\]
\[
= \int_{-\infty}^{t} e^{\lambda_i^- (t-s) - \mu s} ds + \int_{t}^{\infty} e^{\lambda_i^+ (t-s) - \mu s} ds + \int_{t}^{\infty} e^{\lambda_i^+ (t-s) + \mu s} ds
\]
\[
= -\frac{e^{\lambda_i^- t}}{\lambda_i^- + \mu} + \frac{e^{\mu t} - e^{\lambda_i^- t}}{\mu - \lambda_i^-} - \frac{e^{\mu t}}{\mu - \lambda_i^+}
\]
\[
= \frac{2\mu}{(\lambda_i^- - \mu)^2 - \mu^2} e^{\lambda_i^- t} + \frac{\lambda_i^- - \lambda_i^+}{(\lambda_i^- + \mu)(\lambda_i^+ + \mu)} e^{-\mu t} + \frac{2\mu}{(\lambda_i^+ - \mu)^2 - \mu^2} e^{\lambda_i^+ t}
\]
if \( t \geq 0 \) and
\[
\int_{-\infty}^{t} e^{\lambda_i^- (t-s) + \mu s} ds + \int_{t}^{\infty} e^{\lambda_i^+ (t-s) + \mu s} ds
\]
\[
= \int_{-\infty}^{t} e^{\lambda_i^- (t-s) - \mu s} ds + \int_{t}^{\infty} e^{\lambda_i^+ (t-s) - \mu s} ds + \int_{t}^{\infty} e^{\lambda_i^+ (t-s) + \mu s} ds
\]
\[
= \frac{-e^{-\mu t}}{\lambda_i^- + \mu} - \frac{e^{\lambda_i^- t} - e^{-\mu t}}{\lambda_i^- + \mu} - \frac{e^{\lambda_i^+ t}}{\mu - \lambda_i^+}
\]
\[
= \frac{\lambda_i^- - \lambda_i^+}{(\lambda_i^- + \mu)(\lambda_i^+ + \mu)} e^{-\mu t} + \frac{2\mu}{(\lambda_i^+ - \mu)^2 - \mu^2} e^{\lambda_i^+ t}
\]
if \( t < 0 \). In view of (2.7)
\[
\frac{2\mu}{(\lambda_i^-)^2 - \mu^2} > 0, \quad \frac{\lambda_i^- - \lambda_i^+}{(\mu - \lambda_i^-)(\mu - \lambda_i^+)} > 0,
\]
\[
\frac{\lambda_i^- - \lambda_i^+}{(\lambda_i^- + \mu)(\lambda_i^+ + \mu)} < 0, \quad \frac{2\mu}{(\lambda_i^+)^2 - \mu^2} > 0.
\]
Hence, by (2)
\[
|\zeta_i(t) - \chi_i(t)| e^{-\mu t} \leq \frac{L + \beta}{d_i (\lambda_i^+ - \lambda_i^-)} |\xi - \eta|_X \left[ \frac{2\mu}{(\lambda_i^-)^2 - \mu^2} + \frac{\lambda_i^- - \lambda_i^+}{(\mu - \lambda_i^-)(\mu - \lambda_i^+)} \right]
\]
(2.11)
if \( t \geq 0 \), and
\[
|\zeta_i(t) - \chi_i(t)| e^{-\mu t} \leq \frac{L + \beta}{d_i (\lambda_i^+ - \lambda_i^-)} |\xi - \eta|_X \frac{2\mu}{(\lambda_i^+)^2 - \mu^2}
\]
(2.12)
if \( t < 0 \). Let \( M \) be maximum of the coefficients of \( |\xi - \eta|_X \) on the left-hand sides of (2.11) and (2.12), it follows that \( |\zeta - \chi|_X \leq M |\xi - \eta|_X \). This proves the continuity of \( K \) on \( S \).

Next, we show that there is a compact invariant set of \( K \) in \( X \).

**Lemma 2.4.** Suppose that \( (H) \) holds. Let \( \hat{w} \) and \( \tilde{w} \) be a pair of coupled upper and lower solutions, and let
\[
\rho = \sup \{ |f(w)| : \hat{w} \leq w \leq \tilde{w} \}.
\]
Then for any \( \phi \in S \) the function \( \psi = K\phi \) has the property
\[
|\psi'_i(t)| \leq \rho/c \quad \text{for all } t \in \mathbb{R}.
\]
Thus follows from (2.9) that

It can be shown through l’Hôpital’s Rule that

Thus \( \max_{-\infty < t < \infty} |\psi'_i(t)| \) is achieved at certain \( t_0 \in \mathbb{R} \) at which \( \psi''_i(t_0) = 0 \). It follows from (2.9) that

Thus

This proves the lemma.

Now let

\[
\hat{S} = \left\{ \phi \in S : |\phi_i(t) - \phi_i(\tau)| \leq \frac{\rho}{c} |t - \tau| \text{ for } t, \tau \in \mathbb{R}, i = 1, \ldots, n \right\}.
\]

Lemma 2.2 and 2.4 imply that \( \hat{S} \) is invariant for \( K \). We show that \( \hat{S} \) is compact in \( X \).

**Lemma 2.5.** \( \hat{S} \) defined by (2.13) is compact in \( X \).

**Proof.** Let \( (f_n) \) be a sequence of elements in \( \hat{S} \). We need to show that there is a convergent subsequence in \( X \). Using a diagonal selection scheme, we can extract a subsequence \( (f_{nk}) \) that converges at each rational point in \( \mathbb{R} \). We show that this subsequence is Cauchy in \( X \). Let \( \varepsilon > 0 \). There is an \( R > 0 \) such that

\[
|S| e^{-\mu R} < \varepsilon/2
\]

(2.14) where

\[
|S| = \sup \left\{ |\phi|_{C(\mathbb{R})} : \phi \in S \right\}.
\]

Let \( \{r_m\} \) be the set of all rational numbers in the interval \([-R, R]\). By the convergence of \( (f_{nk}) \) on rational numbers, for each \( m \) there is \( N_m \in \mathbb{N} \) such that

|\( f_{nk}(r_m) - f_{n1}(r_m) \)\| < \( \varepsilon/3 \) \text{ for all } k, l > N_m.

Let

\[
U_m = \left\{ t \in \mathbb{R} : |t - r_m| < \frac{\varepsilon}{3\rho} \right\}.
\]

Since

\[
|f_n(t) - f_n(\tau)| < \frac{\rho}{c} |t - \tau|
\]

for each \( n \), it follows that for any \( t \in U_m \),

\[
|f_{nk}(t) - f_{n1}(t)| \leq |f_{nk}(t) - f_{nk}(r_m)| + |f_{nk}(r_m) - f_{n1}(r_m)| + |f_{n1}(r_m) - f_{n1}(t)| \leq 2\frac{\rho}{c} |t - r_m| + \frac{\varepsilon}{3} < \varepsilon.
\]

Choose a finite open cover of \([-R, R]\) by \( U_{m1}, U_{m2}, \ldots, U_{mj} \) and let

\[
N = \max \{N_{m1}, \ldots, N_{mj}\}.
\]
It follows that
\[ |f_{nk}(t) - f_{nl}(t)| e^{-\mu|t|} < |f_{nk}(t) - f_{nl}(t)| < \varepsilon \quad \text{for all } t \in [-R, R], \quad k, l > N. \]

For any \( t \not\in [-R, R] \), by (2.14)
\[ |f_{nk}(t) - f_{nl}(t)| e^{-\mu|t|} \leq (|f_{nk}(t)| + |f_{nl}(t)|) e^{-\mu R} < \varepsilon. \]

Thus
\[ |f_{nk}(t) - f_{nl}(t)| e^{-\mu|t|} < \varepsilon \quad \text{for all } t \in \mathbb{R}, \quad k, l > N. \]

This proves that \((f_{nk})\) is a Cauchy sequence in \( X \).

Finally, we can give the following theorem for existence of wavefront solution (2.2) between the upper solution \( \hat{w} \) and the lower solution \( \tilde{w} \).

**Theorem 2.6.** (Existence-Comparison) Let \((H)\) hold. Suppose that the upper-lower solution pair \( \hat{w}, \tilde{w} \in C^2(\mathbb{R}) \) satisfy (2.3) and (2.5). Then there is a solution \( w \) to (2.2) that satisfies \( \hat{w} \leq w \leq \tilde{w} \) in \( \mathbb{R} \).

**Proof.** This follows from Lemmas 2.2–2.5 and the Schauder Fixed Point Theorem. \( \square \)

3. **Example 1: A ratio-dependent predator-prey model.** In this section we look into the existence of traveling wavefronts in the following model [17] of ratio-dependent interactions between a predator (\( v \)-species) and prey (\( u \)-species).

\[
\begin{align*}
\frac{\partial u}{\partial t} - D_1 \frac{\partial^2 u}{\partial x^2} &= au(1 - u) - \frac{ew}{u + mv}, \\
\frac{\partial v}{\partial t} - D_2 \frac{\partial^2 v}{\partial x^2} &= v \left(-d + \frac{fu}{u + mv}\right)
\end{align*}
\]  

(3.1)

where \( a \) is the prey’s intrinsic growth rate, \( e \) is the capturing rate, \( m \) is the half capturing saturation constant, \( f \) is the conversion rate, and \( d \) is the predator’s death rate. The two-equation reaction-diffusion system in (3.1) is mixed quasi-monotone, and has the prey-only state \((1, 0)\) and the coexistence state \((u^*, v^*)\) with
\[
u^* = \frac{amf - e(f - d)}{amf}, \quad v^* = \left(\frac{f - d}{md}\right) u^*.
\]  

(3.2)

The coexistence state is present under the following conditions:

1. \( d < f < \frac{ed}{e - ma} \) when \( e > ma \);
2. \( f > d \) when \( e \leq ma \).

It is easily seen by the comparison argument that \( v(x, t) \) goes to extinction for \( f \leq d \). Based on the stability analysis done in [17] on the corresponding ordinary differential equation system, when \( f > d \) we have \((1, 0)\) as a saddle point, and the Jacobian matrix \( J(u^*, v^*) \) for \((u^*, v^*)\) has positive determinate while the trace of \( J(u^*, v^*) \) is
\[
-a + \frac{1}{m} \left(1 - \frac{d}{f}\right) \left[e + \frac{d}{f}(e - mf)\right]
\]
which indicates that \((u^*, v^*)\) is asymptotically stable for \( d/f \) close to 1. Throughout this section we assume that

**(H1):** \( e > ma \) and \( d < f < \min\{\frac{ed}{e - ma}, \quad 2d, \quad d + \frac{e - ma}{m}\}\).
Our attention is now given to the existence of wavefront solutions \((u(x, t), v(x, t)) = (u(x + ct), v(x + ct)) = (u(\xi), v(\xi))\) for System (3.1) connecting the prey-only state \((1, 0)\) and the coexistence state \((u^*, v^*)\). The wavefront system has the form
\[
\begin{cases}
D_1 u_{\xi\xi} - cu_{\xi} + au(1-u) - \frac{euv}{u + mv} = 0, \\
D_2 v_{\xi\xi} - cv_{\xi} + v\left(-d + \frac{fu}{u + mv}\right) = 0,
\end{cases}
\]
(3.3)
Using the transformation \((U, V) = (1 - u, v)\), System (3.3) is then changed into
\[
\begin{cases}
D_1 U_{\xi\xi} - cU_{\xi} - aU(1-U) + \frac{e(1-U)V}{1-U+mV} = 0, \\
D_2 V_{\xi\xi} - cV_{\xi} + V\left(-d + \frac{f(1-U)}{1-U+mV}\right) = 0, \\
\left(\begin{array}{c}
U \\
V
\end{array}\right) \bigg|_{-\infty} = \left(\begin{array}{c}
1 \\
0
\end{array}\right), \quad \left(\begin{array}{c}
U \\
V
\end{array}\right) \bigg|_{+\infty} = \left(\begin{array}{c}
u^* \\
v^*
\end{array}\right).
\end{cases}
\]
(3.4)

**Definition 3.1.** A pair of \(C^2(\mathbb{R}) \times C^2(\mathbb{R})\) functions \((\tilde{U}(\xi), \tilde{V}(\xi))\) and \((\hat{U}(\xi), \hat{V}(\xi))\) are coupled upper and lower solutions of (3.4) if they satisfy the following differential inequalities
\[
\begin{cases}
D_1 \tilde{U}_{\xi\xi} - c\tilde{U}_{\xi} - a\tilde{U}(1-\tilde{U}) + \frac{e(1-\tilde{U})\hat{V}}{1-\tilde{U}+m\hat{V}} \leq 0, \\
D_2 \tilde{V}_{\xi\xi} - c\tilde{V}_{\xi} + \hat{V}\left(-d + \frac{f(1-\tilde{U})}{1-\tilde{U}+m\hat{V}}\right) \leq 0, \\
D_1 \hat{U}_{\xi\xi} - c\hat{U}_{\xi} - a\hat{U}(1-\hat{U}) + \frac{e(1-\hat{U})\tilde{V}}{1-\hat{U}+m\tilde{V}} \geq 0, \\
D_2 \hat{V}_{\xi\xi} - c\hat{V}_{\xi} + \tilde{V}\left(-d + \frac{f(1-\hat{U})}{1-\hat{U}+m\tilde{V}}\right) \geq 0
\end{cases}
\]
(3.5)
and the limiting conditions
\[
\left(\begin{array}{c}
\tilde{U} \\
\tilde{V}
\end{array}\right) \bigg|_{-\infty} \geq \left(\begin{array}{c}
0 \\
0
\end{array}\right), \quad \left(\begin{array}{c}
\hat{U} \\
\hat{V}
\end{array}\right) \bigg|_{-\infty} \leq \left(\begin{array}{c}
\hat{U} \\
\hat{V}
\end{array}\right) \bigg|_{-\infty},
\]
\[
\left(\begin{array}{c}
\hat{U} \\
\hat{V}
\end{array}\right) \bigg|_{+\infty} \geq \left(\begin{array}{c}
\frac{e(f-d)}{amf} \\
\frac{1}{v^*}
\end{array}\right), \quad \left(\begin{array}{c}
\tilde{U} \\
\tilde{V}
\end{array}\right) \bigg|_{+\infty} \leq \left(\begin{array}{c}
\tilde{U} \\
\tilde{V}
\end{array}\right) \bigg|_{+\infty}.
\]
(3.6)
We construct a pair of upper and lower solutions for the traveling wave system (3.3) by applying a well-known result on the solution of the K.P.P. equation with limiting conditions. Let \(f\) be a \(C^2\) function on the interval \([0, \beta]\), \(\beta > 0\), with \(f > 0\) on \((0, \beta)\), and \(f(0) = f(\beta) = 0\), \(f'(0) = \alpha_1 > 0\), \(f'(\beta) = -\beta_1 < 0\). We recall the following lemma (see [16], [27] for proof):
Lemma 3.2. Corresponding to every \( c \geq 2\sqrt{\alpha_1} \), the boundary value problem
\[
\begin{align*}
\omega''(\xi) - c\omega'(\xi) + f(\omega(\xi)) &= 0, \\
\omega(-\infty) &= 0, \quad \omega(+\infty) = \beta.
\end{align*}
\] (3.7)
has a unique monotone increasing traveling wave solution \( \omega_c(\xi), \xi \in \mathbb{R} \), where the lower index denotes the dependence of the wave solution \( \omega \) on \( c \).

We denote \( D = \max\{D_1, D_2\} \), \( \bar{D} = \min\{D_1, D_2\} \). For each \( c \geq 2\bar{D}\sqrt{\frac{e-\alpha_1}{mD}} \), we let \( Y \) be the unique (up to a translation of the origin) monotone increasing solution of the following K.P.P. equation:
\[
\begin{align*}
Y_{\xi\xi} - \frac{c}{D}Y_{\xi} + \frac{(e-\alpha_1)Y}{mD}(1 - Y) &= 0, \\
Y(-\infty) &= 0, \quad Y(+\infty) = 1.
\end{align*}
\] (3.8)
Also, for each \( c \geq 2\bar{D}\sqrt{\frac{e-\alpha_1}{mD}} \) (which already holds when \( c \geq 2\bar{D}\sqrt{\frac{e-\alpha_1}{mD}} \)) and some \( 0 < l < 1 \) (to be determined later), let \( Z \) be the solution of the following K.P.P. equation:
\[
\begin{align*}
Z_{\xi\xi} - \frac{c}{D}Z_{\xi} + \frac{(e-\alpha_1)Z}{mD}(l - Z) &= 0, \\
Z(-\infty) &= 0, \quad Z(+\infty) = l.
\end{align*}
\] (3.9)

We next verify that the upper and lower solutions defined by
\[
\begin{align*}
\tilde{U} &= Y, \quad \tilde{V} = \frac{Y}{m}, \\
\hat{U} &= Z, \quad \hat{V} = \frac{Z}{m}
\end{align*}
\] (3.10)
satisfy all the differential inequalities in Definition 3.1. It can be seen that:
\[
\begin{align*}
\tilde{U}_{\xi\xi} - \frac{c}{D_1}\tilde{U}_{\xi} + \frac{(1 - \tilde{U})}{D_1}\left(-a\tilde{U} + \frac{e\tilde{V}}{1 - \tilde{U} + m\tilde{V}}\right) \\
&\leq Y_{\xi\xi} - \frac{c}{D}\tilde{Y}_{\xi} + \frac{1 - Y}{D}\left(-aY + \frac{eY}{Y}\right) \\
&= Y_{\xi\xi} - \frac{c}{D}Y_{\xi} + \frac{(e - ma)Y}{mD}(1 - Y) = 0,
\end{align*}
\]
\[
\begin{align*}
\hat{V}_{\xi\xi} - \frac{c}{D_2}\hat{V}_{\xi} + \frac{\hat{V}}{D_2}\left[-d + \frac{f(1 - \hat{U})}{1 - \hat{U} + m\hat{V}}\right] \\
&\leq \frac{1}{m}\left[Y_{\xi\xi} - \frac{c}{D}Y_{\xi}\right] + \frac{Y}{mD}\left[-d + \frac{f(1 - Z)}{1 - Z + Y}\right] \\
&= \frac{Y}{mD}\left[(f - d)(1 - Z) - \frac{dY - e - ma}{m}(1 - Y)\right] \\
&\leq \frac{Y}{mD}\left[(f - d)(1 - Y) - \frac{e - ma}{m}(1 - Y)\right] \leq 0,
\end{align*}
\]
We are now ready to state the following theorem on the existence of the wavefront solutions.

\[ \begin{align*}
\dot{U} &\leq \frac{e}{D_1} \dot{U} + \frac{(1 - \dot{U})}{D_1} \left( -aU + \frac{e\dot{V}}{1 - U + mV} \right) \\
\geq Z &\leq \frac{c}{D} \dot{Z} + \frac{(1 - Z)}{D} \left[ -aZ + \frac{e}{m} Z \right] \\
\geq -Z &\leq \frac{Z}{D} \left( \frac{e - ma}{m} \right) (l - Z) + \frac{(1 - Z)}{D} \left( \frac{e - ma}{m} \right) Z \\
= &\frac{(1 - l)Z}{D} \left( \frac{e - ma}{m} \right) \geq 0.
\end{align*} \]

And, finally for \( \dot{V} \), we have

\[ \begin{align*}
\dot{V} &\leq \frac{e}{D_2} \dot{V} + \frac{\dot{V}}{D_2} \left[ -d + \frac{f(1 - \hat{U})}{1 - \hat{U} + m\hat{V}} \right] \\
\geq \frac{1}{m} \left[ Z \right] &\leq \frac{c}{D} \left[ Z \right] + \frac{Z}{mD} \left[ -d + \frac{f(1 - \hat{Y})}{1 - \hat{Y} + \hat{Z}} \right] \\
= &\frac{Z}{D} \left[ -\left( \frac{e - ma}{m} \right) (l - Z) - d + \frac{f(1 - \hat{Y})}{1 - \hat{Y} + \hat{Z}} \right].
\end{align*} \]

From the fact that \( f > d \) and \( 0 \leq Z \leq l \), we see that there is an \( l_1 > 0 \) such that \( f(1 - Y)/(1 - Y + Z) > f - \epsilon \) for \( 0 < l < l_1 \), where \( 0 < \epsilon = (f - d)/2 \). There is also an \( l_2 > 0 \) such that \( 0 \leq \left( \frac{e - ma}{m} \right) (l - Z) < \epsilon \) for \( 0 < l < l_2 \). Choosing \( l = \min\{ l_1, l_2 \} \), we now see that

\[ \begin{align*}
\dot{V} &\leq \frac{1}{mD} \dot{V} + \frac{\dot{V}}{mD} \left[ -d + \frac{f(1 - \hat{U})}{1 - \hat{U} + m\hat{V}} \right] \\
\geq &\frac{Z}{mD} (f - d - 2\epsilon) = 0.
\end{align*} \]

By the existence-comparison result in Theorem 2.6, we conclude that there exists a wavefront solution \((U(\xi), V(\xi))\) of system (3.4) for every \( c \geq 2D \sqrt{\frac{e - ma}{mD}} \). We are now ready to state the following theorem on the existence of the wavefront solutions for the ratio-dependent predator-prey model (3.1) connecting the prey-only state \((0,0)\) and the coexistence state \((u^*, v^*)\).

\textbf{Theorem 3.3.} If condition \((H1)\) holds, then for every wave speed \( c \geq 2D \sqrt{\frac{e - ma}{mD}} \), there exists a wavefront solution \((u(x + ct), v(x + ct))\) for (3.1) with \((u(\xi), v(\xi))\) satisfying (3.3).

4. Example 2: Three species food chain model of Lotka-Volterra type. In this section, we apply Theorem 2.6 to show the existence of wavefront solutions of a reaction-diffusion model for three-species food chain, which was studied in several earlier works [5, 18, 23].

\[ \begin{align*}
\frac{\partial u}{\partial t} - D_1 \frac{\partial^2 u}{\partial x^2} &= u \left[ 1 - u - a_1 v - b_1 w \right], \\
\frac{\partial v}{\partial t} - D_2 \frac{\partial^2 v}{\partial x^2} &= rv \left[ 1 + a_2 u - v - b_2 w \right], \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ \quad (4.1) \\
\frac{\partial w}{\partial t} - D_3 \frac{\partial^2 w}{\partial x^2} &= sw \left[ 1 + a_3 u + b_3 v - w \right].
\end{align*} \]

In (4.1), the quantities \( u(x,t), v(x,t) \) and \( w(x,t) \) are scaled population densities of the three species (prey, predator and super-predator) in a food chain at
$t > 0$ and $x \in \mathbb{R}$, with corresponding intrinsic growth rates $1$, $r$ and $s$ that satisfy $0 < s \leq r \leq 1$. In each equation, the population is scaled such that the intra-species competition rate is $1$, while the inter-species consumption coefficients $a_i$ and $b_i$ satisfy $0 < a_i, b_i < 1$. This implies that the Lotka-Volterra model has the following trivial and semi-trivial steady states: $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(\frac{1}{1-a_1}, 1+a_2, 1+a_3)$, $(0, \frac{1}{1-b_2}, \frac{1}{1+b_3}, \frac{1}{1+b_3})$. Through linearization analysis on the corresponding ordinary differential system, we see that there exists a unique coexistence state $(u^*, v^*, w^*)$ with

$$
\begin{align*}
\begin{cases}
u^* & = \frac{1 + b_2b_3 - b_1(1 + a_3) - a_1(1 - b_2)}{1 + a_2b_1 + a_2(1 - b_1) - b_2(1 + a_3)}, \\
v^* & = \frac{1 + a_2b_1 + a_2(1 - b_1) - b_2(1 + a_3)}{1 + a_1b_2 + a_3b_1 + a_3b_3 - a_1a_2b_2}, \\
w^* & = \frac{1 + a_1b_2 + a_3b_1 + a_3b_3 - a_1a_2b_2}{1 + a_1b_2 + a_2b_2 + a_2b_3 + a_3b_1 - a_1a_2b_2},
\end{cases}
\end{align*}
$$

(4.2)

while all the trivial and semi-trivial steady states are unstable. The following assumptions are made throughout this section:

(H2a): $b_1(1 + b_3) + a_3(1 - b_2) < 1 + b_2b_3$, $b_2(1 + a_3) < 1 + a_2b_1 + a_2(1 - b_1)$.

In this section we look into the existence of wavefront solutions of (4.1) connecting the trivial steady state $(0, 0, 0)$ and the coexistence state $(u^*, v^*, w^*)$, in the form $(u(x, t), v(x, t), w(x, t)) = (u(x + ct), v(x + ct), w(x + ct)) = (u(\xi), v(\xi), w(\xi))$ where $\xi \in \mathbb{R}$. We now consider the following equations with limiting conditions:

$$
\begin{align*}
D_1 u_{\xi \xi} - cu_{\xi} + u [1 - u - a_1 v - b_1 w] &= 0, \\
D_2 v_{\xi \xi} - cv_{\xi} + rv [1 + a_2 u - v - b_2 w] &= 0, \\
D_3 w_{\xi \xi} - cw_{\xi} + sw [1 + a_3 u + b_3 v - w] &= 0,
\end{align*}
\tag{4.3}
$$

In system (4.3), the reaction functions are mixed monotone and satisfy the Lipschitz condition in the region $G = \{(u, v, w) | 0 \leq u \leq 1, 0 \leq v \leq 1, 0 \leq w \leq 1\}$. We next define the pair of upper and lower solutions for the wavefront system (4.3).

**Definition 4.1.** A pair of vector functions $(\hat{U}(\xi), \hat{V}(\xi), \hat{W}(\xi))$ and $(\hat{\bar{U}}(\xi), \hat{\bar{V}}(\xi), \hat{\bar{W}}(\xi))$ in $C^2(\mathbb{R}) \times C^2(\mathbb{R}) \times C^2(\mathbb{R})$ are coupled upper and lower solutions of (4.3) if they satisfy the following differential inequalities

$$
\begin{align*}
\begin{cases}
D_1 \hat{U}_{\xi \xi} - c\hat{U}_{\xi} + \hat{U} [1 - \hat{U} - a_1 \hat{V} - b_1 \hat{W}] &\leq 0, \\
D_2 \hat{V}_{\xi \xi} - c\hat{V}_{\xi} + \hat{V} [1 + a_2 \hat{U} - \hat{V} - b_2 \hat{W}] &\leq 0, \\
D_3 \hat{W}_{\xi \xi} - c\hat{W}_{\xi} + \hat{W} [1 + a_3 \hat{U} + b_3 \hat{V} - \hat{W}] &\leq 0, \\
D_1 \hat{\bar{U}}_{\xi \xi} - c\hat{\bar{U}}_{\xi} + \hat{\bar{U}} [1 - \hat{\bar{U}} - a_1 \hat{\bar{V}} - b_1 \hat{\bar{W}}] &\geq 0, \\
D_2 \hat{\bar{V}}_{\xi \xi} - c\hat{\bar{V}}_{\xi} + \hat{\bar{V}} [1 + a_2 \hat{\bar{U}} - \hat{\bar{V}} - b_2 \hat{\bar{W}}] &\geq 0, \\
D_3 \hat{\bar{W}}_{\xi \xi} - c\hat{\bar{W}}_{\xi} + \hat{\bar{W}} [1 + a_3 \hat{\bar{U}} + b_3 \hat{\bar{V}} - \hat{\bar{W}}] &\geq 0,
\end{cases}
\end{align*}
\tag{4.4}
$$
with the limiting conditions

\[
\begin{pmatrix}
\tilde{U} \\
\tilde{V} \\
\tilde{W}
\end{pmatrix}
\bigg|_{-\infty} \geq
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\geq
\begin{pmatrix}
\hat{U} \\
\hat{V} \\
\hat{W}
\end{pmatrix}
\bigg|_{-\infty};
\]

\[
\begin{pmatrix}
\tilde{U} \\
\tilde{V} \\
\tilde{W}
\end{pmatrix}
\bigg|_{+\infty} \geq
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\geq
\begin{pmatrix}
\hat{U} \\
\hat{V} \\
\hat{W}
\end{pmatrix}
\bigg|_{+\infty}.
\]

(4.5)

Under the assumption \(H2(b): a_1(1 + a_2) + b_1(1 + a_3 + b_3 + a_2b_3) < 1, b_2(1 + a_3 + b_3 + a_2b_3) < 1,\) which ensures \(H2(a),\) the construction of upper and lower solutions for the wavefront system (4.2) is again in terms of the wavefront solution of the K.P.P. equations. Denote

\[D = \max\{D_1, D_2, D_3\}, \quad \underline{D} = \min\{D_1, D_2, D_3\}.\]

(4.6)

Let \(Y\) be the wavefront solution of the following K.P.P. equation:

\[
\begin{cases}
Y_{\xi\xi} - c\frac{\underline{D}}{D} Y + Y_\xi(1 - Y) = 0, \\
Y(-\infty) = 0, \quad Y(+\infty) = 1.
\end{cases}
\]

(4.7)

By [27], for each \(c \geq \frac{2\underline{D}}{\sqrt{D}},\) there is a unique solution (up to a translation of the origin) \(Y\) of (4.7). Such solution satisfies \(Y'(\xi) > 0\) for \(\xi \in \mathbb{R}.\) Set

\[m = \max\{a_1(1 + a_2) + b_1(1 + a_3 + b_3 + a_2b_3), b_2(1 + a_3 + b_3 + a_2b_3)\}.\]

(4.8)

Since \(m < 1,\) there exists an \(l\) such that \(0 < l < 1 - m.\) Let \(Z\) be the wavefront solution of the following K.P.P. equation:

\[
\begin{cases}
Z_{\xi\xi} - \frac{\underline{D}}{D} Z + \frac{a_2}{D} (1 - Z) = 0, \\
Z(-\infty) = 0, \quad Z(+\infty) = l.
\end{cases}
\]

(4.9)

Also by [27], corresponding to every \(c \geq \frac{2\underline{D}}{\sqrt{D}},\) (which holds for \(c \geq \frac{2\underline{D}}{\sqrt{D}}),\) there is a unique solution (up to a translation of the origin) \(Z\) of (4.9). Such solution satisfies \(Z'(\xi) > 0\) for \(\xi \in \mathbb{R}.\) From the fact that \(1 - l > m\) and both the K.P.P. systems (4.7) and (4.9) are translation invariant, by a suitable shift of the origin we will have \(\frac{1}{1 - l} Z(\xi) \geq m Y(\xi)\) for all \(\xi \in \mathbb{R}.\)

For \(c \geq \frac{2\underline{D}}{\sqrt{D}},\) we define our upper and lower solutions as follows:

\((\tilde{U}, \tilde{V}, \tilde{W}) = (Y, (1 + a_2)Y, (1 + a_3 + b_3 + a_2b_3)Y); \quad (\hat{U}, \hat{V}, \hat{W}) = (Z, Z, Z).\)

(4.10)

We first verify the inequalities in (4.4) for the upper solution. It can be seen that

\[
\tilde{U}_{\xi\xi} - \frac{c}{D_1} \tilde{U}_\xi + \frac{\tilde{U}}{D_1} \left(1 - \tilde{U} - a_1\tilde{V} - b_1\tilde{W}\right) \\
\leq \quad Y_{\xi\xi} - \frac{c}{D} Y_\xi + \frac{Y}{D} (1 - Y) = 0.
\]
\[
\hat{\eta}_{\xi \xi} - \frac{c}{D_2} \hat{\eta}_{\xi} + \frac{r \hat{\eta}V}{D_2} \left(1 + a_2 \hat{U} - \hat{V} - b_2 \hat{W}\right) \\
\leq (1 + a_2) \left[Y_{\xi \xi} - \frac{c}{D} Y_{\xi}\right] + \frac{r(1 + a_2)Y}{D} \left[1 + a_2 Y - (1 + a_2)Y\right] \\
\leq - (1 - r)(1 + a_2)\frac{Y}{D} (1 - Y) \leq 0,
\]

\[
\hat{w}_{\xi \xi} - \frac{c}{D_3} \hat{w}_{\xi} + \frac{s \hat{W}D_3}{D_3} \left(1 + a_3 \hat{U} + b_3 \hat{V} - \hat{W}\right) \\
\leq (1 + a_3 + b_3 + a_2 b_3) \left[Y_{\xi \xi} - \frac{c}{D} Y_{\xi}\right] + \frac{s(1 + a_3 + b_3 + a_2 b_3)Y}{D} (1 - Y) \\
= - (1 - s)\frac{s(1 + a_3 + b_3 + a_2 b_3)Y}{D} (1 - Y) \leq 0.
\]

We next show that the lower solution \((Z, Z, Z)\) also satisfies the differential inequalities given in (4.4).

\[
\hat{U}_{\xi \xi} - \frac{c}{D_1} \hat{U}_{\xi} + \frac{\hat{U}}{D_1} \left(1 - \hat{U} - a_1 \hat{V} - b_1 \hat{W}\right) \\
\geq Z_{\xi \xi} - \frac{c}{D} Z_{\xi} + \frac{Z}{D} \left[1 - Z - a_1 (1 + a_2)Y - b_1 (1 + a_3 + b_3 + a_2 b_3)Y\right] \\
\geq - \frac{sZ}{lD} (l - Z) + \frac{sZ}{D} (1 - Z - mY) \geq \frac{sZ}{D} \left(\frac{1 - l}{l} Z - mY\right) \geq 0,
\]

\[
\hat{V}_{\xi \xi} - \frac{c}{D_2} \hat{V}_{\xi} + \frac{r \hat{V}V}{D_2} \left(1 + a_2 \hat{U} - \hat{V} - b_2 \hat{W}\right) \\
\geq Z_{\xi \xi} - \frac{c}{D} Z_{\xi} + \frac{sZ}{D} \left[1 - (1 - a_2)Z - b_2 (1 + a_3 + b_3 + a_2 b_3)Y\right] \\
\geq - \frac{sZ}{lD} (l - Z) + \frac{sZ}{D} (1 - Z - mY) \geq \frac{sZ}{D} \left(1 - Z - mY\right) \geq 0,
\]

\[
\hat{W}_{\xi \xi} - \frac{c}{D_3} \hat{W}_{\xi} + \frac{s \hat{W}D_3}{D_3} \left(1 + a_3 \hat{U} + b_3 \hat{V} - \hat{W}\right) \\
\geq Z_{\xi \xi} - \frac{c}{D} Z_{\xi} + \frac{sZ}{D} \left[1 - (1 - a_3 - b_3)Z\right] \\
\geq - \frac{sZ}{D} \left(1 - \frac{Z}{l}\right) + \frac{sZ}{D} (1 - Z) \geq \frac{sZ}{D} \left(\frac{1 - l}{l}\right) Z \geq 0.
\]

Based on the existence-comparison result in Theorem 2.6, we can state the following theorem on the existence of the wavefront solutions for the 3-species food chain model (4.1) connecting the trivial steady state \((0, 0, 0)\) and the coexistence state \((u^*, v^*, w^*)\).
Theorem 4.2. If condition (H2(b)) holds, then for every wave speed \( c \geq 2D/\sqrt{D} \) there exists a wavefront solution \( (u(x + ct), v(x + ct), w(x + ct)) \) for (4.1) with \( (u(\xi), v(\xi), w(\xi)) \) satisfying (4.3).

5. Example 3: Three species competition model of Lotka-Volterra type.

At last, we discuss the existence of wavefront solutions for a reaction-diffusion model for three species competition [7, 18].

Example 3: Three species competition model of Lotka-Volterra type.

Theorem 4.2. If condition (H2(b)) holds, then for every wave speed \( c \geq 2D/\sqrt{D} \) there exists a wavefront solution \( (u(x + ct), v(x + ct), w(x + ct)) \) for (4.1) with \( (u(\xi), v(\xi), w(\xi)) \) satisfying (4.3).

5. Example 3: Three species competition model of Lotka-Volterra type.

At last, we discuss the existence of wavefront solutions for a reaction-diffusion model for three species competition [7, 18].

\[
\begin{align*}
\frac{\partial u}{\partial t} - D_1 \frac{\partial^2 u}{\partial x^2} &= u [1 - u - a_1 v - b_1 w], \\
\frac{\partial v}{\partial t} - D_2 \frac{\partial^2 v}{\partial x^2} &= rv [1 - a_2 u - v - b_2 w], \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\
\frac{\partial w}{\partial t} - D_3 \frac{\partial^2 w}{\partial x^2} &= sw [1 - a_3 u - b_3 v - w].
\end{align*}
\]

(5.1)

where the quantities \( u(x, t) \), \( v(x, t) \), and \( w(x, t) \) represent scaled population densities of the three competing species at \( t > 0 \), and \( x \in \mathbb{R} \), with corresponding intrinsic growth rates \( 1, r \), and \( s \) such that \( 0 < s \leq r \leq 1 \). In each equation, the population is scaled such that the intra-species competition rate is 1, while the inter-species consumption coefficients \( a_i \) and \( b_i \) satisfy \( 0 < a_i, b_i < 1 \). This implies that the Lotka-Volterra competition model (5.1) has the following trivial and semi-trivial states: \( (0,0,0) \), \( (1,0,0) \), \( (0,1,0) \), \( (0,0,1) \), \( (1-a_1,1-a_2,0) \), \( (0,1-b_2,0,1-b_3,0,1-a_3,0) \). In the corresponding ordinary differential system, linearization analysis implies that all the trivial and semi-trivial states are unstable when

**H3(a):** \( a_1 a_2 + a_3 (1 - a_1) + b_3 (1 - a_2) < 1, a_3 b_1 + a_2 (1 - b_1) + b_2 (1 - a_3) < 1, b_2 b_3 + b_1 (1 - a_3) + a_1 (1 - b_2) < 1. \)

In this case, there also exists a unique coexistence state \( (u^*, v^*, w^*) \) with

\[
\begin{align*}
u^* &= \frac{1 - b_2 b_3 - b_1 (1 - a_3) - a_1 (1 - b_2)}{1 - a_1 a_2 - b_3 b_1 - a_2 b_1 b_3 + a_1 a_3 b_2}, \\
v^* &= \frac{1 - a_1 a_2 - b_3 b_1 - a_2 b_1 b_3 + a_1 a_3 b_2}{1 - a_1 a_2 - b_3 b_1 - a_2 b_1 b_3 + a_1 a_3 b_2}, \\
w^* &= \frac{1 - a_1 a_2 - b_3 b_1 - a_2 b_1 b_3 + a_1 a_3 b_2}{1 - a_1 a_2 - b_3 b_1 - a_2 b_1 b_3 + a_1 a_3 b_2}.
\end{align*}
\]

(5.2)

Throughout this section we assume that conditions in **H3(a)** hold, and study the existence of wavefront solutions of (5.1) connecting the trivial steady state \( (0,0,0) \) and the coexistence state \( (u^*, v^*, w^*) \), in the form \( (u(x, t), v(x, t), w(x, t)) = (u(\xi), v(\xi), w(\xi)) \) where \( \xi \in \mathbb{R} \). The function \( (u(\xi), v(\xi), w(\xi)) \) needs to satisfy the following equations with limiting conditions:

\[
\begin{align*}
D_1 u_\xi - c_1 u_\xi + u [1 - u - a_1 v - b_1 w] &= 0, \\
D_2 v_\xi - c_2 v_\xi + rv [1 - a_2 u - v - b_2 w] &= 0, \\
D_3 w_\xi - c_3 w_\xi + sw [1 - a_3 u - b_3 v - w] &= 0, \quad (5.3)
\end{align*}
\]

\[
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix} (-\infty) = \begin{pmatrix} 0 \\
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix} u \\
v \\
w
\end{pmatrix} (+\infty) = \begin{pmatrix} u^* \\
v^* \\
w^*
\end{pmatrix}.
\]

In system (5.3), the reaction functions are mixed monotone and satisfy the Lipschitz condition in the region \( G = \{(u, v, w) \mid 0 \leq u \leq 1, 0 \leq v \leq 1, 0 \leq w \leq 1\}. \)
We next define the upper-solution and lower-solution pair for the wavefront system (5.3).

**Definition 5.1.** A pair of functions \((\tilde{U}(\xi), \tilde{V}(\xi), \tilde{W}(\xi))\) and \((\hat{U}(\xi), \hat{V}(\xi), \hat{W}(\xi))\) in \(C^2(\mathbb{R}) \times C^2(\mathbb{R}) \times C^2(\mathbb{R})\) are coupled upper-lower solutions of (5.3) if they satisfy the following differential inequalities

\[
\begin{align*}
D_1\ddot{U}_{\xi\xi} - c\ddot{U}_\xi + \ddot{U} \left[ 1 - \bar{U} - a_1\hat{V} - b_1\hat{W} \right] &\leq 0 \\
D_2\ddot{V}_{\xi\xi} - c\ddot{V}_\xi + \ddot{V} \left[ 1 - a_2\bar{U} - \hat{V} - b_2\hat{W} \right] &\leq 0 \\
D_3\ddot{W}_{\xi\xi} - c\ddot{W}_\xi + \ddot{W} \left[ 1 - a_3\bar{U} - b_3\hat{V} - \hat{W} \right] &\leq 0, \\
D_1\ddot{U}_{\xi\xi} - c\ddot{U}_\xi + \ddot{U} \left[ 1 - \bar{U} - a_1\hat{V} - b_1\hat{W} \right] &\geq 0 \\
D_2\ddot{V}_{\xi\xi} - c\ddot{V}_\xi + \ddot{V} \left[ 1 - a_2\bar{U} - \hat{V} - b_2\hat{W} \right] &\geq 0 \\
D_3\ddot{W}_{\xi\xi} - c\ddot{W}_\xi + \ddot{W} \left[ 1 - a_3\bar{U} - b_3\hat{V} - \hat{W} \right] &\geq 0,
\end{align*}
\]

(5.4)

with the limiting conditions

\[
\begin{pmatrix}
\ddot{U} \\
\ddot{V} \\
\ddot{W}
\end{pmatrix}
\bigg|_{(-\infty)} \geq \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} \geq \begin{pmatrix}
\ddot{U} \\
\ddot{V} \\
\ddot{W}
\end{pmatrix}
\bigg|_{(+\infty)};
\]

\[
\begin{pmatrix}
\ddot{U} \\
\ddot{V} \\
\ddot{W}
\end{pmatrix}
\bigg|_{(+\infty)} \geq \begin{pmatrix}
u^* \\
v^* \\
w^*
\end{pmatrix} \geq \begin{pmatrix}
\ddot{U} \\
\ddot{V} \\
\ddot{W}
\end{pmatrix}
\bigg|_{(-\infty)}.
\]

(5.5)

Denoting \(\overline{D}, \underline{D}\) as in 4.6 and making an additional assumption that restricts the magnitude of interspecific competitions,

**H3(b):** \(m = \max\{a_1 + b_1, a_2 + b_2, a_3 + b_3\} < 1\),

we again construct a pair of upper and lower solutions for the traveling wave system (5.2) by solutions of the following K.P.P. equations with limiting conditions:

\[
\begin{align*}
Y_{\xi\xi} - \frac{c}{\overline{D}} Y_\xi + \frac{c}{\overline{D}} (1 - Y) &= 0, \\
Y(-\infty) &= 0, \quad Y(+\infty) = 1. \\
(5.6)
\end{align*}
\]

\[
\begin{align*}
Z^{(i)}_{\xi\xi} - \frac{c}{\overline{D}} Z^{(i)}_\xi + \frac{c}{\overline{D}} (1 - Z^{(i)}/l) &= 0, \\
Z^{(i)}(-\infty) &= 0, \quad Z^{(i)}(+\infty) = l.
\end{align*}
\]

(5.7)

where \(0 < l < \min\{1 - m, u^*, v^*, w^*\}\) and \(r_1 = 1, r_2 = r, r_3 = s\). By [27], corresponding to every \(c \geq 2\overline{D}/\sqrt{\overline{D}}\) (which also ensures \(c \geq 2\overline{D}/\sqrt{\overline{D}}\)), there is a unique solution (up to a translation of the origin) \(Y\) of (5.6) and \(Z^{(i)}\) of (5.7). Those solutions are strictly increasing for \(\xi \in \mathbb{R}\). From the fact that \(1 - l > m\) and both the K.P.P. systems (5.6) and (5.7) are translation invariant, by a suitable shift of origin we have \(\frac{1}{l^{1-i}} Z^{(i)}(\xi) \geq mY(\xi)\) for all \(1 \leq i \leq 3\) and \(\xi \in \mathbb{R}\).
For \( c \geq \frac{2D}{\sqrt{D}} \), we set up our upper and lower solutions as follows:

\[
(\tilde{U}, \tilde{V}, \tilde{W}) = (Y, Y, Y); \quad (\hat{U}, \hat{V}, \hat{W}) = (Z^{(1)}, Z^{(2)}, Z^{(3)}).
\]

We first verify the inequalities in (5.4) for the upper solution. It can be seen that

\[
\tilde{U}_{\xi\xi} - \frac{c}{D_1} \tilde{U}_\xi + \frac{\tilde{U}}{D_1} \left( 1 - \tilde{U} - a_1 \tilde{V} - b_1 \tilde{W} \right) \\
\leq Y_{\xi\xi} - \frac{c}{D} Y_\xi + \frac{Y}{D} (1 - Y) = 0.
\]

\[
\tilde{V}_{\xi\xi} - \frac{c}{D_2} \tilde{V}_\xi + \frac{rv \tilde{V}}{D_2} \left( 1 - a_2 \tilde{U} - \tilde{V} - b_2 \tilde{W} \right) \\
\leq Y_{\xi\xi} - \frac{c}{D} Y_\xi + \frac{rv Y}{D} (1 - Y) \leq 0.
\]

\[
\tilde{W}_{\xi\xi} - \frac{c}{D_3} \tilde{W}_\xi + \frac{s \tilde{V}}{D_3} \left( 1 - a_3 \tilde{U} - \tilde{V} - \tilde{W} \right) \\
\leq Y_{\xi\xi} - \frac{c}{D} Y_\xi + \frac{sY}{D} (1 - Y) \leq 0.
\]

We next show that the lower solution \((Z^{(1)}, Z^{(2)}, Z^{(3)})\) also satisfies the differential inequalities given in (5.4). We see that

\[
\hat{U}_{\xi\xi} - \frac{c}{D_1} \hat{U}_\xi + \frac{\hat{U}}{D_1} \left( 1 - \hat{U} - a_1 \hat{V} - b_1 \hat{W} \right) \\
\geq Z^{(1)}_{\xi\xi} - \frac{c}{D} Z^{(1)}_\xi + \frac{Z^{(1)}}{D} \left[ 1 - Z^{(1)} - (a_1 + b_1)Y \right] \\
\geq \frac{Z^{(1)}}{D} \left( 1 - m Y \right) \geq 0,
\]

and the inequalities for \(\hat{V}\) and \(\hat{W}\) can be verified similarly.

Once again, by the existence-comparison result in Theorem 2.6, we have the following result on the existence of the wavefront solutions for the 3-species competition model (5.1) connecting the trivial steady state \((0, 0, 0)\) and the coexistence state \((u^*, v^*, w^*)\).

**Theorem 5.2.** If conditions (H3(a)) and (H3(b)) holds, then for every wave speed \( c \geq \frac{2D}{\sqrt{D}} \) there exists a wavefront solution \((u(x+ct), v(x+ct), w(x+ct))\) to (5.1) with \((u(\xi), v(\xi), w(\xi))\) satisfying (5.3).

6. **Numerical simulations.** Numerical simulations of the wavefront solutions for the models discussed in previous sections can be generated on a rectangular domain \([-L, L] \times [0, T]\). After discretizing the differential equation systems into finite-difference systems, numerical solutions for the reaction-diffusion systems are obtained through the monotone iterative scheme developed in several earlier articles (see for example, [22, 26]). In this section we demonstrate numerical simulations of the wavefront solutions in the ratio-dependent predator-prey model discussed in Section 3 and the Lotka-Volterra food chain model discussed in Section 4.

**Example 1.** - Figure 1. The ratio-dependent predator-prey model (3.1) with a traveling wavefront flowing from the prey-only state \((1, 0)\) to the coexistence state \((u^*, v^*)\) given in (3.2).
As seen in Theorem 3.3, when conditions in \((H1)\) holds, for every \(c \geq \frac{2\bar{D}}{\sqrt{\frac{e - ma}{mD}}}\), the ratio-dependent predator-prey model \((3.1)\) has a wavefront solution \((u(x + ct), v(x + ct))\) connecting the prey-only state \((1, 0)\) to the coexistence state \((u^*, v^*)\) as given in \((3.2)\) for limits at \(\xi = \pm \infty\). We choose the following set of biological parameters that satisfies the conditions in \((H1)\): \(a = 0.4, d = 0.2, e = 0.3, f = 0.25, m = 0.5, D_1 = 0.012, D_2 = 0.01\). Fixing the initial function \((u_0, v_0)\) as a small perturbation of the steady state \((1, 0)\), we demonstrate the traveling wavefront flowing towards the coexistence state \((0.7, 0.35)\) in Figure 1. Observing the wavefront solution at \(t = 500\) and \(t = 900\) in Figure 1(c), one can see that the wave speed \(c = 0.12\) which is larger than the minimal speed \(2\bar{D}\sqrt{\frac{e - ma}{mD}} = 0.048\sqrt{5}\).

**Example 2.** - Figure 2-3. The Lotka-Volterra food chain model \((4.1)\) with a traveling wavefront flowing from the trivial state \((0, 0, 0)\) to the coexistence state \((u^*, v^*, w^*)\) given in \((4.2)\).

As seen in Theorem 4.2, when conditions in \((H2b)\) holds, for every \(c \geq \frac{2\bar{D}}{\sqrt{D}}\), the food-chain model \((4.1)\) has a wavefront solution \((u(x + ct), v(x + ct), w(x + ct))\) connecting the all-extinction steady state \((0, 0, 0)\) to the coexistence state \((u^*, v^*, w^*)\) as given in \((4.2)\) for limits at \(\xi = \pm \infty\). We choose the following set of biological parameters that satisfies the conditions in \((H2b)\): \(a_1 = 0.3, b_1 = 0.3, a_2 = 0.4, b_2 = 0.5, a_3 = 0.3, b_3 = 0.4, r = 0.7, s = 0.5, D_1 = 0.01, D_2 = 0.02, D_3 = 0.03\). Fixing the initial function \((u_0, v_0, w_0)\) as a small perturbation of \((0, 0, 0)\), we demonstrate the traveling wavefront flowing towards the coexistence state \((0.7, 0.35)\) in Figure 1(c).
Figure 2. Traveling wavefront in model (4.1) connecting \((0, 0, 0)\) to \((u^*, v^*, w^*)\), coexistence of the species.

Figure 3. Wavefront solutions in model (4.1), \(u, v, w\)-species.
state \((0.44586, 0.50955, 1.33758)\) in Figure 2 and Figure 3. Observing the wavefront solution at \(t = 30\) and \(t = 90\) in Figure 3, one can see that the wave speed \(c = \frac{7}{3}\) which is larger than the minimum wave speed \(2D/\sqrt{D} = 0.6\).

**Acknowledgments.** The authors would like to thank the anonymous referees for thorough review of the manuscript and very helpful comments for revisions.

**REFERENCES**

[1] S. Ai, S.-N. Chow and Y. Yi, **Traveling wave solutions in a tissue interaction model for skin pattern formation**, *Journal of Dynamics and Differential Equations*, 15 (2003), 517–534.

[2] J. C. Alexander, R. A. Gardner and C. K. R. T. Jones, A topological invariant arising in the stability analysis of traveling waves, *J. Reine Angew Math.*, 410 (1990), 167–212.

[3] A. Boumenir and V. Nguyen, Perron theorem in monotone iteration method for traveling waves in delayed reaction-diffusion equations, *Journal of Differential Equations*, 244 (2008), 1551–1570.

[4] N. Fei and J. Carr, Existence of travelling waves with their minimal speed for a diffusing Lotka-Volterra system, *Nonlinear Analysis: Real World Applications*, 4 (2003), 503–524.

[5] W. Feng, Permanence effect in a three-species food chain model, *Applicable Analysis*, 54 (1994), 195–209.

[6] W. Feng and X. Lu, Traveling waves and competitive exclusion in models of resource competition and mating interference, *J. Math. Anal. Appl.*, 424 (2015), 542–562.

[7] W. Feng and W. Ruan, Coexistence, Permanence, and stability in a three species competition model, *Acta. Math. Appl. Sinica (English Ser.)*, 12 (1996), 443–446.

[8] Y. Hosono, Travelling waves for a diffusive Lotka-Volterra competition model I: Singular Perturbations, *Discrete Continuous Dynamical Systems - B*, 3 (2003), 79–95.

[9] X. Hou and W. Feng, Traveling waves and their stability in a coupled reaction diffusion system, *Communications on Pure and Applied Analysis*, 10 (2011), 141–160.

[10] X. Hou, W. Feng and X. Lu, A mathematical analysis of a public goods games model, *Nonlinear Analysis: Real World Applications*, 10 (2009), 2207–2224.

[11] X. Hou, Y. Li and K. R. Meyer, Traveling wave solutions for a reaction diffusion equation with double degenerate nonlinearities, *Discrete and Continuous Dynamical Systems-A*, 26 (2010), 265–290.

[12] J. I. Kanel, On the wave front of a competition-diffusion system in population dynamics, *Nonlinear Analysis: Theory, Methods & Applications*, 65 (2006), 301–320.

[13] J. I. Kanel and L. Zhou, Existence of wave front solutions and estimates of wave speed for a competition-diffusion system, *Nonlinear Analysis: Theory, Methods & Applications*, 27 (1996), 579–587.

[14] Y. Kan-on, Note on propagation speed of travelling waves for a weakly coupled parabolic system, *Nonlinear Analysis: Theory, Methods & Applications*, 41 (2001), 239–246.

[15] Y. Kan-on, Fisher wave fronts for the lotka-volterra competition model with diffusion, *Nonlinear Analysis: Theory, methods & Applications*, 28 (1997), 145–164.

[16] A. Kolmogorov, A. Petrovskii and N. Piskunov, A study of the equation of diffusion with increase in the quantity of matter, *Bijul. Moskovskovo Gov. Inv.*, 17 (1937), 1–72.

[17] Y. Kuang and E. Beretta, Global qualitative analysis of a ratio-dependent predator-prey system, *Journal of Mathematical Biology*, 36 (1998), 389–406.

[18] A. W. Leung, *Systems of Nonlinear Partial Differential Equations: Applications to Biology and Engineering (Mathematics and Its Applications)*, 1989 Edition, Kluwer Academic Publishers, Dordrecht, 1989.

[19] A. W. Leung, X. Hou and W. Feng, Traveling wave solutions for Lotka-Volterra system revisited, *Discrete and Continuous Dynamical Systems - Series B*, 15 (2011), 171–196.

[20] G. Lin, W. Li and M. Ma, Traveling wave solutions in delayed reaction diffusion system with applications to multi-species models, *Discrete and Continuous Dynamical Systems - B*, 13 (2010), 393–414.

[21] X. Liu and P. Weng, Asymptotic spreading of a three dimensional Lotka-Volterra cooperative-competitive system, *Discrete and Continuous Dynamical Systems - B*, 20 (2015), 505–518.

[22] X. Lu, Monotone method and convergence acceleration for finite-difference solutions of parabolic problems with time delays, *Numer. Meth. Part. Diff. Eqn.s*, 11 (1995), 591–602.
[23] X. Lu and W. Feng, Dynamics and numerical simulations of food-chain populations, *Applied Mathematics and Computations*, 65 (1994), 335–344.

[24] S. W. Ma, Traveling waves for non-local delayed diffusion equations via auxiliary equations, *Journal of Differential Equations*, 237 (2007), 259–277.

[25] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.

[26] C. V. Pao and X. Lu, Block monotone iterative methods for numerical solutions of nonlinear parabolic equations, *SIAM J. Sci. Comput.*, 25 (2003), 164–185.

[27] D. Sattinger, On the stability of traveling waves of nonlinear parabolic systems, *Advances in Mathematics*, 22 (1976), 312–355.

[28] M. M. Tang and P. C. Fife, Propagating fronts for competing species equations with diffusion, *Arch. Rat. Mech. Anal.*, 73 (1980), 69–77.

[29] A. Volpert, V. Volpert and V. Volpert, *Traveling Wave Solutions of Parabolic Systems*, Transl. Math. Monographs, 140, Amer. Math. Soc., Providence, RI, 1994.

[30] Z.-C. Wang, W.-T. Li and S. Ruan, Existence and Stability of traveling wave fronts in reaction-advection diffusion equations with nonlocal delay, *Journal of Differential Equations*, 238 (2007), 153–200.

[31] J. Wu and X. Zou, Traveling wave fronts of reaction-diffusion systems with delay, *Journal of Dynamics and Differential Equations*, 13 (2001), 651–687, and Erratum to traveling wave fronts of reaction-diffusion systems with delays, *Journal of Dynamics and Differential Equations*, 20 (2008), 531–533.

[32] D. Xu and X. Q. Zhao, Bistable waves in an epidemic model, *Journal of Dynamics and Differential Equations*, 16 (2004), 679–707, and Erratum, *Journal of Dynamics and Differential Equations*, 17 (2005), 219–247.

Received July 2015; 1st revision July 2015; 2nd revision September 2015.

*E-mail address: feng@uncw.edu*

*E-mail address: ruanw@purduecal.edu*

*E-mail address: lux@uncw.edu*