Refined enumeration of Alternating Sign Matrices and Descending Plane Partitions

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Introduction

- **Plane Partitions** were introduced by Mac Mahon about a century ago. However **Descending Plane Partitions** (DPPs), as well as other variations on plane partitions (symmetry classes), were considered in the 80s. [Andrews]

- Alternating Sign Matrices (ASMs) also appeared in the 80s, but in a completely different context, namely in Mills, Robbins and Rumsey’s study Dodgson’s condensation algorithm for the evaluation of determinants.

- One of the possible formulations of the **Alternating Sign Matrix conjecture** is that these objects are in bijection (for every size $n$). (proved by Zeilberger in ’96 in a slightly different form)
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Interest in the mathematical physics community because of

1. Kuperberg’s alternative proof of the Alternating Sign Matrix conjecture using the connection to the six-vertex model. (’96)

2. The Razumov–Stroganov correspondence and related conjectures. (’01)

A proof of all these conjectures would probably give a fundamentally new proof of the ASM (ex-)conjecture.

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T. Fonseca and P. Zinn-Justin: proof of the doubly refined Alternating Sign Matrix conjecture (’08).
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J. Propp (’03)

Today’s talk is about the proof of another generalization of the ASM conjecture formulated in ’83 by Mills, Robbins and Rumsey.
Iterative use of the Desnanot–Jacobi identity:

\[
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\vspace{2cm}
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\]

allows to compute the determinant of a $n \times n$ matrix by computing the determinants of the connected minors of size 1, \ldots, $n$.

What happens when we replace the minus sign with an arbitrary parameter?

Laurent phenomenon: the result is still a Laurent polynomial in the entries of the matrix.
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allows to compute the determinant of a $n \times n$ matrix by computing the determinants of the connected minors of size 1, \ldots, $n$.

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\[
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\end{array} & + \lambda \\
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allows to compute the determinant of a $n \times n$ matrix by computing the determinants of the connected minors of size 1, \ldots, $n$.

What happens when we replace the minus sign with an arbitrary parameter?

Laurent phenomenon: the result is still a Laurent polynomial in the entries of the matrix.
Theorem (Robbins, Rumsey, '86)

If $M$ is an $n \times n$ matrix, then

$$\det_{\lambda} M = \sum_{A \in \text{ASM}(n)} \lambda^{\nu(A)} (1 + \lambda)^{\mu(A)} \prod_{i,j=1}^{n} M_{ij}^{A_{ij}}$$

Here $\text{ASM}(n)$ is the set of $n \times n$ Alternating Sign Matrices, that is matrices such that in each row and column, the non-zero entries form an alternation of +1s and −1s starting and ending with +1.
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Refined enumeration of ASMs and DPPs
Example

For $n = 3$, there are 7 ASMs:

\[
ASM(3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \right\}
\]
\[ \det_{\lambda} M = \sum_{A \in \text{ASM}(n)} \chi^{\nu(A)} (1 + \lambda)^{\mu(A)} \prod_{i,j=1}^{n} M_{ij}^{A_{ij}} \]

\( \mu(A) \) is the number of \(-1\)s in \( A \).

\( \nu(A) \) is a generalization of the inversion number of \( A \):

\[ \nu(A) = \sum_{1 \leq i \leq i' \leq n} A_{ij} A_{i'j'} \sum_{1 \leq j' < j \leq n} \]

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\[ \det_{\lambda} M = \sum_{A \in \text{ASM}(n)} \chi^\nu(A) (1 + \lambda)^\mu(A) \prod_{i,j=1}^{n} M_{ij}^{A_{ij}} \]

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Example

For the constant matrix $1_n$, we have the recurrence
\[ \det \lambda 1_{n+1} \det \lambda 1_{n-1} = (1 + \lambda) \det^2 \lambda 1_n \] and therefore:
\[ \det \lambda 1_n = (1 + \lambda)^{n(n-1)/2} = \sum_{A \in \text{ASM}(n)} \lambda^{\nu(A)} (1 + \lambda)^{\mu(A)} \]

In what follows, we shall be interested in more general weighted enumerations of ASMs, of the type $\sum_{A \in \text{ASM}(n)} x^{\nu(A)} y^{\mu(A)}$ and refinements.
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A Descending Plane Partition is an array of positive integers ("parts") of the form

\[
\begin{array}{cccc}
D_{11} & D_{12} & \ldots & D_{1,\lambda_1} \\
D_{22} & \ldots & \ldots & D_{2,\lambda_2+1} \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \\
& & & D_{tt} \ldots D_{t,\lambda_t+t-1}
\end{array}
\]

such that

- The parts decrease weakly along rows, i.e., \( D_{ij} \geq D_{i,j+1} \).
- The parts decrease strictly down columns, i.e., \( D_{ij} > D_{i+1,j} \).
- The first parts of each row and the row lengths satisfy

\[
D_{11} > \lambda_1 \geq D_{22} > \lambda_2 \geq \ldots \geq D_{t-1,t-1} > \lambda_{t-1} \geq D_{tt} > \lambda_t
\]
Let $\text{DPP}(n)$ be the set of DPPs in which each part is at most $n$, i.e., such that $D_{ij} \in \{1, \ldots, n\}$.

**Example**

For $n = 3$, there are 7 DPPs:

$$\text{DPP}(3) = \left\{ \emptyset, 3 \ 3 \ 2, 2, 3 \ 3, 3, 3 \ 2, 3 \ 1 \right\}$$
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$$\text{DPP}(3) = \{\emptyset, \begin{array}{c}3 \\ 2 \end{array}, 2, 3 3, 3, 3 2, 3 1\}$$
Define statistics for each $D \in \text{DPP}(n)$ as:

\[
\nu(D) = \text{number of parts of } D \text{ for which } D_{ij} > j - i,
\]

\[
\mu(D) = \text{number of parts of } D \text{ for which } D_{ij} \leq j - i.
\]
Theorem (Andrews, 79)

*The number of DPPs with parts at most \( n \) is:*

\[
|\text{DPP}(n)| = \prod_{i=0}^{n-1} \frac{(3i + 1)!}{(n + i)!} = 1, 2, 7, 42, 429, \ldots
\]
The Alternating Sign Matrix conjecture

The following result was first conjectured by Mills, Robbins and Rumsey in '82:

**Theorem (Zeilberger, '96; Kuperberg, '96)**

The number of ASMs of size $n$ is

$$|\text{ASM}(n)| = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = 1, 2, 7, 42, 429 \ldots$$

NB: a third family is also known to have the same enumeration as ASMs and DPPs: TSSCPPs. In fact, Zeilberger’s proof consists of a (non-bijective) proof of equienumeration of ASMs and TSSCPPs.
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Theorem (Behrend, Di Francesco, Zinn-Justin, ’11)

The sizes of \( \{A \in \text{ASM}(n) \mid \nu(A) = p, \mu(A) = m\} \) and \( \{D \in \text{DPP}(n) \mid \nu(D) = p, \mu(D) = m\} \) are equal for any \( n, p, m \).

(in fact, an even more general result that was conjectured by Mills, Robbins and Rumsey in ’83 is proved)

Equivalently, if one defines generating series:

\[
Z_{\text{ASM}}(n, x, y) = \sum_{A \in \text{ASM}(n)} x^{\nu(A)} y^{\mu(A)}
\]

\[
Z_{\text{DPP}}(n, x, y) = \sum_{D \in \text{DPP}(n)} x^{\nu(D)} y^{\mu(D)}
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then the theorem states that \( Z_{\text{ASM}}(n, x, y) = Z_{\text{DPP}}(n, x, y) \).
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\]

\[
DPP(3) = \left\{ \emptyset, 3\ 3, 2, 3\ 3, 3, 3\ 2, 3\ 1 \right\}
\]

\[
Z_{ASM/DPP}(3, x, y) = 1 + x^3 + x + x^2 + x + x^2 + xy
\]
Strategy: write the two generating series as determinants:

\[ Z_{\text{ASM}}(n, x, y) = \det M_{\text{ASM}}(n, x, y) \]
\[ Z_{\text{DPP}}(n, x, y) = \det M_{\text{DPP}}(n, x, y) \]

and transform one matrix into another by row/column manipulations.
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and transform one matrix into another by row/column manipulations.
Let $6\text{VDW}(n)$ be the set of all configurations of the six-vertex model on the $n \times n$ grid with DWBC, i.e., decorations of the grid’s edges with arrows such that:

- The arrows on the external edges are fixed, with the horizontal ones all incoming and the vertical ones all outgoing.
- At each internal vertex, there are as many incoming as outgoing arrows.

The latter condition is the “six-vertex” condition, since it allows for only six possible arrow configurations around an internal vertex:
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The bijection from $6\text{VDW}(n)$ to $\text{ASM}(n)$
The bijection from 6VDW\( (n) \) to ASM\( (n) \)

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & -1 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]
Statistics

Statistics also have a nice interpretation in terms of the six-vertex model: if $A \in \text{ASM}(n) \leftrightarrow C \in \text{6VDW}(n)$,

$$
\mu(A) = \frac{1}{2} \left( (\text{number of vertices of type } c \text{ in } C) - n \right)
$$

$$
\nu(A) = \frac{1}{2} (\text{number of vertices of type } a \text{ in } C)
$$
Define the six-vertex partition function of the six-vertex model with DWBC to be:

\[
Z_{6VDW}(u_1, \ldots, u_n; v_1, \ldots, v_n) = \sum_{C \in 6VDW(n)} \prod_{i,j=1}^{n} C_{ij}(u_i, v_j)
\]

where the \( u_i \) (resp. the \( v_j \)) are parameters attached to each row (resp. a column), and \( C_{ij} \) is the type of configuration at vertex \((i, j)\).

\[
a(u, v) = uq - \frac{1}{vq}, \quad b(u, v) = \frac{u}{q} - \frac{q}{v}, \quad c(u, v) = \left(q^2 - \frac{1}{q^2}\right)\sqrt{\frac{u}{v}}
\]
Based on Korepin’s recurrence relations for $Z_{6VDW}$, Izergin found the following determinant formula:

**Theorem (Izergin, ’87)**

$$Z_{6VDW}(u_1, \ldots, u_n; v_1, \ldots, v_n) \propto \frac{\det_{1 \leq i, j \leq n} \left( \frac{c(u_i, v_j)}{a(u_i, v_j)b(u_i, v_j)} \right)}{\prod_{1 \leq i < j \leq n} (u_j - u_i)(v_j - v_i)}$$

Problem: what happens in the homogeneous limit $u_1, \ldots, u_n, v_1, \ldots, v_n \to r$?
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\]

Problem: what happens in the homogeneous limit

\[u_1, \ldots, u_n, v_1, \ldots, v_n \rightarrow r?\]
The “naive” homogeneous limit:

\[
Z_{6\text{VDW}}(r, \ldots, r; r, \ldots, r) \propto \det_{0 \leq i,j \leq n-1} \frac{\partial^{i+j}}{\partial u^i \partial v^j} \left( \frac{c(u,v)}{a(u,v)b(u,v)} \right) \bigg|_{u,v=r}
\]

\[
\propto \det_{0 \leq i,j \leq n-1} \frac{\partial^{i+j}}{\partial u^i \partial v^j} \left( \frac{1}{uv - q^2} - \frac{1}{uv - q^{-2}} \right) \bigg|_{u,v=r}
\]
Define $L_{ij}$ to be the $n \times n$ lower-triangular matrix with entries $\binom{i}{j}$, and $D$ to be the diagonal matrix with entries $\left(\frac{qr-q^{-1}r^{-1}}{q^{-1}r-qr^{-1}}\right)^i$, $i = 0, \ldots, n - 1$.

**Proposition (Behrend, Di Francesco, Zinn-Justin, ’11)**

$$Z_{6VDW}(r, \ldots, r; r, \ldots, r) \propto \det \left( I - \frac{r^2 - q^2}{r^2 - q^{-2}} DLDL^T \right)$$

Proof: write the determinant as $\det(A_+ - A_-)$, note that $A_\pm$ is up to a diagonal conjugation $\frac{1}{r^2 - q^{\pm2}} D_\pm LD_\pm L^T$, pull out $\det A_+$ and conjugate $I - A_-A_+^{-1}$...
Define $L_{ij}$ to be the $n \times n$ lower-triangular matrix with entries $(i^j)$, and $D$ to be the diagonal matrix with entries $(\frac{qr-q^{-1}r^{-1}}{q^{-1}r-qr^{-1}})^i$, $i = 0, \ldots, n - 1$.

Proposition (Behrend, Di Francesco, Zinn-Justin, ’11)

$$Z_{6VDW}(r, \ldots, r; r, \ldots, r) \propto \det \left( I - \frac{r^2 - q^{-2}}{r^2 - q^2} DLDL^T \right)$$

Proof: write the determinant as $\det(A_+ - A_-)$, note that $A_\pm$ is up to a diagonal conjugation $\frac{1}{r^2-q^{\pm2}} D_\pm LD_\pm L^T$, pull out $\det A_+$ and conjugate $I - A_- A_+^{-1} \ldots$
Rewriting the previous proposition in terms of Boltzmann weights $a$, $b$, $c$, and then switching to $x = (a/b)^2$, $y = (c/b)^2$, we finally find $Z_{ASM}(n, x, y) = \det M_{ASM}(n, x, y)$ with

$$M_{ASM}(n, x, y)_{ij} = (1 - \omega)\delta_{ij} + \omega \sum_{k=0}^{\min(i,j)} \binom{i}{k} \binom{j}{k} x^k y^{i-k}$$

with $i, j = 0, \ldots, n-1$ and $\omega$ a solution of

$$y\omega^2 + (1 - x - y)\omega + x = 0$$
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Refined enumeration of ASMs and DPPs
Statistics

Statistics also have a nice interpretation in terms of NonIntersecting lattice Paths (NILPs):

\[
D = \begin{array}{ccc}
6 & 6 & 6 \\
4 & 4 & \ \ \\
3 & \ E_2 & \ \\
\ E_1 & \ & 5 \\
\ & 1 & 2 \\
\ & \ & \ \\
\ & \ & \ \\
\ & \ & \ \\
\ & \ & \ \\
s_1 & s_2 & s_3
\end{array}
\]

\[\nu(D) = 7\]
\[\mu(D) = 2\]
NILPs are (lattice) free fermions:

Number of NILPs from $S_i$ to $E_i$, $i = 1, \ldots, n$

$$= \det_{i,j=1,\ldots,n} \text{(Number of (single) paths from } S_i \text{ to } E_j)$$

and similarly with weighted sums.
NILPs are (lattice) free fermions:

Number of NILPs from $S_i$ to $E_i$, $i = 1, \ldots, n$

$$= \det_{i,j=1,\ldots,n} \left( \text{Number of (single) paths from } S_i \text{ to } E_j \right)$$

and similarly with weighted sums.
Here we are also summing over endpoints and the number of paths ("grand canonical partition function"):

\[
Z_{\text{DPP}}(n, x, y) = \det M_{\text{DPP}}(n, x, y) \text{ with }
\]

\[
M_{\text{DPP}}(n, x, y) = \delta_{ij} + \sum_{k=0}^{i-1} \sum_{\ell=0}^{\min(j,k)} \binom{j}{\ell} \binom{k}{\ell} x^{\ell+1} y^{k-\ell}
\]

Note that the second term is a product of three discrete transfer matrices...
We have

\[(I - S)M_{\text{DPP}}(n, x, y)(I + (\omega - 1)S^T) = (I + (x - \omega y - 1)S)M_{\text{ASM}}(n, x, y)(I - S^T)\]

where \(l_{ij} = \delta_{i,j}\) and \(S_{ij} = \delta_{i,j+1}\).

Therefore,

\[Z_{\text{DPP}}(n, x, y) = Z_{\text{ASM}}(n, x, y)\]
We have

\[(I - S)M_{\text{DPP}}(n, x, y)(I + (\omega - 1)S^T) = (I + (x - \omega y - 1)S)M_{\text{ASM}}(n, x, y)(I - S^T)\]

where \(I_{ij} = \delta_{i,j}\) and \(S_{ij} = \delta_{i,j+1}\).

Therefore,

\[Z_{\text{DPP}}(n, x, y) = Z_{\text{ASM}}(n, x, y)\]
Define refined enumeration by introduction “boundary” statistics:

- For ASMs:
  \[ \rho_1(A) = \text{number of 0's to the left of the 1 in the first row of } A, \]
  \[ \rho_2(A) = \text{number of 0's to the right of the 1 in the last row of } A. \]

- For DPPs:
  \[ \rho_1(D) = \text{number of } n \text{'s in } D, \]
  \[ \rho_2(D) = (\text{number of } (n-1) \text{'s in } D) \]
  \[ + \text{(number of rows of } D \text{ of length } n-1). \]

\[ Z_{\text{ASM/DPP}}(n, x, y, z_1, z_2) = \sum_X x^\nu(X) y^\mu(X) z_1^{\rho_1(X)} z_2^{\rho_2(X)} \]
Define **refined enumeration** by introduction “boundary” statistics:

- For ASMs:
  \[ \rho_1(A) = \text{number of 0’s to the left of the 1 in the first row of } A, \]
  \[ \rho_2(A) = \text{number of 0’s to the right of the 1 in the last row of } A. \]

- For DPPs:
  \[ \rho_1(D) = \text{number of } n’s \text{ in } D, \]
  \[ \rho_2(D) = (\text{number of } (n-1)’s \text{ in } D) \]
  \[ + (\text{number of rows of } D \text{ of length } n-1). \]

\[
Z_{ASM/DPP}(n, x, y, z_1, z_2) = \sum_X x^{\nu(X)} y^{\mu(X)} z_1^{\rho_1(X)} z_2^{\rho_2(X)}
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- For DPPs:
  \[ \rho_1(D) = \text{number of } n \text{’s in } D, \]
  \[ \rho_2(D) = (\text{number of } (n - 1) \text{’s in } D) \]
  \[ + (\text{number of rows of } D \text{ of length } n - 1). \]

\[ Z_{\text{ASM/DPP}}(n, x, y, z_1, z_2) = \sum_X x^\nu(X) y^\mu(X) z_1^{\rho_1(X)} z_2^{\rho_2(X)} \]
Example \((n = 3)\)

\[
\text{ASM}(3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \right\}
\]

\[
\text{DPP}(3) = \left\{ \emptyset, \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}, 2, 3 3, 3, 3 2, 3 1 \right\}
\]

\[
Z_{\text{ASM/DPP}}(3, x, y, z_1, z_2) = 1 + x^3 z_1^2 z_2^2 + xz_2 + x^2 z_1^2 z_2 + xz_1 + x^2 z_1 z_2^2 + xyz_1 z_2
\]
Strategy of proof

1. **Generalize the unrefined proof to a single refinement.**
   
   Involves modifying one row of the matrices...

2. **Show the bilinear identity for both ASMs and DPPs:**

   \[
   (z_1 - z_2)(z_3 - z_4) Z_n(x, y, z_1, z_2) Z_n(x, y, z_3, z_4) -
   (z_1 - z_3)(z_2 - z_4) Z_n(x, y, z_1, z_3) Z_n(x, y, z_2, z_4) +
   (z_1 - z_4)(z_2 - z_3) Z_n(x, y, z_1, z_4) Z_n(x, y, z_2, z_3) = 0.
   \]

   This allows to express the double refinement in terms of the single refinement: \((z_3 = 1, z_4 = 0)\)

   \[
   (z_1 - z_2) Z_n(x, y, z_1, z_2) Z_{n-1}(x, y, 1)
   = (z_1 - 1)z_2 Z_n(x, y, z_1) Z_{n-1}(x, y, z_2) -
   z_1(z_2 - 1) Z_{n-1}(x, y, z_1) Z_n(x, y, z_2).
   \]
Strategy of proof

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(z_1 - z_3) (z_2 - z_4) Z_n(x, y, z_1, z_3) Z_n(x, y, z_2, z_4) +
(z_1 - z_4) (z_2 - z_3) Z_n(x, y, z_1, z_4) Z_n(x, y, z_2, z_3) = 0.
\]

This allows to express the double refinement in terms of the single refinement: \((z_3 = 1, z_4 = 0)\)

\[
(z_1 - z_2) Z_n(x, y, z_1, z_2) Z_{n-1}(x, y, 1)
= (z_1 - 1) z_2 Z_n(x, y, z_1) Z_{n-1}(x, y, z_2) -
z_1 (z_2 - 1) Z_{n-1}(x, y, z_1) Z_n(x, y, z_2).
\]
Strategy of proof

1. Generalize the unrefined proof to a single refinement. Involves modifying one row of the matrices...

2. Show the bilinear identity for both ASMs and DPPs:

\[(z_1 - z_2)(z_3 - z_4) Z_n(x, y, z_1, z_2) Z_n(x, y, z_3, z_4) -
(z_1 - z_3)(z_2 - z_4) Z_n(x, y, z_1, z_3) Z_n(x, y, z_2, z_4) +
(z_1 - z_4)(z_2 - z_3) Z_n(x, y, z_1, z_4) Z_n(x, y, z_2, z_3) = 0.\]

This allows to express the double refinement in terms of the single refinement: \((z_3 = 1, z_4 = 0)\)

\[(z_1 - z_2) Z_n(x, y, z_1, z_2) Z_{n-1}(x, y, 1) = (z_1 - 1)z_2 Z_n(x, y, z_1) Z_{n-1}(x, y, z_2) -
z_1(z_2 - 1) Z_{n-1}(x, y, z_1) Z_n(x, y, z_2).\]
Strategy of proof

1. Generalize the unrefined proof to a single refinement. Involves modifying one row of the matrices...
2. Show the bilinear identity for both ASMs and DPPs:

\[
(z_1 - z_2)(z_3 - z_4) Z_n(x, y, z_1, z_2) Z_n(x, y, z_3, z_4) - \\
(z_1 - z_3)(z_2 - z_4) Z_n(x, y, z_1, z_3) Z_n(x, y, z_2, z_4) + \\
(z_1 - z_4)(z_2 - z_3) Z_n(x, y, z_1, z_4) Z_n(x, y, z_2, z_3) = 0.
\]

This allows to express the double refinement in terms of the single refinement: \((z_3 = 1, z_4 = 0)\)

\[
(z_1 - z_2) Z_n(x, y, z_1, z_2) Z_{n-1}(x, y, 1) \\
= (z_1 - 1)z_2 Z_n(x, y, z_1) Z_{n-1}(x, y, z_2) - \\
z_1(z_2 - 1) Z_{n-1}(x, y, z_1) Z_n(x, y, z_2).
\]
An equivalent form of Desnanot–Jacobi

\[
\begin{array}{cccc}
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\end{array} + \begin{array}{cccc}
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\end{array} \\
\begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} \\
\end{array} = 0
\end{array}
\]

These are also the Plücker relations for \( \text{Gr}(n + 2, n) \).
An equivalent form of Desnanot–Jacobi

\[
\begin{array}{c|c|c|c}
\begin{array}{c}
 & \ & \\
 & \ & \\
 & \ & \\
\end{array} & \begin{array}{c|c}
\begin{array}{c|c}
\begin{array}{c|c}
\begin{array}{c}
 & \ & \\
 & \ & \\
 & \ & \\
\end{array} & \begin{array}{c|c}
\begin{array}{c}
 & \ & \\
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\end{array} & \begin{array}{c}
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\end{array}
\end{array}
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 & \ & \\
\end{array}
\end{array}
\end{array}
+\begin{array}{c|c|c|c}
\begin{array}{c|c|c|c}
\begin{array}{c}
 & \ & \\
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\begin{array}{c}
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\end{array} & \begin{array}{c|c|c|c|c}
\begin{array}{c}
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 & \ & \\
\end{array} & \begin{array}{c}
 & \ & \\
 & \ & \\
 & \ & \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} = 0
\]

These are also the Plücker relations for \( \text{Gr}(n + 2, n) \).
The double refinement of ASMs simply corresponds to keeping two spectral parameters free and letting the others tend to $r$.

→ Apply directly Desnanot–Jacobi to the Izergin matrix

$$\left( \frac{c(u_i, v_j)}{a(u_i, v_j)b(u_i, v_j)} \right)$$

A similar formula appears in [Colomo, Pronko, ’05].
The double refinement of ASMs simply corresponds to keeping two spectral parameters free and letting the others tend to $r$.

→ Apply directly Desnanot–Jacobi to the Izergin matrix

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A similar formula appears in [Colomo, Pronko, ’05].
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A similar formula appears in [Colomo, Pronko, '05].
Direct application of the LGV formula leads to:

\[
Z_n^{\text{DPP}}(x, y, z_1, z_2) = \det_{0 \leq i, j \leq n-1} (-\delta_{i,j+1} + K_n(x, y, z_1, z_2)_{i,j})
\]

with

\[
K_n(x, y, z_1, z_2)_{i,j} = \begin{cases} 
\sum_{k=0}^{\min(i,j+1)} (i-1) (j+1) x^k y^{i-k}, & j \leq n-3 \\
\sum_{k=0}^{i} \sum_{l=0}^{k} (i-1) (n-l-2) x^k y^{i-k} z_1^l, & j = n-2 \\
\sum_{k=0}^{i} \sum_{l=0}^{k} \sum_{m=0}^{l} (i-1) (n-l-2) x^k y^{i-k} z_1^m z_2^{l-m}, & j = n-1.
\end{cases}
\]

but!

\[
(z_1 - z_2)K_n(x, y, z_1, z_2)_{i,n-1} = K_n(x, y, \cdot, z_1)_{i,n-2} - K_n(x, y, \cdot, z_2)_{i,n-2}
\]
Direct application of the LGV formula leads to:

\[
Z_n^{\text{DPP}}(x, y, z_1, z_2) = \det_{0 \leq i, j \leq n-1} (-\delta_{i,j+1} + \mathcal{K}_n(x, y, z_1, z_2)_{i,j})
\]

with

\[
\mathcal{K}_n(x, y, z_1, z_2)_{i,j} =
\begin{cases}
\sum_{k=0}^{\min(i,j+1)} \binom{i-1}{i-k} \binom{j+1}{k} x^k y^{i-k}, & j \leq n - 3 \\
\sum_{k=0}^{i} \sum_{l=0}^{k} \binom{i-1}{i-k} \binom{n-l-2}{k-l} x^k y^{i-k} z_2^l, & j = n - 2 \\
\sum_{k=0}^{i} \sum_{l=0}^{k} \sum_{m=0}^{l} \binom{i-1}{i-k} \binom{n-l-2}{k-l} x^k y^{i-k} z_1^m z_2^{l-m}, & j = n - 1.
\end{cases}
\]

but!

\[
(z_1-z_2)\mathcal{K}_n(x, y, z_1, z_2)_{i,n-1} = \mathcal{K}_n(x, y, \cdot, z_1)_{i,n-2} - \mathcal{K}_n(x, y, \cdot, z_2)_{i,n-2}
\]