Critical Galton-Watson branching processes with countably infinitely many types and infinite second moments

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Abstract
We consider an indecomposable Galton-Watson branching process with countably infinitely many types. Assuming that the process is critical and allowing for infinite variance of the offspring sizes of some (or all) types of particles we describe the asymptotic behavior of the survival probability of the process and establish a Yaglom-type conditional limit theorem for the infinite-dimensional vector of the number of particles of all types.

1 Definition of the process and basic properties of its mean matrix

We consider an indecomposable Galton-Watson branching process $Z(n) := (Z_j(n))_{j \in \mathbb{N}}$ with countably infinitely many types labelled by numbers $j \in \mathbb{N}$. This work was supported from a grant of the Mathematical center in Akademgorodok and from a grant to the Steklov International Mathematical Center in the framework of the national project "Science" of the Russian Federation.

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\( \mathbb{N} := \{1, 2, \ldots\} \). The component \( Z_j(n), n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), of \( Z(n) \) denotes the number of type \( j \) particles in the process at moment \( n \). Let \( \delta_{ij} \) be the Kronecker symbol and \( e_i := (\delta_{ij})_{j \in \mathbb{N}} \) be the vector whose \( i \)th component is equal to one while the remaining are zeros. To specify the evolution of the branching process initiated at moment 0 by a vector \( Z(0) \) of individuals of different types it is sufficient to describe the distributions of the vectors

\[
Z_i = Z_i(1) := \{ Z(1) \mid Z(0) = e_i \} =: (Z_{ij})_{j \in \mathbb{N}} = (Z_{ij}(1))_{j \in \mathbb{N}}.
\]

We suppose that

\[
Z_i := \sum_{j \in \mathbb{N}} Z_{ij} < \infty
\]  

(1)

with probability 1. Let

\[
Z_i(n) := \{ Z(n) \mid Z(0) = e_i \} =: (Z_{ij}(n))_{j \in \mathbb{N}}.
\]

Assuming that \( s = (s_j)_{j \in \mathbb{N}} \in [0, 1]^{\mathbb{N}} \) we specify infinite-dimensional vectors

\[
F(s) := (F_i(s))_{i \in \mathbb{N}} \quad \text{and} \quad F(n; s) := (F_i(n; s))_{i \in \mathbb{N}}
\]

of the offspring generating functions of the process with the components

\[
F_i(s) := \mathbb{E} \left[ \prod_{j \in \mathbb{N}} s_j^{Z_{ij}} \right] =: \mathbb{E}s^{Z_i} =: \sum_{j \in \mathbb{N}_0} p_{ij} s^j,
\]  

(2)

\[
F_i(n; s) := \mathbb{E} \left[ \prod_{j \in \mathbb{N}} s_j^{Z_{ij}(n)} \right] =: \mathbb{E}s^{Z_i(n)},
\]  

(3)

where, for any \( j = (j_i)_{i \in \mathbb{N}} \in \mathbb{N}_0^{\mathbb{N}} \)

\[
p_{ij} := \mathbb{P} \left( Z(1) = j \mid Z(0) = e_i \right) = \mathbb{P} \left( Z_i(1) = j \right).
\]

The probability generating functions are well defined in view of (1).

According to the branching property of the process, each type \( i \) particle belonging to the population has a unit life-length and produces at the end of its life (independently of the prehistory of the process and the reproduction of the others particles existing at this moment) a random number of
children specified by a vector $Z_i$ whose distribution is described by the generating function $F_i(s)$. This property has the following description in terms of iterations of the offspring generating functions:

$$F(n + 1; s) = F(n; F(s))$$

for all $n \in \mathbb{N}$, \hspace{1cm} (4)

where $F(1; s) := F(s)$.

The basic classification of the Galton-Watson branching processes with countably infinitely many types (below we use the short abbreviation GWBP/$\infty$ for representatives of such processes) is given in terms of the mean matrix $M := (M_{ij})_{i,j \in \mathbb{N}} := (\mathbb{E} Z_{ij})_{i,j \in \mathbb{N}} = \left( \frac{\partial F_i(s)}{\partial s_j} \right|_{s=1} \right)_{i,j \in \mathbb{N}}$. \hspace{1cm} (5)

To describe such a classification in detail we recall a number of asymptotic properties of the powers of infinite-dimensional matrices with nonnegative elements borrowed from [1]. We first formulate the desired properties for an abstract matrix $M = (m_{ij})_{i,j \in \mathbb{N}}$ with $m_{ij} \geq 0$ for all $(i, j) \in \mathbb{N}^2$ and then use them in studying GWBP/$\infty$ with matrix $M$ specified by (5).

Let $M^n = (m_{ij}^{(n)})_{i,j \in \mathbb{N}}$ be the $n$th power of the infinite-dimensional matrix $M$. The matrix $M$ with nonnegative elements is called irreducible and aperiodic if for any pair of indices $(i, j)$ there is an $n \in \mathbb{N}$ such that $m_{ij}^{(n)} > 0$ and the greatest common divisor of all $n \in \mathbb{N}$ such that $m_{ij}^{(n)} > 0$ equals 1. According to Theorem A in [1] for any irreducible matrix $M$ there exists a number $R \in [0, \infty)$ such that, for any pair of indices $(i, j)$

$$\lim_{n \to \infty} \left( m_{ij}^{(n)} \right)^{1/n} = 1/R.$$ \hspace{1cm} (6)

The parameter $R$ is a common convergence radius of the functions

$$\mathcal{M}_{ij}(z) := \sum_{n=0}^{\infty} m_{ij}^{(n)} z^n,$$

where $m_{ij}^{(0)} := \delta_{ij}$, and $1/R$ is an analog of the maximal (in absolute value) eigenvalue of a nonnegative matrix in the finite-dimensional case. However, the operator specified by such a matrix is not necessarily bounded.

It follows from (6) that for any pair $(i, j)$ the series

$$\mathcal{M}_{ij}(r) = \sum_{n=0}^{\infty} m_{ij}^{(n)} r^n$$ \hspace{1cm} (7)
is convergent for all $0 < r < R$ and is divergent for $r > R$. The case $r = R$ allows for both possibilities. Moreover, by Theorem B in [1] the series $\mathcal{M}_{ij}(R)$ is either convergent for all pairs $(i, j)$ or is divergent for all $(i, j)$. In addition, the limits of all sequences $\left\{ m^{(n)}_{ij} R^n, n \geq 1 \right\}$ exist and either
\[
\lim_{n \to \infty} m^{(n)}_{ij} R^n = 0 \quad \text{for all} \ (i, j) \in \mathbb{N}^2, \quad (8)
\]
or
\[
\lim_{n \to \infty} m^{(n)}_{ij} R^n > 0 \quad \text{for all} \ (i, j) \in \mathbb{N}^2. \quad (9)
\]

An irreducible matrix $M$ is called

- “$R$–transient” or “$R$–recurrent” depending on the convergence or divergence of the series $\mathcal{M}_{ij}(R)$ in (7);
- “$R$–null” if (8) is valid and “$R$–positive” if (9) holds true.

We now come back to the matrices $M = (M_{ij})_{i,j \in \mathbb{N}}$ and $M^n := (M_{ij}^{(n)})_{i,j \in \mathbb{N}}$ of GWBP/$\infty$, where $M_{ij}^{(n)} := \mathbb{E}Z_{ij}(n)$, $M_{ij} = M_{ij}^{(1)}$.

The properties of nonnegative infinite-dimensional matrices we have listed above allow to introduce the following natural classification of the GWBP/$\infty$’s (see, for instance, [2]):

**Definition 1** A GWBP/$\infty$ is called subcritical {critical, supercritical} and transient {recurrent, null recurrent, positively recurrent} in the type space, if its matrix of the mean offspring numbers $M$ has a convergence radius $R > 1 \{R = 1, R < 1\}$ and is “$R$–transient” {“$R$–recurrent, “$R$–null recurrent, “$R$–positively recurrent}.

In this paper we consider only the critical GWBP/$\infty$’s. We know by [3] or [2] that if a GWBP/$\infty$ is critical then the elements of the sequences $\left\{ M_{ij}^{(n)}, n \geq 1 \right\}$ either vanishes as $n \to \infty$ for all $i$ and $j$, or have positive limits for all $i$ and $j$. Below we analyze only the second option.

If the mean matrix $M$ of a GWBP/$\infty$ is irreducible and “$1$–positive” then (see Theorem D in [1]) there exist unique (up to a positive multiplier) left and right eigenvectors $v := (v_k)_{k \in \mathbb{N}}$ and $u := (u_k)_{k \in \mathbb{N}}$ with positive components such that
\[
vM = v, \quad Mu^T = u^T, \quad vu^T = \sum_{k=1}^{\infty} v_k u_k = 1, \quad v1^T < \infty, \quad (10)
\]
and, as $n \to \infty$

$$M_{ij}^{(n)} \to \frac{u_i v_j}{v^T u} = u_i v_j$$ (11)

for all $(i, j) \in \mathbb{N}^2$.

Observe that if $M$ is a finite-dimensional irreducible aperiodic matrix with its Perron root equal to 1 then $M$ has, according to the Perron-Frobenius theorem, positive left and right eigenvectors $v$ and $u$ of $M$ satisfying (10) and, of course, $v^T < \infty$ in this case. This estimate is, in general, not valid for irreducible and "1"-positive infinite-dimensional matrices. We require in the paper finiteness of the scalar product $v^T$ (matrices with $v^T < \infty$ are called irreducible with finite iterate coefficients, see, for instance, [3]) leaving the case $v^T = \infty$ for the future investigations.

We now introduce an important definition which incorporates major restrictions on the properties of the matrix $M$.

Set $M_i := \mathbb{E} Z_i = \sum_{j \in \mathbb{N}} M_{ij}$.

**Definition 2** We say that the mean matrix $M = (\mathbb{E} Z_{ij})_{i,j \in \mathbb{N}}$ of a critical GWBP/$\infty$ belongs to a class $\mathcal{M}_1$, and write $M \in \mathcal{M}_1$, if

(i) $M$ is irreducible and aperiodic, "1"-positive and "1"-recurrent;

(ii) the left eigenvector $v$ of $M$ has $L_1$-norm equal to 1: $\|v\|_1 = v^T = 1$, and the right eigenvector $u$ has finite $L_\infty$-norm: $U := \|u\|_\infty = \sup_{i \in \mathbb{N}} u_i < \infty$;

(iii) $\lim_{N \to \infty} \sup_{i \in \mathbb{N}} M_{i}^{-1} \sum_{j>N} M_{ij} = 0$ and $\lim_{K \to \infty} \sup_{i \in \mathbb{N}} M_{i}^{-1} \mathbb{E}[Z_i; Z_i > K] = 0$.

We say that $M$ belongs to a subclass $\mathcal{M}^0_1 \subset \mathcal{M}_1$ if, additionally,

(iv) there exist $m \in \mathbb{N}$ and $c, C \in \mathbb{R}^+$ such that

$$M_{ij} < Cu_i v_j, \quad M_{ij}^{(m)} > cv_j, \forall i, j \in \mathbb{N}. \quad (12)$$

We suppose, without loss of generality, that if $M \in \mathcal{M}_1$ then $v^T = 1$ and, therefore, relations (10) and Condition (ii) have the component-wise representations

$$\sum_{j \in \mathbb{N}} M_{ij} u_j = u_i; \sum_{j \in \mathbb{N}} v_j M_{ji} = v_i; \sum_{j \in \mathbb{N}} v_j u_j = 1;$$

$$\sum_{j \in \mathbb{N}} v_j = 1; \sup_{i \in \mathbb{N}} u_i < \infty; \quad v_j u_j > 0, \forall j \in \mathbb{N}. \quad (13)$$
Observe that Condition (iii) has rather transparent meaning. Its first part extracts from all critical GWBP/$\infty$’s those processes in which particles of all types produce with a high probability particles of types with relatively small labels. Thus, our model is, in a sense, close to the so-called lower Hessenberg branching processes [5] in which particles of type $i$ may produce particles of types $j \leq i + 1$ only. The second part of Condition (iii) prevents existence of very productive particles.

We need a number of auxiliary functions related to the generating functions $F_i(n; s)$, $i \in \mathbb{N}$, of the GWBP/$\infty$ $Z_i(n)$:

$$F_{ij}(s) := \mathbb{E} s^{Z_{ij}}; \quad Q_i(n; s) := 1 - F_i(n; s) = 1 - \mathbb{E} s^{Z_i(n)}; \quad \mathbb{Q}(n; s) := (Q_i(n; s))_{i \in \mathbb{N}} = 1 - \mathbb{F}(n; s); \quad Q(s) := Q(1; s) = 1 - \mathbb{F}(s);$$

$$Q_i(n) := Q_i(n; 0) = \mathbb{P}(Z(n) \neq 0|Z(0) = e_i); \quad Q(n) := Q(n; 0), \quad q(n; s) := \mathbb{v} \mathbb{Q}^T(n; s); \quad q(n) := q(n; 0).$$

For $x \geq 0$ and $U = \sup_{i \in \mathbb{N}} u_i$ introduce the function

$$\Phi(x) := \begin{cases} x - v \mathbb{Q}^T(1 - xu) & \text{if } 0 \leq xU \leq 1, \\ x - v \mathbb{Q}^T(0) & \text{if } xU > 1. \end{cases}$$

(14)

We now may formulate the main result of the paper.

**Theorem 3** Let $\{Z_i(n), i \in \mathbb{N}\}$ be a critical GWBP/$\infty$ with mean matrix $\mathbb{M} \in \mathcal{M}_1^0$ and

$$\Phi(x) = x^{\alpha+1} \ell(x),$$

(15)

where $\alpha \in (0, 1]$ and $\ell(x)$ is a slowly varying function as $x \to +0$.

Then

1) for some slowly varying function $\ell_1(n)$

$$q(n) = n^{-1/\alpha} \ell_1(n)$$

(16)

as $n \to \infty$;

2) for any $i \in \mathbb{N}$

$$Q_i(n) = \mathbb{P}(Z(n) \neq 0|Z(0) = e_i) \sim u_i n^{-1/\alpha} \ell_1(n)$$

(17)

as $n \to \infty$;

3) for each vector $\lambda = (\lambda_k)_{k \in \mathbb{N}}$ with bounded coordinates and each $i \in \mathbb{N}$

$$\lim_{n \to \infty} \mathbb{E} \left[ e^{-(\lambda, Z(n))q(n)}|Z(n) \neq 0, Z(0) = e_i \right] = 1 - (1 + (v, \lambda)^{-\alpha})^{-1/\alpha}.$$
In particular, for each vector \((z_1, \ldots, z_m) \in \mathbb{R}^m_+\) the limit

\[
G_m(z_1, \ldots, z_m) := \lim_{n \to \infty} \mathbb{P}(Z_j(n)q(n) \leq z_j, j = 1, \ldots, m; Z(n) \neq 0, Z(0) = e_i)
\]

exists and is independent of \(i\).

**Remark.** It follows from (18) that, as \(n \to \infty\)

\[
\left\{ Z(n)q(n) \big| Z(n) \neq 0, Z(0) = e_i \right\} \overset{d}{\to} \xi v,
\]

where

\[
\mathbb{E}e^{-t \xi} = 1 - \left(1 + t^{-\alpha}\right)^{-1/\alpha}, \quad t \geq 0.
\]

We note that Kolmogorov [6] was the first who investigated the asymptotic behavior of a single-type critical Galton-Watson process. His work was followed by the celebrate Yaglom article [7] who studied the distribution of the number of particles in a single-type critical Galton-Watson process given its survival for a long time. Joffe and Spitzer [8] extended these results to the case of multi-type critical indecomposable Galton-Watson processes. All these papers required finiteness of the second moments of the reproduction laws of the number of particles.

Zolotarev [9], assuming that the variance for the offspring reproduction law of particles may be infinite, had found an asymptotic representation for the survival probability of a single-type continuous-time critical branching processes and proved a Yaglom-type theorem for this case. Zolotarev’s results were complemented by Slack [10], [11] who generalized Kolmogorov’s and Yaglom’s theorems to the case when the offspring generating function of a critical Galton-Watson process has the form

\[
F(s) = s + (1 - s)^{1+\alpha} \ell(1 - s),
\]

where \(\alpha \in (1, 2]\) and \(\ell(x)\) is a slowly varying function as \(x \to +0\).

Slack’s theorems were independently and almost simultaneously extended to the multi-type indecomposable setting by Vatutin [12] and Goldstein and Hoppe [13]. The main assumption of these two papers is just our condition (15) formulated in terms of the critical Galton-Watson processes with finite number of types. Thus, Theorem 3 is a natural generalization of the main results of [12] and [13] to the GWBP/∞’s.

There are several published results for GWBP/∞ (see, for example, [14], [4], [3], [15], [16] and [2]). Sagitov’s article [2] is the most relevant to our
paper. The author analyzed there the case of linear-fractional offspring generating functions. He has established, along with other results, an asymptotic representation for the survival probability of a critical GWBP/$\infty$ and proved a Yaglom-type conditional limit theorem for such processes. Theorem 3 of our paper extends the mentioned Sagitov result in two directions. First, we consider the general form of the reproduction generating functions of particles and, second, we do not assume finiteness of the second moments for the offspring numbers of particles.

The paper is organized as follows. In Section 2 we prove a number of statements describing properties of the offspring generation functions of the GWBP/$\infty$’s and show that the dichotomy property, which states that with probability 1 the population either becomes extinct or drifts to infinity, holds for the processes meeting the conditions of Theorem 3.

One of the basic assumptions of Theorem 3 is condition (15) expressed in terms of the eigenvectors $v$ and $u$ of the mean matrix $M$ and a single variable $x$. The goal of Section 3 is to demonstrate that properties of iterations of the offspring generating functions depending on the unbounded number of arguments may be reduced to considering some function which depends on a single argument only. To this aim we prove a Ratio Theorem showing that the functions $Q_i(n; s)$ may be well approximated by $u_i q(n; s)$ for all $i \in \mathbb{N}$. This approximation allows us to complete the proof of Theorem 3 by the methods similar to those used in [12] for the case of Markov branching processes with finite number of types.

## 2 Properties of generating functions

We prove in this section a number of statements describing properties of the offspring generating functions of a critical GWBP/$\infty$. Some of these statements look evident for the Galton-Watson processes with finite number of types. However, certain efforts and restrictions are needed to check their validity for the infinite type case. The first result of such a kind is the following lemma.

**Lemma 4** If $M \in \mathcal{M}_1$ then

$$\liminf_{n \to \infty} M^n 1^T \in \mathbb{R}^N.$$  \hfill (20)
If $M \in M^0_1$ then there exists a constant $C \in (0, \infty)$ such that
\begin{equation}
M_i^{(n)} := \sum_{j \in N} M_{ij}^{(n)} \leq Cu_i \leq CU =: m < \infty
\end{equation}
for all $i$ and $n$ belonging to the set $N$.

Remark. Observe the difference between the estimates (20) and (21). For the first case the liminf of the row-wise sums of elements is finite while for the second one the sums are uniformly bounded. Clearly, the second statement is not a consequence of the first one.

Proof of Lemma 4. Fix an $\varepsilon \in (0, 0.5)$ and, using Condition (ii) of Definition 2 select a positive integer $N = N(\varepsilon)$ such that
\begin{equation}
\sup_{k \in N} \sum_{j \geq N} M_{kj} \leq \varepsilon.
\end{equation}
Fix now an $i \in N$. Recalling the conditions $\mathbf{v}^T \mathbf{1} = 1$, $\mathbf{u} > 0$, $\mathbf{v} > 0$, and the limiting relation (11), we conclude that for $\delta := \sum_{j \geq N} v_j$ there exists $n_0 = n_0(i, N)$ such that the estimate
\begin{equation}
\sum_{j < N} M_{ij}^{(n)} \leq (1 + \delta)u_i \sum_{j < N} v_j = (1 - \delta^2)u_i \leq u_i
\end{equation}
is valid for all $n \geq n_0$.

Using (22) and (23) for $n \geq n_0$ gives
\begin{equation}
M_i^{(n)} = \sum_{j < N} M_{ij}^{(n)} + \sum_{j \geq N} \sum_{k \in N} M_{ik}^{(n-1)} M_{kj} \leq u_i + \varepsilon M_i^{(n-1)}
\end{equation}
or, for $n \geq 1$
\begin{equation}
M_i^{(n+n_0)} \leq u_i \sum_{l=0}^{n-1} \varepsilon^l + \varepsilon^n M_i^{(n_0)}.
\end{equation}

Note that $M_i = \mathbb{E} Z_i \leq \mathbb{E} [Z_i; Z_i > K] + K$. So the second part of Condition (iii) provides existence of a constant $W < \infty$ such that
\begin{equation}
\sup_{k \in N} M_i \leq W.
\end{equation}
Hence we deduce the following estimate which is valid for all \( i \in \mathbb{N} \):

\[
M_i^{(n_0)} = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} M_{ik}^{(n_0-1)} M_{kj} \leq W M_i^{(n_0-1)} \leq W^{n_0}.
\]

This fact combined with (24) completes the proof of (20).

To check the validity of the second statement of Lemma 4 observe that \( vM^n = v, \ M^n u^T = u^T \) for all \( n \in \mathbb{N} \). Hence, using (12) we conclude that, for all \( i, n \in \mathbb{N} \)

\[
M_i^{(n)} = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} M_{ik}^{(n-1)} M_{kj} \leq C \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} M_{ik}^{(n-1)} u_k v_j = Cu_i. \tag{26}
\]

The last implies (21), since \( \|u\|_\infty = U < \infty \) by Condition (ii). Lemma 4 is proved.

**Lemma 5** If \( M \in \mathcal{M}_1 \) and \( F(s) \neq Ms \) then for each \( i \in \mathbb{N} \) there exists \( n = n(i) \) such that

\[
F_i(n; 0) = \mathbb{P} \left( \|Z_i(n)\|_1 = 0 \right) > 0. \tag{27}
\]

**Proof.** Assume the contrary that there exists \( i \in \mathbb{N} \) such that

\[
F_i(n; 0) = 0 \text{ for all } n \in \mathbb{N}. \tag{28}
\]

We split the set of types of the process into two parts \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \). We assign type \( i \) to the class \( \mathcal{T}_1 \) if condition (28) holds and to the class \( \mathcal{T}_2 \) if relation (27) is valid for some \( n = n(i) \).

Denote

\[
\hat{M}_i := \sum_{k \in \mathcal{T}_1} \mathbb{E} Z_{ik} = \sum_{k \in \mathcal{T}_1} M_{ik}.
\]

Observe that \( \hat{M}_i \geq 1 \) for all \( i \in \mathcal{T}_1 \). By induction it is easy to show that, for any \( i \in \mathcal{T}_1 \)

\[
\hat{M}_i^{(n+1)} := \sum_{k \in \mathcal{T}_1} \mathbb{E} Z_{ik} (n + 1) = \sum_{k \in \mathcal{T}_1} M_{ik}^{(n+1)}
\]

\[
= \sum_{k \in \mathcal{T}_1} \sum_{r \in \mathbb{N}} M_{ir}^{(n)} M_{rk} = \sum_{r \in \mathbb{N}} M_{ir}^{(n)} \hat{M}_r
\]

\[
\geq \sum_{r \in \mathcal{T}_1} M_{ir}^{(n)} \hat{M}_r \geq \sum_{r \in \mathcal{T}_1} M_{ir}^{(n)} = \hat{M}_i^{(n)} \geq 1.
\]
To go further we need to separately consider the cases \( T_2 \neq \emptyset \) and \( T_2 = \emptyset \).

1) Assume first that \( T_2 \neq \emptyset \). Since \( \mathbf{M} \) is an irreducible matrix, it follows that one can find \( i_0 \in T_1 \) and \( j_0 \in T_2 \) such that \( M_{i_0j_0} =: \Delta > 0 \) and, therefore,

\[
M_{i_0} = \hat{M}_{i_0} + \sum_{k \in T_2} M_{i_0k} \geq 1 + \Delta.
\]

For the same reason there exists \( n_0 \) such that

\[
M_{j_0i_0}^{(n_0)} = \mathbb{E} Z_{j_0i_0}(n_0) \geq \mathbb{P}(Z_{j_0i_0}(n_0) > 0) > 0.
\]

Setting \( \Delta_1 := M_{j_0i_0} M_{j_0i_0}^{(n_0)} > 0 \) we have

\[
\begin{align*}
\hat{M}_{i_0}^{(n_0+1)} &= \sum_{k \in T_1} \sum_{r \in \mathbb{N}} M_{i_0k} M_{i_0r} M_{r_0}^{(n_0)} \\
&= \sum_{r \in \mathbb{N}} \sum_{r \in T_1} \hat{M}_{i_0r}^{(n_0)} + \sum_{r \in \mathbb{N} \setminus T_1} \hat{M}_{i_0r}^{(n_0)} \\
&\geq \sum_{r \in T_1} \hat{M}_{i_0r} + M_{i_0j_0} M_{j_0i_0}^{(n_0)} = \hat{M}_{i_0} + \Delta_1 \geq 1 + \Delta_1.
\end{align*}
\]

By the same arguments we conclude that, for any \( q \in \mathbb{N} \)

\[
\begin{align*}
\hat{M}_{i_0}^{(q+1)n_0+q+1)} &= \sum_{k \in T_1} \sum_{r \in \mathbb{N}} M_{i_0r}^{(qn_0+q)} M_{i_0r}^{(n_0+1)} \\
&\geq \sum_{r \in T_1} M_{i_0r}^{(qn_0+q)} \hat{M}_{i_0r}^{(n_0+1)} \\
&\geq \hat{M}_{i_0}^{(qn_0+q)} + \Delta_1 \sum_{t=1}^{q} M_{i_0i_0}^{(tn_0+t)}.
\end{align*}
\]

Since \( \mathbf{M} \) is a “1”-recurrent and “1”-positive matrix, we have by (11)

\[
\sum_{t=1}^{\infty} M_{i_0i_0}^{(tn_0+t)} = \infty.
\]

Hence it follows that, as \( q \to \infty \)

\[
\hat{M}_{i_0}^{(qn_0+q)} \to \infty
\]

contradicting (21). Thus, if (28) holds then \( T_2 \) may be only an empty set.
2) Assume now that (28) holds and \( \mathcal{T}_2 = \emptyset \). In this case \( F_i(1; 0) = F_i(0) = 0 \) for all \( i \in \mathbb{N} \). Therefore, the offspring generating functions may be written for all \( i \in \mathbb{N} \) as
\[
F_i(s) = \sum_{j \in \mathbb{N}_0^1 \setminus \{0\}} p_{ij} s^j, \quad F_i(1) = \sum_{j \in \mathbb{N}_0^1 \setminus \{0\}} p_{ij} = 1.
\]

Hence,
\[
M_i = \hat{M}_i = \sum_{j \in \mathbb{N}_0^1 \setminus \{0\}} \|j\|_1 p_{ij} \geq 1
\]
for all \( i \in \mathbb{N} \), where \( \|j\|_1 = j_1 + j_2 + \ldots \). It is not difficult to see that the case \( M_i = 1 \) for all \( i \in \mathbb{N} \) is possible only if \( p_{ij} = 0 \) for all \( \|j\|_1 \geq 2 \), i.e. for \( F(s) \equiv M_0s \), which is not allowed by our assumptions. Consequently, there exist \( i_0 \) and \( j_0 \) with \( \|j_0\|_1 \geq 2 \) such that \( p_{i_0j_0} > 0 \). Clearly, \( M_{i_0} > 1 \) in this case.

Further, there exists \( n_0 \) such that \( M_i^{(n_0)} = \Delta_1 > 0 \). Repeating now almost literally the arguments used to analyze the case \( \mathcal{T}_2 \neq \emptyset \) we conclude that \( M_i^{(n)} \to \infty \) as \( n \to \infty \). This contradicts to the uniform boundness of \( M_i^{(n)} \) for all \( i \in \mathbb{N} \). Thus, the case \( \mathcal{T}_2 = \emptyset \) is also impossible under the assumption (28). The obtained contradiction proves (27).

Lemma 5 is proved.

The next lemma is a refinement of Lemma 5.

\textbf{Lemma 6} \textit{If} \( \textbf{M} \in \mathcal{M}_1 \) \textit{and} \( \textbf{F}(s) \neq \textbf{M}s \) \textit{then}
\[
\mathbb{P}\left( \lim_{n \to \infty} \|Z_i(n)\|_1 = 0 \right) = 1
\]
\textit{for all} \( i \in \mathbb{N} \).

\textbf{Proof.} First observe that
\[
\mathbb{P}\left( \lim_{n \to \infty} \|Z_i(n)\|_1 = \infty \right) = 0 \quad (29)
\]
for all \( i \in \mathbb{N} \). Indeed, if it would be not the case then
\[
\limsup_{n \to \infty} M_i^{(n)} = \limsup_{n \to \infty} \mathbb{E}\|Z_i(n)\|_1 = \infty
\]
for some \( i \), contradicting (26).
Thus, to prove the lemma it is sufficient to establish that under our conditions the process obeys the so-called dichotomy property (see, for instance, [17]):

\[
\mathbb{P}\left(\lim_{n \to \infty} \|Z_i(n)\|_1 = \infty\right) + \mathbb{P}\left(\lim_{n \to \infty} \|Z_i(n)\|_1 = 0\right) = 1.
\]

It is shown in [18] (see Condition 2.1 and the proof of Proposition 2.2 there) that if, for all \(k \in \mathbb{N}\), there exist an index \(m_k\) and a positive number \(d_k\) such that

\[
\inf_{i \in \mathbb{N}} \mathbb{P}\left(\|Z_i(m_k)\|_1 = 0 \mid 1 \leq \|Z_i(1)\|_1 \leq k\right) \geq d_k
\]

then the respective process possesses the dichotomy property.

Let us check that (30) is valid if there exist an index \(m_0\) and a real number \(d_0 \in (0, 1)\), such that

\[
\inf_{i \in \mathbb{N}} \mathbb{P}\left(\|Z_i(m_0)\|_1 = 0 \mid 1 \leq \|Z_i(1)\|_1 \leq k\right) \geq d_0.
\]

Indeed, take \(r = (r_j)_{j \in \mathbb{N}} \in \mathbb{N}_0^\mathbb{N}\) and introduce the set of events \(A_{i,m} := \{\|Z_i(m)\|_1 = 0\}\) and

\[
B_{i,k} := \{1 \leq \|Z_i(1)\|_1 \leq k\} = \sum_{1 \leq \|r\|_1 \leq k} \{Z_{ij} = r_j\}_{j \in \mathbb{N}} =: \sum_{1 \leq \|r\|_1 \leq k} B_{ir}.
\]

Since

\[
\mathbb{P}(A_{i,m_0+1} | B_{ir}) = \prod_{j \in \mathbb{N}} \mathbb{P}^{r_j}(A_{j,m_0}) \geq d_0^k
\]

for all \(i \in \mathbb{N}\), it follows by the total probability formula that

\[
\mathbb{P}(A_{i,m_0+1} | B_{i,k}) = \frac{\sum_{1 \leq \|r\|_1 \leq k} \mathbb{P}(A_{i,m_0+1} \cap B_{ir})}{\mathbb{P}(B_{i,k})} = \frac{\sum_{1 \leq \|r\|_1 \leq k} \mathbb{P}(A_{i,m_0+1} \cap B_{ir}) \mathbb{P}(B_{i,r})}{\mathbb{P}(B_{i,k})} \geq d_0^k.
\]

This proves (30) with \(m_k = m_0 + 1\) and \(d_k = d_0^k\).

We now show that the estimate (31) indeed holds under the conditions of Lemma 6.
According to the first part of Condition (iii), for each $\varepsilon \in (0,1)$ there exists $N = N(\varepsilon)$ such that

$$\sup_{i \in N} \mathbb{P}\left( \sum_{j > N} Z_{ij} > 0 \right) \leq \sup_{i \in N} \sum_{j > N} M_{ij} \leq \varepsilon \sup_{i \in N} M_i \leq \varepsilon m. \tag{32}$$

We split types of particles into two groups $T_1 := \{ j \leq N \}$ and $T_2 := \{ j > N \}$ and consider the sets

$$\mathcal{A}_{i,m}^{T_1} := \left\{ \sum_{j \leq N} Z_{ij}(m) = 0 \right\} \quad \text{and} \quad \mathcal{A}_{i,m}^{T_2} := \left\{ \sum_{j > N} Z_{ij}(m) = 0 \right\}.$$

By (32) we have

$$P_2(1) := \inf_{i \in N} \mathbb{P}(\mathcal{A}_{i,m}^{T_2}) \geq 1 - \varepsilon m. \tag{33}$$

In view of the second part of Condition (iii), for each $\varepsilon \in (0,1)$ there exists $K = K(\varepsilon)$ such that

$$\sup_{i \in N} \mathbb{E}[Z_i; Z_i > K] \leq \varepsilon M_i \leq \varepsilon m. \tag{34}$$

Similarly to (33) we have

$$\inf_{i \in N} \mathbb{P}(Z_i \leq K) \geq 1 - \varepsilon m. \tag{35}$$

By Lemma 5 for each fixed $i$ there exists $n(i)$ such that $Q_i(n(i);0) < 1$. Thus, there exist $n_0$ and $\theta \in (0,1)$ such that, for all $i \in T_1$ and $n \geq n_0$

$$Q_i(n;0) \leq Q_i(n_0;0) \leq 1 - \theta < 1 \tag{36}$$

or, for all $n \geq n_0$

$$P_1(n) := \inf_{i \in T_1} \mathbb{P}(A_{i,n}) > \theta > 0. \tag{37}$$

Note that if $B_1$ and $B_2$ are two events such that $\mathbb{P}(B_1) > 1 - \sigma_1$ and $\mathbb{P}(B_2) > 1 - \sigma_2$ for some constants $\sigma_1, \sigma_2 \in (0,1)$, then $\mathbb{P}(B_1B_2) > 1 - \sigma_1 - \sigma_2$. Using this simple observation and recalling (33) and (35) gives

$$\inf_{i \in N} \mathbb{P}(A_{i,n_0+1}) \geq \inf_{i \in N} \mathbb{P}(A_{i,n_0+1}; Z_i \leq K, \mathcal{A}_{i,T}^{T_2})$$

$$\geq \inf_{i \in N} \mathbb{P}(A_{i,n_0+1}; Z_i \leq K, \mathcal{A}_{i,T}^{T_1}) (1 - 2\varepsilon m)$$

$$\geq \inf_{i \in T_1} \mathbb{P}(K(1 - 2\varepsilon m)) \geq \theta^K(1 - 2\varepsilon m).$$

Selecting $\varepsilon \in (0, 0.5m^{-1})$ we justify (31) and complete the proof of Lemma 6.
Lemma 7 If $M \in \mathcal{M}_1$ and $F(s) \neq Ms$, then $F_i(n; s) \to 1$ as $n \to \infty$ uniformly in $i \in \mathbb{N}$ and $s \in (0, 1]^\mathbb{N}$.

Proof. Clearly,

$$F_i(n; s) = \mathbb{E} \left[ \prod_{j \in \mathbb{N}} s_j^{Z_{ij}(n)} \right] \geq \mathbb{P}(\|Z_i(n)\|_1 = 0).$$

Recalling Lemma 6 we see that, for each fixed $i \in \mathbb{N}$

$$\sup_{s \in [0, 1]^\mathbb{N}} (1 - F_i(n; s)) = \sup_{s \in [0, 1]^\mathbb{N}} Q_i(n; s) \leq Q_i(n; 0) \to 0 \quad (38)$$

as $n \to \infty$. We now show that convergence in (38) is uniform over $i \in \mathbb{N}$.

Since

$$1 - F(n; s) = 1 - F(1; F(n - 1; s)) \leq M (1 - F(n - 1; s)),$$

it follows that for all $N \in \mathbb{N}$

$$Q_i(n; s) \leq \sum_{j \in \mathbb{N}} M_{ij} Q_j(n - 1; s) \leq \sum_{j \leq N} M_{ij} Q_j(n - 1; s) + \sum_{j > N} M_{ij}. \quad (39)$$

In view of Condition (iii) describing properties of matrices belonging to class $\mathcal{M}_1$, for any $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ meeting estimate (22). On the other hand, for any fixed $N$ and $i \in \mathbb{N}$

$$\sum_{j \leq N} M_{ij} u_j \leq \min_{k \leq N} u_k \leq \frac{u_i}{\min_{k \leq N} u_k} \leq \frac{U}{\min_{k \leq N} u_k}. \quad (40)$$

Lemma 6 and estimates (38)-(40) imply

$$\sup_{s \in [0, 1]^\mathbb{N}, i \in \mathbb{N}} Q_i(n; s) \leq \frac{U}{\min_{k \leq N} u_k} \sup_{j \leq N} Q_j(n - 1; 0) + \varepsilon \leq 2\varepsilon$$

for all sufficiently large $n$.

Lemma 7 is proved.
3 Ratio limit theorem

The next important theorem is an infinite-dimensional analog of Theorem 1 in [19, Ch. VI, §1].

Denote $S := \{ s \in [0,1]^\mathbb{N}, s \neq 1 \}$.

**Theorem 8** Let $\{Z_i(n), i \in \mathbb{N}\}$ be a critical GWBP/$\infty$ with mean matrix $M \in \mathcal{M}^0_1$ and $F(s) \neq Ms$. Then

$$\lim_{n \to \infty} \sup_{s \in S, i \in \mathbb{N}} \left| \frac{Q_i(n; s)}{u_i q(n; s)} - 1 \right| = 0. \quad (41)$$

To justify (41) for the critical GWBP/$\infty$’s we need to attract, along with the standard hypotheses (i) for the mean matrix, additional Conditions (iii) and (iv) which provide the desired uniform convergence in $i \in \mathbb{N}$. These additional conditions automatically fulfill for the Galton-Watson processes with finite number of types.

We split the proof of Theorem 8 into several lemmas.

For an infinite-dimensional vector $s = (s_j)_{j \in \mathbb{N}} \in [0,1]^\mathbb{N}$ introduce the notation $s_j := (s_i \delta_{ij} + 1 - \delta_{ij})_{i \in \mathbb{N}}$. Set also

$$F_{ij}(s_j) := F_i(s_j) = \mathbb{E}s_j^{Z_{ij}}, \quad Q_{ij}(s_j) := Q_i(s_j) = 1 - \mathbb{E}s_j^{Z_{ij}},$$

$$N_{ij}(s) := \sum_{k=0}^{Z_{ij}-1} s_j^k \left( 1 - \prod_{l=j+1}^{\infty} s_j^{Z_{il}} \right) = \frac{1 - s_j^{Z_{ij}}}{1 - s_j} \left( 1 - \prod_{l=j+1}^{\infty} s_j^{Z_{il}} \right),$$

$$N_{ij}(s) := \mathbb{E}N_{ij}(s).$$

**Lemma 9** If all elements of the mean matrix $M = (\mathbb{E}Z_{ij})_{i,j \in \mathbb{N}}$ are finite then, for each $i \in \mathbb{N}$ the following representation is valid

$$Q_i(s) = \sum_{j \in \mathbb{N}} \mathbb{E}(1 - s_j^{Z_{ij}}) - \sum_{j \in \mathbb{N}} (1 - s_j) \mathbb{E} \left[ \sum_{k=0}^{Z_{ij}-1} s_j^k \left( 1 - \prod_{l=j+1}^{\infty} s_j^{Z_{il}} \right) \right]$$

$$= \sum_{j \in \mathbb{N}} Q_{ij}(s_j) - \sum_{j \in \mathbb{N}} (1 - s_j)N_{ij}(s). \quad (42)$$
Proof. Using definitions (2) we perform a chain of evident transformations

\[
1 - \prod_{l=1}^{\infty} s_l^{Z_{li}} = (1 - s_1^{Z_{i1}}) - (1 - s_1^{Z_{i1}}) \left( 1 - \prod_{l=2}^{\infty} s_l^{Z_{li}} \right) + 1 - \prod_{l=1}^{\infty} s_l^{Z_{li}}
\]

\[
= (1 - s_1^{Z_{i1}}) - (1 - s_1) \sum_{k=0}^{Z_{i1}-1} s_1^k \left( 1 - \prod_{l=2}^{\infty} s_l^{Z_{li}} \right) + 1 - \prod_{l=2}^{\infty} s_l^{Z_{li}}
\]

\[
= (1 - s_1^{Z_{i1}}) - (1 - s_1) N_{i1}(s) + \left[ 1 - \prod_{l=2}^{\infty} s_l^{Z_{li}} \right].
\]

(43)

Since all the summands in the chain of identities have finite means, it follows that

\[
Q_i(s) = 1 - F_i(s) = E \left( 1 - \prod_{l=1}^{\infty} s_l^{Z_{li}} \right) = E \left( 1 - \prod_{l=2}^{\infty} s_l^{Z_{li}} \right)
\]

\[+ \quad Q_{i1}(s_1) - (1 - s_1) N_{i1}(s). \quad (44)\]

Repeating the chain of transformations (43) for the first summand at the right-hand side of (44) and doing the same for \(1 - \prod_{l=j+1}^{\infty} s_l^{Z_{li}}\), where the parameter \(j \in \mathbb{N} \setminus \{1\}\) sequentially takes values increasing by 1, we get the desired identity

\[
Q_i(s) = \sum_{j \in \mathbb{N}} Q_{ij}(s_j) - \sum_{j \in \mathbb{N}} N_{ij}(s)(1 - s_j).
\]

The lemma is proved.

Theorem 10 Let \(\{Z_i(n), i \in \mathbb{N}\}\) be a critical GWBP/\(\infty\) with mean matrix \(M \in \mathcal{M}_1\) and \(F(s) \neq Ms\). Then

\[
\limsup_{z \uparrow 1} \sup_{i \in \mathbb{N}} \frac{(1 - z)E Z_i - E(1 - z^{Z_i})}{(1 - z)E Z_i} = 0 \quad (45)
\]

and

\[
\limsup_{z \uparrow 1} \sup_{i \in \mathbb{N}} \frac{(1 - z)E Z_i - \sum_{j \in \mathbb{N}} E(1 - z^{Z_{ij}})}{(1 - z)E Z_i} = 0.
\]

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Remark. The statement of the theorem is always true for the Galton-Watson branching processes with finite number of types. However, this is not always the case for GWBP/$\infty$’s. It is for this reason we are forced to include Conditions (iii) in Definition \[2\].

Proof of Theorem 10 Since

$$
\mathbb{E}(1 - z^{Z_i}) \leq \sum_{j \in \mathbb{N}} \mathbb{E}(1 - z^{Z_{ij}}) \leq \sum_{j \in \mathbb{N}} (1 - z) \mathbb{E}Z_{ij} = (1 - z) \mathbb{E}Z_i,
$$

it sufficient to prove (45) only. According to the second part of Conditions (iii), for any $\varepsilon > 0$ there exists $K = K(\varepsilon)$ such that

$$
0 \leq \sup_{i \in \mathbb{N}} M_i^{-1} \mathbb{E} [(1 - z)Z_i - (1 - z^{Z_i}) ; Z_i > K] \leq \varepsilon (1 - z) / 2. \quad (46)
$$

On the other hand, for any fixed $k \leq K$ the representation

$$
k(1 - z) - (1 - z^k) = (1 - z) \left( k - \sum_{j=0}^{k-1} z^j \right) \leq (1 - z) \left( K - \sum_{j=0}^{K-1} z^j \right) = o(1 - z)
$$

is valid. Since $\mathbb{P}(0 < Z_i \leq K) \leq \mathbb{E}Z_i = M_i$ for each fixed $K \in \mathbb{N}$, it follows that

$$
0 \leq \sup_{i \in \mathbb{N}} M_i^{-1} \mathbb{E} [(1 - z)Z_i - (1 - z^{Z_i}) ; Z_i \leq K] = o(1 - z).
$$

Thus, for $\varepsilon > 0$ and $K = K(\varepsilon)$ selected above one can find a $\Delta > 0$ such that for $0 \leq 1 - z < \Delta$

$$
0 \leq \sup_{i \in \mathbb{N}} M_i^{-1} \mathbb{E} [(1 - z)Z_i - (1 - z^{Z_i}) ; Z_i \leq K] \leq \varepsilon (1 - z) / 2. \quad (47)
$$

Combining (46) and (47) gives (45).

Theorem 10 is proved.

Introduce the notation $Q_n = Q(n; s) := \sup_{i \in \mathbb{N}} Q_i(n; s)$, $F_n = F(n; s) := \inf_{i \in \mathbb{N}} F_i(n; s)$. 

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Lemma 11 Let \( \{Z_i(n), i \in \mathbb{N}\} \) be a critical GWBP/\( \infty \) with mean matrix \( M \in \mathcal{M}_1^n \) and \( F(s) \neq Ms \). Then, for each \( i \)
\[
\sum_{j \in \mathbb{N}} Q_j(n; s)N_{ij}(F(n; s)) =: M_i \epsilon_{1,i}(Q(n; s)),
\]
where
\[
\lim_{n \to \infty} \sup_{s \in \mathbb{S}, i \in \mathbb{N}} \frac{\epsilon_{1,i}(Q(n; s))}{Q(n; s)} = 0.
\]

**Proof.** Recall that \( \sup_{i \in \mathbb{N}} M_i < \infty \) by (21). By definition
\[
Q_j(n; s)N_{ij}(F(n; s)) = \mathbb{E}\left[ \left( 1 - F_{Z_{ij}}(n; s) \right) \left( 1 - \prod_{l=j+1}^{\infty} F_{Z_{il}}(n; s) \right) \right]
\leq \mathbb{E}\left[ (1 - F_{Z_{ij}}) \left( 1 - \prod_{l=j+1}^{\infty} F_{Z_{il}}(n) \right) \right] = \mathbb{E}\left[ (1 - F_{Z_{ij}}) \left( 1 - F_{\sum_{l \geq j} Z_{il}}(n) \right) \right].
\]
Using the equality \( (1 - x)(1 - y) = 1 - x - y + xy \) we see that
\[
Q_j(n; s)N_{ij}(F(n; s)) \leq \mathbb{E}\left[ (1 - F_{Z_{ij}}) \left( 1 - F_{\sum_{l \geq j} Z_{il}}(n) \right) \right] = \mathbb{E}(1 - F_{Z_{ij}}) + \mathbb{E}\left( 1 - F_{\sum_{l \geq j} Z_{il}}(n) \right) - \mathbb{E}\left( 1 - F_{\sum_{l \geq j} Z_{il}}(n) \right).
\]
By (49) we conclude that
\[
0 \leq \sum_{j \in \mathbb{N}} Q_j(n; s)N_{ij}(F(n; s))
\leq \sum_{j \in \mathbb{N}} \left( \mathbb{E}(1 - F_{Z_{ij}}) + \mathbb{E}\left( 1 - F_{\sum_{l \geq j} Z_{il}}(n) \right) - \mathbb{E}\left( 1 - F_{\sum_{l \geq j} Z_{il}}(n) \right) \right)
= \sum_{j \in \mathbb{N}} \mathbb{E}(1 - F_{Z_{ij}}) - \mathbb{E}\left( 1 - F_{\sum_{l \geq j} Z_{il}}(n) \right) \leq \sum_{j \in \mathbb{N}} \mathbb{E}Z_{ij}Q_n - \mathbb{E}(1 - F_{Z_{ij}}).
\]
Observe that the series in the second line of (50) uniformly converges in \( i \in \mathbb{N} \) in view of the estimates
\[
\mathbb{E}\left( 1 - w^{\sum_{l \geq j} Z_{il}} \right) - \mathbb{E}\left( 1 - w^{\sum_{l \geq j} Z_{il}} \right) \leq \mathbb{E}\left( 1 - w^{Z_{ij}} \right)
\]
and $\mathbb{E}(1 - w^{Z_{ij}}) \leq \mathbb{E}Z_{ij}(1 - w)$.

Using (45) for $x = Q(n; s)$ we deduce that

$$0 \leq \lim_{n \to \infty} \sup_{s, i \in N} \frac{\sum_{j \in N} Q_j(n; s) N_{ij}(F(n; s))}{M_i Q(n; s)} \leq \lim_{n \to \infty} \sup_{s, i \in N} \frac{\mathbb{E}Z_{i}Q(n; s) - \mathbb{E}(1 - FZ_{i}(n; s))}{M_i Q(n; s)} = 0,$$

as desired.

Lemma 11 is proved.

**Lemma 12** Let $\{Z_i(n), i \in \mathbb{N}\}$ be a critical GWBP/$\infty$ with mean matrix $M \in M_1$ and $F(s) \neq Ms$. Then, for all $i \in \mathbb{N}$,

$$\sum_{j \in \mathbb{N}} M_{ij}(1 - s_j) - \sum_{j \in \mathbb{N}} Q_{ij}(s_j) =: M_i \epsilon_{2,i}(1 - s),$$

where

$$\lim_{s \in S, \|1 - s\|_{\infty} \to 0} \sup_{i \in \mathbb{N}} \|\epsilon_{2,i}(1 - s)\|_{\infty} = 0.$$

Besides,

$$\sum_{j \in \mathbb{N}} M_{ij}Q_j(n - 1; s) - \sum_{j \in \mathbb{N}} Q_{ij}(F_j(n - 1; s)) = M_i \epsilon_{2,i}(Q(n - 1; s)), \quad (51)$$

where

$$\lim_{n \to \infty} \sup_{s \in S, i \in \mathbb{N}} \frac{|\epsilon_{2,i}(Q(n - 1; s))|}{Q(n - 1; s)} = 0.$$

**Proof.** Consider the function

$$f_{ij}(s_j) := M_{ij}(1 - s_j) - Q_{ij}(s_j) = \mathbb{E} \left[ Z_{ij}(1 - s_j) - 1 + s_j^{Z_{ij}} \right]$$

with partial derivative

$$\frac{\partial f_{ij}(s_j)}{\partial s_j} = \mathbb{E} \left[ Z_{ij} \left( s_j^{Z_{ij} - 1} - 1 \right) \right] \leq 0, \quad j \in \mathbb{N}.$$

Two last relations mean that the function $f_{ij}(s_j)$ is nonincreasing, $f_{ij}(1) = 0$ and, therefore, $f_{ij}(s_j) \geq 0$ for $s_j \in [0, 1]$. As a result,

$$f_{ij}(s_j) \leq f_{ij}(1 - \|1 - s\|_{\infty}) = \mathbb{E} \left[ Z_{ij}\|1 - s\|_{\infty} - 1 + (1 - \|1 - s\|_{\infty})^{Z_{ij}} \right].$$
Now the assertion of Lemma 12 is an evident corollary of Theorem 10.

We now deduce estimates for $Q_i(n; s)$, $i \in \mathbb{N}$, in terms of the function $Q_1(n; s)$.

**Lemma 13** Let $\{Z_i(n), i \in \mathbb{N}\}$ be a critical GWBP/$\infty$ with mean matrix $M \in \mathcal{M}_1^0$ and $F(s) \neq Ms$. Then there exist constants $C_1$ and $C_2$ such that

$$Q_i(n; s) \leq C_1 u_i Q_1(n; s), \quad \forall i \in \mathbb{N},$$

and

$$Q(n; s) \leq C_2 Q_1(n; s)$$

for all sufficiently large $n$.

**Proof.** Taking into account (42) with $s$ replaced by $F(n+1; s)$, (48) with $F(n; s)$ replaced by $F(n+1; s)$, and attracting (51) we get

$$Q_i(n; s) = \sum_{j \in \mathbb{N}} M_{ij} Q_j(n-1; s) + M_i \epsilon_i(Q(n-1; s)), \quad (52)$$

where $\epsilon_i(Q(n-1; s)) = \epsilon_{1,i}(Q(n-1; s)) + \epsilon_{2,i}(Q(n-1; s))$ and

$$\lim_{n \to \infty} \sup_{s \in S, i \in \mathbb{N}} \frac{\epsilon_i(Q(n-1; s))}{Q(n-1; s)} = 0. \quad (53)$$

Multiplying the left and right hand sides of (52) by $v_i$ and summing over $i$ we obtain

$$q(n; s) = vQ^T(n; s) = q(n-1; s) + \epsilon(Q(n-1; s)), \quad (54)$$

where

$$\epsilon(Q(n-1; s)) := \sum_{i \in \mathbb{N}} v_i M_i \epsilon_i(Q(n-1; s)).$$

In view of (53)

$$\lim_{n \to \infty} \sup_{s \in S} \frac{\epsilon(Q(n-1; s))}{Q(n-1; s)} = 0. \quad (55)$$

Representation (52) and estimate (21) imply, for sufficiently large $n$ the inequalities

$$Q_i(n; s) \leq 2M_i Q_{n-1}, \quad Q_n \leq 2m Q_{n-1}. \quad (56)$$
Combining (52) with $n$ sequentially replaced by $n+1, \ldots, n+l$ we obtain

$$Q_i(n + l; s) = M_i\epsilon_i^*(n, l; s) + \sum_{j \in \mathbb{N}} M_{ij}^{(l)} Q_j(n; s),$$

(57)

where

$$M_i\epsilon_i^*(n, 1; s) = M_i\epsilon_i(Q(n; s)),$$

$$M_i\epsilon_i^*(n, k; s) = \sum_{j \in \mathbb{N}} M_{ij} M_j\epsilon_j^*(n, k - 1; s) + M_i\epsilon_i(Q(n + k - 1; s)), k > 1.$$

Clearly,

$$\lim_{n \to \infty} \sup_{s \in S, i \in \mathbb{N}} |\epsilon_i^*(n, l; s)| = 0$$

in view of (53), (56) and (21).

Since $M \in M_1^0$ by our assumptions, applying the first inequality in (12) gives

$$M_{ij}^{(n)} = \sum_{k \in \mathbb{N}} M_{ik} M_k^{(n-1)} \leq Cu_i \sum_{k \in \mathbb{N}} v_k M_{kj}^{(n-1)} = Cu_i v_j, \forall i, j \in \mathbb{N}.$$  

(58)

Further, using the second inequality in (12) for the respective $m \in \mathbb{N}$ we obtain

$$M_{ij}^{(n+m)} = \sum_{k \in \mathbb{N}} M_{ik}^{(m)} M_k^{(n)} \geq c \sum_{k \in \mathbb{N}} v_k M_{kj}^{(n)} = cv_j, \forall j \in \mathbb{N}.$$  

(59)

We fix $l \geq m$. Relations (58) and (59) imply the inequalities

$$M_{ij}^{(l)} \leq Cu_i v_j \leq \frac{CM_{ij}^{(l)} u_i}{c}, \forall i, j \in \mathbb{N}.$$

Hence, using (57) we deduce the estimate

$$Q_i(n + l; s) \leq \frac{2Cu_i}{c} \sum_{j \in \mathbb{N}} M_{ij}^{(l)} Q_j(n; s), \forall i \in \mathbb{N},$$

(60)

which, on account of $\|u\|_\infty = U < \infty$, leads to the inequality

$$Q(n + l; s) = Q_{n+l} \leq \frac{2CU}{c} \sum_{j \in \mathbb{N}} M_{ij}^{(l)} Q_j(n; s).$$  

(61)
Finally, using (57) with $i = 1$ we may transform (60) and (61) as
\[
Q_i(n + l; s) \leq \frac{4Cu_i}{c}Q_1(n + l; s), \quad \forall i \in \mathbb{N}, \quad (62)
\]
and
\[
Q_{n+l} = Q(n + l; s) \leq \frac{4CU}{c}Q_1(n + l; s). \quad (63)
\]

Lemma 13 is proved.

**Lemma 14.** If $M \in M_1^0$ and $F(s) \neq Ms$ then, for any fixed $n \in \mathbb{N}$
\[
\lim_{N \to \infty} \sup_{s \in S} \frac{1}{\sum_{j=N+1}^{\infty} v_j Q_j(n; s)} = 0 \quad (64)
\]
and, for any $\varepsilon > 0$ there exist $l_0 = l_0(\varepsilon) \in \mathbb{N}$ such that
\[
|Q_i(n + l_0; s) - u_i q(n; s)| \leq \varepsilon u_i q(n; s), \quad (65)
\]
\[
|q(n + l_0; s) - q(n; s)| \leq \varepsilon q(n; s) \quad (66)
\]
for all $i \in \mathbb{N}$ and $n > n_0$.

**Proof.** Clearly,
\[
q(n; s) \geq v_1 Q_1(n; s), \quad s \in (0, 1]^N.
\]

On the other hand, estimate (63) and Conditions (13) allow us to deduce for sufficiently large $n$ the estimate
\[
q(n; s) \leq \frac{4CU}{c} Q_1(n; s).
\]

Thus, the functions $q(n; s)$ and $Q_1(n; s)$ have the same order as $n \to \infty$. This fact, estimate (62) and convergence of the series $\sum_{i=1}^{\infty} v_i$ justify (64).

Besides,
\[
0 < \liminf_{n \to \infty} \inf_{s \in S} \frac{q(n; s)}{Q_1(n; s)} \leq \limsup_{n \to \infty} \sup_{s \in S} \frac{q(n; s)}{Q_1(n; s)} < \infty.
\]

We rewrite this relation as
\[
q(n; s) \asymp_{n \to \infty} Q_1(n; s). \quad (67)
\]
The same notation will be used in others similar situations. For instance, in view of (63) and the definition of \( Q(n + l; s) \)

\[
Q_{n+l} = Q(n + l; s) \leq Q_1(n + l; s)
\]

(68)

for each fixed \( l \).

Taking into account Condition (iv) and relation (68) we transform (57) to the form

\[
Q_i(n + l; s) - u_iq(n; s) - M_i\varepsilon_i^*(n, l; s)
= \sum_{j=1}^{N} (M_{ij}^{(l)} - u_{ij}) Q_j(n; s) - \sum_{j=N+1}^{\infty} u_{ij} Q_j(n; s) + \sum_{j=N+1}^{\infty} M_{ij}^{(l)} Q_j(n; s)
= I_1(i, N, l, n; s) + I_2(i, N, n; s) + I_3(i, N, l, n; s).
\]

(69)

Estimates (58), (63), (67), (68) and the equality \( \mathbf{v}^T = 1 \) allow us to claim that

\[
\lim_{N \to \infty} \sup_{i \in \mathbb{N}, n \in \mathbb{N}, s \in S} \left| \frac{I_2(i, N, n; s)}{u_i Q_1(n; s)} \right| = 0,
\]

(70)

and

\[
\lim_{N \to \infty} \sup_{i \in \mathbb{N}, l \in \mathbb{N}, n \in \mathbb{N}, s \in S} \left| \frac{I_3(i, N, l, n; s)}{u_i Q_1(n; s)} \right| = 0
\]

(71)

where \( Q_1(n; s) \) may be replaced by \( q(n; s) \).

We now select an \( \varepsilon > 0 \). In view of (70) and (71) there exists \( N = N(\varepsilon) \) such that

\[
|I_2(i, N(\varepsilon), n; s)| + |I_3(i, N(\varepsilon), l, n; s)| \leq 0.25\varepsilon u_i q(n; s).
\]

(72)

According to (11), conditions (13) and estimate (63), there exist \( C_1 \) and \( l_0 = l_0(\varepsilon) \) such that, for all \( l \geq l_0 \)

\[
|I_1(i, N(\varepsilon), l, n; s)| \leq N(\varepsilon) C_1 Q(n; s) \sup_{j \leq N(\varepsilon)} |M_{ij}^{(l)} - u_{ij} v_j| \leq 0.25\varepsilon u_i q(n; s).
\]

(73)

We know by (67), (68) and (56) that \( Q_1(n + l; s) \leq Q_1(n; s) \) for any fixed \( l \). Therefore, for \( l = l_0 \) and \( n > n_2 = n_2(\varepsilon) \) the third term at the left-hand side of (69) may be evaluated as

\[
|M_i\varepsilon_i^*(n, l_0; s)| = u_i \left| \frac{M_i}{u_i} \varepsilon_i^*(n, l_0; s) \right| \leq C u_i \left| \varepsilon_i^*(n, l_0; s) \right| \leq 0.5\varepsilon u_i q(n; s).
\]

(74)
Combining estimates (72), (73) and (74) and using decomposition (69) we easily obtain (65).

Since $uv^T = 1$, relation (66) immediately follows from (65).

Lemma 14 is proved.

Proof of Theorem 8. Clearly, estimates (65) and (66) imply (11). Theorem 8 is proved.

4 Proof of Theorem 3

We set

$$B(n; s) := vQ^T(n; s) - vQ^T(F(n; s))$$

and first prove an infinite-dimensional analog of Lemma 2 in [12].

Recall that $\Phi(x) = x - vQ^T(1 - xu) = x^{1+\alpha}\ell(x)$ for $xU \leq 1$ as $x \to +0$ by (14) and (15).

Lemma 15 If the conditions of Theorem 3 are valid then

$$\lim_{n \to \infty} \sup_{s \in S} \left| \frac{B(n; s)}{\Phi(q(n; s))} - 1 \right| = 0. \quad (75)$$

The proof of (75) coincides almost literally with the proof of Lemma 2 in [12] and we give it here to only keep the integrity of the presentation.

Introduce the function

$$B(s) := v(1 - s)^T - vQ^T(s).$$

Clearly, for $s = (s_1, s_2, \ldots) \in S$

$$\frac{\partial B(s)}{\partial s_i} = -v_i - \sum_{j=1}^{\infty} v_j \frac{\partial Q_j(s)}{\partial s_i} = -v_i + \sum_{j=1}^{\infty} v_j \frac{\partial F_j(s)}{\partial s_i} \leq -v_i + \sum_{j=1}^{\infty} v_j E_{Z_{ji}} = -v_i + v_i = 0.$$

Thus, $B(s)$ is monotone decreasing with respect to each argument of $s$. By Theorem 8 for any $\varepsilon > 0$ one can find $N = N(\varepsilon)$ such that

$$(1 - \varepsilon)u\ell q(n; s) \leq Q_i(n; s) \leq (1 + \varepsilon)u\ell q(n; s)$$
for all \( n \geq N \) and all \( i \in \mathbb{N} \) and \( s \in S \). Therefore, for \( n \geq N \)

\[
B(1 - (1 - \varepsilon)q(n; s)u) \leq B(1 - Q(n; s)) \leq B(1 - (1 + \varepsilon)q(n; s)u).
\]

Since \( B(1 - Q(n; s)) = B(n; s) \) and \( B(1 - xu) = \Phi(x) \), it follows that, for \( n \geq N \)

\[
\Phi((1 - \varepsilon)q(n; s)) \leq B(n; s) \leq \Phi((1 + \varepsilon)q(n; s)).
\]

By our conditions, \( \ell(x) \) is a slowly varying function as \( x \to +0 \). Therefore (see, for instance, Theorem 1.1 [21, Ch. 1, §1.2]),

\[
\frac{\ell(cx)}{\ell(x)} \to 1
\]
as \( x \to +0 \) uniformly in \( c \in [a, b] \), \( 0 < a < b < \infty \). Fix \( \varepsilon_0 \in (0, 1) \). Since

\[
q(n; s) \leq q(n; 0) \quad \text{and} \quad \lim_{n \to \infty} q(n; 0) = 0,
\]

we see that

\[
\lim_{n \to \infty} \sup_{s \in S} \frac{\Phi((1 \pm \varepsilon)q(n; s))}{\Phi(q(n; s))} = (1 \pm \varepsilon)^{1+\alpha} \lim_{n \to \infty} \sup_{s \in S} \frac{\ell((1 \pm \varepsilon)q(n; s))}{\ell(q(n; s))} = (1 \pm \varepsilon)^{1+\alpha}.
\]

By letting \( \varepsilon \to +0 \) we easily deduce (75).

The next statement is an infinite-dimensional of Lemma 3 in [12].

Lemma 16 Let the conditions of Theorem 3 be valid. Then

\[
\lim_{n \to \infty} \sup_{s \in S} \left| \frac{\Phi(q(n+1, s))}{\Phi(q(n; s))} - 1 \right| = 0. \tag{76}
\]

Proof. In view of \( \Phi(x) = x^{1+\alpha}\ell(x) \), to demonstrate the validity of (76) it is sufficient to show that

\[
\lim_{n \to \infty} \sup_{s \in S} \left| \frac{q(n+1, s)}{q(n; s)} - 1 \right| = 0.
\]

It remains to observe that the desired estimate is a corollary of (54), (55), (57) and (68).

Lemma 16 is proved.

Proof of Theorem 3 Using Lemma 15 we write

\[
B(k; s) = q(k; s) - q(k+1; s) = \Phi(q(k; s)) (1 + \varepsilon(k; s)),
\]
where
\[ \lim_{k \to \infty} \sup_{s \in S} |\varepsilon(k; s)| = 0. \]

Hence, setting \( q(0; s) := v(1 - s)^T \) we obtain that
\[ \sum_{k=0}^{n} \frac{q(k; s) - q(k+1; s)}{\Phi(q(k; s))} = n \left( 1 + \varepsilon_1(n; s) \right), \]
where \( \lim_{n \to \infty} \sup_{s \in S} |\varepsilon_1(n; s)| = 0 \). Lemma 16 and monotonicity of \( \Phi(x) \) in \( x \) allow us to rewrite the previous relation as
\[ \int_{q(n; s)}^{q(0; s)} \frac{dx}{\Phi(x)} = n(1 + \varepsilon_2(n; s)), \quad (77) \]
where \( \lim_{n \to \infty} \sup_{s \in S} |\varepsilon_2(n; s)| = 0 \). Letting \( s = 0 \) and recalling that \( \Phi(x) = x^{1+\alpha} \ell(x) \) as \( x \to +0 \) we deduce by the properties of regularly varying functions (see Theorem 1 [20, Ch. VIII, Section 9]) that
\[ q^\alpha(n) \ell(q(n)) \sim (\alpha n)^{-1}, \quad n \to \infty, \]
or (see property 5° [21, Ch. 1, Section 1.5])
\[ q(n) = n^{-1/\alpha} \ell_1(n) \]
for a function \( \ell_1(n) \) slowly varying as \( n \to \infty \). This proves (16).

Relation (17) follows from (16) and Theorem 8.

We now prove (18) and (19). Let \( \lambda = (\lambda_i)_{i \in \mathbb{N}} \) be an infinite-dimensional vector with nonnegative bounded components. Set
\[ s_i = s(n; \lambda_i) = \exp \{-\lambda_i(q(n))\}, \quad i = 1, 2, ..., \quad (78) \]
and put \( q(0; s) := v(1 - s)^T \). Using the relation \( \Phi(q(n))/q(n) \sim (\alpha n)^{-1}, \quad n \to \infty, \) and making the change of variables \( x \to zq(n) \) we deduce from (77) the representation
\[ \int_{q(n; s)/q(n)}^{q(0; s)/q(n)} \frac{\Phi(q(n))dz}{\Phi(zq(n))} = \frac{1 + \varepsilon_2(n; s)}{\alpha}, \quad (79) \]
where $\varepsilon_2(n; s) \to 0$ as $n \to \infty$. Note that, as $n \to \infty$

$$
\frac{\Phi(q(n))}{\Phi(zq(n))} \to \frac{1}{z^{1+\alpha}}
$$

(80)

uniformly in $z$ from any finite interval $0 < a \leq z \leq b < \infty$.

By assumption, the components of $\lambda$ are bounded and $v_1^T = 1$. Therefore,

$$
\lim_{n \to \infty} \frac{q(0; s)}{q(n)} = (v, \lambda).
$$

Since the right-hand side of (79) has a limit as $n \to \infty$, the same is true for the left-hand side. Consequently,

$$
\lim_{n \to \infty} \frac{q(n; s)}{q(n)/q(n)} =: 1 - \phi(\lambda)
$$

also exists. Moreover, this limit is strictly positive. Indeed, if it would be not the case then the integral at the left-hand side of (79) would be divergent in view of (80). Using (80) once again, passing to the limit in (79) and performing integration we obtain

$$
(1 - \phi(\lambda))^{-\alpha} - (v, \lambda)^{-\alpha} = 1
$$

or

$$
\phi(\lambda) = 1 - (1 + (v, \lambda)^{-\alpha})^{-1/\alpha}.
$$

Finally, recalling Theorem 8 and selecting the same $s$ as in (78) we obtain

$$
\lim_{n \to \infty} \mathbb{E} \left[ e^{-(\lambda, Z(n))q(n)} 1_{Z(n) \neq 0, Z(0) = e_i} \right] = 1 - \lim_{n \to \infty} \frac{Q_i(n; s)}{Q_i(n)} = 1 - \lim_{n \to \infty} \frac{q(n; s)}{q(n)} = \phi(\lambda).
$$

The last is equivalent to (18) which, in turn, implies (19).

Theorem 3 is proved.

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