Symmetric hidden state model for trust-less entanglement witness

Debasis Mondal\textsuperscript{1,7} and Dagomir Kaszlikowski\textsuperscript{1,2,†}

\textsuperscript{1}Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543
\textsuperscript{2}Department of Physics, National University of Singapore, 2 Science Drive 3, 117542 Singapore, Singapore

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We consider two physically separated parties receiving quantum states from an unknown source and introduce symmetric hidden state (SLHS) model for such scenario. A completely new form of nonlocality arises from the model. Our model provides the tightest entanglement witness method for two device-independent/trust-less parties, making it a better alternative to the Bell inequality for self-testing and other device-independent protocols. We also propose an experiment to show the experimental violations of our inequalities. A resource theory of the new form of non-locality is also discussed.

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Introduction.—A quantum state is shared by two parties—Alice and Bob. The state of Alice is correlated to the state Bob but not the state of Bob with that of Alice—sounds as absurd as the statement like: Alice is married to Bob but Bob is not.\textsuperscript{1,2} The violation of the present description of local hidden state (LHS) model\textsuperscript{3,4} sometimes predicts exactly that. There are states for which Alice can steer Bob but Bob cannot steer Alice. Is it then possible to find an alternative and improved local hidden state model for which such scenarios do not arise?

The situation arises primarily because of the game considered in the description of local hidden state model\textsuperscript{3}. We consider two parties—Alice and Bob. Alice prepares a bipartite state. A part of which she sends to Bob and keeps the other part with her. Alice claims, she can steer his state and his state is entangled with her. Bob does not trust Alice. Bob comes up with a strategy to verify her claim by constructing the steering inequality based on the LHS model.

In the existing LHS model\textsuperscript{1,3,6}, Bob assumes that the states, he is receiving from Alice, have single system description and Alice can in principle prepare such states from an ensemble of hidden states \( \{ p(\lambda), \rho_{\lambda} \} \) such that

\[
\rho_{a|\Pi} = \sum_{\lambda} p(\lambda)p(a|\Pi, \lambda)\rho_{\lambda}, \tag{1}
\]

where \( p(a|\Pi, \lambda) \) is Alice’s stochastic map to convince Bob and \( \lambda \) is a hidden variable such that \( \sum_{\lambda} p(\lambda) = 1 \), \( \Pi \) is an observable in which Bob asked Alice to perform measurements and \( a \) is the outcome. Violation of a steering inequality based on the LHS model implies the existence of EPR non-locality.

As it can be observed, the game is asymmetric by construction. A symmetric scenario would be, where Alice (A) and Bob (B) both receive quantum states from an unknown source (S). They do not trust each other and the source. Therefore, they would like to come up with an effective strategy to verify whether they have EPR correlations between their particles or not.

One way to represent the symmetric local hidden state (SLHS) model is of course to express the bipartite state by local hidden states, i.e.,

\[
\rho_{AB} = \sum_{\lambda} p(\lambda)\rho_{A}^\lambda \otimes \rho_{B}^\lambda, \tag{2}
\]

where \( \{ p(\lambda), \rho_{A}^\lambda \otimes \rho_{B}^\lambda \} \) is an ensemble of hidden states like before. Alice and Bob can construct an inequality based on this model and the level of their trust.

Assumption in the above model is that any bipartite state must be expressible in terms of local hidden states. However, this is nothing but a separable state. Any state, which cannot be expressed by the above model is nothing but an entangled state and to witness entangled state in the most effective way, Alice and Bob need to trust each other.

This now raises a question: is it possible to witness quantum nonlocality based on hidden state model in a trust-less manner? Since, local hidden state model is a stronger condition than the local hidden variable model, less quantum states have such local hidden state description and more states violate the model\textsuperscript{1}. In this sense, nonlocality based on the violation of local hidden state model is more accurate description of nonlocality\textsuperscript{7,8} and a new symmetric, trustless description of such nonlocality should be better suited for device independent QKD protocols than the existing protocols based on the Bell nonlocality and the EPR steering\textsuperscript{9,11}.

It turns out that there is indeed another way to represent a symmetric hidden state model. In the local hidden variable (LHV) model\textsuperscript{12}, we consider joint probability distributions and use the principle of locality and determinism to write that in terms of local probability distributions and hidden variable. Here, we assume that both Alice and Bob are receiving quantum states. However, quantum states provide more information than just probability distributions. One can also extract information about certain quantities, which have no classical counterparts unlike probability distributions. We consider a global or joint property with no classical counterpart and
using the principle of locality and hidden states, express in terms of the local property of the hidden states. Ideally, all the elements of a bipartite density matrix with local hidden state description must be expressible by some unknown hidden states $\rho^A_\lambda$ and $\rho^B_\lambda$ generated by the unknown source with a distribution $p(\lambda)$. In a simple mathematical term, this implies,

$$
\langle i^a_j^b | \rho_{AB} | i^a_j^b \rangle = \sum_\lambda p(\lambda) \langle i^a | \rho^A_\lambda | i^a \rangle \langle i^b | \rho^B_\lambda | i^b \rangle \quad (3)
$$

$$
p \left( |i^{a-a'}^A \rangle | j^{b-b'}^B \rangle \right) = \sum_\lambda p(\lambda) p \left( |i^{a-a'}^A \rangle | \lambda \right) p \left( |j^{b-b'}^B \rangle | \lambda \right) \quad (4)
$$

Here, $p(|a-a'|)$ denotes the transition probability of the state from the $a$ Eigenvector to the $a'$ Eigenvector of the observable $i$ on Alice’s side and $p(|a-b|) = |\langle i^a | \rho^B_b | i^b \rangle |^2$. Note however that $p(|a-b|)$ does not really have properties of a probability distribution function. One can indeed show that the Eq. (5) implies the Eq. (4) (see the supplemental material [13]).

**quantum nonlocality.**—In this section, we derive an inequality based on the model for a two-qubit state. We consider the following quantity, where Alice and Bob both measures the transition probabilities in the Eigen bases of the same observable as

$$
\sum_{a,a'} p(|i^{a-a'}^A \rangle | i^{a-a'}^B \rangle) = \sum_{\lambda,a,a'} p(\lambda) p \left( |i^{a-a'}^A \rangle | \lambda \right) p \left( |i^{a-a'}^B \rangle | \lambda \right)
\leq \sum_{a,a', a \neq a'} \sqrt{\sum_{\lambda} p(\lambda' p(\lambda') p^2 \left( |i^{a-a'}^A \rangle | \lambda' \right) \sum_{\lambda} p(\lambda) p^2 \left( |i^{a-a'}^B \rangle | \lambda \right)}
= \sum_{a,a', a \neq a'} \sqrt{p^2 \left( |i^{a-a'}^A \rangle \right) p^2 \left( |i^{a-a'}^B \rangle \right)} \leq \frac{1}{8} \quad (5)
$$

where in the first inequality, we use the Cauchy-Schwarz inequality, the second inequality comes from the fact that for a probability distribution $p(x)$ over a real random variable $x$ and a positive function of the random variable $f(x)$, if the the function is bounded from above by $l$, the $n^{th}$-moment of $f(x)$ is bounded by $l^n$. In this case, the function of the random variable $\lambda$ is $p \left( |i^{a-b}^A \rangle | \lambda \right)$ and is bounded by the maximal value of the transition probability of a local state can possibly take. For a two-qubit scenario, the dimension of the local state is two and the maximum transition probability between two arbitrary orthonormal states labeled by $a$, $b$ such that $a \neq b$ turns out to be $\frac{1}{4}$. Thus, the bound becomes $\frac{1}{8}$. The inequality can also be extended for a general $d \otimes d$ bipartite systems. In this case, the bound turns out to be $\frac{d-1}{ad}$.

As it can be observed from the fig. 1, quantum states indeed violate the inequality. We will leave now the question on how Alice and Bob are going to measure the quantity on the left hand side of the inequality in Eq. (5) and show violation of the inequality in an experiment for the next section and instead, we focus on to ask a more deeper question: what would happen if one considered quantum coherence instead of quantum transition probabilities? One would expect to get similar violation of a bound based on quantum coherence. More so due to the fact that quantum steering and quantum coherence has a deep connection [14][17]. However, it is surprisingly not the case. One can follow similar steps to come
up with a bound, which turns out to be a trivial bound with no violation by the bipartite entangled states.

**Free will**—One can verify that the same Werner state as considered for the plot in fig. 1, will not show violation for a similar inequality as

\[
\sum_{a, a' \neq a} p\left(|a' - a\rangle^A, |a - a'\rangle^B\right) \leq \frac{1}{8}, \quad (6)
\]

The inequality in Eq. (6) is violated by yet another Werner state \(\rho_w = p|\psi_+^A\rangle\langle\psi_+^A| + \frac{1}{d} I_d\), where \(|\psi_+^A\rangle = \sqrt{\alpha}\langle 01\rangle + \sqrt{1 - \alpha}\langle 10\rangle\).

One can derive another inequality based on these two inequalities in Eq. (5) and (6), just by summing both the inequalities. The new inequality is although not tight, to show the violation, one does not need to invoke the assumption of free will. Both the Werner states for \(p > \frac{1}{\sqrt{2}}\), violates the inequality as given below,

\[
\sum_{a, a', b, b' = 0, a \neq a', b \neq b'}^1 p\left(|a' - a\rangle^A, |b - b'\rangle^B\right) \leq \frac{1}{4}, \quad (7)
\]

For a \(d \otimes d\) bipartite system, an equivalent inequality corresponding to the inequality in Eq. (7) will be

\[
\sum_{a, a', b, b' = 0, a \neq a', b \neq b'}^{d-1} p\left(|a' - a\rangle^A, |b - b'\rangle^B\right) \leq \left(1 - \frac{1}{d}\right)^2, \quad (8)
\]

whereas, an equivalent inequality to that given in Eq. (6) will be

\[
\sum_{a, a' = 0, a \neq a'}^{d-1} p\left(|a' - a\rangle^A, |a - a'\rangle^B\right) \leq \frac{d - 1}{d^3}. \quad (9)
\]

The inequality in Eq. (9) is tighter than that in Eq. (8).

Let us now focus on to compare our inequality with that of the existing bounds. For example, it is well-known that a generalized isotropic state belonging to the Hilbert space \(H_d \otimes H_d\) of the form of

\[
\rho_{AB} = (1 - p) \frac{I_d}{d^2} + \frac{p}{d} \sum_{i,j} |ii\rangle\langle jj|
\]

is steerable for \(p > \frac{\sqrt{2} - \sqrt{1 - \frac{1}{d^2}}}{2}\) and entangled for \(p > \frac{1}{\sqrt{d + 1}}\).

We consider a state \(\rho\) and measure its overlap with the state \(|\psi_+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}\) and \(|\phi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}\). We consider the value to be \(p_+\) and \(p_{1+}\) respectively. We can also measure the overlap with the state \(|0\rangle\) and \(|1\rangle\), which we consider to be \(p_0\) and \(p_1\). We define

\[
a_{01} = \text{Tr}(|\psi_+\rangle\langle 1|) \quad \text{and} \quad a_{10} = \text{Tr}(|\phi^+\rangle\langle 0|).
\]

One can easily show that \(p_+ = \frac{1}{2}(p_0 + p_1 + a_{01} + a_{10})\) and \(p_{1+} = \frac{1}{2}(p_0 + p_1 - ia_{01} + ia_{10})\).

Since, we know the values of \(p_+, p_{1+}, p_0\) and \(p_1\) from the measurements of overlaps with the corresponding states,
we should be able to calculate the values of $a_{01} = a_{10}^*$ and $p_{01}$ by solving the two equations.

**Separable state under symmetric local hidden state model**—We consider the transition probability of a bipartite, two-qubit state from $|0\rangle$ to $|1\rangle$ on Alice and as well as Bob’s side as $p(|z^{0-1}\rangle^A,|z^{0-1}\rangle^B)$, where $|z^{0}\rangle$ or $|z^{1}\rangle$ stands for the states $|0\rangle$ or $|1\rangle$ respectively. For a bipartite state $\rho_{AB}$, the transition probability is given by

$$p(|z^{0-1}\rangle^A,|z^{0-1}\rangle^B) = \langle 00|\rho_{AB}|11\rangle \langle 11|\rho_{AB}|00\rangle = \sum_{\lambda,\nu} p_\lambda p_\nu \rho^{A}_{01}(\lambda) \rho^{B}_{01}(\lambda) \rho^{A}_{01}(\nu) \rho^{B}_{01}(\nu) \leq \sum_{\lambda} p_{\lambda} \rho^{A}_{01}(\lambda)^2 \sum_{\nu} p_{\nu} \rho^{B}_{01}(\nu)^2 \leq \frac{1}{16}. \quad (12)$$

Similarly, two terms like $p(|z^{0-1}\rangle^A,|z^{0-1}\rangle^B)$ and $p(|z^{1-0}\rangle^A,|z^{1-0}\rangle^B)$ can at most add up to $\frac{1}{8}$ and cannot go beyond that value. The arguments here can be extended even for the general bipartite separable states. Therefore, separable states cannot violate the inequality given in Eq. $(12)$. Does the same hold for existing asymmetric LHS model as well? We show that the answer to this question to be affirmative.

To prove it, we start with the steering game, where Alice prepares a two-qubit bipartite state. She sends one part of it to Bob and keeps another part with her. She claims that his state is entangled with her and she can steer his state. Bob does not believe Alice. He asks Alice to perform measurements in certain bases. After Alice’s measurements in each basis, Bob measures certain quantities on his system and he claims that the (conditional) states can in principle be explained by hidden state model as laid down in Eq. $(1)$. The overall state of Alice and Bob can be written as (Bob’s assumption)

$$\rho_{AB} = \sum_{a,\theta} p(a,\theta)|a\rangle\langle a| \otimes p_a|\Pi(\theta)\rangle. \quad (13)$$

Now on this state, if Alice and Bob really performs the measurements as prescribed in the article, one can easily show following the previous arguments for separable states that the state cannot violate the inequality based on the symmetric hidden state model.

**Resource theory**—We put forward a resource theory of this new kind of nonlocality in this section. Just like other resources [6, 27, 28] in quantum information theory, the first element we need to dig out is the free operations, i.e., operations under which a state belonging to SLHS model remains in SLHS. Here we show that like entanglement or Bell nonlocality [12, 29, 30], this new kind of nonlocality also cannot be created by local completely positive trace preserving operations (LCPTP) i.e.,

**Theorem 1.** A state $\rho_{AB}$ belonging to the SLHS model remains in the SLHS model under LCPTP operations.

We lay down the proof of the theorem in the supplemental material [13].

Once the free operations are found, the next logical step is to introduce an axiomatic approach to define the measures of the nonlocality. Finding the measure of a resource is a difficult task and even more difficult without knowing the conditions, it must satisfy. In this regard, we lay down a set of axioms, which must be satisfied by the measure $N$ of the new form of nonlocality $- (a) N(\rho_{AB}) = 0 \forall \rho_{AB} \in SLHS$, (b) $\sum_k p_k N(L_k(\rho_{AB})) \leq N(\rho_{AB}) \forall L_k \in LCPTP$, such that $L_k(\cdot) = r_k^A \otimes r_k^B(\cdot) r_k^A \otimes r_k^B(\cdot)$, where $p_k = \text{Tr}(r_k^A \otimes r_k^B(\cdot) r_k^A \otimes r_k^B(\cdot))$ and bipartite states $\rho_{AB}$. Additionally, it must not increase under classical mixing of bipartite states $\rho_{AB}$, i.e., (c) $N(\rho_{AB}) \leq \sum_i p_i N(\rho_{AB}^i)$, such that $\rho_{AB} = \sum_i p_i \rho_{AB}^i$ and $\sum_i p_i = 1$.

In the supplemental material, we show that $|\langle ab|\rho_{AB}|cd\rangle|$ satisfies the axioms (b) and (c) for any bipartite state $\rho_{AB}$. By some numerical adjustments, we turn the quantity into a measure of the new form of nonlocality.

**Conclusion**—We start with the assumptions of EPR locality as laid down in [2] and taking a bit of inspiration from the idea of local hidden variable theory, extend the existing asymmetric LHS model to provide the symmetric local hidden state model. A new form of nonlocality emerges from the model. We consider the transition probabilities of states to provide a nontrivial bound on the symmetric EPR nonlocality. We show that no separable state can violate the inequalities given in the article. We also show that a state with SLHS description remains in the SLHS under the LCPTP operations. This new version of the local hidden state model reveals more nonlocality than the existing LHS or LHV.
models. Moreover, the new formalism is trust-less just like the LHV model. Therefore, it provides a better and more efficient alternative to the Bell inequalities for the self-testing and other device-independent protocols [9–11], a field which needs further investigation.

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* cqtdem@nus.edu.sg
† phykd@nus.edu.sg

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Supplemental Material

STRONGER SLHS MODEL

A stronger SLHS model would be just to express the complex off-diagonal terms corresponding to the transition probabilities of a bipartite state in terms of the complex off-diagonal terms of hidden states,

$$\langle i^a i^b | \rho_{AB} | i'^a i'^b \rangle = \sum_{\lambda} p(\lambda) \langle i^a | \rho^A_{\lambda} | i'^a \rangle \langle i^b | \rho^B_{\lambda} | i'^b \rangle,$$

(14)

where $\rho^A_{\lambda}$ and $\rho^B_{\lambda}$ are hidden states of Alice and Bob respectively with a distribution $p(\lambda)$ prepared by the unknown source to cheat them.

The inequality in Eq. (14) is stronger than the Eq. (4) in the main text is due to the fact that the former implies the later but not the other way around. To prove it, we start with,

$$|\langle i^a i^b | \rho_{AB} | i'^a i'^b \rangle| \leq \sum_{\lambda} p(\lambda) |\langle i^a | \rho^A_{\lambda} | i'^a \rangle| |\langle i^b | \rho^B_{\lambda} | i'^b \rangle|,$$

where we use the Eq. (14) and the triangle inequality.

Now, using the inequality, we get

$$|\langle i^a i^b | \rho_{AB} | i'^a i'^b \rangle|^2 \leq \sum_{\lambda} p(\lambda) |\langle i^a | \rho^A_{\lambda} | i'^a \rangle|^2 |\langle i^b | \rho^B_{\lambda} | i'^b \rangle|^2,$$

where we use the fact that the second moment is greater than the square of the first moment of a random variable $\lambda$ for the distribution $p(\lambda)$. The inequality above implies that there must exist a set of hidden states $\sigma^A_{\lambda}$ and $\sigma^B_{\lambda}$ with a distribution $p'(\lambda)$ such that

$$|\langle i^a i^b | \rho_{AB} | i'^a i'^b \rangle|^2 = \sum_{\lambda} p'(\lambda) |\langle i^a | \sigma^A_{\lambda} | i'^a \rangle|^2 |\langle i^b | \sigma^B_{\lambda} | i'^b \rangle|^2,$$

which is nothing but the Eq. (4).

PROOF OF THEOREM 1

We consider a state $\rho_{AB}$ belongs to the set of SLHS states, such that

$$\text{Tr}(r^A_k \otimes r^B_k \rho_{AB} r^A_{k}^\dagger \otimes r^B_k^\dagger),$$

such that $\sum_k p_k = 1$ and $\sum_k r^A_k r^A_{k}^\dagger \otimes r^B_k r^B_k^\dagger \leq I$ for both the systems. Here, we show that the state $\rho_{AB}$ also belongs to the set of SLHS states. We start with the quantity $|\langle aa | \rho_{AB} | aa' \rangle|^2$ such that $a \neq a'$

for any basis labeled by $i$ and a particular vector from the basis labeled by $a$, such that $a \neq a'$. The state $\rho_{AB}$ under local CPTP operation, transforms to the state $\rho'_{AB}$, such that $\rho'_{AB} = L(\rho_{AB}) = \sum_k r^A_k \otimes r^B_k \rho_{AB} r^A_{k}^\dagger \otimes r^B_k^\dagger$. We define $L_k(\rho_{AB}) = Tr(r^A_k \otimes r^B_k \rho_{AB} r^A_{k}^\dagger \otimes r^B_k^\dagger)$, $p_k = \sum_{i,a} |\langle i^a i^a | \rho_{AB} | i'^a i'^a \rangle|^2$ $\text{SLHS} \equiv \sum_{\lambda} p(\lambda) |\langle i^a | \rho^A_{\lambda} | i'^a \rangle|^2 |\langle i^b | \rho^B_{\lambda} | i'^b \rangle|^2,$

(15)

$$\langle i^a i^a | \rho'_{AB} | i'^a i'^a \rangle = \sum_k \langle i^a i^a | r^A_k \otimes r^B_k \rho_{AB} r^A_{k}^\dagger \otimes r^B_k^\dagger | i'^a i'^a \rangle = \sum_k \text{Tr}(r^A_k \otimes r^B_k \rho_{AB} r^A_{k}^\dagger \otimes r^B_k^\dagger | i'^a i'^a \rangle \langle i^a i^a \rangle )$$

$$= \sum_k \text{Tr}(\rho_{AB} A_k \otimes B_k) = \sum_{k,m,n,p,q} \langle mn | \rho_{AB} | pq \rangle \langle pq | A_k \otimes B_k | mn \rangle$$

$$= \sum_{k,m,n,p,q,\lambda} \langle p | A_k | m \rangle \langle q | B_k | n \rangle p(\lambda) \langle m | \rho^A_{\lambda} | p \rangle \langle n | \rho^B_{\lambda} | q \rangle = \sum_{k,p,q,\lambda} p(\lambda) \langle p | A_k \rho^A_{\lambda} | p \rangle \langle q | B_k \rho^B_{\lambda} | q \rangle$$

$$= \sum_{k,\lambda} p(\lambda) |\langle i^a | r^A_k \rho^A_{\lambda} r^A_{k}^\dagger | i'^a \rangle|^2 |\langle i^b | r^B_k \rho^B_{\lambda} r^B_{k}^\dagger | i'^b \rangle|^2$$

(16)

This proves the stronger version of our SLHS model. To prove the weaker version, we proceed further with the
following,

\[
\langle i^a | r^A_{AB} \rho^A_{AB} | i^{a'} \rangle = \sum_{k, \lambda} p(\lambda) \langle i^a | r^A_k \rho^A_k \lambda r^A_k | i^{a'} \rangle \left| \langle i^a | r^A_k \rho^A_k | i^{a'} \rangle \right|^2 \leq \sum_{k, \lambda} p(\lambda) \langle i^a | r^A_k \rho^A_k \lambda r^A_k | i^{a'} \rangle \left| \langle i^a | r^A_k \rho^A_k \lambda r^A_k | i^{a'} \rangle \right|^2
\]

\[
\left( \sum_{k, \lambda} p(\lambda) \left| \langle i^a | r^A_k \rho^A_k \lambda r^A_k | i^{a'} \rangle \right|^2 \right)^{\frac{1}{2}} \leq \sum_{k, \lambda} p(\lambda) \left| \langle i^a | r^A_k \rho^A_k \lambda r^A_k | i^{a'} \rangle \right|^2 \leq \sum_{k, \lambda} p(\lambda) \left| \langle i^a | r^A_k \rho^A_k \lambda r^A_k | i^{a'} \rangle \right|^2
\]

(17)

which implies that the state \( \rho^A_{AB} \) also belongs to the set of SLHS. In the equation above in Eq. (16), in the second line, we use the fact that \( r^A_k | i^{a'} \rangle \langle i^a | r^A_k \rangle = A_k(B_k) \), in the third line, we use the fact that the state \( \rho^A_{AB} \) is in the set of SLHS, i.e., the Eq. (14). In the first inequality in Eq. (17), we use the triangle inequality for complex numbers and in the last inequality in Eq. (15), we use the fact that the second moment of \( \left| \langle i^a | r^A_k \rho^A_k \lambda r^A_k | i^{a'} \rangle \right|^2 \) is greater than the square of the first moment over the distribution \( p(\lambda) \).

**PROOF OF AXIOMS.**

We consider

\[
\max \left\{ \max_{a \neq a', b \neq b'} \left| \sum_{i} p_{i} | i^{a'} \rangle \langle i^a | \sum_{i} p_{i} | i^{a'} \rangle \right|, \frac{1}{\pi}, 0 \right\} = \pi = N(\rho_{AB})
\]

\[
\sum_{k} p_k |(ab)L^AB_k(\rho_{AB})|cd| \leq \sum_{n,m,p,q,k} |\langle mn|\rho_{AB}|pq\rangle| |\langle pq|A_k \otimes B_k|mn\rangle| |\langle pq|A_k \otimes B_k|mn\rangle|
\]

(19)

where in the inequality, we use the triangle inequality. Now, we start with the quantity \( |\langle p|A_k|m\rangle| \) as

\[
\sum_{k} |\langle p|A_k|m\rangle| = \sum_{k} |\langle p|A_k^A|c\rangle| |\langle c|A_k^A|m\rangle| \leq \sum_{k} |\langle p|A_k^A|c\rangle| |\langle c|A_k^A|m\rangle|
\]

\[
= \sum_{k} |\langle r^A_k |cp|A_k^A |m\rangle| \leq \sum_{k} |\langle r^A_k |cp|A_k^A |m\rangle| \leq \sqrt{\sum_{k} |\langle r^A_k |cp|A_k^A |m\rangle|^2}
\]

(20)

where the first term is zero due to the fact that \( \sum_{k} r^A_k r^A_k = I_d \) and off-diagonal terms of an identity matrix are all zero. We start with the second term as

\[
\sum_{k} |\langle r^A_k |cp|A_k^A |m\rangle| \leq \sum_{k} |\langle r^A_k |cp|A_k^A |m\rangle| \leq \sqrt{\sum_{k} |\langle r^A_k |cp|A_k^A |m\rangle|^2}
\]

(21)

Now, we start with the first term under the square-root, i.e.,

\[
\sum_{k} |\langle r^A_k |cp|A_k^A |m\rangle| \leq \sum_{k} |\langle r^A_k |cp|A_k^A |m\rangle| \leq \sqrt{\sum_{k} |\langle r^A_k |cp|A_k^A |m\rangle|^2}
\]

(22)
We know $\text{Tr}(\sum_k r_k^A\rho_0 r_k^{A\dagger}) = 1$ and $\text{Tr}(r_k^A\rho_0 r_k^{A\dagger}) = p_k$ such that $\sum_k p_k = 1$ for any arbitrary state $\rho_0$. We consider $\rho_0 = |c\rangle\langle c|$. Therefore, we get

$$p_k = \text{Tr}(r_k^A\rho_0 r_k^{A\dagger}) = \sum_m \langle m|r_k^A|c\rangle\langle c|r_k^{A\dagger}|m\rangle$$

or,

$$\langle m|r_k^A|c\rangle\langle c|r_k^{A\dagger}|m\rangle \leq p_k,$$

(23)

where $|m\rangle$ is an arbitrary state and forms a complete basis and the inequality comes from the fact that each term within the summation is positive. Now, we start with the same quantity again to show that

$$p_k = \sum_m \langle m|r_k^A|c\rangle\langle c|r_k^{A\dagger}|m\rangle$$

$$= \sum_m \langle m|r_k^A r_k^A|c\rangle\langle c|m\rangle = \langle c|r_k^{A\dagger} r_k^A|c\rangle.$$  

(24)

Thus, from Eq. (23) and (24), we get

$$|\langle m|r_k^A|c\rangle|^2 \leq p_k = \delta_{mc} \langle m|r_k^{A\dagger} r_k^A|c\rangle$$

(25)

for any states $|m\rangle$ and $|c\rangle$. Now, from Eq. (25) and (22), we get

$$\sum_k |[r_k^{A\dagger}]_{cd}|^2 \leq \delta_{cp}. \quad (26)$$

Using Eq. (19), (20), (21) and (26), we get

$$\sum_k p_k |\langle i^a | i^b |\rho_{AB} | i^c i^d \rangle| \leq |\langle i^a | i^b |\rho_{AB} | i^c i^d \rangle|,$$

(27)

which proves the axiom (b).

Now, we turn to the axiom (c). We consider a bipartite state $\rho_{AB}$ such that $\rho_{AB} = \sum_i p_i \rho_{iAB}$, where $\sum_i p_i = 1$. We start with the following

$$|\langle ab | \sum_i p_i \rho_{iAB} | cd \rangle| = |\sum_i p_i \langle ab | \rho_{iAB} | cd \rangle|$$

$$\leq \sum_i p_i |\langle ab | \rho_{iAB} | cd \rangle| = \sum_i p_i |\langle ab | \rho_{i} | cd \rangle|. \quad (28)$$

Thus, we prove that $N(\rho_{AB}) \leq \sum_i p_i N(\rho_{iAB})$. 

