NONCOMMUTATIVE SPACETIME AND QUANTUM MECHANICS

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April 1, 2022

Abstract

In this paper we will analyze the status of gauge freedom in quantum mechanics (QM) and quantum field theory (QFT). Along with this analysis comparison with ordinary QFT will be given. We will show how the gauge freedom problem is connected with the spacetime coordinates status — the very point at which the difficulties of QM begin. A natural solution of the above mentioned problem will be proposed in which we give a slightly more general form of QM and QFT (in comparison to the ordinary QFT) with noncommutative structure of spacetime playing fundamental role in it. We achieve it by reinterpretation of the Bohr’s complementarity principle on the one hand and by incorporation of our gauge freedom analysis on the other. We will present a generalization of the Bargmann’s theory of exponents of ray representations. It will be given an example involving time dependent gauge freedom describing non-relativistic quantum particle in nonrelativistic gravitational field. In this example we infer the most general Schrödinger equation and prove equality of the (passive) inertial and the gravitational masses of quantum particle.

1 General Introduction

Probably it will be helpful to give a brief outline of this paper providing the general line of reasoning. For details, however, the reader must consult the foregoing sections.

Of late there has been proposed a reformulation of the standard QM (J. Wawrzycki, [math-ph/0301005; Comm. Math. Phys., to appear]), which is slightly more general in comparison to the ordinary form of the theory. This reformulation has emerged in a natural way in description of a quantum particle in the non-relativistic gravitational field with time dependent gauge freedom. Remember, please, that the
states of a physical system do not correspond bi-uniquely to unit vectors $\phi$ of the respective Hilbert space $\mathcal{H}$ but to the rays, say $\phi = \{e^{i\xi}\phi\}$ in the QM and QFT, where $\xi$ is an arbitrary real number.\footnote{Let us recall that all relevant information carried by $\phi$ is contained in the set of numbers}

$$\frac{|(\phi, \varphi)|^2}{(\phi, \phi)(\varphi, \varphi)},$$

where $(\phi, \varphi)$ is the scalar product of the Hilbert space $\mathcal{H}$. As such $\phi$ and $e^{i\xi}\phi$ are equivalent containing exactly the same information.

Observe now, please, that in ordinary (non-relativistic) QM, when using Schrödinger picture, one can go considerably further with this observation. Namely, two Schrödinger waves $\psi$ and $e^{i\xi(t)}\psi$ are indistinguishable even when $\xi$ depends on time, but one have to assume simultaneously that Schrödinger wave equation possess a time dependent gauge freedom. Let us recall that the integral defining the scalar product is over the space coordinates and one can take a time dependent factor over the integral sign. After this, however, the Schrödinger wave functions should constitute the appropriate cross sections of a Hilbert bundle $\mathbb{R}\triangledown\mathcal{H}$ over time $\mathbb{R}$. The representations $T_r$ of a covariance as well as a symmetry groups act in $\mathbb{R}\triangledown\mathcal{H}$ and their exponents $\xi$ in the formula

$$T_rT_s = e^{i\xi(r,s,t)}T_{rs},$$

do depend on time $t \in \mathbb{R}$. Thus, at first sight the natural assumption that two Schrödinger waves differing by a time dependent phase are equivalent leads to a rather strange construction, namely, the Hilbert bundle — an object much more involved then the Hilbert space itself. One can prove, however, that in the non-relativistic Galilean invariant theory, this more general assumption leads exactly to the ordinary QM. The whole structure degenerates due to the specific structure of the Galilean group. Moreover, in the less trivial case of a quantum particle in non-relativistic gravitational field, when the time dependent gauge freedom is indispensable, the results are quite interesting. Namely, one can infer the most general wave equation for that particle and prove equality of the inertial and gravitational mass, the results confirmed by experiments! In the last case the Milne group plays the role of the Galilean group.

This non-relativistic generalization possesses also a very natural relativistic extension which can be incorporated within QFT rather then QM. In QFT one can still go a step further along with this line of generalizing the quantum mechanical principles. Remember, please, that in the Fock construction the Fourier components of the classical field constitute the arguments of the wave function. Anyway the arguments have nothing to do with ordinary spacetime coordinates. In other words the spacetime coordinates are mere parameters or the so-called c-numbers in Heisenberg canonical field quantization — just like the time in ordinary non-relativistic QM. One should, thus, assume the two wave functions to be equivalent whenever they differ by a spacetime dependent wave function. But, when treating...
this assumption seriously, the wave functions should constitute the appropriate cross sections of a Hilbert bundle $\mathcal{M}\Delta\mathcal{H}$ over spacetime $\mathcal{M}$. Accordingly the representations $T_r$ of covariance or symmetry groups possess spacetime dependent exponents $\xi = \xi(r, s, p)$:

$$T_r T_s = e^{i\xi(r, s, p)} T_{rs},$$

with $p \in \mathcal{M}$ (J. Wawrzycki, [math-ph/0301005]; Comm. Math. Phys., to appear).

In particular one is forced to extend the ordinary classification theory of exponents $\xi$ of representations acting in ordinary Hilbert space so as to embrace the above case of representation acting in a Hilbert bundle with spacetime dependent $\xi$. It can be viewed as a generalization of the Bargmann’s theory (V. Bargmann, Ann. Math 54, 1, 1954) of exponents of ray representations acting in ordinary Hilbert space with spacetime-independent $\xi$.

The fact that the simpler form of the theory with time dependent gauge freedom gives the correct form of the Schrödinger equation thought the author to treat seriously also the generalization with spacetime dependent gauge freedom. The natural question emerges if one can find a simple connection of our generalization (those with spacetime dependent gauge freedom) to the ordinary QFT, for example to QED and if the connection is so transparent as in the case of time dependent gauge freedom. At this place one have to note that in particular any representation $T_r$ with spacetime dependent exponent $\xi = \xi(r, s, p)$ in (1) makes any sense if the representation of the algebra of Canonical Commutation Relations (CCR) is reducible and does possesses a nontrivial center. The diagonal algebra over which the above representation of CCR algebra can be decomposed into irreducible components corresponds to the classical spacetime. How one can interpret this strange result along with the ordinary QFT in which, according to “Wightman’s axioms” (A. S. Wightman and L. Gårding, “Fields as Operator Valued Distributions in Relativistic Quantum” Theory, Ark. Fys. 23, No. 13, 1964), the algebra generated by quantum field operators (“smeared with appropriate test functions”) is irreducible?

Answer to this question is by no means trivial. In particular the situation is much less trivial then in the case of the theory with time dependent gauge freedom in non-relativistic QM. In our opinion one is forced to reconsider the fundamental principles of QM and QFT in answering the question. Of late the long-lasting dispute concerning the interpretation of QM is at its renaissance again. We will not go into detail of this dispute but we feel that something is missing in the standard picture and at least some points of the criticism are justified. We assume that the standard QM describes correctly the situation in which we have an atomic system measured with a macroscopic apparatus. In the standard QM the Bohr-Heisenberg cut between the system and the apparatus may range between them, and it is irrelevant how big is the system. It is important only that the cut is “placed” somewhere between the system and the apparatus (in this sense the Bohr-Heisenberg cut does not form any “normal” real physical boundary). This unavoidably implies that a macroscopic body can be in a (quantum) superposition of states with macroscop-
ically different parameters like the center-of-mass position. On the other hand, so long as “big” and “small” are merely relative concepts, it is no help to one who wishes to account for the ultimate structure of matter. We agree with Dirac, that QM should be endowed with ideas in such a way as to give an absolute meaning to the words “small” and “big”. In the common opinion QM is the theory which incorporates the idea of absolute scale of action — the Planck’s quantum of action $\hbar$. Paradoxically, in ordinary QM the only idea, namely that of Bohr-Heisenberg cut, which separates the “small” from the “big” is purely relative such that we lose the possibility of introducing the absolute scale mentioned above. This is the element which in our opinion is missing in the standard interpretation of QM. Thus, we assume that QM with the standard interpretation works good but only within some limits, where the observed system involves only a few quantum particles whereas the apparatus constitute a macroscopic body consisting of an enormous number of quantum particles. But then the QFT is the appropriate scheme within which the systems consisting of many particles are naturally incorporated. Therefore, one has to be careful in applying the QM principles in QFT when the number of particles is too big. The application is justified only if one works within the neighborhood of the vacuum state — which does actually take place in practice e.g. when evaluating the Lamb shift or the anomalous magnetic moment. For the states far removed from the vacuum one has to preserve an open mind on the ordinary QM principles. We maintain that our generalization of QFT is applicable if the number of particles in the system is large, just in opposite to the ordinary QFT applicable when the number is small. Indeed. The “Fock space” $\mathcal{H}_F$ is the direct sum of all $N$-particle spaces $\mathcal{H}_N = \mathcal{H}_1^\otimes N$ (resp. $\mathcal{H}_N = \mathcal{H}_1^\otimes A_N$) i.e. the symmetrized $N$-fold products of the one-particle Hilbert space $\mathcal{H}_1$ (resp. anti-symmetrized $N$-fold products):

$$\mathcal{H}_F = \bigoplus_{N=0}^\infty \mathcal{H}_N.$$ 

A state vector $\psi$ in $\mathcal{H}_F$ is an infinite hierarchy of symmetric (resp. anti-symmetric) wave functions

$$\psi = \begin{pmatrix}
    c \\
    \psi_1(x) \\
    \psi_2(x_1, x_2) \\
    \vdots
\end{pmatrix}$$

where we have written the argument $x$ to denote both position and spin component. $\psi_N$ is the probability amplitude for finding just $N$ particles and those in specified configuration. The complex number $c$ is the probability amplitude to find the vacuum. It is important to note that the argument $x$ in one particle state $\psi_1$ contains the space coordinates whereas $x_1, \ldots, x_N$ in the $N$-particle state have nothing to do with spacetime coordinates being rather the configuration coordinates. In the case of the free field when the number of particles $N$ is conserved our argument that $\psi$ and $e^{i\xi(p)}\psi$ are equivalent is valid (provided $N > 1$). Remember, please, that $p$ stands for spacetime point. In this case the probability amplitude to find a number
of particles different from $N$ is zero (in particular $\psi_1 = 0$). However, relativistic QFT with interaction is a theory describing the phenomenon of creation and annihilation of particles as indispensable effects so as the total number of particles cannot be conserved. Thus, one cannot exclude the amplitude $\psi_1$ when working around the vacuum state. This one-particle amplitude is negligible only if the total number of particles is large. In this case, therefore $|\langle e^{i\xi(p)}\psi, \psi' \rangle| = |e^{i\xi(p)}(\psi, \psi')| = |(\psi, \psi')|$ so that $\psi$ and $e^{i\xi(p)}\psi$ are equivalent; please, compare the first two footnotes — one can take the factor $e^{i\xi(p)}$ over the argument of the inner product in this case. This shows that our generalization becomes adequate when the system consists of an appropriately large number of particles, just in the situation when the ordinary QFT is expected to be somewhat misleading.

Yet the (more) ultimate general theory, the scientist should intend to find, is expected to be adequate to account for systems of a few particles as well as systems consisting of an enormous number of them. We are certain that the ordinary QFT laws\(^3\) works perfectly in these cases of small values $N$ of particles in the system. Should it be possible at all to obtain a theory which can give correct answers also in the cases of small $N$-values? In the cases of small $N$ the ordinary QFT could act as the code-book — namely, one should pick up those general laws or theorems of QFT which can be formulated in terms independent of the actual number of particles. As such theorem which should guide us in further research we take the miracle that: 1) Canonical quantization of a free scalar field leads to Fock space and an interpretation of states in terms of particle configurations, so it explains the wave-particle duality lying at the roots of QM. This theorem contains the general kinematical information that the states of quantized field serve as configuration space for quantum particles which may occupy them. As such this fact does not depend directly on the actual value of $N$. On the other hand, when the number $N$ of particles is large the classical theory is correct and may serve as a guide in this regime exactly as QFT does when $N$ is small, provided the laws we pick up to guide us do not depend on the number of degrees of freedom\(^4\), especially they should not depend on the fact if there is infinite number of them as in classical field or finite as e.g. for a rigid classical body. As the second guiding theorem we take the following theorem: 2) The configuration space for the classical body consists

\(^2\)Probably it will be helpful to recall the formula for the scalar product in $\mathcal{H}_F$

\[
(\psi, \psi') = c^* c' + \int_{\mathbb{R}^3} \psi_1^* \psi_1' \, d^3x + \int_{\mathbb{R}^2 \times \mathbb{R}^3} \psi_2^* \psi_2' \, d^3x_1 d^3x_2 + \ldots
\]

to see the equivalence of $\psi$ and $e^{i\xi(p)}\psi$, compare the preceding footnote.

\(^3\)The term “ordinary” means here the ordinary QM applied to system with infinite number of degrees of freedom.

\(^4\)Remember, please, that the number of particles $N$ in quantum theory corresponds to to the number of degrees of freedom in classical theory, where the degree of freedom is in the sense of the Lagrange-Hamilton theory. The laws serving us as guiding receipt should not depend on $N$ as we have mentioned earlier. Therefore, those laws — if taken from classical theory — should not depend on the number of Lagrange degrees of freedom.
of various space\textsuperscript{5} positions the body or its parts may occupy — a fact which seems to be independent of the actual number of degrees of freedom involved, i.e. independent of the number of independent parts. Next, we should observe that the algebraic formulation of QM and QFT is an approach in which the Hilbert space plays a secondary role and by this fact the number of particles $N$ as well as the number of degrees of freedom plays a very indirect role in it. Thus, the approach is the one we should work with in our research. Comparing now our guiding theorems 1) and 2) one can infer that the (noncommutative) quantum algebra of quantum field operators\textsuperscript{6} plays the role of noncommutative space for quantum particles, just like the classical space (i.e. space-like surface of spacetime) does for a classical body. In this way we arrive at the result that the space the quantum particles live in is a noncommutative space (in the sense of A. Connes, Noncommutative Geometry, Acad. Press 1994) with the noncommutative algebra of quantum field operators corresponding to the noncommutative space.

Still, it would be much better if we were able to find the counterpart of the whole classical spacetime and not only the counterpart of space.\textsuperscript{7} For this purpose let us note that the points of classical spacetime go into play when considering the time evolution of space position of the classical body. \textit{Per analogiam}, if we are to have any hope to find the quantum counterpart of spacetime we must use the quantum substitute for the classical space position evolution. The state in the Fock space $\mathcal{H}_F$ in the Heisenberg picture (which we have used above) does not contain any information about time evolution. One could suppose that the simple replacement of the Heisenberg picture with the Schrödinger picture immediately resolves the problem. Unfortunately this is by no means the case. One has to recall that we still intend to find some general principles valid in general irrespectively if the number $N$ of particles is large or not. Unfortunately the standard connection between the two pictures Schrödinger’s and Heisenberg’s fails down when $N$ is large in general. Remember, please, that when $N$ is large the gauge freedom problem mentioned above goes into play: the two waves $\psi$ and $e^{i\xi(p)}\psi$ are to be considered equivalent. One can prove that instead of normal Schrödinger picture with ordinary Schrödinger waves (Hilbert space vectors parametrically dependent on time) one has to deal with cross sections of an appropriate Hilbert bundle $M\triangle\mathcal{H}$ over whole spacetime $\mathcal{M}$. But, when using the Hilbert bundle $M\triangle\mathcal{H}$ instead of $\mathcal{H}$, the respective algebra of quantum operators acting in $M\triangle\mathcal{H}$ has to be reducible, compare a subsequent section where we give arguments for this fact. The spectrum of the center of this algebra is just equal to the classical spacetime $\mathcal{M}$. According to the above discussion

\textsuperscript{5}Here “space” stands for an appropriate space-like Cauchy section of (classical) spacetime.
\textsuperscript{6}More precisely we have to construct first the specific representation of the algebra with the vacuum plying the role of the cyclic vector of the representation. Then, the the states of the Hilbert space of the representation constitute the analogue of classical space positions.
\textsuperscript{7}The notion of a noncommutative space at a particular classical time is rather a strange mixture of classical and noncommutative properties which cannot serve as a generally valid universal concept.
this algebra corresponds to the noncommutative spacetime — an object plying the
same role for quantum particles as the classical spacetime does for classical bodies.

In this way we have revealed the wave-particle duality as a manifestation of
the noncommutative structure of spacetime. The noncommutative algebra $\mathcal{A}$ corre-
sponding to the noncommutative spacetime is obtained as the smallest von Neumann
algebra $\mathcal{A}_{CCR} \lor \mathcal{A}_{CCR}^1$ containing the von Neumann canonical commutation algebra
$\mathcal{A}_{CCR}$ of field operators and the appropriate maximally Abelian subalgebra $\mathcal{A}_{CCR}^1$ in
the commutant of $\mathcal{A}_{CCR}$. The center $\mathcal{A}_{CCR}^1$ (diagonal algebra) of $\mathcal{A}$ should not
be trivial and corresponds to the classical spacetime.

Apparently, one can formulate a serious objection against our conclusion that
quantum particle lives in a spacetime with a noncommutative structure, with the
structure closely related to the algebra of quantum operators of the field correspon-
ding to the particle. Apparently one can say that each type of particle lives in its
own spacetime related to the corresponding type of field — which is a very strange
idea rather. We maintain that the strange effect of many coexisting spacetimes is
apparent. Indeed. One should note at this place that the representation of the
algebra of quantum field operators $\mathcal{A}_{CCR}$ we are interested in should be reducible 8.
As such the algebra cannot act in the ordinary Fock space. The space is no longer a
direct sum of symmetrized (resp. anti-symmetrized) tensor products of one-particle
Hilbert spaces, in which the tensor product of $N$ factors occurs once and only once
for each natural $N$. This structure of Hilbert space is disturbed in our case. Let us
recall that the symmetrization (resp. anti-symmetrization) is deeply connected with
the irreducibility of the quantum algebra (H. Weyl, Gruppentheorie und Quanten-
mechanik, Leipzig, Verlag von S. Hirzel, 1931). In our case the representation
is reducible so that the symmetrized and anti-symmetrized products may appear
within the same representation space! In the standard theory if the system is in sym-
metrized state it remains symmetrized forever, regardless of what influences may act
upon it — bosons and fermions do not mix each other. In our case (in the limit of
large $N$) the situation is substantially different — the symmetric and anti-symmetric
states do mix each other. Therefore, the algebra $\mathcal{A}$ is much more universal object in
comparison to any particular operator algebra of any specified kind of field in
standard QFT. It, therefore, seems to be capable as to account for different kinds
of particles. Moreover, it is an advantage that in the space of the representation
of our algebra the symmetrization (resp. anti-symmetrization) is disturbed. Re-
member, please, that the symmetrization (resp. anti-symmetrization) reflects the
the Bose-Einstein (resp. Fermi-Dirac) correlations or the so-called entanglement of
quantum states. Thus, in our case the entanglement of states is disturbed which

8Strictly speaking the matter is even more involved but we do not go into detail now in order
to avoid excessive mathematical complexities. We have assumed implicitly that the spacetime is
compact so that the wave functions — the respective cross sections of the Hilbert bundle $M_\Delta \Delta \mathcal{H}$ —
do belong to the so-called direct integral Hilbert space $\int_M \mathcal{H}_p \, d\mu(p), \ p \in M$; compare the respective
section of this Chapter. After this one can think of $\mathcal{A}_{CCR}^1$ and $\mathcal{A}$ as of acting in this direct integral
Hilbert space.
may correspond to the fact that in practice for macroscopic bodies (when \( N \) is large) the entanglement is negligible. It should be stressed that this fact is very difficult to explain within the ordinary QM and QFT.

Some other arguments concerning the apparent problem of many coexisting spacetimes corresponding to the various kinds of fields and particles seem advisable. The generalization of the wave-particle duality to a field-particle duality has dominated thinking in quantum theory for decades and has been heuristically useful in the development of elementary particle theory. Yet the belief in field-particle duality as a general principle, the idea that to each particle there is a corresponding field and to each field a corresponding particle has also been misleading. The number and the nature of different basic fields is related to the charge structure, not to the empirical spectrum of particles. For example, in the presently favored gauge theories the fields are the carriers of charges like colour and flavour but are not directly associated to observed particles like electrons. The biunique field-particle correspondence is therefore broken. Moreover, the spectacular issue of Connes and Lott (\textit{Nucl. Phys.} \textbf{18B}, 29, 1990, see also \textit{A. Connes, loc. cit.}) that the so much a long list of fields in the effective standard model can be considerably reduced with the cost of some noncommutative structure of spacetime involved. They considered the effective standard model, i.e. their analysis was confined to the classical context. The Lagrangean of this effective theory is a combination of a five terms representing to independent contributions. Connes and Lott showed that this artificially complicated Lagrangean is a natural generalization of the Maxwell-Dirac Lagrangean providing the appropriate noncommutative structure of spacetime.

Yet one has to be careful, however, and treat the specific structure of \( A \) mentioned above as a prototype rather then as the ultimate word one can say on this subject.

The fact that the concepts of “spacetime coincidence” and “observation” do require a thorough revision in QM was immediately noticed by the very founders of QM. It was especially evident in Bohr’s writings. In the year 1925 he described the situation in the words: ”From these results it seems to follow that, in the general problem of the quantum theory, one is faced not with a modification of the mechanical and electrodynamical theories describable in terms of the usual physical concepts, but with an essential failure of the pictures in space and time on which the description of natural phenomena has hitherto been based.” (\textit{Nature}, \textbf{116}, p. 535.) Three years later he formulated the complementarity principle. As emphasized by Einstein every observation or measurement ultimately rests on the coincidence of two independent events at the same spacetime point. Now the quantum postulate implies that any observation of atomic phenomena will involve an interaction with the agency of observation not to be neglected. On one hand – Bohr stresses – the definition of the state of a physical system, as ordinary understood, claims the
elimination of all external disturbances. But in that case, according to the quantum postulate, any observation will be impossible, and, above all, the concepts of space and time lose their immediate sense. On the other hand – he concludes – if in order to make observation possible we permit certain interactions with suitable agencies of measurement, not belonging to the system, an unambiguous definition of the state of the system is naturally no longer possible, and there can be no question of causality in the ordinary sense of the word (Heisenberg uncertainty principle). Concluding, we have the complementarity alternative: Either quantum particle is describable in terms of space and time but the state of particle is not well defined or the state of quantum particle is well defined but its description in terms of space and time impossible. We propose to interpret this complementarity principle as indicating that the spacetime event, in the classical sense of the word, cannot be ascribed to quantum particle according to our analysis presented above. The quantum particle lives in a noncommutative spacetime. The event in (classical) spacetime can be ascribed to a body which we would like to call classical – made of many quantum particles.

2 Gauge Symmetry

2.1 Gauge Freedom and Heisenberg Commutation Relations

Let us back to the ordinary QM applied to a system with finite number of degrees of freedom. To be consequent we have to restrict our consideration to non-relativistic case for the moment — please, remember that any relativistic quantum system has to constitute a quantum field system with infinite number of degrees of freedom. The quantum states of our system constitute a Hilbert space $\mathcal{H}$. The inner product $(\cdot, \cdot)$ of the space is an relevant part of structure plying important role in the physical interpretation. Namely, the only contribution of the the vector $\varphi$ of the Hilbert space $\mathcal{H}$ to any measurable effects is inscribed into the set of numbers

$$\frac{|\langle \phi, \varphi \rangle|^2}{(\phi, \phi)(\varphi, \varphi)}$$

$\phi \in \mathcal{H}$.

Suppose we have an ideal source which prepares an ensemble in a pure state, described by an element of $\mathcal{H}$. In particular if in addition we have an ideal detector, giving a yes-answer in a pure state $\varphi$ and the answer no in the orthogonal complement to $\phi$, then the probability of detecting an event in this set up of source and detector is given by

$$\frac{|\langle \phi, \varphi \rangle|^2}{(\phi, \phi)(\varphi, \varphi)}.$$

Therefore, two vectors $\varphi$ and $e^{i\xi}\varphi$ differing by a mere constant phase are indistinguishable. This is very important for the whole structure of quantum theory. For
example, any kind of algebra of some quantities, like the algebra of classical observ-
ables (functions $f(p, q)$ on the phase space with ordinary point wise operations) —
which is commutative — will not necessary be commutative if unitary represented
as operator algebra acting in the Hilbert space of states. Let us explain how it
follows from the constant-phase-equivalence of the two vectors $\varphi$ and $e^{i\xi}\varphi$, $\xi$
being a constant real number. Consider the set of invertible elements of the algebra
in question, which constitutes a commutative group in our case. Because of the
constant-phase-equivalence the two operators $A$ and $e^{i\xi}A$ are equivalent giving the
same average values and having the same spectra. Therefore in general the relation

$$AB = e^{i\xi(A,B)}BA$$

holds for representations acting in $\mathcal{H}$ instead of

$$AB = BA.$$

Such a representation is mostly called ray representation. Let us consider rep-
resentation $U_i$ and $V_i$ of the $2n$-dimensional Abelian group of phase coordinates
$(p_1, \ldots, p_n, q_1, \ldots, q_n)$ in the classical phase space. The algebra of classical phase
coordinates is generated by the one-parameter groups corresponding to $p_i$ and $q_i$
which should hold also for the quantum representation. If one assumes the unitary
representation to be irreducible then one gets the relations

$$U_iV_k = e^{i\xi(p_i, q_k)}V_kU_i,$$

$$U_iU_k = e^{i\xi(p_i, p_k)}U_kU_i,$$

$$V_iV_k = e^{i\xi(q_i, q_k)}V_kV_i,$$

equivalent to the Heisenberg commutation relations and moreover the representation
is unique up to unitary equivalence. That is, denoting the generators of $U_i$ and $V_k$
by $P_i$ and $Q_k$ we get from the above relations the following result

$$Q_iP_k - P_kQ_i = i\delta_{ik},$$

$$Q_iQ_k - Q_kQ_i = 0,$$

$$P_iP_k - P_kP_i = 0,$$

i.e. the Heisenberg commutation relations. This is a well known result noticed
by Weyl (loc. cit.) investigated further by v. Neumann, Rellich and Stone. The
part of the statement concerning the uniqueness is mostly called von Neumann’s
uniqueness theorem.

From the mathematical point of view this Weyl’s result is nothing else but a
special case of Bargmann’s theory of ray representations (loc. cit.) applied to the
Abelian group of canonical coordinates (i.e. to the translation group in the phase
space).
The requirement that the set of $2n$ operators $P_1 \ldots P_n, Q_1, \ldots, Q_n$ should be irreducible is very important in the proof of the above statement of Weyl but also in the whole of QM laws! This postulate is to be added to the Heisenberg commutation rules as an essential supplement. For example in QM there is a standard rule for description of systems composed of several individual parts (i.e. subsystems). Suppose a system $A$ to be composed of two parts $B$ and $C$. The states of the system and the subsystems are to be represented by vectors in the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ and $\mathcal{H}_C$ respectively. Then, the general rule says first that $\mathcal{H}_A = \mathcal{H}_B \otimes \mathcal{H}_C$ is the ordinary tensor product of $\mathcal{H}_B$ and $\mathcal{H}_C$. Second, in accord to the rule, the only hermitian operators which have physical significance depend symmetrically on the two subsystems. Let us consider the very special situation of this kind when both $A$ and $B$ are quantum particles of the same kind. We, thus, have 12 matrices $P_1 \ldots P_6, Q_1, \ldots, Q_6$ fulfilling Heisenberg commutation rules. In general they are reducible in accordance with the decomposition

$$\mathcal{H}_A = \mathcal{H}_B \otimes \mathcal{H}_B = \mathcal{H}_B \otimes \mathcal{S}_B + \mathcal{H}_B \otimes \mathcal{A}_B \mathcal{H}_B$$

of the Hilbert space of the system $A$ into the symmetric and anti-symmetric tensor product of the Hilbert spaces corresponding to the particles $B$ and $C = B$ (remember, please, that all $P_i$ and $Q_k$ depend symmetrically on subsystems $B$ and $C = B$). Experimental evidence tells us that there are only two kind of particles those with states either in $\mathcal{H}_B \otimes \mathcal{S}_B \mathcal{H}_B$ or $\mathcal{H}_B \otimes \mathcal{A}_B \mathcal{H}_B$. Thus the reducibility of the set of operators $P_1 \ldots P_n, Q_1, \ldots, Q_n$ would contradict experiment.

Having given this comment on irreducibility let us back to the constant-phase equivalence and the Weyl’s scheme within which we have obtained the Heisenberg commutation rules from this equivalence. It is remarkable that the quantization of the problem of several particles also falls within this general scheme even for fermions. In dealing with it we are interested in that Abelian group whose basic elements $p_\alpha, q_\alpha$ are all of order 2 in the fermionic case. Such a group consists of the totality of monomials

$$p_1^{n_1} q_1^{n_2} p_2^{n_3} q_2^{n_4} \ldots,$$

where $n_k = 1$ or 0 and $n_1 + n_2 + \ldots \leq N$ (total number of particles), compare (Weyl, loc. cit.). In virtue that the two representations $P_\alpha, Q_\alpha$ and $P'_\alpha = e^{\beta(p_\alpha)} P_\alpha, Q'_\alpha = e^{\beta(q_\alpha)} Q_\alpha$ are equivalent the gauge (exponent $\beta$) can be so chosen that the corresponding operators of the irreducible ray representation satisfy the anti-commutation rules

$$Q_i P_k + P_k Q_i = i\delta_{ik},$$
$$Q_i Q_k + Q_k Q_i = 0,$$
$$P_i P_k + P_k P_i = 0.$$

In passing to the relativistic QM we have to account for systems with infinite degrees of freedom. To obtain the Heisenberg rules within the same general scheme presents a very sophisticated mathematical problem. In the case of a free field,
however, it is quite possible. But now the von Neumann uniqueness theorem no longer holds. There is a vast of inequivalent irreducible representations of Heisenberg commutation rules. We have to impose additional condition that the representation possesses a cyclic vector with some peculiar properties, namely the vacuum.

Because of this results one can see that the constant-phase equivalence of Hilbert space vectors lies at the very heart of QM. On the other hand there is a very natural way of generalizing this Weyl’s scheme. For example in the non-relativistic QM we can use the Schrödinger picture. But then the two wave functions $\psi$ and $e^{i\xi(t)}\psi$ are equivalent. Of course one has to assume that the Schrödinger equation is endowed with the appropriate time dependent gauge freedom, but this is very realistic and one cannot exclude such situation, compare the next subsection. For example when considering wave equation in gravitational field (of course non-relativistic field in this case) the time dependent gauge freedom is unavoidable. Because the constant-phase equivalence is so important one cannot ignore it and the consequences of this generalization should be investigated. Even more, one can go still a step further if starting from QFT instead of QM, as we argued in the first section of this Chapter.

It is,thus, justified to think of this phase-equivalence as of a special kind of gauge symmetry. In this way the Weyl’s constant-phase equivalence is a special kind of gauge symmetry, namely the constant-phase symmetry —the simplest possible one.

2.2 Time Dependent Gauge Freedom

In this subsection we carry out general analysis of the representation $T_r$ of a covariance group, and compare it with the representation of a symmetry group. We also describe correspondence between the space of wave functions $\psi(\vec{x},t)$ and the Hilbert space. Here the analysis is performed in the non-relativistic case.

Before we give the general description, it will be instructive to investigate the problem for a free particle in the flat Galilean spacetime. The set of solutions $\psi$ of the Schrödinger equation, which are admissible in Quantum Mechanics, is precisely given by

$$\psi(\vec{x},t) = (2\pi)^{-3/2} \int \varphi(\vec{k}) e^{-i\frac{\vec{k}\cdot\vec{x}}{2m} + i\vec{k}\cdot\vec{\xi}} d^3k,$$

where $p = \hbar \vec{k}$ is linear momentum and $\varphi(\vec{k})$ is any square integrable function. The functions $\varphi$ (wave functions in the ”Heisenberg picture”) form a Hilbert space $\mathcal{H}$ with the inner product

$$\langle \varphi_1, \varphi_2 \rangle = \int \varphi_1^*(\vec{k})\varphi_2(\vec{k}) d^3k.$$

The correspondence between $\psi$ and $\varphi$ is one-to-one.

In general, however, the construction fails if the Schrödinger equation possesses nontrivial gauge freedom. Let us explain it. For example, the above construction fails for the non-relativistic quantum particle in the curved Newton-Cartan spacetime. Besides, in this spacetime we do not have any plane wave, see (J. Wawrzycki,
Thus, there does not exist any natural counterpart for the Fourier transform. However, we do not need to use the Fourier transform. What is the role of the Schrödinger equation in the above construction of $\mathcal{H}$? Please note that in general

$$\|\psi\|^2 \equiv \int \psi^*(\vec{x},0)\psi(\vec{x},0)\,d^3x = (\varphi,\varphi)$$

$$= \int \psi^*(\vec{x},t)\psi(\vec{x},t)\,d^3x.$$  

This is in accordance with the Born interpretation of $\psi$. Namely, if $\psi^*\psi(\vec{x},t)$ is the probability density, then

$$\int \psi^*\psi\,d^3x$$

has to be preserved over time. In the above construction, the Hilbert space $\mathcal{H}$ is isomorphic to the space of square integrable functions $\varphi(\vec{x}) \equiv \psi(\vec{x},0)$, namely the set of square integrable initial data for the Schrödinger equation, cf. e.g. (D. Giulini, States, Symmetries and Superselection, in: Decoherence: Theoretical, Experimental and Conceptual Problems, (Lecture Notes in Physics, Springer Verlag 2000), page 87.). The connection between $\psi$ and $\varphi$ is given by the time evolution operator $U(0,t)$ (equivalently by the Schrödinger equation):

$$U(0,t)\varphi = \psi.$$  

The correspondence between $\varphi$ and $\psi$ has all formal properties, such as in the Fourier construction above. Of course, the initial data for the Schrödinger equation do not cover the whole Hilbert space $\mathcal{H}$ of square integrable functions, but the time evolution given by the Schrödinger equation can be uniquely extended over the whole Hilbert space $\mathcal{H}$ by the unitary evolution operator $U$.

The construction can be applied to the particle in the Newton-Cartan spacetime. As we implicitly assumed, the wave equation is such that the set of its admissible initial data is dense in the space of square integrable functions (we need this for the uniqueness of the extension). Because of the Born interpretation, the integral

$$\int \psi^*\psi\,d^3x$$

has to be preserved over time. Let us denote the space of the square-integrable initial data $\varphi$ on the simultaneity hyperplane $t(\vec{x},t) = t$ by $\mathcal{H}_t$. Then, the evolution is an isometry between $\mathcal{H}_0$ and $\mathcal{H}_t$. But such an isometry has to be a unitary operator, and the construction is well defined, i.e. the inner product of two states corresponding to the wave functions $\psi_1$ and $\psi_2$ does not depend on the choice of $\mathcal{H}_t$. Let us mention that the wave equation has to be linear in accordance with the Born interpretation of $\psi$ (since any unitary operator is linear the time evolution operator

Int. Jour. of Theor. Phys. 40, 1595 (2001)).
is linear as well). The space of wave functions $\psi(\vec{x}, t) = U(0, t) \varphi(\vec{x})$ isomorphic to the Hilbert space $\mathcal{H}_0$ of $\varphi$'s is commonly called the "Schrödinger picture".

However in general, the connection between $\varphi(\vec{x})$ and $\psi(\vec{x}, t)$ is not unique if the wave equation possesses a gauge freedom. Namely, let us consider two states $\varphi_1$ and $\varphi_2$ and ask when these two states are equivalent, and indistinguishable. The answer is that they are equivalent if

$$\left| \langle \varphi_1, \varphi \rangle \right| = \left| \int \psi_1^*(\vec{x}, t) \psi(\vec{x}, t) \, d^3x \right| = \left| \langle \varphi_2, \varphi \rangle \right|$$

for any state $\varphi$ from $\mathcal{H}$, or for all $\psi = U \varphi$ ($\psi_i$ are defined to be $= U(0, t) \varphi_i$). By substituting $\varphi_1$ and then $\varphi_2$ for $\varphi$ and making use of the Schwarz’s inequality, one gets: $\varphi_2 = e^{i\alpha} \varphi_1$, where $\alpha$ is any constant. The situation for $\psi_1$ and $\psi_2$ is however different. In general, condition (2) is fulfilled if

$$\psi_2 = e^{i\xi(t)} \psi_1$$

and the phase factor can depend on time. Of course, this has to be consistent with the wave equation, that is, together with a solution $\psi$ of the wave equation, the wave function $e^{i\xi(t)} \psi$ is also a solution of the appropriately gauged wave equation. A priori one cannot exclude the existence of such a consistent time evolution. This is not a new observation, as it was noticed by John von Neumann, but it seems that it has never been deeply investigated (probably because the ordinary non-relativistic Schrödinger equation has gauge symmetry with constant $\xi$). The space of waves $\psi$ describing the system cannot be reduced in the above way to any fixed Hilbert space $\mathcal{H}_t$ with a fixed $t$. So, the existence of the nontrivial gauge freedom leads to the following

**Hypothesis.** The two waves $\psi$ and $e^{i\xi(t)} \psi$ are quantum-mechanically indistinguishable.

Moreover, we are obliged to use the whole Hilbert bundle $\mathcal{R} \triangle \mathcal{H} : t \rightarrow \mathcal{H}_t$ over the time instead of a fixed Hilbert space $\mathcal{H}_t$, with the appropriate cross-sections $\psi$ as the waves (see the next section for details).

Let us consider now an action $T_r$ of a group $G$ in the space of waves $\psi$. Before we infer some consequences of the assumption that $G$ is a symmetry group, we need to state a:

9This gives the conception of the ray, introduced to Quantum Mechanics by Hermann Weyl [H. Weyl, *Gruppentheorie und Quantenmechanik*, Verlag von S. Hirzel in Leipzig (1928)]; a physical state does not correspond uniquely to a normed state $\varphi \in \mathcal{H}$, but it is uniquely described by a ray; two states belong to the same ray if they differ by a constant phase factor.

10J. v. Neumann, *Mathematical Principles of Quantum Mechanics*, University Press, Princeton (1955). He did not mention gauge freedom on that occasion. However, gauge freedom is necessary for the equivalence of $\psi_1$ and $\psi_2 = e^{i\xi(t)} \psi_1$. 
Classical-like postulate. Group $G$ is a symmetry group if and only if the wave equation is invariant under the transformation $x' = rx, r \in G$ of independent variables and the transformation $\psi' = T_r \psi$ of the wave function.

The above postulate is indeed commonly accepted in Quantum Mechanics even when the gauge freedom is not excluded. But it is a mere application of the symmetry definition for a classical field equation applied to the wave equation without any change. The wave $\psi$ is not a classical quantity, such as e.g. electromagnetic intensity. The above Hypothesis is not true for classical fields, and we have to be careful in forming the appropriate postulate for the wave equation compatible with the Hypothesis. Namely, the two wave equations differing by a mere gauge are indistinguishable. We call them gauge-equivalent. It is therefore natural to assume the

Quantum postulate. Group $G$ is a symmetry group if and only if the transformation $x' = rx, r \in G$ of independent variables and the transformation $\psi' = T_r \psi$ of the wave function transform the wave equation into a gauge-equivalent one.

Please note that not all possibilities admitted by the Hypothesis are included in the Classical-like postulate.

From the Classical-like postulate it follows that $\psi$ as well as $T_r \psi$ are solutions of exactly the same wave equation, in view of the invariance of the equation. Therefore, $\psi$ and $T_r \psi$ belong to the same "Schrödinger picture", so that

$$T_r T_s \psi = e^{i\xi(r,s)} T_{rs} \psi,$$

with $\xi = \xi(r, s)$ independent of time $t$! This is in accordance with the known theorem that

**Theorem 1** If $G$ is a symmetry group, then the phase factor $\xi$ should be time-independent.

But if we start from the Quantum postulate, we obtain instead

$$T_r T_s \psi = e^{i\xi(r,s,t)} T_{rs} \psi \quad (3)$$

and get

**Theorem 1’** If $G$ is a symmetry group, then the phase factor $\xi = \xi(r, s, t)$ is time-dependent in general.

In this paper we propose to accept the Quantum postulate, which is compatible with the Hypothesis, and is more in spirit of Quantum Mechanics than the Classical-like postulate. It should be noted that in the special case when gauge freedom degenerates to the case of constant phase, the Quantum postulate is equivalent to the Classical-like postulate.
We shall resolve the following paradox. Namely, a natural question arises why
the phase factor $e^{i\xi}$ in (3) is time-independent for the Galilean group (even when the
Galilean group is considered as a covariance group). The explanation of the paradox
is as follows. The Galilean covariance group $G$ induces the representation $T_r$ in the
space $\mathcal{R}\Delta\mathcal{H}$ and fulfills (3). But, as we will show later on, the structure of $G$ is such
that there always exists a function $\zeta(r,t)$ continuous in $r$ and differentiable in $t$,
with the help of which one can define a new equivalent representation $T'_r = e^{i\zeta(r,t)}T_r$
fulfilling
$$T'_r T'_s = e^{i\xi(r,s)} T_{rs}$$
with a time-independent $\xi$. The representations $T_r$ and $T'_r$ are equivalent because
$T'_r \psi$ and $T_r \psi$ are equivalent for all $r$ and $\psi$. However, this is not the case in gen-
eral, when the exponent $\xi$ depends on time, and this time dependence cannot be
eliminated in the same way as for the Galilean group. We have a similar situation
when we try to find the most general wave equation for a non-relativistic quantum
particle in the Newton-Cartan spacetime. The relevant covariance group in this
case is the Milne group which possesses representations with time-dependent $\xi$
not equivalent to any representations with a time-independent $\xi$. Moreover, the only
physical representations of the Milne group are those with time-dependent $\xi$.

2.3 Spacetime Dependent Gauge Freedom

There is a physical motivation to investigate representations $T_r$ fulfilling (3) with $\xi$
depending on spacetime point $p$:
$$T_r T_s = e^{i\xi(r,s,p)} T_{rs}. \tag{4}$$

We have sketched the motivation in the first section. We have argued there, that
the two wave functions $\psi$ and $\psi' = e^{i\xi(p)} \psi$ are indistinguishable in the sense that
they give the same transition probabilities: $|\langle \psi, \phi \rangle|^2 = |\langle \psi', \phi \rangle|^2$ for any $\phi$. One
should provide, however, that we are sufficiently fare away from the vacuum. This
additional assumption is an immediate consequence of the structure of states in the
Fock space as well the form of the inner product in that space:
$$\psi = \begin{pmatrix} c \\ \psi_1(x) \\ \psi_2(x_1, x_2) \\ \vdots \end{pmatrix}$$

and
$$\langle \psi, \psi' \rangle = c^* c' + \int_{\mathbb{R}^3} \psi_1^* \psi'_1 d^3x + \int_{\mathbb{R}^2 \times \mathbb{R}^3} \psi_2^* \psi'_2 d^3x_1 d^3x_2 + \ldots$$
in which the argument $x$ of the one-particle state $\psi_1$ contains the ordinary space
coordinates. If one uses the Schrödinger picture the spacetime-dependent-phase
equivalence effect is of primary importance, as one can expect by comparison with
the previous subsection. Probably it would be superfluous to present in detail that in this case one is forced to use the whole Hilbert bundle $\mathcal{M} \Delta \mathcal{H}$ over spacetime $\mathcal{M}$ and respective cross-sections as the wave functions $\psi$ (see the next section for definitions). We do not present details as the reasoning is a simple analogue of that performed in the previous subsection.

Rather we concentrate on the heart of the whole problem, that is, on the spacetime-dependent-phase equivalence which seems more advisable. As we have said application of that equivalence is justified if working far from the vacuum state, when there is quite a number of particles present. But then the justification of this equivalence principle is the same as that of the fact that spacetime coordinates are $c$-numbers commuting with “everything”, so that the greater the number of particles the more commuting are the spacetime coordinates. This is natural and agrees with the well established knowledge that when dealing with one particle (within QM) the spacetime coordinates are not mere parameters and do not commute with “everything”; but, in passing to quantum field by canonical quantization (appropriate for many particles) the spacetime coordinates are ordinary $c$-numbers commuting with each other and all other quantities. Yet the state of affairs is not quite satisfactory and there is a problem which calls for a further analysis. As we have argued in first section the QFT is expected to work perfectly along with QM laws applied to infinite number of degrees of freedom when there is a very few particles present, i.e. near the vacuum state. This is the case in practice, for example, when computing both the Lamb shift and the anomalous magnetic moment — when we are dealing with one-electron problem. It seems, therefore, that application of our equivalence is not justified, but at the same time the commutativity of spacetime coordinates or their $c$-number character is not justified too! This contradict the canonical field quantization rule in which the coordinates do form a $c$-numbers! In this way we arrive at the puzzle of spacetime coordinate status which we shall try to resolve now.

First of all we should note that the spacetime-coordinates-problem does not exist when the quantum field is free — compare the first section. It goes into play when interaction is taken into account. We confine ourselves to QED in order to be more specific. One should like to work within the ordinary QM perturbation theory considered as causing transitions between the stationary states say of the free field. The ordinary QED in Schrödinger picture, however, presents so much a departure from logic in applying the QM perturbation theory, that it is even impossible to work within this formulation of QED. In general when a realistic interaction is present, so violent in the high frequencies, the ordinary picture of perturbation as causing transitions between ordinary stationary states of the free field (as in the anomalous magnetic moment) is destroyed and does require some extra caution. The Schrödinger picture is unsuited for dealing with QED, because the vacuum fluctuations play such a dominant role in it. Still we shall try to reformulate the QED so as to be as much compatible with ordinary QM laws as possible. Such a
reformulation was proposed by Dirac in his excellent book (*Lectures of Quantum Field Theory*, Academic Press, New York, 1966; see also the last Chap. of the Fourth revised 1981 ed. of *The Principles of Quantum Mechanics*). We have no room here to present the reformulation but we should quote some Dirac’s statements at least, which are of importance in our discussion. Suppose the ket $|Q\rangle$ represents a state for which there are no photons, electrons, or positrons present. One would be inclined to suppose this state to be the perfect vacuum, but it cannot be, because it is not stationary. For it to be stationary we should need to have

$$H|Q\rangle = \lambda|Q\rangle$$

with $\lambda$ a number and $H$ the Hamiltonian of QED. Now $H$ contains the terms (we use the standard notation)

$$-e \int \bar{\psi}_\alpha \gamma^\mu A^\mu \psi \, d^3x + \frac{1}{2} \int \int \frac{\bar{\psi}_\alpha \gamma^\mu \psi_{\alpha'}}{x - x'} \, d^3x \, d^3x', \quad (5)$$

which do not give numerical factors when applied to $|Q\rangle$. If we start with the no-particle state it does not remain the no-particle state. Particles get created where none previously existed, their energy coming from the interaction part of the Hamiltonian. Let us call the no-particle state at a certain time by $|Q\rangle$. In order to study this spontaneous creation of particles, one takes the ket $|Q\rangle$ as initial in the Schrödinger picture and treat the terms (5) as a perturbation giving rise to a probability of the state $|Q\rangle$ jumping into another state, in accordance with the ordinary perturbation theory of QM. The first term resolved into its Fourier components — the photon, electron and positron creation operators — contains a part

$$-e(\alpha_r)^{ab} \int \int A_r^\mu \bar{\xi}_{ap} \zeta_{bp+k\hbar} \, d^3k \, d^3p, \quad (6)$$

causing transitions and corresponding to emission of a photon (creation operator $A_r^\mu$) and simultaneously to creation of an electron-positron pair (creation operators $\bar{\xi}_{ap}$ and $\zeta_{bp+k\hbar}$). After a short time the transition probability is proportional to the squared length of the ket formed by multiplying (6) into the initial ket $|Q\rangle$. But this length is infinite, so the transition probability is infinite. The second term of (5) contains contributions with two electron-positron pairs created simultaneously. Again the transition probability due to this term is also infinite. One can conclude that the state $|Q\rangle$ is is not even approximately stationary. Even with a cutoff the no-particle state $|Q\rangle$ is not approximately stationary. This is why the above procedure presents so much a departure from the ordinary perturbation theory of QM, and seems to be not logically justified. Dirac proposes another way of dealing with QED, which is a less of departure from ordinary QM. From the the above calculations — says Dirac — it follows that the no-particle state $|Q\rangle$ differs very much from the vacuum state. The “vacuum” state must contain many particles, which may be pictured as a state of transient existence with violent fluctuations. Let us introduce
the ket $|V\rangle$ to represent the “vacuum” state. It is the eigenket of $H$ belonging to the lowest eigenvalue. Here and subsequently $H$ denotes the Hamiltonian modified by the cutoff. One might try to calculate $|V\rangle$ as a perturbation of of the ket $|Q\rangle$, but such a method would be of doubtful validity, because the difference between $|V\rangle$ and $|Q\rangle$ is not small. No satisfactory way of calculating $|V\rangle$ is known. In any case the result would depend strongly on the cutoff, and since the cutoff is unspecified the result would not be a definite one. It follows that we must develop the theory without knowing $|V\rangle$. This is not a great hardship — argues further Dirac — because we are not manly interested in the “vacuum”. We are mainly interested in states which differ from the “vacuum” $|V\rangle$ through having a few particles present in addition to those associated with the vacuum fluctuations, and we want to know how this extra particles behave. For this purpose we focus our attention on an operator $K$ representing the creation of the extra particles, so that the state we are interested in appears $K|V\rangle$. We do not now how the ket $|V\rangle$ varies with time in the Schrödinger picture, since we do not now the lowest eigenvalue of $H$. To avoid this difficulty we work in the Heisenberg picture in which $|V\rangle$ is constant. We then require $K|V\rangle$ to represent another state in the Heisenberg picture and thus to be another constant ket. This leads to

$$\frac{dK}{dt} + i\hbar \frac{\partial K}{\partial t} + KH - HK = 0.$$  (7)

We now have each physical state determined by a solution $K$ of (7). Dirac (loc. cit.) proceeded along these lines and built a theory more compatible with the standard QM. He was able to calculate the Lamb shift as well as the anomalous magnetic moment within this theory. Thus it is an open question if we are close to the “true” vacuum $|0\rangle$ when evaluating the anomalous magnetic moment. From the above calculations we expect rather that the state $|V\rangle$ is considerably far removed from the true vacuum $|0\rangle$ (no-particles present). But if we are sufficiently far from the vacuum to ensure the commutativity of spacetime coordinates (as in the canonical field quantization) we will at the same time ensure the justification for our spacetime-dependent-equivalence of states! Is it therefore possible that we should use the Hilbert bundle $\mathcal{M}\Delta\mathcal{H}$ with appropriate cross sections instead of ordinary Hilbert space $\mathcal{H}$ with vectors of $\mathcal{H}$? It depends if the difference between $|V\rangle$ and the true vacuum is large or small — the fact on which we may speculate only. Anyway one has the following alternative:

- Either $\mathcal{H}$ noncommuting spacetime coordinates
- or $\mathcal{M}\Delta\mathcal{H}$ commuting spacetime coordinates.

On the left hand side we have a case in which the spacetime-dependent equivalence is not always justified. In that case one actually works near the true vacuum $|0\rangle$ when considering one-particle problems like the anomalous magnetic moment. Ordinary
QM with states as vectors (or Weyl's rays) in ordinary Hilbert space is justified then, but the spacetime coordinates are not a mere $c$-numbers in it. On the right hand side we have a theory in which the space-time-dependent equivalence is in general justified. States are the appropriate cross sections $\psi : M \ni p \to \psi_p \in \mathcal{H}_p$ in a Hilbert bundle $\mathcal{M} \triangle \mathcal{H} : M \ni p \to \mathcal{H}_p$ over spacetime in which the respective Hilbert spaces $\mathcal{H}_p$ play a role rather. Now, one can see what a peculiar object is the QED. Namely, it is partly on the left hand side of the alternative as it use the basic methods of ordinary QM with $\mathcal{H}$ and at the same time it is at the right hand side of the alternative as a canonically quantized version of the classical field with **commuting spacetime coordinates**. Strictly speaking there are some troubles in constructing ordinary Hilbert space $\mathcal{H}$ for QED, as was pointed by Dirac (*loc. cit.*) even within his method of treatment mentioned above. Moreover, in the canonical quantization one is trying at the outset to implement the ordinary QM to a system with infinite number of degrees of freedom. We believe that the problem just mentioned mirrors an important physical truth. **It strongly suggest that the spacetime coordinates in QED should not be perfectly commuting quantities, and that the less commuting they are the stronger is the cutoff effect.** Indeed, if they are noncommuting, then we are on the left hand side of the alternative, where the difference between $|0\rangle$ and $|V\rangle$ is small. But this is possible only if the cutoff is large. On the other hand if they are commuting, then we are on the right hand side of the alternative where the difference between $|0\rangle$ and $|V\rangle$ is large, i.e. when the cutoff is small and not so violent. As is well known, the infinities in QED originate from the fact that we pass to the limit zero for the space and time intervals involved. But if the spacetime coordinates were noncommuting then the structure of spacetime would be more elaborate and the limit process would be meaning less. **This suggest that the cutoff process reflects some important physical phenomena.**

It is therefore advisable to construct a theory which like QED lies somewhat on the both sides of the above alternative. Namely, it this theory the states as cross sections (looking at the right hand side) should compose at the same time a Hilbert space (looking at the left hand side). There is a natural Hilbert space composed of appropriate cross sections of a Hilbert bundle in mathematics, namely the von Neumann direct integral

$$\mathcal{H} = \int_M \mathcal{H}_p d\mu(p)$$

of Hilbert spaces $\mathcal{H}_p$, compare the next section for definition. But this is possible if the relevant quantum algebra say $\mathcal{A}$ acting in the direct integral $\mathcal{H}$ is decomposable.

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11 People usually think that the cutoff is a man-made process which one has to perform due to the fact that the theory at our disposal is incapable of the high energy processes. This opinion was justified in the early seventies, when the strong interactions were out of our scope. One could hope that the strong interactions will cancel the infinities. But now it seems that this hope turned out to be vain. Strong interactions do not cancel the infinities. Paradoxically QED serves now as the paradigm for a successful quantum field theory.
over a diagonal (commutative) algebra say $A_1^{CCR}$, such that the spectrum of the diagonal algebra is the classical spacetime $M$. Thus we arrive at the following interpretation of the canonical field quantization:

*Already in the classic works of Dirac (loc. cit.) and Heisenberg (The Physical Principles of the Quantum Theory, Dover, New York, 1949), spacetime coordinates are interpreted as representing a special kind of operators, called c-numbers. The spacetime coordinate operators $x^\mu 1$ commute with each other and all other operators of our quantum algebra $A$, which we do not specify here. Here we propose to treat this interpretation seriously. Let us consider an algebra $a$ of functions $f(p)$ on spacetime $M$ which encodes the geometry of $M$. At the moment, the exact structure of $a$ is not important for us. However, it will be a commutative algebra with point-wise operations. For simplicity, we confine ourself to the topological structure of $M$ (i.e. the algebra $a$ of continuous functions), assume $M$ to be compact and $A \subset \mathcal{B}(H)$. Let us form an operator $Df = f(p) 1$ corresponding to the function $p \to f(p)$ in our algebra $a$. It is therefore natural to assume that the set of all $Df, f \in a$ in a natural manner composes an algebra $A$ isomorphic to $a$, i.e. the point-wise function multiplication in $a$ corresponds to the operator composition in $A$. This is precisely what we mean when assuming the spacetime coordinates to be classical. Moreover, it is also natural to assume that $A$ is closed with respect to norm and dense in center of $A$ with respect to the strong operator topology. But this is possible if the operators in $A$ are decomposable and act in a direct integral of Hilbert spaces $H_p, p \in M$, or in the appropriate set of cross sections of a Hilbert bundle $M \triangle H$ over spacetime.*

But the interpretation of the inner products of the Hilbert spaces $H_p, p \in M$ and of their direct integral $H$ is not clear now. One has to preserve an open mind on this subject. Now we know only that these Hilbert spaces are of importance but their role depends on the respective regime (remember, please, our alternative). The problem should be investigated in a subsequent research.

The Dirac’s critique (loc. cit.) inspired Piron in his research who formulated some related ideas cf. ([C. Piron, physics/0204083](physics/0204083)). He brings about within his analysis of Dirac’s work at the conclusion that some spectral families of Hilbert spaces instead of a mere Hilbert space are indispensable. Piron’s reasoning was, however, completely different.

**Appendix.** It should be mentioned an independent argument which shows that the generalized ray representations may play a role in QED. Paradoxically, there should be no zero mass vector particles with helicity = 1, as a consequence of the theory of unitary representations of the Poincaré group, as shown by Lopuszański ([Fortschritte der Physik 26, 261, (1978); Rachunek spinorów, PWN, Warszawa 1985 (in Polish)](Fortschritte der Physik 26, 261, (1978); Rachunek spinorów, PWN, Warszawa 1985 (in Polish))). This is apparently in contradiction to the experiment, because the photon is a vector particle with helicity = 1. What is the solution of this paradox? First, let us describe the solution on the grounds of the existing theory, which constitutes at the same time an orthodox view. We observe that we can build a zero mass vector state with $h = 1$ but we must admit finite-dimensional irreducible and thus non-unitary representations of the small group, that is the two-dimensional non-compact Euclidean group. Next, please note that the representation
of $G$ induced by the non-unitary representation of the small group remains "unitary" if we admit the inner product in the "Hilbert" space to be not positively defined, cf. (S. N. Gupta, Proc. Phys. Soc. 63, 681, (1950); K. Bleuler, Helv. Phys. Acta 23, 567, (1950)), or (S. Weinberg, The Quantum Theory of Fields, volume II, Univ. Press, Cambridge 1996). However, even with the most favorable attitude toward the orthodox view, this solution is rather obscure. We propose to proceed in another way. First, let us observe that the case of the free quantum vector field $A_{\mu}(x)$ with zero mass is exactly the same. As long as the inner product in the Hilbert space is positively defined, we are not able to introduce any vector potential which transforms as a vector field. However, we can introduce a local real electromagnetic field $F_{\mu\nu}(x) = - F_{\nu\mu}(x)$ which is a linear combination of a self-dual and an antiself-dual field with helicity +1 and -1 respectively. If we introduce a vector potential $A_{\mu}$ in some Lorentz frame such that

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$$

then in another Lorentz frame we will have ($\Lambda^r_{\mu}$ is the Lorentz transformation matrix corresponding to the Poincaré transformation $r$)

$$A_{\mu}(x) \to U_r A_{\mu}(x) U^{-1}_r = (\Lambda^{-1})^\nu_{\mu} A_{\nu}(r^{-1} x) + \partial_{\mu} R(r, x), \quad \partial_{\nu} R \neq 0,$$

while $F_{\mu\nu}$ transforms as a tensor field. We infer that gauge transformation of the second kind has to accompany the Poincaré transformation, or that gauge freedom is indispensable in the construction of the vector potential in the quantum field theory, cf. (J. Lopuszański, Fortschrritte der Physik 26, 261, (1978); Rachunek spinorów, PWN, Warszawa 1985 (in Polish).) or (S. Weinberg, The Quantum Theory of Fields, volume II, Univ. Press, Cambridge 1996), vol. I. Let $\Omega$ denote the vacuum state. According to QFT, we should define a photon state $\psi_\mu(x)$ in the following way

$$\psi_\mu(x) = A_{\mu}(x) \Omega.$$

It immediately follows from Eq. (8) that

$$U_r \psi_\mu(x) = (\Lambda^{-1})^\nu_{\mu} \psi_\nu(r^{-1} x) + \partial_{\mu} \Theta(x),$$

where $\partial_{\nu} \Theta(x)$ denotes the vector-valued distribution $\partial_{\nu} R(x) \Omega$. The above representation spanned by the generalized vectors $\psi_\mu(x)$ induces a representation in the appropriate Hilbert space. Indeed, let us write $\varphi_\mu(x)$ and $\theta(x)$ for test functions which "smear" the distributions $\psi_\mu(x)$ and $\Theta(x)$ respectively. From formula (9) we get the transformation law for $\varphi_\mu$

$$T_r \varphi_\mu(x) = (\Lambda^{-1})^\nu_{\mu} \varphi_\nu(r^{-1} x) + \partial_{\mu} \theta(x).$$

By construction, the space of test functions is dense in the corresponding Hilbert space and the above representation $T_r$ can be uniquely extended. As we are dealing with a gauge-invariant theory, the two quantum vector potentials $A_{\mu}(x)$ and $A_{\mu} + \partial_{\nu} \Phi(x)$ are unitary equivalent. Accordingly, two photon states differing by a gradient, as well as their two corresponding vectors $\varphi_\mu(x)$ and $\varphi_\mu(x) + \partial_{\mu} \phi(x)$ should be unitary equivalent. This means that it is more adequate to consider all $\varphi_\mu(x) + \partial_{\mu} \phi(x)$ instead of the respective $\varphi_\mu(x)$ alone. We write $\varphi_\mu(x) + \partial_{\mu} \phi(x)$ as a pair $\{\phi(x), \varphi_\mu(x)\}$. The action of our representation $T_r$ in the space of pairs $\{\phi(x), \varphi_\mu(x)\}$ is as follows

$$T_r \{\phi(x), \varphi_\mu(x)\} = \{\phi(r^{-1} x) + \theta(r, x), U_r \varphi_\mu(x)\},$$

where $U_r$ acts as an ordinary vector transformation:

$$U_r \varphi_\mu(x) = (\Lambda^{-1})^\nu_{\mu} \varphi_\nu(r^{-1} x).$$

Moreover, we have

$$T_r T_s \{\phi(x), \varphi_\mu(x)\} = T_{rs} \{\phi(x), \varphi_\mu(x)\} + \{\xi(r, s, x, 0),$$

This time $A_{\mu}(x)$ is an operator-valued distribution.
where
\[ \xi(r, s, x) = \theta(r, x) + \theta(s, r^{-1}x) - \theta(rs, x). \]

But this seems to be a kind of a (generalized) ray representation of the Poincaré group \( G \) fulfilling \(^4\) with the spacetime-dependent exponent \( \xi(r, s, x) \). Summing up the discussion, we have just seen that the generalized representation in the sense of Eq. (4) seems to be indispensable if we wish to work with ordinary Hilbert spaces with positive norms while having a theory which describes photons.

### 3 Generalization of Bargmann’s Theory

We have shown that we are forced to generalize the Bargmann’s theory of factors to embrace the spacetime-dependent factors of representations acting in a Hilbert bundle over space-time (time) and apply the theory in the simplest non-relativistic case. We have already done it, the results will be presented in the subsequent part of this Chapter.

3.1 Generalized Wave Rays and Operator Rays

In this section we give strict mathematical definitions of the notions of the preceding section, and formulate the problem stated there in an exact way. From the pure mathematical point of view, the analysis of spacetime-dependent \( \xi(r, s, p) \) is more general, so at the outset we confine ourselves to this case \(^{13}\).

Let us recall some definitions, cf. e.g. (G. W. Mackey, Unitary Group Representations in Physics, Probability, and Number Theory. Addison-Wesley Publishing Company, INC. New York, Amsterdam, Wokingham-UK (1989)). Let \( \mathcal{M} \) be a set endowed with an analytic Borel structure.

By a Hilbert bundle over \( \mathcal{M} \) or a Hilbert bundle with base \( \mathcal{M} \) we shall mean an assignment \( \mathcal{H} : p \rightarrow \mathcal{H}_p \) of a Hilbert space \( \mathcal{H}_p \) to each \( p \in \mathcal{M} \). The set of all pairs \((p, \psi)\) with \( \psi \in \mathcal{H}_p \) will be denoted by \( \mathcal{M} \triangle \mathcal{H} \) and called the space of the bundle. By a cross section of our bundle we shall mean an assignment \( \psi : p \rightarrow \psi_p \) of a member of \( \mathcal{H}_p \) to each \( p \in \mathcal{M} \). If \( \psi \) is a cross section and \((p_0, \phi_0)\) a point of \( \mathcal{M} \triangle \mathcal{H} \), we may form a scalar product \( (\phi_0, \psi_{p_0}) \). In this way, every cross-section \( \psi \) defines a complex-valued function \( f_\psi \) on \( \mathcal{M} \triangle \mathcal{H} \). By a Borel Hilbert bundle we shall mean a Hilbert bundle together with an analytic Borel structure in \( \mathcal{M} \triangle \mathcal{H} \) such that the following conditions are fulfilled

1. Let \( \pi(p, \psi) = p. \) Then \( E \subseteq \mathcal{M} \) is a Borel set if and only if \( \pi^{-1}(E) \) is a Borel set in \( \mathcal{M} \triangle \mathcal{H} \).

\(^{13}\)It becomes clear in further analysis that the group \( G \) in question has to fulfill the consistency condition requiring that for any \( r \in G \), \( rt \) is a function of time only in the case of non-relativistic theory with \(^3\).
(2) There exist countably many cross-sections $\psi^1, \psi^2, \ldots$ such that

(a) the corresponding complex-valued functions on $M \triangle \mathcal{H}$ are Borel functions,

(b) these Borel functions separate points in the sense that no two distinct points $(p_i, \phi_i)$ of $M \triangle \mathcal{H}$ assign the same values to all $\psi^j$ unless $\phi_1 = \phi_2 = 0$, and

(c) $p \to (\psi^i(p), \psi^j(p))$ is a Borel function for all $i$ and $j$.

A cross-section is said to be a Borel cross-section if the function on $M \triangle \mathcal{H}$ defined by the cross-section is a Borel function. All Borel cross-sections compose a linear space under the obvious operations, cf. Mackey (loc. cit.). Now let $\mu$ be a measure on $M$. The cross-section $p \to \varphi_p$ is said to be square summable with respect to $\mu$ if

$$\int_M (\varphi_p, \varphi_p) d\mu(p) < \infty.$$ 

The space $L^2(M, \mu, \mathcal{H})$ of all equivalence classes of square-summable cross-sections, where two cross-sections $\varphi$ and $\varphi'$ are in the same equivalence class if $\varphi_p = \varphi'_p$ for almost all $p \in M$, forms a separable Hilbert space with the inner product given by

$$(\varphi, \theta) = \int_M (\varphi_p, \theta_p) d\mu(p),$$

cf. Mackey (loc. cit.). It is called the direct integral of the $\mathcal{H}_p$ with respect to $\mu$ and is denoted by $\int_M \mathcal{H}_p d\mu(p)$.

Identification with the previous section is partially suggested by the notation itself. We shall make this identification more explicit. The set $M$ plays the role of spacetime or real line $\mathbb{R}$ of time $t$ respectively. The wave functions $\psi$ of the preceding section are the Borel cross-sections of $M \triangle \mathcal{H}$ but if they do belong to the subset $L^2(M, \mu, \mathcal{H})$ of cross-sections which are square integrable presents an open question. The separate Hilbert spaces $\mathcal{H}_p$ with their inner products play some role in experiments as well as the inner product in their direct integral product. But would be better to leave unspecified the precise role they play in experiments for now. We have also used $\psi(p)$ and $\psi_p$ as well as $(\psi_p, \theta_p)$ and $(\psi, \theta)_p$ interchangeably.

By an isomorphism of the Hilbert bundle $M \triangle \mathcal{H}$ with the Hilbert bundle $M' \triangle \mathcal{H}'$ we shall mean a Borel isomorphism $T$ of $M \triangle \mathcal{H}$ on $M' \triangle \mathcal{H}'$ such that for each $p \in M$ the restriction of $T$ to $p \times \mathcal{H}_p$ has some $q \times \mathcal{H}'_q$ for its range and is unitary when regarded as a map of $\mathcal{H}_p$ on $\mathcal{H}'_q$. The induced map carrying $p$ into $q$ is clearly a Borel isomorphism of $M$ with $M'$ and we denote it by $T^p$. The above-defined $T$ is said to be an automorphism if $M \triangle \mathcal{H} = M' \triangle \mathcal{H}'$. Please note that for any automorphism $T$ we have $(T\psi, T\phi)_{T^p} = (\psi, \phi)_p$, but in general $(T\psi, T\phi)_p \neq (\psi, \phi)_p$. By this token, any automorphism $T$ is what is frequently called bundle isometry.
The function \( r \to T_r \) from group \( G \) into the set of automorphisms (bundle isometry) of \( M\Delta H \) is said to be a general factor representation of \( G \) associated to the action \( G \times M \ni (r, p) \to r^{-1}p \in M \) of \( G \) on \( M \) if \( T_r\pi(p) = r^{-1}p \) for all \( r \in G \), and \( T_r \) satisfy condition (4).

Of course, \( T_r \) is to be identified with that of the preceding section. Our further specializing assumptions partly following from the above interpretation are as follows. We assume \( M \) to be endowed with the manifold structure inducing a topology associated with the above-assumed Borel structure. We confine ourselves to a finite dimensional Lie group \( G \) which acts smoothly and transitively on spacetime \( M \), such that a \( G \)-invariant measure \( \mu \) exists on \( M \).

By a factor representation of a Lie group we mean a general factor representation with the exponent \( \xi(r, s, p) \) differentiable in \( p \in M \).

Now we define the operator ray \( T \) corresponding to a given bundle isometry operator \( T \) to be the set of operators

\[
T = \{ \tau T, p \to \tau(p) \in D \text{ and } |\tau| = 1 \},
\]

where \( D \) denotes the set of all differentiable real functions on \( M \). Any \( T \in T \) will be called a representative of the ray \( T \). The product \( TV \) is defined as the set of all products \( TV \) such that \( T \in T \) and \( V \in V \).

Please note that not all Borel sections are physically realizable. By interpreting the discussion of the preceding section in the Hilbert bundle language, we see that the role of the Schrödinger equation is essentially to establish all the physical sections. Any two sections \( \psi(p) \) and \( \psi'(p) = e^{i\xi(p)}\psi(p) \) are indistinguishable giving the same probabilities \( |f_\psi|^2 = |f_{\psi'}|^2 \). After this, any group \( G \) acting in \( M \) induces a ray representation of \( G \), i.e. a mapping \( r \to T_r \) of \( G \) into the space of rays of bundle automorphisms (bundle isometries) of \( M\Delta H \), fulfilling the condition

\[
T_rT_s = T_{rs}.
\]

For any cross-section \( \psi \) we define its corresponding ray \( \psi = \{ e^{i\xi(p)}\psi(p), \xi \in D \} \). If \( \psi \) is a physical cross-section, then we get the physical ray of the preceding section. Selecting a representative \( T_r \) for each \( T_r \), we get a factor representation fulfilling (4).

Please note that \( T_r \) transforms rays into rays, and we have \( T_r(e^{i\xi(p)}\psi) = e^{i\xi(r^{-1}p)}T_r\psi \). Further on we assume that that operators \( T_r \) are such that \( \xi_r(p) = \xi(r^{-1}p) \), where \( r^{-1}p \) denotes the action of \( r^{-1} \in G \) on the spacetime point \( p \in M \). This is a natural assumption which does actually take place in practice.

Now we shall make the last assumption, namely that all transition probabilities vary continuously with the continuous variation of the coordinate transformation \( s \in G \):

For any element \( r \) in \( G \), any ray \( \psi \) and any positive \( \epsilon \), there exists a neighborhood \( U \) of \( r \) on \( G \) such that \( d_p(T_s\psi, T_r\psi) < \epsilon \) if \( s \in U \) and \( p \in M \).
where
\[ d_p(\psi_1, \psi_2) = \inf_{\psi_i \in \psi_i} \| \psi_1 - \psi_2 \|_p = \sqrt{2|1 - |(\psi_1, \psi_2)|_p|}. \]

Basing on the continuity assumption, one can prove the following

**Theorem 2** Let \( T_r \) be a continuous ray representation of a group \( G \). For all \( r \) in a suitably chosen neighborhood \( \mathcal{N}_0 \) of the unit element \( e \) of \( G \) one may select a strongly continuous set of representatives \( T_r \in \mathcal{T}_r \). That is, for any compact set \( C \subset M \), any wave function \( \psi \), any \( r \in \mathcal{N}_0 \) and any positive \( \epsilon \) there exists a neighborhood \( \mathcal{N}_0 \) of \( r \) such that \( \| T_s \psi - T_r \psi \|_p < \epsilon \) if \( s \in \mathcal{N}_0 \) and \( p \in C \).

There are numerous possible selections of such factor representations. But many among them merely differ by a differentiable phase factor and are physically indistinguishable. We call them equivalent. Our task then is to classify all possible factor representations with respect to this equivalence.

### 3.2 Local Exponents

The representatives \( T_r \in \mathcal{T}_r \) selected as in Theorem 2 will be called *admissible*, with the representation \( T_r \) obtained in this way referred to as an *admissible* representation. There are infinitely many possibilities of such a selection of admissible representations \( T_r \). We confine ourselves to the local *admissible* representations defined on a fixed neighborhood \( \mathcal{N}_0 \) of \( e \in G \), as in Theorem 2.

Let \( T_r \) be an *admissible* representation. With the help of the phase \( e^{i \zeta(r,p)} \) with a real function \( \zeta(r,p) \) differentiable in \( p \) and continuous in \( r \), we can define

\[ T'_r = e^{i \zeta(r,p)} T_r, \tag{12} \]

which is a new *admissible* representation. This is trivial if one defines the continuity of \( \zeta(r,p) \) in \( r \) appropriately. Namely, from Theorem 2 it follows that the continuity has to be defined in the following way. The function \( \zeta(r,p) \) will be called strongly continuous in \( r \) at \( r_0 \) if and only if for any compact set \( C \subset M \) and any positive \( \epsilon \) there exists a neighborhood \( \mathcal{N}_0 \) of \( r_0 \) such that

\[ |\zeta(r_0,p) - \zeta(r,p)| < \epsilon, \]

for all \( r \in \mathcal{N}_0 \) and for all \( p \in C \). But the converse is also true. Indeed, if \( T'_r \) is also an *admissible* representation, then \( \zeta(r,p) \) has to be fulfilled for a real function \( \zeta(r,p) \) differentiable in \( p \) because \( T'_r \) and \( T_r \) belong to the same ray. Moreover, because both \( T'_r \psi \) and \( T_r \psi \) are strongly continuous (in \( r \) for any \( \psi \)), then \( \zeta(r,p) \) has to be strongly continuous (in \( r \)).

Let \( T_r \) be an *admissible* representation, and thus continuous in the sense indicated in Theorem 2. One can always choose the above \( \zeta \) in such a way that \( T_e \equiv 1 \) as will be assumed from now on.
Because $T_r T_s$ and $T_{rs}$ belong to the same ray, one has
\begin{equation}
T_r T_s = e^{i \xi(r,s,p)} T_{rs}
\end{equation}
with a real function $\xi(r,s,p)$ differentiable in $p$. From the fact that $T_e = 1$, we have
\begin{equation}
\xi(e,e,p) = 0.
\end{equation}
From the associative law $(T_r T_s) T_g = T_r (T_s T_g)$ one gets
\begin{equation}
\xi(r,s,p) + \xi(rs,g,p) = \xi(s,g,r^{-1}p) + \xi(r,sg,p).
\end{equation}
Formula (15) is very important and our analysis largely rests on this relation. From the fact that the representation $T_r$ is admissible follows that the exponent $\xi(r,s,p)$ is continuous in $r$ and $s$. Indeed, let us take a $\psi$ belonging to a unit ray $\psi$. Then, making use of (13), we get
\begin{align*}
e^{i \xi(r,s,p)} (T_r - T_{r's'}) \psi + (T_{r'} (T_s' - T_s) \psi + (T_{r'} - T_r) T_s \psi
&= (e^{i \xi(r',s',p)} - e^{i \xi(r,s,p)}) T_{r's'} \psi.
\end{align*}
Taking norms $\| \cdot \|_p$ of both sides, we get
\begin{equation*}
|e^{i \xi(r',s',p)} - e^{i \xi(r,s,p)}| \leq \| (T_{r's'} - T_{rs}) \psi \|_p +
\end{equation*}
\begin{equation*}
\| T_{r'} (T_s' - T_s) \psi \|_p + \| (T_{r'} - T_r) T_s \psi \|_p.
\end{equation*}
From this inequality and the continuity of $T_r \psi$, the continuity of $\xi(r,s,p)$ in $r$ and $s$ follows. Moreover, from Theorem 2 and the above inequality follows the strong continuity of $\xi(r,s,p)$ in $r$ and $s$.

Formula (12) suggests the following definition. Two admissible representations $T_r$ and $T'_r$ are called equivalent if and only if $T'_r = e^{i \zeta(r,p)} T_r$ for some real function $\zeta(r,p)$ differentiable in $p$ and strongly continuous in $r$. Thus, making use of (13), we get $T'_r T'_s = e^{i \zeta(r,s,p)} T_{rs}$, where
\begin{equation}
\zeta'(r,s,p) = \xi(r,s,p) + \zeta(r,p) + \zeta(s,r^{-1}p) - \zeta(rs,p).
\end{equation}
Then the two exponents $\xi$ and $\xi'$ are equivalent if and only if (16) is fulfilled with $\zeta(r,p)$ strongly continuous in $r$ and differentiable in $p$.

From (14) and (15) it immediately follows that
\begin{equation}
\xi(r,e,p) = 0 \text{ and } \xi(e,g,p) = 0,
\end{equation}
\begin{equation}
\xi(r,r^{-1},p) = \xi(r^{-1},r,r^{-1}p).
\end{equation}
Relation (16) between $\xi$ and $\xi'$ will be written in short by

$$\xi' = \xi + \Delta[\xi].$$

(19)

The relation (16) between exponents $\xi$ and $\xi'$ defines an equivalence relation, which preserves the linear structure.

We introduce now group $H$, a very important notion for our further investigations. It is evident that all operators $T_r$ contained in all rays $T_r$ form a group under multiplication. Indeed, let us consider an admissible representation $T_r$ with a well-defined $\xi(r, s, p)$ in formula (13). Because any $T_r \in T_r$ has the form $e^{i\theta(p)}T_r$ (with a real and differentiable $\theta$), one has

$$\left(e^{i\theta(p)}T_r\right)\left(e^{i\theta'(p)}T_s\right) = e^{i(\theta(p) + \theta'(r^{-1}p) + \xi(r, s, p))}T_{rs}.$$  

(20)

This important relation suggests the following definition of the local group $H$ connected with the admissible representation or with the exponent $\xi(r, s, p)$. Namely, $H$ consists of the pairs $\{\theta(p), r\}$ where $\theta(p)$ is a differentiable real function and $r \in G$. The multiplication rule, suggested by the above relation, is defined as follows

$$\{\theta(p), r\} \cdot \{\theta'(p), r'\} = \{\theta(p) + \theta'(r^{-1}p) + \xi(r, r', p), rr'\}.$$  

(21)

The associative law for this multiplication rule is equivalent to (15) (in complete analogy with the classical Bargmann’s theory). The pair $\hat{e} = \{0, e\}$ plays the role of the unit element in $H$. For any element $\{\theta(p), r\} \in H$ there exists the inverse $\{\theta(p), r\}^{-1} = \{-\theta(rp) - \xi(r, r^{-1}, rp), r^{-1}\}$. Indeed, from (15) it follows that $\{\theta, r\}^{-1} \cdot \{\theta, r\} = \{\theta, r\} \cdot \{\theta, r\}^{-1} = \hat{e}$. The elements $\{\theta(p), e\}$ form an Abelian subgroup $N$ of $H$. Any $\{\theta, r\} \in H$ can be uniquely written as $\{\theta(p), r\} = \{\theta(p), e\} \cdot \{0, r\}$. The same element can be also uniquely expressed in the form $\{\theta(p), r\} = \{0, r\} \cdot \{\theta(rp), e\}$. Thus, we have $H = N \cdot G = G \cdot N$. The Abelian subgroup $N$ is a normal factor subgroup of $H$. But this time, $G$ does not form any normal factor subgroup of $H$ (contrary to the classical case investigated by Bargmann, when the exponents do not depend on $p$). So, this time $H$ is not direct product $N \circ G$, but a semidirect product $N \otimes G$. In this case, however, the theorem that $G$ is locally isomorphic to the factor group $H/N$ is still valid. Then group $H$ composes a semicentral extension of $G$, and not a central extension of $G$ as in the Bargmann’s theory.

The rest of this paper is based on the following reasoning (the author being largely inspired by Bargmann’s work (Ann. Math. 59, 1, 1954)). If the two exponents $\xi$ and $\xi'$ are equivalent, that is $\xi' = \xi + \Delta[\xi]$, then the semicentral extensions $H$ and $H'$ connected with $\xi$ and $\xi'$ are homomorphic. The homomorphism $h : \{\theta, r\} \mapsto \{\theta', r'\}$ is given by

$$\theta'(p) = \theta(p) - \zeta(r, p), \; r' = r.$$  

(22)
Using an Iwasawa-type construction we show that any exponent $\xi(r, s, p)$ is equivalent to a differentiable one (in $r$ and $s$). We can then confine ourselves to the differentiable $\xi$ and $\xi'$. We show that $\zeta(r, p)$ is also a differentiable function of $(r, p)$. Moreover, we show that any $\xi$ is equivalent to the canonical one, that is such $\xi$ which is differentiable and for which $\xi(r, s, p) = 0$ whenever $r$ and $s$ belong to the same one-parameter subgroup. Then we can restrict our investigation to the canonical $\xi$ considering the subgroup of all elements $\{\theta(p), r\} \in H$ with differentiable $\theta(p)$. For simplicity let us denote the subgroup by the same symbol $H$. We embed the subgroup in an infinite dimensional Lie group $D$ with manifold structure modeled on a Banach space. Then we consider the subgroup $\mathcal{H}$ which is the closure of $H$ in $D$. After this, $\mathcal{H}$ turns into a Lie group and the homomorphism (22) becomes an isomorphism of the two Lie groups. Thus, the group $\mathcal{H}$ has the Banach Lie algebra $\mathfrak{h}$. We apply the general theory of analytic groups developed in (G. Birkhoff, Continuous Groups and Linear Spaces, Recueil Mathématique (Moscow) 1(5), 635, (1935); Analytical Groups, Trans. Am. Math. Soc. 43, 61, (1938)) and (E. Dynkin, Uspeki Mat. Nauk 5, (1950), 135; Amer. Math. Soc. Transl. 9(1), (1950), 470).

From this theory it follows that the correspondence between the local $\mathcal{H}$ and $\mathfrak{h}$ is bi-unique and one can construct uniquely the local group $\mathcal{H}$ from the algebra $\mathfrak{h}$ as well. As we will see, the algebra defines a spacetime-dependent anti-linear form $\Xi$ on the Lie algebra $\mathfrak{g}$ of $G$, the so-called infinitesimal exponent $\Xi$. By this we reduce the classification of local $\xi$'s which define $\mathcal{H}$'s to the classification of $\Xi$'s which define $\mathfrak{h}$'s. So, we will simplify the problem of the classification of local $\xi$'s to a largely linear problem. Here are the details.

Iwasawa construction. Let us denote by $dr$ and $d^*r$ the left and right invariant Haar measure on $G$. Let $\nu(r)$ and $\nu^*(r)$ be two infinitely differentiable functions on $G$ with compact supports contained in the fixed neighborhood $\mathfrak{N}_0$ of $e$. By multiplying them by the appropriate constants, we can always obtain: $\int_G \nu(r) \, dr = \int_G \nu^*(r) \, d^*r = 1$. Let $\xi(r, s, p)$ be any admissible local exponent defined on $\mathfrak{N}_0$. We will construct a differentiable (in $r$ and $s$) exponent $\xi''(r, s, p)$ equivalent to $\xi(r, s, p)$ and defined on $\mathfrak{N}_0$, in the following two steps: $\xi' = \xi + \Delta[\zeta]$ and $\xi'' = \xi' + \Delta[\zeta']$, where $\zeta(r, p)$ is the left invariant integral of $l \to -\xi(r, l, p)\nu(l)$, while $\zeta'(r, p)$ is the right invariant integral of $u \to -\xi'(u, r, up)\nu^*(u)$. A rather simple computation in which we use (16) and (15) and the invariance property of the Haar measures shows that $\xi''(r, s, p)$ is a differentiable (up to any order) exponent in all variables. Next we shall show that if two differentiable exponents $\xi$ and $\xi'$ are equivalent, that is, if $\xi' = \xi + \Delta[\zeta]$, then $\zeta(r, p)$ is differentiable in $r$. Clearly, the difference $\xi' - \xi$ is differentiable. Similarly, both $(\xi' - \xi)\nu$ (with $\nu$ defined as above), as well as its left invariant integral $\eta$ are differentiable. It is easy to show that $\zeta' = \eta - \zeta$ is also differentiable. In this way we arrive at differentiability of $\zeta = \eta - \zeta'$ is differentiable. A slightly more complicated argumentation shows that every (local) exponent of one-parameter group is equivalent to zero. However, the argumentation is quite analogous to that of Bargmann. We can treat such a group as the additive
group of real numbers, so that the first two arguments of \( \xi \) are real numbers. Let us set \( \vartheta(\tau, \sigma, p) \) as the derivative of \( \xi \) with respect to the second argument. It is not hard to show that \( \xi + \Delta[\xi] = 0 \), where \( \zeta(\tau, p) \) is the ordinary Riemann integral of \( \mu \rightarrow \tau \vartheta(\mu \tau, 0, p) \) over the unit interval \([0, 1]\). But it means that \( \xi \) is equivalent to zero.

Let us recall that the continuous curve \( r(\tau) \) in a Lie group \( G \) is a one-parameter subgroup if and only if \( r(\tau_1)r(\tau_2) = r(\tau_1 + \tau_2) \), i.e. \( r(\tau) = (r_0)^\tau \) for some element \( r_0 \in G \). (Please note that the real power \( r^\tau \) is well defined on a Lie group, at least on some neighborhood of \( e \).) The coordinates \( \rho^k \) in \( G \) are canonical if and only if any curve of the form \( r(\tau) = \tau \rho^k \) (where the coordinates \( \rho^k \) are fixed) is a one-parameter subgroup (the curve \( r(\tau) = \tau \rho^k \) will be denoted in short by \( \tau a \), with the coordinates of \( a \) equal to \( \rho^k \)). The "vector" \( a \) is called by physicists the generator of the one-parameter subgroup \( \tau a \).

A local exponent \( \xi \) of a Lie group \( G \) is called canonical if \( \xi(r, s, p) \) is differentiable in all variables, and \( \xi(r, s, p) = 0 \) if \( r \) and \( s \) are elements of the same one-parameter subgroup.

Almost the same argument used to show that every \( \xi \) on a one-parameter group is equivalent to zero also shows that everyone local exponent \( \xi \) of a Lie group is equivalent to a canonical local exponent. In order to prove this, we shall apply the argument to the exponent \( \xi_0(\tau, \sigma, p) := \xi(\tau a, \sigma a, p) \), cf. (J. Wawrzycki, [1]). Up to now, the argumentation has been more or less analogous to that of Bargmann. From now on, the argumentation becomes entirely different. Let \( \xi \) and \( \xi' \) be two differentiable and equivalent local exponents of a Lie group \( G \), assuming \( \xi \) to be canonical. Then \( \xi' \) is canonical if and only if \( \xi' = \xi + \Delta[\Lambda] \), where \( \Lambda(r, p) \) is a linear form in the canonical coordinates of \( r \) fulfilling the condition that \( \Lambda(a, (\tau a)p) \) is constant as a function of \( \tau \), i.e. it follows that

\[
a \Lambda(a, p) = \frac{d\Lambda(a, (\tau a)p)}{d\tau} = \lim_{\epsilon \rightarrow 0} \frac{\Lambda(a, (\epsilon a)p) - \Lambda(a, p)}{\epsilon} = 0. \tag{23}
\]

While sufficiency of the condition in the above statement is almost evident, proving its necessity is quite nontrivial. Hereafter we outline the argumentation. Because the exponents are equivalent we have \( \xi'(r, s, p) = \xi(r, s, p) + \Delta[\zeta] \). Since both \( \xi \) and \( \xi' \) are differentiable then \( \zeta(r, p) \) is also a differentiable function, which follows from what has been said above. Let us suppose that \( r = \tau a \) and \( s = \tau' a \). Because both \( \xi \) and \( \xi' \) are canonical, we have \( \xi(\tau a, \tau' a, p) = \xi'(\tau a, \tau' a, p) = 0 \), so that \( \Delta[\zeta](\tau a, \tau' a, p) = 0 \).

\[\text{\footnote{The limit in the expression can be understood in the ordinary point-wise sense with respect to } p, \text{ but also in any linear topology in the function linear space (with obvious addition) of } \vartheta(p), \text{ providing that } p \rightarrow \Lambda(a, p) \text{ is differentiable in the sense of this linear topology. Further on, the simple notation}
\]

\[
a f(p) = \frac{df((\tau a)p)}{d\tau} \bigg|_{\tau = 0} = \lim_{\epsilon \rightarrow 0} \frac{f((\epsilon a)p) - f(p)}{\epsilon},
\]

\[\text{will be used.}\]
Applying the last formula recurrently one gets
\[ \zeta(\tau a, p) = \sum_{k=0}^{n-1} \zeta\left(\frac{\tau}{n}, (-\frac{k}{n})\tau a, p\right). \]

Then we use the Taylor Theorem to each summand in the above expression, and pass to the limit \( n \to +\infty \). In this way, we obtain

\[ \zeta(\tau a, p) = \int_0^\tau \varsigma(a, (-\sigma a) p) \, d\sigma, \tag{24} \]

where \( \varsigma = \varsigma(r, p) \) is a differentiable function, cf. (J. Wawrzycki, math-ph/0301005). If we differentiate now expression (24) with respect to \( \tau \) at \( \tau = 0 \), we will immediately see that the function \( \varsigma(a, p) \) is linear with respect to \( a \). Let us suppose that the spacetime coordinates are chosen in such a way that the integral curves \( p(x) = (xa)p_0 \) are coordinate lines, which is possible for appropriately small \( x \). There are of course three remaining families of coordinate lines besides \( p(x) \), which can be chosen in an arbitrary way, with their parameters denoted by \( y_i \). After this,

\[ \zeta(a, x, y_i) = \frac{1}{\tau} \int_0^\tau \varsigma(a, x - \sigma, y_i) \, d\sigma = \frac{1}{\tau} \int_{x-\tau}^x \varsigma(a, z, y_i) \, dz, \]

for any \( \tau \) (of course, with appropriately small \( |\tau| \), in our case \( |\tau| \leq 1 \)) and for any (appropriately small) \( x \). But this is possible for the function \( \varsigma(a, x, y_k) \) continuous in \( x \) (in our case, differentiable in \( x \)) if and only if \( \varsigma(a, x, y_k) \) does not depend on \( x \). This means that \( \varsigma(a, x, y_k) \) does not depend on \( x \) and the condition of the statement is hereby proved.

**Infinitesimal exponents and embedding of \( H \) in a Lie group \( D \).** According to what has been shown already, we can assume that the exponent is canonical. We also confine ourselves to the subgroup of \( \{ \theta(p), r \} \in H \) with differentiable \( \theta \), and denote this subgroup by the same letter \( H \). We embed this subgroup \( H \) in an infinite dimensional Lie group with the manifold structure modeled on a Banach space. We will extensively use the theory developed by Birkhoff (loc. cit.) and Dynkin (loc. cit.). For the systematic treatment of manifolds modeled on Banach spaces, see e.g. (S. Lang, Differential Manifolds, Springer-Verlag, Berlin, Heidelberg, New York (1985)). By this embedding we ascribe bi-uniquely a Lie algebra to the group \( H \) with the convergent Baker-Hausdorff series.

Please note first that the formula

\[ H \times \mathcal{L}^2(\mathcal{M}, \mu, \mathcal{H}) \ni (\{ \theta(p), r \}, \phi) \to e^{i\theta(p)}T_r\phi \]

(together with (20)) can be viewed as a rule giving the action of \( H \) in the direct integral Hilbert space \( \int_{\mathcal{M}} \mathcal{H}_p \, d\mu(p) \) defined in Section 3. Moreover, this is a unitary action, provided \( \mu \) is \( G \)-invariant. In accordance to Birkhoff (loc. cit.), the group
D of all unitary operators of a Hilbert space is an infinite dimensional Lie group. Hence, \( H = N \otimes G \) can be viewed as a subgroup of a Lie group.

We consider now the closure \( \overline{H} \) of \( H \) in the sense of the topology in \( D \). It is remarkable that the subgroup \( \overline{H} \) has locally the structure of the semi-direct product \( \overline{N} \otimes \overline{G} \) as well. This is a consequence of the following four facts. (1) \( \overline{N} \) is a normal subgroup of \( \overline{H} = \overline{N} \cdot \overline{G} \). (2) \( G \) is finite dimensional, so \( \overline{G} = G \). (3) Locally (in a neighborhood \( \mathcal{O} \)), the multiplication in \( D \) is given by the Baker-Hausdorff formula in the Banach algebra of \( D \). Because \( \overline{N} \) is normal in \( \overline{H} \), then the above exponential mapping converts locally the multiplication \( \overline{N} \cdot S \) of \( \overline{N} \) by any subset \( S \) of \( \overline{H} \) into the sum \( \overline{N} + S \). Because \( G \) is finite-dimensional, and hence locally compact, the neighborhood \( \mathcal{O} \) can be chosen in such a way that locally (in the closure of \( \mathcal{O} + \mathcal{O} \)) the following holds:

\[
\overline{N} + \overline{G} = \overline{N} + \overline{G} = \overline{H}.
\]

(4) The local \( \overline{N} \) (intersected with \( \mathcal{O} \)) has a finite co-dimension in local \( \overline{N} + \overline{G} \) (intersected with \( \mathcal{O} + \mathcal{O} \)) and thus it splits locally \( \overline{N} + \overline{G} \). This is a consequence of the following four facts. (1) \( \overline{N} \) is a normal subgroup of \( \overline{H} = \overline{N}/\overline{squaresmall G} \). (2) \( \overline{G} \) is finite dimensional, so \( \overline{G} = G \). (3) Locally (in a neighborhood \( \mathcal{O} \)), the multiplication in \( D \) is given by the Baker-Hausdorff formula in the Banach algebra of \( D \). Because \( \overline{N} \) is normal in \( \overline{H} \), then the above exponential mapping converts locally the multiplication \( \overline{N} \cdot S \) of \( \overline{N} \) by any subset \( S \) of \( \overline{H} \) into the sum \( \overline{N} + S \). Because \( G \) is finite-dimensional, and hence locally compact, the neighborhood \( \mathcal{O} \) can be chosen in such a way that locally (in the closure of \( \mathcal{O} + \mathcal{O} \)) the following holds:

\[
\overline{N} + \overline{G} = \overline{N} + \overline{G} = \overline{H}.
\]

Because \( \overline{H} = \overline{N} \otimes \overline{G} \), every \( h \in \overline{H} \) is uniquely representable in the form \( ng \), where \( n \in \overline{N} \) and \( g \in \overline{G} \). Please note now that

\[
(n_1g_1(g_1n_2g_1^{-1}g_1g_2) = [n_1(g_1n_2g_1^{-1})](g_1g_2)
\]

and that \( g_1n_2g_1^{-1} \in \overline{N} \) because \( \overline{N} \) is normal in \( \overline{H} \). Let us denote the automorphism \( n \to gng^{-1} \) of \( \overline{N} \) by \( R_g \). The group \( \overline{H} \) can be locally viewed as a topological product of Banach spaces \( \overline{\mathfrak{N}} \times \overline{\mathfrak{G}} \), one of which (namely \( \overline{\mathfrak{G}} \)) is finite-dimensional and isomorphic to the Lie algebra of \( G \). The multiplication in \( \overline{H} \) can be written as \( (n_1g_1)(n_2g_2) = (n_1R_{g_1}(n_2),g_1g_2) \). Moreover, \( \overline{N} \) can be viewed locally as the Banach space \( \overline{\mathfrak{N}} \) with the multiplication law given by the vector addition in \( \overline{\mathfrak{N}} \).

Our task now is to reconstruct the Lie algebra \( \overline{\mathfrak{G}} \) corresponding to the subgroup \( \overline{H} \). Let \( \lambda \to \lambda a \) be a one-parameter subgroup of \( G \). The mapping \( (\lambda,n) \to (R_{\lambda a}n,\lambda a) \) of the Banach space \( \mathfrak{R} \times \overline{\mathfrak{G}} \) into the Banach space \( \overline{\mathfrak{R}} \times \overline{\mathfrak{G}} \) is continuous. In consequence, \( \mathfrak{R} \ni \lambda \to R_{\lambda a}n \in \overline{\mathfrak{R}} \) as well as \( \overline{\mathfrak{R}} \ni n \to R_{\lambda a}n \) are continuous. Therefore, the function \( \lambda \to R_{\lambda a}n \) can be integrated over any compact interval and

\[
\tau \to (n_{\tau a},\tau a) := \left( \int_0^\tau R_{\sigma a}n \, d\sigma, \tau a \right)
\]

is a one-parameter subgroup of \( \overline{H} \) with generator\(^{15} \) \( (n,a) \), cf. Birkhoff (loc. cit.), Dynkin (loc. cit). Having obtained this, we are able to reconstruct the algebra. The

\(^{15}\)The limit process with the help of which the generator is computed refers to the topology in \( D \), of course.
elements of \( H \subset \mathcal{H} \) are representable in the ordinary form \( \{ \alpha, r \} \) with differentiable \( \alpha = \alpha(p), p \in \mathcal{M}, \) and \( r \in G. \) Let us consider the above-defined operator \( R_{\lambda a}. \) Its restriction to \( H \subset \mathcal{H} \) is given by (please remember that \( \xi \) is canonical)

\[
\alpha(p) \rightarrow (R_{\lambda a}\alpha)(p) = \alpha((\lambda a)^{-1}p).
\]

We can now compute explicitly the Lie bracket and the Jacobi identity for all the elements \( \{ \alpha(p), a \} \) of the subalgebra \( \mathfrak{H} \subset \mathfrak{H} \) corresponding to the subgroup \( H. \) The result is as follows\(^{16}\)

\[
[a, b] = \{ a \beta - b \alpha + \Xi(a, b, p), [a, b] \}, \quad (26)
\]

\[
\Xi(a, b, p) = \lim_{\tau \to 0} \tau^{-2} \{ \xi((\tau a)(\tau b), (\tau a)^{-1}(\tau b)^{-1}, p) + \xi(\tau a, \tau b, p) + \xi((\tau a)^{-1}, (\tau b)^{-1}, (\tau a)^{-1}(\tau b)^{-1}p) \}, \quad (27)
\]

From the associative law in \( H \) one gets

\[
\Xi([a, a'], a'', p) + \Xi([a', a''], a, p) + \Xi([a'', a], a', p) = a\Xi(a', a'', p) + a'\Xi(a'', a, p) + a''\Xi(a, a', p), \quad (28)
\]

which can be shown to be equivalent to the Jacobi identity

\[
[[a, a'], a''] + [[a', a''], a] + [[a'', a], a'] = 0. \quad (29)
\]

Thus, in this way we have reconstructed the Lie algebra \( \mathfrak{H} \) giving explicitly \( [a, b] \) for all \( a, b \in \mathfrak{H} \subset \mathfrak{H}. \) Because \( \mathfrak{H} \) is dense in \( \mathfrak{H}, \) the local exponent \( \Xi \) determines the algebra \( \mathfrak{H} \) uniquely. But from the theory of Lie groups the correspondence between the algebras \( \mathfrak{H} \) and local Lie groups \( \mathcal{H} \) is bi-unique, at least locally, cf. e.g. Birkhoff (loc. cit.) and Dynkin (loc. cit.). Therefore, the correspondence \( \mathcal{H} \rightarrow \mathfrak{H} \) between the local group \( \mathcal{H} \) and the algebra \( \mathfrak{H} \) is one-to-one. Because the exponent \( \xi \) determines the multiplication rule in \( H \) and vice-versa, then it follows that the correspondence \( \xi \rightarrow \Xi \) between the local \( \xi \) and the infinitesimal exponent \( \Xi \) is one-to-one. Please note that the term 'local \( \xi = \xi(r, s, p) \)' means that \( \xi(r, s, p) \) is defined for \( r \) and \( s \) belonging to a fixed neighborhood \( \mathfrak{H}_0 \subset G \) of \( e \in G, \) but in our case it is defined globally as a function of the spacetime variable \( p \in \mathcal{M}. \)

**Infinitesimal exponents and local exponents.** Now, let us move on to describing the relation between the infinitesimal exponents \( \Xi \) and the local exponents

\[a \theta(p) = \lim_{\epsilon \to 0} \frac{\theta((\epsilon a)p) - \theta(p)}{\epsilon}, \quad (25)\]

and the limit is in the sense of topology induced from the Lie group \( D. \)
ξ. First, let us compute the infinitesimal exponents Ξ and Ξ′ given by (27), which correspond to the two equivalent canonical local exponents ξ and ξ′ = ξ + Δ[Λ]. Inserting ξ′ = ξ + Δ[Λ] into formula (27), one gets
\[ Ξ′(a, b, p) = Ξ(a, b, p) + aΛ(b, p) - bΛ(a, p) - Λ([a, b], p). \] (30)

According to what has been said, we can confine ourselves to the canonical exponents. Then, as one of our previous statements said, Λ = Λ(a, (τb)p) is a constant function of τ if a = b, and Λ(a, p) is linear with respect to a (we use the canonical coordinates on G). Hence Ξ′(a, b, p) is antisymmetric in a and b and fulfills (28) only if Ξ(a, b, p) is antisymmetric in a and b and fulfills (28). This suggests the definition: two infinitesimal exponents Ξ and Ξ′ will be called equivalent if and only if relation (30) holds. For brevity, we write relation (30) as follows:
\[ Ξ′ = Ξ + d[Λ]. \]

Finally, we maintain that two canonical local exponents ξ and ξ′ are equivalent if and only if the corresponding infinitesimal exponents Ξ and Ξ′ are equivalent. Indeed.

1. Assume ξ and ξ′ to be equivalent. Then, by the definition of equivalence of infinitesimal exponents: Ξ′ = Ξ + d[Λ]. (2) Assume Ξ and Ξ′ to be equivalent: Ξ′ = Ξ + d[Λ] for some linear form Λ(a, t) such that Λ(a, (τa)p) does not depend on τ. Then ξ + Δ[Λ] → Ξ′, and by the uniqueness of the correspondence ξ → Ξ we have ξ′ = ξ + Δ[Λ], i.e. ξ and ξ′ are equivalent. In this way, we arrive at the following:

**Theorem 3** (1) On a Lie group G, every local exponent ξ(r, s, p) is equivalent to a canonical local exponent ξ′(r, s, p) which, on some canonical neighborhood N₀, is analytic in canonical coordinates of r and s, and vanishes if r and s belong to the same one-parameter subgroup. Two canonical local exponents ξ, ξ′ are equivalent if and only if ξ′ = ξ + Δ[Λ] on some canonical neighborhood, where Λ(r, p) is a linear form in the canonical coordinates of r such that Λ(r, sp) does not depend on s if s belongs to the same one-parameter subgroup as r. (2) To every canonical local exponent of G there corresponds uniquely an infinitesimal exponent Ξ(a, b, p) on the Lie algebra θ of G, i.e. a bilinear antisymmetric form which satisfies the identity Ξ([a', a''], p) + Ξ([a', a''], a, p) + Ξ(a'', a', p) = a Ξ(a', a'', p) + a' Ξ(a'', a', p) + a'' Ξ(a, a', p). The correspondence is linear. (3) Two canonical local exponents ξ, ξ′ are equivalent if and only if the corresponding Ξ, Ξ′ are equivalent, i.e. Ξ′(a, b, p) = Ξ(a, b, p) + aΛ(b, p) - bΛ(a, p) - Λ([a, b], p) where Λ(a, p) is a linear form in a on θ such that τ → Λ(a, (τb)p) is constant if a = b. (4) There exist a one-to-one correspondence between the equivalence classes of local exponents ξ (global in p ∈ M) of G and the equivalence classes of infinitesimal exponents Ξ of θ.

### 3.3 Global Extensions of Local Exponents

Theorem 3 provides full classification of exponents ξ(r, s, p) local in r and s, defined for all p ∈ M. But if G is both connected and simply connected, then we have the
following theorems. (1) If an extension $\xi'$ of a given local (in $r$ and $s$) exponent $\xi$ does exist, then it is uniquely determined (up to the equivalence transformation $h$) (Theorem 4). (2) There exists such an extension $\xi'$ (Theorem 5), proved for $G$, which possess finite-dimensional extension $\mathcal{F}'$ only.

We are not able to prove that the (global) homomorphism $\mathcal{F}$ is continuous when $\xi$ is not canonical. Please note that any $\xi$ is equivalent to its canonical counterpart, but only locally! This is why the topology of $H$ induced from $D$ is not applicable in the global analysis. We introduce another topology. Because of the semidirect structure We introduce another topology. Because of the semidirect structure of $H = N \mathcal{S} G$, it is sufficient to introduce it into $N$ and $G$ separately in such a manner that $G$ acts continuously on $N$, cf. e.g. Mackey (loc. cit.). From the discussion of Section 3.2 it is sufficient to introduce the Fréchet topology of almost uniform convergence in the function space $N$. Indeed, from the strong continuity of $\xi$ and $\zeta$ in (22) it follows that the multiplication rule as well as the homomorphism $\mathcal{F}$ are continuous.

**Theorem 4** Let $\xi$ and $\xi'$ be two equivalent local exponents of a connected and simply connected group $G$, so that $\xi' = \xi + \Delta[\xi]$ on some neighborhood, assuming the exponents $\xi_1$ and $\xi'_1$ of $G$ to be extensions of $\xi$ and $\xi'$ respectively. Then, for all $r, s, p \in G$: $\xi_1(r, s, p) = \xi_1(r, s, p) + \Delta[\xi_1]$, where $\xi_1(r, p)$ is strongly continuous in $r$ and differentiable in $p$, and $\xi_1(r, p) = \xi(r, p)$, for all $p \in \mathcal{M}$ and for all $r$ belonging to some neighborhood of $e \in G$.

Here is the proof outline. The two exponents $\xi_1$ and $\xi'_1$ being strongly continuous (by assumption) define two semicentral extensions $H_1 = N_1 \mathcal{S} G$ and $H'_1 = N'_1 \mathcal{S} G$, which are connected and simply connected. Please note that the linear groups $N_1, N_1'$ are connected and simply connected. Because both $H_1$ and $H'_1$ are semi-direct products of two connected and simply connected groups they are both connected and simply connected. Eq. (22) defines a local isomorphism mapping $h : \tilde{r} \to \tilde{r}' = h(\tilde{r})$ of $H_1$ into $H'_1$. Because $H_1$ and $H'_1$ are connected and simply connected, the isomorphism $h$ given by (22) can be uniquely extended to an isomorphism $h_1$ of the entire groups $H_1$ and $H'_1$ such that $h_1(\tilde{r}) = h(\tilde{r})$ on some neighborhood of $H_1$, cf. (L. Pontrjagin, Topological groups, Moscow (1984) (in Russian)), Theorem 80. The isomorphism $h_1$ defines an isomorphism of the two Abelian subgroups $N_1$ and $h_1(N_1)$. By (22), $h_1(\theta, e) = \{\theta, e\}$ locally in $H_1$, that is for $\theta$ lying appropriately close to 0 (in the metric sense defined previously). Both $N_1$ and $h_1(N_1)$ are connected, and $N_1$ is in addition simply connected, so applying once again Theorem 80 of Pontrjagin (loc. cit.), one can see that $h_1(\theta, e) = \{\theta, e\}$ for all $\theta$. A rather simple computation shows that $\zeta_1$ defined by the equality $h_1(0, r) = \{-\zeta_1(p), g(r)\}$ fulfills the conditions of our theorem.

The following theorem is proved for the group $G$ with a finite-dimensional extended algebra $\mathcal{F}'$. 
Theorem 5 Let \( G \) be a connected and simply connected Lie group. Then to every exponent \( \xi(r, s, X) \) of \( G \) defined locally in \((r, s)\) there exists an exponent \( \xi_0 \) of \( G \) defined on the whole group \( G \) which is an extension of \( \xi \). If \( \xi \) is differentiable, \( \xi_0 \) may be chosen differentiable.

Because the proof of Theorem 5 is almost identical to that of Theorem 5.1 in Bargmann (loc. cit.), we do not present it explicitly. Please note that the proof rests largely on the global theory of classical (finite-dimensional) Lie groups. Namely, it rests on the theorem that to any finite dimensional Lie group there always exists a universal covering group. We can use those methods because of the existence of a finite-dimensional extension \( H' \) of \( G \).

We have obtained the full classification of time-dependent \( \xi \) defined on the whole group \( G \) for Lie groups \( G \) which are connected and simply connected in non-relativistic theory. But for any Lie group \( G \) there exists a universal covering group \( G^* \) which is connected and simply connected. Thus, for \( G^* \) the correspondence \( \xi \rightarrow \Xi \) is one-to-one, that is, to every \( \xi \) there exists a unique \( \Xi \) and vice versa, to every \( \Xi \) corresponds a unique \( \xi \) defined on the whole group \( G^* \), and the correspondence preserves the equivalence relation. Because \( G \) and \( G^* \) are locally isomorphic, the infinitesimal exponents \( \Xi \)'s are exactly the same for \( G \) and for \( G^* \). Since to every \( \Xi \) there does exist exactly one \( \xi \) on \( G^* \), so, if to a given \( \Xi \) there exists the corresponding \( \xi \) on the whole \( G \), then such a \( \xi \) is unique. In this way, we have obtained the full classification of \( \xi \) defined on a whole Lie group \( G \) for any Lie group \( G \), in the sense that no \( \xi \) can be omitted in the classification. The set of equivalence classes of \( \xi \) is considerably smaller than that for \( \Xi \); it may happen that to some local \( \xi \) there does not exist any global extension.

3.4 Examples

Example 1: The Galilean Group. According to the conclusions of subsection 2.2. one should a priori investigate such representations of the Galilean group \( G \) which fulfill Eq. (3), with \( \xi \) depending on time. Then, the following paradox arises. Why has the transformation law \( T_r \) under the Galilean group a time-independent \( \xi \) in (3), regardless of whether it is a covariance group or a symmetry group? We will solve the paradox in this subsection. Namely, we will show that any representation of the Galilean group fulfilling (3) is equivalent to a representation fulfilling (3) with time-independent \( \xi \). This is a rather peculiar property of the Galilean group, not valid in general. For example, this is not true for the group of Milne transformations.

\[\text{In this proof we consider the finite-dimensional extension } H' \text{ of } G \text{ instead of the Lie group } H \text{ in the proof presented in Bargmann (loc. cit.). The remaining replacements are rather trivial, but we mark them here explicitly to simplify the reading. (1) Instead of the formula } r' = l(\theta)\bar{r} = \bar{r}l(\theta) \text{ of (5.3) in Bargmann (loc. cit.), we have } r' = l(\theta(r^{-1}p))\bar{r} = \bar{r}l(\theta(p)). \text{ Thus, from the formula } (h_1(r)h_1(s)h_1(g) = h_1(r)(h_1(s)h_1(g)) \text{ see V. Bargmann, loc. cit. it follows that } \xi(rs, g, p) = \xi(s, g, r^{-1}p) + \xi(r, s, p) \text{ instead of (5.8) in Bargmann (loc. cit.). (2) Instead of (4.9), (4.10) and (4.11) we use the Iwasawa-type construction presented in this paper.} \]
In non-relativistic theory $\xi = \xi(r, s, t)$ depends on the time. In this case, according to our assumption about $G$, any $r \in G$ transforms simultaneity hyperplanes into simultaneity hyperplanes. Thus, there are two possibilities for any $r \in G$. First, when $r$ does not change time: $t(rp) = t(p)$, and the second in which time is changed, but in such a way that $t(rp) - t(p) = f(t)$. We assume in addition that the base generators $a_k \in \mathfrak{g}$ can be chosen in such a way that only one acts on time as translation and the remaining ones do not act on time. We can assume that the operators $a_i$ are ordinary differential operators. Hence, the Jacobi identity (28) reads

$$\Xi([a, a'], a'') + \Xi([a', a''], a) + \Xi([a'', a], a') = \partial_t \Xi(a', a''),$$

(31)

if one and only one among $a, a', a''$ is the time-translation generator, namely $a$, and

$$\Xi([a, a'], a'') + \Xi([a', a''], a) + \Xi([a'', a], a') = 0,$$

(32)

in all of the remaining cases. The Jacobi identities (31) and (32) can be treated as a system of ordinary differential linear equations for the finite set of unknown functions $\Xi_{ij}(t) = \Xi(a_i, a_j, t)$, where $a_i$ is the base in the Lie algebra of $G$.

According to Section 3.1, in order to classify all $\xi$ of $G$ we shall determine all equivalence classes of infinitesimal exponents $\Xi$ of the Lie algebra $\mathfrak{g}$ of $G$. The commutation relations for the Galilean group are as follows

$$[a_{ij}, a_{kl}] = \delta_{jk}a_{il} - \delta_{ik}a_{jl} + \delta_{il}a_{jk} - \delta_{jl}a_{ik},$$

(33)

$$[a_{ij}, b_k] = \delta_{jk}b_i - \delta_{ik}b_j,\ [b_i, b_j] = 0,$$

(34)

$$[a_{ij}, d_k] = \delta_{jk}d_i - \delta_{ik}d_j,\ [d_i, d_j] = 0,\ [b_i, d_j] = 0,$$

(35)

$$[a_{ij}, \tau] = 0,\ [b_k, \tau] = 0,\ [d_k, \tau] = b_k,$$

(36)

where $b_i, d_i$ and $\tau$ stand for the generators of space translations, with the proper Galilean transformations and time translation respectively and $a_{ij} = -a_{ji}$ being rotation generators. Please note that the Jacobi identity (32) is identical to that in the ordinary Bargmann’s theory of time-independent exponents see Bargmann (loc. cit., Eqs (4.24) and (4.24a)). Thus, using (33) – (35) we can proceed exactly after Bargmann (loc. cit.), pages 39, 40) and show that any infinitesimal exponent defined on the subgroup generated by $b_i, d_i, a_{ij}$ is equivalent to an exponent equal to zero, with the possible exception of $\Xi(b_i, d_k, t) = \gamma \delta_{ik}$, where $\gamma = \gamma(t)$. Hence, the only components of $\Xi$ defined on the whole algebra $\mathfrak{g}$ which can be a priori not equal to zero are: $\Xi(b_i, d_k, t) = \gamma \delta_{ik}$, $\Xi(a_{ij}, \tau, t)$, $\Xi(b_i, \tau, t)$ and $\Xi(d_k, \tau, t)$. First, we compute function $\gamma(t)$. Substituting $a = \tau$, $a' = b_i, a'' = d_k$ to (31), we get $d\gamma/dt = 0$, so that $\gamma$ is a constant, denoting the constant value of $\gamma$ by $m$. By
inserting \( a = \tau, \ a' = a_i, \ a'' = a_{s'j} \) to (31) and summing up with respect to \( s \), we get \( \Xi(a_{ij}, \tau, t) = 0 \). In the same way, but with the substitution \( a = \tau, \ a' = a_i, \ a'' = b_s \), one shows that \( \Xi(b_i, \tau, t) = 0 \). At last, the substitution \( a = \tau, \ a' = a_i, \ a'' = d_s \) to (31) and summation with respect to \( s \) gives \( \Xi(d_i, \tau, t) = 0 \). In this way, we have proved that any time-dependent \( \Xi \) on \( G \) is equivalent to a time-independent one. In other words, we get a one-parameter family of possible \( \Xi \), with the parameter equal to the inertial mass \( m \) of the system in question. Any infinitesimal time-dependent exponent of the Galilean group is equivalent to the above time-independent exponent \( \Xi \) with some value of the parameter \( m \); and any two infinitesimal exponents with different values of \( m \) are nonequivalent. As was argued in subsection 3.1 (Theorems 3 \( \div \) 5), the classification of \( \Xi \) gives a full classification of \( \xi \). Moreover, it can be shown that the classification of \( \xi \) is equivalent to the classification of possible \( \theta \)-s in the transformation law

\[
T_r \psi(p) = e^{i\theta(r,p)} \psi(r^{-1}p)
\]  

for the spinless non-relativistic particle. On the other hand, the exponent \( \xi(r, s, t) \) of the representation \( T_r \) given by (37) can be easily computed to be equal to \( \theta(r, s, p) - \theta(r, p) - \theta(s, r^{-1}p) \), and the infinitesimal exponent belonging to \( \theta \) defined as \( \theta(r, p) = -m\vec{v} \cdot \vec{x} + \frac{m}{2} \vec{v}^2 t \), covers the whole one-parameter family of the classification (its infinitesimal exponent is equal to that infinitesimal exponent \( \Xi \), which has been found above). Thus, the standard \( \theta(r, p) = -m\vec{v} \cdot \vec{x} + \frac{m}{2} \vec{v}^2 t \), covers the full classification of possible \( \theta \)-s in (37) for the Galilean group. Inserting the standard form for \( \theta \) we see that \( \xi \) does not depend on time but only on \( r \) and \( s \). By this, any time-dependent \( \xi \) on \( G \) is equivalent to its time-independent counterpart. In this way, we have reconstructed the standard result. Using now the formula\(^{18}\)

\[
A_i \psi(p) = \lim_{\sigma \to 0} \frac{(T_{\sigma a_i} - 1)\psi(p)}{\sigma},
\]

for the generator \( A_i \) corresponding to \( a_i \), we get the standard commutation relations for the ray representation \( T_r \) of the Galilean group

\[
[A_{ij}, A_{kl}] = \delta_{jk}A_{il} - \delta_{ik}A_{jl} - \delta_{jl}A_{ik},
\]

\[
[A_{ij}, B_k] = \delta_{jk}B_i - \delta_{ik}B_j, \quad [B_i, B_j] = 0,
\]

\[
[A_{ij}, D_k] = \delta_{jk}D_i - \delta_{ik}D_j,
\]

\[
[D_i, D_j] = 0, \quad [B_i, D_j] = m\delta_{ij},
\]

\(^{18}\)The transformation \( T_r \) does not act in the ordinary Hilbert space but in the Hilbert bundle space \( R \triangle \mathcal{H} \), hence we cannot immediately appeal to the Stone and Gårding Theorems. Nonetheless, \( T_r \) induces a unique unitary representation acting in the Hilbert space \( \int_R \mathcal{H}_t d\mu(t) \) and it can be shown that it is meaningful to talk about the generators \( A \) of \( T_r \).
\[ [A_{ij}, T] = 0, [B_k, T] = 0, [D_k, T] = B_k. \]

Please note that to any \( \xi \) (or \( \Xi \)) there exists a corresponding \( \theta \) (and such a \( \theta \) is unique up to a trivial equivalence relation). As we will see, this is not the case for the Milne group, where some \( \Xi \)'s do exist which cannot be realized by any \( \theta \).

**Example 2: Milne group and equality of inertial and gravitational masses.** In here we apply the theory of Section 3.1 to the Milne transformations group. We proceed like with the Galilean group in the preceding section. The Milne group \( G \) does not form any Lie group, but in the physical application it is sufficient for us to consider some Lie subgroups \( G(m) \) of the Milne group. We will go on according to the following plan. First, we compute the infinitesimal exponents and exponents for each \( G(m) \), \( m = 1, 2, \ldots \), and then the \( \theta \) in (37) for \( G(m) \). Please compare (J. Wawrzycki, math-ph/0301005), where the result is extended on the whole group.

The Milne transformation is defined as follows

\[ (\vec{x}, t) \rightarrow (R\vec{x} + \vec{A}(t), t + b), \tag{38} \]

where \( R \) is an orthogonal matrix, and \( b \) is constant. The extent of arbitrariness of function \( \vec{A}(t) \) in (38) will be left undetermined for now. It is convenient to rewrite the Milne transformations (38) in the following form

\[ \vec{x}' = R\vec{x} + A(t)v, \quad t' = t + b, \]

where \( v \) is a constant vector which does not depend on time \( t \). We define the subgroup \( G(m) \) of \( G \) as the group of the following transformations

\[ \vec{x}' = R\vec{x} + v^{(0)} + t\vec{v}_{(1)} + \frac{t^2}{2!}\vec{v}_{(2)} + \ldots + \frac{t^m}{m!}\vec{v}_{(m)}, \quad t' = t + b, \]

where \( \vec{v} \) is a constant vector which does not depend on time \( t \). We define the subgroup \( G(m) \) of \( G \) as the group of the following transformations

\[ [a_{ij}, a_{kl}] = \delta_{jk}a_{il} - \delta_{ik}a_{jl} + \delta_{il}a_{jk} - \delta_{il}a_{ik}, \tag{39} \]

\[ [a_{ij}, d_k^{(n)}] = \delta_{jk}d_i^{(n)} - \delta_{ik}d_j^{(n)}, \quad [d_i^{(n)}, d_j^{(k)}] = 0, \tag{40} \]

\[ [a_{ij}, \tau] = 0, \quad [d_i^{(0)}, \tau] = 0, \quad [d_i^{(n)}, \tau] = d_i^{(n-1)}, \tag{41} \]

where \( d_i^{(n)} \) is the generator of the transformation \( r(v_{(n)}^i) \):

\[ x'^i = x^i + \frac{t^n}{n!}v_{(n)}^i, \]
which will be called the \( n \)-acceleration, and 0-acceleration in the particular case of the ordinary space translation. All relations (39) and (40) are identical to (33) \({\delta_{ij}}\) with the \( n \)-acceleration instead of the Galilean transformation. Thus, the same argumentation as that used for the Galilean group gives: \( \Xi(\alpha_{ij}, \alpha_{kl}) = 0 \), \( \Xi(\alpha_{ij}, d^{(n)}_k) = 0 \), and \( \Xi(d^{(n)}_i, d^{(n)}_j) = 0 \). Substituting \( a^h_i, a^h_{hi}, \tau \) for \( a, a', a'' \) into Eq. (32), we get \( \Xi(\alpha_{ij}, \alpha_{kl}) = 0 \), making use of the commutation relations and summing up with respect to \( h \), we get \( \Xi(\alpha_{ij}, \tau) = 0 \). In an analogous way, substituting \( a^h_i, d^{(l)}_k, d^{(n)}_k \) for \( a, a', a'' \) into Eq. (22), we get \( \Xi(d^{(l)}_i, d^{(n)}_j) = \frac{1}{3} \Xi(d^{(l)}_h, d^{(n)}_h) \delta_{ik} \). Substituting \( a^h_i, d^{(n)}_h, \tau \) for \( a, a', a'' \) into Eq. (31), making use of the commutation relations, and summing up with respect to \( h \), we get \( \Xi(d^{(n)}_i, \tau) = 0 \). Now, we substitute \( d^{(n)}_i, d^{(0)}_i, \tau \) for \( a, a', a'' \) in (31), and proceed recurrently with respect to \( n \), obtaining in this way \( \Xi(d^{(0)}_i, d^{(n)}_i) = P^{(0,n)}(t) \delta_{ik} \), where \( P^{(0,n)}(t) \) is a polynomial of degree \( n - 1 \), and the time derivation of \( P^{(0,n)}(t) \) has to be equal to \( P^{(0,n)}(t) \). Substituting \( d^{(l)}_k, d^{(l)}_i, \tau \) into (31), in the same way we get \( \Xi(d^{(l)}_i, d^{(n)}_i) = P^{(l,n)}(t) \delta_{ki} \), where \( \frac{d}{dt} P^{(l,n)} = P^{(l-1,n)} + P^{(l,n-1)} \). This allows us to determine all \( P^{(l,n)}(t) \) by the recurrent integration process. Please note that \( P^{(0,0)} = 0 \), and \( P^{(l,n)} = -P^{(n,l)} \), so given the \( P^{(0,n)} \) we can compute all \( P^{(1,n)} \). Indeed, we have \( P^{(1,0)} = -P^{(0,1)} \). \( P^{(1,1)} = 0 \), \( P^{(1,2)}/dt = P^{(0,2)} + P^{(1,1)} \), \( P^{(1,3)}/dt = P^{(0,3)} + P^{(1,2)} \), \( P^{(2,2)} \), and after \( m + 1 \) integrations we compute all \( P^{(1,n)} \). Each elementary integration introduces a new independent parameter (the arbitrary additive integration constant). Exactly in the same way, given all \( P^{(1,n)} \), we can compute all \( P^{(2,n)} \) after \( m - 2 \) elementary integration processes. In general, the \( P^{(l-1,n)} \) allows us to compute all \( P^{(l,n)} \) after \( m - l \) integrations. Thus, \( P^{(l,n)}(t) \) are \( l + n - 1 \)-degree polynomial functions of \( t \), and all are determined by \( m(m+1)/2 \) integration constants. Because \( d[\Lambda](d^{(n)}_i, d^{(l)}_j) = 0 \), the exponents \( \Xi \) defined by different polynomials \( P^{(l,n)} \) are inequivalent. Therefore, the space of nonequivalent classes of \( \Xi \) is \( m(m+1)/2 \)-dimensional.

However, not all \( \Xi \) can be realized by the transformation \( T_r \) of the form (37). It can be seen that any integration constant \( \gamma_{l,q} \) of the polynomial \( P^{(l,q)}(t) \) has to be equal to zero if \( l, q \neq 0 \), provided the exponent \( \Xi \) belongs to the representation \( T_r \) of the form (37). By this, all exponents of \( G(m) \) which can be realized by the transformations \( T_r \) of the form (37) are determined by the polynomial \( P^{(0,m)} \), that is, by \( m \) constants. We omit the proof of this fact, and refer the reader to (J. Wawrzycki, math-ph/0301005), Example 2. In the proof we compute the exponent \( \Xi \) directly for the transformation \( T_r \) of the form (37) and compare it with the classification results above.

Consider the \( \theta \), given by the formula

\[
\theta(r, p) = \gamma_1 \frac{dA}{dt} + \gamma_2 \frac{d^2A}{dt^2} + \ldots + \gamma_m \frac{d^m A}{dt^m} + \bar{\theta}(t),
\]

for \( r \in G(m) \), where \( \gamma_i \) are the integration constants which define the polynomial \( P^{(0,m)} = \gamma_1 \frac{d^{m-1}}{(m-1)!} + \gamma_2 \frac{d^{(m-2)}}{(m-2)!} + \ldots + \gamma_m \), and \( \bar{\theta}(t) \) is any function of time \( t \), and
eventually of the group parameters. A rather simple computation shows that this \( \theta \) covers all possible \( \Xi \) which can be realized by (37). That is, the infinitesimal exponents corresponding to the \( \theta \) given by (42) yield all possible \( \Xi \) with all integration constants \( \gamma(k,n) = 0 \), for \( k, n \neq 0 \). Thus, the most general \( \theta(r,p) \) defined for \( r \in G(m) \) is given by (42).

At this point we make use of the assumption that the wave equation is local. It can be shown then (we leave this without proof) that the \( \theta(r,p) \) can be a function of a finite order derivatives of \( \vec{A}(t) \), say \( k \)-th at most, while the higher derivatives cannot enter into \( \theta \). Therefore, the most general \( \theta(r,p) \) defined for \( r \in G(m) \) has the following form

\[
\theta(r,X) = \gamma_1 \frac{d\vec{A}}{dt} + \ldots + \gamma_k \frac{d^k\vec{A}}{dt^k} + \tilde{\theta}(t).
\] (43)

Having obtained this we can infer the most general Schrödinger equation for a spinless particle in Newton-Cartan spacetime, cf. (J. Wawrzycki, Acta Phys. Polon. B 35, 613 (2004); gr-qc/0301102). The inertial and the gravitational masses are always equal in this equation. Our assumptions are, more precisely, as follows: (i) The quantum particle, when its kinetic energy is small in comparison to its rest energy \( mc^2 \), does not exert any influence on the space-time structure. (ii) The Born interpretation for the wave function is valid, and the transition probabilities in the Newton-Cartan space-time which describes geometrically Newtonian gravity, are equal to the ordinary integral over a simultaneity hyperplane and are preserved under the coordinate transformations. (iii) The wave equation is linear, of second order at most, generally covariant, and can be built in a local way with the help of the geometrical objects describing the space-time structure. (iv) The probability density \( \rho = \psi^* \psi \) is a scalar field (with the scalar transformation rule). In fact the conditions (i), (ii), (iii) and (iv) are somewhat interrelated. The coefficients \( a, b^i, \ldots \) in the wave equation

\[
\left[ a \frac{\partial^2}{\partial t^2} + b^i \frac{\partial}{\partial t} \partial_i + c^{ij} \partial_i \partial_j + f^i \partial_i + d \partial_t + g \right] \psi = 0,
\]

are local functions of the potential and therefore cannot depend on arbitrary high order derivatives of the potential. Within a rather standard analysis one gets the Schrödinger equation from (i), (ii), (iii), and (iv), which after the ordinary notation of constants has the form

\[
\left[ \frac{\hbar^2}{2m} \delta^{ij} \partial_i \partial_j + i\hbar \partial_t - m\phi + \Lambda \right] \psi = 0,
\]

with the \( \theta \) in \( T_r \) given by

\[
\theta = \frac{m}{2\hbar} \int_0^t \vec{A}^2(\tau) \, d\tau + \frac{m}{\hbar} \vec{A}_i x^i.
\]
\( \phi \) is the gravitational potential and \( \Lambda \) is one of the Kronecker’s invariants of the matrix \((\partial_a \partial_b \phi)\) in the above equation.

Note that the inertial mass \( m \) in the equation is equal to the parameter at the gravitational potential. That is, the gravitational mass must be equal to the inertial mass.