Dynamic Programming for One-Sided Partially Observable Pursuit-Evasion Games

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Abstract. We study two player pursuit-evasion games with concurrent moves, infinite horizon, and discounted rewards. The players have partial observability, however, the evader is given an advantage of knowing the current position of the units of the pursuer. We show that (1) value functions of this game depend only on the position of the pursuing units and the belief the pursuer has about the position of the evader, and (2) that they are piecewise linear convex functions in the belief. Finally, we exploit this representation of value functions and design a value iteration algorithm that approximates the value of the game.

1 Introduction

We consider two player finite-state discrete-time zero-sum partially observable stochastic game (POSG) modeling a pursuit-evasion scenario played on a finite graph. In this game a team of centrally controlled pursuing units (a pursuer) aims to locate and capture an evader, while the evader aims for the opposite. Such problems often arise in robotics and security domains [14,2].

We focus on the concurrent setup where both players act simultaneously and we aim to find robust strategies of the pursuer against the worst-case evader. Specifically, we assume that the evader knows the positions of the pursuing units and her only uncertainty is the strategy of the pursuer and the move that will be performed in the current time step. We assume games with infinite horizon and discounted rewards, where the value of the game exists [4], and we approximate optimal strategies of the pursuer.

Our approach extends partially observable models from decision theory, Partially Observable Markov Decision Processes (POMDPs) [11,8,9,12], where a single agent is supposed to find an optimal plan in a stochastic environment. We show that in our pursuit-evasion games the value functions that express the expected utility of the pursuer depend only on the position of the pursuing units and the belief about the position of the evader, but not the exact history of the actions of the pursuer. Despite the fact that players require randomized (or mixed) strategies, the value functions remain piecewise linear and convex, and we can approximate them by means of dynamic programming similar to the one for POMDPs [11,8]. Finally, we show that there is a unique set of value
functions (each corresponding to one initial position of the pursuer) solving the infinite-horizon game.

1.1 Related Work

A similar model with one-sided partial observability where one of the players has a perfect information was presented by McEneaney [7]. Authors assumed that the player with perfect information knows the action the opponent plays at the current stage. Due to the turn-based character of such game, the authors consider only pure strategies.

Our setting with concurrent moves better corresponds to the real-world situations that occur in real time. The evader does not know the action taken by the pursuer in the current stage; hence, the pursuer may need to use randomized strategies. However, allowing randomized strategies provides challenges when designing the dynamic programming operator that we address in this paper.

Another model that uses one-sided partial observability was considered by Chatterjee et al. [1], however with reachability and safety objectives (a player either wants to reach a set of target states or she wants to keep the system in a set of safe states) that do not translate to objectives with discounted rewards.

An algorithm for solving a broader class of POSGs where all players have imperfect information was proposed in [5]. Individual players may have obtained different information throughout the course the game and they may have differing beliefs. The proposed approach uses dynamic programming to iteratively construct pure strategies by extending non-dominated pure strategies with one step; hence the number of pure strategies grows exponentially with the horizon. The one-sided partial observability in our game allows us to avoid such enumeration of pure strategies and define the dynamic programming operator over the expected utility values of subgames.

First we define the finite-horizon version of the pursuit-evasion game, define the value functions, and prove its shape and the convexity. We follow by a dynamic programming method for computing values of the game.

2 Finite-horizon game

We use the notion of finite-horizon POSGs, or extensive-form games, to reason about the infinite-horizon pursuit-evasion game with discounted rewards. An extensive-form game (EFG) is a tuple $G = (\mathcal{N}, \mathcal{H}, \mathcal{Z}, \mathcal{T}, u, \mathcal{I})$. $\mathcal{N}$ is the set of players, in our case $\mathcal{N} = \{p, e\}$ where $p$ stands for pursuer and $e$ for the evader.

Set $\mathcal{H}$ denotes a finite set of histories of actions taken by all players from the beginning of the game. Every history corresponds to a node in the game tree; hence we use the terms history and node interchangeably. Each of the histories may be (1) terminal ($h \in \mathcal{Z} \subseteq \mathcal{H}$) where the game ends and players get utility $u_i(h)$, (2) controlled by the nature player who selects the successor node according to a fixed probability distribution known to all players, or (3) one of the players from $\mathcal{N}$ may be to act. We consider a zero-sum scenario where $u_p(h) = -u_e(h)$. To
simplify the notation we will use $u(h)$ to denote pursuer’s reward. An ordered list of transitions of player $i$ from root to node $h$ is referred to as a sequence. The allowed transitions in the game are modeled using a transition function $T$ that provides a set of successor nodes for each non-terminal history. The imperfect observation of players is modeled via information sets $I_i$ that form a partition over histories $h$ where player $i \in \mathcal{N}$ takes action. We assume perfect recall setting where the players never forget their past actions, i.e. for every $I_i \in I_i$, all histories $h \in I_i$ have the same player $i$’s sequence. Each information set $I_i \in I_i$ corresponds to one decision point of player $i$. A randomized behavioral strategy of player $i$ assigns a distribution over actions to each of the information sets in $I_i$. A Nash equilibrium (NE) in an EFG is a pair of behavioral strategies, in which each player plays a best response to the strategy of the opponent.

We will now use this terminology to construct an EFG for a finite-horizon version of a pursuit-evasion game played on graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ with $N$ pursuing units played for $T$ rounds (we term $T$ as the horizon). At every stage $t \leq T$, pursuer’s units occupy vertices $s^t_p$, where $s^t_p = \{s^t_{p,1}, \ldots, s^t_{p,N}\}$ is an $N$-element multiset of vertices of $\mathcal{G}$, and evader is located in vertex $s^t_e \in \mathcal{V}$. The goal of the pursuer is to achieve a situation where the evader is caught, i.e. $s^t_e \in s^t_p$. In every round, players have to move their units to vertices adjacent to their current positions ($\text{adj}_G(v)$ denotes the set of vertices adjacent to $v$). Position of the evader in round $t + 1$ is thus $s^{t+1}_e \in \text{adj}_G(s^t_e)$. We overload the operator $\text{adj}_G$ to apply it also on multisets representing positions of pursuer’s units, i.e. $s^{t+1}_p \in \text{adj}_G(s^t_p)$, where $\text{adj}_G(s^t_p) = \times_{i=1\ldots N} \text{adj}_G(s^t_{p,i})$.

A horizon-$T$ game $G^T_{s^0_p,b^0}$ is parametrized by an initial position of the pursuer $s^0_p \in \mathcal{V}^N$ and a distribution over evader’s initial positions $b^0 \in \Delta(\mathcal{V})$ (we term $b^0$ as initial belief) known to both players. The game starts with a chance move selecting the initial position of the evader $s^0_e$ (based on $b^0$). Players then alternate in choosing their actions $s^{t+1}_p \in \text{adj}_G(s^t_p)$ and $s^{t+1}_e \in \text{adj}_G(s^t_e)$ – pursuer plays first and the evader plays the second. The structure of information sets ensures that the evader is unable to detect the current action of the pursuer.
A history $h \in \mathcal{H}$ in a game with horizon $T$ corresponds to a list of positions $s^0_e s^1_p \ldots s^T_e$, where $t \leq T$. The utility values are assigned to terminal histories as follows: if $t = T$ and $s^t_e \not\in s^t_p$, the pursuer gets utility $u(h) = 0$; if $t \leq T$ and $s^t_e \in s^t_p$, the pursuer gets the reward $u(h) = \gamma^t$ for capturing the evader in $t$ rounds (where $\gamma \in [0, 1)$ stands for the discount factor). The transition function $T$ complies with the graph (i.e., the adjacency function $\text{adj}_G$), i.e. $s^t_p \in \text{adj}_G(s^{t-1}_p)$ and $s^t_e \in \text{adj}_G(s^{t-1}_p)$ for every $i \geq 1$. For notational simplicity we denote the sequence of pursuer’s actions $s^1_p \ldots s^T_p$ in $h$ as $h|_p$ and the sequence of evader’s actions $s^1_e \ldots s^T_e$ in $h$ as $h|_e$.

Pursuer cannot detect the position of the evader, unless she gets caught, hence due to the perfect recall assumption the game contains one pursuer’s information set $I^h_p$ for each of his histories where $I^h_p = \{h'|h' \in \mathcal{H} \setminus \mathcal{Z} : h'|_p = h|_p\}$.

Evader on the other hand knows the game situation almost perfectly. The only information missing to him relates to the current action being taken by the pursuer. For every history $h = s^0_e s^1_p \ldots s^T_e$ of the pursuer, there is an information set $I^h_e = \{h s^{t+1}_p | s^{t+1}_e \in \text{adj}_G(s^t_e)\}$ containing all the possible continuations of the pursuer (recall that in our formulation of the EFG, the pursuer acts first and the evader follows up while not observing last pursuer’s action).

### 2.1 Shape of the value function

We have discussed that the game $G^T_{s^0_p, b^0}$ is fully determined by the initial position of the pursuer $s^0_p$ and the initial belief $b^0$. It is therefore of natural interest to compute the value of such games parametrized by $s^0_p$ and $b^0$.

**Definition 1.** A value function $v^t(s^0_p) : \Delta(\mathcal{V}) \rightarrow [0, 1]$ is a function assigning the value $v^t(s^0_p)(b^0)$ of the game $G^t_{s^0_p, b^0}$ to every initial belief about the position of the evader. By $v^t$ we mean a set of value functions $v^t(s^0_p)$ for every $s^0_p \in \mathcal{V}^N$.

In the following text we will show that value functions $v^t(s^0_p)$ are piecewise linear and convex (PWLC) for every finite $t$. The term linear will be used also to refer to an affine function. We will show that the pursuer needs to consider only a finite set of randomized behavioral strategies in order to play optimally in game $G^t_{s^0_p, b^0}$ and that this set does not depend on $b^0$. The value of every such strategy (i.e. the expected reward against a best responding evader) is linear in the initial belief $b^0$. The PWLC nature of value functions then follows.

**Theorem 1.** Let $G^t_{s^0_p, b^0}$ be a horizon-$t$ game parametrized by the initial belief $b^0$. There exists a finite set of pursuer’s behavioral strategies $\Sigma^t_{s^0_p}$ such that for every $b^0$ there is a strategy $\sigma_p \in \Sigma^t_{s^0_p}$ that forms part of a NE solution in $G^t_{s^0_p, b^0}$.

**Proof.** From the sequence-form LP for solving EFGs [6] and due to the finiteness of the set of evader’s pure strategies in the finite game $G^t_{s^0_p, b^0}$, the value in every top-level information set $I^0_e$ of the evader is piecewise linear concave function in
the space of pursuer’s realization plans \( r(\sigma_p) \). The set of extreme points of linear segments of this function is finite. Let us denote the set of coordinates of these extreme points in the space of realization plans as \( P[s^0_v] \). The utility of every outcome below \( I^0_{se} \) is multiplied by the probability \( b^0(s^0_v) \) that the evader will be spawned in \( s^0_v \) and this probability can be factored out from the respective constraints. We get therefore the following constraint for the root information set:

\[
v(\text{root}) \leq \sum_{s^0_v \in \mathcal{V}/\mathcal{P}} b^0(s^0_v) \cdot v(I^0_{se}) + \sum_{s^0_v \in \mathcal{P}} b^0(s^0_v) + v(\text{root})
\]

The convex combination of piecewise linear functions does not introduce extreme points at coordinates that were not present in \( P[s^0_v] \) for some \( s^0_v \), hence \( v(\text{root}) \) is a piecewise linear concave function in pursuer’s realization plans with the set of extreme points located on coordinates \( P[\text{root}] \subset \bigcup_{s^0_v \in \mathcal{V}/\mathcal{P}} P[s^0_v] \) (which does not depend on \( b^0 \)). Optimal solution of a LP can be found in vertices of the polytope \( \mathcal{P} \), hence in the finite set \( P[\text{root}] \). Each vertex corresponds to one behavioral strategy of pursuer and this set of strategies \( \Sigma_t^{\mathcal{P}} \) does not depend on \( b^0 \).

**Lemma 1.** Let \( \sigma_p \) be pursuer’s randomized strategy in \( G^t_{\mathcal{P}, b^0} \). The expected reward of playing \( \sigma_p \) against a best responding opponent is then linear in \( b^0 \).

**Proof.** When the strategy \( \sigma_p \) is fixed, the evader chooses a deterministic trail in the graph for every his initial vertex \( s^0_v \) (known to the evader) minimizing pursuer’s expected reward. The initial distribution \( b^0 \) then performs convex combination of these rewards which forms a linear function in \( b^0 \). \( \square \)

**Theorem 2.** Value function \( v^t(s_p) \) is piecewise linear and convex in \( b^0 \).

**Proof.** This result follows from Theorem 1 and Lemma 1. There is a finite set of randomized strategies \( \Sigma_t^{\mathcal{P}} \) that has to be considered by the pursuer and value of every such strategy is linear in \( b^0 \). Thus the value function \( v^t(s_p) \) is a pointwise maximum taken over a finite set of linear functions, which is PWLC. \( \square \)

Every PWLC function can be represented by a finite set of \( \alpha \)-vectors. Every such \( \alpha \)-vector \( \alpha = (\alpha_1, \ldots, \alpha_{|\mathcal{V}|}) \) represents one of the affine functions by assigning an expected reward \( \alpha_i \) to each of the pure beliefs. Due to this fact we will overload the notation and consider a value function \( v^t(s_p) \) from Theorem 2 as a set of such \( \alpha \)-vectors as well.

Lemma 1 and Theorem 2 imply that every linear segment of the PWLC value function corresponds to one randomized strategy of the pursuer. This is similar to the POMDP case where each \( \alpha \)-vector corresponds to one conditional plan. We will be therefore using terms \( \alpha \)-vector and pursuer’s strategy interchangeably.

### 3 Value iteration

The size of the EFG grows exponentially with horizon \( t \). We propose a dynamic programming approach inspired by value iteration algorithms for either
POMDPs [11,8] or perfect information stochastic games [10]. We show that in horizon-t game players choose the starting point of a game with horizon $t - 1$. The pursuer selects a starting location $s^1_p$ in a horizon $t - 1$ game by playing one-shot strategy $\pi_p \in \Delta(V^N)$ in information set $I^0_p$ ($s^1_p \notin \text{adj}_p(s^0_p)$ implies $\pi_p(s^1_p) = 0$), while the evader selects the distribution over his positions by playing a one-shot strategy $\pi_e : V \rightarrow \Delta(V)$ where $s^1_e \notin \text{adj}_e(s^0_e)$ implies $\pi_e(s^0_e, s^1_e) = 0$. She will move to $s^1_e$ from $s^0_e$ with probability $\pi_e(s^0_e, s^1_e)$.

**Theorem 3.** The value of the game $G^{t+1}_{s_p, b}$ can be computed from the solutions of horizon-t games represented by a set of value functions $v^t$ using

$$ v^{t+1}(s_p)(b) = \sum_{s_e \in s_p} b(s_e) + \gamma \left[ \sum_{s_e \in V \setminus s_p} b(s_e) \right] \cdot \max_{\pi_p} \min_{\pi_e} \sum_{s'_e \in V^N} \pi_p(s'_e) \cdot v^t(s'_e)(b') \quad (1) $$

where belief $b'$ depends only on evader’s strategy $\pi_e$ and initial belief $b$:

$$ b'(s'_e) = \frac{1}{\sum_{s_e \in V \setminus s_p} b(s_e)} \sum_{s_e \in V \setminus s_p} b(s_e) \cdot \pi_e(s_e, s'_e) \quad (2) $$

The computation of $v^{t+1}$ by means of Equation (1) forms a dynamic programming operator $H$, such that $v^{t+1} = Hv^t$.

**Proof.** We will prove the correctness of the Equation (1) by computing the value of the game $G^{t+1}_{s_p, b}$ by propagating the value of individual nodes in the tree in a bottom-up fashion. In the course of the proof we will use technical lemmas, the proofs of which can be found in the Appendix. Let us fix the strategy of the players in the first stage of the game (i.e., their behavioral strategy in information sets $I^0_p$ and $I^0_e$ for every $s^0_e \in V \setminus s^0_p$). We will denote the pursuer’s strategy in $I^0_p$ as $\pi_p \in \Delta(\text{adj}_p(s^0_p))$ and evader’s strategy in information set $I^0_e$ as $\pi_e(s^0_e)$.

**Definition 2.** Let $\pi_p$ be pursuer’s behavioral strategy in $I^0_p$ and $\pi_e$ be evader’s behavioral strategy in $I^0_e$ (for every $s^0_e \in V \setminus s_p$). Let $\sigma_p$ and $\sigma_e$ be optimal behavioral strategies of the pursuer and evader with the restriction that the strategy in the first stage of the game is fixed to $\pi_p$ and $\pi_e$. The expected reward of the pursuer when these strategies are followed and node $h$ of the game is reached will be denoted as $u_{\pi_p, \pi_e}(h)$ and termed expected reward in $h$.

**Lemma 2.** The expected reward in the root node equals to:

$$ u_{\pi_p, \pi_e}(\emptyset) = \sum_{s^0_p \in I^0_p} b(s^0_p) + \left[ \sum_{s^0_p \in I^0_p} b(s^0_p) \right] \cdot \sum_{s^1_p \in s^0_p} \pi_p(s^1_p) \left( \gamma \sum_{s^1_e \in s^1_p} b'(s^1_e) + \sum_{s^1_e \notin s^1_p} b'(s^1_e) \sum_{s^0_e \in s^0_p \notin s^1_p} b(s^0_e) \cdot \pi_e(s^0_e, s^1_e) \cdot u_{\pi_p, \pi_e}(s^0_e, s^1_p, s^1_e) \right) \quad (3) $$
Subtrees below $I_p^s$ are completely independent of the rest of the game (pursuer does not forget the action he made, evader always get to know everything except the current move of the pursuer), hence we can treat the subgame below $I_p^s$ separately and let the nature player simulate the belief $b[s_p^1]$ in this information set. Let us denote this game with nature simulating the belief as $G[s_p^1]$. The value of the game $G[s_p^1]$ is equal to the following part of Equation 3 (expected reward from histories in $I_p^s$):

$$
\sum_{s_1 \in s_p^1} \sum_{s_2^0 \in s_p^0} \left[ \sum_{s_3^0 \in s_p^0} b(s_1^0) \cdot \pi_e(s_1^0, s_2^0) \sum_{s_3^1 \in s_p^1} b(s_2^1) \cdot \pi_e(s_2^1, s_3^1) \right]
$$

This game is a horizon-$t$ game that is reminiscent to $G_{b[s_p^1]}^t$ up to two differences. The outcomes of $G[s_p^1]$ are multiplied by $\gamma$ compared to $G_{b[s_p^1]}^t$ (as one stage has already passed), this modification of utilities does not change the strategies and only the value gets multiplied by $\gamma$ as well. More importantly there is not a unique history in $I_p^s$ corresponding to each evader’s position $s_e^1$ as in the case of $G_{b[s_p^1]}^t$. In this case we have a history $s_e^0 s_p^1 s_e^1$ contained in $I_p^s$ for multiple initial positions of the evader $s_e^0$. We will however show that we can eliminate these histories and turn the game into $G_{b[s_p^1]}^t$.

**Definition 3.** Two deterministic game trees over nodes $\mathcal{H}_1, \mathcal{H}_2$ are isomorphic if there exists a bijection $\xi: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that $v \in \mathcal{H}_1$ is a successor of $u \in \mathcal{H}_1$ if and only if $\xi(v)$ is a successor of $\xi(u)$, $n \in \mathcal{H}_1$ is a pursuer’s node if and only if $\xi(n)$ is a pursuer’s node, it is a terminal node if and only if $\xi(n)$ is a terminal node and the utilities $u(n) = u(\xi(n))$. Moreover the trees have the same informational structure so that two nodes $u, v \in \mathcal{H}_1$ are in the same information set if and only if nodes $\xi(u), \xi(v)$ are in the same information set.

We can observe that subtrees of nodes $s_e^0 s_p^1 s_e^1$ and $s_e^0 s_p^1 s_e^1$ (where $s_e^0$ and $s_e^1$ stands for two different initial positions of the evader) are isomorphic. We can establish a bijection $\xi(s_e^0 s_p^1 s_e^1) = s_e^0 s_p^1 s_e^1$. The utility of terminal histories does not depend on the initial position of the evader (only on the time when the pursuer managed to capture the evader). Whenever pursuer’s node $u$ is in information set $I_p$, node $\xi(u)$ is in $I_p$ as well (because pursuer has no way to detect the evader’s initial position. If $s_e^0 s_p^1 s_e^1 \cdots s_e^q$ and $s_e^0 s_p^1 s_e^1 \cdots s_e^q$ are two histories belonging to evader’s information set $I_e$, histories $s_e^0 s_p^1 s_e^1 \cdots s_e^q$ and $s_e^0 s_p^1 s_e^1 \cdots s_e^q$ will belong to the same information set $I_p$ as well (because the uncertainty is related to pursuer’s move $s_p^q$ and it does not depend on evader’s initial position). Thus the subtrees have also same informational structure.

**Lemma 3.** Let $I$ be a top-level information set of game $G[s_p^1]$ and let the belief $b[I]$ over nodes from $I$ be known. Let $n_1, n_2 \in I$ be two nodes whose subtrees are
isomorphic. Then the game $G'$ with the same structure as $G$ with any belief $b'[I]$ satisfying $b[n_1] + b[n_2] = b'[n_1] + b'[n_2]$ has the same value as $G$.

Due to the isomorphism of the subtrees of $s^0_0 s^1_p s^1_e$ and $\pi^0_e s^0_p s^1_e$, Lemma 3 allows us to select a representative history $s^0_0 s^1_p s^1_e$ for all histories $\pi^0_e s^0_p s^1_e$, $\pi^0_e \notin s^0_p$. The belief $b[s^0_0 (s^0_0 s^1_p s^1_e)]$, $\pi^0_e \neq s^0_0$, can be reduced to zero and let the representative history compensate for this change:

$$b'[s^1_p](s^0_e) = b[s^1_p](s^0_0 s^1_p s^1_e) := \sum_{\pi^0_e \notin s^0_p} b(s^0_e) \cdot \pi_e(s^0_0 s^1_p s^1_e)$$

We reduced the probability of reaching non-representative histories in $I_p$ of $G[s^1_p]$ to zero. Histories with zero probability has no contribution and can be eliminated, which leaves us with a single representative history for every possible location $s^1_p$ of the evader. This elimination converted the game $G[s^1_p]$ into a $\gamma$-multiplied version of $G[s^1_p, b'[s^1_p]]$ (in which the evader is never caught immediately as we consider only histories leading towards $I_p$). The value of $G[s^1_p]$ is thus:

$$\sum_{s^1_p \notin s^1_p} \left[ \sum_{\pi^0_e \notin s^0_p} b(s^0_e) \cdot \pi_e(s^0_0 s^1_p s^1_e) \cdot \sum_{s^1_p \notin s^1_p} \sum_{\pi^0_e \notin s^0_p} b(s^0_e) \cdot \pi_e(s^0_0 s^1_p s^1_e) \cdot \alpha_p, \pi_e(s^0_0 s^1_p s^1_e) \right] = \gamma v'(s^1_p) (b'[s^1_p])$$

Together with Equation (3) we get

$$u_{\pi_p, \pi_e}(\emptyset) = \sum_{s^0_e \in s^0_p} b(s^0_e) + \gamma \left[ \sum_{s^0_e \notin s^0_p} b(s^0_e) \right] \cdot \sum_{s^1_p} \pi_p(s^1_p) \cdot v'(s^1_p) (b')$$ (4)

If we allow players to choose $\pi_p$ and $\pi_e$, they will apply the maxmin principle on Equation (4) which gives us Equation (1).

$\square$

### 3.1 From value functions to optimal one-shot strategies

We now show how to compute strategies from value functions $v'(s_p)$. At each stage, players have to decide about their randomized one-shot strategies $\pi_p$ and $\pi_e$ for the next move corresponding to the strategies from Equation (1). Due to the space constraints the LP for finding evader’s optimal strategy was moved to the Appendix and we focus on computing $\pi_p$.

Assume that the evader was not caught (i.e. $\sum_{s_e \notin v(s_p)} b(s_e) = 1$). The pursuer now has to decide what action $s'_p \in \text{adj}(s_p)$ he will play first and what strategy $\alpha'_p \in v'(s'_p)$ he will follow next. He will follow each such strategy with probability $\bar{\pi}(s'_p, i)$. The evader on the other hands best responds his strategy by
deciding action $s'_p$ for every his initial position $s_p$. He chooses the one that minimizes expected reward against pursuer’s randomized strategy (Equation (6)).

The strategy $\pi_p$ is then $\pi_p(s'_p) = \sum_{s''_p \in V} \alpha_i s''_p \cdot \hat{\pi}_p(s'_p, i)$. This program is a dual of the LP for computing evader’s strategy (see Appendix).

**Lemma 4.** For every $t_1 > t_2 \geq 0$ and every initial pursuer’s position $s_p^0$ and every initial belief $b_0$, $v^{t_1} \langle s_p^0 \rangle (b_0) \geq v^{t_2} \langle s_p^0 \rangle (b_0)$.

**Proof.** We can derive a strategy for $G^{t_1}_{s_p^0, b_0}$ from the optimal strategy $\sigma_p$ in the game $G^{t_2}_{s_p^0, b_0}$ as follows. We let the pursuer play for $t_2$ rounds according to $\sigma_p$, and arbitrarily afterwards. This strategy will guarantee him at least expected reward of $v^{t_2} \langle s_p^0 \rangle (b_0)$ if the game $G^{t_1}_{s_p^0, b_0}$ ends in less than $t_2$ rounds, pursuer gets the same reward as if the play occurred in $G^{t_2}_{s_p^0, b_0}$. If the game does not end in less than $t_2$ rounds, pursuer can only get a positive reward, instead of zero reward in $G^{t_2}_{s_p^0, b_0}$.

### 3.2 Computing value functions

Our goal is to approximate value functions of an infinite horizon game by computing value functions of a sequence of finite horizon games $\{G^t_{s_p^0, b_0}\}_{i=0}^\infty$. Assume that value functions $v^{t} \langle s'_p \rangle$ are known for every $s'_p \in V^N$. We are about to compute value functions $v^{t+1} \langle s_p \rangle$ by applying the dynamic programming operator $H$ on value functions $v^t$ (i.e. $v^{t+1} = H v^t$). We will proceed in two steps: (1) Firstly we compute a function $Q_{\pi_p}^{t+1} (s_p)$ corresponding to the expected utility of the pursuer if he plays $\pi_p$ at the first stage of the game; (2) secondly we will show how to compute $v^{t+1} \langle s_p \rangle$ as a combination of multiple $Q_{\pi_p}^{t+1} (s_p)$.

**Definition 4.** Let $\pi_p$ be pursuer’s strategy for the first stage of the game $G^{t+1}_{s_p^0, b_0}$. The value of $\pi_p$ is a function $Q_{\pi_p}^{t+1} (s_p)$ mapping every initial belief to the expected reward of the pursuer against the best responding opponent in $G^{t+1}_{s_p^0, b_0}$ when he plays optimal strategy with the restriction that $\pi_p$ is used in the first stage, i.e.

$$Q_{\pi_p}^{t+1} (s_p) (b) := \sum_{s_c \in s_p} b(s_c) + \gamma \left[ \sum_{s_c \in V \setminus s_p} b(s_c) \right] \cdot \min_{\pi_e} \sum_{s'_p \in V^N} \pi_p(s'_p) \cdot v \langle s'_p \rangle (b')$$
Lemma 5. For every belief $b$ there is a set of strategies $\sigma_p^{s_p'}$ for every $s_p' \in \mathcal{V}^N$ represented by $\alpha$-vectors in $\mathcal{V}^t \langle s_p' \rangle$ such that the strategy $\sigma_p$ restricted to play $\pi_p$ first and then follow with $\sigma_p^{s_p'}$ is optimal in $G_{s_p,b}^{t+1}$. The value of such strategy can be represented by the $\alpha$-vector

$$\alpha^p(s_e) = \begin{cases} 
\gamma \min_{s'_e \in \mathcal{G}(s_e)} \sum_{s_p'} \pi_p(s_p') \cdot \alpha^p(s_p') & s_e \in s_p \\
\text{otherwise} & s_e \not\in s_p
\end{cases}$$

where $\alpha^p(s_p')$ is the value of $\alpha$-vector corresponding to $\sigma_p^{s_p'}$ evaluated at pure belief corresponding to evader’s position $s_e'$.

Proof. Let us assume that pursuer played action $s_p'$ in the first stage (drawn from $\pi_p$). The game moves to a shorter horizon game $G_{s_p',b}^{t+1}$ with a belief updated according to Equation (2). The $\alpha$-vectors in $\mathcal{V}^t \langle s_p' \rangle$ represent all optimal mixed strategies of the pursuer for every possible belief. One of them, $\sigma_p^{s_p'}$, has to be optimal continuation for the pursuer.

We know that the value of strategies is linear in belief (Lemma 1). It is therefore sufficient to compute the value of the strategy in each of the pure beliefs. If the evader is located at the same vertex with some of the pursuers, the game ends immediately and the pursuer’s utility is 1. If the evader is not immediately caught she chooses an adjacent vertex $s'_e$ so that the expected utility of the pursuer is minimized. We know that if evader plays $s'_p$ in the first move and follows with $\sigma_p^{s_p'}$ afterwards, the expected utility of the pursuer is represented by the $\alpha$-vector corresponding to $\sigma_p^{s_p'}$ evaluated at the pure belief $s'_e$. \qed

Lemma 5 gives us a direct algorithm for computation of $Q_{\pi_p}^{t+1}$. We will try all combinations of strategies from $\mathcal{V}^t$ as strategies $\sigma_p^{s_p'}$ and compute their $\alpha$-vectors. The maximization over these linear functions represented by newly constructed $\alpha$-vectors corresponds to the function $Q_{\pi_p}^{t+1} \langle s_p \rangle$ which is piecewise linear and convex because there is a finite number of $\alpha$-vectors in $\mathcal{V}^t$.

The definition of $Q_{\pi_p}^{t+1} \langle s_p \rangle$ implies that we can compute the value function $v^{t+1} \langle s_p \rangle$ by allowing the pursuer to play arbitrary strategy $\pi_p$, when

$$v^{t+1} \langle s_p \rangle (b) = \max_{\pi_p} Q_{\pi_p}^{t+1} \langle s_p \rangle (b)$$

The value function $v^{t+1} \langle s_p \rangle$ can therefore be found by finding a set of strategies $\Pi_p$ for the first stage of the game and computing the pointwise maximum from respective $Q_{\pi_p}^{t+1} \langle s_p \rangle$ functions, $v^{t+1} \langle s_p \rangle = \bigoplus_{\pi_p \in \Pi_p} Q_{\pi_p}^{t+1} \langle s_p \rangle$. The set of such strategies $\Pi_p$ is however unknown. We propose an algorithm (Algorithm 1) that constructs both the set of strategies $\Pi_p$ and the value function $v^t \langle s_p \rangle$ incrementally by iteratively verifying whether the current set of the strategies $\Pi_p$ is sufficient for computing the value function $v^{t+1} \langle s_p \rangle$.

The Algorithm 1 is constructing sets of strategies $\Pi_p$ and a corresponding estimate of value function $\hat{v}^{t+1} \langle s_p \rangle = \bigoplus_{\pi_p \in \Pi_p} Q_{\pi_p}^{t+1} \langle s_p \rangle$, starting with an empty
The algorithm terminates because there is a finite set of beliefs where an optimal strategy of the pursuer for the first stage (see (5)) can be found, all required strategies were considered and \( \hat{\pi}^{t+1} \) is optimal in every belief \( b \in \Delta(\mathcal{V}) \). It is sufficient to consider a finite set of beliefs as Lemma 6 exposes. If it finds a belief \( b \) where the strategy can be improved, i.e. there exists \( \pi_p \) such that \( Q_{\pi_p}^{t+1} (b) > \hat{\pi}^{t+1} (b) \), it updates the set \( \Pi_p \) and recomputes the value function estimate \( \hat{\pi}^{t+1} \). If no such belief is found, all required strategies were considered and \( \hat{\pi}^{t+1} = \pi^{t+1} \).

The algorithm terminates because there is a finite set \( \Pi_p \) sufficient to construct \( \pi^{t+1} \) (consequence of Theorem 1) and we always consider optimal strategies \( \pi_p \) when updating \( \Pi_p \).

**Lemma 6.** If there is a belief \( b \) where \( \pi^{t+1} (b) > \hat{\pi}^{t+1} \), there must be a belief \( b' \) that forms an extreme point of a line segment on the surface of \( \hat{\pi}^{t+1} \) where \( \pi^{t+1} (b') > \hat{\pi}^{t+1} (b') \).

**Proof.** We will prove this by contradiction. Let \( B \) be a set of beliefs where an arbitrary facet on a surface of PWLC function \( \hat{\pi}^{t+1} \) has its corresponding extreme points and assume that the value function \( \hat{\pi}^{t+1} \) is optimal in beliefs \( \hat{b} \in B \), i.e. \( \pi^{t+1} (\hat{b}) = \hat{\pi}^{t+1} (\hat{b}) \) for every \( \hat{b} \in B \). Let us assume that \( b \) satisfying \( \pi^{t+1} (b) > \hat{\pi}^{t+1} (b) \) is a convex combination of beliefs \( B \). The estimate \( \hat{\pi}^{t+1} \) of the value function \( \pi^{t+1} \) is a PWLC function (pointwise maximum from PWLC functions \( Q_{\pi_p}^{t+1} \)) which contradicts \( \pi^{t+1} (b) > \hat{\pi}^{t+1} (b) \) as \( \hat{\pi}^{t+1} (b) \) is a convex combination of values of \( \hat{\pi}^{t+1} \) evaluated in beliefs \( B \) with corresponding coefficients.

### 3.3 Uniqueness of solution

We will show the uniqueness and convergence properties by showing that the dynamic programming operator \( H \) applied to value functions is a contraction and thus there is a unique set of value function satisfying the fixpoint property and the value iteration algorithm (recursive application of \( H \)) converges to it. We will show the contractivity of \( H \) under the following max-norm:

\[
\|v - \pi\| = \max_{s_p \in \mathcal{V} \forall b \in \Delta(\mathcal{V})} |v(s_p)(b) - \pi(s_p)(b)|
\]
Lemma 7. The operator $H$ is a contraction with contractivity factor $\gamma < 1$ under max-norm.

Proof. Let us define $Q^v_{\pi_p} (s_p)$ as the value of $v'$ from Equation [1] where the pursuer’s strategy in the first stage is fixed to $\pi_p (v' (s_p) (b) = \max_{s_p} Q^v_{\pi_p} (s_p) (b))$.

$$Q^v_{\pi_p} (s_p) (b) := \sum_{s_c \in s_p} b(s_c) + \gamma \left[ \sum_{s_c \in \mathcal{V} \setminus s_p} b(s_c) \right] \cdot \min_{s'_p} \sum_{s'_p \in \mathcal{V}} \pi_p(s'_p) \cdot v (s'_p) (b')$$

The proof will closely follow the structure of the proof of Theorem 1 in [12]. First of all we will show that for every $s_p \in \mathcal{V}^N$ and every valid pursuer’s strategy $\pi_p$, the mapping $v \mapsto Q^v_{\pi_p} (s_p)$ has a contractivity factor $\gamma$, then by inspecting all possible $s_p$ and $\pi_p$, we show that the same holds for $H$. Note that difference $|Q^v_{\pi_p} (s_p) (b) - Q^v_{\pi_p} (s_p) (b)|$ is maximized if $b$ is chosen such that the evader is not initially caught according to $b$ (i.e. $\sum_{s_e \in s_p} b(s_e) = 0$), which allows us to simplify the derivation. Belief $b_{\pi_p}$ is a transformed belief $b$ using Equation [2], i.e. $b_{\pi_p} (s'_p) \sim \sum_{s_e \notin s_p} b(s_e) \cdot \pi_e(s_e, s'_e)$.

$$\left| Q^v_{\pi_p} (s_p) - Q^v_{\pi_p} (s_p) \right| = \max_b \left| Q^v_{\pi_p} (s_p) (b) - Q^v_{\pi_p} (s_p) (b) \right|$$

$$= \max_b \left[ \gamma \min_{s'_p} \sum_{s'_p} \pi_p(s'_p) \cdot v (s'_p) (b_{s_e}) - \gamma \min_{s'_p} \sum_{s'_p} \pi_p(s'_p) \cdot v (s'_p) (b_{s_e}) \right]$$

$$= \gamma \max_b \left[ \min_{s'_p} \sum_{s'_p} \pi_p(s'_p) \cdot v (s'_p) (b_{s_e}) - \min_{s'_p} \sum_{s'_p} \pi_p(s'_p) \cdot v (s'_p) (b_{s_e}) \right]$$

$$\leq \gamma \max_b \left[ \sum_{s'_p} \pi_p(s'_p) \cdot \left[ v (s'_p) (b') - \bar{v} (s'_p) (b') \right] \right]$$

$$\leq \gamma \max_b \left[ \sum_{s'_p} \pi_p(s'_p) \cdot \left[ v (s'_p) (b') - \bar{v} (s'_p) (b') \right] \right]$$

$$\leq \gamma \max_b \left[ \sum_{s'_p} \pi_p(s'_p) \cdot \| v - \bar{v} \| \right]$$

Let us now choose arbitrary $s_p \in \mathcal{V}^N$ and $b \in \Delta(\mathcal{V})$. Without loss of generality, let us assume that $(Hv) (s_p) (b) \geq (H\bar{v}) (s_p) (b)$. Let $\pi_p^*$ be an optimal one-shot strategy in $b$ w.r.t. $v$ (i.e. maximizing $Q^v_{\pi_p} (s_p) (b)$) and $\pi_p$ an optimal strategy w.r.t. $\bar{v}$. Note that $(Hv) (s_p) (b) = Q^v_{\pi_p} (s_p) (b)$ and $(H\bar{v}) (s_p) (b) =$
$$Q_{π_p}^v (s_p) (b) \leq Q_{π_p}^v (s_p) (b) \leq Q_{π_p}^v (s_p) (b).$$

Then:

$$| (H v) (s_p) (b) - (H π) (s_p) (b) | = \left| Q_{π_p}^v (s_p) (b) - Q_{π_p}^v (s_p) (b) \right|$$

$$\leq \left| Q_{π_p}^v (s_p) (b) - Q_{π_p}^v (s_p) (b) \right|$$

$$\leq \max_{π_p} \left| Q_{π_p}^v (s_p) (b) - Q_{π_p}^v (s_p) (b) \right|$$

$$\leq \max_{π_p} \max_{s'_p \in V^N} \max_{b' \in \Delta(V)} \left| Q_{π_p}^v (s'_p) (b') - Q_{π_p}^v (s'_p) (b') \right|$$

$$\leq \gamma \cdot \| v - π \|$$

**Theorem 4.** There is a unique set of value functions $v^*$ satisfying $v^* = Hv^*$ and the recursive application of $H$ converges to $v^*$. Series $\{v^t\}_{t=0}^∞$ thus converges.

**Proof.** The operator $H$ is a contraction mapping defined on a metric space of sets of bounded functions defined on the belief space. By applying Banach’s fixed point theorem [3] we get that $H$ has a unique fixed point $v^*$ and the recursive application of $H$ converges to $v^*$. $\square$

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APPENDIX

A Proofs of technical lemmas used in the proof of Theorem 3

A.1 Proof of Lemma 2

The expected reward in the root node $u_{\pi_p, \pi_e}(\emptyset)$ is

$$u_{\pi_p, \pi_e}(\emptyset) = \sum_{s_0^c \in s_0^p} b(s_0^c) + \gamma \sum_{s_0^c \notin s_0^p} \sum_{s_1^p \in s_1^p} \sum_{s_1^c \in s_1^c} b(s_0^c) \cdot \pi_p(s_1^p) \cdot \pi_e(s_1^c, s_1^c) +$$

$$\sum_{s_0^c \notin s_0^p} \sum_{s_1^p \in s_1^p} \sum_{s_1^c \in s_1^c} b(s_0^c) \cdot \pi_p(s_1^p) \cdot \pi_e(s_1^c, s_1^c) \cdot u_{\pi_p, \pi_e}(s_0^c s_1^p s_1^c)$$

The derivation of Equation (3) is then just a technical derivation involving operations with sums and normalization of conditional probability distributions. Throughout the derivation we will use the equation of the transformed belief (Equation (2)).

$$u_{\pi_p, \pi_e}(\emptyset) = \sum_{s_0^c \in s_0^p} b(s_0^c) + \gamma \sum_{s_0^c \notin s_0^p} \sum_{s_1^p \in s_1^p} \sum_{s_1^c \in s_1^c} b(s_0^c) \cdot \pi_p(s_1^p) \cdot \pi_e(s_1^c, s_1^c) +$$

$$\sum_{s_0^c \notin s_0^p} \sum_{s_1^p \in s_1^p} \sum_{s_1^c \in s_1^c} b(s_0^c) \cdot \pi_p(s_1^p) \cdot \pi_e(s_1^c, s_1^c) \cdot u_{\pi_p, \pi_e}(s_0^c s_1^p s_1^c)$$

$$= \sum_{s_0^c \in s_0^p} b(s_0^c) + \gamma \left[ \sum_{s_0^c \notin s_0^p} \sum_{s_1^p \in s_1^p} \sum_{s_1^c \in s_1^c} b(s_0^c) \cdot \pi_p(s_1^p) \cdot \pi_e(s_1^c, s_1^c) \cdot u_{\pi_p, \pi_e}(s_0^c s_1^p s_1^c) \right]$$

$$\cdot \sum_{s_1^p \in s_1^p} \sum_{s_1^c \in s_1^c} \left[ \sum_{s_0^c \in s_0^p} b(s_0^c) \cdot \pi_e(s_1^c, s_1^c) \cdot u_{\pi_p, \pi_e}(s_0^c s_1^p s_1^c) \right] \cdot \sum_{s_0^c \notin s_0^p} \sum_{s_1^p \in s_1^p} \sum_{s_1^c \in s_1^c} b(s_0^c) \cdot \pi_p(s_1^p) \cdot \pi_e(s_1^c, s_1^c) \cdot u_{\pi_p, \pi_e}(s_0^c s_1^p s_1^c)$$
A.2 Proof of Lemma 3

Proof. Without loss of generality, assume that \( \sigma^2 \) is a strategy profile applied in the subtree of \( n_2 \). Let \( \sigma^1 \) be strategy profile applied in the subtree of \( n_1 \) defined as \( \sigma^1(n) = \sigma^2(\xi(n)) \) (this strategy profile will be valid in that subtree due to the fact that the subtrees are isomorphic). The expected reward \( u \) in both subtrees is the same under these strategies as they reach the nodes with the same utility (which is guaranteed by the bijection \( \xi \)). Hence if \( \sigma^2 \) is optimal...
in subtree of $n_2$, $\sigma^t$ will be optimal in the subtree of $n_1$. Both players will play the same strategies in the subtrees of $n_1, n_2$ (up to bijection $\xi$) and the expected contribution of nodes $n_1, n_2$ to the value of $G$ and $G'$ is thus

$$b[I](n_1) \cdot u + b[I](n_2) \cdot u = b'[I](n_1) \cdot u + b'[I](n_2) \cdot u$$

### B Computing optimal strategies

#### B.1 Computing evader’s strategy

The linear program for solving the game $G_{s_p, b}^{t+1}$ from the evader’s perspective is based on the following idea. We know that all optimal pursuer’s strategies in games $G_{s_p, b}^{t+1}$ are represented by $\alpha$-vectors in $v^t(s_p')$. We need not therefore consider all pursuer’s pure strategies in $G_{s_p, b}^{t+1}$, instead we will construct strategies $\sigma_p$ of the form $s_p' \cdot \alpha$ (i.e. play $s_p' \in \text{adj}_G(s_p)$ first and then follow strategy corresponding to $\alpha \in v^t(s_p')$). The utility $u(\sigma_p)(s_p)(b)$ of every such strategy played in $G_{s_p, b}^{t+1}$ is

$$u(\sigma_p)(s_p)(b) = \min_{\pi_e} \left( \sum_{s_e \in s_p} b(s_e) + \gamma \sum_{s_e \in s_p} b(s_e) \cdot \sum_{s_e' \in V} \alpha(s_e') \cdot b'(s_e') \right)$$

Evader searches for $\pi_e$ so that he minimizes the reward of the pursuer, i.e. utility of arbitrary $\sigma_p$. We will assume that the evader has not yet been caught (i.e. $b(s_e) = 0$ if $s_e \in s_p$), otherwise the game would have ended and no strategy would be necessary to compute.

$$\min_{v, b', \pi_e} v$$

s.t. $v \geq \gamma \sum_{s_e' \in V} \alpha(s_e') \cdot b'(s_e') \quad \forall s_e' \in \text{adj}_G(s_p), \forall \alpha \in v^t(s_p')$

$$b'(s_e') = \sum_{s_e \in V \setminus s_p} b(s_e) \cdot \pi_e(s_e, s_e') \quad \forall s_e \in V$$

$$\sum_{s_e' \in \text{adj}_G(s_e)} \pi_e(s_e, s_e') = 1 \quad \forall s_e \in V$$

$$\pi_e(s_e, s_e') \geq 0 \quad \forall s_e, s_e' \in V$$