Non-Commutative Worlds - A Summary

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1 Introduction to Non-Commutative Worlds

Aspects of gauge theory, Hamiltonian mechanics and quantum mechanics arise naturally in the mathematics of a non-commutative framework for calculus and differential geometry. This paper consists in two sections. This first section sketches our results in this domain in general. The second section gives a derivation of a generalization of the Feynman-Dyson derivation of electromagnetism using our non-commutative context and using diagrammatic techniques. The first section is based on the paper [15]. The second section is a new approach to issues in [15].

Constructions are performed in a Lie algebra $\mathcal{A}$. One may take $\mathcal{A}$ to be a specific matrix Lie algebra, or abstract Lie algebra. If $\mathcal{A}$ is taken to be an abstract Lie algebra, then it is convenient to use the universal enveloping algebra so that the Lie product can be expressed as a commutator. In making gen-
eral constructions of operators satisfying certain relations, it is understood that one can always begin with a free algebra and make a quotient algebra where the relations are satisfied.

On $\mathcal{A}$, a variant of calculus is built by defining derivations as commutators (or more generally as Lie products). For a fixed $N$ in $\mathcal{A}$ one defines

$$\nabla_N : \mathcal{A} \rightarrow \mathcal{A}$$

by the formula

$$\nabla_N F = [F, N] = FN - NF.$$ 

$\nabla_N$ is a derivation satisfying the Leibniz rule.

$$\nabla_N (FG) = \nabla_N (F)G + F \nabla_N (G).$$

There are many motivations for replacing derivatives by commutators. If $f(x)$ denotes (say) a function of a real variable $x$, and $\tilde{f}(x) = f(x + h)$ for a fixed increment $h$, define the discrete derivative $Df$ by the formula $Df = (\tilde{f} - f)/h$, and find that the Leibniz rule is not satisfied. One has the basic formula for the discrete derivative of a product:

$$D(fg) = D(f)g + \tilde{f}D(g).$$

Correct this deviation from the Leibniz rule by introducing a new non-commutative operator $J$ with the property that

$$fJ = J\tilde{f}.$$ 

Define a new discrete derivative in an extended non-commutative algebra by the formula

$$\nabla(f) = JD(f).$$
It follows at once that
\[ \nabla(fg) = JD(f)g + J\tilde{f}D(g) = JD(f)g + fJD(g) = \nabla(f)g + f\nabla(g). \]
Note that
\[ \nabla(f) = (J\tilde{f} - Jf)/h = (fJ - Jf)/h = [f, J/h]. \]
In the extended algebra, discrete derivatives are represented by commutators, and satisfy the Leibniz rule. One can regard discrete calculus as a subset of non-commutative calculus based on commutators.

In \( \mathcal{A} \) there are as many derivations as there are elements of the algebra, and these derivations behave quite wildly with respect to one another. If one takes the concept of curvature as the non-commutation of derivations, then \( \mathcal{A} \) is a highly curved world indeed. Within \( \mathcal{A} \) one can build a tame world of derivations that mimics the behaviour of flat coordinates in Euclidean space. The description of the structure of \( \mathcal{A} \) with respect to these flat coordinates contains many of the equations and patterns of mathematical physics.

The flat coordinates \( X_i \) satisfy the equations below with the \( P_j \) chosen to represent differentiation with respect to \( X_j \):
\[
[X_i, X_j] = 0 \\
[P_i, P_j] = 0 \\
[X_i, P_j] = \delta_{ij}.
\]
Derivatives are represented by commutators.
\[ \partial_i F = \partial F/\partial X_i = [F, P_i], \]
\[ \dot{\partial}_i F = \partial F / \partial P_i = [X_i, F]. \]

Temporal derivative is represented by commutation with a special (Hamiltonian) element \( H \) of the algebra:

\[ dF/dt = [F, H]. \]

(For quantum mechanics, take \( i\hbar dA/dt = [A, H]. \)) These non-commutative coordinates are the simplest flat set of coordinates for description of temporal phenomena in a non-commutative world. Note:

**Hamilton’s Equations.**

\[ dP_i/dt = [P_i, H] = -[H, P_i] = -\partial H / \partial X_i \]

\[ dX_i/dt = [X_i, H] = \partial H / \partial P_i. \]

These are exactly Hamilton’s equations of motion. The pattern of Hamilton’s equations is built into the system.

**Discrete Measurement.** Consider a time series \( \{X, X', X'', \ldots\} \) with commuting scalar values. Let

\[ \dot{X} = \nabla X = JDX = J(X' - X) / \tau \]

where \( \tau \) is an elementary time step (If \( X \) denotes a times series value at time \( t \), then \( X' \) denotes the value of the series at time \( t + \tau \).). The shift operator \( J \) is defined by the equation \( XJ = JX' \) where this refers to any point in the time series so that \( X^{(n)}J = JX^{(n+1)} \) for any non-negative integer \( n \). Moving \( J \) across a variable from left to right, corresponds to one tick of the clock. This discrete, non-commutative time derivative satisfies the Leibniz rule.
This derivative $\nabla$ also fits a significant pattern of discrete observation. Consider the act of observing $X$ at a given time and the act of observing (or obtaining) $DX$ at a given time. Since $X$ and $X'$ are ingredients in computing $(X' - X)/\tau$, the numerical value associated with $DX$, it is necessary to let the clock tick once. Thus, if one first observe $X$ and then obtains $DX$, the result is different (for the $X$ measurement) if one first obtains $DX$, and then observes $X$. In the second case, one finds the value $X'$ instead of the value $X$, due to the tick of the clock.

1. Let $\dot{XX}$ denote the sequence: observe $X$, then obtain $\dot{X}$.
2. Let $X\dot{X}$ denote the sequence: obtain $\dot{X}$, then observe $X$.

The commutator $[X, \dot{X}]$ expresses the difference between these two orders of discrete measurement. In the simplest case, where the elements of the time series are commuting scalars, one has

$$[X, \dot{X}] = X\dot{X} - \dot{XX} = J(X' - X)^2/\tau.$$  

Thus one can interpret the equation

$$[X, \dot{X}] = Jk$$

($k$ a constant scalar) as

$$(X' - X)^2/\tau = k.$$  

This means that the process is a walk with spatial step

$$\Delta = \pm \sqrt{k\tau}$$

where $k$ is a constant. In other words, one has the equation

$$k = \Delta^2/\tau.$$
This is the diffusion constant for a Brownian walk. A walk with spatial step size $\Delta$ and time step $\tau$ will satisfy the commutator equation above exactly when the square of the spatial step divided by the time step remains constant. This shows that the diffusion constant of a Brownian process is a structural property of that process, independent of considerations of probability and continuum limits.

**Heisenberg/Schrödinger Equation.** Here is how the Heisenberg form of Schrödinger’s equation fits in this context. Let the time shift operator be given by the equation $J = (1 + H\Delta t/i\hbar)$. Then the non-commutative version of the discrete time derivative is expressed by the commutator

$$\nabla \psi = [\psi, J/\Delta t],$$

and we calculate

$$\nabla \psi = \psi[(1 + H\Delta t/i\hbar)/\Delta t] - [(1 + H\Delta t/i\hbar)/\Delta t]\psi = [\psi, H]/i\hbar,$$

$$i\hbar \nabla \psi = [\psi, H].$$

This is exactly the Heisenberg version of the Schrödinger equation.

**Dynamics and Gauge Theory.** One can take the general dynamical equation in the form

$$dX_i/dt = G_i$$

where $\{G_1, \cdots, G_d\}$ is a collection of elements of $\mathcal{A}$. Write $G_i$ relative to the flat coordinates via $G_i = P_i - A_i$. This is a definition of $A_i$ and $\partial F/\partial X_i = [F, P_i]$. The formalism of gauge theory appears naturally. In particular, if

$$\nabla_i(F) = [F, G_i],$$
then one has the curvature

\[ [\nabla_i, \nabla_j]F = [R_{ij}, F] \]

and

\[ R_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]. \]

This is the well-known formula for the curvature of a gauge connection. Aspects of geometry arise naturally in this context, including the Levi-Civita connection (which is seen as a consequence of the Jacobi identity in an appropriate non-commutative world).

One can consider the consequences of the commutator \([X_i, \dot{X}_j] = g_{ij}\), deriving that

\[ \ddot{X}_r = G_r + F_{rs} \dot{X}^s + \Gamma_{rst} \dot{X}^s \dot{X}^t, \]

where \(G_r\) is the analogue of a scalar field, \(F_{rs}\) is the analogue of a gauge field and \(\Gamma_{rst}\) is the Levi-Civita connection associated with \(g_{ij}\). This decomposition of the acceleration is uniquely determined by the given framework.

One can use this context to revisit the Feynman-Dyson derivation of electromagnetism from commutator equations, showing that most of the derivation is independent of any choice of commutators, but highly dependent upon the choice of definitions of the derivatives involved. Without any assumptions about initial commutator equations, but taking the right (in some sense simplest) definitions of the derivatives one obtains a significant generalization of the result of Feynman-Dyson.
**Electromagnetic Theorem.** (See Section 2.) With the appropriate [see below] definitions of the operators, and taking

\[ \nabla^2 = \partial_1^2 + \partial_2^2 + \partial_3^2, \quad B = \dot{X} \times \dot{X} \quad \text{and} \quad E = \partial_t \dot{X}, \]

one has

1. \( \ddot{X} = E + \dot{X} \times B \)
2. \( \nabla \cdot B = 0 \)
3. \( \partial_t B + \nabla \times E = B \times B \)
4. \( \partial_t E - \nabla \times B = (\partial_t^2 - \nabla^2) \dot{X} \)

The key to the proof of this Theorem is the definition of the time derivative. This definition is as follows

\[ \partial_t F = \dot{F} - \sum_i \dot{X}_i \partial_i(F) = \dot{F} - \sum_i \dot{X}_i[F, \dot{X}_i] \]

for all elements or vectors of elements \( F \). The definition creates a distinction between space and time in the non-commutative world. A calculation (done diagrammatically in Figure 3) reveals that

\[ \ddot{X} = \partial_t \dot{X} + \dot{X} \times (\dot{X} \times \dot{X}). \]

This suggests taking \( E = \partial_t \dot{X} \) as the electric field, and \( B = \dot{X} \times \dot{X} \) as the magnetic field so that the Lorentz force law

\[ \ddot{X} = E + \dot{X} \times B \]

is satisfied.

This result is applied to produce many discrete models of the Theorem. These models show that, just as the commutator \([X, \dot{X}] = Jk\) describes Brownian motion in one dimension, a generalization of electromagnetism describes the interaction of triples of time series in three dimensions.
Remark. While there is a large literature on non-commutative geometry, emanating from the idea of replacing a space by its ring of functions, work discussed herein is not written in that tradition. Non-commutative geometry does occur here, in the sense of geometry occurring in the context of non-commutative algebra. Derivations are represented by commutators. There are relationships between the present work and the traditional non-commutative geometry, but that is a subject for further exploration. In no way is this paper intended to be an introduction to that subject. The present summary is based on [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and the references cited therein.

The following references in relation to non-commutative calculus are useful in comparing with the present approach [2, 3, 4, 17]. Much of the present work is the fruit of a long series of discussions with Pierre Noyes, influenced at critical points by Tom Etter and Keith Bowden. Paper [16] also works with minimal coupling for the Feynman-Dyson derivation. The first remark about the minimal coupling occurs in the original paper by Dyson [1], in the context of Poisson brackets. The paper [5] is worth reading as a companion to Dyson. It is the purpose of this summary to indicate how non-commutative calculus can be used in foundations.

2 Generalized Feynman Dyson Derivation

In this section we assume that specific time-varying coordinate elements $X_1, X_2, X_3$ of the algebra $\mathcal{A}$ are given. We do not assume any commutation relations about $X_1, X_2, X_3$.  

9
In this section we no longer avail ourselves of the commutation relations that are in back of the original Feynman-Dyson derivation. We do take the definitions of the derivations from that previous context. Surprisingly, the result is very similar to the one of Feynman and Dyson, as we shall see.

Here $A \times B$ is the non-commutative vector cross product:

$$(A \times B)_k = \sum_{i,j=1}^{3} \epsilon_{ijk} A_i B_j.$$  

(We will drop this summation sign for vector cross products from now on.) Then, with $B = \dot{X} \times \dot{X}$, we have

$$B_k = \epsilon_{ijk} \dot{X}_i \dot{X}_j = (1/2) \epsilon_{ijk} [\dot{X}_i, \dot{X}_j].$$

The epsilon tensor $\epsilon_{ijk}$ is defined for the indices $\{i, j, k\}$ ranging from 1 to 3, and is equal to 0 if there is a repeated index and is otherwise equal to the sign of the permutation of 123 given by $ijk$. We represent dot products and cross products in diagrammatic tensor notation as indicated in Figure 1 and Figure 2. In Figure 1 we indicate the epsilon tensor by a trivalent vertex. The indices of the tensor correspond to labels for the three edges that impinge on the vertex. The diagram is drawn in the plane, and is well-defined since the epsilon tensor is invariant under cyclic permutation of its indices.

We will define the fields $E$ and $B$ by the equations

$$B = \dot{X} \times \dot{X} \quad \text{and} \quad E = \partial_t \dot{X}.$$  

We will see that $E$ and $B$ obey a generalization of the Maxwell Equations, and that this generalization describes specific discrete models. The reader should note that this means that a
significant part of the *form* of electromagnetism is the consequence of choosing three coordinates of space, and the definitions of spatial and temporal derivatives with respect to them. The background process that is being described is otherwise arbitrary, and yet appears to obey physical laws once these choices are made.

In this section we will use diagrammatic matrix methods to carry out the mathematics. In general, in a diagram for matrix or tensor composition, we sum over all indices labeling any edge in the diagram that has no free ends. Thus matrix multiplication corresponds to the connecting of edges between diagrams, and to the summation over common indices. With this interpretation of compositions, view the first identity in Figure 1. This is a fundamental identity about the epsilon, and corresponds to the following lemma.
Lemma. (View Figure 1) Let $\epsilon_{ijk}$ be the epsilon tensor taking values 0, 1 and $-1$ as follows: When $ijk$ is a permutation of 123, then $\epsilon_{ijk}$ is equal to the sign of the permutation. When $ijk$ contains a repetition from $\{1, 2, 3\}$, then the value of epsilon is zero. Then $\epsilon$ satisfies the following identity in terms of the Kronecker delta.
The proof of this identity is left to the reader. The identity itself will be referred to as the *epsilon identity*. The epsilon identity is a key structure in the work of this section, and indeed in all formulas involving the vector cross product.

The reader should compare the formula in this Lemma with the diagrams in Figure 1. The first two diagrams are two versions of the Lemma. In the third diagram the labels are capitalized and refer to vectors $A, B$ and $C$. We then see that the epsilon identity becomes the formula

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$$

for vectors in three-dimensional space (with commuting coordinates, and a generalization of this identity to our non-commutative context. Refer to Figure 2 for the diagrammatic definitions of dot and cross product of vectors. We take these definitions (with implicit order of multiplication) in the non-commutative context.
Remarks on the Derivatives.

1. Since we do not assume that $[X_i, \dot{X}_j] = \delta_{ij}$, nor do we assume $[X_i, X_j] = 0$, it will not follow that $E$ and $B$ commute with the $X_i$. 
2. We define
\[ \partial_i(F) = [F, \dot{X}_i], \]
and the reader should note that, these spatial derivations
are no longer flat in the sense of section 1 (nor were they in
the original Feynman-Dyson derivation). See Figure 2 for
the diagrammatic version of this definition.

3. We define \( \partial_t = \partial/\partial t \) by the equation
\[ \partial_t F = \dot{F} - \Sigma_i \dot{X}_i \partial_i(F) = \dot{F} - \Sigma_i \dot{X}_i[F, \dot{X}_i] \]
for all elements or vectors of elements \( F \). We take this equa-
tion as the global definition of the temporal partial deriv-
ate, even for elements that are not commuting with the
\( X_i \). This notion of temporal partial derivative \( \partial_t \) is a least
relation that we can write to describe the temporal rela-
tionship of an arbitrary non-commutative vector \( F \) and the
non-commutative coordinate vector \( X \). See Figure 2 for the
diagrammatic version of this definition.

4. In defining
\[ \partial_t F = \dot{F} - \Sigma_i \dot{X}_i \partial_i(F), \]
we are using the definition itself to obtain a notion of the
variation of \( F \) with respect to time. The definition itself
creates a distinction between space and time in the non-
commutative world.

5. The reader will have no difficulty verifying the following
formula:
\[ \partial_t(FG) = \partial_t(F)G + F\partial_t(G) + \Sigma_i \partial_i(F)\partial_i(G). \]
This formula shows that $\partial_t$ does not satisfy the Leibniz rule in our non-commutative context. This is true for the original Feynman-Dyson context, and for our generalization of it. All derivations in this theory that are defined directly as commutators do satisfy the Leibniz rule. Thus $\partial_t$ is an operator in our theory that does not have a representation as a commutator.

6. We define divergence and curl by the equations

$$\nabla \cdot B = \sum_{i=1}^{3} \partial_i(B_i)$$

and

$$(\nabla \times E)_k = \epsilon_{ijk} \partial_i(E_j).$$

See Figure 2 and Figure 4 for the diagrammatic versions of curl and divergence.

Now view Figure 3. We see from this Figure that it follows directly from the definition of the time derivatives (as discussed above) that

$$\ddot{X} = \partial_t \dot{X} + \dot{X} \times (\dot{X} \times \dot{X}).$$

This is our motivation for defining

$$E = \partial_t \dot{X}$$

and

$$B = \dot{X} \times \dot{X}.$$  

With these definition in place we have

$$\ddot{X} = E + \dot{X} \times B,$$

giving an analog of the Lorentz force law for this theory.
Just for the record, look at the following algebraic calculation for this derivative:

\[
\dot{F} = \partial_t F + \Sigma_i \dot{X}_i [F, \dot{X}_i]
\]

\[
= \partial_t F + \Sigma_i (\dot{X}_i F \dot{X}_i - \dot{X}_i \dot{X}_i F)
\]

\[
= \partial_t F + \Sigma_i (\dot{X}_i F \dot{X}_i - \dot{X}_i \dot{X}_i F) + \dot{X}_i F \dot{X}_i - \dot{X}_i \dot{X}_i F
\]

Hence

\[
\dot{F} = \partial_t F + \dot{X} \times F + (\dot{X} \cdot F) \dot{X} - (\dot{X} \cdot \dot{X}) F
\]

(using the epsilon identity). Thus we have

\[
\ddot{X} = \partial_t \dot{X} + \dot{X} \times (\dot{X} \times \dot{X}) + (\dot{X} \cdot \dot{X}) \dot{X} - (\dot{X} \cdot \dot{X}) \dot{X},
\]

whence

\[
\ddot{X} = \partial_t \dot{X} + \dot{X} \times (\dot{X} \times \dot{X}).
\]

In Figure 4, we give the derivation that \( B \) has zero divergence.
Figure 3 - The Formula for Acceleration
\[ E = \nabla \times \mathbf{X} \quad B = \mathbf{X} \times \mathbf{X} \]
\[ \dot{\mathbf{X}} = E + \dot{\mathbf{X}} \times B \]
\[ \nabla \cdot B = [B, \dot{\mathbf{X}}] \]
\[ = B \mathbf{X} - \dot{\mathbf{X}} B = \mathbf{X} \mathbf{X} \mathbf{X} - \mathbf{X} \mathbf{X} \mathbf{X} = 0 \]
\[ \nabla \cdot B = 0 \]

**Figure 4 - Divergence of \( B \)**

Figures 5 and 6 compute derivatives of \( B \) and the Curl of \( E \), culminating in the formula

\[ \partial_t B + \nabla \times E = B \times B. \]

In classical electromagnetism, there is no term \( B \times B \). This term is an artifact of our non-commutative context. In discrete
models, as we shall see at the end of this section, there is no escaping the effects of this term.

\[ \mathcal{O}_t B = \dot{B} + \dot{X} [X, B] \]

\[ B = (1/2)[X, X] = [X, X] \]

\[ = [E, \dot{X}] + [\dot{X} \times B, \dot{X}] \]

\[ = - \nabla X E + [\dot{X} B, \dot{X}] \]

Figure 5 - Computing \( \dot{B} \)
\[ \partial_t B + \nabla \times E = \dot{X} [\dot{X}, B] + [\dot{X} B, \dot{X}] \]

\[ = \dot{X} [\dot{X}, B] + [\dot{X} B, \dot{X}] + [\dot{X} B, \dot{X}] \]

\[ = -\dot{X} \dot{X} B + \dot{X} \dot{X} B \quad \text{(Note that } \dot{X} B = B \dot{X} \text{)} \]

\[ = \dot{X} \dot{X} B = B \times B \]

\[ \partial_t B + \nabla \times E = B \times B \]

Figure 6 - Curl of $E$
Finally, Figure 7 gives the diagrammatic proof that

\[ \partial_t E - \nabla \times B = (\partial_t^2 - \nabla^2) \dot{X}. \]
This completes the proof of the Theorem below.

**Electromagnetic Theorem** With the above definitions of the operators, and taking
\[ \nabla^2 = \partial_1^2 + \partial_2^2 + \partial_3^2, \quad B = \dot{X} \times \dot{X} \quad \text{and} \quad E = \partial_t \dot{X} \] we have

1. \( \ddot{X} = E + \dot{X} \times B \)
2. \( \nabla \cdot B = 0 \)
3. \( \partial_t B + \nabla \times E = B \times B \)
4. \( \partial_t E - \nabla \times B = (\partial_t^2 - \nabla^2) \dot{X} \)

**Remark.** Note that this Theorem is a non-trivial generalization of the Feynman-Dyson derivation of electromagnetic equations. In the Feynman-Dyson case, one assumes that the commutation relations
\[ [X_i, X_j] = 0 \]
and
\[ [X_i, \dot{X}_j] = \delta_{ij} \]
are given, *and* that the principle of commutativity is assumed, so that if \( A \) and \( B \) commute with the \( X_i \) then \( A \) and \( B \) commute with each other. One then can interpret \( \partial_i \) as a standard derivative with \( \partial_i (X_j) = \delta_{ij} \). Furthermore, one can verify that \( E_j \) and \( B_j \) both commute with the \( X_i \). From this it follows that \( \partial_t (E) \) and \( \partial_t (B) \) have standard interpretations and that \( B \times B = 0 \).

The above formulation of the Theorem adds the description of \( E \) as \( \partial_t (\dot{X}) \), a non-standard use of \( \partial_t \) in the original context of Feynman-Dyson, where \( \partial_t \) would only be defined for those \( A \) that commute with \( X_i \). In the same vein, the last formula \( \partial_t E - \nabla \times B = (\partial_t^2 - \nabla^2) \dot{X} \) gives a way to express the remaining Maxwell Equation in the Feynman-Dyson context.
Remark. Note the role played by the epsilon tensor $\epsilon_{ijk}$ throughout the construction of generalized electromagnetism in this section. The epsilon tensor is the structure constant for the Lie algebra of the rotation group $SO(3)$. If we replace the epsilon tensor by a structure constant $f_{ijk}$ for a Lie algebra $\mathcal{G}$ of dimension $d$ such that the tensor is invariant under cyclic permutation ($f_{ijk} = f_{kij}$), then most of the work in this section will go over to that context. We would then have $d$ operator/variables $X_1, \cdots, X_d$ and a generalized cross product defined on vectors of length $d$ by the equation

$$(A \times B)_k = f_{ijk} A_i B_j.$$ 

The Jacobi identity for the Lie algebra $\mathcal{G}$ implies that this cross product will satisfy

$$A \times (B \times C) = (A \times B) \times C + [B \times (A) \times C]$$

where

$$([B \times (A) \times C])_r = f_{klr} f_{ijk} A_i B_k C_j.$$ 

This extension of the Jacobi identity holds as well for the case of non-commutative cross product defined by the epsilon tensor. It is therefore of interest to explore the structure of generalized non-commutative electromagnetism over other Lie algebras (in the above sense). This will be the subject of another paper.

2.1 Discrete Thoughts

In the hypotheses of the Electromagnetic Theorem, we are free to take any non-commutative world, and the Electromagnetic Theorem will satisfied in that world. For example, we can take
each $X_i$ to be an arbitrary time series of real or complex numbers, or bitstrings of zeroes and ones. The global time derivative is defined by

$$\dot{F} = J(F' - F) = [F, J],$$

where $FJ = JF'$. This is the non-commutative discrete context discussed in sections 1. We will write

$$\dot{F} = J\Delta(F)$$

where $\Delta(F)$ denotes the classical discrete derivative

$$\Delta(F) = F' - F.$$

With this interpretation $X$ is a vector with three real or complex coordinates at each time, and

$$B = \dot{X} \times \dot{X} = J^2\Delta(X') \times \Delta(X)$$

while

$$E = \ddot{X} - \dot{X} \times (\dot{X} \times \dot{X}) = J^2\Delta^2(X) - J^3\Delta(X'') \times (\Delta(X') \times \Delta(X)).$$

Note how the non-commutative vector cross products are composed through time shifts in this context of temporal sequences of scalars. The advantage of the generalization now becomes apparent. We can create very simple models of generalized electromagnetism with only the simplest of discrete materials. In the case of the model in terms of triples of time series, the generalized electromagnetic theory is a theory of measurements of the time series whose key quantities are

$$\Delta(X') \times \Delta(X)$$

and

$$\Delta(X'') \times (\Delta(X') \times \Delta(X)).$$
It is worth noting the forms of the basic derivations in this model. We have, assuming that $F$ is a commuting scalar (or vector of scalars) and taking $\Delta_i = X'_i - X_i$,

$$\partial_i(F) = [F, \dot{X}_i] = [F, J\Delta_i] = F J \Delta_i - J \Delta_i F = J(F' \Delta_i - \Delta_i F) = \dot{F} \Delta_i$$

and for the temporal derivative we have

$$\partial_t F = J[1 - J \Delta' \cdot \Delta] \Delta(F)$$

where $\Delta = (\Delta_1, \Delta_2, \Delta_3)$.

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