ACF estimation via difference schemes for a semiparametric model with $m$-dependent errors

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In this manuscript, we discuss a class of difference-based estimators of the autocovariance structure in a semiparametric regression model where the signal is discontinuous and the errors are serially correlated. The signal in this model consists of a sum of the function with jumps and an identifiable smooth function. A simpler form of this model has been considered earlier under the name of Nonparametric Jump Regression (NJRM). The estimators proposed allow us to bypass a complicated problem of prior estimation of the mean signal in such a model. We provide finite-sample expressions for biases and variance of the proposed estimators when the errors are Gaussian. Gaussianity in the above is only needed to provide explicit closed form expressions for biases and variances of our estimators. Moreover, we observe that the mean squared error of the proposed variance estimator does not depend on either the unknown smooth function that is a part of the mean signal nor on the values of difference sequence coefficients. Our approach also suggests sufficient conditions for $\sqrt{n}$- consistency of the proposed estimators.

Keywords: autocovariance estimation, change-point, semiparametric model, difference-based method, quadratic variation.

1. Introduction and discussion of the general case

Let us consider first a general nonparametric regression model with correlated errors

$$y_i = g(x_i) + \epsilon_i, i = 1, \ldots, n$$

(1.1)

where $x_i$ are the fixed sampling points, $g$ is an unknown mean function defined on a closed interval that can be discontinuous, and $(\epsilon_i)$ is a zero mean stationary time-series error process. For such a model, the knowledge of the autocovariance of the errors $\gamma_h = \mathbb{E}[\epsilon_i \epsilon_{i+h}], h = 0, 1, \ldots$ plays an important practical role. More specifically, the autocovariance function is needed to provide efficient estimates and confidence regions for
the location of discontinuities as well as the magnitude of the corresponding jumps. To estimate autocovariance function under the discontinuous and possibly highly fluctuating mean function condition is quite difficult in general, and some prior knowledge of the dependence structure is necessary. (Tecuapetla-Gómez and Munk, 2017) considers the situation where the error process is stationary, zero mean, and $m$-dependent, that is $\gamma_h = 0$ for any $h : |h| > m$. $m$-dependency may be viewed as a somewhat restrictive assumption; however, it can be often thought of as a proxy for more general situations as when, for example, $\gamma_h$ decays exponentially with increasing lag $h$. (Tecuapetla-Gómez and Munk, 2017) also consider a highly fluctuating mean function defined as

$$g(x) = \sum_{j=1}^{K} a_{j-1} 1_{[\tau_{j-1}/n, \tau_j/n]}(x)$$

(1.2)

for any $x \in [0, 1)$ and $a_j \neq a_{j+1}$. Such a function is locally constant and has change-points at $0 = \tau_0 < \tau_1 < \cdots < \tau_{K-1} < \tau_K = 1$. The levels $\{a_j\}$, the location of change points, and their number $K$ are all unknown.

We will take the setting (1.1) one step further and assume that the mean function now has a semiparametric form. More specifically, we now consider a general regression with correlated errors model

$$y_i = f(x_i) + g(x_i) + \epsilon_i, \quad i = 1, \ldots, n,$$

(1.3)

where $x_i = i/n$ are sampling points, $g(x)$ is defined as before in (1.2) while $f$ is a once differentiable and square-integrable function on $[0, 1]$. The errors $\{\epsilon_i\}$ form a zero mean, stationary, $m$-dependent process. To ensure identifiability, we require that $\int_0^1 f(x) \, dx = 0$.

Somewhat similar partial linear regression models (where the function $g(x)$ is a linear function of $x$ that does not contain jumps) with correlated errors have a fairly long history in statistical research. (Engle et al., 1986) already established, in their study of the effect of weather on electricity demand that the data were autocorrelated at order one. (Gao, 1995) was probably the first to study estimation of the partial linear model with correlated errors. (You and Chen, 2007) obtained an improved estimator of the linear component in such a model using the estimated autocorrelation structure of the process $(\epsilon_i)$. In the case where the mean function has jumps, estimation of the jump size and bandwidth selection have been studied mostly in the nonparametric regression setting with i.i.d errors and not so much in the semiparametric setting; as an example, we can mention e.g. (Gijbels and Goderniaux, 2004), (Gijbels et al., 2007) and many others. A model where the mean structure is exactly the same as in (1.3) but the errors are iid is typically called a Nonparametric Jump Regression (NJRM) and was considered in Qiu and Yandell (1998) who were concerned with the jump detection in that model. This model is often appropriate when the mean function in a regression model jumps up or down under the influence of some important random events. A good practical example are stock market indices, physiological responses to stimuli and many others. Regression functions with jumps are typically more appropriate than continuous regression models for such data. Thus, our model (1.3) effectively amounts to the generalization of NJRM
model to the case of serially correlated errors. There has been little work done even with parametric (let alone semiparametric) regression models with jumps in the mean where errors are serially correlated errors. For the most part, it was done in the case where the number of change points is finite but unknown; see, for example, (Davis et al., 2006; Fryzlewicz and Subba Rao, 2014; Preuß et al., 2015; Chakar et al., 2017).

To the best of our knowledge, (Tecuapetla-Gómez and Munk, 2017) were the first to consider having a potentially infinite number of change points in a parametric model with jumps in the mean and with serially correlated errors. Compared to (Tecuapetla-Gómez and Munk, 2017), we take an extra step forward and consider a mean function that consists of both a change-point component and a smooth function \( f \). To the best of our knowledge, estimation of autocovariance structure in such a model has not been attempted before.

When estimating either the mean function or the dependence structure in this model it is preferable to avoid having to estimate the mean function first in order to obtain estimates of the second order structure of the error process i.e. its variance and covariances. To do so we will introduce the so-called difference-based estimators. To be able to introduce them, however, we have to start with some notation. First, for any \( i < j \), we will use the notation \([i : j]\) for an index vector \((i, i+1, \ldots, j)\). Thus, for a generic vector \( v^\top = (v_1 \quad v_2 \quad \cdots \quad v_n) \), for \( 1 \leq i < j \leq n \), we define \( v[^{i:j}] = (v_i \quad v_{i+1} \quad \cdots \quad v_j) \). Let \( F[^{i:j}], G[^{i:j}] \) and \( \varepsilon[^{i:j}] \) denote the vectors \((f(x_i) \quad \cdots \quad f(x_j))\), \((g(x_i) \quad \cdots \quad g(x_j))\) and \((\varepsilon_i \quad \cdots \quad \varepsilon_j)\), respectively. Also, \( f_i \) and \( g_i \) will stand for \( f(x_i) \) and \( g(x_i) \). The quadratic variation of the function \( g(x) \) will be denoted \( J_K := \sum_{j=0}^{K-1} (a_{j+1} - a_j)^2 \).

Let \( l \geq 1 \) be given such that \( n_i = n - l(m+1) \gg 0 \). We start with defining a vector of weights \( d^\top = (d_0 \quad d_1 \quad \cdots \quad d_l) \). Using this vector, we can also define a difference of order \( l \) and a gap \( m + 1 \) as \( \Delta_{l,m+1}(y; d) = \sum_{s=0}^{l} d_s y_{i+s(m+1)} \). Finally, we can define a difference-based estimator of order \( l \) and gap \( m + 1 \) as a quadratic form:

\[
Q_{l,m+1}(y; d) = \frac{1}{p(d)} n_{i_m} \sum_{i=1}^{n_{i_m}} \Delta_{l,m+1}^2(y; d),
\]

where \( p(d) \) is a normalizing factor depending on the vector \( d \) only that we will specify a little later. In the rest of this Section, in order to simplify notation we will omit \( d \) inside the parentheses for observation differences and write simply \( \Delta_{l,m+1}(y) \) unless any confusion results from such an omission.

Next, we denote \( 0_m \) an \( m \times 1 \) vector of zeros. This allows us to define an \( 1 \times (l(m+1)+1) \) row vector \( \mathbf{u}_l^\top = (d_0 \quad 0_m \quad d_1 \quad 0_m \quad \cdots \quad d_{l-1} \quad 0_m \quad d_l) \) and the corresponding \( 2 \times (l(m+1) + 2) \) matrix

\[
\hat{D} = \begin{pmatrix}
\mathbf{u}_l^\top & 0 \\
0 & \mathbf{u}_l^\top
\end{pmatrix}.
\]

Finally, we define an \((l(m+1) + 2) \times (l(m+1) + 2)\) symmetric matrix \( D = \hat{D}^\top \hat{D} \).

It is not difficult to see that for given \( i \), we have \( y[^{(i)}] D y[^{(i)}] = \sum_{l=1}^{i} Q_{l,m+1}(y; d) \).
\[ \sum_{j=i}^{i+1} \Delta_{l,m+1}^2 (y_j) \]. Consequently,

\[ 2 n_m p(d) Q_{l,m+1}(y; d) = \Delta_{l,m+1}^2 (y_i) + \sum_{i=1}^{n_m-1} y_{[i:(i+1+l(m+1)]}^T D y_{[i:(i+1+l(m+1)]} + \Delta_{l,m+1}^2 (y_{n_m}). \]

Note that, for any \( 1 \leq i < j \leq n \), we have \( y_{[i:j]} = F_{[i:j]} + G_{[i:j]} + \varepsilon_{[i:j]} \). Moreover, since \( (\varepsilon_i) \) is a zero mean process, we have, for any \( (j-i+1) \times (j-i+1) \) matrix \( \Sigma \)

\[ E[\varepsilon_{[i:j]}^T \Sigma \varepsilon_{[i:j]}] = \text{tr}(\Sigma \text{VAR}(\varepsilon_{[i:j]})) \] see e.g. (Provost and Mathai, 1992) p.51. Due to this, the following expansion holds:

\[ E[y_{[i:(i+1+l(m+1)]}]^T D y_{[i:(i+1+l(m+1)]}] = G_{[i:(i+1+l(m+1))]}^T D G_{[i:(i+1+l(m+1))]} + 2 G_{[i:(i+1+l(m+1))]}^T D F_{[i:(i+1+l(m+1))]} + F_{[i:(i+1+l(m+1))]}^T D F_{[i:(i+1+l(m+1))]} + \text{tr}(D \text{VAR}(\varepsilon_{[i:(i+1+l(m+1))]})). \]

It can be shown by means of the direct calculation that for any value of \( i \),

\[ E[\Delta_{l,m+1}^2 (y_i)] = \gamma_0 \sum_{s=0}^{l} d_s^2 + \sum_{s=0}^{l} d_s^2 (f_{i+s(m+1)} + g_{i+s(m+1)})^2 + 2 \sum_{s=0}^{l-1} \sum_{t=s+1}^{l} d_s d_t (f_{i+s(m+1)} + g_{i+s(m+1)})(f_{i+t(m+1)} + g_{i+t(m+1)}) \]

\[ = \gamma_0 \sum_{s=0}^{l} d_s^2 + \left( \sum_{s=0}^{l} d_s [f_{i+s(m+1)} + g_{i+s(m+1)}] \right)^2. \]

Also, utilizing Proposition A.1 in the Appendix of (Tecuapetla-Gómez and Munk, 2017) we deduce that for any \( i \),

\[ \text{tr}(D \text{VAR}(\varepsilon_{[i:(i+1+l(m+1))]})) = 2 \gamma_0 \sum_{s=0}^{l} d_s^2. \]

Combining the above results, we obtain

\[ 2 n_m p(d) E[Q_{l,m+1}(y; d)] = 2 n_m \gamma_0 \sum_{s=0}^{l} d_s^2 + 2 \left( \sum_{s=0}^{l} d_s [f_{i+s(m+1)} + g_{i+s(m+1)}] \right)^2 + \sum_{i=1}^{n_m-1} (F_{[i:(i+1+l(m+1))]} + G_{[i:(i+1+l(m+1))]}^T D (F_{[i:(i+1+l(m+1))]} + G_{[i:(i+1+l(m+1))]})). \]
Note that the order \(l\) is finite while the function \(f\) is bounded on \([0, 1]\). Due to this, the term
\[
2 \left( \sum_{s=0}^{l} d_s [f_{i+s(m+1)} + g_{i+s(m+1)}] \right)^2
\]
is clearly bounded. Therefore,
\[
\frac{1}{n^{l_m}} 2 \left( \sum_{s=0}^{l} d_s [f_{i+s(m+1)} + g_{i+s(m+1)}] \right)^2 = o(1)
\]
as \(n \to \infty\). Due to Lemma (1) in the Appendix, we have
\[
\frac{1}{n^{l_m}} \sum_{i=1}^{n^{l_m}-1} F_{[i:(i+1+l(m+1))]}^\top D F_{[i:(i+1+l(m+1))]} \approx \frac{2}{n^{l_m}} \sum_{i=1}^{n^{l_m}} \Delta_{i_m}^2 (f_i)
\]
\[
\lim_{n \to \infty} 2 \left( \sum_{s=0}^{l} d_s \right)^2 \int_0^1 f^2(x) \, dx;
\]
moreover, by the mean value theorem, we also have
\[
\frac{1}{n^{l_m}} \sum_{i=1}^{n^{l_m}-1} F_{[i:(i+1+l(m+1))]}^\top D G_{[i:(i+1+l(m+1))]} \approx \frac{2}{n^{l_m}} \sum_{s=0}^{l} d_s \sum_{i=1}^{n^{l_m}} f_i \langle w_l, G_{[i:(i+1+l(m+1))]} \rangle
\]
\[
+ O \left( \frac{1}{n^{l_m} l^2} \sum_{i=1}^{n^{l_m}} \langle w_l, G_{[i:(i+1+l(m+1))]} \rangle \right).
\]
Finally, the term
\[
\frac{1}{n^{l_m}} \sum_{i=1}^{n^{l_m}-1} G_{[i:(i+1+l(m+1))]}^\top D G_{[i:(i+1+l(m+1))]} D G_{[i:(i+1+l(m+1))]}^\top
\]
is the bias of the original estimator of (Tecuapetla-Gómez and Munk, 2017) for an arbitrary order \(l\) and the gap \(m+1\). As explained in (Tecuapetla-Gómez and Munk, 2017), this term is, therefore, of the order \(l(m+1) J_k / n\). Combining the above results, we obtain the following

**Theorem 1.** Suppose that the conditions of model (1.3) are satisfied, then
\[
p(d) E[Q_{l,m+1}(y; d)] = \gamma_0 \left( \sum_{s=0}^{l} d_s^2 \right) + \left( \sum_{s=0}^{l} d_s \right)^2 \int_0^1 f^2(x) \, dx + O \left( \frac{l(m+1) J_k}{n} \right)
\]
\[
+ \sum_{s=0}^{l} d_s \sum_{i=1}^{n^{l_m}} f_i \langle w_l, G_{[i:(i+1+l(m+1))]} \rangle + O \left( \frac{1}{n^{l_m}} \sum_{i=1}^{n^{l_m}} \langle w_l, G_{[i:(i+1+l(m+1))]} \rangle \right).
\]

The following statement follows immediately from Theorem (1).

**Corollary 1.** Suppose that the conditions of Theorem 1 hold. Furthermore, assume that
\[
\sum_{s=0}^{l} d_s = 0 \quad \text{and let} \quad p(d) = \sum_{s=0}^{l} d_s^2.
\]
Then
\[ E[Q_{l,m+1}(y;d)] = \gamma_0 + O\left( l(m+1) \frac{J_K}{n} \right) + O\left( \frac{1}{n^{l_m}} \sum_{i=1}^{n_{l_m}} (w_i, G_{[i+(l+1)(m+1)]}) \right). \]

The results obtained so far suggest that a prudent course of action at this point is to study in detail a difference-based estimators of error variance and autocovariances of order \( l = 1 \) only. There are two main reasons for this. First, note that so far we have not assumed a particular stationary distribution for the error process \((\varepsilon_i)\). Thus, assuming that the error process has a null mean, is stationary and \(m\)-dependent is sufficient to ensure that, asymptotically, the influence of the smooth part of the regression function, i.e. \(f\), over the bias of the general difference-based variance estimator \(Q_{l,m+1}(y)\) is negligible. Moreover, this fact remains true regardless of the order \(l\). Thus, from the practical viewpoint, it may not make a lot of sense to consider difference-based estimators of the variance and autocovariances of the error process \((\varepsilon_i)\) for \(l > 1\). Second, it is clear from the Corollary (1) that the increase in the order \(l\) of the difference-based estimator will lead to the increase of the bias of the magnitude \(l(m+1)\) which is, of course, undesirable. Finally, note that the full investigation of the influence of the order \(l\) on the bias of the difference-based estimator of the variance requires studying the term \(O\left( \frac{1}{n^{l_m}} \sum_{i=1}^{n_{l_m}} (w_i, G_{[i+(l+1)(m+1)]}) \right)\) whose behavior in general is rather complicated. To give an idea of the problems involved, we investigate a special case of \(l = 1\).

To simplify notation, we denote \(n_{1m} := n_m\). If the function \(g\) obeys a piecewise constant representation (1.2) for \(l = 1\) we obtain the following.

**Corollary 2.** Suppose that the conditions of Theorem 1 and Eq. (1.4) hold. Then,
\[ E[Q_{1,m+1}(y;d)] = \gamma_0 + \frac{m+1}{p(d) n_m} \sum_{j=1}^{K-1} (d_0 a_{j-1} + d_1 a_j)^2 + O(n^{-1}). \]

Moreover, under the constraint \(d_0 + d_1 = 0\) we find that the bias of the estimator \(Q_{1,m+1}(y;d)\) is
\[ \text{BIAS}[Q_{1,m+1}(y;d)] = \frac{m+1}{p(d)} \frac{J_K}{n} + O(n^{-1}). \]

This Corollary will be proved in the next section. Its proof will show a number of technical issues that are already prominent when \(l = 1\).

2. Difference-based estimators of autocovariance structure
In this Section, we continue studying the general model (1.3) that we restate here for convenience. Thus, our model is

\[ y_i = f(x_i) + \sum_{j=1}^{K} a_{j-1} \mathbb{I}[\tau_{j-1}/n, \tau_j/n)(x_i) + \varepsilon_i, \quad i = 1, \ldots, n, \quad (2.1) \]

where \( x_i = i/n \), \( 0 = \tau_0 < \tau_1 < \ldots < \tau_{K-1} < \tau_K = n \). The function \( f \) is assumed to be differentiable on \([0,1]\). The change-point locations \( (\tau_j) \), their levels \( (a_j) \), and the number of jumps \( K \) are assumed to be unknown. Both \( K \) and the change-point levels \( (a_j) \) can be potentially large. The only regularity condition we impose on the change-points is that

\[ \min_{1 \leq j \leq K} |\tau_j - \tau_{j-1}| > 2(m + 1). \quad (2.2) \]

Tecuapetla-Gómez and Munk (2017) considered the model (2.1) when \( f \equiv 0 \). In this simple case the authors showed that the estimation of the autocovariance function (ACF) of the errors can be carried out easily utilizing the family of difference-based estimators (DBE). The authors showed in their Theorem 5 that in order to obtain a mean squared error consistent estimator of the variance \( \gamma_0 \) it is necessary to consider a DBE with the gap at least \( m + 1 \). Since the more general model (2.1) has the same error structure as the special case considered in Tecuapetla-Gómez and Munk (2017), we will also study the second order structure estimation problem for the model (2.1) using a difference-based estimator of order \( l = 1 \) and the gap \( m + 1 \).

We begin with estimation of the variance \( \gamma_0 \). Recall that the first-order difference of observations with the gap \( m + 1 \) is \( \Delta_{1,m+1}(y; d) = d_0 y_i + d_1 y_{i+m+1} \). Note that the difference coefficient vector in this case is \( d = (d_0, d_1) \). Finally, throughout this section, we will use the notation \( C \) for a generic constant that does not depend on \( K \) but may depend on \( m, f, \) change point locations \( (\tau_j) \) and the finite number of change point levels \( (a_j) \). A direct difference-based estimator of \( \gamma_0 \) of order 1 and the gap \( m + 1 \) can then be defined as

\[ \hat{\gamma}_0 = Q_{1,m+1}(y; d) = \frac{1}{n_m p(d)} \sum_{i=1}^{n_m} \Delta_{1,m+1}^2(y_i; d), \quad (2.3) \]

where \( p(d) = d_0^2 + d_1^2 \). The autocovariances \( \gamma_h \) with \( h = 1, \ldots, m \) will be estimated using the following estimator:

\[ \hat{\gamma}_h = Q_{1,m+1}(y; d) - Q_{1,h}(y; (1, -1)), \quad h = 1, \ldots, m. \quad (2.4) \]

We begin the investigation of (2.3) and (2.4) with characterizing the bias of the estimator \( Q_{1,m+1}(y; d) \) This requires us to calculate the expectation \( E[Q_{1,m+1}(y; d)] \) by proving the Corollary 2 stated in the previous section.

2.1. Proof of Corollary 2

The proof of this result is based on the following
**Proposition 1.** Consider the model (2.1) and assume that \( f \) is a function that is differentiable on \((0,1)\) and also square integrable on that interval. Moreover, we assume that the condition (2.2) is satisfied. Then,

\[
\mathbb{E}[p(d) Q_{1,m+1}(y; d)] = (d_0^2 + d_1^2) \gamma_0 + (d_0 + d_1)^2 \int_0^1 f^2(x) dx + o(1)
\]

\[
+ \frac{(d_0 + d_1)^2}{n_m} \sum_{j=1}^K \tau_j - \tau_{j-1} - (m+1) a_{j-1}^2 + \frac{m+1}{n_m} J_K(d)
\]

\[
+ \frac{2(d_0 + d_1)}{n_m} \sum_{j=1}^{K-2} \sum_{i=\tau_{j-1}}^{\tau_j-1} \Delta_m(f_i; d) + O(n^{-1}),
\]

where \( J_K(d) = \sum_{j=1}^{K-1} (d_0 a_{j-1} + d_1 a_j)^2 \).

It is clear that, if Proposition 1 is correct, it implies, along with the conditions imposed in Eq. (1.4) that first statement of the Corollary 2 is established. Next, using the constraint \( d_0 + d_1 = 0 \), we find that

\[
\text{BIAS}^2(Q_{1,m+1}(y; d)) = \left( \frac{1}{p(d)} \frac{1}{n_m} \sum_{i=1}^{n_m} \Delta_m^2(f_i; d) + \frac{m+1}{n_m p(d)} J_K + O(n^{-1}) \right)^2
\]

\[
= \left( \frac{d_0 + d_1}{p(d)} \left[ \int_0^1 f^2(x) dx + o(n^{-2}) \right] + \frac{m+1}{n_m p(d)} J_K + O(n^{-1}) \right)^2
\]

\[
= \left( \frac{m+1}{p(d)} J_K + O(n^{-1}) \right)^2
\]

where the second identity follows from Lemma 1 in the Appendix.

Corollary (2) suggests that, if we assume that \( d_0 + d_1 = 0 \), the smooth function \( f \) does not influence the first order term of the bias expansion for the variance estimator \( Q_{1,m+1}(y; d) \). As before in Tecuapetla-Gómez and Munk (2017), the first order term is proportional to the discrete quadratic variation of the underlying signal of model 2.1 that we denote \( J_K(d) = \sum_{j=1}^{K-1} (d_0 a_{j-1} + d_1 a_j)^2 \). Thus, the discrete quadratic variation \( J_K(d) \) is the quantity that most directly influences the bias.

Next, to analyze the bias of the autocovariance estimator \( \hat{\gamma}_h \) it is necessary first to obtain the bias expression for \( Q_{1,h}(y; d) \). This can be done in a way very similar to that elucidated in the previous proof. To make our presentation reasonably concise, we give the following Proposition without a proof.

**Proposition 2.** Suppose that the conditions of Proposition 1 hold. Furthermore, assume that Eq. (1.4) hold with \( l = 1 \). Then,

\[
\mathbb{E}[Q_{1,h}(y; (1,-1))] = \gamma_0 - \gamma_h + \frac{h}{2n_h} J_K + O(n^{-1}). \quad (2.5)
\]
Proof of Proposition 1. First, for any change-point \( \tau_j \), we can calculate \( \Delta_m(y_i; d) \) in the interval \([\tau_{j-1}, \tau_j]\) directly as follows:

\[
\Delta_m(y_i; d) = \begin{cases}
(d_0 + d_1)a_{j-1} + \Delta_m(f_i; d) + \Delta_m(\varepsilon_i; d) & \text{for } j = 1, \ldots, K, \\
\phantom{\text{for }} \quad \tau_{j-1} \leq i \leq \tau_{j} - (m + 2) \\
d_0a_{j-1} + \Delta_m(f_i; d) + \Delta_m(\varepsilon_i; d) & \text{for } j = 1, \ldots, K - 1, \\
\phantom{\text{for }} \quad \tau_{j-1} \leq i \leq \tau_{j} - 1 \\
d_0(a_{K-1} + f_i + \varepsilon_i) & \text{for } \tau_{K} - m \leq i \leq \tau_{K} - 1, \\
0 & \text{otherwise}.
\end{cases}
\] (2.6)

Set \( c_{n,m,d} = (n_m p(d))^{-1} \). Therefore, we can represent the estimator \( Q_{1,m+1}(y; d) \) as

\[
Q_{1,m+1}(y; d) = c_{n,m,d} \sum_{i=1}^{n_m} \Delta_m^2(y_i; d) = c_{n,m,d} \left( S_{1,K,m,d} + S_{2,K,m,d} \right),
\] (2.7)

where

\[
S_{1,K,m,d} = \sum_{j=1}^{K} \sum_{i=\tau_{j-1}}^{\tau_{j}-(m+2)} \left[ (d_0 + d_1)a_{j-1} + \Delta_m(f_i; d) + \Delta_m(\varepsilon_i; d) \right]^2;
\] (2.8)

and

\[
S_{2,K,m,d} = \sum_{j=1}^{K-1} \sum_{i=\tau_{j}-(m+1)}^{\tau_{j-1}} \left[ d_0a_{j-1} + d_1a_j + \Delta_m(f_i; d) + \Delta_m(\varepsilon_i; d) \right]^2 \\
+ \left[ d_0a_{K-1} + \Delta_m(f_{\tau_{K}-(m+1)}; d) + \Delta_m(\varepsilon_{\tau_{K}-(m+1)}; d) \right]^2 \\
+ d_0^2 \sum_{i=\tau_{K}-m}^{\tau_{K}-1} (a_{K-1} + f_i + \varepsilon_i)^2.
\]

Since \( \mathbb{E}[\varepsilon_i] = 0 \) for all \( i \), and \( \mathbb{E}[\Delta_m^2(\varepsilon_i; d)] = (d_0^2 + d_1^2)\gamma_0 \) for all \( i \leq n_m \), we find that

\[
\mathbb{E}[(d_0 + d_1)a_{j-1} + \Delta_m(f_i; d) + \Delta_m(\varepsilon_i; d)]^2 = (d_0 + d_1)^2a_{j-1}^2 + (d_0^2 + d_1^2)\gamma_0 + \Delta_m^2(f_i; d) \\
+ 2(d_0 + d_1)a_{j-1}\Delta_m(f_i; d) := A_{1,j,i,d},
\] (2.9)

and

\[
\mathbb{E}[d_0a_{j-1} + d_1a_j + \Delta_m(f_i; d) + \Delta_m(\varepsilon_i; d)]^2 = (d_0a_{j-1} + d_1a_j)^2 + (d_0^2 + d_1^2)\gamma_0 + \Delta_m^2(f_i; d) \\
+ 2(d_0a_{j-1} + d_1a_j)\Delta_m(f_i; d) := A_{2,j,i,d}.
\] (2.10)
Analogous calculations also suggest that
\[
E[d_0a_{K-1} + \Delta_m(f_{\tau_K-(m+1)}; d) + \Delta_m(\varepsilon_{\tau_K-(m+1)}; d)]^2 = d_0^2a_{K-1} + (d_0^2 + d_1^2)\gamma_0
\]
\[+ \Delta_m(f_{\tau_K-(m+1)}; d) + 2d_0a_{K-1}\Delta_m(f_{\tau_K-(m+1)}; d)
\]
(2.11)
and
\[
\sum_{i=\tau_K-m}^{\tau_K-1} E(a_{K-1} + f_i + \varepsilon_i)^2 = m(a_{K-1}^2 + \gamma_0) + \sum_{i=\tau_K-m}^{\tau_K-1} f_i^2 + 2a_{K-1}\sum_{i=\tau_K-m}^{\tau_K-1} f_i.
\]
(2.12)

Notice that (2.11)-(2.12) are finite sums. Due to this, in what follows we only use some of their summands while the sum of the rest is viewed as a generic constant \(C\). Hence,
\[
E[S_{1,K,m,d} + S_{2,K,m,d}] = \sum_{j=1}^{K} \sum_{i=\tau_j-1}^{\tau_j-(m+2)} A_{1,j,i,d} + \sum_{j=1}^{K-1} \sum_{i=\tau_j-1}^{\tau_j-(m+1)} A_{2,j,i,d}
\]
\[+ \Delta_m^2(f_{\tau_K-(m+1)}; d) + (d_0^2 + d_1^2)\gamma_0 + C
\]
\[= \sum_{j=1}^{K-1} \sum_{i=\tau_j-1}^{\tau_j-(m+2)} \Delta_m^2(f_i; d) + \sum_{j=1}^{K-1} \sum_{i=\tau_j-1}^{\tau_j-(m+1)} (d_0^2 + d_1^2)\gamma_0 + \sum_{i=\tau_K-1}^{\tau_K-(m+1)} (d_0^2 + d_1^2)\gamma_0
\]
\[+ \sum_{j=1}^{K-1} \left\{ \sum_{i=\tau_j-1}^{\tau_j-(m+2)} (d_0 + d_1)^2_a_{j-1}^2 + \sum_{i=\tau_j-(m+1)}^{\tau_j-1} (d_0a_{j-1} + d_1a_j)^2 \right\} + \sum_{i=\tau_K-1}^{\tau_K-(m+2)} (d_0 + d_1)^2_a_{K-1}^2
\]
\[+ B_{1,K,d} + B_{2,K,d} + 2d_0a_{K-1}\Delta_m(f_{\tau_K-(m+1)}; d) + C
\]
where \(B_{1,K,d} = 2(d_0 + d_1) \sum_{j=1}^{K} a_{j-1} \sum_{i=\tau_j}^{\tau_j-(m+2)} \Delta_m(f_i; d)\) and \(B_{2,K,d} = 2 \sum_{j=1}^{K-1} (d_0a_{j-1} + d_1a_j) \sum_{i=\tau_j}^{\tau_j-(m+1)} \Delta_m(f_i; d)\). Using a telescopic sum over the terms \((\tau_j - \tau_{j-1})\)'s, and recalling that \(\tau_K = n\), we find that
\[
n_{m,p}(d) E[Q_{1,m+1}(y; d)] = n_m(d_0^2 + d_1^2)\gamma_0 + \sum_{i=1}^{n_m} \Delta_m^2(f_i; d)
\]
\[+ (d_0 + d_1)^2 \sum_{j=1}^{K} (\tau_j - \tau_{j-1} - (m + 1)a_{j-1}^2
\]
\[+ (m + 1) \sum_{j=1}^{K-1} (d_0a_{j-1} + d_1a_j)^2 + B_{1,K,d} + B_{2,K,d}
\]
\[+ 2d_0a_{K-1}\Delta_m(f_{\tau_K-(m+1)}; d) + C.
\]
(2.13)
When we sum up the last three terms above (without the constant) we obtain
\[
B_{1,K,d} + B_{2,K,d} + 2d_0 a_{K-1} \Delta_m(f_{\tau_{K-1}(m+1)}; d) = 2(d_0 + d_1) \left\{ \sum_{j=1}^{K-2} a_j \sum_{i=\tau_j-1}^{\tau_j-1} \Delta_m(f_i; d) \right\} 
+ 2d_0 \left\{ a_0 \sum_{i=\tau_0}^{\tau_1-1} \Delta_m(f_i; d) + a_{K-1} \sum_{i=\tau_{K-1}}^{\tau_{K-1}(m+1)} \Delta_m(f_i; d) \right\} 
+ 2d_1 \left\{ a_0 \sum_{i=\tau_0}^{\tau_1-(m+2)} \Delta_m(f_i; d) + a_{K-1} \sum_{i=\tau_{K-2}}^{\tau_{K-1}-1} \Delta_m(f_i; d) \right\}.
\]

It remains to notice that the last two terms are bounded by
\[
2 (m + 1) \max\{|d_0|, |d_1|\} \max\{|a_0|, |a_{K-1}|\} \max_{s \in \{0, \ldots, K-1\}} |\Delta_m(f_{\tau_s}; d)|;
\]
since the function $f$ is differentiable, the bound is a constant. According to (2.13), it only remains to show that $n_m^{-1} \sum_{i=1}^{n_m} \Delta_m^2(f_i; d) \to (d_0 + d_1)^2 \int_0^1 f^2(x) \, dx$ which follows from Lemma 1. This completes the proof. \[\square\]

2.2. On the variance of $Q_{1,m+1}(y; d)$

Now we focus on computing $\text{VAR}(Q_{1,m+1}(y; d))$. To state the needed result, we first need to introduce two additional characteristics of our model: we denote $J_K^\parallel = \sum_{j=1}^{K} |a_{j-1} - a_j|$ and $H_K^\parallel = \sum_{j=1}^{K} (\tau_j - \frac{m+1}{2}) |a_{j-1} - a_j|$.

**Theorem 2.** Let the regularity condition (2.2) be satisfied. Moreover, let $d_0 + d_1 = 0$ and let $p(d) = d_0^2 + d_1^2$. Finally, let both $H_K^\parallel$ and $J_K^\parallel$ be of the order $o(n)$. Then, the variance of the first order estimator $Q_{1,m+1}(y; d)$ is
\[
\text{VAR}(Q_{1,m+1}(y; d)) = \frac{2m + 3}{n} \gamma_0^2 + \frac{(m + 1)(m + 2)}{n^2} J_K \gamma_0 + o(n^{-2}).
\]

**Remark 1.** The theorem (2) implies that the variance of the estimator $Q_{1,m+1}(y; d)$ does not depend on the choice of the difference sequence $d$. This happens because the terms that could potentially introduce such a dependence themselves depend, in turn, on the quantities $J_K^\parallel$ and $H_K^\parallel$. Due to our choice of $H_K^\parallel = o(n)$ and $J_K^\parallel = o(n)$, the dependence on the choice of $d$ is not present in the main terms of the variance expansion.
which implies that

Moreover, it is not difficult to see that

Next, note that for any $d$ whenever this does not cause any confusion. By direct calculation, we find that

In order to make the notation easier, we will suppress the vector $d$ from this point on whenever this does not cause any confusion. By direct calculation, we find that

\begin{align*}
\text{VAR}(\Delta^2_m(y_i)) &= \text{VAR}(\Delta^2_m(f_i + g_i) + 2 \Delta_m(f_i + g_i) \Delta_m(\varepsilon_i) + \Delta^2_m(\varepsilon_i)) \\
&= 4 \Delta^2_m(f_i + g_i) \text{VAR}(\Delta_m(\varepsilon_i)) + \text{VAR}(\Delta^2_m(\varepsilon_i)) \\
&+ 2 \text{COV}(\Delta^2_m(f_i + g_i), 2 \Delta_m(f_i + g_i) \Delta_m(\varepsilon_i)) \\
&+ 2 \text{COV}(\Delta^2_m(f_i + g_i), \Delta^2_m(\varepsilon_i)) \\
&+ 2 \text{COV}(2 \Delta_m(f_i + g_i) \Delta_m(\varepsilon_i), \Delta^2_m(\varepsilon_i)).
\end{align*}

Next, note that for any $i$, \(\text{VAR}(\Delta_m(\varepsilon_i)) = (d_0^2 + d_1^2)\gamma_0^2\), \(\text{VAR}(\Delta^2_m(\varepsilon_i)) = 2 (d_0^2 + d_1^2)^2\gamma_0^2\).

Moreover, it is not difficult to see that

\begin{align*}
\text{COV}(\Delta^2_m(f_i + g_i), 2 \Delta_m(f_i + g_i) \Delta_m(\varepsilon_i)) &= \text{COV}(\Delta^2_m(f_i + g_i), \Delta^2_m(\varepsilon_i)) = 0 \\
\text{COV}(2 \Delta_m(f_i + g_i) \Delta_m(\varepsilon_i), \Delta^2_m(\varepsilon_i)) &= 0
\end{align*}

which implies that

\begin{align*}
\text{VAR}(\Delta^2_m(y_i)) &= 4 \Delta^2_m(f_i + g_i) (d_0^2 + d_1^2)\gamma_0 + 2 (d_0^2 + d_1^2)^2 \gamma_0^2. \\
&= 4 \Delta^2_m(f_i + g_i) \sum_{j=1}^{\tau_{j+1}} a_{j-1} - a_j \sum_{j=1}^{\tau_i} |a_{j-1} - a_j| = \max_{1 \leq j \leq K} |a_{j-1} - a_j| J_{K}^{||}.
\end{align*}

Therefore, it is enough to impose a bound on the growth rate of $J_{K}^{||}$ and on the growth rate of the maximum jump size $\max_{1 \leq j \leq K} |a_{j-1} - a_j|$ to guarantee a reasonably low rate of growth for $J_{K}^{||}$.
\( g_i \) + \( E \Delta_n^2(\varepsilon_i) = \Delta_n^2(f_i + g_i) + (d_0^2 + d_1^2)\gamma_0 \), we find that

\[
\begin{align*}
[\Delta_n^2(y_i) - E[\Delta_n^2(y_i)]] [\Delta_n^2(y_j) - E[\Delta_n^2(y_j)]] &= \Delta_n^2(\varepsilon_i) \Delta_n^2(\varepsilon_j) \\
&+ 4 \Delta_n(f_i + g_i) \Delta_n(f_j + g_j) \Delta_n(\varepsilon_i) \Delta_n(\varepsilon_j) \\
&+ 2[\Delta_n(f_i + g_i) \Delta_n(\varepsilon_i) \Delta_n^2(\varepsilon_j) + \Delta_n(f_j + g_j) \Delta_n(\varepsilon_j) \Delta_n^2(\varepsilon_i)] \\
&- 2(d_0^2 + d_1^2)\gamma_0 [\Delta_n(f_i + g_i) \Delta_n(\varepsilon_i) + \Delta_n(f_j + g_j) \Delta_n(\varepsilon_j)] \\
&- (d_0^2 + d_1^2)\gamma_0 [\Delta_n^2(\varepsilon_i) + \Delta_n^2(\varepsilon_j)] + (d_0^2 + d_1^2)^2\gamma_0^2.
\end{align*}
\tag{2.16}
\]

It is straightforward to verify that for any \( i \) and \( j \), \( E[\Delta_n(\varepsilon_i)] = E[\Delta_n(\varepsilon_i) \Delta_n^2(\varepsilon_j)] = 0 \). In what follows we will need to use the following identities concerning central moments of the multivariate normal distribution (see e.g. Triantafyllopoulos (2003)): for any integers \( r, s, u, \) and \( v \)

\[
\begin{align*}
E[\varepsilon_s^2] &= \gamma_0^2 + 2\gamma_{|r-s|}^2 \\
E[\varepsilon_s \varepsilon_r] &= \gamma_0 \gamma_{|u-v|} + 2\gamma_{|r-u|} \gamma_{|r-v|} \\
E[\varepsilon_s \varepsilon_r \varepsilon_u \varepsilon_v] &= \gamma_{|r-s|} \gamma_{|u-v|} + \gamma_{|r-u|} \gamma_{|s-v|} + \gamma_{|r-v|} \gamma_{|s-u|}.
\end{align*}
\tag{2.17}
\]

First, observe that

\[
E[\Delta_n(\varepsilon_i) \Delta_n(\varepsilon_j)] = E[d_0^2 \varepsilon_i \varepsilon_j + d_0 d_1 \varepsilon_i \varepsilon_{j+m+1} + d_0 d_1 \varepsilon_{i+m+1} \varepsilon_j + d_1^2 \varepsilon_{i+m+1} \varepsilon_{j+m+1}]
\]

\[
= d_0^2 \gamma_{|j-i|} + d_0 d_1 \gamma_{|j-i-m+1|} + d_0 d_1 \gamma_{|j-i-(m+1)|} + d_1^2 \gamma_{|j-i|}
\]

\[
= (d_0^2 + d_1^2) \gamma_{|j-i|} + d_0 d_1 \left( \gamma_{|j-i-(m+1)|} + \gamma_{|j-i+(m+1)|} \right).
\]

Similar considerations allow us to obtain the following:

\[
E[\Delta_n^2(\varepsilon_i) \Delta_n^2(\varepsilon_j)] = (d_0^2 + d_1^2)^2 \gamma_0^2 + 2(d_0^2 + d_1^2)^2 \gamma_{|j-i|}^2 + 2d_0^2 d_1^2 (\gamma_{|j-i-(m+1)|} + \gamma_{|j-i+m+1|})^2
\]

\[
+ 4(d_0^2 d_1 + d_0 d_1^2) \gamma_{|j-i|} (\gamma_{|j-i-(m+1)|} + \gamma_{|j-i+(m+1)|}).
\]

Taking expectation on both sides of (2.16) and utilizing the identities just derived we arrive at

\[
\text{COV}(\Delta_n^2(y_i), \Delta_n^2(y_j)) = \\
4 \Delta_n(f_i + g_i) \Delta_n(f_j + g_j) \left[ (d_0^2 + d_1^2) \gamma_{|j-i|} + d_0 d_1 \left( \gamma_{|j-i-(m+1)|} + \gamma_{|j-i+(m+1)|} \right) \right]
\]

\[
+ 2(d_0^2 + d_1^2)^2 \gamma_{|j-i|}^2 + 2d_0^2 d_1^2 (\gamma_{|j-i+m+1|} + \gamma_{|j-i-(m+1)|})^2
\]

\[
+ 4(d_0^2 d_1 + d_0 d_1^2) \gamma_{|j-i|} \left( \gamma_{|j-i+m+1|} + \gamma_{|j-i-(m+1)|} \right).
\tag{2.18}
\]
Substituting (2.15) and (2.18) into (2.14), we find that

\[
(n_m p(d))^2 \text{VAR}(Q_{1,m+1}(y; d)) = 4(d_0^2 + d_1^2)\gamma_0 \sum_{i=1}^{n_m} \Delta_2^2(f_i + g_i) + 2n_m(d_0^2 + d_1^2)^2 \gamma_0^2 \\
+ 8 \sum_{i,j} \Delta_m(f_i + g_i) \Delta_m(f_j + g_j) \left[(d_0^2 + d_1^2)\gamma_{j-i} + d_0 d_1 \left(\gamma_{j-i+(m+1)} + \gamma_{j-i-(m+1)}\right)\right] \\
+ 4(d_0^2 + d_1^2)^2 \sum_{i,j} \gamma_{j-i}^2 + 4d_0^2 d_1^2 \sum_{i,j} \left(\gamma_{j-i+m+1} + \gamma_{j-i-(m+1)}\right)^2 \\
+ 8d_0^2 d_1 \sum_{i,j} \gamma_{j-i} \left(\gamma_{j-i+m+1} + \gamma_{j-i-(m+1)}\right) + 8d_0 d_1 \sum_{i,j} \gamma_{j-i} \left(\gamma_{j-i+m+1} + \gamma_{j-i-(m+1)}\right).
\]

(2.19)

Above, \(\sum_{i,j}\) is short-hand notation for \(\sum_{i=1}^{n_m-1} \sum_{j=i+1}^{n_m}\). This expression is further simplified when we recall that \(m\)-dependency means that \(\gamma_h = 0\) for \(|h| \geq m + 1\) which implies that whenever \(r = j - i > 0\), \(\gamma_{j-i+m+1} = \gamma_{m+1+r} = 0\). Furthermore, assuming that \(H_K^{||} = o(n)\) and \(J_K^{||} = o(n)\) and using Lemmas 2, 4, 6 and 7 from the Appendix, we find that

\[
p^2(d) \text{VAR}(Q_{1,m+1}(y; d)) = 4(d_0^2 + d_1^2)\gamma_0 \left(\frac{(m+1)d_2^2 J_K}{n^2} + o(n^{-2})\right) + 2(d_0^2 + d_1^2)^2 \gamma_0^2 \\
+ 8(d_0^2 + d_1^2)m(m+1)d_1^2 \gamma_0 \left(\frac{J_K}{2n^2} + \frac{(m+1)M^2}{n^3}\right) \\
+ 8d_0 d_1^2 m(m+1)\gamma_0 \left(\frac{J_K}{2n^2} + \frac{(m+1)M^2}{n^3}\right) \\
+ 4m \gamma_0^2 \frac{m+1}{n} \left((d_0^2 + d_1^2)^2 + \frac{2m+1}{m} d_0^2 d_1^2 + 2(d_0^2 d_1 + d_0 d_1^2)\right) \\
= q_1(d, m) \frac{\gamma_0^2}{n} + q_2(d, m) \frac{\gamma_0 J_K}{n^2} + o(n^{-2}),
\]

where

\[
q_1(d, m) = 2(d_0^2 + d_1^2)^2 + 4m \left((d_0^2 + d_1^2)^2 + \frac{2m+1}{m} d_0^2 d_1^2 + 2(d_0^2 d_1 + d_0 d_1^3)\right) \\
q_2(d, m) = 4(m+1)(d_0^2 + d_1^2)d_1^2 + 4m(m+1)(d_0^2 + d_1^2)d_1^2 + 4m(m+1)d_0 d_1^3.
\]

Under the constraint \(d_0 + d_1 = 0\), and since \(p(d) = d_0^2 + d_1^2\), we find that \(q_1(d, m) = p^2(d)[2m+3]\) and \(q_2(d, m) = p^2(d)\left[(m+1)(m+2)\right]\). Thus, finally, we have

\[
\text{VAR}(Q_{1,m+1}(y; d)) = \frac{2m+3}{n} \gamma_0^2 + \frac{(m+1)(m+2)}{n^2} J_K \gamma_0 + o(n^{-2}).
\]

(2.20)
2.3. On the bias and variance of \( \hat{\gamma}_h \)

Finally, we intend to characterize asymptotic behavior of the autocovariance estimator \( \hat{\gamma}_h, h = 1, \ldots, m \) introduced in (2.4) as well. First of all, we find from Corollary 2 and Proposition 2 that

\[
E[\hat{\gamma}_h] = \gamma_h + [m + 1 - h]J_K \frac{1}{2h} + O(n^{-1}), \quad h = 1, \ldots, m.
\]

(2.21)

Thus, we conclude that, as is the case with \( \hat{\gamma}_0 \), the main terms of the bias do not depend on the difference sequence \( d \). Next, we state an explicit result about the variance of \( \hat{\gamma}_h \). For ease of reading, we recall that \( n_m = n - (m + 1) \) and we denote \( n_h := n - (h + 1), h = 1, \ldots, m. \)

**Proposition 3.** Let the regularity condition (2.2) be satisfied. Moreover, let \( d_0 + d_1 = 0 \) and let \( p(d) = d_0^2 + d_1^2 \). Denote \( M = \sup_{0 \leq x \leq 1} |f'(x)| \), Define \( H_{K,h} = \sum_{j=1}^{K} (\tau_j - \frac{h}{2})|a_j - 1| \). Then, the variance of the estimator \( \hat{\gamma}_h, h = 1, \ldots, m \) can be represented as

\[
\text{VAR}(\hat{\gamma}_h) = \frac{2m + 3}{n} \gamma_0^2 + \frac{(\gamma_0 - \gamma_h)^2}{4n} - \frac{2}{p(d)} O\left(\frac{1}{n}\right)
\]

\[
+ \frac{(m + 1)(m + 2)J_K \gamma_0 + 2(\gamma_0 - \gamma_h)|hJ_K + O(J_K/n)]}{n^2}
\]

\[- \frac{4\gamma_0 |d_1| h(m + 1)}{p(d)n m n_h}\left\{ M^2 + J_K^2 [1 + |d_1| (1 + M)] \right\}
\]

\[+ \frac{2\gamma_0 (7m + h)}{n_h^2} + O\left(\frac{1}{n^2}\right) + \frac{2\gamma_0 h}{n_h^2} \left\{ 3h^2 M^2 + 5h M J_K^2 + M H_{K,h} + \frac{3}{2} n (h + 1) J_K \right\}
\]

\[+ \left\{ O(n^{-1}) + O\left(\frac{1}{n_h^2}\right) \right\} \left\{ \gamma_0 + \frac{(m + 1)}{p(d)m_n} J_K(d) + O(n^{-1}) \right\} + o(n^{-2})
\]

**Remark 3.** Note that the rate at which the variance of \( \hat{\gamma}_h \) converges to zero depends not only on the growth rate of \( J_K \) as \( n \to \infty \) but also on the growth rates of \( J_K^2 \) and \( H_{K,h}^2 \). Thus, the pattern is more complicated than in the case of the variance of estimator \( Q_{1,m+1}(y;d) \). Moreover, note that \( H_{K,h}^2 \) introduced during the discussion of Theorem (2) is a special case of \( H_{K,h}^2 \) when \( h = m \).

**Remark 4.** Note also that, in the same way as for the variance estimator, the variance of \( \hat{\gamma}_h \) does not depend, up to the higher order terms, on the smooth function \( f \).

**Proof.** In order to get the variance of this estimator we just need to observe that for \( h = 1, \ldots, m \),

\[
\text{VAR}(\hat{\gamma}_h) = \text{VAR}(Q_{1,m+1}(y;d)) + \text{VAR}(Q_{1,h}(y)) - 2\text{COV}(Q_{1,m+1}(y;d), Q_{1,h}(y)).
\]

(2.22)
Recall that \( \text{VAR}(Q_{1,m+1}(y; d)) \) was established above, cf. (2.20). For convenience we will use the abbreviated notation \( \sum_{i,j} := \sum_{i=1}^{n-h} \sum_{j=i+1}^{n-h} \). Following the ideas and calculations leading to (2.14)-(2.19) we find that

\[
\text{VAR}(Q_{1,h}(y)) = \left( \frac{\gamma_0 - \gamma_h}{4n_h} \right)^2 + \frac{2(\gamma_0 - \gamma_h)}{n_h^2} \sum_{i=1}^{n_h} \Delta_h^2 (f_i + g_i) \\
+ \frac{2}{n_h^2} \sum_{i,j} \Delta_h(f_i + g_i) \Delta_h(f_j + g_j) \left[ 2\gamma_{j-i} - (\gamma_{j-i+h} + \gamma_{j-i-h}) \right] \\
+ \frac{1}{n_h^2} \sum_{i,j} [4\gamma_{j-i}^2 + (\gamma_{j-i+h} + \gamma_{j-i-h})^2 - 4\gamma_{j-i}^2 (\gamma_{j-i+h} + \gamma_{j-i-h})].
\]

We conclude this calculation by adapting Lemmas 2, 4 and 6 from the Appendix to the current situation. More precisely, following the proof of each of those lemmas line by line we can show first that

\[
\frac{1}{n_h^2} \sum_{i=1}^{n_h} \Delta_h^2 (f_i + g_i) = \frac{hJ_K}{n_h^2} + O(n_h^{-3}J_K^2).
\]

Next, denote \( [\Delta_h(f + g)]_{i,j} = \Delta_h(f_i + g_i)\Delta_h(f_j + g_j) \) and verify directly that

\[
\sum_{i,j} [\Delta_h(f + g)]_{i,j} \gamma_{j-i} = O \left( \frac{h\gamma_0}{n} \left( h^2 M^2 + 2h^2 M J_{\|}^h + n(h + 1)J_K^2 \right) \right),
\]

and

\[
\sum_{i,j} [\Delta_h(f + g)]_{i,j} \gamma_{j-i-h} = O \left( \frac{h\gamma_0}{n} \left( h^2 M^2 + M H_{\|}^h + h^2 M J_{\|}^h + n(h + 1)J_K^2 \right) \right).
\]

Combining the above with an obvious adaptation of Lemma 7 we conclude that

\[
\text{VAR}(Q_{1,h}(y)) = \left( \frac{\gamma_0 - \gamma_h}{4n_h} \right)^2 + \frac{2(\gamma_0 - \gamma_h)}{n_h^2} \left[ hJ_K + O(J_K/n) \right] + \frac{2\gamma_0^2(7M + h)}{n_h^2} \\
+ \frac{2\gamma_0 h}{n_h^3} \left[ 3h^2 M^2 + 5h M J_{\|}^h + M H_{\|}^h + \frac{3}{2}n(h + 1)J_K \right].
\]

We now move on to the computation of the covariance between \( Q_{1,m+1}(y; d) \) and \( Q_{1,h}(y) \). Write \( A_m := E[Q_{1,m+1}(y; d)] = \gamma_0 + \frac{m+1}{p(d)\gamma_m} J_{\|}^d + O(n^{-1}) \) and \( B_h := E[Q_{1,h}(y)] = \gamma_0 - \gamma_h + \frac{h}{2n_h} J_K + O(n^{-1}) \), see Proposition 1 and 2, respectively. Now, it is not difficult to see that

\[
\text{COV}(Q_{1,m+1}(y; d), Q_{1,h}(y)) = C_{m,h} - \frac{B_h}{p(d)\gamma_m} \sum_{i=1}^{n_m} \text{E}\Delta_{m+1}^2(y_i) - \frac{A_m}{2n_h} \sum_{i=1}^{n_h} \text{E}\Delta_h^2(y_i) + A_mB_h,
\]
where, due to Lemma 2,

\[
C_{m,h} = \frac{1}{2p(d)n_h n_m} \sum_{i=1}^{n_m} \sum_{j=1}^{n_h} E \left[ \Delta_{m+1}^2(y_i) \Delta_h^2(y_j) \right] = \gamma_0 (\gamma_0 - \gamma_h) \\
+ \frac{\gamma_0 - \gamma_h}{p(d)} \left\{ \left( m + 1 \right) d_1^2 J_K n_m \right\} + o \left( \frac{J_K n_m^2}{n_h^2} \right) \\
+ \frac{1}{2p(d)} \left\{ \left( m + 1 \right) d_1^2 J_K n_m \right\} + o \left( \frac{J_K n_m^2}{n_h^2} \right) \\
+ \frac{2}{p(d)n_h n_m} \sum_{i,j} \Delta_m(f_i + g_i) \Delta_h(f_j + g_j) \mu_{i,j,h,m}(d) + \mu_{i,j,h,m}^2(d) \right\}, \tag{2.28}
\]

with \( \mu_{i,j,h,m}(d) := [d_0(\gamma_j - |i|) - \gamma_j |i+h|] + d_1(\gamma_{j-i-(m+1)} - \gamma_{j-i-h-(m+1)})]. \)

An application of Eq. (4.9) yields

\[
\sum_{i,j} \Delta_m(f_i + g_i) \Delta_h(f_j + g_j) \mu_{i,j,h,m}(d) = \gamma_0 [d_1 |i| h (m + 1) \left\{ M^2 + J_K^2 \left[ 1 + |d_1| (1 + M) \right]\right\}, \\
\sum_{i,j} \mu_{i,j,h,m}^2(d) = O(\gamma_0 n). \tag{2.29}
\]

The latter follows from \( m \)-dependence.

Observe also that according to Lemma 2,

\[
n_m^{-1} \sum_{i=1}^{n_m} E \Delta_{m+1}^2(y_i) = d_1^2 (m + 1) J_K \left[ 1 + \frac{J_K n_m^2}{n_h^2} \right] + o \left( \frac{J_K n_m^2}{n_h^2} \right) \tag{2.30}
\]

\[
n_h^{-1} \sum_{i=1}^{n_h} E \Delta_h^2(y_i) = h J_K \left[ 1 + \frac{J_K n_m^2}{n_h^2} \right] + 2 (\gamma_0 - \gamma_h). \tag{2.31}
\]

Substituting (2.30) and (2.31) into (2.27) and (2.29) into (2.28), and after some algebra, we obtain that

\[
\text{COV}(Q_{1,m+1}(y; d), Q_{1,h}(y)) = O \left( \frac{\gamma_0^2}{p(d)n} \right) + 2 \gamma_0 [d_1 |h(m + 1) \left\{ M^2 + J_K^2 \left[ 1 + |d_1| (1 + M) \right]\right\}] \\
+ \left\{ O(n^{-1}) + o \left( \frac{J_K n_m^2}{n_h^2} \right) \right\} \left\{ \gamma_0 + (m + 1) \frac{J_K^2}{p(d)n_m} J(d) + O(n^{-1}) \right\}.
\]

\[\Box\]
3. Simulations

Our simulation study assesses the performance of the variance and autocovariance estimators $\hat{\gamma}_0$ and $\hat{\gamma}_h$. This is done both when the stationary distribution of the errors is zero mean Gaussian and when it is a zero mean non-Gaussian. The second case is considered in order to assess the robustness of proposed method against non-normally distributed errors. As an example of a non-Gaussian zero mean error distribution, we choose $t_4$ - a Student distribution with 4 degrees of freedom.

Similarly to Tecuapetla-Gómez and Munk (2017), we consider a 1-dependent error model: $\xi_t = r_0 \xi_t + r_1 \xi_{t-1}$ where $\xi_t$'s are i.i.d and either normally or $t_4$- distributed. It is assumed that $r_0 = [\sqrt{1 + 2\gamma_1} + \sqrt{1 - 2\gamma_1}] / 2$ and $r_1 = [\sqrt{1 + 2\gamma_1} - \sqrt{1 - 2\gamma_1}] / 2$ for a parameter $-\frac{1}{2} \leq \gamma_1 \leq \frac{1}{2}$. The autocorrelation of $\xi_t$ at lag 1 is $\rho_1 = \gamma_1$. Our aim will be to estimate $\rho_1$ (and, therefore, $\gamma_1$) as well as $\rho_2 = 0$.

The simulated signal in this simulation study is a sum of the signal used earlier by Tecuapetla-Gómez and Munk (2017) and a smooth function $f(x)$. Briefly, the first additive component is defined as a piecewise constant function $g$, with six change-points located at fractions $\frac{1}{5}, \frac{4}{5} \pm \frac{1}{36}, \frac{3}{36} \pm \frac{2}{36}$, and $\frac{5}{36} \pm \frac{3}{36}$ of the sample size $n$. In the first segment, $g = 0$, in the second $g = 10$, and in the remaining segments $g$ alternates between 0 and 1, starting with $g = 0$ in the third segment. This type of discontinuous signal was also earlier considered in Chakar et al. (2016). The function $f(x)$ is chosen to ensure that $\int_0^1 f(x) \, dx = 0$ in order to satisfy the identifiability condition. We consider three choices of $f(x)$: a linear function $f_1(x) = 1 - 2x$, a quadratic function $f_2(x) = 4(x - 0.5)^2 - 1/3$, and a periodic function $f_3(x) = \sin(16\pi x)$. All of these functions are defined as zero outside $[0, 1]$ interval. The functions are chosen to range from a simple linear function to a periodic function that may potentially increase the influence of the higher order terms in the risk expansions. Tables (1) and (2) summarize the results for $n = 1600$ observations obtained from 500 replications for Gaussian and $t_4$ errors, respectively. The results seem

| $\gamma_0 = \frac{1}{2}$ | $\gamma_1 = \frac{1}{2}$ | $\rho_2 = 0$ | $\rho_2 = 0.0032$ | $\rho_2 = 0.0034$ | $\rho_2 = 0.0036$ |
|------------------------|------------------------|-------------|------------------|------------------|------------------|
| $\rho_1 = \gamma_1$    | $\rho_1 = 0$           | $\rho_1 = 0$ | $\rho_1 = 0.0033$ | $\rho_1 = 0.0035$ | $\rho_1 = 0.0037$ |
| $\rho_1 = \gamma_1$    | $\rho_1 = 0.0126$      | $\rho_1 = 0.0125$ | $\rho_1 = 0.0128$ | $\rho_1 = 0.0128$ | $\rho_1 = 0.0127$ |
| $\rho_1 = \gamma_1$    | $\rho_1 = 0.0040$      | $\rho_1 = 0.0038$ | $\rho_1 = 0.0038$ | $\rho_1 = 0.0038$ | $\rho_1 = 0.0037$ |
| $\rho_1 = \gamma_1$    | $\rho_1 = 0.0064$      | $\rho_1 = 0.0065$ | $\rho_1 = 0.0064$ | $\rho_1 = 0.0064$ | $\rho_1 = 0.0064$ |
| $\rho_1 = \gamma_1$    | $\rho_1 = 0.0035$      | $\rho_1 = 0.0038$ | $\rho_1 = 0.0037$ | $\rho_1 = 0.0037$ | $\rho_1 = 0.0037$ |
| $\rho_1 = \gamma_1$    | $\rho_1 = 0.0100$      | $\rho_1 = 0.0010$ | $\rho_1 = 0.0010$ | $\rho_1 = 0.0010$ | $\rho_1 = 0.0010$ |
| $\rho_1 = \gamma_1$    | $\rho_1 = 0.0032$      | $\rho_1 = 0.0034$ | $\rho_1 = 0.0034$ | $\rho_1 = 0.0034$ | $\rho_1 = 0.0034$ |
ACF estimation via difference schemes for a semiparametric model

Table 2. The MSE of autocorrelation estimators of $\rho_1 = \gamma_1$ and $\rho_2 = 0$ under the 1-dependent error model where $\delta_i$’s are i.i.d. $t_4$ based on 500 replications of size 1600.

| $\gamma_1$ | $f_1$ | $f_2$ | $f_3$ |
|------------|-------|-------|-------|
| $-0.4$     | 0.0126| 0.0127| 0.0128|
| $0$        | 0.0034| 0.0037| 0.0031|
| $-0.4$     | 0.0108| 0.0102| 0.0102|
| $0$        | 0.0028| 0.0028| 0.0030|
| $-0.2$     | 0.0089| 0.0066| 0.0078|
| $0$        | 0.0024| 0.0022| 0.0024|
| $\gamma_1$| 0.0047| 0.0045| 0.0049|
| $\rho_2 = 0$| 0.0019| 0.0018| 0.0021|
| $0.2$      | 0.0047| 0.0045| 0.0049|
| $\gamma_1 = 0$| 0.0016| 0.0015| 0.0017|
| $\gamma_1 = 0.4$| 0.0010| 0.0011| 0.0011|
| $\rho_2 = 0$| 0.0011| 0.0014| 0.0015|
| $\gamma_1 = 0.5$| 0.0005| 0.0006| 0.0006|
| $\rho_2 = 0$| 0.0014| 0.0015| 0.0015|

to confirm that our estimation method works rather well in both cases. In either case, the MSE of proposed estimators barely depends on the choice of the smooth function $f$. This result seems to confirm our conclusion that the main terms in both the squared bias and the variance of proposed estimators do not depend on the choice of $f$.

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4. Supplementary materials

In this section we will use the partition $\mathcal{P}[0, 1] = \{0, 1/n, 2/n, \ldots, (n - 1)/n, 1\}$ of the interval $[0, 1]$ and denote $t_k = k/n$, $k = 1, \ldots, n$.

Lemma 1. Suppose that $f : [0, 1] \rightarrow \mathbb{R}$ is differentiable with a continuous derivative and squared integrable. Then

$$\frac{1}{n_{lm}} \sum_{i=1}^{n_{lm}} \Delta^2_{lm}(f_i) \rightarrow \left(\sum_{s=0}^{l} d_s\right)^2 \int_0^1 f^2(x) \, dx.$$  

Proof. Observe first that for given $i$,

\[
\Delta^2_{lm}(f_i) = \left(\sum_{s=0}^{l} d_s\right)^2 f_i^2 + \sum_{s=1}^{l} d_s^2(f_{i+s(m+1)} - f_i)^2 \\
+ 2 \sum_{s=0}^{l} d_s \sum_{t=1}^{l} d_t f_i(f_{i+t(m+1)} - f_i) \\
+ 2 \sum_{s=1}^{l-1} \sum_{t=s+1}^{l} d_s d_t (f_{i+s(m+1)} - f_i)(f_{i+t(m+1)} - f_i). \tag{4.1}
\]

Then for $x_i \in \mathcal{P}[0, 1]$,

$$n_{lm}^{-1} \sum_{i=1}^{n_{lm}} f_i^2 \approx \frac{n}{n_{lm}} \sum_{i=1}^{n-1} f^2(t_i)(x_{i+1} - x_i) \overset{n \rightarrow \infty}{\rightarrow} \int_0^1 f^2(x) \, dx.$$  

\[\tag{4.2}\]
ACF estimation via difference schemes for a semiparametric model

Let $M = \sup_{x \in [0,1]} |f'(x)|$. The mean value theorem ensures that for given $1 \leq s \leq l$, for some $\xi_i \in [t_i, t_{i+s(m+1)}]$, $f_{i+s(m+1)}(x) - f_i = s(m+1)f'(\xi_i)/n$, where $i = 1, \ldots, n_m$. This implies that

$$n_m \frac{1}{n_m} \sum_{i=1}^{n_m} l \sum_{s=1}^{l} d_s |f_i (f_{i+s(m+1)} - f_i)| \leq \frac{l M (m+1)}{n_m} \sum_{s=1}^{l} d_s \sum_{i=1}^{n_m} |f(t_i)|(x_{i+1} - x_i) \xrightarrow{n \to \infty} 0,$$

the latter follows since $|f|$ is integrable on $[0,1]$. Similar arguments prove that

$$n_m^{-1} \sum_{i=1}^{n_m} l \sum_{s=1}^{l} d_s^2 (f_{i+s(m+1)} - f_i)^2 \leq \frac{l^2 M^2 (m+1)^2}{n^2} \sum_{s=1}^{l} d_s^2 \xrightarrow{n \to \infty} 0. \tag{4.3}$$

For given $i$, let $1 \leq g \leq l$ be the index such that $|f_{i+g(m+1)} - f_i| = \max_{1 \leq s \leq l} |f_{i+s(m+1)} - f_i|$. Consequently, for any $1 \leq s, t \leq l$,

$$|f_{i+s(m+1)} - f_i||f_{i+t(m+1)} - f_i| \leq (f_{i+g(m+1)} - f_i)^2 \leq \frac{l^2 (m+1)^2 M^2}{n^2}.$$

the latter follows from the mean value theorem. Hence,

$$n_m^{-1} \sum_{i=1}^{n_m} l \sum_{s=1}^{l} d_s d_t |f_{i+s(m+1)} - f_i||f_{i+t(m+1)} - f_i| \leq \frac{l^2 M^2 (m+1)^2}{n^2} \sum_{s=1}^{l} d_s d_t \xrightarrow{n \to \infty} 0. \tag{4.5}$$

The claims follows from (4.1)-(4.2)-(4.3)-(4.4)-(4.5).

**Lemma 2.** Consider the general model 2.1. Assume that $d_0 + d_1 = 0$. Then

$$n_m^{-2} \sum_{i=1}^{n_m} \Delta^2_m(f_i + g_i, d) = \frac{(m+1)d^2 J_K}{n_m^2} + O(n^{-3} f \|K\|),$$

where $J_K = \sum_{j=1}^{K} (a_{j-1} - a_j)^2$ and $J_K = \sum_{j=1}^{K} |a_{j-1} - a_j|$.

**Proof.** We use that $\Delta^2_m(f_i + g_i, d) = \Delta^2_m(f_i; d) + \Delta^2_m(g_i; d) + 2 \Delta_m(f_i; d) \Delta_m(g_i; d)$. Since $d_0 + d_1 = 0$ we can improve the rate of convergence of Lemma 1 and get

$$n_m^{-2} \sum_{i=1}^{n_m} \Delta^2_m(f_i; d) = o(n_m^{-3}). \tag{4.6}$$

Next, it is not difficult to see that for $j = 1, \ldots, K$,

$$g_{i+m+1} - g_i = \begin{cases} 0 & \text{for } \tau_{j-1} \leq i \leq \tau_j - (m+2) \\ a_j - a_{j-1} & \text{for } \tau_j - (m+1) \leq i \leq \tau_j - 1. \end{cases} \tag{4.7}$$
This allows us to get that
\[
\sum_{i=1}^{n_m} \Delta_m^2(g_i; d) = d_1^2 \sum_{j=1}^{K} \sum_{i=\tau_j-(m+1)}^{\tau_j-1} (a_{j-1} - a_j)^2 = d_1^2 (m+1) J_K. \tag{4.8}
\]

Similarly, we can show that
\[
\sum_{i=1}^{n_m} |\Delta_m(g_i; d)| = |d_1| (m+1) \sum_{j=1}^{K} |a_{j-1} - a_j| \tag{4.9}
\]

Next, observe that the mean value theorem yields that uniformly over \(i\),
\[
|\Delta_m(f_i; d)| \leq \frac{|d_1|(m+1)|f'(\xi)|}{n} \leq \frac{(m+1)|d_1|M}{n} \tag{4.10}
\]
Recall that \(M = \sup_{x \in [0,1]} |f'(x)|\). Combining Eqs. (4.9)-(4.10), it follows that
\[
n_m^{-2} \sum_{i=1}^{n_m} |\Delta_m(f_i) \Delta_m(g_i)| \leq \frac{|d_1|^2(m+1)^2M}{n_m^3} \sum_{j=1}^{K} |a_{j-1} - a_j|. \tag{4.11}
\]

The result follows from Eqs. (4.6)-(4.8)-(4.11).

\[\square\]

**Lemma 3.** Consider the general model 2.1, and assume that \(d_0 + d_1 = 0\). Then
\[
\sum_{i=1}^{n-2(m+1)} \sum_{j=i+1}^{i+m} |g_{j+m+1} - g_j| = \mathcal{O}(m(m+1)J_K^1) \tag{4.12}
\]
\[
\sum_{i=1}^{n-2(m+1)} |g_{i+m+1} - g_i| \sum_{j=i+1}^{i+m} |g_{j+m+1} - g_j| = \mathcal{O}\left( \frac{m(m+1)}{2} J_K \right), \tag{4.13}
\]
where \(J_K^1 = \sum_{j=1}^{K} |a_{j-1} - a_j|\).

**Proof.** We begin by establishing (4.12). First note that
\[
\sum_{i=1}^{n-2(m+1)} \sum_{j=i+1}^{i+m} |g_{j+m+1} - g_j| \leq \sum_{j=1}^{K} A_j, \quad \text{where} \quad A_j = \sum_{i=\tau_j-1}^{\tau_j-1} \sum_{l=i+1}^{i+m} |g_{l+m+1} - g_l|. \tag{4.14}
\]
For any \(\tau_j\) with \(j \geq 1\), let \(\kappa_j = \tau_j - 2(m+1)\). We split the computation of \(A_j\) in three ways, first we consider \(i \leq \kappa_j\), then \(\kappa_j + 1 \leq i \leq \kappa_j + m\) and finally \(\kappa_j + m + 1 \leq i \leq \tau_j - 1\).

Assume that \(i \leq \kappa_j\). Note that for \(l = i + 1 \leq \tau_j - (2m+1)\), \(l + m + 1 \leq \tau_j - m\) which implies that \(g_{l+m+1} - g_l = 0\). In the same case, \(l = i + m \leq \tau_j - m - 2\) implies

that $l + m + 1 \leq \tau_j - 1$, i.e., $g_{l+m+1} - g_l = 0$. Since these arguments hold for any $l = i + 1, \ldots, i + m$ when $i \leq \kappa_j$ we have shown that,

$$\sum_{i \leq \kappa_j} \sum_{l=i+1}^{i+m} |g_{l+m+1} - g_l| = 0.$$  

Assume now that $\kappa_j + 1 \leq i \leq \kappa_j + m$. The arguments presented above allow us to get that for $s = 0, 1, \ldots, m - 1$,

$$\sum_{i = \kappa_j + 1 + s}^{\kappa_j + 1 + m} \sum_{l=i+1}^{i+m} |g_{l+m+1} - g_l| = (s + 1)|a_{j-1} - a_j|.$$  

Next, assume that $i \in I_{\tau_j} = \{\kappa_j + (m + 1), \ldots, \tau_j - 1\}$. With the arguments utilized so far (basically inspection case by case), it is not difficult to see that

$$\sum_{i \in I_{\tau_j}} \sum_{l=i+1}^{i+m} |g_{l+m+1} - g_l| = \frac{m(m+1)}{2} |a_{j-1} - a_j|. \quad (4.15)$$  

Eq. (4.12) is established by noticing that the right-hand side of (4.14) is equal to

$$\sum_{j=1}^{K} \sum_{i=\kappa_j + 1}^{\tau_j - 1} \sum_{l=i+1}^{i+m} |g_{l+m+1} - g_l| = [1 + 2 + \cdots + m + \frac{m(m+1)}{2}] \sum_{j=1}^{K} |a_{j-1} - a_j|$$

We can utilize the arguments presented above to show Eq. (4.13). Indeed, note first that the left-hand side of (4.13) is bounded by

$$\sum_{j=1}^{K} \sum_{i=\tau_{j-1}}^{\tau_j - 1} |g_{l+m+1} - g_l| \sum_{l=i+1}^{i+m} |g_{l+m+1} - g_l| \leq \sum_{j=1}^{K} \sum_{i \in I_{\tau_j}} |a_{j-1} - a_j| \sum_{l=i+1}^{i+m} |g_{l+m+1} - g_l|,$$

the latter inequality follows from Eq. (4.7), observe now that according to (4.15) the right-hand side above becomes

$$\sum_{j=1}^{K} |a_{j-1} - a_j| \sum_{i \in I_{\tau_j}} \sum_{l=i+1}^{i+m} |g_{l+m+1} - g_l| = \frac{m(m+1)}{2} \sum_{j=1}^{K} (a_{j-1} - a_j)^2.$$  

This completes the proof. \qed

**Lemma 4.** Consider the general model 2.1 and assume that $d_0 + d_1 = 0$. Then,

$$\sum_{i=1}^{n_m - 1} \sum_{j=i+1}^{n_m} \Delta_m (f_i + g_i) \Delta_m (f_j + g_j) \gamma_{j-i}$$

$$= O \left( \frac{d_1^2 \gamma m (m + 1)}{n} \left\{ (m + 1)M^2 + 2(m + 1)M J_K^\parallel + \frac{n}{2} J_K \right\} \right),$$

where $M = \sup_{x \in [0,1]} |f'(x)|$ and $J_K^\parallel = \sum_{j=1}^{K} |a_{j-1} - a_j|$. 

Proof. Firstly observe that due to $m$-dependency,
\[
\sum_{i=1}^{n_{m-1}} \sum_{j=i+1}^{n_{m}} \Delta_m(f_i + g_i) \Delta_m(f_j + g_j) \gamma_{j-i} = \sum_{i=1}^{n-2(m+1)} \sum_{j=i+1}^{i+m} \Delta_m(f_i + g_i) \Delta_m(f_j + g_j) \gamma_{j-i}.
\]

Then we will utilize that
\[
\Delta_m(f_i + g_i) \Delta_m(f_j + g_j) = \Delta_m(f_i) \Delta_m(f_j) + \Delta_m(f_i) \Delta_m(g_j) + \Delta_m(g_i) \Delta_m(f_j) + \Delta_m(g_i) \Delta_m(g_j).
\]

Because $d_0 + d_1 = 0$, \(\Delta_m(f_i) \Delta_m(f_j) = d^2_i(f_{i+m+1} - f_i)(f_{j+m+1} - f_j)\). The mean-value theorem now yields
\[
|\Delta_m(f_i) \Delta_m(f_j)| \leq d^2_i (m + 1)^2 \frac{M^2}{n^2}.
\]

Similar considerations allow us to see that
\[
|\Delta_m(g_i) \Delta_m(g_j)| = d^2_i |g_{i+m+1} - g_i| |g_{j+m+1} - g_j|.
\]

Consequently,
\[
\sum_{i=1}^{n_{m-1}} \sum_{j=i+1}^{n_{m}} |\Delta_m(f_i + g_i) \Delta_m(f_j + g_j) \gamma_{j-i}| \leq d^2_i m(m + 1)^2 \frac{M^2}{n} + \frac{d^2_i (m + 1) M}{n} \left\{ \sum_{i,j} |g_{j+m+1} - g_j| + m \sum_{i=1}^{n-2(m+1)} |g_{i+m+1} - g_i| \right\}
\]
\[
+ d^2_i \sum_{i=1}^{n-2(m+1)} |g_{i+m+1} - g_i| \sum_{j=i+1}^{i+m} |g_{j+m+1} - g_j|.
\]

The result follows from Eq. (4.9) and Lemma (3).

Lemma 5. Consider the general model 2.1 and assume that $d_0 + d_1 = 0$. Then
\[
\sum_{i=1}^{n_{m-1}} \sum_{j=i+1}^{i+2m+1} |g_{j+m+1} - g_j| = \mathcal{O} \left( m H^H_K \right)
\]

\[
\sum_{i=1}^{n_{m-1}} |g_{i+m+1} - g_i| \sum_{j=i+1}^{i+2m+1} |g_{j+m+1} - g_j| = \mathcal{O} \left( \frac{m(m+1)}{2} J_K \right),
\]

where $H^H_K = \sum_{j=1}^{K} (\tau_j - \frac{m+1}{2})|a_{j-1} - a_j|$. 
**Proof.** We begin by establishing (4.17). As in Lemma 3 here we also need to split the sum in three ways: let $\kappa_j = \tau_j - 2(m + 1)$, first we consider $i \leq \kappa_j$, then $\kappa_j + 1 \leq i \leq \kappa_j + m$ and finally $\kappa_j + m + 1 \leq i \leq \tau_j - 1$.

Assume that $i \leq \kappa_j$. It is not difficult to see that when $s = m + 1, \ldots, 2m + 1$, and $l = i + s$, $g_{l+m+1} - g_l = a_{j-1} - a_j$. Also, when $s \leq m$ and $l = i + s$, $g_{l+m+1} - g_l = 0$. Hence,

$$
\sum_{i=\kappa_j}^{i+2m+1} \sum_{l=i+1}^{i+2m+1} |g_{l+m+1} - g_l| = \sum_{i=\kappa_j}^{i+2m+1} \sum_{l=i+1}^{i+2m+1} |g_{l+m+1} - g_l| = m \sum_{i=\kappa_j}^{i+2m+1} |a_{j-1} - a_j| = \kappa_j m |a_{j-1} - a_j|.
$$

Next, for any $\kappa_j + 1 \leq i \leq \kappa_j + m$, it is straightforward to see that

$$
\sum_{i=\kappa_j+1}^{i+2m+1} |g_{l+m+1} - g_l| = (m + 1)|a_{j-1} - a_j|.
$$

Assume now that $\kappa_j + m + 1 \leq i \leq \tau_j - 1$. We begin by studying the particular case $i = \kappa_j + m + 1 = \tau_j - (m + 1)$. Then we get for $t = 0, \ldots, m - 1$ and $l = i + 1 + t$ that $g_{l+m+1} - g_l = (a_{j-1} - a_j)$. Also, when $t \geq m$ and $l = i + 1 + t$, $g_{l+m+1} - g_l = 0$. Hence,

$$
\sum_{i=\kappa_j+1}^{i+2m+1} |g_{l+m+1} - g_l| = \sum_{i=1}^{t+m} |a_{j-1} - a_j| = m |a_{j-1} - a_j|
$$

Similar arguments allow us to see that for $i = \kappa_j + m + 1 + u$ where $u = 1, \ldots, m - 1$,

$$
\sum_{i=\kappa_j+1}^{i+2m+1} |g_{l+m+1} - g_l| = (m - u)|a_{j-1} - a_j|.
$$

Combining the arguments above, we have shown that the left-hand side of (4.17) is bounded by

$$
\sum_{j=1}^{K} \left[ \sum_{i=\kappa_j}^{i+\kappa_j+m} + \sum_{i=\kappa_j+1}^{\tau_j} + \sum_{i=\kappa_j+m+1}^{i+2m+1} \sum_{l=i+1}^{i+2m+1} |g_{l+m+1} - g_l| \right]
$$

$$
= m \sum_{j=1}^{K} \left[ \tau_j - 2(m + 1) + (m + 1) + \frac{m + 1}{2} \right] |a_{j-1} - a_j|.
$$

In order to establish Eq. (4.18) we follow the proof of Eq. (4.13) and apply Eqs. (4.7) and (4.19). This completes the proof. \qed
Lemma 6. Consider the general model 2.1 and assume that \( d_0 + d_1 = 0 \). Then,

\[
\sum_{i=1}^{n_m-1} \sum_{j=i+1}^{n_m} \Delta_m(f_i + g_i) \Delta_m(f_j + g_j) |\gamma_{j-i-(m+1)}| = \mathcal{O} \left( \frac{d_1^2 \gamma_0 m(m+1)}{n} \left( (m+1)M^2 + MH_K^\| + (m+1)MJ_K^\| + \frac{n}{2}J_K^\| \right) \right).
\]

Proof. For given \( i \), due to \( m \)-dependency we get that \( \gamma_{j-i-(n+1)} \neq 0 \) when \( j < i + 2(m+1) \). Since,

\[
\sum_{i=1}^{n_m-1} \sum_{j=i+1}^{n_m} |\Delta_m(f_i + g_i) \Delta_m(f_j + g_j) | \gamma_{j-i-(m+1)}| \leq \gamma_0 \sum_{i=1}^{n_m-1} \sum_{j=i+1}^{i+2m+1} |\Delta_m(f_i + g_i) \Delta_m(f_j + g_j)|,
\]

we will bound the right-hand side in the expression above. Observe that this expression can be bounded as the left-hand side expression of Eq. (4.16). Hence, this proof is complete by following the proof of Lemma 4 and applying Lemma 5.

Lemma 7. Let \( \gamma_h \) denote the autocovariance function of a stationary, \( m \)-dependent process. Then

1. \( \sum_{i,j} \gamma_{j-i}^2 \leq \gamma_0^2 (n_m - 1)m \),
2. \( \sum_{i,j} |\gamma_{j-i-(m+1)}| \leq \gamma_0^2 (n_m - 1)(2m + 1) \),
3. \( \sum_{i,j} |\gamma_{j-i}| |\gamma_{j-i-(m+1)}| \leq \gamma_0^2 (n_m - 1)m \).

Here \( \sum_{i,j} = \sum_{i=1}^{n_m-1} \sum_{j=i+1}^{n_m} \).

Proof. 1. \( \sum_{i,j} \gamma_{j-i}^2 = \gamma_0^2 \sum_{i,j} \left( \frac{\gamma_j}{\gamma_0} \right)^2 \leq \gamma_0^2 \sum_{i=1}^{n_m-1} \sum_{j=i+1}^{i+m+1} 1 = \gamma_0^2 (n_m - 1)m \). The inequality follows from the \( m \)-dependency.

2. First note that

\[
|j - i - (m + 1)| = \begin{cases} j - i - (m + 1) & \text{for } j \geq i + (m + 1) \\ i + m + 1 - j & \text{for } j < i + m + 1 \end{cases}.
\]

Then, recall that \( \gamma_{j-i-(m+1)} \neq 0 \) when \( |j - i - (m + 1)| \leq m \). Intersecting these two subsets we get that \( \gamma_{j-i-(m+1)} \neq 0 \) for \( i + 1 \leq j \leq i + 2m + 1 \). The rest of the proof is similar to that of 1.

3. Follows from 1 and 2.