HARNACK INEQUALITY FOR DEGENERATE AND SINGULAR OPERATORS
OF $p$-LAPLACIAN TYPE ON RIEMANNIAN MANIFOLDS

SOOJUNG KIM

ABSTRACT. We study viscosity solutions to degenerate and singular elliptic equations of $p$-Laplacian type on Riemannian manifolds. The Krylov-Safonov type Harnack inequality for the $p$-Laplacian operators with $1 < p < \infty$ is established on the manifolds with Ricci curvature bounded from below based on ABP type estimates. We also prove the Harnack inequality for nonlinear $p$-Laplacian type operators assuming that a nonlinear perturbation of Ricci curvature is bounded below.

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1. Introduction

In this paper, we prove the Krylov-Safonov Harnack inequality for viscosity solutions to degenerate and singular elliptic equations of $p$-Laplacian type with $1 < p < \infty$ on Riemannian manifolds. The $p$-Laplacian operator $\Delta_p u = \text{div} \left( |\nabla u|^{p-2} \nabla u \right)$ appears in the Euler-Lagrange equation of the $L^p$-norm of the gradient of functions, and can also be expressed in nondivergence form:

$$\Delta_p u = |\nabla u|^{p-2} \text{tr} \left( \left( I + (p-2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) D^2 u \right),$$

where a tensor product $X \otimes X$ for a vector field $X$ over a Riemannian manifold $M$ is a symmetric bilinear form on $TM$ defined by $X \otimes X (Y,Z) := \langle X,Y \rangle \langle X,Z \rangle$ for $Y,Z \in TM$. Along the lines of a fundamental work of Yau [Y], differential Harnack inequalities for $p$-Laplacian operators have been obtained in [KN, WZ] on Riemannian manifolds with Ricci curvature bounded below. For divergence form operators, the De Giorgi-Nash-Moser Harnack inequality was extended for uniformly parabolic operators in [G, SC], and for the $p$-Laplacian operators in [H, RSV] on Riemannian manifolds satisfying certain properties: a volume doubling property and a weak version of Poincaré’s inequality.
Cabré [Ca] in his remarkable paper investigated the Krylov-Safonov type Harnack inequality for uniformly elliptic operators on Riemannian manifolds by establishing the ABP type estimates. The ABP estimate is a cornerstone of the Krylov-Safonov regularity theory, which is proved using affine functions in the Euclidean space; refer to [CC] for instance. In the Riemannian setting, Cabré used the squared distance function as the appropriate replacement for affine functions to derive the ABP type estimates on Riemannian manifolds with nonnegative sectional curvature. The idea of sliding paraboloids was also used by Savin [S] to show the ABP type measure estimate for small perturbation solutions in $\mathbb{R}^n$; see also [M]. The sectional curvature assumption of Cabré’s result was weakened by estimating an upper bound of the Jacobian determinant of the map $\Phi$ in Theorem 4.2 in [K], which gave in particular a new proof of the Harnack inequality on Riemannian manifolds with nonnegative Ricci curvature. Recently, Wang and Zhang [WZ] studied the ABP type estimates and a locally uniform Harnack inequality for uniformly elliptic operators on the manifolds with sectional curvature bounded below. The Harnack inequality for viscosity solutions to uniformly parabolic equations on Riemannian manifolds has been obtained in [KKL, KL].

This paper deals with the Harnack inequality for viscosity solutions to the $p$-Laplacian type equations with $1 < p < \infty$ on Riemannian manifolds employing the ABP type method. To study the ABP type estimates, we slide the $\frac{p-1}{2}$-th power of the distance function from below, which is the squared distance in the special case $p = 2$, and introduce the $p$-contact set adapted to the $p$-Laplacian operator (see Definition 4.1). Let $d_p$ denote the Riemannian distance from a point $y \in M$. For $u \in C(\overline{\Omega})$ and a compact set $E \subset M$, the $p$-contact set associated with $u$ and the vertex set $E$ is defined by

$$\mathcal{A}^p(E; \Omega; u) := \{x \in \Omega: \exists y \in E \text{ such that } \inf_{\Omega} \left\{ u + \frac{p-1}{p} d_p^{-\frac{p}{2}} \right\} = u(x) + \frac{p-1}{p} d_p^{-\frac{p}{2}}(x) \}.$$ 

Then it is shown in Proposition 4.2 that a vertex point $y \in E$ associated with a contact point $x \in \Omega$ with nonvanishing gradient of $u$ is given by

$$y = \exp_y |\nabla u|^{p-2} \nabla u(x) =: \Phi_p(x).$$

By estimating an upper bound of the Jacobian determinant of the map $\Phi_p(x)$ over the $p$-contact set, we prove the ABP type estimate for the degenerate cases $p \geq 2$ in Theorem 4.2 stating that the measure of the vertex set is bounded by the integral over the $p$-contact set in terms of the $p$-Laplacian operator, provided that Ricci curvature of the underlying manifold is bounded below. This generalizes [WZ, Theorem 1.2], the case $p = 2$. For $1 < p < 2$, the $p$-Laplacian operator becomes singular when the gradient vanishes. To cope with singularities, we make use of a regularized operator $(|\nabla u|^2 + \delta)^{\frac{p-2}{2}} M_{p-1,1}(D^2 u)$ for $\delta > 0$, which has been considered in [DFQ1, ACP] for the Euclidean case. This leads to introduce a regularized map

$$x \mapsto \exp_y \left( |\nabla u|^2 + \delta \right)^{\frac{p-2}{2}} \nabla u(x) =: \Phi_{p,\delta}(x) \quad (\delta > 0),$$

which lies on the minimal geodesic joining $x$ to $y = \exp_y |\nabla u|^{p-2} \nabla u(x) = \Phi_p(x)$ at time

$$\left( \frac{|\nabla u(x)|^2}{|\nabla u|^{p-2} + \delta} \right)^{\frac{p-2}{2}}$$

for the contact point $x$ with nonvanishing gradient of $u$. A uniform Jacobian estimate of $\Phi_{p,\delta}$ over the $p$-contact set with respect to $\delta > 0$ will imply the ABP type estimate for singular cases in Theorem 4.3. Based on the ABP type estimates, a locally uniform Harnack inequality for the $p$-Laplacian operators is established by means of the volume comparison and the Laplacian comparison on Riemannian manifolds with Ricci curvature bounded below. More generally, we are concerned with the nonlinear degenerate
and singular equations of $p$-Laplacian type

$$|\nabla u|^p - F(D^2u) + \langle b, \nabla u \rangle |\nabla u|^{p-2} = f,$$

where $F$ is a uniformly elliptic operator and $b$ is a bounded vector field over $M$. From [K], we recall the Pucci operator of the Ricci transform; for $0 < \lambda \leq \Lambda < \infty$, and for any $x \in M$ and any unit vector $e \in T_xM$,

$$\mathcal{M}_{i,A}(R(e)) := \Lambda \sum_{\kappa_i < 0} \kappa_i + \lambda \sum_{\kappa_i > 0} \kappa_i,$$

where $\kappa_i$ are the eigenvalues of the Ricci transform $R(e)$. Note that $\mathcal{M}_{i,1}(R(e)) = \text{Ric}(e, e)$.

We refer to Section 2 for the definitions. In place of the Ricci curvature bound, we assume that $\mathcal{M}_{i,1}(R(e))$ is uniformly bounded below for any unit vector $e \in TM$ when we study qualitative properties for viscosity solutions to the nonlinear degenerate and singular equations of $p$-Laplacian type.

We would like to mention related results on estimates for the $p$-Laplacian type operators. In the Euclidean space, the ABP estimates for the $p$-Laplacian type operators have been obtained in [DFQ1, I, ACP] on the basis of the estimates over a contact set by the affine functions. A recent work by Imbert and Silvestre [IS] addresses the Hölder estimates and Harnack inequality for viscosity solutions satisfying a uniformly elliptic equation only at points where the gradient is large. In the proof the ABP type measure estimates, they used genuinely a cusp, the square root of the distance which corresponds to the case $p = -1$ in our setting. Assuming the Pucci operator of Ricci transform to be bounded below, their method can be applied to Riemannian manifolds with the use of the argument in Proposition 4.2.

Our approach in this paper yields a different proof involving more intrinsic geometric quantities to the geometry of the $p$-Laplacian operators on Riemannian manifolds with Ricci curvature bounded below. It can also be adapted to show a parabolic analogue of the ABP type estimate by using a generalized Legendre transform with respect to the $\frac{p}{p-1}$-th power of the distance in the singular cases $1 < p \leq 2$ as in [KKL] since $d_y^{\frac{p}{p-1}}$ is twice differentiable on $M \setminus \text{Cut}(y)$.

Now we state our main results as follows. Throughout this paper, let $(M, g)$ be a smooth, complete Riemannian manifold of dimension $n$, and $B_{2R}(z_0)$ denote a geodesic ball of radius $R$ centered at $z_0$.

**Theorem 1.1** (Harnack inequality). Let $1 < p < \infty$, and $\text{Ric} \geq -(n-1)k$ for $k \geq 0$. For $z_0 \in M$ and $0 < R \leq R_0$, let $u$ be a nonnegative viscosity solution to

$$\Delta_p u = f \quad \text{in} \quad B_{2R}(z_0).$$

Then

$$\sup_{B_{R}(z_0)} u \leq C \left( \inf_{B_R(z_0)} u + R^\frac{p}{p-1} \|f\|_{L^p(B_{2R}(z_0))} \right),$$

where a constant $C > 0$ depends only on $n$, $p$, and $\sqrt{n}R_0$.

In particular, when Ricci curvature of the underlying manifold is nonnegative, the above Harnack inequality is a global estimate, which implies the Liouville theorem.

**Corollary 1.2.** Let $1 < p < \infty$, and $\text{Ric} \geq 0$. If $u$ is a viscosity solution to $\Delta_p u = 0$ in $M$, which is bounded from below; then $u$ is a constant function.

As a consequence of the Harnack inequality, we have a locally uniform Hölder estimate for viscosity solutions to the $p$-Laplacian equations.
Corollary 1.3 (Hölder estimate). Let $1 < p < \infty$, and $\text{Ric} \geq -(n-1)\kappa$ for $\kappa \geq 0$. For $z_0 \in M$ and $0 < R \leq R_0$, let $u$ be a viscosity solution to $\Delta_p u = f$ in $B_{2R}(z_0)$. Then
\[
R^a \left[ u \right]_{C^{\alpha}(B_R(z_0))} \leq C \left( \|u\|_{L^\infty(B_{2R}(z_0))} + R^{\frac{n}{p-1}} \|f\|_{L^p(B_{2R}(z_0))} \right),
\]
where the constants $\alpha \in (0,1)$ and $C > 0$ depend only on $n$, $p$, and $\sqrt{\kappa} R_0$.

We also obtain similar results for nonlinear $p$-Laplacian type operators.

Theorem 1.4 (Harnack inequality). Let $1 < p < \infty$, and $\text{M}^p_{-\lambda}(R(e)) \geq -(n-1)\kappa$ with $\kappa \geq 0$ for any unit vector $e \in TM$. Let $z_0 \in M$ and $0 < R \leq R_0$. For $\beta \geq 0$ and $C_0 \geq 0$, let $u$ be a nonnegative viscosity solution to
\[
\begin{align*}
|\nabla u|^p - 2\lambda &\leq C_0 \text{ in } B_{2R}(z_0), \\
|\nabla u|^p - 2\lambda \text{ in } B_{2R}(z_0).
\end{align*}
\]
Then
\[
\sup_{B_R(z_0)} u \leq C \left( \inf_{B_R(z_0)} u + R^{\frac{n}{p-1}} C_0^{1/\beta} \right),
\]
where a constant $C > 0$ depends only on $n$, $p$, $\sqrt{\kappa} R_0$, $\lambda$, and $\beta R_0$.

Corollary 1.5 (Hölder estimate). Let $1 < p < \infty$, and $\text{M}^p_{-\lambda}(R(e)) \geq -(n-1)\kappa$ with $\kappa \geq 0$ for any unit vector $e \in TM$. Let $z_0 \in M$ and $0 < R \leq R_0$. For $\beta \geq 0$ and $C_0 \geq 0$, let $u$ be a viscosity solution to (1). Then
\[
R^a \left[ u \right]_{C^{\alpha}(B_R(z_0))} \leq C \left( \|u\|_{L^\infty(B_{2R}(z_0))} + R^{\frac{n}{p-1}} C_0^{1/\beta} \right),
\]
where the constants $\alpha \in (0,1)$ and $C > 0$ depend only on $n$, $p$, $\sqrt{\kappa} R_0$, $\lambda$, and $\beta R_0$.

The rest of the paper is organized as follows. In Section 3 we collect some of the known results on Riemannian geometry that are used in this paper. In Section 4 we study the properties of the adapted viscosity solutions to singular elliptic equations, and recall a regularization by Jensen’s inf-convolution on Riemannian manifolds. Section 5 is devoted to the proof of the ABP type estimates for degenerate and singular operators of the $p$-Laplacian type with $1 < p < \infty$. Section 6 contains the proof of the $L^p$-estimates by means of a barrier function and the volume comparison. In Section 7 we prove the Harnack inequality.

2. Preliminary: Riemannian geometry

Let $(M, g)$ be a smooth, complete Riemannian manifold of dimension $n$, equipped with the Riemannian metric $g$. A Riemannian metric defines a scalar product and a norm on each tangent space, i.e., $(X, Y)_x := g_x(X, Y)$ and $|X|_x^2 := (X, X)_x$ for $X, Y \in T_xM$, where $T_xM$ is the tangent space at $x \in M$. Let $d(\cdot, \cdot)$ be the Riemannian distance on $M$. For a given point $y \in M$, $d_i(x)$ stands for the distance to $x$ from $y$, namely, $d_i(x) := d(x, y)$. A Riemannian manifold is equipped with the Riemannian measure $\text{Vol} = \text{Vol}_g$ on $M$ which is denoted by $| \cdot |$ for simplicity.

The exponential map $\exp : TM \to M$ is defined as
\[
\exp_x X := \gamma_{x,X}(1),
\]
where $\gamma_{x,X} : \mathbb{R} \to M$ is the unique geodesic starting at $x \in M$ with velocity $X \in T_xM$. We note that the geodesic $\gamma_{x,X}$ is defined for all time since $M$ is complete, but it is not minimizing in general. This leads to define the cut time $t_c(X)$: for $X \in T_xM$ with $|X| = 1$,
\[
t_c(X) := \sup \{ t > 0 : \exp_x tX \text{ is minimizing between } x \text{ and } \exp_x tX \}.
\]
The cut locus of \( x \in M \), denoted by \( \text{Cut}(x) \), is defined by

\[
\text{Cut}(x) := \{ \exp_x t_x(X) : X \in T_xM \text{ with } |X| = 1 \}.
\]

Let \( \mathcal{E}_t := \{ x \in M : 0 \leq t < t_x(X), X \in T_xM \text{ with } |X| = 1 \} \), and \( \mathcal{E} := \{ x \in TM : X \in \mathcal{E}_t, x \in M \} \). In fact, the exponential map \( \exp_i : \mathcal{E} \to M \) is smooth. One can prove that for any \( x \in M \), \( \text{Cut}(x) = \exp_x(\partial \mathcal{E}_t), M = \exp_x(\mathcal{E}_t) \cup \text{Cut}(x) \), and \( \exp_x : \mathcal{E}_t \to \exp_x(\mathcal{E}_t) \) is a diffeomorphism. We recall that \( \text{Cut}(x) \) is closed and of measure zero. Given two points \( x, y \notin \text{Cut}(x) \), there exists a unique minimizing geodesic \( \exp_t t_X \) (for \( X \in \mathcal{E}_t \)) joining \( x \) to \( y = \exp_x X \), and we will denote \( X = \exp_x^{-1}(y) \) in the case. The Gauss lemma implies that

\[
\exp_x^{-1}(y) = -\nabla X^2(x)/2, \quad \forall y \notin \text{Cut}(x).
\]

For a \( C^2 \)-function \( u : M \to \mathbb{R} \), the gradient \( \nabla u \) of \( u \) is defined by \( \langle \nabla u, X \rangle := du(X) \) for any vector field \( X \) on \( M \), where \( du : TM \to \mathbb{R} \) is the differential of \( u \). The Hessian \( D^2u \) of \( u \) is defined as

\[
D^2u(X, Y) := \langle \nabla X \nabla u, Y \rangle,
\]

for any vector fields \( X, Y \) on \( M \), where \( \nabla \) denotes the Riemannian connection of \( M \). We observe that the Hessian \( D^2u \) is a symmetric \( 2 \)-tensor over \( M \), and \( D^2u(x, y) \) at \( x \in M \) depends only on the values \( x, y \) at \( x \), and \( u \) in a small neighborhood of \( x \). By the metric, the Hessian of \( u \) at \( x \) is canonically identified with a symmetric endomorphism of \( T_xM \):

\[
D^2u(X) = \nabla X \nabla u, \quad \forall X, Y \in T_xM.
\]

We will write \( D^2u(x)(X, Y) = \langle D^2u(x) \cdot X, Y \rangle \) for \( X \in T_xM \). If \( x \) is a \( C^1 \)-vector field on \( M \), the divergence of \( x \) is defined as \( \text{div} x := \text{tr} \{ X \mapsto \nabla x \cdot X \} \). For \( u \in C^2(M) \), the Laplacian operator \( \Delta u = \text{tr}(D^2u) \) coincides with \( \text{div}(\nabla u) \).

Denote by \( R \) the Riemannian curvature tensor defined as

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z
\]

for any vector fields \( X, Y, Z \) on \( M \). For two linearly independent vectors \( X, Y \in T_xM \), the sectional curvature of the plane generated by \( X \) and \( Y \) is defined as

\[
\text{Sec}(X, Y) := \frac{\langle R(X, Y)Z, X \rangle}{|Z|^2 - \langle X, Y \rangle^2}.
\]

For a unit vector \( e \in T_xM \), we denote by \( R(e) \) the Ricci transform of \( T_xM \) into itself defined by

\[
R(e)X := R(X, e)e \quad \forall X \in T_xM.
\]

Note that Ricci transform is symmetric. The Ricci curvature is the trace of the Ricci transform, which can be expressed as follows: for a unit vector \( e \in T_xM \) and an orthonormal basis \( \{ e_1, e_2, \ldots, e_n \} \) of \( T_xM \),

\[
\text{Ric}(e, e) = \sum_{j=2}^n \text{Sec}(e, e_j).
\]

As usual, \( \text{Ric} \geq \kappa \) on \( M \) (\( \kappa \in \mathbb{R} \)) stands for \( \text{Ric} \geq \kappa g \) for any \( x \in M \). When dealing with a class of nonlinear elliptic operators, we involve Pucci’s operator of Ricci transform instead of the trace operator; for \( 0 < \lambda \leq \Lambda \), and for any \( x \in M \) and any unit vector \( e \in T_xM \), define

\[
\mathcal{M}_{\lambda, \Lambda}(R(e)) := \Lambda \sum_{\kappa > 0} \kappa_e + \Lambda \sum_{\kappa < 0} \kappa_e,
\]

where \( \kappa_e \) are the eigenvalues of \( R(e) \). In the special case when \( \lambda = \Lambda = 1 \), Pucci’s operator simply coincides with the trace operator, and hence \( \mathcal{M}_{1,1}(R(e)) = \text{Ric}(e, e) \). Notice
that $\lambda \text{Ric}(e,e) \geq M_{\lambda,\Lambda}(R(e))$, so a lower bound of $M_{\lambda,\Lambda}(R(e))$ guarantees one for Ricci curvature.

Now we recall the volume comparison theorem assuming the Ricci curvature to be bounded from below. In terms of polar normal coordinates at $x \in M$, the area element of a geodesic sphere $\partial B_r(x)$ of radius $r$ centered at $x$ is written by $r^{n-1} A(r,\theta)d\theta$, where $A(r,\theta)$ is the Jacobian determinant of the map $\exp_x$ at $r\theta \in T_xM$. With the use of a standard theory of Jacobi fields, the Jacobian determinant of the exponential map has an upper bound depending on a lower bound of Ricci curvature; the proof can be found in [BC] Chapter 11. Bishop-Gromov’s volume comparison theorem relies on the Jacobian estimates, which says that the volume of balls does not increase faster than the volume of balls in the model space (see also [V]). In particular, the volume comparison implies the (locally uniform) volume doubling property.

**Theorem 2.1** (Bishop-Gromov). Assume $\text{Ric} \geq -(n-1)\kappa$ for $\kappa \geq 0$.

(i) Let $x \in M$, $y \notin \text{Cut}(x) \cup \{x\}$, and $\gamma(t) := \exp_x th \xi$ be the minimizing geodesic joining $x = \gamma(0)$ to $y = \gamma(1)$ for $\xi \in E_x$. For $t \in (0,1]$, let $J(t)$ be the differential of $\exp_x$ at $t \xi \in T_xM$, namely, $J(t) := d\exp_x(t\xi)$. Then

$$0 < \det J(t) \leq \mathcal{J}(\sqrt{n}t\xi) = \left(\frac{\sinh(\sqrt{n}t\xi)}{\sqrt{\kappa t\xi}}\right)^{n-1},$$

where $\mathcal{J}(\tau) := \sinh(\tau)/\tau$ for $\tau > 0$ with $\mathcal{J}(0) = 1$. Furthermore, for $t \in (0,1]$

$$0 < \det J(t) \leq \mathcal{J}(\sqrt{n}t\xi) = \left(\frac{\sinh(\sqrt{n}\tau\xi)}{\sqrt{\kappa \tau\xi}}\right)^{n-1}.$$ 

(ii) Let

$$V(r) := \omega_{n,\kappa} \int_0^r \sinh^{n-1}(\sqrt{n}t)dt \quad \forall r > 0,$$

where $V(r)$ denotes the volume of a ball of radius $r$ in the $n$-dimensional space form of constant curvature $-\kappa$. Then

$$r \mapsto \frac{\text{Vol}(B_r(x))}{V(r)}$$

is a nonincreasing function of $r > 0$. In particular, for any $0 < r < R$

$$\frac{\text{Vol}(B_{2r}(x))}{\text{Vol}(B_r(x))} \leq 2^n \cosh^{n-1}(2 \sqrt{n}R) =: D,$$

where $D$ is a so-called doubling constant.

One can see that the doubling property (ii) yields that for any $0 < r < R \leq R_0$,

$$\frac{\text{Vol}(B_{r}(x))}{\text{Vol}(B_{r}(x))} \leq D \left(\frac{R}{r}\right)^{\log_2 D},$$

where $D := 2^n \cosh^{n-1}(2 \sqrt{n}R_0)$. According to the volume comparison, it is not difficult to prove the following lemma by a similar argument to the proof of [IS] Lemma 2.1 (see also [CC]).

**Lemma 2.2.** Let $\text{Ric} \geq -(n-1)\kappa$ for $\kappa \geq 0$ and $0 < R \leq R_0$. Let $E \subset F$ be two open subsets in $B_R(x) \subset M$. Assume that for some $\delta \in (0,1)$,

(a) if any ball $B \subset B_R(x)$ satisfies $|E \cap B| > (1-\delta)|B|$, then $B \subset F$,

(b) $|E| \leq (1-\delta)|B_R(x)|$. 

Then there exists a constant $c_0 \in (0, 1)$, depending only on $n$, and $\sqrt{R_0}$, such that

$$|E| \leq (1 - c_0\delta)|F|.$$  

Using a standard theory of Jacobi fields, the Jacobian determinant of the exponential map involving a vector field has the following formula; we refer to [Ca] Lemma 3.2] for the proof, and see also [CMS] and [V] Chapter 14.

**Lemma 2.3.** Let $\xi$ be a smooth vector field on $M$. Define a map $\phi(z) := \exp_z(\xi(z))$. For a given $x \in M$, assume that $\xi(x) \in \mathcal{E}_n$, and let $y := \phi(x)$. Then

$$d\phi(x) \cdot X = d\exp_z(\xi(x)) \cdot \left(\left(\nabla^2 \xi + D^2 d^2_\xi/2\right)(x) \cdot X\right) \quad \forall X \in T_xM,$$

and

$$\text{Jac} \phi(x) = (\text{Jac} \exp_z(\xi(x))) \cdot \left|\det \left(\nabla^2 \xi + D^2 d^2_\xi/2\right)(x)\right|,$$

where $\text{Jac} \phi(x) := |\det d\phi(x)|$, $\text{Jac} \exp_z(\xi(x))$ is the Jacobian determinant of the map $\exp_z$ at $\xi(x) \in T_xM$, and $\nabla^2 \xi$ denotes the covariant derivative of $\xi$.

With the same notation as the lemma above, we define a map $\phi(t, x) := \exp_x(t \xi(x))$ for $t \in [0, 1]$, and assume that $\nabla^2 \xi(x)$ is symmetric. From the next lemma, we observe that if $\left(\nabla^2 \xi + D^2 d^2_\xi/2\right)(x)$ is positive semi-definite, then $\left(i\nabla^2 \xi + D^2 d^2_\xi/2\right)(x)$ remains positive semi-definite for $t \in [0, 1]$. Making appropriate modifications, this fact will play an important role when we deal with the approximation of singular operators in Section 4. The following lemma can be found in [CMS] Lemma 2.3]; see also [V] Chapter 14, in particular the third appendix.

**Lemma 2.4.** Let $x \in M$ and $y \notin \text{Cut}(x)$. Let $\gamma : [0, 1] \rightarrow M$ be the minimizing geodesic joining $\gamma(0) = x$ to $\gamma(1) = y$. Then the self-adjoint operator

$$D^2 d^2_{\gamma(0)/2}(y) - t D^2(d^2_{\gamma(1)/2})(x)$$

defined on $T_\gamma M$ is positive semi-definite for $t \in [0, 1]$.

An upper Hessian bound for the squared distance function is proved in [CMS] Lemma 3.12] using the formula for the second variation of energy provided that the sectional curvature is bounded from below. We quote it as follows.

**Lemma 2.5.** Let $x, y \in M$. If $\text{Sec} \geq -\kappa$ ($\kappa \geq 0$) along a minimizing geodesic joining $x$ to $y$, then for any $X \in T_xM$ with $|X| = 1$,

$$\limsup_{t \rightarrow 0} \frac{d^2_\gamma \left(\exp_x(tX)\right) + d^2_\gamma \left(\exp_x(-tX)\right) - 2d^2_\gamma(x)}{t^2} \leq 2 \sqrt{\kappa} d_\gamma(x) \coth \left(\sqrt{\kappa} d_\gamma(x)\right).$$

The well-known Laplacian comparison theorem implies that if the Ricci curvature is bounded from below by $-(n - 1)\kappa$ for $\kappa \geq 0$, then

$$\Delta \left(d^2_\gamma/2(x)\right) \leq 1 + (n - 1) \sqrt{\kappa} d_\gamma(x) \coth \left(\sqrt{\kappa} d_\gamma(x)\right) \quad \text{for } x \notin \text{Cut}(y).$$

As a generalization, we are concerned with an upper estimate of Pucci’s maximal operator for the squared distance function when $\mathcal{M}_{\mathcal{L}, \Lambda}(\mathcal{R}(e))$ has a lower bound. The proof for the following lemma uses the formula for the second variation of the energy, and closely follows the argument in the proof of [CMS] Lemma 3.12]; see also [K] Lemma 2.1.

**Lemma 2.6.** For $x \in M$ and $y \notin \text{Cut}(x)$, let $\gamma$ be the minimizing geodesic joining $x$ to $y$. Assume that $\mathcal{M}_{\mathcal{L}, \Lambda}(\mathcal{R}(e)) \geq - (n - 1)\kappa$ with $\kappa \geq 0$ for any unit vector $e \in T_{\gamma(0)}M$. Then

$$\mathcal{M}_{\mathcal{L}, \Lambda} \left(D^2 d^2_\gamma/2(x)\right) \leq \Lambda + (n - 1)\Lambda \mathcal{H} \left(\sqrt{\frac{\kappa}{\Lambda}} d_\gamma(x)\right).$$
where $\mathcal{H}(\tau) := \tau \coth(\tau)$ for $\tau > 0$ with $\mathcal{H}(0) = 1$.

**Proof.** We may assume that $x \notin \text{Cut}(y) \cup \{y\}$ since $D^2 d_x^2/2(y) = 1$. Let $\gamma : [0, l] \to M$ be the minimizing geodesic parametrized by arc length such that $\gamma(0) = x$ and $\gamma(l) = y$. We introduce an orthonormal basis $(e_i)_{i=1}^n$ on $T_x M$ such that $e_1 = \dot{\gamma}(0)$ and $(e_i)_{i=1}^n$ are eigenvectors of $D^2 d_y^2/2(x)$ on $T_y M$. Note that a unit vector $e_1 = -\nabla d_x(x)$ is an eigenvector of $D^2 d_y^2/2(x)$ associated with the eigenvalue 1. By the parallel transport, we extend $(e_i)_{i=1}^n$ to $(e_i)_{i=1}^n$ along $\gamma(t)$, and define the vector fields

$$X_i(t) = \alpha(t)e_i(t), \quad \text{for } \alpha(t) := \frac{\sinh\left(\sqrt{\lambda}(l - t)\right)}{\sinh\left(\sqrt{\lambda}l\right)},$$

which satisfy $X_i(0) = e_i(0)$, and $X_i(l) = 0$. For small $s \in (-\delta, \delta)$, consider

$$\gamma'_i(t) := f_i(s, t) = \exp_{\gamma(t)} sX_i(t),$$

which connects $\exp_x se_i$ to $y$. With the help of the H"{o}lder inequality, we have that for $i = 2, \cdots, n$ and small $s \in (-\delta, \delta),$

$$\frac{1}{2}d^2_e(\exp_x se_i) \leq \frac{1}{2} \left( \int_0^l |\gamma'_i(t)| dt \right)^2 \leq \frac{l}{2} \int_0^l |\gamma'_i(t)|^2 dt =: lE_i(s),$$

where the equality holds for $s = 0$, i.e., $d^2_e/2(x) = lE_0(0)$. This implies that

$$\mathcal{M}_{i,i}^+(D^2 d^2_e/2(x)) = \Lambda + \sup_{\Lambda \leq \Lambda \leq \Lambda} \sum_{i=2}^n a_i \left( D^2 d^2_e/2(x) \cdot e_i, e_i \right) \leq \Lambda + \sup_{\Lambda \leq \Lambda \leq \Lambda} \sum_{i=2}^n a_i \left. \frac{d^2}{ds^2} \right|_{s=0} E_i(s)$$

from the definition of the Pucci operator. Since for each $i = 2, \cdots, n$, and $t \in [0, l]$, a curve $s \mapsto f_i(s, t)$ is also a geodesic, it follows from the formula for the second variation of the energy that

$$\mathcal{M}_{i,i}^+(D^2 d^2_e/2(x)) - \Lambda \leq \sup_{\Lambda \leq \Lambda \leq \Lambda} \sum_{i=2}^n a_i \left. \frac{d^2}{ds^2} \right|_{s=0} E_i(s)$$

$$= \sup_{\Lambda \leq \Lambda \leq \Lambda} \sum_{i=2}^n a_i \int_0^l \left( \langle \dot{X}_i(t), X_i(t) \rangle - \langle R(X_i(t), \dot{\gamma})(t), X_i(t) \rangle \right) dt$$

$$\leq (n - 1) \Lambda \int_0^l \alpha(t)^2 dt - l \inf_{\Lambda \leq \Lambda \leq \Lambda} \int_0^l \alpha(t)^2 \sum_{i=2}^n a_i \langle R(e_i(t), \dot{\gamma})(t), e_i(t) \rangle dt$$

$$= (n - 1) \Lambda \int_0^l \alpha(t)^2 dt - l \inf_{\Lambda \leq \Lambda \leq \Lambda} \int_0^l \alpha(t)^2 \sum_{i=2}^n a_i \langle R(\dot{\gamma}(t), e_i(t), e_i(t) \rangle dt.$$
Using the definition of Pucci’s operator and the assumption, one can check that for any $\lambda \leq a_i \leq \Lambda$, $\sum_{i=2}^{n} a_i \langle R(\hat{y}(t)) \cdot e_i(t), e_i(t) \rangle \geq M_{i,\Lambda}^l(R(\hat{y}(t)) \geq -(n-1)\kappa$, and hence

$$M_{i,\Lambda}^l(D^2 d^2_x/2(x)) - \Lambda \leq (n-1)\Lambda \int_0^t \alpha(t)^2 \, dt - \inf_{\lambda \in \iota \leq \Lambda} \sum_{i=2}^{n} a_i \langle R(\hat{y}(t)) \cdot e_i(t), e_i(t) \rangle \, dt$$

$$\leq (n-1)\Lambda \int_0^t \left\{ \alpha(t)^2 + \frac{\kappa}{\Lambda} \right\} \, dt$$

$$= (n-1)\Lambda \sqrt{\frac{\kappa}{\Lambda}} \sinh^{-2} \left( \sqrt{\frac{\kappa}{\Lambda}} l \right) \sum_{i=1}^{n} \alpha_i \langle R(\hat{y}(t)) \cdot e_i(t), e_i(t) \rangle \, dt$$

$$= (n-1)\Lambda \sqrt{\frac{\kappa}{\Lambda}} l \coth \left( \frac{\sqrt{\kappa}}{\Lambda} l \right),$$

which finishes the proof. □

In [CMS], Proposition 2.5, it is shown that the cut locus of $y \in M$ is characterized as the set of points at which the squared distance function $d_y^2$ fails to be semi-convex.

**Lemma 2.7** (Proposition 2.5 of [CMS]). Let $x, y \in M$. If $x \in \text{Cut}(y)$, then there is a unit vector $X \in T_x M$ such that

$$\lim_{t \to 0} \inf \frac{d^2_x \exp_x tX + d^2_x \exp_x -tx - 2d^2_y(x)}{t^2} = -\infty.$$

**Corollary 2.8.** Let $x, y \in M$ and let $\psi$ be a $C^2$-function in $(0, \infty)$ such that $\psi' > 0$ on $(0, \infty)$. If $x \in \text{Cut}(y)$, then there is a unit vector $X \in T_x M$ such that

$$\lim_{t \to 0} \inf \frac{\psi \left( d^2_x \exp_x tX \right) + \psi \left( d^2_x \exp_x -tx \right) - 2\psi \left( d^2_x(x) \right)}{t^2} = -\infty.$$

**Proof.** By using the Taylor expansion, we have that for any unit vector $X \in T_x M$ and small $|t| \in (0, 1),$

$$\psi \left( d^2_x \exp_x tX \right) + \psi \left( d^2_x \exp_x -tx \right) - 2\psi \left( d^2_x(x) \right)$$

$$= \psi' \left( d^2_x(x) \right) \frac{d^2_x \exp_x tX + d^2_x \exp_x -tx - 2d^2_x(x)}{t^2}$$

$$+ \frac{\psi'' \left( a(t) \right)}{2} \left( \frac{d^2_x \exp_x tX - d^2_x(x)}{t} \right)^2 + \frac{\psi'' \left( b(t) \right)}{2} \left( \frac{d^2_x \exp_x -tx - d^2_x(x)}{t} \right)^2,$$

where $a(t)$ and $b(t)$ converge to $d^2_x(x) > 0$ as $t$ tends to 0. From the triangle inequality, it follows that for any unit vector $X \in T_x M$,

$$\lim_{t \to 0} \sup \left| \frac{d^2_x \exp_x tX - d^2_x(x)}{t} \right| \leq 2d(y, x).$$

Therefore the result follows from Lemma 2.7 since $d^2_x(x) > 0$ and $\psi'(d^2_x(x)) > 0$. □

Lastly, we recall the definition of semiconcavity of functions on Riemannian manifolds.

**Definition 2.9.** Let $\Omega$ be an open set of $M$. A function $u : \Omega \to \mathbb{R}$ is said to be semiconcave at $x_0 \in \Omega$ if there exist a geodesically convex ball $B_r(x_0) \subset \Omega$ with $0 < r < i_M(x_0)$, and a smooth function $\Psi : B_r(x_0) \to \mathbb{R}$ such that $u + \Psi$ is geodesically concave in $B_r(x_0),$. 
where \( i_M(x_0) \) denotes the injectivity radius at \( x_0 \). A function \( u \) is semiconcave in \( \Omega \) if it is semiconcave at each point of \( \Omega \).

For \( C > 0 \), we say that a function \( u : \Omega \to \mathbb{R} \) is \( C \)-semiconcave at \( x_0 \in \Omega \) if there exists a geodesically convex ball \( B_r(x_0) \subset \Omega \) with \( 0 < r < i_M(x_0) \) such that \( u - Cd^2u(x) \) is geodesically concave in \( B_r(x_0) \). A function \( u \) is \( C \)-semiconcave in \( \Omega \) if it is \( C \)-semiconcave at each point of \( \Omega \).

We have the local characterization of semiconcavity from [CMS, Lemma 3.11]. According to Lemma [2.5], the following lemma implies that the squared distance function is locally uniformly semiconcave.

**Lemma 2.10.** Let \( u : \Omega \to \mathbb{R} \) be a continuous function and let \( x_0 \in \Omega \), where \( \Omega \subset M \) is open. Assume that there exist a neighborhood \( U \) of \( x_0 \), and a constant \( C > 0 \) such that for any \( x \in U \) and \( X \in T_xM \) with \( |X| = 1 \),

\[
\limsup_{t \to 0} \frac{u(\exp_t tX) + u(\exp_t -tX) - 2u(x)}{t^2} \leq C.
\]

Then \( u \) is \( C \)-semiconcave at \( x_0 \).

The following result by Bangert [B] is an extension of Aleksandrov’s second differentiability theorem in the Euclidean space that a convex function is twice differentiable almost everywhere [A]; see also [V, Chapter 14].

**Theorem 2.11** (Aleksandrov–Bangert). Let \( \Omega \subset M \) be an open set and let \( u : \Omega \to \mathbb{R} \) be semiconcave. Then for almost every \( x \in \Omega \), \( u \) is differentiable at \( x \), and there exists a symmetric operator \( A(x) : T_xM \to T_xM \) characterized by any one of the two equivalent properties:

(a) For any \( X \in T_xM \), \( A(x) \cdot X = \nabla_X \nabla u(x) \),

(b) \( u(\exp_X X) = u(x) + \langle \nabla u(x), X \rangle + \frac{1}{2} \langle A(x) \cdot X, X \rangle + o(|X|^2) \) as \( X \to 0 \).

The operator \( A(x) \) and its associated symmetric bilinear from on \( T_xM \) are denoted by \( D^2u(x) \) and called the Hessian of \( u \) at \( x \) when no confusion is possible.

We refer to [CMS, V, AF] for more properties of semiconcave functions.

3. Viscosity solutions for singular operators

In this section, we study viscosity solutions to degenerate and singular operators of \( p \)-Laplacian type for \( 1 < p < \infty \) on Riemannian manifolds. As seen before, the \( p \)-Laplacian operator \( \Delta_p u = \text{div} (|\nabla u|^{p-2} \nabla u) \) can be written in nondivergence form:

\[
\Delta_p u = |\nabla u|^{p-2} \text{tr} \left[ \left( 1 + (p-2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) D^2u \right],
\]

where a tensor product \( X \otimes X \) for a vector field \( X \) over \( M \) is a symmetric bilinear form on \( TM \) defined by

\[
(X \otimes X)(Y, Z) := \langle X, Y \rangle \langle X, Z \rangle \quad \text{for} \quad Y, Z \in TM.
\]

The tensor product \( X \otimes X \) is considered as a symmetric endomorphism: \( X \otimes X \cdot Y = \langle X, Y \rangle X \), so we write \( (X \otimes X)(Y, Z) = \langle X \otimes X \cdot Y, Z \rangle = \langle X, Y \rangle \langle X, Z \rangle \). For \( p \geq 2 \), the operator
$G(\nabla u, D^2 u) = \Delta_p u$ is continuous with respect to $\nabla u$ and $D^2 u$, while the $p$-Laplacian operator for $1 < p < 2$ becomes singular at the points with vanishing gradient. More generally, we are concerned with the degenerate and singular fully nonlinear equations given by

$$|\nabla u|^{p-2} F(\nabla^2 u) + \langle b, \nabla u \rangle |\nabla u|^{p-2} = f,$$

where $F$ is a uniformly elliptic operator and $b$ is a bounded vector field over $M$. Let $\text{Sym} T M$ be the bundle of symmetric 2-tensors over $M$. An operator $F : \text{Sym} T M \rightarrow \mathbb{R}$ is said to be uniformly elliptic with the so-called ellipticity constants $0 < \lambda \leq \Lambda$ if the following holds: for any $S \in \text{Sym} T M$, and for any positive semi-definite $P \in \text{Sym} T M$,

$$\lambda \text{trace}(P x) \leq F(S_x + P) - F(S_x) \leq \Lambda \text{trace}(P x), \quad \forall x \in M.$$

As extremal cases of the uniformly elliptic operators, we recall Pucci’s operators: for any $\mu \in (0, +\infty)$ and $\lambda \in \mathbb{R}$,

$$m_{\lambda, \mu}(S_x) := \lambda \sum_{\mu_i < 0} \mu_i + \Lambda \sum_{\mu_i > 0} \mu_i,$$

where $\mu_i = \mu_i(S_x)$ are the eigenvalues of $S_x$. We observe that (4) is equivalent to the following: for any $S, P \in \text{Sym} T M$,

$$m_{\lambda, \mu}(P_x) \leq F(S_x + P) - F(S_x) \leq m_{\lambda, \mu}(P_x) \quad \forall x \in M.$$

Assuming that $F(0) = 0$, we restrict ourselves to the degenerate and singular operators involving the Pucci operators.

For singular elliptic operators, we adopt the concept of viscosity solutions proposed by Birindelli and Demengel; see for instance, [BD], [DFQ2], [ACP] and the references therein. The notion of adapted viscosity solutions takes into account the fact that we can not test functions when the gradient of the test functions vanishes at the testing point.

**Definition 3.1 (Viscosity solutions, [BD]).** Let $(TM \setminus \{0\}) \times_M \text{Sym} T M := \{ (\zeta, A) : \zeta \in T^*_x M \setminus \{0\}, A \in \text{Sym} T M_x, \forall x \in M \}$, where $\emptyset$ denotes the zero section. Let $G : (TM \setminus \{0\}) \times_M \text{Sym} T M \rightarrow \mathbb{R}$, and let $\Omega \subset M$ be an open set. For a function $f : \Omega \rightarrow \mathbb{R}$, we say that $u \in C(\Omega)$ is a viscosity supersolution (respectively subsolution) of the equation

$$G(\nabla u, D^2 u) = f \quad \text{in} \ \Omega$$

if the following holds: for any $x \in \Omega$,

(i) either for any $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum (respectively maximum) at $x$ with $\nabla \varphi(x) \neq 0$, we have

$$G(\nabla \varphi(x), D^2 \varphi(x)) \leq f(x) \quad \text{(respectively \geq)};$$

(ii) or there exists a ball $B_{r}(x) \subset \Omega$ for some $r > 0$ such that $u$ is a constant function on $B_{r}(x)$, and $f \geq 0$ (respectively $f \leq 0$) on $B_{r}(x)$.

We say $u$ is a viscosity solution if $u$ is both a viscosity subsolution and a viscosity supersolution.

The adapted definition is equivalent to the usual viscosity solution when the operator $G$ defined in $TM \times_M \text{Sym} T M$ is continuous. To prove this fact, we modify the Euclidean argument in the proof of [DFQ2], Lemma 2.1.

**Lemma 3.2.** Let $\Omega \subset M$ be an open set, and $f \in C(\Omega)$. Let $G : TM \times_M \text{Sym} T M \rightarrow \mathbb{R}$ be continuous such that

$$G(0, 0) = 0 \quad \text{and} \quad G(\zeta, A) \leq 0$$

for all $\zeta \in T^*_x M \setminus \{0\}$ and $A \in \text{Sym} T M_x$, $x \in \Omega$. Then $u \in C(\Omega)$ is a viscosity solution of the equation

$$G(\nabla u, D^2 u) = f \quad \text{in} \ \Omega$$

if and only if $u \in C(\Omega)$ is a viscosity supersolution of the equation

$$G(\nabla u, D^2 u) = f \quad \text{in} \ \Omega.$$
for any \((\zeta, A) \in T_xM \times \text{Sym}TM_x\) with \(A \leq 0\) for any \(x \in \Omega\). Then \(u\) is a viscosity supersolution of

\[
G \left( \nabla u, D^2 u \right) = f \quad \text{in} \quad \Omega
\]

(6)

in the sense of Definition 3.1 if and only if \(u\) is a viscosity supersolution of (6) in the usual sense.

Proof. First we assume that \(u\) is a viscosity solution in the usual sense. Then it is not difficult to show \(u\) is a viscosity solution in the sense of Definition 3.1 using the assumption that \(G(0,0) = 0\).

We will prove that \(u\) is a viscosity solution in the usual sense if \(u\) is a viscosity solution in the sense of Definition 3.1. Let \(\varphi \in C^2(\Omega)\), and \(x_0 \in \Omega\) be such that \(u - \varphi\) has a local minimum at \(x_0\). We may assume that \(x_0\) is a strict local minimum of \(u - \varphi\). If \(u\) is a constant function in a small ball \(B_\rho(x_0) \subset \Omega\) \((\rho > 0)\), then \(\varphi\) has a local maximum at \(x_0\). Then we have \(D^2 \varphi(x_0) \leq 0\), and hence

\[
G \left( \nabla \varphi(x_0), D^2 \varphi(x_0) \right) = G \left( 0, D^2 \varphi(x_0) \right) \leq 0 \leq f(x_0)
\]

from the assumption of Definition 3.1 and Definition 3.1.

Now we assume that \(u\) is not constant in a small ball \(B_\rho(x_0) \subset \Omega\) and \(\nabla \varphi(x_0) = 0\) since there is nothing to prove in the case \(\nabla \varphi(x_0) \neq 0\). First, we consider the case that \(D^2 \varphi(x_0)\) is nonsingular. We introduce a coordinate map \(\psi : B_r(0) \subset \mathbb{R}^n \to B_\rho(x_0) \subset \Omega\) such that \(\psi(0) = x_0\), where we assume that \(\rho\) is sufficiently small. Define

\[
\tilde{u} := u \circ \psi \quad \text{and} \quad \tilde{\varphi} := \varphi \circ \psi.
\]

Then \(\tilde{u} - \tilde{\varphi}\) has a local minimum at \(0\) in \(B_\rho(0)\), and \(\tilde{u}\) is not constant in \(B_\rho(0)\). One can check that \(\nabla \tilde{\varphi}(0) = 0\), and \(D^2 \tilde{\varphi}(0)\) is nonsingular since \(\nabla \varphi(x_0) = 0\) and \(D^2 \varphi(x_0)\) is nonsingular. Then \(0 \in \mathbb{R}^n\) is the only critical point of \(\nabla \tilde{\varphi}\) in \(B_\rho(0) \subset \mathbb{R}^n\) for a small \(\rho' < \rho\). Using the argument in the proof of [DFQ2 Lemma 2.1], we find sequences \(\{\tilde{\varphi}_k\}_{k=1}^\infty \subset C^2(\overline{B}_{\rho'}(0))\), and \(\{\tilde{x}_k\}_{k=1}^\infty \subset B_{\rho'}(0)\), such that

\[
\tilde{u} - \tilde{\varphi}_k \quad \text{has a local minimum at} \quad \tilde{x}_k \quad \text{in} \quad B_{\rho'}(0) \subset \mathbb{R}^n,
\]

\[
\nabla \tilde{\varphi}_k(\tilde{x}_k) \neq 0 \quad \forall k = 1, 2, 3, \cdots,
\]

\[
\tilde{\varphi}_k \to \tilde{\varphi} \quad \text{in} \quad C^2(\overline{B}_{\rho'}(0)), \quad \text{and} \quad \tilde{x}_k \to 0 \quad \text{as} \quad k \to \infty.
\]

Thus there exist sequences \(\{\varphi_k := \tilde{\varphi}_k \circ \psi^{-1}|_{B_{\rho'}(x_0)}\}_{k=1}^\infty \subset C^2(\overline{B}_{\rho'}(x_0))\), and \(\{x_k := \psi(\tilde{x}_k)\}_{k=1}^\infty \subset B_{\rho'}(x_0)\) for a sufficiently small \(0 < \rho' < \rho\), such that

\[
u - \varphi_k \quad \text{has a local minimum at} \quad x_k \quad \text{in} \quad B_{\rho'}(x_0) \subset \Omega,
\]

\[
\nabla \varphi_k(x_k) \neq 0 \quad \forall k = 1, 2, 3, \cdots,
\]

\[
\varphi_k \to \varphi \quad \text{in} \quad C^2(\overline{B}_{\rho'}(x_0)), \quad \text{and} \quad x_k \to x_0 \quad \text{as} \quad k \to \infty.
\]

From Definition 3.1 it follows that

\[
G \left( \nabla \varphi_k(x_k), D^2 \varphi_k(x_k) \right) \leq f(x_k) \quad \forall k = 1, 2, \cdots,
\]

and hence the continuity of the operator \(G\) and the function \(f\) implies that

\[
G \left( \nabla \varphi(x_0), D^2 \varphi(x_0) \right) \leq f(x_0).
\]

In the case that \(D^2 \varphi(x_0)\) is singular, we consider for \(\delta > 0\),

\[
\varphi_\delta(x) := \varphi(x) - \delta d^2_{x_0}(x)/2 \quad \forall x \in B_{\rho}(x_0),
\]
where we assume that $\rho > 0$ is small enough. For $\delta > 0$, the function $u - \phi_{\delta}$ has a local minimum at $x_0$ in $B_\rho(x_0)$ with $\nabla \phi_{\delta}(x_0) = \nabla \phi(x_0) = 0$. Let $\delta_0 := \min_{\mu \neq 0} |\mu|$ for the eigenvalues $\mu_i$ of $D^2\phi(x_0)$. For $0 < \delta < \delta_0$, the Hessian $D^2\phi_{\delta}(x_0) = D^2\phi(x_0) - \delta I$ is nonsingular. Then we apply the previous argument to $\phi_{\delta}$ to obtain
\[
G\left(\nabla \phi(x_0), D^2\phi(x_0) - \delta I\right) = G\left(\nabla \phi_{\delta}(x_0), D^2\phi_{\delta}(x_0)\right) \leq f(x_0).
\]
Letting $\delta$ go to 0, we conclude that
\[
G\left(\nabla \phi(x_0), D^2\phi(x_0)\right) \leq f(x_0),
\]
which finishes the proof. \qed

Lemma 3.2 asserts that a viscosity solution $u$ to the $p$-Laplacian type equation for $p \geq 2$ with a continuous source term solves the equation in the usual viscosity sense.

For the operators with singularities ($1 < p < 2$), we make use of the following lemma in [ACP, Lemma 2]; see also [DFQ1].

**Lemma 3.3.** (ACP, Lemma 2) Let $1 < p < 2$, and let $u$ be a viscosity supersolution to $\Delta_p u \leq f$ in $\Omega$ with $f \geq 0$. Then $u$ is a viscosity supersolution to
\[
\left(\left|\nabla u(x)\right|^2 + \delta\right)^{\frac{p-2}{2}} \mathcal{M}_{p-1,1}(D^2u) \leq f \quad \text{in } \Omega
\]
for any $\delta > 0$.

According to Lemma 3.2, if $u$ is a viscosity supersolution to the $p$-Laplacian equation for $1 < p < 2$ with a nonnegative, continuous source term, then $u$ solves the regularized equations above in the usual viscosity sense.

Now we recall the sup- and inf-convolutions introduced by Jensen [J] (see also [CC, Chapter 5]) to approximate viscosity solutions. Let $\Omega \subset M$ be a bounded open set, and $u$ be continuous on $\overline{\Omega}$. For $\varepsilon > 0$, let $u_{\varepsilon}$ denote the inf-convolution of $u$ (with respect to $\Omega$) defined as follows: for $x_0 \in \overline{\Omega}$,
\[
u_{\varepsilon}(x_0) := \inf_{y \in \Omega} \left\{ u(y) + \frac{1}{2\varepsilon} d^2(y, x_0) \right\}.
\]
The sup-convolution of $u$ denoted by $u^\varepsilon$ is defined in a similar way. In the following lemmas, we quote the important properties of the inf-convolution from [KL, Section 3]; refer to [J] for the Euclidean case.

**Lemma 3.4.** Let $\Omega \subset M$ be a bounded open set, and $u \in C(\overline{\Omega})$. Assume that for any $x, y \in \overline{\Omega}$, the sectional curvature along the minimizing geodesic joining $x$ to $y$ has a uniform lower bound $-\kappa$ for $\kappa \geq 0$.

(a) $u_{\varepsilon} \uparrow u$ uniformly in $\overline{\Omega}$ as $\varepsilon \downarrow 0$.

(b) $u_{\varepsilon}$ is Lipschitz continuous in $\overline{\Omega}$: for $x_0, x_1 \in \overline{\Omega}$,
\[
|u_{\varepsilon}(x_0) - u_{\varepsilon}(x_1)| \leq \frac{3}{2\varepsilon} \text{diam}(\Omega)d(x_0, x_1).
\]

(c) $u_{\varepsilon}$ is $C_\varepsilon$-semiconcave in $\Omega$, where $C_\varepsilon := \frac{1}{\varepsilon} \sqrt{\kappa} \text{diam}(\Omega)\coth\left(\sqrt{\kappa} \text{diam}(\Omega)\right)$. For almost every $x \in \Omega$, $u_{\varepsilon}$ is differentiable at $x$, and there exists the Hessian $D^2u_{\varepsilon}(x)$ in the sense of Theorem 2.11 such that
\[
u_{\varepsilon}(\exp_\xi x) = u_{\varepsilon}(x) + \langle \nabla u_{\varepsilon}(x), \xi \rangle + \frac{1}{2} \left\{ D^2u_{\varepsilon}(x) \cdot \xi, \xi \right\} + o \left( |\xi|^2 \right)
\]
as $\xi \in T_xM \to 0$. 


(d) \( D^2 u_x \leq \frac{1}{\varepsilon} \sqrt{k \text{diam}(\Omega)} \coth \left( \sqrt{\text{diam}(\Omega)} \right) I \quad \text{a.e. in } \Omega. \)

(e) Let \( H \) be an open set such that \( \overline{H} \subset \Omega. \) Then there exist a smooth function \( \psi \) on \( M \) satisfying

\[
0 \leq \psi \leq 1 \quad \text{on } M, \quad \psi \equiv 1 \quad \text{in } \overline{H} \quad \text{and } \quad \text{supp } \psi \subset \Omega,
\]

and a sequence \( \{w_k\}_{k=1}^{\infty} \) of smooth functions on \( M \) such that

\[
\begin{align*}
\|D w_k\| &\leq C, \quad D^2 w_k \leq CI \\
D w_k &\rightarrow D u_x, \quad D^2 w_k \rightarrow D^2 u_x \quad \text{a.e. in } H \quad \text{as } k \rightarrow \infty,
\end{align*}
\]

where the constant \( C > 0 \) is independent of \( k. \)

We define the second order superjet and subjet of semi-continuous functions.

**Definition 3.5.** Let \( u : \Omega \rightarrow \mathbb{R} \) be a lower semi-continuous function on an open set \( \Omega \subset M. \) We define the second order subjet of \( u \) at \( x \in \Omega \) by

\[
\mathcal{F}^{2-} u(x) := \{ (\nabla \varphi(x), D^2 \varphi(x)) \in T_x M \times \text{Sym} T M_x : \varphi \in C^2(\Omega), \ u - \varphi \text{ has a local minimum at } x \}.
\]

If \((\zeta, A) \in \mathcal{F}^{2-} u(x), \) \( \zeta \) and \( A \) are called a first order subdifferential and a second order subdifferential of \( u \) at \( x, \) respectively.

Similarly, for an upper semi-continuous function \( u : \Omega \rightarrow \mathbb{R}, \) we define the second order superjet of \( u \) at \( x \in \Omega \) by

\[
\mathcal{F}^{2+} u(x) := \{ (\nabla \varphi(x), D^2 \varphi(x)) \in T_x M \times \text{Sym} T M_x : \varphi \in C^2(\Omega), \ u - \varphi \text{ has a local maximum at } x \}.
\]

From the proof of [KL, Proposition 3.3], we deduce the following lemma.

**Lemma 3.6.** Let \( H \) and \( \Omega \) be bounded open sets in \( M \) such that \( \overline{H} \subset \Omega. \) Assume that \( \text{Sec} \geq -\kappa \) on \( \Omega \) for \( \kappa \geq 0. \) Let \( u \in C(\overline{\Omega}) \) and let \( \omega \) denote a modulus of continuity of \( u \) on \( \Omega, \) which is nondecreasing on \((0, +\infty)\) with \( \omega(0+) = 0. \) For \( \varepsilon > 0, \) let \( u_\varepsilon \) be the inf-convolution of \( u \) with respect to \( \Omega. \) Then there exists \( \varepsilon_0 > 0 \) depending only on \( \|u\|_{L^\infty(\Omega)}, H, \) and \( \Omega, \) such that if \( 0 < \varepsilon < \varepsilon_0, \) then the following holds: let \( x_0 \in \overline{H}, \) and let \( y_0 \in \Omega \) be such that

\[
u_\varepsilon(x_0) = u(y_0) + \frac{1}{2\varepsilon} d^2(y_0, x_0)\]

(a) \( y_0 \) is an interior point of \( \Omega, \) and there is a unique minimizing geodesic joining \( x_0 \) to \( y_0, \) which is contained in \( \Omega. \)

(b) If \((\zeta, A) \in \mathcal{F}^{2-} u_\varepsilon(x_0), \) then \( y_0 = \exp_{x_0}(-\varepsilon \zeta), \) and

\[
\left( L_{x_0,y_0} \zeta, L_{x_0,y_0} A - \kappa \min \left\{ \varepsilon |\zeta|^2, 2\omega \left( 2 \sqrt{\varepsilon \|u\|_{L^\infty(\Omega)}} \right) \right\} I \right) \in \mathcal{F}^{2-} u(y_0),
\]

where \( L_{x_0,y_0} \) stands for the parallel transport along the unique minimizing geodesic joining \( x_0 \) to \( y_0, \) and \( L_{x_0,y_0} A \) is a symmetric bilinear form on \( T_{y_0} M \) defined as

\[
\left( (L_{x_0,y_0} A) \cdot \nu, \nu \right)_{y_0} := \left( A \cdot (L_{y_0,x_0} \nu) \right)_{y_0} \quad \forall \nu \in T_{y_0} M.
\]

Using Lemma 3.6 it can be proven that the inf-convolution solves approximated equation in the viscosity sense of Definition 3.1, provided that the sectional curvature is bounded from below, and the operator \( G \) is intrinsically uniformly continuous with respect to \( x. \) The intrinsic uniform continuity of the operator with respect to spatial variables is a natural
extension of the Euclidean concept of uniform continuity of the operator with respect to \( x \); we recall from [AFS] the definition and important examples.

**Definition 3.7.** The operator \( G \) is intrinsically uniformly continuous with respect to \( x \) in \((TM \setminus \{0\}) \times_M \text{Sym} TM\) if there exists a modulus of continuity \( \omega_G : [0, +\infty) \to [0, +\infty) \) with \( \omega_G(0+) = 0 \) such that

\[
G(\zeta, A) - G(L_{x,y}\zeta, L_{x,y}A) \leq \omega_G(d(x,y))
\]

for any \((\zeta, A) \in (T_xM \setminus \{0\}) \times \text{Sym} TM_x\) and \( x, y \in M \) with \( d(x,y) < \min \{i_M(x), i_M(y)\} \), where \( i_M(x) \) denotes the injectivity radius at \( x \) of \( M \).

**Example 3.8.** (a) When \( M = \mathbb{R}^n \), we have \( L_{x,y}\zeta \equiv \zeta, L_{x,y}A \equiv A \) so (7) holds.
(b) Consider the operator \( G \), which depends only on \(|\zeta|\) and the eigenvalues of \( A \in \text{Sym} TM \), of the form :

\[
G(\zeta, A) = \tilde{G}(|\zeta|, \text{eigenvalues of } A) \quad \text{for some } \tilde{G}.
\]

Since the parallel transport preserves inner products, we have \(|\zeta| = |L_{x,y}\zeta|\), and \( A \) and \( L_{x,y}A \) have the same eigenvalues. Thus the operator \( G \) satisfies intrinsic uniform continuity with respect to \( x \) with \( \omega_G \equiv 0 \). The trace and determinant of \( A \) are typical examples of the operator satisfying (8). The Pucci extremal operators also satisfy (8) and hence (7) with \( \omega_G \equiv 0 \).
(c) As seen before, the \( p \)-Laplace operator can be expressed as

\[
G(\nabla u, D^2 u) = |\nabla u|^{p-2} \text{tr} \left[ (I + (p-2)\frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}) D^2 u \right].
\]

One can check that for any \((\zeta, A) \in (T_xM \setminus \{0\}) \times \text{Sym} TM_x\) and \( x, y \in M \) with \( d(x,y) < \min \{i_M(x), i_M(y)\} \),

\[
\langle (L_{x,y}\zeta \otimes L_{x,y}\zeta) L_{x,y}A \cdot \nu, \nu \rangle_y = \langle (\zeta \otimes \zeta) A \cdot L_{x,y} \nu, L_{x,y} \nu \rangle_x \quad \forall \nu \in T_xM.
\]

Thus the \( p \)-Laplace operator is intrinsically uniformly continuous in \((TM \setminus \{0\}) \times_M \text{Sym} TM\) with \( \omega_G \equiv 0 \).

By making use of Lemma 3.6, we have the following lemma whose proof is similar to [KL] Lemma 3.6.

**Lemma 3.9.** Under the same assumption as Lemma 3.6, we also assume that \( G \) is intrinsically uniformly continuous in \((TM \setminus \{0\}) \times_M \text{Sym} TM\). Let \( f \in C(\Omega) \), and \( u \) be a viscosity supersolution to

\[
G(\nabla u, D^2 u) \leq f \quad \text{in } \Omega.
\]

If \( 0 < \varepsilon < \varepsilon_0 \), then \( u_\varepsilon \) is a viscosity supersolution of

\[
G_\varepsilon(\nabla u, D^2 u) := G(\nabla u, D^2 u - \kappa \min \{\varepsilon|\nabla u|^2, 2\omega(2\sqrt{em})\} I) \leq f_\varepsilon \quad \text{in } H,
\]

where \( \varepsilon_0 > 0 \) is the constant as in Lemma 3.6 and

\[
f_\varepsilon(x) := \sup_{\mathcal{P}_x \subset \mathcal{C}(x)} f^+ + \omega_G(2\sqrt{em}) \quad m := ||u||_{L^\infty(\Omega)}.
\]

Moreover, we have

\[
G(\nabla u_\varepsilon, D^2 u_\varepsilon - \kappa \min \{\varepsilon|\nabla u_\varepsilon|^2, 2\omega(2\sqrt{em})\} I) \leq f_\varepsilon \quad \text{a.e. in } H \cap \{||\nabla u_\varepsilon|| > 0\}.
\]
Remark 3.10. Under the same assumption as Lemma 3.9 if
\[ G(∇u, D^2u) := Δ_p u = |∇u|^{p-2} \text{tr} \left[ \left( I + \frac{p-2}{|∇u|} ∇u \otimes ∇u \right) D^2u \right], \]
then \( u_ε \) satisfies
\[ (9) \quad Δ_p u_ε - κ(n + p - 2)|∇u_ε|^{p-2} \min \left\{ ε|∇u_ε|^2, 2ω \left( 2 √|εm| \right) \right\} \leq f_ε \quad \text{in} \ H \]
in the viscosity sense, and almost everywhere in \( H \cap \{|∇u_ε| > 0\} \), where \( ω_G \equiv 0 \) from Example 3.8. When \( p \geq 2 \), the \( p \)-Laplacian operator is continuous, and hence (9) is satisfied almost everywhere in \( H \) by Lemma 3.2.

4. ABP type estimates

In this section, we establish the ABP type estimates for the \( p \)-Laplacian type operators for \( 1 < p < ∞ \). We begin with the definition of the \( p \)-contact set involving the \( \frac{p}{p-1} \)-th power of the distance.

Definition 4.1 (Contact sets). Let \( 1 < p < ∞ \). Let \( Ω \) be a bounded open set in \( M \) and \( u \in C(Ω) \). For a given \( a > 0 \) and a compact set \( E \subset M \), the \( p \)-contact set associated with \( u \) of opening \( a \) with the vertex set \( E \) is defined by
\[ \mathcal{A}_p^u \left( E; \overline{Ω}; u \right) := \left\{ x \in \overline{Ω} : \exists y \in E \text{ such that } \inf_{\overline{Ω}} \left\{ u + a \frac{p-1}{p} d^p_τ \right\} = u(x) + a \frac{p-1}{p} d^p_τ (x) \right\}. \]

When \( p = 2 \), a contact point is touched by a concave paraboloid from below, which is a squared distance function; refer to [WZ, Ca, K]. Notice that the \( p \)-contact set \( \mathcal{A}_p^u \left( E; \overline{Ω}; u \right) \) is a closed set.

4.1. Jacobian estimates. First, we study Jacobian estimates for certain exponential maps which arise in the proof of the ABP type estimate.

Proposition 4.2 (Jacobian estimate on the contact set). Let \( 1 < p < ∞ \), \( E \subset M \) be a compact set, and \( u \in C(Ω) \) be smooth in a bounded open set \( Ω \subset M \). Define \( ϕ_p : \{ z \in Ω : |∇u(z)| > 0 \} \times [0, 1] → M \) as
\[ ϕ_p(z, t) := \exp_z t|∇u(z)|^{p-2} ∇u(z). \]
Assume that \( x \in \mathcal{A}_p^u \left( E; \overline{Ω}; u \right) \cap \{ z \in Ω : |∇u(z)| > 0 \} \). Then the following holds.
(a) If \( y \in E \) satisfies
\[ \inf_{\overline{Ω}} \left\{ u + a \frac{p-1}{p} d^p_τ \right\} = u(x) + a \frac{p-1}{p} d^p_τ (x), \]
then \( y = \exp_x |∇u|^{p-2} ∇u(x) \in \text{Cut}(x) \cup \{ x \} \), and \( |∇u|^{p-2} ∇u(x) \in E_ε \).

The curve \( γ := ϕ_p(x, ·) : [0, 1] → M \) is a unique minimizing geodesic joining \( x \) to \( y = \exp_x |∇u|^{p-2} ∇u(x) \).
(b) \( |∇u|^{p-2} D^2u(x) + \left( I + \frac{2 - p}{p - 1} \frac{∇u}{|∇u|} \otimes \frac{∇u}{|∇u|} \right) D^2 \left( d^p_{ϕ_p(x, ·)}/2 \right)(x) \) is symmetric and positive semi-definite.
(c) 
\[ \text{Jac}_t \phi_p(x, 1) \]
\[ = \text{Jac} \exp_t \left( |\nabla u(x)|^{-2} \nabla u(x) \right) \]
\[ \cdot \det \left( |\nabla u|^{-2} \left( I + (p - 2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2 u(x) + D^2(2d_{\phi_x}(x, t)/2)(x) \right) \]
\[ = \text{Jac} \exp_t \left( |\nabla u(x)|^{-2} \nabla u(x) \right) \cdot \det \left( I + (p - 2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2(2d_{\phi_x}(x, t)/2)(x), \]

where \( \text{Jac} \exp_t \left( |\nabla u(x)|^{-2} \nabla u(x) \right) \) stands for the Jacobian determinant of \( \exp_t \), a map from \( T_xM \) to \( M \), at the point \( |\nabla u(x)|^{-2} \nabla u(x) \in T_xM \).

(d) For \( t \in [0, 1] \), the operator \( t|\nabla u|^{-2} D^2 u(x) + \left( I + t \frac{2 - p}{p - 1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2(2d_{\phi_x}(x, t)/2)(x) \)
is symmetric and positive semi-definite.

(e) If \( 1 < p < 2 \), then for \( t \in (0, 1] \)
\[ t|\nabla u(x)|^{-2} D^2 u(x) + \left( I + \frac{2 - p}{p - 1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2(2d_{\phi_x}(x, t)/2)(x), \]
is symmetric and positive semi-definite, and

\[ \text{Jac}_t \phi_p(x, t) \]
\[ = \text{Jac} \exp_t \left( t|\nabla u(x)|^{-2} \nabla u(x) \right) \]
\[ \cdot \det \left( t|\nabla u|^{-2} \left( I + (p - 2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2 u(x) + D^2(2d_{\phi_x}(x, t)/2)(x) \right) \]
\[ = \text{Jac} \exp_t \left( t|\nabla u(x)|^{-2} \nabla u(x) \right) \cdot \det \left( I + (p - 2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \]
\[ \cdot \det \left( t|\nabla u|^{-2} D^2 u(x) + \left( I + \frac{2 - p}{p - 1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2(2d_{\phi_x}(x, t)/2)(x) \right). \]

Proof. (a) For any \( x \in \mathcal{A}_p^p(E; \overline{\Omega}; u) \), we find \( y \in E \) such that

\[ \inf_{\Omega} \left\{ u + \frac{p - 1}{p} \frac{1}{d^p_{x\gamma}} \right\} = u(x) + \frac{p - 1}{p} d^p_{x\gamma}(x). \]

Then we have

\[ \frac{p - 1}{p} \left( d^p_{x\gamma}(z) - d^p_{x\gamma}(x) \right) \geq -u(z) + u(x) \quad \forall z \in \Omega. \]

We first claim that \( x \notin \text{Cut}(y) \) for \( x \in \mathcal{A}_p^p(E; \overline{\Omega}; u) \cap \Omega \) (see also [WZ]). Suppose to the contrary that \( x \in \text{Cut}(y) \). Note that \( x \neq y \) if \( x \in \text{Cut}(y) \). Letting \( \psi_p(x) := \frac{1}{p} \frac{1}{d^p_{x\gamma}} \), we have \( \psi_p > 0 \) in \((0, \infty)\). According to Corollary [28], there is a unit vector \( \Delta_x \in T_xM \) such that

\[ \liminf_{t \to 0} \frac{\psi \left( d^p_{x\gamma}(\exp_x t\Delta_x) \right) + \psi \left( d^p_{x\gamma}(\exp_x -t\Delta_x) \right) - 2 \psi \left( d^p_{x\gamma}(x) \right)}{t^2} = -\infty, \]
From (10), we have that for 
\[ \lim_{t \to 0} \frac{\psi \left( d_t^2 (\exp_x tX) \right) + \psi \left( d_t^2 (\exp_x -tX) \right) - 2 \psi \left( d_t^2 (x) \right) }{t^2} \geq - \lim_{t \to 0} \frac{u (\exp_x tX) + u (\exp_x -tX) - 2u (x) }{t^2} = - \langle D^2 u(x) \cdot X, X \rangle. \]

Hence \( x \) is not a cut point of \( y \).
Since \( x \not\in \text{Cut}(y) \) and \( p > 1 \), (10) yields that
\[ \nabla u(x) = -\frac{p-1}{p} \nabla d_y^{\frac{1}{p-1}}(x) \]
from which we see that
\[ y = x \iff \nabla u(x) = 0. \]
Thus for \( x \in \mathcal{A}_t^p \left( E; \Omega; u \right) \cap \{ z \in \Omega : |\nabla u(z)| > 0 \} \), we have \( y \not\in \text{Cut}(x) \cup \{ x \} \), and hence
\[ \nabla u(x) = -d_y^{\frac{1}{p-1}}(x) \nabla d_y(x), \quad |\nabla u(x)| = d_y^{\frac{1}{p-1}}(x), \]
and
\[ y = \exp_x \nabla d_y^2(x)/2 = \exp_x |\nabla u(x)|^{p-2} \nabla u(x) \not\in \text{Cut}(x) \cup \{ x \}, \]
which implies (a).

(b) From (10), we have that for \( y = \phi_p(x, 1) \),
\[ D^2 u(x) \geq -\frac{p-1}{p} D^2 d_y^{\frac{1}{p-1}}(x) \]
\[ = -d_y^{\frac{1}{p-1}}(x) \left\{ D^2(d_y^2/2)(x) + \frac{2 - p}{p-1} \nabla d_y(x) \otimes \nabla d_y(x) \right\} \]
\[ = -|\nabla u(x)|^{1-p} \left\{ D^2(d_y^2/2)(x) + \frac{2 - p}{p-1} \nabla d_y(x) \otimes \nabla d_y(x) \right\}, \]
where we recall the symmetric operator \( \nabla d_y(x) \otimes \nabla d_y(x) \) defined by
\[ \langle \nabla d_y(x) \otimes \nabla d_y(x) \cdot X, Y \rangle := \langle \nabla d_y(x), X \rangle \langle \nabla d_y(x), Y \rangle \quad \forall X, Y \in T_x \Omega. \]
Since \( \nabla d_y(x) \) is an eigenvector of \( D^2(d_y^2/2)(x) \) associated with the eigenvalue 1, we obtain that for \( X, Y \in T_x \Omega \),
\[ \langle \nabla d_y(x) \otimes \nabla d_y(x) \circ D^2(d_y^2/2)(x) \cdot X, Y \rangle = \langle \nabla d_y(x), D^2(d_y^2/2)(x) \cdot X \rangle \langle \nabla d_y(x), Y \rangle \]
\[ = \langle X, D^2(d_y^2/2)(x) \cdot \nabla d_y(x) \rangle \langle \nabla d_y(x), Y \rangle \]
\[ = \langle X, \nabla d_y(x) \rangle \langle \nabla d_y(x), Y \rangle \]
\[ = \langle \nabla d_y(x) \otimes \nabla d_y(x) \cdot X, Y \rangle, \]
and hence \( \nabla d_y(x) \otimes \nabla d_y(x) = \nabla d_y(x) \otimes \nabla d_y(x) \circ D^2(d_y^2/2)(x) = \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \circ D^2(d_y^2/2)(x) \).
Therefore, we deduce that
\[ 0 \leq |\nabla u(x)|^{p-2} D^2 u(x) + D^2(d_y^2/2)(x) + \frac{2 - p}{p-1} \nabla d_y(x) \otimes \nabla d_y(x) \]
\[ = |\nabla u(x)|^{p-2} D^2 u(x) + \left( 1 + \frac{2 - p}{p-1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2(d_y^2/2)(x) \]
which is symmetric.
(c) From Lemma 2.3, we have that for \( y = \phi_p(x, 1) \),

\[ \text{Jac}_z \phi_p(x, 1) = \text{Jac} \exp \left( |\nabla u(x)|^{p-2} \nabla u(x) \right) \cdot \det \left( |\nabla u(x)|^{p-2} \left( I + (p - 2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2 u(x) + D^2(\ell^2/2)(x) \right) \]

\[ = \text{Jac} \exp \left( |\nabla u(x)|^{p-2} \nabla u(x) \right) \cdot \det \left( I + (p - 2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \cdot \det \left( |\nabla u(x)|^{p-2} D^2 u(x) + \left( I + \frac{2 - p}{p - 1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2(\ell^2/2)(x) \right) \cdot \det \left( |\nabla u(x)|^{p-2} \right) \cdot \det \left( I + \frac{2 - p}{p - 1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2(\ell^2/2)(x) \right). \]

Thus (11) implies (c).

(d) According to Lemma 2.3, we see that for \( t \in [0, 1] \)

\[ D^2(\ell^2/2)(x) - t D^2(\ell^2/2)(x) \geq 0. \]

Since \( \nabla d_t(x) = -\frac{\nu u(x)}{|\nabla u|} \) is an eigenvector of \( D^2(\ell^2/2)(x) \) associated with the eigenvalue \( 1 \) for any \( t \in [0, 1] \), the above argument yields that

\[ \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \circ D^2(\ell^2/2)(x) = \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \circ D^2(\ell^2/2)(x). \]

From (11), we deduce that for \( t \in [0, 1] \)

\[ 0 \leq t |\nabla u(x)|^{p-2} D^2 u(x) + t \left( I + \frac{2 - p}{p - 1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2(\ell^2/2)(x) \]

\[ \leq t |\nabla u(x)|^{p-2} D^2 u(x) + \left( I + \frac{2 - p}{p - 1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2(\ell^2/2)(x). \]

(e) Since \( 1 < p < 2 \), (d) combined with (12) implies that

\[ t |\nabla u(x)|^{p-2} D^2 u(x) + \left( I + \frac{2 - p}{p - 1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2(\ell^2/2)(x) \geq 0 \quad \forall t \in (0, 1]. \]

Using Lemma 2.3 again, we have

\[ \text{Jac}_z \phi_p(x, t) = \text{Jac} \exp \left( t |\nabla u(x)|^{p-2} \nabla u(x) \right) \cdot \det \left( t |\nabla u(x)|^{p-2} \left( I + (p - 2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2 u(x) + D^2(\ell^2/2)(x) \right) \]

and

\[ \det \left( t |\nabla u(x)|^{p-2} \left( I + (p - 2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2 u(x) + D^2(\ell^2/2)(x) \right) \]

\[ = \det \left( I + (p - 2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \cdot \det \left( t |\nabla u(x)|^{p-2} D^2 u(x) + \left( I + \frac{2 - p}{p - 1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2(\ell^2/2)(x) \right), \]

completing the proof of (e).

\( \Box \)

Proposition 4.2 holds for semiconcave functions for almost all contact points.
Corollary 4.3. Let $1 < p < \infty$, $E \subset M$ be a compact set, and $u \in C(\overline{\Omega})$ be semiconcave in a bounded open set $\Omega \subset M$. Denote by $N$ a subset of $\Omega$ of measure zero such that $u$ is twice differentiable in $\Omega \setminus N$ in the sense of the Theorem 2.11. Define $\phi_p : \{ z \in \Omega \setminus N : \| \nabla u(z) \| > 0 \} \times [0, 1] \to M$ as

$$\phi_p(z,t) := \exp_z \{ t\| \nabla u(z) \|^p - \nabla u(z) \}.$$ 

Then Proposition 4.2 holds true for $x \in \mathcal{A}_p^p \left( E; \overline{\Omega}; u \right) \cap \{ z \in \Omega \setminus N : \| \nabla u(z) \| > 0 \}$.

**Proof.** Using the semi-concavity, note that for any $z \in \Omega$, and a unit vector $X \in T_Mz$,

$$\limsup_{t \to 0} \frac{u(\exp_z tX) - u(\exp_z X) - 2u(z)}{t^2} < \infty.$$ 

Fix a point $x \in \mathcal{A}_p^p \left( E; \overline{\Omega}; u \right) \cap \{ z \in \Omega \setminus N : \| \nabla u(z) \| > 0 \}$. Arguing similarly as in the proof of Proposition 4.2 together with the above property, if $y \in E$ satisfies

$$\inf_{\Omega} \left\{ u + \frac{p-1}{p} d_x^\gamma \right\} = u(x) + \frac{p-1}{p} d_x^\gamma(x),$$

then $y = \phi_p(x, 1) = \exp_x \{ \| \nabla u \| p - \nabla u \} \notin \text{Cut}(x) \cup \{ x \}$, and $\{ \| \nabla u \| p - \nabla u \} \in E_x \setminus \{ 0 \}$. Using Lemma 2.3, we deduce that for $(0, 1)$

$$d\phi_p(x, t) = d\exp_x \{ t\| \nabla u \| p - \nabla u \} \cdot \left\{ t\| \nabla u \| p - \nabla u + \frac{\nabla u}{\| \nabla u \|} \otimes \frac{\nabla u}{\| \nabla u \|} \right\} D^2u(x) + \frac{d^2}{dt^2}(d^2\phi_p(x, t)/2)(x) \right\},$$

where $\gamma(t) := \phi_p(x, t)$ is a minimizing geodesic joining $x$ to $\phi_p(x, 1) \notin \text{Cut}(x) \cup \{ x \}$, and $D^2u(x)$ is the Hessian in the sense of Theorem 2.11; see also the proof of [CMS, Proposition 4.1]. The rest of the proof is the same as for Proposition 4.2. \hfill \Box

4.2. **Degenerate operators.** This subsection is devoted to the ABP type estimate for the $p$-Laplacian type operators with $2 \leq p < \infty$.

**Theorem 4.4 (ABP type estimate).** Assume that $2 \leq p < \infty$ and $\text{Ric} \geq -(n-1)\kappa$ for $\kappa \geq 0$.

For a bounded open set $\Omega \subset M$, let $u \in C(\overline{\Omega})$ be smooth in $\Omega$. For a compact set $E \subset M$,

we assume that

$$\mathcal{A}_p^p \left( E; \overline{\Omega}; u \right) \subset \Omega.$$ 

Then we have

$$|E| \leq \int_{\mathcal{A}_p^p \left( E; \overline{\Omega}; u \right)} \mathcal{H}^{n-1} \left( \sqrt{\nabla u(x)} \| \nabla u \|^p - \nabla u \right) \left\{ \frac{\Delta_p u(x)}{n} + \mathcal{H} \left( \sqrt{\nabla u(x)} \| \nabla u \|^p - \nabla u \right) \right\}^n dx,$$

where $\mathcal{H}(\tau) := \tau \coth(\tau)$, and $\mathcal{H}(\tau) := \sinh(\tau)/\tau$ for $\tau > 0$ with $\mathcal{H}(0) = \mathcal{H}(0) = 1$. In particular, if $\text{Ric} \geq 0$, we have

$$|E| \leq \int_{\mathcal{A}_p^p \left( E; \overline{\Omega}; u \right)} \left\{ \frac{\Delta_p u(x)}{n} + 1 \right\}^n dx.$$ 

**Proof.** For any $x \in \mathcal{A}_p^p \left( E; \overline{\Omega}; u \right) \subset \Omega$, we find $y \in E$ such that

$$\inf_{\Omega} \left\{ u + \frac{p-1}{p} d_x^\gamma \right\} = u(x) + \frac{p-1}{p} d_x^\gamma(x).$$

From the argument as in the proof of Proposition 4.2 we have

$$y = x \iff \nabla u(x) = 0.$$
When \(|\nabla u(x)| = 0\) for \(x \in \mathcal{A}_1^p (E; \overline{\Omega}; u) \subset \Omega\), we have \(y = x = \exp|\nabla u|^p-2 \nabla u(x)\) in (13). If \(|\nabla u(x)| > 0\) for \(x \in \mathcal{A}_1^p (E; \overline{\Omega}; u) \subset \Omega\), then Proposition 4.2 says that
\[
y = \exp|\nabla u|^p-2 \nabla u(x) \notin \text{Cut}(x) \cup \{x\}, \quad \text{and} \quad |\nabla u|^p-2 \nabla u(x) \in \mathcal{E}_x.
\]

Now we define the map \(\Phi_p : \mathcal{A}_1^p (E; \overline{\Omega}; u) \to M\)
\[
\Phi_p(x) := \exp|\nabla u(x)|^p-2 \nabla u(x),
\]
which coincides with \(\phi_p(x, 1)\) in Proposition 4.2. From the argument above and the definition of \(\mathcal{A}_1^p (E; \overline{\Omega}; u)\), we observe that
\[
(14) \quad E = \Phi_p \left( \mathcal{A}_1^p (E; \overline{\Omega}; u) \right).
\]

since we assume \(\mathcal{A}_1^p (E; \overline{\Omega}; u) \subset \Omega\).

By using Proposition 4.2 together with Theorem 2.1, the arithmetic-geometric means inequality yields that for \(x \in \mathcal{A}_1^p (E; \overline{\Omega}; u) \cap \{z \in \Omega : |\nabla u(z)| > 0\},\)
\[
\text{Jac} \Phi_p(x) = \text{Jac} \exp_\nabla \left(|\nabla u(x)|^p-2 \nabla u(x)\right)
\]
\[
\cdot \text{det} \left\{ |\nabla u|^p-2 \left( I + (p - 2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \right\}
\]
\[
= \text{Jac} \exp_\nabla \left(|\nabla u(x)|^p-2 \nabla u(x)\right) \cdot \text{det} \left\{ I + (p - 2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right\}
\]
\[
\cdot \text{det} \left\{ |\nabla u|^p-2 D^2 u(x) + \left( I + \frac{2 - p}{p - 1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \right\}
\]
\[
\leq \mathcal{J}^{n-1} \left( |\nabla \nabla u(x)|^p-1 \right) \left[ \frac{1}{n} \text{tr} \left\{ |\nabla u|^p-2 \left( I + (p - 2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \right\} \right]^n,
\]

where \(y = \Phi_p(x)\), and we used at the last line the arithmetic-geometric means inequality:
\[
\text{det}(AB) \leq \left\{ \frac{1}{n} \text{tr}(AB) \right\}^n
\]
for symmetric nonnegative matrices \(A\) and \(B\). From Lemma 2.6 it follows that for any \(x \in \mathcal{A}_1^p (E; \overline{\Omega}; u) \cap \{z \in \Omega : |\nabla u(z)| > 0\},\)
\[
\text{Jac} \Phi_p(x) \leq \mathcal{J}^{n-1} \left( |\nabla \nabla u(x)|^p-1 \right) \left\{ \frac{\Delta_p u(x) + 1 + (n - 1) \mathcal{H} \left( \frac{1}{n} \right)}{n} \right\}^n
\]
\[
= \mathcal{J}^{n-1} \left( |\nabla \nabla u(x)|^p-1 \right) \left\{ \frac{\Delta_p u(x) + 1 + (n - 1) \mathcal{H} \left( \frac{1}{n} \right)}{n} \right\}^n.
\]

Since \(\Delta_p u\) is continuous in \(\Omega\) for \(p \geq 2\), we deduce that
\[
(15) \quad \text{Jac} \Phi_p \leq \mathcal{J}^{n-1} \left( |\nabla \nabla u(x)|^p-1 \right) \left\{ \frac{\Delta_p u(x) + 1 + (n - 1) \mathcal{H} \left( |\nabla \nabla u(x)|^p-1 \right)}{n} \right\}^n \quad \text{in} \quad \mathcal{A}_1^p (E; \overline{\Omega}; u)
\]

since \(\mathcal{H}(\tau) \geq 1\) for \(\tau \geq 0\). For \(p \geq 2\), the map \(\Phi_p\) is of class \(C^1\), and hence Lipschitz continuous on a compact set \(\mathcal{A}_1^p (E; \overline{\Omega}; u) \subset \Omega\); notice that \(|\nabla u|^p-2 \nabla u(x) \in \mathcal{E}_x\) for any \(x \in \mathcal{A}_1^p (E; \overline{\Omega}; u)\). Therefore, the result follows from the area formula with the help of (13) and (15). \(\Box\)
The following ABP type estimate is concerned with the nonlinear operators of $p$-Laplacian type.

**Corollary 4.5.** Assume that $2 \leq p < \infty$ and $M^*_{1, \Lambda}(R(e)) \geq -(n-1)\kappa$ with $\kappa \geq 0$ for any unit vector $e \in TM$. For a bounded open set $\Omega \subset M$, let $u \in C(\overline{\Omega})$ be smooth in $\Omega$ such that $|\nabla u|^p \geq M^*_{1, \Lambda}(D^2 u) \leq f$ in $\Omega$. For a compact set $E \subset M$, we assume that $A^p_t(E; \overline{\Omega}; u) \subset \Omega$. Then we have

$$|E| \leq \int_{\mathcal{A}^p_t(E; \overline{\Omega}; u)} (p-1) \mathcal{J}^n \left( \frac{\kappa}{\Lambda} |\nabla u|^p \right) \left( f^+ + \frac{\Lambda}{p-1} + (n-1)\mathcal{H} \left( \sqrt{\frac{\kappa}{\Lambda}} |\nabla u|^p \right) \right)^n \, dx.$$

In particular, if $\text{Ric} \geq 0$, we have

$$|E| \leq \int_{\mathcal{A}^p_t(E; \overline{\Omega}; u)} \left( \frac{f^+ (x)}{n\Lambda} + \frac{\Lambda}{n} \right)^n \, dx.$$

**Proof.** Following the proof of Theorem 4.3, it suffices to estimate the Jacobian determinant of $\Phi_p(x) := \exp_x (|\nabla u(x)|^{p-2} \nabla u(x))$ on $\mathcal{A}^p_t(E; \overline{\Omega}; u)$ in terms of the Pucci operator. First we note that $\text{Ric}(e, e) \geq M^*_{1, \Lambda}(R(e))/\Lambda \geq -(n-1)\kappa/\Lambda$ for any unit vector $e \in TM$. According to Proposition 4.2 and Theorem 2.1 we have that for any $x \in \mathcal{A}^p_t(E; \overline{\Omega}; u) \cap \{ z \in \Omega : |\nabla u(z)| > 0 \}$,

$$\text{Jac} \Phi_p(x) = \text{Jac} \exp_x (|\nabla u(x)|^{p-2} \nabla u(x)) \cdot \text{det} \left( I + (p-2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \cdot \text{det} \left( |\nabla u|^p D^2 u \right) + \left( I + \frac{2-p}{p-1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2 (d^2/2)(x) \right) \right)^n$$

$$\leq (p-1) \mathcal{J}^n \left( \frac{\kappa}{\Lambda} |\nabla u|^p \right) \left( \frac{1}{n\Lambda} M_{1, \Lambda} \left( |\nabla u|^p D^2 u \right) + \left( I + \frac{2-p}{p-1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2 (d^2/2)(x) \right) \right)^n$$

$$\leq \mathcal{J}^n \left( \frac{\kappa}{\Lambda} |\nabla u|^p \right) \left( \frac{1}{n\Lambda} M_{1, \Lambda} \left( |\nabla u|^p D^2 u \right) + \left( I + \frac{2-p}{p-1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2 (d^2/2)(x) \right) \right)^n,$$

where $y = \Phi_p(x)$. From (12), we notice that for $x \in \mathcal{A}^p_t(E; \overline{\Omega}; u) \cap \{ z \in \Omega : |\nabla u(z)| > 0 \}$,

$$\left( I + \frac{2-p}{p-1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2 d^2/2(x) = D^2 d^2/2(x) + \frac{2-p}{p-1} \nabla d_i(x) \otimes \nabla d_i(x)$$

since $\nabla d_i(x) = -\frac{\nabla u_i(x)}{|\nabla u|}$ is an eigenvector of $D^2 d^2/2(x)$ associated with the eigenvalue $1$ for $y = \Phi_p(x) = \exp_x (|\nabla u|^p D^2 u)$. By using Lemma 2.6, we deduce that

$$M_{1, \Lambda} \left( I + \frac{2-p}{p-1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2 (d^2/2)(x) \leq \frac{\Lambda}{p-1} + (n-1)\Lambda \mathcal{H} \left( \sqrt{\frac{\kappa}{\Lambda}} d_i(x) \right),$$

and hence for any $x \in \mathcal{A}^p_t(E; \overline{\Omega}; u) \cap \{ z \in \Omega : |\nabla u(z)| > 0 \}$,

$$\text{Jac} \Phi_p(x) \leq \frac{(p-1) \mathcal{J}^n \left( \frac{\kappa}{\Lambda} |\nabla u|^p \right) \left( f^+ + \frac{\Lambda}{p-1} + (n-1)\mathcal{H} \left( \sqrt{\frac{\kappa}{\Lambda}} |\nabla u|^p \right) \right)^n}{n\Lambda \kappa \mathcal{J}^n \left( \frac{\kappa}{\Lambda} |\nabla u|^p \right) \left( f^+ + \frac{\Lambda}{p-1} + (n-1)\mathcal{H} \left( \sqrt{\frac{\kappa}{\Lambda}} |\nabla u|^p \right) \right)^n}.$$

Since $p \geq 2$, this estimate holds for any $x \in \mathcal{A}^p_t(E; \overline{\Omega}; u)$, and then the result follows from (14) and the area formula.
From Theorem 4.4, we obtain the ABP type estimate of viscosity supersolutions for \( p \)-Laplacian operators with the help of regularization by inf-convolution.

**Lemma 4.6.** Assume that \( 2 \leq p < \infty \) and \( \text{Ric} \geq -(n-1)\kappa \) for \( \kappa \geq 0 \). For \( z_0, x_0 \in M \) and \( 0 < r \leq R \leq R_0 \), assume that \( \overline{B}_r(z_0) \subset B_R(x_0) \). Let \( f \in C(B_R(x_0)) \) and \( u \in C(\overline{B}_R(x_0)) \) be such that

\[
\Delta_p u \leq f \quad \text{in} \ B_R(x_0)
\]
in the viscosity sense,

Then

\[
|B_r(z_0)| \leq \mathcal{S}^{n-1}(2\sqrt{\kappa}R_0) \int_{[u < M_p \cap B_{4R_0}(z_0)]} \left\{ \mathcal{H} \left( 2\sqrt{\kappa}R_0 \right) + \frac{r^p \bar{f}^+}{n} \right\}^n
\]

for a uniform constant \( M_p := \frac{p-1}{p} \frac{3^{\frac{1}{p}}}{\kappa} \). Moreover, if \( r^p \bar{f} \leq 1 \) in \( B_{4r}(z_0) \), then there exists a uniform constant \( \delta \in (0, 1) \) depending only on \( n, p \) and \( \sqrt{n}R_0 \), such that

\[
|\{ u \leq \tilde{M}_p \cap B_{4r}(z_0) \} > \delta |B_{4r}(z_0)|.
\]

**Proof.** (i) First, we claim that for \( u \in C^\infty(B_R(x_0)) \cap C(\overline{B}_R(x_0)) \) satisfying (16),

\[
|B_r(z_0)| \leq \mathcal{S}^{n-1}(2\sqrt{\kappa}R_0) \int_{[u < \tilde{M}_p \cap B_{4R_0}(z_0)]} \left\{ \mathcal{H} \left( 2\sqrt{\kappa}R_0 \right) + \frac{r^p \bar{f}^+}{n} \right\}^n
\]

According to Theorem 4.4, it suffices to prove that

\[
\mathcal{A}_p^\gamma \left( \overline{B}_r(z_0); \overline{B}_R(x_0); r\overline{\Delta}_p u \right) \subset B_{4r}(z_0) \cap \left\{ u < \tilde{M}_p, \ r^p |\nabla u|^{p-1} < 2R_0 \right\}
\]

since \( \mathcal{S} \) and \( \mathcal{H} \) are nondecreasing functions in \([0, \infty)\). Indeed, for a fixed \( y \in \overline{B}_r(z_0) \), we consider

\[
w_y := r^{\frac{p}{n-1}} u + \frac{p-1}{p} r^{\frac{p}{n-1}} \frac{1}{\gamma}.
\]

Using (16), we have

\[
r^{\frac{p}{n-1}} w_y \geq 0 + \frac{p-1}{p} r^{\frac{p}{n-1}} \quad \text{outside} \ B_{4r}(z_0),
\]

\[
\inf_{B_{4r}(z_0)} r^{\frac{p}{n-1}} w_y \leq \frac{p-1}{p} \left( r^{\frac{p}{n-1}} \frac{1}{\gamma} \right) < \frac{p-1}{p} \frac{3^{\frac{1}{p}}}{\kappa}.
\]

Then we find a point \( x \in B_{4r}(z_0) \) such that

\[
\inf_{B_{4r}(z_0)} r^{\frac{p}{n-1}} w_y = r^{\frac{p}{n-1}} w_y(x) = u(x) + \frac{p-1}{p} \left( \frac{d_i(x)}{r} \right)^{\frac{p}{n-1}} < \frac{p-1}{p} \frac{3^{\frac{1}{p}}}{\kappa}.
\]

Proposition 4.2 implies that \( r^{\frac{p}{n-1}} |\nabla u|^{p-1} = d_i(x) < 8r \leq 2R_0 \). Then we deduce that

\[
\mathcal{A}_p^\gamma \left( \overline{B}_r(z_0); \overline{B}_{4r}(z_0); r\overline{\Delta}_p u \right) \subset B_{4r}(z_0) \cap \left\{ u < \tilde{M}_p, \ r^p |\nabla u|^{p-1} < 2R_0 \right\}
\]

for \( \tilde{M}_p := \frac{p-1}{p} \frac{3^{\frac{1}{p}}}{\kappa} > 1 \). Thus, (19) follows from Theorem 4.4.
(ii) Now we assume that $u$ is a continuous viscosity supersolution. Let $\eta > 0$. According to Lemma 3.4 and Remark 3.10, the inf-convolution $u_\varepsilon$ of $u$ with respect to $\overline{B}_R(x_0)$ (for small $\varepsilon > 0$) satisfies

$$
\begin{align*}
-u_\varepsilon & \to u \quad \text{uniformly in } B_R(x_0), \\
-u_\varepsilon & \geq -\eta \quad \text{in } B_R(x_0) \setminus B_{4\varepsilon}(z_0), \\
\inf_{B_\varepsilon(z_0)} u_\varepsilon & \leq \frac{p-1}{p} + \eta, \\
\Delta_p u_\varepsilon - \varepsilon \tilde{k}(n + p - 2)|\nabla u_\varepsilon|^p & \leq f_\varepsilon \quad \text{a.e. in } B_{4\varepsilon}(z_0),
\end{align*}
$$

where $-\tilde{k} (\tilde{k} \geq 0)$ is a lower bound of the sectional curvature on $\overline{B}_R(x_0)$, and

$$
f_\varepsilon(z) := \sup_{B_{\varepsilon}(z)} f^+, \quad \forall z \in B_{4\varepsilon}(z_0); \quad m := \|u\|_{L^p(B_{6\varepsilon}(x_0))}.
$$

According to Lemma 3.4, we approximate $u_\varepsilon$ by a sequence $\{w_k\}_{k=1}^\infty$ of smooth functions which satisfy the following:

$$
\begin{align*}
w_k & \to u_\varepsilon \quad \text{uniformly in } B_{4\varepsilon}(z_0), \\
w_k & \geq -2\eta \quad \text{in } B_R(x_0) \setminus B_{4\varepsilon}(z_0), \\
\inf_{B_\varepsilon(z_0)} w_k & \leq \frac{p-1}{p} + 2\eta, \\
|\nabla w_k| & \leq C, \quad D^2 w_k \leq C I \quad \text{uniformly in } M \text{ with respect to } k, \\
\nabla w_k & \to \nabla u_\varepsilon, \quad D^2 w_k \to D^2 u_\varepsilon \quad \text{a.e. in } B_{4\varepsilon}(z_0) \text{ as } k \to \infty,
\end{align*}
$$

for a uniform constant $C > 0$ independent of $k \in \mathbb{N}$. Then we apply (19) to the function

$$
\tilde{w}_k := \frac{(p-1)/p}{(p-1)/p+4\eta}(w_k+2\eta)
$$

to obtain

$$
|B_r(z_0)| \leq \mathcal{S}^{n-1}(2\sqrt{R_0}) \int_{|\bar{u}_k| \leq M_\varepsilon, \ n|\nabla \bar{u}_k|^p \leq 2R_0} \left\{ \mathcal{H} \left( 2\sqrt{R_0} \right) + \frac{r^p \left( \Delta_p \tilde{w}_k \right)^+}{n} \right\}^n.
$$

Letting $k \to \infty$ and $\bar{u}_\varepsilon := \frac{(p-1)/p}{(p-1)/p+4\eta}(u_\varepsilon+2\eta)$, we deduce that

$$
|B_r(z_0)| \leq \mathcal{S}^{n-1}(2\sqrt{R_0}) \int_{|u_\varepsilon| \leq M_\varepsilon, \ n|\nabla u_\varepsilon|^p \leq 2R_0} \left\{ \mathcal{H} \left( 2\sqrt{R_0} \right) + \frac{r^p \left( \Delta_p \bar{u}_\varepsilon \right)^+}{n} \right\}^n
$$

since $\left( \Delta_p \tilde{w}_k \right)^+$ is uniformly bounded with respect to $k$, and converges to $\left( \Delta_p \bar{u}_\varepsilon \right)^+$ almost everywhere in $B_{4\varepsilon}(z_0)$ as $k$ tends to $\infty$. Since

$$
\left( \Delta_p \bar{u}_\varepsilon \right)^+ = \frac{(p-1)/p}{(p-1)/p+4\eta} \left( \Delta_p u_\varepsilon \right)^+ \\
\leq \frac{(p-1)/p}{(p-1)/p+4\eta} \left( \varepsilon \tilde{k}(n + p - 2)|\nabla u_\varepsilon|^p + f_\varepsilon \right) \\
\leq \frac{(p-1)/p+4\eta}{(p-1)/p} \varepsilon \tilde{k}(n + p - 2)(r^{-p}2R_0)^{\frac{2}{p-1}} + \frac{(p-1)/p}{(p-1)/p+4\eta} f_\varepsilon
$$

almost everywhere in $\left\{ r^p |\nabla \bar{u}_\varepsilon|^p \leq 2R_0 \right\} \cap B_{4\varepsilon}(z_0)$, we deduce that

$$
|B_r(z_0)| \leq \mathcal{S}^{n-1}(2\sqrt{R_0}) \int_{(u_\varepsilon \leq M_\varepsilon) \cap B_{4\varepsilon}(z_0)} \left\{ \mathcal{H} \left( 2\sqrt{R_0} \right) + \frac{r^p f^+}{n} \right\}^n
$$

by letting $\varepsilon$ and $\eta$ go to 0. Lastly, the above estimate implies (13) with the help of Bishop-Gromov’s volume comparison. $\square$
We also have the ABP type estimate for viscosity supersolutions to nonlinear $p$-Laplacian type equations.

**Corollary 4.7.** Assume that $2 \leq p < \infty$, and $\mathcal{M}_{\lambda, \Lambda}(R(\varepsilon)) \geq -(n - 1) \kappa$ with $\kappa \geq 0$ for any unit vector $e \in TM$. For $z_0, x_0 \in M$ and $0 < r \leq R \leq R_0$, assume that $\overline{B}_{\delta}(z_0) \subset B_R(x_0)$. Let $\beta \geq 0$, and $u \in C(\overline{B}_{\delta}(x_0))$ be such that

$$|\nabla u|^{p-2} \mathcal{M}_{\lambda, \Lambda}(D^2 u) - \beta|\nabla u|^{p-1} \leq r^{-p} \quad \text{in } B_R(x_0)$$

in the viscosity sense,

(21) \quad $u \geq 0$ on $B_R(x_0) \setminus B_{\delta}(z_0)$ and $\inf_{B_{\delta}(z_0)} u \leq \frac{p - 1}{p}$.

Then there exists a uniform constant $\delta \in (0, 1)$ depending only on $n, p, \sqrt{R_0}, \lambda, \Lambda$, and $\beta R_0$, such that

$$\left| \left\{ u \leq \bar{M}_p \right\} \cap B_{\delta}(z_0) \right| \geq \delta |B_{\delta}(z_0)|$$

for $\bar{M}_p = \frac{p - 1}{p} \frac{3^{\frac{2}{n}}}{R_0^\frac{n}{p}} > 1$.

**Proof.** The proof is similar to one for Lemma 4.6 replacing Theorem 4.4 by Corollary 4.5. We use the same notation as in the proof of Lemma 4.6 and notice that (20) follows from (21). As in the proof of Lemma 4.6 let $\eta > 0$, and let $u_\varepsilon$ be the inf-convolution of $u$ with respect to $\overline{B}_{\delta}(x_0)$ for small $\varepsilon > 0$. Applying Corollary 4.5 to approximating smooth functions for $u_\varepsilon$ from Lemma 4.6 we deduce that for sufficiently small $\varepsilon > 0$,

$$|B_{\delta}(z_0)| \leq \frac{(p - 1)}{r^n} \Lambda^{\delta - 1} \left\{ \left\{ u_\varepsilon \leq \bar{M}_p \right\} \cap B_{\delta}(z_0) \right\} \left( \frac{\Lambda}{(n - 1) \Lambda, \mathcal{H}} 2 \sqrt{\frac{\kappa}{\Lambda}} R_0 + \frac{r^\prime}{|\nabla u_\varepsilon|^{p-2} \left( \mathcal{M}_{\lambda, \Lambda}(D^2 u_\varepsilon) \right)^+} \right)^n,$$

where $u_\varepsilon := \frac{(p - 1)}{r^n} (u_\varepsilon + 2\eta)$. Since the Pucci operators are intrinsically uniformly continuous with $\omega_G \equiv 0$, Lemma 5.9 implies that

$$|\nabla u_\varepsilon|^{p-2} \left( \mathcal{M}_{\lambda, \Lambda}(D^2 u_\varepsilon) \right)^+ = \left( \frac{(p - 1)}{r^n} \right) \frac{(p - 1)}{(p - 1)/p + 4\eta} \frac{(p - 1)/p + 4\eta}{(p - 1)/p + 4\eta}$$

almost everywhere in $\left\{ r^\prime |\nabla u_\varepsilon|^{p-1} \leq 2R_0 \right\} \cap B_{\delta}(z_0)$. Therefore, letting $\varepsilon$ and $\eta$ go to 0, we obtain

$$|B_{\delta}(z_0)| \leq \frac{(p - 1)}{r^n} \Lambda^{\delta - 1} \left\{ \left\{ u_\varepsilon \leq \bar{M}_p \right\} \cap B_{\delta}(z_0) \right\} \left( \frac{\Lambda}{(n - 1) \Lambda, \mathcal{H}} 2 \sqrt{\frac{\kappa}{\Lambda}} R_0 + 2\beta R_0 + \left( \frac{(p - 1)}{(p - 1)/p + 4\eta} \right) \frac{r^\prime}{|\nabla u_\varepsilon|^{p-2} \left( \mathcal{M}_{\lambda, \Lambda}(D^2 u_\varepsilon) \right)^+} \right)^n,$$

from which the result follows by using Bishop–Gromov’s volume comparison. \nobreak \hfill \Box
4.3. **Singular operators.** Now we consider the $p$-Laplacian type operator for $1 < p < 2$. In this case, the operator $\Delta_p u$ becomes singular when its gradient vanishes. To deal with singularities, we make use of a regularized operator

$$
\left( |\nabla u(x)|^2 + \delta \right)^{\frac{p-2}{2}} \mathcal{M}_{p-1,1}(D^2 u)
$$

for any $\delta > 0$; see Lemma 3.3. So we first show the ABP type estimate for nonlinear $p$-Laplacian type operators.

**Theorem 4.8** (ABP type estimate). Assume that $1 < p < 2$ and $\mathcal{M}_{1,\lambda}(R(e)) \geq -(n-1)\kappa$ with $\kappa \geq 0$ for any unit vector $e \in TM$. For a bounded open set $\Omega \subset M$, let $f \in C(\Omega)$, and $u \in C[\bar{\Omega}]$ be smooth in $\Omega$ such that $|\nabla u|^{p-2} \mathcal{M}_{1,\lambda}(D^2 u) \leq f$ in $\Omega$ in the viscosity sense. For a compact set $E \subset M$, we assume that

$$
\mathcal{A}_E^p (E; \bar{\Omega}; u) \subset \Omega.
$$

Then we have

$$
|E| \leq \int_{\mathcal{A}_E^p (E; \bar{\Omega}; u)} \mathcal{H}^{n-1} \left( \sqrt{\nabla u} \right) \left| \frac{1}{n\lambda} \left\{ f^+ \frac{\Lambda}{p-1} + (n-1)\lambda \mathcal{H} \left( \sqrt{\nabla u} \right) \right\} \right|^p.
$$

**Proof.** From Definition 3.1, it is not difficult to see that $u$ is a viscosity supersolution to

$$
\left( |\nabla u|^2 + \delta \right)^{\frac{p-2}{2}} \mathcal{M}_{1,\lambda}(D^2 u) \leq f^+ \quad \text{in } \Omega
$$

for any $\delta > 0$ since $1 < p < 2$; refer to the proof of [ACP, Lemma 2]. Moreover, $u$ is a viscosity supersolution to (22) in the usual sense according to Lemma 3.2 since the regularized operator (for a given $\delta > 0$) is continuous with respect to $\nabla u$ and $D^2 u$. Thus, we deduce that $u$ solves (22) in the classical sense.

For $\delta > 0$, define a regularized map $\Phi_{p,\delta} : \mathcal{A}_E^p (E; \bar{\Omega}; u) \to M$ as

$$
\Phi_{p,\delta}(x) := \exp_x \left( |\nabla u(x)|^2 + \delta \right)^{\frac{p-2}{2}} \nabla u(x).
$$

Note that if $|\nabla u(x)| > 0$, then

$$
\Phi_{p,\delta}(x) = \phi_p (x, t_\delta(x)) \quad \text{for } t_\delta(x) := \left( \frac{|\nabla u(x)|^2}{|\nabla u(x)|^2 + \delta} \right)^{\frac{p-2}{2}} \in (0, 1),
$$

where $\phi_p (x, t) := \exp_x \eta |\nabla u(x)|^{p-2} \nabla u(x)$ for $t \in [0, 1]$ as in Proposition 4.2. We also define $\Phi_{p,0} : \mathcal{A}_E^p (E; \bar{\Omega}; u) \cap \{|\nabla u| > 0\} \to M$ by

$$
\Phi_{p,0}(x) := \exp_x |\nabla u(x)|^{p-2} \nabla u(x).
$$

For any $x \in \mathcal{A}_E^p (E; \bar{\Omega}; u) \subset \Omega$, there is $y \in E$ such that

$$
\inf_{\Omega} \left\{ u + \frac{p-1}{p} d_{\mathcal{F}}(x) \right\} = u(x) + \frac{p-1}{p} d_{\mathcal{F}}(x).
$$

Since $\mathcal{A}_E^p (E; \bar{\Omega}; u) \subset \Omega$, the argument in the proof of Proposition 4.2 asserts that for each $x \in \mathcal{A}_E^p (E; \bar{\Omega}; u)$, a point $y \in E$ satisfying (23) is not a cut point of $x$. In fact, $y$ is uniquely determined, and the map $\mathcal{A}_E^p (E; \bar{\Omega}; u) \ni x \mapsto y \in E$ is well-defined as

$$
y = \begin{cases} 
  x, & \text{for } |\nabla u(x)| = 0, \\
  \exp_x |\nabla u(x)|^{p-2} \nabla u(x) = \Phi_{p,0}(x) \notin \text{Cut}(x) \cup \{ x \}, & \text{for } |\nabla u(x)| > 0,
\end{cases}
$$

for $|\nabla u(x)| = 0$,
which is surjective. Moreover, when $|\nabla u(x)| = 0$ for $x \in A^p_1(E; \Omega; u)$, we have $y = x = \Phi_{p,\delta}(x) \notin \text{Cut}(x)$ in (23) for any $\delta > 0$, and $D^2 u(x) \geq -\frac{p-1}{p} D^2 u^\infty(x) = 0$ since $\frac{p}{p-1} > 2$. If $|\nabla u(x)| > 0$ for $x \in A^p_1(E; \Omega; u)$, then

$$y = \exp_x |\nabla u(x)|^{p-2} \nabla u(x) = \Phi_{p,\delta}(x) = \lim_{\delta \to 0} \Phi_{p,\delta}(x).$$

In the latter case, the curve $\gamma(t) := \phi_p(x, t) = \exp_x t|\nabla u|^p \nabla u(x)$ is a unique minimizing geodesic joining $\gamma(0) = x$ to $\gamma(1) = \exp_x |\nabla u|^p \nabla u(x) \notin \text{Cut}(x) \cup \{x\}$ with velocity $\gamma(0) = |\nabla u|^p \nabla u(x) \in E_x \setminus \{0\}$, and $\gamma(t_\delta(x)) = \Phi_{p,\delta}(x) \notin \text{Cut}(x)$ for $\delta > 0$.

Notice that for each $\delta > 0$, the regularized map $\Phi_{p,\delta}$ is Lipschitz continuous on a compact set $A^p_1(E; \Omega; u) \subset \Omega$ since

$$|x| \leq \limsup_{\delta \to 0} \int_{A^p_1(E; \Omega; u)} \text{Jac} \Phi_{p,\delta}(x)dx.$$

From the argument above, it follows that

$$E = \Phi_{p,0} \left( A^p_1(E; \Omega; u) \cap \{|\nabla u| > 0\} \right) \bigcup \left\{ \Phi_{p,0} \left( A^p_1(E; \Omega; u) \cap \{|\nabla u| = 0\} \right) \right\} = \Phi_{p,0} \left( A^p_1(E; \Omega; u) \cap \{|\nabla u| > 0\} \right) \bigcup \left\{ \Phi_{p,\delta} \left( A^p_1(E; \Omega; u) \cap \{|\nabla u| = 0\} \right) \right\} \quad \forall \delta > 0,$$

where we note $\Phi_{p,\delta} |_{A^p_1(E; \Omega; u) \cap \{|\nabla u| = 0\}}$ is the identity for $\delta > 0$. Letting $A_k := A^p_1(E; \Omega; u) \cap \{|\nabla u| \geq 1/k\}$ for $k \in \mathbb{N}$, we have

$$E = \Phi_{p,0} \left( \bigcup_{k=1}^{\infty} A_k \right) \bigcup \left\{ \Phi_{p,\delta} \left( A^p_1(E; \Omega; u) \cap \{|\nabla u| = 0\} \right) \right\} \quad \forall \delta > 0.$$

Notice that $\left| \Phi_{p,0} \left( \bigcup_{k=1}^{\infty} A_k \right) \right| = \lim_{k \to \infty} \left| \Phi_{p,0} (A_k) \right|$, and the map $\Phi_{p,0} |_{A_k}$ is Lipschitz continuous for each $k \in \mathbb{N}$ since $|\nabla u|^p |\nabla u(x)| \in E_x \setminus \{0\}$ for $x \in A^p_1(E; \Omega; u) \cap \{|\nabla u| > 0\}$. We use the area formula to obtain that for $k \in \mathbb{N}$,

$$\left| \Phi_{p,0} (A_k) \right| \leq \int_{A_k} \text{Jac} \Phi_{p,0}(x)dx.$$

Lemma 3 yields that for $x \in A^p_1(E; \Omega; u) \cap \{|\nabla u| > 0\}$, and for $\delta \geq 0$,

$$\text{Jac} \Phi_{p,\delta}(x) = \text{Jac} \exp_x \left( |\nabla u|^2 + \delta \right) \frac{1}{|\nabla u|^2 + \delta} \nabla u(x) \cdot \det \left( |\nabla u|^2 + \delta \right) \frac{1}{|\nabla u|^2 + \delta} \left( I + (p-2) \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \otimes \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \right) \circ D^2 u(x) + D^2 \left( d^2_{p,\delta}(\cdot)/(2\delta)(x) \right)(x),$$

and hence we use Fatou’s lemma to have

$$\left| \Phi_{p,0} (A_k) \right| \leq \int_{A_k} \text{Jac} \Phi_{p,0} \leq \liminf_{\delta \to 0} \int_{A_k} \text{Jac} \Phi_{p,\delta} \leq \liminf_{\delta \to 0} \int_{A^p_1(E; \Omega; u) \cap \{|\nabla u| > 0\}} \text{Jac} \Phi_{p,\delta}.$$

Thus it follows from (26) and the area formula that
\[ |E| \leq \lim_{\delta \to 0} \sup \left\{ \int_{\mathcal{R}^p_i(E; \overline{\Omega}; u) \cap \{ |\nabla u| = 0 \}} \text{Jac } \Phi_{p,\delta}(x) dx + \left| \Phi_{p,\delta} \big( \mathcal{R}^p_i \left( E; \overline{\Omega}; u \right) \cap \{ |\nabla u| = 0 \} \right) \right\} \]
\[ \leq \lim_{\delta \to 0} \sup \int_{\mathcal{R}^p_i(E; \overline{\Omega}; u)} \text{Jac } \Phi_{p,\delta}(x) dx \]
since \( \Phi_{p,\delta} \) is Lipschitz continuous on \( \mathcal{R}^p_i(E; \overline{\Omega}; u) \) for each \( \delta > 0 \). This finishes the proof of (25).

Now we will prove a uniform estimate for Jacobian determinant of \( \Phi_{p,\delta} \) on \( \mathcal{R}^p_i(E; \overline{\Omega}; u) \) with respect to \( \delta > 0 \). From (24) and Lemma 2.3 we have that for \( x \in \mathcal{R}^p_i(E; \overline{\Omega}; u) \),
\[ d\Phi_{p,\delta}(x) = d \exp_x \left( \left( (|\nabla u|^2 + \delta) \right)^{\frac{1}{2}} \nabla u(x) \right) \]
\[ \circ \left( (|\nabla u|^2 + \delta) \right)^{\frac{1}{2}} \nabla u(x) \circ \left( I + (p-2) \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \otimes \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \right) \circ D^2u(x) + D^2(d_{\Phi_{p,\delta}}^2/2)(x) \}
\[ = d \exp_x \left( \left( |\nabla u|^2 + \delta \right)^{\frac{1}{2}} \nabla u(x) \right) \circ \left( I + (p-2) \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \otimes \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \right) \]
\[ \circ \left( (|\nabla u|^2 + \delta) \right)^{\frac{1}{2}} D^2u(x) + \left( I + \chi_{|\nabla u| > 0} \left( \frac{2-p}{p-1} \frac{|\nabla u|}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2(d_{\Phi_{p,\delta}}^2/2)(x) \right). \]

For \( x \in \mathcal{R}^p_i(E; \overline{\Omega}; u) \), we define the endomorphism \( H_3(x) \) on \( T_x M \) by
\[ H_3(x) := \left( |\nabla u|^2 + \delta \right)^{\frac{1}{2}} D^2u(x) + \left( I + \chi_{|\nabla u| > 0} \left( \frac{2-p}{p-1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2(d_{\Phi_{p,\delta}}^2/2)(x) \right). \]

We prove that \( H_3(x) \) is symmetric, and positive semi-definite for \( x \in \mathcal{R}^p_i(E; \overline{\Omega}; u) \). In fact, for \( x \in \mathcal{R}^p_i(E; \overline{\Omega}; u) \cap \{ z \in \Omega : |\nabla u(z)| > 0 \} \), it follows from (e) of Proposition 4.2 since \( \Phi_{p,\delta}(x) = \phi_p(x, t_0(x)) \) with \( t_0(x) = \left( \frac{|\nabla u|^2 + \delta}{|\nabla u|} \right)^{\frac{1}{2}} \in (0, 1) \). For \( x \in \mathcal{R}^p_i(E; \overline{\Omega}; u) \cap \{ z \in \Omega : |\nabla u(z)| = 0 \} \), \( H_3(x) \) is symmetric and positive definite since \( D^2u(x) \geq \frac{p-1}{p} D^2d_{\Phi_{p,\delta}}^2(x) = 0 \), and \( D^2(d_{\Phi_{p,\delta}}^2/2)(x) = I \).

From (12), recall that if \( |\nabla u(x)| > 0 \) for \( x \in \mathcal{R}^p_i(E; \overline{\Omega}; u) \), then
\[ \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \circ D^2(d_{\Phi_{p,\delta}}^2/2)(x) = \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}(x). \]

Thus we have
\[ H_3(x) = \left( (|\nabla u|^2 + \delta)^{\frac{1}{2}} D^2u(x) + D^2(d_{\Phi_{p,\delta}}^2/2)(x) \right) + \chi_{|\nabla u| > 0} \left( \frac{2-p}{p-1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}(x) \right), \]
which is symmetric and positive semi-definite. Now we claim that for \( x \in \mathcal{R}^p_i(E; \overline{\Omega}; u) \),
\[ \det H_3 \leq \left( \frac{1}{n} \text{tr } H_3 \right)^n \]
\[ \leq \left( \frac{1}{n} \left[ f^+ + \Lambda + (n-1) \frac{\kappa}{\Lambda} \chi_{|\nabla u| > 0} \left( \frac{2-p}{p-1} \right) \right] \right)^n =: h(x). \]

Since \( H_3(x) \) is symmetric and positive semi-definite, the arithmetic-geometric means inequality combined with (22) implies that for \( x \in \mathcal{R}^p_i(E; \overline{\Omega}; u) \),
\[
(\det H_\delta(x))^{\frac{1}{2}} \leq \frac{1}{n} \tr H_\delta(x) = \frac{1}{n} \mathcal{M}_{\delta,\Lambda}(H_\delta(x)) \\
\leq \frac{1}{n} \left\{ \left( |\nabla u(x)|^2 + \delta \right)^{\frac{p}{2}} \mathcal{M}_{\delta,\Lambda} \left( D^2 u(x) \right) + \mathcal{M}_{\delta,\Lambda} \left( D^2 (d_{\delta,p}(x)/2)(x) \right) \right\} \\
+ \frac{1}{n} \chi_{\{|\nabla u| = 0\}} \frac{2 - p}{p - 1} \mathcal{M}_{\delta,\Lambda} \left( \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \\
\leq \frac{1}{n} \left\{ f^+(x) + \mathcal{M}_{\delta,\Lambda} \left( D^2 (d_{\delta,p}(x)/2)(x) \right) + \chi_{\{|\nabla u| = 0\}} \frac{2 - p}{p - 1} \Lambda \right\}.
\]

According to Lemma 2.6, we have
\[
\mathcal{M}_{\delta,\Lambda} \left( D^2 (d_{\delta,p}(x)/2)(x) \right) \leq \Lambda + (n - 1) \Lambda \mathcal{H} \left( \frac{\kappa}{\Lambda} d \left( x, \Phi_{\delta,p}(x) \right) \right) \\
\leq \Lambda + (n - 1) \Lambda \mathcal{H} \left( \frac{\kappa}{\Lambda} |\nabla u(x)|^{p - 1} \right)
\]
since \( \mathcal{H}(\tau) \) is nondecreasing for \( \tau \geq 0 \). This proves (27).

In terms of the operator \( H_\delta \), we have that for \( x \in \mathscr{R}_1^0 \left( E; \overline{\Omega}; u \right) \),
\[
d\Phi_{\delta,p}(x) = d\exp_x \left( \left( |\nabla u|^2 + \delta \right)^{\frac{p}{2}} \nabla u(x) \right) \circ \left( 1 + (p - 2) \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \otimes \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \right) \\
\circ \left( H_\delta(x) - \chi_{\{|\nabla u| = 0\}} \frac{\delta(2 - p)}{(p - 1)(p - 1)|\nabla u|^2 + \delta} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right),
\]
where we used again that \( \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \otimes D^2 (d_{\delta,p}(x)/2)(x) = \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \otimes (x) \) for \( x \in \mathscr{R}_1^0 \left( E; \overline{\Omega}; u \right) \cap \{ z \in \Omega : |\nabla u(z)| > 0 \} \). We notice that
\[
(28) \quad |\det d\Phi_{\delta,p}(x)| = \det H_\delta(x), \quad \forall x \in \mathscr{R}_1^0 \left( E; \overline{\Omega}; u \right) \cap \{ z \in \Omega : |\nabla u(z)| = 0 \}.
\]

Now let \( x \in \mathscr{R}_1^0 \left( E; \overline{\Omega}; u \right) \cap \{ z \in \Omega : |\nabla u(z)| > 0 \} \), and let \( \{e_1, \cdots, e_n\} \) be an orthonormal basis of \( T_xM \) with \( e_1 = \frac{\nabla u(x)}{|\nabla u(x)|} \). Denote
\[
H_{\delta,ij} := \left\langle H_\delta(x) e_j, e_i \right\rangle,
\]
and let $\bar{H}_d$ be the $(n-1) \times (n-1)$-matrix that results from $\left( H_{d,ij} \right)$ by removing the 1-st row and the 1-st column, which is symmetric and positive semi-definite. Then we have

$$
\text{Jac} \Phi_{p,\delta}(x) = \text{Jac} \exp_x \left( \left( |\nabla u|^2 + \delta \right)^{\frac{p-2}{2}} \nabla u(x) \right) \cdot \det \left( I + (p-2) \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \otimes \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \right) \\
\cdot \left( \det \left( H_{d,ij} - \chi_{|\nabla u|>0} \frac{\delta(2-p)}{(p-1)(|p-1| |\nabla u|^2 + \delta)} \delta_{i,j} \right) \right) \\
= \text{Jac} \exp_x \left( \left( |\nabla u|^2 + \delta \right)^{\frac{p-2}{2}} \nabla u(x) \right) \cdot \det \left( I + (p-2) \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \otimes \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \right) \\
\cdot \left( \det H_d - \frac{\chi_{|\nabla u|>0} \delta(2-p)}{(p-1)(|p-1| |\nabla u|^2 + \delta)} \det \bar{H}_d \right) \\
\leq \text{Jac} \exp_x \left( \left( |\nabla u|^2 + \delta \right)^{\frac{p-2}{2}} \nabla u(x) \right) \cdot \det \left( I + (p-2) \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \otimes \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \right) \\
\cdot \left( \det H_d + \frac{\chi_{|\nabla u|>0} \delta(2-p)}{(p-1)(|p-1| |\nabla u|^2 + \delta)} \det \bar{H}_d \right).
$$

Since \( 1 \cdot \det \bar{H}_d \leq \frac{1}{n} \left( 1 + \text{tr} \bar{H}_d \right) \leq \frac{1}{n} \left( 1 + \text{tr} H_d \right) \), it follows from (27) that

$$
\det H_d(x) \leq h(x), \quad \text{and} \quad \det \bar{H}_d(x) \leq \left( \frac{1}{n} + \frac{1}{n} \text{tr} \bar{H}_d(x) \right)^n \leq \left( \frac{1}{n} + h(x) \right)^n, \quad \forall \delta > 0,
$$

and hence (29)

$$
\text{Jac} \Phi_{p,\delta}(x) \leq \text{Jac} \exp_x \left( \left( |\nabla u|^2 + \delta \right)^{\frac{p-2}{2}} \nabla u(x) \right) \cdot \det \left( I + (p-2) \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \otimes \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \right) \\
\cdot \left( \det H_d(x) + \frac{\chi_{|\nabla u|>0} \delta(2-p)}{(p-1)(|p-1| |\nabla u|^2 + \delta)} \left( \frac{1}{n} + \frac{1}{n} \text{tr} H_d(x) \right)^n \right) \\
\leq \text{Jac} \exp_x \left( \left( |\nabla u|^2 + \delta \right)^{\frac{p-2}{2}} \nabla u(x) \right) \left( h(x) + \frac{\chi_{|\nabla u|>0} \delta(2-p)}{(p-1)(|p-1| |\nabla u|^2 + \delta)} \left( \frac{1}{n} + h(x) \right)^n \right)^n.
$$

Using (27), (28) and Theorem 2.1, we deduce that for $x \in \mathcal{M}_x \left( E; \Omega; u \right)$,

$$
\text{Jac} \Phi_{p,\delta}(x) \leq \mathcal{F}^{-1} \left( \sqrt{\frac{k}{\Lambda}} |\nabla u(x)|^{p-1} \right) \left( h(x) + \frac{\chi_{|\nabla u|>0} \delta(2-p)}{(p-1)(|p-1| |\nabla u|^2 + \delta)} \left( \frac{1}{n} + h(x) \right)^n \right)^n
$$

since $\text{Ric}(e, e) \geq \mathcal{M}_x \left( (\text{R}(e)) / \lambda \geq -(n-1) \kappa / \lambda \right)$ for any unit vector $e \in TM$, and $f(\tau)$ is nondecreasing for $\tau \geq 0$. Notice that the last term $\chi_{|\nabla u|>0} \delta(2-p) \left( \frac{1}{n} + h(x) \right)^n$ is uniformly bounded with respect to $\delta > 0$, and converges pointwise to 0 for any $x \in \mathcal{M}_x \left( E; \Omega; u \right)$ as $\delta$ tends to 0. Therefore, the dominated convergence theorem asserts that

$$
\limsup_{\delta \to 0} \int_{\mathcal{M}_x \left( E; \Omega; u \right)} \text{Jac} \Phi_{p,\delta}(x) dx \\
\leq \int_{\mathcal{M}_x \left( E; \Omega; u \right)} \mathcal{F}^{-1} \left( \sqrt{\frac{k}{\Lambda}} |\nabla u(x)|^{p-1} \right) h(x) dx \\
\leq \int_{\mathcal{M}_x \left( E; \Omega; u \right)} \mathcal{F}^{-1} \left( \sqrt{\frac{k}{\Lambda}} |\nabla u|^p \right) \left\{ f^+ + \frac{\Lambda}{p-1} + (n-1) \Lambda \mathcal{H} \left( \sqrt{\frac{k}{\Lambda}} |\nabla u|^p \right) \right\}^n dx,
$$
Lemma 4.9. Assume that $1 < p < 2$ and $\mathcal{M}_{i\Lambda}(R(e)) \geq -(n-1)\kappa$ with $\kappa \geq 0$ for any unit vector $e \in TM$. For a bounded open set $\Omega \subset M$, let $f \in C(\Omega)$ and $u \in C(\overline{\Omega})$ be $C_0$-semiconcave in $\Omega$ for $C_0 > 0$ such that $|\nabla u|^{p-2} \mathcal{M}_{i\Lambda}(D^2 u) \leq f$ in $\Omega$ in the viscosity sense. For a compact set $E \subset M$, we assume that

$$\mathcal{R}_1^p(E; \overline{\Omega}, u) \subset \Omega.$$ 

Then we have

$$|E| \leq \int_{\mathcal{R}_1^p(E; \overline{\Omega}, u)} \mathcal{P}^{-1} \left( \sqrt{\frac{k}{\Lambda}} |\nabla u|^{p-1} \right) \left[ 1 + \frac{1}{p} \left( f^+ + \frac{\Lambda}{p-1} + (n-1)\Lambda \mathcal{P} \left( \sqrt{\frac{k}{\Lambda}} |\nabla u|^{p-1} \right) \right) \right].$$

Proof. First, we prove that $u$ is differentiable on $\mathcal{R}_1^p(E; \overline{\Omega}, u) \subset \Omega$. It suffices to prove subdifferentiability of $u$ on $\mathcal{R}_1^p(E; \overline{\Omega}, u)$ since $u$ is superdifferentiable in $\Omega$ from semiconcavity. Indeed, for any $x \in \mathcal{R}_1^p(E; \overline{\Omega}, u)$, there is $y \in E$ such that

$$\inf_{\Omega} \left( u + \frac{p-1}{p} d_y^{\frac{1}{p-1}} \right) = u(x) + \frac{p-1}{p} d_y^{\frac{1}{p-1}}(x).$$

As seen in the proofs of Corollary 4.3 and Proposition 4.2, $x$ is not a cut point of $y$ since $u$ is semiconcave in $\Omega$. Since $d_y^{\frac{1}{p-1}}$ is of class $C^2$ on $M \setminus \text{Cut}(y)$ for $1 < p < 2$, it follows from (30) that $u$ is subdifferentiable at $x \in \mathcal{R}_1^p(E; \overline{\Omega}, u)$. Hence $u$ is differentiable at any $x \in \mathcal{R}_1^p(E; \overline{\Omega}, u) \subset \Omega$. Notice that $u$ is locally Lipschitz continuous in $\Omega$ from the semiconcavity, and then $|\nabla u|$ is uniformly bounded in a compact set $\mathcal{R}_1^p(E; \overline{\Omega}, u) \subset \Omega$.

As in the proof of Theorem 4.8 for $\delta > 0$, define a regularized map $\Phi_{p, \delta} : \mathcal{R}_1^p(E; \overline{\Omega}, u) \rightarrow M$ as

$$\Phi_{p, \delta}(x) := \exp_x \left( \left| \nabla u(x) \right|^2 + \delta \right)^{-\frac{1}{2p}} \nabla u(x).$$

We also define $\Phi_{p, 0} : \mathcal{R}_1^p(E; \overline{\Omega}, u) \cap \{ |\nabla u| > 0 \} \rightarrow M$ by

$$\Phi_{p, 0}(x) := \exp_x |\nabla u(x)|^{p-2} \nabla u(x).$$

Since $u$ is differentiable on $\mathcal{R}_1^p(E; \overline{\Omega}, u) \subset \Omega$, a similar argument to the proofs of Proposition 4.2 (a), and Theorem 4.8 yields that for each $x \in \mathcal{R}_1^p(E; \overline{\Omega}, u)$, a point $y \in E$ satisfying (30) is not a cut point of $x$. In fact, $y$ is uniquely determined, and the map $\mathcal{R}_1^p(E; \overline{\Omega}, u) \ni x \mapsto y \in E$ is well-defined as

$$y = \begin{cases} x, & \text{for } |\nabla u(x)| = 0, \\ \exp_x |\nabla u(x)|^{p-2} \nabla u(x) = \Phi_{p, 0}(x) \notin \text{Cut}(x) \cup \{ x \}, & \text{for } |\nabla u(x)| > 0, \end{cases}$$

which is surjective. Moreover, when $|\nabla u(x)| = 0$ for $x \in \mathcal{R}_1^p(E; \overline{\Omega}, u)$, we have $y = x = \Phi_{p, \delta}(x) \notin \text{Cut}(x)$ in (30) for any $\delta > 0$. If $|\nabla u(x)| > 0$ for $x \in \mathcal{R}_1^p(E; \overline{\Omega}, u)$, then

$$y = \exp_x |\nabla u(x)|^{p-2} \nabla u(x) = \Phi_{p, 0}(x) = \lim_{\delta \rightarrow 0} \Phi_{p, \delta}(x).$$
In the latter case, the curve \( \gamma(t) := \Phi_p(x, t) = \exp_p t |\nabla u|^p - 2 \nabla u(x) \) is a unique minimizing geodesic joining \( \gamma(0) = x \) to \( \gamma(1) = \exp_p |\nabla u|^p - 2 \nabla u(x) \notin \text{Cut}(x) \cup \{x\} \) with velocity \( \dot{\gamma}(0) = |\nabla u|^p - 2 \nabla u(x) \in E_x' \setminus \{0\} \), and \( \gamma(t_\delta(x)) = \Phi_{p, \delta}(x) \notin \text{Cut}(x) \) for \( \delta > 0 \), where \( t_\delta(x) := \left( \frac{|\nabla u(x)|^2}{\kappa^2} \right)^{\frac{1}{p-1}} \in (0, 1) \). We remark that
\[
(31) \quad (|\nabla u|^2 + \delta)^{\frac{1}{p-2}} \nabla u(x) \in E_x, \quad \forall x \in \mathcal{R}_1^p \left( E; \Omega \right) \quad \text{for any } \delta > 0,
\]
and the above argument implies that
\[
\text{Lipschitz continuity of the maps } \Phi_{p, \delta}(x) \text{ in } \Omega \setminus N \text{ whenever } u \text{ is semiconcave in } \Omega.
\]
In order to prove \( (35) \), we will show that \( \Phi_{p, \delta}(x) \) is Lipschitz continuous on \( \mathcal{R}_1^p \left( E; \Omega \right) \subset \Omega \) by proving the following: there is a uniform constant \( C_1 > 0 \) such that for each \( x_0 \in \mathcal{R}_1^p \left( E; \Omega \right) \), there exists \( 0 < r_0 < \min \left( \iota_{L}^{\text{Lipschitz}}, \iota_{\text{Lipschitz}} \right) \) such that
\[
(34) \quad \limsup_{\delta \to 0} \int_{\mathcal{R}_1^p \left( E; \Omega \right)} \text{Jac} \Phi_{p, \delta}(x) dx \leq \int_{\mathcal{R}_1^p \left( E; \Omega \right)} \text{Jac} \Phi_{p, \delta}(x) dx.
\]
In order to prove \( (35) \), we will show that \( \nabla u \) is Lipschitz continuous on \( \mathcal{R}_1^p \left( E; \Omega \right) \subset \Omega \) by proving the following: there is a uniform constant \( C_1 > 0 \) such that for each \( x_0 \in \mathcal{R}_1^p \left( E; \Omega \right) \), there exists \( 0 < r_0 < \min \left( \iota_{L}^{\text{Lipschitz}}, \iota_{\text{Lipschitz}} \right) \) such that
\[
(36) \quad |\nabla u(x_1) - L_{x_2, x_1} \nabla u(x_2)| \leq C_1 d(x_1, x_2), \quad \forall x_1, x_2 \in B_{r_0}(x_0) \cap \mathcal{R}_1^p \left( E; \Omega \right),
\]
where \( L_{x_2, x_1} \) is the parallel transport along the minimizing geodesic joining \( x_2 \) to \( x_1 \), and \( d(x_1, x_2) \) is the minimum of the injectivity radius and the convexity radius of \( x \in \Omega \), respectively. See [AF, Definition 1.2] for the notions of \( C^{1,1} \) smoothness.

Once Lipschitz continuity of \( \nabla u \) on a compact set \( \mathcal{R}_1^p \left( E; \Omega \right) \) is achieved, we deduce Lipschitz continuity of the maps \( \Phi_{p, \delta}, \Phi_{p, \delta}^{k} \left( E; \Omega \right) \) by using
since the exponential map $\exp_t : E \to M$ is smooth. For the proof of (35), we apply the area formula to Lipschitz maps $\Phi_{p,\alpha}^t|_{\mathcal{A}_k^0(E;\overline{\Omega}^k)} (k \in \mathbb{N})$, and $\Phi_{p,\alpha}^t|_{\mathcal{A}_k^0(\overline{E};\overline{\Omega})}$ as in the proof of Theorem 4.8. Here, we remark that the area formula on Riemannian manifolds follows from the area formula in Euclidean space using a partition of unity. So the above notion of Lipschitz continuity on $\mathcal{A}_k^0(E;\overline{\Omega};u)$, which is uniformly locally Lipschitz, suffices to employ the area formula in our setting. Thus a similar argument to the proof of (33) in Theorem 4.8 with the help of Lipschitz continuity and (32) yields (35), and therefore it follows from (34) that

$$\|E\| \leq \int_{\mathcal{A}_k^0(E;\overline{\Omega};u)} e^{\rho-1} \left( \sqrt{\frac{k}{\rho}} \|\nabla u\|^{\rho-1} \right) \left[ \frac{1}{nA} \left\{ f^+ + (n - 1)\Lambda \mathcal{H} \left( \sqrt{\frac{k}{\rho}} \|\nabla u\|^{\rho-1} + \frac{\Lambda}{p-1} \right) \right\}^\alpha dx. $$

To complete the proof, it remains to show Lipschitz continuity of $\nabla u$ on $\mathcal{A}_k^0(E;\overline{\Omega};u)$. We claim that there is a uniform constant $C_1 > 0$ such that for each $x_0 \in \mathcal{A}_k^0(E;\overline{\Omega};u)$, there is $0 < r_0 < \min(\frac{1}{2}, \frac{1}{2}r/f_0)$ such that for any $x \in B_{r_0}(x_0) \cap \mathcal{A}_k^0(E;\overline{\Omega};u)$,

$$\|u(z) - u(x) - \langle \nabla u(x), \exp_{x}^{-1} z \rangle \| \leq C_1 d^2(x, z), \quad \forall z \in B_{r_0}(x_0).$$

We put off the proof of (37), and first prove that this claim (37) implies (36). Assume that the claim holds. Let us fix $x_0 \in \mathcal{A}_k^0(E;\overline{\Omega};u)$, and let $0 < r_0 < \min(\frac{1}{2}, \frac{1}{2}r/f_0)$ be a constant satisfying (37). Let $x_1, x_2 \in B_{r_0}(x_0) \cap \mathcal{A}_k^0(E;\overline{\Omega};u)$. Let $r := d(x_1, x_2)$ and $V := \exp_{x_1}^{-1} x_2$. From (37), it follows that for any $z = \exp_{x_1} L_{x_1, x_2} X$ with $X \in T_{x_1} M$ and $|X| \leq r/4$,

$$\|\nabla u(x_2), \exp_{x_1}^{-1} z\| - \|\nabla u(x_1), \exp_{x_1}^{-1} z - \exp_{x_1}^{-1} x_2\| \
abla u(x_2) - u(x_1) - \langle \nabla u(x_1), \exp_{x_1}^{-1} z \rangle - \|u(z) - u(x_2) - \langle \nabla u(x_2), \exp_{x_1}^{-1} z \rangle\| \
abla u(x_2) - u(x_1) - \langle \nabla u(x_1), \exp_{x_1}^{-1} x_2 \rangle \| 
abla u(x_2) - u(x_1) - \langle \nabla u(x_1), \exp_{x_1}^{-1} x_2 \rangle \| \leq C_1 \left[ d^2(x_1, z) + d^2(x_2, z) + d^2(x_1, x_2) \right] 
abla u(x_2) - u(x_1) - \langle \nabla u(x_1), \exp_{x_1}^{-1} x_2 \rangle \| \leq C_1 \left[ d^2(x_1, x_2) + d^2(x_2, z) + d^2(x_1, x_2) \right] 
abla u(x_2) - u(x_1) - \langle \nabla u(x_1), \exp_{x_1}^{-1} x_2 \rangle \| \leq C_1 \left[ d^2(x_1, x_2) + d^2(x_2, z) + d^2(x_1, x_2) \right] < 6C_1 r^2$$

since $d(z, x_0) \leq d(z, x_2) + d(x_2, x_0) < r/4 + r_0/2 < r_0$. Setting $z = \exp_{x_1} W = \exp_{x_2} L_{x_1, x_2} X$ for $W \in T_{x_1} M$, we can rewrite that

$$6C_1 r^2 \geq \|\nabla u(x_2), \exp_{x_1}^{-1} z\| - \|\nabla u(x_1), \exp_{x_1}^{-1} z - \exp_{x_1}^{-1} x_2\| \
abla u(x_2) - u(x_1) - \langle \nabla u(x_1), W - V \rangle \
abla u(x_2) - u(x_1) - \langle \nabla u(x_1), X \rangle - \langle \nabla u(x_1), W - X - V \rangle. $$

Recall from [AH] Claim 5.3] that for a fixed point $x_0$, there are $r_2 > 0$ and $C_2 > 0$ such that

$$|W - X - V| \leq C_2 (|V| + |X|)^3$$

for any $x_1, x_2 \in B_{r_2}(x_0)$ and $X \in T_{x_1} M$ with $|X| \leq 2r_2$, where $V = \exp_{x_1}^{-1} x_2$, and $W = \exp_{x_1}^{-1} \left( \exp_{x_2} L_{x_1, x_2} X \right)$. Thus we can choose $0 < r_0 < r_2$ sufficiently small such that

$$\|\nabla u(x_1), W - X - V\| \leq \|\nabla u\|_{L^\infty(\mathcal{A}_k^0(\overline{E};\overline{\Omega}))} \cdot C_2 (|V| + |X|)^3 \leq \|\nabla u\|_{L^\infty(\mathcal{A}_k^0(\overline{E};\overline{\Omega}))} \cdot C_2 \cdot 8r^3 \leq C_1 r^2,$$
since $|\nabla u|$ is uniformly bounded on $\mathcal{A}_{(x)}^{p}(E; \overline{\Omega}, u)$, and $|X| \leq r/4 = |V|/4 < r_0/2$. Therefore, by selecting $r_0 > 0$ sufficiently small, we obtain from (38) that for any $x_1, x_2 \in B_{r_0}(x_0) \cap \mathcal{A}_{(x)}^{p}(E; \overline{\Omega}, u)$ with $r := d(x_1, x_2)$, and $X \in T_{x_1}M$ with $|X| \leq r/4$,

$$\left|\langle L_{x_1, x_2} \nabla u(x_2) - \nabla u(x_1), X \rangle \right| < 7C_1r^2,$$

and hence (36) follows since $X \in B_{r/4}(0) \subset T_{x_1}M$ is arbitrary. We refer to (35 Proposition 3.5) for the Euclidean case.

Lastly, we prove (37) by making use of semi-concavity and (30) combined with Lemma 2.5. Let us fix $x_0 \in \mathcal{A}_{(x)}^{p}(E; \overline{\Omega}, u)$. Due to $C_0$-semiconcavity, there is small $0 < r_0 < \min(\frac{\sqrt{D_{M}}}{\theta'}, \frac{C_p}{\theta'})$ such that $u - C_0d_{x_0}^2$ is geodesically concave in $B_{r_0}(x_0) \subset \Omega$. From differentiability of $u$ on $\mathcal{A}_{(x)}^{p}(E; \overline{\Omega}, u)$, we deduce that for $x \in B_{r_0}(x_0) \cap \mathcal{A}_{(x)}^{p}(E; \overline{\Omega}, u)$ and $z \in B_{r_0}(x_0)$,

$$u(z) - u(x) - \left\langle \nabla u(x), \exp_z^{-1}z \right\rangle \leq C_0 \left\{ d_{x_0}^2(z) - d_{x_0}^2(x) - \left\langle \nabla d_{x_0}^2(x), \exp_z^{-1}z \right\rangle \right\} \leq 2C_0\mathcal{H} \left( \sqrt{\kappa r_0} \right) d^2(x, z) \leq 2C_0\mathcal{H} \left( \sqrt{\kappa} \right) d^2(x, z)$$

by using the Taylor expansion and Lemma 2.5, where $-\kappa$ ($\kappa \geq 0$) is a lower bound of the sectional curvature on $\overline{\Omega}$.

On the other hand, if $x \in B_{r_0}(x_0) \cap \mathcal{A}_{(x)}^{p}(E; \overline{\Omega}, u) \cap \{||\nabla u|| = 0\}$, then (30) is satisfied with $y = x$, and it follows that for any $z \in B_{r_0}(x_0) \subset \Omega$,

$$u(z) - u(x) - \left\langle \nabla u(x), \exp_x^{-1}z \right\rangle = u(z) - u(x) \geq -\frac{1}{p}d_{x_0}^2(z) \geq -(2r_0)^{2/p}d^2(x, z) \geq -(2\kappa)^{2/p}d^2(x, z)$$

since $1 < p < 2$. This proves (37) for $x \in B_{r_0}(x_0) \cap \mathcal{A}_{(x)}^{p}(E; \overline{\Omega}, u) \cap \{||\nabla u|| = 0\}$. Here, the constants $2C_0\mathcal{H} \left( \sqrt{\kappa} \right)$ and $(2\kappa)^{2/p}$ are uniform.

Now, consider $x \in B_{r_0}(x_0) \cap \mathcal{A}_{(x)}^{p}(E; \overline{\Omega}, u) \cap \{||\nabla u|| > 0\}$. Note that the corresponding $y \in E$ in (30) does not belong to $\text{Cut}(x) \cup \{x\}$ as seen before. Let $\tilde{r} := \min(\frac{\sqrt{D_{M}}}{\theta'}, \frac{C_p}{\theta'})/5 > 0$.

We may assume that $0 < r_0 < \tilde{r}$. If $d(x, y) \leq 3\tilde{r}$, then $\frac{p-1}{p}d_{x_0}^2 \sqrt{\mathcal{H}} \left( \frac{x}{\sqrt{5}} \right)$ is of class $C^2$ in $B_{r_0}(x_0)$ since $d(x, y) < 5\tilde{r}$ for $z \in B_{r_0}(x_0) \subset \Omega$ and $\tilde{r} = \min(\frac{\sqrt{D_{M}}}{\theta'}, \frac{C_p}{\theta'})/5$. From (30), we see that for $z \in B_{r_0}(x_0)$,

$$u(z) - u(x) - \left\langle \nabla u(x), \exp_x^{-1}z \right\rangle \geq -\frac{1}{p}d_{x_0}^2(z) + \frac{p-1}{p}d_{x_0}^2(z) + \frac{p-1}{p}\nabla d_{x_0}^2(x) \exp_x^{-1}z \geq -\frac{1}{p}d_{x_0}^2(z) \sqrt{\mathcal{H}} \left( \frac{x}{\sqrt{5}} \right)$$

by using the Taylor expansion and Lemma 2.5. Again, where $-\kappa$ ($\kappa \geq 0$) is a lower bound of the sectional curvature on a bounded set $\{z' \in M : d(z', x') \leq 5\tilde{r}, \ x' \in \overline{\Omega}\}$.

When $d(x, y) > 3\tilde{r}$, we define $\tilde{y} := \exp_{x}^{-1}3\tilde{r} \nabla d_{x}(x) \in \partial B_{\frac{3}{\sqrt{5}\tilde{r}}}(x)$. We observe that

$$d_{x_0}^2(z) \leq \left\{ d_{x}(z) + d_{x}(\tilde{y}) \right\} \frac{1}{\sqrt{5}},$$

$$d_{x_0}^2(z) \leq \left\{ d_{x}(z) + d_{x}(\tilde{y}) \right\} \frac{1}{\sqrt{5}}, \quad \text{and} \quad \tilde{r} < d(z, \tilde{y}) < 5\tilde{r}, \quad \forall z \in B_{r_0}(x_0) \subset B_{2r_0}(x_0)$$

since we assume $0 < r_0 < \tilde{r}$. Let $\rho(z) := \frac{p-1}{p} \left\{ d_{x}(z) + d_{x}(\tilde{y}) \right\} \frac{1}{\sqrt{5}}$. Since $\tilde{r} < d(z, \tilde{y}) < 5\tilde{r}$ for any $z \in B_{r_0}(x_0) \subset \Omega$ and $\tilde{r} = \min(\frac{\sqrt{D_{M}}}{\theta'}, \frac{C_p}{\theta'})/5$, $\rho$ is smooth in $B_{r_0}(x_0)$, and (30) implies that
for $z \in B_{n}(x_0)$,
\[ u(z) - u(x) \geq -\frac{p - 1}{p} d_y^p(z) + \frac{p - 1}{p} d_y^p(x) \geq -\rho(z) + \rho(x), \]
and hence
\[ u(z) - u(x) - \langle \nabla u(x), \exp_x^{-1} z \rangle \geq -\rho(z) + \rho(x) + \langle \nabla \rho(x), \exp_x^{-1} z \rangle \]
\[ \geq -\left( D^2 \rho(\sigma(s)) \cdot \dot{\sigma}(s), \dot{\sigma}(s) \right)^+ d^2(z,x)/2 \]
for some $s \in [0, d(x,z)]$, where $\sigma$ is the minimal geodesic joining $x$ to $z$ parametrized by arc length. Note that $\sigma([0, d(x,z)]) \subset B_{r}(x_0)$, and $\tilde{r} < d(y, \sigma(s)) < 5\tilde{r}$. Using Lemma 2.5 we see that
\[ \left\langle D^2 \rho(\sigma(s)) \cdot \dot{\sigma}(s), \dot{\sigma}(s) \right\rangle \leq \left\{ d_y(\sigma(s)) + d_y(\tilde{\sigma}) \right\} \left\{ (d_y(\sigma(s)) + d_y(\tilde{\sigma})) \left( D^2 \dot{d}_y(\sigma(s)) \cdot \dot{\sigma}(s), \dot{\sigma}(s) \right)^+ + 1 \right\} \]
\[ \leq \left\{ 5\tilde{r} + d_y(\tilde{\sigma}) \right\} \left\{ \left[ 1 + \frac{d_y(\tilde{\sigma})}{d_y(\sigma(s))} \right] d_y(\sigma(s)) \left( D^2 \dot{d}_y(\sigma(s)) \cdot \dot{\sigma}(s), \dot{\sigma}(s) \right)^+ + 1 \right\} \]
and hence
\[ u(z) - u(x) - \langle \nabla u(x), \exp_x^{-1} z \rangle \geq -\left( 5\tilde{r} + d_y(\tilde{\sigma}) \right) \left\{ \left[ 1 + \frac{d_y(\tilde{\sigma})}{d_y(\sigma(s))} \right] d_y(\sigma(s)) \left( D^2 \dot{d}_y(\sigma(s)) \cdot \dot{\sigma}(s), \dot{\sigma}(s) \right)^+ + 1 \right\} \]

Making use of Theorem 4.8 and Lemma 4.9 we prove the ABP type estimate for the $p$-Laplacian operator $(1 < p < 2)$ assuming the Ricci curvature to be bounded from below.

**Lemma 4.10.** Assume that $1 < p < 2$ and $\text{Ric} \geq -(n - 1)\kappa$ for $\kappa \geq 0$. For a bounded open set $\Omega \subset M$, let $f \in C(\Omega)$ and $u \in C(\overline{\Omega})$ be $C_0$-semiconcave in $\Omega$ for $C_0 \geq 0$ such that $\Delta_n u \leq f$ in $\Omega$ in the viscosity sense. For a compact set $E \subset M$, we assume that $\mathcal{A}_1^p \left( E; \overline{\Omega}; u \right) \subset \Omega$. Then
\[ |E| \leq \int_{\mathcal{H}(E; \overline{\Omega}; u)} \frac{\mathcal{H}^{n-1}(\sqrt{n}|\nabla u|^{p-1})}{(p - 1)^n} \left\{ \left[ \mathcal{F}^+ \left( \sqrt{n}|\nabla u|^{p-1} \right) \right]^n \right\} dx. \]

In particular, if $\text{Ric} \geq 0$, then
\[ |E| \leq \int_{\mathcal{H}(E; \overline{\Omega}; u)} \frac{1}{(p - 1)^n} \left\{ \left[ \mathcal{F}^+ \left( \sqrt{n}|\nabla u|^{p-1} \right) \right]^n \right\} dx. \]

**Proof.** First, we recall from Lemma 3.3 that $u$ satisfies
\[ (\nabla u)^2 + \delta \right\} \mathcal{M}_{p-1,1}^n \left( D^2 u \right) \leq f^+ \]
in $\Omega$ for any $\delta > 0$ in the sense of Definition 3.3 and in the usual sense from Lemma 3.2. Using the same notation as in the proof of Lemma 2.2, semiconcavity implies that (40) is satisfied pointwise in $\Omega \setminus N$ with the Hessian $D^2 u$ in the sense of Theorem 2.11 where $u$ is twice differentiable in $\Omega \setminus N$, and a set $N \subset \Omega$ has measure zero.
As in the proof of Lemma 4.9, for \( \delta > 0 \), consider a regularized map \( \Phi_{p,\delta} : \mathcal{R}_1^p \left( E; \overline{\Omega}; u \right) \to M \) defined as \( \Phi_{p,\delta}(x) := \exp_p \left( (|\nabla u(x)|^2 + \delta)^{\frac{p-2}{2}} \nabla u(x) \right) \). From Lemma 4.9, \( \Phi_{p,\delta} \) is Lipschitz continuous in \( \mathcal{R}_1^p \left( E; \overline{\Omega}; u \right) \) for each \( \delta > 0 \), and

\[
\text{(41)} \quad |E| \leq \limsup_{\delta \to 0} \int_{\mathcal{R}_1^p \left( E; \overline{\Omega}; u \right)} \text{Jac} \Phi_{p,\delta}(x) dx.
\]

It suffices to obtain a uniform estimate of \( \text{Jac} \Phi_{p,\delta} \) on \( \mathcal{R}_1^p \left( E; \overline{\Omega}; u \right) \setminus N \) with respect to \( \delta > 0 \) in terms of a lower bound \( -\kappa \) of the Ricci curvature. First, let \( R_0 := \text{diam} (\Omega \cup E) \) and \( z_0 \in \Omega \). We choose \( \tilde{\kappa} \geq 0 \) such that

\[
\text{(42)} \quad \mathcal{M}_{p-1}^-(R(e)) \geq -(n-1)\tilde{\kappa} \quad \text{for any unit vector} \ e \in T_x M \text{ and} \ x \in \overline{\mathcal{B}}_{2R_0}(z_0).
\]

Note from Lemma 4.9 that \( \text{Jac} \Phi_{p,\delta} \) is uniformly bounded for \( \delta > 0 \) in terms of \( -\kappa \) using \( \text{(40)} \) and \( \text{(42)} \). With the same notation as the proofs of Theorem 4.8 and Lemma 4.9, \( \text{(29)} \) yields that for \( x \in \mathcal{R}_1^p \left( E; \overline{\Omega}; u \right) \setminus N \),

\[
\text{(43)} \quad \text{Jac} \Phi_{p,\delta}(x) \leq \text{Jac} \exp_p \left( (|\nabla u(x)|^2 + \delta)^{\frac{p-2}{2}} \nabla u(x) \right) \cdot \det \left( I + (p-2) \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \otimes \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} - \frac{1}{n} \text{tr} H_{\delta}(x) \right)^n,
\]

where we recall

\[
H_{\delta}(x) := \left( |\nabla u|^2 + \delta \right)^{\frac{p-2}{2}} D^2 u(x) + \left( I + \chi_{|\nabla u| > 0} \frac{2 - p}{p-1} \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \circ D^2 (d^2_{\Phi_{p,\delta}(z); 1/2})(x),
\]

which is symmetric and positive semi-definite. Using \( \text{(40)} \) and \( \text{(42)} \), it follows from the proof of \( \text{(27)} \) that for \( x \in \mathcal{R}_1^p \left( E; \overline{\Omega}; u \right) \setminus N \),

\[
\text{(44)} \quad \text{det} H_{\delta}(x) \leq \left( \frac{1}{n} \text{tr} H_{\delta}(x) \right)^n \leq \left( \frac{1}{n(p-1)} \left[ f^+(x) + (n-1)\mathcal{R} \left( \sqrt{\kappa} |\nabla u|^{p-1} \right) + \chi_{|\nabla u| > 0} \frac{2 - p}{p-1} \frac{2 - p}{p-1} \right] \right)^n =: \tilde{h}(x).
\]

From \( \text{(28)} \) and \( \text{(44)} \), we have that for \( x \in \mathcal{R}_1^p \left( E; \overline{\Omega}; u \right) \cap \{ z \in \Omega \setminus N : |\nabla u(z) = 0 \},

\[
\text{(45)} \quad \text{Jac} \Phi_{p,\delta}(x) = \text{det} H_{\delta}(x) \leq \left[ \frac{1}{n(p-1)} \left[ f^+(x) + n \right] \right]^n.
\]

For \( x \in \mathcal{R}_1^p \left( E; \overline{\Omega}; u \right) \cap \{ z \in \Omega \setminus N : |\nabla u(z) > 0 \}, \) let

\[
D_{\delta}(x) := I + (p-2) \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \otimes \frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}}.
\]
Using Theorem 2.1 and the arithmetic-geometric means inequality, we deduce from (43) and (44) that for $\varepsilon > 0$

Assume that

Lemma 4.11.

Then

since $\text{Ric} \geq -(n-1)\kappa$, and $D_\delta$ and $H_\delta$ are symmetric and positive semi-definite. Notice that for $x \in \mathcal{A}_1^n \left( E; \overline{\Omega}, u \right) \cap \{z \in \Omega \setminus N : \|\nabla u(z)\| > 0\},$

$$\text{tr} \left( D_\delta \circ H_\delta \right) (x) \text{ converges to } \Delta_p u(x) + \Delta d_\delta^2(x)/2$$

for $y = \exp_x |\nabla u|^p - 2 \nabla u(x) \notin \text{Cut}(x)$ as $\delta$ tends to 0, and

$$\Delta_\delta u(x) + \Delta d_\delta^2(x)/2 \leq f^+(x) + 1 + (n-1)\mathcal{H} \left( \sqrt{n} |\nabla u(x)|^{p-1} \right)$$

from Lemma 2.6. Since $0 \leq \text{tr} \left( D_\delta \circ H_\delta \right) \leq \text{tr} H_\delta \leq n\delta^2$ in $\mathcal{A}_1^n \left( E; \overline{\Omega}, u \right) \setminus N$, and $N \subset \Omega$ has measure zero, we apply the dominated convergence theorem to deduce that

$$\limsup_{\delta \to 0} \int_{\mathcal{A}_1^n \left( E; \overline{\Omega}, u \right) \setminus \{ |\nabla u| = 0 \}} \text{Jac} \Phi_{p,\delta} \leq \int_{\mathcal{A}_1^n \left( E; \overline{\Omega}, u \right) \setminus \{ |\nabla u| = 0 \}} \mathcal{F}^{n-1} \left( \sqrt{n} |\nabla u|^{p-1} \right) \left\{ \frac{f^+}{n} + \mathcal{H} \left( \sqrt{n} |\nabla u|^{p-1} \right) \right\}^n.$$

This combines with (45) and (41) to complete the proof. \hfill \Box

The following lemma is concerned with the ABP type estimate for viscosity solutions.

**Lemma 4.11.** Assume that $1 < p < 2$ and $\text{Ric} \geq -(n-1)\kappa$ for $\kappa \geq 0$. For $z_0, x_0 \in M$ and $0 < r \leq R \leq R_0$, assume that $\overline{B}_r(z_0) \subset B_R(x_0)$. Let $f \in C \left( B_R(x_0) \right)$ and $u \in C \left( \overline{B}_R(x_0) \right)$ be such that

$$\Delta_p u \leq f \text{ in } B_R(x_0),$$

in the viscosity sense,

(46) \hspace{1cm} u \geq 0 \text{ on } B_R(x_0) \setminus B_{4r}(z_0) \text{ and } \inf_{B_4(z_0)} u \leq \frac{p-1}{p}.

Then

$$|B_r(z_0)| \leq \mathcal{F}^{n-1} \left( \frac{2 \sqrt{R_0}}{p-1} \right) \int_{\{ u \leq \tilde{M}_p \} \cap B_4(z_0)} \left\{ \mathcal{H} \left( 2 \sqrt{R_0} + \frac{r^p f^+}{n} \right) \right\}^n$$

for a uniform constant $\tilde{M}_p := \frac{p-1}{p} \frac{3^{\frac{p-1}{p}}}{p}$. Moreover, if $r^p f \leq 1$ in $B_4(z_0)$, then there exists a uniform constant $\delta \in (0, 1)$ depending only on $n, p$ and $\sqrt{R_0}$, such that

(47) \hspace{1cm} \left| \{ u \leq \tilde{M}_p \} \cap B_{4r}(z_0) \right| > \delta |B_4(z_0)|.$
Lastly, (47) follows from Bishop-Gromov’s volume comparison. Letting \( SOOJUNG~KIM \)

Using (48), we apply Lemma 4.10 to \( \nabla u \) of \( B_{\delta}(\zeta) \), \( C := \frac{1}{\epsilon} \relh \left( \frac{1}{\epsilon^2} R \right) \), and

\[
\int_{\mathcal{A}_1^p \left( B_{\delta}(\zeta); B_{\delta}(\zeta); r^{\frac{p-1}{p}} \tilde{u}_e \right)} \left[ r^p c_{\eta}^{-1} \left( f_e + \tilde{k} n \max \left\{ \epsilon, 2\omega \left( 2\sqrt{\epsilon} \right) \right\} \right) + \relh \left( 2\sqrt{\epsilon} R_0 \right) \right]^n
\]

\[
\leq \frac{\gamma_{n-1} (2 \sqrt{\epsilon} R_0)}{n^p (p-1)^n} \int_{B_{\delta}(\zeta) \cap \{ \tilde{u}_e < M_{p, \epsilon} \}} - \frac{r^p f^+}{n} + \relh \left( 2\sqrt{\epsilon} R_0 \right) \bigg)^n.
\]

Letting \( \epsilon \) go to 0 and then \( \eta \) go to 0, we conclude that

\[
\int_{\mathcal{A}_1^p \left( B_{\delta}(\zeta); B_{\delta}(\zeta); r^{\frac{p-1}{p}} \tilde{u}_e \right)} \left[ r^p c_{\eta}^{-1} \left( f_e + \tilde{k} n \max \left\{ \epsilon, 2\omega \left( 2\sqrt{\epsilon} \right) \right\} \right) + \relh \left( 2\sqrt{\epsilon} R_0 \right) \right]^n.
\]

Lastly, (47) follows from Bishop-Gromov’s volume comparison. □
Corollary 4.12. Assume that $1 < p < 2$, and $M_{1,\Lambda}(R(e)) \geq -(n-1)\kappa$ with $\kappa \geq 0$ for any unit vector $e \in TM$. For $z_0, x_0 \in M$ and $0 < r \leq R \leq R_0$, assume that $\overline{B}_e(z_0) \subset B_e(x_0)$. Let $\beta \geq 0$, and $u \in C(\overline{B}_R(x_0))$ be such that

$$|\nabla u|^{p-2}M_{1,\Lambda}(D^2u) - \beta|\nabla u|^{p-1} \leq r^{-p} \quad \text{in } B_R(x_0)$$

in the viscosity sense,

$$u \geq 0 \quad \text{on } B_R(x_0) \setminus B_e(z_0) \quad \text{and} \quad \inf_{B_e(z_0)} u \leq \frac{p-1}{p} R.$$ 

Then there exists a uniform constant $\delta \in (0, 1)$ depending only on $n, p, \sqrt{R_0}, \lambda, \Lambda$, and $\beta R_0$, such that

$$\left| \{u \leq \tilde{M}_p \} \cap B_{2r}(z_0) \right| > \delta |B_{2r}(z_0)|$$

for $\tilde{M}_p = \frac{p-1}{p} 3^{\frac{1}{p-1}} > 1$. 

Proof. The proof is similar to Lemma 4.11 by replacing Lemma 4.10 by Lemma 4.9. Let $\eta > 0$. Making use of Lemmas 3.4 and 3.9, the inf-convolution $u_\varepsilon$ of $u$ with respect to $\overline{B}_R(x_0)$ (for small $\varepsilon > 0$) satisfies

$$\begin{align*}
&\begin{cases}
  u_\varepsilon \to u & \text{uniformly in } B_R(x_0), \\
  u_\varepsilon \geq -\eta & \text{in } B_R(x_0) \setminus B_e(z_0), \\
  \inf_{B_e(z_0)} u_\varepsilon \leq \frac{p-1}{p} + \eta, \\
  u_\varepsilon \text{ is } C_\varepsilon\text{-semiconcave in } B_R(x_0),
\end{cases}
\end{align*}$$

$$|\nabla u_\varepsilon|^{p-2}M_{1,\Lambda}(D^2u_\varepsilon) - \kappa n \Lambda |\nabla u_\varepsilon|^{p-2} \min \left\{ \varepsilon|\nabla u_\varepsilon|^2, 2\omega \left( 2\sqrt{m\varepsilon} \right) \right\} - \beta |\nabla u_\varepsilon|^{p-1} \leq r^{-p} \quad \text{in } B_{2r}(z_0)$$

in the viscosity sense, where $-\kappa (\kappa \geq 0)$ is a lower bound of the sectional curvature on $\overline{B}_R(x_0), C_\varepsilon := \frac{1}{\varepsilon} \mathcal{H} \left( 2\sqrt{\kappa} \right)$, and $m := |u_{\varepsilon_{\eta}}|_{L^p(B_R(x_0))}$. 

For small $\varepsilon > 0$, consider $\tilde{u}_\varepsilon := \left( \frac{p-1}{p} / (p-1)/p + 2\eta \right) (u_\varepsilon + \eta) = c_\eta (u_\varepsilon + \eta)$. Arguing similarly as in the proof of Lemma 4.11 together with (49), we deduce that

$$\begin{align*}
\mathcal{A}_1^p (\overline{B}_{r}(z_0); \overline{B}_{\varepsilon}(z_0); r^{-\frac{p}{r}} \tilde{u}_\varepsilon) &= \mathcal{A}_1^p (\overline{B}_{r}(z_0); \overline{B}_R(x_0); r^{-\frac{p}{r}} \tilde{u}_\varepsilon) \\
&\quad \subset B_{2r}(z_0) \cap \{\tilde{u}_\varepsilon < M_p, r^p|\nabla \tilde{u}_\varepsilon|^{p-1} < 2R_0\},
\end{align*}$$

and

$$r^p|\nabla \tilde{u}_\varepsilon|^{p-2}M_{1,\Lambda}(D^2\tilde{u}_\varepsilon) \leq c_\eta^{p-1} \left[ r^p \kappa n \Lambda \max \left\{ \varepsilon, 2\omega \left( 2\sqrt{m\varepsilon} \right) \right\} + 1 \right] + 2\beta R_0$$

in $B_{2r}(z_0) \cap \{r^p|\nabla \tilde{u}_\varepsilon|^{p-1} < 2R_0\}$ in the viscosity sense. Applying Lemma 4.9 to $r^p\tilde{u}_\varepsilon$, we obtain

$$\begin{align*}
|B_{2r}(z_0)| &\leq \mathcal{F}^{-1} \left( 2 \sqrt{\frac{k}{\lambda}} R_0 \right) \int_{\mathcal{A}_1(\overline{B}_{2r}(z_0); \overline{B}_{4r}(z_0); r^{-\frac{p}{r}} \tilde{u}_\varepsilon)} \left[ \frac{1}{n\lambda} \left( c_\eta^{p-1} \left( r^p \kappa n \Lambda \max \left\{ \varepsilon, 2\omega \left( 2\sqrt{m\varepsilon} \right) \right\} + 1 \right) \\
+ 2\beta R_0 \right] \frac{\Lambda}{p-1} + (n-1)\Lambda \mathcal{H} \left( 2 \sqrt{\frac{k}{\lambda}} R_0 \right) \right]^n dx \\
&\leq \mathcal{F}^{-1} \left( 2 \sqrt{\frac{k}{\lambda}} R_0 \right) \int_{B_{4r}(z_0) \cap \{\tilde{u}_\varepsilon \leq M_p\}} \left[ \frac{1}{n\lambda} \left( c_\eta^{p-1} \left( r^p \kappa n \Lambda \max \left\{ \varepsilon, 2\omega \left( 2\sqrt{m\varepsilon} \right) \right\} + 1 \right) \\
+ 2\beta R_0 \right] \frac{\Lambda}{p-1} + (n-1)\Lambda \mathcal{H} \left( 2 \sqrt{\frac{k}{\lambda}} R_0 \right) \right]^n dx.
\end{align*}$$
Letting $\epsilon$ and $\eta$ go to 0, we conclude that

$$|B_r(z_0)| \leq C r^{-n-1} \left(2 \sqrt{\frac{K}{A}} R_0 \right) \int_{|x| \leq M_0} \left[ \frac{1}{n!} \left( 1 + 2\beta R_0 + \frac{\Lambda}{p - 1} + (n - 1)\Lambda \mathcal{H}_e \left( 2 \sqrt{\frac{K}{A}} R_0 \right) \right)^p \right].$$

Since $\text{Ric}(e, e) \geq M_0 \lambda (e) / \lambda \geq -(n - 1) \kappa / \lambda$ for any unit vector $e \in TM$, this finishes the proof from Bishop-Gromov’s volume comparison. □

5. $L^p$-estimates

In this section, we derive $L^p$-estimates for $p$-Laplacian type operators by comparing a viscosity solution with a barrier function, and using the volume doubling property in Lemma [22]. We follow a similar argument to [IS; Sections 4 and 5], but it is not straightforward due to the existence of the cut-locus in the setting of Riemannian manifolds.

**Lemma 5.1.** Let $1 < p < \infty$, and $\text{Ric} \geq -(n - 1) \kappa$ for $\kappa \geq 0$. For $z_0, x_0 \in M$ and $0 < r \leq R \leq R_0$, assume that $B_r(z_0) \subset B_2(x_0)$. There exists a large constant $\bar{M} > 1$ such that if for $\beta > 0$, $u \in C \left( \overline{B_r(z_0)} \right)$ is semiconcave in $B_r(z_0)$, and satisfies

$$\begin{cases}
  u \geq 0 & \text{in } \overline{B_r(z_0)}, \\
  u > \bar{M} & \text{in } B_r(z_0), \\
  \Delta_p u - \beta|\nabla u|^{p-1} \leq r^{-p} & \text{in } B_r(z_0) \text{ in the viscosity sense,}
\end{cases}$$

then

$$u > 1 \quad \text{in } B_{4r}(z_0),$$

where $\bar{M} > 1$ depends only on $n, p, \sqrt{R_0}$, and $\beta R_0$.

**Proof.** For $\alpha > 1$ and $\bar{M} > 1$, define

$$v(x) := \bar{M} \left( r^\alpha d(x, z_0)^{-\alpha} - 5^{-\alpha} \right).$$

By selecting uniform constants $\alpha > 1$ and $\bar{M} > 1$, depending only on $n, p, \sqrt{R_0}$ and $\beta R_0$, we have

$$\begin{cases}
  r^\alpha \Delta_p v - \beta r^p |\nabla v|^{p-1} \geq 2 & \text{in } B_r(z_0) \setminus \left( \text{Cut}(z_0) \cup \{z_0\} \right), \\
  v = 0 & \text{in } \partial B_r(z_0), \\
  v > 1 & \text{in } B_{4r}(z_0), \\
  \sup_{\partial B_r(z_0)} v < \bar{M} \\
  |\nabla v(x)| > 0 & \forall x \in \text{Cut}(z_0) \cup \{z_0\},
\end{cases}$$

(50)

In fact, for $x \in B_{5r}(z_0) \setminus \left( \text{Cut}(z_0) \cup \{z_0\} \right)$,

$$\beta r^p |\nabla v(x)|^{p-1} = \left( \frac{r}{d_{z_0}(x)} \right)^p \left( \alpha \bar{M} \left( \frac{r}{d_{z_0}(x)} \right)^\alpha \right)^p \beta d_{z_0}(x) \leq \left( \frac{r}{d_{z_0}(x)} \right)^p \left( \alpha \bar{M} \left( \frac{r}{d_{z_0}(x)} \right)^\alpha \right)^p \beta R_0,$$

and

$$r^\alpha \Delta_p v(x) = r^p |\nabla v|^{p-2} \text{tr} \left( \left( I + (p - 2) \nabla d_{z_0} \otimes \nabla d_{z_0} \right) D^2 v(x) \right)$$

$$= \left( \frac{r}{d_{z_0}(x)} \right)^p \left( \alpha \bar{M} \left( \frac{r}{d_{z_0}(x)} \right)^\alpha \right)^p \left( (\alpha + 1)(p - 1) + 1 - \Delta d_{z_0}^2(x)/2 \right),$$

where we recall that $\nabla d_{z_0}(x)$ is an eigenvector of $D^2 d_{z_0}(x)$ and $D^2 d_{z_0}^2/2(x)$ associated with eigenvalues 0 and 1, respectively. By using Lemma [22,6] we choose $\alpha > 1$ and $\bar{M} > 1$
sufficiently large to obtain
\[ r^p \Delta_p v - \beta \rho |\nabla v|^{p-1} \geq \left( \frac{r}{d_{z_0}} \right)^p \left( \alpha M \left( \frac{r}{d_{z_0}} \right)^{p-1} \right) \left( (\alpha + 1)(p-1) - (n-1) \beta R \right) \]
\[ \geq 5^{-p} \left( \alpha M 5^{-\alpha} \right)^{p-1} \cdot 1 \geq 2 \quad \text{in } B_{5r}(z_0) \setminus (\text{Cut}(z_0) \cup \{z_0\}). \]

It is easy to check other properties in \([50]\) for large \(\alpha > 1\) and \(\tilde{M} > 1\).

Now we prove that if \(u-v\) has a minimum at an interior point \(x \in B_{5r}(z_0) \setminus \overline{B}_r(z_0)\), then \(x\) is not a cut point of \(z_0\). Suppose to the contrary that \(x \in (B_{5r}(z_0) \setminus \overline{B}_r(z_0)) \cap \text{Cut}(z_0)\) is an interior minimum point of \(u-v\). Note that \(x \neq z_0\) if \(x \in \text{Cut}(z_0)\). Define \(\psi(x) := -\tilde{M} \left( r^p s^{-a/2} - 5^{-a} \right) \).

We notice that \(v = -\psi \circ d_{z_0}^2\) and \(\psi' > 0\) in \((0, \infty)\). By Corollary 2.8, there is a unit vector \(X \in T_xM\) such that
\[ \liminf_{t \to 0} \frac{1}{t} \left( \psi \left( d_{z_0}^2 (\exp_x tX) \right) + \psi \left( d_{z_0}^2 (\exp_x -tX) \right) - 2\psi \left( d_{z_0}^2(x) \right) \right) = -\infty. \]

Since \(u-v\) has a minimum at an interior point \(x \in (B_{5r}(z_0) \setminus \overline{B}_r(z_0)) \cap \text{Cut}(z_0)\), we have
\[ -\infty = \liminf_{t \to 0} \frac{1}{t} \left( \psi \left( d_{z_0}^2 (\exp_x tX) \right) + \psi \left( d_{z_0}^2 (\exp_x -tX) \right) - 2\psi \left( d_{z_0}^2(x) \right) \right) \geq \limsup_{t \to 0} \frac{1}{t} \left( u(\exp_x tX) + u(\exp_x -tX) - 2u(x) \right), \]

which is a contradiction due to semi-concavity of \(u\). So \(x\) is not a cut point of \(z_0\).

According to the comparison principle, we conclude that \(u-v \geq 0\) in \(B_{5r}(z_0) \setminus \overline{B}_r(z_0)\) since \(v\) is smooth in \(B_{5r}(z_0) \setminus (\{z_0\} \cup \text{Cut}(z_0))\) with non-vanishing gradient, and \(u-v \geq 0\) on \(\partial B_{5r}(z_0) \cup \partial B_r(z_0)\). Thus \([50]\) implies that \(u > 1\) in \(B_{4r}(z_0)\).

**Corollary 5.2.** Let \(1 < p < \infty\), and \(\kappa \geq -(n-1)\kappa\) for \(\kappa \geq 0\). For \(z_0, x_0 \in M\) and \(0 < r \leq R \leq R_0\), assume that \(B_{6r}(z_0) \subset B_R(x_0)\). There exists a large constant \(\tilde{M} > 1\) such that if \(u \in C(B_{6r}(z_0))\) satisfies
\[
\begin{cases}
  u \geq 0 & \text{in } B_{6r}(z_0), \\
  u > \tilde{M} & \text{in } B_r(z_0), \\
  \Delta_p u \leq r^{-p} & \text{in } B_{6r}(z_0) \text{ in the viscosity sense},
\end{cases}
\]
then
\[ u > 1 \quad \text{in } B_{4r}(z_0), \]
where \(\tilde{M} > 1\) depends only on \(n, p, \text{ and } \sqrt{R_0}\).

**Proof.** We use again Jensen’s inf-convolution to approximate a viscosity supersolution \(u\). Let \(\eta > 0\). According to Lemma 3.4 and Remark 3.10, the inf-convolution \(u_\varepsilon\) of \(u\) with respect to \(B_{6r}(z_0)\) (for small \(\varepsilon > 0\)) satisfies the following:
\[
\begin{cases}
  u_\varepsilon \to u \quad \text{uniformly in } B_{6r}(z_0), \\
  u_\varepsilon \geq -\eta & \text{in } B_{6r}(z_0), \\
  \inf_{B_{6r}(z_0)} u_\varepsilon > \tilde{M} - \eta, \\
  u_\varepsilon \text{ is } C_\varepsilon\text{-semiconcave in } B_{6r}(z_0), \\
  \Delta_p u_\varepsilon - \tilde{k}(n + p - 2) \left( 2\varepsilon \omega \left( \frac{1}{\sqrt{R}} \right) \right) \frac{1}{2} |\nabla u_\varepsilon|^{p-1} \leq r^{-p} \text{ in } B_{5r}(z_0)
\end{cases}
\]
in the viscosity sense, where \(-\tilde{k} (\tilde{k} \geq 0)\) is a lower bound of the sectional curvature on \(\overline{B}_{3R}(x_0), C_\varepsilon := \frac{1}{\varepsilon} \mathcal{H} \left( \frac{1}{\sqrt{R}} \right), m := \|u\|_{L^p(B_{6r}(z_0))}, \) and \(\omega\) denotes a modulus of continuity of
Corollary 5.3. Let $1 < p < \infty$, and $\mathcal{M}_{\mathcal{L}}(R(e)) \geq -(n-1)\kappa$ with $\kappa \geq 0$ for any unit vector $e \in TM$. For $z_0, x_0 \in M$ and $0 < r \leq R \leq R_0$, assume that $\overline{B}_{p}(z_0) \subset B_{R}(x_0)$. There exists a large constant \( \tilde{M} > 1 \) such that if for $\beta \geq 0$, $u \in C(\overline{B}_{p}(z_0))$ satisfies
\[
\begin{cases}
  u \geq 0 & \text{in } B_{p}(z_0), \\
  |\nabla u|^p r^{p-2} \mathcal{M}_{\mathcal{L}}(D^2 u) - \beta |\nabla u|^{p-1} \leq r^{-p} & \text{in } B_{p}(z_0) \text{ in the viscosity sense},
\end{cases}
\]
then
\[
u > 1 \quad \text{in } B_{p}(z_0),
\]
where \( \tilde{M} > 1 \) depends only on $n$, $p$, $\sqrt{R_0}$, $\Lambda$, and $\beta R_0$.

Combined with Lemmas 4.6 and 4.11 we have the following measure estimate.

Corollary 5.4. Let $1 < p < \infty$, and $\text{Ric} \geq -(n-1)\kappa$ for $\kappa \geq 0$. For $z_0, x_0 \in M$ and $0 < r \leq R \leq R_0$, assume that $\overline{B}_{p}(z_0) \subset B_{R}(x_0)$. There exist constants \( \tilde{M} > 1 \) and $0 < \delta < 1$ such that if $u \in C(\overline{B}_{p}(z_0))$ satisfies
\[
\begin{cases}
  u \geq 0 & \text{in } B_{p}(z_0), \\
  r^p \Delta_p u \leq 1 & \text{in } B_{p}(z_0) \text{ in the viscosity sense},
\end{cases}
\]
\[
\left| \left\{ u > \tilde{M} \right\} \cap B_{r}(z_0) \right| > (1 - \delta)|B_{r}(z_0)|,
\]
then
\[
u > 1 \quad \text{in } B_{r}(z_0),
\]
where \( \tilde{M} > 1 \) and $\delta \in (0, 1)$ depend only on $n$, $p$, and $\sqrt{R_0}$.

Proof. Let $\tilde{M}_0 = \frac{p-1}{p} \frac{3}{2}$ be the constants as in Lemmas 4.6 and 4.11 and let $\tilde{M}_1$ be as in Corollary 5.2. Let $\tilde{M} := \frac{p-1}{p} \tilde{M}_0 \tilde{M}_1$. Applying Lemmas 4.6 and 4.11 to $p - \frac{1}{p} \frac{u}{M_1}$ in $B_{r}(z_0)$, we obtain that $\inf_{B_{r}(z_0)} \tilde{M} > \tilde{M}_1$. From Corollary 5.2 it follows that $u > 1$ in $B_{r}(z_0)$.

The homogeneity of the $p$-Laplacian operator implies the following.

Corollary 5.5. Let $1 < p < \infty$, and $\text{Ric} \geq -(n-1)\kappa$ for $\kappa \geq 0$. For $z_0, x_0 \in M$ and $0 < r \leq R \leq R_0$, assume that $\overline{B}_{p}(z_0) \subset B_{R}(x_0)$. There exist constants \( \tilde{M} > 1 \) and $0 < \delta < 1$ such that if $\theta > 0$, $u \in C(\overline{B}_{p}(z_0))$ satisfies
\[
\begin{cases}
  u \geq 0 & \text{in } B_{p}(z_0), \\
  r^p \Delta_p u \leq \theta^{p-1} & \text{in } B_{p}(z_0) \text{ in the viscosity sense},
\end{cases}
\]
\[
\left| \left\{ u > \theta \tilde{M} \right\} \cap B_{r}(z_0) \right| > (1 - \delta)|B_{r}(z_0)|,
\]
then
$$u > \theta \quad \text{in } B_\varepsilon(z_0),$$
where $\tilde{M} > 1$ and $\delta \in (0, 1)$ are the constants in Corollary 5.4.

By using Corollaries 5.4, 4.12, and 5.3, the same argument as the proof of Corollary 5.4 yields the following measure estimate for nonlinear $p$-Laplacian type operators.

**Corollary 5.6.** Let $1 < p < \infty$, and $\mathcal{M}_{L_A}(R(e)) \geq -(n - 1)\kappa$ with $\kappa \geq 0$ for any unit vector $e \in TM$. For $z_0, x_0 \in M$ and $0 < r \leq R \leq R_0$, assume that $B_{2\varepsilon}(z_0) \subset B_{R}(x_0)$. There exist constants $\tilde{M} > 1$ and $0 < \delta < 1$ such that if $u \geq 0$ and $\theta > 0$, $u \in C(B_{2\varepsilon}(z_0))$ satisfies
\[
\begin{aligned}
&\left\{ \begin{array}{ll}
u \geq 0 & \text{in } B_{2\varepsilon}(z_0), \\
|\nabla u|^{p-2} \mathcal{M}_{L_A}(D^2u) - \beta |\nabla u|^{p-1} \leq \theta^{p-1} r^{-p} & \text{in } B_{2\varepsilon}(z_0) \text{ in the viscosity sense,} \\
u \geq 0 & \text{in } B_{2\varepsilon}(z_0)
\end{array} \right.
\end{aligned}
\]
then
$$u > \theta \quad \text{in } B_\varepsilon(z_0),$$
where $\tilde{M} > 1$ and $\delta \in (0, 1)$ depend only on $n, p$, $\sqrt{\kappa}R_0$, $\Lambda$, $\beta$, and $\beta R_0$.

Now we prove $L^\ast$-estimates for viscosity supersolutions by obtaining power decay of measure of super-level sets.

**Theorem 5.7.** Let $1 < p < \infty$, and $\text{Ric} \geq -(n - 1)\kappa$ for $\kappa \geq 0$. Let $x_0 \in M$ and $0 < R \leq R_0$. There exist constants $\tilde{M} > 1$ and $0 < \delta < 1$ such that if $u \in C(B_{2R}(x_0))$ satisfies
\[
\begin{aligned}
&\left\{ \begin{array}{ll}
u \geq 0 & \text{in } B_{2R}(x_0), \\
R^p \Delta_p u \leq 1 & \text{in } B_{2R}(x_0) \text{ in the viscosity sense,} \\
u \geq 0 & \text{in } B_{2R}(x_0)
\end{array} \right.
\end{aligned}
\]
then
$$\left\|u > \tilde{M}^k \right\| \subset B_R(x_0) < (1 - \delta)^k |B_R(x_0)|, \quad \forall k = 1, 2, \cdots.$$
Furthermore,
$$\left\|u > \theta \right\| \subset B_R(x_0) < c t^{-\epsilon} |B_R(x_0)|,$$
where $\tilde{M} > 1$, $\delta \in (0, 1)$, $c > 0$, and $\epsilon \in (0, 1)$ depend only on $n, p$, and $\sqrt{\kappa}R_0$.

**Proof.** We give a sketch of the proof since the proof is similar to [15, Theorem 5.1]. Let $A_k := \left\{ u > \tilde{M}^k \right\} \subset B_R(x_0)$, where $\tilde{M} > 1$ is the constant in Corollary 5.4. According to Corollary 5.4, we have $|A_k| \leq (1 - \delta)|B_R(x_0)|$. We claim that
$$|A_k| \leq (1 - c_0 \delta)^k |B_R(x_0)| \quad \forall k = 1, 2, \cdots,$$
where $0 < c_0 < 1$ is the constant as in Lemma 2.2 and depends on $n$, and $\sqrt{\kappa}R_0$. By induction, suppose that $|A_k| \leq (1 - c_0 \delta)^k |B_R(x_0)|$ for some $k \in \mathbb{N}$. Applying Corollary 5.6 with $\theta = \tilde{M}^k$, it follows that if a ball $B \subset B_R(x_0)$ satisfies the property that $|A_{k+1} \cap B| > (1 - \delta)|B|$, then $B \subset A_k$. Therefore, Lemma 2.2 yields that
$$|A_{k+1}| \leq (1 - c_0 \delta)^k |A_k| \leq (1 - c_0 \delta)^{k+1} |B_R(x_0)|,$$
which finishes the proof. \(\square\)

**Corollary 5.8 ($L^\ast$-estimate).** Let $1 < p < \infty$, and $\text{Ric} \geq -(n - 1)\kappa$ for $\kappa \geq 0$. Let $x_0 \in M$ and $0 < R \leq R_0$. For $C_0 \geq 0$, let $u \in C(B_{2R}(x_0))$ be a nonnegative viscosity supersolution of
$$\Delta_p u \leq C_0 \quad \text{in } B_{2R}(x_0).$$
Then
\[
\left( \int_{B_\varepsilon(x_0)} u^\varepsilon(x) dx \right)^{1/\varepsilon} \leq C \left( \inf_{B_\varepsilon(x_0)} u + R^{-\frac{2}{p-1}} C_0 \right),
\]
where the constants \( \varepsilon \in (0, 1) \) and \( C > 0 \) depend only on \( n, p, \) and \( \sqrt{R_0} \).

**Proof.** We may assume \( C_0 > 0 \). By applying Theorem 5.7 to \( \inf_{B_\varepsilon(x_0)} u + R^{-\frac{2}{p-1}} C_0 \), we deduce power decay estimate for measure of super-level sets, from which the \( L^\varepsilon \)-estimate follows.

Proceeding with the same argument to the proof of Theorem 5.7, with the use of Corollary 5.4, we also obtain \( L^\varepsilon \)-estimates for nonlinear \( p \)-Laplacian type operators; we note that \( \lambda \text{Ric}(e, e) \geq M_{\lambda, \Lambda}(e) \) for any unit vector \( e \in TM \).

**Corollary 5.9** (\( L^\varepsilon \)-estimate). Let \( 1 < p < \infty \), and \( M_{\lambda, \Lambda}(e) \geq -(n - 1)\kappa \) with \( \kappa \geq 0 \) for any unit vector \( e \in TM \). Let \( x_0 \in M \) and \( 0 < R \leq R_0 \). For \( \beta \geq 0 \) and \( C_0 \geq 0 \), let \( u \in C(\bar{B}_2 R_0) \) be a nonnegative viscosity supersolution of
\[
|\nabla u|^{p-2} \lambda_\alpha(D^2 u) - \beta|\nabla u|^{p-1} \leq C_0 \quad \text{in } \bar{B}_2 R_0.
\]
Then
\[
\left( \int_{B_\varepsilon(x_0)} u^\varepsilon(x) dx \right)^{1/\varepsilon} \leq C \left( \inf_{B_\varepsilon(x_0)} u + R^{-\frac{2}{p-1}} C_0 \right),
\]
where the constants \( \varepsilon \in (0, 1) \) and \( C > 0 \) depend only on \( n, p, \sqrt{R_0}, \lambda, \Lambda, \) and \( \beta R_0 \).

6. **Harnack Inequality**

This section is devoted to the proof of Harnack inequality using the scale-invariant \( L^\varepsilon \)-estimates. We follow the method of [13, Theorem 1.3] for the proof.

**Theorem 6.1** (Harnack inequality). Let \( 1 < p < \infty \), and \( \lambda \text{Ric}(e, e) \geq -(n - 1)\kappa \beta \kappa \geq 0 \). For \( x_0 \in M \) and \( 0 < R \leq R_0 \), let \( u \in C(\bar{B}_2 R_0) \) be a nonnegative viscosity solution of
\[
\Delta_p u = f \quad \text{in } \bar{B}_2 R_0.
\]
Then
\[
\sup_{B_1 R_0} u \leq C \left( \inf_{B_1 R_0} u + \frac{R^{-\frac{2}{p-1}} \| f \|_{L^p(\bar{B}_2 R_0)}}{\inf_{B_1 R_0} u + R^{-\frac{2}{p-1}} \| f \|_{L^p(\bar{B}_2 R_0)}} \right),
\]
where a constant \( C > 0 \) depends only on \( n, p, \) and \( \sqrt{R_0} \).

**Proof.** We may assume that \( \inf_{B_1 R_0} u \leq 1 \) and \( \| f \|_{L^\infty(\bar{B}_2 R_0)} \leq R^{-\beta} \) replacing \( u \) by \( \frac{u}{\inf_{B_1 R_0} u + R^{-\frac{2}{p-1}} \| f \|_{L^p(\bar{B}_2 R_0)}} \).

For \( \alpha > 0 \) to be chosen later (depending only on \( n, p, \) and \( \sqrt{R_0} \)), consider a continuous function
\[
h_\tau(x) := \tau \left[ 4/3 - d_\tau(x)/R \right]^{-\alpha}, \quad \forall x \in B_{4R/3} (z_0),
\]
where \( \tau > 0 \) is selected to be the minimal constant such that
\[
u \leq h_\tau \quad \text{in } B_{4R/3} (z_0).
\]
Let \( x_0 \in B_{4R/3} (z_0) \) be a point such that \( u(x_0) = h_\tau(x_0) =: H_0 > 0 \), and let \( r := \frac{4R/3 - d_\tau(x_0)}{2} \in (0, 2R/3) \). Then we have \( B_{2r} (x_0) \subset B_{4R/3} (z_0) \), and \( H_0 = h_\tau(x_0) = \tau(2r/R)^{-\alpha} \). We may assume that \( \tau \geq (4/3)^\alpha \) and then \( H_0 \geq 1 \); otherwise \( \sup_{B_{2r} (x_0)} u \leq \sup_{B_{2r} (x_0)} h_\tau \leq (4/3)^\alpha \cdot 3^\alpha = 4^\alpha \).

After using a standard covering argument, we deduce from Theorem 5.7 that
\[
\left| \{ u > H_0/2 \} \cap B_{4R/3} (z_0) \right| \leq c H_0^{-\alpha} |B_{4R/3} (z_0)|,
\]
where \( c = c(n, p, \lambda, \Lambda, \beta, \kappa) \) is determined.
where the constants \( c > 0 \) and \( \epsilon \in (0, 1) \) depend only on \( n, p \), and \( \sqrt{\kappa R_0} \).

On the other hand, for a given \( \mu \in (0, 1) \),

\[
u \leq h_\tau \leq \tau (r/R)^{-\alpha} (2 - \mu)^{-\alpha} = H_0 \left( \frac{2 - \mu}{2} \right)^{-\alpha} \quad \text{in} \ B_{\mu r}(x_0).
\]

Define

\[
\bar{u}(x) := \frac{\left( \frac{2 - \mu}{2} \right)^{-\alpha} H_0 - u(x)}{\left( \frac{2 - \mu}{2} \right)^{-\alpha} - 1} H_0
\]

which is nonnegative, and satisfies \( \bar{u}(x_0) = 1 \) and

\[
\left( \frac{\mu r}{2} \right)^p \Delta_\mu \bar{u} \leq \mu \left[ \left( \frac{2 - \mu}{2} \right)^{-\alpha} - 1 \right] \left[ \left( \frac{r}{2R} \right)^p \right] \quad \text{in} \ B_{\mu r}(x_0).
\]

Let \( \bar{M} > 1 \) be the constant in Corollary \[5.4\] We select a large constant \( \alpha > 0 \) and a small constant \( \mu \in (0, 1) \) such that

\[
\alpha := \frac{4 \log_2 D}{\epsilon}, \quad \left( \frac{2 - \mu}{2} \right)^{-\alpha} - 1 \leq \frac{4}{\alpha} \leq 1, \quad \text{and} \quad \left\{ \left( \frac{2 - \mu}{2} \right)^{-\alpha} - 1 \right\} \leq \frac{1}{2M}.
\]

for \( D := 2^n \cosh^{-1}\left( 6 \sqrt{\kappa R_0} \right) \) since \( \lim_{\mu \to 0^+} \frac{\mu}{\left( \frac{2 - \mu}{2} \right)^{-\alpha} - 1} = \frac{2}{\alpha} \). Then \( \bar{u} \) satisfies

\[
(\mu r/2)^p \Delta_\mu \bar{u} \leq 1 \quad \text{in} \ B_{\mu r}(x_0),
\]

since \( p > 1, H_0 \geq 1, 0 < \mu < 1, \) and \( 0 < r < R \). By applying Corollary \[5.4\] to \( \bar{u} \) in \( B_{\mu r}(x_0) \) with \( \bar{u}(x_0) = 1 \), we have

\[
||\bar{u} \leq \bar{M}|| \cap B_{\mu r/2}(x_0) || > \delta \left| B_{\mu r/2}(x_0) \right|,
\]

which implies

\[
||u > H_0/2|| \cap B_{\mu r/2}(x_0) || > \delta \left| B_{\mu r/2}(x_0) \right|,
\]

since

\[
H_0 \left( \left( \frac{2 - \mu}{2} \right)^{-\alpha} - \bar{M} \left( \left( \frac{2 - \mu}{2} \right)^{-\alpha} - 1 \right) \right) > \frac{H_0}{2}.
\]

Combined with \[5.1\], we have

\[
\delta |B_{\mu r/2}(x_0)| < ||u > H_0/2|| \cap B_{\mu r/2}(x_0)
\]

\[
\leq ||u > H_0/2|| \cap B_{3R/3}(z_0)|
\]

\[
\leq c H_0^{-\epsilon} |B_{3R/3}(z_0)| \leq c H_0^{-\epsilon} |B_{8R/3}(x_0)| = c \tau^{-\epsilon} 2^{2\alpha} (r/c)^{\alpha} |B_{8R/3}(x_0)|
\]

\[
\leq c \tau^{-\epsilon} 2^{2\alpha} (r/c)^{\alpha} D \left( \frac{16R}{3\mu} \log_2 D \right) |B_{\mu r/2}(x_0)|
\]

for \( D := 2^n \cosh^{-1}\left( 6 \sqrt{\kappa R_0} \right) \) from \[3\]. Therefore, it follows that \( \tau \) is uniformly bounded from above since \( \epsilon \alpha \geq \log_2 D, \) and hence \( \sup u \leq \sup h_\tau \leq \alpha \cdot 3^\alpha \). \( \square \)

By replacing Theorem \[5.7\] and Corollary \[5.4\] by Corollaries \[5.9\] and \[5.6\] we have the following Harnack inequality for nonlinear \( p \)-Laplacian type operators.
**Theorem 6.2.** Let $1 < p < \infty$ and $M_{1,A}^-(R(e)) \geq -(n - 1)k$ with $k \geq 0$ for any unit vector $e \in TM$. Let $z_0 \in M$ and $0 < R \leq R_0$. For $\beta \geq 0$, and $C_0 \geq 0$, let $u$ be a nonnegative viscosity solution to

\[
\begin{cases}
|\nabla u|^{p-2} M_{1,A}^-(D^2 u) - \beta |\nabla u|^{p-1} \leq C_0 & \text{in } B_{2R}(z_0), \\
|\nabla u|^{p-2} M_{1,A}^+(D^2 u) + \beta |\nabla u|^{p-1} \geq -C_0 & \text{in } B_{2R}(z_0).
\end{cases}
\]

Then

\[
\sup_{B_1(z_0)} u \leq C \left( \inf_{B_1(z_0)} u + R^{\frac{1}{p-2}} C_0^{\frac{1}{p-2}} \right),
\]

where a constant $C > 0$ depends only on $n, p, \sqrt{R_0}, \lambda, A$, and $\beta R_0$.

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Soojung Kim: Institute of Mathematics, Academia Sinica, 6F, Astronomy-Mathematics Building No.1, Sec.4, Roosevelt Road, Taipei 10617, Taiwan

E-mail address: soojung26@gmail.com; soojung26@math.sinica.edu.tw