On $PM$-factorizable topological groups *

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Abstract

A topological group $G$ is called $PM$-factorizable if for every continuous real-valued function $f$ on $G$, there exists a perfect homomorphism $\pi : G \to H$ onto a metrizable topological group $H$ such that $f = g \circ \pi$, for some continuous real-valued function $g$ on $H$. It is proved that a topological group is $PM$-factorizable if and only if it is feathered $M$-factorizable, and a topological group is $PR$-factorizable if and only if it is $PM$-factorizable and $\omega$-narrow. The two results deduce that a topological group $G$ is $PR$-factorizable if and only if $G$ is feathered $R$-factorizable. Moreover, some properties about products of $PM$-factorizable topological groups are investigated. In particular, some interesting properties of $M$-factorizable topological groups in \cite{8, 10} are strengthened to $PM$-factorizable topological groups.

Keywords: Topological groups, metrizable, feathered, $PM$-factorizable, $PR$-factorizable.

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1. Introduction

In the field of Topological Algebra, topological groups are standard researching objects and have been studied for many years, see \cite{1}. A topological group is a group equipped with a topology such that the multiplication on the group is jointly continuous and the inverse mapping is also continuous. It is well-known that for every continuous real-valued function $f$ on a compact topological group $G$, there exists a continuous homomorphism $p : G \to L$ onto a second-countable topological group $L$ and a continuous real-valued function $h$ on $L$ such that $f = h \circ p$. Then, Tkachenko posed the concept of $R$-factorizable topological groups, see \cite{4}. A topological group $G$ is called $R$-factorizable if for every continuous real-valued function $f$ on $G$, we can find a continuous homomorphism $\pi : G \to H$ onto a second-countable topological group $H$ such that $f = g \circ \pi$, for some continuous real-valued function $g$ on $H$. We know that $R$-factorizable topological groups are generalizations of compact groups and separable metrizable groups. For more interesting properties about $R$-factorizable topological groups, see \cite{2, 3, 4}. However, since a metrizable topological group need not to be $R$-factorizable, it follows that H. Zhang, D. Peng and W. He in \cite{10} posed the notion of $M$-factorizable topological group. A topological group $G$ is called $M$-factorizable if for every continuous real-valued function $f$ on $G$, there is a continuous homomorphism $\varphi : G \to H$ onto a metrizable topological group $H$ such that $f = g \circ \varphi$, for some continuous real-valued function $g$ on $H$. Since all first-countable topological groups are metrizable, it is trivial to see that all first-countable topological groups are $M$-factorizable. Moreover, it was proved in \cite{10} Theorem 3.2] that a topological group is $R$-factorizable if and only if it is $M$-factorizable and $\omega$-narrow.

By the further research of $R$-factorizable topological groups, L. Peng and Y. Liu introduced the concept of $PR$-factorizable topological groups, that is, a topological group $G$ is called $PR$-factorizable if for every

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continuous real-valued function \( f \) on \( G \), there exists a perfect homomorphism \( \pi : G \to H \) onto a second-countable topological group \( H \) such that \( f = g \circ \pi \), for some continuous real-valued function \( g \) on \( H \). They gave the characterizations of \( \mathbb{PR} \)-factorizable topological groups in \[3\] Theorem 2.6. In particular, a topological group is \( \mathbb{PR} \)-factorizable if and only if it is Lindelöf feathered. Moreover, since every \( \omega \)-narrow feathered topological group is Lindelöf (see \[1\], 4.3.A], it is easy to see that a topological group is \( \mathbb{PR} \)-factorizable iff it is \( \omega \)-narrow and feathered. Then, we introduce the following notion.

**Definition 1.1.** A topological group \( G \) is called \( \mathbb{PM} \)-factorizable if for every continuous real-valued function \( f \) on \( G \), there exists a perfect homomorphism \( \pi : G \to H \) onto a metrizable topological group \( H \) such that \( f = g \circ \pi \), for some continuous real-valued function \( g \) on \( H \).

Clearly, each \( \mathbb{PR} \)-factorizable topological group is \( \mathbb{PM} \)-factorizable. L. Peng and Y. Liu introduced an example \[3\] Example 3.13 which is a \( \mathbb{R} \)-factorizable topological group, but not \( \mathbb{PR} \)-factorizable. Indeed, the topological group \( G \) in \[3\] Example 3.13 is not feathered. Since all \( \mathbb{R} \)-factorizable topological groups are \( \mathcal{M} \)-factorizable, it is a \( \mathcal{M} \)-factorizable topological group. However, by the definition of \( \mathcal{M} \)-factorizability, it is easy to that every \( \mathbb{PM} \)-factorizable topological group is feathered, hence the topological group \( G \) of \[3\] Example 3.13 is not \( \mathbb{PM} \)-factorizable.

In this paper, we give some characterizations of \( \mathbb{PM} \)-factorizable topological groups, such as a topological group \( G \) is \( \mathbb{PM} \)-factorizable if and only if \( G \) is feathered \( \mathcal{M} \)-factorizable. We also shown that a topological group \( G \) is \( \mathbb{PR} \)-factorizable if and only if \( G \) is \( \mathbb{PM} \)-factorizable and \( \omega \)-narrow. Then it is natural to deduce that a topological group \( G \) is \( \mathbb{PR} \)-factorizable if and only if \( G \) is feathered \( \mathbb{R} \)-factorizable. W. He et al. in \[8\] Proposition 2.1 proved the following proposition.

**Proposition 1.2.** Let \( G = \prod_{i \in I} G_i \) be the product of an uncountable family of non-compact separable metrizable topological groups. Then the group \( G \) is \( \mathbb{R} \)-factorizable, but it fails to be feathered.

Therefore, the result also can presents that the product group \( G \) is not \( \mathbb{PR} \)-factorizable. Of course, \( G \) is an \( \mathcal{M} \)-factorizable topological group, but not \( \mathbb{PM} \)-factorizable. Moreover, some interesting properties of \( \mathcal{M} \)-factorizable topological groups in \[8, 10\] are strengthened to \( \mathbb{PM} \)-factorizable topological groups. For example, the product \( G = \prod_{n \in \mathbb{N}} G_n \) of countably many \( \mathbb{PM} \)-factorizable topological groups is \( \mathbb{PM} \)-factorizable if and only if every factor \( G_n \) is metrizable or every \( G_n \) is \( \mathbb{PR} \)-factorizable, the product of a \( \mathbb{PM} \)-factorizable topological group with a compact metrizable topological group is \( \mathbb{PM} \)-factorizable.

2. Preliminary

Throughout this paper, all topological spaces are assumed to be Hausdorff, unless otherwise is explicitly stated. Let \( \mathbb{N} \) be the set of all positive integers and \( \omega \) the first infinite ordinal. The readers may consult \[1\] for notation and terminology not explicitly given here. Next we recall some definitions and facts.

A continuous mapping \( f : X \to Y \) is called **perfect** if it is a closed onto mapping and all fibers \( f^{-1}(y) \) are compact subsets of \( X \) \[2\], p. 182]. A Tychonoff topological space \( X \) is **Čech-complete** if \( X \) is a \( G_\delta \)-set in every compactification \( cX \) of the space \( X \) \[2\], p. 192].

Then we some notions about topological groups. A topological group \( G \) is called **feathered** if it contains a non-empty compact set \( K \) of countable character in \( G \). It is well-known in \[1\], Theorem 4.3.20] that a topological group \( G \) is feathered if and only if it contains a compact subgroup \( H \) such that the left quotient space \( G/H \) is metrizable. Similarly, the group \( G \) is Čech-complete if and only if it contains a compact subgroup \( H \) such that the left quotient space \( G/H \) is metrizable by a complete metric. The class of feathered topological group is countably productive and is closed under closed-heredity. Moreover, all Čech-complete topological groups are feathered and every metrizable (or locally compact) topological group is feathered.

A topological group is **Raïkov complete** if it is complete with respect to its two-sided uniform group structure. Every topological group \( G \) can be embedded into a unique Raïkov complete topological group as a dense subgroup, which is called the **Raïkov completion** of \( G \) and denoted by \( gG \). Topological groups with compact completions are called **precompact**. As we all know, every Čech-complete topological group is
Raikov complete and a feathered topological group is Raikov complete if and only if it is Čech-complete, see [10, Theorem 4.3.15].

We call a topological group $G$ $\omega$-narrow if for any neighborhood $U$ of the identity $e$ in $G$, there exists a countable subset $C$ of $G$ such that $G = UC = CU$. A topological group $G$ is called $\omega$-balanced if for each neighborhood $U$ of the identity $e$ in $G$, there exists a countable family $\gamma$ of open neighborhoods of $e$ such that for each $x \in G$, there exists $V \in \gamma$ satisfying $xVx^{-1} \subseteq U$. [11, Proposition 3.4.10] showed that every $\omega$-narrow topological group is $\omega$-balanced.

3. Some properties of $PM$-factorizable topological groups

In this section, we give some characterizations of $PM$-factorizable topological groups, such as a topological group $G$ is $PM$-factorizable if and only if $G$ is feathered $M$-factorizable. We also shown that a topological group $G$ is $PR$-factorizable if and only if $G$ is $PM$-factorizable and $\omega$-narrow. Then it is natural to deduce that a topological group $G$ is $PR$-factorizable if and only if $G$ is feathered $R$-factorizable.

Recall that a paracompact $p$-space are the preimages of metrizable spaces under perfect mappings. By the definitions of $PM$-factorizable topological groups, the following result is clear.

**Proposition 3.1.** Every $PM$-factorizable topological group is a paracompact $p$-space.

It was proved in [11, Theorem 4.3.35] that a topological group is feathered iff it is a $p$-space, and iff it is a paracompact $p$-space.

**Proposition 3.2.** Every $PM$-factorizable topological group is feathered.

Then according to the concept of $PR$-factorizable topological groups, every $PR$-factorizable topological group is $PM$-factorizable. Therefore, every compact topological group is $PM$-factorizable. Moreover, we show that each $\omega$-narrow $PM$-factorizable topological group is $PR$-factorizable.

**Theorem 3.3.** A topological group $G$ is $PR$-factorizable if and only if $G$ is $PM$-factorizable and $\omega$-narrow.

**Proof.** Since every $PR$-factorizable topological group is $\omega$-narrow, the necessity is trivial.

Let’s prove the sufficiency. Suppose that $G$ is a $PM$-factorizable and $\omega$-narrow topological group, $f : G \rightarrow \mathbb{R}$ is a continuous real-valued function. Then there exists a perfect homomorphism $\varphi : G \rightarrow K$ onto a metrizable topological group $K$ such that $f = g \circ \varphi$, where $g : K \rightarrow \mathbb{R}$ is continuous. Since $G$ is $\omega$-narrow, so is $K$. Since every first-countable $\omega$-narrow topological group is second-countable, we obtain that $G$ is $PR$-factorizable.

It was proved in [3, Theorem 2.6] that a topological group $G$ is $PR$-factorizable if and only if $G$ is feathered and $\omega$-narrow.

**Proposition 3.4.** Let $G$ be an $\omega$-narrow topological group. Then $G$ is feathered if and only if $G$ is $PM$-factorizable if and only if $G$ is $PR$-factorizable.

**Theorem 3.5.** A topological group $G$ is $PM$-factorizable if and only if one of the following holds:

1. $G$ is metrizable;
2. $G$ is $PR$-factorizable.

**Proof.** Since every $PR$-factorizable topological group is $PM$-factorizable and every metrizable topological group is also $PM$-factorizable, the sufficiency is clear.

Then suppose that a $PM$-factorizable topological group $G$ is not metrizable. By Proposition 3.2, $G$ is feathered, then it contains a compact subgroup $N$ such that the quotient space $G/N$ is metrizable. Then $N$ is not metrizable, otherwise, $G$ will be metrizable, this is a contradiction. It follows from [10, Lemma 4.6] that we can find a family $\{V_\alpha : \alpha \in \omega_1\}$ of open neighborhoods of the identity $e_N$ in $N$ such that $\bigcap_{\alpha \in \omega_1} V_\alpha$ is not a $G_\delta$-set in $N$. For every $\alpha \in \omega_1$, we can choose a neighborhood $U_\alpha$ of the identity $e_G$ in $G$ with $V_\alpha = U_\alpha \cap N$. If $\bigcap_{\alpha \in \omega_1} U_\alpha$ contains a $G_\delta$-subgroup $M$ in $G$, then $\bigcap_{\alpha \in \omega_1} V_\alpha$ contains a $G_\delta$-subgroup $M \cap N$. 

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Corollary 3.6. A topological group \( G \) is PR-factorizable if and only if \( G \) is a feathered Lindelöf \( \Sigma \)-group.

Proof. The sufficiency is clear. Indeed, a feathered Lindelöf \( \Sigma \)-group \( G \) is Lindelöf feathered, so \( G \) is PR-factorizable, as a topological group is Lindelöf feathered if and only if it is PR-factorizable by [2, Theorem 2.5].

Then we show the necessity. Let \( G \) be a PR-factorizable topological group. It follows from [2, Theorem 2.6] that \( G \) is \( \omega \)-narrow and feathered. Then, by [3, Theorem 3.4], for a feathered topological group \( G \), \( G \) is \( \omega \)-narrow if and only if \( G \) is a Lindelöf \( \Sigma \)-group.

Theorem 3.7. A topological group \( G \) is PM-factorizable if and only if \( G \) is feathered \( \mathcal{M} \)-factorizable.

Proof. The necessity is trivial, it suffices to claim the sufficiency.

Let \( G \) be a feathered \( \mathcal{M} \)-factorizable topological group and \( f \) a continuous real-valued function on \( G \). Then we can find a continuous homomorphism \( \pi : G \to H \) onto a metrizable topological group \( H \) and a continuous real-valued function \( g \) on \( H \) such that \( f = g \circ \pi \). Since every \( \mathcal{M} \)-factorizable group is \( \omega \)-balanced by [10, Theorem 3.1] and \( G \) is feathered, there exists a perfect homomorphism \( p : G \to K \) onto a metrizable topological group \( K \). Let \( \varphi \) be the diagonal product of the homomorphisms \( \pi \) and \( p \) and \( M = \varphi(G) \subseteq H \times K \).

Since \( p \) is perfect, the homomorphism \( \varphi \) is also perfect by [2, Theorem 3.7.11]. By the definition of \( \varphi \), we can find continuous homomorphisms \( q_H : M \to H \) and \( q_K : M \to K \) satisfying \( \pi = q_H \circ \varphi \) and \( p = q_K \circ \varphi \).

Then from [2, Proposition 3.7.5], it follows that \( q_K \) is perfect.

Then the subgroup \( M \) of \( H \times K \) is metrizable. Indeed, since \( H \) and \( K \) are both first-countable and the property of first-countability is hereditary, it is clear that the subgroup \( M \) of \( H \times K \) is first-countable, hence \( M \) is metrizable.

![Diagram]

We define a continuous real-valued function \( h \) on \( M \) by \( h = g \circ q_H \). Then, for each continuous real-valued function \( f \) on \( G \), we can find a perfect homomorphism \( \varphi : G \to M \) onto a metrizable topological group \( M \) and a continuous function \( h : M \to \mathbb{R} \) such that \( f = h \circ \varphi \). Therefore, we conclude that \( G \) is PM-factorizable.

By Theorems 3.3 and 3.7, we obtain the following result.

Corollary 3.8. A topological group \( G \) is PR-factorizable if and only if \( G \) is feathered \( \mathbb{R} \)-factorizable.

Indeed, it was proved in [3, Theorem 3.4] that for a feathered topological group \( G \), \( G \) is \( \omega \)-narrow if and only if it is \( \mathbb{R} \)-factorizable. Moreover, [3, Theorem 2.6] presented that a topological group \( G \) is PR-factorizable if and only if \( G \) is \( \omega \)-narrow and feathered. As a topological group \( G \) is \( \mathbb{R} \)-factorizable if and only if it is \( \mathcal{M} \)-factorizable and \( \omega \)-narrow by [10, Theorem 3.2], the Corollary 3.8 also can be obtained.

Proposition 3.9. A topological group is PM-factorizable whenever it contains a dense PM-factorizable subgroup.
Proof. Let $G$ be a topological group with a dense subgroup $H$ such that $H$ is $PM$-factorizable. It follows from [10, Theorem 5.8] that whenever a topological group has a dense openly $M$-factorizable topological group, then it is also openly $M$-factorizable. Since $H$ is a $PM$-factorizable group, it is clear that $H$ is openly $M$-factorizable, which deduces that $G$ is openly $M$-factorizable. Then $G$ is a $M$-factorizable topological group. By Proposition 3.2 the dense subgroup $H$ is feathered, and $G$ is also feathered since a topological group is feathered if it contains a dense feathered subgroup. Therefore, $G$ is a $PM$-factorizable topological group by Theorem 3.7.

Since for a feathered topological group $G$, $G$ is Čech-complete if and only if it is Raïkov complete, it is easy to see the following by Proposition 3.9.

Corollary 3.10. Every $PM$-factorizable topological group is a dense subgroup of a $PM$-factorizable Čech-complete topological group.

For a topological group $G$, if $G$ contains a dense $\omega$-narrow subgroup, $G$ is also $\omega$-narrow. Therefore, the following corollary is deduced by Theorem 3.3 and Proposition 3.9.

Corollary 3.11. A topological group is $PR$-factorizable whenever it contains a dense $PR$-factorizable subgroup.

Then from [11, Theorem 4.8], a locally compact group $G$ is $M$-factorizable if and only if $G$ is metrizable or $G$ is $\sigma$-compact. Then, it is well-known that every locally compact topological group is feathered, so the characterization also holds for $PM$-factorizable topological groups by Theorem 3.7.

Proposition 3.12. A locally compact group $G$ is $PM$-factorizable if and only if one of the following conditions holds:

1. $G$ is metrizable;
2. $G$ is $\sigma$-compact.

From [1, Example 8.2.1], there is an Abelian $P$-group $G$ and a dense subgroup $H$ of $G$ such that $G$ is Lindelöf, hence $R$-factorizable, but $H$ is not $R$-factorizable. In particular, $H$ is $\omega$-narrow. Therefore, $H$ is not $M$-factorizable. Moreover, W. He et al. showed that every subgroup of an $M$-factorizable feathered group is $M$-factorizable, it also means that every subgroup of a $PM$-factorizable is $M$-factorizable.

Proposition 3.13. Every closed subgroup of a $PM$-factorizable topological group is $PM$-factorizable.

Proof. Let $G$ be a $PM$-factorizable topological group and $H$ a closed subgroup of $G$. By Theorem 3.7, $G$ is $M$-factorizable and feathered. According to [8, Lemma 4.1], every subgroup of an $M$-factorizable feathered group is $M$-factorizable, so $H$ is a $M$-factorizable topological group. Moreover, it is well-known that a closed subspace of a feathered space is feathered. Hence, $H$ is $M$-factorizable and feathered. We have that $H$ is $PM$-factorizable by Theorem 3.7.

From [1, Theorem 3.4.4], every subgroup of an $\omega$-narrow topological group is $\omega$-narrow, so the following is clear.

Corollary 3.14. Every closed subgroup of a $PR$-factorizable topological group is $PR$-factorizable.

Theorem 3.15. If $f : G \to H$ is an open continuous homomorphism of a $PM$-factorizable topological group onto a topological group $H$, then $H$ is $PM$-factorizable.

Proof. Indeed, it was proved in [10, Corollary 3.8] that a quotient group of a $M$-factorizable topological group is also $M$-factorizable. Moreover, by [1, Corollary 4.3.24], if $f : G \to H$ is an open continuous homomorphism of a feathered topological group onto a topological group $H$, then $H$ is also feathered. Therefore, if $G$ is $PM$-factorizable, that is, feathered and $M$-factorizable by Theorem 3.7, then the topological group $H$ is also $PM$-factorizable as an open continuous homomorphic image.
From [1, Proposition 3.4.2], if a topological group \( H \) is a continuous homomorphic image of an \( \omega \)-narrow topological group \( G \), then \( H \) is also \( \omega \)-narrow. The following corollary is follows from Theorem 3.3.

**Corollary 3.16.** [3, Theorem 2.9] If \( f : G \rightarrow H \) is an open continuous homomorphism of a \( \mathcal{P} \mathcal{R} \)-factorizable topological group onto a topological group \( H \), then \( H \) is \( \mathcal{P} \mathcal{R} \)-factorizable.

### 4. Products of \( \mathcal{P} \mathcal{M} \)-factorizable topological groups

In this section, we investigate some properties about products of \( \mathcal{P} \mathcal{M} \)-factorizable topological groups. In particular, some interesting properties of \( \mathcal{M} \)-factorizable topological groups in [8, 10] are strengthened to \( \mathcal{P} \mathcal{M} \)-factorizable topological groups. For example, the product group \( G = \prod_{n \in \mathbb{N}} G_n \) of countably many \( \mathcal{P} \mathcal{M} \)-factorizable topological groups is \( \mathcal{P} \mathcal{M} \)-factorizable if and only if every factor \( G_n \) is metrizable or every \( G_n \) is \( \mathcal{P} \mathcal{R} \)-factorizable, the product of a \( \mathcal{P} \mathcal{M} \)-factorizable topological group with a compact metrizable topological group is \( \mathcal{P} \mathcal{M} \)-factorizable.

First, according to the result that a locally compact group \( G \) is \( \mathcal{M} \)-factorizable if and only if \( G \) is metrizable or \( G \) is \( \sigma \)-compact, H. Zhang et al. gave an example to show that a product of two \( \mathcal{M} \)-factorizable topological groups may fail to be \( \mathcal{M} \)-factorizable. By further observation about the example, we find that a product of two \( \mathcal{P} \mathcal{M} \)-factorizable topological groups may not be \( \mathcal{M} \)-factorizable, so naturally not be \( \mathcal{P} \mathcal{M} \)-factorizable.

**Example 4.1.** Assume that \( G \) is a metrizable locally compact group which is not \( \sigma \)-compact and \( H \) is a compact and non-metrizable group. Obviously, both \( G \) and \( H \) are \( \mathcal{M} \)-factorizable. Moreover, each locally compact topological group is feathered, then \( G \) and \( H \) both are \( \mathcal{P} \mathcal{M} \)-factorizable. However, the product group \( G \times H \) is neither metrizable nor \( \sigma \)-compact, which deduces that \( G \times H \) is not \( \mathcal{M} \)-factorizable since a locally compact group is \( \mathcal{M} \)-factorizable if and only if it is metrizable or it is \( \sigma \)-compact. Therefore, the product group \( G \times H \) is not \( \mathcal{P} \mathcal{M} \)-factorizable by Proposition 3.12. (Indeed, since \( G \times H \) is feathered but not \( \mathcal{M} \)-factorizable, it can also be yielded that it is not \( \mathcal{P} \mathcal{M} \)-factorizable by Theorem 3.7.)

**Theorem 4.2.** The product \( G = \prod_{n \in \mathbb{N}} G_n \) of countably many \( \mathcal{P} \mathcal{M} \)-factorizable topological groups is \( \mathcal{P} \mathcal{M} \)-factorizable if and only if every factor \( G_n \) is metrizable or every \( G_n \) is \( \mathcal{P} \mathcal{R} \)-factorizable.

**Proof.** It follows from [1, Proposition 4.3.13] that the product space \( G \) is feathered. If \( G \) is \( \mathcal{P} \mathcal{M} \)-factorizable, by Theorem 3.10 \( G \) is either metrizable or \( \mathcal{P} \mathcal{R} \)-factorizable. If \( G \) is metrizable, each \( G_n \) is also metrizable. If \( G \) is \( \mathcal{P} \mathcal{R} \)-factorizable, so is every \( G_n \) by Corollary 3.14.

On the contrary, if every \( G_n \) is metrizable, it is clear that \( G \) is also metrizable. If every \( G_n \) is a \( \mathcal{P} \mathcal{R} \)-factorizable topological group, it is easy to see that \( G \) is \( \mathcal{R} \)-factorizable. Moreover, \( G \) is feathered, we conclude that \( G \) is \( \mathcal{P} \mathcal{R} \)-factorizable by Corollary 3.8. \( \square \)

The product of countably many \( \mathcal{P} \mathcal{R} \)-factorizable topological groups is also \( \mathcal{P} \mathcal{R} \)-factorizable, see [3, Proposition 2.7], then it is clear to achieve the following by Theorems 3.5 and 4.2.

**Proposition 4.3.** If \( G \) is a \( \mathcal{P} \mathcal{M} \)-factorizable topological group, then so is \( G^\omega \).

**Remark 4.4.** Let \( G \) be a compact group with \( w(G) > \omega \) and \( D \) an uncountable discrete group. Since \( G \) and \( D \) both are feathered, it is clear that \( G \times D \) is feathered. However, \( G \) is not metrizable and \( D \) is not \( \mathcal{P} \mathcal{R} \)-factorizable, then \( G \times D \) is not \( \mathcal{P} \mathcal{M} \)-factorizable, hence is also not \( \mathcal{M} \)-factorizable by Theorem 5.7.

By Lemma 3.1 and Theorem 4.7 of [8], an \( \mathcal{M} \)-factorizable topological group which contains a non-metrizable pseudocompact subspace is \( \omega \)-narrow.

**Theorem 4.5.** Let \( G \) and \( H \) be topological groups, where \( G \) contains a non-metrizable pseudocompact subspace. If \( G \times H \) is \( \mathcal{P} \mathcal{M} \)-factorizable, then \( G \times H \) is \( \mathcal{P} \mathcal{R} \)-factorizable.
By [3, Theorem 2.6], a topological group is feathered and is a 
by Theorem 3.8, hence is P
If G
feathered, then K
factorizable if and only if either both G
and P
is not metrizable, it follows from Theorem 3.5 that G
P
factorizable, hence G
R
-factorizable topological group is
P
M
-factorizable if and only if either both G
and K
are metrizable or G
is P
R
-factorizable.

Proof. Let the product group G × K be P
M
-factorizable. Then the factors G and K are P
M
-factorizable
as the open continuous images by Theorem 3.15. If G × K is not metrizable, then either G or K is not
metrizable. If G is not metrizable, it follows from Theorem 3.5 that G is P
R
-factorizable. On the other case, if K is not metrizable, K is a non-metrizable pseudocompact topological group, then G is ω-narrow.

By [3, Theorem 2.6], a topological group is feathered and ω-narrow if and only if it is P
R
-factorizable, hence G is a P
R
-factorizable topological group.

On the contrary, if G and K are metrizable topological groups, it is clear that G × K is P
M
-factorizable. If G is P
R
-factorizable, then G × K is R-factorizable as K is pseudocompact. Moreover, both G and K are feathered, then G × K is also feathered, which deduces that G × K is a P
R
-factorizable topological group by Theorem 3.8 hence is P
M
-factorizable.

Theorem 4.7. Let G and H be topological groups, where the Raïkov completion ρG of G contains a non-metrizable compact subspace. If G × H is P
M
-factorizable, then H is P
R
-factorizable.

Proof. Since every P
M
-factorizable topological group is M-factorizable, it follows from [1, Theorem 3.11] that H is pseudo-ω1-compact, hence H is ω-narrow by [1, Proposition 3.4.31]. Since the product G × H is P
M
-factorizable, so is H by Theorem 3.15. Therefore, we have that H is a P
R
-factorizable topological group by Theorem 3.8.

Proposition 4.8. If the product G × H of topological groups is P
M
-factorizable and the group G is precompact, then either G is second countable or H is P
R
-factorizable.

Proof. The first part that G is second countable follows just from [1, Proposition 3.12] and the second part is deduced by Theorem 4.7.

Theorem 4.9. Let G be a feathered group and H a precompact feathered group. Then G × H is P
M
-factorizable if and only if either both G and H are metrizable or G is Lindelöf Σ-group.

Proof. The necessity is claimed in [1, Theorem 3.13], where H just need to be precompact.

It suffices to prove the sufficiency. On the first case, if both G and H are metrizable, it is trivial that G × H is P
M
-factorizable. On the other case, let G be a feathered Lindelöf Σ-group and H a precompact feathered group. Then the Raïkov completion ρG of H is compact. G × H is R-factorizable as a subgroup of the Lindelöf Σ-group G × ρH. Moreover, since both G and H are feathered, G × H is also feathered, hence is P
R
-factorizable by Theorem 3.8. We obtain that G × H is a P
M
-factorizable topological group.

W. He et al. proved that the product of an M-factorizable topological group with a locally compact separable metrizable topological group is M-factorizable, see [1, Theorem 3.14]. Then we revise the proof to show that the product of a P
M
-factorizable topological group with a locally compact separable metrizable topological group is P
M
-factorizable.

Theorem 4.10. Let G be a P
M
-factorizable topological group and H a locally compact separable metrizable topological group. Then G × H is P
M
-factorizable.

Proof. Since H is a locally compact separable metrizable topological group, H is σ-compact. Then there is an increasing sequence \{H_n : n ∈ N\} of compact subsets of H such that H = \bigcup_{n ∈ N} H_n and H_n is contained in the interior of H_{n+1} for each n ∈ N. Let f be a continuous real-valued function on G × H. Denote by C(H_n) the space of continuous real-valued functions on H_n with sup-norm, for each n ∈ N. Then define a
mapping \( \Psi_n : G \to C(H_n) \) by \( \Psi_n(x)(y) = f(x, y) \) for all \( x \in G \) and \( y \in H_n \). Since \( H_n \) is compact and second countable, \( \Psi_n \) is continuous and \( C(H_n) \) is second countable. Put \( \Psi \) the diagonal product of \( \{\Psi_n : n \in \mathbb{N}\} \). Since \( \prod_{n \in \mathbb{N}} C(H_n) \) is second countable, it is clear that \( \Psi(G) \) is also second countable.

By the hypothesis, \( G \) is \( PM \)-factorizable, we can find a perfect homomorphism \( \pi \) of \( G \) onto a metrizable group \( K \) and a continuous mapping \( \psi \) of \( K \) to \( \Psi(G) \) such that \( \Psi = \psi \circ \pi \). Take \( y \in H \) and choose \( n \in \mathbb{N} \) with \( y \in H_n \). Let \( x, x' \in G \). If \( \pi(x) = \pi(x') \), then \( \Psi(x) = \Psi(x') \). Then \( \Psi_n(x) = \Psi_n(x') \), that is, \( f(x, y) = f(x', y) \). Therefore, we can define a mapping \( h : K \times H \to \mathbb{R} \) such that \( f = h \circ (\pi \times id_H) \). It is not difficult to verify that \( h \) is continuous. Since both \( K \) and \( H \) are metrizable topological groups, \( G \times H \) is also metrizable. Moreover, \( \pi \) is a perfect mapping, so is the mapping \( \pi \times id_H \). Thus, we conclude that the product \( G \times H \) is \( PM \)-factorizable.

**Theorem 4.11.** Let \( G \) be a \( PR \)-factorizable topological group and \( H \) a locally compact separable metrizable topological group. Then \( G \times H \) is \( PR \)-factorizable.

**Proof.** First, \( G \times H \) is \( PM \)-factorizable by Theorem 4.10. Moreover, since both \( G \) and \( H \) are \( \omega \)-narrow, then \( G \times H \) is \( \omega \)-narrow and it is achieved that it is \( PR \)-factorizable by Theorem 3.3.

**Corollary 4.12.** Let \( G \) be a \( PM \)-factorizable topological group and \( H \) a compact metrizable topological group. Then \( G \times H \) is \( PM \)-factorizable.

**Corollary 4.13.** Let \( G \) be a \( PR \)-factorizable topological group and \( H \) a compact metrizable topological group. Then \( G \times H \) is \( PR \)-factorizable.

A topological space \( X \) is called pseudo-\( R_1 \)-compact if every discrete family of open subsets of \( X \) is countable. As we all know, every separable space is pseudo-\( R_1 \)-compact and every pseudo-\( R_1 \)-compact topological group is \( \omega \)-narrow. Therefore, the following follows from Theorem 3.3.

**Corollary 4.14.** A pseudo-\( R_1 \)-compact \( PM \)-factorizable topological group is \( PR \)-factorizable.

**Corollary 4.15.** A product of a pseudo-\( R_1 \)-compact \( PM \)-factorizable topological group and a compact group is \( PR \)-factorizable.

**Proof.** Indeed, let \( G \) be a pseudo-\( R_1 \)-compact \( PM \)-factorizable topological group and \( H \) a compact group. It was proved in [10] Corollary 5.2] that a product of a pseudo-\( R_1 \)-compact \( M \)-factorizable topological group and a compact group is \( IR \)-factorizable. Then the product \( G \times H \) is \( IR \)-factorizable. Moreover, \( G \) is feathered by Proposition 4.2] and \( H \) is also feathered, so \( G \times H \) is a feathered group, which deduces that \( G \times H \) is \( PR \)-factorizable by Theorem 3.3.

It was proved in [11] Theorem 5.4] that if \( G \times K \) is \( M \)-factorizable, where \( G \) is an \( M \)-factorizable group and \( K \) is compact group, then \( G \) is pseudo-\( R_1 \)-compact or \( K \) is metrizable. Then we have the following by Corollaries 4.12 and 4.15.

**Theorem 4.16.** Let \( G \) be a \( PM \)-factorizable topological group and \( K \) a compact group. Then \( G \times K \) is \( PM \)-factorizable if and only if one of the following conditions holds:

1. \( K \) is metrizable;
2. \( G \) is pseudo-\( R_1 \)-compact.

Recall that a mapping \( f : X \to Y \) is \( d \)-open if for every open set \( U \in X \), the image \( f(U) \) is contained in the interior of the closure of \( f(U) \). The following results was proved in [8], see Proposition 6.3 and Theorem 6.5.

**Proposition 4.17.** An image of a feathered topological group under a continuous \( d \)-open homomorphism is feathered.

**Proposition 4.18.** Let \( p \) be a continuous \( d \)-open homomorphism from a topological group \( G \) onto a topological group \( H \). If \( G \) is \( M \)-factorizable, so is \( H \).
Since it is proved in Theorem 3.7 that a topological group $G$ is $PM$-factorizable if and only if $G$ is feathered $M$-factorizable, the following is deduced by two propositions above.

**Corollary 4.19.** If a topological group $H$ is a continuous $d$-open homomorphic image of a $PM$-factorizable topological group $G$, then $H$ is also $PM$-factorizable.

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