RESOLVABILITY IN COMPLEMENT OF THE INTERSECTION GRAPH OF ANNIHILATOR SUBMODULES OF A MODULE

S. B. PEJMAN, SH. PAYROVI* AND S. BABAEI

Abstract. Let $R$ be a commutative ring and $M$ be an $R$-module. The intersection graph of annihilator submodules of $M$, denoted by $GA(M)$, is a simple undirected graph whose vertices are the classes of elements of $Z(M) \setminus \text{Ann}_R(M)$ and two distinct classes $[a]$ and $[b]$ are adjacent if and only if $\text{Ann}_M(a) \cap \text{Ann}_M(b) \neq 0$. In this paper, we study the diameter and girth of $GA(M)$. Furthermore, we calculate the domination number, metric dimension, adjacency metric dimension and local metric dimension of $GA(M)$.

1. Introduction

The intersection graph of ideals of a commutative ring was studied in [5] and rings classified with some specific properties of their intersection graphs in [2, 9, 11]. The intersection graph of submodules of a module defined and studied in [1]. As noted in [1] the intersection graph of submodules of a module, denoted by $G(M)$, is a graph whose vertices are in one to one correspondence with all non-trivial submodules of $M$ and two distinct vertices are adjacent if and only if the corresponding submodules of $M$ have non-zero intersection. The complement of the intersection graph of submodules of a module is considered in [3]. For more work on the intersection graph of modules see [14].

Let $R$ be a commutative ring and $M$ be an $R$-module. For $a, b \in R$, we say that $a \sim b$ whenever $\text{Ann}_M(a) = \text{Ann}_M(b)$. It is easy to see that

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*Corresponding author.
\( \sim \) is an equivalence relation on \( R \). If \([a]\) denotes the class of \( a \), then \([a] = \text{Ann}_R(M)\) and \([a] = R \setminus Z(M)\) whenever \( a \in \text{Ann}_R(M)\) and \( a \in R \setminus Z(M)\) respectively; the other equivalence classes form a partition of \( Z(M) \setminus \text{Ann}_R(M)\). The intersection graph of annihilator submodules of \( M \) studied in [10] and denoted by \( GA(M) \), is a simple undirected graph whose vertices are the classes of elements of \( Z(M) \setminus \text{Ann}_R(M)\) and two distinct classes \([a]\) and \([b]\) are adjacent whenever \( \text{Ann}_M(a) \cap \text{Ann}_M(b) \neq 0 \). In this paper, we study the complement of the intersection graph of annihilator submodules of \( M \) which is denoted by \( \overline{GA(M)} \). In Section 2, we study the diameter and girth of \( \overline{GA(M)} \) and in Section 3 we calculate the domination number, metric dimension, adjacency metric dimension and local metric dimension of \( \overline{GA(M)} \). More precisely; let \( M \) be a Noetherian \( R \)-module, \( GA(M), \overline{GA(M)} \) be connected graphs of order \( m \) and \( |m - \text{Ass}_R(M)| = \omega(GA(M)) = n(n \geq 2) \) we prove that

(i) \( \gamma(GA(M)) = n \)

(ii) If \( GA(M) \) has \( k \) end-vertices, then

\[
\dim(GA(M)) = \dim_A(GA(M)) = m - 2n + k - 1.
\]

(iii) If \( GA(M) \) has no end-vertex, then

\[
\dim(GA(M)) = \dim_A(GA(M)) = m - 2n.
\]

(iv) \( \dim(GA(M)) = n - 1 \).

Let \( G \) be a graph with the vertex set \( V(G) \) and the edge set \( E(G) \). A graph with no edge is called null graph. For every \( u, v \in V(G) \), the distance between \( u \) and \( v \) is defined as the length of a shortest path from \( u \) to \( v \) and is denoted by \( d(u, v) \). We write \( u \sim v \) if \( d(u, v) = 1 \) and \( u \not\sim v \) otherwise. For \( H \subseteq V(G) \), the induced subgraph on \( H \), consists of \( H \) and all edges whose endpoints are contained in \( H \). Assume that \( u \) is a vertex of \( G \). The open neighborhood of \( u \) is defined as \( N(u) = \{v \in V(G) : d(u, v) = 1\} \) and the closed neighborhood of \( u \) is \( N[u] = N(u) \cup \{u\} \). For distinct vertices \( u, v \in V(G) \), if \( N(u) = N(v) \), then \( u \) and \( v \) are non-adjacent twins. The degree of a vertex \( u \), denoted by \( \text{deg}(u) \), is the number of edges incident to \( u \). Also, \( u \) is called end-vertex if \( \text{deg}(u) = 1 \). The diameter of \( G \) is \( \text{diam}(G) = \sup\{d(u, v)\} \) \( u \) and \( v \) are vertices of \( G \). The girth of \( G \), denoted by \( \text{gr}(G) \), is the length of a shortest cycle in \( G \) (\( \text{gr}(G) = \infty \) if \( G \) contains no cycles). A cycle with \( n \) vertices will be denoted by \( C_n \). The complete graph with \( n \) vertices will be denoted by \( K_n \). A complete bipartite graph is a graph \( G \) whose vertex set may be partitioned into two disjoint non-empty vertex sets \( A \) and \( B \) such that two distinct vertices are adjacent if and only if they are in distinct vertex sets which is denoted by \( K_{|A|,|B|} \). A
clique of $G$ is a complete subgraph of $G$ and the number of vertices in the largest clique of $G$, denoted by $\omega(G)$, is called the clique number of $G$.

A dominating set of $G$ is a subset $D$ of $V(G)$ such that every vertex in $V(G) \setminus D$ is adjacent to some vertex in $D$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set. Let $G$ be a connected graph. Assume that $W = \{w_1, w_2, \ldots, w_k\}$ is an ordered subset of $V(G)$. The metric representation (local metric representation) of a vertex $u \in V(G)$ with respect to $W$ is the vector $r(u \mid W) = (d(u, w_1), d(u, w_2), \ldots, d(u, w_k))$. The set $W$ called a resolving (local resolving) set for $G$ if different vertices (adjacent vertices) of $G$ have different representation with respect to $W$. The minimum cardinality of any resolving (local resolving) set of $G$ is the metric dimension (local metric dimension) of $G$ and is denoted by $\dim(G)$ ($\dim_\ell(G)$).

Let $G$ be a connected graph. Assume that $W = \{w_1, w_2, \ldots, w_k\}$ is an ordered subset of $V(G)$. The adjacency representation of a vertex $u \in V(G)$ with respect to an ordered set $W_A = \{w_1, w_2, \ldots, w_k\}$ is the vector $r(u \mid W_A) = (a_G(u, w_1), \ldots, a_G(u, w_k))$, where

$$a_G(u, v) = \begin{cases} 
0 & u = v \\
1 & u - v \\
2 & u \not= v 
\end{cases}$$

for all $v \in V(G)$. The set $W_A$ is an adjacency resolving set for $G$ if the vectors $r(u \mid W_A)$ are distinct for every $u \in V(G)$. The minimum cardinality of an adjacency resolving set is the adjacency dimension of $G$, denoted by $\dim_A(G)$, see [7].

Throughout this paper, $R$ is a commutative ring with non-zero identity and $M$ is an $R$-module. The set of zero-divisors of $M$, denoted by $Z(M)$ is defined to be the set $\{r \in R : \exists m \neq 0 \in M : rm = 0\}$. For $a \in R$, $\text{Ann}_M(a) = \{m \in M : am = 0\}$. A proper submodule $P$ of $M$ is said to be prime whenever for $r \in R$ and $m \in M$, $rm \in P$ implies that $m \in P$ or $r \in \text{Ann}_R(M/P)$. Let $\text{Spec}_R(M)$ denote the set of prime submodules of $M$ and $m - \text{Ass}_R(M) = \{P \in \text{Spec}_R(M) : P = \text{Ann}_M(a) \text{ for some } 0 \neq a \in R\}$. For notations and terminologies not given in this article, the reader is referred to [12, 13].

**Theorem 1.1.** [4, Theorem 5(i)] For all $a \in R$, $aM + \text{Ann}_M(a)$ is an essential submodule of $M$.

**Theorem 1.2.** [10, Theorem 2.6] Let $M$ be a Noetherian $R$-module. Then $GA(M)$ is a disconnected graph if and only if $m - \text{Ass}_R(M) = \{P_1, P_2\}$ and $P_1 \cap P_2 = 0$. 
Theorem 1.3. [6, Corollary 2.4] Suppose that \( u, v \) are twins in a connected graph \( G \) and \( S \) resolves \( G \). Then either \( u \) or \( v \) is in \( S \). Moreover, if \( u \in S \) and \( v \notin S \), then \( (S \setminus \{u\}) \cup \{v\} \) also resolves \( G \).

2. Diameter and Girth of \( \overline{GA(M)} \)

In this section, we study diameter and girth of \( \overline{GA(M)} \). Note that if \( M \) is a Noetherian \( R \)-module and \( m - \text{Ass}_R(M) = \{\text{Ann}_M(a)\} \), then \([a]\) is a universal vertex in \( \overline{GA(M)} \) so \( \overline{GA(M)} \) is a disconnected graph. Hence, \(|m - \text{Ass}_R(M)| \geq 2 \) whenever we assume \( \overline{GA(M)} \) is a connected graph.

Lemma 2.1. Let \( M \) be a Noetherian \( R \)-module and \( \overline{GA(M)} \) be a non-empty connected graph. If \( \text{Ann}_M(a), \text{Ann}_M(b) \in m - \text{Ass}_R(M) \), then \([a]\) and \([b]\) have no common neighbors in \( \overline{GA(M)} \).

Proof. Assume that \( a, b \in Z(M) \setminus \text{Ann}_R(M) \) and \( P_1 = \text{Ann}_M(a), P_2 = \text{Ann}_M(b) \) are two distinct elements of \( m - \text{Ass}_R(M) \). Assume in the contrary that \([a]\) and \([b]\) have a common neighbor in \( \overline{GA(M)} \) such as \([x]\). Thus \( P_1 \cap \text{Ann}_M(x) = 0 = P_2 \cap \text{Ann}_M(x) \). So \( \text{Ann}_M(x) \not\subseteq P_2 \).

Suppose that \( P_1 \not\subseteq P_2 \). Hence, there exist \( m_1 \in \text{Ann}_M(x) \setminus P_2 \) and \( m_2 \in P_1 \setminus P_2 \) such that \( x m_1 = a m_2 = 0 \in P_2 \). So it follows that \( a, x \in \text{Ann}_R(M/\text{Ann}_M(b)) = \text{Ann}_R(bM) \). Therefore, \( bM \subseteq \text{Ann}_M(a) \cap \text{Ann}_M(x) = P_1 \cap \text{Ann}_M(x) \). Thus \( bM = 0 \) and \( b \in \text{Ann}_R(M) \), contrary to the assumption. Hence, \( P_1 \subseteq P_2 \). By a similar argument one can show that \( P_2 \subseteq P_1 \) so \( P_1 = P_2 \) that is a contradiction.

Lemma 2.2. Let \( M \) be a Noetherian \( R \)-module and \( \overline{GA(M)} \), \( \overline{GA(M)} \) be non-empty connected graphs. If \( P_1 = \text{Ann}_M(a), P_2 = \text{Ann}_M(b) \in m - \text{Ass}_R(M) \), then \( d([a],[b]) = 3 \).

Proof. Let \( P_1 = \text{Ann}_M(a) \) and \( P_2 = \text{Ann}_M(b) \) be two distinct elements of \( m - \text{Ass}_R(M) \). By the assumption \( \overline{GA(M)} \) is a connected graph so by Theorem 1.2, \( P_1 \cap P_2 \neq 0 \). Thus \( d([a],[b]) \neq 1 \) also Lemma 2.1 shows that \( d([a],[b]) \neq 2 \). Let \([x],[y]\) be two arbitrary vertices of \( \overline{GA(M)} \) such that \([x] - [a]\) and \([y] - [b]\). Then \( \text{Ann}_M(a) \cap \text{Ann}_M(x) = 0 \subseteq P_2 \) so either \( \text{Ann}_M(a) \subseteq P_2 \) or \( \text{Ann}_M(x) \subseteq P_2 \). If \( \text{Ann}_M(a) \subseteq \text{Ann}_M(b) \), then \([a] - [y] - [b]\) contrary to the Lemma 2.1. Thus \( \text{Ann}_M(x) \subseteq \text{Ann}_M(b) \) and so \([x] - [y]\) which completed the proof.

Lemma 2.3. Let \( M \) be a Noetherian \( R \)-module and \( \overline{GA(M)} \) be a non-empty connected graph. If \( P = \text{Ann}_M(a) \in m - \text{Ass}_R(M) \), then the induced subgraph on the vertices which are adjacent to \([a]\) is empty.
Proof. Let \( P = \text{Ann}_M(a) \in m - \text{Ass}_R(M) \) and let \([x], [y] \in V(\overline{GA(M)})\) be such that \([x] - [a] - [y]\). Suppose on the contrary that \([x] - [y]\). Thus \(\text{Ann}_M(x) \cap \text{Ann}_M(y) = 0 \subseteq P\). Hence, either \(\text{Ann}_M(x) \subseteq P\) or \(\text{Ann}_M(y) \subseteq P\) which contradicts to the assumption \([x] - [a] - [y]\). \(\square\)

**Theorem 2.4.** Let \(M\) be a Noetherian \(R\)-module and \(\overline{GA(M)}\) be a non-empty connected graph. Then \(\text{diam}(\overline{GA(M)}) \leq 3\).

Proof. Let \([a] \text{ and } [b] \) be two distinct vertices of \(\overline{GA(M)}\). If \(\text{Ann}_M(a) \cap \text{Ann}_M(b) = 0\), then \(d([a], [b]) = 1\). Thus assume that \(\text{Ann}_M(a) \cap \text{Ann}_M(b) \neq 0\). If \(ab \notin \text{Ann}_R(M)\), then \([ab] \in V(\overline{GA(M)})\) and by connectivity of \(\overline{GA(M)}\), it follows that there exists \([x] \in V(\overline{GA(M)})\) such that \(\text{Ann}_M(x) \cap \text{Ann}_M(ab) = 0\). Hence, \(\text{Ann}_M(x) \cap \text{Ann}_M(a) = 0\) and \(\text{Ann}_M(x) \cap \text{Ann}_M(b) = 0\). So \(\overline{GA(M)}\) has the path \([a] - [x] - [b]\) as a subgraph. Therefore, \(d([a], [b]) \leq 2\). Now, assume that \(ab \in \text{Ann}_R(M)\). If there exists \(\text{Ann}_M(x) \in m - \text{Ass}_R(M)\) such that \(\text{Ann}_M(a) \subseteq \text{Ann}_M(x)\) and \(\text{Ann}_M(b) \subseteq \text{Ann}_M(x)\), then \(aM \subseteq \text{Ann}_M(b) \subseteq \text{Ann}_M(x)\) so \(aM + \text{Ann}_M(a) \subseteq \text{Ann}_M(x)\). By Theorem 1.1, \([x]\) is a universal vertex of \(\overline{GA(M)}\). Hence, \([x]\) is an isolated vertex of \(\overline{GA(M)}\) that is a contradiction. Hence, \(\text{Ann}_M(a) \subseteq \text{Ann}_M(x)\) and \(\text{Ann}_M(b) \subseteq \text{Ann}_M(y)\) for some \(\text{Ann}_M(x), \text{Ann}_M(y) \in m - \text{Ass}_R(M)\). By Lemma 2.2, there exist \([e], [f] \in V(\overline{GA(M)})\) such that \(\text{Ann}_M(x) \cap \text{Ann}_M(e) = 0\) and \(\text{Ann}_M(y) \cap \text{Ann}_M(f) = 0\) so \([x] - [e] - [f] - [y]\) is a path in \(\overline{GA(M)}\). Hence, \([a] - [e] - [f] - [b]\) is a subgraph of \(\overline{GA(M)}\) which implies that \(\text{diam}(\overline{GA(M)}) \leq 3\). \(\square\)

**Corollary 2.5.** Let \(M\) be a Noetherian \(R\)-module and \(\overline{GA(M)}\) be a disconnected graph. Then \(\overline{GA(M)}\) is a connected graph and \(\text{diam}(\overline{GA(M)}) \leq 2\).

**Theorem 2.6.** Let \(M\) be a Noetherian \(R\)-module and \(\overline{GA(M)}\) be a connected graph. Then either \(\text{gr}(\overline{GA(M)}) \leq 4\) or \(\text{gr}(\overline{GA(M)}) = \infty\).

Proof. Assume that \(n \in \mathbb{N}\) and \(C = ([a_1], \ldots, [a_n])\) is a cycle in \(\overline{GA(M)}\). Suppose that \(\text{Ann}_M(x) = P \in m - \text{Ass}_R(M)\) and \([y]\) is a vertex of \(V(\overline{GA(M)})\) that is adjacent to \([x]\). Since \(\text{Ann}_M(a_1) \cap \text{Ann}_M(a_2) = 0 \subseteq P\), either \(\text{Ann}_M(a_1) \subseteq P\) or \(\text{Ann}_M(a_2) \subseteq P\). If \(\text{Ann}_M(a_1) \subseteq P\) and \(\text{Ann}_M(a_2) \subseteq P\), then \(\overline{GA(M)}\) has the cycle \([a_1] - [y] - [a_2] - [a_1]\) as a subgraph. Now, assume that \(\text{Ann}_M(a_1) \subseteq P\) and \(\text{Ann}_M(a_2) \nsubseteq P\). The fact \(\text{Ann}_M(a_1) \subseteq P\) implies that \([a_1]\) is adjacent to \([y]\). On the other hand, since \(\text{Ann}_M(a_2) \cap \text{Ann}_M(a_3) = 0 \subseteq P\), \(\text{Ann}_M(a_3) \subseteq P\) and so \([a_3]\) is adjacent to \([y]\). Thus, \([a_1] - [a_2] - [a_3] - [y] - [a_1]\) is a cycle of length 4 in \(\overline{GA(M)}\). Therefore, \(\text{gr}(\overline{GA(M)}) \leq 4\).
Assume that \([y] \in V(C)\). Without loss of generality, we may assume that \([y] = [a_1]\). Thus \(\text{Ann}_M(a_1) \cap P = 0\). Since \(\text{Ann}_M(a_1) \cap \text{Ann}_M(a_2) = 0\) and \(\text{Ann}_M(a_1) \not\subseteq P\), \(\text{Ann}_M(a_2) \subseteq P\). If \(\text{Ann}_M(a_3) \subseteq P\), then \(\overline{GA(M)}\) has the cycle \([a_1] - [a_2] - [a_3] - [a_1]\) as a subgraph. If \(\text{Ann}_M(a_3) \not\subseteq P\) as before one can show that \([a_1] - [a_2] - [a_3] - [a_4] - [a_1]\) is a subgraph of \(\overline{GA(M)}\). In the sequel, let \(P = \text{Ann}_M(a_1)\). Then \(\text{Ann}_M(a_2) \not\subseteq P\). So \(\text{Ann}_M(a_3) \subseteq P\) which implies that \(\text{Ann}_M(a_3) \cap \text{Ann}_M(a_n) = 0\), hence \(\overline{GA(M)}\) has the cycle \([a_1] - [a_2] - [a_3] - [a_n] - [a_1]\) as a subgraph. Therefore, either \(\text{gr}(\overline{GA(M)}) \leq 4\) or \(\text{gr}(\overline{GA(M)}) = \infty\). \(\square\)

**Theorem 2.7.** Let \(M\) be a Noetherian \(R\)-module and \(GA(M)\) be a disconnected graph. Then \(\overline{GA(M)}\) is a complete bipartite graph and \(\text{gr}(\overline{GA(M)}) \in \{4, \infty\}\).

**Proof.** It is obvious that \(\overline{GA(M)}\) is a connected graph. By Theorem 1.2, \(m - \text{Ass}_R(M) = \{\text{Ann}_M(a) = P_1, \text{Ann}_M(b) = P_2\}\) and \(P_1 \cap P_2 = 0\) so \([a]\) and \([b]\) are adjacent in \(\overline{GA(M)}\). Furthermore, for every vertex \([x] \in V(GA(M))\) we have either \(\text{Ann}_M(x) \subseteq P_1\) or \(\text{Ann}_M(x) \subseteq P_2\), see [8, Proposition 3.2].

Let \(V_1 = \{[x] : \text{Ann}_M(x) \subseteq P_1\}\) and \(V_2 = \{[y] : \text{Ann}_M(y) \subseteq P_2\}\). By the previous paragraph it is obvious that \(V_1 \cap V_2 = N([b]) \cap N([a]) = \emptyset\) also any vertex in \(V_1\) is adjacent to all vertices in \(V_2\) and conversely any vertex in \(V_2\) is adjacent to all vertices in \(V_1\). On the other hand, Lemma 2.2 shows that the induced subgraph on \(N([a])\) and \(N([b])\) is empty. Thus two distinct vertices in \(V_1\) are not adjacent and the same is true for vertices in \(V_2\). Hence, \(\overline{GA(M)}\) is a complete bipartite graph \(K_{|N([a])|,|N([b])|}\). Therefore, \(\text{gr}(\overline{GA(M)}) = 4\) or \(\text{gr}(\overline{GA(M)}) = \infty\). \(\square\)

### 3. Metric Dimension, Local Metric Dimension and Adjacency Metric Dimension of \(\overline{GA(M)}\)

In this section, we study domination number, metric dimension, adjacency metric dimension and local metric dimension of \(\overline{GA(M)}\).

**Theorem 3.1.** Let \(M\) be a Noetherian \(R\)-module and \(GA(M)\) be a non-empty connected graph. If \(|m - \text{Ass}_R(M)| = \omega(GA(M)) = n\), then \(\gamma(\overline{GA(M)}) = n\).

**Proof.** Let \(m - \text{Ass}_R(M) = \{\text{Ann}_M(a_1), \ldots, \text{Ann}_M(a_n)\}\) and let \(D = \{[a_1], \ldots, [a_n]\}\) we show that \(D\) is a dominating set for \(GA(M)\). By [10, Corollary 2.7], two arbitrary elements of \(m - \text{Ass}_R(M)\) have non-zero intersection. Thus \(D\) is a clique for \(GA(M)\). Let \([x]\) be an arbitrary vertex of \(\overline{GA(M)}\). If \([x]\) is adjacent to any vertices of \(D\) in \(GA(M)\),
then $\omega(GA(M)) \geq n+1$ which contradicts the assumption. Thus there is at least one element $Ann_M(a_i)$ in $m - Ass_R(M)$ with $1 \leq i \leq n$ such that $Ann_M(a_i) \cap Ann_M(x) = 0$. Hence, $[x]$ is adjacent to $[a_i]$ in $GA(M)$. Therefore, all vertices out of $D$ are adjacent in $\overline{GA(M)}$ to at least one vertex in $D$. So $D$ is a dominating set for $\overline{GA(M)}$ which implies that $\gamma(\overline{GA(M)}) \leq n$.

Assume that $\overline{GA(M)}$ has a dominating set with less than $n$ elements such as $D' = \{[b_1], \ldots, [b_t]\}$ ($t < n$). Thus there are two elements in $D$ with a common neighbor in $D'$ this contradict with Lemma 2.1. Hence, $\gamma(\overline{GA(M)}) = n$.

**Theorem 3.2.** Let $M$ be a Noetherian $R$-module and let $GA(M)$ and $\overline{GA(M)}$ be non-empty connected graphs of order $m$. Let $|m - Ass_R(M)| = \omega(GA(M)) = n$ ($n \geq 2$). Then the following statements are true:

(i) If $GA(M)$ has $k$ end-vertices, then $\dim(\overline{GA(M)}) = m - 2n + k - 1$.

(ii) If $GA(M)$ has no end-vertex, then $\dim(\overline{GA(M)}) = m - 2n$.

**Proof.** Let $m - Ass_R(M) = \{Ann_M(a_1), \ldots, Ann_M(a_n)\}$ and $D = \{[a_1], \ldots, [a_n]\}$. Assume that $[b]$ is a vertex of $\overline{GA(M)}$ such that is not adjacent to any $[a_i]$ in $\overline{GA(M)}$. Thus $[b]$ is adjacent to $[a_i]$, for all $1 \leq i \leq n$, in $GA(M)$ and so $\omega(GA(M)) \geq n + 1$ which is a contradiction. Now, assume that $[b] \neq [a_1]$, for all $1 \leq i \leq n$, is a vertex of $\overline{GA(M)}$ we may assume that $[b] - [a_1]$. Since $\overline{GA(M)}$ is connected and $\text{diam}(\overline{GA(M)}) \leq 3$, there is a path as $[a_2] - [c] - [e] - [b]$ that $[c] \neq [a_1]$ and $[e] \neq [a_1]$, see Lemma 2.3. Hence, it follows that the end-vertices of $\overline{GA(M)}$ must belong to $D$. Without loss of generality, suppose that $\{[a_1], \ldots, [a_k]\}$ is the set of end-vertices of $\overline{GA(M)}$, where $1 \leq k \leq n$. Suppose that $N_{\overline{GA(M)}}([a_i]) = \{[u_{i1}], \ldots, [u_{it}]\}$, where $t_i \in \mathbb{N}$ for all $1 \leq i \leq n$.

(i) Consider the ordered set

$$W = \{[u_{11}], \ldots, [u_{(k-1)1}]\} \cup \left( \bigcup_{k+1 \leq i \leq n} (N([a_i]) \setminus \{[u_{i1}]\}) \right)$$

of vertices of $\overline{GA(M)}$. Let $k+1 \leq i \leq n$. Then $r([u_{i1}] \mid W)$ have values 2 and 1 in its components corresponding to $[u_{i2}]$ and $[u_{i2}]$ respectively, where $k+1 \leq j \leq n, i \neq j$; and $r([u_k] \mid W)$ have value 1 in all components. Thus $[u_{i1}]$, for all $k \leq i \leq n$, have distinct representations with respect to $W$. Furthermore, for all $1 \leq i \leq k - 1$, $r([a_i] \mid W)$ have values 1 and 2 in its components corresponding to $[u_{i1}]$ and $[u_{i1}]$, where $1 \leq j \leq k - 1, i \neq j$, and $r([a_i] \mid W)$, for all $k+1 \leq i \leq n$, have values 1.
and 2 in its components corresponding to \([u_{i2}]\) and \([u_{j2}]\), where \(k+1 \leq j \leq n, j \neq i;\) and \(r([a_k] \mid W)\) have value 2 in all components. Hence, all vertices of \(\overline{GA(M)}\) have different representation with respect to \(W\) and therefore it is a resolving set for \(\overline{GA(M)}\). Thus \(\dim(\overline{GA(M)}) \leq m-2n+k-1\).

On the other hand, assume that \(W_0\) is a resolving set for \(\overline{GA(M)}\). Since all vertices contained in \(N([a_i])\) are twins so Theorem 1.3 implies that \(|N([a_i]) \cap W_0| = |N([a_i])|-1\), for all \(i\) with \(k+1 \leq i \leq n\). Thus \(|W_0| \geq m-2n\). We may assume that \([a_1], [u_{11}] \not\in W_0\) since they have distinct representations with respect to \(W_0\) by using \(|N([a_i]) \cap W_0| = |N([a_i])|-1\). For \(2 \leq i \leq k\), if \([u_{i1}], [a_i] \not\in W_0\), then \(r([u_{11}] \mid W_0) = r([u_{i1}] \mid W_0)\) which contradicts the fact that \(W_0\) is a resolving set for \(\overline{GA(M)}\). Hence, either \([u_{i1}] \in W_0\) or \([a_i] \in W_0\) for all \(i\) with \(2 \leq i \leq k\). Thus \(|W_0| \geq m-2n+k-1\) and therefore \(\dim(\overline{GA(M)}) = m-2n+k-1\). The proof is completed.

(ii) Consider the ordered set

\[ W = \bigcup_{1 \leq i \leq n} (N([a_i]) \setminus \{ [u_{i1}] \}) \]

of vertices of \(\overline{GA(M)}\). Let \(1 \leq i \leq n\). Then \(r([u_{i1}] \mid W)\) has values 2 and 1 in its components corresponding to \([u_{i2}]\) and \([u_{j2}]\) respectively, where \(1 \leq j \leq n\) and \(i \neq j\). Thus \([u_{i1}]\) have distinct representations with respect to \(W\), for all \(1 \leq i \leq n\). Also \(r([a_i] \mid W)\), for all \(1 \leq i \leq n\), has values 1 and 2 in its components corresponding to \([u_{i2}]\) and \([u_{j2}]\) respectively, where \(1 \leq j \leq n\) and \(j \neq i\). Hence, every vertex out of \(W\) has an unique representation with respect to \(W\). Therefore, \(W\) is a resolving set for \(\overline{GA(M)}\). Thus, \(\dim(\overline{GA(M)}) \leq m-2n\).

On the other hand, assume that \(W_0\) is a resolving set for \(\overline{GA(M)}\). Since all vertices contained in \(N([a_i])\) are twins so Theorem 1.3 implies that \(|N([a_i]) \cap W_0| = |N([a_i])|-1\), where \(1 \leq i \leq n\). So \(|W_0| \geq m-2n\). Hence, \(W\) is a resolving set for \(\overline{GA(M)}\) and \(\dim(\overline{GA(M)}) = m-2n\). \(\Box\)

From the definitions of the metric and adjacency metric dimensions, it follows that \(\dim(G) \leq \dim_A(G)\). This inequality and Theorem 3.2 give a lower bound for the adjacency metric dimension of \(\overline{GA(M)}\).

**Corollary 3.3.** Let \(M\) be a Noetherian \(R\)-module and let \(GA(M)\) be a non-empty connected graph of order \(m\). Let \(|m - \text{Ass}_R(M)| = \omega(GA(M)) = n\). Then the following statements are true:

(i) If \(GA(M)\) has \(k\) end-vertices, then \(\dim_A(\overline{GA(M)}) = m-2n + k-1\).

(ii) If \(\overline{GA(M)}\) has no end-vertex, then \(\dim_A(\overline{GA(M)}) = m-2n\).
is an adjacency resolving set for $\overline{GA(M)}$. Thus $\dim_A(\overline{GA(M)}) \leq m - 2n + k - 1$ will complete the proof.

(ii) By a similar argument to that of (i) one can show that

$$W_A = \bigcup_{1 \leq i \leq n} (N([a_i]) \setminus \{u_{i1}\})$$

is an adjacency resolving set for $\overline{GA(M)}$. Thus $\dim_A(\overline{GA(M)}) = m - 2n$ will complete the proof. \hfill $\Box$

Theorem 3.4. Let $M$ be a Noetherian $R$-module and let $GA(M)$ and $\overline{GA(M)}$ be non-empty connected graphs. If $|m - \text{Ass}_R(M)| = \omega(GA(M)) = n$, then $\dim_\ell(\overline{GA(M)}) = n - 1$.

Proof. Let $m - \text{Ass}_R(M) = \{P_1 = \text{Ann}_M(a_1), \ldots, P_n = \text{Ann}_M(a_n)\}$ and let $N([a_i]) = \{[u_{i1}], \ldots, [u_{it_i}]\}$, where $t_i \in \mathbb{N}$ for all $1 \leq i \leq n$. Set $W_\ell = \{[u_{11}], [u_{21}], \ldots, [u_{(n-1)1}]\}$. Let $[u] \in N([a_i])$ and $[v] \in N([a_s])$, where $1 \leq r, s \leq n$. If $r = s$, then the vertices $[u]$ and $[v]$ are not adjacent in $\overline{GA(M)}$ by Lemma 2.3 and so there is nothing to prove. Next if $r \neq s$, then Lemma 2.2 implies that $[u] - [v]$. In this case, $r([u] \mid W_\ell)$ has values 2 in its components corresponding to $[u_{r1}]$ and 1 in other components, while $r([v] \mid W_\ell)$ has values 2 in its components corresponding to $[u_{s1}]$ and 1 in other components. Thus $r([u] \mid W_\ell) \neq r([v] \mid W_\ell)$. Also, $r([a_i] \mid W_\ell)$ has values 1 in its components corresponding to $[u_{i1}]$ and 2 in other components, for all $i$ with $1 \leq i \leq n - 1$. Hence, $r([a_i] \mid W_\ell) \neq r([u_{ij}] \mid W_\ell)$, where $1 \leq j \leq t_i$. Finally, $r([a_{n}] \mid W_\ell) \neq r([u_{nj}] \mid W_\ell)$ since all components in the representation of $r([a_{n}] \mid W_\ell)$ are 2 and all components in the representation of $r([u_{nj}] \mid W_\ell)$ are 1, for all $j$ with $1 \leq j \leq t_n$. By the previous arguments, every two adjacent vertices out of $W_\ell$ have a unique representation with respect to $W_\ell$ and so $W_\ell$ is a local resolving set for $\overline{GA(M)}$. Therefore, $\dim_\ell(\overline{GA(M)}) \leq n - 1$.

Suppose that $W_\ell'$ is a local resolving set for $\overline{GA(M)}$ with $|W_\ell'| < n - 1$. Without loss of generality we may assume that $|W_\ell'| = n - 2$. Let $D = \{[a_1], \ldots, [a_n]\}$. Then the following three cases will be considered:

Case 1. $D \cap W_\ell' = \emptyset$.

In this case, there exist at least two indices $i \neq j$ with $1 \leq i, j \leq n$.
such that \( N([a_i]) \cap W'_\ell = N([a_j]) \cap W'_\ell = \emptyset \). Let \([u] \in N([a_i])\) and \([v] \in N([a_j])\). Then Lemma 2.3 shows that \([u]\) and \([v]\) are adjacent and \(r([u] | W'_\ell) = r([v] | W'_\ell) = (1, \ldots, 1)\), which contradicts the fact that \(W'_\ell\) is a local resolving set for \(GA(M)\).

**Case 2.** \(|D \cap W'_\ell| = n - 2\).

Without loss of generality, we may assume that \(W'_\ell = \{[a_1], \ldots, [a_{n-2}]\}\). Let \([u] \in N([a_{n-1}])\) and \([v] \in N([a_n])\). Then Lemma 2.3 shows that \([u]\) and \([v]\) are adjacent and \(r([u] | W'_\ell) = r([v] | W'_\ell) = (2, \ldots, 2)\), which contradicts the fact that \(W'_\ell\) is a local resolving set for \(GA(M)\).

**Case 3.** Suppose \(|D \cap W'_\ell| = t \leq n - 2\).

Assume that \(\{[a_1], \ldots, [a_t]\} \subseteq W'_\ell\) and \(N([a_i]) \cap W'_\ell \neq \emptyset\), where \(t + 1 \leq i \leq n - 2\). Let \([u] \in N([a_{n-1}])\) and \([v] \in N([a_n])\). Then \([u]\) and \([v]\) are adjacent and \(r([u] | W'_\ell) = r([v] | W'_\ell)\) since the first \(t\) components of them are 2 and the other components are 1 and this is a contradiction. Thus \(\dim_\ell(GA(M)) \geq n - 1\), which implies that \(\dim_\ell(GA(M)) = n - 1\) and the proof will be completed.  \(\square\)

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**References**

1. S. Akbari, H. A. Tavallaee and S. Khalashi Ghezelahmad, *Intersection graph of submodules of a module*, J. Algebra Appl, (01) **11** (2012), 1–8.
2. S. Akbari, R. Nikandish and M. J. Nikmehr, *Some results on the intersection graph of ideals of rings*, J. Algebra Appl, (04) **12** (2013), (13 pages).
3. S. Akbari, H. A. Tavallaee and S. Khalashi Ghezelahmad, *On the complement of the intersection graph of submodules of a module*, J. Algebra Appl, (08) **14** (2015), 1550116 (11 pages).
4. S. Babaei, Sh. Payrovi and E. Sengelen Sevim, *On the annihilator submodules and the annihilator essential graph*, Acta Math. Vietnam, (4) **44** (2019), 905–914.
5. L. Chakrabarty, S. Ghosh, T.K. Mukherjee and M.K. Sen, *Intersection graphs of ideals of rings*, Discrete Math. **309** (2009), 5381–5392.
6. C. Hernando, M. Mora, I.M. Pelayo, C. Seara and D.R. Wood, *Extremal graph theory for metric dimension and diameter*, Electron. J. Combin, **17** (2010), 1–28.
7. M. Jannesari, B. Omoomi, *The metric dimension of the composition product of graph*, Discrete Math., **312** (2012), 3349–3356.
8. C.P. Lu, *Union of prime submodules*, Houston J. Math, (2) **23** (1997), 203–213.
9. J. Matczuk, M. Nowakowska and E.R. Puczylowski, *Intersection graphs of modules and rings*, J. Algebra Appl, (07) **17** (2018), (20 pages).
10. S.B. Pejman, Sh. Payrovi and S. Babaei, *The intersection graph of annihilator submodules of a module*, Opuscula Math, (4) **39** (2019), 577–588.
11. K. Porselvi and R. Solomon Jones, *Properties of extended ideal based zero divisor graph of a commutative ring*, Algebra Relat. Topics, (1) 5 (2017), 55–59.
12. R.Y. Sharp, *Steps in Commutative Algebra*, Cambridge University Press, 2000.
13. D.B. West, *Introduction to graph theory*, Prentice Hall, 2001.
14. E. Yaraneri, *Intersection graph of a module*, J. Algebra Appl, (05)12 (2013), (30 pages).

S.B. Pejman  
Department of Mathematics, Imam Khomeini International University, P.O.Box 34149-1-6818, Qazvin, Iran.  
Email: b.pejman@edu.ikiu.ac.ir

Sh. Payrovi  
Department of Mathematics, Imam Khomeini International University, P.O.Box 34149-1-6818, Qazvin, Iran.  
Email: shpayrovi@sci.ikiu.ac.ir

S. Babaei  
Department of Mathematics, Imam Khomeini International University, P.O.Box 34149-1-6818, Qazvin, Iran.  
Email: sbabaei@edu.ikiu.ac.ir