A GEOMETRIC CHARACTERIZATION OF THE EYRING-KRAMERS FORMULA II

BENNY AVELIN AND VESA JULIN

Abstract. This paper is a continuation of [1], where we proved a geometric characterization of Eyring-Kramers formula. Here we extend the capacity estimate in [1] to the case of arbitrary configurations of critical points, which is needed in the extension of the Eyring-Kramers formula to the general setting. We do this by discretizing the problem and relating it to Kirchoff’s theorem from graph theory.

1. Introduction

In this paper we continue the study of the capacity estimate from [1], where we introduce a geometric characterization of the Eyring-Kramers formula. To introduce our setting, we begin by considering the Kolmogorov process

\[ dX_t = -\nabla F(X_t) dt + \sqrt{2\varepsilon} dB_t \]

where \( F \) is a non-convex smooth potential and \( \varepsilon \) is a small positive number. A formula for the expected transition time from one local minimum point to another was proposed independently by Eyring [5] and Kramers [9] in the context of metastability of chemical processes, and can be stated as follows. Assume that \( x \) and \( y \) are quadratic local minimas of \( F \), separated by a unique saddle \( z \) which is such that the Hessian has a single negative eigenvalue \( \lambda_1(z) \). Then the expected transition time from \( x \) to \( y \) satisfies

\[ E^F[\tau] \approx \frac{2\pi}{|\lambda_1(z)|} \sqrt{\frac{\det(\nabla^2 F(z))}{\det(\nabla^2 F(x))}} e^{(F(z) - F(x))/\varepsilon}, \]

(1.1)

where \( \approx \) denotes that the comparison constant tends to 1 as \( \varepsilon \to 0 \). The validity of the above formula has been studied extensively, references can be found in for instance [1, 3, 6, 10]. The first rigorous proof of (1.1) was by [4] using potential theory and this approach has turned out to be fruitful.

In [1] we use the potential theoretic formulation and extended the results of [4] from non-degenerate critical points to a more general ones, including even locally flat critical points. As in [4], the main technical issue is to provide sharp capacity estimates and the main result in [1] is a geometric characterization of newtonian capacity w.r.t. the measure \( e^{-F(x)/\varepsilon} dx \) inspired by the corresponding characterization for conformal capacity originally proved by Gehring, [7]. In [4] we observe that the capacity depends on the configuration of the saddle points which connect the two local minima, but we computed the capacity only in the simple cases when the saddles

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are either parallel or in series, see Fig. 1. However, for an arbitrary smooth potential the situation can be more complex and the configuration of the saddle points can be combinations of both parallel and series cases with essentially arbitrary complexity.

In this paper we extend the capacity estimate from [1] to the case of arbitrary configurations of critical points. We do this by discretizing the problem where the ‘valleys’/‘islands’ around the local minimum points are the vertices and the regions around the saddle points which we call ‘bridges’ are the edges. The local capacity of a bridge can be geometrically characterized using the results from [1] and this defines the weights of the edges, thus turning the problem into a capacitary problem on a graph. We show that this capacitary problem is equivalent to the notion of an electrical network, which was originally defined by Kirchoff in 1840’s in his elegant solution to the problem of replacement resistance for a network of resistors, [8]. For a modern presentation of electrical networks we refer to [2, 11].

1.1. Assumptions. We will have the same assumption on the potential $F$ as in [1]. For the reader’s convenience we recall some of the relevant assumptions and useful notation.

**Definition 1.1.** Let $F \in C^2(\mathbb{R}^n)$, satisfy the quadratic growth condition $F(x) \geq \frac{|x|^2}{C_0} - C_0$, for a constant $C_0 \geq 1$.

We say that $F$ is *admissible* if for every saddle point $z \in \mathbb{R}^n$ of $F$ there are convex functions $g_z : \mathbb{R} \to \mathbb{R}$ and $G_z : \mathbb{R}^{n-1} \to \mathbb{R}$ which have proper minimum at 0, see [1] for the precise definition, such that $g_z(0) = G_z(0) = 0$, and an isometry $T_z : \mathbb{R}^n \to \mathbb{R}^n$ such that, denoting $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$, it holds

$$\left|(F \circ T_z)(x) - F(z) + g_z(x_1) - G_z(x')\right| \leq \omega(g_z(x_1)) + \omega(G_z(x')),$$

(1.2)

where $\omega : [0, \infty) \to [0, \infty)$ is a continuous and increasing function with $\lim_{s \to 0} \omega(s) = 0$.

The assumption (1.2) allows the saddle point to be degenerate, but we do not allow them to have many branches.
Definition 1.2. Let \( F \in C^2(\mathbb{R}^n) \) be admissible, then for every saddle point \( z \) and \( \delta > 0 \), we define the bridge at \( z \) as
\[
O_{z,\delta} := T_z \left( \{ x_1 \in \mathbb{R} : g_z(x_1) < \delta \} \times \{ x' \in \mathbb{R}^{n-1} : G_x(x') < \delta \} \right)
\]
where \( T_z \) is the isometry from Definition 1.1.

We use the definitions of a geodesic length and a minimal cut inspired by [7].

Definition 1.3. Let \( A, B \subset \Omega \subset \mathbb{R}^n \) where \( \Omega \) is a domain and \( A \cap B = \emptyset \). We denote the curve family
\[
C(A, B; \Omega) := \{ \gamma : \gamma \in C^1([0, 1]; \Omega), \gamma(0) \in A, \gamma(1) \in B \}
\]
and the family of separating sets as \( \mathcal{S}(A, B; \Omega) \), where a smooth hypersurface \( S \subset \mathbb{R}^n \) (possibly with boundary) is in \( \mathcal{S}(A, B; \Omega) \) if every \( \gamma \in C(A, B; \Omega) \) intersects \( S \). We define the geodesic distance between \( A \) and \( B \) in \( \Omega \) as
\[
d_s(A, B; \Omega) := \inf \left( \int_0^1 |\gamma'| e^{\frac{F}{\epsilon(t)}} \, dt : \gamma \in C(A, B; \Omega) \right)
\]
and the minimal cut by
\[
V_s(A, B; \Omega) := \inf \left( \int_S e^{-\frac{F}{\epsilon(t)}} d\mathcal{H}^{n-1}(x) : S \in \mathcal{S}(A, B; \Omega) \right).
\]

Finally we have to define some topological quantities

Definition 1.4. Let \( x_u, x_v \) be two local minima of an admissible \( F \). The communication height between \( x_u, x_v \) is defined as
\[
F(x_u; x_v) = \inf_{\gamma \in C(B_u(x_u). B_v(x_v); \mathbb{R}^n)} \sup_{t \in [0, 1]} F(\gamma(t)).
\]

Fixing \( \delta > 0 \) as in [1], we denote the component of the sublevel-set \( \{ F < F(x_u; x_v) + \delta/3 \} \) which contains the points \( x_u \) and \( x_v \) by \( U_{\delta/3} \) and we denote
\[
U_{-\delta/3} := \{ F < F(x_u; x_v) - \delta/3 \} \cap U_{\delta/3}.
\]

We call the components of \( U_{-\delta/3} \) islands. For each island \( U \) we select \( x \) satisfying \( F(x) = \min_U F \), and we will in the following denote \( U_x \) as the island which contains \( x \). Finally we denote all saddle points in \( U_{\delta/3} \setminus U_{-\delta/3} \) by \( Z \).

1.2. Construction of the electrical network.

Definition 1.5. An electrical network is a pair \((G, y)\), where \( G = (V, E) \) is a graph, where \( V \) are the vertices and \( E \) are the edges, the vector \( y \in \mathbb{R}^{|V|} \) is called the admittances.

We will now construct an electrical network based on the islands and bridges from Definitions 1.2 and 1.4. We associate the vertices with the islands and for every vertex \( v \) we denote the corresponding island by \( U_v \). The set of all vertices is \( V \). Furthermore, we associate the edges with the bridges from Definition 1.2, specifically, for every saddle point \( z \in Z \) in Definition 1.4 we associate the edge \( e_z \) with the bridge \( O_{e_z} = O_{e, \delta} \). The set of all edges is \( E \). We say that vertices \( v, v' \in V \) are incident with an edge \( e \), and vice versa, if they are the ends of the edge, or in other words, the associated islands
$U_v, U'_v$ intersect the bridge $O_e$ (there are at most two since $F$ is admissible). An edge which is incident with only one vertex is called a loop. We thus have a graph $G = (V, E)$ and we orient it arbitrarily. In order to have an electrical network we need to define the admittance $y_e$ for $e \in E$. Now, let $e \in E$, which is not a loop, and let $v_-, v_+ \in V$ be its incident vertices. Define the connected set $\Omega_e = O_{v-,v_+} \cup U_{v_+} \cup U_{v_-}$ and the admittance

$$y_e := \varepsilon \frac{V_e(B_e(x_{v_-}), B_e(x_{v_+}); \Omega_e)}{d_e(B_e(x_{v_-}), B_e(x_{v_+}); \Omega_e)} \tag{1.4}$$

From the geometric characterization of capacity in [1], we see that the admittance of the edge $e$ is essentially the capacity of $(B_e(x_{v_-}), B_e(x_{v_+}))$ in $\Omega_e$. If $e$ is a loop we set $y_e = 0$. We have thus constructed our electrical network $(G, y)$.

1.3. The Main Result. We need the definition of a spanning tree for Kirchhoff’s formula.

**Definition 1.6.** Let $G = (V, E)$ be a graph, then we say that $G'$ is a spanning subgraph of $G$ if $V(G') = V(G)$ and $E(G') \subset E(G)$. A tree is a connected graph which does not contain cycles and a spanning tree of $G$ is a spanning subgraph of $G$ that is a tree. We denote the set of all spanning trees of $G$ by $T(G)$. Finally for two vertices $v, w \in V$ we let $G/vw$ denote the graph obtained by merging the vertices $v$ and $w$ together into a single vertex.

Our main result is the sharp characterization of the capacity of the sets $B_e(x_u)$ and $B_e(x_w)$ for local minimum points $x_u, x_w$ via Kirchhoff’s formula. Recall that the capacity of two disjoint sets $A, B$ is defined as

$$\text{cap}(A, B) = \inf \left( \varepsilon \int_{\mathbb{R}^n} |\nabla h|^2 e^{-\frac{1}{\varepsilon} h} \, dx : \text{ h = 1 in } A, h \in W^{1,2}_0(\mathbb{R}^n \setminus B) \right).$$

Finally we use the notation $f \simeq g$ from [1] for nonnegative functions $f$ and $g$ which depend continuously on $\varepsilon > 0$ and satisfy

$$(1 - \hat{\eta}(C\varepsilon))f(\varepsilon) \leq g(\varepsilon) \leq (1 + \hat{\eta}(C\varepsilon))f(\varepsilon),$$

where the constant $C$ depends only on the data of the problem, $\hat{\eta} \to 0$ as $\varepsilon \to 0$ and $\hat{\eta}$ is defined in [1].

**Theorem 1.** Let $F$ be admissible as in Definition 1.1, let $x_u$ and $x_w$ be local minimum points of $F$ and let $(G, y)$ be the electrical network as in Section 1.2. Let $u, w$ be the associated vertices in $V$. Then the capacity is given by

$$\text{cap}(B_e(x_u), B_e(x_w)) \simeq \frac{T(G; y)}{T(G/uw; y)},$$

where

$$T(G; y) = \sum_{T \in T(G)} \left( \prod_{e \in T} y_e \right). \tag{1.5}$$

Theorem 1, together with the result in [1], provides the characterization of the capacity in the general case where the critical points may have any configuration.
2. Preliminaries on graph theory

In this section we recall some basic results in graph theory. For an introduction to the topic we refer to [2, 11].

2.1. Matrix-Tree theorems and graphs. The signed incidence matrix $D$ of the oriented graph $G = (V, E)$ is the $|V| \times |E|$ matrix with entries

$$D_{ve} = \begin{cases} +1 & \text{if } e \text{ points into } v \text{ but not out} \\ -1 & \text{if } e \text{ points out of } v \text{ but not in} \\ 0 & \text{otherwise.} \end{cases}$$

Let $y$ be a vector of admittances defined in Section 1.2. Let $Y$ be the $|E| \times |E|$ diagonal matrix that has $y$ as its entries, i.e., $Y = \text{diag}(y_e : e \in E)$. We also define the weighted Laplacian matrix as $L = DY D^T$.

We begin by recalling the weighted Matrix-Tree theorem, see [11, Theorem 5].

**Proposition 2.1.** Let $G = (V, E)$ be an oriented graph and let $D, Y$ and $L$ be as above. Then, for any $v \in V$

$$T(G; y) = \det L(v \mid v)$$

where $L(v \mid v)$ is $L$ with the row and column corresponding to $v$ removed and $T(G; y)$ is defined in (1.5).

Let us recall Kirchoff’s theorem, see [11, Theorem 8].

**Proposition 2.2.** Let $G = (V, E)$ be an oriented graph and let $D$ and $Y$ be as above. Fix $u, w \in V$ and let the vector $\varphi \in \mathbb{R}^{|V|}$, with the component $\varphi_u = 0$, be the solution to the system

$$DY D^T \varphi = \delta_w,$$

where $\delta_w$ is a vector with 1 in the position of $w$ and 0 otherwise. Then the component $\varphi_w$ is given by

$$\varphi_w = \frac{T(G/\{uw\}; y)}{T(G; y)}.$$

The formula in the statement of Theorem 1 given by Kirchoff’s formula is precise, but if the graph contains many cycles and loops, it may be unnecessary cumbersome to calculate. In the next two lemmas we consider the case when the formula in Theorem 1 can be simplified.

Consider a graph $G = (V, E)$. A cut vertex is a vertex, that when removed from $G$ will increase the number of components. A biconnected graph is a graph with no cut vertices. A biconnected component of a graph $G$ is a maximal biconnected subgraph.

**Lemma 2.3.** Let $G = (V, E)$ be graph with a biconnected component $G_1 = (V_1, E_1)$ and let $G_2 = (V_2, E_2)$ be a subgraph of $G$ such that they intersect in one cut vertex $v \in V$ and $G = G_1 \cup G_2$. Then if $y \in \mathbb{R}^{|E|}$ is the admittance vector, $y_1 = y|_{E_1}$ and $y_2 = y|_{E_2}$, it holds

$$T(G; y) = T(G_1; y_1)T(G_2; y_2).$$
Figure 2. Example of the graph decomposition in Lemma 2.4. Here the subgraph corresponding to the blue edges is the biconnected component and the red edges correspond to the graph $G_1$.

Proof. By the definition of biconnected components, and since $G_1, G_2$ intersect only in $v$, we can by reordering the vertices write the Laplacian matrix $L = DYG^T$ such that the first rows/columns correspond to the vertices in $G_1$. Then $L$ with the column and row corresponding to $v$ removed ($L(v \setminus v)$) has a block diagonal structure with the blocks $L_1 = L_{G_1}(v \setminus v)$ and $L_2 = L_{G_2}(v \setminus v)$. Now, since $\det(L) = \det(L_1) \det(L_2)$ the claim follows by applying Proposition 2.1 on all matrices.

We can use the above lemma to simplify the computation of Kirchoff’s theorem in the presence of irrelevant biconnected components (see Section 2.1).

Lemma 2.4. Consider the graph $G = (V, E)$ and let $Y$ be the admittance matrix. Assume that $G = G_1 \cup G_2$, where $G_2$ is a biconnected component and $G_1, G_2$ intersect in a cut vertex $v \in V$. Then if $u, w \in V_1$, it holds

$$\frac{T(G; y)}{T(G/vw; y)} = \frac{T(G_1; y_1)}{T(G_1/vw; y_1)}.$$ 

The main consequence of Lemma 2.4 is that, using the terminology from [1], only the vertices in $V_1$ charge capacity in Theorem 2.2.

2.2. Preliminaries on electrical networks. We will in this section identify the discrete capacity minimizer with the solution given by Kirchoff’s theorem for electrical networks. See also [2].

Lemma 2.5. Let $(G, y)$ be the electrical network from Section 1.2. Then for $Y = \text{diag}(y)$ and $D$ from Section 2.1

$$\min_{\varphi \in \mathbb{R}^m; \varphi_1 = 1; \varphi_m = 0} (DYD^T \varphi, \varphi) = T(G; y) \frac{T(G/v_1v_m; y)}{T(G/v_1v_m; y)}.$$ 

and the minimizer is given by the unique solution with the boundary conditions $\varphi_m = 0$, $\varphi_1 = 1$ to the linear system

$$DYD^T \varphi = \lambda (\delta_1 - \delta_m)$$

where $\delta_1 = (1, 0, \ldots, 0)$ and $\delta_m = (0, \ldots, 0, 1)$ are vectors of length $m$ and $\lambda$ is the value of the minimum problem.

Proof. Let us first reduce the problem. Note that the constraint $\varphi_m = 0$ implies that we may remove the last row of $D$ (call it $D_\ast$) and the last entry of $\varphi$ (call it $\varphi_\ast$) and note that $D_\ast^T \varphi_\ast = D^T \varphi$. Furthermore, similar
reasoning can be applied for the inner-product $\langle D Y D^T \varphi_-, \varphi_- \rangle$. Thus we obtain that
\[
\langle D Y D^T \varphi_-, \varphi_- \rangle = \langle D Y D^T \varphi, \varphi \rangle.
\]
By the Lagrange multiplier method we get
\[
\begin{aligned}
D Y D^T \varphi_- &= \lambda \delta_1 \\
(\varphi_-)_1 &= 1,
\end{aligned}
\]
where $\delta_1 = (1, 0, \ldots)$. Let $L_- = D_+ Y D^T$ and note that by the matrix-tree theorem we know that $\det(L_-) = T(G; y) \neq 0$ which gives the uniqueness. Finally applying Proposition 2.2 finishes the proof.

We also need the following dual formulation of the minimization problem in Lemma 2.5.

**Lemma 2.6.** Let $G = (V, E)$ be a graph arbitrarily oriented, where $V = (v_1, \ldots, v_m)$, and let $D$ and $Y$ be as above. Then it holds
\[
\min_{\varphi \in \mathbb{R}^m; \varphi_1 = 1; \varphi_m = 0} \langle D Y D^T \varphi, \varphi \rangle = \langle D Y D^T \varphi, \varphi \rangle.
\]

**Proof.** Follows from a similar argument as Lemma 2.5.

A consequence of Lemma 2.5 is that edges with small admittance does not contribute total capacity unless they significantly alter the topology of the graph:

**Lemma 2.7** (Deletion of edge). Let $(G, y)$ be the electrical network as in Lemma 2.5. Let $e \in E$ and define $G' = (V, E \setminus \{e\})$, then it holds
\[
\frac{T(G'; y)}{T(G'/(v_1 v_m); y)} \leq T(G; y) \leq \frac{T(G'; y)}{T(G'/(v_1 v_m); y)} + y_e
\]

**Proof.** Let $Y'$ be the diagonal matrix $Y$ with the entry corresponding to $y_e$ replaced by 0. Then we immediately have
\[
\min_{\varphi \in \mathbb{R}^m; \varphi_1 = 1; \varphi_m = 0} \langle D Y' D^T \varphi, \varphi \rangle \leq \min_{\varphi \in \mathbb{R}^m; \varphi_1 = 1; \varphi_m = 0} \langle D Y D^T \varphi, \varphi \rangle
\]
which proves the first inequality. For the second, note that for any edge $e \in E$, let $v_-, v_+ \in V$ be the incident vertices, then $|\varphi(v_-) - \varphi(v_+)| \leq 1$, hence for any $y$ having each component bounded by 1 satisfies
\[
\langle D Y D^T \varphi, \varphi \rangle \leq \langle D Y' D^T \varphi, \varphi \rangle + y_e
\]
which proves the last inequality.
3. Proof of the main theorem

3.1. Construction. We begin by giving another construction of the network using the domain $U_{\delta/3}$ (see Definition 1.4). To this aim, for a saddle point $z \in Z$, we define the surface

$$S_z := T_z \{ \{0\} \times \{x' \in \mathbb{R}^{n-1} : G_z(x') < \delta\} \},$$  \hspace{1cm} (3.1)

where $T_z$ is from Definition 1.1. The set $U_{\delta/3}$ is connected, but the surfaces $S_z$ in (3.1) divide it into different components, which we will associate with vertices. Define

$$\Omega_{\delta/3} := U_{\delta/3} \setminus \bigcup_{z \in Z} S_z.$$  \hspace{1cm} (3.2)

We prove in Lemma 3.3 that $U_{-\delta/3} \subset \Omega_{\delta/3}$ have the same number of components. Therefore they define exactly the same graph components. Therefore they define exactly the same graph

We will localize the capacity of the sets $A = B_z(x_u)$ and $B = B_z(x_w)$ in $U_{\delta/3}$ by defining

$$\text{cap}(A, B; U_{\delta/3}) := \inf \left( \varepsilon \int_{U_{\delta/3}} |\nabla h|^2 e^{-\frac{F}{\varepsilon}} \, dx : h = 1 \text{ in } A, h \in W^{1,2}_0(B^c) \right).$$  \hspace{1cm} (3.3)

Note that the minimizer of (3.3), denote it by $\hat{h}_{A,B}$, satisfies the Neumann boundary conditions $\nabla \hat{h}_{A,B} \cdot n = 0$ on the smooth part of $\partial U_{\delta/3}$.

It is easy to see that for the localization (3.3) it holds

$$\text{cap}(A, B) \geq \text{cap}(A, B; U_{\delta/3}) \geq (1 - \tilde{\eta}(C \varepsilon)) \text{cap}(A, B).$$  \hspace{1cm} (3.4)

Indeed, the first inequality in (3.4) is trivial. For the second we take $\hat{h}_{A,B}$ to be the minimizer of (3.3). We choose a cut off function $0 \leq \zeta \leq 1$ such that $\zeta = 1$ in $U_{\delta/6}$, $\zeta = 0$ outside $U_{\delta/3}$ and $|\nabla \zeta| \leq C$, where $C$ depends on $\delta$ and on the potential $F$. Using Young’s inequality and the maximum principle we get

$$\begin{align*}
\text{cap}(A, B; U_{\delta/3}) & \geq \varepsilon \int_{U_{\delta/3}} |\nabla \hat{h}_{A,B}|^2 \zeta^2 e^{-\frac{F}{\varepsilon}} \, dx \\
& \geq \varepsilon \int_{U_{\delta/3}} |\nabla \hat{h}_{A,B}\zeta|^2 \zeta^2 e^{-\frac{F}{\varepsilon}} \, dx - \frac{2}{\varepsilon} \int_{U_{\delta/3}} |\nabla \zeta|^2 \hat{h}_{A,B}^2 e^{-\frac{F}{2\varepsilon}} \, dx \\
& \geq (1 - 2\varepsilon) \int_{\mathbb{R}^n} |\nabla \hat{h}_{A,B}\zeta|^2 \zeta^2 e^{-\frac{F}{\varepsilon}} \, dx - \frac{C}{\varepsilon} e^{-\frac{F}{\varepsilon}(x_u,x_w)/\varepsilon} \\
& \geq (1 - \tilde{\eta}(C \varepsilon)) \text{cap}(A, B).
\end{align*}$$

The construction of the network via (3.2) is suitable for the dual definition of the capacity via Thompson’s principle. This is done by defining class of vector fields, denoted by $\mathcal{M}$, where $X \in \mathcal{M}$ if $X \in W^{1,\infty}(U_{\delta/3} \setminus (\bar{A} \cup \bar{B}); \mathbb{R}^n)$ and satisfies

$$\text{div}X = 0 \text{ in } U_{\delta/3} \setminus (\bar{A} \cup \bar{B}), \quad X \cdot n = 0 \text{ on } \partial U_{\delta/3} \quad \text{and} \quad \int_{\partial A} X \cdot n = 1. \hspace{1cm} (3.5)$$

Then we have the following (see e.g. [10])

$$\frac{1}{\text{cap}(A, B; U_{\delta/3})} = \inf \left( \varepsilon \int_{U_{\delta/3} \setminus (\bar{A} \cup \bar{B})} |X|^2 e^{\frac{F}{\varepsilon}} \, dx : X \in \mathcal{M} \right).$$  \hspace{1cm} (3.6)
Let \( G = (V, E) \) be the graph constructed as above using the domain \( \Omega_{\delta/3} \) defined in (3.2) and let \( X \in \mathcal{M} \). We construct a current \( j : E \to \mathbb{R} \) associated with \( X \) as follows. Let us fix a vertex \( v \in V - \{u, w\} \) and let \( \widetilde{U}_v \) be the associated component of the domain \( \Omega_{\delta/3} \). Denote the edges incident with \( v \) by \( e \in E_v \subset E \). Denote the associated bridges by \( O_e \) with \( e \in E_v \) and the surface defined in (3.1) by \( S_v = S_z \). The boundary \( \partial \widetilde{U}_v \) is contained in \( \partial U_{\delta/3} \cup (\bigcup_{e \in E_v} \gamma_e) \). Recall that \( v \neq u, w \), therefore \( \text{div} X = 0 \) in \( \widetilde{U}_v \) and we have by the divergence theorem and by \( X \cdot n = 0 \) on \( \partial U_{\delta/3} \) that

\[
0 = -\int_{\widetilde{U}_v} \text{div}(X) \, dx = \int_{\partial \widetilde{U}_v} X \cdot n \, d\mathcal{H}^{n-1} = \sum_{e \in E_v} \int_{S_e} X \cdot n \, d\mathcal{H}^{n-1}. \tag{3.7}
\]

We define the value of \( j \) at \( e \in E_v \) as

\[
j(e) := \begin{cases} 
\varepsilon \int_{S_e} X \cdot n \, d\mathcal{H}^{n-1}, & \text{if } e \text{ points into } v, \\
-\varepsilon \int_{S_e} X \cdot n \, d\mathcal{H}^{n-1}, & \text{if } e \text{ points out of } v. 
\end{cases} \tag{3.8}
\]

We define the current similarly also at edges incident with \( u \) and \( w \). If we label the edges as \( e_1, \ldots, e_l \) we have a vector \( j \in \mathbb{R}^{|E|} \) which has components \( j_e = j(e_k) \). By construction and by (3.7) \( j \) satisfies the so called Kirchhoff’s current law, which means that at every vertex the current flowing in equals the current flowing out. We may write this simply as (see [11])

\[
Dj = \delta_1 - \delta_m
\]

where we have labelled the vertices as \( v_1, \ldots, v_m \) with \( v_1 = u \) and \( v_m = w \), and \( \delta_1 \) and \( \delta_m \) are as in Lemma 2.5.

### 3.2. Technical lemmas

Before we prove the main theorem we recall the following lemma from [1].

**Lemma 3.1.** Let \( F \) be admissible. Let \( x_u, x_v, \delta, U_{-\delta/3} \) be as in Definition 1.4. If \( U_v \) is a component of \( U_{-\delta/3} \), then

\[
\text{osc} h_{B_{\varepsilon}(x_u), B_{\varepsilon}(x_v)} \leq C \varepsilon,
\]

for small enough \( \varepsilon \).

**Lemma 3.2.** Let \( U_v \) and \( U_w \) be two different components of \( U_{-\delta/3} \) and let \( \gamma \in \mathcal{C}(U_v, U_w; U_{\delta/3}) \). Then there is a critical point \( z \in Z \) such that the intersection \( \gamma([0, 1]) \cap Z \) is non-empty.

**Proof.** W.L.O.G. we assume \( F(x_u; x_w) = 0 \). Fix \( \gamma_0 \in \mathcal{C}(U_v, U_w; U_{\delta/3}) \) and denote \( \gamma \sim \gamma_0 \) when \( \gamma \) is homotopy equivalent to \( \gamma_0 \) in \( U_{\delta/3} \). Define

\[
F_{\gamma_0} := \inf_{\gamma \sim \gamma_0} \sup_{t \in [0, 1]} F(\gamma(t)).
\]

Then there is a critical point \( z \) of \( F \) such that \( F(z) = F_{\gamma_0} \) and a continuous path \( \gamma_1 \sim \gamma_0 \) such that \( \gamma_1(t) = z \) for some \( t \in (0, 1) \). We may choose the coordinates in \( \mathbb{R}^n \) such that \( z = 0 \) and \( S_z = S_0 = \{0\} \times \{x' \in \mathbb{R}^{n-1} : G(x') < \delta\} \).

Note that \( S_0 \) is a convex hypersurface with boundary \( \partial S_0 = \{0\} \times \{x' \in \mathbb{R}^{n-1} : G(x') = \delta\} \), and note that \( \partial S_0 \) is homeomorphic to \( S^{n-2} \). Since \( F \) is admissible it follows that \( F(x) \geq F(0) + \delta/3 \) on \( x \in \partial S_0 \) and therefore \( \partial S_0 \subset \mathbb{R}^n \setminus U_{\delta/3} \). In particular, if \( \gamma \) is a path in \( U_{\delta/3} \) then it does not intersect
\[ \partial S_0, \text{ and if } \gamma \sim \gamma_0 \text{ then } \gamma \text{ has to intersect } S_0. \] The claim then follows from \( \gamma_1 \sim \gamma_0. \)

The next lemma states that the components of \( \Omega_{\delta/3} \) and \( U_{-\delta/3} \) are the same and thus they define the same graph.

**Lemma 3.3.** The set \( \Omega_{\delta/3} \) defined in (3.2) has the same components as \( U_{-\delta/3} \) defined in (1.3). To be more precise, if \( \Omega' \) is a component of \( \Omega_{\delta/3} \) then there is exactly one component, say \( U' \), of \( U_{-\delta/3} \) such that \( U' \subset \Omega' \).

**Proof.** W.L.O.G. we assume \( F(x_u; x_w) = 0 \). Let us fix a component \( \Omega' \) of \( \Omega_{\delta/3} \). Since \( F \) is admissible, then for any \( z \in Z \), we see from the definition of \( S_z \) in (3.1) that \( F(x) \geq F(z) \) for all \( x \in S_z \), and hence \( S_z \cap U_{-\delta/3} = \emptyset \). Thus there is a component \( U' \) of \( U_{-\delta/3} \) such that \( U' \subset \Omega' \). Let us also note that \( U' \) is the only component of \( U_{-\delta/3} \) which is in \( \Omega' \), since if there was another component \( \Omega'' \) then a curve \( \gamma \in \mathcal{C}(U', U''; \Omega') \subset \mathcal{C}(U', U''; U_{-\delta/3}) \) necessarily intersects one \( S_z \) by Lemma 3.2. \( \square \)

### 3.3. Proof of the main theorem.

We prove the main theorem by providing sharp lower bounds for the variation definition of the capacity and for (3.6), which is the opposite to the argument in [10].

**Proof of Theorem 1.** Consider two local minimas \( x_u, x_w \), let \( A = B_\varepsilon(x_u) \) and \( B = B_\varepsilon(x_w) \), and let \( h_{A,B} \) be the capacitary potential for the capacitor \( (A, B) \).

**Lower bound:** Let \( (G, y) \) be the electrical network from Section 1.2, and label the vertices as \( V = \{v_1, \ldots, v_m\} \), where \( v_1 = u, v_m = w \). We need to show that

\[
\text{cap}(A, B) \geq (1 - \hat{\eta}(C\varepsilon)) \frac{T(G; y)}{T(G/uw; y)}.
\]

Let \( \varphi : V \to \mathbb{R} \) be a function such that \( \varphi(v) = h_{A,B}(v) \) where \( v \in V \) and \( x_v \) is the associated minimum point. Note that by Lemma 3.1 we have

\[
\text{osc}_{U_v}(h_{A,B}) \leq C\varepsilon \quad \text{for all } v \in V.
\]

Therefore \( \varphi \) satisfies

\[
|h_{A,B} - \varphi(v)| \leq C\varepsilon \quad \text{in } U_v \quad \text{for } v \in V.
\]

Consider an edge \( e \in E \), which is not a loop, and let \( v_-, v_+ \) be the two incident vertices in \( V \). Denote the associated minimum points as \( x_-, x_+ \), the associated islands as \( U_-, U_+ \) respectively and the saddle point as \( z_e \). We may assume that \( z_e = 0 \), \( F(0) = 0 \) and that the bridge is given by

\[
O_e = O_{z_e, \delta} = \{y_1 : g(y_1) < \delta\} \times \{y' : g(y') < \delta\}.
\]

Then, using the fundamental theorem of calculus and (3.9) (as in the proof of [1, Theorem 1]) we get

\[
|\varphi(v_-) - \varphi(v_+)| - 2C\varepsilon \leq \left( \int_{\{g < \delta\}} |\nabla h_{A,B}(y)|^2 e^{-\frac{F(g)}{2}} \, dy_1 \right)^{\frac{1}{2}} \left( \int_{\{g < \delta\}} e^{\frac{F(g)}{2}} \, dy_1 \right)^{\frac{1}{2}}
\]

for \( (y_1, y') \in \{g < \delta\} \times \{G < \delta/100\} \). As in the proof of [1, Theorem 1: series case] we get from the above that

\[
\int_{O_e} |\nabla h_{A,B}|^2 e^{-\frac{F(g)}{2}} \, dy \geq (1 - \hat{\eta}(C\varepsilon))(\varphi(v_-) - \varphi(v_+))^2 \frac{V_e(B_\varepsilon(x_-), B_\varepsilon(x_+); \Omega_e)}{d_e(B_\varepsilon(x_-), B_\varepsilon(x_+); \Omega_e)}.
\]
By the definition of the admittance $y_e$ in (1.4) we may rewrite the above as
\[ \varepsilon \int_{O_e} |\nabla h_{A,B}|^2 e^{-\frac{E(w)}{\varepsilon}} dy \geq (1 - \tilde{\eta}(C\varepsilon))(\varphi(v_1) - \varphi(v)) y_e. \] (3.10)

Let us rephrase this in terms of the signed incidence matrix $D$ and the admittance matrix $Y$. First, let $\varphi$ be the vector $(\varphi(v_1), \ldots, \varphi(v_m))$, where $v_1 = u$ and $v_m = w$, and for an edge $e \in E$, let $v_e^-$, $v_e^+$ be the incident vertices. Then since $D$ is the $|V| \times |E|$ signed incidence matrix, we have for the edges $(e_1, \ldots, e_k)$
\[ D^T \varphi = (\varphi(v_{e_1}^-) - \varphi(v_{e_1}^+), \ldots, \varphi(v_{e_k}^-) - \varphi(v_{e_k}^+)). \]

Furthermore, by the definition of the admittance matrix $Y$ we have that
\[ YD^T \varphi = ((\varphi(v_{e_1}^-) - \varphi(v_{e_1}^+)) y_{e_1}, \ldots, (\varphi(v_{e_k}^-) - \varphi(v_{e_k}^+)) y_{e_k}). \]

Thus, we see that by repeating the argument leading to (3.10) for every edge $e \in E$, and using the fact that the sets $O_{e_i}$ are disjoint we obtain
\[ \varepsilon \int_{\mathbb{R}^n} |\nabla h_{A,B}|^2 e^{-\frac{E(w)}{\varepsilon}} dx \geq (1 - \tilde{\eta}(C\varepsilon))(D^T \varphi, \varphi). \]

By construction it holds $\varphi_1 = \varphi(u) = 1$ and $\varphi_m = \varphi(w) = 0$, therefore Lemma 2.5 completes the lower bound.

**Upper bound:** We prove the upper bound by a similar argument by providing a lower bound in the dual characterization (3.6). Indeed, by the second inequality in (3.4) this provides an upper bound for the global capacity. Let us fix a vector field $X \in \mathcal{M}$, where $\mathcal{M}$ is defined via conditions (3.5), and construct the associated current $\tilde{j} \in \mathbb{R}^{|E|}$ as in Section 3.1. The construction implies that $\tilde{j}$ satisfies Kirchhoff’s current law $D\tilde{j} = \delta_1 - \delta_m$ and therefore it holds by Lemma 2.5 and Lemma 2.6 that
\[ \langle Y^{-1} \tilde{j}, \tilde{j} \rangle \geq \frac{T(G; uw; y)}{T(G; y)}. \]

In order to conclude the proof, it is enough to show that at every edge $e \in K$ it holds
\[ \varepsilon \int_{O_e} |X|^2 e^{-\frac{E(w)}{\varepsilon}} dx \geq (1 - \tilde{\eta}(C\varepsilon)) \frac{\tilde{j}_e^2}{y_e}. \] (3.11)
where $O_e = O_{x_e, R}$ denotes the associated bridge. To this aim we may choose the coordinates in $\mathbb{R}^n$ such that
\[ O_e = \{ x_1 \in \mathbb{R} : g(x_1) < \delta \} \times \{ x' \in \mathbb{R}^{n-1} : G(x') < \delta \}. \]

For every $|\tau| < \delta$ denote $S_\tau = \{ \tau \} \times \{ x' \in \mathbb{R}^{n-1} : G(x') < \delta \}$, and note that then by definition of $\tilde{j}$ in (3.8) it holds
\[ \varepsilon \int_{S_\tau} X \cdot \dot{e}_1 d\mathcal{H}^{n-1} = |\dot{j}_e|, \] (3.12)
where $\dot{e}_1$ is the first coordinate vector of $\mathbb{R}^n$. Let us consider the domain
\[ \hat{O}_e = \{ x_1 \in \mathbb{R} : g(x_1) < \delta/100 \} \times \{ x' \in \mathbb{R}^{n-1} : G(x') < \delta \} \]
and denote its ‘lateral’ boundary $\Gamma_e \subset \partial \hat{O}_e$ as
\[ \Gamma_e = \{ x_1 \in \mathbb{R} : g(x_1) < \delta/100 \} \times \{ x' \in \mathbb{R}^{n-1} : G(x') = \delta \}. \]
Arguing as in the proof of Lemma 3.2 we deduce that $F > \delta/3$ on $\Gamma_e$ and therefore $\Gamma_e \subset (\partial U_{\delta/3})^c$. Since $X$ satisfies (3.5) $(X \cdot n = 0$ on $\partial U_{\delta/3}$ and $\text{div}(X) = 0$), we obtain by the divergence theorem and by (3.12) that for every $|\tau| < \delta/100$ it holds

$$
\varepsilon \left| \int_{S_\tau^+} X \cdot \partial_1 dH^{n-1} \right| = \varepsilon \left| \int_{S_0} X \cdot \partial_1 dH^{n-1} \right| = |j_e|.
$$

By rescaling the potential we may assume that $F(0) = 0$. Then by the above equality we have

$$
|j_e| = \varepsilon \left| \int_{S_\tau^+} X \cdot \partial_1 dH^{n-1} \right| \leq \varepsilon \left( \int_{S_\tau^+} |X|^2 e^{\frac{F}{\varepsilon}} dH^{n-1} \right)^{\frac{1}{2}} \left( \int_{S_\tau^+} e^{\frac{F}{\varepsilon}} dH^{n-1} \right)^{\frac{1}{2}}.
$$

Arguing as in the proof of [1, Theorem 1] we may estimate

$$
\int_{S_\tau^+} e^{-\frac{F}{\varepsilon}} dH^{n-1} \leq (1 + \hat{\eta}(C \varepsilon)) e^{\frac{\omega(\ell)}{\varepsilon}} e^{-\frac{\omega(\ell)(\tau)}{\varepsilon}} V_e(B_e(x_-), B_e(x_+); \Omega_e).
$$

Hence, by the two previous inequalities we have

$$
\frac{\bar{j}^2_e}{V_e(B_e(x_-), B_e(x_+); \Omega_e)} e^{\frac{\omega(\ell)}{\varepsilon}} e^{-\frac{\omega(\ell)(\tau)}{\varepsilon}} \leq (1 + \hat{\eta}(C \varepsilon)) \varepsilon^2 \int_{S_\tau^+} |X|^2 e^{\frac{F}{\varepsilon}} dH^{n-1}.
$$

Integrating over $\tau \in (-\delta/100, \delta/100)$ and using [1, Proposition 4.1] implies

$$
\frac{\bar{j}^2_e}{V_e(B_e(x_-), B_e(x_+); \Omega_e)} \leq (1 + \hat{\eta}(C \varepsilon)) \varepsilon^2 \int_{\partial \Omega_e} |X|^2 e^{\frac{F}{\varepsilon}} dx.
$$

The inequality (3.11) then follows from the definition of $y_e$ in (1.4). \qed

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**Benny Avelin**, Department of Mathematics, Uppsala University, S-751 06 Uppsala, Sweden  
Email address: [benny.avelin@math.uu.se](mailto:benny.avelin@math.uu.se)

**Vesa Julin**, Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35, 40014 Jyväskylä, Finland  
Email address: [vesa.julin@jyu.fi](mailto:vesa.julin@jyu.fi)