We discuss a procedure of measurement followed by the reproduction of the quantum state of a three-level optical system - a frequency- and spatially degenerate two-photon field. The method of statistical estimation of the quantum state based on solving the likelihood equation and analyzing the statistical properties of the obtained estimates is developed. Using the root approach of estimating quantum states, the initial two-photon state vector is reproduced from the measured fourth moments in the field. The developed approach applied to quantum states reconstruction is based on the amplitudes of mutually complementary processes. Classical algorithm of statistical estimation based on the Fisher information matrix is generalized to the case of quantum systems obeying Bohr’s complementarity principle. It has been experimentally proved that biphoton-qutrit states can be reconstructed with the fidelity of 0.995-0.999 and higher.

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I. INTRODUCTION

The ability of measuring quantum states is of fundamental interest because it provides a powerful tool for the analysis of basic concepts of quantum theory, such as the fundamentally statistical nature of its predictions, the superposition principle, Bohr’s complementarity principle, etc. To measure quantum state one needs to perform some projective measurements on the state and then to apply some computation procedure to the data. The first step is a genuine measurement consisting of a set of operations on the representatives of a quantum statistical (pure or mixed) ensemble. As a result of such operation an experimentalist acquires a set of frequencies at which particular events occur. In the second step a mathematical procedure is applied to the statistical data obtained in the previous step to reconstruct the quantum state. The present paper is devoted to the state reconstruction for the optical three-level systems. The object under study is the polarization state of a frequency- and spatially degenerate biphoton field. The necessity of the adequate measurement of the states of such systems is caused not only by fundamental interest but also by some applications. For example, it has been shown that the security of the key distribution in quantum cryptography is associated with the dimensionality of the Hilbert space for the states in use. From this point of view certain hopes are pinned on the three-level systems or qutrits rather than qubits.

We should mention that there are other implementations of three-level optical systems. The most familiar ones deal with three-arm interferometers and lower-order transverse spatial modes of optical field, realized with holograms. Polarization-entangled four-photon fields, which are equivalent to two entangled spin-1 particles were studied.

The paper is organized as follows. In Sec.II we discuss the main properties of qutrits based on the polarization state of biphoton field. We focus on their preparation, visual representation on Poincaré sphere, unitary transformation by phase plates. Then we consider the coherence matrix, which characterizes completely the properties of biphotons-qutrits in the fourth field moments. Sec.III is devoted to the methods of biphotons-qutrits measurement, in particular, we introduce two quantum tomography protocols and discuss in detail their experimental implementation. We conclude this part with the analysis of statistical reconstruction for qutrits from the outcomes of mutually complementary measurements. Sec. IV deals with the methods of quantum state reconstruction. Namely we consider the least-squares and maximum-likelihood methods and apply these tools to analysis the data obtained in quantum tomography. In Appendix we explore the problem of statistical fluctuations of the state vector which is important for the estimation and control of precision and stability of quantum information.
II. QUTRITS BASED ON BIPHOTONS

A. Preparation

Biphoton field is a coherent mixture of two-photon Fock states and the vacuum state $|\text{vac}\rangle$:

$$\Psi = |\text{vac}\rangle + \frac{1}{2} \sum_{k_i,k'} F_{k_i,k}' |1_{k_i},1_{k}'\rangle,$$  \hspace{1cm} (1)

where $|1_{k_i},1_{k}'\rangle$ denotes the state with one (signal) photon in the mode $k_i$ and one (idler) photon in the mode $k'_i$. The coefficient $F_{k_i,k}'$ is called the biphoton amplitude \cite{12}, because its squared modulus gives a probability to register two photons in modes $k_i$ and $k'_i$.

Let us consider the collinear and frequency degenerate regime, for which $\vec{k}_i \approx \vec{k}'_i$, $\omega_s \approx \omega_i$ and $\omega_s + \omega_i = \omega_p$, where $\omega_p$ is the laser pump frequency. We further restrict our discussion to biphotons that are indistinguishable in terms of spatial, spectral, or temporal parameters. From the point of view of polarization there are three natural states of biphotons, namely, $\Psi_1 = |2,0\rangle$, $\Psi_2 = |1,1\rangle$, and $\Psi_3 = |0,2\rangle$. Here the notation $|2,0\rangle = |2_H,0_V\rangle$, for example, indicates that there are two photons in the horizontal ($H$) polarization mode, while no photons are present in the orthogonal vertical ($V$) mode. These basic states can be generated using type-I (for $\Psi_1$ and $\Psi_3$) and type-II (for $\Psi_2$) phase-matching. Since only two-photon Fock states are considered, for the state $|m,n\rangle$ the condition $m + n = 2$ must be satisfied.

Any arbitrary pure polarization state of biphoton field can be expressed in terms of three complex amplitudes $c_1, c_2,$ and $c_3$:

$$|c\rangle = c_1|2,0\rangle + c_2|1,1\rangle + c_3|0,2\rangle,$$  \hspace{1cm} (2)

where $c_j = |c_j| \exp(i \varphi_j)$, $\sum_{j=1}^3 |c_j|^2 = 1$. The vector $|c\rangle = (c_1, c_2, c_3)$ represents a three-state state or qutrit.

There is an important note concerning the state-vector (2). In principle, one can write the complete polarization state in the form

$$|c\rangle = c_1|2_H,0_V\rangle + c_2|1_V,1_H\rangle + c_2'|1_H,1_V\rangle + c_3|0_H,2_V\rangle,$$  \hspace{1cm} (3)

where the terms $|1_H,1_V\rangle$ and $|1_V,1_H\rangle$ might be distinguishable somehow, for example, if the photon with vertical polarization comes first with respect to the photon with horizontal polarization. However we consider particular two-mode polarization state so photons differ in polarization only and there are no other parameters responsible for their distinguishability.

In general, to generate an arbitrary qutrit state one needs to put three nonlinear crystals separated in space.

![FIG. 1: Preparation of an arbitrary qutrit based on biphotons](image)

Sometimes it is very convenient to use visual representation of a qutrit using the Poincaré sphere. Although the generalization of the Poincaré sphere for qutrits has been discussed earlier \cite{13} we suggest an alternative approach, which allows us to manipulate with qutrits in natural 3-D space rather than in sophisticated 8-D space. Let us map the polarization state of a biphoton into a pair of points on the sphere (but this is not the two-qubit case since the states $|H,V\rangle$ and $|V,H\rangle$ are indistinguishable). In this representation each photon forming the biphoton is plotted as a single point on the Poincare sphere, so the qutrit state vector is represented by

$$|c\rangle = \left[ a_1^\dagger(\vartheta,\phi)a_1^\dagger(\vartheta',\phi') a_1^\dagger(\vartheta,\phi)a_1^\dagger(\vartheta',\phi') \right]|\text{vac}\rangle \left[ a_1^\dagger(\vartheta,\phi)a_1^\dagger(\vartheta',\phi') a_1^\dagger(\vartheta,\phi)a_1^\dagger(\vartheta',\phi') \right]|\text{vac}\rangle$$  \hspace{1cm} (4)

where $a_1^\dagger(\vartheta,\phi)$ and $a_1^\dagger(\vartheta',\phi')$ are the creation operators in idler and signal polarization modes and $a_1^\dagger(\vartheta_m,\phi_m) = \cos(\vartheta_m/2)a^\dagger + e^{i\phi_m}\sin(\vartheta_m/2)b^\dagger$, $m = i,s$. Note that oper-
It is well-known that the number of real parameters characterizing a quantum state is determined by the dimension of the Hilbert space ($s$). For a pure state, 

$$N_{\text{pure}} = 2s - 2,$$  

(5a) 

and for mixed states, 

$$N_{\text{mixed}} = s^2 - 1.$$  

(5b)

According to (5a,b), four real parameters determine completely the pure state of a qutrit, so in the Poincaré sphere representation these parameters are simply the four spherical angles ($\vartheta_i, \phi_i; \vartheta_s, \phi_s$). The links between the angles ($\vartheta, \phi; \vartheta_s, \phi_s$) and the amplitudes $c_j = |c_j| \exp i\varphi_j$ are derived in [14]. As an example three basic states $\Psi_1 = |2, 0\rangle$, $\Psi_2 = |1, 1\rangle$, and $\Psi_3 = |0, 2\rangle$ are shown in Fig. 2. It can be shown that the polarization degree of a qutrit $P = \sqrt{|c_1|^2 - |c_3|^2 + 2|c_1c_2 + c_2c_3|^2}$ has a clear geometrical meaning: it is defined by the angle $\beta$ between the pair of points on the Poincaré sphere as seen from its center:

$$P = \frac{2 \cos(\beta/2)}{1 + \cos^2(\beta/2)}.$$  

(6)

For the states $\Psi_1$ and $\Psi_3$ the polarization degree takes values $P_{1,3} = 1$, since two points coincide on the sphere and $\beta = 0$. For the second state, $\Psi_2$, two points are positioned at the opposite sides of the sphere, that is why $\beta/2 = \pi/2$ and $P_2 = 0$.

C. Transformation

Experimentally a unitary transformation of the polarization state (2) can be achieved by placing any retardation plates, rotators etc. into the biphoton beam. The action of such elements on the state (2) is described by the matrix [15]:

$$G = \begin{pmatrix} t^2 & \sqrt{2tr} & r^2 \\ -\sqrt{2tr^*} & |t|^2 - |r|^2 & \sqrt{2tr^*} \\ -r^2 & \sqrt{2t^*r} & t^2 \end{pmatrix},$$  

(7)

where

$$t = \cos \delta + i \sin \delta \cos 2\alpha, \quad r = i \sin \delta \sin 2\alpha,$$  

(8)

$$\delta = \pi(n_o - n_e)h/\lambda$$

is the optical thickness of the plate, $h$ is its geometrical thickness, $\alpha$ is the orientation angle between the optical axis of the plate and one of the basis, for example, vertical direction.

Let us consider the action of the half-lambda plate on a particular state $\Psi_\perp = \frac{1}{\sqrt{2}}(|2, 0\rangle - |0, 2\rangle)$, when the plate is oriented at 22.5°. For the state $\Psi_\perp$ there are two nonzero amplitudes $c_2 = c_3 = 1/\sqrt{2}$ and there is only one relative phase $\varphi_{13} = \varphi_2 - \varphi_3 = \pi$. Taking into account that for a half-lambda plate $\delta = \pi/2$, the corresponding transmission and reflection coefficients are

$$t = r = \frac{i}{\sqrt{2}}.$$  

(9)

Thus the matrix $G$ has the form

$$G = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$  

(10)

Hence, acting by matrix $G$ on the state $\Psi_\perp$ we get

$$G\Psi_\perp = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \Psi_2.$$  

Note that such kind of transformations cannot change the polarization degree of a qutrit. For the state $\Psi_\perp$ chosen above, as well as for the state $\Psi_2$, the polarization degree $P$ is zero.

In the experiment described below we used a simpler way to generate qutrits. Biphotos were produced via collinear frequency-degenerate spontaneous parametric down conversion in a nonlinear crystal (BBO, type-I or type-II phase matching). For type-I phase matching the polarization of both created photons was vertical; i.e., the state $\Psi_3$ was generated. Then, this state was transformed using a quartz plate with a fixed optical thickness. By changing the angle of the plate, the state $\Psi_2 = |0_H, 2_V\rangle$ is transformed according to the formula $|c_{in}\rangle = G|c_{in}\rangle$.

For the case of type-II phase matching the final state is $|c_{in}\rangle = G|c_{in}\rangle$. Of course the state $|c_{in}\rangle$ does not involve all possible qutrit states because the transformation given by matrix (7) preserves the polarization degree. Anyway
using such a transformation, we select some subset of qutrits to work with.

Such a simple method of the state preparation/ transformation was chosen in order to be able to compare the results of reconstruction with the parameters of the input states, which should be known with a high accuracy. The purpose of this work is the reconstruction of the initial state \(|c_m\rangle\).

We would like to emphasize that only pure qutrit states are accessible by this method. To create a mixed state, some more complicated method is to be used. This method allows one to create arbitrary qutrit states and it implies a possibility to introduce controlled delay between three fundamental states forming the qutrit which could exceed the coherence length of the laser pump.

D. Coherence matrix

We introduced only qualitative description of the qutrits based on biphotons so far. The quantitative measure characterizing the polarization properties of any single-mode state in the forth moment in the field (including biphoton state) was proposed by D. Klyshko in [17]. It is a matrix consisting of six fourth-order moments of the electromagnetic field. An ordered set of such moments can be obtained using the direct product of 2 × 2 coherence matrices for both qubits. After normal ordering, averaging, and crossing out the redundant row and column the matrix takes the following form:

\[
K_4 \equiv \begin{pmatrix}
A & D & E \\
D^* & C & F \\
E^* & F^* & B
\end{pmatrix}.
\] (11)

The diagonal elements are formed by real moments, which characterize the intensity correlation in two polarization modes \(H\) and \(V\):

\[
A \equiv \langle \hat{a}^2 \hat{a}^2 \rangle, \quad B \equiv \langle \hat{b}^2 \hat{b}^2 \rangle, \quad C \equiv \langle \hat{a} \hat{b} \hat{a} \hat{b} \rangle.
\] (12)

Nondiagonal moments are complex:

\[
D \equiv \langle \hat{a}^2 \hat{b} \rangle, \quad E \equiv \langle \hat{b}^2 \hat{a} \rangle, \quad F \equiv \langle \hat{a} \hat{b} \rangle^{\ast} \langle \hat{b} \rangle.
\] (13)

Three real moments (12) and three complex ones (13) completely determine the state under consideration. The elements of the matrix (11) are expressed through the elements of the polarization density matrix. The normalization condition,

\[
A + B + 2C = 2,
\] (14)

reduces the number of independent real parameters, so for a mixed state we get 8 parameters as expected. In the special case of a pure biphoton state, taking the average in Eqs. (12,13) over the state (2), we obtain the matrix components in the following form:

\[
A = 2 |c_1|^2, \quad B = 2 |c_3|^2, \quad C = |c_2|^2.
\] (15)

\[
D = \sqrt{2} c_1^\ast c_2, \quad E = 2 c_1^\ast c_3, \quad F = \sqrt{2} c_2^\ast c_3
\] (16)

So the links between the polarization density matrix and the matrix (11) can be found comparing the corresponding components of \(K_4\) and of \(\rho = |c\rangle \langle c|\); \(\rho_{mk} = c_m^\ast c_k^\ast; m, k = 1, 2, 3\) for a pure state and \(\rho_{mk} = c_m^\ast c_k^\ast\) for a mixed state where the averaging, as usual, is taken over the classical probability distribution. The statistics of the field is assumed to be stationary and ergodic so the time-averaged values of the observed quantities can be described in terms of a quantum statistical ensemble. In this case \(\langle ... \rangle = Tr(\rho ... )\), where \(\rho\) is the polarization density operator.

III. METHODS OF MEASUREMENT

What does it mean to measure the unknown state (2)? From the experimental point of view, it means that the experimentalist has to measure a complete set of real parameters (moments) determining the state. To do this the state must be subject to a set of unitary transformation transformations and projective measurements. By doing this one picks out the outcomes, which are proportional to the corresponding moments (12, 13) or their linear combination. This procedure is known as quantum tomography. The quantum state can be represented using either the wave function, density matrix, or quasi-probability function (Wigner function). Probably the correct way to use the term “quantum tomography” is only for the reconstruction of the quasi-probability function because it gives the graphical representation of the state as a 3D plot. Nevertheless the term “quantum tomography” is also used for a general procedure of complete state reconstruction. For a brief review among the papers where this procedure was realized experimentally, let us mention the works related to states defined by continuous variables. For states characterized by discrete variables, such as two polarization-spatial qubits, quantum tomography was realized in [21]. Recently quantum tomography has been performed for orbital angular momentum entangled qutrits [5] etc.

The physical idea behind the tomography procedure is performing measurements of appropriately complete set of observables called quorum [22] or just “looking” at the state from different positions. The minimal number of such positions might be the number of real parameters determining the state.

According to Bohr’s complementarity principle, it is impossible to measure all moments (12,13) simultaneously, operating with a single qutrit only. So to perform
a complete set of measurements one needs to generate a lot of representatives of a quantum ensemble.

First of all, let us mention that at present, the only realistic way to register single-mode biphoton field is using the Brown-Twiss scheme. This scheme consists of a beam-splitter followed by a pair of detectors connected with the coincidence circuit. It means that registration of a single biphoton, which carries the state (2), can give only a single event at the output of the experimental set-up with some probability. So the statistical treatment of the outcomes becomes extremely important. For correlations between polarization degrees of freedom, which is essential in the case under consideration, the Brown-Twiss scheme must be accomplished with polarization filters introduced into each arm.

A. Qutrit tomography protocols

We proposed two methods to perform polarization reconstruction of a biphoton qutrit state $|c_{in}\rangle$.

1. Protocol 1.

The idea of the first method is splitting the state $|c_{in}\rangle$ into two spatial modes and performing transformations over two photons independently (Fig 3). These transformations can be done using polarization filters placed in front of detectors. Each filter consists of a sequence of quarter- and half-wave plates and a polarization prism, which picks out definite linear polarization, for example, the vertical one. A narrowband filter centered at the doubled pump wavelength $\lambda = 2\lambda_p$ serves to make biphotons emitted from different sources indistinguishable in frequency as well as to reduce the background noise. An event is considered to be detected, if a pulse appears at the output of the coincidence circuit. Approximately in half of trials, one of the photons (signal, by convention) forming a biphoton is going to one of the detectors, while the other one (idler) is going to the other detector. In the remaining cases, both photons appear in the same output beam-splitter arm, and these events are not selected because they do not contribute to coincidences.

In the Heisenberg representation the polarization transformation for each beam-splitter output port is given by:

$$D_{\lambda/2}(\delta = \pi/2, \theta) \times D_{\lambda/4}(\delta = \pi/2, \theta)$$

Four $2 \times 2$ matrixes in the right-hand side of Eq. (17) describe the action of the non-polarizing beam-splitter, $\lambda/4$, $\lambda/2$- plates and vertical polarization prism on the state vector of the signal (idler) photon:

$$D_{\lambda/2,\lambda/4} = \begin{pmatrix} t & r \\ -r^* & t^* \end{pmatrix}$$

where $r$ and $t$ - are the coefficients introduced in Eq. (8), so for a $\lambda/4$-plate ($\delta = \pi/4$),

$$t_{\lambda/4} = \frac{1}{\sqrt{2}}(1 + i \cos 2\chi), r_{\lambda/4} = \frac{i}{\sqrt{2}} \sin 2\chi$$

and for a $\lambda/2$-plate ($\delta = \pi/2$)

$$t_{\lambda/2} = i \cos(2\theta), r_{\lambda/2} = i \sin(2\theta)$$

Thus, there are four real parameters (two for each channel) that determine polarization transformations. Namely, these parameters are orientation angles for two pairs of wave plates: $\theta_1, \chi_1, \theta_2, \chi_2$.

As it was mentioned above, the output of the Brown-Twiss scheme is the coincidence rate of the pulses coming from two detectors $D_s$ and $D_i$. The corresponding moment of the fourth order in the field has the following structure:

$$R_{s, i} \propto \langle b_i^+ b_i^+ b_i^+ b_i^+ \rangle = R(\theta_1, \chi_1, \theta_2, \chi_2)$$

In the most general case this moment contains a linear combination of six moments (12, 13) forming the matrix $K_4$. So the main purpose of the quantum tomography procedure is extracting these six moments from the set-up outcomes by varying the four parameters of the polarization Brown-Twiss scheme.
Consider some special examples, which give the corresponding lines in the complete protocol introduced below (Table I).

First of all, it is obvious that for measuring real moments (12) one needs to make polarization filters transmit both photons with vertical polarizations to measure $A$, both photons with vertical polarization to measure $B$ and one photon with vertical and another one with horizontal polarizations to measure $C$. To do this all quarter-wave plates should be oriented at zero degrees, then to install both half-wave plates at zero degrees for measuring $B$; at $\theta_s = 45^\circ$ and $\theta_i = 45^\circ$ for measuring $A$; and at $\theta_s = 0^\circ$, $\theta_i = 45^\circ$ for measuring $C$. These settings pick out the squared modulus of corresponding amplitudes $c_3$, $c_1$ and $c_2$.

The next example shows how to measure one of the complex moments (13). To measure the real part of the moment $D$, let us set the wave-plates in the Brown-Twiss scheme in the following way.

The idler channel:

$$\frac{\lambda}{4} : \chi_i = 0^\circ, D_{\lambda/4} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + i & 0 \\ 0 & 1 - i \end{pmatrix} ; \quad (20a)$$

$$\frac{\lambda}{2} : \theta_i = 45^\circ, D_{\lambda/2} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} . \quad (20b)$$

The signal channel:

$$\frac{\lambda}{4} : \chi_s = 45^\circ, D_{\lambda/4} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} ; \quad (21a)$$

$$\frac{\lambda}{2} : \theta_s = 22.5^\circ, D_{\lambda/2} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & i \\ i & -i \end{pmatrix} . \quad (21b)$$

Substituting these matrices into Eq. (17) and taking into account the commutation rules for the creation and annihilation operators it is easy to get the final moment to be measured:

$$R = \langle c|b_1^\dagger b_1^\dagger b_s b_i|c\rangle = 1/8(A + C - 2\text{Re}D).$$

A complete set of the measurements called the tomography protocol is presented in Table I. Each row corresponds to the setting of the plates to measure the moment placed in the sixth column. The last one corresponds to the amplitude of the process (see below).

This protocol was suggested and developed in [23, 24]. A similar protocol was considered in details earlier [21].

For estimating polarization state of a biphoton field, generated in a frequency degenerate non-collinear mode. In this case the biphoton field is represented as a pair of polarization qubits.

Before describing the second method of the state measurement let us make some remarks.

*We assume that the source generating qutrits is stationary. Since each measurement eliminates a qutrit one has to be sure that there are a lot of copies of the initial state; each copy must be prepared in the same quantum state. Such ensemble approach guarantees that the experimentalist deals with the same quantum state in all trials. In other words, the outcomes provide him with the information about the same quantum state and elimination of a particular state does not affect the rest ones.

**The outcomes of the set-up are numbers related to the corresponding moments (19). Usually this number is the coincidence counting rate or the number of coincidences in a fixed time interval. Due to the necessity of a proper normalization of the state under investigation, the number of independent real parameters grows up. The normalization is obtained from the measurement of moments $A$, $B$ and $C$. Furthermore, only cosine and sine of the phases $\varphi_{12}$ and $\varphi_{13}$ can be measured in experiment as there is no way to measure the phases directly. That is why the final number of real parameters to be measured in experiment is 7 for a pure qutrit state and 9 for a mixed state.

***To minimize the errors caused by independent statistical fluctuations of the outcomes, the number of moments (12, 13) entering in Eq. (19) should be minimal.

### 2. Protocol 2

In the second method of quantum tomography, a biphoton-qutrit being measured is first subject to a sequence of unitary transformations and, for each of such transformation, it is fed to the Brown-Twiss scheme settled for measuring a fixed moment. Using the wave plate with arbitrary optical thickness, one can achieve the quantum varying the orientation of the plate $\mu$.

In the most general case the coincidence counting rate in this protocol is a periodic function of $\mu$, moreover, its Fourier expansion contains nine harmonics of $\mu : \cos(0\mu)$, $\cos(2\mu)$, $\sin(2\mu)$, $\cos(4\mu)$, $\sin(4\mu)$, $\cos(6\mu)$, $\sin(6\mu)$, $\cos(8\mu)$, $\sin(8\mu)$. These harmonics depend linearly on the nine moments $A$, $B$, $C$, $\text{Re}D$, $\text{Im}D$, $\text{Re}E$, $\text{Im}E$, $\text{Re}F$, $\text{Im}F$. In other words, there is a $9 \times 9$ matrix $T$ that links these nine harmonics to the nine moments as shown below.

$$\begin{pmatrix} \cos 0\mu \\ \cos 2\mu \\ \sin 2\mu \\ \cos 4\mu \\ \sin 4\mu \\ \cos 6\mu \\ \sin 6\mu \\ \cos 8\mu \\ \sin 8\mu \end{pmatrix} = T \begin{pmatrix} A \\ B \\ C \\ \text{Re}F \\ \text{Im}F \\ \text{Re}D \\ \text{Im}D \\ \text{Re}E \\ \text{Im}E \end{pmatrix}.$$
Moments to be measured

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\(\nu\) & \(\chi_{s}\) & \(\theta_{s}\) & \(\chi_{\perp}\) & \(\theta_{\perp}\) & \(R_{s,\perp}\) & \(M_{\nu}\) \\
\hline
1 & 0 & 45° & 0 & -45° & \(A/4\) & \(c_{1}/\sqrt{2}\) \\
2 & 0 & 45° & 0 & 0 & \(C/4\) & \(c_{2}/2\) \\
3 & 0 & 0 & 0 & 0 & \(B/4\) & \(c_{3}/\sqrt{2}\) \\
4 & 45° & 0 & 0 & 0 & \(1/8(B+C+2\text{Im}F)\) & \(2\sqrt{2}c_{2} - c_{3}\) \\
5 & 45° & 22.5° & 0 & 0 & \(1/8(B+C-2\text{Re}F)\) & \(2\sqrt{2}c_{2} - 2c_{3}\) \\
6 & 45° & 22.5° & 0 & -45° & \(1/8(A+C-2\text{Re}D)\) & \(2c_{1} - 2\sqrt{2}c_{2}\) \\
7 & 45° & 0 & 0 & -45° & \(1/8(A+C+2\text{Im}D)\) & \(2\sqrt{2}c_{1} - c_{2}\) \\
8 & -45° & 11.25° & -45° & 11.25° & \(1/16(A+B-2\text{Im}E)\) & \(2\sqrt{2}c_{1} + 2c_{3}\) \\
9 & 45° & 22.5° & -45° & 22.5° & \(1/16(A+B+2\text{Re}E)\) & \(2\sqrt{2}c_{1} - 2\sqrt{2}c_{3}\) \\
\hline
\end{tabular}
\caption{Protocol 1. Each line contains orientation of the half \((\theta_{s},\iota)\) and quarter \((\chi_{s},\iota)\) wave plates in the measurement block. Last two columns show the corresponding moment \(R_{s,\perp}\) and the process amplitude \(M_{\nu}\) \((\nu = 1,...9)\).}
\end{table}

\section*{B. Experimental implementation. Protocol 1}

The experimental set-up for the quantum tomography of qutrits using protocol 1 is shown in Fig.5. The \textit{preparation block} includes a 2-mm BBO crystal with either type-I or type-II degenerate and collinear phase-matching, which is pumped with cw-argon laser operated at 351nm wavelength. In the case of type-II phase-matching, an additional quartz compensator is introduced right after the crystal. The state \(\Psi_{3} = |0,2\rangle\) (for type-I) or \(\Psi_{2} = |1,1\rangle\) (for type-II) generated in the crystal is fed to the \textit{transformation block}. This block consists of the quartz plate with fixed optical thickness \(\delta_{s} = \pi/4, \delta_{\iota} = \pi/2\) and fixed orientation \(\chi_{s},\theta_{s}\) in each channel of the Brown-Twiss scheme. In order to make sure the inverse matrix exists one needs to maximize the determinant of the matrix \(T\) over the orientations of the plates \(\chi_{s},\theta_{s}\). After accomplishing this procedure we obtain \(\chi_{s} \approx 19^\circ, \theta_{s} \approx -28.5^\circ\) (Fig.4).

Instead of finding the links between the harmonics and moments, there is a more elegant method to reconstruct the quantum state using the second protocol (see section III.D). This method is considered in the present work for the first time.

In our experiments the exposure time for measuring each moment is 5 sec. This time is an important experimental parameter. Each measurement consists of 30 runs, after which the scheme is reset. Namely each measurement is performed by setting the angles of wave plates \(\chi_{j},\theta_{j}\) in both arms according to the tomographic protocol (Table I). After 30 runs, a new set of angles is selected and the next moment is measured in the same way. The output data of the set-up are the mean coincidence counting rates. Examples of behavior for some moments \((A,B,C,\text{Re}F,\text{Im}F)\) versus the orientation of the plate \(\text{QP1}\) are plotted in Fig.6.

\section*{C. Quantum states and tomography}

In quantum mechanics the state of a system is described by a density matrix \(\rho\). Each element \(\rho_{ij}\) of this matrix corresponds to the probability of finding the system in the state \(|i\rangle\) when it is measured in the state \(|j\rangle\). The density matrix is Hermitian, i.e., \(\rho = \rho^\dagger\), and its diagonal elements \(\rho_{ii}\) satisfy the normalization condition \(\sum_{i} \rho_{ii} = 1\). The \(\rho_{ij}\) are the probability amplitudes of the state \(|i\rangle\) being measured when the system is in state \(|j\rangle\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Measurement block for Protocol 2. Additional control quartz plate (\textit{QP2}) serves as state \(|c_{s}\rangle\) tomography transformer. Only a single wave plate is introduced in each channel.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5}
\caption{Scheme of qutrits tomograph, consisting of three blocks. Preparation block includes pump laser(s) and nonlinear crystal(s). Transformation block is the quartz plate (\textit{QP1}) which orientation angle \(\alpha\) determines the final state to be measured. The measurement block depends on the protocol to be used (see Fig.3 and Fig.4).}
\end{figure}
C. Experimental implementation. Protocol 2

For the second method we used Ti:Sa laser with pulse duration about 250 fs, operating at 800 nm. After frequency doubling, the UV radiation with 400 nm wavelength was sent into the same set-up as described above. For this protocol we used 2-mm BBO crystal cut for collinear degenerate type-I phase-matching. A quartz plate with the optical thickness $\delta = 0.656$ is placed after the BBO crystal to prepare the qutrit state to be measured. The measurement part of the set-up was slightly changed (Fig.4). Additional control quartz plate introduced in front of the beam splitter accomplishes the protocol. Its orientation angle $\mu$ is a parameter defining the measurement process. The control plate is rotated with 5 degrees step from 0 up to 360 degrees so that Protocol 2 consists of 72 measurements. Each arm of the Brown-Twiss scheme contains either quarter or half-wave plate with fixed orientation. The orientations are $\chi_1 = 18.8^\circ$ for the quarter-wave plate in the first channel and $\theta_2 = -28.5^\circ$ for the half-wave plate in the second channel. Protocol 2 is easier to implement since only a single parameter $\mu$ is changed whereas four $\chi_{1,2}, \theta_{1,2}$ parameters are varied in Protocol 1. In perspective, this kind of protocol allows one to automate the quantum tomography procedure: the control plate can be rotated continuously and the reconstruction of the quantum state can be based on the analysis of coincidence rates corresponding to the respective values of $\mu_i (i = 1, \ldots, 72)$.

D. Statistical reconstruction of biphoton-field qutrits from the outcomes of mutually complementary measurements

Each of the 9 processes from Protocol 1 as well as of the 72 processes from Protocol 2 is described by its amplitude $M_\nu$. From the statistical point of view, the squared modulus of the process amplitude specifies the intensity of the event generation:

$$R_\nu = M_\nu^* M_\nu$$  \hspace{1cm} (22)$$

The considered processes are examples of mutually complementary sets of measurements in the sense of Bohr’s complementarity principle. The event-generation intensities $R_\nu$ for both protocols are the main quantities accessible from the measurement. Making the bridge between statistical and physical description of the process the quantities $R_\nu$ coincide with the fourth moments in the field introduced above in Eq. (19). Their dimension is frequency unit (Hz). The number of events occurring within any given time interval obeys the Poisson distribution. Therefore, the quantities $R_\nu$ specify the intensities of the corresponding mutually complementary Poisson processes and serve as estimates of the Poisson parameters $\lambda_\nu$ (see below).

Although the amplitudes of the processes cannot be measured directly, they are of the greatest interest as these quantities describe fundamental relationships in quantum physics. It follows from the superposition principle that the amplitudes are linearly related to the state-vector components. So the main purpose of quantum tomography is reproduction of the amplitudes and state vectors, which are hidden from direct observation.

The linear transformation of the state vector $c = \{c_1, c_2, c_3\}$ into the amplitude of the process $M$ is described by a certain matrix $X$. For example, considering the first protocol this matrix can be easily obtained from

FIG. 6: Some components of the matrix $K_A$ versus the orientation of the quartz plate QP1. Different angles of the plate correspond to different states sent to the measurement block. The plot at the top corresponds to measured real moments $A$ (squares), $B$ (circles), $C$ (triangles) and theoretical predictions $A(-), B(-), C(-)$ corresponding to different states sent to the measurement block.
Table I (last column in Table I):

\[
X = \begin{pmatrix}
1/\sqrt{2} & 0 & 0 \\
0 & 1/2 & 0 \\
0 & 0 & 1/\sqrt{2} \\
0 & 1/(2\sqrt{2}) & -i/2 \\
0 & 1/(2\sqrt{2}) & -1/2 \\
1/2 & -1/(2\sqrt{2}) & 0 \\
1/2 & -i/(2\sqrt{2}) & 0 \\
1/(2\sqrt{2}) & 0 & i/(2\sqrt{2}) \\
1/(2\sqrt{2}) & 0 & -1/(2\sqrt{2})
\end{pmatrix}
\]  

(23)

Using the matrix \(X\) the complete set of nine amplitudes of the processes can be expressed by a single equation,

\[Xc = M\]  

(24)

The matrix \(X\) is an instrumental matrix for a set of mutually complementary measurements, by analogy with the conventional instrumental function. The implementation of the method to the first protocol has been considered in [25].

Consider an algorithm allowing one to calculate the instrumental matrix \(X\) for Protocol 2. The matrix consists of 72 rows (the number of control plate orientations) and 3 columns (the dimension of Hilbert space for qutrits). Each row is formed in the following way. Using coefficients \(t_{s,i}^{(s,i)}\) and \(r_{s,i}^{(s,i)}\) of the wave plates introduced to the signal and idler channels of the Brown-Twiss scheme (18a,b) the three-element row, which defines the process amplitude right after the control plate, can be written in the form

\[l = \left[\frac{1}{\sqrt{2}}r_{s,i}^{(s,i)}t_{s,i}^{(s,i)} + r_{s,i}^{(s,i)}t_{s,i}^{(s,i)}\right].\]  

(25)

The unitary matrix \(G\) is defined by the control plate according to (7, 8), with a replacement \(\alpha \rightarrow \mu\), where \(\mu\) is the control plate orientation (it takes 72 values from 0° to 355°). We chose the control plate to be a quarter-wave plate, so \(\delta = \pi/4\). Finally,

\[G = G(\mu_i), i = 1, 2, ..., 72.\]  

(26)

Each row of the instrumental matrix \(X\) (72 rows, 3 columns) is defined by the product of the row \(l\) (which is the same for any process) and the matrix \(G\) (which is defined by the control plate orientation angle)

\[X_i = lG(\mu_i), i = 1, 2, ..., 72,\]  

(27)

where \(X_i\) is the \(i\)th row of the matrix \(X\).

IV. METHODS OF QUANTUM STATE RECONSTRUCTION

In the simplest case the density matrix can be estimated directly from the measurements. Since the set of experimental data is limited in this case, the reconstructed density matrix may have non-physical properties like negative eigenvalues. But in the general case of \(s\)-dimensional systems the problem of density matrix reconstruction using direct results of measurements can not be solved since the corresponding inverse problem is ill-posed.

When analyzing the experimental data, we use the so-called root estimator of quantum states [24]. This approach is designed specially for the analysis of mutually complementary measurements (in the sense of Bohr’s complementarity principle). The advantage of this approach consists of the possibility of reconstructing states in a high-dimensional Hilbert space and reaching the accuracy of reconstruction of an unknown quantum state close to its fundamental limit. Below we consider two methods of quantum state root estimation that give similar results. They are the least-squares method (LSM) and maximum-likelihood method (MLM).

A. Least-squares method

In statistical terms, Eq. (24) is a linear regression equation. A distinctive feature of the problem is that only the absolute value of the process amplitude \(M\) is measured in the experiment. The estimate of the absolute value of the amplitude is given by the square root of the corresponding experimentally measured coincidence rate:

\[|M_\nu|^{\text{exp}} = \sqrt{k_\nu/t},\]  

(28)

where \(k_\nu\) is the number of events (coincidences) detected in the \(\nu\)-th process during the measurement time \(t\).

It is worth noting that, by the action of the root-square procedure on a Poissonian random value, one gets at the random variable with a uniform variance, i.e., at the variance stabilization [27]. Note also, since we deal not with event probabilities but with their rates or intensities, it is convenient to use un-normalized state vectors. These vectors allow the coincidence counting rate (event-generation intensities) to be derived directly from Eqs. (22), (24) without introducing the coefficients related to the biphoton generation rate, detector efficiencies, etc. The dimensionality of the vector state obtained in such a way is \(1/\sqrt{t}\). The final state vector obtained by the reconstruction procedure, nevertheless, should be normalized to unity.

Assuming that the variances of different \(|M_\nu|^{\text{exp}}\) are independent and identical, one can apply the standard least-squares estimate to Eq. (24) [27].

\[c = (X^\dagger X)^{-1}X^\dagger M\]  

(29)

Unlike the traditional least-squares method, Eq. (29) cannot be used for the explicit estimation of the state vector \(c\), because it is to be solved by the iteration method.
The absolute value of $M$ is known from the experiment ($|M_{ii}| = |M_{ii}^{\text{exp}}|$). We assume that the phase of vector $X_{ei}$ at the $i$-th iteration step determines the phase of the vector $M$ at the $i + 1$-th step. In other words the phase is determined by the iteration procedure.

It turns out that, for the Gaussian approximation of Poisson’s quantities, this least-squares estimate coincides with a more exact and rigorous maximum-likelihood estimate considered below.

### B. Maximum-likelihood method

The likelihood function is defined by the product of Poissonian probabilities:

$$L = \prod_{i} \frac{(\lambda_{i} t_{i})^{k_{i}}}{k_{i}!} e^{-\lambda_{i} t_{i}},$$

(30)

where $k_{i}$ is the number of coincidences observed in the $i$-th process during the measurement time $t_{i}$, and $\lambda_{i}$ are the unknown theoretical event-generation intensities (expected number of coincidences proportional to the moments in the field), whose estimation is the subject of this section.

The logarithm of the likelihood function is, if we omit an insignificant constant,

$$\ln L = \sum_{i} (k_{i} \ln(\lambda_{i} t_{i}) - \lambda_{i} t_{i}).$$

(31)

Let us introduce the matrices with the elements defined by the following formulas:

$$I_{js} = \sum_{i} t_{i} X_{ij}^{*} X_{is},$$

(32)

$$J_{js} = \sum_{i} \frac{k_{i}}{\lambda_{i}} X_{ij}^{*} X_{is}; \; j, s = 1, 2, 3.$$

(33)

The matrix $I$ is determined from the experimental protocol and, thus, is known a priori (before the experiment). This is the Hermitian matrix of Fisher’s information. The matrix $J$ is determined by the experimental values of $k_{i}$ and by the unknown event-generation intensities $\lambda_{i}$. This is the empirical matrix of Fisher’s information (see also Appendix).

In terms of these matrices, the condition for the extremum of the function (31) can be written as

$$I_{ei} = J_{ei}.$$

(34)

Hence, it follows that

$$I^{-1} J_{ei} = e.$$

(35)

The latest relationship is known as the likelihood equation. This is a nonlinear equation, because $\lambda_{i}$ depends on the unknown state vector $c$. Because of the simple quasi-linear structure, this equation can easily be solved by the iteration method [26]. The quasi-identity operator $I^{-1} J$ acts as the identical operator upon only a single vector in the Hilbert space, namely, on the vector corresponding to the solution of Eq. (35) and representing the maximum possible likelihood estimate for the state vector. The condition for the existence of the matrix $I^{-1}$ is a condition imposed on the initial experimental protocol. The resulting set of equations automatically includes the normalization condition, which is written as

$$\sum_{i} k_{i} = \sum_{i} (\lambda_{i} t_{i}) .$$

(36)

This condition implies that, for all processes, the total number of detected events is equal to the sum of the products of event detection frequencies during the measurement time.

### C. Analysis of the experimental data

#### 1. Pure state reconstruction

The examples of qutrit state reconstruction using both the least-squares and maximum-likelihood methods are given in Table II.

The value of the fidelity parameter $F$ is defined as

$$F = |\langle c_{\text{theory}} | c_{\text{exp}} \rangle|^{2}.$$

(37)

It gives the conventional measure of correspondence between the theoretical and experimental state vectors.

The dependence of fidelity on the amount of experimental data obtained is shown in Fig.7. This figure shows the fidelity achieved in the experiment in comparison with the theoretical range (see Appendix for more details). The lower boundary corresponds to 5% quantile of statistical distribution, while the upper to 95% quantile. It is clearly seen that the fidelity value achieved experimentally for a small volume of experimental data is completely within the limits of the theoretical range, while it goes out for a higher volume. Such behavior of fidelity is due to the existence of two different error types arising under the reconstruction of quantum states. Let us call them statistical and instrumental errors, respectively. The statistical errors are caused by a finite number of quantum system representatives to be measured. As the measurement time increases, the information about the quantum state of interest progressively increases (see Appendix). Accordingly, the statistical error becomes smaller. The instrumental errors are caused by the researcher’s incomplete knowledge of the system; i.e., more exact information exists, in principle, but it is inaccessible to the experimenter. Thus, a comparison between
TABLE II: Results of the state reconstruction. The left column indicates the orientation of the quartz plate QP1, determining the state to be measured. Values of the optical thickness of QP1 are $\delta = 0.656$ for the pulsed regime (Protocols 1, 2), and $\delta = 0.9046$ for the cw regime (Protocol 1). Theoretical state vectors are placed in the right column. The table contains the amplitudes of the reconstructed states ($c_1$, $c_2$, $c_3$) as well as their fidelities, calculated by least-squared (LSM) and maximum-likelihood (MLM) methods.

| $\alpha$ | Pulsed regime, $\delta = 0.656$, Protocol 1 | Pulsed regime, $\delta = 0.9046$, Protocol 2 | CW-regime, $\delta = 0.9046$, Protocol 1 |
|---------|-------------------------------------------|-------------------------------------------|----------------------------------------|
|         | Fidelity | State vector: experiment | Fidelity | State vector: experiment | Fidelity | State vector: experiment |
|         | LSM | MLM | LSM | MLM | LSM | MLM | LSM | MLM |
| 0°      | 0.99981 | 0.99979 | -0.00046+0.0040i | -0.00065+0.0057i | 0  | 0  |
|         | 0.9999  | 0.9999  | -0.0050-0.0115i  | -0.0053-0.0102i  | 1  | 1  |
| 40°     | 0.9989  | 0.9989  | -0.3699-0.0691i  | -0.3669-0.0687i  | -0.3482-0.0948i | -0.00136+0.0413i |
|         | 0.6261  | 0.6261  | -0.0657+0.6814i  | -0.0653+0.6815i  | 0.6392 | 0.6392 |
| 80°     | 0.9993  | 0.9993  | -0.0088+0.0439i  | -0.0091+0.0439i  | 0  | 0  |
|         | 0.9500  | 0.9498  | 0.1697+0.259i    | 0.1697+0.259i    | 0.9565 | 0.9565 |
|         |        |        | -0.0050-0.0115i  | -0.0053-0.0102i  | -0.3482-0.0948i | -0.00136+0.0413i |
|         |        |        | 0.6261  | 0.6261  | 0.6392 | 0.6392 |
|         |        |        |        |        |        |        |
|         |        |        |        |        |        |        |
|         |        |        |        |        |        |        |
|         |        |        |        |        |        |        |
|         |        |        |        |        |        |        |
|         |        |        |        |        |        |        |

Thus, for a small volume of experimental data, statistical errors prevail, whereas for large sample sizes, the setting errors and the instability of protocol parameters dominate. The number of events at which the statistical error becomes smaller than the instrumental error can be called the coherence volume. Numerically coherence volume can be estimated as the intersection point between the experimental fidelity and the lower theoretical fidelity curve. In our case this value is about 25,000-30,000 events. Starting approximately from this value, fidelity is reaching saturation and further growth of experimental data volume does not lead to an increase in the precision of quantum system estimation.

Fig. 7 relates to the state defined by orientation angle of quartz plate QP1 $\alpha = 50^\circ$ (for Protocol 2). To plot Fig. 7 we used the following technique for passing from full-volume experiment to a partial-volume experiment. Let us consider the parameter $0 < f < 1$ that characterizes the volume of experimental data. Suppose that $f = 1$ for a full-volume experiment. A partial volume experiment may be introduced considering the observation time $t'_\nu = ft_\nu$ instead of $t_\nu$. Hence, performing a single full-volume experiment means providing with a large (practically infinite) number of partial-volume ex-

FIG. 7: Fidelity dependence on the sample size. Mean values and standard deviations corresponding to the sample volumes $f = 0.01; 0.04; 0.1; 0.25; 0.5; 1.0$. the state reconstruction result and the fundamental statistical level of accuracy can serve as a guide for the parameter adjustment of the set-up.
FIG. 8: Informational $\chi^2$ criterion for small sample sizes: sample size=4400.

Experiments. For a given volume of experimental data $f$ each event from the full-volume experiment is picked up with the probability $f$ and rejected with the probability $1 - f$. Due to the presence of statistical fluctuations the equation for the number of observations, $k_\nu(t'_\nu) = f k_\nu(t_\nu)$, is violated. Therefore a unique estimate of the state vector corresponds to every partial-volume experiment. Fig.7 shows mean values and standard deviations corresponding to volumes $f = 0.01; 0.04; 0.1; 0.25; 0.5; 1.0$. For each $f < 1$ ten experiments were simulated.

The results of informational fidelity research, introduced in Appendix, are shown in Figs.8, 9. These figures correspond to the same data set as shown in Fig.7. Distribution density of informational fidelity for a small (compared to the coherence volume) sample size closely agrees with the theoretical result given by (A14) (see Fig.8). In this case the instrumental error is negligibly small compared to the statistical one. When the sample volume is close to the coherence volume (Fig. 9) the influence of instrumental and statistical errors is about equal. In other words, the informational losses caused by averaging over instrumental errors are approximately equal to the losses caused by statistical ones. Finally, if the sample size is greater than the coherence volume, instrumental errors predominate. It means that the statistical informational errors are negligibly small compared to the instrumental ones.

2. Mixture separation algorithm

Let us describe the algorithm for reconstructing a two-component mixed state. This algorithm can be easily generalized to an arbitrary number of components.

The total number of events observed in every process is divided between the components proportional to the intensity,

$$k^{(1)}_\nu = k_\nu \frac{\lambda^{(1)}_\nu}{\lambda^{(1)}_\nu + \lambda^{(2)}_\nu}, k^{(2)}_\nu = k_\nu \frac{\lambda^{(2)}_\nu}{\lambda^{(1)}_\nu + \lambda^{(2)}_\nu}$$

where $\nu = 1, 2, ..., \nu_{\text{max}}$ and $\nu_{\text{max}}$ is the total number of processes, $\lambda^{(1)}_\nu$ and $\lambda^{(2)}_\nu$ are the estimates of intensities of processes for a given step of the iteration procedure.

At a certain iteration step, let us represent $k_\nu$ as a sum of two components,

$$k_\nu = k^{(1)}_\nu + k^{(2)}_\nu.$$  (39)

For each component, we can obtain the estimates for the state vector, amplitudes, and intensity of the processes according to the method of pure state analysis described in the previous section. Since we get new intensity estimates, let us again split the total number of events in every process proportionally to the intensities of the components. In such a way, a new iteration is formed and the whole procedure is repeated. The described process is called quasi-Bayesian algorithm $^{[24]}$.

As a result, the iteration process converges to some (non-normalized) components $c^{(1)}$ and $c^{(2)}$. Thus, the mixture separation algorithm reduces to numerous estimations of pure components according to the simple algorithm described above in section IV.B. As a result of the whole algorithm execution, the estimate for the density matrix of the mixture appears:

$$\rho = c^{(1)} c^{(1)\dagger} + c^{(2)} c^{(2)\dagger},$$

$$\rho \rightarrow \frac{\rho}{\text{Tr}(\rho)}.$$  (41)
TABLE III: Example of the mixture separation using quasi-Bayesian algorithm for the given state. CW-regime, protocol 1.

| State vector:theory | Density matrix: experiment |
|---------------------|----------------------------|
| $\alpha = 30^\circ$ | $\delta = 0.9046$ |
| $\rho_{\text{theory}} = (c_1, c_2, c_3)$ | $\rho_{\exp} = (c_1, c_2, c_3)^2$ |
| First principal component weight=0.9238 | Second principal component weight=0.0762 |
| 0.7052 | -0.3027-0.2858i |
| -0.0392-0.0616i | 0.2245-0.6612i |
| 0.2990-0.6387i | 0.5140-0.1668i |
| Fidelity=0.9916 |

The last procedure is normalization of the density matrix.

A remarkable feature of the algorithm is that according to numerical calculations, independent on zero-approximation selection of the zero-approximation.

To verify the reliability, the solution was found twice for each experiment (with random zero approximation selection).

The reconstructed matrix fidelity is $F = 0.999431$. Analysis of the principal components of density matrix is given in Table IV.

This example shows a reasonably high accuracy of mixed state reconstruction. The statistical properties of the proposed algorithm were studied by means of the Monte-Carlo method. One hundred numerical experiments were conducted similar to the one described above. To verify the reliability, the solution was found twice for each experiment (with random zero approximation selection). The solutions appeared to be equal for all cases (within negligible small computational error). The obtained statistical fidelity distribution is shown in Fig.10. Numerical research shows that the fidelity distribution density is well described by the beta-distribution.
TABLE IV: Analysis of the principal components of the density matrix for the state (43, 44): numerical simulation.

| State vector $(c_1, c_2, c_3)$ | Fidelity | First principal component | Experiment | Theory |
|--------------------------------|---------|--------------------------|------------|--------|
|                                |         | weight=0.6188            | -0.3658-0.0448i | -0.3668-0.0211i |
|                                |         |                           | 0.2085+0.4743i | 0.2294+0.4934i |
|                                |         |                           | 0.7718      | 0.7543  |
|                                |         | weight=0.3812            | -0.1208-0.2643i | -0.1490-0.2382i |
|                                |         |                           | -0.1659-0.8150i | -0.1986-0.7942i |
|                                |         |                           | 0.4731      | 0.5009  |

V. CONCLUSION

The procedure of quantum state measurement for a three-state optical system formed by a frequency- and spatially degenerate two-photon field has been considered in this work. The method of the statistical estimation of the quantum state through solving the likelihood equation and examining the statistical properties of the resulting estimates has been developed. Based on the experimental data (fourth-order moments in the field) and the root method of estimating quantum states, the initial wave function of qutrits has been reconstructed.

Experimental data analysis is based on representing the event generation intensity for each one of mutually-complementary quantum processes as a squared module of some amplitude. A complete set of measured processes amplitudes can be compactly described using the instrumental matrix. In the framework of the formalism of a process amplitude one can apply effective tools for the quantum state reconstruction: least-squares and maximum-likelihood methods.

The developed analysis tools provide means of quantum state reconstruction from the experimental data with high accuracy and reliability. The estimate accuracy is determined by concurrence of two types of errors: statistical ones and instrumental ones. For smaller sample sizes statistical errors are dominant, while for greater ones instrumental errors dominate.

Instrumental errors lead to fidelity saturation at less than unity level. In the present work, fidelity for most of performed experiments (more than 20) exceeded level of 0.99. For many cases the level of 0.9998 was achieved.

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APPENDIX A: STATISTICAL FLUCTUATIONS OF THE STATE VECTOR

As it was already mentioned above, an un-normalized state vector provides the most complete information about a quantum system. The usage of un-normalized vector allows us to remove an interaction constant in Eq. (24). The norm of the vector $c$, obtained as a result of quantum system reconstruction, provides one with the information about the total intensity of all the processes considered in the experiment. However, the fluctuations of the quantum state (and norm fluctuations, in particular) in a normally functioning quantum information system should be within certain range defined by the statistical theory. The present section is devoted to this problem.

Practical significance of accounting for statistical fluctuations in a quantum system relates to developing methods of estimation and control of precision and stability of a quantum information system evolution, as well as methods of detecting external interception (Eve’s attack on the quantum channel between Alice and Bob).

The estimate of the un-normalized state vector $c$, obtained by the maximum-likelihood principle, differs from the exact state vector $c^{(0)}$ by random values $\delta c = c^{(0)} - c$. Let us consider the statistical properties of the fluctuation vector $\delta c$ by expansion of the log likelihood function.
near the stationary point,

$$\delta \ln L = - \left[ \frac{1}{2} \left( K_{s \bar{s}} \delta c_s \delta c_{\bar{s}} + K_{s \bar{s}} \delta c_s^* \delta c_{\bar{s}}^* \right) + I_{s \bar{s}} \delta c_s \delta c_{\bar{s}} \right].$$

Together with the Hermitian matrix of the Fisher information $I$ (32), we define the symmetric Fisher information matrix $K$, whose elements are defined by the following equation:

$$K_{s \bar{s}} = \sum_\nu \frac{k_\nu}{M_\nu^2} X_\nu s X_{\nu \bar{s}}, \quad (A2)$$

where $M_\nu$ is the amplitude of the $\nu$-th process. In the general case, $K$ is a complex symmetric non-Hermitian matrix. From all possible types of fluctuations, let us pick out the so-called gauge fluctuations. Infinitesimal global gauge transformations of a state vector are as follows:

$$\delta c_j = i \varepsilon c_j, \quad j = 1, 2, ..., s \quad (A3)$$

where $\varepsilon$ is an arbitrary small real number, $s$ is the Hilbert space dimension.

Evidently, for gauge transformations $\delta \ln L = 0$. It means that two state vectors that differ by a gauge transformation, are statistically equivalent, i.e. they have the same likelihood. Such vectors are physically equivalent since the global phase of the state vector is non-observable. From statistical point of view, the set of mutually complementing measurements should be chosen in such a way that for all other fluctuations (except gauge fluctuations) $\delta \ln L < 0$. This inequality serves as the statistical completeness condition for the set of mutually complementing measurements.

Let us derive some constructive criteria of the statistical completeness of measurements. The complex fluctuation vector $\delta c$ is conveniently represented by a real vector of double length. After extracting the real and the imaginary parts of the fluctuation vector $\delta c_j = \delta c_j^{(1)} + i \delta c_j^{(2)}$ we transfer from the complex vector $\delta c$ to the real one $\delta \xi$:

$$\delta c = \begin{pmatrix} \delta c_1 \\ \vdots \\ \delta c_s \end{pmatrix} \rightarrow \delta \xi = \begin{pmatrix} \delta c_1^{(1)} \\ \vdots \\ \delta c_s^{(1)} \\ \delta c_1^{(2)} \\ \vdots \\ \delta c_s^{(2)} \end{pmatrix}. \quad (A4)$$

In the particular case of qutrits ($s = 3$) this transition provides us with a 6-component real vector instead of a 3-component complex vector.

In the new representation Eq. (A1) becomes:

$$\delta \ln L = - H_{s \bar{s}} \delta \xi_s \delta \xi_{\bar{s}} = - \langle \delta \xi | H | \delta \xi \rangle, \quad (A5)$$

where matrix $H$ is the "complete information matrix" possessing the following block form:

$$H = \begin{pmatrix} \text{Re}(I + K) & -\text{Im}(I + K) \\ \text{Im}(I - K) & \text{Re}(I - K) \end{pmatrix}. \quad (A6)$$

The matrix $H$ is real and symmetric. It is of double dimension respectively to the matrices $I$ and $K$. For qutrits, $I$ and $K$ are $3 \times 3$ matrices, while $H$ is $6 \times 6$.

Using matrix $H$ it is easy to formulate the desired characteristic completeness condition for a mutually complementing set of measurements. For a set of measurements to be statistically complete, it is necessary and sufficient that one and only one eigenvalue of the complete information matrix $H$ is equal to zero, while the other ones are strictly positive.

We would like to stress that checking the condition one not only verifies the statistical completeness of a measurement protocol but also, insures that the obtained extremum is of maximum likelihood.

An eigenvector that has eigenvalue equal to zero corresponds to gauge fluctuation direction. Such fluctuations do not have physical meaning as stated above. Eigenvectors corresponding to the other eigenvalues specify the direction of fluctuations in the Hilbert space.

The principal fluctuation variance is

$$\sigma_j^2 = \frac{1}{2h_j}, \quad j = 1, ... 2s - 1, \quad (A7)$$

where $h_j$ is the eigenvalue of the information matrix $H$, corresponding to the $j$-th principal direction.

The most critical direction in the Hilbert space is the one with the maximum variance $\sigma_j^2$, while the corresponding eigenvalue $h_j$ is accordingly minimal. The knowledge of the numerical dependence of statistical fluctuations allows one to estimate distributions of various statistical characteristics.

The important information criterion that specifies the general possible level of statistical fluctuations in quantum information system is the chi-square criterion. It can be expressed as

$$2 \langle \delta \xi | H | \delta \xi \rangle \propto \chi^2(2s - 1), \quad (A8)$$

where $s$ is the Hilbert space dimension.

The left-hand side of Eq. (A8), which describes the level of state vector information fluctuations, is a chi-square distribution with $2s - 1$ degrees of freedom.

Validity of the analytical expression (A8) is justified by the results of numerical modeling and observed data. Similarly to Eq. (A4), let us introduce the transformation of a complex state vector to a real vector of double
number of observations as 

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} \rightarrow \xi = \begin{pmatrix} c_1^{(1)} \\ \vdots \\ c_N^{(1)} \\ c_1^{(2)} \\ \vdots \\ c_N^{(2)} \end{pmatrix}$$

(A9)

It can be shown that the information carried by a state vector is equal to the doubled total number of observations in all processes:

$$\langle \xi | H | \xi \rangle = 2n,$$

(A10)

where $$n = \sum_{\nu} k_{\nu}$$.

Then, the chi-square criterion can be expressed in the form invariant to the state vector scale (let us remind that we consider a non-normalized state vector).

$$\frac{\langle \delta \xi | H | \delta \xi \rangle}{\langle \xi | H | \xi \rangle} \propto \frac{\chi^2(2s-1)}{4n}$$

(A11)

Relation (A11) describes the distribution of relative information fluctuations. It shows that relative information uncertainty of a quantum state decreases with the number of observations as $$1/n$$.

The mean value of relative information fluctuations is

$$\frac{\langle \delta \xi | H | \delta \xi \rangle}{\langle \xi | H | \xi \rangle} = \frac{2s-1}{4n}$$

(A12)

The information fidelity may be introduced as a measure of correspondence between the theoretical state vector and its estimate:

$$F_H = 1 - \frac{\langle \delta \xi | H | \delta \xi \rangle}{\langle \xi | H | \xi \rangle}.$$  \hspace{1cm} (A13)

Correspondingly, the value $$1 - F_H$$ is the information loss.

The convenience of $$F_H$$ relies on its simpler statistical properties compared to the conventional fidelity $$F$$. For a system where statistical fluctuations dominate, fidelity is a random value, based on the chi-square distribution,

$$F_H = 1 - \frac{\chi^2(2s-1)}{4n},$$

(A14)

where $$\chi^2(2s-1)$$ is a random value of chi-square type with $$2s-1$$ degrees of freedom.

Information fidelity asymptotically tends to unity when the sample size is growing up. Complementary to statistical fluctuations noise leads to a decrease in the informational fidelity level compared to the theoretical level (A14).

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