Solvability of Parabolic Anderson Equation with Fractional Gaussian Noise

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Abstract
This paper provides necessary as well as sufficient conditions on the Hurst parameters so that the continuous time parabolic Anderson model \( \frac{\partial u}{\partial t} = \frac{1}{2} \Delta + u \dot{W} \) on \([0, \infty) \times \mathbb{R}^d\) with \(d \geq 1\) has a unique random field solution, where \(W(t, x)\) is a fractional Brownian sheet on \([0, \infty) \times \mathbb{R}^d\) and formally \(\dot{W} = \frac{\partial^{d+1}}{\partial t \partial x_1 \cdots \partial x_d} W(t, x)\). When the noise \(W(t, x)\) is white in time, our condition is both necessary and sufficient when the initial data \(u(0, x)\) is bounded between two positive constants. When the noise is fractional in time with Hurst parameter \(H_0 > 1/2\), our sufficient condition, which improves the known results in the literature, is different from the necessary one.

Keywords Stochastic heat equation · Fractional Brownian fields · Wiener chaos expansion · Random field solution · Necessary condition · sufficient condition · Moment bounds

Mathematics Subject Classification 60H15 · 60G60 · 60G15 · 60G22 · 35R60

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1 Introduction

Let \( d \geq 1 \). In this paper, we are interested in the following stochastic heat equation (parabolic Anderson model) for \( u = u(t, x) \) with \( t \geq 0 \) and \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \):

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + u \dot{W} \quad \text{on} \ (0, \infty) \times \mathbb{R}^d, \tag{1.1}
\]

where \( W = \{ W(t, x); \ t \in [0, \infty), \ x \in \mathbb{R}^d \} \) is a centered Gaussian process defined on some complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with the covariance given by

\[
\mathbb{E} [W(s, x)W(t, y)] = R_{H_0}(s, t) \prod_{i=1}^d R_{H_i}(x_i, y_i), \quad s, t \geq 0, \ x, y \in \mathbb{R}^d, \tag{1.2}
\]

where \( \frac{1}{2} \leq H_0 < 1 \) and \( 0 < H_i < 1 \) for \( i = 1, \ldots, d \), and where for \( \beta > 0 \),

\[
R_\beta(a, b) := \frac{1}{2} \left( |a|^{2\beta} + |b|^{2\beta} - |a - b|^{2\beta} \right) \quad \text{for all} \ a, b \in \mathbb{R}. \tag{1.3}
\]

Here, we use := as a way of definition. That is, \( W(t, x) = W(t, x_1, \ldots, x_d) \) is a fractional Brownian motion of Hurst parameter \( H_0 \geq 1/2 \) in time variable and is a fractional Brownian motion with Hurst parameter \( H = (H_1, \ldots, H_d) \) in space variables, and formally \( \dot{W}(t, x) = \frac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d} \). Note that \( R_{1/2}(s, t) = s \wedge t \) and so \( \frac{\partial^2}{\partial s \partial t} R_{1/2}(s, t) = \delta_{\{0\}}(s-t) \) in the distributional sense, where \( \delta_{\{0\}} \) is the Dirac measure on \( \mathbb{R} \) concentrated at the origin 0. One-dimensional Brownian motion has covariance function \( R_{1/2} \) so it has Hurst parameter 1/2. For fractional Brownian motions, the smaller the Hurst parameters are, the rougher their sample paths. The precise meaning of the (random field) solution to (1.1) will be given later in Definition 3.1. The product \( u \dot{W} \) in the above equation is in the sense of Wick, which means that in Equation (3.1) of the definition of the random field solution, the stochastic integral is understood as the Itô-Skorohod integral.

The equation (1.1) is one of the simplest stochastic partial differential equations and describes a heat propagation through a random medium. It has a close connection with KPZ equation through the Hopf–Cole transform. Equation (1.1) has been studied by many authors. We refer the reader to a recent survey \([7]\) and references therein.

Throughout the paper, we assume that the Hurst parameter \( H_0 \) in time is always greater than or equal to 1/2. But some of the spatial Hurst parameters \( H_i \) in (1.1) can be less than 1/2, while others are greater than or equal to 1/2. Let \( d_* \) denote the total number of \( H_i \) whose value is strictly less than 1/2. Without loss of generality and for the simplification of notation, we can assume that \( H_k < 1/2 \) when \( 1 \leq k \leq d_* \) and \( H_k \geq 1/2 \) when \( d_* < k \leq d \) for some \( d_* \in \{0, 1, \ldots, d\} \). Let

\[
H_* = H_1 + \cdots + H_{d_*} \quad \text{and} \quad H^* = H_1 + \cdots + H_d.
\]
It is known that when \( H_0 \geq 1/2 \) and \( H_i > 1/2 \) for all \( 1 \leq i \leq d \), the equation (1.1) has a unique random field solution when \( H^* > d - 1 \) (see [10, Example 2.6]; see also [4] for more general Gaussian noises that is white in time, including the fractional Brownian noise with \( H_0 = 1/2 \) and all the spatial Hurst parameters being greater than 1/2). When \( H_0 = 1/2 \) and \( d_\ast = d \), it is a folklore that \( d \) must be 1 for (1.1) to be solvable with bounded initial value. It is shown in [8,9] that when \( d = 1 \), \( H_0 = 1/2 \) and \( 1/4 < H^* < 1/2 \), the equation (1.1) has a unique random field solution.

Xia Chen considered fractional Gaussian noise \( W \) that has \( H_0 \in [1/2, 1) \) but some of \( H_i \in (0, 1), 1 \leq i \leq d \), are greater than or equal to 1/2, while others are allowed to be less than 1/2. He showed [2, Theorems 1.2 and 1.3] that when

\[
\begin{align*}
2(d - H^*) + (d_\ast - 2H_\ast) &< 2 & \text{when } H_0 = 1/2, \\
H^* > d - 1 \text{ and } 4(1 - H_0) + 2(d - H^*) + (d_\ast - 2H_\ast) &< 4 & \text{when } H_0 > 1/2,
\end{align*}
\]

the SPDE (1.1) has a unique global random field solution in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) for any initial value \( u_0(x) \) that is bounded. He also showed that for \( d \geq 2 \), when

\[
H_0 > 1/2, \quad H^* = d - 1 \quad \text{and} \quad 4(1 - H_0) + (d_\ast - 2H_\ast) < 2,
\]

the SPDE (1.1) has a unique local random field solution in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) for any initial value \( u_0(x) \) that is bounded. An interesting and challenging problem is whether the above conditions are also necessary. As we shall see, the answer will be no, and when \( H_0 = 1/2 \), we give a necessary and sufficient condition for the existence of solution to (1.1), and when \( H_0 > 1/2 \), we improve the sufficient condition (1.4) as well as give a necessary condition.

The following are the two main results of this paper, by considering \( H_0 = 1/2 \) and \( 1/2 < H_0 < 1 \) separately. See Definition 3.1 for the precise definition of global and local solution to (1.1). Our main results show that the condition \( H^* > d - 1 \) is sufficient for the existence of global random field solution to (1.1) when \( d \geq 2 \) and \( H_0 \in [1/2, 1) \), and is also necessary for the existence of local random field solution to (1.1) when \( H_0 = 1/2 \). We further show that when \( H_0 \in (1/2, 1) \) and \( d \geq 2 \), there exists a unique local solution in the critical case \( H^* = d - 1 \). Our necessary condition for the case of \( H_0 > 1/2 \) is different from the sufficient condition but it involves \( H^* \) and \( H_0 \) only; we do not need to separate the rougher ones (those with \( H_i < 1/2 \)) from the smoother ones (those with \( H_i \geq 1/2 \)) in spatial Hurst parameters.

**Theorem 1.1** Suppose \( H_0 = 1/2 \); that is, the noise \( W \) is white in time.

(i) Suppose the initial condition satisfies \( |u_0(x)| \leq C \) for some constant \( C > 0 \). If

\[
\begin{align*}
H^* > 1/4 & \quad \text{when } d = 1, \\
H^* > d - 1 & \quad \text{when } d \geq 2,
\end{align*}
\]

then the equation (1.1) has a unique (global) random field solution in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) with \( u(0, x) = u_0(x) \). Moreover, in this case, there is a positive constant \( C_H > 0 \)
so that

$$
\mathbb{E}[u(t, x)^p] \leq C_H \exp \left( C_H t p^{H^*-d+2\varepsilon \over H^*-d+1} \right) \quad \text{for any } t \geq 0 \text{ and } p \geq 2.
$$

(1.7)

(ii) Let the initial condition satisfy \( u_0(x) \geq c \) (or \( u_0(x) \leq -c \)) for some positive constant \( c > 0 \). If the equation (1.1) has a local solution in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \), then (1.6) must be satisfied.

**Remark 1.2**

(i) In the case of \( H_0 = 1/2 \), the condition (1.4) is equivalent to (1.6) when \( d = 1 \) and is stronger than (1.6) when \( d \geq 2 \) since \( d_e - 2H_e > 0 \) when \( d_e \geq 1 \) and \( d_e - 2H_e = 0 \) when \( d_e = 0 \).

(ii) Since \( 0 < H_i < 1 \) for all \( 1 \leq i \leq d \) and \( 1 \leq H_i < 1/2 \) for \( 1 \leq i \leq d_e \), we have \( H^* < d_e \) (that is, \( H^* < d - (d^*/2) \)). Thus, Condition (1.6) implies that \( d - (d_e/2) > d - 1 \), or \( d_e < 2 \). Thus under condition (1.6), \( d_e \) can only be 0 or 1.

**Theorem 1.3**

Suppose \( H_0 \in (1/2, 1) \).

(i) Let \( d \geq 1 \). Suppose the initial condition satisfies \( |u_0(x)| \leq C \) for some constant \( C > 0 \). If

$$
\begin{align*}
H^* > & \frac{3}{4} - H_0 & \text{when } d = 1, \\
H^* > & d - 1 & \text{when } d \geq 2,
\end{align*}
$$

(1.8)

then the equation (1.1) has a unique (global) random field solution in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) with \( u(0, x) = u_0(x) \). Moreover, in this case, there is a positive constant \( C_{H,d} > 0 \) so that

$$
\mathbb{E}[u(t, x)^p] \leq C \exp \left( C_H t^{H^*-d+2H_0} p^{H^*-d+2\varepsilon \over H^*-d+1} \right) \quad \text{for any } t \geq 0 \text{ and } p \geq 2.
$$

(1.9)

Moreover, if

$$
d \geq 2 \quad \text{and} \quad H^* = d - 1,
$$

(1.10)

the equation (1.1) has a unique local random field solution in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) with \( u(0, x) = u_0(x) \).

(ii) Let the initial condition satisfy \( u_0(x) \geq c \) for some positive constant \( c > 0 \). If the equation (1.1) has a local solution in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \), then

$$
H^* + 2H_0 > \begin{cases} 
5/4 & \text{when } d = 1, \\
(3d + 2)/4 & \text{when } d \geq 2.
\end{cases}
$$

(1.11)
Remark 1.4  (i) When $d = 1$ and $H_0 > 1/2$, condition (1.4) is equivalent to $H^* > 3/4 - H_0$. Clearly, when $H_0 > 1/2$, conditions (1.4) and (1.5) are stronger than conditions (1.8) and (1.10), respectively, when $d \geq 2$.

(ii) Condition (1.6) is the same as (1.8) if we take $H_0 = 1/2$ there.

(iii) The necessity of the condition (1.6) when $H_0 = 1/2$ and of the condition (1.11) when $H_0 > 1/2$ for the existence of solutions to SPDE (1.1) is new. There seems no discussions about the necessary conditions for this equation before.

(iv) Note that the sufficient condition and the necessary condition in Theorems 1.1 and 1.3 have different requirements on the initial value $u(0, x)$ of $u$. It is easy to check directly that, when $1/2 < H_0 < 1$, the sufficient condition (1.8) is strictly stronger than the necessary condition (1.11).

(v) Estimate (1.9) coincides with the upper bound part of (6.1) of [10] when all Hurst parameters are greater than $1/2$ (by setting $\eta_i = 2H_i - 2$ and $\beta = 2H_0 - 2$), so we expect our bound (1.9) is sharp.

(vi) As we see from Remark 1.2(ii) that $H^* < d - (d_*/2)$, So condition (1.10) implies that $d_* < 2$, that is, either $d_* = 0$ or $d_* = 1$.

(vii) Again from the fact that $H^* < d - (d_*/2)$, we see that when $d \geq 2$, condition (1.11) implies $d_* < (d/2) + 4H_0 - 1 < (d + 6)/2$. But this condition is not optimal. In fact, when $W$ is time independent (which corresponds to the case $H_0 = 1$) and space white (which corresponds to the case $H_1 = \cdots = H_d = 1/2$), it is known (see [6]) that the equation (1.1) has a global solution when $d = 1$ and has a local solution but has no global solution when $d = 2$, and do not have any local solution when $d \geq 3$. So our conjecture is that even when $H_0 \in (1/2, 1)$ and $H_i \in [1/2, 1)$ for all $1 \leq i \leq d$, to ensure the global unique solution we need $d_* \leq 1$ and to ensure local solution we need $d_* \leq 2$.

In this paper, we do not discuss the solvability of (1.1) when the time Hurst parameter $H_0 \in (0, 1/2)$. We refer the interested reader to [1,3,13] and references therein for recent development when $H_0 < 1/2$. Let us also mention that for the additive noise (namely, replace $u \dot{W}$ by $\dot{W}$ in (1.1), the necessary and sufficient condition is known ([12]) even for more general Gaussian noise.

The rest of the paper is organized as follows. In Sect. 2, we recall some facts on Gaussian random fields, stochastic integrals with respect to them and their properties that will be used in this paper. The proofs of the sufficient part of Theorems 1.1 and 1.3 are presented in Sect. 3, while the proof of the necessary part of these two theorems are given in Sect. 4.

In this paper, for $a, b \in \mathbb{R}$, $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. For two functions $f$ and $g$, notation $f \asymp g$ means that there is a constant $c \geq 1$ so that $c^{-1}g \leq f \leq cg$ on their common domain of definitions.

2 Preliminaries

The noise $W$ can be viewed as a Brownian motion with values in an infinite-dimensional Hilbert space. One might thus think that the stochastic integration theory with respect to $W$ can be handled by classical theories (e.g., [5]). However, the spatial
covariance function of $\dot{W}$, which is formally $\prod_{i=1}^{d} H_i (2 H_i - 1) |x_i - y_i|^{2 H_i - 2}$, is not locally integrable along the diagonals as $H_i < 1/2$ for $1 \leq i \leq d$. The stochastic integral with respect to $W$ then needs to be dealt with through other means. We recall briefly some key points needed in this paper, and we refer to [8,9] for more details.

We start by introducing some basic notation on Fourier transforms. The space of Schwartz functions is denoted by $S$. Its dual, the space of tempered distributions, is denoted by $S'$.

The Fourier transform of a function $u \in S$ is defined by

$$\hat{u}(\xi) := \int_{\mathbb{R}^d} e^{-i \xi x} u(x) dx$$

so that the inverse Fourier transform is given by

$$\check{u}(\xi) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i \xi x} u(x) dx = (2\pi)^{-d} \hat{u}(-\xi).$$

Let $C^\infty_c((0, \infty) \times \mathbb{R}^d)$ denote the space of real-valued infinitely differentiable functions with compact support on $(0, \infty) \times \mathbb{R}^d$. The noise $W$ can be described by a mean zero Gaussian family $\{ W(\varphi), \varphi \in C^\infty_c((0, \infty) \times \mathbb{R}^d) \}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, whose covariance structure is given by

$$\mathbb{E}[W(\varphi) W(\psi)] = c_{d,H} \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \hat{\varphi}(s, \xi) \overline{\hat{\psi}(r, \xi)} \prod_{i=1}^{d} |\xi_i|^{1-2 H_i} \gamma_0(s-r) ds dr d\xi,$$

(2.1)

where

$$\gamma_0(s-r) = H_0(2H_0 - 1)|s-r|^{2H_0 - 2}. \quad (2.2)$$

When $H_0 = 1/2$, we replace $\gamma_0(s-r)$ by the Dirac delta function $\delta_{[0]}(s-r)$:

$$\mathbb{E}[W(\varphi) W(\psi)] = c_{d,H} \int_{\mathbb{R}^*_+ \times \mathbb{R}^d} \hat{\varphi}(s, \xi) \overline{\hat{\psi}(s, \xi)} \prod_{i=1}^{d} |\xi_i|^{1-2 H_i} ds d\xi,$$

(2.3)

where the Fourier transforms $\hat{\varphi}$ and $\hat{\psi}$ are understood as Fourier transforms in spatial variables only and

$$c_{1,H} = \frac{1}{(2\pi)^d} \prod_{i=1}^{d} \Gamma(2H_i + 1) \sin(\pi H_i). \quad (2.4)$$

As above throughout the remaining part of the paper, we will always replace $\gamma_0(s-r)$ by $\delta_{[0]}(s-r)$ when $H_0 = 1/2$. 

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Let
\[
\mathbb{H}_0 = \left\{ \varphi \in \mathcal{S} : \| \varphi \|_{\mathbb{H}} := \left( \int_{\mathbb{R}^2} \tilde{\varphi}(s, \xi) \tilde{\varphi}(r, \xi)|\xi|^{-2H}d\xi \gamma_0(s-r)dsdr \right)^{1/2} < \infty \right\}.
\]

Since \( \gamma_0(s-r) \) is a positive definite kernel it is well-known that \( \| \cdot \|_{\mathbb{H}} \) is a Hilbert norm (in fact it is the \( L^2 \) norm of the stochastic integral \( \int_{\mathbb{R}^+ \times \mathbb{R}^d} \varphi(s, x) W(ds, dx) \)). Let \( \mathbb{H} \) be the completion of \( \mathbb{H}_0 \) under the above norm \( \| \cdot \|_{\mathbb{H}} \). Using this Hilbert norm, we can define the stochastic integration with respect to \( W \).

**Definition 2.1** For any \( t \geq 0 \), let \( \mathcal{F}_t \) be the \( \sigma \)-algebra generated by \( W \) up to time \( t \). An elementary process \( g \) is a process given by
\[
g(s, x) = \sum_{i=1}^n \sum_{j=1}^m X_{i,j} \mathbb{1}_{(a_i, b_i]}(s) \mathbb{1}_{(h_j, l_j]}(x),
\]
where \( n \) and \( m \) are finite positive integers, \( -\infty < a_1 < b_1 < \cdots < a_n < b_n < \infty \), \( h_j = (h_{j1}, \cdots, h_{jd}) \) \( h_{jk} < l_{jk} \), \( \mathbb{1}_{(h_j, l_j]}(x) = \prod_{k=1}^d \mathbb{1}_{(h_{jk}, l_{jk}]}(x_k) \), and \( X_{i,j} \) are \( \mathcal{F}_{a_i} \)-measurable random variables for \( i = 1, \ldots, n \). The integral of such a process with respect to \( W \) is defined as
\[
\int_{\mathbb{R}^+ \times \mathbb{R}^d} g(s, x) W(ds, dx) = \sum_{i=1}^n \sum_{j=1}^m X_{i,j} W(\mathbb{1}_{(a_i, b_i]} \otimes \mathbb{1}_{(h_j, l_j]}).
\]

The following result can be found in [8] when \( d = 1 \).

**Proposition 2.2** Let \( \Lambda_H \) be the space of predictable processes \( g \) defined on \( \mathbb{R}^+ \times \mathbb{R}^d \) such that almost surely \( g \in \mathbb{H} \) and \( \mathbb{E}[\| g \|_{\mathbb{H}}^2] < \infty \). Then, we have the following statements.

(i) The space of elementary processes of the form in Definition 2.1 is dense in \( \Lambda_H \).

(ii) For \( g \in \Lambda_H \), the stochastic integral \( \int_{\mathbb{R}^+ \times \mathbb{R}^d} g(s, x) W(ds, dx) \) is defined as the \( L^2(\Omega) \)-limit of \( \int_{\mathbb{R}^+ \times \mathbb{R}^d} g_n(s, x) W(ds, dx) \) for any \( g_n \) approximating \( g \), and we have
\[
\mathbb{E} \left[ \left( \int_{\mathbb{R}^+ \times \mathbb{R}^d} g(s, x) W(ds, dx) \right)^2 \right] = \mathbb{E} \left[ \| g \|_{\mathbb{H}}^2 \right].
\]
In this section, we shall prove part (ii), the necessary part, of Theorems 1.1 and 1.3.

We can also define the multiple integral using the above definition.

\[
I_n(f)(t) = \int_{0 \leq s_1 < \ldots < s_n \leq t} f(s_1, x_1), \ldots, (s_n, x_n)) W(ds_1, dx_1) \cdots W(ds_n, dx_n)
\]

\[
= \frac{1}{n!} \int_{([0, t] \times \mathbb{R}^d)^n} f(s_1, x_1), \ldots, (s_n, x_n)) W(ds_1, dx_1) \cdots W(ds_n, dx_n),
\]

(2.8)

where \( f((s_1, x_1), \ldots, (s_n, x_n)) \in \mathbb{H}^{\otimes n} \) is symmetric with respect to its \( n \) arguments. We have

\[
\mathbb{E} \left[ (I_n(f)(t))^2 \right] = \frac{1}{n!} \int_{([0, t]^2 \times \mathbb{R}^d)^n} \widehat{f}((s_1, \xi_1), \ldots, (s_n, \xi_n)) \overline{\widehat{f}}((r_1, \xi_1), \ldots, (r_n, \xi_n))
\]

\[
\prod_{i=1}^{n} \gamma_0(s_i - r_i) \prod_{i=1}^{d} \prod_{k=1}^{d} |\xi_k|^{1-2H_k} d\xi_1 \cdots d\xi_n ds_1 \cdots ds_n dr_1 \cdots dr_n,
\]

(2.9)

where \( \widehat{f} \) is the Fourier transform with respect to \( n \) spatial variables \( x_1, \ldots, x_n ; \xi_i = (\xi_{i1}, \ldots, \xi_{id}) ; d\xi_i = d\xi_{i1} \cdots d\xi_{id} ; ds = ds_1 \cdots ds_n, dr = dr_1 \cdots dr_n \). Notice that in (2.9), we do not force an order for \( r_1, \ldots, r_n \).

We also need the following lemma which can be found in [10, Lemma 4.5].

**Lemma 2.3** Let \( \alpha \in (-1 + \varepsilon, 1)^m \) with \( \varepsilon > 0 \). Denote \( |\alpha| = \sum_{i=1}^{m} \alpha_i \) and \( T_m(t) = \{ (r_1, r_2, \ldots, r_m) \in \mathbb{R}^m : 0 < r_1 < \cdots < r_m < t \} \). Then, there is a constant \( \kappa \), depending only on \( \varepsilon \) such that

\[
J_m(t, \alpha) := \int_{T_m(t)} \prod_{i=1}^{m} (r_{(i+1)} - r_{i-1})^{\alpha_i} dr \leq \frac{\kappa^m t^{|\alpha|+m}}{\Gamma(|\alpha|+m+1)},
\]

where by convention, \( r_0 = 0 \).

## 3 Necessary Condition

In this section, we shall prove part (ii), the necessary part, of Theorems 1.1 and 1.3, namely the necessity of (1.6) and (1.11), respectively.

First, we give the meaning of the (random field) solution to equation (1.1) in the following definition.

**Definition 3.1** A real-valued predictable stochastic process \( u = \{ u(t, x), 0 \leq t < \infty, x \in \mathbb{R}^d \} \) is said to be a (global) random field solution of (1.1) if

\[ \text{Springer} \]
(i) for all $t \in [0, \infty)$ and $x \in \mathbb{R}^d$, the process $(s, y) \mapsto p_{t-s}(x-y)u(s, y)1_{[0,t]}(s)$ is an element of $\Lambda_H$, where $p_t(x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right)$ is the heat kernel on the real line associated with $\frac{1}{2}\Delta$.

(ii) for all $t \in [0, \infty)$ and $x \in \mathbb{R}^d$ we have

$$u(t, x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)u(s, y)W(ds, dy) \quad \text{a.s.}, \quad (3.1)$$

where the stochastic integral is understood in the sense of Proposition 2.2.

A real-valued stochastic process $u(t, x)$ is said to be a local (random field) solution of (1.1) if there is some constant $t_0 > 0$ so that $u(t, x)$ is defined on $[0, t_0) \times \mathbb{R}^d$ and satisfies all the above property for $(t, x) \in [0, t_0) \times \mathbb{R}^d$.

We say a random field solution $u(t, x)$ of (1.1) is in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ if $\mathbb{E}[u(t, x)^2] < \infty$ for every $t \geq 0$ and $x \in \mathbb{R}^d$ (for local solution, replace $t \geq 0$ by $t \in [0, t_0)$).

Repeatedly using this definition, we see that if $u(t, x)$ is a random field solution of (1.1), then for each integer $N \geq 1$,

$$u(t, x) = \sum_{n=0}^N u_n(t, x) + R_N(t, x), \quad (3.2)$$

where

$$u_n(t, x) = I_n(f_n^{(t,x)})(t) \quad (3.3)$$

and $R_N(t, x) = I_{N+1}(g_{N+1}^{(t,x)})(t)$ with

$$f_n^{(t,x)} := f_n^{(t,x)}(s_1, x_1, \ldots, s_n, x_n) = f_n^{(t,x)}(s_1, x_1, \ldots, (s_n, x_n))$$

$$:= p_{t-s_1}(x-x_1)p_{s_1-s_2}(x_1-x_2) \cdots p_{s_{n-1}-s_n}(x_{n-1}-x_n)p_{s_n-t}(x_n-x)$$

$$\quad (3.4)$$

when $0 < s_1 < s_2 < \cdots < s_n < t$, and

$$g_{N+1}^{(t,x)} := p_{t-s_{N+1}}(x-x_{N+1})p_{s_{N+1}-s_N}(x_{N+1}-x_N) \cdots p_{s_2-s_1}(x_2-x_1)u(s_1, x_1).$$

Note that (3.2) is the Wiener chaos decomposition of $u(t, x)$ up to order $N + 1$ and

$$\mathbb{E}[u(t, x)^2] = \sum_{n=0}^N \mathbb{E} \left[ I_n(f_n^{(t,x)})(t)^2 \right] + \mathbb{E} \left[ I_{N+1}(g_{N+1}^{(t,x)})(t)^2 \right].$$

Consequently,

$$u(t, x) = \sum_{n=0}^\infty u_n(t, x) \quad (3.5)$$
and \( \mathbb{E}[u(t, x)^2] = \sum_{n=0}^{\infty} \mathbb{E}\left[ I_n(f_n(t, x))(t)^2 \right] \); see, for instance, formula (4.4) in [14] or formula (3.3) in [10]. On the other hand, for any given function \( u_0(x) \), \( u(t, x) \) defined by (3.5) and (3.3) is a random field solution of (1.1) as long as the infinite series in (3.3) converges in \( L^2 \).

By comparison, without loss of generality we assume \( u_0(x) = 1 \) throughout the remaining part of this paper. The Fourier transform of \( f_n(t, x) ((s_1, x_1), \ldots, (s_n, x_n)) \) with respect to the \( n \mathbb{R}^d \)-valued spatial variables \( x_1, \ldots, x_n \) is

\[
\hat{f}_n(t, x, s_1, \xi_1, \ldots, s_n, \xi_n) = e^{-i x (\xi_1 + \cdots + \xi_1)} \prod_{i=1}^{n} e^{\frac{-1}{2} |x_i-s_i|^2} = e^{-i x (\xi_1 + \cdots + \xi_1)} \prod_{i=1}^{n} e^{\frac{-1}{2} |s_{\sigma(i)} - s_{\sigma(i)}|^2},
\]

(3.6)

when \( 0 < s_1 < \cdots < s_n < s_{n+1} = t \), where we denote \( s_{n+1} = t \) (see [11, 3.13-3.14] or [9, p.8]). In general, we have

\[
\hat{f}_n(t, x, s_1, \xi_1, \ldots, s_n, \xi_n) = e^{-i x (\xi_1 + \cdots + \xi_1)} \prod_{i=1}^{n} e^{\frac{-1}{2} |s_{\sigma(i)} - s_{\sigma(i)}|^2},
\]

(3.7)

where \( \sigma \) is a permutation of \( \{1, 2, \cdots, n\} \) so that \( 0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < s_{\sigma(n+1)} = t \).

We will need the following elementary lemma.

**Lemma 3.2** Let \( \varepsilon > 0, \alpha \in (0, 1) \) and \( \beta > 0 \) with \( \alpha + \beta > 1 \). There is a constant \( c \geq 1 \) independent of \( \varepsilon > 0 \) so that for all \( x \in (0, 3\varepsilon) \),

\[
c^{-1} x^{1-(\alpha+\beta)} \leq \int_{0}^{\varepsilon} u^{-\alpha} (u + x)^{-\beta} \, du \leq cx^{1-(\alpha+\beta)}.
\]

(3.8)

**Proof** By a change of variable \( u = x v \),

\[
\int_{0}^{\varepsilon} u^{-\alpha} (u + x)^{-\beta} \, du = x^{1-(\alpha+\beta)} \int_{0}^{\varepsilon/x} v^{-\alpha} (1 + v)^{-\beta} \, dv.
\]

The desired conclusion (3.8) follows from this. \( \square \)

In the following, we use \( C_H \) to denote a positive constant depending on \( H = (H_1, \cdots, H_d) \) as well as the dimension \( d \geq 1 \), whose exact value is unimportant and may change from line to line. For two nonnegative functions \( f \) and \( g \), notation \( f \asymp g \) means that there is a constant \( c \geq 1 \) so that \( c^{-1} f \leq g \leq cf \) on a specified common definition domain of \( f \) and \( g \).

**Proof of Part (ii) of Theorems 1.1 and 1.3.** First, we consider the one dimensional case \( d = 1 \) with \( H_0 > 1/2 \). Denote \( H = H_1 \). Since \( H_0 > 1/2, H + 2H_0 > 5/4 \) if \( H \geq 1/4 \). So it suffices to consider the case when \( 0 < H < 1/4 \).

\( \square \) Springer
Let us consider the second chaos in (3.5). From now on, we denote \( I_n(f_2(t,0)) = I_n(f_2(t,0))(t) \). By (2.9) and (3.6),

\[
\mathbb{E} \left[ I_2(f_2(t,0))^2 \right] = C_{1,H} \int_{0 \leq s_1 < s_2 \leq t} \int_{\mathbb{R}^2} e^{-\frac{1}{2}(t-s_2+t-r_2)s_1 \xi_2 + s_1^2} - \frac{1}{2}(s_2-s_1+r_2-r_1)^2|\xi_1|^2 \times |\xi_1|^{-2H}|\xi_2|^{-2H} d\xi_1 d\xi_2 \gamma_0(s_2-r_2) \gamma_0(s_1-r_1) ds_1 ds_2 dr_1 dr_2
\]

+ \( C_{1,H} \int_{0 \leq s_1 < s_2 \leq t} \int_{\mathbb{R}^2} e^{-\frac{1}{2}(t-s_2+t-r_2)s_1 \xi_2 + s_1^2} - \frac{1}{2}(s_2-s_1+r_2-r_1)^2|\xi_1|^2 \times |\xi_1|^{-2H}|\xi_2|^{-2H} d\xi_1 d\xi_2 \gamma_0(s_2-r_2) \gamma_0(s_1-r_1) ds_1 ds_2 dr_1 dr_2
\]

\[
\geq C_{1,H} \int_{0 \leq s_1 < s_2 \leq t} g(s_1, s_2, r_1, r_2) \gamma_0(s_2-r_2) \gamma_0(s_1-r_1) dr_1 dr_2 ds_1 ds_2,
\]

where

\[
g(s_1, s_2, r_1, r_2) = \int_{\mathbb{R}^2} e^{-\frac{1}{2}(t-s_2+t-r_2)s_1 \xi_2 + s_1^2} - \frac{1}{2}(s_2-s_1+r_2-r_1)^2|\xi_1|^2 \times |\xi_1|^{-2H}|\xi_2|^{-2H} d\xi_1 d\xi_2.
\]

Making substitution \( \eta_1 = \xi_1 \) and \( \eta_2 = \xi_1 + \xi_2 \), we have for \( 0 < s_1 < s_2 < t \) and \( 0 < r_1 < r_2 < t \),

\[
g(s_1, s_2, r_1, r_2) = \int_{\mathbb{R}^2} e^{-\frac{1}{2}(t-s_2+t-r_2)s_1 \xi_2 + s_1^2} - \frac{1}{2}(s_2-s_1+r_2-r_1)^2|\eta_1|^2 \times |\eta_1|^{-2H}|\eta_2|^{-2H} d\eta_1 d\eta_2
\]

\[
= \frac{C_H}{\sqrt{(t-s_2+t-r_2)(s_2-s_1+r_2-r_1)}} \mathbb{E} \left[ \frac{X_1}{\sqrt{s_2-s_1+r_2-r_1}} \right]^{1-2H}
\]

\[
\times \mathbb{E} \left[ \frac{X_2}{\sqrt{t-s_2+t-r_2}} - \frac{X_1}{\sqrt{s_2-s_1+r_2-r_1}} \right]^{1-2H}
\]

\[
= C_H (t-s_2+t-r_2)^{H-1} (s_2-s_1+r_2-r_1)^{2H-3/2}
\]

\[
\times \mathbb{E} \left[ X_1 (\sqrt{s_2-s_1+r_2-r_1} X_2 - \sqrt{t-s_2+t-r_2} X_1) \right]^{1-2H}
\]

\[
= C_H (t-s_2+t-r_2)^{H-1} (s_2-s_1+r_2-r_1)^{2H-3/2} (t-s_1+t-r_1)^{1/2-H}
\]

\[
\times \mathbb{E} \left[ X_1 \left( \frac{\sqrt{s_2-s_1+r_2-r_1}}{\sqrt{t-s_1+t-r_1}} X_2 - \frac{\sqrt{t-s_2+t-r_2}}{\sqrt{t-s_1+t-r_1}} X_1 \right) \right]^{1-2H}
\],

where \( X_1 \) and \( X_2 \) are two independent standard Gaussian random variables. Denote

\[
f(\lambda) := \mathbb{E} \left[ X_1 (\lambda X_1 - \sqrt{1-\lambda^2} X_2) \right]^{1-2H}, \quad \lambda \in [0, 1].
\]

We claim that

\[
\min_{\lambda \in [0,1]} f(\lambda) > 0 \quad \text{for any } 0 < H < 1.
\]
First note that for a standard Gaussian random variable \( Z \) and \( a \in \mathbb{R} \), by the Hardy–Littlewood (symmetric rearrangement) inequality,

\[
\mathbb{E} \left[ |Z - a|^{1-2H} \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z - a|^{1-2H} e^{-|z|^2/2} \, dz \\
\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z|^{1-2H} e^{-|z|^2/2} \, dz = \mathbb{E} \left[ |Z|^{1-2H} \right]. \tag{3.12}
\]

Taking conditional expectation on \( \sigma(X_1) \) and then using (3.12), we have for any \( 0 < H < 1 \),

\[
f(\lambda) = \mathbb{E} \left[ |X_1|^{1-2H} \mathbb{E} \left( |\lambda X_1 - \sqrt{1 - \lambda^2} X_2|^{1-2H} |\sigma(X_1)\right) \right] \\
\leq (1 - \lambda^2)^{1/2-H} \mathbb{E} \left[ |X_1|^{1-2H} \right] \mathbb{E} \left[ |X_2|^{1-2H} \right] < \infty.
\]

We conclude by the dominated convergence theorem that \( f(\lambda) \) is a positive continuous function on \( (0, 1) \) with \( f(0) = \left( \mathbb{E} \left[ |X_1|^{1-2H} \right] \right)^2 \in (0, \infty) \). We need to consider the behavior of \( f(\lambda) \) near \( \lambda = 1 \).

When \( 0 < H < 3/4 \), \( f \) is a continuous function on \( [0, 1] \) with \( f(1) = \mathbb{E} \left[ |X_1|^{2-4H} \right] \in (0, \infty) \) and so \( f \) is bounded between two positive constants on \( [0, 1] \). In particular, we have (3.11). When \( 3/4 \leq H < 1 \), by Fatou’s lemma,

\[
\liminf_{\lambda \to 1^-} f(\lambda) \geq f(1) = \mathbb{E} \left[ |X_1|^{2(1-2H)} \right] = \infty.
\]

This establishes the claim (3.11) for all \( H \in [0, 1] \). Consequently, we have by (3.10) that for any \( H \in (0, 1) \), there is a constant \( C_H > 0 \) so that

\[
g(s_1, s_2, r_1, r_2) \geq C_H (t - s_1 + t - r_1)^{1/2-H} (t - s_2 + t - r_2)^{H-1}(s_2 - s_1 + r_2 - r_1)^{2H-3/2}. \tag{3.13}
\]

In order for \( \mathbb{E} \left[ I_2(f_2(t,0)(t)^2 \right] \) to be finite, by (3.9) and (3.13), the following integral must be finite:

\[
\Upsilon := \int_{0 \leq r_1 < s_1 < s_2 \leq t} \int_{0 \leq r_2 < s_1 < s_2 \leq t} \frac{(t - s_1 + t - r_1)^{1/2-H}(t - s_2 + t - r_2)^{H-1}(s_2 - s_1 + r_2 - r_1)^{2H-3/2}}{r_1 r_2 s_1 s_2} \, dr_1 dr_2 ds_1 ds_2.
\]

Since \( 0 < H < 1/4 \), clearly

\[
\Upsilon \geq \int_{0 \leq r_1 < s_1 < s_2 \leq t} \frac{(t - s_1 + t - r_1)^{1/2-H}(t - s_2 + t - r_2)^{H-1}(s_2 - s_1 + r_2 - r_1)^{2H-3/2}}{r_1 r_2 s_1 s_2} \, dr_1 dr_2 ds_1 ds_2
\]

\[
\times \frac{(t - r_1)^{1/2-H}(t - r_2)^{H-1}(s_2 - r_1)^{2H-3/2}}{r_1 r_2 s_1 s_2} \, dr_1 dr_2 ds_1 ds_2.
\]

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Making substitution from $r_1, s_1, r_2$ and $s_2$ to $u = s_1 - r_1, v = r_2 - s_1, w = s_2 - r_2$ and $x = t - s_2$, we have by Lemma 3.2 that

\[
\Upsilon \geq \int_{u+v+w+x < t, u,v,w,x > 0} (x + w)^{-1/2} (w + v + u)^{2H - 3/2} w^{2H_0 - 2} u^{2H_0 - 2} dudvdwdx
\]

\[
\geq \int_{v+w+x < t/2, v,w,x > 0} (x + w)^{-1/2} w^{2H_0 - 2} \left( \int_0^{t/2} (w + v + u)^{2H - 3/2} u^{2H_0 - 2} du \right) dwdvx
\]

\[
\geq c_1 \int_{v+w+x < t/2, v,w,x > 0} (x + w)^{-1/2} w^{2H_0 - 2} (w + v)^{2H + 2H_0 - 5/2} dwdvx.
\]

\[
\geq \frac{c_1}{\sqrt{t/2}} \int_{v+w+x < t/2, v,w,x > 0} (w + v)^{2H + 2H_0 - 5/2} w^{2H_0 - 2} dudwdx
\]

\[
\geq \frac{c_1}{\sqrt{t}} \int_{v+x < t/4, v,x > 0} \left( \int_0^{t/4} (w + v)^{2H + 2H_0 - 5/2} w^{2H_0 - 2} dw \right) dwdx
\]

\[
\geq c_2 \int_{v+x < t/4, v,x > 0} w^{2H + 4H_0 - 7/2} dwdx.
\]  

\( (3.16) \)

So \( \Upsilon < \infty \) implies the integral in \( (3.16) \) is finite, which happens only if

\[
2H + 4H_0 - 7/2 > -1; \quad \text{that is, if} \quad H + 2H_0 > 5/4.
\]

This proves Theorem 1.3(ii) when \( d = 1 \).

When \( H_0 = 1/2 \), the equation \( (3.14) \) becomes

\[
\Upsilon = 2^{2H-2} \int_{0<s_1<s_2\leq t} (t-s_1)^{1/2-H} (t-s_2)^{H-1} (s_2-s_1)^{2H-3/2} ds_1 ds_2. \]  

\( (3.17) \)

This integral is finite only when the exponent \( 2H - 3/2 \) of \( s_2 - s_1 \) in \( (3.17) \) is larger than \( -1 \). This requires \( H > 1/4 \). Notice that by formally letting \( H_0 = 1/2 \) in \( 4H_0 + 2H > 5/2 \) one also obtains \( H > 1/4 \). This proves Theorem 1.1(ii) when \( d = 1 \).

Next, we consider the case that the dimension \( d \geq 2 \). We still consider the second chaos. As for the one-dimensional case, we have
\[ \mathbb{E}\left[(I_2(f_2))^2\right] \geq \frac{1}{2} \int_{0 \leq s_1, r_2 \leq t} \int_{\mathbb{R}^2} e^{-\frac{1}{2}(t-s_1+t-r_2)|\xi_2|} e^{-\frac{1}{2}(t-s_1+t-r_1)|\xi_1|^2} \times \prod_{k=1}^{d} |\xi_{1k}|^{1-2H_k} |\xi_{2k}|^{1-2H_k} \gamma_0(s_1-r_1) \gamma_0(s_2-r_2) d\xi_1 d\xi_2 ds_1 ds_2 dr_1 dr_2 \]
\[= \int_{0 \leq s_1, s_2 \leq t} \int_{0 \leq r_1, r_2 \leq t} \int_{\mathbb{R}^2} g_k(s_1, s_2, r_1, r_2) \gamma_0(s_1-r_1) \gamma_0(s_2-r_2) d\xi_1 d\xi_2 ds_1 ds_2 dr_1 dr_2, \]
which can be estimated by using (3.13). Thus, we have
\[ \mathbb{E}\left[(I_2(f_2))^2\right] \geq C \int_{0 \leq s_1, s_2 \leq t} \int_{0 \leq r_1, r_2 \leq t} (t-s_1+t-r_1)^{d/2-H^*} (t-s_2+t-r_2)^{H^*-d} \times (s_1-s_2+r_1-r_2)^{2H^*-d/2} \gamma_0(s_1-r_1) \gamma_0(s_2-r_2) d\xi_1 d\xi_2 ds_1 ds_2 dr_1 dr_2. \]

With the same argument as for (3.16), we see that the above integral is finite only if
\[ 2H^* - \frac{3d}{2} + 4H_0 - 2 > -1. \]

This proves part (ii), the necessary part, of Theorem 1.3 for \( d \geq 2 \).

When \( H_0 = 1/2 \), the inequality (3.20) becomes
\[ \mathbb{E}\left[(I_2(f_2))^2\right] \geq C \int_{0 \leq s_1, s_2 \leq t} \int_{0 \leq r_1, r_2 \leq t} (t-s_1+t-r_1)^{d/2-H^*} (s_2-s_1)^{2H^*-3d/2} ds_1 ds_2 \]
\[= C \int_{0}^{t} (t-s_1)^{d/2-H^*} \left[ \int_{s_1}^{t} (t-s_2)^{H^*-d} (s_2-s_1)^{2H^*-3d/2} ds_2 \right] ds_1 \]
\[= C B(H^*-d+1, 2H^*-\frac{3d}{2}+1) \int_{0}^{t} (t-s_1)^{d/2-H^*-\frac{3d}{2}+1} ds_1 \]
\[= C B(H^*-d+1, 2H^*-\frac{3d}{2}+1) \int_{0}^{t} (t-s_1)^{2H^*-2d+1} ds_1, \]
where \( B \) is the beta function and where we see that \( \mathbb{E}\left[(I_2(f_2))^2\right] < \infty \) if and only if when \( 2H^* - 3d/2 > -1 \) and \( H^*-d > -1 \). Note that for \( d \geq 2 \), \( H^*-d > -1 \) implies that \( 2H^*-3d/2+1 = 2(H^*-d) + (d/2)+1 > 0 \). This completes the proof of part (ii), the necessary part, of Theorem 1.1 for \( d \geq 2 \).
4 Sufficient Condition

In this section, we prove part (i), the sufficient part, of Theorems 1.1 and 1.3. It suffices to consider the case that \( d \geq 2 \), as when \( d = 1 \) conditions (1.6) and (1.8) coincide with (1.4) so the result for \( d = 1 \) follows from [2, Theorems 1.2 and 1.3].

Recall that we take \( u_0(x) = 1 \) on \( \mathbb{R}^d \). Let \( u_n(t, x) \) be defined as in (3.3). We compute the \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) norm of each \( u_n(t, x) \). For \( n \geq 0 \), we have by (2.9) that for \( 0 < s_1 < \cdots < s_n < t \) and \( 0 < r_{\sigma(1)} < \cdots < r_{\sigma(n)} < t \),

\[
\mathbb{E} \left[ u_n^2(t, x) \right] = \int_{[0, t]^{2n}} \int_{\mathbb{R}^d} \prod_{i=1}^n e^{-(s_{i+1} - s_i + r_{\sigma(i+1)} - r_{\sigma(i)})|\xi_i + \cdots + \xi_k|^2} \prod_{k=1}^d |\xi_{ik}|^{1-2H_k} d\xi_1 \cdots d\xi_n \\
\times \prod_{i=1}^n \gamma_0(s_i - r_i) ds_1 dr_1 \cdots ds_n dr_n
\]

(4.1)

where \( s_{n+1} = r_{n+1} := t \) and

\[
g_k(s_1, \cdots, s_n, r_1, \cdots, r_n) = \int_{\mathbb{R}^n} \prod_{i=1}^n e^{-(s_{i+1} - s_i + r_{\sigma(i+1)} - r_{\sigma(i)})|\xi_i + \cdots + \xi_k|^2} |\xi_{ik}|^{1-2H_k} d\xi_1 \cdots d\xi_nk
\]

(4.2)

where \( \eta_0 := 0 \). Denote \( u_i = s_{i+1} - s_i + r_{\sigma(i+1)} - r_{\sigma(i)} \) for \( 1 \leq i \leq n \). Let \( u_0 = 1 \), \( X_0 = 0 \), and \( \{X_1, \cdots, X_n\} \) be i.i.d standard Gaussian random variables. Then, we can write for \( 0 < s_1 < s_2 < \cdots < s_n < t \),

\[
g_k(s_1, \cdots, s_n, r_1, \cdots, r_n) = c^n_H \left( \prod_{i=1}^n u_i^{-1/2} \right) \mathbb{E} \left[ \prod_{i=1}^n \frac{X_i}{\sqrt{u_i}} - \frac{X_{i-1}}{\sqrt{u_{i-1}}} \right]^{1-2H_k}
\]

\[
= c^n_H \left( \prod_{i=1}^n u_i^{-1/2} \right) \left( \prod_{i=1}^n (u_i u_{i-1})^{H_k-1/2} \right)
\]

\[
\mathbb{E} \left[ |X_1|^{1-2H_k} \prod_{i=2}^n |\sqrt{u_{i-1}} X_i - \sqrt{u_i} X_{i-1}|^{1-2H_k} \right]
\]
\[ n_H^k = n_{H_k-1} \left( \prod_{i=2}^{n-1} d_i^{2H_k-3/2} \right) \left( \prod_{i=2}^{n} (u_i + u_{i-1})^{1-H_k} \right) \times \mathbb{E} \left[ |X_1|^{1-2H_k} \prod_{i=2}^{n} \left| \frac{u_{i-1}}{u_{i-1} + u_i} X_i - \frac{u_i}{u_{i-1} + u_i} X_{i-1} \right|^{1-2H_k} \right]. \tag{4.3} \]

Denote \( \lambda_i = \sqrt{\frac{u_{i-1}}{u_{i-1} + u_i}} \). The expectation (denoted by \( I_{k,n} \)) in (4.3) is bounded as follows.

\[ I_{k,n} := \mathbb{E} \left[ |X_1|^{1-2H_k} \prod_{i=2}^{d_k} |\lambda_i X_i - \sqrt{1 - \lambda_i^2} X_{i-1}|^{1-2H_k} \prod_{i=d_k+1}^{n} |\lambda_i X_i - \sqrt{1 - \lambda_i^2} X_{i-1}|^{1-2H_k} \right] \leq C_{d_k,H_k} \mathbb{E} \left[ |X_1|^{1-2H_k} \prod_{i=2}^{d_k} (|X_i| \vee |X_{i-1}|)^{1-2H_k} \right] \left( \prod_{i=d_k+1}^{n} |\lambda_i X_i - \sqrt{1 - \lambda_i^2} X_{i-1}|^{1-2H_k} \right) \]

\[ \leq C_{d_k,H_k} \left( \prod_{i=1}^{d_k-1} \left( \mathbb{E} \left[ |X_i|^{1-2H_k} \right] \vee \mathbb{E} \left[ |X_i|^{2-4H_k} \right] \right) \right) \times \mathbb{E} \left[ |X_{d_k}|^{1-2H_k} \prod_{i=d_k+1}^{n} |\lambda_i X_i - \sqrt{1 - \lambda_i^2} X_{i-1}|^{1-2H_k} \right] \]

\[ = C_{d_k,H_k} \mathbb{E} \left[ |X_{d_k}|^{1-2H_k} \prod_{i=d_k+1}^{n} |\lambda_i X_i - \sqrt{1 - \lambda_i^2} X_{i-1}|^{1-2H_k} \right]. \tag{4.4} \]

with the convention that \( \prod_{i=m}^{n} a_i := 1 \) for \( m > n \).

To bound the remaining expectation, we use the following estimate for standard normal random variable \( X \) from [15, Lemma A.1]: there is a constant \( C > 0 \) so that for any \( 0 < \alpha < 1, \lambda > 0 \) and \( b > 0 \),

\[ \mathbb{E} \left[ |\lambda X + b|^{-\alpha} \right] \leq C (\lambda \vee b)^{-\alpha}. \tag{4.5} \]

By taking conditional expectation on the \( \sigma \)-field \( \sigma(X_{d_k}, \cdots X_{n-1}) \) and using ((4.5),

\[ \mathbb{E} \left[ |X_{d_k}|^{1-2H_k} \prod_{i=d_k+1}^{n} |\lambda_i X_i - \sqrt{1 - \lambda_i^2} X_{i-1}|^{1-2H_k} \right] \]

\[ = \mathbb{E} \left[ \mathbb{E} \left[ |X_{d_k}|^{1-2H_k} \prod_{i=d_k+1}^{n} |\lambda_i X_i - \sqrt{1 - \lambda_i^2} X_{i-1}|^{1-2H_k} \bigg| \sigma(X_{d_k}, \cdots X_{n-1}) \right] \right] \]

\[ \leq C_{H_k} \mathbb{E} \left[ |X_{d_k}|^{1-2H_k} \prod_{i=d_k+1}^{n-1} |\lambda_i X_i - \sqrt{1 - \lambda_i^2} X_{i-1}|^{1-2H_k} \left( \left( \sqrt{1 - \lambda_i^2} |X_{n-1}| \right) \vee \lambda_i \right)^{1-2H_k} \right] \]
\[ C_{H_k} \lambda_i^{1-2H_k} \mathbb{E} \left[ |X_{d,a}|^{1-2H_k} \prod_{i=d_a+1}^{n-1} |\lambda_i X_i - \sqrt{1 - \lambda_i^2 X_{i-1}}|^{1-2H_k} \right] \]
\[ \leq \cdots \leq C_{H_k} \prod_{i=d_a+1}^{n} \lambda_i^{1-2H_k}. \]

Thus, we have by (3.6)-(4.4) that

\[ g_k(s_1, \ldots, s_n, r_1, \ldots, r_n) \]
\[ \leq c_{H_k} u_{H_k-1}^{n-1} \left( \prod_{i=1}^{n-1} u_i^{2H_k-3/2} \right) \left( \prod_{i=2}^{n} (u_i + u_{i-1})^{1/2-H_k} \right) \prod_{i=d_a+1}^{n} \lambda_i^{1-2H_k} \]
\[ = c_{H_k} u_{H_k-1}^{n-1} \left( \prod_{i=1}^{n-1} u_i^{2H_k-3/2} \right) \left( \prod_{i=2}^{n} (u_i + u_{i-1})^{1/2-H_k} \right) \prod_{i=d_a+1}^{n} u_{i-1}^{1-H_k}. \]

(4.6)

Consequently,

\[ g(s_1, \ldots, s_n, r_1, \ldots, r_n) = \prod_{k=1}^{d} g_k(s_1, \ldots, s_n, r_1, \ldots, r_n) \]
\[ \leq c_{H_k} u_{H_k-1}^{n-d} \left( \prod_{i=1}^{n-1} u_i^{2H_k-3/2} \right) \left( \prod_{i=2}^{d_a} (u_i + u_{i-1})^{d/H_k} \right) \prod_{i=d_a+1}^{n} u_{i-1}^{d/H_k}. \]

(4.7)

(i) We first consider the case \(1/2 < H_0 < 1\). When \(d \geq 2\) and \(H^* > d - 1\), we have \(H^* > d/2\). We bound \((u_i + u_{i-1})^{d/H_k}\) in (4.7) by \(u_{i-1}^{d/H_k}\). Therefore,

\[ g(s_1, \ldots, s_n, r_1, \ldots, r_n) \leq c_{H_k} u_{H_k-1}^{n-d} \left( \prod_{i=1}^{n-1} u_i^{2H_k-3/2} \right) \left( \prod_{i=2}^{d_a} u_{i-1}^{d/H_k} \right) \prod_{i=d_a+1}^{n} u_{i-1}^{d/H_k} \]
\[ = c_{H_k} \prod_{i=1}^{n} u_i^{H_k^*-d}. \]

(4.8)
It follows then
\[
\mathbb{E} \left[ u_n^2(t, x) \right] = \int_{0 < s_1 < \cdots < s_n < t} g(s_1, \ldots, s_n, r_1, \ldots, r_n) \prod_{i=1}^{n} \gamma_0(s_i - r_i) ds_1 dr_1 \cdots ds_n dr_n
\]
\[
\leq C_H^n \int_{0 < s_1 < \cdots < s_n < t} \prod_{i=1}^{n} (s_i + 1 - s_i + r_{\sigma(i+1)} - r_{\sigma(i)})^{H^*-d}
\times \prod_{i=1}^{n} \gamma_0(s_i - r_i) ds_1 dr_1 \cdots ds_n dr_n.
\]  
(4.9)

We use \((a + b)^{-\beta} \leq a^{-\beta/2} b^{-\beta/2}\) for all \(a, b, \beta > 0\) to get
\[
\mathbb{E} \left[ u_n^2(t, x) \right] \leq C_H^n \int_{0 < s_1 < \cdots < s_n < t} \prod_{i=1}^{n} (s_i + 1 - s_i)^{H^*-d} \prod_{i=1}^{n} (r_{\sigma(i+1)} - r_{\sigma(i)})^{H^*-d}
\times \prod_{i=1}^{n} \gamma_0(s_i - r_i) ds_1 dr_1 \cdots ds_n dr_n
\]
\[
= \frac{C_H^n}{n!} \int_{[0, t]^n} h(s_1, \ldots, s_n) h(r_1, \ldots, r_n) \prod_{i=1}^{d} \gamma_0(s_i - r_i) ds_1 dr_1 \cdots ds_n dr_n.
\]  
(4.10)

where \(h(s_1, \ldots, s_n)\) is the symmetric extension to \([0, t]^n\) of the function \(\prod_{i=1}^{n} (s_i + 1 - s_i)^{H^*-d}\) defined on \(0 < s_1 < \cdots < s_n < t\). Using the multidimensional version of the Hardy–Littlewood inequality (see e.g., [14, (2.4)]), we have
\[
\mathbb{E} \left[ u_n^2(t, x) \right] \leq \frac{C_H^n}{n!} \left[ \int_{[0, t]^n} h(s_1, \ldots, s_n)^{1/H_0} ds_1 \cdots ds_n \right]^{2H_0}
\]
\[
\leq C_H^n(n!)^{2H_0-1} \left[ \int_{0 < s_1 < \cdots < s_n < t} \prod_{i=1}^{n} (s_i + 1 - s_i)^{H^*-d/H_0} ds_1 \cdots ds_n \right]^{2H_0}.
\]  
(4.11)

When
\[
\frac{H^* - d}{2H_0} > -1,
\]  
(4.12)

we may use Lemma 2.3 to bound the above multiple integral to obtain
\[
\mathbb{E} \left[ u_n^2(t, x) \right] \leq C_H^n(n!)^{2H_0-1} \left[ \frac{C_d,H,H_0}{\Gamma \left( \left( \frac{H^*-d}{2H_0} + 1 \right) n + 1 \right)} \right]^{(H^*-d/n)} \cdot \left[ \frac{(H^*-d/n)}{2H_0} + n \right]^{2H_0}.
\]  
(4.13)
For any $p \in [2, \infty)$, by hypercontractivity inequality $\|u_n(t, x)\|_p \leq (p - 1)^{n/2} \|u_n(t, x)\|_2$, we have

$$
\|u_n(t, x)\|_p \\
\leq p^{n/2} \left( \mathbb{E} \left[ u_n^2(t, x) \right] \right)^{p/2} \\
\leq C_H^{n/2} p^{n/2} (n!)^{-1/2} \left( \frac{C_{d, H, H_0}^n \Gamma \left( \frac{H^* - d}{2H_0} + 1 \right)}{t^{(H^* - d + 2H_0)n/2}} \right)^H_0 .
\tag{4.14}
$$

When $H^* > d - 1$, using Stirling’s formula for the gamma function that

$$
\Gamma(z) = \sqrt{2\pi/z} \left( \frac{z}{e} \right)^z (1 + O(1/z)) \quad \text{as } z \to \infty,
\tag{4.15}
$$

we have by (4.14)

$$
\|u_n(t, x)\|_p \leq \frac{C_{d, H, H_0}^{n/2} p^{n/2}}{\Gamma ((H^* - d + 1)n/2 + 1)} t^{(H^* - d + 2H_0)n/2} .
$$

This implies by the asymptotic behavior of the Mittag–Leffler function (e.g., [16, p.41, Formula (1.8.10)]) that for all $t > 0$

$$
\sum_{n=0}^{\infty} \|u_n(t, x)\|_p \leq \sum_{n=0}^{\infty} \frac{C_{H, H_0, d}^{n/2} p^{n/2} t^{(H^* - d + 2H_0)n/2}}{\Gamma ((H^* - d + 1)n/2 + 1)} \\
\leq C \exp \left[ C_{H, t} t^{\frac{H^* - d + 2H_0}{H^* - d + 1}} p^{\frac{1}{H^* - d + 1}} \right] < \infty.
\tag{4.16}
$$

It follows that $u(t, x) := \sum_{n=0}^{\infty} u_n(t, x)$ converges in $L^p(\Omega, \mathcal{F}, \mathbb{P})$ for every $p \in [2, \infty)$, and $u(t, x)$ is a global random field solution to (1.1) with $u(0, x) = 1$ satisfying (1.9).

When $H^* = d - 1$, $H^* > d - 2H_0$ as $H_0 > 1/2$ and we have from (4.14) by the Stirling’s formula (4.15) that

$$
\|u_n(t, x)\|_2 \leq C_{d, H, H_0}^n 2^{n/2} t^{(H^* - d + 2H_0)n/2} \exp \left( a_0(H^* - d + 2H_0)n \right) ,
$$

where $a_0 := \frac{1}{2} (1 + \log(2H_0/(2H_0 - 1))) > 0$. Clearly, there is some positive constant $T_0 = T_0(d, H, H_0)$ so that $\sum_{n=0}^{\infty} \|u_n(t, x)\|_2 < \infty$ for any $t \in [0, T_0)$. It follows that $u(t, x) := \sum_{n=0}^{\infty} u_n(t, x)$ for $(t, x) \in [0, T_0) \times \mathbb{R}^d$ is a local random field solution to (1.1). This completes the proof of part (i), the existence part, of Theorem 1.3.

(ii) When $H_0 = 1/2$ and $H^* > d - 1$, we replace $\gamma_0(s - r)$ in (4.9) by $\delta_{[0]}(s - r)$. Thus, we have
\[ \mathbb{E} \left[ u_n(t, x)^2 \right] \leq C^n_H \int_0^{s_1} \cdots \int_0^{s_n} \prod_{i=1}^n (s_{i+1} - s_i)^{H^* - d} ds_1 \cdots ds_n \leq \frac{C^n_{H,d}}{\Gamma((H^* - d + 1)n + 1)} t^{(H^* - d + 1)n}. \]  

By a similar argument to that of (4.15), we have
\[
\sum_{n=0}^{\infty} \|u_n(t, x)\|^p_p \leq \sum_{n=0}^{\infty} C^n_{H,d} t^{n/2} (H^* - d + 1)n/2 \Gamma(n(H^* - d + 1)/2 + 1) \leq C \exp \left[ C_{H,d} t^{1/2} (H^* - d + 1) \right] < \infty.
\]

It follows that \( u(t, x) = \sum_{n=0}^{\infty} u_n(t, x) \) converges in \( L^p(\Omega, \mathcal{F}, \mathbb{P}) \) for every \( p \in [2, \infty) \), and \( u(t, x) \) is a global random field solution to (1.1) with \( u(0, x) = 1 \) on \( \mathbb{R}^d \) satisfying (1.7). This completes the proof of part (i), the existence part, of Theorem 1.1.

References

1. Chen, L., Hu, Y., Kalbasi, K., Nualart, D.: Intermittency for the stochastic heat equation driven by a rough time fractional Gaussian noise. Probab. Theory Related Fields 171, 431–457 (2018)
2. Chen, X.: Parabolic Anderson model with rough or critical Gaussian noise. Ann. Inst. Henri Poincaré Probab. Stat. 55, 941–976 (2019)
3. Chen, X.: Parabolic Anderson model with a fractional Gaussian noise that is rough in time. Ann. Inst. Henri Poincaré Probab. Stat. 56, 792–825 (2020)
4. Dalang, R.: Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.’s. Electron. J. Probab. 4 (1999), no. 6, 29 pp
5. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. Cambridge University Press (1992)
6. Hu, Y.: Chaos expansion of heat equations with white noise potentials. Potential Anal. 16(1), 45–66 (2002)
7. Hu, Y.: Some recent progress on stochastic heat equations. Acta Math Sci. 39, 874–914 (2019)
8. Hu, Y., Huang, J., Lê, K., Nualart, D., Tindel, S.: Stochastic heat equation with rough dependence in space. Ann. Probab. 45, 4561–4616 (2017)
9. Hu, Y., Huang, J., Lê, K., Nualart, D. and Tindel, S.: Parabolic Anderson model with rough dependence in space. Computation and combinatorics in Dynamics, Stochastics and Control, 477–498, Abel Symp., 13, Springer, Cham, (2018)
10. Hu, Y., Huang, J., Nualart, D., Tindel, S.: Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency. Electron. J. Probab. 20(55), 50 pp (2015)
11. Hu, Y., Le, K.: Joint Hölder continuity of parabolic Anderson model. Acta Math. Sci. l 39, 764–780 (2019)
12. Hu, Y., Liu, Y., Tindel, S.: On the necessary and sufficient conditions to solve a heat equation with general additive Gaussian noise. Acta Math. Sci. 39, 669–690 (2019)
13. Hu, Y., Lu, F., Nualart, D.: Feynman-Kac formula for the heat equation driven by fractional noise with Hurst parameter \( H < 1/2 \). Ann. Probab. 40(3), 1041–1068 (2012)
14. Hu, Y., Nualart, D.: Stochastic heat equation driven by fractional noise and local time. Probab. Theory Related Fields 143(1–2), 285–328 (2009)
15. Hu, Y., Nualart, D., Song, J.: Feynman-Kac formula for heat equation driven by fractional white noise. Ann. Probab. 39(1), 291–326 (2011)
16. Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, (2006)

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