IDEMPOTENT PROBABILITY MEASURES, I

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Abstract. The set of all idempotent probability measures (Maslov measures) on a compact Hausdorff space endowed with the weak* topology determines is functorial on the category Comp of compact Hausdorff spaces. We prove that the obtained functor is normal in the sense of E. Shchepin. Also, this functor is the functorial part of a monad on Comp. We prove that the idempotent probability measure monad contains the hyperspace monad as its submonad. A counterpart of the notion of Milyutin map is defined for the idempotent probability measures. Using the fact of existence of Milyutin maps we prove that the functor of idempotent probability measures preserves the class of open surjective maps. Unlike the case of probability measures, the correspondence assigning to every pair of idempotent probability measures on the factors the set of measures on the product with these marginals, is not open.

1. Introduction

In this paper we investigate the functor of idempotent probability measures in the category of compact Hausdorff spaces. Hopefully, this functor will play in the idempotent analysis a role similar to that of the probability measure functor in the classical functional analysis.

The notion of idempotent (Maslov) measure finds important applications in different part of mathematics, mathematical physics and economics (see the survey article [19] and the bibliography therein). In particular, these measures arise in dynamical optimization [5]. An analogy between Maslov integration and optimization is indicated in [4]. It is noted in [3] that “the use of Maslov measures for encapsulating some aspects of uncertainty can be as relevant to most economic problem than the use of classical probability theory”.

According to an idempotent correspondence principle [19], “there exists a heuristic correspondence between important, interesting, and useful constructions and results of the traditional mathematics over fields and analogous constructions and results over idempotent semirings and semifields (i.e., semirings and semifields with idempotent addition)”.

E. Shchepin [27] introduced the class of normal functors acting in the category Comp of compact Hausdorff spaces and continuous maps and showed that the probability measure functor P is normal (see [11] for a detailed proof of this fact). The aim of this paper is to establish the property of normality for the functor of idempotent probability measures. Moreover, we show that the latter functor determines a monad (see the definition below) in the category Comp. It is also proved that

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the functor of idempotent probability measures contains the hyperspace functor as a subfunctor. Moreover, the hyperspace monad can be embedded as a submonad in the monad of idempotent probability measures. The proofs of these results are based on embeddings of the spaces of idempotent probability measures in the spaces of order-preserving functionals defined by Radul [23]. The monad structure for the functor of idempotent probability measures is tightly connected with the max-plus convex compact subsets in euclidean spaces as these sets endowed with the idempotent barycenter maps turn out to be algebras for the monad of idempotent probability measures.

A Milyutin map is a map of topological spaces that admits an averaging operator. Equivalently, the Milyutin maps are precisely those admitting the probability-measure-value selection. We also define a natural counterpart for the notion of Milyutin map for the spaces of idempotent probability measures. Using this notion we prove that the functor of idempotent probability measures is open, i.e. it preserves the class of open surjective maps. Note that the openness of the functor of probability measures is established by Ditor and Eifler [7].

Eifler [8] proved that the correspondence that assigns to every pair of probability measures on factors the set of probability measures with given marginals, is open (see also [29]). We prove that this is not the case for the idempotent probability measures.

2. Preliminaries

Let $X$ be a compact Hausdorff space. By $C(X)$ we denote the Banach space of continuous functions on $X$ endowed with the sup-norm. For any $c \in \mathbb{R}$ we denote by $c_X$ the constant function on $X$ taking the value $c$. By $w(X)$ we denote the weight of a topological space $X$.

In the sequel, by functor we mean a covariant functor. The probability measure functor acting in the category $\textbf{Comp}$ is denoted by $P$. See, e.g. [11] for the properties of the functor $P$.

By $\text{exp}$ we denote the hyperspace functor acting in the category $\textbf{Comp}$. Given a compact Hausdorff space $X$, the space $\text{exp}X$ is defined as the set of all nonempty closed subsets in $X$ endowed with the Vietoris topology. A base of this topology is formed by the sets of the form

$$\langle U_1, \ldots, U_n \rangle = \{ A \in \text{exp} X \mid A \subset \bigcup_{i=1}^{n} U_i, \ A \cap U_i \neq \emptyset, \ i = 1, \ldots, n \}.$$  

If $f: X \rightarrow Y$ is a continuous map, then $\text{exp} f: \text{exp} X \rightarrow \text{exp} Y$ is defined by $\text{exp} f(A) = f(A), \ A \in \text{exp} X$.

Let $\mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\}$ endowed with the metric $\rho$ defined by $\rho(x, y) = |e^x - e^y|$. Let also $\mathbb{R}_{\text{max}}^n = (\mathbb{R}_{\text{max}})^n$.

A functor in the category $\textbf{Comp}$ is called open if it preserves the class of open surjective maps. Recall that a map of topological spaces is called open if the image of every open set is open. For a surjective map $f: X \rightarrow Y$ of compact Hausdorff spaces the openness of $f$ is equivalent to the continuity of the map $y \mapsto f^{-1}(y): Y \rightarrow \text{exp} X$. If moreover $X$ and $Y$ are metrizable, then $f$ is open if and only if, for any sequence $(y_i)_{i=1}^{\infty}$ converging to $y \in Y$ and every $x \in X$ such that $f(x) = y$, there exists a sequence $(x_i)_{i=1}^{\infty}$ converging to $x$ and such that $f(x_i) = y_i$, for every $i$. 

Following the style of idempotent mathematics (see, e.g., [19, 17, 16]) we denote by
\( \odot : \mathbb{R} \times C(X) \to C(X) \) the map acting by \((\lambda, \varphi) \mapsto \lambda \varphi + c\), and by \( \oplus : C(X) \times C(X) \to C(X) \) the map acting by \((\varphi, \psi) \mapsto \max\{\varphi, \psi\}\).

For each \( c \in \mathbb{R} \) by \( c_X \) we denote the constant function from \( C(X) \) defined by the formula \( c_X(x) = c \) for each \( x \in X \).

**Definition 2.1.** A functional \( \mu : C(X) \to \mathbb{R} \) is called an *idempotent probability measure* (a Maslov measure) if

1. \( \mu(c_X) = c \);
2. \( \mu(c \odot \varphi) = c \odot \mu(\varphi) \);
3. \( \mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi) \),

for every \( \varphi, \psi \in C(X) \).

The number \( \mu(\varphi) \) is the *Maslov integral* of \( \varphi \in C(X) \) with respect to \( \mu \).

Let \( I(X) \) denote the set of all idempotent probability measures on \( X \). We endow \( I(X) \) with the weak* topology. A base of this topology is formed by the sets

\[ \{\nu \in I(X) \mid |\mu(\varphi_i) - \nu(\varphi_i)| < \varepsilon, \ i = 1, \ldots, n\}, \]

where \( \mu \in I(X), \ \varphi_i \in C(X), \ i = 1, \ldots, n, \) and \( \varepsilon > 0 \).

The following is an example of an idempotent probability measure. Let \( x_1, \ldots, x_n \in X \) and \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\max} \) be numbers such that \( \max\{\lambda_1, \ldots, \lambda_n\} = 0 \). Define \( \mu : C(X) \to \mathbb{R} \) as follows:

\[ \mu(\varphi) = \max\{\varphi(x_i) + \lambda_i \mid i = 1, \ldots, n\} \;
\]

As usual, for every \( x \in X \), we denote by \( \delta_x \) (or \( \delta(x) \)) the functional on \( C(X) \) defined as follows:

\[ \delta_x(\varphi) = \varphi(x), \ \varphi \in C(X) \;
\]

the Dirac probability measure concentrated at \( x \). Then one can write \( \mu = \bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \).

In order to establish properties of the set of idempotent probability measures, we use those of the order-preserving functionals. In [23], T. Radul considered the set of order-preserving functionals on compact Hausdorff spaces.

A functional (which is not supposed a priori to be either linear or continuous) \( \nu : C(X) \to \mathbb{R} \) is called

1. *weakly additive* if for each \( c \in \mathbb{R} \) and \( \varphi \in C(X) \) we have \( \nu(\varphi + c_x) = \nu(\varphi) + c \);
2. *order-preserving* if for each \( \varphi, \psi \in C(X) \) with \( \varphi \leq \psi \) we have \( \nu(\varphi) \leq \nu(\psi) \);
3. *normed* if \( \nu(1_X) = 1 \).

The space of real numbers \( \mathbb{R} \) is endowed with the standard metric. The following fact is established in [23].

**Lemma 2.2.** Each order-preserving weakly additive functional is a non-expanding map.

For a compact Hausdorff space \( X \), we denote by \( O(X) \) the set of all order-preserving weakly additive normed functionals in \( C(X) \). It is easy to see that for each \( \nu \in O(X) \) and \( c \in \mathbb{R} \) we have \( \nu(c_X) = c \). Therefore, \( I(X) \subset O(X) \), for every compact Hausdorff space \( X \).

**Proposition 2.3.** The set \( I(X) \) is closed in \( O(X) \).
Proof. Suppose that \( \mu \in O(X) \setminus I(X) \). Then there exist \( \varphi, \psi \in C(X) \) such that \( a = \mu(\varphi \oplus \psi) > \mu(\varphi) \oplus \mu(\psi) = b \). Then \( \mu \in (\mu; \varphi \oplus \psi, \varphi, \psi; \frac{a-b}{2}) \subset O(X) \setminus I(X) \). We see that the complement of \( I(X) \) is an open set in \( O(X) \).

\[ \square \]

Since \( O(X) \) is known to be compact Hausdorff, we conclude that so is the space \( I(X) \).

Given a map \( f: X \to Y \) of compact Hausdorff spaces, the map \( O(f): O(X) \to O(Y) \) by the formula \( O(f)(\mu)(\varphi) = \mu(\varphi f) \), for every \( \varphi \in C(Y) \).

**Proposition 2.4.** Let \( f: X \to Y \) be a continuous map of compact Hausdorff spaces. Then \( O(f)(I(X)) \subset I(Y) \).

Proof. Let \( \mu \in I(X) \) and \( \varphi, \psi \in C(Y) \). Clearly, \( O(f)(\mu)(c_X) = c, c \in \mathbb{R} \), and \( O(f)(\mu)(\lambda \oplus \varphi) = \lambda \oplus O(f)(\mu)(\varphi), \lambda \in \mathbb{R} \). We have also

\[
O(f)(\mu)(\varphi \oplus \psi) = \mu((\varphi \oplus \psi)f) = \mu((f\varphi \oplus f\psi)) = \mu(f\varphi) \oplus \mu(f\psi) = O(f)(\mu)(\varphi) \oplus O(f)(\mu)(\psi).
\]

Thus, \( O(f)(\mu) \in I(Y) \).

\[ \square \]

We denote by \( I(f): I(X) \to I(Y) \) the restriction map \( O(f)|I(X): I(X) \to I(Y) \). Note that, if \( \mu = \bigoplus_{i=1}^{n} \lambda_{i} \odot \delta_{x_{i}} \in I(X) \), then \( I(f)(\mu) = \bigoplus_{i=1}^{n} \lambda_{i} \odot \delta_{f(x_{i})} \in I(Y) \).

It is evident that the construction \( I \) determines a covariant functor in the category \( \text{Comp} \).

**Proposition 2.5.** The functor \( I \) preserves the class of embeddings.

Proof. This directly follows from the fact that the functor \( O \) preserves the embeddings [23].

\[ \square \]

As usual, given a closed subset \( A \) of a compact Hausdorff space \( X \), we identify \( I(A) \) with the subspace \( I(i)(I(A)) \) of \( I(X) \), where by \( i: A \to X \) we denote the inclusion map.

The proof of the following statement involves the max-plus version of the Hahn-Banach theorem [17]. For the sake of completeness, we provide an alternative proof of its special version.

We say that a subset \( L \) of \( C(X) \) is a *max-plus* linear subspace of \( C(X) \) if

1. \( c_X \in L \) for every \( c \in \mathbb{R} \);
2. \( \lambda \odot \varphi \in L \), for every \( \lambda \in \mathbb{R} \) and \( \varphi \in L \);
3. \( \varphi \oplus \psi \in L \), for every \( \varphi, \psi \in L \).

**Lemma 2.6.** Let \( L \) be a max-plus linear subspace of \( C(X) \). Let \( \mu: L \to \mathbb{R} \) be a functional that satisfies the conditions of Definition 2.1 (with \( C(X) \) replaced by \( L \)).

For any \( \varphi_0 \in C(X) \setminus L \), there exists an extension of \( \mu \) onto the minimal linear max-plus subspace \( L' \) containing \( L \cup \{\varphi_0\} \) that satisfies the conditions of Definition 2.1.

Proof. For any \( \varphi \in L' \), define \( \mu(\varphi) = \inf\{\mu(\psi) \mid \psi \in L, \ varphi \leq \psi \} \). It is clear that \( \mu \) is well-defined.
Given $\varphi \in L'$ and $\lambda \in \mathbb{R}$, we see that
\[
\mu(\lambda \circ \varphi) = \inf \{ \mu(\psi) \mid \psi \in L, \lambda \circ \varphi \leq \psi \}
= \inf \{ \mu(\lambda \circ \psi') \mid \psi' \in L, \lambda \circ \varphi \leq \lambda \circ \psi' \}
= \lambda \circ \inf \{ \mu(\psi') \mid \psi' \in L, \varphi \leq \psi' \}
= \lambda \circ \mu(\varphi).
\]

Note also that $\mu(\varphi_1) \leq \mu(\varphi_2)$, whenever $\varphi_1 \leq \varphi_2$, $\varphi_1, \varphi_2 \in L'$.

Now, given $\varphi_1, \varphi_2 \in L'$, we see that
\[
\mu(\varphi_1) \oplus \mu(\varphi_2) = \inf \{ \mu(\psi) \mid \psi \in L, \varphi_1 \leq \psi \} \oplus \inf \{ \mu(\psi') \mid \psi' \in L, \varphi_2 \leq \psi' \}
= \inf \{ \mu(\psi) \oplus \mu(\psi') \mid \psi, \psi' \in L, \varphi_1 \leq \psi, \varphi_2 \leq \psi' \}
\geq \inf \{ \mu(\psi \oplus \psi') \mid \psi, \psi' \in L, \varphi_1 \oplus \varphi_2 \leq \psi \oplus \psi' \}
= \mu(\varphi_1 \oplus \varphi_2).
\]

On the other hand, since $\varphi_i \leq \varphi_1 \oplus \varphi_2$, $i = 1, 2$, from the above remark it follows that $\mu(\varphi_i) \leq \mu(\varphi_1 \oplus \varphi_2)$, $i = 1, 2$, and therefore $\mu(\varphi_1) \oplus \mu(\varphi_2) = \mu(\varphi_1 \oplus \varphi_2)$.

**Lemma 2.7.** Let $L$ be a max-plus linear subspace of $C(X)$. Let $\mu: L \to \mathbb{R}$ be a functional that satisfies the conditions of Definition 2.1 (with $C(X)$ replaced by $L$). Then there exists $\nu \in I(X)$ such that $\nu|L = \mu$.

**Proof.** We apply Lemma 2.6 and the Kuratowski-Zorn lemma to obtain a maximal extension $\mu'$ of $\mu$ onto a max-plus linear subspace $L'$ of $C(X)$. If $L' \neq C(X)$ and $\varphi \in C(X) \setminus L'$, then we can extend $\mu'$ onto the minimal max-plus linear subspace of $C(X)$ containing $L' \cup \{ \varphi \}$, which contradicts the maximality.

**Proposition 2.8.** The functor $I$ preserves the class of the onto maps.

**Proof.** Let $f: X \to Y$ be an onto map of compact Hausdorff spaces. Let $C(f) = \{ \varphi f \mid \varphi \in C(Y) \}$. Given $\mu \in I(X)$, we consider the map $\mu': C(f) \to \mathbb{R}$ defined by the formula $\mu'(\varphi f) = \mu(\varphi)$. Obviously, $\mu$ satisfies the conditions from Definition 2.1 (with $C(X)$ replaced by $C(f)$). By Lemma 2.7, there exists an extension $\nu$ of $\mu'$ onto $C(X)$ satisfying the conditions from Definition 2.1. We then have $\mu = I(f)(\nu)$.

**Proposition 2.9.** The functor $I$ preserves the intersections in the sense that $I(A \cap B) = I(A) \cap I(B)$, for any closed subsets $A, B$ of a compact Hausdorff space $X$.

**Proof.** Since the functor $O$ preserves intersections [23], we see that
\[
I(A \cap B) = O(A \cap B) \cap I(A \cap B) = (O(A) \cap O(B)) \cap I(A \cap B)
= (O(A) \cap I(A \cap B)) \cap (O(B) \cap I(A \cap B))
= I(A) \cap I(B).
\]

If a functor $F$ in $\textbf{Comp}$ preserves the class of embeddings and intersections, one can define the notion of support for it. Namely, given $a \in F(X)$, we call the set
\[
\text{supp}(a) = \cap \{ Y \mid Y \text{ is a closed subset of } X \text{ and } a \in F(Y) \}.
\]
the support of \( a \). Note that if \( \mu = \oplus_{i=1}^{n} \lambda_i \otimes \delta_{x_i} \), then \( \text{supp}(\mu) = \{ x_i \mid \lambda_i > -\infty \} \).

Given a closed subset \( A \) of \( X \) and \( \mu \in I(X) \), we have: \( \text{supp}(\mu) \subset A \) if and only if, for any two functions \( \varphi, \psi \in C(X) \) with \( \varphi|A = \psi|A \), we have \( \mu(\varphi) = \mu(\psi) \).

Let \( S = \{ X_\alpha, p_{\alpha\beta}; \mathcal{A} \} \) be an inverse system over a directed set \( \mathcal{A} \). For any \( \alpha \in \mathcal{A} \), let \( p_\alpha : X = \varprojlim S \to X_\alpha \) denote the limit projection. By \( I(S) \) we denote the inverse system \( \{I(X_\alpha), I(p_{\alpha\beta}); \mathcal{A} \} \).

The following property is a counterpart of the Kolmogorov theorem for probability measures.

**Proposition 2.10.** The map \( h = (I(p_\alpha))_{\alpha \in \mathcal{A}} : I(X) \to \varprojlim I(S) \) is a homeomorphism.

**Proof.** It easily follows from the results of Radul [23] that the map \( h \) is an embedding. We are going to show that \( h \) is an onto map. Let \( (\mu_\alpha)_{\alpha \in \mathcal{A}} \in \varprojlim I(S) \). By [23, Proposition 4], there exists \( \mu \in O(X) \) such that \( O(p_\alpha(\mu)) = \mu_\alpha \), for any \( \alpha \in \mathcal{A} \).

Let \( C' = \{ \varphi_{p_\alpha} \mid \varphi \in C(X_\alpha), \alpha \in \mathcal{A} \} \). Given \( \varphi, \psi \in C' \), one can write \( \varphi = \varphi' p_\alpha, \psi = \psi' p_\alpha \), for some \( \alpha \in \mathcal{A} \), whence

\[
\mu(\varphi \oplus \psi) = \mu((\varphi' p_\alpha) \oplus (\psi' p_\alpha)) = O(p_\alpha)(\mu)(\varphi' \oplus \psi') = \mu_\alpha(\varphi' \oplus \psi') = \mu(\varphi' p_\alpha) \oplus \mu(\psi' p_\alpha) = \mu(\varphi) \oplus \mu(\psi).
\]

Since, by the Stone-Weierstrass theorem, the set \( C' \) is dense in \( C(X) \) and the operation \( \oplus \) is continuous, we conclude that \( \mu(\varphi \oplus \psi) = \mu(\varphi) + \mu(\psi) \) for all \( \varphi, \psi \in C(X) \). One can similarly prove that \( \mu(\lambda \odot \varphi) = \lambda \odot \mu(\varphi) \), for all \( \varphi \in C(X) \) and \( \lambda \in \mathbb{R} \).

Thus, \( \mu \in I(X) \) is as required. \( \square \)

E. Shchepin [27] calls the just established property of the functor \( I \) the continuity of \( I \).

Suppose that a functor \( F \) in the category \( \text{Comp} \) preserves the class of embeddings. We say that \( F \) preserves preimages if, for every map \( f : X \to Y \) and every closed subset \( B \) of \( Y \), we have \( F(f^{-1}(B)) = (F(f))^{-1}(F(B)) \).

**Proposition 2.11.** The functor \( I \) preserves preimages.

**Proof.** Assume the contrary and let \( f : X \to Y \) be a morphism in \( \text{Comp} \), \( \mu \in I(X) \), \( B \) be a closed subset in \( Y \) such that \( I(f)(\mu) \in I(B) \) while \( \mu \notin I(f^{-1}(B)) \). There exist \( \varphi, \psi \in C(X) \) such that \( \varphi|f^{-1}(B) = \psi|f^{-1}(B) \) and \( \mu(\varphi) \neq \mu(\psi) \). Let \( c = |\mu(\varphi) - \mu(\psi)| \).

There exists a neighborhood \( U \) of \( B \) in \( Y \) such that \( ||\varphi|f^{-1}(U)| - |\psi|f^{-1}(U)|| < (c/3) \). There exist functions \( \varphi_1, \psi_1 \in C(X) \) satisfying the properties:

1. \( \varphi_1|f^{-1}(U) = \varphi|f^{-1}(U), \psi_1|f^{-1}(U) = \psi|f^{-1}(U) \);
2. \( \varphi_1 \leq \varphi, \psi_1 \leq \psi \);
3. \( ||\varphi_1 - \psi_1|| < (c/3) \).

One can easily demonstrate that there exist functions \( \varphi_2, \psi_2 \in C(X) \) satisfying the properties:

4. \( \varphi_2(X \setminus f^{-1}(U)) = \varphi(X \setminus f^{-1}(U)), \psi_1(X \setminus f^{-1}(U)) = \psi(X \setminus f^{-1}(U)) \);
5. \( \varphi_2 \leq \varphi, \psi_2 \leq \psi \);
6. \( \varphi_2|f^{-1}(B) = \psi_2|f^{-1}(B) = \min\{\inf \varphi, \inf \psi\} - 1 \).
Since

\[ \mu(\varphi) = \mu(\varphi_1 \oplus \varphi_2) = \mu(\varphi_1) + \mu(\varphi_2), \quad \mu(\psi) = \mu(\psi_1 \oplus \psi_2) = \mu(\psi_1) + \mu(\psi_2), \]

from property (3) and from the choice of \( \varphi, \psi \) it follows that \( \mu(\varphi_2) \neq \mu(\psi_2) \).

There exist functions \( \varphi', \varphi'', \psi', \psi'' \in C(Y) \) such that

\[ (7) \quad \varphi' f \leq \varphi_2 \leq \varphi'' f; \]
\[ (8) \quad \psi' f \leq \psi_2 \leq \psi'' f; \]
\[ (9) \quad \varphi|B = \varphi''|B = \psi|B = \psi''|B = \min\{\inf \varphi, \inf \psi\} - 1. \]

Then we have

\[ I(f)(\mu)(\varphi') = \mu(\varphi' f) \leq \mu(\varphi_2) \leq \mu(\varphi'' f) I(f)(\mu)(\varphi'') \]

and, since

\[ I(f)(\mu)(\varphi') = I(f)(\mu)(\varphi'') = \min\{\inf \varphi, \inf \psi\} - 1, \]

we conclude that \( \mu(\varphi_2) = \min\{\inf \varphi, \inf \psi\} - 1 \). Similarly, one can show that \( \mu(\psi_2) = \min\{\inf \varphi, \inf \psi\} - 1 \). We have obtained a contradiction. \( \square \)

Since the functor \( I \) preserves preimages, for any \( f : X \to Y \) and \( \mu \in I(X) \), we have \( \text{supp}(I(f)(\mu)) = f(\text{supp}(\mu)) \). This follows from general properties of functors in the category \( \text{Comp} \) established in [27].

A functor \( F \) in the category \( \text{Comp} \) is called normal (see [27]) if \( F \) is continuous, preserves weight, singletons, empty set, the onto maps, embeddings, intersections, and preimages.

**Proposition 2.12.** The functor \( I \) is normal.

**Proof.** This follows from Propositions 2.5, 2.9, 2.8, 2.11, and also from the fact that \( I \) is a subfunctor of \( O \) and the latter functor is almost normal in the sense that it satisfies all the properties from the definition of the normal functor excepting the preimage-preserving property (see [23]). As an example, we remark that the functor \( O \) preserves the weight of infinite compact Hausdorff spaces, i.e. \( w(X) = w(O(X)) \), for any infinite \( X \). Since \( I(X) \subset O(X) \), we conclude that \( w(X) = w(I(X)) \). \( \square \)

**Proposition 2.13.** Let \( |X| = n \), then the space \( I(X) \) is homeomorphic to the \((n-1)\)-dimensional simplex.

**Proof.** Let \( X = \{x_1, \ldots, x_n\} \). We first show that, for every \( \mu \in I(X) \), there exist \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\text{max}} \) such that \( \mu(\varphi) = \max\{\varphi(x_i) + \lambda_i \mid i = 1, \ldots, n\} \), for every \( \varphi \in C(X) \). For every \( i = 1, \ldots, n \), define \( \varphi_i \) by the formula \( \varphi_i(x_j) = \delta_{ij} - 1 \). Let \( \lambda_i = \inf\{\mu(\alpha \varphi_j) \mid \alpha \geq 0\} \).

Now, let \( \varphi \in C(X) \). Then, for every \( \alpha \geq 0 \), we have

\[ \varphi \leq \max\{\alpha \varphi_i + \varphi(x_i) \mid i = 1, \ldots, n\}, \]

whence

\[ \mu(\varphi) \leq \max\{\varphi(\alpha \varphi_i) + \varphi(x_i) \mid i = 1, \ldots, n\}. \]

Passing to the limit as \( \alpha \to \infty \), we see that \( \mu(\varphi) \leq \max\{\lambda_i + \varphi(x_i) \mid i = 1, \ldots, n\} \).

Actually, for sufficiently large \( \alpha \), we have the equality. The measure \( \mu \) defined above is denoted by \( \bigoplus_{i=1}^{n} \lambda_i \otimes \delta_{x_i} \).
Let \( \Gamma^{n-1} = \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_{\text{max}} \mid \max \{\lambda_1, \ldots, \lambda_n\} = 0 \} \). It is evident that \( \Gamma^{n-1} \) is homeomorphic to the \((n-1)\)-dimensional simplex. Define the map \( \xi : \Gamma^{n-1} \to I(\{x_1, \ldots, x_n\}) \) by the formula \( \xi(\lambda_1, \ldots, \lambda_n) = \oplus_{i=1}^n \lambda_i \odot \delta_{x_i} \). From what was already proved it follows that \( \xi \) is an onto map.

Given distinct \((\lambda_1, \ldots, \lambda_n), (\lambda'_1, \ldots, \lambda'_n) \in \Gamma^{n-1}\), one can find \( i \) such that \( \lambda_i \neq \lambda'_i \). Define \( \varphi : X \to \mathbb{R} \) as follows:

\[
\varphi(x) = \begin{cases} 
-1 - \max \{\lambda_j, \lambda'_j\}, & \text{if } x = x_j, \ i \neq j, \\
-\max \{\lambda_i, \lambda'_i\}, & \text{if } x = x_i.
\end{cases}
\]

Then

\[
\xi(\lambda_1, \ldots, \lambda_n) = \lambda_i - \max \{\lambda_i, \lambda'_i\} \neq \lambda'_i - \max \{\lambda_i, \lambda'_i\} = \xi(\lambda'_1, \ldots, \lambda'_n),
\]

whence we conclude that the map \( \xi \) is also an embedding.

Since the set \( \Gamma^{n-1} \) is compact, in order to show that \( \xi \) is a homeomorphism, one has to verify that the map \( \xi \) is continuous.

Let \( \mu = \xi(\lambda_1, \ldots, \lambda_n) \) and \( \langle \mu; \varphi; \varepsilon \rangle \) be a subbase neighborhood of \( \mu \) in \( I(X) \), where \( \varphi \in C(X), \varepsilon > 0 \). Define neighborhoods \( U_i \) of \( \lambda_i, \ i = 1, \ldots, n \), in \( \mathbb{R}_{\text{max}} \) as follows:

\[
U_i = \begin{cases} 
(\lambda_i - \varepsilon, \lambda_i + \varepsilon), & \text{if } \lambda_i > -\infty, \\
[-\infty, \min \{\lambda_j \mid \lambda_j > -\infty, \ j = 1, \ldots, n\} - \varepsilon), & \text{if } \lambda_i = -\infty.
\end{cases}
\]

Then \( U = (U_1 \times \cdots \times U_n) \cap \Gamma^n \) is a neighborhood of \((\lambda_1, \ldots, \lambda_n)\) with \( \xi(U) \subset \langle \mu; \varphi; \varepsilon \rangle \) and thus the map \( \xi \) is continuous.

\[
\square
\]

**Proposition 2.14.** The set

\[
I_\omega(X) = \{ \oplus_{i=1}^n \lambda_i \odot \delta_{x_i} \mid \lambda_i \in \mathbb{R}_{\text{max}}, \ i = 1, \ldots, n, \ \oplus_{i=1}^n \lambda_i = 0, \ x_i \in X, \ n \in \mathbb{N} \}
\]

(i.e., the set of idempotent probability measures of finite support) is dense in \( I(X) \).

*Proof.* It follows from Proposition 2.8 and results of general theory of functors in the category \( \textbf{Comp} \) (see [27]) that the set \( I_\omega(X) \) of the idempotent measures with finite supports is dense in \( I(X) \). The statement is now a consequence of Proposition 2.13. \( \square \)

We see that the spaces \( I(X) \) and \( P(X) \) are homeomorphic for every finite \( X \). In forthcoming publications we will show that they are also homeomorphic for infinite metrizable \( X \). However, the following statement holds.

**Proposition 2.15.** The functors \( P \) and \( I \) are not isomorphic.

*Proof.* Let \( X = \{a, b, c\}, \ Y = \{a, b\}, \ Z = \{a, c\} \), where \( a, b, c \) are distinct points. Denote by \( f : X \to Y \) and \( g : X \to Z \) the retractions such that \( f(c) = b \) and \( g(b) = c \). Then the map \((P(f), P(g)) : P(X) \to P(Y) \times P(Z)\) is obviously an embedding while the map \((I(f), I(g)) : I(X) \to I(Y) \times I(Z)\) is not. Indeed, let

\[
\mu = (-1) \odot \delta_a \oplus 0 \odot \delta_b \oplus 0 \odot \delta_c, \\
\nu = (-2) \odot \delta_a \oplus 0 \odot \delta_b \oplus 0 \odot \delta_c,
\]

then...
then
\[ I(f)(\mu) = I(f)(\nu) = 0 \circ \delta_a \oplus 0 \circ \delta_b, \quad I(g)(\mu) = I(g)(\nu) = 0 \circ \delta_a \oplus 0 \circ \delta_c. \]

\[ \square \]

Given \( x, y \in \mathbb{R}^n \) and \( \lambda \in \mathbb{R} \), we denote by \( x \oplus y \) the coordinatewise maximum of \( x \) and \( y \) and by \( \lambda \odot x \) the vector obtained from \( x \) by adding \( \lambda \) to every its coordinate. A subset \( A \) in \( \mathbb{R}^n \) is called \textit{max-plus convex} if \( \lambda_1 \odot x_1 \oplus \lambda_2 \odot x_2 \in A \) whenever \( x_1, x_2 \in A \) and \( \lambda_1, \lambda_2 \in \mathbb{R} \) with \( \lambda_1 \oplus \lambda_2 = 0 \). Note that, in this definition, one can also assume that \( \lambda_1, \lambda_2 \in \mathbb{R}_{\text{max}} \).

One can similarly define, for any \( \mu_1, \mu_2 \in I(X) \) and \( \lambda_1, \lambda_2 \in \mathbb{R}_{\text{max}} \) with \( \lambda_1 \oplus \lambda_2 = 0 \), the \textit{max-plus convex combination} \( \mu = \lambda_1 \odot \mu_1 \oplus \lambda_2 \odot \mu_2 \) as follows:
\[ \mu(\varphi) = \lambda_1 \odot \mu_1(\varphi) \oplus \lambda_2 \odot \mu_2(\varphi), \quad \varphi \in C(X). \]

The following statement is obvious.

**Proposition 2.16.** We have \( \lambda_1 \odot \mu_1 \oplus \lambda_2 \odot \mu_2 \in I(X) \).

**Proposition 2.17.** Let \( A \subset I(X) \), \( A \neq \emptyset \). Then \( \sup A \in I(X) \).

Let \( A \subset \mathbb{R}^n \) be a compact max-plus convex subset. By abusing the language, we denote by \( x_1, \ldots, x_n \) the coordinate functions \( \mathbb{R}^n \to \mathbb{R} \). Given \( \mu \in I(A) \), we let \( \beta_A(\mu) = (\mu(x_1), \ldots, \mu(x_n)) \).

**Proposition 2.18.** The map \( \beta = \beta_A : I(A) \to A \) is continuous.

**Proof.** The continuity of the map \( \mu \mapsto \beta_X(\mu) \) follows from the fact that if \( \mu' \in \langle \mu; x_1, \ldots, x_n; \varepsilon \rangle \), then \( \| \beta(\mu) - \beta(\mu') \| < \varepsilon \).

Given \( \mu = \oplus_{i=1}^k \lambda_i \odot \delta_{a_i} \in I(A) \), we see that \( \beta(\mu) = \oplus_{i=1}^k \lambda_i \odot a_i \in A \). Since \( I_\omega(X) \) is dense in \( I(X) \), we see that \( \beta(\mu) \in A \) for every \( \mu \in I(A) \). Therefore, the map \( \beta \) is well-defined.

\[ \square \]

The map \( \beta : I(A) \to A \) is called the \textit{idempotent barycenter map}.

**Remark 2.19.** One can extend Proposition 2.18 over the case of the compact max-plus convex subsets in arbitrary Tychonov power \( \mathbb{R}^r \).

3. Idempotent Probability Measure Monad

It is known [23] that the functor of order-preserving functionals forms a monad on the category \textbf{Comp}. In this section we are going to show that the functor \( I \) is the functorial part of a submonad of the monad of order-preserving functionals.

A \textit{monad} \( \mathbb{T} = (T, \eta, \mu) \) in the category \( \mathcal{E} \) consists of an endofunctor \( T : \mathcal{E} \to \mathcal{E} \) and natural transformations \( \eta : 1_{\mathcal{E}} \to T \) (unity), \( \mu : T^2 \to T \) (multiplication) satisfying the relations \( \mu \circ T \eta = \mu \circ \eta T = 1_T \) and \( \eta \circ \mu T = \mu \circ T \mu \).

A natural transformation \( \psi : T \to T' \) is called a \textit{morphism} from a monad \( \mathbb{T} = (T, \eta, \mu) \) into a monad \( \mathbb{T}' = (T', \eta', \mu') \) if \( \psi \circ \eta = \eta' \) and \( \psi \circ \mu = \mu' \circ \eta T' \circ T \psi \). If all the components of \( \psi \) are monomorphisms then the monad \( \mathbb{T} \) is called a \textit{submonad} of \( \mathbb{T}' \).
If $T = (T, \eta, \mu)$ is a monad in the category $\mathcal{E}$, then a pair $(X, \xi)$, where $\xi : T(X) \to X$ is a $T$-algebra if $\xi_\eta_X = \text{id}_X$ and $\xi_\mu_X = \xi T(\xi)$. Given $T$-algebras $(X, \xi), (X', \xi')$, we say that a morphism $f : X \to X'$ is a morphism of $T$-algebras if $f \xi = \xi' T(f)$. The $T$-algebras and their morphisms form a category.

The hyperspace monad $\mathbb{H} = (\exp, s, u)$ in the category $\text{Comp}$ is defined as follows. The natural transformation $s : \text{id} \to \exp$ acts by the formula $s_X(x) = \{x\}$, $x \in X$. The natural transformation $u : \exp^2 \to \exp$ is defined by the formula $u(A) = \cup A$, $A \in \exp^2 X$.

Let $X \in |\text{Comp}|$. Given $\varphi \in C(X)$, define $\bar{\varphi} : I(X) \to \mathbb{R}$ as follows: $\bar{\varphi}(\mu) = \mu(\varphi)$, $\mu \in I(X)$.

**Lemma 3.1.** If $\varphi \in C(X)$ and $\lambda \in \mathbb{R}_{\max}$, then $\overline{\lambda \circ \varphi} = \lambda \circ \bar{\varphi}$.

**Proof.** Given $\mu \in I(X)$, we have $\overline{\lambda \circ \varphi}(\mu) = \lambda(\circ \varphi) = \lambda \circ \mu(\varphi) = \lambda \circ \bar{\varphi}(\mu)$. \hfill $\square$

**Lemma 3.2.** If $\varphi, \psi \in C(X)$, then $\overline{\varphi \circ \psi} = \bar{\varphi} \circ \bar{\psi}$.

**Proof.** Given $\mu \in I(X)$, we have $\overline{\varphi \circ \psi}(\mu) = \mu(\varphi \circ \psi) = \mu(\bar{\varphi} \circ \bar{\psi})(\mu) = (\bar{\varphi} \circ \bar{\psi})(\mu)$. \hfill $\square$

Given $M \in I^2(X)$, define the map $\zeta_X(M) : C(X) \to \mathbb{R}$ as follows: $\zeta_X(M)(\varphi) = M(\bar{\varphi})$.

**Proposition 3.3.** We have $\zeta_X(M) \in I(X)$.

**Proof.** Check the conditions from the definition of $I(X)$.

1. $\zeta_X(M)(e_X) = M(\overline{e_X}) = M(e_{I(X)}) = c$.

2. Applying Lemma 3.1 we obtain $\zeta_X(M)(\lambda \circ \varphi) = M(\overline{\lambda \circ \varphi}) = M(\lambda \circ \bar{\varphi}) = \lambda \circ M(\bar{\varphi}) = \lambda \circ \zeta_X(M)(\varphi)$.

3. Applying Lemma 3.2 we obtain $\zeta_X(M)(\varphi \circ \psi) = M(\overline{\varphi \circ \psi}) = M(\bar{\varphi} \circ \bar{\psi}) = M(\bar{\varphi}) \circ M(\bar{\psi}) = \zeta_X(M)(\varphi) \circ \zeta_X(M)(\psi)$. \hfill $\square$

Thus, we obtain a map $\zeta_X : I^2(X) \to I(X)$. It follows from the results of [23] that $\zeta = (\zeta_X)$ is a natural transformation from the functor $I^2$ to the functor $I$. Actually, this natural transformation is the restriction of the natural transformation $O^2 \to O$ defined by Radul.

**Theorem 3.4.** The triple $I = (I, \eta, \zeta)$ is a monad on the category $\text{Comp}$.

**Proof.** As we already remarked, the natural transformation $\zeta$ is the restriction of the natural transformation $O^2 \to O$ and $\delta$ maps the identity functor into $I \subset O$. Therefore, the result follows from the fact that the functor $O$ generates a monad in $\text{Comp}$ (see [23, Theorem 3]). \hfill $\square$

Actually, $I$ is a submonad of the monad $\otimes$ generated by the functor $O$ (see [23]).

**Proposition 3.5.** Let $X$ be a compact max-plus convex subset in $\mathbb{R}^n$. Then the pair $(X, \beta)$ is an $I$-algebra.
Proposition 3.7. The diagonal map 

embeds verifying this equality for \( \mu \).

This follows from the equality 

\[
R \implies \quad \text{Examples of max-plus affine maps are the projections onto coordinate hyperplanes in } \mathbb{R}^n.
\]

Proposition 3.6. Let \( f : X \to Y \) be a continuous affine map of max-plus compact convex subsets in euclidean spaces. Then \( f \) is called a morphism of \( \mathbb{I} \)-algebras.

Proof. One has to show that \( f \beta X(\mu) = \beta_Y I(f)(\mu) \), for any \( \mu \in I(X) \). It suffices to verify this equality for \( \mu \) of finite support. If \( \mu = \bigoplus_{i=1}^n \lambda_i \circ \delta_{x_i} \), then

\[
 f \beta X(\mu) = f \left( \bigoplus_{i=1}^n \lambda_i \circ x_i \right) = \bigoplus_{i=1}^n \lambda_i \circ f(x_i) = \beta_Y \left( \bigoplus_{i=1}^n \lambda_i \circ \delta_{f(x_i)} \right) = \beta_Y I(f)(\mu).
\]

Proposition 3.7. The diagonal map

\[
\Phi = (\varphi)_{\varphi \in C(X)} : I(X) \to \prod_{\varphi} \mathbb{R} = \mathbb{R} = C(X) = \mathbb{R}^{C(X)}
\]

embeds \( I(X) \) as a max-plus convex subset of \( \mathbb{R}^{C(X)} \).

Proof. This follows from the equality

\[
\varphi(\alpha_1 \circ x_i \circ \alpha_2 \circ \mu_2) = \alpha_1 \circ \varphi(\mu_1) \circ \alpha_2 \circ \varphi(\mu_2),
\]

for any \( \mu_1, \mu_2 \in I(X) \), \( \varphi \in C(X) \), and \( \alpha_1, \alpha_2 \in \mathbb{R}_{\text{max}} \) with \( \alpha_1 \circ \alpha_2 = 0 \).

Using the monad structure for the functor \( I \), one can define, for all \( \mu \in I(X) \), \( \nu \in I(Y) \), the tensor product \( \mu \otimes \nu \in I(X \times Y) \) (see, e.g. [30]). For the sake of completeness, we recall its construction. For every \( y \in Y \), let \( i_y : X \to X \times Y \) denote the map defined by the formula \( i_y(x) = (x, y) \), \( x \in X \). Then define the map \( g_\mu : Y \to I(X \times Y) \) by the formula \( g_\mu(y) = I(i_y)(\mu) \), \( y \in Y \). Finally,

\[
\mu \otimes \nu = \zeta_{X \times Y} I(g_\mu)(\nu).
\]

If \( \mu = \bigoplus_{i=1}^n \lambda_i \circ \delta_{x_i} \in I(X) \), \( \nu = \bigoplus_{j=1}^m \kappa_j \circ \delta_{y_j} \in I(Y) \), then

\[
\mu \otimes \nu = \bigoplus_{i=1}^m \bigoplus_{j=1}^n (\lambda_i \circ \kappa_j \circ \delta_{(x_i, y_j)}) \in I(X \times Y).
\]

By induction, one can define the tensor product for arbitrary finite products: if \( \mu_i \in I(X_i) \), \( i = 1, \ldots, n \), then

\[
\mu_1 \otimes \cdots \otimes \mu_n = (\mu_1 \otimes \cdots \otimes \mu_{n-1}) \otimes \mu_n \in I((X_1 \times X_{n-1}) \times X_n) = I(X_1 \times \cdots \times X_n).
\]
The hyperspace monad \( \mathbb{H} \) is a submonad of the monad \( \mathbb{I} \).

**Theorem 3.8.** The hyperspace monad \( \mathbb{H} \) is a submonad of the monad \( \mathbb{I} \).

**Proof.** Given a compact Hausdorff space \( X \), define a map \( j_X : \exp X \to I(X) \) by the condition: \( j_X(A)(\varphi) = \max(\varphi|A) \). It is straightforward to verify that \( j_X \) is well-defined.

We are going to demonstrate that the map \( j_X \) is continuous. Let \( A_0 \in \exp X \) and \( j_X(A_0)_0, \varphi, \varepsilon \) be a subbase neighborhood of \( j_X(A_0) \) in \( I(X) \). There exists a finite open in \( X \) cover \( U = \{U_1, \ldots, U_n\} \) of \( A_0 \) such that, for every \( U_i \in U \), the oscillation of \( \varphi \) on \( U_i \) (i.e. the number \( |\sup(\varphi|U_i) - \inf(\varphi|U_i)| \) is less than \( \varepsilon \). Let \( A \in \{U_1, \ldots, U_n\} \).

We have to show that \( j_X(A) \in j_X(A_0)(\varphi, \varepsilon) \).

Let \( j_X(A)(\varphi) = \varphi(a) \), where \( a \in U_i \cap A \), for some \( i \). Then there is \( a_0 \in U_i \cap A_0 \) and \( |\varphi(a) - \varphi(a_0)| < \varepsilon \), whence \( j_X(A)(\varphi) = \varphi(a) < \varphi(a_0) + \varepsilon \leq j_X(A_0)(\varphi) + \varepsilon \). Proceeding similarly, we prove that \( j_X(A_0)(\varphi) < j_X(A)(\varphi) + \varepsilon \).

Note that the map \( j_X \) is an embedding. Indeed, let \( A, B \in \exp X \) and \( A \neq B \). Without loss of generality, we may assume that \( A \setminus B \neq \emptyset \). Let \( \varphi \in C(X) \) be a function with the following properties: \( \varphi|B \equiv 0 \), \( \varphi(x) > 0 \), for some \( x \in A \setminus B \). Then \( j_X(A)(\varphi) > j_X(B)(\varphi) = 0 \).

Given \( f : X \to Y \) and \( A \in \exp X \), \( \varphi \in C(Y) \), we see that

\[
(I(f)j_X(A))(\varphi) = j_Y(X)(\varphi f) = \max\{\varphi f(a) \mid a \in A\} = \max\{\varphi(b) \mid b \in f(A)\} = j_Y(f(A))(\varphi),
\]

whence \( I(f)j_X = j_Y \exp f \) and we see that \( j : \exp \to I \) is a natural transformation.

We are going to prove that \( j \) is a monad morphism. To this end, show that the diagram

\[
\begin{array}{ccc}
\exp^2 X & \xrightarrow{(I(f)j_X)} & I^2(X) \\
\downarrow u_X & & \downarrow \zeta_X \\
\exp X & \xrightarrow{j_X} & I(X)
\end{array}
\]

is commutative. We prove this for points of with finite supports. Let \( A \in \exp^2 X \), \( A = \{A_1, \ldots, A_k\} \), where \( A_i = \{a_{i1}, \ldots, a_{il}\} \).

Then \( j_{\exp X}(A) = \oplus_{p=1}^k \ominus_0 \delta(A_p) \) and

\[
I(j_X)j_{\exp X}(A) = I(j_X)(\oplus_{p=1}^k \ominus_0 \delta(A_p)) = \oplus_{p=1}^k \ominus_0 \delta(\oplus_{q=1}^l \ominus_0 \delta(a_{pq})),
\]

and

\[
\zeta_X I(j_X)j_{\exp X}(A) = \oplus_{p=1}^k \oplus_{q=1}^l (0 \ominus 0) \ominus \delta(a_{pq}).
\]
On the other hand,  

\[ j_X u_X(A) = j_X \{ a_{pq} \mid 1 \leq p \leq k, 1 \leq q \leq l \} = \bigoplus_{p=1}^{k} \bigoplus_{q=1}^{l} 0 \otimes \delta(a_{pq}). \]

Since the points of finite support are dense in \( \exp^2 X \), we are done.

Also \( j_X s_X(x) = j_X(\{x\}) = \delta_x \), for every \( x \in X \), and we see that \( J \) is a monad morphism. \( \square \)

**Remark 3.9.** Let \( \chi : X \to [0,1] \) be a fuzzy set such that the map \( \chi \) is continuous and \( \chi^{-1}(1) \neq \emptyset \). One can identify \( \chi \) with an element \( j_X(\chi) \) of \( I(X) \) as follows:  

\[ j_X(\chi)(\varphi) = \sup\{ \varphi(x) + \ln \chi(x) \mid x \in X \}, \varphi \in C(X). \]

4. **Milyutin maps of idempotent probability measures**

The notion of Milyutin map was first introduced for the probability measure functor (see, e.g., [22] for the construction).

**Theorem 4.1.** Let \( X \) be a compact metrizable space. Then there exists a zero-dimensional compact metrizable space \( X \) and a continuous map \( f : X \to Y \) for which there exists a continuous map \( s : Y \to I(X) \) such that \( \text{supp}(y) \subset f^{-1}(y) \), for every \( y \in Y \).

**Proof.** One can easily construct a sequence \((W_i)\), where each \( W_i \) is a finite set of pairs of subsets of \( Y \) satisfying the properties:

1. \( U_i = \{U \mid (U, V) \in W_i\} \) and \( V_i = \{U \mid (U, V) \in W_i\} \) are finite closed covers of the space \( Y \);
2. \( U \subseteq \text{Int}_Y(V) \) for every \((U, V) \in W_i\);
3. \( \text{mesh}(V_i) < (1/i) \) for every \( i \) (we assume that some metric is fixed on \( Y \); the mesh of a family of subsets in a metric space is the supremum of the diameters of its members).

We let \( X_i = \coprod\{V \mid (U, V) \in W_i\} \). The map \( f_i : X_i \to Y \) is the map such that \( f_i|V : V \to Y \) is the inclusion map for every \( V \) such that \((U, V) \in W_i\). Let \( \alpha_i : X_i \to [-\infty,0] \) be a continuous function such that, for every \((U, V) \in W_i\), we have \( \alpha_i|U \equiv 0 \) and \( \alpha_i|(V \setminus \text{Int}_Y(V)) \equiv -\infty \).

Let  

\[ X = \left\{ (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i \mid f_i(x_i) = f_j(x_j) \text{ for every } i, j \right\}. \]

Define the map \( f : X \to Y \) by the formula \( f((x_i)_{i=1}^{\infty}) = f_1(x_1) \).

Given \( y \in Y \), define  

\[ s(y) = \bigotimes_{i=1}^{\infty} \{ \alpha_i(x) \otimes \delta_x \mid x \in f_i^{-1}(y) \} \in I(X). \]

It is easy to see that \( s \) is well-defined and continuous. For any \( y \in Y \), we have \( \text{supp}(s(y)) = \prod_{i=1}^{\infty} f_i^{-1}(y) \) and therefore \( f(s(y)) = \delta_y \), for every \( y \in Y \).

Finally, we leave to the reader the verification that the space \( X \) is zero-dimensional and compact metrizable. \( \square \)
Similarly as in the case of probability measures, one can show that the product of idempotent Milyutin maps is Milyutin and than the restriction of a Milyutin map onto a full preimage of a closed set is also Milyutin. This allows us to prove that every compact Hausdorff space is the image of a zero-dimensional compact Hausdorff space under a Milyutin map.

We call a map \( f : X \to Y \) that satisfies the properties of Theorem 4.1 an idempotent Milyutin map.

We will need the following notion introduced by E. Shchepin [26]. A commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{u} \\
Z & \xrightarrow{v} & T
\end{array}
\]

is called \textit{bicommutative} if its characteristic map

\[ \chi = (f, g) : X \to Y \times T = \{(y, z) \in Y \times Z \mid u(y) = v(z)\} \]

is onto.

**Theorem 4.2.** The idempotent probability measure functor is open.

\begin{proof}
We first consider the case of surjective map of finite spaces. Let \( f : X \to Y \) be such a map. Since the composition of any two open maps is open, without loss of generality, one may assume that \( X = \{x_0, x_1, \ldots, x_n\} \), \( Y = \{y_1, \ldots, y_n\} \), and the map \( f : X \to Y \) acts by the formula \( f(x_0) = y_1, f(x_i) = y_i, y = 1, \ldots, n \). Let \( \mu_0 \in I(X) \), \( \mu_0 = \bigoplus_{i=0}^n \alpha_i \odot \delta_{x_i} \), \( \nu_0 = I(f)(\mu_0) \), and \( (\nu_k)_{k=1}^\infty \) be a sequence in \( I(Y) \) converging to \( \nu_0 \). We have \( \nu_k = \bigoplus_{j=1}^n \beta_{kj} \odot \delta_{y_j} \). Then \( \lim_{k \to \infty} \beta_{kj} = \alpha_{0j} \), for \( j = 2, \ldots, n \), and \( \lim_{k \to \infty} \beta_{k1} = \max\{\alpha_{00}, \alpha_{01}\} \). Without loss of generality, we may assume that \( \alpha_{00} \geq \alpha_{01} \). Then let \( \alpha_{k0} = \beta_{k1}, \alpha_{k1} = \min\{\beta_{k1}, \alpha_{01}\} \). Let \( \mu_k = \bigoplus_{i=0}^n \alpha_{ik} \odot \delta_{x_i} \), \( k \in \mathbb{N} \). It is obvious that \( I(f)(\mu_k) = \nu_k \), for every \( k \), and \( \lim_{k \to \infty}(\mu_k) = \mu_0 \). This is equivalent to the openness of the map \( I(f) \).

Let \( C \) denote the Cantor set. Let us prove that the map \( I(pr) \), where \( pr : C \times C \to C \) denotes the projection onto the first factor, is open. To this end, represent \( C \) as \( \lim\{C_i, f_{ij}\} \), where \( C_i \) are finite sets and \( f_{ij} : C_i \to C_j \) are surjections, \( i \geq j \). From the results of [27] it follows that, in order to prove that \( I(pr) \) is open, it is sufficient to prove that the diagram

\[
\begin{array}{ccc}
I(C_i \times C_i) & \xrightarrow{I(f_{ij} \times f_{ij})} & I(C_j \times C_j) \\
\downarrow{I(\pi_i)} & & \downarrow{I(\pi_j)} \\
I(C_i) & \xrightarrow{I(f_{ij})} & I(C_j)
\end{array}
\]
IDEMPOTENT PROBABILITY MEASURES

The product defined by the formula

\[(I(\pi_i), I(f_{ij} \times f_{ij})): I(C_i \times C_i) \to I(C_j \times C_j) \times I(C_i) I(C_i)\]

\[= \{(\mu, \nu) \in I(C_i) \times I(C_j \times C_j) | I(f_{ij})(\mu) = I(\pi_j)(\nu)\}\]

(called the characteristic map of the diagram) is an onto map.

Without loss of generality, one may assume that

\[C_j = \{x_1, \ldots, x_p\}, \ C_i = \{y_0, y_1, \ldots, y_p\}\]

(all the points are assumed to be distinct) and the map \(f_{ij}\) act as follows: \(f_{ij}(y_m) = x_m, m = 1, \ldots, p, f_{ij}(y_0) = y_1\). Thus, given \((\mu, \nu) \in I(C_j \times C_j) \times I(C_j) I(C_i)\), one can write

\[\mu = \bigoplus_{k=0}^{p} \kappa_k \odot \delta_{y_k}, \ \nu = \bigoplus_{m,n=1}^{p} \lambda_{mn} \odot \delta_{(x_{m},x_{n})}.\]

Without loss of generality, we may assume that \(\kappa_0 \leq \kappa_1\). Define

\[\nu' = \bigoplus_{m,n=0}^{p} \lambda'_{mn} \odot \delta_{(y_{m},y_{n})} \in I(C_i \times C_i)\]

by the conditions \(\lambda'_{mn} = \lambda_{mn}\), for \(m \geq 1, n = 0,1,\ldots,p\), \(\lambda'_{0n} = \min\{\kappa_0, \lambda_{1n}\}\), \(n = 0,1,\ldots,p\). We leave to the reader the verification of the fact that \(\nu'\) is as required.

Now, consider an open map \(f: X \to Y\) of compact metrizable spaces. Let \(p: Z \to Y\) be an idempotent Milyutin map, where \(Z\) is a compact metrizable zero-dimensional space. We may assume that \(Z\) is homeomorphic to the Cantor set. \(Z\) is a subset of the product \(T \times Y\) and \(p\) coincides with the restriction of the projection \(\hat{p}: T \times Y \to Y\) onto the second factor.

Denote by \(p: Z \times \times X \to X\) the projection map, \(p(z,x) = x\). We assume that \(Z \times \times X \subset T \times Y \times X\). For every \(x \in X\), let \(i_x: T \times Y \to T \times Y \times X\) be the map defined by the formula \(i_x(t,y) = (t,y,x)\).

Let \(s: Y \to I(Z)\) be a map such that \(\hat{p}s(y) = \delta y\), for every \(y \in Y\). Now, consider a sequence \((\nu_i)\) in \(I(Y)\) converging to \(\nu_0\) and \(\mu_0 \in I(X)\) such that \(I(f)(\mu_0) = \nu_0\).

Define a map \(g: X \to I(T \times Y \times X)\) by the formula \(g(x) = I(i_x)(s(f(x)))\). \(x \in X\).

Let \(\mu_0' = \zeta_{T \times Y \times X}(I(g)(\mu_0))\).

Denote by \(\pi_i\) the projection of \(T \times Y \times X\) onto the \(i\)-th factor and by \(\pi_{ij}\) the projection of \(T \times Y \times X\) onto the product of the \(i\)-th and \(j\)-th factors. We then have

\[I(\pi_3)(\mu_0') = I(\pi_3)\zeta_{T \times Y \times X}(I(g)(\mu_0)) = \zeta_X I^2(\pi_3)(I(g)(\mu_0)) = \zeta_X I(\eta_X)(\mu_0) = \mu_0.\]

For every \(i = 0,1,2,\ldots\), define \(\nu_i' = \zeta_Z I(\nu_i)\).

Then

\[I(p)(\nu_i') = I(p)\zeta_Z I(s)(\nu_i) = \zeta_Y I^2(p)I(s)(\nu_i) = \zeta_Y I(p)(\nu_i) = \zeta_Y I(\eta_Y)(\nu_i) = \nu_i.\]
We have
\[ I(\pi_{12})(\mu_0') = I(\pi_{12}) \zeta_{TX \times Y}(I(f)(\mu_0)) = \zeta_{TX \times Y} I^2(\pi_{12})(I(g)(\mu_0)) = \zeta_{TX \times Y} I(sf)(\mu_0) = \zeta_{TX \times Y} I(s)(I(f)(\mu_0)) \]
\[ = \zeta_{TX \times Y} I(s)(\nu_0) = \nu_0'. \]

Let \( h: K \to \pi_{12}(Z) \) be an open onto map of a zero-dimensional compact metrizable space. Without loss of generality, one may assume that the composition \( \pi_{12}h \) is homeomorphic to the projection map \( pr: C \times C \to C \). Let \( \mu''_0 \in I(K) \) be such that \( I(h)(\mu''_0) = (\mu'_0) \). Then, by the openness of the map \( I(\pi_{12}h) \), there exists a sequence \( (\mu''_i) \) in \( I(K) \) such that \( \lim_{i \to \infty} \mu''_i = \mu''_0 \) and \( I(\pi_{12}h)(\mu''_i) = \nu'_i \).

Let \( \mu_i = I(\pi_{3h})(\mu''_i) \). Then
\[ \lim_{i \to \infty} \mu_i = \lim_{i \to \infty} I(\pi_{3h})(\mu''_i) = I(\pi_{3h})(\mu''_0) = \mu_0. \]
For every \( i \in \mathbb{N} \), we have
\[ I(f)(\mu_i) = I(f \pi_{3h})(\mu'_i) = I(\hat{p} \pi_{12}h)(\mu''_i) = I(\hat{p})(\nu'_i) = \nu_i. \]

This proves that \( I(f) \) is an open map.

A functor \( F \) in the category \( \textbf{Comp} \) is called \textit{bicommutative} if \( F \) preserves the class of bicommutative diagrams.

\textbf{Corollary 4.3.} \textit{The functor \( I \) is bicommutative.}

\textit{Proof.} The fact follows from Theorem 4.2 and the result due to Shchepin [27] that every open functor is bicommutative. \( \square \)

5. \textbf{CORRESPONDENCES OF IDEMPOTENT PROBABILITY MEASURES WITH RESTRICTED MARGINALS}

Given a finite collection \( X_1, \ldots, X_k \) of compact Hausdorff spaces, define a map \( M_{X_1, \ldots, X_k}: I(\prod X_i) \to \prod I(X_i) \) as follows:
\[ M_{X_1, \ldots, X_k}(\mu) = (I(\pi_1)(\mu), \ldots, I(\pi_k)(\mu)), \quad \mu \in I(\prod X_i) \]
(here \( \pi_j: \prod X_i \to X_j \) is the projection onto the \( j \)-th factor). It is proved in [9] that the corresponding map is open for the case of the functor of probability measures. The following simple example shows that this is no true for the functor of idempotent probability measures.

\textbf{Example 5.1.} Let \( X = \{x_1, x_2\}, Y = \{y_1, y_2\} \), and
\[ \mu = 0 \oplus \delta_{(x_1, y_1)} \oplus 0 \oplus \delta_{(x_2, y_2)} \in I(X \times Y). \]

Then
\[ M_{X, Y}(\mu) = (\mu_1, \mu_2) = (0 \oplus \delta_{x_1} \oplus 0 \oplus \delta_{x_2}, 0 \oplus \delta_{y_1} \oplus 0 \oplus \delta_{y_2}) \in I(X) \times I(Y). \]

For every natural \( l \), let
\[ \mu_1^{(l)} = \left( \frac{-1}{l} \right) \oplus \delta_{x_1} \oplus 0 \oplus \delta_{x_2}, \quad \mu_2^{(l)} = 0 \oplus \delta_{y_1} \oplus \left( \frac{-1}{l} \right) \oplus \delta_{y_2}. \]
Then there is no sequence \((\mu^{(i)})_{i=1}^{\infty}\) in \(I(X \times Y)\) with \(M_{X,Y}(\mu^{(i)}) = (\mu_1^{(i)}, \mu_2^{(i)})\) and \(\lim_{i \to \infty} \mu^{(i)} = \mu\). This shows that the map \(M_{X,Y}\) is not open.

A diagram (1) in the category \textbf{Comp} is called \textit{open-bicommutative} if its characteristic map \(\chi\) is an open onto map. A functor \(F\) acting in the category \textbf{Comp} is called \textit{open-bicommutative} if it preserves the class of open-bicommutative diagrams. See [18] for the proof of open-bicommutativity of some functors related to the probability-measure functor.

Clearly, the product diagram

\[
\begin{array}{ccc}
X \times Y & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & \{\ast\}
\end{array}
\]

is open-bicommutative. It is clear from the above example that the diagram obtained by application the functor \(I\) to diagram (2) is not open-bicommutative. This demonstrates that the functor \(I\) is not open-bicommutative.

6. Metrization

Let \((X,d)\) be a compact metric space.

By \(n - \text{LIP} = n - \text{LIP}(X,d)\) we denote the set of Lipschitz functions with the Lipschitz constant \(\leq n\) from \(C(X)\).

Fix \(n \in \mathbb{N}\). For every \(\mu, \nu\), let

\[
\hat{d}_n(\mu, \nu) = \sup\{|\mu(\varphi) - \nu(\varphi)| \mid \varphi \in n - \text{LIP}\}.
\]

**Theorem 6.1.** The function \(\hat{d}_n\) is a continuous pseudometric on \(I(X)\).

**Proof.** We first remark that \(\hat{d}_n\) is well-defined. Indeed, \(\sup \varphi - \inf \varphi \leq n \ \text{diam}\ X\), for every \(\varphi \in n - \text{LIP}\), whence \(|\mu(\varphi) - \nu(\varphi)| \leq 2n \ \text{diam}\ X\).

Obviously, \(\hat{d}_n(\mu, \mu) = 0\) and \(\hat{d}_n(\mu, \nu) = \hat{d}_n(\nu, \mu)\), for every \(\mu, \nu \in I(X)\).

We are going to prove that \(\hat{d}\) satisfies the triangle inequality. Since, for every \(\varphi \in n - \text{LIP}\) and \(\mu, \nu, \tau \in I(X)\),

\[
\hat{d}(\mu, \nu) \geq |\mu(\varphi) - \nu(\varphi)|, \quad \hat{d}(\nu, \tau) \geq |\nu(\varphi) - \tau(\varphi)|,
\]

we have

\[
\hat{d}_n(\mu, \nu) + \hat{d}(\nu, \tau) \geq |\mu(\varphi) - \nu(\varphi)| + |\nu(\varphi) - \tau(\varphi)| \geq |\mu(\varphi) - \tau(\varphi)|,
\]

whence, passing to sup in the right-hand side, we obtain \(\hat{d}_n(\mu, \nu) + \hat{d}(\nu, \tau) \geq \hat{d}(\mu, \tau)\).

Now, we prove that \(\hat{d}\) is continuous. Suppose the contrary. Then one can find a sequence \((\mu_i)_{i=1}^{\infty}\) in \(I(X)\) such that \(\lim_{i \to \infty} \mu_i = \mu \in I(X)\) and \(\hat{d}(\mu_i, \mu) \geq c'\), for some \(c' > 0\). Then there exist \(\varphi_i \in n - \text{LIP}, i \in \mathbb{N}\), such that \(|\mu_i(\varphi_i) - \mu(\varphi_i)| \geq c\), for some \(c > 0\). Since the functionals in \(I(X)\) are weakly additive, without loss of generality, one may assume that \(\varphi_i(x_0) = 0\), for some base point \(x_0 \in X\), \(i \in \mathbb{N}\). By the Arzela-Ascoli theorem, there exists a limit point \(\varphi \in n - \text{LIP}\) of the sequence \((\varphi_i)_{i=1}^{\infty}\). We have \(|\mu_i(\varphi) - \mu(\varphi)| \geq c\), which contradicts to the fact that \((\mu_i)_{i=1}^{\infty}\) converges to \(\mu\). \(\square\)
Remark 6.2. Simple examples demonstrate that \( \hat{d} \) cannot be a metric whenever \( X \) consists of more than one point.

Proposition 6.3. The family of pseudometrics \( \hat{d}_n, n \in \mathbb{N}, \) separates the points in \( I(X) \).

Proof. Let \( \mu, \nu \in I(X), \mu \neq \nu \). There exists \( \varphi \in C(X) \) such that \( |\mu(\varphi) - \nu(\varphi)| > c \), for some \( c > 0 \). There exists \( \psi \in n - \text{LIP} \), for some \( n \in \mathbb{N} \), such that \( \|\varphi - \psi\| \leq (c/3) \). Then, clearly, \( |\mu(\psi) - \nu(\psi)| \geq (c/3) \) and therefore \( \hat{d}_n(\mu, \nu) \geq (c/3) \).

We let \( \tilde{d}_n = (1/n)d_n \).

Proposition 6.4. The map \( \delta = \delta_X, x \mapsto \delta_x: (X, \tilde{d}) \to (I(X), \tilde{d}_n), \) is an isometric embedding for every \( n \in \mathbb{N} \).

Proof. Let \( x, y \in X \) and \( \varphi \in n - \text{LIP} \). Then \( |\delta_x(\varphi) - \delta_y(\varphi)| \leq nd(x, y) \), therefore \( \tilde{d}_n(\delta_x, \delta_y) \leq nd(x, y) \). Thus \( \tilde{d}_n(\delta_x, \delta_y) \leq d(x, y) \).

On the other hand, define \( \varphi_x \in n - \text{LIP} \) by the formula \( \varphi_x(z) = nd(x, z) \), \( z \in X \). Then \( |\delta_x(\varphi_x) - \delta_y(\varphi_x)| \leq nd(x, y) \) and we are done.

Proposition 6.5. Let \( f: (X, \tilde{d}) \to (Y, \rho) \) be a nonexpanding map of compact metric spaces. Then the map \( \hat{f}: (I(X), \hat{d}_n) \to (I(Y), \hat{d}_n) \) is also nonexpanding, for every \( n \in \mathbb{N} \).

Proof. Given \( \varphi \in n - \text{LIP}(Y) \), note that \( \varphi f \in n - \text{LIP}(X) \) and, for any \( \mu, \nu \in I(X) \), we have

\[
|I(f)(\mu)(\varphi) - I(f)(\nu)(\varphi)| = |\mu(\varphi f) - \nu(\varphi f)| \leq \hat{d}_n(\mu, \nu).
\]

Passing to the limit in the left-hand side of the above formula, we are done.

Note that the above construction of \( \hat{d} \) can be applied not only to metrics but also to continuous pseudometrics. Proceeding in this way we obtain the iterations \( (I(X), \hat{d}_n), (I^2(X), \tilde{d}_{nm} = (\hat{d}_n)^m), \ldots \)

Proposition 6.6. For a metric space \( (X, \tilde{d}) \), the map \( \zeta_X: (I^2(X), \tilde{d}_{nm}) \to (I(X), \tilde{d}_n) \) is nonexpanding.

Proof. We first prove that, for any \( \varphi \in n - \text{LIP}(X, \tilde{d}) \), we have \( \bar{\varphi} \in n - \text{LIP}(I(X), \bar{d}) \). Indeed, given \( \mu, \nu \in I(X) \), we see that

\[
nd(\mu, \nu) = \bar{d}(\mu, \nu) \geq |\mu(\varphi) - \nu(\varphi)| = |\bar{\varphi}(\mu) - |\bar{\varphi}(\nu)|
\]

and we are done.

Suppose now that \( M, N \in I^2(X), \mu = \zeta_X(M), \nu = \zeta_X(N) \). Given \( \varphi \in n - \text{LIP}(X, \tilde{d}) \), we obtain

\[
|\mu(\varphi) - \nu(\varphi)| = |\bar{\varphi}(M) - N(\bar{\varphi})| \leq \hat{d}_{nm}(M, N).
\]

Passing to the limit in the left-hand side, we are done.
Remark 6.7. Using the results on existence of the pseudometrics $\tilde{d}_n$, one can define the spaces of idempotent probability measures with compact support for metric and, more generally, uniform spaces. Indeed, let $(X, d)$ be a metric space. We define the set $I(X)$ to be the direct limit of the direct system $\{I(A), I(\iota_{AB}); \text{exp} X\}$ (here, for $A, B \in \text{exp} X$ with $A \subset B$, we denote by $\iota_{AB}: A \to B$ the inclusion map). For every $A \in \text{exp} X$, we identify $I(A)$ with the corresponding subset of $I(X)$ along the map $I(\iota_A)$, where $\iota_A: A \to X$ is the limit inclusion map. For any $\mu \in I(X)$, there exists a unique minimal $A \in \text{exp} X$ such that $\mu \in I(A)$. Then we say that $A$ is the support of $\mu$ and write $\text{supp}(\mu) = A$.

Now, define a family of pseudometrics $\hat{d}_n$, $n \in \mathbb{N}$, on $I(X)$ as follows. Given $\mu, \nu \in I(X)$, we let

$$\hat{d}_n(\mu, \nu) = \hat{d}_n|((\text{supp}(\mu) \cup \text{supp}(\nu)) \times (\text{supp}(\mu) \cup \text{supp}(\nu)))(\mu, \nu).$$

One can prove that, for any uniform space $(X, U)$, if the uniformity $U$ is generated by a family $\{d_\alpha | \alpha \in A\}$ of pseudometrics, then the family $\{\tilde{d}_n^\alpha | \alpha \in A, n \in \mathbb{N}\}$ of pseudometrics on $I(X)$ generates a uniformity on $I(X)$.

7. Remarks and open problems

L. Shapiro [26] remarked that $P$ is the minimal normal functor that admits a factorization through the category of compact convex sets (in locally convex spaces) and affine continuous maps.

Question 7.1. Is $I$ the minimal normal functor that admits a factorization through the category of compact max-plus convex sets?

7.1. Idempotent barycentrically open max-plus convex sets. V. Fedorchuk [13] characterized the barycentrically open compact convex sets, i.e. compact convex sets $X$ for which the barycenter map $P(X) \to X$ is open. Note that some characterization results in this direction are also obtained in [8], [21], [20]. In particular, it is proved in [20] that a compact convex set $K$ in a locally convex space is barycentrically open if the map $(x, y) \mapsto \frac{1}{2}(x + y)$ is open.

Question 7.2. Characterize the class of max-plus convex compact spaces for which the idempotent barycenter map is open. In particular, is the latter property equivalent to the openness of the map $(x, y) \mapsto x \oplus y$?

It is proved in [12] that the product of barycentrically open compact convex sets is again barycentrically open.

Question 7.3. Is an analogous fact true for idempotent barycentrically open max-plus convex sets?

7.2. Idempotent probability measure monad. V. Fedorchuk [14] proved that there exists a unique monad in $\text{Comp}$ with the probability measure functor as its functorial part. It follows from the general properties of normal functors in $\text{Comp}$ that there exists a unique natural transformation $\text{id}_{\text{Comp}} \to I$. This leads to the following question.
Question 7.4. Is $\zeta: I^2 \to I$ the unique natural transformation that determines a monad structure for the functor $I^1$?

T. Świrszcz [28] proved that the category of compact convex sets and affine continuous maps is monadic over the category $\text{Comp}$. This leads to the following question.

Question 7.5. Is the category of (suitably defined) compact max-plus convex sets in locally convex lattices and affine continuous maps monadic over the category $\text{Comp}$?

Question 7.6. Characterize the category of $I$-algebras.

7.3. Milyutin maps. We borrowed the idea of the proof of Theorem 4.1 from [2]. Similarly as in [2], one can prove that one can choose a map $s: Y \to P(X)$ so that $\text{supp}(s(y)) = f^{-1}(y)$, for any $y \in Y$, and, moreover, every idempotent probability measure $s(y)$ is atomless in some appropriate sense.

As we already remarked, it was first proved in [7] that the probability measure functor $P$ preserves the class of open maps. The openness of the functor $O$ is proved in [23]. The method applied in [7] and [23] does not work in our case.

The proof of Theorem 4.2 is based on the properties of Milyutin maps and can be also applied to the proof of openness of the functor $P$ (see [29]) as well as of another related functors.

Similarly like in [25], one can use Milyutin maps in order to prove a counterpart of the Michael selection theorem for the max-plus-convex valued maps. That such a theorem can be proved by methods based on general convexity structures is indicated in [6] (see $\mathbb{B}$-spaces Metatheorem 5.0.19 therein). We return to this topic in another publication.

7.4. Metrization. If $(X, d)$ is a compact metric space, then the space $P(X)$ can be endowed with the Kantorovich metric. It is an open problem whether there exists a natural metrization of the space $I(X)$.

Question 7.7. Is there a metrization of the space $I(X)$, for all compact metric spaces $X$, which makes the monad $I$ perfectly metrizable (see [15] for the definition)?

Note that, for some natural reasons, to the notion of metric in classical analysis and topology there correspond that of ultrametric in the idempotent case (see [10]). (Recall that a metric $d$ on a set $X$ is called an ultrametric if the following strict triangle inequality holds: $d(x, y) \leq \max\{d(x, z), d(z, y)\}$). A counterpart of the Kantorovich metric on the space $I(X)$ can be defined in the case of an ultrametric space $X$. We leave it as an open problem to define a Kantorovich-type metric on the spaces of idempotent probability measures. V. Fedorchuk [15] introduced the notion of perfectly metrizable monad. Roughly speaking, this is a monad $(F, \eta, \mu)$ on $\text{Comp}$ whose functorial part $F$ is metrizable (i.e., it can be lifted to the category of compact metric spaces and nonexpanding maps) and the maps $\eta_X$ and $\mu_X$ are nonexpanding. The results of Section 6 suggest that one can introduce an analogous structure of monad metrizable by a countable family of pseudometrics.

Question 7.8. Is there a counterpart of the Prokhorov metric for the functor of idempotent probability measures?
7.5. Idempotent probability measures of noncompact spaces. Let $X$ be a Tychonov space and $\beta X$ be its Stone-Čech compactification. We define $I(X) = \{ \mu \in I(\beta X) | \text{supp}(\mu) \subseteq X \}$. For any maps $f : X \to Y$ of Tychonov spaces, we have $I(\beta f)(I(X)) \subseteq I(Y)$ and, therefore, we can define the map $I(f) = I(\beta f)|I(X) : I(X) \to I(Y)$. We thus obtain an extension of $I$ onto the category of Tychonov spaces. Note that a base of topology on $I(X)$ can be formed by the sets of the form

$$\langle \mu; \varphi_1, \ldots, \varphi_n; \varepsilon \rangle = \{ \nu \in I(X) | |\mu(\varphi_i) - \nu(\varphi_i)| < \varepsilon, \ i = 1, \ldots, n \},$$

where $\varphi_i \in C(X)$, $i = 1, \ldots, n$, are bounded and $\varepsilon > 0$.

7.6. Generalizations. One can obtain counterparts of the above results for another spaces of pseudo-additive measures. Here we mention only one example. Let $\cdot$ denote the min operation on the set $[-\infty, \infty]$. The set of functionals $\mu : C(X) \to \mathbb{R}$ satisfying $\mu(\varphi) \oplus \mu(\psi) = c$, and $\mu(cX) = c$, and $\mu(\lambda \cdot \varphi) = \lambda \cdot \mu(\varphi)$ can be topologized by the weak* topology and some of the results of this paper have their counterparts for such functionals.

7.7. Geometric properties. Many publications are devoted to geometric properties of the probability measure functor. Some of them have their counterparts for the idempotent probability measures. We will consider some these properties in subsequent publications.

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