LETTER TO THE EDITOR

Conformal off-diagonal boundary density profiles on a semi-infinite strip

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Abstract. The off-diagonal profile $\phi^b_{od}(v)$ associated with a local operator $\hat{\phi}(v)$ (order parameter or energy density) close to the boundary of a semi-infinite strip with width $L$ is obtained at criticality using conformal methods. It involves the surface exponent $x_{\phi}$ and displays a simple universal behaviour which crosses over from surface finite-size scaling when $v/L$ is held constant to corner finite-size scaling when $v/L \to 0$.

The finite-size behaviour of order parameter or energy density profiles has been the subject of much interest during the last two decades following the work of Fisher and de Gennes [1]. These profiles have been studied in the vicinity of the critical point in the mean-field approximation [2], using field-theoretical methods [3] and through exact solutions [4, 5]. Such profiles display universal behaviour at criticality and in two-dimensional systems they can be deduced from ordinary scaling and covariance under conformal transformation [6–17]. A short review can be found in reference [18].

With symmetry-breaking boundary conditions, one may consider diagonal order parameter profiles [6], i.e., ground-state expectation values. Otherwise, off-diagonal profiles can be used with any type of boundary conditions [13]. For the order parameter with Dirichlet boundary conditions, off-diagonal matrix elements must be considered since a diagonal order parameter profile then vanishes for symmetry reasons.

On a strip with fixed boundary conditions at $v = 0$ and $v = L$ the diagonal order-parameter profile $\phi(v)$ associated with an operator $\hat{\phi}$ takes the following form at criticality [6]

$$\phi(v) = \langle 0 | \hat{\phi}(v) | 0 \rangle = A \left[ \frac{L}{\pi} \sin \left( \frac{\pi v}{L} \right) \right]^{-x_{\phi}} 0 < v < L . \quad (1)$$

The exponent $x_{\phi}$ is the bulk scaling dimension of $\hat{\phi}$, $| 0 \rangle$ is the ground state of the Hamiltonian $\mathcal{H} = -\ln T$ where $T$ denotes the row-to-row transfer operator on the strip. When $L \to \infty$ with a fixed value of the ratio $v/L$, one obtains the bulk finite-size scaling behaviour $\phi(L) \sim L^{-x_{\phi}}$. When $L \to \infty$ while keeping $v$ fixed, one obtains the profile $\phi(v) \sim v^{-x_{\phi}}$ on the half-plane with fixed boundary conditions which is a consequence of ordinary scaling. Actually the profile on the strip in (1) follows from the profile on the half-plane through the logarithmic conformal transformation $w = (L/\pi) \ln z$ [6].
The off-diagonal critical profile with general symmetric boundary conditions at \( v = 0 \) and \( L \) is obtained as \([13]\)

\[ \phi_{\text{od}}(v) = \langle \phi | \hat\phi(v) | 0 \rangle \sim \left( \frac{L}{\pi} \right)^{-x_\phi} \left[ \sin \left( \frac{\pi v}{L} \right) \right]^{x_\phi-x_\phi} \quad 0 < v < L, \]  

where \( |\phi\rangle \) is the lowest excited state of \( \mathcal{H} \) leading to a non-vanishing matrix element. Besides the bulk exponent \( x_\phi \) the off-diagonal profile involves the surface scaling dimension \( x_\phi^s \) of the operator \( \hat\phi \). It can be identified by considering the transformation of the connected two-point correlation function \( G_{\phi\phi}^{\text{con}}(z_1, z_2) \) from the half-plane to the strip under the logarithmic conformal mapping. For the order parameter with fixed boundary conditions, \( x_\phi^s = 0 \), and (2) gives an off-diagonal profile in agreement with (1). When \( L \to \infty \), equation (2) shows the crossover from bulk finite-size scaling \( \phi_{\text{od}}(L) \sim L^{-x_\phi} \) when the ratio \( v/L \) is constant, to surface finite-size scaling \( \phi_{\text{od}}(L) \sim L^{-x_\phi^s} \) when \( v \) is constant, i.e., when \( v/L \to 0 \).

Let us now consider a half-strip in the \((u,v)\)-plane with \(0 < u < \infty\), \(0 < v < L\) and uniform boundary conditions. If one crosses the semi-infinite strip at \( u \gg L \) the behaviour of the off-diagonal profile will be the same as for the infinite strip in equation (2). A different behaviour is expected close to the boundary of the semi-infinite strip at \( u \ll L \). The profile should then involve the surface exponent \( x_\phi^s \) and, when \( v \ll L \) or \( L - v \ll L \), the corner exponent \( x_\phi^c \).

In order to obtain the profiles on the semi-infinite strip, we consider the transformation of the connected two-point correlation function \( G_{\phi\phi}^{\text{con}}(z_1, z_2) \) from the half-plane \( z = x + i y, \ y > 0 \) to the half-strip \( w = u + i v, \ 0 < u < \infty, \ 0 < v < L \) as shown in figure 1. The two geometries are related by the conformal transformation \([6]\)

\[ z = \cosh \left( \frac{\pi w}{L} \right) \]  

or

\[ x = \cosh \left( \frac{\pi u}{L} \right) \cos \left( \frac{\pi v}{L} \right) \quad y = \sinh \left( \frac{\pi u}{L} \right) \sin \left( \frac{\pi v}{L} \right). \]
Going from the half-plane to the half-strip, the dilatation factor is given by:

\[ b(z) = \left| \frac{dz}{dw} \right| = \left| \frac{\pi}{L} \sinh \left( \frac{\pi w}{L} \right) \right| = \frac{\pi}{L} \left[ \sinh^2 \left( \frac{\pi u}{L} \right) + \sin^2 \left( \frac{\pi v}{L} \right) \right]^{1/2}. \tag{5} \]

At criticality, the form of \( G_{\phi\phi}^{\text{con}}(z_1, z_2) \) is strongly constrained by conformal invariance. Using an infinitesimal special conformal transformation which preserves the surface geometry, one obtains a system of partial differential equation for the connected two-point correlation function on the half-plane, from which the following scaling form is deduced [19]:

\[ G_{\phi\phi}^{\text{con}}(z_1, z_2) = (y_1 y_2)^{-x_{\phi}} g(\omega) \quad \omega = \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{y_1 y_2}. \tag{6} \]

We are mainly interested in the behaviour of the profile close to the boundary of the half-strip. Thus we may consider the correlations between two points, the first close to the boundary located at \( u = 0 \) and the second far from it:

\[ \frac{u_1}{L} \ll 1 \quad 0 < v_1 < L \quad \frac{u_2}{L} \gg 1 \quad 0 < v_2 < L. \tag{7} \]

Considering (4) and (5) in the limits of equation (7) we have:

\[ x_2 - x_1 \simeq \frac{1}{2} \exp \left( \frac{\pi u_2}{L} \right) \cos \left( \frac{\pi v_2}{L} \right) \quad y_2 - y_1 \simeq \frac{1}{2} \exp \left( \frac{\pi u_2}{L} \right) \sin \left( \frac{\pi v_2}{L} \right) \tag{8} \]
\[ y_1 y_2 \simeq \frac{\pi u_1}{2L} \sin \left( \frac{\pi v_1}{L} \right) \sin \left( \frac{\pi v_2}{L} \right) \exp \left( \frac{\pi u_2}{L} \right) \]
\[ b(z_1) \simeq \frac{\pi}{L} \sin \left( \frac{\pi v_1}{L} \right) \quad b(z_2) \simeq \frac{\pi}{L} \exp \left( \frac{\pi u_2}{L} \right) \]

Thus the crossover variable \( \omega \) defined in (6) takes the form

\[ \omega \simeq \frac{L \exp(\pi u_2/L)}{2\pi u_1 \sin(\pi v_1/L) \sin(\pi v_2/L)} \gg 1. \tag{9} \]

In this limit, ordinary scaling leads to \( g(\omega) \sim \omega^{-x_{\phi}} \) so that, in the half-plane geometry:

\[ G_{\phi\phi}^{\text{con}}(z_1, z_2) \sim \frac{(y_1 y_2)^{x_{\phi}-x_{\phi}}}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{x_{\phi}}} \tag{10} \]

The conformal mapping (3) leads to the correlation function in the half-strip geometry [20]

\[ G_{\phi\phi}^{\text{con}}(w_1, w_2) \sim b(z_1)^{x_{\phi}} b(z_2)^{x_{\phi}} G_{\phi\phi}^{\text{con}}(z_1, z_2). \tag{11} \]

Making use of (8) in equations (10) and (11), we finally obtain:

\[ G_{\phi\phi}^{\text{con}}(w_1, w_2) \sim u_1^{x_{\phi}-x_{\phi}} \left[ \frac{2\pi}{L} \sin \left( \frac{\pi v_1}{L} \right) \right]^{x_{\phi}} \left( \frac{L}{\pi} \right)^{-x_{\phi}} \sin \left( \frac{\pi v_2}{L} \right) \left[ \sin \left( \frac{\pi u_2}{L} \right) \right]^{x_{\phi}-x_{\phi}} \exp \left( -\frac{\pi u_2}{L} \right). \tag{12} \]
In order to identify the different contributions to the two-point correlation function in (12), we can rewrite it using the row-to-row transfer operator $T$ on the strip with width $L$. The two-point correlation function on the semi-infinite strip reads

$$G_{\phi\phi}(w_1, w_2) = \frac{\langle B|\hat{\phi}(v_1)|T^{u_2}\hat{\phi}(v_2)|0\rangle}{\langle B|T^{u_2}|0\rangle} = \sum_n \frac{\langle B|\hat{\phi}(v_1)|n\rangle \exp(-u_2E_n)\langle n|\hat{\phi}(v_2)|0\rangle}{\langle B|0\rangle \exp(-u_2E_0)} \quad (13)$$

where $|0\rangle$ is the ground state of $\mathcal{H}$ which is selected by the transfer from $u = u_2$ to $u = \infty$ and $|B\rangle$ is a state vector appropriate for the boundary conditions at $u = 0$. In the case of free boundary conditions, it describes the free summation over the boundary states. In the last expression the summation is over the complete set of eigenstates $|n\rangle$ of $\mathcal{H}$ with eigenvalues $E_n$.

The connected two-point correlation function is then obtained by substracting the ground-state contribution to the eigenstate expansion:

$$G_{\phi\phi}^{\text{con}}(w_1, w_2) = G_{\phi\phi}(w_1, w_2) - \frac{\langle B|\hat{\phi}(v_1)|0\rangle}{\langle B|0\rangle} \langle 0|\hat{\phi}(v_2)|0\rangle \sim \frac{\langle B|\hat{\phi}(v_1)|\phi\rangle\langle \phi|\hat{\phi}(v_2)|0\rangle \exp[-u_2(E_\phi - E_0)]}{\langle B|0\rangle} \quad (14)$$

In the last expression we took into account the condition $u_2 \gg L$. In this limit, the eigenstate expansion is dominated by the contribution of the lowest excited state $|\phi\rangle$ of $\mathcal{H}$ for which the matrix elements are non-vanishing.

Comparing with the conformal expression in (12), we can read the gap-exponent relation [20] in the exponential factor, $E_\phi - E_0 = \pi x_\phi^s/L$, and the off-diagonal profile at $u_2 \gg L$ in agreement with equation (2). The remaining part can be then identified as the boundary profile on the half-strip at $u_1 \ll L$ and we obtain

$$\phi_{\text{bd}}^b(v) = \frac{\langle B|\hat{\phi}(v)|\phi\rangle}{\langle B|0\rangle} \sim \left[\frac{\pi}{L} \sin \left(\frac{\pi v}{L}\right)\right]^{x_\phi^s} \quad 0 < v < L. \quad (15)$$

In the case of the two-dimensional Ising model with free boundary conditions $\mathcal{H}$ is the Hamiltonian of the Ising model in a transverse field; if one associates the Pauli spin operator $\sigma_i^x$ ($l = 1, L$) with the order parameter, then the boundary state vector $|B\rangle$ is explicitly given by [21]

$$|B\rangle = \prod_{l=1,L} \frac{1}{\sqrt{2}}(|\sigma_i^x = +1\rangle + |\sigma_i^x = -1\rangle) = \prod_{l=1,L} |\sigma_i^z = +1\rangle. \quad (16)$$

Both $|0\rangle$ and $|B\rangle$ are even under the operator $P = \prod_{l=1,L} \sigma_i^z$ [21]. In the expression of the order parameter profile the state $|\phi\rangle = |\sigma\rangle$, which contains a single fermionic excitation, is odd under $P$. At $l = 1$, the order parameter profile coincides with the corner magnetization obtained in reference [21]. The surface magnetic exponent is $x_\sigma^s = 1/2$ [22]. For the energy density profile the state $|\phi\rangle = |\epsilon\rangle$ contains two fermionic excitations and is even under $P$. The surface energy exponent is then $x_\epsilon^s = 2$ [23, 24].

When $L \rightarrow \infty$ with a fixed $v/L$ value, one obtains the surface finite-size scaling behaviour $\phi_{\text{bd}}^b(L) \sim L^{-x_\phi^s}$ while keeping $v$ constant leads to the corner finite-size scaling behaviour

$$\phi_{\text{bd}}^b(L) \sim v^{x_\phi^s} L^{-2x_\phi^s} \quad v \ll L. \quad (17)$$
Thus the corner exponent $x_c^\phi(\pi/2)$ is given by $2x_c^\phi$. This result is in agreement with the general expression $x_c^\phi(\theta) = \pi x_c^\phi/\theta$ for a corner with opening angle $\theta$, which also follows from conformal invariance [19, 21].

The Laboratoire de Physique des Matériaux is Unité Mixte de Recherche CNRS No 7556.

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