TUNNELING FOR A CLASS OF DIFFERENCE OPERATORS: COMPLETE ASYMPTOTICS

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Abstract. We analyze a general class of difference operators $H_\varepsilon = T_\varepsilon + V_\varepsilon$ on $\ell^2((\varepsilon \mathbb{Z})^d)$, where $V_\varepsilon$ is a multi-well potential and $\varepsilon$ is a small parameter. We derive full asymptotic expansions of the prefactor of the exponentially small eigenvalue splitting due to interactions between two “wells” (minima) of the potential energy, i.e., for the discrete tunneling effect. We treat both the case where there is a single minimal geodesic (with respect to the natural Finsler metric induced by the leading symbol $h_0(x, \xi)$ of $H_\varepsilon$) connecting the two minima and the case where the minimal geodesics form an $\ell + 1$ dimensional manifold, $\ell \geq 1$. These results on the tunneling problem are as sharp as the classical results for the Schrödinger operator in [Helffer, Sjöstrand, 1984]. Technically, our approach is pseudodifferential and we adapt techniques from [Helffer, Sjöstrand, 1988] and [Helffer, Parisse, 1994] to our discrete setting.

1. Introduction

The aim of this paper is to derive complete asymptotic expansions for the interaction between two potential minima of a difference operator on a scaled lattice, i.e., for the discrete tunneling effect. We consider a rather general class of families of difference operators $(H_\varepsilon)_{\varepsilon > 0}$ on the Hilbert space $\ell^2((\varepsilon \mathbb{Z})^d)$, as the small parameter $\varepsilon > 0$ tends to zero. The operator $H_\varepsilon$ is given by

$$H_\varepsilon = T_\varepsilon + V_\varepsilon,$$

where $T_\varepsilon$ is given by

$$(\tau_\gamma u)(x) = u(x + \gamma), \quad (a_\gamma u)(x) := a_\gamma(x; \varepsilon)u(x) \quad \text{for} \quad x, \gamma \in (\varepsilon \mathbb{Z})^d$$

and $V_\varepsilon$ is a multiplication operator which in leading order is given by a multiwell-potential $V_0 \in \mathcal{C}^\infty(\mathbb{R}^d)$.

The interaction between neighboring potential wells leads by means of the tunneling effect to the fact that the eigenvalues and eigenfunctions are different from those of an operator with decoupled wells, which is realized by the direct sum of “Dirichlet-operators” situated at the several wells. Since the interaction is small, it can be treated as a perturbation of the decoupled system.

In [K., R., 2012], we showed that it is possible to approximate the eigenfunctions of the original Hamiltonian $H_\varepsilon$ with respect to a fixed spectral interval by (linear combinations of) the eigenfunctions of the several Dirichlet operators situated at the different wells and we gave a representation of $H_\varepsilon$ with respect to a basis of Dirichlet-eigenfunctions.

In [K., R., 2016] we gave estimates for the weighted $\ell^2$-norm of the difference between exact Dirichlet eigenfunctions and approximate Dirichlet eigenfunctions, which are constructed using the WKB-expansions given in [K., R., 2011].

In this paper, we consider the special case, that only Dirichlet operators at two wells have an eigenvalue (and exactly one) inside a given spectral interval. Then it is possible to compute complete asymptotic expansions for the elements of the interaction matrix and to obtain explicit formulæ for the leading order term.

This paper is based on the thesis [R., 2006]. It is the sixth in a series of papers (see [K., R., 2008 - K., R., 2016]); the aim is to develop an analytic approach to the semiclassical eigenvalue problem and tunneling for $H_\varepsilon$ which is comparable in detail and precision to the well known analysis for the Schrödinger operator (see [Simon, 1983] and [Helffer, Sjöstrand, 1984]). We remark that the analysis of tunneling has been extended to classes of pseudodifferential operators in $\mathbb{R}^d$ in [Helffer, 

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Parise, 1994] where tunneling is discussed for the Klein-Gordon and Dirac operator. This article in turn relies heavily on the ideas in the analysis of Harper’s equation in [Helffer, Sjöstrand, 1988] and previous results from [Sjöstrand, 1982] covering classes of analytic symbols. Since our formulation of the spectral problem for the operator in (1.1) is pseudo-differential in spirit, it has been possible to adapt the methods of [Helffer, Parise, 1994] to our case. Since our symbols are analytic only in the momentum variable \( \xi \), but not in the space variable \( x \), the results of [Sjöstrand, 1982] do not all automatically apply.

Our motivation comes from stochastic problems (see [K., R., 2008], [Bovier, Eckhoff, Gayrard, Klein, 2001], [Bovier, Eckhoff, Gayrard, Gayrard, Klein, 2002]). A large class of discrete Markov chains analyzed in [Bovier, Eckhoff, Gayrard, Klein, 2002] with probabilistic techniques falls into the framework of difference operators treated in this article.

We expect that similar results hold in the more general case that the Hamiltonian is a generator of a jump process in \( \mathbb{R}^d \), see [K., Léonard, R., 2014] for first results in this direction.

**Hypothesis 1.1**

1. The coefficients \( a_\gamma(x; \varepsilon) \) in (1.1) are functions

\[
a : (\varepsilon \mathbb{Z})^d \times \mathbb{R}^d \times (0, \varepsilon_0) \rightarrow \mathbb{R}, \quad (\gamma, x, \varepsilon) \mapsto a_\gamma(x; \varepsilon),
\]

satisfying the following conditions:

(i) They have an expansion

\[
a_\gamma(x; \varepsilon) = \sum_{k=0}^{N-1} \varepsilon^k a_\gamma^{(k)}(x) + R_\gamma^{(N)}(x; \varepsilon), \quad N \in \mathbb{N}^*,
\]

where \( a_\gamma \in \mathcal{C}^\infty(\mathbb{R}^d \times (0, \varepsilon_0)) \) and \( a_\gamma^{(k)} \in \mathcal{C}^\infty(\mathbb{R}^d) \) for all \( \gamma \in (\varepsilon \mathbb{Z})^d \) and \( 0 \leq k \leq N - 1 \).

(ii) \( \sum_{\gamma \in (\varepsilon \mathbb{Z})^d} a_\gamma^{(0)} = 0 \) and \( a_\gamma^{(0)} \leq 0 \) for \( \gamma \neq 0 \).

(iii) \( a_\gamma(x; \varepsilon) = a_{-\gamma}(x + \gamma; \varepsilon) \) for all \( x \in \mathbb{R}^d, \gamma \in (\varepsilon \mathbb{Z})^d \).

(iv) For any \( c > 0 \) and \( \alpha \in \mathbb{N}^d \) there exists \( C > 0 \) such that for \( 0 \leq k \leq N - 1 \) uniformly with respect to \( x \in \mathbb{R}^d \) and \( \varepsilon \in (0, \varepsilon_0] \).

\[
\| \frac{\varepsilon^k}{\alpha} \partial_x^\alpha a_\gamma^{(k)}(x) \|_{(\varepsilon \mathbb{Z})^d} \leq C \quad \text{and} \quad \| \frac{\varepsilon^k}{\alpha} \partial_x^\alpha R_\gamma^{(N)}(x) \|_{(\varepsilon \mathbb{Z})^d} \leq C\varepsilon^N.
\]

(v) \( \text{span}\{ \gamma \in (\varepsilon \mathbb{Z})^d \mid a_\gamma^{(0)}(x) < 0 \} = \mathbb{R}^d \) for all \( x \in \mathbb{R}^d \).

2. (i) The potential energy \( V_\varepsilon \) is the restriction to \( (\varepsilon \mathbb{Z})^d \) of a function \( \hat{V}_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}) \) which has an expansion

\[
\hat{V}_\varepsilon(x) = \sum_{t=0}^{N-1} \varepsilon^t V_t(x) + R_N(x; \varepsilon), \quad N \in \mathbb{N}^*.
\]

where \( V_t \in \mathcal{C}^\infty(\mathbb{R}^d), R_N \in \mathcal{C}^\infty(\mathbb{R}^d \times (0, \varepsilon_0]) \) for some \( \varepsilon_0 > 0 \) and for any compact set \( K \subset \mathbb{R}^d \) there exists a constant \( C_K \) such that \( \sup_{x \in K} \| R_N(x; \varepsilon) \| \leq C_K \varepsilon^N \).

(ii) \( V_\varepsilon \) is polynomially bounded and there exist constants \( R, C > 0 \) such that \( V_\varepsilon(x) > 0 \) for all \( x \geq R \) and \( \varepsilon \in (0, \varepsilon_0] \).

(iii) \( V_0(x) \geq 0 \) and it takes the value 0 only at a finite number of non-degenerate minima \( x^j, j \in \{1, \ldots, r\} \), which we call potential wells.

We remark that for \( T_\varepsilon \) defined in (1.1), under the assumptions given in Hypothesis 1.1 one has \( T_\varepsilon = \text{Op}_x^\varepsilon(t(\cdot, \cdot; \varepsilon)) \) (see Appendix A for definition and details of the quantization on the \( d \)-dimensional torus \( \mathbb{T}^d := \mathbb{R}^d/(2\pi \mathbb{Z})^d \)) where \( t \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{T}^d \times (0, \varepsilon_0]) \) is given by

\[
t(x, \xi; \varepsilon) = \sum_{\gamma \in (\varepsilon \mathbb{Z})^d} a_\gamma(x; \varepsilon) \exp\left(-\frac{i}{\varepsilon} \gamma \cdot \xi \right).
\]

Here \( t(x, \xi; \varepsilon) \) is considered as a function on \( \mathbb{R}^{d} \times (0, \varepsilon_0] \), which is \( 2\pi \)-periodic with respect to \( \xi \). By condition (a)(iv) in Hypothesis 1.1 the function \( \xi \mapsto t(x, \xi; \varepsilon) \) has an analytic continuation to \( \mathbb{C}^d \). Moreover for all \( B > 0 \)

\[
\sum_{\gamma} |a_\gamma(x; \varepsilon)| e^{B|x|} \leq C \quad \text{and thus} \quad \sup_{x \in \mathbb{R}^d} |a_\gamma(x; \varepsilon)| \leq C e^{-\frac{B|x|}{K}}
\]
uniformly with respect to $x$ and $\varepsilon$. We further remark that (a)(iv) implies $|a^{(k)}_\gamma(x) - a^{(k)}_\gamma(x + h)| \leq C|h|$ for $0 \leq k \leq N - 1$ uniformly with respect to $\gamma \in (\varepsilon\mathbb{Z})^d$ and $x, h \in \mathbb{R}^d$ and (a)(ii),(iii),(iv) imply that $T_\varepsilon$ is symmetric and bounded and that for some $C > 0$

$$
\langle u, T_\varepsilon u \rangle \geq -C\varepsilon\|u\|^2_{L^2}, \quad u \in \ell^2((\varepsilon\mathbb{Z})^d).
$$

Furthermore, we set

$$
t(x, \xi; \varepsilon) = \sum_{k=0}^{N-1} \varepsilon^k t_k(x, \xi) + \hat{t}_N(x, \xi; \varepsilon) \quad \text{with}
$$

$$
t_k(x, \xi) := \sum_{\gamma \in (\varepsilon\mathbb{Z})^d} a^{(k)}_\gamma(x)e^{-\frac{i}{\varepsilon}\gamma \cdot \xi}, \quad 0 \leq k \leq N - 1,
$$

$$
\hat{t}_N(x, \xi; \varepsilon) := \sum_{\gamma \in (\varepsilon\mathbb{Z})^d} R^{(N)}_\gamma(x; \varepsilon)e^{-\frac{i}{\varepsilon}\gamma \cdot \xi}.
$$

Thus, in leading order, the symbol of $H_\varepsilon$ is $h_0 := t_0 + V_0$. Combining (1.4) and (a)(iii) shows that the $2\pi$-periodic function $\mathbb{R}^d \ni \xi \mapsto t_0(x, \xi)$ is even with respect to $\xi \mapsto -\xi$, i.e.,

$$
a^{(0)}_\gamma(x) = a^{(-0)}_\gamma(x), \quad x \in \mathbb{R}^d, \gamma \in (\varepsilon\mathbb{Z})^d
$$

(see [K., R., 2008], Lemma 1.2) and therefore

$$
t_0(x, \xi) = \sum_{\gamma \in (\varepsilon\mathbb{Z})^d} a^{(0)}_\gamma(x) \cos\left(\frac{1}{\varepsilon}\gamma \cdot \xi\right).
$$

At $\xi = 0$, for fixed $x \in \mathbb{R}^d$ the function $t_0$ defined in (1.10) has by Hypothesis 1.1 a)(ii) an expansion

$$
t_0(x, \xi) = \langle \xi, B(x)\xi \rangle + \sum_{|\alpha| = 2n, n \geq 2} B_\alpha(x)\xi^\alpha \quad \text{as} \quad |\xi| \to 0
$$

where $\alpha \in \mathbb{N}^d$, $B \in C^{\infty}(\mathbb{R}^d, \mathcal{M}(d \times d, \mathbb{R}))$, for any $x \in \mathbb{R}^d$ the matrix $B(x)$ is positive definite and symmetric and $B_\alpha$ are real functions. By straightforward calculations one gets for $1 \leq \mu, \nu \leq d$

$$
B_{\nu\mu}(x) = -\frac{1}{2\varepsilon^2} \sum_{\gamma \in (\varepsilon\mathbb{Z})^d} a^{(0)}_\gamma(x)\gamma_\nu\gamma_\mu.
$$

We set

$$
\tilde{h}_0 : \mathbb{R}^{2d} \to \mathbb{R}, \quad \tilde{h}_0(x, \xi) := -t_0(x, i\xi) - V_0(x).
$$

In order to work in the context of [K., R., 2009], we shall assume

HYPOTHESIS 1.2 At the minima $x^j, j \in C$, of $V_0$, we assume that $t_0$ defined in (1.10) fulfills

$$
t_0(x^j, \xi) > 0 \quad \text{if} \quad \xi \in \mathbb{T}^d \setminus \{0\}.
$$

For any set $D \subset \mathbb{R}^d$, we denote the restriction to the lattice by $D_\varepsilon := D \cap (\varepsilon\mathbb{Z})^d$.

By Hypothesis 1.1, $\tilde{h}_0$ is even and hyperconvex with respect to momentum. We showed in [K., R., 2008], Prop. 2.9, that any function $f \in C^{\infty}(T^*M, \mathbb{R})$, which is hyperconvex in each fibre, is automatically hyperregular (here $M$ denotes a smooth manifold, which in our context is equal to $\mathbb{R}^d$).

We can thus introduce the associated Finsler distance $d = d_\varepsilon$ on $\mathbb{R}^d$ as in [K., R., 2008], Definition 2.16, where we set $\tilde{M} := \mathbb{R}^d \setminus \{x^k, k \in C\}$. Analog to [K., R., 2008], Theorem 1.6, it can be shown that $d$ is locally Lipschitz and that for any $j \in C$, the distance $d^j(x) := d(x, x^j)$ fulfills the generalized eikonal equation and inequality respectively

$$
\tilde{h}_0(x, \nabla d^j(x)) = 0, \quad x \in \Omega^j
$$

(1.16)

$$
\tilde{h}_0(x, \nabla d^j(x)) \leq 0, \quad x \in \mathbb{R}^d
$$

(1.17)

\footnote{For a normed vector space $V$ we call a function $L \in C^2(V, \mathbb{R})$ hyperconvex, if there exists a constant $\alpha > 0$ such that

$$
D^2L|_{v_0}(v, v) \geq \alpha\|v\|^2 \quad \text{for all} \quad v_0, v \in V.
$$

\footnote{We recall from e.g. [Abraham, Marsden, 1978] that $f$ is hyperregular if its fibre derivative $D_f f$ - related to the Legendre transform - is a global diffeomorphism: $T^*M \to TM$.}}
where $\Omega^j$ is some neighborhood of $x_j$. We remark that, assuming only Hypothesis 1.1 it is possible that balls of finite radius with respect to the Finsler distance, i.e. $B_r(x) := \{y \in \mathbb{R}^d \mid d(x, y) \leq r\}, r < \infty$, are unbounded in the Euclidean distance (and thus not compact). In this paper, we shall not discuss consequences of this effect.

Crucial quantities for the subsequent analysis are for $j, k \in C$
\[ S_{jk} := d(x^j, x^k), \quad \text{and} \quad S_0 := \min_{j \neq k} d(x^j, x^k). \quad (1.18) \]

**Remark 1.3** Since $d$ is locally Lipschitz-continuous (see [K., R., 2008]), it follows from (1.8) that for any $B > 0$ and any bounded region $\Sigma \subset \mathbb{R}^d$ there exists a constant $C > 0$ such that
\[ \sum_{\gamma \in \pi \Sigma} \left\| a_{(\cdot ; \varepsilon)} e^{\frac{d(x, \gamma)}{1 - \varepsilon}} \right\|_{L^\infty(\Sigma)} \leq C. \quad (1.19) \]

For $\Sigma \subset \mathbb{R}^d$ we define the space $\ell^2_{\Sigma_\varepsilon} := i\Sigma_\varepsilon (\ell^2(\Sigma_\varepsilon)) \subset \ell^2((\varepsilon \mathbb{Z})^d)$ where $i\Sigma_\varepsilon$ denotes the embedding via zero extension. Then we define the Dirichlet operator
\[ H^\Sigma_\varepsilon := 1_{\Sigma_\varepsilon} H_\varepsilon |_{\Sigma_\varepsilon} : \ell^2_{\Sigma_\varepsilon} \rightarrow \ell^2_{\Sigma_\varepsilon} \quad (1.20) \]
with domain $\mathcal{D}(H^\Sigma_\varepsilon) = \{u \in \ell^2_{\Sigma_\varepsilon} \mid V \varepsilon u \in \ell^2_{\Sigma_\varepsilon}\}$.

For a fixed spectral interval it is shown in [K., R., 2012] that the difference between the exact spectrum and the spectra of Dirichlet realizations of $H_\varepsilon$ near the different wells is exponentially small and determined by the Finsler distance between the two nearest neighboring wells. In the following we give additional assumptions.

The following hypothesis gives assumptions concerning the separation of the different wells using Dirichlet operators and the restriction to some adapted spectral interval $I_\varepsilon$.

**Hypothesis 1.4**
1. There exist constants $\eta > 0$ and $C > 0$ such that for all $x \in \mathbb{R}^d$
\[ \left\| a_{(\cdot)}(x; \varepsilon) e^{\frac{d(x, x^*)}{1 - \varepsilon}} \right\|_{L^\infty(\mathbb{R}^d)} \leq C. \]

2. For $j \in \mathcal{C}$, we choose a compact manifold $M_j \subset \mathbb{R}^d$ with $\mathcal{C}^2$-boundary such that the following holds:
   a. $x^j \in M_j$, $d^j \in \mathcal{C}^2(M_j)$ and $x^k \notin M_j$ for $k \in \mathcal{C}$, $k \neq j$.
   b. Let $X_{\tilde{h}_0}$ denote the Hamiltonian vector field with respect to $\tilde{h}_0$ defined in (1.15). $F_t$ denote the flow of $X_{\tilde{h}_0}$ and set
\[ \Lambda_+ := \{(x, \xi) \in T^*\mathbb{R}^d \mid F_t(x, \xi) \rightarrow (x^j, 0) \text{ for } t \rightarrow \mp \infty\}. \quad (1.21) \]

Then, for $\pi : T^*\mathbb{R}^d \rightarrow \mathbb{R}^d$ denoting the bundle projection $\pi(x, \xi) = x$, we have
\[ \Lambda_+(M_j) := \pi^{-1}(M_j) \cap \Lambda_+ = \{(x, \nabla d^j(x)) \in T^*\mathbb{R}^d \mid x \in M_j\}. \]

Moreover $\pi(F_t(x, \xi)) \in M_j$ for all $(x, \xi) \in \pi^{-1}(M_j) \cap \Lambda_+$ and all $t \leq 0$.

3. Given $M_j, j \in \mathcal{C}$, let $I_\varepsilon := [\alpha(\varepsilon), \beta(\varepsilon)]$ be an interval, such that $\alpha(\varepsilon), \beta(\varepsilon) = O(\varepsilon)$ for $\varepsilon \rightarrow 0$.

Furthermore there exists a function $a(\varepsilon) > 0$ with the property $\| \log a(\varepsilon) \| = o\left(\frac{1}{\varepsilon}\right)$, $\varepsilon \rightarrow 0$.

such that none of the operators $H_\varepsilon, H^M_\varepsilon, \ldots, H^M_M$ has spectrum in $[\alpha(\varepsilon) - 2a(\varepsilon), \alpha(\varepsilon)]$ or $(\beta(\varepsilon), \beta(\varepsilon) + 2a(\varepsilon))$. By [K., R., 2008], Theorem 1.5, the base integral curves of $X_{\tilde{h}_0}$ on $\mathbb{R}^d \setminus \{x_1, \ldots, x_m\}$ with energy 0 are geodesics with respect to $d$ and vice versa. Thus Hypothesis 1.4 (2) implies in particular that there is a unique minimal geodesic between any point in $M_j$ and $x^j$.

Clearly, $\Lambda_+(M_j)$ is a Lagrange manifold (by (2)) and since the flow $F_t$ preserves $\tilde{h}_0$, we have $\Lambda_+(M_j) \subset \pi^{-1}(0)$ by (1.21). Thus the eikonal equation $\tilde{h}_0(x, \nabla d^j(x)) = 0$ holds for $x \in M_j$. It follows from the construction of the solution of the eikonal equation in [K., R., 2011] that in fact $d^j \in \mathcal{C}^\infty(M_j)$. We recall that, in a small neighborhood of $x^j$, the equation $\xi = \nabla d^j$ parametrizes by construction the outgoing manifold $\Lambda_+$ of the hyperbolic fixed point $(x^j, 0)$ of $X_{\tilde{h}_0}$ in $T^*M_j$. Hypothesis 1.4 (2) ensures this globally.

Since the main theorems in this paper treat fine asymptotics for the interaction between two wells, we assume the following hypothesis. It guarantees that neither the wells are too far from each other nor the difference between the Dirichlet eigenvalues is to big (otherwise the main term of the
interaction matrix has the same order of magnitude as the error term).

Given Hypothesis [1.4], we assume in addition

**Hypothesis 1.5**

1. Only two Dirichlet operators $H_{\epsilon}^{M_j}$ and $H_{\epsilon}^{M_k}$, $j, k \in \mathcal{C}$, have an eigenvalue (and exactly one) in the spectral interval $I_{\epsilon}$, which we denote by $\mu_j$ and $\mu_k$ respectively, with corresponding real Dirichlet eigenfunctions $v_j$ and $v_k$.

2. We choose coordinates such that $x^j_0 < 0$ and $x^k_0 > 0$ and set
   \[
   \mathbb{H}_d := \{ x \in \mathbb{R}^d \mid x_d = 0 \} .
   \]

3. For
   \[
   S := \min_{r \in C} \min_{x \in \partial M_r} d(x, x') ,
   \]
   let $0 < a < 2S - S_0$ and $S_{jk} < S_0 + a$ and for all $\delta > 0$
   \[
   |\mu_j - \mu_k| = O \left( e^{- (\frac{a - \delta}{\epsilon})} \right) .
   \]
   We define the closed “ellipse”
   \[
   E := \{ x \in \mathbb{R}^d \mid d^j(x) + d^k(x) \leq S_0 + a \}
   \]
   and assume that $E \subset \mathcal{M}_j \cup \mathcal{M}_k$.

4. For $R > 0$ we set
   \[
   \mathbb{H}_{d,R} := \{ x \in \mathbb{R}^d \mid -R < x_d < 0 \}
   \]
   and choose $R > 0$ large enough such that
   \[
   E \cap \{ x \in \mathbb{R}^d \mid x_d \leq -R \} = \emptyset , \quad E \cap \mathbb{H}_{d,R} \subset \mathcal{M}_j \quad \text{and} \quad E \cap \mathbb{H}_{d,R} \subset \mathcal{M}_k .
   \]

The tunneling between the wells $x^j$ and $x^k$ can be described by the interaction term
   \[
   w_{jk} = \langle v_j, (1 - I_{M_k}) T_{\epsilon} v_k \rangle \}_{L^2} = \langle v_j, [T_{\epsilon}, I_{M_k}] v_k \rangle \}_{L^2}
   \]
introduced in [K., R., 2012], Theorem 1.5 (cf. Theorem B.1).

The main topic of this paper is to derive complete asymptotic expansions for $w_{jk}$, using the approximate eigenfunctions we constructed in [K., R., 2016].

**Remark 1.6**

1. Since the set $\mathbb{H}_{d,R}$ fulfills the assumptions on the set $\Omega$ introduced in [K., R., 2012], it follows from [K., R., 2012], Proposition 1.7, that the interaction $w_{jk}$ between the two wells $x^j$ and $x^k$ (cf. Theorem B.1) is given by
   \[
   w_{jk} = \langle [T_{\epsilon}, \mathbb{H}_{d,R}] I_{E} v_j, I_{E} v_k \rangle \}_{L^2} + O \left( e^{- \frac{S_0 + a - \eta}{\epsilon}} \right) , \quad \eta > 0 .
   \]
   In order to use symbolic calculus to compute asymptotic expansions of $w_{jk}$, we will smooth the characteristic function $1_{\mathbb{H}_{d,R}}$ by convolution with a Gaussian.

2. It follows from the results in [K., R., 2009] that, by Hypothesis [1.5], the Dirichlet eigenvalues $\mu_j$ and $\mu_k$ lie in $\epsilon$-intervals around some eigenvalues of the associated harmonic oscillators at the wells $x^j$ and $x^k$ as constructed in [K., R., 2016], (1.19). Thus we can use the approximate eigenfunctions and the weighted estimates given in [K., R., 2016], Theorem 1.7 and 1.8 respectively.

Next we give assumptions on the geometric setting, more precisely on the geodesics between the two wells given in Hypothesis 1.5. First we consider the generic setting, where there is exactly one minimal geodesic between the two wells. Later on, we consider the more general situation where the minimal geodesics build a manifold.

We recall from [K., R., 2008] that, as usual, geodesics are the critical points of the length functional of the Finsler structure induced by $\hat{h}_0$.

**Hypothesis 1.7** There is a unique minimal geodesic $\gamma_{jk}$ (with respect to the Finsler distance $d$) between the wells $x^j$ and $x^k$. Moreover, $\gamma_{jk}$ intersects the hyperplane $\mathbb{H}_d$ transversally at some point $y_0 = (y_0, 0)$ (possibly after redefining the origin) and is nondegenerate at $y_0$ in the sense that, transversally to $\gamma_{jk}$, the function $d^j + d^k$ changes quadratically, i.e., the restriction of $d^j(x) + d^k(x)$ to $\mathbb{H}_d$ has a positive Hessian at $y_0$. 

Remark 1.9

\[ \text{where we set Hypotheses 1.4, 1.5 and 1.7 are fulfilled. For \( m = j, k \), let} \ b^m \in \mathcal{E}_0^\infty(\mathbb{R}^d \times (0, \varepsilon_0))\text{ and} \ b^m_\ell \in \mathcal{E}_0^\infty(\mathbb{R}^d), \ell \in \mathbb{Z}/2, \ell \geq -N_m \text{ for some} \ N_m \in \mathbb{N} \text{ be such that the approximate eigenfunctions} \ \bar{\nu}_m^\varepsilon \in \ell^2((\varepsilon \mathbb{Z})^d) \text{ of the Dirichlet operators} \ H^\varepsilon_m \text{ constructed in} \ [K., R., 2016], \text{Theorem 1.7, have asymptotic expansions}
\]

\[ \bar{\nu}_m^\varepsilon(x; \varepsilon) = \varepsilon^d e^{-\frac{m(x)}{\varepsilon}} b^m(x; \varepsilon) \text{ with} \ b^m(x; \varepsilon) \sim \sum_{\ell \in \mathbb{Z}/2} \varepsilon^\ell b^m_\ell. \] \hspace{1cm} (1.29)

Then there is a sequence \((I_p)_{p \in \mathbb{N}/2}\) in \( \mathbb{R} \) such that

\[ w_{jk} \sim \varepsilon^\frac{1}{2} (-N_j + N_k) e^{-S_{jk}/\varepsilon} \sum_{p \in \mathbb{N}/2} \varepsilon^p I_p. \]

The leading order is given by

\[ I_0 = \frac{(2\pi)^{d+1}}{\sqrt{\det D^2_\eta(d^d + d^d)(y_0)}} b^k_{-N_k}(y_0) \sum_{\eta \in \mathbb{Z}^d} \bar{a}_\eta(y_0) \eta_d \sinh(\eta \cdot \nabla d^j(y_0)) b^j_{-N_j}(y_0) \] \hspace{1cm} (1.30)

where we set \( \bar{a}_\eta(x) := a^{(0)}_{\eta}(x) \) and

\[ D^2 \tilde{f} := \left( \partial_{x, \eta} \partial_{p, \xi} \right)_{1 \leq r, p \leq d-1}. \] \hspace{1cm} (1.31)

Remark 1.9

1. The sum on the right hand side of (1.30) is equal to the leading order of \( \frac{1}{\varepsilon} \text{Op}_\varepsilon^\dagger(w) \Psi b^j(y_0) \) where

\[ w(x, \xi) := \partial_{\xi, t_0}(x, \xi - i \nabla d^j(x)) = -i \sum_{\gamma \in (\mathbb{Z}^d)^d} a^{(0)}_{\gamma}(x) e^{-\frac{i}{\varepsilon} \gamma \cdot (\xi - i \nabla d^j(x))}. \]

To interpret this term (and formula (1.30)) semiclassically, observe that \( v(x, \xi) := \partial_{x, t_0}(x, \xi) \) is - by Hamilton’s equation - the velocity field associated to the leading order kinetic Hamiltonian \( t_0 \) (or Hamiltonian \( h_0 = t_0 + V_0 \)), evaluated on the physical phase space \( T^* \mathbb{R}^d \). In (1.32), with respect to the momentum variable, the phase space is pushed into the complex domain, over the region \( M_j \subset \mathbb{R}^d \) from Hypothesis 1.4

\[ T^* M_j \ni (x, \xi) \mapsto (x, \xi - i \nabla d^j(x)) \in \Lambda \subset T^* M_j \otimes \mathbb{C} \subset \mathbb{C}^{2d} \]

The smooth manifold \( \Lambda \) lies as a graph over \( T^* M_j \) and projects diffeomorphically. In some sense the complex deformation \( \Lambda \) structurally stays as close a possible to the physical phase space \( T^* M_j \), being both \( \mathbb{R} \)-symplectic and I-langrangian.

We recall the basic definitions (see [Sjöstrand, 1982] or [Helffer, Sjöstrand, 1986]): The standard symplectic form in \( \mathbb{C}^{2d} \) is \( \sigma = \sum_j d\zeta_j \wedge dz_j \) where \( z_j = x_j + iy_j \) and \( \zeta_j = x_j + i\eta_j \).
It decomposes into
\[
\Re \sigma = \frac{1}{2}(\sigma + \bar{\sigma}) = \sum_j d\xi_j \wedge dx_j - d\eta_j \wedge dy_j
\]
\[
\Im \sigma = \frac{1}{2}(\sigma - \bar{\sigma}) = \sum_j d\xi_j \wedge dy_j + d\eta_j \wedge dx_j.
\]

Both \(\Re \sigma\) and \(\Im \sigma\) are real symplectic forms in \(\mathbb{C}^{2d}\), considered as a real space of dimension \(4d\). A submanifold \(\Lambda\) of \(\mathbb{C}^{2d}\) (of real dimension \(2d\)) is called I-Lagrangian if it is Lagrangian for \(\Re \sigma\), and \(\Lambda\) is called \(\mathbb{R}\)-symplectic if \(\Re \sigma|_\Lambda\) - which denotes the pull back under the embedding \(\Lambda \hookrightarrow \mathbb{C}^{2d}\) - is non-degenerate. In our example, one checks in a straightforward way that both \(T^*M_j\) and \(\Lambda\) are \(\mathbb{R}\)-symplectic and I-Lagrangian. In this paper we shall not explicitly use this structure of \(\Lambda\) (it is essential for the microlocal theory of resonances, see \cite{Helffer, Sjöstrand, 1986}); rather, the manifold \(\Lambda\) appears somewhat mysteriously through explicit calculation.

Still, it seems to be physical folklore that both tunneling and resonance phenomena are related to complex deformations of phase space. Our formulae make this precise in the following sense: The leading order \(I_0\) of the tunneling is given by the velocity field \(v|_\Lambda\) (in the direction \(e_d\)) where \(\Lambda\) is the \(\mathbb{R}\)-symplectic, I-Lagrangian manifold obtained as deformation of \(T^*M_j\) through the field \(\nabla d^i\) induced by the Finsler distance \(d^i\), the leading amplitudes \(b^j_N(y_0), b^k_N(y_0)\) of the WKB expansions and the “hydrodynamical factor” \(\sqrt{\det D_+^2(d^i + d^k)(y_0)}\) describing deviations from the shortest path connecting the two potential minima.

Thus, in some sense, tunneling is described by a matrix element of a current (at least in leading order). On physical grounds it is perhaps very plausible that such formulae should hold in the semiclassical limit in any case which exhibits a leading order Hamiltonian. That this is actually true in the case of difference operators considered in this article is conceptually a main result of this paper. For pseudodifferential operators in \(\mathbb{R}^d\) this is proven in \cite{Helffer, Parisse, 1994}. For a less precise, but conceptually related, statement see \cite{K., R., 2012}.

(2) If \(\mu_j\) and \(\mu_k\) correspond to the ground state energy of the harmonic oscillators associated to the Dirichlet operators at the wells (see \cite{K., R., 2016}), we have \(N_j = N_k = 0\). Moreover \(b^j(y_0)\) and \(b^k(y_0)\) are strictly positive. Thus if \(\gamma_{jk}\) intersects \(\mathbb{H}_d\) orthogonal, it follows from Hypothesis 1.1, (1)(ii), that \(I_0 > 0\).

If there are finitely many geodesics connecting \(x^j\) and \(x^k\), separated away from the endpoints, their contributions to the interaction \(w_{jk}\) simply add up (as conductances working in parallel do). This is more complicated (but conceptually similar) in the case where the minimal geodesics form a manifold.

**Hypothesis 1.10** For some \(1 \leq \ell < d\), the minimal geodesics from \(x^j\) to \(x^k\) (with respect to the Finsler distance \(d\)) form an orientable \(\ell + 1\)-dimensional submanifold \(G\) of \(\mathbb{R}^d\) (possibly singular at \(x^j\) and \(x^k\)). Moreover \(G\) intersects the hyperplane \(\mathbb{H}_d\) transversally (possibly after redefining the origin). Then
\[
G_0 := G \cap \mathbb{H}_d
\]
\[
(1.33)
\]
is a \(\ell\)-dimensional submanifold of \(G\).

We shall show in Step 2 of the proof of Theorem 1.12 below (assuming only Hypothesis 1.10) that any system of linear independent normal vector fields \(N_m, m = \ell + 1, \ldots, d\), on \(G_0\) possesses an extension to a suitable tubular neighborhood of \(G_0\) as a family of commuting vector fields. In particular, with such a choice of vector fields \(N_m, m = \ell + 1, \ldots, d\),
\[
D^2_{1,G_0}(d^i + d^k) := \left( N_mN_n(d^i + d^k)|_{G_0}\right)_{0 \leq m, n \leq d - \ell + 1}
\]
\[
(1.34)
\]
is a symmetric matrix. We assume

**Hypothesis 1.11** The transverse Hessian \(D^2_{1,G_0}(d^i + d^k)\) of \(d^i + d^k\) at \(G_0\) defined in (1.34) is positive for all points on \(G\) (which we shortly denote as \(G\) being non-degenerate at \(G_0\)).
The leading order is given by

\[ w_{jk} \sim \varepsilon^{-(N_j + N_k)} \varepsilon^{(1 - \ell)/2} e^{-S_{jk}/\varepsilon} \sum_{p \in \mathbb{N}/2} \varepsilon^p I_p. \]

The leading order is given by

\[ I_0 = (2\pi)^{d-(\ell+1)/2} \int_{G_0} \frac{1}{\sqrt{\det D^2_{L,G_0}(d^j + d^k)}} b^k(y)b^j(y) \sum_{\eta \in \Gamma} \tilde{a}_{\eta}(y)\eta_d \sinh(\eta \cdot \nabla d^j(y)) d\sigma(y) \]

where we used the notation given in Theorem 1.8

We remark that - after appropriate complex deformations - an essential idea in the proof of Theorem 1.8 and Theorem 1.12 is to replace discrete sums by integrals up to a very small error and then apply stationary phase. This replacement of a sum by an integral is considerably more involved in the case of Theorem 1.12 and represents a main difficulty in the proof.

Concerning the case of the Schrödinger operator, results analogous to Theorem 1.12 certainly hold true, but to the best of our knowledge are not published (for the somewhat related case of resonances, see [Helffer, Sjöstrand, 1986]).

The outline of the paper is as follows.

Section 2 consists of preliminary results needed for the proofs of both theorems. The proofs of Theorem 1.8 and Theorem 1.12 are then given in Section 3 and Section 4 respectively. In Section 5 we give some additional results on the interaction matrix. Appendix A consists of some results for the symbolic calculus of periodic symbols. In Appendix B we recall a basic result from [K., R., 2012] about the tunneling where the interaction matrix \( w_{jk} \) is defined.

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2. Preliminary Results on the Interaction Term \( w_{jk} \)

Throughout this section we assume that Hypotheses 1.1, 1.2, and 1.4 are fulfilled and the interaction term \( w_{jk} \) is as defined in (1.27).

Following [Helffer, Sjöstrand, 1988] and [Helffer, Parisse, 1994], we set for some \( C_0 > 0 \)

\[ \phi_0(t) := IC_0 t^2 \quad \text{and} \quad \phi_s(t) := \phi_0(t - s), \quad s, t \in \mathbb{R} \quad (2.1) \]

and define the multiplication operator

\[ \pi_s(x) := \sqrt{C_0}e^{\frac{1}{2} \phi_s(xa)} = \sqrt{C_0}e^{-\frac{C_0}{\pi \varepsilon} (xa - s)^2}, \quad x \in \mathbb{R}^d \quad (2.2) \]

where the factor is chosen such that \( \int_{\mathbb{R}} \pi_s ds = 1 \).

Proposition 2.1

\[ w_{jk} = \int_{-R}^0 \langle [T_\varepsilon, \pi_s] 1_{E} v_j, 1_{E} v_k \rangle_{L^2} ds + O \left( e^{-\frac{2d + s_n}{\varepsilon}} \right), \quad \eta > 0. \quad (2.3) \]

Proof. By [K., R., 2012], Proposition 4.2, we get by arguments similar to those given in the proof of [K., R., 2012], Theorem 1.7, for all \( \eta > 0 \)

\[ w_{jk} = \langle 1_{E} v_j, [T_\varepsilon, \eta_M_{k}] 1_{E} v_k \rangle_{L^2} + O \left( e^{-\frac{2d + s_n}{\varepsilon}} \right). \]

Using \( \int_{\mathbb{R}} \pi_s ds = 1 \) this yields

\[ w_{jk} = \left\langle \int_{-R}^0 \pi_s ds 1_{E} v_j, [T_\varepsilon, \eta_M_{k}] 1_{E} v_k \right\rangle_{L^2} + A + B + O \left( e^{-\frac{2d + s_n}{\varepsilon}} \right) \quad (2.4) \]
where
\[
A := \left( \int_{-\infty}^{-\beta} \pi_s ds \right) E^{1v_j}, \quad \left[ T_\varepsilon, 1_{M_k} \right] \left( 1_{E^{1v_k}} \right)_{\ell^2}
\] and
\[
B := \left( \int_0^\infty \pi_s ds \right) E^{1v_j}, \quad \left[ T_\varepsilon, 1_{M_k} \right] \left( 1_{E^{1v_k}} \right)_{\ell^2}.
\]

By the assumptions on \(E\) and \(R\) in Hypothesis 1.5, we have \(A = 0\). In order to show that 
\(B = O \left( e^{-\frac{2\alpha_+ - \alpha_0}{\varepsilon}} \right)\), we use Lemma 5.1, telling us that for all \(C > 0\) and \(\delta > 0\)
\[
\left[ T_\varepsilon, 1_{M_k} \right] = 1_{\delta M_k} \left[ T_\varepsilon, 1_{M_k} \right] 1_{\delta M_k} + O \left( e^{-\frac{\varepsilon}{\pi}} \right)
\] (2.5)
where, for any \(A \subset \mathbb{R}^d\), we set
\[
\delta A := \{ x \in \mathbb{R}^d \mid \text{dist}(x, \partial A) \leq \delta \}.
\] (2.6)

Setting
\[
b_{\delta,k} := \min\{|x_d| \mid x \in E \cap \delta M_k\},
\] (2.7)
we write
\[
\int_0^\infty \pi_s ds \left[ 1_{E^{1\delta M_k}}(x) = e^{-\frac{C_0}{\varepsilon} b_{\delta,k}^2} e^{\frac{C_0}{\sqrt{\pi \varepsilon}} \int_0^\infty 1_{E^{1\delta M_k}}(x) e^{-\frac{C_0}{\sqrt{\pi \varepsilon}} ((x_d-s)^2 - b_{\delta,k}^2)} ds} \right. (2.8)
\]
Since \(\mathbb{H}^{d,-}\cap E \subset \delta M_k\) by Hypothesis 1.5, it follows that, for \(\delta > 0\) sufficiently small, \(x_d < 0\) for \(x \in E \cap \delta M_k\) and thus \(|x_d - s| \geq |x_d| \geq b_{\delta,k} > 0\) for \(s \geq 0\). Therefore the substitution \(s^2 = \frac{C_0}{\varepsilon} ((x_d - s)^2 - b_{\delta,k}^2)\) on the right hand side of (2.8) yields \(\frac{1}{\sqrt{\varepsilon}} ds \leq \frac{\varepsilon}{\sqrt{C_0}} dz\) and thus by straightforward calculation for some \(C_{\delta} > 0\)
\[
\sup_x \left( \int_0^\infty \pi_s ds \left[ 1_{E^{1\delta M_k}}(x) \right] \right) \leq C_{\delta} \varepsilon e^{-\frac{C_0}{\sqrt{\varepsilon}} b_{\delta,k}^2} \cdot (2.9)
\]
Combining (2.5) and (2.9) and using \(d^j(x) + d^k(x) \geq S_{jk}\) gives for all \(\delta > 0\)
\[
|B| \leq C_{\delta} \varepsilon e^{-\frac{C_0}{\sqrt{\varepsilon}} b_{\delta,k}^2} \cdot \left\| e^{\frac{d^j}{\varepsilon}} \left[ T_\varepsilon, 1_{M_k} \right] 1_{\delta M_k} v_k \right\|_{\ell^2}. (2.10)
\]
The definition of \(T_\varepsilon\) and \(1_{M_k} v_k = v_k\) yield \(\left[ T_\varepsilon, 1_{M_k} \right] v_k(x) = (1 - 1_{M_k})(x) \sum a(x; \varepsilon) v_k(x + \gamma)\). The triangle inequality \(d^j(x) \leq d(x, x + \gamma) + d^k(x + \gamma)\) and the Cauchy-Schwarz-inequality with respect to \(\gamma\) therefore give
\[
\left\| e^{\frac{d^j}{\varepsilon}} \left[ T_\varepsilon, 1_{M_k} \right] 1_{\delta M_k} v_k \right\|_{\ell^2} = \sum_{\gamma \in \mathbb{Z}^d} \sum_{\gamma \in \mathbb{Z}^d} \left( a(x; \varepsilon) e^{\frac{d^j(x)}{\varepsilon}} \right) (1_{\delta M_k} v_k)(x) \gamma \left\| e^{\frac{d^j(x)}{\varepsilon}} \right\|^2 \leq \sum_{\gamma \in \mathbb{Z}^d} \left( a(x; \varepsilon) e^{\frac{d^j(x)}{\varepsilon}} \right) (1_{\delta M_k} v_k)(x + \gamma) \left\| e^{\frac{d^j(x)}{\varepsilon}} \right\|^2 \leq \sum_{\gamma \in \mathbb{Z}^d} \left( a(x; \varepsilon) e^{\frac{d^j(x)}{\varepsilon}} \right) (1_{\delta M_k} v_k)(x + \gamma) \left\| e^{\frac{d^j(x)}{\varepsilon}} \right\|^2 \leq \frac{C}{\varepsilon} \left\| e^{\frac{d^j}{\varepsilon}} v_k \right\|_{\ell^2}. (2.11)
\]

where we set \(\langle \gamma \rangle_{\varepsilon} := \sqrt{\varepsilon^2 + |\gamma|^2}\). By Hypothesis 1.4 for \(\eta > 0\) chosen consistently, the first factor on the right hand side of (2.11) is bounded by some constant \(C > 0\) uniformly with respect to \(x\). Changing the order of summation therefore yields
\[
\left\| e^{\frac{d^j}{\varepsilon}} \left[ T_\varepsilon, 1_{M_k} \right] 1_{\delta M_k} v_k \right\|_{\ell^2} \leq C \sum_{\gamma \in \mathbb{Z}^d} \left( \langle \gamma \rangle_{\varepsilon} \right)^{-(d + \eta)} \left| \sum_{x \in M_k^\delta \cap \delta M_k} e^{\frac{d^j(x)}{\varepsilon}} (1_{\delta M_k} v_k)(x + \gamma) \right|^2 \leq \tilde{C} \left\| e^{\frac{d^j}{\varepsilon}} v_k \right\|_{\ell^2}. (2.12)
\]

We now insert (2.12) into (2.10) and use that, by K. R., 2016, Proposition 3.1, the Dirichlet eigenfunctions decay exponentially fast, i.e. there is a constant \(N_0 \in \mathbb{N}\) such that \(\left\| e^{\frac{d^j}{\varepsilon}} v_i \right\|_{\ell^2} \leq e^{-N_0}\) for \(i = j, k\). This gives for any \(\eta > 0\)
\[
|B| \leq C e^{-\frac{1}{2}((C_0 b_{\delta,k} + S_{j,k} - \eta)}.
\]

Since \(b_{\delta,k} > 0\) we can choose \(C_0\) such that \(C_0 b_{\delta,k} + S_{j,k} \geq S_0 + \alpha\), showing that \(B = O \left( e^{-\frac{2\alpha_+ - \alpha_0}{\varepsilon}} \right)\) for \(C_0\) sufficiently large and therefore by (2.4)
\[
w_{jk} = \left( \int_{-\varepsilon}^0 \pi_s ds \right) E^{1v_j}, \quad \left[ T_\varepsilon, 1_{M_k} \right] \left( 1_{E^{1v_k}} \right)_{\ell^2} + O \left( e^{-\frac{2\alpha_+ - \alpha_0}{\varepsilon}} \right). (2.13)
\]
In order to get the stated result, we use the symmetry of $T_\varepsilon$ to write
\[
\left< \int_{-R}^R \pi_s ds \mathbf{1}_E v_j, [T_\varepsilon, \mathbf{1}_{M} \mathbf{1}_E v_k] \right>_{\ell^2} = \left< T_\varepsilon \int_{-R}^R \pi_s ds \mathbf{1}_E v_j, \mathbf{1}_E v_k \right>_{\ell^2} - \left< \int_{-R}^R \pi_s ds \mathbf{1}_E v_j, \mathbf{1}_{M_k} T_\varepsilon \mathbf{1}_E v_k \right>_{\ell^2} = \left< [T_\varepsilon, \int_{-R}^R \pi_s ds] \mathbf{1}_E v_j, \mathbf{1}_E v_k \right>_{\ell^2} + \sum_{i=1}^5 R_i 
\] 
where by commuting $T_\varepsilon$ and inserting $\mathbf{1}_{M_j} + \mathbf{1}_{M_j}$ in $R_2$ and $R_3$ we have
\[
R_1 := \left< \int_{-R}^R \pi_s ds [T_\varepsilon, \mathbf{1}_E] v_j, \mathbf{1}_E v_k \right>_{\ell^2} 
R_2 := \left< \int_{-R}^R \pi_s ds \mathbf{1}_E \mathbf{1}_{M_j} T_\varepsilon v_j, \mathbf{1}_E v_k \right>_{\ell^2} 
R_3 := \left< \int_{-R}^R \pi_s ds \mathbf{1}_E \mathbf{1}_{M_j} T_\varepsilon v_j, \mathbf{1}_E v_k \right>_{\ell^2} 
R_4 := - \left< \int_{-R}^R \pi_s ds \mathbf{1}_E v_j, \mathbf{1}_E \mathbf{1}_{M_k} T_\varepsilon v_k \right>_{\ell^2} 
R_5 := - \left< \int_{-R}^R \pi_s ds \mathbf{1}_E v_j, \mathbf{1}_{M_k} [T_\varepsilon, \mathbf{1}_E] v_k \right>_{\ell^2} .
\]
We are now going to prove that $\sum_i R_i = O\left( e^{-\frac{n+\eta}{\varepsilon}} \right)$ for all $\eta > 0$.
Since $\mathbf{1}_E(x)(\mathbf{1}_E(x + \gamma) - \mathbf{1}_E(x))$ is equal to $-1$ for $x \in E, x + \gamma \in E^{c}$ and zero otherwise, we have
\[
|R_1| = \left| \sum_{x, \gamma \in (\mathbb{Z})^d} \int_{-R}^0 \pi_s ds v_k(x) a_\gamma(x; \varepsilon)v_j(x + \gamma)\mathbf{1}_E(x)(\mathbf{1}_E(x + \gamma) - \mathbf{1}_E(x)) \right| 
= \left| \sum_{x, \gamma \in (\mathbb{Z})^d} \int_{-R}^0 \pi_s ds v_k(x) a_\gamma(x; \varepsilon)v_j(x + \gamma)\mathbf{1}_E(x)\mathbf{1}_{E^{c}}(x + \gamma) \right| .
\] 
Using for the first step that $d^1(x + \gamma) + d^k(x + \gamma) \geq S_0 + a$ for $x + \gamma \in E^{c}$ and for the second step the triangle inequality for $d$, we get
\[
\text{rhs} (2.15) \leq e^{-\frac{n+\eta}{\varepsilon}} \sum_{x, \gamma \in (\mathbb{Z})^d} \left| \int_{-R}^0 \pi_s ds e^{\frac{d^1(x + \gamma)}{\varepsilon}} v_k(x) a_\gamma(x; \varepsilon)e^{\frac{d^k(x + \gamma)}{\varepsilon}} v_j(x + \gamma)\mathbf{1}_E(x)\mathbf{1}_{E^{c}}(x + \gamma) \right| 
\leq e^{-\frac{n+\eta}{\varepsilon}} \sum_{x \in (\mathbb{Z})^d} \left| \left( \int_{-R}^0 \pi_s ds e^{\frac{d^1}{\varepsilon}} \mathbf{1}_E v_k \right)(x) \right| \sum_{\gamma \in (\mathbb{Z})^d} \left| a_\gamma(x; \varepsilon)e^{\frac{d^k(x + \gamma)}{\varepsilon}} (e^{\frac{d^1}{\varepsilon}} \mathbf{1}_E v_j)(x + \gamma) \right| 
\leq e^{-\frac{n+\eta}{\varepsilon}} \left\| e^{\frac{d^1}{\varepsilon}} v_k \right\|^2_{\ell^2} \left( \sum_{x \in (\mathbb{Z})^d} \sum_{\gamma \in (\mathbb{Z})^d} a_\gamma(x; \varepsilon)e^{\frac{d^1(x + \gamma)}{\varepsilon}} \left( e^{\frac{d^1}{\varepsilon}} v_j \right)(x + \gamma) \right)^2 \right)^{1/2} .
\] 
where in the last step we used the Cauchy-Schwarz-inequality with respect to $x$ and $\int_{R} \pi_s ds = 1$. By Cauchy-Schwarz-inequality with respect to $\gamma$ analog to (2.11) and (2.12) we get
\[
\sum_{x \in (\mathbb{Z})^d} \left( \sum_{y \in (\mathbb{Z})^d} a_\gamma(x; \varepsilon)e^{\frac{d^1(x + \gamma)}{\varepsilon}} \left( e^{\frac{d^1}{\varepsilon}} v_j \right)(x + \gamma) \right)^2 
= \sum_{x \in (\mathbb{Z})^d} \left( \sum_{y \in (\mathbb{Z})^d} |a_\gamma(x; \varepsilon)e^{\frac{d^1(x + \gamma)}{\varepsilon}} (\gamma)(d + \eta)/2)^2 \right) \left( \sum_{y \in (\mathbb{Z})^d} |e^{\frac{d^1(x + \gamma)}{\varepsilon}} u_k)(x + \gamma)(\gamma)^{-d+\eta/2} |^2 \right) 
\leq C \left\| e^{\frac{d^1}{\varepsilon}} v_j \right\|^2_{\ell^2} .
\] 
Inserting (2.17) into (2.16) gives by (2.15) together with [K., R., 2016], Proposition 3.1, for any $\eta > 0$
\[
|R_1| \leq C e^{-\frac{n+\eta}{\varepsilon}} \left\| e^{\frac{d^1}{\varepsilon}} v_k \right\|^2_{\ell^2} \left\| e^{\frac{d^1}{\varepsilon}} v_j \right\|_{\ell^2} \leq C e^{-\frac{n+\eta}{\varepsilon}}.
\] 
Analog arguments show
\[
|R_5| = O\left( e^{-\frac{n+\eta}{\varepsilon}} \right) .
\]
We analyze $|R_2 + R_4|$ together, writing
\[
|R_2 + R_4| \leq \sum_{x \in (\mathbb{Z})^d} \int_{-R}^0 \pi_x \, ds \, 1_E(x) \left| v_k(x) (1_{M_j} T_x v_j)(x) - v_j(x) (1_{M_k} T_x v_k)(x) \right|.
\]

Now using that
\[
v_k 1_{M_j} T_x v_j - v_j 1_{M_k} T_x v_k + V_x v_j v_k - V_x v_j v_k = v_k H_{\epsilon}^M v_j - v_j H_{\epsilon}^M v_k = (\mu_j - \mu_k) v_j v_k
\]
we get by Hypothesis 1.5 Cauchy-Schwarz-inequality and since $\partial^\gamma (x) + \partial^\delta (x) \geq S_{jk}$
\[
|R_2 + R_4| \leq |\mu_j - \mu_k| e^{-\frac{\rho_{jk}}{2}} \sum_{x \in (\mathbb{Z})^d} \int_{-R}^0 \pi_x \, ds \, 1_E(x) \left| e^{\frac{\partial^\gamma (x)}{2}} v_j(x) e^{\frac{\partial^\delta (x)}{2}} v_k(x) \right|
\]
\[
\leq e^{-\frac{\rho_{jk}}{4}} \left\| e^{\frac{\partial^\gamma}{2}} v_j(x) \right\|_2 \left\| e^{\frac{\partial^\delta}{2}} v_k \right\|_2
\]
\[
\leq C e^{-\frac{\rho_{jk}}{4}}
\]
(2.20)
where in the last step we used again [K., R., 2016], Proposition 3.1, and $S_{jk} \geq S_0$.

The term $|R_3|$ can be estimated by methods similar to those used to estimate $|B|$ above. By Hypothesis 1.5 we have $E \cap H_{\epsilon, R} \subset M_j$. Thus $x_d > 0$ for $x \in E \cap M_j$ and, setting $b_j := \min\{|x_d| \mid x \in E \cap M_j\}$, we have $|x_d - s| \geq |x_d| \geq b_j > 0$ for $s \leq 0$. Thus we get analog to (2.8) and (2.9)
\[
\sup_x \left| \int_{-R}^0 \pi_x \, ds \, 1_{E \cap M_j^c}(x) \right| \leq C e^{-\frac{C_0}{2} b_j^2}
\]
(2.21)
and similar to (2.10), using Cauchy-Schwarz-inequality,
\[
|R_3| \leq C e^{-\frac{1}{4}(C_0 b_j^2 + S_{jk})} \left\| e^{\frac{\partial^\gamma}{2}} v_k \right\|_2 \left\| e^{\frac{\partial^\delta}{2}} T_x v_j \right\|_2
\]
(2.22)
As in (2.11) and (2.12), we estimate the last factor in (2.22) as
\[
\left\| e^{\frac{\partial^\gamma}{2}} T_x v_j \right\|_2^2 = \sum_{x \in (\mathbb{Z})^d} \left| \sum_{\gamma \in (\mathbb{Z})^d} a_{\gamma}(x; \epsilon) e^{\frac{\partial^\gamma (x + \gamma)}{2}} v_j(\gamma) \right|^2
\]
\[
\leq \sum_{x \in (\mathbb{Z})^d} \left( \sum_{\gamma \in (\mathbb{Z})^d} \left| a_{\gamma}(x; \epsilon) e^{\frac{\partial^\gamma (x + \gamma)}{2}} v_j(\gamma) \right|^2 \right)\left( \sum_{\gamma \in (\mathbb{Z})^d} \left| e^{\frac{\partial^\gamma (x + \gamma)}{2}} v_j(\gamma) \right|^2 \right)
\]
\[
\leq C \sum_{\gamma \in (\mathbb{Z})^d} \left| \gamma \right| e^{-\frac{1}{4}(d + n)} \sum_{x \in (\mathbb{Z})^d} \left| e^{\frac{\partial^\gamma (x + \gamma)}{2}} v_j(\gamma + 2) \right|^2 \leq \tilde{C} \left\| e^{\frac{\partial^\gamma}{2}} v_j \right\|_2^2.
\]

Thus choosing $C_0$ such that $C_0 b_j^2 + S_{jk} \geq S_0 + a$, we get again by [K., R., 2016], Proposition 3.1, for any $\eta > 0$
\[
|R_3| \leq C e^{-\frac{1}{4}(C_0 b_j^2 + S_{jk})} \left\| e^{\frac{\partial^\gamma}{2}} v_k \right\|_2 \left\| e^{\frac{\partial^\delta}{2}} T_x v_j \right\|_2 \leq C e^{-\frac{1}{4}(S_0 + a - n)}.
\]
(2.23)
Inserting (2.23), (2.20), (2.19) and (2.18) into (2.14) yields (2.3) by (2.13) and interchanging of integration and summation.

In the next step we analyze the commutator in (2.3) using symbolic calculus.

PROPOSITION 2.2 For any $u \in \ell^2((\mathbb{Z})^d)$ compactly supported and $x \in (\mathbb{Z})^d$ we have with the notation $\xi = (\xi', \xi_d) \in \mathbb{T}^d$
\[
\left[ T_x, \pi_x \right] u(x) = \sqrt{\frac{C_0}{\pi^d}} (2\pi)^{-d} \sum_{y \in (\mathbb{Z})^d} e^{\frac{i}{\pi^d} (\phi_s(y_d) + \phi_s(x_d))} u(y)
\]
\[
\times \int_{[-\pi, \pi]^d} e^{i(y - x)\xi} \left( t(x, \xi', \xi_d - \frac{1}{2} \phi_s' \frac{T_x d + y_d}{2}; \epsilon) - t(x, \xi', \xi_d + \frac{1}{2} \phi_s' \frac{T_x d + y_d}{2}; \epsilon) \right) d\xi
\]
(2.24)
where $\phi_s(t) = \frac{d}{\pi^d} \phi_s(t) = 2iC_0(t - s)$ and $T_x = \text{Op}_x^\mathcal{T}$ as given in (A.4).
Proof. By Definition A.1(4), we have

\[(T_επ_su)(x) = \frac{\sqrt{C_0}}{\sqrt{πε}}(2π)^{-d} \sum_{y \in (εz)^d} u(y) \int_{[-π,π]^d} e^{\frac{i}{ε}((y-x)ξ+φ_s(yd))}t(x,ξ;ε)\,dξ \quad (2.25)\]

\[(π_sT_εu)(x) = \frac{\sqrt{C_0}}{\sqrt{πε}}(2π)^{-d} \sum_{y \in (εz)^d} u(y) \int_{[-π,π]^d} e^{\frac{i}{ε}((y-x)ξ+φ_s(xd))}t(x,ξ;ε)\,dξ \quad (2.26)\]

Setting

\[ξ_± := (ξ',ξ_± = \frac{1}{2}φ_s(\frac{x_δ+y_d}{2}) ) \quad (2.27)\]

we have

\[(y-x)ξ + φ_s(yd) = (y-x)ξ_± + \frac{1}{2}(φ_s(yd) + φ_s(xd)) \quad (2.28)\]

\[(y-x)ξ + φ_s(xd) = (y-x)ξ_± + \frac{1}{2}(φ_s(yd) + φ_s(xd)) \]

In fact,

\[(y-x)ξ_± + \frac{1}{2}(φ_s(yd) + φ_s(xd)) = (y-x)ξ_±(yd-xd)iC_0(\frac{x_δ+y_d}{2}-s) + \frac{C_0i}{2}((yd-s)^2+(xd-s)^2). \quad (2.29)\]

Writing \(yd-xd = (yd-s) - (xd-s)\) and \(\frac{x_δ+y_d}{2}-s = \frac{1}{2}((xd-s) + (yd-s))\) gives

\[
\text{rhs}(2.29) = (y-x)ξ_± \pm \frac{iC_0}{2}((yd-s)^2-(xd-s)^2) + \frac{C_0i}{2}((yd-s)^2+(xd-s)^2)
\]

\[
= \begin{cases} 
(y-x)ξ_± + φ_s(yd) & \text{for } + \\
(y-x)ξ_± + φ_s(xd) & \text{for } - 
\end{cases}
\]

Since, with respect to ξ, t has an analytic continuation to \(C^d\), it is possible to combine the integrals in (2.25) and (2.26) using the contour deformation given by the substitution (2.27). To this end, we first need the following Lemma

**Lemma 2.3** Let \(f : C \to C\) be analytic in \(Ω_b := \{z \in C | ∃z < b\}\), for some \(b > 0\) and \(2π\)-periodic on the real axis, i.e., \(f(x+2π) = f(x)\) for all \(x \in R\). Then for any \(a < b\)

\[
\int_{-π+i/2}^{π+i/2} f(z) \, dz = \int_{-π}^{π} f(x) \, dx.
\]

**Proof of Lemma 2.3** If \(f\) is periodic on the real line, it follows that \(f(z) = f(z+2π)\) for \(z \in Ω_b\), by the identity theorem. Then Cauchy’s Theorem yields

\[
\int_{-π+i/2}^{π+i/2} f(z) \, dz - \int_{-π}^{π} f(z) \, dz = \int_{-π}^{π} f(z) \, dz + \int_{-π+i/2}^{π+i/2} f(z) \, dz.
\]

The substitution \(z = z+2π\) in the last integral on the right hand side of (2.30) gives by the periodicity of \(f\)

\[
\text{rhs}(2.30) = \int_{-π}^{π} f(z) \, dz + \int_{-π+i/2}^{π+i/2} f(\bar{z}+2π) \, d\bar{z}
\]

\[
= \int_{-π}^{π} f(z) \, dz + \int_{-π+i/2}^{π+i/2} f(\bar{z}) \, d\bar{z} = 0,
\]

proving the stated result. \(\Box\)

We come back to the proof of Proposition 2.2. For shortening the notation we set

\[
a := \frac{1}{2}φ_0'\left(\frac{x_δ+y_d}{2}\right) = C_0\left(\frac{x_δ+y_d}{2}-s\right).
\]

(2.31)
Inserting the substitution (2.27) in (2.25), we get by (2.28) and (2.31)

\[
(T_\varepsilon \pi_s u)(x) = \sqrt{\frac{C_0}{\varepsilon^d}} (2\pi)^{-d} \sum_{y \in (\varepsilon Z)^d} u(y) \int \left. \left[ e^{\frac{i}{\varepsilon} \left( (y-x)^2 + \frac{1}{4} (\phi_s(y) + \phi_0(x)) \right) t(x, \xi; \varepsilon) \right] \right| d\xi
\]

where in the last step we used Lemma 2.3

By analog arguments for (2.26) we get

\[
(\pi_s T_\varepsilon u)(x) = \sqrt{\frac{C_0}{\varepsilon^d}} (2\pi)^{-d} \sum_{y \in (\varepsilon Z)^d} u(y) \int \left. \left[ e^{\frac{i}{\varepsilon} \left( (y-x)^2 + \frac{1}{4} (\phi_s(y) + \phi_0(x)) \right) t(x, \xi'; \varepsilon) \right] \right| d\xi
\]

and thus combining (2.32) and (2.33) gives (2.24).

\[\square\]

The idea is now to write the \( s \)-dependent terms in (2.24) as \( s \)-derivative of some symbol. To this end, we first introduce some smooth cut-off functions on the right hand side of (2.3).

Let \( \chi_R \in C_0^\infty(\mathbb{R}) \) be such that \( \chi_R(s) = 1 \) for \( s \in [-R, R] \) and \( \chi_E \in C_0^\infty(\mathbb{R}^d) \) such that \( \chi_E(x) = 1 \) for \( x \in E \). Moreover we assume that \( \chi_R(s) = \chi_R(-s) \) and \( \chi_E(x) = \chi_E(-x) \). Then it follows directly from Proposition 2.1 that

\[
w_{jk} = \int_{-R}^{0} \left( \chi_R(s) \right) \left[ T_\varepsilon \pi_s \chi_E 1_{E^c} v_j \right] \chi_E 1_{E^c} v_k \, ds + O \left( e^{-\frac{2n+s-n}{\varepsilon}} \right), \quad \eta > 0. \tag{2.34}
\]

**Proposition 2.4** There are compactly supported smooth mappings

\[
\mathbb{R} \ni s \mapsto q_s \in S_0^1(1)(\mathbb{R}^{2d} \times T^d) \quad \text{and} \quad \mathbb{R} \ni s \mapsto r_s \in S_0^\infty(1)(\mathbb{R}^{2d} \times T^d)
\]

such that \( q_s(x, y, \xi; \varepsilon) \) and \( r_s(x, y, \xi; \varepsilon) \) have analytic continuations to \( \mathbb{C}^d \) with respect to \( \xi \in \mathbb{R}^d \) (identifying functions on \( T^d \) with periodic functions on \( \mathbb{R}^d \)). Moreover, \( q_s \) has an asymptotic expansion

\[
q_s(x, y, \xi; \varepsilon) \sim \sum_{n=0}^\infty \varepsilon^n q_{n,s}(x, y, \xi). \tag{2.35}
\]

and, setting \( \sigma := \frac{x+iy}{2} - s \),

\[
\chi_R(s) \chi_E(x) \chi_E(y) e^{-\frac{C_0}{\varepsilon^2} \sigma^2} \left[ t(x, \xi', \xi_d - iC_0 \sigma; \varepsilon) - t(x, \xi', \xi_d + iC_0 \sigma; \varepsilon) \right]
\]

\[
= \partial_\sigma \left[ e^{-\frac{C_0}{\varepsilon^2} \sigma^2} q_s(x, y, \xi; \varepsilon) \right] + e^{-\frac{C_0}{\varepsilon^2} \sigma^2} r_s(x, y, \xi; \varepsilon). \tag{2.36}
\]

**Proof.** We first remark that by (1.7)

\[
t(x, \xi', \xi_d - iC_0 \sigma; \varepsilon) - t(x, \xi', \xi_d + iC_0 \sigma; \varepsilon)
\]

\[
= \sum_{\gamma \in (\varepsilon Z)^d} a_{\gamma}(x, \varepsilon) e^{-\frac{i}{2} \gamma' \xi} \left[ e^{-\frac{C_0}{\varepsilon^2} \gamma_d} - e^{-\frac{C_0}{\varepsilon^2} \gamma_d} \right]
\]

\[
= \sum_{\gamma \in (\varepsilon Z)^d} a_{\gamma}(x, \varepsilon) e^{-\frac{i}{2} \gamma \xi} 2 \sinh \left( \frac{\gamma_d}{\varepsilon} C_0 \sigma \right). \tag{2.37}
\]

Thus from the assumptions on \( \chi_R \) and \( \chi_E \) it follows that the left hand side of (2.36) is odd with respect to \( \sigma \mapsto -\sigma \). Modulo \( S_0^\infty \), (2.36) is equivalent to

\[
\chi_R(s) \chi_E(x) \left[ t(x, \xi', \xi_d - iC_0 \sigma; \varepsilon) - t(x, \xi', \xi_d + iC_0 \sigma; \varepsilon) \right] = (2C_0 \sigma + \varepsilon \partial_\sigma) q_s(x, y, \xi; \varepsilon). \tag{2.38}
\]
Here $q$ is compactly supported in $x, y$ and $s$ (and thus in $\sigma$) and $q$ is even with respect to $\sigma \mapsto -\sigma$ since $\partial_\sigma = -\partial_\sigma$. We set

$$g_s(x, y, \xi; \varepsilon) := \chi_R(s) \chi_E(x) \chi_E(y) \frac{1}{2C_0 \sigma} \left( t(x, \xi', \xi_d - iC_0 \sigma; \varepsilon) - t(x, \xi', \xi_d + iC_0 \sigma; \varepsilon) \right)$$

(2.39)

$$= \sum_{\ell=0}^\infty \varepsilon^\ell g_{\ell, s}(x, y, \xi)$$

where by (2.37)

$$g_{\ell, s}(x, y, \xi) := -\chi_R(s) \chi_E(x) \chi_E(y) \sum_{\gamma \in \{\varepsilon Z\}^d} a_\gamma^{(\ell)}(x) \varepsilon^{-\gamma} e^{-\varepsilon^\gamma} \frac{1}{C_0 \sigma} \sinh \left( \frac{\gamma d}{\varepsilon} C_0 \sigma \right).$$

(2.40)

Then (2.38) can be written as

$$\left( 1 + \frac{\varepsilon}{2C_0 \sigma} \partial_\sigma \right) g_s(x, y, \xi; \varepsilon) = g_s(x, y, \xi; \varepsilon).$$

(2.41)

Formally (2.41) leads to the von-Neumann-series

$$q_s(x, y, \xi; \varepsilon) = \sum_{m=0}^\infty \varepsilon^m \left( -\frac{1}{2C_0 \sigma} \partial_\sigma \right)^m g_s(x, y, \xi; \varepsilon).$$

(2.42)

Using (2.35), (2.39) and Cauchy-product, (2.42) gives

$$q_{n, s}(x, y, \xi) = \sum_{\ell+m=n} \left( -\frac{1}{2C_0 \sigma} \partial_\sigma \right)^m g_{\ell, s}(x, y, \xi).$$

(2.43)

By (2.39) $g$ and $g_\ell$, $\ell \in \mathbb{N}$, are even with respect to $\sigma \mapsto -\sigma$. Moreover, the operator $\frac{1}{\varepsilon} \partial_\sigma = -\frac{1}{\sigma} \partial_\sigma$ maps a monomial in $\sigma$ of order $2m$ to a monomial of order $\max\{0, 2m-2\}$. Thus, for $x, y \in \text{supp} \chi_E$ and $s \in [-R, R]$, the right hand side of (2.43) is well-defined and analytic and even in $\sigma$ for any $n \in \mathbb{N}$. In particular, it is bounded at $\sigma = 0$ or equivalently at $s = \frac{x_d + y_d}{2}$. Therefore $q_{n, s} \in S_0^0(1)(\mathbb{R}^{2d} \times T^d)$ for any $n \in \mathbb{N}$ and it is $C_0^\infty$ with respect to $s \in \mathbb{R}$.

By a Borel-procedure with respect to $\varepsilon$ there exists a symbol $q_s \in S_0^0(1)(\mathbb{R}^{2d} \times T^d)$ which is $C_0^\infty$ as a function of $s \in \mathbb{R}$ such that (2.35) holds. Moreover, $\partial_\sigma q_s(x, y, \xi; \varepsilon)$ is analytic in $\xi$ by uniform convergence of the Borel procedure and the analyticity of $q_{n, s}$. Thus (2.36) holds for some $r_s \in S_0^0(1)(\mathbb{R}^{2d} \times T^d)$ and since the left hand side of (2.38) has an analytic continuation to $\mathbb{C}^d$ with respect to $\xi$, the same is true for $r_s(x, y, \xi; \varepsilon)$.

$\square$

We remark that by (2.43) and (2.40), the leading order term $q_0$ at the point $s = \frac{x_d + y_d}{2}$ is given by

$$q_{0, \frac{x_d + y_d}{2}}(x, y, \xi) = -\chi_R \left( \frac{x_d + y_d}{2} \right) \chi_E(x) \chi_E(y) \sum_{\gamma \in \{\varepsilon Z\}^d} a_\gamma^{(0)}(x) \varepsilon^{\gamma} e^{-\varepsilon^\gamma}$$

(2.44)

$$\frac{1}{i} \chi_E(y) \chi_E(x) \partial_{\xi_d} t_0(x, \xi)$$

where in the second step we used (1.11) and the fact that $\chi_R \left( \frac{x_d + y_d}{2} \right) = 1$ for $x, y \in \text{supp} \chi_E$.

We now define the operators $Q_s$ and $R_s$ on $L^2((\varepsilon Z)^d)$ by

$$Q_s u(x) := \sqrt{\frac{C_0}{\varepsilon \pi}} (2\pi)^{-d} \sum_{y \in (\varepsilon Z)^d} e^{\frac{i}{2\varepsilon} (\phi(x, y_d) + \phi(y, x_d))} u(y) \int_{\mathbb{R}^d} e^{i \varepsilon (y-x) \xi} q_{s, \frac{x_d + y_d}{2}}(x, y, \xi; \varepsilon) d\xi$$

(2.45)

$$R_s u(x) := \sqrt{\frac{C_0}{\varepsilon \pi}} (2\pi)^{-d} \sum_{y \in (\varepsilon Z)^d} e^{\frac{i}{2\varepsilon} (\phi(x, y_d) + \phi(y, x_d))} u(y) \int_{\mathbb{R}^d} e^{i \varepsilon (y-x) \xi} r_{s, \frac{x_d + y_d}{2}}(x, y, \xi; \varepsilon) d\xi$$

(2.46)

Then we get the following formula for the interaction term $w_{jk}$.

**Proposition 2.5** For $Q_s$ given in (2.45), the interaction term is given by

$$w_{jk} = \varepsilon (Q_0 1_{E} v_j, 1_{E} v_k) e^2 + O \left( \varepsilon^\infty e^{-\frac{i}{\varepsilon} S_{jk}} \right).$$

(2.47)
Proof. We first remark that by the definition (2.1) of $\phi_s$ we have
\[
\frac{i}{2\varepsilon}(\phi_s(y_d) + \phi_s(x_d)) = -\frac{C_0}{\varepsilon} \left[ \left( \frac{x_d + y_d}{2} - s \right)^2 + \frac{1}{4}(y_d - x_d)^2 \right].
\] (2.48)
Combining Proposition 2.2 with Proposition 2.4 and (2.48) gives
\[
\chi_R(s)\chi_E[T_\varepsilon, \pi_s] \chi_E 1_{E} v_j(x) = \sqrt{\frac{C_0}{\varepsilon^2}} (2\pi)^{-d} \sum_{y \in (\varepsilon^2)\mathbb{Z}^d} 1_{E}(y)v_j(y) e^{-\frac{C_0}{\varepsilon^2}(y_d - x_d)^2} 
\times \int_{T^d} e^{i(y - x)\xi} \partial_\xi Q_s + R_s 1_{E} v_j(x)
= \varepsilon \langle \partial_\xi Q_s + R_s, 1_{E} v_j \rangle
\] (2.49)
where the second equation follows from the definitions (2.45) and (2.46). Thus by (2.34) we get for any $\eta > 0$
\[
w_{jk} = \int_{-R}^{0} \langle (\varepsilon \partial_\xi Q_s + R_s) 1_{E} v_j, 1_{E} v_k \rangle ds + O\left( e^{-\frac{s_0 + \eta}{\varepsilon}} \right) \tag{2.50}
= \varepsilon \langle Q_s 1_{E} v_j, 1_{E} v_k \rangle - S_1 + S_2 + O\left( e^{-\frac{s_0 + \eta}{\varepsilon}} \right),
\]
where
\[
S_1 := \varepsilon \langle Q_s 1_{E} v_j, 1_{E} v_k \rangle, \tag{2.51}
S_2 := \varepsilon \int_{-R}^{0} \langle R_s 1_{E} v_j, 1_{E} v_k \rangle ds \tag{2.52}
\]
To analyse $S_2$, we first introduce the following notation, which will be used again later on. We set
(see Definition A.1)
\[
\tilde{u}_s(x) := e^{\frac{i\varepsilon}{2\pi} \phi_s(x_d)} u(x) = e^{-\frac{C_0}{2\pi^2}(x_d - s)^2} u(x) \tag{2.53}
\]
\[
\tilde{Q}_s := \tilde{O}_{\pi} T \left( \frac{C_0}{\pi} q_s \right), \tag{2.54}
\]
\[
\tilde{R}_s := \tilde{O}_{\pi} T \left( \frac{C_0}{\pi} r_s \right), \tag{2.55}
\]
then
\[
\varepsilon \langle Q_s u, v \rangle_{\ell^2} = \langle Q_s \tilde{u}_s, \tilde{v}_s \rangle_{\ell^2} \quad \text{and} \quad \varepsilon \langle R_s u, v \rangle_{\ell^2} = \langle \tilde{R}_s \tilde{u}_s, \tilde{v}_s \rangle_{\ell^2}. \tag{2.56}
\]
To analyse $S_2$ we write, using (2.56)
\[
|S_2| = \left| \int_{-R}^{0} \left| \left( e^{\frac{i\varepsilon}{2\pi} \tilde{R}_s e^{-\frac{i\varepsilon}{2\pi} \tilde{Q}_s} e^{-\frac{C_0}{2\pi^2} \tilde{Q}_s^2} 1_{E} \tilde{v}_{j,s}, e^{\frac{i\varepsilon}{2\pi} 1_{E} \tilde{v}_{k,s}} \right) \right| ds \right|
\leq e^{-\frac{s_{jk}}{2}} \int_{-R}^{0} \left\| e^{\frac{i\varepsilon}{2\pi} \tilde{R}_s e^{-\frac{i\varepsilon}{2\pi} \tilde{Q}_s} e^{-\frac{C_0}{2\pi^2} \tilde{Q}_s^2} 1_{E} \tilde{v}_{j,s}} \right\| \left\| e^{\frac{i\varepsilon}{2\pi} 1_{E} \tilde{v}_{k,s}} \right\| ds \tag{2.57}
\]
Since $r_s \in S_0^\infty(1)(\mathbb{R}^{2d} \times T^d)$, it follows from Corollary A.6 together with Proposition A.7 that for some $C > 0$
\[
|S_2| \leq C\varepsilon e^{-\frac{s_{jk}}{2}} \int_{-R}^{0} \left\| e^{\frac{i\varepsilon}{2\pi} \tilde{Q}_s} 1_{E} \tilde{v}_{j,s} \right\| \left\| e^{\frac{i\varepsilon}{2\pi} 1_{E} \tilde{v}_{k,s}} \right\| ds
\leq O\left( e^{-\frac{s_{jk}}{2}} \right) \tag{2.58}
\]
where for the second step we used weighted estimates for the Dirichlet eigenfunctions given in [K., R., 2016], Proposition 3.1, together with the fact that $|\tilde{u}_s(x)| \leq |u(x)|$.

By (2.51) and (2.56) we get
\[
|S_1| = \left\| \left( \tilde{Q}_s 1_{E} \tilde{v}_{j,-R}, 1_{E} \tilde{v}_{k,-R} \right) \right\| \leq \left\| 1_{E} \tilde{Q}_s 1_{E} \tilde{v}_{j,-R} \right\| \left\| 1_{E} \tilde{v}_{k,-R} \right\| \tag{2.59}
\]
Again by Corollary A.6 together with (2.53), (2.54) and since $q_s \in S_0^\infty(1)(\mathbb{R}^{2d} \times T^d)$ we have for some $C > 0$
\[
|S_1| \leq C\varepsilon \left\| 1_{E} e^{-\frac{C_0}{2\pi^2}(\cdot + R)^2} v_j \right\| \left\| 1_{E} e^{-\frac{C_0}{2\pi^2}(\cdot + R)^2} v_k \right\| \leq \sqrt{\varepsilon} C e^{-\frac{C_0}{2\pi^2} R^2} \tag{2.60}
\]
for $R_E := \min_{x \in E} |x_d - R|$. Thus taking $R$ large enough such that $R_E > S_{jk}$ and inserting (2.60) and (2.58) in (2.50) proves the proposition.

In the next proposition we show that, modulo a small error, the interaction term only depends on a small neighborhood of the point or manifold respectively where the geodesics between $x^j$ and $x^k$ intersect $\mathbb{H}_d$. Since the proof is analogical, we discuss the point and manifold case simultaneously.

**Proposition 2.6** Let $\Psi \in C^\infty_c(M_j \cap M_k \cap E)$ denote a cut-off function near $y_0 \in \mathbb{H}_d$ (or $G_0 \subset \mathbb{H}_d$ respectively) such that $\Psi = 1$ in a neighborhood $U_\Psi$ of $y_0$ (or $G_0$ respectively) and for some $C > 0$

$$\frac{C_0}{2} x_d^2 + d^j(x) + d^k(x) - S_{jk} > C, \quad x \in \text{supp}(1 - \Psi).$$

Then, for the restriction $\Psi^\varepsilon := r_\varepsilon \Psi$ of $\Psi$ to the lattice $(\varepsilon \mathbb{Z})^d$ (see (A.7)),

$$w_{jk} = \varepsilon \langle Q_0 \Psi^\varepsilon v_j, \Psi^\varepsilon v_k \rangle_{L^2} + O\left(\varepsilon^\infty e^{-\frac{1}{2}S_{jk}}\right).$$

**Proof.** Using Proposition 2.5 and the notation (2.53), (2.54) together with (2.56) we have

$$w_{jk} = \left(\hat{Q}_0 1_E \hat{v}_{j,0}, 1_E \hat{v}_{k,0}\right)_{L^2} + O\left(\varepsilon^\infty e^{-\frac{1}{2}S_{jk}}\right)$$

where, using $1_E \Psi = \Psi$,

$$R_1 = \left(\hat{Q}_0 (1 - \Psi^\varepsilon) 1_E \hat{v}_{j,0}, \Psi^\varepsilon \hat{v}_{k,0}\right)_{L^2}$$

$$R_2 = \left(\hat{Q}_0 \Psi^\varepsilon \hat{v}_{j,0}, (1 - \Psi^\varepsilon) 1_E \hat{v}_{k,0}\right)_{L^2}$$

$$R_3 = \left(\hat{Q}_0 (1 - \Psi^\varepsilon) 1_E \hat{v}_{j,0}, (1 - \Psi^\varepsilon) 1_E \hat{v}_{k,0}\right)_{L^2}.$$ 

To estimate $|R_1|$ we write

$$|R_1| = \left|\left\langle e^{-\frac{1}{2}(d^j + d^k)}(1 - \Psi^\varepsilon) e^\frac{d^j}{\varepsilon} 1_E \hat{v}_{j,0}, \chi_E e^{\frac{d^k}{\varepsilon}} e^\frac{d^k}{\varepsilon} \Psi^\varepsilon \hat{v}_{k,0}\right\rangle_{L^2}\right|$$

$$\leq \left\|e^{-\frac{1}{2}(d^j + d^k)}(1 - \Psi^\varepsilon) e^\frac{d^j}{\varepsilon} \hat{v}_{j,0}\right\|_{L^2} \left\|\chi_E e^{\frac{d^k}{\varepsilon}} e^\frac{d^k}{\varepsilon} \Psi^\varepsilon \hat{v}_{k,0}\right\|_{L^2}$$

where $\chi_E$ denotes a cut-off function as introduced above Proposition 2.4. Since by (2.54)

$$\hat{Q}_0^* = \hat{Q}_0^T \left(\sqrt{C_0 \varepsilon^\frac{d}{\varepsilon} \hat{q}_0^*}\right) \text{ for } \hat{q}_0^*(x, y, \xi; \varepsilon) = \hat{q}_0(y, x, \xi; \varepsilon) \in S_0^0(1)(\mathbb{R}^{2d} \times \mathbb{T}^d),$$

it follows from Proposition A.7 that $\chi_E e^{\frac{d^k}{\varepsilon}} e^\frac{d^k}{\varepsilon} \hat{q}_0^*$ is the 0-quantization of a symbol $q_0, d^k, 0 \in S_0^0(1)(\mathbb{R}^{2d} \times \mathbb{T}^d)$. Thus by Corollary A.6 and (2.61), for some $C, C' > 0$,

$$|R_1| \leq e^{-\frac{S_{jk}}{2}} C' \left\|e^\frac{d^j}{\varepsilon} v_j\right\|_{L^2} \left\|e^\frac{d^k}{\varepsilon} v_k\right\|_{L^2} = O\left(\varepsilon^\infty e^{-\frac{S_{jk}}{2}}\right)$$

where the last estimate follows from (K., R., 2016), Proposition 3.1. Similar arguments show $|R_2| = O(\varepsilon^\infty e^{-\frac{S_{jk}}{2}}) = |R_3|$, thus by (2.63) this finishes the proof.

In the next step, we show that modulo the same error term, the Dirichlet eigenfunctions $v_m, m = \ldots, j, k,$ can be replaced by the approximate eigenfunctions $\hat{v}_m$ given in (1.29). We showed in (K., R., 2016), Theorem 1.7, that for some smooth functions $b^m, b^m_0$, compactly supported in a neighborhood of $\tilde{M}_m$, the approximate eigenfunctions $\hat{v}_m \in L^2((\varepsilon \mathbb{Z})^d)$ are given by the restrictions to $(\varepsilon \mathbb{Z})^d$ of

$$\hat{v}_m := \varepsilon^\frac{d}{2} e^{-\frac{S_{jk}}{2}} b^m,$$

(2.68)

(using the notation in (K., R., 2016), these restrictions are $\hat{v}_{m,1,0}$). In (K., R., 2016), Theorem 1.8 we proved that for any $K$ compactly supported in $M_m$ the estimate

$$\left\|e^\frac{d^j}{\varepsilon} (v_m - \hat{v}_m)\right\|_{L^2(K)} = O(\varepsilon^\infty).$$

(2.69)

hold. Using (2.69) we get the following Proposition.
Proposition 2.7 Let $\tilde{\varphi}_n \in L^2((\varepsilon \mathbb{Z})^d)$, $m = j, k$, denote the approximate eigenfunctions of $H_\varepsilon$ in $M_m$ constructed in [K., R., 2016], Theorem 1.7, then, for $\Psi^\varepsilon$ as defined in Proposition 2.6,

$$w_{jk} = \varepsilon \langle Q_0 \Psi^\varepsilon \tilde{\varphi}_j, \Psi^\varepsilon \tilde{\varphi}_k \rangle_{L^2} + O\left(\varepsilon^\infty e^{-\frac{1}{2} S_{jk}}\right).$$ \hspace{1cm} (2.70)

Proof. By Proposition 2.6

$$w_{jk} = \varepsilon \langle Q_0 \Psi^\varepsilon \tilde{\varphi}_j, \Psi^\varepsilon \tilde{\varphi}_k \rangle_{L^2} + \varepsilon \langle Q_0 \Psi^\varepsilon (v_j - \tilde{\varphi}_j), \Psi^\varepsilon v_k \rangle_{L^2} + \varepsilon \langle Q_0 \Psi^\varepsilon (v_k - \tilde{\varphi}_k), \Psi^\varepsilon (v_k - \tilde{\varphi}_k) \rangle_{L^2} + O\left(\varepsilon^\infty e^{-\frac{1}{2} S_{jk}}\right).$$ \hspace{1cm} (2.71)

Using the notation (2.53), (2.54) with $\tilde{u} := \tilde{u}_0$ together with (2.56), we can write

$$|\varepsilon \langle Q_0 \Psi^\varepsilon (v_j - \tilde{\varphi}_j), \Psi^\varepsilon v_k \rangle_{L^2}| = |\langle \tilde{Q}_0 \Psi^\varepsilon (\tilde{v}_j - \tilde{\varphi}_j), \Psi^\varepsilon \tilde{v}_k \rangle_{L^2}|$$

$$= |\langle \chi \varepsilon^{-\frac{1}{2}} \tilde{Q}_0 e^{\frac{1}{\varepsilon}(v_j - \tilde{\varphi}_j)} e^{\frac{1}{\varepsilon}(v_j - \tilde{v}_j)}, e^{\frac{1}{\varepsilon}(v_j - \tilde{v}_j)} \rangle_{L^2}| \leq e^{-\frac{1}{2} S_{jk}} \chi \varepsilon^{-\frac{1}{2}} \tilde{Q}_0 e^{\frac{1}{\varepsilon}(v_j - \tilde{\varphi}_j)} \langle \| \Psi^\varepsilon e^{\frac{1}{\varepsilon}(v_j - \tilde{\varphi}_j)} \rangle_{L^2}, \| \Psi^\varepsilon e^{\frac{1}{\varepsilon}(v_k - \tilde{\varphi}_j)} \rangle_{L^2} \rangle_{L^2}, \quad (2.72)$$

where, analogous to (2.67), the last estimate follows from Proposition 2.4 together with Corollary A.6 for the operator $\chi \varepsilon^{-\frac{1}{2}} \tilde{Q}_0 e^{\frac{1}{\varepsilon}(v_j - \tilde{\varphi}_j)}$. Since $\Psi^\varepsilon$ is compactly supported in $M_j$, we get by (2.69) for any $N \in \mathbb{N}$

$$\| \Psi^\varepsilon e^{\frac{1}{\varepsilon}(v_j - \tilde{\varphi}_j)} \|_{L^2} \leq \| \Psi^\varepsilon e^{\frac{1}{\varepsilon}(v_j - \tilde{v}_j)} \|_{L^2} = O(\varepsilon^N). \hspace{1cm} (2.73)$$

Since by [K., R., 2016], Proposition 3.1

$$\| \Psi^\varepsilon e^{\frac{1}{\varepsilon}(v_k - \tilde{\varphi}_j)} \|_{L^2} \leq C e^{-N_0} \hspace{1cm} (2.74)$$

for some $C > 0$, $N_0 \in \mathbb{N}$, we can conclude by inserting (2.74) and (2.73) in (2.72)

$$|\varepsilon \langle Q_0 \Psi^\varepsilon (v_j - \tilde{\varphi}_j), \Psi^\varepsilon v_k \rangle_{L^2}| = O\left(\varepsilon^\infty e^{-\frac{1}{2} S_{jk}}\right). \hspace{1cm} (2.75)$$

Analog arguments show

$$|\varepsilon \langle Q_0 \Psi^\varepsilon (v_k - \tilde{\varphi}_k), \Psi^\varepsilon (v_k - \tilde{\varphi}_k) \rangle_{L^2}| = O\left(\varepsilon^\infty e^{-\frac{1}{2} S_{jk}}\right). \hspace{1cm} (2.76)$$

Inserting (2.75) and (2.76) in (2.71) gives (2.70).

Proposition 2.7 together with (2.68), (2.53) and (2.56) lead at once to the following corollary.

Corollary 2.8 For $b^j, b^k \in C_0^\infty(\mathbb{R}^d \times (0, \varepsilon_0])$ as given in (1.29), $\Psi$ as defined in Proposition 2.6 and the restriction map $\nu_{\varepsilon}$ given in (A.7) we have

$$w_{jk} = \varepsilon^\frac{4}{\varepsilon} e^{-\frac{1}{2} S_{jk}} \langle \tilde{Q}_0 \nu_{\varepsilon} \Psi b^j, e^{-\frac{1}{\varepsilon}(v_k - \tilde{\varphi}_j)} \rangle_{L^2} + O\left(\varepsilon^\infty e^{-\frac{1}{2} S_{jk}}\right) \hspace{1cm} (2.77)$$

where for $\tilde{Q}_0$ defined in (2.54) we set

$$\varphi(x) := d^i(x) + d^k(x) + C_0 |x_d|^2 - S_{jk} \hspace{1cm} (2.78)$$

$$\tilde{Q}_0 := \varepsilon^\frac{1}{2} e^{\frac{1}{\varepsilon}(v_j - \tilde{\varphi}_j)} e^{-\frac{1}{\varepsilon}(v_j - \tilde{v}_j)} e^{-\frac{1}{\varepsilon}(v_j - \tilde{\varphi}_j)\varepsilon}. \hspace{1cm} (2.79)$$

Remark 2.9 (1) Setting $\psi(x) = \frac{1}{\varepsilon^{2}} C_0 S_{d}^2 + \frac{1}{\varepsilon} d^i(x)$, it follows from Proposition A.7 together with (2.79) and (2.54) that the operator $\tilde{Q}_0$ is the 0-quantization of a symbol $\tilde{q}_0 \in S_0^2(1(\mathbb{R}^d \times \mathbb{T}^d))$, which has an asymptotic expansion, in particular

$$\tilde{Q}_0 = O_{\varepsilon}(\tilde{q}_0), \quad \tilde{q}_0(x, \xi; \varepsilon) \sim \varepsilon^{\frac{1}{2}} \sum_{n=0}^{\infty} \varepsilon^n \tilde{q}_{0,n}(x, \xi). \hspace{1cm} (2.80)$$

Modulo $S^3_0(1(\mathbb{R}^d \times \mathbb{T}^d))$, the symbol $\tilde{q}_0$ is given by

$$\varepsilon^{\frac{1}{2}} \tilde{q}_{0,0}(x, \xi; \varepsilon) = \sqrt{\frac{\varepsilon C_0}{\pi}} q_0(x, \xi - \xi \nabla d^i(x) - iC_0 x_d e_d) \hspace{1cm} (2.81)$$
where \( c_d \) denotes the unit vector in \( d \)-direction (see Proposition 2.4). At the intersection point or intersection manifold, i.e. for \( y = y_0 \) or \( y \in G_0 \) respectively, by (2.44) the leading order of the symbol is given by

\[
\varepsilon^2 \hat{q}_0 \psi(y, \xi) = \frac{1}{i} \sqrt{\frac{\varepsilon C_0}{\pi}} \partial_{\xi^j} t_0 (y, \xi - i \nabla d^j(y)) = -\sqrt{\frac{\varepsilon C_0}{\pi}} \sum_{\eta \in \mathbb{Z}^d} \hat{\eta}(y) \eta e^{-i \eta \cdot (\xi - i \nabla d^j(y))} \]  

(2.82)

where \( \hat{\eta} = \hat{a}(0) \) for \( \eta \in \mathbb{Z}^d \).

(2) By Corollary 2.8 we can write

\[
w_{jk} = \varepsilon^d e^{-\frac{s}{\varepsilon}} \sum_{x \in (\varepsilon \mathbb{Z})^d} e^{-\frac{\omega(x)}{\varepsilon}} (\hat{Q}_0 r_2 \Psi b^j)(x)(\Psi b^k)(x) + O\left(\varepsilon^\infty e^{-\frac{s}{\varepsilon}}\right). \]

(2.83)

(3) In the setting of Hypothesis 1.10 we have \( \varphi|_{G_0} = 0 \) and moreover, since \( d^j + d^k \) is minimal on \( G_0 \), \( \nabla \varphi|_{G_0} = 0 \) and \( \varphi(x) > 0 \) for \( x \in \text{supp} \Psi \setminus G_0 \).

3. Proof of Theorem 1.8

A key element of the proofs of both theorems is replacing the sum on the right hand side of (2.83) by an integral, up to a small error. Here we follow arguments from [di Gesù, 2012], telling us the following: Let \( a \in \mathcal{C}^\infty_c(\mathbb{R}^n, \mathbb{R}) \) and \( \psi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \) be such that \( \psi(x_0) = 0 \), \( D^2 \psi(x_0) > 0 \) and \( \psi(x) > 0 \) for \( x \in \text{supp} a \setminus \{x_0\} \) for some \( x_0 \in \mathbb{R}^n \). Then there exists a sequence \((J_k)_{k \in \mathbb{N}} \) in \( \mathbb{R} \) such that

\[
\varepsilon^d \sum_{x \in (\varepsilon \mathbb{Z})^d} a(x) e^{-\frac{\psi(x)}{\varepsilon}} \sim \sum_{k=0}^\infty \varepsilon^k J_k \quad \text{where} \quad J_0 = \frac{1}{\sqrt{\det D^2 \psi(x_0)}}. \]

(3.1)

We observe that the proof of (3.1) for \( a(x) \) being independent of \( \varepsilon \) immediately generalizes to an asymptotic expansion \( a(x, \varepsilon) \sim \sum \varepsilon^k a_k(x) \).

In order to apply (3.1) to the right hand side of (2.83) we have to verify the assumptions above for \( \psi = \varphi \) defined in (2.78) and for some \( a \in \mathcal{C}^\infty_0 \) which is equal to \( \Psi b^k(\hat{Q}_0 r_2 \Psi b^j) \) on \((\varepsilon \mathbb{Z})^d \) and has an asymptotic expansion in \( \varepsilon \).

It follows directly from its definition that \( \varphi(y_0) = 0 \). Since \( d^j(x) + d^k(x) - S_{jk} > 0 \) in \( E \setminus \gamma_{jk} \) by triangle inequality and \( z_2^j > 0 \) for all \( x \in \gamma_{jk} \), \( x \neq y_0 \), it follows that \( \varphi(x) > 0 \) for \( x \in \text{supp} \Psi \setminus \{y_0\} \).

To see the positivity of \( D^2 \varphi(y_0) \) we first remark that by Hypothesis 1.7 \( d^j + d^k \), restricted to \( \mathbb{H}_d \), has a positive Hessian at \( y_0 \), which we denote by \( D^2_1(d^j + d^k)(y_0) \). Since furthermore \( d^j + d^k \) is constant along the geodesic, it follows that the full Hessian \( D^2(d^j + d^k)(y_0) \) has \( d - 1 \) positive eigenvalues and the eigenvalue zero. The Hessian of \( C_0 x_2^2 \) at \( y_0 \) is diagonal and the only non-zero element is \( \partial^2_2(C_0 x_2^2) = 2C_0 > 0 \). Thus the Hessian \( D^2 \varphi(y_0) \) is a non-negative quadratic form. In order to show that it is in fact positive, we analyze its determinant. Writing the last column as the sum \( \nabla \partial_d(d^j + d^k)(y_0) + v \) where \( v_k = 0 \) for \( 1 \leq k \leq d - 1 \) and \( v_d = 2C_0 \) we get

\[
\det D^2 \varphi(y_0) = \det D^2(d^j + d^k)(y_0) + \det \begin{pmatrix} D^2_1(d^j + d^k)(y_0) & 0 \\ \ast & 2C_0 \end{pmatrix} > 0 \]

(3.2)

where the second equality follows from the fact that one eigenvalue of \( D^2(d^j + d^k)(y_0) \) is zero as discussed above and thus its determinant is zero. This proves that \( D^2 \varphi(y_0) \) is non-degenerate and thus we get \( D^2 \varphi(y_0) > 0 \).

By Proposition A.2, Remark A.3 and (2.80) the operator \( \hat{Q}_0 = \text{Op}_\varepsilon(\hat{q}_0 \psi) \) on \( L^2((\varepsilon \mathbb{Z})^d) \) (multiplied from the right by the restriction operator \( r_\varepsilon \)) is equal to the restriction of the operator \( \text{Op}_\varepsilon(\hat{q}_0 \psi) \) on \( L^2(\mathbb{R}^d) \). Here we consider \( \hat{q}_0 \psi \) as periodic element of the symbol class \( S^0_\varepsilon(1)(\mathbb{R}^d \times \mathbb{R}^d) \). In particular, for \( x \in (\varepsilon \mathbb{Z})^d \) we have

\[
\Psi b^k(x) \hat{Q}_0 r_\varepsilon \Psi b^j(x) = \Psi b^k(x) \text{Op}_\varepsilon(\hat{q}_0 \psi) \Psi b^j(x) \]

(3.3)

where \( r_\varepsilon \) denotes the restriction to the lattice \((\varepsilon \mathbb{Z})^d \) defined in (A.7). We therefore set

\[
a(x; \varepsilon) := \Psi b^k(x) \text{Op}_\varepsilon(\hat{q}_0 \psi) \Psi b^j(x), \quad x \in \mathbb{R}^d. \]

(3.4)
where the last equality follows from the analyticity of $\hat{a}$ (1.30). Note that all $I_a = \Psi$. The functions $\hat{q}_{n,\psi}(x)$ are the coefficients of the expansion of $\hat{q}(x, \cdot)$ into a convergent power series in $\varepsilon$ at zero.

Thus we can apply (3.1) to (2.83), which gives

$$w_{jk} \sim e^{-\frac{\varepsilon}{2} J_0} \sum_{k=0}^{\infty} \varepsilon^k J_k$$

(3.6)

where $J_0$ is the leading order term of

$$J_0 = \frac{(2\pi)^d}{\sqrt{\det D^2\varphi(y_0)}} b^k(y_0) (\text{Op}_\varepsilon(\hat{q}_\psi)\Psi b^j)(y_0; \varepsilon).$$

(3.7)

By (2.82) it follows that

$$\tilde{q}_{0,0,\psi}(y_0) = -\sqrt{\frac{C_0}{\pi}} \sum_{\eta \in \mathbb{Z}^d} \tilde{a}_\eta(y_0) \eta d e^{-\eta \cdot \nabla d'(y_0)}.$$  

(3.8)

Thus, by (3.5) and Fourier inversion formula, the leading order term of $(\text{Op}_\varepsilon(\hat{q}_\psi)\Psi b^j)(y_0; \varepsilon)$ is given by

$$e^{\frac{1}{2} - N_j} \tilde{q}_{0,0,\psi}(y_0)(2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(y - y_0) \cdot \xi} \mathcal{P}^{-N_j} \Psi b^j(y) \, dy \, d\xi$$

$$= -\varepsilon^{\frac{1}{2} - N_j} \sqrt{\frac{C_0}{\pi}} \sum_{\eta \in \mathbb{Z}^d} \tilde{a}_\eta(y_0) \eta d e^{-\eta \cdot \nabla d'(y_0)} \Psi b_{-N_j}(y_0).$$

(3.9)

From (3.9), (3.2), (3.7) and (3.6) it follows that $w_{jk}$ has the stated asymptotic expansion (where $J_0 = I_0 \varepsilon^{\frac{1}{2} - (N_j + N_k)}$) with leading order

$$I_0 = -\frac{(2\pi)^{\frac{d-1}{2}}}{\sqrt{\det D^2(\varphi(y_0))}} b^{-N_k}(y_0) \sum_{\eta \in \mathbb{Z}^d} \tilde{a}_\eta(y_0) \eta d e^{-\eta \cdot \nabla d'(y_0)} b_{-N_j}(y_0).$$

(3.10)

Writing

$$\sum_{\eta \in \mathbb{Z}^d} \tilde{a}_\eta(y_0) \eta d e^{-\eta \cdot \nabla d'(y_0))} = \frac{1}{2} \sum_{\eta \in \mathbb{Z}^d} (\tilde{a}_\eta(y_0) \eta d e^{-\eta \cdot \nabla d'(y_0)} + \tilde{a}_{-\eta}(y_0) (-\eta d) e^{-\eta \cdot \nabla d'(y_0)})$$

$$= \sum_{\eta \in \mathbb{Z}^d} \tilde{a}_\eta(y_0) \eta d \sinh(\eta \cdot \nabla d'(y_0))$$

(3.11)

where in the last step we used $\tilde{a}_\eta(y_0) = \tilde{a}_{-\eta}(y_0)$ (see (1.11)) and inserting (3.11) into (3.10) gives (1.30). Note that all $I_k$ are indeed real (since $w_{jk}$ is real).
4. Proof of Theorem 1.12

Step 1: As in the previous proof, we start proving that the sum in the formula (2.83) for the interaction term $w_{j,k}$ can, up to small error, be replaced by an integral. This can be done using the following lemma, which is proven e.g. in [di Gesù, 2012], Proposition C1, using Poisson's summation formula.

**Lemma 4.1** For $h > 0$ let $f_h$ be a smooth, compactly supported function on $\mathbb{R}^d$ with the property: there exists $N_0 \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^d$, $|\alpha| \geq N_0$ there exists a $h$-independent constant $C_\alpha$ such that

$$
\int_{\mathbb{R}^d} |\partial^\alpha f_h(y)| \, dy \leq C_\alpha .
$$

Then

$$
h^d \sum_{y \in \mathbb{Z}^d} f_h(y) = \int_{\mathbb{R}^d} f_h(y) \, dy + O(h^\infty), \quad (h \to 0).
$$

We shall verify that Lemma 4.1 can be used to evaluate the interaction matrix as given in (2.83). For $\alpha$ given by (3.4) we claim that for any $\alpha_1 \in \mathbb{N}^d$ there is a constant $C_{\alpha_1}$ such that

$$
\sup_{x \in \mathbb{R}^d} |\partial^\alpha_1 a(x; \varepsilon)| \leq C_{\alpha_1} \varepsilon^{\frac{3}{2}}.
$$

Clearly it suffices to prove

$$
\sup_{x \in \mathbb{R}^d} |\Psi(x) \partial^\alpha_1 \text{Op}_\varepsilon(\tilde{q}_\varepsilon) \Psi b_j(x; \varepsilon)| \leq C_{\alpha_1} \varepsilon^{\frac{3}{2}}
$$

or, by Sobolev's Lemma (see i.e. [Folland, 1995]), for all $\beta \in \mathbb{N}^d$ with $|\beta| \leq \frac{d}{2} + 1$

$$
\|\Psi \partial^{\beta + \alpha_1} \text{Op}_\varepsilon(\tilde{q}_\varepsilon) \Psi b_j \|_{L^2} \leq C \varepsilon^{\frac{3}{2}}.
$$

Setting for $0 \leq \ell \leq |\beta + \alpha_1|$

$$
c_{\ell}(\xi) := \sum_{\gamma \in \mathbb{N}^d \mid \gamma \cdot \xi = \ell} \frac{1}{\gamma!} \partial^\gamma \tilde{q}_\varepsilon(\xi; \varepsilon) \quad \text{and} \quad \tilde{q}_{\varepsilon, \ell}(x, \xi; \varepsilon) := \sum_{\gamma \in \mathbb{N}^d \mid \gamma \cdot \xi = \ell} \partial^\gamma \tilde{q}_\varepsilon(x, \xi; \varepsilon),
$$

we have by symbolic calculus (see e.g. [Martinez, 2002], Thm.2.7.4)

$$
\partial^{\beta + \alpha_1} \text{Op}_\varepsilon(\tilde{q}_\varepsilon) = \left(\frac{i}{\varepsilon}\right)^{|\beta + \alpha_1|} \text{Op}_\varepsilon(c_0) \text{Op}_\varepsilon(\tilde{q}_\varepsilon)
$$

$$
= \left(\frac{i}{\varepsilon}\right)^{|\beta + \alpha_1|} \sum_{\ell=0}^{\ell} \text{Op}_\varepsilon(\tilde{q}_{\varepsilon, \ell}) \text{Op}_\varepsilon(c_\ell) \left(\frac{x}{\varepsilon}\right)^\ell
$$

$$
= \sum_{\ell=0}^{\ell} \text{Op}_\varepsilon(\tilde{q}_{\varepsilon, \ell}) c_\ell(\partial \xi)
$$

(4.6)

where in the last step we used that $c_\ell(\xi)$ is homogeneous of degree $|\beta + \alpha_1| - \ell$. Since $\Psi b_j$ is smooth and $\tilde{q}_\varepsilon \in S^1_0(1)(\mathbb{R}^{2d})$, (4.5) (and thus (4.3)) follows from (4.6) together with the Theorem of Calderon and Vaillancourt (see e.g. [Dimassi, Sjöstrand, 1999]).

Then for $\varphi$ and $a$ given by (2.78) and (3.4) respectively and for $h = \sqrt{\varepsilon}$, we set $y = \frac{x}{h}$ and

$$
f_h(y) := h^\ell e^{-\varphi_h(y)} A_h(y) \quad \text{where} \quad \varphi_h(y) := \frac{\varphi(hy)}{h^2} \quad \text{and} \quad A_h(y) := a(hy; h^2).
$$

(4.7)

Then for $\alpha \in \mathbb{N}^d$

$$
\partial^\alpha f_h = h^\ell g_{h, \alpha} e^{-\varphi_h}
$$

(4.8)

where $g_{h, \alpha}$ is a sum of products, where the factors are given by $\partial^{\alpha_1} A_h$ and $\partial^{\alpha_2} \varphi_h, \ldots, \partial^{\alpha_m} \varphi_h$ for partitions $\alpha_1, \ldots, \alpha_m \in \mathbb{N}^d$ of $\alpha$, i.e. $\sum_r \alpha_r = \alpha$. By (4.3) and (4.7) we have for some $C_{\alpha_1}$ independent of $h$

$$
\sup_{y \in \mathbb{R}^d} |\partial^{\alpha_1} A_h(y)| \leq h^{1+|\alpha_1|} C_{\alpha_1}.
$$

(4.9)
In order to analyze \(|\partial^{\nu} \varphi_h|\), we remark that Taylor expansion at \(y_0\) yields for \(\beta \in \mathbb{N}^d\)
\[
\partial^{\beta} \varphi_h(y) = h^{[\beta] - 2}(\partial^{\beta} \varphi)(h y) + h^{[\beta] - 1}(\nabla \partial^{\beta} \varphi)\partial y(y_0) + h^{[\beta]} \int_0^1 (1 - t)^2 (D^2 \partial^{\beta} \varphi)\partial y(t(y_0 + t(y - y_0))) [y - y_0] \, dt. \tag{4.10}
\]
Since for \(y \in \text{supp } A_h, y_0 \in h^{-1} G_0\) the curve \(t \mapsto h(y_0 + t(y - y_0))\) lies in a compact set, it follows from \(4.10\) together with Remark 2.9 (3), that for some \(C_\beta\) and for \(N_\beta = \max\{0, [\beta] - 2\}\)
\[
|h^{[\beta]} \varphi_h(y)| \leq C_\beta h^{N_\beta} (1 + |y - y_0|^2), \quad y_0 \in h^{-1} G_0, \quad y \in \text{supp } A_h. \tag{4.11}
\]
Thus using the above mentioned structure of \(g_{h, \alpha}\) we get
\[
|g_{h, \alpha}(y)| \leq C_\alpha h^{1 + (1 - y_0)^2} \tag{4.12}
\]
where \(C_\alpha\) is uniform for \(y \in \text{supp } A_h\) and \(y_0 \in h^{-1} G_0\). Taking the infimum over all \(y_0\) on the right hand side of \(4.12\) we get
\[
|g_{h, \alpha}(y)| \leq C_\alpha h^{1 + (\text{dist}(y, h^{-1} G_0))^2} \tag{4.13}
\]
Since by Hypothesis 1.11 \(G\) is non-degenerate at \(G_0\) we have for some \(C > 0\)
\[
\varphi(h) \geq C \text{dist}(x, G_0)^2.
\]
and therefore
\[
\varphi_h(y) \geq C \frac{1}{h^2} \text{dist}(h y, G_0)^2 = C \text{dist}(y, h^{-1} G_0)^2. \tag{4.14}
\]
Combining \(4.8\), \(4.13\) and \(4.14\) gives
\[
\int_{\mathbb{R}^d} |\partial^{\nu} f_h(y) | \, dy = h^\nu \int_{\mathbb{R}^d} |g_{h, \alpha} e^{-\varphi_h(y)} | \, dy
\]
\[
\leq C_\alpha h^{\nu + 1} \int_{\text{supp } A_h} e^{-C \text{dist}(y, h^{-1} G_0)^2} \left(1 + (\text{dist}(y, h^{-1} G_0)^2) \right) \, dy
\]
\[
= C_\alpha h^{\nu + 1 - d} \int_{\text{supp } \Psi} e^{-C \text{dist}(x, G_0)^2} \left(1 + h^{-2\alpha} (\text{dist}(x, G_0)^2) \right) \, dx \tag{4.15}
\]
where in the last step we used the substitution \(x = h y\).

Using the Tubular Neighborhood Theorem, there is a diffeomorphism
\[
k : \text{supp } \Psi \to G_0 \times (-\delta, \delta)^d - \ell, \quad k(x) = (s, t). \tag{4.16}
\]
Here \(\delta > 0\) must be chosen adapted to \(\text{supp } \varphi\), which is an arbitrary small neighborhood of \(G_0\). Denoting by \(d\sigma\) the Euclidean surface element on \(G_0\), the right hand side of \(4.15\) can thus be estimated from above by
\[
C_\alpha h^{\nu + 1 - d} \int_{G_0 \times (-\delta, \delta)^d - \ell} e^{-C \sigma^2} \left(1 + \left(\frac{1}{h} \right)^{2\alpha} \right) \, d\sigma(s) \, dt
\]
\[
\leq C_\alpha h \int_{\mathbb{R}^{d - \ell}} e^{-C \sigma^2} \left(1 + |\sigma|^{2\alpha} \right) \, d\tau \leq \tilde{C}_\alpha \tag{4.17}
\]
where in the last step we used that \(G_0\) was assumed to be compact and the substitution \(t = t h\).

By \(4.15\) and \(4.17\) we can use Lemma 4.1 for \(f_h\) given in \(4.7\) and thus we have by \(2.83\) together with \(3.3\) and \(3.4\)
\[
w_{jk} = e^{-\frac{d}{2} \epsilon} e^{-S_{jk} \epsilon} \int_{\mathbb{R}^d} e^{-\frac{d(x)}{2} \epsilon} (\Psi b_k)(x) (\text{Op}_\epsilon(\tilde{q}_\epsilon) \Psi b_j)(x) \, dx + O \left(e^{-\frac{d}{2} \epsilon} \epsilon^\infty \right). \tag{4.18}
\]

**Step 2:** Next we use an adapted version of stationary phase.

On \(G_0\) we choose linear independent tangent unit vector fields \(E_m, 1 \leq m \leq \ell\), and linear independent normal unit vector fields \(N_m, \ell + 1 \leq m \leq d\), where we set \(N_d = e_d\), the normal vector field on \(\mathbb{H}_d\). Possibly shrinking \(\text{supp } \Psi\), the diffeomorphism \(k\) given in \(4.16\) can be chosen such that for each \(x \in \text{supp } \Psi\) there exists exactly one \(s \in G_0\) and \(t \in (-\delta, \delta)^{d - \ell}\) such that
\[
x = s + \sum_{m=\ell+1}^d t_{m-\ell} N_m(s) \quad \text{for} \quad k(x) = (s, t). \tag{4.19}
\]
This follows from the proof of the Tubular Neighborhood Theorem, see e.g. [Hirsch, 1976]. It allows to continue the vector fields $N_m$ from $G_0$ to $\text{supp } \Psi$ by setting $N_m(x) := N_m(s)$, thus $N_m = \partial_{t_{m-1}}$. It follows that these vector fields $N_m(x)$ actually satisfy the conditions above Hypothesis 1.11 (in particular, they commute). We define

$$\tilde{\varphi} := \varphi \circ k^{-1} : G_0 \times (-\delta, \delta)^{d-\ell} \to \mathbb{R} \quad \text{with} \quad \tilde{\varphi}(s, t) := \varphi \circ k^{-1}(s, t) = \varphi(x).$$

Since $\varphi(x) = d^j(x) + d^k(x) + C_0 x_3^2 - S_{jk}$ it follows from the construction above that

$$\tilde{\varphi}|_{k(G_0)} = \varphi|_{G_0} = 0 \quad \text{for} \quad 1 \leq m \leq \ell$$

$$E_m \varphi|_{G_0} = 0, \quad \text{for} \quad 1 \leq m \leq \ell - 1$$

$$\partial_{m} \tilde{\varphi}|_{k(G_0)} = N_{m+1} \varphi|_{G_0} = 0, \quad \text{for} \quad 1 \leq m \leq d - \ell$$

$$D \varphi|_{G_0} = 0.$$

By Hypothesis 1.11, the transversal Hessian of the restriction of $d^j + d^k$ to $\mathbb{H}_d$ at $G_0$ is positive definite, i.e.

$$D^2_{\perp, G_0}(d^j + d^k) = \begin{pmatrix} N_m N_{m'}(d^j + d^k)|_{G_0} \end{pmatrix}_{\ell+1 \leq m, m' \leq d-1} > 0.$$  \hspace{1cm} (4.21)

Analog to the proof of Theorem 1.8, we use that $d^j + d^k$ is constant along the geodesics. Thus, for any $x_0 \in G_0$, the matrix $(N_r N_{p \ell}(d^j + d^k)(x_0))_{\ell+1 \leq r, p \leq d}$ has $d - \ell - 1$ positive eigenvalues and one zero eigenvalue and in particular its determinant is zero. Since

$$N_r N_{p \ell} = \begin{cases} 2C_0 + N_d N_d(d^j + d^k) & \text{for} \quad (r, p) = (d, d) \\ N_r N_p(d^j + d^k) & \text{otherwise} \end{cases} \quad (4.22)$$

the Hessian $(N_m N_{m'} \varphi|_{G_0})_{\ell+1 \leq m, m' \leq d}$ of $\varphi$ restricted to $G_0$ is a non-negative quadratic form. It is in fact positive definite since for any $x_0 \in G_0$

$$\det(N_m N_{m'} \varphi(x_0))_{\ell+1 \leq m, m' \leq d}$$

$$= \det(N_m N_{m'}(d^j + d^k)(x_0))_{\ell+1 \leq m, m' \leq d} + \det\left(\begin{pmatrix} N_m N_{m'}(d^j + d^k)(x_0) \end{pmatrix}_{\ell+1 \leq m, m' \leq d-1} \right)$$

$$= 2C_0 \det(D^2_{\perp, G_0}(d^j + d^k)(x_0)) > 0. \quad (4.23)$$

Thus

$$D^2_{\ell, G_0} \tilde{\varphi}|_{k(G_0)} = \begin{pmatrix} N_m N_{m'} \varphi|_{G_0} \end{pmatrix}_{\ell+1 \leq m, m' \leq d} > 0.$$  \hspace{1cm} (4.24)

The following lemma is an adapted version of the Morse Lemma with parameter (see e.g. Lemma 1.2.2 in [Duistermaat, 1996]).

**Lemma 4.2** Let $\phi \in \mathcal{C}^\infty \left(G_0 \times (-\delta, \delta)^{d-\ell}\right)$ be such that $\phi(s, 0) = 0$, $D_{t_0} \phi(s, 0) = 0$ and the transversal Hessian $D^2_{t_0, \phi(s, \cdot)}|_{t=0} = Q(s)$ is non-degenerate for all $s \in G_0$. Then, for each $s \in G_0$, there is a diffeomorphism $y(s, \cdot) : (-\delta, \delta)^{d-\ell} \to U$, where $U \subset \mathbb{R}^{d-\ell}$ is some neighborhood of 0, such that

$$y(s, t) = t + O(|t|^2) \quad \text{as} \quad |t| \to 0 \quad \text{and} \quad \phi(s, t) = \frac{1}{2} \langle y(s, t), Q(s)y(s, t) \rangle.$$  \hspace{1cm} (4.25)

Furthermore, $y(s, t)$ is $\mathcal{C}^\infty$ in $s \in G_0$.

The proof of Lemma 4.2 follows the proof of the Morse-Palais Lemma in [Lang, 1993] noting that the construction depends smoothly on the parameter $s \in G_0$.

By (4.20) and (4.24), the phase function $\tilde{\varphi}$ satisfies the assumptions on $\phi$ given in Lemma 4.2. We thus can define the diffeomorphism $h := 1 \times y : G_0 \times (-\delta, \delta)^{d-\ell} \to G_0 \times U$ for $y$ constructed with respect to $\tilde{\varphi}$ as in Lemma 4.2. Using the diffeomorphism $k : \text{supp } \Psi \to G_0 \times (-\delta, \delta)^{d-\ell}$ constructed above (see (4.19)), we set $g(x) = h \circ k(x) = (s, y)$ (then $g^{-1}(s, 0) = s$ holds for any $s \in G_0$). Thus

$$\varphi(g^{-1}(s, y)) = \frac{1}{2} \langle y, Q(s)y \rangle.$$  \hspace{1cm} (4.26)
and setting $x = g^{-1}(s, y)$ we obtain by (4.18), using the notation (4.4), modulo $O\left(e^{-\frac{\delta}{x^2}} \varepsilon^\infty\right)$

$$w_{jk} \equiv \varepsilon^{-\frac{1}{2}} e^{-\frac{s_{jk}}{\varepsilon}} \int_{\text{supp} \Psi} e^{-\frac{s_{jk}}{\varepsilon}} a(x; \varepsilon) \, dx$$

$$= \varepsilon^{-\frac{1}{2}} e^{-\frac{s_{jk}}{\varepsilon}} \int_{G_0} \int_{U} e^{-\frac{1}{2\varepsilon} (y, Q(s)y)} a(g^{-1}(s, y); \varepsilon) J(s, y) \, dy \, d\sigma(s)$$

(4.27)

where $d\sigma$ is the Euclidean surface element on $G_0$ and $J(s, y) = \det D_y g^{-1}(s, \cdot)$ denotes the Jacobi determinant for the diffeomorphism $g^{-1}(s, \cdot) : U \to \text{span}(N_{s,1}(s), \ldots, N_d(s))$

and $Q(s) = D^2_t \hat{\phi}(s, \cdot) |_{s=0}$ denotes the transversal Hessian of $\hat{\phi}$ as given in (4.24). From the construction of $g$ and (4.19) it follows that $J(s, 0) = 1$ for all $s \in G_0$.

By the stationary phase formula with respect to $y$ in (4.27), we get modulo $O\left(e^{-\frac{\delta}{x^2}} \varepsilon^\infty\right)$

$$w_{jk} = \varepsilon^{-\frac{1}{2}} e^{-\frac{s_{jk}}{\varepsilon}} (2\pi)^{-\frac{1}{2}} \frac{d}{d} \sum_{\nu=0}^{\infty} \varepsilon^{\nu} \int_{G_0} B_{\nu}(s) \, ds$$

(4.28)

where $\hat{a}(\cdot; \varepsilon) := a(\cdot; \varepsilon) \circ g^{-1}$ and, for any $s \in G_0$, $B_0(s)$ is given by the leading order of

$$\left( \det Q(s) \right)^{-\frac{1}{2}} a(g^{-1}(s, 0; \varepsilon)) = \left| 2C_0 \det D^2_{t, G_0} (d^j + d^k)(s) \right|^{\frac{1}{2}} a(s; \varepsilon),$$

(4.29)

using (4.23), (4.24) and identifying $s \in G_0$ with a point in $\mathbb{R}^d$.

We now use the definition of $a$ in (3.4), the expansion (3.5) of $\text{Op}_{p_0}(\hat{\phi}) \Psi b^j$ and the fact that (3.8) and (3.9) also hold for any $y_0 \in G_0$ in the setting of Hypothesis 1.10 to get for $s \in G_0$

$$B_0(s) = \sqrt{\frac{\varepsilon}{2\pi}} \det D^2_{t, G_0} (d^j + d^k)(s)^{-\frac{1}{2}} e^{-\left(N_j + N_k \right)} b_{-N_j}^k(s) \sum_{\eta \in \mathbb{Z}^d} \hat{a}_\eta(s) \eta \varepsilon^{-\eta \cdot \nabla d^j(s)} b_{-N_j}^j(s).$$

(4.30)

Combining (4.30) and (4.28) and using (3.11) completes the proof.

\[ \square \]

5. Some more results for $w_{jk}$

In this section, we derive some formulae and estimates for the interaction term $w_{jk}$ and its leading order term, assuming only Hypotheses 1.1 to 1.5 i.e. without any assumptions on the geodesics between the potential minima $x^j$ and $x^k$.

We combine the fact that the relevant jumps in the interaction term are those taking place in a small neighborhood of $H \cap E$, proven in [K., R., 2012, Proposition 1.7], with the results on approximate eigenfunctions proven in [K., R., 2016].

**Proposition 5.1** Assume that Hypotheses 1.1 to 1.5 hold and let $\hat{v}_m$, $m = j, k$, denote the approximate eigenfunctions given in (1.29). For $\delta > 0$, we set

$$\delta \Gamma := \delta \mathbb{H}_d, \quad \hat{\delta \Gamma} := \delta \Gamma \cap E$$

where $\delta \mathbb{H}_d$ is defined in (2.6). Then the interaction term is given by

$$w_{jk} = \langle \hat{v}_j, \mathbf{1}_{\delta \Gamma} \nu \mathbf{1}_{\delta \Gamma} \hat{v}_k \rangle_{L^2} - \langle \mathbf{1}_{\delta \Gamma} T \mathbf{1}_{\delta \Gamma} \hat{v}_j, \hat{v}_k \rangle_{L^2} + O\left(\varepsilon^\infty e^{-\frac{\delta}{\varepsilon}}\right).$$

(5.1)

Moreover, setting

$$\hat{\nu}(x, \xi) := - \sum_{\gamma \in \mathbb{Z}^d} \mathbf{1}_{\delta \Gamma}(x + \gamma) a_\nu^{(0)}(x) \cosh \frac{\gamma \cdot \xi}{\varepsilon},$$

(5.2)

the leading order of $w_{jk}$ is can be written as

$$\sum_{x \in \delta \Gamma} \hat{v}_j(x) \hat{v}_k(x) \left( \hat{\nu}(x, \nabla d^j(x)) - \hat{\nu}(x, \nabla d^k(x)) \right).$$

(5.3)
If \( \tilde{\omega}_j \) and \( \tilde{\omega}_k \) are both strictly positive in \( \tilde{\Gamma}_\varepsilon \), we have modulo \( O \left( e^{\infty \varepsilon^{-1/2}} \right) \)
\[
\sum_{x \in \tilde{\Omega}_\varepsilon} \tilde{\omega}_j(x) \tilde{\omega}_k(x) \nabla \hat{\epsilon} \delta(x, \nabla d^k(x))(\nabla d^j(x) - \nabla d^k(x)) \leq w_{jk} \leq \sum_{x \in \tilde{\Gamma}_\varepsilon} \tilde{\omega}_j(x) \tilde{\omega}_k(x) \nabla \hat{\epsilon} \delta(x, \nabla d^j(x))(\nabla d^j(x) - \nabla d^k(x)) \ .
\]
(5.5)

We remark that the translation operator \( 1_{\tilde{\Omega}} \cdot T_x 1_{\tilde{\Omega}} \) is non-zero only for translations mapping points \( x \in E \) with \( 0 \leq x_d \leq \delta \) to points \( x + \gamma \in E \) with \( -\delta \leq x + \gamma < 0 \). Thus each translation crosses the hyperplane \( \mathbb{H}_d \) from right to left.

**Proof.** Since by Hypothesis 1.5 each of the two wells has exactly one eigenvalue within the spectral interval \( I_\varepsilon \), we have \( \tilde{\omega}_j := \tilde{\omega}_{j,1} = \tilde{\omega}_{j,1} \) in the setting of [K., R. 2016], Theorem 1.8. Setting
\[
A := 1_{\tilde{\Omega}} \cdot T_x 1_{\tilde{\Omega}} - 1_{\tilde{\Omega}} \cdot T_x 1_{\tilde{\Omega}} \ ,
\] (5.6)
we have by [K., R., 2016], Proposition 1.7,
\[
\left| w_{jk} - \langle \tilde{\omega}_j, A \tilde{\omega}_k \rangle \right| = \left| \langle v_j, A v_k \rangle \right| + \left| \langle \tilde{\omega}_j, A \tilde{\omega}_k \rangle \right| + O \left( e^{\left(-\frac{(\delta + a - \delta)}{\varepsilon}\right)} \right) \leq \langle v_j - \tilde{\omega}_j, A v_k \rangle + \langle \tilde{\omega}_j, A (v_k - \tilde{\omega}_k) \rangle + O \left( e^{\left(-\frac{(\delta + a - \delta)}{\varepsilon}\right)} \right) \ .
\] (5.7)

From (5.6) and the triangle inequality for the Finsler distance \( d \) it follows that
\[
\left| \langle v_j - \tilde{\omega}_j, A v_k \rangle \right| = \left| \sum_{x \in \tilde{\Omega}_\varepsilon} \sum_{x \in \tilde{\Omega}_{\varepsilon}^c} \left[ 1_{\tilde{\Omega}}(x)1_{\tilde{\Omega}}(x + \gamma) - 1_{\tilde{\Omega}}(x)1_{\tilde{\Omega}}(x + \gamma) \right] \times e^{\frac{x_d(x)}{\varepsilon}} \cdot e^{\frac{x_d(x)}{\varepsilon}} (v_j(x) - \tilde{\omega}_j(x)) \cdot \gamma(x) e^{\frac{x_d(x)}{\varepsilon}} \cdot e^{\frac{x_d(x)}{\varepsilon}} v_k(x + \gamma) \right| \leq e^{\frac{x_d(x + \gamma)}{\varepsilon}} \left( \|v_j - \tilde{\omega}_j\|_{L^2(\tilde{\Gamma})} \|v_k - \tilde{\omega}_k\|_{L^2(\tilde{\Gamma})} \right) \sum_{|\gamma| < B} \|a_{\gamma} e^{\frac{x_d(x + \gamma)}{\varepsilon}} \|_{L^\infty(\tilde{\Gamma})} .
\]
(5.8)

In the last step we used that for some \( B > 0 \) we have \( |\gamma| < B \) if \( x \in \tilde{\Omega}_\varepsilon \) and \( x + \gamma \in \tilde{\Omega}_\varepsilon \) and vice versa. Therefore by [K., R., 2016], Theorem 1.8, Proposition 3.1 and by (1.19) we have
\[
\left| \langle v_j - \tilde{\omega}_j, A v_k \rangle \right| = O \left( e^{\left(-\frac{\delta + a}{\varepsilon}\right)} \right) \ .
\] (5.8)

The second summand on the right hand side of (5.7) can be estimated similarly. This proves (5.2).

For the next step, we remark that by Hypothesis 1.3 as a function on the cotangent bundle \( T^* \tilde{\Gamma} \), the symbol \( \hat{\epsilon} \delta \) is hyperregular (see [K., R. 2008]).

Setting \( \hat{b}^j := b_{-N_j}^j \) for \( \ell \in \{j, k\} \), (5.2) leads to
\[
w_{jk} = \sum_{x \in \tilde{\Gamma}_\varepsilon} \sum_{\gamma \in \tilde{\Omega}_\varepsilon} a_{\gamma}(x) e^{\frac{x_d(x)}{2} - N_j - N_k} \left( \hat{b}^j(x) e^{\frac{d(x + \gamma)}{\varepsilon}} - \hat{b}^j(x + \gamma) e^{\frac{d(x + \gamma)}{\varepsilon}} \right) \ .
\] (5.9)

We split the sum over \( \gamma \) in the parts \( A_1(x) \) with \( |\gamma| \leq 1 \) and \( A_2(x) \) with \( |\gamma| > 1 \). Then it follows at once from (1.8) that for any \( B > 0 \) and some \( C > 0 \)
\[
\left| \sum_{x \in \tilde{\Gamma}_\varepsilon} A_2(x) \right| \leq C e^{-\frac{B}{\varepsilon}} .
\] (5.10)

To analyze \( A_1(x) \), we use Taylor expansion at \( x \), yielding for \( \ell = j, k \)
\[
\sum_{\gamma \in \tilde{\Omega}_\varepsilon} a_{\gamma}(x) e^{\frac{d(x + \gamma)}{\varepsilon}} = \hat{b}^j(x) e^{-\frac{d(x + \gamma)}{2}} \hat{\delta}(x, \nabla d^j(x)) + R_1(x)
\] (5.11)
Thus for \( \hat{\xi} \) periodic with respect to \( \hat{\eta} \), the remainder \( R_1(x) \) can for some \( C > 0 \) and any \( B > 0 \) be estimated by

\[
|R_1(x)| = e^{-\frac{d}{2}x} \sum_{\eta \in \mathbb{Z}^d} 1_{\hat{\Delta}^c}(x + \varepsilon \eta) \varepsilon \eta \nabla \hat{b}^j(x) e^{\eta \nabla d^j(x)} \tilde{a}_\eta(x)(1 + O(1)) \leq \varepsilon C \sum_{\eta \in \mathbb{Z}^d} |\varepsilon| e^{-B|\eta|} \leq \varepsilon C \int_{\mathbb{R}^d} |\varepsilon| e^{-B|\eta|} d\eta \leq C \varepsilon .
\]

Inserting (5.10), (5.11) and (5.12) into (5.9) yields

\[
\gamma \leq \varepsilon C > \frac{\pi}{\varepsilon}
\]

with \( \Gamma \). (5.5) follows from the convexity of \( \tilde{\psi} \).

To show (5.5), we use that for any convex function \( f \) on \( \mathbb{R}^d \)

\[
\nabla f(\eta)(\xi - \eta) \leq f(\xi) - f(\eta) \leq \nabla f(\xi)(\xi - \eta), \quad \eta, \xi \in \mathbb{R}^d .
\]

Thus for \( \tilde{\psi}_\varepsilon^j \) and \( \tilde{\psi}_\varepsilon^k \) both strictly positive in \( \delta \Gamma \), (5.5) follows from the convexity of \( \tilde{\psi} \).

**Appendix A. Pseudo-Differential operators in the discrete setting**

We introduce and analyze pseudo-differential operators associated to symbols, which are \( 2\pi \)-periodic with respect to \( \xi \) (for former results see also K., R., 2009).

Let \( T^d := \mathbb{R}^d / (2\pi \mathbb{Z}^d) \) denote the \( d \)-dimensional torus and without further mentioning we identify functions on \( T^d \) with \( 2\pi \)-periodic functions on \( \mathbb{R}^d \).

**Definition A.1**

1. An order function on \( \mathbb{R}^N \) is a function \( m : \mathbb{R}^N \to (0, \infty) \) such that there exist \( C > 0, M \in \mathbb{N} \) such that

\[
m(z_1) \leq C(z_1 - z_2)^M m(z_2), \quad z_1, z_2 \in \mathbb{R}^N
\]

where \( \langle x \rangle := \sqrt{1 + |x|^2} \).

2. A function \( \tilde{p} \in C^\infty(\mathbb{R}^N \times (0, 1)) \) is an element of the symbol class \( S^k_\delta(m)(\mathbb{R}^N) \) for some order function \( m \) on \( \mathbb{R}^N \), if for all \( \alpha \in \mathbb{N}^N \) there is a constant \( C_\alpha > 0 \) such that

\[
|\partial^\alpha p(z; \varepsilon)| \leq C_\alpha \varepsilon^{k - \delta|\alpha|} m(z), \quad z \in \mathbb{R}^N
\]

uniformly for \( \varepsilon \in (0, 1] \). On \( S^k_\delta(m)(\mathbb{R}^N) \) we define the Fréchet-seminorms

\[
\|p\|_\alpha := \sup_{z \in \mathbb{R}^N, 0 < \varepsilon \leq 1} \frac{|\partial^\alpha p(z; \varepsilon)|}{\varepsilon^{k - \delta|\alpha|} m(z)}, \quad \alpha \in \mathbb{N}^N .
\]

We define the symbol class \( S^k_\delta(m)(\mathbb{R}^N \times \mathbb{T}^d) \) by identification of \( C^\infty(\mathbb{T}^d) \) with the \( 2\pi \)-periodic functions in \( C^\infty(\mathbb{R}^d) \).

3. To \( p \in S^k_\delta(m)(\mathbb{R}^{2d} \times \mathbb{T}^d) \) we associate a pseudo-differential operator \( \tilde{\text{Op}}_\varepsilon^\mathbb{T}(p) : \mathcal{K}(\mathbb{C}^d) \to \mathcal{K}(\mathbb{C}^d) \)

\[
\tilde{\text{Op}}_\varepsilon^\mathbb{T}(p) v(x; \varepsilon) := (2\pi)^{-d} \sum_{y \in \mathbb{Z}^d} \int_{[-\pi, \pi]^d} e^{i(y - x)\xi} p(x, y, \xi; \varepsilon) v(y) d\xi
\]

where

\[
\mathcal{K}(\mathbb{C}^d) := \{ u : (\mathbb{C}^d)^d \to \mathbb{C} \mid u \text{ has compact support} \}
\]

and \( \mathcal{K}'((\mathbb{C}^d)^d) := \{ f : (\mathbb{C}^d)^d \to \mathbb{C} \} \) is dual to \( \mathcal{K}(\mathbb{C}^d) \) by use of the scalar product

\[
\langle u, v \rangle := \sum_{x \in \mathbb{Z}^d} \overline{u}(x) v(x) .
\]

4. For \( t \in [0, 1] \) and \( q \in S^k_\delta(m)(\mathbb{R}^d \times \mathbb{T}^d) \) the associated pseudo-differential operator \( \text{Op}_\varepsilon^\mathbb{T}(q) \) is defined by

\[
\text{Op}_\varepsilon^\mathbb{T}(q) v(x; \varepsilon) := (2\pi)^{-d} \sum_{y \in \mathbb{Z}^d} \int_{[-\pi, \pi]^d} e^{i(y - x)\xi} q((1 - t)x + ty, \xi; \varepsilon) v(y) d\xi
\]

for any \( v \in \mathcal{K}(\mathbb{C}^d) \) and we set \( \text{Op}_\varepsilon^\mathbb{T}(q) := \text{Op}_\varepsilon^\mathbb{T}(q) \).
(5) To \( p \in \mathbb{S}^k_\delta(m)(\mathbb{R}^{3d}) \) we associate a pseudo-differential operator \( \widehat{\text{Op}}_\varepsilon(p) : \mathscr{C}_0^\infty(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d) \) setting
\[
\widehat{\text{Op}}_\varepsilon(p) v(x; \varepsilon) := (2\pi \varepsilon)^{-d} \int_{\mathbb{R}^{3d}} e^{\frac{i}{\varepsilon}(y-x)\xi} p(x, y, \xi; \varepsilon) v(y) \, dy \, d\xi .
\]
\[(A.5)\]

(6) For \( t \in [0,1] \) and \( q \in \mathbb{S}^k_\delta(m)(\mathbb{R}^{2d}) \) the associated a pseudo-differential operator \( \text{Op}_{\varepsilon,t}(q) \) is defined by
\[
\text{Op}_{\varepsilon,t}(q) v(x; \varepsilon) := (2\pi \varepsilon)^{-d} \int_{\mathbb{R}^{2d}} e^{\frac{i}{\varepsilon}(y-x)\xi} q((1-t)x + ty, \xi; \varepsilon) v(y) \, dy \, d\xi , \quad v \in \mathscr{C}_0^\infty(\mathbb{R}^d)
\]
and we set \( \text{Op}_{\varepsilon,0}(q) := \text{Op}_\varepsilon(q) \).

Standard arguments show that \( \widehat{\text{Op}}_\varepsilon(p) \) actually maps \( \mathscr{C}_0^\infty(\mathbb{R}^d) \) into \( \mathscr{C}^\infty(\mathbb{R}^d) \). Moreover, the seminorms given in \[A.1\] induce the structure of a Fréchet-space in \( \mathbb{S}^k_\delta(m)(\mathbb{R}^N) \).

In \[K., R., 2009\] we discussed properties of pseudo-differential operators \( \text{Op}_\varepsilon(.) \). In particular we showed that, for a symbol \( q \in \mathbb{S}^k_\delta(m)(\mathbb{R}^{2d}) \) which is 2\( \pi \)-periodic with respect to \( \xi \), the restriction of \( \text{Op}_\varepsilon(q) \) to \( K(\varepsilon Z)^d \) coincides with \( \text{Op}_\varepsilon^T(q) \).

In the next proposition we show that this statement also holds in the more general case of \( \widetilde{\text{Op}}_\varepsilon \) and \( \widetilde{\text{Op}}_\varepsilon^T \).

**Proposition A.2** For some order function \( m \) on \( \mathbb{R}^{3d} \), let \( p \in \mathbb{S}^k_\delta(m)(\mathbb{R}^{3d}) \) satisfy \( p(x, y, \xi; \varepsilon) = p(x, y, \xi + 2\pi n; \varepsilon) \) for any \( n \in \mathbb{Z}^d \), \( \xi, x, y \in \mathbb{R}^d \) and \( \varepsilon \in (0,1] \). Then \( p \in \mathbb{S}^k_\delta(m)(\mathbb{R}^{2d} \times \mathbb{T}^d) \) and using the restriction map
\[
r_\varepsilon : \mathscr{C}_0^\infty(\mathbb{R}^d) \to K(\varepsilon Z)^d , \quad r_\varepsilon(u) = u|_{(\varepsilon Z)^d}
\]
we have
\[
r_\varepsilon \circ \widetilde{\text{Op}}_\varepsilon(p) u = \widetilde{\text{Op}}_\varepsilon^T(p) r_\varepsilon u , \quad u \in \mathscr{C}_0^\infty(\mathbb{R}^d). \tag{A.8}
\]

**Proof.** For \( x \neq (\varepsilon Z)^d \) both sides of \[A.8\] are zero, so we choose \( x \in (\varepsilon Z)^d \). Then for \( u \in \mathscr{C}_0^\infty(\mathbb{R}^d) \), using the \( \varepsilon \)-scaled Fourier transform
\[
\mathcal{F}_\varepsilon u(x) = \sqrt{2\pi}^{-d} \int_{\mathbb{R}^{3d}} e^{-\frac{i}{\varepsilon}x\xi} u(\xi) \, d\xi ,
\]
we can write
\[
\widetilde{\text{Op}}_\varepsilon(p) u(x; \varepsilon) = (\varepsilon \sqrt{2\pi})^{-d} \int_{\mathbb{R}^{3d}} \left( \mathcal{F}_\varepsilon p(x, y, \cdot; \varepsilon) \right)(x-y) u(y) \, dy .
\]
\[(A.10)\]
Since for any 2\( \pi \)-periodic function \( g \in \mathscr{C}^\infty(\mathbb{R}^d) \) the Fourier transform is given by
\[
\mathcal{F}_\varepsilon g = \left( \frac{\varepsilon}{\sqrt{2\pi}} \right)^d \sum_{z \in (\varepsilon Z)^d} \delta_z c_z , \quad \text{where} \quad c_z := \int_{[-\pi,\pi]^d} e^{-\frac{i}{\varepsilon}z\mu} g(\mu) \, d\mu , \tag{A.11}
\]
(see e.g. \cite{Hörmander 1983}), we formally get
\[
\text{rhs}[A.10] = (\varepsilon \sqrt{2\pi})^{-d} \int_{\mathbb{R}^{3d}} \left( \frac{\varepsilon}{\sqrt{2\pi}} \right)^d \sum_{z \in (\varepsilon Z)^d} \int_{[-\pi,\pi]^d} e^{-\frac{i}{\varepsilon}z\mu} p(x, y, \mu; \varepsilon) \, d\mu \delta_z(x-y) u(y) \, dy
\]
\[
= (2\pi)^{-d} \sum_{z \in (\varepsilon Z)^d} \int_{[-\pi,\pi]^d} \int_{\mathbb{R}^{3d}} e^{-\frac{i}{\varepsilon}z\mu} p(x, y, \mu; \varepsilon) \delta_z(x-y) u(y) \, dy \, d\mu
\]
\[
= (2\pi)^{-d} \sum_{z \in (\varepsilon Z)^d} \int_{[-\pi,\pi]^d} e^{-\frac{i}{\varepsilon}z\mu} p(x, z, \mu; \varepsilon) u(x-z) \, d\mu .
\]
\[(A.12)\]
With the substitution \( y = x-z \) and \( \xi = \mu \) we get by \[A.10\] and \[A.12\]
\[
\widetilde{\text{Op}}_\varepsilon(p) u(x; \varepsilon) = (2\pi)^{-d} \sum_{y \in (\varepsilon Z)^d} \int_{[-\pi,\pi]^d} e^{-\frac{i}{\varepsilon}(x-y)\xi} p(x, y, \xi; \varepsilon) u(y) \, dy \, d\xi = \widetilde{\text{Op}}_\varepsilon^T(p) u(x; \varepsilon)
\]
proving the stated result. \( \Box \)
Remark A.3 Let \( m \) be an order function on \( \mathbb{R}^{2d} \) and \( p \in S^k_\delta(m)(\mathbb{R}^{2d}) \) a symbol. Then, setting
\[
\tilde{\hat{p}}(x,y,\xi;\varepsilon) := p(tx + (1-t)y,\xi;\varepsilon)
\]
for \( t \in [0,1] \), we have \( \text{Op}_\varepsilon(\tilde{\hat{p}}) = \text{Op}_{\varepsilon,t}(p) \). Thus the \( t \)-quantization can be seen as a special case of the general quantization.

Moreover, if \( p \) is periodic in \( \xi \), i.e. if \( p(x,\xi;\varepsilon) = p(x,\xi + 2\pi \eta;\varepsilon) \) for any \( \eta \in \mathbb{Z}^d, \xi \in \mathbb{R}^d \) and \( \varepsilon \in (0,\varepsilon_0] \), then \( p \in S^k_\delta(m)(\mathbb{R}^d \times \mathbb{T}^d) \),
\[
r_\varepsilon \circ \text{Op}_{\varepsilon,t}(p)(u) = \text{Op}_{\varepsilon,t}^T(p) \circ r_\varepsilon(u), \quad u \in \mathcal{C}_c^\infty(\mathbb{R}^d)
\]
and \( \text{Op}_{\varepsilon,t}^T(\tilde{\hat{p}}) = \text{Op}_{\varepsilon,t}^T(p) \).

Remark A.4 For \( a \in S^k_\delta(\langle \xi \rangle^m,\mathbb{R}^{3d}) \) the operator \( \tilde{\text{Op}}_\varepsilon(a) \) is continuous: \( \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d) \) (see e.g. Martinez, 2002) and, similar to Lemma A.2 in \( \text{[K., R., 2009]} \), this result implies that \( \tilde{\text{Op}}_\varepsilon(a) \) is continuous: \( s((\varepsilon \mathbb{Z})^d) \to s((\varepsilon \mathbb{Z})^d) \) by use of Proposition A.2.

The following proposition gives a relation between the different quantizations for symbols which are periodic with respect to \( \xi \). The proof is partly based on Martinez, 2002, where the result is shown for symbols in \( S^0_\delta(\langle \xi \rangle^m,\mathbb{R}^{3d}) \).

Proposition A.5 For \( 0 \leq \delta < \frac{1}{2} \), let \( a \in S^k_\delta(m)(\mathbb{R}^{2d} \times \mathbb{T}^d) \) and \( t \in [0,1] \), then there exists a unique symbol \( a_t \in S^k_\delta(m\tilde{m})(\mathbb{R}^d \times \mathbb{T}^d) \) where \( m(x,\xi) := m(x,\xi) \) such that
\[
\tilde{\text{Op}}_{\varepsilon,t}^T(a) = \text{Op}_{\varepsilon,t}^T(a_t).
\]

Moreover the mapping \( S^k_\delta(m) \ni a \mapsto a_t \in S^k_\delta(m\tilde{m}) \) is continuous in its Fréchet-topology induced from \( \text{\text{[A.1]}} \). \( a_t \) can be written as
\[
a_t(x,\xi;\varepsilon) = (2\pi)^{-d} \sum_{\theta \in (\varepsilon \mathbb{Z})^d} \int_{[-\pi,\pi]^d} e^{\frac{i}{\varepsilon}(\xi-\theta)\mu} a(x+t\theta,x-(1-t)\theta,\mu;\varepsilon) \, d\mu \quad (A.14)
\]
and has the asymptotic expansion
\[
a_t(x,\xi;\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j a_{t,j}(x,\xi), \quad a_{t,j}(x,\xi) := \sum_{\alpha \in \mathbb{N}^d} \frac{i^j}{\alpha!} \partial_{\xi}^\alpha \partial_x^\alpha a(x+tz,x-(1-t)z,\xi;\varepsilon)|_{z=0}. \quad (A.15)
\]

If we write \( a_t(x,\xi;\varepsilon) = \sum_{j \leq N-1} \varepsilon^j a_{t,j}(x,\xi) + S_N(a)(x,\xi;\varepsilon) \) then \( S_N(a) \in S^k_\delta(m\tilde{m}) \) and the Fréchet-seminorms of \( S_N \) depend (linearly) on finitely many \( \|a\|_\alpha \) with \( |\alpha| \leq N \).

Proof. To satisfy \( \text{\text{[A.13]}} \), the symbol \( a_t \) above has to satisfy in \( \mathcal{D}'(\mathbb{R}^{2d}) \)
\[
\int_{[-\pi,\pi]^d} e^{\frac{i}{\varepsilon}(y-x)\mu} a(x,y,\mu;\varepsilon) \, d\mu = \int_{[-\pi,\pi]^d} e^{\frac{i}{\varepsilon}(y-x)\mu} a_t((1-t)x+ty,\mu;\varepsilon) \, d\mu. \quad (A.16)
\]
Setting \( \theta = x - y \) and \( z = (1-t)x + ty = x - t\theta \) in \( \text{\text{[A.16]}} \) gives
\[
\int_{[-\pi,\pi]^d} e^{-\frac{i}{\varepsilon}\theta \mu} a(z+t\theta,z-(1-t)\theta,\mu;\varepsilon) \, d\mu = \sqrt{2\pi} d \left( \mathcal{F}_\varepsilon a_t(z,.;\varepsilon) \right)(\theta) \quad (A.17)
\]
where \( \mathcal{F}_\varepsilon : L^2(\mathbb{T}^d) \to \ell^2((\varepsilon \mathbb{Z})^d) \) denotes the discrete Fourier transform defined by
\[
(\mathcal{F}_\varepsilon f)(\theta) := \frac{1}{\sqrt{2\pi} d} \int_{[-\pi,\pi]^d} e^{-\frac{i}{\varepsilon}\theta \mu} f(\mu) \, d\mu, \quad f \in L^2(\mathbb{T}^d), \quad \theta \in (\varepsilon \mathbb{Z})^d \quad (A.18)
\]
with inverse \( \mathcal{F}_\varepsilon^{-1} : \ell^2((\varepsilon \mathbb{Z})^d) \to L^2(\mathbb{T}^d) \),
\[
(\mathcal{F}_\varepsilon^{-1} v)(\xi) := \frac{1}{\sqrt{2\pi} d} \sum_{\theta \in (\varepsilon \mathbb{Z})^d} e^{\frac{i}{\varepsilon}\theta \xi} v(\theta), \quad v \in \ell^2((\varepsilon \mathbb{Z})^d), \quad \xi \in \mathbb{T}^d \quad (A.19)
\]
where the sum in understood in standard L.I.M-sense. Thus taking the inverse Fourier transform \( \mathcal{F}_\varepsilon^{-1} \) on both sides of \( \text{\text{[A.17]}} \) yields \( \text{\text{[A.14]}} \).
To analyze the right hand side of \( \text{(A.14)} \), we set \( \eta = \mu - \xi \) and introduce a cut-off-function \( \zeta \in \mathcal{K}((\varepsilon \mathbb{Z})^d, [0, 1]) \) with \( \zeta = 1 \) in a neighborhood of 0 to get
\[
\begin{align*}
\eta(x, \xi; \varepsilon) &= b_{1,1}(x, \xi; \varepsilon) + b_{1,2}(x, \xi; \varepsilon) \\
\text{with} \\
\eta_{1,1}(x, \xi; \varepsilon) &= (2\pi)^{-d} \sum_{\theta \in (\varepsilon \mathbb{Z})^d} \int_{[-\pi, \pi]^d} e^{-\frac{i}{2} \eta \theta} (1 - \zeta(\theta)) a(x + t \theta, x - (1 - t) \theta, \xi + \eta; \varepsilon) d\eta \\
\eta_{1,2}(x, \xi; \varepsilon) &= (2\pi)^{-d} \sum_{\theta \in (\varepsilon \mathbb{Z})^d} \int_{[-\pi, \pi]^d} e^{-\frac{i}{2} \eta \theta} \zeta(\theta) a(x + t \theta, x - (1 - t) \theta, \xi + \eta; \varepsilon) d\eta.
\end{align*}
\]

The aim is now to show \( b_{1,1} \in S_{\infty}(\hat{m})(\mathbb{R}^d \times \mathbb{T}^d) \) and \( b_{1,2} \in S_{\delta}^k(\hat{m})(\mathbb{R}^d \times \mathbb{T}^d) \) having the required asymptotic expansion and that the mappings \( a \mapsto b_{1,1} \) and \( a \mapsto b_{1,2} \) are continuous.

Since \( e^{-\frac{i}{2} 2\pi \eta^2} = 1 \) for all \( z \in (\varepsilon \mathbb{Z})^d \) and \( \eta \in \mathbb{Z}^d \), it follows at once from \( \text{(A.14)} \) that \( b_{1,i}(x, \xi + 2\pi \eta; \varepsilon) = b_{1,i}(x, \xi; \varepsilon) \) for \( i = 1, 2 \).

By use of the operator \( L(\theta, \eta) := -\varepsilon^2 \Delta_{\theta/|\theta|^2} \), which is well defined on the support of \( 1 - \zeta(\theta) \) and fulfills \( L(\theta, \eta) e^{-\frac{i}{2} \eta \theta} = e^{-\frac{i}{2} \eta \theta} \), we have for any \( n \in \mathbb{N} \) by partial integration, using the \( 2\pi \)-periodicity of the symbol \( a(x, y, \xi; \varepsilon) \) with respect to \( \xi \),
\[
\begin{align*}
\eta_{1,1}(x, \xi; \varepsilon) &= (2\pi)^{-d} \sum_{\theta \in (\varepsilon \mathbb{Z})^d} \int_{[-\pi, \pi]^d} \left( L^\varepsilon(\theta, \eta) e^{-\frac{i}{2} \eta \theta} \right) (1 - \zeta(\theta)) a(x + t \theta, x - (1 - t) \theta, \xi + \eta; \varepsilon) d\eta \\
&= (2\pi)^{-d} \sum_{\theta \in (\varepsilon \mathbb{Z})^d} \int_{[-\pi, \pi]^d} e^{-\frac{i}{2} \eta \theta} \left( 1 - \zeta(\theta) \right) \left( -\varepsilon^2 \Delta_{\theta/|\theta|^2} \right)^n a(x + t \theta, x - (1 - t) \theta, \xi + \eta; \varepsilon) d\eta.
\end{align*}
\]

Since \( a \in S_{\delta}^k(m) \), the absolute value of the integrand is for some \( C > 0 \) and \( M \in \mathbb{N} \) bounded from above by
\[
C \varepsilon^{k+2n(1-\delta)} M(x + t \theta, x - (1 - t) \theta, \xi + \eta) \leq C \varepsilon^{k+2n(1-\delta)} M - 2n(\eta) M m(x, x, \xi).
\]

This term is integrable and summable for \( n \) sufficiently large yielding
\[
\eta_{1,1}(x, \xi; \varepsilon) = \varepsilon^{k+2n(1-\delta)-d} O(\hat{m}(x, \xi))
\]

The derivatives can be estimated similarly, and thus \( b_{1,1} \in S_{\infty}(\hat{m})(\mathbb{R}^d \times \mathbb{T}^d) \).

To see the continuity of \( S_{\delta}^k(m) \) \( \ni a \mapsto b_{1,1} \in S_{\delta}^{k+2n(1-\delta)-d}(\hat{m}) \) for any \( n \in \mathbb{N} \) large enough, we use \( \text{(A.21)} \) and \( \text{(A.22)} \) to estimate for any \( \alpha, \beta \in \mathbb{N}^d \) and \( x \in \mathbb{R}^d, \xi \in \mathbb{T}^d \)
\[
\begin{align*}
\left| \partial_x^\alpha \partial_{\xi}^\beta \eta_{1,1}(x, \xi; \varepsilon) \right| &\leq C \sum_{\theta \in (\varepsilon \mathbb{Z})^d} \int_{[-\pi, \pi]^d} \left| 1 - \zeta(\theta) \right| \left| (-\varepsilon^2 \Delta_{\theta/|\theta|^2})^n \partial_x^\alpha \partial_{\xi}^\beta a(x + t \theta, x - (1 - t) \theta, \xi + \eta; \varepsilon) \right| \frac{1}{|\theta|^{2n}} d\eta \\
&\quad \times e^{k+2n(1-\delta)-d} m(x + t \theta, x - (1 - t) \theta, \xi + \eta) \\
&\leq C \varepsilon^{k+2n(1-\delta)-d} m(x, x, \xi) a\|_{(\alpha, \beta)} \sum_{\theta \in (\varepsilon \mathbb{Z})^d} \int_{[-\pi, \pi]^d} \left| 1 - \zeta(\theta) \right| M - 2n(\eta) M d\eta \\
&\leq C \varepsilon^{k+2n(1-\delta)-d} \hat{m}(x, \xi) a\|_{(\alpha, \beta)}
\end{align*}
\]

for \( \hat{\beta}(n) = \beta + (2n, \ldots, 2n) \), where the last estimate holds for \( n \) sufficiently large. This gives continuity.

Since in the definition of \( b_{1,2} \) in \( \text{(A.20)} \) the integral and sum range over a compact set, it follows analog to the estimates above that
\[
\left| \partial_x^\alpha \partial_{\xi}^\beta b_{1,2}(x, \xi; \varepsilon) \right| \leq C_{\alpha, \beta} \varepsilon^{k-(|\alpha| + |\beta|)\delta} a\|_{(\alpha, \beta)} m(x, x, \xi)
\]

and thus \( b_{1,2} \in S_{\delta}^k(\hat{m}) \) and the mapping \( S_{\delta}^k(m) \ni a \mapsto b_{1,2} \in S_{\delta}^k(\hat{m}) \) is continuous.

Thus \( S_{\delta}^k(m) \ni a \mapsto a_t \in S_{\delta}^k(\hat{m}) \) is continuous. Using standard arguments, the method of stationary phase (see e.g. \cite{R., 2006}, Lemma B.4) gives the asymptotic expansion \( \text{(A.15)} \).

Since \( a \mapsto S_N = a_t - \sum_{j=0}^{N-1} \varepsilon^j a_{t,j} \) is obviously continuous, each Fréchet-seminorm of \( S_N \) can be estimated by finitely many Fréchet-seminorms of \( a \). To get the more refined statement \( S_N \), we
use (A.14) to write
\[ a_t(x, \xi; \varepsilon) = e^{i \varepsilon \partial_\theta \partial_\eta q(x + t \theta, x - (1 - t) \theta; \xi + \eta; \varepsilon)} \bigg|_{\theta = 0 = \eta}. \]
(A.23)

In fact, by algebraic substitutions, (A.23) is a consequence of the formula
\[ e^{i \varepsilon \partial_\theta \partial_\eta b(\theta, \eta; \varepsilon)} = \sum_{z \in \mathbb{Z}^d} \int_{[-\pi, \pi]^d} e^{i \frac{t}{\varepsilon} \mu} b(\theta - z, \eta - \mu; \varepsilon) \, d\mu \]
for \( b \in S^k_\delta(\widehat{n})(\mathbb{R}^d \times \mathbb{T}^d) \), where, for \( x, \xi \) fixed, we set \( b(\theta, \eta; \varepsilon) = a(x + t \theta, x - (1 - t) \theta; \xi + \eta; \varepsilon). \) (A.24)

may be proved by writing \( e^{i \varepsilon \partial_\theta \partial_\eta q} \) as a multiplication operator in the covariables and applying the Fourier transforms \( \mathcal{F}_\varepsilon, \mathcal{F}_\varepsilon^{-1} \), using that \( e^{i \frac{t}{\varepsilon} \mu z} \) is invariant under \( \mathcal{F}_\varepsilon, \mathcal{F}_\varepsilon^{-1} \).

Using Taylor’s formula for \( e^{i x} \), we get
\[ S_N(a)(x, \xi; \varepsilon) = \frac{(i \varepsilon \partial_\theta \partial_\eta)^N}{(N - 1)!} \int_0^1 (1 - s)^{N-1} e^{i \varepsilon \partial_\theta \partial_\eta q} \, ds \ a(x + t \theta, x - (1 - t) \theta, \eta + \xi; \varepsilon) \bigg|_{\theta = 0 = \eta}, \]
proving that \( S_N \) only depends on Fréchet-seminorms of \( (\partial_\theta \partial_\eta)^N a(x + t \theta, x - (1 - t) \theta, \eta + \xi; \varepsilon) \) and thus not on Fréchet-seminorms \( \|a\|_\alpha \) with \( |\alpha| < N. \)

The norm estimate \( \| \widetilde{\text{Op}_{\text{q}}^T}(a) \| \xi_{((\varepsilon \mathbb{Z})^d)} \leq M \varepsilon^\tau \|u\|_{\ell^2((\varepsilon \mathbb{Z})^d)} \)
Proposition A.6, for operators \( \text{Op}_T^q(a) \) with a bounded symbol \( q \in S^k_\delta(1)(\mathbb{R}^d \times \mathbb{T}^d) \) combined with Proposition A.5 leads at once to the following corollary.

**Corollary A.6** Let \( a \in S^k_\delta(1)(\mathbb{R}^d \times \mathbb{T}^d) \) with \( 0 \leq \delta < \frac{1}{2} \). Then there exists a constant \( M > 0 \) such that, for the associated operator \( \widetilde{\text{Op}_T^q}(a) \) given by (A.2)
the estimate
\[ \left\| \widetilde{\text{Op}_T^q}(a)u \right\|_{\ell^2((\varepsilon \mathbb{Z})^d)} \leq M \varepsilon^\tau \|u\|_{\ell^2((\varepsilon \mathbb{Z})^d)} \]
holds for any \( u \in \ell^2((\varepsilon \mathbb{Z})^d) \) and \( \varepsilon > 0 \). \( \widetilde{\text{Op}_T^q}(a) \) can therefore be extended to a continuous operator:
\[ \ell^2((\varepsilon \mathbb{Z})^d) \rightarrow \ell^2((\varepsilon \mathbb{Z})^d) \] with \( \| \widetilde{\text{Op}_T^q}(a) \| \leq M \varepsilon^\tau \). Moreover \( M \) can be chosen depending only on a finite number of Fréchet-seminorms of the symbol \( a \).

In the next proposition, we analyze the symbol of an operator conjugated with a term \( e^{i x/\varepsilon} \).

**Proposition A.7** Let \( q \in S^k_\delta(1)(\mathbb{R}^d \times \mathbb{T}^d) \), \( 0 \leq \delta < \frac{1}{2} \), be a symbol such that the map \( \xi \mapsto q(x, y, \xi; \varepsilon) \) can be extended to an analytic function on \( \mathbb{C}^d \). Let \( \psi \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}) \) such that all derivatives are bounded.

Then
\[ Q_\psi := e^{i \psi/\varepsilon} \widetilde{\text{Op}_T^q}(q) e^{-i \psi/\varepsilon} \]
is the quantization of the symbol \( q_{\psi} \in S^k_\delta(1)(\mathbb{R}^d \times \mathbb{T}^d) \) given by
\[ q_{\psi}(x, y, \xi; \varepsilon) := q(x, y, \xi - i \Phi(x, y; \varepsilon)) \]
where \( \Phi \) is given in (A.29). In particular, the map \( \xi \mapsto q_{\psi}(x, y, \xi; \varepsilon) \) can be extended to an analytic function on \( \mathbb{C}^d \). If \( q \) has an asymptotic expansion \( q \sim \sum_n c^n q_n \) in \( \varepsilon \), then the same is true for \( q_{\psi} \).

For \( t \in [0, 1] \), the operator \( Q_\psi \) is the \( t \)-quantization of a symbol \( q_{\psi, t} \in S^k_\delta(1)(\mathbb{R}^d \times \mathbb{T}^d) \) with asymptotic expansion \( q_{\psi, t} \sim \sum_n c_n, q_{\psi, t} \) such that \( q_{\psi, t} \sim \sum_{n=0}^{N-1} q_{n, \psi, t} \in S^{k+N(1-2\delta)}(1)(\mathbb{R}^d \times \mathbb{T}^d) \). Moreover, the map \( \xi \mapsto q_{\psi, t}(x, \xi; \varepsilon) \) can be extended to an analytic function on \( \mathbb{C}^d \) and
\[ q_{\psi, t}(x, \xi; \varepsilon) = \tilde{q}_{\psi}(x, \xi; \varepsilon) = q(x, x, \xi - i \nabla \psi(x; \varepsilon)) \mod S^{k+1-2\delta}_\delta(1)(\mathbb{R}^d \times \mathbb{T}^d). \] (A.27)

**Proof.** The integral kernel of \( e^{i \psi/\varepsilon} \widetilde{\text{Op}_T^q}(q) e^{-i \psi/\varepsilon} \) is given by the oscillating integral
\[ K_\psi(x, y) := (2\pi)^{-d} \int_{[-\pi, \pi]^d} e^{i \frac{t}{\varepsilon}[(y-x)\xi + i(\psi(y) - \psi(x))]} q(x, y, \xi; \varepsilon) \, d\xi \]
(A.28)
where we set \( \Phi(x, y) := \int_0^1 \nabla \psi((1 - t)x + ty) \, dt. \) (A.29)
Substituting \( \tilde{\zetas} := \zeta + i\Phi(x, y) \) and iteratively using Lemma 2.3 yields
\[
\text{rhs (A.28)} = (2\pi)^{-d} \int_{[-\pi, \pi]^d + i\Phi(x, y)} e^{\frac{i}{\varepsilon}(y-x)\zetas} q(x, y, \tilde{\zetas} - i\Phi(x, y); \varepsilon) d\tilde{\zetas}
\]
\[
= (2\pi)^{-d} \int_{[-\pi, \pi]^d} e^{\frac{i}{\varepsilon}(y-x)\zetas} q(x, y, \tilde{\zetas} - i\Phi(x, y); \varepsilon) d\tilde{\zetas} \quad \text{(A.30)}
\]
The right hand side of (A.30) is the integral kernel of \( \tilde{\mathcal{O}}_{\Phi}^{\varepsilon}(\tilde{\psi}) \) for \( \tilde{\psi} \) given by (A.26). Since all derivatives of \( \Phi \) are bounded by assumption, it follows that \( \tilde{\psi} \in S_\delta^{k}(1) (\mathbb{R}^{2d} \times \mathbb{T}^{d}) \). The statement on the analyticity of \( \tilde{\psi} \) with respect to \( \zeta \) and on the existence of an asymptotic expansion follow at once from equality (A.26).

Concerning the statement on the \( t \)-quantization we use Proposition A.3, showing that there is a unique symbol \( \tilde{\mathcal{H}}_{t, \varepsilon} \in S_\delta^{k}(1) (\mathbb{R}^{d} \times \mathbb{T}^{d}) \) such that \( \tilde{\mathcal{O}}_{\Phi}^{\varepsilon}(\tilde{\psi}) = \mathcal{O}_{\Phi, t}^{\varepsilon}(\tilde{\mathcal{H}}_{t, \varepsilon}, \tilde{\psi}) \). Moreover, by (A.17), we have in leading order, i.e. modulo \( S_\delta^{k+1-2d}(1) \),
\[
\tilde{\mathcal{H}}_{t, \varepsilon}(x, \xi; \varepsilon) = \tilde{\mathcal{H}}_{t, \varepsilon}(x, \xi; \varepsilon) = q(x, x, \xi - i\Phi(x, x); \varepsilon) = q(x, x, \xi - i\nabla \psi(x); \varepsilon) \quad \text{(A.31)}
\]
and \( q_{t, \varepsilon} \) has an asymptotic expansion with the stated properties.

\[\Box\]

**Remark A.8** Let \( p \in S_\delta^{k}(1) (\mathbb{R}^{d} \times \mathbb{T}^{d}) \) and \( s, t \in [0, 1] \). Then it follows at once from Remark A.3 that \( \varepsilon^{s/t} \tilde{\mathcal{O}}_{\Phi}^{\varepsilon}(p)e^{-\Phi/\varepsilon} \) is the \( \varepsilon \)-quantization of a symbol \( p_{\varepsilon} \in S_\delta^{k}(1) (\mathbb{R}^{d} \times \mathbb{T}^{d}) \) satisfying
\[
p_{\varepsilon}(x, \xi; \varepsilon) = p(x, \xi - i\nabla \psi(x); \varepsilon) \mod S_\delta^{k+1}(1) \, .
\]

**Appendix B. Former results**

In the more general setting, that might be more than two Dirichlet operators with spectrum inside of the spectral interval \( I_\varepsilon \), let
\[
\begin{align*}
\text{spec}(H_\varepsilon) \cap I_\varepsilon &= \{ \lambda_1, \ldots, \lambda_N \}, \\
\mathcal{F} &= \text{span}\{ u_1, \ldots, u_N \} \\
\text{spec}(H_{\varepsilon}^{M_j}) \cap I_\varepsilon &= \{ \mu_{j, 1}, \ldots, \mu_{j, n_j} \}, \\
\mathcal{E}_j &= \text{span}\{ v_{j, 1}, \ldots, v_{j, n_j} \}, \quad j \in C \\
\mathcal{E} &= \bigoplus \mathcal{E}_j
\end{align*}
\]
declare the eigenvalues of \( H_\varepsilon \) and of the Dirichlet operators \( H_{\varepsilon}^{M_j} \) defined in (1.20) inside the spectral interval \( I_\varepsilon \) and the corresponding real orthonormal systems of eigenfunctions (these exist because all operators commute with complex conjugation). We write
\[
v_\alpha \quad \text{with} \quad \alpha = (\alpha_1, \alpha_2) \in C := \{ (j, k) | j \in C, 1 \leq k \leq n_j \} \quad \text{and} \quad j(\alpha) := \alpha_1 .
\]

We remark that the number of eigenvalues \( N, n_j, j \in C \) with respect to \( I_\varepsilon \) as defined in (B.1) may depend on \( \varepsilon \).

For a fixed spectral interval \( I_\varepsilon \), it is shown in [K., R., 2012] that the distance \( \text{dist}(\mathcal{F}, \mathcal{F}) := ||\Pi_\varepsilon - \Pi_\varepsilon \Pi_\varepsilon|| \) is exponentially small and determined by \( S_0 \), the Finsler distance between the two nearest neighboring wells.

The following theorem, proven in [K., R., 2012], gives the representation of \( H_\varepsilon \) restricted to an eigenspace with respect to the basis of Dirichlet eigenfunctions.

**Theorem B.1** In the setting of Hypotheses 1.1, 1.4, and (B.1), (B.2), set \( \mathcal{G}_\varepsilon := \{ (v_\alpha, \nu_\beta) \}_{\alpha, \beta \in \mathcal{F}} \), the Gram-matrix, and \( \tilde{\mathcal{G}} := \tilde{\mathcal{G}}^{1/2} \), the orthonormalization of \( \tilde{\mathcal{G}} := (v_{1, 1}, \ldots, v_{m, n_m}) \). Let \( \Pi_\mathcal{F} \) be the orthogonal projection onto \( \mathcal{F} \) and set \( f_\alpha = \Pi_\mathcal{F} f_\alpha \). For \( \mathcal{G}_f := \{ (f_\alpha, \nu_\beta) \}_{\alpha, \beta \in \mathcal{F}} \), we choose \( \tilde{\mathcal{G}} := \tilde{\mathcal{G}}^{1/2} \) as orthonormal basis of \( \mathcal{F} \).

Then there exists \( \varepsilon_0 > 0 \) such that for all \( \sigma < S \) and \( \varepsilon \in (0, \varepsilon_0) \) the following holds.

1. The matrix of \( H_\varepsilon |_\mathcal{F} \) with respect to \( \tilde{\mathcal{G}} \) is given by
\[
\text{diag}(\mu_{1, 1}, \ldots, \mu_{m, n_m}) + (\tilde{\omega}_{\alpha, \beta})_{\alpha, \beta \in \mathcal{F}} + O\left( e^{-\frac{2}{\varepsilon}} \right)
\]

where
\[
\tilde{\omega}_{\alpha, \beta} = \frac{1}{2}(w_{\alpha, \beta} + w_{\beta, \alpha}) = O\left( e^{-\frac{2}{\varepsilon}} \right)
\]
\[
\begin{align*}
\tilde{\omega}_{\alpha, \beta} &= \frac{1}{2}(w_{\alpha, \beta} + w_{\beta, \alpha}) = O\left( e^{-\frac{2}{\varepsilon}} \right)
\end{align*}
\]
with
\[ w_{\alpha, \beta} = \left\langle v_{\alpha}, (1 - 1_{M_{j(\beta)}}) T_{\varepsilon} v_{\beta} \right\rangle_{L^2} = \sum_{x \in (\varepsilon Z)^d} \sum_{\gamma \in (\varepsilon Z)^d} \alpha_\gamma(x; \varepsilon) v_{\beta}(x + \gamma) v_\alpha(x) \] (B.3)

and \( \tilde{w}_{\alpha, \beta} = 0 \) for \( j(\alpha) = j(\beta) \). The remainder \( O(e^{-\frac{1}{\varepsilon^2}}) \) is estimated with respect to the operator norm.

(2) There exists a bijection
\[ b : \text{spec}(H_\varepsilon |_X) \to \text{spec}((\mu_\alpha \delta_{\alpha, \beta} + \tilde{w}_{\alpha, \beta})_{\alpha, \beta \in \mathcal{J}}) \]
such that \( |b(\lambda) - \lambda| = O\left(e^{-\frac{2\varepsilon}{\varepsilon^2}}\right) \)
where the eigenvalues are counted with multiplicity.

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