p-ADIC QUOTIENT SETS: LINEAR RECURRENCE SEQUENCES

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Abstract

Let \((x_n)_{n \geq 0}\) be a linear recurrence of order \(k \geq 2\) satisfying \(x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_kx_{n-k}\) for all integers \(n \geq k\), where \(a_1, \ldots, a_k, x_0, \ldots, x_{k-1} \in \mathbb{Z}\), with \(a_k \neq 0\). Sanna \('\text{The quotient set of } k\text{-generalised Fibonacci numbers is dense in } \mathbb{Q}_p', \text{Bull. Aust. Math. Soc.} 96(1) (2017), 24–29\) posed the question of classifying primes \(p\) for which the quotient set of \((x_n)_{n \geq 0}\) is dense in \(\mathbb{Q}_p\). We find a sufficient condition for denseness of the quotient set of the \(k\)-th-order linear recurrence \((x_n)_{n \geq 0}\) satisfying \(x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_kx_{n-k}\) for all integers \(n \geq k\) with initial values \(x_0 = \cdots = x_{k-2} = 0, x_{k-1} = 1\), where \(a_1, \ldots, a_k \in \mathbb{Z}\) and \(a_k = 1\). We show that, given a prime \(p\), there are infinitely many recurrence sequences of order \(k \geq 2\) whose quotient sets are not dense in \(\mathbb{Q}_p\). We also study the quotient sets of linear recurrence sequences with coefficients in certain arithmetic and geometric progressions.

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1. Introduction and statement of results

For a set of integers \(A\), the set \(R(A) = \{a/b : a, b \in A, b \neq 0\}\) is called the ratio set or quotient set of \(A\). Several authors have studied the denseness of ratio sets of different subsets of \(\mathbb{N}\) in the positive real numbers (see [3, 5–7, 15, 16–20, 24, 25, 29, 30]). An analogous study has also been done for algebraic number fields (see [12, 28]).

For a prime \(p\), let \(\mathbb{Q}_p\) denote the field of \(p\)-adic numbers. The denseness of ratio sets in \(\mathbb{Q}_p\) has been studied by several authors (see [1, 2, 10, 13, 14, 21–23, 27]). Let \((F_n)_{n \geq 0}\) be the sequence of Fibonacci numbers, defined by \(F_0 = 0, F_1 = 1\) and \(F_n = F_{n-1} + F_{n-2}\) for all integers \(n \geq 2\). In [14], Garcia and Luca showed that the ratio set of Fibonacci numbers is dense in \(\mathbb{Q}_p\) for all primes \(p\). Later, Sanna [27, Theorem 1.2] showed that, for any \(k \geq 2\) and any prime \(p\), the ratio set of the \(k\)-generalised Fibonacci numbers is dense in \(\mathbb{Q}_p\). Sanna remarked that his result could be extended to other linear recurrences over the integers. However, he used some specific properties of the \(k\)-generalised Fibonacci numbers in the proof. Therefore, he asked the following question.

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**Question 1.1** [27, Question 1.3]. Let \((S_n)_{n \geq 0}\) be a linear recurrence of order \(k \geq 2\) satisfying \(S_n = a_1 S_{n-1} + a_2 S_{n-2} + \cdots + a_k S_{n-k}\) for all integers \(n \geq k\), where \(a_1, \ldots, a_k, S_0, \ldots, S_{k-1} \in \mathbb{Z}\), with \(a_k \neq 0\). For which prime numbers \(p\) is the quotient set of \((S_n)_{n \geq 0}\) dense in \(\mathbb{Q}_p\)?

In [13], Garcia et al. studied the quotient sets of certain second-order recurrences: given two fixed integers \(r\) and \(s\), let \((a_n)_{n \geq 0}\) be defined by \(a_n = ra_{n-1} + sa_{n-2}\) for \(n \geq 2\) with initial values \(a_0 = 0\) and \(a_1 = 1\), and let \((b_n)_{n \geq 0}\) be defined by \(b_n = rb_{n-1} + sb_{n-2}\) for \(n \geq 2\) with initial values \(b_0 = 2\) and \(b_1 = r\).

**Theorem 1.2** [13, Theorem 5.2]. With the notation as above, let \(A = \{a_n : n \geq 0\}\) and \(B = \{b_n : n \geq 0\}\).

(a) If \(p | s\) and \(p \nmid r\), then \(R(A)\) is not dense in \(\mathbb{Q}_p\).
(b) If \(p \nmid s\), then \(R(A)\) is dense in \(\mathbb{Q}_p\).
(c) For all odd primes \(p\), \(R(B)\) is dense in \(\mathbb{Q}_p\) if and only if there exists a positive integer \(n\) such that \(p | b_n\).

We study ratio sets of some other linear recurrences over the set of integers. Our results give some answers to Question 1.1. Our first result gives a sufficient condition for the denseness of the ratio sets of certain \(k\)th-order recurrence sequences. Finding a general solution to Question 1.1 seems to be a difficult problem. Hence, in Theorem 1.3, we consider \(k\)th-order recurrence sequences for which \(a_k = 1\) and with initial values \(x_0 = \cdots = x_{k-2} = 0, x_{k-1} = 1\). Recall that a *Pisot number* is a positive algebraic integer greater than 1 all of whose conjugate elements have absolute value less than 1.

**Theorem 1.3.** Let \((x_n)_{n \geq 0}\) be a \(k\)th-order linear recurrence satisfying

\[
x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_{k-1} x_{n-k+1} + x_{n-k}
\]

for all integers \(n \geq k\) with initial values \(x_0 = x_1 = \cdots = x_{k-2} = 0, x_{k-1} = 1\) and \(a_1, \ldots, a_{k-1} \in \mathbb{Z}\). Suppose that the characteristic polynomial of the recurrence sequence has a root \(\pm \alpha\), where \(\alpha\) is a Pisot number. If \(p\) is a prime such that the characteristic polynomial of the recurrence sequence is irreducible in \(\mathbb{Q}_p\), then the quotient set of \((x_n)_{n \geq 0}\) is dense in \(\mathbb{Q}_p\).

If we take \(k = 3\) in Theorem 1.3, then we have the following corollary.

**Corollary 1.4.** Let \((x_n)_{n \geq 0}\) be a third-order linear recurrence satisfying

\[
x_n = ax_{n-1} + bx_{n-2} + x_{n-3}
\]

for all integers \(n \geq 3\) with initial values \(x_0 = x_1 = 0, x_2 = 1\), where the integers \(a\) and \(b\) are such that \((a + b)(b - a - 2) < 0\). If \(p\) is a prime such that the characteristic polynomial of the recurrence sequence is irreducible in \(\mathbb{Q}_p\), then the quotient set of \((x_n)_{n \geq 0}\) is dense in \(\mathbb{Q}_p\).

We discuss two examples as applications of Corollary 1.4.
**Example 1.5.** For \( a \in \mathbb{N} \), let \( \ell \) be an odd positive integer less than \( 2a \). Let \((x_n)_{n \geq 0}\) be a linear recurrence satisfying \( x_n = ax_{n-1} + (a-\ell)x_{n-2} + x_{n-3} \) for all integers \( n \geq 3 \) with initial values \( x_0 = x_1 = 0, x_2 = 1 \). Then \( a \) and \( b := a - \ell \) satisfy \( (a + b)(b - a - 2) < 0 \). The characteristic polynomial \( p(x) = x^3 - ax^2 - (a - \ell)x - 1 \) is irreducible in \( \mathbb{Q}_2 \) because \( p(0) \neq 0 \) and \( p(1) = -2a + \ell \neq 0 \) (mod 2). Therefore, by Theorem 1.3, \( R((x_n)_{n \geq 0}) \) is dense in \( \mathbb{Q}_2 \).

**Example 1.6.** For \( a \in \mathbb{N} \) such that \( 3 \nmid a \), let \( \ell \) be an odd positive integer less than \( 2a \) and such that \( 3 \mid \ell \). Let \((x_n)_{n \geq 0}\) be a linear recurrence satisfying \( x_n = ax_{n-1} + (a-\ell)x_{n-2} + x_{n-3} \) for all integers \( n \geq 3 \) with initial values \( x_0 = x_1 = 0, x_2 = 1 \). Then \( a \) and \( b := a - \ell \) satisfy \( (a + b)(b - a - 2) < 0 \). The characteristic polynomial \( p(x) = x^3 - ax^2 - (a - \ell)x - 1 \) is irreducible in \( \mathbb{Q}_3 \) because \( p(0) \neq 0 \), \( p(1) = -2a + \ell \neq 0 \) (mod 3) and \( p(2) = -6a + 2\ell + 7 \neq 0 \) (mod 3). Therefore, by Theorem 1.3, \( R((x_n)_{n \geq 0}) \) is dense in \( \mathbb{Q}_3 \).

Next, we consider recurrence sequences whose \( n \)th term depends on all the previous \( n - 1 \) terms and obtain the following results.

**Theorem 1.7.** Let \((x_n)_{n \geq 0}\) be a linear recurrence satisfying
\[
x_n = x_{n-1} + 2x_{n-2} + \cdots + (n - 1)x_1 + nx_0
\]
for all integers \( n \geq 1 \) with initial value \( x_0 = 1 \). Then the quotient set of \((x_n)_{n \geq 0}\) is dense in \( \mathbb{Q}_p \) for all primes \( p \).

The recurrence relation given in Theorem 1.7 generates a subsequence of the Fibonacci sequence.

**Theorem 1.8.** Let \((x_n)_{n \geq 0}\) be a linear recurrence satisfying
\[
x_n = ax_{n-1} + arx_{n-2} + \cdots + ar^{n-1}x_0
\]
for all integers \( n \geq 1 \), with \( x_0, a, r \in \mathbb{Z} \). Then the quotient set of \((x_n)_{n \geq 0}\) is not dense in \( \mathbb{Q}_p \) for all primes \( p \).

In Theorem 1.2, Garcia et al. studied second-order recurrence relations with specific initial values. In the following result, we consider a particular second-order recurrence sequence with arbitrary initial values \( x_0 \) and \( x_1 \) in the set of integers.

**Theorem 1.9.** Let \((x_n)_{n \geq 0}\) be a second-order linear recurrence satisfying \( x_n = 2ax_{n-1} - a^2x_{n-2} \) for all integers \( n \geq 2 \), where \( a, x_0, x_1 \in \mathbb{Z} \). Then the quotient set of \((x_n)_{n \geq 0}\) is dense in \( \mathbb{Q}_p \) for all primes \( p \) satisfying \( p \nmid a(x_1 - ax_0) \).

For a prime \( p \), let \( v_p \) denote the \( p \)-adic valuation. The following theorem gives a set of linear recurrence sequences of order \( k \) whose ratio sets are not dense in \( \mathbb{Q}_p \).

**Theorem 1.10.** Let \((x_n)_{n \geq 0}\) be a linear recurrence of order \( k \geq 2 \) satisfying
\[
x_n = a_1x_{n-1} + \cdots + a_kx_{n-k}
\]

for all integers \( n \geq k \), where \( x_0, \ldots, x_{k-1}, a_1, \ldots, a_k \in \mathbb{Z} \). If \( p \) is a prime such that \( p \nmid a_k \) and \( \min\{v_p(a_j) : 1 \leq j < k\} \leq \max\{v_p(x_m) - v_p(x_n) : 0 \leq m, n < k\} \), then the quotient set of \( (x_n)_{n \geq 0} \) is not dense in \( \mathbb{Q}_p \).

The next example is an application of Theorem 1.10. Given a prime \( p \), this example gives infinitely many recurrence sequences of order \( k \geq 2 \) whose quotient sets are not dense in \( \mathbb{Q}_p \).

**Example 1.11.** Let \( (x_n)_{n \geq 0} \) be a linear recurrence of order \( k \geq 2 \) satisfying

\[
x_n = a_1x_{n-1} + \cdots + a_kx_{n-k}
\]

for all integers \( n \geq k \), where \( x_0 = x_1 = \cdots = x_{k-1} = 1 \) and \( a_1, \ldots, a_k \in \mathbb{Z} \). If \( p \) is a prime such that \( p \mid a_j, 1 \leq j \leq k - 1 \), and \( p \nmid a_k \), then by Theorem 1.10, the quotient set of \( (x_n)_{n \geq 0} \) is not dense in \( \mathbb{Q}_p \).

### 2. Preliminaries

Let \( p \) be a prime and \( r \) be a nonzero rational number. Then \( r \) has a unique representation of the form \( r = \pm p^k a/b \), where \( k \in \mathbb{Z}, a, b \in \mathbb{N} \) and \( \gcd(a, p) = \gcd(p, b) = \gcd(a, b) = 1 \). The \( p \)-adic valuation of \( r \) is \( v_p(r) = k \) and its \( p \)-adic absolute value is \( ||r||_p = p^{-k} \). By convention, \( v_p(0) = \infty \) and \( ||0||_p = 0 \). The \( p \)-adic metric on \( \mathbb{Q} \) is \( d(x, y) = ||x - y||_p \). The field \( \mathbb{Q}_p \) of \( p \)-adic numbers is the completion of \( \mathbb{Q} \) with respect to the \( p \)-adic metric. The \( p \)-adic absolute value can be extended to a finite normal extension field \( K \) over \( \mathbb{Q}_p \) of degree \( n \). For \( \alpha \in K \), define \( ||\alpha||_p \) as the \( n \)th root of the determinant of the matrix of the linear transformation from the vector space \( K \) over \( \mathbb{Q}_p \) to itself defined by \( x \mapsto \alpha x \) for all \( x \in K \). Also, define \( v_p(\alpha) \) as the unique rational number satisfying \( ||\alpha||_p = p^{-v_p(\alpha)} \).

The following results will be used in the proofs of our theorems.

**Lemma 2.1** [13, Lemma 2.1]. If \( S \) is dense in \( \mathbb{Q}_p \), then for each finite value of the \( p \)-adic valuation, there is an element of \( S \) with that valuation.

**Lemma 2.2** [13, Lemma 2.3]. Let \( A \subset \mathbb{N} \).

1. If \( A \) is \( p \)-adically dense in \( \mathbb{N} \), then \( RA \) is dense in \( \mathbb{Q}_p \).
2. If \( RA \) is \( p \)-adically dense in \( \mathbb{N} \), then \( RA \) is dense in \( \mathbb{Q}_p \).

**Theorem 2.3** [4, Theorem 1]. Let \( \alpha_1, \ldots, \alpha_n \) be units in \( \Omega_p \), the completion of the algebraic closure of \( \mathbb{Q}_p \), which are algebraic over the rationals \( \mathbb{Q} \) and whose \( p \)-adic logarithms are linearly independent over \( \mathbb{Q} \). These logarithms are then linearly independent over the algebraic closure of \( \mathbb{Q} \) in \( \Omega_p \).

### 3. Proof of the theorems

**Proof of Theorem 1.3.** Let \( p(x) = x^k - a_1x^{k-1} - a_2x^{k-2} - \cdots - a_{k-1}x - 1 \) be the characteristic polynomial of the recurrence. Let \( \alpha_1, \ldots, \alpha_k \) be the \( k \) distinct roots of the
characteristic polynomial in its splitting field, say, $K$ over $\mathbb{Q}_p$. The generating function of the sequence is

$$t(x) = \frac{x^{k-1}}{1 - a_1 x - a_2 x^2 - \cdots - x^k} = \sum_{i=1}^{k} \frac{1}{q(\alpha_i)} \sum_{n=0}^{\infty} \alpha_i^n x^n,$$

where $q(x) := p'(x)$, the derivative of the polynomial $p(x)$. Hence, the $n$th term of the sequence is given by

$$x_n = \sum_{i=1}^{k} \frac{1}{q(\alpha_i)} \alpha_i^n, \quad n \geq 0.$$ 

Since $p(0) = -1$, the roots of $p(x)$ are units in the ring formed by elements in $K$ with $p$-adic absolute value less than or equal to 1. Following Sanna’s proof of [27, Theorem 1.2], we can choose an even $t \in \mathbb{N}$ such that the function

$$G(z) := \exp_p(z \log_p(\alpha_i^n))$$

is analytic over $\mathbb{Z}_p$ and the Taylor series of $G(z)$ around 0 converges for all $z \in \mathbb{Z}_p$. Also, note that $x_{nt} = G(n)$ for $n \geq 0$.

We now use a variant of the following lemma which gives the multiplicative independence of any $k-1$ roots among the $k$ roots $\alpha_1, \ldots, \alpha_k$ of the characteristic polynomial $x^k - x^{k-1} - \cdots - x - 1$ of the $k$-generalised Fibonacci sequence in the field of complex numbers.

**Lemma 3.1** [11, Lemma 1]. With the notation above, each set of $k-1$ different roots $\alpha_1, \ldots, \alpha_{k-1}$ is multiplicatively independent, that is, $\alpha_1^{e_1} \cdots \alpha_{k-1}^{e_{k-1}} = 1$ for some integers $e_1, \ldots, e_{k-1}$ if and only if $e_1 = \cdots = e_{k-1} = 0$.

Let $\sigma(\alpha_1) = \pm \alpha$, where $\alpha$ is a Pisot number with absolute value greater than 1, the other roots, $\sigma(\alpha_2), \ldots, \sigma(\alpha_k)$, having absolute values less than 1, where $\sigma$ is an isomorphism from $\mathbb{Q}(\alpha_1, \ldots, \alpha_k)$ to the splitting field of $p(x)$ over $\mathbb{Q}$ in the field of complex numbers. Therefore, the proof of Lemma 3.1 holds true for the roots of $p(x)$, which are $\sigma(\alpha_1), \ldots, \sigma(\alpha_k)$, since $\log |\sigma(\alpha_1)|$ is positive and $\log |\sigma(\alpha_2)|, \ldots, \log |\sigma(\alpha_k)|$ are negative. Hence, $\sigma(\alpha_1), \ldots, \sigma(\alpha_{k-1})$ are multiplicatively independent, implying that $\alpha_1^{e_1}, \ldots, \alpha_{k-1}^{e_{k-1}}$ are multiplicatively independent. Thus, $\log_p(\alpha_1), \ldots, \log_p(\alpha_{k-1})$ are linearly independent over $\mathbb{Z}$ and hence linearly independent over the algebraic numbers by Theorem 2.3.

Suppose $G'(0) = \sum_{i=0}^{k} (1/q(\alpha_i)) \log_p(\alpha_i) = 0$. Since $\log_p(\alpha_1) = -\log_p(\alpha_1) - \cdots - \log_p(\alpha_k)$ as the product of the roots is $-1$ and $t$ is even, we obtain

$$\sum_{i=1}^{k-1} \left( \frac{1}{q(\alpha_i)} - \frac{1}{q(\alpha_k)} \right) \log_p(\alpha_i) = 0.$$
By linear independence of \( \log_p(\alpha'_1), \ldots, \log_p(\alpha'_{k-1}) \), we have \( 1/q(\alpha_1) = \cdots = 1/q(\alpha_k) = c \), for some \( p \)-adic number \( c \). This gives \( k \) distinct roots \( \alpha_1, \ldots, \alpha_k \) of the \((k-1)\)-degree polynomial \( q(x) - 1/c \), which is not possible. Therefore, \( G'(0) \neq 0 \).

Since

\[
G(z) = \sum_{j=0}^{\infty} \frac{G^{(j)}(0)}{j!} z^j
\]

converges at \( z = 1 \), it follows that \( \|G^{(j)}(0)/j!\|_p \to 0 \). Hence, there exists an integer \( \ell \) such that \( \nu_p(G^{(j)}(0)/j!) \geq -\ell \) for all \( j \). Thus, we obtain \( G(mp^h) = G'(0)mp^h + d \) where \( \nu_p(d) \geq 2h - \ell \) for all \( m, h \geq 0 \). Also, \( G(0) = 0 \) for \( h > h_0 := \ell + \nu_p(G'(0)) \) and hence

\[
\nu_p\left( \frac{G(mp^h)}{G(p^h)} - m \right) \geq h - h_0.
\]

This yields

\[
\lim_{h \to \infty} \left\| \frac{G(mp^h)}{G(p^h)} - m \right\|_p = 0,
\]

and hence \( R(G(n)_{n \geq 0}) \) is \( p \)-adically dense in \( \mathbb{N} \). Since \( x_n = G(n), n \geq 0 \), we find that \( R((x_n)_{n \geq 0}) \) is also \( p \)-adically dense in \( \mathbb{N} \). Therefore, by Lemma 2.2, \( R((x_n)_{n \geq 0}) \) is dense in \( \mathbb{Q}_p \).

\section*{Proof of Corollary 1.4} Since \( p(1)p(-1) = (-a-b)(b-a-2) > 0 \) and \( p(0) = -1 \), by continuity of the polynomial function in \( \mathbb{R} \), \( p(x) \) has one real root with absolute value greater than 1 and two other roots with absolute values less than 1. Hence, the characteristic polynomial has a root \( \pm \alpha \), where \( \alpha \) is a Pisot number, and the corollary follows from Theorem 1.3.

We need the following result to prove Theorem 1.7.

\section*{Corollary 3.2 \cite[Corollary 2.2]{9}} The linear recurrence relation \( x_{n+1} = x_n + 2x_{n-1} + \cdots + nx_1 + (n+1)x_0, n \geq 0 \), with the initial data \( x_0 = 1 \) has the solution

\[
x_n = \frac{1}{\sqrt{5}} \left( \left\lfloor \frac{3 + \sqrt{5}}{2} \right\rfloor - \left\lfloor \frac{3 - \sqrt{5}}{2} \right\rfloor \right)^n, \quad n \geq 1.
\]

\section*{Proof of Theorem 1.7} By Corollary 3.2, for \( n \geq 1 \),

\[
x_n = \frac{1}{\sqrt{5}} \left( \left\lfloor \frac{3 + \sqrt{5}}{2} \right\rfloor - \left\lfloor \frac{3 - \sqrt{5}}{2} \right\rfloor \right)^n = \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{5}} = F_{2n},
\]

where \( \alpha = (1 + \sqrt{5})/2, \beta = (1 - \sqrt{5})/2 \) and \( F_n \) denotes the \( n \)th Fibonacci number which is obtained by the Binet formula. From \cite{14}, the ratio set of the Fibonacci numbers is dense in \( \mathbb{Q}_p \) for all primes \( p \). Therefore, by Lemma 2.1, \( \nu_p(F_n) \) is
not bounded. Hence, for any \( j \in \mathbb{N} \), there exists \( F_m \) such that \( \nu_p(F_m) \geq j \), that is, 
\[
(\alpha^m - \beta^m)/\sqrt{5} \equiv 0 \pmod{p^j}
\]
which gives \( \alpha^m \equiv \beta^m \pmod{p^j} \). This yields 
\[
\alpha^{2mp^j-1}(p-1) = (\alpha^m)^{p^j-1}(p-1) \equiv (\alpha^m \beta^m)^{p^j-1}(p-1) \pmod{p^j}.
\]
Since \( \alpha \beta = -1 \), by using Euler’s theorem, we find that 
\[
\alpha^{2mp^j-1}(p-1) \equiv (\alpha^m \beta^m)^{p^j-1}(p-1) \equiv 1 \pmod{p^j}.
\]
This gives \( \alpha^{2k} \equiv \beta^{2k} \equiv 1 \pmod{p^j} \), where \( k = mp^{j-1}(p-1) \). Hence, 
\[
\frac{x_{kn}}{x_k} = \frac{F_{2kn}}{F_{2k}} = \frac{(\alpha^{2k})^n - (\beta^{2k})^n}{\alpha^{2k} - \beta^{2k}} = (\alpha^{2k})^{n-1} + (\alpha^{2k})^{n-2}\beta^{2k} + \cdots + (\beta^{2k})^{n-1},
\]
which is congruent to \( n \) modulo \( p^j \). Since, for \( n \in \mathbb{N} \), there exists \( k \in \mathbb{N} \) such that 
\[
\|x_{kn}/x_k - n\|_p \leq p^{-j},
\]
\( R((x_n)_{n \geq 0}) \) is \( p \)-adically dense in \( \mathbb{N} \). Therefore, by Lemma 2.2, 
\( R((x_n)_{n \geq 0}) \) is dense in \( \mathbb{Q}_p \).

We need the following results to prove Theorem 1.8.

**Theorem 3.3** [9, Theorem 3.1]. The numbers \( x_n \) are solutions of the linear recurrence relation with constant coefficients in geometric progression \( x_{n+1} = ax_n + aq^n, n \geq 0 \), with initial data \( x_0, \) if and only if they form the geometric progression given by the formula 
\[
x_n = a x_0 (a + q)^{n-1}, n \geq 1.
\]

**Lemma 3.4** [13, Lemma 2.2]. If \( A \) is a geometric progression in \( \mathbb{Z} \), then \( R(A) \) is not dense in any \( \mathbb{Q}_p \).

**Proof of Theorem 1.8.** By Theorem 3.3, \( (x_n)_{n \geq 1} \) forms a geometric progression whose \( n \)th term is \( ax_0 (a + q)^{n-1} \) for \( n \geq 1 \). Hence, by Lemma 3.4, \( R((x_n)_{n \geq 0}) \) is not dense in \( \mathbb{Q}_p \) for any prime \( p \).

To prove Theorem 1.9 we need some results on the uniform distribution of sequences of integers. Recall that a sequence \( (x_n)_{n \geq 0} \) is said to be uniformly distributed modulo \( m \) if each residue occurs equally often, that is, 
\[
\lim_{N \to \infty} \frac{\#\{n \leq N \mid x_n \equiv t \pmod{m}\}}{N} = \frac{1}{m}
\]
for all \( t \in \mathbb{Z} \).

**Proposition 3.5** [8, Proposition 1]. Suppose \( (G_n)_{n \geq 0} \) is the sequence of integers determined by the recurrence relation \( G_{n+1} = AG_n - BG_{n-1} \) with initial values \( G_0, G_1 \) where \( A, B, G_0, G_1 \in \mathbb{Z} \). If \( A = 2a, B = a^2 \), then \( (G_n)_{n \geq 0} \) is uniformly distributed modulo \( a \) prime \( p \) if and only if \( p \nmid a(G_1 - aG_0) \).

**Theorem 3.6** [8, Theorem]. Suppose \( (G_n)_{n \geq 0} \) is the sequence of integers determined by the recurrence relation \( G_{n+1} = AG_n - BG_{n-1} \) with initial values \( G_0, G_1 \) where \( A, B, G_0, G_1 \in \mathbb{Z} \). If \( (G_n)_{n \geq 0} \) is uniformly distributed modulo \( p^h \) with \( h > 1 \) if and only if
(1) \( p > 3; \) or
(2) \( p = 3 \) and \( A^2 \not\equiv B \pmod{9}; \) or
(3) \( p = 2, A \equiv 2 \pmod{4}, B \equiv 1 \pmod{4}. \)

**Proof of Theorem 1.9.** Let \( p \) be a prime. The given recurrence sequence \((x_n)_{n \geq 0}\) satisfies the hypotheses of Proposition 3.5, and hence \((x_n)_{n \geq 0}\) is uniformly distributed modulo \( p \). If \( p > 3 \), then by Theorem 3.6(1), \((x_n)_{n \geq 0}\) is uniformly distributed modulo \( p^k \) with \( k > 1 \), that is,

\[
\lim_{N \to \infty} \frac{\# \{ n \leq N \mid x_n \equiv t \pmod{p^k} \}}{N} = \frac{1}{p^k} > 0.
\]

Therefore, for all \( t \in \mathbb{N} \) and for all \( k > 1 \), there exists \( x_n \) such that \( \|x_n - t\|_p \leq p^{-k} \). Hence, \( R((x_n)_{n \geq 0}) \) is \( p \)-adically dense in \( \mathbb{N} \). Therefore, by Lemma 2.2, \( R((x_n)_{n \geq 0}) \) is dense in \( \mathbb{Q}_p \).

We next consider the remaining primes \( p = 2, 3 \). Since \( p \nmid a(x_1 - ax_0) \), we have \( p \nmid a \). It is easy to check that \( p = 3 \) satisfies the condition given in Theorem 3.6(2) and \( p = 2 \) satisfies the condition given in Theorem 3.6(3). The rest of the proof follows similarly as shown in the case \( p > 3 \). This completes the proof of the theorem. \( \square \)

We need the following lemma to prove Theorem 1.10.

**Lemma 3.7** [26, Lemma 3.3]. Let \((r_n)_{n \geq 0}\) be a linearly recurring sequence of order \( k \geq 2 \) given by \( r_n = a_1 r_{n-1} + \cdots + a_k r_{n-k} \) for each integer \( n \geq k \), where \( r_0, \ldots, r_{k-1} \) and \( a_1, \ldots, a_k \) are all integers. Suppose that there exists a prime number \( p \) such that \( p \nmid a_k \) and \( \min\{\nu_p(a_j) : 1 \leq j < k\} > \max\{\nu_p(r_m) - \nu_p(r_n) : 0 \leq m, n < k\} \).

Then \( \nu_p(r_n) = \nu_p(r_n \pmod{k}) \) for each nonnegative integer \( n \).

**Proof of Theorem 1.10.** By Lemma 3.7,

\[ \nu_p(x_n/x_m) = \nu_p(x_n \pmod{k}) - \nu_p(x_m \pmod{k}) \leq M \]

for all \( n, m \in \mathbb{N} \cup \{0\} \), where \( M = \max\{\nu_p(x_i) : i = 0, 1, \ldots, k-1\} \). Therefore, by Lemma 2.1, \( R((x_n)_{n \geq 0}) \) is not dense in \( \mathbb{Q}_p \). \( \square \)

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