THE TOPOLOGY OF FOLIATIONS FORMED BY THE GENERIC K-ORBITS OF A SUBCLASS OF THE INDECOMPOSABLE MD5-GROUPS

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Abstract

The present paper is a continuation of [13], [14] of the authors. Specifically, the paper considers the MD5-foliations associated to connected and simply connected MD5-groups such that their Lie algebras have 4-dimensional commutative derived ideal. In the paper, we give the topological classification of all considered MD5-foliations. A description of these foliations by certain fibrations or suitable actions of \( \mathbb{R}^2 \) and the Connes’ C*-algebras of the foliations which come from fibrations are also given in the paper.

Key words: Lie group, Lie algebra, MD5-group, MD5-algebra, K-orbits, Foliation, Measured Foliation.

2000AMS Mathematics Subject Classification: Primary 22E45, Secondary 46E25, 20C20.
Introduction

Our point of departure is the problem of finding out of the classes of C*-algebras which can be described by the operator KK-functors. In 1980, studying the Kirillovs Orbit Method, D. N. Diep suggested to consider the class of MD-groups. An MD-group with dimension n (for brevity, an MDn-group) in his terms (see [2, Section 4.1]) is an n-dimensional solvable real Lie group whose orbits in the co-adjoint representation (i.e. the K-representation) are the orbits of zero or maximal dimension. The Lie algebra of each MDn-group is called an MDn-algebra.

The first reason for studying the MD-groups is the fact that the C*-algebras of the MD-groups can be described by KK-functors (see [2, Chapters 3, 5], [4, Section 3]). On the other hand, for every MD-group G, the family of K-orbits of maximal dimension forms a measured foliation in terms of A. Connes ([1]). This foliation is called MD-foliation associated with G. In general, the leaf space of a foliation with the quotient topology is a fairly untractable topological space. To improve upon the shortcoming, A. Connes associates with each measurable foliation a C*-algebra (see [1, Section 2]). In the cases of Reeb foliations, the method of KK-functor has been proved very effective in describing the Connes’ C*-algebras by A. M. Torpe (see [5, Sections 3, 4]).

Combining methods of A. Kirillov (see [3, Section 15]) and A. Connes (see [1, Sections 2, 5]), Vu has considered MD4-foliations associated with all indecomposable connected MD4-groups in [6], [7], [8]. Recently, in [9], [11], [12] Vu together with Nguyen Cong Tri anh Duong Minh Thanh have studied MD5-foliations associated with the MD5-groups such that the first derived ideal of corresponding Lie algebras is \( \mathbb{R}^k \); \( k = 1, 2, 3 \). The present paper is a continuation of [13], [14] of the authors and it is concerned with MD5-foliations associated with the indecomposable connected and simply connected MD5-groups such that MD5-algebras of them have 4-dimensional
We shall begin our discussion in Section 1 by recalling some preliminary results and notations which will be used later. For more details we refer the reader to References [1], [3].

In Section 2 we shall relist all MD5-algebras with 4-dimensional commutative derived ideal and recall the geometric description of the K-orbits of corresponding MD5-groups which have been announced in [13], [14].

Section 3 will be devoted to the discussion of the MD5-foliations associated with considered MD5-groups. We shall give a topological classification of these foliations and describe them by certain fibrations or suitable actions of $\mathbb{R}^2$. In addition, the Connes’ C*-algebras of the MD5-foliations which come from certain fibrations will be also analytically characterized.

1 Preliminaries

1.1 Foliations

Let V be a smooth manifold. We denote by TV its tangent bundle, so that for each $x \in V$, $T_x V$ is the tangent space of V at x.

**Definition 1.1.1.** A smooth subbundle $\mathcal{F}$ of TV is called integrable if and only if the following condition is satisfied: every $x \in V$ is contained in a submanifold W of V such that $T_p(W) = \mathcal{F}_p (\forall p \in W)$.

**Definition 1.1.2.** A foliation $(V, \mathcal{F})$ is given by a smooth manifold V and an integrable subbundle $\mathcal{F}$ of TV. Then, V is called the foliated manifold and $\mathcal{F}$ is called the subbundle defining the foliation. The dimension of $\mathcal{F}$ is also called the dimension of foliation $(V, \mathcal{F})$.
Definition 1.1.3. Each maximal connected submanifolds $L$ of $V$ such that $T_x(L) = \mathcal{F}_x \ (\forall x \in L)$ is called a leaf of the foliation $(V, \mathcal{F})$.

The set of leaves with the quotient topology is denoted by $V/\mathcal{F}$ and called the space of leaves or leaf space of $(V, \mathcal{F})$. In general, it is a fairly untractable topological space.

The partition of $V$ in leaves: $V = \bigcup_{\alpha \in V/\mathcal{F}} L_\alpha$ is characterized geometrically by the following local triviality: Every $x \in V$ has a system of local coordinates $\{U; x^1, x^2, ..., x^n\}(x \in U; n = \dim V)$ so that for any leaf $L$ with $L \cap U \neq \emptyset$, each connected component of $L \cap U$ (which is called a plaque of the leaf $L$) is given by the equations

$$x^{k+1} = c^1, x^{k+2} = c^2, ..., x^n = c^{n-k}, \ k = \dim \mathcal{F} < n;$$

where $c^1, c^2, ..., c^{n-k}$ are constants (depending on each plaque). Each such system $\{U, x^1, x^2, ..., x^n\}$ is called a foliation chart.

A foliation can be given by a partition of $V$ in a family $C$ of its submanifolds if there exist some integrable subbundle $\mathcal{F}$ of $TV$ such that each $L \in C$ is a maximal connected integral submanifold of . Then $C$ is the family of leaves of the foliation $(V, \mathcal{F})$. Sometimes $C$ is identified with $\mathcal{F}$ and we say that $(V, \mathcal{F})$ is formed by $C$.

1.2 Measurable Foliations

Definition 1.2.1. A submanifold $N$ of the foliated manifold $V$ is called a transversal if and only if $T_xV = T_xN \oplus \mathcal{F}_x, \forall x \in N$. Thus, $\dim N = n - \dim \mathcal{F} = \text{codim} \mathcal{F}$.

A Borel subset $B$ of $V$ such that $B \cap L$ is countable for any leaf $L$ is called a Borel transversal to $(V, \mathcal{F})$. 
Definition 1.2.2. A transverse measure $\Lambda$ for the foliation $(V, \mathcal{F})$ is $\sigma$-additive map $B \mapsto \Lambda(B)$ from the set of all Borel transversals to $[0, +\infty]$ such that the following conditions are satisfied:

(i) If $\psi : B_1 \to B_2$ is a Borel bijection and $\psi(x)$ is on the leaf of any $x \in B_1$, then $\Lambda(B_1) = \Lambda(B_2)$.

(ii) $\Lambda(K) < +\infty$ if $K$ is any compact subset of a smooth transversal submanifold of $V$.

By a measurable foliation we mean a foliation $(V, \mathcal{F})$ equipped with some transverse measure $\Lambda$.

Let $(V, \mathcal{F})$ be a foliation with $\mathcal{F}$ is oriented. Then the complement of zero section of the bundle $\Lambda^k(\mathcal{F})$ ($k = \dim \mathcal{F}$) has two components $\Lambda^k(\mathcal{F})^-$ and $\Lambda^k(\mathcal{F})^+$.

Let $\mu$ be a measure on $V$ and $\{U, x_1, x_2, ..., x_n\}$ be a foliation chart of $(V, \mathcal{F})$. Then $U$ can be identified with the direct product $N \times \Pi$ of some smooth transversal submanifold $N$ of $V$ and some plaque $\Pi$. The restriction of $\mu$ on $U \equiv N \times \Pi$ becomes the product $\mu_N \times \mu_\Pi$ of measures $\mu_N$ and $\mu_\Pi$ respectively.

Let $X \in C^\infty(\Lambda^k(\mathcal{F}))^+$ be a smooth k-vector field and $\mu_X$ be the measure on each leaf $L$ determined by the volume element $X$.

Definition 1.2.3. The measure $\mu$ is called $X$-invariant if and only if $\mu_X$ is proportional to $\mu_\Pi$ for an arbitrary foliation chart $\{U, x_1, x_2, ..., x_n\}$.

Let $(X, \mu), (Y, \nu)$ be two pairs where $X,Y \in C^\infty(\Lambda^k(\mathcal{F}))^+$ and $\mu, \nu$ are measures on $V$ such that $\mu$ is $X$-invariant, $\nu$ is $Y$-invariant.

Definition 1.2.4. $(X, \mu), (Y, \nu)$ are equivalent if and only if $Y = \varphi X$ and $\mu = \varphi \nu$ for some $\varphi \in C^\infty(V)$. 

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There is one bijective map between the set of transverse measures for \((V, F)\) and the set of equivalent classes of pairs \((X, \mu)\), where \(X \in C^\infty(\Lambda^k(F))^+\) and \(\mu\) is a \(X\)-invariant measure on \(V\).

Thus, to prove that \((V, F)\) is measurable, we only need to choose some suitable pair \((X, \mu)\) on \(V\).

2 MD5-algebras with 4-dimensional Commutative Derived Ideal and Geometry of K-orbits of Corresponding MD5-Groups

2.1 The list of MD5-algebras with 4-dimensional Commutative Derived Ideal

In [13] the first author have listed all MD5-algebras such that their first derived ideals are \(\mathbb{R}^4\). For completeness, we will relist all these MD5-algebras here.

**Proposition 2.1** (see [13, Theorem 3.2]). Let \(\mathcal{G}\) be an indecomposable MD5-algebra with \(\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}] \cong \mathbb{R}^4\). Then we can choose a suitable basis \((X_1, X_2, X_3, X_4, X_5)\) of \(\mathcal{G}\) such that \(\mathcal{G}^1 = \mathbb{R}.X_3 \oplus \mathbb{R}.X_3 \oplus \mathbb{R}.X_4 \oplus \mathbb{R}.X_5 \equiv \mathbb{R}^4\), \(ad_{X_1} \in \text{End}(\mathcal{G}^1) \equiv \text{Mat}_4(\mathbb{R})\) and \(\mathcal{G}\) is isomorphic to one and only one of the following Lie algebras.

1. \(\mathcal{G}_{5,4,1}(\lambda_1, \lambda_2, \lambda_3)\) :

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix};
\]
\[ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \setminus \{0, 1\}, \quad \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1. \]

2. \( G_{5,4,2(\lambda_1,\lambda_2)} : \)
\[
ad_{X_1} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0, 1\}, \lambda_1 \neq \lambda_2.
\]

3. \( G_{5,4,3(\lambda)} : \)
\[
ad_{X_1} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0, 1\}.
\]

4. \( G_{5,4,4(\lambda)} : \)
\[
ad_{X_1} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0, 1\}.
\]

5. \( G_{5,4,5} : \)
\[
ad_{X_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]
6. $G_{5,4,6(\lambda_1, \lambda_2)}$ :

$$ad_{X_1} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0, 1\}, \lambda_1 \neq \lambda_2.$$

7. $G_{5,4,7(\lambda)}$ :

$$ad_{X_1} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

8. $G_{5,4,8(\lambda)}$ :

$$ad_{X_1} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

9. $G_{5,4,9(\lambda)}$ :

$$ad_{X_1} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

10. $G_{5,4,10}$ :

$$ad_{X_1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
11. $\mathcal{G}_{5.4.11}(\lambda_1, \lambda_2, \varphi)$:

\[
ad_{X_1} = \begin{pmatrix} 
\cos \varphi & -\sin \varphi & 0 & 0 \\
\sin \varphi & \cos \varphi & 0 & 0 \\
0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & \lambda_2 
\end{pmatrix};
\]

$\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}, \lambda_1 \neq \lambda_2, \varphi \in (0, \pi)$.

12. $\mathcal{G}_{5.4.12}(\lambda, \varphi)$:

\[
ad_{X_1} = \begin{pmatrix} 
\cos \varphi & -\sin \varphi & 0 & 0 \\
\sin \varphi & \cos \varphi & 0 & 0 \\
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & \lambda 
\end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0, \pi).
\]

13. $\mathcal{G}_{5.4.13}(\lambda, \varphi)$:

\[
ad_{X_1} = \begin{pmatrix} 
\cos \varphi & -\sin \varphi & 0 & 0 \\
\sin \varphi & \cos \varphi & 0 & 0 \\
0 & 0 & \lambda & 1 \\
0 & 0 & 0 & \lambda 
\end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0, \pi).
\]

14. $\mathcal{G}_{5.4.14}(\lambda, \mu, \varphi)$:

\[
ad_{X_1} = \begin{pmatrix} 
\cos \varphi & -\sin \varphi & 0 & 0 \\
\sin \varphi & \cos \varphi & 0 & 0 \\
0 & 0 & \lambda & -\mu \\
0 & 0 & \mu & \lambda 
\end{pmatrix};
\]

$\lambda, \mu \in \mathbb{R}, \mu > 0, \varphi \in (0, \pi)$.
Remarks

Let us recall that each real Lie algebra $G$ define only one connected and simply connected Lie group $G$ such that $\text{Lie}(G) = G$. Therefore we obtain a collection of fourteen families of connected and simply connected MD5-groups corresponding to the indecomposable MD5-algebras given in Theorem 2.1. For convenience, each MD5-group from this collection is also denoted by the same indices as corresponding MD5-algebra. For example, $G_{5,4,2(\lambda_1, \lambda_2)}$ is the connected and simply connected MD5-group corresponding to $G_{5,4,2(\lambda_1, \lambda_2)}$. All of these groups are indecomposable MD5-groups. In [14], we have described geometry of the K-orbits of them. But we now recall this result in the next subsection for completeness.

2.2 The Picture of K-orbits of Corresponding Connected and Simply Connected Groups

Let $G$ be one of considered MD5-groups, $G = \langle X_1, X_2, X_3, X_4, X_5 \rangle$ be its Lie algebra, $G^* = \langle X_1^*, X_2^*, X_3^*, X_4^*, X_5^* \rangle \equiv \mathbb{R}^5$ be the dual space of $G$, $F = \alpha X_1^* + \beta X_2^* + \gamma X_3^* + \delta X_4^* + \sigma X_5^* \equiv (\alpha, \beta, \gamma, \delta, \sigma)$ be an arbitrary element of $G^*$. The notation $\Omega_F$ will be used to denote the K-orbit of $G$ which contains $F$. The geometrical picture of the K-orbits of considered MD5-groups is given by the following proposition which has proved in [14].

**Proposition 2.2** (see [14, Theorems 3.3.1, 3.3.2, 3.3.3, 3.3.4]). The K-orbit $\Omega_F$ of $G$ is described as follows.

1. Let $G$ is one of $G_{5,4,1(\lambda_1, \lambda_2, \lambda_3)}$, $G_{5,4,2(\lambda_1, \lambda_2)}$, $G_{5,4,3(\lambda)}$, $G_{5,4,4(\lambda)}$, $G_{5,4,5}$, $G_{5,4,6(\lambda_1, \lambda_2)}$, $G_{5,4,7(\lambda)}$, $G_{5,4,8(\lambda)}$, $G_{5,4,9(\lambda)}$, $G_{5,4,10}$, $\lambda_1, \lambda_2, \lambda_3, \lambda \in \mathbb{R}\{0, 1\}$.
   1.1. If $\beta = \gamma = \delta = \sigma = 0$ then $\Omega_F = \{F\}$ (the 0-dimensional orbit).
1.2. If $\beta^2 + \gamma^2 + \delta^2 + \sigma^2 \neq 0$ then $\Omega_F$ is the orbit of dimension 2 and it is one of the following:

- $\{(x, \beta e^{a\lambda_1}, \gamma e^{a\lambda_2}, \delta e^{a\lambda_1}, \sigma e^{a}), x, a \in \mathbb{R}\}$ when $G = G_{5,4,1}(\lambda_1, \lambda_2, \lambda_3)$.
- $\{(x, \beta e^{a\lambda_1}, \gamma e^{a\lambda_2}, \delta e^{a}, \sigma e^{a}), x, a \in \mathbb{R}\}$ when $G = G_{5,4,2}(\lambda_1, \lambda_2)$.
- $\{(x, \beta e^{a\lambda}, \gamma e^{a}, \delta e^{a}, \sigma e^{a}), x, a \in \mathbb{R}\}$ when $G = G_{5,4,2}(\lambda)$.
- $\{(x, \beta e^{a\lambda}, \gamma e^{a}, \delta e^{a}, \sigma e^{a}), x, a \in \mathbb{R}\}$ when $G = G_{5,4,4}(\lambda)$.
- $\{(x, \beta e^{a\lambda}, \gamma e^{a}, \delta e^{a}, \sigma e^{a}), x, a \in \mathbb{R}\}$ when $G = G_{5,4,5}$.
- $\{(x, \beta e^{a\lambda_1}, \gamma e^{a\lambda_2}, \delta e^{a}, \sigma e^{a}), x, a \in \mathbb{R}\}$ when $G = G_{5,4,6}(\lambda_1, \lambda_2)$.
- $\{(x, \beta e^{a\lambda}, \gamma e^{a}, \delta e^{a}, \sigma e^{a}), x, a \in \mathbb{R}\}$ when $G = G_{5,4,7}(\lambda)$.
- $\{(x, \beta e^{a\lambda}, \gamma e^{a}, \delta e^{a}, \sigma e^{a}), x, a \in \mathbb{R}\}$ when $G = G_{5,4,8}(\lambda)$.
- $\{(x, \beta e^{a\lambda}, \gamma e^{a}, \delta e^{a}, \gamma e^{a} + \delta e^{a} + \sigma e^{a}), x, a \in \mathbb{R}\}$ when $G = G_{5,4,9}(\lambda)$.
- $\{(x, \beta e^{a\lambda}, \gamma e^{a}, \delta e^{a}, \gamma e^{a} + \delta e^{a} + \sigma e^{a}), x, a \in \mathbb{R}\}$ when $G = G_{5,4,10}$.

2. Let $G$ be one of $G_{5,4,11}(\lambda, \lambda_2, \varphi), G_{5,4,12}(\lambda, \varphi), G_{5,4,13}(\lambda, \varphi); \lambda_1, \lambda_2, \lambda \in \mathbb{R} \setminus \{0\}$; \varphi \in (0, \pi). Let us identify $G_{5,4,11}(\lambda, \lambda_2, \varphi)$, $G_{5,4,12}(\lambda, \varphi)$, $G_{5,4,13}(\lambda, \varphi)$ with $\mathbb{R} \times \mathbb{C} \times \mathbb{R}^2$ and $F$ with $(\alpha, \beta + i\gamma, \delta, \sigma)$. Then we have

2.1. If $\beta + i\gamma = \delta = \sigma = 0$ then $\Omega_F = \{F\}$ (the 0-dimensional orbit).

2.2. If $|\beta + i\gamma|^2 + \delta^2 + \sigma^2 \neq 0$ then $\Omega_F$ is the orbit of dimension 2 and it is one of the following:

- $\{(x, (\beta + i\gamma)e^{a\lambda_1}, \delta e^{a\lambda_2}), x, a \in \mathbb{R}\}$ when $G = G_{5,4,11}(\lambda, \lambda_2)$.
- $\{(x, (\beta + i\gamma)e^{a\lambda}, \delta e^{a\lambda}), x, a \in \mathbb{R}\}$ when $G = G_{5,4,12}(\lambda, \varphi)$.
- $\{(x, (\beta + i\gamma)e^{a\lambda_1}, \delta e^{a\lambda}, \sigma e^{a\lambda}), x, a \in \mathbb{R}\}$ when $G = G_{5,4,13}(\lambda, \varphi)$.
3. Let $G$ be $G_{5,4,14}$($\lambda,\mu,\varphi$). Let us identify $G_{5,4,14}$(*,$\lambda,\mu,\varphi$) with $\mathbb{R} \times \mathbb{C} \times \mathbb{C}$ and $F$ with $(\alpha, \beta + i\gamma, \delta + i\sigma); \lambda, \mu \in \mathbb{R}; \mu > 0; \varphi \in (0, \pi)$. Then we have

3.1. If $\beta + i\gamma = \delta + i\sigma = 0$ then $\Omega_F = \{F\}$ (the 0-dimensional orbit).

3.2. If $|\beta^2 + i\gamma^2| + |\delta^2 + i\sigma^2| \neq 0$ then

$$
\Omega_F = \{(x, (\beta + i\gamma)e^{ae^{-i\varphi}}, (\delta + i\sigma)e^{a(\lambda-i\mu)}), x, a \in \mathbb{R}\}
$$

(the 2-dimensional orbit). \qed

3 On MD5-foliations Associated to Considered MD5-groups and Their Topological Classification

3.1 Foliations formed by K-orbits of dimension two of considered MD5-groups

In the introductory section we have emphasized that the family of maximal-dimensional K-orbits of every MD-group forms a measured foliation in terms of A. Connes. Namely, this fact has proved in [7], [9], [11], [12] not only for all connected MD4-groups but also for all connected MD5-groups with corresponding Lie algebras have commutative derived ideal of dimension $k$ with $k < 4$. The following is a similar assertion for the MD5-groups considered in this paper. It is also proved by the same method as the one in [7], [9], [11], [12]. So we omit the proof.

**Theorem 3.1.** Let $G$ be one of the connected and simply connected MD5-groups corresponding to the MD5-algebras listed in Theorem 2.1, $\mathcal{F}_G$ be the family of all its K-orbits of dimension two and $V_G := \bigcup \Omega/\Omega \in \mathcal{F}_G$. Then
\((V_G, \mathcal{F}_G)\) is a measurable foliation in the sense of Connes. We call it MD5-foliation associated with MD5-group \(G\). 

Remarks and Notations

It should be noted that \(V_G\) is an open submanifold of the dual space \(G^* \cong \mathbb{R}^5\) of the Lie algebra \(G\) corresponding to \(G\). Furthermore, for all MD5-groups of the forms \(G_{5,4,...}\), the manifolds \(V_G\) are diffeomorphic to each other. So, for simplicity of notation, we shall write \((V, F_{4,...})\) instead of \((V_G{4,...}, F_G{4,...})\).

The following theorem will be fundamental in this paper.

**Theorem 3.2.** (The topological classification of considered MD5-foliations)

1. There exist exactly 3 topological types of fourteen families of considered MD5-foliations as follows:
   
   1.1. \(\{(V, F_{4,1}(\lambda_1, \lambda_2, \lambda_3)), (V, F_{4,2}(\lambda_1, \lambda_3)), (V, F_{4,3}(\lambda))\}
   
   1.2. \(\{(V, F_{4,11}(\lambda_1, \lambda_2, \varphi)), (V, F_{4,12}(\lambda, \varphi)), (V, F_{4,13}(\lambda, \varphi))\}
   
   1.3. \(\{(V, F_{4,14}(\lambda, \mu, \varphi))\}
   
   We denote these types by \(F_1, F_2, F_3\) respectively.

2. Furthermore, we have

2.1. The MD5-foliations of types \(F_1\) are trivial fibration with connected fibres on the 3-dimensional unitary sphere \(S^3\).

2.2. The MD5-foliations of types \(F_2, F_3\) can be given by suitable actions of \(\mathbb{R}^2\) on the foliated manifolds \(V \cong (\mathbb{R}^4)^* \times \mathbb{R}\).
Prove of Theorem 3.2

Let us recall that two foliations \((V, \mathcal{F}), (V, \mathcal{F}')\) are said to be topologically equivalent if there exists a homeomorphism \(h : V \to V\) which takes leaves of \(\mathcal{F}\) onto leaves of \(\mathcal{F}'\). The map \(h\) is called a topological equivalence of considered foliations.

1. Firstly, we prove Assertion 1 in the theorem. Namely, we need to give the topological classification of considered MD5-foliations.

1.1. We consider maps \(h_{4,1}(\lambda_1, \lambda_2, \lambda_3), h_{4,2}(\lambda_1, \lambda_2), h_{4,3}(\lambda), h_{4,4}(\lambda), h_{4,6}(\lambda_1, \lambda_2), h_{4,7}(\lambda), h_{4,8}(\lambda), h_{4,9}(\lambda), h_{4,10}(\lambda)\) from \(V \cong \mathbb{R} \times (\mathbb{R}^4)^*\) to \(V\) which are defined as follows.

\[
h_{4,1}(\lambda_1, \lambda_2, \lambda_3)(x, y, z, t, s) = (x, \text{sign}(y) \cdot |y|^{\frac{1}{\lambda_1}}, \text{sign}(z) \cdot |z|^{\frac{1}{\lambda_2}}, \text{sign}(t) \cdot |t|^{\frac{1}{\lambda_3}}, s).
\]

\[
h_{4,2}(\lambda_1, \lambda_2)(x, y, z, t, s) = (x, \text{sign}(y) \cdot |y|^{\frac{1}{\lambda_1}}, \text{sign}(z) \cdot |z|^{\frac{1}{\lambda_2}}, t, s).
\]

\[
h_{4,3}(\lambda)(x, y, z, t, s) = (x, \text{sign}(y) \cdot |y|^{\frac{1}{\lambda}}, \text{sign}(z) \cdot |z|^{\frac{1}{\lambda}}, t, s).
\]

\[
h_{4,4}(\lambda)(x, y, z, t, s) = (x, \text{sign}(y) \cdot |y|^{\frac{1}{\lambda}}, z, t, s).
\]

\[
h_{4,6}(\lambda_1, \lambda_2)(x, y, z, t, s) = \begin{cases} (x, \text{sign}(y) \cdot |y|^{\frac{1}{\lambda_1}}, \text{sign}(z) \cdot |z|^{\frac{1}{\lambda_2}}, t, s - t \cdot \ln|t|), & t \neq 0; \\ (x, \text{sign}(y) \cdot |y|^{\frac{1}{\lambda_1}}, \text{sign}(z) \cdot |z|^{\frac{1}{\lambda_2}}, 0, s), & t = 0. \end{cases}
\]
\[ h_{4,7}(\lambda)(x, y, z, t, s) := \]
\[ = \begin{cases} 
(x, \text{sign}(y) \cdot |y|^\frac{\lambda}{4}, \text{sign}(z) \cdot |z|^\frac{\lambda}{4}, t, s - t \ln|t|), & t \neq 0; \\
(x, \text{sign}(y) \cdot |y|^\frac{\lambda}{4}, \text{sign}(z) \cdot |z|^\frac{\lambda}{4}, 0, s), & t = 0.
\end{cases} \]

\[ h_{4,8}(\lambda)(x, y, z, t, s) := \]
\[ = \begin{cases} 
(x, \text{sign}(y) \cdot |y|^\frac{\lambda}{4}, \text{sign}(z - \frac{1}{4}y \ln|y|) \cdot |z - \frac{1}{4}y \ln|y||, t, s - t \ln|t|), & y \neq 0, t \neq 0; \\
(x, \text{sign}(z) \cdot |z|^\frac{\lambda}{4}, t, s - t \ln|t|), & y = 0, t \neq 0; \\
(x, \text{sign}(y) \cdot |y|^\frac{\lambda}{4}, \text{sign}(z - \frac{1}{4}y \ln|y|)), & y \neq 0, t = 0; \\
(x, 0, \text{sign}(z) \cdot |z|^\frac{\lambda}{4}, 0, s), & y = 0, t = 0.
\end{cases} \]

\[ h_{4,9}(\lambda)(x, y, z, t, s) := (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{s}), \text{ where} \]
\[ \tilde{x} = x; \]
\[ \tilde{y} = \text{sign}(y) \cdot |y|^\frac{\lambda}{4}; \]
\[ \tilde{z} = z; \]
\[ \tilde{t} = \begin{cases} 
t - z \ln|z|, & z \neq 0; \\
t, & z = 0;
\end{cases} \]
\[ \tilde{s} = \begin{cases} 
s - \frac{1}{2}t \ln|z|, & z \neq 0, t \neq z \ln|z|; \\
- \frac{1}{2}(t - z \ln|z|) \ln|t - z \ln|z||, & z \neq 0, t = z \ln|z|; \\
s - \frac{1}{2}t \ln|z|, & z = 0, t = z \ln|z|.
\end{cases} \]

\[ h_{4,10}(\lambda)(x, y, z, t, s) := (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{s}), \text{ where} \]
\[ \bar{x} = x; \]
\[ \bar{y} = y; \]
\[ \bar{z} = \begin{cases} 
  z - y \cdot \ln|y|, & y \neq 0; \\
  z, & y = 0; \\
  t - \frac{1}{2} z \cdot \ln|y| - \\
  \frac{1}{2} (z - y \cdot \ln|y|) \cdot \ln|z - y \cdot \ln|y||, & y \neq 0, z \neq y \ln|y|; \\
  t - \frac{1}{2} z \cdot \ln|y|, & y \neq 0, z = y \ln|y|; \\
  t, & y = 0, z = y \ln|y|; 
\end{cases} \]
\[ \bar{t} = \begin{cases} 
  s - \frac{1}{3} t \cdot \ln|y| - \\
  \frac{1}{3} (t - \frac{1}{2} z \cdot \ln|y|) \cdot \ln|y| + \\
  \frac{1}{3} (z \cdot \ln|y| - t - \frac{1}{2} y \cdot \ln^2|y|) \cdot \ln|y|, & y \neq 0; \\
  s - \frac{1}{2} t \cdot \ln|z| - \\
  \frac{1}{2} (t - z \cdot \ln|z|) \cdot \ln|t - z \cdot \ln|z||, & y \neq 0 \neq z, t \neq z \ln|z|; \\
  s - \frac{1}{2} t \cdot \ln|z|, & y \neq 0 \neq z, t = z \ln|z|; \\
  s, & y = 0, z = 0. 
\end{cases} \]

It is easy to verify that considered maps are homeomorphisms which take leaves of each foliation listed in Set 1.1 of Theorem 3.2 except \((V, \mathcal{F}_{4,5})\) onto leaves of this one. So these foliations are topologically equivalent to each other.

1.2. The topological equivalence of foliations in Set 1.2 of Theorem 3.2 is also verified similarly by considering homeomorphisms from \(V \cong \mathbb{R} \times (\mathbb{C} \times \mathbb{R}^2)^*\) to oneself as follows

\[
h_{4,11(\lambda_1, \lambda_2, \varphi)}(x, re^{i\theta}, t, s): = \\
= (x, e^{(lnr+i\theta)(-ie^{i\varphi})}, sign(t) \cdot |t|^{\frac{1}{\lambda_1}}, sign(s) \cdot |s|^{\frac{1}{\lambda_2}}); \\
h_{4,12(\lambda, \varphi)}(x, re^{i\theta}, t, s): = \\
= (x, e^{(lnr+i\theta)(-ie^{i\varphi})}, sign(t) \cdot |t|^{\frac{1}{\lambda}}, sign(s) \cdot |s|^{\frac{1}{\lambda}}); \\
h_{4,13(\lambda, \varphi)}(x, re^{i\theta}, t, s): = 
\]
Similarly, the homeomorphisms

\[ h_{4,14(\lambda,\mu,\varphi)} : V \cong \mathbb{R} \times (\mathbb{C} \times \mathbb{C})^* \to V \]

\[
(x, r e^{i\theta}, r' e^{i\theta'}) \mapsto \begin{cases} 
(x, e^{(lnr+i\theta)(-ie^{i\varphi})}, \\ e^{(lnr'+i\theta')(\frac{\mu}{r^2+\mu^2} - i\frac{\lambda}{r^2+\mu^2})}), & \lambda \neq 0; \\
(x, e^{(lnr+i\theta)(-ie^{i\varphi})}, \\ e^{(lnr'+i\theta')\frac{\mu}{r^2}}), & \lambda = 0.
\end{cases}
\]

take leaves of \((V, \mathcal{F}_{4,14(\lambda,\mu,\varphi)})\) onto leaves of \((V, \mathcal{F}_{4,14(0,1,\pi)})\). Thus, foliations listed in Set 1.3 of Theorem 3.2 are topologically equivalent to each other.

2. Now we prove Assertion 2 in Theorem 3.2. Firstly, it is easily seen that the following submersion:

\[ p_{4,5} : V \cong (\mathbb{R}^4)^* \times \mathbb{R} \cong S^3 \times \mathbb{R}_+ \times \mathbb{R} \to S^3; \quad p_{4,5}(s, u, v) := s \]

define Foliation \((V_4, \mathcal{F}_{4,5})\). Hence the foliations of \(\mathcal{F}_1\) are trivial fibrations. Furthermore, the fibres of them are simply connected on \(S^3\).

Nextly, let us consider the actions of the real commutative Lie group \(\mathbb{R}^2\) on \(V \cong (\mathbb{R}^4)^* \times \mathbb{R}\) as follows

\[ \rho_{4,12} : \mathbb{R}^2 \times V \to V \]

\[ \rho_{4,12}((r, a), (x, y + iz, t, s)) = (x + r, (y + iz).e^{-ia}, te^a, se^a), \]

where \((r, a) \in \mathbb{R}^2, (x, y + iz, t, s) \in V \cong \mathbb{R} \times (\mathbb{C} \times \mathbb{R}^2)^*\).

\[ \rho_{4,14} : \mathbb{R}^2 \times V \to V \]
\[ \rho_{4,14}((r,a),(x,y+iaz,t+is)) = (x+r,(y+iaz)e^{-ia},(t+is)e^{-ia}), \]

where \((r,a) \in \mathbb{R}^2, (x,y+iaz,t+is) \in V_4 \cong \mathbb{R} \times (\mathbb{C} \times \mathbb{C})^*\).

It can be verified that the above actions \(\rho_{4,12}, \rho_{4,14}\) generate the foliations \((V, F_{4,12(1,\pi/2)}), (V, F_{4,14(0,1,\pi)})\) respectively. Hence, the foliations of \(F_2, F_3\) can be given by suitable actions of \(\mathbb{R}^2\) on the foliated manifolds \(V \cong (\mathbb{R}^4)^* \times \mathbb{R}\). The proof is complete. \(\square\)

**Concluding Remark**

We close the paper with some remarks as follows.

- The topological structure of considered MD5-foliations will be characterized profoundly when we study the Connes’ C*-algebras of them. In the next paper we shall be concerned with this problem.

- It should be noted that if the foliation \((V, \mathcal{F})\) comes from a fibration \(p : V \to B\) (with connected fibers) then the Connes’ C*-algebra \(C^*(V, \mathcal{F})\) is isomorphic to the tensor product \(C_0(B) \otimes \mathcal{K}\), where \(C_0(B)\) is the algebra of \(\mathbb{C}\)-values continuous functions on \(B\) vanishing at infinity and \(\mathcal{K}\) denotes the C*-algebra of compact operators on an (infinite dimensional) separable Hilbert space (see [1, Section 5]). So we have the following assertion as one direct consequence of Theorem 3.2.

**Corollary 3.3.** *(The C*- algebras of MD5-foliations of the type \(F_1\))*

The Connes’s C*- algebras of all MD5-foliations of the type \(F_1\) are isomorphic to the C*-algebra \(C(S^3) \otimes \mathcal{K}\). \(\square\)
Acknowledgement

The authors would like to take this opportunity to thank Prof. DSc. Do Ngoc Diep for his excellent advice and support. They wish to thank Prof. Nguyen Van Sanh for his encouragement. Thanks are due also to the Scientific Committee and Organizing Committee of The Second International Congress In Algebras and Combinatorics - July 2007, Xi’an, China for inviting the first author to come and give talk on this topic at the congress.

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