A New Proposal for Matrix Theory

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Abstract

We explain the motivation and main ideas underlying our proposal for a Lagrangian for Matrix Theory based on sixteen supercharges. Starting with the pedagogical example of a bosonic matrix theory we describe the appearance of a continuum spacetime geometry from a discrete, and noncommutative, spacetime with both Lorentz and Yang-Mills invariances. We explain the appearance of large $N$ ground states with Dbranes and elucidate the principle of matrix Dbrane democracy at finite $N$. Based on the underlying symmetry algebras that hold at both finite and infinite $N$, we show why the supersymmetric matrix Lagrangian we propose does not belong to the class of supermatrix models which includes the BFSS and IKKT Matrix Models. We end with a preliminary discussion of a path integral prescription for the Hartle-Hawking wavefunction of the Universe derived from Matrix Theory.

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1 Introduction

The pre-eminent task facing string theorists of our time is finding the answer to the question: “What is String Theory?” [1]. We need an answer that is plausible, consistent with all of the known facts about weak-strong-dual effective field theory limits of nonperturbative String/M theory, and that is both mathematically and aesthetically satisfactory [2, 3, 4, 5, 6, 7, 9]. Much work has already been devoted to matrix model frameworks in this context and for reviews we refer the reader to [10]. In recent work, we have proposed a rather different direction of research, and the aim of this paper is to explain the basic features of our proposal.

Our matrix framework is motivated by a beautiful property of the nine-dimensional supergravities [11, 12, 13] that has received insufficient attention in the literature, in our opinion. In nine spacetime dimensions, but not in ten or eleven, it is possible to describe the full spectrum of Dbrane potentials, including the ten-dimensional IIA cosmological constant, in a manifestly covariant Lagrangian. By field redefinitions, and by the action of weak-strong and target-space dualities alone, it is therefore possible to connect all of the known nine-dimensional ground states of M theory. This extends, of course, to 32 supercharges; we have restricted to ground states with sixteen supercharges because of our belief that chirality is a fundamental property of perturbative string theory, worthy of emphasis in any fundamental formulation of the nonperturbative theory. Thus, a somewhat modified view of the well-known “Star” diagram [14] that inspired our matrix framework is as follows. We represent the moduli space of theories with $N=16$ supercharges by a star with six vertices. Place a theory and its $T_9$-dual at opposite vertices. Then, going clockwise around the star, we have the type I string with 32 D9branes, the massive IIA string with 32 D8branes, the heterotic string with gauge group $E_8 \times E_8$, the type I$'$ string— the same as M-theory compactified on $S^1 \times S^1 / \mathbb{Z}_2$, the massive IIB string with 32 D9branes, and the heterotic string with gauge group $Spin(32)/\mathbb{Z}_2$. The weak-strong coupling duals of these theories lie on the same diagram. The fact that the nonperturbative theory has a hidden eleven-dimensional nature is evident, but all of the vertices of the star are nine-dimensional. The matrix action we will propose can be identified in the large $N$ limit with any of the six vertices of this star, up to appropriate field redefinitions.

Our proposed Lagrangian for Matrix Theory assumes a theory based on sixteen supercharges. Spacetime is discrete, and noncommutative, with a full $N^2$ degrees of freedom contained in the $U(N)$ adjoint variable $e^\mu_a$ associated with each point in space. An auxiliary tangent space introduced at each point in spacetime is flat, and assumed to have Lorentzian signature, $(-, +, \cdots, +)$. We identify the minimal $U(N)$ invariant matrix Lagrangian for two independent adjoint variables: $e^\mu_a$ and $A_\mu$, consistent with Lorentz and Yang-Mills invariance at both finite, and infinite, $N$. Since the target symmetries acts noncommutatively on the space of $U(N)$ matrices, provision of a prescription for matrix ordering is crucial in this framework. We should clarify that the supersymmetry partners of the adjoint variables live in the fundamental $N$-representations of the $SU(N)$ subgroup. Thus, unlike both the BFSS and IKKT matrix models, and some proposed extensions, our matrix Lagrangian is not a super-matrix model based on the supersymmetrization of $U(N)$.

In the large $N$ limit, all matrix variables assume diagonal form and play the role of continuum fields. A continuum spacetime emerges and all of the basic elements of a spacetime geometry, including covariant derivatives and the geodesic equation, assume the form taken in Riemannian geometry. From a consideration of the spacetime $\times$ internal symmetries alone, it is evident then
that the large $N$ limit of our matrix Lagrangian with sixteen supercharges coincides with the spacetime Lagrangian of a string theory in the zero slope limit: we hold $g_o$, $g_c$, and $N\alpha'^{1/2}$ fixed in the matrix action, taking $N$ to $\infty$. We should point out that the large $N$ limit is not unique. The reason is that the $1/N$ corrections in the matrix Lagrangian will provide $O(\alpha')$ corrections to the spacetime Lagrangian. These need not, a priori, agree with the known $\alpha'$ expansion of the string Lagrangian. However, when we demand that expansion about the large $N$ limit result in a perturbatively finite and anomaly free spacetime Lagrangian, agreement with the string theory Lagrangian follows because those properties are unique. Finally, for more general quantum backgrounds of String Theory, with Mbranes, NSbranes and extensions, the $1/N$ terms from Matrix Theory make predictions for the unknown $\alpha'$ corrections to these backgrounds. Neither is the large $N$ limit unique in these cases; holding one or more mass scale fixed, in addition to $N\alpha'^{1/2}$, in the large $N$ limit can give inequivalent effective field theory limits. This is familiar from the extensive experience with AdS/CFT large $N$ duals [7] and with the noncommutative large $N$ $N=4$ SYM field theory limits of type II string theory [8, 18].

Our assignment of independent matrix variables to gauge and gravitational degrees of freedom has its origin in the properties of perturbative string theory. At short distances, the noncommutative nature of spacetime manifests itself in the overlap rules for closely spaced Dbranes. At large distances, noncommutativity manifests itself as the existence of an antisymmetric two-form tensor potential, the natural potential of choice in a theory of one-dimensional objects. We emphasize that this has little to do with whether or not there is an antisymmetric tensor background in our four-dimensional world, just one of a myriad of ground states of this theory. Indeed, phenomenological constraints suggest otherwise. But there is no question that the symmetric metric two-form and antisymmetric tensor twoform enter String Theory at a more fundamental level on equal footing. We will therefore preserve this equivalence in our matrix framework, obtaining both from a first order formalism for matrix gravity based on vierbeins.

The duality between short and long distance manifestations of noncommutativity also mirrors the well-known open-closed worldsheet duality of perturbative string theory. We will pause here to clarify an unfortunate misconception that has crept into the recent string literature, with the oft-made remark that “gauge theory contains gravity”. This is incorrect: open strings do produce closed strings at the loop level, but the renormalization of the open and closed string couplings are known to have independent origin. In terms of the worldsheet [17, 18], this is seen in the fact that both the coincidence limit of massless vertex operators, and the limit of vanishing loop lengths, contribute to the renormalization of the open string coupling, but not the closed string coupling. Furthermore, in any open and closed string theory there is always a subsector of pure closed string diagrams. Thus, in the absence of supersymmetric nonrenormalization theorems, the tree-level relation, $g_c=g_o^2$, receives non-trivial correction order-by-order in string perturbation theory. In short, the short-distance degrees of freedom accounting for the renormalization of open (gauge) and closed (gravitational) couplings have independent origin. We have emphasized this point because, unlike both the BFSS and the IIB Matrix Models, where gravity is purely a “derived” effective interaction while gauge theory is fundamental, the open and closed sectors of perturbative string theory contribute on independent footing to the long distance effective interactions. We have taken care to preserve this property in our matrix framework.
As a prologue to discussion of our proposal for Matrix Theory, we begin with the simpler and more pedagogical example of a *Bosonic Matrix Theory*.\(^2\) The gravitational and gauge degrees of freedom in our theory belong to two independent \(U(N)\) adjoint multiplets: \(e^a_{\mu}\) and \(A^\mu\). Here, \(\mu\) labels the directions in spacetime, and \(a\) labels the coordinates in an auxiliary flat tangent space introduced at every point in spacetime. This procedure gives a natural prescription for matrix ordering; an additional consequence is that the full nonlinear part of the gravitational interaction is already present in the classical action. Finally, diffeomorphism invariance is manifest. We describe the emergence of a continuum spacetime and the basic elements of a spacetime geometry, including covariant differentiation and the geodesic equation, from this framework. Important steps are the definition of a volume element, and the definition of partial derivative and integration of matrix variables at finite \(N\). Given these steps, we can obtain expressions for the Riemann curvature, Yang-Mills tensor, and the full tower of higher rank antisymmetric \(p\)-form field strength tensors. Requiring, in addition, invariance under all of the higher rank gauge invariances of the bosonic matrix theory provides a natural prescription for the quantum matrix action.

The supersymmetric matrix theory is a nontrivial extension of the bosonic theory which is not to be confused with the supersymmetrization of a \(U(N)\) matrix model, also known as a super-matrix model. In our framework, matrix variables in the same supermultiplet belong to distinct \(U(N)\) representations. Specifically, the gravitational and gauge degrees of freedom belong to distinct adjoints, \(e^a_{\mu}\) and \(A^\mu\), while their superpartners, \(\chi^a_{\mu}\) and \(\psi^a\), where \(\alpha\) is a spinor index taking values \(1, \cdots, 16\), belong in distinct fundamental representations of \(U(N)\). We will find that this assignment naturally enables chirality in the matrix Lagrangian, which has sixteen supercharges at both finite, and infinite, \(N\). The quantum matrix action will be determined as before by requiring invariance under the full tower of higher rank gauge symmetries. We will find that matrix Dbrane states demonstrate a remarkably simple and elegant phenomenon we refer to as *Dbrane democracy*: closure of the finite \(N\) matrix Lorentz algebra in any matrix theory ground state with Dpbrane charge in the presence of Yang-Mills fields requires that the ground state is simultaneously charged under the full tower of antisymmetric matrix potentials with \(p\leq 26\).\(^3\) We should note that Dbrane democracy has a beautiful large \(N\) remnant in the continuum theory, in the form of mixed Chern-Simons couplings in the Lagrangian when the one-form gauge symmetry is nonabelian.

We close with a matrix path integral representation for the Hartle-Hawking wavefunction, pointing the way to a derivation of the Wheeler De Witt equation for Matrix Theory. We note that a full treatment will require a clarified understanding of the role of compact De Sitter-like 9-geometries in String/M theory, a subject of active ongoing research \[26\]. We conclude with a discussion of possible future directions of research coming out of our work.

### 2 Bosonic Matrix Theory

The fundamental variables in the bosonic matrix Lagrangian are objects living in the \(N^2\)-dimensional adjoint representation of the unitary group \(U(N)\). Notice that although the individual components of a bosonic matrix take value in the field of ordinary real (complex) numbers, the matrix itself

\(^2\)We should clarify at the outset that this bosonic matrix model has no relation to the conjecture put forth in [15].

\(^3\)This is distinct, although not unrelated, to the use of the term *\(p\)-brane democracy* in [16].
is a noncommuting object obeying the rules of $U(N)$ matrix multiplication. Thus, the ordering of matrices within a composite product of $U(N)$ matrices is of crucial importance. This is especially important since we will need to project onto particular tensor products of adjoint representations in order to give matrix expressions for the physical variables, such as the symmetric two-tensor, $g_{\mu\nu}$, antisymmetric two-form, $A_{\mu\nu}$, and scalar, $\Phi$. These are the variables that will appear in the matrix Lagrangian. Thus, an unambiguous prescription for matrix ordering is necessary prior to any meaningful analysis of matrix Lagrangians.

2.1 A Prescription for Matrix Ordering

We will now make the case that an unambiguous prescription for the ordering of individual matrices in a composite operator is given by requiring that each transform simultaneously in an irreducible representation (irrep) of the unitary group, $SU(N)$, and in the $SL(n,\mathbb{C})$ subgroup of the inhomogeneous finite $N$ Lorentz group in $d=2n$ dimensions. The construction of an invariant matrix Lagrangian built out of composite Lorentz scalars then proceeds by the Noether procedure, familiar from analogous manipulations in classical field theory.

We work in the first order formalism for Einstein gravity. The basic objects in our matrix Lagrangian are the vierbein, $e^{a}_{\mu}$, a square array of size $4n^2$, each element of which is a $U(N)$ adjoint, and which is subject to $2n$ constraints, $e^{a}_{\mu}e_{\mu\nu}=\eta_{ab}$. Next, we have the nonabelian vector potential, $A_{\mu}$, a one-dimensional array of size $2n$, each element of which is a $U(N)$ adjoint. Notice that the origin of gravitational degrees of freedom, and of the spacetime continuum, is distinct from the origin of the Yang-Mills sector in this framework. The independent assignment of gauge and gravitational sectors in our construction is directly motivated by the analogous property of perturbative open and closed string theories, as explained in the Introduction.

The dimensionality, $d=2n$, of the auxiliary flat tangent space may be left undetermined in the classical theory at first, allowing for the possibility of bosonic matrix theories with an arbitrary number of noncompact dimensions in the large $N$ continuum limit. We will assume, however, the Minkowskian signature (−, +, ⋯, +) for the tangent space, which is coordinatized by $d$ real-valued parameters, $\xi^a$, and has box-regulated volume $V_d$. Associated with each point in tangent space is a whole $d(d-1)N^2$ unrestricted variables contained in the vierbein, encapsulating information about the background spacetime geometry of some large $N$ ground state of the matrix theory. In what follows, we will work in the first order formalism for gravity. The symmetric twoform metric tensor is the composite, $e^{a}_{\mu}e^{a}_{\nu}\eta^{ab}$, the antisymmetric twoform potential is the composite $e^{a}_{\mu}e^{a}_{\nu}\epsilon^{ab}$. Finally, the dilaton is the scalar, $e^{a}_{a}e^{a}_{\mu}$. Given their $SL(n,\mathbb{C})$ assignments, the kinetic terms for the $U(N)$ matrix variables described above take the manifestly invariant form:

$$\mathcal{L} = -\frac{1}{4} e^{-\Phi} F^{\mu\nu}F_{\mu\nu} - \frac{1}{2} \frac{1}{\kappa} e^{-2\Phi}(\mathcal{R} - 4 \partial^{\mu} \Phi \partial_{\mu} \Phi) - \frac{3}{2} \frac{1}{\kappa} e^{-2\Phi} H^{\mu\nu\lambda} H_{\mu\nu\lambda} \ . \quad (1)$$

Individual terms in the matrix Lagrangian are both $U(N)$ and Lorentz scalars. The Lagrangian has been written in terms of the composite $U(N)$ variables with direct correspondence to the fields appearing in the low energy spacetime Lagrangian of string theory: the scalar dilaton, $\Phi$, symmetric two-form or metric, $g_{\mu\nu}$, and antisymmetric two-form, $A_{\mu\nu}$. At this juncture, it would be helpful to clarify how the spacetime continuum emerges from this framework in the large $N$ limit. We must also give concrete meaning to the various matrix-valued symbols in the Lagrangian. A crucial step
will be the definition of matrix partial differentiation and matrix integration. We will also clarify the origin of spacetime symmetries such as Lorentz invariance and Yang-Mills gauge invariance.

2.2 Emergence of the Spacetime Continuum

Let us put some intuition into the algebraic notions described above by understanding how the spacetime continuum emerges in this framework. The basic idea is to give a suitable definition of length, area, and volume valid in the non-continuum finite $N$ case, clarifying simultaneously the notion of matrix partial differentiation and matrix integration. The nature of the spacetime symmetries such as Lorentz and Yang-Mills invariance outside of the large $N$ limit serves as our guiding principle in arriving at these definitions.

We introduce a continuum flat tangent space coordinatized by the variables $\xi^a$, $a=0, \cdots, d-1$, at every point in space. Spacetime itself is discretized, and there are $N^2$ degrees of freedom associated with each coordinate rather than the expected $N$. Thus, points in spacetime are in one-to-one correspondence with matrices $e_{\mu a}$, where $X_{\mu}=e_{\mu a}(X)d\xi^a$, $\mu=0, \cdots, d-1$. We will now give Lorentz invariant definitions for infinitesimal length and area elements as follows. We define the length of the $d$-dimensional position vector, $X^\mu$, where $X$ is an $N\times N$ $U(N)$ matrix, as follows:

$$|X|^2 = \text{Tr}_{U(N)} e^a_\mu e_\mu b d\xi^a d\xi^b ,$$

where the trace is over $U(N)$ indices. In performing concrete calculations, it will be helpful to work in a proper time gauge in which $X^0$ is taken to be diagonal, and the elements along the diagonal increase smoothly, and monotonically, denoting time. In this gauge, we will identify $X^0=\xi^0$.

In the large $N$ limit, all of the spatial $X^\mu$ will also take diagonal form, and the elements along the diagonal of each $X$ will increase smoothly and monotonically denoting the coordinates of space. Notice that the result of the trace is an ordinary real number denoting the position of some event with respect to an arbitrarily chosen origin. Translation of the origin corresponds to a $U(N)$ transformation. The interval between two neighbouring events in spacetime, $ds^2$, where $X^\mu$, $X'^\mu=X^\mu+(\Delta X)^\mu$, denote events separated by the increment $(\Delta X)^\mu$, is given by:

$$ds^2 = \text{Tr}_{U(N)} \Delta e^\mu_a \Delta e_\mu b d\xi^a d\xi^b ,$$

an invariant length for a given class of inertial observors. The total length along some given curve, $C$, in spacetime, parameterized by a proper time, $\lambda$, with respect to a chosen inertial observer, is given by the integral:

$$l = \int_C |\text{Tr}_{U(N)} \Delta e^\mu_a \Delta e_\mu b d\xi^a d\xi^b|^{1/2} d\lambda .$$

The result for the length is, of course, identical for a class of inertial observers and independent of the choice of parameterization for proper time, or of affine parameter.

Similar definitions can be given for the $p$-th volume form, $p=2, \cdots, d$. Begin with a local Lorentz frame where the $d$-th volume element is simply given by:

$$dV = \text{Tr}_{U(N)} \Delta e^{a_0}_a \cdots \Delta e^{d-1}_{a_{d-1}} d\xi^{a_0} \cdots d\xi^{a_{d-1}} .$$
The result for the volume element in an arbitrary coordinate system follows:

\[
dV = \left[ \det(-g) \right]^{1/2} \text{Tr}_{U(N)} \Delta e_{\alpha_0}^{(d-1)\nu} d\xi_{\alpha_0} \cdots d\xi_{\alpha_{d-1}},
\]

(6)

where \( g \) is the metric tensor:

\[
\left[ \det(-g) \right]^{1/2} = \left[ \det U(N) \left( -e_{\mu a} e_{\nu b} \eta^{ab} \right) \right]^{1/2}.
\]

(7)

The covariant derivative is defined as follows. First, we write down an expression for partial differentiation at a given point labelled by the \( U(N) \) matrix \( X \) by referring to the differentials in the local tangent space:

\[
\frac{\partial}{\partial X^\mu} = \frac{\partial}{\partial \xi^c} \left| \Delta X^\mu_{\lambda} \right|^{-1}.
\]

(8)

The inverse on the R.H.S. of this equation denotes taking the \( U(N) \) inverse of the infinitesimal matrix \( \Delta X \). Matrix integration will correspondingly be defined as multiplication by \( \Delta X^\mu \), such that:

\[
\int_C (\Delta X^\mu)^{-1} \Delta X^\mu = 1,
\]

(9)

and where the integration is understood to be path ordered. The definition of the Christoffel connection takes the form:

\[
\Gamma^\mu_{\nu\lambda} \equiv \frac{1}{2} g^{\mu \delta} \left( \frac{\Delta X^\lambda}{\partial \xi^c} \right)^{-1} g_{\delta \nu, c} + \frac{\Delta X^\nu}{\partial \xi^c} \left( \frac{\Delta X^\delta}{\partial \xi^c} \right)^{-1} g_{\lambda, \delta, c} - \frac{\Delta X^\delta}{\partial \xi^c} \left( \frac{\Delta X^\lambda}{\partial \xi^c} \right)^{-1} g_{\lambda, \nu, c}. \]

(10)

The expressions for covariant differentiation follow. Specifically, given Lorentz tensors \( A^\mu \), \( T^{\mu\nu} \), which are simultaneously \( U(N) \) matrices, we have:

\[
A^\mu_\lambda \equiv \left| \frac{\Delta X^\lambda}{\partial \xi^c} \right|^{-1} \frac{\Delta A^\mu}{\partial \xi^c} + \Gamma^\mu_{\nu\lambda} A^\nu, \quad T^{\mu\nu}_\lambda \equiv \left| \frac{\Delta X^\lambda}{\partial \xi^c} \right|^{-1} \frac{\Delta T^{\mu\nu}}{\partial \xi^c} + \Gamma^\mu_{\delta\lambda} T^{\delta\nu} + \Gamma^{\nu}_{\delta\lambda} T^{\mu\delta}. \]

(11)

**Gauge Covariant Derivative:** We now introduce a different \( U(N) \) matrix variable, \( A_\mu \), also carrying a vector index. A diagonal configuration denotes a smoothly varying classical field, the diagonal entries of which are ordinary continuous functions of spacetime: \((A_\mu)_{ab} = \delta_{ab} a_\mu(x_0, \cdots, x_{d-1})\). In the proper time gauge, the distance along the diagonal will correspond to the field’s progression in time. In particular, static or stationary configurations correspond to a single non-vanishing diagonal element of \( A \), a smooth function of the spatial coordinates alone. In Minkowskian spacetime, the field configuration as measured with respect to an inertial observer’s proper time will differ from that measured by a different inertial observer: the difference is given by the usual Lorentz transformation of the fields. Finally, the vector potential also carries charge under the internal symmetry group, the Yang-Mills symmetry group, \( G \). Thus, the \( U(N) \) adjoint is simultaneously a \( d_G \)-dimensional multiplet under \( G \). Thus, in flat Minkowskian spacetime, the gauge covariant derivative takes the form:

\[
D_\lambda A_\mu \equiv \left| \frac{\Delta X^\lambda}{\partial \xi^c} \right|^{-1} \frac{\Delta A_\mu}{\partial \xi^c} + g[A_\lambda, A_\mu].
\]

(12)

In a general curved spacetime, we must use the Christoffel connection defined earlier to relate the vector potential or field strength as measured by a non-inertial observer.
**Parallel Transport and Geodesics:** Recall that a curve $C$ in spacetime is a progression of $U(N)$ matrices, $V$, labelled by a parameter, the proper time $\lambda$. Thus, the tangent to the curve is given by the progression of matrices, $U=dV/d\lambda$. Here, bold-faced symbols denote $d$-vectors. Given the basic elements that describe the emerging geometry of the spacetime continuum, we can write down the geodesic equation of motion for a test particle. Choose a local inertial system at a given point $P$ such that all components of the Christoffel connection vanish at that point. It follows that $\Delta V^\mu/d\lambda=0=U^\nu V^\mu_\nu$ at $P$. This defines frame invariant parallel transport along the curve $C$. Finally, parallel transport of the tangent vector itself determines the geodesics which satisfy $\nabla_U U=0$, or:

$$U^\mu U_\mu^\nu = U^\mu \left[ \frac{\Delta X^\mu}{\partial \xi_c} \right]^{-1} \frac{\Delta U_\nu}{\partial \xi_c} + U^\mu \Gamma^\nu_{\mu\lambda} U^\lambda = 0 \quad .$$  \hspace{1cm} (13)

**Curvature:** The result of parallel transport of a vector about an infinitesimal closed loop in spacetime at a given point $P$ gives the Riemann curvature tensor, defined in the usual way, and appearing also in the commutator of covariant derivatives:

$$R^\mu_{\nu\lambda\delta} = \Gamma^\mu_{\nu\delta,\lambda} - \Gamma^\mu_{\nu\lambda,\delta} + \Gamma^\mu_{\sigma\lambda} \Gamma^\sigma_{\nu\delta} + \Gamma^\mu_{\sigma\delta} \Gamma^\sigma_{\nu\lambda}, \quad [\nabla_{\mu}, \nabla_{\nu}] V^\lambda = R^\lambda_{\delta\mu\nu} V^\delta \quad .$$  \hspace{1cm} (14)

Partial differentiation with respect to the $X$ is defined as above, with reference to differentials in the local tangent space. The result is the Bianchi identities.

### 2.3 Point Sources and the Newtonian Potential

In common with its predecessors [4, 6], the bosonic matrix theory described above is a second quantized theory in target space, in the sense that a classical matrix configuration may describe one, or multiple, matrix objects. The simplest objects are point sources; the classical equation of motion for a free test particle is simply the geodesic equation given earlier. We will work in the proper time gauge setting $X^0$ equal to $\tau=\xi^0$. As in [4, 6, 10], in the nonrelativistic limit we can consider a block diagonal configuration matrix $U_i \equiv dX_i/d\tau$, where $X_i(\tau)$ gives the location of the $i$th test particle:

$$U = U_1 \quad 0 \quad \ldots \quad 0 \quad U_2 \quad \ldots \quad \ldots \quad 0 \quad U_n \quad .$$  \hspace{1cm} (15)

In this limit, the equation of motion is separable: each test particle satisfies the geodesic equation, $\nabla_{U_i} U_i(\tau)=0$, parameterized by a common proper time, $\tau$, and taking the explicit form:

$$\left( \frac{\Delta X^\mu}{d\tau} \right) \left( \frac{\partial}{\partial \xi_c} \left[ \frac{\Delta X^\nu}{d\tau} \right] \right) + \left( \frac{\Delta X^\mu}{d\tau} \right) \Gamma^\nu_{\mu\lambda} \left( \frac{\Delta X^\lambda}{d\tau} \right) = 0 \quad .$$  \hspace{1cm} (16)

Identifying $\tau/m_i$, where $m_i$ is the $i$th particle’s mass, as affine parameter, we can also express the geodesic equations in terms of the momenta, $p_i=m_i dX_i/d\tau.$
Next, we will extract the Newtonian potential sufficiently far from a nonrelativistic point source in the linearized limit of the Einstein equation $R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi T_{\mu\nu}$. In the limit of weak disturbances, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| << 1$, the curvature tensor takes the form:

$$R_{\mu\nu\lambda\sigma} = h_{\mu\sigma,\nu\lambda} + h_{\nu\lambda,\mu\sigma} - h_{\mu\lambda,\nu\sigma} - h_{\nu\sigma,\mu\lambda}.$$  \hspace{1cm} (17)

Defining $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h$, in the Lorentz gauge, $\bar{h}^{\mu\nu} = 0$, we have the linearized Einstein equations:

$$(-\partial^2_t + \nabla^2)\bar{h}^{\mu\nu} = -16\pi T^{\mu\nu},$$ \hspace{1cm} (18)

The Newtonian limit applies when the disturbances are too small to attain relativistic velocities. Thus, $|T^{00}| >> |T^{0i}| >> |T^{ij}|$, which, from the linearized Einstein equation, implies that, $|\bar{h}^{00}| >> |\bar{h}^{0i}| >> |\bar{h}^{ij}|$. The dominant equation is that for the Newtonian potential, $-4V \equiv \bar{h}^{00}$, and upon setting $T^{00} = \rho + O(\rho v^2)$, and dropping terms of order $v^2\nabla^2$, we have Newton’s equation:

$$\nabla^2 V = 4\pi \rho.$$ \hspace{1cm} (19)

The far field potential for a stationary localized source is given by the solution to $\nabla^2 \bar{h}^{00} = 0$, which is the Laplace equation. Its solution in 26 spacetime dimensions takes the form, $-C/r^{23} + O(1/r^{24})$. The coefficient can be determined on dimensional grounds and by a matching calculation to the known field theory limit. This is the static interaction, with $O(v^2)$ corrections in the bosonic matrix theory. In the 10d supersymmetric case, as is well known, supersymmetry results in cancelation of both the static and $O(v^2)$ corrections; the leading term in the nonrelativistic potential is $O(v^4/r^7)$.

This completes our discussion of the linearized limit of the Einstein equations. However, the full nonlinear part of the gravitational interaction has already been included in the classical theory. Unlike the case of M(atrix) Theory and the IIB Matrix Model, there is no need to invoke quantum effects or the subtleties of the large $N$ limit in order to account for the nonlinear part of the gravitational interaction.

### 2.4 Extended Objects from Matrices

In addition to pointlike gravitational sources, the bosonic matrix theory contains extended objects like Dbranes. Such extended matrix theory objects couple to background antisymmetric tensor potentials of higher rank, $C_{[p]}$, $p \leq 26$. In such a background, the $SO(p-1,1)$ Lorentz subalgebra on the worldvolume of the p-brane extends to an inhomogenous Lorentz algebra— extended by the $\mu = 0, \cdots, p-1$, hermitian generators of spacetime translations, giving the Poincare group in $p$ dimensions:

$$[J_{\mu \nu}, J_{\rho \lambda}] = g_{\rho \nu} J_{\mu \lambda} - g_{\mu \nu} J_{\rho \lambda} - g_{\lambda \mu} J_{\rho \nu} + g_{\lambda \nu} J_{\rho \mu}, \quad [P_{\mu}, J_{\nu \lambda}] = g_{\mu \nu} P_{\lambda} - g_{\mu \lambda} P_{\nu}.$$ \hspace{1cm} (20)

with $[P_{\mu}, P_{\nu}] = 0$. The $J_{\mu \nu}$ are the generators of spatial rotations, hermitian, and antisymmetric in $\mu$, $\nu$. All matrix generators have been denoted by $U(N)$ adjoints. In the classical theory, the commutator denotes the Poisson bracket. It is replaced by the operator-valued Heisenberg commutator in the quantum theory. The remnant $SO(26-p,0)$ Lorentz algebra coordinatizing directions orthogonal to the pbrane does not extend to an inhomogenous algebra.
In the quantum theory, the commutators above are to be understood as operator-valued symbols acting on a ground state with the required properties. We begin with considering stationary, time independent, spacetime geometries. Specifying such a ground state requires a stationary background metric and stationary background gauge potentials. Begin with the spacetime “grid”, the family of \( \{ e_i^a d\xi^a \} \) isomorphic to the spatial coordinates \( X^i \), for all time. In the large \( N \) continuum limit, the \( e_i^a \) are diagonal, with entries along the diagonal displaying a smooth and monotonic increase in the case of noncompact coordinates. For a compact coordinate, the diagonal entries must display the required periodicity. The volume element in the worldvolume of a p-brane takes the form:

\[
dV = \text{Tr} \Delta e_{a_0}^0 \Delta e_{a_1}^1 \cdots \Delta e_{a_p}^p d\xi^{a_0} \cdots d\xi^{a_p} .
\]  

It is evident that the Poincare algebra given above acts as a set of operator identities on the volume element in the worldvolume of the p-brane. The same is true for any function of the \( e^\mu_a \): the \( P \), \( J \), act as derivatives on all such functions, where partial derivatives with respect to \( X^\mu \equiv e^\mu_a d\xi^a \) are as previously defined in section 2.2. For example, we have the usual plane-wave basis for generic eigenfunctions:

\[
\exp[i \mathbf{p} \cdot \mathbf{X}] = \sum_{m=0}^{\infty} \frac{i^m}{m!} (\mathbf{p} \cdot \mathbf{X})^m ,
\]

where each wavefunction is defined by its Taylor series expansion. Such matrix-valued functions can be manipulated in the usual way as long as we keep in mind the rules for matrix partial differentiation and matrix integration.

Let us now construct the tower of extended D-objects that couple to the tower of higher rank matrix potentials, \( C[p] \), in the bosonic matrix theory. We have:

\[
X^{\mu_1 \mu_2} = e_{a_1}^{[\mu_1} e_{a_2}^{\mu_2]} d\xi^{a_1} d\xi^{a_2} \\
X^{\mu_1 \mu_2 \mu_3} = e_{a_1}^{[\mu_1} e_{a_2}^{\mu_2} e_{a_3}^{\mu_3]} d\xi^{a_1} d\xi^{a_2} d\xi^{a_3} \\
\cdots = \cdots \\
X^{\mu_1 \mu_2 \cdots \mu_{26}} = e_{a_1}^{[\mu_1} e_{a_2}^{\mu_2} \cdots e_{a_{26}}^{\mu_{26]}]} d\xi^{a_1} \cdots d\xi^{a_{26}} ,
\]

coupling, respectively, to matrix potentials \( C[2] \), \( \cdots \), \( C[26] \). Each is an \( N \times N \) matrix obtained from the tensor product of \( p \) adjoint irreps of \( SU(N) \). Recall that evidence for the existence of an ordinary vector potential in some region of space is given by the nonvanishing holonomy of the gauge potential around a closed loop threaded by the Yang-Mills field. Likewise, one may verify the existence of a higher rank gauge potential by performing an integration over a suitable hypersurface in space. It is this definition which has a nice extension for extended objects within the finite \( N \) matrix framework.

### 3 The Symmetry Algebra at Finite \( N \)

Having illuminated our admittedly abstract presentation of the bosonic matrix Lagrangian, it is helpful to return to a clearer discussion of the symmetries manifest at finite values of \( N \). Recall the form of the Lagrangian from section 2:

\[
\mathcal{L} = -\frac{1}{4} \frac{1}{\kappa^2} e^{-\Phi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \frac{1}{\kappa^2} e^{-2\Phi} (\mathcal{R} - 4 \partial^\mu \Phi \partial_\mu \Phi) - \frac{3}{2} \frac{1}{\kappa^2} e^{-2\Phi} H^{\mu\nu\lambda} H_{\mu\nu\lambda} .
\]
The gauge covariant derivative has already been defined in section 2.3. The Yang-Mills and antisymmetric threeform field strength may be written more explicitly as follows:

\[ F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + gf^{ijk} A^j_\mu A^k_\nu \]
\[ H_{\mu\nu\lambda} = \partial_{[\mu} A_{\nu\lambda]} - X_{\mu\nu\lambda} \equiv \partial_{[\mu} A_{\nu\lambda]} - \text{tr}_{ijk} (\delta_{ij} A^i_{[\mu} F^j_{\nu\lambda]} - \frac{2}{3} f^{ijk} A^i_{[\mu} A^j_\nu A^k_\lambda) \]  

(25)

Notice the definition of the “shifted” field strength. With this definition, the kinetic terms for both \( F \) and \( H \) take standard form.

It is helpful to verify explicitly the invariance of the matrix Lagrangian under a local Lorentz transformation. Local Lorentz transformations act as tangent space rotations. We introduce an infinitesimal hermitian matrix, \( L_{ab} \), antisymmetric under the interchange of tangent space indices \( a, b \). Keeping terms up to linear in \( L_{ab} \), it is easy to verify that each of the kinetic terms in \( \mathcal{L} \) is invariant under the following transformations:

\[
\begin{align*}
\delta e^\mu_a &= L^c_a e^\mu_c \\
\delta A_a &= L^c_a A_c \\
\delta F_{ab} &= L^c_a F_{cb} + L^c_b F_{ac} \\
\delta A_{ab} &= L^c_a A_{cb} + L^c_b A_{ac} \\
\delta H_{abc} &= L^c_a H_{ebc} + L^c_b H_{eac} + L^c_c H_{abe} 
\end{align*}
\]  

(26)

Likewise, we can verify invariance of the matrix Lagrangian under the Yang-Mills transformation. A Yang-Mills transformation is a rotation in color space. Locality implies the possibility of independent rotations for the elements along the diagonal of the matrix potential \( A_\mu \). We introduce a \( d_G \)-plet of infinitesimal real matrices, \( \{\alpha^j\} \), where \( d_G \) is the dimension of the nonabelian gauge group with hermitian generators \( \{\tau^j\} \). The \( \alpha^j \) are required to take diagonal \( N \times N \) form. With this restriction, it is easy to verify that each term of the Lagrangian is invariant under the Yang-Mills transformation given below:

\[
\begin{align*}
\delta gA^j_\alpha \tau^j &= [D_\alpha, \tau^j \alpha^j] \\
\delta D_\alpha \Phi &= i\tau^j \alpha^j D_\alpha \Phi \\
\delta F_{ab} &= i\tau^j \alpha_j F_{ab} 
\end{align*}
\]  

(27)

Finally, under a gauge transformation mediated by the two-form potential, the gauge potentials transform as follows:

\[
\delta A_{\mu\nu} = \partial_{[\mu} \zeta_{\nu]}, \quad \delta A_\mu = -\zeta_\mu 
\]

(28)

This is also an invariance of the matrix Lagrangian.

In the generic curved spacetime background, the symbol “;” may be used to denote action of the general covariant derivative including Christoffel connection, generalizing the arguments given above. The Riemann curvature scalar may be expressed in the explicit form:

\[
R[E] = (D_b e^{b\lambda})(D_a e^{\lambda}_c) - (D_a e^{b\lambda})(D_b e^{\lambda}_c) + e^{a\lambda}(D_a e^{b}_\sigma)(D_b e^{\lambda}_c)e^\sigma_c - e^{a\lambda}(D_a e^{c}_\lambda)e^\sigma_c(D_b e^{b}_\sigma) 
\]

(29)

Referring back to the Lagrangian, it is clear that the Einstein term as written is invariant under both local Lorentz and Yang-Mills transformations at finite \( N \).
Next, consider expanding about a large $N$ limit of bosonic matrix theory characterized by spatially-extended objects coupled to a $p$-form potential, $C_{[p]}$, $p \leq 26$, namely, Dbranes. The matrix $p$-form transformations have the following nontrivial consequence: under a tensor $p$-form gauge transformation, all $p'$-form gauge potentials, with $p' \leq p$, must transform non-trivially, including the ordinary Yang-Mills potential. For example, consider the 3-form potential, $C_{\mu\nu\lambda}$. We have:

$$\delta C_{\mu\nu\lambda} = \partial_{[\mu} C_{\nu\lambda]} - \partial_{[\nu} C_{\lambda]} - C_{\lambda} = -\zeta_{\lambda} \ . \quad (30)$$

The corresponding kinetic term can be written in standard form:

$$\mathcal{L} = \frac{1}{2} F_4 \wedge F_4, \quad F_{\mu\nu\lambda\sigma} = \partial_{[\mu} C_{\nu\lambda\sigma]} - X_{\mu\nu\lambda\sigma} \ . \quad (31)$$

where the shifted 4-form field strength, $F_4$, is defined as follows:

$$X_{\mu\nu\lambda\sigma} = -C_{[\mu\nu} C_{\lambda\sigma]} - A^i_{[\mu} A^i_{\nu} A^i_{\lambda\sigma]} - f_{ijk} A^i_{[\mu} A^i_{\nu} A^j_{\lambda\sigma]} - A^i_{[\mu} A^i_{\nu} A^j_{\lambda} A^k_{\sigma]} . \quad (32)$$

Is it mandatory that the shift take its most general form inclusive of coupling to all $p$-form potentials with $p \leq 3$? In the case of the matrix transformations, it is indeed the case: if any one $p$-form charge is carried by the matrix theory vacuum, it automatically carries all of the $p$-form charges. The result follows as a consequence of the Lorentz and Yang-Mills invariance of the quantum theory.

Since the matrix potentials are noncommuting objects, the $U(N)$ commutator, $[L_{ab}, C_{[p]}]$, is nontrivial for any value of $p$. This implies coupling to a $(p+2)$-form potential, and, upon iterating this argument, to the chain of $(p\pm 2n)$-form potentials. Conversely, the nontrivial $U(N)$ commutator, $[A_1, C_{[p]}]$, implies coupling to a $(p\pm 1)$-form potential and, by iteration, to all $(p\pm n)$-form potentials. This observation will be termed matrix Dbrane democracy; it follows from Lorentz invariance and the presence of gauge fields in the large $N$ continuum limit. In the special case that the gauge fields are nonabelian, matrix Dbrane democracy has a beautiful remnant in the form of mixed Chern-Simons terms in the low energy spacetime action of perturbative string theory. It implies, in particular, that Dbrane charge conservation must be carefully defined so as to account for the mixing due to the presence of these terms in the action [19].

### 4 Supersymmetric Matrix Theory

We begin discussion of our proposal for Matrix Theory by incorporating a crucial feature absent in previous matrix formulations of String/M theory [4, 6, 10], namely, chirality. We assign bosonic and fermionic members of each supersymmetry multiplet to distinct $U(N)$ representations. Thus, the gaugino, gravitino, and dilatino belong in the fundamental $N$-dimensional representation of the $SU(N)$ subgroup, while their bosonic superpartners, $A_\mu, e^a_\mu$, belong in the $N \times N$ adjoint representation. This is an essential point of difference from previous conjectures for Matrix Theory: matrix variables in the fundamental representations of $U(N)$ have appeared in previous work on matrix formulations of heterotic matrix theory [10], but the fermionic and bosonic superpartners within any multiplet were chosen to belong in the same $U(N)$ irrep. In other words, our finite $N$ symmetry algebra is not simply a supersymmetrization of $U(N)$, and the usual formalism of supergroups and super-matrix models does not apply.
A second point of difference from previous work is that there is no need for a physical gauge fixing in taking the large $N$ limit. Thus, the number of supersymmetries at finite $N$ is the same as in the large $N$ limit, namely, sixteen. The $SU(N)$ matrix variables carry, in addition, both Lorentz and nonabelian group indices. In the discussion that follows, we will denote the finite-dimensional Yang-Mills group as the generic group $G$, of rank $r_G$, and dimension $d_G$. With some guidance from the continuum $N=1$ supergravity-Yang-Mills Lagrangian, we can infer the form of the kinetic terms for the given variables in the matrix Lagrangian. We have:

$$\mathcal{L} = -\frac{1}{2}\kappa e^{-2\Phi} \left( \bar{\psi}^a \Gamma^{\mu\nu\lambda} D_{\mu\nu} \psi^a \chi_{\lambda} - 4\bar{\chi} \Gamma^{\mu\nu} D_{\mu\nu} \psi^a - 4\bar{\psi} D_{\mu} \chi_{\lambda} \right) - \frac{1}{2} \frac{1}{2 \pi^2} e^{-\Phi} \bar{\chi} \Gamma^{\mu} D_{\mu} \chi_{i}^i + \mathcal{L}_{2-\text{fermi}} + \mathcal{L}_{4-\text{fermi}},$$

where the two- and four-fermi terms will be inferred by requiring closure under the supersymmetry transformations. In the expression above, $\chi^{ia}, \bar{\chi}^{i\alpha}$, denote Grassmann-valued fermionic matrices in the $N, \bar{N}$, representations of $SU(N)$. The indices, $i=1, \cdots, r_G$, simultaneously labels a fundamental representation of the Yang-Mills group $G$, while $a=1, \cdots, 16$, labels sixteen distinct Grassmann-valued fermionic matrices. The spinor covariant derivative is both Lorentz, and gauge, covariantized:

$$D_{\mu} \chi = \partial_{\mu} \chi + \frac{1}{2} \Gamma^{ab} \Omega_{ab\mu} \chi + g A_{\mu} \chi,$$

where $\Omega_{ab\mu}$ is the three-index spin connection [20, 21]. In the large $N$ limit, these are diagonal matrices, corresponding to the sixteen components of a Majorana-Weyl spinor field. In the proper time gauge, distance along the diagonal has been mapped to time, and each diagonal element is a smooth function of the spatial coordinates. Thus, we recover the components of a Grassmann field.

Likewise, $\psi_{\mu}^{ia}$, denotes Grassmann-valued fermionic matrices evolving in the large $N$ limit into the components of a Lorentz spinor-vector field in ten dimensions. Finally, we have the matrix representatives of the dilatino field, also living in a Grassmann-valued $SU(N)$ fundamental representations, $\chi^a$. In the continuum limit, $\chi^i, \psi_{\mu}^a$, and $\lambda$, yield, respectively, the gaugino, gravitino, and dilatino fields of the $d=10 \ N'=1$ SYM supergravity Lagrangian. The $SU(N)$ matrices $F_{ab}, H_{abc}, \mathcal{R},$ and $\Phi$ are, respectively, finite $N$ matrix representatives of the Yang-Mills tensor, the shifted antisymmetric three-form field strength corresponding to the two-form potential $C_{[2]}$, plus Chern-Simons term for the Yang-Mills potential, the Ricci curvature, and the dilaton scalar continuum fields.

Closure of the group of transformations that are the finite $N$ manifestation of large $N$ continuum supersymmetry algebra is a nontrivial result. However, as we will see below, with the ordering prescription given earlier, the manipulations required to verify that $\mathcal{S}$ is supersymmetry invariant are well-defined. Consider infinitesimal spinor parameters, $\eta_1, \eta_2$, each of which transforms as a $N$-vector of the unitary group $SU(N)$. We must verify that the commutator of two matrix supersymmetry transformations with arbitrary infinitesimal spinor parameters can always be expressed as the sum of (i) an infinitesimal tangent space translation with parameter, $\xi^a = \eta_1 \Gamma^a \eta_2$, (ii) an infinitesimal local Lorentz transformation with parameter $L_{abc} = \xi_\alpha \omega_{abc}$, and (iii) an infinitesimal local gauge transformation with gauge parameter $\alpha^i = -g \xi^a A_{\mu}^a$ [20].

The form of the supersymmetry transformations is as follows. We consider the following sequence of matrix transformations induced by the infinitesimal spinor parameter, $\eta$, a Grassmann-valued,
\[ N\text{-dimensional vector under } SU(N): \]
\[ \delta e^\mu = \frac{1}{2} \eta \Gamma^a \psi^\mu, \quad \delta \Phi = -\frac{1}{2} \eta \lambda, \quad \delta A_{\mu\nu} = \frac{1}{2} \eta \Gamma^{[\mu} \psi_{\nu]} - \frac{1}{g^2} \text{tr}(A_{\mu} \delta A_{\nu}) \]
\[ \delta A^\mu_i = \frac{1}{2} \eta \Gamma^{\mu} \chi^i, \quad \delta \psi^\mu = \frac{1}{2} \eta \Gamma^{[\mu} \psi_{\nu]} \lambda - \frac{1}{2} (\bar{\psi}^\mu \Gamma^a \eta) \Gamma_a \lambda + \frac{1}{3 \cdot 2^6} \frac{1}{g^4} \text{tr}(\bar{\chi} \Gamma_{abc} \chi) \Gamma^{abc} \Gamma_\mu \eta \]
\[ \delta A^\mu_i = \frac{1}{2} \eta \Gamma^{\mu} \chi^i, \quad \delta \psi^\mu = \frac{1}{2} \eta \Gamma^{[\mu} \psi_{\nu]} \lambda - \frac{1}{2} (\bar{\psi}^\mu \Gamma^a \eta) \Gamma_a \lambda \cdot \quad (35) \]

It may be verified that there is no ambiguity in the ordering of variables in the transformation laws given here. We then complete our expression for the Matrix Theory Lagrangian by including the two-fermion and four-fermion terms required by supersymmetry [20, 21]. With guidance from the continuum \( N=1 \) supergravity-Yang-Mills Lagrangian [21], we infer the following 2-fermi terms:
\[ \mathcal{L}_{2\text{-fermi}} = -\frac{1}{\kappa^2} e e^{-2\Phi} \bar{\psi}^\mu \Gamma^a \psi^\nu (\partial^\nu \Phi) + 2 \frac{1}{\kappa^2} e e^{-2\Phi} \bar{\psi}^\mu \Gamma^a \eta \Gamma_\mu \lambda (\partial^\nu \Phi) \]
\[ + \frac{1}{2^6} \frac{1}{g^4} e e^{-2\Phi} H^\rho \sigma \tau \left[ \bar{\psi}^\mu \Gamma_\rho \Gamma_\sigma \Gamma_\tau \psi^\nu + 4 \bar{\psi}^\mu \Gamma^a \Gamma_\rho \sigma \tau \lambda - 4 \bar{\lambda} \Gamma_\rho \sigma \tau \lambda \right] \]
\[ + \frac{1}{2^6} \frac{1}{g^4} e e^{-2\Phi} H^{ab} \text{tr}(\bar{\chi} \Gamma_{abc} \chi) - \frac{1}{2^6} \frac{1}{g^4} e e^{-2\Phi} \bar{\chi}^i \Gamma^a \Gamma^{ab}(\psi^\mu + \frac{1}{3} \Gamma_\mu \lambda)(F^i_{ab} + \bar{F}^i_{ab}) \quad (36) \]

The 4-fermi terms in the Lagrangian take the form:
\[ \mathcal{L}_{4\text{-fermi}} = -\frac{1}{3 \cdot 2^6} \frac{1}{\kappa^2} e e^{-2\Phi} \bar{\psi}^\mu \Gamma^{abc} \psi^\nu \left( \bar{\psi}^\nu \Gamma^a \Gamma_{abc} \chi + 2 \bar{\psi}^\nu \Gamma^a \psi^\nu - 4 \bar{\lambda} \Gamma_{abc} \psi^\nu \right) \]
\[ + \frac{1}{3 \cdot 2^6} \frac{1}{g^4} e e^{-2\Phi} \text{tr}(\bar{\chi} \Gamma_{abc} \chi) \left( -\frac{1}{2} \bar{\psi}^\nu (4 \Gamma_{abc} \Gamma^\mu + 3 \Gamma^a \Gamma_{abc}) \lambda + 24 \bar{\lambda} \Gamma_{abc} \lambda - 24 \bar{\lambda} \Gamma_{abc} \lambda \right) \]
\[ - \frac{1}{3 \cdot 2^6} \frac{1}{g^4} e e^{-2\Phi} \text{tr}(\bar{\chi} \Gamma_{abc} \chi) \cdot \frac{1}{g^4} \text{tr}(\bar{\chi} \Gamma_{abc} \chi) . \quad (37) \]

The expression for \( \mathcal{L} \) may be simplified and written even more compactly by introducing \( SU(N) \) vectors, \( \Psi, \bar{\Psi}, (d+1+d_G) \times N \)-component \( SU(N) \) vectors. Each transforms simultaneously as, respectively, 16-component right- and left-handed Majorana-Weyl spinors under the inhomogenous Lorentz group. They are denoted as follows:
\[ \bar{\Psi} \equiv (\bar{\chi}, \bar{\psi}_a, \bar{\psi}^i), \quad \Psi \equiv (\chi, \psi_a, \chi^i) . \quad (38) \]

The independent Lorentz structures present in the kinetic and two-fermi terms of \( \mathcal{L} \) may be grouped inside a matrix array of size \( (d+1+d_G)N \times (d+1+d_G)N \), which we denote as \( \mathcal{D} \). The four-fermi terms are likewise expressed in compact form by introducing matrices, \( \mathcal{U}, \mathcal{V} \), of size \( (d+1+d_G)N \times (d+1+d_G)N \), identified by referring to the expression in Eq. (37). In summary, the classical Lagrangian for Matrix Theory takes the remarkably compact form:
\[ \mathcal{L} = -\frac{1}{2} \bar{\Psi} \mathcal{D} \Psi + \frac{1}{4} (\bar{\Psi} \mathcal{U} \Psi)(\bar{\Psi} \mathcal{V} \Psi) - \frac{1}{4^2} e^{-2\Phi} F^{\mu \nu} F_{\mu \nu} - \frac{1}{2^6} \frac{1}{g^4} e^{-2\Phi} (\mathcal{R} - 4 \partial^\mu \Phi \partial_\mu \Phi + 3 H^{\mu \nu \lambda} H_{\mu \nu \lambda}) . \quad (39) \]

We should note that, in principle, \( \mathcal{L} \) belongs to a family of matrix Lagrangians, members of which can differ by \( 1/N \) corrections, thus yielding the same spacetime Lagrangian in the infrared in accordance with the principle of universality classes. However, we can state definitively that the universality class of our theory does not overlap with either the BFSS or IKKT matrix models because of the distinct \( U(N) \) assignments given to the members of a supermultiplet in our framework.
Our procedure for determining $\mathcal{L}$ ensures that all relevant interactions in the large $N$ continuum Lagrangian that are required in order to match correctly with a spacetime Lagrangian that is manifest Yang-Mills invariant, locally supersymmetric, and Lorentz invariant at the scale $\alpha'^{-1/2}$, are already present in the ultraviolet theory defined by $\mathcal{L}$. Thus, the sole source for both nonperturbative and quantum corrections to the spacetime Lagrangian are the quantum corrections from the matrix path integral.

5 The Wheeler De Witt Equation

It should be possible to derive a Wheeler-De Witt equation for Matrix Theory as follows. We will specialize to proper time gauge, where we identify $X^0 = E^0_a(X) d\xi^a$ with tangent space time, $X^0 = \xi^0$, at all points in space, which remains discrete. This implies setting the $E^0_a$ to zero for all spatial $a$, and $E^0_0 = 1$, the unit $N \times N$ matrix. We will work in Euclidean time, and the end-points, $n=1, N$, correspond to the box-regularization, $X^0 = 0, T$. We will formulate the Wheeler-DeWitt equation for the matrix quantum mechanics thus defined, enabling construction of the Hartle-Hawking wavefunction [22, 23]. For convenience, we rename the timelike coordinate, $X^0 = \xi^0 = t$.

Matrix quantum dynamics in the proper time gauge is given by the Schroedinger equation [22], $i \partial \Psi / \partial t = H \Psi$. The wavefunction for the ground state, or state of minimum excitation, $\Psi_0$, is defined by the matrix path integral, made positive definite by a rotation to Euclidean time. As explained in [22, 23], even though there is strictly speaking no minimum energy state in a theory of quantum gravity, our gauge fixing condition makes both the notion of energy and of the minimum energy state well-defined. At the initial time $t=0$, we have:

$$
\Psi_0[e^m_a(0); \phi(0)] = \int d[e^m_a(t)] d[\phi] \exp \{-I[e^m_a; \phi]\},
$$

(40)

where $I$ is the Euclidean action given by the variation of the matrix action described in the previous section. The $e^m_a(0)$ specify a particular spatial nine-geometry, and the wavefunction is the amplitude for that geometry to be created from the zero nine-geometry—a single point, or nothing, at the initial time [22]. The matter and gauge degrees of freedom are the additional data that must be specified on the initial value slice in discrete spacetime, namely, on the spatial nine-geometry valid at $t=0$. The matter and gauge degrees of freedom, $A, C_{[p]}, \phi, \chi, \psi, \lambda$, have been collectively denoted by the symbol $\phi$.

In practice, an object of more direct interest is the probability of finding a certain closed, compact submanifold, $S$, with given 9-geometry and given configuration of regular matter fields, and which divides the spacetime manifold into in- and out- manifolds, $M_\pm$. Such a probability can be factorized into the product of amplitudes, $\Psi_\pm$, where the path integral sums over classes $C_\pm$, of 9-geometries and matter fields on $M_\pm$, which match with the given 9-geometry on $S$. Following [22, 23], the $\Psi_\pm$ may be regarded as wave functions of the Universe. If the classes, $C_\pm$, are identical, we can drop the suffix $\pm$ without ambiguity. The functional differential equation satisfied by $\Psi$ is the Wheeler-De Witt equation. In the case of Matrix Theory, this is defined by introducing a proper time which is constant on $S$, giving the standard lapse-shift decomposition of the 9+1-metric. The Wheeler De Witt equation is the matrix functional differential equation obtained by varying the classical matrix action with respect to the lapse function. We will not obtain its precise form in this paper, but we make the following remarks.
In [22, 23], a major focus of interest is the issue of what constitutes the class of spatial geometries that must be summed over in the path integral. The argument is made that it is the no-boundary geometries based on compact, positive definite metrics that are relevant to the wavefunction of the Universe and hence to quantum cosmology. It is interesting to contrast this with the case of two-dimensional gravity [24], namely, first quantized string theory, where the path integral localizes on a finite dimensional integral and there is no ambiguity about the class of geometries summed in the path integral. The reason for this is Weyl invariance; any metric in two dimensions is gauge equivalent to a constant curvature metric, leaving only a finite dimensional integral over the worldsheet moduli.

Although we will not aim to settle this thorny issue in this paper, we would like to propose that the same is true in a fundamental theory of the Universe. Namely, Matrix Theory is both a dynamical theory and a theory of the ground state [25]. The configuration space of spacetime geometries and background fields leading to a finite, and renormalizable, perturbation theory in the infrared should define a complete set, selected by the high degree of symmetry of the solutions, both kinematic and dynamic. Any other background geometry will be gauge equivalent to a member of this set, where by a gauge equivalence here we mean a transformation falling under the category of spacetime \times internal symmetry transformations. Thus, the higher rank quantum gauge invariances become essential to the understanding of the full configuration space of the matrix path integral. In the large \( N \) continuum limit, the quantum gauge symmetries manifest themselves as the strong-weak and target space dualities linking the different low energy limits of M theory [2, 5, 9, 7].

We should emphasize, however, that we are in broad agreement with the arguments put forth in [22, 23] that asymptotically flat and AdS geometries, while natural in particle physics with its focus on a framework suited to scattering, and also widespread in perturbative string theory [7], are too limiting a class of geometries of likely relevance to cosmology. Fortunately, our understanding of de Sitter-like spacetimes in String/M theory is rapidly undergoing development [26] and there is hope that one will have a clearer perspective on this subject in the future. We reiterate that it is our hope that the choice of boundary condition in Matrix Theory will have an unambiguous origin as in perturbative string theory, without recourse to an independent principle originating outside of the theory.

6 Conclusions

Our proposal for Matrix Theory is reminiscent of a discretization of the \( \alpha' \) expansion of the spacetime Lagrangian for String Theory. The \( \alpha' \) expansion is nonperturbative in the coupling constant and, not surprisingly, has been a major source of insight into strongly coupled string theory and its nonperturbative solutions. However, the discretization or, more precisely, regularization offered by the matrix description is not to be confused with ordinary lattice field theories. The number of degrees of freedom associated with each of the coordinates of space is \( N^2 \), rather than the expected \( N \), and the spacetime geometry is noncommutative. Points in spacetime are in one-to-one correspondence with matrices, and by introducing the auxiliary device of a continuum tangent space, we have achieved a diffeomorphism invariant description capable of accommodating arbitrary curved spacetime geometries. These are significant gains although the resulting matrix Lagrangian is understandably complex. It should be emphasized, however, that the bosonic Lagrangian is
relatively simple, and the fermionic additions to it are mostly a matter of achieving closure of the supersymmetry algebra.

A second theme running through this work has been the notion of gauge symmetry, both classical and quantum. We have stressed the role of the higher rank antisymmetric Lorentz tensors which couple to extended objects, pointing out how, at the level of the finite $N$ algebra, charge under any one gauge potential implies charge under the full tower of potentials. In the large $N$ continuum limit, the quantum gauge symmetries manifest themselves as strong-weak and target space dualities linking the different low energy limits of M theory. The notion of Duality as a gauge symmetry is not a new idea, and has already received considerable attention in the literature. Our work can be taken as further evidence for the validity of this notion. The principle of quantum gauge invariance will be of fundamental importance in any precise treatment of the matrix path integral.

However, we expect that, as with the $\alpha'$ expansion of string theory, the immediate most fruitful directions of work will come from semi-classical analyses of Matrix Theory. The derivation of the Wheeler–De Witt equation, and the possibility of studying quantum cosmology that it opens up, are of the greatest interest here. We leave that effort for future work.
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Note Added (July 2005): This paper continues the stream of conceptual advances in the development of my proposal for nonperturbative String/M theory from hep-th/0201129, 0202138, and 0205306. The notion of emergent spacetime as introduced by me first appears in this paper. For a clearer presentation, the reader should consult hep-th/0408057 [Nucl. Phys B719 (2005) 188].

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