Efficient Classification of Local Problems in Regular Trees

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We give practical, efficient algorithms that automatically determine the distributed round complexity of a given locally checkable graph problem, in two settings. We present one algorithm for unrooted regular trees and another algorithm for rooted regular trees. The algorithms take the description of a locally checkable labeling problem as input, and the running time is polynomial in the size of the problem description. The algorithms decide if the problem is solvable in \( O(\log n) \) rounds. If not, it is known that the complexity has to be \( \Theta(n^{1/k}) \) for some \( k = 1, 2, \ldots \), and in this case the algorithms also output the right value of the exponent \( k \).

In rooted trees in the \( O(\log n) \) case we can then further determine the exact complexity class by using algorithms from prior work; for unrooted trees the more fine-grained classification in the \( O(\log n) \) region remains an open question.

1 INTRODUCTION

We give practical, efficient algorithms that automatically determine the distributed round complexity of a given locally checkable graph problem in rooted or unrooted regular trees, for both LOCAL and CONGEST models (see Section 3 for the precise definitions). In these cases, the distributed round complexity of any locally checkable problem is known to fall in one of the classes shown in Figure 1 [7, 9, 11, 17–19, 28]. Our algorithms are able to distinguish between all higher complexity classes from \( \Theta(\log n) \) to \( \Theta(n) \).

1.1 State of the art

There is a recent paper [6] that presents an algorithm that takes as input the description of a locally checkable graph problem and outputs its distributed round complexity. More precisely, this is applicable to locally checkable problems defined in rooted regular trees, and the algorithm determines the round complexity in the usual LOCAL and CONGEST models of distributed computing, both for randomized and deterministic algorithms.

To illustrate the setting of locally checkable problems in rooted regular trees, consider, for example, the following problem (which is meaningful for rooted binary trees):

Each node is labeled with 1 or 2. If the label of an internal node is 1, exactly one of its two children must have label 1, and if the label of an internal node is 2, both of its children must have label 1.

We can represent it in a concise manner as a problem \( C = \{1 : 12, 2 : 11\} \), where \( a : bc \) indicates that a node of label \( a \) can have its two children labeled with \( b \) and \( c \), in some order. We can take such a description, feed it to the algorithm from [6], and it will output that this problem requires \( \Theta(\log n) \) rounds to solve in a tree with \( n \) nodes.

1.2 What was missing

What the prior algorithm from [6] can do efficiently is classifying a given problem into one of the four main complexity classes \( O(1), \Theta(\log^* n), \Theta(\log n), \) and \( n^{\Theta(1)} \). However, if the complexity is \( n^{\Theta(1)} \), we do not learn whether its complexity
(a) Rooted regular trees (deterministic and randomized CONGEST and LOCAL):

\[
\begin{array}{cccccc}
\Theta(1) & \Theta(\log^* n) & \Theta(\log n) & \ldots & \Theta(n^{1/3}) & \Theta(n^{1/2}) & \Theta(n) \\
\text{prior} & \text{prior} & \text{prior} & \text{this} & \text{this} & \text{this} & \text{this} \\
\text{work} & \text{work} & \text{work} & \text{work} & \text{work} & \text{work} & \text{work} \\
\text{(EXP)} & \text{(EXP)} & \text{(P)} & \text{(P)} & \text{(P)} & \text{(P)} & \text{(P)} \\
\end{array}
\]

(b) Unrooted regular trees (deterministic CONGEST and LOCAL):

\[
\begin{array}{cccccc}
\Theta(1) & \Theta(\log^* n) & \Theta(\log n) & \ldots & \Theta(n^{1/3}) & \Theta(n^{1/2}) & \Theta(n) \\
\text{unknown} & \text{unknown} & \text{this} & \text{this} & \text{this} & \text{this} & \text{this} \\
\text{work} & \text{work} & \text{work} & \text{work} & \text{work} & \text{work} & \text{work} \\
\text{(P)} & \text{(P)} & \text{(P)} & \text{(P)} & \text{(P)} & \text{(P)} & \text{(P)} \\
\end{array}
\]

(c) Unrooted regular trees (randomized CONGEST and LOCAL):

\[
\begin{array}{cccccc}
\Theta(1) & \Theta(\log^* n) & \Theta(\log \log n) & \Theta(\log n) & \ldots & \Theta(n^{1/3}) & \Theta(n^{1/2}) & \Theta(n) \\
\text{unknown} & \text{unknown} & \text{unknown} & \text{this} & \text{this} & \text{this} & \text{this} & \text{this} \\
\text{work} & \text{work} & \text{work} & \text{work} & \text{work} & \text{work} & \text{work} & \text{work} \\
\text{(P)} & \text{(P)} & \text{(P)} & \text{(P)} & \text{(P)} & \text{(P)} & \text{(P)} & \text{(P)} \\
\end{array}
\]

Fig. 1. The most efficient algorithms for the classification of distributed round complexities. In the figure we show all possible complexity classes. Each gap between two classes corresponds to a natural decision problem: given a locally checkable problem, determine on which side of the gap its complexity is. For each gap we indicate whether a practical algorithm was provided already by prior work [6], whether it is first presented in this work, or whether existence of such a routine is still an open question. The figure also indicates whether the algorithms are in \( \mathcal{P} \) (polynomial time in the size of the problem description) or in \( \text{EXP} \) (exponential time in the size of the problem description).

is, say, \( \Theta(n) \) or \( \Theta(\sqrt{n}) \) or maybe \( \Theta(n^{1/10}) \). There are locally checkable problems of complexity \( \Theta(n^{1/k}) \) for every \( k = 1, 2, \ldots \), and there have not been any practical algorithm that would determine the value of the exponent \( k \) for any given problem.

Furthermore, the algorithm from [6] is only applicable in rooted regular trees, while the case of unrooted trees is perhaps even more interesting.

It has been known that the problem of distinguishing between e.g. \( \Theta(n) \) and \( \Theta(\sqrt{n}) \) is in principle decidable, due to the algorithm by Chang [17]. However, this algorithm is best seen as a theoretical construction. To our knowledge, nobody has implemented it, there are no plans of implementing it, and it seems unlikely that one could classify any nontrivial problem with it, using any real-world computer. This is the missing piece that we provide in this work.

1.3 Contributions

We present efficient, practical algorithms that determine not only whether the round complexity of a given locally checkable problem is \( \Theta(n^{1/k}) \) for some \( k \), but they also determine the exact value of \( k \). We give one algorithm for the case of unrooted trees (Section 4) and one algorithm for the case of rooted trees (Section 5).

The algorithms runs in polynomial time (in the size of the description of the problem). We have implemented both algorithms for the case of 3-regular trees, the proof-of-concept implementations are freely available online,\(^1\) and they work fast also in practice.

\(^1\)https://github.com/jendas1/poly-classifier
Together with prior work from [6], there is now a practical algorithm that is able to completely determine the complexity of any local problem in rooted trees; in the case of unrooted trees deciding between the lower complexity classes below $o(\log n)$ remains an open question. We refer to Figure 1 for a summary of where the new algorithms are applicable and what was already known before the present work.

2 RELATED WORK

For many locally checkable problems, researchers have been trying to understand the exact time complexity, and while in many cases upper bounds have been known since the 80s, matching lower bound have been discovered only recently. Examples of this line of research relate to the problems of colorings, matchings, and independent sets (see e.g. [3, 4, 22, 23, 25, 26, 29–31, 35, 36]).

In parallel, there have been many works that tried to understand these problems from a complexity theory point of view, trying to develop general techniques for classifying problems, understanding which complexities can actually exist, and developing generic algorithmic techniques to solve whole classes of problems at once. In particular, a broad class of locally checkable problems, called Locally Checkable Labelings (LCLs), has been studied in the LOCAL model of distributed computing, which will be formally defined in Section 3.

Paths and Cycles. The first graph topologies on which promising results have been proved are paths and cycles. In these graphs, we now know that there are problems with the following three possible time complexities:

- $O(1)$: this class contains, among others, trivial problems, e.g., problems that require every node to output the same label.
- $\Theta(\log^* n)$: this class contains, for example, the 3-coloring problem [22, 29, 32].
- $\Theta(n)$: this class contains hard problems, for example the problem of consistently orient the edges of a cycle, or the 2-coloring problem.

For LCLs, we know that there are no other possible complexities, that is, there are gaps between the above classes. In other words, there are no LCLs with a time complexity that lies between $o(1)$ and $o(\log^* n)$ [33], and no LCLs with a time complexity that lies between $\omega(\log^* n)$ and $o(n)$ [18]. These results hold also for randomized algorithms, and they are constructive: if for example we find a way to design an $O(\log n)$-round randomized algorithm for a problem, then we can automatically convert it into an $O(\log^* n)$-round deterministic algorithm.

Moreover, in paths and cycles, given an LCL problem, we can decide its time complexity. In particular, it turns out that for problems with no inputs defined on directed cycles, deciding the complexity of an LCL is as easy as drawing a diagram and staring at it for few seconds [15, 33]. This result has later been extended to undirected cycles with no inputs [20]. Unfortunately, as soon as we consider LCLs where the constraints of the problem may depend on the given inputs, decidability becomes much harder, and it is now known to be EXPTIME-hard [1, 17].

Trees. Another class of graphs that has been studied quite a lot is the one containing trees. While there are still problems with complexities $O(1)$, $\Theta(\log^* n)$, and $\Theta(n)$, there are also additional complexity classes, and sometimes here randomness can help. For example, there are problems that require $\Theta(\log n)$ rounds for both deterministic and randomized algorithms, while there are problems, like sinkless orientation, that require $\Theta(\log n)$ rounds for deterministic algorithms and $\Theta(\log \log n)$ rounds for randomized ones [14, 18, 27]. Moreover, there are problems with complexity $\Theta(n^{1/k})$, for any natural number $k \geq 1$ [19]. It is known that these are the only possible time complexities in trees [7, 11, 17–19, 28]. In [9], it has been shown that the same results hold also in a more restrictive model of distributed
computing, called CONGEST model, and that for any given problem, its complexities in the LOCAL and in the CONGEST model, on trees, are actually the same.

Concerning decidability, the picture is not as clear as in the case of paths and cycles. As discussed in the introduction, it is decidable, in theory, if a problem requires \( n^{O(1)} \) rounds, and, in that case, it is also decidable to determine the exact exponent \([17, 19]\), but the algorithm is very far to be practical, and in this work we address exactly this issue. Moreover, for lower complexities, the problem is still open. Different works tried to tackle this issue by considering restricted cases. In \([2]\), authors showed that it is indeed possible to achieve decidability in some cases, that is, when problems are restricted to the case of unrooted regular trees, where leaves are unconstrained, and the problem uses only two labels. Then, promising results have been achieved in \([6]\), where it has been shown that, if we consider \([7, 10]\), then any given problem can be represented as a certain directed graph \( G \) contains a node that is.

General Graphs. In general graphs, many more LCL complexities are possible. For example, there is a gap similar to the one between \( \omega(1) \) and \( o(\log^* n) \) of trees, but now it holds only up to \( o(\log \log^* n) \), and we know that there are problems in the region between \( \Omega(\log \log^* n) \) and \( o(\log^* n) \). In fact, for any rational \( \alpha \geq 1 \), it is possible to construct problems with complexity \( \Theta(\log^2 \log^* n) \) \([10]\). A similar statement holds for complexities between \( \Omega(\log n) \) and \( O(n) \) \([7, 10]\).

There are still complexity regions in which we do not know if there are problems or not. For example, while it is known that any problem that has randomized complexity \( o(\log n) \) can be sped up to \( O(T_{LLL}) \) \([19]\), where \( T_{LLL} \) it the distributed complexity of the constructive version of the Lovász Local Lemma, the exact value of \( T_{LLL} \) is unknown, and we only known that it lies between \( \Omega(\log \log n) \) and \( O(\poly \log \log n) \) \([14, 21, 24, 36]\). In fact, there are still natural problems in this region, such as the \( \Delta \)-coloring problem, for which we do not know the exact complexity.

Another open question regards the role of randomness. In general graphs, we know that randomness can also help outside the \( O(\log n) \) region \([8]\), but we still do not know exactly when it can help and how much.

In general graphs, unfortunately, determining the complexity of a given LCL problem is undecidable. In fact, we know that this question is undecidable even on grids \([33]\).

Combinatorial Characterizations. Algorithms for deciding the complexity of a given LCL problem are often based on the following idea: (1) define some combinatorial property \( P \) of LCL problems, (2) show that computing \( P(\Pi) \) for a given problem \( \Pi \) can be done efficiently, and (3) show that \( \Pi \) is in a certain round complexity class if and only if \( P(\Pi) \) holds.

Perhaps the simplest example of this is the decision routine for LCLs on unlabeled directed cycles \([15, 33]\). In that case any given problem \( \Pi \) can be represented as a certain directed graph \( G(\Pi) \). Then \( \Pi \) is solvable in \( O(1) \) rounds if and only if \( G(\Pi) \) contains a self-loop, and \( \Pi \) is solvable in \( O(\log^* n) \) rounds if and only if \( G(\Pi) \) contains a node that is.
part of two cycles with coprime lengths—the latter property is commonly known as “flexibility”. The same ideas have been also generalized to e.g. undirected paths and cycles [20].

Prior work on rooted trees [6] is also based on the same general strategy. They define what they call certificates for each complexity class. For example, the certificate for \( O(\log n) \)-round solvability is a restriction of \( \Pi \) to so-called path-flexible states (which generalizes the idea of flexibility from directed cycles to rooted trees). It was shown that such a certificate exists if and only if \( \Pi \) can be solved in \( O(\log n) \) rounds.

Brandt, Chang, Grebík, Grunau, Rozhon, and Vidnyánszky [13] obtained a characterization of LCL problems with complexity \( O(\log n) \) in unrooted trees. They showed that \( \Pi \) can be solved in \( O(\log n) \) rounds if and only if an \( \ell \)-full set exists for some \( \ell \). Roughly speaking, an \( \ell \)-full set is a set of configurations with which one can complete the labeling along a path of length at least \( \ell \), even if the labels of the endpoints are fixed with arbitrary configurations from the \( \ell \)-full set.

3 PRELIMINARIES

Graphs. Let \( G = (V, E) \) be a graph. We denote with \( n = |V| \) the number of nodes of \( G \), with \( \Delta \) the maximum degree of \( G \), and with \( \deg(v) \), for \( v \in V \), the degree of \( v \). If \( G \) is a directed graph, we denote with \( \deg_{\text{in}}(v) \) and \( \deg_{\text{out}}(v) \), the indegree and the outdegree of \( v \), respectively. The radius-\( r \) neighborhood of a node \( v \) is defined to be the subgraph of \( G \) induced by the nodes at distance at most \( r \) from \( v \).

Model of computing. In the LOCAL model of distributed computing, the network is represented with a graph \( G = (V, E) \), where the nodes correspond to computational entities, and the edges correspond to communication links. In this model, the computational power of the nodes is unrestricted, and nodes can send arbitrarily large messages to each other.

This model is synchronous, and computation proceeds in rounds. Nodes all start the computation at the same time, and at the beginning they know \( n \) (the size of the graph), \( \Delta \) (the maximum degree of the graph), and a unique ID in \( \{1, \ldots, n^c\} \), for some constant \( c \geq 1 \), assigned to them. Then, the computation proceeds in rounds, and at each round nodes can send (possibly different) messages to each neighbor, receive messages, and perform some local computation.

At the end of the computation, each node must produce its own part of the solution. For example, in the case of the \( (\Delta + 1) \)-vertex coloring problem, each node must output its own color, that must be different from the ones of its neighbors. The time complexity is measured as the worst case number of rounds required to terminate, and it is typically expressed as a function of \( n, \Delta \), and \( c \).

4 UNROOTED TREES

In this section, we give a polynomial-time-computable characterization of LCL problems for regular unrooted trees with complexity \( O(\log n) \) or \( \Theta(n^{1/k}) \) for any positive integer \( k \).

4.1 Locally checkable labeling for unrooted trees

A \( \Delta \)-regular tree is a tree where the degree of each node is either 1 or \( \Delta \). An LCL problem for \( \Delta \)-regular unrooted trees is defined as follows.

\textbf{Definition 4.1 (LCL problems for regular unrooted trees).} For unrooted trees, an LCL problem \( \Pi = (\Delta, \Sigma, V, E) \) is defined by the following components.

- \( \Delta \) is a positive integer specifying the maximum degree.
The labeling is a correct solution if it is locally-consistent on all $\forall i$ for any larger $r$ in $S_{T}$ define as rooted trees, they can also be viewed as unrooted trees.

We call a size-$\Delta$ multisets $C$ of labels in $\Sigma$ a node configuration. A node configuration $C$ is correct with respect to $\Pi = (\Delta, \Sigma, V, E)$ if $C \in V$. We call a size-2 multisets $D$ of labels in $\Sigma$ an edge configuration. An edge configuration $D$ is correct with respect to $\Pi = (\Delta, \Sigma, V, E)$ if $D \in E$. We define the correctness criteria for a labeling of a $\Delta$-regular tree in Definition 4.2.

Definition 4.2 (Correctness criteria). Let $G = (V, E)$ be a tree whose maximum degree is at most $\Delta$. For each edge $e = \{u, v\}$ in the tree, there are two half-edges $(u, e)$ and $(v, e)$. A solution of $\Pi = (\Delta, \Sigma, V, E)$ on $G$ is a labeling that assigns a label in $\Sigma$ to each half-edge in $G$.

- For each node $v \in V$ with $\deg(v) = \Delta$ its node configuration $C$ is the multiset of $\Delta$ half-edge labels of $(v, e_1), (v, e_2), \ldots, (v, e_\Delta)$. We say that the labeling is locally-consistent on $v$ if $C \in \Sigma$.
- For each edge $e = \{u, v\} \in E$, its edge configuration $D$ is the multiset of two half-edge labels of $(u, e)$ and $(v, e)$.

We say that the labeling is locally-consistent on $e$ if $D \in E$.

The labeling is a correct solution if it is locally-consistent on all $v \in V$ with $\deg(v) = \Delta$ and all $e \in E$.

In other words, a labeling of $G = (V, E)$ is correct if the edge configuration for each $e \in E$ is correct and the node configuration for each $v \in V$ with $\deg(v) = \Delta$ is correct. All nodes whose degree is not $\Delta$ are unconstrained.

Although $\Pi = (\Delta, \Sigma, V, E)$ is defined for $\Delta$-regular unrooted trees, Definition 4.2 applies to all trees whose maximum degree is at most $\Delta$. Nevertheless, we may focus on $\Delta$-regular unrooted trees without loss of generality. The reason is that for any unrooted tree $G$ whose maximum degree is at most $\Delta$, we may consider the unrooted tree $G^*$ which is the result of appending degree-1 nodes to all nodes $v$ in $G$ with $1 < \deg(v) < \Delta$ to increase the degree of $v$ to $\Delta$. This only blows up the number of nodes by at most a $\Delta$ factor. Any correct solution of $\Pi$ on $G^*$ restricted to $G$ is a correct solution of $\Pi$ on $G$.

4.2 Preliminaries

Definition 4.3 (Complete trees of height $i$). We define the rooted trees $T_i$ and $T^*_i$ recursively as follows.

- $T_0$ is the trivial tree with only one node.
- $T_i$ is the result of appending $\Delta - 1$ trees $T_{i-1}$ to the root $r$.
- $T^*_i$ is the result of appending $\Delta$ trees $T_{i-1}$ to the root $r$.

Observe that $T^*_i$ is the unique maximum-size tree of maximum degree $\Delta$ and height $i$. All nodes within distance $i - 1$ to the root $r$ in $T^*_i$ have degree $\Delta$. All nodes whose distance to $r$ is exactly $i$ are degree-1 nodes. Although $T_i$ and $T^*_i$ are defined as rooted trees, they can also be viewed as unrooted trees.

Definition 4.4 (Trimming). Given an LCL problem $\Pi = (\Delta, \Sigma, V, E)$ and a subset $S \subseteq V$ of node configurations, we define trim$(S)$ as the set of all node configurations $C \in S$ such that for each $i \geq 1$ it is possible to find a correct labeling of $T^*_i$ such that the node configuration of the root is $C$ and the node configurations of the remaining degree-$\Delta$ nodes are in $S$.

In the definition, note that if for some $i \geq 1$ it is not possible to find such a labeling of $T^*_i$, then it is also not possible for any larger $i$. 
Intuitively, \( \text{trim}(S) \) is the subset of \( S \) resulting from removing all node configurations in \( S \) that are not usable in a correct labeling of a sufficiently large \( \Delta \)-regular tree using only node configurations in \( S \).

In fact, given any tree \( G \) of maximum degree \( \Delta \) and a node \( v \) of degree \( \Delta \) in \( G \), after labeling the half-edges surrounding \( v \) using a node configuration in \( \text{trim}(S) \), it is always possible to extend this labeling to a complete correct labeling of \( G \) using only node configurations in \( \text{trim}(S) \). Such a labeling extension is possible due to Lemma 4.5.

**Lemma 4.5 (Property of trimming).** Let \( S \subseteq \mathcal{V} \) such that \( \text{trim}(S) \neq \emptyset \). For each node configuration \( C \in \text{trim}(S) \) and each label \( \sigma \in C \), there exist a node configuration \( C' \in \text{trim}(S) \) and a label \( \sigma' \in C' \) such that the multiset \( \{ \sigma, \sigma' \} \) is in \( \mathcal{E} \).

**Proof.** Assuming that such \( C' \) and \( \sigma' \) do not exist, we derive a contradiction as follows. We pick \( s \) to be the smallest number such that there is no correct labeling of \( T_s^* \) where the node configuration of the root \( r \) is in \( S \setminus \text{trim}(S) \) and the node configuration of each remaining degree-\( \Delta \) node of \( T_s^* \) is in \( S \). Such a number \( s \) exists due to the definition of \( \text{trim} \).

Now consider a correct labeling of \( T_{s+1}^* \) where the node configuration of the root \( r \) is \( C \) and the node configuration of each remaining degree-\( \Delta \) node is in \( S \). Such a correct labeling exists due to the fact that \( C \in \text{trim}(S) \). Our assumption on the non-existence of \( C' \) and \( \sigma' \) implies that the node configuration \( C' \) of one child \( w \) of the root \( r \) of \( T_{s+1}^* \) must be in \( S \setminus \text{trim}(S) \). However, the radius-\( s \)-neighborhood of \( w \) in \( T_{s+1}^* \) is isomorphic to \( T_s^* \) rooted at \( w \). Since the node configuration of \( w \) is in \( S \setminus \text{trim}(S) \), our choice of \( s \) implies that the labeling of the radius-\( s \)-neighborhood of \( w \) cannot be correct, which is a contradiction. \( \square \)

**Path-form of an LCL problem.** Given an LCL problem \( \Pi = (\Lambda, \Sigma, \mathcal{V}, \mathcal{E}) \) and a subset \( S \subseteq \mathcal{V} \) of node configurations, we define

\[
\mathcal{D}_S = \text{the set of all size-2 multisets } D \text{ such that } D \text{ is a sub-multiset of } C \text{ for some } C \in S.
\]

To understand the intuition behind the definition \( \mathcal{D}_S \), define the length-\( k \) hairy path \( H_k \) as the result obtained by starting from a length-\( k \) path \( P = (v_1, v_2, \ldots, v_{k+1}) \) and then adding degree-1 nodes to make \( \deg(v_i) = \Delta \) for all \( 1 \leq i \leq k + 1 \). If our task is to label hairy paths using node configurations in \( S \), then this task is identical to labeling paths using node configurations in \( \mathcal{D}_S \). In other words, the LCL problem \( (\Lambda, \Sigma, \mathcal{V}, \mathcal{E}) \) on hairy paths is equivalent to the LCL problem \( (2, \Sigma, \mathcal{D}_S, \mathcal{E}) \) on paths. Hence \( (2, \Sigma, \mathcal{D}_S, \mathcal{E}) \) is the path-form of \( (\Lambda, \Sigma, \mathcal{V}, \mathcal{E}) \).

**Automaton for the path-form of an LCL problem.** Given a set \( \mathcal{D} \) of size-2 multisets whose elements are in \( \Sigma \), we define the directed graph \( M_\mathcal{D} \) as follows. The node set \( V(M_\mathcal{D}) \) of \( M_\mathcal{D} \) is the set of all pairs \( (a, b) \in \Sigma^2 \) such that the multiset \( \{a, b\} \) is in \( \mathcal{D} \). The edge set \( E(M_\mathcal{D}) \) of \( M_\mathcal{D} \) is defined as follows. For any two pairs \( (a, b) \in V(M_\mathcal{D}) \) and \( (c, d) \in V(M_\mathcal{D}) \), we add a directed edge \( (a, b) \rightarrow (c, d) \) if the multiset \( \{b, c\} \) is an edge configuration in \( \mathcal{E} \). Note that \( M_\mathcal{D} \) could contain self-loops.

The motivation for considering \( M_\mathcal{D} \) is that it can be seen as an automaton recognizing the correct solutions for the LCL problem \( (2, \Sigma, \mathcal{D}, \mathcal{E}) \) on paths, as each length-\( k \) walk \( (a_1, b_1) \rightarrow (a_2, b_2) \rightarrow \cdots \rightarrow (a_{k+1}, b_{k+1}) \) of \( M_\mathcal{D} \) corresponds to a correct labeling of a length-\( k \) path \( (v_1, v_2, \ldots, v_{k+1}) \) where the labeling of half-edge \( (v_i, \{v_{i-1}, v_i\}) \) is \( a_i \) and the labeling of half-edge \( (v_i, \{v_i, v_{i+1}\}) \) is \( b_i \).

**Path-flexibility.** With respect to the directed graph \( M_\mathcal{D} \), we say that \( (a, b) \in V(M_\mathcal{D}) \) is path-flexible if there exists an integer \( K \) such that for each integer \( k \geq K \), there exist length-\( k \) walks \( (a, b) \leadsto (a, b), (a, b) \leadsto (b, a), (b, a) \leadsto (a, b) \), and \( (b, a) \leadsto (b, a) \) in \( M_\mathcal{D} \).
It is clear that \((a, b)\) is path-flexible if and only if \((b, a)\) is path-flexible. Hence we may extend the notion of path-flexibility from \(V(M_D)\) to \(\mathcal{D}\). That is, we say that a size-2 multiset \(\{a, b\} \in \mathcal{D}\) is path-flexible if \((a, b)\) is path-flexible. Lemma 4.6 is useful in lower bound proofs.

**Lemma 4.6 (Property of path-inflexibility).** Suppose that the size-2 multiset \(\{a, b\} \in \mathcal{D}\) is not path-flexible. Then one of the following holds.

- There is no \(s \rightarrow t\) walk for at least one choice of \(s \in \{(a, b), (b, a)\}\) and \(t \in \{(a, b), (b, a)\}\).
- There is an integer \(2 \leq x \leq |\Sigma|^2\) such that for any positive integer \(k\) that is not an integer multiple of \(x\), there are no length-\(k\) walks \((a, b) \leadsto (a, b)\) and \((b, a) \leadsto (b, a)\) in \(M_D\).

**Proof.** Suppose that \(\{a, b\} \in \mathcal{D}\) is not path-flexible. We assume that there are \(s \leadsto t\) walks for all choices of \(s \in \{(a, b), (b, a)\}\) and \(t \in \{(a, b), (b, a)\}\). To prove this lemma, it suffices to show that there is an integer \(2 \leq x \leq |\Sigma|^2\) such that for any positive integer \(k\) that is not an integer multiple of \(x\), there are no length-\(k\) walks \((a, b) \leadsto (a, b)\) and \((b, a) \leadsto (b, a)\).

First of all, we claim that for any integer \(K\) there is an integer \(k \geq K\) such that there is no length-\(k\) walk \((a, b) \leadsto (a, b)\). If this claim does not hold, then there is an integer \(K\) such that there is a length-\(k\) walk \((a, b) \leadsto (a, b)\) for each \(k \geq K\). Combining these walks with existing walks \((a, b) \leadsto (a, b)\) and \((b, a) \leadsto (a, b)\), we infer that there exists an integer \(K'\) such that for each integer \(k \geq K'\), there exist length-\(k\) walks \((a, b) \leadsto (a, b), (b, a) \leadsto (a, b)\), and \((b, a) \leadsto (b, a)\) in \(M_D\), contradicting the assumption that \(\{a, b\} \in \mathcal{D}\) is not path-flexible.

Let \(U\) be the set of integers \(k\) such that there is a length-\(k\) walk \((a, b) \leadsto (a, b)\). Note that by taking reversal, the existence of a length-\(k\) walk \((a, b) \leadsto (a, b)\) implies the existence of a length-\(k\) walk \((b, a) \leadsto (b, a)\), and vice versa. Our assumption on the existence of a walk \((a, b) \leadsto (a, b)\) implies \(U \neq \emptyset\). We choose \(x = \gcd(U)\) to be the greatest common divisor of \(U\), so that for any integer \(k\) that is not an integer multiple of \(x\), there are no length-\(k\) walks \((a, b) \leadsto (a, b)\) and \((b, a) \leadsto (b, a)\) in \(M_D\). We must have \(x \geq 2\) because there cannot be two co-prime numbers in \(U\), since otherwise there exists an integer \(K\) such that \(U\) includes all integers that are at least \(K\), contradicting the claim proved above. We also have \(x \leq |\Sigma|^2\), since the smallest number in \(U\) is at most the number of nodes in \(M_D\), which is upper bounded by \(|\Sigma|^2\). \(\square\)

For the special case of \(|\Sigma| = 1\) and \(\mathcal{D} \neq \emptyset\), we must have \(a = b\) in Lemma 4.6. Since there is no integer \(x\) satisfying \(2 \leq x \leq |\Sigma|^2\) when \(|\Sigma| = 1\), Lemma 4.6 implies that if \(\{a, b\}\) is not path-flexible, then there is no walk \((a, a) \leadsto (a, a)\), where \((a, a)\) is the unique element in \(\mathcal{D}\).

**Path-flexible strongly connected components.** Since each \(\{a, b\} \in \mathcal{D}\) corresponds to two nodes \((a, b)\) and \((b, a)\) in \(M_D\), we will consider a different notion of a strongly connected component. In Definition 4.7, we do not require the elements \(a, b, c,\) and \(d\) to be distinct. For example, we may have \(\{a, b\} = \{c, d\}\) or \(a = b\).

**Definition 4.7 (Strongly connected components).** Let \(\mathcal{D}\) be a set of size-2 multisets of elements in \(\Sigma\). For each \(\{a, b\} \in \mathcal{D}\) and \(\{c, d\} \in \mathcal{D}\), we write \(\{a, b\} \leadsto \{c, d\}\) if there is a walk \(s \leadsto t\) in \(M_D\) for each choice of \(s \in \{(a, b), (b, a)\}\) and \(t \in \{(c, d), (d, c)\}\).

Let \(\mathcal{D}^-\) be the set of all \(\{a, b\} \in \mathcal{D}\) such that \(\{a, b\} \leadsto \{c, d\}\). Then we define the strongly connected components of \(\mathcal{D}\) as the equivalence classes of \(\leadsto\) over \(\mathcal{D}^-\).

By taking reversal, the existence of an \((a, b) \leadsto (c, d)\) walk implies the existence of a \((d, c) \leadsto (b, a)\) walk. Therefore, if there is a walk \(s \leadsto t\) in \(M_D\) for each choice of \(s \in \{(a, b), (b, a)\}\) and \(t \in \{(c, d), (d, c)\}\), then there is also a walk...
where \( D \) satisfies the following conditions. Hence the relation \( \sim \) in Definition 4.7 is symmetric over \( D \). It is clear from the definition of \( \sim \) in Definition 4.7 that it is transitive over \( D \) and it is reflexive over \( D^- \), so \( \sim \) is indeed an equivalence relation over \( D^- \).

For any strongly connected component \( D' \) of \( D \), it is clear that either all \( \{a, b\} \in D' \) are path-flexible or all \( \{a, b\} \in D' \) are not path-flexible. We say that a strongly connected component \( D' \) is path-flexible if all \( \{a, b\} \in D' \) are path-flexible. We define flexibility(\( D' \)) as the minimum number \( K \) such that for each integer \( k \geq K \) there is an \( (a, b) \leadsto (c, d) \) walk of length \( k \) for all choices of \( a, b, c, \) and \( d \) such that \( \{a, b\} \in D' \) and \( \{c, d\} \in D' \). It is clear that such a number \( K \) exists given that \( D' \) is a path-flexible strongly connected component. We define flexible-SCC(\( D \)) = the set of all subsets of \( D \) that are a path-flexible strongly connected component of \( D \).

Clearly, elements in flexible-SCC(\( D \)) are disjoint subsets of \( D \). It is possible that flexible-SCC(\( D \)) is an empty set, and this happens when all nodes in the directed graph \( M_\Theta \) are not path-flexible.

**Restriction of a set of node configurations.** Given an LCL problem \( \Pi = (\Delta, \Sigma, V, \mathcal{E}) \), a subset \( S \subseteq V \) of node configurations, and a set \( D \) of size-2 multisets whose elements are in \( \Sigma \), we define the restriction of \( S \) to \( D \) as follows.

\[
S \upharpoonright D = \{ C \in S \mid \text{ all size-2 sub-multisets of } C \text{ are in } D \}.
\]

Lemma 4.8 shows that if we label the two endpoints of a sufficiently long path using node configurations in \( S \upharpoonright D' \), where \( D' \in \text{flexible-SCC}(D_\mathcal{S}) \), then it is always possible to complete the labeling of the path using only node configurations in \( S \) in such a way that the entire labeling is correct. Specifically, consider a path \( P = (v_1, v_2, \ldots, v_{d+1}) \) of length \( d \geq \text{flexibility}(D') \). Assume that the node configuration of \( v_1 \) is already fixed to be \( C \in S \upharpoonright D' \) where the half-edge \( (v_1, \{v_1, v_2\}) \) is labeled by \( \beta \in \Delta \), and the node configuration of \( v_{d+1} \) is already fixed to be \( C' \in S \upharpoonright D' \) where the half-edge \( (v_{d+1}, \{v_{d+1}, v_d\}) \) is labeled by \( \alpha' \in \Delta \). Lemma 4.8 shows that it is possible to complete the labeling of \( P \) using only node configurations in \( S \), as we may label \( v_i \) using the node configuration \( C_i \) where the two half-edges \( (v_i, \{v_{i-1}, v_i\}) \) and \( (v_i, \{v_i, v_{i+1}\}) \) are labeled by \( \alpha_i \) and \( \beta_i \), for each \( 2 \leq i \leq d \).

**Lemma 4.8 (Property of path-flexible strongly connected components).** Let \( S \subseteq V \) be a set of node configurations, and let \( D' \in \text{flexible-SCC}(D_\mathcal{S}) \). For any choices of \( C \in S \upharpoonright D' \), \( C' \in S \upharpoonright D' \), size-2 sub-multisets \( \{\alpha, \beta\} \subseteq C \), \( \{\alpha', \beta'\} \subseteq C' \), and a number \( d \geq \text{flexibility}(D') \), there exists a sequence

\[
(\alpha_1, C_1, \beta_1, \alpha_2, C_2, \beta_2, \ldots, \alpha_{d+1}, C_{d+1}, \beta_{d+1})
\]

satisfying the following conditions.

- **First endpoint:** \( \alpha_1 = \alpha, \beta_1 = \beta, \text{ and } C_1 = C \).
- **Last endpoint:** \( \alpha_{d+1} = \alpha', \beta_{d+1} = \beta', \text{ and } C_{d+1} = C' \).
- **Node configurations:** for \( 1 \leq i \leq d+1 \), \( \{\alpha_i, \beta_i\} \) is a size-2 sub-multiset of \( C_i \), and \( C_i \in S \).
- **Edge configurations:** for \( 1 \leq i \leq d \), \( \{\beta_i, \alpha_{i+1}\} \in \mathcal{E} \).

**Proof.** By the path-flexibility of \( D' \), there exists a length-\( d \) walk \( (\alpha, \beta) \leadsto (\alpha', \beta') \) in \( M_\Theta \). We fix

\[
(\alpha_1, \beta_1) \leadsto (\alpha_2, \beta_2) \leadsto \cdots \leadsto (\alpha_{d+1}, \beta_{d+1})
\]

to be any such walk. This implies that \( \{\beta_i, \alpha_{i+1}\} \in \mathcal{E} \) for each \( 1 \leq i \leq d \). Since \( \{\alpha_i, \beta_i\} \) is a size-2 multiset of \( D_\mathcal{S} \), there exists a choice of \( C_i \in S \) for each \( 2 \leq i \leq d \) such that \( \{\alpha_i, \beta_i\} \) is a sub-multiset of \( C_i \). \( \square \)
**Good sequences.** Given an LCL problem \( \Pi = (\Delta, \Sigma, \mathcal{V}, \mathcal{E}) \) on \( \Delta \)-regular trees, we say that a sequence

\[
(\mathcal{V}_1, \mathcal{D}_1, \mathcal{V}_2, \mathcal{D}_2, \ldots, \mathcal{V}_k)
\]

is **good** if it satisfies the following requirements.

- \( \mathcal{V}_1 = \text{trim}(\mathcal{V}) \). That is, we start the sequence from the result of trimming the set \( \mathcal{V} \) of all node configurations in the given LCL problem \( \Pi = (\Delta, \Sigma, \mathcal{V}, \mathcal{E}) \).
- For each \( 1 \leq i \leq k-1 \), \( \mathcal{D}_i \in \text{flexible-SCC}(\mathcal{D}_{\mathcal{V}_i}) \). That is, \( \mathcal{D}_i \) is a path-flexible strongly connected component of the automaton associated with the path-form of the LCL problem \( (\Delta, \Sigma, \mathcal{V}_i, \mathcal{E}) \), which is \( \Pi \) restricted to the set of node configurations \( \mathcal{V}_i \).
- For each \( 2 \leq i \leq k \), \( \mathcal{V}_i = \text{trim}(\mathcal{V}_{i-1} \upharpoonright \mathcal{D}_{\mathcal{V}_{i-1}}) \). That is, \( \mathcal{V}_i \) is the result of taking the restriction of the set of node configurations \( \mathcal{V}_{i-1} \) to \( \mathcal{D}_{\mathcal{V}_{i-1}} \) and then performing a trimming.
- \( \mathcal{V}_k \neq \emptyset \). That is, we require that the last set of node configurations is non-empty.

It is straightforward to see that \( \mathcal{V}_1 \supseteq \mathcal{V}_2 \supseteq \cdots \supseteq \mathcal{V}_k \) since \( \mathcal{V}_i = \text{trim}(\mathcal{V}_{i-1} \upharpoonright \mathcal{D}_{\mathcal{V}_{i-1}}) \) is always a subset of \( \mathcal{V}_{i-1} \).

Similarly, we also have \( \mathcal{D}_1 \supseteq \mathcal{D}_2 \supseteq \cdots \supseteq \mathcal{D}_{k-1} \), as \( \mathcal{D}_i \in \text{flexible-SCC}(\mathcal{D}_{\mathcal{V}_i}) \) is a subset of \( \mathcal{D}_{\mathcal{V}_i} \) and \( \mathcal{D}_{\mathcal{V}_i} \) is a subset of \( \mathcal{D}_{i-1} \) due to the definition \( \mathcal{V}_i = \text{trim}(\mathcal{V}_{i-1} \upharpoonright \mathcal{D}_{\mathcal{V}_{i-1}}) \).

**Depth of an LCL problem.** We define the depth \( d_\Pi \) of an LCL problem \( \Pi = (\Delta, \Sigma, \mathcal{V}, \mathcal{E}) \) on \( \Delta \)-regular trees as follows. If there is no good sequence, then we set \( d_\Pi = 0 \). If there is a good sequence \( (\mathcal{V}_1, \mathcal{D}_1, \mathcal{V}_2, \mathcal{D}_2, \ldots, \mathcal{V}_k) \) for each positive integer \( k \), then we set \( d_\Pi = \infty \). Otherwise, we set \( d_\Pi \) as the largest integer \( k \) such that there is a good sequence \( (\mathcal{V}_1, \mathcal{D}_1, \mathcal{V}_2, \mathcal{D}_2, \ldots, \mathcal{V}_k) \). We prove the following results.

**Theorem 4.9 (Characterization of complexity classes).** Let \( \Pi = (\Delta, \Sigma, \mathcal{V}, \mathcal{E}) \) be an LCL problem on \( \Delta \)-regular trees. We have the following.

- If \( d_\Pi = 0 \), then \( \Pi \) is unsolvable in the sense that there exists a tree of maximum degree \( \Delta \) such that there is no correct solution of \( \Pi \) on this tree.
- If \( d_\Pi = k \) is a positive integer, then the optimal round complexity of \( \Pi \) is \( \Theta(n^{1/k}) \).
- If \( d_\Pi = \infty \), then \( \Pi \) can be solved in \( O(\log n) \) rounds.

In Theorem 4.9, all the upper bounds hold in the CONGEST model, and all the lower bounds hold in the LOCAL model. For example, if \( d_\Pi = 5 \), then \( \Pi \) can be solved in \( O(n^{1/5}) \) rounds in the CONGEST model, and there is a matching lower bound \( \Omega(n^{1/5}) \) in the LOCAL model.

**Theorem 4.10 (Complexity of the characterization).** There is a polynomial-time algorithm \( \mathcal{A} \) that computes \( d_\Pi \) for any given LCL problem \( \Pi = (\Delta, \Sigma, \mathcal{V}, \mathcal{E}) \) on \( \Delta \)-regular trees. If \( d_\Pi = k \) is a positive integer, then \( \mathcal{A} \) also outputs a description of an \( O(n^{1/k}) \)-round algorithm for \( \Pi \). If \( d_\Pi = \infty \), then \( \mathcal{A} \) also outputs a description of an \( O(\log n) \)-round algorithm for \( \Pi \).

The distributed algorithms returned by the polynomial-time algorithm \( \mathcal{A} \) in Theorem 4.10 are also in the CONGEST model.

### 4.3 Upper bounds

In this section, we prove the upper bound part of Theorem 4.9. If a good sequence \( (\mathcal{V}_1, \mathcal{D}_1, \mathcal{V}_2, \mathcal{D}_2, \ldots, \mathcal{V}_k) \) exists for some positive integer \( k \), we show that the LCL problem \( \Pi = (\Delta, \Sigma, \mathcal{V}, \mathcal{E}) \) can be solved in \( O(n^{1/k}) \) rounds. If a
good sequence \((\mathcal{V}_1, \mathcal{D}_1, \mathcal{V}_2, \mathcal{D}_2, \ldots, \mathcal{V}_k)\) exists for all positive integers \(k\), then we show that \(\Pi = (\Delta, \Sigma, \mathcal{V}, \mathcal{E})\) can be solved in \(O(\log n)\) rounds. All these algorithms do not require sending large messages and can be implemented in the CONGEST model.

**Rake-and-compress decompositions.** Roughly speaking, a rake-and-compress process is a procedure that decomposes a tree by iteratively removing degree-1 nodes (rake) and removing degree-2 nodes (compress). There are several tree decompositions resulting from variants of a rake-and-compress process. Here we use a variant of decomposition considered in [17] that is parameterized by three positive integers \(y, \ell, \) and \(L\). A \((y, \ell, L)\) decomposition of a tree \(G = (V, E)\) is a partition of the node set

\[
V = V_1^R \cup V_1^C \cup V_2^R \cup V_2^C \cup \cdots \cup V_L^R
\]

satisfying the following requirements.

**Requirements for \(V_i^R\).** For each connected component \(S\) of the subgraph of \(G\) induced by \(V_i^R\), it is required that there is a root \(z \in S\) meeting the following conditions.

- \(z\) has at most one neighbor in \(V_i^C \cup V_{i+1}^R \cup \cdots \cup V_L^R\).
- All nodes in \(S \setminus \{z\}\) have no neighbor in \(V_i^C \cup V_{i+1}^R \cup \cdots \cup V_L^R\).
- All nodes in \(S \setminus \{z\}\) are within distance \(y - 1\) to \(z\).

Intuitively, each connected component \(S\) of the subgraph of \(G\) induced by \(V_i^R\) is a rooted tree of height at most \(y - 1\). The root can have at most one neighbor residing in the higher layers of the decomposition. The remaining nodes in \(S\) cannot have neighbors in the higher layers of the decomposition. For the special case of \(y = 1\), the set \(V_i^R\) is an independent set.

**Requirements for \(V_i^C\).** For each connected component \(S\) of the subgraph of \(G\) induced by \(V_i^C\), \(S\) is a path \((v_1, v_2, \ldots, v_s)\) of \(s \in \lfloor \ell, 2\ell \rfloor\) nodes meeting the following conditions.

- There exist two nodes \(u\) and \(w\) in \(V_{i+1}^R \cup V_{i+1}^C \cup \cdots \cup V_L^R\) such that \(u\) is adjacent to \(v_1\) and \(w\) is adjacent to \(v_s\).
- For each \(1 \leq j \leq s\), \(u_j\) has no neighbor in \((V_{i+1}^R \cup V_{i+1}^C \cup \cdots \cup V_L^R) \setminus \{u, w\}\).

Intuitively, each connected component \(S\) of the subgraph of \(G\) induced by \(V_i^C\) is a path. Only the endpoints of the paths can have neighbors residing in the higher layers of the decomposition. The remaining nodes in \(S\) cannot have neighbors in the higher layers of the decomposition.

**Existing algorithms for rake-and-compress decompositions.** For any \(k = O(1)\) and \(\ell = O(1)\), there is an \(O(n^{1/k})\)-round algorithm computing a \((y, \ell, L)\) decomposition of a tree \(G = (V, E)\) with \(y = O(n^{1/k})\) and \(L = k\) [17]. For any \(\ell = O(1)\), there is an \(O(\log n)\)-round algorithm computing a \((y, \ell, L)\) decomposition of a tree \(G = (V, E)\) with \(y = 1\) and \(L = O(\log n)\) [19]. These algorithms are deterministic and can be implemented in the CONGEST model. We will employ these algorithms as subroutines to prove the upper bound part of Theorem 4.9.

**Lemma 4.11 (Solving \(\Pi\) using rake-and-compress decompositions).** Suppose we are given an LCL problem \(\Pi = (\Delta, \Sigma, \mathcal{V}, \mathcal{E})\) on \(\Delta\)-regular trees that admits a good sequence

\[
(\mathcal{V}_1, \mathcal{D}_1, \mathcal{V}_2, \mathcal{D}_2, \ldots, \mathcal{V}_k).
\]
Suppose we are given a \((γ, ℓ, L)\) decomposition of an \(n\)-node tree \(G = (V, E)\) of maximum degree at most \(Δ\)

\[
V = V^R_1 ∪ V^C_i ∪ V^R_{i+1} ∪ V^C_{i+1} ∪ \cdots ∪ V^R_L
\]

with \(L = k\) and \(ℓ = \max\{1, \text{flexibility}(D_1) − 1, \ldots, \text{flexibility}(D_{k−1}) − 1\}\). Then a correct solution of \(Π\) on \(G\) can be computed in \(O((γ + ℓ)L)\) rounds in the CONGEST model.

Proof. We present an \(O((γ + ℓ)L)\)-round CONGEST algorithm finding a correct solution of \(Π\) on \(G\). The algorithm labels the half-edges surrounding the nodes of the graph in the order \(V^R_L, V^C_L−1, \ldots, V^R_1\). Our algorithm has the property that it only uses the node configurations in \(V_i\) to label the half-edges surrounding the nodes in \(V^R_i\) and \(V^C_i\).

Not all nodes \(v\) in \(G\) have \(\deg(v) = Δ\). In general, for each node \(v\) in \(G\), we say that the node configuration of \(v\) is in \(V_i\) if the multiset of the \(\deg(v)\) half-edge labels surrounding \(v\) is a sub-multiset of some \(C \in V\).

**Labeling \(V^R_i\).** By induction hypothesis, assume the algorithm has finished labeling the half-edges surrounding the nodes in \(V^C_i ∪ V^R_{i+1} ∪ \cdots ∪ V^R_L\) in such a way that their node configurations are in \(V_i\), as we recall that \(V_i ≥ V_{i+1} ≥ \cdots ≥ V_k\). The algorithm then labels each connected component \(S\) of the subgraph of \(G\) induced by \(V^R_i\), in parallel and using \(O(γ)\) rounds in the CONGEST model, as follows.

The set \(S\) has the property that there is at most one node \(z \in S\) that may have a neighbor in \(V^C_i ∪ V^R_{i+1} ∪ \cdots ∪ V^R_L\), and the number of neighbors of \(z\) in \(V^C_i ∪ V^R_{i+1} ∪ \cdots ∪ V^R_L\) is at most one. We claim that it is always possible to complete the labeling of half-edges surrounding the nodes in \(S\) using only node configurations in \(V_i\). To see that this is possible, it suffices to consider the following situation. Given that the existing half-edge labels surrounding a node \(u\) form a node configuration in \(V^C_i\), consider a neighbor \(v\) of \(u\), and we want to label the half-edges surrounding \(u\) in such a way that the edge configuration of \(e = \{u, v\}\) is in \(E\) and the node configuration of \(u\) is in \(V^C_i\). This is always doable due to Lemma 4.5, as we recall from the definition of \(V_i\) that \(V_i = \text{trim}(S)\) for some set \(S \subseteq V\). The round complexity of labeling \(S\) is \(O(γ)\) because \(S\) is a tree rooted at \(z\) of depth at most \(γ − 1\).

**Labeling \(V^C_i\).** Similarly, by induction hypothesis, assume the algorithm has already finished labeling the half-edges surrounding the nodes in \(V^R_{i+1} ∪ V^C_{i+1} ∪ \cdots ∪ V^R_L\) in such a way that their node configurations are in \(V_{i+1}\), as we recall that \(V_{i+1} ≥ V_{i+2} ≥ \cdots ≥ V_k\). The algorithm then labels each connected component \(S\) of the subgraph of \(G\) induced by \(V^C_i\), in parallel and using \(O(ℓ)\) rounds in the CONGEST model, as follows.

The set \(S\) has the property that there are exactly two nodes \(u\) and \(w\) in \(V^R_{i+1} ∪ V^C_{i+1} ∪ \cdots ∪ V^R_L\) adjacent to \(S\), and the subgraph induced by \(S ∪ \{u, w\}\) is a path \((u, v_1, v_2, \ldots, v_ℓ, w)\), with \(s ∈ [ℓ, 2ℓ]\). Hence the length of this path is \(s + 1 ≥ ℓ + 1 ≥ \text{flexibility}(D_ℓ)\) by our choice of \(ℓ\).

Recall that \(D_ℓ \in \text{flexible-SCC}(D_{V_i})\) and the node configurations of \(u\) and \(w\) are in \(V_{i+1} = \text{trim}(V_i \upharpoonright D_ℓ) ⊆ V_i \upharpoonright D_ℓ\), so Lemma 4.8 ensures that we can label the half-edges surrounding the nodes \(v_1, v_2, \ldots, v_ℓ\) using only node configurations in \(V_i\) in such a way that the edge configurations of all edges in the path \((u, v_1, v_2, \ldots, v_ℓ, w)\) are in \(E\). The round complexity of labeling \(S\) is \(O(ℓ)\) because \(S\) is a path of at most \(2ℓ\) nodes.

**Summary.** The number of rounds spent on labeling each part \(V^R_i\) is \(O(γ)\), and the number of rounds spent on labeling each part \(V^C_i\) is \(O(ℓ)\), so the overall round complexity for solving \(Π\) given a \((γ, ℓ, L)\) decomposition is \(O((γ + ℓ)L)\) rounds in the CONGEST model.

Combining Lemma 4.11 with existing algorithms for computing \((γ, ℓ, L)\) decompositions, we obtain the following results.
Lemma 4.12 (Upper bound for the case $d_R = k$). If $d_R = k$ for some positive integer $k$, then $\Pi$ can be solved in $O(n^{1/k})$ rounds in the CONGEST model.

Proof. In this case, a good sequence $(\mathcal{V}_1, D_1, \mathcal{V}_2, D_2, \ldots, \mathcal{V}_k)$ exists. As a $(y, t, L)$ decomposition with $y = O(n^{1/k})$, $t = O(1)$, and $L = k$ can be computed in $O(n^{1/k})$ rounds [17], $\Pi$ can be solved in $O(n^{1/k} + O((y + t)L) = O(n^{1/k})$ rounds using the algorithm of Lemma 4.11. Here both $k$ and $t$ are $O(1)$, as they are independent of the number of nodes $n$. \hfill \Box

Lemma 4.13 (Upper bound for the case $d_R = \infty$). If $d_R = \infty$, then $\Pi$ can be solved in $O(\log n)$ rounds in the CONGEST model.

Proof. In this case, a good sequence $(\mathcal{V}_1, D_1, \mathcal{V}_2, D_2, \ldots, \mathcal{V}_k)$ exists for all positive integers $k$. As a $(y, t, L)$ decomposition with $y = 1$, $t = O(1)$, and $L = O(\log n)$ can be computed in $O(\log n)$ rounds [19], by choosing a good sequence $(\mathcal{V}_1, D_1, \mathcal{V}_2, D_2, \ldots, \mathcal{V}_k)$ with $k = L$, $\Pi$ can be solved in $O(\log n) + O((y + t)L) = O(\log n)$ rounds using the algorithm of Lemma 4.11. Similarly, here $t = O(1)$, as it is independent of the number of nodes $n$. \hfill \Box

4.4 Lower bounds

In this section, we prove the lower bound part of Theorem 4.9. In our lower bound proofs, we pick $y$ to be the smallest integer satisfying the following requirements. For each subset $S \subseteq V$ and each $C \in S \setminus \text{trim}(S)$, there exists no correct labeling of $T^*_y$ where the node configuration of the root $r$ is $C$ and the node configurations of the remaining degree-$\Delta$ nodes are in $S$. Such a number $y$ exists due to the definition of trim.

Lemma 4.14 (Unsolvability for the case $d_R = 0$). If $d_R = 0$, then $\Pi$ is unsolvable in the sense that there exists a tree $G$ of maximum degree $\Delta$ such that there is no correct solution of $\Pi$ on $G$.

Proof. We take $G = T^*_y$. Since $d_R = 0$, we have $\text{trim}(V) = \emptyset$. Our choice of $y$ implies that there is no correct solution of $\Pi$ on $G = T^*_y$. \hfill \Box

For the rest of this section, we focus on the case that $d_R = k$ is a positive integer. We will prove that solving $\Pi$ requires $\Omega(n^{1/k})$ rounds in the LOCAL model. The exact choice of $s = \Theta(t)$ in Definition 4.15 is to be determined later.

Definition 4.15 (Lower bound graphs). Let $t$ be any positive integer and choose $s = \Theta(t)$ to be a sufficiently large integer.

- $G_{R,1}$ is the rooted tree $T_y$, and $G^*_{R,1}$ is the rooted tree $T^*_y$. All nodes in $G_{R,1}$ and $G^*_{R,1}$ are said to be in layer $(R, 1)$.
- For each integer $i \geq 1$, $G_{C,i}$ is the result of the following construction. Start with an $s$-node path $(v_1, v_2, \ldots, v_s)$ and let $v_1$ be the root. For each $1 \leq i < s$, append $\Delta - 2$ copies of $G_{R,i}$ to $v_i$. For $i = s$, append $\Delta - 1$ copies of $G_{R,s}$ to $v_s$. The nodes $v_1, v_2, \ldots, v_s$ are said to be in layer $(C, i)$.
- For each integer $i \geq 2$, $G_{R,i}$ is the result of the following construction. Start with a rooted tree $T_y$. Append $\Delta - 1$ copies of $G_{C,i-1}$ to each leaf in $T_y$. All nodes in $T_y$ are said to be in layer $(R, i)$. The rooted tree $G^*_{R,i}$ is defined analogously by replacing $T_y$ with $T^*_y$ in the construction.

Although the trees $G_{R,i}, G^*_{R,i}$, and $G_{C,i}$ are rooted in their definitions, we may also treat them as unrooted trees. Throughout our lower bound proof in Section 4.4, we fix our main lower bound graph $G = (V, E)$ to be the tree $G^*_{R,k+1}$, see Figure 2 for an example.

Definition 4.16 (Main lower bound graph). Define $G = (V, E)$ as the tree $G^*_{R,k+1}$. 

\hfill \Box
In the subsequent discussion, we focus on the tree $G = G_{R,k+1}^*$. It is clear from its construction that $G$ has $n = O(t^k)$ nodes, if we treat $y$ as a constant independent of $t$. Indeed, $y$ only depends on the underlying LCL problem $\Pi = (\Delta, \Sigma, V', E)$. More generally, the number of nodes in $G_{R,i}$ or $G_{R,i}^*$ is $O(t^{i-1})$, and the number of nodes in $G_{C,i}$ is $O(t^i)$. To show that $\Pi$ requires $\Omega(n^{1/k})$ rounds to solve, it suffices to show that $\Pi$ requires $t$ rounds to solve on $G$, for any positive integer $t$.

The nodes in $G = G_{R,k+1}^*$ are partitioned into layers $(R, 1), (C, 1), (R, 2), (C, 2), \ldots, (R, k + 1)$ according to the rules in the above recursive construction. In the subsequent discussion, we order the layers by $(R, 1) < (C, 1) < (R, 2) < (C, 2) < \cdots < (R, k + 1)$. For example, when we say layer $(R, i)$ or higher, we mean the set of all layers $(R, i), (C, i), \ldots, (R, k + 1)$.

Intuitively, layers $(R, i)$ and $(C, i)$ resemble the parts $V^R_i$ and $V^C_i$ in a rake-and-compress decomposition. Except for some leaf nodes in layer $(R, 1)$, all nodes in the graph have degree $\Delta$. Each connected component of the subgraph of $G$ induced by layer $(C, i)$ nodes is a path of $s$ nodes. Each connected component of the subgraph of $G$ induced by layer $(R, i)$ nodes is a rooted tree $T^R_i$ (if $1 \leq i \leq k$) or a rooted tree $T^*_i$ (if $i = k + 1$). We further classify the nodes in layer $(C, i)$ as follows.
To prove the lemma, it suffices to find a path \( P \) such that all descendants of \( w \) are central nodes in layer of \( G \).

Based on Definition 4.17, we define the following subsets of nodes in \( G = G_{R,k+1}^* \). We assume that \( s = \Theta(t) \) is chosen to be sufficiently large so that central nodes exist.

**Definition 4.17 (Classification of nodes in layer \((C, i)\)).** The nodes in the \( s \)-node path \((v_1, v_2, \ldots, v_s) \) in the construction of \( G_{C,i} \) are classified as follows.

- We say that \( v_j \) is a front node if \( 1 \leq j \leq t \).
- We say that \( v_j \) is a central node if \( t + 1 \leq j \leq s - t \).
- We say that \( v_j \) is a rear node if \( s - t + 1 \leq j \leq s \).

We prove some basic properties of the sets in Definition 4.18.

**Lemma 4.19 (Subset containment).** We have \( S_{R,1} \supseteq S_{C,1} \supseteq \cdots \supseteq S_{R,k+1} \neq \emptyset \).

**Proof.** We have \( S_{C,i} \supseteq S_{R,i+1} \) since it follows from the definition of \( S_{R,i+1} \) that \( v \in S_{C,i} \) is a necessary condition for \( v \in S_{R,i+1} \). The claim that \( S_{R,i} \supseteq S_{C,i} \) follows from the fact that each \( v \in S_{C,i} \) is in layer \((C, i)\) or above: The radius-\( r \) neighborhood of any such node \( v \) is isomorphic to \( T^*_r \) and contains only nodes in layer \((R, i)\) or above, and we know that all nodes in layer \((R, i)\) or above are in \( S_{C,i-1} \). Hence \( v \in S_{C,i} \) implies \( v \in S_{R,i} \).

To see that \( S_{R,k+1} \neq \emptyset \), consider the root \( r \) of \( G = G_{R,k+1}^* \). The radius-\( r \) neighborhood of \( r \) is isomorphic to \( T^*_r \) and contains only nodes in layer \((R, k+1)\). We know that all nodes in layer \((R, k+1)\) are in \( S_{C,k} \), so \( r \in S_{R,k+1} \).

**Lemma 4.20 (Property of \( S_{C,i} \)).** For each node \( v \in S_{C,i} \), one of the following holds.

- \( v \) is a central node in layer \((C, i)\).
- For each neighbor \( u \) of \( v \) such that \( u \in S_{C,i} \), there exists a path \( P = (v, u, \ldots, w) \) such that \( w \) is a central node in layer \((C, i)\) and all nodes in \( P \) are in \( S_{C,i} \).

**Proof.** We assume that \( v \in S_{C,i} \) is not a central node in layer \((C, i)\). Consider any neighbor \( u \) of \( v \) such that \( u \in S_{C,i} \). To prove the lemma, its suffices to find a path \( P = (v, u, \ldots, w) \) such that \( w \) is a central node in layer \((C, i)\) and all nodes in \( P \) are in \( S_{C,i} \). The existence of such a path \( P \) follows from the simple observation that \( S_{C,i} \) induces a connected subree where all the leaf nodes are central nodes in layer \((C, i)\).

More specifically, such a path \( P \) can be constructed as follows. If \( u \) itself is a central node in layer \((C, i)\), then we can simply take \( P = (v, u = w) \).

We first consider the case where \( v \) is the parent of \( u \) in the rooted tree \( G = G_{R,k+1}^* \). In this case, there must exist a descendant \( w \) of \( u \) such that \( w \) is a central node in layer \((C, i)\). This gives us a desired path \( P = (v, u, \ldots, w) \).

Next, we consider the case where \( v \) is a child of \( u \) in the rooted tree \( G = G_{R,k+1}^* \). In this case, it might be possible that all descendants \( w \) of \( u \) such that \( w \) is a central node in layer \((C, i)\) are also descendants of \( v \). Hence we will consider a different approach. Starting from \((v, u)\), we first follow the parent pointers to the root \( r \) of \( G = G_{R,k+1}^* \). There are \( \Delta \) children of \( r \), and it is clear that each child of \( r \) has a descendant \( w \) that is a central node in layer \((C, i)\). Hence we
can extend the current path \((v, u, \ldots, r)\) to a desired \(P = (v, u, \ldots, r, w)\) such that \(w\) that is a central node in layer \((C, i)\). □

Assumptions. We are given an LCL problem \(\Pi = (\Delta, \Sigma, V, E)\) such that \(d_{\Pi} = k\). Hence there does not exist a good sequence

\[(V_1, D_1, V_2, D_2, \ldots, V_{k+1}).\]

Recall that the rules for a good sequence are as follows:

\[
V_i = \begin{cases} \text{trim}(V) & \text{if } i = 1, \\ \text{trim}(V_{i-1} \upharpoonright D_{i-1}) & \text{if } i > 1, \end{cases}
\]

\[D_i \in \text{flexible-SCC}(D_{V_i}).\]

The only nondeterminism in the above rules is the choice of \(D_i \in \text{flexible-SCC}(D_{V_i})\) for each \(i\). The fact that \(d_{\Pi} = k\) implies that for all possible choices of \(D_1, D_2, \ldots, D_k\), we always end up with \(V_{k+1} = \emptyset\).

We assume that there is an algorithm \(A\) that solves \(\Pi\) in \(t\) rounds on \(G = G_{R,k+1}\). As the number \(n\) of nodes in \(G\) satisfies \(t = \Omega(n^{1/k})\), to prove the desired \(\Omega(n^{1/k})\) lower bound, it suffices to derive a contradiction. Specifically, we will prove that the existence of such an algorithm \(A\) forces the existence of a good sequence \((V_1, D_1, V_2, D_2, \ldots, V_{k+1})\), contradicting the fact that \(d_{\Pi} = k\).

Induction hypothesis. Our proof proceeds by an induction on the subsets \(S_{R,1}, S_{C,1}, S_{R,2}, S_{C,2}, \ldots, S_{R,k+1}\). For each \(1 \leq i \leq k\), the choice of \(D_i \in \text{flexible-SCC}(D_{V_i})\) is fixed in the induction hypothesis for \(S_{C,i}\). The choice of \(V_i\) is uniquely determined once \(D_1, D_2, \ldots, D_{i-1}\) have been fixed.

Before defining our induction hypothesis, we recall that the output labels of the half-edges surrounding a node \(v\) are determined by the subgraph induced by the radius-\(t\) neighborhood \(U\) of \(v\), together with the distinct IDs of the nodes in \(U\). For each node \(v\) in \(G\), we define \(V_o^A\) as the set of all possible node configurations of \(v\) that can possibly appear when we run \(A\) on \(G\). In other words, \(C \in V_o^A\) implies that there exists an assignment of distinct IDs to nodes in the radius-\(t\) neighborhood of \(v\) such that the output labels of the half-edges surrounding \(v\) form the node configuration \(C\).

Similarly, for any two edges \(e_1\) and \(e_2\) incident to a node \(v\), we define \(D_{o,e_1,e_2}^A\) to be the set of all size-2 multisets \(\{a, b\}\) of labels such that \(\{a, b\}\) is a possible outcome of labeling the two half-edges \((v, e_1)\) and \((v, e_2)\) when we run the algorithm \(A\) on \(G\).

Definition 4.21 (Induction hypothesis for layer \(S_{R,i}\)). For each \(1 \leq i \leq k+1\), the induction hypothesis for \(S_{R,i}\) specifies that each \(v \in S_{R,i}\) satisfies \(V_o^A \subseteq V_i\).

Definition 4.22 (Induction hypothesis for layer \(S_{C,i}\)). For each \(1 \leq i \leq k\), the induction hypothesis for \(S_{C,i}\) specifies that for each \(v \in S_{C,i}\) and any two incident edges \(e_1 = \{v, u\}\) and \(e_2 = \{v, w\}\) such that \(u\) and \(w\) are in layer \((C, i)\) or higher, we have \(D_{o,e_1,e_2}^A \subseteq D_i\).

Next, we prove that the induction hypotheses stated in Definitions 4.21 and 4.22 hold.

Lemma 4.23 (Base case: \(S_{R,1}\)). The induction hypothesis for \(S_{R,1}\) holds.

Proof. Recall that the number \(\gamma\) satisfies the following property. For each subset \(S \subseteq V\) and each \(C \in S \setminus \text{trim}(S)\), there exists no correct labeling of \(T^*_r\) where the node configuration of the root \(r\) is \(C\) and the node configuration of remaining degree-\(\Delta\) nodes is in \(S\).
To prove the induction hypothesis for $S_{R_1}$, consider any node $v \in S_{R_1}$. By the definition of $S_{R_1}$, the radius-$γ$ neighborhood of $v$ is isomorphic to $T_γ^*$ rooted at $v$, and by setting $S = V$, we infer that there is no correct labeling of $G$ such that the node configuration of $v$ is in $V \setminus \text{trim}(V) = V \setminus V_1$, so we must have $V_0^A \subseteq V_1$, as $A$ is correct. □

**Lemma 4.24 (Inductive step: $S_{R_i}$).** Let $2 \leq i \leq k$. If the induction hypothesis for $S_{R_{i-1}}$ and $S_{C,i-1}$ holds, then the induction hypothesis for $S_{R_i}$ holds.

**Proof.** To prove the induction hypothesis for layer $S_{R_i}$, consider any node $v \in S_{R_i}$. By the definition of $S_{R_i}$, the radius-$γ$ neighborhood of $v$ is isomorphic to $T_γ^*$ rooted at $v$ and contains only nodes from $S_{C,i-1}$. The goal is to prove that $V_0^A \subseteq \text{trim}(V_{i-1}) = \text{trim}(V_{i-1} \upharpoonright D_{i-1})$.

The set of degree-Δ nodes in this $T_γ^*$ is precisely the set of nodes in the radius-$(γ - 1)$ neighborhood of $v$ in $G$. Consider any node $u$ in the radius-$(γ - 1)$ neighborhood of $v$. As $u$ is in $S_{C,i-1} \subseteq S_{R_{i-1}}$, the induction hypothesis for $S_{R_{i-1}}$ implies that

$$V_u^A \subseteq V_{i-1}.$$

Since all neighbors of $u$ are in $S_{C,i-1}$, from the induction hypothesis for $S_{C,i-1}$, we have

$$D_{u,e_1,e_2}^A \subseteq D_{i-1}$$

for any two edges $e_1$ and $e_2$ incident to $u$. Combining these two facts, we infer that

$$V_u^A \subseteq \text{trim}(V_{i-1} \upharpoonright D_{i-1}).$$

Given that all degree-Δ nodes in this $T_γ^*$ satisfy that $V_u^A \subseteq \text{trim}(V_{i-1} \upharpoonright D_{i-1})$, the same argument as in the proof of Lemma 4.23 shows that $V_0^A \subseteq \text{trim}(V_{i-1} \upharpoonright D_{i-1}) = V_i$, as required. □

**Lemma 4.25 (Inductive step: $S_{C,i}$).** Let $1 \leq i \leq k$. If the induction hypothesis for $S_{R,i}$ holds, then the induction hypothesis for $S_{C,i}$ holds.

**Proof.** To prove the induction hypothesis for $S_{C,i}$, we show that there exists a choice $D_i \in \text{flexible-SCC}(D_{V_i})$ such that the following statement holds. For any node $v \in S_{C,i}$ and any two incident edges $e_1 = \{v, u\}$ and $e_2 = \{v, w\}$ such that $u$ and $w$ are in layer $(C, i)$ or higher, we have $D_{v,e_1,e_2}^A \subseteq D_i$.

We first make an observation about central nodes in layer $(C, i)$. Let $v$ be a central node in layer $(C, i)$, and let $u$ and $w$ be the two neighbors of $v$ in layer $(C, i)$. Let $e_1 = \{v, u\}$ and $e_2 = \{v, w\}$. Then it is clear that $D_{v,e_1,e_2}^A$ is the same for each choice of $v$ that is a central node in layer $(C, i)$, as the radius-$t$ neighborhoods of central nodes in the same layer are isomorphic, due to the definition of central nodes and the construction in Definition 4.15. For notational convenience, we write $\tilde{D} = D_{v,e_1,e_2}$ to denote this set.

**Plan of the proof.** Consider any node $v \in S_{C,i}$ and its two incident edges $e_1 = \{v, u\}$ and $e_2 = \{v, w\}$ such that $u$ and $w$ are in layer $(C, i)$ or higher. Due to the induction hypothesis for $S_{R,i}$, we already have $V_v^A \subseteq V_i$ as $v \in S_{C,i} \subseteq S_{R,i}$, and so $D_{v,e_1,e_2}^A \subseteq D_{V_i}$.

To prove that there exists a choice $D_i = \text{flexible-SCC}(D_{V_i})$ such that $D_{v,e_1,e_2}^A \subseteq D_i$ for all such $v$, $e_1$, and $e_2$, we will first show that $\tilde{D}$ must be a subset of a path-flexible strongly connected component of $D_{V_i}$, and then we fix $D_i \in \text{flexible-SCC}(D_{V_i})$ to be this path-flexible strongly connected component.

Next, we will argue that for each $(a, b) \in D_{v,e_1,e_2}$, there exist a walk in $M_{D_{V_i}}$ that starts from $(a, b)$ and ends in $\tilde{D}$ and a walk in $M_{D_{V_i}}$ that starts from $\tilde{D}$ and ends in $(a, b)$. This shows that $(a, b)$ is in the same strongly connected component as the members in $\tilde{D}$, so we conclude that $D_{v,e_1,e_2}^A \subseteq D_i$. 17
Part 1: $\tilde{D}$ is a subset of some $D_i \in \text{flexible-SCC}(D_{V'})$. Consider any path $(u_1, u_2, \ldots, u_s)$ of $s$ nodes in layer $(C, i)$ of $G = G^*_{R,k+1}$. Note that any such a path must be the $s$-node path $(v_1, v_2, \ldots, v_s)$ in the construction of some $G_{C,i}$ in Definition 4.15. We choose $s = \Theta(t)$ to be sufficiently large to ensure that for each integer $0 \leq d \leq |\Sigma|^2$, there exist two nodes $u_j$ and $u_i$ in the path meeting the following conditions.

- $t + 1 \leq j < l \leq s - t$, so $u_j$ and $u_i$ are central nodes.
- The distance $l - j$ between $u_j$ and $u_i$ equals $4t + 3 + d$.

The choice of the number $4t + 3$ is to ensure that the union of the radius-$t$ neighborhoods of the endpoints of any edge in $G$ does not node-intersect the radius-$t$ neighborhood of both $u_j$ and $u_i$. This implies that after arbitrarily fixing distinct IDs in the radius-$t$ neighborhood of $u_j$ and $u_i$, it is possible to complete the ID assignment of the entire graph $G$ in such a way that the union of the radius-$t$ neighborhoods of the endpoints of each edge in $G$ does not contain repeated IDs. If we run $A$ with such an ID assignment, it is guaranteed that the output is correct.

Consider the directed graph $M_{D_{V'}}$ and any two of its nodes $(a, b)$ and $(c, d)$ such that $(a, b) \in \tilde{D}$ and $(c, d) \in \tilde{D}$. Our choice of $\tilde{D}$ implies that there exists an assignment of distinct IDs to the radius-$t$ neighborhood of both $u_j$ and $u_i$ such that the output labels of $(u_j, u_{j-1})$, $(u_i, u_{i-1})$, $(u_i, u_{i+1})$, and $(u_i, u_{i+1})$ are $a, b, c,$ and $d$, respectively. We complete the ID assignment of the entire graph $G$ in such a way that the radius-$t$ neighborhood of each edge in $G$ does not contain repeated IDs. For each node $u_y$ with $j < y < l$, the pair of the two output labels of $(u_y, u_{y-1})$ and $(u_y, u_{y+1})$ resulting from $A$ is also a node in $M_{D_{V'}}$. Hence there exists a walk $(a, b) \leadsto (c, d)$ of length $4t + 3 + d$, for any $0 \leq d \leq |\Sigma|^2$. Since this holds for all choices of $(a, b)$ and $(c, d)$ such that $(a, b) \in \tilde{D}$ and $(c, d) \in \tilde{D}$, all members in $\tilde{D}$ are in the same strongly connected component. Furthermore, Lemma 4.6 implies that this strongly connected component is path-flexible, as the length of the walk can be any integer in between $4t + 3$ and $4t + 3 + |\Sigma|^2$.

Part 2: $D_{v_1, e_1, e_2} \subseteq D_i$. For this part, we use Lemma 4.20, which shows that in the graph $G = G^*_{R,k+1}$ there are a path $P_1$ from $v$ to a central node in layer $(C, i)$ through $e_1$ and a path $P_2$ from $v$ to a central node in layer $(C, i)$ through $e_2$, and these paths use only nodes in $S_{C,i}$.

Consider the output labels resulting from running $A$. Let $a$ be the label of the half-edge $(v, e_1)$ and let $b$ be the label of the half-edge $(v, e_2)$. We have $(a, b) \in D_{v_1, e_1, e_2} \subseteq D_{V'}$. By taking the output labels in $P_1$ and $P_2$ resulting from running $A$, we obtain two walks in the directed graph $M_{D_{V'}}$: $(a, b) \leadsto (c, d)$ and $(b, a) \leadsto (e, f)$, where both $(c, d)$ and $(e, f)$ are nodes in $V(M_{D_{V'}})$ such that $(a, b) \in \tilde{D}$ and $(c, d) \in \tilde{D}$. By taking the opposite directions, we also obtain two walks: $(d, c) \leadsto (b, a)$ and $(f, e) \leadsto (a, b)$. Hence $(a, b)$ is in the same strongly connected component of $D_{V'}$ as the members in $\tilde{D}$.

The same argument can be applied to all $(a, b) \in D_{v, e_1, e_2}$. The reason is that for each $(a, b) \in D_{v, e_1, e_2}$ there is an assignment of distinct IDs such that $(a, b)$ is the multiset of the two labels of $(v, e_1)$ and $(v, e_2)$. Hence we conclude that all members in $D_{v, e_1, e_2}$ are within the same strongly connected component as that of members in $\tilde{D}$, so $D_{v, e_1, e_2} \subseteq D_i$. □

Applying Lemmas 4.23 to 4.25 from $S_{R,1}$ all the way up to the last subset $S_{R,k+1}$, we obtain the following result.

Lemma 4.26 (Lower bound for the case $d_1 = k$). If $d_1 = k$ for a finite integer $k$, then $\Pi$ requires $\Omega(n^{1/k})$ rounds to solve on trees of maximum degree $\Delta$.

Proof. Assume that there is a $t$-round algorithm solving $\Pi$ on $G$. By Lemmas 4.23 to 4.25, we infer that the induction hypothesis for the last subset $S_{R,k+1}$ holds. By Lemma 4.19, $S_{R,k+1} \neq \emptyset$, so there is a node $v$ in $G$ such that $V_v^A \subseteq V_{k+1}$. 

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Therefore, the correctness of $A$ implies that $\mathcal{V}_{k+1} \neq \emptyset$, which implies that $(\mathcal{V}_1, D_1, \mathcal{V}_2, D_2, \ldots, \mathcal{V}_{k+1})$ chosen in the induction hypothesis is a good sequence, contradicting the assumption that $d_\Pi = k$. Hence such a $t$-round algorithm $A$ that solves $\Pi$ does not exist. As this argument holds for all integers $t$ and $t = \Omega(n^{1/k})$, where $n$ is the number of nodes in $G$, we conclude the proof. $\square$

Now we are ready to prove Theorem 4.9.

**Proof of Theorem 4.9.** The upper bound part of the theorem follows from Lemmas 4.12 and 4.13. The lower bound part of the theorem follows from Lemmas 4.14 and 4.26. $\square$

### 4.5 Complexity of the characterization

In this section, we prove Theorem 4.10. We are given a description of an LCL problem $\Pi = (\Delta, \Sigma, \mathcal{V}, \mathcal{E})$ on $\Delta$-regular trees. We assume that the description is given in the form of listing the multisets in $\mathcal{V}$ and $\mathcal{E}$. Therefore, the description length of $\Pi$ is $t = \Theta(\log |\Sigma|) \cdot (|\mathcal{E}| \cdot 2 + |\mathcal{V}| \cdot \Delta)$. Here we allow $\Delta$ to be a non-constant, as a function of $t$. We will design an algorithm that computes all possible good sequences $(\mathcal{V}_1, D_1, \mathcal{V}_2, D_2, \ldots, \mathcal{V}_k)$ in time polynomial in $t$, and this allows us to compute $d_{\Pi}$. As, the main objective of this section is to show that the problem is polynomial-time solvable, we do not aim to optimize the time complexity of our algorithm.

For the case of $d_{\Pi} = \infty$, there are good sequences that are arbitrarily long. Recall that we have $\mathcal{V}_1 \supseteq \mathcal{V}_2 \supseteq \cdots \supseteq \mathcal{V}_k$. Hence if $k > |\mathcal{V}|$, there must exist some index $1 \leq i < k$ such that $\mathcal{V}_i = \mathcal{V}_{i+1}$. This immediately implies that $\mathcal{V}_i = \mathcal{V}_{i+1} = \mathcal{V}_{i+2} = \cdots$, due to the following reasoning. The fact that $\mathcal{V}_i = \mathcal{V}_{i+1}$ implies that $\mathcal{V}_{i+1} = \text{trim}(\mathcal{V}_i \upharpoonright D_i) = \mathcal{V}_i \upharpoonright D_i = \mathcal{V}_i$, so $D_i = D_{\mathcal{V}_i}$. This means that $D_i = D_{\mathcal{V}_i}$ itself is the only element of flexible-SCC($D_{\mathcal{V}_i}$). Therefore, starting from $\mathcal{V}_i$, the multisets $D_i = D_{\mathcal{V}_i}$ and $\mathcal{V}_{i+1} = \mathcal{V}_i$ are uniquely determined. Similarly, we have $D_j = D_i$ and $\mathcal{V}_j = \mathcal{V}_i$ for all $j \geq i$. We conclude that any good sequence with $k > |\mathcal{V}|$ must stabilize at some point $i \leq |\mathcal{V}|$, in the sense that $D_j = D_i$ and $\mathcal{V}_j = \mathcal{V}_i$ for all $j \geq i$.

**High-level plan.** Recall that the rules for a good sequence are as follows:

\[
\begin{align*}
\mathcal{V}_i &= \begin{cases} 
\text{trim}(\mathcal{V}) & \text{if } i = 1, \\
\text{trim}(\mathcal{V}_{i-1} \upharpoonright D_{i-1}) & \text{if } i > 1,
\end{cases} \\
D_i &\in \text{flexible-SCC}(D_{\mathcal{V}_i}).
\end{align*}
\]

To compute all good sequences $(\mathcal{V}_1, D_1, \mathcal{V}_2, D_2, \ldots, \mathcal{V}_k)$, we go through all choices of $D_i \in \text{flexible-SCC}(D_{\mathcal{V}_i})$ and apply the rules recursively until we cannot proceed any further. The process stops when $\mathcal{V}_i = \mathcal{V}_{i-1}$ (the sequence stabilizes) or $\mathcal{V}_i = \emptyset$ (the sequence ends).

In time $O(|\Sigma|^2)$, we build a look-up table that allows us to check whether $(\alpha, \beta) \in \mathcal{E}$ in $O(1)$ time, for any given size-2 multiset $(\alpha, \beta)$. In the subsequent discussion, we assume that such a look-up table is available.

We start with describing the algorithm for computing trim($S$) for a given $S \subseteq \mathcal{V}$.

**Lemma 4.27 (Algorithm for trim).** The set trim($S$) can be computed in $O(\Delta |S||\Sigma|^2)$ time, for any given $S \subseteq \mathcal{V}$.

**Proof.** We write $T'_r$ to denote the tree resulting from adding one extra edge $e'$ incident to the root $r$ of $T_r$. We write $\Sigma_1$ to denote set of all possible $\sigma \in S$ such that there is a correct labeling of $T'_r$ where the node configuration of each degree-$\Delta$ node is in $S$ and the half-edge label of $(r, e')$ is $\sigma$. The set $\Sigma_1$ can be computed recursively as follows:

- For the base case, $\Sigma_1$ is the set of all labels appearing in $S$. 

For the inductive step, each $\sigma \in \Sigma_{i-1}$ is added to $\Sigma_i$ if there exists $C \in \mathcal{S}$ such that $\sigma \in C$ and each of the $\Delta - 1$ labels $\alpha$ in $C \setminus \{\sigma\}$ satisfies that $\{\alpha, \beta\} \in E$ for some $\beta \in \Sigma_{i-1}$.

Using the above look-up table, given that $\Sigma_{i-1}$ has been computed, the computation of $\Sigma_i$ costs $O(\Delta|S||\Sigma_{i-1}|) = O(\Delta|S||\Sigma|)$ time. Clearly, we have $\Sigma_1 \supseteq \Sigma_2 \supseteq \cdots$, and whenever $\Sigma_i = \Sigma_{i+1}$, the sequence stabilizes: $\Sigma_i = \Sigma_{i+1} = \Sigma_{i+2} = \cdots$. We write $\Sigma^*$ to denote the fix point $\Sigma_i$ such that $\Sigma_i = \Sigma_{i+1} = \Sigma_{i+2} = \cdots$. It is clear that $i \leq |\Sigma|$, so the fixed point $\Sigma^*$ can be computed in $O(\Delta|S||\Sigma|^2)$ time.

Given the fix point $\Sigma^*$, the set $\text{trim}(S)$ can be computed as follows. Observe that the tree $T'_i$ is simply the result of merging $\Delta$ trees $T'_{i-1}$ by merging the $\Delta$ degree-1 endpoints of $e^i$ into one node $r$. Therefore, $C \in \text{trim}(S)$ if and only if each of the $\Delta$ labels $\sigma \in C$ satisfies that $\{\alpha, \beta\} \in E$ for some $\beta \in \Sigma^*$. Using this characterization, given the fix point $\Sigma^*$, the set $\text{trim}(S)$ can be similarly computed in $O(\Delta|S||\Sigma|^2)$ time. We conclude the following result.

Next, we give an algorithm that computes all path-flexible strongly connected components $\mathcal{D}' \in \text{flexible-SCC}(\mathcal{D}_S)$ and their corresponding restriction $S \restriction_{\mathcal{D}'}$, for any given $S \subseteq V$.

**Lemma 4.28 (Algorithm for flexible-SCC).** The set of all $\mathcal{D}' \in \text{flexible-SCC}(\mathcal{D}_S)$ and their corresponding restrictions $S \restriction_{\mathcal{D}'}$ can be computed in $O(\Delta^3|S|^4)$ time, for any given $S \subseteq V$.

**Proof.** Observe that $|\mathcal{D}_S| = O(\Delta^2|S|)$, so the directed graph $M_{\mathcal{D}_S}$ has $O(\Delta^2|S|)$ nodes and $O(\Delta^4|S|^2)$ edges.

Using the definition of Definition 4.7, testing whether $\{a, b\} \sim \{c, d\}$ for any $\{a, b\} \in \mathcal{D}_S$ and $\{c, d\} \in \mathcal{D}_S$ costs $O(|E(M_{\mathcal{D}_S})|) = O(\Delta^4|S|^2)$ time by doing four $s$-$t$ reachability computation. By going over all $\{a, b\} \in \mathcal{D}_S$ and $\{c, d\} \in \mathcal{D}_S$, the set of all strongly connected components of $\mathcal{D}_S$ can be computed in time $O(|E(M_{\mathcal{D}_S})|\cdot|V(M_{\mathcal{D}_S})|^2) = O(\Delta^8|S|^4)$.

For each strongly connected component $\mathcal{D}' \in \mathcal{D}_S$, to decide whether $\mathcal{D}'$ is path-flexible, it suffices to pick one element $\{a, b\} \in \mathcal{D}'$ and check if $\{a, b\}$ is path-flexible. Recall that $\{a, b\}$ is path-flexible if there exists an integer $K$ such that for each integer $k \geq K$, there exist length-$s$ walks $(a, b) \leadsto (a, b), (a, b) \leadsto (b, a), (b, a) \leadsto (a, b)$, and $(b, a) \leadsto (b, a)$ in $M_{\mathcal{D}_S}$.

The fact that $(a, b) \in \mathcal{D}'$ and $\mathcal{D}'$ is a strongly connected component of $\mathcal{D}_S$ implies the existence of walks $(a, b) \leadsto (a, b), (a, b) \leadsto (b, a), (b, a) \leadsto (a, b)$, and $(b, a) \leadsto (b, a)$ in $M_{\mathcal{D}_S}$. Therefore, the task for deciding whether $\mathcal{D}'$ is path-flexible is reduced to the following task. Given a node $s = (a, b)$ in the directed graph $M_{\mathcal{D}_S}$, let $L$ be the set of possible lengths of an $s \leadsto s$ walk, check if there exists an integer $K$ such that $L$ contains all integers that are at least $K$. Such an integer $K$ exists if and only if the greatest common divisor $\gcd(L)$ of $L$ is not one. Define $L'$ as the set of numbers in $L$ that are at most $2|V(M_{\mathcal{D}_S})| - 1$. As shown in [20], we have $\gcd(L) = \gcd(L')$.

The computation of $\gcd(L')$ can be done in $O(|V(M_{\mathcal{D}_S})|^3)$ time, as follows. From $i = 0$ up to $i = 2|V(M_{\mathcal{D}_S})| - 1$, we compute a list of nodes $U_i \subseteq V(M_{\mathcal{D}_S})$ such that $v \in U_i$ if there is a walk $s \leadsto v$ of length $i$. Given $U_{i-1}$, it takes $O(|V(M_{\mathcal{D}_S})|^2)$ time to compute $U_i$, as we just need to go over all $O(|V(M_{\mathcal{D}_S})|^2)$ edges between the nodes in $V(M_{\mathcal{D}_S})$. The summation of the time complexity $O(|V(M_{\mathcal{D}_S})|^3)$, over all strongly connected component of $V(M_{\mathcal{D}_S})$, is $O(|V(M_{\mathcal{D}_S})|^3) = O(\Delta^8|S|^4)$.

To summarize, the computation of flexible-SCC($\mathcal{D}_S$) costs $O(\Delta^8|S|^4)$ time. Given that we have computed all path-flexible strongly connected components $\mathcal{D}' \in \text{flexible-SCC}(\mathcal{D}_S)$, the computation of the restriction $S \restriction_{\mathcal{D}'}$ for all path-flexible strongly connected components $\mathcal{D}' \in \text{flexible-SCC}(\mathcal{D}_S)$ costs $O(|\mathcal{D}_S| + \Delta^2|S|) = O(\Delta^2|S|)$ time, as we just need to check for each $C \in \mathcal{S}$ whether there is $\mathcal{D}' \in \text{flexible-SCC}(\mathcal{D}_S)$ such that all $\Delta^2$ size-2 sub-multisets of $C$ belong to the $\mathcal{D}'$. □
Combining Lemmas 4.27 and 4.28, we obtain the following result.

**Lemma 4.29 (Computing all good sequences).** The set of all good sequences can be computed in $O(|\mathcal{V}|^2|\Sigma|^2 + \Delta^8|\mathcal{V}|^5)$ time, for any given LCL problem $\Pi = (\Delta, \Sigma, \mathcal{V}, E)$.

**Proof.** Combining Lemmas 4.27 and 4.28, we infer that given $\mathcal{V}_i \subseteq \mathcal{V}$, the cost of computing all possible $(\mathcal{D}_i, \mathcal{V}_{i+1})$ is $O(|\mathcal{V}_i||\Sigma|^2 + \Delta^8|\mathcal{V}_i|^4)$ time, as the set of $\mathcal{V}_i \upharpoonright \mathcal{D}_i$ over all $\mathcal{D}' \in \text{flexible-SCC} \left(\mathcal{D}_i\right)$ are disjoint subsets of $\mathcal{V}_i$.

Since all the sets $\mathcal{V}_i$ in the depth $i$ of the recursion are disjoint subsets of $\mathcal{V}$, the total cost for the depth $i$ of the recursion is $O(\Delta|\mathcal{V}||\Sigma|^2 + \Delta^8|\mathcal{V}|^4)$.

The recursion stops when $\mathcal{V}_i = \mathcal{V}_{i-1}$ or $\mathcal{V}_i = \emptyset$, so the depth of the recursion is at most $|\mathcal{V}|$. The reason is that we must have $|\mathcal{V}_i| < |\mathcal{V}_{i-1}|$ if $\mathcal{V}_i \neq \mathcal{V}_{i-1}$ and $\mathcal{V}_i \neq \emptyset$. Therefore, the total cost of computing all good sequences is $O(\Delta|\mathcal{V}|^2|\Sigma|^2 + \Delta^8|\mathcal{V}|^5)$.

We are ready to prove Theorem 4.10.

**Proof of Theorem 4.10.** By Lemma 4.29, the set of all good sequences can be computed in polynomial time, and we can compute $d_1$ given the set of all good sequences. If $d_1 = k$ is a positive integer, then from the discussion in Section 4.3 we know how to turn a good sequence $(\mathcal{V}_1, \mathcal{D}_1, \mathcal{V}_2, \mathcal{D}_2, \ldots, \mathcal{V}_k)$ into a description of an $O(n^{1/k})$-round algorithm for $\Pi$. If $d_1 = \infty$, then similarly a good sequence $(\mathcal{V}_1, \mathcal{D}_1, \mathcal{V}_2, \mathcal{D}_2, \ldots, \mathcal{V}_{O(\log n)})$ leads to a description of an $O(\log n)$-round algorithm for $\Pi$.

5 ROOTED TREES

In this section, we give a polynomial-time-computable characterization of LCL problems for regular rooted trees with complexity $O(\log n)$ or $\Theta(n^{1/k})$ for any positive integer $k$.

5.1 Locally checkable labeling for rooted trees

A rooted tree is a tree where each edge is oriented in such a way that the outdegree of each node is at most 1. A $\delta$-regular rooted tree is a rooted tree where the indegree of each node is either 0 or $\delta$. The root of a rooted tree is the unique node $v$ with $\text{deg}_{\text{out}}(v) = 0$. Each node $v$ with $\text{deg}_{\text{in}}(v) = 0$ is called a leaf. For each directed edge $u \to v$, we say that $u$ is a child of $v$ and $v$ is the parent of $u$. An LCL problem for $\delta$-regular rooted trees is defined as follows.

**Definition 5.1 (LCL problems for regular rooted trees).** For rooted trees, an LCL problem $\Pi = (\delta, \Sigma, \mathcal{C})$ is defined by the following components.

- $\delta$ is a positive integer specifying the maximum indegree.
- $\Sigma$ is a finite set of labels.
- $\mathcal{C}$ is a set of pairs $(\sigma, S)$ such that $\sigma \in \Sigma$ and $S$ is a size-$\delta$ multiset of labels in $\Sigma$.

For notational simplicity, we also write $(\sigma : a_1 a_2 \cdots a_k)$ to denote $(\sigma, S)$, where $\sigma \in \Sigma$ and $S = \{a_1, a_2, \ldots, a_k\}$ is a size-$\delta$ multiset of elements in $\Sigma$. We call any such $(\sigma, S)$ a node configuration. A node configuration $(\sigma, S)$ is correct if $(\sigma, S) \in \mathcal{C}$. In this sense, the set $\mathcal{C}$ specifies the node constraint. We define the correctness criteria for a labeling in Definition 5.2.

**Definition 5.2 (Correctness criteria).** Let $G = (V, E)$ be a rooted tree whose maximum indegree is at most $\delta$. A solution of $\Pi = (\delta, \Sigma, \mathcal{C})$ on $G$ is a labeling that assigns a label in $\Sigma$ to each node in $G$. 

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The labeling is a correct solution if it is locally-consistent on all \( v \in V \) with \( \operatorname{deg}_{\text{in}}(v) = \delta \), we define its node configuration \( C = (\sigma : a_1 a_2 \cdots a_\delta) \) by setting \( \sigma \) as the label of \( v \) and setting \( \{a_1, a_2, \ldots, a_\delta\} \) as the multiset of the labels of the \( \delta \) children of \( v \). We say that the labeling is locally-consistent on \( v \) if \( C \in C \).

The labeling is a correct solution if it is locally-consistent on all \( v \in V \) with \( \operatorname{deg}_{\text{in}}(v) = \delta \).

Similarly, although \( \Pi = (\delta, \Sigma, C) \) is defined for \( \delta \)-regular rooted trees, Definition 5.2 applies to all rooted trees whose maximum indegree is at most \( \delta \). We may focus on \( \delta \)-regular rooted trees without loss of generality, as for any rooted tree \( G \) whose maximum indegree is at most \( \delta \), we may consider the rooted tree \( G' \) which is the result of appending leaf nodes to all nodes \( v \) in \( G \) with \( 1 < \operatorname{deg}_{\text{in}}(v) < \delta \) to increase the indegree of \( v \) to \( \delta \). This only blows up the number of nodes by at most a \( \delta \) factor. Any correct solution of \( \Pi \) on \( G' \) restricted to \( G \) is a correct solution of \( \Pi \) on \( G \).

5.2 Preliminaries

Definition 5.3 is the same as Definition 4.3 except that we change \( \Delta - 1 \) to \( \delta \).

**Definition 5.3 (Complete trees of height \( i \)).** We define the rooted trees \( T_i \) recursively as follows.

- \( T_0 \) is the trivial tree with only one node.
- \( T_i \) is the result of appending \( \delta \) trees \( T_{i-1} \) to the root \( r \).

Observe that \( T_i \) is the unique maximum-size rooted tree of maximum indegree \( \delta \) and radius \( i \). All nodes within distance \( i - 1 \) to the root \( r \) in \( T_i \) have indegree \( \delta \). All nodes whose distance to \( r \) is exactly \( i \) are leaf nodes.

**Definition 5.4 (Trimming).** Given a subset \( \tilde{\Sigma} \subseteq \Sigma \) of labels, we define \( \operatorname{trim}(\tilde{\Sigma}) \) as a set of all labels \( \sigma \in \tilde{\Sigma} \) such that for each \( i \geq 1 \) it is possible to find a correct labeling of \( T_i \) such that the label of the root is \( \sigma \) and the label of the remaining nodes are in \( \tilde{\Sigma} \).

Given any rooted tree \( G \) of maximum indegree \( \delta \), after labeling the a node \( v \) with \( \operatorname{deg}_{\text{in}}(v) = \delta \) with a label \( \sigma \in \operatorname{trim}(\tilde{\Sigma}) \), it is always possible to extend this labeling to a complete correct labeling of the subtree rooted at \( v \) using only node configurations in \( \operatorname{trim}(\tilde{\Sigma}) \). Such a labeling extension is possible due to Lemma 5.5.

**Lemma 5.5 (Property of Trimming).** Let \( \Pi = (\delta, \Sigma, C) \) be an LCL problem. Let \( \tilde{\Sigma} \subseteq \Sigma \) such that \( \operatorname{trim}(\tilde{\Sigma}) \neq \emptyset \). For each label \( \sigma \in \operatorname{trim}(\tilde{\Sigma}) \), there exists a node configuration \( (\sigma : a_1 a_2 \cdots a_\delta) \in C \) such that \( a_i \in \operatorname{trim}(\tilde{\Sigma}) \) for all \( 1 \leq i \leq \delta \).

**Proof.** Assuming that such a node configuration \( (\sigma : a_1 a_2 \cdots a_\delta) \in C \) do not exist, we derive a contradiction as follows. We pick \( s \) to be the smallest number such that there is no correct labeling of \( T_s \) where the label of the root \( r \) is in \( \tilde{\Sigma} \setminus \operatorname{trim}(\tilde{\Sigma}) \) and the label of each remaining node of \( T_s \) is in \( \tilde{\Sigma} \). Such a number \( s \) exists due to the definition of \( \operatorname{trim} \).

Now consider a correct labeling of \( T_{s+1} \) where the label of the root \( r \) is \( \sigma \) and the label of each remaining node is in \( \tilde{\Sigma} \). Such a correct labeling exists due to the fact that \( \sigma \in \operatorname{trim}(\tilde{\Sigma}) \). Our assumption on the non-existence of \( (\sigma : a_1 a_2 \cdots a_\delta) \in C \) such that \( a_i \in \operatorname{trim}(\tilde{\Sigma}) \) for all \( 1 \leq i \leq \delta \) implies that the label \( b_i \) of one child \( u_i \) of the root \( r \) of \( T_{s+1} \) must be in \( \tilde{\Sigma} \setminus \operatorname{trim}(\tilde{\Sigma}) \). However, the subtree of \( T_{s+1} \) rooted at \( u_i \) is isomorphic to the rooted tree \( T_s \). Since the label \( b_i \) of \( u_i \) is in \( \tilde{\Sigma} \setminus \operatorname{trim}(\tilde{\Sigma}) \), our choice of \( s \) implies that the labeling of the subtree of \( T_{s+1} \) rooted at \( u_i \) cannot be correct, which is a contradiction. \( \square \)

**Restriction of an LCL problem.** Given a subset \( \tilde{\Sigma} \subseteq \Sigma \) of labels, we define the restriction of \( \Pi \) to \( \tilde{\Sigma} \) as follows.

\[
\Pi \upharpoonright_{\tilde{\Sigma}} = (\tilde{\Sigma}, \tilde{C}),
\]

where \( \tilde{C} = \{ \sigma : a_1 a_2 \cdots a_\delta \in C \text{ such that } \sigma \in \tilde{\Sigma} \text{ and } a_i \in \tilde{\Sigma} \text{ for all } 1 \leq i \leq \delta \}. \]
That is, \( \Pi \upharpoonright \xi \) is simply the result of removing all labels and node configurations in \( \Pi \) that involve labels not in \( \xi \).

**Path-form of an LCL problem.** Given an LCL problem \( \Pi = (\delta, \Sigma, C) \), we define its path-form \( \Pi^{\text{path}} = (1, \Sigma, C^{\text{path}}) \) as follows.

\[
C^{\text{path}} = \text{the set of all } (\sigma : a) \text{ such that there exists } (\sigma : a_1a_2 \cdots a_\delta) \in C \text{ such that } a \in \{a_1, a_2, \ldots, a_\delta\}.
\]

**Automaton for the path-form of an LCL problem.** The path-form \( \Pi^{\text{path}} = (1, \Sigma, C^{\text{path}}) \) of an LCL problem \( \Pi = (\delta, \Sigma, C) \) can be interpreted as a directed graph, where the node set is \( \Sigma \) and the edge set is \( C^{\text{path}} \), by viewing each \((\sigma : a) \in C^{\text{path}} \) as a directed edge \( a \rightarrow \sigma \).

**Path-flexibility.** With respect to \( \Pi^{\text{path}} = (1, \Sigma, C^{\text{path}}) \), we say that \( \sigma \in \Sigma \) is path-flexible if there exists an integer \( K \) such that for each integer \( s \geq K \), there exist a length-\( s \) walk \( \sigma \leadsto \sigma \) in the directed graph of \( \Pi^{\text{path}} = (1, \Sigma, C^{\text{path}}) \).

Lemma 5.6 is useful in lower bound proofs.

**Lemma 5.6 (Property of path-inflexibility).** Suppose that \( \sigma \in \Sigma \) is not path-flexible with respect to \( \Pi^{\text{path}} = (1, \Sigma, C^{\text{path}}) \). Then one of the following holds.

- There is no walk \( \sigma \leadsto \sigma \) in the directed graph of \( \Pi^{\text{path}} = (1, \Sigma, C^{\text{path}}) \).
- There exists an integer \( 2 \leq x \leq |\Sigma| \) such that for any positive integer \( k \) that is not an integer multiple of \( x \), there is no length-\( k \) walk \( \sigma \leadsto \sigma \) in the directed graph of \( \Pi^{\text{path}} = (1, \Sigma, C^{\text{path}}) \).

**Proof.** Since \( \sigma \) is not path-flexible, for any integer \( K \) there is a length-\( k \) walk \( s \leadsto t \). Let \( U \) be the set of integers \( k \) such that there is a length-\( k \) walk \( \sigma \leadsto \sigma \) in the directed graph of \( \Pi^{\text{path}} = (1, \Sigma, C^{\text{path}}) \). If \( U = \emptyset \), then there is no walk \( \sigma \leadsto \sigma \), so the lemma statement holds. For the rest of the proof, we assume \( U \neq \emptyset \). Let \( x = \gcd(U) \) be the greatest common divisor of \( U \). For any integer \( k \) that is not an integer multiple of \( x \), there is no length-\( k \) walk \( \sigma \leadsto \sigma \) in the directed graph of \( \Pi^{\text{path}} = (1, \Sigma, C^{\text{path}}) \). We must have \( x \geq 2 \) because there cannot be two co-prime numbers in \( U \), since otherwise there exists an integer \( K \) such that \( U \) includes all integers that are at least \( K \), implying that \( \sigma \) is path-flexible. We also have \( x \leq |\Sigma| \), since the smallest number of \( U \) is at most the number of nodes in the directed graph of \( \Pi^{\text{path}} = (1, \Sigma, C^{\text{path}}) \), which is \(|\Sigma|\). \( \Box \)

**Path-flexible strongly connected components.** For any strongly connected component \( U \subseteq \Sigma \) of the directed graph of \( \Pi^{\text{path}} = (1, \Sigma, C^{\text{path}}) \), it is clear that either all \( \sigma \in U \) are path-flexible or all \( \sigma \in U \) are not path-flexible. We say that a strongly connected component \( U \) is path-flexible if all \( \sigma \in U \) are path-flexible. We define flexibility(\( U \)) as the minimum number \( K \) such that for each integer \( k \geq K \) there is a walk \( \sigma \leadsto \sigma \) of length \( k \) for all choices of source \( a \in U \) and destination \( b \in U \). It is clear that such a number \( K \) exists given that \( U \) is a path-flexible strongly connected component.

Given any LCL problem \( \Pi = (\delta, \Sigma, C) \) and any \( \Sigma \subseteq \Sigma \), we define flexible-\( \text{SCC}(\Xi) \) as the set of all subsets \( U \subseteq \Xi \) that is a path-flexible strongly connected component of the directed graph of the path-form of \( \Pi \upharpoonright \xi \). It is possible that flexible-\( \text{SCC}(\Xi) \) is an empty set, and this happens when all nodes in the directed graph of the path-form of \( \Pi \upharpoonright \xi \) are not path-flexible.

Lemma 5.7 shows that if we label the two endpoints \( v_1 \) and \( v_{d+1} \) of a sufficiently long directed path \( v_1 \leftarrow v_2 \leftarrow \cdots \leftarrow v_{d+1} \) using only labels in \( U \), where \( U \in \text{flexible-SCC}(\Xi) \), then it is always possible to complete the labeling of the path using only labels in \( U \) in such a way that the entire labeling is correct with respect to \( \Pi \upharpoonright \xi \). Specifically, suppose the label of \( v_1 \) is \( \alpha \) and the label of \( v_{d+1} \) is \( \beta \). Then Lemma 5.7 shows that it is possible to complete the labeling of \( v_1 \leftarrow v_2 \leftarrow \cdots \leftarrow v_{d+1} \) by labeling \( v_i \) with \( \sigma_i \in U \).
Lemma 5.7 (Property of path-flexible strongly connected components). Consider any LCL problem $\Pi = (\delta, \Sigma, C)$ and any $\tilde{\Sigma} \subseteq \Sigma$. Let $U \in \text{flexible-SCC}(\tilde{\Sigma})$. For any choices of $\alpha \in U$, $\beta \in U$, and a number $d \geq \text{flexibility}(U)$, there exists a sequence $\sigma_1, \sigma_2, \ldots, \sigma_{d+1}$ of labels in $U$ satisfying the following conditions.

- First endpoint: $\sigma_1 = \alpha$.
- Last endpoint: $\sigma_{d+1} = \beta$.
- Node configurations: for $1 \leq i \leq d$, there is a node configuration $(\sigma : a_1a_2 \cdots a_8) \in C$ meeting the following conditions.
  - $\sigma_i = \sigma$.
  - There is an index $j$ such that $\sigma_{i+1} = a_j$.
  - $a_i \in \tilde{\Sigma}$ for all $1 \leq i \leq d$.

Proof. By the path-flexibility of $U$, there exists a length-$d$ walk $\beta \rightarrow \alpha$ in the directed graph of the path-form of $\Pi \ideal_{\tilde{\Sigma}}$. We fix

$$\sigma_1 \leftarrow \sigma_2 \leftarrow \cdots \leftarrow \sigma_{d+1}$$

to be any such walk. For $1 \leq i \leq d$, $\sigma_i \leftarrow \sigma_{i+1}$ is a directed edge in the path-flexible strongly connected component $U$, meaning that $(\sigma_i : \sigma_{i+1})$ is a node configuration in the path-form of $\Pi \ideal_{\tilde{\Sigma}}$, so there exists a node configuration $(\sigma_i : a_1a_2 \cdots a_8) \in C$ in $\Pi \ideal_{\tilde{\Sigma}}$ such that $\sigma_{i+1} \in \{a_1, a_2, \ldots, a_8\}$. □

Good sequences. Given an LCL problem $\Pi = (\delta, \Sigma, C)$ on $\delta$-regular rooted trees, we say that a sequence

$$(\Sigma_R^1, \Sigma_C^1, \Sigma_R^2, \Sigma_C^2, \ldots, \Sigma_R^k)$$

is good if it satisfies the following requirements.

- $\Sigma_R^1 = \text{trim}(\Sigma)$. That is, we start the sequence from the result of trimming the set $\Sigma$ of all labels in the given LCL problem $\Pi = (\delta, \Sigma, C)$.
- For each $1 \leq i \leq k - 1$, $\Sigma_C^i \in \text{flexible-SCC}(\Sigma_R^i)$. That is, $\Sigma_C^i$ is a path-flexible connected component of the automaton associated with the path-form of the LCL problem $\Pi \ideal_{\Sigma_R^i}$.
- For each $2 \leq i \leq k$, $\Sigma_R^i = \text{trim}(\Sigma_C^{i-1})$. That is, $\Sigma_R^i$ is the result of trimming the set $\Sigma_C^{i-1}$.
- $\Sigma_R^k \neq \emptyset$. That is, we require that the last set of labels is non-empty.

It is straightforward to see that $\Sigma \supseteq \Sigma_R^1 \supseteq \Sigma_C^1 \supseteq \Sigma_R^2 \supseteq \Sigma_C^2 \supseteq \cdots \supseteq \Sigma_R^k \neq \emptyset$.

Depth of an LCL problem. We define the depth $d_{1\Pi}$ of an LCL problem $\Pi = (\delta, \Sigma, C)$ on $\delta$-regular rooted trees as follows. If there is no good sequence, then we set $d_{1\Pi} = 0$. If there is a good sequence $(\Sigma_R^1, \Sigma_C^1, \Sigma_R^2, \Sigma_C^2, \ldots, \Sigma_R^k)$ for each positive integer $k$, then we set $d_{1\Pi} = \infty$. Otherwise, we set $d_{1\Pi}$ as the largest integer $k$ such that there is a good sequence $(\Sigma_R^1, \Sigma_C^1, \Sigma_R^2, \Sigma_C^2, \ldots, \Sigma_R^k)$. We prove the following results.

Theorem 5.8 (Characterization of complexity classes). Let $\Pi = (\delta, \Sigma, C)$ be an LCL problem on $\delta$-regular rooted trees. We have the following.

- If $d_{1\Pi} = 0$, then $\Pi$ is unsolvable in the sense that there exists a rooted tree of maximum indegree $\delta$ such that there is no correct solution of $\Pi$ on this rooted tree.
- If $d_{1\Pi} = k$ is a positive integer, then the optimal round complexity of $\Pi$ is $\Theta(n^{1/k})$. 

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If \( d_\Pi \equiv \infty \), then \( \Pi \) can be solved in \( O(\log n) \) rounds.

In Theorem 5.8, all the upper bounds hold in the CONGEST model, and all the lower bounds hold in the LOCAL model. For example, if \( d_\Pi = 5 \), then \( \Pi \) can be solved in \( O(n^{1/5}) \) rounds in the CONGEST model, and there is a matching lower bound \( \Omega(n^{1/5}) \) in the LOCAL model.

**Theorem 5.9 (Complexity of the Characterization).** There is a polynomial-time algorithm \( \mathcal{A} \) that computes \( d_\Pi \) for any given LCL problem \( \Pi = (\delta, \Sigma, C) \) on \( \delta \)-regular rooted trees. If \( d_\Pi = k \) is a positive integer, then \( \mathcal{A} \) also outputs a description of an \( O(n^{1/k}) \)-round algorithm for \( \Pi \). If \( d_\Pi = \infty \), then \( \mathcal{A} \) also outputs a description of an \( O(\log n) \)-round algorithm for \( \Pi \).

The distributed algorithms returned by the polynomial-time algorithm \( \mathcal{A} \) in Theorem 5.9 are also in the CONGEST model.

### 5.3 Upper bounds

In this section, we prove the upper bound part of Theorem 5.8. If a good sequence \( (\delta, \Sigma, C) \) exists for some positive integer \( k \), we show that the LCL problem \( \Pi = (\delta, \Sigma, C) \) can be solved in \( O(n^{1/k}) \) rounds. If a good sequence \( (\delta, \Sigma, C) \) exists for all positive integers \( k \), we show that \( \Pi = (\delta, \Sigma, C) \) can be solved in \( O(\log n) \) rounds. All these algorithms do not require sending large messages and can be implemented in the CONGEST model.

**Rake-and-compress decompositions.** Similar to the case of unrooted trees, we will use a variant of rake-and-compress decomposition for rooted trees. Our rake-and-compress decomposition for rooted trees is also parameterized by three positive integers \( \gamma, \ell, \) and \( L \). A \((\gamma, \ell, L)\) decomposition of a rooted tree \( G = (V, E) \) is a partition of the node set \[ V = V^R_1 \cup V^C_1 \cup V^R_2 \cup V^C_2 \cup \cdots \cup V^R_L \] satisfying the following requirements. The requirements will be different from the ones in Section 4.3 due to the difference between rooted trees and unrooted trees.

**Requirements for** \( V^R_1 \). For each connected component \( S \) of the subgraph of \( G \) induced by \( V^R_1 \), it is required that \( S \) is a rooted tree meeting the following conditions.

- All nodes in \( S \) have no in-neighbor in \( V^C_1 \cup V^R_{i+1} \cup \cdots \cup V^R_L \).
- All nodes in \( S \) are within distance \( \gamma - 1 \) to \( z \).

Here we only require that the nodes in \( S \) do not have in-neighbors from the higher layers of the decomposition. Only the root \( z \) of \( S \) can possibly have an out-neighbor outside of \( S \) and we do not restrict anything about this out-neighbor.

**Requirements for** \( V^C_1 \). For each connected component \( S \) of the subgraph of \( G \) induced by \( V^C_1 \), \( S \) is a directed path \( v_1 \leftarrow v_2 \leftarrow \cdots \leftarrow v_s \) of \( s \in [\ell, 2\ell] \) nodes meeting the following conditions.

- There is a node \( u \in V^R_{i+1} \cup \cdots \cup V^R_L \) such that \( u \leftarrow v_1 \).
- There is a node \( w \in V^R_1 \cup \cdots \cup V^R_{i+1} \) such that \( v_s \leftarrow w \).
- Other than \( u \) and \( w \), all the remaining neighbors of \( S \) are not in \( V^R_{i+1} \cup V^C_1 \cup \cdots \cup V^R_L \).

The rest of the section is organized as follows. In Section 5.3.1, we will design distributed algorithms that efficiently compute a \((\gamma, \ell, L)\) decomposition, for certain choices of parameters. In Section 5.3.2, we use our \((\gamma, \ell, L)\) decomposition algorithms to prove the upper bound part of Theorem 5.8.
5.3.1 Algorithms for rake-and-compress decomposition. Given a rooted tree \(G = (V, E)\), we define the two operations rake and compress as follows, where the operation compress depends on a parameter \(\ell\), which is a positive integer.

- The operation rake: Remove all nodes \(v \in V\) with \(\deg_{\text{in}}(v) = 0\).
- The operation compress: Remove all nodes \(v \in V\) such that \(v\) belongs to an \(\ell\)-node directed path \(P = v_1 \leftarrow v_2 \leftarrow \cdots \leftarrow v_s\) such that \(\deg_{\text{in}}(v_i) = \deg_{\text{out}}(v_i) = 1\) for each \(1 \leq i \leq s\).

Intuitively, the rake operation removes the set of all leaf nodes, and the compress operation removes the set of all nodes that belong to an \(s\)-node directed path consisting of only degree-2 nodes.

The decomposition algorithm. Recall that our goal is to find a \((\gamma, \ell, L)\) decomposition of a rooted tree \(G = (V, E)\), which is a partition \(V = V^R \cup V^C \cup V^R_2 \cup V^C_2 \cup \cdots \cup V^R_L\) meeting all the requirements. We will first describe our algorithm, and then we analyze for which combinations of parameters \((\gamma, \ell, L)\) our algorithm works.

Our algorithm for finding such a decomposition is as follows. For \(i = 1, 2, \ldots\), perform \(\gamma\) rake operations and then perform one compress operation. We initially set \(V^R_i\) to be the set of nodes removed during a rake operation in the \(i\)th iteration. Similarly, we initially set \(V^C_i\) to be the set of nodes removed during the compress operation in the \(i\)th iteration.

It is clear that these sets \(V^R_i\) and \(V^C_i\) already satisfy all the specified requirements, except that a connected component of the subgraph induced by \(V^C_i\) may be a path whose number of nodes exceeds \(2\ell\). Similar to existing rake-and-compress decomposition algorithms [6, 17, 19], to fix this, we will do a post-processing step which promotes some nodes from \(V^C_i\) to \(V^R_i\) to break long paths of \(V^C_i\) into small paths, for each \(i\). We need the following definition.

Definition 5.10 (Ruling set). Let \(P\) be a path. A subset \(I \subset V(P)\) is called an \((\alpha, \beta)\)-independent set if the following conditions are met: (i) \(I\) is an independent set that does not contain either endpoint of \(P\), and (ii) each connected component of the subgraph induced by \(V(P) \setminus I\) has at least \(\alpha\) node and at most \(\beta\) node, unless \(|V(P)| < \alpha\), in which case \(I = \emptyset\).

It is a well-known [1, 19, 22] that an \((\ell, 2\ell)\)-independent set of a path \(P\) can be computed in \(O(\log^* n)\) rounds deterministically in the CONGEST model when \(\ell = O(1)\). In our post-processing step, we simply compute an \((\ell, 2\ell)\)-independent set \(I\) in each connected component induced by \(V^C_i\), in parallel for all \(i\). Then we promote the nodes in \(I\) from layer \(V^C_i\) to layer \(V^R_i\). After this promotion, it is clear that the decomposition

\[ V = V^R_1 \cup V^C_1 \cup V^R_2 \cup V^C_2 \cup \cdots \cup V^R_L \]

is a \((\gamma, \ell, L)\) decomposition meeting all the requirements, where \(L\) can be any number such that no node remains after the \(\gamma\) rake operations in the \(L\)th iteration.

Assuming that \(\ell = O(1)\), the round complexity of computing the decomposition is clearly \(O(\gamma L) + O(\log^* n)\) in the CONGEST model, where \(O(\gamma L)\) is the round complexity for executing \(L\) iterations of the rake-and-compress process, and \(O(\log^* n)\) is the cost for the post-processing step.

Number of layers. Next, we consider the following question. Given two positive integers \(\gamma\) and \(\ell\), what is the smallest number \(L\) such that no node remains after the \(\gamma\) rake operations in the \(L\)th iteration, for any given input \(n\)-node rooted tree \(G = (V, E)\)?

To answer this question, we consider the following notation. For each node \(v \in V\), we write \(U^R_{R,i}\) to denote the set of nodes in the subtree rooted at \(v\) right after we finish the \(\gamma\) rake operation in the \(i\)th iteration. Similarly, we write \(U^C_{C,i}\) to denote the set of nodes in the subtree rooted at \(v\) right after we finish the compress operation in the \(i\)th iteration.
In these definitions, the notion of subtree is with respect to the subgraph induced by the nodes that are not yet removed, not with respect to the original rooted tree. In particular, if \( v \) is already removed before the \( i \)th iteration, then we have \( U_{R,i}^v = U_{C,i}^v = \emptyset \). For notational simplicity, we write \( U_{C,0}^v \) to denote the set of nodes in the subtree rooted at \( v \) in the original rooted tree \( G \).

**Lemma 5.11 (Shrinkage rate).** For each \( v \in V \) and \( i \geq 1 \), we have \(|U_{C,i}^v| < |U_{C,i-1}^v| \cdot \frac{2\ell}{\gamma + 2\ell} \).

**Proof.** We consider the \( i \)th iteration of the rake-and-compress process and focus on the set of nodes \( U_{R,i}^v \). We partition \( U_{R,i}^v = A \cup B \cup C \cup D \) into four parts as follows.

- \( A \) is the set of nodes \( u \) in \( U_{R,i}^v \) such that \( u \) belongs to an \( \ell \)-node directed path in \( U_{R,i}^v \) consisting of only nodes whose indegree in \( U_{R,i}^v \) is exactly one.
- \( B \) is the set of nodes \( u \) in \( U_{R,i}^v \) such that the indegree of \( u \) in \( U_{R,i}^v \) is exactly one and \( u \notin A \).
- \( C \) the set of nodes \( u \) in \( U_{R,i}^v \) whose indegree in \( U_{R,i}^v \) is greater than one.
- \( D \) the set of nodes \( u \) in \( U_{R,i}^v \) whose indegree in \( U_{R,i}^v \) is zero.

We prove the following inequalities.

- We have \(|A| \leq |U_{R,i}^v| - |U_{C,i}^v|\), since \( A \) is precisely the set of nodes in \( U_{R,i}^v \) that will subsequently be removed during the compress operation in the \( i \)th iteration. The reason that we have an inequality rather than an equality is that all the descendants of \( A \) in \( U_{R,i}^v \) are also not included in \( U_{C,i}^v \).
- We have \(|C| + 1 \leq |D|\), since the number of leaf nodes in a rooted tree is at least one plus the number of nodes with more than one child.
- We have \(|B| \leq (\ell - 1)(|C| + |D|)\), since the number of connected components induced by indegree-1 nodes in a rooted tree is at most the number of nodes whose indegree is not one, and \( B \) is the union of all these connected components of size at most \( \ell - 1 \).
- We have \( \gamma|D| \leq |U_{C,i-1}^v| - |U_{R,i}^v| \), since the fact that each leaf node of \( U_{R,i}^v \) is not removed during the \( \gamma \) rake operations in the \( i \)th iteration implies that it has at least \( \gamma \) descendants removed during these \( \gamma \) rake operations. That is, the number \(|U_{C,i-1}^v| - |U_{R,i}^v| \) of nodes in \( U_{C,i-1}^v \) removed during the \( \gamma \) rake operations in the \( i \)th iteration is at least \( \gamma \) times the number \(|D|\) of leaf nodes of \( U_{R,i}^v \).

Combining these four inequalities, we have

\[
|U_{C,i}^v| \leq |U_{R,i}^v| - |A| \\
= |B| + |C| + |D| \\
\leq \ell(|C| + |D|) \\
< 2\ell|D| \\
\leq \frac{2\ell}{\gamma} (|U_{C,i-1}^v| - |U_{R,i}^v|) \\
\leq \frac{2\ell}{\gamma} (|U_{C,i-1}^v| - |U_{C,i}^v|).
\]

Hence \(|U_{C,i}^v| < |U_{C,i-1}^v| \cdot \frac{2\ell}{\gamma + 2\ell} \). \( \square \)

**Lemma 5.12 (Number of layers).** If the inequality \( n \cdot \left( \frac{2\ell}{\gamma + 2\ell} \right)^{L-1} \leq \gamma \) holds, then we have \( V = V_1^R \cup V_1^C \cup V_2^R \cup V_2^C \cup \ldots \cup V_L^R \). In particular, we have the following.
If $t = O(1)$ and $\gamma = 1$, then we may set $L = O(\log n)$ to satisfy the inequality.

If $t = O(1)$ and $L = k$, then we may set $\gamma = O(n^{1/k})$ to satisfy the inequality.

Proof. For any $v \in V$, we have $|U^0_{C,0}| \leq n$, as $U^0_{C,0}$ is the set of nodes in the subtree rooted at $v$ in the original rooted tree $G$. By Lemma 5.11, we have $|U^0_{C,L-1}| < n \cdot \left(\frac{2t}{\log n}\right)^{L-1} \leq y$, which implies that $v$ must be removed during the $y$ rake operations in the $L$th iteration, if $v$ has not been removed by the time the $L$th begins. Hence $V = V^R_1 \cup V^C_1 \cup V^R_2 \cup V^C_2 \cup \ldots \cup V^R_L$.

We are ready to prove the main results of Section 5.3.1.

**Lemma 5.13 (O(\log n)-round rake-and-compress algorithm).** For any positive integer $t = O(1)$, a $(\gamma, t, L)$ decomposition of an $n$-node rooted tree with $\gamma = 1$ and $L = O(\log n)$ can be computed in $O(\log n)$ rounds in the CONGEST model.

Proof. By Lemma 5.12, we may set $\gamma = 1$ and $L = O(\log n)$ in such a way that we always have $V = V^R_1 \cup V^C_1 \cup V^R_2 \cup V^C_2 \cup \ldots \cup V^R_L$. The round complexity for computing the decomposition is $O(\gamma L) + O(\log^* n) = O(\log n)$. □

**Lemma 5.14 (O(n^{1/k})-round rake-and-compress algorithm).** For any positive integers $t = O(1)$ and $k = O(1)$, a $(\gamma, t, L)$ decomposition of an $n$-node rooted tree with $\gamma = O(n^{1/k})$ and $L = k$ can be computed in $O(\log n)$ rounds in the CONGEST model.

Proof. By Lemma 5.12, we may set $\gamma = O(n^{1/k})$ and $L = k$ in such a way that we always have $V = V^R_1 \cup V^C_1 \cup V^R_2 \cup V^C_2 \cup \ldots \cup V^R_L$. The round complexity for computing the decomposition is $O(\gamma L) + O(\log^* n) = O(n^{1/k})$. □

### 5.3.2 Distributed algorithms via rake-and-compress decompositions.

In this section, we use Lemmas 5.13 and 5.14 to design distributed algorithms solving a given LCL problem $\Pi = (\delta, \Sigma, C)$ on $\delta$-regular rooted trees.

**Lemma 5.15 (Solving $\Pi$ using rake-and-compress decompositions).** Suppose we are given an LCL problem $\Pi = (\delta, \Sigma, C)$ that admits a good sequence $$(\Sigma^R_1, \Sigma^C_1, \Sigma^R_2, \Sigma^C_2, \ldots, \Sigma^R_k).$$

Suppose we are given a $(\gamma, t, L)$ decomposition of an $n$-node rooted tree $G = (V, E)$ of maximum indegree at most $\delta$

$$V = V^R_1 \cup V^C_1 \cup V^R_2 \cup V^C_2 \cup \ldots \cup V^R_L$$

with $L = k$ and $t = \max\{1, \text{flexibility}(\Sigma^C_1), \ldots, \text{flexibility}(\Sigma^C_{k-1})\}$. Then a correct solution of $\Pi$ on $G$ can be computed in $O((\gamma + t)L)$ rounds in the CONGEST model.

Proof. We present an $O((\gamma + t)L)$-round CONGEST algorithm finding a correct solution of $\Pi$ on $G$. In this proof, whenever we say the algorithm labels a node $v$, we mean picking a node configuration $(\sigma : a_1, a_2, \ldots, a_\delta) \in C$ and fixing the labels of $v$ and its $\delta$ children according to the chosen node configuration.

The algorithm processes the nodes of the graph in the order $V^R_L, V^C_L, \ldots, V^R_1$. We require that when the algorithm processes $V^R_L$, the algorithm only uses node configurations $(\sigma : a_1, a_2, \ldots, a_\delta) \in C$ such that $\sigma$ and all of $a_1, a_2, \ldots, a_\delta$ are in $\Sigma^R_1$. We also require that when the algorithm processes $V^C_1$, the algorithm uses node configurations $(\sigma : a_1, a_2, \ldots, a_\delta) \in C$ such that $\sigma \in \Sigma^C_1$ and all of $a_1, a_2, \ldots, a_\delta$ are in $\Sigma^R_1$. 

Labeling $V_i^R$. Suppose the algorithm has finished labeling the nodes in $V_i^C \cup V_{i+1}^R \cup \cdots \cup V_L^R$. The algorithm then labels each connected component $S$ of the subgraph of $G$ induced by $V_i^R$, in parallel and using $O(\gamma)$ rounds in the CONGEST model, as follows.

Recall that the set $S$ induces a rooted tree of height at most $\gamma - 1$ such that no node in $S$ has an in-neighbor in $V_i^C \cup V_{i+1}^R \cup \cdots \cup V_L^R$. The root $z$ of $S$ may have an out-neighbor $u$.

If $u \notin V_i^C \cup V_{i+1}^R \cup \cdots \cup V_L^R$, then the label of $z$ is fixed now. In this case, we choose any node configuration $(\sigma : a_1, a_2, \ldots, a_\delta) \in C$ such that $\sigma$ and all of $a_1, a_2, \ldots, a_\delta$ are in $\Sigma_i^R$ to label $z$ and its children. Such a node configuration exists because of Lemma 5.5, as we recall that $\Sigma_i^R = \text{trim}(\Sigma_i^{\alpha})$ (if $i > 1$) and $\Sigma_i^R = \Sigma_i$ (if $i = 1$).

If $u \in V_i^C \cup V_{i+1}^R \cup \cdots \cup V_L^R$, then the label of $u$ is fixed to be some label $\sigma \in \Sigma_i^R$, due to the above requirement of our algorithm for labeling $V_i^C \cup V_{i+1}^R \cup \cdots \cup V_L^R$, as we recall that $\Sigma_i^R \triangleright \Sigma_i^C \triangleright \cdots \triangleright \Sigma_k^R$. In this case, we choose any node configuration $(\sigma : a_1, a_2, \ldots, a_\delta) \in C$ such that all of $a_1, a_2, \ldots, a_\delta$ are in $\Sigma_i^R$ to label $z$ and its children. Similarly, the existence of such a node configuration is due to Lemma 5.5 and the fact that $\sigma \in \Sigma_i^R$.

The node configuration for the remaining nodes in $S$ can be fixed similarly. We start processing a node $v \in S$ once the node configuration of its parent $u$ is fixed. Our requirement for labeling $V_i^R$ ensures that the label of $v$ is fixed to be some $\sigma \in \Sigma_i^R$, so we can choose any node configuration $(\sigma : a_1, a_2, \ldots, a_\delta) \in C$ where all of $a_1, a_2, \ldots, a_\delta$ are in $\Sigma_i^R$ and use this node configuration for $v$ to label its children. The round complexity of labeling $S$ is $O(\gamma)$ because $S$ is a rooted tree of depth at most $\gamma - 1$.

Labeling $V_i^C$. Suppose the algorithm has finished labeling the nodes in $V_{i+1}^R \cup V_{i+1}^C \cup \cdots \cup V_L^R$. The algorithm then labels each connected component $S$ of the subgraph of $G$ induced by $V_i^C$, in parallel and using $O(\ell)$ rounds in the CONGEST model, as follows.

The set $S$ has the property that there are two nodes $u$ and $v$ in $V_{i+1}^R \cup V_{i+1}^C \cup \cdots \cup V_L^R$ adjacent to $S$ such that the subgraph induced by $S \cup \{u, v\}$ is a directed path $u \leftarrow v_1 \leftarrow v_2 \leftarrow \cdots \leftarrow v_s \leftarrow v$, with $s \in [\ell, 2\ell]$.

Similarly, the requirement of the choice of node configurations for $V_{i+1}^R \cup V_{i+1}^C \cup \cdots \cup V_L^R$ ensures that the labels of $u, s_1,$ and $v$ have been fixed to be some labels in $\Sigma_i^C$, as we recall that $\Sigma_i^C \triangleright \Sigma_i^R \triangleright \cdots \triangleright \Sigma_k^R$.

Now, our task is to assign node configurations to $v_1, v_2, \ldots, v_s$ such that the labels used to label $v_1, v_2, \ldots, v_s$ are in $\Sigma_i^C$ and the labels used to label their children are in $\Sigma_i^R$.

To find such a labeling, we use Lemma 5.7. Specifically, recall that the length of the path $v_1 \leftarrow v_2 \leftarrow \cdots \leftarrow v_s \leftarrow v$ is $s \geq \ell \geq \text{flexibility}(\Sigma_i^C)$ by our choice of $\ell$. We let $\alpha$ be the existing label of $v_1$ and let $\beta$ be the existing label of $v$. Recall that $\Sigma_i^C \in \text{flexible-SCC}(\Sigma_i^R)$, so we may apply Lemma 5.7 with $U = \Sigma_i^C, \Sigma = \Sigma_i^R, d = s$, and our choices of $\alpha \in U$ and $\beta \in U$.

Lemma 5.7 returns a sequence of labels $\alpha = \sigma_1, \sigma_2, \ldots, \sigma_s = \beta$. For each $1 \leq j \leq s$, we use $\sigma_j$ to label $v_j$. Moreover, for each $1 \leq j \leq s$, Lemma 5.7 guarantees that there is a node configuration $(\sigma_j : \sigma_{j,1}, \sigma_{j,2}, \ldots, \sigma_{j,\ell}) \in C$ such that all of $\sigma_{j,1}, \sigma_{j,2}, \ldots, \sigma_{j,\ell}$ are in $\Sigma = \Sigma_i^R$ and there exists an index $l$ such that $\sigma_{j,l} = \sigma_{j+1}$. Therefore, we may use the labels in this size-$(\delta - 1)$ multiset $(\sigma_{j,1}, \sigma_{j,2}, \ldots, \sigma_{j,\ell}) \setminus \{\sigma_{j,l}\}$ to label the remaining $\delta - 1$ children of $v_j$, so that the node configuration of $v_j$ is $(\sigma_j : \sigma_{j,1}, \sigma_{j,2}, \ldots, \sigma_{j,\ell}) \in C$. The round complexity of labeling $S$ is $O(\ell)$ because $S$ is a path of at most $2\ell$ nodes.

Summary. The number rounds spent on labeling each part $V_{i+1}^R$ is $O(\gamma)$, and the number rounds spent on labeling each part $V_i^C$ is $O(\ell)$, so the overall round complexity for solving $\Pi$ given a $(\gamma, \ell, L)$ decomposition is $O((\gamma + \ell) L)$ rounds in the CONGEST model.
Combining Lemma 4.11 with existing algorithms for computing \((γ, ℓ, L)\) decompositions, we obtain the following results.

**Lemma 5.16 (Upper bound for the case \(d_Γ = k\)).** If \(d_Γ = k\) for some positive integer \(k\), then \(Π\) can be solved in \(O(n^{1/k})\) rounds in the CONGEST model.

**Proof.** In this case, a good sequence \((\Sigma R, Σ C, \Sigma R, \Sigma C, ..., \Sigma R)_k\) exists. By Lemma 5.14, a \((γ, ℓ, L)\) decomposition with \(γ = O(n^{1/k})\) and \(L = k\) can be computed in \(O(n^{1/k})\) rounds, \(Π\) can be solved in \(O(n^{1/k}) + O((γ + ℓ)L) = O(n^{1/k})\) rounds using the algorithm of Lemma 5.15. Here both \(k\) and \(ℓ\) are \(O(1)\), as they are independent of the number of nodes \(n\). □

**Lemma 5.17 (Upper bound for the case \(d_Π = ∞\)).** If \(d_Π = ∞\), then \(Π\) can be solved in \(O(\log n)\) rounds in the CONGEST model.

**Proof.** In this case, a good sequence \((\Sigma R, Σ C, \Sigma R, Σ C, ..., \Sigma R)_k\) exists for all positive integers \(k\). By Lemma 5.13, a \((γ, ℓ, L)\) decomposition with \(γ = 1\) and \(L = O(\log n)\) can be computed in \(O(\log n)\) rounds. By choosing a good sequence \((\Sigma R, Σ C, \Sigma R, Σ C, ..., \Sigma R)_k\) with \(k = L\), \(Π\) can be solved in \(O(\log n) + O((γ + ℓ)L) = O(\log n)\) rounds using the algorithm of Lemma 5.15. Similarly, here \(ℓ = O(1)\), as it is independent of the number of nodes \(n\). □

### 5.4 Lower bounds

In this section, we prove the lower bound part of Theorem 5.8. Similar to the case of unrooted trees, in our lower bound proofs, we pick \(γ\) to be the smallest integer satisfying the following requirements. For each subset \(\tilde{Σ} \subseteq Σ\) and each \(σ \in \tilde{Σ} \setminus \text{trim}(\tilde{Σ})\), there exists no correct labeling of \(T_r\) where the label of the root \(r\) is \(σ\) and the label of remaining nodes is in \(\tilde{Σ}\). Such a number \(γ\) exists due to the definition of \(\text{trim}\). Recall that \(T_r\) is defined in Definition 5.3.

**Lemma 5.18 (Unsolvability for the case \(d_Τ = 0\)).** If \(d_Τ = 0\), then \(Π\) is unsolvable in the sense that there exists a rooted tree \(G\) of maximum indegree \(δ\) such that there is no correct solution of \(Π\) on \(G\).

**Proof.** We take \(G = T_r\). Since \(d_Τ = 0\), we have \(\text{trim}(Σ) = 0\). Our choice of \(γ\) implies that there is no correct solution of \(Π\) on \(G = T_r\). □

For the rest of this section, we focus on the case \(d_Γ = k\) is a positive integer. We will prove that \(Π\) needs \(Ω(n^{1/k})\) rounds to solve in the LOCAL model. Definition 5.19 is similar to Definition 4.15. The exact choice of \(s = Θ(t)\) in Definition 5.19 is to be determined.

**Definition 5.19 (Lower bound graphs).** We let \(t\) be any positive integer, and we let \(s = Θ(t)\).

- Define \(G_{Γ,1}\) as the rooted tree \(T_r\). All nodes in \(G_{Γ,1}\) are said to be in layer \((R, 1)\).
- For each integer \(i ≥ 1\), define \(G_{Γ,i}\) as the result of the following construction. Start with an \(s\)-node directed path \(v_1 ← v_2 ← ... ← v_s\). For each \(1 ≤ i ≤ s - 1\), append \(δ - 1\) copies of \(G_{Γ,i-1}\) to \(v_i\). Append \(δ\) copies of \(G_{Γ,i-1}\) to \(v_s\). The nodes \(v_1, v_2, ..., v_s\) are said to be in layer \((C, i)\). We call \(v_1 ← v_2 ← ... ← v_i\) the core path of \(G_{Γ,i}\).
- For each integer \(i ≥ 1\), define \(G_{C,i}\) as the result of the construction of \(G_{Γ,i}\), with a difference that we append \(δ - 1\) copies of \(G_{Γ,i-1}\) to \(v_s\).
- For each integer \(i ≥ 2\), define \(G_{R,i}\) as follows. Start with a rooted tree \(T_r\). Append \(δ\) copies of \(G_{C,i-1}\) to each leaf in \(T_r\). All nodes in \(T_r\) are said to be in layer \((R, i)\).

The only difference between Definition 5.19 and Definition 4.15 is that we here define a new rooted tree \(G_{C,i}\) where we append only \(δ - 1\) copies of \(G_{R,i-1}\) to the last node \(v_δ\) of the core path, so \(\text{deg}_{in}(v_δ) = δ - 1\). The purpose of this
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modification is to allow us to concatenate the rooted trees together without violating the maximum indegree bound $\delta$. Specifically, for our lower bound proof in Section 5.4, we define our main lower bound graph $G = (V, E)$ as follows.

**Definition 5.20 (Main lower bound graph).** Define the rooted tree $G = (V, E)$ as the result of the following construction.

- The construction starts with the rooted trees $G^0_{C,1}, G^0_{C,2}, \ldots, G^0_{C,k}$ and $G_{R,k+1}$.
- Let $P_i = v_1^i \rightarrow v_2^i \rightarrow \cdots \rightarrow v_s^i$ be the core path of $G^0_{C,i}$ and let $r$ be the root of $G_{R,k+1}$.
- Add the directed edges $v_1^1 \rightarrow v_2^1, v_2^1 \rightarrow v_3^1, \ldots, v_{s-1}^1 \rightarrow v_s^1$, and $v_s^1 \rightarrow r$.

It is clear that all nodes in the rooted tree $G$ have indegree either 0 or $\delta$, and the total number of nodes in $G$ is $O(t^k)$, if we treat $\gamma$ as a constant independent of $t$. Therefore, to show that $\Pi$ requires $\Omega(n^{1/k})$ rounds to solve, it suffices to show that there is no algorithm that solves $\Pi$ within $t$ rounds on $G$.

Similar to Section 4.4, the nodes in $G$ are partitioned into layers $(R, 1), (C, 1), (R, 2), (C, 2), \ldots, (R, k+1)$ according to the rules in the above recursive construction, and we order the layers by $(R, 1) < (C, 1) < (R, 2) < (C, 2) < \cdots < (R, k+1)$. Recall that in the graph $G$, the nodes in layer $(C, i)$ form directed paths of $s$ nodes. We consider the following classification of nodes in layer $(C, i)$. Again, we will choose $s = \Theta(i)$ to be sufficiently large to ensure that central nodes exist.

**Definition 5.21 (Classification of nodes in layer $(C, i)$).** The nodes in the $s$-node directed path $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_s$ in the construction of $G^0_{C,i}$ are classified as follows.

- We say that $v_j$ is a front node if $1 \leq j \leq t$.
- We say that $v_j$ is a central node if $t + 1 \leq j \leq s - t$.
- We say that $v_j$ is a rear node if $s - t + 1 \leq j \leq s$.

Based on Definition 5.21, we define the following subsets of nodes in $G$. In Definition 5.22, recall that $P_i$ is the core path of the rooted tree $G^0_{C,i}$ in the construction of $G$ in Definition 5.20.

**Definition 5.22 (Subsets of nodes in $G$).** We define the following subsets of nodes in $G$.

- Define $S_{R,1}$ as the set of nodes $v$ in $G$ such that the subgraph induced by $v$ and its descendants within radius-$\gamma$ neighborhood of $v$ is isomorphic to $T_{\gamma}$.
- For $2 \leq i \leq k + 1$, define $S_{R,i}$ as the set of nodes $v$ in $G$ such that the subgraph induced by $v$ and its descendants within radius-$\gamma$ neighborhood of $v$ is isomorphic to $T_{\gamma}$ and contains only nodes in $S_{C,i-1}$.
- For $1 \leq i \leq k$, define $S_{C,i}$ as the set of nodes $v$ in $G$ meeting one of the following conditions.
  - $v$ is in layer $(R, i+1)$ or above.
  - $v \in P_i$ is a central or rear node in layer $(C, i)$.
  - $v \not\in P_i$ is a central or front node in layer $(C, i)$.

We prove some basic properties of the sets in Definition 5.22.

**Lemma 5.23 (Subset containment).** We have $S_{R,1} \supseteq S_{C,1} \supseteq \cdots \supseteq S_{R,k+1} \neq \emptyset$.

**Proof.** The claim that $S_{C,i} \supseteq S_{R,i+1}$ follows from the definition of $S_{R,i+1}$ that $v \in S_{C,i}$ is a necessary condition for $v \in S_{R,i+1}$. To prove claim that $S_{R,i} \supseteq S_{C,i}$, we recall that $v \in S_{C,i}$ implies that $v$ is in layer $(C, i)$ or above. By the construction of $G$, the subgraph induced by $v$ and its descendants within the radius-$\gamma$ neighborhood of $v$ is isomorphic to $T_{\gamma}$ and contains only nodes in layer $(R, i)$ or above. Since all nodes in layer $(R, i)$ or above are in $S_{C,i-1}$, we infer that $v \in S_{C,i}$ implies $v \in S_{R,i}$.
To see that $S_{R,k+1} \neq \emptyset$, consider the node $r$ in the construction of $G$. Since $r$ is the root of $G_{R,k+1}$, the subgraph induced by $r$ and its descendants within radius-$\gamma$ neighborhood of $r$ is isomorphic to $T_\gamma$ and contains only nodes in layer $(R, k+1)$. We know that all nodes in layer $(R, k+1)$ are in $S_{C,k}$, so $r \in S_{R,k+1}$. □

**Lemma 5.24 (Property of $S_{C,i}$).** For each node $v \in S_{C,i}$, either one of the following holds.

- $v$ is a central node in layer $(C, i)$.
- For each child $u$ of $v$ such that $u \in S_{C,i}$, there exists a directed path $P = w_1 \leftarrow \cdots \leftarrow v \leftarrow u \leftarrow \cdots w_2$ such that $w_1 \in P_i$ and $w_2 \notin P_i$ are central nodes in layer $(C, i)$ and all nodes in $P$ are in $S_{C,i}$.

**Proof.** We assume that $v \in S_{C,i}$ is not a central node in layer $(C, i)$. Consider any child $u$ of $v$ such that $u \in S_{C,i}$. The goal of the proof is to find a path $P = w_1 \leftarrow \cdots \leftarrow v \leftarrow u \leftarrow \cdots w_2$ such that all nodes of $P$ are in $S_{C,i}$, and $w_1 \in P_i$ and $w_2 \notin P_i$ are central nodes in layer $(C, i)$. The existence of such a directed path $P$ follows from a simple observation that $S_{C,i}$ induces a connected subtree where all the leaf nodes are central nodes in layer $(C, i)$ that are not in $P_i$ and the root node is a central node in layer $(C, i)$ that is in $P_i$.

Specifically, the directed path $P$ can be constructed as follows. To construct the part $w_1 \leftarrow \cdots \leftarrow v$, we simply start from $v$ and follow the parent pointers until we reach a node $w_1$ that is a central node in $P_i$. The correctness of the construction of this part follows from the definition of $G$ and the fact that either $v$ is a rear node in $P_i$ or $v \notin P_i$ is in layer $(C, i)$ or above.

To construct the remaining part $v \leftarrow u \leftarrow \cdots \leftarrow w_2$, we simply observe that either $u$ itself is a central node in layer $(C, i)$ or there is a descendant $w_2$ of $u$ such that $w_2$ is a central node in layer $(C, i)$. Hence we can always extend $v \leftarrow u$ to a desired path $v \leftarrow u \leftarrow \cdots \leftarrow w_2$. □

We note that the reason for attaching the rooted trees $G_{C,1}^o, G_{C,2}^o, \ldots, G_{C,k}^o$ to $G_{R,k+1}$ in the definition of $G$ is precisely that we want to define $S_{C,i}$ in such a way that allows us to have Lemma 5.24. That is, the design objective is to ensure that for each node $v \in S_{C,i}$ that is not a central node in layer $(C, i)$, there is a directed path in $S_{C,i}$ passing through $v$ and starting and ending at central nodes in layer $(C, i)$.

**Assumptions.** We are given an LCL problem $\Pi = (\delta, \Sigma, C)$ such that $d_\Pi = k$. Hence there does not exist a good sequence

$$(\Sigma_1^R, \Sigma_1^C, \Sigma_2^R, \Sigma_2^C, \ldots, \Sigma_k^R).$$

Recall that the rules for a good sequence are as follows:

$$\Sigma_i^R = \begin{cases} \text{trim}(\Sigma) & \text{if } i = 1, \\ \text{trim}(\Sigma_{i-1}^C) & \text{if } i > 1, \end{cases}$$

$$\Sigma_i^C \in \text{flexible-SCC}(\Sigma_i^R).$$

The only nondeterminism in the above rules is the choice of $\Sigma_i^C \in \text{flexible-SCC}(\Sigma_i^R)$ for each $i$. The fact that $d_\Pi = k$ implies that for all possible choices of $\Sigma_1^C, \Sigma_2^C, \ldots, \Sigma_k^C$, we always end up with $\Sigma_{k+1}^C = \emptyset$.

We assume that there is an algorithm $\mathcal{A}$ that solves $\Pi$ in $t = O(n^{1/k})$ rounds on $G = G_{R,k+1}^*$, where $n$ is the number of nodes in $G$. To prove the desired $\Omega(n^{1/k})$ lower bound, it suffices to derive a contradiction. Specifically, we will prove that the existence of such an algorithm $\mathcal{A}$ forces the existence of a good sequence $(\Sigma_1^R, \Sigma_1^C, \Sigma_2^R, \Sigma_2^C, \ldots, \Sigma_k^R)$, contradicting the fact that $d_\Pi = k$. 32
Induction hypothesis. Our proof proceeds by an induction on the subsets \( S_{R,1}, S_{C,1}, S_{R,2}, S_{C,2}, \ldots, S_{R,k+1} \). For each \( 1 \leq i \leq k \), the choice of \( \Sigma_i^R \) is fixed in the induction hypothesis for \( S_{C,i} \). The choice of \( \Sigma_i^C \) is uniquely determined once \( \Sigma_{i-1}^R \) has been fixed.

Similar to Section 4.4, before defining our induction hypothesis, we recall that the output label a node \( v \) is determined by the subgraph induced by the radius-\( t \) neighborhood \( U \) of \( v \), together with the distinct IDs of the nodes in \( U \). For each node \( v \) in \( G \), we define \( \Sigma_v^A \) as the set of all possible output labels of \( v \) that can possibly appear when we run \( A \) on \( G \). In other words, \( \sigma \in \Sigma_v^A \) implies that there exists an assignment of distinct IDs to nodes in the radius-\( t \) neighborhood of \( v \) such that the output label of \( v \) is \( \sigma \).

Definition 5.25 (Induction hypothesis for layer \( S_{R,i} \)). For each \( 1 \leq i \leq k + 1 \), the induction hypothesis for \( S_{R,i} \) specifies that each \( v \in S_{R,i} \) satisfies \( \Sigma_v^A \subseteq \Sigma_i^R \).

Definition 5.26 (Induction hypothesis for layer \( S_{C,i} \)). For each \( 1 \leq i \leq k \), the induction hypothesis for \( S_{C,i} \) specifies that there exists a choice \( \Sigma_i^C \in \text{flexible-SCC}(\Sigma_i^R) \) such that each \( v \in S_{C,i} \) satisfies \( \Sigma_v^A \subseteq \Sigma_i^C \).

Next, we prove that the induction hypotheses stated in Definitions 5.25 and 5.26 hold.

Lemma 5.27 (Base case: \( S_{R,1} \)). The induction hypothesis for \( S_{R,1} \) holds.

Proof. Recall that the number \( \gamma \) satisfies the following property. For each subset \( \tilde{\Sigma} \subseteq \Sigma \) and each \( \sigma \in \tilde{\Sigma} \setminus \text{trim}(\tilde{\Sigma}) \), there exists no correct labeling of \( T_\gamma \), where the label of the root \( r \) is \( \sigma \) and the label of remaining nodes is in \( \tilde{\Sigma} \).

To prove the induction hypothesis for \( S_{R,1} \), consider any node \( v \in S_{R,1} \). By the definition of \( S_{R,1} \), the subgraph induced by \( v \) and its descendants within radius-\( \gamma \) of \( v \) is isomorphic to \( T_\gamma \) rooted at \( v \). By setting \( \tilde{\Sigma} = \Sigma \), we infer that there is no correct labeling of \( G \) such that the label of \( v \) is in \( \Sigma \setminus \text{trim}(\Sigma) = \Sigma \setminus \Sigma_1 \), so we must have \( \Sigma_v^A \subseteq \Sigma_1^R \), as \( A \) is correct.

Lemma 5.28 (Inductive step: \( S_{R,i} \)). Let \( 2 \leq i \leq k \). If the induction hypothesis for \( S_{C,i-1} \) holds, then the induction hypothesis for \( S_{R,i} \) holds.

Proof. To prove the induction hypothesis for \( S_{R,i} \), consider any node \( v \in S_{R,i} \). By the definition of \( S_{R,i} \), the subgraph \( S \) induced by \( v \) and its descendants within radius-\( \gamma \) of \( v \) is isomorphic to \( T_\gamma \) rooted at \( v \) and contains only nodes in \( S_{C,i-1} \). Our goal is to prove that \( \Sigma_v^A \subseteq \Sigma_i^R = \text{trim}(\Sigma_{i-1}^C) \).

Consider any node \( u \) in the subgraph \( S \) induced by \( v \) and its descendants within radius-\( \gamma \) of \( v \). As \( u \in S_{C,i-1} \), the induction hypothesis for \( S_{C,i-1} \) implies that

\[ \Sigma_u^A \subseteq \Sigma_{i-1}^C. \]

Hence the same argument in the proof of Lemma 5.27 shows that \( \Sigma_v^A \subseteq \text{trim}(\Sigma_{i-1}^C) = \Sigma_i^R \), as required.

Lemma 5.29 (Inductive step: \( S_{C,i} \)). Let \( 1 \leq i \leq k \). If the induction hypothesis for layer \( S_{R,i} \) holds, then the induction hypothesis for layer \( S_{C,i} \) holds.

Proof. To prove the induction hypothesis for \( S_{C,i} \), we show that there exists a choice \( \Sigma_i^C \in \text{flexible-SCC}(\Sigma_i^R) \) such that each \( v \in S_{C,i} \) satisfies \( \Sigma_v^A \subseteq \Sigma_i^C \).

We first consider the case that \( v \) is a central node in layer \((C, i)\). Then it is clear that \( \Sigma_v^A \) is the same for all \( v \) that is a central node in layer \((C, i)\), as the radius-\( t \) neighborhood of these nodes \( v \) are isomorphic, due to the definition of central nodes and the construction in Definition 5.19. For notational convenience, we write \( \tilde{\Sigma} \) to denote the this set \( \Sigma_v^A \) for any central node \( v \) in layer \((C, i)\).
Plan of the proof. Consider any node $v \in S_{C,i}$, we claim that the node configuration ($\sigma : a_1 a_2 \cdots a_\delta$) of $v$ resulting from running $\mathcal{A}$ uses only labels in $\Sigma_i^\delta$. To see this, observe that $v$ and all $\delta$ children of $v$ are in $S_{R,i}$, so the induction hypothesis for $S_{R,i}$ implies that their output labels must be in $\Sigma_i^\delta$. Hence all labels in ($\sigma : a_1 a_2 \cdots a_\delta$) are in $\Sigma_i^\delta$.

We write $M$ to denote the directed graph representing the automaton associated with the path-form of the LCL problem $\Pi_{\gamma_i^\gamma}$. The above claim implies the following. Consider any directed edge $u \leftarrow v$ such that both $u$ and $v$ are in $S_{C,i}$. Let $a$ be the output label of $u$ and let $b$ be the output label of $v$. Then $a \leftarrow b$ must be a directed edge in $M$.

To prove that there exists a choice $\Sigma_i^\delta$ of flexible-SCC($\Sigma_i^\delta$) such that $\Sigma_i^\delta \subseteq \Sigma_i^\delta$ for all $v \in S_{C,i}$. We will first show that $\Sigma_i$ must be a subset of a path-flexible strongly connected component of $\Sigma_i^\delta$, and then we fix $\Sigma_i^\delta \subseteq$ flexible-SCC($\Sigma_i^\delta$) to be this path-flexible strongly connected component.

Next, we will argue that for each $\sigma \in \Sigma_i^\delta$, there exist a walk in $M$ that starts from $\sigma$ and ends in $\delta$ and a walk in $M$ that starts from $\delta$ and ends in $\sigma$. This shows that $\sigma$ is in the same strongly connected component as the members in $\delta$, so we conclude that $\Sigma_i^\delta \subseteq \Sigma_i^\delta$.

Part 1: $\delta$ is a subset of some $\Sigma_i^\delta \subseteq$ flexible-SCC($\Sigma_i^\delta$). Consider a path $u_1 \leftarrow u_2 \leftarrow \cdots \leftarrow u_s$ of $s$ nodes in layer $(C, i)$ of $G$. We choose $s = \Theta(t)$ to be sufficiently large to ensure that for each integer $0 \leq d \leq |\Sigma|$, there exist two nodes $u_j$ and $u_t$ in the path meeting the following conditions.

- $t + 1 \leq j < l \leq s - t$, so $u_j$ and $u_t$ are central nodes.
- The distance $l - j$ between $u_j$ and $u_t$ equals $4t + 3 + d$.

Similar to the proof of Lemma 4.25, the choice of the number $4t + 3$ is to ensure that the union of the radius-$t$ neighborhood of any node $v$ and the radius-$t$ neighborhood of children of $v$ in $G$ does not node-intersect the radius-$t$ neighborhood of both $u_j$ and $u_t$. This implies that after arbitrarily fixing distinct IDs in the radius-$t$ neighborhood of $u_j$ and $u_t$, it is possible to complete the ID assignment of the entire graph $G$ in such a way that the union of the radius-$t$ neighborhood of any node $v$ and the radius-$t$ neighborhood of children of $v$ in $G$ does not contain repeated IDs. If we run $\mathcal{A}$ with such an ID assignment, it is guaranteed that the output is correct.

Consider the directed graph $M$ and any of its two nodes $a$ and $b$ such that $a \in \delta$ and $b \in \delta$. Our choice of $\delta$ implies that there exists an assignment of distinct IDs to the radius-$t$ neighborhood of both $u_j$ and $u_t$ such that the output labels of $u_j$ and $u_t$ are $a$ and $b$, respectively. We complete the ID assignment of the entire graph $G$ in such a way that the union of the radius-$t$ neighborhood of any node $v$ and the radius-$t$ neighborhood of children of $v$ in $G$ does not contain repeated IDs.

We write $\sigma_y$ to denote the output label of $u_y$ resulting from running $\mathcal{A}$. Hence $a = \sigma_j \leftarrow \sigma_{j+1} \leftarrow \cdots \leftarrow \sigma_1 \leftarrow b$ is a walk $b \leftarrow a$ in $M$ of length $4t + 3 + d$. Therefore, for all choices of $a \in \delta$ and $b \in \delta$ and any $0 \leq d \leq |\Sigma|$, we may find a walk $b \leftarrow a$ in $M$ of length $4t + 3 + d$. This implies that all members in $\delta$ are in the same strongly connected component of $M$. Furthermore, Lemma 5.6 implies that this strongly connected component is path-flexible, as the length of the walk can be any integer in between $4t + 3$ and $4t + 3 + |\Sigma|$.

Part 2: $\Sigma_i^\delta \subseteq \Sigma_i^\delta$ for each $v \in S_{C,i}$. For this part, we use Lemma 5.24, which shows that for each $v \in S_{C,i}$ in the graph $G$, there is a directed path $P = v_1 \leftarrow \cdots \leftarrow v \leftarrow \cdots \leftarrow v_2$ such that $v_1 \in P_i$ and $v_2 \notin P_i$ are central nodes in layer $(C_i, i)$ and all nodes in $P$ are in $S_{C,i}$.

Consider the output labels of the nodes in $P$ resulting from running $\mathcal{A}$. We write $\sigma_v$ to denote the output label of $v$. Then $\sigma_{v_1} \leftarrow \cdots \leftarrow \sigma_v$ is a walk in $M$ from $\sigma_v$ to a node in $\delta$ and $\sigma_v \leftarrow \cdots \leftarrow \sigma_{v_2}$ is a walk in $M$ from a node in $\delta$ to $\sigma_v$. This shows that $\sigma_v$ is in the same strongly connected component of $M$ as the members in $\delta$. 
The same argument can be applied to all $\sigma_0 \in \Sigma^A_n$. The reason is that for each $\sigma_0 \in \Sigma^A_n$ there is an assignment of distinct IDs such that $\sigma_0$ is the output label of $v$. Hence we conclude that all members in $\Sigma^A_n$ are in the same strongly connected component of $M$ as the members in $\tilde{\Sigma}$, so $\Sigma^A_n \subseteq \Sigma^C_i$. \qed

Applying Lemmas 5.27 to 5.29 from $\text{SR}_1$ all the way up to the last subset $\text{SR}_{k+1}$, we obtain the following result.

**Lemma 5.30 (Lower bound for the case $d_{\Pi} = k$).** If $d_{\Pi} = k$ for a finite integer $k$, then $\Pi$ requires $\Omega(n^{1/k})$ rounds to solve on rooted trees of maximum indegree $\delta$.

**Proof.** Assume that there is a $t$-round algorithm solving $\Pi$ on $G$. By Lemmas 5.27 to 5.29, we infer that the induction hypothesis for the last subset $\text{SR}_{k+1}$ holds. By Lemma 5.23, $\text{SR}_{k+1} \neq \emptyset$, so there is a node $v$ in $G$ such that $\Sigma^A \subseteq \Sigma^R_{k+1}$. Therefore, the correctness of $A$ implies that $\Sigma^R_{k+1} \neq \emptyset$, which implies that $(\Sigma^R_1, \Sigma^C_1, \Sigma^C_2, \ldots, \Sigma^C_{k+1})$ chosen in the induction hypothesis is a good sequence, contradicting the assumption that $d_{\Pi} = k$. Hence such a $t$-round algorithm $A$ that solves $\Pi$ does not exist. As $t$ can be any positive integer and $t = \Omega(n^{1/k})$, where $n$ is the number of nodes in $G$, we conclude the proof. \qed

Now we are ready to prove Theorem 5.8.

**Proof of Theorem 5.8.** The upper bound part of the theorem follows from Lemmas 5.16 and 5.17. The lower bound part of the theorem follows from Lemmas 5.18 and 5.30. \qed

### 5.5 Complexity of the characterization

In this section, we prove Theorem 5.9. We are given a description of an LCL problem $\Pi = (\delta, \Sigma, C)$ on $\delta$-regular rooted trees. We assume that the description is given in the form of listing all the node configurations in $C$. Therefore, the description length of $\Pi$ is $t = O(|C|\delta \log |\Sigma|)$. We allow $\delta$ to be a non-constant as a function of $t$. We will design an algorithm that computes all possible good sequences $(\Sigma^R_1, \Sigma^C_1, \Sigma^C_2, \ldots, \Sigma^C_k)$ in time polynomial in $t$.

For the case of $d_{\Pi} = \infty$, there are good sequences that are arbitrarily long. Recall that we have $\Sigma^R_1 \supseteq \Sigma^C \supseteq \cdots \supseteq \Sigma^R_k$. Hence if $k > |\Sigma|$, there must exists some index $1 \leq i < k$ such that $\Sigma^R_i = \Sigma^C_i = \Sigma^R_{i+1}$. This immediately implies that $\Sigma^C_i = \Sigma^R_i = \Sigma^C_{i+1} = \cdots$. The reason is that $\Sigma^C_i = \Sigma^C_i$ implies that $\Sigma^C_i$ itself is the only element of flexible-SCC($\Sigma^R_i$). We conclude that any good sequence with $k > |\Sigma|$ must stabilizes at some point $i \leq |\Sigma|$, in the sense that $\Sigma^C_i = \Sigma^R_i = \Sigma^C_{i+1} = \cdots$.

**High-level plan.** Recall that the rules for a good sequence are as follows.

$$
\Sigma^R_i = \begin{cases} 
\text{trim}(\Sigma) & \text{if } i = 1, \\
\text{trim}(\Sigma^C_{i-1}) & \text{if } i > 1,
\end{cases}
\Sigma^C_i \in \text{flexible-SCC}(\Sigma^R_i).
$$

To compute all good sequences $(\Sigma^R_1, \Sigma^C_1, \Sigma^C_2, \ldots, \Sigma^C_k)$, we go through all choices of $\Sigma^C_i \in \text{flexible-SCC}(\Sigma^R_i)$ and apply the rules recursively until we cannot proceed any further. The process stops when $\Sigma^C_i = \Sigma^R_i$ (the sequence stabilizes) or $\Sigma^C_i = \emptyset$ or $\Sigma^R_i = \emptyset$ (the sequence ends).

We start with describing the algorithm for computing trim($\Sigma$) for a given $\Sigma \subseteq \Sigma$.

**Lemma 5.31 (Algorithm for trim).** The set trim($\Sigma$) can be computed in $O(|C|\delta|\Sigma| + |\Sigma|^2)$ time, for any given $\Sigma \subseteq \Sigma$. 35
Proof. We write $\Sigma_i$ to denote set of all possible $\sigma \in \hat{\Sigma}$ such that there is a correct labeling of the rooted tree $T_i$ where the label of each node is in $\hat{\Sigma}$ and the label of the root $r$ is $\sigma$. The set $\Sigma_i$ can be computed recursively as follows.

- For the base case, $\Sigma_0$ is the set of all labels appearing in $\hat{\Sigma}$.
- For the inductive step, each $\sigma \in \Sigma_{i-1}$ is added to $\Sigma_i$ if there exists a node configuration $(\sigma : a_1a_2 \cdots a_3) \in C$ such that $a_j \in \Sigma_{i-1}$ for all $1 \leq j \leq \delta$.

The above recursive computation implies that given $\Sigma_{i-1}$ has been computed, the computation of $\Sigma_i$ costs $O(|C|\delta + |\hat{\Sigma}|)$ time. The algorithm simply goes over each $\Sigma_{i-1}$ as binary string of length $|\hat{\Sigma}|$, testing whether $a_j \in \Sigma_{i-1}$ costs $O(1)$ time. Therefore, the process of going through all $(\sigma : a_1a_2 \cdots a_3) \in C$ costs $O(|C|\delta)$ time. After that, using $O(|C| + |\hat{\Sigma}|)$ time, we may calculate $\Sigma_i$ and store it as a binary string of length $|\hat{\Sigma}|$.

Clearly, we have $\Sigma_1 \supseteq \Sigma_2 \supseteq \cdots$. Once $\Sigma_i = \Sigma_{i+1} = \Sigma_{i+2} = \cdots$. It is clear that the sequence stabilizes at some $i \leq |\hat{\Sigma}|$. We write $\Sigma^*$ to denote the fix point $\Sigma_i$ such that $\Sigma_i = \Sigma_{i+1} = \Sigma_{i+2} = \cdots$. It is clear that $\operatorname{trim}(\hat{\Sigma}) = \Sigma^*$, and it can be computed in $O(|C|\delta + |\hat{\Sigma}|)$ time. $\boxtimes$

Next, we give an algorithm that computes all path-flexible strongly connected components $\Sigma' \in \mathsf{flexible-SCC}(\hat{\Sigma})$, for any given $\hat{\Sigma} \subseteq \Sigma$.

Lemma 5.32 (Algorithm for flexible-SCC). The set of all $\Sigma' \in \mathsf{flexible-SCC}(\hat{\Sigma})$ can be computed in $O(|\hat{\Sigma}|^3 + |C|\delta)$ time, for any given $\hat{\Sigma} \subseteq \Sigma$.

Proof. Let $M$ be the directed graph representing the automaton associated with the path-form of the LCL problem $\Pi \upharpoonright \hat{\Sigma}$. The directed graph $M$ has $|\hat{\Sigma}|$ nodes and $O(|\hat{\Sigma}|^2)$ directed edges, and it can be constructed in $O(|C|\delta)$ time by going through all node configurations in $C$. The strongly connected components of $M$ can be computed in time linear in the number of nodes and edges of $M$, which is $O(|\hat{\Sigma}|^2)$.

By the proof of Lemma 4.28, for each strongly connected component $U$ of $M$, deciding whether $U$ is path-flexible costs $O(|U|^3)$ time. The summation of the time complexity $O(|U|^3)$, over all strongly connected component $U$ of $M$, is $O(|\hat{\Sigma}|^3)$. To summarize, all $\Sigma' \in \mathsf{flexible-SCC}(\hat{\Sigma})$ can be computed in $O(|\hat{\Sigma}|^3 + |C|\delta)$ time. $\boxtimes$

Combining Lemmas 5.31 and 5.32, we obtain the following result.

Lemma 5.33 (Computing all good sequences). The set of all good sequences can be computed in $O(|\Sigma|^4 + |C|\delta|\Sigma|^2)$ time, for any given LCL problem $\Pi = (\delta, \Sigma, C)$.

Proof. By Lemma 5.31, given $\Sigma_i \subseteq \Sigma$, the cost of computing $\Sigma_{i+1}$ is $O(|C|\delta|\Sigma_i| + |\Sigma_i|^2)$ time. Since all sets $\Sigma_i$ in the depth $i$ of the recursion are disjoint, the total cost for this step of the recursion is $O(|C|\delta|\Sigma| + |\Sigma|^2)$.

By Lemma 5.32, given $\Sigma_i \subseteq \Sigma$, the cost of computing all possible $\Sigma_i$ is $O(|\Sigma|^3 + |C|\delta)$ time. Since all sets $\Sigma_i$ in the depth $i$ of the recursion are disjoint, the total cost for this step of the recursion is $O(|\Sigma|^3 + |C|\delta|\Sigma|)$.

The depth of the recursion is at most $|\Sigma|$, so the total cost of computing all good sequences is $O(|\Sigma|^4 + |C|\delta|\Sigma|^2)$. $\boxtimes$

We are ready to prove Theorem 5.9.

Proof of Theorem 5.9. By Lemma 5.33, the set of all good sequences can be computed in polynomial time, and we can compute $d_{\Pi}$ given the set of all good sequences. If $d_{\Pi} = k$ is a positive integer, then from the discussion in Section 5.3 we know how to turn a good sequence $(\Sigma_1^R \Sigma_2^C \Sigma_2^R \Sigma_3^C \cdots \Sigma_k^R)$ into a description of an $O(n^{1/k})$-round
algorithm for $\Pi$. If $d_1 = \infty$, then similarly a good sequence $(\Sigma^R_1, \Sigma^C_1, \Sigma^R_2, \Sigma^C_2, \ldots)^{O(\log n)}$ leads to a description of an $O(\log n)$-round algorithm for $\Pi$. \qed

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