ON THE $p$-DIVISIBILITY OF CLASS NUMBERS OF AN INFINITE FAMILY OF IMAGINARY QUADRATIC FIELDS $\mathbb{Q}(\sqrt{d})$ AND $\mathbb{Q}(\sqrt{d+1})$.

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Abstract. For any odd prime $p$, we construct an infinite family of pairs of imaginary quadratic fields $\mathbb{Q}(\sqrt{d}), \mathbb{Q}(\sqrt{d+1})$ whose class numbers are both divisible by $p$. One of our theorems settles Iizuka’s conjecture for the case $n = 1$ and $p > 2$.

1. INTRODUCTION

The ideal class group of a number field $K$ is defined to be the quotient group $J_K/P_K$, where $J_K$ is the group of fractional ideals of $K$ and $P_K$ is the group of principal fractional ideals of $K$. It is denoted by $Cl_K$. It is well known that $Cl_K$ is finite. The class number $h_K$ of a number field $K$ is the order of $CL_K$. The ideal class group is one of the most basic and mysterious objects in algebraic number theory. This has drawn the attention of several authors. The divisibility properties of the class numbers of number fields play very important role in understanding the structure of the ideal class groups of number fields. Cohen-Lenstra heuristics are a set of conjectures about this structure. In general, given a positive integer $n$, there is no formula to find all quadratic number fields whose class numbers are divisible by $n$. For $n = 3$, these fields were characterized in [14]. For a given integer $n > 1$, the Cohen–Lenstra heuristic [4] predicts that a positive proportion of imaginary quadratic number fields have class number divisible by $n$. It has been proved by several authors that for every $n > 1$, there exist infinitely many quadratic number fields whose class number is divisible by $n$. Refer to ([13], [11], [5], [3], [7], [19]).

B. H. Gross and D. E. Rohrlich proved that for any odd $n > 1$, there are infinitely many imaginary quadratic number fields $(\mathbb{Q}(\sqrt{1-4U^n}), U > 1)$ whose class numbers are divisible by $n$. Furthermore Stéphane Louboutin in [19] gave simple proof of the same result for $U > 2$. Murty proved that the class number of $\mathbb{Q}(\sqrt{1-U^n})$ is divisible by $n$ if $1-U^n$ is square-free. We study the families $\mathbb{Q}(\sqrt{1-2m^p})$ for all odd primes $p$. The following result on 3-divisibility of class number (Theorem 3.2. [8]) is proved by K. Chakraborty and A. Hoque.

Theorem 1. The class number of $\mathbb{Q}(\sqrt{1-2m^3})$ is divisible by 3 for any odd integer $m > 1$.

We generalize the above result for all odd primes $p$, and prove the $p$-divisibility of class numbers by using the results of Bugeaud, Yann and Shorey, T. N from [2].
Theorem 2. For prime numbers $p, q \geq 3$ and $m = q^r, r \in \mathbb{N}$ such that $Q(\sqrt{1-2mp}) \neq Q(i)$, the class number of $Q(\sqrt{1-2mp})$ is divisible by $p$.

Birch Swinnerton-Dyer conjecture is elliptic curve analogue of the analytic class number formula. For any elliptic curve defined over $Q$ of rank zero and square-free conductor $N$, if $p | |E(Q)|$, under certain conditions on the Shafarevich-Tate group $|\sha|$, the second author [18] showed that $p | |\sha| \Leftrightarrow p | h_K, K = Q(\sqrt{-D})$. Some pairs of infinite families of quadratic fields with class numbers divisible by a fixed integers were given by Komatsu ([15],[16],[17]), Ito [12], Aoki and Kishi [1], Iizuka, Konomi and Nakano [10] and Y.Iizuka [9]. Komatsu ([15],[16]) and Ito [12] gave infinite families of pairs of quadratic fields with both class numbers divisible by 3 by using solutions of diophantine equations. Iizuka,Konomi and Nakano [10] constructed pairs of quadratic fields whose class numbers are divisible by 3, 5 or 7 by associating the problem to the study of points on elliptic curves. Aoki and Kishi [1] gave an infinite family of pairs of imaginary quadratic fields, parametrized in terms of Fibonacci numbers $F_n$, with both class numbers divisible by 5.

The following result on class numbers of imaginary quadratic fields is recently proved by Y.Iizuka in [9].

Theorem 3. There is an infinite family of pairs of imaginary quadratic fields $Q(\sqrt{d})$ and $Q(\sqrt{d+1})$ with $d \in \mathbb{Z}$ whose class numbers are both divisible by 3.

In this article, we generalize the above result for all odd prime numbers $p$ by using theorem 2 and prove the following theorem.

Theorem 4. For every odd prime number $p \geq 3$, there is an infinite family of pairs of imaginary quadratic fields $Q(\sqrt{d})$ and $Q(\sqrt{d+1})$ with $d \in \mathbb{Z}$ whose class numbers are both divisible by $p$.

The above theorem settles Iizuka’s conjecture [12] for the case $n = 1$ and $p > 2$.

2. Preliminaries

We recall some known results and prove some lemmas which are necessary for proving our main theorem.

Definition 5. ([21]) Let $K$ be number field and let $S$ be a finite set of valuations on $K$, containing all the archimedean valuations. Then

$$R_S = \{\alpha \in K \mid \nu(\alpha) \geq 0 \text{ for all } \nu \notin S\}$$

is called the set of $S$-integers.

Lemma 6. (Siegel’s theorem, [21]) Let $K$ be a number field and let $S$ be a finite set of valuations on $K$, containing all the archimedean valuations. Let $f(X) \in K[X]$ be a polynomial of degree $d \geq 3$ with distinct roots in the algebraic closure $\overline{K}$ of $K$. Then the equation $y^2 = f(x)$ has only finitely many solutions in $S$-integers $x, y \in R_S$. 


On the $p$-divisibility of class numbers of $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{d+1})$.

We recall some results of Yann Bugeaud and T.N. Shorey from [2] on solutions of Diophantine equation $D_1x^2 + D_2 = \lambda^2 k^y$, where $\lambda = \sqrt{2}$, 2 and $D_1, D_2$ are coprime.

Let us denote $F_k$ the fibonacci sequence defined by $F_0 = 0, F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for all $k \geq 2$. Let $L_k$ be the Lucas sequence defined by $L_0 = 2, L_1 = 1$ and satisfying $L_k = L_{k-1} + L_{k-2}$ for all $k \geq 2$. define the subsets $F, G, H$ of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ by

$$F := \{(F_{k-2\epsilon}, L_{k+r}, F_k) \mid k \geq 2, \epsilon \in \{\pm 1\}\},$$

$$G := \{(1, 4k^r - 1, k) \mid k \geq 2, r \geq 1\},$$

$$H := \{(D_1, D_2, k) \mid \text{there exist positive integers } r \text{ and } s \text{ such that } \lambda < D_1s^2 + D_2 = \lambda^2 k^y \text{ and } 3D_1s^2 - D_2 = \pm \lambda^2\}. $$

Define $N(\lambda, D_1, D_2, p)$ be the number of $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ of the Diophantine equation $D_1x^2 + D_2 = \lambda^2 p^y$.

**Theorem 7.** ([2], Theorem 1) Let $p$ be a prime number. then we have $N(\lambda, D_1, D_2, p) \leq 1$ expect for $N(2, 13, 3, 2) = N(\sqrt{2}, 7, 11, 3) = N(1, 2, 1, 3) = N(2, 7, 1, 2) = N(\sqrt{2}, 1, 1, 5) = N(\sqrt{2}, 1, 1, 13) = N(2, 1, 3, 7) = 2$ and when $(D_1, D_2, p)$ belongs to one of the infinite families $F, G$ and $H$.

**Lemma 8.** For any odd prime $q$, an integer $D > 3$, the equation $Dx^2 + 1 = 2q^y$ has atmost one solution $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$.

**Proof.** Let $D_1 = D$ and $D_2 = 1$. Observe that $(D_1, D_2, p) \not\in F$ since $(F_{k-\epsilon}, L_{k+\epsilon}, F_k) = (D_1, D_2, p)$ gives $k = 2, \epsilon = -1$ and $D = 3$. This is not possible because $D > 3$. $(D_1, D_2, p) \not\in G$ since $(1, 4k^r - 1, k)$ gives $D_1 = 1$ this is not possible. $(D_1, D_2, p) \not\in H$ since $D_1x^2 + 1 = 2q^y$, then $3D_1x^2 - 1 \geq D_1x^2 - 1 \geq 2q^y - 2 \geq 4$. Hence $(D_1, D_2, p) \not\in F \cup G \cup H$. Note $(\lambda, D_1, D_2, q) \not\in \{(2, 13, 3, 2), (\sqrt{2}, 7, 11, 3), (1, 2, 1, 3), (2, 7, 1, 2), (\sqrt{2}, 1, 1, 5), (\sqrt{2}, 1, 1, 13), (2, 1, 3, 7)\}$ since $\lambda = \sqrt{2}$.

Theorem 7 implies that $Dx^2 + 1 = 2q^y$ has atmost one solution $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. 

**Proposition 9.** For prime numbers $p, q \geq 3$ and $m = q^r, r \in \mathbb{N}$, let $\alpha = \pm 1 + \sqrt{1 - 2mp}$, then $\pm 2^{\frac{p-1}{2}} \alpha$ is not a $p$th power of an algebraic integer in $\mathbb{Q}(\sqrt{1 - 2mp})$.

**Proof.** Let $d$ be the square free part of $\sqrt{1 - 2mp}$ with signature. Let $K = \mathbb{Q}(\sqrt{d})$ and $\mathcal{O}_K$ be the ring of integers of $K$. Note that $-2^{\frac{p-1}{2}} \alpha$ is a $p$th power in $\mathcal{O}_K$ if and only if $2^{\frac{p-1}{2}} \alpha$ is $p$th power in $\mathcal{O}_K$. It is enough to show $2^{\frac{p-1}{2}} \alpha$ is not $p$th power. Suppose

$$2^{\frac{p-1}{2}} \alpha = \beta^p \text{ for some } \beta = a + b\sqrt{d} \in \mathcal{O}_K. \quad (2.1)$$

$$2^{\frac{p-1}{2}} (1 + \sqrt{1 - 2mp}) = \sum_{j=0}^{\frac{p-1}{2}} \binom{p-1}{2j} a^{p-2j} b^{2j} d^j + \gamma \sqrt{d} \text{ for some } \gamma \in \mathbb{Z}. \quad (2.2)$$

By comparing constant terms on both sides, we have

$$2^{\frac{p-1}{2}} = \sum_{j=0}^{\frac{p-1}{2}} \binom{p-1}{2j} a^{p-2j} b^{2j} d^j. \quad (2.3)$$
This implies that

\[ 2^{\frac{p-1}{2}} = a \left( \sum_{j=0}^{\frac{p-1}{2}} \binom{p}{2j} a^{p-2j-1} b^{2j} d \right). \]

Hence \( a \) divides \( 2^{\frac{p-1}{2}} \).

**Case 1:** \( a \) is even.

Since \( a, b \) have same parity implies that \( b \) is also even. We look at \( (p1) \)

\[ 2^{\frac{p-1}{2}} \alpha = \left( a + b\sqrt{d} \right)^p, \]

applying norm map on the both sides

\[ (2m)^p = (a^2 - b^2 d)^p. \]

Hence we have

\[ 2m = a^2 - b^2 d. \]

Since \( 2 \mid a \), we obtain \( 2 \mid bd^2 \). We deduce that \( 2 \mid b \) since \( d \) is odd. Taking divisibility of \( a^2, b^2 \) by 4 into consideration, we conclude that \( 4 \mid 2m \) but \( m \) is odd which contradicts the assumption that \( a \) is even.

**Case 2:** \( a \) is odd.

Suppose \( a \) is odd. Since \( a \mid 2^{\frac{p-1}{2}} \), this implies that \( a = \pm 1 \). Putting in Equation \( (2.3) \) we have

\[ 2^{\frac{p-1}{2}} (1 + \sqrt{1 - 2m^p}) = \left( \pm 1 + b\sqrt{d} \right)^p, \]

applying Norm on both sides we get

\[ (2m)^p = (1 - b^2 d)^p. \]

Rewriting the above equation using \( D = -d \) and \( m = q^r \), we get

\[ 1 + Db^2 = 2q^r. \]

We observe \( 1 - 2m^p = 1 - 2(q^r)^p = b^2 d \) for some \( b' \in \mathbb{Z} \) Rephrasing this equation we have

\[ 1 + b^2 D = 2q^{rp}. \]

Thus from Equations \( (2.5) \) and \( (2.6) \) we get \((b, r), (b', rp)\) are solutions of the equation

\[ Dx^2 + 1 = q^y, \]

which is a contradiction to Lemma 8. Hence \( \pm 2^{\frac{p-1}{2}} \alpha \) is not \( p \)th power in \( \mathcal{O}_K \).

\[ \square \]

**3. Proof of the theorems**

**Proof of Theorem 4** Let \( d \) be the square free part of \( 1 - 2m^p \) with signature then \( d \equiv 3 \mod 4 \) and \( K = \mathbb{Q}(\sqrt{d}) \). Put \( \alpha := 1 + \sqrt{1 - 2m^p} \). Thus \( N(\alpha) = 2m^p \). Since \( d \equiv 3 \mod 4 \), the ideal \( (2) \) is ramified, there exist an ideal \( \mathcal{P} \) such that \( (2) = \mathcal{P}^2 \). Let the prime decomposition of \( (\alpha) \) be

\[ (\alpha) = \mathcal{P} \prod_i \mathcal{P}_i^{t_i}. \]
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Then $N_{K/\mathbb{Q}}((\alpha)) = 2 \prod_i \mathcal{P}_i^{a_i t_i}$, where $N(\mathcal{P}_i) = p_i^{a_i}$, $a_i = 1, 2$. Hence $p \mid t_i$.

Set $I := \mathcal{P} \prod_i \mathcal{P}_i^{t_i}$, an ideal of $K$. Observe that

$$I^p = \mathcal{P}^p \prod_i \mathcal{P}_i^{t_i} = (2)^{\frac{p-1}{2}} \mathcal{P} \prod_i \mathcal{P}_i^{t_i} = (2)^{\frac{p-1}{2}} (\alpha) = (2)^{\frac{p-1}{2}} \alpha.$$

We claim that the order of the ideal $I$ in ideal class group is $p$. Suppose not, let $(\beta) = I$ for some $\beta$ in $\mathcal{O}_K$. Then

$$(\beta^p) = (\beta)^p = I^p = (2)^{\frac{p-1}{2}} \alpha.$$

Since only units of $K$ are $\{1, -1\}$, this implies that $I$ is principal ideal if and only if $\pm 2^{\frac{p-1}{2}} \alpha$ is power of $p$ in $\mathcal{O}_K$. From Lemma 9, we know that $\pm 2^{\frac{p-1}{2}} \alpha$ is not a $p$th power in $\mathcal{O}_K$. Hence class group of $\mathbb{Q}(\sqrt{1-2q^p})$ has an element $I$ of order $p$. \qed

Stéphane Louboutin in [19] proved the following result ( Theorem 1 ) on divisibility of class number of $\mathbb{Q}(\sqrt{1-4UF})$ for $U > 2$.

**Lemma 10.** If $k \in \mathbb{Z}^+$ be odd number, then for any integer $U \geq 2$ the ideal class groups of the imaginary quadratic fields $\mathbb{Q}(\sqrt{1-4UF})$ contain an element of order $k$.

We now prove the main result of this article.

**Proof of Theorem 4.**

Fix an odd prime $p$. Consider the set

$$S = \{ m | m = q^r, \text{ for some prime } q, r \in \mathbb{N}, p \mid \text{cl} \left( \mathbb{Q} \left( \sqrt{1-2mp} \right) \right) \}.$$

By Siegel’s theorem ( Lemma 6 ) the equation $1 - 2x^p \neq -y^2$ has finitely many solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$. Hence $S$ is an infinite set. In fact, $S$ doesn’t contain atmost finitely many odd primes’ powers. It follows from Theorem 2 any odd prime $p$ divides the class number of

$$K = \mathbb{Q}(\sqrt{4(1-2mp)^p}) = \mathbb{Q} \left( \sqrt{(1-2mp)^p} \right) \text{ for } m \in S.$$

Let $U = 2mp^2 - 1$. Then $U \geq 2$. Furthermore Lemma 10 implies that $p$ divides the class number of $\mathbb{Q} \left( \sqrt{1-4UP} \right)$. Now look at

$$\mathbb{Q}(\sqrt{1-4UP}) = \mathbb{Q} \left( \sqrt{1-4(2q^p - 1)^p} \right) = \mathbb{Q} \left( \sqrt{4(1-2q^p)^p + 1} \right).$$

Form $\in S$, let $d = 4(1-2m^p)^p$. The prime $p$ divides class numbers of

$$\mathbb{Q}(\sqrt{d}), \mathbb{Q}(\sqrt{d+1}).$$

Now to conclude the theorem, we need to prove the set $A = \left\{ \mathbb{Q} \left( \sqrt{1-2mp} \right), m \in S \right\}$ is an infinite set. For every square-free integer $d_0 \neq 0$, let $f(x) = \frac{1-2x^p}{d_0^p}$. This polynomial $f(x)$ has distinct roots in $\overline{\mathbb{Q}}$. Thus by Lemma 6 $y^2 = f(x)$ has finitely many integral solutions. Hence $A$ is infinite. Hence the theorem follows.

We prove a corollary of Theorem 2.

**Corollary 11.** For every odd prime $p$, there exist infinitely many quartic field with Galois group isomorphic to Klein 4-group whose class number is divisible by $p$. 

Proof. Let $m \in S$, $m = q^r$, $q$-prime, $r$ is odd and $m \equiv 1 \pmod{4}$. Denote $K_m = \mathbb{Q}(\sqrt{1-2m^2}, \sqrt{m})$, $L^1_m := \mathbb{Q}(\sqrt{1-2m^2})$, $L^2_m := \mathbb{Q}(\sqrt{m})$ and $L^3_m = \mathbb{Q}\left(\sqrt{(1-2m^2)(\sqrt{m})}\right)$. Observe $K_m$ is compositum of $L^1_m$, $L^2_m$. Let $h_m, h^1_m, h^2_m$ and $h^3_m$ be the class numbers of $K_m, L^1_m, L^2_m$ and $L^3_m$, respectively. Then by Lemma 2 in [6], we have $h_m = h^1_m h^2_m h^3_m$ where $i = 0, 1$. Since $m \in S$, by Theorem 2 $p \mid h^1_m$. Hence $p \mid h_m$ as $p$ is odd. Therefore the class numbers of the infinite family

$$\{K_m \mid m \in S, m = q^r, q \text{ prime}, r \text{ is odd and } m \equiv 1 \pmod{4}\}$$

are divisible by $p$. □

CONCLUDING REMARKS

Our theorems prove that the following conjecture is true for $n = 1$ and for all $p > 2$.

**Conjecture 12. (Iizuka)** For any prime $p$ and any positive integer $n$, there is an infinite family of $n+1$ successive real (or imaginary) quadratic fields

$$Q(\sqrt{D}), Q(\sqrt{D+1}), \ldots, Q(\sqrt{D+n})$$

with $D \in \mathbb{Z}$ whose class numbers are divisible by $p$.

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