1. Introduction

For a topological group \( \Gamma \), by a \( \Gamma \)-flow we mean pair \((X, \rho)\), where \( X \) is a topological space and \( \rho \) is a continuous action of \( \Gamma \) on \( X \). For any two \( \Gamma \)-flows \((X, \rho)\) and \((X', \sigma)\), a continuous map \( f : X \to X' \) is said to be \( \Gamma \)-equivariant if \( f \circ \rho(\gamma) = \sigma(\gamma) \circ f \), \( \forall \gamma \in \Gamma \). Two \( \Gamma \)-flows \((X, \rho)\) and \((X', \sigma)\) are said to be topologically conjugate if there exists a \( \Gamma \)-equivariant homeomorphism \( f : X \to X' \) and they are said to be orbit equivalent if there exists a homeomorphism \( f : X \to X' \) which takes orbits under \( \rho \) to orbits under \( \sigma \).

When \( X, X' \) are topological groups, an affine map from \( X \) to \( X' \) is a map of the form \( x \mapsto c \theta(x) \), where \( c \in X' \) and \( \theta \) is a continuous homomorphism from \( X \) to \( X' \). A \( \Gamma \)-flow \((X, \rho)\) is said to be affine if for all \( \gamma \) in \( \Gamma \), \( \rho(\gamma) \) is an affine map. \((X, \rho)\) is said to be an automorphism flow (resp. a translation flow) if each \( \rho(\gamma), \gamma \in \Gamma \), is an automorphism on \( X \) (resp a translation on \( X \)). \((X, \rho)\) and \((X', \sigma)\) are said to be algebraically conjugate if there exists a continuous isomorphism \( \theta : X \to X' \) such that \( \theta \circ \rho(\gamma) = \sigma(\gamma) \circ \theta \), \( \forall \gamma \in \Gamma \).

In this note we prove certain results concerning classification of affine flows on compact connected metrizable abelian groups, upto orbit equivalence and topological conjugacy.

We will denote by \( S^1 \) the usual circle group. For any locally compact abelian group \( G \), we denote by \( \widehat{G} \) the dual group of \( G \). For a compact connected metrizable abelian group \( G \), we denote by \( L(G) \) the topological vector space consisting of all homomorphisms from \( \widehat{G} \) to \( \mathbb{R} \), under pointwise addition and scalar multiplication and the topology of pointwise convergence. We define a map \( E \) from
\[ (\phi \circ E)(p) = e^{2\pi i p(\phi)} \quad \forall p \in L(G), \phi \in \hat{G}. \]

From the defining equation it is easy to see that \( E \) is a continuous homomorphism from \( L(G) \) to \( G \). The kernel of \( E \) can be identified with the set of all homomorphisms from \( \hat{G} \) to \( \mathbb{Z} \); it is a totally disconnected subgroup of \( L(G) \). Note that using the duality theorem we can realise \( L(G) \) with set of all one-parameter subgroups of \( G \) and \( E \) with the map \( \alpha \mapsto \alpha(1) \). In particular when \( G \) is a torus, \( L(G) \) can be identified with \( \mathbb{R}^n \), the Lie algebra of \( G \), and \( E \) can be identified with the standard exponential map. However, in general \( E \) is not surjective, e.g for \( G = \hat{\mathbb{Q}} \), where \( \mathbb{Q} \) is the group of rational numbers equipped with discrete topology, \( L(G) \) is isomorphic to \( \mathbb{R} \) and consequently \( E \) is not surjective.

Now let \( \Gamma \) be a discrete group and \((G, \rho)\) be an affine \( \Gamma \)-flow on \( G \). Note that \( \rho \) induces an automorphism flow \( \rho_a \) and a map \( \rho_t : \Gamma \rightarrow G \) defined by
\[
\rho(\gamma)(x) = \rho_a(\gamma)(x) \rho_t(\gamma) \quad \forall x \in G.
\]
We define an automorphism action \( \rho_* \) of \( \Gamma \) on \( L(G) \) by
\[
\rho_*(\gamma)(p)(\phi) = p(\phi \circ \rho_a(\gamma)) \quad \forall \phi \in \hat{G}, \gamma \in \Gamma.
\]
In [1] and [2] it was proved that any topological conjugacy between two ergodic automorphisms of \( n \)-torus is affine. In [8] this was generalized to certain class of affine transformations on compact connected metrizable abelian groups. Here we prove the following

**Theorem 1 :** Let \( \Gamma \) be a discrete group and \( G \) and \( H \) be compact connected metrizable abelian groups. Let \( \rho, \sigma \) be affine actions of \( \Gamma \) on \( G \) and \( H \) respectively. Let \( f : G \rightarrow H \) be a \( \Gamma \)-equivariant continuous map. Then there exist \( c \in H \), a continuous homomorphism \( \theta : G \rightarrow H \) and a continuous map \( S : G \rightarrow L(H) \) such that

\[
\text{a) } S(e) = 0 \text{ and for all } x \in G, \text{ the orbit of } S(x) \text{ under } \sigma_* \text{ is bounded.}
\]

\[
\text{b) } f(x) = c \, \theta(x)(E \circ S)(x), \quad \forall x \in G.
\]

Moreover if \( \rho \) and \( \sigma \) are automorphism actions then \( S \) is a \( \Gamma \)-equivariant map
from \((G, \rho)\) to \((L(H), \sigma_*)\).

We show that under various additional conditions \(S\) can be concluded to be identically 0, which means that \(f\) is affine. This will be shown to be the case, for instance, if the \(\Gamma\)-action on \(H\) is expansive (see corollary 1), see also Corollary 2 and Remark 1 for other applications of the theorem.

In section 3 we classify translation flows up to orbit equivalence and topological conjugacy (in the continuous and discrete parameter cases respectively). Firstly, for one-parameter flows of translations, generalizing a classical result in the case of tori (see [3]) we prove the following

**Theorem 2:** Let \(G\) and \(H\) be compact connected metrizable abelian groups and \(\alpha\) and \(\beta\) be one-parameter subgroups of \(G\) and \(H\) respectively. Then the translation flows on \(G\) and \(H\) induced by \(\alpha\) and \(\beta\) respectively are orbit equivalent if and only if there exist a continuous isomorphism \(\theta : G \to H\) and a nonzero \(c \in R\) such that \(\theta\alpha(t) = \beta(ct)\) \(\forall t \in R\).

We will also prove the following result which previously seems to have been noted only for ergodic translations (see [2]).

**Theorem 3:** Let \(G, H\) be two compact connected metrizable abelian groups and \(\rho, \sigma\) be two translation flows of a discrete group \(\Gamma\) on \(G\) and \(H\) respectively. Then \((G, \rho)\) and \((H, \sigma)\) are topologically conjugate if and only if they are algebraically conjugate.

2. Rigidity of affine actions

In this section we freely use various results from duality theory of locally compact abelian groups; the reader is referred to [5] for details. We will also use the following result due to VanKampen; for a proof see [2], [7].

**Theorem** (VanKampen) Let \(G\) be a compact connected metrizable abelian group and \(f : G \to S^1\) be a continuous map. Then there exist \(c \in S^1\), \(\phi \in \hat{G}\) and a
continuous map \( h : G \to \mathbb{R} \) such that

\[ h(0) = 0, \quad f(x) = c \phi(x) e^{2\pi i h(x)} \quad \forall x \in G. \]

Moreover, \( c, \phi \) and \( h \) are uniquely defined.

Now for any two groups \( G, H \) and any continuous map \( f : G \to H \) we will define a continuous homomorphism \( \theta(f) : G \to H \) as follows. For each character \( \phi \) of \( H \) let \( c_\phi \in S^1, \hat{\theta}(\phi) \in \hat{G} \) and \( f_\phi : G \to \mathbb{R} \) be such that

\[ f_\phi(0) = 0, \quad \phi \circ f(x) = c_\phi \hat{\theta}(\phi)(x) e^{2\pi i f_\phi(x)} \quad \forall x \in G; \]

note that by VanKampen’s theorem there exist \( c_\phi, \hat{\theta}(\phi) \) and \( f_\phi \) satisfying the conditions and they are unique. From the uniqueness one can deduce that \( \phi \mapsto \hat{\theta}(\phi) \) is a homomorphism from \( \hat{H} \) to \( \hat{G} \). By the duality theorem there exists a continuous homomorphism \( \theta(f) : G \to H \) such that

\[ \hat{\theta}(\phi) = \phi \circ \theta(f) \quad \forall \phi \in \hat{H}. \]

Using the uniqueness part of VanKampen’s theorem it is easy to see that

i) If \( f \) is a continuous homomorphism then \( \theta(f) = f \).

ii) If \( f : G_1 \to G_2 \) and \( g : G_2 \to G_3 \) be two continuous maps then

\[ \theta(g \circ f) = \theta(g) \circ \theta(f). \]

The following corollary of VanKampen’s theorem is essentially due to Arov (see [2]).

**Proposition 1:** Let \( \Gamma \) be a discrete group and \( \rho : \Gamma \to Aut(G) \) and \( \sigma : \Gamma \to Aut(H) \) be automorphism actions of \( \Gamma \) on \( G \) and \( H \) respectively. Then \( \rho \) and \( \sigma \) are topologically conjugate if and only if they are algebraically conjugate.

**Proof:** Let \( f \) be a topological conjugacy between \( \rho \) and \( \sigma \). Using i) and ii) we see that

\[ \theta(f) \circ \rho(\gamma) = \sigma(\gamma) \circ \theta(f) \quad \forall \gamma \in \Gamma. \]

Since \( f \) is a homeomorphism, it follows that \( \theta(f) \) is an isomorphism. Hence \( \rho \) and \( \sigma \) are algebraically conjugate.
The following lemma generalizes of VanKampen’s theorem.

**Lemma 1 :** Let $G$, $H$ be two compact connected metrizable abelian groups and $f$ be a continuous map from $G$ to $H$. Then there exist $c \in H$, a continuous homomorphism $\theta : G \to H$ and a continuous map $S : G \to L(H)$ such that

\[ S(0) = 0 \text{ and } f(x) = c \theta(x)(E \circ S)(x), \quad \forall x \in G. \]

Moreover $c$, $\theta$ and $S$ are unique.

**Proof :** For each character $\phi$ of $H$ define $c_{\phi} \in S^1, \theta'(\phi) \in \widehat{G}$ and $f_{\phi} : G \to R$ by the condition

\[ f_{\phi}(0) = 0, \quad \phi \circ f(x) = c_{\phi} \theta'(\phi)(x) e^{2\pi i f_{\phi}(x)} \quad \forall x \in G; \]

note that by VanKampen’s theorem there exist uniquely defined $c_{\phi}, \theta'(\phi)$ and $f_{\phi}$ satisfying the condition. From the uniqueness it follows that

\[ f_{\phi \psi} = f_{\phi} + f_{\psi}, \quad f_{\phi^{-1}} = -f_{\phi}. \]

Define $S : G \to L(H)$ by $S(x)(\phi) = f_{\phi}(x)$. Since each $f_{\phi}$ is continuous, $S$ is continuous. Similarly using uniqueness of $c_{\phi}$ we see that the map $\phi \mapsto c_{\phi}$ is a homomorphism from $\widehat{H}$ to $S^1$. By the duality theorem there exists $c \in H$ such that $c_{\phi} = \phi(c)$ $\forall \phi \in \widehat{H}$. Also putting $\theta = \theta(f)$ we see that $\theta'(\phi) = \phi \circ \theta$, $\forall \phi \in \widehat{H}$. Hence for all $x \in G$ and $\phi \in \widehat{H}$, we have

\[ \phi \circ f(x) = c_{\phi} \theta'(\phi)(x) e^{2\pi i f_{\phi}(x)} = \phi(c) (\phi \circ \theta)(x) (\phi \circ E \circ S)(x). \]

Since characters separate points, $f(x) = c \theta(x)(E \circ S)(x), \quad \forall x \in G$. Using VanKampen’s theorem we see that for a fixed $\phi \in \widehat{H}, \phi(c), \phi \circ \theta$ and the map $x \mapsto S(x)(\phi)$ are determined by the equation

\[ (\phi \circ f)(x) = \phi(c) (\phi \circ \theta)(x) e^{2\pi i S(x)(\phi)}. \]

Hence $c, \theta$ and $S$ are unique.

**Proof of Theorem 1 :** Suppose $f = c \theta(E \circ S)$ where $c, \theta$ and $S$ are as in Lemma 1. Fix any $\gamma \in \Gamma$. Note that for all $x \in G$,

\[ f \circ \rho(\gamma)(x) = c_1 \theta_1(x)(E \circ S)(x), \]

where $c_1, \theta_1$ are determined by the equation.

\[ (\phi \circ f)(x) = \phi(c) (\phi \circ \theta)(x) e^{2\pi i S(x)(\phi)}. \]
where \( c_1 = f \circ \rho(\gamma)(e) \), \( \theta_1 = \theta \circ \rho_a(\gamma) \) and \( S_1(x) = S \circ \rho(\gamma)(x) - S \circ \rho(\gamma)(e) \). Also for all \( x \in G \),

\[
\sigma(\gamma) \circ f(x) = c_2 \theta_2(x)(E \circ S_2)(x),
\]

where \( c_2 = \sigma(\gamma) \circ f(e) \), \( \theta_2 = \sigma_a(\gamma) \circ \theta \) and \( S_2 = \sigma_s(\gamma) \circ S \). From the uniqueness part of Lemma 1 it follows that \( S_1 = S_2 \) i.e.

\[
S \circ \rho(\gamma)(x) - S \circ \rho(\gamma)(e) = \sigma_s(\gamma) \circ S(x), \quad \forall x \in G.
\]

Since for a fixed \( x \in G \) the left hand side is contained in a bounded subset of \( L(H) \), it follows that for all \( x \) in \( G \), the \( \sigma_s \)-orbit of \( S(x) \) is bounded. Also it is easy to see from the previous identity that when \( \rho \) and \( \sigma \) are automorphism actions, \( S \) is a \( \Gamma \)-equivariant map.

When \( X \) is a topological group, \((X, \rho)\) is said to be \textit{expansive} if there exists a neighbourhood \( U \) of the identity such that for any two distinct elements \( x, y \in X \) there exists a \( \gamma \in \Gamma \) such that \( \rho(\gamma)(x) \rho(\gamma)(y)^{-1} \) is not contained in \( U \); such a neighbourhood is called an expansive neighbourhood. For various characterizations of expansiveness of automorphism actions on compact abelian groups the reader is referred to [6].

**Corollary 1**: Let \( \Gamma \) be a discrete group and \( G \) and \( H \) be compact connected metrizable abelian groups. Let \( \rho, \sigma \) be affine actions of \( \Gamma \) on \( G \) and \( H \) respectively such that \((H, \sigma)\) is expansive. Then every \( \Gamma \)-equivariant continuous map \( f : (G, \rho) \to (H, \sigma) \) is an affine map.

**Proof**: Since \( \sigma \) is expansive, \( \sigma_a \) is an expansive automorphism action on \( H \). We claim that for every nonzero point \( p \in L(H) \), the orbit of \( p \) under \( \sigma_s \) is unbounded. Suppose not. Choose a non-zero \( p \) and a compact set \( C \subset L(H) \) such that orbit of \( p \) under \( \sigma_s \) is contained in \( C \). Since kernel of \( E \) is totally disconnected, there exists a sequence \( \{t_i\} \) such that \( t_i \to 0 \) as \( i \to \infty \) and \( E(t_i p) \neq e \ \forall i \).

Let \( U \) be an expansive neighbourhood of \( e \) in \( H \). Since \( e \) is fixed by \( \sigma_a \), \( e \) is the only element in \( G \) whose orbit under \( \sigma_a \) is contained in \( U \). Since \( t_i \to 0 \), it is easy to see that \( \cup t_i^{-1}E^{-1}(U) = L(H) \). From the compactness of \( C \) it follows that there exists \( n \) such that \( t_n C \subset E^{-1}(U) \) i.e. \( E(t_n C) \subset U \). Since the orbit of \( t_n p \) under \( \sigma_s \) is contained in \( t_n C \) and \( E \circ \sigma_s(\gamma) = \sigma_a(\gamma) \circ E \), \( \forall \gamma \), this implies
that orbit of \( E(t_n p) \) under \( \sigma_* \) is contained in \( U \). This contradicts the fact that \( E(t_n p) \neq e \).

Now suppose \( f = c \ \theta(E \circ S) \), where \( c, \theta \) and \( S \) are as in Theorem 1. From Theorem 1 and the previous argument it follows that \( S = 0 \) i.e. \( f \) is an affine map.

For a set \( A \) we denote by \(|A|\) the cardinality of \( A \).

**Corollary 2**: Let \( \Gamma \) be a discrete group and \( G \) and \( H \) be compact connected metrizable abelian groups. Let \( \rho, \sigma \) be automorphism actions of \( \Gamma \) on \( G \) and \( H \) respectively such that

\begin{itemize}
  \item [a)] \( \{ g \in G \mid |\rho(\Gamma)(g)| < \infty \} \) is dense in \( G \).
  \item [b)] for any natural number \( k \), the set \( \{ h \in H \mid |\sigma(\Gamma)(h)| = k \} \) is totally disconnected.
\end{itemize}

Then every continuous \( \Gamma \)-equivariant map \( f : (G, \rho) \rightarrow (H, \sigma) \) is an affine map.

**Proof**: Suppose \( f = c \ \theta(E \circ S) \), where \( c, \theta \) and \( S \) are as in Theorem 1. Let \( g \in G \) be such that the orbit of \( g \) under \( \rho \) is finite. Since \( S \) is \( \Gamma \)-equivariant by Theorem 1, the orbit of \( S(g) \) under \( \sigma_* \) is also finite. Since \( \sigma_* \) is a linear action on \( L(H) \) and \( E \) is a \( \Gamma \)-equivariant map from \( (L(H), \sigma_*) \) to \( (H, \sigma) \), the orbit of \( E(tS(g)) \) under \( \sigma \) is finite for all \( t \in \mathbb{R} \). Now from b) it follows that \( S(g) = 0 \). Since \( \{ g \in G \mid |\rho(\Gamma)(g)| < \infty \} \) is dense in \( G \), \( S = 0 \) i.e. \( f \) is an affine map.

**Remark 1**: It is easy to see that condition (a) as in corollary 2, holds when \( G = T^n \) for some \( n \). Various other conditions under which the set of periodic orbits of an automorphism action on a compact abelian group is dense, viz condition (a) as in corollary 2 holds, are described in [4]. Condition (b) holds in the case of \( H = T^n \) if \( \Gamma \) contains an element acting ergodically; more generally this holds for any finite dimensional compact abelian group \( H \).

3. **Classification of translation flows**

**Lemma 2**: Let \( G \) be a compact connected metrizable abelian group and \( \alpha \) be a one-parameter subgroup of \( G \). Then there exists a \( p \in L(G) \) such that
\[ E(tp) = \alpha(t) \quad \forall t \in \mathbb{R}. \]

**Proof:** For each \( \phi \in \hat{G} \), we define \( \alpha_\phi \in \mathbb{R} \) by

\[ \phi \circ \alpha(t) = e^{2\pi i \alpha_\phi t} \quad \forall t \in \mathbb{R}. \]

Since \( \phi \circ \alpha \) is a continuous homomorphism from \( \mathbb{R} \) to \( S^1 \), \( \alpha_\phi \) is well defined. We define \( p \in L(G) \) by \( p(\phi) = \alpha_\phi \forall \phi \in \hat{G} \). Fix any \( t \in \mathbb{R} \). From the defining equation of \( E \) it follows that for all \( \phi \in \hat{G} \),

\[ \phi \circ E(tp) = e^{2\pi itp(\phi)} = e^{2\pi i \alpha_\phi t} = \phi \circ \alpha(t). \]

Since characters separate points, it follows that \( \alpha(t) = E(tp) \forall t \in \mathbb{R} \).

The following lemma is needed to prove Theorem 2. The main idea of the Proof is derived from \[3\].

**Lemma 3:** Let \( G \) be a compact connected metrizable abelian group and \( p,q \in L(G) \), with \( p \neq 0 \). Let \( f : \mathbb{R} \to L(G) \) be a bounded continuous function such that \( f(0) = 0 \), \( \{ E(tp + f(t)) \mid t \in \mathbb{R} \} \) = \( \{ E(tq) \mid t \in \mathbb{R} \} \).

Then \( p = cq \) for some nonzero \( c \in \mathbb{R} \).

**Proof:** First we will prove the special case when \( G = T^2 \), the two-dimensional torus. After suitable identifications we have

\[ L(G) = \mathbb{R}^2, \quad E(x_1, x_2) = Exp(x_1, x_2) = (e^{2\pi i x_1}, e^{2\pi i x_2}). \]

Define a function \( d : \mathbb{R}^2 \to \mathbb{R}^+ \) by

\[ d(x) = \text{distance between the point } x \text{ and the line } \{ t q \mid t \in \mathbb{R} \} \]

\[ = \inf \{ ||y|| \mid x + y = t q \text{ for some } t \in \mathbb{R} \}. \]

By our hypothesis, for all \( t \in \mathbb{R} \), \( tp + f(t) = z + t'q \) for some \( t' \in \mathbb{R}, z \in \mathbb{Z}^2 \).

This implies

\[ \{ d(tp + f(t)) \mid t \in \mathbb{R} \} \subset \{ d(z) \mid z \in \mathbb{Z}^2 \} \]

Since the map \( t \mapsto d(tp + f(t)) \) is continuous, the left hand side is a connected subset of \( \mathbb{R} \) containing 0. Since the right hand side is countable, \( d(tp + f(t)) = 0, \forall t \).

Since \( f \) is bounded this implies that \( d(tp) \) is bounded by a constant \( M \), for all
$t \in R$. Since distinct lines in $R^2$ diverge from each other we conclude that $p = cq$ for some $c \neq 0$.

To prove the general case choose $\phi$ such that $p(\phi) \neq 0$. For each $\psi \in \hat{G}$ define $h : G \rightarrow T^2$ and $h^* : L(G) \rightarrow R^2$ by

$$h(x) = (\phi(x), \psi(x)), \quad h^*(r) = (r(\phi), r(\psi)).$$

Now for all $r \in L(G)$,

$$h \circ E(r) = (\phi \circ E(r), \psi \circ E(r)), \quad Exp \circ h^*(r) = (e^{2\pi i r(\phi)}, e^{2\pi i r(\psi)}).$$

From the defining equation of $E$ it follows that $h \circ E = Exp \circ h^*$. Define $p', q' \in R^2$ and $f' : G \rightarrow R^2$ by $p' = h^*(p)$, $q' = h^*(q)$ and $f' = h^* \circ f$. From our hypothesis it follows that

$$\{Exp(tp' + f'(t)) \mid t \in R\} = \{Exp(tq') \mid t \in R\}$$

Now applying the special case we see that $(p(\phi), p(\psi)) = c (q(\phi), q(\psi))$ for some nonzero real number $c$. Therefore $q(\phi) \neq 0$ and, since $\psi$ is arbitrary, $p = c_0q$ where $c_0 = p(\phi)/q(\phi)$.

**Proof of Theorem 2:** Let $h$ be an orbit equivalence between the translation flows induced by $\alpha$ and $\beta$. Define $f : G \rightarrow H$ by $f(x) = h(x)h(e)^{-1}$. Then $f(e) = e$ and it is easy to check that $f$ is also an orbit equivalence. Suppose $f = \theta(E \circ S)$, where $\theta$ and $S$ are as in Lemma 1. By Lemma 2 there exists $p, q$ in $L(H)$ such that $E(tq) = \beta(t)$ and $E(tp) = \theta(\alpha(t))$. Since $f$ is an orbit equivalence,

$$\{f(\alpha(t)) \mid t \in R\} = \{\beta(t) \mid t \in R\}$$

Now for all $t \in R$, $\beta(t) = E(tp)$ and

$$f(\alpha(t)) = (\theta \circ \alpha)(t)(E \circ S \circ \alpha)(t) = E(tp + S \circ \alpha(t)).$$

Since $f$ is a homeomorphism, $\theta$ is an isomorphism. Hence $\theta(\alpha) \neq 0$, i.e. $p \neq 0$. By applying Lemma 3 we see that $p = cq$ for some nonzero $c$. Therefore for some $c \neq 0$, $\theta(\alpha(t)) = \beta(ct)$, $\forall t \in R$. This proves the theorem.

**Proof of Theorem 3:** Let $f$ be a topological conjugacy between the induced translation flows. Suppose $f = c \theta (E \circ S)$, where $c, \theta$ and $S$ are as in
Lemma 1. Since $f$ is a homeomorphism, $\theta$ is an isomorphism. We claim that $\theta$ is an algebraic conjugacy between $(G, \rho)$ and $(H, \sigma)$. To see this fix any $\gamma \in \Gamma$. Define $x_0 \in G$, $y_0 \in H$ by

$$\rho(\gamma)(x) = x_0 x, \quad \sigma(\gamma)(y) = y_0 y \quad \forall x \in G, \ y \in H.$$  

Then for all $x \in G$,

$$f(x_0 x) = c \theta(x_0 x)(E \circ S)(x_0 x) = c_1 \theta(x)(E \circ S_1)(x),$$

where $c_1 = c \theta(x_0)(E \circ S)(x_0)$, $S_1(x) = S(x_0 x) - S(x_0)$. Also for all $x \in G$,

$$y_0 f(x) = c_2 \theta(x)(E \circ S)(x),$$

where $c_2 = c \theta(x_0)$. From the uniqueness part of Lemma 1 it follows that $S_1 = S$ i.e.

$$S(x_0 x) = S(x) + S(x_0) \quad \forall x \in G.$$  

Putting $x = x_0, x_0^2, \ldots$ and using the above recursion relation we obtain

$$S(x_0^n) = n S(x_0) \quad \forall n \in \mathbb{Z}^+.$$  

Since the left hand side is contained in the image of $S$, which is compact, it follows that $S(x_0) = 0$. Since $c_1 = c_2$, this implies $\theta(x_0) = y_0$. Hence $\theta \circ \rho(\gamma) = \sigma(\gamma) \circ \theta$.

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School of Mathematics, T.I.F.R, Mumbai 400005, India

E-mail address: siddhart@math.tifr.res.in