A low multiplicative complexity fast recursive DCT-2 algorithm

Maxim Vashkevich  
Computer Engineering Department  
Belarusian State University  
of Informatics and Radioelectronics  
Minsk, Belarus, 220013  
Email: vashkevich@bsuir.by

Alexander Petrovsky  
Computer Engineering Department  
Belarusian State University  
of Informatics and Radioelectronics  
Minsk, Belarus, 220013  
Email: palex@bsuir.by

Abstract—A fast Discrete Cosine Transform (DCT) algorithm is introduced that can be of particular interest in image processing. The main features of the algorithm are regularity of the graph and very low arithmetic complexity. The 16-point version of the algorithm requires only 32 multiplications and 81 additions (that is only 17 multiplications greater than [7]), but the computational core has advantages from an algorithmic point of view. It reveals the algorithm structure and simplifies manipulation with it to derive new variants.

In ASP, the theory has been called algebraic signal processing theory (ASP). MATLAB implementation of the algorithm can be found in the public repository https://github.com/Mak-Sim/Max_Sim_Fast_recursive_DCT.

I. INTRODUCTION

The Discrete Cosine Transform (DCT) has found many applications in image processing, data compression and other fields due to its decorrelation property [1]. Despite the fact that a number of fast DCT algorithms have been proposed already, designing new efficient schemes is still of great interest [2]–[4]. Majority of proposed fast DCT algorithms have been obtained using graph transformation, equivalence relation or sophisticated manipulation of the transform coefficients. Recently an algebraic approach to derivation of fast DCT has been presented [5]. The approach uses polynomial algebra associated with DCT to obtain fast algorithms. Subsequently this theory has been called algebraic signal processing theory (ASP) [6]. The theory provides consistent algebraic interpretation of fast DCT algorithms.

The paper presents derivation of fast DCT-2 n-point algorithm (n is a power of two) based on ASP. The algorithm is recursive and has a regular graph. Another feature of the algorithm is very low arithmetic complexity: 16-point DCT requires only 32 multiplications and 81 additions (that is only one multiplication greater than [7]), but the computational core of algorithm contains only 17 multiplication while other 15 are scaling factors that can be compensated in the post-processing. Because of the mentioned properties the algorithm is a very attractive choice for hardware DCT implementations.

II. ALGEBRAIC APPROACH TO DCT

In this section the fundamentals of algebraic signal processing theory [6] are considered that are used further for derivation of the fast DCT-2 algorithm.
Permutation matrices that has exactly one entry 1 in row \( i \) at position \( f(i) \) and each column and 0 elsewhere is defined as:

\[
P: i \mapsto f(i), \quad 0 \leq i < n.
\]

One important is the \( n \times n \) stride permutation matrix defined for \( m|n \) as

\[
L^n_m : \begin{bmatrix} \frac{n}{m} \end{bmatrix}_i + i_1 \rightarrow i_1m + i_2
\]

for \( 0 \leq i_1 < \frac{n}{m} \), \( 0 \leq i_2 < m \).

ASP states that every DCT corresponds to some polynomial algebra \( \mathbb{F}[x]/p(x) \) with basis \( b \). In this case DCT is given by the CRT (2) and its matrix takes the form of polynomial transform (3) or a scaled polynomial transform (4). From (2) it can be seen that \( \mathcal{F} \) decomposes \( \mathbb{F}[x]/p(x) \) into one-dimensional polynomial algebras. Fast algorithm is obtained by complying this decomposition in step using an intermediate subalgebras.

One possible way to perform decomposition of \( \mathbb{F}[x]/p(x) \) in step is to use factorization \( p(x) = q(x) \cdot r(x) \). If \( \deg(q) = k \) and \( \deg(r) = m \) then

\[
\mathcal{F}[x]/p(x) \rightarrow \mathcal{F}[x]/q(x) \oplus \mathcal{F}[x]/r(x) \quad \text{for} \quad \beta \quad \text{and} \quad \gamma
\]

respectively, then (5)-(7) are expressed in the following matrix form [5]:

\[
P_{\alpha,\beta} = P_{\alpha,\beta} \oplus P_{\alpha,\gamma}
\]

where \( \beta \) and \( \gamma \) are the zeros of \( q(x) \) and \( r(x) \) correspondingly. If \( c \) and \( d \) are the bases of \( \mathcal{F}[x]/q(x) \) and \( \mathcal{F}[x]/r(x) \), respectively, then (5)-(7) are expressed in the following matrix form [5]:

\[
\mathbb{P}_{\alpha,\beta} = \mathbb{P}_{\alpha,\beta} \oplus \mathbb{P}_{\alpha,\gamma}
\]

where \( A \oplus B = \begin{bmatrix} A & \vdots \\ \vdots & B \end{bmatrix} \) denotes the direct sum of matrices. Step (6) uses the CRT to decompose \( \mathcal{F}[x]/q(x) \) and \( \mathcal{F}[x]/r(x) \). This step corresponds to the direct sum of matrices \( \mathbb{P}_{\alpha,\beta} \) and \( \mathbb{P}_{\alpha,\gamma} \). Finally permutation matrix \( \mathbb{P} \) maps the concatenation \( \beta, \gamma \) to the ordered list of zeros \( \alpha \) in (7). Given that \( B \) is sparse (8) leads to a fast algorithm.

C. Polynomial algebras for DCT-2 and DCT-4

This subsection introduces polynomial algebras which is connected with DCT-4 and DCT-2. Let us first consider the polynomial algebra associated with DCT-4:

\[
\mathcal{A}_\mathbb{F} = \mathbb{F}[x]/2T_n(x), \quad b = (V_0, \ldots, V_{n-1})
\]

where \( T \) and \( V \) are Chebyshev polynomials of the first and third kind, respectively. This Chebyshev polynomials have the following closed form expressions (\( \cos(\theta) = x \))

\[
T_n(x) = \cos(n\theta), \quad V_n(x) = \frac{\cos((n+1)\theta)}{2}\cos(\frac{\theta}{2}).
\]

\[
\alpha_k = \cos(k + \frac{1}{2})\frac{\pi}{n}, \quad 0 \leq k < n \quad \text{are zeros of} \quad 2T_n(x). \quad \text{in accordance with (3) polynomial transform for algebra (9) is defined as}
\]

\[
P_{\alpha,\beta} = [V_0(\alpha_k)]_{0 \leq k, \ell < n} = \begin{bmatrix} \cos(k + \frac{1}{2})((\ell + 1)\frac{\pi}{2n}) \\ \cos(k + \frac{1}{2})((\ell + 1)\frac{\pi}{2n}) \end{bmatrix}
\]

In order to get the matrix of DCT-4 \( n \) (10) is multiplied from the left by scaling diagonal matrix

\[
D_{n}^{(C4)} = \operatorname{diag}_{0 \leq k < n}(\cos(k + \frac{1}{2})\frac{\pi}{2n})
\]

that yields

\[
\text{DCT-4}_n = \begin{bmatrix} \cos(k + \frac{1}{2})(\ell + \frac{1}{2})\frac{\pi}{2n} \end{bmatrix}_{0 \leq k, \ell < n}. \quad (11)
\]

Eq. (10)-(11) show that DCT-4 is a scaled polynomial transform of the form (4) for the specified polynomial algebra (9).

DCT-2\( n \) is obtained from polynomial algebra

\[
\mathcal{A}_\mathbb{F} = \mathbb{F}[x]/(x - 1)U_{n-1}(x), \quad b = (V_0, \ldots, V_{n-1})
\]

where \( U \) is Chebyshev polynomial of the second kind that can be written as (\( \cos \theta = x \))

\[
U_n(x) = \frac{\sin((n+1)\theta)}{\sin \theta}
\]

Since zeros of \( U_n(x) \) is given by \( \alpha_k = \cos((k+1)\pi)/n, \quad 0 \leq k < n \)

polynomial transform for (12) takes the form

\[
P_{\alpha,\beta} = [V_0(\alpha_k)]_{0 \leq k, \ell < n} = \begin{bmatrix} \cos(k\ell + \frac{\pi}{n})/(\cos \frac{\pi}{n/2}) \end{bmatrix}
\]

To obtain DCT-2 matrix (13) need to be multiplied from the left by the scaling diagonal

\[
D_{n}^{(C2)} = \operatorname{diag}_{0 \leq k < n}(\cos(k + \frac{\pi}{2n}))
\]

Polynomial transform corresponding to discrete trigonometric transform (DTT) is denoted as DTT, for instance DCT-4 stands for the matrix in (10).

In what follows we need skew DCT-4\( r \). In [6] this transform was introduced since it appears to be important building blocks of Cooley-Tukey type of algorithms for DCT. Skew DCT-4\( r \) associates with polynomial algebra

\[
\mathcal{A}_\mathbb{F} = \mathbb{F}[x]/(2T_n(x) - 2\cos r \pi)
\]

with the basis \( b = (V_0, \ldots, V_{n-1}) \), where \( 0 < r < 1 \). The conventional DCT-4\( n \) is the special case of skew DCT-4\( r \) for \( r = 1/2 \).

III. DERIVATION OF FAST DCT-2\( 2^k \) ALGORITHM

In this section the procedure of algebraic derivation of fast DCT-2\( 2^k \) algorithm is given in detail. According to (12) the polynomial algebra corresponding to DCT-2\( 2^k \) is given by

\[
\mathcal{A}_\mathbb{F} = \mathbb{F}[x]/(x - 1)U_{2k-1}(x), \quad b = (V_0, \ldots, V_{2k-1})
\]

Important issue is to choose the base field \( \mathbb{F} \). Since Chebyshev polynomials \( V \) and \( U \) which is included in definition (12) have integer coefficients (for example \( V_2(x) = 4x^2 - 2x - 1 \)), the base field \( \mathbb{F} \) is set to the field of rational numbers \( \mathbb{Q} \). The filed is extended during factorization of polynomial \( U_{2^k-1}(x) \), since the polynomial is not factored over \( \mathbb{Q} \).

It is well known [11] that fast DCT-2\( 2^n \) algorithm can be reduced to fast DCT-2\( 2^k \) and DCT-4\( n \) algorithms. Using factorization for the Chebyshev polynomial of the second kind

\[
U_{2n-1}(x) = U_{n-1}(x) \cdot 2T_n(x),
\]
the algebra \( \mathbb{Q}[x]/(x - 1)U_{2n-1}(x) \) with basis \( b = (V_0, \ldots, V_{2n-1}) \) can be decomposed as
\[
\mathbb{Q}[x]/(x - 1)U_{2n-1}(x) \\
\leftrightarrow \mathbb{Q}[x]/(x - 1)U_{n-1}(x) \oplus \mathbb{Q}[x]/2T_n(x),
\]
that according to (5)–(7) leads to the following fast algorithm [6]
\[
\text{DCT-}2_{2n} = L_{2n}^n(\text{DCT-}2_n \oplus \text{DCT-}4_n)B_{2n},
\]
where \( L_{2n}^n \) is the stride permutation matrix and \( B_{2n} \) is change of basis matrix. \( B_{2n} \) maps basis \( b \) to the concatenation \( (c, d) \), where \( c = d = (V_0, \ldots, V_{n-1}) \) are the basis for subalgebras in the right-hand side of (14). The first \( n \) columns of \( B_{2n} \) are
\[
B_{2n} = \begin{bmatrix} I_n & J_n \\ I_n & -J_n \end{bmatrix},
\]
since the elements \( V_\ell \in b \) for \( 0 \leq \ell < n \) are already contained in \( c \) and \( d \). The rest entries are determined by the following expressions
\[
V_{n+\ell} \equiv V_{n-\ell-1} \mod (x - 1)U_n \quad (16) \\
V_{n+\ell} \equiv -V_{n-\ell-1} \mod 2T_n, \quad (17)
\]
which yields
\[
B_{2n} = \begin{bmatrix} I_n & J_n \\ I_n & -J_n \end{bmatrix}.
\]
(16)–(17) can be induced using the following relation
\[
2T_n = V_n + V_{n-1}, \quad (x - 1)U_{n-1} = V_n - V_{n-1} \quad \text{and} \quad V_n = 2xV_{n-1} - V_{n-2}.
\]
Note that decomposition (14) does not require extension of based field \( \mathbb{Q} \). This leads to multiplication-free change of basis matrix \( B_{2n} \).

When the size of DCT-2 is power of two (15) can be applied recursively to obtain fast algorithm. Thus, the problem of derivation of fast DCT-2^k algorithm reduces to derivation of fast DCT-4^2k-1 algorithm. From the ASP point of view the question is how to factor polynomial \( 2T_n \) (when \( n \) is power of 2) in step. We propose to use the following general recursive formula
\[
2T_n(x) - 2 \cos r\pi = (2T_n(x) - 2 \cos \frac{r\pi}{2}) \\
\times (2T_n(x) - 2 \cos \pi(1 - \frac{r}{2})),
\]
that can be proved using the closed form of \( T_n \), parameter \( r \in (0, 1) \). The special case of (18) for \( r = 1/2 \) specify factorization of \( 2T_n \). Using (18) polynomial algebra related to DCT-4^2n(r) is decomposed as
\[
\mathbb{Q}[\cos \frac{r\pi}{2}] \cdot (2T_n(x) - 2 \cos r\pi) \\
\rightarrow \mathbb{Q}[\cos \frac{r\pi}{2}] \cdot (2T_n(x) - 2 \cos \frac{r\pi}{2}) \oplus \mathbb{Q}[\cos \frac{r\pi}{2}] \cdot (2T_n(x) - 2 \cos \pi(1 - \frac{r}{2})).
\]
The decomposition leads to the following fast algorithm
\[
\text{DCT-}4_{2n}(r) = P \cdot (\text{DCT-}4_n(\frac{r}{2}) \\
\oplus \text{DCT-}4_n(1 - \frac{r}{2})) \cdot B_{2n}^{(C4)}(r),
\]
where \( P \) is a permutation matrix of the form
\[
P = \begin{bmatrix} 1 & I_2 \\ I_2 & \ddots \\ \vdots & \ddots & I_2 \\ 1 & \end{bmatrix},
\]
and \( B_{2n}^{(C4)}(r) \) is the change of basis matrix
\[
B_{2n}(r) = \begin{bmatrix} I_m & (2 \cos \frac{r\pi}{2}I_m - J_m) \\ I_m & (-2 \cos \frac{r\pi}{2}I_m - J_m) \end{bmatrix} \\
= \begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix} \cdot \begin{bmatrix} I_m & -J_m \\ I_m & 2 \cos \frac{r\pi}{2}I_m \end{bmatrix},
\]
which is determined by
\[
V_{n+\ell} \equiv V_{n-\ell-1} + 2 \cos \frac{r\pi}{2}V_n \mod 2T_n - 2 \cos \frac{r\pi}{2} \\
V_{n+\ell} \equiv -V_{n-\ell-1} - 2 \cos \frac{r\pi}{2}V_n \mod 2T_n - 2 \cos \pi(1 - \frac{r}{2}).
\]

Decomposition (19) requires extension of the based field \( \mathbb{Q}[\cos \frac{r\pi}{2}] \) to \( \mathbb{Q}[\cos \frac{r\pi}{2}] \). New elements of the field appears in matrix \( B_{2n}^{(C4)}(r) \).

Joint use of factorizations (15) and (20) leads to the new fast DCT-2_2^k algorithm. The basic operation of the algorithm is multiplication by the matrix \( B_{2n}^{(C4)}(r) \). All nontrivial multiplication concentrate in it that is very similar to butterfly operation in FFT algorithm.

IV. FAST DCT-2_{16} ALGORITHM

In this section the proposed approach is applied to derivation of fast DCT-2_{16} algorithm. At first the transform expressed as a product
\[
\text{DCT-}2_{16} = D_{16}^{(C2)} \cdot \text{DCT-}2_{16},
\]
Then factorization (15) and (20) is applied recursively to obtain fast transform algorithm. Flow graph of this algorithm is shown in Fig. 1 (for simplicity scaling of the output is omitted). Fig. 2 explains the basic building block (BB) of the algorithm that performs the multiplication by matrix (21). All operations inside the BB are implemented on the input \( m \) components vectors. Evaluation of one BB requires \( 3m \) addition and \( m \) multiplication.

The presented 16-point DCT-2 algorithm uses 32 multiplication and 81 addition. However only 17 multiplication constitute the core of algorithm while other 15 is scaling factors that can be compensate in the post-processing. Also the Fig. 1 shows that algorithm include computation of 8-point DCT-2 that requires only 5 multiplication. It is the same result as in [8]. In fact, proposed algorithm can be considered as generalization of Arai’s DCT algorithm since the resulting computational scheme has very low multiplicative complexity and scaling outputs.

\[\text{2You can find MATLAB implementation of the algorithm in the public repository } https://github.com/Mak-Sim/Fast_recursive_DCT\]
V. CONCLUSION

A fast $2^k$-point algorithm of DCT-2 based on ASP is presented. The key features of the algorithm are regularity of the graph (DCT-$2^n/2$ available inside of a DCT-$2^n$) and very low arithmetic complexity (computational core of the DCT-$2^n$ algorithm contains only $\sum_{p=1}^{k-1} 2^p p$ multiplications). Regular graph of proposed algorithm is well suited for development of new parallel-pipeline architecture of DCT processor. Also, the algorithm extends existing space of alternative fast algorithms of the DCT. It can be used by automatic code generation programs that search alternative implementations for the same transform to find the one that is best tuned to desired platform [9], [10].

ACKNOWLEDGMENT

This work was supported by the Belarusian Fundamental Research Fund (F11MS-037).

REFERENCES

[1] K. Rao and P. Yip, Discrete cosine transform: algorithms, advantages, applications, Academic Press, 1990.
[2] M. Parfieniuk and A. Petrovsky, “Structurally orthogonal finite precision implementation of the eight point DCT,” in Proc. Int. Conf. Acoustics, Speech, Signal Processing (ICASSP), 2006.
[3] J. Liang and T. D. Tran, “Fast multiplierless approximations of the DCT with the lifting scheme,” IEEE Trans. on Signal Processing, vol. 49, pp. 3032–3044, 2001.
[4] M. Vashkevich, M. Parfieniuk, and A. Petrovsky, “FPGA implementation of short critical path CORDIC-based approximation of the eight-point DCT,” in Proc. Int. Conf. on Pattern Recognition and Information Processing, 2009, pp. 161–164.
[5] M. Püschel and J. M. F. Moura, “The algebraic approach to the discrete cosine and sine transforms and their fast algorithms,” SIAM J. Comput., vol. 32, no. 5, pp. 1280–1316, 2003.
[6] ——, “Algebraic signal processing theory: Cooley-Tukey type algorithms for DCTs and DSTs,” IEEE Trans. Signal Process., vol. 56, no. 4, pp. 1502–1521, 2008.
[7] C. Loeffler, A. Ligtenberg, and G. S. Moschytz, “Practical fast 1-D DCT algorithms with 11 multiplications,” in Proc. Int. Conf. Acoustics, Speech, Signal Processing (ICASSP), 1989, pp. 988–991.
[8] Y. Arai, T. Agui, and M. Nakajima, “A fast DCT-SQ scheme for images,” Trans. IEICE, vol. 71, no. 11, pp. 1095–1097, 1988.
[9] L. Wanhammer, DSP integrated circuits. Academic Press, 1999.
[10] M. Püschel, J. M. F. Moura, J. Johnson, D. Padua, M. Veloso, B. Singer, J. Xiong, F. Franchetti, A. Gacic, Y. Voronenko, K. Chen, R. W. Johnson, and N. Rizzolo, “SPIRAL: Code generation for DSP transforms,” Proceedings of the IEEE, special issue on “Program Generation, Optimization, and Adaptation”, vol. 93, no. 2, pp. 232–275, 2005.