Exponential stability of impulsive neutral stochastic functional differential equation driven by fractional Brownian motion and Poisson point processes

Brahim Boufoussi¹ · Salah Hajji² · El Hassan Lakhel³

Received: 1 January 2017 / Accepted: 13 October 2017 / Published online: 4 November 2017 © African Mathematical Union and Springer-Verlag GmbH Deutschland 2017

Abstract In this paper we consider a class of impulsive neutral stochastic functional differential equations with variable delays driven simultaneously by a fractional Brownian motion and a Poisson point processes in a Hilbert space. We prove an existence and uniqueness result and we establish some conditions ensuring the exponential decay to zero in mean square for the mild solution by means of the Banach fixed point theory. Finally, an illustrative example is given to demonstrate the effectiveness of the obtained result.

Keywords Mild solution · Impulsive neutral stochastic differential equations · Fractional powers of closed operators · Fractional Brownian motion · Poisson point processes

Mathematics Subject Classification 60H15 · 60G22 · 60J75

1 Introduction

In this paper, we study the existence, uniqueness and asymptotic behavior of mild solutions for a class of impulsive neutral functional stochastic differential equations with poisson jumps described in the form:
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{d[x(t) + g(t, x(t - r(t)))]}{dt} = [Ax(t) + f(t, x(t - \rho(t)))dt + \sigma(t)dB_H(t) \\
\quad + \int_t^\infty h(t, x(t - \theta(t)), y)\tilde{N}(dt, dy), \ t \geq 0, \ t \neq t_k \\
\Delta x(t_k) = I_kx(t^-_k), \ t = t_k, \ k = 1, 2, \ldots, \\
x(t) = \varphi(t), \ -\tau \leq t \leq 0 \ a.s.
\end{array} \right.
\end{align*}
\]

where \(\varphi \in D := D([-\tau, 0])\) the space of càdlàg functions from \([-\tau, 0]\) into \(X\) equipped with the supremum norm \(|\varphi|_D = \sup_{t \in [-\tau, 0]}|\varphi(t)|_X\) and \(A\) is the infinitesimal generator of an analytic semigroup of bounded linear operators, \((S(t))_{t \geq 0}\), in a Hilbert space \(X\), \(B_H^t\) is a fractional Brownian motion on a real and separable Hilbert space \(Y\), \(\rho, \theta : [0, +\infty) \rightarrow [0, \tau]\) \((\tau > 0)\) are continuous and \(f, g : [0, +\infty) \times X \rightarrow X\), \(\sigma : [0, +\infty) \rightarrow L_2^0(Y, X)\), \(h : [0, +\infty) \times X \times \mathcal{U} \rightarrow X\) are appropriate functions. Here \(L_2^0(Y, X)\) denotes the space of all \(Q\)-Hilbert–Schmidt operators from \(Y\) into \(X\) (see Sect. 2 below). Moreover, the fixed moments of time \(t_k\) satisfy \(0 < t_1 < t_2 < \cdots < t_k < \cdots\) and \(\lim_{k \to \infty} t_k = \infty\) \(x(t_k^-)\) and \(x(t_k^+)\) represent the left and right limits of \(x(t)\) at time \(t_k\), \(k = 1, 2, \ldots\) respectively. \(\Delta x(t_k) = x(t_k^+) - x(t_k^-)\) denotes the jump in the state \(x\) at time \(t_k\) with \(I(.) : X \rightarrow X\) determining the size of the jump.

It is known that the theory of the impulsive neutral functional differential equations has become an active area of investigation due to their applications in the fields such as mechanics, electrical engineering, economics, neural networks, medicine, and so forth (see [1,14,20]). On the other hand, stochastic modelling has come to play an important role in many branches of science and industry because any real world system and natural process may be disturbed by many stochastic factors. Therefore, stochastic impulsive systems arise naturally from a wide variety of applications and can be used as an appropriate description of these phenomena of abrupt qualitative dynamical changes of essentially continuous time systems which are disturbed by stochastic factors (see [18,19] and the references therein).

As a generalization of the classical Brownian motion, fBm heavily depends on a parameter \(H \in (0, 1)\) called as the Hurst index [15]. When \(H = 1/2\), the fBm is a standard Brownian motion. When \(H \neq 1/2\) the fBm \(B_H^t\) is not a semimartingale (see Biagini et al. [2]), we can not use the classical Itô theory to construct a stochastic calculus with respect to fBm. Especially, when \(H > 1/2\), fBm has a long range dependence. This property makes this process as a useful driving noise in models appeared in telecommunications networks, finance markets, biology, and other fields (see [8,10]). Since some physical phenomena are naturally modeled by stochastic partial differential equations and the randomness can be described by a fBm, it is important to study the existence and exponential stability of infinite dimensional equations with a fBm. Many studies of the solutions of stochastic equations in an infinite dimensional space with a fBm have been emerged recently, see [3–7,12,13,19].

Motivated by the previously mentioned papers, we will study the existence, uniqueness and exponential stability in mean square of a mild solution of the impulsive neutral stochastic partial functional differential equations. On the other hand, to the best of our knowledge, there is no paper which investigates the study of impulsive neutral stochastic functional differential equations with delays driven both by fractional Brownian motion and by Poisson point processes. Thus, we will make the first attempt to study such problem in this paper.

The paper is organized as follows. In Sect. 2 we introduce some notations, concepts, and basic results about fractional Brownian motion, Poisson point processes, Wiener integral over Hilbert spaces and we recall some preliminary results about analytic semi-groups and fractional power associated to its generator. In Sect. 3 by the Banach fixed point theorem we
consider a sufficient condition for the existence, uniqueness and exponential decay to zero in mean square for mild solutions of Eq. (1.1). In Sect. 4 we give an example to illustrate the efficiency of the obtained result.

2 Preliminaries

In this section, we collect some notions, conceptions and lemmas on Wiener integrals with respect to an infinite dimensional fractional Brownian and we recall some basic results about analytical semi-groups and fractional powers of their infinitesimal generators, which will be used throughout the whole of this paper.

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete probability space satisfying the usual condition, which means that the filtration is right continuous increasing family and \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets.

Let \((\mathcal{U}, \mathcal{E}, \nu(du))\) be a \(\sigma\)-finite measurable space. Given a stationary Poisson point process \((p_t)_{t > 0}\), which is defined on \((\Omega, \mathcal{F}, \mathbb{P})\) with values in \(\mathcal{U}\) and with characteristic measure \(\nu\) (see \([11]\)). We will denote by \(N(t, du)\) be the counting measure of \(p_t\) such that \(\tilde{N}(t, A) := \mathbb{E}(N(t, A)) = t\nu(A)\) for \(A \in \mathcal{E}\). Define \(\tilde{N}(t, du) := N(t, du) - t\nu(du)\), the Poisson martingale measure generated by \(p_t\).

Consider a time interval \([0, T]\) with arbitrary fixed horizon \(T\) and let \(\{\beta^H(t), t \in [0, T]\}\) the one-dimensional fractional Brownian motion with Hurst parameter \(H \in (1/2, 1)\). This means by definition that \(\beta^H\) is a centered Gaussian process with covariance function:

\[
R_H(s, t) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right).
\]

Moreover \(\beta^H\) has the following Wiener integral representation:

\[
\beta^H(t) = \int_0^t K_H(t, s) \, d\beta(s),
\]

where \(\beta = \{\beta(t) : t \in [0, T]\}\) is a Wiener process, and \(K_H(t; s)\) is the kernel given by

\[
K_H(t, s) = c_H s^{\frac{1}{2} - H} \int_s^t (u - s)^{H - \frac{3}{2}} u^{H - \frac{1}{2}} \, du,
\]

for \(t > s\), where \(c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H,H-H/2)}}\) and \(\beta(\cdot, \cdot)\) denotes the Beta function. We put \(K_H(t, s) = 0\) if \(t \leq s\).

We will denote by \(\mathcal{H}\) the reproducing kernel Hilbert space of the fBm. In fact \(\mathcal{H}\) is the closure of set of indicator functions \(\{1_{[0,t]}, t \in [0, T]\}\) with respect to the scalar product

\[
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).
\]

The mapping \(1_{[0,t]} \rightarrow \beta^H(t)\) can be extended to an isometry between \(\mathcal{H}\) and the first Wiener chaos and we will denote by \(\beta^H(\varphi)\) the image of \(\varphi\) by the previous isometry.

We recall that for \(\psi, \varphi \in \mathcal{H}\) their scalar product in \(\mathcal{H}\) is given by

\[
\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H - 1) \int_0^T \int_0^T \psi(s) \varphi(t) |t - s|^{2H-2} \, ds \, dt.
\]
Let us consider the operator $K_H^\alpha$ from $\mathcal{H}$ to $L^2([0, T])$ defined by

$$(K_H^\alpha \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K}{\partial r}(r, s) dr.$$ 

We refer to [16] for the proof of the fact that $K_H^\alpha$ is an isometry between $\mathcal{H}$ and $L^2([0, T])$. Moreover for any $\varphi \in \mathcal{H}$, we have

$$\beta_H^\alpha(\varphi) = \int_0^T (K_H^\alpha \varphi)(t) d\beta(t).$$

It follows from [16] that the elements of $\mathcal{H}$ may be not functions but distributions of negative order. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions $\psi$ such that

$$\|\psi\|^2_{|\mathcal{H}|} := \alpha_H \int_0^T \int_0^T |\psi(s)||\psi(t)||s - t|^{2H-2} ds dt < \infty,$$

where $\alpha_H = H(2H - 1)$. We have the following Lemma (see [16]).

**Lemma 1** The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$ and we have the following inclusions

$$L^2([0, T]) \subseteq L^{1/H}([0, T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H},$$

and for any $\varphi \in L^2([0, T])$, we have

$$\|\psi\|^2_{|\mathcal{H}|} \leq 2HT^{2H-1} \int_0^T |\psi(s)|^2 ds.$$
Now, let \( \phi(s); s \in [0, T] \) be a function with values in \( L^0_2(Y, X) \). The Wiener integral of \( \phi \) with respect to \( B^H \) is defined by
\[
\int_0^t \phi(s)dB^H(s) = \sum_{n=1}^\infty \int_0^t \sqrt{\bar{\lambda}_n} \phi(s)e_n dB^H_n(s) = \sum_{n=1}^\infty \int_0^t \sqrt{\bar{\lambda}_n}(K_H^\ast(\phi e_n))(s)dB_n(s),
\]
where \( \beta_n \) is the standard Brownian motion used to present \( \beta_n^H \) as in (2.1).

Now, we end this subsection by stating the following result which is fundamental to prove our result. It can be proved by similar arguments as those used to prove Lemma 2 in [5].

Lemma 2 If \( \psi: [0, T] \to L^0_2(Y, X) \) satisfies \( \int_0^T \| \psi(s) \|_{L^2_2}^2 ds < \infty \) then the above sum in (2.2) is well defined as a \( X \)-valued random variable and we have
\[
\mathbb{E}\| \int_0^t \psi(s)dB^H(s) \|_2^2 \leq 2Ht^{2H-1} \int_0^t \| \psi(s) \|_{L^2_2}^2 ds.
\]

Now we turn to state some notations and basic facts about the theory of semi-groups and fractional power operators.

Let \( A : D(A) \to X \) be the infinitesimal generator of an analytic semigroup, \( (S(t))_{t \geq 0} \), of bounded linear operators on \( X \). For the theory of strongly continuous semigroup, we refer to [9,17]. We will point out here some notations and properties that will be used in this work. It is well known that there exist \( M \geq 1 \) and \( \lambda \in \mathbb{R} \) such that \( \|S(t)\| \leq Me^{\lambda t} \) for every \( t \geq 0 \). If \( (S(t))_{t \geq 0} \) is a uniformly bounded and analytic semigroup such that \( 0 \in \rho(A) \), where \( \rho(A) \) is the resolvent set of \( A \), then it is possible to define the fractional power \( (-A)^{\alpha} \) for \( 0 < \alpha \leq 1 \), as a closed linear operator on its domain \( D(-A)^{\alpha} \). Furthermore, the subspace \( D(-A)^{\alpha} \) is dense in \( X \), and the expression
\[
\|h\|_{\alpha} = \|(-A)^{\alpha}h\|
\]
defines a norm in \( D(-A)^{\alpha} \). If \( X_\alpha \) represents the space \( D(-A)^{\alpha} \) endowed with the norm \( \| \cdot \|_{\alpha} \), then the following properties are well known (cf. [17, p. 74]).

Lemma 3 Suppose that the preceding conditions are satisfied.

1. Let \( 0 < \alpha \leq 1 \). Then \( X_\alpha \) is a Banach space.
2. If \( 0 < \beta \leq \alpha \) then the injection \( X_\alpha \hookrightarrow X_\beta \) is continuous.
3. For every \( 0 < \alpha \leq 1 \) there exists \( M_\alpha > 0 \) such that
\[
\|(-A)^{\alpha}S(t)\| \leq M_\alpha t^{-\alpha}e^{-\lambda t}. \quad t > 0, \quad \lambda > 0.
\]

3 Main results

In this section, we consider existence, uniqueness and exponential stability of mild solution to Eq. (1.1). Our main method is the Banach fixed point principle. First we define the space \( S_\varphi \) of the càdlàg processes \( x(t) \) as follows:

Definition 4 Let the space \( S_\varphi \) denote the set of all càdlàg processes \( x(t) \) such that \( x(t) = \varphi(t) \) \( t \in [-\tau, 0] \) and there exist some constants \( M^* = M^*(\varphi, a) > 0 \) and \( a > 0 \)
\[
\mathbb{E}\|x(t)\|_{X}^2 \leq M^* e^{-at}, \quad \forall \ t \geq 0.
\]

Definition 5 \( \| \cdot \|_{S_\varphi} \) denotes the norm in \( S_\varphi \) which is defined by
\[
\|x\|_{S_\varphi} := sup_{t \geq 0} \mathbb{E}\|x(t)\|_{X}^2 \quad \text{for} \ x \in S_\varphi.
\]
Remark 6 It is routine to check that $S_{\phi}$ is a Banach space endowed with the norm $\| \cdot \|_{S_{\phi}}$.

In order to obtain our main result, we assume that the following conditions hold.

($H.1$) $A$ is the infinitesimal generator of an analytic semigroup, $(S(t))_{t \geq 0}$, of bounded linear operators on $X$. Further, to avoid unnecessary notations, we suppose that $0 \notin \rho(A)$, and that, see Lemma 3,
\[
\|S(t)\| \leq M e^{-\lambda t} \quad \text{and} \quad \|(-A)^{1-\beta} S(t)\| \leq \frac{M_{1-\beta}}{t^{1-\beta}}
\]
for some constants $M$, $\lambda$, $M_{1-\beta}$ and every $t \in [0, T]$.

($H.2$) There exist positive constant $K_1 > 0$ such that, for all $t \in [0, T]$ and $x, y \in X$
\[
\|f(t, x) - f(t, y)\|^2 \leq K_1 \|x - y\|^2.
\]

($H.3$) There exist constants $0 < \beta < 1$, $K_2 > 0$ such that the function $g$ is $X_{\beta}$-valued and satisfies for all $t \in [0, T]$ and $x, y \in X$
\[
\|(-A)^{\beta} g(t, x) - (-A)^{\beta} g(t, y)\|^2 \leq K_2 \|x - y\|^2.
\]

($H.4$) The function $(-A)^{\beta} g$ is continuous in the quadratic mean sense:

For all $x \in D([0, T], L^2(\Omega, X)), \quad \lim_{t \to \bar{t}} \|(-A)^{\beta} g(t, x(t)) - (-A)^{\beta} g(s, x(s))\|^2 = 0.$

($H.5$) There exists some $\gamma > 0$ such that the function $\sigma : [0, +\infty) \to L^2_0(Y, X)$ satisfies
\[
\int_0^\infty e^{2\gamma s} \|\sigma(s)\|^2_{L^2_0} ds < \infty.
\]

($H.6$) There exist positive constant $K_3 > 0$ such that, for all $t \in [0, T]$ and $x, y \in X$
\[
\int_{\mathcal{U}} \|h(t, x, z) - h(t, y, z)\|^2_{\chi} v(dz) \leq K_3 \|x - y\|^2_{\chi}.
\]

We further assume that $g(t, 0) = f(t, 0) = h(t, 0, z) = 0$ for all $t \geq 0$ and $z \in \mathcal{U}$.

($H.7$) $I_k \in C(X, X)$ and there exists a positive constant $q_k$ such that $\|I_k(x) - I_k(y)\| \leq q_k \|x - y\|$ and $I_k(0) = 0, k=1,2,\ldots,$ for each $x, y \in X$.

Moreover, we assume that $\varphi \in C([-\tau, 0], L^2(\Omega, X))$.

Similar to the deterministic situation we give the following definition of mild solutions for Eq. (1.1).

Definition 7 An $X$-valued process $\{x(t), \ t \in [-\tau, T]\}$, is called a mild solution of equation (1.1) if
(i) $x(.)$ has càdlàg path, and \( \int_0^T \|x(t)\|^2 dt < \infty \) almost surely;
(ii) $x(t) = \varphi(t), \ -\tau \leq t \leq 0 \ a.s.$
(iii) For arbitrary $t \in [0, T], x(t)$ satisfies the following integral equation
Theorem 8 Suppose that (H.1) – (H.7) hold. If the following conditions are satisfied,

(i) there exists a constant q such that 

\[ q_k \leq q(t_k - t_{k-1}), \quad k = 1, 2, \ldots, \]

(ii) \( K_2 \|(-A)^{-\beta}\|^2 + K_2 M_1^{-\beta} \lambda^{-2\beta} \Gamma(\beta)^2 + K_1 M_2 \lambda^{-2} + M_3 (2\lambda)^{-1} + M_4 \frac{q^2}{\lambda^2} < \frac{1}{\xi}, \)

where \( \Gamma(.) \) is the Gamma function, \( M_{1-\alpha} \) is the corresponding constant in Lemma 3. If the initial value \( \phi(t) \) satisfies

\[ E\|\phi(t)\|^2 \leq M_0 E|\phi|^2 e^{-at}, \quad t \in [-\tau, 0], \]

for some \( M_0 > 0, a > 0; \) then, for all \( T > 0, \) the Eq. (1.1) has a unique mild solution on \([-\tau, T]\) and is exponential decay to zero in mean square, i.e., there exists a pair of positive constants \( a > 0 \) and \( M^* = M^*(\phi, a) \) such that

\[ E\|x(t)\|^2 \leq M^* e^{-at}, \quad \forall t \geq 0. \]

Proof Define the mapping \( \Psi \) on \( S_\phi \) as follows:

\[ \Psi(x)(t) := \phi(t), \quad t \in [-\tau, 0], \]

and for \( t \in [0, T] \)

\[ \Psi(x)(t) = S(t)(\phi(0) + g(0, \phi(-r(0)))) - g(t, x(t - r(t))) \]

\[ - \int_0^t AS(t - s)g(s, x(s - r(s)))ds + \int_0^t S(t - s)f(s, x(s - \rho(s)))ds \]

\[ + \int_0^t S(t - s)\sigma(s)dB^H(s) \]

\[ + \int_0^t \int_{\mathcal{U}} S(t - s)h(s, x(s - \theta(s)), y)\tilde{N}(ds, dy) + \sum_{0 < \tau_k < t} S(t - \tau_k)I_k(x(t_{\tau_k}^-)). \]

Then it is clear that to prove the existence of mild solutions to Eq. (1.1) is equivalent to find a fixed point for the operator \( \Psi. \)

We will show by using Banach fixed point theorem that \( \Psi \) has a unique fixed point. First we show that \( \Psi(S_\phi) \subset S_\phi. \)
Let $x(t) \in S_\psi$, then we have
\[
\mathbb{E}\|\Psi(x(t))\|^2 \leq 7\mathbb{E}\|S(t)(\varphi(0) + g(0, \varphi(-r(0))))\|^2 \\
+ 7\mathbb{E}\|g(t, x(t - r(t)))\|^2 + 7\mathbb{E}\|\int_0^t A S(t - s) g(s, x(s - r(s)))ds\|^2 \\
+ 7\mathbb{E}\|\int_0^t S(t - s) f(s - \rho(s))ds\|^2 + 7\mathbb{E}\|\int_0^t (t - s)\sigma(s)dB^H(s)\|^2 \\
+ 7\mathbb{E}\|\int_0^t \int_{I_k} S(t - s)h(s, x(s - \theta(s)), y)\tilde{N}(ds, dy)\|^2 \\
+ 7\mathbb{E}\|\sum_{0 \leq t_k < t} S(t - t_k)I_k(x(t_k^-))\|^2 \\
:= 7 \sum_{k=1}^n J_k. \tag{3.1}
\]

Now, let us estimate the terms on the right of the inequality (3.1).

Let $M^* = M^*(\varphi, a) > 0$ and $a > 0$ such that
\[
\mathbb{E}\|x(t)\|^2 \leq M^*e^{-at}, \quad \forall \ t \geq 0.
\]

Without loss of generality we may assume that $0 < a < \lambda$. Then, by assumption (H.1) we have
\[
J_1 \leq M^2\mathbb{E}\|\varphi(0) + g(0, \varphi(-r(0)))\|^2 e^{-\lambda t} \leq C_1 e^{-\lambda t} \tag{3.2}
\]
where $C_1 = M^2\mathbb{E}\|\varphi(0) + g(0, \varphi(-r(0)))\|^2 < +\infty$.

By using assumption (H.3) and the fact that the operator $(-A)^{-\beta}$ is bounded, we obtain that
\[
J_2 \leq \|(A)^{-\beta}\|^2\mathbb{E}\|(A)^\beta g(t, x(t - r(t))) - (A)^\beta g(t, 0)\|^2 \\
\leq K_2 \|(A)^{-\beta}\|^2\mathbb{E}\|x(t - r(t))\|^2 \\
\leq K_2 \|(A)^{-\beta}\|^2(\mathbb{E}\|x(t - r(t))\| + M_0\mathbb{E}\|\varphi(0)\|^2) \\
\leq K_2 \|(A)^{-\beta}\|^2(\mathbb{E}\|x(t - r(t))\| + M_0\mathbb{E}\|\varphi(0)\|^2) \\
\leq C_2 e^{-at} \tag{3.3}
\]
where $C_2 = K_2 \|(A)^{-\beta}\|^2(\mathbb{E}\|x(t - r(t))\| + M_0\mathbb{E}\|\varphi(0)\|^2) e^{at} < +\infty$.

To estimate $J_3$, we use the trivial identity
\[
c^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1}e^{-ct}dt, \quad \forall \ c > 0. \tag{3.4}
\]

Using Hölder’s inequality, Lemma 3 together with assumption (H.3) and the identity (3.4), we get
\[
J_3 \leq \mathbb{E}\|\int_0^t A S(t - s) g(s, x(s - r(s)))ds\|^2 \\
\leq \int_0^t \|(A)^{1-\beta} S(t - s)\|ds \int_0^t \|(A)^{1-\beta} S(t - s)\| \mathbb{E}\|(-A)^\beta g(s, x(s - r(s)))\|^2 ds \\
\leq M_{1-\beta}^2 K_2 \int_0^t (t - s)^{\beta-1}e^{-\lambda(t-s)}ds \int_0^t (t - s)^{\beta-1}e^{-\lambda(t-s)}\mathbb{E}\|x(s - r(s))\|^2 ds
\]
\[
\leq M_{1-\beta}^2 K_2 \lambda^{-\beta} \Gamma(\beta) \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} (M^* + M_0 \mathbb{E}|\varphi|_{D}^2) e^{-as} e^{at} ds
\]

\[
\leq M_{1-\beta}^2 K_2 \lambda^{-\beta} \Gamma(\beta) (M^* + M_0 \mathbb{E}|\varphi|_{D}^2) e^{-at} e^{at} \int_0^t (t-s)^{\beta-1} e^{(a-\lambda)(t-s)} ds
\]

\[
\leq M_{1-\beta}^2 K_2 \lambda^{-\beta} \lambda^2 (\lambda - \alpha)^{-1} (M^* + M_0 \mathbb{E}|\varphi|_{D}^2) e^{-at} e^{at}
\]

\[
\leq C_3 e^{-at}
\]

where \( C_3 = M_{1-\beta}^2 K_2 \lambda^{-\beta} \lambda^2 (\lambda - \alpha)^{-1} (M^* + M_0 \mathbb{E}|\varphi|_{D}^2) e^{-at} < +\infty. \)

Similar computations can be used to estimate the term \( J_4. \)

\[
J_4 \leq \mathbb{E}\left\| \int_0^t S(t-s) f(s, x(s - \rho(s))) ds \right\|^2
\]

\[
\leq M^2 K_1 \int_0^t e^{-\lambda(t-s)} ds \int_0^t e^{-\lambda(t-s)} \mathbb{E}\|x(s - \rho(s))\|^2 ds
\]

\[
\leq M^2 K_1 \lambda^{-1} \int_0^t e^{-\lambda(t-s)} (M^* + M_0 \mathbb{E}|\varphi|_{D}^2) e^{-as} e^{at} ds
\]

\[
\leq M^2 K_1 \lambda^{-1} (M^* + M_0 \mathbb{E}|\varphi|_{D}^2) e^{-at} e^{at} \int_0^t e^{(a-\lambda)(t-s)} ds
\]

\[
\leq M^2 K_1 \lambda^{-1} (\lambda - \alpha)^{-1} (M^* + M_0 \mathbb{E}|\varphi|_{D}^2) e^{-at} e^{at}
\]

\[
\leq C_4 e^{-at}.
\]

By using Lemma 2, we get that

\[
J_5 \leq \mathbb{E}\left\| \int_0^t S(t-s) \sigma(s) d B^H (s) \right\|^2
\]

\[
\leq 2 M^2 H t^{2H-1} \int_0^t e^{-2\lambda(t-s)} \| \sigma(s)\|_{L_2}^2 ds,
\]

(3.7)

If \( \gamma < \lambda, \) then the following estimate holds

\[
J_5 \leq 2 M^2 H t^{2H-1} \int_0^t e^{-2\lambda(t-s)} e^{-2\gamma(t-s)} e^{2\gamma(t-s)} \| \sigma(s)\|_{L_2}^2 ds
\]

\[
\leq 2 M^2 H t^{2H-1} e^{-2\gamma t} \int_0^t e^{-2(\lambda-\gamma)(t-s)} e^{2\gamma s} \| \sigma(s)\|_{L_2}^2 ds
\]

\[
\leq 2 M^2 H t^{2H-1} e^{-2\gamma t} \int_0^t e^{-2(\lambda-\gamma)(t-s)} e^{2\gamma s} \| \sigma(s)\|_{L_2}^2 ds
\]

\[
\leq 2 M^2 H t^{2H-1} e^{-2\gamma t} \int_0^t e^{2\gamma s} \| \sigma(s)\|_{L_2}^2 ds,
\]

(3.8)

If \( \gamma > \lambda, \) then the following estimate holds

\[
J_5 \leq 2 M^2 H t^{2H-1} e^{-2\lambda t} \int_0^t e^{2\gamma s} \| \sigma(s)\|_{L_2}^2 ds
\]

(3.9)

In virtue of (3.7), (3.8) and (3.9) we obtain

\[
J_5 \leq C_5 e^{-\min(\lambda, \gamma) t}
\]

(3.10)

where \( C_5 = 2 M^2 H \left( \sup_{t \geq 0} t^{2H-1} e^{-\min(\lambda, \gamma) t} \right) \int_0^\infty e^{2\gamma s} \| \sigma(s)\|_{L_2}^2 ds < +\infty. \)
On the other hand, by assumptions (H.1) and (H.6), we get

\[ J_6 \leq \mathbb{E} \left[ \int_0^t \int_{U^t} S(t-s)h(s, x(s - \theta(s)), y) \tilde{N}(ds, dy) \right]^2 \]

\[ \leq M^2 \mathbb{E} \int_0^t e^{-2\lambda(t-s)} \int_{U^t} \|h(s, x(s - \theta(s)), y)\|^2 \nu(dy) ds \]

\[ \leq M^2 K_3 \int_0^t e^{-2\lambda(t-s)} \mathbb{E} \|x(s - \theta(s))\|^2 ds \]

\[ \leq M^2 K_3 \int_0^t e^{-2\lambda(t-s)} (M^* + M_0 \mathbb{E} \|\varphi\|_D^2) e^{-at} e^{\sigma_t} ds \]

\[ \leq M^2 K_3 (M^* + M_0 \mathbb{E} \|\varphi\|_D^2) e^{-at} \int_0^t e^{(-2\lambda+a)(t-s)} ds \]

\[ \leq C_6 e^{-at}, \quad (3.11) \]

where \( C_6 = M^2 K_3 (M^* + M_0 \mathbb{E} \|\varphi\|_D^2) e^{\sigma} (2\lambda - a)^{-1} < +\infty \).

Now, we estimate the impulsive term, by assumption (H.7) and from condition (i) of the Theorem 8, we obtain

\[ J_7 \leq \mathbb{E} \left( \sum_{0 < t_k < t} Me^{-\lambda(t-t_k)} q_k \|x(t_k^-)\| \right)^2 \]

\[ \leq \mathbb{E} \left( \sum_{0 < t_k < t} Me^{-\lambda(t-t_k)} q(t_k - t_{k-1}) \|x(t_k^-)\| \right)^2 \]

\[ \leq \mathbb{E} \left( \int_0^t Me^{-\lambda(t-s)} q \|x(s)\| ds \right)^2 \]

\[ \leq M^2 q^2 \left( \int_0^t e^{-\lambda(t-s)} ds \right) \left( \int_0^t e^{-\lambda(t-s)} \mathbb{E} \|x(s)\|^2 ds \right) \]

\[ \leq M^2 q^2 \lambda^{-1} \left( \int_0^t e^{-\lambda(t-s)} M^* e^{-as} ds \right) \]

\[ \leq M^* M^2 q^2 \lambda^{-1} e^{-at} \left( \int_0^t e^{(a-\lambda)(t-s)} ds \right) \]

\[ \leq M^2 q^2 \lambda^{-1} M^* (\lambda - a)^{-1} e^{-at} \]

\[ \leq C_7 e^{-at}. \quad (3.12) \]

Inequalities (3.2), (3.3), (3.5), (3.6),(3.10), (3.11) and (3.12) together imply that

\[ \mathbb{E} \|\Psi(x)(t)\|^2 \leq \overline{M} e^{-at}, \quad \text{for all } t \geq 0 \text{ and for some constants } \overline{M} > 0, \overline{a} > 0. \]

Next we show that \( \Psi(x)(t) \) is càdlàg process on \( S_\varphi \). Let \( 0 < t < T \) and \( h > 0 \) be sufficiently small. Then for any fixed \( x(t) \in S_\varphi \), we have

\[ \mathbb{E} \|\Psi(x)(t+h) - \Psi(x)(t)\|^2 \]

\[ \leq 7 \mathbb{E} \| (S(t+h) - S(t)) (\varphi(0) + g(0, \varphi(-r(0)))) \|^2 \]

\[ + 7 \mathbb{E} \| g(t + h, x(t + h - r(t + h))) - g(t, x(t - r(t))) \|^2 \]

\( \mathbb{E} \) Springer
+7\mathbb{E}\left[ \int_0^{t+h} A S(t + h - s)g(s, x(s - r(s)))ds - \int_0^t A S(t - s)g(s, x(s - r(s)))ds \right]^2

+7\mathbb{E}\left[ \int_0^{t+h} S(t + h - s)f(s - \rho(s))ds - \int_0^t S(t - s)f(s - \rho(s))ds \right]^2

+7\mathbb{E}\left[ \int_0^{t+h} (S(t + h - s)\sigma(s)dB^H(s) - \int_0^t S(t - s)\sigma(s)dB^H(s)) \right]^2

\quad +7\mathbb{E}\left[ \int_0^{t+h} \int_U S(t + h - s)h(s, x(s - \theta(s)), y)\tilde{N}(ds, dy) \right] - \int_0^t \int_U S(t - s)h(s, x(s - \theta(s)), y)\tilde{N}(ds, dy)\right]^2

+7\mathbb{E}\left[ \sum_{0 < t_k < t+h} S(t + h - t_k)I_k(x(t_k^-)) - \sum_{0 < t_k < t} S(t - t_k)I_k(x(t_k^-)) \right]^2

= 7 \sum_{1 \leq i \leq 7} F_i(h).

It can be easily obtained that \( \mathbb{E}\|F_i(h)\|^2 \rightarrow 0 \), for \( i = 1, 2, \ldots, 5 \), as \( h \rightarrow 0 \).

Moreover, for the term \( I_6(h) \), by Hölder’s inequality and assumption (H.1), we obtain

\[
I_6(h) \leq 2\mathbb{E}\left[ \int_0^t \int_U (S(t + h - s) - S(t - s))h(s, x(s - \theta(s)), y)\tilde{N}(ds, dy) \right]^2
\]

\[+2\mathbb{E}\left[ \int_0^{t+h} \int_U S(t + h - s)h(s, x(s - \theta(s)), y)\tilde{N}(ds, dy) \right]^2 \]

\[\leq 2M^2\|S(h) - I\|\mathbb{E}\left[ \int_0^t \int_U e^{-2\lambda(t-s)}\|h(s, x(s - \theta(s)), y)\|^2v(dy)ds \right]
\]

\[+2M^2\mathbb{E}\int_0^{t+h} \int_U e^{-2\lambda(t+h-s)}\|h(s, x(s - \theta(s)), y)\|^2v(dy)ds \quad (3.13)\]

By assumption (H.6), we have

\[
\mathbb{E}\int_0^t \int_U e^{-2\lambda(t-s)}\|h(s, x(s - \theta(s)), y)\|^2v(dy)ds \leq K_3 \int_0^t e^{-2\lambda(t-s)}\mathbb{E}\|x(s - \theta(s))\|^2ds
\]

\[\leq K_3 \int_0^t e^{-2\lambda(t-s)}(M^* + M_0\mathbb{E}|\varphi|^2_D)e^{-as}e^{a\tau}ds
\]

\[\leq K_3(M^* + M_0\mathbb{E}|\varphi|^2_D)e^{a\tau}(2\lambda - a)^{-1}e^{-at} \quad (3.14)\]

Inequality (3.14) imply that there exist a constant \( B > 0 \) such that

\[
\mathbb{E}\int_0^t \int_U e^{-2\lambda(t-s)}\|h(s, x(s - \theta(s)), y)\|^2v(dy)ds \leq B \quad (3.15)
\]

Using the strong continuity of \( S(t) \) together with inequalities (3.13) and (3.15) we obtain that \( I_6(h) \rightarrow 0 \) as \( h \rightarrow 0 \).

Similarly, we can verify that \( \mathbb{E}\|F_7(h)\|^2 \rightarrow 0 \), as \( h \rightarrow 0 \).

The above arguments show that \( \Psi(x)(t) \) is càdlàg process. Then, we conclude that \( \Psi(S_\varphi) \subset S_\varphi \).
Now, we are going to show that $\Psi : S_\varphi \to S_\varphi$ is a contraction mapping. For this end, fix $x, \, y \in S_\varphi$, we have

$$
\mathbb{E}\|\Psi(x)(t) - \Psi(y)(t)\|^2 \\
\leq 5\mathbb{E}\|g(t, x(t - r(t))) - g(t, y(t - r(t)))\|^2 \\
+ 5\mathbb{E}\int_0^t AS(t - s)(g(s, x(s - r(s))) - g(s, y(s - r(s))))ds \|^2 \\
+ 5\mathbb{E}\int_0^t S(t - s)(f(s, x(s - \rho(s))) - f(s, y(s - \rho(s))))ds \|^2 \\
+ 5\mathbb{E}\int_0^t S(t - s)\int_{t_d} h(s, x(s - \theta(s)), z) - h(s, y(s - \theta(s)), z)\tilde{N}(ds, dz) \|^2 \\
+ 5\mathbb{E}\sum_{0 < t_k < t} S(t - t_k)[I_k(x(t^-_k)) - I_k(y(t^-_k))] \|^2 \\
:= 5(J_1 + J_2 + J_3 + J_4 + J_5). \tag{3.16}
$$

We estimate the various terms of the right hand of (3.16) separately.

For the first term, we have

$$
J_1 \leq \mathbb{E}\|g(t, x(t - r(t))) - g(t, y(t - r(t)))\|^2 \\
\leq K_2\|(-A)^{-\beta}\|^2 \mathbb{E}\|x(s - r(s)) - y(s - r(s))\|^2 \\
\leq K_2\|(-A)^{-\beta}\|^2 \sup_{s \geq 0} \mathbb{E}\|x(s) - y(s)\|^2. \tag{3.17}
$$

For the second term, combing Lemma 3 and Hölder’s inequality, we get

$$
J_2 \leq \mathbb{E}\int_0^t AS(t - s)(g(s, x(s - r(s))) - g(s, y(s - r(s))))ds \|^2 \\
\leq K_2 M^2_{1-\beta} \int_0^t (t - s)^{\beta - 1}e^{-\lambda(t-s)}ds \int_0^t (t - s)^{\beta - 1}e^{-\lambda(t-s)}\mathbb{E}\|x(s - r(s)) \\
- y(s - r(s))\|^2ds \\
\leq K_2 M^2_{1-\beta} \lambda^{-\beta} \Gamma(\beta) \int_0^t (t - s)^{\beta - 1}e^{-\lambda(t-s)}ds (\sup_{s \geq 0} \mathbb{E}\|x(s) - y(s)\|^2) \\
\leq K_2 M^2_{1-\beta} \lambda^{-\beta} \Gamma(\beta) \sup_{s \geq 0} \mathbb{E}\|x(s) - y(s)\|^2. \tag{3.18}
$$

For the third term, by assumption (H.2), we get that

$$
J_3 \leq \mathbb{E}\int_0^t S(t - s)(f(s, x(s - \rho(s))) - f(s, y(s - \rho(s))))ds \|^2 \\
\leq K_1 M^2 \int_0^t e^{-\lambda(t-s)}ds \int_0^t e^{-\lambda(t-s)}\mathbb{E}\|x(s - \rho(s)) - y(s - \rho(s))\|^2ds \\
\leq K_1 M^2 \lambda^{-\beta} \sup_{s \geq 0} \mathbb{E}\|x(s) - y(s)\|^2. \tag{3.19}
$$
For the fourth term, by using assumption (\(H.6\)), we get

\[
J_4 \leq \mathbb{E} \left\| \int_0^t S(t-s) \int_{t_k} \left( h(s, x(s-\theta(s)), z) - h(s, y(s-\theta(s)), z) \right) \tilde{N}(ds, dz) \right\|^2
\]

\[
\leq M^2 \mathbb{E} \left\| \int_0^t e^{-2\lambda(t-s)} \left( \int_{t_k} \left( h(s, x(s-\theta(s)), z) - h(s, y(s-\theta(s)), z) \right) \right) \nu(dz)ds \right\|^2
\]

\[
\leq M^2 K_3(2\lambda)^{-1} \sup_{s \geq 0} \mathbb{E} \| x(s) - y(s) \|^2. \tag{3.20}
\]

For the last term, we have

\[
J_5 \leq \mathbb{E} \left\| \sum_{0 < t_k < t} S(t-t_k)[I_k(x(t_k^-)) - I_k(y(t_k^-))] \right\|^2
\]

\[
\leq M^2 q^2 \int_0^t e^{-\lambda(t-s)} ds \int_0^t e^{-\lambda(t-s)} \mathbb{E} \| x(s) - y(s) \|^2 ds
\]

\[
\leq M^2 q^2 \lambda^{-2} \left( \sup_{t \geq 0} \mathbb{E} \| x(t) - y(t) \|^2 \right). \tag{3.21}
\]

Thus, inequality (3.17), (3.18), (3.19), (3.20) and (3.21) together imply

\[
\sup_{t \geq 0} \mathbb{E} \| \Psi(x)(t) - \Psi(y)(t) \|^2 \leq 5(K_2\|\|A\|-\|\|\|A\|\|) + K_2 M_1^2 \lambda^{-2}\beta^2 + K_1 M^2 \lambda^{-2}
\]

\[
+ M^2 K_3(2\lambda)^{-1} + M^2 q^2 \lambda^{-2} \left( \sup_{t \geq 0} \mathbb{E} \| x(t) - y(t) \|^2 \right).
\]

Therefore by the condition (ii) of the Theorem 8 it follows that \(\Psi\) is a contractive mapping. Thus by the Banach fixed point theorem \(\Psi\) has the fixed point \(x(t) \in S_\varphi\), which is a unique mild solution to (1.1) satisfying \(x(s) = \varphi(s)\) on \([-\tau, 0]\).

By the definition of the space \(S_\varphi\) this solution is exponentially stable in mean square. This completes the proof. \(\square\)

**Remark 9** In Eq. (1.1) provided \(h(.) = 0\) and \(\Delta x(t_k) = 0\), Eq. (1.1) becomes stochastic neural partial differential equations, which is investigated in [3]. In the sense, the results of this paper are generalized.

**4 Example**

We consider the following impulsive neutral stochastic partial differential equation with delays and Poisson jumps driven by a fractional Brownian motion of the form:

\[
\begin{cases}
    d \begin{bmatrix} x(t, \xi) + \frac{\alpha_i}{M_{1/2}} x(t-r(t), \xi) \\ M_{1/2} x(t-r(t), \xi) \end{bmatrix} = \begin{bmatrix} \partial_2 x(t, \xi) + \alpha_2 x(t-r(t), \xi) \\ \alpha_2 x(t-r(t), \xi) \end{bmatrix} dt + \sigma(t) d\beta^H(t) \\
    + \int_Z \alpha_3 x(t, \xi) \tilde{N}(dt, dy), \quad t \geq 0, \quad t \neq t_k \\
    \Delta x(t_k, \xi) = I_k x(t_k^-), \quad k = 1, 2, \ldots, \quad t = t_k \\
    x(t, 0) = x(t, \pi) = 0, \quad t \geq 0, \quad \alpha_i > 0, \quad i = 1, 2, 3, 4 \\
    
\end{cases}
\]

\[
x(s, \xi) = \varphi(s, \xi), \quad \varphi(s, \cdot) \in L^2([0, \pi]); \quad -\tau \leq s \leq 0 \quad a.s.,
\]

\[
\tag{4.1}
\]

where \(M_{1/2}\) is the corresponding constant in Lemma 3, \(\beta^H(t)\) is a standard one-dimensional fractional Brownian motion and \(Z = \{z \in \mathbb{R} : 0 < |z| \leq c, c > 0\}\). For the convenience of writing, in the following, the variable \(\xi\) of \(x(t, \xi)\) is omitted.
We rewrite (4.1) into abstract form of (1.1). Let $X = L^2([0, \pi])$. Define the operator $A : D(A) \subset X \longrightarrow X$ given by $A = \frac{\partial^2}{\partial x^2}$ with

$$D(A) = \left\{ y \in X : y' \text{ is absolutely continuous}, \ y'' \in X, \ y(0) = y(\pi) = 0 \right\},$$

then we get

$$Ax = \sum_{n=1}^{\infty} n^2 x_n > X e_n, \ x \in D(A),$$

where $e_n = \frac{\sqrt{2}}{\pi} \sin nx, \ n = 1, 2, \ldots$ is an orthogonal set of eigenvector of $-A$.

The bounded linear operator $(-A)^{\frac{3}{4}}$ is given by

$$(-A)^{\frac{3}{4}} x = \sum_{n=1}^{\infty} n^2 x_n > X e_n,$$

with domain

$$D((-A)^{\frac{3}{4}}) = X^{\frac{3}{4}} = \left\{ x \in X, \sum_{n=1}^{\infty} n^2 x_n > X e_n \in X \right\}.$$

It is well known that $A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t \geq 0}$ in $X$, and is given by (see [17])

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} x_n > X e_n$$

for $x \in X$ and $t \geq 0$, that satisfies $\|S(t)\| \leq e^{-\pi^2 t}$ for every $t \geq 0$.

In order to define the operator $Q : Y := L^2([0, \pi], \mathbb{R}) \longrightarrow Y$, we choose a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^+$, set $Qe_n = \lambda_n e_n$, and assume that

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty.$$ 

Define the fractional Brownian motion in $Y$ by

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n^H(t)e_n,$$

where $H \in (\frac{1}{2}, 1)$ and $\{\beta_n^H\}_{n \in \mathbb{N}}$ is a sequence of one-dimensional fractional Brownian motions mutually independent. Let us assume the previous hypotheses on operator $\sigma$.

Let $g(t, x(t - r(t))) = \frac{\alpha_1}{M_4^1 \|(-A)^{\frac{3}{4}}\|} x(t - r(t))$, $f(t, x(t - \rho(t))) = \alpha_2 x(t - \rho(t))$, $h(t, x(t - \theta(t), y)) = \alpha_3 y x(t - \theta(t))$, $I_k x(t_k^-) = \alpha_4 x(t_k^-)$, $t = t_k$, $k = 1, 2, \ldots$, and $t_{k+1} = t_k + 1$.

It is obvious that all the assumptions are satisfied with

$$\lambda = \frac{\pi^2}{2}, \ M = 1, \ K_1 = \alpha_2^2, \ K_2 = \frac{\alpha_1}{M_4^1}, \ K_3 = \alpha_3^2 \int_0^\pi y^2 \nu(dy), \ \tilde{q} = \alpha_4, \ \|(-A)^{\frac{3}{4}}\| = 1,$$

and $\|(-A)^{\frac{3}{4}}\| \leq \frac{1}{\Gamma(\frac{3}{2})} \int_0^{\infty} t^{-\frac{1}{2}} \|S(t)\| dt \leq \frac{1}{\pi^2}$.
Thus, by Theorem 8, if the initial value $\varphi(t)$ satisfies
\[ E\|\varphi(s)\|^2 \leq M_0 E|\varphi|^2 e^{-as}, \quad s \in [-\tau, 0], \]
for some $M_0 > 0$, $a > 0$; then, the Eq. (4.1) has one unique mild solution and is exponential stable in mean square provided that the following inequality
\[ \frac{\alpha_1^2}{M_1^2} + \frac{\Gamma^2 (3/4 \alpha_1^2)}{\pi} + \frac{\alpha_2^2}{\pi^2} + \frac{-\int z y^2 v(dy)}{2} + \frac{\alpha_4^2}{\pi^2} < \frac{\pi^2}{5} \]
holds.

References

1. Benchohra, M., Ouahab, A.: Impulsive neutral functional differential equations with variable times. Nonlinear Anal. 55(6), 679–693 (2003)
2. Biagini, F., Hu, Y., Øksendal, B., Zhang, T.: Stochastic calculus for Fractional brownian motion and Application. Springer, Berlin (2008)
3. Boufoussi, B., Hajji, S.: Neutral stochastic functional differential equation driven by a fractional Brownian motion in a Hilbert space. Stat. Probab. Lett. 82, 1549–1558 (2012)
4. Boufoussi, B., Hajji, S., Lakhel, E.: Functional differential equations in Hilbert spaces driven by a fractional Brownian motion. Afrika Matematika 23(2), 173–194 (2011)
5. Caraballo, T., Garrido-Atienza, M.J., Taniguchi, T.: The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion. Nonlinear Anal. 74, 3671–3684 (2011)
6. Caraballo, T., Diop, M.A., Ndiaye, A.A.: Asymptotic behavior of neutral stochastic partial functional integro-differential equations driven by a fractional Brownian motion. J. Nonlinear Sci. Appl. 7, 407–421 (2014)
7. Dung, T.N.: Stochastic Volterra integro-differential equations driven by fractional Brownian motion in a Hilbert space. Stochastics 87(1), 142–159 (2015)
8. Feyel, D., De la Pradelle, A.: On fractional Brownian processes. Potential Anal. 10, 273–288 (1999)
9. Goldstein, J.A.: Semigroups of linear operators and applications. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York (1985)
10. Hernandez, E., Keck, D. N., McKibben, M. A.: On a class of measure-dependent stochastic evolution equations driven by fBm. J. Appl Math Stoch Anal., Art ID 69747, p. 26 (2007)
11. Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes. Nort-Holland/Kodansha, Amsterdam/New York (1989)
12. Lakhel, E.E., McKibben, M.A.: Controllability of Impulsive Neutral Stochastic Functional Integro-Differential Equations Driven by Fractional Brownian Motion. Chapter 8; McKibben, M.A., Webster, M. (eds.) Brownian Motion: Elements, Dynamics, and Applications, pp. 131–148. Nova Science Publishers, New York (2015)
13. Lakhel, E., Hajji, S.: Existence and uniqueness of mild solutions to neutral SIFDEs driven by a fractional Brownian motion with non-Lipschitz coefficients. J. Numer. Math. Stoch. 7(1), 14–29 (2015)
14. Lakshmikantham, V., Bainov, D.D., Simeonov, P.S.: Theory of Impulsive Differential Equations, Series in Modern Appl. Math., vol. 6. World Scientific Publ., Teaneck (1989)
15. Mandelbrot, B., Ness, V.: Fractional Brownian motion, fractional noises and applications. SIAM Rev. 10(4), 422–437 (1968)
16. Nualart, D.: The Malliavin Calculus and Related Topics, 2nd edn. Springer, Berlin (2006)
17. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences, vol. 44. Springer, New York (1983)
18. Ren, Y., Hu, L., Sakthivel, R.: Controllability of impulsive neutral stochastic functional differential inclusions with infinite delay. J. Comput. Appl. Math. 235(8), 2603–2614 (2011)
19. Ren, Y., Cheng, X., Sakthivel, R.: Impulsive neutral stochastic functional integro-differential equations with infinite delay driven by fBm. Appl. Math. Comput. 247, 205–212 (2014)
20. Xu, D., Yang, Z., Yang, Z.: Exponential stability of nonlinear impulsive neutral differential equations with delays. Nonlinear Anal. 67(5), 1426–1439 (2006)