SYK behaviors generated by Hubbard-type four-fermion on-site random interactions and the related phase transitions

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In this paper, we discuss the possible emergence of SYK non-Fermi liquid in the incoherent critical metal phase generated by the Hubbard-type four-fermion on-site random interactions which follow the Gaussian distribution. The SYK behaviors here are similar to the $q = 4$ SYK model except an additional factor $N$. The phase transitions to the Fermi liquid regime and high temperature DMFT insulting regime are possible through the turning of related parameters, including the temperature, coherence scale, SYK coupling, and Coulomb boson field. Taking the nonlocal Coulomb interaction (with pseudospin interaction) as an example, we also discuss the existence of stable gapless bosonic excitations in SYK compressible non-Fermi liquid phase away from the quantum critical point (and thus prevent the condensation), which is due to the preserved SU(2) local gauge symmetry in incoherent critical metal phase with strong SYK coupling and at intermediate temperature. Note that here the compressibility plays the role of effective coupling in the low-energy SYK bosonic action, and is a measurable quantity in non-fermi liquid phase. We found that in this special case, the global U(1) and U(1)$\otimes$U(1) symmetries are also preserved. Our results are useful in detecting the effect of Hubbard parameter on the SYK behaviors in non-Fermi liquid incoherent metal phase and how the on-site interacting model enters the holographic picture at intermediate and high temperature regimes, as well as the relation between the phase transitions with the symmetry breaking.

1 Introduction

For boson mode described by a two-point fermion-loop, its interaction term’s expectation value can be represented by a product of boson field operators (or order parameters) $\Delta^\dagger = c^\dagger d$ ($c$ denotes the impurity while $d$ denotes the fixed atomic electrons), as $U\langle \Delta^\dagger \Delta \rangle$, where $U$ is the interaction vertex. We firstly consider a model with impurities in dynamical mean field theory with large-$N$, interacting with the localized electrons in the bath when they occupy the same lattice site. In this model, we consider only the on-site interspecies interaction between impurities ($c$) and localized electrons ($d$). In SYK phase, the Hamiltonian reads

$$H = \frac{1}{\sqrt{N}} \sum_{ij} t_{ij} c^\dagger_i c_j - \mu \sum_i c^\dagger_i c_i + \frac{1}{N} \sum_{\alpha\beta} U_{\alpha\beta\alpha\beta} c^\dagger_{\alpha i} c_{\alpha i'} d^\dagger_{\beta j} d_{\beta j'},$$

where $ij$ are the site indexes and $\alpha\beta$ are the orbital indexes. We note that, here the on-site Hubbard term (third term) can be viewed as a special case of SYK-term, as long as it does not
contribute to the formation of the Cooper pairs. For SYK model with inverse scaling dimension \( q = 4 \), the Hamiltonian reads

\[
H_{\text{SYK}} = \frac{1}{(2N)^{1/2}} \sum_{ijkl} U_{ijkl} c_i^\dagger c_j^\dagger c_k c_l,
\]

with \( i, j, k, l = 1, 2, \ldots, N \) are the orbital indices or site indices. In our model, we refer them as the orbital indices which describe different fermion modes with four-fermion random on-site interactions between them.

In Eq. (1), \( U_{\alpha\beta\alpha'\beta'} \) is a real Gaussian random variable with zero mean \( \langle U_{\alpha\beta\alpha\beta} \rangle = 0 \), and variance \( \langle U_{\alpha\beta\alpha\beta}^2 \rangle / N^2 = \frac{U^2}{N^2} \) (details about the prefactor is given in Sec.4). And we have the symmetry constraint in SYK phase \( U_{\alpha\beta\alpha'\beta'} = U_{\beta\alpha\beta'\alpha'} = -U_{\beta\alpha'\alpha\beta'} = -U_{\alpha\beta'\alpha'\beta} = U_{\alpha'\beta\alpha\beta'} \).

Here \( \alpha (\alpha') \) is independent of \( \beta (\beta') \), but clearly \( \alpha, \beta, \alpha', \beta' \) are not four independent indices (the transition mechanism from orbital \( \alpha \) to \( \alpha' \) is the same as that from orbital \( \beta \) to \( \beta' \)), thus the above SYK model should not be interpreted as SYK\(_4\) system but a SYK\(_2\times\text{SYK}_2\) system. While when \( \alpha = \alpha' \) and \( \beta = \beta' \), it becomes a simple SYK\(_2\) model, which corresponds a square of fermion bilinear term and with the effect similar to large chemical potential. This is the most important premise of this paper.

In fact, the four fermions in a single site can be of the same energy by making all orbitals degenerate exactly with the on-site energy. The SYK-feature mainly contributed by the normal-state incoherency, for example, in the phase of locally incoherent critical metal with \( U \gg \omega \) (and in the mean time \( \Sigma \gg \omega \)). The anomalous component of the local Green’s function, as well as the anomalous self-energy (correlation function) in the presence of nonzero boson field will suppress the SYK non-Fermi-liquid, and supports the coherent spectral peaks. In the mean time, the resulting gap in many-body spectrum (as well as the finite expectation of boson field \( \langle i\phi_r \rangle \) will be maximized in the presence of maximum condensation energy. In this case, the gauge symmetry of system ground state is broken by the condensation of matter field and the existence of Fermi liquid phase rely on the existence of well-defined Fermi surface and vanishing fermionic anomalous dimension. Thus in the presence of strong bosonic fluctuation (e.g., enhanced by large density-of-states), the SYK non-Fermi-liquid is suppressed by the the large many-body spectral gap, or overwhelmed by the hopping-induced hybridization obtained in dynamical mean-field theory with \( N \to \infty \). Note that, however, such many-body spectrum gap is usually hard to found in large-N limit due to the Nambu-Goldstone theory.

We also discuss the related phase transitions in this paper. For example, the parameter \( U_{\alpha\beta\alpha'\beta'} \) behaves exactly like a constant on-site Hubbard \( U \) when the system leaves the intermediate temperature (or frequency) regime \( \omega_c \ll \max[T, \omega] \ll U \), and enters the high temperature regime \( (T \gg U) \), the phase transition is possible as orbitals of the two species \( (c \text{ and } d) \) can exactly degenerate with the on-site energy when the on-site energy is large enough. In strong coupling limit, the non-Fermi-liquid as well as the strange metal behavior may arise from the long-range interaction (like the Coulomb interaction close to half-filling, as we discuss in Sec.4) between gapless fermions and the critical boson (with \( \langle i\phi(\tau) \rangle = 0 \)). An incoherent critical metal phase with local random all-to-all interaction is possible to observed in strong coupling limit and at temperature (or frequency) much higher than the coherence scale, which means the strange metal behavior can arise without turning the order parameter mass term \( \tau |i\phi(\tau)|^2 \) to zero. Such phenomenon usually appears in the Bose metal. Usually, the stable gapless bosonic excitations are related to continuous symmetry breaking which can be achieved by turning the order parameter mass term across the quantum critical point at zero temperature, but the boson excitation may remains gapless without the quantum critical point in the incoherent regime of Bose metal (incoherent critical metal phase) and thus prevent the condensation. In the latter case a sharp fermi surface can be observed and without the Landau quasiparticle. The incoherent regime can be extended to zero temperature when the coherent scale is much larger than the temperature but much smaller than the on-site interaction, in which case the conformal symmetry is absent. Then the non-Fermi liquid arise as a stable zero temperature phase, with the gapless compressible SYK modes away from its ground state. This relies on the preservation of symmetry of system (or subsystem) in a lattice (sublattice), like the Goldstone
modes in momentum space (for nonlocal case). When the temperature is close to (but still lower than) the coherence scale, the system exhibits disordered Fermi liquid phase, where the two-point correlation is suppressed by the low condensate fraction\[19, 20\]. We will show in this article that in the absence of symmetry breaking and condensation, for system in non Fermi liquid phase, the equal-time two-point correlation and the thermodynamical quantities will not affected by the boson potential as long as the potential is spatially-uniform (in mean-field treatment). As the disorder from anomalous self-energy breaks the SU(2) symmetry of Coulomb interaction (when we treat the fermion flavors as spins), the bosonic gap term will suppress the non Fermi liquid phase, and leads to exponentially decayed two-point correlation. Such effect can also be observed in other lattice systems with magnetic disorder or condensed defects, e.g., the superconducting pairing system or pair hopping system which will explicitly break the U(1) gauge symmetry.

The incoherent critical metal phase in the narrow bandwidth limit, according to SYK theory, it is approximately conformal invariant as it extends to low temperature and low frequency region. We further note that, in Fermi liquid phase, large kinetic energy and chemical potential of single fermion will suppress the quantum fluctuation induced by strong non-Fermi-liquid correlations. But in the mean time, the hopping (i.e., the tunneling term at zero phase) can be competitive with the chemical potential (or the U(1) charge density), e.g., in small bandwidth limit, the SYK in coherent regime with gapless non Fermi liquid feature can be extended to zero temperature. In the presence of comparable chemical potential and tunning (|µ| = |t|) the system exhibits two kinds of gapless SYK modes: one with finite compressibility and one with zero compressibility (since the renormalized chemical potential becomes µ − t = 0). As the disorder of tunneling overlaps the chemical potential, the entropy becomes zero at low temperature regime, which corresponds to the absence of compressibility or an unique gapped ground state energy that is independent of the chemical potential (see Eq.(64)). If the tunneling parameter further increases and leads to coherent scala much larger than the temperature (but smaller than the on-site interaction), the system enters the disordered Fermi liquid phase. Fininallly, we stress that, the on-site interaction (as a random complex Gaussian variable) will not leads to the two-fermion term (or bilinear fermion operator) which implies the nonzero boson mass and condensation. That is in contrast to the pair hopping or interaction-assisted tunneling, which will leads to anomalous component of self-energy and destroy the non Fermi liquid phase. We will also discuss this effect by taking the Coulomb interaction (in the absence or presence of SU(2) symmetry breaking) as an example.

2 Green’s functions and correlations in each phase

The local Green’s function for impurities has the form

\[
g_{loc}(i\omega) \sim \begin{cases} \frac{1-n_c}{\omega+\mu} + \frac{n_c}{\omega+\mu-\Sigma_{ii}} & \text{insulator phase at half - filling; } t \ll 1, \\ \frac{1}{\omega+\mu-\Sigma_{DMFT}} & \text{Fermi - liquid phase at half - filling; } \max[\omega, T] \ll \omega_c, \\ \frac{1}{\omega Z_{DMFT} \pm \Delta_h(0)} & \text{incoherent metal phase; } \omega_c \ll \omega \ll U, \end{cases}
\]

where the noninterating chemical potential reads \(\mu_0 = \mu - \Sigma_{DMFT} = \mu - U/2 = 0\) at half-filling and \(\omega_c = W^2/U\) is the coherence scale. \(t = \sum_k e^{i k r} \xi_k\) is the hopping integral, and in second line of above equation, we approximately treat the hybridization function as \(\Delta_h(0)\) which is nonzero and frequency-independent.

Note that we use \(n_c\) to denote the number operator of localized impurity, which becomes a classical variable \((n_c = 0 \text{ or } 1)\) in the atomic limit (cf. the Falicov-Kimball model in the absence...
of spin-flip terms). The incoherent metal phase has not quasiparticle excitations and it depends on the largest low-energy cutoff which is not necessarily the coherence scale. These features are shared with the Sachdev-Ye-Kitaev (SYK) regime. We note that the SYK model additionally requires particle-hole asymmetries due to the finite charge away from the half-filling, which cannot be seen from the insulating phase and fermi-liquid phase of \( g_{\text{loc}} \). Further, we note that for \( U_{\alpha\beta\alpha'\beta'} \gg \omega_c \gg T \), we obtain a crossover from the incoherent metal phase to the fermi-liquid phase, but since we consider the local Green’s function in this paper, the formation (or destruction) of fermi surface cannot be seen as the signature of this phase transition, instead, we can use the temperature-dependence of resistivity as the signature, i.e., the linear-in-\( T \) resistivity in strongly interacting incoherent metal phase and the one scales as \( \sim T^2 \) in fermi-liquid phase. The linear-in-\( T \) feature of resistivity in weakly dispersive SYK model is also referred to the Planckian dissipative transport [17, 18]. However, we note that the Wiedemann-Franz (WF) law with linear-in-\( T \) behavior can be found also in some fermi-liquid cases when the temperature is much lower than the Fermi temperature, in which case the quenched impurities’ elastic scattering (with energy conservation but not momentum conservation) is dominating (play a main role during both the charge and heat transports). And in this case the WF law is valid as long as the quasiparticle is existing and long-lived[?].

\( \Sigma_{ii} \) is the site diagonal self-energy which reads

\[
\Sigma_{ii} = \Sigma_{DMFT} + \Sigma(i\omega) = \begin{cases} U, & n_c = 1, \\ \frac{1}{2} g_{\text{loc}} (1 - \sqrt{1 + g_{\text{loc}}^2 U^2 + 2 g_{\text{loc}} U(2n_d - 1)}), & n_c \neq 1, \end{cases}
\]

and here we focus on the commutate case \([\Delta, \Delta^\dagger] = 1 - n_c - n_d = 0\). \( n_d = \frac{e^{-\beta E_{dZ}(\mu - U)}}{Z(\mu) + e^{-\beta E_{dZ}(\mu - U)}} \) which reduced to 1/2 in zero-temperature limit. Note that unlike the SYK-like incoherent critical metal phase or the gapless thermal insulating phase or the quantum phase fluctuation-induced pseudogap insulating phase, the induced Mott band insulating phase here locates in the low-frequency and low-temperature regime, and that is also different from the pair hopping or Coulomb boson induced bosonic insulating phase.

The hybridization function is related to the spectral function (or density-of-state) \( \rho(\omega) = -\text{Im} G_0^{-1}(\omega + i\eta) = -\Delta_\uparrow(\omega) \), and has \( \Delta_\uparrow(0) = \text{Im} G_0^{-1}(i\eta) \neq 0 \). Here \( G_0 \) is the Weiss function, which is the bare version of the above local Green’s function

\[
G_0 = \frac{1}{i\omega + \mu - \Delta_\uparrow(i\omega)} = \frac{1}{i\omega + \mu - t^2 g_{\text{loc}}(i\omega)} = \begin{cases} \frac{1}{i\omega + \mu - t^2 g_{\text{loc}}(i\omega)}, & \text{insulator phase}; t \ll 1, \\ \frac{1}{i\omega + \mu - \frac{1}{\omega^2} t^2 - \Delta_\uparrow}, & \text{Fermi - liquid phase at half - filling}; \omega \ll \omega_c, \\ \frac{1}{i\omega + \mu - \frac{1}{\omega^2} \text{sgn}(\omega)}, & \text{metal phase}; \omega_c \ll \omega \ll U, \end{cases}
\]

and self-consistently

\[
g_{\text{loc}}(i\omega) = \frac{1}{G_0^{-1} - \Sigma(i\omega) - \Sigma_{DMFT}}.
\]
Note that the single particle local Green’s function can be written in spectral representation as

\[ g_{\text{loc}}(i\omega) = \sum_k \frac{1}{i\omega + \mu - \varepsilon_k - \Sigma(i\omega) - \Sigma_{\text{DMFT}}} \]

which, in atomic limit \( t \to 0 \) (i.e., \( \Delta_h = 0 \)), becomes

\[ g_{\text{loc}}(i\omega) = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \frac{\delta(\Omega)}{[i\omega + \mu - \Sigma(i\omega) - \Sigma_{\text{DMFT}}] - \Omega} \]

or in noninteracting limit becomes

\[ g_{\text{loc}}(i\omega) = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \frac{1}{[i\omega + \mu - \Sigma(i\omega)] - \Omega}, \]

which is divergent. While in the intermediate region with \( t \neq 0 \) and \( U \neq 0 \) and in the limit of infinite dimensions (or lattice sites), we have

\[ -\Delta_h = g_{\text{loc}}^{-1}(i\omega) - \mathcal{R}[g_{\text{loc}}(i\omega)] \]

\[ = \left[ \sum_k \frac{1}{i\omega + \mu - \varepsilon_k - \Sigma(i\omega) - \Sigma_{\text{DMFT}}} \right]^{-1} - (i\omega + \mu - \Sigma(i\omega) - \Sigma_{\text{DMFT}}) \]

\[ = -t^2 g_{\text{loc}}(i\omega) \]

\[ = \sum_k \frac{-t^2}{i\omega + \mu - \varepsilon_k - \Sigma(i\omega) - \Sigma_{\text{DMFT}}}, \]

where \( \mathcal{R}[\cdot] \) denotes the reciprocal function of Hilbert transform. Then the local Green’s function in the second line of Eq. (1) can be obtained.

From the above equation, we also have the spectral function of conduction bath

\[ \rho_{\text{bath}}(\omega) = -\text{Im} G_0^{-1}(\omega + i\eta) \]

\[ \approx \begin{cases} 0, & \text{insulator phase,} \\ 0, & \text{Fermi – liquid phase at half – filling; } \omega \ll \omega_c, \\ \frac{\varepsilon_{\text{sgn}(\omega)}}{U/2 \omega_{1/2}}, & \text{metal phase; } \omega_c \ll \omega \ll U, \end{cases} \]

where we use the analytical continuation \( i\omega \to \omega + i\eta \) \((\omega \to -i\omega + \eta)\). And that of the local electron reads

\[ \rho_{\text{loc}}(\omega) = -\text{Im} g_{\text{loc}}(\omega + i\eta) \]

\[ \approx \begin{cases} 0, & \text{insulator phase,} \\ 0, & \text{Fermi – liquid phase at half – filling; } \omega \ll \omega_c, \\ \frac{-\text{sgn}(\omega)}{U/2 \omega_{1/2}}, & \text{metal phase; } \omega_c \ll \omega \ll U, \end{cases} \]

3 SYK phase generated by the on-site coupling \( U_{\alpha\beta\alpha'\beta'} \)

In large-N limit, the SYK theory shows that the saddle-point is also the conformal fixed point in infrared limit, which owns a low-energy conformal symmetry. We find that the model
above can also related to the results of SYK model (in $U = 0$)\textsuperscript{3} as well as its generalized one (in $U \neq 0$)\textsuperscript{4}. Firstly, the hybridization function (dynamical mean field) $\Delta_h = t^2 g_{\text{loc}}(\tau)$ can be regarded as the SYK-type self-energy in IR limit with scaling dimension of fermion operator $1/q = 1/2$ in conformal limit; while for $1/q = 1/4$, we will obtain $\Sigma(\tau) = -t^2 g_{\text{loc}}^2(\tau)g_{\text{loc}}(-\tau)$. Here the mean-field can be obtained through the cavity method in a lattice, in which case the action of impurity within the term $\int D[c^\dagger, c]e^{-S_{\text{imp}}}$ reads

$$S_{\text{imp}} = -\int_0^\beta d\tau d\tau' \left[ \sum_{ia} c_{ia}^\dagger(\tau) g_{\text{loc}}^{-1}_{ia}(\tau - \tau') c_{ia}(\tau') + t^2 \lim_{N \to \infty} \sum_i g_{\text{loc}}^{ii}(\tau - \tau') \right]$$

$$+ \frac{1}{N} \sum_{ia\beta} U_{a\beta a'\beta'} c_{ia}^\dagger(\tau) c_{ia'}(\tau) d_{i\beta}(\tau) d_{i\beta'}(\tau)$$

$$= -\int_0^\beta d\tau d\tau' \left[ \sum_{ia} c_{ia}^\dagger(\tau) g_{\text{loc}}^{-1}_{ia}(\tau - \tau') c_{ia}(\tau') + t^2 g_{\text{loc}}(\tau - \tau') \right]$$

$$+ \frac{1}{N} \sum_{ia\beta} U_{a\beta a'\beta'} c_{ia}^\dagger(\tau) d_{i\beta}(\tau) d_{i\beta'}(\tau) c_{ia'}(\tau).$$

Here both the hybridization and local Green’s function $g_{\text{loc}}^{ii}$ have been self-averaged, and the hopping $t_{ij}$ (with factor $1/\sqrt{N}^{i-j} = 1$) is restricted to $i = j$ since neighbors of cavity are disconnected and the off-diagonal term with $i \neq j$ averages out to zero in thermodynamic limit $N \to \infty$. Also, we note that a next nearest neighbor hopping will breaks the particle-hole symmetry. Thus the hybridization is site-independent (translationally invariant).

For $1/q = 1/2$, the effective action in IR limit reads (the term $i\omega = -\partial_\tau$ is omitted)

$$S = -N \text{Tr} \text{ln}(-g_{\text{loc}}^{-1}(\tau)) - N \int_0^\beta d\tau \left[ \Sigma(\tau) g_{\text{loc}}(\tau) + \frac{t^2}{2}(-g_{\text{loc}}(\tau)g_{\text{loc}}(-\tau)) \right]$$

$$= -N \text{Tr} \text{ln}(-g_{\text{loc}}(\tau)) - N \int_0^\beta d\tau \left[ \Sigma(\tau) g_{\text{loc}}(-\tau) + \frac{t^2}{2}(-g_{\text{loc}}(\tau)g_{\text{loc}}(-\tau)) \right].$$

in $N \to \infty$ limit, the saddle-point approximation leads to

$$\frac{\delta S}{\delta \Sigma(\tau)} = 0,$$

i.e.,

$$-N \frac{1}{\Sigma(\tau)} = N \int_0^\beta d\tau g_{\text{loc}}(-\tau)$$

$$\frac{1}{-\Sigma(\tau)} = \int_0^\beta d\tau g_{\text{loc}}(-\tau)$$

$$\frac{1}{-\Sigma(\tau)} = \int_0^\beta d\tau \frac{1}{N} \sum_i c_i^\dagger(\tau)c_i(0)$$

$$\frac{1}{-\Sigma(\tau)} = \frac{\beta}{N} \frac{1}{\Sigma(\tau)} \sum_i c_i^\dagger(\tau)c_i(0)$$

$$\frac{1}{-\Sigma(\tau)} = \sum_i c_i^\dagger(\tau)c_i(0).$$

Here we use the particle-hole symmetry relation $g_{\text{loc}}(\tau) = -g_{\text{loc}}(-\tau)$ and the definition of bilocal field (or Lagrange multiplier field which is related to the fermion number constraint).
where we stated in another work\cite{31}. where the inverse scaling dimension \(q\) here also corresponds to the number of independent orbital indices. Thus every fermion operator in above expression contributes a factor \(1/N^{1/4}\). This discussion will be valid even for intraband interaction. For intraband interaction

\[
S = - \int_0^\beta d\tau \left[ \sum_i c_i^\dagger(\tau)g_{\text{loc}}^{-1}(\tau)c_i(0) + \sum_j d_j^\dagger(\tau)G_d^{-1}(\tau)d_j(0) \right] + \int_0^\beta d\tau \sum_{ij\alpha} \frac{t_{ij}}{\sqrt{N}} c_{i\alpha}^\dagger(\tau)c_{j\alpha}(\tau) + \frac{1}{N} \sum_{i\alpha\beta} U_{\alpha\beta\alpha'\beta'} c_{i\alpha}^\dagger(\tau)d_{i\alpha}(\tau)d_{i\beta}(\tau)c_{i\alpha'}(\tau),
\]

Here we conclude that, for even \(q > 1\), each single operator within each term contributes a leading factor of \(1/N^{(q-1)/(2q)}\) (while for \(q = 1\), each fermion operator contributes \(1/N^{1/(2q)}\)) as we stated in another work\cite{31}. where the inverse scaling dimension \(q\) here also corresponds to the number of independent orbital indices. Thus every fermion operator in above expression contributes a factor \(1/N^{1/4}\). This discussion will be valid even for intraband interaction. For intraband interaction

\[
\frac{1}{N^2} \sum_{i\alpha} U_{\alpha\beta} c_{i\alpha}^\dagger(\tau)c_{i\alpha}(\tau)d_{i\alpha}(\tau),
\]

since each fermion operator contributes \(1/N^{1/2}\). And similarly , for \(q = 4\) the leading factor reads \((1/N^{(q-1)/(2q)})^4 = 1/N^{3/2}\), for \(q = 6\) the leading factor reads \((1/N^{(q-1)/(2q)})^4 = 1/N^{5/2}\). While for SU(M=N) spin degree of freedom is taken into account, e.g., the term

\[
\sum_{\alpha\beta} \sum_{\sigma\sigma'} U_{\alpha\beta\sigma'\sigma} c_{i\alpha}^\dagger(\tau)c_{i\alpha}(\tau)d_{i\beta}(\tau)d_{i\sigma}(\tau)
\]

has a leading factor \(1/N\).

Thus it can be verified that \(U_{\alpha\beta\sigma'\sigma}\) can be related to the SYK \(_2\) coupling by \(J_{\text{SYK}}^{\alpha\beta} U_{\alpha\beta\sigma'\sigma} = (J_{\text{SYK}}^{\alpha\beta})^2\), and the last term can be rewritten as \(-\frac{1}{N^2} \sum_{i\alpha\beta} (J_{\text{SYK}}^{\alpha\beta})^2 (c_{i\alpha}^\dagger(\tau)d_{i\beta}(\tau))(c_{i\alpha}^\dagger(\tau)d_{i\beta}(\tau))^*,\) with each factor \(\frac{1}{N^{1/2}}\) given by \(J_{\alpha\beta}^{\text{SYK}}\). We further note that, the random all-to-all Gaussian variables have the following variance after disorder average, \(\langle t_{ij} \rangle^2 = \frac{t^2}{(2g/(2+1)/2q)^2} = \frac{t^2}{2},\)

\(\langle U_{\alpha\beta\sigma'\sigma}^2 \rangle = \frac{U^2}{(2g/(2+1)/2q)^4} = \frac{U^2}{4}.\) The variation of \(U_{\alpha\beta\sigma'\sigma}\) is thus \(\sigma^2 = \langle U_{\alpha\beta\sigma'\sigma}^2 \rangle = \frac{U^2}{(4N^2)},\) which satisfies \(\int dU_{\alpha\beta\sigma'\sigma} dU_{\alpha\beta'\sigma'\sigma'}^* e^{-U_{\alpha\beta\sigma'\sigma} U_{\alpha\beta'\sigma'\sigma'}^*/\sigma^2} = 1.\)

Unlike the above disordered complex Gaussian random variable \(U_{\alpha\beta\sigma'\sigma}\) with orbital index-dependence, the pure site-local Hubbard-B-induced pair hopping term, which is an non-random double-trace boson operator \(\frac{1}{N} \sum_{\alpha\beta} \sum_{\sigma\beta} c_{i\alpha}^\dagger c_{i\beta} O_{\beta\sigma} c_{i\alpha} O_{\beta\sigma},\) will gives rise to the anomalous self-energy. We will discuss the effect of such anomalous self-energy induced by the Coulomb boson in Sec.4.

The SYK feature can be preserved by using the disorder averaging in Gaussian unitary ensemble (instead of the orthogonal ensemble which will leads to the anomalous component),

\[
e^{-\int_0^\beta d\tau \sum_i U_{O_i}(\tau)} = e^{2U^2 \int_0^\beta d\tau d\tau' \sum_i O_i(\tau)O_i(\tau)} = e^{2U^2 \int_0^\beta d\tau d\tau' \sum_i O_i(\tau)O_i(\tau)},
\]

where \(O_i\) denotes a bosonic physical observable at site \(i\) which satisfies \([O_i, O_j^\dagger] = 0\). Then the effective action becomes (we consider the case that the impurities and localized electrons
occupy same number of sites, i.e., $N$

\[
\mathcal{S} = -\int^\beta_0 d\tau \left[ \sum_i c_{i\alpha}^\dagger (\tau) g_{loc}^{-1}(\tau) c_{i\alpha} (0) + \sum_j d_j^\dagger (\tau) G_d^{-1}(\tau) d_j(0) \right] \\
- \int^\beta_0 d\tau d\tau' \left[ \sum_{ij} \frac{2t^2}{4N} c_{i\alpha}(\tau) c_j(\tau') c_{j\alpha}^\dagger (\tau') c_{i\alpha} (\tau') \right] \\
+ \frac{2U^2}{8N^2} \sum_i c_{i\alpha}(\tau) c_{i\alpha}(\tau') d_{i\beta}(\tau) d_{j\beta}(\tau) d_{j\beta}(\tau') d_{i\beta}(\tau') c_{i\alpha}(\tau'),
\]

(20)

after integrating over the fermion operators $c, d$, using the relation

\[
1 = \int D[g_{loc}] \delta (Ng_{loc}(\tau, \tau') - \sum_\alpha c_{\alpha}^\dagger (\tau') c_{\alpha} (\tau))
\]

\[
= \int D[g_{loc}, \Sigma] \text{Exp} \left[ \int d\tau d\tau' \Sigma(\tau, \tau') (Ng_{loc}(\tau, \tau') - \sum_\alpha c_{\alpha}^\dagger (\tau') c_{\alpha} (\tau)) \right],
\]

(21)

we obtain the effective action

\[
S_{eff} = -N[\text{Tr} \ln (-g_{loc}^{-1}(\tau)) + \text{Tr} \ln (-G_d^{-1}(\tau))] \\
- \int^\beta_0 d\tau d\tau' \left[ -\frac{t^2}{2} g_{loc}(-\tau) g_{loc}(\tau) + \frac{U^2}{4} N g_{loc}(-\tau) g_{loc}(\tau) G_d(-\tau) G_d(\tau) \right] \\
+ \Sigma(\tau) g_{loc}(-\tau) + \Sigma_d(\tau) G_d(-\tau),
\]

(22)

where $\Sigma_d$ and $G_d$ are the self-energy and Green’s function of localized electrons, respectively. Here we note that in mean field approximation both the hopping $t$ and interspecies interaction $U$ are independent of the site index. In particle-hole symmetry case ($g_{loc}(-\tau) = -g_{loc}(\tau)$), the effective action becomes

\[
S_{eff} = -N[\text{Tr} \ln (-g_{loc}^{-1}(\tau)) + \text{Tr} \ln (-G_d^{-1}(\tau))] \\
- N \int^\beta_0 d\tau d\tau' \left[ t^2 g_{loc}(\tau) + \frac{U^2}{2} N g_{loc}^2(\tau) G_d^2(\tau) \right] \\
+ \Sigma(\tau) g_{loc}(-\tau) + \Sigma_d(\tau) G_d(-\tau),
\]

(23)

Then we obtain the saddle-point solutions for the self-energies,

\[
\Sigma(\tau) = t^2 g_{loc}(\tau) - U^2 N g_{loc}(\tau) G_d(\tau) G_d(-\tau),
\]

\[
\Sigma_d(\tau) = -U^2 N g_{loc}(\tau) G_d(\tau) g_{loc}(-\tau),
\]

(24)

which satisfy $\Sigma(-\tau) = -\Sigma(\tau), \Sigma_d(-\tau) = -\Sigma_d(\tau)$ at half-filling. Comparing to the $q = 4$ results[16 11], the only difference is the factor $N$ accompanied by the $U^2$, while in other aspect, there are no difference between the SYK behaviors of $q = 2$ on-site $U_{ijij}$ with the $q = 4$
$U_{ijkl}$. This is one of the most important result of this paper. That also leads directly to the ansatz

$$g_{loc}(\tau) = \frac{\text{sgn}[\tau]}{\sqrt{U N^{1/2} |\tau|}},$$

$$\Sigma(\tau) = -\frac{\text{sgn}[\tau] \sqrt{U N^{1/2}}}{|\tau|^{3/2}},$$

(25)

which means the self-energy (Green function) scales as $\sim N^{1/4}$ ($\sim N^{-1/4}$), unlike the $q = 4$ case where both the Green’s function and self-energy are independent of the factor $N$.

Note that the variables here ($\Sigma, \Sigma_D, g_{loc}, G_d$) are not the integration variables, but the solutions of the classical equation (like the free energy). While those within the above actions are the integration variables (within functional integrals), which have the replica indices although being omitted by us since in SYK theory no replica-off-diagonal term contributes to zeroth order in $1/N$. By substituting these classical solutions into the above action, the free energy density reads

$$f = \frac{F}{2N}$$

$$= \frac{-T \ln Z}{2N}$$

$$= -\frac{1}{2\beta} [\text{Tr} \ln(-g_{loc}^{-1}(\tau)) + \text{Tr} \ln(-G_d^{-1}(\tau))]$$

$$- \frac{1}{2} \int_0^\beta d\tau \left[-\frac{t^2}{2} g_{loc}(\tau) g_{loc}(\tau) + \frac{U^2}{4} N g_{loc}(\tau) G_d(\tau) G_d(\tau) - U^2 N g_{loc}(\tau) G_d(\tau) G_d(\tau) g_{loc}(\tau) + ( - U^2 N g_{loc}(\tau) G_d(\tau) g_{loc}(\tau) G_d(\tau)) \right]$$

$$= -\frac{1}{2\beta} [\text{Tr} \ln(-g_{loc}^{-1}(\tau)) + \text{Tr} \ln(-G_d^{-1}(\tau))]$$

$$- \frac{1}{2} \int_0^\beta d\tau \left[\frac{t^2}{2} g_{loc}(\tau) g_{loc}(\tau) - \frac{7U^2}{4} N g_{loc}(\tau) G_d(\tau) G_d(\tau) g_{loc}(\tau) G_d(\tau) \right];$$

(26)

which reduced to

$$f = -\frac{1}{2\beta} [\text{Tr} \ln(-g_{loc}^{-1}(\tau)) + \text{Tr} \ln(-G_d^{-1}(\tau))] - \frac{1}{2} \int_0^\beta d\tau \frac{7}{4} g_{loc}(\tau) \Sigma(\tau)$$

$$= -\frac{1}{2\beta} \sum_{i\omega} [\ln(-\beta g_{loc}^{-1}(i\omega)) + \ln(-\beta G_d^{-1}(i\omega))] - \frac{1}{2} \int_0^\beta d\tau \frac{7}{4} g_{loc}(\tau) \Sigma(\tau)$$

(27)

$$= -\frac{1}{2\beta} [2\ln 2 + \sum_{i\omega} \ln(1 + \frac{-\Sigma(i\omega)}{i\omega}) + \sum_{i\omega} \ln(1 + \frac{-\Sigma_d(i\omega)}{i\omega})] - \frac{1}{2} \int_0^\beta d\tau \frac{7}{4} g_{loc}(\tau) \Sigma(\tau),$$

in the SYK non-Fermi-liquid (incoherent metallic) phase (with $\tau \to \infty$ and $t \to 0$) at half-filling. This is in the same form with Luttinger-Ward formula. Here we use the relation $\sum_{i\omega} \ln(-\beta g_{0}^{-1}(i\omega))e^{i\omega}\mu = \ln(1 + e^{\beta\mu})$. In the limit of zero hopping (or tunneling), the entropy $S = -\partial f / \partial T$ is a negative constant through the whole temperature range.

We can also see that the hybridization plays the role of part of the self-energy at low-frequency, and then the above low-energy fermi-liquid result of $g_{loc} = 1/(i\omega Z^{-1} - \Delta_h)$ can be
rewritten as (after Fourtier transform)

\[
g_{\text{loc}}(i\omega) = \frac{1}{i\omega - t^2 \int_0^\beta d\tau e^{i\omega\tau} g_{\text{loc}}(\tau) - U^2 N \int_0^\beta d\tau e^{i\omega\tau} g_{\text{loc}}(\tau) G_d(\tau) G_d(-\tau)}.
\]

The action of SYK non-Fermi-liquid (incoherent metallic) phase contains a finite size correction, like \(1/N\), while in the Fermi-liquid phase with \(\omega_c = W^2/U \gg \omega\), we have \(t \gg U\) (thus the on-site \(U\)-term can be omitted), and after taking the disorder average with \(N \to \infty\) to drive the hybridization and local Green’s function be self-averaged, the impurity action can be rewritten as

\[
S_{\text{imp}} = -\int_0^\beta d\tau d\tau' \sum_{\alpha} c_{\alpha}(\tau) g_{\text{loc}}^{-1}(\tau - \tau') c_{\alpha}(\tau') + \int_0^\beta d(\tau - \tau') \frac{1}{\sqrt{N}} \sum_{ij} \sum_{\alpha} t_{ij} c_{\alpha}(\tau - \tau') c_{\alpha}(\tau - \tau')
\]

\[
= -\int_0^\beta d\tau d\tau' \sum_{\alpha} c_{\alpha}(\tau) [g_{\text{loc}}^{-1}(\tau - \tau') - t^2 g_{\text{loc}}(\tau - \tau')] c_{\alpha}(\tau')
\]

\[
= -\int_0^\beta d\tau d\tau' \sum_{\alpha} c_{\alpha}(\tau) [g_{\text{loc}}^{-1}(\tau - \tau') - \Delta h(\tau - \tau')] c_{\alpha}(\tau'),
\]

where here \(g_{\text{loc}}^{-1}(\tau)\) reads

\[
g_{\text{loc}}^{-1}(\tau) = -\partial_{\tau} Z^{-1} + \mu_{\text{int}} - \Sigma_{DMFT}
\]

\[
Z \approx 1 - D_{N \to \infty}^0(E_c) U^2,
\]

\[
D_{N \to \infty}^0(E) = \frac{1}{2\pi t^2} \sqrt{4t^2 - E},
\]

where the second line provides the quasiparticle residue in weak coupling limit. Note that for \(U = 0\), \(Z = 1\). \(D_{N \to \infty}^0(E)\) is the bare density-of-states which follows the semi circular distribution after being self-averaged in \(N \to \infty\) limit, and \(E_c\) is given in Appendix.A.

4 Effect of Coulomb boson

From Eq.(2) we see that the boson self-energy depends on the low-energy scales, and at low temperature limit, for fermion frequency lower than the coherence scale \(\omega_c\), the system is in fermi liquid phase, but if the temperature is higher than the related low-energy scales other than coherence scale, e.g., when temperature becomes the Thouless correlation energy \((T > \omega)\), the system becomes disordered fermi liquid. When the boson frequency (induced by external source) larger than the \(\omega_c\), the Green’s function decorated by it can be obtained by adding a time-dependent auxiliary boson field \(i\phi(\tau)\) whose effect can be absorbed into the renormalization of chemical potential, i.e., even the noninteracting Green’s function contains
this boson potential, and then the local Green’s function reads
\[ g_{\text{loc}}^{\phi}(\tau) = \frac{1}{Z(\mu)} \int d\phi e^{-\beta H_0} g_{\text{loc}}(\tau, \mu - i\phi(\tau)) \]
\[ = \frac{1}{Z(\mu)} \int d\phi_0 e^{-\beta H_{\phi_0}} \prod_{\phi \neq \phi_0} \int d\phi_m e^{-\beta H_{\phi_m}} g_{\text{loc}}(\tau, \mu - i\phi(\tau)) \]
\[ = \frac{1}{Z(\mu)} \int d\phi_0 e^{-\beta(\phi_0 \Delta_b \phi_0 - i\phi_0 N_0 + H_0(\mu - i\phi_0))} \prod_{\phi \neq \phi_0} \int d\phi_m e^{-\beta H_{\phi_m}} g_{\text{loc}}(\tau, \mu - i\phi(\tau)) \]
\[ = \frac{1}{Z(\mu)} \int d\phi_0 e^{-\beta(\phi_0 \Delta_b \phi_0 - i\phi_0 N_0 + H_0(\mu - i\phi_0))} \left( \prod_{\phi \neq \phi_0} \right) \int d\phi_m e^{-\beta \sum_m H_{\phi_m}} g_{\text{loc}}(\tau, \mu - i\phi(\tau)) \]
\[ = \frac{1}{Z(\mu)} \int d\phi_0 e^{-\beta(\phi_0 \Delta_b \phi_0 - i\phi_0 N_0 + H_0(\mu - i\phi_0))} \left( \prod_{\phi \neq \phi_0} \right) \int d\phi_m e^{-\beta \sum_m H_{\phi_m}} g_{\text{loc}}(\tau, \mu - i\phi(\tau)), \tag{31} \]
where \( Z(\mu) = e^{-\beta H_0(\mu)} \) is the partition function of noninteracting particles, \( \Delta_b \) is the bosonic gap equation which controls the excitation gap of the boson. \( \phi_0 = \beta^{-1} \int_0^\beta d\tau \phi(\tau) \). The decomposition of \( H_\phi \) into \( H_{\phi_0} \) and \( H_{\phi_m} \) is reminiscent of the Trotter-Suzuki decomposition, where at half-filling there is a canonical transformation \( c \to e^{-iq\tau c^\dagger} = -c^\dagger \) with checkerboard pattern \( q = (\pi, \pi) \). We can see that \( H_{\phi_m} \) reduced to \( H_{\phi_m} = \phi_m \Delta_b \phi_m \) for \( \tau = 0 \) and \( \tau = \beta \), since \( \frac{1}{\beta} \int_0^\beta d\tau [\phi(\tau) - \phi_0] = \frac{1}{\beta} \int_0^\beta d\tau [\phi(\tau) - \frac{1}{\beta} \int_0^\beta d\tau \phi(\tau)] = 0 \). For large enough number of fermions \( N \), in which case
\[ -\frac{\partial H_0(\mu_0)}{\partial \mu} = N \gg 1, \]
\[ -\frac{\partial^2 H_0(\mu_0)}{\partial \mu^2} \gg \tau, \tag{32} \]
we can use the saddle-point approximation to simplify our following calculations. Note that we denote \( \mu_0 \equiv \mu - i\phi_0 \) as the saddle point, and since the bosonic Matsubara frequency has \( \Omega_m = 2\pi m/\beta \), we also have, equivalently, \( -\frac{\partial^2 H_0(\mu_0)}{\partial \mu^2} \gg \beta \). For integral over \( \phi_0 \), the saddle point can be determined by
\[ \frac{\partial (\beta H_{\phi_0})}{\partial \mu} = -\beta \frac{\partial(\phi_0 \Delta_b \phi_0 - i\phi_0 N_0 + H^0(\mu_0))}{\partial \mu} \]
\[ = -\beta \frac{\partial(\mu - \mu_0)^2 \Delta_b - (\mu - \mu_0)N_0 + H^0(\mu_0))}{\partial \mu} \]
\[ = \beta(2\Delta_b(\mu - \mu_0) + N_0 - \frac{\partial H_0(\mu_0)}{\partial \mu}) = 0, \tag{33} \]
and we also have
\[ \frac{\partial^2 (-\beta H_{\phi_0})}{\partial \mu^2} = \frac{\partial}{\partial \mu} \left[ \beta(2\Delta_b(\mu - \mu_0) + N_0 - \frac{\partial H_0(\mu_0)}{\partial \mu}) \right] \]
\[ = (2\Delta_b - \frac{\partial^2 H_0(\mu_0)}{\partial \mu^2}) < 0. \tag{34} \]
In saddle-point approximation the integral over \( \phi_m(\mu \neq 0) \) becomes Gaussian, and we note that the retardation effect emerges once the non-Gaussian degree-of-freedom is integrated.\[5\].
In Gaussian form, the local Green’s function reads \( g^\phi_{\text{loc}}(\tau) = g_{\text{loc}}(\tau, \mu_0)e^{-S(\tau)} \), where the action reads

\[
S(\tau) = \frac{1}{\beta} \sum_{m \neq 0} \frac{1}{2\Delta_b \Omega_m} (1 - e^{i\Omega_m \tau})
\]

\[
= \frac{\beta}{48\pi^2 \Delta_b} (\pi^2 - 3\text{Li}_2 e^{-2i\pi/\beta} - 3\text{Li}_2 e^{2i\pi/\beta}),
\]

which is a periodic function of imaginary time and can be approximately rewritten as

\[
S(\tau) = \frac{1}{4\Delta_b} (|\tau| - \frac{\tau^2}{\beta}),
\]

within the range \(|\tau| < \beta\). Thus \( g^\phi_{\text{loc}}(\tau) = g_{\text{loc}}(\tau, \mu_0) e^{-\frac{1}{4\Delta_b} \tau^2} \) where \( g^\phi_{\text{loc}}(0) = g_{\text{loc}}(0, \mu_0) \) implies that \( V_0 \) will not affect the equal-time fermion propagator. Thus in UV or IR limits with \( \tau \to 0 \) and \( \tau \to \infty \), respectively, we can ignore the effect of Coulomb boson. Next we prove that, in high-temperature limit, the saddle-point approximation fail to obtain the local Green’s function decorated by the spatially uniform boson potential, and the spectral weight of the boson propagator is finite; while at zero-temperature limit, the spectral function behaves as a delta function. Firstly, the contour integral of boson propagator reads

\[
D(\Omega_m) = \int_0^\beta d\tau e^{i\Omega_m \tau} e^{-\frac{1}{4\Delta_b} \tau^2}.
\]

As the saddle-point approximation is exact at high-frequency (temperature) or in the presence of large flavor number, we firstly focus on the high-temperature case. At infinite temperature system, we have \( \beta \to 0 \) and \( \tau \equiv t + is \approx is \), then

\[
D(\Omega_m) = i \int_0^\infty ds e^{-\Omega_m s} e^{-\frac{s^2}{4\Delta_b}} + i \int_0^\infty ds e^{-\Omega_m s} e^{-\frac{4\Delta_b}{s^2} e^{-\frac{s^2}{4\Delta_b}}}
= i \int_0^\infty ds e^{-\Omega_m s} e^{-\frac{s^2}{4\Delta_b}} (e^{-\frac{4\Delta_b}{s^2}} - e^{-\frac{s^2}{4\Delta_b}})
\]

Thus the spectral weight can be obtained through

\[
\rho_B(\Omega_m) = 2\text{Im} D(\Omega_m) = 2\text{Im} D(-i\Omega_m + \eta).
\]

In such high temperature limit, the bosonic spectral weight \( \rho_B \) is shown in Fig.1, where we see that the DOS is exponentially suppressed with increasing temperature or \( |V_0| \) (for a large scale), which is due to the Coulomb blockade. That is in consistent with Ref.13 in hierarchy of energy scales \( U/N \ll V_0 \ll T \ll U \).

While in zero-temperature limit \( (\Omega_m \to 0) \),

\[
D(\Omega_m) = \int_0^\infty dt e^{i\Omega_m t} e^{-\frac{1}{4\Delta_b} \frac{1}{t}} = \frac{-4\Delta_b}{-1 + 4\Delta_b i \Omega_m},
\]

thus \( D(\Omega_m) = 4\Delta_b \) in this limit. And then \( \rho_B = 0 \). We can write the local Green’s function with spatially uniform boson potential as

\[
g^\phi_{\text{loc}}(\omega) = T \sum_m g^0_{\text{loc}}(i\omega + i\Omega_m) D(\Omega_m)
= T \sum_m \frac{1}{i\omega + i\Omega_m + \mu - i\phi_0} D(\Omega_m)
\]

\[
= \frac{2(\Delta_b \coth[1/(8\Delta_b T)] + \Delta_b \coth[\mu + i\omega - i\phi_0]/2T]}{1 + 4\Delta_b \mu + 4i\Delta_b \omega - 4i\Delta_b \phi_0}.
\]
Further, in this limit, at largest imaginary time difference \[2\] with \( \text{Re} \tau = \beta/2 \), we have

\[
D(\tau = \beta/2) = -\int_{-\infty}^{\infty} d\Omega \frac{e^{-\Omega \tau}}{2\pi} \frac{1}{1 + e^{-\beta \Omega}} \rho_B(\Omega)
\]

\[
= -\int_{-\infty}^{\infty} d\Omega \frac{\rho_B(\Omega)}{2\pi e^{\beta \Omega}}
\]

\[
\approx -\int_{-\infty}^{\infty} d\Omega \frac{\rho_B(\Omega)}{2\pi e^{\beta \Omega}}
\]  

(42)

The single boson residue can be obtained as

\[
Z = \rho_B(\Omega_m = 0) = 0.
\]

Since

\[
D(\tau = \beta/2) = -\int_{-\infty}^{\infty} d\Omega \frac{e^{-\Omega \tau}}{2\pi} \frac{1}{1 + e^{-\beta \Omega}} \rho_B(\Omega)
\]

\[
\approx -\int_{-\infty}^{\infty} d\Omega \frac{\rho_B(\Omega)}{2\pi e^{\beta \Omega}}
\]

\[
= -\frac{\rho_B(0)}{2\beta},
\]

(43)

we have

\[
\rho_B(\Omega_m = 0) = -2\beta D(\tau = \beta/2) \approx 0,
\]

which is in consistent of the above result of residue. The detail form of \( \rho_B(0) \sim \rho_B(0)/\Omega \beta \) is unimportant here. For the spatially uniform boson potential induced by zero-mode interaction in nonperturbation treatment, which only gives rise to the renormalization of chemical potential, the pairing order parameter will breaks the U(1) gauge symmetry and leads to zero mode as well as the zero bias anomalous of single-fermion quantities, like the density-of-states and spectrum of single-fermion propagator, but the gauge-invariant quantities like thermodynamical quantities or other two-particle quantities (like the density-density correlation or conductivity) will not be affected by zero-mode interaction since it does not leads to any interaction effects i.e., \( g_{\text{loc}}(\tau - \tau' = 0, \mu) = g_{\text{loc}}(\tau - \tau' = 0, \mu_0) \).

We note that for one-dimensional chain of spinless fermions with finite size, the due to the vanishing espectation value of pairing order parameter, the zero-bias anomalous exists in the end of chain even in the absence of gauge symmetry breaking. The gauge transformation of observable \( \mathcal{P} \) reads \( \mathcal{P}' = \mathcal{U} \mathcal{P} \mathcal{U}^\dagger - \partial_\tau S(\tau) \). In the case of \( \beta \to 0 \) (or \( \Delta_b \to \infty \)), the linear response theory works well in revealing the effect of Coulomb potential. The potential \( i\phi(\tau) \) here will not affects the equal time Green’s function as long as it only changes the chemical potential but not the self-energy, and this can also be verified by using the Baym-Kadanoff equation

\[
(-\partial_\tau + \mu) g_{\text{loc}}(\tau, \tau') = \int d\tau'' \Sigma(\tau, \tau'') g_{\text{loc}}(\tau'', \tau'),
\]

\[
(-\partial_\tau + \mu) g_{\text{loc}}(\tau, \tau) = 1 + \int d\tau'' \Sigma(\tau, \tau'') g_{\text{loc}}(\tau'', \tau),
\]

(45)

using the above ansatz,

\[
g_{\text{loc}}(\tau) \approx i \text{sgn}[\tau] \sqrt{UN^{1/2}/|\tau|},
\]

\[
\Sigma(\tau) \approx i \text{sgn}[\tau] \sqrt{UN^{1/2}/|\tau|^{5/2}},
\]

(46)

w we obtain \( \int d\tau'' \Sigma(\tau, \tau'') g_{\text{loc}}(\tau'', \tau) = \int d\tau'' \frac{1}{1 + e^{-\beta \tau''}} = 0 \), thus \( g_{\text{loc}}(\tau, \tau) = (-\partial_\tau + \mu)^{-1} \). This is in consistent with the results in the large \( N \) saddle point. However, the gauge-invariant (two-particle) quantities will be affected by this spatially uniform field when the anomalous
self-energy is nonzero, in which case the many-body spectrum is gapped out by the bosonic mass term. This will be discussed in the following.

Next we discuss the effect of anomalous self-energy in SYK non-Fermi-liquid phase at half-filling ($\mu = 0$) induced by the Coulomb boson through the term

$$Q = -\frac{V_0}{N} \sum_{\alpha \beta} \sum_{ij} \int_0^\beta d\tau c^\dagger_{i\alpha}(\tau)d_{j\beta}(\tau)c_{i\alpha}(\tau),$$

(47)

where $V_0 = -\Delta_b^{-1} < 0$ is the zero mode attractive Coulomb interaction. Here we consider the long-range Coulomb interaction since at half-filling the static screening induced by particle-hole excitations is weak[12], $i\phi(\tau) = |i\phi_0|e^{i\phi(\tau)}$, and the action about the spatially local phase $\psi(\tau)$ in mean field approximation reads[11, 29]

$$S[\psi(\tau)] = \int d\tau [N\frac{K}{2} |\partial_\tau \psi(\tau)|^2 - \frac{\kappa_\phi}{N} \sum_{ij} \cos(\varphi_i - \varphi_j)]$$

$$= \int d\tau [2N - \frac{\partial^2 E_0}{\partial \mu^2} \mu^2 - |i\phi(\tau)|^2 \frac{\partial^2 E_0}{\partial (i\phi(\tau))(i\phi(\tau))^\ast} \cos(\varphi_i - \varphi_j)],$$

(48)

where $E_0$ is the ground state energy of system, and $\kappa_\mu = \frac{\partial^2 E_0}{\partial \mu^2} = \frac{\partial (2N)}{\partial \mu}$ is the system compressibility (local susceptibility). $\kappa_\phi$ is the nonlocal susceptibility. Note that the boson charge, i.e., the difference between conjugate boson number density and the mean boson density, has the relation, $n_c n_d - \frac{1}{2} = \partial_\tau \psi(\tau) = 2\mu$, and the boson number operator satisfies the canonical conjugation $[\varphi, n_c n_d] = i$. Here the mean boson density at half-filling is $1/2$. While the globally conserved U(1) charge has the form $n_c - 1/2$ even for the case of finite chemical potential.

Here the Coulomb interaction $V_0$ is indeed more similar to the pair hopping instead of the bare charging energy[13, 20, 1] which reads (in high frequency unscreened limit) $E_c = e^2/C$ with $C$ the effective electrostatic capacitance, and the term describe interaction reads $E_c = \frac{2}{\pi^2} (\sum_i c_i^\dagger c_i - N_0)^2$ where $N_0 = 1/2$ is the background charge at half-filling. Obviously this term contains on-site Coulomb interaction which is not being considered in this paper due to the screening. Both $V_0$ and $E_c$ will suppress the single particle density-of-states and such suppression effect fades away with the increasing temperature. For example, the system enters disordered fermi liquid phase when $V_0 \ll T$.

With the auxiliary boson field $i\phi_m$, for local Green’s function in boson potential to first order (i.e., the Hartree-Fock solution to first order in $V_0[4]$), we have

$$g_{loc} = \frac{1}{i\omega - \Sigma(i\omega) - i\phi_m},$$

(49)

or in matrix form,

$$g_{loc} = \begin{pmatrix} g_{loc}(\tau) \\ g_{loc}^\dagger(\tau) \end{pmatrix},$$

$$g_{loc}^{11}(\tau) = \frac{i\omega + \Sigma_{22}(i\omega) - i\phi_m}{(i\omega - \Sigma_{11} + i\phi_m)[-(-i\omega - \Sigma_{22} + i\phi_m)] - i\phi_m(i\phi_m)^\ast}$$

$$= \frac{i\omega + \Sigma_{22}(i\omega) - i\phi_m}{(i\omega - \Sigma_{11} + i\phi_m)(i\omega + \Sigma_{22} - i\phi_m) - \phi_m\phi_m}$$

$$= -g_{loc}^{22}(\tau),$$

(50)

$$g_{loc}^{12}(\tau) = \frac{i\phi_m}{(i\omega - \Sigma_{11} + i\phi_m)(i\omega + \Sigma_{22} - i\phi_m) - \phi_m\phi_m}$$

$$= g_{loc}^{21}(\tau).$$

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$G_{d}^{11}$ and $G_{d}^{12}$ have the similar expressions and they share the same fermionic frequency with $g_{loc}$. The phase of $i\phi_{0}$ can be fixed artificially (since the phase will not be affected by the saddle-point approximation) to make the anomalous Green’s function real, and thus the anomalous self-energy is also real, and is even function of imaginary time. Thus in the presence of particle-hole symmetry, we have $g_{11}(-\tau) = g_{11}(\tau)$, $g_{12}(-\tau) = g_{12}(\tau)$, $g_{21}(\tau) = g_{22}(\tau)$, $g_{12}(\tau) = -g_{21}(\tau)$, $\Sigma(-\tau) = -\Sigma(\tau)$, $\Sigma_{A}(-\tau) = \Sigma_{A}(\tau)$.

In the presence of Copper pair source field $i\phi_{m}$ within mean-field approximation, by decoupling the $V_{0}$ term using the Hubbard-Stratonovich transformation,

$$Z^{\phi}(\mu)g_{loc}^{\phi}(\tau - \tau', \mu)\exp[-Q] = Z^{\phi}(\mu)g_{loc}^{\phi}(\tau - \tau', \mu)\exp\left[\frac{V_{0}}{N}\sum_{\alpha \beta} \int_{0}^{\beta} d\tau c_{\alpha}^{\dagger}(\tau)d_{\beta}(\tau)\right]$$

$$= \int D[c_{\alpha}^{\dagger}c_{\beta}]\exp\left[-\int_{0}^{\beta} d\tau d\tau' \sum_{\alpha \beta} c_{\alpha}^{\dagger}(\tau)(\partial_{\tau} + \Sigma(\tau - \tau') + i\phi(\tau - \tau'))c_{\alpha}\right.\right.$$$$

$$+ d_{\beta}^{\dagger}(\tau)(\partial_{\tau} + \Sigma_{d}(\tau - \tau') + i\phi(\tau - \tau'))d_{\beta}^{\dagger}(\tau')]$$

$$\left. \int D[\phi(\tau)]\exp\left[-\frac{N}{V_{0}} \int_{0}^{\beta} d\tau i\phi(\tau)(i\phi(\tau))^{*}\right.\right.$$

$$- \sum_{\alpha \beta} \int_{0}^{\beta} d\tau[(i\phi(\tau))^{*}c_{\alpha}(\tau)d_{\beta}(\tau) + i\phi(\tau)c_{\alpha}^{\dagger}(\tau)d_{\beta}(\tau)] + i\phi(\tau)N_{0}],$$

where $N_{0}$ is the background charge. Then the effective action for the impurity reads

$$S_{eff} = -N \int_{0}^{\beta} d\tau[\text{Trln}(\partial_{\tau} + \Sigma + i\phi_{m}) - \frac{|i\phi(\tau)|^{2}}{V_{0}}]$$

$$- N \int_{0}^{\beta} d\tau[\Sigma(\tau)g_{loc}(\tau)]$$

$$- \frac{l^{2}}{2}(g_{loc}^{11}(\tau)g_{loc}^{22}(\tau) + g_{loc}^{12}(\tau)g_{loc}^{21}(\tau))$$

$$+ \frac{U^{2}N}{4}[g_{loc}^{11}(\tau)g_{d}^{11}(\tau)G_{d}^{11}(\tau)G_{d}^{22}(\tau) + g_{loc}^{12}(\tau)g_{loc}^{21}(\tau)G_{d}^{12}(\tau)G_{d}^{21}(\tau)]$$

$$= N \int_{0}^{\beta} d\tau\left[\sum_{\alpha \beta} c_{\alpha}^{\dagger}(\tau)(\partial_{\tau} + \Sigma_{11}(\tau))c_{\alpha} + \Sigma_{12}c_{\alpha}^{\dagger}(\tau)d_{\beta}(0) + \Sigma_{21}c_{\alpha}(\tau)d_{\beta}(0)\right]$$

$$+ (i\phi(\tau))^{*}c_{\alpha}(\tau)d_{\beta}(\tau) + i\phi(\tau)c_{\alpha}^{\dagger}(\tau)d_{\beta}(\tau) + \frac{N}{V_{0}} \int_{0}^{\beta} d\tau i\phi(\tau)(i\phi(\tau))^{*}$$

$$- N \int_{0}^{\beta} d\tau[\Sigma(\tau)g_{loc}(\tau)]$$

$$- \frac{l^{2}}{2}(g_{loc}^{11}(\tau)g_{loc}^{22}(\tau) + g_{loc}^{12}(\tau)g_{loc}^{21}(\tau))$$

$$+ \frac{U^{2}N}{4}[g_{loc}^{11}(\tau)g_{d}^{11}(\tau)G_{d}^{11}(\tau)G_{d}^{22}(\tau) + g_{loc}^{12}(\tau)g_{loc}^{21}(\tau)G_{d}^{12}(\tau)G_{d}^{21}(\tau)]$$
For impurity, in saddle-point approximation with $i\phi_m \approx i\phi_0$ and in $t \to 0$ limit,

$$S_{\text{eff}} = -N \sum_{\omega} \ln[\beta^2(-i\omega + \Sigma)^2 - \beta^2(\Sigma^{12} + i\phi_0)(\Sigma^{21} - i\phi_0)] + \frac{N\beta}{V_0}\phi_0^2$$

$$- N \int_0^\beta d\tau \Sigma^{12}(\tau)g_{\text{loc}}^{21}(-\tau) + \Sigma^{21}(\tau)g_{\text{loc}}^{12}(-\tau) + \Sigma^{11}(\tau)g_{\text{loc}}(-\tau)$$

$$- N \int_0^\beta d\tau U^2N/4 [g_{\text{loc}}^{11}(-\tau)g_{\text{loc}}^{22}G_d^{11}(-\tau)G_d^{22}(\tau) + g_{\text{loc}}^{12}(-\tau)g_{\text{loc}}^{21}(\tau)G_d^{12}(-\tau)G_d^{21}(\tau)]$$

$$= -N \sum_{\omega} \ln[\beta^2(-i\omega + \Sigma)^2 - \beta^2(\Sigma^{12} + i\phi_0)(\Sigma^{21} - i\phi_0)] + \frac{N\beta}{V_0}\phi_0^2$$

$$- N \int_0^\beta d\tau \Sigma^{12}(\tau)g_{\text{loc}}^{21}(-\tau) + \Sigma^{21}(\tau)g_{\text{loc}}^{12}(-\tau) + \Sigma^{11}(\tau)g_{\text{loc}}(-\tau)$$

$$- N \int_0^\beta d\tau U^2N/4 [(g_{\text{loc}}^{11}(\tau)G_d^{11}(\tau))^2 + (g_{\text{loc}}^{12}(\tau)G_d^{12}(\tau))^2].$$

(53)

Here we use the formula

$$\int D[c^\dagger, c] e^{\beta d\tau c^\dagger(-g_{\text{loc}}^{-1}(i\omega)c)} = \int D[c^\dagger, c] e^{\beta d\tau c^\dagger(-g_{\text{loc}}^{-1}(i\omega)c)} = \text{Det}[-\beta g_{\text{loc}}^{-1}(i\omega)].$$

(54)

We note that the anomalous self-energy is an even function of imaginary time $\tau$, in contrast with the normal self-energy. Since in long-time limit $\Sigma^{12} = (\Sigma^{21})^* \propto (g_{\text{loc}}^{12})^q \propto (i\phi_0)^q$, for $q = 4$, we omit the term $\Sigma^{12}(i\phi_m)$ with $m \neq 0$, thus we obtain

$$S_{\text{eff}} = -N \sum_{\omega} \ln[\beta^2(-i\omega + \Sigma)^2 - \beta^2\phi_0^2] + \frac{N\beta}{V_0}\phi_0^2$$

$$- N \int_0^\beta d\tau \Sigma^{11}(\tau)g_{\text{loc}}(-\tau)$$

$$- N \int_0^\beta d\tau U^2N/4 [(g_{\text{loc}}^{11}(\tau)G_d^{11}(\tau))^2 + (g_{\text{loc}}^{12}(\tau)G_d^{12}(\tau))^2].$$

(55)

By solving $\delta S_{\text{eff}}/\delta\phi_0$, we obtain

$$i\phi_0 \frac{-\beta}{V_0} = \sum_{\omega} g_{\text{loc}}^{12}(i\omega)e^{i\omega 0^+},$$

$$i\phi_0 \frac{1}{V_0} = -g_{\text{loc}}^{12}(\tau).$$

(56)

Similarly, we can obtain the self-energies

$$\Sigma^{11} = -U^2N g_{\text{loc}}^{11}(\tau)G_d^{11}(\tau)G_d^{22}(-\tau)$$

$$= U^2N g_{\text{loc}}^{11}(\tau)G_d^{11}(\tau),$$

$$\Sigma^{12} = -U^2N g_{\text{loc}}^{12}(\tau)G_d^{12}(\tau)G_d^{21}(-\tau).$$

(57)

For $U \gg V_0$ in IR limit, we further obtain

$$i\phi_0 \frac{1}{V_0} = -g_{\text{loc}}^{12}(\tau) = \frac{-i\phi_0}{(-\Sigma^{11} + i\phi_0)(\Sigma^{22} - i\phi_0) + \phi_0^2} \propto -\frac{4i\phi_0}{U^4N^2},$$

(58)

since $|i\phi_0|$ is exponentially small in large $U$, and thus $\Delta_b = \frac{1}{V_0} \propto \frac{4}{U^4N^2}$. 

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In the presence of small static bosonic gap term $i \phi_0 \ll U$ and at $\omega_c \ll T \ll \omega \ll U$, the Green’s function decays exponentially. Based on the variational ansatz of Ref. [22], the anomalous Green’s function is related to on-site SYK coupling by the following relation

$$g_{\text{loc}}^A(\tau) \sim -i \phi_0 \frac{e^{-|\tau| \phi_0^2 / U}}{\pi U} \ln(1 + c \frac{U}{\phi_0^2 |\tau|}) \equiv -\frac{|i \phi_0|}{V_0}, \quad (59)$$

where $c$ is a tunable parameter. It is easy to find that term $\ln(1 + c \frac{U}{\phi_0^2 |\tau|})$ is related to the Gaussian fluctuation part of the thermodynamical potential, which reads in frequency domain $\ln(1 - g_{\text{loc}}(\omega) \Sigma(\omega))$, and thus the effect of bosonic gap can be neglected in the case $|U| \gg |\phi_0|$. In UV and IR limits, the $g_{\text{loc}}^A$ in non-Fermi liquid phase reads

$$g_{\text{loc}}^{UV}(\tau) \sim -|i \phi_0| \frac{1}{\pi U} \ln\left(\frac{U}{\phi_0^2 |\tau|}\right) \sim -|i \phi_0| \frac{2}{\pi U} \ln\left(\frac{U}{\phi_0}\right),$$

$$g_{\text{loc}}^{IR}(\tau) \sim -|i \phi_0| \frac{1}{\pi U} e^{-1} \ln 2 \approx -0.255 |i \phi_0| \frac{1}{\pi U}, \quad (60)$$

respectively. Note that in IR limit, we have $|i \phi_0| \gg \phi_0^2 / U \gg \omega$, which is a signature of superconductivity order in Cooper channel.

5 Emergent SU(2) symmetry in case of $|U| = 4|V_0|$

Next we discuss the emergence of gapless boson mode away from quantum critical point due to the conserved boson number in each subsystem (or each pair of neighbor sites in the lattice) with SU(2) symmetry in the special case of $|U| = 4|V_0|$. For nonlocal version of SYK interaction term, we have

$$S_{\text{int}} = \int_0^\beta d\tau \sum_{i,j;\alpha,\beta,\alpha',\beta'} U_{\alpha \beta \alpha' \beta'} c_{i\alpha}(\tau) d_{j\beta}^\dagger(\tau) d_{j'\beta'}(\tau) c_{i'\alpha'}(\tau)$$

$$= \int_0^\beta d\tau \left[ \frac{1}{4N} \sum_{i,j;\alpha,\beta,\alpha',\beta'} U c_{i\alpha}(\tau) d_{j\beta}^\dagger(\tau) d_{j'\beta'}(\tau) c_{i'\alpha'}(\tau) \right], \quad (61)$$

where we use the new basis $c_{\alpha, \alpha' = \pm} = \frac{1}{\sqrt{2}} (c_{\alpha} \pm ic_{\alpha})$. Thus in the special case of $|U| = 4|V_0|$, the Coulomb $Q$ term, where we consider only the interaction within each pair of the neighbor sites $i, j = 1, 2$, can be incorporated into the above Hamiltonian, which is SU(2) invariant due to the conserved boson number in each subsystem (with at most $N(N-1)/2$ pairs, i.e., the case that each $c$ ($d$) fermion is surrounded by four $d$ ($c$) fermions in the square lattice consist of $N$ $c$ fermions and $N$ $d$ fermions, and thus there are at most $N(N-1)/2$ associated symmetries).

In the new basis, the local Green’s function can be written as

$$g_{\text{loc}}^{x=\pm}(\tau) = \frac{1}{N} \sum_{\alpha} \langle c_{i\alpha}(\tau) c_{i\alpha}^\dagger(0) \rangle$$

$$= \frac{1}{N} \sum_{\alpha} \left\{ \frac{1}{2} (c_{\alpha}(\tau) c_{\alpha}^\dagger(0) + c_{\alpha}(\tau) c_{\alpha}^\dagger(0)) + i \frac{1}{2} (c_{\alpha}(\tau) c_{\alpha}^\dagger(0) \pm c_{\alpha}(\tau) c_{\alpha}^\dagger(0)) \right\}. \quad (62)$$

Due to the SU(2) symmetry preserved by the conservation of boson number in sublattice (pseudospin) space, the expectation of $Q$ term is a conserved quantity (good quantum number) and thus $[Q, H] = 0$, where $H = S_{\text{int}} + Q$ (and thus $[S_{\text{int}}, H] = 0$). This can be proved by rewriting
the expectation value of $Q$ in new basis (in $\tau \to 0$ limit) as

$$
\langle Q \rangle = -\frac{V_0}{N} \sum_{i'j'} \sum_{\alpha\beta} c_{i'\alpha}^+ d_{j'\beta}^\dagger d_{j'\beta} c_{i'\alpha}
$$

$$
\quad = -\frac{V_0}{N} \sum_{i,j} \frac{1}{2} (c_i^+ + ic_j^+)(d_i^+ \pm id_j)(c_i \mp id_j) + H.c.
$$

$$
\quad = -\frac{V_0}{N} \frac{1}{2} (c_i^+ - ic_j^+)(d_i^+ + id_j)(d_i - id_j)(c_i + ic_j) + H.c.
$$

$$
\quad = \frac{V_0}{N} 4(1 + g_{loc}^{11})(1 + G_d^{11}),
$$

which is independent of imaginary time. Here $g(\tau \to 0^+) = -c(\tau)c^\dagger(0)$. The last line use the result of mirror symmetry (when the intersite and intrasite SYK interactions are with the same strength) $g_{11}(\tau) = g_{22}(\tau), g_{12}(\tau) = -g_{21}(\tau)$. Thus $\langle Q \rangle$ can be tuned by changing the $V_0$ (or equivalently, $U$, in this special case), just like the turning of globally conserved U(1) charge by changing the chemical potential. We note that the mirror symmetry here can be enforced by the tunneling amplitude with phase factor $\pi/2$ or simply by the particle-hole symmetry with $g_{12}(\tau)^* = g_{12}(-\tau)$ (i.e., ignore the dynamic part of anomalous self-energy).

Next we conclude some situations related to value of $\langle Q \rangle$ in the presence of mirror symmetry,

$$
\begin{align*}
\langle Q \rangle & \sim g_{loc}^{12} G_d^{12}, & V_0 \neq 0, \text{without SU}(2), \ U(1) \otimes U(1), \ \text{but with U}(1), \\
\langle Q \rangle & \sim g_{loc}^{12} G_d^{12} V_0 \to 0, & V_0 \to 0, \ \text{without SU}(2), \ U(1) \otimes U(1), \ \text{but with U}(1), \\
\langle Q \rangle & = 0, & V_0 = 0, \ \text{with SU}(2) \ \text{and U}(1) \otimes U(1) \ \text{and U}(1), \\
\langle Q \rangle & = \frac{1}{N} 4(1 + g_{loc}^{11})(1 + G_d^{11}), & V_0 \neq 0, \ \text{with SU}(2) \ \text{and U}(1) \otimes U(1) \ \text{and U}(1),
\end{align*}
$$

In first case, a gap opens through continuous spontaneous breaking of $U(1) \otimes U(1)$ symmetry (through Higgs mechanism) with superconducting instability, since $Q$ term does not conserves the charge of each subsystem. In second case, $g_{loc}^{12} \neq 0$ but $[Q, H] \approx 0$, thus a gap opens through continuous spontaneous breaking of $U(1) \otimes U(1)$ symmetry down to $U(1)$ symmetry, and the gapless Goldstone mode exists due to the conserved total gauge charge carried by bosons. In the mean time, the electronic fluctuation destroys the Fermi surface and lead to non-Fermi liquid phase. This is the case we discussed in Sec.4. Note that a finite $|V_0|$ must leads to nonzero anomalous component $g_{loc}^{12}$, but a nonzero $g_{loc}^{12}$ does not necessarily requires a finite $|V_0|$ (it can also be generated through spontaneous symmetry breaking). In third case, both the local gauge symmetry and global U(1) symmetry are preserved, and the $Q$ term cannot breaks any symmetries. In fourth case, which is what we discuss in this section, the stable gapless Goldstone mode exists which do not require continuous symmetry breaking, and in the mean time, the global U(1) symmetry is also preserved since the total boson number is conserved as the $[Q, H_{SU(2)}] = 0$ (it contains the hermitian conjugate terms within itself even in the old basis). In this case the $U(1) \otimes U(1)$ symmetry is also conserved since $\langle Q \rangle$ does not depend on the anomalous component although $g_{loc}^{12} \neq 0$. Also, in the presence of SU(2) symmetry (at half-filling), the hopping term vanishes, since

$$
H_{hop} = \frac{1}{\sqrt{N}} \sum_{i'j'} t_{i'j'} c_{i'}^+ c_j^\dagger + \frac{1}{\sqrt{N}} \sum_{i'j'} t_{i'j'} c_{j'}^+ c_i^\dagger
$$

$$
\quad = \frac{1}{\sqrt{N}} \sum_{ij} t_{ij} \frac{1}{2} (c_i^+ \mp ic_j^+)(c_i \mp ic_j) + \frac{1}{\sqrt{N}} \sum_{ij} t_{ij} \frac{1}{2} (c_i^\dagger \pm ic_j^\dagger)(c_i \pm ic_j)
$$

$$
\quad = \frac{1}{\sqrt{N}} \sum_{ij} t_{12}(g_{loc}^{11} - g_{loc}^{22}) = 0.
$$
This implies that in the presence of SU(2) symmetry there is no exchanges between subsystems of 1(c) and 2(d) subsystems. This is necessary for the preservation of U(1)⊗U(1) symmetry, and is also why U(1)⊗U(1) symmetry is broken in the absence of SU(2) symmetry (first and second cases). Note that in last two cases the U(2)~U(1)×SU(2) is also preserved, and the conformal solution can be obtained even in the presence of finite small |V0|, which turns out to be power law decaying gloc(τ) and Σ(τ). We note that the Green’s functions with the same indexes (g_{11}, g_{22}) enable only the intrasite SYK coupling, while the ones with different indexes (g_{12}, g_{21}) enable the intersite SYK coupling which breaks the U(1)⊗U(1) symmetry. Note that the propagators appearing in Eq.(65) are all at limit τ → 0⁺ thus are time-independent. Also, the existence of U(1) symmetry here only appears in the pure SYK states, i.e., with W = 0 and ωc = ω = 0 and thus the SYK regime is being extended to zero temperature limit. Thus the time-derivative term −∂τ = iω must vanishes in the presence of U(1) symmetry.

The conserved ⟨Q⟩ can be proved as following.

\[ [Q, H] = [Q, Q + H_{SYK}] = [Q, H_{SYK}] = [e^{τH_{SYK}} (0)e^{-τH_{SYK}}, H_{SYK}] = [e^{τH_{SYK}}, Q(0)] / Q(0), H_{SYK} = 0, \]

where \( H_{SYK} = S_{\text{int}} \) is the nonlocal SYK Hamiltonian, and we have \( ⟨H_{SYK}⟩ = \frac{U}{2N} 4(1 + g_{loc}^{11})(1 + G_{d}^{11}) = −⟨Q⟩ \). While for the case of local SYK interaction (and the part of screened Coulomb interaction that incorporated into the SYK term is also of on-site type), than after SU(2) Hamiltonian reads (with now the local SYK term)

\[ H = \int_{0}^{β} dτ \frac{1}{2N} \sum_{i,j'} \sum_{αβ} U_{iα} d_{iα}(τ)d_{jβ}(τ)c_{iα}(τ)c_{jβ}(τ) + Q. \]  

It is easy to obtain through calculation that the expectation value of local SYK term reads −8(1 + g_{loc}^{11})(1 + G_{d}^{11}). Due to the relation (at τ → 0⁺)

\[ \[c_{i}d_{i}d_{i}c_{i}, c_{i}d_{i}d_{i}c_{i}\] = (1 + g_{loc}^{11})(−g_{loc}^{11})[G_{d}^{21}(−G_{d}^{12}) − G_{d}^{12}(−G_{d}^{21})] = 0, \]

we conclude that for the local SYK non Fermi liquid with SU(2), [Q, H] = 0, i.e., the Coulomb term is still a conserved quantity just like in the non-local SYK non Fermi liquid.

Finally, we note that the chemical potential term will not breaks the SU(2) or U(1) or U(1)⊗U(1) symmetries (in pure SYK regime). As proved in the new basis

\[ \langle H_μ \rangle = \sum_{i'} μc_{i'}^½ c_{i'} + \sum_{j'} μc_{j'}^½ c_{j'} = \sum_{ij} \mu \frac{1}{2}(c_{i} ± ic_{j})(c_{i} ± ic_{j}) + \sum_{ij} \mu \frac{1}{2}(c_{i} ± i c_{j})(c_{i} ± i c_{j}) = \mu \sum_{12}(1 + g_{loc}^{11} ± ig_{loc}^{21}) + μ \sum_{12}(1 + g_{loc}^{11} ± ig_{loc}^{21}) = 2μ \sum_{1}(1 + g_{loc}^{11}) \]

which contains only the intrasite process. Note that the chemical potential term will contributes to localize the electrons, however strong enough quantum fluctuation induced by SYK coupling will makes this single-particle term irrelevant. In conclusion, the chemical term and hopping term will not breaks the U(1)⊗U(1) and global U(1) symmetries; while in the absence of SU(2) symmetry, the hopping term as well as the Q term will breaks the U(1)⊗U(1) symmetry. The
6 Conclusion

Next we conclude the phase transitions and the main results of this paper. As shown in the phase diagram of Fig.2, the Fermi-liquid phase appears in $\max[T, \omega] \ll \omega_c$ regime, where the Landau Fermi liquid appears at zero temperature and the disordered Fermi liquid appears at temperature lower than coherence scale $W^2/U$ but higher that other low energy cutoffs. The stable gapless boson excitations appears either in the quantum critical point (with a sharp critical fermi surface) of the SYK non-Fermi liquid regime which is unperturbed, i.e., there without the Coulomb interaction or interaction between SYK modes, or in the low frequency (long time) SU(2) regimes where the nonlocal part of $|V_0| = |U|/2$ are incorporated into the SYK term, and the expectation value of the remaining Coulomb term with $|V_0| \ll U$ becomes a conserved quantity, and thus will not breaks the symmetry in subspace. A signature of the stable gapless boson excitation is the algebraic boson density correlations, which also falls off algebraicallywith time, insteads of exponentially which is the case in the $|V_0| \ll U$ gapped SYK regime where the SU(2) symmetry is absent. In fact, in the presence of SU(2) symmetry, the finite $V_0$ term will not breaks the global U(1) symmetry unless the matter field (like charges) condensed, while in $|V_0| \ll U$ gapped SYK phase the SU(2) symmetry is absent and the expectation value of Coulomb term is nonzero, thus the U(1) gauge symmetry is broken since the total particle number of lattice is changed as long as the hermitian conjugate term is missing within the Coulomb term, and thus the two-point correlation function decays exponentially and finally to zero at Gaussian fixed point. Due to the existence of Fermi liquid regime and the finite fremonic frequency term $i\omega$, it is obvious that the U(1) global symmetry is broken in this phase diagram.

Further away from the quantum critical point the system enters into gapped phase again once the SU(2) symmetry is broken by condensation of charges or defects or by breaks the equality between the strength of intersite and intrasite SYK couplings\[30, 16\]. Finally, there is a high-temperature insulating phase which does not shown in the phase diagram (where the hopping becomes perturbative $t \ll U, T$), with its frequency-independent DMFT self-energy $\Sigma = Un_c \ll |\omega|$ (obtained by Hartree-Fock theory\[26\]) in the UV limit with $\omega \rightarrow \infty (\tau \rightarrow \beta^-)$ and $T \gg U \gg t$. This is unlike the Mott insulating phase in zero-temperature limit which is a spin liquid and the emergence of gapless critical excitations do not rely on symmetry breaking. In high-temperature limit, the Green’s function in DMFT insulating phase reads $g_{loc}(i\omega) = 1/i\omega$ as we obtained in Eq.(88). In this paper, we discuss the possible emergence of SYK non-Fermi liquid in the incoherent critical metal phase generated by the Hubbard-type four-fermion on-site random interactions which follow the Gaussian distribution. The SYK behaviors here are similar to the $q = 4$ SYK model except an additional factor $N$. The phase transitions to the Fermi liquid regime and high temperature DMFT insulating regime are possible through the turning of related parameters, including the temperature, coherence scale, SYK coupling, and Coulomb boson field. Taking the nonlocal Coulomb interaction (with pseudospin interaction) as an example, we also discuss the existence of stable gapless bosonic excitations in SYK compressible non-Fermi liquid phase away from the quantum critical point (and thus prevent the condensation), which is due to the preserved SU(2) local gauge symmetry in the special case of $|V_0| = |U|/2$ in incoherent critical metal phase with strong SYK coupling and at intermedia temperature $U \gg T \gg \omega_c$. We found that in this special case, the global U(1)
and $U(1) \otimes U(1)$ symmetries are also preserved.

As we stated in the beginning, the most important premise of this paper is the same mechanism for the transitions from orbital $\alpha$ to $\alpha'$ and that from orbital $\beta$ to $\beta'$, which make the Hamiltonian (Eq.(1)) describing the on-site four fermion interaction become a SYK$_2 \times$SYK$_2$ model. However, for the emergence of SYK physics, the bosonic pair condensation, which will breaks gauge charge conservation, should be avoided. In Hamiltonian shown in Eq.(1), the term $C_{\alpha \alpha'} D_{\beta \beta'} \equiv \langle c^\dagger_{\alpha \beta} d_{\beta'} c_{\alpha'} \rangle$ can be treated as a $N^2 \times N^2$ diagonal matrix. This is the SYK case, with the relation $\langle c^\dagger_{\alpha \beta} d_{\beta'} c_{\alpha'} \rangle = \langle c^\dagger_{\alpha \beta} \rangle \langle d_{\beta'} \rangle - \langle c^\dagger_{\alpha \beta} d_{\beta'} \rangle$ being satisfied at zero temperature. While when the pair condensation happen, i.e., $\langle c^\dagger_{\alpha \beta} d_{\beta'} \rangle$, the eigenvalues split to create discrete spectrum and lead to the formation of off-diagonal long-range order. This breaks the continuous distribution of SYK spectrum and lead to level statistics agree with Poisson distribution, which supports the Fermi liquid phase instead of SYK non-Fermi liquid. For pair condensate case, the eigenvalues of $\langle c^\dagger_{\alpha \beta} d_{\beta'} \rangle$ satisfy $\sum n^2 \lambda_l = \min[|n_c|, n_d] \leq N^2$ thus $\lambda_l \leq 1$. However, for positive-definite matrix $\langle c^\dagger_{\alpha \beta} d_{\beta'} \rangle$, due to the existence of off-diagonal term, there is a largest eigenvalue $\lambda_{\text{max}}$ being of order of $N^2$, and the rest $N^2 - 1$ eigenvalues being of order of 1. A simple example is the unit matrix, whose eigenvalues contain the largest one and the rest are zeros. Similarly, for matrix $N^2 \times N^2$ matrix $U_{\alpha \beta \alpha' \beta'}$, we have $\lambda^2 = U_{\alpha \beta \alpha' \beta'}^2 = \frac{U^2}{4N^2}$ in Gaussian unitary ensemble (GUE), and $\lambda^2 = U_{\alpha \beta \alpha' \beta'}^2 = \frac{U^2}{4N^2} + \frac{O(\delta_{\alpha \beta \alpha' \beta'})}{N^2}$ in Gaussian orthogonal ensemble (GOE). Thus in GUE we have $\sum \lambda^2 = \frac{U^2}{4}$, and in GOE we have $\sum \lambda^2 = \frac{U^2}{4} + O(\delta_{\alpha \beta \alpha' \beta'})$. The matrix $U_{\alpha \beta \alpha' \beta'}$ in GUE does not admit the anomalous term, thus it has a set of eigenvalues in ascending order. While the matrix $U_{\alpha \beta \alpha' \beta'}$ in GOE admits the anomalous terms (which is necessary for the pair condensation), and thus has the largest eigenvalue $\lambda_{\text{max}} = \pm \frac{U}{2\sqrt{2}}$, and $\frac{N^2-2}{2}$ eigenvalues $\pm \frac{1}{\sqrt{N^2-2}}$ and $\frac{N^2-2}{2}$ eigenvalues $\pm \frac{1}{\sqrt{N^2-2}}$, which forms a discrete spectrum.

7 Appendix A: High temperature insulating phase in DMFT

In atomic limit where the interaction $U$ between impurity and the conduction electron is much larger than the hopping of impurity, the exact self-energy and Green’s function $g_{\text{loc}}$ of impurity can be obtained through iterative algorithm. Firstly assuming an initial $\Sigma(i \omega)$ and substituting it into expression $g_{\text{loc}} = \int_0^\infty d\epsilon \frac{\rho(\epsilon)}{\epsilon - \Sigma(i \omega)}$, then at half-filling we have $g_{\text{loc}}^{-1} = g_{\text{loc}}^{-1} + \Sigma(i \omega)$, and then $g_{\text{loc}}^{-1} = g_{\text{loc}}^{-1} + \left(\frac{\rho(\epsilon)}{\epsilon - \Sigma(i \omega)}\right)^{-1}$, then the new self-energy is obtained as $\Sigma(i \omega)' = (g_{\text{loc}}^{-1})^{-1} - (g_{\text{loc}}^{-1})^{-1}$. We need to substitute this new self-energy to the begining and repeat the procedure until converges. There is another simpler way to obtain the self-energy contributed by on-site Hubbard repulsion $U$ between impurity and the localized electrons, which is the Baym-Kadanoff method. For $n_d$ as a quantum variable, the effect of strong on-site interaction between impurity and localized electrons can be represented by a time-dependent (spatially uniform) auxiliary field which is added into the exponential of partition function.

$$Z = \text{Tr}_d \text{Tr}_c \langle e^{-\beta H(\mu) + \int_0^\beta d\tau \lambda(\tau\tau')c(\tau)c(\tau')} \rangle = \text{Tr}_c \langle e^{-\beta H_0(\mu) + \int_0^\beta d\tau \lambda(\tau\tau')c(\tau)c(\tau')} \rangle + e^{-\beta E_d} \text{Tr}_c \langle e^{-\beta H_0(\mu - U) + \int_0^\beta d\tau \lambda(\tau\tau')c(\tau)c(\tau')} \rangle,$$

where in insulating phase ($t = 0$) we define $\lambda(\tau, \tau') = \lambda(\tau - \tau') = \sum_{\omega} e^{-i\omega(\tau - \tau')} (\mu - \Sigma_{\text{loc}})$ with $\Sigma_{\text{loc}} = \sum_{\alpha} \Sigma_{\alpha}$ the local self-energy of impurity (Eq.(3)). When $\mu = \Sigma_{\text{loc}}$, $Z_0 = (1 + e^{\beta\mu})\exp[\sum_{\omega} \ln(i \omega + \mu) e^{i \omega 0^+}]$ reduces to $(1 + e^{\beta\mu})$ which is the well known noninteracting result.
and $H_0(\mu) = -\mu \sum_i c_i^\dagger c_i$. Then the local Green’s function reads
\begin{equation}
\begin{aligned}
g_{\text{loc}}(\tau, \tau') &= -\frac{1}{Z(\lambda)} \frac{\delta Z(\lambda)}{\delta \lambda(\tau, \tau')} \\
&= -\text{Tr}_e \langle e^{-\beta H_0(\mu-U)} + \int_0^\beta d\tau' \int_0^\beta d\tau' c(\tau) \chi(\tau, \tau') \delta_{\tau', \tau'} c(\tau') \rangle \\
&= \frac{-\text{Tr}_e \langle e^{-\beta H_0(\mu-U)} + \int_0^\beta d\tau' \int_0^\beta d\tau' c(\tau) \chi(\tau, \tau') \delta_{\tau', \tau'} c(\tau') \rangle}{Z(\lambda)} \\
\end{aligned}
\end{equation}
where $\chi(\tau) = U \Theta(\tau' - \tau)$. In large-N limit with saddle-point approximation, the saddle point $\lambda_0(\tau, \tau')$ can be obtained by solving
\begin{equation}
\frac{\delta S(\tau')}{\delta \lambda(\tau, \tau')} = \frac{\delta \int_0^\beta d\tau \lambda(\tau, \tau') d(\tau)}{\delta \int_0^\beta d\tau \lambda(\tau, \tau')} = 0,
\end{equation}
where we denotes $-S(\tau') = -\beta H_0 + \int_0^\beta d\tau \lambda(\tau, \tau') c(\tau) c(\tau)$.

In spirit of Baym-Kadanoff method, firstly we use the many-body local consistent relation in high-frequency limit with short time propagation,
\begin{equation}
\lim_{\tau \to -\eta} \int_k \Sigma(k) G(k) e^{-ik\tau} = \langle \Delta^\dagger \Delta \rangle = U \langle n_{c} n_{d} \rangle,
\end{equation}
where $\Delta^\dagger = c^d d^c$ is a field operator in Cooper channel. In imaginary-time domain it becomes
\begin{equation}
\langle \Delta^\dagger \Delta \rangle = \langle d^\dagger(\tau) c(\tau) c(\tau) d(\tau + \tau) \rangle = \langle d^\dagger(\tau) c(\tau) c(\tau) c(\tau + \tau) \rangle,
\end{equation}
In the absence of anomalous contribution, and using the definition of generalized susceptibility, we have (we simply denote $\tau_i \equiv i$)
\begin{equation}
\begin{aligned}
\Sigma_c(1, 1') G_c(1', 2) &= -U \langle d^\dagger(\tau) c(\tau) c(\tau + \tau) \rangle \\
&= -U \chi_{\text{gs}}(1, 1, 1, 2) - G_d(1, 1') G_c(1, 2) \\
&= -U \Delta_0 G_c(1, 2) - G_d(1, 1') G_c(1, 2) \\
&= -U \Delta_c G_c(1, 1') G_d(2^- + 2) - G_d(1, 1') G_c(1, 2),
\end{aligned}
\end{equation}
by multiplying $G_c^{-1}(2, 1)$ in both sides we obtain
\begin{equation}
\Sigma_c(1, 1') = U \Gamma(1, 1, 3, 4) c(\tau) G_c(1, 2) + U G_d(1, 1') \\
= U \Gamma(1, 1, 3, 4) c(\tau) G_d(2^- + 2) + U \langle n_d \rangle,
\end{equation}
which becomes
\begin{equation}
\Sigma_c(1, 1') = U \Gamma(1, 1, 3, 4) c(\tau) G_d(2^- + 2) + U \frac{1}{2},
\end{equation}
in the case of half-filling and in the presence of commutation relation $[\Delta, \Delta^\dagger] = 1 - n_c - n_d = 0$.

For the localized electrons around the impurity, the bare charge susceptibility reads
\begin{equation}
\chi_d^0 = -\sum_k G(k, i\nu_m) G(k + q, i\nu_{m'})
\end{equation}
Considering the vertex function $\Gamma_d = \delta \Sigma_m / \delta G_m$ in the limit of vanishing bosonic frequency, the dressed susceptibilities reads

$$\chi_d = \frac{1}{\chi_0^{-1} + \Gamma_d} = \begin{cases} \frac{-G(\nu_m) - G(\nu_m')}{i\nu_m - i\nu_m'}, & q = 0, \\ \frac{2G(\nu_m)}{-i\nu_m + i\nu_m' + 2(\Sigma_m + G(\nu_m)\Gamma_d)}, & q = \pi. \end{cases} \quad (80)$$

The hybridization function can be written in spectral representation as

$$\Delta_h = t^2 g_{loc}(i\omega) = \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} \frac{\rho(\varepsilon)}{\varepsilon} \frac{\varepsilon^2}{i\omega - \varepsilon} = i\omega \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} \frac{\rho(\varepsilon)}{i\omega - \varepsilon} = -i\omega + (i\omega)^2 \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} \frac{\rho(\varepsilon)}{i\omega - \varepsilon} = -i\omega + (i\omega)^2 g_{loc}. \quad (81)$$

The $g_{loc}$ can be expanded in IR and UV limits as

$$g_{loc}^{IR} = \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} \frac{\rho(\varepsilon)}{i\omega - \varepsilon} = \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} \rho(\varepsilon) \left[ \frac{1}{\varepsilon} - \frac{i\omega}{\varepsilon^2} - \frac{(i\omega)^2}{\varepsilon^3} - O(\omega^3) \right], \quad (82)$$

and

$$g_{loc}^{UV} = \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} \rho(\varepsilon) \left[ \frac{1}{i\omega} + \frac{\varepsilon}{(i\omega)^2} + \frac{(\varepsilon)^2}{(i\omega)^3} + O(\omega^{-4}) \right], \quad (83)$$

respectively. Thus in IR limit, we can approximately use $\Delta_h = -i\omega$ and $g_{loc} = \frac{1}{i\omega(z+1)}$. A more precise one can be obtained by calculating the self-consistent loop with a initial $g_{loc} = \frac{1}{i\omega(z+1)}$ substituted into the denominator of $g_{loc} = \frac{1}{i\omega(z+1)} - (i\omega)^2 g_{loc}$, and then repeat until it converges to a fixed point. In IR limit, there are two tendency of real part of self-energy for metallic phase and insulating phase, due to the different behavior of spectral functions. In metallic phase, especially for the incoherent metal which has a large imaginary part of self-energy, the spectral density is a constant in IR limit, thus

$$\text{Re} \Sigma(\omega + i0^+) - \frac{U}{2} = \left[ \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} \rho(\varepsilon) \frac{\omega}{\varepsilon^2} \right]^{-1} \approx [\omega \rho \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} \frac{1}{\omega^2 + \varepsilon^2}]^{-1} = \frac{2}{\rho \omega^{1-b/2}} \to 0, \quad (84)$$

where $b/2 \gg 1$. In insulating phase, the spectral density (as well as the imaginary part of
self-energy) is a delta-function in IR limit, thus

\[ \text{Re}\Sigma(\omega + i0^+) - \frac{U}{2} = \lim_{\epsilon \to 0} \left[ \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \frac{\delta(\omega)\omega}{\epsilon^2} \right] - 1 \approx \left[ \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \frac{\delta(\omega)}{\omega^2 + \epsilon^2} \right] - 1 \]

(85)

Thus the cancellation of interacting chemical potential by the shift of self-energy at half-filling is only possible in incoherent states.

While in UV limit (but not in the incoherent critical metal phase due to the finite value of hybridization \(\Delta_h\)), there exist the following sum rules for uniform case \(q = 0\) (see Appendix.A)

\[
\int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \rho(\epsilon) = 1,
\]

\[
\int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \rho(\epsilon) \epsilon = \{c(\tau), H, c^\dagger(0)\},
\]

\[
\rho(\tau) = -i\{c(\tau), c^\dagger(0)\} \bigg|_{\tau = 0} = e^{-\tau E_c} E_c \bigg|_{\tau = 0} = E_c \equiv \sum_k \varepsilon_k - \mu + \Sigma^{DMFT} + \Sigma(i\omega),
\]

we obtain

\[ g_{loc} = \frac{1}{i\omega + \mu - \Sigma^{DMFT} - \Sigma^{UV}(i\omega) - \Delta_h} \]

(86)

which becomes \(g_{loc} = \frac{1}{i\omega - \Sigma^{UV}(i\omega)}\) at half-filling but with \(\Sigma(i\omega)\) no more follows the expression of Eq.(83). In above, the spectral function reads \(\rho(\epsilon) = -\text{Im}g_{loc}(\epsilon + i0^+)\), \(\rho(t) = \{c(t), c^\dagger(0)\}\) and \(\rho(\tau) = -i\{c(\tau), c^\dagger(0)\}\) in frequency, real time, and imaginary time domains, respectively.

At half-filling, the weak-coupling expansion (in iterated perturbation theory approximation) of self-energy with shifted Weiss function, \(\Sigma(i\omega) = \frac{U}{2} + \frac{U^2}{2}\Sigma^G_0 = \frac{U}{2} + \frac{U^2}{4} \frac{1}{i\omega}\), can still be applied in atomic limit \((U \gg \rho(0))\). Then in high-frequency non-Fermi-liquid phase, the local Green’s function reads \(g_{loc} = \frac{1}{i\omega - U/2}\), and for high energy excitations with \(\omega \gg U\), we further have \(g_{loc} = \frac{1}{i\omega}\), which is exactly the same with the shifted Weiss function \(\Sigma^G_0 = \Sigma^G - U/2\). This local Green’s function can also be obtained by

\[ g_{loc} = \left\{ \frac{1}{i\omega + \mu - \Sigma^{DMFT} - \Sigma^{UV}(i\omega)} + \frac{1}{i\omega - U/2} \right\} = \frac{i\omega}{(i\omega)^2 - U^2/4} \approx \frac{1}{i\omega}, \]

(87)

(88)

The identity between DMFT results and the SYK self-consistency in this high temperature insulating phase can be seen through the series expansion of small \(U_{SYK}\) in the iterative process as we shown in Appendix.G.

Some formulas are required to obtain Eq.(refsumrule). The first one is the local Green’s function \(g_{loc}(\tau) = -\langle c(\tau)c^\dagger(0)\rangle\) (or equivalently in real time domain \(g_{loc}(t) = -i\langle c(t)c^\dagger(0)\rangle\)).
The second one is the relation \( \frac{\partial}{\partial \tau} c(\tau) = \frac{\partial}{\partial \tau} e^{\tau H_0} e^{-\tau H_0} = -[e^{\tau H_0} c e^{-\tau H_0}, H_0] \) (or in real time domain \(-i \frac{\partial}{\partial t} c(t) = \frac{\partial}{\partial t} e^{i t H_0} e^{-i t H_0} = -[e^{i t H_0} c e^{-i t H_0}, H_0] \)). The third one is the Baker-Campbell-Hausdorff formula \( e^X e^{-X} = A_{dX} Y = e^{adX} Y = e^{[X,Y]/Y} Y = Y e^{[X,Y]/[i\omega]} \). From the third formula, we have

\[
c(\tau) = e^{\tau H_0} c(0) e^{-\tau H_0} = e^{\tau E_c c} c(0) e^{-\tau E_c c} = e^{-\tau E_c c}(0),
\]

where \( E_c = \sum \epsilon_k - \mu + \Sigma^{DMFT} + \Sigma(i\omega) \).

In terms of the interaction effect between impurity and localized electrons, as well as the distance between them in Hilbert space \([10]\), we next discuss the term \( \langle \Delta^2 \rangle - \langle \Delta \rangle^2 \) (compared to the term \( \langle \epsilon^2 \rangle - \langle \epsilon \rangle^2 \)). With Eq.(86), we can further write

\[
\int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \rho(\epsilon) \epsilon^2 = \{ [c(\tau), H], H], c(0) \} \\
= \left\{ -\frac{\partial}{\partial \tau} E_c c(\tau), c(0) \right\} \bigg|_{\tau=0} \\
= E_c^2 \langle c(0), c(0) \rangle
\]

thus \( \langle \epsilon^2 \rangle - \langle \epsilon \rangle^2 = 0 \).

For \( \chi = \langle \Delta(\tau) \Delta^\dagger(0) \rangle \) (or \( \chi(i\Omega) = \int \frac{d\omega}{2\pi} \frac{\rho_B(\omega)}{\Omega-\omega} \) in frequency domain), then

\[
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \rho_B(\omega) \omega = \{ [\Delta(\tau), H], \Delta^\dagger(0) \} \\
= \left\{ -\frac{\partial}{\partial \tau} \Delta(\tau), \Delta^\dagger(0) \right\} \\
= \{ (E_c + E_d) d c, c^\dagger d^\dagger \} \\
= (E_c + E_d)(1 - n_c - n_d),
\]

\[
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \rho_B(\omega) \omega^2 = \{ [\Delta(\tau), H], \Delta^\dagger(0) \} \\
= \{ [(E_c + E_d) d(\tau) c(\tau), H], \Delta^\dagger(0) \} \\
= \left\{ -\frac{\partial}{\partial \tau} (E_c + E_d) d(\tau) c(\tau), \Delta^\dagger(0) \right\} \\
= \{ (E_c + E_d)^2 d c, c^\dagger d^\dagger \} \\
= (E_c + E_d)^2 (1 - n_c - n_d),
\]

thus \( \langle \Delta^2 \rangle - \langle \Delta \rangle^2 = (E_c + E_d)^2[-(n_c + n_d)^2 + n_c + n_d] \), where \( E_d = -\mu + \Sigma^{DMFT} + \Sigma_d(i\omega) \). Since we consider the on-site interaction \( U \), the term \( \langle \Delta^2 \rangle - \langle \Delta \rangle^2 \) is in fact extensive and scale linearly with \( N \).

8 Appendix.B: Zero mode

The existence of zero-mode discussed in Ref.[8] is different from the one studied in this paper. In this paper, the zero mode (zero bosonic gap and \( \mu = 0 \)) vanish when \( U \gg V_0 \) since \( |i\phi_m| \) is exponentially small in large \( U \). Thus in the presence of Colomb boson, small \( V_0 \) corresponds to large boson mass term \( \Delta_b = \frac{1}{V_0} \), this is highly possible in the presence of large \( U \) and finite \( V_0 \). When \( |\Delta_b| > \mu \), the system enters the insulating phase. While for finite size one-dimensional
spinless fermion chain as discussed in Ref. [8], the zero mode exists at arbitrarily small attractive intersite interaction and with the U(1) gauge symmetry been preserved, since the interaction term consist of, instead of the charge \(Q = \frac{1}{N} \sum_i c_i^\dagger c_i\) (the fermion number constrain implies \(0 < Q < 1\)), the globally conserved U(1) charge per site \(\hat Q = \frac{1}{N} \sum_i c_i^\dagger c_i - \frac{1}{2}\), which has only two possible values: \(\frac{1}{2}\) and \(-\frac{1}{2}\) \((\sum_i c_i^\dagger c_i + \sum_i c_i^\dagger c_i = N)\). We note that (at \(\tau \to 0^+\)) \(g_{\text{loc}}(\tau) = -\langle c(\tau)c^\dagger(0)\rangle = \hat Q - \frac{1}{2}\), \(g_{\text{loc}}(-\tau) = -\langle c(0)c^\dagger(\tau)\rangle = \hat Q + \frac{1}{2}\).

9 Appendix.C: Related correlations

In the presence of single-site SYK Green function \(g_{\text{loc}}(\tau) = \frac{\text{sign}[\tau]}{\sqrt{UN^{1/2}|\tau|}}\), then for hopping \(t \to 0\), since the incoherent regime extends to zero temperature and frequency as stated above, we have \(\tau \approx is \approx t\) and \(\beta \to \infty\)

\[
g_{\text{loc}}(i\omega) = \int_0^\beta d\tau e^{i\omega\tau}g_{\text{loc}}(\tau)
= \int_{-\infty}^\infty dt e^{i\omega t} \frac{i\text{sign}[t]}{\sqrt{UN^{1/2}|t|}}
= -\sqrt{\frac{\pi}{2}}(\frac{1}{\sqrt{UN^{1/2}|\omega|}} + \frac{1}{\sqrt{UN^{1/2}|\omega|}})
= -\sqrt{\frac{2\pi}{UN^{1/2}} \frac{\text{sign}[\omega]}{|\omega|}}.
\]

While in high temperature limit we have \(\tau \approx is\), \(t \to 0^+\) and \(\beta \to 0\)

\[
g_{\text{loc}}(i\omega) = i \int_0^\infty ds e^{-\omega s} \frac{i}{\sqrt{UN^{1/2}|s|}} + i \int_{-\infty}^0 dt e^{-\omega s} \frac{-i}{\sqrt{UN^{1/2}|s|}}
= i \int_0^\infty ds e^{-\omega s} \frac{2i}{\sqrt{UN^{1/2}|s|}}
= -2\sqrt{\frac{\pi}{2\pi}} \frac{\text{sign}[\omega]}{|\omega|}.
\]

The polarization in incoherent metal phase reads

\[
\Pi_{e}^{NFL}(i\Omega) = \int_{\omega_c}^{\infty} d\omega \frac{i\text{sign}[\omega]}{2\pi \sqrt{UN^{1/2}\omega}} \frac{i\text{sign}[\omega + \Omega]}{\sqrt{UN^{1/2}(\omega + \Omega)}}
\approx -\frac{1}{UN^{1/2}} \ln \frac{UN^{1/2}}{\max[|\Omega|, \omega_c]}
= \Pi_{d}^{NFL}(i\Omega).
\]
Thus for case that both the $c$ and $d$ particles are of the non-Fermi-liquid phase, the self-energy reads

$$
\Sigma_c^{NN}(i\Omega) = -U^2 \int \frac{d\omega}{2\pi} g_{loc}^{NFL}(i\omega) \Pi_d^{NFL}(i\omega + i\Omega)
$$

$$
= U^2 \int_{\omega_c}^{U} \frac{d\omega}{2\pi} \frac{\text{sgn}[\omega]}{i\omega} \frac{1}{2U N^{1/2} |\omega + \Omega|} \left( \ln |\omega| - \ln |\omega + \Omega| \right)
$$

$$
= \frac{U^2}{N^{3/4}} \left[ \int_{\omega_c}^{U} \frac{d\omega}{2\pi} \frac{\text{sgn}[\omega]}{i\omega} \lnN \right] \left( \ln |\omega| - \ln |\omega + \Omega| \right)
$$

$$
\approx \frac{1}{N^{3/4}} \frac{2i\sqrt{U}(2\sqrt{U} - 2\sqrt{\omega_c} - 2\sqrt{\Omega}) \arctan(1/\sqrt{\Omega/U})}{2\sqrt{\omega_c} - 2\sqrt{\Omega} + \sqrt{U} \ln |\omega| - \sqrt{\omega_c} \ln |U| - \sqrt{\Omega} \ln |\omega + \Omega|} \left( \ln |\omega| - \ln |\omega + \Omega| \right)
$$

$$
+ \frac{i\sqrt{U}(2\sqrt{U} - 2\sqrt{\omega_c}) \lnN}{2\sqrt{\omega_c} - 2\sqrt{\Omega} + \sqrt{U} \ln |\omega| - \sqrt{\omega_c} \ln |U| - \sqrt{\Omega} \ln |\omega + \Omega|} \left( \ln |\omega| - \ln |\omega + \Omega| \right)
$$

$$
\approx \frac{1}{N^{3/4}} \frac{2i\sqrt{U}(2\sqrt{U} - 2\sqrt{\omega_c} - 2\sqrt{\Omega}) \arctan(1/\sqrt{\Omega/U})}{2\sqrt{\omega_c} - 2\sqrt{\Omega} + \sqrt{U} \ln |\omega| - \sqrt{\omega_c} \ln |U| - \sqrt{\Omega} \ln |\omega + \Omega|} \left( \ln |\omega| - \ln |\omega + \Omega| \right)
$$

In low frequency IR-limit (second line of above equation), the self-energy of local heavy impurity which is very closes to the fermi surface can be written in the form of

$$
\Sigma^{FL}(i\omega) = i\omega(1 - Z^{-1}),
$$

thus we have the local fermion Green’s function $g_{loc}(i\omega) = (i\omega Z^{-1})^{-1}$. We focus on the fermi-liquid phase and metal phase at low- and high-frequency limits, respectively, in this paper. In fermi-liquid phase, the boson self-energy (polarization function) of impurity in low-frequency Fermi-liquid phase reads

$$
\Pi_c^{FL}(i\Omega) = \int_0^{\omega_c} \frac{d\omega}{2\pi} g_{loc}^{FL}(i\omega) g_{loc}^{FL}(i\omega + i\Omega)
$$

$$
\approx \int_{\eta}^{\infty} \frac{d\omega}{2\pi} \frac{1}{i\omega(1 + Z^{-1})} \frac{1}{(i\omega + i\Omega)(1 + Z^{-1})} \left[ -Z^2 \ln \left| \frac{\eta + i\Omega}{\eta} \right| \right]
$$

$$
= \frac{-Z^2 \ln \left| \frac{\eta + i\Omega}{\eta} \right|}{2\pi \Omega(1 + Z)^2},
$$

where $\eta$ a infinitely small positive number (IR cutoff). Note that in weak coupling limit, $Z \approx 1 - D^2 U^2$.

For case that the $c$ particles are of the Fermi-liquid phase, while $d$ particles are of the non-fermi-liquid phase, the self-energy reads ($\omega_c < \Omega < U$)

$$
\Sigma_c^{FN}(i\Omega) = -U^2 \int \frac{d\omega}{2\pi} g_{loc}^{FL}(i\omega + i\Omega) \Pi_d^{NFL}(i\omega)
$$

$$
= U^2 \int \frac{d\omega}{2\pi} \frac{1}{i\omega(1 + Z^{-1})} \frac{1}{U N^{1/2} |\omega|} \left( \ln |\omega| - \ln |\omega + \Omega| \right)
$$

$$
= \frac{U^2}{N^{1/2}(1 + Z^{-1})} \int_{\omega_c}^{U - \Omega} \frac{d\omega}{2\pi} \frac{1}{i\omega + i\Omega} \ln |\omega|\n$$

$$
= -\frac{iU^2 Z (\ln |\omega_c|/\Omega) (\ln |U| - \ln (-\Omega + \Re[\omega_c])) - \ln 2 - \Re[\omega_c]/\Omega)}{N^{1/2}(U + UZ)}
$$

$$
- \frac{iU^2 Z \ln N (-\ln \Omega + \ln \omega_c)}{(2U + 2UZ) \sqrt{N}}.
$$
While for the case that $c$ particles are of the non-Fermi-liquid phase, and $d$ particles are of the Fermi-liquid phase, the self-energy of $d$ particle reads ($\omega_c \ll \omega + \Omega \ll U$, $\omega \ll \omega_c$, and thus $\omega \ll \Omega$)

$$
\Sigma^{NF}_d(i\Omega) = -U^2 \int \frac{d\omega}{2\pi} G^{NFL}_d(i\omega + i\Omega) \Pi_c^{FL}(i\omega) \\
\approx -U^2 \int_{\eta}^{\omega_c} \frac{d\omega}{2\pi} \frac{isgn(\Omega)}{\sqrt{U N^{1/2}|\Omega|}} \frac{\omega^2}{\eta} Z^2 \ln\left(\frac{\omega + i\Omega}{\eta}\right) \omega(1 + Z)^2 \\
= \frac{iU^2 Z^2 \left(-\frac{\pi^2}{12} - Li_2(-\omega_c/\eta)\right) \text{sgn}[\omega]}{(1 + Z)^2 \sqrt{|\omega|} \sqrt{U N^{1/2}}}. 
$$

(99)

While for the case that both $c$ and $d$ particles are of the Fermi-liquid phase, the self-energy of reads ($\omega_c \ll \omega + \Omega \ll U$, $\omega \ll \omega_c$, and thus $\omega \ll \Omega$)

$$
\Sigma^{FF}_c(i\Omega) = -U^2 \int \frac{d\omega}{2\pi} g^{FL}_{loc}(i\omega + i\Omega) \Pi_c^{FL}(i\omega) \\
= -U^2 \int_{\eta}^{\omega_c-\Omega(\Omega)} \frac{d\omega}{2\pi} \frac{1}{\omega + i\Omega(1 + Z^{-1})} \frac{\omega^2}{\eta} Z^2 \ln\left(\frac{\omega + i\Omega(1 + Z^{-1})}{\eta}\right) \\
= \frac{1}{12\Omega(1 + Z)^3} iU^2 Z^3 \left(\pi^2 - 6\ln[\eta]^2 + 12\ln[\eta] \ln\left[\frac{1}{\Omega}\right] - 12\ln[2\eta] \ln[1/(\eta + \Omega)]ight) \\
- \frac{1}{12\ln[2\eta] \ln[\eta + \Omega] + 12\ln[\eta] \ln[\eta/(\eta + \Omega)] - 12\ln[\eta] \ln[-\Omega(\Omega)]} \\
+ \omega_c \right) \ln\left[\frac{\Omega - \Omega(\Omega) + \omega_c}{\Omega(\Omega) - \omega_c}\right] - 12\ln\left[\frac{2\eta}{\eta - \Omega(\Omega)} + 12\ln\left[\frac{-\eta}{\Omega(\Omega) - \omega_c}\right]ight] \\
+ 12\ln\left[\frac{\Omega - \Omega(\Omega) + \omega_c}{\eta - \Omega(\Omega) - \omega_c}\right] \\
= \frac{1}{12\ln[2\eta] \ln[\eta + \Omega] + 12\ln[\eta] \ln[\eta/(\eta + \Omega)] - 12\ln[\eta] \ln[-\Omega(\Omega)]} \\
+ \omega_c \right) \ln\left[\frac{\Omega - \Omega(\Omega) + \omega_c}{\Omega(\Omega) - \omega_c}\right] - 12\ln\left[\frac{2\eta}{\eta - \Omega(\Omega)} + 12\ln\left[\frac{-\eta}{\Omega(\Omega) - \omega_c}\right]ight] \\
+ 12\ln\left[\frac{\Omega - \Omega(\Omega) + \omega_c}{\eta - \Omega(\Omega) - \omega_c}\right].
$$

(100)

In Fig.3, we show the calculated self-energies. We found that only $\Sigma^{FN}_{c}$ exhibits Fermi liquid feature in the positive frequency region, while the $\Sigma^{NN}_{c}$ and $\Sigma^{NF}_{d}$ show exponential and power law decays, respectively. In Fig.4, the corresponding spectral functions are presented. Although not shown, we found, by varying the parameters of $N$ and $U$, the width of spectral functions approximately scale as $\sim \frac{1}{N}$ which is consistent with Ref. [24], and the peaks are always locating around $\omega = 0$ due to $\mu = 0$ setted here.

In the presence of finite chemical potential, the symmetry relations stated above (below Eq.(67)) are invalid, and by using the conformal symmetry for $U \gg \omega \gg T \gg \omega_c$ and $i\phi = 0$, the SKY local Green’s function at negative long time ($\tau \gg -|U|^{-1}$) reads $g_{loc}(\tau) = e^{-2\pi\xi \text{sgn}[\tau]}/\sqrt{|U|\tau}$ where $\xi$ is the spectral asymmetry parameter [3]. Then in low temperature, we have

$$
g_{loc}(i\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \frac{isgn[t]}{\sqrt{U N^{1/2}|t|}} + \int_{0}^{\infty} dt e^{-2\pi\xi} e^{i\omega t} \frac{i}{\sqrt{U N^{1/2}|t|}} \\
e^{\pi/2(\pi/2)(-1 + e^{2\pi\xi^2})} \frac{1}{\sqrt{U N^{1/2} \ln[\omega]}}. 
$$

(101)

In high temperature limit, when the integral path is still the along the upper half plane from zero to $\beta$, the results is the same with Eq.(92), while for integral path in the lower half plane,
we have now
\[
 g_{\text{loc}}(i\omega) = \int_{-\infty}^{0} ds \left[-e^{-2\pi\varepsilon}e^{-\omega s} \frac{i}{\sqrt{UN^{1/2}|s|}} + i \int_{-\infty}^{0} ds \left[-e^{-2\pi\varepsilon}e^{-\omega s} \frac{-i}{\sqrt{UN^{1/2}|s|}}\right]\right]
\]
\[
\frac{-2\sqrt{i\varepsilon}e^{-2\pi\varepsilon}}{\sqrt{UN^{1/2}(-\omega)}} \Theta(-\omega),
\]

which, compared to Eq.(92), implies the broken particle-hole symmetry.

10 Appendix.D: Extention to nonlocal regime

Except the DMFT with the resulting on-site quantites, the disorder averaging also leads to local Green’s function and self-energy in the presence of perturbative SYK coupling. In fact, even the lattice Green’s function can also be local as long as the hopping is perturbative[21].

Next we discuss the extention to nonlocal regime. The extention to nonlocal scheme can be processed by using the dual fermion approach. The bare dual Green’s function which is nonlocal can be introduced through the relation

\[
 G_{\text{dual}}^{0} = G_{\text{lat}}^{0} - g_{\text{loc}}
\]

\[
\frac{1}{g_{\text{loc}}^{-1} + (\Delta_{h} - \varepsilon_{k})^{-1}} - g_{\text{loc}}^{-1} (\Delta_{h} - \varepsilon_{k})^{-1}
\]

(103)

where \( G_{\text{lat}}^{0} \) is the bare lattice Green’s function. The bare and dressed lattice Green’s function can be obtained through the bare and dressed dual Green’s function as

\[
 G_{\text{lat}}^{0} = (\Delta_{h} - \varepsilon_{k})^{-1} + (\Delta_{h} - \varepsilon_{k})^{-1} g_{\text{loc}}^{-1} G_{\text{dual}}^{0} g_{\text{loc}}^{-1} (\Delta_{h} - \varepsilon_{k})^{-1}
\]

\[
= (\Delta_{h} - \varepsilon_{k})^{-1} + (\Delta_{h} - \varepsilon_{k})^{-1} g_{\text{loc}}^{-1} \left(\frac{\Delta_{h} g_{\text{loc}}^{-2}}{1 + g_{\text{loc}} \Delta_{h}}\right) g_{\text{loc}}^{-1} (\Delta_{h} - \varepsilon_{k})^{-1}
\]

\[
= \frac{1}{g_{\text{loc}}^{-1} + (\Delta_{h} - \varepsilon_{k})^{-1}}
\]

(104)

and

\[
 G_{\text{lat}} = (\Delta_{h} - \varepsilon_{k})^{-1} + (\Delta_{h} - \varepsilon_{k})^{-1} g_{\text{loc}}^{-1} \left(\frac{-(\Delta_{h} - \varepsilon_{k}) g_{\text{loc}}^{-2}}{1 + g_{\text{loc}} (\Delta_{h} - \varepsilon_{k})}\right)^{-1} - \Sigma_{\text{dual}}^{-1} g_{\text{loc}}^{1} (\Delta_{h} - \varepsilon_{k})^{-1}
\]

\[
= \frac{1}{g_{\text{loc}}^{-1} (1 + g_{\text{loc}} \Sigma_{\text{dual}})^{-1} + (\Delta_{h} - \varepsilon_{k})^{-1}}
\]

(105)

respectively. The spectral function of bare dual Green’s function reads

\[
 \rho_{\text{dual}}(\omega) = -\text{Im} G_{\text{dual}}(\omega + i\eta)
\]

\[
\approx \begin{cases} 0, & \text{insulator phase at half – filling}, \\
0, & \text{Fermi – liquid phase at half – filling}; \omega \ll \omega_{c}, \\
\frac{\text{sgn}(\omega)}{U^{1/2}\omega^{1/2}}, & \text{metal phase}; \omega_{c} \ll \omega \ll U.
\end{cases}
\]

(106)

Comparing to the above spectral functions of Weiss and local Green’s functions, we found that only the spectral function of dual Green’s function is an odd function, \( \rho_{\text{dual}}(\omega) = -\rho_{\text{dual}}(-\omega) \).
similar to the odd bosonic functions consisting of the commuting observables. Thus the spectral representation of dual Green’s function

\[ G_{\text{dual}}(i\omega) = \int_{-\infty}^{\infty} d\Omega \frac{\rho_{\text{dual}}(\Omega)}{i\omega - \Omega}, \]  

(107)
can be transformed to

\[ G_{\text{dual}}(\omega) = \int_{0}^{\infty} d\Omega \frac{-2\Omega \rho_{\text{dual}}(\Omega)}{\omega^2 + \Omega^2}. \]  

(108)
Then this dual Green’s function is purely real and is proportional to \( \omega^{-2} \) in UV limit and consistent with its original expression \( G_{\text{dual}} = -\frac{1}{g_{\text{loc}} - \frac{1}{2} g_{\text{loc}}(\Delta - \epsilon_k)} \), i.e., more delocalized in time domain than the \( g_{\text{loc}} \) and \( G_{0 \text{lat}}^{0} \), which have \( g_{\text{loc}} \propto \omega^{-1} \) and \( G_{0 \text{lat}}^{0} \propto \omega^{-1} \) in non-metal phase and \( g_{\text{loc}} \propto \omega^{-1/2} \) and \( G_{0 \text{lat}}^{0} \propto \omega^{-1/2} \) in metal phase.

11 Appendix.E: Identity between DMFT result and the SYK consistency in high temperature limit

The large \( N \) (electron degrees of freedom) limit in SYK incoherent metal enforce the self-energy and Green’s function be local (and \( \text{Im} \Sigma \sim \sqrt{U_{\text{SYK}}T} \) in pure SYK regime), and this is similar to the DMFT theory where the infinite dimension enforce the lattice translational symmetry and thus the local propagator as well as self-energy.

In high temperature insulating phase with \( |\omega| \gg \Sigma \) and \( U \gg U_{\text{SYK}} = U_{\alpha\beta\alpha\beta} \) (on-site hubbard U much larger than the SYK coupling), since the DMFT results become identical with the SYK self-consistent set of equations (as we present in Sec.4), and thus we can obtain a set of equations of Green’s function and self-energy base on the series expansion of small \( U_{\text{SYK}} \). We start by the above-obtained result of Green’s function in high temperature limit, \( g_{0}(\omega) = 1/(i\omega) \) with \( \omega \gg 0 \), using Fourier transform we have

\[ g_{0}(\tau) \approx -i\ln \frac{\tau}{\tau_{c}}, \]  

(109)
where \( \tau_{c} = \frac{U_{\text{SYK}}}{\phi_{0}} \) is the long time cutoff (note that in high temperature limit \( \phi_{0} \) can be replaced by \( \Sigma^{\text{DMFT}} = U/2 \) here which shifts the zero noninteracting chemical potential to the nonzero
effective one). Though iterative calculation, we have (at $\mu_{\text{eff}} = \mu_{\text{int}} = \mu_0 - U/2 = 0$)

\begin{align*}
\Sigma_0(\tau) &= U_{\text{SYK}}^2 N \left(-i \ln \frac{\tau_c}{\tau}\right)^3, \\
\Sigma_0(\omega) &= U_{\text{SYK}}^2 N \frac{1}{(i\omega)^3}, \\
g_1(\omega) &= \frac{1}{i\omega - U_{\text{SYK}}^2 N \frac{1}{(i\omega)^3}} \\
&= \frac{1}{i\omega} + \frac{U_{\text{SYK}}^2 N}{(i\omega)^3}, \\
g_1(\tau) &= -i \ln \frac{\tau_c}{\tau} + U_{\text{SYK}}^2 N \left(\frac{i}{4\tau_c^4} - \frac{i}{4\tau^4}\right) \\
&= -i \ln \frac{\tau_c}{\tau} + U_{\text{SYK}}^2 N \frac{i(\tau_c^4 - \tau^4)}{\tau^4}, \\
\Sigma_1(\omega) &= U_{\text{SYK}}^2 N \frac{1}{(i\omega)^3} + U_{\text{SYK}}^4 N^2 \frac{4}{(i\omega)^5}, \\
g_2(\omega) &= \frac{1}{i\omega - U_{\text{SYK}}^2 N \frac{1}{(i\omega)^3}} - \frac{U_{\text{SYK}}^2 N^2}{4 \frac{1}{(i\omega)^5}} = \frac{1}{i\omega} + \frac{U_{\text{SYK}}^2 N}{(i\omega)^3} + \frac{1}{(i\omega)^9} N^2 U_{\text{SYK}}^4,
\end{align*}

\ldots

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Figure 1: Bosonic spectral weight as a function of $V_0$ at high temperature limit or as a function of $\beta$ at different $\Delta_b$. 
Figure 2: Phase diagram of this paper with Coulomb bosonic potential $i\phi$. $\alpha$ is the turning parameter which can be treated as the effective coupling between fermion and bosonic sectors, and $\alpha_c$ is the quantum critical point. Note that there will be not separation between fermionic and bosonic sectors for $|V_0| \ll U$. The red zone is the regime of stable gapless boson excitations. For the SYK regime in this phase diagram, there does not exist $U(1)$ symmetry as it is not the pure SYK regime which requires $\omega_c = \omega = 0 \ll U$. However, it is possible to find the $U(N)$ symmetry in the low-temperature Fermi liquid regime with $N$ degenerated Fermi surfaces\cite{27}.

Figure 3: Self-energies contributed by $c$-$d$ interaction with $c$ and $d$ fermions in the same or different phases as described in detail in Appendix.D. Note that the self-energies considered here can be shown by the sunset diagram instead of the tadpole diagram which is of the high-temperature limit.
Figure 4: Spectral functions corresponding to Fig.3.