Max/Min Puzzles in Geometry†

James M Parks
Dept. of Math.
SUNY Potsdam
Potsdam, NY
parksjm@potsdam.edu
January 4, 2022

Abstract. The objective here is to find the maximum polygon, in area, which can be enclosed in a given triangle, for the polygons: parallelograms, rectangles and squares. It is initially assumed that the choices are inscribed polygons, that is all vertices of the polygon are on the sides of the triangle. This concept is generalized later to include wedged polygons.

One of the earliest known examples of a max/min inscribed polygon puzzle is called Euclid’s Maximum Problem [7].

Puzzle 1. Inscribe a parallelogram in a given triangle, with its sides parallel to two sides of the triangle, such that the parallelogram has maximum area.

Since all vertices of the parallelogram must be on the triangle, and the sides must be parallel to 2 sides of the triangle, the picture you should be imagining looks something like one of the parallelograms BDEF or CHIG in Fig. 1.

Figure 1

Using Sketchpad* makes studying such examples a simple process. It then becomes apparent that the prime candidate for the maximum parallelogram for the configuration of BDEF looks like Fig. 2, where the vertices D, E, F are at or very near the midpoints of sides BC, CA, and AB, respectively.

† Some of these results have appeared in [4] & [5].
* All graphs were made with Sketchpad v.5.10 BETA.
It can then be conjecture that the maximum inscribed parallelograms are determined by the midpoints of the sides of the triangle and the vertices of the triangle. A proof of this conjecture follows from Euclid’s Proposition 27 in Book VI of *The Elements* [3], but there exists also a proof which is based on algebra.

Let \( h \) be the height at \( A \), and \( a \) the length of the base \( BC \), Fig.3. Then \( (ABC) = ha/2 \), where parentheses denote area. Assume \( BDEF \) is the parallelogram with \( D, E, F \) the midpoints of the sides of \( \Delta ABC \) as shown. Then \( (BDEF) = (h/2)(a/2) = ha/4 \).

Let \( BGHI \) be another inscribed parallelogram with side \( BI \) on \( AB \), and let \( x \) equal the length of \( JE \), \( J \) the intersection of \( GH \) with \( FE \), and \( y \) the height of \( \Delta JEH \) at \( H \). Then \( (BGHI) = (a/2 - x)(h/2 + y) = ha/4 - hx/2 + ay/2 - xy \). But \( \Delta ABC \sim \Delta HJE \), since the angles are equal, so \( x/y = a/h \), and thus \( hx = ay \). Therefore, \( (BGHI) = (BDEF) - xy \), and \( (BDEF) \) is clearly the maximum parallelogram.

Similarly, if one chooses \( H \) on \( EC \), and \( G \) on \( DC \), and similarly if one chooses vertex \( C \) or vertex \( A \). Thus the max parallelogram occurs when 3 of the vertices are midpoints of the triangle sides, and the forth vertex is a vertex of the triangle. Hence there exist 3 max parallelogram solutions for \( \Delta ABC \) regardless of whether \( \Delta ABC \) is acute, right, or obtuse. There are obviously no min. parallelograms.

These 3 parallelograms are all equal in area, since any two of them share a base and a height, as can be clearly seen in Fig. 4.

The next puzzle, which is related to Puzzle 1, studies the same problem for rectangles.

**Puzzle 2.** Inscribe a rectangle in a given triangle, such that the rectangle has maximum area.
The relationship between inscribed rectangles and inscribed parallelograms on the same vertices $E, F$ of sides $AB, AC$, resp., is illustrated in Fig. 5, where the base angles are acute. It is well known that inscribed maximum area rectangles must have one side on a side of the triangle [2].

Also, the maximum rectangle must ‘coincide’ with the maximum parallelogram, in the sense shown in Fig. 5, since if not, then there would be a larger rectangle or parallelogram, which would contradict the maximum parallelogram or rectangle, as the case may be. Thus the maximum rectangle area is achieved when the side opposite the side which is on $\Delta ABC$ is connecting the midpoints of the other 2 sides of $\Delta ABC$.

However, there is one difference between the two types of polygons. For the parallelograms, there are 3 equal maximum ones in all types of triangles. But for rectangles, there are 3 equal maximum rectangles in acute triangles, 2 equal maximum rectangles in right triangles (the max rectangles on sides $AB$ and $AC$ coincide), and one maximum rectangle in obtuse triangles, Fig. 6. There are obviously no min. rectangles. You can’t inscribe a rectangle in a triangle if one of the base angles is obtuse.

As with parallelograms, the area of each of the inscribed rectangles is $ha/4$, where $h$ is the height, and $a$ is the length of the respective base side. So in the acute triangle in Fig. 6, $(GHEF) = (DLIF) = (KLDE) = ha/4$, and similarly for right triangles, and obtuse triangles.

Since we are studying inscribed rectangles in triangles, we will also consider the case of inscribed squares. This turns out not to just be a special case of rectangles. But the first problem is “how do you inscribe a square in a triangle”?

With a rectangle the height and the width are independent, not so with a square.

Here’s a method, due to Polya [6], for constructing an inscribed square on side $BC$ of a given triangle $\Delta ABC$ with no obtuse angle on the base. Of course there are also other methods [2].
Let $D$ be a point on side $AB$, near $B$, and construct a square $DEFG$ with $EF$ on side $BC$, Fig. 7L. Construct a line from $B$ through $G$ to $AC$ at $H$. The desired inscribed square $D'E'F'H$ is obtained by the dilation of $DEFG$ about point $B$ by the ratio $BH/BG$, Fig. 7R.

As with rectangles, the number of inscribed squares depends on the type of triangle. There are 3, 2, or 1 inscribed squares, if the triangle is acute, right, or obtuse, respectively. However, unlike rectangles, the area of the squares may differ with the base side of the triangle, since the height equals the width, Fig. 8.

**Puzzle 3.** Determine a formula for the length of the side of an inscribed square.

The formula for computing the length $s$ of the side of an inscribed square is determined as follows [2]. Let $\triangle ABC$ be a triangle with height $h$, and inscribed square $HIJK$ on base $BC$ of length $a$, where the base angles are acute, Fig. 8. Let $s$ be the length of the side of the square, and note that $\triangle ABC \sim \triangle AKJ$, (equal angles). Then you have the following equation, $s/a = (h-s)/h$. Solving this equation for $s$, determines the solution $s = ha/(h+a)$. Thus, the area of the inscribed square with side $s$ is $s^2 = [ha/(h+a)]^2$.

Notice that the area of the inscribed square seems to be smaller than the area of the maximum inscribed rectangle on the same side, Fig. 8.

This leads to a new puzzle.

**Puzzle 4.** Show that the area of a maximum inscribed rectangle is always larger than the area of the inscribed square on the same side of a given triangle which has no obtuse angles on the base, with one exception, if the height equals the base the areas are equal.

Consider $\triangle ABC$ with inscribed square $HIJK$, and the inscribed rectangle $DEFG$ added, Fig. 8. Then show that $(ha/(h+a))^2 \leq ha/4$. 

![Figure 7](image)  
![Figure 8](image)
But this inequality is equivalent to the inequality $0 \leq (h - a)^2$, which is obviously true, with the added bonus that equality holds only when $h = a$.

**Example 1.** When looking at examples of acute triangles notice that the largest inscribed squares are always on the smallest side, and the smallest inscribed square is always on the largest side.

Let $\Delta ABC$ be the acute triangle with $m<A = 75^\circ$, $m<B = 60^\circ$, and $m<C = 45^\circ$, Fig. 9.

Then $a > b > c$, since the largest side is opposite the largest angle.

Let $c = 2u.$, then $h_a = \sqrt{3} \sim 1.73u.$, $a = 1+\sqrt{3} \sim 2.73u.$, and $s_a = \sqrt{3(1+\sqrt{3})/(1+2\sqrt{3})} \sim 1.060u.$; $b = \sqrt{6} \sim 2.45u.$, $h_b \sim 1.93u.$, so $s_b \sim 1.080u.$; and $h_c \sim 2.37u.$, so $s_c \sim 1.084u.$      Figure 9

Thus $s_a < s_b < s_c$.

This observation is the basis for Puzzle 5.

**Puzzle 5.** If $\Delta ABC$ is an acute scalene triangle, with sides $a > b > c$, show that the sides of the inscribed squares satisfy $s_a < s_b < s_c$.

It is only necessary to show that whenever $a > b$, then $s_a < s_b$.

So, let $\Delta ABC$ is an acute triangle, with sides $a > b$, $h_b$ the height of $\Delta ABC$ at $B$, Fig. 10.

First show that $a + h_a > b + h_b$.

By the assumption, $a - b > 0$, and $h_a < h_b$, since $(ABC) = ah_a/2 = bh_b/2$. Plus it follows that $b > h_a$, since $b$ is the hypotenuse of a right triangle with one leg of length $h_a$.

Thus $b - h_a > 0$, so $(a + h_a) - (b + h_b) = (a - b) + (2(ABC)/a - 2(ABC)/b) - (a - b)(1 - 2(ABC)/ab) = (a - b)(b - h_a)/b > 0$. Figure 10

Thus $s_a = ah_a/(a+h_a) < ah_b/(b+h_b) = bh_b/(b+h_b) = s_b$, so $s_a < s_b$.

Similar arguments show that if $b > c$, then $s_b < s_c$.

Thus the max. inscribed square is on the shortest side, and the min. inscribed square is on the longest side.

But what about the obtuse triangles?

The problem with obtuse triangles is that if there is a square at the obtuse angle, there is no way that all of the vertices can be on the triangle, Fig. 11.

So instead of inscribed squares, you have something more general, called *wedged squares*, named by E. Calabi [1], [2].
There is a short-cut to constructing such squares at an obtuse angle, such as at $A$

Construct a line at the vertex $A$ at a 45° angle to the base leg $CA$. The intersection point $D$ of this line with $BC$ determines a diagonal of the square $AFDE$, Fig. 11.

Of course the area of wedged squares is computed by a different formula.

**Puzzle 6.** Let $\triangle ABC$ be an obtuse triangle, with obtuse angle at vertex $A$. Let $AEFG$ be the largest wedged square on side $CA = b$. Find a formula for the length of the side of the square in terms of $\phi$, the angle at $C$.

By the given, \( \frac{s_b}{b - s_b} = \tan \phi = \sin \phi / \cos \phi \). Solve for $s_b$, to find $s_b = b \sin \phi / (\sin \phi + \cos \phi)$, where $\phi$ is angle $C$, Fig. 11.

Once the existence of wedged squares is allowed, there will be 3 wedged squares in every type of triangle, as with parallelograms. However, the areas of wedged squares differ with the length of the base side!

The concept of a wedged square can also be applied to rectangles in an obtuse triangle $\triangle ABC$. Given a rectangle on a base side $AC$ with an obtuse angle at $A$, let one vertex of the rectangle be on $A$, and another vertex $F$ on side $AC$. Then vary the vertex $E$, opposite vertex $A$, on side $BC$. The max area of the rectangle $(ADEF)$ appears to be when vertex $F$ is at or near the midpoint of side $AC$, Fig. 12. We conjecture that this is the actual max area of a wedged rectangle in an obtuse triangle on a side with an obtuse angle.

**Puzzle 7.** If $\triangle ABC$ is an obtuse triangle, with obtuse angle at $A$, show that the area of a maximum wedged rectangle on side $AC$, say $ADEF$, occurs when the length of the base $AF = b/2$, for $b = AC$.

Assume $ADEF$ is the wedged rectangle on $AC$ with $F$ on the midpoint of $AC$. Then $(ADEF) = (b^2/4) \tan \phi$, $\phi = \text{angle } C$, Fig. 12. Suppose $AD'E'F'$ is another wedged rectangle on $AC$, with length $b/2 + e$, $e \geq 0$, and height $w' = (b/2 - e) \tan \phi$.

Solving the equation $(ADEF) \leq (AD'E'F')$ for $e$, determines $e = 0$. Thus $ADEF$ is the max wedged rectangle on side $AC$.

Similarly, for $AD'E'F'$ with length $b/2 - e$, $e \geq 0$. 

\[ \text{Figure 11} \]

\[ \text{Figure 12} \]
The areas of wedged rectangles depend on the length of the base side and the angle opposite the obtuse angle on the base side, so they vary with the side of the triangle, as seen in the case for wedged squares in Example 1 above. However, the larger side has the larger area for wedged rectangles, contrary to the case for squares, Fig. 13.

**Puzzle 8.** Show that for the obtuse $\triangle ABC$, if $a > b > c$, with obtuse angle at A, then $(GHFJ) \geq (ADEF) \geq (ALKJ)$, if F and J are on the midpoints M and I of AC and AB, respectively.

That $(ADEF) \geq (ALKJ)$ follows from the observation that angle $JAD = angle LAF$, so $FL > JD$, since $b > c$, Fig.13. Then, this is equivalent to $(ADF) > (AJL)$, since $(ADF) = (a/2 - JD)h$, and $(AJL) = (a/2 - LF)h$. But $(ALKJ) = 2(AJL)$, and $(ADEF) = 2(ADF)$, so $(ADEF) \geq (ALKJ)$.

To show $(GHFJ) \geq (ADEF)$, consider the triangle $\triangle ADF$, and the rectangle $GHFJ$, Fig.13. The rectangle $ADEF$ is divided into 2 copies of $\triangle ADF$ by diagonal $DF$, one which is in $GHFJ$, since $D$ is on $JF$. The vertical line $AP$ cuts $\triangle ADF$ into 2 right triangles. One is $\triangle ADP$, which is congruent to $\triangle FEH$, with heights $h/2$, and sides $AD//FE$, and $AP//FH$. The other one is $\triangle APF$ which is congruent to $\triangle DOE$, where $O$ is the foot of the vertical line at $D$ to $GH$. Again, the heights are $h/2$, and sides $AP//DO$, and $AF//DE$.

Thus, rectangle $DOHF$ equals 2 copies of $\triangle ADF$, and since $JGHF$ contains $DOHF$, it follows that $(GHFJ) > (DOHF) = (ADEF)$.

The relation between the areas of maximum wedged rectangles and wedged squares is the same as for rectangles and squares, Puzzle 4, Fig. 14.

**Puzzle 9.** Let $\triangle ABC$ be obtuse at A, and let $ADEF$ be the maximum wedged rectangle on side AC with vertex F at the midpoint of AC, and $AHIJ$ the wedged square on AC. Show that $(ADEF) \geq (AHIJ)$.  

Figure 14
Observe that \((ADEF) \geq (AHIJ)\) is equivalent to \((b^2/4) \tan \theta \geq (b \sin \theta/(\sin \theta + \cos \theta))^2\) which is equivalent to \((\sin \theta - \cos \theta)^2 \geq 0\), with equality at \(\theta = 45^\circ\), which means \(\triangle ABC\) is a right triangle with base \(b\) equal to the height \(c\).

It should be clear that the areas of wedged squares (and wedged rectangles) on the legs of isosceles triangles are equal, as are the areas of wedged squares (and wedged rectangles) in equilateral triangles. By allowing this more general type of enclosed squares in triangles, a new and unexpected result appears. The best way to discover it is with Sketchpad.

**Example 2.** Let \(\triangle ABC\) be an obtuse isosceles triangle with obtuse angle at \(A\), and assume sides \(AB\) and \(AC\) are fixed at length 2. Construct the 3 wedged squares, then the area of the squares on the legs \(AB\) and \(AC\) are equal, Fig. 15.

Now move the vertex \(A\) up and down in the range 95° to 105°. The chart in Fig. 15 gives some values of the areas of the wedged squares as \(A\) is moved.

| Angle   | Area of 2 eq. sq.s | Area of DEFG |
|---------|--------------------|--------------|
| 105.66° | 0.7438u²           | 0.7677u²     |
| 102.64° | 0.7908u²           | 0.7968u²     |
| 101.61° | 0.8070u²           | 0.8061u²     |
| 98.50°  | 0.8569u²           | 0.8325u²     |
| 95.31°  | 0.9095u²           | 0.8565u²     |

Figure 15

Observe that in the range 101.61° to 102.64° there is a reversal of the sizes from the 2 equal squares being larger than \(DEFG\), to them being smaller.

By continuity, there must be an angle in between 101.61° and 102.64° where the 3 areas are all equal! It was discovered by E. Calabi, sometime in pre1996, that there exists a triangle which is not equilateral, but which has 3 equal area wedged squares. He calls it a catastrophe, in the sense of R. Thom [1].

It is possible to compute the exact values of the angle at \(A\), and the ratio of the long side to the short side of \(\triangle ABC\), Fig. 15.

**Puzzle 10.** Determine the angle at \(A\), and the ratio of the long side to the short side of Calabi’s triangle \(\triangle ABC\).

Assume \(AC = AB = 1\) unit. From above we have \(s_a = ah_a/(a+h_a)\), and \(s_c = s_b = b \sin \theta/(\sin \theta + \cos \theta)\), where \(\theta = \text{angle C}\). Then \(\sin \theta = h_a\) and \(\cos \theta = a/2\), so \(s_a = s_b\) is equivalent to \(a/(a+h_a) = 1/(h_a+a/2)\). Solving for \(h_a\) determines that \(h_a = (a^2 - 2a)/(2 - 2a)\).
Also, by the Pythagorean Theorem, \( h_a = \sqrt{1 - a^2/4} \), so substituting this into the equation above and squaring out the radical determines \( 1 - a^2/4 = [(a^2 - 2a)/(2 - 2a)]^2 \), which simplifies to the quartic equation \( 2a^4 - 6a^3 + a^2 + 8a - 4 = 0 \). This equation has the extraneous root \( a = 2 \), since 2 is a zero, but \( a < 2 \) by the assumptions.

Divide out the factor \((a - 2)\) to obtain the cubic equation \( 2a^3 - 2a^2 - 3a + 2 = 0 \).

The largest positive solution of this equation is \( a = 1.5513875... \), and since \( b = 1 \), this is the ratio of the longest side \( a \) to the shortest side \( b \) of \( \triangle ABC \).

Since \( \cos \theta = a/2 \), \( \theta = 39.132...^\circ \), and the obtuse angle at \( A \) is \( 101.736...^\circ \).

There are 8 different combinations of possible sizes of wedged squares in triangles, and they can be categorized as follows (cf. [8]).

Assume that the triangle \( \triangle ABC \) satisfies the inequality \( a \geq b \geq c \), with \( a = 1 \). Let \( C = (0, 0) \), \( B = (-1, 0) \), and \( D = (-1/2, 0) \), Fig. 16.

There are 3 possible triangles shown.
If angle \( A \) is a right angle, then it is on the circle with radius \( a/2 \) and center \( D \).
If angle \( A \) is obtuse, then \( A \) is inside this circle, and if angle \( A \) is acute, then \( A \) is outside this circle, but inside the circle centered at \( C \), with radius \( a \).

Whenever \( A \) is on \( DG \), \( \triangle ABC \) is isosceles, and if \( A = G \), then \( \triangle ABC \) is an equilateral triangle.

All values of \( A \) must be inside or on the circle about \( C \) of radius \( a \), since \( a \geq b \geq c \).

If \( s_a = s_b \), then \( h_a = (\sin \theta + \cos \theta - 1) \), for \( a = 1 \), since \( s_a = h_a/(1 + h_a) \), \( s_b = b\sin \theta/(\sin \theta + \cos \theta) \), and \( h_a = b\sin \theta \), \( \theta = \angle C \). Simplifying and formulating in polar coordinates \( (r, \mu) \) determines the equation \( r = (1 - \cot(\mu/2)) \), \( \mu = 180^\circ - \theta \). Transforming this equation to rectangular coordinates determines the cubic equation \( y^3 + x^2y + 2x^2 + 2y^2 + 2x = 0 \). The graph of this equation contains the dashed curve \( BHE \) under the semicircle centered at \( D \), with radius \( a/2 \), and \( s_b \) is the max for \( A \) just above this curve, and \( s_a \) is the max for \( A \) just below it, Fig. 17.

If \( s_a = s_c \) we have the analogous case. This can be solved by reflecting \( \triangle ABC \) about \( DG \), solving for the cubic equation, then reflecting this curve about \( DG \) again to get the solution for \( \triangle ABC \).

The reflected cubic equation is \( y^3 + (x+1)^2y + 2(x+1)^2 + 2y^2 - 2(x+1) = 0 \). The graph of this equation contains the dashed curve \( BIE \) between the semicircle and the cubic just below it. If \( A \) is just above this curve, then \( s_c \) is the larger, and \( s_a \) is the min., and when \( A \) is just below it, then \( s_c \) is the min., and \( s_a \) is the larger.

The two cubics intersect at \( E \) on \( DG \), which is the point where Calabi’s triangle occurs [1].
Starting at the top yellow region in Fig. 17, the results are summarized in the following list.

- $s_a < s_b < s_c$, for acute triangles: $A$ in the yellow region,
- $s_a < s_b = s_c$, for right triangles: $A$ on circle $BJF$,
- $s_a < s_c < s_b$, for obtuse triangles, and $A$ in the blue region,
- $s_a = s_c < s_b$, for $A$ on the cubic curve $BIE$,
- $s_c < s_a < s_b$, for obtuse triangles, and $A$ in the green region,
- $s_c < s_a = s_b$, for $A$ on the cubic curve $BHE$,
- $s_c < s_b < s_a$, for obtuse triangles, and $A$ in the magenta region,
- $s_a = s_b = s_c$, for Calabi’s triangle, $A$ on $E$.

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