COMPACTNESS OF THE COMPLEX GREEN OPERATOR ON NON-PSEUDOCONVEX CR MANIFOLDS

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Abstract. In this paper, we investigate the compactness theory of the complex Green operator on smooth, embedded, orientable CR manifolds of hypersurface type that satisfy the weak $Y(q)$ condition. The sufficient condition that we define is an adaptation of the CR-$P_q$ property for weak $Y(q)$ manifolds and does not require that the CR manifold is the boundary of a domain.

We also provide several non-pseudoconvex examples (and a level $q'$) for which the complex Green operator is compact.

1. Introduction. The Kohn Laplacian is the natural complex Laplacian on CR manifolds $M$, and its regularity theory fundamentally depends on the geometry and potential theory of $M$. The $L^2$-minimizing inverse to the Kohn Laplacian on $(0,q)$-forms (when it exists) is called the complex Green operator and denoted by $G_q$. In this paper, we study the $L^2$-compactness theory of $G_q$ on not necessarily pseudoconvex CR manifolds. Solvability of $\square_b$ in $L^2_0(M)$ is currently known on smooth, orientable CR manifolds of hypersurface type that satisfy a geometric condition called weak $Y(q)$ \cite{10, 11, 6}, so these manifolds are our starting point. In this paper, we introduce a condition named weak $Z(q')$-(CR-$P_{q'}$) that is a natural generalization of Catlin's Property $P_q$ \cite{4, 20} and reduces to the (CR-$P_q$) condition introduced by Raich \cite{18} in the case that $M$ is pseudoconvex. We show that if $M^{2n-1}$ satisfies weak $Z(q')$-(CR-$P_{q'}$) at the symmetric levels $q' = q$ and $n - 1 - q$, then $G_q$ and $G_{n-1-q}$ are compact operators. We also provide examples showing that our condition admits new examples. This paper is the first to study the compactness theory of the tangential Cauchy-Riemann operator $\bar{\partial}_b$ on not necessarily pseudoconvex CR manifolds.

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The $L^2$ theory for the tangential Cauchy-Riemann, or $\bar{\partial}_b$, operator is only established in any generality of CR manifolds of hypersurface type. These manifolds are generalizations of and share many properties with the boundaries of domains. In this way, they represent the most important class of examples to investigate, and the analysis of $\bar{\partial}_b$ depends fundamentally the geometry and potential theory of $M$. The Kohn Laplacian $\square_b = \partial_b \partial^*_b + \partial^*_b \partial_b$ is neither elliptic nor constant coefficient and its coefficients depend on the defining functions of $M$ (when $M$ is embedded).

The complex Green operator may or may not satisfy subelliptic estimates, compactness estimates, global regularity, continuity in a given $L^2$-Sobolev space, or $L^2$-continuity. It is classical that $G_q$ gains one derivative if and only if $M$ satisfies the geometric condition $Y(q)$ [9], but there are no known sufficient conditions that are necessary for any of the other regularity conditions. In the pseudoconvex case, $Y(q)$ is simply strict pseudoconvexity. Also, in the case that $M$ is the boundary of a pseudoconvex domain, $G_q$ gains a fractional derivative if and only if $M$ satisfies a D’Angelo finite type condition [7, 8]. Necessary and sufficient conditions are not known for either global regularity or compactness of $G_q$ (except for compactness when $M$ is the boundary of a convex domain [19]) or even subellipticity when $M$ lacks pseudoconvexity (let alone is no longer the boundary of a domain in $\mathbb{C}^n$).

There are two general conditions that imply compactness of $G_q$ in the case that $M$ is pseudoconvex: Property $P_q$ and Straube’s vector field condition [17]. Raich and Straube proved that if $M$ is the boundary of a pseudoconvex domain, then $P_q$ and $P_{n-1-q}$ suffice to show that $G_q$ is compact [19]. Form level symmetry is built into $\bar{\partial}_b$ and symmetric conditions are always required for regularity results due the presence of a Hodge-* like operator [16]. In the case that $M$ is a smooth, embedded, orientable CR manifold of hypersurface type, Raich showed that $G_q$ is compact if $M$ satisfies $(CR-P_q)$ and $(CR-P_{n-1-q})$ where $(CR-P_q)$ is a CR version of $P_q$. Straube later relaxed the orientability condition [21] and also showed the equivalence of $P_q$ and $(CR-P_q)$ if $M$ is orientable and pseudoconvex. Khanh, Pinton, and Zampieri later relaxed the embeddedness requirement [15]. Straube’s equivalence of $P_q$ and $(CR-P_q)$ uses in an essential way that $M$ is pseudoconvex as the strict pseudoconvexity naturally degenerates into points of weak pseudoconvexity, but $Y(q)$ degenerates into a condition that Basener calls weak $q$-pseudoconvexity [1]. Unfortunately, it is unknown whether or not weak $q$-pseudoconvexity is a sufficient condition for $L^2$ solvability of $\bar{\partial}_b$. Consequently, we develop a $(CR-P_q)$ condition in analog to Raich [18] and this has orientability as a requirement to formulate the definition. Roughly speaking, $P_q$ and $(CR-P_q)$ both hypothesize the existence of bounded (plurisubharmonic) functions that have Hessians with arbitrarily large eigenvalues. This is the right condition for pseudoconvex CR manifolds, and the weak $Y(q)$ manifolds are not necessarily pseudoconvex; hence we need to formulate an appropriate condition for the manifolds under consideration here.

Our main result is the following.

**Theorem 1.1.** Let $M^{2n-1} \subset \mathbb{C}^n$ be a smooth, compact, embedded, orientable CR manifold of hypersurface type. Let $1 \leq q \leq n-2$. If $M$ satisfies weak $Y(q)$-(CR-$P_q$). Then the complex Green operator $G_q$ is a compact operator on $L^2_{0,q}(M)$.

The definition for weak $Y(q)$-(CR-$P_q$) includes the assumption that $M$ satisfies weak $Y(q)$. We prove an auxiliary compactness result that is very useful when constructing examples. The motivation behind the next result is that if $M$ is a weak $Y(q)$ manifold that is $Y(q)$ everywhere except for a small set $S$ and there
exist good weight functions on $S$, then we can still prove $G_q$ is a compact operator. The definition of weak $Z(q)$-compatible functions is in Section 2, but are essentially functions that have Hessians with enough large eigenvalues.

**Theorem 1.2.** Let $M$ be a smooth, compact, embedded, orientable weak $Y(q)$ manifold. Let $S$ be the set of weak $Y(q)$ points. Suppose that $S$ is compact and for each $B > 0$, there exist functions $\lambda_B^+$ and $\lambda_B^-$ so that $0 \leq \lambda_B^+ \leq 1$ and $\lambda_B^- \leq 2B$ on $S$. Then $G_q$ is a compact operator.

To prove the Theorem 1.2 we use a slight modification of the microlocal norm used to prove the Theorem 1.1. We also address a technical issue, namely, that there were no prior results about this modified norm, so we prove a harmonic analysis used to prove the Theorem 1.1. We also address a technical issue, namely, that there were no prior results about this modified norm, so we prove a harmonic analysis that we developed to proved closed range of $\partial_b$ when $M$ satisfies weak $Y(q)$ in the manner of Raich’s earlier argument to prove compactness in the pseudoconvex case [18].

The technique that we use to prove compactness strengthens the microlocal analysis of [6] that we developed to proved closed range of $\partial_b$ when $M$ satisfies weak $Y(q)$ in the manner of Raich’s earlier argument to prove compactness in the pseudoconvex case [18].

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The outline of the paper is as follows. In Section 2, we carefully define our notation, including the weak $Y(q)$ and weak $Y(q)$-(CR-$P_q$) conditions. In Section 3 we set up the microlocal decomposition in local coordinates. In Section 4 we define the norms involved in this article. In Section 5 we establish a family of basic estimates that we use in Section 6 when we prove Theorem 1.1. In Section 7 we prove Theorem 1.2 and provide a new (and non-pseudoconvex) example on which $G_q$ is compact.

2. Preliminaries. Throughout the paper, we use the following notation: constants that contain no subscript do not depend on any choices that we make, such as the choice of weight functions, scaling constants in the microlocal analysis, etc. Constants that have a subscript are allowed to depend on global objects and the element in the subscript. We closely follow the setup and notation of [6].

2.1. The CR structure on $M$. Let $M^{2n-1} \subset \mathbb{C}^N$ be a smooth, compact, CR manifold of hypersurface type. Since $M$ is embedded, we take the CR structure on $M$ to be the induced CR structure $T^{1,0}(M) = T^{1,0}(\mathbb{C}^N) \cap T(M) \otimes \mathbb{C}$. Let $T^{p,q}(M)$ be the exterior algebra generated by $T^{1,0}(M)$ and $T^{0,1}(M)$ and $\Lambda^{p,q}(M)$ the bundle of $(p, q)$-forms on $T^{p,q}(M)$. Since $M \subset \mathbb{C}^N$, our calculations do not depend on $p$, so we take $p = 0$ for the entirety of the paper. For additional background on CR geometry, please see [3, 5].

2.2. $\partial_b$ on embedded manifolds. Since $M \subset \mathbb{C}^N$, we use the induced metric on $\mathbb{C}T(M)$, denoted by $\langle \cdot, \cdot \rangle_x$ for $x \in M$. The associated inner product on $\Lambda^{0,q}(M)$ is then given by

$$(\varphi, \psi)_0 = (\varphi, \psi) = \int_M \langle \varphi, \psi \rangle_x dV$$
where $dV$ is the volume element on $M$. The tangential Cauchy-Riemann operator $\partial_b$ is the restriction of the de Rham exterior derivative $d$ to $\Lambda^{0,q}(M)$. To our inner product, we have an associated $L^2$-norm denoted by $\| \cdot \|_0$. Additionally, we abuse notation slightly and continue to call the closure of $\partial_b$ in this norm by $\partial_b$. As such, $\partial_b : L^2_{0,q}(M) \rightarrow L^2_{0,q+1}(M)$ is a well-defined, closed, densely defined operator with $L^2$ adjoint $\partial_b^* : L^2_{0,q+1}(M) \rightarrow L^2_{0,q}(M)$. We define the Kohn Laplacian $\Box_b : L^2_0(M) \rightarrow L^2_0(M)$ by

$$\Box_b := \partial_b^* \partial_b + \partial_b \partial_b^*.$$  

### 2.3. The Levi form

Since $M$ is orientable and of hypersurface type, there is a unit vector $T$, taken purely imaginary, so that $T$ is orthogonal to $T^{1,0}(M) \oplus T^{0,1}(M)$. Let $\gamma$ be the globally defined 1-form that annihilates $T^{1,0}(M) \oplus T^{0,1}(M)$ and is normalized so that $\langle \gamma, T \rangle = -1$.

**Definition 2.1.** The Levi form at a point $x \in M$ is the Hermitian form $\langle d\gamma_x, L \wedge \bar{L}' \rangle$ for $L, L' \in T^{1,0}_x(U)$ and $U$ is a small neighborhood of $x \in M$.

By Cartan’s formula,

$$\langle d\gamma, L \wedge \bar{L}' \rangle = -\langle [L, \bar{L}'], \gamma \rangle$$  

(2.1)

for any $L, L' \in T^{1,0}_x(U)$. If $U$ is a small neighborhood in $M$, then there exists a local basis $\{L_1, \ldots, L_{n-1}\}$ of $T^{1,0}(U)$, so that for any $1 \leq j, k \leq n-1$,

$$[L_j, L_k] = c_{jk}T \mod T^{1,0}(U) \oplus T^{0,1}(U).$$  

(2.2)

This means $\langle d\gamma, L_j \wedge \bar{L}_k \rangle = c_{jk}$, and we call the matrix $[c_{jk}]_{1 \leq j, k \leq n-1}$ the Levi matrix with respect to $L_1, \ldots, L_{n-1}, T$.

Let $\mu_1 \leq \cdots \leq \mu_{n-1}$ be the eigenvalues of $[c_{jk}]$. We say that $M$ is (strictly) pseudoconvex at a point $p \in M$ if $\mu_1(p) \geq 0$ (resp., $\mu_1(p) > 0$). $M$ is said to be (strictly) pseudoconvex if $M$ is (strictly) pseudoconvex at every point.

The next definition is the current most general known sufficient condition for the closed range of $\partial_b$ on $L^2_0(M)$, and it is given in Harrington and Raich in [11]. We use it as a basic geometric hypothesis that we require on our CR manifolds.

**Definition 2.2.** For $1 \leq q \leq n-1$ we say $M$ satisfies weak $Z(q)$ if there exists a real $\Upsilon \in T^{1,1}(M)$ satisfying

(A) $|\theta|^2 \geq (i \theta \wedge \bar{\theta})(\Upsilon) \geq 0$ for all $\theta \in \Lambda^{1,0}(M)$

(B) $\mu_1 + \mu_2 + \cdots + \mu_q - i \langle d\gamma_x, \Upsilon \rangle \geq 0$ where $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of the Levi form at $x$ in increasing order.

(C) $\omega(\Upsilon) \neq 0$ where $\omega$ is the $(1,1)$-form associated to the induced metric on $\mathbb{C}T(M)$.

We say that $M$ satisfies weak $Y(q)$ if $M$ satisfies both $Z(q)$-weakly and $Z(n-q-1)$-weakly.

It is immediate that pseudoconvexity implies weak $Z(q)$ for any $1 \leq q \leq n-1$ with $\Upsilon = 0$. For a discussion of the history and significance of weak $Z(q)$, please see [11, 12, 13]. The existence of a Hodge* like operator forces the symmetric hypotheses on form levels on $q$ and $n-1-q$ [19, 2].

**Remark 1.** To understand Definition 2.2, it is very helpful to consider the case when $M$ is the boundary of a domain in $\mathbb{C}^n$ and $p \in M$ is a fixed point. We adapt the discussion in Harrington and Raich [11, p.1718]. We can choose local
coordinates that are orthogonal at \( p \) and satisfy \( \frac{\partial \rho(p)}{\partial z_j} = 0 \) for \( 1 \leq j \leq n-1 \). At \( p \), we can represent coordinatize the Levi form (denoted by \( \mathcal{L} \)) and \( \Upsilon \) by

\[
\Upsilon = i \sum_{j,k=1}^{n-1} u^{kj} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_j} \quad \text{and} \quad \mathcal{L} = i \sum_{j,k=1}^{n-1} c_{jk} dz_j \wedge d\bar{z}_k. \tag{2.3}
\]

We may further assume that either \( \Upsilon \) or \( \mathcal{L} \) is diagonalized at \( p \) (there is no reason to assume they can be simultaneously diagonalized). If we assume that \( \Upsilon \) is diagonalized at \( p \) with eigenvalues \( \alpha_j \), then \( u^{kj} = \delta_{jk} \alpha_j \) and the conditions at \( p \) in Definition 2.2 reduce to

(A) \( 0 \leq \alpha_j \leq 1 \) for \( 1 \leq j \leq n-1 \);
(B) \( \mu_1 + \cdots + \mu_q - (\alpha_1 c_{11} + \cdots + \alpha_{n-1} c_{n-1,n-1}) \geq 0 \);
(C) \( \alpha_1 + \cdots + \alpha_{n-1} = q. \)

**Remark 2.** If \( M \) satisfies weak \( Y(q) \), there is no \textit{a priori} reason to believe that the \( \Upsilon \) implied by \( Z(q) \) (denoted by \( \Upsilon_q \)) is related to the \( \Upsilon \) implied by weak \( Z(n-1-q) \) (denoted by \( \Upsilon_{n-1-q} \)).

Following \[6, 18, 10\], we let \( \nu \) be the real part of the complex normal to \( M \) and for a function \( \varphi \) defined near \( M \), we define the real \((1,1)\)-form

\[
\Theta^\varphi = \frac{1}{2} \left( \delta_b \delta_b \varphi - \delta_b \partial \varphi + \frac{1}{2} \nu(\varphi) \right) d\gamma
\]

In local coordinates, we associate \( \Theta^\varphi \) with the matrix \( \Theta^\varphi_{jk} = \langle \Theta^\varphi, L_j \wedge \bar{L}_k \rangle \). By \[18,\ Proposition 3.2\]

\[
\left\langle \frac{1}{2} \left( \delta_b \delta_b \varphi - \delta_b \partial \varphi \right), L \wedge \bar{L} \right\rangle = \langle \Theta^\varphi, L \wedge \bar{L} \rangle.
\]

Thus, \( \Theta^{|z|^2} = \partial \bar{\partial} |z|^2 = -i\omega \) \[10,\ Proposition 3.1\].

### 2.4. Weak \( Z(q)\)-(CR-\( P_q \)) and weak \( Z(q) \) compatible functions.

**Definition 2.3.** Let \( M^{2n-1} \) be a smooth, compact, oriented CR-manifold of hypersurface type satisfying weak \( Z(q) \), for some \( 1 \leq q \leq n-2 \). Let \( \lambda \) be a smooth function in \( M \). We say \( \lambda \) is \textit{weak} \( Z(q) \) compatible if there exists a constant \( B_\lambda > 0 \) such that

- \( b_1 + \cdots + b_q - i \langle \Theta^\lambda, \Upsilon \rangle \geq B_\lambda \) if \( q > \omega(\Upsilon) \),
- \( b_{n-q} + \cdots + b_{n-1} - i \langle \Theta^\lambda, \Upsilon \rangle \leq -B_\lambda \) if \( q < \omega(\Upsilon) \),

where \( b_1, \ldots, b_{n-1} \) are the eigenvalues of \( \Theta^\lambda \) in increasing order and \( \Upsilon \) is a real \((1,1)\)-form associated to weak \( Z(q) \). We will call \( B_\lambda \) the \textit{positivity constant} of \( \lambda \).

**Remark 3.** Keeping the notation and setup of Remark 1, the conditions from Definition 2.3 mean that at \( p \),

- \( b_1 + \cdots + b_q - (\Theta^\lambda_{11} \alpha_1 + \cdots + \Theta^\lambda_{n-1,n-1} \alpha_{n-1}) \geq B_\lambda \) if \( q > \text{Tr}(\Upsilon) \),
- \( b_{n-q} + \cdots + b_{n-1} - (\Theta^\lambda_{11} \alpha_1 + \cdots + \Theta^\lambda_{n-1,n-1} \alpha_{n-1}) \leq -B_\lambda \) if \( q < \text{Tr}(\Upsilon) \).

It is worth reiterating that \( \Upsilon = 0 \) on pseudoconvex neighborhoods and \( \Upsilon = I \) on pseudoconcave neighborhoods. Given that every compact manifold in \( \mathbb{R}^n \) has a point of convexity (think about the point farthest away from the origin), the \( q > \text{Tr}(\Upsilon) \) case is the relevant case for a general \( M \). Of course, if \( M = b\Omega \) and \( \Omega \) is an annular region, then the case \( q < \text{Tr}(\Upsilon) \) describes the inner boundary.
Definition 2.4. Let $M$ be a $2n - 1$ real dimensional, smooth, compact, oriented CR-manifold of hypersurface type of real dimension $2n - 1$ satisfying weak $Z(q)$. We say $M$ satisfies weak $Z(q)$-(CR-$P_q$) if for every $B > 0$, there exists a smooth function $\lambda_B$ near $M$ so that $0 \leq \lambda_B \leq 1$ and $\lambda_B$ is weak $Z(q)$ compatible with positivity constant $B$. We say that $M$ is weak $Y(q)$-(CR-$P_q$) if $M$ is both weak $Z(q)$-(CR-$P_q$) and weak $Z(n - 1 - q)$ (CR-$P_{n-1-q}$).

When $M$ is pseudoconvex, $M$ satisfies the Definition 2.2 with $Y = 0$, and if in addition $M$ satisfies property (CR-$P_q$) (see [18, Definition 2.6]), then $M$ will be weak $Z(q)$-(CR-$P_q$).

3. Pseudodifferential operators. We use the standard construction of pseudodifferential operators in several complex variables and continue to follow [6]. Let $\xi = (\xi_1, \ldots, \xi_{2n-1}) = (\xi', \xi_{2n-1})$ denote the dual coordinates so that $\xi'$ denote the dual to the maximal complex subspace in $\mathbb{C}T(M)$, $T^{1,0}(M) \oplus T^{0,1}(M)$, and $\xi_{2n-1}$ is dual to the remaining direction $T$. Let

$$C^+ = \left\{ \xi : \xi_{2n-1} \geq \frac{1}{2} |\xi'| \quad \text{and} \quad |\xi| \geq 1 \right\}; \quad C^0 = \left\{ \xi : -\xi \in C^+ \right\}; \quad C^- = \left\{ \xi : -\xi \in C^+ \right\};$$

$$C_0 = \left\{ \xi : -\frac{3}{4} |\xi'| \leq \xi_{2n-1} \geq \frac{3}{4} |\xi'| \right\} \cup \left\{ \xi : |\xi| \leq 1 \right\}.$$

Define $\psi^+, \psi^-$ and $\psi^0$ to be smooth functions on the unit sphere so that

$$\psi^+(\xi) = 1 \quad \text{when} \quad \xi_{2n-1} \geq \frac{3}{4} |\xi'| \quad \text{and} \quad \text{supp} \psi^+ \subset \left\{ \xi : \xi_{2n-1} \geq \frac{1}{2} |\xi'| \right\};$$

$$\psi^-(-\xi) = \psi^+(\xi); \quad \psi^0(\xi) = 1 - \psi^+(\xi)^2 - \psi^-(\xi)^2.$$

Extend $\psi^+, \psi^-$, and $\psi^0$ homogeneously of degree zero outside of the unit ball and smoothly inside the unit ball so that $(\psi^+)^2 + (\psi^-)^2 + (\psi^0)^2 = 1$. Finally, for a constant $A > 0$, set

$$\psi^+_A(\xi) = \psi^+(\xi/A), \quad \psi^-_A(\xi) = \psi^-(\xi/A), \quad \text{and} \quad \psi^0_A(\xi) = \psi^0(\xi/A).$$

Next, let $\Psi^+_A, \Psi^-_A$, and $\Psi^0_A$ be the pseudodifferential operators of order zero with symbols $\psi^+_A, \psi^-_A$, and $\psi^0_A$, respectively. It follows from the equality $(\psi^+_A)^2 + (\psi^-_A)^2 + (\psi^0_A)^2 = 1$ that

$$(\Psi^+_A)^* \Psi^+_A + (\Psi^-_A)^* \Psi^-_A + (\Psi^0_A)^* \Psi^0_A = Id.$$
The space of weighted harmonic forms $H_\partial$ inner product $(\cdot,\cdot)_{\phi^+.,\phi^-}$ will be bounded in $L^2$. Let $\theta$ be smooth functions on $M$. As done in [18, 10, 6], we define an inner product and norm

$$(f,g)_{\phi^+.,\phi^-} := \sum\left(\tilde{\zeta}_\mu \Psi_{\mu, A}^+ \zeta_{\mu} f^\mu, \tilde{\zeta}_\mu \Psi_{\mu, A}^+ \zeta_{\mu} g^\mu\right)_{\phi^+} + \left(\tilde{\zeta}_\mu \Psi_{\mu, A}^0 \zeta_{\mu} f^\mu, \tilde{\zeta}_\mu \Psi_{\mu, A}^0 \zeta_{\mu} g^\mu\right)_0 + \left(\tilde{\zeta}_\mu \Psi_{\mu, A}^- \zeta_{\mu} f^\mu, \tilde{\zeta}_\mu \Psi_{\mu, A}^- \zeta_{\mu} g^\mu\right)_{\phi^-}$$

and

$$\|f\|^2_{\phi^+.,\phi^-} = \|f\|^2_2 := \sum\left(\|\tilde{\zeta}_\mu \Psi_{\mu, A}^+ \zeta_{\mu} f^\mu\|_{\phi^+}^2 + \|\zeta_{\mu} f^\mu\|_0^2 + \|\zeta_{\mu} f^\mu\|_{\phi^-}^2\right)$$ (4.1)

where $f^\mu$ and $g^\mu$ are the representations of $f$ and $g$ in the local coordinates on $U_\mu$. We usually omit the superscript $\mu$. For a form $f$ on $M$, we define the $L^2$-Sobolev norm of order $s$ by

$$\|f\|_s := \sum\|\tilde{\zeta}_\mu \Lambda^s \zeta_{\mu} f^\mu\|_0^2$$

where $\Lambda$ is the pseudodifferential operator with symbol $(1 + |\xi|^2)^{1/2}$. The next theorem is essentially [18, Lemma 3.5].

**Theorem 4.1.** Let $\phi^+$ and $\phi^-$ be smooth functions on $M$ with $|\phi^+| \leq 1$ and $|\phi^-| \leq 1$. Then there exist constant $C_1$ and $C_2$ so that

$$C_1 \|\varphi\|_0^2 \leq \|\varphi\|_{\phi^+.,\phi^-}^2 \leq C_2 \|\varphi\|_0^2$$ (4.2)

where $C_1$ and $C_2$ does not depend either $\phi^+$ or $\phi^-$, (assuming $A \geq 1$).

This last result establishes the equivalence of unweighted $L^2$ norm and the microlocal norm, where the constants do not depend on $A$ (for $A \geq 1$). As a result of this equivalence, the self adjoint operator $E_{\phi^+.,\phi^-} = E_\pm$ satisfying

$$(f,g)_0 = (f,E_{\phi^+.,\phi^-} g)_{\phi^+.,\phi^-} = (f,E_\pm g)_\pm$$

will be bounded in $L^2(M)$, independently of $A \geq 1$ [18, Corollary 3.6].

5. $\partial_b$ and the basic estimate. Let $\partial_b^{\pm,\pm}$ be the adjoint of $\partial_b$ with respect to the inner product $(\cdot,\cdot)_{\phi^+.,\phi^-}$. We also define the weighted Kohn Laplacian by $\Box_b := \partial_b \partial_b^* + \partial_b^* \partial_b$.

Define the energy form $Q_{b,\pm}(\cdot,\cdot)$ by

$$Q_{b,\pm}(f,g) := (\partial_b f, \partial_b g)_{\pm} + (\partial_b^* f, \partial_b^* g)_{\pm}$$

The space of weighted harmonic forms $H_{\pm}^q$ is defined by

$$H_{\pm}^q := \{ f \in \text{Dom}(\partial_b) \cap \text{Dom}(\partial_b^*) : \partial_b f = 0, \partial_b^* f = 0 \} = \{ f \in \text{Dom}(\partial_b) \cap \text{Dom}(\partial_b^*) : Q_{b,\pm}(f,f) = 0 \} .$$

The argument of [6, Proposition 4.1] also proves the following basic estimate.
Proposition 1. Let $M^{2n-1}$ be a smooth, compact, orientable CR manifold of hypersurface type embedded in $\mathbb{C}^N$, that satisfies weak $Y(q)$ for some fixed $1 \leq q \leq n-2$. Let $\lambda_1$ be weak $Z(q)$ compatible and $\lambda_2$ be weak $Z(n-1-q)$ compatible. Set

$$
\phi^+(z) = \begin{cases} 
\lambda_1 & \text{if } \omega(T_q) < q \\
-\lambda_1 & \text{if } \omega(T_{n-1-q}) > q
\end{cases}
\quad \text{and} \quad
\phi^-(z) = \begin{cases} 
-\lambda_2 & \text{if } \omega(T_{n-1-q}) < n-1-q \\
\lambda_2 & \text{if } \omega(T_{n-1-q}) > n-1-q.
\end{cases}
$$

When $\Lambda$ is sufficiently large (e.g., $\Lambda \geq \frac{1}{2} |\nu(\lambda_j)|$, $j = 1, 2$), there exist constants $K$, $K_{\pm}$, and $K_{\pm, A}$ so that

$$
B_{\lambda_1, \lambda_2} \|f\|_\pm^2 
\leq KQ_{b, \pm}(f, f) + K \|f\|_\pm^2 + K_{\pm} \sum_j \sum_{\nu} \|\tilde{\zeta}_\nu \tilde{\Psi}^0_{\nu, A}\zeta_\nu f_j\|_0 + K_{\pm, A} \|f\|_{-1}^2.
$$

(5.2)

The constant $B_{\lambda_1, \lambda_2} > 0$ is the minimum of the positivity constants $B_{\lambda_1}$ and $B_{\lambda_2}$.

Remark 4. The microlocal norm, the basic estimate, and the weak $Y(q)$-(CR-$P_q$) are all technical conditions and it is not clear how they function together, so we would like to sketch the idea of the proof of Proposition 1. In the case that $M = \mathbb{B}^\omega$, the basic identity for a smooth form $f$ defined in a small neighborhood of a boundary point with weight function $e^{-\varphi}$ is

$$
\|\partial_\nu \tilde{f}\|_\varphi^2 + \|\partial_\nu^\varphi \tilde{f}\|_\varphi^2 = \sum_{j=1}^{n-1} \sum_{l \in I_{q-1}, k=1}^{n-1} \|\tilde{L}_{jl} f_l\|_{\varphi}^2 + \sum_{l \in I_{q-1}, k=1}^{n-1} \sum_{j=1}^{n-1} \text{Re}(c_{jk} TF_{jl}, f_k) \varphi \sum_{l \in I_{q-1}, k=1}^{n-1} \sum_{j=1}^{n-1} (\varphi_{jk} f_j, f_k) \varphi + \cdots.
$$

(5.3)

There are three terms here – the derivative terms, the Levi form terms, and the $L^2$ norm of $f$ terms multiplied by the “Levi form” of $\varphi$. If the matrix $(c_{jk})$ is positive semidefinite, then we can use sharp Gårding inequality to remove the $T$ from the Levi form term (with an acceptable error). Microlocally, $\sigma(T)$ has a positive and a negative direction, and (5.3) works well if $\sigma(T) > 0$ and the Levi form is positive semidefinite on $\text{supp } \tilde{f}$ or if $\sigma(T) < 0$ and the Levi form is negative semidefinite on $\text{supp } \tilde{f}$. In the mixed signature (or more complicated cases), we integrate the derivative terms by parts, and $\bar{\Upsilon}$ tells us how to integrate by parts. For example, the first condition in the definition of $\bar{\Upsilon}$ guarantees that the derivative terms, post integration by parts, remain nonnegative. When we integrate by parts, the commutators $[L_k, L_k]$ produce both additional $-c_{jj} T f_j$ terms and $-\varphi_{jj} f_j$. Properly interpreted, the second part of the definition of $\bar{\Upsilon}$ is exactly what we need so that the Levi form terms are positive semidefinite and we can apply Gårding’s inequality. What is complicated and the reason that we need a symmetric condition on the form levels is that the $\sigma(T) > 0$ and $\sigma(T) < 0$ regions require separate arguments. Typically, the hypothesis for level $q$ handles $\sigma(T) > 0$ and the hypothesis for level $n-1-q$ handles the $\sigma(T) < 0$ case. Finally, the third piece of the definition of $\bar{\Upsilon}$ guarantees that $L^2$ norm of $f$ terms actually bound the $L^2$ norm of $f$ and assorted detritus that we accumulate from the integration by parts.

The reason for the complicated norm is that we want the weight to look like $\varphi^+$ on the part of the form supported where $\sigma(T) > 0$ and dominant and like $\varphi^-$ for the part of the form supported where $\sigma(T) < 0$ and dominant. When $\sigma(T)$ is not the dominant term in the microlocal decomposition, $\square_b$ behaves elliptically the estimates are better. The microlocal decomposition that breaks apart the form $f$ into these pieces (up to controllable error terms) naturally leads to $\|\cdot\|_\pm$ norm.
The preceding argument explains how to obtain $L^2$ boundedness as in [6]. The strengthening from $L^2$ estimates to compactness estimates in the $P_q$ realm of results involves hypothesizing that bounded weight functions with very large Hessians exist nearby the weakly pseudconvex points (or in a neighborhood of $M$). This is the genesis of the weak $Y(q)$-(CR-$P_q$). Essentially, instead of using a generic strictly plurisubharmonic function like $|z|^2$ (as one does with $L^2$ estimates), we instead have the weight functions from weak $Y(q)$-(CR-$P_q$) which allow us to prove a basic estimate where the $L^2$ norm of $f$ terms are multiplied by an arbitrarily large number.

**Remark 5.** The functions $\phi^+$ and $\phi^-$ are well-defined as a consequence of (C) from Definition 2.2. For example, $\phi^+$ is well defined because the signs of $q - \omega(\Upsilon_q)$ and $(n - 1 - q) - \omega(\Upsilon_{n-1-q})$ are preserved on connected components in $M$.

Forms supported on $C^0$ up to smooth terms satisfy elliptic estimates, as seen by the next proposition, [18, Proposition 4.11].

**Proposition 2.** For any $\epsilon > 0$ there exists $C_{\epsilon,\pm,A} > 0$ so that

\[
\|\tilde{z}_\nu^B \Psi_{\nu,\pm,A}^B f^\nu\|^2_{\|\cdot\|_2} \leq \epsilon Q_{b,\pm}(f^\nu, f^\nu) + C_{\epsilon,\pm,A} \|f^\nu\|^2_{-1}.
\]

6. **Compactness estimates.**

**Proposition 3.** Let $M^{2n-1}$ be a smooth, compact, orientable CR manifold of hypersurface type embedded in $\mathbb{C}^N$ and fix $1 \leq q \leq n-2$. If $M$ satisfies weak $Y(q)$-(CR-$P_q$), then given $B > 0$ (sufficiently large) there exist constants $K$ and $K_B$ independent of $B$, and a microlocal norm $\|\cdot\|_B$ such that for any $f \in \text{Dom}(\partial_b) \cap \text{Dom}(\partial_b^*)$

\[
B \|f\|^2_B \leq K Q_{b,B}(f, f) + K_B \|f\|_{-1}^2.
\]  

(6.1)

To each $B > 0$, there are functions $\phi^+$ and $\phi^-$ from the weak $Y(q)$-(CR-$P_q$) property. The microlocal norm $\|\cdot\|_B$ is simply the microlocal norm $\|\cdot\|_{\pm}$ corresponding to $\phi^+$ and $\phi^-$.\[\]

**Proof.** Given $B > 0$ choose two smooth CR-plurisubharmonic functions $\lambda_1$ and $\lambda_2$ on $(0, q)$-forms and $(0, n - 1 - q)$-forms resp. compatible with $\Upsilon_q$ and $\Upsilon_{n-1-q}$, and with positivity constant equal to $B$ (for both $\lambda_1$ and $\lambda_2$). Choose a constant $A$ sufficiently large (e.g., $A > \frac{1}{2} |\nu(\lambda_j)|$, $j = 1, 2$) and define the microlocal norm $\|\cdot\|_B$ following (5.1) and (4.1). By Proposition 1 and Proposition 2, there exists a constant $K$ and $K_{B,A}$ such that

\[
B \|f\|^2_B \leq K Q_{b,B}(f, f) + \|f\|^2_B + K_{B,A} \|f\|_{-1}^2.
\]  

(6.2)

The independence of our constant $K$ from $B$ allows us to choose $B$ sufficiently large such that we can absorb the term $K \|f\|_B$ in the left side of (6.2) and so obtain (6.1). \[\]

6.1. **The compactness of the unweighted complex Green operator $G_q$.** In [6], we proved the existence of $G_q$ follows from (6.1). We will show that family of estimates (6.2) suffices to establish compactness of $G_q$, but we cannot use the argument of [18] or many of the other arguments from the pseudoconvex case because they use the existence of $G_{q+1}$ which may or may not be a continuous operator (or even exist) on $L^2_{\omega,q+1}(M)$.\[\]

Let $G_q$ and $S_q$ denote the unweighted complex Green operator and the Szegö projection (with respect to $\|\cdot\|_0$) at level $q$.

We follow the general argument of [20, Theorem 4.29], which uses the following well-known lemma [20, (ii) in Lemma 4.3].
Lemma 6.1. Let X and Y be Hilbert spaces (over \( \mathbb{C} \)), \( T : X \to Y \) is linear. Assume that for all \( \varepsilon > 0 \) there are a Hilbert space \( Z_\varepsilon \), a linear compact operator \( S_\varepsilon : X \to Z_\varepsilon \), and a constant \( C_\varepsilon \) such that
\[
\|Tx\|_Y \leq \varepsilon\|x\|_X + C_\varepsilon\|S_\varepsilon x\|_{Z_\varepsilon}.
\]
Then \( T \) is compact.

Proposition 4. Let \( M \) be as in Theorem 1.1 and \( B \) be sufficiently large.

1. For all \( v \in \ker(\tilde{\partial}_h) \cap \text{Dom}(\tilde{\partial}_h^*) \)
\[
B\|v\|_0^2 \leq C\|\tilde{\partial}_h^*v\|_0^2 + C_B\|S_{q,B}E_Bv\|_2^2
\]
where \( S_{q,B} \) is the Szegő projection (respect to \( \|\cdot\|_B \)).

2. For all \( u \in \ker(\tilde{\partial}_h) \cap \text{Dom}(\tilde{\partial}_h) \)
\[
B\|u\|_0^2 \leq C\|\tilde{\partial}_h u\|_0^2 + C_B\|Z_B u\|_2^2
\]
for some continuous operator \( Z_B \) on \( L_{0,q}(M) \).

Proof. Proof of (6.3). Let \( v \in \ker(\tilde{\partial}_h) \cap \text{Dom}(\tilde{\partial}_h^*) \). By Proposition 3, given \( B \)
\[
B\|v\|_0^2 \leq C\|\tilde{\partial}_h^*v\|_B^2 + C_B\|v\|_2^2.
\]
Observe that, for every \( \alpha \in \text{Dom}(\tilde{\partial}_h^*) \) and \( \beta \in \text{Dom}(\tilde{\partial}_h) \)
\[
(\tilde{\partial}_h^*\alpha, \beta)_B = (\tilde{\partial}_h^*\alpha, \tilde{\partial}_h\beta)_B = (F_B\alpha, \tilde{\partial}_h\beta)_B = (\tilde{\partial}_h^*F_B\alpha, \beta)_B
\]
so \( \tilde{\partial}_h^*B = E_B\tilde{\partial}_h^*F_B \) on \( \text{Dom}(\tilde{\partial}_h^*) \). Note that this also means
\[
\tilde{\partial}_h^*B = E_B\tilde{\partial}_h^*F_B \tag{6.5}
\]
In what follows, \( C \) can change from line to line but is independent of \( B \). Then
\[
B\|F_B v\|_0^2 \leq B\|v\|_B^2 \leq C\|E_B\tilde{\partial}_h^*F_B v\|_B^2 + C_B\|v\|_2^2 \leq C\|\tilde{\partial}_h^*F_B v\|_0^2 + C_B\|v\|_2^2 \tag{6.6}
\]
Also if \( w, g \in \ker(\tilde{\partial}_h) \), then
\[
(w, g) = (E_Bw, g)_B = (S_{q,B}E_Bw, g)_B = (F_BS_{q,B}E_Bw, g).
\]
This means \( w - F_BS_{q,B}E_Bw, g = 0 \) for every \( g \in \ker(\tilde{\partial}_h) \), and since \( w \in \ker(\tilde{\partial}_h) \), it follows that \( w = S_q(F_BS_{q,B}E_Bw) \).

Applying (6.6) to \( S_{q,B}E_Bv \) (this is justified because \( v \in \text{Dom}(\tilde{\partial}_h) \) which means \( E_Bv \in \text{Dom}(\tilde{\partial}_h^*) \)) and therefore \( S_{q,B}E_Bv \in \text{Dom}(\tilde{\partial}_h^*) \), we observe that \( S_{q,B}E_Bv \in \ker(\tilde{\partial}_h) \cap \text{Dom}(\tilde{\partial}_h^*) \) and use that \( \tilde{\partial}_h^*g = \tilde{\partial}_h^*S_qg \) to obtain
\[
B\|v\|_0^2 = B\|S_q(F_BS_{q,B}E_Bv)\|_0^2 \leq B\|F_BS_{q,B}E_Bv\|_0^2 \\
\leq C\|\tilde{\partial}_h^*F_BS_{q,B}E_Bv\|_0^2 + C_B\|S_{q,B}E_Bv\|_2^2 \\
= C\|\tilde{\partial}_h^*S_qF_BS_{q,B}E_Bv\|_0^2 + C_B\|S_{q,B}E_Bv\|_2^2 \\
= C\|\tilde{\partial}_h^*v\|_0^2 + C_B\|S_{q,B}E_Bv\|_2^2.
\]
Proof of (6.4). Let \( u \in \ker(\tilde{\partial}_h) \cap \text{Dom}(\tilde{\partial}_h) \). By Proposition 3, given \( B \) and Theorem 4.1,
\[
B\|S_{q,B}^*u\|_0^2 \leq C\|\tilde{\partial}_hS_{q,B}^*u\|_B^2 + C_B\|S_{q,B}^*u\|_2^2
\]
where $S_{q,B}^*$ denotes the orthogonal projection (with respect to $\|\cdot\|_B$) onto $\ker(\bar{\partial}_b^*).$

Note that the previous calculation is justified because $S_{q,B}^* = Id - \bar{\partial}_b \bar{\partial}_b^* G_{q,B},$ so it follows that if $u \in \operatorname{Dom}(\bar{\partial}_b)$ then so is $S_{q,B}^* u \in \operatorname{Dom}(\bar{\partial}_b).$ Then it follows that $\bar{\partial}_b S_{q,B}^* u = \bar{\partial}_b u.$

On the other hand, it can be proved also that if $f \in \ker(\bar{\partial}_b^*)$ then $f = S_{q,B}^* S_{q,B} f,$ because if $g \in \ker(\bar{\partial}_b^*),$ then

$$\langle f, g \rangle = \langle f, E_B g \rangle_B = \langle S_{q,B}^* f, E_B g \rangle_B = \langle S_{q,B}^* f, g \rangle$$

where the second equality follows from (6.5). Thus,

$$B\|u\|_B^2 = B\|S_{q,B}^* S_{q,B} u\|_0^2 \leq B\|S_{q,B}^* u\|_0^2$$

$$\leq C\|\bar{\partial}_b S_{q,B}^* u\|_2^2 + C_B\|S_{q,B}^* u\|_{-1}^2 = C\|\bar{\partial}_b u\|_0^2 + C_B\|S_{q,B}^* u\|_{-1}^2.$$  \hfill $\square$

6.2. Proof of Theorem 1.1.

Proof. If $v \in \mathcal{H}^q,$ then $G_q v = 0,$ so we may assume that $v \in \ker(\bar{\partial}_b) \cap \perp \mathcal{H}^q.$ Next, since $G_q(\ker(\bar{\partial}_b) \cap \perp \mathcal{H}^q) \subset \ker(\bar{\partial}_b),$ by (6.3), given $B$ large

$$B\|G_q v\|_0^2 \leq C\|\bar{\partial}_b G_q v\|_0^2 + C_B\|S_{q,B} E_B G_q v\|_{-1}^2$$

$$= C\left(\bar{\partial}_b \bar{\partial}_b^* G_q v, G_q v\right) + C_B\|S_{q,B} E_B G_q v\|_{-1}^2$$

$$\leq B\|G_q v\|_0^2 + \frac{C^2}{2B}\|v\|_0^2 + C_B\|S_{q,B} E_B G_q v\|_{-1}^2,$$  \hfill (6.7)

After absorbing terms we obtain the compactness estimate for $G_q$ (Lemma 6.1) restricted to $\ker(\bar{\partial}_b) \cap \perp \mathcal{H}^q.$

Finally, suppose $u \in \perp \ker(\bar{\partial}_b) \cap \perp \mathcal{H}^q = \perp \ker(\bar{\partial}_b).$ By the Hodge theory for $\bar{\partial}_b,$ $u = \bar{\partial}_b^* \bar{\partial}_b G_q u \in \ker(\bar{\partial}_b^*).$ Since $G_q(\operatorname{Range}(\bar{\partial}_b^*)) \subset \ker(\bar{\partial}_b^*)$ [10, Section 6.2], we can proceed as before using (6.4) and so obtain the compactness of $G_q$ on $\perp \ker(\bar{\partial}_b).$ \hfill $\square$

7. Examples and the proof of Theorem 1.2. A weak $Y(q)$ manifold has two types of points: points where $Y(q)$ holds and points where $Y(q)$ does not hold. If the structure of the points where $Y(q)$ fails is good (e.g., a lower dimensional manifold) and $Y$ is well-understood, then the norm that we constructed in Section 4 is unwieldy. The $\bar{\partial}_b$-operator satisfies 1/2-estimates on the set of $Y(q)$ points and hence we only need the weight functions and microlocal norms in a neighborhood of the set where $Y(q)$ fails.

Condition (B) of Definition 2.2 invites the horribly (or wonderfully?) named strict weak $Z(q)$ if the inequality is strict. Fortunately, by the argument immediately preceding Lemma 2.8 in [11], if $Y$ exists so that the inequality in Condition B strict at $p \in M,$ then $Z(q)$ holds at $p.$ Thus, in a weak $Z(q)$ manifold, points $p \in M$ are either $Z(q),$ or there exists $Y$ that makes Condition B an equality. If $p \in M$ is a point where $Z(q)$ does not hold but weak $Z(q)$ holds, we call $p$ a weak $Z(q)$ point. Also we say $p \in M$ is a weak $Y(q)$ point if $p$ is a weak $Z(q)$ point or a weak $Z(n-1-q)$ point.

7.1. Norms, revisited. Let $M$ be a weak $Y(q)$ manifold and $S$ a compact subset of $M.$ Suppose that the functions $\lambda^+$ and $\lambda^-$ are defined in a neighborhood $U$ of $S.$ Let $\{U_u\}$ be the neighborhoods from Section 4 that intersect $S.$ Let $\chi$ be a smooth cutoff function so that $\chi(s) = 1$ if $s \leq \frac{1}{2}$ and $\chi(s) \equiv 0$ if $s \geq \frac{3}{4}.$ Let $\tilde{\chi}$ be a
Lemma 7.1. The integral kernel $K$ satisfies

\begin{align}
(f, g)_{\pm} & := \sum_{\mu} \left[ \left( \tilde{\chi}_r \xi_{\mu} \Phi_{\lambda, A} \xi_r f^\mu, \tilde{\chi}_r \xi_{\mu} \Phi_{\lambda, A} \xi_r g^\mu \right)_{\lambda^+} + \left( \tilde{\chi}_r \xi_{\mu} \Phi_{\lambda, A} \xi_r f^\mu, \tilde{\chi}_r \xi_{\mu} \Phi_{\lambda, A} \xi_r g^\mu \right)_{\lambda^-} \right] + \left( (1 - \chi_r) f, (1 - \chi_r) g \right)_0.
\end{align}

Proposition 5. If $A$ is sufficiently large and $0 \leq \lambda^+, \lambda^- \leq 1$, then the norm associated to the inner product (7.1), denoted by $\| f \|_\pm$ satisfies

$$c \| f \|_\pm \leq \| f \|_0 \leq C \| f \|_\pm$$

for constants $c$ and $C$ that are independent of $\lambda^+, \lambda^-$ and $A$.

Before we prove Proposition 5, we need the following estimate.

Lemma 7.1. Let $\zeta$ and $\tilde{\zeta}$ be cutoff functions so that $\tilde{\zeta}$ is identically 1 on $\text{supp} \zeta$. The integral kernel $K(x, y)$ in $\mathbb{R}^n$ defined by

$$K(x, y) := (1 - \tilde{\zeta}(x))\zeta(y) \int_{\mathbb{R}^n} e^{2\pi i \xi (x-y)} \psi^+_A(\xi) d\xi$$

satisfies

$$|K(x, y)| \leq C_k \frac{(1 + |\xi|)^k}{A^{k-n}} |1 - \tilde{\zeta}(x)| |\zeta(y)|$$

for each $k > n$ where $d = \text{dist}(\text{supp} \zeta, \text{supp}(1 - \tilde{\zeta}))$ and $C_k$ is such that

$$\sup_{\eta} \left| (1 + |\eta|)^k D^k \psi^+(\eta) \right| n^k \leq C_k.$$

Remark 6. The integral defining $K(x, y)$ needs to be interpreted in the sense of distributions. In the absence of the cutoff functions, the kernel would be a standard Calderón-Zygmund kernel and agree with a smooth function away from the singularity $x = y$. Also, $d > 0$ since $1 - \tilde{\zeta}$ and $\zeta$ have disjoint supports.

Proof. The proof is a more careful version of the calculation in [18, Lemma 3.5]. Let $k > n$.

\begin{align}
(1 - \tilde{\zeta}(x))\Phi^+_{\lambda, A} F(x) &= (1 - \tilde{\zeta}(x)) \int_{\mathbb{R}^n} e^{2\pi i \xi x} \psi^+_A(\xi) \widetilde{F}(\xi) d\xi \\
&= \int_{\mathbb{R}^n} F(y)(1 - \tilde{\zeta}(x))\zeta(y) \int_{\mathbb{R}^n} e^{2\pi i \xi (x-y)} \psi^+_A(\xi) d\xi dy. \quad (7.2)
\end{align}

As in Remark 6, the inverse Fourier transform of $\psi^+_A$ needs to be understood in the sense of distributions. There exists $j$ so that $|x_j - y_j| \geq \frac{1}{A} |x - y|$. Integrating by parts in $\xi_j$ yields

$$|K(x, y)| \leq \frac{n^k |1 - \tilde{\zeta}(x)| |\zeta(y)|}{|x - y|^k} \int_{\mathbb{R}^n} A^k \frac{D^k \psi^+_A}{D\xi_j^k} \left( \frac{\xi}{A} \right) d\xi$$

$$\leq \frac{|1 - \tilde{\zeta}(x)| |\zeta(y)|}{|x - y|^k} C_k \int_{\mathbb{R}^n} A^n \left( 1 + |w| \right) d\omega \leq \frac{|1 - \tilde{\zeta}(x)| |\zeta(y)|}{|x - y|^k} A^{k-n} C_k.$$
Finally, $1 + |x - y| = \frac{d + d|x - y|}{d} \leq |x - y| (1 + 1/d)$, so that $|x - y|^{-k} \leq (1 + 1/d)^k (1 + |x - y|)^{-k}$ and the result follows. \hfill \Box

**Proof of Proposition 5.** That $c \|u\| \leq \|u\|_0$ for a constant $c > 0$ independent of $\lambda^\pm$ and $A$ is immediate. For the other equivalence, observe that

$$
\|u\|_0^2 = \sum_{\mu} \|\nabla u\|_{\mu,A}^2 \leq c \sum_{\mu} \|u\|_{\mu,A}^2 = c \sum_{\mu} \|\nabla u\|_{\mu}^2
$$

We claim that

$$
\|\nabla u\|_{\mu}^2 \leq C \sum_{\mu} \left( \|\nabla u\|_{\mu}^2 + \|\nabla u\|_{\mu,A}^2 \right).
$$

We (momentarily) suppress the $\mu$ subscript and estimate that

$$
\|\nabla \cdot u\|^2 = \left( \nabla \cdot \nabla \cdot u \right)^2 + \left( \nabla \cdot \nabla \cdot u \right)_0^2 = \left( \nabla u \cdot \nabla \cdot u \right)^2 + \left( \nabla u \cdot \nabla \cdot u \right)_0^2
$$

We claim that

$$
\|\nabla u\|_{\mu,A}^2 \leq C \sum_{\mu} \left( \|\nabla u\|_{\mu}^2 + \|\nabla u\|_{\mu,A}^2 \right).
$$

We (momentarily) suppress the $\mu$ subscript and estimate that

$$
\|\nabla \cdot u\|^2 = \left( \nabla \cdot \nabla \cdot u \right)^2 + \left( \nabla \cdot \nabla \cdot u \right)_0^2 = \left( \nabla u \cdot \nabla \cdot u \right)^2 + \left( \nabla u \cdot \nabla \cdot u \right)_0^2
$$

The key to proving Theorem 1.2 is the following refinement of our basic estimate.

**Proposition 6.** Assume the hypotheses of Theorem 1.2. Then there exists a microlocal norm $\|\cdot\|_B$ and constants $K$ and $K_B$ so that for any $f \in \text{Dom}(\bar{\partial}_b) \cap \text{Dom}(\bar{\partial}_b^*)$

$$
B \|f\|_B^2 \leq KQ_{b,B}(f,f) + K_B \|f\|_{-1}^2.
$$

(7.4)
Proof. Derivatives of $\chi_{rB}$ and $\bar{\chi}_{rB}$ are supported away from $S$, as is $(1 - \chi_{rB})$, and hence are supported in the set of $Y(q)$ points. Since $1/2$-estimates hold nearby $Y(q)$ points, the argument leading to Proposition 1 and (6.1) establishes the result. □

Proof of Theorem 1.2. The argument of Section 6 relies on the family of basic estimates proven in Proposition 1. Substituting in the basic estimates proven in Proposition 6 and rerunning the compactness argument establishes the results. □

7.2. An example. The next example demonstrates the situation when $M \subset \mathbb{C}^5$ is a weak $Y(2)$ manifold that is $Y(2)$ except at one point. We could formulate a more general proposition in the spirit of [20, Proposition 4.15] but we would need hypotheses both on the structure of the weak $Y(q)$ points and the form of $T$, and we think providing an explicit example demonstrates the type of new examples that are easily checkable with our hypotheses.

Example. Let $M \subset \mathbb{C}^5$ be the bounded CR manifold with defining function

$$
\rho(z) = |z_1|^4 - |z_1|^2 + |z_2|^2(|z'|^2 + x_3^2) + |z_3|^2 + |z_4|^2 + |z_5|^2 - 1.
$$

where $z' = (z_1, \ldots, z_4)$. Away from the “north pole” $(0, 0, 0, 0, i)$ and “south pole” $(0, 0, 0, 0, -i)$, we can quickly check that $M$ is $Y(2)$. In particular, if we write the complex Hessian of $\rho$ as a diagonal $D$ plus the matrix $E$ of off-diagonal terms, then the diagonal elements of $D$ are

$$(D_{11}, \ldots, D_{55}) = (4|z_1|^2 + |z_2|^2 - 1, 3|z_2|^2 + |z'|^2 + x_3^2, 1 + |z_3|^2, 1 + |z_2|^2, 1 + \frac{1}{2}|z_1|^2).$$

The matrix has nonzero off-diagonal terms $E_{2k} = \overline{E_{k2}}, k = 1, 3, 4, 5$ and

$$(E_{21}, E_{23}, E_{24}, E_{25}) = (z_1 \overline{z}_2, z_3 \overline{z}_2, z_4 \overline{z}_2, x_5 \overline{z}_2).$$

Away from the north and south poles, $D$ has at most one negative and at least four positive eigenvalues, and the second smallest eigenvalue of $D$ is $D_{22}$. The smallest eigenvalue of $E$ is $-|z_2|\sqrt{|z_1|^2 + |z_2|^2 + |z_3|^2 + x_3^2}$, so it follows from the Weyl inequalities that the complex Hessian has at least four positive eigenvalues. Next, for a fixed point $p \in M$, we can view $M$ in coordinates so that the Levi form is a principal $4 \times 4$ minor, and an interlacing theorem for eigenvalues of principal minor of Hermitian matrices (e.g., [14, Theorem 4.3.15]) that the Levi form has at least three positive eigenvalues. Therefore, away from the poles, $M$ is $Y(2)$.

We now show that $M$ is weak $Z(2)$ in a convenient metric near the north pole $(0, 0, 0, 0, i)$ and prove that it admits a good weight function. The argument for the south pole is similar. Near the north pole $P$, the vector fields

$$L_j = \frac{\partial \rho}{\partial z_j} - \frac{\partial \rho}{\partial \bar{z}_j}, \quad j = 1, \ldots, 4,$$

for a basis of $T^{1,0}(M \cap U)$. We also take the bad direction to be

$$T = \frac{\partial \rho}{\partial \bar{z}_5} - \frac{\partial \rho}{\partial z_5} = (z_5 + x_5|z_2|^2) \frac{\partial}{\partial z_5} - (\bar{z}_5 + x_5|z_2|^2) \frac{\partial}{\partial \bar{z}_5}.$$

We pick the metric so that $\{L_1, \ldots, L_4, T\}$ are orthonormal. The Levi matrix $L$ with entries $c_{jk}$ satisfying (2.2) is given by

$$L = \begin{pmatrix}
-1 + 4|z_1|^2 + |z_2|^2 & z_2 \bar{z}_1 & 0 & 0 \\
3|z_2|^2 + |z'|^2 + x_3^2 + O & z_3 \bar{z}_2 & z_5 \bar{z}_2 & z_4 \bar{z}_2 \\
0 & z_2 \bar{z}_3 & 1 + |z_2|^2 & 0 \\
0 & 2z_4 & 0 & 1 + |z_2|^2
\end{pmatrix}.$$
where \( O = O(x_5(|z'|^2 + x_5^2)|z|^2) \). At the north pole, \( z' = 0 \) and \( x_5 = 0 \), so that \( \mathcal{L} \) is diagonal with eigenvalues \(-1, 0, 1, 1\) and \( Y(2) \) fails. We now show that \( M \) is weak \( Y(2) \) in a neighborhood of the north pole. We take \( T \) to be the matrix with 1 in the (1, 1) slot and zeros elsewhere. Although it is difficult to compute the eigenvalues of \( \mathcal{L} \) away from the diagonal, we decompose

\[
\mathcal{L} = A + B
\]

where \( A \) is the diagonal matrix whose entries are the diagonal elements of \( \mathcal{L} \) and \( B \) is the matrix of the off-diagonal terms. The eigenvalues of \( Y \) are simply the diagonal entries and the matrix \( B \) has eigenvalues \( \{\pm|z_2|\sqrt{|z_1|^2 + |z_3|^2 + |z_4|^2}, 0, 0\} \). For an \( n \times n \) Hermitian matrix \( C \), we denote the eigenvalues of \( C \) by \( \lambda_1(C) \leq \lambda_2(C) \leq \cdots \leq \lambda_n(C) \). The eigenvalue inequality

\[
\lambda_1(\mathcal{L}) \geq \lambda_1(A) + \lambda_1(B) = -1 + 4|z_1|^2 + |z_2|^2 - |z_2|\sqrt{|z_1|^2 + |z_3|^2 + |z_4|^2}
\]

is immediate and the Weyl inequalities yield that

\[
\lambda_2(\mathcal{L}) \geq \lambda_2(A) + \lambda_1(B) = 4|z_1|^2 + |z_1|^2 + |z_3|^2 + |z_4|^2 + x_5^2 + O(x_5(|z'|^2 + x_5^2)|z_2|^2) - |z_2|\sqrt{|z_1|^2 + |z_3|^2 + |z_4|^2}.
\]

Since \( 2ab \leq 2a^2 + \frac{b^2}{2} \) for any positive number \( a \) and \( b \), it follows that

\[
\mu_1 + \mu_2 - i(d\tau_x, \Theta) = \lambda_1(\mathcal{L}) + \lambda_2(\mathcal{L}) - (-1 + 4|z_1|^2 + |z_2|^2) \geq 4|z_2|^2 + |z_1|^2 + |z_3|^2 + |z_4|^2 + x_5^2 + O(x_5(|z'|^2 + x_5^2)|z_2|^2) - 2|z_2|\sqrt{|z_1|^2 + |z_3|^2 + |z_4|^2} \geq 0
\]

with equality only at the north pole.

It is immediate that this example satisfies the hypotheses of Theorem 1.2 with weight functions \( \alpha(z) = \rho(z) + |z'|^2 + |z_5 - i|^2 \) near \((0, 0, 0, 0, i)\).

Although we use a different metric in the example than we used earlier, our compactness estimates will apply equally in this metric as the crucial calculations were all done locally.

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