Braids of the N-body problem by cabling a body in a central configuration

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Abstract. We prove the existence of periodic solutions of the \( N = (n+1) \)-body problem starting with \( n \) bodies whose reduced motion is close to a non-degenerate central configuration and replacing one of them by the center of mass of a pair of bodies rotating uniformly. When the motion takes place in the standard Euclidean plane, these solutions are a special type of braid solutions obtained numerically by C. Moore. The proof uses blow-up techniques to separate the problem into the \( n \)-body problem, the Kepler problem, and a coupling which is small if the distance of the pair is small. The formulation is variational and the result is obtained by applying a Lyapunov-Schmidt reduction and by using the equivariant Lyusternik-Schnirelmann category.

Keywords. N-body problem, periodic solutions, perturbation theory.

1. Introduction

The discovery of braids and choreographies are linked since the appearance of the original work [28] which contains the first choreography solution differing from the classical Lagrange circular one. In this choreography, three bodies follow one another along the now famous figure-eight orbit. The result was obtained numerically by finding minimisers of the classical Euler functional with a topological constraint associated with a braid. Later on, the first rigorous mathematical proof of the existence of the figure-eight orbit was obtained in [8] by minimising the Euler functional over paths that connect a collinear and an isosceles triangle configuration. However, the name choreography was adopted after the numerical work [36] to describe \( n \) masses that follow the same path. The study of choreographies has attracted much attention in recent years, while the study of braids has been relatively less explored. The purpose of our paper is to obtain new results on the existence of braids by cabling of central configurations (Figure 1).

Concretely, we investigate the motion of \( n \) bodies interacting under a general homogeneous potential. The motion takes place in an even dimensional Euclidean space \( E \) equipped with a compatible complex structure \( J \). Denote by \( Q_\ell(t) \in E \) the position of the \( \ell \)th body at time \( t \) and let \( M_\ell > 0 \) be its mass. Newton’s equations are given by

\[
M_\ell \ddot{Q}_\ell = - \sum_{k \neq \ell} M_\ell M_k \frac{Q_\ell - Q_k}{\|Q_\ell - Q_k\|^{\alpha+1}}, \quad \ell = 1, \ldots, n
\]

where \( \alpha \geq 1 \). The case \( \alpha = 2 \) corresponds to the problem of \( n \) bodies moving under the influence of the gravitation. A central configuration \( a \in E^n \) is a configuration which gives rise to a solution of the form \( Q(t) = \exp(tJ)a \). We construct braids of the \( N = n + 1 \)-body problem starting with a central configuration \( a \) of \( n \) bodies. Without loss of generality we may assume that \( M_1 = 1 \). The main idea is to replace one body \( Q_1 \) by the center of mass of a pair of bodies \( q_0, q_1 \) rotating uniformly, with masses \( m_0, m_1 > 0 \) such that \( m_0 + m_1 = 1 \). We assume that the central configuration \( a \) is non-degenerate (definition 4.2). The non-degeneracy of the Lagrange triangular configuration and the Maxwell configuration (consisting of a central body and \( n \)-bodies of equal masses attached to the
vertices of a regular polygon) follows a consequence of the stability analysis in [24, 23, 34, 15] except for a finite number of mass parameters.

Specifically, our main results (Theorems 5.1 and 5.3) state that, when the central configuration $a = (a_1, \ldots, a_n)$ is non-degenerate, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, Newton’s equations of the $N = n + 1$-body problem admit at least two solutions $q(t) = (q_0(t), \ldots, q_n(t))$ such that

$$
q_0(t) = \exp(t,\mathcal{J})u_1(\nu t) - m_1\varepsilon \exp(t,\mathcal{J})u_0(\nu t)
$$
$$
q_1(t) = \exp(t,\mathcal{J})u_1(\nu t) + m_0\varepsilon \exp(t,\mathcal{J})u_0(\nu t)
$$
$$
q_\ell(t) = \exp(t,\mathcal{J})u_\ell(\nu t), \quad \ell = 2, \ldots, n,
$$

where the components $u_\ell = a_\ell + O(\varepsilon)$ are $2\pi$-periodic paths in $E$, $a_0 \in E$ is a vector of unit length, $O(\varepsilon)$ is $2\pi$-periodic of order $\varepsilon$ with respect to a Sobolev norm, and $\nu$ and $\omega$ are functions of $\varepsilon$ through the relations $\omega = \pm \varepsilon^{-(\alpha+1)/2}$ and $\nu = \omega - 1$. The sign of the frequency $\omega$ represents whether the binary pair has a prograde or retrograde rotation. That is, prograde ($\omega > 0$) refers to the case that the pair rotates in the same direction as the relative equilibrium, while retrograde ($\omega < 0$) refers to the case that the pair rotates in the opposite direction. These solutions are quasi-periodic if $\omega \notin \mathbb{Q}$, and periodic if $\omega \in \mathbb{Q}$.

When $E$ is the plane and the frequency $\omega = \pm p/q$ is rational, there is, for any fixed integer $q \in \mathbb{Z} \setminus \{0\}$, some $p_0 > 0$ such that, for each $p > p_0$, the components $q_\ell(t)$ of (2) are $2\pi q$-periodic. In these solutions $n - 1$ bodies (close to $a_\ell$ for $\ell = 2, \ldots, n$) and the center of mass of the pair $q_0, q_1$ (close to $a_1$) wind around the origin $q$ times, while the bodies $q_0, q_1$ wind around their center of mass $p$ times (see Corollary 5.2 and Figure 1). These solutions are called braid solutions in [28] and the process of replacing a body by a pair is called cabling. In the braid formalism this means replacing a strand of a braid by another braid. For example, in Figure 2, the rigid motion obtained by rotating the central configuration of three equal masses located at the vertices of an equilateral triangle corresponds to the braid $b_1$, and this motion is $2\pi$-periodic. Replacing one of the bodies by the center of mass of two bodies rotating around their center of mass two times after a complete period of $2\pi$ amounts to perform the cabling of the braid $b_1$ with the braid of two strands $b_2$. The result is a new braid $b_1 \odot b_2$ with four strands.
Figure 2: The picture illustrates the solution of Figure 1 as a braid. The black strand in the braid $b_1$ on the left side is replaced by the braid $b_2$ to form a new braid. The cabling operation is denoted by $b_1 \odot b_2$.

For the case of the gravitational potential $\alpha = 2$, the result for the 3-body problem ($N = 2 + 1$) has been obtained separately by Moulton [29] and Siegel [35]. They establish the existence of periodic solutions of the 3-body problem by combining two circular motions of the 2-body problem. This problem, which includes Hill’s moon problem as a special case, enjoys a large literature and has been treated from various point of views in the original works [18] by Hill and [19] by Hopf. The case $N = 3 + 1$ has been studied in [10]. The methods used in [29, 35] and [10] to prove the existence of solutions are quite different from ours.

Our method starts by writing the Euler-Lagrange equations with respect to the Euler functional $A$ of the $N$-body problem, with $N = n + 1$. By changing the variables in the configuration space, the Euler functional splits into two terms $A = A_0 + H$, where $A_0$ is the uncoupled Euler functional of the $n$-body problem and the Kepler problem. The part $H$ represents the interaction of the pair with the $n$-body problem. Using the parameter $\varepsilon$, representing the radius of the circular orbit of the Kepler problem, the coupling term $H = O(\varepsilon)$ is small in order of $\varepsilon$, and the functional $A_0$ explodes as $\varepsilon \to 0$.

If $\dim(E) = 2d$, the functional $A_0$ is invariant under the group $U(d)^2$ acting diagonally on the Kepler component $u_0 \in E$ and the $n$ bodies component $u \in E^n$, while the coupling term $H$ is invariant only by the action of the diagonal subgroup $U(d)$ that rotates the $N$-body problem. Let $x_a = (a_0, a)$ where $a_0$ represents the orientation of the circular orbit of the Kepler problem with respect to the central configuration $a$. The $U(d)^2$-orbit of $x_a$ consists of critical points of the unperturbed functional $A_0$. In the gravitational case $\alpha = 2$, even if the central configuration $a$ is non-degenerate, the group orbit of $x_a$ is degenerate due to the existence of elliptic orbits. A similar problem arises when $E$ has at least dimension four, due to resonances of the circular orbit of the Kepler problem with extra dimensions. To deal with this issue, we need an extra assumption on the symmetries of the central configuration $a$. Thus the functional $A$ is invariant under the action of a discrete group $\Gamma$ and we can restrict the study of critical points to the fixed point space of $\Gamma$. The advantage is that, in the fixed point space of $\Gamma$, the problem of resonances can be avoided.

The symmetry group of $A_0$ will thus be taken to be a subgroup $G = G_1 \times G_2 \subset U(d) \times U(d)$ such that it leaves the fixed point space of $\Gamma$ invariant; similarly for the symmetry group $H \subset U(d)$ of the coupling term $H$. Then the orbit $G(x_a)$ is non-degenerate in the space of periodic paths fixed by $\Gamma$ when $a$ is a non-degenerate central configuration. The core of the proof (section 3 and 4) relies on several Lyapunov-Schmidt reductions in a neighbourhood of $G(x_a)$ such that one can solve the normal components to the orbit $G_2(a)$. In this manner, finding critical points of $A$ in a neighbourhood of $G(x_a)$ is equivalent to finding critical $H_{a_0}$-orbits of the regular functional $\Psi^\varepsilon_1 : G_2(a) \to \mathbb{R}$ defined on the compact manifold $G_2(a)$. The delicate part of the proof consists in finding uniform estimates in $\varepsilon$ because the functional $A$ explodes when $\varepsilon \to 0$. The main theorem is obtained by computing the $H_{a_0}$-equivariant Lyusternik-Schnirelmann category of the compact manifold $G_2(a)$, which gives a
lower bound for the number of $H_{0_{\epsilon}}$-orbits of critical points of $\Psi_{\epsilon}'$ along the lines of [13].

Besides our interest in gravitational potentials ($\alpha = 2$), we are interested in the case $\alpha = 1$ corresponding to solutions of steady near-parallel vortex filaments in fluids. The equations for $\alpha = 1$ govern the interaction of steady vortex filaments in fluids (Euler equation) [30], Bose-Einstein condensates (Gross-Pitaevskii equation) [20] and superconductors (Ginzburg-Landau equation) [9]. Specifically, the positions of the steady near-parallel vortex filaments are determined in space by

$$(q_j(s), s) \in C \times \mathbb{R} \simeq \mathbb{R}^3.$$ 

Therefore, the solutions that we construct correspond to $N = n + 1$ vortex filaments forming helices, where one of the vortices is replaced by a pair of vortices forming another helix (Figure 1).

The existence of braids has been investigated previously under the assumption that the force is strong (case $\alpha \geq 3$) in [17, 27] and references therein. In the case of strong forces, the Euler functional blows up at any orbit belonging to the boundary of a braid class because it contains collisions. This allows to prove the existence of minimisers for most braid classes by the direct method of calculus of variation for tied braids (which excludes the lack of coercitivity caused by the possibility that groups of bodies escape to infinity). Similar results hold for the existence of choreographic classes under the assumption of strong forces. In [25] and references therein the symmetry groups of choreographic classes have been classified. A short exposition of different methods used to prove the existence of choreographies can be found in [7] and references therein.

However, the relevant cases from the physical point of view are the $N$-body problem ($\alpha = 2$) and the $N$-vortex filament problem ($\alpha = 1$). The difficulty to obtain minimisers on braid classes is that the minimiser of the Euler functional may have collisions. In [12] a method was developed to obtain choreographies of the $N$-body problem ($\alpha = 2$) as minimisers. But finding braids of the $N$-body problem ($\alpha = 2$) as minimisers is a more difficult task. Furthermore, finding braids of the $N$-vortex filament problem ($\alpha = 1$) is more difficult than the $N$-body case ($\alpha = 2$). In this paper we propose a new method based on blow up methods (similar to [3, 4]) to approach these problems. The blow-up method described in this manuscript is part of a series of applications, namely (a) replacing one body in a central configuration by $k$ bodies, (b) replacing each body in a central configuration by $k_j$ bodies.

In section 2 we set the problem of finding solutions of the $N$-body problem arising as critical points of the Euler functional defined on a Sobolev space and we discuss the symmetries of the problem. In section 3 we perform a Lyapunov-Schmidt reduction to a finite dimensional problem by using a decomposition of paths in Fourier series. In section 4 we perform a second Lyapunov-Schmidt reduction to solve the normal components to the group orbit and we obtain a lower bound for the critical points by using Lyusternik-Schnirelmann methods. In section 5 we discuss the existence of braids (Theorem 5.1 and Corollary 5.2) by cabling central configurations. We also discuss the solutions in higher dimension (Theorem 5.3).

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2. Problem setting

Let $E$ be a real Euclidean space with inner product $\langle \cdot, \cdot \rangle$. Denote by $q := (q_0, q_1, \ldots, q_n) \in E^N$ a configuration of $N = n + 1$ bodies in $E$ with masses $m_0, \ldots, m_n > 0$. We work only with configurations whose center of mass is fixed at the origin, which amounts to say that the configuration space has been reduced by translations. Define the kinetic energy and the potential function

$$K = \frac{1}{2} \sum_{j=0}^{n} m_j \|q_j\|^2 \quad \text{and} \quad U = \sum_{0 \leq j < k \leq n} m_j m_k \phi_{\alpha}(\|q_j - q_k\|),$$
where \( \|\dot{q}_j\|^2 = (\dot{q}_j, \dot{q}_j) \) and \( \phi_\alpha \) is a function such that \( \phi_\alpha'(r) = -r^{-\alpha} \). The Newtonian potential corresponds to \( \phi_2(r) = 1/r \) and the vortex filament potential corresponds to \( \phi_1(r) = -\ln(r) \). The system of equations of motion of the \( N \)-body problem reads

\[
me \ddot{q}_\ell = \nabla_{q_\ell} U = -\sum_{k \neq \ell} m_k m_k \frac{q_\ell - q_k}{\|q_\ell - q_k\|^{\alpha + 1}}, \quad \ell = 0, \ldots, n.
\]

Let \( L = K + U \) be the Lagrangian of the system. The **Euler functional**

\[
\mathcal{A}(q) = \int_0^T L((q(t), \dot{q}(t))dt
\]

is taken over the Sobolev space \( H^1([0,T], E^N) \) of paths \( q : [0,T] \to E^N \) such that \( q \) and its first derivative \( \dot{q} \) are square integrable in the sense of distributions.

### 2.1 Jacobi-like coordinates

Define fictional mass parameters \( M_0 = m_0 m_1, M_1 = m_0 + m_1 \) and \( M_\ell = m_\ell \) otherwise. After a rescaling we suppose that \( M_1 = 1 \). Define new variables in the configuration space namely, \( Q_0 = q_1 - q_0 \), \( Q_1 = m_0 q_0 + m_1 q_1 \), and \( Q_\ell = q_\ell \) otherwise. Setting \( \mu_0 = m_1 \) and \( \mu_1 = -m_0 \) we can write \( q_j = Q_1 - \mu_j Q_0 \) for \( j = 0, 1 \). Observe that the center of mass of the configuration

\[
Q = (Q_1, \ldots, Q_n),
\]

with respect to the fictional masses \( M_1, \ldots, M_n \), remains at the origin.

**Proposition 2.1.** In the new coordinates \((Q_0, Q)\), the kinetic energy and the potential energy become

\[
K = \frac{1}{2} \sum_{j=0}^n M_j \|\dot{Q}_j\|^2 \quad \text{and} \quad U = M_0 \phi_\alpha(\|Q_0\|) + \sum_{1 \leq j < k \leq n} M_j M_k \phi_\alpha(\|Q_j - Q_k\|) + h(Q_0, Q)
\]

with

\[
h(Q_0, Q) = \sum_{k=2}^n \sum_{j=0,1} m_k m_j \left( \phi_\alpha(\|Q_1 - \mu_j Q_0 - Q_k\|) - \phi_\alpha(\|Q_1 - Q_k\|) \right).
\]

**Proof.** Using that \( m_0 + m_1 = 1 \), \( q_0 = Q_1 - m_1 Q_0 \) and \( q_1 = Q_1 + m_0 Q_0 \), we have

\[
\sum_{j=0,1} m_j \|\dot{q}_j\|^2 = \|\dot{Q}_1\|^2 + (m_0 n_1^2 + m_0^2 m_1) \|\dot{Q}_0\|^2 = M_1 \|\dot{Q}_1\|^2 + M_0 \|\dot{Q}_0\|^2.
\]

Then \( K = \frac{1}{2} \sum_{j=0}^n M_j \|\dot{Q}_j\|^2 \). For the potential energy we have

\[
U = \sum_{j < k} m_j m_k \phi_\alpha(\|q_j - q_k\|)
\]

\[
= m_0 m_1 \phi_\alpha(\|q_0 - q_1\|) + \sum_{k=2}^n \sum_{j=0,1} m_k m_j \phi_\alpha(\|q_j - q_k\|) + \sum_{2 \leq j < k \leq n} m_j m_k \phi_\alpha(\|q_j - q_k\|)
\]

\[
= M_0 \phi_\alpha(\|Q_0\|) + \sum_{1 \leq j < k \leq n} M_j M_k \phi_\alpha(\|Q_j - Q_k\|) + h(Q_0, Q),
\]

where

\[
h(Q_0, Q) = \sum_{k=2}^n \sum_{j=0,1} m_k m_j \phi_\alpha(\|q_j - Q_k\|) - \sum_{k=2}^n M_1 M_k \phi_\alpha(\|Q_1 - Q_k\|).
\]
Since $M_1 = m_0 + m_1 = 1$, and $q_k = Q_k$ and $m_k = M_k$ for $k \geq 2$, we obtain

$$h(Q_0, Q) = \sum_{k=2}^{n} \sum_{j=0,1} m_k m_j (\phi_\alpha(\|q_j - Q_k\|) - \phi_\alpha(\|Q_1 - Q_k\|)).$$

The result for $h$ follows from the fact that $q_j = Q_1 - \mu_j Q_0$ for $j = 0, 1$. □

The Euler functional splits into two terms

$$\mathcal{A}(Q_0, Q) = \mathcal{A}_0(Q_0, Q) + \mathcal{H}(Q_0, Q).$$

They are explicitly given by

$$\mathcal{A}_0(Q_0, Q) = \int_0^T \frac{1}{2} \sum_{j=0}^{n} M_j \|q_j(t)\|^2 + M_0 \phi_\alpha(\|Q_0(t)\|) + \sum_{1 \leq j < k \leq n} M_j M_k \phi_\alpha(\|q_j(t) - q_k(t)\|) \, dt$$

and $\mathcal{H}(Q_0, Q) = \int_0^T h(Q_0(t), Q(t)) \, dt$ with $h$ as in (3). Notice that $h(Q_0, Q)$ is an analytic function in a neighbourhood of $Q_0 = 0$ with $h(Q_0, Q) = O(\|Q_0\|)$. Furthermore $h$ is invariant under linear isometries

$$h(gQ_0, gQ) = h(Q_0, Q)$$

where $g \in O(E)$ and $gQ = (gQ_1, \ldots, gQ_n)$.

### 2.2 Rotating-like coordinates

Since we already reduced the space by translations, a relative equilibrium of the $n$-body problem is now a solution of the Newton’s equations which is an equilibrium after reducing the configuration space by the group of linear isometries $O(E)$ acting diagonally on $E^n$. That is, the motion is of the form $Q(t) = \exp(t\Lambda)a$ for a fixed configuration $a \in E^n$ and a skew-symmetric matrix $\Lambda$. Since $\Lambda$ is non-degenerate on the space of motion (see (1)), we may suppose from the beginning that $E$ is even dimensional and is endowed with a compatible almost complex structure. We set $\dim(E) = 2d$ and pick a basis such that the complex structure is block diagonal

$$\mathcal{J} := J \oplus \cdots \oplus J,$$

where $J$ is the standard symplectic matrix on $\mathbb{R}^2$. We define rotating-like coordinates

$$Q_j(t) = \exp(t\mathcal{J})v_j(t).$$

In the coordinates $v_j$, the two terms of the Euler functional (4) become

$$\mathcal{A}_0(v_0, v) = \int_0^T \frac{1}{2} \sum_{j=0}^{n} M_j \|\partial_t + \mathcal{J} v_j(t)\|^2 + M_0 \phi_\alpha(\|v_0(t)\|) + \sum_{1 \leq j < k \leq n} M_j M_k \phi_\alpha(\|v_j(t) - v_k(t)\|) \, dt$$

and $\mathcal{H}(v_0, v) = \int_0^T h(v_0(t), v(t)) \, dt$ remains unchanged because of its invariance under linear isometries (5). The Euler-Lagrange equations for $\mathcal{A}_0$ are

$$\frac{\delta \mathcal{A}_0}{\delta v_0} = -M_0 (\partial_t + \mathcal{J})^2 v_0 - M_0 \frac{v_0}{\|v_0\|^{\alpha+1}} = 0$$

and

$$\frac{\delta \mathcal{A}_0}{\delta v_0} = -M_\ell (\partial_t + \mathcal{J})^2 v_\ell - \sum_{k=1, k \neq \ell}^{n} M_\ell M_k \frac{v_\ell - v_k}{\|v_\ell - v_k\|^{\alpha+1}} = 0.$$

Equation (6) is the Kepler problem in rotating coordinates. Equations (7) are Newton’s equations for $n$ bodies with masses $M_1, \ldots, M_n$ in rotating coordinates. A central configuration $a = (a_1, \ldots, a_n) \in E^n$ satisfies the equations.
\[ a_\ell = \sum_{k \neq \ell} M_k \frac{a_\ell - a_k}{\|a_\ell - a_k\|^{\alpha+1}}. \]

Therefore, \( a \) is an equilibrium of equations [7], and the motion \( Q(t) = \exp(tJ)a \) is a relative equilibrium. Central configurations can also be defined as critical points of the \textit{amended potential} of the \( n \)-body problem

\[ V(v) = \frac{1}{2} \sum_{j=1}^{n} M_j \|v_j\|^2 + \sum_{1 \leq k < j \leq n} M_j M_k \phi_\alpha(\|v_j - v_k\|). \]  

Then \( a \in E^n \) is a central configuration if and only if \( \nabla V(a) = 0 \).

### 2.3 Time and space scaling

Equation (6) is the Kepler problem for homogeneous potentials in rotating coordinates. This equation has solutions corresponding to circular orbits. We consider a special type of circular orbits of the form

\[ v_0(t) = \varepsilon \exp((\omega - 1)Jt)a_0, \]

where

\[ a_0 \in E \quad \text{is of unit length and} \quad \omega = \pm \varepsilon^{-(\alpha+1)/2}. \]

The case \( \omega > 0 \) corresponds to a prograde rotation of the pair and \( \omega < 0 \) to a retrograde rotation.

We introduce a change of coordinates which is particularly useful to continue the circular solution of [6] and the equilibrium of [7]. This change of coordinates is defined by

\[ v_0(t) = \varepsilon \exp((\omega - 1)Jt)u_0(\nu t), \]
\[ v_\ell(t) = u_\ell(\nu t), \quad \ell = 1, \ldots, n \]

where \( \nu \in \mathbb{R} \) is a frequency. We shall now introduce a new time parameter \( s = \nu t \) and write

\[ x(s) = (u_0(s), u(s)). \]

In the proposition below, we express the Euler functional \( \mathcal{A}(v_0, v) \) in terms of the new coordinates \((u_0, u)\). This functional is referred to as a \textit{normalised} functional because it involves a scaling in time of the old functional by a factor \( \nu \). By making an abuse of notation it is still denoted \( \mathcal{A}(x) = \mathcal{A}_0(x) + \mathcal{H}(x) \).

Note that the old and the new functionals have the same critical points. Moreover, for any central configuration \( a \in E^n \) and any unit length vector \( a_0 \in E \), the constant path

\[ x_a(s) = (a_0, a) \]  

is a critical point of the unperturbed functional \( \mathcal{A}_0(x) \).

**Proposition 2.2.** Suppose \( \alpha \geq 1 \). In the coordinates \( x(s) = (u_0(s), u(s)) \), the normalised Euler functional \( \mathcal{A}(x) = \mathcal{A}_0(x) + \mathcal{H}(x) \) is given by the two terms

\[
\mathcal{A}_0(x) = \varepsilon^{1-\alpha} M_0 \int_0^{2\pi} \frac{1}{2} \left( \frac{\nu}{\omega} \partial_s + J \right) u_0(s)^2 + \phi_\alpha(\|u_0(s)\|) \, ds \\
+ \int_0^{2\pi} \frac{1}{2} \sum_{j=1}^{n} M_j \|\nu \partial_s + J u_j(s)\|^2 + \sum_{1 \leq j < k \leq n} M_j M_k \phi_\alpha(\|u_j(s) - u_k(s)\|) \, ds \\
\mathcal{H}(x) = \int_0^{2\pi} h \left( \varepsilon \exp \left( \frac{\omega - 1}{\nu} s J \right) u_0(s), u(s) \right) \, ds.
\]

Explicitly the integrand \( h \) is

\[
h(u_0, u) = \sum_{k=1}^{n} \sum_{j=0,1} M_k m_j \left( \phi_\alpha(\|u_1(s) - \mu_j \varepsilon \exp \left( \frac{\omega - 1}{\nu} s J \right) u_0(s) - u_k(s)\|) - \phi_\alpha(\|u_1(s) - u_k(s)\|) \right).
\]
\textbf{Proof.} When $\alpha > 1$ the potential $\phi_\alpha$ is homogeneous of degree $1 - \alpha$, then

$$\phi_\alpha(\|v_0(t)\|) = \varepsilon^{1-\alpha}\phi_\alpha(\|u_0(s)\|).$$

Moreover

$$\|(\partial_t + J) v_0(t)\|^2 = \|\varepsilon (\nu \partial_\alpha + \omega J) u_0(s)\|^2 = \varepsilon^{1-\alpha} \left\| \left( \frac{\nu}{\omega} \partial_\alpha + J \right) u_0(s) \right\|^2$$

and the result follows by rescaling $A$ by $\nu$.

The case $\alpha = 1$ is similar, but now $\phi_\alpha(\|v_0(t)\|) = \phi_\alpha(\|u_0(s)\|) - \ln(\varepsilon)$ and

$$\|(\partial_t + J) v_0(t)\|^2 = \left\| \left( \frac{\nu}{\omega} \partial_\alpha + J \right) u_0(s) \right\|^2.$$

The result for $\alpha = 1$ follows by rescaling $A$ by $\nu$ and adding the constant $-2\pi M_0 \ln(\varepsilon)$.

Finally the term $h$ is given by changing the coordinates in (3). Since the term is invariant by rotations we may replace each term $Q_k$ by $v_k$. In terms of the coordinates $(u_0, u)$ it becomes

$$h = \sum_{k=1}^n \sum_{\ell=0}^m M_k m_j (\phi_\alpha(\|u_1(\nu t) - \mu_j \varepsilon \exp ((\omega - 1) t J) u_0(\nu t) - u_k(\nu t))\| - \phi_\alpha(\|u_1(\nu t) - u_k(\nu t))\|).$$

By changing the time parameter as $s = \nu t$, we obtain the expression written in the statement. \ \Box

\subsection{2.4 Gradient formulation}

Let $S^1 = \mathbb{R}/2\pi \mathbb{Z}$ and consider the subset of $2\pi$-periodic paths,

$$X := H^1(S^1, E^N) \subset H^1([0, 2\pi], E^N).$$

The space $X$ is a real Hilbert space with inner product

$$(x_1, x_2)_X = (x_1, x_2)_{L^2} + (\hat{x}_1, \hat{x}_2)_{L^2} = \int_0^{2\pi} (x_1(s), x_2(s)) + (\dot{x}_1(s), \dot{x}_2(s)) ds.$$

The topological dual $X'$ is identified with the Sobolev space of distributions $X^{-1}$ defined by

$$X^* := H^*(S^1, E^N) = \left\{ (\hat{x}_t)_{t \in \mathbb{Z}} | \sum_{\ell \in \mathbb{Z}} (\ell^2 + 1)^s \|\hat{x}_\ell\|^2 < \infty \right\},$$

where $(\hat{x}_\ell)$ is the sequence of Fourier coefficients in $(E^N)^{\ast} = (E \oplus i E)^{\ast}$ of $x$ satisfying $\hat{x}_\ell = \overline{x}_{-\ell}$. In particular, an element $x \in X^*$ for $s \geq 0$ can be written as a Fourier series for the function $x(s) = \sum_{\ell \in \mathbb{Z}} \hat{x}_\ell e^{i\ell s}$. On the other hand, an element $y \in X^{-s}$ for $s > 0$ is a distribution that acts on a test function $x(s) = \sum_{\ell \in \mathbb{Z}} \hat{x}_\ell e^{i\ell s} \in X^s$ by the formula $(y, x) = \sum_{\ell \in \mathbb{Z}} \hat{y}_\ell \cdot \hat{x}_\ell$.

For a given open collision-less subset $\Omega \subset X$ we denote by $dA : \Omega \subset X \to X'$ the differential of $A$. Using the identification between the dual space $X'$ and $X^{-1}$ we define the operator of first variation $dA : \Omega \subset X \to X^{-1}$ satisfying $dA(x)(y) = (\delta A(x), y)$. On the other hand, by the Riesz representation theorem, the gradient operator $\nabla A : \Omega \subset X \to X$ is uniquely defined by $dA(x)(y) = (\nabla A(x), y)_X$. Using an integration by parts and the fact that the paths are periodic, we obtain that

$$(\delta A(x), y) = dA(x)(y) = (\nabla A(x), y)_X = ((-\partial^2_x + 1)\nabla A(x), y)$$

where $(-\partial^2_x + 1)^{-1} : X^{-1} \to X$ is the Riesz map. Thus we conclude that

$$\nabla A = (-\partial^2_x + 1)^{-1} \delta A : \Omega \subset X \to X.$$
For $x \in X$ the Euler-Lagrange equations of the unperturbed functional $A_0$ in gradient formulation are
\begin{align}
\nabla_{u_0} A_0(x) &= (-\partial_x^2 + 1)^{-1} \frac{1}{\varepsilon^{1-\alpha}} M_0 \left( -\left( \frac{\nu}{\omega} \partial_x + J \right)^2 u_0 - \frac{u_0}{\|u_0\|^{\alpha+1}} \right) = 0 \quad (11) \\
\nabla_{u_\ell} A_0(x) &= (-\partial_x^2 + 1)^{-1} M_\ell \left( -\left( \nu \partial_x + J \right)^2 u_\ell - \sum_{k \neq \ell} M_k \frac{u_\ell - u_k}{\|u_\ell - u_k\|^{\alpha+1}} \right) = 0. \quad (12)
\end{align}

Note that the operators $\left( \frac{\nu}{\omega} \partial_x + J \right)^2$ and $\left( \nu \partial_x + J \right)^2$ are defined from $X$ to $X^{-1}$ whereas $(-\partial_x^2 + 1)^{-1}$ is defined from $X^{-1}$ to $X$. Furthermore, it is possible to consider the composition of these operators as operators from $X$ to $X$ without passing by the dual space $X^{-1}$. For instance, given $x \in X$,
\begin{align}
(-\partial_x^2 + 1)^{-1} (\nu \partial_x + J)^2 x &= \sum_{\ell \in \mathbb{Z}} \frac{1}{\ell^2 + 1} (i\ell \nu \mathcal{I} + J)^2 \hat{x}_\ell e^{i\ell s}. \quad (13)
\end{align}

The above equations admit the solution path $x_\alpha \in X$ given by
\begin{align}
x_\alpha(s) = (u_0, a), \quad \forall s \in S^1. \quad (14)
\end{align}

We want to prove that there are critical solutions $x(s) = (u_0(s), u(s))$ close to $x_\alpha$ that persist as critical solutions for the perturbed functional $A(x) = A_0(x) + \mathcal{H}(x)$.

To ensure that $\exp\left( \frac{\omega}{\nu} J s \right)$ is $2\pi$-periodic, and $\mathcal{H}$ is well defined in $X$, we need to impose the condition $\omega = 1 + m\nu$ for some $m \in \mathbb{Z}$. In particular, we imposed the following conditions on the set of parameters:

(A) $\omega = \pm \varepsilon^{-(\alpha+1)/2}$.

(B) $\omega = 1 + \nu$.

We will prove that $\nabla \mathcal{H}(x) = \mathcal{O}_X(\varepsilon)$ in the space $X$. Condition (A) implies that $x_\alpha$ is a critical point of $A_0(x)$ and condition (B) that $\mathcal{H}(x)$ is well defined in the space of $2\pi$-periodic paths $X$. The critical solutions of $A(x)$ provide solutions of the $N$-body problem. We prove the existence of a continuum of solutions when $\varepsilon \to 0$. Conditions (A)-(B) determine $\omega$ and $\nu$ as functions of $\varepsilon$ such that $\omega, \nu \to +\infty$ when $\varepsilon \to 0$ for the prograde rotation, and $\omega, \nu \to -\infty$ for the retrograde rotation. In principle, we do not need to assume that the parameter $\omega$ is rational. Braids are particular solutions such that $d = 1$ and $\omega \in \mathbb{Q}$.

### 2.5 Discrete and continuous symmetries

Since $U(d)$ is the centraliser of $J$ in $O(E)$, the unperturbed functional $A_0$ is invariant with respect to the product group $U(d) \times U(d)$. The first factor acting on the component $u_0$, and the second factor acting diagonally on the $n$ last components $u \in E^n$. The action of this group extends on $X$ by rotating non simultaneously the Kepler orbit and the central configuration; that is,
\begin{align}
(g_1, g_2)(u_0, u) = (g_1 u_0, g_2 u), \quad (g_1, g_2) \in U(d) \times U(d)
\end{align}

where $g_2 u = (g_2 u_1, \ldots, g_2 u_n)$. Observe that the coupling term $\mathcal{H}$ in the functional breaks the symmetry of $A_0$ and the perturbed functional $A = A_0 + \mathcal{H}$ is only invariant with respect to the diagonal subgroup
\begin{align}
\widehat{U(d)} = \{(g_1, g_1) \in U(d) \times U(d) \mid g_1 \in U(d)\},
\end{align}

acting by rotating the $N = n + 1$ bodies with respect to the origin. We now distinguish the three following cases:
(C1) \( E \) is the plane \( (d = 1) \) and \( \alpha \neq 2 \).

(C2) \( E \) is the plane \( (d = 1) \) and \( \alpha = 2 \) (Newtonian case).

(C3) \( E \) is of higher dimension \( (d \geq 2) \) and \( \alpha \geq 1 \).

Those cases need to be treated separately in Lemma 3.1 in order to perform a reduction of dimension. Indeed, the reduction relies on the invertibility of a regularised hessian operator at the critical point on some slice in \( X \). The invertibility fails in cases (C2) and (C3). In case (C2) this is due to the appearance of resonances given by elliptic orbits, and in case (C3) this is due to the presence of resonances in higher dimension. To deal with this issue, we make use of an extra discrete symmetry subgroup \( \Gamma \) of the perturbed functional \( \mathcal{A} \). The problem of resonances can be avoided when working on the fixed point space \( X^\Gamma \) instead of \( X \). This is allowed by the principle of symmetric criticality of Palais \[33\]. In this case \( x_a \) needs to be chosen such that \( x_a \in X^\Gamma \) and, similarly, the symmetry group \( G \) of \( \mathcal{A}_0 \) must be chosen so that it leaves \( X^\Gamma \) invariant. Note that there may be other solutions outside of this fixed point space. We discuss below which discrete symmetry is relevant for each case and which symmetry group \( G \) must be taken. The discrete symmetry also restricts the type of central configurations we can braid, at least in the case (C2) and (C3).

(C1) No restriction is needed in this case, there are no resonances. We may take \( \Gamma \) to be the trivial group, \( G = U(1) \times U(1) \) and \( H = \widehat{U}(1) \). We then study the critical points of \( \mathcal{A} \) in \( X^\Gamma = X \).

(C2) The bodies are now moving in the plane under the influence of the Newtonian gravitational force. We can braid the central body of symmetric central configurations which include symmetric configurations we can braid, at least in the case (C2) and (C3).

Let \( S_n \) be the permutation group of \( n \) letters and consider the discrete subgroup \( \Gamma < \mathbb{Z}_m \times S_n \) generated by a non-trivial element \((\theta, \sigma)\) such that

\[
\theta = 2\pi/m \in \mathbb{Z}_m, \quad \sigma^m = (1) \in S_n, \quad \sigma(1) = 1.
\]

This group acts on \( X \) as follows: for \( x \in X \) we have

\[
(\theta, \sigma)x(s) = (u_0(s + \theta), \exp(-\theta J)u_{\sigma(1)}(s + \theta), \ldots, \exp(-\theta J)u_{\sigma(n)}(s + \theta)).
\]

(C2a) The first assumption on the central configuration is that the masses satisfy

\[
M_\ell = M_{\sigma(\ell)}.
\] (15)

The functional \( \mathcal{A}_0 \) is \( \Gamma \)-invariant because, in its expression, the variables \( u_0(s) \) and \( u_\ell(s) \) are uncoupled. Furthermore, in the next proposition we show that the coupling term \( \mathcal{H} \) is \( \Gamma \)-invariant. Thus the functional \( \mathcal{A} \) is \( \Gamma \)-invariant and we can restrict the study of its critical points to the fixed point set \( X^\Gamma \).

**Proposition 2.3.** Under condition (C2a) the action functional \( \mathcal{H}(x) \) is \( \Gamma \)-invariant

**Proof.** Consider the term \( h \) given in (10) with \((\omega - 1)/\nu = 1\). Given \((\theta, \sigma) \in \Gamma\), we first write explicitly \( h((\theta, \sigma)(\varepsilon \exp(sJ)u_0(s), u(s))) \). Using (10) this gives

\[
\sum_{k=2}^{n} \sum_{j=0,1} M_k m_j \phi_\alpha(\|\exp(-\theta J)u_{\sigma(1)}(s + \theta) - \mu_j \varepsilon \exp(sJ)u_0(s + \theta) - \exp(-\theta J)u_{\sigma(k)}(s + \theta)\|)
\]

\[
- \sum_{k=2}^{n} \sum_{j=0,1} M_k m_j \phi_\alpha(\|\exp(-\theta J)u_{\sigma(1)}(s + \theta) - \exp(-\theta J)u_{\sigma(k)}(s + \theta)\|).
\]
We now consider the higher dimensional case; that is when the space of motion and symmetric by 2 \( \sigma \) satisfies the property group action of \( U \).

One can now use the \( SO(2) \)-invariance of the norms and \( \sigma(1) = 1 \) to rewrite this as

\[
\sum_{k=2}^{n} \sum_{j=0,1} M_k m_j \phi_\alpha(\|u_1(s+\theta) - \mu_j \varepsilon \exp((s+\theta)J)u_0(s+\theta) - u_{\sigma(k)}(s+\theta)\|)
\]

\[
- \sum_{k=2}^{n} \sum_{j=0,1} M_k m_j \phi_\alpha(\|u_1(s+\theta) - u_{\sigma(k)}(s+\theta)\|).
\]

In particular, changing the indices in the summation with respect to \( k \) implies that

\[
h((\theta, \sigma)(\varepsilon \exp(sJ)u_0(s), u(s))) = h(\varepsilon \exp((s+\theta)J)u_0(s+\theta), u(s+\theta)).
\]

Since \( \mathcal{H} \) is defined in the space of 2\( \pi \)-periodic functions, we get

\[
\mathcal{H}((\theta, \sigma)x) = \int_{0}^{2\pi} h(\varepsilon \exp((s+\theta)J)u_0(s+\theta), u(s+\theta)) \, ds
\]

\[
= \int_{0}^{2\pi+\theta} h(\varepsilon \exp(s'J)u_0(s'), u(s')) \, ds' = \mathcal{H}(x).
\]

\((C2b)\) The second assumption (to ensure that \( x_a \in X^\Gamma \)) is that the central configuration \( a \in E^n \) satisfies the property

\[
a_x = \exp(-\theta J)a_{\sigma(1)}.
\]

Since \( \sigma^m = 1 \) and \( \theta = 2\pi/m \), conditions \((C2a)-(C2b)\) imply that the central configuration \( a \) is symmetric by \( 2\pi/m \)-rotations in the plane and that \( a_1 = 0 \).

Symmetric configurations that satisfy this condition are discussed in section \ref{uniqueness}. In this case the group action of \( U(1) \times U(1) \) on \( X \) commutes with the action of \( \Gamma \), then we can take \( G = U(1) \times U(1) \) and \( H = \tilde{U}(1) \).

\((C3)\) We now consider the higher dimensional case; that is when the space of motion \( E \) is at least four dimensional. Let \( \Gamma \) be the finite subgroup isomorphic to \( \mathbb{Z}_2 \) whose generator \( \zeta \) acts on \( x \) as follows:

\[
\zeta x(s) = (-R u_0(s + \pi), R u(s + \pi)),
\]

where

\[
R = -I_2 \oplus I_2 \oplus \ldots \oplus I_2 \in \text{End}(E).
\]

The functional \( A_0 \) is \( \Gamma \)-invariant because \( R \) commutes with \( J \). Similarly, the functional \( H \) is invariant because

\[
\mathcal{H}(\zeta x) = \int_{0}^{2\pi} h(-\varepsilon \exp((s-\pi)J)R u_0(s), R u_1(s), \ldots, R u_n(s)) \, ds
\]

\[
= \int_{0}^{2\pi} h(R \varepsilon \exp(sJ)u_0(s), R u_1(s), \ldots, R u_n(s)) \, ds = \mathcal{H}(x).
\]

Therefore, the functional \( A \) is \( \Gamma \)-invariant and we can restrict the study of critical points to the fixed point space \( X^\Gamma \). In this case we choose the symmetry group \( G \) to be the maximal subgroup of \( U(d) \times U(d) \) acting on \( X^\Gamma \). The groups are thus of the form \( G = G_1 \times G_2 \) and \( H = G_1 \), where each \( G_i \) is the centraliser of \( R \) in \( U(d) \); that is

\[
G_i = U(1) \times U(d-1).
\]

Note that \( x_a = (a_0, a) \in X^\Gamma \) if and only if \( -Ra_0 = a_0 \) and \( Ra_j = a_j \) for \( j = 1, \ldots, n \). Therefore \( x_a \) must be taken such that \( a_0 \) lies in the plane

\[
\Pi = \{(x, y, 0, \ldots, 0)\} \subset E,
\]
and the central configuration \( a \) consists of points lying in the orthogonal complement \( \Pi^\perp \subset E \). The choice of symmetry group \( G \) ensures that \( G(x_a) \subset X^\Gamma \). In dimension four \((d = 2)\) the Kepler orbit is located in a plane and the central configuration lies in an orthogonal plane.

Choosing the symmetry group \( G \) and the path \( x_a \) accordingly to one of the assumptions \((C1)\) or \((C2),(C3)\), the equations \((\text{II})\) vanish along the orbit \( G(x_a) \) and the real question to answer is whether some orbits of solutions along the orbit persist in the space \( X^\Gamma \) when considering the perturbation term \( \mathcal{H} = O(\varepsilon) \) for small \( \varepsilon \). For this purpose, we suppose that the collision-less neighbourhood \( \Omega \) is of the form

\[
\Omega = \{ x \in X \mid \exists g \in U(d) \times U(d), \quad \| x - g x_a \|_X < \rho \}
\]

for some \( \rho > 0 \). Then in further applications, one shall take an open subset \( \Omega^\Gamma \subset \Omega \cap X^\Gamma \) which is a \( \rho \)-neighbourhood of radius \( \rho \) around the group orbit \( G(x_a) \).

**Proposition 2.4.** The functional \( \mathcal{A} = \mathcal{A}_0 + \mathcal{H} \) is well defined in \( \Omega \subset X \).

**Proof.** Since \( \| x \|_{C^\alpha} \leq \gamma \| x \|_X \) by Sobolev embedding, the paths \( x \in \Omega \) do not leave the pointwise neighbourhood of the orbit

\[
\bar{\Omega} = \{ y \in E^N \mid \exists g \in U(d) \times U(d), \quad \| y - g x_a \|_{E^N} < \gamma \rho \}.
\]

The potential energy \( U \) and the nonlinear term \( h \) are pointwise analytic functions defined in \( \bar{\Omega} \) if \( \rho \) is small enough. Since paths in \( \Omega \) do not leave \( \bar{\Omega} \), i.e. \( x \in \Omega \) implies \( x(s) \in \bar{\Omega} \) for all \( s \in \mathbb{S}^1 \), the Euler functional \( \mathcal{A} \) is well defined in the region \( \Omega \subset X \) if \( \rho \) is small enough. \( \blacksquare \)

Hereafter, we use the Banach algebra property of \( X \) and the analyticity of \( \mathcal{A} \) to obtain functional estimates of its derivatives. In particular, we have the following estimate:

**Lemma 2.5.** There is a constant \( N_2 > 0 \) such that the compact operator \( \nabla \mathcal{H} : X \to X \) satisfies

\[
\| \nabla \mathcal{H}(x) \|_X \leq N_2 \varepsilon.
\]

uniformly for \( x \in \bar{\Omega} \) and small \( \varepsilon \).

**Proof.** After setting \( \varepsilon^{-1} = 1 \), the integrand term in \( \mathcal{H}(x) = \int_0^{2 \pi} h(\varepsilon \exp(s \mathcal{J}) u_0(s), u(s)) ds \) is

\[
h(\varepsilon \exp(s \mathcal{J}) u_0(s), u(s)) = \sum_{k=2}^n \sum_{j=0,1} M_k m_j (\phi_\alpha(\| u_1(s) - \mu_j \varepsilon \exp(\mathcal{J} s) u_0(s) - u_k(s) \|) - \phi_\alpha(\| u_1(s) - u_k(s) \|))
\]

by using \((10)\). A straightforward calculation yields

\[
\frac{\delta \mathcal{H}}{\delta u_0} = \varepsilon \sum_{k=2}^n \sum_{j=0,1} M_k m_j \mu_j \exp(\mathcal{J} s) \left( \frac{u_1(s) - u_k(s) - \mu_j \varepsilon \exp(\mathcal{J} s) u_0(s)}{\| u_1(s) - u_k(s) - \mu_j \varepsilon \exp(\mathcal{J} s) u_0(s) \|^{\alpha+1}} \right).
\]

Notice that the term

\[
\frac{u_1(s) - u_k(s) - \mu_j \varepsilon u_0}{\| u_1(s) - u_k(s) - \mu_j \varepsilon u_0 \|^{\alpha+1}}
\]

is real analytic for \( x = (u_0, \ldots, u_n) \in \bar{\Omega} \) and small \( \varepsilon \), i.e. it satisfies

\[
\left\| \frac{u_1(s) - u_k(s) - \mu_j \varepsilon u_0}{\| u_1(s) - u_k(s) - \mu_j \varepsilon u_0 \|^{\alpha+1}} \right\| \leq C_E, \quad x \in \bar{\Omega}.
\]

Notice that \( \exp(\mathcal{J} s) u_0(s), u_1(s), \ldots, u_n(s) \) is \( \bar{\Omega} \) pointwise for any \( x \in \Omega \) by the embedding \( X \subset C^0 \).

By the Banach algebra property of \( X \), we conclude that

\[
\left\| \frac{u_1(s) - u_k(s) - \mu_j \varepsilon \exp(\mathcal{J} s) u_0(s)}{\| u_1(s) - u_k(s) - \mu_j \varepsilon \exp(\mathcal{J} s) u_0(s) \|^{\alpha+1}} \right\| \leq C_X, \quad x \in \Omega.
\]
Therefore, we have that $\nabla u_0 \mathcal{H}(x) = (-\partial_x^2 + 1)^{-1} \delta \mathcal{H} / \delta u_0$ is a compact operator of order $\varepsilon$. That is, $\|\nabla u_0 \mathcal{H}(x)\|_X \leq N_2 \varepsilon$ with the constant $N_2$ independent of $x \in \Omega$.

Similarly, one obtains

$$\frac{\delta \mathcal{H}}{\delta u_1} = - \sum_{k=2}^n \sum_{j=1}^m M_k m_j \left( \frac{u_1(s) - \mu_j \varepsilon \exp(\mathcal{J} s) u_0(s) - u_k(s)}{\|u_1(s) - \mu_j \varepsilon \exp(\mathcal{J} s) u_0(s) - u_k(s)\|^{\alpha+1}} - \frac{u_1(s) - u_k(s)}{\|u_1(s) - u_k(s)\|^{\alpha+1}} \right).$$

Notice that the function

$$\frac{u_1 - \mu_j \varepsilon u_0 - u_k}{\|u_1 - \mu_j \varepsilon u_0 - u_k\|^{\alpha+1}} - \frac{u_1 - u_k}{\|u_1 - u_k\|^{\alpha+1}} = \mathcal{O}(\varepsilon)$$

is real analytic for $x = (u_0, ..., u_n) \in \overline{\Omega}$ and its Taylor expansion with respect to $\varepsilon$ has vanishing constant term. We conclude by a similar argument that $\|\nabla u_1 \mathcal{H}(x)\|_X \leq N_2 \varepsilon$ with the constant $N_2$ independent of $x \in \Omega$. The result follows by noticing that $\frac{\delta \mathcal{H}}{\delta u_k} = - \frac{\delta \mathcal{H}}{\delta u_1}$ for $k = 2, ..., n$. 

**Remark 2.1.** The functions (17) and (18) are real analytic for $x \in \overline{\Omega}$ and small $\varepsilon$. By the Banach algebra property of $X$, all the successive derivatives of $\nabla \mathcal{H}(x) = (-\partial_x^2 + 1)^{-1} \delta \mathcal{H}(x)$ are bounded operators with operator norms of order $\varepsilon$ for all $x \in \Omega$. In particular, the operator norm of $\nabla^2 \mathcal{H}(x) : X \to X$ is of order $\varepsilon$, $\|\nabla^2 \mathcal{H}(x)\| \leq C \varepsilon$ for $x \in \Omega$. Actually, the operator $\mathcal{A}$ and its components $\mathcal{A}_0$ and $\mathcal{H}$ are analytic functionals in the domain $\Omega \subset X$ in the sense of definition 2.3.1 in [3].

### 3. Lyapunov-Schmidt reduction

As before, we take the standard parametrisation $S^1 = \mathbb{R}/2\pi \mathbb{Z}$ and we identify

$$X = H^1(S^1, E^N) = \left\{ x \in L^2(S^1, E^N) \mid \sum_{\ell \in \mathbb{Z}} (\ell^2 + 1) \|\hat{x}_\ell\|^2 < \infty \right\},$$

where $(\hat{x}_\ell)$ is the sequence of Fourier coefficients in $(E^C)^N = (E \oplus iE)^N$ satisfying $\hat{x}_\ell = \overline{\hat{x}}_{-\ell}$. Write an element $x \in X$ as a Fourier series $x = \sum_{\ell \in \mathbb{Z}} \hat{x}_\ell e_\ell$ where $e_\ell : S^1 \to \mathbb{C}$ is given by $e_\ell(s) = e^{i\ell s}$. Then we can write $X = X_0 \oplus W$, where $X_0$ is the subspace of loops in $X$ and $W$ is the subspace of loops in $X$ having zero mean. Thus any element $x \in X$ decomposes uniquely as $x = \xi + \eta$, where

$$\xi = \hat{x}_0, \quad \eta = \sum_{\ell \neq 0} \hat{x}_\ell e_\ell.$$

Denote by $P : X \to X_0$ the canonical projection onto $X_0$, then $Px = \xi$ and $(I - P)x = \eta$, where $I$ denotes the identity on $X$. The system of equations $\nabla \mathcal{A}(\xi + \eta) = 0$ splits into

$$\nabla_\xi \mathcal{A}(\xi + \eta) = P \nabla \mathcal{A}(\xi + \eta) = 0 \in X_0,$$

$$\nabla_\eta \mathcal{A}(\xi + \eta) = (I - P) \nabla \mathcal{A}(\xi + \eta) = 0 \in W.$$

Reducing the system to finite dimension by mean of the Lyapunov-Schmidt reduction requires to solve the equation $\nabla_\eta \mathcal{A}(\xi + \eta) = 0$. For this purpose, we define $F_\varepsilon : \Omega \subset X \to W$ as the operator

$$F_\varepsilon(\xi, \eta) := D_\varepsilon \nabla_\eta \mathcal{A}(\xi + \eta),$$

where $D_\varepsilon \in \text{End}(E^N)$ is the block diagonal matrix

$$D_\varepsilon = \varepsilon^{\alpha-1} I \oplus \varepsilon^{\alpha+1} I \oplus \cdots \oplus \varepsilon^{\alpha+1} I,$$

(19)
where \( \mathcal{I} \) denotes the identity on \( E \). Solving the second equation is equivalent to solving \( F_\varepsilon(\xi, \eta) = 0 \) for \( \varepsilon \neq 0 \) because \( \mathcal{D}_\varepsilon \) is an isomorphism. While \( \nabla_\eta A(\xi + \eta) \) explodes as \( \varepsilon \to 0 \), the function \( F_\varepsilon(\xi, \eta) \) is continuous at \( \varepsilon = 0 \) because \( \lim_{\varepsilon \to 0} (\nu/\omega)^2 = 1 \). Therefore,

\[
F_0(\xi, \eta) = \lim_{\varepsilon \to 0} (\mathcal{D}_\varepsilon \nabla_\eta A_0(\xi + \eta))
\]

is well defined. Furthermore, \( F_0(g(x), 0) = 0 \) for all \( g \in G \). Solving \( F_\varepsilon(\xi, \eta) = 0 \) requires the functional derivative \( \partial_\eta F_0[(g(x), 0)] \) to be invertible on \( W \). Although this is true when working under condition (C1), the operator is not invertible on the whole space \( W \) under condition (C2) – (C3) (see the lemma below). However, in those bad cases, the operator is invertible on \( W^\Gamma \).

We use the notations \( X_0^\Gamma \) and \( W^\Gamma \) to denote the projections of \( X^\Gamma = (X_0 \oplus W)^\Gamma \) on the first and second factor, respectively.

**Lemma 3.1.** Assume conditions (A) – (B). Under assumption (C1), the operator \( \partial_\eta F_0[(g(x), 0)] \) is invertible on \( W \) for all \( g \in G \), i.e. there is a constant \( c > 0 \) such that

\[
\|\partial_\eta F_0[(g(x), 0)]^{-1}\eta\| \leq c\|\eta\| \quad \text{for every} \quad \eta \in W, \quad g \in G.
\]

Under assumptions (C2) or (C3), the same result holds when the operator \( \partial_\eta F_0[(g(x), 0)] \) is restricted to the fixed point space \( W^\Gamma \), with \( \Gamma \) and \( G \) chosen accordingly to those assumptions.

**Proof.** We first write the Hessian of \( A_0 \) at \( x_0 \) as the block diagonal matrix

\[
\nabla^2 A_0[x_0] = \nabla^2_{\omega_0} A_0[x_0] \oplus \nabla^2_{\omega_0} A_0[x_0].
\]

A straightforward calculation yields

\[
\nabla^2_{\omega_0} A_0[x_0] = (-\partial^2_{s} + 1)^{-1} M_0 e^{1-\alpha} (-\nu^2/\omega)^2 \mathcal{I} \partial^2_x - 2(\nu/\omega) \mathcal{J} \partial_x + (\alpha + 1) a_0 a_0^T,
\]

where \( a_0^T \) denotes the transpose of \( a_0 \). Similarly,

\[
\nabla^2_{\omega_0} A_0[x_0] = (-\partial^2_{s} + 1)^{-1} (-\nu^2 M \partial^2_x - 2\nu M J_0 \partial_x + \nabla^2 V[a]),
\]

where \( \mathcal{M} = M_1 \mathcal{I} \oplus \cdots \oplus M_n \mathcal{I} \) and \( J_0 = J \oplus \cdots \oplus J \) are block diagonal matrices, both with \( n \) blocks of size \( 2d \).

Let \( \eta = \sum_{s \neq 0} x_s e_s \in W \) and write

\[
\partial_\eta F_0[(x, 0)] \eta = \sum_{s \neq 0} \hat{T}_s x_s e_s
\]

where the matrix \( \hat{T}_s \) is block diagonal of the form

\[
\hat{T}_s = \hat{T}_{s,u_0} \oplus \hat{T}_{s,u}.
\]

Since the coefficients \( x_s \) do not depend on \( s \) we get

\[
\partial_x \eta = \sum_{s \neq 0} i s x_s e_s \quad \text{and} \quad \partial^2_{s} \eta = \sum_{s \neq 0} \partial^2_{s} x_s e_s.
\]

Since \( \lim_{\varepsilon \to 0} (\nu/\omega) = 1 \), the first block in (20) is given by

\[
\hat{T}_{s,u_0} = \frac{M_0}{\ell^2 + 1} (\ell^2 \mathcal{I} - 2i\ell \mathcal{J} + (\alpha + 1) a_0 a_0^T).
\]

Without loss of generality, suppose \( a_0 = (1, 0, \ldots, 0) \in E \). Hence the block \( \hat{T}_{s,u_0} \) is diagonal of the form

\[
\hat{T}_{s,u_0} = \frac{M_0}{\ell^2 + 1} \left( \begin{array}{cc}
\ell^2 + (\alpha + 1) & -2i\ell \\
2i\ell & \ell^2
\end{array} \right) \bigoplus_{d-1} \left( \begin{array}{cc}
\ell^2 & -2i\ell \\
2i\ell & \ell^2
\end{array} \right)
\]
The matrix $\hat{T}_{\ell,u_0}$ has eigenvalues
\[
\lambda^\pm_{1,\ell} = \frac{M_0}{\ell^2 + 1} \left( \ell^2 + \alpha + 1 \pm \frac{1}{2} \sqrt{16\ell^2 + (\alpha + 1)^2} \right),
\]
which appear with multiplicity one, and
\[
\lambda^\pm_{2,\ell} = \frac{M_0}{\ell^2 + 1} \ell (\ell \pm 2)
\]
which appear with multiplicity $d - 1$. We now study the invertibility for each assumption (C1), (C2) and (C3).

(C1) Since we are working on the plane, the matrix $\hat{T}_{\ell,u_0}$ has only the two eigenvalues $(22)$. Since $\ell \neq 0$ and $\alpha \neq 2$, these eigenvalues never vanish. This proves invertibility.

(C2) In Fourier components, $x$ is fixed by $\Gamma < \mathbb{Z}_m \times S_n$ if and only if $x(s) = (\theta, \sigma)x(s)$. This enforces $u_0$ to be $2\pi/m$-periodic. Therefore, the Fourier expansion of $u_0$ is fixed by $\Gamma$ only if
\[
u_{0,\ell} = 0 \text{ for } \ell \neq 0, \pm m, \pm 2m, ...
\]
Since the eigenvalues of the matrix $\hat{T}_{\ell,u_0}$ are not singular for $\alpha = 2$ as long as $\ell \neq \pm 1$, then the operator $\partial_\eta F_0(x,0)$ restricted to $W^\Gamma$ is invertible.

(C3) In Fourier components, $x$ is fixed by $\Gamma$ if an only if
\[
\sum_{\ell \in \mathbb{Z}} (u_{0,\ell}, u_\ell)e^{i\ell s} = x(s) = \zeta x(s) = \sum_{\ell \in \mathbb{Z}} (-\mathcal{R} u_{0,\ell}, \mathcal{R} u_\ell)e^{i(\ell s + \pi \ell)}.
\]
Set
\[
u_{0,\ell} = u_{\ell,\ell}^1 \oplus u_{\ell,\ell}^2.
\]
This implies that $\hat{x}_\ell = (u_{0,\ell}, u_\ell) \in E^N$ is fixed by $\Gamma$ only if
\[
u_{0,\ell}^1 = 0 \text{ for } \ell \neq 0, \pm 2, \pm 4, ...
\]
\[
u_{0,\ell}^2 = 0 \text{ for } \ell \neq \pm 1, \pm 3, \pm 5, ...
\]
Since the eigenvalues of the matrix $\hat{T}_{\ell,u_0}$ for the component $u_{0,\ell}^1$ are non-zero as long as $\ell \neq \pm 1$ (in the case $\alpha = 2$) and for the component $u_{0,\ell}^2$ if $\ell \neq \pm 2$, then the operator $\partial_\eta F_0(x,0)$ restricted to $W^\Gamma$ is invertible.

The limits of the eigenvalues of $\hat{T}_{\ell,u_0}$ tends to $M_0$ when $\ell \to \infty$. Since $\lim_{\varepsilon \to 0}(\varepsilon^{\alpha+1} \nu^2) = 1$, the second block in (22) is
\[
\hat{T}_{\ell,u} = \frac{\ell^2}{\ell^2 + 1} M.
\]
Therefore, there is a constant $c > 0$ (depending only on the masses) such that any eigenvalue $\lambda$ of $\hat{T}_{\ell}$ satisfies $|\lambda| \geq c^{-1}$. We conclude that the matrix $\hat{T}_{\ell}$ in (20) is invertible and we write
\[
\partial_\eta F_0((x,0))^{-1} \eta = \sum_{\ell \neq 0} \hat{T}_{\ell}^{-1} \hat{x}_\ell \epsilon \ell, \quad \eta \in W^\Gamma.
\]
It follows that
\[
\|\partial_\eta F_0((x,0))^{-1} \eta\| \leq c\|\eta\|.
\]
Note that the Hessian $\nabla^2 A_0|gx_\alpha|$ is conjugated to $\nabla^2 A_0|g x_\alpha|$ because $\nabla A_0$ is $G$-equivariant. Hence $\partial_\eta F_0((gx_\alpha,0))$ and $\partial_\eta F_0((x,0))$ are conjugated. Therefore, the estimate for $\partial_\eta F_0((g x_\alpha,0))$ holds independently of $g$ because the group $G$ acts by isometries. ■
Remark 3.1. In the plane \((d = 1)\) and for the Newton gravitational force \((\alpha = 2)\), the operator \(\partial_\eta F_0[(gx_{x_a},0)]\) is not invertible because \(\sqrt{3-\alpha} = 1\) and \(\lambda_{1,1} = 0\), which is a consequence of the fact that circular orbits of the Kepler problem with gravitational potential are never isolated due to the existence of elliptic orbits. In the case of more dimensions \((d > 1)\), the operator \(\partial_\eta F_0[(gx_{x_a},0)]\) is never invertible in \(W\) due to resonances of the circular orbit of the generalized Kepler problem with its rotations in more dimensions. In both cases, the operators are invertible only when we restrict the operator to \(W^T\).

Theorem 3.2 (Lyapunov-Schmidt reduction). Assume conditions (A) – (B). Under one of the assumptions \((C1) – (C3)\), there is \(\varepsilon_0 > 0\) such that, for every \(\varepsilon \in (0,\varepsilon_0)\), there is an open neighbourhood \(\mathcal{V} \subset X^T_0\) of the orbit \(G(x_a)\) and a smooth \(H\)-equivariant mapping \(\varphi_\varepsilon : \mathcal{V} \to W^T\) such that solving \(\nabla \mathcal{A}(\xi + \eta) = 0\) for \(\xi \in \mathcal{V}\) is equivalent to solve the finite dimensional system of equations \(\nabla \Psi_\varepsilon(\xi) = 0\) for \(\xi \in \mathcal{V}\), where

\[
\Psi_\varepsilon(\xi) = \mathcal{A}(\xi + \varphi_\varepsilon(\xi))
\]

is the reduced functional. The fact that, for \(\varepsilon \in (0,\varepsilon_0)\), the operator \(F_\varepsilon(\xi,\eta)\) is analytic implies that the implicit function \(\varphi_\varepsilon(\xi)\) is also analytic.

Proof. Lemma 3.1 ensures that, for every \(g \in G\), the operator \(\partial_\eta F_0[(gx_{x_a},0)]\) restricted to \(W^T\) has bounded inverse. The implicit function theorem assures the existence of open neighbourhoods \(\mathcal{I}^g \subset \mathbb{R}\) of \(0\) and \(\mathcal{V}^g \subset X^T_0\) of \(gx_{x_a}\) such that, for every \(\varepsilon \in \mathcal{I}^g\), there is a unique smooth mapping \(\varphi^g_\varepsilon : \mathcal{V}^g \to W^T\) such that the solutions of

\[
F_\varepsilon(\xi,\varphi^g_\varepsilon(\xi)) = 0, \quad \xi \in \mathcal{V}^g
\]

lie on \(\varphi^g_\varepsilon(\xi)\). Since this argument is valid for every \(g \in G\), we can repeat this procedure until we obtain a cover of the orbit \(G(x_a)\) by open sets \(\mathcal{V}^g \subset X^T_0\) from which we can extract a finite cover \(\{\mathcal{V}^g\}_{i = 1}^n\), by compactness of the group orbit. We define open sets \(\mathcal{V} = \bigcup_{i=1}^n \mathcal{V}^g\) and \(\mathcal{I} = \bigcap_{i=1}^n \mathcal{I}^g\). We take \(\varepsilon_0\) small enough such that \((0,\varepsilon_0) \subset \mathcal{I}\). Hence for \(\varepsilon \in (0,\varepsilon_0)\) there is a smooth mapping \(\varphi_\varepsilon : \mathcal{V} \to W^T\), defined by \(\varphi_\varepsilon(\xi) = \varphi^g_\varepsilon(\xi)\) whenever \(\xi \in \mathcal{V}^g\), such that the solutions of

\[
F_\varepsilon(\xi,\varphi_\varepsilon(\xi)) = 0, \quad \xi \in \mathcal{V}
\]

(25)

lie on \(\eta = \varphi_\varepsilon(\xi)\). Since \(F_\varepsilon\) is \(H\)-equivariant, both functions \((g^{-1}\varphi_\varepsilon(\xi))\) and \((\xi,\varphi_\varepsilon(\xi))\) are solutions of (25) for any \(g \in H\). By uniqueness of solutions, \(\varphi_\varepsilon\) is \(H\)-equivariant. Note that we may have to take \(\mathcal{V}\) smaller such that if \(\xi \in \mathcal{V}\) then \(\xi + \varphi_\varepsilon(\xi)\) lies in the open set \(\Omega^F\), which is the open neighbourhood in \(X^T\) of \(G(x_a)\) we started with.

For fixed \(\varepsilon \in (0,\varepsilon_0)\) define the reduced functional \(\Psi_\varepsilon : \mathcal{V} \subset X^T_0 \to \mathbb{R}\) by

\[
\Psi_\varepsilon(\xi) := \mathcal{A}(\xi + \varphi_\varepsilon(\xi)).
\]

Then

\[
\nabla \Psi_\varepsilon(\xi) = P \nabla \mathcal{A}(\xi + \varphi_\varepsilon(\xi)) + \nabla_\eta \mathcal{A}(\xi + \varphi_\varepsilon(\xi))D_\xi \varphi_\varepsilon(\xi) = P \nabla \mathcal{A}(\xi + \varphi_\varepsilon(\xi)).
\]

Hence \(\nabla \mathcal{A}(\xi + \eta) = 0\) with \(\xi \in \mathcal{V}\) if and only if \(\eta = \varphi_\varepsilon(\xi)\) and \(\nabla \Psi_\varepsilon(\xi) = 0\). \(\blacksquare\)

3.1 Estimate for the reduced functional

Fix \(\varepsilon \in (0,\varepsilon_0)\) and write the reduced functional \(\Psi_\varepsilon : \mathcal{V} \to \mathbb{R}\) as \(\Psi_\varepsilon(\xi) = A_0(\xi) + \mathcal{N}(\xi)\), where

\[
\mathcal{N}(\xi) = A_0(\xi + \varphi_\varepsilon(\xi)) - A_0(\xi) + H(\xi + \varphi_\varepsilon(\xi)).
\]

The terms \(A_0(\xi)\) and \(A_0(\xi + \varphi_\varepsilon(\xi))\) blow up as \(\varepsilon \to 0\) for \(\alpha > 1\). The core of the main theorem resides in obtaining uniform estimates for \(\nabla \mathcal{N}(\xi)\). While the matrix \(D_\varepsilon\) scales correctly the equation \(\nabla_\eta \mathcal{A}(\xi + \eta) = 0\), we need to define another matrix that scales correctly the equation \(\nabla_\varepsilon \mathcal{A}(\xi + \eta) = 0\). Let

\[
C_\varepsilon := \varepsilon^{\alpha - 1} I \oplus I \oplus \cdots \oplus I.
\]

(26)
Lemma 3.3. Assume conditions (A) – (B). Under one of the assumptions (C1) – (C3), there is a constant $N_1 > 0$, independent of the parameter $\varepsilon \in (0, \varepsilon_0)$, such that
\[ \| \varphi_\varepsilon (\xi) \| \leq N_1 (\varepsilon + \| \xi - gx_a \|^2) \] for every $\xi \in \mathcal{V}$, $g \in G$, where we may have to take a smaller neighborhood $\mathcal{V}$ of $G(x_a)$.

**Proof.** By theorem 3.2 the implicit mapping $\varphi_\varepsilon (\xi)$ solves the equation
\[ \nabla_\eta A_0 (\xi + \varphi_\varepsilon (\xi)) = -\nabla_\eta \mathcal{H} (\xi + \varphi_\varepsilon (\xi)). \]
for $\xi \in \mathcal{V}$. Since we can take $\varepsilon_0 < 1$ and $(I - P)$ is a projection, there is a constant $N_2 > 0$ such that
\[ \| \nabla_\eta A_0 (\xi + \varphi_\varepsilon (\xi)) \| = \| \nabla_\eta \mathcal{H} (\xi + \varphi_\varepsilon (\xi)) \| \leq N_2 \varepsilon \] (27)
by Lemma 2.5. Define the operator $L : X \to X$ by
\[ L = M_0 \left( \frac{\nu}{\omega} \partial_x + \mathcal{J} \right)^2 \oplus M_1 (\nu \partial_x + \mathcal{J})^2 \oplus ... \oplus M_n (\nu \partial_x + \mathcal{J})^2. \]
For $x \in X$ given by $x(s) = (u_0(s), u(s))$, set
\[ U_0(x) = M_0 \phi_\alpha (\| u_0 \|) + \sum_{1 \leq j < k \leq n} M_j M_k \phi_\alpha (\| u_j - u_k \|). \]
We have that
\[ C_x \nabla_\eta A_0 (x) = (I - P) (-\partial^2_x + 1)^{-1} (-\mathcal{L}x + \nabla U_0(x)). \] (28)

Hereafter we use the fact that the differential operator $(-\partial^2_x + 1)^{-1}$ commutes, because they are block diagonal operators in Fourier components (13). Thus $(I - P) (-\partial^2_x + 1)^{-1} L \xi = 0$ for any $\xi \in \mathcal{V}$ and
\[ C_x \nabla_\eta A_0 (\xi + \varphi_\varepsilon (\xi)) = (I - P) (-\partial^2_x + 1)^{-1} (-\mathcal{L} \varphi_\varepsilon (\xi) + \nabla U_0 (\xi + \varphi_\varepsilon (\xi))) , \]
for any $\xi \in \mathcal{V}$.

Since $X$ is a Banach algebra and $U_0(x)$ is analytic in $\Omega \subset X$, we can perform a Taylor expansion of $\nabla U_0(\xi + \varphi_\varepsilon (\xi))$ around $\xi = x_a$ in $X$. In particular, there is a ball $B_3 \subset \mathcal{V}$ of radius $\delta > 0$ (independent of the parameter $\varepsilon$ because $U_0$ does not depend on $\varepsilon$) centered at $x_a$ such that, if $\xi \in B_3$, the following inequality holds
\[ \left\| \nabla U_0(\xi + \varphi_\varepsilon (\xi)) - \nabla^2 U_0[x_a] (\xi - x_a + \varphi_\varepsilon (\xi)) \right\| \leq N_3 \| \xi - x_a + \varphi_\varepsilon (\xi) \|^2 \]
for some positive constant $N_3$. Since the norms of the operator $(-\partial^2_x + 1)^{-1} : X \to X$ and $(I - P) : X \to W$ are smaller or equal to 1 then, for $\xi \in B_3$,
\[ \left\| C_x \nabla_\eta A_0 (\xi + \varphi_\varepsilon (\xi)) - C_x \nabla^2_\eta A_0 [x_a] \varphi_\varepsilon (\xi) \right\| \leq N_3 \| \xi - x_a + \varphi_\varepsilon (\xi) \|^2. \] (29)

By the triangle inequality,
\[ \left\| C_x \nabla^2_\eta A_0 [x_a] \varphi_\varepsilon (\xi) \right\| \leq \left\| C_x \nabla_\eta A_0 [\xi + \varphi_\varepsilon (\xi)] \right\| + N_3 \| \xi - x_a + \varphi_\varepsilon (\xi) \|^2. \]

Since $\| D_x \| \leq \| C_x \| \leq 1$ if $\varepsilon_0 < 1$, we conclude using (27) that
\[ \| D_x \nabla^2_\eta A_0 [x_a] \varphi_\varepsilon (\xi) \| \leq N_2 \varepsilon + N_3 \| \xi - x_a \|^2 + N_3 \| \varphi_\varepsilon (\xi) \|^2. \] (30)
In lemma 3.1 we obtained a uniform bound $c > 0$ for the inverse of the operator $\partial_\eta F_0 [x_a] = \lim_{\varepsilon \to 0} D_x \nabla^2_\eta A_0 [x_a]$. Since $D_x \nabla^2_\eta A_0 [x_a]$ is continuous at $\varepsilon = 0$, then
\[ \| (D_x \nabla^2_\eta A_0 [x_a])^{-1} \| \leq 2 \| \partial_\eta F_0 [x_a]^{-1} \| \leq 2c \]
for $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0$ small enough. Taking $\eta = D_\varepsilon \nabla^2_{a} A_0[x_a] \varphi_\varepsilon(\xi)$, we conclude that 
\[
\|\varphi_\varepsilon(\xi)\| = \left\| \left( D_\varepsilon \nabla^2_{a} A_0[x_a] \right)^{-1} \eta \right\| \leq 2c\|\eta\| = 2c\|D_\varepsilon \nabla^2_{a} A_0[x_a] \varphi_\varepsilon(\xi)\|.
\]
By (30) and the previous inequality we obtain
\[
\frac{1}{2c} \|\varphi_\varepsilon(\xi)\| \leq N_2 \varepsilon + N_3 \|\xi - x_a\|^2 + N_3 \|\varphi_\varepsilon(\xi)\|^2.
\]
By choosing the ball radius $\delta$ small enough such that $N_3 \|\varphi_\varepsilon(\xi)\| < \frac{1}{4e}$ we get 
\[
\|\varphi_\varepsilon(\xi)\| \leq 4c (N_2 \varepsilon + N_3 \|\xi - x_a\|^2),
\]
whenever $\xi \in B_\delta$. We obtain the result with $N_1 := 4c \max(N_2, N_3)$.

This procedure gives the constant $N_1$ of the statement independent of $\varepsilon \in (0, \varepsilon_0)$. This estimate holds on a neighbourhood of the orbit $G(x_a)$ and not only in a neighbourhood of $x_a$. Indeed, since the constants $N_2$ and $c$ do not depend on the point of the orbit, we could work around another point $gx_a$ of the orbit and obtain the same estimates in a ball $B_{\delta g} \subset \mathcal{V}$. By compactness of the orbit, there is $\delta > 0$ such that the orbit can be covered by balls of radius $\delta$ and the estimate (29) holds at each point of the orbit. Therefore, all the estimates are valid in the union of balls of radius $\delta$ that we rename $\mathcal{V}$.

**Theorem 3.4 (Uniform estimate).** Assume conditions (A) – (B). Under one of the assumptions (C1) – (C3), the reduced functional $\Psi_\varepsilon : \mathcal{V} \to \mathbb{R}$ can be written as $\Psi_\varepsilon(\xi) = A_0(\xi) + \mathcal{N}(\xi)$, where $\mathcal{N}(\xi)$ is $H$-equivariant and satisfies the uniform estimate
\[
\|C_\varepsilon \nabla \mathcal{N}(\xi)\| \leq N(\varepsilon + \|\xi - gx_a\|^2),
\]
for all $\varepsilon \in (0, \varepsilon_0)$ and $g \in G$, with $N > 0$ a constant independent on the parameters.

**Proof.** Note that
\[
C_\varepsilon \nabla \mathcal{N}(\xi) = C_\varepsilon P \left( \nabla A_0(\xi + \varphi_\varepsilon(\xi)) - \nabla A_0(\xi) \right) + C_\varepsilon P \nabla H(\xi + \varphi_\varepsilon(\xi)).
\]
Since the operator norms of $C_\varepsilon$ and $P$ are bounded by 1, there is a constant $N_2 > 0$ such that $\|P C_\varepsilon \nabla H(\xi + \varphi_\varepsilon(\xi))\| \leq N_2 \varepsilon$. By the triangle inequality
\[
\|C_\varepsilon \nabla \mathcal{N}(\xi)\| \leq \|C_\varepsilon P \left( \nabla A_0(\xi + \varphi_\varepsilon(\xi)) - \nabla A_0(\xi) \right)\| + N_2 \varepsilon.
\]
Applying the mean value theorem, there is some $\mu \in [0, 1]$ such that 
\[
C_\varepsilon \left[ \nabla A_0(\xi + \varphi_\varepsilon(\xi)) - \nabla A_0(\xi) \right] = C_\varepsilon \nabla^2 A_0[\xi + \mu \varphi_\varepsilon(\xi)] \varphi_\varepsilon(\xi).
\]
Using the notations of the previous lemma, the Hessian reads
\[
C_\varepsilon \nabla^2 A_0[\xi + \mu \varphi_\varepsilon(\xi)] = \left( -\partial^2_\xi + 1 \right)^{-1} \left( -\mathcal{L} + \nabla^2 U_0[\xi + \mu \varphi_\varepsilon(\xi)] \right).
\]
Since the operator $\mathcal{L}$ commutes with $P$ and $P \varphi_\varepsilon(\xi) = 0$, then
\[
P C_\varepsilon \nabla^2 A_0[\xi + \mu \varphi_\varepsilon(\xi)] \varphi_\varepsilon(\xi) = P \left( -\partial^2_\xi + 1 \right)^{-1} \nabla^2 U_0[\xi + \mu \varphi_\varepsilon(\xi)] \varphi_\varepsilon(\xi).
\]
Therefore by (31) and the fact that the norm of $\left( -\partial^2_\xi + 1 \right)^{-1}$ is bounded by 1, we obtain
\[
\|P C_\varepsilon \left( \nabla A_0(\xi + \varphi_\varepsilon(\xi)) - \nabla A_0(\xi) \right)\| \leq \|P \nabla^2 U_0[\xi + \mu \varphi_\varepsilon(\xi)] \varphi_\varepsilon(\xi)\| \leq c \|\varphi_\varepsilon(\xi)\|,
\]
for some constant $c > 0$ independent of $\mu$, which exists because the operator $P \nabla^2 U_0[\xi + \mu \varphi_\varepsilon(\xi)]$ is bounded independently of the parameter $\varepsilon$ because $U_0$ does not depend on $\varepsilon$. The result of the statement follows from lemma 3.3 by setting $N := cN_1 + N_2$. ■
4. Critical points of the reduced functional

Let us summarise what we achieved so far. Suppose \( ε \in (0, \varepsilon_0) \) and conditions (A) – (B) are satisfied. Then, under one of the assumptions (C1) – (C3), there is a neighbourhood \( V \subset X_0^f \) of the orbit \( G(x_0) \) such that the problem of finding a solution \( x = \xi + \eta \in X^f \) of the Euler-Lagrange equations (11) is reduced to finding a solution \( \xi \in V \subset X_0^f \) of \( \nabla \Psi_ε(\xi) = 0 \). Furthermore, the reduced functional is given by

\[
\Psi_ε(\xi) = A_0(\xi) + \mathcal{N}(\xi),
\]

where \( A_0(\xi) \) is \( G \)-invariant, \( \mathcal{H}(\xi) \) is \( H \)-invariant, and \( \varphi_ε(\xi) \) is \( H \)-equivariant, where \( H \subset G \).

The critical points of \( \Psi_ε(\xi) \) cannot be obtained directly by a continuation of solutions of \( \nabla \Psi_ε(\xi) = 0 \) using the parameter \( \varepsilon \in (0, \varepsilon_0) \) because \( \varepsilon \) encodes the distance between the pair of bodies, and the function \( \nabla \Psi_ε(\xi) \) explodes as \( \varepsilon \to 0 \) when \( \alpha > 1 \). Before proceeding with the continuation of solutions we need to solve first the singular part of \( \nabla \Psi_ε(\xi) \). For logarithm potentials (case \( \alpha = 1 \)), it is still possible to continue the solutions directly from \( \Psi_ε(\xi) \) for \( \varepsilon = 0 \). For instance, in [8], this approach is used for a Hamiltonian system corresponding to the \( n \)-vortex problem.

4.1 The regular functional

In this section we obtain a regular functional by passing to the quotient space \( \xi \in V \subset X_0^f \) under the action of the group \( H \). Let

\[
\xi = (\xi_0, \xi_1) \in X_0^f = E_0 \times E'.
\]

and recall that the group \( G = G_1 \times G_2 \) acts diagonally on \( X_0^f \). Under the conditions (C1) – (C3), we have that \( G_1(a_0) \) is a unit circle \( S(E_0) \) in \( E_0 \). Under the conditions (C1) – (C2) this follows from the fact that \( E_0 = E \) is the plane and \( G_1 \) acts as \( U(1) \) on the plane. Under the conditions (C3) we have that \( E_0 = E \) is the plane \( \Pi \subset E \) and \( G_1 = U(1) \times U(d - 1) \) acts as \( U(1) \) on the plane \( E_0 = \Pi \).

Notice that we chose \( a_0 = (1, 0, \ldots, 0) \) for \( d \geq 1 \). Thus for every \( \xi_0 \in E_0 \) we can find \( h \in G_1 \) such that \( \xi_0 = rh_0 \) for some \( r \in \mathbb{R} \). Since \( H = \hat{G}_1 \), we obtain

\[
\Psi_ε(\xi_0, \xi_1) = \Psi_ε(h^{-1}\xi_0, h^{-1}\xi_1) = \Psi_ε(ra_0, h^{-1}\xi_1),
\]

by using \( H \)-invariance. Setting \( \xi' = h^{-1}\xi_1 \) one obtains that \( \Psi_ε(\xi_0, \xi_1) = \Psi_ε(ra_0, \xi') \) depends only on the variables \((r, \xi')\). In particular the solutions of \( \nabla \Psi_ε(\xi_0, \xi_1) = 0 \) are in one to one correspondence with the solutions of

\[
\partial_r \Psi_ε(ra_0, \xi') = 0 \quad \text{and} \quad \nabla_{\xi'} \Psi_ε(ra_0, \xi') = 0.
\]

Furthermore, we observe that the function \( \Psi_ε(ra_0, \xi') \) is \( H_{a_0} \)-invariant, where

\[
H_{a_0} := \{ g \in G_2 \mid g \in (G_1)_{a_0} \}
\]

is the stabiliser of \( a_0 \) in \( H \). Note that \( H_{a_0} \) is only acting on the second component because it is a subgroup of \( G_2 \).

Remark 4.1. In the case \( d = 1 \), we can use polar coordinates to write \( \xi_0 = re^{i\theta} \). Similarly \( \xi_1 = (\rho_1e^{i\theta_1}, \ldots, \rho_ne^{i\theta_n}) \). Then the ‘reduced’ variable is \( \xi' = (\rho_1e^{i\theta_1}, \ldots, \rho_ne^{i\theta_n}) \) where \( \theta_j' = \theta_j - \theta \).

Theorem 4.1. Under conditions (C1) – (C3), for \( \varepsilon \in (0, \varepsilon_0) \), the critical points of \( \Psi_ε(\xi) \) in the (possibly smaller) neighbourhood \( V \subset X_0^f \) are in one to one correspondence with the critical points of the \( H_{a_0} \)-invariant function \( \Psi' \colon V' \subset E' \to \mathbb{R} \) given by

\[
\Psi'_ε(\xi') = V(\xi') + \mathcal{N}(\xi'),
\]

where \( V' \subset E' \) is a neighbourhood of \( G_2(a) \), \( V(\xi') \) is the amended potential as in [8] and

\[
\mathcal{N}(\xi') = A_0(r_ε(\xi')a_0, \xi') - V(\xi') + \mathcal{N}(r_ε(\xi')a_0, \xi')
\]
where \( r_x : \mathcal{V} \subset E' \to \mathbb{R} \) is the unique \( H_{a_0} \)-invariant function that solves the equation \( \partial_r \Psi(r_x(\xi'))a_0, \xi' = 0 \). Furthermore there are constants \( N', N'_1 > 0 \) such that for each \( g \in G_2 \),
\[
||r_x(\xi')|| \leq N'_1(\varepsilon + ||\xi' - ga||^2) \quad \text{and} \quad ||\nabla_{\xi'}N'(\xi')|| \leq N'(\varepsilon + ||\xi' - ga||^2).
\]

**Proof.** The function \( \Psi(\varepsilon r_0, \xi') \) reads
\[
\Psi(\varepsilon r_0, \xi') = A_0(\varepsilon r_0, \xi') + N(\varepsilon r_0, \xi')
\]
where \((\varepsilon r_0, \xi') \in \mathcal{V} \subset X_0^\Gamma\) and
\[
A_0(\varepsilon r_0, \xi') = 2\pi \left( \varepsilon^{1-\alpha} M_0 \left( \frac{1}{2} r^2 + \phi_n(r) \right) \right).
\]
We want to express \( r \) as a function of \( \xi' \) from the equation \( \partial_r \Psi(\varepsilon r_0, \xi') = 0 \). Using the same strategy as before, we consider the regularised \( r \)-gradient
\[
f(\varepsilon, \xi') = \varepsilon^{\alpha-1} \partial_r \Psi(\varepsilon r_0, \xi').
\]
Observe that
\[
f(\varepsilon, \xi') := 2\pi M_0 \left( r - \frac{1}{r^\alpha} \right) + \varepsilon^{\alpha-1} \partial_r N(\varepsilon r_0, \xi').
\]
By Theorem 3.3 the regularised \( r \)-gradient extends continuously at \( \varepsilon = 0 \) and \( f_0(1, a) = 0 \). Thus, in order to apply the implicit function theorem we only need to show that the derivative
\[
\partial_r f_0(1, a) = 2\pi M_0(\alpha + 1) + \lim_{\varepsilon \to 0} \varepsilon^{\alpha-1} \partial_r^2 N(x_a) \tag{32}
\]
is non zero. Since \( \varepsilon r_0 \) is \( \varepsilon \)-invariant, i.e. \( \Psi \) is analytic in \( \varepsilon \) and the uniform estimate in Theorem 3.4 imply that \( ||C_\varepsilon \nabla^2 N(x_a) || \leq N ||\xi - gx_a||^2 \). This inequality implies that \( \nabla_{\xi'}N(\xi) \) has no linear term at \( \xi = x_a \), i.e. its linearisation is zero \( \lim_{\varepsilon \to 0} C_\varepsilon \nabla^2 N(x_a) = 0 \). Thus (32) is non-vanishing.

By the implicit function theorem we conclude that there is a smooth function \( r_x \) defined on a neighbourhood \( \mathcal{V}' \subset E' \) of \( a \) such that
\[
f(\varepsilon r_x(\xi'), \xi') = \varepsilon^{\alpha-1} \partial_r \Psi(\varepsilon r_x(\xi'), \xi') = 0
\]
on this neighbourhood. As before this argument can be repeated at any point of the orbit \( G_2(a) \) in \( E' \) and we can assume that \( \mathcal{V}' \) is a neighbourhood of \( G_2(a) \) in \( E' \). Hence, when we fix \( \varepsilon \in (0, \varepsilon_0) \) and take a smaller neighbourhood \( \mathcal{V} \subset X_0^\Gamma \), the critical points of \( \Psi(\varepsilon r_0, \xi') \) in \( \mathcal{V} \) are in one to one correspondence with the critical points of the function \( \Psi' : \mathcal{V}' \subset E' \to \mathbb{R} \) given by
\[
\Psi'(\xi') = A_0(r_x(\xi') a_0, \xi') + N(r_x(\xi') a_0, \xi') = V(\xi') + N'(\xi').
\]
By uniqueness of \( r_x \) and \( H_{a_0} \)-equivariance of \( \Psi'(\varepsilon r_0, \xi') \), we have that \( r_x \) is \( H_{a_0} \)-invariant, i.e. \( \Psi' \) is \( H_{a_0} \)-invariant. By Theorem 3.4 we have for \( g \in G_2 \) the uniform estimates
\[
|\varepsilon^{\alpha-1} \partial_r N(\varepsilon r_0, \xi')| \leq N(\varepsilon + ||\xi' - ga||^2 + |r - 1|^2),
\]
\[
||\nabla_{\xi'}N(\varepsilon r_0, \xi')|| \leq N(\varepsilon + ||\xi' - ga||^2 + |r - 1|^2).
\]
Using these estimates and an argument analogous to Lemma 3.3 it is possible to obtain the uniform estimates for \( r_x(\xi') \) and \( N'(\xi') \).
4.2 Critical points of the regular functional

In this section we find the critical points of the regular functional $\Psi'_\epsilon(\xi') = V(\xi') + N'(\xi')$, where $V(\xi')$ is $G_2$-invariant and $N'(\xi')$ is $H_{a_0}$-invariant. The potential $\Psi'_0(\xi') = V(\xi')$ has the orbit of critical points $G_2(a)$. Thus, we encounter a similar situation to the case studied in \[32, 31\] where the term $N'$ breaks the symmetry from $G_2$ to the subgroup $H_{a_0}$.

Next we use Palais slice coordinates for $\xi'$. Let $K := (G_2)_a$ be the stabiliser of $a$ and $G_2(a) \subset E'$ be the group orbit of $a$. Let $W = E'/T_aG_2(a)$ be a $K$-invariant complement in $E'$. By the Palais slice theorem, there is a $K$-invariant neighbourhood of $0$ denoted $W_0 \subset W$, and a $G_2$-invariant neighbourhood of $G_2(a)$ denoted $V' \subset E'$, such that $V'$ is isomorphic to the associated bundle $G_2 \times_K W_0$ \[32, 31\]. We can then shrink $W_0$ such that $V'$ is contained in $V$. This provides slice coordinates $\xi' = [(g, w)] \in G_2 \times_K W_0$ near $G_2(a)$ with respect to which $a$ corresponds to the class $[(e, 0)]$. We can thus write the $H_{a_0} \times K$-invariant lift $\Psi'_0(g, w)$ of $\Psi'_\epsilon(\xi')$ with respect to the variables $(g, w) \in G_2 \times W_0$, where the twisted action of $H_{a_0} \times K$ on $G_2 \times W_0$ is given by

$$(h, k) \cdot (g, w) = (h g k^{-1}, k \cdot w) \quad (h, k) \in H_{a_0} \times K.$$

By $G_2$-equivariance of $\Psi'_0(g, w) = V(\xi')$, we have

$$\nabla_w \Psi'_0(g, 0) = 0 \quad \text{for every} \quad g \in G_2$$

where $\nabla_w \Psi'_0 : G \times W_0 \to W$ denotes the projection of $\nabla \Psi'_0$ to the slice $W$. In the previous section we performed a finite-dimensional Lyapunov-Schmidt reduction and a second Lyapunov-Schmidt to solve the singular part of $\nabla \Psi'_0(\tau \alpha, \xi')$. Now we perform a third Lyapunov-Schmidt reduction to express the (normal) variables $w \in W_0$ in terms of the variables along the group orbit $g \in G_2$. For this purpose we also need the following non-degeneracy condition on the central configuration:

Definition 4.2. We say that $a$ is **non-degenerate** if the only zero eigenvalues of the Hessian $\nabla^2 V(a)$ correspond to the eigenvectors belonging to the tangent space $T_a U(d)(a)$.

Remark 4.3. In the case (C1)-(C2) we have $G_2 = U(1)$. In the case (C3) the group $G_2 \subset U(d)$ is lower-dimensional than the group $U(d)$. The central configuration $a$ in the fixed point space of $\Gamma$ consists of points lying in the orthogonal complement $\Pi^\perp$, i.e. $a \in E' = (\Pi^\perp)^n$. The orbit $U(d)(a) \subset E^n$ intersects $(\Pi^\perp)^n$ in the $G_2$-orbit $G_2(a) \subset (\Pi^\perp)^n$. Since $a \in E^n$ is non-degenerate, the hessian $\nabla^2 V(a) : E' \to E'$ is non-singular when restricted to a complement of the tangent space of the group orbit $G_2(a) \subset E'$. Note that we could have considered a degenerate central configuration $a$ such that the hessian restricted to the fixed point space of $\Gamma$, $\nabla^2 V(a) : E' \to E'$, has only zero eigenvalues with eigenvectors belonging to the tangent space of the orbit $T_a G_2(a) \subset E'$. Although, we ignore if any degenerate central configuration $a$ satisfies this weaker condition.

Before concluding the proof of the existence of critical points for the regular functional $\Psi'_0(g, w)$, we briefly recall some tools of Lyusternik-Schnirelmann theory \[21\]. Given a compact Lie group $G$ acting on a compact manifold $M$ and a smooth $G$-invariant function $f : M \to \mathbb{R}$, the equivariant version of the Lyusternik-Schnirelmann theorem states that the number of $G$-orbits of critical points of $f$ is bounded below by $\text{Cat}_G(M)$ \[11\]. The latter is defined as being the least number of $G$-categorical open subsets required to cover $M$. Those are the $G$-invariant open subsets which are contractible onto a $G$-orbit by mean of a $G$-equivariant homotopy.

Theorem 4.2. Assume conditions (A) – (B) and (C1) – (C3) and suppose that the central configuration $a \in E^n$ is non-degenerate. For each $\epsilon \in (0, \epsilon_0)$ there is a neighbourhood $V' \subset E'$ of the orbit $G_2(a)$ so that the number of $H_{a_0}$-orbits of critical points of the reduced potential $\Psi'_\epsilon$ defined on $V'$ is bounded below by

$$\text{Cat}_{H_{a_0}}(G_2/K).$$

Furthermore, we have that the $H_{a_0}$-orbits of solutions have an element of the form $\xi' = g \cdot a + O_{E'}(\epsilon)$ for some $g \in G_2$.
We now work out the solutions that we obtain for the $N$-body problem. Solutions of the ordinary differential equation (ODE) $\ddot{\mathbf{q}} = F(\mathbf{q})$, where $\mathbf{q} = (q_1, q_2, \ldots, q_N)$ represents the positions of the $N$ bodies, can be obtained by integrating the ODE numerically. The initial conditions for $\mathbf{q}$ and its time derivative $\dot{\mathbf{q}}$ determine the solutions. The equations of motion for the $N$-body problem in the plane can be given by

$$m_\ell \ddot{q}_\ell = -\sum_{k \neq \ell} m_\ell m_k \frac{q_\ell - q_k}{\|q_\ell - q_k\|^{n+1}}, \quad \ell = 0, \ldots, n$$

(33)

according to the three cases (C1)-(C2)-(C3) that we discussed earlier. The solutions are now written in components

$$q(t) = (q_0(t), q_1(t), \ldots, q_n(t)) \in \mathbb{R}^N.$$

5. Solutions of the $N$-body problem

We now work out the solutions that we obtain for the $N = (n+1)$-body problem

$$m_\ell \ddot{q}_\ell = -\sum_{k \neq \ell} m_\ell m_k \frac{q_\ell - q_k}{\|q_\ell - q_k\|^{n+1}}, \quad \ell = 0, \ldots, n$$

(33)

according to the three cases (C1)-(C2)-(C3) that we discussed earlier. The solutions are now written in components

$$q(t) = (q_0(t), q_1(t), \ldots, q_n(t)) \in \mathbb{R}^N.$$

5.1 Solutions in the plane (C1)-(C2)

If $E$ is two dimensional, we set $J = J$. In this case we obtain solutions that in some particular cases correspond to braids. In this case, $G = U(1) \times U(1)$ and $H = U(1)$ is diagonal in $G$. The group $H$ is isomorphic to the circle $U(1)$, and $H$ is trivial. Furthermore, the groups $H_{n0}$ and $K$ are trivial, the orbit is $G_2(a) = S^1$ and

$$\text{Cat}(G_2/K) = 2.$$

By Theorem 4.2, the regular functional $\Psi_\lambda(\xi')$ has at least two critical points near $G_2(a)$. We can identify the critical points of $\Psi_\lambda(\xi')$ by an element of the form $g = e^{i\vartheta} \in G_2 = U(1)$ for some $\vartheta \in [0, 2\pi]$. Then for the critical points of $\Psi(\xi)$ we have $\xi = (a_0, e^{i\vartheta} a) + \mathcal{O}_\lambda(\varepsilon)$. Therefore, the critical points of $\mathcal{A}(\xi + \eta)$ are given by

$$u = \xi + \eta = (a_0, e^{i\vartheta} a) + \mathcal{O}_\lambda(\varepsilon),$$

where $\mathcal{O}_\lambda(\varepsilon)$ is a function in $X^\Gamma$ such that $\|\mathcal{O}_\lambda(\varepsilon)\|_X \leq c\varepsilon$ for some constant $c$. Then we have,
Theorem 5.1. Suppose that \( \dim(E) = 2 \) and the conditions (A) – (B) are satisfied. Let \( a \in E^n \) be a central configuration, satisfying the conditions (C1) or (C2), and such that \( \nabla^2 V(a) \) has kernel of real dimension 1. Then the following occurs:

(C1) If \( \alpha \neq 2 \), then for every \( \varepsilon \in (0, \varepsilon_0) \), there are at least two solutions \( q(t) \) of (33) with components of the form

\[
\begin{align*}
q_0(t) &= \exp(tJ) u_1(\nu t) - m_1 \varepsilon \exp(t\omega J) u_0(\nu t) \\
q_1(t) &= \exp(tJ) u_1(\nu t) + m_0 \varepsilon \exp(t\omega J) u_0(\nu t) \\
q_\ell(t) &= \exp(tJ) u_\ell(\nu t), \quad \ell = 2, \ldots, n,
\end{align*}
\]

where \( \omega = \pm \varepsilon^{-(\alpha+1)/2} \), \( \nu = \omega - 1 \), \( u_0(s) = a_0 + O_X(\varepsilon) \) and \( u_\ell(s) = e^{\theta J} a_\ell + O_X(\varepsilon) \) for some phase \( \theta \in [0, 2\pi] \). The case \( \omega > 0 \) corresponds to a prograde rotation of the pair and \( \omega < 0 \) to a retrograde rotation.

(C2) If \( \alpha = 2 \), the same result holds with the addition that \( u_0(s) \) is \( 2\pi/n \)-periodic and

\[
uq_{\ell}(s) = \exp(-\theta J) u_{\sigma(\ell)}(s + \theta), \quad \ell = 1, \ldots, n,
\]

where \( (\theta, \sigma) \) is the generator of the discrete symmetry group \( \Gamma \) defined in Section 2.5.

For such solutions, the bodies \( \ell = 0, 1 \) rotate in a circular Kepler orbit whose center of mass follows the position determined by a body in a rigid motion of \( n \) bodies. If \( \varepsilon \in (0, \varepsilon_0) \) is such that \( \omega \in \mathbb{Q} \), then \( \nu = 1 - \omega \in \mathbb{Q} \) and the solution is periodic. Otherwise the solution \( q(t) \) is quasi-periodic. Furthermore, if the frequency \( \omega = \pm p/q \) is rational, then \( \nu = (\pm p - q)/q \) is rational and the functions \( u_j(\nu t) \) and \( e^{\omega t J} \) are \( 2\pi q \)-periodic. Therefore, the solutions \( q(t) \) are \( 2\pi q \)-periodic.

Corollary 5.2 (Braid solutions). Suppose that \( \dim(E) = 2 \) and the conditions (A) – (B) are satisfied. Let \( a \in E^n \) be a central configuration, satisfying the conditions (C1) or (C2), and such that \( \nabla^2 V(a) \) has kernel of real dimension 1. Fix an integer \( q \in \mathbb{Z} \setminus \{0\} \). Set \( \varepsilon = (p/q)^{-2/(\alpha+1)} \), where \( p \) is relatively prime to \( q \). Then there is \( p_0 \) such that, for each \( p > p_0 \), there are at least two solutions \( q(t) \) of (33) with components of the form

\[
\begin{align*}
q_0(t) &= \exp((t + \vartheta J) a_1 - m_1 \varepsilon \exp(\pm (pt/q) J) a_0 + O(\varepsilon), \\
q_1(t) &= \exp((t + \vartheta J) a_1 - m_0 \varepsilon \exp(\pm (pt/q) J) a_0 + O(\varepsilon), \\
q_\ell(t) &= \exp((t + \vartheta J) a_\ell + O(\varepsilon), \quad \ell = 2, \ldots, n.
\end{align*}
\]

where \( \vartheta \) represents a phase, and \( O(\varepsilon) \) is a \( 2\pi q \)-periodic function of order \( \varepsilon \).

In these solutions the bodies \( \ell = 0, 1 \) wind around their center of mass \( p \) times in the period \( 2\pi q \), while the center of mass of the bodies \( \ell = 0, 1 \) and the bodies \( \ell = 2, \ldots, n \) wind around the origin \( q \) times. The case \( \omega = p/q \) corresponds to a prograde rotation of the pair and \( \omega = -p/q \) to a retrograde rotation.

5.2 Examples of solutions satisfying conditions (C2)

Given that we need the symmetric conditions (C2a)-(C2b) in the gravitational case, we now present examples of configurations that we can braid: the Maxwell configuration and configurations symmetric through the origin. For each case, we find a symmetry \( \sigma \in S_n \) that allows to deal with the resonances.

- Maxwell configuration. The Maxwell configuration is proposed by Maxwell as a model of Saturn and its ring. This central configuration consists of a polygonal configuration of unitary
masses with a central body of different mass $\mu$. The central body is at the origin $a_1 = 0$ with mass $M_1 = \mu$. The other bodies have masses $M_\ell = 1$ and coordinates

$$a_\ell = (\mu + S_{n-1})^{3/2} e^{J_\ell \theta} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \theta = \frac{2\pi}{n-1}$$

for $\ell = 1, \ldots, n-1$, where

$$S_{n-1} = \frac{1}{4} \sum_{\ell=1}^{n-1} \frac{1}{\sin\left(\frac{\pi\ell}{n-1}\right)}$$

(see [15] for details). We consider the discrete symmetry generated by $(\theta, \sigma)$, where $\sigma \in S_n$ is such that $\sigma^{n-1} = (1)$. We only need to verify conditions (C2a)-(C2b). The masses satisfy condition (C2a) because $\sigma(1) = 1$ and $M_\ell = 1$ for $\ell = 2, \ldots, n$. The positions satisfy condition (C2b) because $\sigma(1) = 1$ with $a_n = 0$ and $a_{\sigma(\ell)} = \exp(\theta J_\ell) a_\ell$ for $\ell = 2, \ldots, n$.

![Figure 3: Maxwell configuration for seven bodies.](image)

- **Symmetric configuration with respect to the origin.** In this case we assume that $\theta = \pi$ and that there is an involution $\sigma \in S_n$ such that $\sigma^2 = (1)$ and $\sigma(1) = 1$. That is, the central configuration $a$ and its associated masses $M_\ell$ need to be invariant under the involution $\sigma$. Explicitly we require

$$M_\ell = M_{\sigma(\ell)}, \quad a_{\sigma(\ell)} = -a_\ell,$$

for $\ell = 1, \ldots, n$, i.e. $a_1 = 0$. This class of central configuration are symmetric with respect to the origin.

### 5.3 Solutions in more dimensions (C3)

For $d \geq 2$ the symmetry group is $G = G_1 \times G_2$ where $G_1 = G_2 = U(1) \times U(d-1)$. Since $a_0$ is in the plane $E_0 = \Pi := \{(x, y, 0, \ldots, 0)\}$, the group orbit of $x_0$ is identified with

$$G(x_0) = G_1(a_0) \times G_2(a)$$

where $G_1(a_0) = S^1$. Note that, in this case, $H_{a_0} = \{e\} \times U(d-1)$. By assumption (C3), the central configuration $a$ lies in the subspace orthogonal to the plane, $E_1 = \Pi^\perp$, then $G_2(a) = S^{2d-3}$. It follows that

$$\text{Cat}_{H_{a_0}}(G_2(a)) = \text{Cat}(pt) = 1.$$

We can identify the critical orbit of $\Psi'(g, \phi_1(g))$ with any element $g \in G_2$. Therefore, the critical point of $A(\xi + \eta)$ is given by

$$u = \xi + \eta = (a_0, ga) + O_{X^r}(\varepsilon),$$

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Figure 4: A central configuration with $D_2$ symmetry (see [26] for the existence of such configurations).

where $O_{X^Γ}(ε)$ is a function in $X^Γ$ such that $∥O_{X^Γ}(ε)∥_X ≤ cε$ for some constant $c$.

If the central configuration $a$ is non-degenerate, then the Hessian of the amended potential $V$ is invertible in the orthogonal complement to the tangent space to the orbit $G_2(a)$ in the fixed point space of $Γ$.

**Theorem 5.3.** Assume conditions (A)-(B) and (C3). Suppose that $a ∈ E^n$ is not-degenerate. Then, for every $ε ∈ (0, ε_0)$, the $N = n + 1$-body problem has at least one solutions $q(t)$ of the form

$$
\begin{align*}
q_0(t) &= \exp (tJ) u_1(νt) - m_1ε \exp(tωJ)u_0(νt) \\
q_1(t) &= \exp (tJ) u_1(νt) + m_0ε \exp(tωJ)u_0(νt) \\
q_ℓ(t) &= \exp (tJ) u_ℓ(νt), \quad ℓ = 2, ..., n,
\end{align*}
$$

where $ω = ±ε^{-(α+1)/2}$, $ν = ω - 1$, $u_0(s) = a_0 + O_{X^Γ}(ε)$ and $u_ℓ(s) = ga_ℓ + O_{X^Γ}(ε)$ with $g ∈ \{ε\} × U(d-1) ⊂ U(d)$. Furthermore, in this case $u_0(s)$ and $u_ℓ(s)$ have the symmetries $u_0(s) = -Ru_0(s + π)$ and $u_ℓ(s) = Ru_ℓ(s + π)$.

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