Tight LP Approximations for the Optimal Power Flow Problem

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Abstract—DC power flow approximations are ubiquitous in the electricity industry. However, these linear approximations fail to capture important physical aspects of power flow, such as the reactive power and voltage magnitude, which are crucial in many applications to ensure voltage stability and AC solution feasibility. This paper proposes two LP approximations of the AC optimal power flow problem, founded on tight polyhedral approximations of the SOC constraints, in the aim of retaining the good lower bounds of the SOCP relaxation and relishing the computational efficiency of LP solvers. The high accuracy of the two approximations is corroborated by rigorous computational evaluations on systems with up to 9,241 buses and different operating conditions. This performance is ideal for MILP extensions of these LP models since MILP is computationally more efficient than MIQCP.

Index Terms—LP approximations, convex relaxations, optimal power flow, second-order cone programming.

NOTATION

A. Input data and operators

\( \mathcal{B} \) Set of buses in the power network.
\( \mathcal{B}_i \) Set of buses connected to bus \( i \).
\( b_{ij}^h \) Shunt susceptance (p.u.) in the \( \pi \)-model of line \( ij \).
\( c0^g_i \) Constant coefficient ($) term of generator \( g \)'s cost function.
\( c1^g_i \) Coefficient ($/MW) of the linear term of generator \( g \)'s cost function.
\( c2^g_i \) Coefficient ($/MW^2) of the quadratic term of generator \( g \)'s cost function.
\( \mathcal{G} \) Set of all generators \( (g, i) \) in the power network such that \( g \) is the generator and \( i \) is the bus connected to it.
\( i \) Imaginary unit.
\( \mathcal{L} \) Set of all transmission lines \( ij \) where \( i \) is the "from" bus.
\( \mathcal{L}_t \) Set of all transmission lines \( ij \) where \( i \) is the "to" bus.
\( P_i^d/Q_i^d \) Active/reactive power demand (MW/MVAR) at bus \( i \).
\( \overline{S}_{ij} \) Apparent power rating (MVA) of line \( ij \).

\( \theta^\Delta_{ij} \) Lower limit of the difference of voltage angles at buses \( i \) and \( j \).
\( \overline{\theta}_{ij} \) Upper limit of the difference of voltage angles at buses \( i \) and \( j \).
\( \theta_i^{\text{shift}} \) Phase shift (Radians) of phase shifting transformer connected between buses \( i \) and \( j \) \((\theta_i^{\text{shift}} = 0 \) for a transmission line\).
\( \tau_{ij} \) Tap ratio magnitude of phase shifting transformer connected between buses \( i \) and \( j \) \((\tau_{ij} = 1 \) for a transmission line\).
\( y_{ij} \) Series admittance (p.u.) in the \( \pi \)-model of line \( ij \).
\( \Re{\bullet} \) Imaginary value operator.
\( \Re{\bullet} \) Real value operator.
\( \min{\bullet} \) Minimum/maximum magnitude operator.
\( |\bullet| \) Magnitude operator/Cardinality of a set.
\( \ast \) Conjugate operator.
\( \succeq \) Matrix inequality sign in the positive semidefinite sense.

B. Decision variables

\( P_i^g/Q_i^g \) Active/reactive power (MW/MVAR) generation of generator \( g \) at bus \( i \).
\( P_{ij}/Q_{ij} \) Active/reactive power (MW/MVAR) flow along transmission line \( ij \).
\( V_i \) Complex phasor voltage (p.u.) at bus \( i \) \((V_i = |V_i| / \angle \theta_i|) \).
\( \theta_i \) Voltage angle (Radians) at bus \( i \).

I. INTRODUCTION

The alternating current (AC) power flow equations, which model the steady-state physics of power flows, are the linchpins of a broad spectrum of optimization problems in electrical power systems. Unfortunately, these nonlinear equations are the main source of nonconvexity in these problems and are notorious for being extremely challenging to solve using global nonlinear programming (GNLP) solvers. Therefore, the research community has focused on improving interior-point based nonlinear optimization methods to compute feasible solutions efficiently [1], [2]. However, these methods only guarantee local optimality and therefore provide no bounds on the optimal solution.
Due to these challenges, the electricity industry resorted to two main approaches for finding a good tradeoff between computational complexity and quality of lower bound. The first approach consists of methods for approximating the power flow equations, such as the direct current optimal power flow (DC OPF). The DC OPF exploits some physical properties of power flows in typical power systems, such as small bus voltage magnitude ranges and small bus voltage angle differences, to approximate the AC OPF by a linear program (LP). Under normal operating conditions and some adjustments of the lines losses, the DC OPF can approximate the AC active power flow equations with reasonable accuracy [3]. Moreover, the DC OPF can be extended to a mixed-integer linear programming (MILP) model to suit a wide variety of optimization applications in power system operations such as optimal transmission switching (OTS), capacitor placement, transmission and distribution network expansion planning, optimal feeder reconfiguration, power system restoration, and vulnerability analysis, to name a few. In summary, the DC OPF is particularly attractive because it leverages the high computational efficiency of LP and MILP solvers. On the downside, the DC OPF fails to capture important physical aspects of power flow, such as the reactive power and voltage magnitude, which are crucial in many applications to ensure voltage stability and AC power flow feasibility. Additionally, the accuracy and feasibility of the DC OPF under congested or unstable operating conditions are questionable. For these reasons, the DC OPF can return solutions that are infeasible in the original space and is proven to be inadequate in applications such as optimal transmission switching [4], [5].

The second, more recent approach consists of developing computationally efficient convex relaxations. In particular, the second-order cone programming (SOCP) and the semidefinite programming (SDP) relaxations have garnered considerable attention in the electricity industry. The increased interest in this line of research stems from the fact that the SDP relaxation is proven to be exact (i.e., yields a zero optimality gap) on a variety of case studies [6]. However, in many practical OPF instances, the SDP relaxation yields inexact solutions [7], [8]. In these scenarios, an AC feasible solution cannot be recovered from the SDP relaxed solution. The SDP relaxation can be strengthened by solving a hierarchy of moment relaxations at the cost of larger SDP problems [9], [10]. The main drawback of the SDP relaxation is that it cannot be readily embedded in mixed-integer programming (MIP) models as easily as LP models. Furthermore, mixed-integer SDP technology is still in its infancy compared to the more mature MILP technology.

Even more recently, increased attention was given to the computationally less demanding SOCP relaxation initially proposed in [11]. The SOCP relaxation in its classical form [11] is shown to be dominated by the SDP relaxation but recent strengthening techniques [12]–[14] have shifted this paradigm. The attractiveness of the SOCP relaxation is also due to the fact that SOCP models can be easily extended to mixed-integer quadratically constrained programming (MIQCP) models to suit applications with discrete variables, mentioned earlier.

Against this background, this paper aims at narrowing the gap between LP approximations and convex relaxations of AC power flow equations by retaining the good lower bounds of the SOCP relaxation and relishing the computational efficiency of LP solvers. In more detail, this paper proposes two LP approximations for the OPF problem based on tight polyhedral approximations of the second-order cone (SOC) constraints [15]. The first LP model is a direct LP approximation of the classical SOCP relaxation in [11], whereas the second LP model employs strengthening techniques inspired by [12] which aim at preserving stronger links between the voltage variables through convex envelopes of the polar representation. As shown in [12], a model adopting these strengthening techniques neither dominates nor is dominated by the SDP relaxation. It is important to note that in this context the term “tight” designates the high accuracy of the LP approximation of the OPF compared to its respective parent SOCP relaxation.

This paper is not the first attempt to approximate both active and reactive power flow equations in the OPF problem. The LP approximation in [16] is based on outer approximations which are strengthened by incorporating several different types of valid inequalities. However, both the computation time and the accuracy of the approximation seem to vary arbitrarily with system size. In contrast to [16], the accuracy of the LP models in this paper does not exceed $10^{-2}\%$ in the worst case and the computational efficiency is comparable to, if not better than, that of the SOCP models in most test instances.

In summary, the contributions of this paper are twofold:

- The two LP models are tested on instances from MATPOWER [2] and NESTA v0.5.0 archive [17] with up to 9241 buses and different operating conditions and are shown to consistently produce high approximation accuracies in the order of $10^{-4}\%$ on average.
- Numerical results show that the computational efficiency of the LP models is comparable to, if not better than, that of the SOCP models in most instances. This performance is ideal for MILP extensions of these LP models since MILP is computationally more efficient than MIQCP.

The paper progresses with the OPF problem formulation in Section II, followed by a review of the different types of relaxations proposed in the literature in Section III. Sections IV and V describe the polyhedral formulations of the OPF problem and Section VI showcases the numerical results. Finally, the paper concludes in Section VII.

II. OPTIMAL POWER FLOW PROBLEM

In a power network, the OPF problem consists of finding the most economic dispatch of power from generators to satisfy the load at all buses in a way that is governed by physical laws, such as Ohm’s Law and Kirchhoff’s Law, and other technical restrictions, such as transmission line thermal limit constraints. More specifically, the OPF problem is written as in Model II where $T_{ij} = r_{ij}e^{\theta_{ij}}$ is the complex tap ratio of a phase shifting transformer.
Model 1: AC OPF

\[
\begin{align*}
\text{minimize} & \quad \sum_{(g,i) \in G} c_2^{ii} (P_i^g)^2 + c_1^{ii} (P_i^g) + c_0^{ii} \\
\text{subject to} & \quad P_i^g \leq P_i^d, \quad Q_i^g \leq Q_i^d, \quad (g,i) \in G \\
& \quad V_i \leq |V_i| \leq \overline{V}_i, \quad i \in B \\
& \quad \theta_i - \theta_j \leq \overline{\theta}_{ij}, \quad ij \in L \\
& \quad \sum_{j \in B_i} P_i^g - \sum_{j \in B_i} P_{ij} = \sum_{j \in B_i} Q_i^g - \sum_{j \in B_i} Q_{ij}, \quad i \in B \\
& \quad P_{ij} = \Re \left\{ \frac{y_{ij} - \frac{b_{ij}}{2}}{|T_{ij}|} \right\} |V_i|^2 - \Re \left\{ \frac{y_{ij}}{T_{ij}} \right\} \Re \{ V_i V_j^* \} \\
& \quad Q_{ij} = \Im \left\{ \frac{y_{ij} - \frac{b_{ij}}{2}}{|T_{ij}|} \right\} |V_i|^2 - \Im \left\{ \frac{y_{ij}}{T_{ij}} \right\} \Re \{ V_i V_j^* \} \\
& \quad \sqrt{P_{ij}^2 + Q_{ij}^2} \leq \overline{\Phi}_{ij}, \quad ij \in \mathcal{L} \cup \mathcal{L}_t.
\end{align*}
\]

Problem (1) is a nonconvex nonlinear optimization problem that is proven to be NP-hard [6]. Therefore, solving large-scale instances of this problem to optimality is intractable. Consequently, applying interior-point methods (IPM) [2] to this problem provides no bounds or guarantees on the optimality of the solution, which incited researchers to channel considerable effort on convex relaxation methods.

The next section presents two of the most extensively studied relaxations of problem (1), namely, the SDP and the SOCP relaxations.

III. THE SDP AND SOCP RELAXATIONS

The SDP relaxation was first introduced in [18] and later formalized in [6]. An equivalent formulation of problem (1), described in [6], starts by setting

\[
W = \begin{bmatrix}
|V_i|^2 & V_i V_j^* & \cdots & V_i V_{|G|}^* \\
V_j^* V_j & |V_j|^2 & \cdots & V_j V_{|G|}^* \\
\vdots & \vdots & \ddots & \vdots \\
V_{|G|} V_1^* & V_{|G|} V_2^* & \cdots & |V_{|G|}|^2
\end{bmatrix}
\]

and requiring that $W \succeq 0$ and rank($W$) = 1. The SDP relaxation is then obtained by dropping the rank constraint. The main setback of applying the SDP relaxation to very large systems is that the matrix $W$ is dense even when all the data matrices are sparse. To this end, sparsity exploiting methods have been proposed in [10], [19–21] to reduce the computational burden. However, even after applying sparsity exploiting techniques, the computational efficiency of current primal-dual interior-point methods for large-scale SDP is still substantially lower than that of state-of-the-art SOCP solvers. Therefore, in an effort to exploit the sparsity of the power network and leverage the higher computational efficiency of SOCP solvers, [22] proposes further relaxing some selected positive semidefinite (PSD) conditions in the PSD constraint matrix $W$ to SOC constraints [23]. The first condition is that every $2 \times 2$ principal submatrix of a PSD matrix is also a PSD matrix. The second condition is that the positive semidefiniteness of each $2 \times 2$ symmetric matrix can be represented by a SOC constraint. More specifically, $W \succeq 0$ is replaced by $|\mathcal{L}|$ constraints of the form

\[
|W_{ij}|^2 \leq W_{ii} W_{jj}, \quad (W_{ii}, W_{jj} \geq 0), \quad ij \in \mathcal{L}.
\]

It was also observed in [22] that the resulting SOCP relaxation [3] is tantamount to the SOCP relaxation proposed earlier in [11] for radial networks. The SOC representation of the power flow constraints (1)–(11) was initially introduced in [24] as follows:

\[
\begin{align*}
W_{ij} &= V_i V_j^*, \\
W_{ij} W_{ij}^* &= V_i V_j^* V_j^* V_j, \\
|W_{ij}|^2 &= W_{ii} W_{jj}.
\end{align*}
\]

However, (4c) is not convex because it describes the surface of a rotated SOC. Therefore a convex relaxation of (4c) was proposed in [11] by relaxing the equality into an inequality as in (3). By defining

\[
\begin{align*}
W_{ii} &= |V_i|^2, \\
W_{ij} &= \Re \{ W_{ij} \} = |V_i| |V_j| \cos(\theta_i - \theta_j), \\
W_{ij}^* &= \Im \{ W_{ij} \} = |V_i| |V_j| \sin(\theta_i - \theta_j),
\end{align*}
\]

the SOCP relaxation of problem (1) can be written as in Model 2 where (6d), introduced in [21], is the equivalent of (1d).

Next, the SOCP relaxation in Model 2 can be strengthened by adding constraints that define tight convex envelopes of the nonlinear terms in (5a), (5b) and (5c) [12], [25]. As shown in [26], the convex hull of a bilinear term $\{ w = xy \mid (x, y) \in [\overline{\alpha}, \overline{\beta}] \times [\overline{\gamma}, \overline{\delta}] \}$ is given by

\[
\text{conv} \mathcal{M} := \begin{bmatrix}
w \geq \overline{\alpha} \overline{\beta} + \overline{\gamma} x + \overline{\delta} y \\
w \geq \overline{\alpha} \overline{\delta} + \overline{\gamma} y + \overline{\beta} x \\
w \leq \overline{\alpha} \overline{\gamma} + \overline{\beta} x - \overline{\delta} y \\
w \leq \overline{\alpha} \overline{\delta} + \overline{\beta} y - \overline{\gamma} x
\end{bmatrix}
\]

and the convex hull of $\{ w_2 = x^2 \mid x \in [\overline{\alpha}, \overline{\beta}] \}$ is given by

\[
\text{conv} C := \begin{bmatrix}
w_2 \geq x^2 \\
w_2 \leq (\overline{\alpha} + \overline{\beta}) x - \overline{\beta} \overline{\beta}
\end{bmatrix}
\]
Model 2: SOCP-0

\[
\begin{align*}
\text{minimize} & \quad \sum_{(y, i) \in \tilde{\psi}} c_{yi}^0 (P_{yi}^0)^2 + c_{yi}^1 (P_{yi}^0) + c_{yi}^2 \\
\text{subject to} & \quad \begin{cases} 
\Lambda_j^2 \leq W_{ij} & i \in \mathcal{I}_y, \\
\tan \left( \frac{\theta_{ij}}{2} \right) \leq W_{ij} & i \in \mathcal{I}_y, \\
\tan \left( \frac{\theta_{ij}}{2} \right) \geq W_{ij} & i \in \mathcal{I}_y,
\end{cases} \\
P_{ij} &= \Re \left\{ \frac{\theta_{ij}}{T_{ij}} \right\} W_{ij}, & i \in \mathcal{L} \\
Q_{ij} &= \Im \left\{ \frac{\theta_{ij}}{T_{ij}} \right\} W_{ij}, & i \in \mathcal{L}
\end{align*}
\]

Under the assumption that \( \theta^\Delta \) does not exceed the range \((-\pi, \pi] \) the convex envelopes of \( \{x_c = \cos(x) | x \in [-\pi, \pi] \} \) and \( \{x_s = \sin(x) | x \in [-\pi, \pi] \} \) are given by

\[
\begin{align*}
\text{conv}_c &:= \left\{ x_c \leq 1 - \cos \left( \frac{x}{2} \right) \right\} \\
\text{conv}_s &:= \left\{ x_s \leq \cos \left( \frac{x}{2} \right) \left( x - \frac{x}{2} \right) + \sin \left( \frac{x}{2} \right) \right\} \\
\end{align*}
\]

A. Approximation of the 3-dimensional SOC

A 3-dimensional SOC \( \mathbf{L}^2 \) is a subset of \( \mathbb{R}^3 \) defined by

\[
\mathbf{L}^2 = \left\{ (r,x_0,y_0) \in \mathbb{R}^3 | \sqrt{x_0^2 + y_0^2} \leq r \right\}.
\]

As proposed by \cite{23}, the key to decreasing the number of inequalities is to lift the approximating polyhedron into a higher dimension by introducing several additional variables and projecting it into the 3-dimensional subspace of the original variables \( (r,x_0,y_0) \). This polyhedral formulation is modified in \cite{15} to require fewer variables and linear inequalities. In more detail, for an integer \( k \geq 2 \), let \( \mathcal{P}_k \in \mathbb{R}^{2k+3} \) be defined as

\[
\begin{align*}
\mathcal{P}_k := \left\{ x_{i+1} = x_i \cos \left( \frac{\pi}{2} \right) + y_i \sin \left( \frac{\pi}{2} \right), 0 \leq i < k, \\
y_{i+1} \geq -y_i \cos \left( \frac{\pi}{2} \right) + x_i \sin \left( \frac{\pi}{2} \right), 0 \leq i < k, \\
y_{i+1} \geq -y_i \cos \left( \frac{\pi}{2} \right) - x_i \sin \left( \frac{\pi}{2} \right), 0 \leq i < k, \\
x_{k+1} = x_k \cos \left( \frac{\pi}{2} \right) + y_k \sin \left( \frac{\pi}{2} \right) \right\}
\end{align*}
\]

The next section describes how to tightly approximate Models \( 2 \) and \( 3 \) by LPs.

IV. POLYHEDRAL FORMULATIONS

This section describes how to tightly approximate a 3-dimensional SOC by a polyhedral set. This formulation is also extended to approximate a 4-dimensional rotated SOC. Additionally, this section also describes a polyhedral approximation of the cosine term.

In practice, \( \theta^\Delta \) typically does not exceed \( \pm 10^6 \).
B. Approximation of the 4-dimensional rotated SOC

The formulation in Section IV-A can be further extended to approximate a 4-dimensional rotated SOC $L^3$ [29], which is a subset of $R^4$ defined by $L^3 = \{(r_1, r_2, x_0, y_0) \in R^4 | x_0^2 + y_0^2 \leq r_1 r_2\}$. The rotated SOC $L^3$ can also be expressed as

$$r \geq \sqrt{x_0^2 + y_0^2}, \quad (13)$$

$$r' \geq \sqrt{(x_0')^2 + (y_0')^2}, \quad (14)$$

$$r' = \frac{(r_1 + r_2)}{2}, r = \frac{(r_1 - r_2)}{2}, y_0' = r. \quad (15)$$

Now, (13) and (14) can each be approximated by $P_k$ and coupled by (15) to construct a polyhedron $P_k^r$ with $2k + 6$ variables $(r_1, r_2, x_0, y_0, \ldots, y_{k-1}, r, r', x_0', y_0', \ldots, y_{k-1}')$, 4$k$ linear inequalities and 3 linear equalities to approximate $L^3$.

C. Approximation of the square of a variable

A function of the form $w^2 \geq x^2$ can be approximated by a polyhedron $P_k^r$ as described in Section IV-B. However, numerical simulations have shown that the increased accuracy of approximating (8a) by a polyhedron $P_k^r$ has very little effect on the accuracy of the overall solution. This stems from a combination of two factors. The first is the size of the range of the voltage magnitude (e.g. $[0.95, 1.05]$ in practice) and the second is that (11c) might be dominated by (3). To this end, a simpler polyhedral approximation of (8) is constructed as follows: $l$ points $x_1, \ldots, x_l$ are selected in the interval $[\pi, \pi]$, which allows adding $l + 1$ constraints of the form

$$P_l^s := \begin{cases} w_2 \geq (2x_h) x - x_h^2, & h = \{1, \ldots, l\}, \quad (16a) \\ w_2 \leq (\pi + x) x - \pi \pi, & \quad (16b) \end{cases}$$

The approximation in (16) requires no additional variables, and numerical simulations have shown that it can result in a higher overall accuracy for $l = 20$.

D. Approximation of the cosine

The cosine term in (51h) can be approximated by a convex affine set provided that $\theta^A$ does not exceed the range $(-\frac{\pi}{2}, \frac{\pi}{2})$ [30]. One obvious way is to approximate the quadratic term in (51a) by a polyhedron $P_l^s$ as described in Section IV-C. However, since the benefit of relaxing the cosine into its convex hull becomes more prominent when $\theta^A$ is small, a direct approximation of the cosine can still achieve a high accuracy under these conditions (i.e. when the domain of the cosine function is small). In more detail, a direct polyhedral approximation of the cosine term is constructed as follows: $s$ points $\theta_1^A, \ldots, \theta_s^A$ are selected in the interval $[\bar{\theta}^A, \theta^A]$ and each cosine term $\cos(\theta^A)$ is replaced with a corresponding new variable $x_c$, which allows adding $s + 1$ constraints of the form

$$x_c \leq -\sin (\theta_0^A) (\theta^A - \theta_0^A)$$

$$x_c \geq \cos (\theta_0^A), \quad a = \{1, \ldots, s\}, \quad (17a)$$

$$x_c \geq \cos (\bar{\theta}^A). \quad (17b)$$

The approximation in (17) requires no additional variables, and numerical simulations have shown that it can result in a high overall accuracy for $s = 20$.

V. LP OPTIMAL POWER FLOW

Given the building blocks in Section IV, a tight LP approximation of the OPF problem is now possible. The only remaining step is to substitute the quadratic terms in the objective function by corresponding variables and rotated SOC constraints. This substitution now enables leveraging the techniques in Section IV to tightly approximate the quadratic terms in the objective function by polyhedrons and thereby obtaining a LP approximation of the OPF problem. Specifically, $|G|$ variables and constraints of the form

$$p_g = \sqrt{e^{2P_p} r_g}, \quad (g, i) \in G, \quad (18)$$

are introduced along with $N = \lfloor |G| / 2 \rfloor + \lfloor |G| / 2 \rfloor$ variables $\alpha_n$ and constraints of the form

$$\alpha_n \geq p_{2n-1}^2 + p_{2n}^2, \quad n = \{1, \ldots, N\}. \quad (19)$$

Finally, the LP approximations of SOCP-0 and SOCP-S are shown in Models [4] and [5], respectively, and their accuracy and computational efficiency are evaluated in the next section.

VI. NUMERICAL EVALUATION

This section evaluates the accuracy and computational efficiency of the LP approximations in Models [4] and [5] as compared to their respective parent SOCP relaxations in Models [2] and [3]. The models are tested on standard IEEE instances available from the IPM-based OPF solver MATPOWER [2] as well as more challenging instances from NESTA v0.5.0.
archive [17], with \( k = 16 \) for the LP models. All simulations are carried out on an Intel Core i7, 3.7GHz, 64-bit, 128GB RAM computing platform. MATPOWER is used to solve the original nonconvex AC model in problem (1), which provides an upper bound on the optimal solution. Additionally, IPOPT [31] via the MATLAB toolbox OPTI [32] is used to compute upper bounds on instances where MATPOWER diverges or fails to compute a solution. These locally optimal solutions are not shown in this paper due to space limitations.

Interested readers are instead referred to [17] for a complete list of AC (locally optimal) solutions with the exception of MATPOWER’s case1354pegase and case9241pegase whose solutions are $74069.354$ and $83159.128$ respectively.

On the other hand, both CPLEX 12.6 [33] and Gurobi 6.0.5 [34] are considered for solving the SOCP and the LP models. An interesting observation is that the polyhedral approximations described in Sections IV-A and IV-B which are the cornerstones of LP-0 and LP-S, make these models particularly difficult to solve using the primal or dual simplex methods. This could be due to the large coefficient ranges and/or due to the irrational coefficients (cos(\( \pi/2 \)) and sin(\( \pi/2 \))) introduced by these polyhedral formulations. For the LP models, both CPLEX and Gurobi use their default concurrent optimization algorithms which invoke multiple methods (primal simplex, dual simplex and parallel barrier) simultaneously on multiple cores, and return the optimal solution from the method that finishes first. Therefore, in this scenario, only the parallel barrier method is chosen instead of the default concurrent optimization algorithm to solve the LP models. Ultimately, CPLEX is chosen to solve the LP models due to a better performance of its parallel barrier method, for these specific LP models, as compared to Gurobi’s parallel barrier method.

By letting \( S^{AC} \) denote the best known AC solution and \( S^{conv} \) denote the solution from the corresponding relaxation, the optimality gap can be measured as \( (S^{AC} - S^{conv} / S^{AC}) \times 100 \). The optimality gaps and the computation times of the four models are summarized in Tables I and II for MATPOWER and NESTA instances respectively. It is evident from Tables I and II that both LP-0 and LP-S tightly approximate their parent SOCP models, SOCP-0 and SOCP-S respectively. However, the values marked by * designate instances where the SOCP relaxation’s solution does not match the LP one despite the “optimal” exitflag or vice versa. In these cases (*), both Gurobi and IPOPT are used to ascertain that the LP solution is in fact the accurate one in most cases. This is also corroborated by results in the literature, namely in [17] for SOCP-0 and

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Table I. Model Comparison on MATPOWER Instances.

| Case | SOCP-0 | LP-0 | SOCP-S | LP-S |
|------|--------|------|--------|------|
| 118  | 0.25   | 0.25 | 0.25   | 0.25 |
| 300  | 0.15   | 0.15 | 0.15   | 0.15 |
| 1354 | 0.09*  | 0.08 | 0.08   | 0.08 |
| 3375wp | 0.27* | 0.26 | 0.26   | 0.25 |
| 9241 | 2.02*  | 2.01 | 2.02*  | 2.01 |

Table II. Model Comparison on NESTA Instances.

| Case | SOCP-0 | LP-0 | SOCP-S | LP-S |
|------|--------|------|--------|------|
| 24   | 0.01   | 0.01 | 0.01   | 0.01 |
| 29   | 0.14   | 0.14 | 0.12   | 0.12 |
| 30   | 0.30   | 0.30 | 0.28   | 0.28 |

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\( * \) denotes a coefficient of \( \epsilon = 1.15 \times 10^{-9} \).

\(^2\) Computed using IPOPT via OPTI.
for SOCP-S. These discrepancies are due to numerical stability issues despite the solver reporting reaching an optimal solution. This is not surprising since it was also pointed out in [12] that IPOPT is numerically more stable than both CPLEX and Gurobi’s QCP for large systems, and was ultimately used for solving their SOCP models. However, CPLEX is still used to solve the SOCP models in this paper for the sake of comparison. Also, the fact that Gurobi and CPLEX are both state-of-the-art LP (and MILP) solvers, it would not make sense to use IPOPT to solve the LP models. In fact, the approximation accuracy is in the order $10^{-5}\%$ when the solution of both SOCP models and LP models does not run into numerical stability issues.

Moreover, Tables I and II also show that the computational efficiency of the LP models is comparable to, if not better than, that of the SOCP models in most cases. This performance is ideal for MILP extensions of these LP models, which gives more edge over the MIQCP extensions of the SOCP models because state-of-the-art MIQCP technology is still not as mature as state-of-the-art MILP technology.

VII. CONCLUSION

Two tight LP approximations of the OPF problem, founded on tight polyhedral approximations of the SOC constraints, are proposed in this paper. The first LP model is a direct LP approximation of the classical SOCP relaxation whereas the second LP model employs strengthening techniques that preserve stronger links between the voltage variables through convex envelopes of the polar representation. Rigorous computational tests on systems with up to 9241 buses and different operating conditions have shown that the proposed LP models consistently produce high approximation accuracies of $10^{-4}\%$ on average compared to their respective parent SOCP relaxations. Moreover, the computational efficiency of the two proposed LP models is shown to be comparable to, if not better than, that of the SOCP models in most instances, which makes them ideal for MILP extensions knowing that MILP technology is more mature than the MIQCP technology. Finally, the LP models in this paper can easily be extended to handle any convex generator cost function.

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