ELLiptic FActors in the JacoBian Variety of Riemann Surfaces

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Abstract. For given \( r \geq 1 \) elliptic curves \( E_1, \ldots, E_r \), there exists a closed Riemann surface of the minimal genus \( e(E_1, \ldots, E_r) \) whose jacobian variety is isogenous to a product \( E_1 \times \cdots \times E_r \times A \), where \( A \) is also a product of jacobian varieties. Let \( e'(r) \) be the maximum of the values \( e(E_1, \ldots, E_t) \), where \( E_1, \ldots, E_t \) runs over the space of elliptic curves. If \( r \leq 3 \), then it is known that \( e'(r) = r = e(r) \); we provide explicit equations of such a minimal genus Riemann surface. If \( r \geq 4 \), then we obtain that \( e'(r) \leq \hat{e}(r) \), where (i) \( \hat{e}(r) = 1 + 2(r-2)/2 \) (for even) and (ii) \( \hat{e}(r) = 1 + 2(r-3)/2(r-1) \) (for odd). Our constructions also permit to obtain a 2-dimensional family of Riemann surfaces of genus \( g \in \{5, 9\} \) and also of a 1-dimensional family of genus \( g = 13 \) whose jacobian varieties are isogenous to the product of elliptic curves.

1. Introduction

An abelian variety is called simple if it is not isogenous to a product of two abelian varieties of smaller dimensions. A non-simple abelian variety is then isogenous to a non-trivial product of abelian varieties of smallest positive dimension. Such a decomposition is not unique, but if we, moreover, require all the factors to be simple, then Poincaré reducibility theorem asserts that the decomposition is unique up to permutation of factors. Examples of (principally polarized) abelian varieties are given by the jacobian variety of a closed Riemann surface (the polarization is induced by the intersection form of the first integral homology group of the surface). In this paper we should restrict to these types of principally polarized abelian varieties and isogenous decompositions of them into a product of jacobian varieties.

Let \( S \) be a closed Riemann surface and \( JS \) be its jacobian variety. If \( JS \) is a non-simple abelian variety, then it is isogenous to a product \( A_1^{n_1} \times \cdots \times A_s^{n_s} \), where \( A_j \) is an abelian variety of positive dimension, \( n_j \geq 1 \) is an integer and \( s \geq 2 \). If \( g \leq 4 \), then it possible to assume all the factors \( A_j \) to be jacobian varieties. For \( g \geq 5 \), this seems to be a difficult problem to see if one may choose such a decomposition so that all its factors are jacobian varieties. Somehow related to the above is the following converse. Assume we are given \( r \geq 1 \) closed Riemann surfaces \( S_1, \ldots, S_r \), where \( S_j \) has genus \( g_j \geq 1 \).

(a) Is there a closed Riemann surface whose jacobian variety is isogenous to the product \( JS_1 \times \cdots \times JS_r \times A \), for \( A \) being also a suitable product of jacobian varieties?

(b) If the answer to (a) is affirmative, then which is minimal genus \( e(S_1, \ldots, S_r) \) of such a closed Riemann surfaces? Clearly, \( e(S_1, \ldots, S_r) \geq g_1 + \cdots + g_r \).

In this paper, we consider the previous problems in the particular case that each \( S_j \) is a genus one (i.e., an elliptic curve) Riemann surface \( E_j \). In this case, we also set \( e(r) \) as

\[ e(r) = \max \{ e(E_1, \ldots, E_t) : E_1, \ldots, E_t \text{ run over the space of elliptic curves} \} \]

\[ e'(r) = \max \{ e(E_1, \ldots, E_t) : E_1, \ldots, E_t \text{ run over the space of elliptic curves} \} \]

\[ \hat{e}(r) = 1 + 2(r-2)/2 \] (for even)

\[ \hat{e}(r) = 1 + 2(r-3)/2(r-1) \] (for odd)

\[ e'(r) \leq \hat{e}(r) \]

\[ e(S_1, \ldots, S_r) \geq g_1 + \cdots + g_r \]

\[ e(r) \text{ as} \]

2000 Mathematics Subject Classification. 30F10, 30F20, 14H40, 14H37.

Key words and phrases. Riemann surface, Jacobian variety, Fiber product.

Partially supported by Project Fondecyt 1150003 and Anillo ACT 1415 PIA-CONICYT.
the maximum of the values \( e(E_1, \ldots, E_r) \), where \( E_1, \ldots, E_r \) run over the space of elliptic curves. Since \( e(E_1) = e(1) = 1 \), the interesting case is \( r \geq 2 \).

In [5] Ekedah and Serre constructed examples of elliptic curves \( E_1, \ldots, E_r \) (for almost every \( r \leq 1297 \)) so that \( e(E_1, \ldots, E_r) = r \). It is an open problem to determine if there are similar examples for infinitely many values of \( r \). Another similar constructions can be found, for instance, in [4, 5, 7, 11, 12, 13, 14].

The space of isomorphism classes of principally polarized abelian varieties of dimension \( r \) is the upper half Siegel space \( \tilde{H}_r \); of complex dimension \( r(r + 1)/2 \). The rational symplectic group \( \text{Sp}_{2g}(\mathbb{Q}) \) acts on \( \tilde{H}_r \) as a group of holomorphic automorphism and, moreover, if \( z \in \tilde{H}_r \) and \( T \in \text{Sp}_{2g}(\mathbb{Q}) \), then \( T(z) \) and \( z \) represent isogenous principally polarized abelian varieties.

If \( E_1, \ldots, E_r \) are elliptic curves and \( z_0 \) represents the abelian variety \( E_1 \times \cdots \times E_r \), then the \( \text{Sp}_{2g}(\mathbb{Q}) \)-orbit of \( z_0 \) is dense in \( \tilde{H}_r \). As for \( r \in \{2, 3\} \), the jacobian locus \( \mathcal{A}_r \) in \( \tilde{H}_r \) (i.e. the principally polarized abelian varieties obtained as jacobian of Riemann surfaces) is an open dense set, the density of the previous orbit asserts that there are infinitely many closed Riemann surfaces of genus \( r \) whose jacobian variety is isogenous to \( E_1 \times \cdots \times E_r \); in particular, \( e(r) = r \). In Theorems 1 and 4 we construct explicit equations for such Riemann surfaces (for \( r = 2 \) this was already done by Gaudry and Schost in [6]). If \( r \geq 4 \), then the dimension of \( \mathcal{A}_r \) is \( 3(r-1) \); which is strictly smaller that of \( \tilde{H}_r \). So the density property of the \( \text{Sp}_{2g}(\mathbb{Q}) \)-orbit of \( z_0 \) does not ensure intersection of it with \( \mathcal{A}_r \) (note that the density asserts that the jacobian variety of any closed Riemann surface of genus \( r \) is very near, in some sense, to be isogenous to \( E_1 \times \cdots \times E_r \)).

In Theorem 5 we observe that, for \( r \geq 4 \),

\[
    e(r) \leq \begin{cases} 
        1 + 2^{(r-2)/2}, & \text{for } r \text{ even}, \\
        1 + 2^{(r-3)/2} (r-1), & \text{for } r \text{ odd}. 
    \end{cases}
\]

We believe these upper bounds are sharp.

The proof of Theorem 5 is based in an explicit construction. Given \((2s-3)\) elliptic curves, where \( s \geq 3 \), we explicitly construct a Riemann surface of genus \( 1 + 2^{s-2}(s-2) \) whose jacobian variety is isogenous to the product of \( s(s-1)/2 \) elliptic curves and some other jacobian varieties of elliptic/hyperelliptic curves (Theorem 2). Such a Riemann surface happens to be one of the two irreducible component of a certain fiber product of \( s \) elliptic curves (theses two components being isomorphic). The construction also permits to obtain a 2-dimensional family of Riemann surfaces of genus \( g \in \{5, 9\} \) whose jacobian varieties are isogenous to the product of \( g \) elliptic curves (Theorem 3).

We also provide a similar construction, a fiber product of \( r \) given elliptic curves \( E_1, \ldots, E_r \), which turns out to be irreducible. Such a fiber product is a closed Riemann surface of genus \( g = 1 + 2^{r-2}(r-1) \) whose jacobian variety is isogenous to the product \( E_1 \times \cdots \times E_r \times A_{g-r} \), where \( A_{g-r} \) is again the product of jacobian varieties of elliptic/hyperelliptic Riemann surfaces. This construction also permits to obtain a 2-dimensional family of Riemann surfaces of genus \( g = 5 \) and a 1-dimensional family of Riemann surfaces of genus \( g = 13 \) whose jacobian varieties are isogenous to the product of elliptic curves (see Corollaries 1 and 2).

2. Main results

Before to state the main results, we need to recall some facts on elliptic curves. Set \( \Delta_1 = \mathbb{C} - \{0, 1\} \) and, for \( s \geq 2 \), set \( \Delta_s = \{ (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s : \lambda_j \in \Delta_1; \lambda_i \neq \lambda_j, i \neq j \} \). If \( \lambda \in \Delta_1 \), then we set the elliptic curve

\[
    E_\lambda : y^2 = x(x-1)(x-\lambda).
\]
It is known that $E_1$ and $E_2$ are isomorphic if and only if there is some $T \in \mathbb{G} = \langle u \lambda \rangle = 1/\lambda, V(\lambda) = 1 - \lambda \rangle \cong \mathbb{Z}_2$ so that $\mu = T(\lambda)$. It is also well known that for every $\lambda \in \Delta_1$ there exist infinitely many values $\mu \in \Delta_1$ (in fact a dense subset) so that $E_\mu$ and $E_\lambda$ are isogenous. In particular, given $s \geq 2$ elliptic curves $E_1, \ldots, E_s$, there are infinitely many tuples $(\lambda_1, \ldots, \lambda_s) \in \Delta_s$ so that $E_{\lambda_j}$ and $E_j$ are isogenous for each $j = 1, \ldots, s$. For $s = 2$, we may even replace “isogenous” by “isomorphic”.

2.1. Genus two case: the known situation. As already noted in the introduction, given any pair $(E_1, E_2)$ of elliptic curves, there is a closed Riemann surface $S$ of genus two whose jacobian variety $JS$ is isogenous to $E_1 \times E_2$. The following describes explicit equations for one of them (an argument is provided in Section 4). This is not new and it can be tracked back to, for instance, [4, 6, 7, 8].

**Theorem 1** ($e(2) = 2$). Let $E_1$ and $E_2$ two elliptic curves. Choose $(\lambda_1, \lambda_2) \in \Delta_2$ so that $E_j$ is isomorphic (or isogenous) to $E_{\lambda_j}$, for $j = 1, 2$, and set

$$\eta_1 = \frac{\lambda_1 - 1}{\lambda_2 - 1}, \quad \eta_2 = \frac{\lambda_2 - 1}{\lambda_1 - 1}.$$  

If $S$ is the genus two Riemann surface defined by the hyperelliptic curve

$$y^2 = (x^2 - 1)(x^2 - \eta_1)(x^2 - \eta_2),$$

then $JS$ is isogenous to $E_1 \times E_2$.

**Remark 1.** The constructed genus two Riemann surface $S$ in Theorem 1 admits a non-hyperelliptic involution $\iota$ so that $S/\langle \iota \rangle$ is isomorphic to $E_1$ and $S/\langle \iota \rangle$ is isomorphic to $E_2$, where $\iota$ is the hyperelliptic involution. But, there are also Riemann surfaces of genus two with no extra automorphisms (with the exception of the hyperelliptic involution) whose jacobian variety is also isogenous to $E_1 \times E_2$; these can be obtained by considering non-constant holomorphic maps $h : S \to E$, where $S$ is a closed Riemann surface of genus two and $E$ being some elliptic curve.

2.2. A general construction. Next, given $2s - 3$ elliptic curves, for $s \geq 3$, we make an explicit construction of a closed Riemann surface of genus $g = 1 + 2^{s-2}(s-2)$, whose jacobian variety is isogenous to the product of at least $s(s-1)/2$ elliptic curves and jacobian varieties of some elliptic/hyperelliptic Riemann surfaces. Let us consider the set of cardinality $2^{s-1} - 1$ defined as

$$V_s = \{ \alpha = (\alpha_1, \ldots, \alpha_s) \in \{0, 1\}^s - \{(0, \ldots, 0)\} : \alpha_1 + \cdots + \alpha_s \text{ is even} \}.$$  

As previously noted, given $2s - 3$ elliptic curves, $E_1, \ldots, E_{2s-3}$, we may find a tuple $(\lambda_1, \ldots, \lambda_{2s-3}) \in \Delta_{2s-3}$ so that $E_j$ is isogenous to the elliptic curve $y^2 = x(x-1)(x-\lambda_j)$.

**Theorem 2.** Let $s \geq 3$ and $(\lambda, \mu_1, 1, \mu_1, 2, \mu_2, 2, \ldots, \mu_{s-2}, 1, \mu_{s-2}, 2) \in \Delta_{2s-3}$. Set

$$\eta_0 = \frac{-1}{\mu_{s-2, 2}}, \quad \eta_1 = \frac{1}{1 - \mu_{s-2, 2}}, \quad \eta_2 = \frac{1}{\lambda - \mu_{s-2, 2}}, \quad \eta_3 = \frac{1}{\mu_{s-2, 1} - \mu_{s-2, 2}},$$

where $\mu_{s-2, 2}$, $\mu_{s-2, 1}$, $\lambda$ are parameters such that $\mu_{s-2, 2}, \mu_{s-2, 1} \in \Delta_s$, and $\lambda \in \Delta_1$. Then $JS$ is isogenous to $E_1 \times E_2 \times \cdots \times E_{2s-3}$.
\[ \eta_{j,t} = \frac{1}{\mu_{j,t} - \mu_{s-2,2}}, \quad t = 1, 2, \quad j = 1, \ldots, s - 3. \]

For each \( \alpha = (\alpha_1, \ldots, \alpha_s) \in V_s \), set \( K_\alpha \in \mathbb{C}^s = \mathbb{C} - \{0\} \) given by

\[ K_\alpha = (-\mu_{s-2,2})^{a_1} (\mu_{s-2,2} - 1)^{a_2} (\mu_{s-2,2} - \lambda)^{a_3} (\mu_{s-2,2} - \mu_{s-1,2})^{a_4} \prod_{k=1}^{s-2} (\mu_{s-2,2} - \mu_{k,1})^{a_k} (\mu_{s-2,2} - \mu_{k,2})^{a_k+1}. \]

If \( X \subset \mathbb{C}^{2s-1} \) is the affine curve defined by the following \( 2^{s-1} - 1 \) equations

\[ \left\{ w_\alpha^2 = K_\alpha z^{a_1}(z - \eta_0)z^{a_2}(z - \eta_2)z^{a_3}(z - \eta_3)z^{a_4} \prod_{k=1}^{s-2} (z - \eta_k)z^{a_k}(z - \eta_{k+2})z^{a_k+1}, \right\}, \]

then \( X \) defines a closed Riemann surface \( S \) of genus \( g = 1 + 2^{s-2}(s - 2) \) whose jacobian variety \( JS \) is isogenous to the product of the jacobian varieties of the following \( \Sigma_{j=1}^{[s/2]} \binom{s}{2} \) elliptic/hyperelliptic curves

\[ C_{i_1, \ldots, i_k} : v^2 = (u - \rho_{i_1,1})(u - \rho_{i_2,1}) \cdots (u - \rho_{i_k,1})(u - \rho_{i_k,2}), \]

where \( 2 \leq k \leq s \) is even, the tuples \((i_1, \ldots, i_k)\) satisfy

\[ 1 \leq i_1 < i_2 < \cdots < i_k \leq s, \]

and

\[ \rho_{i,1} = \begin{cases} \infty, & \text{if } i_j = 1 \\ 1, & \text{if } i_j = 2 \\ \mu_{s-2,1}, & \text{if } i_j \geq 3 \end{cases} \]

\[ \rho_{i,2} = \begin{cases} 0, & \text{if } i_j = 1 \\ \lambda, & \text{if } i_j = 2 \\ \mu_{s-2,2}, & \text{if } i_j \geq 3 \end{cases} \]

In the case that \( \rho_{i,1} = \infty \), then the factor \((u - \rho_{i,1})\) is deleted from the above expression.

**Remark 2.**

1. The above provides a \((2s - 3)\)-dimensional family of closed Riemann surfaces of genus \( g = 1 + 2^{s-2}(s - 2) \).
2. The Riemann surface \( S \), constructed in Theorem 2 has the following properties.
   a) \( JS \) contains at least \( s(s - 1)/2 \) elliptic curves in its isogenous decomposition.
   b) Some of the elliptic curves factors of \( JS \) are

\[ E_1 : y^2 = x(x - 1)(x - \lambda) \]
\[ E_2 : y^2 = (x - 1)(x - \lambda)(x - \mu_{1,2}) \]
\[ E_3 : y^2 = (x - \mu_{1,1})(x - \mu_{1,2})(x - \mu_{2,1})(x - \mu_{2,2}) \]
\[ E_4 : y^2 = (x - \mu_{2,1})(x - \mu_{2,2})(x - \mu_{3,1})(x - \mu_{3,2}) \]
\[ \vdots \]
\[ E_{s-1} : y^2 = (x - \mu_{s-3,1})(x - \mu_{s-3,2})(x - \mu_{s-2,1})(x - \mu_{s-2,2}) \]
\[ E_s : y^2 = x(x - \mu_{s-2,1})(x - \mu_{s-2,2}) \]

(c) It happens that \( S \) is an irreducible connected component of the fiber product of the pairs \((E_1, \pi_1), \ldots, (E_s, \pi_s)\), where \( \pi_j(x, y) = x \).
2.3. **An example of genus nine.** The construction provided in Theorem 2 permits to obtain Riemann surfaces whose jacobian variety is isogenous to the product of elliptic curves. Next, as an example, we describe a 2-dimensional family of genus nine Riemann surfaces whose jacobian is isogenous to the product of nine elliptic curves.

**Theorem 3.** Let \((\lambda, \mu) \in \Delta_2\) and set

\[
\mu_{1,1} = \mu, \mu_{1,2} = \frac{\lambda}{\mu}, \mu_{2,1} = \frac{\lambda(\mu - 1)}{\mu - \lambda}, \mu_{2,2} = \frac{\mu - \lambda}{\mu - 1},
\]

\[
K_1 = (\mu_{2,2} - \mu_{1,1})(\mu_{2,2} - \mu_{1,2})(\mu_{2,2} - \mu_{2,1}), \quad K_2 = (\mu_{2,2} - 1)(\mu_{2,2} - \lambda)(\mu_{2,2} - \mu_{2,1}),
\]

\[
K_3 = (\mu_{2,2} - 1)(\mu_{2,2} - \lambda)(\mu_{2,2} - \mu_{1,1})(\mu_{2,2} - \mu_{1,2}), \quad K_4 = -\mu_{2,2}(\mu_{2,2} - \mu_{2,1}),
\]

\[
K_5 = -\mu_{2,2}(\mu_{2,2} - \mu_{1,1})(\mu_{2,2} - \mu_{1,2}), \quad K_6 = -\mu_{2,2}(\mu_{2,2} - 1)(\mu_{2,2} - \lambda).
\]

If \(S\) is the genus nine Riemann surface defined by the curve

\[
\begin{align*}
    w_1^2 &= K_1 \left( z - \frac{1}{\mu_{1,1} - \mu_{2,2}} \right) \left( z - \frac{1}{\mu_{1,2} - \mu_{2,2}} \right) \left( z - \frac{1}{\mu_{2,1} - \mu_{2,2}} \right), \\
    w_2^2 &= K_2 \left( z - \frac{1}{\mu_{2,2}} \right) \left( z - \frac{1}{\mu_{1,2} - \mu_{2,2}} \right) \left( z - \frac{1}{\mu_{2,1} - \mu_{2,2}} \right), \\
    w_3^2 &= K_3 \left( z - \frac{1}{\mu_{1,2} - \mu_{2,2}} \right) \left( z - \frac{1}{\mu_{1,1} - \mu_{2,2}} \right) \left( z - \frac{1}{\mu_{1,2} - \mu_{2,2}} \right), \\
    w_4^2 &= K_4 \left( z - \frac{1}{\mu_{2,2}} \right) \left( z - \frac{1}{\mu_{1,2} - \mu_{2,2}} \right) \left( z - \frac{1}{\mu_{2,1} - \mu_{2,2}} \right), \\
    w_5^2 &= K_5 \left( z - \frac{1}{\mu_{2,2}} \right) \left( z - \frac{1}{\mu_{1,2} - \mu_{2,2}} \right) \left( z - \frac{1}{\mu_{2,1} - \mu_{2,2}} \right), \\
    w_6^2 &= K_6 \left( z - \frac{1}{\mu_{2,2}} \right) \left( z - \frac{1}{\mu_{1,2} - \mu_{2,2}} \right) \left( z - \frac{1}{\mu_{2,1} - \mu_{2,2}} \right), \\
    w_7^2 &= w_3^2 w_5^2.
\end{align*}
\]

then \(JS\) is isogenous to the product of nine elliptic curves.

2.4. **Genus three case.** As previously noted in the introduction, for every triple \((E_1, E_2, E_3)\) of elliptic curves there is some genus three closed Riemann surface \(S\) whose jacobian variety is isogenous to the product \(E_1 \times E_2 \times E_3\). We may use the constructed Riemann surface in Theorem 2, for \(s = 3\), to construct explicitly equations for such surface \(S\).

**Theorem 4** \((e(3) = 3)\). Let \(E_1, E_2\) and \(E_3\) be three elliptic curves. Choose \((\lambda_1, \lambda_2, \lambda_3) \in \Delta_3\) so that \(E_j\) is isogenous to \(E_{3j}\), for \(j = 1, 2, 3\), and let \(\mu\) be a root of

\[
\lambda_1 \lambda_2 \lambda_3 \mu^2 - (\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3 - \lambda_1 - \lambda_3 + 1) \mu + \lambda_1 \lambda_2 = 0.
\]

If \(S\) is the genus three Riemann surface defined by the curve

\[
\begin{align*}
    w_1^2 &= \mu(\lambda_3 \mu - 1)(\lambda_3 \mu - \lambda_1)(\lambda_3 - 1) z^2 \left( z - \frac{1}{\lambda_1 - \lambda_3 \mu} \right) \left( z - \frac{1}{\mu(1 - \lambda_3)} \right), \\
    w_2^2 &= -\lambda_3 \mu^2 (\lambda_3 - 1) z \left( z + \frac{1}{\lambda_3 \mu} \right) \left( z - \frac{1}{1 - \lambda_3 \mu} \right), \\
    w_3^2 &= -\lambda_3 \mu^2 (\lambda_3 - 1)(\lambda_3 - 1) z^2 \left( z + \frac{1}{\lambda_3 \mu} \right) \left( z - \frac{1}{\mu(1 - \lambda_3)} \right)
\end{align*}
\]

then \(JS\) is isogenous to the product \(E_1 \times E_2 \times E_3\).
2.5. **Upper bounds for** $e(r)$, $r \geq 4$. Another direct consequence of the construction provided by Theorem 2 is the following upper bound for $e(r)$.

**Theorem 5.**

$$e(r) \leq \begin{cases} 1 + 2^{(r-2)/2}r, & r \geq 4 \text{ even} \\ 1 + 2^{(r-3)/2}(r-1), & r \geq 5 \text{ odd} \end{cases}$$

We conjecture that the above inequalities are in fact equalities.

2.6. **Another fiber product construction.** In Section 9 we describe another similar construction which permits to obtain the following construction.

**Theorem 6.** If $(\lambda_1, \ldots, \lambda_r) \in \Delta_r$, then

$$X = \left\{ (x, y_1, \ldots, y_r) \in \mathbb{C}^{r+1} : y_j = x(x-1)(x-\lambda_j); \ j = 1, \ldots, r \right\}$$

defines a closed Riemann surface of genus $g = 1 + 2^{r-2}(r-1)$ whose jacobian variety is isogenous to a product $E_{\lambda_1} \times \cdots \times E_{\lambda_r} \times A_{g-r}$, where $A_{g-r}$ is the product of certain explicit elliptic/hyperelliptic Riemann surfaces.

3. **Preliminaries: The jacobian variety of a closed Riemann surface**

3.1. **Abelian varieties.** A polarized abelian variety of dimension $g$ is a pair $A = (T, Q)$, where $T = \mathbb{C}^g/L$ is a complex torus of dimension $g$ and $Q$ (called a principal polarization of $A$) is a positive-definite Hermitian product in $\mathbb{C}^g$ with $\text{Im}(Q)$ having integral values over elements of the lattice $L$. There is basis of $L$ for which $\text{Im}(Q)$ can be represented by the matrix

$$
\begin{pmatrix}
0 & D \\
-D & 0
\end{pmatrix}
$$

where $D$ is a diagonal matrix, whose diagonal entries are $d_1, \ldots, d_g$, where $d_j \geq 1$ divides $d_{j+1}$. The tuple $(d_1, \ldots, d_g)$ is called the polarization type. When $d_1 = \cdots = d_g = 1$, we say that the polarization is principal and that the abelian variety is principally polarized.

A non-constant surjective morphism $h : A_1 \to A_2$ between abelian varieties is called an isogeny if it has a finite kernel. In this case we say that $A_1$ and $A_2$ are isogenous.

An abelian variety $A$ is called decomposable if it is isogenous to the product of abelian varieties of smaller dimensions. It is called completely decomposable if it is the product of elliptic curves (varieties of dimension 1). If the abelian variety is not isogenous to a product of lowest dimensional abelian varieties, then we say that it is simple.

**Theorem 7** (Poincaré reducibility theorem). If $A$ is an abelian variety, then there exist simple polarized abelian varieties $A_1, \ldots, A_s$ and positive integers $n_1, \ldots, n_s$ such that its jacobian variety $J_S$ is isogenous to the product $A_1^{n_1} \times \cdots A_s^{n_s}$. Moreover, the factors $A_j$ and the integers $n_j$ are unique up to isogeny and permutation of the factors.
In general, to describe these simple factors of an abelian variety seems to be a very difficult problem. When the abelian variety $A$ admits a non-trivial group $G$ of automorphisms, then there is a method to compute factors (non-necessarily simple ones) by using the rational representations of $G$ (the isotypical decomposition) \cite{2, 10, 15}.

3.2. The Jacobian variety. Let $S$ be a closed Riemann surface of genus $g \geq 1$. The first homology group $H_1(S, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{2g}$ and its complex vector space $H^{1,0}(S)$ of its holomorphic 1-forms is isomorphic to $\mathbb{C}^g$. There is a natural injective map

$$\iota : H_1(S, \mathbb{Z}) \hookrightarrow \left(H^{1,0}(S)\right)^\ast$$

(The dual space of $H^{1,0}(S)$)

$$\alpha \mapsto \int_\alpha.$$

The image $\iota(H_1(S, \mathbb{Z}))$ is a lattice in $\left(H^{1,0}(S)\right)^\ast$ and the quotient $g$-dimensional torus

$$JS = \left(H^{1,0}(S)\right)^\ast / \iota(H_1(S, \mathbb{Z}))$$

is called the jacobian variety of $S$. The intersection product in $H_1(S, \mathbb{Z})$ induces a principal polarization on $JS$; that is, $JS$ is a principally polarized abelian variety.

If we fix a point $p_0 \in S$, then there is a natural holomorphic embedding

$$\rho_{p_0} : S \to J(S)$$

defined by $\rho(p) = \int_\alpha$, where $\alpha \in S$ is an arc connecting $p_0$ with $p$.

If we choose a symplectic homology basis for $S$, say $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ (i.e. a basis for $H_1(S, \mathbb{Z})$ such that the intersection products $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ and $\alpha_i \cdot \beta_j = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta function), we may find a dual basis $\{\omega_1, \ldots, \omega_g\}$ (i.e. a basis of $H^{1,0}(S)$ such that $\int_{\alpha_i} \omega_j = \delta_{ij}$). We may consider the Riemann matrix

$$Z = \left(\int_{\alpha_i} \omega_j\right)_{g \times g}.$$

If we now consider the Riemann period matrix $\Omega = (I \ Z)_{2g \times 2g}$, then its 2g columns define a lattice in $\mathbb{C}^g$. The quotient torus $\mathbb{C}^g / \Omega$ is isomorphic to $JS$.

3.3. A decomposition result of the jacobian variety. Next, we recall the following decomposition result due to Kani-Rosen \cite{9} which will be used in our constructions.

**Proposition 1** \cite{9}. Let $S$ be a closed Riemann surface of genus $g \geq 1$ and let $H_1, \ldots, H_s \subset \text{Aut}(S)$ such that:

1. $H_i H_j = H_j H_i$ for all $i, j = 1, \ldots, s$;
2. $g_{H_i H_j} = 0$, for $1 \leq i < j \leq s$
3. $g = \sum_{i=1}^s g_{H_i}$

Then

$$JS \cong_{\text{isog}} \prod_{j=1}^s J(S_{H_j}).$$
4. Proof of Theorem 1

Let us consider the fiber product \( C \) of \((E_{\lambda_1}, \pi_1)\) and \((E_{\lambda_2}, \pi_2)\), where \( \pi_j(x, y) = x \), that is,

\[
C : \{(x, y_1, y_2) : y_1 = x(x - 1)(x - \lambda_1), \ y_2 = x(x - 1)(x - \lambda_2)\}.
\]

We consider the projection \( \pi : C \to \hat{\mathbb{C}} \), where \( \pi(x, y_1, y_2) = x \). On \( C \) we have the automorphisms

\[
A_1(x, y_1, y_2) = (x, -y_1, y_2), \quad A_2(x, y_1, y_2) = (x, y_1, -y_2),
\]

with \( H = \langle A_1, A_2 \rangle \cong \mathbb{Z}_2^2 \), \( H \) being the deck group of \( \pi \), \( C/H \) being of signature \((0; 2, 2, 2, 2)\) and branch values being \( \infty, 0, 1, \lambda_1 \) and \( \lambda_2 \). The Riemann surface \( S \) defined by \( C \) has genus two.

If \( H_j = \langle A_j \rangle \), it can be seen that the orbifold \( C/H_j \) has underlying Riemann surface \( E_{\lambda_{j+}} \). We may apply Proposition 1, using \( H_1 \) and \( H_2 \), to obtain that \( JS \) is isogenous to \( E_1 \times E_2 \).

The automorphism \( A_2 \circ A_1(x, y_1, y_2) = (x, -y_1, -y_2) \) is the hyperelliptic one. Let us consider a two-fold branched cover \( P : C \to \hat{\mathbb{C}} \) with deck group \( \langle A_2 \circ A_1 \rangle \). We may assume that the involution \( A_1 \) descents to the involution \( T(x) = -x \) (this because all involutions are conjugated in the Möbius group). So the branch values of \( P \) may be assumed to be \( \pm 1, \pm \rho_1 \) and \( \pm \rho_2 \). In this way, \( C \) can be described by the hyperelliptic curve

\[
y^2 = (x^2 - 1)(x^2 - \rho_1^2)(x^2 - \rho_2^2).
\]

Let us consider the two-fold branched cover \( Q(x) = x^2 \) whose deck group is \( \langle T \rangle \). The branch values of \( Q \) are \( 1, \rho_1^2 \) and \( \rho_2^2 \). The images of \( \infty \) and \( \pm 1 \) are \( \infty \) and \( 1 \). Let us consider the Möbius transformation

\[
L(x) = \frac{(1 - \rho_1^2)(x - \rho_2^2)}{(1 - \rho_2^2)(x - \rho_1^2)}.
\]

Then \( L(1) = 1, L(\rho_1^2) = \infty, L(\rho_2^2) = 0, L(\infty) = (1 - \rho_1^2)/(1 - \rho_2^2) \) and \( L(0) = \rho_2^2L(\infty)/\rho_1^2 \). By making \( L(\infty) = \lambda_1 \) and \( L(0) = \lambda_2 \), we obtain that

\[
\rho_1^2 = \frac{\lambda_1 - 1}{\lambda_2 - 1}, \quad \rho_2^2 = \frac{\lambda_2}{\lambda_1^2}.
\]

5. Proof of Theorem 2

5.1. Let us consider the following \( s \) elliptic curves

\[
E_1 : \quad y^2 = x(x - 1)(x - \lambda)
\]
\[
E_2 : \quad y^2 = (x - 1)(x - \lambda)(x - \mu_1)(x - \mu_2)
\]
\[
E_3 : \quad y^2 = (x - \mu_1)(x - \mu_2)(x - \mu_3)(x - \mu_4)
\]
\[
E_4 : \quad y^2 = (x - \mu_2)(x - \mu_3)(x - \mu_4)(x - \mu_5)
\]
\[
E_{s-1} : \quad y^2 = (x - \mu_{s-3})(x - \mu_{s-2})(x - \mu_{s-1})(x - \mu_s)
\]
\[
E_s : \quad y^2 = x(x - \mu_{s-1})(x - \mu_s)
\]

If we consider the degree two maps \( \pi_j : E_j \to \hat{\mathbb{C}} \) defined as \( \pi_j(x, y) = x \), then we may perform the fiber product of the \( s \) pairs \((E_1, \pi_1), \ldots, (E_s, \pi_s)\). Such a fiber product is given by the curve \( \hat{C} \) formed of the tuples \((x, y_1, \ldots, y_s)\) so that \((x, y_j) \in E_j\), for \( j = 1, \ldots, s \). This curve is reducible and contains two irreducible components, both of them
being isomorphic. The curve \( \hat{C} \) admits the group of automorphisms \( N = \langle f_1, \ldots, f_s \rangle \cong \mathbb{Z}_2^s \), where

\[
f_j(x, y_1, \ldots, y_s) = (x, y_1, \ldots, y_{j-1}, -y_j, y_{j+1}, \ldots, y_s).
\]

The two irreducible factors are permuted by some elements of \( N \) and each one is invariant under a subgroup isomorphic to \( \mathbb{Z}_2^{s-1} \).

In what follows we will construct a Riemann surface \( S \) which is isomorphic to the irreducible components of \( \hat{C} \).

5.2. Let us now consider the (affine) generalized Humbert curve (see [1] for details)

\[
D : \begin{cases}
\mu_1 z_1^2 + z_2^2 + z_3^2 = 0 \\
\mu_2 z_1^2 + z_2^2 + z_4^2 = 0 \\
\mu_3 z_1^2 + z_5^2 + z_3^2 = 0 \\
P \cdot D \to \hat{C} : (z_1, \ldots, z_{2s-1}) \mapsto -\left( \frac{z_3}{z_1} \right)^2
\end{cases}
\]

The conditions on the parameters ensure that \( D \) is a non-singular algebraic curve, that is, a closed Riemann surface. On \( D \) we have the abelian group \( (b_1, \ldots, b_{2s-1}) = H_0 \cong \mathbb{Z}_2^{s-1} \) of conformal automorphisms, where

\[
b_j(z_1, \ldots, z_{2s-1}) = (z_1, \ldots, z_{j-1}, -z_j, z_{j+1}, \ldots, z_{2s-1}), \quad j = 1, \ldots, 2s - 1.
\]

Inside the group \( H_0 \), the only non-trivial elements acting with fixed points are \( b_1, \ldots, b_{2s-1} \) and \( b_{2s} = b_1 b_2 \cdots b_{2s-1} \). The degree \( 2^{s-1} \) holomorphic map

\[
P : D \to \hat{C} : (z_1, \ldots, z_{2s-1}) \mapsto -\left( \frac{z_3}{z_1} \right)^2
\]

is a branched regular cover with deck group being \( H_0 \). The projection under \( P \) of the set of fixed points is as follows:

\[
P(\text{Fix}(b_1)) = \infty, \quad P(\text{Fix}(b_2)) = 0, \quad P(\text{Fix}(b_3)) = 1, \quad P(\text{Fix}(b_4)) = \lambda_1,
\]

\[
P(\text{Fix}(b_{2k+3})) = \mu_{k,1}, \quad P(\text{Fix}(b_{2k+4})) = \mu_{k,2}, \quad k = 1, \ldots, s.
\]

In particular, the branch locus of \( P \) is the set

\[
\{ \infty, 0, 1, \lambda, \mu_1, 1, \mu_2, \ldots, \mu_{s-2,1}, \mu_{s-2,2} \}.
\]

By the Riemann-Hurwitz formula, \( D \) has genus \( g_D = 1 + 2^{s-2}(s - 2) \).

5.3. Let us consider the surjective homomorphism

\[
\theta : H_0 \to H = \langle a_1, \ldots, a_{s-1} \rangle \cong \mathbb{Z}_2^{s-1}
\]
5.4. In order to write equations for \( S \), we need to compute a set of generators of \( \mathbb{C}[z_1, \ldots, z_{2s-1}]^K \), the algebra of \( K \)-invariant polynomials. Since the linear action of \( K \) is given by diagonal matrices, a set of generators can be found to be

\[
\begin{align*}
  t_1 &= z_1^2, \\
  t_2 &= z_2^2, \\
  &\ldots \\
  t_{2s-1} &= z_{2s-1}^2,
\end{align*}
\]

together the monomials of the form

\[
t_\alpha = (z_1 z_2)^{\alpha_1} (z_3 z_4)^{\alpha_2} \cdots (z_{2s-3} z_{2s-2})^{\alpha_{s-1}} z_{2s-1}^{\alpha_s},
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_s) \in V_s \). As \( V_s \) has cardinality \( 2^{s-1} - 1 \), the number of the above set of generators is \( N = 2^{s-1} + 2s - 2 \).

Using the map \( \Phi : D \to \mathbb{C}^N \), whose coordinates are \( t_1, \ldots, t_{2s-1} \) and the monomials \( t_\alpha \), \( \alpha \in V_s \), one obtains that the Riemann surface induced by \( \Phi(D) \) is isomorphic to \( S \) and that its equations are given by

\[
\Phi(D) = \begin{pmatrix}
  t_1 + t_2 + t_3 &= 0 \\
  \lambda t_1 + t_2 + t_4 &= 0 \\
  \mu_{1,1} t_1 + t_2 + t_5 &= 0 \\
  \mu_{1,2} t_1 + t_2 + t_6 &= 0 \\
  \vdots & \vdots \\
  \mu_{s-1,1} t_1 + t_2 + t_{2k+3} &= 0 \\
  \mu_{s-2,2} t_1 + t_2 + t_{2k+4} &= 0 \\
  \vdots & \vdots \\
  \mu_{s-2,1} t_1 + t_2 + t_{2s-1} &= 0 \\
  \mu_{s-2,2} t_1 + t_2 + t_{2s-1} &= 0 \\
  t_3^2 &= (t_3 t_4)^{\alpha_2} \cdots (t_{2s-3} t_{2s-2})^{\alpha_{s-1}} t_{2s-1}^{\alpha_s}, \quad \alpha = (\alpha_1, \ldots, \alpha_s) \in V_s
\end{pmatrix}.
\]

The first linear equations permit to write \( t_2, \ldots, t_{2s-1} \) in terms of \( t_1 \) as follows:

\[
\begin{align*}
  t_2 &= -1 - \mu_{s-2,2} t_1 \\
  t_3 &= 1 + (\mu_{s-2,2} - 1) t_1 \\
  t_4 &= 1 + (\mu_{s-2,2} - \lambda) t_1 \\
  t_{2k+3} &= 1 + (\mu_{s-2,2} - \mu_{k,1}) t_1, \quad t_{2k+4} = 1 + (\mu_{s-2,2} - \mu_{s,2}) t_1, \quad k = 1, \ldots, s - 3,
\end{align*}
\]
t_{2s-1} = 1 + (\mu_{s-2,2} - \mu_{s-2,1})t_1.

We may then eliminate the variables $t_2, \ldots, t_{2s-1}$ and just keep the variables $t_1$ and $t_{a_1, \ldots, a_r}$.

Let us set $t_1 = \zeta$ and $t_\alpha = w_\alpha$, for $\alpha \in V_s$. In these new $2^{s-1}$ coordinates, the above curve is isomorphic to the one given by

$$C = \left\{ \begin{array}{l}
  w_\alpha^2 = \zeta^{a_1}((1 + (\mu_{s-2,2} - 1)\zeta)^a_2(1 + (\mu_{s-2,2} - \lambda)\zeta)^a_3 \cdots \\
  \cdots (1 + (\mu_{s-2,2} - \mu_{s-3,1})\zeta)^a_3 \cdots (1 + (\mu_{s-2,2} - \mu_{s-3,2})\zeta)^a_4 \cdots (1 + (\mu_{s-2,2} - \mu_{s-2,1})\zeta)^a_{s-1}, \\\n  \alpha = (a_1, \ldots, a_s) \in V_s.
\end{array} \right.$$ 

By making the choices as described in the hypothesis of the theorem for $K_s$, and the values of $\eta_0, \eta_1, \eta_2, \eta_3$ and $\eta_{a,j}$, then the above curve can be written in the desired algebraic form.

5.5. If $\Phi_1 : D \to \mathbb{C}^{2^{s-1}}$ is the map whose coordinates are $z$ and $w_\alpha$, where $\alpha \in V_s$, then $\Phi_1(D) = C$.

If $\alpha = (a_1, \ldots, a_s)$ and $j = 1, \ldots, s-1$, then the induced automorphisms $a_j$ acts by multiplication by $-1$ at coordinates $w_\alpha$ if $a_j = 1$ and acts by the identity on the rest of coordinates.

The map

$$\pi : C \to \mathbb{C} : (z, \{w_\alpha\in V_s\}) \mapsto \frac{1 + \mu_{s-2,2}z}{\zeta}$$

is a regular branched cover with $H$ as its deck group and its satisfies that $P = \pi \circ \Phi_1$. The branch locus of $\pi$ is the set

$$\{\infty, 0, 1, \lambda, \mu_{1,1}, \mu_{1,2}, \ldots, \mu_{s-2,1}, \mu_{s-2,2}\}.$$

**Lemma 1.** The only non-trivial elements of $H$ acting with fixed points are $a_1, \ldots, a_{s-1}$ and $a_s = a_1a_2 \cdots a_{s-1}$. Moreover,

$$\pi(\text{Fix}(a_1)) = \{\infty, 0\}, \quad \pi(\text{Fix}(a_2)) = \{1, \lambda\},$$

$$\pi(\text{Fix}(a_k)) = \{\mu_{s-2,1}, \mu_{s-2,2}\}, \quad k = 3, \ldots, s-1,$$

$$\pi(\text{Fix}(a_s)) = \{\mu_{s-2,1}, \mu_{s-2,2}\}.$$ 

It can be seen that, for $j = 1, \ldots, s$, $\text{Fix}(a_j)$ has cardinality $2^{s-1}$.

**Proof.** This follows directly from the above. \qed

5.6. If $2 \leq k \leq s$ is even and $1 \leq i_1 < i_2 < \cdots < i_k \leq s$, then we consider the subgroup

$$H_{i_1, i_2, \ldots, i_k} = \langle a_i a_{i_1}, a_i a_{i_2}, \ldots, a_i a_{i_k}, a_j ; j \in \{1, \ldots, s\} - \{i_1, \ldots, i_k\} \rangle \cong \mathbb{Z}^{2^{s-2}}.$$ 

In the case $k = 2$ we have $s(s-1)/2$ such subgroups. Between them are the following ones

$$H_{1,2} = \langle a_1 a_2, a_3, \ldots, a_s \rangle$$

$$H_{2,3} = \langle a_2 a_3, a_4, \ldots, a_1 \rangle$$

$$H_{3,4} = \langle a_3 a_4, a_5, \ldots, a_2 \rangle$$

$$\vdots$$
Lemma 2. With the above notations, the following hold for the above defined subgroups.

1. Any two such subgroups \( H_{i_1,j_1,\ldots,j_l} \) and \( H_{i_2,j_2,\ldots,j_m} \) commute.
2. The quotient \( C/H_{i_1,j_1,\ldots,j_l} \) has genus \( k-1 \) and its underlying Riemann surface is given by the (elliptic) hyperelliptic curve
   \[
   y^2 = (u - \rho_{i_1,1})(u - \rho_{i_1,2}) \cdots (u - \rho_{i_l,1})(u - \rho_{i_l,2}),
   \]
   where
   \[
   \rho_{i_1,1} = \begin{cases} 
   \infty, & i_j = 1 \\
   1, & i_j = 2 \\
   \mu_{r-2,1}, & i_j = r \geq 3
   \end{cases}
   \]
   \[
   \rho_{i_2,2} = \begin{cases} 
   0, & i_j = 1 \\
   \lambda, & i_j = 2 \\
   \mu_{r-2,2}, & i_j = r \geq 3
   \end{cases}
   \]
   In the case that \( \rho_{i_1,1} = \infty \), then the factor \( (u - \rho_{i_1,1}) \) is deleted from the above expression.
3. The group generated by any two different such subgroups is \( H \).

Proof. Property (1) holds trivially as \( H \) is an abelian group. Property (2) follows from Riemann-Hurwitz formula and Lemma 1. Property (3) is clear as in the product we obtain all the generators. \( \square \)

The next result states that the sum of the genera appearing in all quotients of the form \( C/H_{i_1,i_2,\ldots,i_l} \) is equal to the genus of \( C \) (that is, the genus of \( S \)). Recall that we are considering \( k \) even and \( 2 \leq k \leq s \).

Lemma 3.

\[
\sum_{k=2}^{s} \binom{s}{k}(k-1)\left(1 + (-1)^k \right)^2 = 1 + 2^{r-2}(s-2).
\]

Proof. Consider the function

\[
f(x) = \frac{(1 + x)^r}{2x} = \frac{1}{2} \sum_{k=0}^{s} \binom{s}{k}x^{k-1}
\]

and its derivative

\[
f'(x) = \frac{(1 + x)^{r-1}((s-1)x - 1)}{2x^2} = \frac{1}{2} \sum_{k=0}^{s} \binom{s}{k}(k-1)x^{k-2}.
\]

Next, we evaluate at \( x = 1 \) and \( x = -1 \) and then we add the results to obtain

\[
2^{r-2}(s-2) = f'(1) + f'(-1) = \frac{1}{2} \sum_{k=0}^{s} \binom{s}{k}(k-1)(1 + (-1)^k) = -1 + \sum_{k=2}^{s} \binom{s}{k} \frac{(1 + (-1)^k)}{2}(k-1).
\]

\( \square \)
5.7. We may apply Proposition 1 for $C$ using all the subgroups $H_{i_{1},\ldots,i_{k}}$ in order to obtain that $JC$ (so $JS$) is isogenous to the product of the jacobian varieties of all Riemann surfaces $C/H_{i_{1},\ldots,i_{k}}$ as desired. The equations of these curves are provided in Lemma 2.

6. Proof of Theorem 3

As a consequence of Theorem 2 (and Lemma 2), the jacobian variety of $S$ is isogenous to the product of the following six elliptic curves

$E_1 : y^2 = x(x - 1)(x - \lambda)$
$E_2 : y^2 = (x - 1)(x - \lambda)(x - \mu_{1,1})(x - \mu_{1,2})$
$E_3 : y^2 = (x - \mu_{1,1})(x - \mu_{1,2})(x - \mu_{2,1})(x - \mu_{2,2})$
$E_4 : y^2 = x(x - \mu_{2,1})(x - \mu_{2,2})$
$E_5 : y^2 = x(x - \mu_{1,1})(x - \mu_{1,2})$
$E_6 : y^2 = (x - 1)(x - \lambda)(x - \mu_{2,1})(x - \mu_{2,2})$

and the jacobian variety of the following genus three hyperelliptic Riemann surface

$R : y^2 = x(x - 1)(x - \lambda)(x - \mu_{1,1})(x - \mu_{1,2})(x - \mu_{2,2})$.

Let us observe that the group $J = \langle f_1(x) = \lambda/x, f_2(x) = \lambda(x - 1)/(x - \lambda) \rangle \cong \mathbb{Z}_2^2$ keeps invariant the collection

$\infty, 0, 1, \lambda, \mu_{1,1}, \mu_{1,2}, \mu_{2,1}, \mu_{2,2}$.

In this way, $R$ admits the following automorphisms

$F_1(x, y) = \left( \frac{A}{x}, \frac{A^2 y}{x^2} \right)$
$F_2(x, y) = \left( \frac{\lambda(x - 1)}{x - \lambda}, \frac{\lambda^2 (\lambda - 1)^2 y}{(x - \lambda)^2} \right)$

We may see that $\langle F_1, F_2 \rangle \cong \mathbb{Z}_2^2$ and that each of the involutions $F_1, F_2$ and $F_1 \circ F_2$ acts with exactly 4 fixed points on $R$.

The quotients $R/(F_1), R/(F_2)$ and $R/(F_1 \circ F_2)$ have genus one. We may apply Proposition 1 to $R$ using the three cyclic groups of order two in order to see that $JR$ is isogenous to the product of three elliptic curves.

7. Proof of Theorem 4

Given the triple $(\lambda_1, \lambda_2, \lambda_3) \in \Delta_3$ and $\mu$ a root of

$\lambda_2 \lambda_3 \mu^2 - (\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3 - \lambda_1 - \lambda_3 + 1) \mu + \lambda_1 \lambda_2 = 0,$

then we set $\lambda = \lambda_1, \mu_{1,1} = \mu$ and $\mu_{1,2} = \lambda_3 \mu$. It can be seen that $(\lambda, \mu_{1,1}, \mu_{1,2}) \in \Delta_3$ and that

$F_1 : y^2 = x(x - 1)(x - \lambda_1) : E_{\lambda_1},$
$F_2 : y^2 = (x - 1)(x - \lambda_1)(x - \lambda_2)(x - \lambda_3 \mu) : E_{\lambda_2},$
$F_3 : y^2 = x(x - \mu)(x - \lambda_3 \mu) : E_{\lambda_3}.$

Theorem 2, applied to the triple $(\lambda, \mu_{1,1}, \mu_{1,2})$, asserts that $JS$ is isogenous to the product $F_1 \times F_2 \times F_3$, so isogenous to the product $E_1 \times E_1 \times E_3$. 


In this case the Riemann surface $S$ is described by the curve

$$
\begin{aligned}
  w_1^2 &= \mu(\lambda_3 \mu - 1)(\lambda_3 \mu - \lambda_1)(\lambda_3 - 1)z \left( z - \frac{1}{\lambda_1 - \lambda_3 \mu} \right) \left( z - \frac{1}{\mu(1 - \lambda_3)} \right), \\
  w_2^2 &= -\lambda_3 \mu^2(\lambda_3 - 1)z \left( z + \frac{1}{\lambda_3 \mu} \right) \left( z - \frac{1}{1 - \lambda_3 \mu} \right), \\
  w_3^2 &= -\lambda_3 \mu^2(\lambda_3 \mu - 1)(\lambda_3 - 1)z^2 \left( z + \frac{1}{\lambda_3 \mu} \right) \left( z - \frac{1}{\mu(1 - \lambda_3)} \right),
\end{aligned}
$$

the group $H = \langle a_1, a_2 \rangle \cong \mathbb{Z}_2^3$ is generated by

$$
\begin{aligned}
  a_1(z, w_1, w_2, w_3) &= (z, w_1, w_2, -w_3), \\
  a_2(z, w_1, w_2, w_3) &= (z, -w_1, w_2, -w_3),
\end{aligned}
$$

the three automorphisms $a_1$, $a_2$ and $a_3 = a_1 a_2$ acts with exactly four fixed points each one, and the corresponding regular branched cover with $H$ as deck group is

$$
\pi : S \rightarrow \hat{C} : (z, w_1, w_2, w_3) \mapsto \frac{\lambda_3 \mu z + 1}{z}.
$$

The subgroups of $H$, in this case ($k = 2$ is the only option) are

$$
H_{1,2} = \langle a_1 a_2 \rangle, \\
H_{1,3} = \langle a_1 a_3 \rangle = \langle a_2 \rangle, \\
H_{2,3} = \langle a_2 a_3 \rangle = \langle a_1 \rangle.
$$

The quotients $S/\langle a_1 \rangle$, $S/\langle a_2 \rangle$ and $S/\langle a_1 a_2 \rangle$ are of genus one and they correspond, respectively, to the elliptic curves $E_{a_1}$, $E_{a_2}$ and $E_{a_1 a_2}$.

8. Proof of Theorem 5

Let us first consider the case $r \geq 3$ odd. Write $r = 2s - 3$, where $s \geq 3$, and let us fix $(\lambda_1, \ldots, \lambda_r) \in \Delta_r$. Set $\lambda = \lambda_1$ and, for $j = 1, \ldots, s-2$, we set $\mu_{j,2} = \lambda_{j+1} \mu_{j,1}$ and $\mu_{j,1}$ a root of the polynomial

$$
\lambda_{j+1}(1 - \lambda_{j-2s})\mu_{j,1}^2 + (\lambda_{j-2s} - 1 - \lambda_{j+1} - \lambda_1 \lambda_{j+1} \lambda_{j-2s}) \mu_{j,1} + (1 - \lambda_1 \lambda_{j-2s}) = 0.
$$

Note that a value $\lambda_j$ may be changed to some other value $\lambda_j'$ (inside an infinite set of values) so that $E_{\lambda_j}$ and $E_{J_j}$ are isogenous. This observation permits to ensure that the constructed tuple $(\lambda, \mu_{1,1}, \mu_{1,2}, \mu_{2,1}, \mu_{2,2}, \ldots, \mu_{s-2,1}, \mu_{s-2,2}) \in \Delta_{2s-3}$.

Let us consider the Riemann surface $S$ constructed in Theorem 2. The jacobian variety of $S$ is isogenous to a product of certain explicit jacobian varieties. It can be seen, from the proof of that theorem, that some of these factors are (isogenous to) the elliptic curves

$$
\begin{aligned}
  y^2 &= x(x-1)(x-\lambda), \\
  y^2 &= x(x-\mu_{j,1})(x-\mu_{j,2}), \quad j = 1, \ldots, s-2, \\
  y^2 &= (x-1)(x-\lambda)(x-\mu_{j,2}), \quad j = 1, \ldots, s-2.
\end{aligned}
$$

The choice we have done for the tuple $(\lambda, \mu_{1,1}, \mu_{1,2}, \mu_{2,1}, \mu_{2,2}, \ldots, \mu_{s-2,1}, \mu_{s-2,2}) \in \Delta_{2s-3}$ ensures that they are isomorphic to the elliptic curves

$$
y^2 = x(x-1)(x-\rho)
$$

where $\rho \in \{\lambda_1, \ldots, \lambda_r\}$.

The case $r \geq 4$ even can be worked similarly, but in this case we add an extra elliptic curve to the $r$ given ones in order to obtain the result as a consequence of the odd situation.
9. An irreducible fiber product of elliptic curves: Theorem 6

In the above main construction, the fiber product turned out to be reducible; it has two isomorphic irreducible components. In this section we describe a similar construction, but the fiber product we obtain is irreducible. Unfortunately, the genus is bigger than the previous one.

9.1. The fiber product construction. Let \((\lambda_1, \ldots, \lambda_r) \in \Delta_r, E_{\lambda_j} : y_j^2 = x_j(x_j - 1)(x_j - \lambda_j)\) and \(\pi_j : E_{\lambda_j} \to \mathbb{C}\) defined by \(\pi_j(x_j, y_j) = x_j\). The locus of branch values of \(\pi_j\) is the set \(\{\infty, 0, 1, \lambda_j\}\). Let \(X\) be the fiber product of these \(r\) pairs \((E_{\lambda_1}, \pi_1), \ldots, (E_{\lambda_r}, \pi_r)\), that is,
\[
X = \{(x, y_1, \ldots, y_r) : y_j^2 = x(x - 1)(x - \lambda_j), \ j = 1, \ldots, r\}.
\]

The curve \(X\) is irreducible but it has singular points at those points with first coordinate \(x \in \{\infty, 0, 1\}\). The Riemann surface defined by \(X\) (after desingularization) admits a group of conformal automorphisms \(H \cong \mathbb{Z}_2^r\) so that \(X/H\) is the Riemann sphere with conical points (each one of order two) at \(\infty, 0, 1, \lambda_1, \ldots, \lambda_r\). In particular, it has genus \(g = 1 + 2^{r-2}(r-1)\).

We should to proceed to see that \(JX\) is isogenous to a product of the form \(E_{\lambda_1} \times \cdots \times E_{\lambda_r} \times A_{g-r}\), where \(A_{g-r}\) is a suitable product of Jacobian varieties of elliptic/hyperelliptic Riemann surfaces (Theorem 8).

9.2. Another way to describe \(X\). Let us consider the generalized Humbert curve (see [1])
\[
F : \begin{cases}
x_1^2 + x_2^2 + x_3^2 = 0 \\
\lambda_1 x_1^2 + x_2^2 + x_3^2 = 0 \\
\lambda_2 x_1^2 + x_2^2 + x_3^2 = 0 \\
\vdots \\
\lambda_r x_1^2 + x_2^2 + x_3^2 = 0
\end{cases} \subset \mathbb{P}^9.
\]

The conditions on the parameters \(\lambda_j\) ensure that \(F\) is a non-singular projective algebraic curve, that is, a closed Riemann surface. On \(F\) we have the Abelian group \(\langle b_1, \ldots, b_{r+2} \rangle = H_0 \cong \mathbb{Z}_2^{r+2}\) of conformal automorphisms, where
\[
b_j([z_1 : \cdots : z_{r+3}]) = [z_1 : \cdots : z_{j-1} : -z_j : z_{j+1} : \cdots : z_{r+3}], \ j = 1, \ldots, r+2.
\]

Inside the group \(H_0\), the only non-trivial elements acting with fixed points are \(b_1, \ldots, b_{r+2}\) and \(b_{r+3} = b_1 b_2 \cdots b_{r+2}\). The degree \(2^{r+2}\) holomorphic map
\[
P : F \to \widehat{\mathbb{C}} : [z_1 : \cdots : z_{r+3}] \mapsto -\left(\frac{z_2}{z_1}\right)^2,
\]
is a branched regular cover with deck group being \(H_0\). The projection under \(P\) of the set of fixed points are as follows:
\[
P(\text{Fix}(b_1)) = \infty, \quad P(\text{Fix}(b_2)) = 0, \quad P(\text{Fix}(b_3)) = 1,
\]
\[
P(\text{Fix}(b_{r+2})) = \lambda_j, \quad j = 1, \ldots, r.
\]

In particular, the branch locus of \(P\) is the set \(\{\infty, 0, 1, \lambda_1, \ldots, \lambda_r\}\).

By the Riemann-Hurwitz formula, \(F\) has genus \(1 + 2^{r}(r - 1)\).

Let us consider the subgroup \(K^* = \langle b_1 b_2, b_2 b_3 \rangle \cong \mathbb{Z}_2^2\). It can be seen that \(X = F/K^*\) and, in particular, that \(X\) has genus \(g_X = 1 + 2^{r-2}(r - 1)\).
The quotient group $H_0/\langle b_1 b_2, b_2 b_1 \rangle$ induces the group of automorphisms of $X$ given by
\[
L = \langle c_1, \ldots, c_r \rangle \cong \mathbb{Z}_2^r,
\]
where
\[
c_j(x, y_1, \ldots, y_r) = (x, y_1, \ldots, y_{j-1}, -y_j, y_{j+1}, \ldots, y_r), \quad j = 1, \ldots, r.
\]

The map
\[
\pi : X \rightarrow \overline{C} : (x, y_1, \ldots, y_r) \mapsto x
\]
is a regular branched cover with $H$ as its deck group. The branch locus of $\pi$ is the set
\[
\{\infty, 0, 1, A_1, \ldots, A_r\}.
\]

**Remark 3.** With the above description, we obtain another set of equations for $X$ as
\[
\begin{align*}
  w_1^2 &= (\lambda_1 - 1)u + 1 \\
  w_2^2 &= (\lambda_2 - 1)u + 1 \\
  & \vdots \\
  w_{r-1}^2 &= (\lambda_{r-1} - 1)u + 1 \\
  w_r^2 &= -u(\lambda_ru + 1)((\lambda_r - 1)u + 1)
\end{align*}
\]
The equations for $E_{A_j}$ can also be written as (for $j = 1, \ldots, r - 1$)
\[
\begin{align*}
  w_1^2 &= (\lambda_1 - 1)u + 1 \\
  & \vdots \\
  w_{j-1}^2 &= (\lambda_{j-1} - 1)u + 1 \\
  w_j^2 &= (\lambda_j - 1)u + 1 \\
  & \vdots \\
  w_{r-1}^2 &= (\lambda_{r-1} - 1)u + 1 \\
  w_r^2 &= -u(\lambda_ru + 1)((\lambda_r - 1)u + 1)
\end{align*}
\]
and for $E_{A_j}$ as
\[
\begin{align*}
  w_1^2 &= (\lambda_1 - 1)u + 1 \\
  & \vdots \\
  w_{j-1}^2 &= (\lambda_{j-1} - 1)u + 1 \\
  w_j^2 &= (\lambda_{j+1} - 1)u + 1 \\
  & \vdots \\
  w_{r-1}^2 &= (\lambda_{r-1} - 1)u + 1 \\
  w_r^2 &= -u(\lambda_ru + 1)((\lambda_r - 1)u + 1)
\end{align*}
\]

**Lemma 4.** The only non-trivial elements of $L$ acting with fixed points are $c_1, \ldots, c_r$ and $c_{r+1} = c_1 c_2 \cdots c_r$. Moreover,
\[
\pi(\text{Fix}(c_j)) = \lambda_j, \quad j = 1, \ldots, r,
\]
\[
\pi(\text{Fix}(c_{r+1})) = \{\infty, 0, 1\}.
\]

**Proof.** A non-trivial element of $L$ has the form
\[
c(x, y_1, \ldots, y_r) = (x, (-1)^{\alpha_1}y_1, \ldots, (-1)^{\alpha_r}y_r),
\]
where $\alpha_1, \ldots, \alpha_r \in \{0, 1\}$ and $\alpha_1 + \cdots + \alpha_r > 0$.

A point $(x, y_1, \ldots, y_r) \in C$ is a fixed point of $c$ if and only if $y_j = 0$ for $\alpha_j = 1$. The equality $y_j = 0$ is equivalent to have $x \in \{\infty, 0, 1, \lambda_j\}$. The values $x \in \{\infty, 0, 1\}$ produce
fixed points for $c_{r+1}$. Also, as we are assume the values $\lambda_i$ to be different, it follows that the only possibility is to have only one $j$ with $\alpha_j = 1$.

It can be seen that, for $j = 1, \ldots, r$, $\text{Fix}(c_j)$ has cardinality $2^{r-1}$ and that $\text{Fix}(c_1 \cdots c_r)$ has cardinality $3 \times 2^{r-1}$. In particular, for $r = 3$, the surface $X$ is hyperelliptic with $c_1c_2c_3$ as its hyperelliptic involution.

9.3. **Some subgroups of $L$.** If either $1 \leq k \leq r$ is odd or $4 \leq k \leq r$ is even, and $\{i_1, \ldots, i_k\} \subset \{1, \ldots, r\}$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq r$, then we consider the subgroup

$$L_{i_1, i_2, \ldots, i_k} = < c_{i_1}, c_{i_2}, c_{i_1}c_{i_2}, \ldots, c_{i_1}c_{i_2} \cdots c_{i_k} ; j \in \{1, \ldots, r\} - \{i_1, \ldots, i_k\} > \cong \mathbb{Z}_2^{r-1}.$$

Note that for $k = 1$ we have the subgroups

$$L_j = < c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_r > \cong \mathbb{Z}_2^{r-1}.$$

If

$$Q_j : X \to E_{\lambda_j} : (x, y_1, \ldots, y_r) \mapsto (x, y_j),$$

then $Q_j$ is a regular branched cover with deck group being $L_j$. The branch locus of $Q_j$ is the set

$$\{(\lambda_j, y_j) : y_j^2 = \lambda_j(\lambda_j - 1)(\lambda_j - \lambda_i), \; i = 1, \ldots, r, \; i \neq j\}.$$

**Lemma 5.** With the above notations, the following hold for the above defined subgroups.

1. Any two such subgroups commute.
2. The quotient $X/L_{i_1, i_2, \ldots, i_k}$ is an orbifold of genus $(k + 1)/2$ if $k$ is odd and genus $(k - 2)/2$ if $k$ is even. Moreover, the underlying Riemann surface is given by the hyperelliptic curve

$$w^2 = z(z - 1)(z - \lambda_{i_1})(z - \lambda_{i_2}) \cdots (z - \lambda_{i_k}),$$

if $k$ is odd,

$$w^2 = (z - \lambda_{i_1})(z - \lambda_{i_2}) \cdots (z - \lambda_{i_k}),$$

if $k$ is even.

3. The group generated by any two different such subgroups is $L$.

**Proof.** Property (1) holds trivially as $L$ is an abelian group. Property (2) follows from the Riemann-Hurwitz formula and Lemma 4. Property (3) is clear. \hfill \square

The next result states that the sum of the genera appearing in all quotients of the form $X/L_{i_1, i_2, \ldots, i_k}$ is equal to the genus of $X$.

**Lemma 6.**

$$\sum_{k=1}^{r} \left( \frac{r}{k} \right) \frac{(1 - (-1)^k)}{2} \frac{(k + 1)}{2} + \sum_{k=4}^{r} \left( \frac{r}{k} \right) \frac{(1 + (-1)^k)}{2} \frac{(k - 2)}{2} = 1 + 2^{r-2}(r - 1).$$

**Proof.** Consider the functions

$$f_1(x) = x \frac{(1 + x)^r}{4} = \frac{1}{4} \sum_{k=0}^{r} \left( \frac{r}{k} \right) x^{k+1}.$$
\[ f_2(x) = \frac{(1 + x)^r}{4x^2} = \frac{1}{4} \sum_{k=0}^{r} \binom{r}{k} x^{k-2}, \]

and their derivatives
\[ f_2'(x) = \frac{(1 + x)^r(1 + (1 + r)x)}{4} = \frac{1}{4} \sum_{k=0}^{r} \binom{r}{k} (k + 1) x^k, \]
\[ f_2''(x) = \frac{(1 + x)^{r-1}(r - 2x(1 + x))}{4x^2} = \frac{1}{4} \sum_{k=0}^{r} \binom{r}{k} (k - 2) x^{k-3}. \]

Next, we evaluate at \( x = \pm 1 \) to obtain
\[ 2^{r-3}(r + 2) = f_2'(1) - f_2'(-1) = \frac{1}{4} \sum_{k=0}^{r} \binom{r}{k} (k + 1)(1 - (-1)^k) = \sum_{k=1}^{r} \binom{r}{k} \frac{(1 - (-1)^k)(k + 1)}{2}. \]
\[ 2^{r-3}(r - 4) = f_2''(1) + f_2''(-1) = \frac{1}{4} \sum_{k=0}^{r} \binom{r}{k} (k - 2)(1 + (-1)^k) = -1 + \sum_{k=0}^{r} \binom{r}{k} \frac{(1 + (-1)^k)(k - 2)}{2}. \]

By adding the above equalities we obtain the desired result. \( \square \)

**9.4. Decomposition of \( JX \).** We may apply Proposition 1 for \( X \) using the subgroups \( L_{i_1, \ldots, i_k} \) in order to obtain the following.

**Theorem 8.** \( JX \) is isogenous to a product of the form \( E_{h_1} \times \cdots \times E_{h_r} \times A_{g-r} \), where \( A_{g-r} \) is the product of the jacobian varieties of all elliptic/hyperelliptic Riemann surfaces \( X/L_{i_1, \ldots, i_k} \), for \( k \geq 2 \).

**9.5. Case \( r = 3 \): A construction of a 2-dimensional family of curves \( X \) of genus five with \( JX \) isogenous to the product of five elliptic curves.** In this case, the jacobian variety of the fiber product \( X \) (being of genus 5) of the three elliptic curves
\[ E_{h_1} : y^2 = x(x - 1)(x - \lambda_1), \quad E_{h_2} : y^2 = x(x - 1)(x - \lambda_2), \quad E_{h_3} : y^2 = x(x - 1)(x - \lambda_3), \]
is isogenous to the product
\[ E_{h_1} \times E_{h_2} \times E_{h_3} \times J\mathcal{S}_0, \]
where
\[ S_0 : y^2 = x(x - 1)(x - \lambda_1)(x - \lambda_2)(x - \lambda_3). \]

The corresponding subgroups of \( H \) are in this case
\[ L_1 = \langle c_2, c_3 \rangle, \quad L_2 = \langle c_1, c_3 \rangle, \quad L_3 = \langle c_1, c_2 \rangle, \quad L_{1,2,3} = \langle c_1c_2, c_1c_3 \rangle. \]

**Corollary 1.** If \( \lambda_3 = \lambda_1/\lambda_2 \), then \( JX \) isogenous to the product of 5 elliptic curves.

**Proof.** If we assume that \( \lambda_3 = \lambda_1/\lambda_2 \), then \( S \) admits the involution \((x, y) \mapsto (\lambda_1/\lambda_2, \lambda_1^{1/2} y/\lambda_2^3)\). Such an involution has exactly two fixed points. It follows that, under this restriction, \( J\mathcal{S}_0 \) is isogenous to the product of two elliptic curves. \( \square \)
**Remark 4.** Corollary 1 provide a 2-dimensional family of curves \( X \) of genus five with \( JX \) isogenous to the product of five elliptic curves.

### 9.6. Case \( r = 4 \): A construction of a 1-dimensional family of curves \( X \) of genus 13 with \( JX \) isogenous to the product of 13 elliptic curves

In this case, the jacobian variety of the fiber product \( X \) (being of genus 13) of the four elliptic curves

\[
E_{A_1} : y^2 = x(x-1)(x-A_1), \quad E_{A_2} : y^2 = x(x-1)(x-A_2),
\]

\[
E_{A_3} : y^2 = x(x-1)(x-A_3), \quad E_{A_4} : y^2 = x(x-1)(x-A_4),
\]

is isogenous to the product

\[
E_{A_1} \times E_{A_2} \times E_{A_3} \times E_{A_4} \times JS_1 \times JS_2 \times JS_3 \times JS_4 \times E_5,
\]

where

\[
S_1 : y^2 = x(x-1)(x-A_1)(x-A_2)(x-A_3),
\]

\[
S_2 : y^2 = x(x-1)(x-A_1)(x-A_2)(x-A_4),
\]

\[
S_3 : y^2 = x(x-1)(x-A_1)(x-A_3)(x-A_4),
\]

\[
S_4 : y^2 = x(x-1)(x-A_2)(x-A_3)(x-A_4),
\]

\[
E_5 : y^2 = (x-A_1)(x-A_2)(x-A_3)(x-A_4).
\]

The corresponding subgroups of \( H \) are in this case

\[
L_1 = \langle c_2, c_3, c_4 \rangle, \quad L_2 = \langle c_1, c_3, c_4 \rangle, \quad L_3 = \langle c_1, c_2, c_4 \rangle, \quad L_4 = \langle c_1, c_2, c_3 \rangle,
\]

\[
L_{1,2,3} = \langle c_1 c_2, c_1 c_3, c_4 \rangle, \quad L_{1,2,4} = \langle c_1 c_2, c_1 c_4, c_3 \rangle, \quad L_{1,3,4} = \langle c_1 c_3, c_1 c_4, c_2 \rangle,
\]

\[
L_{2,3,4} = \langle c_2 c_3, c_2 c_4, c_1 \rangle, \quad L_{1,2,3,4} = \langle c_1 c_2, c_1 c_3, c_1 c_4 \rangle.
\]

**Corollary 2.** If \( A_3 = A_1 / A_2, A_4 = A_1 (A_2 - 1) / (A_2 - A_1) \) and \( A_5 (1 + A_1) - 4 A_1 A_2 + A_1 (1 + A_1) = 0 \), then \( JX \) isogenous to the product of 13 elliptic curves.

**Proof.** If \( a_1(z) = A_1 / z \) and \( a_2(z) = A_1 (z - 1) / (z - A_1) \), then the group generated by them is isomorphic to \( \mathbb{Z}_2^2 \). Since \( a_1 \) permutes in pairs the elements in \( \{ \infty, 0, 1, A_1, A_2, A_3 \} \), it follows that \( JS_1 \) is isogenous to the product of two elliptic curves. Similarly, as \( a_2 \) permutes in pairs the elements in \( \{ \infty, 0, 1, A_1, A_2, A_4 \} \), it follows that \( JS_2 \) is isogenous to the product of two elliptic curves and as \( a_5 a_1 \) permutes in pairs the elements in \( \{ \infty, 0, 1, A_1, A_3, A_4 \} \), it follows that \( JS_3 \) is isogenous to the product of two elliptic curves. In this way, under the above assumptions, \( JX \) is isogenous to the product of 11 elliptic curves and \( JS_4 \). If we also assume that \( A_5 (1 + A_1) - 4 A_1 A_2 + A_1 (1 + A_1) = 0 \), then \( a_5(z) = A_2 (z - A_3) / (z - A_2) \) permutes in pairs the elements of the set \( \{ \infty, 0, A_2, A_3, A_4 \} \). In this case, \( JS_4 \) is also isogenous to the product of two elliptic curves \( \square \).

**Remark 5.** Corollary 2 provides a 1-dimensional family of curves \( X \) of genus 13 with \( JX \) isogenous to the product of 13 elliptic curves. Examples of values as in the above proposition are \( A_1 = 2 \) and \( A_2 = (4 + i \sqrt{2})/3 \); so \( A_3 = (4 - i \sqrt{2})/3 \) and \( A_4 = -i \sqrt{2} \).
9.7. **Case** \( r = 5 \). In this case, the jacobian variety of the fiber product \( X \) (being of genus 33) of the four elliptic curves

\[
E_{\lambda_1} : y^2 = x(x - 1)(x - \lambda_1), \quad E_{\lambda_2} : y^2 = x(x - 1)(x - \lambda_2), \quad E_{\lambda_3} : y^2 = x(x - 1)(x - \lambda_3),
\]

is isogenous to the product

\[
E_{\lambda_1} \times E_{\lambda_2} \times E_{\lambda_3} \times E_{\lambda_4} \times JS_1 \times \cdots \times JS_{10} \times JS_{11} \times E_6 \times \cdots \times E_{10},
\]

where

\[
S_1 : y^2 = x(x - 1)(x - \lambda_1)(x - \lambda_2)(x - \lambda_3), \\
S_2 : y^2 = x(x - 1)(x - \lambda_1)(x - \lambda_2)(x - \lambda_4), \\
S_3 : y^2 = x(x - 1)(x - \lambda_1)(x - \lambda_2)(x - \lambda_5), \\
S_4 : y^2 = x(x - 1)(x - \lambda_1)(x - \lambda_3)(x - \lambda_4), \\
S_5 : y^2 = x(x - 1)(x - \lambda_1)(x - \lambda_3)(x - \lambda_5), \\
S_6 : y^2 = x(x - 1)(x - \lambda_1)(x - \lambda_4)(x - \lambda_5), \\
S_7 : y^2 = x(x - 1)(x - \lambda_2)(x - \lambda_3)(x - \lambda_4), \\
S_8 : y^2 = x(x - 1)(x - \lambda_2)(x - \lambda_3)(x - \lambda_5), \\
S_9 : y^2 = x(x - 1)(x - \lambda_2)(x - \lambda_4)(x - \lambda_5), \\
S_{10} : y^2 = x(x - 1)(x - \lambda_3)(x - \lambda_4)(x - \lambda_5), \\
S_{11} : y^2 = x(x - 1)(x - \lambda_1)(x - \lambda_2)(x - \lambda_3)(x - \lambda_4)(x - \lambda_5),
\]

The corresponding subgroups of \( H \) are in this case

\[
L_1 = \langle c_2, c_3, c_4, c_5 \rangle, \quad L_2 = \langle c_1, c_3, c_4, c_5 \rangle, \quad L_3 = \langle c_1, c_2, c_4, c_5 \rangle, \\
L_4 = \langle c_1, c_2, c_3, c_5 \rangle, \quad L_5 = \langle c_1, c_2, c_3, c_4 \rangle,
\]

\[
L_{1,2,3} = \langle c_1 c_2, c_1 c_3, c_4, c_5 \rangle, \quad L_{1,2,4} = \langle c_1 c_2, c_1 c_4, c_3, c_5 \rangle, \quad L_{1,2,5} = \langle c_1 c_2, c_1 c_5, c_3, c_4 \rangle, \\
L_{1,3,4} = \langle c_1 c_3, c_1 c_4, c_2, c_5 \rangle, \quad L_{1,3,5} = \langle c_1 c_3, c_1 c_5, c_2, c_4 \rangle, \quad L_{1,4,5} = \langle c_1 c_4, c_1 c_5, c_2, c_3 \rangle, \\
L_{2,3,4} = \langle c_2 c_3, c_2 c_4, c_1, c_5 \rangle, \quad L_{2,3,5} = \langle c_2 c_3, c_2 c_5, c_1, c_4 \rangle, \quad L_{2,4,5} = \langle c_2 c_4, c_2 c_5, c_1, c_3 \rangle, \\
L_{3,4,5} = \langle c_3 c_4, c_3 c_5, c_1, c_2 \rangle, \quad L_{1,2,3,4} = \langle c_1 c_2, c_1 c_3, c_1 c_4 \rangle, \quad L_{1,2,3,5} = \langle c_1 c_2, c_1 c_3, c_1 c_5 \rangle, \\
L_{1,2,4,5} = \langle c_1 c_2, c_1 c_4, c_1 c_5 \rangle, \quad L_{1,3,4,5} = \langle c_1 c_3, c_1 c_4, c_1 c_5 \rangle, \quad L_{2,3,4,5} = \langle c_2 c_3, c_2 c_4, c_2 c_5 \rangle.
\]
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References

[1] A. Carocca, V. González, R. A. Hidalgo and R. Rodríguez. Generalized Humbert Curves. Israel Journal of Mathematics 64, No. 1 (2008), 165–192.
[2] A. Carocca, R. E. Rodríguez. Jacobians with group actions and rational idempotents. J. Algebra 30 (2006), 322–343.
[3] M. Carvacho, R. A. Hidalgo and S. Quispe. Isogenous decomposition of the Jacobian of generalized Fermat curves. http://arxiv.org/abs/1507.02903
[4] Clifford J. Earle. Some Jacobian varieties which split. Lecture Notes in Mathematics 747 (1979), 101–107.
[5] T. Ekedahl and J.-P. Serre. Exemples de courbes algébriques à jacobienne complètement décomposable. C. R. Acad. Sci. Pari Sér. I Math. 317 No. 5 (1993), 509–513.
[6] P. Gaudry and É. Schost. On the Invariants of the Quotients of the Jacobian of a Curve of Genus 2. Applied Algebra, Algebraic Algorithms and Error-Correcting Codes. Lecture Notes in Computer Science 2227 (2001), 373–386.
[7] T. Hayashida and M. Nishi. Existence of curves of genus two on a product of two elliptic curves. J. Math. Soc. Japan 17 (1965), 1–16.
[8] C. Hermite. Sur un exemple de réduction d’intégrales abéliennes aux fonctions elliptiques. Ann. Soc. Sci. Bruxelles 1 (1876), 1–16.
[9] E. Kani and M. Rosen. Idempotent relations and factors of Jacobians. Math. Ann. 284 No. 2 (1989), 307–327.
[10] H. Lange and S. Recillas. Abelian varieties with group action. J. reine angew. Math. 575 (2004), 135–155.
[11] Nakajima, Ryo. On splitting of certain Jacobian varieties. J. Math. Kyoto Univ. 47 No. 2 (2007), 391–415.
[12] J. Paulhus. Decomposing Jacobians of curves with extra automorphisms. Acta Arith. 132 No. 3 (2008), 231–244.
[13] J. Paulhus. Elliptic factors in Jacobians of hyperelliptic curves with certain automorphism groups. THE OPEN BOOK SERIES 1 (2013). Tenth Algorithmic Number Theory Symposium msp dxdoi.org/10.2140/obs.2013.1.487
[14] G. Riera and R. E. Rodríguez. The period matrix of Brings curve. Pacific J. Math. 154 No. 1 (1992), 179–200.
[15] A. M. Rojas. Group actions on Jacobian varieties. Rev. Mat. Iber. 23 (2007), 397–420.

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