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Bertrand RÉMY : Topological simplicity, commensurator super-rigidity and non-linearities of Kac-Moody groups.

Appendix by Patrick BONVIN : Strong boundaries and commensurator super-rigidity.

Résumé : Nous donnons de nouveaux arguments pour voir les groupes de Kac-Moody topologiques comme des généralisations de groupes semi-simples sur des corps locaux : ils sont produits directs de groupes topologiquement simples et leurs sous-groupes d’Iwahori sont les normalisateurs de leur pro-p sous-groupes maximaux. Nous utilisons une caractérisation dynamique des sous-groupes paraboliques pour montrer que certains groupes de Kac-Moody de type fini à immeubles fuchsiens ne sont pas linéaires. Nous montrons pour cela que la linéarité d’un groupe de Kac-Moody de type fini implique l’existence d’un plongement du groupe topologique correspondant dans un groupe de Lie simple non archimédien. Nous utilisons un théorème de super-rigidité du commensurateur prouvé en appendice par P. Bonvin.

Mots-clés : groupe de Kac-Moody, système de Tits raffiné, groupe pro-p, réseau, groupe de Lie simple non archimédien, super-rigidité du commensurateur, non linéarité, immeuble de Bruhat-Tits, immeuble hyperbolique.

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Topological simplicity, commensurator super-rigidity and non-linearities of Kac-Moody groups

Appendix by P. Bonvin: Strong boundaries and commensurator super-rigidity

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Abstract.— We provide new arguments to see topological Kac-Moody groups as generalized semisimple groups over local fields: they are products of topologically simple groups and their Iwahori subgroups are the normalizers of the pro-$p$ Sylow subgroups. We use a dynamical characterization of parabolic subgroups to prove that some countable Kac-Moody groups with Fuchsian buildings are not linear. We show for this that the linearity of a countable Kac-Moody group implies the existence of a closed embedding of the corresponding topological group in a non-Archimedean simple Lie group, thanks to a commensurator super-rigidity theorem proved in the Appendix by P. Bonvin.

Keywords: Kac-Moody group, refined Tits system, pro-$p$ group, lattice, non-Archimedean Lie group, commensurator super-rigidity, non-linearity, Bruhat-Tits building, hyperbolic building

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Introduction

This paper contains two kinds of results, according to which definition of Kac-Moody groups is adopted. The main goal is to prove non-linearities of countable Kac-Moody groups as defined by generators and relations by J. Tits [44]. The strategy we follow, based on continuous extensions of abstract group homomorphisms, leads us to prove structure results on topological Kac-Moody groups as previously defined in [40]. We first give the statements; we will provide below details and motivations for each theorem individually, and then explain the relationship between them. The first theorem basically says that infinitely many countable Kac-Moody groups are not linear over any field (Theorem 4.C.1). Its proof uses super-rigidity and dynamical properties of generalized parabolic subgroups.

**Theorem.**— Let Λ be a generic countable Kac-Moody group over a finite field with right-angled Fuchsian buildings. Then Λ is not linear over any field. Moreover for each prime number p, there are infinitely many such non-linear groups defined over a finite field of characteristic p.

The second theorem says that topological Kac-Moody groups should be seen as generalizations of semisimple groups over local fields of positive characteristic. We learnt from O. Mathieu that R.V. Moody has proved some topological simplicity results in Kac-Moody theory; unfortunately we couldn’t find any written version of this work. The statement below roughly sums up Theorem 2.A.1 and Proposition 1.B.2.

**Theorem.**— (i) A topological Kac-Moody group over a finite field is the direct product of topologically simple groups, with one factor for each connected component of its Dynkin diagram.

(ii) The Iwahori subgroups, i.e. the chamber fixators for the natural action on the building of the group, are characterized as the normalizers of the pro-p Sylow subgroups.

The motivation for both results is based on an analogy with well-known classical cases. Namely, Kac-Moody theory is usually introduced via the specific affine type. For Lie algebras, this case corresponds to semisimple Lie algebras tensorised by Laurent polynomials over the groundfield [25]. For countable Kac-Moody groups over finite fields, it corresponds to {0; ∞}-arithmetic subgroups of semisimple groups G over function fields. That general countable Kac-Moody groups over finite fields provide a generalization of arithmetic groups was observed some years ago. The main argument is that there is a natural diagonal action of such a group A on the product of two isomorphic buildings, and when the groundfield is large enough, A is a lattice of the product of (the automorphism groups of) the buildings [17], [36]. In the affine case, when A = G(F_q[t, t^{-1}]), the two buildings are the Bruhat-Tits buildings of G over the completions F_q((t)) and F_q((t^{-1})).

In the general case, it is obvious that some buildings are new, e.g. because they admit natural CAT(-1) metrics. Still, on the group side, we could imagine a situation where a well-known discrete group operates on exotic geometries. A first step to prove that this is not the case was to prove that for a countable Kac-Moody group over F_q, the only possible linearity is over a field of the same characteristic p [39]. The first quoted theorem above shows that there exist countable Kac-Moody groups over finite fields which are not linear, even in equal characteristic. Of course, the affine example of arithmetic groups shows that the answer to the linearity question cannot be stated in the equal characteristic case as for unequal characteristics. Arithmetic groups also make expect that the proof of the former case is not as easy as in the latter case.

This is indeed the case, but the proof is fruitful since it gives structure results for topological Kac-Moody groups. Such a group is defined in [30] as the closure of the non-discrete action of a countable Kac-Moody group on only one of the two buildings, forgetting the other. In [loc. cit.] it
was proved that these groups satisfy the axioms of a sharp refinement of Tits systems and that the parahoric subgroups, i.e. the facet fixators, are virtually pro-$p$. Once again, these results are *a posteriori* not surprising when considering the affine case: then the topological Kac-Moody groups correspond to groups of the form $G_F\left(\mathbb{F}_q((t))\right)$ with $G$ as above. The second quoted theorem makes deeper the analogy between topological Kac-Moody groups and semisimple groups over local fields of positive characteristic – see [1.3] for further arguments.

We can now explain the connection between the two results. The general strategy of the proof consists in strengthening the linearity assumption for a countable Kac-Moody group to obtain a closed embedding of the corresponding topological Kac-Moody group into a simple non-Archimedean Lie group. The main tool is a very general commensurator super-rigidity, stated by M. Burger [13] according to ideas of G. Margulis [29] and proved in the Appendix by P. Bonvin. We found the idea to use super-rigidity in order to disprove linearity in a paper written by A. Lubotzky, Sh. Mozes and R.J. Zimmer [27]. In Sect. 3, we prove the following dichotomy.

**Theorem.**— Let $\Lambda$ be a countable Kac-Moody group with connected Dynkin diagram and large enough groundfield. Under mild assumptions, either $\Lambda$ is not linear or the corresponding topological group is a closed subgroup of a simple non-Archimedean Lie group $G$, with equivariant embedding of the vertices of the Kac-Moody building to the vertices of the Bruhat-Tits building of $G$.

To prove this result we need further structure results about closures of Levi factors: these groups have a Tits sub-system, are virtually products of topologically simple factors and their buildings naturally appear, via an inessential geometric realization, in the building of the ambient Kac-Moody group [2.C.1].

This criterion is used to obtain complete non-linearities by proving that some topological Kac-Moody groups are not closed subgroups of Lie groups. We concentrate at this stage on Kac-Moody groups with right-angled Fuchsian buildings, the most well-understood hyperbolic buildings. We define generalized parabolic subgroups as boundary point stabilizers, in complete analogy with the symmetric space or Bruhat-Tits building case. The dynamical approach to these groups is the last argument. We use G. Prasad’s work showing that to each suitable element of a non-Archimedean semisimple group is attached a proper parabolic subgroup [35]. These results, as well as their analogues for groups with right-angled Fuchsian buildings, enable us to exploit the incompatibility between the geometries of CAT($-1$) Kac-Moody and Euclidean Bruhat-Tits buildings [4.C.1]. This is then geometrically explained in the framework of dynamics of translations on group-theoretical (or Furstenberg) compactifications of buildings [4.C.2].

Discrete group-theorists may object that a direct way to disprove linearity of finitely generated groups is to disprove their residual finiteness. We chose the opposite point of view, at least because it led us to prove structure results for a new class of topological groups. Now that we know that some Kac-Moody groups are non-linear, we at last feel comfortable to ask ourselves whether some of them are non-residually finite, or even simple. For this question, the work by M. Burger and Sh. Mozes [15] on lattices of products of trees may be relevant. Moreover the work by R. Pink on compact subgroups of non-Archimedean Lie groups [33] may lead to a complete answer to the linearity question of countable Kac-Moody groups, purely in terms of the Dynkin diagram (and the size of the finite groundfield).

This article is written as follows. In Sect. 1, we provide some references on Kac-Moody groups as defined by J. Tits, and recall some facts, mainly of combinatorial nature, on topological Kac-Moody groups, as well as the analogy with algebraic groups. The new result here is the intrinsic characterization of Iwahori subgroups. The discrete group point of view is also briefly discussed. In Sect. 2,
we prove the topological simplicity theorem, based on Tits system and pro-$p$ group arguments. We explain why the proof doesn’t work for countable groups, but also why some other simplicity results in this case are not excluded. In Sect. 3, we use the commensurator super-rigidity theorem proved in the Appendix by P. Bonvin to strengthen the linearity assumption as in the last theorem above: the linearity of a countable Kac-Moody groups leads to a closed embedding of the corresponding topological group in an algebraic one. In Sect. 4, we define and analyze generalized parabolic subgroups of topological Kac-Moody groups with right-angled Fuchsian buildings. After proving a decomposition involving a generalized unipotent radical, we turn to the dynamical study of these groups, in the spirit of G. Prasad’s work. We conclude with the proof of the non-linearity theorem, followed by a geometric explanation in terms of compactifications of buildings. P. Bonvin was kind enough to write an appendix to this paper. He wrote down the proof of a general commensurator super-rigidity which is used in Sect. 3.

Let us finally state a convention for group actions. If a group $G$ acts on a set $X$, the (pointwise) stabilizer of a subset $Y \subset X$ is called its fixator and is denoted by $\text{Fix}_G(Y)$. The classical (global) stabilizer is denoted by $\text{Stab}_G(Y)$. When $Y$ is a facet of a building $X$ and $G$ is a type preserving group of automorphisms, the two notions coincide. The notation $G|_Y$ refers to the group obtained from $G$ by factoring out the kernel of the action on a $G$-stable subset $Y \subset X$.

I would like to thank M. Bourdon, M. Burger and Sh. Mozes for their constant interest in this approach to Kac-Moody groups, as well as the audiences of talks at ETH Zürich (Advances in Rigidity Theory, June 2002, M. Burger and A. Katok organizers) and at CIRM, Luminy (Geometry and Dynamics of Groups, July 2002, M. Bourdon and L. Potyagailo organizers) for motivating questions and hints. At last I am glad to express my deep gratitude to G. Rousseau; his careful reading of a previous version of this paper enabled me to remove many inaccuracies and mistakes.

1 Structure theorem

We review some properties of topological Kac-Moody groups, keeping in mind the analogy with semisimple groups over local fields of positive characteristic. We quote results of combinatorial nature, to be used to prove the topological simplicity theorem, and we prove some results on pro-$p$ Sylow subgroups. Results on lattices are also shortly recalled.

1.A Topological Kac-Moody groups. — Topological Kac-Moody groups were introduced in [40, 1.B]. Their geometric definition uses the buildings naturally associated to the countable Kac-Moody groups defined over finite fields.

1.A.1 According to J. Tits [41, 3.6], a split Kac-Moody group is defined by generators and Steinberg relations once a Kac-Moody root datum and a groundfield $K$ are given. A generalized Cartan matrix is the main ingredient of a Kac-Moody root datum, the other part determining the maximal split torus of the group. More precisely, what J. Tits defines by generators and relations is a functor on rings. When the generalized Cartan matrix is a Cartan matrix in the usual sense (which we call of finite type), the functor coincides over fields with the functor of points of a Chevalley-Demazure group scheme.

An almost split Kac-Moody group is the group of fixed points of a Galois action on a split group [37, §11]. Let $\Lambda$ be an almost split Kac-Moody group. Any such group satisfies the axioms of a twin root datum [37, §12]. This is a group combinatorics sharply refining the structure of a $BN$-pair, and the associated geometry is a pair of twinned buildings, conventionnally one for each sign [46].
We denote by $X$ (resp. $X_-$) the positive (resp. negative) building of $A$. The group $A$ acts diagonally on $X \times X_-$ in a natural way. We choose a pair of opposite chambers $R \subset X$ and $R_- \subset X_-$, which defines a pair of opposite apartments $A \subset X$ and $A_- \subset X_-$. We call $R$ the standard positive chamber and $A$ the standard positive apartment.

**Examples.**— 1) The group $SL_n(K[t,t^{-1}])$ is a Kac-Moody group of affine type. The generalized Cartan matrix is $\tilde{A}_{n-1}$ and the groundfield is $K$. More generally, values of Chevalley schemes over rings of Laurent polynomials are Kac-Moody of affine type [11 I.1 example 3]. 2) The group $SU_3(F_q[t,t^{-1}])$ is an almost split Kac-Moody group of rank one. The so-obtained geometry is a semi-homogeneous twin tree of valencies $1 + q$ and $1 + q^3$ [35 3.5].

**Definition.**— (i) We call $\Gamma$ the fixator of the negative chamber $R_-$, i.e. $\Gamma := \text{Fix}_A(R_-)$. (ii) We call $\Omega$ the fixator of the positive chamber $R$, i.e. $\Omega := \text{Fix}_A(R)$.

**Remark.**— In the above quoted references, the groups $\Lambda$, $\Gamma$ and $\Omega$ are denoted by $G$, $B_-$ and $B$, respectively. The reason is that the group $\Omega$ (resp. $\Gamma$) is the Borel subgroup of the positive (resp. negative) Tits system of the twin root datum of $\Lambda$ [40, 37 §7].

**Example.**— For $\Lambda = SL_n(K[t,t^{-1}])$, a natural choice of $R$ and $R_-$ defines the group $\Omega$ as the subgroup of $SL_n(K[t])$ which reduces to upper triangular matrices modulo $t$, and $\Gamma$ is the subgroup of $SL_n(K[t^{-1}])$ which reduces to lower triangular matrices modulo $t^{-1}$.

In this article, we are not interested in Tits group functors whose generalized Cartan matrices are of finite type, whose values over finite fields are finite groups of Lie type. Kac-Moody groups of affine type are seen as a guideline to generalize of classical results about algebraic and arithmetic groups. We are mainly interested in Kac-Moody groups which do not admit any natural matrix interpretation, and we want to understand to what extent these new groups can be compared to the obviously linear ones.

**1.A.2** We now define topological groups – see [40 1.B] for a wider framework.

**Assumption.**— In this article, $\Lambda$ is a countable Kac-Moody group over the finite field $F_q$ of characteristic $p$ with $q$ elements, i.e. $\Lambda$ is the group of rational points of an almost split Kac-Moody group with infinite Weyl group $W$.

The kernel of the $\Lambda$-action on $X$ is the finite center $Z(\Lambda)$, and the group $\Lambda/Z(\Lambda)$ still enjoys the structure of twin root datum [40 Lemma 1.B.1]. Another consequence of the finiteness of the groundfield is that the full automorphism group $Aut(X)$ is an uncountable totally disconnected locally compact group.

**Definition.**— (i) We call topological Kac-Moody group (associated to $\Lambda$) the closure in $Aut(X)$ of the group $\Lambda/Z(\Lambda)$. We denote it by $\overline{\Lambda}$.

(ii) We call parahoric subgroup (associated to $F$) the fixator in $\overline{\Lambda}$ of a given facet $F$. We denote it by $\overline{\Lambda}_F$. When the facet is a chamber, we call the corresponding group an Iwahori subgroup.

**Remark.**— The group $\overline{\Lambda}$ is so to speak a completion of the group $\Lambda$. Recall that the $\Lambda$-action on the single building $X$ is not discrete since the stabilizers are parabolic subgroups with infinite unipotent radical [37 Theorem 6.2.2].

**Example.**— For $\Lambda = SL_n(K[t,t^{-1}])$, $X$ is the Bruhat-Tits building of $SL_n(F_q((t)))$, that is a Euclidean building of type $\tilde{A}_{n-1}$. If $\mu_n(F_q)$ denotes the $n$-th roots of unity in $F_q$, the image $\Lambda/Z(\Lambda)$ of $SL_n(F_q[t,t^{-1}])$ under the action on $X$ is $SL_n(F_q[t,t^{-1}])/\mu_n(F_q)$ and the completion $\overline{\Lambda}$ is $\text{PSL}_n(F_q((t))) = SL_n(F_q((t)))/\mu_n(F_q)$. 
1.B Refined Tits system and virtual pro-p-ness of parahoric subgroups.— The reference for this subsection is [40] 1.C.

1.B.1 Let us state [40] Theorem 1.C.2] showing that topological Kac-Moody groups generalize semisimple groups over local fields of positive characteristic.

Theorem.— Let $\Lambda$ be an almost split Kac-Moody group over $F_q$ and $\overline{\Lambda}$ be its associated topological group. Let $R \subset A$ be the standard chamber and apartment of the building $X$ associated to $\Lambda$. We denote by $B$ the standard Iwahori subgroup $\overline{A}_R$ and by $W_R$ the Coxeter group associated to $A$, generated by reflections along the panels of $R$.

(i) The topological Kac-Moody group $\overline{\Lambda}$ enjoys the structure of a refined Tits system with abstract Borel subgroup $B$ and Weyl group $W_R$, which is also the Weyl group of $\Lambda$.

(ii) Any spherical facet-fizaror $\overline{A}_F$ is a semidirect product $M_F \ltimes \hat{U}_F$ where $M_F$ is a finite reductive group of Lie type over $F_q$ and $\hat{U}_F$ is a pro-p group. In particular, any parahoric subgroup is virtually pro-p. □

Remarks.— 1) The group $\Lambda$ is strongly transitive on the building $X$, i.e. transitive on the pairs of chambers at given combinatorial (or $W$-) distance from one another [41] §5]. This implies that $\overline{\Lambda}$ is strongly transitive on $X$ too, and that $X$ is also the building associated to the above Tits system of $\overline{\Lambda}$.

2) Refined Tits system were defined by V. Kac and D. Peterson [21]. For twin root data, there are two relevant standard Borel subgroups playing symmetric roles. For refined Tits systems, only one conjugacy class of Borel subgroups is introduced. The latter structure is abstractly implied by the former one [37] 1.6], but it applies to a strictly wider class of groups, e.g. $SL_n(F_q((t)))$ satisfies the axioms of a refined Tits system while it doesn’t admit a twin root datum structure.

Let us also briefly mention how the groups $M_F$ and $\hat{U}_F$ are defined geometrically. We first note that in the CAT(0) realization of buildings only spherical facets appear. We denote by $St(F)$ the star of a facet $F$, that is the set of chambers whose closure contains $F$. Theorem 6.2.2 of [37] applies and we have a Levi decomposition $A_F := \text{Stab}_A(F) = M_F \ltimes \hat{U}_F$, where $M_F$ is a Kac-Moody subgroup for a Cartan submatrix of finite type and $U_F$ fixes pointwise $St(F)$. The group $\hat{U}_F$ is the closure of $U_F$ in $\overline{\Lambda}$, hence it fixes $St(F)$ too. Moreover by [loc. cit., Proposition 6.2.3], $St(F)$ is a geometric realization of the finite building attached to $M_F$, and the action by $M_F$ is the standard one. Therefore the image of the surjective homomorphism $\pi_F : \overline{A}_F \to M_F$ associated to $\overline{A}_F = M_F \ltimes \hat{U}_F$ gives the local action of $\overline{A}_F$ on $St(F)$.

These facts are analogues of classical results in Bruhat-Tits theory [12] Proposition 5.1.32]. Namely, any facet in the Bruhat-Tits building of a semisimple group $G$ over a local field $k$ defines an integral structure over the valuation ring $O$ of $k$. The reduction of the $O$-structure modulo the maximal ideal is a semisimple group over the residue field, whose building is the star of the facet. The integral points of the $O$-structure act on it via the natural action of the reduction. The splitting $\overline{A}_F = M_F \ltimes \hat{U}_F$ of a parahoric subgroup as a semidirect product is specific to the case of valuated fields in equal characteristic, and in the case of locally compact fields this only occurs in characteristic $p$.

Example.— Let $v$ be a vertex in the Bruhat-Tits building of $SL_n(F_q((t)))$. Then its fixator is isomorphic to $SL_n(F_q[[t]])$ and its star is isomorphic to the building of $SL_n(F_q)$. The subgroup $\hat{U}_v$ is the first congruence subgroup of $SL_n(F_q[[t]])$, i.e. the matrices reducing to the identity modulo $t$. The above reduction corresponds concretely to moding out by $\hat{U}_v$, and the Iwahori subgroups fixing a chamber in $St(v)$ are the subgroups reducing to a Borel subgroup of $SL_n(F_q)$, e.g. reducing to the upper triangular matrices.
In our case, the Bruhat decomposition and the rule to multiply double classes [6 IV.2] have topological consequences.

**Corollary.**— A topological Kac-Moody group is compactly generated.

**Proof.**— Let $\Lambda$ be such a group and let $B = \overline{\Lambda}_R$ be the standard Iwahori subgroup. Then $\Lambda$ is generated by $B$ and by the compact double classes $BsB$, when $s$ runs over the finite set of reflections along the panels of $R$. $\Box$

**1.B.2** Pro-$p$ Sylow subgroups of totally disconnected groups are defined in [42 I.1.4] for instance. The following proposition is suggested by classical results on pro-$p$ subgroups of non-Archimedean Lie groups, e.g. [31 Theorem 3.10].

**Proposition.**— (i) Given any chamber $R$, the group $\hat{U}_R$ of the decomposition $\Lambda_R = M_R \rtimes \hat{U}_R$ is the unique pro-$p$ Sylow subgroup of the Iwahori subgroup $\overline{\Lambda}_R$.

(ii) Let $K$ be a pro-$p$ subgroup of $\Lambda$. Then there is a chamber $R$ such that $K$ lies in the pro-$p$ Sylow subgroup $\hat{U}_R$ of $\overline{\Lambda}_R$.

(iii) The pro-$p$ Sylow subgroups of $\Lambda$ are precisely the subgroups $\hat{U}_R$ when $R$ ranges over the chambers of the building $X$; they are all conjugate.

(iv) The Iwahori subgroups of $\Lambda$ are intrinsically characterized as the normalizers of the pro-$p$ Sylow subgroups of $\Lambda$.

**Proof.**— (i). By quasi-splitness of an almost split Kac-Moody group $\Lambda$ over $F_q$ [37 13.2], the Levi factor $M_R$ of a chamber fixator in $\Lambda$ is the $F_q$-points of a torus. Therefore its order is prime to $p$ and we conclude by the decomposition $\Lambda_R = M_R \rtimes \hat{U}_R$.

(ii). Let $K$ be a pro-$p$ subgroup of $\Lambda$. Since it is compact, it fixes a spherical facet $F$ [37 4.6] and by **1.B.1** (ii) we can write $K < \Lambda_F = M_F \rtimes \hat{U}_F$. Let us look at the local action $\pi_F : \Lambda_F \to M_F$ [1.B.1]. By the Bruhat decomposition in split $BN$-pairs, the $p$-Sylow subgroups of finite reductive groups of Lie type are the unipotent radicals of the Borel subgroups [4 B Corollary 3.5], so the $p$-subgroup $\pi_F(K)$ of $M_F$ is contained in some Borel subgroup of $M_F$, hence fixes a chamber $R$ in $\text{St}(F)$. Since $\hat{U}_F$ fixes pointwise $\text{St}(F)$, the whole subgroup $K$ fixes $R$ and we conclude by (i).

(iii). The first assertion follows immediately from (ii), and the second one follows from the transitivity of $\Lambda$ on the chambers of $X$.

(iv). According to (iii), it is enough to show that we have $B = N_{\Lambda}(\hat{U}_R)$ for the Iwahori subgroup $B = \overline{\Lambda}_R$ fixing the standard positive chamber $R$. By **1.B.1** (ii), we know that $B < N_{\Lambda}(\hat{U}_R)$. By [6 IV.2.5] the normalizer $N_{\Lambda}(\hat{U}_R)$ is an abstract parabolic subgroup of the Tits system of $\Lambda$ with abstract Borel subgroup $B$. If $N_{\Lambda}(\hat{U}_R)$ were bigger than $B$, it would contain a reflection conjugating a positive root group to a negative one. But this is in contradiction with axiom (RT3) of refined Tits systems [24], so we have $B = N_{\Lambda}(\hat{U}_R)$. $\Box$

**Remark.**— M. Sapir pointed out to us that knowing whether the pro-$p$ Sylow subgroups of some $\Lambda$ are new is an interesting question too. Note that it is a very hard problem (solved by E. Zelmanov) to show that the free pro-$p$ group is non-linear in low dimension.

**1.C Lattices and generalized arithmeticity.**— We briefly discuss existence and generalized arithmeticity of lattices in topological Kac-Moody groups.
1.C.1 We keep the almost split Kac-Moody group $\Lambda$ over $F_q$. The Weyl group $W$ is infinite, and we denote by $W(t)$ its growth series $\sum_{w \in W} t^\ell(w)$.

**Theorem.**— Assume that $W(1/q) < \infty$. Then $\Lambda$ is a lattice of $X \times X_-$ for its diagonal action, and for any point $x_- \in X_-$ the subgroup $\Lambda_{x_-} = \text{Fix}_\Lambda(x_-)$ is a lattice of $X$. These lattices are not uniform. □

This result was independently proved in [17] and in [36]. In the case of $\text{SL}_n(F_q[t, t^{-1}])$, the lattices of the form $\Lambda_{x_-}$ are all commensurable to the arithmetic lattice $\text{SL}_n(F_q[t^{-1}])$ in $\text{SL}_n(F_q(t))$. Recall that by Margulis commensurator criterion [17], Theorem 6.2.5, a lattice in a semisimple Lie group $G$ is arithmetic if and only if its commensurator is dense in $G$. Taking this characterization as a definition for general situations, [40], Lemma 1.B.3 (ii) says that the groups $\Lambda_{x_-}$ are arithmetic lattices of $\overline{\Gamma}$ by the very definition of this topological group. Here is the statement, whose proof is based on refined Tits system arguments.

**Lemma.**— For any $x_- \in X_-$, we have: $\Lambda < \text{Comm}_{\overline{\Gamma}}(\Lambda_{x_-})$. □

**Remark.**— Some lattices may have big enough commensurators to be arithmetic in $\text{Aut}(X)$, meaning that the commensurators are dense in $\text{Aut}(X)$. This is the case for the Nagao lattice $\text{SL}_2(F_q(t^{-1}))$ in the full automorphism group of the $q + 1$-regular tree [32]. The proof can be formalized and extended to exotic trees admitting a Moufang twinning [2].

1.C.2 A way to produce lattices in automorphism groups of cell-complexes is to take fundamental groups of complexes of groups [10] III.C], the point being then to recognize the covering space. A positive result is the following – see [8]: let $R$ be a regular right-angled polygon in the hyperbolic plane $H^2$ and $q := \{q_i\}_{1 \leq i \leq r}$ be a sequence of integers $\geq 2$. (When $q$ is constant, we replace $q$ by its value $q$.) Then there exists a unique right-angled Fuchsian building $I_{r,1+q}$ with apartments isomorphic to the tiling of $H^2$ by $R$, and such that the link at any vertex of type $\{i; i + 1\}$ is the complete bipartite graph of parameters $(1 + q_i, 1 + q_{i+1})$. The so-obtained lattices are uniform and abstractly defined by: $I_{r,1+q}^+ = \langle \{\gamma_i\}_{i \in \mathbb{Z}/r} : \gamma_i^{q_i+1} = 1 \text{ and } [\gamma_i, \gamma_{i+1}] = 1 \rangle$. This uniqueness is a key argument to prove [40], Proposition 5.C.]

**Proposition.**— For any prime power $q$, there exists a non-uniquely defined Kac-Moody group $\Lambda$ over $F_q$ whose building is $I_{r,1+q}$. Moreover we can choose $\Lambda$ such that its natural image in $\text{Aut}(I_{r,1+q})$ contains the uniform lattice $I_{r,1+q}^+$. □

This result says that the buildings $I_{r,1+q}$ are relevant to both Kac-Moody theory, and generalized hyperbolic geometry since they carry a natural CAT(−1)-metric. They provide the most well-understood infinite family of exotic Kac-Moody buildings (indexed by $r \geq 5$ when $q$ is fixed). The corresponding countable groups $\Lambda$ are actually typical groups to which our non-linearity result applies (4.C). We study these buildings more carefully in 4.A. Finally, combining [39], Theorem 4.6 and [7] leads to a somewhat surprising situation, with coexistence of lattices with sharply different properties [40], Corollary 5.C].

**Corollary.**— Whenever $q$ is large enough and $r$ is even and $\gg q$, the topological group $\overline{\Lambda}$ associated to the above $\Lambda$ contains both uniform Gromov-hyperbolic lattices embedding convex-cocompactly into real hyperbolic spaces, and non-uniform Kac-Moody lattices which are not linear in characteristic $\neq p$, containing infinite groups of exponent dividing $p^2$. □

**Remark.**— Determining the commensurator of the uniform lattice $I_{r,1+q}^+$ hence deciding its arithmeticity in $\text{Aut}(I_{r,1+q})$, is an open question.
2 Topological simplicity theorem

We prove that topological Kac-Moody groups as previously defined are direct products of topologically simple groups. In view of existence of congruence subgroups in the affine case, the proof is expected not to work for countable Kac-Moody groups. A failing argument and some open problems are discussed, and some results on Levi factors and homomorphisms to non-Archimedean groups are proved.

2.A Topological simplicity. — Here is a further argument supporting the analogy with semisimple algebraic groups over local fields of positive characteristic.

2.A.1 As for simplicity of classical groups, we need to assume the groundfield large enough, because in our proof we need simplicity of some rank-one finite groups of Lie type.

Theorem. — Let $\Lambda$ be a countable Kac-Moody group which is almost split over the finite field $\mathbf{F}_q$, with $q > 4$. We assume that $\Lambda$ is generated by its root groups, e.g. simply connected.

(i) If the Dynkin diagram of $\Lambda$ is connected, the associated topological Kac-Moody group $\bar{\Lambda}$ is topologically simple.

(ii) For an arbitrary Dynkin diagram, the group $\bar{\Lambda}$ is a direct product of topologically simple groups, each factor being the topological Kac-Moody group associated to a connected component of the Dynkin diagram.

Proof. — (ii). The building of a Kac-Moody group $\Lambda$ is defined as a gluing $X := A \times A \sim$, where $A$ is the model for an apartment, i.e. the CAT(0) realization of the Coxeter complex of the Weyl group $W$. The $A$-action comes from factorizing the map $A \times A \times A \to A \times A$ which sends $(\lambda', \lambda, x)$ to $(\lambda' \lambda, x)$ [37, §4]. The rule by which the Coxeter diagram of $W$ is deduced from the Dynkin diagram [44, 3.1] implies that irreducible factors correspond to connected components of the diagram. The model $A$ is then the direct product of the models for the Coxeter complexes of the irreducible factors of $W$. By the defining relations of $\Lambda$ [44, 3.6], a root group indexed by a root in the subsystem associated to a given connected component of the Dynkin diagram centralizes a root group arising from another connected component. By the relationship between buildings and Tits systems [41, §5], the Kac-Moody subgroup defined by a given connected component acts trivially on a factor of the building $X$ arising from any other connected component. Therefore proving (ii) is reduced to proving (i).

(i). Let $B = \bar{\Lambda}_R$ be the standard Iwahori subgroup. By Theorem 1.B.1(i), it is the Borel subgroup of a Tits system with the same Coxeter system as the one for $\Lambda$. By the Kac-Moody analogue of Lang’s theorem [37, 13.2.2], the Levi factor $M_R$ in $B = M_R \ltimes \hat{U}_R$ is a maximally split maximal torus $T$ of $\Lambda$, i.e. the rational points of a finite $\mathbf{F}_q$-torus.

Let $I$ be the indexing set of the simple roots of $\Lambda$ and let $G_i$ be the standard semisimple Levi factor of type $i \in I$. The group $G_i$ is a finite almost simple group of Lie type generated by the root groups attached to the simple root $a_i$ and its opposite [37, 6.2]. By our assumption they generate $\Lambda$ as an abstract group, hence $\bar{\Lambda}$ as a topological group. Note that $G_i$ has no non-trivial abelian quotient, and it has no quotient isomorphic to a $p$-group either – see Remark 2 below.

We isolate the remaining arguments in a lemma also used to prove 2.C.1(iv).

Lemma. — Let $G$ be a topological group acting continuously and strongly transitively by type-preserving automorphisms on a building $X$ with irreducible Weyl group. We denote by $\mathcal{B}$ a chamber fixator and we assume that it is an abelian-by-pro-$p$ extension. We also assume that $G$ is topologically generated by finite subgroups admitting no non-trivial quotient isomorphic to any abelian or $p$-group. Then any proper closed normal subgroup of $G$ acts trivially on $X$.
This proves the theorem by setting \( G := \Lambda, \mathcal{B} := \Lambda_R = T \rtimes \hat{U}_R \) and by choosing \( \{G_i\}_{i \in I} \) as generating family of subgroups, since by definition \( \Lambda \) acts faithfully on \( X \). \( \square \)

Let us now prove the lemma.

**Proof.**— By [11] §5, \( G \) admits an irreducible Tits system with \( \mathcal{B} \) as abstract Borel subgroup. Assume we are given a closed normal subgroup \( H \triangleleft \mathcal{G} \). Then by [11] IV.2.7, we have either \( H.\mathcal{B} = \mathcal{B} \) or \( H.\mathcal{B} = G \). The first case implies \( H < \mathcal{B} \), and since \( H \) is normal in \( G \):

\[
H < \bigcap_{g \in G} g \text{Fix}_G(R)g^{-1} = \text{Fix}_G(\bigcup_{g \in G} gR) = \text{Fix}_G(\bigcup_{g \in G} g\overline{R}) = \text{Fix}_G(X),
\]

because \( G \) is type-preserving and transitive on the chambers of the building \( X \).

From now on, we assume that \( H \) doesn’t act trivially on \( X \). By the previous point, this implies that we have the identification of compact groups: \( G/H \cong \mathcal{B}/(\mathcal{B} \cap H) \). We denote by \( 1 \rightarrow \hat{\mathcal{U}} \rightarrow \mathcal{B} \rightarrow \mathcal{T} \rightarrow 1 \) the extension of the assumption, and we consider the homomorphism \( \hat{\mathcal{U}} \rightarrow \mathcal{B}/(\mathcal{B} \cap H) \) sending \( u \) to \( u(\mathcal{B} \cap H) \). Its kernel is \( \hat{\mathcal{U}} \cap H \), so we have an injection \( \hat{\mathcal{U}}/(\hat{\mathcal{U}} \cap H) \rightarrow \mathcal{B}/(\mathcal{B} \cap H) \), and since \( \hat{\mathcal{U}} \) is normal in \( \mathcal{B} \), so is \( \hat{\mathcal{U}}/(\hat{\mathcal{U}} \cap H) \) in \( \mathcal{B}/(\mathcal{B} \cap H) \). Then we consider the composition of surjective homomorphisms

\[
\mathcal{B} \rightarrow \mathcal{B}/(\mathcal{B} \cap H) \rightarrow \frac{\mathcal{B}/(\mathcal{B} \cap H)}{\hat{\mathcal{U}}/(\hat{\mathcal{U}} \cap H)}.
\]

Its kernel contains \( \hat{\mathcal{U}} \), which shows that \( \frac{\mathcal{B}/(\mathcal{B} \cap H)}{\hat{\mathcal{U}}/(\hat{\mathcal{U}} \cap H)} \) is a quotient of the abelian group \( \mathcal{T} \). Therefore if we denote by \( \{G_i\}_{i \in I} \) the generating family of subgroups of the assumptions, the image of each \( G_i \) under \( G \rightarrow G/H \cong \mathcal{B}/(\mathcal{B} \cap H) \rightarrow \frac{\mathcal{B}/(\mathcal{B} \cap H)}{\hat{\mathcal{U}}/(\hat{\mathcal{U}} \cap H)} \) is trivial.

In other words, the image of \( G_i \) under \( G \rightarrow G/H \cong \mathcal{B}/(\mathcal{B} \cap H) \) is a finite subgroup of the group \( \hat{\mathcal{U}}/(\hat{\mathcal{U}} \cap H) \). But the group \( \hat{\mathcal{U}}/(\hat{\mathcal{U}} \cap H) \) is pro-\( p \) since it is the quotient of a pro-\( p \) group by a closed normal subgroup [19] 1.11. Therefore, once again in view of the possible images of \( G_i \), we must have: \( G_i/(G_i \cap \hat{\mathcal{U}}) = \{1\} \), that is \( G_i < H \). Since the groups \( G_i \) topologically generate \( G \), we have: \( H = G \). \( \square \)

**Example.**— In order to see which classical result is generalized here, we take \( A = \text{SL}_n(\mathbb{F}_q[[t, t^{-1}]]) \). Let \( \mu_n(\mathbb{F}_q) \) be the \( n \)-th roots of unity in \( \mathbb{F}_q \). The image of \( \text{SL}_n(\mathbb{F}_q[[t, t^{-1}]]) \) for the action on its positive Euclidean building is \( \text{SL}_n(\mathbb{F}_q[[t, t^{-1}]])/\mu_n(\mathbb{F}_q) \). Though \( \text{SL}_n(\mathbb{F}_q[[t, t^{-1}]]) \) does not come from a simply connected Kac-Moody root datum, it is generated by standard rank-one Levi factors. The lemma says that the closed normal subgroups of \( \text{SL}_n(\mathbb{F}_q[[t]]) \) are central, i.e. in \( \mu_n(\mathbb{F}_q) \).

**Remarks.**— 1) An argument in the proof is that the \( \mathbb{F}_q \)-almost simple finite Levi factors of the panel fixators generate the countable group \( A \) and don’t admit any quotient isomorphic to an abelian or a \( p \)-group. The same proof works if the panel fixators are replaced by a generating family of facet fixators with the same property on quotients. For \( \text{SL}_3(\mathbb{F}_2[[t]]) \) the Levi factors of the panels fixators are solvable \( \simeq \text{SL}_2(\mathbb{F}_2) \), but the proof works with the Levi factors for the three standard vertices, because these groups are isomorphic to \( \text{SL}_3(\mathbb{F}_2) \).

2) Assuming \( q > 4 \) is a way to have this property on quotients for any facet fixator since nonsimplicity of rational points of adjoint \( \mathbb{F}_q \)-simple groups occurs only for \( q \leq 4 \) [13] 11.2.2 and 14.4.1. Indeed, let \( G \) be the group generated by the \( p \)-Sylow subgroups of the rational points of an almost simple \( \mathbb{F}_q \)-group with \( q > 4 \), and let \( H = \text{G}/\text{K} \) be a quotient. By simplicity, \( K/\text{Z}(G) \) equals \( \{1\} \) or \( G/\text{Z}(G) \). The first case implies that \( H \) surjects onto the non-abelian simple group \( G/\text{Z}(G) \). In the second case we have \( G = K \cdot \text{Z}(G) \), showing that \( H \) is a quotient of \( \text{Z}(G) \) of order prime to \( p \). If \( U \) denotes a \( p \)-Sylow subgroup of \( G \) then \( U/(U \cap K) \) is trivial in \( H \), implying that \( K \) contains the \( p \)-Sylow subgroups of \( G \), hence equals \( G \).
In [39] it is proved that the group $\Lambda$ cannot be linear in characteristic $\neq p$ (e.g. in characteristic 0), meaning that no homomorphism $\Lambda \to \text{GL}_N(k)$ hence no homomorphism $\overline{\Lambda} \to \text{GL}_N(k)$ can be injective. Combined with the topological simplicity of $\overline{\Lambda}$, this leads to:

**Corollary.**— Let $k$ be a local field of characteristic $\neq p$ and let $G$ be a linear algebraic group over $k$. Then any continuous homomorphism $\varphi : \overline{\Lambda} \to G(k)$ has trivial image. $\square$

**Remark.**— According to [40, Theorem 4.E.2], there exist generalizations of Kac-Moody groups with Weyl groups of arbitrarily large rank and with several groundfields. The consequence of the latter point is the complete non-linearity of some of these groups [loc. cit., Theorem 5.B]. It is expectable that the corresponding topological groups are topologically simple, but the pro-$p$-ness argument in the proof of Lemma 2.A.1 is not available. This would lead to the same corollary, yet without any restriction on the target characteristic.

**2.B Discussion of the proof for countable groups.**— The proof of Lemma 2.A.1 is inspired by [6, IV.2.7], where an abstract simplicity criterion is derived from properties of Tits systems. We show why the proof doesn’t work for countable Kac-Moody groups, which will lead us to natural questions about the latter groups.

**2.B.1** The basic idea is to require the irreducibility of a Tits system, and two further assumptions involving the unipotent radical $U$ of the abstract Borel subgroup. The first one is a generation assumption on $U$ and commutator subgroups. The second one – property (R) of [6, IV.2.7] – is technical, and satisfied by solvable or simple groups. A consequence of this result is a conceptual proof of the abstract simplicity of $\text{PSL}_n$ over (large enough) fields.

**Remark.**— In our situation, the Borel subgroup of the Tits system is $\mathcal{B}$ and the pro-$p$ group $\hat{U}_R \lhd \overline{\Lambda}$ is the abstract unipotent radical. Virtual pro-$p$-ness of the Iwahori subgroup $\mathcal{B} = \overline{\Lambda}_R$ replaces property (R). We don’t know if checking the assumptions of [6, IV.2.7] is doable; which would provide an abstract simplicity theorem.

Back to $\Lambda = \text{SL}_n(F_q[[t, t^{-1}]]$, we see that we cannot apply the arguments of the above proof to it since the conclusion is false: this arithmetic group is residually finite, which means that the intersection $\bigcap_{A' \lhd A} A'$ of its finite index normal subgroups is trivial. A key point for $\overline{\Lambda}$ is:

$$\Lambda = \text{SL}_n(F_q[t, t^{-1}]) \to \text{SL}_n(F_q[t, t^{-1}]/a).$$

Congruence subgroups enable to see on this example what goes wrong in the above proof when $\overline{\Lambda}$ is replaced by $\Lambda$. A key point for $\overline{\Lambda}$ is:

(*) a finite rank-one simple Levi factor cannot be mapped non-trivially into $\hat{U}/(\hat{U} \cap H)$.

This is proved by using the facts that the quotient of a pro-$p$ group by a normal closed subgroup is pro-$p$ and that pro-$p$-ness implies strong restrictions on closed, e.g. finite, subgroups. Here the pro-$p$ group $\hat{U}_R \lhd \overline{\Lambda}$ is:

$$\hat{U}_R = \text{SL}_n\left(\begin{pmatrix} 1 + tF_q[[t]] & F_q[[t]] \\ tF_q[[t]] & 1 + tF_q[[t]] \end{pmatrix} \right).$$

It is the closure of the subgroup:
\[ U = \text{SL}_n \left( \begin{array}{cc} 1 + tF_q[t] & F_q[t] \\ tF_q[t] & 1 + tF_q[t] \end{array} \right), \]

which replaces \( \hat{U}_R \) in the discrete case. The ideal \( \mathfrak{a} = (t - 1)F_q[t] \) provides a congruence subgroup \( K(\mathfrak{a}) \). Moding out by \( K(\mathfrak{a}) \) amounts to applying the identification \( t \sim 1 \), for which the quotient of \( U \) is \( \text{SL}_n(F_q) \), hence contains several copies of \( \text{SL}_2(F_q) \). This shows that \( U \) contains normal subgroups \( H \) such that \( U/(U \cap H) \) contains finite rank-one simple groups of Lie type. Thus the above quoted key fact \((\ast)\) doesn’t hold when replacing \( \widetilde{\mathcal{A}} \) (resp. \( \hat{U}_R \)) by \( \Lambda \) (resp. \( U \)).

**2.B.2** In view of the existence of congruence subgroups e.g. in \( \text{SL}_n(F_q[t,t^{-1}]) \), it is worth discussing for a general \( \Lambda \) a group-theoretic property that is opposite to the notion of simplicity. The affine case thus leads to the following

**Question.**— Which countable Kac-Moody groups are residually finite?

**Remark.**— This question was asked by M. Burger, who notes that taking fixators of combinatorial balls centered at a chamber \( R \) provides a family of finite index normal subgroups of \( \Omega = \text{Fix}_A(R) \). Since the intersection of these groups is trivial, \( \Omega \) is residually finite, and the same argument works for \( \Gamma \) using the negative building.

According to [25, Proposition 1.7], if a generalized Cartan matrix is indecomposable and invertible, then the corresponding Kac-Moody algebra over \( \mathbb{C} \) is simple, but the relation between Kac-Moody algebras and groups is very loose. Still, the fact that affine Kac-Moody algebras are very specific for simplicity suggests the following

**Question.**— Are there simple infinite countable Kac-Moody groups?

**Remark.**— According to Mal’cev theorem [28, §3], proving the non residual finiteness (and a fortiori the simplicity) of an infinite Kac-Moody group \( \Lambda \) is a way to disprove any linearity. The previous subsection shows that the method of [2.4.1] cannot be applied to countable Kac-Moody groups, but for non residual finiteness, another strategy could be to use analogues of the criterion introduced by M. Burger and Sh. Mozes in the context of products of trees [15, §2].

**2.C** Closures of Levi factors and maps to non-Archimedean groups.--- We investigate the structure of closures of Levi factors. This enables us to prove a result on continuous homomorphisms from topological Kac-Moody groups to Lie groups over non-Archimedean fields.

**2.C.1** We keep our Kac-Moody group \( \Lambda \) and the inclusion of the chamber \( R \) in the apartment \( A \). We denote by \( \{a_i\}_{i \in J} \) the finite family of simple roots, which we see as half-spaces of \( \Lambda \) whose intersection equals \( R \). Let us now choose a subset \( J \) of \( I \). We can thus introduce the standard parabolic subgroup \( A_J \) in \( \Lambda \), which is the union \( A_J = \bigcup_{w \in W_J} \Omega w \Omega \) indexed by the Coxeter group generated by the reflections along the walls \( \partial a_i \) for \( i \in J \). By [37, Theorem 6.2.2], the group \( A_J \) admits a Levi decomposition \( A_J = M_J \ltimes U_J \), where the Levi factor \( M_J \) is the Kac-Moody subgroup generated by the maximal split torus \( T \) and the root groups indexed by the roots \( w.a_i \) for \( w \in W_J \) and \( i \in J \).

**Definition.**— (i) We denote by \( \overline{M}_J \) the closure of \( M_J \) in \( \overline{\mathcal{A}} \), and by \( \overline{G}_J \) the topological group generated by the root groups \( U_{\pm a_i} \) when \( i \) ranges over \( J \).

(ii) The intersection of roots \( R_J := \bigcap_{i \in J} a_i \) is called the inessential chamber of \( \overline{M}_J \) in \( X \).

(iii) We denote by \( \mathcal{B}_J \) the stabilizer of \( R_J \) in \( \overline{M}_J \).

(iv) The union of closures \( \overline{X}_J := \bigcup_{g \in M_J} g.R_J \) is called the inessential building of \( \overline{M}_J \) in \( X \).
What supports the terminology is the following

**Proposition.**— (i) The closure group $\overline{M}_J$ admits a natural refined Tits system with Weyl group $W_J$ and abstract Borel subgroup $B_J$.
(ii) The subgroup $B_J$ fixes pointwise $R_J$ and admits a decomposition $B_J = T \ltimes \hat{U}_J$, where $T$ is the maximally split maximal $F_q$-torus attached to $A$ and $\hat{U}_J$ is pro-$p$.
(iii) The space $X_J$ is a geometric realization of the building of $\overline{M}_J$ arising from the above Tits system structure.
(iv) The group $\overline{G}_J$ is of finite index in $\overline{M}_J$ and when $q > 4$ it admits a direct product decomposition into topologically simple factors.

**Proof.**— (ii). By [37, 6.2], the group $M_J$ has a twin root datum structure with positive Borel subgroup $\Omega_J := \Omega \cap M_J$, negative Borel subgroup $\Gamma_J := \Gamma \cap M_J$ and Weyl group $W_J$. Moreover the group $\Omega_J$ is generated by $T$ and the root groups $U_a$ where $a$ is a positive root of the form $w.a_i$ with $i \in J$ and $w \in W_J$. By the proof of Corollary 1 in [30, 5.7], such a root contains $R_J$, so the corresponding group $U_a$ fixes $R_J$ pointwise. Since $T$ fixes pointwise the whole apartment $A$ and $B_J = \Omega_J$, we proved the first assertion. We have $T < \Omega_J$ and $B_J < B$ since $B_J$ fixes $R$: this proves the second point by setting $\hat{U}_J := B_J \cap \hat{U}_R$.

The Borel subgroup $\Omega_J$ of the positive Tits system of $M_J$ is the fixator of $R_J$, so $X_J$ is a geometric realization of the corresponding building and $\overline{M}_J$ is strongly transitive on $X_J$. This proves (iii) and the fact that $\overline{M}_J$ admits a natural Tits system with Weyl group $W_J$ and abstract Borel subgroup $B_J$. We are now in position to apply the same arguments as in [40, 1.C] to prove that we actually have a refined Tits system, which proves (i).

(iv). The first assertion follows from the fact that we have $\overline{M}_J = \overline{G}_J \cdot T$. The same argument as for Theorem [2.A.1] (ii) implies that we are reduced to considering the groups corresponding to the connected components of the Dynkin subdiagram obtained from the one of $A$ by removing the vertices of type $\notin J$ (and the edges containing one of them). Such groups commute with one another, and they are topologically simple by Lemma 2.A.1.

**Remarks.**— 1) Any proper submatrix of an irreducible affine generalized Cartan matrix is of finite type [25, Theorem 4.8]. Consequently, the Levi factors of the parabolic subgroups of $A$ are all finite groups of Lie type in this case.
2) Conversely, if the submatrix of type $J$ of the generalized Cartan matrix of $A$ is not spherical (i.e. of finite type), then $\overline{M}_J$ contains an infinite topologically simple subgroup. Such a group comes from a non spherical connected component of the Dynkin subdiagram given by $J$.

2.C.2 The virtual topological simplicity of closures of Levi factors enables to prove the following result about actions of Kac-Moody groups on Euclidean buildings.

**Proposition.**— Let $A$ be a countable Kac-Moody group which is almost split over the finite field $F_q$ with $q > 4$, which is generated by its root groups and which has connected Dynkin diagram. We assume we are given a continuous group homomorphism $\mu : A \to G(k)$, where $G$ is a semisimple group defined over a non-Archimedean local field $k$. We denote by $\Delta$ the Bruhat-Tits building of $G$ over $k$, and for each point $x$ in the Kac-Moody building $X$, we denote by $A_x$ the fixator $\text{Fix}_{A_x}(x)$.

(i) For any point $x$ in the Kac-Moody building $X$, the set of fixed points $\Delta^{\mu(A_x)}$ is a non-empty closed convex union of facets in the Euclidean building $\Delta$.
(ii) If the image $\mu(A)$ is non-trivial, then for any pair of distinct vertices $v \neq v'$ in $X$, the sets of fixed points $\Delta^{\mu(A_v)}$ and $\Delta^{\mu(A_{v'})}$ are disjoint.
Remark.— As W is infinite, (ii) implies that \( \mu(\overline{A}) \) is unbounded in \( G(k) \) since it has no global fixed point in \( \Delta \).

Proof.— (i). Let \( x \in X \). By the very definition of the topology on Aut(\( X \)), the group \( \overline{A}_x \) is compact, and so is its continuous image \( \mu(\overline{A}_x) \). By non-positive curvature of \( \Delta \) and the Bruhat-Tits fixed point lemma [11 VI.4], the set of \( \mu(\overline{A}_x) \)-fixed points in \( \Delta \) is non-empty. The \( G(k) \)-action on \( \Delta \) is simplicial, and since \( \overline{A} \) is topologically simple (2.A.1) and \( \mu \) is continuous, the \( \mu(\overline{A}) \)-action is type-preserving. Therefore each time a subgroup of \( \mu(\overline{A}) \) fixes a point, it fixes the closure of the facet containing it. The convexity of \( \Delta^\mu(\overline{A}_x) \) comes from the intrinsic definition of a geodesic segment in \( \Delta \) [11 VI.3A], and from the fact that \( G(k) \) acts isometrically on \( \Delta \).

(ii). The type of a vertex in the CAT(0)-realization of a building defines a subdiagram in the Dynkin diagram of \( \Lambda \) which is spherical and maximal for this property [37 4.3]. Therefore the fixator of a vertex \( v \) in the building \( X \) is a maximal spherical parabolic subgroup of the Tits system of \( \Lambda \) with Borel subgroup the Iwahori subgroup \( B \). Now let \( v' \) be another vertex in \( X \). By [30 IV.2.5], the group generated by \( \overline{A}_v \) and \( \overline{A}_{v'} \) is a parabolic subgroup of the latter Tits system, which cannot be spherical since it is strictly bigger than \( \overline{A}_v \). By the second remark in [2.C.1] the closed subgroup \( (\overline{A}_v, \overline{A}_{v'}) \) generated by \( \overline{A}_v \) and \( \overline{A}_{v'} \) contains an infinite topologically simple group \( H \); up to conjugacy, this group is a factor of Proposition [2.C.1](iv).

We assume now that there exist two vertices \( v \neq v' \) in \( X \) such that \( \Delta^\mu(\overline{A}_v) \cap \Delta^\mu(\overline{A}_{v'}) \) contains a point \( y \in \Delta \). Then the image of \( (\overline{A}_v, \overline{A}_{v'}) \) by \( \mu \) lies in the compact fixator of \( y \) in \( G(k) \). Restricting \( \mu \) to the topologically simple group \( H \), and composing with an embedding of \( k \)-algebraic groups \( G < GL_m(k) \), we are in position to apply the lemma below: \( \mu(H) \) is trivial since it is bounded. But then the kernel of \( \mu \) is non trivial hence equal to \( \overline{A} \) by topological simplicity (2.A.1). Finally \( \mu(\overline{A}) \neq \{1\} \) implies \( \Delta^\mu(\overline{A}_v) \cap \Delta^\mu(\overline{A}_{v'}) = \emptyset \) whenever \( v \neq v' \).

We finally prove the following quite general and probably well-known lemma.

Lemma.— Let \( H \) be an infinite topological group all of whose proper closed normal subgroups are finite. Let \( k \) be a non-Archimedean local field and let \( \mu : H \to GL_m(k) \) be a continuous homomorphism for some \( m \geq 1 \). Then \( \mu(H) \) is either trivial or unbounded.

Proof.— We denote by \( O \) the valuation ring and chose a uniformizer \( \varpi \). Let us assume that \( \mu(H) \) is bounded. After conjugation, we may – and shall – assume that \( \mu(H) < GL_m(O) \) [11 1.12]. For each integer \( N \geq 1 \), the group \( K := GL_m(O) \) has an open finite index congruence subgroup \( K(N) \), by definition \( \ker(GL_m(O) \to GL_m(O/\varpi^N)) \). For each \( N \geq 1 \), \( \mu^{-1}(K(N)) \) is a closed normal finite index subgroup of \( H \). Since \( H \) is infinite, so is \( \mu^{-1}(K(N)) \), and by the hypothesis on closed normal subgroups of \( H \) we have \( H = \mu^{-1}(K(N)) \). Since \( \bigcap_{N \geq 1} K(N) = \{1\} \), we have \( \mu(H) = \{1\} \).

3 Embedding theorem

We use the commensurator super-rigidity to prove that the linearity of a countable Kac-Moody group implies that the corresponding topological group is a closed subgroup of a non-Archimedean Lie group. More precisely:

Theorem.— Let \( \Lambda \) be an almost split Kac-Moody group over the finite field \( \mathbb{F}_q \) of characteristic \( p \) with \( q > 4 \) elements, with infinite Weyl group \( W \) and buildings \( X \) and \( X_\cdot \). Let \( \overline{A} \) be the corresponding Kac-Moody topological group. We make the following assumptions:

(TS) the group \( \overline{A} \) is topologically simple;

(NA) the group \( \overline{A} \) is not amenable.
(LT) the group \( \Lambda \) is a lattice of \( X \times X_- \) for its diagonal action.

Then, if \( \Lambda \) is linear over a field of characteristic \( p \), there exists:
- a local field \( k \) of characteristic \( p \) and a connected adjoint \( k \)-simple group \( G \),
- a topological embedding \( \mu : \overline{\Lambda} \to G(k) \) with Hausdorff unbounded and Zariski dense image,
- and a \( \mu \)-equivariant embedding \( \iota : V_X \to V_\Delta \) from the set of vertices of the Kac-Moody building \( X \) of \( \Lambda \) to the set of vertices of the Bruhat-Tits building \( \Delta \) of \( G(k) \).

We now discuss the hypotheses. When \( q > 4 \) and \( \Lambda \) is generated by its root groups, condition (TS) is equivalent to the connectedness of the Dynkin diagram of \( \Lambda \) 2.A.1. Condition (LT) is equivalent to \( \Gamma \) being a lattice of \( \overline{\Lambda} \), which holds whenever \( q \gg 1 \) (1.C.1). For condition (NA) we note that \( \Lambda \) is chamber-transitive; in particular, it is non-compact and \( \overline{\Lambda} \) is cocompact in \( \text{Aut}(X) \), so that \( \overline{\Lambda} \) is amenable (resp. Kazhdan) if and only if \( \text{Aut}(X) \) is. Therefore, since amenable groups with property (T) are compact, condition (NA) is satisfied whenever the latter group \( \text{Aut}(X) \) has property (T). By [20] the question of property (T) for automorphism groups of buildings admits a fairly complete answer. Another case when (NA) is fulfilled is when the building \( X \) has property (T).

**Example.**— The above three conditions are satisfied for instance whenever the Kac-Moody building \( X \) has apartments isomorphic to a hyperbolic tiling and the thicknesses at panels are large enough, a widely available situation that we will meet in Sect. 4.

The remainder of this section is devoted to the proof of the above embedding theorem.

**3.A Semisimple Zariski closure and injectivity.**— The linearity assumption says that there is a field \( F \) of characteristic \( p \) and an injective group homomorphism \( \eta : \Lambda \to \text{GL}_N(F) \) to a general linear group over \( F \). We choose an algebraic closure \( \overline{F} \) of \( F \), and denote by \( H \) the Zariski closure \( \overline{\eta(\Lambda)} \) of the image of \( \eta \) in \( \text{GL}_N \). Let \( RH^\circ \) be the radical of the identity component \( H^\circ \) of \( H \) in the Zariski topology. The group \( H^\circ / RH^\circ \) is connected normal and of finite index in \( H / RH^\circ \), hence it is its (semisimple) identity component. We denote by \( L \) the quotient of \( H / RH^\circ \) by its finite center \( Z(H / RH^\circ) \), and by \( q \) the natural quotient map \( H \to L \). Note that \( L^\circ \) is adjoint semisimple and that the identity component of \( \ker(q) \) is solvable. We consider the composed homomorphism:

\[
\varphi : \Lambda \twoheadrightarrow H \xrightarrow{q} L.
\]

Let \( L < \text{GL}_M \) be an embedding of linear algebraic groups. Since \( \Lambda \) is finitely generated, there is a finitely generated field extension \( E \subset \overline{F} \) of \( \mathbb{Z} / p \mathbb{Z} \) such that \( q(\Lambda) < \text{GL}_M(E) \). The group \( \varphi(\Lambda) \) is Zariski dense in \( L \), so the group \( L \) is defined over \( E \) [5 AG 14.4], and we have \( \varphi(\Lambda) < L(E) \).

**Lemma.**— The group homomorphism \( \varphi : \Lambda \to L(E) \) is injective.

**Proof.**— Let us assume that the kernel of \( \varphi \) is non-trivial, so that its closure is a non-trivial closed normal subgroup of \( \overline{\Lambda} \). By topological simplicity 2.A.1 we have: \( \ker(\varphi) = \overline{\Lambda} \). But since \( \varphi = q \circ \eta \) and since \( \eta \) is injective, we have: \( \ker(\varphi) = \eta^{-1}(\eta(\Lambda) \cap \ker(q)) \simeq \eta(\Lambda) \cap \ker(q) \). This would imply that \( \ker(\varphi) \) is virtually solvable, hence amenable for the discrete topology [47 4.1.7]. Then \( \ker(\varphi) = \overline{\Lambda} \) and [37 4.1.13] would imply that \( \overline{\Lambda} \) is amenable, which is excluded by the assumption (NA). \( \square \)
3. B  Unbounded image and continuous extension.— Our next goal is to check that we are in position to apply the commensurator super-rigidity theorem.

Proposition.— (i) There exist an infinite order element $\gamma$ in the lattice $\Gamma$ and a field embedding $\sigma : \mathbb{E} \to k$ into a local field of characteristic $p$ such that $\sigma(\varphi(\gamma))$ is semisimple with an eigenvalue of absolute value $> 1$ in the adjoint representation of $L(k)$.

(ii) There is a connected adjoint $k$-simple group $G$ and an injective continuous group homomorphism $\mu : \overline{\alpha} \to G(k)$. The map $\mu$ coincides on a finite index subgroup of $\Lambda$ with the composition of $\varphi$ with the projection onto a $k$-simple factor of $L^0$; its image is Zariski dense and Hausdorff unbounded.

Proof.— (i). Let us fix a reflection $s$ in a wall $H_s$ containing an edge of the standard chamber $R$, and let us denote by $a_s$ the simple root bordered by $H_s$. The condition (TS) implies that the Weyl group $W$ of $\Lambda$ (or $\overline{\Lambda}$) is indecomposable (of non-spherical type). Therefore by [23] Proposition 8.1 p. 309 there is a reflection $r$ in a wall $H_r$ such that $rs$ has infinite order, implying that $H_r$ and $H_s$ don’t meet in the interior of the Tits cone of $W$ [27 5.2]. We call $-b$ the negative root bordered by $H_r$. If $-a_s \cap -b = \emptyset$ then by definition $\{ -a_s, -b \}$ is a non-prenilpotent pair of roots [31 3.2]; otherwise $\{ -a_s, -s, b \}$ is. In any case the group generated by the corresponding root groups is isomorphic to $F_q \ast F_q$ [35] Proposition 5. Therefore it contains an element $\tilde{\gamma} \in \Gamma$ of infinite order, and by injectivity $\varphi(\tilde{\gamma})$ has infinite order too. Since the field $E$ has characteristic $p > 0$, a suitable power $p^r \gamma$ kills the unipotent part of the Jordan decomposition of $\varphi(\tilde{\gamma})$. Let us set $\gamma := \frac{1}{r} \varphi(\tilde{\gamma})$, so that $\varphi(\gamma)$ is semisimple. It has an eigenvalue $\lambda$ of infinite multiplicative order, so by [43] Lemma 4.1 there is a local field $k$ endowed with a valuation $v$ and a field embedding $\sigma : \mathbb{E}[\lambda] \to k$ such that $v(\sigma(\lambda)) \neq 0$. Up to replacing $\gamma$ by $\gamma^{-1}$, this proves (i).

(ii). By (i) the composed map $(L(\sigma) \circ \varphi) : \Lambda \to L(E[\lambda]) \to L(k)$, which for short we still denote by $\varphi$, is such that $\Gamma$ has unbounded image in $L(k)$. Let us introduce the preimage $\Lambda^0 := \varphi^{-1}(L^0)$. It is a normal finite index subgroup of $\Lambda$, which we denote by $\Lambda^0 \triangleleft \Lambda$. We also set $\Gamma^0 := \Gamma \cap \Lambda^0$. Since $\Gamma^0 \triangleleft \Gamma$, $\varphi(\Gamma^0)$ is not relatively compact in $L^0(k)$. The connected adjoint semisimple $k$-group $L^0$ decomposes as a direct product of adjoint connected $k$-simple factors. One of them, which we denote by $G$, is such that the projection of $\varphi(\Gamma^0)$ is not relatively compact. The abstract group homomorphism we consider now, and which we denote by $\varphi|_{\Lambda^0}$, is obtained by composing with the projection onto $G$. Therefore we obtain $\varphi|_{\Lambda^0} : \Lambda^0 \to G(k)$ such that $\varphi(\Gamma^0)$ is unbounded in $G(k)$. We also have: $\varphi(\Lambda^0)^{-1} = G$.

By Lemma 1.3.1 the group $\Lambda$ is contained in the commensurator $\text{Comm}_{\overline{\Lambda}}(\Gamma)$. Since $\Gamma^0 \triangleleft \Gamma$, we have: $\text{Comm}_{\overline{\Lambda}}(\Gamma) = \text{Comm}_{\overline{\Lambda}}(\Gamma^0)$, so we are in position to apply the commensurator superrigidity theorem of the Appendix in order to extend $\varphi|_{\Lambda^0}$ to a continuous homomorphism $\mu : \overline{\Lambda^0} \to G(k)$, where $\overline{\Lambda^0}$ denotes the closure of $\Lambda^0$ in $\text{Aut}(X)$. The non-trivial closed subgroup $\overline{\Lambda^0}$ is normal in $\overline{\Lambda}$, hence it is $\overline{\Lambda}$ by topological simplicity [2.3.1]. Therefore there is a map $\mu : \overline{\Lambda} \to G(k)$ which coincides with the abstract group homomorphism $\varphi$ on $\Lambda^0$. By topological simplicity of $\overline{\Lambda}$, $\mu$ is either injective or trivial. By Zariski density of the image, the only possible case is that $\mu$ be injective. Summing up, we have obtained an injective continuous group homomorphism $\mu : \overline{\Lambda} \to G(k)$ such that $\mu(\Gamma^0)$ is unbounded in $G(k)$ and $\mu(\Lambda^0)^{-1} = G$. $\square$

3. C  Embedding of vertices and closed image.— We can finally conclude in view of the following lemma.

Lemma.— (i) There is a $\mu$-equivariant injective unbounded map $\iota : \mathcal{V}_X \to \mathcal{V}_\Delta$ from the vertices of the Kac-Moody building $X$ into the vertices of the Bruhat-Tits building $\Delta$ of $G(k)$. 

(ii) The continuous homomorphism $\mu$ sends closed subsets of $\overline{\Lambda}$ to closed subsets of $G(k)$.

This is the part of the proof which most uses Kac-Moody and Tits system theories, so let us briefly recall some facts. See [11, §5] for the general connection between Tits systems and buildings. For our specific case, we keep the inclusion $R \subset A$ of the standard chamber in the standard apartment. Let $W_R$ be the quotient of the stabilizer $N_A = \text{Stab}_{\mathbb{R}}(A)$ by the fixator $\Omega_A = \text{Fix}_{\mathbb{R}}(A)$, which is the Weyl group of the building $X$, and of the groups $A$ and $\overline{\Lambda}$. It is generated by the reflections along the panels of $R$, and simply transitive on the chambers of $A$. We denote by $B$ the standard Iwahori subgroup $\overline{\Lambda}_R$. By Theorem 1.3.1, $B$ is the Borel subgroup of a Tits system in $\overline{\Lambda}$ with Weyl group $W_R$. The Tits system structure implies a Bruhat decomposition [6, IV.2.3]: $\overline{\Lambda} = \bigcup_{w \in W_R} BwB$.

Proof. — (i) By [2, C.2], (i) we choose for each vertex $v$ in the closure of the chamber $R$ a $\mu(\overline{\Lambda}_v)$-fixed vertex $\iota(v) \in \Delta$. We can extend $\mu$-equivariantly this choice $\overline{\Lambda}_v \cap V_X \to V_\Delta$ to obtain a map $\iota : V_X \to V_\Delta$, where $\iota(v)$ is a $\mu(\overline{\Lambda}_v)$-fixed vertex in $\Delta$ for each vertex $v$ in $X$. By [2, C.2], (ii), the sets of fixed points $\Delta^\mu(\overline{\Lambda}_v)$ are mutually disjoint when $v$ ranges over $V_X$, so $\iota$ is injective. By discreteness of the vertices in $\Delta$, the unboundedness of $\iota$ follows from its injectivity because $V_X$ is infinite (since so is $W$).

(ii). Let $F$ be a closed subset of $\overline{\Lambda}$; we must show that $\overline{\mu(F)} < \mu(F)$. Let $g = \lim_{n \to \infty} \mu(h_n)$ be in $\overline{\mu(F)}$, with $h_n \in F$ for each $n \geq 1$. It is enough to show that $\{h_n\}_{n \geq 1}$ has a converging subsequence. By the Bruhat decomposition $\overline{\Lambda} = \bigcup_{w \in W_R} BwB$, we can write $h_n = k_nw_nk'_n$ with $k_n, k'_n \in B$ and $w_n \in N_A$. Since by compactness of $B$ the sequences $\{k_n\}_{n \geq 1}$ and $\{k'_n\}_{n \geq 1}$ admit cluster values, we are reduced to the situation where $g = \lim_{n \to \infty} \mu(w_n)$ with $w_n \in N_A$ for each $n \geq 1$.

Let us assume that the union of chambers $\bigcup_{n \geq 1} w_n.R$ is unbounded in $A$. Then there is an injective subsequence of chambers $\{w_{n_j}, R\}_{j \geq 1}$. Let us fix a vertex $v \in R$. Since its stabilizer in $W_R$ is finite, possibly after extracting again a subsequence, we get an injective sequence of vertices $\{w_{n_j}, v\}_{j \geq 1}$. But $\mu(w_{n_j}), \iota(v) = \iota(w_{n_j}, v)$ where $\iota : V_X \to V_\Delta$ is the $\mu$-equivariant embedding of vertices of (i).

Since $g = \lim_{n \to \infty} \mu(w_n)$, the continuity of the $G(k)$-action on $\Delta$ implies: $\lim_{j \to \infty} \mu(w_{n_j}), \iota(v) = g, \iota(v)$.

By discreteness of the vertices in $\Delta$, the sequence $\{\iota(w_{n_j}, v)\}_{j \geq 1}$ hence the sequence $\{w_{n_j}, v\}_{j \geq 1}$ is eventually constant: a contradiction.

We henceforth know that the sequence $\{w_n, R\}_{n \geq 1}$ is bounded, hence takes finitely many values. So there is a subsequence $\{w_{n_j}\}_{j \geq 1}$ and $w \in N_A$ such that $\{w_{n_j}\}_{j \geq 1}$ is constant equal to $w$ modulo $\Omega_A$. This proves the lemma, possibly after extracting a converging subsequence in the compact group $\Omega_A < B$. $\square$

4 Some concrete non-linear examples

We prove that most of countable Kac-Moody groups with right-angled Fuchsian buildings are not linear over any field. This requires to settle structure results for boundary point stabilizers and generalized unipotent radicals. Dynamical arguments due to G. Prasad play a crucial role. The geometry of compactifications of buildings sheds some light on ideas of the proof.

4.A Groups with right-angled Fuchsian buildings. — We prove further structure results on topological Kac-Moody groups with right-angled Fuchsian buildings. Let $R$ be a regular hyperbolic right-angled $r$-gon, so that $r \geq 5$. To obtain a countable Kac-Moody group with a building covered by chambers $\simeq R$, we need to lift the Coxeter diagram of the corresponding Fuchsian Weyl group $W_R$ to a Dynkin diagram. The Coxeter diagram of the latter group is connected and all its edges are labelled by $\infty$, so according to the rule [11, 3.1] infinitely many Dynkin diagrams are suitable.
Henceforth, $\Lambda$ denotes a Kac-Moody group over $\mathbb{F}_q$ whose positive building $X$ is isomorphic to some $I_{r,1+q}$ $\text{[1.C.2]}$. We choose a standard positive chamber $R$ in a standard positive apartment $A \simeq H^2$ $\text{[1.A.1]}$. We denote by $d$ the natural CAT($-1$) distance on $X$ and by $\ell$ the length of any edge. We fix a numbering $\{E_i\}_{i \in \mathbb{Z}/r}$ by $\mathbb{Z}/r$ of the edges of $R$, and we denote by $a_i$ the simple root containing $R$ whose wall contains $E_i$. Note that since all wall intersections are orthogonal, all the edges in a given wall $L$ have the same type, which we also call the type of $L$ $\text{[40, 4.A]}$.

**4.A.1** A geodesic ray in a geodesic CAT($-1$) metric space $(X,d)$ is an isometry $r : [0; \infty) \to X$. The Busemann function of $r$ is the function $f_r : X \to \mathbb{R}$ defined by $f_r(x) := \lim_{t \to \infty} (d(x,r(t)) - t)$. The horosphere (resp. horoball) associated to $r$ is the level set $H(r) := \{f_r = 0\}$ (resp. $Hb(r) := \{f_r \leq 0\}$). Let $\xi \in \partial_\infty X$ be a boundary point of $X \simeq I_{r,1+q}$, i.e. an asymptotic class of geodesic ray $\text{[21] \S7}$.

**DEFINITION.** (i) We call parabolic subgroup attached to $\xi$ the stabilizer $P_\xi := \text{Stab}_\Lambda(\xi)$.
(ii) We call horospheric subgroup attached to $\xi$ the subgroup of $P_\xi$ stabilizing each horosphere centered at $\xi$. We denote it by $D_\xi$.

**REMARKS.**— 1) The terminology mimicks the geometric definition of proper parabolic subgroups in semisimple Lie groups, but there are differences. There are two kinds of boundary points, according to whether the point is the end of a wall or not. A point of the first kind is called singular; otherwise, it is called regular. This provides two kinds of parabolic subgroups which are both amenable by $\text{[4]}$ Proposition 1.6]. The only classical case when all proper parabolic subgroups are amenable is when they are minimal, i.e. when the split rank equals one. But then all proper parabolics are conjugate.

2) Defining a suitable notion of rank for the buildings $I_{r,1+q}$ is an interesting question. On the one hand, they contain sharply different kinds of lattices, which makes them close to trees $\text{[1.C.2]}$. On the other hand, they enjoy remarkable rigidity properties, which makes them close to higher-rank buildings $\text{[9]}$.

**PICTURE.**

![Diagram](image)

We are now interested in singular boundary points. Let $L$ be a wall and let $a$ and $b$ be two roots with $a \supset b$ whose walls $\partial a$ and $\partial b$ intersect $L$. The reflection along $\partial a$ (resp. $\partial b$) is denoted by $r_a$ (resp. $r_b$) and $\tau := r_b.r_a$ is a hyperbolic translation along $L$ with attracting point $\xi$ contained in...
We denote by $(ii)$ a closed edges are at distance $\geq\ell$. By germ clearly define the same boundary point, and the converse is true because any two disjoint closed edges are at distance $\geq\ell$, so we can talk of the germ of a singular boundary point. Since any element of $(iii)$ we have:

Let us denote by $D_{\xi}$ the root groups indexed by the roots $\xi, A$. We set:

Two half-walls define the same germ if they intersect along a half-wall. Two half-walls in the same germ clearly define the same boundary point, and the converse is true because any two disjoint closed edges are at distance $\geq\ell$, so we can talk of the germ of a singular boundary point. Since any element of $\overline{A}$ sends a wall onto a wall, we have the following characterizations.

**Lemma.** The group $P_{\xi}$ is the stabilizer of the germ of $\xi$, and $D_{\xi}$ is the fixator of this germ, meaning that there is a half-wall in it which is fixed under $D_{\xi}$. □

**4.A.2** Thanks to the Moufang property and the language of horoballs, we can say more about the groups $V_n$. Let us denote by $E$ the intersection of the wall $L$ with the strip $a \cap (-b)$. By the previously assumed minimality of $a \cap (-b)$, $E$ is reduced to the edge of a chamber $w.R$. Transforming the objects above by $w^{-1} \in W_R$, we may – and shall – assume that we are in the case where $L$ is the wall $\partial a_i$ (where $i$ the type of $E$), and either $a = a_{i-1}$ and $b = -a_{i+1}$, or $a = a_{i+1}$ and $b = -a_{i-1}$. These two situations are completely analogous, and we assume that $a = a_{i-1}$ and $b = -a_{i+1}$. We denote by $r_j$ the reflection in the edge $E_j$ of $\overline{T}$, we set $J := \{i - 1; i + 1\}$, $W_J := \langle r_{i-1}, r_{i+1} \rangle$ and we use notions and notation of 2.C.1. Then $V_{\xi, A}$ is a subgroup of the topologically simple group $\overline{G}_J$, and the inessential building $X_J$ is a combinatorial tree. Its vertices (resp. edges) are the $\overline{G}_J$-transforms of a line $\partial a_{i-1}$ or $\partial a_{i+1}$ (resp. of the strip $a_{i-1} \cap a_{i+1}$). The root groups in $\overline{G}_J$ are those indexed by the roots $a = w.a_j$ with $w \in W_J$ and $j \in J$. They are automorphisms of the tree $X_J$ fulfilling the Moufang condition [11] §6.4. Selecting in each strip of $X_J$ the $\overline{G}_J$-transform of the edge $E_i$ of $R$ provides a bijection between the inessential tree $X_J$ and the tree-wall attached to $E_i$ [9] 1.2.C]. We henceforth adopt the tree-wall viewpoint when dealing with $X_J$, so that $v_i$ are vertices and $\xi$ is a boundary point of it. Moreover the alluded to below geodesic rays, horospheres and horoballs are those defined in the tree-wall $X_J$.

**Lemma.** We assume that all the root groups indexed by the roots $a = w.a_j$ with $w \in W_J$ and $j \in J$, and containing $\xi$, commute with one another:

(i) The group $V_n$ acts trivially on the horoball defined by $[v_n\xi]$.
(ii) For each vertex $v$ on the horosphere defined by $[v_n\xi]$, the group $U_{a_n}$ is simply transitive on the edges containing $v$ which are outside the corresponding horoball.
(iii) We have: $\bigcap_{n \geq 1} V_n = \{1\}$.
(iv) Any $g \in V_{\xi, A}$ stabilizing $[v_n\xi] = a_n \cap L$ belongs to $V_n$.

**Remark.** The roots $a = w.a_j$ with $w \in W_J$ and $j \in J$ are the real roots of a rank 2 Kac-Moody root system, and the corresponding root groups generate a rank 2 countable Kac-Moody group with generalized Cartan matrix $\begin{pmatrix} 2 & A_{i-1,j+1} \\ A_{i-1,j+1} & 2 \end{pmatrix}$. As explained in 4.A these off-diagonal
coefficients are $\leq -1$, and their product is $\geq 4$. According to the explicit commutator relations due to J. Morita [31 §3 (6)], the group generated by the root groups $U_a$ for $\bar{a} \supset \xi$ is abelian whenever the off-diagonal coefficients are both $\leq -2$ (otherwise it may be metabelian), so the assumption made in the lemma is quite not restrictive.

**Proof.** Let $v$ be a vertex in $X_J$ and let $v_N$ be the projection of $v$ on the geodesic $\{v_n\}_{n \in \mathbb{Z}}$ for some $N \in \mathbb{Z}$. By the Moufang property, there are uniquely defined $m \geq 0$ and $u_i \in U_{a_i}$ for $N - m < i \leq N$ such that $v = (u_N u_{N-1} \ldots u_{N-m+1}).v_{N-m}$. Denoting by $f_{\rho}$ the Busemann function of $\rho = [v_0 \xi]$, we have: $f_{\rho}(v) = (m - N)\ell$. The Moufang property thus provides a parametrization of the horoballs centered at $\xi$ in $X_J$ since $H([v_j \xi]) = \{[u_N u_{N-1} \ldots u_{N-m+1}].v_{N-m} : N \in \mathbb{Z}, m \geq 0, N - m = j, u_N \neq 1 \text{ and } u_i \in U_{a_i} \text{ for } N - m < i \leq N\}$. The commutation of all the root groups $U_{a_k}$, along with the parametrization of the horoballs by means of root groups, proves (i) and (ii). Moreover if $h \in \bigcap_{n \geq 1} V_n$, then (i) implies that $h$ fixes all the horoballs centered at $\xi$ in $X_J$, so $h$ belongs to the kernel of the $\overline{G}_J$-action on $X_J$. We have $h = 1$ by topological simplicity (2.4.1), which proves (iii).

(iv). By (i) any $g \in V_{\xi,A}$ stabilizes the horospheres centered at $\xi$, so if $g$ stabilizes $[v_n \xi]$, it fixes it. By definition of $V_{\xi,A}$ as an increasing union of groups, it is enough to show that $\text{Fix}_{\text{V}_n}([v_n \xi]) = V_n$ for each $N > n$. Let $\{K_j\}_{j \geq 1}$ be an increasing exhaustion of the tree-wall $X_J$ by finite unions of closed facets, and such that $K_1 = [v_n ; v_N]$. For each $j \geq 1$, the subset $C_j := V_n . K_j$ is $V_n$-stable by construction and is still a finite union of closed facets since $V_n$ is compact. By definition of $V_n$ as a closure, we have: $\text{Fix}_{\text{V}_n}([v_n \xi]) = \lim_{j \to \infty} \text{Fix}_{\{u_i;i \leq N\}}([v_n \xi]) |c_j$, and an element in $\text{Fix}_{\{u_i;i \leq N\}}([v_n \xi]) |c_j$ can be written $\prod_{i \leq N} u_i |c_j$ with $u_i \in U_{a_i}$ and finitely many non-trivial $u_i$'s. By (ii), we have $u_i = 1$ for $n + 1 \leq i \leq N$, which implies: $\text{Fix}_{\text{V}_n}([v_n \xi]) |c_j = \text{Fix}_{\{u_i;i \leq n\}}([v_n \xi]) |c_j = V_n |c_j$. Passing to the projective limit shows (iv). □

4.A.3 We can now state the main properties of the boundary point stabilizers and of their generalized unipotent radicals in the commutative case, keeping the previous notation.

**Proposition.**— We assume that all the root groups indexed by the roots $a = w . a_j$ with $w \in W_J$ and $j \in J$, and containing $\xi$, commute with one another.

(i) The group $V_{\xi,A}$ is closed, normalized by $\langle \tau \rangle$ but not by $K_L$. Each group $V_n$ is abelian of exponent $p$, hence so is $V_{\xi,A}$.

(ii) The group $K_L$ is normalized but not centralized by $\langle \tau \rangle$. It admits a semidirect product decomposition $M_{L,A} \ltimes \hat{U}_L$, where $\hat{U}_L$ is a pro-$p$ group. In particular, $K_L$ is virtually pro-$p$.

(iii) The following decompositions hold: $P_\xi = K_L . \langle \tau \rangle . V_{\xi,A}$ and $D_\xi = K_L . V_{\xi,A}$, with trivial pairwise intersections of the factors: $\langle \tau \rangle \cap K_L = \langle \tau \rangle \cap V_{\xi,A} = K_L \cap V_{\xi,A} = \{1\}$.

**Proof.**— (i). Let $u \in \overline{V}_{\xi,A}$. We write: $u = \lim_{j \to \infty} u_j$ with $u_j \in V_{\xi,A}$ for each $j \geq 1$, and we have $u \in P_\xi$. By Lemma 4.A.1, there is an $n \in \mathbb{Z}$ such that $u$ sends the geodesic ray $[v_n \xi]$ to a geodesic ray contained in $L$ and ending at $\xi$. For $j \gg 1$ we have: $u_j([v_n;v_{n+1}]) = u([v_n;v_{n+1}])$. Since $u([v_n;v_{n+1}])$ is an edge in $L$ and since $u_j$ stabilizes the horospheres centered at $\xi$, we have $u_j([v_n;v_{n+1}]) = [v_n;v_{n+1}]$ for $j \gg 1$. Therefore $u_j$ fixes $[v_n \xi]$ for $j \gg 1$, implying by Lemma 4.A.2 (iv) that for $j \gg 1$ the elements $u_j$ lie in the compact group $V_n$. This implies $u \in V_n < V_{\xi,A}$, hence (i).

We turn now to the group-theoretic properties of $V_{\xi,A}$. By assumption all the groups $U_{a_k}$ commute, so the continuous commutator map $[.,.]$ is trivial on a topologically generating set for each $V_n$. This proves the commutativity of each group $V_n$. Any of the commuting root groups $U_{a_k}$ is isomorphic to $(\mathbb{F}_q, +)$ so replacing $[,]$ by $\rho$ shows that each $V_n$ is of exponent $p$. By definition, $V_{\xi,A}$ is normalized but not centralized by $\tau$. Pick a root $a$ containing $L$ so that $U_a < K_L$ and choose $n \in \mathbb{Z}$ such that $-a \cap -a_n = \emptyset$, i.e. $\{a;a_n\}$ is not prenilpotent. By [15] Proposition 5, the free product $U_a * U_{a_n}$
injects in $A$, so for any $u \in U_a \setminus \{1\}$ and $u' \in U_{a_n} \setminus \{1\}$ the order of $[u, u']$ is infinite whereas it would divide $p$ if $K_L$ normalized $V_{\xi,A}$.

(ii). If $\Pi$ is a panel in the wall $L$, we have $K_L < \overline{\Pi} = M_{\Pi} \ltimes \hat{U}_{\Pi}$ (1.B.1) and $M_{\Pi} = M_{L,A}$ by the precise version of the Levi decomposition [37] Theorem 6.2.2). The group $M_{L,A}$ fixes $L$, so $M_{L,A} \subset K_L$. Moreover the kernel $\tilde{U}_L := \hat{U}_{\Pi} \cap K_L$ of the restricted map $K_L \to M_{L,A}$ is pro-$p$, and we have: $M_{L,A} \cap \tilde{U}_L = \{1\}$ by the same argument as for [40] Lemma 1.C.5 (ii). The group $K_L = \text{Fix}(\tau(L))$ is normalized by $\langle \tau \rangle$ because $\tau$ stabilizes $L$. Now we pick a root $a \neq \pm c$ with $a \supset L$, so that $U_a < K_L$. For $M \gg 1$, the root $\tau^M.a$ contains $a$ but $a \cap \tau^M.a$ is a strip in $A$, implying that $\{a; \tau^M.a\}$ is not prenilpotent. As for (i) the free product $U_a * U_{\tau^M.a}$ injects in $A$, so for $u \in U_a \setminus \{1\}$ the commutator $[\tau^M, u]$ has infinite order whereas it would be trivial if $K_L$ were centralized by $\tau$.

(iii). Let $\tau : [0; \infty) \to X$ be the geodesic ray such that $\tau(n\ell) = v_n$ for each $n \geq 0$, and let $g \in P_{\xi}$. By Lemma [4.A.1] there are integers $N \geq 1$ and $t \in \mathbb{Z}$ such that $(\ell, g.r(n\ell)) = \tau((n + t)\ell)$ for $n \geq N$. Since the $\mathcal{A}$-action on $X$ is type-preserving, $t$ is an even number, say $2m$, and we have: $(\tau^{-m}.g.r)(n\ell) = \tau(n\ell)$ for each $n \geq N$. Thus $d := \tau^{-m}g$ fixes the geodesic ray $\{v_n\xi\}$, hence belongs to $D_{\xi}$, and we are reduced to decompose $D_{\xi}$.

Let $d \in D_{\xi}$, which fixes a geodesic ray $\{v_n\xi\}$ by Lemma [4.A.1]. The link of the vertex $v_N$ is complete bipartite, so there is a chamber $R'$ whose closure contains both $d.\{v_N; v_{N-1}\}$ and an edge $E'$ contained in the wall $\partial N$. By the Moufang property, there exists $u_N \in U_{a_N}$ such that $(u_N^{-1}d).R'$ is the chamber in $A$ whose closure contains both $\{v_N; v_{N-1}\}$ and $E'$; in particular, $u_N^{-1}d$ fixes the geodesic ray $\{v_{N-1}\xi\}$. By a downwards induction, for each $m < N$ we pick $u_m \in U_{a_m}$ such that $u_m^{-1}u_{m+1}^{-1}u_N^{-1}d$ fixes the geodesic ray $\{v_{m-1}\xi\}$. By compactness of $V_N$, the sequence $\{u_Nu_{N-1}\ldots u_m1\} < N$ has a cluster value $u \in V_{\xi,A}$ such that $u^{-1}d$ fixes the geodesic $L$, hence belongs to $K_L$. Taking inverses, we proved: $D_{\xi} = K_L \cdot V_{\xi,A}$, and along with the previous paragraph: $P_{\xi} = K_L \cdot (\tau) \cdot V_{\xi,A}$ since $\tau$ normalizes $V_{\xi,A}$ and $K_L$.

The trivial intersection $\langle \tau \rangle \cap V_{\xi,A} = \{1\}$ follows from the fact that $\langle \tau \rangle \simeq \mathbb{Z}$ whereas any non-trivial element in $V_{\xi,A}$ has order $p$, and $\langle \tau \rangle \cap K_L = \{1\}$ follows from the fact that no non-trivial power $\tau^m$ fixes $L$. An element in $K_L \cap V_{\xi,A}$ lies in any $\text{Fix}_{V_{\xi,A}}(\{v_n\xi\}) (n \in \mathbb{Z})$, hence in any $V_n$ by Lemma [4.A.2] (iv), so $K_L \cap V_{\xi,A} = \{1\}$ follows from (iii) in the same lemma. $\square$

**Remark.**— 1) Horoball arguments as in [4.A.1] show that each group $V_n$ is isomorphic to $(\mathbb{F}_q[[t]], +)$ and that there is an isomorphism $V_{\xi,A} \simeq (\mathbb{F}_q((t)), +)$ under which conjugation by $\tau$ corresponds to multiplication by $t^{-2}$ and the $t$-valuation corresponds to the index $n$.

2) Let us denote by $-\xi$ the other end of $L$, so that $L = (\xi, -\xi)$. By definition, $D_{\xi} \cap D_{-\xi}$ stabilizes $L$ and actually fixes it since $D_{\xi}$ stabilizes the horospheres centered at $\xi$. Therefore we have: $D_{\xi} \cap D_{-\xi} = K_L$.

**4.B Dynamics and parabolics.**— Let us have a dynamical viewpoint on the above groups. The prototype for parabolics, used in [4.C.1] is G. Prasad’s work in the algebraic group case [35].

**4.B.1** A first consequence of the existence of many hyperbolic translations is the connection with Furstenberg boundaries – see [29] VI.1.5) for a definition, where this notion is simply called a boundary. We denote by $\mathcal{M}^1(\partial_{\infty}X)$ the space of probability measures on $\partial_{\infty}X$; it is compact and metrizable for the weak*-topology. This subsection owes its existence to discussions with M. Bourdon and Y. Guivarc’h.

**Lemma.**— The asymptotic boundary $\partial_{\infty}X$ is a Furstenberg boundary for $\mathcal{A}$.

**Proof.**— Let us prove that the $\mathcal{A}$-space $\partial_{\infty}X$ is both minimal and strongly proximal [29] VI.1).

Strong proximality. Let $\mu \in \mathcal{M}^1(\partial_{\infty}X)$. Since some unions of walls in $X$ are trees, the set of singular points is uncountable. Therefore there is a hyperbolic translation $\tau$ along a wall whose repelling
point is not one of the at most countably many atoms for \( \mu \). By dominated convergence, the sequence converges in \( \mathcal{M}^1(\partial_\infty X) \) to the Dirac mass centered at the attracting point of \( \tau \).

Minimality. Let \( \xi \in \partial_\infty X \). We write it \( \xi = r(\infty) \) for a geodesic ray \( r : [0; \infty) \rightarrow X \) with \( r(0) \in R \). For each \( n \geq 1 \), \( r(n) \) is in the closure of a chamber \( g_n R \) with \( g_n \in \mathcal{A} \). By the Bruhat decomposition \( \mathcal{A} = \bigcup_{w \in W_R} B w B \), we have: \( r(n) = k_n w_n R \), hence \( k_n^{-1} r(n) \in A \). Let us denote by \( r_n \) the geodesic ray in \( A \cong \mathbf{H}^2 \) starting at \( r(0) \) and passing through \( k_n^{-1} r(n) \). By compactness of \( \partial_\infty \mathbf{H}^2 \cong S^1 \) and \( B \), there is an extraction \( \{ n_j \}_{j \geq 1} \) such that \( r_{n_j}(\infty) \) converges to some \( \eta \in \partial_\infty A \) and \( k_{n_j} \) converges to some \( k \in B \) as \( j \rightarrow \infty \). Thus in the \( \mathcal{A} \)-compification \( X \sqcup \partial_\infty X \), we have: \( \xi = \lim_{j \rightarrow \infty} r(n_j) \). Since the action of the Weyl group \( W_R \), a lattice of \( \text{PSL}_2(\mathbf{R}) \), is minimal on \( \partial_\infty A \), we proved the minimality of the \( \mathcal{A} \)-action on \( \partial_\infty X \). \( \square \)

REMARK. — Note that the group \( \mathcal{A} \) admits a Furstenberg boundary on which it doesn’t act transitively, whereas any such boundary for a semisimple algebraic group is an equivariant image of the maximal flag variety \([14, \text{§}5]\).

4.B.2 Iteration of hyperbolic translations along walls also leads to computations of limits of later use for the non-linearity theorem.

PROPOSITION. — Let \( \tau \) be a hyperbolic translation along a wall \( L \), with attracting point \( \xi \). Let \( v \) be a vertex on the wall \( L \).

(i) We have: \( \lim_{n \to \infty} \tau^n L \tau^{-n} = \mathcal{D}_\xi \) in the compact metrizable space \( \mathcal{S}_\mathcal{A} \) of closed subgroups of \( \mathcal{A} \), endowed with the Chabauty topology.

(ii) For any \( u \in \mathcal{A} \), we have: \( \lim_{n \to \infty} \tau^n u \tau^{-n} = 1 \).

(iii) For any \( g \in \mathcal{D}_\xi \), the sequence \( \{ \tau^n g \tau^{-n} \}_{n \geq 1} \) is bounded in \( \mathcal{A} \).

REMARK. — Point (i) about the Chabauty topology on closed subgroups is used in the final discussion 4.C.2. Recall that the Chabauty topology on the closed subsets of a topological space \( S \) is the topology defined by a sub-base consisting of the subsets \( O(K) := \{ A : A \cap K = \emptyset \} \) for all compact subsets \( K \), and \( O(U) := \{ A : A \cap U \neq \emptyset \} \) for all open subsets \( U \) ([16, 3.1.1]). This topology is always compact, and when \( S \) is Hausdorff, locally compact and second countable, it is separable and metrizable ([16, 3.1.2]). When \( S \) is locally compact, a sequence \( \{ A_n \}_{n \geq 1} \) of closed subsets converges in the Chabauty topology on a closed subset \( A \) if and only if:

1) Any limit \( x = \lim_{k \to \infty} x_n(k) \) for an increasing \( \{ n(k) \}_{k \geq 1} \) of \( x_n(k) \in A_n(k) \), satisfies \( x \in A \).

2) Any \( x \in A \) is the limit of a sequence \( \{ x_n \}_{n \geq 1} \) of \( x_n \in A_n \) for each \( n \geq 1 \).

This characterization of convergence is referred to as geometric convergence ([16, 3.1.3]); it implies that for a locally compact group \( G \), the subset \( \mathcal{S}_G \) of closed subgroups is closed, hence compact, for the Chabauty topology.

Proof. — (i). By compactness of \( \mathcal{S}_\mathcal{A} \), it is enough to show that any cluster value \( D \) of \( \{ \tau^n \mathcal{A}, \tau^{-n} \}_{n \geq 1} \) is equal to \( \mathcal{D}_\xi \). In one direction, the very definition of the Chabauty topology implies that \( D \) contains \( K_L \) and \( V_{\xi, A} \), hence \( \mathcal{D}_\xi \) by Proposition 4.A.3 (iii). Indeed, the group \( K_L \) lies in \( D \) since it fixes all the vertices in \( L \), hence lies in all the conjugates \( \tau^n \mathcal{A}, \tau^{-n} \). The limit group \( D \) also contains \( V_{\xi, A} \) since for each \( m \in \mathbf{Z} \) there is \( N \in \mathbf{N} \) such that \( \tau^n v \in a_m \), hence \( v_m < \tau^n \mathcal{A}, \tau^{-n} \) for any \( n \geq N \).

We are thus reduced to proving that any cluster value \( D \) lies in \( \mathcal{D}_\xi \). Let \( \nu_\xi \in \mathcal{M}^1(\partial_\infty X) \) be a \( \mathcal{A}, \mathcal{D}_\xi \)-invariant measure such that the repelling point of \( \tau \) is not an atom for it. By dominated convergence, we have: \( \lim_{n \to \infty} \tau^n \nu_\xi = \delta_\xi \), where \( \delta_\xi \) is the Dirac mass at \( \xi \), so by [22, Lemma 3]: \( D < \mathcal{P}_\xi = \text{Stab}_\mathcal{A}(\delta_\xi) \).
Let \( g \in D \), which by the previous paragraph and Proposition\(^{4.A.3}\) (iii) we write \( g = u\tau^N k \) with \( u \in V_{\xi,A} \), \( N \in \mathbb{Z} \) and \( k \in K_L \). We choose this order to forget the factor \( k \) when \( g \) acts on \( v \). Since \( D \) is a limit group, we also have: \( g = \lim_{j \to \infty} \tau^{n_j} k_j \tau^{-n_j} \) for a sequence \( \{k_j\}_{j \geq 1} \subset \overline{A}_v \) and integers \( n_j \to \infty \) as \( j \to \infty \). Therefore there is an index \( J \geq 1 \) for which \( j \geq J \) implies \( (u\tau^N)_v = (\tau^{n_j} k_j \tau^{-n_j})_v \). Since \( u \) stabilizes all the horospheres centered at \( \xi \), there is a vertex \( z \in L \) with \((u\tau^N)_v \) and \( \tau^N \) at the same distance from \( z \). We choose \( j \gg 1 \) to have 
\[
 d(\tau^{n_j} v, (u\tau^N)_v) = (n_j - N)\delta,
\]
where \( \delta \) is the translation length of \( \tau \). But the group \( \tau^{n_j} \overline{A}_v \tau^{-n_j} \) stabilizes the spheres centered at \( \tau^{n_j} v \), so that 
\[
 d(\tau^{n_j} v, (\tau^{n_j} k_j \tau^{-n_j}) v) = n_j\delta.\]
In order to have \( (u\tau^N)_v = (\tau^{n_j} k_j \tau^{-n_j})_v \), we must have \( N = 0 \), i.e. \( g = uk \): this shows that \( D \) lies in \( D_\xi \).

(ii). Let \( u \in V_{\xi,A} \). Then \( u \in V_m \) for some \( m \in \mathbb{Z} \). For each \( N \geq 1 \), the sequence \( \{\tau^{-n} u\tau^n\}_{n \geq 1} \) lies in the compact group \( V_m^{-N} \), so that any cluster value of \( \{\tau^{-n} u\tau^n\}_{n \geq 1} \) belongs to \( V_m^{-N} \). By Lemma\(^{4.A.2}\) (iii), this shows that the only cluster value of the sequence \( \{\tau^{-n} u\tau^n\}_{n \geq 1} \) in the compact subset \( V_m \) is the identity element, which proves (ii).

(iii). Let \( g \in D_\xi \), which we write \( g = ku \) with \( k \in K_L \) and \( u \in V_{\xi,A} \) by Proposition\(^{4.A.3}\) (iii). Since \( u \in V_m \) for some \( m \in \mathbb{Z} \), we have \( \tau^{-n} u\tau^n \in V_m \) for each \( n \geq 1 \). By Proposition\(^{4.A.3}\) (i) \( K_L \) is normalized by \( \tau \), so we finally have \( \tau^{-n} g\tau^n \in K_L \cdot V_m \) for each \( n \geq 1 \). \( \square \)

4.C Non-linearity in equal characteristic.— We finally state and prove the non-linearity theorem for some countable Kac-Moody groups with hyperbolic buildings. It applies to an infinite family of groups, the Weyl group of which being of arbitrarily large rank.

4.C.1 The previous dynamical results from 4.B as well as the embedding theorem from Sect. 3 provide the main arguments to prove the result below.

Theorem.— Let \( A \) be a countable Kac-Moody group over a finite field \( \mathbb{F}_q \) of characteristic \( p \), and let \( r \) be an integer \( \geq 5 \). We assume that the geometry of \( A \) is a twinned pair of right-angled Fuchsian buildings \( I_{r,q+1} \) with \( q \geq \max\{r-2; 5\} \), and that a generalized unipotent radical of \( \mathbb{A} \) is abelian. Then \( A \) is not linear over any field.

Remark.— According to 4.A.2 the assumption of commutativity of a generalized unipotent radical is mild, since it amounts to requiring that for some \( i \in \mathbb{Z}/r \), both negative coefficients \( A_{i-1,i+1} \) and \( A_{i+1,i-1} \) be \( \leq -2 \) (their product must always be \( \geq 4 \) to have \( X \simeq I_{r,q+1} \) see 4.A).

Proof.— By 39 Proposition 4.3], it is enough to disprove linearity in equal characteristic. Let us assume that there is an abstract injective homomorphism from \( A \) to a linear group in characteristic \( p \), in order to obtain a contradiction. Up to replacing \( A \) by a finite index subgroup, we may – and shall – assume that \( A \) is generated by its root groups.

We first check that we can apply the embedding theorem of Sect. 3. By the comments in the introduction of this Section, (NA) holds because \( X \) is CAT(-1) and \( \mathbb{A} \) is chamber-transitive. By 1.C.1 last remark] the growth series of the Weyl group is \( W(t) = \frac{(1 + t)^2}{(1 - (r - 2)t + t^2)} \in \mathbb{Z}[[t]] \), and \( W(A_6) \) is finite if and only if \( q \geq r - 2 \); so (LT) holds by 1.C.1. Theorem 2.A.1 implies that (TS) is satisfied because the Dynkin diagrams leading to the buildings \( I_{r,q+1} \) are connected [37 13.3.2] and we have assumed that \( q > 4 \). Consequently, we have a closed embedding \( \mu : \mathbb{A} \to G(k) \) of topological groups.

Let \( L \) be a wall with end \( \xi \) such that \( V_{\xi,A} \) is abelian. We pick a hyperbolic translation \( \tau \) along \( L \) with attracting point \( \xi \) as in 4.A.1, and \( u \in V_{\xi,A} \setminus \{1\} \). We set \( B := (\text{Ad} \circ \mu)(\tau) \) and \( Y := (\text{Ad} \circ \mu)(u) \), where \( \text{Ad} \) is the adjoint representation of \( G \). Then Lemma 4.B.2 (ii) implies that \( \{B^{-i}YB^i\}_{i \geq 1} \)
contains the identity element in its closure, so \([29, \text{Lemma II.1.4}]\) says that the element \(B\) has two eigenvalues with different absolute values. We can thus use \([35, \text{Lemma 2.4}]\), which provides us parabolic subgroups: by \([\text{loc. cit. (i)}]\) and \([4.B.2, \text{iii}]\), there is a proper parabolic \(k\)-subgroup \(P_{\mu(\tau)}\) whose \(k\)-points \(P_{\mu(\tau)}\) contain \(\mu(D_\xi)\). Replacing \(\tau\) by \(\tau^{-1}\), the attracting point becomes the boundary point \(-\xi\) such that \((-\xi_\xi) = L\). By \([\text{loc. cit. (ii)}]\), the corresponding parabolic subgroup \(P_{\mu(\tau^{-1})}\) is opposite \(P_{\mu(\tau)}\), so that \(\mu(D_\xi) \cap \mu(D_{-\xi})\) lies in the Levi factor \(P_{\mu(\tau)} := P_{\mu(\tau)} \cap P_{\mu(\tau^{-1})}\).

By the second remark following Proposition \([4.A.3]\), we have \(K_L = D_\xi \cap D_{-\xi}\), so we finally obtain: \(\mu(K_L) < \mu(\tau)\).

The contradiction comes when we look at the image \(\mu(V_{\xi,A})\). By \([\text{loc. cit. (i)}]\) and \([4.B.2, \text{ii}]\), it lies in the unipotent radical \(R_uP_{\mu(\tau)}\). The decomposition \(D_\xi = K_L \cdot V_{\xi,A}\) of Proposition \([4.A.3, \text{iii}]\) then implies: \(\mu(K_L) = \mu(D_\xi) \cap M_{\mu(\tau)}\) and \(\mu(V_{\xi,A}) = \mu(D_\xi) \cap R_uP_{\mu(\tau)}\). But according to Proposition \([4.A.3, \text{i}]\), the group \(V_{\xi,A}\) is not normalized by \(K_L\).

\[\square\]

### 4.C.2
Let us give a geometric flavour to the above proof by using the framework of group-theoretic compactifications of buildings \([3]\). We keep the notation of the previous proof, choose a \(k\)-embedding \(G < \text{GL}_r\) of algebraic groups and still call \(\mu\) the composed closed embedding \(\mu: \Lambda \rightarrow \text{GL}_r(k)\). Replacing \(\tau\) by \(\tau^{(r)}\) for \(r \gg 1\) and taking a finite extension which we still denote by \(k\), we may and shall assume that \(t := \mu(\tau)\) is diagonal with respect to a basis \(\{e_i\}_{1 \leq i \leq r}\) of \(k^r\). We write: \(t.e_i = u_i \varphi^{(i)}(t)e_i\) where \(\varphi\) is the uniformizer of \(k\), \(u_i \in O^\circ\) and \(\nu_i(t) \in Z\). Composing \(\mu\) with a permutation matrix enables to assume that \(\nu_1(t) \leq \nu_2(t) \leq \ldots \leq \nu_r(t)\). The basis \(\{e_i\}_{1 \leq i \leq r}\) defines a maximal flat \(F \simeq R^{r-1}\) in the Bruhat-Tits building \(\Delta\) of \(\text{GL}_r(k)\), whose vertices are the homothety classes of \(O\)-lattices \(\{\bigoplus \varphi^nOe_i\}\) when \(L := \{\nu_i\}_{1 \leq i \leq r}\) ranges over \(Z^r\). We denote by \(o\) the origin \(\bigoplus Oe_i\). The same use of \([29, \text{Lemma II.1.4}]\) as in \([4.C.1]\) shows that there is \(i \in \{1; 2; \ldots; r - 1\}\) such that \(\nu_i(t) < \nu_{i+1}(t)\). Geometrically, this means that \(\{tn.o\}_{n \geq 1}\) is a sequence of vertices in the Weyl chamber \(\{\nu_1 \leq \nu_2 \leq \ldots \leq \nu_r\}\) which goes to infinity, staying in the intersection of the fundamental walls indexed by the indices \(i\) for which \(\nu_i(t) = \nu_{i+1}(t)\).

Sequences of points staying at given distance from some walls in a Weyl chamber while leaving the others typically converge in Furstenberg compactifications of symmetric spaces \([22]\). This is a hint to consider compactifications of Bruhat-Tits buildings in our context \([3]\). We denote by \(S_G\) the space of closed subgroups of a locally compact metrizable group \(G\), and we endow \(S_G\) with the compact metrizable Chabauty topology \([16, 3.1.1]\). The vertices \(V_{\Delta}\) in \(\Delta\) are seen as the maximal compact subgroups in \(\text{SL}_r(k)\). Therefore we have \(V_{\Delta} \subset S_{\text{SL}_r(k)}\) and we can sum up some results from \([3]\):

**Theorem.**— The above procedure leads to a \(\text{GL}_r(k)\)-compactification of \(\Delta\) where the boundary points are the following closed subgroups of \(\text{SL}_r(k)\). Start with a Levi decomposition \(M \ltimes U\) of some proper parabolic subgroup and select \(K < M\) a maximal compact subgroup. Then \(K \ltimes U\) is a limit group, and any limit group is of this form. \(\square\)

**Remark.**— In higher rank (i.e. for \(r \geq 3\)), this compactification is not the one obtained by asymptotic classes of geodesic rays.

Now Proposition \([4.B.2, \text{i}]\) says that \(\lim_{n \rightarrow \infty} \tau^n \tau^{-n} = D_\xi\) in \(\mathcal{S}_T\). It follows from the geometric characterization of convergence in the Chabauty topology \([4.B.2]\) that \(\mu\) induces an embedding \(\mu: \mathcal{S}_T \rightarrow \mathcal{S}_{\text{SL}_r(k)}\). Applying \(\mu\) to the above limit and using the theorem imply: \(\mu(D_\xi) < K \ltimes U\) and \(\mu(D_{-\xi}) < K \ltimes U^\circ\), with \(U^\circ\) opposite \(U\). This leads to \(\mu(K_L) < K\), an important step in the previous proof.

The comparison of hyperbolic and Euclidean apartments emphasizes a sharp difference between Fuchsian and affine root systems. In a Euclidean apartment, there is a finite number of parallelism
classes of walls, whereas in the hyperbolic tiling there are arbitrarily large families of roots pair-wise intersecting along strips. This explains why there are so many non prenilpotent pairs of roots (hence free products $F_q \ast F_q$) in the latter case. This is used to prove that $K_L$ doesn’t normalize $V_{\xi,A}$, a key fact for non-linearity. Another crucial difference is the dynamics of the Weyl groups on the boundaries of apartments: in the hyperbolic case, there are infinitely many hyperbolic translations with strong dynamics [21 §8], whereas the finite index translation subgroup of a Euclidean Weyl group acts trivially on the boundary of a maximal flat. This makes the computation of limit groups easier in the former case [4.B.2], but the boundary of the Furstenberg compactification of a Bruhat-Tits building has a much richer group-theoretic structure since it contains compactifications of smaller Euclidean buildings [26 §14].

Picture.— [on the right hand-side: a compactified apartment for $SL_3$]

\begin{figure}
\centering
\includegraphics[width=0.7\textwidth]{compactified_apartment.png}
\caption{Compactified apartment for $SL_3$}
\end{figure}

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Appendix: Strong boundaries and commensurator super-rigidity

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Introduction

Let $G$ be a locally compact second countable group, and let $\Gamma$ be a lattice of $G$, i.e. a discrete subgroup such that $G/\Gamma$ carries a finite $G$-invariant measure. The commensurator of $\Gamma$ in $G$ is the group: \[ \text{Comm}_G(\Gamma) = \{ g \in G \mid \Gamma \text{ and } g\Gamma g^{-1} \text{ has finite index in both } \Gamma \text{ and } g\Gamma g^{-1} \} \].

Our purpose is to use the recent double ergodicity theorem by V. Kaimanovich on Poisson boundaries, in order to show that G. Margulis’ proof of the commensurator super-rigidity – as analyzed and generalized by N. A’Campo and M. Burger – extends to a quite general setting.

**Theorem 1** Let $G$ be a locally compact second countable topological group, $\Gamma < G$ be a lattice and $\Lambda$ be a subgroup of $G$ with $\Gamma < \Lambda < \text{Comm}_G(\Gamma)$. Let $k$ be a local field and $H$ be a connected almost $k$-simple algebraic group. Assume $\pi: \Lambda \to H_k$ is a homomorphism such that $\pi(\Lambda)$ is Zariski dense in $H$ and $\pi(\Gamma)$ is unbounded in the Hausdorff topology on $H_k$. Then $\pi$ extends to a continuous homomorphism $\Lambda \to H_k/Z(H_k)$, where $Z(H_k)$ is the center of $H_k$.

This theorem is basically due to G. Margulis, who proved it in the case where $G$ is a semisimple group over a locally compact field [Mar91, VII.5.4]. A deep idea in the proof is that the existence of the continuous extension follows from the existence of a $\Lambda$-equivariant map from the maximal Furstenberg boundary of $G$ to a homogeneous space $H_k/L_k$, where $L$ is a proper $k$-subgroup of $H$. N. A’Campo and M. Burger extended the result to the case where $G$ is as above [AB94], assuming the existence of a closed subgroup $P$ playing the same measure-theoretic role as a minimal parabolic subgroup. This led M. Burger to state the above result assuming the existence of a substitute for a maximal Furstenberg boundary rather than a minimal parabolic subgroup [Bur95]. The assumption that $k$ be of characteristic 0, made so far, was removed too. M. Burger and N. Monod then constructed suitable boundaries for compactly generated groups (up to finite index, see [BM02, Theorem 6]), which implied the above result for a compactly generated group $G$ ([BM02, Remark 7]). The last step was made by V. Kaimanovich [Kai02] (Theorem 2 below), who proved that the Poisson boundary for a nice measure on any locally compact second countable group is a strong hence a suitable boundary. We couldn’t finish this historical summary without mentioning the work of T.N. Venkataramana [Ven88], who was the first to prove super-rigidity and arithmeticity theorems in arbitrary characteristics.

This note, which relies heavily on the proof given by N. A’Campo and M. Burger in [AB94], shows how to use the previously quoted references to prove the above commensurator super-rigidity. It is organized as follows. We first quote the results about boundaries of groups. Then we recall how the existence of the continuous extension is reduced to finding a $\Lambda$-equivariant map from a boundary of $G$ to a non-trivial homogeneous space $H_k/L_k$. We finally sketch the steps to construct the required $\Lambda$-equivariant map, taking care of the fact that $k$ is possibly of positive characteristic.

Poisson boundaries and strong boundaries

Given a topological group $G$, a Banach $G$-module is a pair $(\pi, E)$ where $E$ is a Banach space and $\pi$ is an isometric linear representation of $G$ on $E$. The module $(\pi, E)$ is continuous if the action map
$G \times E \to E$ is continuous. A coefficient $G$-module is a Banach $G$-module $(\pi, E)$ contragredient to some separable continuous Banach $G$-module, i.e., $E$ is the dual of some separable Banach space $E^\circ$ and $\pi$ consists of operators adjoint to a continuous action of $G$ on $E^\circ$ (see [Mon01 chapter 1]). Denote by $\mathcal{X}^{sp}$ the class of all separable coefficient modules.

Let $G$ be a locally compact group, and $(S, \mu)$ be a Lebesgue space endowed with a measure class preserving action of $G$. Given any class of coefficient Banach modules $\mathcal{X}$, the action of $G$ on $S$ is called doubly $\mathcal{X}$-ergodic if for every coefficient $G$-module $E$ in $\mathcal{X}$, any weak-* measurable $G$-equivariant function $f : S \times S \to E$ (with respect to the diagonal action) is a.e. constant ([Mon01 11.1.1]).

Recall ([Zim84 4.3.1]) that the $G$-action on $S$ is called amenable if for every separable Banach space $E$ and every measurable right cocycle $\alpha : S \times G \to \text{Iso}(E)$ the following holds for $\alpha^*$, the adjoint of the $\alpha$-twisted action on $L^1(S, E)$: any $\alpha^*$-invariant measurable field $\{A_s\}_{s \in S}$ of non-empty convex weak-* compact subsets $A_s$ of the closed unit ball in $E^*$ admits a measurable $\alpha^*$-invariant section. We can now state V. Kaimanovich’s result.

**Theorem 2** [Kai02] Let $G$ be a locally compact $\sigma$-compact group. There exists a Lebesgue space $(S, \mu)$ endowed with a measure class preserving action of $G$ such that:

(i) The $G$-action on $S$ is amenable.

(ii) The $G$-action on $S$ is doubly $\mathcal{X}^{sp}$-ergodic.

Such a space $(S, \mu)$ is called a strong $G$-boundary [MS02 Def. 2.3].

The space $S$ is a Poisson boundary for a suitable measure on $G$. As mentioned before, this theorem strengthens a result of M. Burger and N. Monod [BM02], who proved that any compactly generated locally compact group possesses a finite index open subgroup which has a strong boundary.

Note that the double $\mathcal{X}^{sp}$-ergodicity of the $G$-action on $S$ implies that the $\Gamma$-action on $S$ is doubly $\mathcal{X}^{sp}$-ergodic [BM02 Prop. 3.2.4] and that $G$ (as well as any finite index subgroup of $\Gamma$) acts ergodically on $S$ and on $S \times S$.

**Reduction to finding a suitable equivariant map**

First, in view of the conclusion of the theorem, we may – and shall – assume until the end of the note that the group $H$ is adjoin. Recall also that if $f : X \to Y$ is a measurable map from a Lebesgue space $(X, \lambda)$ to a topological space $Y$, its essential image is the closed subset of $Y$ defined by: $\text{Essv}(f) := \{y \in Y \mid \lambda(f^{-1}(V)) > 0 \text{ for any neighbourhood } V \text{ of } y\}$, and $f$ is called essentially constant if $\text{Essv}(f)$ reduces to a point. Here is the reduction theorem.

**Theorem 3** Let $G$ be a locally compact second countable topological group, $\Gamma < G$ be a lattice and $\Lambda$ be a subgroup of $G$ with $\Gamma < \Lambda < \text{Comm}_G(\Gamma)$. Let $k$ be a local field and $H$ be a connected almost $k$-simple algebraic group. Assume that $\pi : \Lambda \to H_k$ is a homomorphism such that $\pi(\Lambda)$ is Zariski dense and that there exists a $\Lambda$-equivariant non-essentially constant map $S \to H_k/L_k$, where $L$ is a proper $k$-subgroup of $H$. Then there exists a continuous extension $\overline{\Lambda} \to H_k$ of $\pi$.

The proof uses a simple and powerful ergodic argument [AB94 Sect. 2.3], used many times in the full proof of super-rigidity. We will often deal with maps $\Theta : B \to M$ where $B$ is an ergodic $\Gamma$-space, $M$ is a space with a continuous $H_k$-action and $\Theta$ is equivariant with respect to a group homomorphism $\Gamma \to H_k$. Then if $M$ is a separable complete metrizable space and if the $H_k$-orbits are locally closed in $M$, there is a $H_k$-orbit $O$ in $M$ such that a conull subset of $B$ is sent to $O$ by
\( \Theta \). This is to be combined with the fact that a \( k \)-algebraic action of a \( k \)-group \( G \) on a \( k \)-variety \( V \) induces a continuous action of \( G_k \) on \( V_k \) in the Hausdorff topology, and with the following crucial result, due to I. Bernstein and A. Zelevinski.

**Theorem 4** [BZ76 6.15] Let \( k \) be a local field, \( V \) be a \( k \)-variety and \( G \) be a \( k \)-group acting \( k \)-algebraically on \( V \). Then the orbits of \( G_k \) in \( V_k \) are locally closed.

This theorem has a wide range of application because it implies local closedness of orbits in many spaces. Let \( F(S, W_k) \) be the space of classes of measurable maps from \( S \) to \( W_k \), endowed with the topology of convergence in measure on \( F(S, W_k) \). It is metrizable by a complete separable metric. Then, according to [AB94 Lemma 6.7], the \( H_k \)-orbits in \( F(S, W_k) \) are locally closed. The proof is given for a characteristic zero local field \( k \), but it goes through once the stabilizer \( \text{Stab}_H(w) \) of any \( k \)-rational point \( w \in W_k \), only \( k \)-closed in general, is replaced by the \( k \)-subgroup \( \text{Stab}_{H_k}(w) \).

The ergodic argument applied to the function space \( F(S, W_k) \) is a key point in the proof of the above reduction theorem, whose proof can now be sketched (see [AB94 Sect. 7] for further details).

**Proof.** Since \( \pi(\Lambda)^{\mathbb{Z}} = H \) and since \( \theta \) is \( \Lambda \)-equivariant, we have \( \text{Essv}(\theta)^{\mathbb{Z}} = W \) where \( W := H/L \). We define \( \overline{\theta} : \overline{\Lambda} \rightarrow F(S, W_k) \) by \( \overline{\theta}(\lambda)(s) := \theta(\lambda s) \). It is \( \Lambda \)-equivariant and continuous. Since \( \Lambda \) acts ergodically on \( \overline{\Lambda} \) and since the \( H_k \)-orbits in \( F(S, W_k) \) are locally closed, there is a \( H_k \)-orbit \( O \subset F(S, W_k) \) such that \( \overline{\theta}(\lambda) \in O \) for almost all \( \lambda \in \overline{\Lambda} \). One deduces then from the fact that \( O \) is open in \( \overline{\Lambda} \) and \( \overline{\theta} \) is continuous, that \( \overline{\theta}(\overline{\Lambda}) \subset O \). In particular \( O = H_k \Lambda \). Then it follows from \( \text{Essv}(\overline{\theta})^{\mathbb{Z}} = W \) that \( \text{Stab}_{H_k}(\theta) \) fixes pointwise \( W \) and thus is trivial since \( H \) is adjoint. Therefore the map \( h : \overline{\Lambda} \rightarrow H_k \) defined by \( h(\lambda) = h(\lambda)_s \theta \) for any \( \lambda \in \overline{\Lambda} \), is a continuous homomorphism. Since \( \theta \) is \( \Lambda \)-equivariant, \( h \) is the desired extension of \( \pi \). \( \square \)

### Constructing the required equivariant map

We now sketch the proof of the existence of a \( \Lambda \)-equivariant map as above under the hypotheses of Theorem 4. Since \( k \) is of arbitrary characteristic, the adjoint representation \( \text{Ad} \) of \( H \) need no longer be irreducible. Still, we can choose \( \rho : H \rightarrow \text{GL}(V) \) a faithful rational representation of \( H \), defined and irreducible over \( k \), on a finite-dimensional \( k \)-vector space \( V \). The induced map \( \rho : H_k \rightarrow \text{PGL}(V_k) \) is injective because \( H \) is adjoint, and by [Mar91 1.2.1.3] it is a closed embedding. We have a homomorphism \( \rho \pi : \Gamma \rightarrow \text{PGL}(V_k) \), so that \( \Gamma \) acts by homeomorphisms on the compact metric space \( PV \). This induces a continuous action \( \Gamma \times M^1(PV_k) \rightarrow M^1(PV_k) \), where \( M^1(PV_k) \) is the space of probability measures on \( PV \) endowed with the compact metrizable weak-* topology.

**Proposition 1** Let \( G \) be a locally compact group and \((S, \mu)\) be a Lebesgue space on which \( G \) acts amenably. Then, possibly after discarding an invariant null set in \( S \), there exists a measurable \( \Gamma \)-equivariant map \( \phi : S \rightarrow M^1(PV_k) \).

**Proof.** This follows immediately from Theorem 4.3.5 and Proposition 4.3.9 in [Zim84]. \( \square \)

At this stage, we have a measurable map \( \phi : S \rightarrow M^1(PV_k) \) which is equivariant for the \( \Gamma \)-action only, and which goes to a space of probability measures. The next step provides a \( \Gamma \)-equivariant map to a homogeneous space \( H_k/L_k \).

We denote by \( \text{Var}_k(PV) \) the set of algebraic subvarieties of \( PV \) defined over \( k \) and by \( \text{supp}_Z : M^1(PV_k) \rightarrow \text{Var}_k(PV) \) the map which to a probability measure \( \mu \) associates \( \text{supp}(\mu)^Z \), the Zariski closure of its support. For any \( n \)-dimensional \( k \)-vector space \( W_k \) we set \( \text{Gr}(W_k) := \bigsqcup_{l=0}^n \text{Gr}_l(W_k) \), where \( \text{Gr}_l(W_k) \) is the compact Grassmannian of \( l \)-planes in \( W_k \). By attaching to each projective
variety $X \subset \mathcal{P}V$ its graded defining ideal $I_X$, we see $\text{Var}_k(\mathcal{P}V)$ as a subspace of the compact space $\prod_{d=0}^{\infty} \text{Gr}(k[V]_d)$, where $k[V]_d$ is the space of $d$-homogeneous polynomials on $V_k$. This induces a topology on $\text{Var}_k(\mathcal{P}V)$, and it is proved in [AB94 Sect. 5], by characteristic free arguments, that the map $\text{supp}_\mathcal{Z}$ is measurable and $\text{PGL}(V_k)$-equivariant. Therefore we obtain by composition a $\Gamma$-equivariant measurable map $\Phi : S \to \text{Var}_k(\mathcal{P}V)$, sending each $s \in S$ to the Zariski closure of the support of $\phi(s)$. We denote it by $\Phi$, and call it boundary map.

**Theorem 5 [AB94 Theorem 5.1]** The boundary map $\Phi$ is not essentially constant.

This result follows from the arguments in [AB94 Sect. 5]. To see this, we first note that since $H$ is $k$-simple, $\pi(\Gamma)$ is unbounded and $\pi(A)$ is Zariski dense, the inclusion $A < \text{Comm}_G(\Gamma)$ and the fact that the identity component of an algebraic group is always a finite index subgroup imply that $\pi(\Gamma)$ is Zariski dense in $H$. The other facts needed in [AB94 Sect. 5] are the ergodicity of $\Gamma$ on $S$ and on $S \times S$, and the Furstenberg lemma, all available in our context.

From the result, there is a $d$ for which $\Phi : S \to \text{Gr}(k[V]_d)$ is not essentially constant. The ergodic argument of the previous section and the ergodicity of the $\Gamma$-action on $S$ imply that $\Phi$ essentially sends $S$ to a $H_k$-orbit in $\text{Gr}(k[V]_d)$, which is homeomorphic to a space $H_k/L_k$ for some proper algebraic subgroup $L$ of $H$: we have obtained a $\Gamma$-equivariant measurable map $\phi : S \to H_k/L_k$.

The very last step consists in passing from $\Gamma$- to $\Lambda$-equivariance. Once maps as above are known to exist, the descending chain condition for algebraic subgroups and Zorn’s lemma, as used in [AB94 Sect.7], prove the existence of a couple $(\phi, H_k/L_k)$ satisfying a universal property. The normalizer of $L_k$ in $H$ may only be $k$-closed, but if we denote by $L'$ the Zariski closure of the normalizer of $L_k$ in $H_k$, we get a $k$-subgroup, which is proper by $k$-simplicity of $H$ and such that:

**Theorem 6 [AB94 Corollary 7.2]** The composed map $\theta : S \to H_k/L_k \to H_k/L'_k$ is $\Lambda$-equivariant and measurable.

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