Self-consistent spin waves in magnetized BEC

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We obtain equations of quantum hydrodynamic (QHD) for magnetized spin-1 neutral Bose-Einstein condensate (BEC). System of QHD equations contains the equation of magnetic moment evolution (an analog of the Bloch equation). We account spin-spin interaction along with the short range interaction. We consider self-consistent field approximation of QHD equations. Starting from QHD equation we derive the Gross-Pitaevskii equation for magnetized BEC. We show that Gross-Pitaevskii equation exists under condition that the magnetic moment direction is not change. Using obtained QHD equations we study the dispersion of collective excitation. As in electrically polarized BEC [P. A. Andreev, L. S. Kuz’menkov, arXiv: 1201.2440], in the magnetically polarized BEC there is second wave mode (polarization mode or spin wave), in addition to the Bogoliubov’s mode. Second wave solution appears due to the magnetic moment evolution. The influence of magnetization on dispersion of Bogoliubov’s mode is found. We found strong difference of dispersion properties of waves in magnetized BEC from electrically polarized BEC.

I. INTRODUCTION

Special attention at studying Bose-Einstein condensation (BEC) in ultracold atomic vapors gives to polarized BEC [1]- [14]. The term a polarized BEC is used in literature as for electrically [15], [16] as for magnetically [17], [18] polarized BEC. In the recent years, an attention increases to the BEC of atoms with magnetic polarization because of the BEC of \( ^{52}\text{Cr} \) was experimentally realized [17]. These atoms have a large magnetic moment \( 6\mu_B \), where \( \mu_B \) is the Bohr magneton. There are a lot of attempts to realize electrically polarized BEC [15], [16], for such aim two nuclear molecules are usually used. Attention to electrically polarized BEC is particularly caused by the fact that molecules have electrical dipole moment of order of 1 Debye and this means that the force of dipole-dipole interaction of molecules on four orders more than the force of spin-spin interaction of such atom as \( ^{52}\text{Cr} \).

Nevertheless, in the chromium atoms the quantity of spin-spin interaction is comparable with the value of the short range interaction. Thereby, a spin-spin interaction makes anisotropy. Consequently, spin-spin interaction has essential contribution in chromium atoms static and dynamic properties.

Usually, for polarized BEC studying a generalized Gross-Pitaevskii (GP) equation is used [1] - [13]. A non-linear Schrödinger equation is usually used for quantum gases studying. One is also used for dipoles BEC description. In the last case, in GP equation a new term is added. However, this approximation includes interaction among parallel dipoles, and does not account spatial and temporal dipoles evolution. Different approximations based on scattering process were analyzed in Ref. [19]. Particularly, where were considered condition there can be used first Born approximation and presented generalization of GP equation for the scattering of polarized atoms beyond first Born approximation. In Ref. [12] authors developed method accounted spatially inhomogeneous distribution of dipoles directions. This method includes the potential of interaction unparallel dipoles in GP equation, and the second equation determining dipoles moment of a volume unit. In paper [13] were suggested a three fluid hydrodynamic model for description of a spin-1 Bose gas at finite temperatures, three sounds are evaluated there in a mean-field approximation. The method of many-particle quantum hydrodynamic for electrically polarized BEC was developed in Ref. [14]. This approximation allows describe as spatial as temporal evolution of electric dipole moment. For magnetized BEC studying in this paper we use the method of quantum hydrodynamics (QHD). This method was developed in the recent years for various physical systems. Namely, it was quantum plasma [20], [21], plasma of particles with the own magnetic moment [22]- [28], relativistic quantum plasma [29], ultracold Bose and Fermi gases with nonlocal interaction [30] and three-particle interaction at nonzero temperatures [31], quantum particles with electrical polarization [32], particularly being in the BEC state [14], [33], the graphene electrons [34] and BEC of graphene excitons [35]. Influence of magnetic moment evolution on the collective excitation dispersion for quantum plasma were considered in Ref. [36], where were found that temporal evolution of magnetic moment leads to existence of new branches in dispersion dependence. More detailed analysis of spin waves in quantum plasma and methods of these waves generation were considered in Ref. [26].

In this paper we develop a self-consistent theory for a magnetic moment evolution of the BEC of neutral spin-1 atoms. We derive the QHD equations: continuity equation, momentum balance equation (Euler equation) and magnetic moment evolution equation. The last one describe the temporal evolution of the BEC magnetization and it’s influence on BEC properties. In Ref. [14] were shown that temporal evolution of collective electric polarization in BEC of particles having electric dipole moment leads to existence of new type waves in BEC - polarization waves,
where supposed that BEC located in external uniform electric field. Dispersion dependence of collective excitations shows no dependence on direction of wave propagation and exhibits comparably stable excitation spectrum. Here, we study the linear dispersion in three dimensional magnetized BEC located in external magnetic field. We find the contribution of equilibrium magnetization in dispersion dependence of Bogoliubov’s mode and obtain that magnetization leads to destabilization of Bogoliubov’s mode. The last fact coincide to the other authors results [5]- [8]. We show existence of magnetic moment waves (spin waves). The spin waves propagate along direction of external magnetic field.

Our paper is organized as follows. In Sec. II we describe the used model and present the QHD equations for magnetized BEC. One of the equations is the magnetic moment evolution equation, this equation describe evolution of magnetic moment in space and time due to the interaction particles magnetic moments each other and with the external field. Corresponding GP equation is also presented in Sec. II. Condition for GP equation derivation is described. In Sec. III we presented the method of QHD equations solution and the method of obtaining of the dispersion dependence. In Sec. IV the dispersion spectrum properties of the collective excitations are described. In Sec. V brief summary of obtained results is presented.

II. MODEL

We do not consider here the details of the QHD equation derivation, but we present and use in the paper the QHD equation for the neutral Bose particles with the own magnetic moment being in the BEC state. All details for equation derivation can be found in Refs. [20], [22], [30], [32]. Here we present the Schrodinger equation used for QHD equation derivation

\[ \text{i} \hbar \partial_t \psi_s(R, t) = \left( \sum_i \left( \frac{p_i^2}{2m_i} - \gamma_i \hat{s}_i^\alpha B_i^\alpha(R_{\text{ext}}) \right) + \frac{1}{2} \sum_{i,j \neq i} \left( U_{ij} - \gamma_i \gamma_j G_{ij}^{\alpha \gamma} \hat{s}_i^\alpha \hat{s}_j^\gamma \right) \right) \psi_s(R, t), \tag{1} \]

where we include short range and spin-spin interactions, and action of external magnetic field on particles spin. In the Schrodinger equation (1) we use following designations: \( \gamma_i \) is the gyromagnetic ratio, \( p_i \) is the operator of momentum, \( B_{\text{ext}}(R) \) is the external magnetic field acting on \( i \)-th particle, \( U_{ij} \) present the short range interaction (SRI), the Green function of the spin-spin interaction (SSI) has form

\[ G_{ij}^{\alpha \beta} = \partial_\alpha \partial_\beta (1/r_{ij}) + 4\pi \delta_\alpha_\beta \delta(R_{ij}). \tag{2} \]

For spin matrices \( \hat{s}_i^\alpha \) the commutation relations are

\[ [\hat{s}_i^\alpha, \hat{s}_j^\gamma] = i\delta_{ij} \varepsilon^{\alpha \beta \gamma} \hat{s}_i^\beta. \]

Thereby, we consider spin-1 Bose particles we present here the evident form of the spin matrixes \( \hat{s}_i^\alpha \) for particles with spin equal to 1:

\[ \hat{s}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{s}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \]

\[ \hat{s}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

At derivation of system of QHD equations we need to write an evident form of Hamiltonian of dipole-dipole interaction. Usually using expressions for a Hamiltonian of dipole-dipole interaction are the same as for electric for magnetic dipoles:

\[ H_{dd} = \frac{\delta^{\alpha \beta} - 3r_{\alpha \beta} r_{\gamma \delta}}{r^2}, \tag{3} \]

but in this paper for SSI we used (2). In Ref. [14] were shown that for electric dipoles interaction we need to use the first term in formula (2).

There is well-known identity

\[ -\partial_\alpha \partial_\beta \frac{1}{r} = \frac{\delta^{\alpha \beta} - 3r_{\alpha \beta} r_{\gamma \delta}}{r^3} + \frac{4\pi}{3} \delta^{\alpha \beta} \delta(r), \tag{4} \]

so we can see the difference between usually using Hamiltonian and one’s used in this paper (2). The corrections of our choice followed from the fact that the equations obtained in the paper coincide to the Maxwell equations.

It has been shown by Breit [37]- [40] that a Hamiltonian for spin-spin interaction, and, as a consequence, for the interaction of magnetic moments contains a term that is proportional to Dirac \( \delta \)-function \( \delta(r_1 - r_2) d_1^2 d_2^2 \) along with (3). The coefficient of the \( \delta \)-function has been refined later [22] by using idea that the Hamiltonian is in accord with Maxwell’s free equations, such as \( div B = 0 \). The resultant expression for the spin-spin interaction Hamiltonian is:

\[ H_{\mu \nu} = \left( \frac{\delta^{\alpha \beta} - 3r_{\alpha \beta} r_{\gamma \delta}}{r^3} - \frac{8\pi}{3} \delta_{\alpha \beta} \delta(r) \right) \mu_1^\alpha \mu_2^\beta. \tag{5} \]

Formula (5) is obtained at substitution of relation (4) in to the formula (2).

The system of QHD equations for magnetized BEC consists of:

- continuity equation
  \[ \partial_t n(r, t) + \partial_\alpha (n(r, t) v^\alpha(r, t)) = 0, \tag{6} \]
  where \( n(r, t) \) is the concentration of particles, \( v(r, t) \) is the velocity field. In this equation describing the whole particle number invariance, and evolution of concentration owing to particles displacement;
- momentum balance equation (Euler equation)
  \[ mn(r, t)(\partial_t + v \nabla) v^\alpha(r, t) + \partial_\beta p^{\alpha \beta}(r, t) \]
\[-\frac{\hbar^2}{4m} \partial^\alpha \Delta n(r, t) + \frac{\hbar^2}{4m} \partial^\beta \left( \frac{\partial^\alpha n(r, t) \cdot \partial^\beta n(r, t)}{n(r, t)} \right) \]

\[= \Upsilon n(r, t) \partial^\alpha n(r, t) + \frac{1}{2} \Upsilon_2 \partial^\alpha \Delta n^2(r, t) + M^\beta(r, t) \partial^\alpha B^\beta(r, t), \tag{7} \]

where $\nabla$ is the gradient operator, $\Delta$ is the Laplace operator, $M(r, t)$ is the density of magnetic moment (magnetization), $B(r, t)$ is the magnetic field, $p^{\alpha \beta}(r, t)$ is the kinetic pressure describing the temperature effects and can be vanished for the BEC,

\[\Upsilon = \frac{4\pi}{3} \int dr(r)^3 \frac{\partial U(r)}{\partial r}, \tag{8} \]

and

\[\Upsilon_2 = \frac{\pi}{30} \int dr(r)^3 \frac{\partial U(r)}{\partial r}, \tag{9} \]

describing the SRI influence on particles dynamic. The momentum balance equation for the BEC with SRI up to TOIR approximation were obtained in Ref. [30] in the absence of the magnetization contribution. The second term in the right-hand side of equation (7) is the term appearing in the TOIR approximation. This term is an example of non-local SRI, analogous terms were obtained in Ref.s [41], [42]:

balance equation of magnetic moment (Bloch equation)

\[\partial_t M^\alpha(r, t) + \nabla^\beta J^\beta_M(r, t) = \frac{\gamma}{\hbar} \varepsilon^{\alpha \beta \gamma} M^\beta(r, t) B^\gamma(r, t), \tag{10} \]

where a tensor of spin current $J^\beta_M$ arises. Vanishing by thermal motion we have $J^\alpha_M = M^\alpha v^\beta$. This equation describes the dynamic of magnetization and influence of external magnetic field and inter-particle interaction on magnetization evolution. For the first time the many-particle QHD equations for spinning particles were derived in Ref. [22] for the case spinning charged Fermi particles. It was made in 2001.

During the derivation of QHD equation we do not consider the scattering problem and do not use conception of scattering for interpretation of the SRI and SSI. However, for simplicity of comparison of our results with the results of other authors we reduce connection of $\Upsilon$ and $\Upsilon_2$ with the scattering amplitude [30].

The first order of the interaction radius interaction constant for dilute gases has the form $\Upsilon = -g = -4\pi \hbar^2 a/m$, where $g$ is the interaction constant usually used in GP equation, $a$ is the scattering length [30, 43]. The value $\Upsilon_2$ may be expressed approximately as $\Upsilon_2 = -\theta a^2 \Upsilon / 8$ [30], where $\theta$ is a constant positive value about 1, which depends on the interatomic interaction potential. Finally, $\Upsilon_2$ takes the form $\Upsilon_2 = -\pi \theta \hbar^2 a^3 / 2m$.

We consider the three-dimensional (3D) system of Bose particles. It allows us to introduce in equations (7) and (10) the magnetic field caused by particles magnetic moments. Spin-spin interaction in these equations at derivation appears in the integral form (see for example [22]). In self-consistent approximation in 3D particles system we can represent interaction via magnetic field which has form

\[B^\alpha(r, t) = \int dr' G^\alpha\beta(|r - r'|)M^\beta(r', t), \tag{11} \]

which satisfies to the Maxwell equation:

\[\text{curl} B(r, t) = 4\pi \text{curl} M(r, t). \tag{11} \]

From equations (6)- (10) we can see that short range interaction, the terms proportional to $\Upsilon$ and $\Upsilon_2$, gives contribution in the momentum balance equation only.

In the absence of magnetic moment direction evolution $M^\alpha(r, t) = n(r, t) \mu^\alpha(r, t)$, $\mu^\alpha(r, t) = \text{const}$ (supposing that magnetization change because of the concentration changing only), we can derive the GP equation for spinning neutral particles in the BEC state. We notice that for the $\mu^\alpha(r, t) = \text{const}$ the magnetic moment evolution equation does not exist (It does not need). In this case the magnetic moment evolution reduces to the concentration evolution $\mathbf{M}(r, t) \sim n(r, t)$. Particles dynamic is described by two equations: the continuity equation (6) and simplification of the Euler equation (7).

The method of GP equation derivation from QHD equation is described in Ref.s [30] and [32]. Resulting GP equation, in the absence of terms appeared up to TOIR approximation, has form

\[\text{th} \partial_t \Phi(r, t) = \left( -\frac{\hbar^2}{2m} \nabla^2 + g \cdot \Phi(r, t) \right)^2 + \mu_0^3 \mu_0 \int dr'|G^\alpha\beta(|r - r'|)|\Phi(r', t)|^2 \Phi(r, t), \tag{12} \]

where $G^\alpha\beta$ the Green function of SSI [2], using evident form of the Green function we can rewrite GP equation

\[\text{th} \partial_t \Phi(r, t) = \left( -\frac{\hbar^2}{2m} \nabla^2 + g \cdot \Phi(r, t) \right)^2 + \mu_0^3 \mu_0 \int dr'|1 - 3 \cos^2 \theta' \cdot |\Phi(r', t)|^2 \]

\[-8\pi \mu_0 \frac{\mu_0^3}{3} |\Phi(r, t)|^2 \Phi(r, t). \tag{13} \]

This equation has difference from usually used for magnetized BEC [2], [3], [9], [11]. This difference appear because of second term in GP proportional to delta function. The last term in equation (13) appear because of the second term in Green function of SSI [5].

At low temperature in the external magnetic field the atoms magnetic moments directed along the external field,
because of absence disordered factors as a thermal motion. However, the perturbation of dipoles direction might propagate there. These perturbations can exhibit as waves. In this case we have deal with the spin waves. It is very spread phenomenon in the system of particles with magnetic moment. For example we can admit that spin waves there are in ferromagnetic, but there magnetic moments bound with the unit of crystalline lattice. In gases atoms can move through the system. Thus, in gases there is one more mechanism of magnetization changing, compare with ferromagnetic. On the other hand, the Bose-Einstein condensation of new wave mode, it is the wave of electrical polarization. Shown that electric dipoles evolution leads to the existence in the BEC, it is the Bogoliubov mode. In Ref. [14] were considered magnetic moments motion there is the collective excitation included in GP equation if they have a simple enough form. 

In fact, quantum spin-spin correlations might be included in GP equation also described the magnetic moment evolution. Consequently, this equation contains information about spin waves considered in our paper. At derivation of the hydrodynamic equations from the spinor nonlinear Schrodinger equation the SSI appears in self-consistent Euler equations only. Thus, the spinor generalization of GP equation also described the magnetic moment evolution. In the case of spin-0 particles or if we do not account magnetic moments motion there is the collective excitation in the BEC, it is the Bogoliubov mode. In Ref. [14] were shown that electric dipoles evolution leads to the existence of new wave mode, it is the wave of electrical polarization. In this paper, we show the existence of the spin waves (magnetic moment waves) in magnetized BEC. Below we consider wave dispersion calculations based on above described model.

III. METHOD OF SOLUTION

We can analyze the linear dynamics of elemental excitations in the magnetized 3D BEC using the QHD equations \[6\], \[7\], \[10\] and \[11\]. Let’s assume the system is placed in an external magnetic field \(B_0 = B_0\hat{e}_z\). The values of equilibrium concentration \(n_0\) and equilibrium magnetization \(M_0 \approx B_0\) for the system in an equilibrium state are constant and uniform and its velocity field \(v^\alpha(r,t)\) values equal to zero.

We consider the small perturbation of equilibrium state \(n = n_0 + \delta n, v^\alpha = 0 + v^\alpha\) and \(M^\alpha = M_0^\alpha + \delta M^\alpha\). Substituting these relations into system of equations \[6\], \[7\], \[10\] and \[11\] and neglecting nonlinear terms, we obtain a system of linear homogeneous equations in partial derivatives with constant coefficients. Passing to the following representation for small perturbations \(\delta f\)

\[
\delta f = f(\omega, k)e^{i(\omega t + ikr)}
\]

one yields the homogeneous system of algebraic equations. The magnetic field strength is assumed to have a nonzero value. Expressing all the quantities entering the system of equations in terms of the magnetic field, we come to the equation \(\Lambda^{\alpha\beta}(\omega, k) \cdot B^\beta(\omega, k) = 0\), where \(\Lambda^{\alpha\beta}(\omega, k)\) is the dispersion matrix. The evident form of the dispersion matrix \(\Lambda^{\alpha\beta}(\omega, k)\) is presented in the Appendix. In this case, the dispersion equation is \(det \Lambda(\omega, k) = 0\). Solving this equation with respect to \(\omega^2\) we obtain following results.

IV. ELEMENTARY EXCITATIONS IN THE POLARIZED BEC

The dispersion characteristic for EE in BEC can be expressed in the form of

\[\omega^2 = \left(\frac{\hbar^2}{4m^2} + \frac{n_0 \Upsilon}{m}\right)k^4 - \frac{n_0}{m}(\Upsilon + 4\pi\mu_0^2)k^2\], \hspace{1cm} (14)

where \(\mu_0 = \chi B_0/n_0\), where \(\chi\) is the ratio between equilibrium magnetic susceptibility and magnetic permeability. In the case when all magnetic dipoles directed parallel to external field (in equilibrium state) we have \(\mu_0 = \gamma\). From formula (14) we can see that for the repulsive SRI (\(\Upsilon < 0\)) at long wave length limit magnetization lead to instability of the Bogoliubov’s mode because the frequency square become negative:

\[\omega^2 = \frac{n_0}{m}(\Upsilon + 4\pi\mu_0^2)k^2 < 0\]

at condition \(4\pi\mu_0^2 > |\Upsilon|\).

We obtained three solutions for spin waves. They frequencies are

\[\omega = \Omega,\]

\[\omega = \Omega(1 + 4\pi\chi),\]

where \(\Omega = \gamma B_0/h\) is the cyclotron frequency. It is the frequency of the single magnetic moment precession in the external uniform magnetic field. For the case when all magnetic dipoles directed parallel to external field fully magnetized BEC, which can be considered as ferromagnetic state, we have

\[\omega = \Omega(1 + 4\pi\chi) = \begin{pmatrix} \Omega(1 + 8\pi) \\ \Omega \end{pmatrix}.\] \hspace{1cm} (16)

Thus, we have only two solutions in this case because one of solutions \(\omega = \Omega(1 + 4\pi\chi)\) equal to \(\Omega\).
V. CONCLUSION

We researched dispersion properties of magnetized 3D BEC. We obtained the contribution of equilibrium magnetization in the Bogoliubov’s mode dispersion dependence. We found that time evolution of magnetic moments lead to existence of new type of waves in magnetized BEC, it is spin waves. We found that SRI has no influence of spin wave dispersion. At comparison dispersion properties magnetized BEC with the electrically polarized BEC we got large difference between them.

We studied dispersion of collective excitations by means of QHD equations derived in this paper for magnetized BEC.

We have found two types of branches of dispersion curve. These are Bogoliubov’s mode and spin waves.

In correspondence with the other authors we obtained that magnetization leads to destabilizing of Bogoliubov’s mode. For attractive SRI at small wave vector we have \( \omega^2 = -n_0 \mid \Upsilon \mid /m \cdot k^2 = -4\pi\hbar^2 /a \mid n_0/m^2 \cdot k^2 \), this spectrum is unstable. Addition of magnetization leads to instability growing \( \omega^2 = -(n_0/m) \mid \Upsilon \mid + 4\pi\mu_0^2/k^2 \). For repulsive SRI \( \Upsilon = - \mid \Upsilon \mid \) and in the absence of the magnetization we have stable sound wave \( \omega^2 = n_0 \mid \Upsilon \mid /m \cdot k^2 \), but magnetization disturbs stability \( \omega^2 = (n_0/m) \mid \Upsilon \mid - 4\pi\mu_0^2/k^2 \). This spectrum remains stable in two cases. The first one while magnetization is small compare with the constant of SRI \( \Upsilon \). The second case corresponds to the large constant of SRI \( \Upsilon \) which can be reached by the Feshbach resonance [47], [48].

In general case in magnetized BEC exist three spin waves with frequencies: \( \omega = \Omega \) and \( \omega = \Omega(1 \pm 4\pi\chi) \). But, for fully magnetized ferromagnetic BEC \( \Omega(1 - 4\pi\chi) \) equal to \( \Omega \), so we have only two solutions \( \omega = \Omega \) and \( \omega = \Omega(1 + 8\pi) \). As any physical system of particles with magnetic moments, especially ferromagnets, the magnetized BEC shows existence of the spin waves. In this paper we consider the macroscopic spin dynamic described by QHD, but there are a lot of well-known method’s based on second quantization [49], [53]. These methods are close to the Bese-Habbard model and can be used together for studying of microscopic influence of magnetic moment dynamic on quantum gases properties.

APPENDIX: EVIDENT FORM OF DISPERSION MATRIX

We describe \( \Lambda^{\alpha\beta}(\omega, k) \) by presenting of the evident form of each component of the matrix

\[
\begin{align*}
\Lambda^{xx}(\omega, k) &= 4\pi\omega k_z n_0 \frac{\gamma\mu_0}{h} \frac{1}{\omega^2 - 2 \frac{\gamma B_0^2}{\hbar^2}}; \\
\Lambda^{xy}(\omega, k) &= ik_z + 4\pi ik_x n_0 \frac{\gamma B_0}{h} \frac{\gamma \mu_0}{\omega^2 - 2 \frac{\gamma B_0^2}{\hbar^2}}; \\
\Lambda^{xz}(\omega, k) &= -ik_y + 4\pi ik_y n_0 \mu_0 \Xi; \\
\Lambda^{yz}(\omega, k) &= \Lambda^{xy}(\omega, k)^*; \\
\Lambda^{yy}(\omega, k) &= \Lambda^{xy}(\omega, k); \\
\Lambda^{yz}(\omega, k) &= ik_x - 4\pi ik_x n_0 \mu_0 \Xi; \\
\Lambda^{zx}(\omega, k) &= -i k_y \\
\Lambda^{zz}(\omega, k) &= 0,
\end{align*}
\]

where

\[
\Xi = \frac{-\mu_0 k^2}{m\omega^2 - \frac{\mu_0^2}{4m^2} + (\Upsilon + 4\pi\mu_0^2)n_0 k^2 - n_0 \Upsilon_2 k^4}
\]

Quantity \( \Xi \) contains whole information about SRI.

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