Magnon magic angles in 2D twisted ferromagnetic bilayers

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Abstract. Twisted 2D magnetic bilayers currently constitute an active research field parallel to their fermionic counterparts. The single-particle theory applied to excitations in honeycomb monolayers and AA/AB stacked bilayers yields similar properties in bosonic and fermionic materials. Despite the extensive experimental and theoretical studies on the band structure of twisted bilayer electronic systems, the spin wave (or magnon) theory have not yet been developed for their magnetic counterpart. We here present the first formalism of the linear spin wave theory for weakly coupled twisted ferromagnetic honeycomb bilayers (tFBL), including exchange and Dzyaloshinskii-Moriya interactions (DMI). We highlight the similarities and differences between the magnon spectrum in tFBL and the electronic structure of twisted bilayer graphene (tBLG). Unlike the tBLG case, we demonstrate that the spin wave dispersion in tFBL depends on the relative translation between the layers. tFBL hence presents energy dispersion profiles that are not possible in tBLG. We further demonstrate the existence of magnon magic angles, at which the spin wave spectrum presents flat moiré superlattice minibands, in analogy with twisted bilayer graphene (tBLG). We additionally analyze the consequences of the DMI, known to be present in 2D magnetic material. This DMI is found to induce a band gap while increasing the number of flat bands. We hope the present study will motivate further the field of twisted 2D magnetic materials, which might open new horizons in the field of 2D magnetic materials.

Introduction. Magnetism in two-dimensional (2D) materials has recently been realized [1, 2] which fueled the field of 2D magnetism [3-32]. In these bosonic Dirac materials, magnetic anisotropy is found to overcome thermal fluctuations and hence stabilize the magnetically ordered ground state at finite temperatures. To a large extent, the theoretical investigation and experimental realization of bosonic Dirac materials was motivated by their fermionic counterpart. The first experimental observation of fermionic Dirac energy dispersion was realized in graphene [33], prior to bosonic 2D material realization. Research studies in graphene demonstrated that the electronic properties change drastically in bilayers compared to single layers [34 - 38]. A particularly interesting class of bilayer graphene is the twisted bilayer graphene (tBLG), presenting moiré patterns and bands as a result of the twist angle. The tBLG was found to present fascinating electronic and optical properties, giving rise to novel physics that is completely absent in single graphene sheets or graphene bilayers in AA/AB configurations [39-59].
Very recently, a rapidly increasing interest in the exotic physics of twisted 2D magnetic bilayers emerged [60-64]. The current studies focus on the ability to control the interlayer exchange coupling via the relative twist angle, offering the opportunity to manipulate the magnetic ground states in these bilayers.

In the exchange interaction limit, the linear spin wave theory for a single layer honeycomb ferromagnet is mathematically identical to the linear electronic tight-binding theory in graphene. The analogy was also demonstrated in 2D bilayer magnetic materials, where the existing spin wave studies are restricted to AB or AA stacking [8, 10, 16, 28]. The topic of spin wave excitations in twisted 2D magnetic bilayers, to our knowledge, has not yet been initiated. In view of the groundbreaking physics present in tBLG, its magnonic analogue should be of interest for the rapidly developing field of 2D magnetic Dirac materials. This motivated the present work to develop the spin wave theory on weakly coupled 2D twisted honeycomb ferromagnetic bilayers (tFBL). These systems should be easy to realize in the family of chromium trihalides CrX₃ (X = F, Cl, Br and I). Our formalism focusses on the region near the corners of the moiré superlattice Brillouin zone (BZ) where most of the exotic physics takes place. We predict the existence of magnon magic angles and flat spin wave bands in the spin wave spectrum of tFBL. Beyond the moiré physics in tBLG, we demonstrate the possibility to manipulate the spin wave spectrum in tFBL via the intralayer DMI and the relative translation between the layers. For example, the magnon spectrum for rotated AA ferromagnetic bilayers, with zero DMI, is qualitatively identical to the electronic structure of tBLG. Those obtained by rotating AB stacked bilayers, however, a significantly different gapped band structure. tFBL hence present spin wave dispersion profiles absent in their fermionic counterparts.

**Heisenberg Hamiltonian.** Consider first a single ferromagnetic honeycomb sheet. The coordinate system (Fig. 1) is chosen such that the zigzag and armchair directions are along the x-axis and the y-axis respectively. The spins on the A and B sublattices are aligned parallel to the z-axis. Each unit cell contains two spins that belong to different sublattices. The lattice basis vectors are \( \vec{a}_1 = \alpha (1/2, \sqrt{3}/2) \) and \( \vec{a}_2 = \alpha (-1/2, \sqrt{3}/2). \) with the distance between two neighboring sites \( d = \alpha/\sqrt{3}. \) The reciprocal lattice basis vector are \( \vec{b}_1 = \frac{2\pi}{3d} (\sqrt{3}, 1) \) and \( \vec{b}_2 = \frac{2\pi}{3d} (-\sqrt{3}, 1). \) The three nearest neighbors for an A-site are at relative positions \( \vec{\delta}_1^A, \) with \( \vec{\delta}_1^A = \alpha (0,1/\sqrt{3}) \), \( \vec{\delta}_2^A = \alpha (1/2,-\sqrt{3}/6) \) and \( \vec{\delta}_3^A = \alpha (-1/2,-\sqrt{3}/6). \) For a B-site, \( \vec{\delta}_i^B = -\vec{\delta}_i^A. \)
We add another layer to form a ferromagnetic bilayer in AB configuration (Bernal stacking). Sites in layer $l = 1, 2$ are denoted $A_l$ and $B_l$. The $B_2$-sublattice is placed exactly above the $A_1$-sublattice, with a constant ferromagnetic interlayer exchange $J_\perp$ between them. The interlayer exchange for $A_2$ and $B_1$ are neglected.

To form the tFBL, we translate layer 2 by a vector $\vec{\tau}_0$ followed by opposite rotations of layer 1 and layer 2 by $\theta/2$ (clockwise for 1 and anti-clockwise for 2). With $R_\theta$ representing a 2D anticlockwise rotation, the lattice vectors $\vec{a}_{l,1}$ and $\vec{a}_{l,2}$ are given as $\vec{a}_{l,\alpha} = R_{\theta/2}(\vec{a}_\alpha + \vec{\tau}_0)$, $\vec{a}_{1,\alpha} = R_{-\theta/2} \vec{a}_\alpha$, whereas the reciprocal vectors read $\vec{b}_{l,\alpha} = R_{\theta/2} \vec{b}_\alpha$ and $\vec{b}_{1,\alpha} = R_{-\theta/2} \vec{b}_\alpha$.

The positions of the atoms on the four different sublattices can then be expressed as

\begin{align}
\vec{R}_{A_1} &= \vec{R}_1 + \vec{\tau}_{1,A} \\
\vec{R}_{B_1} &= \vec{R}_1 + \vec{\tau}_{1,B} \\
\vec{R}_{A_2} &= \vec{R}_2 + \vec{\tau}_{2,A} \\
\vec{R}_{B_2} &= \vec{R}_2 + \vec{\tau}_{2,B}
\end{align}

With $\vec{R}_l = n_1 \vec{a}_{l,1} + n_2 \vec{a}_{l,2}$ ($n_1, n_2 \in \mathbb{Z}$), $\vec{\tau}_{1,A} = (0,0)$, $\vec{\tau}_{1,B} = R_{-\theta/2}(0,d)$, $\vec{\tau}_{2,A} = R_{\theta/2}[(0,-d) + \vec{\tau}_0]$, and $\vec{\tau}_{2,B} = R_{\theta/2} \vec{\tau}_0$.

The twist gives rise to a moiré superlattice with reciprocal basis vectors.
The semi-classical Heisenberg Hamiltonian, including only nearest neighbors exchange interaction, can be expressed as

\[
\mathcal{H} = -J \sum_{\vec{R}_{A_1}\delta_i^A} \vec{S}^{A_1}(\vec{R}_{A_1}, t) \cdot \vec{S}^{B_1}(\vec{R}_{A_1} + \delta_i^A, t) - J \sum_{\vec{R}_{A_2}\delta_i^A} \vec{S}^{A_2}(\vec{R}_{A_2}, t) \cdot \vec{S}^{B_2}(\vec{R}_{A_2} + \delta_i^A, t)
\]

\[
- \sum_{\vec{R}_{A_1}\vec{R}_{A_2}} J_\perp(\vec{R}_{A_1}, \vec{R}_{A_2}) \vec{S}^{A_1}(\vec{R}_{A_1}, t) \cdot \vec{S}^{A_2}(\vec{R}_{A_2}, t)
\]

\[
- \sum_{\vec{R}_{A_1}\vec{R}_{B_2}} J_\perp(\vec{R}_{A_1}, \vec{R}_{B_2}) \vec{S}^{A_1}(\vec{R}_{A_1}, t) \cdot \vec{S}^{B_2}(\vec{R}_{B_2}, t)
\]

\[
- \sum_{\vec{R}_{B_1}\vec{R}_{A_2}} J_\perp(\vec{R}_{B_1}, \vec{R}_{A_2}) \vec{S}^{B_1}(\vec{R}_{B_1}, t) \cdot \vec{S}^{A_2}(\vec{R}_{A_2}, t)
\]

\[
- \sum_{\vec{R}_{B_1}\vec{R}_{B_2}} J_\perp(\vec{R}_{B_1}, \vec{R}_{B_2}) \vec{S}^{B_1}(\vec{R}_{B_1}, t) \cdot \vec{S}^{B_2}(\vec{R}_{B_2}, t)
\]

(3)

\( J > 0 \) is the in-plane exchange interaction coefficient and \( \vec{S}(\vec{R}, t) \) denotes the spin at site \( \vec{R} \) and time \( t \). The coefficient \( J_\perp(\vec{R}, \vec{R}') \) denotes the inter-layer exchange interaction coefficient.

**Landau-Lifshitz equations.** We first derive the Landau-Lifshitz (LL) equations for the sublattice \( A_1 \). In the semi-classical treatment of spin waves [11, 29-32, 64-70], the effective exchange fields \( \vec{H}^{A_1} \) acting on the magnetic moment \( \vec{M}^{A_1} \) is deduced from the Heisenberg Hamiltonian as

\[
\vec{H}^{A_1}(\vec{R}_{A_1}, t) = -J \sum_{\delta_i^A} \vec{M}^{B_1}(\vec{R}_{A_1} + \delta_i^A, t) - \sum_{\vec{R}_{A_2}} J_\perp(\vec{R}_{A_1}, \vec{R}_{A_2}) \vec{M}^{A_2}(\vec{R}_{A_2}, t)
\]

\[
- \sum_{\vec{R}_{B_2}} J_\perp(\vec{R}_{A_1}, \vec{R}_{B_2}) \vec{M}^{B_2}(\vec{R}_{B_2}, t)
\]

(4)
The LL equations of motion are given by $\partial_t \vec{M}^{A_1} = \vec{M}^{A_1} \times \vec{H}^{A_1}$ (for simplicity, we have suppressed the gyromagnetic ratio from the equation). The x-component yields

$$\partial_t M^{A_1}_x (\vec{R}_{A_1}, t) = - \left[ 3JM + M \sum_{\vec{R}_{A_2}} J_\perp (\vec{R}_{A_1}, \vec{R}_{A_2}) + M \sum_{\vec{R}_{B_2}} J_\perp (\vec{R}_{A_1}, \vec{R}_{B_2}) \right] M^{A_1}_y (\vec{R}_{A_1}, t)$$

$$+ JM \sum_{\vec{R}_{A_2}} M^{B_1}_y (\vec{R}_{A_1} + \vec{\delta}^{A_1}_i, t) + M \sum_{\vec{R}_{A_2}} J_\perp (\vec{R}_{A_1}, \vec{R}_{A_2}) M^{A_2}_y (\vec{R}_{A_2}, t)$$

$$+ M \sum_{\vec{R}_{B_2}} J_\perp (\vec{R}_{A_1}, \vec{R}_{B_2}) M^{B_2}_y (\vec{R}_{B_2}, t)$$

while the y-component gives

$$\partial_t M^{A_1}_y (\vec{R}_{A_1}, t) = \left[ 3JM + M \sum_{\vec{R}_{A_2}} J_\perp (\vec{R}_{A_1}, \vec{R}_{A_2}) + M \sum_{\vec{R}_{B_2}} J_\perp (\vec{R}_{A_1}, \vec{R}_{B_2}) \right] M^{A_1}_x (\vec{R}_{A_1}, t)$$

$$- JM \sum_{\vec{R}_{A_2}} M^{B_1}_x (\vec{R}_{A_1} + \vec{\delta}^{A_1}_i, t) - M \sum_{\vec{R}_{A_2}} J_\perp (\vec{R}_{A_1}, \vec{R}_{A_2}) M^{A_2}_x (\vec{R}_{A_2}, t)$$

$$- M \sum_{\vec{R}_{B_2}} J_\perp (\vec{R}_{A_1}, \vec{R}_{B_2}) M^{B_2}_x (\vec{R}_{B_2}, t)$$

We assume harmonic time dependence of the magnetizations, with frequency $\omega$. Defining $M^{\alpha_i} = M^{\alpha_i}_x + i M^{\alpha_i}_y$ we can combine the x and y components of the LL equation to get

$$\omega M^{A_1}(\vec{R}_{A_1}) = \left[ 3JM + M \sum_{\vec{R}_{A_2}} J_\perp (\vec{R}_{A_1}, \vec{R}_{A_2}) + M \sum_{\vec{R}_{B_2}} J_\perp (\vec{R}_{A_1}, \vec{R}_{B_2}) \right] M^{A_1}(\vec{R}_{A_1})$$

$$- JM \sum_{\vec{R}_{A_2}} M^{B_1}(\vec{R}_{A_1} + \vec{\delta}^{A_1}_i) - M \sum_{\vec{R}_{A_2}} J_\perp (\vec{R}_{A_1}, \vec{R}_{A_2}) M^{A_2}(\vec{R}_{A_2})$$

$$- M \sum_{\vec{R}_{B_2}} J_\perp (\vec{R}_{A_1}, \vec{R}_{B_2}) M^{B_2}(\vec{R}_{B_2})$$

(5)
Similar equations hold for $M^B_1$, $M^A_2$, and $M^B_2$. It is now possible to write a general equation which yields the four LL equations as follows

$$\omega M^\alpha_l(\vec{R}_\alpha) = \left[ 3JM + M \sum_{\vec{R}_{\alpha_i}} J_\perp(\vec{R}_{\alpha_i}, \vec{R}_\alpha) + M \sum_{\vec{R}_{\alpha_i}} J_\perp(\vec{R}_{\alpha_i}, \vec{R}_{\alpha_i}) \right] M^\alpha_l(\vec{R}_\alpha)$$

$$-JM \sum_{\delta^\alpha_i} M^{\bar{\alpha}_l}(\vec{R}_\alpha + \delta^\alpha_i) - M \sum_{\vec{R}_{\alpha_i}} J_\perp(\vec{R}_{\alpha_i}, \vec{R}_{\alpha_i}) M^{\bar{\alpha}_l}(\vec{R}_{\alpha_i})$$

$$-M \sum_{\vec{R}_{\alpha_i}} J_\perp(\vec{R}_{\alpha_i}, \vec{R}_{\alpha_i}) M^{\bar{\alpha}_l}(\vec{R}_{\alpha_i})$$

(6)

where $\alpha$ stands for $A$ or $B$. If $\alpha = A$ then $\bar{\alpha} = B$ and vice versa. Same relation holds for $l$ and $\bar{l}$.

We next expand the amplitudes in equation 6 in terms of Bloch waves

$$\frac{\omega}{\sqrt{N_l}} \sum_{k'_{\bar{l}}} e^{-ik'_{\bar{l}} \cdot \vec{R}_{\bar{l}} u_{\bar{l}}(k'_{\bar{l}})} =$$

$$\frac{1}{\sqrt{N_l}} \left[ 3JM + M \sum_{\vec{R}_{\alpha_i}} J_\perp(\vec{R}_{\alpha_i}, \vec{R}_\alpha) + M \sum_{\vec{R}_{\alpha_i}} J_\perp(\vec{R}_{\alpha_i}, \vec{R}_{\alpha_i}) \right] \sum_{k'_{\bar{l}}} e^{-ik'_{\bar{l}} \cdot \vec{R}_{\bar{l}} u_{\bar{l}}(k'_{\bar{l}})}$$

$$-\frac{JM}{\sqrt{N_l}} \sum_{k'_{\bar{l}}} f(k'_{\bar{l}}) e^{-ik'_{\bar{l}} \cdot \vec{R}_{\bar{l}} u_{\bar{l}}(k'_{\bar{l}})}$$

$$-\frac{M}{\sqrt{N_l}} \sum_{\vec{R}_{\alpha_i}, \vec{k}_{\bar{l}}} J_\perp(\vec{R}_{\alpha_i}, \vec{R}_{\alpha_i}) e^{-ik_{\bar{l}} \cdot \vec{R}_{\bar{l}} u_{\bar{l}}(k_{\bar{l}})}$$

$$-\frac{M}{\sqrt{N_l}} \sum_{\vec{R}_{\alpha_i}, \vec{k}_{\bar{l}}} J_\perp(\vec{R}_{\alpha_i}, \vec{R}_{\alpha_i}) e^{-ik_{\bar{l}} \cdot \vec{R}_{\bar{l}} u_{\bar{l}}(k_{\bar{l}})}$$

(7)
where \( N_l \) and \( N_{\bar{l}} \) are the number of unit cells in layer \( l \) and \( \bar{l} \). The vectors \( \vec{k}_l' \) and \( \vec{k}_{\bar{l}}' \) are wave vectors in layers \( l \) and \( \bar{l} \). We have also used the fact that

\[
\sum_{\vec{\delta}_l} e^{-ik_l' \cdot \vec{\delta}_l} = e^{-ik_{l'}} \frac{a}{\sqrt{2}} + 2e^{i\frac{\sqrt{3}a}{3}k_{l'y}} \cos\left(\frac{a}{2}k_{lx}\right) = f(\vec{k}_l')
\]

Next, we multiply equation 7 by \( e^{i\vec{k}_l \cdot \vec{R}_{al}} \) and sum the whole equation over \( \vec{R}_{al} \). Substituting \( \sum_{\vec{k}_l', \vec{R}_{al}} e^{i(\vec{k}_l - \vec{k}_l') \cdot \vec{R}_{al}} = \sum_{\vec{k}_l', \vec{a}_l} \delta_{\vec{k}_l - \vec{k}_l'} \cdot \vec{a}_l \cdot \vec{R}_{al} \), where \( \vec{G}_l \) being a lattice vector of layer \( l \), we arrive at

\[
\omega \sum_{\vec{k}_l', \vec{a}_l} \delta_{\vec{k}_l - \vec{k}_l'} \cdot \vec{a}_l \ u_{\alpha_l}(\vec{k}_l') = 3J\sum_{\vec{k}_l', \vec{a}_l} \delta_{\vec{k}_l - \vec{k}_l'} \cdot \vec{a}_l \ u_{\alpha_l}(\vec{k}_l') - J\sum_{\vec{k}_l', \vec{a}_l} f(\vec{k}_l') \delta_{\vec{k}_l - \vec{k}_l'} \cdot \vec{a}_l \ u_{\alpha_l}(\vec{k}_l') \\
+ M \sum_{\vec{k}_l'} \mathcal{J}^{\alpha_l, \alpha_l}_l(\vec{k}_l, \vec{k}_l') u_{\alpha_l}(\vec{k}_l') + M \sum_{\vec{k}_l'} \mathcal{J}^{\alpha_l, \bar{\alpha}_l}_l(\vec{k}_l, \vec{k}_l') u_{\alpha_l}(\vec{k}_l') \\
- M \sum_{\vec{k}_l} \mathcal{J}^{\alpha_l, \alpha_l}_l(\vec{k}_l, \vec{k}_l) u_{\alpha_l}(\vec{k}_l) - M \sum_{\vec{k}_l} \mathcal{J}^{\alpha_l, \bar{\alpha}_l}_l(\vec{k}_l, \vec{k}_l) u_{\alpha_l}(\vec{k}_l)
\]

(8)

with the interlayer coefficients defined as

\[
\mathcal{J}^{\alpha_l, \alpha_l}_l(\vec{k}_l, \vec{k}_l') = \frac{1}{\sqrt{N_l N_{\bar{l}}}} \sum_{\vec{R}_{al}, \vec{R}_{\bar{a}l}} e^{i\vec{k}_l \cdot \vec{R}_{al}} f_{\perp}(\vec{R}_{al}, \vec{R}_{\bar{a}l}) e^{-i\vec{k}_{l'} \cdot \vec{R}_{al}}
\]

(9a)

\[
\mathcal{J}^{\alpha_l, \bar{\alpha}_l}_l(\vec{k}_l, \vec{k}_l') = \frac{1}{\sqrt{N_l N_{\bar{l}}}} \sum_{\vec{R}_{al}, \vec{R}_{\bar{a}l}} e^{i\vec{k}_l \cdot \vec{R}_{al}} f_{\perp}(\vec{R}_{al}, \vec{R}_{\bar{a}l}) e^{-i\vec{k}_{l'} \cdot \vec{R}_{\bar{a}l}}
\]

(9b)

while the intralayer coefficients read

\[
\mathcal{J}^{\alpha_l, \alpha_l}_l(\vec{k}_l, \vec{k}_l') = \frac{1}{N_l} \sum_{\vec{R}_{al}, \vec{R}_{\bar{a}l}} e^{i(\vec{k}_l - \vec{k}_l') \cdot \vec{R}_{al}} f_{\perp}(\vec{R}_{al}, \vec{R}_{\bar{a}l})
\]

(9c)
\[ J^{\alpha_l \bar{\alpha}_l}(\vec{k}_l, \vec{k}'_l) = \frac{1}{N_l} \sum_{\vec{R}_{\alpha_l}, \vec{\alpha}_l} e^{i(\vec{k}_l - \vec{k}'_l) \cdot \vec{R}_{\alpha_l}} J_{\perp}(\vec{R}_{\alpha_l}, \vec{\alpha}_l) \]

(9d)

For small twist angles, \(|\vec{k}_l - \vec{k}'_l|\) is very small near the Dirac points and \(\vec{k}_l - \vec{k}'_l\) does not match any of the non-zero \(\vec{G}_l\), hence

\[
\sum_{\vec{k}'_l, \vec{\alpha}_l} \delta_{\vec{k}_l - \vec{k}'_l, \vec{\alpha}_l} = \sum_{\vec{k}'_l} \delta_{\vec{k}_l - \vec{k}'_l, \vec{\alpha}_l}
\]

Equation 8 then reduces to

\[
\omega u_{\alpha_l}(\vec{k}_l) = 3JM u_{\alpha_l}(\vec{k}_l) - Jmf(\vec{k}_l) u_{\bar{\alpha}_l}(\vec{k}_l)
\]

\[
+ M \sum_{\vec{k}'_l} \left[ J^{\alpha_l, \bar{\alpha}_l}(\vec{k}_l, \vec{k}'_l) + J^{\alpha_l, \bar{\alpha}_l}(\vec{k}_l, \vec{k}'_l) \right] u_{\alpha_l}(\vec{k}_l)
\]

\[
- M \sum_{\vec{k}_l, \vec{\alpha}_l} J^{\alpha_l, \bar{\alpha}_l}(\vec{k}_l, \vec{k}_l) u_{\alpha_l}(\vec{k}_l) - M \sum_{\vec{k}_l, \vec{\alpha}_l} J^{\alpha_l, \bar{\alpha}_l}(\vec{k}_l, \vec{k}_l) u_{\alpha_l}(\vec{k}_l)
\]

(10)

The interlayer coefficients in 10 are qualitatively identical to those encountered in the tight-binding electronic theory for twisted bilayer graphene. We can hence benefit from the approaches developed in the tight binding theory to evaluate \(J^{\alpha_l, \bar{\alpha}_l}_{\perp}\) and \(J^{\alpha_l, \bar{\alpha}_l}_{\perp}\). In particular, we follow the approach by Bistritzer and MacDonald [44], valid for commensurate and incommensurate structures at small twist angles.

To start, the interlayer coefficients \(J_{\perp}(\vec{R}_{\alpha_l}, \vec{R}_{\alpha_l})\) is assumed function of \(\vec{R}_{\alpha_l} - \vec{R}_{\alpha_l}\). With the help of equations 1, the Fourier transform of \(J_{\perp}(\vec{R}_{\alpha_l}, \vec{R}_{\alpha_l})\) can then be written as

\[
J_{\perp}(\vec{R}_{\alpha_l}, \vec{R}_{\alpha_l}) = \int_{\mathbb{R}^2} \frac{d^2 \vec{p}}{(2\pi)^2} e^{-i\vec{p} \cdot (\vec{R}_{\alpha_l} - \vec{R}_{\alpha_l})} J_{\perp}(\vec{p})
\]

which when substituted in equation 9a yields
\[ J_{\perp}^{\alpha_l \alpha_l}(\tilde{k}_l, \tilde{k}_l) = \frac{1}{\sqrt{N_l N_l}} \int_{\mathbb{R}^2} \frac{d^2 \tilde{p}}{(2\pi)^2} J_{\perp}(\tilde{p}) \sum_{\tilde{R}_l} e^{i(\tilde{k}_l - \tilde{p}) \cdot (\tilde{R}_l - \tilde{R}_l)} \sum_{\tilde{R}_l} e^{-i(\tilde{k}_l - \tilde{p}) \cdot (\tilde{R}_l + \tilde{R}_l)} = \sqrt{N_l N_l} \int_{\mathbb{R}^2} \frac{d^2 \tilde{p}}{(2\pi)^2} J_{\perp}(\tilde{p}) \sum_{\tilde{G}_l, \tilde{G}_l} e^{i\tilde{G}_l \cdot \tilde{R}_l} e^{-i\tilde{G}_l \cdot \tilde{R}_l} \delta_{\tilde{k}_l - \tilde{p}, \tilde{G}_l} \delta_{\tilde{k}_l - \tilde{p}, -\tilde{G}_l} \]

Replacing the \( \delta - \)Kronecker with \( \delta - \)Dirac and performing the integral yields

\[ J_{\perp}^{\alpha_l \alpha_l}(\tilde{k}_l, \tilde{k}_l) = \frac{1}{A} \sum_{\tilde{G}_l, \tilde{G}_l} e^{-i\tilde{G}_l \cdot \tilde{R}_l} e^{i\tilde{G}_l \cdot \tilde{R}_l} \delta_{\tilde{k}_l + \tilde{G}_l, \tilde{k}_l + \tilde{G}_l} \]

(11)

with the unit cell area \( A = |\tilde{a}_1 \times \tilde{a}_2| = \sqrt{3}a^2/2 \). In equation 11, we replaced \( \tilde{G}_l \) and \( \tilde{G}_l \) with \( -\tilde{G}_l \) and \( -\tilde{G}_l \). The equation hence imposes the generalized umklapp condition [71], namely \( \tilde{k}_l + \tilde{G}_l = \tilde{k}_l + \tilde{G}_l \).

Close to the Dirac points \( K_{l/l} \), we write \( \tilde{k}_{l/l} = \tilde{K}_{l/l} + \tilde{q}_{l/l} \) with \( |\tilde{q}_{l/l}| \ll |\tilde{K}_l| = 4\pi/3a \). Therefore, \( J_{\perp}(\tilde{k}_l + \tilde{G}_l) \approx J_{\perp}(\tilde{K}_l + \tilde{G}_l) \). In addition, \( J_{\perp}(\tilde{K}_l + \tilde{G}_l) \) is expected to be isotropic in the \( \tilde{k} \)-space and hence \( J_{\perp}(\tilde{K}_l + \tilde{G}_l) = J_{\perp}(|\tilde{K}_l + \tilde{G}_l|) \).

Equation 11 then yields

\[ J_{\perp}^{\alpha_l \alpha_l}(\tilde{q}_b, \tilde{q}_l) = \frac{1}{A} \sum_{\tilde{G}_l, \tilde{G}_l} e^{-i\tilde{G}_l \cdot \tilde{R}_l} e^{i\tilde{G}_l \cdot \tilde{R}_l} \delta_{\tilde{q}_l - \tilde{q}_l, -(\tilde{k}_l - \tilde{K}_l + \tilde{G}_l - \tilde{G}_l)} \]

(12)

For the weakly coupled ferromagnetic planes, the interlayer coefficient \( J_{\perp}(|\tilde{K}_l|) \) is expected to decay rapidly with \( |\tilde{K}_l| \) and the most relevant contributions to \( J_{\perp}^{\alpha_l \alpha_l}(\tilde{q}_l, \tilde{q}_l) \) correspond to \( \tilde{g}_{l/l} = \tilde{b}_{l/l, 2}, \) and \( -\tilde{b}_{l/l, 1} \). For these values of \( \tilde{G}_l \), it is easy to prove that \( |\tilde{k}_l + \tilde{G}_l| = |\tilde{K}_l| \). Consequently, equation 12 yields

\[ J_{\perp}^{\alpha_l \alpha_l}(\tilde{q}_l, \tilde{q}_l) = \frac{J_{\perp}(|\tilde{K}_l|)}{A} \left[ \delta_{\tilde{q}_l - \tilde{q}_l, -(\tilde{k}_l - \tilde{K}_l)} + e^{-i\tilde{b}_{l/l, 2} \cdot \tilde{R}_l} e^{i\tilde{b}_{l/l, 2} \cdot \tilde{R}_l} \delta_{\tilde{q}_l - \tilde{q}_l, -(\tilde{k}_l - \tilde{K}_l + \tilde{b}_{l/l, 2} - \tilde{b}_{l/l, 2})} + e^{i\tilde{b}_{l/l, 2} \cdot \tilde{R}_l} e^{-i\tilde{b}_{l/l, 2} \cdot \tilde{R}_l} \delta_{\tilde{q}_l - \tilde{q}_l, -(\tilde{k}_l - \tilde{K}_l + \tilde{b}_{l/l, 1} + \tilde{b}_{l/l, 1})} \right] \]

(13a)
We can now readily deduce

\[
\mathcal{J}_{\perp}^{a_l, \tilde{a}_l}(\tilde{q}_b, \tilde{q}_l) = \frac{J_\perp(|\tilde{K}_l|)}{A} \left[ \delta_{\tilde{q}_l - \tilde{q}_l, -(\tilde{K}_l - \tilde{K}_l)} + e^{-i\tilde{b}_{l2}} e^{i\tilde{b}_{l2}} \delta_{\tilde{q}_l - \tilde{q}_l, -(\tilde{K}_l - \tilde{K}_l + \tilde{b}_{l2})} 
+ e^{i\tilde{b}_{l2}} e^{-i\tilde{b}_{l2}} \delta_{\tilde{q}_l - \tilde{q}_l, -(\tilde{K}_l - \tilde{K}_l - \tilde{b}_{l2})} \right] 
\]

(13b)

Equations (13) determine the interlayer coefficients in the LL equations (equation 10). We see that the three transferred momenta in 13 include reciprocal lattice vectors of the moiré pattern. Moreover, the Bernal stacking can be retrieved from 13 by setting \( \theta = 0 \), which allows us to deduce \( \frac{J_\perp(|\tilde{K}_l|)}{A} = \frac{J_\perp}{3} \).

We next consider the intralayer coefficients \( \mathcal{J}^{a_l, a_l}(\tilde{k}_l, \tilde{k}'_l) \), which can be treated similar to the interlayer coefficients. For the present case, \( \tilde{k}_l \) and \( \tilde{k}'_l \) are expanded near \( \tilde{K}_l \). Following the previous steps, we arrive at

\[
\mathcal{J}^{a_l, a_l}(\tilde{q}_l, \tilde{q}'_l) = \frac{1}{A} \sum_{\tilde{g}_l, \tilde{g}'_l} e^{-i\tilde{g}_l} e^{i\tilde{g}'_l} J_{\|}(|\tilde{q}_l - \tilde{q}'_l + \tilde{g}_l|) e^{i\tilde{g}_l} \delta_{\tilde{q}_l - \tilde{q}'_l, -(\tilde{g}_l - \tilde{g}_l)} 
\]

(14a)

while

\[
\mathcal{J}^{a_l, \bar{a}_l}(\tilde{q}_b, \tilde{q}_l) = \frac{1}{A} \sum_{\tilde{g}_l, \tilde{g}_l} e^{-i\tilde{g}_l} e^{i\tilde{g}_l} J_{\|}(|\tilde{q}_l - \tilde{q}_l + \tilde{g}_l|) e^{i\tilde{g}_l} \delta_{\tilde{q}_l - \tilde{q}_l, -(\tilde{g}_l - \tilde{g}_l)} 
\]

(14b)

Equations 14 pose the condition \( \tilde{q}_l - \tilde{q}'_l = \tilde{g}_l - \tilde{g}_l \). Since \( |\tilde{q}_l - \tilde{q}'_l| \) is small, the last condition is satisfied only if \( \tilde{g}_l - \tilde{g}_l \) represents a moiré reciprocal lattice vector. Any deviation from the moiré reciprocal lattice vectors will yield a large vector \( \tilde{g}_l - \tilde{g}_l \) which can’t match \( \tilde{q}_l - \tilde{q}'_l \). Consequently, \( u_{a_l}(\tilde{q}'_l) \) reduces to \( u_{a_l}(\tilde{q}_l) \). The intralayer coefficients are, as expected, independent of the twist angle.

We can now substitute equations 13 in 10 to express the LL equations in details. We note that near \( \tilde{K}_l, f(\tilde{K}_l + \tilde{q}) \approx -\sqrt{2} (q_x + i q_y) e^{-i\theta/2} = -\sqrt{2} |\tilde{q}| e^{i\theta} e^{-i\theta/2}. \) Moreover, the in-plane DMI can be
easily included in the LL equations, contributing only to the diagonal terms by $\pm 3\sqrt{3} D$ for A and B sublattices respectively ($D$ is the DMI coefficient).

The LL equations for the two sites in layer 1 then reads

$$\Omega \ u_{A_1}(\vec{K}_1 + \bar{q}) = \frac{\sqrt{3} a}{2} |\bar{q}| e^{i(\theta_q - \theta/2)} u_{B_1}(\vec{K}_1 + \bar{q}) +$$

$$\left[ 3 + 3 \sqrt{3} D + \frac{J_1}{3} \left( 2 + e^{-i\varphi} e^{i\vec{b}_{1,2} \vec{\tau}_0} + e^{i\varphi} e^{-i\vec{b}_{1,1} \vec{\tau}_0} + e^{i\vec{b}_{1,2} \vec{\tau}_0} + e^{-i\vec{b}_{1,1} \vec{\tau}_0} \right) \right] u_{A_1}(\vec{K}_1 + \bar{q})$$

$$- \frac{J_1}{3} \left[ u_{A_2}(\vec{K}_2 + \bar{q} + \bar{q}_b) + e^{-i\varphi} e^{i\vec{b}_{1,2} \vec{\tau}_0} u_{A_2}(\vec{K}_2 + \bar{q} + \bar{q}_{\theta}) + e^{i\varphi} e^{-i\vec{b}_{1,1} \vec{\tau}_0} u_{A_2}(\vec{K}_2 + \bar{q} + \bar{q}_{\theta}) \right]$$

$$- \frac{J_1}{3} \left[ u_{B_2}(\vec{K}_2 + \bar{q} + \bar{q}_b) + e^{i\vec{b}_{1,2} \vec{\tau}_0} u_{B_2}(\vec{K}_2 + \bar{q} + \bar{q}_{\theta}) + e^{-i\vec{b}_{1,1} \vec{\tau}_0} u_{B_2}(\vec{K}_2 + \bar{q} + \bar{q}_{\theta}) \right]$$

(15a)

$$\Omega \ u_{B_1}(\vec{K}_1 + \bar{q}) = \frac{\sqrt{3} a}{2} |\bar{q}| e^{-i(\theta_q - \theta/2)} u_{A_1}(\vec{K}_1 + \bar{q})$$

$$\left[ 3 - 3 \sqrt{3} D + \frac{J_1}{3} \left( 2 + e^{i\varphi} e^{i\vec{b}_{1,2} \vec{\tau}_0} + e^{-i\varphi} e^{-i\vec{b}_{1,1} \vec{\tau}_0} + e^{i\varphi} e^{i\vec{b}_{1,2} \vec{\tau}_0} + e^{-i\varphi} e^{-i\vec{b}_{1,1} \vec{\tau}_0} \right) \right] u_{B_1}(\vec{K}_1 + \bar{q})$$

$$- \frac{J_1}{3} \left[ u_{A_2}(\vec{K}_2 + \bar{q} + \bar{q}_b) + e^{i\varphi} e^{i\vec{b}_{1,2} \vec{\tau}_0} u_{A_2}(\vec{K}_2 + \bar{q} + \bar{q}_{\theta}) + e^{-i\varphi} e^{-i\vec{b}_{1,1} \vec{\tau}_0} u_{A_2}(\vec{K}_2 + \bar{q} + \bar{q}_{\theta}) \right]$$

$$- \frac{J_1}{3} \left[ u_{B_2}(\vec{K}_2 + \bar{q} + \bar{q}_b) + e^{-i\varphi} e^{i\vec{b}_{1,2} \vec{\tau}_0} u_{B_2}(\vec{K}_2 + \bar{q} + \bar{q}_{\theta}) + e^{i\varphi} e^{-i\vec{b}_{1,1} \vec{\tau}_0} u_{B_2}(\vec{K}_2 + \bar{q} + \bar{q}_{\theta}) \right]$$

(15b)

with $\varphi = 2\pi/3$ and $\Omega = \frac{\omega}{jM}$. We have also defined the momenta

$$\bar{q}_b = \frac{4\pi \sin(\theta/2)}{3\sqrt{3}d} (0,-1)$$

$$\bar{q}_{\theta} = \frac{4\pi \sin(\theta/2)}{3\sqrt{3}d} (\sqrt{3}/2,1/2)$$

$$\bar{q}_{\theta} = \frac{4\pi \sin(\theta/2)}{3\sqrt{3}d} (-\sqrt{3}/2,1/2)$$

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Each amplitude in layer 1 is hence coupled to 6 amplitudes in layer 2. Writing the LL equations for these 6 amplitudes, however, will not yield a close system of equations. In other words, there is no closed set of coupled amplitudes from layers 1 and 2 that can describe the spin dynamics in the tFBL. It is hence necessary to truncate the set of amplitudes involved in the formalism in order to form the system’s Hamiltonian and perform numerical calculations. For completeness, we present the LL equations for the two sites in layer 2

\[
\frac{ω}{JM} u_{A2}(\vec{K}_2 + \vec{q}) = \frac{\sqrt{3}a}{2} |\vec{q}| e^{i(\theta_q/2)} u_{B2}(\vec{K}_2 + \vec{q}) \\
[3 + 3\sqrt{3} D + \frac{I_1}{3J} \left( 2 + e^{i\varphi} e^{-i\vec{b}_{1.2} \cdot \vec{r}_0} + e^{-i\varphi} e^{i\vec{b}_{1.1} \cdot \vec{r}_0} + e^{-i\varphi} e^{i\vec{b}_{1.2} \cdot \vec{r}_0} + e^{i\varphi} e^{-i\vec{b}_{1.1} \cdot \vec{r}_0} \right)] u_{A2}(\vec{K}_2 + \vec{q}) \\
- \frac{I_1}{3J} \left[ u_{A2}(\vec{K}_1 + \vec{q} + \vec{q}_b) + e^{i\varphi} e^{-i\vec{b}_{1.2} \cdot \vec{r}_0} u_{A1}(\vec{K}_1 + \vec{q} + \vec{q}_f) + e^{-i\varphi} e^{i\vec{b}_{1.1} \cdot \vec{r}_0} u_{A1}(\vec{K}_1 + \vec{q} + \vec{q}_f) \right] \\
- \frac{I_1}{3J} \left[ u_{B2}(\vec{K}_1 + \vec{q} + \vec{q}_b) + e^{-i\varphi} e^{i\vec{b}_{1.2} \cdot \vec{r}_0} u_{B1}(\vec{K}_1 + \vec{q} + \vec{q}_f) + e^{i\varphi} e^{-i\vec{b}_{1.1} \cdot \vec{r}_0} u_{B1}(\vec{K}_1 + \vec{q} + \vec{q}_f) \right] 
\]

(15c)

\[
\frac{ω}{JM} u_{B2}(\vec{K}_2 + \vec{q}) = \frac{\sqrt{3}a}{2} |\vec{q}| e^{-i(\theta_q/2)} u_{A2}(\vec{K}_2 + \vec{q}) \\
[3 - 3\sqrt{3} D + \frac{I_1}{3J} \left( 2 + e^{-i\vec{b}_{1.2} \cdot \vec{r}_0} + e^{i\varphi} e^{i\vec{b}_{1.1} \cdot \vec{r}_0} + e^{-i\varphi} e^{-i\vec{b}_{1.1} \cdot \vec{r}_0} \right)] u_{B2}(\vec{K}_2 + \vec{q}) \\
- \frac{I_1}{3J} \left[ u_{A2}(\vec{K}_1 + \vec{q} + \vec{q}_b) + e^{-i\vec{b}_{1.2} \cdot \vec{r}_0} u_{A1}(\vec{K}_1 + \vec{q} + \vec{q}_f) + e^{i\varphi} e^{i\vec{b}_{1.1} \cdot \vec{r}_0} u_{A1}(\vec{K}_1 + \vec{q} + \vec{q}_f) \right] \\
- \frac{I_1}{3J} \left[ u_{B2}(\vec{K}_1 + \vec{q} + \vec{q}_b) + e^{i\varphi} e^{-i\vec{b}_{1.2} \cdot \vec{r}_0} u_{B1}(\vec{K}_1 + \vec{q} + \vec{q}_f) + e^{-i\varphi} e^{-i\vec{b}_{1.1} \cdot \vec{r}_0} u_{B1}(\vec{K}_1 + \vec{q} + \vec{q}_f) \right] 
\]

(15d)

**Numerical results.** In figure 2 left, we present the magnon spectrum along the high symmetry axes in the moiré superlattice for a weakly coupled tFBL with \( \vec{r}_0 = (0, d) \), \( \frac{I_1}{J} = 0.05 \), \( \theta = 5^\circ \) and \( D = 0 \). The present case corresponds to a rotated AA configuration. The spectrum is found to be qualitatively identical to the electronic structure of tBLG, with Dirac dispersion at \( K_m \) and \( K'_m \). In
figure 2 right, the spectrum is plotted at the first magic angle, found to be around $\theta = 0.91^\circ$. Again, the spectrum is very similar to that of tBLG.

![Figure 2: The numerically calculated magnon spectra for tFBL with $\bar{\tau}_0 = (0, d), \frac{\mu}{J} = 0.05$, and zero DMI. The left figure corresponds to $\theta = 5^\circ$ whereas the right figure corresponds to the magic angle $\theta = 0.91^\circ$.](image)

We next demonstrate the effect of the relative translation on the spin wave spectrum. In figure 3 left, we plot the magnon spectrum for $\bar{\tau}_0 = (0,0)$ and $\theta = 5^\circ$. The rest of the parameters are kept the same. The present case corresponds to a rotated AB configuration. The spectrum shows a band gap throughout the region. The translation is also found to alter the value of the magic angle in tFBL. The right figure shows the spectrum at the magic angle $\theta \approx 0.79^\circ$ in the present case. Interestingly, the rotated AB configuration presents 2 perfectly flat and gapped bands at the magic angle. These exotic dispersion profiles are indeed absent in the tBLG case.

![Figure 3: Gapped magnon spectra for tFBL with $\bar{\tau}_0 = (0,0), \frac{\mu}{J} = 0.05$, and zero DMI. The left and right figures respectively correspond to $\theta = 5^\circ$ and the magic angle $\theta = 0.79^\circ$.](image)
We next analyze the drastic effect induced by weak in-plane DMI. Figure 4 presents the magnon spectrum for $J_z = 0.05$, $\theta = 5^\circ$ and $D = 0.05$, with $\vec{r}_0 = (0, d)$ (left) and $\vec{r}_0 = (0, 0)$ (right). As expected, the weak DMI induces a relatively large band gap in the spin wave spectrum. The magic angles, however, are found to be robust against the DMI as illustrated in figure 5. The left and right figures respectively correspond to $[\vec{r}_0 = (0, d), \theta = 0.91^\circ]$ and $[\vec{r}_0 = (0, 0), \theta = 0.79^\circ]$. The DMI is found to induce additional flat bands in the spectra.

**Figure 4:** Gapped magnon spectra for $J_z = 0.05$, $\theta = 5^\circ$ and $D = 0.05$, with $\vec{r}_0 = (0, d)$ (left) and $\vec{r}_0 = (0, 0)$ (right)

**Figure 5:** Gapped magnon spectra for $J_z = 0.05$ and $D = 0.05$, with $[\vec{r}_0 = (0, d), \theta = 0.91^\circ]$ (left) and $[\vec{r}_0 = (0, 0), \theta = 0.79^\circ]$ (right)
Conclusion. We developed the first theoretical formalism of spin dynamics in weakly coupled honeycomb tFBL, including exchange and Dzyaloshinskii-Moriya interactions (DMI). The theory is based on a semi-classical Heisenberg Hamiltonian, LL equations of motion and the reciprocal moiré superlattice techniques. We demonstrate the existence of magnon magic angles and flat spin wave bands. tFBL is found to host spin wave dispersion profiles absent in magnetic monolayers, AB/AA stacked magnetic bilayers and in twisted fermionic counterparts. Weakly coupled tFBL hence presents novel moiré physics and important potentials for technological applications.

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