A characterization and an improved inequality for warped product submanifolds in Kenmotsu manifolds

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Abstract

In this paper, we prove the existence of warped product semi-slant submanifolds in a Kenmotsu manifold by its characterization. Also, we obtain an inequality for the square norm of second fundamental form in terms of the warping function and the slant angle of the immersion. The equality case is considered.

Key words: Warped product manifold; slant submanifolds; semi-slant submanifolds; semi-slant warped products; Kenmotsu manifolds.

1 Introduction

Warped product manifolds were studied by Bishop, and O'Neill in 1969 as a natural generalization of the Riemannian product manifolds [4]. Later on, the geometrical aspects of these manifolds have been studied by many researchers [7, 11]. Recently, B.Y. Chen [7] has given the idea of warped product submanifolds. Motivated by Chen’s papers, many geometers studied warped product submanifolds for different structures on Riemannian manifolds [1, 10, 12].

On the other hand, in [6] the authors studied slant immersions in K-contact and Sasakian manifolds. They introduced many interesting examples of slant submanifolds in both almost contact metric manifolds and Sasakian manifolds. They characterized slant submanifolds by means of the covariant derivative of the square of the tangent projection on the submanifold of almost contact structure of a K-contact manifold. Later on, in [5], they have defined and studied semi-slant submanifolds of Sasakian manifolds.

Recently, Atceken studied warped product semi-slant submanifolds in Kenmotsu manifolds and Kenmotsu space forms, he has shown the non-existence cases of the warped product semi-slant submanifolds in a Kenmotsu manifold [2, 3]. Later on, S. Uddin et al. studied warped product semi-slant submanifolds of Kenmotsu manifolds and showed that the warped product submanifolds exist except the case when the structure vector field $\xi$ is tangent to the fiber [12]. They have obtained some existence results which we will use in this paper. In this paper, we study semi-slant submanifolds in brief not in details as our aim is to discuss the warped products. We prove the existence of warped product semi-slant submanifolds by a characterization and obtain an inequality for the squared norm of the second fundamental form in terms of the warping function and the slant angle for the warped product semi-slant submanifolds of a Kenmotsu manifold.

2 Preliminaries

Let $\tilde{M}$ be an almost contact metric manifold with structure $(\varphi, \xi, \eta, g)$ where $\varphi$ is a $(1,1)$ tensor field, $\xi$ a vector field, $\eta$ is a 1-form and $g$ is a Riemannian...
metric on $\tilde{M}$ satisfying the following properties

$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1.$$  

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$ (2.2)

If in addition to the above relations,

$$(\tilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$$ (2.3)

holds, then $\tilde{M}$ is said to be Kenmotsu manifold, where $\tilde{\nabla}$ is the Levi-Civita connection of $g$. We shall use the symbol $\Gamma(TM)$ to denote the Lie algebra of vector fields on the manifold $\tilde{M}$.

Let $M$ be submanifold of an almost contact metric manifold $\tilde{M}$ with induced metric $g$ and if $\nabla$ and $\nabla^\perp$ are the induced connections on the tangent bundle $TM$ and the normal bundle $T^\perp M$ of $M$, respectively then Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$$ (2.4)

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N,$$ (2.5)

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where $h$ and $A_N$ are the second fundamental form and the shape operator (corresponding to the normal vector field $N$) respectively for the immersion of $M$ into $\tilde{M}$. They are related as $g(A_N X, Y) = g(h(X, Y), N)$.

For any $X \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, we write

$$(a) \quad \varphi X = PX + FX, \quad (b) \quad \varphi N = tN + fN$$ (2.6)

where $PX$ and $tN$ are the tangential components of $\varphi X$ and $\varphi N$ respectively and $FX$ and $fN$ are the normal components of $\varphi X$ and $\varphi N$, respectively. The submanifold $M$ is said to be invariant if $F$ is identically zero, that is, $\varphi X \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. On the other hand $M$ is said to be anti-invariant if $F$ is identically zero, that is, $\varphi X \in \Gamma(T^\perp M)$, for any $X \in \Gamma(TM)$.

For a Riemannian submanifold $M$ of a Kenmotsu manifold $\tilde{M}$, we have

$$(\tilde{\nabla}_X P)Y = AfY X + \theta h(X, Y) - g(X, PY)\xi - \eta(Y)PX$$ (2.7)

and

$$(\tilde{\nabla}_X F)Y = fh(X, Y) - h(X, PY)\xi - \eta(Y)FX$$ (2.8)

where $(\tilde{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y$ and $(\tilde{\nabla}_X F)Y = \nabla^\perp_X FY - F\nabla_X Y$ are the covariant derivative of the tensor field $P$ and $F$, respectively.

Let $M$ be a submanifold tangent to the structure vector field $\xi$ isometrically immersed into an almost contact metric manifold $\tilde{M}$. Then $M$ is said to be contact CR-submanifold if there exists a pair of orthogonal distributions $\mathcal{D} : p \rightarrow \mathcal{D}_p$ and $\mathcal{D}^\perp : p \rightarrow \mathcal{D}^\perp_p$, $\forall p \in M$ such that

(i) $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \langle \xi \rangle$, where $\langle \xi \rangle$ is the 1-dimensional distribution spanned by the structure vector field $\xi$.

(ii) $\mathcal{D}$ is invariant, i.e., $\varphi \mathcal{D} = \mathcal{D}$
(iii) $D^\perp$ is anti-invariant, i.e., $\varphi D^\perp \subseteq T^\perp M$.

Invariant and anti-invariant submanifolds are the special cases of a contact CR-submanifold. If we denote the dimensions of the distributions $D$ and $D^\perp$ by $d_1$ and $d_2$, respectively. Then is $M$ is invariant (resp. anti-invariant) if $d_2 = 0$ (resp. $d_1 = 0$).

There is another class of submanifolds that is called the slant submanifold. For each non zero vector $X$ tangent to $M$ at $x$, such that $X$ is not proportional to $\xi$, we denotes by $\theta(X)$, the angle between $\varphi X$ and $PX$.

$M$ is said to be slant [6] if the angle $\theta(X)$ is constant for all $X \in TM - \{\xi\}$ and $x \in M$. The angle $\theta$ is called slant angle or Wirtinger angle. Obviously if $\theta = 0$, $M$ is invariant and if $\theta = \pi/2$, $M$ is an anti-invariant submanifold. If the slant angle of $M$ is different from 0 and $\pi/2$ then it is called proper slant.

A characterization of slant submanifolds is given by the following result.

**Theorem 2.1** [6]. Let $M$ be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in \Gamma(TM)$. Then $M$ is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$P^2 = \lambda(-I + \eta \otimes \xi).$$  \hspace{1cm} (2.9)

Furthermore, in such case, if $\theta$ is slant angle, then $\lambda = \cos^2 \theta$.

Following relations are straight forward consequence of equation (2.9)

$$g(PX, PY) = \cos^2 \theta g(X, Y) - \eta(X)\eta(Y)$$ \hspace{1cm} (2.10)

$$g(FX, FY) = \sin^2 \theta g(X, Y) - \eta(X)\eta(Y)$$ \hspace{1cm} (2.11)

for any $X, Y$ tangent to $M$.

### 3 Semi-slant submanifolds

Semi-slant submanifolds were defined and studied by N. Papaghiuc [11] as a natural generalization of CR-submanifolds of almost Hermitian manifolds in terms of slant distribution. Later on, Cabrerizo et al. [4] studied these submanifolds in contact setting. They defined these submanifolds as follows:

**Definition 3.1** [5]. A Riemannian submanifold $M$ of an almost contact manifold $\tilde{M}$ is said to be a semi-slant submanifold if there exist two orthogonal distributions $D$ and $D_\theta$ such that $TM = D \oplus D_\theta \oplus \langle \xi \rangle$, the distribution $D$ is invariant i.e., $\varphi D = D$ and the distribution $D_\theta$ is slant with slant angle $\theta = \frac{\pi}{2}$.

If we denote the dimensions of $D$ and $D_\theta$ by $d_1$ and $d_2$, respectively, then it is clear that contact CR-submanifolds and slant submanifolds are semi-slant submanifolds with $\theta = \frac{\pi}{2}$ and $d_1 = 0$, respectively.

A semi-slant submanifold $M$ is said to be mixed geodesic if $h(X, Z) = 0$, for any $X \in \Gamma(D)$ and $Z \in \Gamma(D_\theta)$. Moreover, if $\nu$ is the $\varphi$–invariant subspace of the normal bundle $T^\perp M$, then in case of semi-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as $T^\perp M = FD_\theta \oplus \nu$.

In this section, we prove the integrability conditions of involved distributions of semi-slant submanifolds of a Kenmotsu manifold which we required for the characterization of warped products.
Theorem 3.1. Let $M$ be a semi-slant submanifold of a Kenmotsu manifold $\widetilde{M}$. Then the distribution $\mathcal{D} \oplus \langle \xi \rangle$ is integrable if and only if
\[
g(\nabla_Y Z, X) = \csc^2 \theta \{g(h(X, Y), FPZ) - g(h(X, \varphi Y), FZ)\}
\]
for any $X, Y \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ and $Z \in \Gamma(\mathcal{D}_0)$.

Proof. From the definition of Lie bracket and (2.2), we have
\[
g([X, Y], Z) = g(\varphi \nabla_X Y, \varphi Z) - g(\nabla_Y X, Z),
\]
for any $X, Y \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ and $Z \in \Gamma(\mathcal{D}_0)$. Using the definition of covariant derivative of $\varphi$ and the relation (2.6)-(a), we get
\[
g([X, Y], Z) = g(\nabla_X \varphi Y, PZ) + g(\nabla_X \varphi Y, FZ) - g(\nabla_X \varphi Y, FZ) - g(\nabla_Y X, Z).
\]
Then from (2.3), (2.4) and the fact that $\varphi Y$ and $PZ$ are orthogonal vector fields, we derive
\[
g([X, Y], Z) = -g(\varphi Y, \nabla_X PZ) + g(h(X, \varphi Y), FZ) - g(\nabla_Y X, Z).
\]
Again using (2.6)-(a) and then by (2.9) and (2.5), we obtain
\[
g([X, Y], Z) = -\cos^2 \theta g(Y, \nabla_X Z) - g(A_{FPZ} X, Y) + g(h(X, \varphi Y), FZ) - g(\nabla_Y X, Z).
\]
By the orthogonality of two distributions ones get
\[
g([X, Y], Z) = \cos^2 \theta g(Y, \nabla_X Z) - g(A_{FPZ} X, Y) + g(h(X, \varphi Y), FZ) - g(\nabla_Y X, Z).
\]
Again, by the definition of Lie bracket, we derive
\[
g([X, Y], Z) = \cos^2 \theta g([X, Y], Z) + \cos^2 \theta g(\nabla_Y X, Z) - g(h(X, Y), FPZ) + g(h(X, \varphi Y), FZ) - g(\nabla_Y X, Z).
\]
Finally, the above equation can be written as
\[
\sin^2 \theta g([X, Y], Z) = g(h(X, \varphi Y), FZ) - g(h(X, Y), FPZ) - \sin^2 \theta g(\nabla_Y X, Z).
\]
Hence the result follows from the last relation. \qed

Now, we prove the integrability of the slant distribution.

Theorem 3.2. On a semi-slant submanifold $M$ of a Kenmotsu manifold $\widetilde{M}$, the slant distribution $\mathcal{D}_0$ is integrable if and only if
\[
g(\nabla_W X, Z) = \csc^2 \theta \{g(h(\varphi X, Z), FW) - g(h(X, Z), FPW)\} + \eta(X)g(Z, W)
\]
for any $X \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ and $Z, W \in \Gamma(\mathcal{D}_0)$. 

4
For a warped product manifold $N$ on a Riemann metric $g$, the idea of warped product manifolds was given by Bishop and O’Neill [4].

**Definition 4.1** [4]. They defined these manifolds as follows:

Thus from (2.6)-(a) and (2.3), we derive

$$ g([Z, W], X) = -g(\tilde{\nabla}_Z\varphi P W, \varphi X) - g(A_{FW} Z, \varphi X) - \eta(X)g(Z, W) - g(\tilde{\nabla}_W Z, X). $$

Then, by the definition of covariant derivative of $\varphi$, we get

$$ g([Z, W], X) = -g(\tilde{\nabla}_Z\varphi P W, X) + g((\tilde{\nabla}_Z\varphi) P W, X) $$

$$ - g(A_{FW} Z, \varphi X) - \eta(X)g(Z, W) - g(\tilde{\nabla}_W Z, X). $$

Thus from (2.6)-(a) and (2.3), we derive

$$ g([Z, W], X) = -g(\tilde{\nabla}_Z P^2 W, X) - g(\tilde{\nabla}_Z P F W, X) + g(P Z, P W)\eta(X) $$

$$ - g(h(\varphi X, Z), F W) - \eta(X)g(Z, W) - g(\tilde{\nabla}_W Z, X). $$

By the Theorem 2.1 and the relation (2.5), we obtain

$$ g([Z, W], X) = \cos^2 \theta g(\tilde{\nabla}_Z W, X) + g(A_{FW} Z, X) + \cos^2 \theta g(Z, W)\eta(X) $$

$$ - g(h(\varphi X, Z), F W) - \eta(X)g(Z, W) - g(\tilde{\nabla}_W Z, X). $$

Hence, by the definition of Lie bracket the above equation simplified as

$$ \sin^2 \theta g([Z, W], X) = g(h(Z, X), P F W) - g(h(\varphi X, Z), F W) $$

$$ - \sin^2 \theta \eta(X)g(Z, W) - \sin^2 \theta g(\tilde{\nabla}_W Z, X), \quad (3.3) $$

which proves our assertion. 

**4 warped product semi-slant submanifolds**

The idea of warped product manifolds was given by Bishop and O’Neill [4]. They defined these manifolds as follows:

**Definition 4.1** [4]. Let $(N_1, g_1)$ and $(N_2, g_2)$ be two Riemannian manifolds with Riemann metric $g_1$ and $g_2$, respectively, and $f$ be a positive smooth function on $N_1$. The warped product of $N_1$ and $N_2$ is the Riemannian manifold $M = N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

$$ g = g_1 + f^2 g_2. \quad (4.1) $$

For a warped product manifold $M = N_1 \times_f N_2 = (N_1 \times N_2, g)$ is said to be trivial if the warping function $f$ is constant. More explicitly, if the vector fields $X$ and $Y$ are tangent to $M = N_1 \times_f N_2$ at $(p, q)$, then

$$ g(X, Y) = g_1(\pi_1 \ast X, \pi_1 \ast Y) + f^2(p) g_2(\pi_2 \ast X, \pi_2 \ast Y) $$

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where $\pi_1$ and $\pi_2$ are the canonical projections of $M = N_1 \times N_2$ onto $N_1$ and $N_2$ respectively and $*$ is the symbol for the tangent map.

Now, let us recall the following results on warped products for later use.

Lemma 4.1 [4]. Let $M = N_1 \times f N_2$ be a warped product manifold with the warping function $f$. Then

(i) $\nabla X Y \in \Gamma(T N_1)$

(ii) $\nabla X Z = \nabla Z X = (X \ln f) Z$

(iii) $\nabla Z W = \nabla Z N_2 W - (g(Z, W)/f) \nabla f$,

for any $X, Y \in \Gamma(T N_1)$ and $Z, W \in \Gamma(T N_2)$, where $\nabla$ and $\nabla^{N_2}$ denote the Levi-Civita connections on $M$ and $N_2$, respectively and $\nabla f$ is the gradient of $f$ defined by $g(\nabla f, X) = X(f)$, for any $X \in \Gamma(TM)$.

As a consequence, we have the equality

$$\|\nabla f\|^2 = \sum_{i=1}^n (e_i(f))^2$$

for the orthonormal frame $\{e_1, \cdots, e_n\}$ of the tangent space of $M$.

The following result is useful to prove our main theorem.

Lemma 4.2 [12]. Let $M = N_T \times j N_\theta$ be a warped product semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that $\xi$ is tangent to $N_T$ where $N_T$ and $N_\theta$ are invariant and proper slant submanifolds of $\tilde{M}$, respectively. Then

(i) $\xi \ln f = 1$,

(ii) $g(h(X, Z), FPZ) = g(h(X, PZ), FZ) = \{X \ln f - \eta(X)\} \cos^2 \theta \|Z\|^2$,

(iii) $g(h(X, Z), FZ) = -(\varphi X \ln f) \|Z\|^2$,

for any $X \in \Gamma(T N_T)$ and $Z \in \Gamma(T N_\theta)$.

In the above lemma, if we replace $X$ by $\varphi X$ in the third part, then we have

$$g(h(\varphi X, Z), FZ) = \{(X \ln f - \eta(X))\} \|Z\|^2.$$  \hspace{1cm} (4.3)

Also, for any $X, Y \in \Gamma(T N_T)$ and $Z \in \Gamma(T N_\theta)$, we have

$$g(h(X, Y), FZ) = g(\bar{\nabla}_X Y, FZ) = -g(Y, \bar{\nabla}_X FZ).$$

Using (2.6)-(a), we obtain

$$g(h(X, Y), FZ) = g(Y, \bar{\nabla}_X PZ) - g(Y, \bar{\nabla}_X \phi Z).$$

Then from (2.4) and Lemma 4.1 (ii), the first term of right hand side is identically zero. Also, by (2.3) and the fact that $\xi$ is tangent to $N_T$, the second term vanishes identically, hence

$$g(h(X, Y), FZ) = 0.$$  \hspace{1cm} (4.4)
Now, we prove the existence of warped product semi-slant submanifolds $\tilde{M} = N_T \times_f N_\theta$ in a Kenmotsu manifold $\tilde{M}$ by a characterization result.

**Theorem 4.1** Let $M$ be a semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$. Then $M$ is locally a warped product submanifold of invariant and slant submanifolds if and only if

$$A_{FW}X = \{(X(\mu) - \eta(X))W\} \quad \text{and} \quad A_{FPW}X = \cos^2 \theta \{X(\mu) - \eta(X)\}W$$ \hspace{1cm} (4.5)

for any $Z \in \Gamma(D\oplus < \xi >)$ and $W \in \Gamma(D_\theta)$ and for some function $\mu$ on $M$ satisfying $Z\mu = 0$, for every $Z \in D_\theta$ where $D\oplus < \xi >$ and $D_\theta$ are invariant and proper slant distributions of $M$, respectively.

**Proof.** Let $M = N_T \times_f N_\theta$ be a warped product submanifold of a Kenmotsu manifold $\tilde{M}$, then the direct part follows from Lemma 4.2 (ii) and the relation (4.3).

Conversely, if $M$ is a semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that (4.5) holds, then by the Theorem 3.1, the distribution $D\oplus < \xi >$ is integrable if and only if

$$g(\nabla_X Y, W) = \csc^2 \theta \{g(A_{FW}X, Y) - g(A_{FPW}X, Y)\}.$$  

for any $X, Y \in \Gamma(D\oplus < \xi >)$ and $Z \in \Gamma(D_\theta)$. Using (4.5), we arrive at

$$g(\nabla_X Y, W) = \{X(\mu) - \eta(X)\}g(Y, W) = 0.$$

Hence $D\oplus < \xi >$ is integrable and its leaves are totally geodesic in $M$. Also, from Theorem 3.2, the slant distribution $D_\theta$ is integrable if and only if

$$g(\nabla_W Z, X) = \csc^2 \theta \{g(A_{FPW}X, Z) - g(A_{FW}X, Z)\} - \eta(X)g(Z, W)$$

for any $Z, W \in \Gamma(D_\theta)$ and $X \in \Gamma(D\oplus < \xi >)$. If $h^\theta$ be the second fundamental form of a leaf $N_\theta$ in $M$, then from (4.5), we derive

$$g(h^\theta(Z, W), X) = \cot^2 \theta \{X(\mu) - \eta(X)\}g(Z, W) - \csc^2 \theta \{X(\mu) - \eta(X)\}g(Z, W) - \eta(X)g(Z, W)$$

$$= -X(\mu)g(Z, W).$$

Thus, from the definition of gradient and last relation, we get

$$h^\theta(Z, W) = -\nabla_\mu g(Z, W),$$ \hspace{1cm} (4.6)

which means that $N_\theta$ is totally umbilical in $M$ with mean curvature vector $H^\theta = -\nabla_\mu$. Now we prove that $H^\theta$ is parallel corresponding to the normal connection $D$ of $N_\theta$ in $M$. For this consider any $Y \in \Gamma(D\oplus < \xi >)$ and $Z \in \Gamma(D_\theta)$,

$$g(D_Z \nabla \lambda, Y) = g(\nabla_Z \nabla \lambda, Y)$$

$$= Zg(\nabla \lambda, Y) - g(\nabla \lambda, \nabla_Z Y)$$

$$= Z(Y(\lambda)) - g(\nabla \lambda, [Z, Y]) - g(\nabla \lambda, \nabla_Y Z)$$

$$= Y(\lambda) + g(\nabla_Y \nabla \lambda, Z).$$
responding invariant and slant distributions \( D \). Proof. Let \( n \) then reeb vector field \( \xi \) is tangent to \( N_\theta \) is parallel. Thus the leaves of \( D_\theta \) are totally umbilical with parallel mean curvature. Hence, by a result of Hiepko [8], \( M \) is a warped product submanifold, which completes the proof. ■

Now, we have the following result for a mixed geodesic warped product semi-slant submanifold of aKenmotsu manifold.

Now, using the above mentioned results we are able to prove our main theorem which generalizes the Theorem 3.1 in [1].

**Theorem 4.2.** Let \( M = N_T \times_f N_\theta \) be a warped product semi-slant submanifold of a Kenmotsu manifold \( \tilde{M} \). Then

(i) The second fundamental form of \( M \) satisfies the following inequality

\[
||h||^2 \geq 4\beta (\sec^2 \theta + \cot^2 \theta)[||\nabla \ln f||^2 - 1]
\]

where \( \nabla \ln f \) is the gradient of the warping function \( \ln f \) and \( 2\beta \) is the real dimension of the tangent space of \( N_\theta \).

(ii) If the equality sign in (i) holds, then \( N_T \) is totally geodesic submanifold and \( N_\theta \) is totally umbilical submanifold of \( \tilde{M} \). Moreover, \( M \) is minimal submanifold of \( \tilde{M} \).

**Proof.** Let \( \tilde{M} \) be \((2m+1)\)-dimensional Kenmotsu manifold and \( M = N_T \times f N_\theta \) be \( n \)-dimensional warped product semi-slant submanifold of \( \tilde{M} \) such that the reeb vector field \( \xi \) is tangent to \( N_T \). If \( \dim N_T = 2\alpha + 1 \) and \( \dim N_\theta = 2\beta \), then \( n = 2\alpha + 1 + 2\beta \). Let us consider the orthonormal frames of corresponding invariant and slant distributions \( D \) and \( D_\theta \) of \( N_T \) and \( N_\theta \) as:

\[\{e_1, \cdots, e_\alpha, e_{\alpha+1} = \varphi e_1, \cdots, e_{2\alpha} = \varphi e_\alpha, e_{2\alpha+1} = \xi\}\]

is the orthonormal frame for \( D \) and \( \{e_1^*, \cdots, e_\beta, e_{\beta+1}^* = \sec \theta Pe_1^*, \cdots, e_{2\beta}^* = \sec \theta Pe_\beta^*\}\) is the orthonormal frame for \( D_\theta \). Then the orthonormal frames in the normal bundle \( T^\perp \tilde{M} \) of \( F D_\theta \) and invariant normal subbundle \( \mu \) are respectively \( \{\tilde{e}_1 = \csc \theta Pe_1^*, \cdots, \tilde{e}_\beta = \csc \theta Pe_\beta^*, \tilde{e}_{\beta+1} = \sec \theta \sec \theta Pe_1^*, \cdots, \tilde{e}_{2\beta} = \sec \theta \sec \theta Pe_\beta^*\}\) and \( \{\tilde{e}_{n+2\beta+1}, \cdots, \tilde{e}_{2m+1}\} \). Then from the definition of second fundamental form and the above frame we have

\[
||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j))
\]

\[
= \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{n} g(h(e_i, e_j), \tilde{e}_r)^2
\]

\[
= \sum_{r=n+1}^{n+2\beta} \sum_{i,j=1}^{n} g(h(e_i, e_j), \tilde{e}_r)^2 + \sum_{r=n+2\beta+1}^{2m+1} \sum_{i,j=1}^{n} g(h(e_i, e_j), \tilde{e}_r)^2.
\]

The first term in the right hand side of the above equation is the \( FD_\theta \)-component and the second term is \( \nu \)-component. Let us compute \( F D_\theta \)-component, then we
have
\[ \|h\|^2 \geq \sum_{r=n+1}^{n+2\beta} \sum_{i,j=1}^{n} g(h(e_i, e_j), \tilde{e}_r)^2. \]

Using the frames of \( D \) and \( D_\theta \), the above inequality will be
\[
\|h\|^2 \geq \sum_{r=1}^{2\beta} \sum_{i,j=1}^{2\alpha+1} g(h(e_i, e_j), \tilde{e}_r)^2 + 2 \sum_{r=1}^{2\beta} \sum_{i=1}^{2\beta} \sum_{j=1}^{2\beta} g(h(e_i, e_j^*), \tilde{e}_r)^2
\]
\[ + \sum_{r=1}^{2\beta} \sum_{i,j=1}^{2\beta} g(h(e_i^*, e_j^*), \tilde{e}_r)^2. \quad (4.9) \]

Then from (4.4), the first term in the right hand side of above inequality is identically zero. Let us compute just next term
\[
\|h\|^2 \geq 2 \sum_{j=1}^{2\beta} \sum_{i=1}^{2\alpha+1} \sum_{j=1}^{2\beta} g(h(e_i, e_j^*), \tilde{e}_j)^2.
\]
\[= 2 \sum_{j=1}^{2\beta} \sum_{i=1}^{2\alpha} g(h(e_i, e_j^*), \tilde{e}_j)^2 + 2 \sum_{j=1}^{2\beta} g(h(\xi, e_j^*), \tilde{e}_j)^2. \]

Since for a Kenmotsu manifold \( \widetilde{M} \), we have \( h(X, \xi) = 0 \), for any \( X \in \Gamma(T\widetilde{M}) \). Using this fact the last term of above inequality is identically zero. Then using the assumed frames of \( D \), \( D_\theta \) and \( FD_\theta \), we derive
\[
\|h\|^2 \geq 2 \csc^2 \theta \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} g(h(e_i, e_j^*), Fe_j^*)^2
\]
\[+ 2 \csc^2 \theta \sec^2 \theta \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} g(h(e_i, Pe_j^*), FP e_j^*)^2
\]
\[+ 2 \csc^2 \theta \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} g(h(\varphi e_i, e_j^*), Fe_j^*)^2
\]
\[+ 2 \csc^2 \theta \sec^2 \theta \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} g(h(e_i, Pe_j^*), FP e_j^*)^2
\]
\[+ 2 \csc^2 \theta \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} g(h(e_i, e_j^*), Fe_j^*)^2
\]
\[+ 2 \csc^2 \theta \sec^2 \theta \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} g(h(\varphi e_i, Pe_j^*), Fe_j^*)^2
\]
\[+ 2 \csc^2 \theta \sec^2 \theta \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} g(h(e_i, Pe_j^*), FP e_j^*)^2
\]
\[+ 2 \csc^2 \theta \sec^2 \theta \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} g(h(\varphi e_i, Pe_j^*), Fe_j^*)^2
\]
\[ + 2 \csc^2 \theta \sec^2 \theta \sum_{i=1}^{2\alpha} \sum_{j=1}^{\beta} g(h(\varphi e_i, e_j^*), FP e_j^*)^2. \]

Now, from the relation (4.3), Lemma 4.2 and the fact that for an orthonormal frame of \(D\), \(\eta(e_i) = 0\), for \(i = 1, \cdots, 2\alpha\), we derive

\[ \|h\|^2 \geq 4 \csc^2 \theta \sum_{i=1}^{2\alpha} \sum_{j=1}^{\beta} (e_i \ln f)^2 g(e_j^*, e_j^*)^2 \]

\[ + 4 \csc^2 \theta \cos^2 \theta \sum_{i=1}^{2\alpha} \sum_{j=1}^{\beta} (e_i \ln f)^2 g(e_j^*, e_j^*)^2. \]

Hence, to satisfy (4.2), we add and subtract the same term, we get

\[ \|h\|^2 \geq 4 \sum_{i=1}^{2\alpha+1} \sum_{j=1}^{\beta} \{\csc^2 \theta + \cot^2 \theta\} (e_i \ln f)^2 g(e_j^*, e_j^*)^2 \]

\[ - 4 \sum_{j=1}^{\beta} \xi \ln f)^2 g(e_j^*, e_j^*)^2. \]

Then from Lemma 4.2 (i), we derive

\[ \|h\|^2 \geq 4\beta (\csc^2 \theta + \cot^2 \theta) \|\nabla \ln f\|^2 - 1 \]

which is the inequality (i). If the equality holds in (i), then from (4.8) and (4.9), we get

\( h(D, D) = 0, \quad h(D_\theta, D_\theta) = 0 \) and \( h(D, D_\theta) \subset FD_\theta \). (4.10)

If \( h^\theta \) is the second fundamental form of \( N_\theta \) in \( M \), then from (4.6) we have

\[ h^\theta(Z, W) = -\nabla \ln fg(Z, W) \]

for any \( Z, W \in \Gamma(D_\theta) \). Since \( N_\theta \) is totally geodesic submanifold in \( M \), using this fact with the first condition of (4.10), we get \( N_\theta \) is totally geodesic in \( \tilde{M} \). Also, since \( N_\theta \) is totally umbilical in \( M \), using this fact with (4.6) and the second condition of (4.10), we get \( N_\theta \) is totally umbilical in \( \tilde{M} \). Moreover all conditions of (4.10) with the above fact show the minimality of \( M \) in \( \tilde{M} \). This proves the theorem completely. 

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