NOTE ON THE BLOWUP CRITERION OF SMOOTH SOLUTION TO THE INCOMPRESSIBLE VISCOELASTIC FLOW

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Abstract. We study the blowup criterion of smooth solution to the Oldroyd models. Let \((u(t, x), F(t, x))\) be a smooth solution in \([0, T]\), it is shown that the solution \((u(t, x), F(t, x))\) does not appear breakdown until \(t = T\) provided \(\nabla u(t, x) \in L^1([0, T]; L^\infty(\mathbb{R}^n))\), \(n = 2, 3\).

AMS Subject Classification 2000: 76A10, 76A05, 35B05.

Key words and phrases: Incompressible viscoelastic fluids, Oldroyd model, blowup criterion of smooth solution.

1. Introduction

In this paper, we consider the blowup criterion of smooth solution to the incompressible Oldroyd model in the two and three dimensional space:

\[
\begin{align*}
\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p &= \nabla \cdot (FF^t), \\
\partial_t F + u \cdot \nabla F &= \nabla u F, \\
\text{div} u &= 0,
\end{align*}
\]

for any \(t > 0, \ x \in \mathbb{R}^n, \ n = 2, 3\), where \(u(t, x)\) is the velocity field, \(p\) is the pressure, \(\mu\) is the viscosity and \(F\) the deformation tensor. We denote \((\nabla \cdot F)_i = \partial_{x_j} F_{ij}\) for a matrix \(F\). The Oldroyd model \((1.1)\) describes an incompressible non-Newtonian fluid, which bears the elastic property. For the details on this model see [7].

The local existence and uniqueness of the Oldroyd model on entire space \(\mathbb{R}^n\) or a periodic domain was established by Lin etc. in [7], where the global existence and uniqueness of smooth solution with small initial data was also established see also [5]. The wellposedness on a bounded smooth domain with Dirichlet conditions was established by Lin and Zhang in [5].

We remark some properties of the deformation tensor. Let \(x\) be the Euler coordinate and \(X\) the Lagrangian coordinate. For a given velocity field \(u(t, x)\) the flow map \(x(t, X)\) is defined by the following ordinary differential equation

\[
\begin{align*}
\frac{d}{dt} x(t, X) &= u(t, x(t, X)), \\
x(0, X) &= X.
\end{align*}
\]

The deformation tensor is \(\tilde{F}(t, X) = \frac{dx}{dt}(t, X)\). In the Eulerian coordinate, the corresponding deformation tensor is defined as \(F(t, x(t, X)) = \tilde{F}(t, X)\). Differentiating its both sides with respect to \(t\) by chain rule one obtain the second equation of \((1.1)\), which says that \(\partial_t F_{ij} + u_k \cdot \partial_{x_k} F_{ij} = \partial_{x_k} u_i F_{kj}\) for \(i, j = 1, 2, \cdots, n\), in the \((i, j)\)-th entries, where we use the Einstein summation convention that the repetition index denotes sum over 1 to \(n\).

If \(\text{div} F(0, x) = 0\), then from the second equation of Oldroyd \((1.1)\) we have

\[
\partial_t (\nabla \cdot F^t) + u \cdot \nabla (\nabla \cdot F^t) = 0.
\]

Therefore, \(\nabla \cdot F^t = 0\) for any \(t > 0\).
Denote the ith column of $F$ as $F_i$, then $\nabla \cdot (FF^t) = F_i \cdot \nabla F_i$ by the fact $\nabla \cdot F^t = 0$. So the system (1.4) can be rewritten in an equivalent form

\[
\begin{aligned}
\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p &= F_i \cdot \nabla F_i, \\
\partial_t F_k + u \cdot \nabla F_k &= F_k \cdot \nabla u, \quad k = 1, \cdots, n, \\
\text{div} u &= 0, \text{div} F = 0.
\end{aligned}
\]

(1.3)

In reference [7], Lin, Liu and Zhang obtained the local existence and uniqueness of smooth solution for smooth initial data, and had a blowup criterion.

**Theorem** (Lin, Liu and Zhang) For smooth initial data $(u_0, F_0) \in H^2(\mathbb{R}^n)$, there exists a positive time $T = T(\|u_0\|_{H^2}, \|F_0\|_{H^2})$ such that the system (1.4) possesses a unique smooth solution on $[0, T]$ with

$$
(u, F) \in L^\infty([0, T]; H^2(\mathbb{R}^n)) \cap L^2([0, T]; H^3(\mathbb{R}^n)).
$$

Moreover, if $T^*$ is the maximal time of existence, then

$$
\int_0^{T^*} \left\| \nabla u(t) \right\|^2_{H^2}dt = +\infty.
$$

In reference [3], Hu and Hynd study the blowup criterion for the ideal viscoelastic flow, which is the Oldroy system (1.1) in the case of $\mu = 0$. They showed an Beale-Kato-Majda [1] type blowup criterion that the smooth solution to the Oldroy flow do not develop singularity for $t \leq T$ provided that

$$
\int_0^T \left\| \nabla \times u \right\|_{L^\infty(\mathbb{R}^n)}ds + \sum_{k=1}^3 \int_0^T \left\| \nabla \times F_k \right\|_{L^\infty(\mathbb{R}^n)}ds < +\infty.
$$

From the modeling of Oldroy system we know that the deformation tensor can be determined by the velocity $u$ of the flow. Therefore we consider the blowup criterion of smooth solution by means of only $\|\nabla u\|_{\infty}$. In fact, Zhao, Guo and Huang [12] constructed a set of finite time blowup solution in two dimension case:

$$
u(t, x) = \left( \frac{x_1 f_0}{1 + \frac{x_2 f_0}{\beta - \alpha}} \right)^{\frac{x_1 f_0}{\beta - \alpha}}, \quad p(t, x) = \frac{(\alpha x_1^2 - \beta x_2^2)f_0}{(\beta - \alpha)(1 - \frac{x_1^2 + x_2^2 f_0 t}{\beta - \alpha})^{\beta - \alpha}},
$$

$$
F(t, x) = \text{diag} \left( \left| 1 - \frac{\alpha + \beta}{\alpha - \beta} f_0 t \right|^{\frac{\beta - \alpha}{\alpha + \beta}}, \left| 1 - \frac{\alpha + \beta}{\alpha - \beta} f_0 t \right|^{\frac{\alpha - \beta}{\alpha + \beta}} \right).
$$

If $\frac{\alpha - \beta}{\alpha + \beta} f_0 > 0$, $\alpha + \beta \neq 0$ and $\alpha - \beta \neq 0$, then the above solution will blow up at time $T^* = \frac{\alpha - \beta}{(\alpha + \beta) f_0}$. We see that

$$
\int_0^{T^*} \| \nabla u(t) \|_{\infty} dt = +\infty.
$$

There are other types of blowup criteria of smooth solutions to the Oldroy models, for example [6, 2]. To this end, we state our main results.

**Theorem 1.1.** Let $u_0 \in H^2(\mathbb{R}^n)$ and $F_0 \in H^2(\mathbb{R}^n)$ with $\nabla \cdot u_0 = \nabla \cdot F_{k,0} = 0$ for $k = 1, \cdots, n$. Assume the pair $(u, F) \in L^\infty([0, T]; H^2(\mathbb{R}^n)) \cap L^2([0, T]; H^3(\mathbb{R}^n))$ is a smooth solution to the Oldroy system (1.3). Then the smooth solution do not appear breakdown until $T^* > T$ provided that

(1.4) \[
\int_0^{T^*} \| \nabla u(t) \|_{\infty} dt < +\infty.
\]

**Remark 1.1.** For the local smooth solution $(u, F) \in L^\infty([0, T]; H^2(\mathbb{R}^n)) \cap L^2([0, T]; H^3(\mathbb{R}^n))$, if $T^*$ is its maximum existence time, then $\int_0^{T^*} \| \nabla u(t) \|_{\infty} dt = +\infty.$
In the second section we will prove the Theorem [1.1] for the case \( n = 2 \), which can be done by energy estimates. The \( L^2 \) and \( H^1 \) energy estimates are the same for the case \( n = 2 \) and \( n = 3 \). In the \( H^2 \) energy estimate, we use the Sobolev interpolation inequality \( \| \nabla F \|^2 \leq C \| \nabla F \|^2 \| \Delta F \| \). In case \( n = 3 \), however, the inequality is \( \| \nabla F \|^2 \leq C \| \nabla F \|^2 \| \Delta F \| \) which does not match the \( H^2 \) energy estimate, because it will result in the appearance of the term \( \| \Delta F \| \) that the power is higher than the left hand side. We obtain the \( H^2 \) energy estimate of \( u \) by virtue of the momentum equation, combining the \( H^2 \) estimate of \( u \) and \( F \) again with the estimate of \( \| \nabla F \| \), we grasp the \( H^2 \) energy estimate of \( u \) and \( F \) finally. The section three will devote to the proof of the case \( n = 3 \).

In this paper \( C \) denote a harmless constant which may be dependent on dimension \( n \), the norm of initial data, the viscosity \( \mu \), but not dependent on the estimated quantity. We denote the \( L^p \) norm of a function \( f \) by \( \| f \|_p \) or \( \| f \|_{L^p} \). We denote the derivative with respect to \( x_i \) by \( \partial_i \) or \( \partial_{x_i} \). We also use \( f_t \) to denote the derivative of \( f \) with respect to \( t \).

2. Proof of the case \( n = 2 \)

(1) \( L^2 \)-energy estimate and \( L^p \) estimate of the deformation tensor \( F \)

The \( L^2 \)-energy estimate can be easily obtained by the standard \( L^2 \) inner product process.

\[
\frac{1}{2} \frac{d}{dt} (\| u \|^2 + \| F \|^2) + \mu \| \nabla u \|^2 = (F_i \cdot \nabla F_i, u) + (F_k \cdot \nabla u, F_i) = 0.
\]

So we have

\[
\frac{d}{dt} (\| u \|^2 + \| F \|^2) + 2\mu \int_0^t \| \nabla u \|^2 ds = \| u_0 \|^2 + \| F(0) \|^2.
\]

Multiplying both sides of the second equation of [1.3] by \( p |F_k|^p F_k \) for \( 2 \leq p < \infty \) and integrating both sides on \( \mathbb{R}^n \) it follows that

\[
\frac{d}{dt} \| F \|_p \leq p \| \nabla u \| \| F \|_p.
\]

Summing up the estimate (2.2) with respect to \( k \) one has

\[
\| F \|_p \leq \| F_0 \|_p \exp \left\{ C(n) \int_0^t \| \nabla u(s) \| \| s \| ds \right\}.
\]

Let \( p \to \infty \), we have

\[
\| F \|_\infty \leq \| F_0 \|_\infty \exp \left\{ C(n) \int_0^t \| \nabla u(s) \| \| s \| ds \right\}.
\]

(2) \( H^1 \)-energy estimate

We differentiate the equations [1.3] with respect to \( x_i \), then multiply the resulting equations by \( \partial_i u \) and \( \partial_i F_j \) for \( i = 1, 2 \), integrate with respect to \( x \) and sum them up. It follows that

\[
\frac{1}{2} \frac{d}{dt} (\| \partial_i u \|^2 + \| \partial_i F \|^2) + \mu \| \partial_i \nabla u \|^2 \leq
\]

\[
\| (\partial_i u \cdot \nabla u, \partial_i u) \| + \| (\partial_i F_k \cdot \nabla F_k, \partial_i u) \| + \| (\partial_i u \cdot \nabla F_j, \partial_i F_j) \| + \| (\partial_i F_j \cdot \nabla u, \partial_i F_j) \|,
\]

where use has been made of the facts

\[
(\partial_i u \cdot \nabla F_j, \partial_i u) = (\partial_i F_j, \partial_i u) = (\nabla \partial_i p, \partial_i u) = 0,
\]

\[
(F_k \cdot \nabla \partial_i F_k, \partial_i u) + (F_j \cdot \nabla \partial_i u, \partial_i F_j) = 0.
\]

Noting that

\[
| (\partial_i u \cdot \nabla u, \partial_i u) | \leq \| \nabla u \| \| \nabla u \|^2,
\]

\[
| (\partial_i F_k \cdot \nabla F_k, \partial_i u) |, \quad | (\partial_i u \cdot \nabla F_j, \partial_i F_j) |, \quad | (\partial_i F_j \cdot \nabla u, \partial_i F_j) | \leq \| \nabla u \| \| \nabla F \|^2.
\]
that Gronwall’s inequality implies
\[ \| \nabla u \|_2^2 + \| \nabla F \|_2^2 \leq C \| \nabla u \|_\infty (\| \nabla u \|_2^2 + \| \nabla F \|_2^2). \]

Gronwall’s inequality implies
\[ (2.5) \quad \| \nabla u \|_2^2 + \| \nabla F \|_2^2 + 2\mu \int_0^t \| D^2 u \|_2^2 ds \leq (\| \nabla u_0 \|_2^2 + \| \nabla F(0) \|_2^2) \exp \left\{ \int_0^t C \| \nabla u(s) \|_\infty ds \right\}. \]

(3) \( H^2 \)-energy estimate

Applying operator \( \Delta \) on both sides of (1.3), we have
\[ (2.6) \quad \begin{aligned}
\partial_t \Delta u - \mu \Delta^2 u + \Delta u \cdot \nabla u + u \cdot \Delta \nabla u + 2\partial_t u \cdot \nabla \partial_t u + \nabla \Delta p &= \Delta F_k \cdot \nabla F_k + F_k \Delta F_k + 2\partial_t F_k \cdot \nabla \partial_t F_k \\
\partial_t \Delta F_k + \Delta u \cdot \nabla F_k + u \cdot \nabla \Delta F_k + 2\partial_t u \cdot \nabla \partial_t F_k &= \Delta F_k \cdot \nabla u + F_k \cdot \nabla u + 2\partial_t F_k \cdot \nabla \partial_t u.
\end{aligned} \]

Taking the \( L^2 \) inner of equation (2.6) with \( \Delta u \) and \( \Delta F_k \) and summing them up, one can obtain that
\[ (2.7) \quad \begin{aligned}
\frac{1}{2} \frac{d}{dt} (\| \Delta u \|_2^2 + \| \Delta F \|_2^2) &+ \mu \| \Delta \nabla u \|_2^2 \\
&\leq (\| \Delta u \cdot \nabla u, \Delta u \|) + 2(\| \partial_t u \cdot \nabla \partial_t u, \Delta u \|) + (\| \Delta F_k \cdot \nabla F_k, \Delta u \|)
\end{aligned} \]

Taking the \( L^2 \) inner of equation (2.6) with \( \Delta u \) and \( \Delta F_k \) and summing them up, one can obtain that
\[ (2.8) \quad \begin{aligned}
\| \Delta u \|_2^2 + \| \Delta F \|_2^2 + \mu \int_0^t \| \Delta \nabla u(s) \|_2^2 ds &\leq (\| \Delta u_0 \|_2^2 + \| \Delta F(0) \|_2^2) \exp \left\{ C \exp \left( \int_0^t C t \| \nabla u \|_\infty ds \right) \right\}. \end{aligned} \]

(4) Higher derivative estimates.
Next we derive the higher derivative estimate of $u$ and $F$. For this purpose we need the following commutator estimate.

**Proposition 2.1.** (Kato and Ponce [4, 9]) Let $1 < p < \infty$ and $0 < s$. Assume that $f$, $g \in W^{s,p}$, then there exists a constant $C$ such that

\begin{equation}
\| [J^s, f]g \|_p \leq C(\| \nabla f \|_{p_1} \| g \|_{W^{s-1,p_2}} + \| f \|_{W^{s,p_3}} \| g \|_{p_4})
\end{equation}

and

\begin{equation}
\| [\Lambda^s, f]g \|_p \leq C(\| \nabla f \|_{p_1} \| \Lambda^{s-1}g \|_{p_2} + \| \Lambda^s f \|_{p_3} \| g \|_{p_4})
\end{equation}

with $1 < p_2, p_3 < \infty$ such that

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},
\]

where $[\Lambda^s, f]g = \Lambda^s(fg) - f \Lambda^s g$ and $\Lambda^s = (-\Delta)^{\frac{s}{2}}$, $J = (1 - \Delta)^{1/2}$.

Applying $\Lambda^s$ on both sides of (1.3) and taking the inner product with $\Lambda^s u$ and $\Lambda^s F$, it can be derived that

\begin{equation}
\frac{1}{2} \frac{d}{dt} (\| \Lambda^s u \|_2^2 + \| \Lambda^s F_k \|_2^2) + \mu \| \Lambda^{s+1} u \|_2^2 \leq (\Lambda^s(u \cdot \nabla) - u \cdot \nabla \Lambda^s u, \Lambda^s u) + (\Lambda^s(F_k \cdot \nabla F_k) - F_k \cdot \nabla \Lambda^s F_k, \Lambda^s u) + (\Lambda^s(u \cdot \nabla F_k) - u \cdot \nabla \Lambda^s u, \Lambda^s F_k) + (\Lambda^s(F_k \cdot \nabla u) - F_k \cdot \nabla \Lambda^s u, \Lambda^s F_k),
\end{equation}

where we have used the facts

\[
(F_k \cdot \nabla \Lambda^s F_k, \Lambda^s u) + (F_k \cdot \nabla \Lambda^s u, \Lambda^s F_k) = 0,
\]

\[
(u \cdot \nabla \Lambda^s F_k, \Lambda^s u) = (u \cdot \nabla \Lambda^s u, \Lambda^s u) = 0.
\]

The commutator estimate (2.10) implies that

\[
\| \Lambda^s(u \cdot \nabla) - u \cdot \nabla \Lambda^s u \|_2 \leq \| \nabla u \|_{\infty} \| \Lambda^s u \|_2,
\]

\[
\| \Lambda^s(F_k \cdot \nabla F_k) - F_k \cdot \nabla \Lambda^s F_k \|_2 \leq \| \nabla F \|_{\infty} \| \Lambda^s F \|_2 \leq \| \nabla F \|_{H^{s-1}} \| \Lambda^s F \|_2,
\]

\[
\| \Lambda^s(u \cdot \nabla F_k) - u \nabla \Lambda^s u \| \leq \| \nabla u \|_{\infty} \| \Lambda^s F \|_2 + \| F \|_{\infty} \| \Lambda^{s+1} u \|_2,
\]

\[
\| \Lambda^s(F_k \cdot \nabla u) - F_k \nabla \Lambda^s u \| \leq \| \nabla u \|_{\infty} \| \Lambda^s F \|_2 + \| F \|_{\infty} \| \Lambda^{s+1} u \|_2,
\]

where the Sobolev embedding $H^{s-1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ for $s > 1 + \frac{n}{2}$ is applied.

Inserting the above estimates into estimate (2.11), it follows

\begin{equation}
\frac{1}{2} \frac{d}{dt} (\| \Lambda^s u \|_2^2 + \| \Lambda^s F_k \|_2^2) + \frac{\mu}{2} \| \Lambda^{s+1} u \|_2^2 \leq C(\| \nabla u \|_{\infty} + \| \nabla F \|_2 + \| \Lambda^s u \|_2 + \| F \|_{L^\infty}) (\| \Lambda^s u \|_2^2 + \| \Lambda^s F \|_2^2),
\end{equation}

where we have used the fact

\[
\| \nabla F \|_{H^{s-1}} \| \Lambda^s F \|_2 \| \Lambda^s u \|_2 \leq \| \nabla F \|_2 (\| \Lambda^s F \|_2^2 + \| \Lambda^s u \|_2^2) + \| \Lambda^s F \|_2^2 \| \Lambda^s u \|_2.
\]

So, for $s \geq 3$, applying Gronwall’s inequality to (2.12), by induction for $u$’s estimate, we obtain the higher derivative estimate:

\[
\| \Lambda^s u \|_2^2 + \| \Lambda^s F \|_2^2 + \mu \int_0^t \| \Lambda^{s+1} u \|_2^2 ds \leq (\| u_0 \|_{H^s}^2 + \| F(0) \|_{H^s}^2) \exp \left\{ \int_0^t C(\| \nabla u \|_{\infty} + \| \nabla F \|_2 + \| \Lambda^s u \|_2 + \| F \|_{L^\infty}) ds \right\}.
\]

Therefore, we complete the proof of the case $n = 2$. 
3. Proof of the case $n = 3$

In the three dimensional case the $L^2$ and $H^1$ energy estimates are the same as the case of dimension two. To estimate the $H^2$ energy estimate we need the following estimates.

Multiplying the first equation of (1.3) by $u_t$ and integrating both sides over $\mathbb{R}^3$ with respect to $x$, and noting $\text{div } u = 0$, it follows

$$\frac{\mu}{2} \frac{d}{dt} \| \nabla u \|^2 + \| u_t \|^2 \leq |(u \cdot \nabla u, u_t)| + |(F_k \cdot \nabla F_k, u_t)|$$

$$\leq \frac{1}{2} \| u_t \|^2 + C \| u \|_{L^\infty}^2 \| \nabla u \|^2 + C \| F \|^2 \| F \|_{L^\infty}^2.$$  

Integrating both sides with respect to $t$ it yields

$$\mu \| \nabla u \|^2 + \int_0^t \| u_t \|^2 ds \leq \mu \| \nabla u_0 \|^2 + \sup_{0 \leq s \leq t} \| \nabla u \|^2 + \int_0^t \| u \|^2 \| F \|^2 \| F \|_{L^\infty}^2 ds$$

where the Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ has been used.

Differentiating the first equation of (1.3) with respect to $t$, we arrive at

$$u_{tt} - \mu \Delta u_t + u_t \cdot \nabla u + \mu \nabla u_t + \nabla p_t = F_{ktt} \cdot \nabla F_k + F_k \cdot \nabla F_{kt}.$$  

Taking $L^2$ inner product of the equation (3.2) with respect to $u_t$, it can be similarly derived that

$$\frac{1}{2} \frac{d}{dt} \| u_t \|^2 + \mu \| \nabla u_t \|^2 \leq \| \nabla u \|_{L^\infty} \| u_t \|^2 + 2 \| F \|_{L^\infty} \| \nabla u_t \| \| F_t \|_2$$

$$\leq \frac{\mu}{2} \| u_t \|^2 + \| \nabla u \|_{L^\infty} \| u_t \|^2 + C \| F \|^2 \| F_t \|_2^2.$$  

Applying the Gronwall' inequality, it yields

$$\| u_t \|^2 + \mu \int_0^t \| u_t \|^2 ds \leq (\| u_t(0) \|^2 + C \int_0^t \| F \|^2 \| F \|_{L^\infty}^2 ds) \exp \left\{ \int_0^t \| \nabla u \|_{L^\infty} ds \right\}.$$  

It need still to estimate $\| F_t \|^2_2$. From the second equation of (1.3) it can be derived that

$$\| F_t \|^2_2 \leq \| F_t \|_2 \| u \|_{L^\infty} \| \nabla F \|_2 + \| F_t \|_2 \| F \|_{L^\infty} \| \nabla u \|_2$$

$$\leq \frac{1}{2} \| F_t \|^2_2 + C \| u \|^2 \| \nabla F \|^2_2 + C \| F \|^2 \| \nabla u \|^2_2.$$  

So we arrive at

$$\| F_t \|^2_2 \leq C \| u \|^2 \| \nabla F \|^2_2 + C \| F \|^2 \| \nabla u \|^2_2.$$  

Inserting it to the estimate (3.3) we obtain the estimate of $\| u_t \|_2$:

$$\| u_t \|^2_2 + \mu \int_0^t \| u_t \|^2 ds \leq C(t) < \infty,$$

where $C(t)$ is explicit increasing function of $t$ dependent on $\int_0^t \| \nabla u \|_{L^\infty} ds$. From the first equation of (1.3), $\nabla p$ can be solved by Riesz transformation $R = (R_1, R_2, R_3)^t$, with $R_j = -i\partial_{x_j} (-\Delta)^{-\frac{1}{2}}$ being the jth Riesz transformation.

$$\nabla p = RR \cdot (u \cdot \nabla u) - RR \cdot (F_k \cdot \nabla F_k).$$

In virtue of the boundedness of Riesz operator $R$ in $L^p$ space for $1 < p < \infty$, we obtain that

$$\| \nabla p \|_2 \leq C \| \nabla u \|_2 \| u \|_{L^\infty} + C \| \nabla F \|_2 \| F \|_{L^\infty}.$$  

For details about Riesz transformation see [10] [11].

Thus from the first equation of (1.3) we have

$$\frac{\mu}{2} \| \Delta u \|_2 \leq \| u_t \|_2 + \| u \cdot \nabla u \|_2 + \| \nabla p \|_2 + \| F_k \cdot \nabla F_k \|_2$$

$$\leq \| u_t \|_2 + \frac{\mu}{2} \| \Delta u \|_2 + C \| u \|_2 \| \nabla u \|^2_2 + C \| F \|_{L^\infty} \| \nabla F \|^2_2,$$
where the interpolation inequality \(\|u\|_\infty \leq C\|u\|_2^{\frac{1}{2}}\|\Delta u\|_2^{\frac{3}{2}}\) has been used. So we derive

\[
(3.5) \quad \|\Delta u\|_2 \leq C(\|u\|_2 + \|\Delta u\|_2 + \|u\|_2 \|\nabla u\|_2^4 + \|F\|_\infty \|\nabla F\|_2).
\]

Next we derive the estimate of \(\|\Delta F\|_2\). Applying \(\Delta\) on both sides of equation (1.3) and taking the \(L^2\) inner product with \(\Delta u\) and \(\Delta F\) respectively, we have

\[
(3.6) \quad \frac{1}{2} \frac{d}{dt} \|\Delta u\|_2^2 + \mu \|\Delta \nabla u\|_2^2 \leq |(\Delta (u \cdot \nabla u) - u \cdot \nabla \Delta u, \Delta u)| + |(\Delta (F \cdot \nabla F) - F \cdot \nabla \Delta F, \Delta u)|,
\]

\[
(3.7) \quad \frac{1}{2} \frac{d}{dt} \|\Delta F\|_2^2 \leq |(\Delta (u \cdot \nabla F) - u \cdot \nabla \Delta F, \Delta F)| + |(\Delta (F \cdot \nabla u) - F \cdot \nabla \Delta u, \Delta F)|,
\]

where use has been of the facts

\[
(u \cdot \nabla \Delta u, \Delta u) = (u \cdot \nabla \Delta F, \Delta F) = 0,
\]

\[
(F \cdot \nabla \Delta F, \Delta u) + (F \cdot \nabla \Delta u, \Delta F) = 0.
\]

Next we estimate the right hand sides. By the communicator estimate (2.10) one has

\[
|(\Delta (u \cdot \nabla u) - u \cdot \nabla \Delta u, \Delta u)| \leq \|\Delta u\|_2\|\Delta (u \cdot \nabla u) - u \cdot \nabla \Delta u\|_2 \leq \|\nabla u\|_\infty \|\Delta u\|_2^2,
\]

\[
|(\Delta (u \cdot \nabla F) - u \cdot \nabla \Delta F, \Delta F)| \leq \|\Delta F\|_2(\|\nabla u\|_\infty \|\Delta F\|_2^2 + \|F\|_\infty \|\nabla \Delta u\|_2^2),
\]

\[
|(\Delta (F \cdot \nabla u) - F \cdot \nabla \Delta u, \Delta F)| \leq \|\nabla u\|_\infty \|\Delta F\|_2^2 + C\|F\|_\infty \|\Delta F\|_2^2 + \frac{\mu}{8} \|\nabla \Delta u\|_2^2.
\]

For the second term on the right hand side of (3.6) we estimate as follows

\[
|(\Delta (F \cdot \nabla F) - F \cdot \nabla \Delta F, \Delta u)| \leq \|\Delta u\|_6 \|\Delta (F \cdot \nabla F) - F \cdot \nabla \Delta F\|_{6/5},
\]

and

\[
\|\Delta (F \cdot \nabla F) - F \cdot \nabla \Delta F\|_{6/5} \leq \|\nabla F\|_6 \|\Delta F\|_{3/2} \leq \|\nabla u\|_6 \|\nabla \Delta F\|_2^2 \|\Delta F\|_2^2.
\]

So one has the estimate

\[
|(\Delta (F \cdot \nabla F) - F \cdot \nabla \Delta F, \Delta u)| \leq \frac{\mu}{4} \|\nabla \Delta u\|_2^2 + C\|\nabla F\|_6^4 + C\|\nabla \Delta F\|_2^4 \|\Delta F\|_2^2.
\]

Summing up (3.6) and (3.7), and inserting the above estimates into the summation, we arrive at

\[
(3.8) \quad \frac{d}{dt}(\|\Delta u\|_2^2 + \|\Delta F\|_2^2) + \mu \|\Delta \nabla u\|_2^2 \leq C(\|\nabla u\|_\infty \|\nabla \Delta u\|_2^2 + \|\nabla F\|_\infty \|\nabla \Delta F\|_2^2) + C\|\nabla F\|_6^6.
\]

We still have to estimate \(\|\nabla F\|_6\). Differentiating the second equation of (1.3) with respect to \(x_i\), one has

\[
\partial_i \partial_i F_k + \partial_i u \cdot \nabla F_k + u \cdot \nabla \partial_i F_k = \partial_i F_k \cdot \nabla u + F_k \cdot \nabla \partial_i u.
\]

Multiplying both sides of the above equation by \(6|\partial_i F_k|^4 \partial_i F_k\), and integrating both sides with respect to \(x\) over \(\mathbb{R}^3\), it can be derived that

\[
(3.9) \quad \frac{d}{dt}\|\nabla F\|_6^6 \leq C(\|\Delta u\|_\infty \|\nabla \Delta u\|_6^2 + \|\nabla F\|_\infty \|\nabla \Delta F\|_6^2).
\]

Next we have to derive an estimate of \(\|\Delta u\|_6\). Using an argument similar to deriving the \(L^2\) estimate \(\|\Delta u\|_2\) in (3.5) we have

\[
(3.10) \quad \mu \|\Delta u\|_6 \leq \|\Delta u\|_6 + \|\nabla u\|_\infty \|\nabla \Delta u\|_6 + C\|\nabla F\|_\infty \|\nabla F\|_6 \leq \|\partial_i \nabla u\|_2 + C\|\nabla u\|_6 \|\nabla \Delta u\|_6 + C\|\nabla F\|_6^2.
\]

Inserting estimates (3.10) to (3.9) one has

\[
(3.11) \quad \frac{d}{dt}\|\nabla F\|_6^6 \leq C(\|\nabla u\|_\infty \|\nabla F\|_6^2 + \|\partial_i \nabla u\|_6^4 + 1)\|\nabla F\|_6^4 + C\|\nabla F\|_\infty \|\nabla \Delta u\|_6^2 + C,
\]

where use has been of the facts

\[
(u \cdot \nabla \Delta u, \Delta u) = (u \cdot \nabla \Delta F, \Delta F) = 0,
\]

\[
(F \cdot \nabla \Delta F, \Delta u) + (F \cdot \nabla \Delta u, \Delta F) = 0.
\]
Combining the estimates (3.8) and (3.11) we arrive at
\[
\frac{d}{dt}(\|\Delta u\|_2^2 + \|\Delta F \cdot k\|_2^2 + \|F\|_4^4) + \mu \|\nabla \Delta u\|_2^2 \leq C(\|\nabla u\|_\infty^2 + \|\nabla F\|_2^2 + \|\partial_t \nabla u\|_2^2 + 1)(\|\Delta u\|_2^2 + \|\Delta F\|_2^2 + \|F\|_4^4) + C \|F\|_\infty^4 \|u\|_2 \|\Delta u\|_2^2 + C.
\]

Gronwall’s inequality implies the $H^2$ estimates:
\[
\|\Delta u\|_2^2 + \|\Delta F \cdot k\|_2^2 + \|F\|_4^4 + \mu \int_0^t \|\nabla \Delta u\|_2^2 ds \leq \exp\left\{C(t) \int_0^t (\|\nabla u\|_\infty + \|\partial_t \nabla u\|_2^2) ds \right\} \times
\left(\|\Delta u(0)\|_2^2 + \|\Delta F \cdot k(0)\|_2^2 + \|F(0)\|_4^4 + C \int_0^t (\|F\|_\infty^4 \|u\|_2 \|\Delta u\|_2^2 + 1) ds \right) < \infty.
\]

Based on the $H^2$ energy estimate the higher energy estimate can be obtained by bootstrap method as we did in section two. Thus the proof of the case $n = 3$ is completed.

**Acknowledgements** This work was done when the author was visiting the Courant Institute of Mathematical Sciences at New York University. The author would like to thank Professor Fanghua Lin for stimulating discussion on this topic. The research of B Yuan was partially supported by the National Natural Science Foundation of China (No. 11071057), Innovation Scientists and Technicians Troop Construction Projects of Henan Province (No. 104100510015), Program for Science&Technology Innovation Talents in Universities of Henan Province (No. 2009 HASTIT007) and Doctor Fund of Henan Polytechnic University (No. B2008-62).

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