On Dubins Paths to a Circle
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Abstract
This paper is concerned with characterizing the shortest path of a Dubins vehicle from an initial position with a prescribed initial heading angle to a target circle with the final heading tangential to the target circle. Such a shortest path is of significant importance as it is usually required in many scenarios, such as to take a snapshot of an adversarial radar or to loiter above a ground sensor to collect data by a fixed-wing unmanned aerial vehicle in a minimum time. In this paper, by applying Pontryagin’s maximum principle, some geometric properties for the shortest path are first established, showing that the shortest path must lie in a sufficient family of 12 types. In addition, by employing those geometric properties, the analytical solution to each type is devised, which allows computing the shortest path in a constant time through checking the values of at most 12 analytical functions. Finally, some numerical simulations are presented, illustrating and validating the developments of the paper.

1 Introduction
When planning a minimum-time path for an Unmanned Aerial Vehicle (UAV), it is common to consider that the UAV flies in altitude hold mode and that its cruise speed is a constant. If taking into account a nonholonomic constraint that the turning rate is bounded from below, the UAV can be simply considered as a typical nonholonomic vehicle that moves only forward at a constant speed with a minimum turning radius. Such a nonholonomic vehicle has been dubbed the Dubins vehicle since L. E. Dubins studied its shortest path in 1957 [5]. For this reason, the problem of minimum-time path planning for a large class of vehicles, such as UAVs [14], fixed-wing aircrafts [10], and thrusted skates [11], is usually simplified in the literature to the problem of finding the shortest path of Dubins vehicles (notice that the shortest path is equivalent to the minimum-time path as the speed is constant).

By basic geometric analyses, it is shown in [5] that the shortest path of Dubins vehicle between two configurations (a configuration consists of a position point and a heading orientation angle) is a \( C^1 \) path which is a concatenation of circular arcs and straight line segments, and that it can be computed in a constant time by checking at most 6 possible types of paths. This result was proven later in [1, 17] using the optimal control theory. Moreover, the shortest Dubins path from a configuration to a point with the terminal heading angle left free was studied as well, and this problem is now dubbed the relaxed Dubins problem [2].

Following the aforementioned papers, the shortest Dubins paths in many scenarios have been studied in the literature. For instance, the shortest Dubins paths through three points were studied in [3,12,16]. To be specific, the work in [3,12] presents a nature extension of the relaxed Dubins problem in [2], which consists of moving from a configuration, via a fixed intermediate point, to a fixed final point with free heading angles at both the intermediate and final points; in [16], the three-point Dubins problem (consisting of three points with prescribed heading angles at initial and final points) was studied with an assumption that the distance between any two consecutive points is at least four times the minimum turning radius. More recently, by releasing the assumption in [16], the three-point Dubins problem was thoroughly studied in [4], showing that the shortest path of three-point Dubins problem must be in a sufficient family of 18 types, and a polynomial-based method was proposed to compute each of the 18 types, allowing to solve the three-point Dubins problem efficiently. Based on the solutions of the three-point Dubins problem presented in [4,12,16], some algorithms have been proposed to address a more complex variant of the Dubins problem, namely the Dubins traveling salesman problem [9], for which there are multiple targets and the Dubins vehicle has to visit each target exactly once and finally returns to the initial point with a minimum time.

In all the aforementioned papers, the shortest Dubins paths were studied in scenarios where the Dubins vehicle moves between or among fixed points with either free or fixed heading angles at those points. In fact, there is another significant scenario for which a UAV has to fly from an initial configuration to a target circle...
with the terminal velocity (or heading) tangential to the circle. This problem is motivated by surveillance missions requiring to take a snapshot of an adversarial radar, for which the UAV should fly to a circle with its heading tangential to the circle. There are some other scenarios that may also require planning a minimum-time path from a configuration to a target circle with the terminal heading tangential to the target circle. For instance as proposed in [13], if a UAV collects data from a sensor on the ground, it has to loiter above a sensor while a reliable communication network is established, and the data transfer is done; to loiter above a ground sensor, the UAV would travel along a circle with smallest radius possible with the center of the circle above the sensor location. In addition, if a UAV is to patrol a circular-like border from an airport, the same scenario appears because the UAV needs to fly from an initial configuration to a circle with the heading tangential to the circle [14]. As pointed out in [15], another application that is closely related to this problem is the Dubins traveling salesman problem with neighborhoods [6–8, 18].

As stated above, if a UAV flies in altitude hold mode with a constant cruise speed, the problem of minimum-time path planning for the UAV to a circle can be simplified to the problem of finding the shortest Dubins path to a circle. In fact, with a strict assumption that the distance between the initial position and the target circle is greater than four times the minimum turning radius, the shortest Dubins path to a circle has been primarily studied in [13]. However, to the author’s best knowledge, if the distance assumption is released, the properties for the shortest Dubins path to a circle are not known in the literature.

In this paper, the optimal control theory will be used to characterize the shortest Dubins path to a circle without any assumptions. First of all, the shortest Dubins path to a circle is formulated as the solution path of an optimal control problem. Then, by using Pontryagin’s maximum principle [15], the necessary conditions for optimality are synthesized so that some geometric properties for the solution path are established, showing that the solution path is a smooth concatenation of circular arcs and straight line segments. Further analyses of those geometric properties not only show that some results in [15] are not correct but also restrict the solution path into a sufficient family of 12 types. In addition, the analytical solution to each of all the 12 types is devised by using the geometric properties. As a result, the shortest Dubins path from any initial configuration to any target circle can be computed in a constant time by checking the values of at most 12 analytical functions.

This paper is organized as follows: In Section 2 the optimal control problem is formulated and some necessary conditions for optimality are presented by applying Pontryagin’s maximum principle. In Section 3 some geometric properties for the shortest Dubins path to a circle are established. As a result of those geometric properties, the analytical solution to the shortest path is devised in Section 4. Finally, some numerical simulations are presented in Section 5 to illustrate the developments of the paper.

2 Problem formulation

In this section, the problem of minimum-time path planning from an initial configuration to a target circle via a kinematic UAV model is formulated as an optimal control problem, and the necessary conditions for optimality are established by applying Pontryagin’s maximum principle.

2.1 Optimal control problem

Consider the two-dimensional engagement geometry in Fig. 1 with a fixed target circle with radius \( r > 0 \). Without lose of generality, the coordinate system \( Oxy \) has its origin located at the center of the target circle, and the \( x- \) and \( y- \) axis are defined in inertially fixed directions. In the horizontal plane, the positive \( x- \)axis points to the east, and the \( y- \)axis is aligned with the north. It is obvious that the state of the UAV in the two-dimensional plane consists of a position vector and a heading orientation angle. Throughout, we denote by \((x, y) \in \mathbb{R}^2\) the position of the UAV in frame \( Oxy \), and denote by \( \theta \in [0, 2\pi] \) the heading orientation angle of the UAV with respect to the \( x \) axis, which is positive when measured counter-clockwise, as presented in Fig. 1.

The kinematic UAV model assumes that the speed is a constant and that the turning radius is lower bounded. Then, by normalizing the position \((x, y)\) so that the speed is one, the equations of motion for the UAV can be expressed as

\[
(S) : \begin{cases} 
\dot{x}(t) = \cos \theta(t), \\
\dot{y}(t) = \sin \theta(t), \\
\dot{\theta}(t) = u(t)/\rho
\end{cases}
\]  \( \tag{1} \)

where \( t \geq 0 \) is the time, \( u \in [-1, 1] \) is the control input, and the scalar \( \rho > 0 \) denotes the minimum turning radius of the UAV. At the initial time \( t = 0 \), the state is given as \((x(0), y(0), \theta(0)) = (x_0, y_0, \theta_0)\).

Note that the radius of the target circle has to be no less than the minimum turning radius, i.e., \( r \geq \rho \), if the UAV is required to finally travel on the target circle.
Define two real-valued functions in terms of $x$, $y$, and $\theta$ as
\[
\phi_1(x, y, \theta) = \frac{1}{2}(x^2 + y^2 - r^2) \tag{2}
\]
\[
\phi_2(x, y, \theta) = x \cos(\theta) + y \sin(\theta) \tag{3}
\]
Then, the minimum-time path of the UAV, modeled as $\Sigma$, from the initial configuration $(x_0, y_0, \theta_0)$ to the target circle with the terminal heading tangential to the circle is equivalent to the solution path of the following Optimal Control Problem (OCP).

**Problem 1 (OCP)** The OCP consists of finding a minimum time $t_f > 0$ so that there exists a path $(x(\cdot), y(\cdot), \theta(\cdot))$ subject to $\Sigma$ and $u \in [-1, 1]$ with the following boundary conditions satisfied:
\[
(x(0), y(0), \theta(0)) = (x_0, y_0, \theta_0)
\]
and
\[
\phi_1(x(t_f), y(t_f), \theta(t_f)) = 0 \tag{4}
\]
\[
\phi_2(x(t_f), y(t_f), \theta(t_f)) = 0 \tag{5}
\]
In the next subsection, the necessary conditions for the OCP will be presented.

### 2.2 Necessary conditions

Denote by $p_x$, $p_y$, and $p_\theta$ the costate variables of $x$, $y$, and $\theta$, respectively. Then, the Hamiltonian is
\[
H = p_x \cos(\theta) + p_y \sin(\theta) + p_\theta u/\rho - 1. \tag{6}
\]
According to Pontryagin’s maximum principle \cite{15}, for $t \in [0, t_f]$ it holds that
\[
\dot{p}_x(t) = -\frac{\partial H}{\partial x} = 0, \tag{7}
\]
\[
\dot{p}_y(t) = -\frac{\partial H}{\partial y} = 0, \tag{8}
\]
\[
\dot{p}_\theta(t) = -\frac{\partial H}{\partial \theta} = p_x(t) \sin(\theta) - p_y(t) \cos(\theta(t)). \tag{9}
\]
Note that $p_x$ and $p_y$ are constant according to Eq. (7) and Eq. (8), hence we hereafter use $p_x$ and $p_y$ to denote $p_x(t)$ and $p_y(t)$, respectively. Directly integrating Eq. (9) leads to
\[
p_\theta(t) = p_x y(t) - p_y x(t) + c_0, \tag{10}
\]
where the scalar $c_0 \in \mathbb{R}$ is an integral constant. If $p_\theta \equiv 0$ on a nonzero interval, then Eq. (10) implies that the graph of $(x, y)$ on this interval forms a straight line segment, further indicating $u \equiv 0$ on this interval. Thus, applying Pontryagin’s maximum principle, the switching of $u$ is totally determined by $p_\theta$, i.e.,
\[
u = \begin{cases} 
1, & p_\theta > 0, \\
0, & p_\theta \equiv 0, \\
-1, & p_\theta < 0.
\end{cases} \tag{11}
\]
The transversality conditions imply that at the final time $t_f$ it holds

$$p_x = \frac{\partial \phi_1}{\partial x} \nu_1 + \frac{\partial \phi_2}{\partial x} \nu_2$$
$$= \nu_1 x(t_f) + \nu_2 \cos[\theta(t_f)]$$

$$p_y = \frac{\partial \phi_1}{\partial y} \nu_1 + \frac{\partial \phi_2}{\partial y} \nu_2$$
$$= \nu_1 y(t_f) + \nu_2 \sin[\theta(t_f)]$$

$$p_\theta(t_f) = \frac{\partial \phi_1}{\partial \theta} \nu_1 + \frac{\partial \phi_2}{\partial \theta} \nu_2$$
$$= \nu_2 \{ -x(t_f) \sin[\theta(t_f)] + y(t_f) \cos[\theta(t_f)] \}$$

(12) (13) (14)

where the two real scalars, $\nu_1$ and $\nu_2$, are Lagrangian multipliers.

In the next section, the necessary conditions from Eq. (6) to Eq. (14) will be employed to establish some geometric properties for the solution path of the OCP.

3 Characterization of the shortest path

Note that the solution path of the OCP is a circular arc with its radius being $\rho$ if $||u|| = 1$. Furthermore, if $u = 1$ (resp. $u = -1$), the circular arc has a left (resp. right) turning direction. Therefore, the relations in Eq. (14) indicate that the solution path of the OCP is a concatenation of straight line segments and circular arcs. Hereafter, we denote by “S” and “C” a straight line segment and a circular arc with a radius of $\rho$, respectively, and denote by “R” and “L” circular arcs with right and left turning directions, respectively.

Property 1 The solution path of the OCP is of type CCC or CSC or a substring thereof, where

- $CCC = \{RLR, LRL\}$,
- $CSC = \{RSR, RSL, LSR, LSL\}$.

Proof. Denote by $(x(t), y(t), \theta(t))$ for $t \in [0, t_f]$ the solution path of the OCP. By contradiction, assume that the type of $(x(t), y(t), \theta(t))$ for $t \in [0, t_f]$ does not belong to either CCC or CSC or a substring thereof. Then, according to [5] and [17], there exists another shorter path along which the Dubins vehicle can move from $(x_0, y_0, \theta_0)$ to $(x(t_f), y(t_f), \theta(t_f))$. This contradicts with the fact that the solution path of the OCP is the shortest path from $(x_0, y_0, \theta_0)$ to $(x(t_f), y(t_f), \theta(t_f))$, hence completing the proof.

Although Property 1 shows that the solution path of the OCP shares the same types as the shortest Dubins path between two configurations (see, e.g., [5, 17] for the details of the shortest Dubins path between two configurations), it is not straightforward to compute the solution path of the OCP because one does not know which point on the target circle will be the final point of the solution path. By the following lemmas, some geometric properties for the solution path of the OCP will be established so that the final point on the target circle can be found analytically.

Lemma 1 If the solution path of the OCP is of type CSC, then the heading angle along the straight line segment $S$ must point to the center of the target circle.

Proof. Rearranging the equations in Eqs. (12) and (13) to eliminate the Lagrangian multipliers $\nu_1$ and $\nu_2$, we have

$$p_\theta(t_f) = p_x y(t_f) - p_y x(t_f).$$

(15)

Combining Eq. (15) with Eq. (10) indicates $c_0 = 0$ and

$$p_\theta(t) = p_x y(t) - p_y x(t), \forall t \in [0, t_f].$$

(16)

Without loss of generality, assume that $[t_1, t_2] \subset [0, t_f]$ is the interval of the straight line segment $S$. Then, since $p_\theta \equiv 0$ along the straight line segment $S$, it follows

$$p_x y(t) - p_y x(t) = 0, \forall t \in [t_1, t_2].$$

As $p_x$ and $p_y$ are constant, it is apparent that for any point $(x, y)$ such that $p_x y - p_y x = 0$, it will be on (the extension of) the straight line segment $S$. Note that the center of the target circle (it is actually the origin of frame $Oxy$, as defined in Section 2) is such a point, completing the proof.

As a result of Lemma 1, we immediately have the following result.
Lemma 2 If the solution path of the OCP is of type CSC and if the length of the straight line segment S is not zero, then the radian of the circular arc C after S is non-zero.

Proof. By contradiction, let us assume that the radian of the circular arc after S is zero. Then, the straight line segment S is tangential to the target circle instead of passing through the center of the target circle. This contradicts with Lemma 1, completing the proof. □

This lemma indicates that the solution path of the OCP cannot be a type of the substring CS. As a consequence, we immediately have the following result.

Corollary 1 The solution path of the OCP must be of a type in $\mathcal{F} = \{CCC, CSC, SC, CC, C\}$, where

- $CCC = \{RLR, LRL\}$,
- $CSC = \{RSR, RSL, LSL, LSR\}$,
- $SC = \{SR, SL\}$,
- $CC = \{RL, LR\}$,
- $C = \{R, L\}$.

It should be noted that the total number of possible types (including substrings) for the shortest Dubins path between two configurations is up to 15 [5, 17]. However, this corollary means that, including the substrings, the total number of possible types for the solution path of the OCP is 12.

Lemma 3 Assume that the minimum turning radius $\rho$ is at least half of the radius of the target circle, i.e., $\rho \geq r/2$. If the solution path is of type $C_1S_2C_3$, then the final circular arc $C_3$ and the target circle are externally tangent to each other.

Proof. By contradiction, assume that the target circle and $C_3$ are not externally tangent to each other. We first consider the case of $\rho = r$. Then, the contradicting assumption implies that $C_3$ lies on the target circle, and the straight line segment $S_2$ will be tangent to the target circle instead of pass through the center of the target circle, contradicting with Lemma 1. Hence, if $r = \rho$, this lemma holds.

We then consider the case of $r/2 \leq \rho < r$ for which $C_3$ lies in the target circle. In this case, the direction of the straight line $S_2$ cannot path through the center of the target circle because of $\rho \geq r/2$, which contradicts with Lemma 1. Hence, by contraposition, the target circle and $C_3$ are externally tangent to each other if $r/2 \leq \rho < r$.

From now on, we consider the case of $\rho > r$, for which the contradicting assumption implies that the target circle lies in $C_3$. In this case, it is immediate that the direction of the straight line $S_2$ cannot path through the center of the target circle, which contradicts with Lemma 1. Hence, by contraposition, the proof is completed. □

Lemma 4 Assume that the solution path of the OCP is of type $C_1S_2C_3$, and let $\alpha_c > 0$ be the radian of the final circular arc $C_3$. Then, the following two statements hold:

1. If $C_3$ is externally tangent to the target circle, we have
   \[ \alpha_c = \arccos \frac{\rho}{\rho + r}. \] (17)

2. If $C_3$ is internally tangent to the target circle, we have
   \[ \alpha_c = \pi - \arccos \frac{\rho}{r - \rho}. \] (18)

Proof. (1) According to Lemma 1, the final circular arc $C_3$ must be tangent to both the straight line segment $S$ and the target circle, as illustrated by the specific example in Fig. 2. Hence, the radian $\alpha_c > 0$ of the final circular arc $C_3$ takes a value such that
   \[ \cos \alpha_c = \frac{\rho}{r + \rho}. \]

Note that $\alpha_c < \pi$. Hence, the equation in (17) holds, completing the proof of the first statement.

(2) Analogously, the final circular arc $C_3$ must be tangent to both the straight line segment $S$ and the target circle, as illustrated by the specific example in Fig. 3. It is apparent that the radian $\alpha_c > 0$ of the final circular arc $C_3$ takes a value such that
   \[ \cos(\pi - \alpha_c) = \frac{\rho}{r - \rho}. \]

Since $\alpha_c < \pi$, we immediately have Eq. (18), completing the proof of the second statement. □
Lemma 5 If the solution path of the OCP is of type $C_1C_2C_3$, then the concatenating point from $C_1$ to $C_2$, the concatenating point from $C_2$ to $C_3$, and the center of the target circle lie on one straight line.

Proof. Let $(x_1, y_1) \in \mathbb{R}^2$ be the concatenating point from $C_1$ to $C_2$, and let $(x_2, y_2) \in \mathbb{R}^2$ be the concatenating point from $C_2$ to $C_3$. According to Eq. (11), we have $p_\theta > 0$ (resp. $p_\theta < 0$) on any left (resp. right) turning circular arc.

Note that the type $CCC$ can be either RLR or LRL. Because the costate variable $p_\theta$ is continuous, we have that $p_\theta = 0$ at the two concatenating points $(x_1, y_1)$ and $(x_2, y_2)$. As it has been proved in the proof of Lemma 1 that $c_0 = 0$, writing the expression of $p_\theta$ in Eq. (10) explicitly at the two points $(x_1, y_1)$ and $(x_2, y_2)$ leads to

$$p_x y_1 - p_y x_1 = p_x y_2 - p_y x_2 = 0.$$  

It is apparent that for any point $(x, y) \in \mathbb{R}^2$ such that $p_x y - p_y x = 0$ it will be on the straight line formed by the two concatenating points $(x_1, y_1)$ and $(x_2, y_2)$. Note that the center of the target circle (the origin of frame $Oxy$) is such a point, which completes the proof. $\square$

In the next section, all the above geometric properties will be used to devise analytical solutions for all the 12 types in $\mathcal{F}$.

4 Analytical solution for each type in $\mathcal{F}$

Once the final heading angle $\theta_f$ is known, then the final position is available because it lies on the target circle. With the final angle and the final position, one can compute the solution path according to the geometric method in [5]. In this section, analytical solutions of $\theta_f$ for all the types in $\mathcal{F}$ will be presented based on the geometric properties established in Lemmas 1–5.

4.1 Analytical solutions for CSC

By the following lemma, we present the relation between the final heading angle $\theta_f$ and the known variables $(x_0, y_0, \theta_0, \rho, \alpha)$ for the paths of type CSC.

Lemma 6 Given any $r > 0$, $\rho > 0$, and any initial configuration $(x_0, y_0, \theta_0)$, if the solution path of the OCP is of type CSC, then the optimal heading orientation angle $\theta_f$ at the final time holds as follows:
(1) If the path is of type RSR (resp. RSL), then \( \theta_f = \alpha_s - \alpha_c \) (resp. \( \theta_f = \alpha_s + \alpha_c \)) where \( \alpha_c > 0 \) is the radian of \( C_3 \) given in Lemma 4 and \( \alpha_s \in [0, 2\pi] \) is the heading angle along the straight line segment \( S \) such that

\[
(x_0 + \rho \sin \theta_0) \sin \alpha_s - (y_0 - \rho \cos \theta_0) \cos \alpha_s = \rho. \tag{19}
\]

(2) If the path is of LSL (resp. LSR), then \( \theta_f = \alpha_s + \alpha_c \) (resp. \( \theta_f = \alpha_s - \alpha_c \)) where \( \alpha_c > 0 \) is the radian of \( C_3 \) given in Lemma 4 and \( \alpha_s \in [0, 2\pi] \) is the heading angle along the straight line segment \( S \) such that

\[
-(x_0 - \rho \sin \theta_0) \sin \alpha_s + (y_0 + \rho \cos \theta_0) \cos \alpha_s = \rho.
\]

Proof. Let us first consider the types RSR and LSR, as shown in Fig. 2a. It is apparent that \( \theta_f = \alpha_s + \alpha_c \) if the final circular arc is L and \( \theta_f = \alpha_s - \alpha_c \) if the final circular arc is R. Hence, we just need to prove that Eq. (19) holds for the type of RSC.

Denote by \((x_1, y_1)\) the center of the first circular arc with right turning direction, then we have

\[
\begin{align*}
x_1 &= x_0 + \rho \cos(\theta_0 - \pi/2) \\
y_1 &= y_0 + \rho \sin(\theta_0 - \pi/2).
\end{align*}
\tag{20}
\]

Denote by \((x_2, y_2)\) the concatenating point from the first circular arc to the straight line segment. Then, we have

\[
\begin{align*}
x_2 &= x_1 + \rho \cos(\alpha_s + \pi/2) \\
y_2 &= y_1 + \rho \sin(\alpha_s + \pi/2).
\end{align*}
\tag{21}
\]

According to Lemma 4, the direction of the straight line segment \( S \) passes through the center of the target circle. Thus, the vector \([\cos(\alpha_s + \pi/2), \sin(\alpha_s + \pi/2)]\) is perpendicular with the vector from the origin of \( Oxy \) to the point \((x_2, y_2)\), indicating

\[
0 = x_2 \cos(\alpha_s + \pi/2) + y_2 \sin(\alpha_s + \pi/2).
\tag{22}
\]

Substituting Eq. (20) and Eq. (21) into Eq. (22) and simplifying the resulting equation will yield Eq. (19), which completes the proof of the first statement.

The LSL and LSR paths are symmetrical to the RSR and RSL paths, respectively, and hence the results for LSL and LSR can be obtained by the similar way as done for the RSR and RSL paths. So, the proof for the second statement is omitted here. \( \square \)

As a result of this lemma, the final heading angle \( \theta_f \) can be analytically found if the solution path of the OCP is of type CSC. It also should be noted from Lemma 6 that the length of LSL (resp. RSR) is equal to that of LSR (resp. RSL). Hence, if the solution path of the OCP is of type CSC, then the solution path is not unique.

4.2 Analytical solution for CCC

If the solution path of OCP is of type CCC, it is also enough to compute the heading angle \( \theta_f \) at the final time in order to compute the whole CCC path. By the following lemma, we shall show how to compute the final heading angle \( \theta_f \) if the solution path of OCP is of type CCC.

**Lemma 7** If the solution path of the OCP is of type \( C_1C_2C_3 \) and if \([x_1, y_1]^T\) is the center of the initial circular arc, then the following two statements hold:

1. If the solution path is of type RLR, we have

\[
x_1 = x_0 + \rho \cos(\theta_0 - \pi/2) \quad \text{and} \quad y_1 = y_0 + \rho \sin(\theta_0 - \pi/2),
\]

and \( \tan(\frac{\theta_f}{2} - \frac{\pi}{4}) \) is a zero of the following quartic polynomial:

\[
A_1 x^4 + A_2 x^3 + A_3 x^2 + A_4 x + A_5 = 0
\]

(23)
where

\[
A_1 = 2x_1^2 (\rho^2 + 9r^2 + 18\rho r - 2y_1 (\rho + r) + y_1^2) + \\
(\rho + r - y_1)^2 (5\rho + r - y_1) (-3\rho + r - y_1) + x_1^4
\]
\[
A_2 = (-7\rho^2 + r^2 + 2\rho r + 6y_1 (\rho + r) + x_1^4 + y_1^2) \\
\times \\
[\rho - 8x_1 (\rho + r)]
\]
\[
A_3 = 2[-2x_1^2 (11\rho^2 + 3r^2 + 6\rho r) + x_1^4 + y_1^4 + \\
2y_1^2 (7\rho^2 + 15r^2 + 30\rho r + x_1^2) + \\
(\rho + r)^2 (5\rho + r) (r - 3\rho)]
\]
\[
A_4 = (-7\rho^2 + r^2 + 2\rho r - 6y_1 (\rho + r) + x_1^2 + y_1^2) \\
\times \\
[\rho - 8x_1 (\rho + r)]
\]
\[
A_5 = 2x_1^2 (\rho^2 + 9r^2 + 18\rho r + 2y_1 (\rho + r) + y_1^2) + \\
(\rho + r + y_1)^2 (5\rho + r + y_1) (-3\rho + r + y_1) + x_1^4
\]

(2) If the solution path is of type \textit{LRL}, we have

\[x_1 = x_0 + \rho \cos(\theta_0 - \pi/2) \text{ and } y_1 = y_0 + \rho \sin(\theta_0 - \pi/2),\]

and \(\tan(\theta_T + \frac{\pi}{3})\) is a zero of the quartic polynomial in Eq. (25).

Proof. Let us first consider the type of \textit{RLR}, as shown by the the specific example in Fig. 4a. Denote by

\[
e_{3}^{r} = [x_{3}, y_{3}]^{T}\]

the center of the final circular arc. Then, according to the geometry in Fig. 4a we have

\[
\begin{align*}
x_3 = (r + \rho) \cos(\theta_f - \pi/2) \\
y_3 = (r + \rho) \sin(\theta_f - \pi/2)
\end{align*}
\]

(24)

As \(e_{1}^{r} := [x_{1}, y_{1}]^{T}\) is denoted as the center of the initial circular arc, we immediately have

\[
\begin{align*}
x_1 = x_0 + \rho \cos(\theta_0 - \pi/2) \\
y_1 = y_0 + \rho \sin(\theta_0 - \pi/2)
\end{align*}
\]

Denote by \(e_{2}^{r} = [x_{2}, y_{2}]^{T}\) the center of the second circular arc, by \(A = [x_{A}, y_{A}]^{T}\) and \(B = [x_{B}, y_{B}]^{T}\) the concatenating points from \(C_1\) to \(C_2\) and from \(C_2\) to \(C_3\), respectively. Furthermore, denote by \(\alpha_A\) and \(\alpha_B\) the heading angle at \(A\) and \(B\), respectively. Then, we have

\[
e_{3}^{l} = e_{1}^{r} + 2\rho [\cos(\alpha_A + \pi/2), \sin(\alpha_A + \pi/2)]^{T}
\]

and

\[
e_{2}^{l} = e_{3}^{r} + 2\rho [\cos(\alpha_B + \pi/2), \sin(\alpha_B + \pi/2)]^{T}
\]

Combining the two equations indicates

\[
x_3 + 2\rho \cos(\alpha_B + \pi/2) = x_1 + 2\rho \cos(\alpha_A + \pi/2) \tag{25}
\]
\[
y_3 + 2\rho \sin(\alpha_B + \pi/2) = y_1 + 2\rho \sin(\alpha_A + \pi/2) \tag{26}
\]
These two equations can be simplified to

\[
0 = (x_3 - x_1)^2 + 4(x_3 - x_1)\rho \cos(\alpha_B + \pi/2) + (y_3 - y_1)^2 + 4(y_3 - y_1)\rho \sin(\alpha_B + \pi/2)
\]  

(27)

According to the geometry in Fig. 4a, we also have

\[
\begin{bmatrix}
  x_A \\
  y_A \\
\end{bmatrix} = \begin{bmatrix}
  x_1 + \rho \cos(\alpha_A + \pi/2) \\
  y_1 + \rho \sin(\alpha_A + \pi/2) \\
\end{bmatrix}
\]  

(28)

\[
\begin{bmatrix}
  x_B \\
  y_B \\
\end{bmatrix} = \begin{bmatrix}
  x_3 + \rho \cos(\alpha_B + \pi/2) \\
  y_3 + \rho \sin(\alpha_B + \pi/2) \\
\end{bmatrix}
\]  

(29)

Note that \(A, B,\) and the origin \(O\) lie on one straight line according to Lemma 5. Hence, it follows

\[
y_A/x_A = y_B/x_B.
\]  

(30)

Substituting Eq. (28) and Eq. (29) into Eq. (30) leads to

\[
\frac{y_1 + \rho \sin(\alpha_A + \pi/2)}{x_1 + \rho \cos(\alpha_A + \pi/2)} = \frac{y_3 + \rho \sin(\alpha_B + \pi/2)}{x_3 + \rho \cos(\alpha_B + \pi/2)}.
\]  

(31)

Substituting Eq. (23) and Eq. (26) into Eq. (31), we have

\[
\frac{y_1 + y_3 + 2\rho \sin(\alpha_B + \pi/2)}{x_1 + x_3 + 2\rho \cos(\alpha_B + \pi/2)} = \frac{y_3 + \rho \sin(\alpha_B + \pi/2)}{x_3 + \rho \cos(\alpha_B + \pi/2)},
\]  

(32)

which can be rearranged as

\[
0 = y_1 x_3 - x_1 y_3 + (x_3 - x_1)\rho \sin(\alpha_B + \pi/2) + (y_1 - y_3)\rho \cos(\alpha_B + \pi/2)
\]  

(33)

Combining Eq. (27) with Eq. (33) leads to

\[
0 = (x_1 y_3 - y_1 x_3)^2 + [(x_3 - x_1)^2 + (y_3 - y_1)^2]^2/16 - \rho^2[(x_3 - x_1)^2 + (y_3 - y_1)^2].
\]  

(34)

Substituting Eq. (23) into this equation yields

\[
0 = (r + \rho)^2[x_1 \sin(\theta_f - \pi/2) - y_1 \cos(\theta_f - \pi/2)]^2 + [-2(r + \rho)[x_1 \cos(\theta_f - \pi/2) + y_1 \sin(\theta_f - \pi/2)]
+ (r + \rho)^2 + x_1^2 + y_1^2)^2/16 - \rho^2[x_1^2 + y_1^2 + (r + \rho)^2
- 2(r + \rho)[x_1 \cos(\theta_f - \pi/2) + y_1 \sin(\theta_f - \pi/2)].
\]  

(35)

By taking into account the half-angle formulas

\[
\sin \theta = \frac{2 \tan(\theta/2)}{1 + \tan^2(\theta/2)} \quad \text{and} \quad \cos \theta = \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)}
\]

Then, Eq. (35) can be written as the 4-th degree polynomial in Eq. (23), which completes the proof of the first statement.

For the case of LRL, it can be proved in the same way as proving the type of RLR, so the proof for the type of LRL is omitted here. \(\Box\)

Notice that the roots of a quartic polynomial can be efficiently found by either radicals or a standard polynomial solver. Therefore, Lemma 7 implies that the solution path of OCP can be efficiently obtained if it is of type CCC.

4.3 Analytical solutions for the substrings of CCC and CSC

According to Corollary 1, the substring of CCC and CSC only contains C, SC, and CC. By the following lemmas, we present the analytical solutions for all the three types of substrings.

**Lemma 8** Given any \(r > 0, \rho > 0,\) and \((x_0, y_0, \theta_0),\) assume that the solution path of the OCP is of a single circular arc \(C.\) Then, the final heading angle \(\theta_f\) takes a value such that

\[
(x_0 + \delta \rho \sin \theta_0) \cos \theta_f + (y_0 + \delta \rho \cos \theta_0) \sin \theta_f = 0
\]  

(36)

where \(\delta = 1\) (resp. \(\delta = -1\)) if \(C = R\) (resp. \(L\)), and the following statements holds:
(1) If the circular arc $C$ and the target circle are externally tangent to each other, then we have

$$[x_0 + \delta \rho \sin \theta_0]^2 + [y_0 - \delta \rho \cos \theta_0]^2 = (r + \rho)^2$$  \hspace{1cm} (37)$$

(2) If the circular arc $C$ and the target circle are internally tangent to each other, we have

$$[x_0 + \delta \rho \sin \theta_0]^2 + [y_0 - \delta \rho \cos \theta_0]^2 = (r - \rho)^2$$  \hspace{1cm} (38)$$

Proof. Set $\delta = 1$ (resp. $\delta = -1$) if $C = R$ (resp. $C = L$). Then, it is clear that

$$c_1(\delta) = [x_0 + \cos(\theta_0 - \delta \pi/2), y_0 + \sin(\theta_0 - \delta \pi/2)]^T$$

is the center of the circular arc $C$. Note that the vector $[\cos \theta_j, \sin \theta_j]^T$ is perpendicular with the vector from the center of the target circle (or the origin of $Oxy$) to $c_1(\delta)$. Then, we have $c_1(\delta)^T[\cos \theta_j, \sin \theta_j]^T = 0$, indicating that Eq. (38) holds. Then, we prove the two statements one by one.

(1) If $C$ and the target circle are externally tangent to each other, then the distance from the center of $C$ to the origin (or the center of the target circle) is $r + \rho$, indicating $\|c_1(\delta)\| = r + \rho$. Explicitly writing $\|c_1(\delta)\| = r + \rho$ leads to Eq. (37), which completes the proof of the first statement.

(2) If $C$ and the target circle are internally tangent to each other, then the distance from the center of $C$ to the origin (or the center of the target circle) is $\|r - \rho\|$, indicating $\|c_1(\delta)\| = \|r - \rho\|$. Explicitly writing $\|c_1(\delta)\| = \|r - \rho\|$ leads to Eq. (38), which completes the proof of the second statement. □

As a consequence of this lemma, one can first check the satisfaction of Eq. (37) and Eq. (38) in order to determine if a single circular arc is a candidate solution of the OCP. If either Eq. (37) or Eq. (38) is satisfied, we can obtain the final heading angle $\theta_f$ by solving Eq. (40).

**Lemma 9** Given any $r > 0$, $\rho > 0$, and $(x_0, y_0, \theta_0)$, assume that the solution path of the OCP is of type $S_1 C_2$. Then, we have

$$y_0 \cos \theta_0 - x_0 \sin \theta_0 = 0$$  \hspace{1cm} (39)$$

and

$$\theta_f = \theta_0 + \delta \alpha_c$$  \hspace{1cm} (40)$$

where $\delta = 1$ (resp. $\delta = -1$) if $C_2 = L$ (resp. $C_2 = R$) and $\alpha_c > 0$ is the radian of $C_2$ which is given in Lemma 4.

Proof. If the solution path is of type SC, the heading angle along the straight line segment $S$ is the same as the initial heading angle $\theta_0$, and $(x_0, y_0)$ is the initial point of the straight line segment $S$, indicating the satisfaction of Eq. (39).

According to the definition of $\alpha_c$ in Lemma 4, we immediately have Eq. (40), which completes the proof. □

As a result of this lemma, it is enough to check the satisfaction of Eq. (39) in order to determine if the solution path of the OCP is of type SC or not. Once it is of type SC, the optimal final heading angle can be computed by Eq. (40).

**Lemma 10** Given any $r > 0$, $\rho > 0$, and $(x_0, y_0, \theta_0)$, assume the solution path is of type $C_1 C_2$. Then, we have

$$4\rho^2 = [x_0 + \delta \rho \sin \theta_0 + \delta(r + \rho) \sin \theta_f]^2 + [y_0 - \delta \rho \cos \theta_0 - \delta(r + \rho) \cos \theta_f]^2$$  \hspace{1cm} (41)$$

where $\delta = 1$ (resp. $\delta = -1$) if $CC = RL$ (resp. $CC = LR$).

Proof. We use $c_1$ and $c_2$ to denote the centers of $C_1$ and $C_2$, respectively. In any case, we have

$$\|c_1 - c_2\| = 2\rho.$$

(42)$$

We also have

$$c_1 = [x_0 + \rho \cos(\theta_0 - \delta \pi/2), y_0 + \rho \sin(\theta_0 - \delta \pi/2)]^T$$

and

$$c_2 = [(r + \rho) \cos(\theta_f + \delta \pi/2), (r + \rho) \sin(\theta_f + \delta \pi/2)]^T$$

where $\delta = 1$ (resp. $\delta = -1$) if $CC = RL$ (resp. $CC = LR$). Substituting $c_1$ and $c_2$ into Eq. (42) leads to Eq. (41), indicating that Eq. (41) holds, which completes the proof. □

According to this lemma, once $CC$ is a candidate type, we can compute the optimal final heading angle by solving the equation in Eq. (41).

**Remark 1** All the equations in Lemmas 8-10 can be analytically solved so that an analytical solution to $\theta_f$ is available if the solution path of OCP is a substring of CCC or CSC.
values of \( d \) from Table 1 that the improvement factors of the analytic method compared with the DBM(360) for different uniformly discretize the angular position of the target circle. Note that given any final angle without the study of this paper, a straightforward way to compute the solution path of the OCP is to

5 Numerical simulations

In the following two subsections, we present some numerical simulations to illustrate the developments of this paper.

5.1 Computational complexity

Without the study of this paper, a straightforward way to compute the solution path of the OCP is to uniformly discretize the angular position of the target circle. Note that given any final angle \( \theta_f \), the final position on the target circle is readily available as

\[
[r \cos(\theta_f + \delta \pi/2), r \sin(\theta_f + \delta \pi/2)]^T
\]

where \( \delta = 1 \) (resp. \( \delta = -1 \)) if the rotational direction of \( \theta_f \) is clockwise (resp. counter-clockwise) with respect to the center of the target circle. Let us denote by \( D(\theta_f, \delta) \) the shortest Dubins path from \((x_0, y_0, \theta_0)\) to \((r \cos(\theta_f + \delta \pi/2), r \sin(\theta_f + \delta \pi/2), \theta_f)\). Then, if the discretization level is denoted by \( l > 0 \), the Discretization-Based Method (DBM) is to select an angle \( \theta \) in \( \{\theta = 2\pi \times i/l : i = 0, 1, \ldots, l\} \) so that \( D(\theta, \delta) \) is the smallest, i.e.,

\[
D(\theta, \delta) = \min_{i=0,1\ldots,l} D(2\pi \times i/l, \delta), \quad \delta = \pm 1.
\]

For notational simplicity, we denote hereafter by DBM\((l)\) the DBM with a discretization level of \( l \in \mathbb{N} \).

Let the parameters \((x_0, y_0, \theta_0), \rho > 0, r > 0\) be generated randomly by uniform distribution. Both the analytical solutions in Section 4 and the DBM(360) are tested on 10000 randomly generated OCPs. Table 1 presents the time complexity of the analytical method in comparison with the DBM(360) for different \( \rho \), where \( d_m \geq 0 \) denotes the distance from initial point \((x_0, y_0)\) to the center of the target circle. We can see from Table 1 that the improvement factors of the analytic method compared with the DBM(360) for different values of \( d_m \) are greater than around 2000.

Notice that the DBM can only generate an approximate solution path for the OCP. If a more accurate solution is required, a higher level of discretization is needed, which however results in a higher computational complexity. As the analytic solution to each type in \( \mathcal{F} \) is devised, for any \((x_0, y_0, \theta_0), r > 0, \rho > 0\), the accurate solution of the OCP can be obtained in a constant time by checking at most 12 analytic functions.

5.2 Specific examples

In this subsection, we present some specific examples to illustrate the geometric properties of the shortest Dubins paths to a circle.

5.2.1 Case A

For case A, set \( \rho = 1, r = 1, \) and \((x_0, y_0, \theta_0) = (-0.2, -0.5, \pi/2)\). The analytical results in Section 4 are applied to computing the shortest path from \((x_0, y_0, \theta_0)\) to the target circle. Fig. 5 shows the solution paths with two different final rotational directions. It is apparent to see from Fig. 5 that the two concatenating points and the center of the target circle lie on a straight line, coinciding with Lemma 5.

5.2.2 Case B

For case B, the initial condition is set as \((x_0, y_0, \theta_0) = (-3, 0, \pi)\). Set \( \rho = 1 \) and \( r = 1 \). The shortest paths are computed by directly applying the analytical results in Section 4 and presented in Fig. 6. We can see from Fig. 6 that the direction of the straight line segment passes through the center of the target circle, as predicted by Lemma 6. Also notice that the length of the path of RSR is the same as that of the path of RSL, as shown by Lemma 7.
Figure 5: The solution paths of the OCP for different rotational directions for case A.

(a) Clockwise rotational direction  
(b) Counter-Clockwise rotational direction

Figure 6: The shortest paths for case B.
5.2.3 Case C

For case C, set $r = 2$ and $\rho = 0.5$. We choose $(x_0, y_0, \theta_0) = (-0.5, 0, \pi/2)$. Employing the analytical results in Section 4 once again, the shortest paths are computed and presented in Fig. 7. We can see that the final circular arc is internally tangent to the target circle. In this case, we still have that the direction of the straight line segment passes through the center of the target circle, as predicted by Lemma 1, and that the length of the path of LSL is the same as that of the path of LSR.

5.2.4 Case D

For case D, let $r = 1$ and $\rho = 2$, and set the initial condition as

$$
\begin{bmatrix}
  x_0 \\
  y_0 \\
  \theta_0
\end{bmatrix} = \begin{bmatrix}
  r \cos(269.5 \times \pi/180) + \rho \cos(179.5 \times \pi/180) \\
  r \sin(269.5 \times \pi/180) + \rho \sin(179.5 \times \pi/180) \\
  89.5 \times \pi/180
\end{bmatrix}
$$

This initial condition is tailored so that the solution path of the OCP is a single circular arc. Using the analytical solutions in Section 4, the solution is computed and presented in Fig. 8a. However, it should be noted that the DBM(360) cannot find the solution path. In fact, the path computed by the DBM(360) is quite different from the analytical solution path, as shown in Fig. 8b.

Taking into account all the numerical examples presented in this section, it is concluded that the analytic method not only can compute the solution path of the OCP in a constant time but also can generate more accurate solutions, in comparison with the DBM.
6 Conclusions

The shortest Dubins paths from a fixed initial configuration to a target circle with the terminal heading (or velocity) tangential to the circle was studied by applying Pontryagin’s maximum principle. Through synthesizing the necessary conditions for optimality, some geometric properties for the shortest path were presented. To be more specific, once the shortest path is of type CSC, then the direction of the straight line segment S passes through the center of the target circle; if the shortest path is of type CCC, then the two concatenating points between the circular arcs and the center of the target circle lies on one single straight line. These geometric properties rule out the substring CS so that the shortest path must lie in a sufficient family of 12 types. In addition, the geometric properties allow to devise an analytical solution for each of all the 12 types. Comparing with the straightforward discretization-based method, employing the analytical solution to compute the shortest path could not only reduce the computing time (cf. Table I) but also generate more accurate solutions.

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