A Partially Distributed Fixed-Time Economic Dispatch Algorithm With Kron’s Modeled Power Transmission Losses

Shivanshu Tripathi, Anoop Jain, Member, IEEE, and Abhisek K. Behera, Member, IEEE

Abstract—A partially distributed economic dispatch algorithm, which renders optimal value in fixed time with the objective of supplying the load requirement as well as the power transmission losses, is proposed in this article. The transmission losses are modeled using Kron’s B-loss formula, under a few (standard) assumptions on the values of B-coefficients. The total power supplied by the generators is subject to time-varying equality constraints due to the time-varying nature of the transmission losses. Using Lyapunov stability and optimization theory, we prove the convergence of the proposed algorithm to the optimal value within a fixed time, which depends on the B-loss coefficients, cost function parameters associated with each generator, and the interaction topology among them. Finally, two test cases, including an IEEE-57 bus system, are simulated to illustrate the performance of the proposed algorithm.

Index Terms—B-loss coefficients, distributed optimization, economic dispatch, fixed-time convergence, transmission loss.

I. INTRODUCTION

A. Motivation and Literature Survey

The economic dispatch problem (EDP) has been a celebrated problem in the optimal operation and management of power systems. With the rapid integration of renewable energy sources in the microgrid, solving EDP became a challenging task due to the scalability of the power network, in transition from a centralized grid structure to distributed microgrids. Eventually, the issues pertaining to the robust operation, reliability, and efficiency become critical in designing a distributed algorithm [1]. The primary goal of EDP in a distributed infrastructure is to seek the minimum value of a collective cost function defined over a network of generators. This led to the requirement of an algorithm that works with renewable energy resources in a distributed manner. In this direction, many algorithms were proposed in the literature, e.g., consensus-based algorithms [2], [3]; distributed Laplacian-gradient-based algorithms [4], [5]; initialization-free privacy-guaranteed distributed algorithm [6]; gossip-based distributed algorithm [7]; adaptive event-triggering-based distributed algorithm [8], etc.

In addition to supplying the load demand, it is equally important that the distributed algorithm must cater to the constraints posed by the time-varying power transmission losses. Especially, in a large-scale power grid network, transmission losses play a critical role in affecting both the optimal solution of the EDP and the convergence rate of the algorithm [9]. Two approaches, namely, B-coefficient-based (also known as Kron’s model based) EDP and power-flow-based EDP, are commonly seen in the literature addressing the power transmission losses [10]. It has been argued that the former approach is suitable in practical applications, as the latter one is time-consuming and even leads to convergence issues [11]. Yet, only simplified models of B-coefficient-based transmission losses are considered and the convergence to the optimal solution is guaranteed in the asymptotic or exponential sense, as can be seen in [12], [13], [14], [15], [16], and [17]. Furthermore, the works [18], [19], [20], [21] do address the issue of convergence speed by proposing finite-time or fixed-time algorithms, however, they do not deal with power transmission losses. Unlike these works, this article proposes an algorithm that accounts for Kron’s modeled power transmission losses (without any simplification) and reaches the optimal solution of the EDP in a fixed time.

B. Our Approach and Main Contributions

Aggregation of Kron’s modeled power transmission losses, by nature, poses an additional requirement of globally sharing the generated power information for obtaining the transmission losses for individual generators. Addressing this fact, the proposed algorithm in this article considers that the generators have a two-layered communication topology—the power generated by each generator is shared to a central server to obtain the power transmission losses for each unit while the auxiliary
variables are shared locally. Such multilayered topological considerations are motivated by many engineering problems, such as multiagent systems [22, 23, 24]; smart grids [25]; learning algorithms [26, 27, 28]; etc., in catering present demands of privacy and security.

The introduction of a centralized (or global) network helps in restricting the sharing of critical information like power generated by each generator to other generators. In our algorithm, the central server processes the power information of each generator and, based on this, it supplies back the information about the individual transmission losses. However, no private information such as cost function and/or intermediate (or auxiliary) variables of the generators is given to the central server and is shared locally during the process. The conventional distributed algorithms that do not employ a centralized layer for privacy may reveal extensive information or the private data of the generators, as has been discussed for the learning algorithms in [26, 27, and 28]. To prevent such information leakage, the proposed partially distributed algorithm ensures that the critical information of each generator is kept private, both from other generators and the central server. The communication system employs a similar algorithm to protect the privacy of the individual mobile users [29, 30]. Furthermore, our analysis is based on certain assumptions relying on an interplay between the eigenvalues of Kron’s B-loss matrix and parameters characterizing the convexity of the cost functions associated with each generator.

The main contributions of this work can be summarized as follows: 1) We propose a novel consensus-based partially distributed algorithm, which solves the EDP in the presence of power transmission losses characterized by the accurate Kron’s B-loss formula [31]; and 2) using tools from Lyapunov stability and optimization theory, we rigorously show that the optimal solution of the EDP is rendered in a fixed time. An analytical expression of the upper bound on the convergence time is obtained, which is independent of initial values of power and dependent on the eigenvalues of the matrix containing B-loss coefficients, with the convexity of the cost function associated with each generator and network topology among them.

C. Paper Structure

Section II describes Kron’s transmission loss formula, formulates the problem, and presents some preliminary results on fixed-time stability. Section III derives a few introductory results, describes the proposed algorithm, and obtains an upper bound on the convergence time. Theoretical results are illustrated through two case studies in Section IV. Finally, Section V concludes the article and presents future research directions.

Notations and Graph Theory: Throughout the article, \( \mathbb{R} \) and \( \mathbb{R}_+ \) denote the set of real and nonnegative real numbers, respectively. For any \( x \in \mathbb{R} \), we define function \( \text{sign}^\mu : \mathbb{R} \to \mathbb{R} \) as \( \text{sign}^\mu(x) = |x|^{\mu} \text{sign}(x) \) for \( \mu > 0 \), where \( \text{sign}(x) \) is the signum function of \( x \). The Hadamard product (or elementwise product) of two matrices \( X \) and \( Y \) of the same dimension \( m \times n \) is defined as \( (X \odot Y)_{ij} := [X]_{ij}[Y]_{ij} \). For any \( \psi = [\psi_1 \cdots \psi_N]^T \in \mathbb{R}_N \), \( \text{diag}\{\psi\} \) denotes the diagonal matrix with the entries of \( \psi \) along its principal diagonal. For a given function \( f : \mathbb{R}^n \to \mathbb{R} \), \( \nabla f(x) \) and \( \nabla^2 f(x) \) represent the gradient and Hessian of the function \( f(x) \) at \( x \), respectively. The Jacobian of a function \( g : \mathbb{R}^n \to \mathbb{R}^m \) is defined to be an \( m \times n \) matrix, whose \((i,j)\)th entry is \( J_{ij}(x) = (\partial g_i/\partial x_j)(x) \), evaluated at point \( x \in \mathbb{R}^n \). We represent \( I_N = [1 \cdots 1]^T \in \mathbb{R}^N \) and \( 0_N = [0 \cdots 0]^T \in \mathbb{R}^N \), respectively. \( I_N \) denotes the identity matrix of order \( N \times N \). We use symbols \( \geq, \leq \) to represent the elementwise comparison between two matrices of the same size.

An undirected graph \( G = (V, E, A) \) is a collection of node set \( V \) with \( V = \{1, \ldots, N\} \), the edge set \( E \subseteq V \times V \), along with edge weights captured by the adjacency matrix \( A = [a_{ij}] \in \mathbb{R}^{N \times N} \) with \( a_{ij} = a_{ji} > 0 \) if the undirected edge \( \{i, j\} \in E \), and \( a_{ij} = 0 \) otherwise. The Laplacian of \( G \) is defined as \( L = [\ell_{ij}] \in \mathbb{R}^{N \times N} \) with \( \ell_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij} \) and \( \ell_{ij} = -a_{ij}, \forall i \neq j \), where \( \mathcal{N}_i \) is the set of neighboring vertices of vertex \( i \). For an undirected and connected graph, 0 is a simple eigenvalue of \( L \) with the corresponding eigenvector \( 1_N \), and all the other eigenvalues are positive.

II. KRON’S TRANSMISSION LOSS FORMULA, PROBLEM DESCRIPTION, AND PRELIMINARIES

This section reviews Kron’s B-loss formula for power transmission losses, formulates the problem considered in this article, and discusses some introductory results from previous literature.

A. Transmission Losses

Consider a power grid network consisting of \( N \) generators. The power output of the \( i \)th generator is represented by \( P_i \), where \( i = 1, \ldots, N \). By convention, we assume that \( P_i \) is positive when the power (by the \( i \)th generator) is supplied to the grid or negative otherwise. Since power is transmitted over the transmission network, the actual power transmitted from the generators to the load incurs some losses. In other words, the power received at the load is not equal to that of the transmitted end, which greatly influences the optimal solution of the associated EDP and the convergence rate of the distributed algorithm. We model these losses using Kron’s loss formula; an expression for which in terms of source loading and a set of loss coefficients (usually referred to as B-coefficients) is given by the quadratic form [31, 32]

\[
P_L = P^T B P + B_0^T P + B_{00}
\]

where \( P = [P_1 \cdots P_N]^T \) is the vector of all generator power outputs and \( B = [B_{ij}] \in \mathbb{R}^{N \times N} \), \( B_0 = [B_{10} \cdots B_{N0}]^T \in \mathbb{R}^N \), and \( B_{00} \in \mathbb{R} \) are the loss coefficient square matrix, vector, and constant, respectively. These coefficients can be evaluated using methods discussed in [31] and [32]. The expression (1) can also be written in the form of summation as

\[
P_L = \sum_{i=1}^{N} \sum_{j=1}^{N} P_i B_{ij} P_j + \sum_{i=1}^{N} P_i B_{0i} + \sum_{i=1}^{N} B_{00i} =: \sum_{i=1}^{N} P_L_i
\]

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where \( \sum_{i=1}^{N} B_{00i} = B_{00} \), and

\[
P_L_i = \sum_{j=1}^{N} P_i B_{ij} P_j + P_i B_{i0} + B_{00i}
\]

is the power transmission loss associated with the \( i \)th generator. Note that the constant factor \( B_{00i} \) corresponding to the \( i \)th generator is defined for the convenience of analysis; however, our main result depends only on the overall constant factor \( B_{00} \), as appears in Kron’s model (1).

**Remark 1 (A comparison with linear power flow models):** Unlike the linear power flow models in obtaining transmission losses [33, 34, 35], in this work, we consider Kron’s modeled power transmission losses due to several reasons. First, it expresses the transmission losses in terms of the active power, which makes it feasible to analyze an EDP where the cost function depends upon the power supplied by each generator. Second, a large body of literature on linear power flow models uses the iterative approach to consider the effect of transmission losses in an EDP. According to this, the optimization problem is updated at each iteration with new information about the value of the losses, which may not lead to a correct value of the optimal cost. We believe that the correct procedure would be to update the expression of the losses instead of the values themselves. There also exist other approaches, for instance, the piecewise linear model where the transmission losses are modeled in terms of a squared function of the change in phase angle \((\Delta \theta_i)\) at the two ends of the \( i \)th bus and are approximated as \( P_L_i \approx g_i \Delta \theta_i^2 \), where \( g_i \) is line conductance. However, in this case, it is challenging to prove the convergence of the algorithm or comment anything about its time of convergence, as discussed in [35]. In summary, although the linear power flow models might be computationally efficient in obtaining the transmission losses, these neither accurately model the transmission losses nor express them in terms of the generated power \( P_i \) by each generator. On the other hand, Kron’s \( B \)-loss formula gives transmission losses without solving or approximating the nonlinear power flow equations. It is an efficient way to calculate the power losses as power generations vary [11, 36, 37, 38], giving the transmission losses in terms of power supplied by each generator and is adapted in this work.

### B. Problem Formulation

Let the cost function associated with the individual generator in the grid be \( C_i(P_i) \), where \( i = 1, \ldots, N \). The main objective here is to cooperatively minimize the total cost, that is, the sum of all individual local objective functions \( C_i(P_i) \) while maintaining an equality constraint, defined in terms of the load demand and power transmission losses \( P_L \). Let \( D \) and \( P_T \) be the total load demand and total power supplied by the system of generators, respectively. With this description, the EDP can be formulated as

\[
\min_P C(P) = \min_P \sum_{i=1}^{N} C_i(P_i)
\]

subject to

\[
\sum_{i=1}^{N} P_i = D + P_L = \sum_{i=1}^{N} D_i + \sum_{i=1}^{N} P_{Li} = P_T
\]

where \( D_{i0} \) is the initial value of the time-varying load demand \( D_i \), which corresponds to the \( i \)th generator–load pair. It is assumed that the load demand is constant for all time, that is, \( \sum_{i=1}^{N} D_i = \sum_{i=1}^{N} D_{i0} \), which is often a standard assumption in power system networks [19]. It is worth noticing that the inclusion of power transmission losses \( P_L \) does not result in trivially regularizing the overall cost function \( (4a) \), instead, it affects the equality constraints \((4b)\) of the optimization problem 4. Unlike [19], where the constraint (described in terms of the power balance between supply and demand as \( \sum_{i=1}^{N} P_i = D \)) is constant, the constraints proposed in our article change with time because of the time-varying term \( P_L \) associated with Kron’s modeled transmission losses and, hence, makes it a novel and challenging problem.

### C. Some Useful Results

In the following, we describe some useful results that will be helpful in the sequel.

**Lemma 1 (Fixed-Time Convergence [39]):** Consider the dynamical system \( x = f(x) \), where \( x \in \mathbb{R}^N \), \( f : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is a continuous function with \( x(0_N) = x_0 \), for some \( x_0 \in \mathbb{R}^N \). If there exists a continuous radially unbounded Lyapunov function \( V : \mathbb{R}^N \rightarrow \mathbb{R}_+ \cup \{0\} \) such that \( V(x) = 0 \Rightarrow x = 0 \) and any solution \( x(t) \) of the system satisfy the inequality \( \dot{V}(x) \leq -x^T \sum_{i=1}^{N} C_i(p_i) \beta x^T \) for some \( \alpha, \beta > 0 \), \( pk < 1, qk > 1 \), then the origin of the system is globally fixed-time stable, and the following estimates of the settling time holds:

\[
T_s \leq \frac{1}{\alpha k (1 - pk)} + \frac{1}{\beta k (qk - 1)}. \tag{5}
\]

**Lemma 2 (Power Series Inequalities [40]):** For any set of scalars \( \zeta_i > 0 \), where \( i = 1, \ldots, N \), it holds that

\[
\sum_{i=1}^{N} \zeta_i^m \geq \left( \sum_{i=1}^{N} \zeta_i \right)^m, \text{ if } 0 < m \leq 1 \tag{6a}
\]

\[
\sum_{i=1}^{N} \zeta_i^m \geq N^{1-m} \left( \sum_{i=1}^{N} \zeta_i \right)^m, \text{ if } 1 < m < \infty \tag{6b}
\]

### III. MAIN ASSUMPTIONS AND RESULTS

This section presents our main results by proposing an algorithm to solve the optimization problem \((4a)\) in the presence of equality constraints \((4b)\) with transmission losses \((2)\). The
The proposed algorithm is as follows:

\[
\begin{align*}
\lambda_i &= \frac{\partial C_i(P_i)}{\partial P_i} \\
H_i &= 1 + \frac{\partial P_{i_L}}{\partial P_i} \\
\dot{z}_i &= -k_1 \text{sign} \left( \sum_{j \in N_i} a_{ij} \left( H_j \lambda_j - H_i \lambda_i \right) \right) \\
& \quad -k_2 \text{sign} \left( \sum_{j \in N_i} a_{ij} \left( H_j \lambda_j - H_i \lambda_i \right) \right) \\
P_i &= \sum_{j \in N_i} a_{ij} (z_j - z_i) + D_{i0} + P_{iL_i},
\end{align*}
\]  

(7)

where \(k_1, k_2\) are some positive constants, \(0 < \mu < 1\) and \(\nu > 1\), and \(a_{ij} = 1\) if \(\{i, j\} \in E\), and \(a_{ij} = 0\) otherwise. Furthermore, \(\lambda_i\) and \(H_i\) are intermediate variables used in the dynamics of the auxiliary variable \(z_i\). The initial values of \(P_{i_L}\) and \(P_i\) are considered as \(P_{i_L}(0) = 0\) and \(P_i(0) = \sum_{l=1}^{m} d_{ik} P_{i_k}\), where \(m\) is the number of load buses in the power system. \(P_{i_0}\) denotes the power demanded by the \(k\)th load bus, and \(d_{ik}\) is used to represent the connection between the generator bus \(i\) and the load bus \(k\) such that \(d_{ik} = 1\), if buses \(i\) and \(k\) are neighbors, else \(d_{ik} = 0\). Unlike \(18\) and \(19\), the proposed algorithm \(7\) also accounts for transmission losses \(P_{i_L}\) by assimilation of an additional term \(H_i\), which influences the dynamics of the auxiliary variable \(z_i\). In fact, the work in \(19\) is a special case of the problem addressed in this article with \(P_{i_L} = 0\) and \(H_i = 1\) in algorithm \(7\).

**Remark 2:** As will be shown in Lemma 3, the computation of term \(\partial P_{i_L}/\partial P_i\) in algorithm \(7\) requires the information of power generated by all the generators. Thereby, the implementation of \(7\) requires a global network for obtaining \(H_i\), and local topology for sharing the information about the auxiliary variable \(\lambda_i, z_i\) for each \(i\). This is the reason we call it a partially distributed consensus algorithm. Although the problem can be solved in a completed distributed way by considering only a single local network for the simplified Kron’s modeled transmission losses \(P_{i_L} = \sum_{i=1}^{N} B_i P_i^2\) as discussed in \(12\) and \(15\), this, however, would be a main problem of the problem addressed in this article.

### A. Main Assumptions

Before proceeding further, we state the assumptions incorporated in our analysis.

**Assumption 1 (Boundedness):** There exists a positive constant \(\rho\) such that \(|z_j - z_i| \leq \rho\) for all \(i, j = 1, \ldots, N\).

Assumption 1 essentially ensures that the power generated by individual generators is always bounded for finite initial demand and transmission losses [see the last equation in \(7\)]. Similar bounds on the intermediate variables have been used in \(41\) and \(42\).

**Assumption 2 (Initial Demand):** The initial demand is such that \(D_{i0} > \rho \sum_{j \in N_i} a_{ij}\) for all \(i = 1, \ldots, N\), where \(\rho > 0\) is defined in Assumption 1.

Assumption 2 provides the lower bound on the value of initial demand, which helps in maintaining the positivity of the power generated by each generator, as discussed in the subsequent Remark 3.

**Assumption 3 (Network Topology):** The generators have a two-layered network topology—The information (only) about generated powers \(P_i\) is shared to a global network for processing of the individual transmission losses, and the information about auxiliary variables \(z_i\) and \(\lambda_i\) \((7)\) is shared locally, according to an undirected and connected topology.

**Assumption 4 (Cost Function):** For any \(i \in \{1, \ldots, N\}\), the cost function \(C_i(P_i)\) is a strongly convex function such that \(\nabla^2 C_i(P_i) \geq \sigma > 0\) for some constant \(\sigma \in \mathbb{R}^+\). Furthermore, there exists a \(\delta \in \mathbb{R} \setminus \{0\}\) such that \(\nabla C_i(P_i) \geq \delta, \forall i\).

**Assumption 5 (\(B\)-Loss Coefficients):** Kron’s \(B\)-loss coefficient matrix \(B = [B_{ij}]\) for \(i, j = 1, \ldots, N\) in \(2\) is symmetrical with \(B_{ij} \geq 0, \forall i, j\), such that

\[A1 \ 0 \leq \partial P_{i_L}/\partial P_i < 1, \forall i \in \{1, \ldots, N\}.\]

A2 Let \(b_1 \leq b_2 \leq \cdots \leq b_N\) be the eigenvalues of \(B\). Denote \(\rho = \min_i \{B_{i0}\}\). There exist constants \(\sigma, \delta, \rho\) such that \((1 + \rho)\sigma + b_1 \delta > 0, \text{ if } \delta > 0, \text{ and } (1 + \rho)\sigma + b_N \delta > 0, \text{ if } \delta < 0.\)

**Remark 3:** It can be easily verified that Assumptions 1 and 2 ensure that \(P_i > 0\) for all \(i\) at \(t = 0\) for some \(\rho > 0\) (since \(P_{iL}(0) = 0, \forall i\), as per our consideration). Since \(B\)-coefficients are always positive by Assumption 5 and \(P_i\) is positive as calculated in the previous iteration, \(P_{iL}\) is positive for the next iteration. Following this regrettively, it can be concluded that \(P_i > 0\) for all \(i\) and \(t \in [0, \infty)\). Note that this argument requires prior information about \(\rho\), which may not be available. However, for sufficiently large load demands, this has been a well-accepted assumption for all practical purposes \([13], [19], [43];\) otherwise, the value of \(P_i\) may become unbounded and reach \(\infty\), which is not feasible.

**Remark 4:** Assumption \((A1)\) (of Assumption 5) is common in practical power system networks (for instance, refer to \([32], [44], [45],\) and \([46]\)). In fact, the term \(\partial P_{i_L}/\partial P_i\) can be obtained from the well-known notion of penalty factor, defined by \(q_i = 1/(1 - (\partial P_{i_L}/\partial P_i))\), as in \([44], [45], [46], [47], [48]\), which usually satisfies \(1 \leq q_i < \infty\) (refer to \([47, pg. 223]\) and \([48, pg. 283]\)) in most practical cases. This, in turn, implies that the value of \(\partial P_{i_L}/\partial P_i\) satisfies \(0 \leq \partial P_{i_L}/\partial P_i < 1\), which justifies our Assumption \((A1)\).

### B. Preliminary Results

We now discuss the following lemmas before stating the main result.

**Lemma 3:** Under Assumption 5, the following relation holds:

\[
\frac{\partial P_{i_L}}{\partial P_i} = \frac{1}{2} \frac{\partial P_{i_L}}{\partial P_i} + B_{ii} P_i + \frac{B_{i0}}{2} \quad \forall i.
\]  

\[8\]

**Proof:** Differentiating \((2)\) and \((3)\) with respect to \(P_i\), we have

\[
\frac{\partial P_{i_L}}{\partial P_i} = 2 \sum_{j=1}^{N} B_{ij} P_j + 2B_{ii} P_i + B_{i0}
\]  

\[9\]

\[
\frac{\partial P_{i_L}}{\partial P_i} = \sum_{j=1,j \neq i}^{N} B_{ij} P_j + 2B_{ii} P_i + B_{i0}
\]  

\[10\]

which leads to required result, as \([B_{ij}] = [B_{ji}], \forall i, j\), by Assumption 5.
Lemma 4: Define \( F = \nabla P_L(P) \odot \nabla C(P), M = \nabla R(P) \odot \nabla C(P) \), and \( Q = \nabla^2 C(P) + 0.5 \nabla F + \nabla M \), where
\[
\nabla R(P) = \left[ \left( B_{11}P + \frac{B_{10}}{2} \right) \cdots \left( B_{NN}P + \frac{B_{N0}}{2} \right) \right]^T.
\]
Furthermore, let \( S \) be an \( N \times N \) matrix with diagonal entries \( [S_{ii}] = (1 + B_{ii})\sigma + 2B_{ii}\delta \) and off-diagonal entries \( [S_{ij}] = B_{ij}\delta \). Considering that \( P_i > 0, \forall i \), the following properties hold under Assumptions 4 and 5:

\( R1 \) \( [\nabla F_{ij}] \geq 2B_{ij}\delta + B_{ij}\sigma [\nabla M_{ij}] \geq 2B_{ij}\delta \).
\( R2 \) \( \nabla M \) is a diagonal matrix with \( [\nabla M_{ii}] \geq (B_{ii}\delta + \frac{B_{i0}}{\nu} \sigma) \).
\( R3 \) \( [Q_{ii}] \geq \sigma + 2B_{ii}\delta + B_{i0}\sigma \) and \( [Q_{ij}] \geq B_{ij}\delta \).
\( R4 \) \( S \) is a symmetric matrix satisfying \( \tilde{S} \preceq Q \).
\( R5 \) Let \( \tau_1 \leq \tau_2 \leq \cdots \leq \tau_N \) be the eigenvalues of \( S \). Then, \( b_1\delta \leq \tau_i - \sigma(1 + B_{i0}) \leq b_N\delta \), if \( \delta > 0 \); and \( b_N\delta \leq \tau_i - \sigma(1 + B_{i0}) \leq b_1\delta \), if \( \delta < 0 \), for each \( i \), where \( b_1 \) and \( b_N \) are the smallest and largest eigenvalues of \( B \), as defined in Assumption (A2) of Assumption 5.

Proof: Refer to the Appendix for the proof. \( \square \)

C. Main Result

We are now ready to state the main result.

Theorem 1: Algorithm (7), under Assumptions 1–5, solves the EDP (4) in a fixed time. Furthermore, the time of convergence of the proposed algorithm satisfies
\[
T_s \leq \frac{1}{k_1(1 + \rho)\tau_0^{2(1+\frac{1}{\nu})}} \frac{1}{(1 + \frac{1}{\nu})} (1 - \mu)
\]
where \( k_1, k_2, \mu, \) and \( \nu \) are the positive constants as defined in (7), \( \phi_0 \) denotes the second largest eigenvalue of Laplacian \( \mathcal{L} \) of the underlying local network, \( \rho \) is defined in Assumption 5, \( \tau_0 \) is obtained using (R5) in Lemma 4, and \( N \) denotes the number of total generators.

Proof: The sum of power supplied by each generator satisfies
\[
\sum_{i=1}^{N} P_i = \sum_{i=1}^{N} \left( \sum_{j \in N_i} a_{ij}(z_j - z_i) + D_{i0} + P_{Li} \right) = \sum_{i=1}^{N} D_{i0} + \sum_{i=1}^{N} P_{Li} = P_T
\]
as \( \sum_{i=1}^{N} \sum_{j \in N_i} a_{ij}(z_j - z_i) = 0 \) for an undirected and connected graph with \( a_{ij} = a_{ji} \). Clearly, (12) satisfies the desired equality constraint (4b). Substituting for \( P_i \) from (7), the optimization problem (4) can be represented as the following unconstrained optimization problem:
\[
\min z C(z) = \min z \sum_{i=1}^{N} C_i \left( \sum_{j \in N_i} a_{ij}(z_j - z_i) + D_{i0} + P_{Li} \right)
\]
where \( z = [z_1 \cdots z_N]^T \). Based on the cost function (13), the proof invokes the following three steps.

\section{Gradient and Hessian of \( C(z) \):} From (7), the derivative of \( P_i \) with respect to \( z_j \) is obtained as
\[
\frac{\partial P_i}{\partial z_j} = \left\{ \begin{array}{ll}
- \sum_{j=1}^{N} a_{ij} + \frac{\partial P_i}{\partial z_j} & \text{if } i = j \\
\frac{\partial P_i}{\partial z_j} & \text{if } i \neq j
\end{array} \right.
\]
(14)

Once again using (14) for \( \frac{\partial P_i}{\partial z_j} \), we have
\[
\frac{\partial P_i}{\partial z_j} = \left\{ \begin{array}{ll}
- \sum_{j=1}^{N} a_{ij} + \frac{\partial P_i}{\partial z_j} & \text{if } j = i \\
\frac{\partial P_i}{\partial z_j} & \text{if } j \neq i
\end{array} \right.
\]
(15)

Continuing the substitution in each step, an infinite series is formed for \( \frac{\partial P_i}{\partial z_j} \), as follows:
\[
\frac{\partial P_i}{\partial z_j} = \left\{ \begin{array}{ll}
- \sum_{j=1}^{N} a_{ij} + \frac{\partial P_i}{\partial z_j} & \text{if } j = i \\
\frac{\partial P_i}{\partial z_j} & \text{if } j \neq i
\end{array} \right.
\]
(16)

Following Assumption (A1) of Assumption 5, the higher-order terms are neglected to obtain
\[
\frac{\partial P_i}{\partial z_j} = \left\{ \begin{array}{ll}
- \sum_{j=1}^{N} a_{ij} \left( 1 + \frac{\partial P_i}{\partial z_j} \right) & \text{if } j = i \\
\frac{\partial P_i}{\partial z_j} & \text{if } j \neq i
\end{array} \right.
\]
(17)

which on substitution for \( \frac{\partial P_L}{\partial P_i} \), from Lemma 3 results in
\[
\frac{\partial P_i}{\partial z_j} = \left\{ \begin{array}{ll}
- \sum_{j=1}^{N} a_{ij} \left( 1 + \frac{1}{2} \frac{\partial P_i}{\partial z_j} + B_i P_i + \frac{B_{i0}}{2} \right) & \text{if } j = i \\
\frac{\partial P_i}{\partial z_j} & \text{if } j \neq i
\end{array} \right.
\]
Similarly, it can be written from (7) about the cost function that
\[
\frac{\partial C_i}{\partial z_j} = \frac{\partial C_i}{\partial P_i} \frac{\partial P_i}{\partial z_j} = \left\{ \begin{array}{ll}
- \sum_{j=1}^{N} a_{ij} \left( 1 + \frac{1}{2} \frac{\partial P_i}{\partial z_j} + B_i P_i + \frac{B_{i0}}{2} \right) & \text{if } j = i \\
\frac{\partial C_i}{\partial z_j} & \text{if } j \neq i
\end{array} \right.
\]
Note that the gradient of \( P_L \) is \( \nabla P_L(P) = \left[ \frac{\partial P_i}{\partial P_j} \cdots \frac{\partial P_i}{\partial P_N} \right]^T \), using which (19) can be expressed in the form of Jacobian
\[
\mathbf{J}_P = \left[ \frac{\partial P_i}{\partial z_j} \right]_{ij}, \text{ } i, j \in \{1, \ldots, N\},
\]
as
\[
\mathbf{J}_P = - \left( I_N + \frac{1}{2} \text{diag}(\nabla P_L(P)) + \text{diag}(\nabla R(P)) \right) \mathcal{L}
\]
(21)
where \( \nabla R(P) \) is defined in Lemma 4 and \( \mathcal{L} \) is the Laplacian of the local interaction network required for sharing the cost function and auxiliary variable in (7). Furthermore, the gradient
of \( C(z) \), using (20), is given by

\[
\nabla C(z) = -\mathcal{L}\left[ \left( I_N + \frac{1}{2}\text{diag}(\nabla P_L(P)) \right) \nabla C(z) \right].
\]

(22)

We emphasize here that \( \mathcal{J}_P \) is an \( N \times N \) matrix while \( \nabla C(z) \) is an \( N \times 1 \) vector, as \( \nabla C(z) = \left[ \frac{\partial C}{\partial z_1}, \ldots, \frac{\partial C}{\partial z_N} \right]^T \). Now, evaluating the Hessian matrix of \( C(z) \) using (22), we obtain

\[
\nabla^2 C(z) = -\mathcal{L}\left[ \left( I_N + \frac{1}{2}\text{diag}(\nabla P_L(P)) \right) \nabla^2 C(z) \right] \nabla C(z) \mathcal{J}_P.
\]

(23)

which, further simplifying the term inside the bracket and using (21), yields

\[
\nabla^2 C(z) = \mathcal{L}\left[ \nabla^2 C(z) + \frac{1}{2}\nabla(\nabla P_L(P) \circ \nabla C(z)) + \nabla\nabla(\nabla P_L(P) \circ \nabla C(z)) \right] \mathcal{J}_P.
\]

(24)

2) Convexity and Optimality of \( C(z) \): Let the optimal solution of convex optimization problem (13) be given as \( z^* = [z_{1}^* \ldots z_{N}^*]^T \), which belongs to a compact and convex set \( Z \subset \mathbb{R}^N \). Now, recall from the work in [49] that given any strongly convex function defined on \( Z \) and any \( z, \xi \in Z \), there exists an \( \eta \in [0,1] \) such that

\[
C(z) = C(\xi) + \nabla^T C(\xi)(z - \xi) + \frac{1}{2}(z - \xi)^T \nabla^2 C(\xi)(z - \xi)
\]

(25)

where \( \tilde{z} = \xi + \eta(z - \xi) \). Replacing \( z, \xi \) by \( z^*, z \in Z \), respectively, (25) becomes

\[
C(z^*) = C(z) + \nabla^T C(z)(z^* - z) + \frac{1}{2}(z^* - z)^T \nabla^2 C(z)(z^* - z)
\]

(26)

where \( \tilde{z} = z + \eta(z^* - z) \) with \( \eta \in [0,1] \). Rearranging (26) as

\[
C(z) - C(z^*) = \nabla^T C(z)(z^* - z) + \frac{1}{2}(z^* - z)^T \nabla^2 C(z)(z^* - z)
\]

and using Assumption 4, it holds that \( C(z) - C(z^*) \leq \nabla^T C(z)(z^* - z) \). Let \( \phi_1, \phi_2, \ldots, \phi_N \) be the eigenvalues of \( \omega(z) \) such that 0 = \( \phi_1 \leq \phi_2 \leq \cdots \leq \phi_N \) with corresponding orthogonal eigenvectors \( 1_N, v_2, \ldots, v_N \), where \( \|v_i\| = 1, i = 1, 2, \ldots, N \). The vector \( z^* - z \) can be expressed as \( z - z^* = \kappa_1 1_N + \kappa_2 v_2 + \cdots + \kappa_N v_N \) for some constants \( \kappa_i, i = 1, 2, \ldots, N \). Using this, we have

\[
C(z) - C(z^*) \leq \nabla^T C(z)(\kappa_1 1_N + v)
\]

(27)

where \( v = \kappa_2 v_2 + \cdots + \kappa_N v_N \). Note that \( \|v\|^2 = \kappa_2^2 + \cdots + \kappa_N^2 \). From (22) and (27), it follows that:

\[
C(z) - C(z^*) \leq -\kappa_1 [(I_N + \text{diag}(\nabla P_L(P)) \nabla C(P)) \nabla^T C(z)] + \nabla^T C(z)v
\]

(28)

as \( \mathcal{L} = \mathcal{L}_F \) and \( \mathcal{L}_F = 0 \), for an undirected and connected graph. Now, substituting \( \xi = z^* \) in (25) and noting that \( \nabla C(z^*) = 0 \), we have

\[
C(z) - C(z^*) = (1/2)(z - z^*)^T \nabla^2 C(z)(z - z^*)
\]

(29)

which, upon substitution from (24), gives

\[
C(z) - C(z^*) = 0.5(\nabla(z - z^*)^T \nabla^2 C(z) + 0.5 \nabla^2 C + \nabla M)
\]

(30)

According to Assumption 5, and Lemmas 3 and 4, it is clear that

\[
1 + 0.5 \nabla P_L(P) + \nabla R(P) = 1 + \sum_{j=1, j \neq i}^N B_{ij} P_j + 2B_{ii} P_i + B_{i0} \geq (1 + \rho) I_N \text{ for each } i, \text{ where } \rho = \min_{i} B_{i0}, \text{ and } \rho < 1.
\]

(31)

Consequently, it holds from (30) that \( C(z) - C(z^*) \geq 0.5(1 + \rho)(\nabla(z - z^*)^T \nabla^2 C + \nabla M)(\nabla(z - z^*)) = 0.5(1 + \rho)(\nabla(z - z^*))^T \nabla^2 C(z) \), where \( \nabla^2 C(z) + \nabla^2 C + \nabla M \), according to Lemma 4. Furthermore, one can write using result (R4) from Lemma 4 that

\[
C(z) - C(z^*) \geq 0.5(1 + \rho)(\nabla(z - z^*))^T \nabla^2 C(z) \S(z - z^*)
\]

(31)

where \( \S \) is a symmetric matrix with eigenvalues \( \tau_1 \leq \tau_2 \leq \cdots \leq \tau_N \), as per Lemma 4. Using Courant–Fischer theorem [50, Ch. 4, p. 236], for the symmetric matrix \( \S \), it holds for \( z \in Z \) that

\[
C(z) - C(z^*) \geq 0.5(1 + \rho) \tau_1(\nabla(z - z^*))^T \nabla^2 C(z) \S(z - z^*) \geq 0.5(1 + \rho) \tau_1(\kappa_2 \phi_2^2 + \kappa_3 \phi_3^2 + \cdots + \kappa_N \phi_N^2) \nabla^2 C(z) \S(z - z^*) \geq 0.5(1 + \rho) \tau_1 \phi_2^2 \|v\|^2 \geq 0.5(1 + \rho) \tau_1 \phi_2^2 \|v\|^2.
\]

(32)

Now, it follows from (28) and (31) that

\[
\|\nabla^T C(z)\|^2 \leq 0.5(1 + \rho) \tau_1 \phi_2^2 \|v\|^2(C(z) - C(z^*)) \Leftrightarrow \|\nabla^T C(z)\|^2 \geq 0.5(1 + \rho) \tau_1 \phi_2^2 (C(z) - C(z^*)) \|v\|^2 \neq 0 \text{ for nontrivial optimal solution. It is worth noticing that } \tau_1 > 0, \forall i, \text{ under Assumption (A2). This follows from the fact (R5) in Lemma 4 that }\tau_i \geq (1 + B_{i0}) \sigma + b_i \delta \geq (1 + \rho) \sigma + b_i \delta > 0 \text{ if } \delta > 0 \text{ and } \tau_i \geq (1 + B_{i0}) \sigma + b_i \delta \geq (1 + \rho) \sigma + b_i \delta > 0 \text{ if } \delta < 0, \text{ as per Assumption (A2).}
\]

3) Convergence Analysis: Consider the candidate Lyapunov function

\[
V = \frac{1}{2}(C(z) - C(z^*))^2
\]

(33)
whereas the time derivative is $\dot{V} = (C(z) - C(z^*))\nabla^T C(z)\dot{z}$. According to Lemma 3, (20) can be written as

$$\frac{\partial C_i}{\partial z_j} = \left\{ \begin{array}{ll}
-\sum_{j=1}^{N} a_{ij} \left( 1 + \frac{\partial P_{ij}}{\partial z_j} \right) \lambda_i if \ j = i \\
\quad \quad a_{ij} \left( 1 + \frac{\partial P_{ij}}{\partial P} \right) \lambda_i if \ j \neq i.
\end{array} \right. \quad (33)
$$

Since $H_i = (1 + (\partial P_{ij}/\partial P))$ as per algorithm (7), (33) becomes

$$\frac{\partial C_i}{\partial z_j} = \left\{ \begin{array}{ll}
-\sum_{j=1}^{N} a_{ij} H_i \lambda_i if \ j = i \\
\quad \quad a_{ij} H_i \lambda_i if \ j \neq i.
\end{array} \right. \quad (34)
$$

using which, the dynamics $\dot{z}_i$ in algorithm (7), can be expressed in terms of $\partial C_i/\partial z_j$ as

$$\dot{z}_i = -k_1 \text{sig} \left( \frac{\partial C_i}{\partial z_i} + \sum_{j=1}^{N} \frac{\partial C_i}{\partial z_j} \right) \mu$$

$$- k_2 \text{sig} \left( \frac{\partial C_i}{\partial z_i} + \sum_{j=1}^{N} \frac{\partial C_i}{\partial z_j} \right) \nu$$

$$= -k_1 \text{sig} \left( \frac{\partial C_i}{\partial z_i} \right) \mu - k_2 \text{sig} \left( \frac{\partial C_i}{\partial z_i} \right) \nu \forall i. \quad (35)
$$

It can be rewritten in vector notations that

$$\dot{z} = -k_1 \text{sig} \left( \nabla C(z) \right)^\mu - k_2 \text{sig} \left( \nabla C(z) \right)^\nu$$

which implies that $\dot{V} = (C(z) - C(z^*))\nabla^T C(z) \times (-k_1 \text{sig} \left( \nabla C(z) \right)^\mu - k_2 \text{sig} \left( \nabla C(z) \right)^\nu)$. Note that [19]

$$\nabla^T C(z) \text{sig} \left( \nabla C(z) \right)^\mu = \sum_{i=1}^{N} \left( \frac{\partial C_i}{\partial z_i} \right)^{\nu+1} = \sum_{i=1}^{N} \left( \frac{\partial C_i}{\partial z_i} \right)^{2 \frac{\nu+1}{\nu}} \quad (36)
$$

Using these relations, we have

$$\dot{V} = (C(z) - C(z^*))$$

$$\times \left[ -k_1 \sum_{i=1}^{N} \left( \frac{\partial C_i}{\partial z_i} \right)^{2 \frac{\nu+1}{\nu}} - k_2 \sum_{i=1}^{N} \left( \frac{\partial C_i}{\partial z_i} \right)^{2 \frac{\nu+1}{\nu}} \right]. \quad (36)
$$

From Lemma 2, we have that $\sum_{i=1}^{N} \left( \frac{\partial C_i}{\partial z_i} \right)^2 \geq \sum_{i=1}^{N} \left( \frac{\partial C_i}{\partial z_i} \right)^2 \geq \sum_{i=1}^{N} \left( \frac{\partial C_i}{\partial z_i} \right)^2 \geq \sum_{i=1}^{N} \left( \frac{\partial C_i}{\partial z_i} \right)^2 \frac{\nu+1}{\nu}$, implying that

$$\dot{V} \leq -k_1 (C(z) - C(z^*)) \sum_{i=1}^{N} \left( \frac{\partial C_i}{\partial z_i} \right)^2 \frac{\nu+1}{\nu}$$

$$- k_2 (C(z) - C(z^*)) N \frac{\nu+1}{\nu} \sum_{i=1}^{N} \left( \frac{\partial C_i}{\partial z_i} \right)^2 \frac{\nu+1}{\nu}$$

$$\leq -k_1 (C(z) - C(z^*)) (\| \nabla C(z) \|^2 \frac{\nu+1}{\nu}$$

$$- k_2 (C(z) - C(z^*)) N \frac{\nu+1}{\nu} (\| \nabla C(z) \|^2 \frac{\nu+1}{\nu}. \quad \Box
$$

**Remark 5:** Note that the right-hand side of the inequality (11) is well-defined as $\tau_1 > 0$ (and hence, the fractional powers of $\tau_1$ are real and positive), under Assumption A2 (of Assumption 5). According to Lemma 4, since $\tau_1$ depends upon the eigenvalues of matrix $B$, coefficients $B_{ij}$, and the constants $\sigma, \delta$ associated with the cost functions $C_i$, the settling time $T_s$ shows a dependence on these parameters, and network topology because of the occurrence of $\phi_2$ (the second smallest eigenvalue of the Laplacian $\mathcal{L}$ associated with the local network). In fact, the inequality (11) provides an estimate of the upper bound for the convergence time. Its value is robust to the changes in the initial conditions and power transmission losses. The actual convergence time may be much less than this estimated value.

**IV. CASE STUDIES**

In this section, we simulate two different test cases to illustrate the effectiveness of the proposed algorithm. Initially, we consider a power system network comprising four generators as shown in Fig. 1. Next, we consider an IEEE-57 bus system to show the applicability of the proposed algorithm.
A. Case 1: Power System Network in Fig. 1

Consider a power system network of four generators comprising a two-layered interaction topology, as shown in Fig. 1. The generators share power outputs to a central server to obtain $H_i$ and the auxiliary variables $\lambda_i, \xi_i$ in algorithm (7) are shared locally. The power generation cost associated with each generator is characterized by the quadratic function $C_i(P_i) = c_i P_i^2 + b_i P_i + a_i$, where $a_i, b_i, c_i$ are the cost coefficients and $i = 1, \ldots, 4$. The EDP can be described as: $\min_P C(P) = \min_{[P_n, P]} \sum_{i=1}^4 c_i P_i^2 + b_i P_i + a_i$, subject to $\sum_{i=1}^4 P_i = D + P_L$. The values of $a_i, b_i, c_i$ are given in Table I. Clearly, $\sigma = 2 \min_i \{c_i\} = 0.164$ and $\delta = \min_i \{b_i\} = 1.21$. Let the total load demand be $D = 600$ MW. The initial power supplied by the generators is given by $P(0) = 170$ MW, $P_2(0) = 110$ MW, $P_3(0) = 140$ MW, and $P_4(0) = 180$ MW. The power transmission losses (2) are obtained by setting the $B$-loss coefficients as

$$B = \begin{bmatrix} 0.1200 & 0.0286 & 0.0481 & 0.0321 \\ 0.0286 & 0.1341 & 0.0511 & 0.1251 \\ 0.0481 & 0.0511 & 0.1539 & 0.1463 \\ 0.0321 & 0.1251 & 0.1463 & 0.1612 \end{bmatrix} \times 10^{-3}$$

which is symmetric with all positive entries and $B_0 = [2.0, 1.0, 2.5, 1.5]^T \times 10^{-3}$, $B_{00} = 4$. Algorithm (7) is simulated with control parameters $k_1 = k_2 = 5$, and $\mu = 0.5, \nu = 2$. The optimal power supplied by the generators is obtained as $P_1^* = 161.4$ MW, $P_2^* = 171.3$ MW, $P_3^* = 170.4$ MW, and $P_4^* = 138.1$ MW, as shown in Fig. 2(a). The total power supplied is $P_T = 641.2$ MW, meeting the load demand $D = 600$ MW and the power transmission losses 41.2 MW at the optimal solution [see Fig. 2(b) and (c)]. One can observe from Fig. 2 that the demand and transmission losses are supplied by the generators at every instant of time. The optimal cost is plotted in Fig. 2(d), and is evaluated to be $110,093$. Furthermore, we verify the convergence time in these plots by evaluating the settling time $T_s$ in (11). For the given values of $\sigma, \delta$, and $B$-coefficients, the matrix $S$ in Lemma 4 is obtained as (considering each entry with four significant decimal places)

$$S = \begin{bmatrix} 0.1646 & 0.0000 & 0.0001 & 0.0000 \\ 0.0000 & 0.1645 & 0.0001 & 0.0002 \\ 0.0001 & 0.0001 & 0.1648 & 0.0002 \\ 0.0000 & 0.0002 & 0.1648 & 0.1646 \end{bmatrix}.$$ 

The minimum eigenvalues of $B$ and $S$ are $b_1 = -0.0161 \times 10^{-3}$ and $\tau_1 = 0.1644$, respectively. Furthermore, $\rho = \min_i \{B_{i0}\} = 1.0 \times 10^{-3}$ and $\phi_2 = 0.5858$. It can be easily verified that $(1 + \rho)\sigma + b_1\delta = (1 + 1.0 \times 10^{-3}) \times 0.164 + (-0.0161 \times 10^{-3}) \times 1.21 = 0.1641 > 0$, satisfying Assumption (A2) (of Assumption 5). Using the above values, the settling time is obtained as $T_s = 154.47$ s, which supports our simulation results in Fig. 2.

B. Case 2: An IEEE 57-Bus System

We consider an IEEE 57-bus system from [51]. It comprises 7 generators (locally connected as a line graph) and 52 loads with the relevant parameters as $a_i = $20/\$/h, $b_i = $3.0/\$/MWh, $c_i = $0.3/\$/MW$^2$ h for all $i = 1, \ldots, 7$. The following $B$-loss coefficient matrix is used to obtain the value of transmission losses

$$B = \begin{bmatrix} 1.20 & 0.28 & 0.48 & 0.32 & 1.20 & 0.28 & 0.48 \\ 0.28 & 1.34 & 0.51 & 1.25 & 1.34 & 0.51 & 1.25 \\ 0.48 & 0.51 & 1.53 & 1.46 & 0.51 & 1.53 & 1.46 \\ 0.48 & 1.25 & 1.46 & 1.61 & 1.25 & 1.46 & 1.61 \end{bmatrix} \times 10^{-4}$$

which is symmetric with all positive entries and $B_0 = [2.0, 1.0, 2.5, 1.5, 1.0, 2.5, 1.5]^T \times 10^{-3}$, $B_{00} = 4$. Here, $\sigma = 2 \min_i \{c_i\} = 0.6$ and $\delta = \min_i \{b_i\} = 3$. The initial power

---

**Table I** Generator Cost Parameters for System in Fig. 1

| Bus | $a_i$ ($$/h) | $b_i$ ($$/MWh) | $c_i$ ($$/MW^2*h) |
|-----|--------------|----------------|-------------------|
| G1  | 53           | 1.21           | 0.094             |
| G2  | 34           | 3.47           | 0.082             |
| G3  | 45           | 2.24           | 0.086             |
| G4  | 78           | 2.55           | 0.105             |

---

**Fig. 2.** Evolution of generated power, transmission losses, total power, and cost during $0 - 200$ s. (a) $P_i$ versus time. (b) $P_L$ versus time. (c) $\sum_{i=1}^N P_i$ versus time. (d) $\sum_{i=1}^N C_i(P_i)$ versus time.
supplied by the generators is considered as $P_1(0) = 170$ MW, $P_2(0) = 110$ MW, $P_3(0) = 140$ MW, $P_4(0) = 180$ MW, $P_5(0) = 170$ MW, $P_6(0) = 210$ MW, and $P_7(0) = 220$ MW. Using the values of $\sigma$, $\delta$, and $B$-coefficients, the matrix $S$ in Lemma 4 is obtained as

$$S = \begin{bmatrix}
0.001 & 0.001 & 0.001 & 0.004 & 0.001 & 0.001 & 0.001 \\
0.001 & 6.014 & 0.002 & 0.004 & 0.004 & 0.002 & 0.004 \\
0.001 & 0.002 & 6.024 & 0.004 & 0.002 & 0.005 & 0.004 \\
0.001 & 0.004 & 0.004 & 6.019 & 0.004 & 0.004 & 0.005 \\
0.004 & 0.004 & 0.015 & 0.004 & 6.014 & 0.015 & 0.004 \\
0.001 & 0.002 & 0.005 & 0.004 & 0.015 & 6.024 & 0.015 \\
0.001 & 0.004 & 0.004 & 0.005 & 0.004 & 0.015 & 6.023
\end{bmatrix} \times 10^{-1}.$$

Algorithm (7) is again simulated with control parameters $k_1 = k_2 = 5$, and $\mu = 0.5$, $\nu = 2$. Depending upon the load variation, we further examine the following two cases to show the effectiveness of our algorithm in the IEEE 57-bus system.

1) First, we consider a constant load of 1200 MW, which is the total load demand of all the generator buses. In this case, the algorithm (7) obtains the optimal values of power as $P_1^* = 224.68$ MW, $P_2^* = 224.77$ MW, $P_3^* = 224.86$ MW, $P_4^* = 224.96$ MW, $P_5^* = 225.05$ MW, $P_6^* = 225.08$ MW, and $P_7^* = 225.11$ MW, as shown in Fig. 3. The total power supplied by the generators is $P_T = 1568.87$ MW. The transmission losses are $368.87$ MW at optimal power generation. Furthermore, the optimal cost is obtained as $108729$ and is plotted in Fig. 3(d). The minimum eigenvalues of $B$ and $S$ are $b_1 = 0.0063 \times 10^{-3}$ and $\tau_1 = 0.6$, respectively. Furthermore, $\rho = \min \{B_{00}\} = 1.0 \times 10^{-3}$ and $\phi_2 = 0.1981$. It can be easily verified that $(1+\rho)\sigma + b_1\delta = (1 + 1.0 \times 10^{-3}) \times 0.6 + (0.0063 \times 10^{-3}) \times 3 = 0.6066 > 0$, satisfying Assumption (A2). Using the above values, the settling time is obtained as $T_s = 550.21$ s, which supports our simulation results in Fig. 3.

2) Next, we consider a random change in load demand with time at every 250 s. In this case, the plot of $P_i$ versus time is shown in Fig. 4. The plots for transmission losses and cost function were found to be well-behaving and are omitted due to limited space. Furthermore, we compare the proposed fixed-time convergent algorithm with the classical incremental cost consensus (ICC) algorithm [52]. Fig. 5 shows the performance of the two algorithms for a net load demand of 1200 MW with the system parameters specified in [53]. The simulations are carried out for the same setting of the initial conditions with a fixed step size of 0.1 s. It can be seen in Fig. 5 that, despite being a partially distributed algorithm, the convergence of the proposed fixed-time algorithm is much faster, as compared to the ICC algorithm.
From the above case studies, it is clear that the locations of generators and loads are not important as far as the problem in our article is considered since we are calculating the transmission losses using Kron’s B-loss formula for the given B-coefficients.

V. CONCLUSIONS AND FUTURE REMARK

In this article, we investigated the EDP with Kron’s modeled power transmission losses, under a few assumptions on the B-loss coefficients, network topology, initial demand, and the convexity of the cost functions associated with each generator. The time-varying power transmission losses are incorporated in the equality constraints of the considered EDP. It is shown that the proposed partially distributed consensus-based algorithm solves the EDP in a finite time, which is upper bounded by a term relying on the eigenvalues of the matrix B, local Laplacian matrix, and the parameters describing the convexity of the cost functions.

Building on this work with consideration of Kron’s modeled power transmission losses, we identify the following directions for future research.

1) Although the proposed algorithm in this article can be implemented in a fully distributed manner for the simplified Kron’s modeled power transmission losses (see Remark 2), it remains a challenging problem to come up with an algorithm accounting for these losses without any approximations.

2) The consideration of constant B-coefficients is motivated by certain classical assumptions, e.g., fixed power factor at each generator bus, the ratio of load current to the total load current, etc. A detailed discussion on this can be found in [11] and [46]. It may be possible to relax these assumptions and incorporate the time-varying nature of the B-coefficients in Kron’s formula.

3) In our problem, either the inclusion of an additional dynamics for the auxiliary variable λ, [43] or the idea of a penalty function [54] might be useful in achieving the generation limits. However, in the context of the present problem, these approaches are restricted to a special class of cost functions that might not accurately characterize the cost function of an actual power generation system. An extension of the algorithm to also account for the generation limits in the presence of Kron’s modeled power transmission is an interesting and challenging scope for future research.

4) One would also like to examine the robustness of the proposed algorithm (7) in the presence of any disturbances on the dynamics z_i of the auxiliary variable z_i. Although, some works, for instance, [43], are reported in this direction for the additive bounded disturbance with zero mean but without consideration of Kron’s modeled transmission losses.

5) In our analysis, we have considered the positivity of the power generated by each generator under Assumptions 1 and 2, which requires prior knowledge of the parameter φ and may not be available. One can further come up with an algorithm that relaxes these assumptions and generalize the results.

APPENDIX

Proof of Lemma 4: The proof is provided separately for each claim.

R1) By definition, \( F \in \mathbb{R}^N \) is a column vector (notice the use of Hadamard product ⊙) with ith entry \( F_i = (∂P_i/∂P_i)(∂C_i/∂P_i) \), where \( ∂P_i/∂P_i \) depends on both \( P_i \) and \( P_j \), according to Lemma 3, and \( ∂C_i/∂P_i = (∂C_i/∂C_i)(∂P_i/∂P_i) = ∂C_i/∂P_i \) depends only on \( P_i \) (as \( ∂C_i/∂C_i = 1 \), according to (4a)). Therefore, \( ∇F \) is an \( N \times N \) matrix, whose diagonal entries can be obtained using chain rule as

\[
[∇F_i] = \frac{∂}{∂P_i} \left[ ∂P_i \frac{∂C_i}{∂P_i} \right] = \frac{∂}{∂P_i} ∂C_i + ∂P_i \frac{∂^2 C_i}{∂P_i^2} \]

which, using Lemma 3, can be written as

\[
[∇F_i] = 2B_{ii} \left( \frac{∂C_i(P_i)}{∂P_i} + P_i \frac{∂^2 C_i(P_i)}{∂P_i^2} \right) + \sum_{j=1,j\neq i}^{N} 2B_{ij}P_j \]

\[
+ B_{ii} \frac{∂^2 C_i(P_i)}{∂P_i^2} = 2B_{ii} \frac{∂C_i(P_i)}{∂P_i} + \sum_{j=1,j\neq i}^{N} 2B_{ij}P_j \]

\[
≥ 2B_{ii} \frac{∂C_i(P_i)}{∂P_i} + B_{ii} \frac{∂^2 C_i(P_i)}{∂P_i^2} \]

under Assumptions 4 and 5, as \( P_i, P_j ≥ 0, ∀i, j \) and \( t \in [0, ∞) \), according to Remark 3. Similarly, the off-diagonal entries are given by

\[
[∇F_{ij}] = \frac{∂}{∂P_i} \frac{∂C_i(P_i)}{∂P_j} = \frac{∂C_i}{∂P_i} \frac{∂}{∂P_j} \left[ ∂P_i \right] \]

\[
= 2B_{ij} \frac{∂C_i(P_i)}{∂P_j} \]

R2) Clearly, \( M \in \mathbb{R}^N \) is a column vector with ith entry \( M_i = |B_{ii}P_i + (B_{ii}/2)|/|∂C_i(P_i)/∂P_i| \), which depends only on \( P_i \) for each \( i \). As a result, \( ∇M \) is an \( N \times N \) diagonal matrix with diagonal entries

\[
[∇M_{ii}] = B_{ii} \frac{∂C_i(P_i)}{∂P_i} + B_{ii} \frac{∂^2 C_i(P_i)}{∂P_i^2} \]

\[
≥ \left[ B_{ii}P_i + \frac{B_{ii}/2}{2} \right] \frac{∂C_i(P_i)}{∂P_i} \]

for \( P_i ≥ 0, ∀i \), and following Assumptions 4 and 5.
R3) The proof of this statement is straightforward and follows the similar steps as above.

R4) From Assumption 5, it is obvious that $S$ is a symmetric matrix. Furthermore, using (R3), it trivially holds that $S \preceq Q$.

R5) Since $S$ is a symmetric matrix, its eigenvalues are real and can be arranged as $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_N$. By construction, $S$ can be written as the summation of two symmetric matrices $\sigma(B_1 \sigma) + \delta B$, where the constants $\sigma$ and $\delta$ are defined in Assumption 4. Notice that the eigenvalues of $\sigma(B_1 \sigma) + \delta B$ are $[\sigma(1 + B_{\sigma})]_i \leq b_i \delta = b_2 \delta \leq \cdots \leq b_N \delta$, if $\delta > 0$; and $b_N \delta \leq b_{N-1} \delta \leq \cdots \leq b_1 \delta$, if $\delta < 0$. Now, the results immediately follow by applying Weyl’s theorem [50, Ch. 4, p. 239].

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Shivanshu Tripathi received the bachelor’s degree in electrical engineering from the National Institute of Technology, Srinagar, India, in 2020, and the master’s degree in electrical engineering from the Indian Institute of Technology (IIT) Jodhpur, Jodhpur, India, in 2022. He is currently working toward the Ph.D. degree in electrical and computer engineering with the University of California, Riverside, CA, USA. His research interests include distributed optimization and multiagent systems.

Anoop Jain (Member, IEEE) received the B.Tech. degree in electrical and electronics engineering from the Krishna Institute of Engineering and Technology, Ghaziabad, India, affiliated with Uttar Pradesh Technical University, in 2009, and the M.E. and Ph.D. degrees in aerospace engineering from the Department of Aerospace Engineering, Indian Institute of Science (IISc), Bangalore, Bengaluru, India, in 2011 and 2017, respectively. He is currently an Assistant Professor with the Department of Electrical Engineering, Indian Institute of Technology (IIT) Jodhpur, Jodhpur, India. From 2017 to 2019, he was a Postdoctoral Fellow with the Faculty of Aerospace Engineering, Technion—Israel Institute of Technology, where his research was funded by the Lady-Davis Fellowship. His research interests include cooperative control of multiagent systems, event-triggered control, cyber-physical systems, and nonlinear control theory.

Abhisek K. Behera (Member, IEEE) received the bachelor’s degree in electrical engineering from the Biju Patnaik University of Technology, Rourkela, India, in 2008, the master’s degree in electrical engineering from the National Institute of Technology Rourkela, Rourkela, in 2011, and the Ph.D. degree in systems and control engineering from the Indian Institute of Technology (IIT) Bombay, Mumbai, India, in 2017. After holding postdoctoral positions with Seoul National University, Seoul, Korea, and Technion—Israel Institute of Technology, Haifa, Israel, he joined IIT Roorkee as an Assistant Professor in 2020. His current research interests include sliding mode control and event-triggered control. Dr. Behera was the recipient of the Award for Excellence in Thesis Work from IIT Bombay in 2018.