Sharp Gaussian Estimates for Heat Kernels of Schrödinger Operators

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Abstract. We characterize functions \( V \leq 0 \) for which the heat kernel of the Schrödinger operator \( \Delta + V \) is comparable with the Gauss–Weierstrass kernel uniformly in space and time. In dimension 4 and higher the condition turns out to be more restrictive than the condition of the boundedness of the Newtonian potential of \( V \). This resolves the question of V. Liskevich and Y. Semenov posed in 1998. We also give specialized sufficient conditions for the comparability, showing that local \( L^p \) integrability of \( V \) for \( p > 1 \) is not necessary for the comparability.

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1. Introduction and Main Results

Let \( d = 1, 2, \ldots \). We consider the Gauss–Weierstrass kernel,

\[
g(t, x, y) = (4\pi t)^{-d/2} e^{-|y-x|^2/(4t)}, \quad t > 0, \ x, y \in \mathbb{R}^d.
\]

It is well known that \( g \) is the fundamental solution of the equation \( \partial_t u = \Delta u \), and time-homogeneous probability transition density – the heat kernel of \( \Delta \).

For Borel measurable function \( V : \mathbb{R}^d \to \mathbb{R} \) we call \( G : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty] \) the heat kernel of \( \Delta + V \) or the Schrödinger perturbation of \( g \) by \( V \), if the following Duhamel or perturbation formula holds for \( t > 0, \ x, y \in \mathbb{R}^d \),

\[
G(t, x, y) = g(t, x, y) + \int_0^t \int_{\mathbb{R}^d} G(s, x, z)V(z)g(t-s, z, y)dzds.
\]
Under appropriate assumptions on $V$, explicit definition of $G$ may be given by means of the Feynmann-Kac formula [5, Section 6], the Trotter formula [29, p. 467], the perturbation series [5], or by means of quadratic forms on $L^2$ spaces [10, Section 4]. In particular the assumption $V \in L^p(\mathbb{R}^d)$ with $p > d/2$ was used by Aronson [2], Zhang [29, Remark 1.1(b)] and by Dziubański and Zienkiewicz [11]. Aizenman and Simon [1,23] proposed functions $V(z)$ from the Kato class, which contains $L^p(\mathbb{R}^d)$ for every $p > d/2$ [1, Chapter 4], see also Chung and Zhao [9, Chapter 3, Example 2]. An enlarged Kato class was used by Voigt [25] in the study of Schrödinger semigroups on $L^1$ [25, Proposition 5.1]. For perturbations by time-dependent functions $V(u, z)$, Zhang [26,28] introduced the so-called parabolic Kato condition. The condition was then generalized and employed by Schnaubelt and Voigt [21], Liskevich and Semenov [18], Milman and Semenov [20], Liskevich et al. [19], and Gulisashvili and van Casteren [14].

Given the function $V : \mathbb{R}^d \rightarrow \mathbb{R}$ we ask if there are positive numbers, i.e., constants $0 < c_1 \leq c_2 < \infty$ such that the following two-sided bound holds,

$$c_1 \leq \frac{G(t, x, y)}{g(t, x, y)} \leq c_2, \quad t > 0, \ x, y \in \mathbb{R}^d. \quad (1)$$

One can also ponder a weaker property – if for a given $T \in (0, \infty)$,

$$c_1 \leq \frac{G(t, x, y)}{g(t, x, y)} \leq c_2, \quad 0 < t \leq T, \ x, y \in \mathbb{R}^d. \quad (2)$$

We call (1) and (2) sharp Gaussian estimates or bounds, respectively global (or uniform) and local in time. We observe that the inequalities in (1) and (2) are stronger than the plain Gaussian estimates:

$$c_1 (4\pi t)^{-d/2} e^{-\frac{|y-x|^2}{4t}} \leq G(t, x, y) \leq c_2 (4\pi t)^{-d/2} e^{-\frac{|y-x|^2}{4t}}, \ x \in \mathbb{R}^d,$$

where $0 < \varepsilon_1, c_1 \leq 1 \leq \varepsilon_2, c_2 < \infty$, which can also be global or local in time.

Berenstein proved the plain Gaussian estimates for $V \in L^p$ with $p > d/2$ (see [17]). Simon [23, Theorem B.7.1] resolved them for $V$ in the Kato class, Zhang [28] and Milman and Semenov [20] applied the parabolic Kato class for this purpose. For further discussion we refer the reader to [18–20,29] and [6, Lemma 4]. We also refer to Bogdan and Szczypkowski [7, Section 1, 4] for a survey of the plain Gaussian bounds for Schrödinger heat kernels along with a streamlined approach, new results and explicit constants based on the so-called 4G inequality.

The plain Gaussian estimates are ubiquitous in analysis but (1) and (2) provide precious qualitative information, if they hold for $V$. It is intrinsically difficult to characterize (1) and (2) for those $V$ that take on positive values, while the case of $V \leq 0$ is more manageable. Arsen’ev proved (2) for $V \in L^p + L^\infty$ with $p > d/2, d \geq 3$. Van Casteren [24] proved (2) for $V$ in the intersection of the Kato class and $L^{d/2} + L^\infty$ for $d \geq 3$ (see [20]). Arsen’ev also obtained (1) for $V \in L^p$ with $p > d/2$ under additional smoothness assumptions (see [17]). Liskevich and Semenov stated sufficient conditions for (1) and (2) in [17, Theorem 1, Corollary 1, Theorem 2]. Zhang [29, Theorem 1.1] and Milman
and Semenov [20, Theorem 1C, Remark (2)] gave sufficient integro-supremal conditions for (1) and (2) for general \( V \) and characterized (1) and (2) for \( V \leq 0 \). It will be convenient to state the conditions by means of

\[
S(V, t, x, y) = \int_0^t \int_{\mathbb{R}^d} g(s, x, z) g(t - s, z, y) |V(z)| |V(z)| dz ds, \quad t > 0, \ x, y \in \mathbb{R}^d.
\]

The motivation for using this quantity comes from Zhang [29, Lemma 3.1 and Lemma 3.2] and from Bogdan et al. [5, (1)]. We often write \( S(V) \) if we do not need to specify \( t, x, y \). As explained in Sect. 4, \( S(V) \) is the potential of \(|V|\) for the so-called Gaussian bridges. We also note that [5, Section 6] uses \( S(V) \) for general transition densities. The next two results indicate why \( S(V) \) is important. Their proofs are given in Sect. 2.

**Lemma 1.1.** Let \( V \leq 0 \). Then (1) is equivalent to

\[
\sup_{t > 0, x, y \in \mathbb{R}^d} S(V, t, x, y) < \infty. \tag{3}
\]

Also, for each \( T \in (0, \infty) \), (2) is equivalent to

\[
\sup_{0 < t \leq T, x, y \in \mathbb{R}^d} S(V, t, x, y) < \infty. \tag{4}
\]

We say that \( V \) satisfying (3) or (4) has bounded potential for bridges (is bridge-potential bounded) globally or locally in time, respectively.

**Lemma 1.2.** If for some \( h > 0 \) and \( 0 \leq \eta < 1 \) we have

\[
\sup_{0 < t \leq h, x, y \in \mathbb{R}^d} S(V^+, t, x, y) \leq \eta,
\]

and if \( S(V^-) \) is bounded on bounded subsets of \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\), then

\[
e^{-S(V^-, t, x, y)} \leq \frac{G(t, x, y)}{g(t, x, y)} \leq \left( \frac{1}{1 - \eta} \right)^{1+t/h}, \quad t > 0, \ x, y \in \mathbb{R}^d. \tag{5}
\]

We also notice the following consequence of the Duhamel formula.

**Remark 1.3.** If \( V \geq 0 \), then (1) implies (3) and (2) implies (4).

For clarity we note that \( S(V) \) is unbounded for all nontrivial \( V \) in dimensions \( d = 1 \) and \( 2 \), hence (1) is impossible for nontrivial \( V \geq 0 \) and nontrivial \( V \leq 0 \) in these dimensions. This is explained at the end of Sect. 2 below.

### 1.1. Characterization of Sharp Gaussian Estimates

The conditions in Lemmas 1.1 and 1.2 may be cumbersome to verify. For this reason we propose a simpler integro-supremal test for (1). For \( d \geq 3 \) and \( x, y \in \mathbb{R}^d \) we define

\[
K(V, x, y) = \int_{\mathbb{R}^d} |V(z)| K(z - x, y) dz,
\]

where

\[
K(x, y) = \frac{e^{-(|x|y|y|x-y)/2}}{|x|^{d-2}} (1 + |x||y|)^{d/2-3/2}, \tag{6}
\]
and $x \cdot y$ is the usual scalar product. We denote, as usual,

$$\|S(V)\|_\infty = \sup_{t>0, x,y \in \mathbb{R}^d} S(V, t, x, y),$$
$$\|K(V)\|_\infty = \sup_{x,y \in \mathbb{R}^d} K(V, x, y).$$

These two integro-supremal quantities turn out to be comparable, as follows.

**Theorem 1.4.** There are constants $M_1, M_2$ depending only on $d$, such that

$$M_1 \|K(V)\|_\infty \leq \|S(V)\|_\infty \leq M_2 \|K(V)\|_\infty. \quad (7)$$

The proof of Theorem 1.4 is given in Sect. 3. By (7) and Lemma 1.1 we get the following characterization of the sharp global Gaussian estimates.

**Corollary 1.5.** If $V \leq 0$, then (1) holds if and only if $K(V)$ is bounded.

Similarly, for general (signed) $V$ we get (1) provided $\|K(V^-)\|_\infty < \infty$ and $\|K(V^+)\|_\infty < 1/M_2$. This follows from Lemma 1.2 and Theorem 1.4.

We next elaborate on more specific applications of $K(V)$ to sharp global Gaussian estimates. In particular we resolve a long-standing open problem posed by Liskevich and Semenov. For $d \geq 3$ we let $C_d = \Gamma(d/2 - 1)/(4\pi^{d/2})$. The Newtonian kernel is $\int_0^\infty g(s, x, z) ds = C_d |z - x|^{2-d}, x, z \in \mathbb{R}^d$, and the Newtonian potential of (a nonnegative) function $f$ at $x \in \mathbb{R}^d$ is denoted

$$-\Delta^{-1} f(x) := \int_0^\infty \int_{\mathbb{R}^d} g(s, x, z) f(z) dz ds = \int_{\mathbb{R}^d} \frac{C_d}{|z - x|^{d-2}} f(z) dz.$$

For $d = 3$ the formula (6) considerably simplifies and we easily obtain

$$\|K(V)\|_\infty = C_d^{-1} \|\Delta^{-1} |V|\|_\infty \quad \text{for } d = 3. \quad (8)$$

Thus if $d = 3$ and $V \leq 0$, then by Corollary 1.5 the sharp global Gaussian bounds (1) are equivalent to the condition of potential-boundedness, namely $\|\Delta^{-1} V\|_\infty < \infty$. This is classical [20, Remark (3) on p. 4] but remarkable, because the Newtonian kernel is isotropic, that is rotation-invariant, while $K(V)$ and $S(V)$ have a certain anisotropy-sensitivity.

Putting aside the exceptional, the main focus of the present paper is on $d \geq 4$. As usual, we let

$$\|V\|_{d/2} = \left( \int_{\mathbb{R}^d} |V(z)|^{d/2} dz \right)^{2/d}.$$

By [17, Theorem 2] and [20, Remark (1) and (4) on p. 4] we have (1) for $d \geq 4$ if $\|\Delta^{-1} V^-\|_\infty + \|V^-\|_{d/2} < \infty$, $\|\Delta^{-1} V^+\|_\infty < 1$ and $\|V^+\|_{d/2}$ is small enough. A long-standing open problem for (1) with $V \leq 0$ posed in 1998 by Liskevich and Semenov [17, p. 602] reads as follows: “The validity of the two-sided estimates for the case $d > 3$ without the additional assumption $V \in L^{d/2}$ is an open question.” In view of Theorem 1.4 and Lemma 1.1 the question is whether for $d \geq 4$ the finiteness of $\|\Delta^{-1} V\|_\infty$ implies the finiteness of $\|K(V)\|_\infty$. Our next estimate is a variant of [17, Corollary 1] and closely relates $\|K(V)\|_\infty$ to $\|\Delta^{-1} V\|_\infty$ and $\|V\|_{d/2}$ for $d \geq 4$:
\[ C_d^{-1} \| \Delta^{-1} |V| \|_{\infty} \leq \| K(V) \|_{\infty} \leq 2^{(d-3)/2} \left( C_d^{-1} \| \Delta^{-1} |V| \|_{\infty} + \kappa_d |V|_{d/2} \right). \]  

(9)

In Sect. 3 we prove (9) and the following result, which points out a gap between \( \| K(V) \|_{\infty} \) and \( \| \Delta^{-1} |V| \|_{\infty} \) in (9).

**Proposition 1.6.** Let \( d \geq 4 \). For \( z = (z_1, z_2, \ldots, z_d) \in \mathbb{R}^d \) we write \( z = (z_1, z_2) \), where \( z_2 = (z_2, \ldots, z_d) \in \mathbb{R}^{d-1} \). We define

\[ A = \{(z_1, z_2) \in \mathbb{R}^d : z_1 > 4, \ |z_2| \leq \sqrt{z_1}\}, \quad \text{and} \]

\[ V(z_1, z_2) = -\frac{1}{z_1} 1_A(z_1, z_2). \]

Then \( \| \Delta^{-1} V \|_{\infty} < \infty \) but \( \| K(V) \|_{\infty} = \infty \). There is even a function \( V \leq 0 \) with compact support such that \( \| \Delta^{-1} V \|_{\infty} < \infty \) but \( \| K(V) \|_{\infty} = \infty \).

From Lemma 1.1 we conclude that for \( d \geq 4 \) neither finiteness nor smallness of \( \| \Delta^{-1} V \|_{\infty} \) are sufficient for (1). Therefore the answer to the question of Liskevich and Semenov is negative.

Here are a few more comments that relate our result to existing literature and serve as preparation for the proofs. Due to the work of Zhang [29], the following quantity is a proxy for \( S(V) \),

\[ N(V, t, x, y) := \int_0^{t/2} \int_{\mathbb{R}^d} \frac{e^{-|z-y+(\tau/t)(y-x)|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau \]

\[ + \int_{t/2}^t \int_{\mathbb{R}^d} \frac{e^{-|z-y+(\tau/t)(y-x)|^2/(4(t-\tau))}}{(t-\tau)^{d/2}} |V(z)| dz d\tau = N(V, t, y, x), \]

(10)

where \( t > 0, \ x, y \in \mathbb{R}^d \). Indeed, by [29, Lemma 3.1, Lemma 3.2, and line 11 on p. 469] there are constants \( m_1, m_2 \) depending only on \( d \) such that

\[ S(V, t, x, y) \geq m_1 N(V, t/2, x, y), \quad t > 0, \ x, y \in \mathbb{R}^d, \]

\[ S(V, t, x, y) \leq m_2 N(V, t, x, y), \quad t > 0, \ x, y \in \mathbb{R}^d. \]

We also let \( \| N(V) \|_{\infty} = \sup_{t>0, x, y \in \mathbb{R}^d} N(V, t, x, y) \). By (L) and (U) we get

\[ m_1 \| N(V) \|_{\infty} \leq \| S(V) \|_{\infty} \leq m_2 \| N(V) \|_{\infty}. \]

In [20, Theorem 1C] and [17, (8)] another quantity was used to study (1) and (2),

\[ e_*(V, \lambda) = \sup_{\alpha \in \mathbb{R}^d} \| V(\lambda - \Delta + 2\alpha \cdot \nabla)^{-1} \|_{1 \rightarrow 1} \]

\[ = \sup_{\alpha \in \mathbb{R}^d} \| (\lambda - \Delta + 2\alpha \cdot \nabla)^{-1} |V| \|_{\infty}, \]

where \( \lambda \geq 0 \). It may be given in terms of the Gauss–Weierstrass kernel, e.g.,

\[ e_*(V, 0) = (4\pi)^{-d/2} \sup_{x, y \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(z - x, y) |V(z)| dz, \]

where

\[ J(x, y) = \int_0^{\infty} \tau^{-d/2} e^{-\frac{|x-y|^2}{4\tau}} d\tau, \quad x, y \in \mathbb{R}^d. \]
There is a certain anisotropy-sensitivity of \( e_*(V,0) \) due to \( 2\alpha \cdot \nabla \) above, which is similar to that of \( K(V) \). In fact, in Lemma 3.2 below we prove that there are constants \( n_1, n_2 \) depending only on \( d \geq 3 \) such that

\[
n_1 e_*(V,0) \leq \|S(V)\|_\infty \leq n_2 e_*(V,0).
\]

In view of Theorem 1.4, the quantities \( \|S(V)\|_\infty, \|K(V)\|_\infty, \|N(V)\|_\infty \) are all comparable, which makes them equivalent for studying (1) with \( V \leq 0 \).

We add a few comments on the exceptional case \( d = 3 \). By [20, (3) in Remark on p. 4] and (8) we have \( e_*(V,0) = \|\Delta^{-1}\|_\infty = C_d \|K(V)\|_\infty \).

Also, for \( d = 3 \) by [20, Remark (1) and (3) on p. 4], the condition

\[
\|\Delta^{-1}V^-\|_\infty < \infty, \quad \|\Delta^{-1}V^+\|_\infty < 1,
\]

suffices for (1). Furthermore, if \( V \leq 0 \), then the condition \( \|\Delta^{-1}V\|_\infty < \infty \) characterizes the plain global Gaussian bounds, see [22] and [30, p. 556 and Corollary A]. Therefore by (8), for \( d = 3 \) the plain global Gaussian bounds hold for \( V \leq 0 \) if and only if the sharp global Gaussian bounds hold. In contrast, for \( d \geq 4 \) by Proposition 1.6 the plain global Gaussian bounds may occur in the absence of the sharp global Gaussian bounds (1).

We recall that \( \sup_{0<t<\Delta, x,y\in\mathbb{R}^d} S(V,t,x,y) \) with \( T < \infty \) is useful for the local in time sharp Gaussian estimates (2), see Lemmas 1.1 and 1.2. In a similar fashion \( \sup_{0<t<\Delta, x,y\in\mathbb{R}^d} N(V,t,x,y) \) is used in [29, Theorem 1.1], while in [20, Theorem 1C] the authors make use of \( e_*(V,\lambda) \) for \( \lambda > 0 \). In this connection see also Corollary 2.4 below.

### 1.2. Sufficient Conditions for Sharp Gaussian Estimates

In this section we propose sufficient conditions for (1) and (2) for functions \( V \) which have a form of the tensor product. Such conditions are the second main topic of the paper—they culminate in Theorem 1.8 below. We also show that \( L^p \) integrability for \( p > 1 \) is not necessary for (1) or (2). Let \( p, p_1, p_2 \in [1, \infty] \).

**Definition 1.7.** We write \( f \in L^{p_1}(\mathbb{R}^{d_1}) \times L^{p_2}(\mathbb{R}^{d_2}) \) if there are \( f_1 \in L^{p_1}(\mathbb{R}^{d_1}) \) and \( f_2 \in L^{p_2}(\mathbb{R}^{d_2}) \), such that

\[
f(x_1, x_2) = f_1(x_1)f_2(x_2), \quad x_1 \in \mathbb{R}^{d_1}, \quad x_2 \in \mathbb{R}^{d_2}.
\]

We note that \( L^p(\mathbb{R}^{d_1}) \times L^p(\mathbb{R}^{d_2}) \subset L^p(\mathbb{R}^{d_1+d_2}) \), in fact \( \|f\|_p = \|f_1\|_p\|f_2\|_p \) if \( f \) is the tensor product \( f(x_1, x_2) = f_1(x_1)f_2(x_2) \).

**Theorem 1.8.** Let \( d_1, d_2 \in \mathbb{N}, d = d_1 + d_2 \), \( V : \mathbb{R}^d \to \mathbb{R}, p_1, p_2 \in [1, \infty] \) and

\[
\frac{d_1}{2p_1} + \frac{d_2}{2p_2} = 1.
\]

(a) If \( r \in (p_1, \infty] \) and \( V \in L^r(\mathbb{R}^{d_1}) \times L^{p_2}(\mathbb{R}^{d_2}) \), then

\[
\sup_{x,y\in\mathbb{R}^d} S(V,t,x,y) \leq c t^{1-d_1/(2r)-d_2/(2p_2)},
\]

where \( c = C(d_1, r)C(d_2, p_2) \frac{[1-r/(2r)-d_2/(2p_2)]^2}{(1-2-d_1/(2r)-d_2/(2p_2))} \|V_1\|_r\|V_2\|_{p_2} \).

(b) If \( 1 \leq q < p_1 < r \leq \infty \) and \( V \in \left[L^q(\mathbb{R}^{d_1}) \cap L^r(\mathbb{R}^{d_2})\right] \times L^{p_2}(\mathbb{R}^{d_2}) \), then (3) holds.
The proof of Theorem 1.8 is given in Sect. 4, where we use in a crucial way the tensorization of the Gauss–Weierstrass kernel and its bridges. Lemmas 1.1 and 1.2 provide the following conclusion.

**Corollary 1.9.** Under the assumptions of Theorem 1.8(a), $G$ satisfies the sharp local Gaussian bounds (2). If $V \leq 0$ and the assumptions of Theorem 1.8(b) hold, then $G$ has the sharp global Gaussian bounds (1).

Clearly, if $|U| \leq |V|$, then $S(U) \leq S(V)$. This may be used to extend the conclusions of Theorem 1.8 and Corollary 1.9 beyond tensor products.

**Proposition 1.10.** For every $d \geq 3$ there is a function $V \leq 0$ such that (1) holds but $V \notin L^1(\mathbb{R}^d) \cup \bigcup_{p>1} L^p_{\text{loc}}(\mathbb{R}^d)$.

In particular (1) does not necessitate $\|V\|_{d/2} < \infty$, i.e., the finiteness of $\|K(V)\|_{\infty}$ does not imply that of $\|V\|_{d/2}$; see also (9) in this connection. We note in passing that local $L^1$ integrability is necessary for (2) if $V$ does not change sign, cf. Lemmas 1.1 and 2.1, and Remark 1.3. The function $V$ in Proposition 1.10 is constructed in Sect. 5 from highly anisotropic tensor products of power functions.

The structure of the remainder of the paper is as follows. In Sect. 2 we provide definitions and preliminaries, in particular we prove Lemmas 1.1 and 1.2. In Sect. 3 we prove Theorem 1.4 and Proposition 1.6. In Sect. 4 we prove Theorem 1.8. In Sect. 5 we prove Proposition 1.10 and give examples which illustrate and comment on our results.

## 2. Preliminaries

We let $\mathbb{N} = \{1, 2, \ldots\}$, $f^+ = \max\{0, f\}$ and $f^- = \max\{0, -f\}$. Recall that $d \in \mathbb{N}$ and $V$ is an arbitrary Borel measurable function: $\mathbb{R}^d \rightarrow \mathbb{R}$.

We begin with the following observations on integrability and potential-boundedness (14) of functions $V$ which are bridges potential-bounded.

**Lemma 2.1.** If $S(V, t, x, y) < \infty$ for some $t > 0$, $x, y \in \mathbb{R}^d$, then $V \in L^1_{\text{loc}}(\mathbb{R}^d)$. If (4) holds, then

$$\sup_{x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} g(s, x, z)|V(z)|dzds < \infty. \quad (13)$$

If (3) even holds, then

$$\sup_{x \in \mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} g(s, x, z)|V(z)|dzds < \infty. \quad (14)$$

**Proof.** The first statement follows, because $g(t, x, y)$ is locally bounded from below on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ (see [13, Lemma 3.7] for a quantitative general result). Since $\int_{\mathbb{R}^d} S(V, t, x, y)g(t, x, y)dy = \int_0^t \int_{\mathbb{R}^d} g(s, x, z)|V(z)|dzds$, we see that (4) implies (13) and (3) implies (14). $\square$

We shall use the following functions:
Lemma 2.2. For all \( t \in (0, \infty) \),
\[
f(t) = \sup_{x, y \in \mathbb{R}^d} S(V, t, x, y),
\]
\[
F(t) = \sup_{0 < s < t} f(s) = \sup_{x, y \in \mathbb{R}^d} S(V, s, x, y), \quad t \in (0, \infty].
\]
We fix \( V \) and \( x, y \in \mathbb{R}^d \). For \( 0 < \varepsilon < t \), we consider
\[
S(V, t - \varepsilon, x, y) = \int_0^t \int_{\mathbb{R}^d} g(s, x, z) g(t - s, x, y) |V(z)| 1_{[0, t - \varepsilon]}(u) \, dzds.
\]
By Fatou’s lemma we get
\[
S(V, t, x, y) = \lim_{\varepsilon \to 0} \inf S(V, t - \varepsilon, x, y),
\]
meaning that \((0, \infty) \ni t \mapsto S(V, t, x, y)\) is lower semicontinuous on the left.

It follows that \( f \) is lower semi-continuous on the left, too. In consequence, \( f(t) \leq F(t) \) and \( F(t) = \sup_{0 < s \leq t} f(s) \) for \( 0 < t < \infty \).

We next claim that \( f \) is sub-additive, that is,
\[
f(t_1 + t_2) \leq f(t_1) + f(t_2), \quad t_1, t_2 > 0.
\]
This follows from the Chapman–Kolmogorov equations for \( g \). Indeed, we have
\[
S(V, t_1 + t_2, x, y) = I_1 + I_2,
\]
where
\[
I_1 = \int_0^{t_1} \int_{\mathbb{R}^d} g(s, x, z) g(t_1 + t_2 - s, z, y) |V(z)| \, dzds
\]
\[
= \int_0^{t_1} \int_{\mathbb{R}^d} g(s, x, z) g(t_1 - s, z, w) g(t_2, w, y) g(t_1, x, w) |V(z)| \, dwdzds
\]
\[
\leq \int_{\mathbb{R}^d} g(t_1, w, y) g(t_1, x, w) S(V, t_1, x, w) \, dw \leq f(t_1),
\]
and
\[
I_2 = \int_{t_1}^{t_1 + t_2} \int_{\mathbb{R}^d} g(s, x, z) g(t_1 + t_2 - s, z, y) |V(z)| \, dzds
\]
\[
= \int_{t_1}^{t_1 + t_2} \int_{\mathbb{R}^d} g(t_1, x, w) g(s - t_1, w, z) g(t_2 - (s - t_1), z, y) g(t_2, w, y) |V(z)| \, dwdzds
\]
\[
\leq \int_{\mathbb{R}^d} g(t_1, x, w) g(t_2, w, y) S(V, t_2, w, y) \, dw \leq f(t_2).
\]
This yields (15).

Lemma 2.2. For all \( t, h > 0 \) we have \( f(t) \leq F(h) + tf(h)/h \).

Proof. Let \( k \in \mathbb{N} \) be such that \((k - 1)h < t \leq kh\), and let \( \theta = t - (k - 1)h \). Then \( t = \theta + (k - 1)h \), and by (15) we get
\[
f(t) \leq f(\theta) + t f(h)/h \leq F(h) + tf(h)/h,
\]
since \( 0 < \theta \leq h \).

\[ \square \]

Corollary 2.3. \( F(t) \leq F(h) + tF(h)/h \) and \( F(2t) \leq 2F(t) \) for \( t, h > 0 \).

We may now prove Lemmas 1.1 and 1.2.
Proof of Lemma 1.2. If $V \leq 0$ then $0 \leq G \leq g$ is constructed in [29, p. 470], and the Duhamel formula follows from the finiteness of $S(V^-)$ and the discussion after [29, (3.3)]. Then the left-hand side of (5) follows from [29, pp. 467-468], or we can use [5, (41)], which results therein from Jensen’s inequality and the second displayed formula on page 252 of [5]. For general, i.e., for signed $V$, the kernel $G$ is constructed by applying the above procedure to $g$ and $-V^-$, and then perturbing the resulting kernel by $V^+$. The latter is done by means of the perturbation series, cf. [5, Lemma 2]; then the Duhamel formula is obtained without further conditions. We now prove the right hand side of (5), and without loss of generality we may assume that $V \geq 0$. For $0 < s < t$, $x, y \in \mathbb{R}^d$, we let $p_0(s, x, t, y) = g(t - s, x, y)$ and $p_n(s, x, t, y) = \int_s^t \int_{\mathbb{R}^d} p_{n-1}(s, x, u, z)V(z)p_0(u, z, t, y)dz du$, $n \in \mathbb{N}$. Let $Q : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ satisfy $Q(u, r) + Q(r, v) \leq Q(u, v)$. By [15, Theorem 1] (see also [6, Theorem 3]) if there is $0 < \eta < 1$ such that $p_1(s, x, t, y) \leq \left[ \eta + Q(s, t) \right] p_0(s, x, t, y)$, \hspace{1cm} (16) then

\[ \tilde{p}(s, x, t, y) := \sum_{n=0}^{\infty} p_n(s, x, t, y) \leq \left( \frac{1}{1 - \eta} \right)^{1 + \frac{Q(s, t)}{\eta}} p_0(s, x, t, y). \] \hspace{1cm} (17)

Corollary 2.3 and the assumptions of the lemma imply that (16) is satisfied with $\eta = F(h) < 1$ and $Q(s, t) = (t - s)F(h)/h$. Since $G(t, x, y) = \tilde{p}(0, x, t, y)$, the proof of (5) is complete (see also [5, (17)]).

Proof of Lemma 1.1. If (2) holds then Duhamel formula and nonnegativity of $G$ yield (4). Similarly, (1) implies (3). The reverse implications follow from (5).

As a consequence of Corollary 2.3 we also obtain the following result.

Corollary 2.4. Let $V \leq 0$ and $T > 0$. Then (2) holds if and only if

\[ Ce^{-ct}g(t, x, y) \leq G(t, x, y), \hspace{1cm} t > 0, x, y \in \mathbb{R}^d, \] \hspace{1cm} (18)

for some constants $C$ and $c$. In fact we can take

\[ \ln C = - \sup_{x, y \in \mathbb{R}^d} S(V, t, x, y) \quad \text{and} \quad c = \frac{1}{T} \sup_{x, y \in \mathbb{R}^d} S(V, T, x, y). \]

Proof. (18) implies (2) for every fixed $T > 0$. Conversely, if (2) holds for fixed $T > 0$, then by Lemmas 1.2 and 2.2 we have

\[ \frac{G(t, x, y)}{g(t, x, y)} \geq e^{-S(V, t, x, y)} \geq e^{-f(t)} \geq e^{-F(T)e^{-tf(T)/T}}. \]

We note in passing that the above proof shows that (2) is determined by the behavior of $\sup_{x, y \in \mathbb{R}^d} S(V, t, x, y)$ for small $t > 0$. We also see that (14) and thus (3) fail in dimensions $d = 1$ and $d = 2$, because then $\int_0^\infty g(s, x, z)ds \equiv \infty$, unless $V = 0$ a.e. From Lemma 1.1 and Remark 1.3 it follows that (1) fails for nontrivial $V \leq 0$ and for nontrivial $V \geq 0$ if $d = 1$ or 2.
3. Characterization of the Sharp Global Gaussian Estimates

In this section we prove our main result, i.e., Theorem 1.4. We start by using $N(V, t)$, (U) and (L), to estimate $S(V, t)$.

Lemma 3.1. Let $t > 0$. We have

$$
\int_0^{t/2} \int_{\mathbb{R}^d} e^{-|z-y+(\tau/t)(y-x)|^2/(4\tau)} \frac{dzd\tau}{\tau^{d/2}} |V(z)| \leq N(V, t, x, y), \quad x, y \in \mathbb{R}^d,
$$

and

$$
\sup_{x, y} N(V, t, x, y) \leq 2 \sup_{x, y} \int_0^{t/2} \int_{\mathbb{R}^d} e^{-|z-y+(\tau/t)(y-x)|^2/(4\tau)} \frac{dzd\tau}{\tau^{d/2}} |V(z)|.
$$

Proof. The first inequality follows by the definition of $N(V, t, x, y)$. For the proof of the second one we note that

$$
\int_t^{t/2} \int_{\mathbb{R}^d} e^{-|z-y+(\tau/t)(y-x)|^2/(4\tau)} \frac{(t-\tau)^{d/2}}{\tau^{d/2}} |V(z)| d\tau
= \int_0^{t/2} \int_{\mathbb{R}^d} e^{-|z-x+(\tau/t)(x-y)|^2/(4\tau)} \frac{dzd\tau}{\tau^{d/2}} |V(z)|. \quad \Box
$$

Lemma 3.2. We have

$$
\sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) \geq m_1 (4\pi)^{d/2} e_*(V, 0),
$$

and

$$
\sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) \leq 2 m_2 (4\pi)^{d/2} e_*(V, 0).
$$

Proof. By (L) and Lemma 3.1,

$$
\sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) \geq m_1 \sup_{t>0, x, y \in \mathbb{R}^d} \int_0^{t/4} \int_{\mathbb{R}^d} e^{-|z-y+(2\tau/t)(y-x)|^2/(4\tau)} \frac{dzd\tau}{\tau^{d/2}} |V(z)|
= m_1 \sup_{t>0, x, y \in \mathbb{R}^d} \int_0^{t/4} \int_{\mathbb{R}^d} e^{-|z-y+\tau w|^2/(4\tau)} \frac{dzd\tau}{\tau^{d/2}} |V(z)|
= m_1 \sup_{x, y, w \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(z-y, w) |V(z)| dz = m_1 (4\pi)^{d/2} e_*(V, 0).
$$

By (U) and Lemma 3.1,

$$
\sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) \leq m_2 \sup_{t>0, x, y \in \mathbb{R}^d} N(V, t, x, y)
\leq 2 m_2 \sup_{t>0, x, y \in \mathbb{R}^d} \int_0^{t/2} \int_{\mathbb{R}^d} e^{-|z-y+(\tau/t)(y-x)|^2/(4\tau)} \frac{dzd\tau}{\tau^{d/2}} |V(z)|
\leq 2 m_2 \sup_{t>0, x, y \in \mathbb{R}^d} \int_0^{t/2} \int_{\mathbb{R}^d} e^{-|z-y+\tau w|^2/(4\tau)} \frac{dzd\tau}{\tau^{d/2}} |V(z)|.
$$
\[ = 2m_2 \sup_{y,w \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(z - y, w)|V(z)|\,dz = 2m_2(4\pi)^{d/2}e_* (V, 0). \] 

**Proof of Theorem 1.4.** For \( x, w \in \mathbb{R}^d \) and \( \tau > 0 \) we have
\[
\frac{|x - \tau w|^2}{4\tau} = \frac{|x|^2}{4\tau} - \frac{x \cdot w}{2} + \frac{\tau|w|^2}{4}.
\]
We change the variables \( \tau = 4u/|w|^2 \) and use \([12, 8.432, \text{formula 6.}]\) to get
\[
J(x, w) = \int_{0}^{\infty} \tau^{-d/2} e^{-\frac{|x - \tau w|^2}{4\tau}}\,d\tau = 2e^{\frac{x \cdot w}{2}} \left( \frac{|x|}{|w|} \right)^{-d/2+1} \frac{x}{|w|} \left( \frac{|x|}{|w|} \right)^{d/2-3/2} K_{\nu} (\frac{|x||w|}{2}) K_{\nu} (\frac{|x||w|}{2}).
\]
Here, as usual, \( K_{\nu} \) is the modified Bessel function of the second kind. We claim that for each \( \nu \geq 1/2, \)
\[
K_{\nu} (z) \approx z^{-\nu} e^{-z}(1 + z)^{\nu-1/2}, \quad z > 0.
\]
Here \( \approx \) means that the ratio of both sides is bounded above and below by constants independent of \( z \). The comparison is obtained by putting \( x = 1 \) in \([12, 8.432, \text{formulas 9. and 8.}]\) and considering cases \( z \leq 1 \) and \( z > 1 \), correspondingly. Therefore,
\[
J(x, w) \approx e^{-|x||w| - x \cdot w}/2 \left( \frac{|x|}{|w|} \right)^{-d/2+1} \left( \frac{|x||w|}{2} \right)^{-d/2+1} \left( 1 + \frac{|x||w|}{2} \right)^{d/2-3/2} K_{\nu} (\frac{|x||w|}{2}),
\]
and so \( K (V) \approx e_* (V, 0) \). The result follows by Lemma 3.2. \( \square \)

**Proof of (9).** The left hand side inequality follows from the identity \( K (V, x, 0) = C_d^{-1}(-\Delta^{-1})|V|(x) \). If \( y = 0 \), then the upper bound trivially holds. For \( y \neq 0 \) we consider two domains of integration. We have
\[
\int_{|z - x||y| \leq 1} K(z - x, y)|V(z)|\,dz \leq 2^{(d-3)/2} \int_{|z - x||y| \leq 1} \frac{1}{|z - x|^{d-2}} |V(z)|\,dz \leq \frac{2^{(d-3)/2}}{C_d} |\Delta^{-1}|V||_{\infty}.
\]
Furthermore, by a change of variables and the H"older inequality,
\[
\int_{|z - x||y| \geq 1} K(z - x, y)|V(z)|\,dz \leq \int_{|z - x||y| \geq 1} e^{-\frac{1}{2}(|z - x||y| - (z - x)\cdot y)} |y|^{d-2} |V(z)|\,dz \leq 2^{(d-3)/2} \kappa_d \|V\|_{d/2},
\]
where
\[
\kappa_d = \left( \int_{|w| \geq 1} \left( e^{-\frac{1}{2}(|w| - w_1)|w|^{-\frac{d-1}{2}}} \right)^{d/(d-2)} \,dw \right)^{(d-2)/d} < \infty.
\]
We skip the proof of the finiteness of \( \kappa_d \); it can be found in the first version of the paper on \texttt{arXiv:1706.06172v1}. \( \square \)
Proof of Proposition 1.6. We first prove that $\|K(V)\|_{\infty} = \infty$. Let $y = (1, 0) \in \mathbb{R}^d$. For $z = (z_1, z_2) \in A$ we have
\[
0 \leq |z| |y| - z \cdot y = |z| - z_1 = \frac{|z_2|^2}{\sqrt{z_1^2 + |z_2|^2} + z_1} \leq \frac{z_1}{\sqrt{z_1^2 + |z_2|^2} + z_1} \leq 1
\]
and thus also $z_1 \leq |z| \leq 2z_1$. Then,
\[
\|K(V)\|_{\infty} \geq \int_{\mathbb{R}^d} e^{-\frac{1}{2}(|y| - z \cdot y)}|V(z)| \left(1 + \frac{|z| |y|}{|y|^2} \right) \frac{1}{|z|^{d-2}} dz \geq c \int_A \frac{1}{z_1} \frac{1}{|z_1|^2} dz
\]
\[
= c \int_{\mathbb{R}^d} \int_{|z_2| < \sqrt{z_1}} z_1^{1+2-d} \left|\frac{1}{z_1} - z_1 \right|^{\frac{d}{2}} dz_2 dz_1
\]
\[
= c \int_{\mathbb{R}^d} \int_{|z_2| < \sqrt{z_1}} z_1^{1+2-d} \left(z_1 + x_1 \right)^{\frac{d}{2}-\frac{3}{2}} dz_1 = c \left(1 + x_1 \right)^{\frac{d}{2}-\frac{3}{2}} \left(\int_{\mathbb{R}^d} \int_{|z_2| < \sqrt{z_1}} z_1^{1+2-d} \frac{1}{z_1} dz_2 dz_1\right)
\]
\[
\leq c' \int_{\mathbb{R}^d} \int_{|z_2| < \sqrt{z_1}} z_1^{1+2-d} \frac{1}{z_1} dz_2 dz_1 + c' \int_{\mathbb{R}^d} \int_{|z_2| < \sqrt{z_1}} z_1^{1+2-d} \frac{1}{z_1} dz_2 dz_1 \leq c'' < \infty.
\]
The second integral we consider is
\[
I_2 = \int_{\mathbb{R}^d} \int_{|z_2| < \sqrt{z_1}} z_1^{1+2-d} \frac{1}{z_1} dz_2 dz_1
\]
\[
\leq c \int_{\mathbb{R}^d} \int_{|z_2| < \sqrt{z_1}} z_1^{1+2-d} \frac{1}{z_1} dz_2 dz_1
\]
\[
= c \int_{\mathbb{R}^d} \int_{|z_2| < \sqrt{z_1}} z_1^{\frac{d}{2}-\frac{3}{2}} dz_1
\]
\[
= c \int_{\mathbb{R}^d} \int_{|z_2| < \sqrt{z_1}} z_1^{\frac{d}{2}-\frac{3}{2}} dw \leq c \int_{\mathbb{R}^d} \int_{|w| > \frac{1}{2}} \frac{1}{w^{d-2}} \frac{1}{z_1^{\frac{d}{2}-\frac{3}{2}}} dw \leq c' < \infty.
\]
The remaining integral is
\[
I_3 = \int_{x_1 - \sqrt{x_1}}^{x_1 + \sqrt{x_1}} \int_{|z_2| < \sqrt{x_1}} \frac{1}{(z_1 - x_1)^2 + |z_2|^2} \frac{1}{z_1} dz_2 \, dz_1
\]
\[
\leq 2 \int_{x_1 - \sqrt{x_1}}^{x_1 + \sqrt{x_1}} \int_{|z_2| < 2\sqrt{x_1}} \frac{1}{(z_1 - x_1)^2 + |z_2|^2} \frac{1}{z_1} dz_2 \, dz_1
\]
\[
\leq 2 \int_{B(0,3\sqrt{x_1})} \frac{1}{|z|^d - x_1} \, dz \leq c < \infty.
\]

To prove the second statement of Proposition 1.6, for \(s > 0\) we let \(d_s f(x) = sf(\sqrt{s}x)\). Note that the dilatation does not change the norms:
\[
\|\Delta^{-1}(d_s f)\|_\infty = \|\Delta^{-1}f\|_\infty, \quad \|K(d_s f)\|_\infty = \|K(f)\|_\infty.
\]
Furthermore, \(\text{supp}(d_s f) \subseteq B(0, r/\sqrt{s})\) if \(\text{supp}(f) \subseteq B(0, r), \ r > 0\). Since \(\|\Delta^{-1}V\|_\infty = C < \infty\) and \(\|K(V)\|_\infty = \infty\), therefore \(\|\Delta^{-1}(V 1_{B_r})\|_\infty \leq C\) for every \(r > 0\) and \(\|K(V 1_{B_r})\|_\infty \to \infty\) as \(r \to \infty\). For \(n \in \mathbb{N}\) we define
\[
V_n = d_{r_n}(V 1_{B_{r_n}}),
\]
where \(r_n\) is chosen such that \(\|K(V 1_{B_{r_n}})\|_\infty \geq 4^n\). Also, \(\text{supp}(V_n) \subseteq B(0, 1)\).

We define \(\tilde{V} = \sum_{n=1}^\infty V_n/2^n\). Then,
\[
\|K(\tilde{V})\|_\infty \geq \|K(V_n)\|_\infty/2^n \geq 2^n \to \infty,
\]
as \(n \to \infty\), and
\[
\|\Delta^{-1}\tilde{V}\|_\infty \leq \sum_{n=1}^\infty \|\Delta^{-1}V_n\|_\infty/2^n \leq C. \quad \square
\]

Similarly, (1) fails for \(-\varepsilon \tilde{V} \geq 0\) with any \(\varepsilon > 0\), cf. Remark 1.3.

4. Sufficient Conditions for the Sharp Gaussian Estimates

Recall from \([8, (2.5)]\) that for \(p \in [1, \infty)\),
\[
\|P_t f\|_\infty \leq C(d, p) t^{-d/(2p)} \|f\|_p, \quad t > 0,
\]
where \(P_t f(x) = \int_{\mathbb{R}^d} g(t, x, z) f(z) dz, \ f \in L^p(\mathbb{R}^d)\) and
\[
C(d, p) = \begin{cases} 
(4\pi)^{-d/2}, & \text{if } p = 1, \\
(4\pi)^{-d/(2p)} (1 - p^{-1})(1-p^{-1})^{-d/2}, & \text{if } p \in (1, \infty].
\end{cases}
\]
We will give an analogue for the bridges \(T_s^{t,y}\). Here \(t > 0, y \in \mathbb{R}^d\), and
\[
T_s^{t,y} f(x) = \int_{\mathbb{R}^d} \frac{g(s, x, z) g(t - s, z, y)}{g(t, x, y)} f(z) dz, \quad 0 < s < t, \quad x \in \mathbb{R}^d.
\]
Clearly,
\[
T_s^{t,y} f(x) = T_{t-s}^{t,x} f(y), \quad 0 < s < t, \quad x, y \in \mathbb{R}^d. \quad (19)
\]
By the Chapman–Kolmogorov equations (the semigroup property) for the kernel $g$, we have $T^{t,y}_s 1 = 1$. We also note that $S(V)$ is related to the potential (0-resolvent) operator of $T$ as follows,

$$S(V, t, x, y) = \int_0^t T^t_s |V|(x) \, ds.$$ 

**Lemma 4.1.** For $p \in [1, \infty]$ and $f \in L^p(\mathbb{R}^d)$ we have

$$\|T^{t,y}_s f\|_\infty \leq C(d, p) \left[ \frac{(t-s)s}{t} \right]^{-d/(2p)} \|f\|_p, \quad 0 < s < t, \ y \in \mathbb{R}^d.$$ 

**Proof.** We note that

$$\frac{g(s, x, z)g(t-s, z, y)}{(4\pi)^{-d/2}g(t, x, y)} = \left[ \frac{(t-s)s}{t} \right]^{-\frac{d}{4}} \exp \left[ -\frac{|z-x|^2}{4s} - \frac{|y-z|^2}{4(t-s)} + \frac{|y-x|^2}{4t} \right].$$

As in [27, (3.4)], we have

$$\frac{|z-x|^2}{4s} + \frac{|y-z|^2}{4(t-s)} \geq \frac{|y-x|^2}{4t}.$$ (20)

Indeed, (20) follows from the triangle and Cauchy-Schwarz inequalities:

$$|y-x| \leq \sqrt{s} \frac{|z-x|}{\sqrt{s}} + \sqrt{t-s} \frac{|y-z|}{\sqrt{t-s}} \leq \sqrt{t} \left( \frac{|z-x|^2}{s} + \frac{|y-z|^2}{t-s} \right)^{1/2}.$$ 

For $p = 1$, the assertion of the lemma results from (20). For $p \in (1, \infty)$, we let $p' = p/(p-1)$, apply Hölder’s inequality, the identity $g(s, x, z)^{p'} = g(s/p', x, z)(4\pi s)^{(1-p')d/2}(p')^{-d/2}$, and the semigroup property, to get

$$|T^{t,y}_s f(x)| \leq g(t, x, y)^{-1} \left[ \int_{\mathbb{R}^d} g(s, x, z)^{p'} g(t-s, z, y)^{p'} \, dz \right]^{1/p'} \|f\|_p$$

$$= g(t, x, y)^{-1} \left[ (4\pi)^{(1-p')d} (p')^{-d} [s(t-s)]^{(1-p')d/2} \right]$$

$$\times \int_{\mathbb{R}^d} g(s/p', x, z)g((t-s)/p', z, y) \, dz \right]^{1/p'} \|f\|_p$$

$$= C(d, p) \left[ \frac{s(t-s)}{t} \right]^{-d/(2p)} \|f\|_p.$$ 

For $p = \infty$, the assertion follows from the identity $T^{t,y}_s 1 = 1$. \hfill \Box

Zhang [29, Proposition 2.1] showed that (1) and (2) hold for $V$ in specific $L^p$ spaces (see also [29, Theorem 1.1 and Remark 1.1]). In Proposition 4.2 and Corollary 4.3 below we prove his result by a different method.

**Proposition 4.2.** Let $V : \mathbb{R}^d \to \mathbb{R}$ and $p, q \in [1, \infty]$.

(a) If $V \in L^p(\mathbb{R}^d)$, $p > d/2$ and $c = C(d, p) \frac{\Gamma(1-d/(2p))}{\Gamma(2-d/p)}$, then

$$\sup_{x, y \in \mathbb{R}^d} S(V, t, x, y) \leq ct^{1-d/(2p)}, \quad t > 0.$$ 

(b) If $V \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and $q < d/2 < p$, then (3) holds.
Proof. Part (a) follows from Lemma 4.1, so we proceed to (b). For \( t > 2 \),
\[
\int_0^t T_s^{t,y} |V|(x) \, ds = \int_0^{t/2} T_s^{t,y} |V|(x) \, ds + \int_0^{t/2} T_s^{t,x} |V|(y) \, ds.
\]
(21)
By Lemma 4.1, the first term of the sum can be estimated as follows:
\[
\int_0^{t/2} T_s^{t,y} |V|(x) \, ds \leq c \|V\|_p \int^1_0 \left[ (t-s) s \right]^{-d/(2p)} \, ds + c \|V\|_q \int^{t/2}_0 \left[ (t-s) s \right]^{-d/(2q)} \, ds.
\]
(22)
By (19), the second term has the same bound. For \( t \in (0, 2] \) we use (a). \( \square \)

Lemmas 1.1 and 1.2 provide the following conclusion:

**Corollary 4.3.** Under the assumptions of Proposition 4.2(a), \( G \) satisfies the sharp local Gaussian bounds (2). If \( V \leq 0 \) and the assumptions of Proposition 4.2(b) hold, then \( G \) has the sharp global Gaussian bounds (1).

Recall from Sect. 1 that (1) holds for \( d = 3 \) if \( \| \Delta^{-1} V^- \|_\infty < \infty \) and \( \| \Delta^{-1} V^+ \|_\infty < 1 \), and it holds for \( d \geq 4 \) if \( \| \Delta^{-1} V^- \|_\infty + \| V^- \|_{d/2} < \infty \), \( \| \Delta^{-1} V^+ \|_\infty \) < 1 and \( \| V^+ \|_{d/2} \) is small enough. This yields another proof of the second statement of Corollary 4.3, because of the following observation:

**Lemma 4.4.** The assumptions of Proposition 4.2(b) necessitate that \( d \geq 3 \), \( \| \Delta^{-1} V^- \|_\infty < \infty \) and \( \| V^- \|_{d/2} < \infty \).

Proof. Plainly, the assumptions of Proposition 4.2(b) imply \( d > 2 \) and \( V \in L^{d/2}(\mathbb{R}^d) \). We now verify that \( \| \Delta^{-1} V \|_\infty < \infty \). By Hölder’s inequality,
\[
\sup_{x \in \mathbb{R}^d} \int_{B(0,1)} \frac{|V(z+x)|}{|z|^{d-2}} \, dz \leq \| |z|^{2-d} 1_{B(0,1)}(z) \|_{p'} \| V \|_p < \infty,
\]
\[
\sup_{x \in \mathbb{R}^d} \int_{B(0,1)} \frac{|V(z+x)|}{|z|^{d-2}} \, dz \leq \| |z|^{2-d} 1_{B^c(0,1)}(z) \|_{q'} \| V \|_q < \infty,
\]
where \( p', q' \) are the exponents conjugate to \( p, q \), respectively. \( \square \)

In what follows, we develop sufficient conditions for (1) and (2). Let \( d_1, d_2 \in \mathbb{N} \) and \( d = d_1 + d_2 \).

**Remark 4.5.** The Gauss–Weierstrass kernel \( g(t, x) \) in \( \mathbb{R}^d \) can be represented as a tensor product:
\[
g(t, x) = (4\pi t)^{-d_1/2} e^{-|x_1|^2/(4t)} (4\pi t)^{-d_2/2} e^{-|x_2|^2/(4t)},
\]
where \( x_1 \in \mathbb{R}^{d_1}, x_2 \in \mathbb{R}^{d_2} \) and \( x = (x_1, x_2) \). The kernels of the bridges factorize accordingly:
\[
g(s, x, z) g(t-s, z, y) g(t, x, y) = (4\pi s)^{-d_1/2} e^{-|z_1-x_1|^2/(4s)} (4\pi (t-s))^{-d_1/2} e^{-|y_1-z_1|^2/(4(t-s))}
\]
\[
\times (4\pi s)^{-d_2/2} e^{-|z_2-x_2|^2/(4s)} (4\pi (t-s))^{-d_2/2} e^{-|y_2-z_2|^2/(4(t-s))}.
\]
Corollary 4.6. Let $V_1: \mathbb{R}^d_1 \to \mathbb{R}$, $V_2: \mathbb{R}^d_2 \to \mathbb{R}$, and $V(x) = V_1(x_1)V_2(x_2)$, where $x = (x_1, x_2) \in \mathbb{R}^d$, $x_1 \in \mathbb{R}^d_1$ and $x_2 \in \mathbb{R}^d_2$. Assume that $V_1 \in L^\infty(\mathbb{R}^d_1)$ and $\sup_{t>0} \sup_{x_2, y_2 \in \mathbb{R}^d_2} S(V_2, t, x_2, y_2) < \infty$. Then (3) holds.

Proof. In estimating $S(V, t, x, y)$ we first use the factorization of the bridges and the boundedness of $V_1$, and then the Chapman–Kolmogorov equations and the boundedness of $S(V_2)$. $\square$

Lemma 4.7. For $f(x_1, x_2) = f_1(x_1)f_2(x_2) \in L^{p_1}(\mathbb{R}^d_1) \times L^{p_2}(\mathbb{R}^d_2)$, $0 < s < t$ and $y \in \mathbb{R}^d$, we have

$$\|T_{s,y}^t f\|_\infty \leq C(d_1, p_1) C(d_2, p_2) \left[\frac{(t-s)s}{t}\right]^{-d_1/(2p_1) - d_2/(2p_2)} \|f_1\|_{p_1} \|f_2\|_{p_2}.$$ 

Proof. We proceed as in the proof of Lemma 4.1, using Remark 4.5. $\square$

Proof of Theorem 1.8. We follow the proof of Proposition 4.2, replacing Lemma 4.1 by Lemma 4.7.

We note in passing that Theorem 1.8 is an extension of Proposition 4.2.

5. Examples

Let $1_A$ denote the indicator function of $A$. In what follows, $G$ in (1) is the Schrödinger perturbation of $g$ by $V$.

Example 5.1. Let $d \geq 3$ and $1 < p < \infty$. For $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}^{d-1}$ we let $V(x_1, x_2) = -|x_1|^{-1/p} 1_{|x_1|<1} 1_{|x_2|<1}$. Then (1) holds but $V \notin L_{loc}^p(\mathbb{R}^d)$.

Indeed, $V(x_1, x_2) = V_1(x_1)V_2(x_2)$, where

$$V_1(x_1) = -|x_1|^{-1/p} 1_{|x_1|<1}, \quad x_1 \in \mathbb{R},$$

$$V_2(x_2) = 1_{|x_2|<1}, \quad x_2 \in \mathbb{R}^{d-1}.$$ 

Let

$$1 \leq q < p_1 < r < p,$$

and

$$p_2 = \frac{d-1}{2} \frac{p_1}{p_1 - 1/2}.$$ 

Since $d \geq 3$, $p_2 > 1$. In the notation of Theorem 1.8 we have $d_1 = 1$, $d_2 = d-1$, and indeed $d_1/(2p_1) + d_2/(2p_2) = 1$. Since $V_1 \in L^r(\mathbb{R}) \cap L^q(\mathbb{R})$ and $V_2 \in L^{p_2}(\mathbb{R}^{d-1})$, the assumptions of Theorem 1.8(b) are satisfied, and (1) follows by Corollary 1.9. Clearly, $V \notin L_{loc}^p(\mathbb{R}^d)$.

Example 5.2. For $d \geq 3$, $n = 2, 3, \ldots$, let $V_n(x) = |x_1|^{-1+1/n} 1_{|x_1|<1} 1_{|x_2|<1}$, where $x = (x_1, x_2)$, $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}^{d-1}$. Let $a_n = \sup_{t>0} \sup_{x, y \in \mathbb{R}^d} S(V_n, t, x, y)$,

$$V(x) = -\sum_{n=2}^\infty \frac{1}{n^2} \frac{V_n(x)}{a_n}, \quad x \in \mathbb{R}^d.$$ 

Then (1) holds but $V \notin \bigcup_{p>1} L_{loc}^p(\mathbb{R}^d)$. 
Indeed, $0 < a_n < \infty$ by Example 5.1, and so
\[
\sup_{t > 0, x, y \in \mathbb{R}^d} S(V, t, x, y) \leq \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty.
\]
This yields the global sharp Gaussian bounds. Fix $p > 1$. Since the function $x_1 \to |x_1|^{-1+1/n}$ is not in $L^p(-1, 1)$ for large $n$, we get that $V \notin L^p(B(0, 1))$.

**Example 5.3.** Let $d \geq 3$ and $V(x_1, x_2) = \frac{-1}{(|x_2|+1)^2}$ for $x_1 \in \mathbb{R}^{d-3}$, $x_2 \in \mathbb{R}^3$. Then (1) holds but $V \notin L^1(\mathbb{R}^d)$.

Indeed, $V \notin L^1(\mathbb{R}^d)$. We let $V_2(x_2) = \frac{-1}{(|x_2|+1)^2}$, $x_2 \in \mathbb{R}^3$. By the symmetric rearrangement inequality [16, Chapter 3], in dimension $d = 3$ we have
\[
0 \leq \Delta^{-1} V_2 \leq C_3 \int_{\mathbb{R}^3} \frac{1}{|z|(|z|+1)^3} \, dz < \infty.
\]
By (7) and (8),
\[
\sup_{t > 0, x_2, y_2 \in \mathbb{R}^3} S(V_2, t, x_2, y_2) < \infty.
\]
By Corollary 4.6 and Lemma 1.1 we see that (1) holds for $V$.

**Proof of Proposition 1.10.** Add the functions from Examples 5.2 and 5.3. □

We can have nonnegative examples, too. Namely, let $V \leq 0$ be as in Proposition 1.10. Then $M = \sup_{t > 0, x, y \in \mathbb{R}^d} S(V, t, x, y) < \infty$. We let $\tilde{V} = |V|/(M + 1)$. Then $\tilde{V} \geq 0$, $\tilde{V} \notin L^1(\mathbb{R}^d) \cup \bigcup_{p > 1} L^p_{loc}(\mathbb{R}^d)$ and
\[
\sup_{t > 0, x, y \in \mathbb{R}^d} S(\tilde{V}, t, x, y) = M/(M + 1) < 1.
\]
Therefore (5) holds for $\tilde{V}$ with $h = \infty$ and $\eta = M/(M + 1)$, which yields (1).

Let $d_1, d_2 \in \mathbb{N}$, $d = d_1 + d_2$, $V_1 : \mathbb{R}^{d_1} \to \mathbb{R}$, $V_2 : \mathbb{R}^{d_2} \to \mathbb{R}$, and $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$, where $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$. Let $G_1(t, x_1, y_1)$, $G_2(t, x_2, y_2)$ be the Schrödinger perturbations of the Gauss–Weierstrass kernels on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ by $V_1$ and $V_2$, respectively. Then $G(t, (x_1, x_2), (y_1, y_2)) := G_1(t, x_1, y_1)$ $G_2(t, x_2, y_2)$ is the Schrödinger perturbation of the Gauss–Weierstrass kernel on $\mathbb{R}^d$ by $V$. Clearly, if the sharp Gaussian estimates hold for $G_1$ and $G_2$, then they hold for $G$. Our next example shows that the situation is quite different for tensor products.

**Example 5.4.** Let $V(x_1, x_2) = -V_1(x_1)V_2(x_2)$, where $x_1, x_2 \in \mathbb{R}^3$,
\[
V_1(x) = V_2(x) = -\frac{1 - \varepsilon}{2} |x|^{-1-\varepsilon} 1_{|x| < 1},
\]
and $\varepsilon \in [0, 1)$. Then the heat kernels in $\mathbb{R}^3$ of $\Delta + V_1$ and $\Delta + V_2$ satisfy (1) and (2), but that of $\Delta + V$ in $\mathbb{R}^6$ satisfies neither (1) nor (2).

Indeed, by the symmetric rearrangement inequality [16, Chapter 3],
\[
0 \leq -\Delta^{-1} V_1(x) \leq -\Delta^{-1} V_1(0) = \frac{1 - \varepsilon}{8\pi} \int_{\{z \in \mathbb{R}^3 : |z| < 1\}} \frac{1}{|z|} |z|^{-1-\varepsilon} \, dz = 1/2,
\]
for all $x \in \mathbb{R}^3$. Thus, $\|\Delta^{-1}V_1\|_\infty = \|\Delta^{-1}V_2\|_\infty < \infty$. Using the comment following (8), we get (1) for the heat kernels in $\mathbb{R}^3$ of $\Delta + V_1$ and $\Delta + V_2$. However, the heat kernel in $\mathbb{R}^6$ of $\Delta + V$ fails even (2). Indeed, if we let $T \leq 1$, $a \in \mathbb{R}^6$, $|a| = 1$, and $c = \int_0^1 p(s, 0, a)ds$, then by [9, Lemma 3.5],

$$\int_0^T \int_{\mathbb{R}^6} g(s, 0, x)|V(x)| \, dx \, ds \geq \int_{\{x \in \mathbb{R}^6 : |x|^2 \leq T\}} \int_0^T g(s, 0, x)|V(x)| \, dx \, ds \geq c \int_{\{x \in \mathbb{R}^6 : |x|^2 \leq T\}} \frac{1}{|x|^4} |V(x)| \, dx \geq c \int_{\{x_1 \in \mathbb{R}^3 : |x_1|^2 < T/2\}} |V_1(x_1)| \int_{\{x_2 \in \mathbb{R}^3 : |x_2|^2 < T/2\}} \frac{|V_2(x_2)|}{(|x_1|^2 + |x_2|^2)^2} \, dx_2 \, dx_1 \geq \frac{c(1 - \varepsilon)}{2} \int_{\{x_1 \in \mathbb{R}^3 : |x_1|^2 < T/2\}} |V_1(x_1)| \int_{\{x_2 \in \mathbb{R}^3 : |x_2|^2 < T/2\}} \frac{|x_2|^{-1}}{(|x_1|^2 + |x_2|^2)^2} \, dx_2 \, dx_1 = \frac{c(1 - \varepsilon)}{2} \int_{\{x_1 \in \mathbb{R}^3 : |x_1|^2 < T/2\}} |V_1(x_1)| \frac{\pi T}{|x_1|^2(T/2 + |x|^2)} \, dx_1 $$

$$= \pi^2 c T(1 - \varepsilon)^2 \int_0^{\sqrt{T/2}} \frac{r^{-1-\varepsilon}}{T/2 + r^2} \, dr = \infty.$$

Therefore by Lemma 2.1, (4) fails, and so does (2), according to Lemma 1.1. Thus, the sharp Gaussian estimates may hold for the Schrödinger perturbations of the Gauss–Weierstrass kernels by $V_1$ and $V_2$ but fail for the Schrödinger perturbation of the Gauss–Weierstrass kernel by $V(x_1, x_2) = -V_1(x_1)V_2(x_2)$.

In passing we note that the functions $-V_1$, $-V_2$ and $-V$ give a similar counterexample with nonnegative factors, because $1/2 < 1$, cf. (12). Let us also remark that the sharp global Gaussian estimates may hold for $V(x_1, x_2) = V_1(x_1)V_2(x_2)$ but fail for $V_1$ or $V_2$. Indeed, it suffices to consider $V_1(x_1) = -1_{|x_1| < 1}$ on $\mathbb{R}^3$ and $V_2 \equiv 1$ on $\mathbb{R}$, and to apply Theorem 1.8. We see that it is indeed the combined effect of the factors $V_1$ and $V_2$ that matters—as captured in Sect. 4.

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