Competition between the Mott transition and the Anderson localization in 1D disordered interacting electron systems

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(August 18, 2021)

The competition between the Mott transition and the Anderson localization in one dimensional electron systems is studied based upon the bosonization and the renormalization group method. The beta function is calculated up to the second order in the strength of diagonal disorder by using a replica trick. It is found that the sufficiently strong forward scattering by random impurities destroys the Mott-Hubbard gap, and the backward scattering gives rise to the Anderson localization for the resulting gapless state. On the other hand, if the Umklapp interaction is strong enough, the Mott insulating state still overwhelms the Anderson localization.

There has been continuous interest in the studies of correlated electron systems in the presence of disorder [1–4]. Although in the case of weak localization the effect of electron-electron interaction has been extensively studied in the Hartree-Fock approximation, it is not well understood how the strongly correlated electron systems such as the Hubbard model are affected by the presence of disorder [5,6]. In particular, the effect of disorder on electron systems with the excitation gap, such as the Mott insulator, has received considerable attention recently.

In one-dimensional (1D) electron systems, some elaborate techniques like bosonization and renormalization group were successfully applied to disordered systems and revealed the role of electron-electron interaction for the localization-delocalization transition [7–10]. In the previous studies, however, the Umklapp interaction which is essential to cause the Mott transition was not taken into account. To investigate the disorder effect on the Mott insulating phase, it is crucial to study how the Umklapp interaction should affect low-energy properties.

Motivated by this, we investigate 1D disordered interacting electron systems putting emphasis on the competition between the Mott transition and the Anderson localization. We clarify the interplay between the disorder and the Umklapp interaction by using the bosonization and the renormalization group method. It is found that the Mott-Hubbard gap collapses and a gapless metallic state appears when the random forward scattering due to impurities is strong enough. The impurity backward scattering then drives this metallic state to the Anderson localized state. We should mention here a recent numerical work for the 1D Hubbard model with disorder, which indicates the transition from the Mott insulator to the Anderson localization [11]. We note that our field-theoretic approach not only provides the knowledge complementary to the above numerical work, but also reveals a new feature in the role played by the forward and backward impurity scatterings.

We start by introducing the effective Hamiltonian to describe 1D interacting electron systems with random potentials. Applying standard bosonization rules, we write down the effective Hamiltonian in field theory limit [11],

\[
H = H_c + H_s + H_{dis}
\]

\[
H_c = \int dx \left[ \frac{v_c}{2K_c} \left( \frac{\partial}{\partial x} \phi_c(x) \right)^2 + \frac{v_c K_c}{2} \left( \Pi_c(x) \right)^2 \right]
\]

\[
+ \frac{U}{\alpha^2} \int dx \cos(\sqrt{8\pi} \phi_c(x) + \delta x)
\]

\[
H_s = \int dx \frac{2\pi v_s}{3} [J_L(x) \cdot J_L(x) + J_R(x) \cdot J_R(x)]
\]

\[
+ \lambda \int dx J_L(x) \cdot J_R(x)
\]

\[
H_{dis} = \sqrt{\frac{2}{\pi}} \int dx \eta(x) \partial_x \phi_c(x)
\]

\[
+ \frac{1}{\alpha} \int dx \{ \xi(x) e^{i(\sqrt{8\pi} \phi_c(x) + 2k_F x)} \text{tr}(g(x)) + h.c. \}.
\]

Here \( \phi_c \) is a boson phase field for the charge degrees of freedom and \( \Pi_c \) is its canonical conjugate momentum field, and \( \delta \equiv 4k_F - 2\pi \). The \( U \)-term in eq.(2) represents the Umklapp interaction, which may cause the Mott transition. For the spin degrees of freedom, we have used non-abelian bosonization [12,13] to preserve SU(2) symmetry: \( J_{L(R)} \) is the left(right)-going current operator of level-1 SU(2) Kac-Moody algebra, and \( g(x) \) is a fundamental representation of SU(2) Lie algebra. The \( \lambda \)-term in eq.(3) is a marginally irrelevant interaction which arises from SU(2)×SU(2) symmetry. We have introduced real and complex random fields \( \eta(x) \) and \( \xi(x) \) for forward and backward scatterings by impurities,
respectively, which obey the gaussian distribution law,
\( \langle \eta(x)\eta(x') \rangle = D_\eta \delta(x-x') \),
\( \langle \xi(x)\xi(x') \rangle = D_\xi \delta(x-x') \),
and \( \langle \eta(x) \rangle = \langle \xi(x) \rangle = 0 \). In addition to the above interactions, we should consider the following term
\[ H_{rg} = -\sqrt{\frac{2}{\pi}} \int dx A(x) \partial_x \theta_c, \]  
(5)
where \( \partial_x \theta_c \equiv \Pi_c \) and \( A(x) \) is a random gauge field with the gaussian distribution,
\( \langle A(x) A(x') \rangle = D_A \delta(x-x') \),
\( \langle A(x) \rangle = 0 \). As will be seen later, this term is generated in the process of renormalization due to the backward scattering by random potentials. We note that apart from the \( U \)-term, the charge part of the Hamiltonian is similar to that of the 2D random phase sine-Gordon model [4] for which randomness is introduced for the 2D plane in contrast to the present 1D system.

We now consider the quenched disorder with the use of a replica trick. Introducing \( n \) species of replicas and integrating over the random variables [10], we have the following effective action arising from disorder,
\[ S_{dis} = -\frac{2DA}{\pi} \int dx \int dt \int d\tau' \sum_{i,j} \partial_x \phi^i_c(x, \tau) \partial_x \phi^j_c(x, \tau') \]
\[ -\frac{2DA}{\pi} \int dx \int dt \int d\tau' \sum_{i,j} \frac{1}{v_c^2} \partial_x \phi^i_c(x, \tau) \partial_x \phi^j_c(x, \tau') \]
\[ -\frac{D_\xi}{\alpha^2} \int dx \int dt \int d\tau' \sum_{i,j} \text{tr}(g^i(x, \tau) \text{tr}(g^j(x, \tau'))) \]
\[ \times \cos \sqrt{2\pi}(\phi^i_c(x, \tau) - \phi^j_c(x, \tau')), \]  
(6)
where \( i, j \) are replica indices. In the following argument, we calculate the beta functions up to the second order in \( D_{\xi, \eta, A} \). To this end, it is quite useful to exploit the operator product expansions for the \( U(1) \) gaussian model [10].

\[ e^{i\alpha \phi(x, \tau)} e^{-i\alpha \phi(0,0)} \sim \left( \frac{\partial_x \phi(0,0)}{z} + \frac{\partial_z \phi(0,0)}{\bar{z}} \right) \]
\[ + \frac{1}{|z|^2} \frac{\alpha^2}{\zeta^2 K_c/2\pi} \left( \frac{\partial_x \phi(0,0) \partial_z \phi(0,0)}{z} \right) \]
\[ - \frac{\alpha^2}{2|z|^2 K_c/2\pi} \left( \frac{z^2 (\partial_x \phi)^2 + \bar{z}^2 (\partial_z \phi)^2}{\partial_x \phi(0,0) \partial_z \phi(0,0)} \right) + \cdots, \]
\[ \partial_x \phi(x, \tau) e^{i\alpha \phi(0,0)} \sim \left( \frac{i\alpha K_c}{8\pi \zeta^2} e^{i\alpha \phi(0,0)} \right) + \cdots, \]
(7)
with \( z = x + iv_s \tau, \bar{z} = x - iv_s \tau \), and also those for the level-1 SU(2) Wess-Zumino-Witten model [17][19].

\[ J^0_L(x, \tau) J^0_L(0,0) \sim \frac{\delta_{ab}}{z^2} + \frac{\varepsilon_{abc}}{z} J^a_L(0,0) + \cdots, \]
\[ J^b_L(x, \tau) g(0,0) \sim \frac{\varepsilon^{bac}}{z} g(0,0) + \cdots, \]
\[ \text{tr}(g(x, \tau)) \text{tr}(g(0,0)) \sim |z| J^0_L(0,0) \cdot J^0_R(0,0) + \frac{1}{|z|} \]
\[ + \frac{1}{|z|} (z^2 J^0_L(0,0) \cdot J^0_L(0,0) + \cdots, \]
\[ + z^2 J^0_R(0,0) \cdot J^0_R(0,0) + \cdots, \]
(11)
with \( z = x + iv_s \tau, \bar{z} = x - iv_s \tau \), and \( t^a \), the generator of the SU(2) Lie algebra. By expanding the action in terms of the interactions and using the above operator product expansions, we end up with the following scaling equations up to the second order in the strength of randomness and the lowest order in \( U \) and \( \lambda \) after taking the replica limit \( n \to 0 \),

\[ \frac{dD_\xi}{dl} = (2 - K_c - 3\lambda) \tilde{D}_\xi, \]
\[ \frac{dD_\eta}{dl} = \tilde{D}_\eta + 4\pi^2 g(u) \tilde{D}_\xi, \]
\[ \frac{dD_A}{dl} = \tilde{D}_A + 4\pi^2 g(u) \tilde{D}_\xi, \]
\[ \frac{d\tilde{D}_\xi}{dl} = (2 - 2K_c) \tilde{U} - \frac{4D_\eta K_c^2}{\pi^2} \tilde{U}, \]
\[ \frac{d\tilde{D}_\eta}{dl} = -\frac{\tilde{D}_\xi}{2} - \tilde{D}_\xi, \]
\[ \frac{dK_c}{dl} = -2\pi K_c^2 \tilde{U} J_0(\delta \alpha) - \frac{K_c^2 \tilde{D}_\xi}{2u}, \]
\[ \frac{dv_s}{dl} = -\frac{\pi K_c \tilde{D}_\xi}{2u}, \]
\[ \frac{du}{dl} = -u \tilde{D}_\xi, \]
(12-19)
where \( \tilde{D}_{\xi,\eta,A} = D_{\xi,\eta,A}/v_c^2, \tilde{U} = U/v_c, \tilde{\lambda} = \lambda/v_s, u = v_s/v_c \),

\[ g(u) = \left( \int_{-\infty}^{\infty} \frac{dy}{\sqrt{1 + u^2 y^2 (1 + y^2 K_c/2\pi)^2}} \right)^2, \]
(20)
and \( J_0(x) \) is the Bessel function. We see from eq. (14) that \( D_A \)-term is indeed generated by the second order contribution of \( D_\xi \)-term, even when \( D_A \) is initially equal to zero as mentioned before. We can see that \( D_A \) does not couple to the Umklapp term, so that the random gauge field may not cause any essential change in the Mott insulating phase. The effect of the random gauge field manifests itself in the metallic phase away from half-filling, as will be discussed later again. Up to the first order in \( D_\xi \) and the zeroth order in \( D_\eta \) and \( D_A \), these scaling equations coincide with those obtained by Giamarchi and Schulz except that they did not take into account the Umklapp interaction [10][21].

In what follows, we restrict our arguments to the case of half-filling. We then put \( \delta = 0 \) and \( J_0(0) = 1 \) in eq. (17). In the absence of randomness, the Umklapp interaction is relevant, resulting in the low-energy fixed point of the Mott insulator with the charge excitation gap. Let us first study the effects of the random forward scattering \( \tilde{D}_\eta \) on this Mott insulating phase. For
a while we neglect the backward scattering, by putting $D_\xi = 0$ and $D_A = 0$ ($D_A$-term is generated by $D_\xi$-term). Then $\lambda$ in (16) is scaled to zero after the renormalization, preserving spin excitations still massless (the interplay between $\lambda$ and $D_\xi$ will be discussed later). Also, from eq.(13), we have $D_\eta(l) = D_\eta(0)e^l$. Substituting this expression into eq.(13), and solving eqs. (15) and (17) numerically, we obtain the renormalization flow for $\hat{U}$ and $D_\eta$ as shown in Fig. 1. We find that for sufficiently large values of $\hat{D}_\eta(0)$, the Umklapp term, $\hat{U}(l)$, becomes irrelevant and thus the Mott-Hubbard gap disappears, resulting in massless charge excitations. Note, however, that this massless state is not a conventional Tomonaga-Luttinger liquid but is a disordered metallic state for which some spatial correlation functions show exponential decay, as discussed in [10]. On the other hand, if $\hat{U}(0)$ is sufficiently large compared to $\hat{D}_\eta(0)$, we can see that $\hat{U}(l)$ is scaled to a strong-coupling value as $l \to +\infty$, and thus the Mott-Hubbard gap should still persist. The transition between the Mott insulator and the gapless metallic phase is of Kosterlitz-Thouless type [21]. Also, the numerical results indicate that at the critical point the magnitude of the Mott-Hubbard gap is roughly proportional to the inverse of the correlation length, $1/\ln(D_{\eta}^{-1}(0))$.

Having noticed that the strong forward scattering can drive the Mott insulator to a disordered metallic state, let us now discuss the effects of the backward scattering $D_\xi$ which may bring about the Anderson localization. The scaling equation (12) implies that for the repulsive $U$ ($K_c < 1$), $D_\xi$ is renormalized to a larger value even if its initial value is small, and hence the backward scattering becomes relevant. Therefore, if there is no intermediate fixed point between the weak-coupling and strong-coupling regimes, the low-energy fixed point is classified by $\hat{U}_* = 0$ and $\hat{D}_\xi \to +\infty$, or by $\hat{U}_* \to \infty$ and $\hat{D}_\xi \to +\infty$. In both cases, the fixed-point is identified with the insulator, since the Drude weight $D = v_cK_c$ is scaled to zero by $\hat{U}$ or $\hat{D}_\xi$. In order to see whether this is the Mott insulating state or the Anderson localized state, we examine the behavior of the compressibility. The compressibility is given by $K_c/v_c$, which satisfies,

$$
\frac{d}{dt} \left( \frac{K_c}{v_c} \right) = -2\pi \hat{U}^2 K_c/v_c, \quad (21)
$$

as easily seen from eqs.(17) and (18). Thus if $\hat{U}$ is irrelevant, the compressibility takes a finite value, resulting in the fixed point of the Anderson localization. On the other hand, if $\hat{U}$ is relevant, the compressibility vanishes at the low-energy fixed point, characterizing the Mott insulator. Therefore when the random forward scattering $D_\eta(0)$ is in the region where $\hat{U} \to 0$ in Fig. 1, the backward scattering $\hat{D}_\xi(0)$ drives the system to the Anderson localized state, whereas for sufficiently large values of $\hat{U}(0)$ compared to $\hat{D}_\eta(0)$, the Mott insulating state still persists. Therefore, we end up with the conclusion that there can occur the transition driven by disorder from the Mott insulator to the Anderson insulator for 1D disordered electron systems, although the Mott insulator still overwhelms the Anderson localization if the Umklapp interaction $\hat{U}(0)$ is large enough. This is consistent with the numerical results for the disordered Hubbard model [7].

The above characteristic behavior found for correlated electron systems shows sharp contrast to the results for 1D spinless fermion systems [24]. A remarkable point for the spinless fermion model is that the Mott transition is always accompanied by the order of $2k_F$-charge density wave (CDW). Since this state can be mapped to the Ising ordered state, one can apply Imry-Ma’s statement [23] that infinitesimally small disorder destroys this Ising ordered state. This indeed leads to the conclusion that the Mott insulator for the spinless fermion model is changed to the Anderson localized state even if infinitesimally small disorder is introduced [22]. Obviously, this argument cannot be directly applied to the Mott insulator for the present electron systems, which is not accompanied by the CDW order.

The difference between the spinless model and the electron model can also be seen in the scaling equations (13) and (14): for the spinless model $g(u)$ is replaced by $(\Gamma(K_c - 1/2)/\Gamma(K_c))^2$, where $\Gamma(x)$ is the gamma function. While $g(u)$ is always finite for the electron model, it diverges for $K_c \leq 1/2$ for the spinless fermion model, and the renormalization group procedure breaks down. Thus as $K_c$ approaches this singular point, $D_\eta$ becomes large, and at last the right-hand side of eq.(15) becomes negative, even for infinitesimally small $D_\xi(0)$. Note that the Mott transition for the pure spinless model occurs at $K_c = 1/2$. As a result, the Umklapp term $\hat{U}$ becomes always irrelevant and the Mott gap closes for a disordered spinless model. The above comparison with the spinless fermion model naturally leads us to claim that the stability of the Mott insulator for electron systems against weak diagonal disorder may be closely related to the absence of the long-range order.

So far, we have been concerned with the charge degrees of freedom. Here we briefly mention the effect of disorder on the spin degrees of freedom. The scaling equation (10) indicates that the backward scattering $D_\xi$ may renormalize $\hat{\lambda}$ to a negative large value, hence freezing the spin degrees of freedom. This is essentially the same as that observed for the case away from half filling [10]. Therefore, the Mott-insulating phase with frozen spins may be realized if the spin degrees of freedom is frozen after the charge-gap formation due to the Umklapp term (the large-$U$ region in Fig.1). Also, for the small-$U$ region, the Anderson localized state with frozen spins may be expected.

Finally some comments are in order for the random
gauge field $A(x)$ generated by the impurity backward scattering. Although this term does not cause a serious change in the half-filled Mott insulator, it plays an important role in a delocalized phase. To see this clearly, we consider the system away from half-filling. We can then put $\tilde{U} = 0$ in the scaling equations because the Umklapp interaction is irrelevant. We see from eq. (22) that if the electron-electron interaction is strongly attractive and the condition, $2 - K_\eta - 3\lambda < 0$, is satisfied, the backward scattering $\tilde{D}_\eta$ becomes irrelevant and then a delocalized metallic phase realizes. It has been naively expected that in this phase the correlation for singlet superconductivity is dominant. It turns out, however, the random gauge field may change properties of this phase. The random gauge field $A(x)$ as well as $\eta(x)$ can be incorporated into the shift of the phase fields \[ \tilde{\theta}_c(x) \equiv \theta_c(x) + \tilde{A}(x), \] (22)\[ \tilde{A}(x) \equiv \frac{1}{K_\eta v_c} \sqrt{\frac{2}{\pi}} \int x' A(x'), \] (23) etc., we can cast the change part of the Hamiltonian into a gaussian type. Although the random field, $A(x)$, is now eliminated from the Hamiltonian, it may change long-range behaviors of some correlation functions. For example, the random gauge field $A(x)$ brings about the exponential decay of the correlation function for singlet superconducting pairing:\[ \langle c^\dagger_{cL}(x)c_{cR}(x)c^\dagger_{cR}(0)c_{cL}(0) \rangle \sim \frac{1}{|x|^{1+3/2} |x| e^{-\sqrt{D_\eta x^2}}}. \] (24) This is due to the random phase shift caused by the scattering due to random gauge fields. Eq. (24) implies that the susceptibility for singlet superconductivity does not diverge in the thermodynamic limit, $\chi_{ss}(k = 0, \omega) \sim \omega^{1/K_\eta}$. Thus the fluctuation toward superconductivity is much suppressed by random gauge fields.

In conclusion, the transition between the Mott insulator and Anderson insulator occurs according to the strength of disorder and the Umklapp interaction. We have found that the random forward scattering by impurities plays a central role to change the Mott insulating state to a metallic state, whereas the backward scattering drives the resulting metallic state to the Anderson localized state.

This work was partly supported by a Grant-in-Aid from the Ministry of Education, Science and Culture.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The renormalization flow for $\tilde{U}$ and $\tilde{D}_\eta$. The flow for some initial values of $\tilde{U}$ is shown: $K_c(0) = 0.8$, $\tilde{D}_\eta(0) = 0.05$. The transition point exists around $\tilde{U}(0) = 0.0463$.}
\end{figure}