Resonance spectrum of near-extremal Kerr black holes in the eikonal limit

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Abstract

The fundamental resonances of rapidly rotating Kerr black holes in the eikonal limit are derived analytically. We show that there exists a critical value, \( \mu_c = \sqrt{\frac{15-\sqrt{193}}{2}} \), for the dimensionless ratio \( \mu \equiv m/l \) between the azimuthal harmonic index \( m \) and the spheroidal harmonic index \( l \) of the perturbation mode, above which the perturbations become long lived. In particular, it is proved that above \( \mu_c \) the imaginary parts of the quasinormal frequencies scale like the black-hole temperature: \( \omega_I(n; \mu > \mu_c) = 2\pi T_{BH}(n + \frac{1}{2}) \). This implies that for perturbations modes in the interval \( \mu_c < \mu \leq 1 \), the relaxation period \( \tau \sim 1/\omega_I \) of the black hole becomes extremely long as the extremal limit \( T_{BH} \to 0 \) is approached. A generalization of the results to the case of scalar quasinormal resonances of near-extremal Kerr-Newman black holes is also provided. In particular, we prove that only black holes that rotate fast enough (with \( M\Omega \geq \frac{2}{5} \), where \( M \) and \( \Omega \) are the black-hole mass and angular velocity, respectively) possess this family of remarkably long-lived perturbation modes.
The response of a black hole to external perturbations is characterized by ‘quasinormal ringing’, damped (complex) oscillations with a discrete frequency spectrum (see [1, 2] for excellent reviews and detailed lists of references). This implies that radiative perturbations of the black-hole spacetime fade away over time in a manner reminiscent of the last pure dying tones of a ringing bell [3]. This characteristic decay of black-hole perturbations is in accord with the no-hair conjecture [4] which asserts that the external field of a perturbed black hole should relax into a Kerr-Newman spacetime, characterized solely by three observable (conserved) parameters: the black-hole mass, charge, and angular momentum.

This relaxation phase in the dynamics of perturbed black holes is characterized by a temporal decay of the perturbation fields of the form $e^{-i\omega t}$, where the characteristic black-hole resonance frequencies are complex numbers ($\omega = \omega_R - i\omega_I$ with $\omega_I \geq 0$) that depend on the black-hole physical parameters. These damped oscillations are then followed by late-time decaying tails that depend on the asymptotic properties of the spacetime [5, 6].

The black-hole quasinormal modes (QNMs) correspond to solutions of the perturbations equations with the physical boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves crossing the black-hole horizon [7]. These boundary conditions single out a discrete and infinite family of black-hole resonances [8] {\omega(n; m, l)}, where $l$ and $m$ are the multipolar indexes of the angular eigenfunctions which correspond to the QNMs [see Eqs. (2)-(3) below]. The resonance parameter $n$ is a non-negative integer which characterizes the overtone number.

Quasinormal resonances are expected to play a prominent role in gravitational radiation emitted by a variety of astrophysical scenarios involving black holes. Given the fact that these damped oscillations are the characteristic ‘sound’ of the black hole itself, they have attracted much attention from both physicists and mathematicians. In particular, the spectrum of black-hole QNMs is of great importance from both the theoretical [9, 10] and astrophysical points of view [1, 2]. These black-hole characteristic oscillations provide a direct way of identifying the black-hole parameters. This fact has motivated a flurry of research during the last four decades aiming to compute the resonance spectrum of various types of black holes [1, 2].

It turns out that for fixed values of the multipolar indexes $m$ and $l$ there exist an infinite
number of QNMs, characterizing oscillations with decreasing relaxation times (increasing imaginary part), see [9–13] and references therein. The mode with the smallest imaginary part — known as the fundamental mode — determines the characteristic dynamical timescale for generic perturbations to decay [14–17].

It is worth emphasizing that in most situations of physical interest the spectrum of QNMs must be computed numerically by solving the black-hole perturbations equations supplemented by the appropriate physical boundary conditions [18]. However, Mashhoon [19] has developed an analytical technique for calculating the equatorial QNMs of rotating Kerr black holes in the eikonal (geometric-optics) limit \( l = m \gg 1 \), see also [20–23].

Recently, Yang et. al. [24] have generalized the large-\( l \) analysis to include non-equatorial modes with \( l \neq m \). The analysis of Yang et. al. [24] is remarkably elegant and intuitive. Yet, their final expressions for the quasinormal frequencies are rather complicated: see Eqs. (2.35)-(2.36) of [24] for the real parts of the frequencies and Eqs. (2.36) and (2.40) of [24] for the imaginary parts of the frequencies. These equations must then be solved numerically in order to obtain the corresponding quasinormal frequencies; the numerical values of these quasinormal frequencies are presented in Fig. 3 (real parts) and Fig. 5 (imaginary parts) of [24].

One of the most remarkable conclusions of [24] is that near-extremal Kerr black holes are characterized by a significant fraction of QNMs that have nearly zero imaginary part (see also [15–17]). These resonances thus correspond to black-hole perturbations which may survive for relatively long times as compared to the dynamical timescale set by the mass of the black hole. In particular, it was observed numerically in [24] that such long-lived modes exist for near-extremal black holes in the finite interval

\[
\mu_c \equiv 0.74 < \frac{m}{l} \leq 1 \quad \text{for} \quad l \gg 1 .
\]

Thus, not only for equatorial modes with \( l = m \) [19] does \( \omega_I \) vanish in the extremal limit! Below we shall provide a fully analytical explanation for this phenomena. Furthermore, we shall obtain an analytical expression for the exact value of the critical ratio \( \mu_c \) above which the long-lived modes appear.
II. DESCRIPTION OF THE SYSTEM

In order to determine the black-hole quasinormal resonances we shall study the scattering of massless fields in the Kerr black-hole spacetime. The dynamics of a perturbation field $\Psi$ in the rotating Kerr spacetime is governed by the Teukolsky master equation \[25\]. As we shall show below, the Teukolsky equation is amenable to an analytical treatment in the near-extremal limit $(M^2 - a^2)^{1/2} \ll a \lesssim M$ (we use units in which $G = c = \hbar = 1$), where $M$ and $a$ are the black-hole mass and angular momentum per unit mass, respectively.

One may decompose the field as

$$\Psi_{slm}(t, r, \theta, \phi) = e^{im\phi} S_{slm}(\theta; a\omega) \psi_{slm}(r)e^{-i\omega t},$$  \(2\)

where $(t, r, \theta, \phi)$ are the Boyer-Lindquist coordinates, $\omega$ is the (conserved) frequency of the mode, $l$ is the spheroidal harmonic index, and $m$ is the azimuthal harmonic index with $-l \leq m \leq l$. The parameter $s$ is called the spin weight of the field, and is given by $s = \pm 2$ for gravitational perturbations, $s = \pm 1$ for electromagnetic perturbations, $s = \pm \frac{1}{2}$ for massless neutrino perturbations, and $s = 0$ for scalar perturbations \[25\]. (We shall henceforth omit the indices $s, l, m$ for brevity.) With the decomposition (2), $\psi$ and $S$ obey radial and angular equations, both of confluent Heun type \[25–28\], coupled by a separation constant $A(a\omega)$.

The angular functions $S(\theta; a\omega)$ are the spin-weighted spheroidal harmonics which are solutions of the angular equation \[25–27\]

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial S}{\partial \theta} \right) + \left[ a^2 \omega^2 \cos^2 \theta - 2a\omega s \cos \theta - \frac{(m + s \cos \theta)^2}{\sin^2 \theta} + s + A \right] S = 0.$$  \(3\)

The angular functions are required to be regular at the poles $\theta = 0$ and $\theta = \pi$. These boundary conditions pick out a discrete set of eigenvalues $sA_{lm}$ labeled by the integers $m$ and $l$. [In the $a\omega \ll 1$ limit these angular functions become the familiar spin-weighted spherical harmonics with the corresponding angular eigenvalues $A = l(l+1) - s(s+1) + O(a\omega)$.] The angular equation (3) can be solved analytically in the $l \gg 1$ limit (with $\omega_R \gg \omega_I$) to yield \[24\]

$$A = l(l+1) - \frac{1}{2} a^2 \omega_R^2 (1 - \mu^2) + O(1),$$  \(4\)

where here

$$\mu = \frac{m}{l}$$  \(5\)
is the dimensionless ratio between the azimuthal harmonic index \(m\) and the spheroidal harmonic index \(l\).

The radial Teukolsky equation is given by \(\Delta^{-s} \frac{d}{dr} (\Delta^{s+1} \frac{d\psi}{dr}) + \left[ \frac{K^2 - 2is(r - M)K}{\Delta} - a^2 \omega^2 + 2ma\omega - A + 4is\omega r \right] \psi = 0\), (6)

where \(\Delta \equiv r^2 - 2Mr + a^2\) and \(K \equiv (r^2 + a^2)\omega - am\). The zeroes of \(\Delta\), \(r_\pm = M \pm (M^2 - a^2)^{1/2}\), are the black hole (event and inner) horizons.

For the problem of wave-scattering in a black-hole spacetime one should impose physical boundary conditions of purely ingoing waves at the black-hole horizon and a mixture of both ingoing and outgoing waves at spatial infinity (these correspond to incident and scattered waves, respectively). That is,

\[
\psi \sim \begin{cases} 
e^{-i\omega y} + R(\omega)e^{i\omega y} & \text{as } r \to \infty \ (y \to \infty) ; \\ T(\omega)e^{-i(\omega - m\Omega)y} & \text{as } r \to r_+ \ (y \to -\infty) , \end{cases}
\]

where the “tortoise” radial coordinate \(y\) is defined by \(dy = [(r^2 + a^2)/\Delta]dr\). Here \(\Omega\) is the angular velocity of the black hole [see Eq. (8) below]. The coefficients \(T(\omega)\) and \(R(\omega)\) are the transmission and reflection amplitudes for a wave incident from infinity. They satisfy the usual probability conservation equation \(|T(\omega)|^2 + |R(\omega)|^2 = 1\).

**III. THE QUASINORMAL RESONANCES**

The discrete family of quasinormal frequencies describes the scattering resonances of the fields in the black-hole spacetime. These resonances correspond to poles of the transmission and reflection amplitudes. (The pole structure reflects the fact that the QNMs correspond to purely outgoing waves at spatial infinity.) These resonances determine the ringdown response of a black hole to external perturbations. As we shall now show, the spectrum of quasinormal frequencies can be studied analytically in the near-extremal limit \(a \to M\), see also \[15–17\].

Teukolsky and Press \[29\] and also Starobinsky and Churilov \[30\] have studied the black-hole scattering problem in the double limit \(a \to M \ (T_{\text{BH}} \to 0)\) and \(\omega \to m\Omega\), where

\[
T_{\text{BH}} \equiv \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)} ; \quad \Omega \equiv \frac{a}{r_+^2 + a^2} \quad (8)
\]
are the black-hole temperature and angular momentum, respectively. Detweiler \[31\] then used the analysis of \[29, 30\] to obtain a resonance condition for near-extremal Kerr black holes. It is convenient to define a set of dimensionless variables:

\[
\sigma \equiv \frac{r_+ - r_-}{r_+} ; \quad \tau \equiv M(\omega - m\Omega) ; \quad \hat{\omega} \equiv \omega r_+ ,
\]

in terms of which the resonance condition obtained in \[31\] for \(\sigma \ll 1\) and \(\tau \ll 1\) is:

\[
- \Gamma(2i\delta)\Gamma(1+2i\delta)\Gamma(1/2+s-2i\hat{\omega}-i\delta)\Gamma(1/2-s-2i\hat{\omega}-i\delta)
\]

\[
\Gamma(-2i\delta)\Gamma(1-2i\delta)\Gamma(1/2+s-2i\hat{\omega}+i\delta)\Gamma(1/2-s-2i\hat{\omega}+i\delta)
\]

\[
= (-2i\hat{\omega}\sigma)^{2i\delta}\frac{\Gamma(1/2+2i\hat{\omega}+i\delta-4i\tau/\sigma)}{\Gamma(1/2+2i\hat{\omega}-i\delta-4i\tau/\sigma)} ,
\]

where

\[
\delta^2 \equiv 4\hat{\omega}^2 - 1/4 - A - a^2\omega^2 + 2m\omega .
\]

We shall assume without loss of generality that \(\Re \delta \geq 0\). Taking cognizance of Eq. (4), one finds

\[
\delta = l \times F(\mu) + O(1)
\]

for near-extremal Kerr black holes in the eikonal limit \(l \gg 1\), where

\[
F(\mu) \equiv \sqrt{-1 + \frac{15}{8}\mu^2 - \frac{1}{8}\mu^4} .
\]

Here we have used the relations \(a \simeq r_+ \simeq M\) and \(\omega \simeq m\Omega \simeq m/2M\) for near-extremal Kerr black holes. Note that the function \(F(\mu)\) is real and positive in the interval

\[
\mu_c \equiv \sqrt{\frac{15 - \sqrt{193}}{2}} < \mu \leq 1 .
\]

The resonance condition (10) can be solved analytically in the regime \(\sigma \ll 1\) with \(\omega \simeq m\Omega\) \[15-17\]. The l.h.s of it has a well defined limit as \(a \to M\) and \(\omega \to m\Omega\). We denote that limit by \(\mathcal{L}\). Given the fact that \(\delta \simeq l\mathcal{F} \gg 1\) is purely real and large for \(l \gg 1\) in the interval \(\mu_c < \mu \leq 1\) [see Eq. (14)], one finds \((-i)^{-2i\delta} = e^{(-i\pi/2)(-2i\delta)} = e^{-\pi l\mathcal{F}} \ll 1\), which implies \(\epsilon \equiv (-2i\hat{\omega}\sigma)^{-2i\delta} \ll 1\). Thus, a consistent solution of the resonance condition, Eq. (10), may be obtained if \(1/\Gamma(1/2 + 2i\hat{\omega} - i\delta - 4i\tau/\sigma) = O(\epsilon)\). Suppose

\[
1/2 + 2i\hat{\omega} - i\delta - 4i\tau/\sigma = -n + \eta \epsilon + O(\epsilon^2) ,
\]

where
where \( n \geq 0 \) is a non-negative integer and \( \eta \) is an unknown constant to be determined below. Then one has
\[
\Gamma \left( 1/2 + 2i\hat{\omega} - i\delta - 4i\tau/\sigma \right) \simeq \Gamma(-n + \eta \epsilon) \simeq (-n)^{-1} \Gamma(-n + 1 + \eta \epsilon) \simeq \cdots \simeq \left[ (-1)^n n! \right]^{-1} \Gamma(\eta \epsilon),
\]
where we have used the relation \( \Gamma(z + 1) = z\Gamma(z) \) \[32\]. Next, using the series expansion
\[
1/\Gamma(z) = \sum_{k=1}^{\infty} c_k z^k \quad \text{with} \quad c_1 = 1 \quad \text{\[32\]},
\]
one obtains
\[
1/\Gamma \left( 1/2 + 2i\hat{\omega} - i\delta - 4i\tau/\sigma \right) = (-1)^n n! \eta \epsilon + O(\epsilon^2). \quad \text{(17)}
\]
Substituting \((17)\) into the resonance condition \((10)\) one finds
\[
\eta = L / \left[ (-1)^n n! \Gamma(-n + 2i\delta) \right].
\]
Finally, substituting \(4\tau/\sigma = (\omega - m\Omega)/2\pi T_{BH}, 2i\hat{\omega} = im + O(mMT_{BH})\) and \(\delta = \ell F + O(1)\) for \(\omega = m\Omega + O(mT_{BH})\) into Eq. \((15)\), one obtains the resonance condition
\[
(\omega - m\Omega)/2\pi T_{BH} = i[-n + \eta \epsilon - 1/2] + m - \ell F. \quad \text{(18)}
\]
Thus, the spectrum of black-hole quasinormal resonances in the eikonal limit \(\ell \gg 1\) within the interval \(\mu_c < \mu \leq 1\) is described by the compact analytical formula \[33\]
\[
\omega(n; \mu > \mu_c) = m\Omega + 2\pi T_{BH} \left[ (1 - F/\mu) m - i(n + \frac{1}{2}) \right] + O(MT_{BH}^2) \quad ; \quad n = 0, 1, 2, \ldots, \quad \text{(19)}
\]
where the dimensionless function \(F(\mu)\) is defined in \((13)\). Note that the spectrum \((19)\) corresponds to black-hole perturbation modes with relaxation times \(\tau \sim 1/\omega_l\) that become extremely long as the extremal limit \(T_{BH} \to 0\) is approached. We also note that the numerically computed value \(\mu_c \simeq 0.74\) \[24\] is astonishingly close to the analytical expression \((14)\) for the critical ratio \(\mu_c\).

### IV. SCALAR QNMS OF KERR-NEWMAN BLACK HOLES

Our analysis can readily be generalized to the case of scalar quasinormal resonances of charged and rotating Kerr-Newman (KN) black holes \[2, 34\] of mass \(M\), angular momentum per unit mass \(a\), and charge \(Q\). Substituting the relations \(r_+ \simeq M\) and \(\omega \simeq m\Omega \simeq ma/(M^2 + a^2)\) \[35\] into Eqs. \((11), (11)\) and \((12)\), one finds
\[
F(\mu; \alpha) \equiv \sqrt{-1 + \frac{6\alpha^2 (1 + \frac{1}{2}\alpha^2)}{(1 + \alpha^2)^2} \mu^2 - \frac{\alpha^4}{2(1 + \alpha^2)^2} \mu^4} \quad \text{(20)}
\]
for near-extremal KN black holes, where

$$\alpha \equiv a/M$$

(21)

is the rescaled (dimensionless) angular momentum of the black hole. Note that Eq. (20) is merely a generalization of (13) and reduces to it in the limit $a \to M$ ($\alpha \to 1$). The critical ratio $\mu_c(\alpha)$ is obtained from the limiting case $\mathcal{F} = 0$. One obtains

$$\mu_c(\alpha) = \frac{1}{2\alpha} \sqrt{6\alpha^2 + 24 - \sqrt{4\alpha^4 + 224\alpha^2 + 544}}$$

(22)

for near extremal KN black holes.

Using an analysis along the same lines as before, one finds that the scalar quasinormal mode spectrum of near-extremal KN black holes in the eikonal limit $l \gg 1$ is described by the formula

$$\omega(n; \mu > \mu_c(\alpha)) = m\Omega + 2\pi T_{BH} \left[ (1 - \mathcal{F}(\mu; \alpha)/\mu) m - i(n + \frac{1}{2}) \right] + O(MT_{BH}^2) \quad ; \quad n = 0, 1, 2, \ldots ,$$

(23)

where the functions $\mathcal{F}(\mu; \alpha)$ and $\mu_c(\alpha)$ are defined in Eqs. (20) and (22), respectively.

Inspection of Eq. (22) reveals that the interval $\mu_c(\alpha) < \mu \leq 1$ shrinks as the ratio $\alpha$ decreases [For example, for near-extremal black holes with $\alpha = 0.75$ one finds $\mu_c(\alpha = 0.75) \simeq 0.807$ instead of the value $\mu_c(\alpha = 1) = \sqrt{\frac{15 - \sqrt{193}}{2}} \simeq 0.744.$] Remarkably, we find that the condition $\mu_c(\alpha) < \mu \leq 1$ can only be satisfied by rotating black holes in the interval

$$\frac{1}{2} \leq \alpha \leq 1 .$$

(24)

One therefore concludes that, within the eikonal approximation, only KN black holes that rotate fast enough ($a \geq \frac{1}{2}M$, or equivalently $M\Omega \geq \frac{2}{9}$) can be characterized by relaxation periods $\tau \sim 1/\omega_I$ which become infinitely long as the extremal limit $T_{BH} \to 0$ is approached.

V. SUMMARY

In summary, the quasinormal mode spectrum of near-extremal rotating Kerr black holes was studied analytically within the eikonal approximation $l \gg 1$. It was shown that the fundamental resonances can be expressed in terms of the black-hole physical parameters: the temperature $T_{BH}$ and the angular velocity $\Omega$. In particular, we have proved the existence of a critical value, $\mu_c$, for the dimensionless ratio $\mu \equiv m/l$ between the azimuthal harmonic
index $m$ and the spheroidal harmonic index $l$ of the perturbation mode, above which the perturbations become long lived – for modes in the interval $\mu_c < \mu \leq 1$ the imaginary parts of the quasinormal frequencies scale like the black-hole temperature: $\omega_I(n; \mu > \mu_c) = 2\pi T_{BH}(n + \frac{1}{2})$. For these perturbation modes the relaxation period $\tau \sim 1/\omega_I$ of the black hole becomes extremely long as the extremal limit $T_{BH} \to 0$ is approached. Our analytical expression (14) for the critical ratio, $\mu_c = \sqrt{15 - \sqrt{193}}/2$, is remarkably close to the numerically computed [24] value $\mu_c \simeq 0.74$.

We have also generalized the results to the case of charged and rotating Kerr-Newman black holes, proving that the interval $\mu_c(\alpha) < \mu \leq 1$ in which the long-lived modes exist shrinks as the dimensionless ratio $\alpha \equiv a/M$ decreases. In particular, we have shown that only near-extremal black holes that rotate fast enough (with $M\Omega \geq 2/5$) possess this family of extremely long-lived perturbation modes.

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