The integral cohomology rings of some \( p \)-groups

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1. Introduction

We determine the integral cohomology rings of an infinite family of \( p \)-groups, for odd primes \( p \), with cyclic derived subgroups. Our method involves embedding the groups in a compact Lie group of dimension one, and was suggested independently by P. H. Kropholler and J. Huebschmann. This construction has also been used by the author to calculate the mod-\( p \) cohomology of the same groups and by B. Moselle to obtain partial results concerning the mod-\( p \) cohomology of the extra special \( p \)-groups [7], [9].

2. The method and the groups

Given a finite group \( G \) and a central cyclic subgroup \( C \), we fix an embedding of \( C \) into \( S^1 \), and define a Lie group \( \hat{G} \) as the product of \( S^1 \) and \( G \) amalgamating \( C \), that is

\[ \hat{G} = S^1 \times G / \langle (c^{-1}, c) : c \in C \rangle. \]

Then we have a commutative diagram

\[
\begin{array}{ccc}
C & \longrightarrow & G \\
\downarrow & & \downarrow \\
S^1 & \longrightarrow & \hat{G}
\end{array}
\]

If \( M \) is a \( G \)-module on which \( C \) acts trivially, we may consider \( M \) as a \( \hat{G} \)-module by letting \( S^1 \) act trivially, and the Lyndon–Hochschild–Serre spectral sequence for the second extension is often simpler than that for the first. To find \( H^*(BG;M) \), given \( H^*(B\hat{G};M) \), we use the Serre spectral sequence of the fibration

\[ S^1/C \cong \hat{G}/G \rightarrow BG \rightarrow B\hat{G}. \]

This spectral sequence has \( E_2^{i,j} = 0 \) for \( j > 1 \), so the only possible non-zero differential is \( d_2 \). The above was first suggested to the author by P. Kropholler. A similar idea occurs in J. Huebschmann’s papers [5] and [6]. In the case where \( M \) is a commutative ring on which \( G \) acts trivially, it appears that we may obtain another filtration of \( H^*(BG;M) \) by examining the Eilenberg–Moore spectral sequence for the pullback square:

\[
\begin{array}{ccc}
BG & \longrightarrow & B\hat{G} \\
\downarrow & & \downarrow \\
\{\ast\} & \longrightarrow & B\hat{G}/G,
\end{array}
\]
but it can be shown that the two filtrations are identical. These spectral sequences are just alternative ways to view the Gysin sequence for the $S^1$-bundle $BG$ over $B\hat{G}$.

It is possible for non-equivalent extensions of $C$ by $Q$ to yield equivalent extensions of $S^1$ by $Q$. In fact this happens if and only if their extension classes in $H^2(Q; C)$ map to the same element of $H^3(Q; \mathbb{Z})$ under the Bockstein associated with the coefficient sequence $\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow C$.

The groups we shall consider are central extensions of $C_{p^{n-2}}$ by $C_p \oplus C_p$ where $p$ is an odd prime, and $n \geq 3$. They may be presented as

$$P(n) = \langle A, B, C | A^p = B^p = C^{p^{n-2}} = [A, C] = [B, C] = 1, [A, B] = C^{p^{n-2}} \rangle.$$  

We shall let $\hat{P}$ be the corresponding central extension of $S^1$ by $C_p \oplus C_p$, that is the group obtained from $S^1 \times P(n)$ by amalgamating the subgroup of $P(n)$ generated by $\langle C \rangle$ and the $C_{p^{n-2}}$ subgroup of $S^1$. There are four central extensions of $C_{p^{n-2}}$ by $C_p \oplus C_p$; two abelian ones, $P(n)$, and a metacyclic group $M(n)$ containing a cyclic subgroup of index $p$. This may be checked by verifying that the action of $\text{Aut}(C_p \oplus C_p)$ on $H^2(B(C_p \oplus C_p); C_p^{n-2})$ has only four orbits, and then explicitly constructing four non-isomorphic groups. There are however only two central extensions of $S^1$ by $C_p \oplus C_p$; the direct product, which is abelian, and $\hat{P}$ which is not. This follows from the fact that $\text{Aut}(C_p \oplus C_p)$ acts transitively on the non-zero elements of $H^2(B(C_p \oplus C_p); \mathbb{Z})$, which may be identified with $H^2(B(C_p \oplus C_p); S^1)$ via the Bockstein associated with the coefficient sequence $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow S^1$. Hence we see that $BM(n)$ is also an $S^1$-bundle over $B\hat{P}$, and in fact $H^*(BM(n); \mathbb{Z})$ could easily be determined from the results of this paper. This cohomology ring has already been calculated using other methods [11].

3. Calculations

We now begin our calculation of $H^*(B\hat{P})$ by examining the spectral sequence with integer coefficients for $\hat{P}$ considered as an extension of $S^1$ by $C_p \oplus C_p$. The $E_2$ page is readily seen to be generated by elements $\alpha, \beta \in E_2^{0,0}, \gamma \in E_2^{0,2}$ and $\tau \in E_2^{0,5}$ subject only to the relations $p\alpha = p\beta = 0$, $p\gamma = 0$ and $\gamma^2 = 0$. Note that $\tau$ has infinite order. Since $E_2^{0,j}$ is trivial for $j$ odd, we see that all the even differentials must vanish. The behaviour of the differentials is summarized in the following lemma.

**Lemma 1.** In the above spectral sequence there are exactly two non-zero differentials, $d_3$ and $d_{2p-1}$. $d_3(\tau)$ is a non-zero multiple of $\gamma$, and $E_4$ is generated by the classes of the elements $\alpha, \beta, \tau, \ldots, \tau^{p-1}, \tau^p$ and $\tau^{p-1}\gamma$. All of these generators are universal cycles except for $\tau^{p-1}\gamma$, which is mapped by $d_{2p-1}$ to a non-zero multiple of $\alpha^p\beta - \beta^p\alpha$. The $E_\infty$ page is generated by the elements $\alpha, \beta, \tau, \ldots, \tau^{p-1}, \tau^p$ subject only to the relations they satisfy as elements of $E_2$, and the relation $\alpha^p\beta = \beta^p\alpha$.

**Proof.** The derived subgroup of $\hat{P}$ consists of the subgroup of its central $S^1$ of order $p$, so there can be no homomorphism from $\hat{P}$ to $S^1$ that restricts to an isomorphism from the centre to $S^1$. It follows by considering the natural isomorphism $H^2(BG; \mathbb{Z}) \cong \text{Hom}(BG, S^1)$ that the element $\tau$ cannot survive to $E_\infty$, so we must have $d_3(\tau)$ a non-zero multiple of $\gamma$. This determines $d_3$ completely. It may be checked that $E_4$ is isomorphic to the subring of $E_2$ generated by $\alpha, \beta, \tau, \ldots, \tau^{p-1}, \tau^p$ and $\tau^{p-1}\gamma$. All these elements must be universal cycles, with the possible exception of $\tau^{p-1}\gamma$, because...
the groups in which their images under \( d_n \) lie are already trivial. The only remaining potentially non-zero differential is \( d_{p-1}(r^{p-1} \gamma) \). To complete this proof it suffices to show that in the \( E_\infty \) page the relation \( \alpha^p \beta = \beta^p \alpha \) must hold.

Let \( Q \) be the quotient of \( \tilde{P} \) by its \( S^1 \) subgroup, and take generators \( \alpha' \), \( \beta' \) for \( H^2(BQ; \mathbb{Z}) \) and \( \gamma' \) for \( H^3(BQ; \mathbb{Z}) \). The statement that \( \gamma \) does not survive to \( E_\infty \) in the spectral sequence is equivalent to the statement that \( \gamma' \) is mapped to zero by the inflation map from \( Q \) to \( \tilde{P} \). Now we calculate \( \phi(\gamma') \), where \( \phi \) is the integral cohomology operation \( \delta_p \gamma^\ast \), where \( \gamma^\ast \) is the map induced by the change of coefficients from \( \mathbb{Z} \) to \( \mathbb{F}_p \). \( P^1 \) is a reduced power, and \( \delta_p \) is the Bockstein for the sequence \( \mathbb{Z} \to \mathbb{Z} \to \mathbb{F}_p \).

Taking \( y, y' \in H^1(BQ; \mathbb{F}_p) \) such that \( \delta_p(y') = \gamma' \), and \( \delta_p(y') = \beta' \), we see that

\[
\phi(\gamma') = \delta_p P^1 \gamma^\ast (y') = \delta_p P^1 (\alpha' y - \beta' y') y
= \delta_p (\beta_p(y') y' - \beta_p(y') y) = \alpha^{p} \beta' - \beta^{p} \alpha'.
\]

It follows that

\[
\alpha^{p} \beta - \beta^{p} \alpha = \inf (\alpha^{p} \beta' - \beta^{p} \alpha') = \inf (\phi(\gamma')) = \phi \inf (\gamma') = 0.
\]

We are now ready to state our theorem on \( H^*(\tilde{P}) \).

**Theorem 2.** Let \( p \) be an odd prime, and let \( \tilde{P} \) be the group defined above. Then \( H^*(\tilde{P}; \mathbb{Z}) \) is generated by elements \( \alpha, \beta, \chi_1, \ldots, \chi_{p-1}, \zeta, \) with

\[
\deg(\alpha) = \deg(\beta) = 2 \quad \deg(\chi_i) = 2i \quad \deg(\zeta) = 2p,
\]

subject to the following relations:

\[
\begin{align*}
\alpha \chi_i &= \chi_{i+1} \quad \text{for } i < p-1 \\
\alpha^p \beta &= \beta^p \alpha \\
-\alpha^p \beta \chi_i &= 0 \quad \text{for } i = p-1 \\
X_i X_j &= \begin{cases}
-\beta^p & \text{for } i+j = p \\
p^{i+j} \zeta & \text{for } i+j < p \\
p^i \chi_{i+p-j} & \text{for } p < i+j < 2p-2 \\
p^i \chi_{p-2+i} + \alpha^2 \beta^{p-2} - \alpha^{p-1} \beta^{p-1} & \text{for } i = j = p-1.
\end{cases}
\end{align*}
\]

Chern classes of representations of \( \tilde{P} \) generate the whole ring. An automorphism of \( \tilde{P} \) sends \( \chi_i \) to \( \chi_i \) (resp. \( -1 \chi_i \)) and \( \zeta \) to \( \zeta \) (resp. \( \zeta \)) if it fixes (resp. reverses) \( S^1 \). The effect of an automorphism on \( \alpha, \beta \) may be determined from their definition. Considered as elements of \( \text{Hom}(\tilde{P}, S^1) \), \( \alpha \) has kernel \( \langle S^1, B \rangle \) and sends \( A \) to \( e^{2\pi i/p} \), and \( \beta \) has kernel \( \langle S^1, A \rangle \) and sends \( B \) to \( e^{2\pi i/p} \). If we let \( H \) be the subgroup generated by \( B \) and elements of \( S^1 \) we may define \( \chi \) as above where \( r \) is any element of \( H^2(H; \mathbb{Z}) \) restricting to \( S^1 \) as the generator \( r \). Similarly, \( \zeta = c_p(\rho) \), where \( \rho \) is an irreducible representation of \( \tilde{P} \) restricting to \( S^1 \) as \( p \) copies of the representation \( \xi \) with \( c_1(\xi) = r \).

**Proof.** First we note that in the \( E_\infty \) page of the above spectral sequence all the group extensions that we need to examine are extensions of finite groups by the
infinite cyclic group, so are split. The elements $\alpha$ and $\beta$ defined in the statement above clearly yield generators for $E_{\infty}^{0,0}$, and the relations between them are exactly the relations that hold between the corresponding elements in the spectral sequence. Let $\beta' \in H^2(BH)$ be the restriction to $H$ of $\beta$, and take any choice of $\tau'$ as in the statement. We may show by considering $\beta'$ and $\tau'$ as homomorphisms from $H$ to $S^1$ that conjugation by $A^i$ induces the map on $H^2(BH)$ that fixes $\beta'$ and sends $\tau'$ to $\tau' - i\beta'$. Now applying the formula for $\text{Res}_{\alpha}^{\beta} \text{Cor}_{\beta}^{\alpha}$ (see for example [4]) it follows that $\chi_i$ restricts to $S^1$ as $p^i\chi_i$, so yields a generator for $E_{\infty}^{0,2i}$.

Any irreducible representation of $\tilde{\rho}$ has degree 1 or $p$, because $\tilde{\rho}$ has an abelian subgroup of index $p$. Let $\rho$ be the representation of $\tilde{\rho}$ induced from a 1-dimensional representation of $H$ with first Chern class $\tau'$. $\rho$ restricts to $S^1$ as $p$ copies of the representation with first Chern class $\tau$, so its total Chern class restricts to $S^1$ as $(1+\tau)^p$, and so $c_p(\rho)$ yields a generator for $E_{\infty}^{0,2p}$, and $c_1(\rho) = 1/p(\rho)\chi_i + P_i(\chi_i)$ for some polynomial $P_i$. We shall show later that $P_i = 0$.

The restriction to $H$ of $\alpha$ is trivial, so by Frobenius reciprocity
\[ \alpha \text{Cor}_{\alpha}^{\beta}(\tau') = \text{Cor}_{\alpha}^{\beta}(\text{Res}_{\alpha}^{\beta}(\alpha) \tau') = 0, \]
and the expressions given for $\alpha\chi_i$ follow. By calculating $\alpha(\beta\chi_i) = \beta(\alpha\chi_i)$, we may deduce that $\beta\chi_i = 0$ for $i < p-1$ and $\beta\chi_{p-1} = \lambda(\alpha^{p-1}\beta - \beta^p) - \alpha^{p-1}\beta$ for some scalar $\lambda$. To show that $\lambda = 1$ we use the restriction map to $H$, and the formula for corestriction followed by restriction.
\[
\text{Res}_{\alpha}^{\beta}(\beta\chi_{p-1}) = \beta' \sum_{i=0}^{p-1} (\tau' + i\beta')^{p-1} \\
= \beta' \sum_{j=0}^{p-1} \tau'^{p-1-j} \beta'^j \sum_{i=0}^{p-1} \beta^i.
\]
Newton’s formula tells us that
\[
\sum_{i=1}^{p-1} \beta^i = \begin{cases} 
0(p) & \text{for } j \not\equiv 0(p-1) \\
-1(p) & \text{for } j \equiv 0(p-1) 
\end{cases}
\]
so $\text{Res}_{\alpha}^{\beta}(\beta\chi_{p-1}) = -\beta'^p$, and the required relation follows.

We now know $\text{Res}_{\alpha}^{\beta}(\chi_i\chi_j) = \alpha\chi_i\chi_j$, and $\beta\chi_i\chi_j$, which together imply the relations given for $\chi_i\chi_j$. To complete the proof of the theorem we must determine the effect of automorphisms of $\tilde{\rho}$ on the $\chi_i$. We know that an automorphism sends $c_i(\rho)$ to itself or $(-1)^i$ times itself depending whether or not it reverses the sense of $S^1$, so it will suffice to show that $\chi_i = 1/p(\rho) c_i(\rho)$. The character of $\rho$ is zero except on $S^1$, so if $\theta$ is a 1-dimensional representation of $\tilde{\rho}$ restricting trivially to $S^1$, then $\rho \otimes \theta$ is isomorphic to $\rho$. If we apply the formula expressing $c.(\rho \otimes \theta)$ in terms of $c.(\rho)$ and $c.(\theta)$ (see [3]) we obtain
\[
c_i(\rho) = c_i(\rho \otimes \theta) = \sum_{j=0}^{i} \binom{p-i+j}{j} c_1(\theta)^jc_{i-j}(\rho),
\]
and hence inductively
\[
c_i(\rho) c_1(\theta) = \begin{cases} 
0 & \text{for } i < p-1 \\
-c_1(\theta)^p & \text{for } i = p.
\end{cases}
\]
Since $\alpha$ and $\beta$ are possible values for $c_1(\theta)$ the required result follows. We may show inductively that $\chi_i$ is in the subring generated by Chern classes because $\chi_1$ is, and $\chi_1 \chi_{i-1}, 1/p(\xi) \chi_i$ are coprime multiples of $\chi_i$.

We are now ready to state our theorem on the integral cohomology of $BP(n)$.

**Theorem 3.** Let $p$ be an odd prime and let $P(n)$ be as defined above. Then $H^*(BP(n); \mathbb{Z})$ is generated by elements $\alpha, \beta, \mu, \nu, \chi_1, \ldots, \chi_p, \xi$, with

\[
\deg(\alpha) = \deg(\beta) = 2 \quad \deg(\mu) = \deg(\nu) = 3 \quad \deg(\chi_i) = 2i \quad \deg(\xi) = 2p
\]

subject to the following relations:

\[
\begin{align*}
\alpha \mu &= \beta \nu \\
\alpha^p \beta &= \beta^p \alpha \\
\alpha^p \mu &= \beta^p \mu \\
\alpha X_i &= \begin{cases} 0 & \text{for } i < p-1 \\ -\alpha^p & \text{for } i = p-1 \end{cases} \\
\beta X_i &= \begin{cases} 0 & \text{for } i < p-1 \\ -\beta^p & \text{for } i = p-1 \end{cases} \\
\mu X_i &= \begin{cases} 0 & \text{for } i < p-1 \\ -\mu^{p-1} & \text{for } i = p-1 \end{cases} \\
\nu X_i &= \begin{cases} 0 & \text{for } i < p-1 \\ -\nu^{p-1} & \text{for } i = p-1 \end{cases} \\
XX &= \begin{cases} pX_{i+j} & i+j < p \\ p^2 \xi & i+j = p \\
p^p \xi X_{i+j-p} & p < i+j < 2p-2 \\
p^p \xi X_{p-2} + \alpha^{2p-2} + \beta^{2p-2} - \alpha^{p-1} \beta^{p-1} & i = j = p-1 \end{cases} \\
\mu \nu &= \begin{cases} 0 & \text{for } n > 3 \\ \lambda \chi_3 & \text{for } n = 3, p > 3, \lambda \in \mathbb{F}_p^* \\ 3\lambda \xi & \text{for } n = 3, p = 3, \lambda = \pm 1. \end{cases}
\end{align*}
\]

Chern classes of representations of $P(n)$ generate $H^{even}(BP(n); \mathbb{Z})$. Under an automorphism of $P(n)$ which restricts to the centre as $C \mapsto C$, $X_i$ is mapped to $j^iX_i$, and $\xi$ is mapped to $j^n\xi$. The effect of automorphisms on $\alpha$ and $\beta$ is determined by the natural isomorphism $H^2(BG; \mathbb{Z}) \cong \text{Hom}(G, \mathbb{R}/\mathbb{Z})$, under which

\[
\begin{align*}
\alpha &: A \mapsto 1/p \\
\beta &: A \mapsto 0 \\
\chi_1 &: A \mapsto 0 \\
B &: \mapsto 1/p \\
B &: \mapsto 0 \\
C &: \mapsto 0 \\
C &: \mapsto 1/p^{n-3}.
\end{align*}
\]

An automorphism of $P(n)$ which sends $\alpha$ to $n_1 \alpha + n_2 \beta$, $\beta$ to $n_3 \alpha + n_4 \beta$ and restricts to the centre as $C \mapsto C'$ sends $\mu$ to $j(n_4 \mu + n_3 \nu)$ and $\nu$ to $j(n_2 \mu + n_1 \nu)$. If $\gamma'$ in $H^2(B\langle B, C \rangle; \mathbb{Z})$ is such that it maps to the following element of $\text{Hom}(\langle B, C \rangle, \mathbb{R}/\mathbb{Z})$

\[
\gamma' &: B \mapsto 0 \\
& \mapsto 1/p^{n-2},
\]

then $\chi_i$ is defined as follows:

\[
\chi_i = \begin{cases} \text{Cor}_{\langle B, C \rangle}^p(\gamma'^i) & \text{for } i < p-1 \\ \text{Cor}_{\langle B, C \rangle}^p(\gamma'^{p-1}) - \alpha^{p-1} & \text{for } i = p-1. \end{cases}
\]
These are, up to scalar multiples, equal to $c_t(p)$, where $p$ is a $p$-dimensional irreducible representation of $P(n)$, whose restriction to $\langle C \rangle$ is a sum of $p$ copies of the representation $\theta$, with $c_t(\theta) = \text{Res}_{\langle C \rangle}^\theta(y')$. In fact, $c_t(p) = 1/p\chi_t$. Also, we may define $\xi = c_p(p)$.

**Proof.** We examine the spectral sequence for $BP(n)$ as an $S^1$-bundle over $BP$. Now $E^*_2.0$ is isomorphic to $H^*(BP(n); \mathbb{Z})$ and $E^*_2.*$ is freely generated by $E^*_2.0$ and an element $\xi$ of infinite order in $E^*_2.1$. We know that

$$H^p(BP(n)) \cong \text{Hom}(P(n), S^1) \cong C_{p^n-3} \oplus C_p \oplus C_p,$$

so $d_2(\xi)$ must be $\pm p^{n-3}\chi_t$. If we wanted to calculate the cohomology of the metacyclic groups $M(n)$ described above, the differential in this spectral sequence would send $\xi$ to $\pm p^{n-3}\chi_t + \gamma$ for some non-zero $\gamma$ in $\langle \alpha, \beta \rangle$. It is now easy to see that $E^\infty$ is generated by the elements $\alpha, \beta, \mu = \beta_\xi$, $\nu = a\xi, \chi_1, \ldots, \chi_{p-1}$ and $\xi$ subject to the relations they satisfy as elements of $E^*_2.*$ together with $p^{n-3}\chi_t = 0, p^{n-2}\chi_t = 0,$ and $p^{n-1}\xi = 0$. For each $m$, the filtration of $H^m(BP(n))$ given by the $E^\infty$ page is trivial, so we may use the same symbols to denote elements of $H^m(BP(n))$, and the relations that hold in $E^\infty$ determine all the relations that hold in $H^m(BP(n))$ except for the product of the two odd-dimensional generators.

We know that $p\mu\nu = 0$, and the relation $\alpha\mu = \beta\nu$ implies that $\alpha\mu = \beta\mu\nu = 0$, and so $\mu\nu$ must be a multiple of $p^{n-3}\chi_t$ for $p \geq 5$ (resp. $3\xi$ for $p = 3$). Note that these elements restrict to zero on all proper subgroups of $P(n)$. In the case of $P(3)$, Lewis\[8\] shows that $\alpha\gamma$ is not zero by considering the spectral sequence for $P(n)$ considered as an extension of a maximal subgroup by $P$. A similar method will work in general, but we offer an alternative proof that involves expressing $\mu, \nu$ as Bocksteins of elements of $H^2(BP(n))$. This proof is contained in Lemma 4 and Corollary 5.

The effect of automorphisms on $\chi_t$ and $\xi$ is easily seen to be as claimed from their alternative definitions as Chern classes. To determine the effect of automorphisms on $\mu$ and $\nu$, we note that an automorphism of $P(n)$ restricting to the centre as $C \mapsto C'$ extends to an endomorphism of $\tilde{P}$ which wraps the central circle $j$ times around itself, so induces a map of the above spectral sequence to itself sending $\xi$ to $j\xi$. This completes the proof of Theorem 3 modulo Lemma 4 and its corollary.

We now examine the spectral sequence with $\mathbb{F}_p$ coefficients for the central extension $C_{p^n-1} \mapsto P(n) \mapsto C_p \oplus C_p$. Take generators so that

$$H^*(BC_{p^n} \oplus C_p; \mathbb{F}_p) \cong \mathbb{F}_p[x, x'] \otimes \Lambda[y, y'],$$

where $\beta_p(y) = x$, $\beta_p(y') = x'$, and

$$H^*(BC_{p^n-1}; \mathbb{F}_p) \cong \mathbb{F}_p[t] \otimes \Lambda[u],$$

where $\beta_p(u) = t$ for $n = 3$ (resp. $\beta_p(u) = 0$ for $n \geq 4$). Then the $E_2$ page is isomorphic to $\mathbb{F}_p[x, x', t] \otimes \Lambda[y, y', u]$, and the first two differentials are as described in the following lemma.

**Lemma 4.** With notation as above, identify the elements $x, x', y, y'$ in the spectral sequence with their images in $H^*(BP(n); \mathbb{F}_p)$ under the inflation map.

1. Let $n \geq 4$. Then $d_2$ is trivial, and $d_2(t)$ is a non-zero multiple of $xy' - x'y$. The elements $x, x', yy', u'y, w'y$ form a basis for $H^2(BP(n))$, where $u'$ is any element of $H^1(BP(n))$ restricting to $C_{p^n-1}$ as $u$.

2. Let $n = 3$. Then $d_2(u)$ is a non-zero multiple of $yy'$, $d_2(t) = 0$, and $E_3$ is generated by $y, y', x, x', [uy], [uy']$, and $t$ subject to the relation $yy' = 0$ and those implied by the
relations in $E_2$. In particular $[uy']y' = -[uy']y$ but this element is non-zero. As in the case $n \geq 4$, $d_2(t)$ is a non-zero multiple of $xy' - x'y$. Let $Y, Y'$ be elements of $H^2(BP(n))$ such that $x, x', Y, and Y'$ form a basis for $H^2(BP(3))$, and let $X = \beta_p(Y), X' = \beta_p(Y')$. Then $yY', xy, xy', x'y, X, and X'$ form a basis for $H^3(BP(3))$ and $xX, x'X, x'y, x'y', xx'y', xx'y', and X'Y$ form a basis for $H^5(BP(3))$.

Proof. (1) In this case $H^1$ has order $p^3$, so $u$ must survive. The element $xy' - x'y$ is the image under $\pi_*$ of a generator for $H^3(B(C_p \oplus C_p); \mathbb{Z})$, so must be killed by some differential. We have already shown that it cannot be killed by $d_2$, so the only possibility is that $t$ survives until $E_3$ and kills it. The rest of the statement follows easily.

(2) In this case $H^1$ has order $p^2$, so $d_2(u)$ must be non-zero. It is true in general that if $G$ is a central extension of $C_p$ by $Q$, then in the corresponding spectral sequence with $\mathbb{F}_p$ coefficients $d_2: E_2^{2,0} \rightarrow E_2^{2,1}$ must kill the extension class. This follows by naturality, since one may regard the extension class as defining a homotopy class of maps from $BQ$ to $K(C_p, 2)$ such that $BG$ is the $BC_p$-bundle induced by the path-loop fibration over $K(C_p, 2)$. Since all subgroups of $P(3)$ of order $p^2$ are copies of $C_p \oplus C_p$, the extension class of $P(3)$ must restrict to zero on all cyclic subgroups, so must be a multiple of $yy'$. The transgression commutes with the Bockstein so $d_2(t) = 0$ and $d_2(u) = \beta_p d_2(u)$.

Given the values of these differentials it is routine to compute the $E_4$ page of the spectral sequence. If we write $E_4^{i,j} = \bigoplus_{i+j=n} E_4^{i,j}$, then $\{[uy], [uy'], x, x'\}$ forms a basis for $E_4^{2,2}$, and $\{[ty], [ty'], [uy]y', xy, xy', x'y'\}$ forms a basis for $E_4^{2,0}$. The spectral sequence operation $\beta$ introduced by Araki[2] and Vasquez[12] maps $[uy]$ to $[ty]$ and $[uy']$ to $[ty']$, so if $Y$ and $Y'$ are chosen to yield the generators for $E_4^{1,1}$ their Bocksteins yield generators for $E_4^{1,2}$. A basis for $E_4^{1,2}$ is given by the eight elements of the statement, which we know to be universal cycles, and the elements $[t^2y], [t^2y']$. $E_4^{1,2}$ consists of universal cycles, and the universal coefficient theorem tells us that $H^5$ has order $p^3$, so $[t^2y]$ and $[t^2y']$ cannot be universal cycles.

Corollary 5. In $H^*(BP(n); \mathbb{Z})$ the product $uv$ is non-zero if and only if $n = 3$.

Proof. In the notation of Lemma 4 it suffices to determine $\delta_p(u'y)\delta_p(u'y')$ in the case $n \geq 4$, and $\delta_p(Y)\delta_p(Y')$ in the case $n = 3$. In the case when $n = 3$,

$$\delta_p(Y)\delta_p(Y') = \delta_p(Y\beta_p(Y')) = \delta_p(YX').$$

The kernel of $\delta_p: H^i(BP(3); \mathbb{F}_p) \rightarrow H^i(BP(3); \mathbb{Z})$ is equal to $\pi_*(H^i(BP(3); \mathbb{Z}))$, which is generated by $xX$, $xX'$, and $x'X'$, so by Lemma 4, $\delta_p(YX')$ is non-zero.

In the case when $n = 4$, $H^i(BP(4); \mathbb{Z})$ has exponent $p$ for $i = 2, 3$, so $\pi_*$ is injective from these groups, and $\ker \beta_p: H^i(BP(4)) \rightarrow H^i(BP(4))$ is equal to $\beta_p(H^i(BP(4)))$. $\beta_p(yy') = xy' - x'y = 0$, so we may choose the element $u'$ in Lemma 4 so that $\beta_p(u') = \lambda yy'$ for some non-zero $\lambda$. Then we have

$$\delta_p(u'y)\delta_p(u'y') = \delta_p(u'y\beta_p(u'y')) = \delta_p(u'y(\lambda yy'y' - u'x')) = 0.$$

The case when $n \geq 5$ is similar but simpler, since $u'$ may be chosen so that $\delta_p(u') = p^{n-1}X_1$, which implies that $\beta_p(u') = 0$.

Remarks. Theorem 3 contains independent proofs of Thomas' result that the even degree subring of $H^*(BP(n); \mathbb{Z})$ is generated by Chern classes [10], and Lewis'
calculation of $H^*(BP(3); \mathbb{Z})$. Our notation differs slightly from that of Lewis[8]. We have renumbered the generators $\chi_i$ (note that $\chi_1$ vanishes for $n = 3$). Also our $\chi_{p-1}$ and Lewis' $\chi_{p-2}$ are related by the formula
\[ \chi_{p-2}^{\text{Lewis}} = \chi_{p-1} + \alpha^{p-1} + \beta^{p-1}. \]
Our result disagrees with that of AlZubaidy[1].

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