Blackbody Radiation in Classical Physics: A Historical Perspective

Timothy H. Boyer

Department of Physics, City College of the City University of New York, New York, New York 10031

Abstract

We point out that current textbooks of modern physics are a century out-of-date in their treatment of blackbody radiation within classical physics. Relativistic classical electrodynamics including classical electromagnetic zero-point radiation gives the Planck spectrum with zero-point radiation as the blackbody radiation spectrum. In contrast, nonrelativistic mechanics cannot support the idea of zero-point energy; therefore if nonrelativistic classical statistical mechanics or nonrelativistic mechanical scatterers are invoked for radiation equilibrium, one arrives at only the low-frequency Rayleigh-Jeans part of the spectrum which involves no zero-point energy, and does not include the high-frequency part of the spectrum involving relativistically-invariant classical zero-point radiation. Here we first discuss the correct understanding of blackbody radiation within relativistic classical physics, and then we review the historical treatment. Finally, we point out how the presence of Lorentz-invariant classical zero-point radiation and the use of relativistic particle interactions transform the previous historical arguments so as now to give the Planck spectrum including classical zero-point radiation. Within relativistic classical electromagnetic theory, Planck’s constant $\hbar$ appears as the scale of source-free zero-point radiation.
I. INTRODUCTION

During the first two decades of the 20th century, the blackbody radiation spectrum appeared prominently in physics research. Debates over the correct theoretical interpretation of the experimental measurements reached a climax around 1909. The view that emerged from the debates was that classical physics led inevitably to the Rayleigh-Jeans spectrum and was incapable of explaining the experimentally-measured blackbody spectrum. In this article, we use historical accounts to go back to the debates of a century ago, but now with the awareness of the developments within classical physics of the intervening years. We apply the new knowledge to the old controversies. We point out why the old arguments fail; in some cases the arguments must be abandoned, and in other cases the arguments are easily corrected in the light of new information. A reanalysis of the conflict of the previous century leads to the conclusion that classical physics can indeed give an accurate account of the Planck spectrum appearing in the experimental data.

The textbooks of modern physics have not caught up with current knowledge regarding blackbody radiation within classical physics and instead provide students with an incorrect classical view.[1] An accurate statement of the situation is as follows: the use of relativistic physics with the inclusion of classical electromagnetic zero-point radiation leads to the Planck spectrum including classical zero-point radiation, whereas the use of nonrelativistic classical statistical mechanics or of nonrelativistic nonlinear scattering systems leads only to the low-frequency, Rayleigh-Jeans part of the spectrum. The erroneous view that classical physics leads inevitably to the Rayleigh-Jeans form for the entire spectrum corresponds to the mistaken conclusion reached by the physicists of a century ago. However, the physicists of the early 20th century were unaware of two crucial elements of classical physics, namely 1) the presence of classical electromagnetic zero-point radiation with a scale set by Planck’s constant $\hbar$, and 2) the importance of special relativity.

Many of the readers of this article (having been trained using misleading textbooks) may be thoroughly skeptical of the claims made in the preceding two paragraphs. Therefore let me point out the obvious problem with the views which are presented in contemporary treatments of blackbody radiation within classical physics. Classical electrodynamics is a relativistic theory. Nonrelativistic mechanics is not. However, electromagnetic radiation can be brought to thermal equilibrium only by the interaction of electromagnetic radiation
with charged mechanical systems. How can a nonrelativistic mechanical system which is inconsistent with special relativity be expected to give the equilibrium spectrum of relativistic classical electrodynamics? Thus classical statistical mechanics with its energy equipartition ideas is a nonrelativistic theory which cannot support the concept of zero-point energy for either mechanical particles or wave phenomena. In classical statistical mechanics, a non-relativistic particle of mass $m$ in one spatial dimension in thermal equilibrium has a kinetic energy $KE = (1/2)mv^2 = (1/2)k_BT$. However, this means that for fixed temperature $T$ and sufficiently small mass $m$, the velocity $v$ of the particle will exceed the speed of light in vacuum $c$. Such a situation is nonsense in a relativistic theory. The application of nonrelativistic classical statistical mechanics or of nonrelativistic scatterers to the problem of thermal radiation with the expectation of deriving the full radiation spectrum is a fundamental error which has persisted for over a century and still appears in the textbooks of modern physics. Indeed, we are aware that thermal radiation involves two regimes depending upon whether the ratio $\hbar \omega / (k_B T)$ is small or large. When $\hbar \omega / (k_B T)$ is small, we are in the Rayleigh-Jeans regime where nonrelativistic energy-equipartition ideas are satisfactory. When $\hbar \omega / (k_B T)$ is large, we are in the region dominated by Lorentz-invariant and scale-invariant classical zero-point radiation with a scale set by Planck’s constant $\hbar$. Lorentz-invariant mechanics (which depends on the constant $c$) provides the appropriate transition between the two regimes for scatterers of radiation, with the ratio $mc^2/(k_BT)$ providing the appropriate limits for a charged mass in a Coulomb potential.

The outline of our presentation is as follows. Before discussing the historical situation, we first summarize our current understanding of some basic classical ideas. We point out the experimental basis for classical zero-point radiation. Then we discuss relativistic mechanical systems, and we note the convenience of using action variables when discussing thermal equilibrium. Next we turn to the historical aspects of blackbody radiation. We summarize the thermodynamic aspects leading to the Wien displacement theorem. We review the thermodynamics of a single radiation mode and note that thermodynamics allows the asymptotic limits corresponding to zero-point energy and energy equipartition. Next we describe the historical blackbody controversies of the early 20th century, including Planck’s search for a place for his constant $\hbar$ within classical electrodynamics, and concluding with the Rayleigh-Jeans consensus. In the last section, we review a number of classical electromagnetic calculations of the early 20th century which led to the Rayleigh-Jeans spectrum,
and we show how the inclusion of classical zero-point radiation and appropriate relativistic treatments transforms the calculations into derivations of the Planck spectrum with zero-point radiation within classical physics. A closing summary ends the discussion.

II. CLASSICAL ELECTROMAGNETIC ZERO-POINT RADIATION

Before our discussion of the historical controversy involving blackbody radiation within classical electromagnetic theory, we wish to emphasize the current understanding of the two aspects which the early investigators missed: zero-point radiation and the importance of relativity. We start with the idea of classical electromagnetic zero-point radiation.

A. Experimental Evidence for Classical Zero-Point Radiation

The experimentalists who investigated the blackbody radiation spectrum around the turn of the 20th century were able to measure only the thermal radiation from their sources in excess of the ambient radiation surrounding their detectors. If their sources were at the same temperature as their detectors, they measured no signal at all. However, today experimenters have a great advantage over the earlier researchers. By using the Casimir effect, it is possible to measure not only the excess electromagnetic radiation arriving from a hot source, but indeed to measure the entire spectrum of radiation surrounding an object.

The Casimir effect involves the force between two uncharged conducting parallel plates. The conducting boundary conditions at the plates lead to forces associated with the radiation normal modes interacting with the plates. From the magnitude of the force and from the dependence of the force on the separation between the plates, it is possible to determine the entire spectrum of random classical radiation surrounding the plates. At high temperature $T$ or large separations $l$, the Casimir force expected from the Rayleigh-Jeans spectrum of random classical radiation surrounding conducting plates of area $A$ and separation $l$ is

$$F_{RJ} = -\frac{\zeta(3) k_B T A}{4\pi l^3} \quad (1)$$

At low temperatures, this force falls to zero along with the temperature $T$. However, experimental measurements of the Casimir force show that the force between the plates does not vanish with vanishing temperature $T \to 0$, but rather goes to the form which is
temperature independent,
\[ F_{zp} = -\frac{\pi^2 \hbar c A}{240 l^4} \]  

(2)

where \( \hbar \) is a constant which must be fitted from experiment, and which indeed takes the same numerical value as Planck’s constant. \[6\] Interpreting the zero-temperature Casimir force within classical electromagnetic theory,\[4\] we conclude that, surrounding the conducting plates, there must be a spectrum of random classical radiation corresponding to an average energy per normal mode

\[ U_{zp}(\omega) = \frac{1}{2} \hbar \omega. \]  

(3)

This is the classical electromagnetic zero-point radiation of which the physicists of a century ago were unaware. The experimentalist of that earlier period were unable to measure this temperature-independent random radiation, and, during the crucial period of decision, the theoretical physicists of that earlier era did not anticipate the possibility of such zero-point radiation.

Of course today, the Casimir force is usually interpreted through quantum theory, and some physicist wish to claim that classical zero-point energy cannot be used within a classical electromagnetic theory.\[7\] However, the classical electromagnetic calculations for the Casimir forces are perfectly valid classical calculations.\[4\] Indeed, the physicists at the turn of the 20th century treated thermal radiation as random classical radiation.

B. Properties of Classical Zero-Point Radiation

The zero-point radiation spectrum measured in Casimir experiments and appearing in Eq. (3) leads to an energy spectrum

\[ \rho_{zp}(\omega) = [\omega^2/(\pi^2 c^3)]U_{zp}(\omega), \]  

(4)

where the factor \([\omega^2/(\pi^2 c^3)]\) is the number of normal modes per unit (angular) frequency interval, and leads to a divergent energy density

\[ u_{zp} = \int_0^\infty d\omega \rho_{zp}(\omega) \to \infty. \]  

(5)

Despite the divergence of the energy density \( u_{zp} \), we anticipate no electromagnetic problems because each electromagnetic system interacts with radiation within only a limited range of frequencies.
At thermal equilibrium at positive temperature \( T > 0 \), we expect the thermal energy spectrum \( \rho_T(\omega, T) \) to be in addition to the zero-point energy \( \rho_{zp}(\omega) \) which exists at zero temperature. The total spectrum is the sum \( \rho(\omega, T) = \rho_T(\omega, T) + \rho_{zp}(\omega) \), or, in terms of the average energy per normal mode,

\[
U(\omega, T) = U_T(\omega, T) + U_{zp}(\omega),
\]

where \( U(\omega, T) \) is the total electromagnetic energy in the radiation mode. The energy density \( u_T \) due to thermal radiation \( \rho_T(\omega, T) \) is indeed finite

\[
u_T = \int_0^\infty d\omega \rho_T(\omega) < \infty.
\]

When we remove all possible thermal radiation from a container by going to zero temperature, what is left is the zero-point radiation.

Crucially, the spectrum of zero-point radiation appearing in Eq. (3) is Lorentz invariant; it takes the same spectral form in any inertial frame. Indeed, only a spectrum leading to a divergent energy density can look the same in every inertial frame. On the other hand, thermal radiation above the zero-point radiation has a finite energy density and a preferred inertial frame; the preferred frame is that of the container in which the radiation is at equilibrium.

It will be useful in the subsequent analysis to deal with the \( \sigma_{ltU^{-1}} \)-scale invariance of classical electromagnetism which leaves invariant the fundamental constants \( c, e, \) and \( \hbar \). In addition to being Lorentz-invariant, the spectrum of classical zero-point radiation in Eq. (3) is also \( \sigma_{ltU^{-1}} \)-scale invariant. By this we mean that if all lengths are transformed by the multiplicative factor \( \sigma \) so that \( l \to l' = \sigma l \), while all times are transformed as \( t \to t' = \sigma t \), and all energies are transformed as \( U \to U' = \sigma^{-1}U \), then the random zero-point radiation is unchanged. Indeed, the spectrum is unchanged, since

\[
U_{zp}'(\omega') = U_{zp}(\omega)/\sigma = (1/2)\hbar\omega/\sigma = (1/2)\hbar\omega'.
\]

Under an adiabatic expansion or compression of the thermal radiation in a spherical conducting-walled cavity, the total spectrum is transformed, but remains a blackbody spectrum at a new temperature. However, there is no mixing of the average zero-point and thermal contributions during the adiabatic change of the modes’ frequency, wavelength, and energy; the zero-point spectrum is \( \sigma_{ltU^{-1}} \)-invariant and \( U_{zp}(\omega) \) is mapped onto itself, \( U_{zp}(\omega) \to U'(\omega') = U_{zp}(\omega') \), while the thermal radiation \( U_T(\omega, T) \) is mapped onto thermal radiation at a new scale-transformed temperature \( U_T(\omega, T) \to U_T(\omega, T') \), where \( T' = T/\sigma \).
C. How is Zero-Point Radiation Different from Thermal Radiation?

One may ask, “What is the difference between zero-point radiation and thermal radiation within classical physics?” The answer is that there is no difference at all, except for the spectrum of the radiation. Thus for classical radiation in equilibrium in an enclosure, if we are told the average energy of a radiation mode without being told its frequency, then we do not know how much of the average energy is zero-point energy and how much is thermal energy. However, the spectrum of zero-point radiation is \( \sigma_{TU-1} \)-scale invariant so that the ratio \( U_{zp}(\omega)/\omega \) between the average energy of the radiation mode and the frequency is the same for all modes, \( U_{zp}(\omega)/\omega = U_{zp}(\omega')/\omega' \). On the other hand, for positive temperature \( T > 0 \), the ratios involving the total energy \( U \) of the radiation modes are related as \( U(\omega, T)/\omega > U(\omega', T)/\omega' \) for \( \omega < \omega' \) and \( T > 0 \), since the thermal contribution to the total energy of a radiation mode decreases with increasing frequency. In thermal equilibrium, the radiation energy above the zero-point energy is thermal energy. By going to sufficiently high frequency where the thermal energy contribution becomes ever smaller, we can determine the zero-point radiation spectrum which underlies the thermal contribution.

This classical idea that both zero-point radiation and thermal radiation are part of a single spectrum of random radiation is quite different from the prevailing quantum view that quantum zero-point energy (involving no photons) is quite different from the photons existing at positive temperature. No such distinction exists within classical physics, nor in the measurements of Casimir forces.

III. RELATIVISTIC MECHANICAL SYSTEMS FOR USE IN RELATIVISTIC ELECTRODYNAMICS

We have stressed that the misapprehension that classical physics leads inevitably to the Rayleigh-Jeans spectrum for thermal equilibrium arises because of the erroneous use of nonrelativistic mechanical systems as agents of radiation equilibrium for not only the low-frequency portion of the spectrum, but for the entire spectrum. In order to understand classical radiation equilibrium, we must demand that the mechanical agents of equilibrium are compatible with relativistic electrodynamics. In the present article, we will consider only relativistic point charges in Coulomb potentials (regarded as part of relativistic classical
electrodynamics), and point charges in harmonic-oscillator potentials in the zero-amplitude limit. The harmonic oscillator system appears frequently in historical accounts and can be regarded as the low-velocity limit of a relativistic mechanical system.

Also, it is convenient to introduce the action variables $J_i$ which appear in graduate courses in mechanics. In the early 20th century, the action variables appeared in Bohr-Sommerfeld quantization and played a role in Ehrenfest’s adiabatic theorem. These variables are also ideal parameters for use in thermodynamic systems. The action variables $J_i$ are both adiabatic invariants and also $\sigma_{ltU^{-1}}$-scale invariants. The dimensions of the action variables are $\text{energy} \times \text{time}$, and hence are invariant under a $\sigma_{ltU^{-1}}$-scaling transformation.\[11\]

A. Harmonic Oscillator in the Point Limit

A point charge in a harmonic-oscillator potential can be fitted into relativistic electrodynamics in the limit of zero size for the oscillation excursion. The harmonic oscillator is treated in the same fashion as the electromagnetic radiation modes, except that the spatial extent of the mechanical oscillator is taken as vanishingly small. In this case, the energy $\mathcal{E}_{osc}$ of a one-dimensional harmonic oscillator $\mathcal{E}_{osc} = \frac{m\dot{x}^2}{2} + \frac{\kappa x^2}{2} = \frac{m\dot{x}^2}{2} + m\omega_0^2 x^2/2$ can be given in terms of action-angle variables as\[12\]

$$\mathcal{E}_{osc}(\omega_0, J) = J\omega_0.$$  \hspace{1cm} (8)

The mass $m$ and spring-constant $\kappa$ are not discussed separately from the characteristic frequency $\omega_0 = (\kappa/m)^{1/2}$, while the displacement $x$ and velocity $v$ are regarded as so small as to be negligible. The system is $\sigma_{ltU^{-1}}$-scale covariant because energy and time are connected as in Eq. (8) while lengths do not appear. As pointed out by Planck, the point dipole oscillator in random classical radiation acquires an average energy $U_{osc}(\omega_0) = \langle \mathcal{E}_{osc}(\omega_0) \rangle$ equal to the average energy $U(\omega) = \langle \mathcal{E}(\omega) \rangle$ of the radiation normal modes at the same frequency $\omega = \omega_0$ as the oscillator.\[13\]\[14\] Also, the probability distribution for the action variable $J$ of the mechanical harmonic oscillator is the same as that for the radiation modes at the same frequency as the oscillator frequency.
B. Point Charge in a Coulomb Potential

The far more important relativistic system is that of a point charge in a Coulomb potential, since, in relativistic classical electrodynamics, point charges interact through electromagnetic fields. The relativistic energy \( E_C = mc^2 - Ze^2/r \) (with \( \gamma = (1 - v^2/c^2)^{-1/2} \)) of a point charge \( e \) in a Coulomb potential \( V_C(r) = -Ze^2/r \) can be written in terms of action variables as

\[
E_C(m, J_2, J_3) = mc^2 \left[ 1 + \left( \frac{Ze^2/c}{J_3 - J_2 + \sqrt{J_2^2 - (Ze^2/c)^2}} \right)^2 \right]^{-1/2},
\]

where \( J_1 = J_\phi, J_2 = J_\phi + J_\theta \), and \( J_3 = J_\phi + J_\theta + J_r \). We notice that the energy \( E_C(m, J_2, J_3) \) is a product of the particle rest energy \( mc^2 \) and a dimensionless function of \( J_i/(Ze^2/c) \). This system is \( \sigma_{\text{ltU}^{-1}} \)-scale covariant with the mass \( m \) as the one scaling parameter. The quantities \( J_i, Z, e, \) and \( c \) are all \( \sigma_{\text{ltU}^{-1}} \)-invariant. All lengths, times, and energies can be found from the fundamental length \( e^2/(mc^2) \), fundamental time \( e^2/(mc^3) \), and fundamental energy \( mc^2 \), multiplied by a function of \( J_i/(Ze^2/c) \). The existence of two different regimes associated with relativistic particle behavior is vividly illustrated in the unfamiliar trajectories of relativistic particles in a Coulomb potential, which can be strikingly different from the nonrelativistic orbits associated with conic sections.

IV. THERMAL BEHAVIOR IN TERMS OF ACTION-ANGLE VARIABLES IN RELATIVISTIC CLASSICAL ELECTRODYNAMICS

For the case of a circular orbit where \( J_r = 0 \) and \( J_2 = J_3 = J \), the energy becomes \( E_C(m, J) = mc^2\{1 - [Ze^2/(Jc)]^2\}^{1/2} \) with velocity \( v = Ze^2/J \) and the velocity ratio becomes \( v/c = Ze^2/(Jc) \). We notice that the range of \( J_2 = J_3 = J \) is limited, \( Ze^2/c < J < \infty \), corresponding to the particle velocity which is less than \( c \).

Having mentioned some aspects of relativistic classical systems, we now turn to the thermal behavior of classical systems.
A. Probability Distribution for Disordered Systems

The classical thermodynamics of both particles and radiation can be treated as involving random magnitudes for the action variables and random phases for the angle variables. When a periodic system is discussed in terms of action-angle variables, the system has an energy expression containing mechanical parameters characterizing the intrinsic system itself (such as mass $m$ or radiation-mode frequency $\omega$) and also the action variables $J_i$ which characterize the particular state of the system (such as the system’s angular momentum or energy). An ensemble of identical mechanical systems with the same mechanical parameters and with differing energies will be described by a probability distribution in the action variables $J_i$, analogous to the distribution on phase space used in statistical thermodynamics. For example, a one-spatial-degree-of-freedom system in thermal equilibrium with a heat bath at temperature $T$ has a probability distribution $P(J, T)dJ$ associated with the randomness of one action variable $J$. The average energy $U(T)$ of the system is found by integrating the energy $E(J)$ over the probability distribution $P(J, T)dJ$ for the action variable

$$U(T) = \int dJ \, E(J) \, P(J, T).$$

(10)

B. Probability Distribution for Radiation Modes

For a radiation normal mode of frequency $\omega$ in an enclosure at temperature $T$, the quantity $P(\omega, J, T)dJ$ can be determined by the fundamental properties of waves. Within classical physics, the fluctuations for waves can be described in terms of interference between waves of differing frequency, leading to a probability distribution for random wave behavior at frequency $\omega$ in the form of a Gaussian distribution in the wave amplitude, which corresponds to an exponential distribution in the action variable.[16] Thus the probability distribution for a radiation mode in a thermal bath involves the mode energy $\mathcal{E}(J) = J\omega$ divided by the average mode energy $U(\omega, T)$ and takes the form[17]

$$P(\omega, J, T)dJ = \exp \left[ \frac{-J\omega}{U(\omega, T)} \right] \frac{\omega}{U(\omega, T)} dJ.$$

(11)

The distribution is normalized so that $\int_0^\infty dJ \, P(\omega, J, T) = 1$. It is from this probability distribution that the entropy of the system would be evaluated, if the entropy function were known. We notice that the probability distribution satisfies $\sigma_{U^{-1}}$-scale covariance for the
theory, but the distribution (11) is not $\sigma_{ltU^{-1}}$-scale invariant, since, under a $\sigma_{ltU^{-1}}$-scale transformation, the energy is mapped to a new energy at a new temperature.

In the case of classical zero-point radiation, we know the explicit form for the average energy from Casimir force measurements, namely $U(\omega, 0) = (1/2)\hbar\omega$. Therefore equation (11) becomes

$$P(\omega, J, 0)dJ = \exp\left[\frac{-J\omega}{(1/2)\hbar}\right] \frac{\omega}{(1/2)\hbar}\omega dJ = \exp\left[\frac{-J}{\hbar/2}\right] \frac{2}{\hbar} dJ. \quad (12)$$

Thus at zero temperature, the zero-point radiation probability distribution for the action variable $J$ depends upon Planck’s constant $\hbar$, and is exactly the same for every radiation mode, independent of the frequency $\omega$ of the mode. The distribution (12) is $\sigma_{ltU^{-1}}$-scale invariant since $J$ and $\hbar$ are each $\sigma_{ltU^{-1}}$-scale invariant.

During an adiabatic change for a system, the distribution of the action variables $J_i$ remains unchanged, since the $J_i$ are adiabatic invariants. On the other hand, when heat energy is added to a system, the transfer of heat energy without work involves a change in the distribution of action variables while the physical dimensional parameters of the system remain unchanged. At zero temperature, there can be no transfer of heat and therefore no change in the distribution of action variables, even when the physical dimensional parameters of the system are changed and work is done. We notice in Eq. (12) (which holds at $T = 0$) that indeed the distribution of the action variable $J$ for a radiation mode of frequency $\omega$ does not change when the frequency of the mode is changed.

C. Probability Distribution for a Point Charge in a Coulomb Potential

We have mentioned two possible simple mechanical systems which can be regarded as part of relativistic classical electrodynamics: the point dipole oscillator (no spatial extent) and the point charge in a Coulomb potential. A mechanical harmonic oscillator (taken in the point-size limit with negligible velocity) is regarded as a passive mechanical system since it cannot change the frequency spectrum of random radiation. It behaves exactly as a radiation mode as concerns its distribution of action variables in thermal radiation, and so involves no new information.

The more interesting situation involves a relativistic point charge $e$ of mass $m$ in a Coulomb potential $V_C(r) = -Ze^2/r$ at temperature $T$. In this case, the probability dis-
tribution for the mechanical action variables $J_i$ may depend upon such quantities as the mass $m$, the potential constant $Ze^2$, the temperature $T$, and the speed of light $c$. Because probability is a dimensionless quantity, the probability distribution must depend upon dimensionless ratios, such as $\mathcal{E}_C(J_i)/(k_B T)$ involving the system energy $\mathcal{E}_C(J_i)$ given in Eq. (9) divided by $k_B T$. However, we note from Eq. (9) that the ratio $\mathcal{E}_C(J_i)/(k_B T)$ can be simplified so as to involve a product of the ratio $mc^2/(k_B T)$ and a function of $J_i/(Ze^2/c)$. Indeed, from dimensional considerations alone, the functional form of the probability distribution for the $J_i$ must depend on the dimensionless ratios $J_i/(Ze^2/c)$ and $mc^2/T$, giving a functional form $P_C[J_i/(Ze^2/c), mc^2/(k_B T)][c/(Ze^2)]dJ_i$.

We have noted that thermal radiation involves two different regimes depending upon the ratio $\hbar \omega / (k_B T)$ determining whether Lorentz-invariant zero-point radiation or thermal radiation provides the dominant energy at a given frequency. The two-regime distinction between zero-point energy and thermal energy for the relativistic particle in a Coulomb potential is also clear. The distinction involves the ratio $mc^2/(k_B T)$. This ratio reflects the ratio of mechanical frequency to temperature, since (from the energy expression in Eq. (9)) the frequency of the orbital motion $\omega_i(J_i) = \partial \mathcal{E}_C(J_i)/\partial J_i$ is proportional to the mass $m$ of the charged particle. The dominance of zero-point mechanical energy involves the situation of large mass $m$, where the frequency $\omega_i$ of orbital motion is high, the ratio $mc^2/k_B T$ is large, and the contribution of thermal energy is small. The high-frequency high-velocity limit of mechanical energy is associated with relativistic zero-point energy, just as the high-frequency limit of blackbody radiation is associated with relativistic zero-point radiation. On the other hand, the thermal energy becomes important in the opposite nonrelativistic limit where the mass $m$ is small, the particle velocity is small, the orbital frequency $\omega$ is small, and the ratio $mc^2/k_B T$ is small, in complete analogy with the electromagnetic radiation situation where the thermal energy dominates at low frequencies where the Rayleigh-Jeans form is indeed appropriate.

Crucially, at zero temperature, there is the possibility of non-zero energy where the thermodynamic probability distribution $P_C[J_i/(Ze^2/c)][c/(Ze^2)]dJ_i$ depends upon fundamental constants $Ze^2/c$ with no dependence upon the mass $m$ because there is no dimensionless ratio available which involves $m$. This $\sigma_{W-1}$-scale-invariant situation is exactly analogous to the situation for relativistic zero-point radiation where, at zero temperature, the radiation modes have non-zero energy while the thermodynamic probability distribution for the action
variables (given in Eq. 12) depends upon the fundamental constant \( \hbar \) with no dependence upon the frequency \( \omega \). The Coulomb potential of relativistic classical electrodynamics can indeed support relativistic zero-point energy through the fundamental constant \( e^2/c \), just as electromagnetic radiation can support relativistic zero-point radiation with an amplitude given by the fundamental scale factor \( \hbar \).

D. Nonrelativistic Classical Theory Cannot Support Zero-Point Energy

We emphasize that within classical theory, zero-point particle energy which is compatible with relativistic zero-point radiation exists only within relativistic electrodynamics, where, for the Coulomb potential, large mass is associated with high orbital frequency, and, for a circular orbit, the velocity increases to \( c \), \( v \rightarrow c \), as \( J \) decreases to \( Ze^2/c \), \( J \rightarrow Ze^2/c \). On the other hand, in the nonrelativistic mechanics of bounded potentials, large mass is ordinarily associated with low frequency, and the fundamental velocity \( c \) does not appear. Now, for a point harmonic oscillator, the particle amplitude and velocity can be regarded as negligible, and there is no conflict with a finite value of \( c \). However, there is an obvious conflict for nonlinear nonrelativistic mechanical systems where the particle amplitudes and velocities cannot be regarded as zero. The assumptions ordinarily made for a multiply periodic system with action variables \( J_i \) in (nonrelativistic) classical statistical mechanics are that \( J \) becomes infinite with each of the \( J \)'s, and . . . the amplitudes and energy vanish with the \( J \)'s.” This suggests that the velocity of the particle of the system can increase without limit as either the temperature of the system becomes large or the frequency of the system becomes large when adiabatic work is done on the system. The possibility of arbitrarily large system velocity is unacceptable in classical electrodynamics; there is no way in which a charge moving with velocity in excess of \( c \) can be reconciled with classical radiation equilibrium.

V. THERMODYNAMICS OF BLACKBODY RADIATION

Having reviewed some basic aspects for the current understanding of blackbody radiation within relativistic classical physics, we now turn to a historical perspective on blackbody radiation.
A. Thermodynamics in the History of Blackbody Radiation

Both thermodynamics and classical electrodynamics were developed during the 19th century. The theoretical treatment of blackbody radiation is founded upon Kirchoff’s blackbody analysis of 1860. The original thermodynamics discussion involved the emissive and absorptive properties of materials. Later the idea of cavity radiation was introduced. It was noted that the blackbody energy density $u_T(T)$ and the blackbody radiation spectrum $\rho_T(\omega, T)$ were universal functions independent of the material forming the walls of the cavity enclosing the radiation. A single “black particle” which scattered radiation within a conducting-walled cavity would change an arbitrary radiation distribution over to the spectrum of thermal equilibrium. Thus a “black particle” encodes within itself and its interactions the laws of thermodynamic equilibrium for radiation.

In 1879, based upon experimental measurements, Stefan suggested that the total thermal radiation energy $U_T$ in a container with volume $V$ at temperature $T$ was given by

$$U_T = a_S V T^4$$  \hspace{1cm} (13)

where $a_S$ was constant. In 1884, Boltzmann derived this Stefan-Boltzmann relation \((13)\) by combining ideas from thermodynamics and classical electrodynamics. With the appearance of the Stefan-Boltzmann law, blackbody radiation was easily recognized to involve a new fundamental constant $a_S$ (Stefan’s constant). Today Stefan’s constant is restated in terms of more familiar constants as \((20)\)

$$a_S = \frac{\pi^2 k^4}{15 \hbar^3 c^3}.$$  \hspace{1cm} (14)

Finally, in 1893, Wien applied thermodynamics to an adiabatic compression of thermal radiation and derived the displacement law. This law indicated that the blackbody radiation spectrum was of the form $\rho_T(\omega, T) = \text{const} \times \omega^3 f(\omega/T)$. Subsequent work showed that Wien’s displacement law was equivalent to the statement that the average energy per normal mode of radiation was given by $U_T(\omega, T) = \omega f(\omega/T)$ where $f(\omega/T)$ was an unknown function.

B. Thermodynamics of a Single Radiation Mode

Although Kirchoff, Stefan, Boltzmann, and Wien all thought in terms of thermal radiation having a finite energy density, we now know that measurements of Casimir forces indicate the...
existence of classical zero-point energy with its divergent energy density. Cole has reviewed the thermodynamics of blackbody radiation when zero-point radiation is included. It also seems helpful to reconsider the thermodynamics associated with a single radiation mode in order to clarify the situation in the presence of zero-point radiation.

The thermodynamics of an electromagnetic radiation mode (or indeed of a harmonic oscillator) is particularly simple since it involves only two thermodynamic variables: frequency \( \omega \) and temperature \( T \). In equilibrium with a heat bath, the average mode energy is denoted by \( U = \langle \mathcal{E} \rangle = \langle J \rangle \omega \) (where \( J \) is the action variable), and satisfies the first law of thermodynamics, \( dQ = dU + dW \), with the entropy \( S \) satisfying \( dS = dQ/T \). Now since \( J \) is an adiabatic invariant, the work done by the system is given by \( dW = -\langle J \rangle d\omega = -(U/\omega)d\omega \). Combing these equations, we have \( dS = dQ/T = [dU - (U/\omega)d\omega]/T \). Writing the differentials for \( S \) and \( U \) in terms of \( T \) and \( \omega \), we have \( dS = (\partial S/\partial T)dT + (\partial S/\partial \omega)d\omega \) and \( dU = (\partial U/\partial T)dT + (\partial U/\partial \omega)d\omega \). Therefore \( \partial S/\partial T = (\partial U/\partial T)/T \) and \( \partial S/\partial \omega = [(\partial U/\partial \omega) - (U/\omega)]/T \). Now equating the mixed second partial derivatives \( \partial^2 S/\partial T \partial \omega = \partial^2 S/\partial \omega \partial T \), we have \( (\partial^2 U/\partial \omega \partial T)/T = -(\partial U/\partial T)/(\omega T) + (\partial^2 U/\partial T \partial \omega)/T + [(U/\omega) - (\partial U/\partial \omega)]/T^2 \) or \( 0 = (\partial U/\partial T)/(\omega T) - [(U/\omega) - (\partial U/\partial \omega)]/T^2 \). The general solutions of the differential equations for \( U \) and \( S \) involve one unknown function of the single variable \( \omega/T \). Indeed, the whole thermodynamic system can be described by the unknown thermodynamic potential function \( \phi(\omega/T) \), a function of one variable \( \omega/T \), where the energy is given by

\[
U(\omega,T) = T^2(\partial \phi/\partial T)_\omega = -\omega \phi'(\omega/T)
\]

and the entropy corresponds to

\[
S(\omega/T) = \phi(\omega/T) + U(\omega,T)/T = \phi(\omega/T) - (\omega/T)\phi'(\omega/T).
\]

The result (15), obtained here purely from thermodynamics, corresponds to the familiar Wien displacement law of classical physics.

C. Thermodynamic Limits: Zero-Point Radiation and Equipartition

Although the thermodynamic analysis culminating in the Wien displacement theorem of 1893 simplified the classical blackbody problem down to an unknown function of one variable \( \omega/T \), the blackbody spectrum was still undetermined. However, it is noteworthy that the
unknown thermodynamic potential function $\phi(\omega/T)$ allows two natural limits which make the energy $U(\omega, T)$ independent of one of its variables.  

1. **Equipartition Limit at High Temperature**

If the function $\phi(\omega/T) \to -\text{const}_1 \times \ln(\omega/T)$ for small arguments $\omega/T \ll 1$, then the average energy $U(\omega, T)$ becomes independent of $\omega$, $U(\omega, T) = -\omega \phi'(\omega/T) \to \omega \times \text{const}_1 \times (T/\omega) = T \times \text{const}_1$. Choosing the constant $\text{const}_1$ as Boltzmann’s constant $k_B$, this limit becomes

$$U(\omega, T) \to k_B T \quad \text{for large } T.$$  

(17)

This is the familiar equipartition result of nonrelativistic kinetic theory and nonrelativistic classical statistical mechanics. It agrees with the experimentally-measured thermal radiation at high temperatures and low frequencies.

2. **Zero-Point Energy Limit at Low Temperature**

On the other hand, if the function $\phi(\omega/T) \to -\text{const}_2 \times (\omega/T)$ for large arguments $\omega/T \gg 1$, then the average energy $U(\omega, T)$ becomes independent of the temperature $T$, $U(\omega, T) = \omega \times \text{const}_2$. This energy corresponds to relativistic zero-point energy for the radiation mode provided that the constant $\text{const}_2$ is taken as half Planck’s constant $\hbar/2$,

$$U(\omega, T) \to (1/2)\hbar \omega \quad \text{for small } T.$$  

(18)

This corresponds to the relativistic-invariant and scale-invariant zero-point radiation which is measured experimentally from Casimir forces.

3. **Zero-Point Radiation Has Zero Entropy**

We see here that zero-point radiation fits in naturally with the Wien displacement theorem for thermal radiation. Furthermore, the entropy associated with zero-point radiation vanishes, since, for $\phi(\omega/T) = -(\hbar/2)(\omega/T)$, the entropy following from Eq. (16) is

$$S(\omega, T) = \phi(\omega/T) - (\omega/T)\phi'(\omega/T) = -(\hbar/2)(\omega/T) - (\omega/T)(-\hbar/2) = 0.$$  

(19)
Thus zero-point radiation makes no contribution to the thermodynamic entropy $S(\omega, T)$. The derivation of the Stefan-Boltzmann law in Eq. (13) involves only the total thermal energy $U_T(T)$ energy obtained by summing over the mode thermal energies $U_T(\omega, T)$ associated with changes in thermal mode entropies $S(\omega, T)$, and so refers to the thermal radiation energy above the zero-point radiation energy, $U_T(\omega, T) = U(\omega, T) - U(\omega, 0)$.

D. The Planck Spectrum is the “Smoothest Interpolation” Between Energy Equipartition and Zero-Point Energy

The thermodynamic potential can be rewritten in terms of the constants $k_B$ and $\hbar$ appearing in Eqs. (17) and (18) as a function $\phi[\hbar\omega/(k_B T)]$ of the single dimensionless variable $\hbar\omega/(k_B T)$ with asymptotic forms

$$\phi[\hbar\omega/(k_B T)] \rightarrow -k_B \ln(\hbar\omega/k_B T) \text{ for } \hbar\omega/k_B T << 1,$$

$$\phi[\hbar\omega/(k_B T)] \rightarrow -k_B \frac{\hbar\omega}{2 k_B T} \text{ for } \hbar\omega/k_B T >> 1,$$  \hspace{1cm} (20)

where the energy relation (15) now becomes

$$U(\omega, T) = -\frac{\hbar\omega}{k_B} \phi'[\hbar\omega/(k_B T)].$$ \hspace{1cm} (21)

The physicists of the 1890s were unaware of zero-point radiation and so did not discuss the asymptotic limits of the thermodynamic potential. Thus the physicists of this earlier era moved on to aspects which did not involve thermodynamics. However, since we are now working at a time when the asymptotic limits are known, and also we are used to the basic idea of thermodynamics involving smooth functions, we can try to evaluate the full blackbody radiation spectrum by making the “smoothest possible” interpolation between the known asymptotic limits.

Now the idea of a “smoothest possible” interpolation seems ambiguous except in the case where we are told that a function $f(z)$ takes the value $a$ at $z = 0$, has first derivative $-b$ at $z = 0$, ($a, b > 0$), and goes to zero as $z$ goes to infinity. In this case, it seems clear that the smoothest possible interpolation from $f(z) \approx a - bz$ near $z = 0$ to $f(\infty) = 0$ involves $f(z) = a \exp[-bz/a]$, since this function meets all the asymptotic requirements, introduces no new parameters or special points, and has all derivatives related back to the function itself.
Evaluation of the “smoothest possible” interpolation can be carried out as follows. First we rescale the thermodynamic potential by taking \( \phi[\hbar \omega/(k_B T)]/k_B \) and work with the scale-invariant \( \phi(z) \) where \( z = \hbar \omega/(k_B T) \). Next we remove the logarithmic behavior by taking the exponential of \(-\phi(z)\). Now we try to arrange for the difference between \( \exp[-\phi(z)] \) and some known function to have exactly the asymptotic properties of a “smoothest” function of the form \( a \exp[-zb/a] \). Clearly the required form appears for the combination \( \exp[z/2] - \exp[-\phi(z)] \), since we have the asymptotic limits

\[
\exp[z/2] - \exp[-\phi(z)] \rightarrow 1 + z/2 - z = 1 - z/2 \quad \text{for} \quad z \rightarrow 0,
\]
\[
\exp[z/2] - \exp[-\phi(z)] \rightarrow 0 \quad \text{for} \quad z \rightarrow \infty. \tag{22}
\]

Thus the “smoothest possible” interpolation between the thermodynamic limits is \( \exp[z/2] - \exp[-\phi(z)] = \exp[-z/2] \), so that \( \exp[-\phi(z)] = \exp[z/2] - \exp[-z/2] = 2 \sinh[z/2] \). Now restoring the scale for \( \phi \) and, inserting \( z = \hbar \omega/(k_B T) \), we have

\[
\phi[\hbar \omega/(k_B T)] = -k_B \ln\{2 \sinh[\hbar \omega/(2k_B T)]\}. \tag{23}
\]

But then the average energy per normal mode in Eq. (21) is

\[
U_{\text{exp}}(\omega, T) = -\frac{\hbar \omega}{k_B} \phi'[\hbar \omega/(k_B T)] = \frac{1}{2} \hbar \omega \coth\left(\frac{\hbar \omega}{2k_B T}\right) = \frac{\hbar \omega}{\exp[\hbar \omega/(k_B T)] - 1} + \frac{1}{2} \hbar \omega \tag{24}
\]

which is precisely the Planck spectrum including zero-point energy. Thus an easy interpolation between the thermodynamic limits (including the zero-point energy limit) gives the experimentally-measured spectrum.

VI. DETERMINING THE UNKNOWN SPECTRAL FUNCTION: HISTORICAL ASPECTS

Historically, the thermodynamic analysis of blackbody radiation reached its limit in the Wien displacement theorem. At the turn of the 20th century, there was no thought of relativistic zero-point radiation, and hence there was no attempt at any interpolation between the zero-point radiation and equipartition asymptotic forms. What was recognized was that electromagnetic radiation within a conducting-walled enclosure would not bring itself to thermal equilibrium. Therefore it was the interaction of matter with radiation...
in the sense of Kirchoff’s “black particles” which brought about thermal radiation equilibrium. Since the physics of matter was described in terms of classical mechanics, radiation equilibrium somehow involved the interaction of radiation with mechanical systems.

A. Planck’s Resonators

At the end of the 19th century, it was Planck who introduced linear electromagnetic “resonators” in hopes that they would serve as black particles and so determine the black-body radiation spectrum theoretically. Today we think of Planck’s resonators in terms of point electric dipole oscillators interacting with the surrounding radiation. Planck found that the average energy $U_{osc}(\omega_0, T)$ of an oscillator of natural frequency $\omega_0$ in random radiation at temperature $T$ is the same as the average energy $U(\omega, T)$ per normal mode of the random radiation in the modes at the same frequency $\omega = \omega_0$ as the natural frequency of the oscillator. Here was a first equilibrium connection between matter and radiation. However, despite Planck’s original hope, a harmonic oscillator does not act as a black particle. The average energy of the oscillator matches the average energy of the random radiation modes at $\omega_0$, but the oscillator does not determine the spectrum of blackbody radiation. The charged dipole oscillator scatters radiation so as to make the random radiation spectrum more nearly isotropic, but the frequency spectrum of the scattered radiation is exactly the same as the frequency spectrum of the incident radiation.

B. Conflict Between Classical Mechanics and Electrodynamics

Even during the 19th century, it was recognized that classical mechanics and classical electrodynamics had different, indeed clashing aspects. By the turn of the 20th century, “the relationship between electrodynamics and mechanics had, for a generation, been growing increasingly problematic.” Classical mechanics has no fundamental velocity, whereas classical electrodynamics incorporates a fundamental speed $c$ corresponding to the speed of light in vacuum. Furthermore, classical statistical mechanics (which is based upon nonrelativistic classical mechanics) leads to the idea of kinetic energy equipartition though point collisions of particles, whereas classical electrodynamics does not consider point particle collisions but rather involves long-range Coulomb forces which do not fit into nonrelativis-
tic classical statistical mechanics. Sharing energy by point collisions is satisfactory for nonrelativistic mechanics but becomes dubious for a relativistic theory with radiation.

In addition to the speed of light \( c \), classical electrodynamics was found to involve two new fundamental constants. Stefan’s constant was introduced in 1879, and the electron charge \( e \) was found around 1897. Thus the Coulomb force involved a smallest characteristic charge \( e \), corresponding to the electron charge. The conflicts between nonrelativistic classical mechanics and classical electrodynamics became intense in the last years of the 19th century when increasingly accurate experiments searched for the inertial frame of the electrical ether, and also when increasingly accurate experiments measured the heat transfer of blackbody radiation. Michelson and Morley carried out what is today the most famous of the ether-search experiments, and Paschen, Lummer and Pringsheim, Rubens and Kurlbaum provided the most famous blackbody measurements. The conflict between the theories of mechanics and electrodynamics reached a climax in the early years of the 20th century. It was realized that classical electrodynamics was a relativistic theory satisfying Lorentz transformations whereas Newtonian classical mechanics satisfied Galilean transformations between inertial frames. However, in the historical accounts of early 20th century physics, there appears to be no hint that relativity might have any relevance for the problems of thermal radiation.

C. Experimental Measurements in the Long-Wavelength Region

In 1896, shortly after his derivation of the displacement law, Wien suggested the expression for the blackbody spectrum \( \rho_W(\omega, T) = \alpha \omega^3 \exp[-\beta \omega/T] \) (\( \alpha, \beta \) positive constants) based on the current experimental data and some vague ideas from the Maxwell velocity distribution of classical particles in thermal equilibrium. However, new experimental measurements in the long-wavelength (low frequency) region by Lummer and Pringsheim and by Rubens and Kurlbaum in 1899 clearly disagreed with Wien’s suggestion, although Wien’s distribution continued to represent the high-frequency region well. Informed of this experimental disagreement, Planck attempted an interpolation between the new experimentally-determined low-frequency spectrum and the satisfactory high-frequency behavior appearing in Wien’s distribution. The interpolation involved the energy \( U \) and entropy \( S \) of the
radiation modes. Planck’s interpolation of 1900 led to the average energy per normal mode

\[ U_P(\omega, T) = \frac{\hbar \omega}{\exp[\hbar \omega/(k_B T)] - 1}. \]  

(25)

Planck’s radiation spectrum in Eq. (25) fitted the experimental measurements of thermal radiation so well that it became a focus of theoretical analysis thereafter.

D. Direct Use of the Equipartition Theorem of Classical Statistical Mechanics

In the early years of the 20th century, Rayleigh and Jeans gave the derivation of the classical radiation spectrum which is still quoted in all the textbooks of modern physics. In 1900, Rayleigh suggested that the “Maxwell-Boltzmann doctrine of the partition of energy” be applied to the thermal radiation in a cavity. He came up with the average energy per radiation normal mode \( U(\omega, T) = k_B T \). Since this spectrum clearly did not fit the high-frequency part of the spectrum, he introduced an exponential factor which cut off the spectrum at high frequency. In 1905, Jeans extended the equipartition analysis on a carefully-argued basis, so that today the radiation energy per normal mode

\[ U_{RJ}(\omega, T) = k_B T \]  

(26)

is known as the Rayleigh-Jeans spectrum. This spectrum agreed with experimental measurements in the low-frequency region, but was clearly wrong at high frequency. Jeans felt so strongly about the firm theoretical basis for his equipartition analysis that he suggested that the high-frequency regions of the experimentally-measured spectrum might not correspond to thermal equilibrium.

It is noteworthy that Planck’s constant does not appear in the Rayleigh-Jeans spectrum (26). Rather, the Rayleigh-Jeans spectrum allows a completely independent scaling in energy, unconnected with other quantities, as is typical in nonrelativistic mechanics.

E. Lorentz’s Use of Maxwell’s Velocity Distribution from Classical Kinetic Theory

In 1903, Lorentz gave a derivation\[28\] of the Rayleigh-Jeans law from classical physics based upon the interaction of radiation with the electrons in a thin slab of material. The motion of the electrons was described in terms of the Drude model involving free-particle
motion with collisions where the velocity distribution for the electrons was taken as the
Maxwell velocity distribution of nonrelativistic classical kinetic theory. The approximations
applied were all appropriate for long wave-length radiation, and indeed Lorentz arrived
at exactly the Rayleigh-Jeans result in Eq. (26). Lorentz looked upon his work as a
confirmation of the understanding of the long-wavelength (low-frequency) portion of the
spectrum.

F. Climax of the Classical Statistical Mechanical Arguments

Although the use of classical statistical mechanical arguments to arrive at the Rayleigh-
Jeans spectrum is repeated in all the modern physics texts today, the physicists of the first
decade of the 20th century did not regard these arguments as compelling. The classical
statistical mechanical arguments were regarded as merely providing an understanding for
the long-wavelength region of the blackbody spectrum.

It was Lorentz’s speech in Rome in 1908 which precipitated the change in the accepted
views regarding blackbody radiation. Lorentz reviewed the derivations of the Rayleigh-Jeans
spectrum from classical statistical mechanics, and then raised the possibility that Jeans’ view
was correct; perhaps the experimentally-measured high-frequency portions of the spectrum
did not represent thermal equilibrium. The experimentalists, who had previously ignored
Jeans’ suggestions regarding the failure of thermal equilibrium, did not ignore Lorentz’s
remarks; they reacted with outraged ridicule to the idea that their careful measurements
did not correspond to thermal equilibrium. Lorentz then quickly retreated. “Only after
Lorentz’s Rome lecture does the physics profession at large seem to have been confronted
by what shortly came to be called the ultraviolet catastrophe...” [29]

VII. THE PLACE FOR PLANCK’S CONSTANT IN CLASSICAL PHYSICS

In the reanalysis of the theoretical situation following Lorentz’s speech and the experi-
menters’ contradictions, attention focused on the apparent absence of any place for Planck’s
constant \( \hbar \) within classical theory.
A. Blackbody Radiation Introduces a New Constant into Physics

The experimental measurements of blackbody radiation indicated clearly that the radiation involved two constants, not simply one. Indeed, Wien’s suggestion for the blackbody radiation spectrum \( \rho_W(\omega, T) = \alpha \omega^3 \exp[-\beta \omega/T] \) involved two constants from fitting the experimental data. Planck’s work (relating the radiation spectrum \( \rho \) to the average energy \( U \) for an oscillator) gave for Wien’s suggested spectrum \( \rho_W(\omega, T) \) the average oscillator energy \( U_W(\omega, T) = \hbar \omega \exp[-\beta \omega/T] \) where Planck introduced the constant \( h = 2\pi \hbar \). In May 1899, Planck determined the value of the constant as \( h = 2\pi \hbar = 6.885 \times 10^{-27} \) erg-sec from the experimental blackbody data.\(^30\) Planck’s subsequent work (attempting to give a statistical mechanical basis for his own interpolated fit to the experimental data) separated the constant \( \beta \) as \( \beta = \hbar/k_B \) involving Boltzmann’s constant \( k_B \) (related to the gas constant \( R \) and Avogadro’s number \( N_A \) of classical kinetic theory) and his constant \( h = 2\pi \hbar \). Although \( k_B = R/N_A \) involved familiar constants from classical kinetic theory, the constant \( h \) was unfamiliar. Clearly, the constant \( h \) was a fundamental constant since it appeared in the universal blackbody radiation spectrum. Indeed, Planck’s constant \( h \) is an alternative parameter to Stefan’s constant \( a_S \) which was introduced in 1879, since Stefan’s constant can be written in the form given in Eq. \(^14\). However, Planck’s constant \( h \) was a new fundamental constant in the sense that it was unknown in classical statistical mechanics or in classical electrodynamics.

Indeed, in the struggle to understand blackbody radiation at the beginning of the 20th century, one of the most compelling arguments for the turn to a new quantum theory was the apparent absence of any place for the new Planck constant \( h \) within classical mechanics or classical electrodynamics. If Planck’s constant is taken to zero, \( \hbar \to 0 \), then Planck’s spectrum in Eq. \(^25\) becomes the Rayleigh-Jeans spectrum \(^26\) to leading order in \( \hbar \omega/(k_B T) \). Within classical physics, there seemed to be no possible entry point for Planck’s constant which was so vital to Planck’s blackbody radiation spectrum. It is noteworthy that Planck, the “reluctant revolutionary” of the early 20th century, was desperately seeking a role for his constant \( h \) within electrodynamics, and felt he could find none.\(^31\)


B. Planck’s Constant in Classical Electrodynamics

Today, with the advantages of hindsight, we know exactly the classical electrodynamical role for Planck’s constant that eluded the physicists of the early 20th century. Planck’s constant appears naturally as the scale factor in relativistic classical electromagnetic zero-point radiation. Yet it is fascinating that our current textbooks of electrodynamics and modern physics either omit mention of this natural role for \( \hbar \) or hide the possibility of a classical electrodynamical role for \( \hbar \).

It was clear to the physicists of 1900, as it is clear to the physicists of today, that Planck’s constant \( \hbar \) does not appear in Maxwell’s equations, the fundamental differential equations for the relativistic classical electromagnetic fields. On the other hand, the speed of light \( c \) is at least implicit in Maxwell’s equations. The general solution of Maxwell’s differential equations when written in terms of the electromagnetic potentials \( V(r, t) \) and \( A(r, t) \) and using the retarded Green function, takes the form

\[
\begin{align*}
V(r, t) &= V^\text{in}(r, t) + \int d^3r' \int dt' \frac{\delta(t - t' - |r - r'|/c)}{|r - r'|} \rho(r', t') \\
A(r, t) &= A^\text{in}(r, t) + \int d^3r' \int dt' \frac{\delta(t - t' - |r - r'|/c)}{|r - r'|} \frac{J(r', t')}{c}
\end{align*}
\] (27)

where \( \rho(r, t) \) and \( J(r, t) \) are the charge density and current density of the sources, and \( V^\text{in}(r, t) \) and \( A^\text{in}(r, t) \) are the (homogeneous) source-free terms. All the authors of the electromagnetism textbooks agree that these are indeed the correct solutions. However, in current textbooks of electrodynamics, the source-free terms \( V^\text{in}(r, t) \) and \( A^\text{in}(r, t) \) are always omitted.\[32\] Indeed, it is precisely as the scale factor in these source-free terms that Planck’s constant makes a natural entry into relativistic classical electromagnetic theory.\[24\]

Thus in a shielded spacetime region at zero temperature where only source-free contribution is zero-point radiation, the scalar potential can be taken to vanish and the vector potential can be written as\[25\]

\[
A(r, t) = \sum_{\lambda=1}^{\lambda=2} \int d^3k \, \hat{\epsilon}(k, \lambda) \left( \frac{\hbar}{2\pi^2\omega} \right)^{1/2} \sin[k \cdot (r - \omega t + \theta(k, \lambda))] \\
+ \int d^3r' \int dt' \frac{\delta(t - t' - |r - r'|/c)}{|r - r'|} \frac{J(r', t')}{c}.
\]

The appearance of Planck’s constant \( \hbar \) in the source-free term corresponds to the presence of relativistic classical electromagnetic zero-point radiation.
From a historical perspective, we can see why it was so hard for the physicists of the early 20th century to find a place for Planck’s constant within classical electrodynamics. Planck’s constant does not enter the fundamental differential equations for classical electrodynamics but rather enters only in the (homogeneous) source-free solution to the differential equations. Indeed, most current physicists of the early 21st century are probably just as unaware of the possibility of a source-free contribution to the general solution of Maxwell’s equations as were the physicists of the previous century. Our classical electromagnetism textbooks all omit the (homogeneous) source-free contribution in their statement of the general solution of Maxwell’s equations, and so physicists are unaware of the possibility. However, as has been pointed out before, the choice of this (homogenous) source-free boundary condition is “as much a part of the postulates of the theory as the form of the Lagrangian or the value of the electron charge.”[33]

C. Contrasting Roles for Planck’s Constant in Classical and Quantum Theories

Planck’s constant enters both classical and quantum theories, but the roles played in the theories are strikingly different. Within quantum theory, Planck’s constant is embedded in the foundations of the theory. The existence of fundamental commutators such as for position and momentum operators \( [\hat{x}, \hat{p}_x] = i\hbar \) depends upon the non-zero value for \( \hbar \). From the commutators, one can derive the zero-point energy \( U_{\text{quantum}} \) for a quantum harmonic oscillator of natural frequency \( \omega_0 \) giving \( U_{\text{quantum}} = (1/2)\hbar \omega_0 \). Within quantum theory, the quantum character of the theory and ideas of zero-point energy both collapse with the limit \( \hbar \to 0 \) which removes Planck’s constant from the theory.

On the other hand, Planck’s constant enters classical electrodynamics only as a scale factor in the (homogeneous) source-free part of the general solution of Maxwell’s differential equations. Therefore there are two natural versions of classical electrodynamics. In one form, Planck’s constant \( \hbar \) is taken as non-zero, and zero-point radiation exists. In the other form, the source-free contribution to the general solution of Maxwell’s equations is taken to vanish, and Planck’s constant does not appear in the theory. It is only this last version which is presented in the current textbooks of classical electrodynamics and modern physics.
VIII. ZERO-POINT RADIATION, NONRELATIVISTIC PHYSICS, AND THE RAYLEIGH-JEANS CONSENSUS

Unable to find a role for Planck’s constant within classical mechanics or classical electrodynamics, and confronted with classical derivations leading to the Rayleigh-Jeans spectrum, the physicists of 1910 concluded that classical physics led inevitably to the “ultraviolet catastrophe” contained within the Rayleigh-Jeans spectrum. This consensus regarding blackbody radiation, which was reached a century ago, is still repeated in all the textbooks. [1]

A. Collapse of the Arguments of 1910

The thermal experiments of the early 20th century involved a crucial limitation which confused the classical electromagnetic theorists at the beginning of the 20th century. The experimentalists measured only the random radiation of their sources which was above the random radiation surrounding their measuring devices. If their sources were at the same temperature as their measuring devices, the measuring devices registered no signal at all. Today, in contrast to the end of the 19th century, random classical radiation measurements are available which are of an entirely different character from those of a century earlier.

Today, awareness of the possibility of relativistic classical electromagnetic zero-point radiation (and indeed of its experimental detection by Casimir force measurements), leads to the collapse of all the arguments of 1910 in favor of the Rayleigh-Jeans spectrum being the inevitable conclusion of classical physics. All the theoretical analysis for blackbody radiation in the early 20th century was in terms of nonrelativistic classical statistical theory. However, nonrelativistic physics cannot support the idea of a zero-point energy. Nonrelativistic classical statistical mechanics cannot include zero-point radiation. Therefore it is not surprising that application of classical statistical mechanics gave agreement with experiment only in the long-wavelength (low frequency) region of the spectrum where the relativistic zero-point contribution is negligible and can safely be ignored.

B. Nonrelativistic Scattering Calculations and the Importance of Relativity

There is a further set of calculations which did not influence the analysis of the early 20th century but appeared subsequently and seemed to confirm the earlier conclusion. These
are the scattering experiments involving nonrelativistic mechanical scatterers in random radiation. As was remarked earlier, radiation equilibrium is achieved when some mechanical scatterer (black particle) interacts with radiation in a cavity. A point dipole oscillator does not act as a black particle since it only changes the directions of the radiation and not the frequency spectrum. However, a charged nonlinear oscillator is not passive, and will indeed change the frequency spectrum of random radiation.

In 1925, van Vleck calculated the scattering of random radiation by a nonrelativistic nonlinear multiply periodic system in the dipole approximation which couples radiation to particle motion only through the frequency of the motion, and not through a spatially-extended orbital motion. Although he published only a preliminary report on his work, he concluded that the nonlinear multiply periodic system assumed the Boltzmann distribution and was in equilibrium with the Rayleigh-Jeans spectrum of random radiation. This same conclusion was reached in subsequent calculations. Indeed, in 1983, Blanco, Pesquera, and Santos went further and showed that a charged particle with relativistic linear momentum \( p = m\gamma v \) in a general class of potentials also came to equilibrium at the Boltzmann distribution for the particle in the presence of the Rayleigh-Jeans spectrum for the radiation. Significantly, the general class of potentials excluded the Coulomb potential, the only potential in relativistic electrodynamics.

All of these calculations fail to represent nature in the high-frequency region of the spectrum because they are not relativistic calculations. None of these calculations allow zero-point energy in the mechanical system. Indeed, the one calculation which attempted to be relativistic excluded the one potential, the Coulomb potential, which, when incorporated into relativistic classical electrodynamics, is indeed relativistic. Crucially, the Coulomb potential allows relativistic mechanical zero-point energy; large mass \( m \) is associated with high velocity and high frequency so that the mechanical zero-point energy associated with large values of \( mc^2/(k_B T) \) is matched consistently with high frequency radiation where \( \hbar \omega/(k_B T) \) is large and relativistic zero-point radiation dominates the thermal radiation spectrum. Contrary to this relativistic situation, in nonrelativistic potentials \( V(\mathbf{r}) \) where \( V(0) \) is finite, a large mass is associated with low frequency oscillations.
IX. CLASSICAL CALCULATIONS GIVING THE PLANCK SPECTRUM WITH ZERO-POINT RADIATION

In the section above, we noted that the classical theoretical analyses leading to the Rayleigh-Jeans law are valid only in the low-frequency portion of the spectrum; the calculations use nonrelativistic classical mechanics and/or do not allow the presence of relativistic classical zero-point radiation. However, there are several historical calculations which can be transformed by the introduction of zero-point radiation in a valid context. Here we review these historical calculations, and also mention additional analyses. The calculations come from a wide variety of views, but all contribute to the overwhelming evidence that relativistic classical electrodynamics including relativistic classical zero-point radiation predicts the Planck spectrum with zero-point radiation for blackbody radiation.

A. Einstein’s Fluctuation Analysis for Radiation Modes

1. Original Einstein Calculation

We consider first Einstein’s radiation-fluctuation analysis\(^{[37]}\) of 1909. Einstein considered the fluctuations of radiation in a box, and, connecting entropy with probability, arrived at the relation between the entropy \(S\) of a radiation normal mode and the average energy \(U\) of the mode

\[
\frac{\partial^2 S}{\partial U^2} = -\frac{k_B}{\langle \varepsilon^2 \rangle} \tag{28}
\]

where \(\langle \varepsilon^2 \rangle\) corresponds to the mean square fluctuation in the energy of the radiation normal mode. Now for radiation, the mean-square energy fluctuation can be obtained from purely classical wave theory\(^{[16]}\)\(^{[38]}\) as \(\langle \varepsilon^2 \rangle = U^2\). Introducing this result into Eq. (28), we have

\[
\frac{\partial^2 S}{\partial U^2} = -\frac{k_B}{U^2} \tag{29}
\]

If we integrate once with regard to \(U\), we have

\[
\frac{\partial S}{\partial U} = \frac{k_B}{U} = \frac{1}{T} \tag{30}
\]

which leads to exactly the Rayleigh-Jeans result \(U = k_B T\). This was Einstein’s result.
2. Modification to Include Zero-Point Radiation

However, if relativistic classical electromagnetic zero-point radiation is present, making the total energy $U = U_T + U_{zp}$, then the entropy $S$ should be associated with only the thermal energy $U_T$ and not with the zero-point energy $U_{zp} = (1/2)\hbar\omega$ in the normal mode. As noted above in Section V C3, zero-point radiation involves vanishing entropy. Nevertheless, both sources of energy contribute to the amplitude of the radiation field and so to the fluctuations of the radiation associated with the normal mode. Thus we should associate the entropy change with the fluctuations $\langle \varepsilon_T^2 \rangle$ above the zero-point fluctuations:

$$\langle \varepsilon_T^2 \rangle = U^2 - U_{zp}^2 = (U_T + U_{zp})^2 - U_{zp}^2 = U_T^2 + 2U_T U_{zp}. \quad (31)$$

Therefore the connection of Eq. (28) becomes

$$\frac{\partial^2 S}{\partial U^2} = -k_B U^2 - U_{zp}^2. \quad (32)$$

Now integrating once with respect to $U$, we have

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{k_B}{2U_{zp}} \ln \left( \frac{U + U_{zp}}{U - U_{zp}} \right). \quad (33)$$

Simplifying and taking the exponential so as to remove the logarithm while inserting $U_{zp} = (1/2)\hbar\omega$, we find exactly the Planck spectrum with zero-point radiation as in Eq. (24) above. Thus simply adding relativistic classical electromagnetic zero-point radiation to Einstein’s analysis immediately gives us the full Planck spectrum with zero-point radiation. Since Einstein (and his contemporaries) did not consider classical electromagnetic zero-point radiation, he instead reinterpreted Eq. (31) in terms of photons of energy $2U_{zp} = 2[(1/2)\hbar\omega] = \hbar\omega$.

Only relativistic classical electromagnetic radiation enters this calculation; there are no considerations involving relativistic particle behavior. The inclusion of classical zero-point radiation leads directly to the Planck spectrum with zero-point radiation.

B. Moving Oscillator-Particle of Einstein and Hopf

1. Original Einstein-Hopf Calculation

In 1910, Einstein continued his work on blackbody radiation beyond the radiation-fluctuation analysis mentioned above. Wishing to drive home the point that classical
physics predicted the Rayleigh-Jeans spectrum for the equilibrium between radiation and matter, Einstein and Hopf\cite{40} considered a particle of mass $m$ moving with one translational degree of freedom and containing a small dipole oscillator of natural frequency $\omega_0$ interacting with random classical radiation. The idea of the calculation was to connect the translational motion of the oscillator to the spectrum of random radiation which was providing impulses to the particle through the interaction with the dipole oscillator. Einstein and Hopf required that the average kinetic energy of the oscillator must be $(1/2)k_BT$, corresponding to the kinetic energy equipartition theorem. This kinetic-energy equipartition for the translational motion of point particles was regarded as the most secure aspect of classical statistical mechanics.

The change $mv(t + \tau) - mv(t)$ in the translational momentum $mv(t)$ of the particle during a short time interval $\tau$ was taken as due to a random impulse $\Delta$ due to the random radiation interacting with the oscillator, and a velocity-dependent damping $-Pv(t)\tau$ due to the retarding force on the oscillator arising from the motion of the oscillator through the doppler-shifted random radiation. Thus at the end of the short time interval $\tau$, the momentum was

$$mv(t + \tau) = mv(t) + \Delta - Pv(t)\tau$$

(34)

In equilibrium, the average translational energy of the particle should not change in time, so that through first order in the time $\tau$

$$\langle [mv(t + \tau)]^2 \rangle = \langle [mv(t) + \Delta - Pv(t)\tau]^2 \rangle$$

$$= \langle [mv(t)]^2 \rangle + \langle \Delta^2 \rangle + 2(m - P\tau)\langle v(t)\Delta \rangle - 2mP\tau\langle [v(t)]^2 \rangle$$

(35)

Now the random impulse $\Delta$ is as often positive as negative so that $\langle v(t)\Delta \rangle = 0$, and we find

$$0 = \langle \Delta^2 \rangle - 2mP\tau\langle [v(t)]^2 \rangle.$$ 

Einstein and Hopf calculated\cite{40,9} the square of the average random impulse $\langle \Delta^2 \rangle$ and also the retarding force coefficient $P$ for a general spectrum of random radiation and found the connection

$$\frac{4\Gamma^4c^4}{5\omega^2} [\rho(\omega, T)]^2 - 2 \left\{ \frac{6c^2\Gamma^2}{5} \left[ \rho(\omega, T) - \frac{\omega \partial \rho(\omega, T)}{3 \partial \omega} \right] \right\} \tau m\langle [v(t)]^2 \rangle = 0$$

(36)

Now assuming that the equilibrium translational kinetic energy of the particle was $(1/2)mv^2 = (1/2)k_BT$, they derived $\rho_T(\omega, T) = [\omega^2/(\pi^2c^3)]k_BT$, the Rayleigh-Jeans spectrum.
2. Einstein-Stern Modification

It is striking that Einstein was among the first to realize that the inclusion of zero-point energy sharply changed the equilibrium between matter and radiation. In the years 1910-1912, Planck developed his “second theory” of blackbody radiation which included a zero-point energy \((1/2)\hbar\omega\) for oscillators in equilibrium with radiation, but excluded any zero-point energy for the radiation field itself. Planck and the other physicists of the time apparently made no connection between the oscillator zero-point energy and relativity. Einstein and Stern\[41\] picked up the idea of zero-point energy for an oscillator and modified the earlier Einstein-Hopf calculation by including a zero-point energy \(\hbar\omega\) (no factor of 1/2) for the oscillator. Thus instead of \([\rho(\omega, T)]^2\) in \(\langle \Delta^2 \rangle\), they introduced \(\rho(\omega, T)\{\rho(\omega, T) + \hbar\omega^3/(\pi^2c^3)\}\) where the last term correspond to the zero-point energy \(\hbar\omega\) of the oscillator.

The differential equation (36) from the work of Einstein and Hopf was modified to

\[
\frac{4\pi^4c^4\tau}{5\omega^2}\rho(\omega, T) \left[ \rho(\omega, T) + \frac{\hbar\omega^3}{\pi^2c^3} \right] - 2 \left\{ \frac{6c\pi^2\tau}{5} \left[ \rho(\omega, T) - \frac{\omega}{3} \frac{\partial \rho(\omega, T)}{\partial \omega} \right] \right\} \tau m \langle [v(t)]^2 \rangle = 0. \tag{37}
\]

Now introducing \((1/2)mv^2 = (1/2)k_BT\), Einstein and Stern found that the modified relation (37) gave the Planck spectrum without any zero-point energy for the radiation field, \(\rho(\omega, T) = \left[ \omega^2/(\pi^2c^3) \right] U_P(\omega, T)\) with \(U_P(\omega, T)\) as given in Eq. (25). The Einstein-Stern calculation attracted little notice as a classical calculation since physicists had already decided that classical physics led to the Rayleigh-Jeans spectrum, and zero-point energy for an oscillator was regarded as part of the new physics of discrete quanta.\[42\] As far as classical physics is concerned, this calculation (with zero-point energy \(\hbar\omega\) for the oscillator but not for the radiation) is actually inconsistent; an oscillator comes to equilibrium with ambient random radiation when the average energy of the oscillator matches that of the radiation modes at the same frequency as the oscillator.

Einstein and Stern also pointed out that the Planck spectrum for the oscillator as given in Eq. (25) did not go over fully to the Rayleigh-Jeans asymptotic form at high temperature, but rather involved

\[
U_P(\omega, T) = \frac{\hbar\omega}{\exp[\hbar\omega/(k_BT)] - 1} = k_BT - \frac{1}{2} \hbar\omega + O\left( \frac{\hbar\omega}{k_BT} \right). \tag{38}
\]

The inclusion of a zero-point energy \((1/2)\hbar\omega\) for the oscillator removed the temperature-independent term on the right-hand side of Eq. (38).
3. Recent Modification

Another modified version of the Einstein and Hopf calculation, but now involving purely classical electromagnetic theory, appeared in 1969. This time zero-point energy \((1/2)\hbar\omega\) was introduced in the classical radiation field with only the thermal part of the radiation field regarded as unknown. Due to the zero-point radiation alone, there is no velocity-dependent retarding force, since zero-point radiation is relativistically invariant and takes the same form in every inertial frame. Thus in the presence of zero-point radiation, the particle of Einstein and Hopf (which involved no radiation damping of the translational motion) would undergo a random walk in velocity without any retarding force, and would never come to equilibrium. In this case, the Einstein-Hopf analysis required modification associated with radiation damping of the translational motion as the particle was accelerated at the walls of the container. Introducing the radiation-damping impulse into Eq. (34) corresponds to removing the impulse of the zero-point radiation; the Einstein-Hopf equation is then changed to

\[
0 = \frac{4\pi^4 c^4 \tau}{5\omega^2} \left[ [\rho(\omega, T)]^2 - [\rho(\omega, 0)]^2 \right] - 2 \left\{ \frac{6c\pi^2 \Gamma}{5} \left[ \rho(\omega, T) - \frac{\omega}{3} \frac{\partial \rho(\omega, T)}{\partial \omega} \right] \right\} \tau m \langle [v(t)]^2 \rangle
\]

(39)

where \(\rho(\omega, 0) = [\omega^2/(\pi^2 c^3)](1/2)\hbar\omega\) corresponds to the zero-point radiation spectrum. Now, the solution for the radiation field giving the average kinetic energy \((1/2)mv^2 = (1/2)k_B T\) for the particle of large mass was found to be the Planck spectrum including zero-point radiation \(\rho(\omega, T) = [\omega^2/(\pi^2 c^3)]U_{P_{zp}}(\omega, T)\) with \(U_{P_{zp}}(\omega, T)\) as given in Eq. (24). Since this modified Einstein-Hopf calculation involves both relativistic classical electromagnetic zero-point radiation and can be regarded as a relativistic calculation involving negligible velocity in the large-mass limit, the calculation meets all the criteria suggested here for giving an experimentally-correct radiation spectrum, and indeed the correct spectrum is what is found.

It is also noteworthy that Marshall has provided a modified version of Einstein’s 1909 analysis of the Brownian motion of a moving mirror in the presence of random radiation. Again Einstein’s derivation of the Rayleigh-Jeans spectrum becomes a derivation of the Planck spectrum with zero-point radiation when zero-point radiation is included in the random thermal radiation.
C. Comparing Diamagnetism of a Free Particle and Paramagnetic Behavior

According to the form of classical electrodynamics which ignores classical zero-point radiation and which appears in all the textbooks, classical physics does not allow diamagnetic behavior. However, if classical electromagnetic zero-point radiation is included, then classical electromagnetism indeed shows diamagnetic behavior for a charged particle in an isotropic harmonic-oscillator potential in the presence of a magnetic field.

Although diamagnetic behavior for a charged particle does indeed appear within classical physics including classical zero-point radiation, the diamagnetic behavior disappears if the random radiation spectrum is taken as the Rayleigh-Jeans spectrum. Thus we again encounter distinct asymptotic forms associated with low- and high-temperature limits. We expect that in thermal radiation, the diamagnetic behavior is a continuous function of temperature $T$. The behavior is most striking for the free-particle case involving a large magnetic field $B$ and a small isotropic oscillator potential $V(r) = (1/2)m\omega_0 r^2$, $eB/(mc) >> \omega_0$. In this case at zero temperature, the average particle orbital angular momentum $\langle L \rangle$ takes the magnitude $\hbar$ and is oriented antiparallel to the direction of the magnetic field. As the mass $m$ is increased while maintaining the frequency ratio $eB/(mc)$ as constant, the situation becomes that of negligible velocity, and so can be regarded as the low-velocity limit of a relativistic system.

Now the existence of a non-zero average magnetic moment at low temperature and its disappearance at high temperature is exactly the sort of behavior which is found for a paramagnetic magnetic moment treated using classical statistical mechanics with a Boltzmann factor $\exp[-\mu B \cos \theta/(k_bT)]$ taken over all possible orientations for the magnetic moment $\hat{\mu}$. If the paramagnetic moment is embedded in a spherical particle of very large moment of inertia, then all frequencies should be very small, and hence zero-point energy should not play any role; therefore applying nonrelativistic classical statistical mechanics to the paramagnetic particle should be justified. If we require that the ratio of the diamagnetic and paramagnetic magnetic moments should be the same at all temperatures $T$, and solve for the spectrum of random radiation which gives this behavior, then we find exactly the Planck spectrum with zero-point radiation in Eq. 24.

In this example yet again, we see the importance of including relativistic classical electromagnetic zero-point radiation and also of being sure that the behavior fits with relativistic
electrodynamics. Again the Planck spectrum with zero-point radiation appears naturally as the spectrum of radiation equilibrium.

D. Fully Relativistic Analysis for Radiation in a Non-Inertial Rindler Frame

A further derivation of the Planck spectrum within classical physics with classical zero-point radiation involves the use of ideas of conformal transformations of free electromagnetic fields. Classical zero-point radiation corresponds to the spectrum of random classical radiation which has the least possible information. Indeed, the correlation function involving zero-point radiation depends upon only the geodesic separation between the spacetime points at which it is evaluated. In an inertial frame, Lorentz-invariant zero-point radiation is invariant under conformal transformation. However, in a non-inertial frame, such as a uniformly accelerating Rindler coordinate frame, a time-dilating conformal transformation carries the zero-point radiation spectrum at $T = 0$ into a thermal spectrum at positive temperature $T > 0$. It is then possible to go to the asymptotic region of the coordinate frame where the spacetime becomes Minkowskian, and so to recover Planck’s spectrum including zero-point radiation as the blackbody radiation spectrum in an inertial frame. In this case, the entire analysis is fully relativistic at every step.

X. CLOSING SUMMARY

Blackbody radiation appeared in the physics research literature during the first two decades of the 20th century, and accurate experimental measurements of transferred heat energy made possible a comparison between theory and experiment which seemed convincing. However, the physicists of the period were unaware of two aspects which today are regarded as crucial to an understanding of blackbody radiation within classical theory. One crucial element is the presence of classical electromagnetic zero-point radiation with its Lorentz-invariant spectrum. Significantly, the experimental work of that early period did not measure the zero-point radiation, the theorists of the period did not consider the possibility of classical zero-point radiation, and accurate Casimir force measurements making clear the need for zero-point radiation within classical theory did not occur until nearly a century later. Furthermore, the scale of zero-point radiation is associated with Planck’s
constant $\hbar$ which did not appear in 19th century mechanics or electrodynamics. Indeed, the physicists of the early 20th century did not see any way that Planck’s constant could appear within a classical theory.

Today we are aware that Planck’s constant can appear naturally within classical electrodynamics as the scale factor for the source-free part of the general solution of Maxwell’s equations. However, today most students see Planck’s constant as exclusively an element of quantum theory and are completely unaware of its role as a scale for a Lorentz-invariant spectrum of random classical radiation. In sharp contrast with quantum theory which depends upon Planck’s constant $\hbar$ for its basic algebraic structure, classical electromagnetism makes use of Planck’s constant only as the scale-factor of the (homogeneous) source-free part of the general solution of Maxwell’s differential equations. Thus classical electromagnetic theory can exist either with or without Planck’s constant. In order to describe as many aspects as possible in nature while using classical theory, we must include classical zero-point radiation. Indeed, Planck’s constant appears in classical derivations of Casimir forces, van der Waals forces, low-temperature specific heats of solids, diamagnetism, atomic structure, and blackbody radiation.

The second crucial element of classical theory unrecognized by the physicists of a century ago is the importance of relativistic behavior. Although it became clear during the 19th century that Newtonian mechanics and classical electrodynamics were in conflict, it was only at the turn of the 20th century that classical electrodynamics was recognized as satisfying Lorentz invariance whereas nonrelativistic mechanics satisfied Galilean invariance. However, in the histories of the blackbody problem, there is no suggestion that physicists ever considered any connection between relativity and the blackbody problem. The same situation holds true today in the textbooks of modern physics where special relativity is presented without making any connection to the problem of blackbody radiation. Attempts to use nonrelativistic classical statistical mechanics or the scattering of radiation by nonrelativistic systems all led simply to the low-frequency Rayleigh-Jeans portion of the blackbody spectrum because nonrelativistic classical mechanical systems cannot allow the presence of relativistic zero-point radiation which is required for an understanding of the full Planck blackbody spectrum within classical physics. Although point harmonic oscillators can be fitted into relativistic classical electrodynamics as mechanical systems involving negligible velocity, nonlinear mechanical systems cannot be regarded as relativistic unless they cor-
respond to the Coulomb interaction which is part of relativistic classical electrodynamics. Only the relativistic Coulomb potential with the fundamental constant $e^2/c$ allows a relativistic mechanical zero-point energy which fits with the relativistic zero-point radiation of classical electromagnetic theory.

Because of the failure to consider relativity and zero-point radiation, during the first two decades of the 20th century, physicists came to an erroneous conclusion regarding the classical physics of blackbody radiation, and this erroneous conclusion is still repeated in the textbooks today. The physicists of the early 20th century concluded that classical physics led inevitably to the Rayleigh-Jeans spectrum for the full spectrum of thermal radiation. This false conclusion arose because physicists did not (and still do not) realize the important implications of classical zero-point radiation and special relativity.

[1] See, for example, R. Eisberg and R. Resnick, *Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles* 2nd ed. (Wiley, New York 1985) or K. S. Krane, *Modern Physics* 2nd ed. (Wiley, New York 1996) or J. R. Taylor, C. D. Zafiratos, and M. A. Dubson, *Modern Physics for Scientists and Engineers* 2nd ed. (Pearson, New York, 2003) or S. T. Thornton and A. Rex, *Modern Physics for Scientists and Engineers* 4th ed. (Brooks/Cole, Cengage Learning, Boston, MA 2013).

[2] H. B. G. Casimir, “On the attraction between two perfectly conducting plates,” Proc. Kon. Ned. Akad. Wetensch. 51, 793-795 (1948).

[3] S. K. Lamoreaux, “Resource letter CF-1: Casimir Force,” Am. J. Phys. 67, 850-861 (1999).

[4] T. H. Boyer, “Van der Waals Forces and Zero-Point Energy for Dielectric and Permeable Materials,” Phys. Rev. A9, 2078-2084 (1974).

[5] M. J. Sparnaay, “Measurement of the attractive forces between flat plates,” Physica 24, 751-764 (1958). S. K. Lamoreaux, “Demonstration of the Casimir force in the 0.6 to 6 $\mu$m range,” Phys. Rev. Lett. 78, 5-8 (1997). S. K. Lamoreaux, “Demonstration of the Casimir force in the 0.6 to 6 $\mu$m range,” Phys. Rev. Lett. 81, 5475-5478 (1998). U. Mohideen, “Precision measurement of the Casimir force from 0.1 to 0.9 $\mu$m,” Phys. Rev. Lett. 81, 4549-4552 (1998). H. B. Chan, V. A. Aksyuk, R. N. Kleiman, D. J. Bishop, and F. Capasso, “Quantum mechanical actuation of microelectromechanical systems by the Casimir force,” Science 291, 1941-1944
(2001). G. Bressi, G. Carugno, R. Onofrio, and G. Ruoso, “Measurement of the Casimir force between parallel metallic surfaces,” Phys. Rev. Lett. 88, 041804 (2002).

[6] We will refer to both $\hbar$ and $\hbar = 2\pi h$ as “Planck’s constant.”

[7] See the discussion by T. H. Boyer, “The Contrasting Roles of Planck’s Constant in Classical and Quantum Theories,” arXiv 1710.01616.

[8] T. W. Marshall, “Statistical Electrodynamics,” Proc. Camb. Phil. Soc. 61, 537-546 (1965).

[9] T. H. Boyer, “Derivation of the Blackbody Radiation Spectrum without Quantum Assumptions,” Phys. Rev. 182, 1374-1383 (1969).

[10] T. H. Boyer, “Scaling symmetries of scatterers of classical zero-point radiation,” J. Phys. A: Math. Theor. 40, 9635-9642 (2007).

[11] It is noteworthy that within quantum mechanics, there is no thought of consistent $\sigma_{\nu U^{-1}}$-scaling between mechanical systems and zero-point radiation. Thus quantum mechanics involves only a $\sigma_{\nu U^{-1}}$-scaling connecting time and energy, with a completely independent scaling in length.

[12] H. Goldstein, C. Poole, and J. Safko, Classical Mechanics 3rd ed. (Addison-Wesley, New York, 2002), pp. 452-482. There is a factor of $2\pi$ between the action variables in Goldstein’s text and those used here, corresponding to the change from Planck’s constant $\hbar$ over to $\hbar$.

[13] M. Planck, The Theory of Heat Radiation (Dover publications, New York 1959).

[14] See for example, B. H. Lavenda, Statistical Physics: A Probabilistic Approach (Wiley, New York 1991), pp., 67-70.

[15] T. H. Boyer, “Unfamiliar trajectories for a relativistic particle in a Kepler or Coulomb potential,” Am. J. Phys. 75, 992-997 (2004).

[16] S. O. Rice, “Mathematical analysis of random noise,” in Selected Papers on Noise and Stochastic Processes, edited by N. Wax (Dover, New York 1954), pp.133-294.

[17] T. H. Boyer, “A Connection Between the Adiabatic Hypothesis of Old Quantum Theory and Classical Electrodynamics with Classical Electromagnetic Zero-Point Radiation,” Phys. Rev. A18, 1238-1245 (1978).

[18] M. Born, The Mechanics of the Atom (Ungar, New York 1970), pp., 66-71.

[19] An extensive account of the history of blackbody radiation is given by T. S. Kuhn, Black-Body Theory and the Quantum Discontinuity 1894-1912 (Oxford U. Press, New York 1978).

[20] P. H. Morse, Thermal Physics (Benjamin-Cummins, Reading, MA 1969), p. 339.
[21] D. C. Cole, “Reinvestigation of the thermodynamics of blackbody radiation via classical physics,” Phys. Rev. A 45, 8471-8489 (1992).
[22] T. H. Boyer, “Thermodynamics of the harmonic oscillator: Wien’s displacement law and the Planck spectrum,” Am. J. Phys. 71, 866-870 (2003).
[23] C. Garrod, Statistical Mechanics and Thermodynamics (Oxford U.P., New York, 1995), p. 128.
[24] T. H. Boyer, “Understanding zero-point energy in the context of classical electromagnetism,” Eur. J. Phys. 37, 055206(14) (2016).
[25] See the appendix of T. H. Boyer, “Random electrodynamics: The theory of classical electrodynamics with classical electromagnetic zero-point radiation,” Phys. Rev. D 11, 790-808 (1975).
[26] J. H. van Vleck, “A correspondence principle for absorption,” Jour. Opt. Soc. Amer. 9, 27-30 (1924).
[27] Quoted from ref. 19, p. 112.
[28] H. A. Lorentz, The Theory of Electrons (Dover, New York 1952), pp. 80-96.
[29] Quoted from ref. 19, p. 195.
[30] A. Hermann (translated by C. W. Nash) The Genesis of Quantum Theory (1899-1913) (MIT Press, Cambridge, MA 1971), pp. 8-11.
[31] M. J. Klein, “Planck, Entropy, and Quanta 1901-1906,” The Natural Philosopher I, pp. 81-108 (Blaisdell Publishing Co. New York 1963).
[32] See for example, D. J. Griffiths, Introduction to Electrodynamics 4th ed. (Pearson, New York 2013), section 10.2.1 Eq. (10.26); or L. Eyges, The Classical Electrodynamic Field (Cambridge University Press 1972), p. 186, Eqs. (11.45) and (11.46); or J. D. Jackson, Classical Electrodynamics 3rd ed. (John Wiley & Sons, New York, 1999), 3rd ed., p. 246, Eq. (6.48); or A. Zangwill, Modern Electrodynamics (Cambridge U. Press, 2013), p. 724-725, Eqs. (20.57) and (20.58); or A. Garg, Classical Electromagnetism in a Nutshell (Princeton U. Press, Princeton, NJ 08450, 2012), p. 204, Eqs.(54.16) and (54.17).
[33] Quoted from S. Coleman, “Classical Electron Theory from a Modern Standpoint,” (RAND Corp. Report, Santa Monica, CA 1961), p. 19; reprinted in Doris Teplitz, ed, Electromagnetism: Paths to Research (Spring Science, New York 1982), Chapter 6, p. 183-210.
[34] J. H. van Vleck, “The absorption of radiation by multiply periodic orbits, and its relation to
the correspondence principle and the Rayleigh-Jeans law: Part II. Calculation of absorption by multiply periodic orbits,” Phys. Rev. 24, 347-365 (1924). See also ref. 26. Although he published a preliminary report, van Vleck never completed his work because he was sidetracked by the appearance of quantum mechanics.

[35] T. H. Boyer, “Equilibrium of random classical electromagnetic radiation in the presence of a nonrelativistic nonlinear electric dipole oscillator,” Phys. Rev. D 13, 2832-2845 (1976); “Statistical equilibrium of nonrelativistic multiply periodic classical systems and random classical electromagnetic radiation,” Phys. Rev. A 18, 1228-1237 (1978).

[36] R. Blanco, L. Pesquera, and E. Santos, “Equilibrium between radiation and matter for classical relativistic multiperiodic systems. Derivation of Maxwell-Boltzmann distribution form Rayleigh-Jeans spectrum,” Phys. Rev. D 27, 1254-1287 (1983); “Equilibrium between radiation and matter for classical relativistic multiperiodic systems. II. Study of radiative equilibrium with Rayleigh-Jeans radiation,” Phys. Rev. D 29, 2240-2254 (1984).

[37] A. Einstein, “Zur gegenwärtigen Stand des Strahlungsproblems,” Phys. Zeits. 10, 185-193 (1909).

[38] See H. A. Lorentz, Les Théories Statistiques en Thermodynamique (B. B. Teubner Verlag, Leipzig 1916), p.114. Also, S. Tomonaga, Quantum Mechanics (North-Holland Publishing Co., Amsterdam 1962), p. 298.

[39] T. H. Boyer, “Classical Statistical Thermodynamics and Electromagnetic Zero-Point Radiation,” Phys. Rev. 186, 1304-1318 (1969).

[40] A. Einstein and L. Hopf, “Statistische Untersuchung der Bewegung eines Resonators in einem Strahlungsfeld,” Ann. Physik (Leipzig) 33, 1105-1115 (1910).

[41] A. Einstein and O. Stern, “Einigen Argumente für die Annahme einer molekularen Agitation beim absoluten Nullpunkt,” Ann. Physik 40, 551-560 (1913).

[42] See ref. 19, p. 319, note 29. Kuhn mentions this Einstein-Stern calculation only in the notes where he includes Einstein and Stern’s closing remark. At the end of their successful calculation of Planck’s spectrum, they comment, “It seems doubtful that the other difficulties will be overcome without the hypothesis of quanta.”

[43] T. W. Marshall, “Brownian motion of a mirror,” Phys. Rev. D 24, 1509-1515 (1981).

[44] See for example, Griffiths’ text in ref. 32, p. 272.

[45] T. W. Marshall, “Random electrodynamics,” Proc. R. Soc A276, 475-491 (1963).
[46] T. H. Boyer, “Diamagnetism of a free particle in classical electron theory with classical electromagnetic zero-point radiation,” Phys. Rev. A 21, 66-72 (1980); “Derivation of the Planck radiation spectrum as an interpolation formula in classical electrodynamics with classical electromagnetic zero-point radiation,” Phys. Rev. D 27, 2906-2911 (1983); “Understanding the Planck blackbody spectrum and Landau diamagnetism within classical electromagnetism,” Eur. J. Phys. 37, 065102(17) (2016).

[47] T. H. Boyer, “Derivation of the Planck spectrum for relativistic classical scalar radiation from thermal equilibrium in an accelerating frame,” Phys. Rev. D 81, 105024(10) (2010); “Classical physics of thermal scalar radiation in two spacetime dimensions,” Am. J. Phys. 79, 644-656 (2011); “The blackbody radiation spectrum follows from zero-point radiation and the structure of relativistic spacetime in classical physics,” Found. Phys. 42, 595-614 (2012).

[48] A recent brief review is given by T. H. Boyer, “Any classical description of nature requires classical electromagnetic zero-point radiation,” Am. J. Phys. 79, 1163-1167 (2011). See also, D. C. Cole and Y. Zou, “Quantum Mechanical Ground State of Hydrogen Obtained from Classical Electrodynamics,” Phys. Lett. A 317, 14-20 (2003). A review of the work on classical electromagnetic zero-point radiation up to 1996 is provided by L. de la Pena and A. M. Cetto, The Quantum Dice - An Introduction to Stochastic Electrodynamics (Kluwer Academic, Dordrecht 1996).

(revised November 3, 2017)