Abstract. In the context of wave propagation on a manifold $X$, the characteristic functions are real valued functions on cotangent bundle of $X$ that specify the allowable phase velocities of the waves. For certain classes of differential operators (e.g. Maxwell’s Equations) the associated characteristic functions have singularities. These singularities account for phenomena like conical refraction and the transformation of longitudinal waves into transversal ones (or vice-versa). For a specific class of differential operators on surface, we prove that the singularities of the characteristic functions can be accounted from purely topological considerations. We also prove that there is a natural way to desingularize the characteristic functions, and observe that this fact and Morse Theory establishes a specific connection between singularities and critical points of these functions. The relation between characteristic functions and differential operators is obtained through what is known as the symbol of the operator. We establish a connection between these symbols and holomorphic vector fields, which will provide us with symbols whose characteristic functions have interesting singularity sets.

1. Introduction

In an appropriate context, certain aspects of wave propagation on a Riemannian manifold $X$ can be described by means of what are known as characteristic functions $\{\lambda_i : S(T^*X) \to \mathbb{R}\}_{1 \leq i \leq k}$, where $S(T^*X) = \{\xi \in T^*X | <\xi, \xi> = 1\}$ and the metric in $T^*X$ is the one induced by that in $TX$. For a covector $\xi \in S(T^*X)$, the set $\{\lambda_i(\xi)\}_{1 \leq i \leq k}$ consists of the set of allowable wave speeds in the direction of $\xi$.

For certain systems of hyperbolic differential operators it is a well known fact that the associated characteristic functions have singular points (by which we mean that they are not differentiable at these points). For example, each of the characteristic functions associated to Maxwell’s equations in biaxial crystals have four singularities. These singularities account for the phenomenon of conical refraction discovered by Hamilton (see [2, pg. 571]). For hyperbolic partial differential equations which correspond to generic non-homogeneous media the existence of singularities in the characteristic functions account for a phenomena known as wave transformation, in which longitudinal waves convert into transversal waves or vice-versa (see [1, Ch. 8] and [3]). It is then of interest to find methods for detecting singularities of the characteristic functions associated to general linear hyperbolic operators of the form $P : C^\infty(E) \to C^\infty(F)$, where $C^\infty(E)$ and $C^\infty(F)$ are the spaces of sections of the vector bundles $E$ and $F$ over space-time $X \times \mathbb{R}$. For certain constant
coefficients differential operators it is well know that the singularities of the characteristic functions occur for topological reasons (see for example [5, 7]). We are interested in generalizing these results to the case of operators of vector bundles. More concretely, we will consider the following problems.

**Problem 1.** For a hyperbolic differential operator $P : C^\infty(E) \to C^\infty(F)$, can we relate the existence of singularities of its characteristic functions to the characteristic classes of $E$ and $F$?

**Problem 2.** Is there a “natural” way desingularise the characteristic functions?

As we have explained before, the points at which the characteristic function are non-smooth explain interesting physical phenomena. Other points of interests are the critical points of the characteristic functions, i.e points at which their derivative vanish. Assuming $X$ is compact, among the critical values of the characteristic functions are the extremum values (i.e maximum and minimum) of wave fronts velocities, which are of physical interest. By giving answers to the above problems and using Morse theory we will be able to establish a specific connection between singularities and critical points of the characteristic functions.

The purpose of this paper is to solve the the above problems in a very specific case: that of first order hyperbolic operators with symmetric symbols, and acting on a rank two orientable vector bundle over a surface. The reason for restricting to this case is that it already provides us with interesting mathematics. Furthermore, in this case we will be able to provide complete answers to the above two problems, and this will pave the way for the more general cases. The restriction to operators with symmetric symbols can be justified on the grounds that they appear in most of the physical problems derived from variational principles (see [1, Ch. 8]).

For the rest of the paper we will let $X$ stand for a riemmanian compact, connected and oriented surface (whose metric we will denote by $\langle , \rangle_X$). The cotangent bundle $T^*X$ inherits a metric from that of $TX$, defined by letting the map $v \mapsto \langle v, \cdot \rangle$ be an isometry. We will use the same notation for the metric in $TX$ and that in $T^*X$. We will also assume that $F$ is an orientable rank two vector bundle over $X$ with a metric (that we will denote by $\langle , \rangle_F$). The differential operators that interest us have the form

$$P = \left( \frac{\partial}{\partial t} + L \right),$$

where $L : C^\infty(F) \to C^\infty(F)$ is a differential operator of order one with symmetric symbol. The symbol associated to $L$ is an homomorphism $\sigma : T^*X \to S^2F$ from the bundle $T^*X$ to the bundle of symmetric linear operators in $F$ (see Section 2).

The outline of the paper is as follows:

**In section 2:** We recall how to construct the characteristic function associated to the differential operator $P$ by means of the symbol of $L$.

**In section 3:** We find topological obstructions to the the existence of non-singular characteristic functions (see Proposition 8 and its Corollary). We spend some time proving that two apparently unrelated notions of genericity for symbols are actually the same (see Propositions 12 and its Corollaries), and use this result to show that generically the set of singularities of the characteristic functions form torus knots on $S(T^*X)$. 
In section 4: We show how to desingularise the characteristic functions associated to a symbol and use this to relate the singularities and critical points of these functions (see Proposition 24).

In section 5: We show how to construct symbols from holomorphic vector fields, and use this to explore what kind of torus knots can appear as singularities of the characteristic functions.

Notation. Unless otherwise stated we will use the following conventions. For any bundle $E$ over a manifold $M$ we will always denote the projection map from $E$ to $M$ by $\pi : E \to M$, where it will be clear from the context which bundle does $\pi$ has as its domain. If $E$ is a bundle with a metric we will denote this metric by $<.,>_E$ but for the case where $E = TX$ in which case we will use the notation $<.,>_X$. For elements $v \in E$ and $v \in TX$ we will their norms as $|v|_E$ and $|v|_X$. The metric in $TX$ induces a metric in $T^*_X$ which we will always denote by $<.,>_X$.

2. Motivation - The Characteristic Equation

For a linear differential operator of degree one $L : C^\infty(F) \to C^\infty(F)$, consider the asymptotic partial differential equation

$$\left(\frac{\partial}{\partial t} + L\right) \left(u_0 e^{ix\varphi(x,t)}\right) = 0 \quad \text{as} \quad s \to \infty,$$

where $u_0$ is a section of $F$ and $\varphi$ is a smooth function in $X \times \mathbb{R}$. The above equation leads to the the characteristic equation (see [4, pg. 31])

$$\det \left(\frac{\partial \varphi}{\partial t} + \sigma(d_x \varphi)\right) = 0,$$

where $I : F \to F$ is the identity morphism and $\sigma : T^*_X \to \text{End}(F)$ is the principal symbol of $L$. Since $L$ is a differential morphism of degree one, for every $x \in X$, we have that $\sigma$ is a linear map from $T^*_x X$ to $\text{End}(F_x)$. We will assume that for all $\xi \in T^*_x X$ the endomorphism $\sigma(\xi)$ is a symmetric operator with respect to the metric in $F$, i.e for all $x \in X$ and $\xi \in T^*_x X$ we have that

$$< \sigma(\xi)v, w >_F = < v, \sigma(\xi)w >_F \quad \text{for all} \quad v, w \in F_x.$$

When the above condition holds we will say that $\sigma$ is symmetric (being understood that the symmetry is with respect to the metric in $F$). If we denote the bundle of symmetric endomorphisms of $F$ by $S^2 F$, then $\sigma$ is a map of the form $\sigma : T^*_X \to S^2 F$. In this case, the characteristic equation (2.1) is equivalent to the equations

$$\frac{\partial \varphi}{\partial t}(x,t) + \lambda_{\sigma,i}(d_x \varphi(x,t)) = 0 \quad \text{for} \quad i = 1, 2,$$

where the functions $\lambda_{\sigma,1}, \lambda_{\sigma,2} : T^*_X \to \mathbb{R}$ are the eigenvalue functions of $\sigma$, i.e for each $\xi \in T^*_X$ we have that $\lambda_{\sigma,1}(\xi)$ and $\lambda_{\sigma,2}(\xi)$ are the two real eigenvalues of the symmetric operator $\sigma(\xi)$. From now on we will assume that we have chosen $\lambda_{\sigma,1}$ and $\lambda_{\sigma,2}$ so that $\lambda_{\sigma,1} \leq \lambda_{\sigma,2}$. The functions $\lambda_{\sigma,1}$ and $\lambda_{\sigma,2}$ are positively homogeneous of degree one in the fibre variable, i.e $\lambda_{\sigma,i}(r\xi) = r \lambda_{\sigma,i}(\xi)$ for $r > 0$, and their physical interpretation is as follows. Consider a time-dependent curve $x : \mathbb{R} \to X$ satisfying $\varphi(x(t), t) = \text{constant}$, so that

$$d_x \varphi(x(t), t) \dot{x}(t) + \frac{\partial \varphi}{\partial t}(x(t), t) = 0,$$
where \( \dot{x} \) is the derivative of \( x \) with respect to \( t \). From this last equation and (2.2) we obtain

\[
d_x \varphi(x(t), t) \dot{x}(t) = \lambda_{\sigma,i}(d_x \varphi(x(t), t)).
\]

We conclude that for a point \( x \in X \) traveling on the wave front

(2.3)

\[ W^c_t = \{ x \in X | \varphi(x, t) = c \} \]

the component of \( \dot{x}(t) \) normal to the front (known as phase velocity) is equal to

\[
\lambda_{\sigma,1}(d_x \varphi(x(t), t)) \frac{d_x \varphi(x(t), t)}{|d_x \varphi(x(t), t)|_X} \]

(assuming \( d_x \varphi(x(t), t) \neq 0 \)). For this reason, we will refer to \( \lambda_{\sigma,1} \) and \( \lambda_{\sigma,2} \) as the characteristic velocity functions of \( \sigma \). These functions can become non-smooth at certain points.

Example 3. For Maxwell’s equations on a crystal (see [2, pg. 571]) the symbol \( \sigma : (\mathbb{R}^3)^* \to \text{End}(\mathbb{R}^6) \) in the characteristic equation (2.1) is given by

\[
\sigma(\xi)(E, B) = (\xi \times B, -\xi \times (\epsilon^{-1}E))
\]

where \( E, B \in \mathbb{R}^3 \) correspond to the electric and magnetic fields, the dielectric tensor \( \epsilon \) is a \( 3 \times 3 \) positive definite symmetric matrix which describes the properties of the crystal, and \( \times \) is the standard cross product in \( \mathbb{R}^3 \) (notice we have identified \( (\mathbb{R}^3)^* \) with \( \mathbb{R}^3 \)). Figure 2.1 shows the Fresnel surface

\[ \mathcal{F}_\epsilon = \{ \tau \xi | \xi \in S^2 \text{ and } \det(\tau I + \sigma(\xi)) = 0 \} \]

where \( S^2 = \{ \xi \in (\mathbb{R}^3)^* | <\xi, \xi> = 1 \} \), for the case in which \( \epsilon \) has non-multiple eigenvalues. In this case, the crystal associated to \( \epsilon \) is said to be bi-axial (e.g. aragonite) and the four singularities of \( \mathcal{F}_\epsilon \) account for a phenomenon known as conical refraction.

### 3. Symbols and Their Multiplicities

#### 3.1. Preliminaries

From now on and for rest of the paper we will assume that \( F \) is an orientable real vector bundle over a riemannian compact, connected and oriented surface \( X \). Motivated by the discussion in Section 2, we introduce our basic objects of study.

**Definition 4.** A symbol on a bundle \( F \to X \) is a morphism \( \sigma : T^*X \to S^2F \).
The bundle $S^2 F$ has a riemmanian metric given by
\[(A, B) = \frac{1}{2} \text{tr}(AB) \text{ for all } x \in X \text{ and } A, B \in S^2 F_x.\]
This metric induces a norm function on $S^2 F$, which for an element $A \in S^2 F$ we will denote by $||A||$.

**Lemma 5.** The eigenvalue functions of a symbol $\sigma$ are given by the formulas
\[
\begin{align*}
\lambda_{\sigma,1} &= \frac{1}{2} \text{tr}(\sigma) - ||\sigma_0||, \\
\lambda_{\sigma,2} &= \frac{1}{2} \text{tr}(\sigma) + ||\sigma_0||
\end{align*}
\]
where $\sigma_0 = \sigma - \frac{1}{2} \text{tr}(\sigma) I$ is the traceless part of $\sigma$.

**Proof.** Let $\xi \in T^* X$. A direct computation shows that the eigenvalues of $\sigma_0(\xi)$ are $-||\sigma_0(\xi)||$ and $||\sigma_0(\xi)||$. The eigenvalues of $\sigma(\xi) = \frac{1}{2} \text{tr}(\sigma(\xi)) I + \sigma_0(\xi)$ are the solutions of the equation
\[
\det \left( \left( \lambda - \frac{1}{2} \text{tr}(\sigma(x)) \right) I - \sigma_0(x) \right) = 0,
\]
and hence the result. \qed

If we define spherisation $S(T^* X)$ of $T^* X$ as the set
\[S(T^* X) = \{ \xi \in T^* X | <\xi, \xi> = 1 \},\]
then both $\sigma$ and its eigenvalue functions $\lambda_{\sigma,1}$ and $\lambda_{\sigma,2}$ are completely determined by the values they take on $S(T^* X)$.

**Definition 6.** The multiplicity set of a symbol $\sigma$ is the set
\[\mathcal{M}_\sigma = \{ \xi \in S(T^* X) | \lambda_{\sigma,1}(\xi) = \lambda_{\sigma,2}(\xi) \}.\]

By lemma 5 we have that
\[\mathcal{M}_\sigma = \{ \xi \in S(T^* X) | \sigma_0(\xi) = 0 \}.\]
We will let $S^2_\sigma F$ stand for the bundle of traceless elements in $S^2 F$. The following result characterize (generically) the multiplicity set of $\sigma$ as the set at which the eigenvalue functions are non-smooth.

**Proposition 7.** Consider symbol $\sigma$ and let $\sigma_0$ be its traceless part. If $\sigma_0|S(T^* X)$ is transversal to the zero section of $S^2_\sigma F$ then functions $\lambda_{\sigma,1}|S(T^* X)$ and $\lambda_{\sigma,2}|S(T^* X)$ are non-smooth over the set $\mathcal{M}_\sigma$.

**Proof.** From the transversality assumption on $\sigma_0$ it follows that function $\xi \mapsto ||\sigma_0(\xi)||$ is non-differentiable for points $\xi \in S(T^* X)$ such that $\sigma_0(\xi) = 0$, i.e over points $\xi$ in the set $\mathcal{M}_\sigma$. The result then follows from formulas (3.1) and (3.2). \qed

### 3.2. The Multiplicity Set

In this section we will find obstructions for $\mathcal{M}_\sigma$ to be empty (see the next Proposition and its Corollary) and we will see that if $\mathcal{M}_\sigma$ is non-empty then it is generically the intersection of $S(T^* X)$ with line bundle $\mathcal{K}_\sigma$ over a one dimensional sub-manifold $S_\sigma$ of $X$ (see the discussion after Corollary 9).

**Proposition 8.** Consider a symbol $\sigma$ on the bundle $F \to X$, and let $e(F), e(TX) \in H^2(X, \mathbb{R})$ denote the Euler classes of $F$ and $TX$, respectively. If $e(TX) - 2e(F) \neq 0$ in $H^2(X, \mathbb{R})$ then $\mathcal{M}_\sigma \neq \emptyset$. 
Proof. Consider a rotation matrix
\[ R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in \text{SO}(2) \]
and
\[ A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \in \mathbb{S}_0^2 \mathbb{R}^2. \]
It easy to see that
\[ R_\theta A R_\theta^T = \begin{pmatrix} p & q \\ q & -p \end{pmatrix}, \]
where
\[ \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \]
This means that the transition functions of the bundle \( \mathbb{S}_0^2 F \) are the same as those of the bundle \( F \otimes \mathbb{C} F \), so these bundles must be isomorphic. If \( M_\sigma \) were empty then we would have that \( \sigma_0(\xi) \neq 0 \) for all \( \xi \in S(T^*X) \), which means that \( \sigma_0 \) is an isomorphism between \( T^*X \) and \( \mathbb{S}_0^2(TX) \). Hence \( TX \) and \( F \otimes \mathbb{C} F \) are isomorphic, which implies that.
\[ e(TX) = e(F \otimes \mathbb{C} F) = 2e(F). \]
\[ \square \]

Corollary 9. If the genus of \( X \) is different from one, then the multiplicity set of any symbol \( \sigma \) on \( TX \) is non-empty.

Proof. Follows from the previous proposition and the fact that
\[ \int_X (e(TX) - 2e(TX)) = 2(g(X) - 1) \neq 0, \]
where \( g(X) \neq 1 \) stands for the genus of \( X \). \[ \square \]

Definition 10. We will say that a symbol \( \sigma \) over \( F \rightarrow X \) is traceless if \( \sigma(\xi) \in \mathbb{S}_0^2 F \) for all \( \xi \in T^*X \).

We will now give a more detailed description of the multiplicity set. The symbol \( \sigma \) can be considered as a section of the bundle \( \text{Hom}(T^*X, \text{End}(F)) \), and we will denote such a section as \( \hat{\sigma} : X \rightarrow \text{Hom}(T^*X, \text{End}(F)) \)

Definition 11. Let \( \sigma \) be a traceless symbol on \( F \rightarrow X \). We define the singular set \( \mathcal{S}_\sigma \) and kernel bundle \( \mathcal{K}_\sigma \) by
\[ \mathcal{S}_\sigma = \{ x \in X | \dim(\ker(\hat{\sigma}(x))) > 0 \}; \]
\[ \mathcal{K}_\sigma = \{ \xi \in T^*_x X | x \in \mathcal{S}_\sigma \text{ and } \sigma(\xi) = 0 \}. \]
For a general (not necessarily traceless) symbol \( \sigma \) we define \( \mathcal{S}_\sigma = \mathcal{S}_{\sigma_0} \) and \( \mathcal{K}_\sigma = \mathcal{K}_{\sigma_0} \).

Since we have that
\[ \mathcal{M}_\sigma = S(T^*X) \cap \mathcal{K}_{\sigma_0}, \]
it is then of interest to find conditions under which \( \mathcal{S}_\sigma \) is a smooth submanifold of \( X \) and \( \mathcal{K}_\sigma \) is a smooth bundle over \( \mathcal{S}_\sigma \). To study this problem, observe that \( \text{Hom}(T^*X, \mathbb{S}_0^2 F) \) stratifies as follows
\[ \text{Hom}(T^*X, \mathbb{S}_0^2 F) = \bigcup_{i=0}^2 [\text{Hom}(T^*X, \mathbb{S}_0^2 F)], \]
where
\[ [\text{Hom}(T^* X, S^2_0 F)]_i = \{ A \in \text{Hom}(T^* X, S^2_0 F) \mid \dim(\ker(A)) = i \}. \]

We then have that for a traceless symmetric symbol \( \sigma \)
\[ S_\sigma = \hat{\sigma}^{-1} \left( \bigcup_{i=1}^{2} [\text{Hom}(T^* X, S^2_0 F)]_i \right). \]

The manifold \( S(T^* X) \) has dimension 3 and \( [\text{Hom}(T^* X, S^2_0 F)]_i \) is a smooth submanifold of \( \text{Hom}(T^* X, S^2_0 F) \) of codimension \( i^2 \) (see [10, pg. 28]). Hence, transversality of \( \hat{\sigma} \) to \( [\text{Hom}(T^* X, S^2_0 F)]_2 \) and \( [\text{Hom}(T^* X, S^2_0 F)]_1 \) imply that
\[ \hat{\sigma}^{-1} ([\text{Hom}(T^* X, S^2_0 F)]_2) = \emptyset \]
and that
\[ S_\sigma = \hat{\sigma}^{-1} ([\text{Hom}(T^* X, S^2_0 F)]_1) \]
is a smooth one dimensional submanifold of \( X \).

**Proposition 12.** If \( \sigma \) is a traceless symmetric symbol \( \sigma \) then \( \hat{\sigma} \) is transversal to \( [\text{Hom}(T^* X, S^2_0(TX))]_1 \) and \( [\text{Hom}(T^* X, S^2_0(TX))]_2 \) iff \( \sigma|S(T^* X) \) is transversal to the zero section of \( S^2_0 F \).

**Proof.** The proof is local. We can locally trivialize the bundles \( T^* X \) and \( F \) by choosing local orthonormal frames over appropriate open sets \( U \subset X \). In these trivializations the map \( \sigma \) can be written in the form
\[ \sigma(x, \xi) = M(x)\xi \]
where
\[ M(x) = \begin{pmatrix} a_1(x) & b_1(x) \\ a_2(x) & b_2(x) \end{pmatrix} \quad \text{and} \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \]

We have that
\[ d\sigma_0(x, \xi)(\dot{x}, \dot{\xi}) = (dM(x)\dot{x})\xi + M(x)\dot{\xi}. \]
The condition \( \sigma_0(x, \xi) = 0 \) (for \( \xi_1^2 + \xi_2^2 = 1 \)) is equivalent to the existence scalars \( \alpha_1, \alpha_2 \in \mathbb{R} \) such that
\[ (a_i, b_i) = \alpha_i(-\xi_2, \xi_1) \]
(3.3)

The condition that \( \xi \) is unitary implies that \( \dot{\xi} = \dot{\beta}(\xi_2, -\xi_1) \) for a scalar \( \dot{\beta} \in \mathbb{R} \).

Using these identities we obtain
\[ d\sigma_0(x, \xi)(\dot{x}, \dot{\xi}) = \begin{pmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{pmatrix} + \beta \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \]
where
\[ dM(x)\dot{x} = \begin{pmatrix} \dot{a}_1 & \dot{b}_1 \\ \dot{a}_2 & \dot{b}_2 \end{pmatrix} \quad \text{and} \quad \dot{\gamma}_i = \dot{a}_i \xi_1 + \dot{b}_i \xi_2. \]

We conclude that the transversality condition \( \sigma_0 \) means that there exists a \( \dot{x} \in \mathbb{R}^2 \) such that
\[ (3.4) \quad \det \begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} \neq 0. \]

Let \( f = \det(M) = a_1 b_2 - a_2 b_1. \) The condition \( \delta_0(x) \in \bigcup_{i=1}^2 [\text{Hom}(T^* X, S^2_0(TX))]_i \)
is equivalent to the condition \( f(x) = 0. \) The transversality conditions on \( \delta_0 \) are
equivalent to $0 \in \mathbb{R}$ being a regular value of $f$, i.e. for each point $x \in f^{-1}(0)$ there must exist $\dot{x}$ such that

$$df(x)\dot{x} = \dot{a}_1b_2 + a_1\dot{b}_2 - \dot{a}_2b_1 - a_2\dot{b}_1 \neq 0$$

If $f(x) = 0$ then we must have that $\ker(M(x)) > 0$. This means that there exists $(\xi_1, \xi_2)$ such that $\xi_1^2 + \xi_2^2 = 1$ and such that $|\xi_1| > |\xi_2|$. A simple computation shows that

$$df(x)\dot{x} = \text{det} \begin{pmatrix} \dot{\gamma}_1 & \dot{\gamma}_2 \\ \alpha_1 & \alpha_2 \end{pmatrix}.$$ 

Hence, the condition $df(x)\dot{x} \neq 0$ is the same as $(3.4)$. \qed

In the space of traceless symbols the condition of $\sigma|S(T^*X)$ being transversal to the zero section of $S_0^*F$ is clearly a stable condition, but not so clearly a generic condition since $\sigma$ lives within the class of maps from $T^*X$ to $S_0^*F$ that are morphisms (i.e. linear maps on each fiber).

**Corollary 13.** In the space of traceless symbols the condition of $\sigma|S(T^*X)$ being transversal to the zero section of $S_0^*F$ is a generic condition.

**Proof.** That the condition is open follows from the general results regarding the stability of the transversality of a map to a given smooth manifold. That the condition is dense follows from Proposition 12 and the density of the condition: $\sigma_0$ is transversal to the stratified manifold $\bigcup_{i=1}^{2}[\text{Hom}(T^*X, S_0^*(TX))]_i$ (see [10, Ch. 2]). \qed

**Corollary 14.** If $\sigma$ is a traceless symbol such that $\sigma|S(T^*X)$ is transversal to the zero section of $S_0^*F$, we have that $S_\sigma$ is a smooth one dimensional submanifold of $X$ and $K_\sigma$ is a smooth real line bundle over $S_\sigma$.

**Proof.** The smoothness of $S_{\sigma_0}$ follow immediately from the Proposition 12. Also by Proposition 12 we have that for all $x \in S_{\sigma_0}$ the kernel of $\dot{\sigma}_0(x)$ is one dimensional, i.e the fibers $K_{\sigma_0}$ are one dimensional vector spaces. The transversality condition on $\sigma_0$ implies that $\sigma_0$ is a smooth submanifold of $S(T^*X)$. From this and the the fact that $\mathcal{M}_\sigma = S(T^*X) \cap K_{\sigma_0}$ we obtain $K_{\sigma_0}$ is a smooth real line bundle over $S_{\sigma_0}$. \qed

Consider then a generic symbols $\sigma$. Over every connected component $C$ of $S_\sigma$ the multiplicity set over $C$, which we will denote by $\mathcal{M}_\sigma|C$, is a one dimensional compact submanifold of the torus $S(T^*X)|C$. The projection $\pi : T^*X \to X$ when restricted to $\mathcal{M}_\sigma|C$ is two to one, which intuitively this means that $\mathcal{M}_\sigma$ “winds” twice around the “$C$-direction” of the torus $S(T^*X)|C$. The winding number of $\mathcal{M}_\sigma|C$ in the “fiber-direction” is given by

$$m_C = \frac{1}{2\pi} \int_{\mathcal{M}_\sigma|C} a_C^*(d\theta) \in \mathbb{Z} \quad (3.5)$$

where $T : C \to TC \subset TX$ is the unit tangential vector field to $C$ with counterclockwise orientation, the map $\alpha : \mathcal{M}_\sigma|C \to S^1$ is given by $a_C(\xi) = \exp(i\xi(T(\pi(\xi))))$ and $d\theta \in H^1[S^1, \mathbb{R}]$ is the angular form. The parity of $m_C$ determines whether the real line bundle $K_\sigma|C$ is trivial or not. More concretely, $\mathcal{M}_\sigma|C$ is connected if $m_C$ is odd and it consists of exactly two connected components if $m_C$ is even.
Example 15. If $X = S^2$ then $T^*X$ is homeomorphic to the three dimensional projective space $\mathbb{R}P^3$. Over every connected component $C$ of $\mathcal{M}_s$ such that $m_C$ odd, we have that $\mathcal{M}_s|C$ is a torus knot of the type $2, m_C$ with ambient space $\mathbb{R}P^3$. (see [6, pg. 216] for a definition torus knots with ambient space $\mathbb{R}^3$.

Problem 16. Can any integer appear as a number of the form $m(C)$?

Using the results in the next section we will be able to give an answer to this problem for the case when $X$ is the two dimensional sphere.

4. Desingularising Characteristic Velocities

In this section we will develop tools for desingularising the eigenvalue functions $\lambda_{s,1}$ and $\lambda_{s,2}$ of the previous sections. We start by generalizing some of the notions introduced earlier. Let $E$ be a rank 2 vector bundle over a differentiable connected manifold $M$, and assume that $E$ is endowed with a metric $<,>_E$. We will denote by $S^2E$ the set of symmetric morphisms of $E$ (symmetric with respect to the metric $<,>_E$). For a given section $s : M \to S^2E$ its traceless part $s_0 : M \to S^0_2E$ is given by $s_0 = s - (1/2)\text{tr}(s)I$. The bundle $S^2E$ has a Riemannian metric given by

\[(A, B) = \frac{1}{2}\text{tr}(AB) \quad \text{for any} \quad p \in M \quad \text{and} \quad A, B \in S^2E_p.\]

The eigenvalues of $s$ are then given by

\[\lambda_{s,i} = \frac{1}{2}\text{tr}(s) + (-1)^i||s_0|| \quad \text{for} \quad i = 1, 2,\]

where the norm is the one induced by the metric in $S^0_2E$. The multiplicity set $\mathcal{M}_s$ associated to $s$ is given by

\[\mathcal{M}_s = \{p \in M|\lambda_{s,0}(p) = \lambda_{s,1}(p)) = \{p \in M|s_0(p) = 0\}.\]

Recall that the projectivisation $PE$ of a real vector bundle $E$ is the space of its real one-dimensional subspaces. For point $p \in M$, we will denote a given line in a fiber $E_p$ by $l_p$.

Definition 17. For a given section $s : M \to S^2E$ we define its space of eigenlines $\mathcal{E}_s \subset PE$ by

\[\mathcal{E}_s = \bigcup_{p \in M} \{l_p \in PE|l_p \text{ is spanned by a non-zero eigenvector of } s(p)\}.\]

The eigenvalue function of $s$ is defined as the function $\lambda_s : \mathcal{E}_s \to \mathbb{R}$ given by

\[\lambda_s(l_p) = \text{eigenvalue of } s(p) \text{ corresponding to } l_p\]

The projection map $\pi : PE \to M$ restricts to a projection map $\pi_s : \mathcal{E}_s \to M$, that when restricted to $\mathcal{E}_s - \pi^{-1}(\mathcal{M}_s)$ is a double covering map onto $M - \mathcal{M}_s$. For points $p \in \mathcal{M}_s$ we have that $\pi_s^{-1}(p) = PE_p$. Since $s$ and $s_0$ have the same eigenspaces, we have that $\mathcal{E}_s = \mathcal{E}_{s_0}$. The set $\pi_s^{-1}(\mathcal{M}_s)$ can be characterized in terms of $\lambda_{s_0}$ as follows.

\[\pi_s^{-1}(\mathcal{M}_s) = \lambda_{s_0}^{-1}(0).\]

Proposition 18. Let $E$ be a rank 2 riemannian bundle real over a manifold $M$ and $s : M \to S^2E$ be a smooth section such that its traceless part $s_0$ is transversal to the zero section of $S^0_2E$. We have that $\mathcal{E}_s$ is a smooth submanifold of $PE$, the functions $\lambda_s$ and $\lambda_{s_0}$ are a smooth functions, and $0 \in \mathbb{R}$ is a regular value of $\lambda_{s_0}$.
Corollary 19. The set $\mathcal{M}_s$ has a tubular neighborhood $\mathcal{N}_s$ such that $\pi^{-1}_s(\mathcal{N}_s)$ is diffeomorphic to $P(E|M_s) \times (-\epsilon, \epsilon)$ (for some $\epsilon > 0$).

Proof. Since $0 \in \mathbb{R}$ is a regular value of $\lambda_{s0}$ there exists and $\epsilon > 0$ such that $\lambda_{s0}^{-1}(-\epsilon, \epsilon)$ is diffeomorphic to $\lambda_{s0}^{-1}(0) \times (-\epsilon, \epsilon) = P(E|M_s) \times (-\epsilon, \epsilon)$. If we choose a positive small enough we can then ensure that $\pi_s(\lambda_{s0}^{-1}(-\epsilon, \epsilon))$ is a tubular neighborhood of $\mathcal{M}_s$. \hfill \Box

Example 20. Some simple calculations show that the Fresnel surface $\mathcal{F}_s$ defined in Example 3 can be given as

$$\mathcal{F}_s = \{(\lambda_s, 1)(\xi) \mid \xi \in S^2\} \cup \{(\lambda_s, 2)(\xi) \mid \xi \in S^2\},$$

where $\lambda_{s,1} \leq \lambda_{s,2}$ are the eigenvalue functions of the section $s : S^2 \to S^2(TS^2)$ defined by the formula.

$$<s(x)v, w> = ||s^{-1}v, w||.$$

The singularities of $\mathcal{F}_s$ coincide with the non-smoothness points of the functions $\lambda_{s,1}$ and $\lambda_{s,2}$, i.e. with the zero set of the section $s_0 : S^2 \to S^0(TS^2)$. From the proof of Proposition 3 we have that

$$\int_{S^2} e(S^0_0(TS^2)) = \int_{S^2} e(TS^2 \otimes CTS^2) = 2 \int_{S^2} e(TS^2) = 4.$$

This last computations is the topological explanation of the four singularities of $\mathcal{F}_s$. It is also easy to verify that for a biaxial crystal (i.e. $\epsilon$ has non-multiple eigenvalue) the section $s_0$ is transversal to $S^0_0(TS^2)$ at each of the four points in $\mathcal{M}_s$. The tubular neighborhood $\mathcal{N}_s$ (described in the previous corollary) is given by $\cup_{i=1}^4 D_i$ where the family $\{D_i\}_{i=1}^A$ consists of four disjoint circles in $S^2$. We then have that $\mathcal{E}_s - \pi_s(\mathcal{N}_s)$ is the same as two disjoint copies of $S^2 - \mathcal{N}_s$ (see part A of the figure). By the previous corollary $\pi^{-1}_s(\mathcal{N}_s)$ consists of four cylinders. Part B of the figure illustrates the projection map $\pi_s$ over one of the disks $D_i$. By joining the two copies of $S^2 - \cup_{i=1}^4 D_i$ along this four cylinders we obtain a surface of genus 3 (see part C of the figure).

The critical points of $\lambda_s$ on $\mathcal{E}_s - \pi^{-1}_s(\mathcal{M}_s)$ are the inverse image under $\pi_s$ of the critical points of $\lambda_{s,1}$ and $\lambda_{s,2}$. It remains to study the critical points of $\lambda_s$ over the set $\pi^{-1}_s(\mathcal{M}_s)$. The condition for $\lambda_s$ to have a critical point on $\pi^{-1}_s(\mathcal{M}_s)$ will be given in terms of a Riemannian metric on the co-normal bundle of $\mathcal{M}_s$ in $X$. More concretely, we have the following result.

Proposition 21. Consider a section $s : X \to S^2 E$ such as the one in Proposition 7 and let $s_{B} = (1/2) tr(s)$. A necessary condition for an element $l_p \in \pi^{-1}_s(\mathcal{M}_s)$ to be a critical point of $\lambda_s$ is that $ds_B(p) \in C(\mathcal{M}_s)$, where

$$C(\mathcal{M}_s) = \{ \alpha \in T_p^*M \mid p \in \mathcal{M}_s \text{ and } \alpha(v) = 0 \text{ for all } v \in T_p\mathcal{M}_s \}.$$
Remark 22. Observe that the necessary condition for \( \lambda_s \) to have a critical point at \( l_p \in \pi_s^{-1}(M_s) \), is simply that \( s_R|_{M_s} \) (the restriction of \( s_R \) to \( M_s \)) has a critical point on \( p \). This will always happen at certain points if \( M_s \) is a compact manifold.

To apply the previous results to a symbol \( \sigma : T^*X \to S^2F \) consider the section \( s_\sigma : S(T^*X) \to S^2(\pi^*F) \) (where \( \pi : S(T^*X) \to X \) is the natural projection) defined by

\[
s_\sigma(\xi)v = \sigma(\pi(\xi))\tilde{\pi}(v) \quad \text{for } v \in \pi^*F
\]

where \( \tilde{\pi} \) is the natural projection \( \tilde{\pi} : \pi^*F \to F \).

Definition 23. For a symbol \( \sigma : T^*X \to S^2F \) we define

\[
E_\sigma = E_{s_\sigma} \subset \mathbb{P}(\pi^*F) \quad \text{and} \quad \lambda_\sigma = \lambda_{s_\sigma} : E_\sigma \to \mathbb{R}.
\]

There is a natural projection map \( \pi_\sigma : E_\sigma \to S(T^*X) \) given by \( \pi_\sigma = \pi_{s_\sigma} \). Applying the \( \mathbb{R} \) to \( s_\sigma \) we obtain the following result.

Proposition 24. For a generic morphism \( \sigma : T^*X \to S^2F \) we have that \( E_\sigma \) is a smooth manifold and \( \lambda_\sigma \) is a smooth function. Furthermore, if the multiplicity set \( M_\sigma \) is non-empty then it is the disjoint union of set \( S_1, \ldots, S_m \subset S(T^*X) \) each one of them diffeomorphic to a circle. For each \( 1 \leq i \leq m \) we have that each \( S_i \) has a tubular neighborhood \( N_i \) such that for an appropriate \( \epsilon > 0 \) the set \( \pi_\sigma^{-1}(N_i) \) is diffeomorphic to \( S^1 \times S^1 \times (-\epsilon, \epsilon) \).
Proof. We have the $s_{\sigma} : S(T^*X) \to S^2(\pi^*F)$ defined above is transversal to the zero section of $S^2(\pi^*F)$ if $\sigma_0 : T^*X \to S^2_0 F$ is transversal to the zero section of $S^2_0 F$. By Corollary 13, we have that this last condition is generic. The first two assertions of the proposition follow from this, the fact that $M = \sigma_0^{-1}(0)$ and Proposition 15. From Corollary 19, we have that for each $1 \leq i \leq n$ we can find a tubular neighborhood of $S_i$ such that $\pi_0^{-1}(N_i)$ is diffeomorphic to $P(\pi^*F|S_i) \times (-\epsilon, \epsilon)$. Since $F$ is orientable, we have that $\pi^*F$ is orientable and hence $\pi^*F|S_i$ is a trivial bundle. From this we conclude that $P(\pi^*F|S_i)$ is diffeomorphic to $S^1 \times S^1$. □

For symbols $\sigma$ such that $\lambda_\sigma$ is a Morse function, this theorem and the cell decomposition of $E_\sigma$ provided by the critical points of $\lambda_\sigma$ establishes the connection between the critical points of wave speeds and the “conical-refraction” type behavior induced by multiplicity set $M_\sigma$. This can be expressed in terms homology groups of $E_\sigma$ by using the same techniques as in [3] pgs. 20,28-31.

5. Symbols on the Riemann Sphere

5.1. Algebraic preliminaries. A linear map from $(\mathbb{R}^2)^* \to S^2_0 \mathbb{R}^2$ can be seen as an element $S^2_0 \mathbb{R}^2 \otimes \mathbb{R}^2$. An orthonormal change of coordinates in $S^2_0 \mathbb{R}^2 \otimes \mathbb{R}^2$ is given in terms of the action of a rotation $R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ which acts on elements $A \otimes v \in S^2_0 \mathbb{R}^2 \otimes \mathbb{R}^2$ as

$$R \cdot (A \otimes v) = RAR^T \otimes Rv.$$ 

We will identify both $S^2_0 \mathbb{R}^2$ and $\mathbb{R}^2$ with $\mathbb{C}$ by letting

$$\begin{pmatrix} p \\ q \end{pmatrix} \sim p + iq \text{ and } (v_1, v_2) \sim v_1 + iv_2.$$ 

By Proposition 5, the above action can be seen as the action of $\exp(i\theta) \in \mathbb{C} - \{0\}$ on $(p + iq) \otimes (v_1 + iv_2)$ given by

$$\exp(i\theta) \cdot ((p + iq) \otimes (a + ib)) = (\exp(i2\theta)(p + iq)) \otimes (\exp(i\theta)(v_1 + iv_2))$$

This action is better understood in the following basis

$$g_1 = \frac{1}{\sqrt{2}}(1 \otimes 1 + i \otimes i),$$

$$g_2 = \frac{1}{\sqrt{2}}(i \otimes 1 - 1 \otimes i),$$

$$g_3 = \frac{1}{\sqrt{2}}(1 \otimes 1 - i \otimes i),$$

$$g_4 = \frac{1}{\sqrt{2}}(i \otimes 1 + 1 \otimes i),$$

since a simple calculation shows that

$$\exp(i\theta) \cdot g_1 = \cos(\theta)g_1 + \sin(\theta)g_2,$$

$$\exp(i\theta) \cdot g_2 = -\sin(\theta)g_1 + \cos(\theta)g_2,$$

$$\exp(i\theta) \cdot g_3 = \cos(3\theta)g_3 + \sin(3\theta)g_4,$$

$$\exp(i\theta) \cdot g_4 = -\sin(3\theta)g_3 + \cos(3\theta)g_4.$$
We conclude that if we have that
\[ \alpha g_1 + \beta g_2 + \gamma \cdot g_3 + \delta \cdot g_4 = \exp(i\theta) \cdot (a g_1 + b g_2 + c g_3 + d g_4) \]
then
\[ \alpha + i\beta = \exp(i\theta)(a + ib) \]
\[ \gamma + i\delta = \exp(i3\theta)(c + id). \]
The morphism \(\sigma = a g_1 + b g_2 + c g_3 + d g_4 : (\mathbb{R}^2)^* \to S^2_0 \mathbb{R}^2\) acts on cotangent vectors \(\xi = (\xi_1, \xi_2)\) as follows
\[
(5.1) \quad \sigma(\xi) = \begin{pmatrix} p & q \\ q & -p \end{pmatrix} \xi_1 + \begin{pmatrix} r & s \\ s & -r \end{pmatrix} \xi_2
\]
where
\[ p + iq = \frac{1}{\sqrt{2}}((a + ib) + (c + id)) \]
\[ r + is = \frac{i}{\sqrt{2}}((a + ib) - (c + id)). \]
Inversely, if we are given \(\sigma\) as in 5.1 then the corresponding coefficients \(a, b, c, d\) must satisfy
\[ a + ib = \frac{1}{\sqrt{2}}((p + iq) - i(r + is)) \]
\[ c + id = \frac{1}{\sqrt{2}}((p + iq) + i(r + is)). \]
From the above formulas it is easy to see that the \(\sigma : (\mathbb{R}^2)^* \to S^2_0 \mathbb{R}^2\) is invertible iff
\[
\det \begin{pmatrix} p & r \\ q & s \end{pmatrix} = |a + ib|^2 - |c + id|^2 \neq 0,
\]
where \(|z|\) is the standard norm of a complex number \(z \in \mathbb{C}\). The above discussion proves the following.

**Proposition 25.** Let \(F\) be an \(SO(2)\) vector bundle of rank 2 over a manifold \(M\). We have that \(\text{Hom}(F^*, S^2_0 F)\) is isomorphic to \(F \oplus (F \otimes \mathbb{C} F \otimes \mathbb{C} F)\).

In particular, to any quadruple of sections \(V, V_1, V_2, V_3 : M \to F\) we can associate a section \(\sigma(V, V_1, V_2, V_3) : M \to \text{Hom}(F^*, S^2_0 F)\) that corresponds to the section \(V + V_1 \otimes V_2 \otimes V_3\). We can write a local expression for \(\sigma(V, V_1, V_2, V_3)\) by considering orthonormal vector fields \(e_1, e_2 : U \to F\) over an open set \(U \subset M\), and such that \(e_2 = ie_1\) (multiplications by \(i\) in \(F\) is well defined since it is a \(SO(2) = U(1)\) bundle). We can write (for \(k = 1, 2, 3\))
\[ V = ve_1, V_k = v_k e_1 \]
for smooth function \(v, v_k : U \to \mathbb{C}\) so that
\[ V + V_1 \otimes V_2 \otimes V_3 = ve_1 + (v_1 v_2 v_3)e_1 \otimes \mathbb{C} e_1 \otimes \mathbb{C} e_1. \]
By formulas 5.2 we have that \(\sigma = \sigma(V, V_1, V_2, V_3)\) is given by the formulas
Observe that we have identified $F^*$ with $F$ by using the metric in $<,>$ in $F$.

5.2. Holomorphic fields and symbols. Consider the unit sphere $$S^2 = \{ p \in \mathbb{R}^3 | < p, p > = 1 \}$$
with the metric it inherits as a subspace of $\mathbb{R}^3$. We can parametrize the regions
$$D_i = S^2 - \{ 0, 0, (-1)^{i+1} \} \text{ for } i = 1, 2$$
using the stereographic projection maps $\varphi_i : \mathbb{R}^2 \rightarrow D_i$ given by
$$\varphi_i(x, y) = (1 + x^2 + y^2)^{-1}(2x, (-1)^{i+1}, 2y, (-1)^{i+1}(x^2 + y^2 - 1)).$$
If we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the natural way, then $\varphi_2 \circ \varphi_1^{-1} : \mathbb{C} - \{ 0 \} \rightarrow \mathbb{C} - \{ 0 \}$ is given by $\varphi_1 \circ \varphi_2^{-1}(z) = z^{-1}$. If we denote coordinates in $D_1$ by $z = x + iy$ and those in $D_2$ by $w = u + iv$, then change of variables formula is $w = z^{-1}$ and hence
$$\frac{\partial}{\partial x} = -z^{-2} \frac{\partial}{\partial u}.$$ The metric in $S^2$ is such that if we let
$$e_1 = \frac{1 + |z|^2}{2} \frac{\partial}{\partial x}, \quad e_2 = \frac{1 + |z|^2}{2} \frac{\partial}{\partial y},$$
$$f_1 = \frac{1 + |w|^2}{2} \frac{\partial}{\partial u}, \quad f_2 = \frac{1 + |w|^2}{2} \frac{\partial}{\partial v},$$
then $e_1, e_2$ are orthonormal with $e_2 = ie_1$, and similarly for $f_1, f_2$. If $V, V_1, V_2, V_3$ are smooth vector fields in $S^2$ then we can write
$$V = v \frac{\partial}{\partial x}, V_k = v_k \frac{\partial}{\partial x}$$
for smooth a function $v, v_k : \mathbb{C} \rightarrow \mathbb{C}$ and if $\sigma = \sigma(V, V_1, V_2, V_3)$ is the symbol corresponding to $V + V_1 \otimes V_2 \otimes V_3$ then from formulas [5.3] and [5.4] we obtain that
$$< \sigma(e_1)e_1, e_1 > + i < \sigma(e_1)e_1, e_2 > = \frac{1}{\sqrt{2}}(\lambda(|z|)v(z) + (\lambda(|z|))^3s(z))$$
$$< \sigma(e_2)e_1, e_1 > + i < \sigma(e_2)e_1, e_2 > = \frac{i}{\sqrt{2}}(\lambda(|z|)v(z) - (\lambda(|z|))^3s(z)),$$
where
$$s(z) = v_1(z)v_2(z)v_3(z) \text{ and } \lambda(r) = \frac{2}{1 + r^2}.$$ In the $w$ variables the above equation becomes
$$< \sigma(f_1)f_1, f_1 > + i < \sigma(f_1)f_1, f_2 > = -\frac{1}{\sqrt{2}}(\lambda(|w|)|w|^2v(\bar{w}^{-1}) + \lambda^3(|w|)|w|^6s(\bar{w}^{-1}))$$
$$< \sigma(f_2)f_1, f_1 > + i < \sigma(f_2)f_1, f_2 > = \frac{i}{\sqrt{2}}(\lambda(|w|)|w|^2v(\bar{w}^{-1}) - \lambda^3(|w|)|w|^6s(\bar{w}^{-1})).$$
We conclude that singular set of $\sigma$ on the $z$ variables is then given by
$$S_z = \{ z \in \mathbb{C} | (|v(z)| = \lambda^2(|z|)|s(z)|) \}$$
and over the $w$ variables is given by

\begin{equation}
S_w = \{ w \in \mathbb{C} | |w^2 v_0(w^{-1})| = \lambda^2(|w|)|w^6 s(w^{-1})| \}.
\end{equation}

We are now ready to construct traceless symmetric symbols from holomorphic vector fields on $S^2$. Recall the following simple result.

Lemma 26. Consider a quadratic polynomial $P(z) = a_0 + a_1 z + a_2 z^2$. The holomorphic vector field $V_P(z) = P(z) \frac{\partial}{\partial x}$ extends to a holomorphic vector field over the whole sphere $S^2$.

Proof. We only need to check that the $V_P$ is smooth at the point at infinity. By the change of variables formula for vector fields, we have that

\[
P(z) \frac{\partial}{\partial x} = -w^2 P(w^{-1}) \frac{\partial}{\partial u} = -(a_2 + a_1 z + a_0 w^2) \frac{\partial}{\partial u}
\]

which is holomorphic at 0. Hence $V_P$ is smooth at infinity. \qed

Given a second degree polynomial $P$ and a sixth degree polynomial $S = P_1 P_2 P_3$ (where the $P_i$'s are also second degree polynomials) we will let

\[\sigma_{P,S} = \sigma(V_P, V_{P_1}, V_{P_2}, V_{P_3}).\]

We will now study the symbols $\sigma_{m,n} = \sigma_{z^n, z^m}$. To compute the singular set of $\sigma_{m,n}$ we will need the following result.

Lemma 27. For $0 \leq m \leq 2$ and $0 \leq n \leq 6$ let

\[Z_{m,n} = \{ r \in \mathbb{R} | r^m - (\lambda(r))^2 r^n = 0 \}.\]

We have that

\[Z_{m,0} = Z_{0,n} = \{ 1 \} \text{ iff } n \neq 1, 3\]

and

\[Z_{0,1} = \{ \alpha, 1 \}, Z_{0,3} = \{ 1, 1/\alpha \}\]

where $\alpha$ is the only real root of $r^3 + r^2 + 3r - 1$. Furthermore, for $0 < m \leq 2$ and $0 < n \leq 6$ we have that

\[Z_{m,n} = \begin{cases} Z_{m-n,0} \cup \{ 0 \} & \text{if } n \leq m \\ Z_{0,n-m} \cup \{ 0 \} & \text{if } m \leq n \end{cases}.\]

Proof. Compute! \qed

Proposition 28. For $0 \leq m \leq 2$ and $0 \leq n \leq 6$, the singular set $S_{m,n}$ of the symbol $\sigma_{m,n}$ is given by

\[S_{m,n} = \begin{cases} \{ z \in \mathbb{C} | |z| \in Z_{m,n} \} & \text{if } m = 2 \text{ or } n = 6 \\ \{ z \in \mathbb{C} | |z| \in Z_{m,n} \} \cup \{ \infty \} & \text{if } m < 2 \text{ and } n < 6 \end{cases}.\]

The traceless symbol $\sigma_{m,n}$ is transversal to the zero section of $\mathcal{S}_0^2(TS^2)$ at $\mathcal{M}_{\sigma_{m,n}}|S^1$, where $\{ z \in \mathbb{C} | |z| = 1 \}$, if $n - m \neq 2$. 
Proof. For the first part apply Lemma 27 to formulas 5.5 and 6.6. The determinant function of \( \sigma: T^*S^2 \to S^2_{0}(TS^2) \) is given at \( |z| = r \) is by

\[
h(r) = (\lambda(r)r^m)^2 - (\lambda(r)^3r^n)^2.
\]

Is easy to verify that

\[
\frac{\partial h}{\partial r}(1) = 2(2 + m - n).
\]

From this and Proposition 12 we obtain the result of the second part of this proposition. \( \square \)

The above results shows that the only symbols everywhere transversal to the zero section of \( S^2_{0}(TS^2) \) are \( \sigma_{0,0}, \sigma_{2,0} \). We will now study winding numbers \( m_{S^1} \) defined in 3.5 associated to the multiplicity sets

\[
\mathcal{M}_{m,n} = \mathcal{M}_{\sigma_{m,n}} \subset S(T^*S^2)
\]

over the circle \( S^1 = \{|z| = 1\} \).

**Proposition 29.** For \( \sigma = \sigma_{m,n} \) we have that \( m_{S^1} = n - m \).

Proof. At the point \( e^{i\theta} \in S^1 \) we have that

\[
\sigma_{m,n}(v_1e_1 + v_2e_2) = (pv_1 + rv_2)e_1 + (qv_1 + sv_2)e_2,
\]

where

\[
p + iq = \frac{1}{\sqrt{2}}(e^{im\theta} + e^{in\theta})
\]

\[
r + is = \frac{i}{\sqrt{2}}(e^{i\theta} - e^{in\theta}).
\]

Hence \( v_1e_1 + v_2e_2 \) is in \( \ker(\sigma_{m,n}) \) (at the point \( e^{i\theta} \)) iff

\[
e^{im\theta}(v_1 + iv_2) + e^{in\theta}(v_1 - iv_2) = 0.
\]

If we choose \( v_1 + iv_2 \) to be of unit norm then we can write \( v_1 + iv_2 = e^{i\varphi} \), and the above equation leads to the solutions

\[
\varphi_1 = \frac{1}{2}(n - m)\theta + \frac{\pi}{2}
\]

\[
\varphi_2 = \varphi_1 + \pi,
\]

and hence the result of the proposition. \( \square \)

**Remark 30.** The above results gives an answer to the problem 16 when \( 0 \leq m \leq 2 \) and \( 0 \leq n \leq 6 \). These restrictions on \( m, n \) can be removed by using partitions of unity to smooth out the symbols at 0 and \( \infty \).

**References**

[1] V.I. Arnold. *Singularities of Caustics and Wave Fronts*. Kluwer, 1991.

[2] M. Born and E. Wolf. *Principles of Optics*. Pergamon Press, 1959.

[3] P.J. Braam and J.J. Duistermaat. Normal forms of real symmetric systems with multiplicity. *Indag. Mathem.*, 4(4):69–72, 1993.

[4] V. Guillemin and S. Sternberg. *Geometric Asymptotics*. Number 14 in Mathematicals Surveys and Monographs. American Mathematical Society, 1977.

[5] F. John. Algebraic conditions for hyperbolicity of systems of partial differential operators. *Communications in Pure and Applied Mathematics*, 31:89–106, 1978.

[6] C. Kosniowski. *A First Course in Algebraic Topology*. Cambridge University Press, 1980.
[7] P.D. Lax. The multiplicity of eigenvalues. *Bulletin of the American Mathematical Society*, 6:213–215, 1982.

[8] J.W. Milnor. *Morse Theory*. Number 51 in Annals of Mathematics Studies. Princeton University Press, 1973.

[9] Carlos Valero. Morse theory for the eigenvalue functions of symmetric tensors. *Journal of Topology and Analysis*, 1-4:417–429, 2009.

[10] A.N. Varchenko V.I. Arnold, S.M. Gusain-Zade. *Singularities of Differentiable Maps, Volume I*. Monographs in Mathematics Vol. 82, Birkhauser, 1985.