COMPUTING EIGENPAIRS OF TWO-PARAMETER STURM-LIOUVILLE SYSTEMS USING THE BIVARIATE SINC-GAUSS FORMULA

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Abstract. The use of sampling methods in computing eigenpairs of two-parameter boundary value problems is extremely rare. As far as we know, there are only two studies up to now using the bivariate version of the classical and regularized sampling series. These series have a slow convergence rate. In this paper, we use the bivariate sinc-Gauss sampling formula that was proposed in [6] to construct a new sampling method to compute eigenpairs of a two-parameter Sturm-Liouville system. The convergence rate of this method will be of exponential order, i.e. $O(e^{-\delta N}/\sqrt{N})$ where $\delta$ is a positive number and $N$ is the number of terms in the bivariate sinc-Gaussian formula. We estimate the amplitude error associated to this formula, which gives us the possibility to establish the rigorous error analysis of this method. Numerical illustrative examples are presented to demonstrate our method in comparison with the results of the bivariate classical sampling method.

1. Introduction. The issue of computing eigenvalues of one-parameter eigenvalue problems using sinc methods has attracted many researchers. During the period 1996-2018, six sampling methods have been developed to compute the eigenvalues of boundary value problems of various types. These are the classical sinc (1996) [13], the regularized sinc (2005) [14], the sinc-Gaussian (2008) [2], the Hermite (2012) [3], the Hermite-Gauss (2016) [4], and the generalized sinc-Gaussian method (2018) [7]. The use of sampling methods in computing eigenpairs of two-parameter boundary value problems is extremely rare. As far as we know, there are only two studies, cf. [1, 15]. In [1], the authors used the bivariate classical sampling series of Whittaker, Kotelnikov and Shannon (WKS) to find a representation for the eigencurves of a two-parameter Sturm-Liouville eigenvalue system. The convergence rate of this method is of order $O(\ln(N)/\sqrt{N})$. The authors of [15] used the regularized sampling method to compute the eigenpairs of a two-parameter Sturm-Liouville eigenvalue problem with three-point boundary conditions, without studying the error analysis.

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The convergence rate of the regularized sampling series is not much better than that of the classical sampling method and it still is of polynomial order. This paper is concerned with constructing a new sampling method to compute eigenpairs of a two-parameter Sturm-Liouville eigenvalue system with separate boundary conditions using a bivariate sinc-Gauss sampling formula. This formula is established in [6] and it is used for the first time to compute eigenpairs of two-parameter eigenvalue problems. The convergence rate of this formula is of exponential order, so it will give us higher accuracy results. Indeed, we consider the regular two-parameter Sturm-Liouville eigenvalue system

\[ y''_{\lambda}(x_r) + \left[ \lambda^2 p_r(x_r) + \mu^2 q_r(x_r) + w_r(x_r) \right] y_r(x_r) = 0, \quad x_r \in [0, b], \tag{1.1} \]

with the separated boundary conditions

\[ \cos(\alpha_r)y_r(0) - \sin(\alpha_r)y'_r(0) = 0, \quad \alpha_r \in [0, \pi), \tag{1.2} \]

\[ \cos(\beta_r)y_r(b) - \sin(\beta_r)y'_r(b) = 0, \quad \beta_r \in [0, \pi), \tag{1.3} \]

where

\[ w_r \in L^1[0, b], \quad p_r, q_r \in C'[0, b], \quad p'_r, q'_r \in AC[0, b], \quad \text{and} \quad \inf \{|p_r|, |q_r|\} > 0. \tag{1.4} \]

Here \( AC[0, b] \) is the class of absolutely continuous functions. We will need the assumption (1.4) to use the Liouville transformation, see (3.2) below, but it is not necessary for the definition of the system (1.1)-(1.3). Throughout this paper, we assume that \( r = 1, 2 \). The parameters \( \lambda \) and \( \mu \) can be complex but will be assumed to be real in most of what follows. The solution of the system (1.1)-(1.3) exists and is an entire function in \( \lambda \) and \( \mu \) for each fixed \( x_r \in [0, b] \). For more details on the system (1.1)-(1.3) see e.g. [8, 11].

A pair \((\lambda^2_n, \mu^2_n)\) is called an eigenpair of (1.1)-(1.3) if this system has a pair of nontrivial solutions, \( y_r(\cdot) \neq 0 \) for \( r = 1, 2 \). Under the definiteness condition, cf. e.g. [12, Theorem 1.1] and [8, Theorem 10.7.1],

\[ \delta(x_1, x_2) := \frac{p_1(x_1)}{p_2(x_1)} \frac{q_1(x_2)}{q_2(x_2)} > 0, \quad \text{for all} \ (x_1, x_2) \in [0, b]^2, \tag{1.5} \]

the system (1.1)-(1.3) has an infinite sequence of eigenpairs lying in \( \mathbb{R}^2 \). For a fixed real value of \( \mu \), each Sturm-Liouville problem in (1.1)-(1.3) has a sequence of eigenvalues \( \{\lambda^2_{r,n}(\mu)\}_{n \geq 1} \). From the one-parameter Sturm-Liouville theory it follows that

\[ \lambda^2_{r,1}(\mu) < \lambda^2_{r,2}(\mu) < \lambda^2_{r,3}(\mu) < \ldots \]

and these sequences converge to \( \pm \infty \) as \( \mu \to \mp \infty \) if the functions \( p_r \) and \( q_r \) have the same sign in \([0, b]\) and to \( \pm \infty \) as \( \mu \to \pm \infty \) if \( p_r, q_r \) have different signs, [8, pp. 110-111]. For each \( n \), \( \lambda_{r,n}(\mu) \) describes a curve in the \( \lambda \mu \)-plane and each \( \lambda_{r,n}(\mu) \) will be monotonically decreasing or increasing depending on the sign of \( p_r \) and \( q_r \). The curves \( \{\lambda^2_{r,n}(\mu)\}_{r=1,2} \) are called the \( n \)-th eigencurves of system (1.1)-(1.3) and they are analytic in \(-\infty < \mu < \infty\), cf. e.g. [8, 11]. Evidently, the eigenpair \((\lambda^2_n, \mu^2_n)\) of system (1.1)-(1.3) is the intersection of the \( n \)-th eigencurves. The asymptotic analysis of eigencurves for problem (1.1)-(1.3) is investigated in [9, 10]. A review and bibliography of eigencurves for two-parameter Sturm-Liouville equations can be found in [11].

Let \( y_r(\cdot, \lambda, \mu) \) denote the solution of (1.1) satisfying the initial conditions

\[ y_r(0, \lambda, \mu) = \sin(\alpha_r), \quad y'_r(0, \lambda, \mu) = \cos(\alpha_r). \]
By a theorem on analytic parameter dependence, the function
\[ \Delta_r(\lambda, \mu) = \cos(\beta_r) y_r(b, \lambda, \mu) - \sin(\beta_r) y'_r(b, \lambda, \mu), \quad r = 1, 2, \] (1.6)
is an entire function in \( \lambda \) and \( \mu \), cf. e.g. [11]. The couples \((\lambda_n, \mu_n)\) are the eigenpairs of system (1.1)-(1.3) if and only if \((\lambda_n, \mu_n)\) are the common zeros of the equations (1.6), [8, p. 106]. Those zeros cannot be computed exactly except for extremely rare cases. If \( \{y_r(\cdot, \lambda_n, \mu_n)\}_{r=1,2} \) is a corresponding set of simultaneous solutions of (1.1)-(1.3), then the product \( \prod_{r=1}^2 y_r(\cdot, \lambda_n, \mu_n) \) is an eigenfunction of this system corresponding to the eigenpair \((\lambda_n^2, \mu_n^2)\), and it is unique up to a multiplicative factor. Under the condition (1.5), the set of eigenfunctions is complete in \( L^2([0, b]^2) \) with respect to the weight function \( \delta(x_1, x_2) \), cf. [8, Theorem 10.6.1].

The rest of the paper has been organized as follows: The next section is devoted to briefly describe the bivariate sinc-Gauss sampling formula. Since alternative samples will be used in our sampling formula, the amplitude error appears in our method. For this reason, we will derive estimates for the amplitude error associated with the bivariate sinc-Gauss sampling formula, which gives us the possibility to establish the rigorous error analysis of this method. The method is constructed in Section 3 and 4, where we also establish a rigorous error analysis associated with this method. Section 5 deals with illustrative examples and comparisons. Lastly, Section 6 concludes the paper.

2. Bivariate sinc-Gauss sampling formula. This section is devoted to describe briefly the bivariate sinc-Gauss sampling formula and investigate the amplitude error associated with it. Let \( \gamma = (\gamma_1, \gamma_2), \gamma_r > 0, \quad r = 1, 2 \). The Bernstein space \( B_{\gamma,\infty}([0, b]^2) \) is the class of entire functions of two variables satisfying the following growth condition
\[ |f(z)| \leq \|f\|_{\infty} \exp \left( \sum_{r=1}^2 \gamma_r |3z_r| \right), \quad z := (z_1, z_2) \in \mathbb{C}^2, \]
which belong to \( L^\infty([0, b]^2) \) when restricted to \( \mathbb{R}^2 \). For \( h_r \in (0, \pi/\gamma_r) \), set \( \delta_r := (\pi - \gamma_r h_r)/2 \) and let \( \mathcal{E}^2 \) be the class of all entire functions on \( \mathbb{C}^2 \). For the Bernstein space \( B_{\gamma,\infty}([0, b]^2) \), in [6] we defined a localization operator \( \mathcal{G}_{h,N} : B_{\gamma,\infty}([0, b]^2) \to \mathcal{E}^2 \cap L^p([0, b]^2) \) via
\[ \mathcal{G}_{h,N}[f](z) := \sum_{k \in \mathbb{Z}^2_0(z)} f(k_1 h_1, k_2 h_2) \prod_{r=1}^2 \sin(\pi h_r^{-1} z_r - k_r, \pi) \exp \left( - \frac{\delta_r(z_r - k_r h_r)^2}{Nh_r^2} \right), \] (2.1)
where \( N \) is a positive integer, \( h := (h_1, h_2), k := (k_1, k_2), z := (z_1, z_2) \in \mathbb{C}^2 \) and \( \mathbb{Z}^2_0(z) := \{ k \in \mathbb{Z}^2 : |[h_r^{-1} R z_r + 1/2] - k_r| \leq N, \quad r = 1, 2 \} \).

The sinc function is defined by
\[ \text{sinc}(t) := \begin{cases} \frac{\sin t}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases} \]
This operator is generalized in [5] including samples from the function and its partial derivatives. Let us mention here that formula (2.1) is defined for wider classes than the Bernstein space \( B_{\gamma,\infty}([0, b]^2) \), cf. [6], but here this space is sufficient to treat our problem. There, we estimated the absolute error \( |f(z) - \mathcal{G}_{h,N}[f](z)| \) where \( f \in B_{\gamma,\infty}([0, b]^2) \) and bounds of exponential order were found. Since the eigenpairs of
the system (1.1)-(1.3) are a subset of $\mathbb{R}^2$, we state a bound for $|f(z) - G_{h,N}[f](z)|$ only on real domain. If $f \in B_{\gamma,\infty}(\mathbb{R}^2)$, then for all $x \in \mathbb{R}^2$, we have [6, Corollary 3.4]

$$|f(x) - G_{h,N}[f](x)| \leq 2\|f\|_{\infty}A_{\delta,N}(x)\frac{e^{-\delta N}}{\sqrt{\pi\delta N}},$$

(2.2)

where $\delta := \min\{\delta_1, \delta_2\}$ and $A_{\delta,N}$ is the real-valued function defined by

$$A_{\delta,N}(x) := \left(1 + \frac{2}{\sqrt{\pi\delta N}} + \frac{1}{e^{2\pi N} - 1}\right)\sum_{r=1}^{2}\left|\sin(\pi h_r^{-1}x_r)\right| + \sum_{r=1}^{2}2\prod_{r=1}^{2}\left|\sin(\pi h_r^{-1}x_r)\right|,$$

(2.3)

such that $x := (x_1, x_2) \in \mathbb{R}^2$. As the function $A_{\delta,N}(x)$ is bounded for all $x \in \mathbb{R}^2$, it does not affect the convergence rate of the error bound in (2.2). Formula (2.1) has a convergence rate of exponential order. In Table 1, we compare between three methods with the same numbers of samples $(2N + 1)^2$

Table 1. Comparisons

| Methods               | Region of approximation | Convergence rate   |
|-----------------------|-------------------------|--------------------|
| WKS sampling          | $[-N, N]^2$             | $\ln N/\sqrt{N}$   |
| Regularized sampling  | $[-N, N]^2$             | $\ln N/N^{m+1/2}$  |
| Sinc-Gaussian sampling| $\prod_{j=1}^{2}[(n_j - 1/2)h_j, (n_j + 1/2)h_j]$ | $e^{-\delta N/\sqrt{N}}$ |

where $n_j, m \in \mathbb{N}$. Thus, the bivariate sinc-Gaussian sampling will give us higher accuracy results when we use it in computing eigenpairs of a two-parameter Sturm-Liouville system with the additional cost that the function is approximated on a smaller domain or using additional samples. It is approximating the function from the Bernstein space $B_{\gamma,\infty}(\mathbb{R}^2)$ using only a finite number of samples of the function. However, sometimes these samples cannot be computed explicitly. This is why the amplitude error associated with formula (2.1) appears. This amplitude error arises when the exact values $f(k_1h_1, k_2h_2)$ of (2.1) are replaced by close approximate ones. We assume that the approximated samples $\tilde{f}(k_1h_1, k_2h_2)$ are close to the original samples $f(k_1h_1, k_2h_2)$, i.e. there is $\varepsilon > 0$ sufficiently small such that

$$\sup_{k \in Z_N^{2}}\left|f(k_1h_1, k_2h_2) - \tilde{f}(k_1h_1, k_2h_2)\right| < \varepsilon.$$

(2.4)

The amplitude error associated with (2.1) is defined by $G_{h,N}[f](x) - G_{h,N}[\tilde{f}](x), x \in \mathbb{R}^2$. In the following theorem, we will show a bound for $|G_{h,N}[f](x) - G_{h,N}[\tilde{f}](x)|$ on the real domain, which will be used in the investigation of the error analysis of our method in Section 4.

**Theorem 2.1.** Let $\gamma_r > 0, h_r \in (0, \pi/\gamma_r)$ and $\delta_r = (\pi - \gamma_r h_r)/2$. Assume that (2.4) holds. Then we have for $x \in \mathbb{R}^2$

$$\left|G_{h,N}[f](x) - G_{h,N}[\tilde{f}](x)\right| < 4\varepsilon \prod_{r=1}^{2}e^{-\delta_r/4N}\left(1 + \sqrt{N/\delta_r}\right).$$

(2.5)

**Proof.** Using condition (2.4), we obtain for $x = (x_1, x_2) \in \mathbb{R}^2$

$$\left|G_{h,N}[f](x) - G_{h,N}[\tilde{f}](x)\right| < \varepsilon \prod_{r=1}^{2}\sum_{k_r \in Z_N(x_r)}\exp\left(-\frac{\delta_r x_r - k_r h_r}{Nh_r^2}\right).$$

(2.6)
Here we have used the fact $|\text{sinc}(\frac{\pi h_r}{r}x_r - k_r\pi)| \leq 1$ for all $x_r \in \mathbb{R}$, and the index $Z_N(x_r)$ is defined as follows

$$Z_N(x_r) := \left\{ k_r \in \mathbb{Z} : |(h_r^{-1}x_r + 1/2) - k_r| \leq N \right\}.$$ 

Let $|h_r^{-1}x_r + 1/2| - k_r = l_r$. Then

$$\sum_{k_r \in Z_N(x_r)} \exp\left(-\frac{\delta_r (x_r - k_r h_r)^2}{N h_r^2}\right) \leq \sum_{|l_r| \leq N} \exp\left(-\frac{\delta_r (l_r - 1/2)^2}{N h_r^2}\right)$$

$$\leq 2e^{-\delta_r/4N} + 2 \int_0^N e^{-\frac{\delta_r}{N}(l_r+1/2)^2} dl_r$$

$$\leq 2e^{-\delta_r/4N} + 2\sqrt{N/2}\int_{\delta_r/2N}^\infty e^{-\frac{1}{2}t^2} dt_r. \quad (2.7)$$

Combining Mills’ Ratio inequality, cf. [17],

$$\int_x^\infty e^{-\frac{1}{2}t^2} dt < \frac{4e^{-\frac{1}{2}x^2}}{3x + \sqrt{x^2 + 8}}, \quad x > 0,$$

with (2.7) and (2.6) implies (2.5). \qed

3. The method. This section is devoted to the construction of our method. The main concept of this approximation is to split the two simultaneous equations in (1.6) into two parts. The first is known and the second is unknown, but it belongs to the Bernstein space $B_{\gamma, \infty}(\mathbb{R}^2)$. Therefore, we approximate the unknown parts of both equations (1.6) using our bivariate sinc-Gauss sampling formula (2.1). We distinguish between the two cases $\alpha_r = 0$ and $\alpha_r \neq 0$ because the representation of solutions of system (1.1)-(1.3) is different in these cases, cf. [12, Theorem 6.1]. From now on, unless otherwise stated, $r = 1, 2$.

3.1. The case $\alpha_r = 0$. Let us start with the following result which we will use in the sequel.

Lemma 3.1. If $a, b > 0$ and $\lambda, \mu \in \mathbb{C}$, then we have

$$\Im\left(\sqrt{a \lambda^2 + b \mu^2}\right) \leq \sqrt{a(3\lambda)^2 + b(3\mu)^2}. \quad (3.1)$$

Proof. Applying the identity $(\Im z)^2 = \frac{1}{2}(|z| - \Re z)$ for $z \in \mathbb{C}$ twice, we obtain

$$\left(\Im\sqrt{a \lambda^2 + b \mu^2}\right)^2 = \frac{1}{2}\left(\Im(a \lambda^2 + b \mu^2) - \Re(a \lambda^2 + b \mu^2)\right)$$

$$\leq \frac{a}{2}\left(|\lambda|^2 - \Re(\lambda^2)\right) + \frac{b}{2}\left(|\mu|^2 - \Re(\mu^2)\right)$$

$$= a\left(3\sqrt{\lambda^2}\right)^2 + b\left(3\sqrt{\mu^2}\right)^2$$

$$= a(3\lambda)^2 + b(3\mu)^2,$$

from which the assertion follows immediately. \qed

To use the Liouville transformation, denote by

$$\rho_r(x_r) = \int_0^{x_r} \sigma_r^2(\tau_r) d\tau_r, \quad \text{where} \quad \sigma_r(\tau_r) = (\lambda^2 p_r(\tau_r) + \mu^2 q_r(\tau_r))^{1/4}, \quad (3.2)$$
and let $S_n$ be a family of non-empty subsets

$$S_n = \left\{ (\lambda^2, \mu^2) \in \mathbb{C}^2 : \inf_{x_r \in [0,b]} |\sigma_r(x_r)| \geq n, \quad n \in \mathbb{N} \right\}.$$  

Under the assumptions (1.4) and for $(\lambda^2, \mu^2) \in S_n$ as $n \to \infty$ and $\alpha_r \neq 0$, the solutions of the system (1.1)-(1.3) are given by, cf. [12, Theorem 6.1],

$$y_r(x_r, \lambda, \mu) = \frac{1}{\sigma_r(0)\sigma_r(x_r)} \left[ \sin(\rho_r(x_r)) + \int_0^{x_r} \frac{w^*_r(\tau_r) - \rho_r^*(\tau_r)}{\sigma_r^2(\tau_r)} \sin[\rho_r(x_r) - \rho_r(\tau_r)] \sin(\rho_r(\tau_r)) d\tau_r + O\left(\frac{e^{\left|3\rho_r(x_r)\right|}}{n^2}\right) \right],$$

where $\sigma_r, \rho_r, w^*_r$ are functions of $\lambda, \mu$ and the function $w^*_r$ is defined as

$$w^*_r(x_r) = \frac{\sigma_r''(x_r)}{\sigma_r(x_r)} - \frac{\left(\sigma_r'(x_r) - \frac{\sigma_r'(0)}{\sigma_r(0)}\right)^2}{\sigma_r^2(x_r)}$$

satisfying $\sup_{(\lambda^2, \mu^2) \in S_n} \|w^*_r\|_{L^1} < \infty$.  

The derivative of the solutions $y_r(\cdot, \lambda, \mu)$ is also given by, cf. [12, Theorem 6.1],

$$y'_r(x_r, \lambda, \mu) = \frac{\sigma_r(x_r)}{\sigma_r(0)} \left[ \cos(\rho_r(x_r)) - \frac{\sigma_r'(x_r)}{\sigma_r^2(x_r)} \sin(\rho_r(x_r)) \right] + \int_0^{x_r} \frac{w^*_r(\tau_r) - \rho_r^*(\tau_r)}{\sigma_r^2(\tau_r)} \cos[\rho_r(x_r) - \rho_r(\tau_r)] \sin(\rho_r(\tau_r)) d\tau_r + O\left(\frac{e^{\left|3\rho_r(x_r)\right|}}{n^2}\right).$$

Observe that the solutions $y_r(\cdot, \lambda, \mu)$ and their derivatives can be split into

$$y_r(x_r, \lambda, \mu) = \frac{\sin(\rho_r(x_r))}{\sigma_r(0)\sigma_r(x_r)} + R_r(x_r, \lambda, \mu),$$

$$y'_r(x_r, \lambda, \mu) = \frac{\sigma_r(x_r)}{\sigma_r(0)} \left[ \cos(\rho_r(x_r)) - \frac{\sigma_r'(x_r)}{\sigma_r^2(x_r)} \sin(\rho_r(x_r)) \right] + \mathcal{R}_r(x_r, \lambda, \mu),$$

where $R_r$ and $\mathcal{R}_r$ are the remaining terms containing the integrals. By substituting from (3.4) and (3.5) into (1.6), we obtain

$$\Delta_r(\lambda, \mu) = \cos(\beta_r) \frac{\sin(\rho_r(b))}{\sigma_r(0)\sigma_r(b)} - \sin(\beta_r) \frac{\sigma_r(b)}{\sigma_r(0)} \left[ \cos(\rho_r(b)) - \frac{\sigma_r'(b)}{\sigma_r^2(b)} \sin(\rho_r(b)) \right] + \cos(\beta_r) \mathcal{R}_r(b, \lambda, \mu) - \sin(\beta_r) \mathcal{R}_r(b, \lambda, \mu).$$

Observe that the simultaneous functions $\Delta_r(\lambda, \mu)$ can be split into two parts as follows

$$\Delta_r(\lambda, \mu) := K_r(\lambda, \mu) + U_r(\lambda, \mu),$$

where $K_r(\lambda, \mu)$ is the known part

$$K_r(\lambda, \mu) := \cos(\beta_r) \sin(\rho_r(b)) - \sin(\beta_r) \frac{\sigma_r(b)}{\sigma_r(0)} \left[ \cos(\rho_r(b)) - \frac{\sigma_r'(b)}{\sigma_r^2(b)} \sin(\rho_r(b)) \right],$$

and the second part, which will be denoted by $U_r(\lambda, \mu)$, consists of the remaining terms in (3.6), i.e.

$$U_r(\lambda, \mu) := \cos(\beta_r) \mathcal{R}_r(b, \lambda, \mu) - \sin(\beta_r) \mathcal{R}_r(b, \lambda, \mu).$$
In the following theorem, we will show that the unknown part $\mathcal{U}_r$ belongs to the Bernstein space $B_{\gamma_r,\infty}(\mathbb{R}^2)$. That gives us the possibility to approximate $\mathcal{U}_r$ via the bivariate sinc-Gauss sampling formula (2.1).

**Theorem 3.2.** Assume that (1.4) holds. Let $\gamma_r = (\gamma_{r1}, \gamma_{r2})$ be such that

$$\gamma_{r1} = \int_0^b \sqrt{p_r(t)} \, dt, \quad \gamma_{r2} = \int_0^b \sqrt{q_r(t)} \, dt. \quad (3.9)$$

Then $\mathcal{U}_r \in B_{\gamma_r,\infty}(\mathbb{R}^2)$.

**Proof.** Applying inequality (3.1) implies

$$3 \left( \sqrt{\lambda^2 p_r(\tau_r)} + \mu^2 q_r(\tau_r) \right) \leq \sqrt{p_r(\tau_r)(3\lambda)^2 + q_r(\tau_r)(3\mu)^2} \leq \sqrt{p_r(\tau_r)[3\lambda] + \sqrt{q_r(\tau_r)[3\mu]}}.$$  

It is easy to see that

$$|\sin[\rho_r(b) - \rho_r(\tau_r)] - \sin[\rho_r(\tau_r)]| = O(e^{(3\rho_r(b))}),$$

$$|\cos[\rho_r(b) - \rho_r(\tau_r)] - \cos[\rho_r(\tau_r)]| = O(e^{(3\rho_r(b))}), \quad (3.10)$$

for all $\tau_r \in [0, b]$ where $\rho_r$ is defined in (3.2). Combining (3.10) and (3.9), using the fact $w_r \in L^1[0, b]$, $\sigma_r(\tau_r) \geq n$ for all $\tau_r \in [0, b]$ and $\sup_{(\lambda^2, \mu^2) \in S_n} \|w_r\|_{L^1} < \infty$, cf. (3.3), we get a positive constant $A$, independent on $\lambda, \mu$, such that

$$|\mathcal{U}_r(\lambda, \mu)| \leq Ae^{(3\lambda)|+\gamma_{r2}|}\mu], \quad (\lambda^2, \mu^2) \in S_n, \quad (3.11)$$

where $\gamma_{r1}$ and $\gamma_{r2}$ are defined above. It follows from (3.11) that $\mathcal{U}_r$ belongs to the Bernstein space $B_{\gamma_r,\infty}(\mathbb{R}^2)$.

Since $\mathcal{U}_r \in B_{\gamma_r,\infty}(\mathbb{R}^2)$, we can approximate it using the bivariate sinc-Gauss sampling formula (2.1), i.e. $\mathcal{U}_r(\lambda, \mu) \approx G_{h, N}[\mathcal{U}_r](\lambda, \mu)$ where the samples are given by

$$\mathcal{U}_r(k_1h_1, k_2h_2) = \cos(\beta_r) y_r(b, k_1h_1, k_2h_2) - \sin(\beta_r) y'_r(b, k_1h_1, k_2h_2) - K_r(k_1h_1, k_2h_2), \quad (3.12)$$

where $k \in \mathbb{Z}^2_\lambda(\lambda, \mu)$ and $\beta_r \in [0, \pi)$. Unfortunately, the samples $\mathcal{U}_r(k_1h_1, k_2h_2)$ cannot be determined explicitly in the general case. That is why the amplitude error usually appears. Let $\hat{\mathcal{U}}_r(k_1h_1, k_2h_2)$ be the approximation of the samples $\mathcal{U}_r(k_1h_1, k_2h_2)$ when the solution $y_r(b, k_1h_1, k_2h_2)$ and its derivative $y'_r(b, k_1h_1, k_2h_2)$ are computed numerically at the nodes $(k_1h_1, k_2h_2)$, $k \in \mathbb{Z}^2_\lambda(\lambda, \mu)$. Now, let us define the following interesting function

$$\hat{\Delta}_{r, h, N}(\lambda, \mu) := K_r(\lambda, \mu) + G_{h, N}[\hat{\mathcal{U}}_r](\lambda, \mu),$$

where $K_r$ is defined in (3.8) and $G_{h, N}[\hat{\mathcal{U}}_r]$ is the bivariate sinc-Gauss sampling formula (2.1) which is constructed by the approximated samples $\hat{\mathcal{U}}_r(k_1h_1, k_2h_2)$. The simultaneous functions $\hat{\Delta}_{r, h, N}(\lambda, \mu)$ are determined explicitly and will be very close to the functions $\Delta_r(\lambda, \mu)$ which are defined in (1.6), as we will see in the next result. Therefore, the zeros of the simultaneous equations $\hat{\Delta}_{r, h, N}(\lambda, \mu)$ will be very close to the desired zeros, which are precisely the eigenpairs of the system (1.1)-(1.3), of $\Delta_r(\lambda, \mu)$.
3.2. **The case** $\alpha_r \neq 0$. In this case, we describe our method briefly. Under the assumptions (1.4), $p_r, q_r > 0$ on $[0, b]$ and for $(\lambda^2, \mu^2) \in S_n$ as $n \to \infty$ and $\alpha_r \neq 0$, the solutions of system (1.1)-(1.3) are given by, cf. [12, Theorem 6.1],

$$
y_r(x_r, \lambda, \mu) = \frac{\sigma_r(0)\sin(\alpha_r)}{\sigma_r(x_r)} \left[ \cos(\rho_x(x_r)) \left( \frac{\cot(\alpha_r)}{\sigma^2_r(0)} + \frac{\sigma_r'(0)}{\sigma^2_r(0)} \right) \sin(\rho_x(x_r)) \right]
\quad + \int_0^{x_r} \frac{w^*_r(\tau_r) - w_r(\tau_r)}{\sigma^2_r(\tau_r)} \sin[\rho_x(x_r) - \rho_x(\tau_r)] \cos(\rho_x(\tau_r)) d\tau_r
\quad + O\left( \frac{\|e_{3,\rho_x(x_r)}\|}{n^2} \right),
$$

(3.14)

where the function $w^*_r$ is defined in (3.3). In this case, the derivative of the solution $y_r(\cdot, \lambda, \mu)$ is also given by, cf. [12, Theorem 6.1],

$$
y'_r(x_r, \lambda, \mu)
\quad = \frac{\sigma_r(x_r)\sigma_r(0)\sin(\alpha_r)}{\sigma_r(b)} \left[ -\sin(\rho_x(x_r)) \left( \frac{\cot(\alpha_r)}{\sigma^2_r(0)} + \frac{\sigma_r'(0)}{\sigma^2_r(0)} \right) \cos(\rho_x(b)) \right]
\quad + \int_0^{x_r} \frac{w^*_r(\tau_r) - w_r(\tau_r)}{\sigma^2_r(\tau_r)} \cos[\rho_x(x_r) - \rho_x(\tau_r)] \cos(\rho_x(\tau_r)) d\tau_r
\quad + O\left( \frac{\|e_{3,\rho_x(x_r)}\|}{n^2} \right),
$$

(3.15)

Combining (3.14), (3.15) and (1.6) and splitting the function $\Delta_r$ into two parts, as we have done in the last case, the known part will be

$$
\mathcal{K}_{r,\alpha_r}(\lambda, \mu) = \cos(\beta_r) \frac{\sigma_r(0)\sin(\alpha_r)}{\sigma_r(b)} \left[ \cos(\rho_x(b)) \left( \frac{\cot(\alpha_r)}{\sigma^2_r(0)} + \frac{\sigma_r'(0)}{\sigma^2_r(0)} \right) \sin(\rho_x(b)) \right]
\quad - \sin(\beta_r) \sigma_r(b) \sigma_r(0) \sin(\alpha_r)
\quad \times \left[ -\sin(\rho_x(b)) + \left( \frac{\cot(\alpha_r)}{\sigma^2_r(0)} + \frac{\sigma_r'(0)}{\sigma^2_r(0)} - \frac{\sigma'(\beta_r)}{\sigma^2(\beta_r)} \right) \cos(\rho_x(b)) \right],
$$

(3.16)

and the unknown part will be

$$
\mathcal{U}_{r,\alpha_r}(\lambda, \mu) = \cos(\beta_r) \mathcal{R}_{r,\alpha_r}(b, \lambda, \mu) - \sin(\beta_r) \mathcal{R}_{r,\alpha_r}(b, \lambda, \mu),
$$

where $\mathcal{R}_{r,\alpha_r}$ and $\mathcal{R}_{r,\alpha_r}$ are the remaining terms of (3.14) and (3.15), respectively, containing the integrals with $x_r = b$. Applying the same technique as in the proof of Theorem 3.2, we can prove the following result.

**Theorem 3.3.** Assume that (1.4) holds. Then $\mathcal{U}_{r,\alpha_r}$ belongs to the Bernstein space $B_{\gamma_r,\infty}(\mathbb{R}^2)$ where $\gamma_r = (\gamma_{r1}, \gamma_{r2})$ and $\gamma_{rs}, s = 1, 2$, is defined in (3.9).

Since $\mathcal{U}_{r,\alpha_r} \in B_{\gamma_r,\infty}(\mathbb{R}^2)$, we can approximate it using the bivariate sinc-Gauss sampling formula (2.1). Here, we complete the method in the same way as we have done in the last case.

**Remark 1.** If $\alpha_1 = 0$ and $\alpha_2 \neq 0$, the solutions $y_1$ and $y_2$ will be as given in (3.4) and (3.14), respectively. The cases $\alpha_1 \neq 0$ and $\alpha_2 = 0$ are similar. In these cases, the method will be completed as indicated above.

4. **Error analysis.** In this section, we show that the function $\tilde{\Delta}_{r,h,N}$ is very close to the function $\Delta_r$. Furthermore, we find a bound for the standard Euclidean norm $\| (\lambda^*, \mu^*) - (\lambda_{\varepsilon,N}, \mu_{\varepsilon,N}) \|_{\mathbb{R}^2}$, where $((\lambda^*)^2, (\mu^*)^2)$ is an eigenpair of the system (1.1)-(1.3) and $(\lambda_{\varepsilon,N}^2, \mu_{\varepsilon,N}^2)$ is its desired approximation.
Theorem 4.1. For \((\lambda, \mu) \in \mathbb{R}^2\) and \(N \in \mathbb{N}\), we have
\[
\left| \Delta_r(\lambda, \mu) - \tilde{\Delta}_{r,h,N}(\lambda, \mu) \right| < T_{r,h,N}(\lambda, \mu) + A_{\varepsilon,r,N},
\]
(4.1)
where \(\Delta_r\) and \(\tilde{\Delta}_{r,h,N}\) are given in (1.6) and (3.13), respectively. The functions \(T_{r,h,N}\) and \(A_{\varepsilon,r,N}\) are defined by
\[
T_{r,h,N}(\lambda, \mu) := 2 B_r A_{\delta_{r,N}}(\lambda, \mu) e^{-\delta_r N \sqrt{\pi \delta_r N}},
\]
(4.2)
\[
A_{\varepsilon,r,N} := 4 \varepsilon^2 \Pi_{r=1}^2 e^{-\delta_r / 4N} \left(1 + \sqrt{N / \delta_r} \right),
\]
(4.3)
where \(\delta_r = \min\{\delta_{rs}, \delta_{rs}\}\), \(\delta_{rs} = (\pi - \gamma_s h_s) / 2, h_s \in (0, \pi / \gamma_s], s = 1, 2\) and
\[
B_r := \begin{cases} 
\|U_r\|_{\infty}, & \alpha_r = 0, \\
\|U_{r,\alpha_r}\|_{\infty}, & \alpha_r \neq 0,
\end{cases}
\]
where \(\|f\|_{\infty} := \sup \{|f(x)|, x \in \mathbb{R}^2\}\). The functions \(A_{\delta_{r,N}}, U_r\) and \(U_{r,\alpha_r}\) are given in (2.3), (3.9) and (3.17), respectively. Moreover, \(\tilde{\Delta}_{r,h,N} \rightarrow \Delta_r\) holds uniformly on \(\mathbb{R}^2\) for \(\varepsilon \rightarrow 0\) and \(N \rightarrow \infty\).

Proof. According to (3.7) and (3.13), we have for all \((\lambda, \mu) \in \mathbb{R}^2\)
\[
\left| \Delta_r(\lambda, \mu) - \tilde{\Delta}_{r,h,N}(\lambda, \mu) \right| \leq |U_r(\lambda, \mu) - G_{h,N}[U_r](\lambda, \mu)|
\]
\[
+ \left| G_{h,N}[U_r](\lambda, \mu) - G_{h,N}[\tilde{U}_r](\lambda, \mu) \right|.
\]
(4.4)
Since \(U_r \in B_{r,\infty}(\mathbb{R}^2)\), cf. Theorem 3.2, we have, cf. (2.2),
\[
|U_r(\lambda, \mu) - G_{h,N}[U_r](\lambda, \mu)| \leq T_{r,h,N}(\lambda, \mu),
\]
(4.5)
where \(T_{r,h,N}\) is defined as above. Assume that \(\varepsilon\) is sufficiently small such that condition (2.4) holds, then we have, cf. Theorem 2.1,
\[
\left| G_{h,N}[U_r](\lambda, \mu) - G_{h,N}[\tilde{U}_r](\lambda, \mu) \right| < A_{\varepsilon,r,N},
\]
(4.6)
where \(A_{\varepsilon,r,N}\) is defined as above. Combining (4.6), (4.5) and (4.4), we obtain (4.1) in the case \(\alpha_r = 0\). The proof of the case \(\alpha_r \neq 0\) is similar. In view of (4.2) and (4.3), the right-hand side of (4.1) goes to zero uniformly when \(\varepsilon \rightarrow 0\) and \(N \rightarrow \infty\), and therefore \(\tilde{\Delta}_{r,h,N} \rightarrow \Delta_r\) uniformly on \(\mathbb{R}^2\). \(\square\)

Denote by \(J_{\Delta}\) the Jacobian matrix
\[
J_{\Delta}(\lambda, \mu) = \begin{pmatrix} \frac{\partial}{\partial \lambda} \Delta_1(\lambda, \mu) & \frac{\partial}{\partial \mu} \Delta_1(\lambda, \mu) \\ \frac{\partial}{\partial \lambda} \Delta_2(\lambda, \mu) & \frac{\partial}{\partial \mu} \Delta_2(\lambda, \mu) \end{pmatrix}.
\]
If \((\lambda^*, \mu^*)\) is a zero of both equations in (1.6), then the determinant \(|J_{\Delta}(\lambda^*, \mu^*)|\) is nonzero, cf. [16, Lemma 4.1]. This fact will be used in the proof of the following theorem.
Theorem 4.2. Let \((\lambda^*, \mu^*)\) be an eigenpair of system (1.1)-(1.3) and denote by \((\lambda_{e,N}^*, \mu_{e,N}^*)\) the corresponding approximation. In the notation of (4.2) and (4.3), we have the following estimate for the Euclidean norm

\[
\| (\lambda^*, \mu^*) - (\lambda_{e,N}, \mu_{e,N}) \|_{\mathbb{R}^2} < D_{e,N} \sum_{r=1}^{2} (T_{r,h,N}(\lambda_{e,N}, \mu_{e,N}) + A_{e,r,N}),
\]

where \(D_{e,N} := \max_{(\xi_1, \xi_2) \in N_{e,N}(\lambda^*, \mu^*)} | J_\Delta^{-1}(\xi_1, \xi_2) |_F \), \(J_\Delta^{-1} \) is the inverse Jacobian matrix, \(N_{e,N}(\lambda^*, \mu^*) \) is a neighborhood of \((\lambda^*, \mu^*) \) depending on \(\varepsilon, N \) and \(\| \cdot \|_F \), \(\| \cdot \|_{\mathbb{R}^2} \) are the Frobenius and standard Euclidian norm, respectively. Furthermore, \(\| (\lambda^*, \mu^*) - (\lambda_{e,N}, \mu_{e,N}) \|_{\mathbb{R}^2} \) approaches 0 as \(\varepsilon \to 0 \) and \(N \to \infty \).

Proof. Replacing \((\lambda, \mu) \) by \((\lambda_{e,N}, \mu_{e,N}) \) in (4.1) implies

\[
| \Delta_r(\lambda_{e,N}, \mu_{e,N}) | < T_{r,h,N}(\lambda_{e,N}, \mu_{e,N}) + A_{e,r,N}. \tag{4.8}
\]

Expanding the function \(\Delta_r \) using bivariate Taylor expansion at the point \((\lambda^*, \mu^*)\) and replacing \((\lambda, \mu) \) by \((\lambda_{e,N}, \mu_{e,N}) \), we obtain

\[
\Delta_r(\lambda_{e,N}, \mu_{e,N}) = (\lambda_{e,N} - \lambda^*) \frac{\partial}{\partial \lambda} \Delta_r(\xi_1, \xi_2) + (\mu_{e,N} - \mu^*) \frac{\partial}{\partial \mu} \Delta_r(\xi_1, \xi_2), \tag{4.9}
\]

where \((\xi_1, \xi_2) \) is sufficiently close to the point \((\lambda^*, \mu^*) \). Indeed, \((\xi_1, \xi_2) \) is an unknown point on the line joining \((\lambda_{e,N}, \mu_{e,N}) \) and \((\lambda^*, \mu^*) \). System (4.9) can be represented as

\[
\begin{pmatrix}
\Delta_1(\lambda_{e,N}, \mu_{e,N}) \\
\Delta_2(\lambda_{e,N}, \mu_{e,N})
\end{pmatrix}
= J_\Delta \begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix}
\begin{pmatrix}
\lambda_{e,N} - \lambda^* \\
\mu_{e,N} - \mu^*
\end{pmatrix}. \tag{4.10}
\]

Since the Jacobian determinant is an entire function and \(| J_\Delta(\lambda^*, \mu^*) | \neq 0 \), for a sufficiently large \(N \) and a sufficiently small \(\varepsilon \) we can find a neighborhood \(N_{e,N}(\lambda^*, \mu^*) \) such that the inverse of the Jacobian matrix \(J_\Delta^{-1}(\xi_1, \xi_2) \) exists for all points in this neighborhood. Now (4.10) implies that

\[
\begin{pmatrix}
\lambda_{e,N} - \lambda^* \\
\mu_{e,N} - \mu^*
\end{pmatrix}
= J_\Delta^{-1}(\xi_1, \xi_2) \begin{pmatrix}
\Delta_1(\lambda_{e,N}, \mu_{e,N}) \\
\Delta_2(\lambda_{e,N}, \mu_{e,N})
\end{pmatrix}. \tag{4.11}
\]

Therefore

\[
\| (\lambda^*, \mu^*) - (\lambda_{e,N}, \mu_{e,N}) \|_{\mathbb{R}^2} \leq \| J_\Delta^{-1}(\xi_1, \xi_2) \|_F \| \Delta_r(\lambda_{e,N}, \mu_{e,N}) \|_{\mathbb{R}^2}. \tag{4.11}
\]

Combining (4.11) and (4.8) yields (4.7). The right-hand side of (4.7) approaches 0 as \(\varepsilon \to 0 \) and \(N \to \infty \), so \(\| (\lambda^*, \mu^*) - (\lambda_{e,N}, \mu_{e,N}) \|_{\mathbb{R}^2} \to 0 \).

5. Numerical illustrations. This section is devoted to the presentation of three numerical examples. In all examples, our results are compared with the results for the bivariate WKS sampling method, cf. [1]. Our method produces results which are more accurate than the bivariate WKS method because its convergence rate is higher than that of the bivariate WKS method. All numerical computations were carried out using Mathematica 12 on a personal computer. For the sake of simplicity, we set \(\nu_r := \max_{r=1,2} (\gamma_1, \gamma_2) \) and \(h := h_1 = h_2 \). Denote by \(\| (\lambda^*_k, \mu^*_k) - (\lambda_{k, e,N}, \mu_{k, e,N}) \|_{\mathbb{R}^2} \) the norm of the error where \((\lambda^*_k, \mu^*_k) \) is an exact zero of system (1.1)-(1.3) and \((\lambda_{k, e,N}, \mu_{k, e,N}) \) is its desired approximation. In Examples 5.1 and 5.2, we choose \(\alpha_r = \beta_r = 0 \), while in Example 5.3 \(\alpha_r = \pi/2 \) and \(\beta_r = 0 \).

Example 5.1. Consider the following system

\[
y''_1(x) + (2\lambda^2 + \mu^2 + 1) y_1(x) = 0, \quad x \in [0, 1], \tag{5.1}
\]
\[
y''_2(x) + (\lambda^2 + 2\mu^2 + 2) y_2(x) = 0, \quad x \in [0, 1]. \tag{5.2}
\]
It is easily checked that conditions (1.4)-(1.5) hold. In this example, \( \alpha_r = \beta_r = 0 \), \( U_r,0 \in B(\nu_1,\nu_2,\infty)(\mathbb{R}^2) \) where \( \nu_1 = \nu_2 = \sqrt{2} \), and it is a simple task to compute the functions \( \Delta_r \):

\[
\Delta_1(\lambda,\mu) = \text{sinc} \left( \sqrt{2\lambda^2 + \mu^2 + 1} \right), \quad \Delta_2(\lambda,\mu) = \text{sinc} \left( \sqrt{\lambda^2 + 2\mu^2 + 2} \right).
\]

The function \( K_r \), defined in (3.8), is computed as

\[
K_1(\lambda,\mu) = \text{sinc} \left( \sqrt{2\lambda^2 + \mu^2} \right)
\]

and

\[
K_2(\lambda,\mu) = \text{sinc} \left( \sqrt{\lambda^2 + 2\mu^2} \right).
\]

Since the functions \( \Delta_r \) are given explicitly, the amplitude error does not appear in this example, i.e. \( \varepsilon = 0 \). Figure 1 shows the eigencurves of system (5.1)-(5.3) in the region \([-15, 15]^2\). Their intersections are the eigenpairs of the system and we can compute few of them using our method described above. Note that eigencurves for the two equations in system (5.1)-(5.3) are shaped like ellipses. Table 2 compares results of our method and results of the bivariate WKS sampling method. In Table 3, we show the norm error \( \| (\lambda^*_k,\mu^*_k) - (\lambda_{k,\varepsilon,N},\mu_{k,\varepsilon,N}) \|_{\mathbb{R}^2} \) associated with the two methods. In Figures 2a and 2b, we illustrate the behaviour of our method with respect to \( h \) and \( N \).

**Example 5.2.** Consider the following system

\[
y''_1(x) + (2(x + 1)\lambda^2 + (x + 1)\mu^2 + x) y_1(x) = 0, \quad x \in [0, 1], \quad (5.5)
\]

\[
y''_2(x) + (\lambda^2 + 2\mu^2 + 3) y_2(x) = 0, \quad x \in [0, 1], \quad (5.6)
\]

\[
y_r(0) = 0, \quad y_r(1) = 0, \quad r = 1, 2. \quad (5.7)
\]

This system is a special case of system (1.1)-(1.3) and satisfies the conditions (1.4)-(1.5). In this example, \( U_r,0 \in B(\nu_1,\nu_2,\infty)(\mathbb{R}^2) \) where \( \nu_1 = \nu_2 = 2(4 - \sqrt{2})/3 \), and the function \( \Delta_1 \) can be expressed in terms of Airy functions \( \text{Ai}, \text{Bi} \) and their first derivatives. Let

\[
\omega_1 = \frac{\sqrt[3]{1}(2 + 4\lambda^2 + 2\mu)}{(1 + 2\lambda^2 + \mu)^{2/3}} \quad \text{and} \quad \omega_2 = \frac{\sqrt[3]{1}(2\lambda^2 + \mu)}{(1 + 2\lambda^2 + \mu)^{2/3}}.
\]

Then

\[
\Delta_1(\lambda,\mu) = \frac{\text{Ai}(\omega_1)\text{Bi}(\omega_2) - \text{Ai}(\omega_2)\text{Bi}(\omega_1)}{(1 + 2\lambda^2 + \mu)^{2/3}(\text{Ai}'(\omega_2)\text{Bi}(\omega_2) - \text{Ai}(\omega_2)\text{Bi}'(\omega_2))}, \quad (5.8)
\]
Table 2. Approximation of eigenpairs with $h = 1$

| k | $\lambda_{k,0.15}$ | $\mu_{k,0.15}$ |
|---|---|---|
| Bivariate WKS sampling | | |
| 1 | 1.813797507802172 | 1.513239555736101 |
| 2 | 3.627597850186581 | 3.487076569018237 |
| 3 | 5.441403076170987 | 5.34879863878748 |
| 4 | 7.25520877727408 | 7.18607252549332 |
| 5 | 9.06906371615362 | 9.0137773520938 |
| 6 | 10.88280054723645 | 10.836602869705539 |

| Bivariate sinc-Gauss sampling | | |
| 1 | 1.813799364683959 | 1.513231023664942 |
| 2 | 3.627598728958227 | 3.487043523167927 |
| 3 | 5.441398093112097 | 5.348720707354780 |
| 4 | 7.255197457187725 | 7.185950886348564 |
| 5 | 7.255197457187725 | 9.013695321137181 |
| 6 | 10.882796185506988 | 10.83675471755794 |

Table 3. The norm error $\| (\lambda_k^*, \mu_k^*) - (\lambda_{k,0.15}, \mu_{k,0.15}) \|_{R^2}$

| k | Bivariate WKS sampling | Bivariate sinc-Gauss sampling |
|---|---|---|
| | $h = 1$ | $h = 0.5$ |
| 1 | $8.73082 \times 10^{-6}$ | $1.00333 \times 10^{-9}$ | $6.94810 \times 10^{-12}$ |
| 2 | $3.30572 \times 10^{-5}$ | $5.91952 \times 10^{-10}$ | $1.23606 \times 10^{-11}$ |
| 3 | $7.93134 \times 10^{-5}$ | $4.50801 \times 10^{-10}$ | $3.57188 \times 10^{-11}$ |
| 4 | $1.22826 \times 10^{-4}$ | $2.82605 \times 10^{-10}$ | $2.22197 \times 10^{-11}$ |
| 5 | $6.3407 \times 10^{-5}$ | $2.54946 \times 10^{-10}$ | $2.27273 \times 10^{-12}$ |
| 6 | $1.51911 \times 10^{-4}$ | $2.33596 \times 10^{-10}$ | $7.64573 \times 10^{-12}$ |

Figure 2. (a) The logarithm of the norm error $\| (\lambda_k^*, \mu_k^*) - (\lambda_{k,0.20}, \mu_{k,0.20}) \|_{R^2}$ for $k = 1, \ldots, 6$ in Example 1. (b) The logarithm of the norm error $\| (\lambda_N^*, \mu_N^*) - (\lambda_{3,0,N}, \mu_{3,0,N}) \|_{R^2}$ for $N = 10, 15, 20, 25$ in Example 1.
while the function $\Delta_2$ is given in explicit form as

$$
\Delta_2(\lambda, \mu) = \text{sinc}\left(\sqrt{\lambda^2 + 2\mu^2 + 3}\right).
$$

(5.9)

In this example, we have $K_1(\lambda, \mu) = \frac{\nu_1}{2\mu^3}\text{sinc}(\frac{\nu_1}{\sqrt{2}}\sqrt{2\lambda^2 + \mu^2})$ and $K_2(\lambda, \mu) = \text{sinc}(\sqrt{\lambda^2 + 2\mu^2})$. To compute the norm error in this example, the exact common zeros of equations (5.8) and (5.9) are computed sufficiently exactly with Mathematica. We apply our method and summarize the result in Tables 4 and 5. Figure 3 demonstrates the eigencurves of system (5.5)-(5.7) in the region $[-15, 15]^2$. Here, too, the eigencurves are shaped like ellipses.

**Figure 3.** The eigencurves in Example 2.

**Table 4.** Approximation of eigenpairs with $h = 1$

| $k$ | $\lambda_{k,0.15}$ | $\mu_{k,0.15}$ |
|-----|-----------------|-----------------|
|     |                 |                 |
| Bivariate WKS sampling |                 |                 |
| 1   | 1.359821568195881 | 1.584365124779384 |
| 2   | 2.294869574533618 | 3.950449753437753 |
| 3   | 5.235270180088456 | 2.129614898841256 |
| 4   | 6.477462750322421 | 4.683407147326487 |
| 5   | 7.667060430768335 | 6.932768257873039 |
| 6   | 8.825101408478323 | 9.106626719982193 |
| Bivariate sinc-Gauss sampling |                 |                 |
| 1   | 1.359811348447286 | 1.584379611568847 |
| 2   | 2.294859272608290 | 3.950445447061610 |
| 3   | 5.235258286227501 | 2.12961255383138 |
| 4   | 6.477390507161662 | 4.683473685527605 |
| 5   | 7.666946848415846 | 6.932950168963926 |
| 6   | 8.825213824551480 | 9.106473269299752 |

**Example 5.3.** The following system

\begin{align*}
 y_1''(x) + (3\lambda^2 + 2\mu^2 + \sqrt{x}) y_1(x) &= 0, \quad x \in [0, 1], \quad (5.10) \\
 y_2''(x) + (2\lambda^2 + 3\mu^2 + x\sqrt{x}) y_2(x) &= 0, \quad x \in [0, 1], \quad (5.11) \\
 y_r'(0) &= 0, \quad y_r'(1) = 0, \quad r = 1, 2, \quad (5.12)
\end{align*}
Table 5. The norm error \( \| (\lambda^*_k, \mu^*_k) - (\lambda_{k,0.15}, \mu_{k,0.15}) \|_{\mathbb{R}^2} \) with \( h = 1 \)

| \( k \) | Bivariate WKS sampling | Bivariate sinc-Gauss sampling |
|------|----------------|--------------------------|
| 1    | 1.77230 \times 10^{-5} | 6.16607 \times 10^{-9} |
| 2    | 1.11696 \times 10^{-5} | 8.90886 \times 10^{-9} |
| 3    | 1.24318 \times 10^{-5} | 7.90168 \times 10^{-9} |
| 4    | 9.82270 \times 10^{-5} | 1.60926 \times 10^{-8} |
| 5    | 2.14480 \times 10^{-4} | 2.13712 \times 10^{-8} |
| 6    | 1.90214 \times 10^{-4} | 8.33743 \times 10^{-9} |

is a special case of system (1.1)-(1.3). Here, we cannot compute the function \( \Delta_r \) and then the eigenpairs cannot be computed exactly. To find the samples \( U_r(nh,mh) \), \((n,m) \in \mathbb{Z}^2_N(\lambda,\mu) \) in this example, we compute the solution of system (1.1)-(1.3), i.e. \( y_r(1,nh,mh) \), numerically at the nodes \((n,m) \in \mathbb{Z}^2_N(\lambda,\mu) \) and then use (3.12). Therefore, we approximate the function \( \Delta_r \) as in (3.13). Here, \( K_1(\lambda,\mu) = \cos(\sqrt{3\lambda^2 + 2\mu^2}) \) and \( K_2(\lambda,\mu) = \cos(\sqrt{2\lambda^2 + 3\mu^2}) \). Few eigenpairs of this system are given in Figure 6 and the eigencurves are shown in Figure 4, as well.

Figure 4. The eigencurves in Example 3.

6. Conclusions. This work is devoted to constructing a new sampling method to compute eigenpairs of a two-parameter Sturm-Liouville eigenvalue system with separate boundary conditions. This method is built by using the bivariate sinc-Gauss sampling formula which was established by the authors in 2016. Due to the exponential order of convergence, the method described here, i.e. the bivariate sinc-Gauss method, is superior to comparable methods for computing the eigenpairs of the boundary value system. The accuracy of the method increases without additional cost when the parameter \( N \) is fixed and \( h \) is decreasing, except that the function is approximated on a smaller domain. Examples are provided to illustrate the effectiveness of the approximation. This method can be applied to compute eigenpairs for various types of two-parameter boundary eigenvalue systems. This will be studied in future works.

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Table 6. Approximation of eigenpairs with $h = 1$ and $\varepsilon = 10^{-8}$

| $k$ | $\lambda_{k,\varepsilon,15}$ | $\mu_{k,\varepsilon,15}$ |
|-----|----------------------------|-------------------------|
| Bivariate WKS sampling | | |
| 1   | 0.515656277786066          | 0.762177530812667       |
| 2   | 2.051784932534724          | 2.114071060975086       |
| 3   | 3.478736174942723          | 3.516120586250193       |
| 4   | 4.893280857549082          | 4.91999452785837        |
| 5   | 6.30363755607843           | 6.32437664274745        |
| 6   | 7.12016456380302           | 7.72878003793143        |
| 7   | 9.119198704295004          | 9.133705129467295       |
| 8   | 10.525522861700562         | 10.53851564514548       |
| Bivariate sinc-Gauss sampling | | |
| 1   | 0.515671212590693          | 0.762173604088073       |
| 2   | 2.051799490194234          | 2.11406851491409        |
| 3   | 3.478721556147922          | 3.51612305356794        |
| 4   | 4.893200190982574          | 4.92001111879944        |
| 5   | 6.303486384409280          | 6.324392419791860       |
| 6   | 7.11851697945461           | 7.728986314987891       |
| 7   | 9.119176716920121          | 9.133693454689087       |
| 8   | 10.525875454418745         | 10.538468065160703      |

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