POINT-INvariant CLASSES OF THE THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS.

V. V. Dmitrieva

Abstract. Point transformations of the 3-rd order ordinary differential equations are considered. Special classes of ordinary differential equations that are invariant under the general point transformations are constructed.

1. Introduction.

Let us consider a class of ordinary differential equations of the third order resolved with respect to the leading derivative

\[ y''' = f(x, y, y', y''). \]

(1.1)

Under the general point transformation

\[
\begin{align*}
\tilde{x} &= \tilde{x}(x, y), \\
\tilde{y} &= \tilde{y}(x, y).
\end{align*}
\]

(1.2)

some equation from the class (1.1) is transformed into the following one

\[ \tilde{y}''' = g(\tilde{x}, \tilde{y}, \tilde{y}', \tilde{y}''). \]

(1.3)

Let us choose the special subclass of ODE’s (1.1) such that the function \( f \) from the right hand side of (1.1) is in the certain preassigned set of functions \( F \) (\( f \in F \)).

Definition 1. If \( \forall f \in F \) the function \( g \) from the right hand side of transformed equation (1.3) is also in the set \( F \) (\( g \in F \)) then this subclass (1.1) is called a point-invariant subclass.

As usual one consider the class \( F \) consisting of polynomials or rational functions in derivatives \( y' \) and \( y'' \).

In XIX-th century the following ODE class

\[ y'' = P(x, y) + 3Q(x, y) y' + 3 R(x, y) (y')^2 + S(x, y) (y')^3. \]

(1.4)

was investigated by R. Liouville [1], S. Lie [2]-[3] and A. Tresse [4]-[5]. E. Cartan considered (1.4) as the equation of geodesic line in the projectively connected space

Author is grateful to Prof. R. A. Sharipov for stating of the problem and for useful discussions.

This research was supported by RFBR, grant no. 00-01-00068, coordinator Prof. Y. T. Sultanaev.

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They called it the point-expansion.

In particular, coordinates (2.1):

\[
y'' = \frac{P(x, y) + 4Q(x, y)y' + 6R(x, y)(y')^2 + 4S(x, y)(y')^3 + L(x, y)(y')^4}{Y(x, y) - X(x, y)y'}.
\]

They called it \textit{the point-expansion} for the class of equations (1.4).

Another class was mentioned L.A. Bordag in [17]:

\[(y'')^2 = P_3(y'; x, y).\]

Here \(P_3 = P_3(y'; x, y)\) is a polynomial of the fifth order in \(y'\).

2. Point transformations.

We can interpret the change of variables (1.2) as the change of some curvilinear coordinates on the plane for another ones. Let’s denote by \(T\) and \(S\) the direct and inverse Jacoby matrices for this change of variables:

\[
S = \begin{vmatrix} x_{1.0} & x_{0.1} \\ y_{1.0} & y_{0.1} \end{vmatrix}, \quad T = \begin{vmatrix} \tilde{x}_{1.0} & \tilde{x}_{0.1} \\ \tilde{y}_{1.0} & \tilde{y}_{0.1} \end{vmatrix}. \tag{2.1}
\]

The point transformations (1.2) are supposed to be non-degenerate, hence \(\det S \neq 0\).

Here and below the notation \(\Phi_{i,j}\) denotes the partial derivative:

\[
\Phi_{i,j} = \frac{\partial^i + \partial^j \Phi}{\partial x^i \partial y^j}.
\]

In particular

\[
x_{1.0} = \frac{\partial x(\tilde{x}, \tilde{y})}{\partial \tilde{x}}.
\]

The derivative \(y'\) obeys the following transformation rule under the change of coordinates (2.1):

\[
y' = \frac{y_{1.0} + y_{0.1} \tilde{y}'}{x_{1.0} + x_{0.1} \tilde{y}'} \tag{2.2}
\]

The corresponding rule for the \(y''\) is more complicated:

\[
y'' = \frac{y_{2.0} + 2y_{1.1} \tilde{y}' + y_{0.2} \tilde{y}'' + y_{0.1} \tilde{y}'''}{(x_{1.0} + x_{0.1} \tilde{y}')^2} - \frac{(y_{1.0} + y_{0.1} \tilde{y}')(x_{2.0} + 2x_{1.1} \tilde{y}' + x_{0.2} \tilde{y}'' + x_{0.1} \tilde{y}'')}{(x_{1.0} + x_{0.1} \tilde{y}')^3} \tag{2.3}
\]

Appropriate formula for the \(y'''\):

\[
y''' = \frac{a_1 \tilde{y}'''}{(x_{1.0} + x_{0.1} \tilde{y}')^5} + \frac{a_2 \tilde{y}'''}{(x_{1.0} + x_{0.1} \tilde{y}')^5} + \frac{a_3 \tilde{y}'''}{(x_{1.0} + x_{0.1} \tilde{y}')^5} + \frac{a_4 \tilde{y}'''}{(x_{1.0} + x_{0.1} \tilde{y}')^5} + \frac{a_5 \tilde{y}'''}{(x_{1.0} + x_{0.1} \tilde{y}')^5} + \frac{a_6 \tilde{y}'''}{(x_{1.0} + x_{0.1} \tilde{y}')^5} + \frac{a_7 \tilde{y}'''}{(x_{1.0} + x_{0.1} \tilde{y}')^5} + \frac{a_8 \tilde{y}'''}{(x_{1.0} + x_{0.1} \tilde{y}')^5} + \frac{a_9 \tilde{y}'''}{(x_{1.0} + x_{0.1} \tilde{y}')^5} + \frac{a_{10} \tilde{y}'''}{(x_{1.0} + x_{0.1} \tilde{y}')^5} + \frac{a_{11} \tilde{y}'''}{(x_{1.0} + x_{0.1} \tilde{y}')^5}. \tag{2.4}
\]
All coefficients \(a_2, \ldots, a_{11}\) in (2.4) are some certain functions in \(x, y\) and their derivatives with respect to \(\tilde{x}, \tilde{y}\) except for the coefficient \(a_1\) which contains the dependence on \(\tilde{y}'\). Let’s write the explicit formulae for \(a_1, \ldots, a_{11}\).

The coefficient of \(\tilde{y}''\):

\[
a_1 = -(x_{1.0} + x_{0.1} \tilde{y}') (y_{1.0}x_{0.1} - x_{1.0} y_{0.1})
\]  

(2.5)

The coefficient of \(\tilde{y}'\):

\[
a_2 = 3x_{0.1} (y_{1.0}x_{0.1} - x_{1.0} y_{0.1})
\]  

(2.6)

The coefficient of \(\tilde{y}''\) \(\tilde{y}'^2\):

\[
a_3 = -6y_{0.1} x_{1.0} x_{0.2} - 3y_{1.1} x_{0.1}^2 + 3y_{1.0} x_{0.2} x_{0.1} + 3y_{0.2} x_{1.0} x_{0.1} + 3y_{0.1} x_{1.1} x_{0.1}
\]  

(2.7)

The coefficient of \(\tilde{y}''\) \(\tilde{y}'\):

\[
a_4 = 9y_{1.0} x_{0.1} x_{1.1} - 3y_{2.0} x_{0.1}^2 + 3y_{0.1} x_{2.0} x_{0.1} + 3y_{0.2} x_{1.0}^2 - 3y_{1.0} x_{0.2} x_{1.0} - 9y_{0.1} x_{1.1} x_{1.0}
\]  

(2.8)

The coefficient of \(\tilde{y}''\):

\[
a_5 = -3y_{1.0} x_{1.1} x_{1.0} + 6y_{1.0} x_{2.0} x_{0.1} - 3y_{0.1} x_{1.0} x_{2.0} + 3y_{1.1} x_{0.1}^2 - 3y_{2.0} x_{1.0} x_{0.1}
\]  

(2.9)

The coefficient of \(\tilde{y}'^5\):

\[
a_6 = 3y_{0.1} x_{0.2}^2 - 3y_{0.2} x_{0.1} x_{0.2} + y_{0.3} x_{0.1}^2 - y_{0.1} x_{0.3} x_{0.1}
\]  

(2.10)

The coefficient of \(\tilde{y}'^4\):

\[
a_7 = -3y_{0.1} x_{1.2} x_{0.1} - 3y_{0.2} x_{1.0} x_{0.2} - 3y_{0.1} x_{0.3} x_{1.0} - 6y_{1.1} x_{0.2} x_{0.1} + 2y_{0.3} x_{1.0} x_{0.1} + 3y_{1.0} x_{0.2}^2 - y_{1.0} x_{0.3} x_{0.1} - 6y_{0.2} x_{1.1} x_{0.1} + 12y_{0.1} x_{1.1} x_{0.2} + 3y_{1.2} x_{0.1}^2
\]  

(2.11)

The coefficient of \(\tilde{y}'^3\):

\[
a_8 = -3y_{0.1} x_{2.1} x_{0.1} - 6y_{0.2} x_{1.1} x_{1.0} - 3y_{1.0} x_{0.3} x_{1.0} - 3y_{0.1} x_{1.2} x_{1.0} + 12y_{0.1} x_{1.1}^2 - 3y_{0.2} x_{2.0} x_{0.1} - 3y_{2.0} x_{0.2} x_{0.1} - 6y_{1.1} x_{0.2} x_{1.0} + 6y_{0.1} x_{2.0} x_{0.2} - 12y_{1.1} x_{1.1} x_{0.1} + 6y_{1.2} x_{1.0} x_{0.1} + y_{0.3} x_{1.0}^2 + 3y_{2.1} x_{0.1}^2 - 3y_{1.0} x_{1.2} x_{0.1} + 12y_{1.0} x_{1.1} x_{0.2}
\]  

(2.12)
The coefficient of $\tilde{y}^2$:
\[
a_9 = -3y_1ox_{1.2}x_{1.0} + 12y_0ox_{2.0}x_{1.1} + 3y_1ox_{1.0}^2 + \\
y_3ox_{1.0}^2 - 6y_2ox_{1.1}x_{1.0} - 3y_0ox_{2.1}x_{1.0} - \\
6y_1ox_{2.0}x_{0.1} - 3y_0ox_{2.1}x_{0.1} - \\
y_0ox_{3.0}x_{1.0} + 2y_3ox_{1.0}x_{0.1} + \\
3y_0ox_{2.0}^2 + 3y_1ox_{1.0}^2 - y_0ox_{3.0}x_{0.1}. 
\]

(2.13)

The coefficient of $\tilde{y}'$
\[
a_{10} = -3y_2ox_{0.1}x_{2.0} + 12y_1ox_{2.0}x_{1.1} - \\
6y_2ox_{1.1}x_{1.0} - 6y_1ox_{2.0}x_{1.0} - 3y_1ox_{2.1}x_{1.0} - \\
y_0ox_{3.0}x_{1.0} + 2y_3ox_{1.0}x_{0.1} + \\
3y_0ox_{2.0}^2 + 3y_1ox_{1.0}^2 - y_0ox_{3.0}x_{0.1}. 
\]

(2.14)

The last term:
\[
a_{11} = -y_{1.0}x_{3.0}x_{1.0} + y_{3.0}x_{1.0}^2 + \\
y_3ox_{2.0}^2 - 3y_2ox_{1.0}x_{2.0}. 
\]

(2.15)

Formula (2.3) and the series of formulae (2.4) – (2.15) determine the transformation rule for $y'''$ under the change of coordinates (2.1).

3. POINT-INARIANT CLASSES OF EQUATIONS OF THE FORM (1.1).

Below we shall seek the point-invariant classes of the equations of the form (1.1) such that the functions $f$ in their right hand sides are rational in $y'$ and polynomial in $y'''$. At first we apply the transformation rules (2.4)–(2.15) to the simplest equation
\[
y''' = 0. 
\]

(3.1)

The shape of transformed equation (3.1)
\[
y''' = \frac{3x_{0.1}\tilde{y}'\tilde{y}^2}{x_{1.0} + x_{0.1}\tilde{y}} + \frac{a_3\tilde{y}'\tilde{y}^2 + a_4\tilde{y}'\tilde{y}^2 + a_5\tilde{y}' + a_6\tilde{y}'''}{x_{1.0} + x_{0.1}\tilde{y}'(y_{1.0}x_{0.1} - x_{1.0}y_{0.1})} 
\]

(3.2)

allows us to assume that the point-invariant class of the equations (1.1) should have the following form
\[
y''' = \frac{B(x, y)y''' + P(x, y)y'y'^2 + Q(x, y)y''y' + R(x, y)y'' + S(x, y)y'''}{Y(x, y) - X(x, y)y'} + \\
\frac{L(x, y)y'^4 + K(x, y)y'^3 + M(x, y)y'^2 + N(x, y)y' + T(x, y)}{Y(x, y) - X(x, y)y'} 
\]

(3.3)

Let’s transform an arbitrary equation (3.3) using formulae (2.2)–(2.15). In general the transformed equation do not belong to the class (3.3). The coefficient $B/(Y - Xy')$
of $y''^2$ in the right hand side of (3.3) as a result of coordinate transformations (1.2) generates a new term with $y'y''^2$.

$$\ddot{y}'' = \frac{(\ddot{B}_1 + \ddot{B}_2 \dot{y})y''^2}{(\dot{Y} - \dot{X} \dot{y})'(x_{1.0} + x_{0.1} \dot{y})} + \ldots \quad (3.4)$$

Let us introduce a new function $f(z)$ of the variable $z = y'$ associated with the equation (3.4):

$$f(z) = \frac{\ddot{B}_1 + \ddot{B}_2 z}{x_{1.0} + x_{0.1} z}, \quad (3.5)$$

where

$$\ddot{B}_1 = 3x_{0.1}(Yx_{0.1} - Xy_{0.1})$$
$$\ddot{B}_2 = 3x_{0.1}(Yx_{1.0} - Xy_{1.0}) + B(x_{1.0}y_{0.1} - x_{0.1}y_{1.0}) \quad (3.6)$$

The rational function $f(z)$ in (3.5) has a first order pole at the point $z_0 = -x_{1.0}/x_{0.1}$. Let's calculate the residue of this function at the point $z_0$ and denote it by $\Omega$:

$$\Omega = \text{Res}_{z=z_0} f(z). \quad (3.7)$$

It is easy to get the explicit formula for $\Omega$ using the expression (3.6):

$$\Omega = (B + 3X) \det S. \quad (3.8)$$

The condition $\Omega = 0$ is the necessary condition for the class of equations (3.2) to be point-invariant. Let's remember that the point transformations (1.2) are non-degenerate, hence the condition (3.8) is equivalent to additional relation between functions $B$ and $X$:

$$B = -3X. \quad (3.9)$$

The direct calculations show that the condition (3.9) is also sufficient condition for the class of equations (3.2) to be closed under the transformations (1.2). This completes the proof of Theorem 1 below.

**Theorem 1.** The class of equations

$$y''' = \frac{-3X(x,y)y''^2 + P(x,y)y''y'^2 + Q(x,y)y'' y' + R(x,y)y''' + S(x,y)y''^3 + L(x,y)y'^4 + K(x,y)y'^6 + M(x,y)y'^2 + N(x,y)y' + T(x,y)}{Y(x,y) - X(x,y)y'}$$

is invariant under point transformations (1.2).

**References**

1. R. Liouville, *Sur les invariants de certaines equations differentielles et sur leurs applications*, J. de L’Ecole Polytechnique 59 (1889), 7–76.
2. S. Lie, *Vortlesungen über continuirliche Gruppen*, Teubner Verlag, Leipzig, 1893.
3. S. Lie, *Theorie der Transformationsgruppen III*, Teubner Verlag, Leipzig, 1930.
4. A. Tresse, *Sur les invariants differentiales des groupes continus de transformations*, Acta Math. 18 (1894), 1–88.
5. A. Tresse, *Determination des Invariants ponctuels de l’Equation différentielle ordinaire de second ordre: \(y'' = u(x,y,y')\)*, Preisschriften der fürtlichen Jablonowski’schen Gesellschaft XXXII, S. Hirzel, Leipzig, 1896.

6. E. Cartan, *Sur les varietes a connection projective*, Bulletin de Soc. Math. de France 52 (1924), 205–241.

7. E. Cartan, *Sur les varietes a connexion affine et la theorie de la relatifive generalisee*, Ann. de l’Ecole Normale, 40 (1923), 325–412; 41 (1924), 1–25; 42 (1925), 17–88.

8. E. Cartan, *Spaces of affine, projective and conformal connection*, Platon, Volograd, 1997.

9. G. Thomsen, *Über die topologischen Invarianten der Differentialgleichung \(y'' = f(x,y)\), Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität, 7 (1930), 301–328.

10. C. Grissom, G. Thompson and G. Wilkens, *Linearisation of Second Order Ordinary Differential Equations via Cartan’s Equivalence Method*, Diff. Equations 77 (1989), 1-15.

11. N. Kh. Ibragimov, *Elementary Lie group analysis and ordinary differential equations*, Wiley series in mathematical methods in practice; Vol.4, John Wiley & Sons Ltd, England, 1999.

12. V.S. Dryuma, *Geometrical theory of nonlinear dynamical system*, Preprint of Math. Inst. of Moldova, Kishinev, 1986.

13. V.S. Dryuma, *On the theory of submanifolds of projective spaces given by the differential equations*, Sbornik statey, Math. Inst. of Moldova, Kishinev, 1989, pp. 75–87.

14. V.S. Dryuma, *Geometrical properties of multidimensional nonlinear differential equations and phase space of dynamical systems with Finslerian metric*, Theor. and Math. Phys., 99 (1994), no. 2, 241-249.

15. M.V. Babich and L.A. Bordag, *Projective Differential Geometrical Structure of the Painleve Equations*, J. of Diff. Equations 157 (1999), no. 2, September, 452–485.

16. Yu.R. Romanovsky, *Calculation of local symmetries of second order ordinary differential equations by means of Cartan’s method of equivalence*, Manuscript, 1–28.

17. S. Bacso and M. Matsumoto, *On Finsler spaces of Douglas type. A generalization of the notion of Berwald space*, Publ. Math. Debrecen 51 (1997), no. 3-4, 385–406.

18. Dmitrieva V. V., Sharipov R. A., *On the point transformations for the second order differential equations*, Electronic archive at LANL (1997), solv-int #9703003, pp. 1–14.

19. Sharipov R. A., *On the point transformations for the equation \(y'' = P + 3Qy' + 3Ry'^2 + Sy'^3\)*, Electronic archive at LANL (1997), solv-int #9706003, pp. 1–35.

20. Sharipov R. A., *Effective procedure of point classification for the equations \(y'' = P + 3Qy' + 3Ry'^2 + Sy'^3\)*, Electronic archive at LANL (1998), Math.DG #9802027, pp. 1–35.

21. Mikhailov O. N., Sharipov R. A., *On the point expansion for certain class of differential equations of second order*, Electronic archive at LANL (1997), solv-int #9712001, pp. 1–8.

22. Dmitrieva V. V., Sharipov R. A., *On the point transformations for the second order differential equations*, Electronic archive at LANL (1997), solv-int #9703003, pp. 1–14.

23. Dmitrieva V. V., *Determination des Invariants ponctuels de l’Equation différentielle ordinaire de second ordre: \(y'' = u(x,y,y')\)*, Preisschriften der fürtlichen Jablonowski’schen Gesellschaft XXXII, S. Hirzel, Leipzig, 1896.

24. V. V. Dmitrieva, *Determination of Invariants of The Second Order Ordinary Differential Equation: \(y'' = u(x, y, y')\)*, Preisschriften der fürtlichen Jablonowski’schen Gesellschaft XXXII, S. Hirzel, Leipzig, 1896.

25. Y. V. Dmitrieva, *Linearisation of Second Order Ordinary Differential Equations via Cartan’s Equivalence Method*, Diff. Equations 77 (1989), 1-15.

26. N. Kh. Ibragimov, *Elementary Lie group analysis and ordinary differential equations*, Wiley series in mathematical methods in practice; Vol.4, John Wiley & Sons Ltd, England, 1999.

27. V.S. Dryuma, *Geometrical theory of nonlinear dynamical system*, Preprint of Math. Inst. of Moldova, Kishinev, 1986.

28. V.S. Dryuma, *On the theory of submanifolds of projective spaces given by the differential equations*, Sbornik statey, Math. Inst. of Moldova, Kishinev, 1989, pp. 75–87.

29. V.S. Dryuma, *Geometrical properties of multidimensional nonlinear differential equations and phase space of dynamical systems with Finslerian metric*, Theor. and Math. Phys., 99 (1994), no. 2, 241-249.

30. L.A. Bordag and V.S. Dryuma, *Investigation of dynamical systems using tools of the theory of invariants and projective geometry*, NTZ-Preprint 24/95, Leipzig, 1995; J. of Applied Mathematics (ZAMP) in appear.; Electronic archive at LANL (1997), solv-int #9705006, pp. 1–18.

31. L.A. Bordag, *Symmetries of the Painleve equations and the connection with projective differential geometry*, VIIth EWM Meeting Proceedings, Hindawi Publishing Corporation, Trieste, Italy, 1997, pp. 145–159.

32. M.V. Babich and L.A. Bordag, *Projective Differential Geometrical Structure of the Painleve Equations*, J. of Diff. Equations 157 (1999), no. 2, September, 452–485.

33. Yu.R. Romanovsky, *Calculation of local symmetries of second order ordinary differential equations by means of Cartan’s method of equivalence*, Manuscript, 1–28.

34. S. Bacso and M. Matsumoto, *On Finsler spaces of Douglas type. A generalization of the notion of Berwald space*, Publ. Math. Debrecen 51 (1997), no. 3-4, 385–406.

35. V.N. Gusiatnikova and V.A. Yumaguzhin, *Point transformations and linearizability of second-order ordinary differential equations*, Mathem. Zametki, 49 (1991), no. 1, 146-148 (in Russian); English transl. in Soviet Math. Zametki.

36. Dmitrieva V. V., Sharipov R. A., *On the point transformations for the second order differential equations*, Electronic archive at LANL (1997), solv-int #9703003, pp. 1–14.

37. Sharipov R. A., *On the point transformations for the equation \(y'' = P + 3Qy' + 3Ry'^2 + Sy'^3\)*, Electronic archive at LANL (1997), solv-int #9706003, pp. 1–35.

38. Sharipov R. A., *Effective procedure of point classification for the equations \(y'' = P + 3Qy' + 3Ry'^2 + Sy'^3\)*, Electronic archive at LANL (1998), Math.DG #9802027, pp. 1–35.

39. Mikhailov O. N., Sharipov R. A., *On the point expansion for certain class of differential equations of second order*, Electronic archive at LANL (1997), solv-int #9712001, pp. 1–8.

**Department of Mathematics, Bashkir State University, Frunze str. 32, 450074 Ufa, Russia.**

**E-mail address:** DmitrievaVV@ic.bashedu.ru