Non-Ladder Extended Renormalization Group Analysis of the Dynamical Chiral Symmetry Breaking

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The order parameters of dynamical chiral symmetry breaking in QCD, the dynamical mass of quarks and the chiral condensates, are evaluated by numerically solving the non-perturbative renormalization group (NPRG) equations. We employ an approximation scheme beyond “the ladder”, that is, beyond the (improved) ladder Schwinger-Dyson equations. The chiral condensates are enhanced in comparison with the ladder approximation, which is phenomenologically favorable. The gauge dependence of the order parameters is reduced significantly in this scheme.

§1. Introduction

Dynamical chiral symmetry breaking plays an important role in the field of particle physics. In particular, hadron dynamics provide the most important example of dynamical chiral symmetry breaking phenomena. At present, the low energy phenomenology of pions and kaons is well understood by regarding them as the Nambu-Goldstone bosons of the dynamical chiral symmetry breaking \( SU(3)_L \times SU(3)_R \to SU(3)_V \) caused by the non-vanishing quark pair condensate \( \langle \bar{q}q \rangle \neq 0 \) in QCD.

There have been many studies concerning dynamical chiral symmetry breaking. In particular the ladder Schwinger-Dyson (SD) equations in the Landau gauge have been extensively used.\(^1\)\(^-\)\(^10\) In strongly coupled QED, the chiral critical behavior has been explored, and the anomalous dimension of operator \( \bar{\psi}\psi \), as well as the critical exponents of the phase transition, have been obtained.\(^2\)\(^-\)\(^6\) The SD approach has been applied also to various models beyond the Standard Model.\(^4\)\(^-\)\(^6\) The chiral order parameters for QCD, the dynamical mass of quarks, the chiral condensate, and the decay constant of \( \pi \) mesons \( f_\pi \), have also been calculated by using the improved ladder approximation in the SD equations.\(^7\)\(^,\)\(^8\) They incorporate the running effects of the asymptotically free gauge coupling constant in the ladder self-consistent equations (thus referred to as the “improved ladder approximation”) and provide good results, even quantitatively. However, it should be noted that the improved ladder approximation has no theoretical justification, and it has been just an artificial “model.” Therefore there has been no way to further improve the “improved ladder” until the non-perturbative renormalization group (NPRG)\(^11\)\(^-\)\(^14\) analyses shed new light on it.\(^15\),\(^16\) Furthermore, it has been shown that the ladder SD equations are plagued by a strong gauge parameter dependence.\(^9\) Also, it is difficult to proceed beyond the ladder approximation so as to overcome this unpleasant problem.\(^10\)

The NPRG is a powerful analytical technique for the study of non-perturbative
phenomena in various fields of physics.\textsuperscript{17} It has many favorable properties: The NPRG equations can be formulated exactly and there are systematic methods of approximation. The exact NPRGs are given in the form of nonlinear functional differential equations for the Wilsonian effective action. Therefore, it is necessary to approximate them for practical calculations. We expand the Wilsonian effective action in powers of derivatives and truncate the series at a certain order. We often use the lowest order of this approximation, the so-called local potential approximation (LPA).\textsuperscript{18} Although the LPA appears to be a very crude approximation, it gives very good results indeed, for example, for the second order phase transition in $O(N)$ scalar field theories, compared with other non-perturbative methods, the $1/N$-expansion and the $\epsilon$-expansion.\textsuperscript{20}

In previous papers,\textsuperscript{15, 16, 21} we studied chiral critical behavior mainly using Abelian gauge theory with strongly coupled massless fermions. We proposed a set of gauge independent NPRG equations describing the chiral phase transition. Gauge independence is achieved by including the non-ladder type diagrams to the $\beta$ function of the four-fermi operators, as well as the ladder type diagrams. The gauge independent values of the critical exponent and of the anomalous dimension of the mass operator $\bar{\psi}\psi$ were obtained. When we restrict the $\beta$ functions to the “ladder parts,” the NPRG equations exactly reproduce the ladder SD results. It was analytically shown in the ladder approximation that not only the critical behavior but also the order parameters are identical to those resulting from the SD equations. It is noteworthy that this exact equivalence between the ladder part NPRG and the ladder SD holds also for the improved ladder SD with the running gauge coupling constant. Thus our NPRG method has given for the first time a definite physical meaning to the improved ladder SD, and now we know how to improve the improved ladder, which is the challenge we address in this article.

In Ref. 21) we proposed a new scheme for the NPRG equations, incorporating the composite operators so as to definitely evaluate the order parameters of the dynamical chiral symmetry breaking. This formulation was also applied to the non-Abelian gauge theories by taking account of the asymptotically free running of the gauge coupling constant. However, the practical calculations of the order parameters were demonstrated only in the ladder approximation. Therefore the results obtained there should depend on the gauge parameter just as the ladder SD equations. Note here that our calculational method of incorporating the composite operators should be distinguished from other schemes of constructing the effective meson theory at some scale in QCD, although they should be compared with each other. Also non-perturbative renormalization group analyses of QCD with effective meson components are done in the context of the hadronic matter.\textsuperscript{22}

In this paper we evaluate the chiral order parameters of QCD in a new approximation scheme which is a minimal extension including the non-ladder type diagrams so as to compensate for the serious gauge dependence of the ladder parts. The results obtained in this scheme, however, are not completely free of the gauge parameter dependence. There are still a few sources in our approximation causing the gauge dependence. We examine the amount of the gauge dependence appearing in the order parameters. It is found that the gauge dependence in the observable quantity,
the quark condensate $\langle \bar{\psi}\psi \rangle$, is significantly reduced compared with the results in the ladder approximation.

The outline of this paper is as follows. Section 2 contains a brief review of the formalism upon which our work is based, the method of NPRG and its approximation. We present our model in §3, where we show how to treat the infrared divergences occurring in the dynamical chiral symmetry breaking. In §4 we consider the origin of the gauge dependence in the ladder approximation and construct a new approximation scheme beyond the ladder approximation, which should reduce the gauge dependence. In §5 we describe the practical calculation procedures and present numerical results. We discuss the remaining gauge dependence of the order parameters in our approximation in §6. A summary is given in §7.

§2. The NPRG equation and its approximation

There are several formulations of NPRG.11)-14) In this paper we consider the Wegner-Houghton (W-H) equation.12) First let us briefly review its formulation. The starting point is the Euclidean path integral with the controlled momentum cutoff $\Lambda(t) = e^{-t}A_0$:

$$Z = \int Z_{A(t)} D\phi \exp[-S_{\text{eff}}(\phi; t)], \quad (2.1)$$

where $S_{\text{eff}}$ is called the Wilsonian effective action. The NPRG equation describes how the Wilsonian effective action $S_{\text{eff}}$ should change as the higher momentum degrees of freedom are integrated out. It is obtained by reducing $A(t)$ infinitesimally while fixing the partition function $Z$. Simultaneously, we rescale the momentum variables and the fields by the cutoff $\Lambda(t)$, since the changes of the dimensionless quantities are of our physical interest. We obtain the differential equation

$$\frac{dS_{\text{eff}}(\phi; t)}{dt} = DS_{\text{eff}} - \int_{|p| \leq 1} \frac{d^D p}{(2\pi)^D} \frac{\partial}{\partial p^\mu} \left( D_\phi - p^\mu \frac{\partial}{\partial p^\mu} \right) \frac{\delta}{\delta \phi_i(p)} S_{\text{eff}}$$

$$- \frac{1}{2dt} \int' \frac{d^D p}{(2\pi)^D} \left\{ \frac{\delta S_{\text{eff}}}{\delta \phi_i(p)} \left( \frac{\delta^2 S_{\text{eff}}}{\delta \phi_i(p) \delta \phi_j(-p)} \right)^{-1} \frac{\delta S_{\text{eff}}}{\delta \phi_j(-p)} - \text{str ln} \left( \frac{\delta^2 S_{\text{eff}}}{\delta \phi_i(p) \delta \phi_j(-p)} \right) \right\}, \quad (2.2)$$

where $D$ is the space-time dimension, and $D_\phi$ is the dimension of $\phi$ including its anomalous dimension. The second primed integral denotes integration over the infinitesimal shell modes of momenta $e^{-dt} \leq p \leq 1$, and the prime in the derivative indicates that it does not act on the $\delta$ function in $\frac{\delta}{\delta \phi_i(p)} S_{\text{eff}}$. The subscript $i$ represents every Lorentz and internal symmetry index. This equation is known as a sharp cutoff version of the NPRG and is called the Wegner-Houghton (W-H) equation.12) It is inevitable to approximate this for practical calculations. To do so, we expand the Wilsonian effective action in powers of derivatives. We employ the local potential approximation, which is regarded as the lowest order of this derivative expansion.
Any derivative couplings are dropped, except for the fixed kinetic terms,

\[ S_{\text{eff}} = \int \frac{d^D p}{(2\pi)^D} \left\{ \frac{1}{2} \phi_i K_{ij}(p) \phi_j + V_{\text{eff}}(\phi) \right\}, \quad (2.3) \]

where \( K_{ij}(p) \) is a matrix of the canonical kinetic terms, and \( V_{\text{eff}} \) is called the Wilsonian effective potential. As a simple example, we consider a theory of one scalar, \( \varphi \), and one Dirac fermion, \( \psi \), and its conjugate, \( \bar{\psi} \). The matrix \( K(p) \) in \((\varphi, \psi, \bar{\psi})\)-space is written as

\[ K(p) = \begin{pmatrix} p^2 & 0 & 0 \\ 0 & 0 & -i p^T \\ 0 & -i p & 0 \end{pmatrix}. \quad (2.4) \]

In this approximation, the W-H equation is reduced to a nonlinear partial differential equation for the Wilsonian effective potential \( V_{\text{eff}}(\phi, t) \),

\[ \frac{\partial V_{\text{eff}}(\phi; t)}{\partial t} = D V_{\text{eff}} - D'_\phi \frac{\partial V_{\text{eff}}}{\partial \phi_i} + \frac{1}{2} \int d^D p (2\pi)^D \text{str} \ln \left( K_{ij} + \frac{\partial^2 V_{\text{eff}}}{\partial \phi_i \partial \phi_j} \right), \quad (2.5) \]

where \( D'_\phi \) denotes the canonical dimension of the field \( \phi \). The anomalous dimension of the field \( \phi \) vanishes, because the kinetic term is not renormalized in the LPA.

We should note that any newly generated operators with derivatives are ignored in this approximation. We take account of the generated operators that do not depend on the external momenta. Precisely speaking, we evaluate amplitudes of such local non-derivative operators by setting all external momenta to 0.

This partial differential equation may be solved numerically. However, actually it is not easy to obtain its solution with enough precision. In addition, it would not be practical for more complicated models. Therefore here we expand the effective potential into polynomials in the field \( \phi \). With this approximation, we solve a system of coupled ordinary differential equations for various coupling constants.

Let us characterize these approximations from the viewpoint of the NPRG formulation. The basic logic of the approximation in the NPRG formalism is to restrict the theory space to a subspace of the original full theory space. Namely, the approximation in the NPRG formalism is to analyze the NPRG equation projected onto a subspace that is actually finite dimensional, so as to get results numerically within finite computation time. In order to improve the approximation, we enlarge the subspace, step by step, expecting the results will converge to certain values. The above two approximations, the local potential approximation and the polynomial expansion, are just two consecutive steps of the subspace projection. It should be noted that the ladder approximation itself cannot be regarded as a projection to any subspace, and therefore it has some pathological features indeed.

Finally, we should mention an intrinsic problem of the NPRG formulation. The momentum space cutoff is indispensable for almost any formulation of the NPRG. It turns out that the NPRG does not manifestly respect the gauge invariance. There have been several approaches for restoration of the gauge invariance.\(^{23} - ^{25}\) Our purpose here is not to overcome this gauge invariance problem but to improve the
improved ladder approximation. Therefore we use a simple approximation scheme for the gauge interactions. We evaluate the $\beta$ function of the gauge coupling constant using the one-loop perturbative $\beta$ function. Of course, the running of the gauge coupling constant is automatically derived from the original NPRG equation. When we adopt a sub-theory space with lower dimensional operators, then the NPRG equation effectively reproduces the perturbative renormalization group equation.\(^{13,15}\) Therefore, as the first stage, we work with this level of the small sub-theory space, and we adopt the running gauge coupling constant controlled by the perturbative $\beta$ function. We should note that any newly generated operators including the gauge fields are irrelevant in this scheme. Therefore the Faddeev-Popov ghosts are also irrelevant.

§3. NPRG equations for the dynamical chiral symmetry breaking

Now we apply Eq. (2.5) to QCD with three massless quarks. We take the local potential effective action,

$$S_{\text{eff}}[\psi, \bar{\psi}, A_\mu; t] = \int d^4x \left[ V_{\text{eff}}(\psi, \bar{\psi}; t) + \bar{\psi}(\partial - gA)\psi + \frac{1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2\alpha}(\partial_\mu A^a_\mu)^2 \right],$$

(3.1)

where $\alpha$ is the gauge parameter, and $\psi$ denotes massless triplet quarks. Here, as mentioned at the end of the previous section, the gauge coupling constant $g$ follows the one-loop RG equation. We start with the general form of the effective potential consistent with the chiral symmetry $SU(3)_L \times SU(3)_R$ and the parity. Let us first consider four-fermi operators. We regard the operators corresponding to the Fierz transformation as the identical operators. Furthermore, we do not consider the flavor and/or color changing multi-fermi operators. Then there are two independent four-fermi operators in our theory space:

$$O_1 = (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 = -\frac{1}{2}\left\{ (\bar{\psi}\gamma_\mu\psi)^2 - (\bar{\psi}\gamma_5\gamma_\mu\psi)^2 \right\},$$

$$O_2 = (\bar{\psi}\gamma_\mu\psi)^2 + (\bar{\psi}\gamma_5\gamma_\mu\psi)^2.$$  

(3.2)

To approximate the NPRG equation, we specify a subspace in the full theory space. Here we take a subspace spanned by polynomials in the scalar operator $O_1$ only up to some maximum power $N$,

$$V_{\text{eff}}(\psi, \bar{\psi}; t) = \sum_{n=1}^N \frac{G_{2n}(t)}{n} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right]^n.$$  

(3.3)

The NPRG equation for the scalar four-fermi operator is obtained from the diagrams in Fig. 1:

$$\frac{dG_2}{dt} = -2G_2 + \frac{1}{2\pi^2}(G_2)^2 + \frac{3g^2}{4\pi^2}G_2 + \frac{9g^4}{32\pi^2}.$$  

(3.4)
Fig. 1. The NPRG $\beta$ function of the four-fermi operators. The diagrams in the solid box correspond to the ladder part, and the diagrams in the dashed box are crucial for the gauge independence. Other possible diagrams are ignored in our approximation.

Fig. 2. The flow diagram in the $G_2$-$g$ plane in QCD. The phase structure does not appear, and all flows diverge at a certain finite scale.

Generally, the NPRG equation is a set of coupled equations of various operators. In this case, however, the NPRG $\beta$ function of the four-fermi operator $O_1$ consists of $O_1$ itself and the gauge coupling constant $g$. Therefore the NPRG equation for $O_1$ operators can be solved without recourse to other higher multi-fermi operators. When ignoring the running of the gauge coupling constant, there is a fixed point for the flow of the $O_1$ operator given by the zero of the right-hand side of Eq. (3.4), which is nothing but the critical point of the dynamical chiral symmetry breaking, and we have two-phase structure of the standard ferromagnet type phase transition.\(^{(15),(16)}\)

The strong coupling phase is the symmetry breaking phase, where the four-fermi coupling constant diverges at a finite scale.

We make the gauge coupling constant run according to the asymptotically free $\beta$ function. The flow diagram in the $G_2$-$g$ plane in QCD is depicted in Fig. 2. There appears no phase boundary. All the flows diverge at a certain finite scale. Also, we see the renormalized trajectory in the flow diagram, which assures that the bare four-fermi interactions are irrelevant to the infrared physics. This behavior suggests that the entire region is to be in the broken phase of the chiral symmetry. This is due to the infrared slavery behavior of the QCD gauge coupling constant and it is believed to hold naturally.

Renormalization group flows have also been analyzed using the SD equation method,\(^{(26)}\) where fixing the quark mass obtained, the relation between the four-fermi coupling constant and the gauge coupling constant is calculated assuming the cutoff dependence of the gauge coupling constant. This procedure is justified within the SD formalism and it actually gives something resembling the results in Fig. 2. However, there are critical differences between these two calculations. The main difference comes from the fact that in the NPRG formalism the bare four-fermi interactions turn out to be irrelevant, while in the SD formalism there is
no mechanism to automatically generate effective four-fermi interactions by gluon exchanges.

Correspondence between the divergence of the four-fermi operator and the dynamical chiral symmetry breaking is not trivial. All the coupling constants in the polynomial expansion keep growing in the infrared region and diverge at some finite scale. This corresponds to the fact that at this scale the Wilsonian effective potential exhibits non-analytical behavior at its origin, which is actually observed as a jump of the first derivative by direct analysis of the Wilsonian effective potential using the partial differential equation. This singularity at the origin clearly reveals the spontaneous symmetry breakdown, since it guarantees a non-vanishing magnetization at zero external field limit.

This singular behavior invalidates the renormalization group calculation of the evolution of the effective potential with polynomial expansion at the origin. Introducing a composite operator $\phi$ corresponding to the order parameter $\langle \bar{\psi}\psi \rangle$ enables us to carry out the calculation of the Wilsonian effective potential even in the infrared region.\(^\text{21}\) Our theory space is extended to include the composite operator $\phi$. First, we introduce the composite operator $\phi$ as an auxiliary field in the original path integral without changing the dynamics. The partition function $Z$ in this extended theory space is written as

$$Z = \int D\psi D\bar{\psi} D\tilde{A} \exp \left[ - \int d^4x \mathcal{L}_{\text{org}} [\psi, \bar{\psi}, A] \right] = \int D\psi D\bar{\psi} D\tilde{A} D\phi \exp \left[ - \int d^4x \left\{ \mathcal{L}_{\text{org}} [\psi, \bar{\psi}, A] + \frac{1}{2} (\phi - y \bar{\psi}\psi)^2 \right\} \right].$$ (3.5)

Here we have abbreviated the counter part $(\bar{\psi} i \gamma_5 \psi)^2$ for simplicity, and it should be pointed out that the original chiral symmetry is maintained. Then the bare Lagrangian is modified as follows:

$$\mathcal{L} = \mathcal{L}_{\text{org}} [\psi, \bar{\psi}, A; t] + \frac{1}{2} (\phi - y \bar{\psi}\psi)^2$$

$$= \frac{1}{4} F^{\alpha}_{\mu\nu} F^{\alpha}_{\mu\nu} + \frac{1}{2\alpha} (\partial_{\alpha} A^{\alpha}_{\mu})^2 + \bar{\psi} (\tilde{\phi} - g A - y \phi) \psi + \frac{1}{2} \phi^2 + \frac{y^2}{2} (\bar{\psi}\psi)^2. \quad (3.6)$$

Even in the case of omitting the chiral symmetric representation, this Lagrangian has discrete chiral symmetry:

$$\bar{\psi} \rightarrow \gamma_5 \bar{\psi}, \quad \psi \rightarrow -\bar{\psi} \gamma_5, \quad \phi \rightarrow -\phi. \quad (3.7)$$

Then we may analyze the NPRG equations for the following Wilsonian effective potential:

$$\mathcal{L}_{\text{eff}} = \frac{1}{4} F^{\alpha}_{\mu\nu} F^{\alpha}_{\mu\nu} + \bar{\psi} (\tilde{\phi} - g A) \psi + V_{\text{eff}} (\phi, \sigma; t), \quad (3.8)$$

$$V_{\text{eff}} (\phi, \sigma; t) = \tilde{G}_0 (\phi; t) + \sum_{n=1}^{N} \frac{1}{n} \tilde{G}_n (\phi; t) \sigma^n$$

$$= \tilde{G}_0 (\phi; t) + \tilde{G}_1 (\phi; t) \sigma + \frac{1}{2} \tilde{G}_2 (\phi; t) \sigma^2 + \cdots, \quad (3.9)$$
where the notation $\sigma = \bar{\psi}\psi$ has been introduced. Note again that we are actually working with the chiral symmetric effective potential, and we take a particular direction of the scalar operator $\sigma$ condensation to get the effective potential represented by Eq. (3.9). Since we do not consider the propagation of composite operator modes, Eq. (3.9) is sufficient to define the NPRG evolution of the chiral symmetric system.

In this formalism, it was shown that the chiral condensate $\langle \bar{\psi}\psi \rangle$ is proportional to the minimum position of the scalar potential $\tilde{G}_0(\phi)$, denoted by $\langle \phi \rangle$, which is the order parameter of the dynamical chiral symmetry breaking:

$$\langle \bar{\psi}\psi \rangle = \frac{1}{y} \langle \phi \rangle. \quad (3.10)$$

Then the dynamical mass of quarks $\Sigma(0)$ is given by

$$\Sigma(0) = \frac{\partial V_{\text{eff}}}{\partial \sigma} \bigg|_{\sigma = 0} = \tilde{G}_1(\phi = \langle \phi \rangle). \quad (3.11)$$

In the usual argument of introducing the auxiliary field, the four-fermi interaction is removed from the action by tuning the Yukawa coupling constant $y$. Then this action may be regarded as the gauged Yukawa system with the compositeness condition.\(^6\) There are several studies of dynamical chiral symmetry breaking that use this realization. However, we should note that our purpose for introducing the auxiliary field $\phi$ is not to eliminate the four-fermi interaction. Also, our results obtained in Eqs. (3.10) and (3.11) turn out to be independent of the Yukawa coupling constant $y$,\(^{21}\) since it should not change the dynamics at all.

§4. Compensation of the gauge dependence

As noted above, the ladder part NPRG exactly reproduces the results obtained by the ladder SD equations. That is, the results from the ladder part NPRG depend strongly on the gauge parameter $\alpha$, as do the ladder SD approaches. In order to improve the gauge dependence, we must proceed beyond the ladder approximation. Thus it is required to develop a non-ladder extended approximation in the course of the systematic approximation of NPRG.

To begin with, we discuss the origin of the gauge dependence in the ladder approximation. Let us consider a set of diagrams summed up by the ladder approximation. For this purpose we define the “massive” quark propagator

$$\frac{1}{i\not{\!p} - m(\phi, \sigma)}, \quad (4.1)$$

where

$$m(\phi, \sigma) \equiv \frac{\partial V}{\partial \sigma} = \frac{\partial}{\partial \sigma} \sum_{n=1}^{N} \frac{\tilde{G}_n(\phi)}{n} \sigma^n = \sum_{n=1}^{N} \tilde{G}_n(\phi) \sigma^{n-1}$$

$$= \tilde{G}_1(\phi) + \tilde{G}_2(\phi)\sigma + \tilde{G}_3(\phi)\sigma^2 + \tilde{G}_4(\phi)\sigma^3 + \cdots. \quad (4.2)$$

Using the Feynman diagrams, the “massive” quark propagator can be represented
Fig. 3. The “massive” quark propagator. The heavy full line on the left-hand side is a “massive” fermion propagator. The light full lines are massless fermion operators, and the dashed lines are the auxiliary fields $\phi$.

Fig. 4. The ladder part $\beta$ function. These diagrams do not contain any crossed ladder type diagrams. The wavy lines are gluons, the heavy full lines are the “massive” quarks, and the light full lines are external quark operators.

Fig. 5. The set of diagrams satisfying the Ward-Takahashi identity which involves two fermions and two gauge bosons. The wavy lines are gauge bosons and the full lines are fermions.

as shown in Fig. 3. The ladder part $\beta$ functions are defined by summing up a set of diagrams which do not contain any crossed ladder type diagrams (Fig. 4).

We now consider the Abelian Ward-Takahashi (WT) identities assuring the gauge independence. To satisfy the WT identities we must sum over the diagrams for the S-matrix at any given order. When the gauge boson is inserted at a certain point along the fermion line, we must sum over all possible insertion points. Let us consider the simple case of the WT identity involving two fermions and two gauge bosons at $g^2$ order. The sum of the diagrams in Fig. 5 is gauge independent at on-shell. The right diagram of Fig. 5, called the crossed diagram, is not involved in the ladder approximation. This is one of the reasons for the strong gauge dependence in the ladder approximation. Actually, as seen in Fig. 1, adding the crossed box diagram to the $\beta$ function has wiped out the gauge dependence of the critical behavior. 15)
Fig. 6. The corrected vertex. The wavy lines are gluons, and the heavy full lines are “massive” quarks, the light full lines are external quark operators. The curved arrows denote the direction of the shell-mode momentum $p$.

\[
\frac{dV}{dt} = \frac{d}{dt} \left\{ \tilde{G}_0 + \tilde{G}_1(\bar{\psi}\psi) + \frac{1}{2} \tilde{G}_2(\bar{\psi}\psi)^2 + \frac{1}{3} \tilde{G}_3(\bar{\psi}\psi)^3 + \frac{1}{4} \tilde{G}_4(\bar{\psi}\psi)^4 + \cdots \right\}
\]

Fig. 7. The $\beta$ function in our new approximation. The wavy lines are gluons, the heavy full lines are “massive” quarks, and the light full lines are external quark operators.

Now we consider a generalization of the crossed box diagrams in Fig. 1 for the $\beta$ functions of the higher multi-fermi operators. First we define a corrected vertex,

\[
\frac{-2g^2}{p^2 + m^2} \left[ ip_\alpha \epsilon^{\mu\nu\alpha\beta} \gamma_5 \gamma_\beta + mg^{\mu\nu} \right],
\]

which is composed of two diagrams, the ladder and crossed, using the “massive” quark propagator (Fig. 6). Therefore the corrected vertex itself comprises an infinite number of diagrams. Then we replace double vertices in Fig. 4 with the corrected vertex and sum up the diagrams just as the ladder part (Fig. 7).

Just as in Fig. 4, every diagram in Fig. 7 contributes to an infinite number of coupling constants in the polynomial expansion. Accordingly, a certain diagram with $n$ external quarks contributes to the $\beta$ functions of coupling constants with more than $n$ external quarks. For instance, the $\beta$ function of the mass $\tilde{G}_1$ (the generalized Yukawa coupling) is given by the first and the second diagrams in Fig. 7. Also, the $\beta$ function of the four-fermi operator comes from the first, the second and the third diagrams, etc. Note that the third diagram has a symmetry factor of 1/2.
Now we discuss the method of "projection" of the newly generated operators onto our target subspace. Here we consider the effective potential $V_{\text{eff}}$ composed of the polynomials only in the scalar operator $O_1$. Therefore we pick up only the parts proportional to $(1)^n$ in the spinor space from the generated total operators. For example, let us consider $g^8$ terms in the eight-fermi $\beta$ function (Fig. 8). The form of the generated operators are in general

$$\sum_{a,b,c,d} C_{abcd}(\bar{\psi}_1 \Gamma^a \psi_1) \times (\bar{\psi}_2 \Gamma^b \psi_2) \times (\bar{\psi}_3 \Gamma^c \psi_3) \times (\bar{\psi}_4 \Gamma^d \psi_4), \quad (4.4)$$

where the $\Gamma^i$ are the 16 independent matrices for spinor indices and $C_{abcd}$ are amplitudes. We should consider all the parts proportional to $(\bar{\psi} \psi)^4$ from the generated operators. Instead of picking up all of them, however, we take only one simple combination of the $\Gamma^i$,

$$(\bar{\psi}_1 \mathbf{1} \psi_1) \times (\bar{\psi}_2 \mathbf{1} \psi_2) \times (\bar{\psi}_3 \mathbf{1} \psi_3) \times (\bar{\psi}_4 \mathbf{1} \psi_4). \quad (4.5)$$

This part is represented by the left figure in Fig. 9. Of course there are other combinations of the $\Gamma^i$ contributing to the $(\bar{\psi} \psi)^4$ term, for example, the contraction shown by the right figure in Fig. 9, which is omitted. We adopt this approximation, since the left figure in Fig. 9 exactly coincides with the ladder part approximation, when the corrected vertices are replaced with the ladder type. We now attempt to obtain the minimal extension of the ladder approximation. We should mention that this restriction of the manner of picking up $\sigma$ fields does not correspond to a general procedure of projection in the systematic approximation method of NPRG, and it might cause a problem.

Now let us consider the non-Abelian effects. The quarks belong to the 3 dimensional representation. For example, the $g^4$ term in the four-fermi $\beta$ function from

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*Fig. 8.* The newly generated $g^8$ operator. The numbers denote the suffixes of the quark fields (see Eq. (4.4)).

*Fig. 9.* The manner of picking up $\sigma$. The left figure corresponds to the ladder part approximation, when the corrected vertices are replaced with the ladder type. We ignore the manner of picking up $\sigma$ shown in the right figure.
\[ \sigma \Rightarrow \text{Tr} (T^a T^b T^a T^b) = 3C_2(3)^2. \]

Fig. 10. The gauge group factor of the box diagram in the four-fermi \( \beta \) function.

\[ \sigma \Rightarrow \text{Tr} (T^a T^b T^c T^c) = 3C_2(3)^2 + i f^{abc} \text{Tr} (T^a T^b T^c). \]

Fig. 11. The gauge group factor of the crossed box diagram in the four-fermi \( \beta \) function.

the ladder-type diagram (Fig. 10) is identical to the Abelian estimate, except that we must add the color Casimir eigenvalue. As for the crossed ladder type diagrams (Fig. 11), there appears the commutator term \( i f^{abc} \) in addition to the Casimir term. Here we simply ignore this commutator term and take account of the \( C_2(3)^2 \) term only. Of course this is a significant truncation, which might break the gauge independence, and it will be discussed later. Due to this additional approximation, the non-Abelian nature is absorbed into the Casimir factor, which defines the effective gauge coupling constant \( C_2(3)g \). With this effective gauge coupling constant, the \( \beta \) function of the effective potential in QCD is evaluated just as in the Abelian case.

Now we are able to write down general formulae of \( g^{2n} \) terms in the NPRG equations as follows:

\[ \sigma^n = -\frac{1}{8\pi^2} \frac{g^{2n}}{2^{n-2}} \frac{1}{n!} \frac{1}{(1 + m^2)^n} \frac{1}{1 + \delta_{n,1} + \delta_{n,2}} \]
\[ \times \left[ m^n (3 + \alpha^n) + \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \frac{n!}{(2k)! (n-2k)!} (2 + 4k) m^{n-2k} \right] \sigma^n. \]  

(4.6)

For example, the \( \beta \) function for the four-fermi operator reads

\[ \frac{d\tilde{G}_2}{dt} = -2\tilde{G}_2 + \frac{1}{2\pi^2} \frac{\tilde{G}_2^2}{1 + \tilde{G}_2^2} \left[ 1 - \frac{2\tilde{G}_1^2}{1 + \tilde{G}_1^2} \right]. \]
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\[ + \frac{g^2}{4\pi^2} \frac{3 + \alpha}{1 + G_1^2} \left[ \tilde{G}_2 - \frac{2\tilde{G}_1^2\tilde{G}_2}{1 + G_1^2} \right] + \frac{g^4}{16\pi^2} \frac{1}{\left[1 + \tilde{G}_1^2\right]^2} \left[ 6 - (3 + \alpha^2)\tilde{G}_1^2 \right]. \]

(4.7)

Let us compare the above equation with the ladder approximated one,

\[ \frac{d\tilde{G}_2}{dt} = -2\tilde{G}_2 + \frac{1}{2\pi^2} \frac{\tilde{G}_2^2}{1 + G_1^2} \left[ 1 - \frac{2\tilde{G}_1^2}{1 + G_1^2} \right] \]

\[ + \frac{g^2}{4\pi^2} \frac{3 + \alpha}{1 + G_1^2} \left[ \tilde{G}_2 - \frac{2\tilde{G}_1^2\tilde{G}_2}{1 + G_1^2} \right] + \frac{g^4}{32\pi^2} \frac{(3 + \alpha)^2}{\left[1 + \tilde{G}_1^2\right]^2} \left[ 1 - \tilde{G}_1^2 \right]. \]  

(4.8)

The gauge-dependent terms in Eq. (4.7) should be almost compensated by the gauge-dependent anomalous dimension of the quark fields when we go beyond the LPA. This point will be discussed separated article.

§5. Numerical calculation and results

Now we describe how to get the chiral order parameters in QCD with our approximation scheme. We work with the Wilsonian effective potential defined in Eq. (3.9) with some finite highest powers \( N \), and we numerically integrate its NPRG equation. The NPRG equation is defined by the \( \beta \) function given in Eq. (4.6); that is, we take only the quantum loops of the quarks and gluons and not of the scalar composites. The initial effective potential is taken from Eq. (3.6). During evolution, the scalar field \( \phi \) is fixed to be a certain value, just as an external source field. The gauge coupling constant is set to follow the one-loop perturbative \( \beta \) function with three flavor quarks. We take the QCD scale parameter \( \Lambda_{\text{QCD}} \) to be 490 MeV, and we adopt the same infrared cutoff scheme of the gauge coupling constant divergence as in Ref. 8, since our results should first be compared with the previous ladder SD results. 8)

Integrating the NPRG equation, the effective potential finally stops to move, except for the canonical scaling behavior, where the cutoff scale has been lowered well below the quark mass scale. Then we get the scalar potential \( \tilde{G}_0(\phi) \) at the fixed \( \phi \) value. To solve the NPRG equation with various values of \( \phi \), we obtain the scalar potential function \( \tilde{G}_0(\phi) \) and its minimum point \( \langle \phi \rangle \). Then we estimate the chiral condensates and the quark mass using Eqs. (3.10) and (3.11). The chiral condensates obtained above should be regarded as the bare operator condensation at the initial highest cutoff scale. It should be renormalized through the standard procedure to obtain the renormalized condensates at the 1 GeV scale.

We give the results (\( \alpha = 0 \) case) in Fig. 12. First, we check the \( N \) dependence of the results. Though there still remain some small fluctuations, we may claim that we have obtained the results for our total subspace of \( N = \infty \). We have checked also that the dependence on the initial cutoff \( \Lambda_0 \) is negligible; that is, our results are guaranteed to be on the renormalized trajectory, and we can regard them as those of an infinite initial cutoff limit.
Fig. 12. The chiral condensates and the dynamical mass of quarks with their truncation dependence in the Landau gauge. The dashed line is $\Lambda_{\text{QCD}}$. Non-ladder results are enhanced compared to the ladder results.

Fig. 13. The gauge parameter dependence of the chiral condensates and the dynamical mass of quarks are plotted for $\alpha = 0$ (Landau gauge), $\alpha = 1$ (Feynman gauge) and $\alpha = 2$.

Now we compare our non-ladder extended results with the ladder results. The ladder results exactly coincide with those of the ladder SD equation, which assures the total consistency of our calculational machinery. The chiral condensates and the quark mass are both enhanced by including the non-ladder contributions. Though we do not argue here in detail about the phenomenological implications of this enhancement, this enhancement is actually favorable for phenomenology, since our setting of the QCD scale parameter is much higher than the current estimate, even considering that it is a one-loop $\beta$ estimate.

The gauge parameter dependence of the results is depicted in Fig. 13. Compared with the ladder results, the improvement of the gauge dependence of $\langle \bar{\psi} \psi \rangle$ is clearly
seen in the left figure of Fig. 13. On the other hand, the right figure of Fig. 13 shows that the gauge dependence of $\Sigma(0)$ still remains significant, even in the non-ladder approximation. We understand the different situations for these quantities as follows. The chiral condensate $\langle \bar{\psi}\psi \rangle$ is a measurable physical quantity, and it should therefore not depend on the gauge. On the other hand, the dynamical “mass” of the quark $\Sigma(0)$ is an off-shell quantity, and it is not directly related to a measurable quantity. Therefore, it may depend on the gauge. It should be noted also that the quark mass $\Sigma(0)$ strongly depends on the infrared cutoff scheme of the gauge coupling constant divergence, while the chiral condensates do not.

### §6. Issues of the gauge dependence

In this section we discuss the origin of the gauge dependence in the NPRG method. First of all, the Wilsonian effective action $S_{\text{eff}}$ itself depends on the gauge, because it is not directly related to any measurable quantities. Therefore the NPRG equations (or the $\beta$ function) describing the evolution of the Wilsonian effective action also depend on the gauge. Furthermore, even in the infrared limit, the effective action is not totally gauge independent, except for the on-shell quantities. For example, the effective potential is not gauge independent, except for the position of the minimum. Thus, in general, the $\beta$ function, which depends on the gauge parameter, finally gives gauge independent results only for physical quantities in the infrared limit. Actually, in our approximation scheme of evaluating the effective potential, there is no way of eliminating all the gauge parameter dependence in the $\beta$ function.

We discuss here the gauge dependence due to the approximation we adopted, that is, the local potential approximation. It ignores any corrections to the derivative couplings, including the kinetic terms, and therefore no anomalous dimension is taken into account. This seems to be the largest source of the gauge dependence. We will report elsewhere the results obtained by taking account of the quark anomalous dimension, where we will see the reduced gauge dependence. Before obtaining these new results, we may evaluate the physical quantities as follows. In the one-loop approximation, the quark anomalous dimension is proportional to $\alpha$, and therefore we have vanishing anomalous dimension in the Landau gauge $\alpha = 0$. Therefore we may claim that the Landau gauge results in our scheme are most significant, and they would be very close to the results with the quark anomalous dimension. Then our main results should read

$$
\frac{\langle \bar{\psi}\psi \rangle^{1/3}}{\Lambda_{\text{QCD}}} \bigg|_{\text{non-ladder}} = 0.512 \pm 0.014,
$$

which is compared with the previous ladder results,

$$
\frac{\langle \bar{\psi}\psi \rangle^{1/3}}{\Lambda_{\text{QCD}}} \bigg|_{\text{ladder}} = 0.439.
$$

There are other subtleties of the gauge dependence due to the LPA. In the above calculation we have made the further approximation of ignoring some parts
contributing to the $\beta$ functions. In Eq. (4.6) we omitted the non-Abelian commutator parts of the gauge effective vertex. This omission itself does not bring about the gauge dependence. Rather it “hides” the gauge dependence of the LPA. Consider the four-fermi amplitude, for example. Such commutator parts should be summed up with diagrams in Fig. 14 (second derivative part of the gluon field, $\partial_\mu F^{\mu\nu} \bar{\psi} \gamma^\nu \psi$, as an operator form) to generate the gauge independent (on-shell) four-fermi amplitudes. This situation is quite similar to that of the penguin diagram to give the local four-fermi effective operators. Of course we also have to add all related diagrams in Fig. 15 and ghost diagrams to get totally gauge independent results with the properly renormalized gauge coupling constant. Therefore we must take account of the derivative couplings $\partial_\mu F^{\mu\nu} \bar{\psi} \gamma^\nu \psi$ to compensate for the gauge dependence appearing in the four-fermi box diagrams.

All these extensions require for higher order derivative couplings in our sub-theory space. Then we have to proceed to use smooth cutoff scheme NPRG equations, since the sharp cutoff NPRG equations suffer singularities when applied to the derivative couplings.

§7. Summary and discussion

In this article we attempted to go beyond the ladder calculation of the dynamical chiral symmetry breaking in QCD by using a non-ladder extension in the non-perturbative renormalization group method. The ladder approximation of the NPRG local potential $\beta$ function has been integrated to give exactly the same results as the (improved) ladder Schwinger-Dyson equation for the chiral condensates.
and the dynamical mass of the quark $\Sigma(0)$. Extension beyond the ladder has strong motivation of reducing the inevitable gauge dependence of the ladder approximation.

We added non-ladder diagrams to the NPRG ladder $\beta$ function, attempting to reduce the gauge dependence of the physical results. We developed a set of $\beta$ functions using the effective gluon vertex defined by the sum of the ladder and the crossed couplings. We numerically solved this new $\beta$ function to obtain the chiral condensates and quark mass function at zero momentum. They are enhanced compared with the previous ladder results. This situation is favorable phenomenologically.

Also, we evaluated the gauge parameter dependence of our results and found it is significantly reduced compared to the ladder case.

We stress here again that our results are the first results in the long history of analyzing the dynamical chiral symmetry breaking in gauge theories that go beyond the (improved) ladder in a systematic approximation method. This is realized by a new viewpoint of the NPRG method for dynamical chiral symmetry breaking.

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