A BAYESIAN SEQUENTIAL TEST FOR THE DRIFT OF A FRACTIONAL BROWNIAN MOTION

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Abstract

We consider a fractional Brownian motion with linear drift such that its unknown drift coefficient has a prior normal distribution and construct a sequential test for the hypothesis that the drift is positive versus the alternative that it is negative. We show that the problem of constructing the test reduces to an optimal stopping problem for a standard Brownian motion obtained by a transformation of the fractional Brownian motion. The solution is described as the first exit time from some set, and it is shown that its boundaries satisfy a certain integral equation, which is solved numerically.

Keywords: Sequential test; Chernoff’s test; fractional Brownian motion; optimal stopping

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1. Introduction

Suppose one observes a fractional Brownian motion process with linear drift and unknown drift coefficient. We are interested in sequentially testing the hypotheses that the drift coefficient is positive or negative. By a sequential test we mean a procedure which continues to observe the process until a certain time (which generally depends on a path of the process, so it is a stopping time), and then decides which of the two hypotheses should be accepted. We consider a Bayesian setting where the drift coefficient has a prior normal distribution, and we use an optimality criterion of a test which consists of a linear penalty for the duration of observation and a penalty for a wrong decision proportional to the true value of the drift coefficient. The goal of this paper is to describe the structure of the exact optimal test in this problem, i.e. to specify a stopping time and a rule to choose between the two hypotheses.

The main novelty of our work compared to the large body of literature related to sequential tests (for an overview of the field, see e.g. [14, 28]) is that we work with fractional Brownian motion. To the best of our knowledge, this is the first non-asymptotic solution of a continuous-time sequential testing problem for this process. Recall that fractional Brownian motion is a Gaussian process which generalizes the standard Brownian motion and allows for dependent increments; see the definition in Section 2. It was first introduced by Kolmogorov [13] and gained much attention after the work of Mandelbrot and van Ness [16]. Recently, this process has been used in various models in applied areas, including, for example, modeling of traffic in computer networks and modeling of stock market prices and their volatility; a review can be found in the preface to the monograph [17].

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It is well known that a fractional Brownian motion is neither a Markov process nor a semimartingale (except when it is the standard Brownian motion), which may present difficulties in using it in applications. In relation to our problem, recall that a general method to construct exact sequential tests, especially in Bayesian problems, consists in reduction to optimal stopping problems for processes of sufficient statistics; see e.g. Chapter VI in [24]. In the majority of problems considered in the literature, sufficient statistics are Markov processes, and so the well-developed general theory of Markov optimal stopping problems can be applied. Fortunately, in the problem we consider it turns out to be possible to change the original problem for a fractional Brownian motion so that it reduces to a tractable problem for a standard Brownian motion (see [19] for a general result about this transformation). This is achieved by integrating an appropriate kernel with respect to the observable process and using the known fact that a fractional Brownian motion can be expressed as an integral with respect to a standard Brownian motion, and vice versa.

In the literature, the result which is most closely related to ours is the sequential test proposed by Chernoff [5], which has exactly the same setting and uses the same optimality criterion, but considers a standard Brownian motion. For a prior normal distribution of the drift coefficient, Breakwell and Chernoff [2, 6] found asymptotically optimal sequential tests when the variance of the drift tends to zero or infinity. In the paper [29], we extended their result and constructed an exact optimal test. An important step was a transformation of the problem that reduced the optimal stopping problem for the sufficient statistic process, as studied by Breakwell and Chernoff, to an optimal stopping problem for a standard Brownian motion with nonlinear observation cost. A similar transformation is used in the present paper as well; see Section 4.1.

Let us mention two other recent results in the sequential analysis of fractional Brownian motion, related to estimation of its drift coefficient. Cetin, Novikov, and Shiryaev [4] considered a sequential estimation problem assuming a normal prior distribution of the drift with a quadratic or a δ-function penalty for a wrong estimate and a linear penalty for observation time. They proved that in their setting the optimal stopping time is non-random. Gapeev and Stoev [10] studied sequential testing and changepoint detection problems for Gaussian processes, including fractional Brownian motion. They showed how those problems can be reduced to optimal stopping problems and found asymptotics of optimal stopping boundaries. There are many more results related to fixed-sample (i.e. non-sequential) statistical analysis of fractional Brownian motion. See, for example, Part II of the recent monograph [27], which discusses statistical methods for this process in detail.

Our paper is organized as follows. Section 2 formulates the problem. Section 3 states the main theorem, which describes the structure of the optimal sequential test. Its proof is provided in Section 4. The appendix contains some technical results used in the paper.

2. Decision rules and their optimality

Suppose one observes the stochastic process

$$X_t = \theta t + B^H_t,$$

where $B^H_t$, $t \geq 0$, is a fractional Brownian motion with known Hurst parameter $H \in (0, 1)$ and unknown drift coefficient $\theta$. Recall that a fractional Brownian motion is a continuous zero-mean Gaussian process with the covariance function

$$\text{cov}(B^H_t, B^H_s) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}), \quad t, s \geq 0.$$
In the particular case $H = 1/2$ this process is a standard Brownian motion and has independent increments; its increments are positively correlated in the case $H > 1/2$ and negatively correlated in the case $H < 1/2$.

We will consider a Bayesian setting and assume that $\theta$ is a random variable defined on the same probability space as $B^H_t$, independent of it, and having a normal distribution with known mean $\mu \in \mathbb{R}$ and known variance $\sigma^2 > 0$.

It is assumed that neither the value of $\theta$ nor the values of $B^H_t$ can be observed directly, but the observer wants to determine whether the value of $\theta$ is positive or negative based on the information conveyed by the combined process $X_t$. We will look for a sequential test for the hypothesis $\theta > 0$ versus the alternative $\theta \leq 0$. By a sequential test we mean a pair $\delta = (\tau, d)$ which consists of a stopping time $\tau$ of the completed right-continuous filtration $\mathcal{F}^X_t$ generated by $X$, and an $\mathcal{F}^X_\tau$-measurable function $d$ assuming values $\pm 1$. The stopping time is the moment of time when observation is terminated and a decision about the hypotheses is made; the value of $d$ shows which of them is accepted.

We will use the criterion of optimality of a decision rule consisting in minimizing the linear penalty for observation time and the penalty for a wrong decision proportional to the absolute value of $\theta$. Namely, with each decision rule $\delta$ we associate the risk functional

$$ R(\delta) = \mathbb{E}(\tau + |\theta|I(d \neq \text{sgn}(\theta))), $$

where $\text{sgn}(\theta) = -1$ if $\theta \leq 0$ and $\text{sgn}(\theta) = 1$ if $\theta > 0$. The problem that we consider consists in finding a decision rule with the smallest risk $R(\delta)$; i.e. it can be formulated as follows:

$$ \text{find } \delta^* \text{ such that } R(\delta^*) = \inf_{\delta} R(\delta), $$

where the infimum is taken over all decision rules. Such a decision rule $\delta^*$ will be called optimal.

Note that one can consider a more general setting in which the penalty for observation time is equal to $c\tau$ with some constant $c > 0$ (or the penalty for a wrong decision is $c|\theta|I(d \neq \text{sgn}(\theta))$), but this case can be reduced to the one we consider by a change of the parameters $\mu, \sigma$ (see [5]), and so we will focus only on $c = 1$.

This problem was proposed by Chernoff in [5] for a standard Brownian motion, and we refer the reader to that paper and the subsequent papers [2, 6, 7] for a rationale for this setting. Those papers contain results about the asymptotics of the optimal test and its other properties, including a comparison with Wald’s sequential probability ratio test. In our paper [29], an exact (non-asymptotic) optimal test was obtained, which can be constructed by a relatively simple numerical procedure.

### 3. The main result

The main theorem formulated in this section provides a representation of the optimal decision rule $\delta^* = (\tau^*, d^*)$ in the form where $\tau^*$ is the first time when a certain process $Z_t$, obtained by a transformation of the observable process $X_t$, exits from a region bounded by two symmetric curves.

In order to formulate the theorem, we need to introduce auxiliary objects. In what follows, we will assume that the constants $H, \mu, \sigma$ are fixed and omit them in notation. Let

$$ K(t, s) = C(t - s)^{\frac{1}{2} - H}_2F_1 \left( \frac{1}{2} - H, \frac{1}{2} - H, \frac{3}{2} - H, \frac{s - t}{t} \right). $$
with the constant

\[ C = \left( \frac{\Gamma(2 - 2H)}{2H\Gamma\left(\frac{3}{2} - H\right)} \right)^{\frac{1}{2}}, \]

where \( \Gamma(x) \) is the gamma function and \( _2F_1(a, b, c, x) \) is the Gauss hypergeometric function (the definitions of these special functions can be found in, e.g., Chapters 6.1 and 15.1 of [1]).

Define the process

\[ Y_t = \int_0^t K(t, s)dX_s = \theta \int_0^t K(t, s)ds + \int_0^t K(t, s)dB^H_s. \] (1)

The second integral in the right-hand side is defined as the \( L^2 \)-limit of appropriate approximations of the integrand by simple functions, and it is well known that it is a standard Brownian motion if considered as a random process in time \( t \) (for details, see [11] or the earlier results [18, 20]). Denoting it by \( B_t \) and computing the first integral in the right-hand side of (1) (see the appendix, Part (a)), we obtain the representation

\[ Y_t = \theta L \int_0^t s^{\frac{1}{2} - H}ds + B_t \] (2)

with

\[ L = \left( 2H\left(\frac{3}{2} - H\right)B\left(\frac{1}{2} + H, 2 - 2H\right) \right)^{-\frac{1}{2}}, \]

where \( B(x, y) \) is the beta function (see, e.g., Chapter 6.2 in [1]). Note that the transformation (1) of \( X_t \) into \( Y_t \) can be inverted, \( X_t = \int_0^t K^*(t, s)dY_s \) with an appropriate kernel \( K^*(t, s) \) (such that \( B^H = \int_0^t K^*(t, s)dB_s \); see [11]), so the completed right-continuous natural filtrations \( \mathcal{F}_t^X \) and \( \mathcal{F}_t^Y \) generated by the processes \( X_t \) and \( Y_t \) coincide.

Next, we define the process \( Z_t \) that will be used as the observable statistic, based on which the optimal decision rule will be constructed:

\[ Z_t = \frac{\mu}{\sigma^2} + L \int_0^t s^{\frac{1}{2} - H}dY_s = \frac{\mu}{\sigma^2} + \int_0^t \tilde{K}(t, s)dX_s \]

with

\[ \tilde{K}(t, s) = Ls^{\frac{1}{2} - H}K(t, s). \]

From the representation (2), one can see that the integral in the above formula is well-defined. Moreover, the filtrations \( \mathcal{F}_t^Z \) and \( \mathcal{F}_t^Y \) (and, hence, \( \mathcal{F}_t^X \)) coincide. So, if \( \tau \) is a stopping time of the filtration \( \mathcal{F}_t^Z \), then it is also a stopping time of the filtration \( \mathcal{F}_t^X \), and vice versa.

Finally, we introduce several non-random constants and functions which will be used to characterize the boundary of the region such that, when \( Z_t \) leaves it, one should stop the observation (below, the symbol \( \vee \) denotes the maximum of two numbers):

\[ r_0 = 0 \vee \frac{1 - 2H}{4(1 - H)}, \] (3)

\[ \psi(t) = 1 - \left( \frac{\sigma^2L^2t^{2-2H}}{2 - 2H} + 1 \right)^{-1}, \quad t \geq 0, \] (4)

\[ t_0 = \psi^{-1}(r_0) = \left( 0 \vee \frac{(2 - 2H)(1 - 2H)}{\sigma^2L^2(3 - 2H)} \right)^{\frac{1}{2H}}, \] (5)

\[ b(t) = \frac{1}{\sigma^2} + \frac{L^2}{2 - 2H}t^{2-2H}, \quad t \geq 0, \] (6)
and for a standard normal random variable $\zeta$ we introduce the functions

$$G(r, x) = \mathbb{E}[\zeta \sqrt{1 - r + x}] - x,$$

$$F(r, x, u, y) = f'(u)\mathbb{P}(\zeta \sqrt{1 - r + x} \leq y),$$

(7)

where $f(u) = \frac{2}{\sigma} \left( \frac{(2 - 2H)u}{\sigma^2 L^2(1-u)} \right)^{-\frac{1}{2}}$, for $r \in [0, 1], x \geq 0, u \in [r, 1], y \geq 0$.

Observe that the function $\psi(t)$ is strictly increasing and maps $\mathbb{R}_+$ on $[0, 1]$. If $H \in (1/2, 1)$, then $r_0 = 0$ and $t_0 = 0$. If $H \in (0, 1/2)$, then $r_0 \in (0, 1/4)$ and $t_0 > 0$.

**Theorem 1.** 1) There exists a function $a^*(t) > 0$ defined on $\mathbb{R}_+$ such that the following decision rule $\delta^* = (\tau^*, d^*)$ is optimal:

$$\tau^* = \inf\{t \geq 0 : |Z_t| \geq a^*(t)\}, \quad d^* = \text{sgn} \, Z_{\tau^*}.$$

2) For $t > t_0$, the function $a^*(t)$ admits the representation

$$a^*(t) = b(t)A(\psi(t)),$$

(8)

where the function $A(r)$, defined on $(r_0, 1]$, is the unique continuous nonnegative solution of the equation

$$G(r, A(r)) = \int_r^1 F(r, A(r), u, A(u))du, \quad r \in (r_0, 1),$$

(9)

with the value $A(1) = 0$.

Thus, the theorem states that the observation should be stopped as soon as the process $Z_t$ crosses the boundaries $\pm a^*(t)$. The function $A(r)$, and hence the boundary $a^*(t)$, can be found by solving Equation (9) numerically. In the explicit form, the functions $G$ and $H$ from (9) can be written as follows (for $r < 1$ and $u > r$):

$$G(r, x) = 2\sqrt{1 - r}\varphi\left(\frac{x}{\sqrt{1 - r}}\right) - 2x\Phi\left(\frac{-x}{\sqrt{1 - r}}\right),$$

$$F(r, x, u, y) = \frac{2(2u(1 - H))^{2H-1}}{\sigma(\sigma L)^{1/2} \left(1 - u\right)^{2-2H}} \left( \Phi\left(\frac{y - x}{\sqrt{u - r}}\right) - \Phi\left(\frac{-y - x}{\sqrt{u - r}}\right) \right),$$

where $\Phi$, $\varphi$ are the standard normal distribution and density functions. Then the solution of Equation (9) can be obtained by backward induction on a discrete set of points from $(r_0, 1]$ starting with $r = 1$ and going towards $r = r_0$, using that the expression under the integral depends only on the values of $A(u)$ for $u \geq r$; the method is described in detail in, for example, [21]. Figure 1 shows the function $A(r)$ and the boundary $a^*(t)$ for different values of $H$.

**Remark 1.** As the reader can observe, the theorem states that the representation (8)–(9) holds for all $t > 0$ in the case $H \geq 1/2$ (when $t_0 = 0$), but only for $t > t_0 > 0$ in the case $H < 1/2$. Let us clarify why the region $t \leq t_0$ is not covered if $H < 1/2$.

The main reason is that the method of proof we use to show that $A(r)$ satisfies the integral equation requires $A(r)$ to be of bounded variation (at least locally). This condition is needed
as a sufficient condition to apply the Itô formula with local time on curves [22], on which the proof is based. In the case $H \geq \frac{1}{2}$ and for $r \geq r_0$ in the case $H < \frac{1}{2}$, by a direct probabilistic argument we can prove that $A(r)$ is monotone and therefore has bounded variation; however, this argument does not work for $r \leq r_0$ in the case $H < \frac{1}{2}$, and, as a formal numerical solution shows, the boundary $A(r)$ indeed seems to be non-monotone in that case. Of course, the assumption of bounded variation can be relaxed while the Itô formula can still be applied (see e.g. [8, 9, 22]); however, verification of weaker sufficient conditions seems not to be easily achievable. Although the general scheme to obtain integral equations of the type (9) and prove uniqueness of their solutions was discovered quite a while ago (the first full result was obtained by Peskir [23] for the optimal stopping problem for American options) and has been used many times in the literature for various optimal stopping problems (a large number of examples can be found in [24]), we are unaware of any nontrivial applications of it in the case when stopping boundaries are not monotone. Nevertheless, in our problem, the numerical procedure of computing the function $A(r)$ works well also in the case $H < 1/2$ and $r < r_0$, and, as Figure 1 shows, the function $A(r)$ indeed ‘looks smooth’ on the whole interval.

Also, it is worth mentioning that it is not clear whether the function $A(r)$ and the boundary $a^*(t)$ have a finite limit as $r \to 0$ and $t \to 0$ when $H > 1/2$. However, since the process $Z_t$ and the boundary $a^*(t)$ are continuous, the value $a^*(0)$ is not essential for specification of the stopping time $\tau^*$. For $H \leq 1/2$, it can be shown that $A(0)$ and $a^*(0)$ are finite; this follows from the inequality (24) in the proof of the theorem below.

4. Proof of the theorem

In order to prove Theorem 1, we will first transform the problem of finding an optimal decision rule by eliminating the function $d$ from it and reducing it to an optimal stopping problem for the process $Z_t$. Then, by changing time and space coordinates, we will reduce it to an optimal stopping problem for a standard Brownian motion, which allows application of well-developed methods.

4.1. Reduction to an optimal stopping problem for a standard Brownian motion

The function $\psi(t)$, defined in (4), strictly monotonically maps $[0, \infty)$ on $[0, 1)$. We will use its inverse

$$\psi^{-1}(r) = \left(\frac{(2 - 2H)r}{\sigma^2L^2(1 - r)}\right)^{\frac{1}{2-2H}}, \quad r \in [0, 1),$$

FIGURE 1: Left: the function $A(r)$ for different values of $H$ and $\sigma = 1$. Right: the stopping boundary $a^*(t)$ for different values of $H$ and $\sigma = 1$. 

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as a time-change from the time \( r \in [0, 1) \) into the time \( t \in [0, \infty) \). The next proposition shows that the problem of finding the optimal decision rule can be reduced to an optimal stopping problem for the time-changed process \( Z \).

For brevity, we will define
\[
\gamma = \frac{1}{2 - 2H}, \quad M = \frac{2}{\sigma} \left( \frac{2 - 2H}{\sigma^2 L^2} \right)^\gamma. \tag{10}
\]

**Proposition 1.** 1) The process
\[
W_r = \frac{Z_{\psi^{-1}(r)}}{\sigma b(\psi^{-1}(r))} - \frac{\mu}{\sigma}, \quad r \in [0, 1),
\]
is a standard Brownian motion in \( r \in [0, 1) \), and the filtrations \( \mathcal{F}_r^W \) and \( \mathcal{F}_{\psi^{-1}(r)}^X \) coincide.

2) If \( \rho^* \) is a stopping time which solves the optimal stopping problem
\[
V = \inf_{\rho < 1} \mathbb{E} \left( M \left( \frac{\rho}{1 - \rho} \right)^\gamma - W_\rho + \frac{\mu}{\sigma} \right), \tag{11}
\]
where the infimum is taken over all stopping times \( \rho < 1 \) of \( \mathcal{F}_r^W \), then the optimal decision rule \( \delta^* = (\tau^*, d^*) \) is given by
\[
\tau^* = \psi^{-1}(\rho^*), \quad d^* = \text{sgn}(Z_{\tau^*}). \tag{12}
\]

**Proof.** 1) From (2), one can find that the conditional distribution of \( \theta \) is normal (see the appendix, Part (b)):
\[
\text{Law}(\theta | \mathcal{F}_t^X) = \text{Law}(\theta | \mathcal{F}_t^Y) = N \left( \frac{Z_t}{b(t)}, \frac{1}{b(t)} \right), \tag{13}
\]
where \( b(t) \) is defined in (6).

Define the process \( \widetilde{B}_t, t \geq 0, \) by
\[
\widetilde{B}_t = Y_t - L \int_0^t \mathbb{E}(\theta | \mathcal{F}_s^Y) \frac{1}{\sigma^2} d^{\frac{1}{\sigma^2} - H} L_s ds = Y_t - L \int_0^t \frac{Z_s}{b(s)} \frac{1}{\sigma^2} d^{\frac{1}{\sigma^2} - H} ds.
\]
The innovation representation (see Chapter 7.4 in [15]) implies that \( \widetilde{B}_t \) is a standard Brownian motion. Then it is not hard to see that the process \( Z_t \) satisfies the stochastic differential equation
\[
dZ_t = \frac{Z_t}{b(t)} db(t) + L \frac{1}{\sigma^2} d^{\frac{1}{\sigma^2} - H} \widetilde{B}_t, \quad Z_0 = \frac{\mu}{\sigma^2}.
\]
Applying the Itô formula, one can see that \( m_t = Z_t / (\sigma b(t)) \) is a continuous martingale satisfying the stochastic differential equation
\[
dm_t = \frac{L \frac{1}{\sigma^2} - H}{\sigma b(t)} d\widetilde{B}_t, \quad m_0 = \frac{\mu}{\sigma}.
\]
Let \( \langle m \rangle_t \) denote the quadratic characteristic of \( m_t \), which turns out to be the non-random function
\[
\langle m \rangle_t = \int_0^t \frac{L^2 s^{1-2H}}{\sigma^2 b^2(s)} ds = \psi(t).
\]
The Dambis–Dubins–Schwarz theorem (see, e.g., Theorems 1.6 and 1.7 in Chapter V of [25]) states that a continuous martingale vanishing at time 0 can be represented as a time-changed standard Brownian motion. Applying it to $m_t$, we obtain that the process

$$W_t = m_{t(r)} - m_0,$$

where $t(r) = \inf\{ t \geq 0 : \langle m \rangle_t \geq r \} = \psi^{-1}(r)$,

is a standard Brownian motion for $r < 1$ (the condition $r < 1$ ensures that $\langle m \rangle_t \geq r$ for some $t \geq 0$). Moreover, we have already established that $\mathcal{F}_t^X = \mathcal{F}_t^Y = \mathcal{F}_t^Z$, and from the construction of $W_t$ it follows that $\mathcal{F}_t^Z = \mathcal{F}_t^{W}$. This proves the first part of the theorem.

2) Recall that $\mathcal{R}(\delta) = \mathbb{E}(\tau + |\theta|I(d \neq \text{sgn}(\delta)))$. From the relation $|\theta|I(d \neq \text{sgn}(\delta)) = \theta^+I(d=1) + \theta^-I(d=1)$, where $\theta^+ = \max(\theta, 0)$, $\theta^- = -\min(\theta, 0)$, and by using that $d$ is $\mathcal{F}_t^X$-measurable, one can see that the optimal decision rule should be looked for among rules $(\tau, d)$ with $d = 1$ if $\mathbb{E}(\theta^-|\mathcal{F}_\tau^X) < \mathbb{E}(\theta^+|\mathcal{F}_\tau^X)$ and $d = -1$ otherwise. Hence, by (13), it will be enough to solve the optimal stopping problem which consists in finding a stopping time $\tau^*$ such that $\mathcal{R}(\tau^*) = \inf_\tau \mathcal{R}(\tau)$, where

$$\mathcal{R}(\tau) = \mathbb{E}(\tau + h(Z_\tau, b(\tau)))$$

with the function $h(z, b) = \min(\mathbb{E}\xi^+, \mathbb{E}\xi^-)$ for a normal random variable $\xi \sim N(\frac{z}{b}, \frac{1}{b^2})$. Then the optimal decision rule will be $\delta^*(\tau^*, d^*)$ with $d^* = 1$ if $\mathbb{E}(\theta^-|\mathcal{F}_\tau^X) < \mathbb{E}(\theta^+|\mathcal{F}_\tau^X)$ and $d^* = -1$ otherwise, which is equivalent, by (13), to $d^* = \text{sgn}(Z_{\tau^*})$.

We will now transform $\mathcal{R}(\tau)$ by applying the Itô formula to $h(Z_\tau, b(\tau))$. In the explicit form, we have

$$h(z, b) = \frac{1}{\sqrt{b}} \varphi \left( \frac{z}{\sqrt{b}} \right) - \frac{|z|}{b} \Phi \left( -\frac{|z|}{\sqrt{b}} \right).$$

In order to avoid problems caused by the fact that $h(z, b)$ is not smooth at $z = 0$, consider the function

$$\tilde{h}(z, b) = h(z, b) + \frac{|z|}{2b}.$$ 

It can be easily verified that $\tilde{h} \in C^{2,1}(\mathbb{R} \setminus \{0\} \times \mathbb{R}_+)$, the derivative $\tilde{h}_z(z, b)$ is continuous at $z = 0$ for any $b > 0$, and

$$\tilde{h}_b(z, b) + \frac{1}{2} \tilde{h}_{zz}(z, b) + \frac{z}{b} \tilde{h}_z(z, b) = 0 \quad \text{for } z \in \mathbb{R} \setminus \{0\}, \ b > 0.$$ 

Using this identity and applying the Itô formula, we obtain

$$d\tilde{h}(Z_t, b(t)) = \tilde{h}_z(Z_t, b(t))L_t^{\frac{1}{2} - H}d\tilde{B}_t.$$ 

Using the fact that

$$\tilde{h}_z(z, b) = \frac{1}{b} \Phi \left( \frac{z}{\sqrt{b}} \right) - \frac{1}{2b},$$

one can see that $\mathbb{E} \int_0^\infty (\tilde{h}_z(Z_t, b(t))L_t^{\frac{1}{2} - H})^2 dt$ is finite, so $\tilde{h}(Z_t, b(t))$ is a square-integrable martingale. Therefore, for any stopping time $\tau$ we have $\mathbb{E}\tilde{h}(Z_\tau, b(\tau)) = \tilde{h}(Z_0, b(0))$, and (14) transforms into

$$\mathcal{R}(\tau) = \mathbb{E} \left( \tau - \frac{|Z_\tau|}{2b(\tau)} \right) + \tilde{h}(Z_0, b(0)).$$
(In the explicit form,
\[ \tilde{h}(Z_0, b(0)) = \sigma \varphi \left( \frac{\mu}{\sigma} \right) + |\mu| \left( \frac{1}{2} - \Phi \left( -\frac{|\mu|}{\sigma} \right) \right), \]
but we will not need this value.) Using the process \( W_\rho \) defined in Proposition 1, for any stopping time \( \tau \) of the filtration \( \mathcal{F}_t^X \) and the corresponding stopping time \( \rho = \psi^{-1}(\tau) < 1 \) of the filtration \( \mathcal{F}_r^W \) we have
\[ \tilde{R}(\tau) = \frac{\sigma}{2} \mathbb{E} \left( M \left( \frac{\rho}{1-\rho} \right)^\gamma - |W_\rho + \frac{\mu}{\sigma}| \right) + \tilde{h}(Z_0, b(0)), \]
where the constants \( M \) and \( \gamma \) are defined in (10). Thus, the optimal stopping problem for \( Y \) in \( t \)-time is equivalent to the optimal stopping problem (11) for \( W \) in \( r \)-time. That is, if \( \rho^* \) is the optimal stopping time in (11), then the optimal decision rule \( \delta^* = (\tau^*, d^*) \) is given by (12). \( \square \)

4.2. Solution of the optimal stopping problem for a standard Brownian motion

Now we will solve the optimal stopping problem (11). It is well known that under general conditions the solution of an optimal stopping problem for a Markov process can be represented as the first time when the process enters some set—a stopping set. To define this set, let us first rewrite the problem (11) in the Markov setting by allowing the process \( W_\rho \) to start from any point \((r, x) \in [0, 1) \times \mathbb{R} \) and introduce the value function
\[ V(r, x) = \inf_{\rho < 1-r} \mathbb{E}(f(r + \rho) - |W_\rho + x| - f(r)), \tag{15} \]
where \( f(r) = M(r/(1-r))^\gamma \) (this is the same function as in (7)), and the infimum is taken over all stopping times \( \rho \) of the standard Brownian motion \( W \) such that \( \rho < 1 - r \) almost surely (a.s.). In particular, for the quantity \( V \) from (11) we have \( V = V(0, \mathcal{F}_t^Z) \). We subtract \( f(r) \) in the definition of \( V(r, x) \) to make the function \( V(r, x) \) bounded. For \( r = 1 \) we define \( V(1, x) = -|x| \).

Let the constant \( r_0 \) be as defined in (3).

**Proposition 2.** 1) There exists a function \( A(r) \) defined on \((r_0, 1] \), which is continuous, non-increasing, and strictly positive for \( r < 1 \) with \( A(1) = 0 \), such that for any \( r > r_0 \) and \( x \in \mathbb{R} \) the infimum in the optimal stopping problem (15) is attained at the stopping time
\[ \rho^*(r, x) = \inf \{ s \geq 0 : |W_s + x| \geq A(r + s) \}. \]

2) The function \( A(r) \) on \((r_0, 1] \) is the unique continuous nonnegative solution of the integral equation (9).

**Proof.** The proof will be conducted in several steps: (i) prove the monotonicity of the function \( V(r, x) + |x| \) and the continuity of \( V(r, x) \); (ii) analyze the stopping set and its boundaries; (iii) formulate a free-boundary problem for \( V(r, x) \); (iv) derive the integral equation for \( A(r) \); (v) prove the uniqueness of its solution.

It will be more convenient to rewrite the value function \( V(r, x) \) in the following form containing the so-called Lagrange integral functional (see [24, Chapter III, Section 7.2]):
\[ V(r, x) = \inf_{\rho < 1-r} \mathbb{E} \left( \int_0^\rho f'(s + r) ds - |W_\rho + x| \right) . \]
(i) Observe that the function \( V(r, x) + |x| \) is non-decreasing in \( r \in [r_0, 1] \) for any fixed \( x \in \mathbb{R} \) and is non-decreasing in \( x \in [0, \infty) \) for any fixed \( r \in [0, 1] \) (and hence is non-increasing in \( x \in (-\infty, 0] \) since \( V(r, x) = V(r, -x) \)). The first property follows from the fact that \( f'(r) \) is increasing when \( r \in [r_0, 1] \) (note that \( r_0 \) is the point of minimum of \( f'(r) \) on \((0, 1))\). The second property follows from the fact that the function \( x \mapsto |x| - |y + x| \) is increasing for any \( y \in \mathbb{R} \) (take \( y = W_\rho \)).

Let us prove that \( V(r, x) \) is continuous on \([0, 1) \times \mathbb{R} \). We will first prove the continuity on \([0, 1) \times \mathbb{R} \). Consider two arbitrary points \((r_1, x_1)\) and \((r_2, x_2)\) such that \( 0 \leq r_1 \leq r_2 < 1 \) and \( x_1, x_2 \in \mathbb{R} \). Observe that for any \( r \in [0, 1), x_1, x_2 \in \mathbb{R} \) we have

\[
|V(r, x_1) - V(r, x_2)| \leq \sup_{\rho < 1-r} \mathbb{E} \left| W_\rho + x_1 \right| - \left| W_\rho + x_2 \right| \leq |x_1 - x_2|.
\]

Therefore,

\[
|V(r_1, x_1) - V(r_2, x_2)| \leq |V(r_1, x_1) - V(r_2, x_1)| + |V(r_2, x_1) - V(r_2, x_2)|
\leq |V(r_1, x_1) - V(r_2, x_1)| + |x_1 - x_2|.
\]

To estimate the first term of the sum in the right-hand side, consider the two cases \( V(r_1, x_1) < V(r_2, x_1) \) and \( V(r_1, x_1) > V(r_2, x_1) \) (the latter case can occur only if \( H < 1/2 \) and \( r_2 < r_1 \)).

Suppose \( V(r_1, x_1) < V(r_2, x_1) \). Let \( \varepsilon = r_2 - r_1 \). Then for any stopping time \( \rho < 1 - r_1 \),

\[
\mathbb{E} \left[ \int_0^\rho f'(r_1 + s)ds - |W_\rho + x_1| \right]
\geq \mathbb{E} \left[ \int_0^{\rho \vee \varepsilon} f'(r_1 + s)ds - |W_{\rho \vee \varepsilon} + x_1| + |W_\varepsilon + x_1| - |W_{\rho \wedge \varepsilon} + x_1| \right] - \int_0^\varepsilon f'(r + s)ds
\geq \mathbb{E}V(r_1 + \varepsilon, W_\varepsilon + x_1) + \mathbb{E}(|W_\varepsilon + x_1| - |W_{\rho \wedge \varepsilon} + x_1|),
\]

where the first inequality follows from straightforward algebraic transformations, and to prove the second one the strong Markov property of a standard Brownian motion was used. We have

\[
|\mathbb{E}(|W_\varepsilon + x_1| - |W_{\rho \wedge \varepsilon} + x_1|)| \leq \mathbb{E}|W_\varepsilon - W_{\rho \wedge \varepsilon}| \leq \mathbb{E}|W_\varepsilon| + \mathbb{E}|W_{\rho \wedge \varepsilon}| \leq 2\sqrt{\varepsilon},
\]

where in the right inequality we used the well-known bound

\[
\mathbb{E}|W_\nu| \leq \sqrt{\mathbb{E}\nu}
\]

applied to the stopping time \( \nu = \rho \wedge \varepsilon \). This bound follows from the fact that \( (\mathbb{E}|W_\nu|)^2 \leq \mathbb{E}W_\nu^2 = \mathbb{E}\nu \) (for any stopping time \( \nu \) with \( \mathbb{E}\nu < \infty \)), where the last equality here is Wald’s identity (see, e.g., Section 3.2 in [24]).

Taking in (17) the infimum over all stopping times \( \rho < 1 - r_1 \), we obtain

\[
V(r_1, x_1) \geq \mathbb{E}V(r_1 + \varepsilon, W_\varepsilon + x_1) - 2\sqrt{\varepsilon}.
\]

According to (16),

\[
|\mathbb{E}V(r_2, W_\varepsilon + x_1) - V(r_2, x_1)| \leq \mathbb{E}|W_\varepsilon| = \sqrt{2\varepsilon}/\pi.
\]
Therefore, from (19) and (20), we obtain
\[ V(r_2, x_1) - V(r_1, x_1) \leq 2\sqrt{\varepsilon + \sqrt{2\varepsilon/\pi}} \tag{21} \]
in the case \(V(r_2, x_1) > V(r_1, x_1)\).

In the opposite case, \(V(r_1, x_1) > V(r_2, x_1)\), we use the inequality
\[
V(r_1, x_1) \leq \inf_{\varepsilon \leq \rho \leq 1-r} \mathbb{E} \left( \int_0^\rho f'(s + r) ds - |W_\rho + x_1| \right)
\leq \mathbb{E}V(r_2, W_\varepsilon + x_1) - \int_0^\rho f'(r_1 + s) ds,
\]
where the second line can be obtained using the strong Markov property. This inequality and (20) imply that
\[
V(r_1, x_1) - V(r_2, x_1) \leq \sqrt{2\varepsilon / \pi} + f(r_1) - f(r_2). \tag{22}
\]

Thus, from (21) and (22), we find that for any \(r_1, r_2 \in [0, 1]\) and \(x_1, x_2 \in \mathbb{R}\) we have
\[
|V(r_1, x_1) - V(r_2, x_2)| \leq 3\sqrt{|r_1 - r_2|} + |f(r_1) - f(r_2)| + |x_1 - x_2|,
\]
so \(V(r, x)\) is continuous on \([0, 1] \times \mathbb{R}\). Also, \(V(r, x)\) is continuous at any point \((r, x)\) with \(r = 1\) (recall that we define \(V(1, x) = -|x|\)), since we can estimate
\[
-|x| \geq V(r, x) \geq -|x| - \sup_{\rho < 1-r} \mathbb{E}|W_\rho| \geq -|x| - \sqrt{1-r},
\]
where the first inequality follows from considering the stopping time \(\rho = 0\) in the definition of \(V(r, x)\); the second one follows from the fact that, obviously, \(\int_0^\rho f'(r + s) ds \geq 0\) and \(|W_\rho + x| \leq |W_\rho| + |x|\); and the third inequality is due to (18).

The proof of the continuity of \(V(r, x)\) on \([0, 1] \times \mathbb{R}\) is finished.

(ii) Define the stopping set \(D = \{(r, x) \in [0, 1] \times \mathbb{R} | V(r, x) = -|x|\}\). The continuity of \(V(r, x)\) implies that \(D\) is closed. Also, it is clear that if \((r, x) \in D\), then \((r, -x) \in D\). From this fact and the monotonicity of \(V(r, x) + |x|\), it follows that \(D\) can be represented in the form
\[
D = \{(r, x) : r \in [0, 1], |x| \geq A(r)\}, \tag{23}
\]
where \(A(r)\) is some function on \([0, 1]\) which is non-increasing on \([r_0, 1]\). At this moment, in the representation (23) we do not exclude the possibility that \(A(r)\) may assume the value \(+\infty\), but now we will show that \(A(r)\) is actually finite-valued for \(r > 0\).

Obviously, \(A(1) = 0\), and one can easily see that \(A(r) > 0\) for any \(r < 1\), since for any \(r < 1\) it is possible to find a sufficiently small non-random \(s\) such that \(\mathbb{E}(f(s + r) - |W_s|) < f(r)\), and hence \(V(r, 0) < 0\).

Let us prove the inequality
\[
A(r) \leq \frac{(1 - r)^\gamma}{2Mr^\gamma - 1}, \quad r > 2r_0. \tag{24}
\]

By using the inequality
\[
f'(s) \geq \frac{M\gamma r^\gamma - 1}{(1 - s)^{\gamma + 1}}
\]
for any \( s \geq r \), one can see that for any \( r \in (0, 1) \) and any stopping time \( \rho < 1 - r \),
\[
f(r + \rho) - f(r) \geq M \rho^{\gamma - 1} \left( \frac{(1 - r)^\gamma}{(1 - \rho - r)^\gamma} - 1 \right).
\]
Denote the expression in the large brackets in the right-hand side by \( \nu(\rho) \). Observe that if \( r > 2r_0 \), then \( \nu(\rho) \geq \rho \) and hence \( \nu(\rho) \) is a stopping time, and \( \mathbb{E}|W_{\nu(\rho)} + x| \geq \mathbb{E}|W_\rho + x| \) provided that \( \mathbb{E}\nu(\rho) < \infty \) (since \( |W_r + x| \) is a submartingale). Then
\[
V(r, x) \geq \inf_{\nu \geq 0} \mathbb{E}(cv - |W_\nu + x|),
\]
where
\[
c = \frac{M \rho^{\gamma - 1}}{(1 - r)^\gamma}
\]
and the infimum is taken over all stopping times of \( W \). For the optimal stopping problem in the right-hand side of the inequality (25), the solution is well known (see e.g. Section 16 in [24]): the optimal stopping time is unbounded, a finer argument is needed.

In the general theory of optimal stopping for Markov processes, it is well known that the first entry into the stopping set is an optimal stopping time under mild conditions. In our probability, if \( H \geq 1/2 \), then \( A(r) < \infty \) for \( r > 0 \). Note that, unfortunately, our method does not allow us to say whether \( A(0) \) is finite for \( H > 1/2 \) (for \( H = 1/2 \), it is finite thanks to the inequality (24)), which becomes \( A(r) \leq (1 - r)/(2M) \), since \( \gamma = 1 \); however, we will not need this information.

In order to show that \( A(r) \) is finite-valued for \( r \in [0, 2r_0] \) when \( H < \frac{1}{2} \), one can use that \( V(r, x) \geq \inf_{\rho \geq 0} \mathbb{E}(f'(r_0)\rho - |W_\rho + x|) \), since \( r_0 \) is the point of minimum of the function \( f'(r) \) when \( H < \frac{1}{2} \). Again, all the points \( (r, x) \) with
\[
|x| \geq \frac{1}{2f'(r_0)}
\]
should be in the stopping set, so \( A(r) \) is bounded by \( 1/(2f'(r_0)) \).

Thus, we have shown that \( A(r) < \infty \) for \( r > 0 \). Next, for any point \( (r, x), r > r_0 \), define the candidate optimal stopping time \( \rho^*(r, x) \)—the first entry into the stopping set:
\[
\rho^* = \rho^*(r, x) = \inf\{s \geq 0 : (r + s, W_s + x) \in D\}.
\]
In the general theory of optimal stopping for Markov processes, it is well known that the first entry into the stopping set is an optimal stopping time under mild conditions. In our problem this fact can be proved similarly to the corresponding fact in [30]. (One subtlety here is that general conditions for the optimality of \( \rho^* \) typically require boundedness of the payoff function; see, for example, Chapter 1 in [24]. Since the payoff function in our problem is unbounded, a finer argument is needed.)

Now we will prove that \( A(r) \) is continuous on \((r_0, 1]\). Since it does not increase on \((r_0, 1]\), and the set \( D \) is closed, it is clear that \( A(r) \) is right-continuous on \((r_0, 1]\). Let us prove that it is left-continuous. Using that \( \mathbb{E}W_{\rho^*} = 0 \) we can write
\[
V(r, x) = \mathbb{E}\left(f(\rho^* + r) + W_{\rho^*} - |W_{\rho^*} + x|\right) - f(r).
\]
Suppose \( A(r - \epsilon) > A(r) \) for some \( r \in (r_0, 1] \). Consider points \((r - \epsilon, x)\) with \( x = (A(r - \epsilon) + A(r))/2 \) and sufficiently small \( \epsilon > 0 \). Let
\[
\Omega_\epsilon = \{\omega : W_s(\omega) + x \in [A(r), A(r - \epsilon)] \text{ for all } s \leq \epsilon\}
denote the random event that the path \((r, W_r)\) exits the rectangle \([r - \varepsilon, r] \times [A(r), A(r) - \varepsilon]\)
through the right boundary. For brevity, define \(f_\varepsilon = f(r) - f(r - \varepsilon) > 0\). Then from (26) we have
\[
V(r - \varepsilon, x) \geq \mathbb{E}
\left[
(f_\varepsilon - x) \mathbb{I}(\Omega_\varepsilon)\right] + \mathbb{E}
\left[
(W_{\rho^*}^x - |W_{\rho^*}^x + x|) \mathbb{I}(\Omega \setminus \Omega_\varepsilon)\right]
\geq f_\varepsilon - x - \sqrt{\mathbb{P}(\Omega \setminus \Omega_\varepsilon)} \left(|f_\varepsilon - x| + \sqrt{\mathbb{E}(W_{\rho^*}^x - |W_{\rho^*}^x + x|)^2}\right),
\]
where in the first inequality we used that \(W_{\rho^*}^x - |W_{\rho^*}^x + x| = -x\) on \(\Omega_\varepsilon\), and in the second one we applied the Cauchy–Schwarz inequality to \(\mathbb{E}(W_{\rho^*}^x - |W_{\rho^*}^x + x|) \mathbb{I}(\Omega \setminus \Omega_\varepsilon)\). We also have
\[
\mathbb{P}(\Omega \setminus \Omega_\varepsilon) \leq \mathbb{P}
\left(
\sup_{s \leq \varepsilon} W_s > \frac{A(r - \varepsilon) - A(r)}{2}\right) + \mathbb{P}
\left(
\inf_{s \leq \varepsilon} W_s < -\frac{A(r - \varepsilon) - A(r)}{2}\right)
\leq 2 \exp\left(-\frac{(A(r - \varepsilon) - A(r))^2}{8\varepsilon}\right),
\]
where the second inequality follows from Doob’s martingale inequality (see, e.g., Proposition II.1.8 in [25]). Since \(f_\varepsilon \geq f'(r - \varepsilon)\varepsilon\) when \(r - \varepsilon \geq r_0\), from (27) and (28) it follows that there exists a sufficiently small \(\varepsilon > 0\) such that \(V(r - \varepsilon, x) > -x\), which contradicts the definition of \(V(r, x)\). This proves the continuity of \(A(r)\) on \((r_0, 1)\). The continuity at \(r = 1\) follows from the inequality (24).

(iii) As follows from the general theory of optimal stopping for Markov processes, inside the continuation set \(C = [0, 1] \times \mathbb{R} \setminus D\), the value function \(V(r, x)\) is \(C^{1,2}\) and satisfies the following partial differential equation (see Section 7 in [24]):
\[
V'(r, x) + \frac{1}{2}V''(r, x) = -f'(r), \quad r \in (0, 1), \; |x| < A(r)
\]  
Together with the condition \(V(r, x) = -|x|\) in the set \(D\), this constitutes a free boundary problem for the value function \(V(r, x)\) with the unknown free boundary \(A(r)\).

The continuity of \(V(r, x)\) implies the so-called condition of instantaneous stopping:
\(V(r, A(r) - \varepsilon) = V(r, A(r) + \varepsilon)\), i.e. \(V(r, x)\) is continuous at the stopping boundary. Let us now prove the smooth-fit condition, which states that the \(x\)-derivative of \(V(r, x)\) is continuous at the stopping boundary:
\[
V'(r, A(r) - \varepsilon) = V'(r, A(r) + \varepsilon) = -1, \quad r \in (0, 1].
\]
The function \(V(r, x)\) is concave in \(x\), since it is the infimum (over \(\rho\)) of concave functions. Therefore, the left- and right-hand derivatives \(V'_{x+}(r, A(r))\) exist. Clearly, \(V'_{x+}(r, A(r)) = -1\), since \(V(r, x) = -|x|\) for \(x \geq A(r)\). Moreover, for any sufficiently small \(\varepsilon > 0\) we have
\[
\frac{V(r, A(r) - \varepsilon) - V(r, A(r))}{-\varepsilon} \geq -1,
\]
since \(V(r, A(r) - \varepsilon) \leq -(A(r) - \varepsilon)\) and \(V(r, A(r)) = -A(r)\). Therefore, \(V'_{x-}(r, A(r)) \geq -1\). Let us prove the converse inequality.

Fix \(r \in (r_0, 1)\). Set \(x = A(r)\) and let \(\varepsilon > 0\) be sufficiently small. Then for the optimal time \(\rho^* = \rho^*(r, x - \varepsilon)\) we have
\[
\frac{V(r, x - \varepsilon) - V(r, x)}{-\varepsilon} \leq \frac{\mathbb{E}|W_{\rho^*}^x + x| - \mathbb{E}|W_{\rho^*}^x + x - \varepsilon|}{-\varepsilon},
\]
where we used that \( V(r, x) \leq \mathbb{E}[f(\rho^* + r) - |W_{\rho^* + x}|] - f(r) \) and \( V(r, x - \varepsilon) = \mathbb{E}[f(\rho^* + r) - |W_{\rho^* + x - \varepsilon}|] - f(r) \). We transform the obtained expression:

\[
\mathbb{E}|W_{\rho^* + x}| - \mathbb{E}|W_{\rho^* + x - \varepsilon}| = \varepsilon \mathbb{P}(W_{\rho^* + x} - \varepsilon = A(r + \rho^*)) \\
+ \mathbb{E}[(|W_{\rho^* + x}| - |W_{\rho^* + x - \varepsilon}|)I(W_{\rho^* + x} - \varepsilon = -A(r + \rho^*))].
\]

The second term can be bounded in absolute by \( \varepsilon \mathbb{P}(W_{\rho^* + x} - \varepsilon = -A(r + \rho^*)) = o(\varepsilon) \).

Then

\[
\lim_{\varepsilon \downarrow 0} \frac{V(r, x - \varepsilon) - V(r, x)}{-\varepsilon} \leq -1,
\]

which proves the inequality \( V'_x(r, A(r)) \leq -1 \). Hence, \( V'_x(r, A(r)) = -1 \).

(iv) So far we have established the following properties: (a) \( V(r, x) \) is continuous on \([0, 1] \times \mathbb{R}\) and is \( C^{1,2} \) in the interior of \( C \) and \( D \); (b) \( A(r) \) is continuous and non-increasing on \((r_0, 1]\); (c) \((V'_r + \frac{1}{2} V''_{\rho^*})(r, x)\) is locally bounded in the interior of \( C \) and \( D_0 \), which follows from (29); (d) the function \( x \mapsto V(r, x) \) is concave and the function \( r \mapsto V'_{\pm}(r, A(r)) (\equiv \mp 1) \) is continuous.

These properties allow us to apply the Itô formula with local time on curves (see [22] and Section 2.6 in [23]) to \( V(r, x) \): for any \( r_0 < r < R < 1 \) and \( x \in \mathbb{R} \) we have

\[
\mathbb{E}V(R, W_{R-r} + x) - V(r, x) \\
= \mathbb{E} \int_0^{R-r} \left(V'_r + \frac{1}{2} V''_{\rho^*}\right)(r + s, W_s + x)I(W_s + x \neq \pm A(r + s)) \, ds \\
+ \mathbb{E} \int_0^{R-r} V'_s(r + s, W_s + x)I(W_s + x \neq \pm A(r + s)) \, dW_s \tag{31} \\
+ \frac{1}{2} \mathbb{E} \int_0^{R-r} \Delta V'_s(r + s, A(r + s))I(W_s + x = A(r + s)) \, dL^A_s \\
+ \frac{1}{2} \mathbb{E} \int_0^{R-r} \Delta V'_s(r + s, -A(r + s))I(W_s + x = -A(r + s)) \, dL^{-A}_s,
\]

where \( L^\pm A \) denotes the local time processes of \( W \) on the curves \( \pm A \) (see [22]), and \( \Delta V'_s(r, x) = V'_s(r, x+) - V'_s(r, x-) \).

The smooth-fit condition (30) implies that the last two terms in (31) are equal to zero. Also, the derivative \( V'_x(r, x) \) is uniformly bounded according to (16), and therefore the expectation of the stochastic integral in (31) is also zero.

From (29) and the fact that \( (V'_r + \frac{1}{2} V''_{\rho^*})(r, x) = 0 \) for \(|x| > A(r)\) we obtain

\[
V(r, x) = \mathbb{E}V(R, W_{R-r} + x) + \int_0^{R-r} f'(r + s)\mathbb{P}(|W_s + x| < A(r + s)) \, ds.
\] (32)

By passing to the limit as \( R \to 1 \), we have \( \mathbb{E}V(R, W_{R-r} + x) \to -\mathbb{E}|W_{1-r} + x| \) from the dominated convergence theorem. Finally, to obtain the integral equation (9), it remains to put \( x = A(r) \) and use the identity \( V(r, A(r)) = -A(r) \).
(v) To prove that $A(r)$ is the unique solution of the integral equation (9), suppose $\tilde{A}(r)$ is another nonnegative continuous solution satisfying (24). Define the function

$$
\tilde{V}(r, x) = -E[W_{1-r} + x] + E \int_0^{1-r} f'(r + s)I(|W_s + x| < \tilde{A}(r + s))ds
$$

where $r \in (r_0, 1), x \in \mathbb{R}$ (cf. (32)). We do not exclude the possibility $\tilde{V}(r, x) = +\infty$; however, obviously, $\tilde{V}(r, x) > -\infty$ for all $r, x$. Using the strong Markov property, one can show that for any $r \in (r_0, 1), x \in \mathbb{R}$, and any stopping time $\rho < 1 - r$ we have

$$
\tilde{V}(r, x) = E\tilde{V}(r + \rho, W_\rho + x) + \mathbb{E} \int_0^\rho f'(r + s)I(|W_s + x| < \tilde{A}(r + s))ds.
$$

(33)

Consider the stopping time $\rho A = \inf\{s \geq 0 : |W_s + x| = \tilde{A}(r + s)\} \wedge (1 - r)$. Since $\tilde{A}$ satisfies the integral equation (9), one can see that $\tilde{V}(r + \rho A, W_{\rho A} + x) = -|W_{\rho A} + x|$. Together with (33), this implies

$$
\tilde{V}(r, x) = -|x|,
$$

|x| \geq \tilde{A}(r),

$$
\tilde{V}(r, x) = E\left[\int_0^{\rho A} f'(r + s)ds - |W_{\rho A} + x|\right],
$$

|x| < \tilde{A}(r).

Consequently $\tilde{V}(r, x) \geq V(r, x)$ for all $r \in (r_0, 1), x \in \mathbb{R}$.

Suppose $A(r) > A(r)$ for some $r \in (r_0, 1)$. Set $x = \tilde{A}(r)$ and consider the corresponding optimal stopping time $\rho^*$. Then from (33), using that $\tilde{V}(r, x) = -|x|$ and $\tilde{V}(r + \rho^*, W_{\rho^*}) \geq V(r + \rho^*, W_{\rho^*}) = -|W_{\rho^*} + x|$, we get

$$
-x \geq E\left[-|W_{\rho^*} + x| + \int_0^{\rho^*} f'(r + s)I(|W_s + x| < \tilde{A}(r + s))ds\right].
$$

However, the expectation of the integral in the above formula is strictly positive since the process $W_s + x$ spends a.s. strictly positive time between the boundaries $\pm \tilde{A}$ (we use the assumption that $A$ is continuous). Moreover, $E|W_{\rho^*} + x| = E(W_{\rho^*} + x) = x$, since $W_s + x$ remains positive until time $\rho^*$. Thus, we get a contradiction, implying that $A(r) \leq A(r)$.

Suppose now that $A(r) < A(r)$ for some $r \in (r_0, 1)$, and set $x = A(r)$. Then $\tilde{V}(r, x) = -x$ and $\tilde{V}(r + \rho^*, W_{\rho^*}) = -|W_{\rho^*} + x|$. From (33) we get

$$
-x = E\left[-|W_{\rho^*} + x| + \int_0^{\rho^*} f'(r + s)I(|W_s + x| < \tilde{A}(r + s))ds\right] < V(r, x),
$$

where we used that the indicator function under the integral is not identically 1 a.s. Again, we get a contradiction to the fact that $V(r, x) \leq -x$, which finishes the proof of Proposition 2. □

The proof of Theorem 3 directly follows from Propositions 1 and 2.

Appendix A

This appendix contains some technical details.

(a) Evaluation of the integral $\int_0^t K(t, s)ds$. By the change $u = \frac{t-s}{s}$ we have

$$
\int_0^t K(t, s)ds = Ct^{\frac{3}{2} - H} \int_0^\infty \frac{u^{\frac{1}{2} - H}}{(u + 1)^{\frac{5}{2} - H}} \text{2F1}\left(\frac{1}{2} - H, \frac{1}{2} - H, \frac{3}{2} - H, -u\right)du.
$$

(34)
Next one can use the formula
\[
\int_0^\infty \frac{u^{c-1}}{(u+z)^r} \, _2F_1(a, b, c, -u) \, dx = \frac{\Gamma(c)\Gamma(a-c+r)\Gamma(b-c+r)}{\Gamma(r)\Gamma(a+b-c+r)} \, _2F_1(a-c+r, b-c+r, a+b-c+r, 1-z)
\]
(see Formula 2.21.1.16 in [3]), which holds for complex \(a, b, c, r, z\) such that \(\text{Re}(a+r), \text{Re}(b+r) > \text{Re}(c) > 0\) and \(|\text{Arg}(z)| < \pi\). Clearly, these conditions are satisfied in (34), and using that \(_2F_1(a, b, c, 0) = 1\) we obtain
\[
\int_0^t K(t, s) ds = \frac{C(\Gamma(\frac{3}{2} - H))^2}{\Gamma(2-2H)} \int_0^t s^{\frac{1}{2}-H} ds.
\]
Substituting the constant \(C\), we obtain \(L\) in front of the integral.

(b) **The conditional distribution** \(\text{Law}(\theta \mid \mathcal{F}_T^Y)\). Although the formula (13) can be obtained using general results from the filtration theory for Gaussian processes, let us show its direct derivation. From the general Bayes theorem (see e.g. Section II.7 in [26]), one can find the conditional density
\[
\mathbb{P}(\theta \in du \mid \mathcal{F}_T^Y) = \frac{d\mathbb{P}_u^\mu}{d\mathbb{P}_0^\mu} \phi_{\mu, \sigma}(u) \times \left( \int \frac{d\mathbb{P}_v^\mu}{d\mathbb{P}_0^\mu} \phi_{\mu, \sigma}(v) dv \right)^{-1}, \quad u \in \mathbb{R},
\]
where \(\frac{d\mathbb{P}_u^\mu}{d\mathbb{P}_0^\mu}\) denotes the density process of the measure generated by \(Y_u^\mu = B_t + uL \int_0^t s^{\frac{1}{2}-H} ds\) with respect to the measure generated by \(Y_0^0 = B_t\), both restricted to the \(\sigma\)-algebra \(\mathcal{F}_T^Y\). By
\[
\phi_{\mu, \sigma}(u) = \frac{1}{\sigma} \phi\left( \frac{u - \mu}{\sigma} \right)
\]
we denote the density function of the normal distribution \(N(\mu, \sigma^2)\). From the Cameron–Martin or Girsanov theorem (see, e.g., Section 3.5 in [12]),
\[
\frac{d\mathbb{P}_u^\mu}{d\mathbb{P}_0^\mu} = \exp\left( uL \int_0^t s^{\frac{1}{2}-H} ds - \frac{u^2L^2}{2} \int_0^t s^{1-2H} ds \right),
\]
and the remaining step to obtain \(\text{Law}(\theta \mid \mathcal{F}_T^Y)\) is a straightforward integration.

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**References**

[1] **Abramowitz, M. and Stegun, I. A.** (1972). *Handbook of Mathematical Functions*, 10th printing. U.S. National Bureau of Standards, Washington, D.C.

[2] **Breakwell, J. and Chernoff, H.** (1964). Sequential tests for the mean of a normal distribution II (large t). *Ann. Math. Statist.* 35, 162–173.
[3] Brychkov, Y. A., Marichev, O. I. and Prudnikov, A. P. (1986). Integrals and Series, Vol. 3: More Special Functions. Nauka, Moscow.

[4] Çetin, U., Novikov, A. and Shiryaev, A. N. (2013). Bayesian sequential estimation of a drift of fractional Brownian motion. *Sequential Anal.* 32, 288–296.

[5] Chernoff, H. (1961). Sequential tests for the mean of a normal distribution. In *Proc. 4th Berkeley Symp. Math. Statist. Prob.* University of California Press, Berkeley, pp. 79–91.

[6] Chernoff, H. (1965). Sequential tests for the mean of a normal distribution III (small t). *Ann. Math. Statist.* 36, 28–54.

[7] Chernoff, H. (1965). Sequential tests for the mean of a normal distribution IV (discrete case). *Ann. Math. Statist.* 36, 55–68.

[8] Eisenbaum, N. (2006). Local time–space stochastic calculus for Lévy processes. *Stoch. Process. Appl.* 116, 757–778.

[9] Föllmer, H., Protter, P. and Shiryaev, A. N. (1995). Quadratic covariation and an extension of Itô’s formula. *Bernoulli* 1, 149–169.

[10] Gapeev, P. V. and Stoëve, Y. I. (2017). On the sequential testing and quickest change-point detection problems for Gaussian processes. *Stochastics* 89, 1143–1165.

[11] Jost, C. (2006). Transformation formulas for fractional Brownian motion. *Stoch. Process. Appl.* 116, 1341–1357.

[12] Karatzas, I. and Shreve, S. E. (1998). *Brownian Motion and Stochastic Calculus*, 2nd edn. Springer, New York.

[13] Kolmogorov, A. N. (1940). Wienerische Spiralen und einige andere interessante Kurven im Hilbertschen Raum. *Dokl. Akad. Nauk. SSSR* 26, 115–118.

[14] Lai, T. L. (1997). On optimal stopping problems in sequential hypothesis testing. *Statistica Sinica* 7, 33–51.

[15] Liptser, R. S. and Shiryaev, A. N. (2001). *Statistics of Random Processes I: General Theory*, 2nd edn. Springer, Berlin, Heidelberg.

[16] Mandelbrot, B. B. and Van Ness, J. W. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* 10, 422–437.

[17] Mishura, Y. S. (2008). *Stochastic Calculus for Fractional Brownian Motion and Related Processes*. Springer, Berlin, Heidelberg.

[18] Molchan, G. M. and Golosov, Y. I. (1969). Gaussian stationary processes with asymptotic power spectrum. *Dokl. Akad. Nauk. SSSR* 184, 546–549.

[19] Muravlev, A. A. (2013). Methods of sequential hypothesis testing for the drift of a fractional Brownian motion. *Russian Math. Surveys* 58, 577.

[20] Norros, I., Valkeila, E. and Virtamo, J. (1999). An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli* 5, 571–587.

[21] Pedersen, J. L. and Peskir, G. (2002). On nonlinear integral equations arising in problems of optimal stopping. In *Proc. Functional Analysis VII (Dubrovnik 2001)* (Various Publ. Ser. 46), University of Aarhus, pp. 159–175.

[22] Peskir, G. (2005). A change-of-variable formula with local time on curves. *J. Theoret. Prob.* 18, 499–535.

[23] Peskir, G. (2005). On the American option problem. *Math. Finance* 15, 169–181.

[24] Peskir, G. and Shiryaev, A. (2006). *Optimal Stopping and Free-Boundary Problems*. Birkhäuser, Basel.

[25] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd edn. Springer, Berlin, Heidelberg.

[26] Shiryaev, A. N. (1996). *Probability*, 2nd edn. Springer, New York.

[27] Tanaka, K. (2017). *Time Series Analysis: Nonstationary and Noninvertible Distribution Theory*, 2nd edn. John Wiley, Hoboken.

[28] Tartakovsky, A., Nikiforov, I. and Basseville, M. (2014). *Sequential Analysis: Hypothesis Testing and Change-Point Detection*. Chapman & Hall/CRC, Boca Raton.

[29] Zhitlukhin, M. V. and Muravlev, A. A. (2012). On Chernoff’s hypotheses testing problem for the drift of a Brownian motion. *Theory Prob. Appl.* 57, 708–717.

[30] Zhitlukhin, M. V. and Shiryaev, A. N. (2014). On the existence of solutions of unbounded optimal stopping problems. *Proc. Steklov Inst. Math.* 287, 299–307.