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Abstract. We conjecture an explicit formula for a cyclic analog of the Formality $L_\infty$-morphism [K]. We prove that its first Taylor component, the cyclic Hochschild-Kostant-Rosenberg map, is in fact a morphism (and a quasiisomorphism) of the complexes. To prove it we construct a cohomological version of the Connes-Tsygan bicomplex in cyclic homology. As an application of the cyclic Formality conjecture, we obtain an explicit formula for cyclically invariant deformation quantization. We show that (a more precise version of) the Connes-Flato-Sternheimer conjecture [CFS] on the existence of closed star-products on a symplectic manifold also follows from our conjecture.

1. Introduction
1.1. Here we recall the definition of the cohomological Hochschild complex of an associative algebra $A$, and the definitions of the associative product on this complex and the Gerstenhaber bracket. Let $A$ be an associative algebra over $\mathbb{C}$. The Hochschild complex is complex

$$0 \to C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \ldots$$

where $C^k = \text{Hom}_\mathbb{C}(A^\otimes k, A)$, $k \geq 0$, and for $\varphi \in C^k$ the differential $d\varphi \in C^{k+1}$ is defined as follows:

$$(d\varphi)(a_1 \otimes \ldots \otimes a_{k+1}) = a_1 \cdot \varphi(a_2 \otimes \ldots \otimes a_{k+1}) - \varphi((a_1 \cdot a_2) \otimes a_3 \otimes \ldots \otimes a_{n}) + \varphi(a_1 \otimes (a_2 a_3) \otimes a_4 \otimes \ldots \otimes a_{n+1}) - \ldots \pm \varphi(a_1 \otimes a_2 \otimes \ldots \otimes (a_n \cdot a_{n+1}))$$

$$+ \varphi(a_1 \otimes \ldots \otimes a_n) \cdot a_{n+1}.$$  (1)

There exists a nice interpretation of the last formula as well as of the Gerstenhaber bracket. Let $A$ be the cofree coalgebra, cogenerated by the
vector space $A[1]$. As a vector space, $\mathcal{A} = \mathbb{C} \oplus A[1] \oplus A[1] \otimes A[1] \otimes A[1] \oplus \ldots$, and the comultiplication $\Delta$ is defined as follows:

\[
\Delta(a_1 \otimes \ldots \otimes a_k) = \\
1 \otimes (a_1 \otimes \ldots \otimes a_k) + a_1 \otimes (a_2 \otimes \ldots \otimes a_k) + \ldots \\
\ldots + (a_1 \otimes \ldots \otimes a_{k-1}) \otimes a_k + (a_1 \otimes \ldots \otimes a_k) \otimes 1.
\]

Any map $D : A \otimes^k \to A$ defines uniquely a coderivation of the coalgebra $\mathcal{A}$ (as well as a map $B \to B \otimes^k$ defines uniquely the derivation of the tensor algebra $B$, generated by the vector space $B$). We still denote by $D$ the corresponding coderivation of the coalgebra $\mathcal{A}$. The bracket $[D_1, D_2]$ of the two coderivations defined by maps $D_1 : A \otimes^k \to A$ and $D_2 : A \otimes^\ell \to A$ defines a map $[D_1, D_2] : A \otimes^{(k+\ell-1)} \to A$ and the last map is called the Gerstenhaber bracket of $D_1$ and $D_2$. Moreover, if the space $A$ has an algebra structure, there exists the canonical coderivation $m : A \otimes^2 \to A$, $m(a_1 \otimes a_2) = a_1 \cdot a_2$. One can check that $[m, m] = 0$. Therefore, the map $d : C^k \to C^{k+1}$,

\[
d(\psi) = [m, \psi]
\]

satisfies the differential equation $d^2 = 0$. One can check that this is exactly the Hochschild differential given by formula (1). One can check that the Gerstenhaber bracket defines a $\mathbb{Z}$-graded Lie algebra structure on the Hochschild complex $C^\bullet$. Then it follows from the definition (3) that $C^\bullet$ equipped with the differential $d$ and the Gerstenhaber bracket defines a $dg$ Lie algebra structure on the complex $C^\bullet[+1]$.

In the sequel we will need the explicit formula for the Gerstenhaber bracket. Let $\psi_1 \in C^{k_1+1}$, $\psi_2 \in C^{k_2+1}$. We have:

\[
[\psi_1, \psi_2] = \psi_1 \circ \psi_2 - (-1)^{k_1 k_2} \psi_2 \circ \psi_1
\]

where

\[
(\psi_1 \circ \psi_2)(a_0 \otimes \ldots \otimes a_{k_1+k_2}) = \\
\sum_{i=0}^{k_1} (-1)^{i k_2} \psi_1(a_0 \otimes \ldots \otimes a_{i-1} \otimes \psi_2(a_i, a_{i+1}, \ldots, a_{i+k_2}) \otimes \\
\otimes a_{i+k_2+1} \otimes \ldots \otimes a_{k_1+k_2}).
\]

It is a standard fact that the cohomology of the complex $C^\bullet$ are equal to

\[
H^k(C^\bullet) = \text{Ext}^k_{\mathcal{A} \otimes \mathcal{A}^0}(A, A),
\]

the $k$-th Ext’s group in the category of $A$-bimodules, where the algebra $A$ is considered as an $A$-bimodule. According to (6), there exists a canonical associative product on the cohomology $H^\bullet(C^\bullet)$. In fact, this product also can be defined on the complex $C^\bullet$ as follows: for $\varphi \in C^{k_1}$, $\psi \in C^{k_2}$

\[
(\varphi \cdot \psi)(a_1 \otimes \ldots \otimes a_{k_1+k_2}) = \varphi(a_1 \otimes \ldots \otimes a_{k_1}) \cdot \psi(a_{k_1+1} \otimes \ldots \otimes a_{k_2}).
\]
The associative product and the Lie bracket on the complex $C^\bullet$ are not compatible, i.e.
$$[\psi_1 \cdot \psi_2, \psi_3] \neq [\psi_1, \psi_3] \cdot \psi_2 \pm \psi_1 \cdot [\psi_2, \psi_3].$$

1.2. Let $A = \mathbb{C}[x_1, \ldots, x_d]$ or $A = C^\infty(M)$ where $M$ is a smooth manifold. Then the cohomology $H^1(C^\bullet)$ (the cohomology of the polydifferential part of $C^\bullet$ in the smooth case) are equal to the space of polynomial polyvector fields on $\mathbb{R}^d$ or smooth polyvector fields on the manifold $M$. The induced associative product and bracket are exactly the Λ-product of polyvector fields and the Schouten-Nijenhuis bracket of polyvector fields, defined as follows:

$$[\xi_0 \wedge \ldots \wedge \xi_k, \eta_0 \wedge \ldots \wedge \eta_\ell] = \sum_{i=0}^k \sum_{j=0}^\ell (-1)^{i+j+k} [\xi_i, \eta_j] \wedge \xi_0 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \xi_k \wedge \eta_0 \wedge \ldots \wedge \hat{\eta}_j \wedge \ldots \eta_\ell$$

where $\{\xi_i\}$ and $\{\eta_j\}$ are vector fields.

Denote by $T^\bullet_{\text{poly}}$ the space of $(i+1)$-vector fields, and by $D^\bullet_{\text{poly}}$ the polydifferential part of the space $\text{Hom}_\mathbb{C}(A^\otimes(i+1), A)$.

The Hochschild-Kostant-Rosenberg map $\varphi_{\text{HKR}} : T^\bullet_{\text{poly}} \to D^\bullet_{\text{poly}}$ is defined as follows:

$$\varphi_{\text{HKR}}(\xi_1 \wedge \ldots \wedge \xi_k)(f_1 \otimes \ldots \otimes f_k) = \frac{1}{k!} \text{Alt}^{\ell}_{\xi_1, \ldots, \xi_k} \xi_1(f_1) \cdot \ldots \cdot \xi_k(f_k).$$

**Theorem.** (Hochschild-Kostant-Rosenberg)

(i) the map $\varphi_{\text{HKR}} : T^\bullet_{\text{poly}} \to D^\bullet_{\text{poly}}$ is a quasiisomorphism of the complexes;

(ii) the induced map $T^\bullet_{\text{poly}} \to H^\bullet(D^\bullet_{\text{poly}})$ is an (iso)morphism of both associative and Lie algebras.

It follows from this theorem that for any $\eta_1, \eta_2 \in T^\bullet_{\text{poly}}$ one has

$$[\varphi_{\text{HKR}}(\eta_1), \varphi_{\text{HKR}}(\eta_2)] = \varphi_{\text{HKR}}([\eta_1, \eta_2]) \mod \text{coboundaries}.$$ 

The Formality theorem of M. Kontsevich [K] states that the dg Lie algebras $T^\bullet_{\text{poly}}$ and $D^\bullet_{\text{poly}}$ are quasiisomorphic dg Lie algebras, i.e. there exists a dg Lie algebra $\mathcal{U}$ and diagram

$$\begin{array}{ccc}
\mathcal{U} & \xrightarrow{s} & T^\bullet_{\text{poly}} \\
\xleftarrow{t} & & \\
& D^\bullet_{\text{poly}} & \\
\end{array}$$

where the maps $s$ and $t$ are quasiisomorphisms of the dg Lie algebras. In fact, this result was proved using the language of the homotopical algebra, and it was constructed an $L_\infty$-quasiisomorphism $\mathcal{U} : T^\bullet_{\text{poly}} \to D^\bullet_{\text{poly}}$ (see [K]).

The analogous result also holds for the associative dg algebras $T^\bullet_{\text{poly}}$ and $D^\bullet_{\text{poly}}$ (see [Sh]).
1.3. Let $A = \mathbb{C}[x_1, \ldots, x_d]$. To formulate the cyclic Formality conjecture, we need an additional data – a volume form $\Omega$ on the space $\mathbb{R}^d$. Let us suppose that this form is fixed.

1.3.1. The form $\Omega$ defines an isomorphism $T^{i-1}_{\text{poly}} \to \Omega^{d-i}_{\text{DR}}$ (recall, that $T^{i}_{\text{poly}}$ is the space of $i$-polyvector fields).

**Definition.** (The divergention operator.) The map $\text{div} : T^i_{\text{poly}} \to T^{i-1}_{\text{poly}}$ is defined as follows: $T^i_{\text{poly}} \xrightarrow{\Omega} \Omega^{d-i}_{\text{DR}} \xrightarrow{d\Omega} \Omega^{d-i} \xrightarrow{\Omega} T^{i-1}_{\text{poly}}$.

The analog of the $dg$ Lie algebra $T^*_{\text{poly}}$ for the cyclic Formality conjecture is $T^*_{\text{poly}} \otimes \mathbb{C}[u]$, where $\deg u = 2$, equipped with the differential

$$d_{\text{div}}(\gamma \otimes u^k) = \text{div}(\gamma) \otimes u^{k+1}$$

and the bracket

$$[\gamma_1 \otimes u^{k_1}, \gamma_2 \otimes u^{k_2}] = [\gamma_1, \gamma_2] \otimes u^{k_1+k_2}. \quad (10)$$

It follows from the Lemma below that this is actually a $dg$ Lie algebra.

**Lemma.** For $\eta_1 \in T^i_{\text{poly}}, \eta_2 \in T^j_{\text{poly}}$ one has

$$\text{div} [\eta_1, \eta_2] = [\text{div} \eta_1, \eta_2] + (-1)^{i+1} [\eta_1, \text{div} \eta_2].$$

**Proof.** One can prove that

$$[\eta_1, \eta_2] = \pm (\text{div} (\eta_1 \wedge \eta_2) - (\text{div} \eta_1) \wedge \eta_2 - (-1)^{i+1} \eta_1 \wedge (\text{div} \eta_2)).$$

The statement of Lemma is a direct consequence of this formula.

1.3.2. The cyclic analog of the $dg$ Lie algebra $D^*_{\text{poly}}$ is defined in a bit more tricky way.

Let us define an operator of the cyclic shift $C : D^*_{\text{poly}} \to D^*_{\text{poly}}$.

**Definition.**

$$\int_{\mathbb{R}^d} \psi(f_1, \ldots, f_n) \cdot f_{n+1} \cdot \Omega = (-1)^n \cdot \int_{\mathbb{R}^d} C(\psi)(f_2, \ldots, f_{n+1}) \cdot f_1 \cdot \Omega, \quad (11)$$

where $f_1, \ldots, f_{n+1}$ are functions with a compact support. The operator $C : T^*_{\text{poly}} \to D^*_{\text{poly}}$ is defined by the continuity for any functions $f_1, \ldots, f_{n+1}$.

**Definition.**

$$[D^*_{\text{poly}}]_{\text{cycl}} = \{ \psi \in D^*_{\text{poly}} \mid C(\psi) = \psi \}.$$

**Lemma.** $[D^*_{\text{poly}}]_{\text{cycl}}$ is a $dg$ Lie algebra with respect to the Hochschild differential and the Gerstenhaber bracket.
We prove the statement of the Lemma for the differential, the proof for
the Gerstenhaber bracket is analogous. We have:
\[
\int (d_{\text{Hoch}} \psi)(f_1, \ldots, f_{n+1}) \cdot f_{n+2} \cdot \Omega \\
= \int (f_1 \psi(f_2, \ldots, f_{n+1}) - \psi(f_1 \cdot f_2, f_3, \ldots, f_{n+1}) + \ldots - \ldots \\
\pm \psi(f_1, \ldots, f_n) f_{n+1} f_n \\
= \int [f_1 f_{n+2} \psi(f_2, \ldots, f_{n+1}) - (-1)^n C(\psi)(f_3, \ldots, f_{n+1}, f_{n+2}) f_1 f_2 \\
+ (-1)^n C(\psi)(f_2 f_3, f_4, \ldots, f_{n+2}) \cdot f_1 \ldots \\
\pm (-1)^n C(\psi)(f_2, \ldots, f_n, f_{n+1} f_{n+2}) f_1] \cdot \Omega = \\
= (-1)^{n+1} \int (d_{\text{Hoch}} \psi)(f_2, \ldots, f_{n+2}) \cdot f_1 ,
\]
because
\[C(\psi) = \psi.\]

1.4. Cyclic Formality conjecture

Conjecture. The dg Lie algebras \(T_{\text{poly}}^\bullet \otimes \mathbb{C}[u]\) and \([D_{\text{poly}}^\bullet]_{\text{cycl}}\) are quasiisomorphic for any volume form \(\Omega\).

In this form the Conjecture is due to M. Kontsevich (private communication). In the present paper we construct explicitly set of maps:
\[C_1 : T_{\text{poly}}^\bullet \otimes \mathbb{C}[u] \to [D_{\text{poly}}^\bullet]_{\text{cycl}}\]
\[C_2 : \Lambda^2(T_{\text{poly}}^\bullet \otimes \mathbb{C}[u]) \to [D_{\text{poly}}^\bullet]_{\text{cycl}} [-1]\]
\[C_3 : \Lambda^3(T_{\text{poly}}^\bullet \otimes \mathbb{C}[u]) \to [D_{\text{poly}}^\bullet]_{\text{cycl}} [-2]\]
and conjecture that these maps are the components of an \(L_\infty\)-morphism (see (34) below), and also an \(L_\infty\)-quasiisomorphism.

The map \(C_1\) is a cyclic analog of the Hochschild-Kostant-Rosenberg map \(\varphi_{\text{HKR}}\) (see Section 1.2). Let us note, that even this map is quite nontrivial. We prove that the map \(C_1\) is in fact a (quasiiso)morphism of the complexes.

Let us note that the map \(C : D_{\text{poly}}^i \to D_{\text{poly}}^i\) satisfies the equality
\[C^{i+2} = 1.\] (12)
We denote
\[\sum = 1 + C + \ldots + C^{i+1}.\] (13)
It is clear that
\[C \left( \sum \psi \right) = \sum \psi\] (14)
for any \(\psi\). The map \(\sum\) is not compatible with the differential and the Gerstenhaber bracket, but, in a sense, it almost is. We will discuss the compatibility with the Hochschild differential in the next Section. Compatibility, in
a certain sense, the map \(\sum\) with the Gerstenhaber bracket is the most mysterious part of the cyclic Formality conjecture, and the main Conjecture 3.2.2 can be considered as an expression of this compatibility.

2. Cyclic Hochschild-Kostant-Rosenberg Theorem

2.1. We consider the \(dg\) Lie algebras

\[ \{T_{\text{poly}}^\bullet \otimes \mathbb{C} \langle u \rangle, d_{\text{div}}\} \quad \text{and} \quad \{[D_{\text{poly}}^\bullet]_{\text{cyc}}, d_{\text{Hoch}}\} \]

constructed from the algebra \(A = \mathbb{C} [x_1, \ldots, x_d]\) (see Section 1.3). We denote \([T_{\text{poly}}^\bullet]_{\text{div}} = \{\gamma \in T_{\text{poly}}^\bullet \mid \text{div} \gamma = 0\}\). It follows from the Poincaré lemma that

\[
H^i(T_{\text{poly}}^\bullet \otimes \mathbb{C} \langle u \rangle) \cong \begin{cases} [T_{\text{poly}}^i]_{\text{div}}, & i \leq d - 1 \\ \mathbb{C}, & i = d + 2k - 1, \quad k \geq 0 \end{cases}. \tag{15}
\]

The main result of this Section is the following

**Theorem.**

\[
H^i([D_{\text{poly}}^\bullet]_{\text{cyc}}) \cong \begin{cases} [T_{\text{poly}}^i]_{\text{div}}, & i \leq d - 1 \\ \mathbb{C}, & i = d + 2k - 1, \quad k \geq 0 \end{cases}. \tag{16}
\]

2.1.1. The analogous result in cyclic homology was proved using the spectral sequence, connected with the Connes-Tsygan bicomplex. It turns out that there exists an explicit analog of this construction in our situation.

Let \(A\) be an associative algebra. We define the complex \(\{K^\bullet, d_K\}\) as follows:

\[
K^i = \text{Hom}_\mathbb{C}(A^\otimes i, A), \quad i \geq 0, \tag{16}
\]

and for \(\psi \in K^i\)

\[
(d_K \psi)(a_1 \otimes \ldots \otimes a_{i+1}) = a_1 \psi(a_2 \otimes \ldots \otimes a_{i+1}) - \psi(a_1 a_2 \otimes a_3 \otimes \ldots \otimes a_{i+1}) + \ldots \pm \psi(a_1 \otimes \ldots \otimes a_i a_{i+1}). \tag{17}
\]

Note that it is exactly the Hochschild differential without the last term.

**Lemma.**

\[
d_{K}^i : K^i \to K^{i+2}, \quad i \geq 0, \quad \text{is equal to 0 for any associative algebra} \ A. \tag{18}
\]

**Proof.** Is straightforward.

2.1.2. **Lemma.** For any associative algebra \(A\) with unit the complex \(\{K^\bullet, d_K\}\) is acyclic.

**Proof.** It is sufficient to construct a homotopy \(h : K^\bullet \to K^{\bullet-1}\) such that

\[
d_K h \pm h d_K = \pm \text{Id}. \tag{18}
\]
We set

\[(h \psi)(a_1 \otimes \ldots \otimes a_{k-1}) = \psi(a_1 \otimes \ldots \otimes a_{k-1} \otimes 1). \tag{19}\]

It is clear that (18) is satisfied.

2.1.3. Let \( A \) be the algebra of functions on a smooth manifold \( M \), \( A = C^\infty(M) \), and \( \Omega \) be a volume form on \( M \). Then formula (11) defines the operator \( C : D^i_{\text{poly}}(M) \to D^i_{\text{poly}}(M) \) such that \( C^{i+2} = 1 \), and we set \( \sum = 1 + C + C^2 + \ldots + C^{i+1} \).

The results of Sections 2.1.1, 2.1.2 are true also for polydifferential part of the complex \( \{K^\bullet, d_K\} \). We denote by \( \text{Hom}_{C}^{\text{poly}}(A \otimes^k, A) \) the polydifferential part of \( \text{Hom}_{C}(A \otimes^k, A) \).

**Key-lemma.** The following is a bicomplex:

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Hom}_{C}^{\text{poly}}(A \otimes^3, A) & \overset{1-C}{\rightarrow} & \sum & \text{Hom}_{C}^{\text{poly}}(A \otimes^3, A) & \overset{1-C}{\rightarrow} & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & \text{Hom}_{C}^{\text{poly}}(A \otimes^2, A) & \overset{1-C}{\rightarrow} & \sum & \text{Hom}_{C}^{\text{poly}}(A \otimes^2, A) & \overset{1-C}{\rightarrow} & 0 \\
\uparrow & & \uza_1 & & \uparrow & & \uza_2 & \uparrow & \uparrow & \uparrow \\
0 & \rightarrow & \text{Hom}_{C}^{\text{poly}}(A, A) & \overset{1-C}{\rightarrow} & \sum & \text{Hom}_{C}^{\text{poly}}(A, A) & \overset{1-C}{\rightarrow} & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & \uparrow & \uparrow \\
0 & \rightarrow & A & \overset{1-C}{\rightarrow} & A & \overset{1-C}{\rightarrow} & A & \overset{1-C}{\rightarrow} & 0 \\
\end{array}
\]

**Proof.** It follows from (12) that \((1 - C) \sum = \sum (1 - C) = 0\). The columns are complexes by the definitions. It remains to prove the commutativity of squares.

(i) Let us prove the commutativity of square \( \star_1 \). For \( \psi \in \text{Hom}_{C}^{\text{poly}}(A \otimes^k, A) \) one has:

\[
\int f_{k+2} \cdot (1 - C) \cdot d_{\text{Hoch}} \psi \cdot \Omega =
\]

\[
= \int f_{k+2} \cdot (1 - C) \cdot \{f_1 \cdot \psi(f_2 \otimes \ldots \otimes f_{k+1}) - \psi(f_1 f_2 \otimes f_3 \otimes \ldots) + \ldots \pm \psi(f_1 \otimes f_2 \otimes \ldots \otimes f_k f_{k+1}) \mp \psi(f_1 \otimes \ldots \otimes f_k) f_{k+1}\} \Omega. \tag{20}
\]

The right-hand side of (20) is equal to

\[
\int f_{k+2} \cdot \{f_1 \psi(f_2 \otimes \ldots \otimes f_{k+1}) - \psi(f_1 f_2 \otimes \ldots \otimes f_{k+1}) + \ldots
\]
We see that $21 = 22$, and the commutativity of square $\ast$ is proved.

(ii) the proof of the commutativity of the square $\ast$

The second summand in the r.h.s. of (22) is equal to

$$\int f_{k+2} \cdot d_K (1 - C) \psi \cdot \Omega =$$

$$= \int f_{k+2} (d_K \psi - d_K C \psi) \cdot \Omega =$$

$$= \int f_{k+2} \{ f_1 \psi (f_2 \circ \ldots \circ f_{k+1}) - \psi (f_1 f_2 \circ \ldots \circ f_{k+1}) + \ldots$$

$$\pm \psi (f_1 \circ f_2 \circ \ldots \circ f_k f_{k+1}) \} \cdot \Omega$$

$$- \int f_{k+2} \{ f_1 \cdot (C \psi) (f_2 \circ \ldots \circ f_{k+1}) - (C \psi) (f_1 f_2 \circ \ldots \circ f_k)$$

$$+ \ldots \pm (C \psi) (f_1 \circ f_2 \circ \ldots \circ f_k f_{k+1}) \} \cdot \Omega. \tag{22}$$

The second summand in the r.h.s. of (22) is equal to

$$- (-1)^k \int \{ f_{k+1} \psi (f_{k+2} f_1 \circ \ldots \circ f_k) - f_{k+1} \psi (f_{k+2} \circ f_1 f_2 \circ \ldots \circ f_k)$$

$$+ \ldots \pm f_k f_{k+1} \psi (f_{k+2} \circ f_1 \circ f_2 \circ \ldots \circ f_{k-1}) \} \Omega.$$

We see that $21 = 22$, and the commutativity of square $\ast$ is proved;

(ii) the proof of the commutativity of the square $\ast$ is analogous.

**Remark.** The results of this Subsection hold for any associative algebra $A$

with unit equipped with a trace functional $\int : A \to \mathbb{C}$, i.e. $\int a \cdot b = \int b \cdot a$

for any $a, b \in A$, provided the condition that for any cochain $\psi(a_1 \circ \ldots \circ a_k)$

there exists the unique cochain $(C \psi)(a_1 \circ \ldots \circ a_k) \cdot \psi(\cdots) 

\int \psi(a_1 \circ \ldots \circ a_n) \cdot a_{n+1} = (-1)^n \int (C \psi)(a_2 \circ \ldots \circ a_{n+1}) \cdot a_1.$

2.1.4. Here we prove the following statement.

**Theorem.** Let $M$ be a smooth manifold.

$$H^i([\mathcal{D}_{poly}(M)]_{\text{div}}) \cong \left\{ \begin{array}{ll} \left[H^i_{poly}(M) \right]_{\text{div}} & i \leq d - 1 \\ H^{i-1}_{\text{DR}}(M) & i = d - 1 + k, \text{ even} \\ H^i_{\text{DR}}(M) & i = d - 1 + k, \text{ odd} \end{array} \right.$$
Proof. The rows of the bicomplex 2.1.3 are acyclic, except degree 0, because its cohomology are equal to the group cohomology $H^* \mathbb{Z} / (n + 1) \mathbb{Z}$, $\text{Hom}_C(A^\otimes n, A)$, and it is clear that the last cohomology is zero except $H^0(\mathbb{Z} / (n + 1) \mathbb{Z}, \text{Hom}_C(A^\otimes n, A)) \cong [\text{Hom}_C(A^\otimes n, A)]_{\text{cycl}} = [D^\text{poly}_n(M)]_{\text{cycl}}$. Therefore, the bicomplex 2.1.3 is quasiisomorphic to the complex $[D^\text{poly}_n(M)]_{\text{cycl}}$.

On the other hand, the second filtration of the bicomplex 2.1.3 gives us the spectral sequence with second term

$$\cdots \rightarrow T^2_{\text{poly}} 0 T^2_{\text{poly}} 0 T^2_{\text{poly}} \cdots \rightarrow d_2 \rightarrow T^1_{\text{poly}} 0 T^1_{\text{poly}} 0 T^1_{\text{poly}} \cdots \rightarrow d_2 \rightarrow T^0_{\text{poly}} 0 T^0_{\text{poly}} 0 T^0_{\text{poly}} \cdots \rightarrow d_2 \rightarrow T^{-1}_{\text{poly}} 0 T^{-1}_{\text{poly}} 0 T^{-1}_{\text{poly}} \cdots \rightarrow 0 \rightarrow 0$$

because of Lemma 2.1.2.

Here $d_2 = \text{div} : T^i_{\text{poly}} \rightarrow T^{i-1}_{\text{poly}}$. One can prove that the spectral sequence collapses in the second term. Therefore, we obtain the statement of the Theorem.

2.2. Cyclic Hochschild-Kostant-Rosenberg map (I)

We want to construct a map $\varphi^\text{cycl}_{\text{HKR}} : \{T^*_{\text{poly}} \otimes \mathbb{C}[u], d_{\text{div}}\} \rightarrow \{[D^*_{\text{poly}}]_{\text{cycl}}, d_{\text{Hoch}}\}$ which is a quasiisomorphism of the complexes, in the case $A = \mathbb{C}[x_1, \ldots, x_d]$. Here we consider first examples.

(i) for $f \in T^{-1}_{\text{poly}}$ we set

$$\varphi^\text{cycl}_{\text{HKR}}(f) = f \in D^{-1}_{\text{poly}} = [D^{-1}_{\text{poly}}]_{\text{cycl}}$$

(ii) for $\gamma = \xi_1 \wedge \ldots \wedge \xi_k \in T^{k-1}_{\text{poly}}$ we want to define $\varphi^\text{cycl}_{\text{HKR}}(\gamma)$ such that for $\gamma \in [T^{k-1}_{\text{poly}}]_{\text{div}}$ one has

$$\varphi^\text{cycl}_{\text{HKR}}(\gamma) = \frac{1}{k!} \text{Alt}_{\xi_1, \ldots, \xi_k} \xi_1(f_1) \cdot \ldots \cdot \xi_k(f_k) = \varphi_{\text{HKR}}(\gamma).$$

Let us consider the first case, $\gamma = \xi \in T^0_{\text{poly}}$. We set:

$$\varphi^\text{cycl}_{\text{HKR}}(\xi)(f) = \xi(f) + \frac{1}{2} \text{div}(\xi) \cdot f = \frac{1}{2}(\varphi_{\text{HKR}}(\xi) + C(\varphi_{\text{HKR}}(\xi))(f)).$$
It is easy to see that
\[
\int f_1 \cdot \varphi_{\text{HKR}}^{\text{cycl}}(f_2) \cdot \Omega = - \int f_2 \cdot \varphi_{\text{HKR}}^{\text{cycl}}(f_1) \cdot \Omega.
\]

(iii) We have to define \( \varphi_{\text{HKR}}^{\text{cycl}}(f \otimes u) \) such that
\[
\varphi_{\text{HKR}}^{\text{cycl}}(d_{\text{div}}(\xi)) = d_{\text{Hoch}}(\varphi_{\text{HKR}}^{\text{cycl}}(\xi)). \tag{23}
\]
We set:
\[
\varphi_{\text{HKR}}^{\text{cycl}}(f \otimes u)(f_1 \otimes f_2) = \frac{1}{2} f \cdot f_1 \cdot f_2.
\]

It is clear that (23) is true.

(iv) \( \gamma = \xi_1 \wedge \xi_2 \in T_{\text{poly}}^1 \) we set:
\[
\varphi_{\text{HKR}}^{\text{cycl}}(\gamma) = \frac{1}{3} \sum (\varphi_{\text{HKR}}(\gamma)).
\]
We have:
\[
\varphi_{\text{HKR}}^{\text{cycl}}(\gamma)(f_1 \otimes f_2) = \varphi_{\text{HKR}}(\gamma)(f_1 \otimes f_2) - \frac{1}{6} \left( \text{div}(\xi_1 \wedge \xi_2)(f_1) \cdot f_2 + f_1 \cdot \text{div}(\xi_1 \wedge \xi_2)(f_2) \right).
\]

(v) We want to define
\[
\varphi_{\text{HKR}}^{\text{cycl}}(\xi \otimes u), \text{ where } \xi \in T_{\text{poly}}^0,
\]
such that
\[
\varphi_{\text{HKR}}^{\text{cycl}}(d_{\text{div}}(\xi_1 \wedge \xi_2)) = d_{\text{Hoch}}(\varphi_{\text{HKR}}^{\text{cycl}}(\xi_1 \wedge \xi_2)). \tag{24}
\]
We set:
\[
\varphi_{\text{HKR}}^{\text{cycl}}(\xi \otimes u) (f_1 \otimes f_2 \otimes f_3) =
\frac{1}{6} (\xi(f_1) \cdot f_2 \cdot f_3 + f_1 \cdot f_2 \cdot \xi(f_3)) + \frac{1}{12} \text{div}(\xi) \cdot f_1 \cdot f_2 \cdot f_3.
\]

2.3. Cyclic Hochschild-Kostant-Rosenberg map (II)

Here we define \( \varphi_{\text{HKR}}^{\text{cycl}}(\gamma \otimes u^k) \) for arbitrary \( \gamma \otimes u^k \in T_{\text{poly}}^* \otimes \mathbb{C}[u] \). It is very convenient to use the language of graphs of M. Kontsevich [K].

Let \( \Gamma(\ell, k) \) be the set of all the graphs with \( \ell + 2k \) vertices on the line \( \mathbb{R} \) (vertices of the second type in [K]), and the unique additional vertex (of the first type) and \( \ell \) oriented edges started at this vertex such that:

- **between any two consecutive endpoints of edges there is an even number of other vertices of the second type** \( \tag{25} \)

(a vertex of the second type is the endpoint for not more than 1 edge).
Example. The set $\Gamma(2, 1)$ is shown on the Figure 1

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example_figure}
\caption{The set $\Gamma(2, 1)$}
\end{figure}

We attach to any $\Gamma \in \Gamma(\ell, k)$ a polydifferential operator $\varphi_\Gamma(f_1 \otimes \ldots \otimes f_{\ell+2k})$ as follows. Let $\xi_1, \ldots, \xi_\ell$ be vector fields. We set:

$$
\varphi_\Gamma(f_1 \otimes \ldots \otimes f_{\ell+2k}) = \text{Alt}_{\xi_1 \ldots \xi_\ell} \prod_{i=1}^{\ell+2k} \varphi_i^\Gamma(f_i)
$$

(26)

where

$$
\varphi_i^\Gamma(f) = \begin{cases} 
    f, & \text{if the } i\text{-th vertex of the} \\
    \text{second type is not an endpoint of an edge in the graph } \Gamma \\
    \xi_j(f), & \text{if the } i\text{-th vertex of the} \\
    \text{second type is the } j\text{-th endpoint (from left to right)}
  \end{cases}
$$

We set ($\gamma \in T_{\text{poly}}^{\ell-1}$):

$$
\tilde{\varphi}_{\text{HKR}}^{\text{cycl}}(\gamma \otimes u^k) = \frac{k!}{(\ell+2k)!} \sum_{\Gamma \in \Gamma(\ell, k)} \varphi_\Gamma
$$

(27)

and

$$
\varphi_{\text{HKR}}^{\text{cycl}} = \frac{1}{\ell+2k+1} \sum \varphi_{\text{HKR}}^{\text{cycl}}(f_1 \otimes \ldots \otimes f_{\ell+2k})
$$

(28)

where $\sum = 1 + C + \ldots + C^{\ell+2k}$.

One can see that the definition (28) coincides with Examples 2.2 (i) – (v) in the simplest cases.

**Theorem.** The map $\varphi_{\text{HKR}}^{\text{cycl}} : T_{\text{poly}}^\bullet \otimes \mathbb{C}[u] \to [D_{\text{poly}}^\bullet]_{\text{cycl}}$ is a map of the complexes

$$
\varphi_{\text{HKR}}^{\text{cycl}} : \{T_{\text{poly}}^\bullet \otimes \mathbb{C}[u], d_{\text{div}}\} \to \{[D_{\text{poly}}^\bullet]_{\text{cycl}}, d_{\text{Hoch}}\}
$$

and also a quasiisomorphism of the complexes.
We prove this Theorem in Sections 2.4 – 2.6.

2.4. To prove Theorem 2.3 we need some preparations.

2.4.1. Lemma. For any \( k > 0, \ell \geq 0 \), and \( \Gamma \in \Gamma(\ell, k) \), the cochain \( \varphi_\Gamma(f_1 \otimes \ldots \otimes f_{\ell+2k}) \) is a Hochschild coboundary.

Proof. For the graph \( \Gamma \in \Gamma(\ell, k) \), \( k > 0 \), we define a graph \( \tilde{\Gamma} \) as follows: the graph \( \tilde{\Gamma} \) has 1 vertex of the first type, \( \ell + 2k - 1 \) vertices of the second type, and \( \ell \) edges. We just short the first maximal sequence of consecutive free (= not endpoints of edges) vertices of second type on 1 vertex, see Figure 2.

\[ \Gamma = \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \begin{array}{c} 4 \end{array} \begin{array}{c} 5 \end{array} \begin{array}{c} 6 \end{array} \begin{array}{c} 7 \end{array} \begin{array}{c} 8 \end{array} \begin{array}{c} 9 \end{array} \begin{array}{c} 10 \end{array} \begin{array}{c} 11 \end{array} \begin{array}{c} 12 \end{array} \rightarrow \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \begin{array}{c} 4 \end{array} \begin{array}{c} 5 \end{array} \begin{array}{c} 6 \end{array} \begin{array}{c} 7 \end{array} \begin{array}{c} 8 \end{array} \begin{array}{c} 9 \end{array} \begin{array}{c} 10 \end{array} \begin{array}{c} 11 \end{array} \begin{array}{c} 12 \end{array} \]

\[ \tilde{\Gamma} = \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \begin{array}{c} 4 \end{array} \begin{array}{c} 5 \end{array} \begin{array}{c} 6 \end{array} \begin{array}{c} 7 \end{array} \begin{array}{c} 8 \end{array} \begin{array}{c} 9 \end{array} \begin{array}{c} 10 \end{array} \begin{array}{c} 11 \end{array} \begin{array}{c} 12 \end{array} \]



Figure 2
\( \Gamma \in \Gamma(3, 2) \) and \( \tilde{\Gamma} \)

We define the cochain \( \varphi_{\tilde{\Gamma}}(f_1 \otimes \ldots \otimes f_{\ell+2k-1}) \) analogously to (26). It is easy to see that

\[
(d_{\text{Hoch}} \varphi_{\tilde{\Gamma}})(f_1 \otimes \ldots \otimes f_{\ell+2k}) = \varphi_{\Gamma}(f_1 \otimes \ldots \otimes f_{\ell+2k}).
\]

\[ \square \]

Corollary. For any \( k, \ell \geq 0 \), and any \( \Gamma \in \Gamma(\ell, k) \) the cochain \( \varphi_{\Gamma}(f_1 \otimes \ldots \otimes f_{\ell+2k}) \) is a Hochschild cocycle.

2.4.2. Here we study in which sense the map \( \sum = 1 + C + C^2 + \ldots + C^{i+1} : \mathcal{D}^i_{\text{poly}} \to \mathcal{D}^i_{\text{poly}} \) is compatible with the Hochschild differential \( d_{\text{Hoch}} \). The results of this Subsection hold for any associative algebra \( A \) with a trace functional \( \int : A \to \mathbb{C} \), i.e. \( \int a \cdot b = \int b \cdot a \) for any \( a, b \in A \), provided the condition that for any cochain \( \psi(a_1 \otimes \ldots \otimes a_k) \) there exists the unique cochain \( (C\psi)(a_1 \otimes \ldots \otimes a_k) \) such that

\[
\int \psi(a_1 \otimes \ldots \otimes a_n) \cdot a_{n+1} = (-1)^n \int (C\psi)(a_2 \otimes \ldots \otimes a_{n+1}) \cdot a_1.
\]

Lemma. For \( \psi \in \text{Hom}_\mathbb{C}(A^{\otimes k}, A) \) one has

\[
\int f_{k+2} \cdot (d_{\text{Hoch}} \sum \psi - \sum d_{\text{Hoch}} \psi)(f_1 \otimes \ldots \otimes f_{k+1}) = (-1)^{k-1} \int \phi(f_1 \otimes \ldots \otimes f_{n+2})
\]
where

\[
\phi(f_1 \otimes \ldots \otimes f_{n+2}) = \\
= \psi(f_1 \otimes \ldots \otimes f_n) \cdot f_{n+1} \cdot f_{n+2} + (-1)^{k-1} f_1 \cdot \psi(f_2 \otimes \ldots \otimes f_{n+1}) \cdot f_{n+2} + \\
+ f_1 \cdot f_2 \cdot \psi(f_3 \otimes \ldots \otimes f_{n+2}) + (-1)^{k-1} f_2 \cdot f_3 \cdot \psi(f_4 \otimes \ldots \otimes f_{n+2} \otimes f_1) \\
+ \ldots \pm f_n \cdot f_{n+1} \cdot \psi(f_{n+2} \otimes f_1 \otimes \ldots \otimes f_{n-1}).
\]

(29)

Proof. It is a straightforward calculation.

2.5. Here we prove that the map

\[
\varphi_{\text{HKR}}^{\text{cycl}} : T_{\text{poly}}^\bullet \otimes \mathbb{C}[u] \rightarrow [D_{\text{poly}}^\bullet]_{\text{cycl}}
\]

is a morphism of the complexes.

We have to prove that

\[
\varphi_{\text{HKR}}^{\text{cycl}}(\text{div}(\gamma) \otimes u^{k+1}) = d_{\text{Hoch}} \varphi_{\text{HKR}}^{\text{cycl}}(\gamma \otimes u^k)
\]

for any \(\gamma = \xi_1 \wedge \ldots \wedge \xi_\ell \in T_{\text{poly}}^{\ell-1}\), and any \(k\). We have:

\[
\int f_{\ell+2k+2} \cdot d_{\text{Hoch}} \varphi_{\text{HKR}}^{\text{cycl}}(\gamma \otimes u^k)(f_1 \otimes \ldots \otimes f_{\ell+2k+1}) \cdot \Omega = \\
= \int f_{\ell+2k+2} \cdot \left( d_{\text{Hoch}} \cdot \frac{1}{\ell+2k+1} \sum \varphi_{\text{HKR}}^{\text{cycl}} \right) (f_1 \otimes \ldots \otimes f_{\ell+2k+1}) \Omega = 
\]

by Lemma 2.4.2

\[
\int f_{\ell+2k+2} \cdot \left( \frac{1}{\ell+2k+1} \sum d_{\text{Hoch}} \right) (f_1 \otimes \ldots \otimes f_{\ell+2k+1}) \cdot \Omega \\
+ \int \phi(f_1 \otimes \ldots \otimes f_{\ell+2k+2}) \cdot \Omega =
\]

by Corollary 2.4.1

\[
\int \phi(f_1 \otimes \ldots \otimes f_{\ell+2k+2}) \cdot \Omega.
\]

On the other hand, for any \(\Gamma \in \Gamma(\ell-1, k+1)\) and any vector fields \(\xi_1, \ldots, \xi_\ell\) we define a cochain \(\varphi_\Gamma(f_1 \otimes \ldots \otimes f_{\ell+2k+2})\) as follows:

\[
\varphi_\Gamma(f_1 \otimes \ldots \otimes f_{\ell+2k+2}) = \sum_{\text{over all free}} \varphi_\Gamma^{(h)}(f_1 \otimes \ldots \otimes f_{\ell+2k+2})
\]

(31)

where

\[
\varphi_\Gamma^{(h)}(f_1 \otimes \ldots \otimes f_{\ell+2k+2}) = \text{Alt}_{\xi_1, \ldots, \xi_\ell} \prod_{i=1}^{\ell+2k+2} \varphi_\Gamma^{(j, i)}(f_i)
\]

(32)
and
\[ \varphi^{(j)}_i(f) = \begin{cases} f, & \text{if } i \text{ is a free vertex of } \Gamma \text{ and } i \neq j \\ \xi(f) & \text{if } i = j \\ \xi_s(f) & \text{if } i \text{ is } s\text{-th endpoint.} \end{cases} \]

We set
\[ \varphi = \sum_{\Gamma \in \Gamma(\ell-1,k+1)} \varphi_\Gamma. \tag{33} \]

2.5.1. Lemma. Let \( \gamma = \xi_1 \wedge \ldots \wedge \xi_\ell \)
\[ \int \varphi_{\ell,k}(f_1 \otimes \ldots \otimes f_{\ell+2k+2}) \cdot \Omega = \]
\[ = \pm \int \sum_{\Gamma \in \Gamma(\ell-1,k+1)} \varphi_\Gamma((\text{div } \gamma) \otimes u^{k+1})(f_1 \otimes \ldots \otimes f_{\ell+2k+1}) \cdot f_{\ell+2k+2} \cdot \Omega. \]

Proof. It is clear. ■

2.5.2. Lemma. Let \( \phi(f_1 \otimes \ldots \otimes f_{\ell+2k+2}) \) is defined from the cochain
\[ \sum_{\Gamma \in \Gamma(\ell,k)} \varphi_\Gamma(f_1 \otimes \ldots \otimes f_{\ell+2k}) \]
by (29). Then
\[ (\ell + 2k + 1) \cdot \int \phi(f_1 \otimes \ldots \otimes f_{\ell+2k+2}) \cdot \Omega = \]
\[ = \pm k \cdot \int \varphi_{\ell,k}(f_1 \otimes \ldots \otimes f_{\ell+2k+2}) \cdot \Omega. \]

Proof. It is straightforward. ■

Now (30) follows from Lemma 2.5.1 and Lemma 2.5.2.

2.6. We have proved that \( \varphi_{\text{cycl}}^{\text{HKR}} : T^\bullet_{\text{poly}} \otimes \mathbb{C} [u] \to [D^\bullet_{\text{poly}}]_{\text{cycl}} \) is a map of the complexes. It remains to prove that it is a quasiisomorphism.

We know that both complexes have the same cohomology:
\[ H^i(T^\bullet_{\text{poly}} \otimes \mathbb{C} [u], d_{\text{div}}) = H^i([D^\bullet_{\text{poly}}]_{\text{cycl}}, d_{\text{Hoch}}) \]
\[ = \begin{cases} [T^i_{\text{poly}}]_{\text{div}}, & i \leq d - 1 \\ \mathbb{C}, & i = d + 2k - 1, \ k \geq 0. \end{cases} \]

We know that \( \varphi_{\text{HKR}} | [T^\bullet_{\text{poly}}]_{\text{div}} = \varphi_{\text{HKR}} | [T^\bullet_{\text{poly}}]_{\text{div}}, \) and therefore the map
\[ \varphi_{\text{cycl}}^{\text{HKR}} | [T^\bullet_{\text{poly}}]_{\text{div}} : [T^\bullet_{\text{poly}}]_{\text{div}} \to ([D^\bullet_{\text{poly}}]_{\text{cycl}}, d_{\text{Hoch}}), \]
is an embedding on the level of cohomology. Only what remains to prove is that the map
\[ \varphi_{\text{HKR}}^{\text{cycl}} : \mathbb{C} = H^{d+2k-1}(T^\bullet_{\text{poly}} \otimes \mathbb{C} [u], d_{\text{div}}) \to H^{d+2k-1}([D^\bullet_{\text{poly}}]_{\text{cycl}}, d_{\text{Hoch}}) \]
is an embedding. We assume that \( \Omega = dx_1 \wedge \ldots \wedge dx_d \). It is clear that \( \theta_k = \left( \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_d} \right) \otimes u^k \) is a cocycle in \( \{ T_{\mathrm{poly}} \otimes \mathbb{C} [u], d_{\mathrm{div}} \} \), and it is not cohomologous to zero. We have to prove that \( \varphi_{\mathrm{HKR}}(\theta_k) \) is not cohomologous to zero for any \( k \geq 0 \).

### 2.6.1. Long exact sequence, associated with bicomplex 2.1.3.

Let us denote by \( C^{\bullet \bullet} \) the bicomplex 2.1.3. Then there exists the following short exact sequence of the bicomplexes:

\[
0 \to C^{\bullet \bullet}[-2] \to C^{\bullet \bullet} \to (\text{first 2 columns}) \to 0.
\]

The cohomology of the last term are equal to the Hochschild cohomology, because the complex \( K^\bullet \) is acyclic (Lemma 2.1.2). We obtain the following long exact sequence:

\[
\ldots \to H^{i-1} \to HC^{i-2} \xrightarrow{S} HC^i \to H^i \to HC^{i-1} \xrightarrow{S} HC^{i+1} \to H^{i+1} \to \ldots
\]

where \( H^\bullet \) stands for the Hochschild cohomology and \( HC^\bullet \) stands for the cyclic cohomology (i.e., the total cohomology of bicomplex 2.1.3). It is clear from the Hochschild-Kostant-Rosenberg theorem that the map \( S: HC^{i-2} \to HC^i \) is an isomorphism for \( i \geq d + 1 \) \( (A = \mathbb{C} [x_1, \ldots, x_d]) \).

### 2.6.2. Lemma.

\[
S [\varphi_{\mathrm{HKR}}^{\mathrm{cycl}}(\theta_k)] = \mu \cdot [\varphi_{\mathrm{HKR}}^{\mathrm{cycl}}(\theta_{k+1})] \quad \text{where} \quad \theta_k = \left( \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_d} \right) \otimes u^k, \quad \text{where} \quad \mu \neq 0.
\]

It follows from this Lemma that \( \varphi_{\mathrm{HKR}}^{\mathrm{cycl}}(\theta_k) \) is not cohomologous to 0 for any \( k \geq 0 \), and that the map \( \varphi_{\mathrm{HKR}}^{\mathrm{cycl}}: \{ T_{\mathrm{poly}} \otimes \mathbb{C} [u], d_{\mathrm{div}} \} \to \{ D_{\mathrm{poly}}^{\mathrm{cycl}}, d_{\mathrm{Hoch}} \} \) is a quasiisomorphism of the complexes.

**Proof of Lemma.** We assume that \( \Omega = dx_1 \wedge \ldots \wedge dx_d \). It is clear that \( \mathrm{div} \left( \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_d} \right) = 0 \). We have to prove that \( S [\varphi_{\mathrm{HKR}}^{\mathrm{cycl}}(\theta_k)] = \mu \cdot [\varphi_{\mathrm{HKR}}^{\mathrm{cycl}}(\theta_{k+1})] \).

We prove here the last formula for \( k = 0 \), the general case is analogous.

We consider \( \varphi_{\mathrm{HKR}}^{\mathrm{cycl}}(\theta_0) \) as an element of the third column of the bicomplex 2.1.3. We have: \((1 - C)(\varphi_{\mathrm{HKR}}^{\mathrm{cycl}}(\theta_0)) = 0 \) and \( d_{\mathrm{Hoch}}(\varphi_{\mathrm{HKR}}^{\mathrm{cycl}}(\theta_0)) = 0 \). The rows of the bicomplex 2.1.3 are exact, and it follows from the first equation that there exists \( \alpha \) such that \( \sum \alpha = \varphi_{\mathrm{HKR}}^{\mathrm{cycl}}(\theta_0) \) (the element \( \alpha \) lies in the second column of the bicomplex 2.1.3). We set \( \beta = d_K(\alpha) \) where \( d_K \) is the differential in the complex \( K^\bullet \). We have:

1. \( d_K \beta = d_K^2 \alpha = 0 \)
2. \( \sum \beta = \sum d_K \alpha = d_{\mathrm{Hoch}} \sum \alpha = d_{\mathrm{Hoch}} \varphi_{\mathrm{HKR}}^{\mathrm{cycl}}(\theta_0) = 0 \).

We need to know explicit formulas for \( \alpha \) and \( \beta \).

We can choose \( \alpha = \frac{1}{k+1} \varphi_{\mathrm{HKR}}^{\mathrm{cycl}}(\theta_0) \).

Then, up to a constant factor,

\[
\beta = d_K \alpha = \left\{ f_1 \otimes \ldots \otimes f_{d+1} \to \mathrm{Alt} \left( \frac{\partial}{\partial x_1} \ldots \frac{\partial}{\partial x_d} \right) (f_1) \cdot \ldots \cdot (f_d) \cdot f_{d+1} \right\}.
\]
Furthermore, there exists $\gamma$ such that $(1 - C) \gamma = \beta$. We set $\theta = d_{\text{Hoch}} \gamma$. It is clear that $\theta$ is a cocycle in the bicomplex, cohomologous to the element $\varphi_{\text{HKR}}(\theta_0)$ in the third column. The element $\theta$ lies in the first column. We have to express $\theta$ explicitly.

First of all, let us find $\gamma$ such that $(1 - C) \gamma = \beta$.

Denote by $\beta(k)$, $1 \leq k \leq d + 1$, the cochain

$$
\beta(k) = \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_d} (f_1) \cdots \frac{\partial}{\partial x_{k-1}} (f_{k-1}) \cdot f_k \cdot \frac{\partial}{\partial x_k} (f_{k+1}) \cdots \frac{\partial}{\partial x_d} (f_{d+1}) .
$$

Lemma.

$$
C(\beta(k)) = \begin{cases} 
- \beta(k-1) + (-1)^{k+d-1} \beta(d+1), & k > 1 \\
\beta(d+1), & k = 1.
\end{cases}
$$

Proof. It is a direct calculation.

It follows from this lemma that, up to a nonzero constant, one can choose

$$
\gamma = \beta^{(1)} - \beta^{(2)} + \beta^{(3)} - \beta^{(4)} + \ldots \pm \beta^{(d+1)}.
$$

Now it is clear that $\theta = d_{\text{Hoch}} \gamma$ is equal, up to a nonzero constant factor, to $\varphi_{\text{HKR}}(\theta_1)$ (see formulas (27), (28)).

The case $k > 0$ is analogous.

We have proved that the map

$$
\varphi_{\text{HKR}} : \{T_{\text{poly}} \otimes C[u], d_{\text{div}}\} \rightarrow \{[D_{\text{poly}}^\bullet]_{\text{cycl}}, d_{\text{Hoch}}\}
$$

is a quasiisomorphism of the complexes.

3. Cyclic Formality morphism

In this section we construct explicitly maps

$$
\varphi_{\text{HKR}}^\text{cycl} = C_1 : T_{\text{poly}}^\bullet \otimes C[u] \rightarrow [D_{\text{poly}}^\bullet]_{\text{cycl}}
$$

$$
C_2 : \Lambda^2(T_{\text{poly}}^\bullet \otimes C[u]) \rightarrow [D_{\text{poly}}^\bullet]_{\text{cycl}}[-1]
$$

$$
C_3 : \Lambda^3(T_{\text{poly}}^\bullet \otimes C[u]) \rightarrow [D_{\text{poly}}^\bullet]_{\text{cycl}}[-2]
$$

and our main Conjecture states that these maps are the Taylor components of an $L_\infty$-morphism $C : \{T_{\text{poly}}^\bullet \otimes C[u], d_{\text{div}}\} \rightarrow \{[D_{\text{poly}}^\bullet]_{\text{cycl}}, d_{\text{Hoch}}\}$, i.e. for any $\eta_1, \ldots, \eta_m \in T_{\text{poly}}^\bullet \otimes C[u]$ and any functions $f_1, \ldots, f_m$ one has:

$$
\sum_{i=1}^n \pm C_n(\eta_1 \wedge \ldots \wedge d_{\text{div}} \eta_i \wedge \ldots \wedge \eta_n)(f_1 \otimes \ldots \otimes f_m) + \\
\quad + f_1 \cdot C_n(\eta_1 \wedge \ldots \wedge \eta_m)(f_2 \otimes \ldots \otimes f_m)
$$
\[\pm C_n(\eta_1 \wedge \ldots \wedge \eta_n)(f_1 \otimes \ldots \otimes f_{m-1}) \cdot f_m + \sum_{i=1}^{m-1} \pm C_n(\eta_1 \wedge \ldots \wedge \eta_n)(f_1 \otimes \ldots \otimes f_i f_{i+1} \otimes \ldots \otimes f_m) + \]
\[\sum_{i \neq j} \frac{1}{k! \ell!} \sum_{\sigma \in \Sigma_n} \pm [C_k(\eta_{\sigma_1} \wedge \ldots \wedge \eta_{\sigma_k}), C_\ell(\eta_{\sigma_{k+1}} \wedge \ldots \wedge \eta_{\sigma_n})](f_1 \otimes \ldots \otimes f_m) = 0. \quad (34)\]

(We refer reader to [K], Section 4 for the general theory of \(L_\infty\)-algebras).

The idea of the construction of the maps \(C_1, C_2, C_3, \ldots\) is the following. We consider \([T_{\text{poly}}]_{\text{div}} \otimes \mathbb{C}[u]\) as a \(\text{dg} \text{ Lie subalgebra in } \{T_{\text{poly}} \otimes \mathbb{C}[u], d_{\text{div}}\}\) (with zero differential), and construct (with proofs) an \(L_\infty\)-morphism

\[\tilde{C} : [T_{\text{poly}}]_{\text{div}} \otimes \mathbb{C}[u] \to \mathcal{D}_{\text{poly}}.\]

Here the idea is very close to [K].

In the next step, we apply the map \(\sum\) to \(\mathcal{D}_{\text{poly}}\). The map \(\sum\) is not a map of \(\text{dg} \text{ Lie algebras (see Lemma 2.4.2), but it turns out that the composition }\]

\[\left[\sum\right] \circ \tilde{C} : [T_{\text{poly}}]_{\text{div}} \otimes \mathbb{C}[u] \to [\mathcal{D}_{\text{poly}}]_{\text{cycl}}\]

where \([\sum] : \mathcal{D}_{\text{poly}}^i \to [\mathcal{D}_{\text{poly}}^i]_{\text{cycl}}\) is equal to

\[\left[\sum\right] = \frac{1}{i + 2} \sum\]

is still an \(L_\infty\)-morphism (it is a conjecture), and, moreover, the same formulas define an \(L_\infty\)-morphism \(\{T_{\text{poly}} \otimes \mathbb{C}[u], d_{\text{div}}\} \to \{[\mathcal{D}_{\text{poly}}]_{\text{cycl}}, d_{\text{Hoch}}\}\). We have checked this fact only for \(C_1 = \psi_{\text{HKR}}\) in Section 2, and this check is quite nontrivial.

### 3.1. \(L_\infty\)-map \(\tilde{C} : [T_{\text{poly}}]_{\text{div}} \otimes \mathbb{C}[u] \to \mathcal{D}_{\text{poly}}\)

#### 3.1.1. Admissible graphs

**Definition.** Admissible graph \(\Gamma\) is an oriented graph with labels and two types of edges: usual edges, and **dashed edges**, such that

1) the set of vertices \(V_\Gamma\) is \(\{1, \ldots, n\} \cup \{\overline{1}, \ldots, \overline{m}\}\) where \(n, m \in \mathbb{Z}_{\geq 0}\); vertices from the set \(\{1, \ldots, n\}\) are called vertices of the first type, vertices from \(\{\overline{1}, \ldots, \overline{m}\}\) are called vertices of the second type,

2) every edge \((v_1, v_2) \in E_\Gamma\) starts at a vertex of the first type, \(v_1 \in \{1, \ldots, n\}\),

3) there are no loops,

4) dashed edges end only on the vertices of second type and appear only in pairs, i.e. it is a pair of edges \((v, \overline{i})\) and \((v, \overline{i + 1})\) where \(v \in \{1, \ldots, n\}\).
and \(i, i+1\) are elements from \(\{1, 2, \ldots, m\}\); we will speak about a dashed pair,

5) for every vertex \(\ell \in \{1, \ldots, n\}\) the set of usual (not dashed) edges

\[
\text{Star}(\ell) := \{(v_1, v_2) \in E_\Gamma \mid v_1 = \ell\}
\]

is labelled by symbols \((e_1^\ell, \ldots, e_\text{Star}(\ell))^\ell\).

A typical admissible graph is shown on Figure 3

![Figure 3](image_url)

**Figure 3**

An admissible graph with 3 dashed pairs

### 3.1.2. Configuration spaces

We work with the same configuration spaces as in [K]. Let us recall the definitions.

Let \(n, m\) be non-negative integers satisfying the inequality \(2n + m \geq 2\). We denote by \(\text{Conf}_{n, m}\) the product of configuration space of the upper half-plane with the configuration space of real line:

\[
\text{Conf}_{n, m} = \{(p_1, \ldots, p_n; q_1, \ldots, q_m) \mid p_i \in \mathbb{H}, q_j \in \mathbb{R}, p_i \neq p_j \text{ for } i \neq j \}
\]

The group

\[
G^{(1)} = \{z \mapsto az + b \mid a, b \in \mathbb{R}, a > 0\}
\]

acts on the space \(\text{Conf}_{n, m}\). It follows from the condition \(2n + m \geq 2\) that this action is free. The quotient space \(C_{n, m} = \text{Conf}_{n, m}/G^{(1)}\) is a smooth manifold of dimension \(2n + m - 2\).

Analogously, we introduce simpler spaces \(\text{Conf}_{n}\) and \(C_{n}\) for any \(n \geq 2\):

\[
\text{Conf}_{n} = \{(p_1, \ldots, p_n) \mid p_i \in \mathbb{C}, p_i \neq p_j \text{ for } i \neq j\}
\]
\[ C_n = \text{Conf}_n/G^{(2)} , \quad \dim C_n = 2n - 3 \]

where \(G^{(2)}\) is a group

\[ G^{(2)} = \{ z \mapsto az + b \mid a \in \mathbb{R} , \ b \in \mathbb{C} , \ a > 0 \} . \]

We construct (following [K], Section 5) compactifications \(\overline{C}_{n,m}\) of \(C_{n,m}\) (and compactifications \(\overline{C}_n\) of \(C_n\)). These compactifications are “manifolds with corners”.

Let us describe the strata of codimension 1. There are two types of strata of codimension 1 in \(\overline{C}_{A,B}\):

S1) points \(p_i \in \mathcal{H}\) for \(i \in S \subseteq A\) where \(\# S \geq 2\) move close to each other and far from \(\mathbb{R}\), the corresponding boundary stratum

\[ \partial_S \overline{C}_{A,B} = C_S \times (A \setminus S) \sqcup \{pt\} , \ B \]

S2) points \(p_i \in \mathcal{H}\) for \(i \in S \subseteq A\) and points \(q_j \in \mathbb{R}\) for \(j \in S' \subseteq B\), where \(2 \# S + \# S' \geq 2\), all move close to each other and to \(\mathbb{R}\), with at least one point left outside \(S\) and \(S'\), i.e. \(\# S + \# S' \leq \# A + \# B - 1\). The corresponding boundary stratum is

\[ \partial_{S,S'} \overline{C}_{A,B} = C_S S' \times (A \setminus S) , \ (B \setminus S') \sqcup \{pt\} . \]

It is instructional to draw low-dimensional spaces \(C_{n,m}\). The simplest one, \(C_{1,0} = \overline{C}_{1,0}\) is just a point. The space \(C_{0,2} = \overline{C}_{0,2}\) is a two-element set. The space \(C_{1,1}\) is an open interval, and its closure \(\overline{C}_{1,1}\) is a closed interval.

The space \(C_{2,0}\) is diffeomorphic to \(\mathcal{H}\setminus\{0 + 1 \cdot i\}\). The closure \(\overline{C}_{2,0}\) is shown on Fig. 4.

![Figure 4](image)

**Figure 4**

The space \(\overline{C}_{2,0}\) (“the Eye”)

See [K], Section 5 for more details.

### 3.1.3. Differential forms on configuration spaces

The space \(\overline{C}_{2,0}\) is homotopy equivalent to the standard circle \(S^1 \simeq \mathbb{R}/2\pi \mathbb{Z}\). Moreover, one of its boundary components, the space \(C_2 = \overline{C}_2\) is naturally \(S^1\). The other component of the boundary is the union of two closed intervals (copies of \(\overline{C}_{1,1}\)) with identified endpoints.
Definition. An angle map is a smooth map \( \phi : \mathcal{C}_{2,0} \to \mathbb{R}/2\pi\mathbb{Z} \) such that the restriction of \( \phi \) to \( C_2 \simeq S^1 \) is the angle measure in anti-clockwise direction from the vertical line, and \( \phi \) maps the whole upper interval \( \mathcal{C}_{1,1} \simeq [0,1] \) of \( \mathcal{C}_{2,0} \) to a point in \( S^1 \).

We denote by \( G_{n,m,2k} \) the set of all the admissible graphs with \( n \) vertices of the first type, \( m \) vertices of the second type, \( k \) dashed pairs and \( 2n + m - 2k - 2 \) usual edges. Let \( \Gamma \in G_{n,m,2k} \).

We define the weight \( W_\Gamma \) of the graph \( \Gamma \) by the following formula:

\[
W_\Gamma = \prod_{\ell=1}^{n} (k_\ell)! \cdot \prod_{\ell=1}^{n} \frac{1}{(\# \text{Star}(\ell))!} \cdot \frac{1}{(2\pi)^{2n+m-2}} \int_{\mathcal{C}_{n,m}} \wedge d\varphi_\epsilon. \tag{35}
\]

Let us explain what is written here. The domain of integration \( \mathcal{C}_{n,m}^+ \) is a connected component of \( \mathcal{C}_{n,m} \) which is the closure of configurations for which points \( q_j, 1 \leq j \leq m \) on \( \mathbb{R} \) are placed in the increasing order: \( q_1 < \ldots < q_m \). Every edge \( e \) of \( \Gamma \) defines a map from \( \mathcal{C}_{n,m} \) to \( \mathcal{C}_{2,0} \) or to \( \mathcal{C}_{1,1} \subset \mathcal{C}_{2,0} \) (we consider \( \mathcal{C}_{1,1} \subset \mathcal{C}_{2,0} \) as the lower interval of the Eye). The pull-back of the angle function \( \phi \) by the map \( \mathcal{C}_{n,m} \to \mathcal{C}_{2,0} \) corresponding to edge \( e \) is denoted by \( \phi_\epsilon \). The number \( k_\ell \) is the number of dashed pairs starting at the vertex \( \ell \). Finally, the ordering in the wedge product of 1-forms \( d\phi_\epsilon \) is fixed by enumeration of the set of sources of edges and by the enumeration of the set of edges with a given source (the dashed pairs can be counted in any order).

3.1.4. Pre-\( L_\infty \)-morphism associated with graphs

For any admissible graph \( \Gamma \in G_{n,m,2k} \) (it has \( n \) vertices of the first type, \( m \) vertices of the second type, \( k \) dashed pairs, and \( 2n - 2k + m - 2 \) usual edges) we define a linear map \( \tilde{C}_\Gamma : \bigotimes^n ([T_{\text{poly}}]_{\text{div}} \otimes \mathbb{C}[u]) \to D_{\text{poly}}[1-n] \). This map has only one nonzero component \( (\tilde{C}_\Gamma)_{(\ell_1,k_1),\ldots,(\ell_n,k_n)} ((\ell_j,k_j) \text{ here stands for } [T_{\text{poly}}]_{\text{div}} \otimes u^{k_j}) \), where \( \ell_j = \# \text{Star}(j) - 1 \), and \( k_j \) is the number of dashed pairs starting at the vertex \( j \) of the first type.

Let \( \eta_1, \ldots, \eta_n \in [T_{\text{poly}}]_{\text{div}} \otimes \mathbb{C}[u] \), \( \eta_j = \gamma_j \otimes u^{k_j} \) and \( \gamma \in [T_{\text{poly}}]_{\text{div}} \), and let \( f_1, \ldots, f_m \) be functions on \( \mathbb{R}^d \). We are going to write a formula for function \( \Phi \) on \( \mathbb{R}^d \):

\[
\Phi = \tilde{C}_\Gamma(\eta_1 \otimes \ldots \otimes \eta_n)(f_1 \otimes \ldots \otimes f_m). 
\]

The formula for \( \Phi \) is the sum over all configurations of indices running from 1 to \( d \), labeled by \( \overline{E}_\Gamma \) where \( \overline{E}_\Gamma \) is the set of usual (not dashed) edges of the graph \( \Gamma \):

\[
\Phi = \sum_{I : \overline{E}_\Gamma \{1, \ldots, d\} \backslash I} \Phi_I \tag{36} \]

where \( \Phi_I \) is the product over all \( n+m \) vertices of \( \Gamma \) of certain partial derivatives of functions \( f_j \) and of polyvector fields \( \gamma_i \).

Namely, with each vertex \( i \), \( 1 \leq i \leq n \) of the first type we associate function \( \psi_i \) on \( \mathbb{R}^d \), where

\[
\psi_i = \langle \gamma_i, dx_{I(e_i^1)} \otimes \ldots \otimes dx_{I(e_i^{i+1})} \rangle.
\]
Here we use the identification of polyvector fields with skew-symmetric tensor fields as
\[ \xi_1 \wedge \ldots \wedge \xi_{\ell+1} \mapsto \sum_{\sigma \in \Sigma_{\ell+1}} \text{sgn}(\sigma) \xi_{\sigma_1} \otimes \ldots \otimes \xi_{\sigma_{\ell+1}}. \]

For each vertex \( \overline{j} \) of the second type the associated function is defined \( \psi_{\overline{j}} \) as \( f_{\overline{j}} \).

Now, at each vertex of graph \( \Gamma \) we put a function on \( \mathbb{R}^d \) (i.e. \( \psi_v \) of \( \psi_{\overline{j}} \)). Also, on edges of graph \( \Gamma \) there are indices \( I(e) \) which label coordinates in \( \mathbb{R}^d \). In the next step we put into each vertex \( v \) instead of the function \( \psi_v \) its partial derivative
\[
\left( \prod_{e \in E_{\Gamma}, e = (\ast, v)} \partial_{I(e)} \right) \psi_v,
\]
and take the product over all vertices \( v \) of \( \Gamma \). The result is by definition the summand \( \phi_I \).

**Remark.** The graphs we have considered in Section 2 are exactly graphs \( \Gamma \in G_{1,m,2\ell} \). Indeed, the dashed pair is by the definition a pair of edges \((1, j)\) and \((1, j+1)\) for some \( j \in \{1, \ldots, m\} \), and the graph \( \Gamma \) has \( 2\ell \) dashed edges and \( m-2\ell \) usual edges. Every vertex of the second type is the endpoint of exactly one edge (usual or dashed), because otherwise the corresponding weight \( W_{\Gamma} = 0 \).

3.1.5. **\( L_\infty \)-morphism** \( \tilde{C} : [T_{\text{poly}}^\bullet \otimes \mathbb{C}[u]] \to D_{\text{poly}}^\bullet \)**

**Theorem.** The maps
\[
\tilde{C}_n = \sum_{m \geq 0} \sum_{k \geq 0} \sum_{\Gamma \in G_{n,m,2k}} W_{\Gamma} \cdot \tilde{C}_{\Gamma}
\]
are the Taylor components of an \( L_\infty \)-morphism
\[
\tilde{C} : [T_{\text{poly}}^\bullet \otimes \mathbb{C}[u]] \to D_{\text{poly}}^\bullet,
\]
(or \( \tilde{C} : \{[T_{\text{poly}}^\bullet] \otimes \mathbb{C}[u], 0 \} \to D_{\text{poly}}^\bullet \)).

**Proof.** The proof is analogous to proof of the \( L_\infty \)-Formality conjecture in [K], Section 6. The left-hand side of (34) can be written as a linear combination
\[
\sum_{\Gamma} C_{\Gamma} \tilde{C}_{\Gamma}(\eta_1 \otimes \ldots \otimes \eta_n)(f_1 \otimes \ldots \otimes f_m)
\]
over admissible graphs with \( n \) vertices of the first type, \( m \) vertices of the second type, and \( 2n + m - 3 \) edges (usual and dashed).

Coefficients \( C_{\Gamma} \) in (38) are equal to quadratic-linear combinations of the weights \( W_{\Gamma}, \Gamma' \in G_{n,m,\ast} \).

We want to check that \( C_{\Gamma} \) vanishes for each \( \Gamma \).
The idea is to identify $C_\Gamma$ with the integral over the boundary $\partial C_{n,m}$:

$$
\int_{\partial C_{n,m}} \bigwedge_{e \in E_{\Gamma}} d\phi_e = \int_{C_{n,m}} d \left( \bigwedge_{e \in E_{\Gamma}} d\phi_e \right) = 0. \tag{39}
$$

We have:

$$
0 = \int_{\partial C_{n,m}} \bigwedge_{e \in E_{\Gamma}} d\phi_e = \sum_S \int_{\partial_S C_{n,m}} \bigwedge_{e \in E_{\Gamma}} d\phi_e + \sum_{S,S'} \int_{\partial_{S,S'} C_{n,m}} \bigwedge_{e \in E_{\Gamma}} d\phi_e. \tag{40}
$$

The first summand in the r.h.s. of (40) does not vanish only for $\# S = 2$ (see [K], Section 6.4.1), and this case corresponds to the summands in (34) with Schouten-Nijenhuis bracket. The second summand in the r.h.s. of (40), corresponds to summands in (34) with the Hochschild coboundary and with the Gerstenhaber bracket (see [K], Section 6.4.2). The only new thing is the dashed pairs.

3.1.5.1.

When a vertex of the first type $v \in S$, in the boundary component $\partial_S S' C_{n,m}$ the both endpoints of all the dashed pairs, starting at the vertex $v$, lie in $S'$; otherwise, the corresponding integral vanishes. The situation, shown on Figure 5, corresponds to zero integral.

3.1.5.2. When two points of the first type with $k_1$ and $k_2$ dashed pairs move close to each other (case (S1) with $\# S = 2$) we obtain a vertex with $k_1 + k_2$ dashed pairs. The same final graph is corresponded to $(k_1 + k_2)!$ graphs (i.e. we can select any $k_1$ dashed pairs from $k_1 + k_2$ dashed pairs as dashed pairs of the first vertex from two vertices which move close to each other). This is the cause of the appearance of the product $\prod_{\ell=1}^{n} (k_\ell)!$ in the formula (35) for weight $W_\Gamma$. 

![Figure 5](image-url)
3.2. The cyclic \(L_\infty\)-morphism

We denote by \(\sum\) the operator \(\frac{1}{i+2} \cdot \sum: \mathcal{D}^{i}_{\text{poly}} \to [\mathcal{D}^{i}_{\text{poly}}]_{\text{cycl}}, i \geq -1.\)

**Conjecture 1.** The composition \(C = \sum \circ \tilde{C}\) defines an \(L_\infty\)-morphism

\[ C : \{T_{\text{poly}}^{\bullet} \otimes \mathbb{C} [u], d_{\text{div}}\} \to \{[\mathcal{D}^{\bullet}_{\text{poly}}]_{\text{cycl}}, d_{\text{Hoch}}\}. \]

**Conjecture 2.** The composition \(C = \sum \circ \tilde{C}\) defines an \(L_\infty\)-morphism

\[ C : \{T_{\text{poly}}^{\bullet} \otimes \mathbb{C} [u], d_{\text{div}}\} \to \{[\mathcal{D}^{\bullet}_{\text{poly}}]_{\text{cycl}}, d_{\text{Hoch}}\}. \]

Conjecture 2 is the cyclic Formality conjecture. We have proved (34) for \(n = 1\) and \(C = \sum \circ \tilde{C}\) in Section 2. Let us note that it follows from Conjecture 2 and Theorem 2.3 that the \(L_\infty\)-morphism \(C\) is in fact an \(L_\infty\)-quasiisomorphism.

4. Globalization

**Corollary.** Assuming Conjecture 2 the following is true. Let \(M\) be a smooth manifold, \(\Omega\) be a volume form on \(M\). Then there exists an \(L_\infty\)-quasiisomorphism

\[ C_M : \{T_{\text{poly}}^{\bullet}(M) \otimes \mathbb{C} [u], d_{\text{div}}\} \to \{[\mathcal{D}^{\bullet}_{\text{poly}}(M)]_{\text{cycl}}, d_{\text{Hoch}}\}. \]

The proof is analogous to the proof of the globalization of the \(L_\infty\)-Formality morphism ([K], Section 7), and we omit it here.

5. Applications

Here we consider first applications of the previous results to the deformation quantization.

5.1. Maurer-Cartan equation

Let \(g^{\bullet}\) be a dg Lie algebra.

**Definition.** An element \(\gamma \in g^1\) satisfies the Maurer-Cartan equation iff

\[ d\gamma + \frac{1}{2}[\gamma, \gamma] = 0.\]

**Lemma.** Let \(g^{\bullet}_1, g^{\bullet}_2\) be two dg Lie algebras, and let \(F : g^{\bullet}_1 \to g^{\bullet}_2\) be an \(L_\infty\)-morphism. Let \(\gamma \in g^1_1\) satisfies the Maurer-Cartan equation in \(g^{\bullet}_1\). Then

\[ F_1(\gamma) + \frac{1}{2} F_2(\gamma, \gamma) + \frac{1}{6} F_3(\gamma, \gamma, \gamma) + \ldots + \frac{1}{n!} F_n(\gamma, \ldots, \gamma) + \ldots = F(\gamma) \quad (41)\]

also satisfies the Maurer-Cartan equation in \(g^{\bullet}_2\): \(d(F(\gamma)) + \frac{1}{2} [F(\gamma), F(\gamma)] = 0.\)

**Example.** Let \(U : T_{\text{poly}}^{\bullet} \to \mathcal{D}^{\bullet}_{\text{poly}}\) be an \(L_\infty\)-quasiisomorphism of M. Kontsevich [K]. A solution of the Maurer-Cartan equation in \(T_{\text{poly}}^{\bullet}\) is a bivector field \(\gamma\)
such that \([\gamma, \gamma] = 0\). Formula (41) produces a solution of the Maurer-Cartan equation in \(D^\bullet\text{poly}\).

**Lemma.** Let \(A = C^\infty(\mathbb{R}^d)\). An element \(\varphi \in D^1_{\text{poly}}(\mathbb{R}^d) \subset \text{Hom}_\mathbb{C}(A^\otimes 2, A)\) satisfies the Maurer-Cartan equation iff \(f \ast g = f \cdot g + \varphi(f \otimes g)\) defines an associative star-product.

This Lemma and formula (41) produce a deformation quantization of the Poisson structure on \(\mathbb{R}^d\), given by the bivector field \(\gamma\).

### 5.2. Cyclically-invariant deformation quantization

A solution of the Maurer-Cartan equation in the \(dg\) Lie algebra \(\{T^\bullet_{\text{poly}} \otimes \mathbb{C}[u], d_{\text{div}}\}\) is a bivector field \(\gamma\) such that:

1. \([\gamma, \gamma] = 0\) \hspace{1cm} (42)
2. \(\text{div}(\gamma) = 0\) \hspace{1cm} (43)

The cyclic \(L_\infty\)-Formality morphism

\[
\mathcal{C} : \{T^\bullet_{\text{poly}} \otimes \mathbb{C}[u], d_{\text{div}}\} \to \{[D^\bullet_{\text{poly}}]_{\text{cycl}}, d_{\text{Hoch}}\}
\]

given in Conjecture 2 of Section 3.2 produces by formula (41) a solution \(\psi\) of the Maurer-Cartan equation in \([D^\bullet_{\text{poly}}]_{\text{cycl}}\), i.e. an element \(\psi : A^\otimes 2 \to A\) such that

1. \(f \ast g = f \cdot g + \psi(f \otimes g)\) is an associative star-product on \(\mathbb{R}^d\) \hspace{1cm} (44)
2. \(\int_{\mathbb{R}^d} (f \ast g) \cdot h \cdot \Omega\) is invariant with respect to the cyclic permutation of \((f, g, h)\) \hspace{1cm} (45)

for any functions \(f, g, h \in C^\infty(\mathbb{R}^d)\) with compact support.

Moreover, the fact that \(\tilde{\mathcal{C}}\) is an \(L_\infty\)-quasiisomorphism (Section 2) coupled with standard deformation theory [K] allows us to deduce the following statement:

**Corollary.** Assuming Conjecture 2 the following is true. Let \(A\) be the set of all Poisson structures on \(\mathbb{R}^d\) satisfying (43) modulo diffeomorphisms of \(\mathbb{R}^d\) generating by vector fields \(\xi\) such that \([\gamma, \xi] = 0\) and \(\text{div} \xi = 0\). Let \(B\) be the set of all star-products on \(\mathbb{R}^d\)

\[
f \ast g = f \cdot g + \hbar B_1(f, g) + \hbar^2 B_2(f, g) + \ldots
\]

which satisfy (45) for any three functions \(f, g, h\) with compact support modulo transformations

\[
f \ast g \to T(T^{-1}(f) \ast T^{-1}(g))
\]

where \(T(f) = f + \hbar T_1(f) + \hbar^2 T_2(f) + \ldots\) and \(\int_{\mathbb{R}^d} T(f) \cdot g \cdot \Omega = \int_{\mathbb{R}^d} f \cdot T^{-1}(g) \cdot \Omega\) for any two functions \(f, g\) on \(\mathbb{R}^d\) with compact support.

Then the sets \(A\) and \(B\) are canonically isomorphic, and the isomorphism is given by formula (41).
5.3. After the globalization of the cyclic $L_\infty$-Formality conjecture (Section 4) one can deduce from Conjecture 2 the statement analogous to Corollary 5.2 in the case of arbitrary $C^\infty$-manifolds.

Example. Let $M$ be a symplectic manifold of dimension $2d$ with the symplectic form $\omega$. Then there exists a deformation quantization of the algebra of smooth functions on this manifold such that for any 3 functions $f, g, h$ with compact support one has:

$$\int_{(M)} (f * g) \cdot h \cdot \omega^d = \int_{(M)} (g * h) \cdot f \cdot \omega^d.$$

When we put $h = 1$ we obtain

$$\int_{(M)} f * g \cdot \omega^d = \int_{(M)} f \cdot g \cdot \omega^d.$$

This formula means, in particular, that the functional $\int_{(M)} f \cdot \omega^d$ is a trace functional on the deformed algebra. Such a star-products on the algebra of functions on a symplectic manifold were called closed star-products in [CFS].

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