Exotic subfactors of finite depth with Jones indices \( (5 + \sqrt{13})/2 \) and \( (5 + \sqrt{17})/2 \)

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Abstract

We prove existence of subfactors of finite depth of the hyperfinite II\(_1\) factor with indices \( (5 + \sqrt{13})/2 = 4.302 \cdots \) and \( (5 + \sqrt{17})/2 = 4.561 \cdots \). The existence of the former was announced by the second named author in 1993 and that of the latter has been conjectured since then. These are the only known subfactors with finite depth which do not arise from classical groups, quantum groups or rational conformal field theory.

1 Introduction

In the theory of operator algebras, subfactor theory has been developing dynamically, involving various fields in mathematics and mathematical physics since its foundation by V. F. R. Jones in 1983 [J]. Above all, the classification of subfactors is one of the most important topics in the theory. In the celebrated Jones index theory [J], Jones introduced the Jones index for subfactors of type II\(_1\) as an invariant. Later, he also introduced a principal graph and a dual principal graph as finer invariants of subfactors. Since Jones proved in the middle of 1980’s that subfactors with index less than 4 have one of the Dynkin diagrams as their (dual) principal graphs, the classification of the hyperfinite II\(_1\) subfactors, has been studied by A. Ocneanu and S. Popa, and also by M. Izumi, Y. Kawahigashi, and a number of other mathematicians.

In this process, Ocneanu’s paragroup theory [O1] has been quite effective. He penetrated the algebraic, or rather combinatorial, nature of subfactors and constructed a paragroup from a subfactor of type II\(_1\). A paragroup is a set of data consisting of four graphs made of (dual) principal graph and assignment of complex numbers to “cells” arising from four graphs, called a connection. Thanks to the “generating property” for subfactors of finite depth proved by Popa in [P1], it has turned out that the correspondence between paragroups and subfactors of the hyperfinite II\(_1\) factor

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with finite index and finite depth is bijective, therefore the classification of hyperfinite II_1 subfactors with finite index and finite depth is reduced to that of paragroups. By checking the flatness condition for the connections on the Dynkin diagrams, Ocneanu has announced in [O1] that subfactors with index less than 4 are completely classified by the Dynkin diagrams A_n, D_{2n}, E_6, and E_8. (See also [BN], [I1], [I2], [K], [SV].) After that, Popa ([P2]) extended the correspondence between paragroups and subfactors of the hyperfinite II_1 factor to the strongly amenable case, and gave a classification of subfactors with indices equal to 4. (In all the above mentioned cases, the dual principal graph of a subfactor is the same as the principal graph. See also [IK].) We refer readers to [EK], [GHJ] for algebraic aspects of a general theory of subfactors.

The second named author then tried to find subfactors with index a little bit beyond 4. Some subfactors with index larger than 4 had been already constructed from other mathematical objects. For example, we can construct a subfactor from an arbitrary finite group by a crossed product with an outer action, and this subfactor has an index equal to the order of the original finite group. Trivially, the index is at least 5 if it is larger than 4. We also have subfactors constructed from quantum groups \( U_q(\mathfrak{sl}(n)) \), \( q = e^{2\pi i/k} \) with index \( \frac{\sin^2(\frac{n\pi}{k})}{\sin^2(\frac{\pi}{k})} \) as in [W] and these index values do not fall in the interval (4,5). Unfortunately or naturally, these subfactors do not contain more information about the algebraic structure than the original mathematical objects themselves such as groups or quantum groups. "Does there exist any subfactor not arising from (quantum) groups ?" If it is the case, we have a subfactor as a really new object producing new mathematical structures. We expect that subfactors with index slightly larger than 4 would be indeed those with exotic nature and they do not to arise from other mathematical objects. The second named author gave in 1993 a list of possible candidates of graphs which might be realized as (dual) principal graphs of subfactors with index in \((4,3+\sqrt{3}) = (4,4.732\cdots)\) in [H]. We see four candidates of pairs of graphs, including two pairs with parameters, in §7 of [H]. At the same time, the second named author announced a proof of existence of the subfactor with index \((5+\sqrt{13})/2\) for the case \( n = 3 \) of (2) in the list in [H], but the proof has not been published until now. Ever since, nothing had been known for the other cases for some years, until D. Bisch recently proved that a subfactor with (dual) principal graph (4) in §7 of [H] does not exist [E].

About case (3) in §7 of [H] as in Figure 2, as well as the case \( n = 3 \) of (2), we can easily determine a bimunitary connection uniquely on the four graphs consisting of the graphs (3), and we thus have a hyperfinite II_1 subfactor with index \((5+\sqrt{17})/2\) constructed from the connection by commuting square as in [S]. The problem is whether this subfactor has (3) as (dual) principal graphs or not. This amounts to verifying the flatness condition of the connection. In 1996, K. Ikeda made a numerical check of the flatness of this connection by approximate computations on a computer in [IK] and showed that the graphs (3) are very "likely" to exist as (dual) principal graphs. (He also made a numerical verification of the flatness for the case \( n = 7 \) of (2) in §7 of [H].)
In this paper, we will give the proof of the existence for the case of index \((5 + \sqrt{13})/2\) previously announced by the second named author, and give the proof of the existence for the case of index \((5 + \sqrt{17})/2\). The proof in the latter case was recently obtained by computations of the first named author based on a strategy of the second named author. Our main result in this paper is as follows.

**Theorem 1** \(((5 + \sqrt{13})/2\) case\)

The two graphs in Figure 1 are realized as a pair of (dual) principal graphs of a subfactor with index equal to \(\frac{5+\sqrt{13}}{2}\) of the hyperfinite II\(_1\) factor.

**Theorem 2** \(((5 + \sqrt{17})/2\) case\)

The two graphs in Figure 2 are realized as a pair of (dual) principal graphs of a subfactor with index equal to \(\frac{5+\sqrt{17}}{2}\) of the hyperfinite II\(_1\) factor.

Figure 1: The case \(n = 3\) of the pair of graphs (2) in the list of Haagerup

In Section 2, we will give two key lemmas to prove our two main results respectively. In Section 3, we will give a construction of *generalized open string bimodules* which is a generalization of Ocneanu’s open string bimodules in [D1], [S], and we will give a correspondence between bimodules and general biunitary connections on finite graphs. In Sections 4 and 5, we will prove our two main theorems respectively.

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2 Key lemmas to the main results

In this section, we will give the key lemmas which have been proved by the second named author.

First of all, we will explain the motivation to the lemmas. Proofs of our main theorems presented in Section 1 are reduced to verifying “flatness” of the biunitary connections which exist on the four graphs made of the pairs of the graphs in Figure 1, 2 respectively. However, it is well known that, to verify flatness exactly is almost impossible, except for some easy cases such as the biunitary connections arising from the subfactors of crossed products of finite groups ([EK, 10.6]). So far, in the history of classification of subfactors, several methods have been introduced to prove flatness/nonflatness of biunitary connections. Finding inconsistency of the fusion rule on the graph of a given biunitary connection has been sometimes effective to prove nonflatness, e.g. $D_{2n+1}, E_7$ ([I1, [SV], ...). On the other hand, since consistency of fusion rule does never mean flatness of given biunitary connection, several ideas have been introduced to prove flatness ([EK1, [I2, [IK, [K] ...). The second named author, however, inspected the fusion rules of the upper graphs in Figures 1 and 2 and noticed that if we can construct bimodules satisfying a part of the fusion rule, we can conclude that there exists a subfactor having the desired principal graph. These are the lemmas which we explain in this section. Now we introduce the notation used in the lemmas.

**Definition 1** Let $N$ and $M$ be $II_1$ factors and $X = X_N = X_M$ be an $N$-$M$ bimodule (see [EK]). We denote by $R_X(M)$ the right action of $M$ on $X$, and by $L_X(N)$ the left action of $N$. We have the subfactor $R_X(M)' \supset L_X(N)$. We denote its Jones index by $[X]$. We define the principal graph of the bimodule $X$ as that of the subfactor.
\[ R_X(M)' \supset L_X(N). \]

**Definition 2** For bimodules \( X \) and \( Y \) with common coefficient algebras, we define
\[
\langle X, Y \rangle = \dim \text{Hom}(X, Y).
\]

A formal \( \mathbb{Z} \)-linear combination \( Y \) of bimodules (of finite index) will be called positive if it is an actual bimodule, i.e. of \( \langle Y, Z \rangle \geq 0 \) for any irreducible bimodule \( Z \) which appears in the direct sum decomposition of \( X \).

When \( Y \cong X \oplus Z \) for some positive bimodule \( Z \), we write \( X \ll Y \).

Hereafter we use the expression as follows, so far as it does not cause misunderstanding.

\[
\begin{align*}
1_N &= N N_N, \\
2X &= X \oplus X, \\
XY &= X \otimes_N Y, \\
X^2 &= X \otimes_N X,
\end{align*}
\]
where \( N \) is a \( \text{II}_1 \) factor and \( X \) and \( Y \) are suitable bimodules.

**Lemma 1** Let \( X = N X_M \) be a bimodule with finite Jones index larger than or equal to four. Then,

1) \( XX - 1_N \) and \( (XX)^2 - 3XX \oplus 1_N \) are positive \( N-N \) bimodules.

2) \( XXXX - 2X \) and \( (XX)^2 \otimes_N X - 4XXX \oplus 3X \) are positive \( N-M \) bimodules.

**Proof**

Let \( \mathcal{G} \) be the principal graph of \( X \). We set
\[
\mathcal{G}^{(0)}_{\text{even}} := \text{set of all irreducible components of } 1_N, \ (XX)^n, \quad n = 1, 2, \ldots,
\]

\[
\mathcal{G}^{(0)}_{\text{odd}} := \text{set of all irreducible components of } X, \ (XX)^n X, \quad n = 1, 2, \ldots,
\]
where \( \mathcal{G}^{(0)}_{\text{even}} \) (resp. \( \mathcal{G}^{(0)}_{\text{odd}} \)) means the even (resp. odd) vertices of \( \mathcal{G} \), and \( G = (G_{Y,Z})_{Y \in \mathcal{G}^{(0)}_{\text{even}}, Z \in \mathcal{G}^{(0)}_{\text{odd}}} \) to be the incidence matrix for \( \mathcal{G} \), i.e.,
\[
G_{Y,Z} = \langle YX, Z \rangle, \quad Y \in \mathcal{G}^{(0)}_{\text{even}}.
\]

Since \( 4 \leq [X] < \infty \), we have \( 2 \leq ||G|| < \infty \). Put
\[
\Delta = \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix},
\]
then $\Delta$ is the adjacency matrix of $G^{(0)} = G^{(0)}_{\text{even}} \cup G^{(0)}_{\text{odd}}$ and $||\Delta|| \geq ||G|| \geq 2$. Let $P_0, P_1, P_2, \ldots$ be the sequence of the polynomials given by

$$P_0(x) = 1, \quad P_1(x) = x, \ldots, P_{n+1}(x) = P_n(x)x - P_{n-1}(x).$$

Then, by [HW], all of $P_2(\Delta), P_3(\Delta), \ldots$ have non-negative entries. For $n = 2, 3, 4, 5$, we get, in particular, that

$$GG^t - 1, GG^tG - 2G, (GG^t)^2 - 3GG^t, \text{ and } (GG^t)^2G - 4GG^tG + 3G$$

have non-negative entries. Hence, for $W \in G^{(0)}_{\text{even}}$,

$$\langle XX - 1_N, W \rangle = (GG^t - 1)_{1_N, W} \geq 0,$$

$$\langle (XX)^2 - 3XX \oplus 1_N, W \rangle = ((GG^t)^2 - 3GG^t + 1)_{1_N, W} \geq 0,$$

namely, $XX - 1_N$ and $(XX)^2 - 3XX \oplus 1_N$ are positive $N$-$N$ bimodules. The same argument (with $W \in G^{(0)}_{\text{odd}}$) shows that $XX - 2X$ and $(XX)^2X - 4XX \oplus 3X$ are positive $N$-$M$ bimodules.

q.e.d.

### 2.1 Key lemma for the case of index $(5 + \sqrt{13})/2$

In this subsection, we present the key lemma given by the second named author to which the construction of the finite depth subfactor with index $(5 + \sqrt{13})/2$ is reduced.

**Lemma 2** Let $M$ and $N$ be II$_1$ factors. Assume the following:

1) We have an $N$-$M$ bimodule $X = NX_M$ of index $(5 + \sqrt{13})/2$,

2) We have an $N$-$N$ bimodule $S = NS_N$ of index 1 satisfying $S^3 \cong 1_N$ and $S \not\cong 1_N$,

i.e. $S$ is given by an automorphism of $N$ of outer period 3,

3) The six bimodules

$$1_N, S, S^2, XX - 1_N, S(XX - 1_N), S^2(XX - 1_N)$$

are irreducible and mutually inequivalent.

4) The four bimodules

$$X, SX, S^2X, XX - 2X$$

are irreducible and mutually inequivalent.

5) (the most important assumption)

$$S(XX - 1_N) \cong (XX - 1_N)S^2.$$

Then the principal graph of $X$ and bimodules corresponding to the vertices on the graph are as follows;
2 Key lemmas to the main results

\[ \begin{align*}
\mathbf{1}_N & \quad X \quad X \mathbf{1}_N \\
\ast & \quad a \quad b \quad \mathbf{1}_N \\
S X & \quad S(X \mathbf{1}_N)
\end{align*} \]

**Remark**

By lemma 1, all the above bimodules are well-defined (i.e. positive in the sense of Def. 2).

**Proof**

We have

\[
\langle (X \mathbf{1}_N - X^2)X, X \mathbf{1}_N - 1 \rangle \\
= \langle X \mathbf{1}_N - X^2, (X \mathbf{1}_N - 1)X \rangle \quad \text{(by Frobenius reciprocity)} \\
= \langle X \mathbf{1}_N - X^2, X \mathbf{1}_N - X^2 \rangle + \langle X \mathbf{1}_N - X^2, X \rangle \\
= 1
\]

because \( X \) and \( X \mathbf{1}_N - X^2 \) are irreducible and inequivalent. Hence, by irreducibility of \( X \mathbf{1}_N - 1 \), we have

(i) \( X \mathbf{1}_N - 1 \prec (X \mathbf{1}_N - X^2)X \).

We have

(ii) \( (X \mathbf{1}_N - X^2)X \cong (X \mathbf{1}_N - 1)^2 - 1 \).

and 5) says

(iii) \( S(X \mathbf{1}_N - 1) \cong (X \mathbf{1}_N - 1)S^2 \).

Hence

(iv)

\[
S^2(X \mathbf{1}_N - 1) \cong S(X \mathbf{1}_N - 1)S^2 \\
\cong (X \mathbf{1}_N - 1)S^4 \\
\cong (X \mathbf{1}_N - 1)S.
\]

Therefore,

\[
S(X \mathbf{1}_N - 1)^2 \cong (X \mathbf{1}_N - 1)S^2(X \mathbf{1}_N - 1) \cong (X \mathbf{1}_N - 1)^2S
\]

and

\[
S^2(X \mathbf{1}_N - 1)^2 \cong (X \mathbf{1}_N - 1)^2S^2.
\]

Hence by (ii),

\[
S(X \mathbf{1}_N X - 2X)X^2 \cong (X \mathbf{1}_N X - 2X)X^3S \\
\cong (X \mathbf{1}_N X - 2X)X^3.
\]
and similarly
\[ S^2(XXXX - 2X)XS \cong (XXXX - 2X)X. \]

So, by (i),
(v) \[ S(XXXX - 2X)XS^2 \prec (XXXX - 2X)X, \]
(vi) \[ S^2(XXXX - 2X)XS \prec (XXXX - 2X)X, \]
by (iii) and (iv). Hence, by (i), (v), (vi) and 3),
\[ (XX - 1_N), S(XX - 1_N), S^2(XX - 1_N) \]
are mutually inequivalent subbimodules of \((XXXX - 2X)\), i.e.,
\[ (XXXX - 2X)X \cong (XX - 1_N) \oplus S(XX - 1_N) \oplus S^2(XX - 1_N) \oplus Y \]
where \(Y\) is an \(N-N\) bimodule (possibly zero).

Since \(X = _N X_M\) is irreducible, the subfactor \(R_X(M)' \supset L_X(N)\) has the trivial relative commutant, hence extremal (see \([P2]\), p.176). Therefore, the square root of the Jones index of a bimodule \([\cdot]^{1/2}\) is additive and multiplicative on the bimodules expressed in terms of \(X\) and \(X\) (see \([P2]\)). Thus, we have
\[ \sqrt{[(XXXX - 2X)X]} = 3\sqrt{[(XX - 1_N)]} + \sqrt{[Y]}, \]
where the index of a zero bimodule is defined to be 0. Hence with \(\lambda = [X] = \sqrt{\frac{5 + \sqrt{13}}{2}}\), we get
\[ \sqrt{[Y]} = \lambda(\lambda^3 - 2\lambda) - 3(\lambda^2 - 1) \]
\[ = \lambda^4 - 5\lambda^2 + 3 \]
\[ = 0. \]

We must finally prove that
(vii) \[ S(XXXX - 2X) \cong XXXX - 2X. \]
To see this, we compute
\[ \langle S(XXXX - 2X), XXXX - 2X \rangle \]
\[ = \langle S(XXXX - X), XXXX - 2X \rangle - \langle SX, XXXX - 2X \rangle \]
\[ = \langle S(XXX - 1_N), (XXX - 2X)X \rangle - \langle SX, XXXX - 2X \rangle. \]
The first bracket is 1 because \(S(XXX - 1_N)\) is contained in \((XXXX - 2X)X\) with multiplicity 1, and the second bracket is 0 because \(SX\) and \(XXXX - 2X\) are irreducible and inequivalent by 4), hence
\[ \langle S(XXXX - 2X), XXXX - 2X \rangle = 1, \]
thus, the equality of the irreducible bimodules (vii) holds.
From all the above, it follows easily that
(a) $1_N \in \mathcal{G}^{(0)}$,
(b) $\mathcal{G}$ is connected,
(c) Multiplication by $X$ (resp. $\overline{X}$) from the right (resp. left) on any bimodule $U$ in $\mathcal{G}^{(0)}_{\text{odd}}$ (resp. $\mathcal{G}^{(0)}_{\text{even}}$) gives a direct sum of the bimodules in $\mathcal{G}^{(0)}_{\text{even}}$ (resp. $\mathcal{G}^{(0)}_{\text{even}}$) connected to $U$ by edges,

namely, we find that $\mathcal{G}$ is the principal graph of $X$.

q.e.d.

2.2 Key lemma for the case of index $(5 + \sqrt{17})/2$

In this subsection we present the key lemma similar to the previous one for the construction of the finite depth subfactor with index $(5 + \sqrt{17})/2$.

**Lemma 3** Let $M$, $N$ be $II_1$ factors. Assume the following.
1) We have an $N$-$M$ bimodule $X$ of index $\frac{5 + \sqrt{17}}{2}$.
2) We have an $N$-$N$ bimodule $S$ of index 1 satisfying $S^2 \cong 1_N$ and $S \not\cong 1_N$, i.e., $S$ is given by an automorphism of $N$ of outer period 2,
3) The eight $N$-$N$ bimodules $1_N, S, X\overline{X} - 1_N, S(X\overline{X} - 1_N), (X\overline{X} - 1_N)S, S(X\overline{X} - 1_N)S, (X\overline{X})^2 - 3X\overline{X} \oplus 1_N, S((X\overline{X})^2 - 3X\overline{X} \oplus 1_N)$

are irreducible and mutually inequivalent.
4) The six $N$-$M$ bimodules

$X, SX, X\overline{X}X - 2X, S(X\overline{X}X - 2X), (X\overline{X})^2 X - 4X\overline{X}X \oplus 3X, (X\overline{X} - 1_N)SX$

are irreducible and mutually inequivalent.
5) (The most important assumption)

$S(X\overline{X} - 1_N)SX \cong (X\overline{X} - 1_N)SX$.

Then the principal graph of $X$ and the bimodules corresponding to the vertices on the graph are as follows:
where,
\[b \cdots XX - 1_N\]
\[c \cdots XXX - 2X,\]
\[d \cdots (XX)^2 - 3XX \oplus 1_N,\]
\[e \cdots (XX)^2X - 4XXX \oplus 3X,\]
\[f \cdots (XX - 1_N)S(XX - 1_N) - S(XX - 1_N)S,\]
\[\tilde{d} \cdots S((XX)^2 - 3XX \oplus 1_N),\]
\[\tilde{c} \cdots S(XXX - 2X),\]
\[\tilde{b} \cdots S(XX - 1_N)\]

**Remark**
By lemma 1, we know that all the bimodules above except for the “bimodule” corresponding to \(f\) are well-defined (i.e., positive in the sense of Def. 2). The well-definedness of the bimodule at \(f\) will come out of the proof below.

**Proof**

In the proof we will sometimes use formal computations in the \(\mathbb{Z}\)-linear span of the \(N\)-\(M\) bimodules or \(N\)-\(N\) bimodules considered. The symbol \(\langle \cdot, \cdot \rangle\) for computing the dimension of the space of intertwiners can be extended to \(\mathbb{Z}\) bilinear maps and \(\langle Z, Z \rangle = 0\) implies \(Z = 0\) also for these generalized bimodule.

Since \(1_N \vartriangleleft XX\), we have by 5) that both \((XX - 1_N)S\) and \(S(XX - 1_N)S\) are (equivalent to) subbimodules of \((XX - 1_N)SX\).

By 3), \((XX - 1_N)S\) and \(S(XX - 1_N)S\) are two non-equivalent irreducible bimodules. Hence we can write

\[(XX - 1_N)SX \cong (XX - 1_N)S \oplus S(XX - 1_N)S \oplus R\]

where \(R\) is an \(N\)-\(N\) bimodule, and we have

\[R \cong (XX - 1_N)S(XX - 1_N) - S(XX - 1_N)S.\]

Note that \(R \not\cong 0\) because \(R \cong 0\) would imply \([XX - 1_N] = 1\), which is impossible since \(X\) is irreducible and \([X] > 4\).

Next we will show the following.

(i) The bimodule \(R\) is irreducible,
(ii) \(S((XX)^2 - 3XX \oplus 1_N)S \cong (XX)^2 - 3XX \oplus 1_N,\)
(iii) \(S((XX)^2X - 4XXX \oplus 3X) \cong (XX)^2X - 4XXX \oplus 3X.\)

We have

\[
\langle R, R \rangle = \langle (XX - 1_N)S(XX - 1_N), (XX - 1_N)S(XX - 1_N) \rangle - 2\langle S(XX - 1_N)S, (XX - 1_N)S(XX - 1_N) \rangle + \langle S(XX - 1_N)S, S(XX - 1_N)S \rangle
\]

\[= t_1 + t_2 + t_3,\]
where
\[ t_1 = \langle S(X\overline{X} - 1_N)^2 S, (X\overline{X} - 1_N)^2 \rangle, \]
\[ t_2 = \langle S(X\overline{X} - 1_N) S, (X\overline{X} - 1_N) S (X\overline{X} - 1_N) \rangle, \]
\[ t_3 = \langle (X\overline{X} - 1_N), (X\overline{X} - 1_N) \rangle. \]

Note first that \( t_3 = 1 \) because \((X\overline{X} - 1_N)\) is irreducible. Next
\[ t_2 = \langle S(X\overline{X} - 1_N) S, (X\overline{X} - 1_N) S (X\overline{X} - 1_N) \rangle - \langle S(X\overline{X} - 1_N) S, (X\overline{X} - 1_N) S \rangle. \]

The last term is 0 because \( S(X\overline{X} - 1_N) S \) and \((X\overline{X} - 1_N) S \) are irreducible and inequivalent by 3). Hence, using 4) and 5), we get
\[ t_2 = \langle S(X\overline{X} - 1_N) S X, (X\overline{X} - 1_N) S X \rangle = 1. \]

To compute \( t_1 \), set irreducible bimodules \( Y, Z \) as
\[ Y = X\overline{X} - 1_N, \quad Z = (X\overline{X})^2 - 3X\overline{X} \oplus 1_N. \]

Then
\[ t_1 = \langle S(1_N \oplus Y \oplus Z) S, 1_N \oplus Y \oplus Z \rangle = \langle 1_N, 1_N \rangle + \langle SY S, Y \rangle + \langle SZ S, Z \rangle + 2\langle 1_N, Y \rangle + 2\langle 1_N, Z \rangle + 2\langle SY S, Z \rangle. \]

By 3), \( 1_N, Y, SY S, \) and \( Z \) are irreducible and mutually inequivalent. Hence
\[ t_1 = 1 + \langle SZ S, Z \rangle. \]

Altogether, we have shown that
\[ \langle R, R \rangle = (1 + \langle SZ S, Z \rangle) - 2 + 1 = \langle SZ S, Z \rangle. \]

Since \( R \neq 0 \), we have \( \langle R, R \rangle \geq 1 \). Moreover, since \( Z \) is irreducible, so is \( SZ S \). Hence \( \langle SZ S, Z \rangle \leq 1 \). Therefore
\[ \langle R, R \rangle = \langle SZ S, Z \rangle = 1, \]

which shows that \( R \) is irreducible, and using that \( Z \) and \( SZ S \) are irreducible, we also get that \( SZ S \cong Z \). Hence we have verified (i) and (ii).

To prove (iii), put
\[ G = X\overline{X} X - 2X, \quad E = (X\overline{X})^2 X - 4X\overline{X} X \oplus 3X. \]

Note that \( E = ((X\overline{X})^2 - 3X\overline{X} \oplus 1_N) X - (X\overline{X} X \oplus 2X) \), then, by (ii)
\[ E \cong S((X\overline{X})^2 - 3X\overline{X} \oplus 1_N) SX - X\overline{X} X \oplus 2X \]
\[ \cong S(X\overline{X} - 1_N)^2 SX - S(X\overline{X} - 1_N) SX - (X\overline{X} - 1_N) X. \]
Using 5), we have
\[
E \cong S(X\overline{X} - 1_N)S(X\overline{X} - 1_N)SX - (X\overline{X} - 1_N)SX - (X\overline{X} - 1_N)X
\cong S(X\overline{X} - 1_N)SX\overline{X}SX - S(X\overline{X} - 1_N)X - (X\overline{X} - 1_N)SX - (X\overline{X} - 1_N)X.
\]
Hence, again using 5) we get
\[
E \cong (X\overline{X} - 1_N)SX\overline{X}SX - S(X\overline{X} - 1_N)X - (X\overline{X} - 1_N)SX - (X\overline{X} - 1_N)X
\cong (X\overline{X} - 1_N)SX(X\overline{X} - 1_N) - (1_N \oplus S)(X\overline{X} - 1_N)X.
\]
From this expression of \(E\) and 5), we clearly have \(SE \cong E\), which proves (iii).
We next prove (iv) \(RX \cong (X\overline{X} - 1_N)SX \oplus E\), where \(R \cong (X\overline{X} - 1_N)S(X\overline{X} - 1_N) - S(X\overline{X} - 1_N)S\) is irreducible by (i). We put
\[
E' = RX - (X\overline{X} - 1_N)SX.
\]
By 5), we have
\[
E' \cong (X\overline{X} - 1_N)S(X\overline{X} - 1_N)X - 2(X\overline{X} - 1_N)SX
\cong (X\overline{X} - 1_N)S(X\overline{X} - 31N)X.
\]
To prove (iv), we just have to show that \(E \cong E'\), namely
\[
\langle E' - E, E' - E \rangle = 0.
\]
Note that
\[
\langle E' - E, E' - E \rangle = s_1 - 2s_2 + s_3,
\]
where \(s_1 = \langle E', E' \rangle\), \(s_2 = \langle E', E \rangle\), and \(s_3 = \langle E, E \rangle\).
First, \(s_3 = 1\) because \(E\) is irreducible. Next,
\[
s_2 = \langle (X\overline{X} - 1_N)S(X\overline{X} - 31N)X, (X\overline{X} - 1_N)(X\overline{X} - 31N)X \rangle
= \langle S, (X\overline{X} - 1_N)^2(X\overline{X} - 31N)X(X\overline{X} - 31N)(X\overline{X} - 1_N) \rangle
= \langle S(X\overline{X} - 1_N)(X\overline{X} - 31N)X, (X\overline{X} - 1_N)(X\overline{X} - 31N)X \rangle
= \langle SE, E \rangle
= 1 \quad \text{(by (iii))}.
\]
Finally,
\[
s_1 = \langle E', E' \rangle
= \langle S(X\overline{X} - 1_N)^2S, (X\overline{X} - 31N)X(X\overline{X} - 31N) \rangle
= \langle S(1_N \oplus Y \oplus Z), (X\overline{X} - 31N)^2X\overline{X} \rangle.
\]
Here, using \(SYS = S(X\overline{X} - 1_N)S\) and \(SZS \cong Z\) by (ii), we have
\[
s_1 = \langle 1_N \oplus S(X\overline{X} - 1_N)S \oplus Z, (X\overline{X} - 31N)^2X\overline{X} \rangle
= \langle X \oplus S(X\overline{X} - 1_N)SX \oplus ZX, (X\overline{X} - 31N)^2X \rangle
= \langle X \oplus (X\overline{X} - 1_N)SX \oplus ZX, (X\overline{X} - 31N)^2X \rangle.
\]
where we have used 5) again.

We expand $ZX$ and $(X\overline{X} - 31_N)^2X$ in terms of the irreducible bimodules $X$, $G = X\overline{X}X - 2X$, and $E = (X\overline{X})^2X - 4X\overline{X}X \oplus 3X$, and get

$$ZX \cong G \oplus E$$

and

$$(X\overline{X} - 31_N)^2X \cong 2X - 2G \oplus E.$$ 

Hence

$$s_1 = \langle X \oplus (X\overline{X} - 1_N)SX \oplus G \oplus E; 2X - 2G \oplus E \rangle.$$ 

By 4), $X$, $G$, $E$, and $(X\overline{X} - 1_N)SX$ are irreducible and mutually inequivalent, hence

$$s_1 = 2\langle X, X \rangle - 2\langle G, G \rangle + \langle E, E \rangle = 1.$$ 

Altogether,

$$\langle E' - E, E' - E \rangle = s_1 - 2s_2 + s_3 = 1 - 2 + 1 = 0,$$

which proves (iv).

We need to prove one more relation

(v) $E\overline{X} \cong Z \oplus SZ \oplus R$.

To prove (v), note first that $X$, $Z$, and $E$ all correspond to the vertices in $G_{\text{odd}}^{(0)}$, the set of the odd vertices of the principal graph of $X$. (We write $G_{\text{even}}^{(0)}$ for the even vertices.) Hence, $E\overline{X} \in G_{\text{even}}^{(0)}$, and since

$$\langle S, E\overline{X} \rangle = \langle SE, E \rangle = 1,$$

$S$ is an irreducible subbimodule of $E\overline{X}$, so also $S \in G_{\text{even}}^{(0)}$. Therefore, every irreducible $N$- $N$ bimodule or $N$-$M$ bimodule that can be expressed in terms of $X$, $\overline{X}$, and $S$, belong to the principal graph $G$ of $X$. Therefore, by the same argument in the proof of the previous lemma, the square root of the Jones index is additive and multiplicative on the $N$-$N$ bimodules or $N$-$M$ bimodules which can be expressed in terms of $X$, $\overline{X}$, and $S$, because it will occur as a submodule of

$$(X\overline{X})^n, \ n \geq 0, \ \text{or} \ \ (X\overline{X})^nX, \ n \geq 1.$$ 

Since $ZX \cong G \oplus E$, we have $E \prec ZX$ and therefore

(vi) $Z \prec E\overline{X}$.

By (iii), also

(vii) $SZ \prec E\overline{X}$.

Moreover, in the same way, we have

(viii) $R \prec E\overline{X}$ by (iv).

We know that $Z$, $SZ$, and $S$ are irreducible and $Z \not\cong SZ$ by 3). Moreover, by a simple computation using the additivity and multiplicativity of $[\cdot]^{1/2}$, we have

$$[R]^{1/2} = [Z]^{1/2} - 1 = [SZ]^{1/2} - 1.$$
Hence all of $R$, $Z$, $SZ$ are mutually inequivalent. Thus
\[ E \mathbb{X} \cong Z \oplus SZ \oplus R \oplus T, \]
where $T$ is an $N$-$N$ bimodule. By $[X] = (5 + \sqrt{17})/2$, we easily get
\[ [E \mathbb{X}]^{1/2} = [Z]^{1/2} + [SZ]^{1/2} + [R]^{1/2}, \]

hence, $T = 0$.

Putting everything together, we see that conditions (a), (b), (c) in the proof of the previous lemma hold, namely, $\mathcal{G}$ is the principal graph of $X$.

q.e.d.

3 Generalized open string bimodules

In section 2, we have reduced our construction problem to verification of certain fusion rules, but we still have a problem of handling bimodules in a concrete way. For example, we do not know how to represent $X$ or $S$, or how to verify equalities of infinite dimensional bimodules. In this section, we will introduce the item to make full use of the lemmas. Consider a biunitary connection $\alpha$, as in Figure 3, on the four graphs with upper graph $\mathcal{K}$, lower graph $\mathcal{L}$ and the sets of vertices $V_0, \ldots, V_3$. Note that, by the definition of biunitary connection, the graphs $\mathcal{K}$ and $\mathcal{L}$ should be connected, and the vertical graphs are not necessary to be connected. We fix the vertices $*_{\mathcal{K}} \in V_0$ and $*_{\mathcal{L}} \in V_2$. We will now construct the bimodule corresponding to $\alpha$.

First we construct AFD II$_1$ factors from the string algebras
\[
K = \bigcup_{n=1}^{\infty} \text{String}^{(n)}_{*_{\mathcal{K}}} \mathcal{K},
\]
\[
L = \bigcup_{n=1}^{\infty} \text{String}^{(n)}_{*_{\mathcal{L}}} \mathcal{L},
\]
by the GNS construction using the unique trace, where
\[
\text{String}_{*_{\mathcal{G}}}^{(n)} \mathcal{G} = \text{span}\{ (\xi, \eta) | \text{a pair of paths on the graph } \mathcal{G} \}
\]
\[
s(\xi) = s(\eta) = *_{\mathcal{G}}, \quad r(\xi) = r(\eta), \quad |\xi| = |\eta| = n\}.
\]
Here for a path $\zeta$, we denote the initial vertex, the final vertex and the length of the path by $s(\zeta)$, $r(\zeta)$ and $|\zeta|$ respectively. We define its $*$-algebra structure as

$$(\xi, \eta) \cdot (\xi', \eta') = \delta_{\eta, \xi'} (\xi, \eta'),$$

$$(\xi, \eta)^* = (\eta, \xi).$$

Now we have another AFD II$_1$ factor

$$\tilde{L} = \bigcup_{n=0}^{\infty} \text{span} \left\{ \left( \begin{array}{c} \ast_K x \ast_K y \\ x \end{array} \right) \mid \text{a pair of paths, } x \in L^{(0)}, \text{horizontal paths are in } L, \text{length } n. \right\}_{\text{weak}},$$

where $L^{(0)}$ denotes the set of vertices on $L$. We identify elements in $K$ with elements in $\tilde{L}$ by the embedding using connection $\alpha$, and then have an AFD II$_1$ subfactor $K \subset \tilde{L}$. (See [EK], Chapter 11).

Next we construct the $K$-$L$ bimodule corresponding to $\alpha$. Consider a pair of paths as follows:

$$\left( \begin{array}{c} \ast_K x \\ \ast_L y \end{array} \right),$$

here the horizontal part of the left (resp. right) path consists of edges of the graph $K$ (resp. $L$), the vertical edge is from one of the two vertical graphs of the four graphs of the connection $\alpha$, and the paths have a common final vertex. In general, a pair of paths, as above, with a common final vertex, not necessary with a common initial vertex, is called an open string. It was first introduced by Ocneanu in [O1] in more restricted situations. We embed an open string of length $k$ into the linear span of open strings of length $k + 1$ in a similar way to the embedding of string algebras as follows:

$$\sum_{|\xi| = 1} \left( \begin{array}{c} \ast_K x \\ \ast_L y \end{array} \right) = \sum_{\eta' \eta' \xi' | \xi| = 1} \eta' \xi' \left( \begin{array}{c} \ast_K x \\ \ast_L y \end{array} \right),$$

here the square marked with $\alpha$ means the value given by the connection $\alpha$.

We define the vector space spanned by the above open strings with the above embedding as follows.

$$X^\alpha = \bigcup_n \text{span} \{ (\xi, \eta) \mid s(\xi) = \ast_K, s(\eta) = \ast_L, r(\xi) = r(\eta) \}$$

$$= \bigcup \text{span} \{ \left( \begin{array}{c} \ast_K x \\ \ast_L y \end{array} \right) \}. $$
We define an inner product of $\tilde{X}^\alpha$ as the sesqui-linear extension of the following:

\[
\langle (\xi \cdot \zeta, \eta), (\xi', \zeta', \eta') \rangle = \langle \left( \begin{array}{ll}
\mu_K^* (s(\zeta)), & \mu_L^* \eta \\
\mu_K^* (\xi), & \mu_L^* \eta'
\end{array} \right), \left( \begin{array}{ll}
\mu_K^* (\xi'), & \mu_L^* \eta' \\
\mu_K^* (\zeta'), & \mu_L^* \eta
\end{array} \right) \rangle = \mu_K^* (s(\zeta)) \delta_{\xi, \xi'} \cdot \xi, \xi \cdot \eta = \mu_L^* (\eta') \delta_{\eta, \eta'} \cdot \eta, \eta',
\]

where $\xi \cdot \zeta$ denotes the concatenation of $\xi$ and $\zeta$, $\mu_K$ and $\mu_L$ denotes the Perron-Frobenius eigen vector of the graphs, $\text{tr}_K$ is the unique trace on $K$, and $\text{tr}_L$ is as well. We set $(\xi, \xi')$ and $(\eta, \eta')$ are the elements 0 of $\tilde{L}$ and $L$ respectively if the end points of each pair of paths do not coincide.

By this inner product, $\tilde{X}^\alpha$ is regarded as a pre-Hilbert space, and then we complete it and denote the completion by $X^\alpha$.

We have the natural left action of $K$ and the right action of $L$ as follows; for

\[
x = \left( \begin{array}{ll}
\mu_K^* (\xi), & \mu_L^* \eta \\
\mu_K^* (\zeta), & \mu_L^* \eta'
\end{array} \right) \in \tilde{X}^\alpha,
\]

\[
k = \left( \begin{array}{ll}
\mu_K^* (\sigma), & \mu_K^* (\sigma') \\
\mu_L^* (\xi), & \mu_L^* (\zeta)
\end{array} \right) \in K,
\]

\[
l = \left( \begin{array}{ll}
\mu_L^* (\rho), & \mu_L^* (\rho') \\
\mu_L^* (\sigma), & \mu_L^* (\sigma')
\end{array} \right) \in L,
\]

we have

\[
k \cdot x = \sum_{|\xi| = 1} \left( \begin{array}{ll}
\mu_K^* (\sigma), & \mu_K^* (\sigma') \\
\mu_L^* (\xi), & \mu_L^* (\eta)
\end{array} \right) \cdot x
\]

\[
= \sum_{|\xi| = 1} \delta_{\xi', \zeta, \xi} \left( \begin{array}{ll}
\mu_K^* (\sigma), & \mu_K^* (\zeta) \\
\mu_L^* (\eta), & \mu_L^* (\xi)
\end{array} \right),
\]

\[
x \cdot l = \delta_{\eta, \rho} \left( \begin{array}{ll}
\mu_K^* (\xi), & \mu_K^* (\zeta) \\
\mu_L^* (\rho'), & \mu_L^* (\rho)
\end{array} \right).
\]

By the extension of this action, the Hilbert space $X^\alpha$ is considered as a Hilbert $K$-$L$ bimodule $K X_L^\alpha$. Then we have a $K$-$L$ bimodule $K X^\alpha L$ constructed from $\alpha$. (We call this bimodule made of open strings an open string bimodule. This is a generalization of open string bimodules in [O1] and [Sa], which are the bimodules constructed from flat connections.)

We make the correspondence between direct sums, relative tensor products, and the contragredient map of bimodules and "sums", "products", and the renormalization of connections, so that fusion rules on open string bimodules reduced to the operations of connections. ([O3])
First we introduce the sum of two connections. Consider $\alpha$ and $\beta$ as connections on the four graphs with upper graph $K$, lower graph $L$ and sets of vertices $V_0, \ldots, V_3$ as in Figure 3 (The side graphs of $\alpha$ and $\beta$ need not to be identical), then they give rise two $K\text{-}L$ bimodules. We define the sum of the connections as follows:

$$(\alpha + \beta)(m, n) = \begin{cases} 
\alpha(m, n), & \text{if both } m, n \text{ are edges appearing in } \alpha, \\
\beta(m, n), & \text{if both } m, n \text{ are edges appearing in } \beta, \\
0, & \text{otherwise.}
\end{cases}$$

Obviously it satisfies the biunitarity. We denote the bimodule constructed from a connection $\gamma$ by $X^\gamma$. By considering the action of $K$ from the left, it is easy to see that

$$K\cdot X^\alpha_L \oplus K\cdot X^\beta_L = K\cdot X^{\alpha + \beta}_L,$$

thus, we can use the summation of connections instead of the direct sum of bimodules.

Next we define the product of connections ([O3], [Sa]). Consider the connections $\alpha$ and $\beta$, as in Figure 4, which give rise to $K\text{-}L$ bimodule (resp. $L\text{-}M$ bimodule).

![Figure 4](image-url)

Note that $L$ is appears in the both four graphs. Then we can define the product connection $\alpha \beta$ on the four graphs with upper graph $K$, lower graph $M$ and the sets of vertices $V_0, V_1, V_4, V_5$. The side graphs consist of the edges $\{p - q \mid p \in V_0(\text{resp. } V_1), q \in V_4(\text{resp. } V_5)\}$ with multiplicity

$$\#\{p - x - q \mid \text{a path of length 2 from } p \text{ to } q, x \in V_2(\text{resp. } V_3)\}.$$

We have the connection $\alpha \beta$ as follows:

$$(\alpha \beta)(n, m) = (\alpha \beta) \left( \begin{pmatrix} n_1 & k \\ n_2 & m \end{pmatrix} \phi_1 \right) = \sum \alpha(n_1, k) \phi_1 \beta(n_2, m) \phi_2,$$
where \( n_1 \) and \( n_2 \) are edges such that their concatenation \( n_1 \cdot n_2 \) is \( n \), and \( o_1 \) and \( o_2 \) are as well. We observe that this process corresponds to the following process of composing commuting squares of finite dimension.

\[
A \subset B \quad C \subset D \quad A \subset B \\
\cap \quad \cap \quad \cap \quad \cap \quad \cap \quad \cap \\
C \subset D \quad E \subset F \quad E \subset F
\]

where these three squares are finite dimensional commuting squares. We will show that \( K^{X^\alpha} \otimes_{L} X^\beta_M \) is isomorphic to \( K^{X^\alpha\beta}_M \). We define the map \( \varphi \) from \( K^{X^\alpha} \otimes_{L} X^\beta_M \) to \( K^{X^\alpha\beta}_M \) as follows; For

\[
x = \left( \begin{array}{cc}
* \kappa & \xi \\
\zeta & * \eta
\end{array} \right) \in K^{X^\alpha}_L,
\]

\[
y = \left( \begin{array}{cc}
* \zeta & \rho \\
\sigma & * \sigma
\end{array} \right) \in L^{X^\beta}_M,
\]

we define

\[
\varphi(x \otimes_L y) = x \cdot y
\]

\[
= \delta_{\eta, \zeta, \rho} \left( \begin{array}{cc}
* \kappa & \xi \\
\zeta & * \sigma
\end{array} \right) \in K^{X^\alpha\beta}_M.
\]

Since

\[
(x \otimes_L y, x \otimes_L y) = (x(y, y)_L, x) = \text{tr}_M(x^* \cdot (x^* \cdot x)_L),
\]

\[
(x \cdot y, x \cdot y) = \text{tr}_M(y^* \cdot x^* \cdot x \cdot y)
\]

\[
= \text{tr}_M(y^* \cdot (x^* \cdot x)_L) \quad ((x^* \cdot x) \in L)
\]

\[
= \text{tr}_M(x^* \cdot x(y, y)_L) = (x \otimes_L y, x \otimes_L y),
\]

\[
\quad \text{where } x^* \text{ and } y^* \text{ means that we reverse the order of the pairs of paths and also take the complex conjugate of their coefficients, we see that } \varphi \text{ is an isometry, so it is well-defined as a linear map from } K^{X^\alpha} \otimes_{L} X^\beta_M \text{ to } K^{X^\alpha\beta}_M, \text{ and it is also injective. Moreover, it is surjective because, for an element}
\]

\[
x = \left( \begin{array}{cc}
* \kappa & \rho \\
\zeta & * \sigma
\end{array} \right) \in K^{X^\alpha}_M,
\]

where we assume without loss of generality that $x$ is long enough that there is a path connecting $*\mathcal{L}$ and $s(\rho)$,

$$
x = \sum_\xi \delta_\rho, \xi \left( \begin{array}{c} *\mathcal{K} \\ \xi, *\mathcal{M} \end{array} \right)
= \sum_\xi \left( \begin{array}{c} *\mathcal{K} \\ \xi, *\mathcal{L} \end{array} \eta \right) \cdot \left( \begin{array}{c} *\mathcal{L} \\ \rho, *\mathcal{M} \end{array} \right),
= \varphi\left( \left( \begin{array}{c} *\mathcal{K} \\ \xi, *\mathcal{L} \end{array} \eta \right) \otimes \left( \begin{array}{c} *\mathcal{L} \\ \rho, *\mathcal{M} \end{array} \right) \right)
$$

for some $\eta$ with $s(\eta) = *\mathcal{L}$, $r(\eta) = s(\rho)$. Therefore, we have the isomorphism

$$
\mathcal{K}X_\alpha^\alpha \otimes_L X^\beta_M \cong \mathcal{K}X^\alpha_\beta.
$$

Next we prove

$$
\overline{\mathcal{K}X_L^\alpha} = L\mathcal{X}_K^\tilde{\alpha},
$$

here we denote the renormalization of the connection $\alpha$ by $\tilde{\alpha}$.

Take an element

$$
x = \left( \begin{array}{c} *\mathcal{K} \\ \xi, *\mathcal{L} \end{array} \eta \right) \in \mathcal{K}X_L^\alpha.
$$

For $x$, we easily see that its image by the contragredient map is given as

$$
\bar{x} = \left( \begin{array}{c} *\mathcal{L} \\ \eta, *\mathcal{K} \end{array} \zeta \right) \in \overline{\mathcal{K}X_L^\alpha},
$$

here $\tilde{\zeta}$ means the upside down edge of $\zeta$. Since

$$
\bar{x} = \sum_\sigma \left( \begin{array}{c} *\mathcal{K} \\ \xi, *\mathcal{L} \end{array} \eta \sigma \right)
= \sum_{\sigma, \sigma', \xi'} \xi' \alpha' \xi' \left( \begin{array}{c} *\mathcal{K} \\ \xi', *\mathcal{L} \end{array} \eta' \sigma' \right),
= \sum_{\sigma, \sigma', \xi'} \tilde{\xi}' \beta' \xi' \sigma' \left( \begin{array}{c} *\mathcal{K} \\ \xi', *\mathcal{L} \end{array} \eta' \sigma' \right),
$$

$\bar{x}$ is regarded as the element of $L\mathcal{X}_K^\tilde{\alpha}$, thus we have

$$
\overline{\mathcal{K}X_L^\alpha} \cong L\mathcal{X}_K^\tilde{\alpha}.
$$

Now we have a good correspondence between the operations of certain bimodules and those of connections. To complete it, we should check that the construction of bimodules from connections is a one to one correspondence of the equivalent classes.
Theorem 3 Let $\alpha$ and $\beta$ be two connections as below:

\[
\begin{array}{c}
V_0 & K & V_1 & V_0 & K & V_1 \\
S_1 & \alpha & T_1 & S_2 & \beta & T_2 \\
V_2 & L & V_3 & V_2 & L & V_3
\end{array}
\]

then the K-L bimodules $KX^\alpha_L$ and $KX^\beta_L$ are isomorphic if and only if $\alpha$ and $\beta$ are equivalent to each other up to gauge choice for the vertical edges, in particular the pairs $(S_1, T_1)$ and $(S_2, T_2)$ of the vertical graphs must coincide.

Remark

In [O3], the same correspondence of bimodules and equivalent classes of connections has been introduced for limited objects, and there an equivalent class of connections is defined as that of a gauge transform not only by vertical gauges but also horizontal ones. If the horizontal graphs are “trees”, the equivalent class by total gauges is the same as that by vertical gauges, however, for general bimunitary connections, we should limit the gauge choices only to vertical ones.

Proof.

First assume that $\alpha$ and $\beta$ are equivalent up to gauge choice for the vertical edges. Now $\alpha$ and $\beta$ are on the common four graphs, namely $S_1 = S_2 = S$, $T_1 = T_2 = T$. From the assumption, we have two unitary matrices $u_S$, $u_T$ corresponding to the graphs $S$, $T$ respectively, such that

\[
u^*_S \alpha u_T = \beta,
\]

where $\alpha$ and $\beta$ represent the matrices corresponding to the connections. Now we define the isomorphism $\Phi$ from $KX^\alpha_L$ to $KX^\beta_L$ as follows:

\[
x = \left( \begin{array}{c}
\star_K \\
\star_L 
\end{array} \right) \in KX^\alpha_L, \quad |x| = n
\]

\[
\Phi(x) = \begin{cases}
(id^{(n)} \cdot u_S) \left( \begin{array}{c}
\star_K \\
\xi \\
\star_L 
\end{array} \right), & \text{if } n \text{ is even,} \\
(id^{(n)} \cdot u_T) \left( \begin{array}{c}
\star_K \\
\xi \\
\star_L 
\end{array} \right), & \text{if } n \text{ is odd,}
\end{cases}
\]

\[
\in KX^\beta_L,
\]

where $id^{(n)}$ represents the identity of $\text{String}_{(n)}^K\mathcal{K}$, and $id^{(n)} \cdot u_S$ is the concatenation, regarding $u_S$ as an element of $\bigoplus_{p \in V_0} \text{String}_{(p)}^K\mathcal{K}$, and $u_T$ is as well. Note that this map changes only the vertical part of the elements of bimodule. Now we check that $\Phi$ is
a well-defined linear map, i.e., does not depend on the length of the expression of $x$. Here we assume $n$ is even. We have

$$\Phi(x) = \sum_{\eta} u_S^{\eta,\xi} \left( \begin{array}{c} \eta' \\ \eta, \sigma \end{array} \right)$$

$$= \sum_{\eta} u_S^{\eta,\xi} \sum_{\sigma} \left( \begin{array}{c} \eta' \\ \eta, \sigma \end{array} \right)$$

$$= \sum_{\eta} u_S^{\eta,\xi} \sum_{\sigma,\sigma',\eta'} \eta' \beta(\eta', \eta') \left( \begin{array}{c} \eta' \\ \eta, \sigma \end{array} \right),$$

where $u_S^{\eta,\xi}$ denotes the $\eta,\xi$ entry of the matrix $u_S$. On the other hand, we have

$$\Phi(x) = \Phi(\sum_{\sigma} \left( \begin{array}{c} \eta' \\ \eta, \sigma \end{array} \right))$$

$$= \text{id}^{(n+1)} \cdot u_T \sum_{\sigma,\sigma',\eta'} \xi(\sigma') \xi(\eta') \left( \begin{array}{c} \sigma' \\ \eta', \sigma \end{array} \right)$$

$$= \sum_{\sigma,\sigma',\eta'} \xi(\sigma') \xi(\eta') \sum_{\eta'} u_T^{\eta',\eta'} \left( \begin{array}{c} \eta' \\ \eta, \sigma \end{array} \right),$$

By $u_S^{\xi} \alpha u_T = \beta$, the above two expressions of $\Phi(x)$ coincide. When $n$ is odd, it follows from the same argument. Therefore, $\Phi$ is a well-defined linear map.

Here, $\Phi$ is obviously a right $L$-homomorphism, and, since $\text{id} \cdot u_S$ (resp. $\text{id} \cdot u_T$) of any length commutes with the element of $K$ of the same length, $\Phi$ is a left $K$-homomorphism, too. Since $u_S$ and $u_T$ are unitaries, $\Phi$ is an isomorphism. Then, we have $K X_\alpha^\beta \cong K X_\beta^\alpha$.

Next we prove the converse. Assume $K X_\alpha^\beta \cong K X_\beta^\alpha$. Then we have a partial isometry

$$u \in \text{End}(K X_\alpha^\beta \oplus K X_\beta^\alpha) = \text{End}(K X_\alpha^{\alpha+\beta})$$

such that

$$u : K X_\alpha^\alpha \xrightarrow{\sim} K X_\beta^\beta, \quad uu^* + u^*u = \text{id}.$$

Our aim is to prove $S_1 = S_2$, $T_1 = T_2$ and construct a gauge transform between $\alpha$ and $\beta$ from $u$.

Claim 1 Consider a connection $\gamma$ with four graphs as below.

$$\begin{array}{c}
V_0 & \xrightarrow{\gamma} & V_1 \\
S & \gamma & T \\
V_2 & \xrightarrow{\gamma} & V_3
\end{array}$$
and three AFD $II_1$ factors as in the beginning of this section. Then we have

$$\text{End}(KX_L^\gamma) = K' \cap \tilde{L},$$

where the embedding of $K \subset \tilde{L}$ is given by $\gamma$.

**Proof**

First we have

$$\text{End}(KX_L^\gamma) = (\text{the left action of } K \text{ on } X^\gamma)^' \cap (\text{the right action of } L \text{ on } X^\gamma)^'.$$

We have a natural left action of $\tilde{L}$ on $KX_L^\gamma$. Now we prove

$$(\text{the right action of } L \text{ on } X^\gamma)^' = (\text{the left action of } \tilde{L} \text{ on } X^\gamma).$$

(♠)

Obviously we have the inclusion $\subset$, so we prove the equality by comparing dimensions of $X^\gamma$ as modules of both algebras. Take a vertex $x$ on $L$ and consider projections as below;

$$p = \left( \begin{array}{c} \ast \ast \ast \\ \ast \ast \ast \\ \ast \ast \ast \end{array} \right) \in \tilde{L}$$

$$q = \left( \begin{array}{c} \ast \ast \ast \\ \ast \ast \ast \\ \ast \ast \ast \end{array} \right) \in L.$$ 

We see that $p\tilde{L}p$ consists of the strings such as

$$p \cdot \left( \begin{array}{c} x \\ \ast \ast \ast \end{array} \right),$$

where $\cdot$ means the concatenation. Namely, $p\tilde{L}p$ essentially consists of the strings of $L$ with the initial vertex $x$. It is the case for $pX^\gamma q$ and $qLq$ by similar argument, thus we have

$$\dim_{p\tilde{L}p}(pX^\gamma q) = 1.$$ 

On the other hand, we have

$$\dim_{p\tilde{L}p}(pX^\gamma q) = \frac{\text{tr}_{Lq}}{\text{tr}_{\tilde{L}p}} \dim_{\tilde{L}}X^\gamma,$$

then we have

$$\dim_{\tilde{L}}X^\gamma = \frac{\text{tr}_{\tilde{L}p}}{\text{tr}_{Lq}}.$$ 

By the same argument, we have

$$\dim(pX^\gamma q)_{qLq} = 1.$$
and
\[ \dim X^\gamma_L = \frac{\text{tr} L q}{\text{tr} L p}. \]
Thus, we have
\[ \dim \tilde{L} X^\gamma = \frac{1}{\dim X^\gamma_L} = \dim X^\gamma_L, \]
and the equality in (♠) holds.

By applying this claim to \( \alpha + \beta \), we see that the partial isometry \( u \) is in \( K' \cap \tilde{L} \) and the map \( X^\alpha \rightarrow X^\beta \) is given by the natural left action of \( \tilde{L} \) on \( X^\alpha \). To construct the gauge matrices which transfer \( \alpha \) to \( \beta \), we use the compactness argument of Ocneanu. ([O2], [EK, Section 11.4]) We introduce some necessary notions and facts.

**Definition 3** (Flat element, Flat field, Ocneanu [O2], [EK])
Consider a connection \( \gamma \) on the four graphs as in the previous claim, and three AFD II \(_1\) factors \( K, L, \) and \( \tilde{L} \) as at the beginning of this section. Take an element \( \xi \in \bigoplus_{p \in V_0} \text{String}^{(1)}_p S \). It is called a flat element if
\[ \xi \bigg|_{\text{id}^{(2)}} = \frac{\text{id}^{(2)}}{\xi}, \quad l \in \mathbb{N}, \]
under the identification by the connection \( \gamma \), where \( \text{id}^{(2)} \) denotes the string \( \sum_{|\sigma|=2l} (\sigma, \sigma) \) on the graph \( K \) (resp. \( L \)). We use this notation often hereafter under similar conditions.

It is known that, for a flat element \( \xi \), there is the element \( \eta \in \bigoplus_{p \in V_1} \text{String}^{(1)}_p T \) such that
\[ \xi \bigg|_{\text{id}^{(1)}} = \frac{\text{id}^{(1)}}{\eta}, \]
and
\[ \eta \bigg|_{\text{id}^{(2)}} = \frac{\text{id}^{(2)}}{\eta}. \]
by the connection \( \gamma \). We call \( \eta \) a flat element, too. This “couple” of flat elements represents an element of the string algebra with identification by \( \gamma \), namely,
\[ *\kappa \frac{\text{id}^{(k)}}{\xi} = *\kappa \frac{\text{id}^{(l)}}{\eta} \]
for any sufficiently large \( k \): even and \( l \): odd, that is, large enough that the set of the end points of \( \text{id}^{(k)} \) (resp. \( \text{id}^{(l)} \)) coincides with \( V_0 \) (resp. \( V_1 \)). Now we define \( z \) to be a function on \( V_0 \cup V_1 \) such that \( z(p) \in \text{String}^{(1)}_p S \) (resp. \( \text{String}^{(1)}_p T \)) for \( p \in V_0 \) (resp. \( V_1 \)) and \( \bigoplus_{p \in V_0} z(p) = \xi \) (resp. \( \bigoplus_{p \in V_1} z(p) = \eta \)), and call it flat field. Let \( V_n' \) to be a proper subset of \( V_n \), where \( n = 0, 1 \), and put \( \xi_0 = \bigoplus_{p \in V_0'} z(p) \) (resp. \( \eta_0 = \bigoplus_{p \in V_1'} z(p) \)). It is known that
\[ \xi_0 (\text{resp. } \eta_0) = \frac{\text{id}^{(2j)}}{\xi} (\text{resp. } \eta), \]
for sufficiently large \( j \). We call such elements as \( \xi_0 \) and \( \eta_0 \) flat, too, though they are not flat elements by the definition above.
Theorem 4 (Ocneanu [O2, EK])
Let $K \subset \tilde{L}$ be the AFD $II_1$ subfactor constructed from the connection $\gamma$. Then,

$$K' \cap \tilde{L} = \{\text{flat field}\}.$$

The correspondence of elements as follows;
Take a flat field $z$ and let $\xi = \bigoplus_{p \in V_0} z(p)$, then

$$\mathcal{K}^{(2k)}_z \xi \in K' \cap \tilde{L},$$

and conversely, for $x \in K' \cap \tilde{L}$, it turns out that $x$ is written as

$$x = \mathcal{K}_z(\xi) \in \text{String}^{(1)}_x \mathcal{S} \subset \tilde{L}$$

for some flat field $z$.

This theorem is proved by the compactness argument of Ocneanu, see [O2] and [EK]. (Generally, the length of flat field/element can be arbitrary.)

Now we continue the proof of Theorem 3, using the above notions. Let $\gamma = \alpha + \beta$ and $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. By the above theorem, we consider the partial isometry $u \in K' \cap \tilde{L}$ which gives the isometry $X^\alpha \longrightarrow X^\beta$ as a flat field for the connection $\gamma$. Take $p \in V_0$ and $q \in V_2$ so that they are connected in $\mathcal{S}$. Since $uu^* + u^*u = 1$, we have

$$u(p,q)u(p,q)^* + u(p,q)^*u(p,q) = 1$$

in the algebra $\text{String}^{(1)}_{(p,q)} \mathcal{S} = \text{span}\{(\sigma, \rho) \mid |\sigma| = |\rho| = 1, \quad s(\sigma) = s(\rho) = p, r(\sigma) = r(\rho) = q\}$, where $u(p,q) \in \text{String}^{(1)}_{(p,q)} \mathcal{S}$ such that $\bigoplus_{q \in V_2} u(p,q) = u(p)$. Take an element of $X^\alpha$

$$x = \left( \begin{array}{c}
\mathcal{K}^p_z \\
q \xi \end{array} \right), \quad \xi \in \mathcal{S}_1.$$

From the definition of $u$, we have $ux \in X^\beta$, then

$$(id \cdot u(p,q)) \cdot x = \mathcal{K}_z \xi \cdot u(p,q) \cdot \left( \begin{array}{c}
\mathcal{K}^p_z \\
q \xi \end{array} \right)$$

$$= \sum_{\eta} u(p,q)^\eta \xi \left( \begin{array}{c}
\mathcal{K}^p_z \\
q \eta \xi \end{array} \right) \in X^\beta, \quad u(p,q)^\eta \xi \in \mathcal{C}.$$

Note that $\eta \in \mathcal{S}_2$ if $u(p,q)^\eta \xi \neq 0$. Since $u$ gives an isometry of $X^\alpha$ and $X^\beta$,

$$\dim \text{span}\left\{ \left( \begin{array}{c}
\mathcal{K}^p_z \\
q \xi \end{array} \right), \quad \xi \in \mathcal{S}_1 \right\}$$

$$= \dim \text{span}\left\{ \left( \begin{array}{c}
\mathcal{K}^p_z \\
q \eta \xi \end{array} \right), \quad \eta \in \mathcal{S}_2 \right\}$$
for each $\zeta$ and $\varepsilon$. This means

$$\sharp \left\{ \begin{array}{c} p \\ q \end{array} \in S \right\} = \sharp \left\{ \begin{array}{c} p \\ q \end{array} \in S \right\}.$$  

By seeing all the possible pairs of vertices $p$ and $q$, we have

$$S_1 = S_2.$$  

By the same discussion, we have also

$$T_1 = T_2,$$  

then we know that $\alpha$ and $\beta$ are on the same four graphs.

Now we see that $u(p, q)$ gives the gauge matrix for the edges which connect $p$ and $q$. Let $u_S$ and $u_T$ be “stable” flat elements on $S$ and $T$ corresponding to the flat field $u$. Since the isomorphism $x \in X^\alpha \rightarrow u \cdot x \in X^\beta$ is well-defined, from the same deformation as we proved the well-definedness of $\Phi$ in the first half proof of our main statement here,

$$u^*_S \alpha u_T = \beta$$  

follows. Under the identification of $S_1 = S_2$ and $T_1 = T_2$, $u_S$ and $u_T$ are considered as unitary matrices corresponding to the gauge transform action of $\alpha$ and $\beta$. Thus we have

$$\alpha \cong \beta \quad \text{up to vertical gauge choice.}$$  

**Corollary 1** $KX^\alpha_L$ is irreducible if and only if $\alpha$ is indecomposable.

**Proof**

Assume $\alpha$ is decomposable, i.e. there exist gauge unitaries $u_S, u_T$ and connections $\beta, \gamma$ such that

$$u^*_S \alpha u_T = \beta + \gamma.$$  

Then we have

$$KX^u_{L, \alpha} \cong KX^\beta_L \oplus KX^\gamma_L \cong KX^\alpha_L,$$  

namely, $KX^\alpha_L$ is reducible.

Conversely, assume $KX^\alpha_L$ is reducible. then we have bimodules $KY_L$ and $KZ_L$ such that

$$KX^\alpha_L \cong KY_L \oplus KZ_L$$  

and a projection

$$p \in \text{End}(KX^\alpha_L) = K' \cap \tilde{L} \quad \text{with} \quad p : KX^\alpha_L \longrightarrow KY_L.$$  

Along the same argument as in the proof of the previous theorem, we consider $p$ as a flat field and make the projections $p_S$ and $p_T$ which project elements of $KX^\alpha_L$ to $KY_L$. 
at the finite level, and they act as the “projections” of the connection matrix, and we have a “sub connection” of $\alpha$

$$\beta = p_S^* \alpha p_T$$

so that

$$K X^\beta_L \cong K Y_L,$$

thus, $\alpha$ is decomposable.

q.e.d.

**Corollary 2** Let $\gamma$ be a connection as in Claim 1, i.e.,

\[
\begin{array}{cccc}
V_0 & K & V_1 \\
\gamma & & T \\
S & & \\
V_2 & L & V_3
\end{array}
\]

If there exists a vertex $p$ to which only one vertical edge is connected. Then the bimodule $K X^\gamma_L$ is irreducible.

**Proof**

Assume $p \in V_0$ without missing generality. Let $\xi$ be the only one vertical edge in $S$ connected to $p$. Assume $K X^\gamma_L$ is not irreducible. Then, by the above argument, we have connections $\gamma_1$ and $\gamma_2$ with the four graphs

\[
\begin{array}{cccc}
V_0 & K & V_1 \\
\gamma_1 & & T_1 \\
S_1 & & \\
V_2 & L & V_3 \\
V_0 & K & V_1 \\
\gamma_2 & & T_2 \\
S_2 & & \\
V_2 & L & V_3
\end{array}
\]

respectively, so that

$$\gamma \cong \gamma_1 + \gamma_2, \quad S = S_1 \cup S_2, \quad T = T_1 \cup T_2.$$  

$\xi$ should be contained either in $S_1$ or $S_2$. Assume $\xi \in S_2$, then no edge in $S_1$ connects to $p$. This contradicts to the unitarity of $\gamma_1$.

q.e.d.

**Remark**

Corollary 2 is a generalization of Wenzl’s Criterion for irreducibility of subfactors obtained from a periodic sequence of commuting squares (c.f. [W]).

4 Main theorem for the case of $(5 + \sqrt{13})/2$

In this section, we give a proof for our main theorem for the case of index $(5 + \sqrt{13})/2$ due to the second named author.
Theorem 5 A subfactor with principal graph and dual principal graph as in Figure 1 exists.

From the key lemma, we know that the above theorem follows from the next proposition. We define the connection $\sigma$ as

$$\sigma \left( \begin{array}{c} p \\ r \\ \hline q \\ s \end{array} \right) = \delta_{\sigma(p),r} \delta_{\sigma(q),s},$$

where $p, q, r, s$ are the vertices on the upper graph in Figure 1, and we define $\sigma(\cdot)$ as $\sigma(x) = x_\sigma$, $\sigma(x_\sigma) = x_{\sigma^2}$, and $x_{\sigma^3} = x$. Note that, for the vertex $c$ we put $\sigma(c) = c$.

Proposition 1 Let $\alpha$ be the unique connection on the four graphs consisting of the pair of the graphs appearing in Figure 1, and $\sigma$ be the connection defined above. Then, the following hold.

1) The six connections

$$1, \sigma, \sigma^2, (\alpha \tilde{\alpha} - 1), \sigma(\alpha \tilde{\alpha} - 1), \sigma^2(\alpha \tilde{\alpha} - 1)$$

are indecomposable and mutually inequivalent.

2) The four connections

$$\alpha, \sigma \alpha, \sigma^2 \alpha, \alpha \tilde{\alpha} \alpha - 2 \alpha$$

are irreducible and mutually inequivalent.

3) $\sigma(\alpha \tilde{\alpha} - 1) \cong (\alpha \tilde{\alpha} - 1)\sigma^2$.

Proof

The four graphs of the connection $\alpha$ are as in Figure 3.
The Perron-Frobenius weights of the vertices can easily be computed as follows,

\[ \begin{align*}
\mu(*) &= 1, & \mu(a) &= \mu(a_\sigma) = \mu(a_{\sigma^2}) = \lambda, \\
\mu(b) &= \mu(b_\sigma) = \mu(b_{\sigma^2}) = \lambda^2 - 1, & \mu(c) &= \lambda^3 - 2\lambda, \\
\mu(1) &= 1, & \mu(2) &= \lambda^2 - 1, & \mu(3) &= \lambda^2 - 2, & \mu(4) &= \lambda^2,
\end{align*} \]

where \( \lambda = \sqrt{\frac{5+\sqrt{13}}{2}} \). One can check that the Table 1 defines a connection \( \alpha \) on the four graphs (Figure 5) which satisfies Ocneanu’s biunitary conditions, i.e.,

\[ \begin{pmatrix} p' & \eta' \\ \xi & \xi' \end{pmatrix} \]

is a unitary matrix for each fixed \( p, s \), (unitarity)

and

\[ \frac{x \eta' z}{y \eta w} = \sqrt{\frac{\mu(y)\mu(z)}{\mu(x)\mu(w)}} \cdot \frac{y \eta w}{x \eta' z}, \]

(renormalization)

see [O1] and [EK, chapter 10]. We see that such a biunitary connection \( \alpha \) on these four graphs is determined uniquely up to complex conjugate arising from the symmetricity of the graphs, namely it is essentially unique. The connection \( \alpha \) is as in Table 1. Note

\[ (xy, zw) \text{-entry in the table} = \alpha \begin{pmatrix} x & z \\ y & w \end{pmatrix}, \]
where we note that, since all the graphs which consist the four graphs in Figure 3 are “tree”, all the edges are expressed by the both ends. For example, in the Table 4 one can find

\[(*a, a2)\text{-entry} = \alpha \left( \begin{array}{c} * \\ a \\ 2 \end{array} \right) = 1.\]

We also note that blank entries are all 0’s, and

\[|\rho| = |\tau| = 1, \quad (\bar{\tau}^3 = \rho).\]

Now we display the table of the connection \(\tilde{\alpha}\) computed by “renormalization” in Table 1: Connection \(\alpha\)

|     | a1 | a2 | c2 | c3 | c4 | a_\sigma 4 | a_\sigma 4 |
|-----|----|----|----|----|----|-----------|-----------|
| *a  | 1  | 1  |    |    |    |           |           |
| ba  | 1  | \frac{-1}{\sqrt{\lambda^2 - 1}} | \frac{\sqrt{\lambda^2 - 2}}{\lambda^2 - 1} |    |    |           |           |
| bc  | \frac{1}{\sqrt{\lambda^2 - 1}} | \frac{\lambda^2 - 2}{\lambda^2 - 1} | 1  | 1  |    |           |           |
| b_\sigma c | \bar{\rho} | \bar{\tau} | \frac{1}{\sqrt{\lambda^2 - 1}} | \frac{\lambda^2 - 2}{\lambda^2 - 1} | -1 | \bar{\lambda}^2 | \bar{\lambda}^2 - 1 |
| b_\sigma a_\sigma | \rho | \tau | \frac{1}{\sqrt{\lambda^2 - 1}} | \frac{\lambda^2 - 2}{\lambda^2 - 1} | -1 | \bar{\lambda}^2 | \bar{\lambda}^2 - 1 |
| b_\sigma^2 c | \rho | \tau | \frac{1}{\sqrt{\lambda^2 - 1}} | \frac{\lambda^2 - 2}{\lambda^2 - 1} | -1 | \bar{\lambda}^2 | \bar{\lambda}^2 - 1 |
| b_\sigma^2 a_\sigma | \rho | \tau | \frac{1}{\sqrt{\lambda^2 - 1}} | \frac{\lambda^2 - 2}{\lambda^2 - 1} | -1 | \bar{\lambda}^2 | \bar{\lambda}^2 - 1 |
| *_\sigma a_\sigma |   |   |   |   |   |           |           |
| *_\sigma^2 a_\sigma^2 |   |   |   |   |   |           |           |

Table 2 for use of the later computations.

First we check condition 3), namely we prove

\[\sigma(\alpha\tilde{\alpha} - 1) \cong (\alpha\tilde{\alpha} - 1)\sigma^2\]

up to vertical gauge choice. It is enough to show

\[(\alpha\tilde{\alpha} - 1) \cong \sigma(\alpha\tilde{\alpha} - 1)\sigma,\]

so now we will prove this equivalence.
First we compute the connection \(\alpha\tilde{\alpha}\). The four graphs on which the connection \(\alpha\tilde{\alpha}\) exists are as in Figure 3. The vertical graphs are constructed as in Figure 6, where we explain it only by \(\bar{G}\bar{G}'\).
4 Main theorem for the case of \((5 + \sqrt{13})/2\)

| \(a^*\) | \(\frac{1}{\lambda}\) | \(\sqrt{\lambda^2 - 1}\) | \(\frac{-1}{\lambda}\) | \(1\) |
| \(ab\) | \(\sqrt{\lambda^2 - 1}\) | \(\frac{-1}{\lambda}\) | \(1\) |  |
| \(cb\) | 1 | \(\frac{1}{\lambda(\lambda^2 - 2)}\) | \(\frac{1}{\sqrt{3}}\) | \(\frac{\sqrt{\lambda^2 - 1}}{\lambda^2 - 2}\) |
| \(cb_\sigma\) | \(\frac{\lambda^2 - 1}{\lambda(\lambda^2 - 2)}\) \(\rho\) | \(\frac{1}{\sqrt{3}}\) | \(\frac{\lambda^2 - 2}{\lambda^2 - 2}\) | 1 |
| \(cb_{\sigma^2}\) | \(\frac{\lambda^2 - 1}{\lambda(\lambda^2 - 2)}\) \(\overline{\rho}\) | \(\frac{1}{\sqrt{3}}\) | \(\frac{\lambda^2 - 2}{\lambda^2 - 2}\) | 1 |

| \(a_\sigma b_\sigma\) | 1 | -1 |  |
| \(a_\sigma b_{\sigma^2}\) | 1 | -1 |  |
| \(a_{\sigma^2} b_{\sigma^2}\) | 1 | -1 |  |
| \(a_{\sigma^2} b_{\sigma}\) | 1 | -1 |  |

Table 2: Connection \(\tilde{\alpha}\)

Figure 6: construction of the vertical graphs of \(\alpha\tilde{\alpha}\).
Main theorem for the case of \((5 + \sqrt{13})/2\)

To obtain the connection \(\alpha \tilde{\alpha} - 1\), we multiply the entries of the connections \(\alpha\) and \(\tilde{\alpha}\) properly. (We call this sort of computations of the multiplication of the connections “actual” multiplication.), transform it by vertical gauge so that the entries corresponding to the trivial connection \(1\) are 1, and subtract 1. (In Figure 7, the broken lines corresponds to this trivial summand.) Here, in the Table 3, we show the landscape of \(\alpha \tilde{\alpha}\) with 1’s in the entries corresponding to 1.

First we will compute the entries marked \(\odot\) in the Table 3. We assume that \(1 \times 1\) gauge transform unitaries corresponding to single vertical edges which connect different vertices in the graph \(\mathcal{G}\) to be 1 without losing generality, because they are not involved in the trivial connection 1. We compute such entries by “actual” multiplication.

\[
\begin{align*}
\begin{array}{ccc}
* & a & \alpha \\
b & c & \tilde{\alpha}
\end{array}
\begin{array}{c}
\alpha \\
\tilde{\alpha}
\end{array}
& =
\begin{array}{ccc}
* & a & 2 \\
a & b & c
\end{array}
\begin{array}{c}
\alpha \\
\tilde{\alpha}
\end{array}
& = 1 \cdot 1 = 1,
\end{align*}
\]

\[
\begin{align*}
\begin{array}{ccc}
b & c & \alpha \\
a & \alpha & 2
\end{array} & = \lambda \sqrt{\lambda^2 - 2} \cdot \frac{\sqrt{\lambda^2 - 1}}{\lambda} = \frac{\sqrt{\lambda^2 - 2}}{\sqrt{\lambda^2 - 1}},
\end{align*}
\]
Main theorem for the case of $(5 + \sqrt{13})/2$

From here, we only write the result of multiplication.

\[
\begin{align*}
\left[ \begin{array}{c|c|c|c|c}
\alpha^1 & \alpha^2 & \alpha^3 & \alpha^4 \\
\hline
\beta & \beta & \beta & \beta \\
\hline
\end{array} \right] &= \left[ \begin{array}{c|c|c|c|c}
\alpha^1 & \alpha^2 & \alpha^3 & \alpha^4 \\
\hline
\beta & \beta & \beta & \beta \\
\hline
\end{array} \right] = \frac{\rho}{\sqrt{\lambda^2 - 2}},
\end{align*}
\]

\[
\begin{align*}
\left[ \begin{array}{c|c|c|c|c}
\alpha^1 & \alpha^2 & \alpha^3 & \alpha^4 \\
\hline
\beta & \beta & \beta & \beta \\
\hline
\end{array} \right] &= \left[ \begin{array}{c|c|c|c|c}
\alpha^1 & \alpha^2 & \alpha^3 & \alpha^4 \\
\hline
\beta & \beta & \beta & \beta \\
\hline
\end{array} \right] = 1,
\end{align*}
\]

\[
\begin{align*}
\left[ \begin{array}{c|c|c|c|c}
\alpha^1 & \alpha^2 & \alpha^3 & \alpha^4 \\
\hline
\beta & \beta & \beta & \beta \\
\hline
\end{array} \right] &= \left[ \begin{array}{c|c|c|c|c}
\alpha^1 & \alpha^2 & \alpha^3 & \alpha^4 \\
\hline
\beta & \beta & \beta & \beta \\
\hline
\end{array} \right] = \frac{\bar{\rho}}{\sqrt{\lambda^2 - 2}},
\end{align*}
\]

\[
\begin{align*}
\left[ \begin{array}{c|c|c|c|c}
\alpha^1 & \alpha^2 & \alpha^3 & \alpha^4 \\
\hline
\beta & \beta & \beta & \beta \\
\hline
\end{array} \right] &= \left[ \begin{array}{c|c|c|c|c}
\alpha^1 & \alpha^2 & \alpha^3 & \alpha^4 \\
\hline
\beta & \beta & \beta & \beta \\
\hline
\end{array} \right] = 1,
\end{align*}
\]
\[
\begin{align*}
\frac{b_\sigma}{a} \cdot \frac{c}{b} &= \frac{b_\sigma}{c} \cdot \frac{a_\sigma}{b}, \\
\frac{b_\sigma}{a} \cdot \frac{c}{b} &= \frac{b_\sigma}{c} \cdot \frac{\alpha}{b} = \bar{\rho}, \\
\frac{b_\sigma}{a} \cdot \frac{c}{b} &= \frac{b_\sigma}{c} \cdot \frac{\alpha}{b} = \frac{1}{\sqrt{\lambda^2 - 2}}, \\
\frac{b_\sigma}{a} \cdot \frac{c}{b} &= \frac{b_\sigma}{c} \cdot \frac{\alpha}{b} = \frac{\lambda^2 - 4}{\sqrt{\lambda^2 - 2}}, \\
\frac{b_\sigma}{a} \cdot \frac{c}{b} &= \frac{b_\sigma}{c} \cdot \frac{\alpha}{b} = \frac{\lambda^2 - 1}{\sqrt{\lambda^2 - 1}}, \\
\frac{b_\sigma}{a} \cdot \frac{c}{b} &= \frac{b_\sigma}{c} \cdot \frac{\alpha}{b} = \frac{\lambda^2 - 2}{\lambda^2 - 1}, \\
\frac{b_\sigma}{a} \cdot \frac{c}{b} &= \frac{b_\sigma}{c} \cdot \frac{\alpha}{b} = \frac{\lambda^2 - 1}{\sqrt{\lambda^2 - 1}}, \\
\frac{b_\sigma}{a} \cdot \frac{c}{b} &= \frac{b_\sigma}{c} \cdot \frac{\alpha}{b} = \frac{\lambda^2 - 2}{\lambda^2 - 1}, \\
\frac{b_\sigma}{a} \cdot \frac{c}{b} &= \frac{b_\sigma}{c} \cdot \frac{\alpha}{b} = \frac{\lambda^2 - 1}{\sqrt{\lambda^2 - 1}}, \\
\frac{b_\sigma}{a} \cdot \frac{c}{b} &= \frac{b_\sigma}{c} \cdot \frac{\alpha}{b} = \frac{\lambda^2 - 2}{\lambda^2 - 1}, \\
\frac{b_\sigma}{a} \cdot \frac{c}{b} &= \frac{b_\sigma}{c} \cdot \frac{\alpha}{b} = \frac{\lambda^2 - 1}{\sqrt{\lambda^2 - 1}}.
\end{align*}
\]
Next, we will obtain the entries marked $\sigma$. We have two vectors of connection $a\tilde{\alpha}$ concerning to $b-b$ double edges by “actual” multiplication as follows:

\[
\begin{pmatrix}
  b & c \\
  b & \tilde{\alpha}
\end{pmatrix}
\begin{pmatrix}
  b & c \\
  a & 2
\end{pmatrix}
\begin{pmatrix}
  a & \tilde{\alpha} \\
  b & a
\end{pmatrix}
\begin{pmatrix}
  a & 2 \\
  b & a
\end{pmatrix}
= \begin{pmatrix}
  -\frac{\sqrt{\lambda^2-2}}{\lambda^2-1} \\
  \frac{1}{\lambda^2-1}
\end{pmatrix},
\]

\[
\begin{pmatrix}
  b & a \\
  b & \tilde{\alpha}
\end{pmatrix}
\begin{pmatrix}
  b & a \\
  a & 2
\end{pmatrix}
\begin{pmatrix}
  a & \tilde{\alpha} \\
  b & a
\end{pmatrix}
\begin{pmatrix}
  a & 2 \\
  b & a
\end{pmatrix}
= \begin{pmatrix}
  -\frac{1}{\lambda^2-1} \\
  \frac{1}{(\lambda^2-1)\sqrt{\lambda^2-2}}
\end{pmatrix}.
\]

Since these two vectors are proportional, they are transformed to two proportional vectors by a left vertical gauge transform for the double edges $b-b$, i.e., multiplication from the left by an element of $U(2)$. Since we should have 1’s in the $(bb^1, aa^1)$-entry and the $(bb^1, cc^1)$-entry, they can be transformed into the following pair,

\[
\begin{pmatrix}
  0 \\
  \frac{1}{\sqrt{\lambda^2-1}}
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
  0 \\
  \frac{\sqrt{\lambda^2-4}}{\sqrt{\lambda^2-2}}
\end{pmatrix},
\]

then we have $(bb^2, ca) = \frac{1}{\sqrt{\lambda^2-1}}$, $(bb^2, ac) = \frac{\sqrt{\lambda^2-4}}{\sqrt{\lambda^2-2}}$ respectively, where we have omitted “-entry”. The same procedure for the entries with vertical double edges $b_\sigma-b_\sigma$ and $b_\sigma^2-b_\sigma^2$ gives two pairs of vectors as follows.

\[
\begin{pmatrix}
  b_\sigma & a_\sigma \\
  b_\sigma & \tilde{\alpha}
\end{pmatrix}
\begin{pmatrix}
  b_\sigma & a_\sigma \\
  c & 4
\end{pmatrix}
\begin{pmatrix}
  c & \tilde{\alpha} \\
  b_\sigma & c
\end{pmatrix}
\begin{pmatrix}
  c & \tilde{\alpha} \\
  b_\sigma & c
\end{pmatrix}
= \begin{pmatrix}
  \frac{1}{\sqrt{(\lambda^2-1)(\lambda^2-2)}} \\
  -\frac{1}{\sqrt{\lambda^2-1}}
\end{pmatrix},
\]

\[
\begin{pmatrix}
  b_\sigma & a_\sigma \\
  b_\sigma & \tilde{\alpha}
\end{pmatrix}
\begin{pmatrix}
  b_\sigma & a_\sigma \\
  a_\sigma & 4
\end{pmatrix}
\begin{pmatrix}
  a_\sigma & \tilde{\alpha} \\
  b_\sigma & c
\end{pmatrix}
\begin{pmatrix}
  a_\sigma & \tilde{\alpha} \\
  b_\sigma & c
\end{pmatrix}
= \begin{pmatrix}
  -\frac{1}{\sqrt{\lambda^2-1}} \\
  \frac{1}{\sqrt{(\lambda^2-1)(\lambda^2-2)}}
\end{pmatrix}.
\]
4 Main theorem for the case of \((5 + \sqrt{13})/2\)

\[
\begin{align*}
\alpha \begin{pmatrix}
0 & \sqrt{\lambda^2 - 2} & 1 & \sqrt{\lambda^2 + 2} - 1
\end{pmatrix}
\end{align*}
\]
4 Main theorem for the case of \((5 + \sqrt{13})/2\)

| \(aa\) | \(aa^2\) | \(aa^3\) | \(aa^4\) | \(aa^5\) | \(aa^6\) | \(aa^7\) | \(aa^8\) | \(aa^9\) | \(aa^{10}\) | \(aa^{11}\) |
|---|---|---|---|---|---|---|---|---|---|---|
| \(*\mathbf{a}\) | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \(*\mathbf{b}\) | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**Table 4:** Connection \(\alpha \bar{\alpha} \ (\lambda_n = \sqrt{\lambda^2 - n})

Now we will obtain the entries marked \(\diamond\) in Table 3. Denote the vectors of entries of “actual” multiplication \(\alpha \bar{\alpha}\) corresponding to \((0, g??)\) by \(f??\). We use the following data of \(f??\)'s.

\[
f_{bb_\sigma} = \frac{b}{\alpha_\sigma} c = \left( \frac{b}{c} \alpha_\sigma \cdot \frac{c}{2} \hat{\alpha} , \frac{b}{c} \alpha_\sigma \cdot \frac{c}{3} \hat{\alpha} , \frac{b}{c} \alpha_\sigma \cdot \frac{c}{4} \hat{\alpha} \right) c = \left( \frac{\rho}{\lambda(\lambda^2 - 2)} \cdot \frac{\tau}{\sqrt{3}} \cdot \frac{1}{\lambda^2 - 2} \right),
\]

\[
f_{bb_\sigma 2} = \left( \frac{\bar{\rho}}{\lambda(\lambda^2 - 2)} \cdot \frac{\bar{\tau}}{\sqrt{3}} \cdot \frac{1}{\lambda^2 - 2} \right),
\]

\[
f_{b_\sigma b} = \left( \frac{\rho}{\lambda(\lambda^2 - 2)} \cdot \frac{\tau}{\sqrt{3}} \cdot \frac{1}{\lambda^2 - 2} \right),
\]
4 Main theorem for the case of \((5 + \sqrt{13})/2\)

\[
\begin{align*}
\mathbf{f}_{b_a b} &= \left( \frac{(\lambda^2 - 1) \bar{\rho}^2}{\lambda(\lambda^2 - 2)} , \frac{\bar{\tau}^2}{\sqrt{3}} , \frac{1}{(\lambda^2 - 2)\sqrt{\lambda^2 - 1}} \right), \\
\mathbf{f}_{b_a b} &= \left( \frac{\rho}{\lambda(\lambda^2 - 2)} , \frac{\tau}{\sqrt{3}} , \frac{1}{\lambda^2 - 2} \right), \\
\mathbf{f}_{b_a b} &= \left( \frac{(\lambda^2 - 1) \bar{\rho}^2}{\lambda(\lambda^2 - 2)} , \frac{\bar{\tau}^2}{\sqrt{3}} , \frac{1}{(\lambda^2 - 2)\sqrt{\lambda^2 - 1}} \right).
\end{align*}
\]

Note that \(\mathbf{f}_{b b} = \mathbf{f}_{a b}\) and \(\mathbf{f}_{b a} = \mathbf{f}_{b b}\), so they are transformed with keeping equality by the gauge transform of the triple edges \(-c-c\). Therefore, we see only \(\mathbf{f}_{b a}, \mathbf{f}_{b a}, \mathbf{f}_{b a} b\) and \(\mathbf{f}_{b a} b\). We have the following lemma.

**Lemma 4** The three vectors

\[
\begin{align*}
\mathbf{u}_1 &= \left( \frac{1}{\sqrt{3}} , \frac{1}{\lambda} , \frac{1}{\sqrt{\lambda^2 - 2}} \right), \\
\mathbf{u}_2 &= \left( \frac{\sqrt{\lambda^2 - 2}}{3} , \frac{\sqrt{\lambda^2 - 2}}{\lambda\sqrt{3}} , \frac{-\lambda^2 - 3}{\sqrt{3}} \right), \\
\mathbf{u}_3 &= \left( \frac{\lambda^2 - 2}{\lambda\sqrt{3}} , \frac{\lambda^2 - 2}{3}, 0 \right)
\end{align*}
\]

form an orthonormal basis for \(\mathbb{C}^3\) and

\[
\begin{align*}
\mathbf{f}_{b a} b &= \mathbf{f}_{b a} = -\frac{1}{\sqrt{3}} \mathbf{u}_2 + \left( \frac{\sqrt{\lambda^2 - 2}}{2\lambda} - \frac{\lambda^2 - 3}{2\lambda} i \right) \mathbf{u}_3, \\
\mathbf{f}_{b b} a &= \mathbf{f}_{b b} = -\frac{1}{\sqrt{3}} \mathbf{u}_2 + \left( \frac{\sqrt{\lambda^2 - 2}}{2\lambda} + \frac{\lambda^2 - 3}{2\lambda} i \right) \mathbf{u}_3, \\
\mathbf{f}_{b a} b &= -\sqrt{\frac{\lambda^2 - 4}{3}} \mathbf{u}_2 + \left( -\frac{\lambda^2 - 2}{2\sqrt{3}} + \frac{\sqrt{\lambda^2 - 3}}{2} i \right) \mathbf{u}_3, \\
\mathbf{f}_{b a} b &= -\sqrt{\frac{\lambda^2 - 4}{3}} \mathbf{u}_2 + \left( -\frac{\lambda^2 - 2}{2\sqrt{3}} - \frac{\sqrt{\lambda^2 - 3}}{2} i \right) \mathbf{u}_3.
\end{align*}
\]

**Proof**

Checked by elementary, but heavy computations, using \(\lambda^4 - 5\lambda^2 + 3 = 0\).

\(\text{q.e.d.}\)
From Lemma 4, we have \( g_{\gamma} \)'s as the expression of \( f_{\gamma} \)'s by the orthonormal basis \( u_2 \) and \( u_3 \) as follows,

\[
\begin{align*}
  g_{b_\sigma} &= g_{b_2 b_\sigma} = \left( \frac{-1}{\sqrt{3}}, \frac{\sqrt{\lambda^2 - 2}}{2\lambda} + \frac{\lambda^2 - 3}{2\lambda}, \frac{\lambda^2 - 3}{2\lambda} \right), \\
  g_{b_\sigma b} &= g_{b_\sigma b_2} = \left( \frac{-1}{\sqrt{3}}, \frac{\sqrt{\lambda^2 - 2}}{2\lambda} - \frac{\lambda^2 - 3}{2\lambda}, \frac{\lambda^2 - 3}{2\lambda} \right), \\
  g_{b_\sigma b_2} &= \left( -\sqrt{\frac{\lambda^2 - 4}{3}}, -\frac{\lambda^2 - 2}{2\sqrt{3}} + \frac{\sqrt{\lambda^2 - 3}}{2}, \frac{\lambda^2 - 3}{2} \right), \\
  g_{b_\sigma b_2 b_\sigma} &= \left( -\sqrt{\frac{\lambda^2 - 4}{3}}, -\frac{\lambda^2 - 2}{2\sqrt{3}} - \frac{\sqrt{\lambda^2 - 3}}{2}, \frac{\lambda^2 - 3}{2} \right).
\end{align*}
\]

\( g_{bb}, g_{b_\sigma b_\sigma} \) and \( g_{b_\sigma b_2 b_\sigma} \) are uniquely determined so that the matrices

\[
\begin{bmatrix}
  b \\
  \bar{\alpha} \\
  \sigma
\end{bmatrix}_c = \begin{pmatrix}
  \sqrt{\frac{\lambda^2 - 4}{\lambda^2 - 2}} & g_{bb} & \sqrt{\frac{\lambda^2 - 2}{\lambda^2 - 2}} \\
  \sqrt{\frac{\lambda^2 - 2}{\lambda^2 - 2}} & g_{b_\sigma b_\sigma} & g_{b_\sigma b_2 b_\sigma} \\
  \sqrt{\frac{\lambda^2 - 2}{\lambda^2 - 2}} & g_{b_\sigma b_2} & \sqrt{\frac{\lambda^2 - 2}{\lambda^2 - 2}}
\end{pmatrix},
\]

\[
\begin{bmatrix}
  b_\sigma \\
  \bar{\alpha} \\
  \sigma
\end{bmatrix}_c = \begin{pmatrix}
  g_{b_\sigma b} & 1 \\
  g_{b_\sigma b_\sigma} & \sqrt{\frac{\lambda^2 - 2}{\lambda^2 - 2}} \\
  g_{b_\sigma b_2 b_\sigma} & \sqrt{\frac{\lambda^2 - 2}{\lambda^2 - 2}}
\end{pmatrix},
\]

and

\[
\begin{bmatrix}
  b_\sigma b_2 \\
  \bar{\alpha} \\
  \sigma
\end{bmatrix}_c = \begin{pmatrix}
  g_{b_\sigma b_2 b} & \frac{1}{\sqrt{\lambda^2 - 2}} \\
  g_{b_\sigma b_2 b_\sigma} & \sqrt{\frac{\lambda^2 - 2}{\lambda^2 - 2}} \\
  g_{b_\sigma b_2 b_\sigma b_2} & \frac{1}{\sqrt{\lambda^2 - 2}}
\end{pmatrix}
\]

are unitaries, hence we have

\[
\begin{align*}
  g_{bb} &= \left( \frac{-1}{\sqrt{3}}, \frac{\lambda^2 - 2}{\lambda}, \frac{\lambda^2 - 3}{2\lambda} \right), \\
  g_{b_\sigma b_\sigma} &= \left( \frac{\lambda^2 - 3}{\sqrt{3}}, 0 \right), \\
  g_{b_\sigma b_2 b_\sigma} &= \left( \frac{\lambda^2 - 3}{\sqrt{3}}, 0 \right).
\end{align*}
\]

Now, the connection \( (\alpha \bar{\alpha} - 1) \) is as in the Table 4.

Our aim is to show \( (\alpha \bar{\alpha} - 1) \cong \sigma (\alpha \bar{\alpha} - 1) \sigma \). For this purpose, an expression of \( \alpha \bar{\alpha} - 1 \) with symmetry up to \( \sigma \) is useful. We will re-choose another gauge as in Table 5, where \( s = \bar{r}, \lambda_n = \sqrt{\lambda^2 - n} \), and numbers beside the name of edges denote \( 1 \times 1 \) unitaries corresponding to the edges, namely, we have multiplied these numbers to the corresponding rows (resp. columns) in the previous table, and \( g'_{ss} \)'s
Table 5: Connection $\alpha\tilde{\alpha} - 1$ ($\lambda_n = \sqrt{\lambda^2 - n}$)

denote the vectors corresponding to $g'_{**}$’s after being multiplied by suitable gauge numbers respectively. By seeing this table, we easily see that entries other than $g'_{**}$’s are invariant to the transformation of

$$(\alpha\tilde{\alpha} - 1) \rightarrow \sigma(\alpha\tilde{\alpha} - 1)\sigma,$$

which acts on the table as the relabeling $xy \rightarrow \sigma(x)\sigma(y)$. The remaining problem is whether we have a gauge unitary matrix $u_{(c)}^{(c)}$ corresponding to double edges $c-c$ such that

$$g'_{**} \xrightarrow{u_{(c)}^{(c)}} g'_{\sigma(\ast)\sigma(\ast)}$$

or not. We can check by a simple computation that

$$u_{(c)}^{(c)} = \begin{pmatrix}
-\frac{1}{2} - \frac{(\lambda^2 - 2)^{\frac{3}{2}}}{6} & \frac{\lambda^2 - 2}{\lambda\sqrt{3}} \\
\frac{\lambda^2 - 2}{\lambda\sqrt{3}} & -\frac{1}{2} + \frac{(\lambda^2 - 2)^{\frac{3}{2}}}{6}
\end{pmatrix}$$

gives rise to the transformation

$$g'_{bb} \rightarrow g'_{b_{a}b_{a}2} \rightarrow g'_{b_{a}2b_{a}} \rightarrow g'_{bb},$$

$$g'_{ba} \rightarrow g'_{b_{a}2b_{a}} \rightarrow g'_{b_{a}2} \rightarrow g'_{ba}.$$
4 Main theorem for the case of $(5 + \sqrt{13})/2$

|        | 1 | 1 | 1 | s | s | s | s | s | s |
|--------|---|---|---|---|---|---|---|---|---|
| $aa^2$ | $ca$ | $ac$ | $cc^2$ | $ce^3$ | $a_{\sigma}c$ | $a_{\sigma^2}c$ | $ca_{\sigma}$ | $a_{\sigma^2}a_{\sigma}$ | $ca_{\sigma^2}$ | $a_{\sigma}a_{\sigma^2}$ |

| 1 | $sb$ | 1 | 1 |   |   |   |   |   |   |

| 1 | $b^s$ |   |   | $\frac{1}{\lambda_1}$ | $\frac{1}{\lambda_2}$ | $\bar{s}$ | $b_{\sigma}$ | $\sigma$ | $\bar{s}$ |
| 1 | $bb^s$ | $\frac{1}{\lambda_2}$ | $\frac{1}{\lambda_1}$ | $\frac{1}{\lambda_2}$ | $\bar{s}$ | $bb_{\sigma}$ | $s$ | $b_{\sigma}$ | $\bar{s}$ |

| $s^2$ | $bb_{\sigma}$ | $\frac{1}{\lambda_2}$ | $\bar{s}$ | $bb_{\sigma^2}$ | $\frac{1}{\lambda_2}$ | $b_{\sigma}$ | $s$ | $bb_{\sigma^2}$ | $\bar{s}$ |

| $s^2$ | $b_{\sigma}b$ | $s$ | $g_{bb_{\sigma}}$ | $\frac{1}{\lambda_2}$ |   |   |   |   |   |

| 1 | $b_{\sigma}b_{\sigma^2}$ | $\frac{1}{\lambda_2}$ | $s$ | $b_{\sigma}b_{\sigma^2}$ | $\frac{1}{\lambda_2}$ | $\frac{1}{\lambda_1}$ | $\frac{1}{\lambda_2}$ | $\frac{1}{\lambda_2}$ | $\frac{1}{\lambda_1}$ |

| $s$ | $b_{\sigma}b_{\sigma^2}$ | $\frac{1}{\lambda_2}$ | $s$ | $b_{\sigma}b_{\sigma^2}$ | $\frac{1}{\lambda_2}$ | $\frac{1}{\lambda_2}$ | $\frac{1}{\lambda_1}$ | $\frac{1}{\lambda_2}$ | $\frac{1}{\lambda_1}$ |

| 1 | $*_{\sigma}b_{\sigma^2}$ |   |   |   |   |   |   |   |   |
| 1 | $*_{\sigma}b_{\sigma}$ |   |   |   |   |   | 1 | 1 | 1 |

| 1 | $*_{\sigma}b_{\sigma}$ |   |   |   |   |   | 1 | 1 | 1 |

Table 6: Connection $\alpha\tilde{\alpha} - 1$ after taking symmetric gauge choice

and

$$g'_{bb} \rightarrow g'_{b_{\sigma}b_{\sigma}} \rightarrow g'_{b_{\sigma}b_{\sigma^2}b_{\sigma^2}}.$$  

Thus, we have proved the equivalence of connections

$$\alpha\tilde{\alpha} - 1 \cong \sigma(\alpha\tilde{\alpha} - 1)\sigma.$$  

Finally we will check conditions 1) and 2). Mutual inequivalence is obvious by seeing four graphs of the connections appearing there. Namely, connections producing the bimodules of different indices are trivially mutually inequivalent. To prove inequivalence of connections which produce the bimodules of the same index, it is sufficient to show the existence of the unitary matrices of the connection of the form

$$x_{\alpha\tilde{\alpha}} - 1$$

which have different sizes in each connection. We can check it only by seeing the four graphs. About the indecomposability, since it was irreducibility of the bimodules in our original lemma, all we must see is the irreducibility of bimodules made of connections here. The bimodule $X^1 = N M$ is trivially irreducible, and indecomposability of $\sigma$ and $\sigma^2$ follows. To see the irreducibility of $X^\alpha$, consider the subfactor $N \subset M$ constructed from the connection $\alpha$. Then $X^\alpha = N M$. By Ocneanu’s compactness argument, (see Section 3, Theorem 4)

$$\text{End}(N X^\alpha_M) = \text{End}(N M_M)$$
5 Main theorem for the case of \((5 + \sqrt{17})/2\)

In this section, we will give a proof for our main theorem for the case of index \((5 + \sqrt{17})/2\) due to the first named author.

**Theorem 6** A subfactor with principal graph and dual principal graph as in Figure 2 exists.

From the key lemma, we know that the above theorem follows from the next proposition. We define the connection \(\sigma\) as

\[
\sigma \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \delta_{p,\tilde{r}}\delta_{q,\tilde{s}},
\]

here \(p, q, r, s\) are the vertices on the upper graph in Figure 2 and we consider \(\tilde{x}\) as \(x\) and if \(x\) is one of \(e, f, g\), \(\tilde{x} = x\).

**Proposition 2** Let \(\alpha\) be the unique connection on the four graphs consisting of the pair of the graphs appearing in Figure 2 and \(\sigma\) be the connection defined above. Then, the following hold.

1) The eight connections

\[
1, \sigma, \sigma^2, (\alpha\bar{\alpha} - 1), \sigma(\alpha\bar{\alpha} - 1), (\alpha\bar{\alpha} - 1)\sigma, \sigma(\alpha\bar{\alpha} - 1)\sigma,
(\alpha\bar{\alpha})^2 - 3\alpha\bar{\alpha} + 1, \sigma((\alpha\bar{\alpha})^2 - 3\alpha\bar{\alpha} + 1)
\]

are indecomposable and mutually inequivalent.

2) The six connections

\[
\alpha, \sigma\alpha, \alpha\bar{\alpha} - 2\alpha, \sigma(\alpha\bar{\alpha} - 2\alpha), (\alpha\bar{\alpha})^2\alpha - 4\alpha\bar{\alpha}\alpha + 3\alpha, (\alpha\bar{\alpha} - 1)\sigma\alpha
\]
are irreducible and mutually inequivalent.

3) 

$$\sigma(\alpha \tilde{\alpha} - 1) \sigma \alpha \cong (\alpha \tilde{\alpha} - 1) \sigma \alpha.$$ 

**Proof**

The four graphs of the connection \(\alpha\) and the Perron-Frobenius weights are as in Figure 8.

The Perron-Frobenius weights:

- \(\mu(\ast) = \mu(\tilde{\ast}) = 1\), \(\mu(a) = \mu(\tilde{a}) = \beta\), \(\mu(b) = \mu(\tilde{b}) = \mu(h) = \mu(\tilde{h}) = \beta^2 - 1\),
- \(\mu(c) = \mu(\tilde{c}) = \beta^3 - 2\beta\), \(\mu(d) = \mu(\tilde{d}) = 2\beta^2 - 1\), \(\mu(e) = \beta^3 + \beta\), \(\mu(f) = 2\beta^2\),
- \(\mu(g) = \beta^3 - \beta\), \(\mu(2) = \beta^2 - 1\), \(\mu(3) = 2\beta^2 - 1\), \(\mu(4) = \beta^2 + 1\),
- \(\mu(5) = 3\beta^2 - 2\), \(\mu(6) = \beta^2\).

Figure 8: Four graphs of the connection \(\alpha\)

Note that the Perron-Frobenius weights of the vertices in \(V_3\) are the same as that of the vertices in \(V_1\), and here we used \(\beta^4 - 5\beta^2 + 2 = 0\). The biunitary connection \(\alpha\) on these four graphs is determined uniquely as in Table 7, as in \((5 + \sqrt{13})/2\) case. We will also display the Table of the connection \(\tilde{\alpha}\) for use of later computations.
### Table 7: Connections $\alpha$ (upper) and $\tilde{\alpha}$ (lower)

|      | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $c_5$ | $g_5$ | $g_6a_6$ |
|------|-------|-------|-------|-------|-------|-------|-------|----------|
| $A*$ | 1     | 1     |       |       |       |       |       |          |
| $bA$ | 1     | $\frac{\beta_3}{\beta_3}$ | $\frac{\beta_5}{\beta_5}$ |       |       |       |       |          |
| $bC$ | $\frac{\beta_2}{\beta_2}$ | $\frac{1}{\beta_2}$ | 1     |       |       |       |       |          |
| $dC$ | 1     | $-\frac{1}{\gamma}$ | $\frac{\beta_3}{\gamma}$ |       |       |       |       |          |
| $dE$ | $\frac{-\gamma'}{\gamma'}$ | $\frac{1}{\gamma}$ | 1     | 1     |       |       |       |          |
| $fE$ | 1     | $-1$  | $\frac{\beta_2}{\beta_2}$ |       |       |       |       |          |
| $fG$ |       | $\frac{\beta_2}{\beta_2}$ | $\frac{1}{\beta_2}$ | 1     |       |       |       |          |
| $hG$ |       |       | 1     | 1     |       |       |       |          |
| $hC$ |       |       |       |       |       |       |       |          |
| $hA$ |       |       |       |       |       |       |       |          |
| $dE$ | 1     | 1     | $-\frac{\gamma}{\gamma}$ | $\frac{\beta_2}{\gamma}$ |       |       |       |          |
| $dC$ |       |       | $\frac{1}{\gamma}$ | $\frac{\beta_2}{\gamma}$ |       |       |       |          |
| $bG$ |       |       | 1     | 1     |       |       |       |          |
| $\tilde{A}$ |       |       |       |       |       |       |       |          |

where,

$\beta_n = \sqrt{\beta^2 - n}$,  \hspace{0.2cm} \beta'_n = \beta^2 - n,$

$\gamma = \sqrt{2\beta^2 - 1}$,  \hspace{0.2cm} \gamma' = 2\beta^2 - 1.$
First we check condition 3), namely we prove
\[ \sigma(\alpha \tilde{\alpha} - 1) \sigma = (\alpha \tilde{\alpha} - 1) \sigma. \]
up to vertical gauge choice. Now we compute the connection \( \alpha \tilde{\alpha} \). The four graphs on which the connection \( \alpha \tilde{\alpha} \) exists are as in Figure 9.

The broken edges correspond to the trivial connection 1. We will now compute the connection \( \alpha \tilde{\alpha} - 1 \), which is determined only up to vertical gauges. As in the \((5 + \sqrt{13})/2\)-case, we assume that \( 1 \times 1 \) gauge transform unitaries corresponding to single vertical edges which connect different vertices in the graph \( G_1G_1^t \cup G_3G_3^t \) to be 1. Then we easily find 38 entries of \( \alpha \tilde{\alpha} - 1 \) by “actual” multiplication of the connections \( \alpha \) and \( \tilde{\alpha} \). Next, as in the \((5 + \sqrt{13})/2\)-case, we can compute all the entries of \( \alpha \tilde{\alpha} - 1 \) which involve the double edges in the graph of \( \alpha \tilde{\alpha} \) by a simple gauge transform, then we have 14 entries listed in the Tables 8–9 other than the entries marked “(?)”, where \( \beta_n = \sqrt{\beta^2 - n} \) and \( \gamma = \sqrt{2\beta^2 - 1} \).
5 Main theorem for the case of \((5 + \sqrt{17})/2\)

| \(\alpha\alpha\) | \(cc\) | \(ee\) | \(ee^1\) | \(ee^2\) | \(\check{ce}\) | \(ge\) |
|------------------|--------|--------|--------|--------|--------|--------|
| \(ab\) | 1 | 1 | | | | |
| \(bb\) | \(\frac{\delta}{\beta_1} \frac{\beta_2}{\beta_1}\) | 1 | \(\sqrt{2\delta}\) | | | |
| \(bd\) | \(\sqrt{2\delta}\) | \(\frac{1}{\beta_1}\) | 1 | | | |
| \(db\) | 1 | \(\frac{1}{\beta_2}\) | \(\sqrt{\beta^2 - 1}\) | | | |
| \(dd\) | \(\sqrt{\beta^2 - 1}\) | \(\frac{1}{\beta_2}\) | \(\frac{\beta_2}{\beta_1}\) | \(l_1\) | \(l_2\) | | |
| \(df\) | \(\frac{\beta^2 - 1}{\beta^2 - 1}\) | \(\frac{1}{\beta_2}\) | \(m_1\) | \(m_2\) | | |
| \(dd\) | \(\frac{\beta^2}{\beta^2 - 1}\) | \(n_1\) | \(n_2\) | | | |
| \(fd\) | 1 | \(p_1\) | \(p_2\) | | | |
| \(ff\) | | | \(q_1\) | \(q_2\) | | |
| \(fh\) | | | | | | |
| \(fd\) | | | \(r_1\) | \(r_2\) | | |
| \(fb\) | | | | | | |
| \(hf\) | | | | | | |
| \(hb\) | | | | | | |
| \(hh\) | | | | | | |
| \(hd\) | | | | | | |
| \(hs\) | | | | | | |
| \(dd\) | 1 | \(s_1\) | \(s_2\) | \(\frac{\beta_2}{\beta - 1}\) | | |
| \(df\) | | | \(t_1\) | \(t_2\) | | |
| \(dh\) | | | | | | |
| \(dd\) | | | \(u_1\) | \(u_2\) | \(\frac{\beta_2}{\beta - 1}\) | | |
| \(bf\) | | | | | | |
| \(bh\) | | | | | | |
| \(sh\) | | | | | | |

Table 8: Connection \(\alpha\alpha - 1\) (left part of diagram)
Table 9: Connection $\alpha\tilde{\alpha} - 1$ (right part of diagram)
The four entries marked (?) in Table 8 can easily be computed by the unitarity of the $2 \times 2$ matrices which they are part of, and the entry marked (?) in Table 9 can be put equal to 1, because a gauge choice corresponding to the $\tilde{h}h$-edge in the vertical left graph will only be concerned with $(\tilde{h}h, gg)$-entry. Together with the fact that all the entries of the connection are real scalars, all the gauge choices involved in decomposing the connection into $(\alpha \tilde{\alpha} - \mathbf{1})$ can be chosen to be matrices with real entries. Hence, $l_1, l_2, m_1, m_2, . . . , u_1, u_2$ become real numbers. We still have a possibility of making a gauge choice of the double edges $e-e$ with an orthogonal matrix, i.e., we can make the following change:

$$(l_1, l_2) \rightarrow (l_1, l_2)v, \quad (m_1, m_2) \rightarrow (m_1, m_2)v, \ldots, (u_1, u_2) \rightarrow (u_1, u_2)v$$

for some $v \in O(2)$. (A common orthogonal matrix for all the vectors in $\mathbb{R}^2$.) Then, we can assume $l_2 = 0$ and $m_2 \geq 0$, thus, we obtain

$$d \begin{bmatrix} e & e \\ f & f \end{bmatrix} \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix} = 
\begin{bmatrix}
\frac{1}{\sqrt{\beta^2 + 1}} \sqrt{\frac{\beta^2 - 2}{\beta^2 + 1}} & \sqrt{\frac{\beta^4 + 4}{\beta^2(\beta^2 + 1)}} \\
\beta^2 - 2 & -\sqrt{\frac{\beta^4 - 4}{\beta^2(\beta^2 + 1)}} \\
2\sqrt{\beta^2 - 1} & \sqrt{\frac{\beta^4 - 4}{(\beta^2 - 1)(\beta^2 + 1)}} \\
2\sqrt{\beta^2 - 1} & -\sqrt{\frac{\beta^4 + 4}{(\beta^2 - 1)(\beta^2 + 1)}}
\end{bmatrix}
$$

by the orthogonality of the matrix.

Now, all the gauge choices have been used up. We know that there is an orthogonal matrix $V \in O(3)$ such that

$$d \begin{bmatrix} e & e \\ f & f \end{bmatrix} \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix} = (0, m_1, m_2)V, \quad f \begin{bmatrix} e & e \\ d & d \end{bmatrix} \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix} = (0, r_1, r_2)V,$$

$$d \begin{bmatrix} e & e \\ f & f \end{bmatrix} \begin{bmatrix} n_1 & n_2 \\ m_1 & m_2 \end{bmatrix} = (0, n_1, n_2)V, \quad \tilde{d} \begin{bmatrix} e & e \\ d & d \end{bmatrix} \begin{bmatrix} n_1 & n_2 \\ m_1 & m_2 \end{bmatrix} = (0, s_1, s_2)V,$$

$$d \begin{bmatrix} e & e \\ f & f \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ m_1 & m_2 \end{bmatrix} = (0, p_1, p_2)V, \quad \tilde{d} \begin{bmatrix} e & e \\ d & d \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ m_1 & m_2 \end{bmatrix} = (0, s_1, s_2)V$$

where

$$\alpha \tilde{\alpha}$$

denotes the $1 \times 3$ matrices obtained by “actual” multiplication of $\alpha$ and $\tilde{\alpha}$. It is clear from the definition of the renormalization of connection, that $(\alpha \tilde{\alpha} - \mathbf{1})^\sim = \alpha \tilde{\alpha} - \mathbf{1}$ without any gauge transformation. Together with the fact that all the entries of the connection $\alpha \tilde{\alpha}$ by “actual” multiplication are real numbers, we have
5 Main theorem for the case of \((5 + \sqrt{17})/2\)

\[
f \begin{bmatrix} \alpha \beta \\ d \end{bmatrix} e = \sqrt{\frac{\mu(d)}{\mu(f)}} \begin{bmatrix} \alpha \beta \\ d \end{bmatrix} e,
\]

\[
\bar{d} \begin{bmatrix} \alpha \beta \\ d \end{bmatrix} e = \sqrt{\frac{\mu(d)}{\mu(d)}} \begin{bmatrix} \alpha \beta \\ d \end{bmatrix} e,
\]

\[
\frac{\bar{d} \begin{bmatrix} \alpha \beta \\ f \end{bmatrix}}{e} = \sqrt{\frac{\mu(f)}{\mu(d)}} \frac{\bar{d} \begin{bmatrix} \alpha \beta \\ d \end{bmatrix}}{e},
\]

hence,

\[
(p_1, p_2) = \sqrt{\frac{2\beta^2 - 1}{2\beta^2}} (m_1, m_2)
\]

\[
= \sqrt{\frac{2\beta^2 - 1}{2\beta^2}} \left( \frac{-\sqrt{\beta^4 - 4}}{\beta(\beta^2 - 1)\sqrt{\beta^4 + 4}} \right),
\]

\[
(s_1, s_2) = (n_1, n_2)
\]

\[
= \left( -\sqrt{\frac{\beta^2(\beta^2 - 2)}{(\beta^2 + 1)(2\beta^2 - 1)(\beta^4 + 4)}}, \frac{-4\sqrt{\beta^2(\beta^2 - 1)}}{(\beta^2 - 1)\sqrt{(\beta^2 - 1)(\beta^4 + 4)}}, \right),
\]

and

\[
(t_1, t_2) = \sqrt{\frac{2\beta^2}{2\beta^2 - 1}} (r_1, r_2). \text{ (51)}
\]

We next determine \((r_1, r_2)\) and \((t_1, t_2)\). In the text, we denote the connection matrix, e.g.,

\[
\begin{array}{c}
\begin{bmatrix}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{bmatrix}
\end{array}
\end{array}
\]

by \(M(b/5)\) for the convenience of space. By orthogonality of the first and the last row in the 3 \times 3 matrix \(M(f/e)\) in the Tables 8, 9, we have

\[
p_1r_1 + p_2r_2 = \frac{\sqrt{3\beta^2 - 1}}{2(\beta^2 + 1)}, \text{ (52)}
\]

and by orthogonality of the first two rows in the 3 \times 3 matrix \(M(\bar{d}/e)\) in Table 9, we have

\[
s_1t_1 + s_2t_2 = -\frac{1}{\beta^2 - 1} \sqrt{\beta^2 - 2},
\]

too, and together with (51) we have

\[
(s_1r_1 + s_2r_2) = \frac{-1}{\beta^2 - 1} \sqrt{\frac{(\beta^2 - 2)(2\beta^2 - 1)}{(\beta^2 + 1)2\beta^2}}. \text{ (53)}
\]
Solving (22) and (23) with respect to \((r_1, r_2)\) using the known values of \(p_1, p_2, s_1,\) and \(s_2\) gives

\[
(r_1, r_2) = \left(\frac{-\beta^3}{\sqrt{(\beta^2 + 1)(\beta^4 + 4)}}, \frac{4\beta^2}{(\beta^2 + 1)(\beta^4 + 4)}\right),
\]

and therefore,

\[
(t_1, t_2) = \sqrt{\frac{2\beta^2}{2\beta^2 - 1}} \left(\frac{-\beta^3}{\sqrt{(\beta^2 + 1)(\beta^4 + 4)}}, \frac{4\beta^2}{(\beta^2 + 1)(\beta^4 + 4)}\right).
\]

The four remaining entries \(q_1, q_2, u_1, u_2\) can now be computed using the orthogonality of the \(3 \times 3\) matrices \(M(f/e)\) and \(M(\tilde{d}/e)\). We have

\[
(q_1, q_2) = \left(\frac{2\beta^4}{\sqrt{(\beta^2 + 1)(\beta^4 + 4)}}, \frac{2(\beta^2 + 1)}{\beta^4 + 4}\right),
\]

\[
(u_1, u_2) = \left(\frac{\beta^2(\beta^2 + 2)}{(\beta^2 + 1)(\beta^4 + 4)}, \frac{2\sqrt{2}}{\sqrt{\beta^4 + 4}}\right).
\]

Now, we have obtained all the entries of \(\alpha \tilde{\alpha} - 1\). We can obtain \((\alpha \tilde{\alpha} - 1)\sigma\) only by exchanging the vertices at the bottom of the connection \(\alpha \tilde{\alpha} - 1\) as below.

\[
\begin{array}{cc}
p & q \\
r & s
\end{array} \quad \text{(in \((\alpha \tilde{\alpha} - 1)\sigma\))} = \begin{array}{cc}
p & q \\
r & s
\end{array} \quad \text{(in \((\alpha \tilde{\alpha} - 1))\)}
\]

Together with the information of \(\alpha\), we obtain all the entries of the connection \((\alpha \tilde{\alpha} - 1)\sigma\). Now we show the landscape of them in Table[10]. The four graphs of this connection are as in Figure[10].
5 Main theorem for the case of \((5 + \sqrt{17})/2\)

\[ \begin{array}{c}
V_0 & G_0 & V_1 \\
| & \square & | \\
| & H_0 & H_1 \\
V_2 & G_2 & V_3
\end{array} \]

Figure 10: Four graphs of the connection \((\alpha \tilde{\alpha} - 1)\sigma\alpha\)
Table 10: Landscape of $(\alpha \tilde{\alpha} - 1)\sigma \alpha$
Since the exact values of the connection take up too much room to be listed up in a Table, we will show them in the shape of unitary matrices. Table 10 gives also an overview of the connection \( \sigma(\alpha\tilde{\alpha} - 1)\sigma\alpha \), because it is easy to check that \( \sigma(\alpha\tilde{\alpha} - 1)\sigma\alpha \) has exactly the same vertical edges as \( (\alpha\tilde{\alpha} - 1)\sigma\alpha \).

Below we list all the entries of \( (\alpha\tilde{\alpha} - 1)\sigma\alpha \). These entries can be obtained by direct multiplication of the connections \( (\alpha\tilde{\alpha} - 1)\sigma \) and \( \alpha \) as explained in section 3. In the list we have labeled rows and columns of the unitary matrices according to those entries that have to be used in the direct multiplication, for instance, in the \( 2 \times 2 \)-matrix below, the entry with row-label \( G^b \) and column-label \( a\tilde{a} \) is computed as follows:

\[
\begin{align*}
  b & \quad a\tilde{a} \\
G^b & \quad 6 \\
\end{align*}
\]

\[
\begin{align*}
  b & \quad a\tilde{a} \\
G^b & \quad 6 \\
\end{align*} = \begin{align*}
  \overline{b} & \quad \sigma(\alpha\tilde{\alpha} - 1)\sigma \cdot \tilde{b} \quad \tilde{\alpha} \\
  \sigma & \quad \alpha \quad G^a \quad 6 \\
\end{align*}
\]

\[
= \begin{align*}
  b & \quad \alpha\tilde{\alpha} - 1 \quad a \\
\sigma & \quad \alpha \quad G^a \quad 6 \\
\end{align*} = \begin{align*}
  \overline{b} & \quad \tilde{\alpha} \\
\tilde{b} & \quad \tilde{\alpha} \\
\end{align*} \cdot \begin{align*}
  \sigma & \quad \alpha \\
G^a & \quad 6 \\
\end{align*} = ( - \sqrt{\beta^2 - 2} / \sqrt{\beta^2 - 1}) \cdot 1,
\]

where the last equality is obtained from the tables 8 and 1. Sometimes the entries listed below appear at first glance to be different from the entries obtained by direct multiplication. However in all those cases, it is just a different representation of the same algebraic number. This can easily be checked using the following identities for \( \beta = \sqrt{5 + \sqrt{17}} / 2 \):

\[
\begin{align*}
  \beta^2 + 1 &= \frac{2(\beta^2 - 1)^2}{\beta^4}, & \beta^2 - 4 &= \frac{2}{\beta^2 - 1}, \\
  \beta^2 + 2 &= \frac{4\beta^2}{\beta^4}, & 5 - \beta^2 &= \frac{2}{\beta^2}, \\
  \beta^2 + 3 &= \frac{(\beta^4 - 1)^2}{2\beta^4} (\beta^4 + 4), & 2\beta^2 - 1 &= \frac{\beta^2(\beta^2 - 1)}{2}, \\
  \beta^2 - 2 &= \frac{2\beta^2}{\beta^2 - 1}, & 3\beta^2 - 1 &= (\beta^2 - 1)^2, \\
  \beta^2 - 3 &= \frac{2(\beta^2 - 1)^2}{\beta^4}, & 3\beta^2 - 4 &= \frac{\beta^2 - 1}{2\beta^2} (\beta^4 + 4).
\end{align*}
\]

Here comes the list of entries of \( (\alpha\tilde{\alpha} - 1)\sigma\alpha \):

\[
\begin{align*}
  * & \quad a\tilde{a} \\
G^b & \quad 6 \\
\end{align*} \quad \begin{align*}
  * & \quad a \\
\tilde{b} & \quad \tilde{a} \\
G^a & \quad 6 \\
\end{align*} = \begin{align*}
  \overline{b} & \quad \tilde{\alpha} \\
\tilde{b} & \quad \tilde{\alpha} \\
\end{align*} \cdot \begin{align*}
  \sigma & \quad \alpha \\
G^a & \quad 6 \\
\end{align*} = 1,
\]

\[
\begin{align*}
  \overline{6} & \quad \tilde{a}\tilde{g} \\
G^g & \quad 6 \\
\end{align*} = 1,
\]
5 Main theorem for the case of \((5 + \sqrt{17})/2\)

\[
\begin{align*}
\begin{array}{c}
\boxed{5} \\
\begin{array}{c}
\boxed{5} \\
\boxed{G^b} \\
\end{array}
\end{array} & \quad \begin{array}{c}
\boxed{6} \\
\begin{array}{c}
\boxed{6} \\
\boxed{G^b} \\
\end{array}
\end{array} \\
\vdots \\
\begin{array}{c}
\boxed{5} \\
\begin{array}{c}
\boxed{5} \\
\boxed{G^h} \\
\end{array}
\end{array} & \quad \begin{array}{c}
\boxed{6} \\
\begin{array}{c}
\boxed{6} \\
\boxed{G^h} \\
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\star \quad a_{\tilde{e}} & = 1, \\
\star \quad \tilde{a}_g & = 1, \\
\begin{array}{c}
\boxed{6} \\
\begin{array}{c}
\boxed{6} \\
\boxed{G^b} \\
\end{array}
\end{array} & \quad \begin{array}{c}
\boxed{5} \\
\begin{array}{c}
\boxed{5} \\
\boxed{G^h} \\
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
b & = \begin{bmatrix} a_{\tilde{a}} & c_{\tilde{a}} \\ \frac{\sqrt{\beta^2 - 1}}{\sqrt{\beta^2}} & \frac{\sqrt{\beta^2 - 1}}{\sqrt{\beta^2}} \end{bmatrix}, \\
\tilde{b} & = \begin{bmatrix} \tilde{a}_g & \tilde{c}_g \\ \frac{\sqrt{\beta^2 - 1}}{\sqrt{\beta^2}} & \frac{\sqrt{\beta^2 - 1}}{\sqrt{\beta^2}} \end{bmatrix} \end{align*}
\]

\[
\begin{align*}
b & = G^b \begin{bmatrix} a_{\tilde{e}} & c_{2\tilde{e}} & c_e \\ \frac{1}{\sqrt{\beta^2 + 1}} & \frac{\sqrt{\beta^2 + 1}}{\sqrt{\beta^2} \beta} & \frac{\sqrt{\beta^2 - 1}}{\sqrt{\beta^2} \beta} \\ \frac{\sqrt{\beta^2 - 1}}{\sqrt{\beta^2}} & \frac{1}{\sqrt{\beta^2} \beta} & \frac{1}{\sqrt{\beta^2} \beta} \end{bmatrix} \begin{bmatrix} e_g \\ c_{\tilde{a}} \\ 0 \end{bmatrix}, \\
\tilde{b} & = G^f \begin{bmatrix} \tilde{a}_g & \tilde{c}_g & \tilde{c}_e \\ \frac{\sqrt{\beta^2 + 1}}{\sqrt{\beta^2}} & \frac{\sqrt{\beta^2 + 1}}{\sqrt{2\beta^2 (\beta^2 - 2)}} & \sqrt{\frac{\beta^2 - 1}{2\beta^2 (\beta^2 - 2)}} \\ \frac{1}{\sqrt{\beta^2}} & \frac{\sqrt{\beta^2 - 1}}{\sqrt{2\beta^2 (\beta^2 - 2)}} & 0 \end{bmatrix} \end{align*}
\]

\[
\begin{align*}
d & = G^f \begin{bmatrix} e_g & c_{\tilde{a}} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
\tilde{d} & = G^f \begin{bmatrix} e_g & c_{\tilde{a}} \\ \frac{\sqrt{\beta^2 + 1}}{\sqrt{2\beta^2 (\beta^2 - 2)}} & \frac{\sqrt{\beta^2 - 1}}{\sqrt{2\beta^2 (\beta^2 - 2)}} \end{bmatrix} \end{align*}
\]
\[
\begin{align*}
\tilde{d} & \equiv 5 \\
\tilde{C}^d & = \begin{pmatrix}
-\frac{\sqrt{\beta^4 - 1}}{2\beta^2 - 1} & \frac{1}{\beta^2 - 1} & \sqrt{\frac{\beta^4 + 4}{\beta^2 (2\beta^2 - 1)}} & 0 & \frac{1}{\beta^2 - 1} & 0 \\
0 & \frac{1}{\beta^2 - 1} & -\frac{\sqrt{2(\beta^2 + 4)}}{(\beta^4 - 1)^2} & 0 & \frac{1}{\beta^2 \sqrt{\frac{\beta^4 + 1}{2\beta^2 - 1}}} & 0 \\
0 & \frac{\beta^2}{\sqrt{(\beta^2 - 1)^2}} & -\frac{2(\beta^2 - 2)}{\sqrt{2(\beta^2 - 1)(\beta^4 + 4)}} & \sqrt{\frac{\beta^2 - 1}{\sqrt{2(\beta^4 + 4)}}} & 0 & \frac{\sqrt{\beta^2 + 1}}{\beta^2 - 2} \\
0 & \sqrt{\frac{2}{\beta^2}} & -\frac{\beta^2}{\sqrt{2(\beta^2 - 1)(\beta^4 + 4)}} & \sqrt{\frac{\beta^2 (\beta^2 + 1)}{\sqrt{2(\beta^4 + 4)}}} & 0 & -\frac{1}{\beta^2} \\
\frac{1}{\beta^2} & 0 & 0 & 0 & \sqrt{\frac{\beta^2 - 1}{\beta^2}} & 0 \\
0 & \sqrt{\frac{\beta^2 - 1}{\beta^2}} & -\frac{\sqrt{\beta^2 (\beta^2 + 1)(\beta^4 + 4)}}{\sqrt{2(\beta^4 + 4)}} & 0 & 0 & \sqrt{\frac{\beta^2 + 1}{2\beta^2 - 1}}
\end{pmatrix},
\end{align*}
\]

\[
\tilde{d} & \equiv 5 \\
\tilde{C}^h & = \begin{pmatrix}
\frac{\beta^2 - 2}{\beta^2} & 0 & 0 & 0 & 0 \\
0 & \frac{\beta^2 - 2}{\beta^2} & 0 & 0 & 0 \\
-\frac{1}{\beta^2} & 0 & 0 & 0 & \sqrt{\frac{\beta^2 - 1}{\beta^2}} \\
-\frac{1}{\beta^2} & 0 & 0 & 0 & \sqrt{\frac{\beta^2 - 1}{\beta^2}} \\
\beta^2 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
f & \equiv 2 \\
A^b & = \begin{pmatrix}
\frac{e_c}{\sqrt{\frac{\beta^2 + 1}{2(\beta^2 - 1)}}} & -\frac{1}{\beta^2 - 1} & -\frac{1}{\beta^2} \\
\frac{1}{\sqrt{\frac{\beta^2 - 1}{2\beta^2}}} & 0 & \frac{1}{\sqrt{2(\beta^2 - 1)}} \\
\frac{\sqrt{\beta^2 (\beta^2 - 2)}}{\beta^4 - 1} & \frac{\sqrt{\beta^2 (\beta^2 - 2)}}{\beta^2 - 1} & -\frac{1}{2\beta^2}
\end{pmatrix},
\]

\[
C^d & = \begin{pmatrix}
e_c & 0 & 0 \\
-\frac{1}{\beta^2 - 1} & 0 & 0 \\
\frac{1}{\beta^2} & \frac{1}{\beta^2} & 0
\end{pmatrix},
\]

\[
C^b & = \begin{pmatrix}
\sqrt{\frac{\beta^2 + 1}{2(\beta^2 - 1)}} & -\frac{1}{\beta^2} & -\frac{1}{\beta^2} \\
\frac{1}{\sqrt{\frac{\beta^2 - 1}{2\beta^2}}} & 0 & \frac{1}{\sqrt{2(\beta^2 - 1)}} \\
\frac{\sqrt{\beta^2 (\beta^2 - 2)}}{\beta^4 - 1} & \frac{\sqrt{\beta^2 (\beta^2 - 2)}}{\beta^2 - 1} & -\frac{1}{2\beta^2}
\end{pmatrix}.
\]
5 Main theorem for the case of \((5 + \sqrt{17})/2\)

\[
\begin{align*}
\begin{bmatrix}
\tilde{d} & 3 \\
E^d & = \\
E^d & \begin{pmatrix}
\sqrt{\beta^4+4} & e_{3e} & e_c & c_c \\
\beta^2 \sqrt{(\beta^2-1)(\beta^4+4)} & 0 & 0 & \frac{\beta^2}{\beta^2-1} \sqrt{(\beta^2-1)} \\
\frac{1}{\beta^2-1} \sqrt{\beta^4(\beta^4+4)} & \beta^2 \sqrt{(\beta^2-1)(\beta^4+4)} & 0 & \frac{\beta^2}{\beta^2-1} \\
\frac{2}{(\beta^2-1)(\beta^2-1)} \sqrt{2(\beta^4+4)} & \frac{4(\beta^2-2)}{(\beta^4+4)(\beta^2-1)} & -\frac{1}{2(\beta^2-1)} & \frac{(2\beta^2-2)}{(2\beta^2-1)^3} \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
\tilde{d} & 3 \\
E^d & = \\
E^d & \begin{pmatrix}
\sqrt{\beta^2(\beta^2-2)} & e_{3e} & e_c & \tilde{c}_c \\
-\frac{2(\beta^2-2)}{\beta^2+1} \sqrt{2(\beta^4+4)} & 0 & 0 & \sqrt{\beta^2+1} \\
-\frac{2}{2(\beta^2-1)} \frac{\beta^3}{(\beta^4+1)(\beta^4+4)} & \frac{4}{2(\beta^2-1)} \frac{\beta^4}{(\beta^4+1)(\beta^4+4)} & 0 & \sqrt{\beta^2+1} \\
\frac{2}{2(\beta^2-1)} \frac{\beta^2(\beta^2-1)}{(\beta^4+1)(\beta^4+4)} & \frac{4}{2(\beta^2-1)} \frac{\beta^4}{(\beta^4+1)(\beta^4+4)} & 2\sqrt{\frac{2}{2(\beta^2-1)}} & \frac{1}{2(\beta^2-1)} \sqrt{(\beta^2-1)} \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
h & 3 \\
C^b & = \\
E^f & \begin{pmatrix}
g_c & g_e \\
1 & 0 \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
\tilde{h} & 3 \\
C^d & = \\
E^d & \begin{pmatrix}
g_c & g_e \\
-\frac{1}{2(\beta^2-1)} & \frac{2\sqrt{2}}{2(\beta^2-1)} \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
f & 3 \\
f & =
\end{align*}
\]
Table 5: Main theorem for the case of \((5 + \sqrt{17})/2\)
5 Main theorem for the case of $(5 + \sqrt{17})/2$

\[
\begin{align*}
E^d & \left( \begin{array}{ccccccc} g_g & g_e & e_{2e} & e_{3e} & e_{\tilde{e}} & e_g \\
0 & -\frac{(\beta^2 - 2)}{\sqrt{8\beta^2}} & \frac{\sqrt{\beta^2(\beta^2 - 2)\beta^2 - 4}}{\beta^2(\beta^2 - 1)2(\beta^4 + 4)} & -\frac{\sqrt{\beta^2(\beta^2 - 2)}}{(\beta^2 - 1)\beta^2 + 4} & \frac{\sqrt{\beta^2(\beta^2 - 2)}}{2\beta^2 - 1} & 0 \\
\frac{\beta^2 + 1}{2\beta^2} & \frac{\sqrt{\beta^2(\beta^2 - 1)}}{\beta^2 - 2} & -\frac{1}{2(\beta^2 - 2)} & \frac{\beta}{2\beta^4 + 4} & \frac{\sqrt{(\beta^2 + 1)(\beta^2 - 1)}}{(\beta^2 - 2)\beta^4 + 4} & 0 & -\frac{(\beta^2 - 1)}{2\beta^2} \frac{\sqrt{\beta^2 + 1}}{\beta^2 - 2} \\
0 & -\frac{1}{\sqrt{\beta^2 + 1}} & \frac{\sqrt{(\beta^2 + 1)(\beta^2 - 1)}}{\beta^2(\beta^2 - 1)} & -\frac{\sqrt{(\beta^2 + 1)(\beta^2 - 1)}}{\beta^2(\beta^2 - 1)2(\beta^4 + 4)} & \frac{\sqrt{(\beta^2 - 1)(\beta^2 + 1)}}{\beta^2 - 1} & 0 & 0 \\
0 & \frac{1}{\beta^2 - 2} & -\frac{\sqrt{(\beta^2 + 1)(\beta^2 - 1)}}{\beta^2(\beta^2 - 1)} & \frac{2(\beta^2 - 1)}{\beta^4 + 4} & \frac{\sqrt{(\beta^2 - 1)(\beta^2 + 1)}}{\beta^2 - 1} & 0 & \frac{\sqrt{\beta^2 + 1}}{2\beta^2} \\
\frac{\beta^2 - 1}{2\beta^2} & \frac{\beta^2 - 1}{2\beta^2} & \frac{\sqrt{\beta^2(\beta^2 - 1)}}{\beta^2(\beta^2 - 1)2(\beta^4 - 2)} & \frac{\sqrt{\beta^2(\beta^2 - 1)}}{\beta^2(\beta^4 - 2)(\beta^4 - 2)} & 0 & \frac{\beta^2 - 1}{2\beta^2} & 1 \\
\end{array} \right),
\end{align*}
\]

\[
\begin{align*}
& b_\Box \ c_e = 1, \\
& \bar{b}_\Box \tilde{c}_e = 1, \\
& b_\Box \tilde{c}_e = 1, \\
& \bar{b}_\Box \tilde{c}_e = -1, \\
& d_\Box \quad = E^d \left( \begin{array}{cccc}
\frac{1}{\beta^2} & \frac{\sqrt{\beta^2(\beta^2 - 2)}}{\beta^2(\beta^2 + 1)} & \frac{\sqrt{\beta^2(\beta^2 - 2)}}{\beta^2(\beta^2 + 1)} & 0 \\
-\frac{\sqrt{\beta^2 + 2}}{\beta^2 - 1} & -\frac{1}{\beta^2 - 1}\frac{\sqrt{\beta^2(\beta^2 - 2)}}{\beta^2(\beta^4 + 4)} & -\beta^2 & -\frac{2}{\beta^2(\beta^2 - 1)(\beta^4 + 4)} \\
\frac{\beta^2}{\beta^2 + 1} & -\frac{\sqrt{\beta^2(\beta^2 - 2)}}{\beta^2(\beta^2 + 1)} & -\frac{4}{\beta^2 - 1}\frac{\sqrt{2\beta^2 - 1}}{2(\beta^4 - 2)} \\
\end{array} \right), \\
& \tilde{d}_\Box \quad = E^d \left( \begin{array}{cccc}
\tilde{c}_e & e_{2e} & e_{3e} \\
-\frac{\beta^2(\beta^2 - 2)}{(\beta^2 + 1)(2\beta^2 - 1)(\beta^4 + 4)} & -\frac{\beta^2}{\beta^2 + 1} & -\frac{4}{\beta^2 - 1}\frac{\sqrt{2\beta^2 - 1}}{2(\beta^4 - 2)} \\
\frac{\beta^2(\beta^2 - 2)}{\beta^2(\beta^2 + 1)} & \frac{\beta^3}{\beta^2 + 1} & -\frac{2\beta^2}{\beta^2(\beta^2 + 1)(\beta^4 + 4)} \\
\end{array} \right).
\end{align*}
\]
5 Main theorem for the case of \((5 + \sqrt{17})/2\)

\[
\begin{align*}
\begin{array}{c}
h_g \\
G^f \\
6 \\
\end{array} & = 1, \\
\begin{array}{c}
h_g \\
G^h \\
6 \\
\end{array} & = 1, \\
f_a \begin{array}{c}
g_a \\
a^b \\
1 \\
\end{array} & = *, \\
\begin{array}{c}
f \\
6 \\
\end{array} & = G_f^f \begin{pmatrix} e_g & g_g \\
\end{pmatrix}, \\
(\text{We do not use the values of these entries.})
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
h \\
2 \\
\end{array} & = A^b \begin{pmatrix} -\frac{1}{\beta^2 - 1} & \frac{\sqrt{\beta^2(\beta^2 - 2)}}{\beta^2 - 1} \\
\frac{\sqrt{\beta^2(\beta^2 - 2)}}{\beta^2 - 1} & \frac{1}{\beta^2 - 1} \\
\end{pmatrix}, \\
\tilde{h} \begin{array}{c}
g_a \\
1 \\
\end{array} & = A^* \begin{pmatrix} 1 & g_c \\
g_c & 0 \\
\end{pmatrix}, \\
\begin{array}{c}
h \begin{array}{c}
g_a \\
a^b \\
1 \\
\end{array} & = 1, \\
\tilde{h} \begin{array}{c}
g_a \\
a^b \\
1 \\
\end{array} & = 1, \\
h \begin{array}{c}
g_e \\
E^f \\
4 \\
\end{array} & = -1, \\
\tilde{h} \begin{array}{c}
g_e \\
E^a \\
4 \\
\end{array} & = 1,
\end{align*}
\]
5 Main theorem for the case of \((5 + \sqrt{17})/2\)

Here we will display three matrices of the connection \(((\alpha \bar{\alpha} - 1)\sigma \alpha) \sim = \alpha \sigma (\alpha \bar{\alpha} - 1)\) for easiness of later procedure. These matrices are computed from the entries of \((\alpha \bar{\alpha} - 1)\sigma \alpha\) and the Perron-Frobenius weights of the horizontal graphs by renormalization rule as in section 4.

\[
\begin{pmatrix}
\begin{array}{cccccc}
G & e & 5_{e_2} & 5_{e_3} & 5_{e_c} & 5_g & 6_g \\
\alpha & 0 & \sqrt{\beta^2(\beta^2 - 1)/\beta^2} & \sqrt{\beta^2(\beta^2 - 1)/\beta^2} & 1 & \sqrt{\beta^2(\beta^2 - 1)/\beta^2} & \sqrt{\beta^2(\beta^2 - 1)/\beta^2} \\
\end{array}
\end{pmatrix}
\]

\[
A \begin{pmatrix}
\begin{array}{cccc}
\begin{array}{cccc}
\alpha & 0 & \sqrt{\beta^2(\beta^2 - 1)/\beta^2} & \sqrt{\beta^2(\beta^2 - 1)/\beta^2} \\
\end{array}
\end{array}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\begin{array}{cccc}
\begin{array}{cccc}
\alpha & 0 & \sqrt{\beta^2(\beta^2 - 1)/\beta^2} & \sqrt{\beta^2(\beta^2 - 1)/\beta^2} \\
\end{array}
\end{array}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\begin{array}{cccc}
\begin{array}{cccc}
\alpha & 0 & \sqrt{\beta^2(\beta^2 - 1)/\beta^2} & \sqrt{\beta^2(\beta^2 - 1)/\beta^2} \\
\end{array}
\end{array}
\end{pmatrix}
\]
Now we will prove that \((\alpha \tilde{\alpha} - 1)\sigma \alpha\) and \(\sigma(\alpha \tilde{\alpha} - 1)\sigma \alpha\) are equivalent up to vertical gauge choice. What we should do is to construct gauge transformation matrices for each vertical edges. We write \(u(p/q)_{m,l}\) for the \(m \times m\) unitary gauge transformation matrix coming from the edges \(p-q\) of multiplicity \(m\) in the left vertical graphs \(\mathcal{H}_0\) and \(u(r/s)_{n,r}\) for the \(n \times n\) unitary gauge transformation matrix coming from the edges \(r-s\) of multiplicity \(n\) in the right vertical graph \(\mathcal{H}_1\). Let

\[
\begin{pmatrix}
x \\
\xi \\
y \\
z \\
w
\end{pmatrix}
\]

\(\xi, \eta\)

to be a \(n \times m\) matrix of the connection \((\alpha \tilde{\alpha} - 1)\sigma \alpha\), where \(n\) and \(m\) is the multiplicities of the edges \(x-y\) and \(z-w\) respectively, and

\[
\begin{pmatrix}
x \\
\xi \sim \eta \\
y \\
z \\
w
\end{pmatrix}
\]

\(\xi, \eta\)

to be a \(n \times m\) matrix of the connection \(\sigma(\alpha \tilde{\alpha} - 1)\sigma \alpha\). Then, the gauge matrices which we are going to construct should satisfy the equality

\[
\begin{pmatrix}
x \\
\xi \sim \eta \\
y \\
z \\
w
\end{pmatrix}
\]

\(\xi, \eta\)

\[
= u(x)_{n,l} u(x)_{z} u(w)_{m,r}
\]

for all pair of vertical edges \((xy, zw)\). Notice that multiplying the connection \(\sigma\) from the left means simply changing the upper vertices of the connection as \(x \leftrightarrow \tilde{x}\), and then the above equality is equivalent to

\[
\begin{pmatrix}
\tilde{x} \\
\xi \\
\tilde{z} \\
y \\
w
\end{pmatrix}
\]

\(\xi, \eta\)

\[
= u(x)_{n,l} u(x)_{z} u(w)_{m,r}
\]

Note that the vertices \(e, f\) and \(g\) are fixed by taking \(\sim\). We easily know that

\[
u(z)_{w} = u(z)_{w}^{t} = u(z)_{w}\]

Note that the multiplicity \(n\) of the edges \(z-w\) is equal to that of the edges \(\tilde{z}-w\). Now we begin to construct a candidate for the list of gauge transformation matrices. First, for the connections \(M(*/6)\) and \(M(*/6)\), we fix the gauges for the simple edges as

\[
u(G)_{1,l} = u(a)_{6} = u(\tilde{a})_{6} = u(G)_{1,r} = (1)_{1,1}
\]
here the matrices are all $1 \times 1$. From the next matrices, we always fix the gauges for simple edges to $1 \times 1$ matrices $(1)_{1,1}$, unless otherwise specified. We denote it simply by $1$.

Next we fix the gauges for the connections $M(\star/5)$, $M(\tilde{\star}/5)$ and $M(b/6)$, $M(\tilde{b}/6)$. We put $u(\tilde{b})_{1,l} = u(b)_{1,l} = 1$.

For $M(b/5)$ and $M(\tilde{b}/5)$, we fix gauges as follows:

\[
\begin{align*}
\tilde{b} & = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} b, \\
& = \begin{pmatrix} 1 \\ u(\tilde{c})_{2,r} \end{pmatrix}, \\
& = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\beta^2}} & \frac{0}{\beta^2 - 1} \\ 0 & \frac{\sqrt{\beta^2 - 1}}{\beta^2} & \frac{-1}{\beta^2} \end{pmatrix}.
\end{align*}
\]

In the same way we get

\[
u(\tilde{c})_{2,r} = \begin{pmatrix} \frac{1}{\beta^2} & \frac{\sqrt{\beta^2 - 1}}{\beta^2} \\ \frac{\beta^2 - 1}{\beta^2} & \frac{-1}{\beta^2} \end{pmatrix}.
\]

For $M(d/6)$ and $M(\tilde{d}/6)$,

\[
\begin{align*}
\tilde{d} & = u(d)_{2,l} \\
& = \begin{pmatrix} -\frac{\sqrt{\beta^2 - 2}}{2\beta^2 - 1} & \frac{\sqrt{\beta^2 + 1}}{2\beta^2 - 1} \\ -\frac{\sqrt{\beta^2 - 2}}{2\beta^2 - 1} & \frac{\sqrt{\beta^2 + 1}}{2\beta^2 - 1} \end{pmatrix} = u(d)_{2,l},
\end{align*}
\]

by symmetricity of this matrix. To check $M(d/5)$ and $M(\tilde{d}/5)$, we use $M(G/e)$. See the matrix (1). Note that the multiplication by $\sigma$ from the left to $(\alpha \tilde{\alpha} - 1)\sigma\alpha$ corresponds to that by $\sigma$ from the right on $\tilde{\alpha}\sigma(\alpha\tilde{\alpha} - 1)$, which causes the permutation of the entries of connection matrix $M(G/e)$ as follows.

\[
\begin{array}{ccc}
G & e \\
 & d & e \\
\end{array} 
\begin{array}{ccc}
G & e \\
 & d & e \\
\end{array}
\]

We denote the connection matrix made from $M(G/e)$ by multiplying $\sigma$ by $M(G/e)_\sim$. Since the vertex $e$ is fixed by multiplying $\sigma$, we should fix the gauge matrix so that $M(G/e)$ and $M(G/e)_\sim$ are transferred each other. By the effect of multiplying $\sigma$,
we see that $M(G/e)_{\sim}$ is made from $M(G/e)$ by exchanging $d^f$ (resp. $\tilde{d}^h$)-row and $\tilde{d}^f$ (resp. $d^h$)-row, i.e., we have the following relation:

$$G \sim e = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \end{pmatrix} = G \sim e,$$

here $\sim$ at the lower left corner in the left hand side square means the changing of the labels and replacing of columns according to the labels. Now we get the gauge as usual. Note that gauge matrices for upside down edges are the same to those of normal position.

$$G \sim e = \begin{pmatrix} u_{(d)_{2,l}} & u_{(d)_{2,l}} \\
st & s \end{pmatrix} = \begin{pmatrix} u_{(e)_{5,r}} \\
1 \end{pmatrix} G \sim e \begin{pmatrix} u_{(e)_{5,r}} \\
1 \end{pmatrix}$$

$$= \begin{pmatrix}
\frac{(\beta^2 - 2)^3}{2\beta^4(\beta^4 + 1)} & -\frac{(\beta^2 - 2)}{2\beta^4} & \frac{\sqrt{\beta^2 - 2}}{2\beta^2 - 1} & \frac{\sqrt{\beta^2 - 2}}{2\beta^2} & -\frac{(\beta^2 - 2)}{2(\beta^2 - 1)} \\
-\frac{\beta^2(\beta^2 - 2)}{2(\beta^2 - 1)(\beta^4 + 1)} & \frac{\sqrt{\beta^2 - 2}}{\beta^4} & \frac{\sqrt{\beta^2 - 2}}{2\beta^2 - 1} & \frac{1}{\sqrt{\beta^2 - 1}} & \frac{1}{\sqrt{\beta^2 - 1}} \\
\frac{\sqrt{\beta^2 - 2}}{(\beta - 1)(\beta^2 + 1)} & -\frac{\sqrt{\beta^2 - 2}}{(\beta^2 - 1)(\beta^4 + 1)} & \frac{\sqrt{\beta^2 - 2}}{\beta^4} & \frac{-1}{2\sqrt{2} - 1} & \frac{-1}{\beta^2 + 1} \\
\frac{\beta^2}{\beta^2 + 4} & \frac{\beta^2}{(\beta^2 - 1)(\beta^2 + 1)} & \frac{\sqrt{\beta^2}}{\beta^4} & \frac{\sqrt{\beta^2}}{\beta^4} & 0 \\
-\frac{\sqrt{\beta^2}}{\beta^4} & \frac{\sqrt{\beta^2}}{\beta^4} & \frac{2\beta}{\beta^4} & \frac{\sqrt{\beta^2}}{\beta^4} & 0 \\
\frac{-2}{\beta^4} & \frac{-2}{(\beta^2 - 1)(\beta^4 + 1)} & \frac{2\beta}{\beta^4} & \frac{\sqrt{\beta^2}}{\beta^4} & \frac{-1}{\beta^2 + 1} \\
\frac{-2}{\beta^4} & \frac{-2}{(\beta^2 - 1)(\beta^4 + 1)} & \frac{2\beta}{\beta^4} & \frac{\sqrt{\beta^2}}{\beta^4} & \frac{-1}{\beta^2 + 1} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

It is too hard to obtain above gauge matrix by calculating all the elements by the multiplication of matrices. Note that,
5. Main theorem for the case of \((5 + \sqrt{17})/2\)

\[
G_e \sim \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} G_e
\]

\[
= \begin{pmatrix}
u(d_{G,2,l}) & u(d_{G,2,l}) \\
u(e_{G,2,l}) & 1
\end{pmatrix} \begin{pmatrix}
u(e_{5,4,r}) \\
1
\end{pmatrix}
\]

\[
\begin{pmatrix}
u(d_{G,2,l})
\end{pmatrix} = \begin{pmatrix}
u(d_{G,2,l})
\end{pmatrix}
\]

and \(u(d_{G,2,l}) = u(d_{G,2,l}) = u(d_{G,2,l})^t\), by comparing the above two equations we know that the matrix for the gauge \(u(e_{5,4,r})\) is symmetric. Thus it is enough first to check the \((5,5)\)-entry is 1, and then to calculate the \((1,1), (2,1), (3,1), (4,1), (2,2), (3,2), (3,3), (2,4), (3,4)\) and \((4,4)\)-entries.

We continue to fix gauge transformation matrices.

\[
\tilde{d}_{5} = \begin{pmatrix}
-1 & u(e_{3,l}) & u(d_{G,2,l}) \\
u(d_{E,3,l}) & u(d_{G,2,l}) & 1
\end{pmatrix} \begin{pmatrix}
u(e_{5,2,r}) \\
u(e_{5,4,r})
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-1 & u(e_{3,l}) & u(d_{G,2,l})
\end{pmatrix}
\]
above calculation, we do not use the entries and (5,6)-entries by unitarity and signs obtained by the numerical calculation of the product of matrices by Mathematica. Then we have (3,3) and (3,4)-entries by the orthogonal relation of the second column and the third, and forth. Now we know that the matrix is block diagonal, and we have the rest (5,5) and (5,6)-entries by unitarity and signs obtained by Mathematica. To execute the above calculation, we do not use the entries * in the multiplication matrix of $M(d/5)$ and the right gauges. Note $u(d)_{E} = u(d)_{C} = -1$.

We see that $u(d)_{C}$ is the same matrix as we have already gotten. From now, we consider that we are always checking it when the matrices for the gauges whose matrices have been already obtained appear with the matrices of new gauges.

We check for the rest connections as before.
\[
A \sim g = A \begin{pmatrix} 1 \\ u(\frac{g}{2}, r) \end{pmatrix},
\]

\[
\begin{pmatrix} 1 \\ u(\frac{g}{2}, r) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\beta^2(\beta^2 - 2)}{\beta^2 - 1} \end{pmatrix},
\]

\[
\begin{pmatrix} 1 & u(\frac{f}{2}, l) & f \end{pmatrix} \begin{pmatrix} 1 & u(\frac{g}{2}, r) \end{pmatrix} = \begin{pmatrix} 1 & u(\frac{f}{2}, l) & f \end{pmatrix} \begin{pmatrix} 1 & u(\frac{g}{2}, r) \end{pmatrix} = \begin{pmatrix} u(\frac{f}{2}, l) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\beta^2(\beta^2 - 2)}{\beta^2 - 1} \end{pmatrix},
\]

\[
C \sim e = \begin{pmatrix} 1 & u(\frac{f}{2}, l) \end{pmatrix} \begin{pmatrix} 1 & u(\frac{e}{3}, r) \end{pmatrix} = \begin{pmatrix} 1 & u(\frac{f}{2}, l) \end{pmatrix} \begin{pmatrix} 1 & u(\frac{e}{3}, r) \end{pmatrix} = \begin{pmatrix} u(\frac{e}{3}, r) \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix},
\]

\[
\begin{pmatrix} u(\frac{e}{3}, r) \end{pmatrix} = \begin{pmatrix} 2(\beta^2 - 2)^3 & -2(\beta^2 - 2)^2 & -\sqrt{(\beta^2 - 2)^3} \\ \beta^2(\beta^2 - 2)^3 & -\beta(\beta^2 - 1)(\beta^2 + 1)^2 & \beta^2(\beta^2 - 2) \\ 1 & \sqrt{2(\beta^2 - 2)} & \sqrt{2}\sqrt{2(\beta^2 - 2)} \\ \sqrt{2(\beta^2 - 2)} & \sqrt{2(\beta^2 - 2)} & \sqrt{2}\sqrt{2(\beta^2 - 2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
\[
\begin{align*}
\tilde{d}_{3} & = \left( u_{(E)}_{3,l} \right) 1 \left( \begin{array}{c}
\frac{2(\beta^2-2)}{\sqrt{2(\beta^2+1)(\beta^2-1)}} \\
\frac{-\sqrt{(\beta^2-2)^2}}{\beta^2(\beta^2+1)^3} \\
\frac{-8\sqrt{2}\beta^2}{\sqrt{2(\beta^2-1)}} \\
\frac{-4\sqrt{2(\beta^2-1)}}{\beta^2(\beta^2+1)} \\
\end{array} \right) \left( \begin{array}{c}
\frac{-\beta^2}{\beta^2+1} \\
\frac{\beta^2}{2\sqrt{(\beta^2-1)^3}} \\
\frac{-\beta^2}{4\beta^2-1} \\
\frac{-\beta^2}{4\beta^2-1} \\
\end{array} \right) \\
& = \left( u_{(E)}_{3,l} \right) 1 \left( \begin{array}{c}
\frac{\beta^2-2}{\beta^2+1} \\
\frac{2\beta^2}{\sqrt{(\beta^2-1)^3}} \\
\frac{-\beta^2}{4\beta^2-1} \\
\frac{-\beta^2}{4\beta^2-1} \\
\end{array} \right),
\end{align*}
\]

\[
\tilde{h}_{3} = h_{3} u_{(g)}_{3,r},
\]

\[
u_{(g)}_{3,r} = \left( \begin{array}{c}
\frac{-1}{2\beta^2-1} \\
\frac{2\sqrt{2}}{2\beta^2-1} \\
\frac{1}{\beta^2+1} \\
\frac{2\sqrt{2}}{2\beta^2-1} \\
\end{array} \right),
\]

\[
f_{3} = \left( u_{(C)}_{2,l} \right) u_{(E)}_{3,l} \left( \begin{array}{c}
\frac{1}{\beta^2} \\
-\frac{(\beta^2-1)\sqrt{\beta^2+1}}{\beta^2} \\
-\frac{1}{\beta^2(\beta^2-1)} \\
\frac{1}{\beta^2} \\
\end{array} \right) \left( \begin{array}{c}
\frac{-2\sqrt{2}\beta^2-1}{\beta^2(\beta^2-2)} \\
\frac{4\beta^2-1}{\beta^2} \\
\frac{-\beta^2}{\beta^2(\beta^2-2)} \\
\frac{-2\sqrt{2}\beta^2+3}{\beta^2\beta^4+4} \\
\end{array} \right) \\
= \left( u_{(C)}_{2,l} \right) u_{(E)}_{3,l} \left( \begin{array}{c}
\frac{1}{\beta^2} \\
-\frac{(\beta^2-1)\sqrt{\beta^2+1}}{\beta^2} \\
-\frac{1}{\beta^2(\beta^2-1)} \\
\frac{1}{\beta^2} \\
\end{array} \right) \left( \begin{array}{c}
\frac{-2\sqrt{2}\beta^2-1}{\beta^2(\beta^2-2)} \\
\frac{4\beta^2-1}{\beta^2} \\
\frac{-\beta^2}{\beta^2(\beta^2-2)} \\
\frac{-2\sqrt{2}\beta^2+3}{\beta^2\beta^4+4} \\
\end{array} \right),
\]

\[
\left( \begin{array}{c}
u_{(C)}_{2,l} \\
u_{(E)}_{3,l} \end{array} \right) = \left( \begin{array}{cc}
\frac{1}{\beta^2-2} & \frac{\sqrt{\beta^2+1}}{\beta^2-2} \\
\frac{-1}{\beta^2-2} & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{array} \right).
\]
\[
\begin{align*}
f_4 & = u(E_{3,1}) f_4 \begin{pmatrix}
\frac{\beta}{2\sqrt{2}} & -\frac{\beta^2}{\sqrt{2}(\beta^4+4)} \\
-\frac{1}{2} & \frac{2\beta}{\sqrt{\beta^4+4}} \\
1 & \frac{-2}{\sqrt{2}(\beta^2-2)} \\
\end{pmatrix} \begin{pmatrix}
-1 \\
0 \\
0 \\
\end{pmatrix}
\begin{pmatrix}
u(4)_{2,r} \\
u(e)_{2,r} \\
u(4)_{2,r} \\
\end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix}
-1 \\
u(e)_{2,r} \\
\end{pmatrix}
& = \begin{pmatrix}
-1 & 0 & 0 \\
0 & \frac{-4\beta^2}{\beta^4+4} & \frac{-\beta^3}{\beta^4+4} \\
0 & \frac{-4\beta^2}{\beta^4+4} & \frac{-\beta^3}{\beta^4+4} \\
\end{pmatrix},
\end{align*}
\]

here note \(u(g-4) = -1\)

\[
\begin{align*}
\tilde{h}_5 & = h_5 u(g)_{5,2,r},
\end{align*}
\]

\[
\begin{align*}
u(g)_{5,2,r} & = \begin{pmatrix}
\frac{-1}{\beta^2-2} & \frac{\sqrt{\beta^2+1}}{\beta^2-2} \\
\frac{\sqrt{\beta^2+1}}{\beta^2-2} & \frac{1}{\beta^2-2} \\
\end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
f_5 & = \begin{pmatrix}
u(f)_{3,1} & u(f)_{2,1} & 1 \\
u(e)_{2,1} & u(e)_{2,1} & 1 \\
\end{pmatrix}
\begin{pmatrix}
u(g)_{5,2,r} & u(g)_{5,4,r} \\
u(e)_{5,4,r} & u(e)_{5,4,r} \\
\end{pmatrix},
\end{align*}
\]

\[
\begin{pmatrix}
\begin{pmatrix}
* & \frac{3}{2(\beta^2-2)} & \beta^2 & \beta^2 \sqrt{\beta^4+4} & \beta \sqrt{\beta^4+4} & \beta \sqrt{\beta^4+4} \\
* & \frac{3}{2(\beta^2-2)} & \beta^2 & \beta^2 \sqrt{\beta^4+4} & \beta \sqrt{\beta^4+4} & \beta \sqrt{\beta^4+4} \\
* & \frac{-\beta}{(\beta^2-1)\sqrt{\beta^4+4}} & \frac{-\beta}{(\beta^2-1)\sqrt{\beta^4+4}} & \frac{-\beta}{(\beta^2-1)\sqrt{\beta^4+4}} & \frac{-\beta}{(\beta^2-1)\sqrt{\beta^4+4}} & \frac{-\beta}{(\beta^2-1)\sqrt{\beta^4+4}} \\
* & \frac{-\beta}{(\beta^2-1)\sqrt{\beta^4+4}} & \frac{-\beta}{(\beta^2-1)\sqrt{\beta^4+4}} & \frac{-\beta}{(\beta^2-1)\sqrt{\beta^4+4}} & \frac{-\beta}{(\beta^2-1)\sqrt{\beta^4+4}} & \frac{-\beta}{(\beta^2-1)\sqrt{\beta^4+4}} \\
* & \frac{2\beta^2-1}{\sqrt{2}(\beta^2-2)} & \beta^2 & \beta \sqrt{\beta^4+4} & \beta \sqrt{\beta^4+4} & \beta \sqrt{\beta^4+4} \\
* & \frac{2\beta^2-1}{\sqrt{2}(\beta^2-2)} & \beta^2 & \beta \sqrt{\beta^4+4} & \beta \sqrt{\beta^4+4} & \beta \sqrt{\beta^4+4} \\
\end{pmatrix}
\end{pmatrix},
\end{align*}
\]
$$\begin{pmatrix} u(f_3)_3, l & u(f_4)_{2, l} & 1 \\ \end{pmatrix} = \begin{pmatrix} -\frac{1}{\beta^2+1} & -\frac{(\beta^2-2)^2}{4\sqrt{\beta}} & \frac{\beta^2}{\beta^2+1} & 0 & 0 & 0 \\ -\frac{(\beta^2-2)^2}{4\sqrt{2}\beta} & -\frac{(\beta^2-1)}{\beta^2+1} & -\frac{(\beta^2-2)}{2\sqrt{\beta^2+1}} & 0 & 0 & 0 \\ \frac{\beta^2}{\beta^2+1} & -\frac{(\beta^2-2)}{2\sqrt{\beta^2+1}} & -\frac{1}{\beta^2+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta^2+1} & \frac{\beta^2+1}{2\sqrt{2}\beta} & 0 \\ 0 & 0 & 0 & \frac{2\sqrt{\beta}}{\beta^2+1} & \frac{1}{\beta^2+2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \end{pmatrix},$$

$$\begin{pmatrix} b & c_e \\ E^3 & 3 \\ \end{pmatrix} = \begin{pmatrix} \tilde{b} & \tilde{c}_e \\ E^d & 3 \\ \end{pmatrix} \times (-1),$$

$$u(c-4) = u(\tilde{c}-4) = -1.$$
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\[
\begin{pmatrix}
-1 & 0 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & \frac{\beta^4-4}{\beta^4+4} & \frac{\beta^2}{\beta^4+4} & \frac{4\beta^2}{\beta^4+4}
\end{pmatrix}
\]

We can see that \(u_{(4),2,r}^{(e)}\) here is the same matrix as when it appeared first. The latter equalities are of scalar matrices or \(2 \times 2\) matrices which we can check at a glance that the gauge matrix is common with which we already used, in either case it is easy enough not to write down.

All the above identities, we have checked using Mathematica and of course we have made repeatably use of the identity \(\beta^4 - 5\beta^2 + 2 = 0\). At last, we have obtained the equivalence of the connections

\[(\alpha\bar{\alpha} - 1)\sigma\alpha \cong \sigma(\alpha\bar{\alpha} - 1)\sigma\alpha\]

up to the vertical gauge choice.

Finally we will check conditions 1) and 2). Along the same argument of the proof of the previous theorem for \((5 + \sqrt{13})/2\) case, we see the indecomposability other than for \((\alpha\bar{\alpha} - 1)\sigma\alpha\) and mutually inequivalence of all. In Figure 10, \(*\) in \(V_0\) is vertex of only one edge in \(\mathcal{K}\). Thus, using Cororally 3, we have indecomposability of the connection \((\alpha\bar{\alpha} - 1)\sigma\alpha\). Now, the proposition holds and thus we have proved the theorem.

q.e.d.
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