Inverted Mass Hierarchy from Scaling in the Neutrino Mass Matrix: Low and High Energy Phenomenology

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Abstract

Best-fit values of recent global analyzes of neutrino data imply large solar neutrino mixing, vanishing $U_{e3}$ and a non-maximal atmospheric neutrino mixing angle $\theta_{23}$. We show that these values emerge naturally by the hypothesis of “scaling” in the Majorana neutrino mass matrix, which states that the ratios of its elements are equal. It also predicts an inverted hierarchy for the neutrino masses. We point out several advantages and distinguishing tests of the scaling hypothesis compared to the $L_e - L_\mu - L_\tau$ flavor symmetry, which is usually assumed to provide an understanding of the inverted hierarchy. Scenarios which have initially vanishing $U_{e3}$ and maximal atmospheric neutrino mixing are shown to be unlikely to lead to non-maximal $\theta_{23}$ while keeping simultaneously $U_{e3}$ zero. We find a peculiar ratio of the branching ratios $\mu \to e\gamma$ and $\tau \to e\gamma$ in supersymmetric seesaw frameworks, which only depends on atmospheric neutrino mixing and results in $\tau \to e\gamma$ being unobservable. The consequences of the scaling hypothesis for high energy astrophysical neutrinos at neutrino telescopes are also investigated. Then we analyze a seesaw model based on the discrete symmetry $D_4 \times Z_2$ leading to scaling in the low energy mass matrix and being capable of generating the baryon asymmetry of the Universe via leptogenesis. The relevant CP phase is identical to the low energy Majorana phase and successful leptogenesis requires an effective mass for neutrinoless double beta decay larger than 0.045 eV.

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1 Introduction

Observed lepton mixings are consequences of a non-trivial structure of the neutrino mass matrix $\mathcal{M}_\nu$. This symmetric matrix for Majorana neutrinos (having entries $m_{\alpha\beta}$ with $\alpha, \beta = e, \mu, \tau$) is in the charged lepton basis diagonalized by the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) neutrino mixing matrix $U$. The very different structure of $U$ compared to the quark sector for all possible neutrino mass orderings is indicative of an unexpected texture of the mass matrix, and could hold important clues to our understanding of the physics of fundamental constituents of matter. To unravel this new physics, various Ans"atze for $\mathcal{M}_\nu$ have been made in the literature [1] and their associated symmetries have been sought after. One particular proposal, recently proposed by two of us (R.N.M. and W.R.), on which we will focus in this note, is called “scaling” [2]. The scaling hypothesis demands that the ratio $\frac{m_{\alpha\beta}}{m_{\alpha\gamma}}$ is independent of the flavor $\alpha$:

$$\frac{m_{e\beta}}{m_{e\gamma}} = \frac{m_{\mu\beta}}{m_{\mu\gamma}} = \frac{m_{\tau\beta}}{m_{\tau\gamma}} = c \text{ for fixed } \beta \text{ and } \gamma.$$  \hspace{1cm}(1)

There are three possibilities and the only one phenomenologically allowed is when $\beta = \mu$ and $\gamma = \tau$. We shall call this case scaling henceforth. The resulting mass matrix reads

$$\mathcal{M}_\nu = m_0 \begin{pmatrix} A & B & B/c \\ B & D & D/c \\ B/c & D/c & D/c^2 \end{pmatrix}.$$  \hspace{1cm}(2)

Similar matrices have been found independently in the context of specific models (see Ref. [3]). The most important phenomenological prediction of scaling is that Eq. (2) leads to an inverted hierarchy with $m_3 = 0$ and $U_{e3} = 0$. Atmospheric neutrino mixing is governed by the “scaling factor” $c$ via $\tan^2 \theta_{23} = 1/c^2$, i.e., is in general non-maximal because $c$ is naturally of order, but not equal to, one. It is interesting to note that current data analyzes (though at the present stage statistically not very significant) yield non-maximal $\tan^2 \theta_{23} = 0.89$ as the best-fit point [4] (see also [5], where the best-fit value is $\tan^2 \theta_{23} = 0.82$). The reason is that in the SuperKamiokande experiment there is an excess of sub-GeV electron events, but no excess either of sub-GeV muon events or of multi-GeV electrons. In a realistic 3-flavor analysis this prefers $\cos \theta_{23} > \sin \theta_{23}$ [4] [5]. The best-fit value of $U_{e3}$ is zero.

If these values, $\theta_{23} \neq \pi/4$ and $U_{e3} = 0$, are indeed confirmed by future data then one should look for symmetries and/or models which are capable of predicting such a situation. Ideally, such a candidate should be rather insensitive to radiative corrections and should not require much, if any, breaking to achieve the values sought for. Scaling is one such appealing possibility, several general aspects of which will be discussed in Section 2. Typical models are shown to predict – when constructed in a supersymmetric seesaw framework – a characteristic ratio of the branching ratios $\mu \rightarrow e\gamma$ and $\tau \rightarrow e\gamma$. The latter decay is then too rare to be observable in presently foreseen experiments. Simple phenomenology of
fluxes of high energy astrophysical neutrinos at neutrino telescopes is predicted and studied in Section 3. We argue further in Section 3 that it is difficult to obtain $\theta_{23} \neq \pi/4$ and $U_{e3} = 0$ in scenarios in which initially $\theta_{23} = \pi/4$ and $U_{e3} = 0$ holds. Stressing that scaling predicts an inverted hierarchy leads us to compare the Ansatz with the flavor symmetry $L_e - L_\mu - L_\tau$ [3]. The latter is usually assumed to be the origin of an inverted hierarchy. We show in Section 5 that scaling possesses several advantages over $L_e - L_\mu - L_\tau$. If future experiments indeed show that neutrinos obey an inverted hierarchy, then one needs a full list of possible scenarios that can predict it. Necessarily, these are unusual symmetries as typical GUT models do not lead to an inverted ordering. Accordingly, we investigate in Section 6 a seesaw model leading to scaling based on the discrete symmetry $D_4 \times Z_2$. We show that the baryon asymmetry of the Universe via leptogenesis can be reproduced and analyze the connection to the low energy parameters. We summarize in Section 7.

2 General Properties of Scaling

The neutrino mass matrix giving rise to scaling is given in Eq. (2). It appears in the Lagrangian

$$\mathcal{L} = \frac{1}{2} \bar{\nu}_L \mathcal{M}_\nu \nu_L + \bar{\ell}_R \mathcal{M}_\ell \ell_L ,$$

where $\mathcal{M}_\ell$ is the charged lepton mass matrix. Diagonalizing it via $\mathcal{M}_\ell = V_\ell \mathcal{M}_\ell^{\text{diag}} U_\ell^\dagger$ and the neutrino mass matrix with $U_\nu^* \mathcal{M}_\nu^{\text{diag}} U_\nu^\dagger = \mathcal{M}_\nu$, gives us the PMNS matrix

$$U = U_\ell^\dagger U_\nu = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13} \end{pmatrix} P ,$$

where $c_{ij} = \cos \theta_{ij}$, $s_{ij} = \sin \theta_{ij}$, and $P = \text{diag}(1, e^{i\alpha}, e^{i(\beta+\delta)})$ contains the Majorana phases [7]. The best-fit values as well as the allowed 1$\sigma$ and 3$\sigma$ ranges of the oscillation parameters are [4]:

$$\Delta m^2_{\odot} = \begin{pmatrix} 7.9^{+0.3}_{-0.3}, & 1.0 \end{pmatrix} \cdot 10^{-5} \text{ eV}^2 ,$$

$$\sin^2 \theta_{12} = 0.31^{+0.02}_{-0.02},$$

$$\Delta m^2_{\text{atm}} = \begin{pmatrix} 2.6^{+0.2}_{-0.2}, & 0.6 \end{pmatrix} \cdot 10^{-3} \text{ eV}^2 ,$$

$$\tan^2 \theta_{23} = 0.89^{+0.31}_{-0.21},$$

$$|U_{e3}|^2 < 0.008 (0.040) ,$$

where $\Delta m^2_{\odot} = m^2_2 - m^2_1$ and $\Delta m^2_{\text{atm}} = |m^2_3 - m^2_1|$.

We will most of the time assume that scaling holds in the charged lepton basis, i.e., $U_\ell = 1$. We nevertheless stress here the following interesting property of scaling: if $U_\ell$ is non-trivial, then in the charged lepton basis we have $\mathcal{M}_\nu = U_\ell^T \mathcal{M}_\nu U_\ell$. In case $U_\ell$ is only given by a
23-rotation it is easy to show that the $ee$ entry of $\mathcal{M}_\nu$ is not affected, and in addition we have:
\[ \frac{\bar{m}_{e\mu}}{m_{e\tau}} = \frac{\bar{m}_{\mu\mu}}{m_{\mu\tau}} = \frac{\bar{m}_{\mu\tau}}{m_{\tau\tau}} = \frac{c \cos \theta_{23} - \sin \theta_{23}^\ell}{\cos \theta_{23} + c \sin \theta_{23}^\ell} \equiv \tilde{c}, \]
where $\theta_{23}^\ell$ is the rotation angle of $U_\ell$ and $\bar{m}_{\alpha\beta}$ are the entries of $\bar{\mathcal{M}}_\nu$. Consequently, the texture of the neutrino mass matrix is not changed at all, it still obeys scaling and predicts $m_3 = U_{e3} = 0$ but now with $\tan^2 \theta_{23} = 1/\tilde{c}^2$. If for some reason in $\mathcal{M}_\nu$ the scaling factor $c$ is much larger or smaller than 1, then a 23-rotation from the charged lepton sector can still save the scaling hypothesis. For $c \gg 1$ we find $\tan^2 \theta_{23} \simeq \tan^2 \theta_{\ell 23}$, while for $c \ll 1$ we have $\tan^2 \theta_{23} \simeq \cot^2 \theta_{\ell 23}$. This observation can simplify the construction of models.

Interestingly, the characteristic predictions $m_3 = U_{e3} = 0$ are not subject to any radiative corrections when going from high scale down to low scale. This can be understood by letting RG effects directly modify the mass matrix, as done in Ref. [2], or by glancing at the RG equations of $\theta_{13}$ and $m_3$. For both of them it holds that
\[ \dot{m}_3 \propto m_3 \quad \text{and} \quad \dot{\theta}_{13} \propto m_3, \]
i.e., $\theta_{13} = m_3 = 0$ is stable under RG evolution.

What are other phenomenological predictions of scaling? First of all, there will be no CP violation in oscillation experiments because of $\theta_{13} = 0$. Then we note that from $m_3 = 0$ it follows that one Majorana phase is unphysical. The other one appears in the effective mass to which neutrinoless double beta decay is sensitive [9]. This parameter takes a very simple form for the inverted hierarchy with $m_3 = \theta_{13} = 0$:
\[ \langle m \rangle = \sqrt{\Delta m^2_{A}} \sqrt{1 - \sin^2 2\theta_{12} \sin^2 \alpha}. \]

The range of $\langle m \rangle$ lies for the best-fit parameters from Eq. (5) between 0.019 and 0.051 eV, while at 1(3)σ it ranges between 0.017 and 0.053 eV (0.011 and 0.057 eV). If the parameters do not conspire to render $\langle m \rangle$ at the very low end of this range then next-generation experiments will definitely observe neutrinoless double beta decay [9]. The conditions under which one can extract $\alpha$ from an observation of neutrinoless double beta decay are given in Ref. [10]. Unlike many other approaches, the scaling Ansatz can therefore be completely reconstructed. Of course, the scaling Ansatz is easier to disprove than to prove. However, if future experiments give very strong limits on $|U_{e3}|$ and the inverted hierarchy is present, then this would be a very strong hint towards the realization of scaling.

How can a low energy mass matrix like Eq. (2) be achieved? We work of course in the framework of the seesaw mechanism [11] in which $\mathcal{M}_\nu = -M_D^T M_R^{-1} M_D$, where $M_D$ is the Dirac and $M_R$ the heavy Majorana neutrino mass matrix. One remarkable property of scaling is the following: if the Dirac mass matrix obeys scaling, i.e.,
\[ M_D = \begin{pmatrix} a_1 & b & b/c \\ a_2 & d & d/c \\ a_3 & e & e/c \end{pmatrix}, \]

then $M_\nu$ takes the form obeying scaling\footnote{Note that there is no need for $M_R$ to obey scaling because it would be singular in this case, invalidating simple seesaw.} from Eq. (2) regardless of the structure of $M_R$! Note that $M_D$ is not necessarily required to be symmetric.

Within supersymmetrized seesaw models one has an interesting connection to lepton flavor violating (LFV) decays of charged leptons such as $\mu \rightarrow e\gamma$ \cite{12}. RG evolution within theories of universal boundary (mSUGRA) conditions leads to off-diagonal entries in the slepton mass matrix, which trigger effects of LFV. In particular, branching ratios of the decays $\ell_i \rightarrow \ell_j \gamma$ with $(\ell_i, \ell_j) = (\mu, e)$, $(\tau, e)$, $(\tau, \mu)$ are given by \cite{12}

$$\frac{BR(\ell_i \rightarrow \ell_j \gamma)}{BR(\ell_i \rightarrow e \nu \nu)} = \frac{\alpha^3}{G_F^2 v_{wk}^4 m_S^2} \left| \frac{(3 + a_0^2)}{8 \pi^2} \right|^2 \left| (M_D^\dagger M_D)_{ij} \right|^2 \tan^2 \beta,$$

$$\text{where } (L)_{ij} = \delta_{ij} \ln \frac{M_X}{M_i}.$$  

Here $v_{wk} = 174 \text{ GeV}$, $M_i$ are the heavy Majorana neutrino masses and $M_X > M_i$ is the scale at which universal boundary conditions are implemented. The SUSY parameters are $a_0 = A_0/m_0$ with $m_0$ the universal scalar mass, $A_0$ the universal trilinear coupling, and $m_S$ is a typical SUSY mass. Neglecting the only logarithmic dependence on the heavy Majorana masses, the branching ratios are proportional to the modulus-squared of the off-diagonal entries of $M_D^\dagger M_D$. It follows with Eq. (9) that

$$\frac{1}{BR(\tau \rightarrow e \nu \nu)} \frac{BR(\tau \rightarrow e\gamma)}{BR(\mu \rightarrow e\gamma)} = \left| \frac{(M_D^\dagger M_D)_{31}}{(M_D^\dagger M_D)_{21}} \right|^2 = \frac{1}{c^2} = \tan^2 \theta_{23},$$

with $BR(\tau \rightarrow e\nu \nu) \simeq 0.1784$. The two branching ratios are therefore simply related by the atmospheric neutrino mixing angle. Consequently, such models predict them within (see Eq. (5)) a factor of less than four. The current limit of $BR(\mu \rightarrow e\gamma) \leq 1.2 \times 10^{-11}$ \cite{13}, implies therefore that $BR(\tau \rightarrow e\gamma)$ will always be close to this number and consequently at least two orders of magnitude below the future limits (between $10^{-8}$ and $10^{-9}$) which are currently foreseen.

To be more precise, the branching ratios as defined in Eq. (10) have to be evaluated in the basis in which the charged leptons and the heavy Majorana neutrinos are diagonal. In this case $M_D$ has to be replaced with $\tilde{M}_D = V_R^T M_D U_\ell$, where $M_R = V_R^* M_R^{\text{diag}} V_R^T$. Obviously, $V_R$ drops out of $\tilde{M}_D^\dagger \tilde{M}_D$ and does not influence LFV. Now consider again the case that $U_\ell$ is non-trivial and given by a 23-rotation, which we showed above to keep the scaling predictions of $m_3 = U_{e3} = 0$ and to change $c$ to $\tilde{c}$ given in Eq. (3). One easily finds that

$$\left| \frac{(M_D^\dagger M_D)_{31}}{(M_D^\dagger M_D)_{21}} \right|^2 = \frac{\cos \theta_{23}^\ell + c \sin \theta_{23}^\ell}{c \cos \theta_{23}^\ell - \sin \theta_{23}^\ell}.$$  

This expression is nothing but $1/c^2$ and therefore we recover the relation Eq. (11) between the “double ratio” $BR(\tau \rightarrow e\gamma)/BR(\mu \rightarrow e\gamma)$ and atmospheric neutrino mixing.
It is interesting to ask whether scaling can be applied to non-standard neutrino scenarios. In particular, the possibility of neutrinos being Dirac neutrinos and the presence of additional light sterile neutrino species is discussed frequently.

Dirac neutrinos could be accommodated by scaling if the neutrino mass matrix would take the form given in Eq. (9). The resulting neutrino oscillation phenomenology (obtained by diagonalizing $M_D^* M_D$) would be again an inverted hierarchy with $\tan^2 \theta_{23} = 1/c^2$ and $U_{e3} = m_3 = 0$. The reason is simply because the resulting mass matrix possesses an eigenvalue 0 with a corresponding eigenvector of $(0, -1/c, 1, 0, 0, \ldots)^T$.

Sterile neutrinos would require enlarging $M_\nu$ from being a $3 \times 3$ matrix to a $(3 + n_s) \times (3 + n_s)$ matrix, where $n_s$ is the number of additional sterile neutrinos. The recent results of the MiniBooNE experiment \cite{14} seem to indicate that $n_s \geq 2$ \cite{15}. We can modify the scaling condition from Eq. (1) to include also the sterile neutrinos:

$$
\frac{m_{e\mu}}{m_{e\tau}} = \frac{m_{\mu\mu}}{m_{\mu\tau}} = \frac{m_{\tau\mu}}{m_{\tau\tau}} = \frac{m_{s1\mu}}{m_{s1\tau}} = \frac{m_{s2\mu}}{m_{s2\tau}} = \ldots = c.
$$

The result is a mass matrix with a zero eigenvalue having an eigenvector $(0, -1/c, 1, 0, 0, \ldots)^T$.

The mixing scenario is described by $U_{e3} = 0$, $|U_{\mu3}/U_{\tau3}|^2 = 1/c^2$ and $U_{s13} = U_{s23} = \ldots = 0$. Leptonic CP violation is important in order to allow scenarios with two sterile neutrinos to survive constraints from current data \cite{15}. In total there are five CP phases in this case, and only three of them are unphysical due to the vanishing mixing matrix elements. For two sterile neutrinos, the resulting scenario would correspond to an inverted hierarchy of the three mostly active neutrinos and two heavier, mostly sterile neutrinos. It is in the language of Ref. \cite{16} the scenario “SSI”, which has from all possible mass orderings the smallest predictions for the various mass-related observables (neutrinoless double beta decay, neutrino mass in KATRIN and the sum of masses in cosmology) \cite{16}.

### 3 Scaling Predictions for Astrophysical Neutrinos

It has recently been recognized that measuring flux ratios of high energy astrophysical neutrinos \cite{17} is an alternative method to determine neutrino mixing parameters \cite{18, 19, 20}. In particular, one expects from astrophysical $pp$ or $p\gamma$ processes, which generate pions and kaons, an initial flux composition of the form $\Phi^0_e : \Phi^0_\mu : \Phi^0_\tau = 1 : 2 : 0$, where $\Phi^0_\alpha$, with $\alpha = e, \mu, \tau$ is the flux of neutrinos with flavor $\alpha$. Neutrino mixing modifies the flavor composition and in terrestrial neutrino telescopes such as IceCube \cite{21} one can then measure flux ratios and thereby obtain information on the neutrino parameters. The measurable neutrino flux is given by

$$
\Phi_\alpha = \sum_{\beta} P_{\alpha\beta} \Phi^0_\alpha \quad \text{with} \quad P_{\alpha\beta} = \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2.
$$

In the limit of maximal atmospheric neutrino mixing and vanishing $U_{e3}$, the composition $1 : 2 : 0$ is transformed into $1 : 1 : 1$, independent of the solar neutrino mixing angle. Small
deviations from $\theta_{23} = \pi/4$ and $\theta_{13} = 0$ lead to \cite{19, 20}
\[
\Phi_e : \Phi_\mu : \Phi_\tau = 1 + 2 \Delta : 1 - \Delta : 1 - \Delta ,
\]
where $\Delta = \frac{1}{4} \sin 4\theta_{12} |U_{e3}| \cos \delta + \frac{1}{2} \sin^2 2\theta_{12} \left( \frac{1}{2} - \sin^2 \theta_{23} \right)$ .
\hspace{1cm} (15)

Thus, there is a universal first order correction in terms of the small parameters $|U_{e3}|$ and $\frac{1}{2} - \sin^2 \theta_{23}$. With the current $1\sigma$ ($3\sigma$) ranges of the oscillation parameters one finds that $-0.036 \leq \Delta \leq 0.057$ ($-0.097 \leq \Delta \leq 0.113$). The ratio of electron neutrinos to the other flavors is therefore a probe of $\theta_{23}$, $\theta_{13}$ and $\cos \delta$. Note that in the definition of $\Delta$ in Eq. (15) the factor in front of $\frac{1}{2} - \sin^2 \theta_{23}$ is larger and has a smaller range than the one in front of $|U_{e3}| \cos \delta$. To be precise, for the allowed $3\sigma$ range of solar neutrino mixing, $\frac{1}{4} \sin 4\theta_{12}$ ranges from 0.12 to 0.21, whereas $\frac{1}{2} \sin^2 2\theta_{12}$ ranges from 0.38 to 0.48. Consequently \cite{20}, the sensitivity to deviations from maximal atmospheric neutrino mixing is better than the sensitivity to deviations from $|U_{e3}| = 0$, which in addition gets smeared by the dependence on the CP phase $\delta$.

We conclude that scaling – predicting only non-maximal $\theta_{23}$ and neither $\delta$ nor $\theta_{13}$ – will have particularly simple, interesting and potentially testable phenomenology at neutrino telescopes. To go into more detail, let us introduce the small parameter
\[
\epsilon = \frac{\pi}{4} - \theta_{23} ,
\]\hspace{1cm} (16)
which can be linked to the scaling parameter $c$ via $\frac{1}{4} (c^2 - 1)/(c^2 + 1) = \frac{1}{2} - \sin^2 \theta_{23} \simeq \epsilon$.

The result for the flux ratios is
\[
\Phi_e : \Phi_\mu : \Phi_\tau = 1 + 2 c^2 s^2_{12} (c^2_{23} - s^2_{23}) : 2 (1 - 2 c^2_{23} s^2_{23} - c^2_{12} s^2_{12} (c^2_{23} - s^2_{23}) c^2_{23}) : 2 s^2_{23} (1 + (1 - c^2_{12} s^2_{12}) (c^2_{23} - s^2_{23})) \\
\simeq 1 + 4 c^2_{12} s^2_{12} \epsilon : 1 - 2 c^2_{12} s^2_{12} \epsilon : 1 - 2 c^2_{12} \epsilon ,
\]\hspace{1cm} (17)
where we have given the exact expression and the expansion in terms of $\epsilon$ to first order.

The electron neutrino flux $\Phi_e$ receives no quadratic correction and the second order term for $\Phi_\mu$ is $4 (1 - c^2_{12} s^2_{12}) \epsilon^2$, while that for $\Phi_\tau$ is identical but with opposite sign. In Fig. [1] we plot – using the exact probabilities – the flux ratios $\Phi_e/\Phi_\mu$ and $\Phi_\mu/\Phi_{tot}$, where $\Phi_{tot}$ is the total neutrino flux, as a function of the scaling parameter $c$. For $c = 1$ we obtain $\Phi_e/\Phi_\mu = 1$ and $\Phi_\mu/\Phi_{tot} = \frac{1}{3}$. Quite large deviations from these values are allowed, and the dependence on solar neutrino mixing is very weak.

Finally, we note that if there are sufficiently fast non-standard decay modes of the neutrinos, and only the lightest state $\nu_i$ ($i = 1$ for normal ordering, $i = 3$ for inverted ordering) survives and is detected, the fluxes obey the relation \cite{22} $\Phi_e : \Phi_\mu : \Phi_\tau = |U_{ei}|^2 : |U_{\mu i}|^2 : |U_{\tau i}|^2$. In case of scaling (inverted hierarchy and $\theta_{13} = 0$), this simplifies to
\[
\Phi_e : \Phi_\mu : \Phi_\tau = 0 : \sin^2 \theta_{23} : \cos^2 \theta_{23} = 0 : 1 : c^2 .
\]\hspace{1cm} (18)

There are no electron neutrinos and the ratio of muon to tau neutrinos is $\tan^2 \theta_{23} = 1/c^2$.

We plot for the case of decaying neutrinos in Fig. [1] the ratio of muon neutrinos to the total flux, which is simply equal to $\sin^2 \theta_{23}$. 

7
Flux Ratio

Figure 1: Flux ratios of high energy astrophysical neutrinos as a function of the scaling parameter $c$. The red (solid) lines are $\Phi_e/\Phi_\mu$ for the standard case of an initial $1:2:0$ composition, the green (dashed) lines are $\Phi_\mu/\Phi_{\text{tot}}$ in the standard case and the violet (dot-dashed) line is for neutrino decay. The first two cases are plotted for the best-fit values and the upper and lower $3\sigma$ values of the solar neutrino mixing angle $\theta_{12}$.

4 Non-maximal atmospheric Neutrino Mixing and vanishing $U_{e3}$ from other Scenarios

The question arises if we can obtain the values $\theta_{13} = 0$ and $\theta_{23} \neq \pi/4$ by breaking or modifying scenarios with initial $\theta_{13} = 0$ and $\theta_{23} = \pi/4$. Explicit breaking of the symmetry in a mass matrix, RG effects and contributions from the charged lepton sector are appealing possibilities, which we will now comment on.

Considering first radiative corrections, we note here that $\dot{\theta}_{13}$ and $\dot{\theta}_{23}$ are inversely proportional to $\Delta m^2_{3\alpha}$ [8], and due to this one would expect that in general RG corrections to them are of the same order and in addition small. Therefore, if initially $\theta_{13} = \theta_{23} - \pi/4 = 0$, then keeping at low energy $\theta_{13}$ zero but having simultaneously $\theta_{23}$ non-maximal would require rather special values of the other parameters and unnatural cancellations in particular between the CP phases. Hence, $\theta_{23} \neq \pi/4$ and $\theta_{13} = 0$ should hold initially and we end up again with the question of how these peculiar values arose, which leads us back to the scaling hypothesis.

Turning to explicit breaking of the symmetry in a mass matrix leading to $\theta_{13} = 0$ and $\theta_{23} = \pi/4$ requires a glance at $\mu-\tau$ symmetry [23], which imprints the following form on
the mass matrix:

\[ \mathcal{M}_\nu = m_0 \begin{pmatrix} A & B & B \\ B & D & E \\ B & E & D \end{pmatrix} \]

(19)

Maximal \( \theta_{23} \) and zero \( U_{e3} \) are predicted, but only if the eigenvalue belonging to the eigenvector \((0, -1, 1)^T\) is the largest or smallest one, which would then correspond to the normal or inverted mass ordering. In case of scaling, the eigenvector \((0, -1/c, 1)^T\) belongs automatically to the zero eigenvalue, therefore there is no such ambiguity. For a normal hierarchy of the neutrino masses the parameters in Eq. (19) have to fulfill \( D, E \ll A, B \). If in this case the \( \mu-\tau \) symmetry is broken such that only the \( \mu\mu \) and \( \mu\tau \) entries differ, but the \( e\mu \) and \( e\tau \) elements stay identical, then it turns out that \( U_{e3} \) is small but non-zero (to be precise, it is of order \( \Delta m_{23}^2/\Delta m_{12}^2 \)). This way of breaking has no analogue for an inverted hierarchy or quasi-degenerate neutrinos, for which rather tuned breaking scenarios are required in order to end up with \( \theta_{13} = 0 \) but \( \theta_{23} \neq \pi/4 \).

Contributions from the charged leptons arise when \( \mathcal{M}_\nu \) is \( \mu-\tau \) symmetric and in the limit of a diagonal charged lepton mass matrix \( \theta_{13} = 0 \) and \( \theta_{23} = \pi/4 \) would result from \( U = U_\ell \dagger U_\nu = U_\nu \). It is commonly assumed that \( U_\ell \) contains only small angles. Introducing for the sines of these angles the abbreviations \( \sin \theta_{ij} \equiv \lambda_{ij} \), one finds in first order of these small parameters that [25, 26]

\[ |U_{e3}| \simeq \frac{1}{\sqrt{2}} |\lambda_{12} - \lambda_{13} e^{i\phi_1}|, \]

\[ \sin^2 \theta_{23} \simeq \frac{1}{2} + \lambda_{23} \cos \phi_2 - \frac{1}{4} (\lambda_{12}^2 - \lambda_{13}^2) + \frac{1}{2} \cos \phi_1 \lambda_{12} \lambda_{13}, \]

(20)

where \( \phi_1 \) and \( \phi_2 \) are CP phases. Consequently, \( \theta_{13} = 0 \) and \( \theta_{23} \neq \pi/4 \) would require delicate interplay of the angles and phases in \( U_\ell \) and would lead to a rather unnatural form for it. In particular, for the natural case of \( U_\ell \) being CKM-like, i.e., \( \lambda_{12} \gg \lambda_{13,23} \), the result \( \theta_{13} = 0 \) and \( \theta_{23} \neq \pi/4 \) can not be achieved. Similar statements hold for the opposite case in which in the limit \( U_\nu = \mathbb{1} \) the charged lepton sector would suffice to generate \( \theta_{13} = 0 \) and \( \theta_{23} = \pi/4 \) in \( U = U_\ell \dagger U_\nu = U_\ell \) [25].

We conclude that it seems rather unnatural to obtain \( U_{e3} = 0 \) and \( \theta_{23} \neq \pi/4 \) within broken \( \mu-\tau \) symmetry. These predictions for the mixing angles also occur in models based on the flavor symmetry \( L_e - L_\mu - L_\tau \). However, as we will show in the next Section, various tuning problems show up for this Ansatz.

5 Inverted Hierarchy: Scaling vs. the \( L_e - L_\mu - L_\tau \) Flavor Symmetry

Note that scaling is an Ansatz for the inverted hierarchy which is fundamentally different from the flavor symmetry \( L_e - L_\mu - L_\tau \) [6], which is usually “blamed” for it. A detailed
comparison is therefore a worthy exercise. A mass matrix obeying the flavor symmetry 
$L_e - L_\mu - L_\tau$ reads

$$
\mathcal{M}_\nu = m_0 \begin{pmatrix}
0 & \cos \theta_{23} & -\sin \theta_{23} \\
\cos \theta & 0 & 0 \\
-\sin \theta_{23} & 0 & 0
\end{pmatrix}, \text{ where } m_0 = \sqrt{\Delta m_A^2},
$$

(21)

and generates $m_3 = U_{e3} = 0$ as well as non-maximal atmospheric neutrino mixing given by $\theta_{23}$. However, it also predicts *maximal* solar neutrino mixing and vanishing $\Delta m^2_\odot = m_2^2 - m_1^2$, which is in contradiction to observation. Therefore, in contrast to scaling, the symmetry needs to be broken to achieve correct phenomenology, which imposes three problems [27]:

(i) the breaking terms in the mass matrix have to have at least 30% the magnitude of the terms allowed by the symmetry. The reason is that in the inverted hierarchy the
$ee$ entry of $\mathcal{M}_\nu$ (the effective mass) is required to be larger than $\sqrt{2 \Delta m_A^2 \cos 2\theta_{12}}$;

(ii) the (large) breaking of the symmetry in the mass matrix is always connected with
fine-tuning because usually the required large deviation from maximal $\theta_{12}$ is connected with the small ratio of the solar and atmospheric mass-squared differences. For instance, let us add to the matrix in Eq. (21) the following ($\mu-\tau$ symmetric) perturbation:

$$
\epsilon \begin{pmatrix}
a & b & b \\
b & d & e \\
b & e & d
\end{pmatrix}.
$$

If for simplicity we set $\theta_{23} = \pi/4$ in Eq. (21) then maximal atmospheric neutrino mixing and $U_{e3} = 0$ will remain, but the ratio of mass-squared differences is now $\Delta m^2_\odot/\Delta m_A^2 \simeq \sqrt{2} (a + d + e) \epsilon$ while solar neutrino mixing is governed by $\sin \theta_{12} \simeq 1/\sqrt{2} - (a - d - e) \epsilon/8$. Hence, if one wants to reproduce the best-fit values of the oscillation parameters in Eq. (11), the fine-tuned condition $(a + d + e)/(a - d - e) \simeq 0.0027$ has to be fulfilled;

(iii) if the symmetry is broken by contributions from the charged lepton sector (note that in this case in addition breaking in the neutrino sector is necessary to generate non-vanishing $\Delta m^2_\odot$) a CP violating phase appears in the expression for the now non-maximal solar neutrino mixing angle, which is required to be close to zero: using again the natural choice of a CKM-like $U_\ell$ (see also Eq. (20)) leads to the formula

$$
\sin^2 \theta_{12} \simeq \frac{1}{2} - \cos \phi \cos \theta_{23} \lambda_{12},
$$

(22)

The fact that $L_e - L_\mu - L_\tau$ predicts an effective mass near the lower end of its allowed range, whereas scaling admits $\langle m \rangle$ to take any of its allowed values, represents a possibility to distinguish the two Ansätze.
where $\lambda_{12}$ is the leading 12-rotation in $U_\ell$. From the experimentally observed $\sin^2 \theta_{12} \simeq 0.3$ it follows for the natural value of $\lambda_{12} \simeq 0.2$ that $\cos \phi$ has to be tuned to be very close to one. Leptonic CP violation in oscillation experiments is proportional to $\sin \phi$ \cite{26, 27} and very much suppressed.

All these fine-tuning problems occurring in $L_e - L_\mu - L_\tau$ do not occur in scaling, which therefore represents a presumably better Ansatz for the inverted hierarchy. Moreover, as we will elaborate upon in the next Section, scaling can easily be obtained in models based on discrete flavor symmetries, which are currently intensively studied \cite{28}.

### 6 A Model for Scaling and Phenomenological Consequences

We consider now a seesaw model based on the $D_4 \times Z_2$ flavor symmetry which was proposed in Ref. \cite{2} to generate scaling. The particle content together with the quantum numbers under $D_4 \times Z_2$ is shown in Table 1. The superscripts $+, -$ refer to the transformation under $Z_2$ and the rest are the $D_4$ representations. For the mathematical details of the $D_4$ group see for instance Ref. \cite{29}. Apart from the usual Majorana neutrinos $N_{e,\mu,\tau}$, the right-handed charged leptons $e_R, \mu_R, \tau_R$ and the lepton doublets $L_{e,\mu,\tau}$, one has to introduce five Higgs doublets $\phi_{1,2,3,4,5}$. In the Appendix we show as a proof of principle that the $D_4 \times Z_2$-invariant Higgs potential can be minimized with Higgs masses having values above current limits. From the assignment in Table 1 the following Lagrangian is obtained:

$$L = k_1 \tau_R \phi_1 L_e + \bar{\tau_R} (k_2 \phi_1 - k_3 \phi_3) L_\mu + \bar{\tau_R} (k_2 \phi_1 + k_3 \phi_3) L_\tau + h_1 \bar{N_e} \phi_1 L_e + h_2 \bar{N_\mu} \phi_2 L_e + h_3 \bar{N_\mu} (\phi_4 L_\tau + \phi_5 L_\mu) + \frac{1}{2} (\bar{N_e} N_e^c M_1 + \bar{N_\mu} N_\mu^c M_2 + \bar{N_\tau} N_\tau^c M_3) + h.c.$$  \( (23) \)

Hence, the neutrino Dirac mass matrix can be written as

$$M_D = \begin{pmatrix} a e^{i\phi} & 0 & 0 \\ b & d & e \\ 0 & 0 & 0 \end{pmatrix} v_{wk} ,$$  \( (24) \)

and both the charged lepton and Majorana mass matrix are diagonal. Note that multi-Higgs models as the one analyzed here typically predict flavor changing neutral currents and LFV in the charged lepton sector at dangerous levels. Here the model has a diagonal charged lepton mass matrix which renders processes like $\mu \rightarrow e\gamma$ suppressed either by the GIM mechanism or by the masses of the heavy right-handed neutrinos. Note that the model presented here is non-supersymmetric. A supersymmetric version would have LFV.

\footnote{There is another model based on $D_4 \times Z_2$ presented in Ref. \cite{2}. The heavy Majorana mass matrix $M_R$ is arbitrary in this case, which has therefore little predictivity for leptogenesis.}
| Field                | $D_4 \times Z_2$ quantum number |
|----------------------|----------------------------------|
| $L_e$                | $1_1^-$                          |
| $e_R, N_e, \phi_1$   | $1^-_1$                          |
| $N_\mu, \phi_2$     | $1^+_2$                          |
| $N_\tau$            | $1^-_2$                          |
| $\phi_3$            | $1^-_4$                          |
| $(L_\mu, L_\tau)$   | $2^+$                            |
| $(\phi_4, \phi_5)$  |                                  |
| $(\mu_R, \tau_R)$   | $2^-$                            |

Table 1: Transformation properties under $D_4 \times Z_2$ of the particle content of the model.

via off-diagonal slepton mass matrices generated by the mechanism described in Section 2 (see Eq. (10)). Because $M_D$ from Eq. (24) is a special case (recall that $e = d/c$) of the form given in Eq. (9) one would in this case obtain the characteristic relation between the LFV charged lepton decays and atmospheric neutrino mixing from Eq. (11).

In $M_D$ shown in Eq. (24) we have already included one complex phase. It is easy to show that with diagonal charged lepton and Majorana mass matrices there is only one complex phase in the model. Using all this, we can calculate the neutrino mass matrix using the type I seesaw formula to obtain

$$
\mathcal{M}_\nu = -\frac{v_{wk}^2}{M_2} \begin{pmatrix} M_2 \frac{a^2}{M_1} e^{2i\phi} + b^2 & b d & b e \\ b d & d^2 & d e \\ b e & d e & e^2 \end{pmatrix}.
$$

Note that the third heavy neutrino mass $M_3$ does not appear in $\mathcal{M}_\nu$, i.e., effectively we are dealing with a $2 \times 3$ seesaw. The low energy mass matrix apparently obeys scaling with $c = d/e$. We therefore have $m_3 = U_{e3} = 0$ and $\tan^2 \theta_{23} = e^2/d^2$. By evaluating the rephasing invariant $\text{Im} \{m_{ee} m_{\tau\tau} m_{e\tau}^* m_{\tau e}^*\}$ both with Eq. (25) and with the usual parametrization of the PMNS matrix, one finds a compact relation between the parameters in $M_D$ and $M_R$ and the low energy observables:

$$
\frac{1}{4} \Delta m^2_\alpha \Delta m^2_\odot \sin^4 \theta_{23} \sin^2 2\theta_{12} \sin 2\alpha = \left(\frac{v_{wk}^2}{M_2} \right)^4 \frac{M_2}{M_1} a^2 b^2 e^4 \sin 2\varphi.
$$

The effective mass governing neutrinoless double beta decay is

$$
\langle m \rangle = \frac{v_{wk}^2}{M_2} \left| \frac{M_2}{M_1} a^2 e^{2i\phi} + b^2 \right| = \sqrt{\Delta m^2_\alpha} \sqrt{1 - \sin^2 2\theta_{12} \sin^2 \alpha}.
$$

Let us focus now on high energy phenomenology in terms of leptogenesis [31]. The decay
asymmetries of the heavy neutrinos $N_{1,2,3}$ into final states with flavor $\alpha = e, \mu, \tau$ are

$\varepsilon^\alpha_i = \frac{\Gamma(N_i \to \phi \bar{l}_\alpha) - \Gamma(N_i \to \phi^\dagger l_\alpha)}{\Gamma(N_i \to \phi l_\alpha) + \Gamma(N_i \to \phi^\dagger l_\alpha)} \approx \left(\frac{1}{2} \varepsilon_i^{e,\mu,\tau} M_2 \right) \sin 2\varphi f(M_2^2/M_1^2), \quad (28)$

where $f(x) = \sqrt{x} \left(1 + \frac{1}{1-x} - (1+x) \ln \left(\frac{1+x}{x}\right)\right)$.

With the very restricted form of the Dirac mass matrix from Eq. (24) it turns out that only two of the nine possible $\varepsilon^\alpha_i$ are non-zero. Those are

$\varepsilon^e_1 = \frac{1}{8\pi v_{wk}^2} \frac{1}{(M_D M_D^\dagger)^{ii}} \sum_{j \neq i} \text{Im} \left\{ (M_D)_{i\alpha} (M_D^\dagger)_{\alpha j} \left( M_D M_D^\dagger \right)_{ij} \right\} f(M_2^2/M_1^2), \quad (29)$

$\varepsilon^e_2 = -\frac{1}{8\pi} \frac{a\bar{b}^2}{b^2 + d^2 + e^2} \sin 2\varphi f(M_1^2/M_2^2) \approx \frac{3}{16\pi} \frac{a^2 b^2}{b^2 + d^2 + e^2} \sin 2\varphi \frac{M_1}{M_2}$,

where we also gave the limits for $M_2^2 \gg M_1^2$. Again the third heavy neutrino does not play any role. The leptogenesis phase $\varphi$ is identical to the low energy Majorana phase $\alpha$, so that we can expect a correlation between the baryon asymmetry and the effective mass governing neutrinoless double beta decay. Let us focus in the following on the very typical case that $\varepsilon_1 = \varepsilon_1^e$ governs the baryon asymmetry. We need to specify the magnitude of the heavy neutrino masses. The common mass scale $v_{wk}^2/M_2$ in the light neutrino mass matrix from Eq. (25) should be $\sqrt{\Delta m^2_\Lambda} \simeq 0.05$ eV, which brings us to the choice $M_2 = (5 \cdot 10^{13} \div 5 \cdot 10^{15})$ GeV, when the entries of the mass matrix lie between 0.1 and 10. We further note that in case of an inverted hierarchy all entries of $M_D$ are of the same order of magnitude, so the ratio $M_2/M_1$ should not be too large. Nevertheless, there should be a moderate hierarchy in order to avoid the complications and tuning issues of heavy neutrinos close in mass, so for definiteness we choose $M_1 = (\frac{1}{6} \div \frac{1}{2}) M_2$. This in turn means that flavor effects in leptogenesis [32] do not play a role. In this limit, we can estimate the baryon asymmetry as

$Y_B \simeq c_{SP} \frac{g^*}{g^*} \kappa(\bar{m}_1), \quad (30)$

where $g^* = 122.75$, $c_{SP} = -44/87$ and $\kappa$ can be parameterized as [33]

$\kappa(\bar{m}_1) = \left(\frac{0.55 \cdot 10^{-3} \text{ eV}}{\bar{m}_1}\right)^{1.16}$,

with $\bar{m}_1 = (M_D M_D^\dagger)_{11}/M_1$. In Fig. 2 we show the results of an analysis in which we searched with the ranges of $M_1$ and $M_2$ specified above for values of $a, b, d, e$ and $\phi$ which generate the correct neutrino mixing phenomenology as defined in Eq. (5). Having found such parameters we evaluate the baryon asymmetry, which should lie in the range $(8 \div 10) \cdot 10^{-11}$ [34]. We see that there is as expected a correlation between $Y_B$ and $\langle m \rangle$ and that several of the points generate the correct baryon asymmetry, both in sign and magnitude. The particular choice of the parameters demands for successful leptogenesis the effective mass to lie around its maximal allowed value, $\langle m \rangle \gtrsim 0.045$ eV.
Figure 2: Scatter plot of the effective mass against the baryon asymmetry of the Universe in the model from Section 6.

7 Summary

In summary, we present a detailed investigation of the hypothesis that the Majorana neutrino mass matrix obeys a scaling law as a way to understand current neutrino observations. Two consequences of this hypothesis are that (i) the neutrino mass ordering is inverted rather than normal and (ii) both $U_{e3}$ and the lightest neutrino mass vanish. These results are invariant under renormalization group extrapolation and are therefore stable under radiative corrections, which distinguishes the scaling proposal from many others motivated by family symmetries or texture zeros. The effective mass governing neutrinoless double beta decay can be as large as $\sqrt{\Delta m^2_A} \simeq 0.06$ eV. Another distinguishing prediction of scaling is that the value of the atmospheric mixing angle is not necessarily maximal even though $U_{e3} = 0$. This is in contrast to models with $\mu-\tau$ symmetry, which provide a simple way to understand both maximal atmospheric mixing with a very small $U_{e3}$. Models with approximate or broken $\mu-\tau$ symmetry always correlate non-vanishing of $U_{e3}$ with deviations from maximal $\theta_{23}$. We note that recent analyzes of the available neutrino data tend to favor non-maximal atmospheric neutrino mixing. We also compare the scaling hypothesis to the $L_e - L_\mu - L_\tau$ flavor symmetry which is very often utilized to understand an inverted hierarchy. Unlike the scaling Ansatz, fitting observations requires a fine-tuning of mass matrix elements/perturbation parameters. An interesting aspect of our hypothesis that it is invariant under any possible rotation of the basis in $\mu-\tau$ space, e.g., coming from the
charged lepton sector. This may make it easier to construct models that obey scaling. We also note ways to test scaling using high energy astrophysical neutrino fluxes. We discuss a particular class of seesaw models based on the $D_4 \times Z_2$ flavor symmetry that realize the scaling hypothesis. There are (i) no dangerous flavor changing neutral currents in the lepton sector, (ii) the leptogenesis phase is identical to the low energy Majorana phase and (iii) successful leptogenesis requires the effective mass in neutrinoless double beta decay to be larger than 45 meV.

Acknowledgments

R.N.M. was supported by the National Science Foundation grant no. Phy–0354401 and by the Alexander von Humboldt Foundation (the Humboldt Research Award). The work was also supported by the EU program ILIAS N6 ENTApP WP1 and by the Deutsche Forschungsgemeinschaft in the DFG-Sonderforschungsbereich Transregio 27 “Neutrinos and beyond – Weakly interacting particles in Physics, Astrophysics and Cosmology” (A.B. and W.R.), and under project number RO–2516/3–2 (W.R.). A.B. acknowledges support from the Studienstiftung des Deutschen Volkes.

A Scalar Potential

The model we study in Section 6 is defined by the transformation properties given in Table 1. It includes five Higgs doublets $\phi_1$, $\phi_2$, $\phi_3$, $\phi_4$ and $\phi_5$. We will show now that the potential involving all these fields can be minimized with a realistic vev configuration leading to a phenomenologically viable spectrum of scalar bosons below mass bounds from direct production. Five Higgs fields correspond to ten charged and ten neutral ones, or to 20 physical degrees of freedom. Of the ten charged degrees of freedom, two are eaten by the electroweak $W$ bosons, leaving eight degrees of freedom, i.e., four complex charged scalars. Of the ten neutral degrees of freedom five are even under charge conjugation. These give five real physical scalars. The other five neutral degrees of freedom are odd under charge conjugation. One of these is eaten by the electroweak $Z$ boson, leaving four physical pseudoscalars. With the assignment given in Table 1 the most general $D_4 \times Z_2$-invariant potential with real coefficients reads:

$$
V = -\sum_{i=1}^{3} \mu_i^2 \phi_i^\dagger \phi_i - \mu_4^2 (\phi_4^\dagger \phi_4 + \phi_5^\dagger \phi_5) + \sum_{i=1}^{3} \lambda_i (\phi_i^\dagger \phi_i)^2 + \lambda_4 (\phi_4^\dagger \phi_4 + \phi_5^\dagger \phi_5)^2 \\
+ \lambda_{12} (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + \lambda_{13} (\phi_1^\dagger \phi_1)(\phi_3^\dagger \phi_3) + \lambda_{23} (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_3) \\
+ \sum_{i=1}^{3} \kappa_i (\phi_i^\dagger \phi_i)(\phi_4^\dagger \phi_4 + \phi_5^\dagger \phi_5) + \alpha_1 [(\phi_1^\dagger \phi_2)^2 + h.c.] + \alpha_2 |\phi_1^\dagger \phi_2|^2 \\
+ \alpha_3 [(\phi_2^\dagger \phi_3)^2 + h.c.] + \alpha_4 |\phi_2^\dagger \phi_3|^2 + \alpha_5 (\phi_4^\dagger \phi_5 + \phi_5^\dagger \phi_4)^2 + \alpha_6 (\phi_1^\dagger \phi_5 - \phi_5^\dagger \phi_1)^2 \\
+ \alpha_7 (\phi_4^\dagger \phi_4 - \phi_5^\dagger \phi_5)^2 + \alpha_8 (\phi_4^\dagger \phi_4 - \phi_5^\dagger \phi_5)(\phi_1^\dagger \phi_3 + h.c.) + \alpha_9 [(\phi_4^\dagger \phi_4)^2 + (\phi_2^\dagger \phi_5)^2 + h.c.] \\
+ \alpha_{10} [(\phi_2^\dagger \phi_4)|^2 + |\phi_2^\dagger \phi_5|^2)]
$$

(A1)
If one uses the following numerical values for the coefficient $\alpha$:

$$
\alpha_{11}[(\phi_1^\dagger \phi_4)^2 + (\phi_1^\dagger \phi_5)^2 + h.c.] + \alpha_{12}((\phi_1^\dagger \phi_4)^2 + (\phi_1^\dagger \phi_5)^2) + \\
\alpha_{13}[(\phi_3^\dagger \phi_4)^2 + (\phi_3^\dagger \phi_5)^2 + h.c.] + \alpha_{14}((\phi_3^\dagger \phi_4)^2 + (\phi_3^\dagger \phi_5)^2) + \\
\alpha_{15}[(\phi_1^\dagger \phi_4)(\phi_3^\dagger \phi_4) - (\phi_1^\dagger \phi_5)(\phi_3^\dagger \phi_5) + h.c.] + \alpha_{16}[(\phi_1^\dagger \phi_4)(\phi_3^\dagger \phi_3) - (\phi_1^\dagger \phi_5)(\phi_3^\dagger \phi_3) + h.c.]
$$

This potential can for instance be minimized by choosing the VEV configuration:

$$
\langle \phi_i \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{5}} \end{pmatrix} \text{ for } i = 1, 2, 4, 5 \text{ and } \langle \phi_3 \rangle = \begin{pmatrix} 0 \\ -\frac{v}{\sqrt{5}} \end{pmatrix}, \quad (A2)
$$

if one uses the following numerical values for the coefficients:

$$
\begin{align*}
\lambda_1 &= 2.62904; & \lambda_2 &= 2.91805; & \lambda_3 &= 2.53936; \\
\lambda_4 &= 2.93754; & \lambda_{12} &= 2.94576; & \lambda_{13} &= -1.29472; \\
\lambda_{23} &= -0.455254; & \kappa_1 &= 0.190918; & \kappa_2 &= 2.25757; \\
\kappa_3 &= -0.778689; & \alpha_1 &= -1.58251; & \alpha_2 &= -0.77218; \\
\alpha_3 &= -2.649; & \alpha_4 &= 2.32869; & \alpha_5 &= 0.690183; \\
\alpha_6 &= -1.82076; & \alpha_7 &= -0.097017; & \alpha_8 &= -1.38637; \\
\alpha_9 &= -0.853325; & \alpha_{10} &= 0.393294; & \alpha_{11} &= -0.404394; \\
\alpha_{12} &= 0.68049; & \alpha_{13} &= -1.29704; & \alpha_{14} &= 0.764254; \\
\alpha_{15} &= 1.70881; & \alpha_{16} &= -0.322439; & \mu_1^2 &= 0.619434 v_{wk}^2; \\
\mu_2^2 &= 0.661705 v_{wk}^2; & \mu_3^2 &= -0.971516 v_{wk}^2; & \mu_4^2 &= 1.4775 v_{wk}^2. \\
\end{align*} \quad (A3)
$$

One finds masses of 552 GeV, 390 GeV, 362 GeV, 240 GeV and 147 GeV for the scalars, 682 GeV, 607 GeV, 460 GeV and 322 GeV for the pseudoscalars and 404 GeV, 363 GeV, 273 GeV and 155 GeV for the charged scalars. These scalars are in general superpositions of all five Higgs fields, except for the 240 GeV scalar, which is only a linear combination of $\phi_4$ and $\phi_5$. Note that our model is only focussing on the lepton sector. Confronting the obtained Higgs mass values with limits stemming from rare meson decays would mean to construct a full model including also the quark sector and carefully perform a lengthy study of the diagrams leading to flavor changing neutral currents taking into account all five Higgs doublets. Within our model the specific choice of Eq. (A2) leads in the Dirac mass matrix from Eq. (24) to $d = e$ and therefore maximal atmospheric neutrino mixing. Other values are of course possible, which would lead to slightly different scalar masses and parameters in Eq. (A3).

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