We consider the problem of soliton generation in PT-symmetric optical fiber networks, where soliton dynamics is governed by nonlocal nonlinear Schrödinger equation on metric graphs. Exact formulae for the number of generated solitons are derived for the cases, when the problem is integrable. Numerical solutions are obtained for the case, when integrability is broken.

I. INTRODUCTION

The problem of soliton generation in optical fibers is of fundamental and practical importance for modern optoelectronics and information technologies. It was Hasegawa and Tappert [1], who proposed first to use optical solitons as carriers of information in high-speed communication systems in early seventies of the last century. Further development of the idea later led to advanced optoelectronic and information technologies based on use of solitons in optical fibers (see, e.g., Refs. [5]-[10] for review). Dynamics of generated solitons strongly depends on the shape of the initial pulse profile. This makes choosing initial pulse profile effective tool for tuning the soliton propagation. Mathematically, the problem of soliton generation is reduced to the Cauchy problem for the nonlinear evolution equation, governing the dynamics of soliton. An important task arising in this context, besides soliton dynamics, is finding the number of generated solitons using given initial condition. In case of long (unbranched) fibers such problem was studied in the Refs. [12]-[19]. In [12], where an effective method for computing the number of generated solitons is proposed. Extension of the approach for other initial pulse profiles was proposed later in [13]. Mathematical treatment of soliton generation on a half line was considered [15]. Generation in optical solitons in fibers with a dual-frequency input was considered in [16]. Soliton generation and their instability are investigated in a system of two parallel-coupled fibers, with a pumped (active) nonlinear dispersive core and a lossy (passive) linear one in [20]. In this paper we address the problem of generation PT-symmetric solitons described in terms of nonlocal nonlinear Schrödinger (NNLS) equation. The latter has attracted much attention in different contexts. It can be written as

$$i \frac{\partial}{\partial t} q(x,t) = \frac{\partial^2}{\partial x^2} q(x,t) + 2q^2(x,t)q^*(-x,t).$$ (1)

Introducing the PT-symmetric self-induced potential, $V = -2q(x,t)q^*(-x,t)$, one can write Eq.(1) in form of the following linear Schrodinger equation:

$$\frac{\partial}{\partial t} q(x,t) = -i \frac{\partial^2}{\partial x^2} q(x,t) + iV(x,t)q(x,t),$$ (2)

Due to the PT-symmetry of potential $V(x,t)$, given by the relation $V(x,t) = V^*(-x,-t)$, Eq.(2) can be considered as the PT-symmetric Schrodinger equation. We note that from the physical viewpoint, Eq.(2) describes the PT-symmetric optical solitons propagating in optical waveguide having "gain-and-loss" structure. A single-soliton solution of Eq.(1) was derived in [23] and can be written as

$$q(x,t) = -\frac{2(\eta_1 + \eta_2)e^{i\theta_1}e^{-4i\theta_1 t}e^{-2i\eta_1 x}}{1 + e^{i(\theta_1 + \theta_2)}e^{4i(\eta_1^2 - \eta_2^2)}e^{-2i(\eta_1 + \eta_2)x}}.$$ (3)
An important task of soliton generation problem is finding the number of solitons generated for a given initial pulse profile. From the mathematical viewpoint, such a task represents initial value (Cauchy) problem for a given initial pulse profile. An effective method for solving such task was proposed in [12], which was later applied for different types of pulse profile in [13, 16]. Starting point in calculation the number of solitons generated for a given initial pulse profile is the Zakharov-Shabat problem. For Eq. (1) Zakharov-Shabat problem is given in terms of the following AKNS system:

\[
\frac{\partial v^{(1)}}{\partial x} = -ikv^{(1)} + q(x,0)v^{(2)}, \\
\frac{\partial v^{(2)}}{\partial x} = ikv^{(2)} - q^*(-x,0)v^{(1)},
\]

where \(q(x,0)\) is the initial condition (initial pulse profile) for NNLS equation. Let us consider the special family of the initial potentials

\[
q(x,0) = Q(x,0)e^{i(\delta + \pi/2)}, \\
q^*(-x,0) = Q(-x,0)e^{-i(\delta + \pi/2)},
\]

where \(Q(x,0)\) is the real function and \(\delta (0 \leq \delta \leq 2\pi)\) is arbitrary constant. One can show that the transformations

\[
v^{(1)} \rightarrow V^{(1)}e^{i\gamma}, \\
v^{(2)} \rightarrow V^{(2)}e^{i(\gamma - \delta)}
\]

lead to the following eigenvalue problem

\[
\frac{\partial V^{(1)}}{\partial x} = -ikV^{(1)} + iQ(x,0)V^{(2)}, \\
\frac{\partial V^{(2)}}{\partial x} = ikV^{(2)} + iQ(-x,0)V^{(1)}.
\]

Following the Ref. [22], one can define the number of the zeros of the Jost coefficients \(a(k)\) at \(k = 0\).

If the initial condition is symmetric to the point \(x = 0\):

\(Q(x,0) = Q(-x,0)\) then the formal solution of Eq. (7) with \(k = 0\) are

\[
V^{(1)}(x,0) = \exp(-iS(x)) \left[ C^{(1)} \int_{-\infty}^{x} Q(x',0) \exp(2iS(x')) dx' + C^{(2)} \right], \\
V^{(2)}(x,0) = -iC^{(1)} \exp(iS(x)) - V^{(1)},
\]

where

\[
S(x) = \int_{-\infty}^{x} Q(x',0) dx',
\]

If one chooses \(V^{(1)}(x,0) \to 0\) for \(x \to -\infty\) and \(V^{(2)}(x,0) \to 0\) for \(x \to +\infty\), then \(C^{(2)} = 0\), and we have

\[
a(0) = \lim_{x \to +\infty} V^{(2)}(x,0) = \\
-\exp(iS_0) - i \exp(-iS_0) \int_{-\infty}^{+\infty} Q(x,0) \exp(2iS(x)) dx = -iC^{(1)} \cos S_0,
\]

where

\[
S_0 = \int_{-\infty}^{+\infty} Q(x,0) dx.
\]

From Eqs. (9) for the soliton number we get

\[
N = \frac{1}{2} + \frac{S_0}{\pi}.
\]
Noting that for the initial pulses given by Eq. (15) for any x and with $Q(x,0) > 0$

$$S_0 \equiv \int_{-\infty}^{+\infty} Q(x,0)dx = \int_{-\infty}^{+\infty} |q(x,0)|dx = F.$$  \hspace{1cm} (12)

we have from Eqs. (11) and (12)

$$N = \left\langle \frac{1}{2} + \frac{F}{\pi} \right\rangle. \hspace{1cm} (13)$$

Here we consider number of generated solitons for the rectangular initial pulse profile:

$$q(x,0) = \begin{cases} 0, & \text{for } |x| > \frac{1}{2}a \\ b, & \text{for } |x| \leq \frac{1}{2}a \end{cases} \hspace{1cm} b > 0.$$ 

Using the above approach for this profile leads to

$$F = \int_{-\infty}^{+\infty} |q(x,0)|dx = ab, \hspace{1cm} N = \left\langle \frac{1}{2} + \frac{ab}{\pi} \right\rangle.$$ 

This equation provides relation between the initial pulse profile and number of generated solitons, described by the PT-symmetric nonlocal nonlinear Schrodinger equation (11).

III. SOLITON GENERATION IN STAR-SHAPED OPTICAL WAVEGUIDE NETWORK

The above approach can be applied for soliton generation in branched waveguides, by modelling these lattices in terms of so-called metric graphs, which are the systems of wires connected to each other at the nodes (vertices) according to some rule, called topology of a graph. Such process is described in terms of the nonlocal nonlinear Schrodinger equation on graphs studied recently in [34]. We note that evolution equations on metric graphs attracted much attention during the last decade (see, Refs. [35] - [52]). Such NNLS equation on a six-bond, star branched graph (see, Fig. 2) can be written as

$$i\frac{\partial}{\partial t}q_{\pm j}(x,t) = \frac{\partial^2}{\partial x^2}q_{\pm j}(x,t) + \sqrt{\beta_{j} \beta_{\pm j}} q_{j}(x,t)q_{\mp j}^*(x,-t), \hspace{1cm} (14)$$

where $q_{\pm j}(x,t)$ at $x \in b_{\pm j}$ and $j = 1, 2, 3$. Eq. (14) is written on the each bond of the star graph with six bonds $b_{\pm j}$ (see, Fig. 2), for which a coordinate $x_{\pm j}$ is assigned. The origin of coordinates is chosen at the vertex; for bond $b_{-j}$ we put $x_{-j} \in (-\infty, 0]$ and for $b_{j}$ we fix $x_{j} \in [0, +\infty)$. An important feature of Eq. (14) comes from the fact that it is a system of coupled nonlocal nonlinear Schrodinger equations in which components of $q_{\pm j}$ are mixed in nonlinear term. In usual NLSE on graphs, such mixing does not appear explicitly, but caused by the vertex boundary conditions. Complete task formulation for NNLS equation on metric star graph requires imposing the boundary conditions at the node (vertex). Such boundary conditions can be derived, e.g., from physically relevant conservation laws. A set of the vertex boundary conditions following from the norm and energy conservation can be written as [34]:

$$\begin{align*}
\alpha_1 q_1(x,t)|_{x=0} &= \alpha_1 q_{-1}(x,t)|_{x=0} = \alpha_2 q_2(x,t)|_{x=0} = \alpha_2 q_{-2}(x,t)|_{x=0} = \alpha_3 q_3(x,t)|_{x=0} = \alpha_3 q_{-3}(x,t)|_{x=0}, \\
\frac{1}{\alpha_1} \frac{\partial}{\partial x} q_1(x,t)|_{x=0} &= \frac{1}{\alpha_2} \frac{\partial}{\partial x} q_2(x,t)|_{x=0} = \frac{1}{\alpha_3} \frac{\partial}{\partial x} q_3(x,t)|_{x=0}, \\
\frac{1}{\alpha_{-1}} \frac{\partial}{\partial x} q_{-1}(x,t)|_{x=0} &= \frac{1}{\alpha_{-2}} \frac{\partial}{\partial x} q_{-2}(x,t)|_{x=0} = \frac{1}{\alpha_{-3}} \frac{\partial}{\partial x} q_{-3}(x,t)|_{x=0}. \hspace{1cm} (15)
\end{align*}$$

The problem given by Eqs. (14) - (15) were recently studied in detail in the Ref. [34], where constraints providing the integrability of the NNLS equation on graphs have been derived in terms of the nonlinearity coefficients, $\beta_{\pm j}$. Here we briefly recall these results, which will be utilized for solving of soliton generation problem. Below, using
the approach applied in the previous section, we demonstrate derivation of expression for the number of solitons generated in a metric star graph. Let \( q(x,t) \) is the solution of Eq.\( (1) \) and the following constraints are fulfilled:

\[
\frac{\alpha_{\pm j}}{\alpha_1} = \sqrt{\frac{\beta_{\pm j}}{\beta_1}}, \\
\frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3} = \frac{1}{\beta_{-1}} + \frac{1}{\beta_{-2}} + \frac{1}{\beta_{-3}}.
\]

Then solution of NNLS equation (14), on metric star graph fulfilling the boundary conditions (15) can be written as

\[
q_{\pm j}(x,t) = \sqrt{\frac{2}{\beta_{\pm j}}} q(x,t)
\]

For the soliton solution given by Eq. (3), the solution of NNLS equation a star graph can be written as

\[
q_{\pm j}(x,t) = -\sqrt{\frac{2}{\beta_{\pm j}}} \frac{4\varphi^2 e^{-4\varphi^2} e^{-2\eta x}}{1 + e^{(\varphi + \varphi') e^{-4\varphi^2}}}
\]

\( \varphi, \bar{\varphi}, \eta \) are arbitrary complex constants.

Here we will provide brief derivation of the relation between the number of generated solitons and the initial pulse profile in a branched optical waveguide, which is modeled in terms of the star graph presented in Fig. 3. Consider the following Zakharov-Shabat problem for NNLS equation (14):

\[
\frac{\partial v_{\pm j}^{(1)}}{\partial x} = -ikv_{\pm j}^{(1)} + \sqrt{\frac{\beta_{\pm j}}{2}} q_{\pm j}(x,0)v_{\pm j}^{(2)}, \\
\frac{\partial v_{\pm j}^{(2)}}{\partial x} = ikv_{\pm j}^{(2)} - \sqrt{\frac{\beta_{\pm j}}{2}} q_{\pm j}^*(x,0)v_{\pm j}^{(1)},
\]

where \( q_{\pm j}(x,0) \) are the initial conditions (initial pulse profiles) for Eq. (14). Introducing the special family of the initial potentials given by

\[
q_{\pm j}(x,0) = Q_{\pm j}(x,0) e^{i(\delta_{\pm j} + \pi/2)}, \\
q_{\pm j}^*(x,0) = Q_{\pm j}(x,0) e^{-i(\delta_{\pm j} + \pi/2)}
\]

where \( Q_{\pm j}(x,0) \) are the real functions and \( \delta_{\pm j} (0 \leq \delta_{\pm j} \leq 2\pi) \) are arbitrary constants, one can show that the transformations

\[
v_{\pm j}^{(1)} \to V_{\pm j}^{(1)} e^{i\gamma_{\pm j}}, \\
v_{\pm j}^{(2)} \to V_{\pm j}^{(2)} e^{i(\gamma_{\pm j} - \delta_{\pm j})}
\]

lead to the following eigenvalue problem

\[
\frac{\partial V_{\pm j}^{(1)}}{\partial x} = -ikV_{\pm j}^{(1)} + i\sqrt{\frac{\beta_{\pm j}}{2}} Q_{\pm j}(x,0)V_{\pm j}^{(2)}, \\
\frac{\partial V_{\pm j}^{(2)}}{\partial x} = ikV_{\pm j}^{(2)} + i\sqrt{\frac{\beta_{\pm j}}{2}} Q_{\pm j}(x,0)V_{\pm j}^{(1)}.
\]

From a physical viewpoint, the generation of the single quiescent soliton will occur with a smaller energy than the soliton pair. Therefore, following the Ref. [22], we will define the number of the zeros of the Jost coefficients \( a_{\pm j}(k) \) at \( k = 0 \). If the initial condition is symmetric with respect the point \( x = 0 \): \( Q_{\pm j}(x,0) = \sqrt{\frac{2}{\beta_{\pm j}}} Q_{\pm j}(x,0) \).

The formal solutions of Eq. (21) with \( k = 0 \) are

\[
V_{-j}^{(1)}(x,0) = \exp(-iS_{-j}(x)) \left( C_{-j}^{(1)} \int_{-\infty}^{x} Q_{-j}(x',0) \exp(2iS_{-j}(x')) dx' + C_{-j}^{(2)} \right), \\
V_{-j}^{(2)}(x,0) = -iC_{-j}^{(1)} \exp(iS_{-j}(x)) - V_{-j}^{(1)}, \\
V_{j}^{(1)}(x,0) = \exp(-iS_{j}(x)) \left( C_{j}^{(1)} \int_{0}^{x} Q_{j}(x',0) \exp(2iS_{j}(x')) dx' + C_{j}^{(2)} \right), \\
V_{j}^{(2)}(x,0) = -iC_{j}^{(1)} \exp(iS_{j}(x)) - V_{j}^{(1)},
\]

\( C^{(1)}_{-j}, C^{(1)}_{j} \) are the Jost coefficients.
\[ S_{-j}(x) = \sqrt{\frac{\beta_{-j}}{2}} \int_{-\infty}^{x} Q_{-j}(x',0)dx' \]

and

\[ S_j(x) = \sqrt{\frac{\beta_j}{2}} \int_{0}^{x} Q_j(x',0)dx'. \]

If one chooses \( V_j^1(x,0) \to 0 \) for \( x \to -\infty \) and \( V_j(x,0) \to 0 \) for \( x \to +0 \), then \( C_{-j}^{(2)} = 0 \), and we have

\[ a_{-j}(0) = \lim_{x \to -0} V_{-j}^{(2)}(x,0) = -iC_{-j}^{(1)} \left( \exp(iF_{-j}) - i \exp(-iF_{-j}) \int_{-\infty}^{0} Q_{-j}(x,0) \exp(2iS_{-j}(x))dx \right) = -iC_{-j}^{(1)} \cos F_{-j}, \] (22)

\[ a_j(0) = \lim_{x \to +\infty} V_{j}^{(2)}(x,0) = -iC_{j}^{(1)} \left( \exp(iF_j) - i \exp(-iF_j) \int_{0}^{+\infty} Q_j(x,0) \exp(2iS_j(x))dx \right) = -iC_{j}^{(1)} \cos F_j. \] (23)

Noting that for the initial pulses given by Eq. (19) for any \( x \) and with \( Q_{\pm j}(x,0) > 0 \)

\[ F_{\pm j} = \sqrt{\frac{\beta_{\pm j}}{2}} \int_{b_{\pm j}}^{x} Q_{\pm j}(x,0)dx = \sqrt{\frac{\beta_{\pm j}}{2}} \int_{b_{\pm j}}^{x} |q_{\pm j}(x,0)|dx. \] (24)

From Eqs. (22) and (23) for the soliton number we get

\[ N = \left\langle 3 + \sum_{j=1}^{3} (F_{-j} + F_j) \right\rangle. \] (25)

Now consider the star graph with rectangle initial pulse (see, Fig 3). For such profile, the initial condition is given at the vertex and can be written as \( q_{\pm j}(x,0) = \sqrt{\frac{2}{\beta_{\pm j}}} \psi_{\pm j}(x) \):

\[ \psi_{-j}(x) = \begin{cases} 0, & \text{for } x < -\frac{1}{2}a \\ b, & \text{for } -\frac{1}{2}a \leq x \leq 0 \end{cases} \]

\[ \psi_{j}(x) = \begin{cases} 0, & \text{for } x > \frac{1}{2}a \\ b, & \text{for } 0 \leq x \leq \frac{1}{2}a \end{cases} \]

where \( b > 0 \).

The number of generated solitons

\[ F = \sum_{j=1}^{3} (F_{-j} + F_j) = \sum_{j=1}^{3} \left( \sqrt{\frac{\beta_{-j}}{2}} \int_{b_{-j}}^{x} |q_{-j}(x,0)|dx + \sqrt{\frac{\beta_j}{2}} \int_{b_{j}}^{x} |q_j(x,0)|dx \right) = 3ab, \]

\[ N = \left\langle 3 + \frac{3ab}{\pi} \right\rangle. \]

Another initial pulse profile is the Gaussian one given by

\[ q_{\pm j}(x,0) = \sqrt{\frac{2}{\beta_{\pm j}}} \exp \left[ -\frac{1}{2} (1 - i\alpha) \left( \frac{x}{\sigma} \right)^{2m} \right]. \] (26)

Using the above approach for this profile leads to

\[ F = \sum_{j=1}^{3} \left( \sqrt{\frac{\beta_j}{2}} \int_{b_{j}}^{x} |q_{j}(x,0)|dx + \sqrt{\frac{\beta_j}{2}} \int_{b_{j}}^{x} |q_{j}(x,0)|dx \right) = \frac{3 \cdot 2 \pi^{m} A\sigma}{m} \Gamma \left( \frac{1}{2m} \right), \]

\[ N = \left\langle 3 + \frac{F}{\pi} \right\rangle. \]

We note that Eq. (25) for the number of generated solitons is derived under the assumption that the sum rule in Eqs. (10), which is equivalent to the integrability of NNLS equation on graph (see, the Ref. [34]). For the case, when sum rule is broken, one needs to solve the problem numerically, by imposing the initial conditions given by Eq. (20). The plots of the numerically obtained solution are presented in Fig. 4 for the time moments, \( t = 0 \) and \( t = 0.0022 \). Discretization scheme proposed in [24] is used for numerical solution of the initial value problem for NNLS equation on a star graph. An important feature of the soliton generation, i.e., breaking of the initial pulse profile due to the radiation can be observed from the plots of Fig. 4.

IV. EXTENDING FOR THE TREE GRAPH

The central branch, i.e. the branch at the middle of the graph is chosen as an origin of coordinates. Then
the bonds can be determined as 
\[ b_{-1}, b_{-1n}, b_{-1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n} \sim (-\infty; 0), \]
\[ b_{-1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n} \sim [0; L_{1n}], \]
\[ b_{1}, b_{1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n}, b_{1n} \sim (0; \infty), \]
where \( L_{1n} \) are the lengths of \( b_{\pm 1n} \) bonds and \( n = 1, 2, m = 1, 2 \). Here the “+” sign is for right-handed bonds and the “−” sign is for left-handed bonds from the center of the tree graph. Soliton solutions on each bond can be written as

\[
q_{\pm 1}(x, t) = \sqrt{\frac{2}{\beta_{\pm 1}}} q(x + S_{\pm 1}, t),
\]
\[
q_{\pm 1n}(x, t) = \sqrt{\frac{2}{\beta_{\pm 1n}}} q(x + S_{\pm 1n}, t),
\]
\[
q_{\pm 1nm}(x, t) = \sqrt{\frac{2}{\beta_{\pm 1nm}}} q(x + S_{\pm 1nm}, t).
\]

\[ N = \left( 7 + \sum_{s=-2}^{2} F_s \right), \]
where
\[ F_0 = 3aA, \quad F_{\pm 1} = \frac{3aA_1}{2}, \quad F_{\pm 2} = \frac{3aA_2}{2}. \]

Again, for the case, when the constraints given by Eq. (27), NNLS equation should be solved numerically and Eq. (29) cannot be used for finding the number of solitons generated.

V. CONCLUSIONS

In this paper we studied the problem of soliton generation for PT-symmetric optical waveguides and their networks described in terms of NNLS equation. The problem of finding the number of generated solitons for a given initial pulse profile is reduced to the Cauchy problem for
NNLS equation on a line and on metric graphs, where the initial condition is give in terms of the initial pulse. Exact expression for the number of solitons generated is derived. In case of optical waveguide networks, the problem is solved for star- and tree-branched networks. The results obtained in this paper and proposed models can be applied for the problem of tunable generation of solitons in branched optical fiber networks providing PT-symmetry via "gain-loss" property. Experimental realization of such a model is of importance for engineering and practical implementation of PT-symmetric optical fiber networks, capable to generate solitonic pulses and tunable signal propagation. Although the above treatment deals with star- and tree graphs, the method can be extended for arbitrary graph topologies having semi-infinite incoming and outgoing bonds.

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