ESTIMATING POINT EXPOSURE EFFECTS ON COUNT OUTCOMES WITH OBSERVATIONAL DATA

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ABSTRACT

Causal inference methods can be applied to estimate the effect of a point exposure or treatment on an outcome of interest using data from observational studies. When the outcome of interest is a count, the estimand is often the causal mean ratio, i.e., the ratio of the counterfactual mean count under exposure to the counterfactual mean count under no exposure. This paper considers estimators of the causal mean ratio based on inverse probability of treatment weights, the parametric g-formula, and doubly robust estimation, each of which can account for overdispersion, zero-inflation, and heaping in the measured outcome. Methods are compared in simulations and are applied to data from the Women’s Interagency HIV Study to estimate the effect of incarceration in the past six months on two count outcomes in the subsequent six months: the number of sexual partners and the number of cigarettes smoked per day.

Keywords Data heaping; Doubly robust estimation; Inverse probability weighting; Overdispersion; Parametric g-formula; Zero-inflation.

1 Introduction

Researchers often seek to estimate the causal effect of a point exposure or treatment on an outcome of interest. Randomized experiments are infeasible for many exposures, and thus inference often relies on data from observational studies. Associations measured from such studies can be subject to confounding, so various methods have been developed to consistently estimate causal effects from observational data. Three commonly-used methods are inverse probability of treatment weight (IPTW) estimators, (Robins, 1998; Robins et al., 2000; Hernán et al., 2000), the parametric g-formula (Robins, 1986), and doubly robust estimators that incorporate both exposure and outcome model estimators (Bang and Robins, 2005; Hernán and Robins, 2020; Funk et al., 2011; Kang and Schafer, 2007). In practice, these estimators are frequently applied to observational data when the outcome of interest is continuous, binary, or categorical (Hernán et al., 2000; Bodnar et al., 2004; Cole and Hernán, 2008; Taubman et al., 2009; Young et al., 2011; Garcia-Aymerich et al., 2013; Funk et al., 2011; Waernbaum, 2012).

Count outcomes are common in observational studies, as researchers often seek to estimate measures over a fixed period of time such as numbers of sexual partners (Wiederman, 1997), pill counts to assess treatment adherence (Bangsberg
et al., 2001), or the number of cigarettes smoked (Singh et al., 1994). To estimate the effect of a binary point exposure on a count outcome, the estimand is often either the causal mean ratio or the causal rate ratio. The causal mean ratio contrasts the counterfactual mean count under exposure to the counterfactual mean count under no exposure over a fixed period of time. When follow-up time varies across members of the population, an alternative estimand to the causal mean ratio is the causal rate ratio, i.e., the ratio of the counterfactual rate under exposure to the counterfactual rate under no exposure. Rates are commonly estimated using similar modeling approaches to those used for counts, but incorporating an offset to account for varied follow-up time. This paper focuses on estimation of the causal mean ratio.

Applying causal methods to count outcomes poses challenges unique to count data that must be accounted for to yield valid inference. The Poisson distribution is commonly used to model count outcomes, but the observed variance of a count outcome often exceeds the variance assumed under the Poisson model, i.e., there may be overdispersion. Zero-inflation occurs when the number of observed zero counts exceeds the number expected under the Poisson distribution (Böhning et al., 1999). Count outcomes are also susceptible to data heaping, a form of measurement error which occurs when reported counts are rounded to different levels of precision (Wang and Heitjan, 2008). This phenomenon is commonly observed when collecting self-reported retrospective counts or measures of duration, including cigarette usage (Klesges et al., 1995), duration of breastfeeding (Singh et al., 1994), and number of sexual partners (Wiederman, 1997; Roberts and Brewer, 2001). Data heaping is often attributed to cognitive processes in respondents, including choosing round numbers or approximations (digit preference) or using estimation methods to aid in recall (Roberts and Brewer, 2001). Data heaping distorts the true underlying distributions of counts, which makes point and variance estimators that ignore this measurement error biased when applied to the observed data (Wang and Heitjan, 2008).

This paper considers three types of estimators for the causal mean ratio, each of which can account for overdispersion, zero-inflation, and data heaping. Section 2 presents the estimators in detail and describes their large sample properties. Section 3 demonstrates and compares the empirical properties of the estimators with a simulation study, and Section 4 applies the methods to Women’s Interagency HIV Study (WIHS) data to estimate the effect of incarceration on two count outcomes: the number of sexual partners in a six-month period and the average number of cigarettes smoked per day among smokers in a six-month period. Section 5 concludes with a discussion of the results. The Appendix includes proofs of the results appearing in the main text, additional details regarding the analysis of WIHS data, and supplemental tables and figures from the simulation study. R code for computing the different estimators along with the corresponding standard error estimators is available on GitHub.

2 Methods

2.1 Preliminaries

Consider an observational study where the aim is to assess the effect of a binary exposure (or treatment) \( A \) on an outcome \( Y \in \mathbb{N}^0 \), where \( \mathbb{N}^0 \) is the set of non-negative integers. Let \( L \) denote a vector of baseline covariates. Unless noted otherwise, all vectors are assumed to be row vectors. Assume \( n \) independent and identically distributed copies of \((A, Y, L)\) are observed, denoted \((A_i, Y_i, L_i)\) for \( i = 1, \ldots, n \). Let \( Y_i^a \) denote the potential outcome if individual \( i \) is exposed to \( A = a \), possibly counter to fact, is exposed. Similarly, let \( Y_i^0 \) denote the potential outcome if individual \( i \) is not exposed, such that \( Y_i = A_i Y_i^1 + (1 - A_i) Y_i^0 \). Assume that conditional exchangeability holds, i.e., \( Y^a \perp A \mid L, a \in \{0, 1 \} \). Also assume that positivity holds such that \( \Pr(A = a \mid L = l) > 0 \) for all \( l \) such that \( dF_l(l) > 0 \) and \( a \in \{0, 1 \} \), where \( F_L \) is the cumulative distribution function of \( L \). Let \( E(Y^a) = \lambda^a \) for \( a \in \{0, 1\} \). The goal is to draw inference about the causal mean ratio, \( CMR = \lambda^1/\lambda^0 \).

2.2 Estimators: Correctly measured outcome

This section presents three estimators of the \( CMR \) that are consistent and asymptotically normal when the outcome is measured without error.

2.2.1 Inverse Probability of Treatment Weighting

Consider the (saturated) marginal structural model (MSM)

\[
\log(\lambda^a) = \beta_0 + \beta_1 a
\]

Under the assumptions specified in Section 2.1, the parameters of (1), and hence \( CMR = \exp(\beta_1) \), can be consistently estimated using IPTW as follows. First, the propensity score for each participant, \( e_i = \Pr(A_i = 1 \mid L_i) \), is estimated using a finite dimensional parametric model. For example, \( A \) can be regressed on \( L \) using logistic regression, i.e., the model \( \logit(e_i) = L_i^\top \alpha \) is fit, where \( \alpha \) is the column vector of regression coefficients from the weight model. Predicted propensity scores are calculated as \( \hat{e}_i = e(L_i, \hat{\alpha}) = \logit^{-1}(L_i^\top \hat{\alpha}) \) where \( \hat{\alpha} \) is the maximum likelihood estimate (MLE)
of α. Participant i’s IPTW is estimated as \( \hat{W}_i = A_i \hat{e}_i^{-1} + (1 - A_i)(1 - \hat{e}_i)^{-1} \). Then, the IPTW estimator of the CMR is

\[
CMR_{IPTW} = \frac{\sum_{i=1}^{n} \hat{W}_i Y_i A_i}{\sum_{i=1}^{n} \hat{W}_i A_i} / \frac{\sum_{i=1}^{n} \hat{W}_i Y_i (1 - A_i)}{\sum_{i=1}^{n} \hat{W}_i (1 - A_i)}
\]

(2)

The estimator (2) is equal to \( \exp(\hat{\beta}_1) \), where \( \hat{\beta}_1 \) is the weighted least squares estimator of the exposure coefficient when regressing \( Y \) on \( A \) with weights \( \hat{W} \) and a log link.

If the assumed \( A \mid L \) weight model is correctly specified, then (2) is consistent and asymptotically normal with asymptotic variance \( \Sigma I_{IPTW} \), which can be consistently estimated with the empirical sandwich variance estimator as discussed in Section 2.4. Alternatively, if the weights are known functions of \( A \) and \( L \), then (2) is consistent and asymptotically normal with asymptotic variance \( \Sigma I_{IPTW} \) where \( \Sigma I_{IPTW} \leq \Sigma I_{IPTW} \). This is analogous to the classic result about the IPTW estimator of the average treatment effect (Lunceford and Davidian, 2004). Note standard statistical software can be used to estimate \( \Sigma I_{IPTW} \) by the empirical sandwich variance estimator from weighted least squares regression. In practice, the weights are rarely if ever known in the observational setting. The derivations of the asymptotic variance of (2), both when treating the weights as fixed and when treating the weights as estimated, are included in Section A.1 of the Appendix.

### 2.2.2 Parametric g-formula

Robins (1986) introduced the parametric g-formula as a type of standardization that allows for the estimation of causal effects by directly modeling the outcome as a function of the exposure and covariates \( L \) and then integrating over the distribution of \( L \). The parametric g-formula estimator of the CMR is

\[
CMR_{PG} = \frac{\sum_{i=1}^{n} \hat{E}(Y_i \mid L_i, A_i = 1)}{\sum_{i=1}^{n} \hat{E}(Y_i \mid L_i, A_i = 0)}
\]

(3)

where \( \hat{E}(Y_i \mid L_i, A_i = a) \) is computed for \( a \in \{0, 1\} \) from the MLEs of the parameters for an assumed model for \( Y \mid L, A \). If the assumed parametric model is correctly specified, then (3) is consistent and asymptotically normal (see proof in Section A.2 of the Appendix).

Count outcomes are commonly modeled using the Poisson, negative binomial (NB), zero-inflated Poisson (ZIP), and zero-inflated negative binomial (ZINB) distributions. These distributions can be used to model \( Y \mid L, A \) which allows for computation of \( \hat{E}(Y_i \mid L_i, A_i = a) \) for \( a \in \{0, 1\} \) in (3). For the Poisson and NB distributions, a generalized linear model (GLM) is fit of the form: \( \log(\mu_i) = X_i \gamma \) for \( i = 1, \ldots, n \), where \( \mu_i = E(Y_i \mid A_i, L_i) \) and \( X_i = g(L_i, A_i) \) is a vector of predictors for participant \( i \) for some user-specified function \( g \) of \( L_i \) and \( A_i \), and \( \gamma \) is a column vector of regression coefficients. The MLE \( \hat{\gamma} \) is obtained for the GLM and \( \hat{E}(Y_i \mid L_i, A_i = a) = \exp\{g(L_i, a)\hat{\gamma}\} \) is calculated for each participant for \( a \in \{0, 1\} \).

The ZIP and ZINB distributions account for excess zeros in the count outcome without and with overdispersion, respectively, by assuming that only a portion of the population is susceptible to having a non-zero count while the remaining are not (Mullahy, 1986). When the outcome follows a ZIP or ZINB distribution, models for the probability of individual \( i \) not being susceptible (\( \nu_i \)) and the expected count for individual \( i \) within the susceptible population (\( \eta_i \)) are simultaneously fit: \( \logit(\nu_i) = X_i \gamma_1 \) and \( \logit(\eta_i) = X_i \gamma_2 \), where \( X_i = g_1(L_i, A_i) \) and \( X_i = g_2(L_i, A_i) \) for user-specified functions \( g_1 \) and \( g_2 \), and \( \gamma_1 \) and \( \gamma_2 \) are corresponding column vectors of regression coefficients. MLEs \( \hat{\gamma}_1 \) and \( \hat{\gamma}_2 \) are obtained for the model and \( \hat{E}(Y_i \mid L_i, A_i = a) = [1 - \logit^{-1}\{g_1(L_i, a)\hat{\gamma}_1\}] \exp\{g_2(L_i, a)\hat{\gamma}_2\} \) is calculated for each participant for \( a \in \{0, 1\} \). For the ZIP and ZINB models above, regression coefficients have latent interpretations; alternatively, marginalized ZIP and ZINB models may be assumed where parameters have marginal interpretations (Long et al., 2014; Preissier et al., 2016b). Albert et al. (2014) propose and apply (3) to estimate the CMR assuming a ZINB or zero-inflated beta binomial parametric model for the outcome.

### 2.2.3 Doubly Robust Estimation

Next consider doubly robust estimators which incorporate both IPTW and parametric g-formula estimators and are consistent when either the weight or outcome model, but not necessarily both, are correctly specified (Bang and Robins, 2005; Hernán and Robins, 2020; Funk et al., 2011). Specifically, the following is a doubly robust estimator for the CMR:

\[
CMR_{DR} = \frac{\hat{\lambda}^1_{DR}}{\hat{\lambda}^0_{DR}}
\]

(4)

where \( \hat{\lambda}^1_{DR} = n^{-1} \sum_{i=1}^{n} \hat{e}_i^{-1}\{A_i Y_i - (A_i - \hat{e}_i) m_1(L_i, \hat{\gamma})\} \) and \( \hat{\lambda}^0_{DR} = n^{-1} \sum_{i=1}^{n} (1 - \hat{e}_i)^{-1}\{1 - A_i\} Y_i + (A_i - \hat{e}_i) m_0(L_i, \hat{\gamma})\}, \hat{\gamma} \) is the estimated propensity score for participant \( i \) from the weight model as described in Section
2.2.1, and \( m_a(L_i, \gamma) = \hat{E}(Y_i \mid L_i, A_i = a) \) is the predicted potential outcome for participant \( i \) for \( a \in \{0, 1\} \) from the outcome model, based on the Poisson, NB, ZIP, or ZINB distribution. The causal mean estimators \( \hat{\lambda}_{HI}^a \) for \( a \in \{0, 1\} \) are of the form considered in Lunceford and Davidian (2004) that were originally proposed by Robins et al. (1994). When either the weight or outcome model is correctly specified, (4) is consistent and asymptotically normal (see proof in Section A.3 of the Appendix).

2.3 Estimators: Data Heaping

In many settings, the true outcome of interest \( Y \) is measured with error. Count outcomes are particularly susceptible to a type of measurement error known as data heaping, which can occur when some participants round or approximate their reported count outcomes rather than reporting exact counts. For example, self-reported cigarette counts are frequently rounded to the nearest multiple of 10 or 20 (Klesges et al., 1995; Wang and Heitjan, 2008). When data heaping is present, statistical methods which ignore heaping will in general not lead to valid inference (Wang and Heitjan, 2008).

2.3.1 Inverse Probability of Treatment Weighting

Consider the following heaping model. First, let \( Y_{hi}^a \) denote the heaped potential outcome for participant \( i \), i.e., the outcome they would report if, possibly counter to fact, they received treatment \( a \). Define the observed (heaped) count as \( Y_{hi} = A_i Y_{hi}^a + (1 - A_i) Y_{hi}^0 \). Because \( Y_{i}^a \neq Y_{i}^0 \) for some \( i \), \( E(Y_{i}^a) \neq \lambda^a \) in general. Suppose that some individuals report heaped outcomes and other individuals report their outcome exactly. In particular, suppose \( Y_{hi}^a = \Delta_i Y_{i}^a + (1 - \Delta_i) h_{i}(Y_{i}^a) \) for \( a \in \{0, 1\} \), where \( \Delta_i \) is 1 if participant \( i \) reported the outcome exactly and 0 otherwise, and \( h_{i}(Y_{i}^a) \) is a function which rounds \( Y_{i}^a \) to the nearest multiple of the known constant \( \eta \). Under this model, two participants may report the same value of the outcome for different reasons. For example, consider two smokers who are asked to report the number of cigarettes they smoked the previous day. One woman recalls the exact number, i.e., \( \Delta_i = 1 \), and reports \( Y_{hi} = 20 \). Another woman does not recall the exact number, i.e., \( \Delta_i = 0 \), but estimates approximately one pack of cigarettes and reports \( Y_{hi} = 20 \). Note in the latter setting, it is possible that \( Y_i = Y_{hi} = 20 \). Because \( \Delta_i \) is unobserved, one cannot distinguish between these two cases from the observed data.

Below, three estimators of CMR are described for this heaping model under the non-informative heaping assumption \( Y_{i}^a \perp \Delta_i \), that is, whether the outcome is reported exactly or rounded is independent of the potential outcome itself. Each of the estimators can be computed by fitting finite dimensional parametric models of \( Y \mid Z \), where \( Z \) is defined in Sections 2.3.1-2.3.3 for each method. Let \( f(y; \delta) \) denote the probability mass function (PMF) for the conditional distribution of \( Y \mid Z \) with parameter vector \( \delta = (\delta_1, \delta_2, ..., \delta_k) \). Note \( f(y; \delta) \) depends on \( Z \) but this is left implicit for notational simplicity. Define \( l(Y_{hi}) = \pi f(Y_{hi}; \delta) + (1 - \pi) \sum_y h_{i}(y) = Y_{i}, f(y; \delta) \), where \( \pi = \Pr(\Delta_i = 1) \). Then, conditional on \( Z \), the log-likelihood is

\[
\mathcal{L} = \sum_{i=1}^{n} \log l(Y_{hi})
\]

Sections 2.3.1-2.3.3 present estimators of the CMR based on (5) that account for data heaping using IPTW, parametric g-formula, and doubly robust methods.

2.3.1 Inverse Probability of Treatment Weighting

To estimate the parameters of (1) in the presence of data heaping, a parametric distribution is assumed for the marginal distribution of \( Y_{i}^a \) for \( a \in \{0, 1\} \). Let \( f_a(y; \delta) \) denote the assumed parametric density for \( Y_{i}^a \) and suppose the parameterization is such that \( CMR = \exp(\delta_1) \). Then the IPTW estimator is computed by maximizing the log-likelihood (5) conditional on \( Z = A \) with \( f(y; \delta) = \sum_{a=0}^{1} f_a(y; \delta) I(A = a) \) and individual contributions to the log-likelihood weighted by \( \hat{W}_i \) as defined in Section 2.2.1. This is equivalent to finding the estimators \( \hat{\pi} \) and \( \hat{\delta} = (\delta_1, \delta_2, ..., \delta_k) \) which solve the score equations

\[
\sum_{i=1}^{n} \hat{W}_i \left[ \frac{\partial \log l(Y_{hi})/\partial \delta}{\partial \log l(Y_{hi})/\partial \pi} \right] = 0
\]

When the \( A \mid L \) and \( Y_{i}^a \) models are correctly specified, \( CMR_{IPTW, heap} = \exp(\delta_1) \) is consistent and asymptotically normal for the CMR, as shown in Section A.4 of the Appendix.

2.3.2 Parametric g-formula

The parametric g-formula estimator of the CMR can be modified to accommodate data heaping by replacing the log-likelihood function for \( Y \mid A, L \) with (5) where \( f(y; \delta) \) is the PMF for the assumed \( Y \mid A, L \) parametric model, i.e.,
When the \( \hat{Y} \) where the MLEs for the parameters in the heaping model are used to calculate \( \hat{Y} \) (Section 3.1), and with data heaping, where the observed count was rounded to the nearest ten for some participants (Stefanski and Boos, 2002). In addition, the asymptotic variance of these estimators can be consistently estimated using the empirical sandwich variance estimator, which can in turn be used to construct Wald type confidence intervals (CIs).

Simulation studies were conducted to examine and compare the empirical properties of the IPTW, parametric g-formula, and doubly robust estimators of \( CMR \) proposed in Section 2.3.2, a DR estimator of the form evaluated by Bang and Robins (2005) is considered:

\[
CMR_{DR, heap} = \frac{\sum_{i=1}^{n} \hat{E}_{heap}(Y_i | L_i, A_i = 1, \hat{r}_i)}{\sum_{i=1}^{n} \hat{E}_{heap}(Y_i | L_i, A_i = 0, \hat{r}_i)}
\]

where \( \hat{r}_i = A_i \hat{W}_i - (1 - A_i) \hat{W}_i \), i.e., \( \hat{r}_i \) is the estimated IPTW for treated participants and the negative of the estimated IPTW for untreated participants. The estimated potential outcomes \( \hat{E}_{heap}(Y_i | L_i, A_i = a, \hat{r}_i) \) for \( a \in \{0, 1\} \) are obtained from a heaping model as specified in Section 2.3.2, but also including \( \hat{r} \) as a covariate in the outcome model. That is, the heaping model is fit by maximizing (5) with \( Z = \{A, L, \hat{r}\} \). The estimator (9) is consistent and asymptotically normal when either the weight or outcome model is correctly specified.

2.4 Variance Estimation and Confidence Intervals

Each of the proposed \( CMR \) estimators from Sections 2.2 - 2.3 can be expressed as solutions to a vector of unbiased estimating equations, and therefore are consistent and asymptotically normal under certain regularity conditions (Stefanski and Boos, 2002). In addition, the asymptotic variance of these estimators can be consistently estimated using the empirical sandwich variance estimator, which can in turn be used to construct Wald type confidence intervals (CIs).

3 Simulation Study

Simulation studies were conducted to examine and compare the empirical properties of the IPTW, parametric g-formula, and doubly robust estimators of \( CMR \) proposed in Section 2 for a binary exposure \( A \) and a count outcome \( Y \) in the presence of covariates \( L \). Simulations were conducted both without data heaping, where the true outcome was observed (Section 3.1), and with data heaping, where the observed count was rounded to the nearest ten for some participants (Section 3.2).

3.1 Without Data Heaping

The first set of simulations were designed based on the motivating example in Section 4.1, which aimed to estimate the effect of incarceration on the number of sexual partners in a six-month period, controlling for covariates such as age, drug use, and sex exchange practices. A sample of \( n = 500 \) participants was simulated. Simulations were also conducted for \( n = 2000 \), with the results presented in the Appendix. Three covariates \( L_1, L_2, \) and \( L_3 \) were generated. Representing a participant’s baseline age, \( L_1 \) was simulated from Uniform(20, 40). The covariate \( L_2 \) represented baseline drug use status and was Bernoulli with mean \( p_1 = \text{logit}^{-1}(- (L_1 - 0.5)/100 + \epsilon_1) \) where \( \epsilon_1 \) was Uniform(-1, 1), and \( L_3 \) represented the baseline sex exchange variable and was Bernoulli with mean \( p_2 = \text{logit}^{-1}(-3 - (L_1 - 0.5)/100 + 1.2L_2 + \epsilon_2) \).
where \( \epsilon_2 \) was Uniform(-0.5, 0.5). The exposure \( A \) represented the binary incarceration status at the visit following baseline and was Bernoulli with mean \( p_3 = \logit^{-1}(0.5 - L_1/100 + 0.5L_2 + 0.5L_3) \).

The outcome of interest \( Y \) represented the number of total male sexual partners in the six-month period following measurement of the exposure, with \( Y^a \mid L \) generated under the four assumed parametric distributions: Poisson, NB, ZIP, and ZINB. The parameters of the four distributions from Section 2.2 were equalled \( \mu^a = \eta^a = \exp(-1 - 0.005L_1 + 0.7L_2 + 3.5L_3 + 0.5a) \) and \( \nu^a = \logit^{-1}(-2.5 + L_1/100 - 0.3L_2 - 2L_3) \), where superscripts denote the values of parameters under exposure \( a \in \{0, 1\} \). The dispersion parameter \( \theta = 0.5 \) was defined such that \( \text{Var}(Y_i \mid A_i, L_i) = \mu_i + \mu_i^2 \theta \) for the NB distribution. For each scenario, \( \log(CMR) = 0.5 \).

The estimated causal mean ratios \( \hat{CMR}_{IPTW} \), \( \hat{CMR}_{PG} \), and \( \hat{CMR}_{DR} \) and their estimated variances were calculated for each scenario both under correct model specification and when the weight and/or outcome model were incorrectly specified by excluding \( L_2 \). Standard errors for \( \hat{CMR}_{IPTW} \) were estimated both conservatively treating the weights as fixed or known, and appropriately treating the weights as estimated. Standard error estimates were computed using the geex package in R (Saul and Hudgens, 2020). Corresponding 95% Wald confidence intervals (CIs) were computed throughout.

The results of the simulation for \( n = 800 \) are presented in Figure 1, with more detailed results in Table A.1 and Table A.2. These results demonstrate minimal empirical bias regardless of the method or underlying distribution of the data when models were correctly specified. Empirical bias was even smaller when the sample size was increased to \( n = 2000 \) (see Table A.3). The IPTW estimator, empirical coverage was close to the nominal 95% level when the weight model was correctly specified and weights were treated as estimated, but was at or near 100% when weights were treated as known. This aligns with the inflated median estimated standard error (MSE) relative to the empirical standard error (ESE) when the weights are treated as known, resulting in the standard error ratio (SER) being above one. The IPTW estimator with weights treated as estimated, parametric g-formula, and doubly robust estimators all had SERs close to one, demonstrating the consistency of the empirical sandwich variance estimator. The parametric g-formula and doubly robust estimators yielded more precise estimates than IPTW, with the parametric g-formula having the smallest MSEs (Table A.1).

As anticipated, the doubly robust estimators yielded minimal bias when either the weight or outcome model was correctly specified, while the IPTW and parametric g-formula estimators were biased under weight and outcome model misspecification, respectively (see Figure 1, Table A.2, and Table A.4). The doubly robust estimators were biased when both models were misspecified. For the doubly robust estimators, MSEs were smaller when the outcome model was correctly specified than when it was misspecified: the MSEs were similar when the weight model was misspecified compared to correctly specified (Table A.2 and Table A.4). These findings are consistent with the empirical results in Funk et al. (2011).

ZIP and ZINB models failed to converge for between 0.1% and 3.4% of simulations when \( n = 800 \), and between 0.1% and 1.1% of simulations when \( n = 2000 \) (Tables A.3 - A.4). This amount of non-convergence is in line with empirical findings from other studies using mixture models (Preisser et al., 2016a; Benecha et al., 2017).

### 3.2 With Data Heaping

To demonstrate the empirical properties of the estimators that account for data heaping, data were simulated where the outcome was heaped, such that the estimators from Section 2.2 were expected to be biased. This simulation study was designed based on the motivating example in Section 4.2, which aimed to estimate the effect of incarceration on the number of cigarettes smoked per day among smokers in a six-month period, controlling for covariates such as income. As with the simulations without data heaping, a sample of \( n = 800 \) participants was simulated, with additional simulations conducted for \( n = 2000 \) presented in the Appendix. The covariate \( L_4 \) was simulated such that \( \exp(L_4) \sim \text{Gamma}(a = 5, s = 2) \) where the density function for the Gamma distribution was \( f(x) = \{s^a\Gamma(a)\}^{-1}x^{a-1}\exp(-x/s), \) and \( \Gamma(z) = \int_0^\infty t^{a-1}\exp(-t)dt \). The exposure \( A \) was simulated from a Bernoulli distribution with mean \( p_4 = 1 - \logit^{-1}(-0.8 + 0.65L_4) \). The potential outcomes for the number of cigarettes smoked under exposure and no exposure were simulated as \( Y^a \mid L_4 \sim \text{Poisson}(\mu^a) \) for \( a \in \{0, 1\} \), where \( \mu^a = \exp(-0.9 + L_4 + 0.25a) \). Thus, \( \log(CMR) = 0.25 \). Under this data generating mechanism, the conditional distribution of \( Y^a \mid L \) was Poisson and the marginal distribution of \( Y^a \) was NB. Thus, the NB distribution was assumed for the correctly specified IPTW heaping estimator, while the Poisson distribution was assumed for the correctly specified PG and DR heaping estimators.

Data heaping were induced with \( \eta = 10 \) and \( \pi = 0.4 \), such that the exact count was observed with probability 0.4 and the count rounded to the nearest multiple of ten was observed with probability 0.6. True and heaped counts for a single iteration of the simulation are presented in Figure A.1.
Figure 1: Results of the simulation study by method and distribution across 2000 samples with correct and incorrect model specification, $n = 800$, without data heaping. Percent empirical bias and 95% confidence interval coverage calculated for the causal mean ratio. ZIP and ZINB g-formula and doubly robust results exclude between 0.1% and 3.4% of simulations where models did not converge.

The estimated $CMRs$ based on the IPTW, parametric g-formula, and doubly robust estimators and their estimated variances were calculated for each scenario both ignoring data heaping, using the estimators described in Section 2.2 (referred to as naive in the results below), and accounting for data heaping using the methods described in Section 2.3. As with the simulations presented in Section 3.1, the $CMR$ was estimated under correct model specification and when the weight and/or outcome model were incorrectly specified. For incorrectly specified models, the covariate $L_5$ was included in the model(s) instead of $L_4$, where $L_5 = \logit^{-1}(-3 + L_4 + 2\epsilon_3)$ and $\epsilon_3$ is simulated from a standard normal distribution.

The results of the data heaping simulations are presented in Figure 2 with more detailed results in Tables A.5 - A.6. Detailed results for simulations with $n = 2000$ are presented in Tables A.7 - A.8. When data heaping was accounted for, the results were similar to those presented in Section 3.1. That is, the IPTW, parametric g-formula, and doubly robust methods all had low empirical bias and close to nominal CI coverage under correct model specification when the weights were treated as estimated. The doubly robust heaping estimator demonstrated low bias under incorrect specification of one (but not both) of the weight or outcome models. When data heaping was ignored and the naive estimators from Section 2.2 were applied to heaped data, the estimates exhibited considerable bias and 95% CI coverage was below the nominal level.

4 Example: Women’s Interagency HIV Study

The WIHS is a multicenter cohort study of women living with HIV or at risk of acquiring HIV (Adimora et al., 2018). At each six-month visit, the WIHS collects data regarding women’s self-reported incarceration status, sexual behaviors, and substance use behaviors during the prior six month period. The methods from Section 2 were applied to data from the WIHS to estimate the effect of incarceration in the past six months on two outcomes in the subsequent six months: the number of sexual partners (Section 4.1) and the number of cigarettes smoked per day among smokers (Section 4.2).
Figure 2: Results of the data heaping simulation study by method across 2000 samples with correct and incorrect model specification, \( n = 800 \). The IPTW heaping estimator assumes a negative binomial distribution while the parametric g-formula and doubly robust heaping and naïve estimators assume a Poisson distribution. Percent empirical bias and 95% confidence interval coverage calculated for the causal mean ratio.

4.1 Number of Partners

The methods in Section 2.2 were applied to estimate the effect of incarceration in the past six months on the total number of male sexual partners (subsequently referred to as partners) during the following six-month period. The effect of incarceration on the number of partners has been estimated when the number of partners was categorized as 0, 1, 2, or 3 or more partners (Knittel et al., 2020). Here, the number of partners is treated as a count outcome for estimation of the \( CMR \). As shown in Figure 3, many participants reported no partners over a six-month period and the number of partners exhibited potential overdispersion, with some participants reporting large numbers of partners relative to the mean.

Baseline was defined as the visit in which the exposure, incarceration status, was measured. The outcome was measured at the visit following baseline and is defined as the number of partners in the six-month period following the baseline visit. As in Knittel et al. (2020), it was assumed that potential outcomes were independent of the exposure conditional on the following covariates, measured at the visit prior to baseline: age, educational attainment (high school or more versus less than high school), race (Black, White, or other), six-level collapsed WIHS site (Bronx and Brooklyn, NY; Washington, DC; Los Angeles, CA; San Francisco, CA; Chicago, IL; Southern Sites - Chapel Hill, NC, Atlanta, GA, Miami, FL, Birmingham, AL, and Jackson, MS), HIV status (positive or negative), binary prior incarceration status, unstable housing (living in a rooming/boarding/halfway house versus other housing), sex exchange practices (exchanging sex for drugs, money, or shelter versus not), alcohol use (none, 1-7 drinks/week, or > 7 drinks/week), binary marijuana use, and illicit drug use (use of crack cocaine, cocaine, heroin, methamphetamines, other opioids, or any injection use versus none). When fitting zero-inflated models to these data, the susceptibility model included the vector of covariates consisting of age, marital status (legally married/common-law married/living with a partner or widowed/divorced/marriage annulled/separated/never married/other), sex exchange practices, HIV status, and sexual orientation (lesbian/gay or heterosexual/straight/bisexual/other), also measured at the visit prior to baseline. These variable classifications were made to predict a woman's potential to have one or more male sexual partners in subsequent study visits. Additional details regarding the data set analyzed are provided in Appendix B.

The estimators \( \hat{CMR}_{IPTW} \), \( \hat{CMR}_{PG} \), and \( \hat{CMR}_{DR} \) were calculated as described in Section 2.2 and standard error estimates were computed using the geex package in R. For each estimate, 95% CIs were constructed. When calculating standard errors for \( \hat{CMR}_{IPTW} \), the weights were treated as estimated in the computation of standard errors. To compare the fit of parametric models to these data, the Akaike information criterion (AIC) was computed for each parametric model. AIC values for the Poisson, NB, ZIP, and ZINB distributions were 2321, 2125, 2281, and 2094, respectively, indicating that the ZINB distribution provided the best fit and was thus used for parametric g-formula and DR estimation.
Figure 3: Distribution of male sexual partners (left, \(n=882\)) and cigarettes smoked per day among smokers (right, \(n=716\)) during the six months following baseline reported by Women’s Interagency HIV Study (WIHS) participants.

The estimated CMR (95% CIs) for the IPTW, parametric g-formula, and doubly robust methods were 1.27 (0.68, 1.85), 1.34 (0.91, 1.77), and 1.24 (0.70, 1.78), respectively. The expected number of partners if incarcerated is estimated to be about 1.3 times the expected number of partners if not incarcerated. Precision estimates were similar across methods, with the parametric g-formula having the smallest estimated standard error and the IPTW having the largest estimated standard error.

4.2 Number of Cigarettes

The CMR for the effect of incarceration in the past six months on the number of cigarettes smoked per day among smokers in the subsequent six-month period was estimated using the methods in Section 2.3. As shown in Figure 3, cigarette counts exhibited overdispersion and data heaping at multiples of ten, and ignoring this measurement error could lead to biased estimates of the CMR. The same set of covariates was assumed to provide conditional exchangeability for the number of cigarettes outcome as the number of partners outcome, except that the sex exchange variable was removed and household income (\(\leq \$12,000\) per year or less versus > \$12,000) was included. The analytic data set was derived similarly to the data set for the partners outcome, with details included in Appendix B. Participants could report cigarettes smoked per day in cigarette or pack counts; it was assumed that one pack of cigarettes equated to 20 cigarettes.

The NB distribution was used for estimation to account for overdispersion in the reported number of cigarettes. An estimated 61%-65% of participants reported exact counts based on the three estimation approaches. The estimated CMR (95% CI) across the three methods were similar, with \(\hat{CMR}_{IPTW,heap}=1.01\) (0.75, 1.26), \(\hat{CMR}_{PG,heap}=1.04\) (0.88, 1.20), and \(\hat{CMR}_{DR,heap}=1.03\) (0.81, 1.25). Under the given assumptions, there is little evidence of an effect of incarceration on the number of cigarettes smoked per day among smokers, as the expected number of cigarettes smoked if incarcerated is estimated to be about the same as the expected number of cigarettes smoked if not incarcerated.

5 Discussion

This paper considers estimators of the causal mean ratio based on marginal structural modeling with IPTWs, the parametric g-formula, and doubly robust estimation. Estimators are developed for outcomes measured without error or subject to data heaping, and each class of estimators accommodates overdispersion or zero-inflation in the outcome. In the absence of measurement error, consistency and asymptotic normality holds for the IPTW and parametric g-formula estimators under correct exposure and outcome model specification, respectively, and for the doubly robust estimators when either the exposure or the outcome model is correctly specified. When data heaping is present, the IPTW estimator also requires a correctly specified MSM for consistent estimation. Simulations demonstrate that all estimators were
empirically unbiased under correct model specification and Wald confidence intervals based on the empirical sandwich variance estimator generally had nominal coverage. The parametric g-formula and doubly robust estimators were more precise than the IPTW estimator in the absence of data heaping but required correct specification of the parametric distribution of the outcome. One notable advantage of the IPTW estimator is that it does not require specification of a parametric model for the outcome when data heaping is not present. The IPTW variance estimator was overly conservative when weights were treated as fixed and thus the use of standard software which treats the weights as known is not recommended. Instead, the sandwich variance estimator which accounts for estimation of the weights can be used.

The results in this manuscript apply to causal mean ratios, i.e., the ratios of counterfactual mean counts over a fixed period of time. Future research could consider extending these methods to allow for inference about rate ratios in settings where follow-up time varies across individuals. These methods could be further extended to accommodate more complex data heaping structures. Semiparametric or non-parametric methods, e.g., using machine learning, could also be considered to relax the parametric modeling assumptions of the estimators considered here. Methods are also needed to estimate causal estimands when the exposure is a count potentially subject to data heaping.

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**Code and Data Availability Statement**

Access to individual-level data from the MACS/WIHS Combined Cohort Study Data (MWCCS) may be obtained upon review and approval of a MWCCS concept sheet. Links and instructions for online concept sheet submission are on the study website (http://mwccs.org/).
Appendix A: Proofs of Main Results

A.1 Section 2.2.1 Derivations

A.1.1 Derivation of the asymptotic variance of (2) when the weights are treated as fixed

Let \( O_i = (Y_i, A_i, L_i) \) and \( \Lambda = (\lambda^0, \lambda^1) \). The estimating equations for \( \hat{\Lambda}^{IPTW} \) and \( \tilde{\Lambda}^{IPTW} \) are

\[
\sum_{i=1}^{n} \psi(O_i, \Lambda) = \left[ \sum_{i=1}^{n} \psi_1(O_i, \Lambda) \right] = \left[ \sum_{i=1}^{n} e_i^{-1} (Y_i - \lambda^1 I(A_i = 1)) \sum_{i=1}^{n} (1 - e_i) (Y_i - \lambda^0) I(A_i = 0) \right] = 0 \quad (A.1)
\]

Assuming causal consistency and conditional exchangeability:

\[
E\{\psi_1(O_i, \Lambda)\} = E\{e_i^{-1} (Y_i - \lambda^1) I(A_i = 1)\} = E_L \{ e_i^{-1} E_{A|L}(A_i | L_i) E_{Y|L}(Y_i | L_i, \lambda^1 - \lambda^1) \} = E(Y_i^1) - \lambda^1 = 0
\]

Similarly, \( E\{\psi_0(O_i, \Lambda)\} = 0 \). Therefore, (A.1) is an unbiased set of estimating equations. It follows that under certain regularity conditions (Stefanski and Boos, 2002), as \( n \to \infty \),

\[
\sqrt{n} \left[ \hat{\Lambda}^{IPTW} - \lambda^1 \right] \xrightarrow{d} N(0, V(\Lambda))
\]

where \( V(\Lambda) = A(\Lambda)^{-1} B(\Lambda) (A(\Lambda)^{-1})^T \), \( A(\Lambda) = E(-\dot{\psi}) \), \( B(\Lambda) = E[\psi(O_i, \Lambda) \dot{\psi}(O_i, \Lambda)^T] \), and

\[
\dot{\psi}(O_i, \Lambda) = \frac{\partial \psi(O_i, \Lambda)}{\partial \Lambda} = \begin{bmatrix} -e_i^{-1} I(A_i = 1) \\ 0 \\ -(1 - e_i)^{-1} I(A_i = 0) \end{bmatrix}
\]

Note \( A(\Lambda) = I_{2 \times 2} \) where \( I_{2 \times 2} \) is the identity matrix. By causal consistency, iterated expectation, and conditional exchangeability it is straightforward to show that

\[
B(\Lambda) = E[\psi(O_i, \Lambda) \dot{\psi}(O_i, \Lambda)^T] = E \begin{bmatrix} e_i^{-2} (Y_i - \lambda^1)^2 I(A_i = 1) & 0 \\ 0 & (1 - e_i)^{-2} (Y_i - \lambda^0)^2 I(A_i = 0) \end{bmatrix}
\]

The delta method can then be used to obtain the asymptotic variance of \( \tilde{\Lambda}^{IPTW} = \hat{\Lambda}^{IPTW} / \lambda^0 \). Specifically, let \( g(\Lambda) = \lambda^1 / \lambda^0 \) such that \( \partial g / \partial \Lambda = (1 / \lambda^0, -\lambda^1 / (\lambda^0)^2) \). Then,

\[
\sqrt{n} \left( \frac{\hat{\Lambda}^{IPTW}}{\lambda^0} - \frac{\lambda^1}{\lambda^0} \right) \xrightarrow{d} N(0, \Sigma_{IPTW})
\]

where

\[
\Sigma_{IPTW} = \frac{\partial g}{\partial \Lambda} V(\Lambda) \left( \frac{\partial g}{\partial \Lambda} \right)^T = E \left[ e_i^{-1} \left( \frac{Y_i - \lambda^1}{\lambda^0} \right)^2 + (1 - e_i)^{-1} \left( \frac{\lambda^1 (Y_i - \lambda^0)}{(\lambda^0)^2} \right)^2 \right]
\]

A.1.2 Derivation of the asymptotic variance of (2) when the weights are treated as estimated

When the weights are treated as estimated rather than fixed, consider the set of estimating equations

\[
\sum_{i=1}^{n} \psi(O_i, \Lambda) = \left[ \sum_{i=1}^{n} \psi_1(O_i, \Lambda) \right] = \left[ \sum_{i=1}^{n} \psi_0(O_i, \Lambda) \right] = 0
\]

where the parameter vector \( \Lambda^T = (\alpha^T, \lambda^1, \lambda^0) \) includes the \( p \) parameters from the logistic regression weight model \( (\alpha) \) and the two causal means \( (\lambda^1 \text{ and } \lambda^0) \), and \( \psi_\alpha \) is the vector of score functions from the logistic regression weight model.

Let the solutions to the estimating equations be denoted by \( \hat{\Lambda} = [\hat{\alpha}, \hat{\lambda}^{IPTW}, \tilde{\lambda}^0] \), where \( \tilde{\lambda}^{IPTW} = \sum_{i=1}^{n} W_i(\hat{\alpha}) Y_i I(A_i = a) / \sum_{i=1}^{n} W_i(\hat{\alpha}) I(A_i = a) \) for \( a \in \{0, 1\} \). When the weight model is correctly specified, \( \hat{\Lambda} \) is the solution to an unbiased set of estimating equations. Thus, \( \sqrt{n}(\hat{\Lambda} - \Lambda) \xrightarrow{d} N(0, V(\Lambda)) \), where
When the outcome model is correctly specified, these estimating equations are unbiased based on maximum likelihood
\[
\hat{\psi}(O_i, \Lambda) = \left[ \begin{array}{c}
\partial \psi_\alpha / \partial \alpha \\
\partial \psi_\alpha / \partial \lambda_1 \\
\vdots \\
\partial \psi_\alpha / \partial \lambda_p 
\end{array} \right] = \left[ \begin{array}{c}
\partial \psi_\alpha / \partial \alpha \\
\partial \psi_\alpha / \partial \lambda_1 \\
\vdots \\
\partial \psi_\alpha / \partial \lambda_p 
\end{array} \right] \left[ \begin{array}{c}
0_{p \times 1} \\
0_{p \times 1} \\
\vdots \\
0_{p \times 1} 
\end{array} \right]
\]
where \( \partial \psi_\alpha / \partial \alpha \) is the \( p \times p \) Jacobian matrix of partial derivatives for \( \psi_\alpha \), \( \partial \psi_\alpha / \partial \lambda \) are gradient vectors for \( \alpha \in \{0, 1\} \), and \( 0_{p \times 1} \) are vectors of 0. Then,
\[
A(\Lambda) = \begin{bmatrix}
A_1 & 0_{p \times 2} \\
A_2 & I_{2 \times 2}
\end{bmatrix}
\]
where \( A_1 = E(-\partial \psi_\alpha / \partial \alpha) \) and \( A_2 = [E(-\partial \psi_1 / \partial \alpha), E(-\partial \psi_0 / \partial \alpha)]^T \). Let
\[
B(\Lambda) = \begin{bmatrix}
B_{11} & B_{12}^T \\
B_{21} & B_{22}
\end{bmatrix}
\]
where \( B_{11} \) is \( (p \times p) \), \( B_{21} \) is \( 2 \times p \), and \( B_{22} \) is \( (2 \times 2) \). By Lemma 7.3.11 in Casella and Berger (2002), \( A_1 = B_{11} \). It is straightforward to show that \( A_2 = B_{21} \). Thus,
\[
V(\Lambda) = \begin{bmatrix}
A_1^{-1} & 0_{p \times 2} \\
a_{2 \times 2}
\end{bmatrix} \begin{bmatrix}
A_1 & A_2^T \\
A_2 & B_{22}
\end{bmatrix} \begin{bmatrix}
A_1^{-1} & 0_{p \times 2} \\
0_{2 \times p} & I_{2 \times 2}
\end{bmatrix} = \begin{bmatrix}
A_1^{-1} & 0_{p \times 2} \\
0_{2 \times p} & B_{22} - A_2 A_1^{-1} A_2^T
\end{bmatrix}
\]
Letting \( g(\lambda) = \lambda^1 / \lambda^0 \), it then follows from the delta method that
\[
\sqrt{n} \left( \begin{array}{c}
\lambda_{IPTW} \\
\lambda_0
\end{array} \right) \sim N(0, \Sigma_{IPTW})
\]
where
\[
\Sigma_{IPTW} = \frac{\partial g}{\partial \lambda} \left( \begin{array}{c}
\lambda_{IPTW} \\
\lambda_0
\end{array} \right)^T = g^T B_{22} g^* - g^T (A_2 A_1^{-1} A_2^T) g^* = \Sigma_{IPTW} - g^T (A_2 A_1^{-1} A_2^T) g^*
\]
and \( g^* = [1/\lambda^0 - \lambda^1/(\lambda^0)^2] \). The final equality holds because \( B_{22} = V(\Lambda) \) from A.1.1. Note that \( g^T (A_2 A_1^{-1} A_2^T) g^* \geq 0 \) because \( B_{22} \) is positive semi-definite. Thus, \( \Sigma_{IPTW} \leq \Sigma_{IPTW} \).

A.2 Section 2.2.2 Derivations

Assume that \( \hat{E}(Y_i \mid L_i, A_i = a) \) is estimated based on one of the four models described in Section 2.2.2. Define the set of estimating equations:
\[
\sum_{i=1}^{n} \psi(Y_i, A_i, L_i; \gamma, \lambda) = \sum_{i=1}^{n} \psi_i(Y_i, A_i, L_i; \gamma) = 0 \quad (A.2)
\]
where \( \psi_i \) is the derivative of the log-likelihood function for the model with respect to the regression coefficients. When the outcome model is correctly specified, these estimating equations are unbiased based on maximum likelihood theory, with solutions \( \hat{\gamma} \). Now we define the estimating equations for the causal means. Define \( \hat{\lambda}_{PG} = \hat{E}(Y^a) = \int \hat{E}(Y \mid L, A)dF_L(l) = \int \hat{E}(Y \mid L, A)dF_L(l) = \int \hat{E}(Y \mid L, A, A_i = a) \) where \( F_L \) is the empirical distribution function of \( L \). Then, \( \sum_{i=1}^{n} \psi(Y_i, A_i, L_i; \lambda^a) = \sum_{i=1}^{n} \{ \psi(Y_i, A_i, L_i; \lambda^a) - \lambda^a \} = 0 \) for \( a \in \{0, 1\} \), where \( E(Y_i \mid L_i, A_i = a) \) is estimated by the predicted count for observation \( i \) based on the outcome model. When the model is correctly specified and based on causal consistency and conditional exchangeability,
\[
E(\psi(Y_i, L_i; \gamma, \lambda^a)) = E(E(Y_i \mid L_i, A_i = a) - \lambda^a) = E(E(Y_i \mid L_i, A_i = a) - \lambda^a)
\]
Thus, (A.2) is an unbiased set of estimating equations, implying
\[
\sqrt{n} \left( \begin{array}{c}
\hat{\lambda}_{PG} - \lambda^1 \\
\hat{\lambda}_{PG} - \lambda^0
\end{array} \right) \sim N(0, \Sigma_{PG})
\]
where \( \Sigma_{PG} = A(\Lambda)^{-1} B(\Lambda) \{ A(\Lambda)^{-1} \}^T \), with \( \Lambda^T = (\gamma^T, \lambda^1, \lambda^0) \).
When the weight model is correctly specified, \( E = \{ \psi(Y, A, L, \Lambda) \psi(Y, A, L, \Lambda)^T \} \), and \( \dot{\psi}(Y, A, L, \Lambda) = \partial \psi(Y, A, L, \Lambda) / \partial \Lambda^T \). The delta method can then be applied to obtain the asymptotic distribution of \( \hat{\lambda}_{MR}^\ast = \hat{\lambda}_{PG}^\ast / \hat{\lambda}_{PG}^0 \). Specifically, let \( g(\Lambda) = \lambda^1 / \lambda^0 \) such that \( \partial g(\Lambda) / \partial \Lambda = [0_{1 \times p}, 1/\lambda^0, -\lambda^1 / (\lambda^0)^2]^T \). Then,

\[
\sqrt{n} \left( \frac{\hat{\lambda}_{PG}^\ast - \lambda^1}{\lambda_{PG}^0} - \frac{\lambda^1}{\lambda^0} \right) \overset{d}{\rightarrow} N(0, \Sigma_{PG})
\]

where

\[
\Sigma_{PG} = \frac{\partial g(\Lambda)}{\partial \Lambda} \Sigma_{PG} \left( \frac{\partial g(\Lambda)}{\partial \Lambda} \right)^T
\]

### A.3 Section 2.2.3 Derivations:

Define the set of estimating equations:

\[
\sum_{i=1}^n \psi(Y_i, A_i, L_i; \alpha, \gamma, \lambda) = \left[ \sum_{i=1}^n \psi_n(A_i, L_i; \alpha) \right]
\]

where \( \psi_n \) and \( \psi_m \) are defined in Sections A.1 and A.2 and

\[
\psi(Y, A, L; \alpha, \gamma, \lambda) = [\psi_1(Y, A, L; \alpha, \gamma, \lambda)]^T = \left[ \begin{array}{c} \psi(Y, A, L; \alpha, \gamma, \lambda) \\ \psi(Y, A, L; \alpha, \gamma, \lambda) \end{array} \right]
\]

are the estimating equations for \( \hat{\lambda}_{DR}^1 \) and \( \hat{\lambda}_{DR}^0 \), respectively.

When either the weight model or the outcome model is correctly specified, the solutions \( \hat{\lambda}_{DR}^1 \) and \( \hat{\lambda}_{DR}^0 \) to the estimating equations \( \psi(Y_i, A_i, L_i; \alpha, \gamma, \lambda) \) and \( \psi_0(Y_i, A_i, L_i; \alpha, \gamma, \lambda) \) are consistent estimators of the causal means \( \lambda^1 \) and \( \lambda^0 \), respectively. This is shown as follows. Suppose \( \hat{\alpha} \overset{p}{\rightarrow} \alpha_0 \) and \( \hat{\gamma} \overset{p}{\rightarrow} \gamma_0 \), where \( p \) denotes convergence in probability. When the weight model is correctly specified, \( e(L_i, \alpha_0) = E(A_i = 1 \mid L_i) \). Similarly, when the outcome model is correctly specified, \( m_0(L_i, \gamma_0) = E(E_i = 1 \mid L_i) \). By causal consistency and with algebraic manipulation, \( \hat{\psi}_1(Y_i, A_i, L_i; \alpha, \gamma, \lambda) = (1 - Y_i) \) and \( \hat{\psi}_0(Y_i, A_i, L_i; \alpha, \gamma, \lambda) = (1 - (\alpha_0)) \). Note that by conditional exchangeability:

\[
E[\{A_i - e(L_i, \alpha)\} \{Y_i^1 - m_1(L_i, \gamma)\}^T] = E_L \left( \{e(L_i, \alpha)\}^T A_i - e(L_i, \alpha) \right) E_{Y_1 \mid L} \{Y_i^1 - m_1(L_i, \gamma)\}
\]

When the model is correctly specified, \( E_{A_i \mid L} \{A_i - e(L_i, \alpha)\} = E_{A_i \mid L} \{A_i\} = e(L_i, \alpha_0) = 0 \) and when the outcome model is correctly specified \( E_{Y_1 \mid L} \{Y_i^1 - m_1(L_i, \gamma)\} = E_{Y_1 \mid L} \{Y_i^1\} = m_1(L_i, \gamma_0) = 0 \). Then, \( E \{\hat{\psi}_1(Y_i, A_i, L_i; \alpha, \gamma, \lambda)\} = E(Y_i^1) = \lambda^1 \). Thus,

\[
\hat{\psi}_1(Y_i, A_i, L_i; \alpha, \gamma, \lambda) \text{ is unbiased when the weight or outcome model is correctly specified. Similarly,}
\]

\( \hat{\psi}_0(Y_i, A_i, L_i; \alpha, \gamma, \lambda) \text{ is unbiased when either model is correctly specified. Therefore,}
\]

\[
\sqrt{n} \left[ \begin{array}{c} \hat{\alpha} - \alpha \\ \hat{\gamma} - \gamma \\ \hat{\lambda}_{DR}^1 - \lambda^1 \\ \hat{\lambda}_{DR}^0 - \lambda^0 \end{array} \right] \overset{d}{\rightarrow} N(0, \Sigma_{DR})
\]

where \( \Sigma_{DR} = A(\Lambda)^{-1}B(\Lambda)^{-1} \). The delta method is applied to obtain the asymptotic distribution of \( \hat{\lambda}_{DR}^1 / \hat{\lambda}_{DR}^0 \). Specifically, let \( g(\Lambda) = \lambda^1 / \lambda^0 \) such that \( \partial g(\Lambda) / \partial \Lambda = [0_{1 \times p}, 1/\lambda^0, -\lambda^1 / (\lambda^0)^2]^T \). Then,

\[
\sqrt{n} \left( \frac{\hat{\lambda}_{DR}^1}{\lambda_{DR}^0} - \frac{\lambda^1}{\lambda^0} \right) \overset{d}{\rightarrow} N(0, \Sigma_{DR}^*)
\]

where

\[
\Sigma_{DR}^* = \frac{\partial g(\Lambda)}{\partial \Lambda} \Sigma_{DR} \left( \frac{\partial g(\Lambda)}{\partial \Lambda} \right)^T
\]
We first show that the estimating equations for \( \hat{\delta} \) in (6) are unbiased. Note \( \hat{CMR}_{IPTW,heap} = \exp(\hat{\delta}_i) \), so \( \hat{CMR}_{IPTW,heap} \) will be consistent and asymptotically normal if these estimating equations are unbiased. The estimating equations for \( \hat{\delta} \) are

\[
\sum_{i=1}^{n} W_i \frac{\partial}{\partial \delta_j} \log l(Y_{ih}) = 0
\]

Note

\[
E(W \frac{\partial}{\partial \delta_j} \log l(Y_{ih})) = EW \frac{\partial}{\partial \delta_j} \log \left\{ \pi f(Y_{ih}; \delta) + (1 - \pi) \sum_{y:h_n(y) = Y_{ih}} f(y; \delta) \right\}
\]

\[
= EAW \frac{\partial}{\partial \delta_j} \log \left\{ \pi f(Y_{ih}; \delta) + (1 - \pi) \sum_{y:h_n(y) = Y_{ih}} f(y; \delta) \right\}
\]

\[
+ E(1 - A)W \frac{\partial}{\partial \delta_j} \log \left\{ \pi f(Y_{ih}; \delta) + (1 - \pi) \sum_{y:h_n(y) = Y_{ih}} f(y; \delta) \right\}
\]

\[
= E \frac{\partial}{\partial \delta_j} \log \left\{ \pi f_1(Y_{ih}; \delta) + (1 - \pi) \sum_{y:h_n(y) = Y_{ih}} f_1(y; \delta) \right\} + E \frac{\partial}{\partial \delta_j} \log \left\{ \pi f_0(Y_{ih}; \delta) + (1 - \pi) \sum_{y:h_n(y) = Y_{ih}} f_0(y; \delta) \right\}
\]

where the last equality holds by causal consistency and conditional exchangeability. Then under the assumed MSM

\[
E \frac{\partial}{\partial \delta_j} \log \left\{ \pi f_1(Y_{ih}; \delta) + (1 - \pi) \sum_{y:h_n(y) = Y_{ih}} f_1(y; \delta) \right\}
\]

\[
= E \left[ \left\{ \pi f_1(Y_{ih}; \delta) + (1 - \pi) \sum_{y:h_n(y) = Y_{ih}} f_1(y; \delta) \right\} \right]^{-1} \times \frac{\partial}{\partial \delta_j} \left\{ \pi f_1(Y_{ih}; \delta) + (1 - \pi) \sum_{y:h_n(y) = Y_{ih}} f_1(y; \delta) \right\}
\]

\[
= \sum_{y_{ih} = 0}^{\infty} \frac{\partial}{\partial \delta_j} \left\{ \pi f_1(y_{ih}; \delta) + (1 - \pi) \sum_{y:h_n(y) = y_{ih}} f_1(y; \delta) \right\}
\]

\[
= \pi \frac{\partial}{\partial \delta_j} \sum_{y_{ih} = 0}^{\infty} f_1(y_{ih}; \delta) + (1 - \pi) \frac{\partial}{\partial \delta_j} \sum_{y_{ih} = 0}^{\infty} \sum_{y:h_n(y) = y_{ih}} f_1(y; \delta)
\]

\[
= \pi \frac{\partial}{\partial \delta_j} (1) + (1 - \pi) \frac{\partial}{\partial \delta_j} (1) = 0
\]

Similarly, under the assumed MSM

\[
E \frac{\partial}{\partial \delta_j} \log \left\{ \pi f_0(Y_{ih}; \delta) + (1 - \pi) \sum_{y:h_n(y) = Y_{ih}} f_0(y; \delta) \right\} = 0
\]

Thus, (6) is an unbiased estimating equation vector and \( \hat{CMR}_{IPTW,heap} \) is a consistent and asymptotically normal estimator of the \( CMR \).

Appendix B: WIHS Data Analysis

The WIHS sample used in the Section 4.1 analysis was created by restricting the longitudinal WIHS data set of 4,982 women to women who attended at least one visit between 2007-2017, as 2007 is when incarceration questions were added to the WIHS questionnaire. As in Knittel et al. (2020), the data set was further restricted to include only women without missing covariates following implementation of last value carried forward and next value carried back imputation, excluding the history of incarceration covariate which was only asked at a single timepoint and thus could not be imputed using this method. For each woman who reported being incarcerated between 2007-2017, her first incarcerated visit following a non-incarcerated visit was selected as her baseline visit. This allowed for an appropriate run-in period in which to measure covariates at the visit preceding baseline. The outcome was measured at the visit following baseline. This resulted in \( n = 294 \) incarcerated women after excluding the 28 women missing outcome
data at the visit following baseline. A sample of one visit from each of \( n = 588 \) women who did not report being incarcerated between 2007-2017 was randomly selected, ensuring the same distribution of baseline visits over calendar time as the incarcerated women and restricting sampling to women with non-missing outcome data at the visit following the sampled baseline visit. Missing values for prior incarceration for \( n = 14 \) participants were imputed with the mode (no history of incarceration).

Similar procedures were used to create the WIHS sample used in the Section 4.2 analysis, with modifications to the covariate set and an added requirement that women were current smokers at the visit preceding baseline. This resulted in \( n = 179 \) incarcerated women with complete outcome data. A sample of \( n = 537 \) women who reported no incarcerations during the study period were selected with the same distributions of baseline visits over calendar time as the incarcerated women, for a total sample size of \( n = 716 \). Missing prior incarcerations were imputed to the mode for \( n = 6 \) participants.

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Table A.1: Results of the simulation study by distribution and method across 2000 samples with correct model specification, $n = 800$. Percent empirical bias, MSE, ESE, SER, and empirical 95% CI coverage calculated for the CMR. ZIP PG and DR results exclude 3.0% and 2.9% of simulations, respectively, where models did not converge. ZINB PG and DR results exclude 3.4% of simulations where models did not converge.

| Distribution | Method          | Empirical Bias (%) | MSE   | ESE   | SER   | 95% CI Coverage (%) |
|--------------|-----------------|--------------------|-------|-------|-------|---------------------|
| Poisson      | IPTW, fixed     | 0.5                | 0.34  | 0.12  | 2.86  | 100                 |
|              | IPTW, estimated | 0.5                | 0.12  | 0.12  | 0.97  | 95                  |
|              | PG              | 0.1                | 0.08  | 0.08  | 0.97  | 95                  |
|              | DR              | 0.1                | 0.08  | 0.08  | 0.96  | 95                  |
| NB           | IPTW, fixed     | 1.6                | 0.41  | 0.27  | 1.53  | 99                  |
|              | IPTW, estimated | 1.6                | 0.26  | 0.27  | 0.97  | 94                  |
|              | PG              | 0.1                | 0.16  | 0.16  | 1.00  | 95                  |
|              | DR              | 1.2                | 0.25  | 0.25  | 0.98  | 94                  |
| ZIP          | IPTW, fixed     | 0.3                | 0.35  | 0.13  | 2.75  | 100                 |
|              | IPTW, estimated | 0.3                | 0.12  | 0.13  | 0.97  | 95                  |
|              | PG              | 0.0                | 0.08  | 0.09  | 0.95  | 94                  |
|              | DR              | 0.0                | 0.09  | 0.09  | 0.93  | 94                  |
| ZINB         | IPTW, fixed     | 1.7                | 0.43  | 0.29  | 1.45  | 99                  |
|              | IPTW, estimated | 1.7                | 0.27  | 0.29  | 0.92  | 93                  |
|              | PG              | 0.4                | 0.19  | 0.18  | 1.05  | 96                  |
|              | DR              | 1.4                | 0.30  | 0.28  | 1.07  | 95                  |

Abbreviations: IPTW=Inverse Probability of Treatment Weight; PG=Parametric g-formula; DR=Doubly Robust Estimator; MSE=Median Estimated Standard Error; ESE=Empirical Standard Error; SER=Standard Error Ratio (MSE/ESE); CI=Confidence Interval; CMR=Causal Mean Ratio; NB=Negative Binomial; ZIP=Zero-Inflated Poisson; ZINB=Zero-Inflated Negative Binomial
Table A.2: Results of the simulation study by distribution and method across 2000 samples with the weight model misspecified (MW), the outcome model misspecified (MO), or both models misspecified (MB), $n = 800$. Percent empirical bias, MSE, ESE, SER, and empirical 95% CI coverage calculated for the CMR. ZIP PG and DR results exclude 0.1%-2.9% of simulations where models did not converge. ZINB PG and DR results exclude 1.8%-3.4% of simulations where models did not converge.

| Distribution | Method | Empirical Bias (%) | MSE | ESE | SER | 95% CI Coverage (%) |
|--------------|--------|---------------------|-----|-----|-----|---------------------|
| Poisson      | IPTW, fixed, MW | 7.5 | 0.36 | 0.13 | 2.73 | 100 |
|              | IPTW, estimated, MW | 7.5 | 0.13 | 0.13 | 0.98 | 87 |
|              | DR, MW | 0.1 | 0.08 | 0.08 | 0.96 | 95 |
| NB           | IPTW, fixed, MW | 8.6 | 0.44 | 0.29 | 1.52 | 100 |
|              | IPTW, estimated, MW | 8.6 | 0.28 | 0.29 | 0.97 | 94 |
|              | DR, MW | 1.1 | 0.25 | 0.25 | 0.98 | 94 |
| ZIP          | IPTW, fixed, MW | 7.4 | 0.37 | 0.14 | 2.58 | 100 |
|              | IPTW, estimated, MW | 7.4 | 0.14 | 0.14 | 0.96 | 89 |
|              | DR, MW | 0.0 | 0.09 | 0.09 | 0.93 | 94 |
| ZINB         | IPTW, fixed, MW | 8.9 | 0.45 | 0.32 | 1.43 | 100 |
|              | IPTW, estimated, MW | 8.9 | 0.29 | 0.32 | 0.93 | 94 |
|              | DR, MW | 1.4 | 0.30 | 0.28 | 1.07 | 95 |
| Poisson      | PG, MO | 7.5 | 0.13 | 0.13 | 0.97 | 87 |
|              | DR, MO | 0.6 | 0.12 | 0.12 | 0.96 | 95 |
| NB           | PG, MO | 8.7 | 0.18 | 0.18 | 1.00 | 92 |
|              | DR, MO | 1.6 | 0.26 | 0.27 | 0.97 | 94 |
| ZIP          | PG, MO | 7.3 | 0.13 | 0.14 | 0.96 | 88 |
|              | DR, MO | 0.4 | 0.13 | 0.13 | 0.97 | 95 |
| ZINB         | PG, MO | 9.1 | 0.21 | 0.20 | 1.05 | 95 |
|              | DR, MO | 1.8 | 0.31 | 0.29 | 1.07 | 96 |
| Poisson      | DR, MB | 7.5 | 0.13 | 0.13 | 0.97 | 87 |
| NB           | DR, MB | 8.6 | 0.28 | 0.29 | 0.97 | 94 |
| ZIP          | DR, MB | 7.4 | 0.14 | 0.14 | 0.96 | 89 |
| ZINB         | DR, MB | 8.9 | 0.33 | 0.31 | 1.06 | 97 |

Abbreviations: IPTW=Inverse Probability of Treatment Weight; PG=Parametric g-formula; DR=Doubly Robust Estimator; MSE=Median Estimated Standard Error; ESE=Empirical Standard Error; SER=Standard Error Ratio (MSE/ESE); CI=Confidence Interval; CMR=Causal Mean Ratio; NB=Negative Binomial; ZIP=Zero-Inflated Poisson; ZINB=Zero-Inflated Negative Binomial
Table A.3: Results of the simulation study by distribution and method across 2000 samples with correct model specification, $n = 2000$. Percent empirical bias, MSE, ESE, SER, and empirical 95% CI coverage calculated for the $CMR$. ZIP PG and DR results exclude 1.1% of simulations where models did not converge. ZINB PG and DR results exclude 1.0% of simulations where models did not converge.

| Distribution | Method      | Empirical Bias (%) | MSE     | ESE     | SER     | Coverage (%) |
|--------------|-------------|---------------------|---------|---------|---------|--------------|
| Poisson      | MSM, fixed  | 0.2                 | 0.21    | 0.08    | 2.80    | 100          |
|              | MSM, estimated | 0.2              | 0.07    | 0.08    | 0.96    | 94           |
|              | PG          | 0.1                 | 0.05    | 0.05    | 0.98    | 94           |
|              | DR          | 0.1                 | 0.05    | 0.05    | 0.97    | 94           |
| NB           | MSM, fixed  | 0.4                 | 0.26    | 0.18    | 1.48    | 100          |
|              | MSM, estimated | 0.4              | 0.17    | 0.18    | 0.95    | 93           |
|              | PG          | 0.2                 | 0.10    | 0.10    | 0.98    | 95           |
|              | DR          | 0.3                 | 0.16    | 0.17    | 0.96    | 93           |
| ZIP          | MSM, fixed  | 0.1                 | 0.22    | 0.08    | 2.78    | 100          |
|              | MSM, estimated | 0.1              | 0.08    | 0.08    | 1.00    | 94           |
|              | PG          | 0.0                 | 0.05    | 0.05    | 0.98    | 94           |
|              | DR          | 0.0                 | 0.06    | 0.06    | 0.95    | 94           |
| ZINB         | MSM, fixed  | 0.6                 | 0.27    | 0.18    | 1.51    | 99           |
|              | MSM, estimated | 0.6              | 0.17    | 0.18    | 0.98    | 95           |
|              | PG          | 0.1                 | 0.12    | 0.11    | 1.05    | 96           |
|              | DR          | 0.5                 | 0.18    | 0.17    | 1.07    | 97           |

Abbreviations: IPTW=Inverse Probability of Treatment Weight; PG=Parametric $g$-formula; DR=Doubly Robust Estimator; MSE=Median Estimated Standard Error; ESE=Empirical Standard Error; SER=Standard Error Ratio (MSE/ESE); CI=Confidence Interval; CMR=Causal Mean Ratio; NB=Negative Binomial; ZIP=Zero-Inflated Poisson; ZINB=Zero-Inflated Negative Binomial
### Table A.4: Results of the simulation study by distribution and method across 2000 samples with the weight model misspecified (MW), the outcome model misspecified (MO), or both models misspecified (MB), n = 2000. Percent empirical bias, MSE, ESE, SER, and empirical 95% CI coverage calculated for the CMR. ZINB PG and DR results exclude 0.4%-1.0% of simulations where models did not converge. ZINB PG and DR results exclude 0.4%-1.0% of simulations where models did not converge.

| Distribution | Method                  | Empirical Bias (%) | MSE   | ESE   | SER   | 95% CI Coverage (%) |
|--------------|-------------------------|--------------------|-------|-------|-------|---------------------|
| Poisson      | MSM, fixed, MW          | 7.2                | 0.23  | 0.08  | 2.69  | 100                 |
|              | MSM, estimated, MW      | 7.2                | 0.08  | 0.08  | 0.98  | 72                  |
|              | DR, MW                  | 0.1                | 0.05  | 0.05  | 0.97  | 94                  |
| NB           | MSM, fixed, MW          | 7.4                | 0.28  | 0.19  | 1.47  | 100                 |
|              | MSM, estimated, MW      | 7.4                | 0.18  | 0.19  | 0.96  | 92                  |
|              | DR, MW                  | 0.2                | 0.16  | 0.17  | 0.96  | 93                  |
| ZIP          | MSM, fixed, MW          | 7.2                | 0.23  | 0.09  | 2.66  | 100                 |
|              | MSM, estimated, MW      | 7.2                | 0.09  | 0.09  | 1.00  | 76                  |
|              | DR, MW                  | 0.0                | 0.06  | 0.06  | 0.95  | 94                  |
| ZINB         | MSM, fixed, MW          | 7.8                | 0.29  | 0.19  | 1.48  | 100                 |
|              | MSM, estimated, MW      | 7.8                | 0.19  | 0.19  | 0.97  | 93                  |
|              | DR, MW                  | 0.5                | 0.18  | 0.17  | 1.07  | 96                  |

**Abbreviations:** IPTW=Inverse Probability of Treatment Weight; PG=Parametric g-formula; DR=Doubly Robust Estimator; MSE=Median Estimated Standard Error; ESE=Empirical Standard Error; SER=Standard Error Ratio (MSE/ESE); CI=Confidence Interval; CMR=Causal Mean Ratio; NB=Negative Binomial; ZIP=Zero-Inflated Poisson; ZINB=Zero-Inflated Negative Binomial

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### Table A.5: Results of the data heaping simulation study by method across 2000 samples with correct model specification, n = 800. Percent empirical bias, MSE, ESE, SER, and empirical 95% CI coverage calculated for the CMR.

| Method  | Estimator | Empirical Bias (%) | MSE   | ESE   | SER   | 95% CI Coverage (%) |
|---------|-----------|--------------------|-------|-------|-------|---------------------|
| IPTW, fixed | Naïve     | 6.6                | 0.10  | 0.08  | 1.22  | 93                  |
|          | Heaping   | 0.0                | 0.07  | 0.06  | 1.23  | 98                  |
| IPTW, estimated | Naïve   | 6.6                | 0.08  | 0.08  | 1.01  | 84                  |
|          | Heaping   | 0.0                | 0.06  | 0.06  | 1.01  | 95                  |
| PG       | Naïve     | 7.2                | 0.08  | 0.08  | 1.01  | 82                  |
|          | Heaping   | 0.0                | 0.05  | 0.05  | 1.03  | 96                  |
| DR       | Naïve     | 6.7                | 0.08  | 0.08  | 1.01  | 84                  |
|          | Heaping   | 0.0                | 0.05  | 0.05  | 1.02  | 96                  |

**Abbreviations:** IPTW=Inverse Probability of Treatment Weight; PG=Parametric g-formula; DR=Doubly Robust Estimator; MSE=Median Estimated Standard Error; ESE=Empirical Standard Error; SER=Standard Error Ratio (MSE/ESE); CI=Confidence Interval; CMR=Causal Mean Ratio
Table A.6: Results of the data heaping simulation study by method across 2000 samples with the weight model misspecified (MW), the outcome model misspecified (MO), or both models misspecified (MB), \( n = 800 \). Percent empirical bias, MSE, ESE, SER, and empirical 95% CI coverage calculated for the CMR.

| Method               | Empirical Bias (%) | MSE   | ESE   | SER   | 95% CI Coverage (%) |
|----------------------|--------------------|-------|-------|-------|--------------------|
| IPTW, fixed, MW      | -11.5              | 0.06  | 0.06  | 1.04  | 33                 |
| IPTW, estimated, MW  | -11.5              | 0.06  | 0.06  | 1.03  | 33                 |
| DR, MW               | 0.1                | 0.05  | 0.05  | 1.03  | 97                 |
| PG, MO               | -12.6              | 0.05  | 0.05  | 1.03  | 17                 |
| DR, MO               | -0.6               | 0.05  | 0.05  | 1.01  | 95                 |
| DR, MB               | -12.6              | 0.05  | 0.05  | 1.03  | 18                 |

Abbreviations: IPTW=Inverse Probability of Treatment Weight; PG=Parametric g-formula; DR=Doubly Robust Estimator; MSE=Median Estimated Standard Error; ESE=Empirical Standard Error; SER=Standard Error Ratio (MSE/ESE); CI=Confidence Interval; CMR=Causal Mean Ratio

Table A.7: Results of the data heaping simulation study by method across 2000 samples with correct model specification, \( n = 2000 \). Percent empirical bias, MSE, ESE, SER, and empirical 95% CI coverage calculated for the CMR.

| Method               | Estimator | Empirical Bias (%) | MSE   | ESE   | SER   | 95% CI Coverage (%) |
|----------------------|-----------|--------------------|-------|-------|-------|--------------------|
| IPTW, fixed          | Naïve     | 6.5                | 0.06  | 0.05  | 1.20  | 79                 |
|                      | Heaping   | 0.0                | 0.05  | 0.04  | 1.20  | 98                 |
| IPTW, estimated      | Naïve     | 6.5                | 0.05  | 0.05  | 0.99  | 62                 |
|                      | Heaping   | 0.0                | 0.04  | 0.04  | 0.98  | 95                 |
| PG                   | Naïve     | 7.0                | 0.05  | 0.05  | 0.99  | 58                 |
|                      | Heaping   | 0.0                | 0.03  | 0.03  | 0.99  | 95                 |
| DR                   | Naïve     | 6.5                | 0.05  | 0.05  | 0.99  | 62                 |
|                      | Heaping   | 0.0                | 0.03  | 0.03  | 0.99  | 95                 |

Abbreviations: IPTW=Inverse Probability of Treatment Weight; PG=Parametric g-formula; DR=Doubly Robust Estimator; MSE=Median Estimated Standard Error; ESE=Empirical Standard Error; SER=Standard Error Ratio (MSE/ESE); CI=Confidence Interval; CMR=Causal Mean Ratio

Table A.8: Results of the data heaping simulation study by method across 2000 samples with the weight model misspecified (MW), the outcome model misspecified (MO), or both models misspecified (MB), \( n = 2000 \). Percent empirical bias, MSE, ESE, SER, and empirical 95% CI coverage calculated for the CMR.

| Method               | Empirical Bias (%) | MSE   | ESE   | SER   | 95% CI Coverage (%) |
|----------------------|--------------------|-------|-------|-------|--------------------|
| IPTW, fixed, MW      | -11.6              | 0.04  | 0.04  | 0.99  | 4                  |
| IPTW, estimated, MW  | -11.6              | 0.04  | 0.04  | 0.98  | 4                  |
| DR, MW               | 0.0                | 0.03  | 0.03  | 0.99  | 95                 |
| PG, MO               | -12.7              | 0.03  | 0.03  | 0.98  | 1                  |
| DR, MO               | -0.8               | 0.03  | 0.03  | 0.97  | 93                 |
| DR, MB               | -12.7              | 0.03  | 0.03  | 0.98  | 1                  |

Abbreviations: IPTW=Inverse Probability of Treatment Weight; PG=Parametric g-formula; DR=Doubly Robust Estimator; MSE=Median Estimated Standard Error; ESE=Empirical Standard Error; SER=Standard Error Ratio (MSE/ESE); CI=Confidence Interval; CMR=Causal Mean Ratio