I. INTRODUCTION

Black objects (holes, strings, rings etc.) in higher dimensional spacetimes have attracted a lot of attention recently. The existence of higher than 4 dimensions of the spacetime is a natural consequence of the consistency requirement in the string theory. Models with large extra dimensions, originally proposed to solve such long-standing fundamental ‘puzzles’ as the hierarchy and cosmological constant problems, became very popular recently. In these models mini black holes and other black objects play a special role serving as natural probes of extra dimensions, originally proposed to solve such long-standing fundamental ‘puzzles’ as the hierarchy and cosmological constant problems, became very popular recently. In these models mini black holes and other black objects play a special role serving as natural probes of extra dimensions.

Higher dimensional generalizations of the Kerr metric for a rotating black holes were obtained quite long time ago by Myers and Perry. In a 4-dimensional spacetime the MP metrics besides the mass M contain also [(D − 1)/2] parameters connected with the independent components of the angular momentum of the black hole. The event horizon of the MP black holes has the spherical topology S^{D−2}. This makes them in many aspects similar to the 4D Kerr black hole. According to the Hawking theorem any stationary black hole in a 4D spacetime obeying the dominant energy condition has the topology of the horizon S^2. Black hole surface topologies distinct from S^2 are possible if the dominant energy condition is violated. Moreover, a vacuum stationary black hole is uniquely specified by its mass and angular momentum. Recent discover of black ring solutions demonstrated that both the restriction on the topology of the horizon and the uniqueness property of black holes are violated in the 5D spacetime.

In this paper we discuss the geometry of the horizon surfaces of 5D black rings and a 5D black holes with one rotation parameter. A similar problem for the 4D rotating black holes was studied in detail by Smarr. We generalize his approach to the 5D case. After a brief summary of known properties of 3D round spheres and tori in the flat 4D space (Section 2) we consider a geometry of 3D space which admits 2 orthogonal commuting Killing vectors (Section 3). In particular we calculate its Gauss curvature. In Section 4 we apply these results to the horizon surface of 5D rotating black hole with one rotation parameter. The embedding of this 3D surface into the flat spacetime is considered in Section 5. The horizon surface geometry for a 5D rotating black ring is discussed in Section 6. This Section considers also a Kaluza-Klein reduction of the black ring metric along the direction of its rotation which maps this solution onto a black hole solution of 4D Einstein equations with the dilaton and ‘electromagnetic’ fields. The geometry and embedding of the horizon in the E^3 for this metric is obtained. Section 7 contains the discussion of the results.

II. SPHERE S^3 AND TORUS S^2 × S^1 IN E^4

A. Sphere S^3

In this section we briefly remind some known properties of a 3D sphere and a torus in a flat 4D space.

Consider 4-dimensional Euclidean space E^4 and denote by X_i (i = 1, . . . , 4) the Cartesian coordinates in it. A 3-sphere consists of all points equidistant from a single point X_0 = 0 in R^4. A unit round sphere S^3 is a surface defined by the equation \sum_{i=1}^{4} X_i^2 = 1. Using complex coordinates z_1 = X_1 + iX_2 and z_2 = X_3 + iX_4 one can also equivalently define the unit 3-sphere as a subset of C^2

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1 \}.$$  \hspace{1cm} (1)

We use the embedding of S^3 in \mathbb{C}^2 to introduce the Hopf co-ordinates (t, ϕ, ψ) as,

$$z_1 = \text{sin}(\theta)e^{i\phi}; \quad z_2 = \text{cos}(\theta)e^{i\psi}.$$ \hspace{1cm} (2)

Here θ runs over the range [0, π/2], and ϕ and ψ can take any values between 0 and 2π. In these co-ordinates the
metric on the 3-sphere is
\[ ds^2 = \theta d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta dv^2. \]  
(3)

The volume of the unit 3-sphere is \( 2\pi^2 \). Coordinate lines of \( \phi \) and \( \psi \) are circles. The length of these circles take the maximum value \( 2\pi \) at \( \theta = \pi/2 \) for \( \phi \)-line and at \( \theta = 0 \) for \( \psi \)-line, respectively. These largest circles are geodesics. Similarly, the coordinate lines of \( \theta \) coordinate are geodesics. For the fixed values of \( B = \phi_0 \) and \( \psi = \psi_0 \) and \( \theta \in [0, \pi/2) \) this line is a segment of the length \( \pi/2 \) connecting the fixed points of the Killing vectors \( \partial_\phi \) and \( \partial_\psi \). Four such segments \( \phi = \phi_0, \phi_0 + \pi, \psi = \psi_0, \psi_0 + \pi \) form the largest circle of the length \( 2\pi \).

The surfaces of constant \( \theta \) are flat tori \( T^2 \). For instance, \( \theta = \theta_0 \) can be cut apart to give a rectangle with horizontal edge length \( \cos \theta_0 \) and vertical edge length \( \sin \theta_0 \). These tori are called Hopf tori and they are pairwise linked. The fixed points of the vectors \( \partial_\phi \) and \( \partial_\psi \) \( (\theta = 0 \) for \( \partial_\phi \) and \( \theta = \pi/2 \) for \( \partial_\psi \) \) form a pair of linked great circles. Every other Hopf torus passes between these circles. The equatorial Hopf torus is the one which can be made from a square. The others are all rectangular. Also we can easily see that the surfaces of constant \( \phi \) or constant \( \psi \) are half 2-spheres or topologically disks.

B. Torus \( T^3 = S^2 \times S^1 \)

The equation of a torus \( T^3 = S^2 \times S^1 \) in \( \mathbb{E}^4 \) is
\[ X_1^2 + X_2^2 + \sqrt{X_3^2 + X_4^2 - a^2} = b^2. \]  
(4)

The surface \( T^3 \) is obtained by the rotation of a sphere \( S^2 \) of the radius \( b \) around a circle \( S^1 \) of the radius \( a \) \( (a > b) \). Let us define toroidal coordinates as
\[ X_1 = \frac{a \sin \theta}{B} \cos \phi, \quad X_2 = \frac{a \sin \theta}{B} \sin \phi, \quad X_3 = \frac{a \sin \eta}{B} \cos \psi, \quad X_4 = \frac{a \sin \eta}{B} \sin \psi, \]  
(5)

where \( B = \cosh \eta - \cos \theta \). The toroidal coordinates \( (\eta, \theta, \phi, \psi) \) change in the following intervals
\[ 0 < \eta < \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi, \psi \leq 2\pi. \]  
(6)

The flat metric in this coordinates takes the form
\[ ds^2 = \frac{a^2}{B^2} \left( d\eta^2 + \sin^2 \eta d\psi^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right). \]  
(7)

In these coordinates the surface of constant \( \eta = \eta_0 \) is a torus \( T^3 \) and one has
\[ \alpha = \sqrt{a^2 - b^2}, \quad \cosh \eta_0 = a/b. \]  
(8)

Introducing new coordinates \( y = \cosh \eta \) and \( x = \cos \theta \) one can also write the metric \( \mathbb{E}^4 \) in the form \( \mathbb{R}^3 \)
\[ ds^2 = \frac{\alpha^2}{(y - x)^2} \left[ \frac{dy^2}{y^2 - 1} + (y^2 - 1) d\psi^2 \right. \] \[ + \left. \frac{dx^2}{1 - x^2} + (1 - x^2) d\phi^2 \right]. \]  
(9)

The points with \( \eta < \eta_0 \) lie in the exterior of \( T^3 \). The induced geometry on the 3-surface \( \eta = \eta_0 \) is
\[ ds^2 = \frac{a^2 - b^2}{(a - b \cos \theta)^2} \left[ (a^2 - b^2) d\psi^2 + b^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \]  
(10)

This metric has 2 Killing vectors, \( \partial_\phi \) and \( \partial_\psi \). The first one has 2 sets of fixed points, \( \theta = 0 \) and \( \theta = \pi \), which are circles \( S^1 \). The second Killing vector, \( \partial_\psi \) does not have fixed points. The 3-volume of the torus \( T^3 \) is \( 8\pi^2 ab^2 \).

Since the sections \( \psi = \text{const} \) are round spheres, instead of \( \theta \) it is convenient to use another coordinate, \( \theta \in [0, \pi] \),
\[ \sin \theta = \frac{\sqrt{a^2 - b^2} \sin \theta}{a - b \cos \theta}. \]  
(11)

Using this coordinate one can rewrite the metric \( \mathbb{E}^4 \) in the form
\[ ds^2 = (a + b \cos \theta)^2 d\psi^2 + b^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]  
(12)

Once again we can easily see that the surfaces of constant \( \theta \) are flat tori \( T^2 \) except for \( \theta = 0 \) or \( \theta = \pi \), which are circles. The surfaces of constant \( \psi \) are 2-spheres whereas the surfaces of constant \( \phi \) are 2-tori.

Sometimes it is convenient to consider special foliations of \( T^3 \). This foliation is a kind of “clothing” worn on a manifold, cut from a stripy fabric. These stripes are called plaques of the foliation. On each sufficiently small piece of the manifold, these stripes give the manifold a local product structure. This product structure does not have to be consistent outside local patches, a stripe followed around long enough might return to a different, nearby stripe. As an example for foliations let us consider the manifold \( \mathbb{R}^3 \). The foliations are generated by two dimensional leaves or plaques with one co-ordinate as constant. That is, the surfaces \( z = \text{constant} \), would be the plaques of the foliations and in this case there is a global product structure. Similarly one can consider the foliations of \( S^2 \times S^1 \). Fig.(1) shows the transverse Reeb foliations of the cylindrical section of \( S^2 \times S^1 \). We can see the stacking of spherical shaped plaques giving rise to the cylindrical section.

III. GEOMETRY OF 3-DIMENSIONAL SPACE WITH 2 ORTHOGONAL COMMUTING KILLING VECTORS

As we shall see both metrics of the horizon surface of a 5D black ring and a black hole with one rotation parameter can be written in the form
\[ ds^2_H = f(\zeta) d\zeta^2 + g(\zeta) d\phi^2 + h(\zeta) d\psi^2. \]  
(13)

Here \( f, g \) and \( h \) are non-negative functions of the coordinate \( \zeta \). One can use an ambiguity in the choice of the coordinate \( \zeta \) to put \( f = 1 \). For this choice \( \zeta \) has the meaning of the proper distance along \( \zeta \)-coordinate.
line. We call such a parametrization canonical. The coordinates \( \phi \) and \( \psi \) have a period of \( 2\pi \) and \( \zeta \in [\zeta_0, \zeta_1] \). \( \partial_{\zeta} \) and \( \partial_{\psi} \) are two mutually orthogonal Killing vectors. If \( g(\zeta) (h(\zeta)) \) vanishes at some point then the Killing vector \( \partial_{\zeta} \) has a fixed point at this point. The metric (13) does not have a cone-like singularity at a fixed point of \( \partial_{\zeta} \) if at this point the following condition is satisfied

\[
\frac{1}{2\sqrt{hf}} \frac{dh}{d\zeta} = 1.
\]

A condition of regularity of a fixed point of \( \partial_{\psi} \) can be obtained from (14) by changing \( h \) to \( g \).

By comparing the metric (13) with the metric for the 3-sphere (3) one can conclude that (13) describes the geometry of a distorted 3D sphere if \( g \) and \( h \) are positive inside some interval \( (\zeta_1, \zeta_2) \), while \( g \) vanishes at one of its end point (say \( \zeta_1 \)) while \( h \) vanishes at the other (say \( \zeta_2 \)). Similarly, by comparison (13) with (12) one concludes that if for example \( g \) is positive in the interval \( (\zeta_1, \zeta_2) \) and vanishes at its ends, while \( h \) is positive everywhere on this interval, including its ends, the metric (13) describes a topological torus.

For the metric (13), the non-vanishing components of the curvature tensor are,

\[
R_{\zeta\phi\phi\psi} = \frac{g''(fg)' - \frac{1}{2} g'}{4fg},
\]

\[
R_{\zeta\psi\psi\psi} = \frac{h''(fh)'}{4fh} - \frac{1}{2} h'',
\]

\[
R_{\phi\psi\psi\psi} = -\frac{g'h'}{4f}.
\]

Here (’) denotes the differentiation with respect to coordinate \( \zeta \).

Denote by \( e^i_1 \) \((i, a = 1, 2, 3)\) 3 orthonormal vectors and introduce the Gauss curvature tensor as follows

\[
K_{ab} = -R_{ijkl}e^i_1e^j_1e^k_1e^l_1.
\]

The component \( K_{ab} \) of this tensor coincides with the curvature in the 2D direction for the 2D plane spanned by \( e^1_a \) and \( e^2_b \). One has

\[
\sum_{b=1}^{3} K_{ab} = R.
\]

For the metric (13) the directions of the coordinate line \( \theta, \phi \) and \( \psi \) are eigen-vectors of \( K_{ab} \) and the corresponding eigen-values are \( K_a \)

\[
K_\psi = \frac{R_{\zeta\phi\psi\psi}}{fg}, \quad K_\phi = \frac{R_{\zeta\psi\psi\psi}}{fh}, \quad K_\zeta = \frac{R_{\phi\psi\psi\psi}}{gh}.
\]

These quantities are the curvatures of the 2D sections orthogonal to \( \psi, \phi \) and \( \zeta \) lines, respectively. For brevity, we call these 2D surfaces \( \psi-, \phi- \) and \( \zeta-\)sections.

For the unit sphere \( S^3 \), from (3) one can easily see that

\[
K_\psi = K_\phi = K_\zeta = 1.
\]

However for the torus \( S^2 \times S^1 \), from (12) we have

\[
K_\psi = \frac{1}{b^2}; \quad K_\phi = K_\theta = \frac{\cos \theta}{b(a + b \cos \theta)}.
\]

Thus we see that \( K_\psi \) always remains positive, while \( K_\phi \) and \( K_\theta \) is positive in the interval \( (0 \leq \theta < \pi/2) \). Thus the equitorial plane \( (\theta = \pi/2) \), divides the torus in two halves, one in which all the sectional curvatures are positive while the other has two of the sectional curvatures negative. In fact the surface \( \theta = \pi/2 \) is topologically \( S^1 \times S^1 \) with the metric,

\[
ds^2 = a^2 d\psi^2 + b^2 d\phi^2
\]

The equations (19) imply

\[
R_\zeta = K_\psi + K_\phi, \quad R_\phi = K_\theta + K_\zeta, \quad R_\psi = K_\zeta + K_\phi.
\]

\[
R = 2(K_\zeta + K_\phi + K_\psi).
\]

From the above expression it is clear that \( K_\psi < 0 \) if \( g' \) and \( \ln[fg/(g')^2] \) have the opposite signs. Similarly \( K_\phi < 0 \) imply \( h' \) and \( \ln[fh/(h')^2] \) have opposite sign. For \( K_\zeta < 0 \), \( g' \) and \( h' \) must have the same sign.

Let us consider now Euler characteristics of the two dimensional sections of the horizon surface. We denote by \( \chi_0 \) the Euler characteristic for the 2-surface \( x^a = \text{const} \).

By using the Gauss Bonnet theorem we have,

\[
2\pi \chi_0 = \int_{\Sigma} K_a dA + \int_{\partial\Sigma} k_g ds.
\]

Here \( dA \) is the element of area on the surface and \( k_g \) is the geodesic curvature on the boundary. If the surface has no boundary or the boundary line is a geodesic, then the last
term vanishes. For the metric \[13\] simple calculations give

\[
2\pi \chi_\psi = -\pi \left[ \frac{g'}{\sqrt{f}} \right]_{\zeta_0}^{\zeta_1} + \int_{\partial M} k_\psi ds , \tag{27}
\]

\[
2\pi \chi_\phi = -\pi \left[ \frac{h'}{\sqrt{f}} \right]_{\zeta_0}^{\zeta_1} + \int_{\partial M} k_\phi ds , \tag{28}
\]

\[
\chi_\zeta = 0 . \tag{29}
\]

Thus we see that the Gaussian curvatures of sections, completely describe the topology and geometry of the 3-horizons.

IV. A 5D ROTATING BLACK HOLE WITH ONE ROTATION PARAMETER

A. Volume and shape of the horizon surface

For the 5 dimensional MP black hole with a single parameter of rotation, the induced metric on the horizon is \[1\],

\[
ds^2 = r_0^2 ds_H^2 , \tag{30}
\]

\[
d s_H^2 = f(\theta)d\theta^2 + \frac{\sin^2 \theta}{f(\theta)} d\phi^2 + (1 - \alpha^2) \cos^2 \theta d\psi^2 . \tag{31}
\]

Here \( f(\theta) = (1 - \alpha^2 \sin^2 \theta) \) and \( r_0 \) is length parameter related with the mass \( M \) of the black hole as

\[
r_0^2 = \frac{8\sqrt{\pi} GM}{3} \tag{32}
\]

The metric \[31\] is in the Hopf coordinates and hence the co-ordinate \( \theta \) varies from 0 to \( \pi/2 \). The rotation is along the \( \phi \) direction. The quantity \( \alpha = |a/r_0| \) characterize the rapidity of the rotation. It vanishes for a non-rotating black hole and take the maximal value \( \alpha = 1 \) for an extremely rotating one. In what follows we put \( r_0 = 1 \), so that \( \alpha \) coincides with the rotation parameter. Different quantities (such as lengths and curvature components) can be easily obtained from the corresponding dimensionless expressions by using their scaling properties.

For \( \alpha = 0 \) the horizon is a round sphere \( S^3 \) of the unit radius. In the presence of rotation this sphere is distorted. Its 3-volume is \( V_3 = 2\pi^2 \sqrt{1 - \alpha^2} \). In the limiting case of an extremely rotating black hole, \( \alpha = 1 \), \( V_3 \) vanishes.

The coordinate lines of \( \phi \) and \( \psi \) on this distorted sphere are remain closed circles. The length of the circle corresponding to the \( \phi \)-coordinate changes from 0 (at \( \theta = 0 \)) to its largest value (at \( \theta = \pi/2 \))

\[
l_\phi = \frac{2\pi}{\sqrt{1 - \alpha^2}} . \tag{33}
\]

Similarly, the length of the circles connected with \( \psi \)-coordinate changes from its maximal value (at \( \theta = 0 \)) to 0 at \( \theta = \pi/2 \). A line \( \phi, \psi = \text{const} \) on the disported sphere is again a geodesic.

\[
l_\psi = 2\pi \sqrt{1 - \alpha^2} \tag{34}
\]

\[
l_\theta = 4E(\alpha) . \tag{35}
\]

The lengths \( l_\psi, l_\theta \) and \( l_\phi \) as the functions of the rotation parameter \( \alpha \) are shown at figure 2 by the lines 1, 2, and 3, respectively. All these lines start at the same point \((0, 2\pi)\). In the limit of the extremely rotating black hole \( (\alpha = 1) \) the horizon volume vanishes, \( l_\psi = 0, l_\theta = 4, \) and \( l_\phi \) infinitely grows.

B. Gaussian curvature

Calculations of the eigen-values \( K_\alpha \) of the Gaussian curvatures give

\[
K_\psi = \frac{[1 - \alpha^2(1 + 3 \cos^2 \theta)]}{f(\theta)^3} , \quad K_\phi = K_\theta = \frac{1}{f(\theta)^2} . \tag{36}
\]

From these relations it follows that the quantity \( K_\psi \) is negative in the vicinity of the 'pole' \( \theta = 0 \) for \( 1/2 < \alpha < 1 \), while the other two quantities, \( K_\theta \) and \( K_\phi \) are always positive. This is similar to the 4D Kerr black hole where the Gaussian curvature of the two dimensional horizon becomes negative near the pole for \( \alpha > 1/2 \). This is not surprising since the 2D section \( \psi = \text{const} \) of the metric is isometric to the geometry of the horizon surface of the Kerr black hole.

The Ricci tensor and Ricciscalar for the metric on the
surface of the horizon of 5D black hole are
\[
R^\theta_\theta = R_\phi^\phi = \frac{2[1 - \alpha^2(1 + \cos^2 \theta)]}{f(\theta)^{3}}, \tag{37}
\]
\[
R^\phi_\psi = \frac{2}{f(\theta)^{2}}, \quad R = \frac{2[3 - \alpha^2(3 + \cos^2 \theta)]}{f(\theta)^{3}}. \tag{38}
\]

The components of Ricci tensor \(R^\theta_\theta\) and \(R_\phi^\phi\) becomes negative for certain values near the ‘pole’ \(\theta = 0\), when \(\alpha > 1/\sqrt{2}\), while the Ricci scaler is negative when \(\alpha > \sqrt{3}/4\).

It is interesting to note that the surfaces of constant \(\phi\) or constant \(\psi\) are topologically disks with Euler Characteristics equal to unity. The boundary of these disks are on \(\theta = \pi/2\). It is easy to check from equations \(27\) and \(28\) that boundary terms of Gauss Bonnet equation vanishes on this boundary. This shows the boundary, which is the equatorial line on the deformed hemisphere, is a geodesic of the induced metric. Another important point is, while approaching the naked singularity limit (\(\alpha = 1\)), the Gaussian curvatures of all the three sections, as well as the negative Ricci scalar blows up along the ‘equator’ (\(\theta = \pi/2\)). This shows the extreme flattening of the horizon along equatorial plane, before the horizon shrinks to zero volume.

V. EMBEDDING

A. Embedding of the horizon in 5D pseudo-Euclidean space

Let us discuss now the problem of the embedding of the horizon surface of a rotating 5D black hole into a flat space. We start by reminding that a similar problem for a 4D (Kerr) black hole was considered long time ago by Smarr [12]. He showed that if the rotation parameter of the Kerr metric \(\alpha < 1/2\), then 2D surface of the horizon can be globally embedded in \(\mathbb{E}^4\) as a rotation surface. For \(\alpha > 1/2\) such an embedding is possible if the signature of the 3D flat space is \((-+,+,+\)). In a recent paper [13] there was constructed a global embedding of the horizon of a rapidly rotating black hole into \(\mathbb{E}^4\).

Since the 3D surface of a rotation 5D black hole has 2 commuting orthogonal Killing vectors it is natural to consider its embedding into the flat space which has at least two independent orthogonal 2-planes of the rotation. In this case the minimal number of the dimensions of the space of embedding is 5. We write the metric in the form
\[
dS^2 = \varepsilon dz^2 + dx_1^2 + dx_2^2 + dy_1^2 + dy_2^2, \tag{39}
\]
where \(\varepsilon = \pm 1\). By introducing polar coordinates \((\rho, \phi)\) and \((r, \psi)\) in the 2-planes \((x_1, x_2)\) and \((y_1, y_2)\), respectively, we obtain
\[
dS^2 = \varepsilon dz^2 + d\rho^2 + \rho^2 d\phi^2 + dr^2 + r^2 d\psi^2. \tag{40}
\]

Using \(\mu = \cos \theta\) as a new coordinate one can rewrite the metric on the horizon \(\mathbb{S}^3\) in the form
\[
ds^2 = f d\mu^2 + \rho^2 d\phi^2 + r^2 d\psi^2, \tag{41}
\]
\[
f = \frac{1 - \alpha^2 \mu^2}{1 - \mu^2}, \quad \rho = \frac{\mu}{\sqrt{1 - \alpha^2 \mu^2}}, \tag{42}
\]
\[
r = \sqrt{(1 - \alpha^2)(1 - \mu^2)}. \tag{43}
\]

Assuming that \(z\) is a function of \(\mu\), and identifying \(\rho\) and \(r\) in \(40\) with \(41\) one obtains the metric \(41\) provided the function \(z(\mu)\) obeys the equation
\[
\left(\frac{dz}{d\mu}\right)^2 = \varepsilon \left[f - \left(\frac{d\rho}{d\mu}\right)^2 - \left(\frac{dr}{d\mu}\right)^2\right]. \tag{44}
\]

By substituting \(43\) into \(44\) one obtains
\[
\left(\frac{dz}{d\mu}\right)^2 = \varepsilon \frac{\alpha^2 \mu^2(3 \alpha^2 \mu^2 - 4 \alpha^4 \mu^4 - 3)}{(1 - \alpha^2 \mu^2)^3}. \tag{45}
\]

It is easy to check that for \(|\alpha| \leq 1\) and \(0 \leq \mu \leq 1\) the expression in the right hand side of \(45\) always has the sign opposite to the sign of \(\varepsilon\). Thus one must choose \(\varepsilon = -1\) and one has
\[
z = \frac{1}{2\alpha} \int_{y_0}^{1} \frac{dy}{\sqrt{1 - \alpha^2 \mu^2}} \sqrt{1 + y + y^2}. \tag{46}
\]

Let us emphasize that this result is valid both for the slowly and rapidly rotating black holes.

B. Global embedding into \(\mathbb{E}^6\)

1. Construction of an embedding

It is possible, however, to find a global isometric embedding of the 3-horizon of a rotating black hole in a flat space with positive signature, if the number of dimensions is 6. This embedding is analogous to the one discussed in [13] for the rapidly rotating Kerr black hole.

Let us denote by \(X_i, (i = 1...6)\) the Cartesian coordinates in \(\mathbb{E}^6\). We write the embedding equations in the form
\[
X_i = \frac{\eta(\theta)}{\rho_0} n^i(\tilde{\phi}), \quad (i = 1, 2, 3), \tag{47}
\]
\[
X_4 = \nu(\theta) \cos \psi, \quad X_5 = \nu(\theta) \sin \psi, \quad X_6 = \chi(\theta). \tag{48}
\]

Here the functions \(n^i\) obey the condition
\[
\sum_{i=1}^{3} (n^i(\tilde{\phi}))^2 = 1. \tag{49}
\]

In other words, the 3D vector \(n^i\) as a function of \(\tilde{\phi}\) describes a line on the unit round sphere \(S^2\). We require
this line to be a smooth closed loop \((n(0) = n(2\pi))\) without self interactions. We denote

\[
\rho(\tilde{\phi}) = \left[\sum_{i=1}^{3} (n^i_{\tilde{\phi}})^2\right]^{1/2}.
\]

(50)

Then \(dl = \rho(\tilde{\phi})d\tilde{\phi}\) is the line element along the loop. The total length of the loop is

\[
l_0 = 2\pi \rho_0 = \int_0^{2\pi} \rho(\tilde{\phi})d\tilde{\phi}.
\]

(51)

We define a new co-ordinate \(\phi\) as

\[
\phi = \frac{1}{\rho_0} \int_0^{\tilde{\phi}} \rho(\tilde{\phi})d\tilde{\phi}.
\]

(52)

It is a monotonic function of \(\tilde{\phi}\) and has the same period \(2\pi\) as \(\tilde{\phi}\). The induced metric for the embedded 3D surface defined by \(l_0\) becomes

\[
ds^2 = \frac{n_{\theta}^2}{\rho_0^2} + \nu \theta^2 + \chi \phi^2
d\theta^2 + \eta^2 d\phi^2 + \nu^2 d\psi^2.
\]

(53)

Now comparing equations \((51)\) and \((52)\) we get

\[
\eta(\theta) = \frac{\sin \theta}{\sqrt{1 - \alpha^2 \sin^2 \theta}}, \quad \nu(\theta) = \sqrt{1 - \alpha^2 \cos \theta} \sin \theta,(54)
\]

\[
\chi(\theta) = \int_0^\theta \cos \theta \sqrt{1 - \frac{1}{\rho_0^2(1 - \alpha^2 \sin^2 \theta)^2}} d\theta. \quad (55)
\]

We choose the functions \(n^i\) in such a way that

\[
\rho_0^2 \geq \frac{1}{(1 - \alpha^2)^3},
\]

(56)

so that the function \(\chi(\theta)\) remains real valued for all \(\theta\) and hence we can globally embed the horizon in \(\mathbb{R}^6\).

2. A special example

To give an explicit example of the above described embedding let us put

\[
n^1 = \frac{\cos \tilde{\phi}}{F}, \quad n^2 = \frac{\sin \tilde{\phi}}{F}, \quad n^3 = \frac{a \sin(N \tilde{\phi})}{F},
\]

(57)

\[
F = \sqrt{1 + a^2 \sin^2(N \tilde{\phi})}.
\]

(58)

Here \(N \geq 1\) is a positive integer. For this choice the value of \(\rho_0\) is

\[
\rho_0 \approx \frac{aN}{2\pi \sqrt{1 + a^2 \sin^2(\tilde{\phi})}}.
\]

(59)

\[\text{FIG. 3: } \rho_0 \text{ as a function of } a \text{ for the values of } N \text{ from 2 (line 2) to 7 (line 7)}\]

\[\text{FIG. 4: } \rho_0 \text{ as a function of } N \text{ for } a = 10.\]

For \(N = 1 \rho_0 = 1\). For \(N > 1\) the above integral can be exactly evaluated to give

\[
\rho_0 = \frac{2k_1}{\sqrt{1 + a^2}} \left[ N^2 \Pi(a^2 k_1^2, ik_2) - (N^2 - 1) K(ik_2) \right],
\]

(60)

\[
k_1 = \frac{1}{\sqrt{1 + a^2}}, \quad k_2 = a \sqrt{\frac{N^2 - 1}{1 + a^2}}.
\]

(61)

Here \(K\) and \(\Pi\) are elliptic integrals of first and third kind, respectively. For a fixed value of \(N \rho_0\) is monotonically growing function of \(a\) (see Fig. 3). For a fixed value of \(a\) the value of \(\rho_0\) increases monotonically with \(N\) (see Fig. 4). The asymptotic form of \(\rho_0\) for large values of \(a\) can be easily obtained as follows. Notice that for large \(a\) the denominator in the integral \((59)\) is large unless \(\tilde{\phi}\) is close to 0, \(\pi\), or 2\(\pi\). Near these points \(\cos \tilde{\phi}\) can be approximated by 1, and the expression for \(\rho_0\) takes the form

\[
\rho_0 \approx \frac{aN}{2\pi} \int_0^{2\pi} \frac{d\tilde{\phi}}{1 + a^2 \sin^2(\tilde{\phi})} = \frac{aN}{\sqrt{1 + a^2}}.
\]

(62)

Using these properties of \(\rho_0\), one can show that for large enough values of \(N\) and \(a\) the quantity \(\rho_0\) can be made arbitrary large, so that the condition \((50)\) is satisfied and we have the global embedding of the horizon surface for any \(\alpha < 1\).
VI. A 5D ROTATING BLACK RING

A. Horizon surface of a black ring

Now we consider properties of horizon surfaces of stationary black strings in an asymptotically 5D flat spacetime [2]. In this paper we would only consider the balanced black ring in the sense that there are no angular deficit or angle excess causing a conical singularity. The ring rotates along the $S^3$ and this balances the gravitational self attraction. The metric of the rotating black ring is \[ ds^2 = -[F(x)/F(y)] \left[ dt + r_0 \frac{\sqrt{2}y}{\sqrt{1 + \nu^2}} (1 + y) d\psi \right]^2 + \frac{r_0^2}{(x - y)^2} \left[ -F(x) \left( G(y) d\psi^2 + [F(y)/G(y)] dy \right) + F(y)^2 \left( [dx^2/G(x)] + [G(x)/F(x)] d\theta^2 \right) \right], \] (63)

where

\[ F(\zeta) = 1 - \frac{2\nu}{1 + \nu^2} \zeta, \quad G(\zeta) = (1 - \zeta^2)(1 - \nu^2). \] (64)

The quantity $r_0$ is the radius scale of the ring. The parameter $\nu \in [0, 1]$ determines the shape of the ring. The coordinate $x$ changes in the the interval $-1 \leq x \leq 1$, while $y^{-1} \in [-1, (2\nu)/(1 + \nu^2)]$. The black ring is rotating in $d\psi$-direction. The positive ‘y’ region is the ergosphere of the rotating black ring while the negative ‘y’ region is lies outside the ergosphere with the spatial infinity at $x = y = -1$.

The metric (63) has a co-ordinate singularity at $y = 1/\nu$. However after the transformation

\[
\begin{align*}
    d\psi &= d\tilde{\psi} + J(y) dy, \\
    dv &= dt - r_0 \frac{\sqrt{2}y}{\sqrt{1 + \nu^2}} (1 + y) J(y) dy,
\end{align*}
\] (65)

with $J(y) = \sqrt{-F(y)/G(y)}$, the metric is regular at $y = 1/\nu$. In these regular coordinates one can show that the surface $y = 1/\nu$ is the horizon.

The induced metric on the horizon of a rotating black ring is given by

\[
\begin{align*}
    ds^2 &= r_0^2 d\tilde{s}_H^2, \\
    d\tilde{s}_H^2 &= \frac{p}{k(\theta)} \left[ d\theta^2 + \frac{\sin^2 \theta d\omega^2}{l(\theta)} \right] + q l(\theta) d\psi^2, \quad (66)
\end{align*}
\]

\[
\begin{align*}
    k(\theta) &= 1 + \nu \cos \theta, \\
    l(\theta) &= 1 + \nu^2 + 2\nu \cos \theta, \quad (67)
\end{align*}
\]

\[
\begin{align*}
    p &= \nu^2(1 - \nu^2)^2, \\
    q &= \frac{1}{1 + \nu^2}. \quad (68)
\end{align*}
\]

In this metric the co-ordinates $\phi$ and $\psi$ have a period of $2\pi$ and $\theta \in [0, \pi]$. $\theta = 0$ is the axis pointing outwards (i.e. increasing $S^1$ radius), while $\theta = \pi$ points inwards. The volume of the horizon surface for the metric (67) is

\[
V = 8\sqrt{2}\pi^2 \nu^2 \sqrt{1 - \nu} \left[ \frac{\sqrt{1 + \nu}}{\sqrt{1 + \nu^2}} \right]^3. \quad (69)
\]
Let us emphasize that, because of the distortions due to the rotation, both $K_\phi$ and $K_\theta$ do not become negative at the same value of $\theta$ as it was for the flat torus case. To get an invariant measure of distortion produced due to rotation, let us define two invariant lengths in the following way. Let $\theta = \theta_i$ ($i = \theta, \phi$), be the point where $F_i(\cos \theta)$ vanishes. Then the two invariant lengths are,

$$
\lambda_{i1} = 2 \sqrt{p} \int_0^{\theta_i} \frac{d\theta}{\sqrt{k(\theta)}}, \quad \lambda_{i2} = 2 \sqrt{p} \int_{\theta_i}^{\pi} \frac{d\theta}{\sqrt{k(\theta)}}.
$$

It is easy to check from (12) that in the case of flat tori we have $\lambda_{i1} = \lambda_{i2}$. However for the rotating black rings they are different functions of the parameter $\nu$. Figures 5 and 6 shows the invariant lengths for $(i = \theta, \phi)$ as function of $\nu$ respectively. We see that for $i = \theta$, $\lambda_{i2} < \lambda_{i1}$ for small $\nu$. However as we increase $\nu$ the difference between them reduces and ultimately at $\nu \approx 0.615$, $\lambda_{i2}$ overtakes $\lambda_{i1}$. Whereas for $i = \phi$, $\lambda_{i2}$ is always greater than $\lambda_{i1}$.

It is evident that both $\psi$- and $\phi$-sections are closed and do not have a boundary. Calculating the Euler numbers for these surface we get

$$
\chi_\psi = 2, \quad \chi_\phi = 0.
$$

This shows that the $\psi$-section is a deformed 2-sphere with positive Gaussian curvature. Its rotation in the $\psi$ direction generates the horizon surface of the rotating black ring.

C. Kaluza-Klein Reduction of Rotating Black Ring

The absence of the cone-like singularities in the black ring solution (63) is a consequence of the exact balance between the gravitational attraction and the centrifugal forces generated by the ring’s rotation. We discuss the effects connected with the ring rotation from a slightly different point of view. Let us write the metric (63) in the following Kaluza-Klein form (see e.g. [19])

$$
ds_5^2 = \Phi^{-\frac{1}{2}} \left[ h_{\alpha \beta} dx^\alpha dx^\beta + \Phi(A_t dt + d\phi)^2 \right].
$$

The 4D reduced Pauli metric in this space is $(a, b = 0, \ldots, 3)$

$$
ds_4^2 = h_{ab} dx^a dx^b = \Phi^{\frac{1}{2}} \left[ g_{ab} dx^a dx^b - \Phi A_{t}^2 dt^2 \right].
$$

Here $g_{ab}$ is the four dimensional metric on the $\tilde{\psi}$-section of the 5D black ring. By the comparison of (75) and (76) one has

$$
\Phi_{t} = \xi_\phi = -\frac{F(x)}{F(y)} L(x, y),
$$

$$
A_t = \frac{\xi_\phi}{\xi_\theta} = \sqrt{\frac{2}{1 + \nu^2}} \frac{(1 + y)}{L(x, y)},
$$

$$
L(x, y) = \left[ \frac{2\nu^2}{1 + \nu^2}(1 + y)^2 + \frac{F(y)G(y)}{(x - y)^2} \right],
$$

where $\xi_\ell = \partial_\ell$, $\xi_\phi = \partial_\phi$ and $\xi_\theta = \partial_\theta$ are the Killing vectors of (75). The quantities $\ln(\Phi)$ and $A_t$ can be interpreted as a dilaton field and a ‘electromagnetic potential’ in the 4D spacetime.

The horizon for the 4D metric (76) is defined by the condition

$$
h_H = \xi_\phi^2 - (\xi_\phi^2)^{-1}(\xi_\ell^2)\xi_\phi^2 = 0.
$$

It is easy to show that this condition is equivalent to the condition defining the horizon of the 5D metric. Thus both horizons are located at $y = 1/\nu$.

To summarize, the 4D metric (76) obtained after the reduction describes a static 4D black hole in the presence of an external dilaton and ‘electromagnetic’ field. The dilaton field $\ln \Phi$ (as well as the metric (76)) has a singularity at the points where $\xi_\phi^2$ either vanish (at the axis of symmetry, $x = 1, y = -1$) or infinitely grows (at the spatial infinity, $x = y = -1$) (see Fig. 4). Outside these regions the dilaton field is regular everywhere including the horizon where it takes the value

$$
\Phi_H = \left[ \frac{2}{1 + \nu^2} \frac{(1 + \nu)}{(1 - \nu)} l(\theta) \right]^\frac{2}{7}
$$

The ‘electromagnetic field strength’, which has non zero components

$$
F_{tx} = -A_{t, x}, \quad F_{ty} = -A_{t, y},
$$

is regular everywhere and vanishes at the spatial infinity. However the $F^2 = F_{ab}F^{ab}$ invariant is well defined throughout the space time but drops (towards negative infinity) at the axis of symmetry. Figure 8 illustrates the behavior of the invariant $F^2$ for $\nu = 2/3$.

The metric $ds_5^2$ on 2D horizon surface for the reduced metric (70) is conformal to the metric $ds_5^2$ of 2D section $\ddot{\psi} = \text{const}$ of the black ring horizon (67). These metrics are of the form $(k = k(\theta), \ell = l(\theta))$

$$
ds_5^\epsilon = \Phi_{H}^{\epsilon/3} \frac{d\theta^2}{k^2 + \sin^2 \theta d\phi^2}, \quad \epsilon = 0, 1.
$$

Both metrics, $ds_5^2$ can be embedded in $\mathbb{E}^3$ as rotation surfaces. The embedding equations are

$$
X_1 = m\beta \cos \phi; \quad X_2 = m\beta \sin \phi; \quad X_3 = m\gamma,
$$

FIG. 7: $\ln(\Phi)$ as a function of $x$ and $z = 1/y$ for $\nu = 2/3$. 
FIG. 8: The invariant \( F^2 \) as a function of \( x \) and \( z = 1/y \) for \( \nu = 2/3 \). It can be seen that it is well defined everywhere but drops (towards negative infinity) at the axis of symmetry, \( x = 1, y = -1 \).

FIG. 9: The embedding diagrams for the metrics \( ds_0^2 \) (to the left) and \( ds_1^2 \) (to the right) for \( \nu = 2/3 \).

where,

\[
\beta = k^{-1/2} l^{-1/2 + \epsilon/4} \sin \theta ,
\gamma = \int_0^\theta (k^{-3/2} - \beta_3^2)^{1/2} d\theta ,
\m = \sqrt{p} \left[ \frac{2}{1 + \nu} \left( 1 + \nu \right)^{1/2} \right]^{1/2} .
\]

The embedding diagrams for the metrics \( ds_0^2 \) and \( ds_1^2 \) are shown in figure 9 by the left and right plots, respectively. Both rotation surfaces are deformed spheres. The surface with the geometry \( ds_1^2 \) is more flattened at poles.

VII. DISCUSSION

In this paper we discussed and analyzed the surface geometry of five dimensional black hole and black rings with one parameter of rotation. We found that the sectional Gaussian curvature and the Ricci scalar of the horizon surface of the 5D rotating black hole are negative if the rotation parameter is greater than some critical value, similarly to the case of 4D the Kerr black hole. However, there is an important difference between the embeddings of the horizon surfaces of 5D and 4D black holes in the flat space. As was shown in [12], a rotating 2-horizon can be embedded as a surface of rotation in a 3-dimensional Euclidean space only when the rotation parameter is less than the critical value. For the ‘super-critical’ rotation, the global embedding is possible either in 3D flat space with the signature if the metric \((-, +, +, +, +)\) or in \(E^4\) with the positive signature [13]. For the 5D black hole for any value of its rotation parameter the horizon surface cannot be embedded in 5D Euclidean space as a surface of rotation. Such an embedding requires that the signature of the flat 5D space is \((-, +, +, +, +, +)\). However we found a global embedding of this surface in 6D Euclidean space.

We calculated the surface invariants for the rotating black ring and analyzed the effect of rotation on these invariants. Finally we considered the Kaluza-Klein reduction of rotating black ring which maps its metric onto the metric of 4D black hole in the presence of external dilaton and ‘electromagnetic’ fields. Under this map, the horizon of the 5D black ring transforms into the horizon of 4D black hole. The ‘reduced’ black hole is static and axisymmetric. Distorted black holes in the Einstein-Maxwell-dilaton gravity were discussed in [23]. This paper generalizes the well known results of [21, 22] for vacuum distorted black holes. It would be interesting to compare the ‘reduced’ distorted black hole discussed in this paper with solutions presented in [23].

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