1. INTRODUCTION

With the merit of the lateral thrusts, Ryll’s tilt-rotor [1] attracted great attention in the past decade. While controlling this system utilizing feedback linearization, the tilting angles can change in unexpected over-intensive ways, which intrigues the birth of the gait plan of the tilt-rotor [1]–[4].

The gait plan is a procedure to pre-define the tilting angles (time-specified functions) of the tilt-rotor. Since there are eight inputs in the tilt-rotor, there will be only four magnitudes of the thrusts left to be specified by the controller after the gait plan procedure. A feedback linearization method is subsequently adopted to assign these four magnitudes of the thrusts.

Despite the fact that this control scenario successfully avoids the over-intensive change in the inputs while applying to stabilize Ryll’s tilt-rotor. Our previous research thus puts the extra procedure named gait plan forward to suppress the unexpected changes in the tilting angles. Accompanying with Two Color Map Theorem, the tilting angles are planned robustly and continuously. The designed gaits are robust to the change of the attitude. However, this is not a complete theory before applying to the tracking simulation test. This paper further discusses some gaits following Two Color Map Theorem and simulates a tracking problem for a tilt-rotor. A uniform circular moving reference is designed to be tracked by the tilt-rotor equipped with the designed robust gait and the feedback linearization controller. The planned gaits satisfying Two Color Map Theorem in this research show the robustness. The results from the simulation show the success in tracking of the tilt-rotor.

2. DYNAMICS OF THE TILT-ROTOR

Ryll’s tilt-rotor is sketched in Fig. 1 [2].

The position of the tilt-rotor [5], [12] is given by

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z} \\
\dot{\gamma}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4
\end{bmatrix} + \frac{1}{m} \cdot w \cdot R \cdot F(\alpha) \cdot \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4
\end{bmatrix}
\]

Fig. 1. Ryll’s tilt-rotor.
where \( \mathbf{P} = [X \ Y \ Z]^T \) represents the position with respect to the earth frame, \( m \) represents the total mass, \( g \) represents the gravitational acceleration, \( \alpha_i \), \( (i = 1,2,3,4) \) represents the angular velocity of the propeller \( \omega = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]^T \), and the term \( \mathbf{w}^T \mathbf{R} \) represents the rotational matrix.

The tilting angles \( \alpha = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4] \). \( F(\alpha) \) is given by

\[
F(\alpha) = \begin{bmatrix}
0 & K_f \cdot s2 & 0 & -K_f \cdot s4 \\
K_f \cdot s1 & 0 & -K_f \cdot s3 & 0 \\
-K_f \cdot c1 & K_f \cdot c2 & -K_f \cdot c3 & K_f \cdot c4 \\
\end{bmatrix},
\]

where \( si \) is \( \sin(\alpha_i) \), \( ci \) is \( \cos(\alpha_i) \), and \( (i = 1,2,3,4) \). \( K_f \) \( \left( 8.048 \times 10^{-6} N \cdot s^2/rad^2 \right) \) is the coefficient of the thrust.

The angular velocity of the body [5], [13] with respect to \( \mathbf{F}_B \), \( \omega_B = [p \ q \ r]^T \), is given by

\[
\omega_B = I_B^{-1} \cdot \tau(\alpha) \cdot \mathbf{w},
\]

where \( I_B \) is the matrix of moments of inertia, \( K_m \) \( \left( 2.423 \times 10^{-7} N \cdot m \cdot s^2/rad^2 \right) \) is the coefficient of the drag, and \( L \) is the length of the arm,

\[
\tau(\alpha) = \begin{bmatrix}
0 & L \cdot K_f \cdot c2 - K_m \cdot s2 & 0 & -L \cdot K_f \cdot c4 + K_m \cdot s4 \\
- L \cdot K_f \cdot c1 + K_m \cdot s1 & 0 & -L \cdot K_f \cdot c3 - K_m \cdot s3 & 0 \\
L \cdot K_f \cdot c1 - K_m \cdot s1 & -L \cdot K_f \cdot c3 - K_m \cdot s3 & 0 & 0 \\
\end{bmatrix}
\]

We refer the details of the dynamics and the feedback linearization control method to our previous research [5].

In our control scenario, the tilting angles, \( \alpha \), are defined in advanced in a separated procedure called gait plan [2], [14]. The tilt-rotor is controlled using the four magnitudes of the thrusts by feedback linearization. Note that the decoupling matrix is singular for some attitude. The gait plan is supposed to generate a robust gait, which enlarges the acceptable attitude region.

### 3. GAIT PLAN

The gait of a tilt-rotor is a time-specified tilting angles, \( \alpha_1(t), \alpha_2(t), \alpha_3(t), \) and \( \alpha_4(t) \). Our previous research [11] puts forward a theorem, Two Color Map Theorem, to design robust gaits. The robust gait is a gait that has a large region of the acceptable attitudes, which introduces the invertible decoupling matrix.

#### 3.1 Two Color Map Theorem

Firstly, design \( \alpha_1(t) \) and \( \alpha_2(t) \) on the \( \alpha_1 - \alpha_2 \) plane. For example, if we expect a periodic gait, then \( (\alpha_1(t), \alpha_2(t)) \) is an enclosed rectangular with direction.

For a given time point at \( t_1 \), \( (\alpha_1(t_1), \alpha_2(t_1)) \) is subsequently determined. Then, \( (\alpha_3(t_1), \alpha_4(t_1)) \) is to be determined to finish designing a gait.

It is proved [11] that there are two \( (\alpha_3(t_1), \alpha_4(t_1)) \) corresponding to a defined \( (\alpha_1(t_1), \alpha_2(t_1)) \), in general, to design a robust gait.

These two corresponding \( (\alpha_3, \alpha_4) \), corresponding to \( (\alpha_1, \alpha_2) \) on the entire \( \alpha_1 - \alpha_2 \) diagram, can be classified into two groups, ‘red \( (\alpha_3, \alpha_4) \)’ and ‘blue \( (\alpha_3, \alpha_4) \)’.

Interestingly, \( \alpha_3 \) in the same color, all the red \( \alpha_3 \) or all the blue \( \alpha_3 \), lies on the same plane in the \( 0\alpha_1\alpha_2\alpha_3 \) coordinate system. Similarly, \( \alpha_4 \) in the same color, all the red \( \alpha_4 \) or all the blue \( \alpha_4 \), lies on the same plane in the \( 0\alpha_1\alpha_2\alpha_3 \) coordinate system.

Obviously, to design a robust gait, the adjacent \( (\alpha_3, \alpha_4) \) must be in the same color for the interested \( (\alpha_1, \alpha_2) \). This is the simplified Two Color Map Theorem without crossover. Note that the cases considering crossover is also considered in our previous research [11], which is beyond the scope of this paper. Fig. 2 and Fig. 3 display the entire map for the first time.
3.2 Robustness analysis

In this research, the robustness of four gaits is analyzed. They are Gait 1, Gait 2, and Gait 3, illustrated in Fig. 4, where \((\alpha_3, \alpha_4)\) of Gait 1 is blue \((\alpha_3, \alpha_4)\), \((\alpha_3, \alpha_4)\) of Gait 2 and Gait 3 are red \((\alpha_3, \alpha_4)\).

Fig. 4. Three gaits analyzed in this paper. \((\alpha_3, \alpha_4)\) of Gait 1 is blue \((\alpha_3, \alpha_4)\). \((\alpha_3, \alpha_4)\) of Gait 2 and Gait 3 are red \((\alpha_3, \alpha_4)\).

Figs. 5 ~ 7 plot the unacceptable attitudes (red curves) for Gait 1 ~ Gait 3, respectively. The unacceptable attitudes will introduce the singular decoupling matrix in feedback linearization, which should be prohibited. The tilt-rotor can only maneuver in the attitude region, which is not occupied by these attitude curves.

In comparison, we create the biased gait for each gait; the biased gait is generated by scaling \(\alpha_3\) and \(\alpha_4\) to their 80%. The unacceptable attitudes (blue curves) for the biased Gait 1 ~ biased Gait 3 are also displayed in Figs 5 ~ 7, respectively.

Fig. 5. The red curves represent the unacceptable attitude of Gait 1. The blue curves represent the unacceptable attitude of the biased Gait 1.

In general, the acceptable attitude region enlarges if the gait following Two Color Map Theorem, especially for the gaits with the red \((\alpha_3, \alpha_4)\). The decoupling matrix introduced by the biased Gait 2 is very sensitive to the attitude. The tilt-rotor can be less likely to be stabilized by feedback linearization for these two biased gaits.

4. TRACKING SIMULATION

4.1 Reference and feasible gait

The reference set in the simulation is a uniform circular moving reference, the position of which is define in Eqs. (5) ~ (7).

\[
\begin{align*}
    x_r &= 5 \cdot \cos(0.1 \cdot t) \quad (5) \\
    y_r &= 5 \cdot \sin(0.1 \cdot t) \quad (6) \\
    z_r &= 0 \quad (7)
\end{align*}
\]

The velocities of the reference along each dimension are received by calculating the derivatives of the velocities. Note that the acceleration of the reference is set as zero in the simulation test. We refer the detail of the controller as well as the modified attitude-position
decoupler to our previous research [5], [7].

Besides, although we planned 3 gaits, Gait 1, Gait 2, and Gait 3. The result shows that only Gait 1 works in tracking. The rest gaits receive singular decoupling matrix, which is caused by the saturation of the input (non-zero saturation). Further discussions on the saturations of the input and the stability of feedback linearization can be referred to [14].

The initial position of the tilt-rotor is at \((x_i, y_i, z_i) = (0, 0, 0)\). The initial angular velocities of the propellers are insufficient to compensate the gravity. The tilt-rotor is expected to stabilize its altitude as well as to track the reference.

4.2 Tracking simulation result

Fig. 8 displays the gait that we adopted (Gait 1) in one period. And, the angular velocity history of each propeller is in Fig. 9.

![Fig. 8. Gait 1 within the first period.](image)

![Fig. 9. Angular velocity history.](image)

The trajectory (red curve) of the tilt-rotor in the simulation is illustrated in Fig. 10, where the reference is represented by the blue curve.

![Fig. 10. Tracking result of the tilt-rotor in the simulation. The blue circle is the reference. The red curve represents the actual trajectory of the tilt-rotor while tracking.](image)

The dynamic state error is presented in Fig. 11. It is close to zero after sufficient time, indicating that the tilt-rotor approaches the reference.

![Fig. 11. The dynamic state error of the tilt-rotor. It is defined by the difference between the reference and the actual position of the tilt-rotor.](image)

5. CONCLUSION

In this paper, we further explore Two Color Map theorem. The complete map is displayed for the first time.

It is found that the gaits on the Two Color Map show greater robustness comparing with the adjacent biased gaits not on the map. In general, the roll-pitch diagrams demonstrate that robust gaits receive larger admissible region in attitudes, which result in invertible decoupling matrices, near \((\text{roll}, \text{pitch}) = (0, 0)\). While the adjacent biased gaits receive the roll-pitch diagrams where the unacceptable attitude curves, which introduce the singular decoupling matrices, highly occupy the attitude region near \((\text{roll}, \text{pitch}) = (0, 0)\).

The tilt-rotor, starting from an initial state of non-equilibrium, tracks the designed uniform circular reference successfully with little state error while adopting one robust gait on Two Color Map. No saturations or negative constraints of the angular velocities of the propeller are activated in the simulation. All the inputs, the angular velocities of the propellers and the gaits (tilting angles), are continuous. The decoupling matrix is invertible throughout the simulation with the adopted robust gait.

One of the further steps can be the exploration on the dynamics of the tilt-rotor and the gaits.

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