Joint probability generating function for degrees of active/passive random intersection graphs

Yilun Shang

Abstract

Correlations of active and passive random intersection graphs are studied in this letter. We present the joint probability generating function for degrees of $G^{\text{active}}(n, m, p)$ and $G^{\text{passive}}(n, m, p)$, which are generated by a random bipartite graph $G^*(n, m, p)$ on $n + m$ vertices.

Keywords: combinatorial problems; random intersection graph; degree; generating function.

1. Introduction

Consider a set $V$ with $n$ vertices and another set of objects $W$ with $m$ objects. Define a bipartite graph $G^*(n, m, p)$ with independent vertex sets $V$ and $W$. Edges between $v \in V$ and $w \in W$ exist independently with probability $p$. The active random intersection graph $G^{\text{active}}(n, m, p)$ derived from $G^*(n, m, p)$ is defined on the vertex set $V$ with vertices $v_1, v_2 \in V$ adjacent if and only if there exists some $w \in W$ such that both $v_1$ and $v_2$ are adjacent to $w$ in $G^*(n, m, p)$. Analogously, the passive random intersection graph $G^{\text{passive}}(n, m, p)$ is defined on the vertex set $W$ with vertices $w_1, w_2 \in W$ adjacent if and only if there is some $v \in V$ such that both $w_1$ and $w_2$ are adjacent to $v$ in $G^*(n, m, p)$.

The models of random intersection graphs defined above were first introduced in [6, 9]. Observe that the degree of a given vertex $v \in V$ (or $w \in W$, respectively) in $G^*(n, m, p)$ is binomial $\text{Bin}(m, p)$ (or $\text{Bin}(n, p)$, respectively) distributed, and by interchanging the roles of $V$ and $W$, the active and passive graphs are dual to each other essentially. The random intersection graphs are generalized in [3] by allowing an arbitrary distribution for vertex degree in $G^*(n, m, p)$ instead of merely binomial. Extended in such a way, the active and passive graphs reveal different properties including degree distributions; see [3] for details. Intersection graphs are relationship graphs and widely applied in various fields such as investigation of secure wireless sensor networks [1], modeling of social networks [8], and statistical tests in cluster analysis for non-metric data [4].

An interesting topic in the study of random intersection graphs regards the interrelations between the active and passive graphs. For example, if $V$ and $W$ represent researchers and research papers, respectively; and $v \in V$ and $w \in W$ are adjacent if researcher $v$ is an author of paper $w$. Thus, the resulting active graph is a collaboration graph on $V$ of researchers, and accordingly, the passive graph is a relation graph on $W$ of papers. The correlation of these two graphs shall shed light on the in-depth architectures and patterns in scientific collaboration [7]; e.g. in what way and to what extent, the appearance of certain configurations in $G^{\text{active}}(n, m, p)$ may influence the topology of $G^{\text{passive}}(n, m, p)$ and vice versa.

In the current paper, as a first step towards this research, we explore the correlation of vertex degrees between $G^{\text{active}}(n, m, p)$ and $G^{\text{passive}}(n, m, p)$, and the corresponding joint
probability generating function is provided by exploiting the sieve method. Some related works dealing with degree distributions of random intersection graphs have been done. [10] treats the degree distribution of $G^{\text{active}}(n,m,p)$ with parameters $m = \lfloor n^\alpha \rfloor$ for some $\alpha > 0$ and $p = \sqrt{cn^-(1+\alpha)/2}$ for some $c > 0$, and the distribution of the degree of a given vertex is shown to converge to a point mass at 0, a Poisson distribution or a compound Poisson distribution depending on whether $\alpha < 1$, $\alpha = 1$ or $\alpha > 1$. [4] examines the sufficient and necessary conditions for generalized active random intersection graph to have a Poisson limiting degree distribution. The related conditions which imply a limiting degree distribution are given in [5] for generalized passive random intersection graph. To the best of our knowledge, our work is the first one devoted to joint degree distributions between the active and passive graphs. It is obvious that the number of edges in active and passive graphs should be positively correlated.

The rest of the paper is organized as follows. The main result (Theorem 1) is presented and proved in Section 2. In Section 3, we conclude the paper with some further remarks.

2. Joint probability generating function

Let $X$ be the number of vertices of $V\{v\}$ adjacent in $G^{\text{active}}(n,m,p)$ to a vertex $v \in V$, and $Y$ be the number of vertices of $W\{w\}$ adjacent in $G^{\text{passive}}(n,m,p)$ to a vertex $w \in W$. That is, $X$ and $Y$ are typical vertex degrees in $G^{\text{active}}(n,m,p)$ and $G^{\text{passive}}(n,m,p)$, respectively. The joint probability generating function of $X$ and $Y$ is defined to be $F(x,y) := E x^X y^Y$ for $x,y \in \mathbb{R}$. The main result in this paper is stated as follows.

**Theorem 1.** The joint probability generating function $F(x,y)$ is given by

$$F(x,y) = \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \binom{n-1}{k} \binom{m-1}{l} x^{n-1-k}(1-x)^k y^{m-1-l}(1-y)^l \cdot [1 - p + p(1-p)^k]^{n-1-k} \cdot [1 - p + p(1-p)^l]^{m-1-l} \cdot \left[ (1-p)^{k+l}p + (1-p) \sum_{i=0}^{l} \binom{l}{i} p^i (1-p)^{l-i} \right]^{n-1-k} \cdot \left[ (1-p)^{k+l}p + (1-p) \sum_{i=0}^{l} \binom{l}{i} p^i (1-p)^{l-i} \right]^{m-1-l}.$$

Let $F(x)$ and $F(y)$ denote the probability generating functions for random variables $X$ and $Y$, respectively. Since $F(x) = F(x,1)$ and $F(y) = F(1,y)$, we have the following corollary which is consistent with Theorem 1 in [10].

**Corollary 1.** The probability generating functions $F(x)$ and $F(y)$ are given by

$$F(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k}(1-x)^k [1 - p + p(1-p)^k]^m$$

and

$$F(y) = \sum_{l=0}^{m-1} \binom{m-1}{l} y^{m-1-l}(1-y)^l [1 - p + p(1-p)^l]^n$$

respectively.

To prove Theorem 1, we will invoke a lemma which is a probability generating function version of the sieve method, whose proof is similar to Lemma 1 in [10]. In fact, the following
lemma can be viewed as a high dimensional extension of Lemma 1 in [10]. Let $P_1$ and $P_2$ be two disjoint sets of properties that a random object can take on. Let $p_{k,l}$ be the probability that the object takes on exactly $k$ properties in $P_1$ and $l$ properties in $P_2$. The related probability generating function is defined as $F(x,y) := \sum_{k,l \geq 0} p_{k,l} x^k y^l$.

**Lemma 1.** Given $S_1 \subset P_1$ and $S_2 \subset P_2$, we define $N_{S_1,S_2}$ to be the event that the random object possesses properties $S_1$ and $S_2$. Define

$$N_{k,l} := \sum_{|S_1|=k, |S_2|=l} P(N_{S_1,S_2}) \quad \text{and} \quad N(x,y) := \sum_{k,l \geq 0} N_{k,l} x^k y^l.$$ 

Then we have

$$F(x,y) = N(x-1, y-1).$$

**Proof.** Let $Y_1$ and $Y_2$ be the numbers of properties that the random object possesses in $P_1$ and $P_2$, respectively. Let $1_{N_{S_1,S_2}}$ be the indicator function of event $N_{S_1,S_2}$. We clearly obtain

$$N_{k,l} = \sum_{|S_1|=k, |S_2|=l} E(1_{N_{S_1,S_2}}) = E\left( \sum_{|S_1|=k, |S_2|=l} 1_{N_{S_1,S_2}} \right) = E\left( \binom{Y_1}{k} \binom{Y_2}{l} \right).$$

Hence,

$$N(x,y) = \sum_{k,l \geq 0} E\left( \binom{Y_1}{k} \binom{Y_2}{l} \right) x^k y^l = E\left[ \sum_{k \geq 0} \binom{Y_1}{k} x^k \sum_{l \geq 0} \binom{Y_2}{l} y^l \right] = E[(x+1)^{Y_1} \cdot (y+1)^{Y_2}] = F(x+1, y+1).$$

\[ \square \]

In the following proof, we will take $P_1$ as the set of $n-1$ properties consisting of the non-adjacency of a fixed vertex to the other $n-1$ vertices in $V$; and take $P_2$ as the set of $m-1$ properties regarding $W$ similarly.

**Proof of Theorem 1.** For $v \in V$ and $w \in W$, let $p_{k,l}^{v,w}$ be the probability that exactly $k$ vertices in $V \setminus \{v\}$ are not adjacent to $v$ and $l$ vertices in $W \setminus \{w\}$ are not adjacent to $w$. Let $G(x,y)$ be the corresponding probability generating function. The probability that the fixed vertices $v$ and $w$ are adjacent to none of the vertices represented by $S_1 \subset V \setminus \{v\}$ and $S_2 \subset W \setminus \{w\}$ respectively is given by

$$P(N_{S_1,S_2}) = P(N_{S_1,S_2} | v \sim w) \cdot p + P(N_{S_1,S_2} | v \not\sim w) \cdot (1-p),$$

where

$$P(N_{S_1,S_2} | v \sim w) = \sum_{i=1}^{n-|S_1|} \sum_{j=1}^{|S_2|} \binom{m-1-|S_2|}{j-1} \binom{n-1-|S_1|}{i-1} \cdot p^{i-1}(1-p)^{m-j}(1-p)^{(j-1)|S_1|} p^{j-1}(1-p)^{n-i}(1-p)^{(i-1)|S_2|} \cdot (1)$$

and

$$P(N_{S_1,S_2} | v \not\sim w) = \sum_{i_o=0}^{|S_1|} \sum_{i_s=0}^{n-|S_1|} \sum_{j_o=0}^{|S_2|} \sum_{j_s=0}^{m-1-|S_2|} \binom{m-1-|S_2|}{j_o} \binom{|S_2|}{j_s} \binom{n-1-|S_1|}{i_o} \binom{|S_1|}{i_s} \cdot p^{i_o+j_s}(1-p)^{m-j_o-j_s} p^{i_o+i_s}(1-p)^{n-1-i_o-i_s} \cdot (1-p)^{(j_o+j_s)|S_1|+(i_o+i_s)|S_2|-i_o-j_s} \cdot (2)$$
3. Concluding remarks

In the above expression (1), the index \( i \) counts the number of vertices of \( V \) adjacent to \( w \) in \( G^*(n, m, p) \) and similarly, \( j \) counts the number of vertices of \( W \) adjacent to \( v \) in \( G^*(n, m, p) \); c.f. Fig. 1. For the expression (2), \( i_o + i_s \) counts the number of vertices of \( V \) adjacent to \( w \) in \( G^*(n, m, p) \), where \( i_o \) counts vertices outside \( S_1 \) while \( i_s \) inside. The roles of indices \( j_o \) and \( j_s \) can be interpreted likewise; c.f. Fig. 2. The inclusion and exclusion principle is utilized here.

Hence we have

\[
N_{k,l} = \binom{n-1}{k} \binom{m-1}{l} \left[ \sum_{i=0}^{k-1} \sum_{j=0}^{m-1-l} \binom{m-1-l}{j} \binom{n-1-k}{i} \cdot (1-p)^{j-i} \cdot (1+p)^{m-1-l} \right]
\]

Consequently, Lemma 1 yields \( N(x, y) = N(x-1, y-1) \), where

\[
N(x, y) = \sum_{k=0}^{n-m-1} \sum_{l=0}^{m-1} N_{k,l} x^k y^l
\]

\[
= \sum_{k=0}^{n-m-1} \sum_{l=0}^{m-1} x^k y^l \binom{n-1}{k} \binom{m-1}{l} \left[ 1-p+p(1-p)^k \right]^{m-1-l} \left[ 1-p+p(1-p)^l \right]^{n-1-k}
\]

\[
\cdot \left[ (1-p)^{k+l} + (1-p) \sum_{i=0}^{l} \binom{l}{i} p^i (1-p)^{l-i} \right].
\]

The result then follows from the fact that

\[
F(x, y) = \sum_{k=0}^{n-m-1} \sum_{l=0}^{m-1} p_{k,l} x^k y^l = \sum_{k=0}^{n-m-1} \sum_{l=0}^{m-1} p'_{k,l} x^{n-1-k} y^{m-1-l} = x^{n-1} y^{m-1} G(x^{-1}, y^{-1}).
\]
our work is only a preliminary step to the interesting and meaningful topic of correlation of active/passive random intersection graphs. Corollary 1 gives the probability generating functions of the marginal distributions \( P(X = k) \) and \( P(Y = l) \), respectively. As is known, the random variables \( X \) and \( Y \) are independent if and only if \( F(x, y) = F(x)F(y) \) for all \( x, y \) (c.f. \[2\] pp. 279). Hence, we may tackle the interdependence between the active graph and passive graph through their probability generating functions. For the future research, the method developed in this paper may be applied to the joint distribution of degrees of generalized active and passive graphs in \[3\]. A more hard question is to investigate the joint distribution of clusters in these two models.

References

[1] L. Eschenauer, V. D. Gilgor, A key-management scheme for distributed sensor networks. *Proc. 9th ACM Conference of Computer and Communications Security*, 2002 pp. 41–47

[2] W. Feller, *An Introduction to Probability Theory and Its Applications, Vol. 1*. Wiley, New York, 1968

[3] E. Godehardt, J. Jaworski, Two models of random intersection graphs for classification. In: M. Schwaiger, O. Opitz (Eds.), *Exploratory Data Analysis in Empirical Research*. Springer-Verlag, Berlin, 2003 pp. 67–81

[4] J. Jaworski, M. Karoński, D. Stark, The degree of a typical vertex in generalized random intersection graph models. *Discrete Mathematics*, 306(2006) pp. 2152–2165

[5] J. Jaworski, D. Stark, The vertex degree distribution of passive random intersection graph models. *Combinatorics, Probability and Computing*, 17(2008) pp. 549–558

[6] M. Karoński, E. R. Scheinerman, K. B. Singer-Cohen, On random intersection graphs: the subgraph problem. *Combinatorics, Probability and Computing*, 8(1999) pp. 131–159

[7] M. E. J. Newman, The structure of scientific collaboration networks. *Proc. National Academy of Sciences, USA*, 2001 pp. 404–409

[8] M. E. J. Newman, Properties of highly clustered networks. *Phys. Rev. E*, 68(2003) 026121

[9] K. B. Singer-Cohen, Random intersection graphs. Dissertation, Johns Hopkins University, Baltimore, MD, 1995

[10] D. Stark, The vertex degree distribution of random intersection graphs. *Random Structures and Algorithms*, 24(2004) pp. 249–258
Figure 1: $G^*(n, m, p)$ in the case of $v \sim w$. Solid line represents edges and dashed line represents non-edges.
Figure 2: \( G^*(n, m, p) \) in the case of \( v \not\sim w \). Solid line represents edges and dashed line represents non-edges.