Nonlocality, singularity, and elastic scattering in quantum fields

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Using path integrals we express the quantum nonlocality of AB-effect type in the form of singularity. The gauge-fixing term in path integrals induce the AB effect in ordinary scattering processes.

This means that all scattering processes are accompanied by nonlocal effect. The formulae are then extended to theory of fields that additionally include a scalar potential. It turns out that the degree of freedom of nonlocality in quantum fields is just the degree of the ghosts. Furthermore, renormalization method can be related to this type of nonlocal effect.

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Introduction. The nonlocality in quantum mechanics has long been a hot topic in the past decades, and up to date there has been no experiment contradicting the nonlocality; It refers to the correlation between two particles separated in space such as entanglement derived from the Bell theory [1] and well confirmed in many experiments [2]. All these experiments used massless photons as carriers of the states, and the nonlocality is of the Bell type. The study of nonlocality has been also extended to a single photon, which is in a superposition state of two space-separated states such as |a⟩ and |b⟩, as in the situation when a photon passes through a two-slit plane (diffraction). This type of nonlocality is called Hardy type [3, 4]. The conventional method of studying this type of states is to introduce two vacuum states |0⟩_A and |0⟩_B at two local regions for A and B [5-7]. Then the correlation of states |a⟩ and |b⟩ is obvious when one couples them to entangled state |Ψ⟩ = \(\frac{1}{\sqrt{2}}(|a⟩|0⟩_A + e^{i\phi}|b⟩|0⟩_B⟩\). However, some authors argued that [8] for a single fermion, the above method is not effective any longer because the massive fermion manifests its nonlocality in a completely different way.

Since its nonlocality can’t be transformed to explicit correlation as above, one can’t measure it at location A and B and then evaluate how it violates the Bell inequalities. Through a tedious analysis, the authors arrived at the conclusion that the only type of nonlocality for a fermion (except the collapse) wave is of the AB-effect [9] type.

The AB effect for fermions has been well demonstrated by experiments [10, 11, 12]. It appears when a charged particle winds around a magnetic flux completing closed n (integer) loops. To understand the effect in more general circumstances [13], one may ask what will happen if a charged particle is scattered by a very thin flux such as a spin moment. The answer to this question provides us with some more useful observables of nonlocality other than the familiar scattering results. The scattering amplitudes and phase shifts have been studied quantitatively [14, 15], and distinguishing feature is that the scattering result loses the axile symmetry and dependence on the magnetic component appears. But, how does the nonlocal effect take place simultaneously with the scattering process remains elusive as does not make sense to require the charged particle to move around the flux for exactly n closed loops in a certain plane and then come back to fulfill the scattering process involving other interactions.

In this paper we will demonstrate that the nonlocal effect takes place when and only when scattering occurs. First, we prove that the Aharonov-Bohm effect can be reproduced if we introduce a singularity to the Feynman path integral for (2+1)-dimensional quantum electrodynamics. Secondly, the condition of (2+1)-dimension can be removed by considering the gauge transformation and gauge-fixing condition. To this end, the nonlocality like AB-effect can be described using the same Lagrangian with only an additional singularity. In above expression, we have viewed the fermion that is with magnetic moment as rest singularity, the other fermion is initial and final particle that being scattered. This view makes us aware that any scattering process is accompanied by the AB-like nonlocality. The resultant formulae can be easily generalized to quantum field and thus to the non-Abelian situation. It is showed that the degree of nonlocality is relevant to the degree of freedom of ghost fields. The application to the renormalization group is sketched.

In what follows, we employ the path integral method of Feynman to interpret the propagating process of the wave function [16]. For instance, using the kernel \(K(x_2, t_2; x_1, t_1)\) to describe how the wave function \(ψ(x_2, t_2)\) has evolved from all states \(ψ(x_1, t_1)\) at the moment \(t_1\):

\[
ψ(x_2, t_2) = \int K(x_2, t_2; x_1, t_1)ψ(x_1, t_1)d^3x_1
\]

and

\[
K(x_2, t_2; x_1, t_1) = \int [dq]e^{iS}
\]

where \([dq]\) denotes all possible paths, \(S = \int_{t_1}^{t_2} L(q, ˙q)dt\) is the action along a certain path, and \(L(q, ˙q)\) is the corresponding Lagrangian.
In calculation, the following two properties of the kernel are very useful. In quantum mechanics level, the main contribution to the kernel comes from the paths that nearly make the action \( S \) in an classical extremum \( S_{el} \) [16] up to a normalizing factor, i.e. \( K \sim e^{iS_{el}/\hbar} \) if the \( L(q,\dot{q}) \) in \( S \) is a quadratic form. And due to the kinetic energy \( \frac{1}{2}m\dot{q}^2 \), few situations violate the quadratic requirement. Although this simplification originally appeared only as a mathematical technique, it can also simplify our physical consideration evidently, which will be validated later.

Another feature of kernel is that "Amplitudes for events occurring in succession in time multiply." [16]. For two such succeeding events \( a \rightarrow c \) and \( c \rightarrow b \), we have

\[
K(b,a) = \int_{x_c} K(b,c)K(c,a)dx_c
\]

(I) Expressing AB effect in Feynman Integral with a singularity: we do the integral of kernel as usual for any Lagrangian, and then evaluate the contribution from the singularity and its neighborhood. By virtue of Eq. (3), we can perceive the removal of singularity as a succeeding event after the just finished integral. This is equal to the case that a particle walks backward along all the original paths near the singularity after the particle has already completed all the possible paths. Henceforth we call this sort of walking backward as a return mechanism. So, in physical sense, the redundant integral performed around the singularity is removed not by subtracting but by multiplying another technique, it can also simplify our physical consideration evidently, which will be validated later.

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\[
K(b,a)_{total} = K_s(\text{singular region})K(b,a)
\]

= \[
\int [dq']e^{i\int_b^c L(q',\dot{q}')dt}\int [dq]e^{i\int_a^b L(q,\dot{q})dt}
\]

= \[
\int [dq][dq']e^{iS+iS'}
\]

the measure \([dq]\) stands for all the paths and \([dq']\) represents all the paths covering the neighborhood of the singularity, see Graph 1. It will be clear later that the action \( S \) is associated with scattering calculation and \( S' \) associated with nonlocal effect.

In general, we are only concerned with the effect of singularity on the wave function in Eq.(1). In such a case we only need to evaluate the first kernel \( K_s = \int [dq']e^{iS'} \). Now suppose an ideal dimensional (2+1) situation in which a low energy electron is scattered by a neutron: a plane is predetermined and the magnetic moment of neutron is vertical to the plane. A general Lagrangian has a simple form as \( L = \frac{1}{2}m\dot{q}^2 - e(\vec{q} - \vec{v} \cdot \vec{A}) \), in which \( \varphi \) and \( \vec{A} \) are scalar and vector potential respectively. For neutron, we know \( \varphi = 0 \) and make \( \vec{A} = (x, y, 0) \). Then it is straightforward to carry out the integral \( K_s \sim e^{iS_{el}} \) using the above-mentioned first property. But the result is not what we have expected for the singularity. In a plane with singularity, the contribution to the kernel can’t be approximated using only one classical extremum of \( S'_{el} \). It is obvious that the paths encompassing a singularity can’t be topologically invariably shrunk to one classical path. This point will be clearer with the consideration of gauge fixing condition. Using the above mentioned \( K \sim e^{iS_{el}}, \) we may get

\[
K_s = \int ([dq']_{\text{left}} + [dq']_{\text{right}})e^{iS'_{el}} \sim e^{iS'_{el}} + e^{iS'_{el}}.
\]

In the integral of action \( S_{el} = \int L(q,\dot{q})dt = \int (\frac{1}{2}m\dot{q}^2 + \vec{v} \cdot \vec{A})dt \), the second term \( \int \vec{v} \cdot \vec{A} dt = \int \vec{A} \cdot \vec{d} \cdot \vec{r} \) can just afford the phase difference \( K_s \sim e^{iS'_{el}} (1 + e^{i/h\int \vec{A} \cdot \vec{d} \cdot \vec{r}}) \) required by AB-effect. However, there yet left the first term to be treated in consistency. By a similar process, we obtain

\[
\int \frac{1}{2}m\dot{q}^2 dt = \frac{1}{2}\int \vec{p} \cdot \vec{d} \cdot \vec{r},
\]

where \( \vec{p} \) is the momentum, and thus the phase difference is

\[
\frac{1}{2}\int \vec{p} \cdot \vec{d} \cdot \vec{r} = \frac{1}{2}\int (\nabla \times \vec{p}) \cdot \vec{d} \cdot \vec{\sigma}
\]

(\( \vec{d} \cdot \vec{\sigma} \) is differential area element), which actually vanishes when the momentum is constant. Obviously when a momentum is split into two on one side and merge on another side of the singularity, the direction of momenta along
the two paths will change. Using the method of difference, we can prove that \((\nabla \times \vec{p})\) is non-vanishing (Graph 2), so is the nontrivial difference. This difference from \((\nabla \times \vec{p})\) is expected to be responsible for the phase shift due to diffraction, which should be held even when the interaction \(\vec{v} \cdot \vec{A}\) is absent. The interaction of scalar potential \(\varphi\) can be included (neutron replaced by proton) without affecting the above result since it is path-independent.

(II) Generalizing the expression of nonlocality. In order to generalize the expression, it is necessary to consider the gauge transformation \(\vec{A} \rightarrow \vec{A}' = \vec{A} + \nabla \theta(x)\), and gauge fixing condition, e.g. coulomb gauge \(\nabla \cdot \vec{A} = 0\). Concerning the gauge transformation, since the scalar function \(\theta(x)\) can be arbitrarily determined, it can directly reduce the degree of freedom of \(\vec{A}\) by one. Therefore, the vector \(\vec{A}\) can directly be chosen as \(\vec{A} = (\vec{A}_x, \vec{A}_y, 0)\), and thus in a plane. On the other hand, we know that any two-particle scattering must occur in a plane which can’t be predetermined but actually exists. So as an effective choice (equivalent way), make all the \(\vec{A}\)’s are parallel to the scattering plane. So, the above assumption for AB effect that the scattering must happen in a plane can actually be removed. Additionally, \(\nabla \cdot \vec{A} = 0\) suggests that the vector \(\vec{A}\) behaves like a tangent of magnetic lines without divergence; and hence form closed loops. To this end, the condition of AB-effect, in a plane and close loop, is automatically satisfied by two-particle scattering with regard to the gauge transformation and gauge fixing condition. These two constraints reduce the freedoms of vector \(\vec{A}\) from three to one, identical to the freedom along a loop. We note that the freedom for gauge transformation is the very nonlocal degree of the fermion.

On an alternative viewpoint, since the choice of function \(\theta(x)\) for the gauge transformation \(\vec{A} \rightarrow \vec{A}' = \vec{A} + \nabla \theta(x)\) is arbitrary, \(\vec{A}\) can be replaced by \(\nabla \theta(x)\) at every point in a continuous manner. Thus the discussion of \(\vec{A}\) also applies to \(\nabla \theta(x)\), i.e. the \(\nabla \theta(x)\) is in the scattering plane with \(\nabla \cdot \nabla \theta(x) = 0\). A change of \(\nabla \theta(x)\) in \(\vec{A}\) transform the wave function \(\psi\) to \(\psi e^{i\theta(x)}\). The movement will not be trivial if a singularity exists for it produces observable interference.

In QED the coulomb gauge is equivalent to the Lorentz gauge \(\partial_\mu A^\mu = 0\), so the above argument can be extended to include scalar potential. We do this at the cost of giving up the time arrow, and the ”plane” in which scattering and diffraction occur is now a 3-dimensional ”super plane”. The condition \(\nabla \cdot \nabla \theta(x) = 0\) becomes \(\partial_\mu \partial^\mu \theta(x) = 0\), accordingly. Similarly, the two constrains \(A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \theta(x)\) and \(\partial^\mu A_\mu = 0\) will reduce the freedoms of \(A_\mu\) from four to two. In this way, the scattering considered here is automatically extended to the general cases of a two-fermion scattering.

To extend the above argument to Field theory, let’s write down the gauge condition in terms of Faddeev and Popov form [17]. In QED, the form is

\[
\Delta[A] \int [d\theta] \delta(\partial_\mu (A^\mu)^\theta) = 1, \tag{8}
\]

where \(\Delta[A]\) is the Jacobian

\[
\frac{\delta(\partial_\mu (A^\mu)^\theta)}{\delta \theta} = \partial_\mu \partial^\mu \delta(x-y) = \Box \delta(x-y), \tag{9}
\]

which is independent of \(A^\mu\) in Abelian case, and hence gauge invariant. Eq. (8) is demanded to be gauge invariant, so the gauge invariance of \(\int [d\theta] \delta(\partial_\mu (A^\mu)^\theta)\) automatically restores the result \(\partial_\mu \partial^\mu \theta(x) = 0\). The gauge condition \(\partial_\mu A^\mu = 0\) can be controlled in experiments by specifying the final outcome, but the resultant path of \(\partial^\mu \theta(x)\) is a pure gauge property that can’t be fixed by using local experiments, i.e. the function \(\theta(x)\) can’t be determined locally. If considering the integral for the Grassman variables \(c_i\), \((\prod_{i=1}^4 \int dc_i^* dc_i) e^{-c_i^* B_{ij} c_j} = \det B\ [18], \Delta[A]\) in eq. (8) can be expressed using degree [18] of freedom of the ghost as

\[
\int [D\bar{c}] [Dc] e^{i \int d^4 x \bar{c}(x) D(x) c(x)} \tag{10}
\]

in the case of QED the above integral only contains dynamical term that can be absorbed into normalization constant. We reserve its explicit form here to see clearly that the freedom of ghosts (\(\bar{c}(x)\) and \(c(x)\)) is just the nonlocal degree of freedom. And the degree of freedom of an electron is 4, larger than the ghosts.

From amplitude \(K_{\text{total}}\) it is straightforward to extend the above discussion to field theory. Notice the form of Eq.(1) and the expansion property of quantum mechanics \(\psi = \sum c_i \varphi_i\), every element of the S-matrix can be written as

\[
\langle \text{in} | \text{out} \rangle \sim K_{\text{total}} = K_{\text{normal}}. \tag{11}
\]

Now the integral measurement should be replaced by the fermion and boson fields instead of configuration space. For QED, \(K_{\text{normal}}\) has the form \(\int [D\bar{\psi}] [D\psi] [D\vec{A}] e^{iS}\), where the Lagrangian in \(S\) now becomes \(S = -1/4 F_{\mu\nu} F^{\mu\nu} + \ldots\).
\( \bar{\psi}(iD-m)\psi \) with \( F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \) and \( D_{\mu} = \partial_{\mu} - ieA_{\mu} \). A gauge-invariant Lagrangian made by imposing gauge fixing condition and adding the ghost \([19, 20]\) has a general form

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(iD-m)\psi + \frac{1}{2} \xi (\partial^{\mu}A_{\mu})^{2} + \bar{c}(x)(\partial^{\mu}\partial_{\mu})c(x)
\]  

(12)

A Lagrangian with ghost field is convenient to do BRST transformation to get identities used in renormalization. Here we skip the details along this context. Notice that the term \( c(x)(\partial^{\mu}\partial_{\mu})c(x) \) in Eq. (12) and the term \( \partial^{\mu}\partial_{\mu}\theta(x) = 0 \) in Eq. (8) are both responsible for gauge invariance after the gauge fixing term having been added [19]. Comparing the freedom of \( \partial_{\mu}\theta(x) \) and ghost field \( c(x) \), it is found that the ghost degree is very the nonlocal degree. In QED, this degree of freedom is 2, one for scalar, one for vector.

The above formulae can be further extended to the non-Abelian case (e.g. QCD) by only changing \( F_{\mu\nu} \) and \( D_{\mu} \) to \( F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + gf^{abc}A_{\mu}^{a}A_{\nu}^{b} \) and \( D_{\mu} = \partial_{\mu} - igA_{\mu}^{a}\lambda^{a} \), in which the \( \lambda^{a} \) is the generator of the gauge group and \( f^{abc} \) is the corresponding structure constant. The terms in Lagrangian of \( K_{s} \) is the same as \( K_{\text{normal}} \); however, the integral measurement should be independent. Furthermore, at the neighborhood of singularity, the fermion wave function and measurement should be independent. Here let’s again concentrate on the degree of freedom for nonlocality.

For non-Abelian field, Eq.(8) should be changed to

\[
\int [d\alpha] \delta(G(A^{\alpha})) \text{det}(\frac{\delta G(A^{\alpha})}{\delta \alpha}) = 1,
\]

(13)

in which the Jacobian \( \text{det}(\frac{\delta G(A^{\alpha})}{\delta \alpha}) \) is related to \( A^{\alpha} \) and can’t be taken out of the integral as \( \Delta[A] \) in QED. For QCD, the gauge transformation is

\[
A_{\mu}^{a} \rightarrow (A^{\alpha})_{\mu}^{a} = A_{\mu}^{a} + \frac{1}{g} \partial_{\mu}\alpha^{a} + f^{abc}A_{\mu}^{b}\alpha^{c} = A_{\mu}^{a} + \frac{1}{g} D_{\mu}\alpha^{a}
\]

(14)

from this expression we can get the analogous result of QED in Lorentz gauge:

\[
\frac{\delta G(A^{\alpha})}{\delta \alpha} = \frac{1}{g} \partial^{\mu}D_{\mu}.
\]

(15)

Similarly Eq.(10) takes the form:

\[
\text{det}(\frac{\delta G(A^{\alpha})}{\delta \alpha}) = \int [D\bar{\psi}] [D\psi] e^{i \int d^{4}x \bar{\psi}(\partial^{\mu}D_{\mu})\psi(x)}
\]

(16)

Accordingly gauge-invariant Lagrangian becomes [18]

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(iD-m)\psi + \frac{1}{2} \xi (\partial^{\mu}A_{\mu})^{2} + \bar{c}(x)(\partial^{\mu}D_{\mu})c(x)
\]

(17)

for \( a, c = 1, 2, \ldots, 8 \).

The fact that the degrees of the ghost fields are responsible for the nonlocal degrees is directly extended from QED. In QCD, the degree of freedom for ghost fields is 16. Here the discussion of nonlocality is more complex for the ghost fields are coupled with the vector field, which is one of the main features in non-Abelian field theory. The relationship between the wilson loop and Berry phase [21] belongs to 2-dimensional out of 4-dimension case of the spatial degrees. And the degree of quarks here is \( 4 \times 3 = 12 \), smaller than 16. It may be of this reason that quarks are confined.

(III) New Hamiltonian. Let’s start from the Eq.(4). Since the two Lagrangians in actions \( S \) and \( S' \) have the same forms, the same integrand will certainly induce the same integral resultant function, in which only boundary values of \( q \) and \( q' \) are different. For example, if resultant function is \( S(q) \) and \( S'(q') = S(q') \), then for the first action, \( S_{1} = S(q) = S(q(t_{b})) - S(q(t_{a})) \), and for the second, \( S_{2} = S(q') = S(q'(t_{c})) - S(q'(t_{a})) \). To finish the integral of \([dq] \) and \([dq'] \), let’s divide the time interval into many parts as, \([t_{a}, t_{b}] \sim [t_{1}, t_{2}, \ldots, t_{m}, \ldots, t_{n}] \). Then the integral measurement \([dq] \) changes with the time division to \([dq] = dq_{2}dq_{3} \ldots dq_{l} \ldots dq_{m} \ldots dq_{n-1} \). For the division of \([t_{b}, t_{c}] \), we determine the intervals of time so coincident with the time interval between the \([t_{l}, t_{m}] \) as to reflect the return mechanism: If \([t_{b}, t_{c}] \sim [t_{1}, t_{2}, \ldots, t_{n}] \), then make \( q'(t_{c}) = q(t_{l}), q'(t_{n-1}) = q(t_{l+1}), q'(t_{n-2}) = q(t_{l+2}), \ldots, q'(t_{b}) = q(t_{m}) \), which means \([dq'] = dq_{m-1}dq_{m-2} \ldots dq_{1} \). And thus applying the second property of Feynman integral and multiplying every integral results together yield
In the singular region, then there will be no paths between the region \( q_i \), and thus nothing to the Kernel will only add a factor 1 to the Kernel. So the mechanism and our design of division, \( S_{l-1,i} \), is really different from the scattering. The former is sensitive to measurement and thus unobservable, and the latter can be detected anyway. Now the dissertation can be unified using the gauge transformation (or AB-effect) is really different from the scattering. The former is sensitive to measurement and thus unobservable locally, but the latter can be detected anyway.

Here we will not reiterate the lengthy procedure of renormalization calculations. It is easy to recognize that the cutoff for renormalization are determined by the nonlocal region \( S_d \) as the counter terms for that in action \( S \) in \( K_{\text{normal}} \), and the Lagrangian has the same form as that in \( S \) with only some renormalization constants multiplied to the corresponding terms. The new Lagrangian means a new Hamiltonian, which contributes only to nonlocal phase as in Berry phase, and doesn’t contribute to the transitional amplitude. In this sense, the renormalization and the nonlocality have been unified to an expression.

Now the integral measurement is over all the region of space, and \( S_d \) is the part which is nontrivial only in singular region. The same analysis applies equal to quantum fields: changing the integral measurement and extending the variables in \( \mathcal{L} \) and \( \mathcal{L}' \) directly to fields quantities, and make fermion fields vanish at the singularity point correspondingly.

Eq.(11) suggests that future calculations may consist of two steps. First, we calculate the scattering amplitude \( K_{\text{normal}} \) as usually done; then, we consider the \( K_s \) to include nonlocal effect. To calculate the correction of higher order than the tree level, then, the amplitude \( K_{\text{normal}} \) and \( K_s \) have to be considered together. For the case of ultraviolet divergence in QED and the infrared behavior in QCD, the "\( \text{return} \)" region for \( K_s \) may just be the forbidden region for \( K_{\text{normal}} \), whose contribution should be cut off for its non-physical meaning. In this sense, we make the Lagrangian in \( S_d \) as the counter terms for that in action \( S \) in \( K_{\text{normal}} \), and the Lagrangian has the same form as that in \( S \) with only some renormalization constants multiplied to the corresponding terms. The new Lagrangian means a new Hamiltonian, which contributes only to nonlocal phase as in Berry phase, and doesn’t contribute to the transitional amplitude. In this sense, the renormalization and the nonlocality have been unified to an expression.

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\[ a \times b = a \times b \]

Return part

Singularity
\[ \nabla \times \vec{p} = \Delta \vec{p} \times \vec{p} \]