Cosmological parameters in noncommutative inflation

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Abstract

We investigate how the uncertainty of noncommutative spacetime could explain the WMAP data. For this purpose, the spectrum is divided into the IR and UV region. We introduce a noncommutative parameter of $\gamma_0$ in the IR region and a noncommutative parameter of $\mu_0$ in the UV region. We calculate cosmological parameters using the slow-roll expansion in the UV region and a perturbation method in the IR region. The power-law inflation is chosen to obtain explicit forms for the power spectrum, spectral index, and running spectral index. Further, these are used to fit the data.

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I. INTRODUCTION

String theory as a candidate for the theory of everything can say something about cosmology [1]. Focusing on a universal property of string theory, it is very interesting to study its connection to cosmology. The universal property which we consider is a stringy spacetime uncertainty relation (SSUR) of $\Delta t_p \Delta x_p \geq l_s^2$ where $l_s$ is the string length scale [2]. This implies that spacetime is noncommutative. It is compared to a stringy uncertainty relation of $\Delta x_p \Delta p \geq 1 + l_s^2 \Delta p^2$. The former is considered as a universal property for strings as well as D-branes, whereas the latter is suitable only for strings. Spacetime noncommutativity does not affect the evolution of the homogeneous background. However, this leads to a coupling between the fluctuations generated in inflation and the flat background of Friedmann-Robertson-Walker (FRW) spacetime [3].

On the other hand, it is generally accepted that curvature perturbations produced during inflation are considered to be the origin of CMB anisotropies and inhomogeneities for large-scale structure. The first year results of WMAP put forward more constraints on cosmological models and confirm the emerging standard model of cosmology, a flat $\Lambda$-dominated universe seeded by scale-invariant adiabatic gaussian fluctuations [4]. In other words, these results coincide with predictions of the slow-roll inflation using a single inflaton. Also WMAP brings about some new intriguing results: a running spectral index of scalar perturbations and an anomalously low quadrupole of the CMB power spectrum [5]. If inflation is affected by physics at a short distant close to string scale, one expects that the spacetime uncertainty must be encoded in the CMB power spectrum [6]. For example, the noncommutative power-law inflation may produce a negative running spectral index to fit the data of WMAP [7–9].

In the noncommutative inflation, there exist the two regions: IR and UV region. The former covers a small energy scale ($k < k_s$), while the latter covers a large energy scale ($k > k_c$). In the IR region we expect to find a strongly noncommutative effect on the inflation but we could not use a conventional approach such as the slow-roll approximation to obtain cosmological parameters. Although we find a weakly noncommutative effect in the UV region, the slow-roll approximation is employed in computing cosmological parameters. If one chooses the IR critical scale $k_s$ and UV critical scale $k_c$ appropriately, these parameters could be used to fit the data. It is necessary to introduce both the IR and UV regions to cover the whole noncommutative effect on the inflation.

Recently the UV cosmological parameters have been calculated with the slow-roll parameters $\epsilon_1$ and $\delta_n$ and a noncommutative parameter $\mu_0$ [10]. It was shown that the noncommutative parameter $\mu_0$ could be regarded as a zeroth order slow-roll parameter in the UV region [11,12]. In this work, we make a further progress in this direction. We study the IR region parallel with the UV region by making use of $\gamma_0$. We show how the uncertainty of noncommutative spacetime could explain the WMAP data.

The organization of this work is as follows. In Section II we review the framework for a perturbative inflation. In Section III, we calculate cosmological parameters in the UV region. Section IV is devoted to computing cosmological parameters in the IR region. Finally we discuss our results in Section V.
II. FRAMEWORK FOR INFLATION

A. Commutative inflation

Our starting point is an effective action based on an inflaton minimally coupled to gravity

\[ A = \int d^4x \sqrt{-g} \left[-\frac{M_P^2}{2} R + \frac{1}{2} (\partial_{\mu} \phi)^2 - V(\phi) \right], \]  

(1)

where \( M_P^2 \) is the reduced Planck mass defined by \( M_P = 1/(8\pi G)^{1/2} = 1/l_P \) and its length scale is given by \( l_P = 8.101 \times 10^{-33} \text{cm} \). Considering a flat FRW background of \( ds_{\text{FRW}}^2 = dt^2 - a(t)^2 d\mathbf{x} \cdot d\mathbf{x} \), one finds the Friedmann equations

\[ H^2 = \frac{\rho}{3M_P^2}, \quad \dot{H} = -\frac{1}{2M_P^2}(\rho + p) \]  

(2)

with the Hubble parameter \( H = \dot{a}/a \). From the action (1), one finds the equation

\[ \ddot{\phi} + 3H \dot{\phi} = -V', \]  

(3)

where dot and prime denote derivative with respect to a cosmic time \( t \) and \( \phi \), respectively. Its energy density and pressure are given by \( \rho = \dot{\phi}/2 + V \) and \( p = \dot{\phi}/2 - V \).

We briefly review the slow-roll approximation. This approximation means that an inflation (\( \ddot{a} > 0 \)) is driven by a single scalar field slowly rolling down its potential toward a local minimum. Then Eqs.(2) and (3) take the following form approximately:

\[ H^2 \approx \frac{V}{3M_P^2}, \quad \dot{H} \approx -\frac{\dot{\phi}^2}{2M_P^2}, \quad \dot{\phi} \approx -V'/3H. \]  

(4)

In order to take this approximation into account, we introduce slow-roll parameters (H-SR towers) as

\[ \epsilon_1 \equiv -\frac{\dot{H}}{H^2} = \frac{1}{2M_P^2} \left( \frac{\dot{\phi}}{H} \right)^2, \quad \delta_n \equiv \frac{1}{H^n} \frac{d^{n+1}\phi}{dt^{n+1}} \]  

(5)

which satisfy the slow-roll condition: \( \epsilon_1 < \xi, |\delta_n| < \xi^n \) for some small perturbation parameter \( \xi \). Here the subscript denotes slow-roll (SR)-order in the slow-roll expansion. A scalar metric perturbation to the flat FRW background is expressed in the longitudinal gauge as [13]

\[ ds_{\text{con-}p}^2 = a^2(\eta) \left\{ (1 + 2A) d\eta^2 - (1 + 2\psi) d\mathbf{x} \cdot d\mathbf{x} \right\} \]  

(6)

with a conformal time \( \eta \) defined by \( d\eta = dt/a \). We get a relation of \( \psi = A \) because the stress-energy tensor does not have any off-diagonal component. It is convenient to express the scalar perturbations in terms of the curvature perturbation \( \mathcal{R} \) [14]

\[ \mathcal{R} = \psi - \frac{H}{\dot{\phi}}\delta \phi, \]  

(7)

where \( \delta \phi \) is a perturbation of inflaton: \( \phi(\mathbf{x}, \eta) = \phi(\eta) + \delta \phi(\mathbf{x}, \eta) \).
Introducing a parameter $z$ and a canonical scalar $\varphi$ as

$$z \equiv \frac{a \dot{\phi}}{H} \text{ and } \varphi \equiv a \left( \delta \phi - \frac{\dot{\phi}}{H} \psi \right) = -z \mathcal{R},$$

the bilinear action for the scalar perturbations leads to a canonical form [15]

$$S = \frac{1}{2} \int d\eta d^3x \left[ \left( \frac{\partial \varphi}{\partial \eta} \right)^2 - (\nabla \varphi)^2 + \left( \frac{1}{z} \frac{d^2z}{d\eta^2} \right) \varphi^2 \right].$$

(9)

We note that $z$ encodes all information about an expanding universe of inflation. Because the background is isotropic and homogeneous, we can expand $\varphi$ in terms of Fourier modes as

$$\varphi(x, \eta) = \int d^3k \left( \frac{2\pi}{k} \right)^{3/2} \tilde{\varphi}_k(\eta) e^{i k \cdot x}$$

with $\varphi_k(\eta) = a_k \varphi_k(\eta) + a^*_k \varphi^*_k(\eta)$ [16]. Its Fourier-transformed action can be expressed in terms of $\mathcal{R}_k = -\varphi_k/z$ as

$$\mathcal{F}[S] = \frac{V_T}{2} \int d\eta d^3k z^2(\eta) \left[ d\mathcal{R}_{-k} d\mathcal{R}_k \right. - k^2 \mathcal{R}_{-k} \mathcal{R}_k],$$

(10)

where $V_T$ is the total spatial volume. This action is a useful form because replacing $z^2$ by $a^2$ in the integrand leads to that for the tensor perturbation and its form persists to the noncommutative case. From this action one finds the Mukhanov equation

$$\frac{d^2 \varphi_k}{d \eta^2} + \left( k^2 - \frac{1}{z} \frac{d^2z}{d\eta^2} \right) \varphi_k = 0$$

(11)

which governs the dynamics for evolution of a $k$-th scalar mode ($\varphi_k$).

**B. Noncommutative inflation**

We introduce another time coordinate $\tau$ defined by $d\tau = adt$ to incorporate the noncommutative spacetime appropriately [3]. Its connection to the conformal time is given by $d\eta/d\tau = 1/a^2$. The other noncommutative approach appeared in ref. [17]. Then Eq.(6) can be rewritten as

$$ds^2_{\text{non}-p} = a^{-2}(\tau)(1 + 2A)d\tau^2 - a^2(\tau)(1 + 2\psi)dx \cdot dx.$$  

(12)

Using this time, the SSUR of $\Delta t_p \Delta x_p \geq l_s^2$ becomes

$$\Delta \tau \Delta x \geq l_s^2.$$  

(13)

Considering $\Delta x \sim 1/k$, a range of the time-uncertainty is given by $\Delta \tau = l_s^2 k$. We propose the transition to noncommutative spacetime obeying Eq.(13) by taking the operator appearing in the bilinear action in Eq.(9) and replacing all multiplications by $\ast$-products [18]. Performing the Fourier transform, the SSUR modifies its action minimally as

$$\mathcal{F}[\tilde{S}] = \frac{V_T}{2} \int d\tilde{\eta} d^3k z^2(\tilde{\eta}) \left[ d\tilde{\mathcal{R}}_{-k} d\tilde{\mathcal{R}}_k \right. - k^2 \tilde{\mathcal{R}}_{-k} \tilde{\mathcal{R}}_k].$$

(14)
Here $z_k$ a smeared version of $z$ and $\tilde{\eta}$ a new conformal time are given by

$$z_k^2(\tilde{\eta}) = z^2 y_k^2(\tilde{\eta}), \quad y_k^2 = \sqrt{\beta_k^+ \beta_k^-} = \frac{a_0^2 + a_-^2}{2a_+ a_-}, \quad \frac{d\tilde{\eta}}{d\tau} = \sqrt{\frac{\beta_k^-}{\beta_k^+}} \equiv \frac{1}{a_{\text{eff}}^2} = \frac{1}{a_+ a_-}.$$ \hspace{1cm} (15)

where

$$\beta_k^\pm = \frac{1}{2} \left[ a_+^2 + a_-^2 \right]$$ \hspace{1cm} (16)

with $a_\pm \equiv a(\tau \pm \Delta \tau)$. We note that the SSUR does not affect an evolution of the homogeneous background. However, this leads to a coupling between $\ddot{\varphi}_k(\tilde{\eta})$ generated during inflation and the flat FRW background through $z_k$. Actually the SSUR induces the uncertainty of time in defining $a$. If one takes a limit of $\Delta \tau \to 0 (l_s \to 0)$, one finds $\beta_k^\pm \to a^\pm_2$. This implies that the commutative case is recovered from the noncommutative formalism: $y_k \to 1$, $a_{\text{eff}} \to a^2$, $z_k \to z$, $\tilde{\eta} \to \eta$, $d\tilde{\eta}/d\tau \to d\eta/d\tau$, $\ddot{\varphi}_k(\tilde{\eta}) \to \varphi_k(\eta)$.

From Eq. (14), we derive the Mukhanov equation for the noncommutative inflation

$$\frac{d^2 \ddot{\varphi}_k}{d\tilde{\eta}^2} + \left( k^2 - \frac{1}{z_k^2} \frac{d^2 z_k}{d\tilde{\eta}^2} \right) \ddot{\varphi}_k = 0.$$ \hspace{1cm} (17)

Our task is to solve Eq. (17) in the UV and IR regions. The key step to calculate cosmological parameters is to use the horizon crossing time at $\tilde{\eta} = \tilde{\eta}_s$ in the UV region and the saturation time at $\tilde{\eta} = \tilde{\eta}_0$ in the IR region. For this purpose we consider a power-law inflation of $a(t) = a_0 t^p$ with $p > 1$. Taking $a_0 = [1/(p+1)]^p$, we have $a(\tau) \equiv a_0 \tau^{s_{\text{eff}}} = (\tau/l)^{s_{\text{eff}}}$ and $H = [p/(p+1)](p\tau)^{-1/(p+1)}$. From $k = a_{\text{eff}}/l_s$, one finds an important relation between $\tau$ and $k$ [9]

$$\tau(k) = \Delta \tau \left[ 1 + \left( \frac{k}{k_s} \right)^{s_{\text{eff}}} \right]^{1/2}$$ \hspace{1cm} (18)

with a critical scale $k_s = l_{s_{\text{eff}}}^{-1}/l^p = 1/(l^p M_s^{p-1})$. This scale is the ratio of a string scale $l_s$ to other scale $l$ related to $a_0$.

### III. UV COSMOLOGICAL PARAMETERS

In this section we compute cosmological parameters in the UV region. In order to calculate these, we have to specify both the time at $\tilde{\eta} = \tilde{\eta}_0$ when the $k$-mode is generated and the later time at $\tilde{\eta} = \tilde{\eta}_s > \tilde{\eta}_0$ when it crosses the Hubble horizon. However, in the UV region, the cosmological energy scale when a perturbation is generated is much smaller than the string energy scale $M_s = 1/l_s$: $H(\tilde{\eta}) = \tilde{\eta}_0 \ll H(\tilde{\eta}_0) \ll M_s$. Hence the time $\tilde{\eta}_0$ is not crucial because in the UV region, all modes are generated inside the horizon. Further, noncommutative effects are soft and thus $a_{\pm}$ could be Taylor-expanded up to first-order as [18]

$$a(\tau \pm \Delta \tau) = a(\tau)[1 \pm \sqrt{\mu_0} + \{\pm \sqrt{\mu_0} - (1 \pm \sqrt{\mu_0}) \ln(1 \pm \sqrt{\mu_0})\} \epsilon_1] + \mathcal{O}(\epsilon_1^2)$$ \hspace{1cm} (19)

with a noncommutative parameter in the UV region [10].
\[
\mu_0(t, k) = \left( \frac{\Delta \tau H}{a} \right)^2 = \left( \frac{kH}{aM_s^2} \right)^2.
\]  
(20)

Its total time derivative is given by [11,12]

\[
\frac{d\mu_0}{dt} = -4H\mu_0\epsilon_1.
\]  
(21)

Hence we interpret \(\mu_0\) to be a slow-roll parameter in addition to \(\epsilon_1, \delta_n\) and its subscript denotes the zeroth-order in the slow-roll expansion. \(\mu_0(t, k)\) is a function of \(t\) and \(k\) at the beginning, but one finds two interesting forms: \(\mu_0(t) = 2(H/M_s)^4\) at pivot scale \(k = k_s = \sqrt{2aH}\) and \(\mu_0(k) = (k_c/k)^{4\epsilon_1}\) for power-law inflation. Although there is no direct relation between \(\mu_0\) and the slow-roll potential \(V(\phi)\), we don’t doubt that \(\mu_0\) is regarded as a slow-roll parameter. At this stage we note a procedure of realizing the noncommutative effect on the cosmological parameters: the SSUR \((\Delta \tau = kl_s^2) \to a_\pm \to \mu_0(t, k) \to z_k \to \text{cosmological parameters.}\)

We start with the slow-roll approximation to calculate cosmological parameters. This means that \(\mu_0, \epsilon_1, \delta_1\) are taken to be approximately constant in calculation of the noncommutative power spectrum. To this end we obtain a potential-like term up to first order

\[
\frac{1}{z_k} \frac{d^2 z_k}{d\tilde{\eta}^2} \simeq 2(aH)^2 \left( 1 - 2\mu_0 + \epsilon_1 + \frac{3}{2}\delta_1 \right)
\]  
(22)

and relations from Eqs.(15) and (19)

\[
aH \simeq -\frac{1}{\tilde{\eta}}(1 + \mu_0 + \epsilon_1), \; y_k \simeq 1 + \mu_0.
\]  
(23)

Then Eq.(17) takes the same form as in the commutative case [20]

\[
\frac{d^2 \tilde{\phi}_k}{d\tilde{\eta}^2} + \left( k^2 - \frac{(\nu^2 - \frac{1}{4})}{\tilde{\eta}^2} \right) \tilde{\phi}_k = 0
\]  
(24)

with the same index \(\nu = \frac{3}{2} + 2\epsilon_1 + \delta_1\) except replacing \(\eta\) by \(\tilde{\eta}\). Its asymptotic solution to Eq.(17) in the limit of \(-k\tilde{\eta} \to \infty\) takes a plane-wave

\[
\tilde{\phi}_k = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}.
\]  
(25)

In the limit of \(-k\tilde{\eta} \to 0\), one finds an asymptotic form of the Hankel function \(H^{(1)}_{\nu}(-k\tilde{\eta})\)

\[\text{In order to show it, we consider the horizon-flow approximation [19] whose parameters are defined as } \tilde{\epsilon}_0 = H_{in f}/H \text{ and } \tilde{\epsilon}_{i+1} = d\tilde{\epsilon}_i/dN \text{ with } H_{in f} \text{ the Hubble parameter at some chosen time and } N = \int Hdt \text{ the } e\text{-folding number since the horizon-crossing time. Here } \tilde{\epsilon}_0 \text{ is a geometric quantity. One takes this quantity by hand to obtain higher-order slow-roll parameters } \tilde{\epsilon}_{i+1}. \mu_0 \text{ arises from an effect of the noncommutative spacetime and it belongs to a geometric quantity involving a string scale. Similarly we could include } \mu_0 \text{ as another slow-roll parameter to carry with a noncommutative effect.}\]
\[ \varphi_k \simeq e^{i(\nu-\frac{1}{2})}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\sqrt{2k(-k\bar{\eta})^{\nu}}. \] (26)

Then the noncommutative power spectrum is defined by

\[ P_{RV}^U(k) = \left( \frac{k^3}{2\pi^2} \right) \lim_{-k\eta \to 0} \frac{|\varphi_k|^2}{z_k}. \] (27)

One finds a scale-dependent power spectrum

\[ P_{RV}^U(k) = \frac{H^4}{(2\pi)^2} \left\{ 1 - 4\mu_0 - 2\epsilon_1 + 2 \left( \alpha - \ln \left( \frac{k}{aH} \right) \right) (2\epsilon_1 + \delta_1) \right\}. \] (28)

which leads to by making use of the Taylor expansions

\[ P_{RV,1st}^U(k) = \frac{H^4}{(2\pi)^2} \left\{ 1 - 4\mu_0 - 2\epsilon_1 + 2 \left( \alpha - \ln \left( \frac{k}{aH} \right) \right) (2\epsilon_1 + \delta_1) \right\} \bigg|_{k=k_*}. \] (29)

up to first order in slow-roll parameters. Here \( \alpha = 2 - \ln 2 - \bar{\gamma} = 0.729637 \). In the limit of \( \mu_0 \to 0 \), \( P_{RV,1st}^U(k) \) reduces to the commutative power spectrum [21,22], while in the extreme slow-roll limit of \( \epsilon_1, \delta_1 \to 0 \), one finds the de Sitter result including \( \mu_0 \)[10]. In the noncommutative approach the horizon crossing occurs at \( k^2 = \frac{1}{2aH} \eta \frac{d^2z}{d\eta^2} \)[3]. Hence, from Eq.(22) we use the pivot scale \( k_* = \sqrt{2aH} \) instead of a pivot scale of \( k_* = aH \) for a commutative inflation. Finally, we obtain the noncommutative power spectrum up to first order as

\[ P_{RV,1st}^U(k) = \frac{H^4}{(2\pi)^2} \left\{ 1 - 4\epsilon_1 - 4\mu_0 + 2\alpha_*(2\epsilon_1 + \delta_1) \right\} \bigg|_{k=k_*}. \] (30)

with \( \alpha_* = \alpha - \ln 2/2 \). Let us compare Eq.(30) with the commutative power spectrum. A change of the pivot scale from \( k_* = aH \) to \( k_* = \sqrt{2aH} \) amounts to replacing \( \alpha = 0.7296 \) by \( \alpha_* = 0.3831 \) in the first-order calculation [22]. Thus the SSUR imprints on cosmological parameters by means of \( \alpha \to \alpha_* \) and \( \mu_0 \neq 0 \)[11,12].

In order to calculate the power spectrum even for first-order correctly, one has to use the slow-roll expansion based on Green’s function technique [23–25]. The key step in the slow-roll expansion is to use Eq.(21) in deriving the power spectrum. In the case of \( \mu_0=0 \), the slow-roll approximation and slow-roll expansion give the same power spectrum up to first order. However, in the case of \( \mu_0 \neq 0 \), two provide different results. The details appeared in ref. [11,12]. The slow-roll approximation is not generally suitable for the noncommutative case. From now on we obtain cosmological parameters using the slow-roll expansion. Also we wish to compare the noncommutative cosmological parameters with the WMAP data.

As an example, we choose the power-law inflation like \( a(t) = a_0 t^p \), \( H = p/t \), \( z = a\sqrt{2/p}M_P \) whose potential is given by

\[ V(\phi) = V_0 \exp \left( -\sqrt{\frac{2}{p}} \frac{\phi}{M_P} \right). \] (31)
Then slow-roll parameters are given by
\[ \mu_0(k) \equiv \left( \frac{k_c}{k} \right)^{\frac{4}{p}} \simeq 2 \left( \frac{H}{M_s} \right)^4 \equiv \mu_s(t), \quad \epsilon_1 = \frac{1}{p}, \quad \epsilon_1 = -\frac{1}{p}, \quad \delta_1 = \frac{2}{p^2}, \quad \delta_3 = -\frac{6}{p^2}, \quad \delta_4 = \frac{24}{p^3} \quad (32) \]
where \( \mu_0(k) \) is given by a solution to \( d\mu_0/d\ln k = -(4/p)\mu_0 \). Also \( \mu_s(t) \) satisfies Eq. (21). A UV critical scale \( k_c \) is given by \( k_c \approx \frac{2}{p/4}k_s \) approximately. A UV region of \( \mu_0 \) means \( \mu_s < 1(H < M_s) \). In the UV region, we calculate power spectrum using \( H(\tilde{\eta}_s) \) which is a solution to \( \mu_0(k = k_s) = \mu_s(\tilde{\eta} = \tilde{\eta}_s) \). Then the noncommutative power spectrum takes the form in the slow-roll expansion
\[ \tilde{P}_{r,1}^{UV}(k) = P_{r,1}^{c}(k) + \mu_0(k)H^2 \left\{ \frac{1}{(2\pi z)^2} \left\{ -4 + \frac{12(1 - 2\alpha_s)}{p} \right\} \right\}_{k=k_s}, \quad (33) \]
where the commutative spectrum is given by
\[ \tilde{P}_{r,1}^{c}(k) = \frac{H^2}{(2\pi z)^2} \left\{ 1 + \frac{2(\alpha_s - 1)}{p} \right\}_{k=k_s}. \quad (34) \]
Comparing with Eq. (30), the last term in Eq. (33) is new. The noncommutative spectral index can be easily calculated up to second-order
\[ n_s^{UV}(k) = n_s^{c}(k) + \mu_0(k) \left\{ \frac{16}{p} + \frac{64\alpha_s}{p^2} \right\}_{k=k_s} \quad (35) \]
with the commutative contribution
\[ n_s^{c}(k) = 1 - \frac{2}{p} - \frac{2}{p^2}. \quad (36) \]
Here one finds the last term in Eq. (35) from the slow-roll expansion. Finally the running spectral index is found to be
\[ \frac{dn_s^{UV}}{d\ln k} = \frac{dn_s^{c}}{d\ln k} - \mu_0(k) \left\{ \frac{64}{p^2} + \frac{8(32\alpha_s + 8)}{p^2} \right\}_{k=k_s}, \quad (37) \]
but the commutative contribution is zero up to third-order,
\[ \frac{dn_s^{c}}{d\ln k} = 0. \quad (38) \]
The last term in Eq. (37) comes from the slow-roll expansion.

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2From Eq. (18), we have \( \tau \simeq t(kl_s)^{\frac{p+1}{p}} \) in the UV region of \( k > k_s \). Then we obtain a relation \( k \simeq \left( \frac{p}{M_s H(p+1)} \right)^{\frac{1}{p+1}} \). With \( k_c = \left( \frac{p(p+1)^{\frac{1}{p}}}{M_s (p+1)^2} \right)^{\frac{1}{p+1}} k_s \), one finds \( (k_c/k)^p \simeq 2(H/M_s)^4 \) for a large \( p \) which satisfies \( p \pm 1 \simeq p \).
We obtain a UV critical scale $k_c = 0.998 \times 10^{-5}$Mpc$^{-1}$ and a IR critical scale $k_s = 1.05 \times 10^{-6}$Mpc$^{-1}$ by choosing $l_s = 3.49 \times 10^{-29}$cm, $l = 1.19 \times 10^{-24}$cm, and $p = 13^3$ [9,10]. For simplicity we choose $k_c = 10^{-5}$Mpc$^{-1}$ and $k_s = 10^{-6}$Mpc$^{-1}$. In this case the critical scale $k_t$ is chosen by $k_t = 10^{-3}$Mpc$^{-1}$ which is slightly larger than $H_0 = 4.6 \times 10^{-4}$Mpc$^{-1}$. The relevant pivot scales in the UV region should satisfy $k_s \gg k_c$ and $k_s > k_t$. Hence we choose $k_s = 0.05$Mpc$^{-1}$ for a small length scale and $k_s = 0.002$Mpc$^{-1}$ for a large length scale to compare with the WMAP data. In this case $\mu_0 = 0.07275$ at $k_s = 0.05$Mpc$^{-1}$ and $\mu_0 = 0.19588$ at $k_s = 0.002$Mpc$^{-1}$. We trust more the data at $k_s = 0.05$Mpc$^{-1}$ than that at $k_s = 0.002$Mpc$^{-1}$ because of $\mu_0$ as a slow-roll parameter should be comparable with $\epsilon_1 < 0.08$ in the UV region [10]. Here our power spectrum normalization $A$ at $k_s = 0.05$Mpc$^{-1}$ is defined by $P^{UV}_{R} = \left(\frac{\alpha H}{2\pi^2}\right)^2 \times A = 1.69 \times 10^{-9} \times A$ with $A = 0.629$, while the WMAP provides $P_{R} = 2.95 \times 10^{-9} \times A$ with $A = 0.833^{+0.086}_{-0.083}$ [4]. Approximately, there exists a difference of “2” in power spectrum normalization in 10$^{-9}$ order. For reference, we have $P^{UV}_{R} = 4.55 \times 10^{-10}$ at $k_s = 0.002$Mpc$^{-1}$. In Table I, we show noncommutative spectral index at two different scales up to first-order and second-order. Noncommutative running spectral index at two different scales up to second-order and third-order appear in Table II. A negatively large running spectral index could be obtained even in the UV approach.

### IV. IR COSMOLOGICAL PARAMETERS

In the IR region, the situation is quite different from the UV case [3,8,7,26]. The perturbed modes are generated outside the Hubble horizon. Their magnitude depends on the time when they are generated because they are frozen as soon as they are generated. When the SSUR is saturated, this corresponds to the time $\tilde{\eta} = \tilde{\eta}_0$ ($\tau = \tau_0$). Actually a saturation time $\tilde{\eta}_0$ in the IR region plays a similar role of a pivot scale $\tilde{\eta}_s$ in the UV region. Further,

\footnote{In this work, we consider only an integer $p$. The two scales depend on $p$ critically. For $p = 12$, we have large energy scales of $k_c = 0.28$Mpc$^{-1}$, $k_s = 0.035$Mpc$^{-1}$, whereas for $p = 14$, we have small energy scales of $k_c = 3.48 \times 10^{-10}$Mpc$^{-1}$, $k_s = 0.28 = 3.07 \times 10^{-11}$Mpc$^{-1}$. All of these are not suitable for describing a noncommutative inflation.}
the perturbed modes start out with their vacuum amplitude. In the case of \( \tau \simeq \Delta \tau \) for \( k < k_s \), we have

\[
H = \frac{p}{p + 1} \left( \frac{1}{\tau} \right)^{p+1} \simeq \frac{p}{p + 1} \left( \frac{1}{l_s^2 k} \right)^{p+1}.
\]  

(39)

One finds for \( k < k_s \), \( p + 1 \simeq p \)

\[
\gamma_0(k) \equiv \left( \frac{k}{k_s} \right)^{p2} = \left( \frac{p}{p + 1} \right)^{\frac{2(\epsilon + 1)}{p}} \left( \frac{M_s}{H} \right)^{\frac{2(\epsilon + 1)}{p}} \simeq \left( \frac{M_s}{H} \right)^2 \equiv \tilde{\gamma}_0(t).
\]  

(40)

where a newly noncommutative zeroth-order parameter \( \gamma_0(k)(\tilde{\gamma}_0(t)) \) is suitable for describing the IR region of either \( \gamma_0(k) < 1(k < k_s) \) or \( \tilde{\gamma}_0 = \sqrt{2/\mu_s} < 1(H > M_s) \). We use a relation of \( \gamma_0(k = k_0) = \tilde{\gamma}_0 \tau = \tau_0 \) to obtain \( H(\tau_0) \). Then Eq.(18) leads to

\[
\tau = \Delta \tau \left[ 1 + \gamma_0 \right]^{\frac{1}{2}}
\]  

(41)

where at the IR end of \( \gamma_0 \rightarrow 0 \), one has \( \tau_0 \rightarrow \Delta \tau \), as was shown previously. It seems that one is not easy to achieve this limit with \( k_s = 10^{-6}\text{Mpc}^{-1} \) because \( \gamma_0 = 0.70 \) at \( k = 10^{-1}k_s \), \( \gamma_0 = 0.34 \) at \( k = 10^{-3}k_s \), \( \gamma_0 = 0.17 \) at \( k = 10^{-5}k_s \), and \( \gamma_0 = 0.08 \) at \( k = 10^{-7}k_s \). However, the last corresponds to a length scale of \( l = 10^{-20}\text{cm} \) slightly smaller than the string scale \( l_s = 3.49 \times 10^{-20}\text{cm} \). We define the IR region to be \( \gamma_0 \leq 0.12 \) which is equivalent to \( k \leq 10^{-6}k_s \) for clarity. In the IR region we have \( k_0 = \tau_0 M_s^2 = a H(M_s/H)^2 = a H \tilde{\gamma}_0 = k_s \tilde{\gamma}_0 / \sqrt{2} < k_s \), which is smaller than the corresponding UV pivot scale \( k_s = \sqrt{2aH} \). In the IR region, we have \( a = \alpha_0 \tau^{-\frac{1}{p+1}} \simeq H \tau = H \Delta \tau \sqrt{1 + \tilde{\gamma}_0 / 2} \) with \( H \simeq \alpha_0 \tau^{-\frac{1}{p+1}} \). Furthermore, we have \( a_\pm \simeq H \Delta \tau (\sqrt{1 + \gamma_0} \pm 1) \). From \( \tilde{\gamma}_0(t) = (M_s/H)^2 \) for \( H < M_s \), we find a relation

\[
\tilde{\gamma}_0 = 2H\tilde{\gamma}_0 \epsilon_1.
\]  

(42)

Similarly from \( \gamma_0(k) = (k/k_s)^{2/p} \) for \( k < k_s \), one finds \( d\gamma_0/d\ln k = (2/p)\gamma_0 \). These show that \( \gamma_0 \) could be treated as a zeroth-order parameter to describe the IR region. Hereafter in order to evaluate all of cosmological parameters, we have to use a pivot scale \( k = k_0 \) for IR instead of a pivot scale \( k = k_s \) for UV in the previous section. Then the noncommutative power spectrum from Eqs.(27) and (26) takes the form \[18]\]

\[
P^{IR}_s(k) = \frac{k^3}{2\pi^2} \frac{1}{k\eta} \frac{1}{z^2 y_k^2} \left( \frac{aH}{2\pi z} \right)^2 \frac{\gamma_0^3}{(1 + \gamma_0)^2(2 + \gamma_0)} \simeq \frac{H^4}{(2\pi)^2 \phi^2} \left[ \frac{\gamma_0^3(k)}{2} \right]_{k = k_0}
\]  

(43)

with \( \tilde{\eta} = -(1/aH)(a/a_\text{eff})^2 \). Here we find that the IR power spectrum is the product of the commutative contribution by \( \gamma_0^3/3 \). The latter can be interpreted as a normalization. For \( \gamma_0 = 0.12(k_0 = 10^{-6}k_s) \) and \( p = 13 \), we have \( (aH/2\pi z)^2 = 7.39 \times 10^{-8} \) and \( \gamma_0^3/2 = 8.6 \times 10^{-4} \). Then we have \( P^{IR}_s = 6.3 \times 10^{-11} \) at \( k_0 = 10^{-12}\text{Mpc}^{-1} \) which is smaller than \( P^{UV}_s = 1.198 \times 10^{-9} \) at \( k_s = 0.05\text{Mpc}^{-1} \). The noncommutative spectral index can be easily calculated as

\[
n_s^{IR}(k) = n_s^c(k) + \left[ \frac{4(3 + 2\gamma_0)}{(2 + \gamma_0)(1 + \gamma_0)} \right]_{k = k_0}
\]  

(44)
TABLE III. Comparison between UV and IR for power-law inflation $a(t) \equiv a_0 t^p = [t/l(p+1)]^p$ for a large $p$ such that $p \pm 1 \simeq p$. Here we distinguish between a length scale $l$ and string scale $l_s = 1/M_s$.

| region | SR parameter | critical scale | pivot scale | condition |
|--------|--------------|----------------|-------------|-----------|
| UV     | $\mu_0 = (k_c/k)^{4/p} \simeq 2(H/M_s)^{4} = \mu_*$ | $k_c = 2^{p/4}k_s$ | $k_0 = \sqrt{2aH}$ | $k_c < k, \ H < M_s$ |
| IR     | $\gamma_0 = (k/k_s)^{2/p} \simeq (M_s/H)^2 = \gamma_0$ | $k_s = l_s^{p-1}/p$ | $k_0 = aH\gamma_0$ | $k_s > k, \ H > M_s$ |

with the commutative contribution

$$n^s_c(k) = 1 - 4\epsilon_1 - 2\delta_1 = 1 - \frac{2}{p}. \quad (45)$$

Also the last term in Eq.(44) could be approximated as $(6 - 5\gamma_0)\epsilon_1$. From the above one finds

$$n^s_{IR}(k) \simeq 1 + \frac{4}{p} \left(1 - \frac{5}{4} \frac{\gamma_0(k)}{k} \right) \bigg|_{k=k_0}, \quad (46)$$

where we find a blue spectral index of $n_s > 1$ for $\gamma_0 < 4/5$ in the IR region. For $\gamma_0 = 0.12$ and $p = 13$, this leads to a blue spectral index of $n^s_{IR} = 1.26$.

The IR running spectral index is found to be

$$\frac{dn^s_{IR}}{d\ln k} = \frac{dn^s_c}{d\ln k} + (6 - 5\gamma_0)\epsilon_1[(2 - \bar{\sigma})\epsilon_1 + 2\delta_1] \bigg|_{k=k_0} \quad (47)$$

with

$$\bar{\sigma} = \frac{2\gamma_0(5 + 6\gamma_0 + 2\gamma_0^2)}{(1 + \gamma_0)(2 + \gamma_0)(3 + 3\gamma_0)} \simeq \frac{5}{3} \gamma_0. \quad (48)$$

Here the commutative contribution is zero up to third order,

$$\frac{dn^s_c}{d\ln k} = 0. \quad (49)$$

Finally we have

$$\frac{dn^s_{IR}}{d\ln k} \simeq \left(-10 + \frac{25}{3} \frac{\gamma_0(k)}{p^2} \right) \frac{\gamma_0(k)}{p^2} \bigg|_{k=k_0}. \quad (50)$$

For $\gamma_0 < 1$ IR region, we find a negative running spectral index. For $\gamma_0 = 0.12$ and $p = 13$, this leads to a negatively small running spectral index of $\frac{dn^s_{IR}}{d\ln k} = -0.0064$.

V. DISCUSSION

We summarize important information about UV and IR cases in Table III. In the UV region with the power-law inflation, we mainly use a noncommutative parameter $\mu_0(k)$ to
compute cosmological parameters at the pivot scale \( k = k_\ast = \sqrt{2aH} > k_c \) with a UV critical scale \( k_c \). \( \mu_s(t) \) is employed only to calculate the UV power spectrum of \( (aH/2\pi)^2 \) at the horizon-crossing time \( \tilde{\eta} = \tilde{\eta}_s \) with \( H \). On the other hand, in the IR region, we mainly use a noncommutative parameter \( \gamma_0(k) \) to compute cosmological parameters at the pivot scale \( k = k_0 = \tau_0 M_\ast^2 = aM_\ast^2/H < k_\ast \) with a IR critical scale \( k_\ast \). \( \tilde{\gamma}_0(t) \) is employed only to calculate the IR power spectrum of \( (aH/2\pi)^2 \) at the saturation time \( \tilde{\eta} = \tilde{\eta}_0 \).

At the pivot scale \( k_\ast = 0.05\text{Mpc}^{-1} \), the UV power spectrum is given by \( P_R^{\text{UV}} = 1.198 \times 10^{-9} \) and \( P_R^{\text{WMAP}} \approx 2.46 \times 10^{-9} \) from the WMAP data, while \( P_R^{\text{UV}} = 4.55 \times 10^{-10} \) at \( k_\ast = 0.002\text{Mpc}^{-1} \). At the very small pivot scale \( k_0 = 10^{-12}\text{Mpc}^{-1} \), the IR power spectrum is \( P_R^{\text{IR}} = 6.3 \times 10^{-11} \). In general the power spectrum decreases when the energy scale \( k \) decreases. The discrepancy with the data is not so important because the normalization of the noncommutative power spectrum depends on the string length scale \( l_s \) and a scale \( l \) related to \( a_0 \). Concerning the second-order spectral index, we have a red one of \( n_s^{\text{UV}} = 0.935692 < 1 \) at \( k_\ast = 0.05\text{Mpc}^{-1} \) and a blue one \( n_s^{\text{UV}} = 1.10382 > 1 \) at \( k_\ast = 0.002\text{Mpc}^{-1} \). Fortunately these are close to the data of \( 0.93^{+0.03}_{-0.05} \) at \( k_\ast = 0.05\text{Mpc}^{-1} \) and \( 1.20^{+0.12}_{-0.11} \) at \( k_\ast = 0.002\text{Mpc}^{-1} \). At the very small pivot scale \( k_0 = 10^{-12}\text{Mpc}^{-1} \), the IR spectral index is largely blue \( (n_s^{\text{IR}} = 1.26) \). Here we find that the spectral index increases when the energy scale \( k \) decreases. For the third-order running spectral index, we have a negative one of \( dn_s^{\text{UV}}/d\ln k = -0.0329171 \) at \( k_\ast = 0.05\text{Mpc}^{-1} \) and a negative one \( dn_s^{\text{UV}}/d\ln k = -0.086296 \) at \( k_\ast = 0.002\text{Mpc}^{-1} \). These are close to the data of \( -0.031^{+0.016}_{-0.017} \) at \( k_\ast = 0.05\text{Mpc}^{-1} \) and \( -0.077^{+0.05}_{-0.052} \) at \( k_\ast = 0.002\text{Mpc}^{-1} \). But the IR spectral index is a negatively small quantity \( (dn_s^{\text{IR}}/d\ln k = -0.0064) \) at the very small pivot scale \( k_0 = 10^{-12}\text{Mpc}^{-1} \).

In the IR end \( (k \ll k_\ast) \) with a power-law inflation, one finds the scale-dependent power spectrum \( P(k) \approx k^{4/(p+1)} \) which implies a blue spectral index \( n_s = 1 + 4/(p+1) > 1 \) \([8,26]\). In the UV end \( (k \gg k_c) \) one finds \( P(k) \approx k^{-2/(p-1)} \) which implies a red spectral index \( n_s = 1 - 2/(p-1) < 1 \). In this work we find \( n_s^{UV} \rightarrow 1 - 2/p \) in the limit of \( \mu_0 \rightarrow 0 \) and \( n_s^{IR} \rightarrow 1 + 4/p \) in the limit of \( \gamma_0 \rightarrow 0 \). Considering a large \( p \) such as \( p \pm 1 \approx p \), the two results are nearly the same.

In conclusion, we show that the unfamiliar IR region is treated as in the familiar UV region by introducing a noncommutative parameter \( \gamma_0(\tilde{\gamma}_0) \). But at the scales of \( l_s \sim 10^{-29}\text{cm} \) and \( l \sim 10^{-24}\text{cm} \), the IR region of \( k < k_\ast = 1.05 \times 10^{-6}\text{Mpc}^{-1} \) is too small to cover the cosmologically relevant scales of \( 10^{-4}\text{Mpc}^{-1} < k < 10^{-1}\text{Mpc} \). In this case the UV region is relevant to fitting the data, instead of the IR region. The other choice of the string scale was introduced to show the relevance of the IR region \([8]\).

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