On a class of problems on interaction of stress concentrators of different types with an elastic semi-infinite plate

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Abstract. A class of mixed boundary-value problems of mathematical theory of elasticity dealing with interaction between stress concentrators of different types (such as cracks, absolutely rigid thin inclusions, punches, and stringers) and an elastic semi-infinite plate is considered. The method of Mellin integral transformation is used to reduce solving these problems to solving singular integral equations (SIE). After the governing SIE are solved, the following characteristics of the problem are determined: tangential contact stresses under stringers, dislocation density on the crack edges, breaking stresses outside the cracks on their line of location, the stress intensity factor (SIF), crack openings, jumps of contact stresses on the edges of inclusions.

1. Introduction
In many contact and mixed boundary-value problems of mechanics of deformable solids, the deformable foundations are often simulated as the classical foundation of half-plane type, for example, as an elastic half-plane or an elastic semi-infinite plate. In fracture mechanics, numerous problems on stress strain state of an elastic half-plane containing cracks of various geometric forms and in various combinations are investigated. In theoretical and practical aspects, the problems for an elastic half-plane containing a collinear system of cracks and a vertical edge crack are especially important [1–4]. In [5], a more general problem where a piecewise homogeneous elastic half-plane contains a crack, issuing perpendicularly to the direct section of the material, is considered. In [1, 2, 4, 5], the Wiener–Hopf method is used to construct a closed solution of the problem and to obtain exact explicit expressions of SIF in the cases of constant or polynomial loadings acting on the crack edges. The main results of studying this range of problems are presented sufficiently fully in monographs [6, 7] and in handbooks [8, 9].

The problems of interaction of absolutely rigid thin inclusions with massive deformable bodies of different shapes were considered in [10, 11]. In [12, 13], the crack development near the apex of a rigid inclusion is studied and the SIFs near the crack on the continuation of a linear rigid inclusion are determined.

The problems on stress state of an elastic half-plane, strip or half-plane reinforced by stringers and weakened by cracks were considered in [14–18].

In the present paper, a class of mixed boundary-value problems of elasticity dealing with interaction of stress concentrators (such as cracks, absolutely thin inclusions, punches, and stringers) with an elastic semi-infinite plate is considered. The following problems are considered:
(i) A problem on stress strain state of an elastic semi-infinite plate occupying the lower half-plane in the right rectangular system of coordinates when it contains arbitrarily many cracks on the vertical axis. It is assumed that the crack edges and the plate on its boundary are loaded by arbitrary asymmetric non-self-balanced distributed normal and tangential forces. The formulation of the problem is here somewhat wider than that in [3, 4, 7]. On the basis of the first boundary-value problem for the quarter-plane wedge with a right angle of opening, constructed by using the Mellin integral transformation, solving the given problem is reduced to solving the SIE for the unknown complex combination of dislocation densities on the crack edges. The governing SIE is solved by the well-known numeral-analytical method [7, 19–20] after which the following basic characteristics of the problem are determined: the dislocation density on the crack edges, the crack openings, the breaking stresses outside the cracks system on their lines of location, and SIF. The following important particular cases of the posed problem are discussed.

(a) The plate contains only one inner crack located on the vertical axis at a distance from its boundary. It is noted that, in this case, the effective solution to the governing SIE of the problem can be obtained by using the Lagrange interpolation polynomial at Chebyshev nodes [21] or the Multhopp method [22]. It is also shown that when the upper tip of the crack approaches the plate boundary, the SIF at this tip increases infinitely taking also the value of the brittle fracture limit for the given material.

(b) As a result, it is necessary to consider another important specific case where the crack vertically approaches the plate boundary. In this case, it is shown that the dislocation density at that upper tip of the crack has a finite value, and a simple dependence is established between this value and the load normal to the crack edges at the same point. Finally, the governing SIE is solved in the class of functions unbounded at the inner tip of the crack and bounded at its upper tip on the boundary.

(c) The problem where the plate boundary is reinforced by an absolutely rigid broken infinite stringer and the crack vertically comes to the plate boundary at the point of the stringer fracture. In this case, it is shown that the dislocation density at the upper boundary point of the crack has a logarithmic singularity if, at the same point, the forces normal to the crack edges are different from zero.

(d) Problem (b) where the crack edges near its inner tip are reinforced by a thin inclusion, which is simulated by continuously distributed linearly elastic springs [23]. The presence of such an inclusion decreases SIF and, therefore, prevents the crack from propagation.

(ii) Problem (i) where the system of cracks is replaced by a system of absolutely rigid thin inclusions. Particular cases, as well as the case of one inclusion approaching the plate boundary (reinforcement problem), are discussed.

(iii) The symmetric problem on stress state of an elastic infinite plate or a half-space under a plane deformation when, on the plate boundary, there is a system of arbitrarily many indented punches and the plate contains a system of cracks on the vertical axis.

(iv) The previous problem when the system of cracks is replaced by a system of absolutely rigid thin inclusions.

The results of investigations of the last two problems can used in calculations of the foundations. They are generalizations of classical contact problems of the theory of elasticity. All problems are mathematically formulated as governing SIE and are solved exactly or by the numerical-analytical method mentioned above.
2. Solution of the first boundary-value problem for an elastic quarter-plane

First, we write the basic equations of elasticity in the case of plane deformation in cylindrical coordinates \( r, \vartheta, z \) under the assumption that the polar axis \( O\vartheta \) coincides with the positive axis \( O\varpi \) and is directed vertically down and the axis \( Oy \) is horizontally directed to the right. In this system of coordinates, the Lame equations in the absence of volume forces have the form [24]

\[
(\lambda + 2\mu) \frac{\partial \epsilon}{\partial r} - \frac{\mu}{r^2} \frac{\partial^2 \vartheta}{\partial \vartheta^2} = 0, \quad (\lambda + 2\mu) \frac{1}{r} \frac{\partial \vartheta}{\partial \vartheta} + \frac{\mu}{r} \frac{\partial}{\partial r} \left( \frac{\vartheta}{r} \right) = 0, \tag{1}
\]

\[
e = \frac{\partial u_r}{\partial r} + \frac{1}{r} \left( u_r + \frac{\partial u_\vartheta}{\partial \vartheta} \right), \quad \chi = r \frac{\partial u_\vartheta}{\partial r} + u_\vartheta - \frac{\partial u_r}{\partial \vartheta} = 2r \omega_z,
\]

and Hook’s law becomes

\[
\sigma_r = 2\mu \frac{\partial u_r}{\partial r} + \lambda \epsilon, \quad \sigma_\vartheta = 2\mu \left( \frac{\partial u_\vartheta}{\partial \vartheta} + u_r \right) + \lambda \epsilon, \quad \tau_{r\vartheta} = \mu \left( \frac{\partial u_\vartheta}{\partial r} - \frac{u_\vartheta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \vartheta} \right). \tag{2}
\]

Here \( \sigma_r, \sigma_\vartheta, \tau_{r\vartheta} \) are the stress components, \( u_r \) and \( u_\vartheta \) are correspondingly the components of point displacements of an elastic body in the radial and circular directions, \( \epsilon \) is the relative volume expansion, \( \omega_z \) is the angle of rotation about the axis \( O\varpi \) perpendicular to the elastic plane, and \( \lambda \) and \( \mu \) are Lame constants of the elastic body expressed in terms of the Young module \( E \) and Poisson ratios by the formulas

\[
\lambda = \frac{E \nu}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)}.
\]

Now for the right elastic quarter-plane wedge \( \Omega_+ = \{0 < r < \infty, \ 0 < \vartheta < \pi/2 \} \) with a right angle of opening, we consider the first boundary-value problem

\[
\sigma_{\vartheta} \big|_{\vartheta=0} = -q_{0+}^+(r), \quad \tau_{r\vartheta} \big|_{\vartheta=0} = -\tau_{0+}^+(r), \\
\sigma_{\vartheta} \big|_{\vartheta=\pi/2} = q^+(r), \quad \tau_{r\vartheta} \big|_{\vartheta=\pi/2} = \tau^+(r) \quad (0 < r < \infty), \tag{3}
\]

where \( q_{0+}^+(r), q^+(r), \tau_{0+}^+(r), \tau^+(r) \) are functions of the load given on the wedge faces \( \Omega_+ \). They are considered to be a prespecified integrable function.

The solution of the boundary-value problem (1)–(3) will be constructed by the method of Mellin integral transformation for which we introduce the Mellin transforms:

\[
\{ \bar{\sigma}_r(\vartheta, p); \bar{\sigma}_\vartheta(\vartheta, p); \bar{\tau}_{r\vartheta}(\vartheta, p); \bar{\epsilon}(\vartheta, p) \} = \int_0^\infty \{ \sigma_r(r, \vartheta); \sigma_\vartheta(r, \vartheta); \tau_{r\vartheta}(r, \vartheta); \epsilon(r, \vartheta) \} r^p dr,
\]

\[
\{ \bar{u}_r(\vartheta, p); \bar{u}_\vartheta(\vartheta, p); \bar{\chi}(\vartheta, p) \} = \int_0^\infty \{ u_r(r, \vartheta); u_\vartheta(r, \vartheta); \chi(r, \vartheta) \} r^{p-1} dr
\]

\((-1 + \varepsilon < \text{Re} p < 0, \ r > 0). \tag{4}
\]

Applying the Mellin transformation to equations (1), after simple manipulations, we obtain the general solution of these equations in Mellin transforms represented by the formulas

\[
\bar{\epsilon}_+ = A_1^+ \sin(p + 1)\vartheta + A_2^+ \cos(p + 1)\vartheta \quad (0 \leq \vartheta \leq \frac{\pi}{2}), \quad A_1^+ = \frac{1}{4\mu p}[(\lambda + \mu)(p + 1) + 2\mu], \quad A_2^+ = \frac{1}{4\mu p}[(\lambda + \mu)(p - 1) - 2\mu],
\]

\[
\bar{u}_r = -A^* [A_1^+ \sin(p + 1)\vartheta + A_2^+ \cos(p + 1)\vartheta] + A_3^+ \sin(p - 1)\vartheta + A_4^+ \cos(p - 1)\vartheta, \quad A^* = \frac{1}{4\mu p}[(\lambda + \mu)(p + 1) - 2\mu],
\]

\[
\bar{u}_\vartheta = B^* [A_1^+ \cos(p + 1)\vartheta - A_2^+ \sin(p + 1)\vartheta] + A_3^+ \sin(p - 1)\vartheta - A_4^+ \cos(p + 1)\vartheta, \tag{5}
\]

\[
(A^*)^2 + (B^*)^2 = 1.
\]
Further, the Mellin transformation is also applied to Hook’s law (2) and boundary conditions (3). From them, with the help of (5), we determine the constants \( A^+_j \) (\( j = 1, 4 \)):

\[
A^+_1 = \frac{1}{(\lambda + \mu)D(p)} \left\{ 2p - 1 + \cos(\pi p)\bar{\tau}^+_0(p) - \sin(\pi p)\bar{q}^+_0(p) \right. \\
- 2p \cos\left( \frac{\pi p}{2} \right) \bar{q}^+(p) + 2(p - 1) \sin\left( \frac{\pi p}{2} \right) \bar{\tau}^+(p) \right\};
\]

\[
A^+_2 = -\frac{1}{(\lambda + \mu)D(p)} \left\{ \sin(\pi p)\bar{\tau}^+_0(p) - [2p + 1 - \cos(\pi p)]\bar{q}^+_0(p) \right. \\
- 2p \cos\left( \frac{\pi p}{2} \right) \bar{q}^+(p) - 2(p + 1) \sin\left( \frac{\pi p}{2} \right) \bar{\tau}^+(p) \right\}, \quad D(p) = 2p^2 - 1 + \cos(\pi p),
\]

\[
A^+_3 = -\frac{1}{4\mu p D(p)} \left\{ (p - 1)[2p + 1 - \cos(\pi p)]\bar{\tau}^+_0(p) \right. \\
+ (p + 1) \left[ \sin(\pi p)\bar{q}^+_0(p) - 2(p - 1) \sin\left( \frac{\pi p}{2} \right) \bar{\tau}^+(p) + 2p \cos\left( \frac{\pi p}{2} \right) \bar{q}^+(p) \right\],
\]

\[
A^+_4 = -\frac{1}{4\mu p D(p)} \left\{ (p + 1)[2p - 1 + \cos(\pi p)]\bar{q}^+_0(p) \right. \\
+ (p - 1) \left[ \sin(\pi p)\bar{\tau}^+_0(p) - 2p \cos\left( \frac{\pi p}{2} \right) \bar{q}^+(p) - 2(p + 1) \sin\left( \frac{\pi p}{2} \right) \bar{\tau}^+(p) \right\}.
\]

Here, according to (4), \( \bar{\tau}^+_0(p), \bar{q}^+_0(p), \bar{\tau}^+(p), \) and \( \bar{q}^+(p) \) are Mellin transformants of the corresponding functions from boundary conditions (3).

Substituting the constants (6) into (5), we finally obtain \( \bar{u}_r(\vartheta, p) \) and \( \bar{u}_\vartheta(\vartheta, p) \). In particular, we have

\[
\bar{u}^+_r(0, p) = \frac{1}{2\mu(\lambda + \mu)pD(p)} \left\{ (\lambda + 2\mu)\sin(\pi p)\bar{\tau}^+_0(p) - (2\lambda + \mu)p(p + 1) \right. \\
+ \mu[2p + 1 - \cos(\pi p)]\bar{\tau}^+_0(p) - 2(\lambda + 2\mu)p \cos\left( \frac{\pi p}{2} \right) \bar{\tau}^+(p) - 2(\lambda + 2\mu)(p + 1) \sin\left( \frac{\pi p}{2} \right) \bar{q}^+(p) \right\};
\]

\[
\bar{u}^+_\vartheta(0, p) = \frac{1}{2\mu(\lambda + \mu)pD(p)} \left\{ [2(\lambda + \mu)p(p - 1) - \mu[2p - 1 + \cos(\pi p)]]\bar{\tau}^+_0(p) \right. \\
+ (\lambda + 2\mu) \left[ \sin(\pi p)\bar{q}^+_0(p) - 2(p - 1) \sin\left( \frac{\pi p}{2} \right) \bar{\tau}^+(p) + 2p \cos\left( \frac{\pi p}{2} \right) \bar{q}^+(p) \right\}.
\]

The first boundary-value problem will also be considered for the left elastic quarter-plane, i.e. for the wedge \( \Omega_- = \{ 0 < r < \infty, -\pi/2 < \vartheta < 0 \} \), assuming that

\[
\sigma^+_{\vartheta|\vartheta = -\pi/2} = -q^+(r), \quad \tau^+_{r\vartheta|\vartheta = -\pi/2} = -\tau^+_0(r),
\]

\[
\sigma^+_{\vartheta|\vartheta = -\pi/2+0} = -q^+(r), \quad \tau^+_{r\vartheta|\vartheta = -\pi/2+0} = -\tau^+_0(r) \quad (0 < r < \infty).
\]

In this case, using the Mellin integral transformation, we again obtain equalities of the type of (5), where all “+” sign must be replaced by “−” signs. The constants \( A^-_j \) (\( j = 1, 4 \)) are determined by the above-mentioned method based on Hook’s law (2) and boundary conditions (8) in Mellin
i.e., according to (3) and (8), we have

\[
\bar{u}_r(0, p) = -\frac{1}{2\mu(\lambda + \mu)pD(p)} \left\{ 2(\lambda + \mu)p(p + 1) + \mu[2p + 1 - \cos(\pi p)] \right\} \dot{q}_0^-(p) \\
+ (\lambda + 2\mu) \left[ \sin(\pi p)\dot{\tau}_0^-(p) + 2p\cos\left(\frac{\pi p}{2}\right)\dot{\tau}^-(p) - 2(p + 1)\sin\left(\frac{\pi p}{2}\right)\dot{q}^-(p) \right],
\]

\[
\bar{u}_\theta(0, p) = -\frac{1}{2\mu(\lambda + \mu)pD(p)} \left\{ 2(\lambda + \mu)p(p - 1) - \mu[2p - 1 + \cos(\pi p)] \right\} \dot{\tau}_0^-(p) \\
- (\lambda + 2\mu) \left[ \sin(\pi p)\dot{q}_0^-(p) - 2(p - 1)\sin\left(\frac{\pi p}{2}\right)\dot{\tau}^-(p) - 2p\cos\left(\frac{\pi p}{2}\right)\dot{q}^-(p) \right],
\]

where the notation is similar to that introduced above.

3. Stress state of an elastic half-plane with a collinear system of vertical cracks

Let an elastic half-plane or an elastic semi-infinite plate \( x \geq 0 \), on its vertical axis \( Ox \) (or), contain a collinear system of cracks \( L = \bigcup_{k=1}^{n} [a_k, b_k] \) consisting of arbitrarily many cracks along the segments \([a_k, b_k] \) \((k = 1, n)\). Denote the set of the right and left edges of this system of cracks by \( L_\pm \), respectively. Later, adhering to the general case, let us consider that the edges of the system of cracks \( L_\pm \) are loaded by asymmetric non-self-balanced normal and tangential forces, i.e., according to (3) and (8), we have

\[
(\tau_{r\theta} + i\sigma_\theta)\big|_{L_\pm} = -[\dot{\tau}_0^+(r) + i\dot{q}_0^+(r)] \quad (r \in L).
\]

In addition, we assume that the boundary of the half-plane is also loaded by forces of this nature, i.e., according to (3) and (8), we have

\[
(\tau_{r\theta} + i\sigma_\theta)\big|_{\theta=\pm\pi/2} = \pm[\tau^\pm(r) + iq^\pm(r)] \quad (0 < r < \infty).
\]

It is required to determine the following characteristics of the problem: the dislocation density on the edges of the system of cracks \( L \), the breaking stresses outside the system of cracks on their line of location, SIF, and the crack openings.

For their determination, we derive the governing SIE of the problem. With this aim in mind, we first introduce the notation:

\[
\Phi_\pm(p) = -p\bar{u}_\pm^+(0, p), \quad \Psi_\pm(p) = -p\bar{u}_\pm^+(0, p), \quad X_\pm(p) = \frac{1}{2}[\dot{\tau}_0^+(p) \pm \dot{\tau}_0^-(p)],
\]

\[
\Omega_\pm(p) = \frac{1}{2}[\dot{q}_0^+(p) \pm \dot{q}_0^-(p)], \quad \varphi_\pm(p) = \frac{1}{2}[\dot{\Phi}_+(p) \pm \dot{\Phi}_-(p)], \quad \psi_\pm(p) = \frac{1}{2}[\dot{\Psi}_+(p) \pm \dot{\Psi}_-(p)].
\]

Then we have

\[
\Phi_\pm(p) = \varphi_\pm(p) \pm \psi_\pm(p), \quad \Psi_\pm(p) = \psi_\pm(p) \pm \varphi_\pm(p),
\]

\[
\dot{\tau}_0^\pm(p) = \dot{X}_\pm(p) \pm \dot{X}_\mp(p), \quad \dot{q}_0^\pm(p) = \dot{\Omega}_\pm(p) \pm \dot{\Omega}_\mp(p).
\]
Later, we write formulas (7) and (9) in the new notation as follows:

\[
\begin{align*}
\tilde{\varphi}_+(p) + \tilde{\varphi}_-(p) &= \frac{1}{\Delta(p)} \left\{ - (\lambda + 2\mu) \sin(\pi p) [\bar{X}_+(p) + \bar{X}_-(p)] \\
+ \{2[(\lambda + \mu)p(p + 1) + \mu[2p + 1 - \cos(\pi p)]\} [\bar{\Omega}_+(p) + \bar{\Omega}_-(p)] \\
+ 2(\lambda + 2\mu) p \cos \left( \frac{\pi p}{2} \right) \bar{q}^+(p) + 2(\lambda + 2\mu)(p + 1) \sin \left( \frac{\pi p}{2} \right) \bar{q}^+(p) \right\}, \\
\tilde{\psi}_+(p) + \tilde{\psi}_-(p) &= \frac{1}{\Delta(p)} \left\{ - (2(\lambda + \mu)p(p - 1) - \mu[2p - 1 + \cos(\pi p)] \right\} \bar{X}_+(p) + \bar{X}_-(p) \\
+ (1 + 2\mu) \left\{ \sin(\pi p)[\bar{\Omega}_+(p) + \bar{\Omega}_-(p)] - 2(p - 1) \sin \left( \frac{\pi p}{2} \right) \bar{q}^+(p) \right\}, \\
\varphi_+(p) - \varphi_-(p) &= \frac{1}{\Delta(p)} \left\{ (2(\lambda + \mu)p(p + 1) + \mu[2p + 1 - \cos(\pi p)] \right\} \bar{\Omega}_+(p) - \bar{\Omega}_-(p) \\
+ (\lambda + 2\mu) \left\{ \sin(\pi p)[\bar{X}_+(p) - \bar{X}_-(p)] + 2p \cos \left( \frac{\pi p}{2} \right) \bar{q}^-(p) - 2(p + 1) \sin \left( \frac{\pi p}{2} \right) \bar{q}^-(p) \right\}, \\
\psi_+(p) - \psi_-(p) &= \frac{1}{\Delta(p)} \left\{ (2(\lambda + \mu)p(p - 1) - \mu[2p - 1 + \cos(\pi p)] \right\} \bar{X}_+(p) - \bar{X}_-(p) \\
- (\lambda + 2\mu) \left\{ \sin(\pi p)[\bar{\Omega}_+(p) - \bar{\Omega}_-(p)] - 2(p - 1) \sin \left( \frac{\pi p}{2} \right) \bar{q}^-(p) - 2p \cos \left( \frac{\pi p}{2} \right) \bar{q}^-(p) \right\}.
\end{align*}
\]

Now from (12) we determine \(\tilde{\varphi}_+(p)\), from (13) we determine \(\tilde{\psi}_+(p)\), and substitute their expressions, correspondingly, in (11) and (10). After some calculations and transformations, we come to the equation

\[
\begin{align*}
\bar{X}_+(p) + i\bar{\Omega}_+(p) &= - \frac{2\mu(\lambda + \mu)[2p^2 - 1 + \cos(\pi p)]}{(\lambda + 2\mu) \sin(\pi p)} [\varphi_-(p) + i\psi_-(p)] + \frac{1}{(\lambda + 2\mu) \sin(\pi p)} \\
&\times \left\{ 2(\lambda + \mu)p^2 + \mu[1 - \cos(\pi p)] \right\} [\bar{\Omega}_-(p) - i\bar{X}_-(p)] + 2(\lambda + 2\mu)p [\bar{\Omega}_-(p) + i\bar{X}_-(p)] \\
&+ \frac{1}{\sin(\pi p)} \left\{ p \left[ \cos \left( \frac{\pi p}{2} \right) + i \sin \left( \frac{\pi p}{2} \right) \right] [\bar{q}^+(p) - i\bar{q}^-(p)] - p \left[ \cos \left( \frac{\pi p}{2} \right) - i \sin \left( \frac{\pi p}{2} \right) \right] \right\} \\
&\times [\bar{q}^-(p) - i\bar{q}^-(p)] - i \sin \left( \frac{\pi p}{2} \right) [\bar{q}^+(p) + i\bar{q}^+(p)] - i \sin \left( \frac{\pi p}{2} \right) [\bar{q}^-(p) + i\bar{q}^-(p)].
\end{align*}
\]

Equations (14) in the whole form the key equation of the posed problem for cracks in Mellin transformants. To write it in original form, we apply the inverse Mellin transform formula to (14). The line of integration over the variable \(p\) lies in the strip of regularity \(-1 + \varepsilon < \Re p < 0\). However, in our case, the strip of regularity of the Mellin transformation is actually wider than the above-indicated strip and includes the imaginary axis in the plane of the complex variable \(p = u + iv\), since the functions in (14) do not have poles on the imaginary axis and are analytic on it. Transferring the line of integration in the Mellin inverse transform to the imaginary axis and calculating the required integrals, we obtain the following key equation of
the problem under study:

\[
X_\pm(r) + i\Omega_\pm(r) = -\frac{16\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)} \int_L s^2 r[\varphi_-(s) + i\psi_-(s)] \, ds \\
+ \frac{2}{\pi} \int_L \left[ \frac{1}{s + r} - \frac{1}{\lambda + 2\mu s - r} \right] s\Omega_-(s) \, ds \\
+ \frac{2i}{\pi} \int_L \left[ \frac{1}{s + r} + \frac{1}{\lambda + 2\mu s - r} \right] sX_-(s) \, ds \\
+ \frac{2}{\pi} \int_0^\infty \frac{r - is}{(s^2 + r^2)^2} s^2 \tau^+(s) \, ds + \frac{2}{\pi} \int_0^\infty \frac{r + is}{(s^2 + r^2)^2} rsq^+(s) \, ds \\
- \frac{2}{\pi} \int_0^\infty \frac{r + is}{(s^2 + r^2)^2} s^2 \tau^-(s) \, ds + \frac{2}{\pi} \int_0^\infty \frac{r - is}{(s^2 + r^2)^2} rsq^-(s) \, ds \quad (0 < r < \infty). 
\]

Note that, according to the above notation,

\[
X_\pm(r) = \frac{1}{2}[\tau_0^+(r) \pm \tau_0^-(r)], \quad \Omega_\pm(r) = \frac{1}{2}[q_0^+(r) \pm q_0^-(r)], \\
\varphi_-(r) = \frac{1}{2} \frac{d}{dr}[u_0^+(r,0) - u_r^-(r,0)], \quad \psi_-(r) = \frac{1}{2} \frac{d}{dr}[u_0^+(r,0) - u_r^-(r,0)],
\]

and equation (15) was derived with the stresses and displacements continuity outside the system of cracks on \( L' = [0, \infty) \setminus L \) taken into account. Under these conditions, we have

\[
X_-(r) = \Omega_-(r) \equiv 0, \quad \varphi_-(r) = \psi_-(r) \equiv 0 \quad (r \in L').
\]

The functions \( \varphi_-(r) \) and \( \psi_-(r) \) are halves of the dislocation density components on the cracks edges.

Now, considering the key equation (15) on the system of cracks \( L \), we obtain the following governing SIE for the complex combination of the dislocation density components:

\[
\frac{2\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)} \int_L K(r, s)[\varphi_-(s) + i\psi_-(s)] \, ds = f(r) \quad (r \in L), 
\]

\[
K(r, s) = \frac{8s^2 r}{(s^2 - r^2)(s + r)^2} = \frac{1}{s - r} - \frac{1}{s + r} + \frac{6r}{(s + r)^2} - \frac{4r^2}{(s + r)^3}; \\
f(r) = -[X_+(r) + i\Omega_+(r)] + g(r),
\]

\[
g(r) = \frac{2}{\pi} \int_L \left[ \frac{1}{s + r} - \frac{1}{\lambda + 2\mu s - r} \right] s\Omega_-(s) \, ds \\
+ \frac{2i}{\pi} \int_L \left[ \frac{1}{s + r} + \frac{1}{\lambda + 2\mu s - r} \right] sX_-(s) \, ds \\
+ \frac{2}{\pi} \int_0^\infty \frac{r - is}{(s^2 + r^2)^2} s^2 \tau^+(s) \, ds + \frac{2}{\pi} \int_0^\infty \frac{r + is}{(s^2 + r^2)^2} rsq^+(s) \, ds \\
- \frac{2}{\pi} \int_0^\infty \frac{r + is}{(s^2 + r^2)^2} s^2 \tau^-(s) \, ds + \frac{2}{\pi} \int_0^\infty \frac{r - is}{(s^2 + r^2)^2} rsq^-(s) \, ds \quad (r \in L),
\]

\[
X_\pm(r) = \frac{1}{2}[	au_0^+(r) \pm \tau_0^-(r)], \quad \Omega_\pm(r) = \frac{1}{2}[q_0^+(r) \pm q_0^-(r)] \quad (r \in L).
\]

The solution of SIE (16) for \( a_1 \neq 0 \) must satisfy the conditions

\[
\int_{ak}^{bh} [\varphi_-(s) + i\psi_-(s)] \, ds = 0 \quad (k = 1, n),
\]

\[
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\]

\[
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\]
which express the continuity conditions for displacements at the end points (tips) of the cracks.

The solution of SIE (16)–(17) can be constructed by the well-known numerical-analytical method for solving SIE [7, 19, 20], where the Gauss quadrature formulas at Chebyshev nodes are used to calculate the integrals. To this end, we must first introduce dimensionless quantities in (16), (17) and transform each integral on \((a_k, b_k)\) in the system \(L\) into an integral on \((-1, 1)\). As a result, SIE (16), (17) is reduced to a system of linear algebraic equations.

However, if the key equation (16) is considered outside \(L\) on \(L' = [0, \infty) \setminus L\), then we obtain a complex combination of breaking normal and tangential forces out of \(L\) on \(L'\).

\[
X_+(r) + i\Omega_+(r) = \tau(r) + iq(r) = \frac{1}{2}[\tau_0^+(r) + \tau_0^-(r)] + \frac{i}{2}[q_0^+(r) + q_0^-(r)]
\]

\[
= -\frac{2\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)} \int_L K(r, s)[\varphi_-(s) + i\psi_-(s)] ds + g(r) \quad (r \in L' = [0, \infty) \setminus L),
\]

where we assumed that the solution of SIE (16)–(17) has already been found. From (18) and also from the expression for \(\varphi_-(r) + i\psi_-(r)\), we can determine SIF by the well-known formula \([8, 9]\) and find the crack openings.

### 3.1. Particular cases

Consider a particular case where there is one crack along the segment \([a, b]\). In this case, the Multhopp method [22] or the method based on application of Lagrange interpolation polynomials at Chebyshev nodes [21] can successfully be applied to solve SIE (16), (17). At the same time, it turns out that SIF infinitely increases at the endpoint (tip) \(r = a\) of the crack as \(a \to +0\) exceeding the brittle fracture limit for the given material. As a result, the crack vertically approaches the boundary of the elastic half-plane. To prevent the crack from propagation, it is possible to connect its edges together near the endpoint \(r = a\) by an elastic inclusion which is modelled as continuously distributed linear springs [23]. Let us consider the case of an edge crack \([0, b]\) in more detail. For simplicity, we here assume that the boundary of the elastic half-plane is free from external forces, i.e., \(\tau_0^+(r) = q_0^+(r) \equiv 0\) \((0 < r < \infty)\), and identical normal forces of intensity \(p(r)\) (opening displacement) act on the crack edges, i.e., \(\tau_0^+(r) = 0\), \(q_0^+(r) = q_0^-(r) = p(r)\) \((0 \leq r < b)\). Then SIE (16) has the form

\[
\frac{1}{\pi} \int_0^b K(r, s)\psi(s) ds = -\Lambda p(r) \quad (0 < r < b),
\]

\[
\psi(r) = 2\psi_- (r) = 2 \frac{d}{dr} [u_0^+ (r, 0)] = -2 \frac{d}{dr} u_0^- (r, 0),
\]

where

\[
\Lambda = \begin{cases} 
\frac{4(1 - \nu^2)}{E} & \text{under the plane deformation,} \\
\frac{4}{E} & \text{in generalized plane stress state.} 
\end{cases}
\]

Due to its theoretical and practical importance, the problem on the edge crack in an elastic half-plane, as was noted above, was discussed in many papers, and the Wiener–Hopf rigorous analytical method was applied to solve this problem. At the same time, for a detailed numerical analysis of the problem by simpler computational means, it is necessary to construct effective approximate but sufficiently accurate solutions of the problem. In [7], an approximate solution of the above-discussed problem is constructed by passing to the limit in the solution of a similar problem for the inner crack when the crack tip close to the boundary of the half-plane approaches it in the limit. But though very effective, such a solution does not sufficiently reflect the true behavior of the body stresses and deformations near the boundary point of the crack.
Here the solution of governing SIE (19) is constructed by the well-known numerical-analytical method for solving SIE [7, 19, 20]. With this aim in mind, we first note that, according to well-known results [25], the solution of SIE (19) is bounded at the point $\eta = 0$ not always, as was stated in [27].

Now, on the basis of (20), SIE (19) is transformed as

$$
\frac{1}{\pi} \int_{-1}^{1} \left[ \frac{1}{\eta - \xi} + K_0(\xi, \eta) \right] \chi_0(\eta) d\eta = f_0(\xi) \quad (-1 < \xi < 1),
$$

(22)

where $\chi_0(\xi) = \varphi_0(\xi) + \omega_0(\xi) \quad (-1 < \xi < 1),$

and $\omega_0(\xi)$ is the solution of SIE

$$
\frac{1}{\pi} \int_{-1}^{1} \frac{\omega_0(\eta)}{\eta - \xi} d\eta = \frac{1}{4} p_0(0) \ln(1 - \xi) \quad (-1 < \xi < 1).
$$

(23)

The solution of SIE (22), bounded at the point $\xi = -1$ and unbounded at the point $\xi = 1,$ is given by the formula [26]

$$
\omega_0(\xi) = -\frac{1}{4 \pi} p_0(0) \int_{-1}^{1} \sqrt{\frac{1 + \xi}{1 - \xi}} v_0(\xi) d\tau, \quad v_0(\xi) = \int_{-1}^{1} \sqrt{1 - \tau \ln(1 - \tau) \frac{d\tau}{\tau - \xi}}.
$$

(24)

Taking into account (23) and (24), we can rewrite SIE (22) in the form

$$
\int_{-1}^{1} \left[ \frac{1}{\eta - \xi} + K_0(\xi, \eta) \right] \varphi_0(\eta) d\eta = h_0(\xi) \quad (-1 < \xi < 1),
$$

(25)

where

$$
h_0(\xi) = h_1(\xi) + h_2(\xi), \quad h_1(\xi) = \frac{1}{4} p_0(0) \left[ \frac{4(\xi + 5)}{(\xi + 3)^2} - \ln(\xi + 3) \right] - p_0(\xi),
$$

$$
h_2(\xi) = \frac{1}{4 \pi^2} p_0(0) \int_{-1}^{1} K_0(\xi, \eta) \sqrt{\frac{1 + \eta}{1 - \eta}} v_0(\eta) d\eta, \quad v_0(\eta) = \int_{-1}^{1} \sqrt{\frac{1 - \tau \ln(1 - \tau)}{1 - \tau^2(\tau - \eta)}} d\tau.
$$
Now the solution of SIE (25) is given by the formula

\[ \varphi_0(\xi) = \sqrt{\frac{1 + \xi}{1 - \xi}} \Phi_0(\xi) \quad (-1 < \xi < 1), \tag{26} \]

where \( \Phi_0(\xi) \) is a Hölder function on the interval \(-1 \leq \xi \leq 1\). From (26), the Jacobi polynomials \( P_n^{(-1/2,1/2)}(\xi) \) with the weight function \( w(\xi) = \sqrt{(1 + \xi)/(1 - \xi)} \) are orthogonal on the interval \((-1,1)\) and besides [28]

\[ P_n^{(-1/2,1/2)}(\xi) = \frac{\Gamma(n + \frac{1}{2})}{\sqrt{n!}} \frac{T_{n+1}(\xi) + T_n(\xi)}{1 + \xi} \quad (-1 < \xi < 1, \ n = 0, 1, \ldots), \]

where \( \Gamma(x) \) is the Euler gamma function and \( T_n(x) \) are Chebyshev polynomials of the first kind. The integral relationship with the Cauchy kernel, necessary in this case to implement the procedure outlined in [7, 19, 20], has the form

\[ \frac{1}{\pi} \int_{-1}^{1} \frac{T_{n+1}(\eta) + T_n(\eta)}{1 + \eta} \frac{1}{1 - \eta} \sqrt{\frac{1 + \eta}{1 - \eta}} \, d\eta = U_n(\xi) + U_{n-1}(\xi) \]

\[ (-1 < \xi < 1, \ n = 0, 1, 2, \ldots, U_{-1}(\xi) \equiv 0), \]

where \( U_n(\xi) \) are Chebyshev polynomials of the second kind. This relationship immediately follows from the well-known relationship with the Cauchy kernel, which transforms Chebyshev polynomials of the first kind to Chebyshev polynomials of the second kind [28]. In the notation of [7], in our case, we have

\[ P_M(\xi) = \frac{T_{M+1}(\xi) + T_M(\xi)}{1 + \xi}, \quad Q_M(\xi) = -\frac{\pi}{2} [U_M(\xi) + U_{M-1}(\xi)], \]

where \( M \) is an arbitrary natural number. The roots of the equations \( P_M(\eta) = 0 \) and \( Q_M(\xi) = 0 \) are, respectively,

\[ \eta_m = \cos \left( \frac{2m - 1}{2M + 1} \pi \right), \quad \xi_r = \cos \left( \frac{2\pi r}{2M + 1} \right) \quad (m, r = \overline{1, M}). \]

Further, following the well-known procedure [7], solving SIE (25) is reduced to solving the final system of linear algebraic equations

\[ \frac{4}{2M + 1} \sum_{m=1}^{M} \cos^2 \left[ \frac{(2m - 1)\pi}{2(2M + 1)} \right] \left[ \frac{1}{\eta_m - \xi_r} + K_0(\eta_m, \xi_r) \right] \Phi_0(\eta_m) = h_0(\xi_r) \quad (r = \overline{1, M}) \tag{27} \]

for the unknowns \( \Phi_0(\eta_m) \). Note that the function \( v_0(\eta) \) in (25) is calculated at Chebyshev ordinary nodes by the formula

\[ v_0(\eta^*_r) = \frac{\pi}{M} \sum_{m=1}^{M} \frac{(1 - \tau_m) \ln(1 - \tau_m)}{\tau_m - \eta^*_r}, \quad (r = \overline{1, M - 1}), \tag{28} \]

\[ \tau_m = \cos \left[ \frac{(2m - 1)\pi}{2M} \right], \quad (m = \overline{1, M}), \quad \eta^*_r = \cos \left( \frac{\pi r}{M} \right), \quad (r = \overline{1, M - 1}). \]
The values \( v_0(\eta_m) \) in the expression for \( h_2(\xi) \) are calculated by the Lagrange interpolation polynomial at Chebyshev nodes using (28). For dimensionless SIF \( K_I^{(0)} \), we have
\[
K_I^{(0)} = \frac{\Lambda}{\sqrt{2\pi b}} K_I = -\left[ \Phi_0(1) - \frac{p_0(0)}{4\pi} \nu_0(1) \right].
\]
The crack opening and the breaking stresses outside the crack can also be calculated easily.

Let us consider another type of the edge crack in an elastic half-plane whose boundary is reinforced by an infinite broken stringer.

Let an elastic half-plane or an elastic semi-infinite plate \( x \geq 0 \) be reinforced on its boundary \( x = 0 \) by an infinite absolutely rigid stringer broken at the origin. On its vertical axis \( Ox \), the plane contains an edge crack coinciding with the segment \( 0 \leq x \leq l \). Further, let the edges of the crack be loaded only by normal forces of identical intensities \( p(r) \), i.e.,
\[
\sigma_\vartheta|_{\vartheta=\pm 0} = -p(r), \quad \tau_\vartheta|_{\vartheta=\pm 0} = 0 \quad (0 < r < l).
\]

It is assumed that the stringer does not resist bending in the vertical direction, i.e., it is an absolutely flexible thin-walled element, and in the horizontal direction, it behaves as an absolutely rigid element with respect to tension or compression.

Under these assumptions, due to symmetry, for the right quarter-plane \( \Omega_+ \), we come to the mixed boundary-value problem
\[
\begin{align*}
\tau_\vartheta|_{\vartheta=0} &= 0 \quad (0 < r < \infty); \quad \sigma_\vartheta|_{\vartheta=0} = -q_0(r) \quad (0 < r < \infty), \quad q_0(r) = p(r) \text{ for } (0 < r < l); \\
u_\vartheta|_{\vartheta=0} &= 0 \quad (l < r < \infty); \quad \sigma_\vartheta|_{\vartheta=\pi/2-0} = 0; \quad u_\tau|_{\vartheta=\pi/2-0} = 0 \quad (0 < r < \infty). \\
\end{align*}
\]

The solution of the problem is again obtained using the Mellin integral transformation based on the concepts (7). Here we take into account that the condition \( \bar{u}_r(\vartheta, p)|_{\vartheta=\pi/2} = 0 \) implies the relation
\[
\bar{\tau}^+(p) = -\frac{p + 1}{\cos(\pi p/2)} \bar{q}_0^+(p) \quad (-1 < \text{Re} p < 1).
\]

Further, let us introduce the Mellin transformant of dislocation density on the crack edges
\[
\bar{\varphi}(p) = -2p\bar{u}_\vartheta|_{\vartheta=0}
\]
and express the transformant \( \bar{q}_0(p) \) from the second boundary condition of problem (29) in terms of \( \bar{\varphi}(p) \) as follows:
\[
\bar{q}_0(p) = -\frac{1}{\Lambda} \frac{1 + \cos(\pi p)}{\sin(\pi p)} \bar{\varphi}(p) \quad (-1 < \text{Re} p < 0).
\]

Now we apply the Mellin inverse transformation to (31), taking the imaginary axis of complex plane \( p \) as the line of integration over the variable \( p \) and subtracting the half-residue of the integrand at the zero point \( p = 0 \). As a result, after some simple transformations, we obtain the key equation of problem (29)
\[
q_0(r) = -\frac{1}{\Lambda \pi} \int_0^l \left( \frac{1}{s-r} - \frac{1}{s+r} \right) \varphi(s) \, ds \quad (0 < r < \infty).
\]

Considering the key equation (32) on the crack, we come to the governing SIE of the mixed boundary-value problem (29):
\[
\frac{1}{\pi} \int_0^l \left( \frac{1}{s-r} - \frac{1}{s+r} \right) \varphi(s) \, ds = -\Lambda p(r) \quad (0 < r < l).
\]
The solution of SIE (33) must satisfy the continuity condition for displacements at the crack tips:

$$\int_0^l \varphi(s) \, ds = 0. \quad (34)$$

Let us investigate the behavior of the solution $\varphi(r)$ of SIE (33) as $r \to +0$. For this we use the well-known results on behavior of a Cauchy-type integral near the endpoints of the interval of integration [27] (pp. 74–75).

In application to this case, we consider the Cauchy-type integral

$$\Phi(z) = \frac{1}{2\pi i} \int_0^l \frac{\varphi(\tau) \, d\tau}{\tau - z}$$

and assume that the density of this integral at the end point $\tau = 0$ of the interval $0 < \tau < l$ has a logarithmic singularity: $\varphi(\tau) = \varphi^*(\tau) \ln \tau$, where the function $\varphi^*(\tau)$ satisfies the Hölder condition on $[0, l]$. Then near the zero point, we have the representations

$$\Phi(z) = \varphi^*(0) \omega_2(z, 0) + \Phi_0(z), \quad \Phi(t) = \frac{1}{2} \varphi^*(0) [\omega^+_2(t, 0) + \omega^-_2(t, 0)] + \Phi_0(t), \quad (35)$$

where $\Phi_0(z)$ is a single-valued bounded function near $0$ and tends to a certain limit as $z \to 0$.

Here

$$\omega_1(z, 0) = -\frac{1}{2\pi i} \ln z, \quad \omega_2(z, 0) = -\frac{1}{4\pi i} \ln^2 z + \frac{1}{2} \ln z.$$  

We put

$$\omega^+_1(t, 0) = -\frac{1}{2\pi i} \ln t \quad (t \in (0, l)),$$

where any single-valued branch of the logarithmic function is taken in the complex plane $z$ cut along $[0, l]$. Then

$$\omega^-_1(t, 0) = -\frac{1}{2\pi i} \ln t - \frac{1}{2\pi i} 2\pi i = \omega^+_1(t, 0) - 1,$$

which implies

$$\omega^+_1(t, 0) - \omega^-_1(t, 0) = 1, \quad \omega^+_1(t, 0) + \omega^-_1(t, 0) = -\frac{1}{\pi i} \ln t + 1 \quad (t \in (0, l)).$$

In the same way, we obtain

$$\omega^+_2(t, 0) - \omega^-_2(t, 0) = \ln t, \quad \omega^+_2(t, 0) + \omega^-_2(t, 0) = -\frac{1}{2\pi i} \ln^2 t \quad (t \in (0, l)),$$

whence we have

$$\omega^-_2(t, 0) = -\frac{1}{4\pi i} \ln^2 t - \frac{1}{2} \ln t \quad (t \in (0, l)).$$

Therefore,

$$\omega^-_2(-t, 0) = -\frac{1}{4\pi i} \ln^2(-t) - \frac{1}{2} \ln(-t) = -\frac{1}{4\pi i} \ln^2 t + \frac{i\pi}{4}$$

and from (35), it follows that

$$\Phi(t) = \frac{1}{2\pi i} \int_0^l \frac{\varphi^*(\tau) \ln \tau \, d\tau}{\tau + t} = \varphi^*(0) \omega^-_2(-t, 0) + \Phi_0(-t)$$

$$= \left(-\frac{1}{4\pi i} \ln^2 t + \frac{i\pi}{4}\right) \varphi^*(0) + \Phi_0(-t) \quad (t \to +0). \quad (36)$$
Substituting (35) and (36) into SIE (33), we now get an important relation

$$\varphi^*(0) = -\frac{2}{\pi} \Lambda p(+0),$$

(37)

which implies that if $p(+0) \neq 0$, then the function $\varphi(r)$ is the solution of SIE (33) and has a logarithmic singularity at the point $r = 0$.

On the basis of just stated results, as in [15], it is possible to show that the solution of SIE (33) cannot have power singularity at the point $r = 0$.

The SIE (33), (34) allow one to obtain an exact solution which obviously contains a term with logarithmic singularity. Actually, in (33), we continue the function $\varphi(r)$ to the interval $(-l, 0)$ in the even way, while the right-hand side $p(r)$ is continued in the odd way. Then (33) transforms as

$$\frac{1}{\pi} \int_{-l}^{l} \frac{\varphi(s) ds}{s - r} = -\Lambda q(r) \quad (q(r) = p(|r|)\text{sgn } r),$$

(38)

and condition (34) takes the form

$$\int_{-l}^{l} \varphi(s) ds = 0.$$  

(39)

Now SIE becomes [26]

$$\varphi(r) = \frac{\Lambda}{\pi \sqrt{l^2 - r^2}} \int_{-l}^{l} \frac{\sqrt{l^2 - s^2} q^*_0(s) ds}{s - r} + \frac{C}{\sqrt{l^2 - r^2}} \quad (-l < r < l).$$

(40)

Integrating both sides of this equality with respect to $r$ over the interval $(-l, l)$ and taking into account the expression of the indefinite integral with the Cauchy kernel from [29] (p. 111), we obtain $C = 0$. Then (40), where $C = 0$, becomes

$$\varphi(r) = \frac{\Lambda}{\pi \sqrt{l^2 - r^2}} \int_{-l}^{l} \frac{\sqrt{l^2 - s^2} q^*_0(s) ds}{s - r} + \frac{\Lambda p(+0)}{\pi \sqrt{l^2 - r^2}} \int_{-l}^{l} \frac{\sqrt{l^2 - s^2} \text{sgn } s ds}{s - r} \quad (-l < r < l).$$

(41)

After calculating the second integral in this relation, we have

$$\varphi(r) = -\frac{2\Lambda p(+0)}{\pi \sqrt{l^2 - r^2}} + \frac{\Lambda p(+0)}{\pi} \ln \left( \frac{l + \sqrt{l^2 - r^2}}{l - \sqrt{l^2 - r^2}} \right)$$

$$+ \frac{\Lambda}{\pi \ln r \sqrt{l^2 - r^2}} \int_{0}^{l} \frac{\sqrt{l^2 - s^2} q^*_0(s) ds}{s^2 - r^2} \quad (0 < r < l).$$

The second term in (41) has a logarithmic singularity at the point $r = 0$. After simple transformations, (41) can be represented as

$$\varphi(r) = \varphi^*(r) \ln r \quad (r \rightarrow +0),$$

$$\varphi^*(r) = -\frac{2\Lambda p(+0)}{\pi \ln r \sqrt{l^2 - r^2}} + \frac{2\Lambda p(+0)}{\pi \ln r} \ln(r + \sqrt{l^2 - r^2})$$

$$+ \frac{2\Lambda}{\pi \ln r \sqrt{l^2 - r^2}} \int_{0}^{l} \frac{s \sqrt{l^2 - s^2} q^*_0(s) ds}{s^2 - r^2} \quad (0 < r < l).$$

From here, again relationship (37) is immediately obtained.
Further, the Mellin inverse transformation is applied to (30) as well. Taking into account (32), after some simple transformations and calculations of necessary integrals, we have

\[
\tau^+(r) = -\frac{4r^2}{\pi} \int_0^1 \frac{s \varphi(s)}{s^2 + r^2} ds + \frac{4r^3}{\Delta^2} \int_0^1 \left[ \frac{1}{t^2 + r^2} + \frac{1}{s^2 + r^2} \ln \left( \frac{t^2 - s^2}{t^2 + r^2} \right) \right] \varphi(s) ds \quad (0 < r < \infty).
\]

Whence, using the above-described results on the behavior of Cauchy-type integrals, we obtain from [26]

\[
\tau^+(+0) = -\frac{2}{\pi} \mu(+0).
\]

4. On the stress state of an elastic half-plane containing a collinear system of absolutely rigid thin inclusions on the vertical axis

Let an elastic half-space or an elastic semi-infinite plate \( x \geq 0 \) be loaded on its boundary according to boundary conditions (3) and (8) and, on the vertical axis \( Ox \) (\( Or \)), contain a collinear system of absolutely rigid thin inclusions along the system of the segments \( L = \bigcup_{k=1}^n [a_k, b_k] \). Starting from (10)–(13) and using a method similar to that applied to the problem for a system of cracks, we reduce solving the considered problem to solving the following governing SIE:

\[
\begin{align*}
&\frac{1}{2(\lambda + 2\mu)} \int_L \left\{ \frac{2(\lambda + \mu)}{\mu} K_1(r, s) - \frac{2[3\mu(\lambda + \mu) + \lambda^2]}{\mu(\lambda + \mu)} K_2(r, s) - 2K_3(r, s) \right. \\
&\left. + \frac{\mu}{\lambda + \mu} K_4(r, s) + \frac{\lambda^2 + 4\lambda \mu + 3\mu^2}{2\mu(\lambda + \mu)} K_5(r, s) \right\} [X_-(s) + i\Omega_-(s)] ds = rf(r) \quad (r \in L), \quad (42)
\end{align*}
\]

\[
\begin{align*}
K_1(r, s) &= \frac{1}{\pi} \int_0^\infty \frac{\sigma^4 \sin(\alpha a_0) d\alpha}{\cosh(\pi \alpha) - 1 - 2\alpha^2 \sinh(\pi \alpha)} \left( a_0 = \ln \frac{a}{\rho} \right); \\
K_2(r, s) &= \frac{1}{\pi} \left\{ \int_0^\infty \frac{\sigma^2 \sin(\alpha a_0) d\alpha}{2\alpha^2 + 1 - \cosh(\pi \alpha) \sinh(\pi \alpha)} + \frac{1}{\pi^2 - 4} \right\}, \\
K_3(r, s) &= \frac{1}{\pi} \left\{ \int_0^\infty \frac{\sigma \cosh(\pi \alpha) \sin(\alpha a_0) d\alpha}{2\alpha^2 + 1 - \cosh(\pi \alpha) \sinh(\pi \alpha)} + \frac{1}{\pi^2 - 4} \right\}, \\
K_4(r, s) &= \frac{1}{\pi} \left\{ \int_0^\infty \frac{\tanh(\pi/2) \sin(\alpha a_0) d\alpha}{2\alpha^2 + 1 - \cosh(\pi \alpha)} + \frac{\pi^2}{2(\pi^2 - 4)} \right\}, \\
K_5(r, s) &= -\frac{\sqrt{\sigma}}{\pi(s - r)} - \frac{1}{\pi} \left\{ \int_0^\infty \frac{(1 + 2\alpha^2) \tanh(\pi \alpha) \sin(\alpha a_0) d\alpha}{\cosh(\pi \alpha) - 1 - 2\alpha^2} - \frac{\pi^2}{\pi^2 - 4} \right\};
\end{align*}
\]

where the function \( f(r) \) depends on the loads acting on the boundary of the elastic half-plane and the angles of rotation of the inclusions. Besides, here the unknown functions \( X_-(r) \) and \( \Omega_-(r) \) characterize the jumps of the tangential normal stresses on the edges of the inclusions:

\[
X_-(r) = -[\tau \varphi_{r=+0} - \tau \varphi_{r=0}], \quad \Omega_-(r) = -[\sigma \varphi_{r=+0} - \sigma \varphi_{r=0}] \quad (r \in L).
\]

The solution of SIE (42) must satisfy the equilibrium conditions for the inclusions

\[
\int_{a_k}^{b_k} [X_-(r) + i\Omega_-(r)] dr = 0, \quad \int_{a_k}^{b_k} r[X_-(r) + i\Omega_-(r)] dr = 0. \quad (43)
\]

Writing SIE (42), (43) in dimensionless variables and then transforming each interval \( (a_k, b_k) \) \((k = 1, n)\) into the interval \([-1, 1]\), we can apply the above-mentioned numerical-analytical method for solving SIE [7, 19, 20].
5. Indentation of a system of punches into an elastic half-plane containing a collinear system of absolutely rigid thin inclusions

Let the system of $2n + 1$ punches be indented into an elastic half-plane $x \geq 0$ along the segments on the half-plane boundary defined as $L = [-a, a] \cup \bigcup_{k=-n,k \neq 0}[a_k, b_k]$ ($k = \Gamma, n$). Further, let the elastic half-plane contain a collinear system of absolutely rigid thin inclusions on its vertical axis $Ox$ ($Or$) on the set of segments $l = \bigcup_{k=1}^{n}[c_k, d_k]$. We assume that the punches are located symmetrically with respect to the origin of coordinates and a centrally applied vertical concentrated force $P_k$ acts on the $k$th punch contacting with the elastic foundation along the segment $[a_k, b_k]$ ($k = -n, n, a_0 = a, b_0 = a, a_{-k} = -b_k, b_{-k} = -a_k$); furthermore, $P_0 = P, P_{-k} = P_k$ ($k = 1, n$). The bases of the punches are geometrically characterized by the function $f(r)$ ($r = y$). This problem generalizes the classical contact problem of elasticity for indenting a system of punches into an elastic half-plane and is encountered in foundation engineering in construction of buildings on elastic foundations reinforced by piles.

To reduce the above problem to the system of integral equations, we consider the following mixed boundary-value problem for the elastic right quarter-plane $\Omega^+$:

$$
\begin{align*}
\sigma_\varphi|_{\varphi=\pi/2-0} &= -p(r) \quad (0 < r < \infty, \ r = y); \\
\tau_\varphi|_{\varphi=\pi/2-0} &= 0 \quad (0 < r < \infty); \\
u_r|_{\varphi=+0} &= 0, \quad u_\varphi|_{\varphi=+0} = 0 \quad (r \in l); \\
\tau_\varphi|_{\varphi=+0} &= u_\varphi|_{\varphi=+0} = 0 \quad (r \in [0, \infty) \setminus l).
\end{align*}
$$

(44)

Here the function $p(r)$ is the contact pressure under the punches and is considered as a specified function.

The solution of the problem (44) is again constructed by the Mellin integral transformation which permits reducing the above-formulated contact problem to solving the following system of two SIEs:

$$
\begin{align*}
\int_l K_{11}(r, s) \tau_\varphi^+(s) \, ds + \int_{L_1} K_{12}(r, s) p(s) \, ds &= 0 \quad (r \in l), \\
\int_l K_{21}(r, s) \tau_\varphi^+(s) \, ds + \int_{L_1} K_{22}(r, s) p(s) \, ds &= r f'(r) \quad (r \in L_1, \ L_1 = [0, a] \cup \bigcup_{k=1}^{n}[a_k, b_k]),
\end{align*}
$$

(45)

$$
\begin{align*}
K_{11}(r, s) &= -\frac{\mu}{(\lambda + \mu)(\lambda + 2\mu)} \frac{r}{s + r} \\
&+ \frac{2}{\pi \mu(\lambda + 2\mu)} \left\{ \frac{\lambda + 3\mu}{\pi^2 - 4} + \int_0^{\infty} \frac{1}{\alpha^2 + 1 - \cosh(\pi\alpha)} \sin(\alpha a_0) \, d\alpha \right\} \\
&- \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)\pi} \left[ \frac{\pi^2}{\pi^2 - 4} - \frac{\sqrt{s}}{s - r} + \int_0^{\infty} \frac{(1 + 2\alpha^2) \tanh(\pi\alpha) \sin(\alpha a_0) \, d\alpha}{2\alpha^2 + 1 - \cosh(\pi\alpha)} \right],
\end{align*}
$$

and

$$
\begin{align*}
K_{12}(r, s) &= -\frac{2}{\pi \mu} \int_0^{\infty} \frac{\alpha^3 \cosh(\pi\alpha/2) \cos(\alpha a_0) \, d\alpha}{[\cosh(\pi\alpha) - 1 - 2\alpha^2] \sinh(\pi\alpha)} \\
&+ \frac{2(\lambda + 2\mu)}{\pi \mu(\lambda + \mu)} \left\{ \frac{1}{\pi^2 - 4} + \int_0^{\infty} \frac{\alpha^2 \cosh(\pi\alpha/2) \sin(\alpha a_0) \, d\alpha}{2\alpha^2 + 1 - \cosh(\pi\alpha)} \right\} \\
&+ \frac{1}{\pi(\lambda + \mu)} \int_0^{\infty} \frac{\alpha \cosh(\pi\alpha/2) \cos(\alpha a_0) \, d\alpha}{[2\alpha^2 + 1 - \cosh(\pi\alpha)] \sinh(\pi\alpha)} \\
&- \frac{\lambda + 2\mu}{\pi \mu(\lambda + \mu)} \left[ \int_0^{\infty} \frac{\alpha \sin(\pi\alpha/2) \cos(\pi a_0) \, d\alpha}{2\alpha^2 + 1 - \cosh(\pi\alpha)} + \frac{\pi^2}{2(\pi^2 - 4)} + \int_0^{\infty} \frac{\sin(\pi\alpha/2) \sin(\alpha a_0) \, d\alpha}{2\alpha^2 + 1 - \cosh(\pi\alpha)} \right],
\end{align*}
$$

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\[ K_{21}(r,s) = \frac{2}{\pi \mu} \int_0^\infty \frac{\alpha^3 \cosh(\pi \alpha/2) \cos(\alpha a_0)}{2\alpha^2 + 1 - \cosh(\pi \alpha) \sinh(\pi \alpha)} \, d\alpha \]

\[ - \frac{2(\lambda + 2\mu)}{\pi \mu(\lambda + \mu)} \left\{ \frac{1}{\pi^2 - 4} + \int_0^\infty \frac{\alpha^2 \cosh(\pi \alpha/2) \sin(\alpha a_0)}{2\alpha^2 + 1 - \cosh(\pi \alpha) \sinh(\pi \alpha)} \, d\alpha \right\} \]

\[ + \frac{1}{\pi \mu(\lambda + \mu)} \int_0^\infty \left\{ \mu \cos(\pi \alpha/2)[1 - \cosh(\pi \alpha)] + (\lambda + 2\mu) \sinh(\pi \alpha/2) \sin(\pi \alpha) \right\} \frac{\alpha \cos(\alpha a_0) \, d\alpha}{[\cosh(\pi \alpha) - 1 - 2\alpha^2] \sinh(\pi \alpha)} \]

\[ + \frac{\lambda + 2\mu}{\pi \mu(\lambda + \mu)} \left\{ \frac{\pi^2}{2(\pi^2 - 4)} + \int_0^\infty \frac{\sin(\pi \alpha/2) \sin(\pi \alpha) \sin(\alpha a_0) \, d\alpha}{[2\alpha^2 + 1 - \cosh(\pi \alpha) \sinh(\pi \alpha)]} \right\}, \]

\[ K_{22}(r,s) = - \frac{\lambda + 2\mu}{\mu(\lambda + \mu) \pi} \frac{r^2}{s^2 - r^2}. \]

The solution of the system of SIEs (45) must satisfy the equilibrium conditions for the punches and inclusions

\[ \int_0^a p(s) \, ds = \frac{P}{2}, \quad \int_{a_k}^{b_k} p(s) \, ds = P_k \quad (k = 1,n), \quad 2 \int_{c_k}^{d_k} \tau_0^+(s) \, ds = Q_k \quad (k = 1,m). \] (46)

Here \( \tau_0^+(r) \) are tangential contact stresses acting on the edges of the \( k \)th inclusion and \( Q_k \) is the axial force applied to the inclusion.

To simplify the study and the procedure of solving the system of SIEs (45), (46), we set \( m = n = 1, L = [0,a], L_0 = [b,c] \). Assuming that \( p(r) = P \delta(r)/2 \), where \( \delta(r) \) is the Dirac delta function, and passing to the limit as \( a \to +0 \), we obtain the following SIE from the system of SIEs (45):

\[ \int_b^c K_{11}(r,s)\tau_0^+(s) \, ds = \frac{\lambda + 2\mu}{2\pi \mu(\lambda + \mu)} P \] (47)

which describes the contact problem for the reinforcement in an elastic matrix. In this connection, the case \( b = 0 \) is practically important.

Equations (45), (46), and (47) can efficiently be solved by the above-mentioned method proposed in [7, 19, 20].

Note that after solving SIEs (45), (46), the distribution of normal stresses on the vertical axis of the elastic half-plane is determined by the formula

\[ \sigma_0^+(r) = \frac{4r}{\pi} \int_0^\infty \frac{s^2 p(s) \, ds}{(s^2 + r^2)^2} + \frac{16(\lambda + \mu)}{\pi(\lambda + 2\mu)} \int_0^\infty \frac{s^2(s^2 - r^2)\tau_0^+(s) \, ds}{(s^2 + r^2)^3} \]

\[ + \frac{8r}{\pi} \int_0^\infty \frac{s^2 \tau_0^+(s) \, ds}{(s^2 + r^2)^2} + \frac{2\mu}{\pi(\lambda + 2\mu)} \int_0^\infty \frac{s \tau_0^+(s) \, ds}{s^2 - r^2} \] (0 < r < \infty).

Equations similar to (45), (46), and (48) can describe the contact problem, where a system of \( 2n + 1 \) punches is indented into an elastic half-plane which contains a system of collinear cracks on its vertical axis, i.e., the case where, in the contact problem described above, the system of inclusions is replaced by a system of cracks.

**Conclusions**

In this paper, in a fairly general formulation, a class of boundary-value problems related to the interaction of stress concentrators (such as cracks, absolutely rigid thin inclusions and punches) with an elastic semi-infinite plate is studied by a unified method of SIE. Some results of other authors concerning the problems of an edge crack in an elastic plate, when the plate boundary is
free from external forces or when it is reinforced by an infinite absolutely rigid broken stringer, are simultaneously clarified. The classical contact problems of indenting a punch system into an elastic half-plane, which are of both theoretical and practical interest, are generalized. The results and approaches discussed in the paper can be used to study related boundary-value problems of the theory of elasticity.

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