When and how much the altruism impacts your privileged information? Proposing a new paradigm in game theory: The “boxers game”

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In this work, we propose a new $N$-person game in which the players can bet on two boxers. Some of the players have privileged information about the boxers and part of them can provide this information for uninformed players. However, this information may be true if the informed player is altruist or false if he is selfish. So, in this game, the players are divided in three categories: informed and altruist players, informed and selfish players, and uninformed players. By considering the matchings ($N/2$ distinct pairs of randomly chosen players) and that the payoff of the winning group follows aspects captured from two important games, the public goods game and minority game, we show quantitatively and qualitatively how the altruism can impact on the privileged information. We localized analytically the regions of positive payoff which were corroborated by numerical simulations performed for all values of information and altruism densities given that we know the information level of the informed players.

Keywords: Theoretical game theory, privileged information, boxers game

Just over half a decade ago, J. von Neumann and O. Morgenstern [1] probably had no idea of the extent that their work would have, not only on applications in the economic behavior but also in several areas of biology, ecology, and mainly in the evolutionary theory of the species [2] culminating in the replicator dynamics [3, 4] and its branches (see for example [5, 6]). In fact, the game theory still persists as an important theory of mathematical models which study the choices of optimal decisions under conflict or combat conditions. However, long before the evolutionary aspects were studied in this context, the economical applications were the primordial motivations of this theory, and leveraged the construction of suitable concepts as Nash equilibrium, that contemplating the existence of an equilibrium of mixed strategies in non-cooperative games [7]. The idea was very simple but brilliant: in such equilibrium, when there are two or more players, no player will gain by unilaterally changing its strategy. J. Nash was beyond and extended the concept to cooperative games by performing a reduction to non-cooperative games. His important works culminated in the Nobel Memorial Prize in Economic Sciences jointly to R. Selten and J. Harsany [8].

Games where the profit depends on the number of investors has a direct similarity with the reality in the business world. Here, the paradigmatic El farol bar problem (EFBP), which is a simple model that shows how (selfish) players cooperate with each other in the absence of communication, arises as an important starting point [9]. In that problem, a set of people must decide if they go to a bar or stay at home. If they decide to go out and the bar is too crowded, they probably will not have a funny night and stay at home would be more interesting. They will have (or not have) a funny night according to a threshold on the number of people in the bar. If less than a fraction (a threshold) $x$ of the population decide to go to the bar, they will have a better time than ones which decided to remain in their houses. On the other hand, if more than a fraction $x$ of the population go to the bar, they will have bad times when compared with the comfort of their homes. Two physicists saw the potential of this idea, and a formalization of the EFBP was given by Challet and Zhang [10], the so-called Minority Game (MG) which was proposed as a rough model to describe the price fluctuation of stock markets. In that game, an odd number of players have to choose one of two options independently at each turn. The players who end up on the minority side win the game. Following this line, another multiple player game (strongly based on experimental economics of the public goods), is the public goods game (PGG). In that game, multiple players can contribute to a common fund with or without the presence of communication in order to obtain benefits, and all money collected is doubled, or triplicated, or simply corrected according to some multiplicative factor. This game has been studied in many contexts by including optional participation obtaining a fixed income by opting not to invest in the public good, spatial diffusion effects and many other ingredients, both when studying many public goods games [11–14] and when considering only one public good (see, for example, [15–18]). Differently from MG where the payoff players are inversely proportional to the number of persons which choose the same group, in PGG the payoff depends on how many persons invest in the fund.

The important feature of the MG is the absence of
communication and noise. However, in general, real situations have properties such as misinformation, noise, selfishness, altruism, and some of these ingredients can coexist in more complex scenarios. In this work, we propose a new game that captures some aspects of both PGG and MG by taking into account some of those properties as well as multiplayer games. However, we consider very different constraints, with payoff determined probabilistically and according to the information (and its propagation) of some players. We call it the boxers game, since it was partly based on the idea of the payoff obtained by gamblers in boxing matches.

In this approach, we consider a $n$-person game in a scenario where each player must choose between two boxers (groups): $A$ or $B$. Some players (with density $\rho$) know that the boxer $A$ wins the boxer $B$ with probability $p > 1/2$ (we denote as $A > B$) given, for example, its skills, and therefore the boxer $B$ wins the boxer $A$ (we denote as $A < B$) with probability $1 - p - q$ such that $q$ is the draw probability. Here, it is important to mention that the information is incomplete since $p$ is not exactly equal to 1 (in this case $q$ should be necessarily equal to 0). The misinformation, which has an important role in the payoff of the players, was firstly formalized by the Nobel prize laureate in Economic Sciences, John C. Harsanyi [19]. However, we propose a different approach: by considering the similarities with a boxing match, we establish that the payoff of the player in a group $A$ or $B$ depends on some features considering some protocols that remember both PGG and MG. So, we set up three important fundamentals that govern our game:

1. MG protocol: in the case of victory, the payoff of the players (winners) depends on how many persons have bet in the losing group;
2. PGG protocol: the profit obtained for the group must be divided equally among the players of that group;
3. The gain is proportional to the invested amount.

By denoting $m^{(i)}_A$ as the payoff of the $i$-th player that has chosen the group $A$, which, without loss of generality, is the group where a number of people have privileged information ($A > B$ with probability $p > 1/2$), we have

$$m^{(i)}_A = \begin{cases} s_i(A) \frac{\sum_{s_i = 1}^{n_{A, B}} s_i(B)}{\sum_{s_i = 1}^{n_{A, B}} s_i(A)} & \text{with probability } p \\ -s_i(A) & \text{with probability } 1 - p - q \\ 0 & \text{elsewhere} \end{cases}$$

and naturally

$$m^{(i)}_B = \begin{cases} s_i(B) \frac{\sum_{s_i = 1}^{n_{A, B}} s_i(A)}{\sum_{s_i = 1}^{n_{A, B}} s_i(B)} & \text{with probability } 1 - p - q \\ -s_i(B) & \text{with probability } p \\ 0 & \text{elsewhere} \end{cases}$$

In this game, the reader should observe that the return, in case of a win, is the sum of the investment of all players in the losing group multiplied by a factor that distributes equally the gain between the participants of the winning group: $\alpha_{A,B} = s_i(A,B) / \sum_{s_i = 1}^{n_{A, B}} s_i(A,B)$. It is important to mention that for equal investments, $s_i(A,B) = s_i$, for all players, the payoff for each player is $\alpha_{A,B} = 1/n_{A,B}$.

As a partition and the sequences of "bets" are given, respectively, by $\{n_A, n_B\}, s_1(A), \ldots, s_{n_A}(A)$ and $s_1(B), \ldots, s_{n_B}(B)$, a simple calculation can be performed to provide us the expected payoff of the $i$-th player:

$$E\left[m^{(i)}_A\right] = ps_i(A) \frac{\sum_{s_i = 1}^{n_{A, B}} s_i(B)}{\sum_{s_i = 1}^{n_{A, B}} s_i(A)} - (1 - p - q)s_i$$

and

$$E\left[m^{(i)}_B\right] = (1 - p - q)s_i(B) \frac{\sum_{s_i = 1}^{n_{A, B}} s_i(A)}{\sum_{s_i = 1}^{n_{A, B}} s_i(B)} - ps_i(B)$$

In this regard, it is important to think of a mathematical point of view. Without taking into consideration the complex interaction that exists among players and how some of them ($n\rho$) can use their privileged information to make money, the problem is: giving the partition $\{n_A, n_B\}$ and the bets $\{s_i(A)\}_{i=1}^{n_A}$ and $\{s_i(B)\}_{i=1}^{n_B}$ conditioned to this partition, how to compute the average payoffs?

For example, if, by hypothesis, we know the probability distribution $Pr(n_{A,B})$ that determines how many individuals choose the group $A(B)$ as well as the conditional $f(s|A,B)$ probability distribution function (pdf) that governs the probability of a player which invests in the group $A(B)$ to perform a proposal between $s$ and $s + ds$, we would have the average payoff of an individual in the groups $A(B)$ which is given, respectively, by

$$\langle \langle m_A \rangle \rangle = p \langle s(B) \rangle \frac{n_B}{n_A} - (1 - p - q) \langle s(A) \rangle$$

$$\langle \langle m_B \rangle \rangle = (1 - p - q) \langle s(A) \rangle \frac{n_A}{n_B} - p \langle s(B) \rangle$$

where $n_{A,B} = \sum_{n_{A,B}=0}^{n} n_{A,B} Pr(n_{A,B})$ and $\langle s \rangle_{A,B} = \int_0^\infty s f(s|A,B) ds$.

At this point, the interactions among players, misinformation, and other relevant features, should lead to more exciting studies for this game since no indication produces the following distributions: $Pr(n_{A,B})$ and $f(s|A,B)$. So, once we showed that our very different approach avoids the Bayesian formalism of the incomplete information game theory [19], now we are able to define the dynamics for this complex scenario by presenting how the players propagate their privileged information. First, in real situations the information can be propagated with
Players | Decision
--- | ---
1 versus I (A or S) | Both players go to the group A
U versus U | Each player choose a group with probability 1/2
IA versus U | Both players go to the group A
IS versus U | The player IS goes to the group A and the player U goes to the group B

Table I. Possible decisions according to the different encounters among the players.

Good or bad intentions. Among the players that know the bias to a group (there are \( n_I = p n \) players with privileged information), a fraction \( \alpha \) of them propagate this information with good intention (i.e., saying the truth). Therefore, on the other hand, there are \( (1 - \alpha)p n \) players which only wish to maximize their profit (sincerity is not their main feature) and will influence all the other players with good intention (i.e., saying the truth). Thus we can write

\[
\langle \langle m_A \rangle \rangle = \frac{n_B}{n_A} p - (1 - p - q) \]

\[
\langle \langle m_B \rangle \rangle = (1 - p - q) \frac{n_A}{n_B} - p
\]

According to Table [I] along with the fraction of players, we have

\[
\pi_A = N \rho + (1 - \rho) \alpha \rho N + \frac{1}{2} (1 - \rho)^2 N
\]

\[
\pi_B = (1 - \rho)(1 - \alpha) \rho N + \frac{1}{2} (1 - \rho)^2 N
\]

where \( \pi_A + \pi_B = N \), i.e., the number of players is held constant.

The rate \( \frac{\pi_A}{\pi_B} \) yields

\[
\frac{\pi_A}{\pi_B} = f(\alpha, \rho) = \frac{\rho + (1 - \rho) \alpha \rho + \frac{1}{2} (1 - \rho)^2}{(1 - \rho)(1 - \alpha) \rho + \frac{1}{2} (1 - \rho)^2}
\]

Thus we can write

\[
\langle \langle m_A \rangle \rangle = \frac{(1 - \rho)(1 - \alpha) \rho \frac{1}{2} (1 - \rho)^2}{\rho + (1 - \rho) \alpha \rho + \frac{1}{2} (1 - \rho)^2} p - (1 - p - q)
\]

and

\[
\langle \langle m_B \rangle \rangle = \frac{\rho + (1 - \rho) \alpha \rho + \frac{1}{2} (1 - \rho)^2}{(1 - \rho)(1 - \alpha) \rho + \frac{1}{2} (1 - \rho)^2} (1 - p - q) - p
\]

It is interesting to observe that for \( \langle \langle m_A \rangle \rangle > 0 \), i.e., the informed players have profit in average. This situation leads to

\[
p > p_c = (1 - q) \frac{\rho + (1 - \rho) \alpha \rho + \frac{1}{2} (1 - \rho)^2}{\rho + (1 - \rho) \alpha \rho + \frac{1}{2} (1 - \rho)^2 + (1 - \rho)} \]

This means that \( p_c \) is the minimum information level necessary to obtain profit, given the density of informed players, \( \rho \), and the density of altruist informed ones, \( \alpha \).

Another alternative is to think that we have in hands the theoretical result predicted by Eqs. (3) and (4).

In this work, we performed numerical simulations based on turns. In one turn, \( N/2 \) pairs of players are randomly matched and all players necessarily participate once (such as performing a matching in a graph). In each turn, the players take a decision according to the information and nature of their partners determined by Table [I]. Figure [I] shows the payoff obtained by Monte Carlo (MC) simulations and these estimates are compared with our theoretical result predicted by Eqs. (3) and (4).
Since the altruism influences the information, we are wondering how the payoff vary for all possible values of information, $\rho$, and altruism, $\alpha$ for a fixed value of $p$, for instance, $p = 0.8$. Figure 2 shows the payoff of players of the group A whereas this is the group of informed players. We can observe iso-payoff curves which depend on values of $\rho$ and $\alpha$.

One can observe a perfect agreement between the simulations and our analytical approach. Finally, we focus our attention on the influence of the information level ($p$) on the transition curves $\alpha_c$ versus $\rho$ according to Eq. 6. Figure 4 shows $\alpha_c \times \rho$ for different values of $p$. As $p$ increases, we observe the evolution of the curves $\alpha_c \times \rho$. It is interesting to observe that for $\rho > \rho_c = (2p - 1)^{1/2}$ (this value is obtained by making $\alpha_c = 0$ in Eq. 6), even for a population entirely formed by non-altruist players, the payoff is always negative. So, our work shows that in a population with people that possess some privileged information, the payoff of the informed players is deeply changed by the altruism level. In addition, there is a critical altruism level which separates the profit from the loss in the payoff of the players. Our analytical results are in complete agreement with MC simulations.

In conclusion, we proposed a new game which presents an alternative social paradigm in economic scenarios with incomplete information. We showed our analytical approach whose results are corroborated by MC simulations. Our findings suggest that a lot of new applications may be considered in future investigations, including, for strong damages for the group and the information is not a trump for the game. In this case, it is interesting to obtain the minimal altruism level required to reach a positive payoff in the scenario predicted by Eq. 6. In Fig. 3, we show the diagram for all possible values of $\rho$ and $\alpha$ for $p = 0.8$. The positive (gray) and negative (purple) payoff regions predicted by MC simulations are regions where each set $(\rho, \alpha)$ yields a profit or loss for the players, respectively. The continuous curve (in black) shows the theoretical prediction of the threshold between the profit and loss obtained by Eq. 6.
Figure 4. Plots of $\alpha_c \times \rho$, for different values of $\rho$, predicted by Eq. (6). From the left to the right, we show curves which correspond to the following values of $\rho$: 0.51, 0.52, 0.53, 0.54, 0.55, 0.60, 0.70, 0.80, and 0.90.

instance, the study of spacial effects through the analysis of this game in different networks or even considering the evolutionary aspects on the strategies.

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