Energy Distribution associated with Static Axisymmetric Solutions

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Abstract
This paper has been addressed to a very old but burning problem of energy in General Relativity. We evaluate energy and momentum densities for the static and axisymmetric solutions. This specializes to two metrics, i.e., Erez-Rosen and the gamma metrics, belonging to the Weyl class. We apply four well-known prescriptions of Einstein, Landau-Lifshitz, Papaterou and Möller to compute energy-momentum density components. We obtain that these prescriptions do not provide similar energy density, however momentum becomes constant in each case. The results can be matched under particular boundary conditions.

Keywords: Energy-momentum, axisymmetric spacetimes.

1 Introduction
The problem of energy-momentum of a gravitational field has always been an attractive issue in the theory of General Relativity (GR). The notion of energy-momentum for asymptotically flat spacetime is unanimously accepted. Serious difficulties in connection with its notion arise in GR. However, for gravitational fields, this can be made locally vanish. Thus one is always able to find the frame in which the energy-momentum of gravitational field is zero, while in other frames it is not true. Noether’s theorem

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and translation invariance lead to the canonical energy-momentum density tensor, $T^b_a$, which is conserved.

$$T^b_{a,b} = 0, \quad (a, b = 0, 1, 2, 3).$$ (1)

In order to obtain a meaningful expression for energy-momentum, a large number of definitions for the gravitation energy-momentum in GR have been proposed. The first attempt was made by Einstein who suggested an expression for energy-momentum density [1]. After this, many physicists including Landau-Lifshitz [2], Papapetrou [3], Tolman [4], Bergman [5] and Weinberg [6] had proposed different expressions for energy-momentum distribution. These definitions of energy-momentum complexes give meaningful results when calculations are performed in Cartesian coordinates. However, the expressions given by Möller [7,8] and Komar [9] allow one to compute the energy-momentum densities in any spatial coordinate system. An alternate concept of energy, called quasi-local energy, does not restrict one to use particular coordinate system. A large number of definitions of quasi-local masses have been proposed by Penrose [10] and many others [11,12]. Chang et al. [13] showed that every energy-momentum complex can be associated with distinct boundary term which gives the quasi-local energy-momentum.

There is a controversy with the importance of non-tensorial energy-momentum complexes whose physical interpretation has been a problem for the scientists. There is a uncertainty that different energy-momentum complexes would give different results for a given spacetime. Many researchers considered different energy-momentum complexes and obtained encouraging results. Virbhadra et al. [14-18] investigated several examples of the spacetimes and showed that different energy-momentum complexes could provide exactly the same results for a given spacetime. They also evaluated the energy-momentum distribution for asymptotically non-flat spacetimes and found the contradiction to the previous results obtained for asymptotically flat spacetimes. Xulu [19,20] evaluated energy-momentum distribution using the Möller definition for the most general non-static spherically symmetric metric. He found that the result is different in general from those obtained using Einstein’s prescription. Aguirregabiria et al. [21] proved the consistency of the results obtained by using the different energy-momentum complexes for any Kerr-Schild class metric.

On contrary, one of the authors (MS) considered the class of gravitational waves, Gödel universe and homogeneous Gödel-type metrics [22-24] and used
the four definitions of the energy-momentum complexes. He concluded that
the four prescriptions differ in general for these spacetimes. Ragab [25,26] ob-
tained contradictory results for Gödel-type metrics and Curzon metric which
is a special solution of the Weyl metrics. Patashnick [27] showed that dif-
f erent prescriptions give mutually contradictory results for a regular MMaS-
class black hole. In recent papers, we extended this procedure to the non-null
Einstein-Maxwell solutions, electromagnetic generalization of Gödel solution,
singularity-free cosmological model and Weyl metrics [28-30]. We applied
four definitions and concluded that none of the definitions provide consistent
results for these models. This paper continues the study of investigation of
the energy-momentum distribution for the family of Weyl metrics by using
the four prescriptions of the energy-momentum complexes. In particular, we
would explore energy-momentum for the Erez-Rosen and gamma metrics.

The paper has been distributed as follows. In the next section, we shall de-
scribe the Weyl metrics and its two family members Erez-Rosen and gamma
metrics. Section 3 is devoted to the evaluation of energy-momentum densities
for the Erez-Rosen metric by using the prescriptions of Einstein, Landau-
Lifshitz, Papapetrou and Möller. In section 4, we shall calculate energy-
momentum density components for the gamma metric. The last section
contains discussion and summary of the results.

2 The Weyl Metrics

Static axisymmetric solutions to the Einstein field equations are given by the
Weyl metric [31,32]

\[ ds^2 = e^{2\psi} dt^2 - e^{-2\psi} [e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\phi^2] \] (2)

in the cylindrical coordinates \((\rho, \phi, z)\). Here \(\psi\) and \(\gamma\) are functions of
coordinates \(\rho\) and \(z\). The metric functions satisfy the following differential
equations

\[ \psi_{\rho\rho} + \frac{1}{\rho} \psi_\rho + \psi_{zz} = 0, \] (3)

\[ \gamma_\rho = \rho (\psi_\rho^2 - \psi_z^2), \quad \gamma_z = 2 \rho \psi_\rho \psi_z. \] (4)
It is obvious that Eq.(3) represents the Laplace equation for $\psi$. Its general solution, yielding an asymptotically flat behaviour, will be

$$\psi = \sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} P_n(\cos \theta),$$

(5)

where $r = \sqrt{\rho^2 + z^2}$, $\cos \theta = z/r$ are Weyl spherical coordinates and $P_n(\cos \theta)$ are Legendre Polynomials. The coefficients $a_n$ are arbitrary real constants which are called *Weyl moments*. It is mentioned here that if we take

$$\psi = -\frac{m}{r}, \quad \gamma = -\frac{m^2 \rho^2}{2r^4}, \quad r = \sqrt{\rho^2 + z^2}$$

(6)

then the Weyl metric reduces to special solution of Curzon metric [33]. There are more interesting members of the Weyl family, namely the Erez-Rosen and the gamma metric whose properties have been extensively studied in the literature [32,34].

The Erez-Rosen metric [32] is defined by considering the special value of the metric function

$$2\psi = \ln\left(\frac{x - 1}{x + 1}\right) + q_2(3y^2 - 1)\left[\frac{1}{4}(3x^2 - 1)\ln\left(\frac{x - 1}{x + 1}\right) + \frac{3}{2}x\right],$$

(7)

where $q_2$ is a constant.

### 3 Energy and Momentum for the Erez-Rosen Metric

In this section, we shall evaluate the energy and momentum density components for the Erez-Rosen metric by using different prescriptions. To obtain meaningful results in the prescriptions of Einstein, Ladau-Lifshitz’s and Papapetrou, it is required to transform the metric in Cartesian coordinates. This can be done by using the transformation equations

$$x = \rho \cos \theta, \quad y = \rho \sin \theta.$$ 

(8)

The resulting metric in these coordinates will become

$$ds^2 = e^{2\psi} dt^2 - \frac{e^{2(\gamma - \psi)}}{\rho^2} (xdx + ydy)^2 - \frac{e^{-2\psi}}{\rho^2} (xdy - ydx)^2 - e^{2(\gamma - \psi)} dz^2.$$ 

(9)
3.1 Energy and Momentum in Einstein’s Prescription

The energy-momentum complex of Einstein [1] is given by

$$\Theta^b_a = \frac{1}{16\pi} H^{bc}_{a,c},$$

(10)

where

$$H^{bc}_{a} = \frac{g_{ad}}{\sqrt{-g}}[-g\left(g^{bd}g^{ce} - g^{be}g^{cd}\right)], \quad a, b, c, d, e = 0, 1, 2, 3.$$  

(11)

Here $\Theta^0_0$ is the energy density, $\Theta^i_0 (i = 1, 2, 3)$ are the momentum density components and $\Theta^b_a$ are the energy current density components. The Einstein energy-momentum satisfies the local conservation laws

$$\frac{\partial \Theta^b_a}{\partial x^b} = 0.$$  

(12)

The required components of $H^{bc}_{a}$ are the following

$$H^{01}_0 = \frac{4y}{\rho^2} e^{2\gamma} (y\psi_x - x\psi_y) + \frac{4x}{\rho^2} (x\psi_x + y\psi_y)$$

$$- \frac{x}{\rho^2} - 2x\psi^2 + \frac{x}{\rho^2} e^{2\gamma},$$

(13)

$$H^{02}_0 = \frac{4x}{\rho^2} e^{2\gamma} (x\psi_y - y\psi_x) + \frac{4y}{\rho^2} (x\psi_x + y\psi_y)$$

$$- \frac{y}{\rho^2} - 2y\psi^2 + \frac{y}{\rho^2} e^{2\gamma}.$$  

(14)

Using Eqs.(13)-(14) in Eq.(10), we obtain the energy and momentum densities in Einstein’s prescription

$$\Theta^0_0 = \frac{1}{8\pi \rho^2} \left[e^{2\gamma} \left(\rho^2 \psi_{,\rho}^2 + 2(x^2 \psi_{,yy} + y^2 \psi_{,xx} - x\psi_{,x} - y\psi_{,y})\right)\right]$$

$$+ 2\{x^2 \psi_{,xx} + y^2 \psi_{,yy} + x\psi_{,x} + y\psi_{,y} - \rho^2 \psi_{,\rho} (\psi_{,\rho} + \rho \psi_{,\rho\rho})\}.$$  

(15)

All the momentum density components turn out to be zero and hence momentum becomes constant.
3.2 Energy and Momentum in Landau-Lifshitz’s Prescription

The Landau-Lifshitz [2] energy-momentum complex can be written as

$$L^{ab} = \frac{1}{16\pi} \ell^{abcd},$$  \hspace{1cm} (16)

where

$$\ell^{abcd} = -g^{ab}g^{cd} - g^{ad}g^{cb}. \hspace{1cm} (17)$$

$L^{ab}$ is symmetric with respect to its indices. $L^{00}$ is the energy density and $L^{0i}$ are the momentum (energy current) density components. $\ell^{abcd}$ has symmetries of the Riemann curvature tensor. The local conservation laws for Landau-Lifshitz energy-momentum complex turn out to be

$$\frac{\partial L^{ab}}{\partial x^b} = 0. \hspace{1cm} (18)$$

The required non-vanishing components of $\ell^{abcd}$ are

$$\ell^{0101} = -\frac{y^2}{\rho^2} e^{2\gamma - 4\psi} - \frac{x^2}{\rho^2} e^{2\gamma - 4\psi}, \hspace{1cm} (19)$$

$$\ell^{0202} = -\frac{x^2}{\rho^2} e^{2\gamma - 4\psi} - \frac{y^2}{\rho^2} e^{2\gamma - 4\psi}, \hspace{1cm} (20)$$

$$\ell^{0102} = \frac{xy}{\rho^2} e^{2\gamma - 4\psi} - \frac{xy}{\rho^2} e^{2\gamma - 4\psi}. \hspace{1cm} (21)$$

Using Eqs.(19)-(21) in Eq.(16), we get

$$L^{00} = \frac{e^{2\gamma - 4\psi}}{8\pi \rho^2} [e^{2\gamma} \{2\rho^2 \psi^2_{,\rho} - 8(y^2 \psi^2_{,x} + x^2 \psi^2_{,y}) + 2(x^2 \psi_{,xx} + y^2 \psi_{,yy}) - x\psi_{,x} - y\psi_{,y} + 16xy\psi_{,x}\psi_{,y} - 4xy\psi_{,xy}\}$$

$$- \rho^2 \psi_{,\rho}(3\psi_{,\rho} + 2\rho^2 \psi^3_{,\rho} + 2\rho\psi_{,\rho\rho}) - 8\rho^2 \psi^2_{,\rho}(x\psi_{,x} + y\psi_{,y})$$

$$- 8(x^2 \psi^2_{,x} + y^2 \psi^2_{,y}) + 2(x^2 \psi_{,xx} + y^2 \psi_{,yy})$$

$$+ x\psi_{,x} + y\psi_{,y}) - 16xy\psi_{,x}\psi_{,y} + 4xy\psi_{,xy}] \hspace{1cm} (22)$$

The momentum density vanishes and hence momentum becomes constant.
3.3 Energy and Momentum in Papapetrou’s Prescription

We can write the prescription of Papapetrou [3] energy-momentum distribution in the following way

\[ \Omega^{ab} = \frac{1}{16 \pi} N^{abcd}, \]  

(23)

where

\[ N^{abcd} = \sqrt{-g}(g^{ab} \eta^{cd} - g^{ac} \eta^{bd} + g^{cd} \eta^{ab} - g^{bd} \eta^{ac}), \]  

(24)

and \( \eta^{ab} \) is the Minkowski spacetime. It follows that the energy-momentum complex satisfies the following local conservation laws

\[ \frac{\partial \Omega^{ab}}{\partial x^b} = 0. \]  

(25)

\( \Omega^{00} \) and \( \Omega^{0i} \) represent the energy and momentum (energy current) density components respectively.

The required components of \( N^{abcd} \) are

\[ N^{0011} = -\frac{y^2}{\rho^2} e^{2\gamma} - \frac{x^2}{\rho^2} - e^{2\gamma-4\psi}, \]  

(26)

\[ N^{0022} = -\frac{x^2}{\rho^2} e^{2\gamma} - \frac{y^2}{\rho^2} - e^{2\gamma-4\psi}, \]  

(27)

\[ N^{0012} = -\frac{xy}{\rho^2} e^{2\gamma} - \frac{xy}{\rho^2}. \]  

(28)

Substituting Eqs.(26)-(28) in Eq.(23), we obtain the following energy density

\[ \Omega^{00} = \frac{e^{2\gamma}}{8\pi} \left[ \psi^4_{,\rho} - e^{-4\psi} \left\{ \psi^2_{,\rho} + 2\rho^2 \psi^4_{,\rho} + 2\rho \psi_{,\rho} \psi_{,\rho} \right\} ight. \right. \]

\[ - \left. 8\psi^2_{,\rho} (x \psi_{,x} + y \psi_{,y}) + 8(\psi^2_{,x} + \psi^2_{,y}) - 2(\psi_{,xx} + \psi_{,yy}) \right\}. \]  

(29)

The momentum density vanishes.

3.4 Energy and Momentum in Möller’s Prescription

The energy-momentum density components in Möller’s prescription [7,8] are given as

\[ M^a_b = \frac{1}{8\pi} K^{bc}_{a,c}, \]  

(30)
where

\[ K_{a}^{bc} = \sqrt{-g} (g_{ad,e} - g_{ae,d}) g^{be} g^{cd}. \]  

(31)

Here \( K_{a}^{bc} \) is symmetric with respect to the indices. \( M_{0}^{0} \) is the energy density, \( M_{0}^{i} \) are momentum density components, and \( M_{i}^{0} \) are the components of energy current density. The Möller energy-momentum satisfies the following local conservation laws

\[ \frac{\partial M_{a}^{b}}{\partial x^{b}} = 0. \]  

(32)

Notice that Möller’s energy-momentum complex is independent of coordinates.

The components of \( K_{a}^{bc} \) for Erez-Rosen metric is the following

\[ K_{0}^{01} = 2|\rho\psi_{,\rho}. \]  

(33)

Substitute Eq.(33) in Eq.(30), we obtain

\[ M_{0}^{0} = \frac{1}{4\pi} [\psi_{,\rho} + \rho \psi_{,\rho\rho}] \]  

(34)

Again, we get momentum constant.

The partial derivatives of the function \( \psi \) are given by

\[ \psi_{,x} = \frac{1}{x^2 - 1} + \frac{q_{2}}{4} (3y^2 - 1)[3xln(\frac{x - 1}{x + 1}) + \frac{3x^2 - 1}{x^2 - 1} + 3], \]  

(35)

\[ \psi_{,y} = \frac{3yq_{2}}{4} [(3x^2 - 1)ln(\frac{x - 1}{x + 1}) + 6x], \]  

(36)

\[ \psi_{,xx} = -\frac{2x}{(x^2 - 1)^2} + \frac{q_{2}}{4} (3y^2 - 1)[3ln(\frac{x - 1}{x + 1}) + 2x \frac{3x^2 - 5}{(x^2 - 1)^2}], \]  

(37)

\[ \psi_{,yy} = \frac{3q_{2}}{4} [(3x^2 - 1)ln(\frac{x - 1}{x + 1}) + 6x], \]  

(38)

\[ \psi_{,xy} = U_{,yx} = \frac{3yq_{2}}{4} [3xln(\frac{x - 1}{x + 1}) + 2\frac{3x^2 - 2}{x^2 - 1}], \]  

(39)

\[ \psi_{,\rho} = \frac{\rho}{x(x^2 - 1)} + \frac{\rho q_{2}}{4x} [3x(3\rho^2 - 2)ln(\frac{x - 1}{x + 1}) + 2(3x^2 - 1)(3y^2 - 1)] \frac{1}{x^2 - 1} + 18x^2], \]  

(40)
\[ \psi_{,\rho\rho} = \frac{1}{x(x^2 - 1)} - \frac{2\rho^2}{x(x^2 - 1)^2} + \frac{q_2}{4x^2}(3y^2 - 1)[3(\rho^2 + x^2)\ln\left(\frac{x - 1}{x + 1}\right) \\
+ \frac{2x}{x^2 - 1}(3x^2 - 2 + \rho^2(3x^2 - 5)) + \frac{3\rho q_2}{4}(1 + \rho^2)] \\
\times [(3x^2 - 1)\ln\left(\frac{x - 1}{x + 1}\right) + 6x] + \frac{3\rho^2 q_2}{x}[3ln\left(\frac{x - 1}{x + 1}\right) + 2 \frac{3x^2 - 1}{x^2 - 1}]. \quad (41) \]

4 Energy and Momentum for the Gamma Metric

A static and asymptotically flat exact solution to the Einstein vacuum equations is known as the gamma metric. This is given by the metric [34]

\[ ds^2 = (1 - \frac{2m}{r})\gamma dt^2 - (1 - \frac{2m}{r})^{-\gamma}[(\frac{\Delta}{\Sigma})^{\gamma - 1} dr^2 + \frac{\Delta \gamma^2}{\Sigma^{\gamma^2 - 1}} d\theta^2 + \Delta \sin^2 \theta d\phi^2], \quad (42) \]

where

\[ \Delta = r^2 - 2mr, \]
\[ \Sigma = r^2 - 2mr + m^2 \sin^2 \theta, \]

\(m\) and \(\gamma\) are constant parameters. \(m = 0\) or \(\gamma = 0\) gives the flat spacetime. For \(|\gamma| = 1\) the metric is spherically symmetric and for \(|\gamma| \neq 1\), it is axially symmetric. \(\gamma = 1\) gives the Schwarzschild spacetime in the Schwarzschild coordinates. \(\gamma = -1\) gives the Schwarzschild spacetime with negative mass, as putting \(m = -M(m > 0)\) and carrying out a non-singular coordinate transformation \((r \rightarrow R = r + 2M)\) one gets the Schwarzschild spacetime (with positive mass) in the Schwarzschild coordinates \((t, R, \theta, \Phi)\).

In order to have meaningful results in the prescriptions of Einstein, Landau-Lifshitz and Papapetrou, it is necessary to transform the metric in Cartesian coordinates. We transform this metric in Cartesian coordinates by using

\[ x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta. \quad (45) \]

The resulting metric in these coordinates will become

\[ ds^2 = (1 - \frac{2m}{r})\gamma dt^2 - (1 - \frac{2m}{r})^{-\gamma}[(\frac{\Delta}{\Sigma})^{\gamma - 1} \frac{1}{r^2} (xdx + ydy + zdz)^2 \\
+ \frac{\Delta \gamma^2}{\Sigma^{\gamma^2 - 1}} \left(\frac{xyz + yzdy - (x^2 + y^2)dz}{r^2 \sqrt{x^2 + y^2}}\right)^2 + \frac{\Delta(xdy - ydx)^2}{r^2(x^2 + y^2)}]. \quad (46) \]
Now we calculate energy-momentum densities using the different prescriptions given below.

### 4.1 Energy and Momentum in Einstein’s Prescription

The required non-vanishing components of \( H_{bc}^a \) are

\[
H_{01}^0 = 4\gamma m \frac{x}{r^3} + \left( \frac{\Delta}{\Sigma} \right)^{\gamma^2-1} \frac{x}{x^2 + y^2} - (\gamma^2 + 1)(1 - \frac{m}{r}) \frac{2x}{r^2} \\
+ (\gamma^2 - 1)(1 - \frac{m}{r}) \frac{2\Delta x}{\Sigma r^2} + \frac{2\Delta x}{r^4} + (\gamma^2 - 1) \frac{2m^2 x z^2}{\Sigma r^4} \\
- \frac{x z^2}{r^2(2 + y^2)} + \frac{x}{r^2}. \tag{47}
\]

\[
H_{02}^0 = 4\gamma m \frac{y}{r^3} + \left( \frac{\Delta}{\Sigma} \right)^{\gamma^2-1} \frac{y}{x^2 + y^2} - (\gamma^2 + 1)(1 - \frac{m}{r}) \frac{2y}{r^2} \\
+ (\gamma^2 - 1)(1 - \frac{m}{r}) \frac{2\Delta y}{\Sigma r^2} + \frac{2\Delta y}{r^4} + (\gamma^2 - 1) \frac{2m^2 y z^2}{\Sigma r^4} \\
- \frac{y z^2}{r^2(x^2 + y^2)} + \frac{y}{r^2}. \tag{48}
\]

\[
H_{03}^0 = 4\gamma m \frac{z}{r^3} - (\gamma^2 + 1)(1 - \frac{m}{r}) \frac{2z}{r^2} + (\gamma^2 - 1)(1 - \frac{m}{r}) \frac{2\Delta z}{\Sigma r^2} \\
+ \frac{2\Delta z}{r^4} - (\gamma^2 - 1)(x^2 + y^2) \frac{2m^2 z^2}{\Sigma r^4} + \frac{2z}{r^2}. \tag{49}
\]

Using Eqs. (47)-(49) in Eq. (10), we obtain non-vanishing energy density in Einstein’s prescription given as

\[
\Theta_0^0 = \frac{1}{8\pi \Sigma r^6} \left[ (\gamma^2 - 1) \Sigma \Delta \gamma^2 - 2 r^5 (r - m) - (\gamma^2 - 1) \Delta \gamma^2 - 1 r^2 \\
\times \left\{ r^4 - m r^3 + m^2 r^2 - m^2 (x^2 + y^2) \right\} - (\gamma^2 + 1) \Sigma^2 r^4 \\
+ 2(\gamma^2 - 1) \Sigma \Delta \gamma^2 - 2 \Delta r^4 (r - m)^2 + (\gamma^2 - 1) m \Delta \Sigma \gamma^2 - 1 r^2 \\
- (\gamma^2 - 1) \Delta \gamma^2 - 2 \Delta \Sigma r^4 (r - m)^2 + (\gamma^2 - 1) \Delta \gamma^2 - 1 r^2 \\
\times \Delta r^4 (r - m) + 2 \Sigma \gamma^2 r^3 (r - m) - \Sigma \Delta \gamma^2 - 2 \Sigma^2 r^4 \\
+ 3(\gamma^2 - 1) \Sigma^2 r^2 m^2 z^2 - (\gamma^2 - 1) \Sigma \gamma^2 - 1 r^4 m^2 \\
- 2(\gamma^2 - 1) \Sigma \gamma^2 - 2 m^4 z^2 (x^2 + y^2) \right]. \tag{50}
\]

The momentum density components become zero and consequently momentum is constant.
4.2 Energy and Momentum in Landau-Lifshitz’s Prescription

The required non-vanishing components of $\ell^{abcd}$ are

$$
\ell^{0101} = -(1 - \frac{2m}{r})^{-2} \gamma \left[ \frac{y^2 \Delta \gamma^2 - 1}{r^2 (x^2 + y^2) \Sigma^2 \gamma^2 - 1} \right] \quad \text{and} \quad \ell^{0202} = -(1 - \frac{2m}{r})^{-2} \gamma \left[ \frac{x^2 \Delta \gamma^2 - 1}{r^2 (x^2 + y^2) \Sigma^2 \gamma^2 - 1} \right] + \frac{\Delta \gamma^2 x^2 y^2}{r^2 (x^2 + y^2) \Sigma^2 \gamma^2 - 1},
$$

(51)

$$
\ell^{0303} = -(1 - \frac{2m}{r})^{-2} \gamma \left[ \frac{x z \Delta \gamma^2 - 1}{r^2 (x^2 + y^2) \Sigma^2 \gamma^2 - 1} \right] - \frac{\Delta \gamma^2 x z y^2}{r^2 (x^2 + y^2) \Sigma^2 \gamma^2 - 1},
$$

(52)

$$
\ell^{0102} = (1 - \frac{2m}{r})^{-2} \gamma \left[ \frac{x y \Delta \gamma^2 - 1}{r^2 (x^2 + y^2) \Sigma^2 \gamma^2 - 1} \right] - \frac{\Delta \gamma^2 x y z^2}{r^2 (x^2 + y^2) \Sigma^2 \gamma^2 - 1},
$$

(53)

$$
\ell^{0203} = -(1 - \frac{2m}{r})^{-2} \gamma \left[ \frac{y z \Delta \gamma^2 - 1}{r^2 (x^2 + y^2) \Sigma^2 \gamma^2 - 1} \right] - \frac{\Delta \gamma^2 x z y^2}{r^2 (x^2 + y^2) \Sigma^2 \gamma^2 - 1},
$$

(54)

$$
\ell^{0103} = -(1 - \frac{2m}{r})^{-2} \gamma \left[ \frac{x z \Delta \gamma^2 - 1}{r^2 (x^2 + y^2) \Sigma^2 \gamma^2 - 1} \right] - \frac{\Delta \gamma^2 x z y^2}{r^2 (x^2 + y^2) \Sigma^2 \gamma^2 - 1}.
$$

(55)

$$
\ell^{0203} = -(1 - \frac{2m}{r})^{-2} \gamma \left[ \frac{y z \Delta \gamma^2 - 1}{r^2 (x^2 + y^2) \Sigma^2 \gamma^2 - 1} \right] - \frac{\Delta \gamma^2 x z y^2}{r^2 (x^2 + y^2) \Sigma^2 \gamma^2 - 1}.
$$

(56)

When we substitute these values in Eq.(16), it follows that the energy density remains non-zero while momentum density components vanish. This is given as follows

$$
L^{00} = \frac{(1 - \frac{2m}{r})^{-2} \gamma}{8\pi} \left[ - \frac{4\gamma m^2}{r^4} \left( \frac{\Delta}{\Sigma} \right) \gamma^2 - 1(2\gamma + 1) + \frac{2\gamma m}{r^2} \left( \frac{\Delta}{\Sigma} \right)^2(\gamma^2 - 1) r^4 
\right.
$$

$$
+ \frac{4(\gamma^2 + 1)(1 - \frac{m}{r})\left( \frac{\Delta}{\Sigma} \right) \gamma^2 - 1 r^4 - 4(\gamma^2 - 1)(1 - \frac{m}{r})\left( \frac{\Delta}{\Sigma} \right)^2 r^4 
\right]
$$

$$
- \frac{\Delta \gamma^2}{\Sigma \gamma^2 - 1} r^2 - \frac{1}{r^2} \left( 2\gamma^2 - 1 \right)(1 - \frac{m}{r})\left( \frac{\Delta}{\Sigma} \right)^2(\gamma^2 - 1) - \frac{2}{r^2} \left( \gamma^2 - 1 \right)(1 - \frac{m}{r})
$$

$$
+ \frac{m^2}{r^2} - \frac{m^2 (x^2 + y^2)}{r^4} \left( \frac{\Delta}{\Sigma} \right)^2 \gamma^2 - 1 = \frac{2\Delta^2 \gamma^2 - 1}{r^4 \Sigma \gamma^2 - 1}.
$$
\[- \frac{2\gamma^2}{r^2} (\gamma^2 + 1)(1 - \frac{m}{r})^2 (\frac{\Delta}{\Sigma})^{\gamma^2-1} + \frac{4}{r^2} (\gamma^4 - 1)(1 - \frac{m}{r})^2 (\frac{\Delta}{\Sigma})^\gamma \]
\[- \frac{2}{r^2} (\gamma^2 - 1)(1 - \frac{m}{r})^2 (\frac{\Delta}{\Sigma})^{\gamma^2+1} - 2(\gamma^2 - 1)(1 - \frac{m}{r} + \frac{m^2}{r^2}) \]
\[- \frac{m^2 (x^2 + y^2)}{r^4} \frac{\Delta^{\gamma^2+1}}{\Sigma^{\gamma^2}} - (\gamma^2 + 1) \frac{m \Delta \gamma^2}{r^5 \Sigma^{\gamma^2} - 1} \]
\[+ 3(\gamma^2 + 1) \frac{\Delta \gamma^2}{r^4 \Sigma^{\gamma^2-1}} (1 - \frac{m}{r}) + (\gamma^2 - 1) \frac{m \Delta \gamma^2 + 1}{r^5 \Sigma^{\gamma^2}} (1 - \frac{2m}{r}) \]
\[- \frac{m^2 (x^2 + y^2)}{r^3} - 2(\gamma^2 - 1)(x^2 + y^2) \frac{m^2 z^2 \Delta^{\gamma^2+1}}{r^{10} \Sigma^{\gamma^2}} \]
\[- \frac{3 \Delta \gamma^2}{r^6 \Sigma^{\gamma^2-1}} + \gamma^2 (\frac{\Delta}{\Sigma})^{\gamma^2-1} (1 - \frac{m}{r}) \frac{1}{r^2} + \frac{2z^2}{r^4} - 2\gamma^2 (\gamma^2 - 1) \]
\[\times (x^2 + y^2) \frac{m^4 z^2 \Delta^{\gamma^2}}{r^8 \Sigma^{\gamma^2+1}} + 2(\gamma^2 - 1)(x^2 + y^2) \frac{\Delta \gamma^2}{r^8 \Sigma^{\gamma^2}} \]
\[- (\gamma^2 - 1) (\frac{\Delta}{\Sigma})^{\gamma^2} (1 - \frac{m}{r} + \frac{m^2}{r^2}) \frac{1}{r^2}. \] 

(57)

As momentum density vanishes hence it is constant.

4.3 Energy and Momentum in Papapetrou’s Prescription

The required non-vanishing components of \( N^{abcd} \) are given by

\[ N^{0011} = -(\frac{\Delta}{\Sigma})^{\gamma^2-1} \frac{y^2}{x^2 + y^2} - \frac{\Delta x^2}{r^4} - \frac{x^2 z^2}{r^2 (x^2 + y^2)} \]
\[ - (1 - \frac{2m}{r})^{-2\gamma} \frac{\Delta \gamma^2}{r^2 \Sigma^{\gamma^2-1}}, \] 

(58)

\[ N^{0022} = -(\frac{\Delta}{\Sigma})^{\gamma^2-1} \frac{x^2}{x^2 + y^2} - \frac{\Delta y^2}{r^4} - \frac{y^2 z^2}{r^2 (x^2 + y^2)} \]
\[ - (1 - \frac{2m}{r})^{-2\gamma} \frac{\Delta \gamma^2}{r^2 \Sigma^{\gamma^2-1}}, \] 

(59)

\[ N^{0033} = -\frac{\Delta z^2}{r^4} - \frac{x^2 + y^2}{r^2} - (1 - \frac{2m}{r})^{-2\gamma} \frac{\Delta \gamma^2}{r^2 \Sigma^{\gamma^2-1}}, \] 

(60)

\[ N^{0012} = (\frac{\Delta}{\Sigma})^{\gamma^2-1} \frac{xy}{x^2 + y^2} - \frac{\Delta xy}{r^4} - \frac{xy}{r^2 (x^2 + y^2)} \]
\begin{align}
\text{Substituting Eqs.}(58)-(63) \text{ in Eq.}(23), \text{ we obtain the following energy density and momentum density components}
\end{align}

\begin{align}
\Omega_{00} &= \frac{1 - \frac{2m}{r}}{8\pi} \left[ -4\gamma m^2(2\gamma + 1) \frac{\Delta \gamma^2 - 2}{r \Sigma \gamma^2 - 1} + 8\gamma m \left\{ \frac{\Delta \gamma^2 - 2}{r \Sigma \gamma^2 - 1} \right\} \right] \\
N^{0013} &= -\frac{\Delta x z}{r^4} + \frac{x z}{r^2} \\
N^{0023} &= -\frac{\Delta y z}{r^4} + \frac{y z}{r^2}.
\end{align}

4.4 Energy and Momentum in Möller’s Prescription

For the gamma metric, we obtain the following non-vanishing components of $K_{a}^{bc}$

\begin{align}
K_{0}^{01} &= -2m \gamma \sin \theta.
\end{align}

When we make use of Eq.(65) in Eq.(30), the energy and momentum density components turn out to be

\begin{align}
M_{0}^{0} &= 0.
\end{align}
and

\[ M_0^i = 0 = M_i^0. \]  \hspace{1cm} (67)

This shows that energy and momentum turn out to be constant.

## 5 Conclusion

Energy-momentum complexes provide the same acceptable energy-momentum distribution for some systems. However, for some systems [22-30], these prescriptions disagree. The debate on the localization of energy-momentum is an interesting and a controversial problem. According to Misner et al. [35], energy can only be localized for spherical systems. In a series of papers [36] Cooperstock et al. has presented a hypothesis which says that, in a curved spacetime, energy and momentum are confined to the regions of non-vanishing energy-momentum tensor \( T_{\alpha}^{\beta} \) of the matter and all non-gravitational fields. The results of Xulu [19,20] and the recent results of Bringley [37] support this hypothesis. Also, in the recent work, Virbhadra and his collaborators [14-18] have shown that different energy-momentum complexes can provide meaningful results. Keeping these points in mind, we have explored some of the interesting members of the Weyl class for the energy-momentum distribution.

In this paper, we evaluate energy-momentum densities for the two solutions of the Weyl metric, i.e., Erez-Rosen and the gamma metrics. We obtain this target by using four well-known prescriptions of Einstein, Landau-Lifshitz, Papapetrou and Möller. From Eqs.(15), (22), (29), (34), (50), (57), (64) and (67), it can be seen that the energy-momentum densities are finite and well defined. We also note that the energy density is different for the four different prescriptions. However, momentum density components turn out to be zero in all the prescriptions and consequently we obtain constant momentum for these solutions. The results of this paper also support the Cooperstock’s hypothesis [36] that energy is localized to the region where the energy-momentum tensor is non-vanishing.

We would like to mention here that the results of energy-momentum distribution for different spacetimes are not surprising rather they justify that different energy-momentum complexes, which are pseudo-tensors, are not covariant objects. This is in accordance with the equivalence principle [35] which implies that the gravitational field cannot be detected at a point. These examples indicate that the idea of localization does not follow the
lines of pseudo-tensorial construction but instead it follows from the energy-momentum tensor itself. This supports the well-defined proposal developed by Cooperstock [36] and verified by many authors [22-30]. In GR, many energy-momentum expressions (reference frame dependent pseudo-tensors) have been proposed. There is no consensus as to which is the best. Hamiltonian’s principle helps to solve this enigma. Each expression has a geometrically and physically clear significance associated with the boundary conditions.
Acknowledgment

We would like to thank for the anonymous referee for his useful comments.

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