Leadership Exponent in the Pursuit Problem for 1-D Random Particles

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Abstract
For $n + 1$ particles moving independently on a straight line, we study the question of how long the leading position of one of them can last. Our focus is the asymptotics of the probability $p_{T,n}$ that the leader time will exceed $T$ when $n$ and $T$ are large. It is assumed that the dynamics of particles are described by independent, either stationary or self-similar, Gaussian processes, not necessarily identically distributed. Roughly, the result for particles with stationary dynamics of unit variance is as follows: $L := -\ln p_{T,n}/(T \ln n) = 1/d_0 + o(1)$, where $d_0/(2\pi)$ is the power of the zero frequency in the spectrum of the leading particle, and this value is the largest in the spectrum. Previously, in some particular models, the asymptotics of $L$ was understood as a sequential limit first over $T$ and then over $n$. For processes that do not necessarily have non-negative covariances, the limit over $T$ may not exist. To overcome this difficulty, the growing parameters $T$ and $n$ are considered in the domain $c \ln T < n \leq CT$ where $c > 1$. The Lamperti transform allows us to transfer the described result to self-similar processes by changing the $\ln p_{T,n}$ normalization to the value $\ln T \ln n$.

Keywords Exit time · Capture problem · Persistence probability

1 Introduction and the Main Result

In this paper we consider the pursuit problem in an ensemble of particles with random dynamics. The problem involves a population of particles on a straight line consisting of a single “pursued” particle and $n$”pursuing” particles. The motion of the particles is described by independent random Gaussian processes. It is assumed the pursued particle is ahead of the others at the start. The main problem is the distribution of the time $\tau_n$ that it takes to catch up with the pursued particle.
The problem was the subject of a lively discussion for particles with uniform Brownian dynamics in the 1990-2010s. The issue as to the finiteness of the mean $E \tau_n$ has turned out to be a nontrivial. For the Brownian particles De Blassie [7, 8] (see also [4]) has shown that

$$P(\tau_n > T) \sim m T^{-\gamma_n}, \ T \to \infty,$$

where $m$ depends on the initial particle positions,

$$2\gamma_n = \sqrt{\lambda_1 + (n-1)^2/4 - (n-1)/2}$$

and $\lambda_1$ is the first (principal) eigenvalue of the Dirichlet problem (with zero boundary conditions) for the Laplace–Beltrami operator on the subset $G_{n+1} \cap S^n$ of the unit sphere $S^n \subset R^{n+1}$:

$$G_{n+1} = \{x = (x_0, \ldots, x_n) : x_i - x_0 < 0, i = 1, \ldots, n\}.$$ 

The estimation of $\lambda_1$ is a technically complicated problem; nevertheless, it has been shown working on these lines that the mean $E \tau_n$ is infinite when $n \leq 3$ [5] and that it is finite when $n = 4$ [22]. Earlier, the finiteness of $E \tau_n$ for $n \geq 5$ has been proved by another method, [15].

Kesten [12] raised the issue of the asymptotics of $\gamma_n$ and showed that in the Brownian case

$$P(\tau_n > T) > T^{-\ln n(1+\epsilon)/4}, \ T > T_0(n), n > n_0(\epsilon),$$

for any $\epsilon > 0$.

For diffusing particles, Krapivsky and Redner [13] made strong arguments in favor of the asymptotics $\gamma_n \approx \ln(4Rn)/(4R)$, where $R = \sigma_0/\sigma$ is the ratio of the diffusivity of the leading particle, $\sigma_0$, to the corresponding parameter for the other particles, $\sigma$.

Li and Shao [16, 17] considered a non-Markov model in which the particle dynamics is described by the fractional Brownian motion (FBM) $B_H(t)$, i.e., by the centered Gaussian process with stationary increments of the form

$$E(B_H(t) - B_H(s))^2 = (t - s)^{2H}, \ 0 < H < 1, \ B_H(0) = 0.$$ 

The case $H = 1/2$ corresponds to Brownian motion.

The Li and Shao result is as follows: for independent processes $B^{(i)}_H(t), i \geq 0$,

$$P\{B^{(i)}_H(t) - B^{(0)}_H(t) < 1, 0 < t < T, 1 \leq i \leq n\} = T^{-\gamma_n,H + o(1)},$$ 

as $T \to \infty$ and

$$1/d_H \leq \lim_{n \to \infty} \inf \gamma_{n,H}/\ln n \leq \lim_{n \to \infty} \sup \gamma_{n,H}/\ln n < \infty,$$

where

$$d_H = 2 \int_0^\infty \left[e^{tH} + e^{-tH} - (e^{t/2} - e^{-t/2})^2\right] dt = \frac{2(1 - H)\Gamma(2H)}{\Gamma(1 + H)}.$$ 

The physical meaning of the constant will become clear later (see (1.14)).

The two-sided estimates of the exponent for the Brownian particles coincided, namely $\gamma_{n,1/2} = \ln n/(4 + o(1))$. This enabled Li and Shao to conjecture that for the general case of $H$ we must have

$$\gamma_{n,H} = \ln n/(d_H + o(1)).$$ 

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Our goal is to consider the case of particles with heterogeneous dynamics, and to use the limit transition in which \( T \) and \( n \) are varied simultaneously. In this way, we can prove the asymptotics \((1.4, 1.6)\).

More specifically, we will assume the following.

**Assumption A** The dynamics of any, say \( j \)-th, particle is described by a continuous centered Gaussian process \( X^{(j)}(t), t \geq 0 \), all processes are independent, and the set of its covariance functions, \( EX^{(j)}(t)X^{(j)}(s) = r_j(t, s), \ r_j(1, 1) = 1, \) is finite.

In addition, all elements of the ensemble \( \{X^{(j)}(t), j \geq 0\} \) can be either stationary or self-similar of a fixed index \( H \). This means that in the stationary case \( r_j(t, s) = r_j(t-s) = \int_{-\infty}^{\infty} e^{i(t-s)\lambda} dF_j(\lambda) \), where \( dF_j(\lambda) \) is the (symmetric) spectral measure of the process, whereas in the self-similar case \( r_j(t, s) = s^{2H} r_j(t/s, 1) \) and \( X^{(j)}(0) = 0 \).

An example of a self-similar process is the fractional Brownian motion of the index \( 0 < H < 1 \).

**The Problem** Our problem is the asymptotics of the probability

\[
p_{T,n}(u) = P\{X^{(i)}(t) - X^{(0)}(t) < u, i = 1, \ldots, n, 0 \leq t \leq T\}
\]

as \( T, n \to \infty \). We will consider \( u = 1 \) in the conditional situation \( \{X^{(i)}(0) = 0, i \geq 0\} \) and \( u = 0 \) in the unconditional one.

The expected order of the \( p_{T,n}(0) \) asymptotics in the stationary case can be illustrated by the following simple example. Consider \((n + 1)\) i.i.d. sequences \( \{X^{(i)}(k), 1 \leq k \leq S\} \) with independent (even non-Gaussian) components. Then

\[
p_{S,n}(0) = P\{X^{(i)}(k) < X^{(0)}(k), i = 1, \ldots, n, 1 \leq k \leq S\}
\]

\[
= (n + 1)^{-S} \approx e^{-S \ln n}
\]

In the general (stationary) case we can expect that for large \( T,n \) our problem can be approximated using the array of random variables described above with \( S = T/d_0 \).

To implement this idea, we relied on the works [3, 10, 16] and on the theory of extreme values for random Gaussian processes, [18].

Let us formulate the main result.

**Theorem 1** Consider an ensemble of stationary processes \( \{X^{(i)}(t), i \geq 0\} \) with the property

A, assuming that \( r_0(t) \in L_1 \) and \( d_0 = \int r_0(t)dt > 0 \) (in this case \( X^{(0)}(t) \) has a continuous spectral density \( f_0(\lambda) = F_0(d\lambda)/d\lambda \)).

**The Lower Bound.** Let

\[
(a) \quad 1 - r_0(t) \leq c_0|t|^{2H_0}, \ |t| \leq \varepsilon_0, \text{for some } 0 < H_0 < 1, c_0 > 0.
\]

Then

\[
\lim \inf_{T,n \to \infty} \ln p_{T,n}(0)/(T \ln n) \geq -1/d_0
\]

provided that \( T, n \) increase so that \( \ln n \leq CT \).

**The Upper Bound.** Let for any \( i \geq 1 \)

\[
(b) \quad 1 - |r_i(t)| \leq c_i |t|^{2h_i} \land \rho_i \text{ for some } 0 < \rho_i < 1, h_i > 0, c_i > 0;
\]

\[
(c) \quad |r_i(t)| \leq \tilde{r}_i(t) \in L_1, \text{where } \tilde{r}_i(t) \text{ is a monotone function};
\]

\[
(d) \quad f_0(\lambda) \leq f_0(0) < \infty;
\]

\[
(e) \quad f_0(\lambda) \leq \tilde{f}_0(\lambda) \in L_1, \text{where } \tilde{f}_0(\lambda) \text{ is a monotone function};
\]
(f) \( \ln f_0(\lambda) \in L_{1, \text{loc}} \).

Then
\[
\lim_{T,n \to \infty} \sup \ln p_{T,n}(0)/(T \ln n) \geq -1/d_0
\]
provided that \( T,n \) increase so that \( c \ln T \leq \ln n \), with any fixed \( c > 1 \).

**Remark 1** The quantity \( 1/d_0 \) is determined by the dynamics of the leading particle. Therefore, it is quite natural to call it a leadership exponent.

The following assumptions were crucial for proving Theorem 1: the relation \( f_0(\lambda) \leq f_0(0) \), according to which the leading particle has peak value of the spectral density at zero frequency; independence of the particles and weakening of its own dependence after a small time delay: \( |r_i(t)| \leq \bar{\rho} < 1, |t| > \varepsilon, i \geq 1 \) and the variance equality: \( r_i(0) = \text{const}, i \geq 0 \).

The other conditions of Theorem 1 are more technical in nature and most likely not optimal: (a, b) are related to local smoothness of processes, since \( 1 - r_i(t) = E(X^{(i)}(t+s) - X^{(i)}(s))^2/2 \); and (f) is related to the unpredictability of \( X^{(0)}(t) \) based on discrete data \( \{X^{(0)}(t+k\theta), k \geq 1\} \) for any small \( \theta \geq 0 \).

**Remark 2** Obviously, \( d_0 = \int r_0(t)dt = 2\pi f_0(0) \), that shows the role of the low-frequency spectral component of the leading particle in our problem.

Since \( 2\pi f_0(\lambda) \leq \int |r_0(t)|dt \) the assumption (d): \( f_0(\lambda) \leq f_0(0) \) is automatically met if \( r_0(t) \geq 0 \). In addition, in this case a modified normal comparison inequality proposed in [16] allows us to prove the upper bound (1.8) without any restrictions on the growth of the parameters \( T \) and \( n \). The proof is omitted because it does not require new ideas but complicates the text.

The peculiarity of the problem under consideration is that the low-frequency part of the spectrum of a Gaussian stationary process expanding is controlled by a one-dimensional parameter, say, based on the similarity of the regions. This restriction is covered by the conditions of Theorem 1.

Finiteness of the set of covariance functions \( \{r_i(t), i \geq 0\} \) is not essential. This condition is used only in order to have uniform restrictions of the parameters in the stochastic dynamics of particles.

**Remark 3** The above pursuit problem can be interpreted as a persistence problem for the random field \( \xi(t, i) = X^{(i)}(t) - X^{(0)}(t) \) on the expanding region \( [0, T] \times [1, n] \).

Only a few nontrivial examples of Gaussian fields are known for which the persistence exponent (in our case \( 1/d_0 \)) is obtained explicitly (see e.g. [21]). In these examples the expanding is controlled by a one-dimensional parameter, say, based on the similarity of the regions. In our case \( \log p_{T,n}(0) = O(T \log n) \). Therefore the restriction of the type \( 0 < r < \log n/T < R \) for the stationary case can be considered as an analogous of the similarity of the regions. This restriction is covered by the conditions of Theorem 1.

The background of the persistence problem for random processes can be found in the surveys [1, 6]. The current state of the problem is well presented in [2, 9, 10]. In [10], it is shown how the low-frequency part of the spectrum of a Gaussian stationary process determines the order of growth, of the log-probability: \( -\ln P[X(t) \leq 0, 0 \leq t \leq T] \) as \( T \to \infty \). The peculiarity of the problem under consideration is that the low-frequency component of the leading particle determines not only the order \( O(T \ln n) \) of growth of \( -\ln p_{T,n} \) but also the constant \( 1/d_0 \) in this asymptotics.

### 1.1 H-Self-similar (H-ss) Processes

Originally, the pursuit problem was considered for (fractional) Brownian particles with a fixed index \( H \), initial conditions \( \{X^{(i)}(0) = 0, i \geq 0\} \) and the threshold \( u = 1 \).
In this case it can be useful the Lamperti transformation: \( \tilde{X}(\tau) = X(e^\tau)e^{-\tau H} \), because it maps any \( H \)-ss process \( \{X(t), 1 \leq t \leq T\} \) to a stationary process \( \{\tilde{X}(\tau), 0 \leq \tau \leq \tilde{T} := \ln T\} \).

In addition, for the threshold value \( u = 0 \), we shall have

\[
p_{(1,T),n}(0) = \tilde{p}_{(0,\tilde{T}),n}(0),
\]

where it is used a slightly modified notation

\[
p_{\Delta,n}(u) = P\{X^{(i)}(t) - X^{(0)}(t) \leq u, t \in \Delta, 1 \leq i \leq n\},
\]

and the similar one \( \tilde{p}_{\Delta,n}(u) \) for the transformed processes \( \tilde{X}^{(i)}(\tau) \).

The original asymptotic problem for \( p_{T,n}(1) = p_{(0,T),n}(1) \) can be solved using the described reduction. To do this, we need to show that the log-asymptotics of \( p_{(0,T),n}(1) \) and \( p_{(1,T),n}(0) \) normalized by the factor \( [\ln T \ln n]^{-1} = [\ln T \ln n]^{-1} \) are identical. The following statement provides sufficient conditions for this purpose.

**Theorem 2** Let \( \{X^{(i)}(t), i \geq 0\} \) be an ensemble of \( H \)-ss processes with the property A.

1. Assume that \( T, n(T) \to \infty \) in such a way that \( \ln n \leq C \ln T, C > 0 \), then

\[
\liminf \ln p_{(1,T),n}(0)/\psi_{T,n} \leq \liminf \ln p_{(0,T),n}(1)/\psi_{T,n}, \tag{1.10}
\]

where \( \psi_{T,n} = \ln T \ln n \).

2. Assume that

\[
r_0(t, 1)t^{-H-1} \in L_1(0, \infty) \text{ and } d_0 = \int_0^\infty r_0(t, 1)t^{-H-1}dt > 0. \tag{1.11}
\]

Then

\[
\limsup \ln p_{(1,T),n}(0)/\psi_{T,n} \geq \limsup \ln p_{(0,T),n}(1)/\psi_{T,n}, \tag{1.12}
\]

where both limits are related to an arbitrary but common sequence \( T, n(T) \to \infty \).

Obviously, the coincidence of the left-hand parts of inequalities (1.10) and (1.12) entails equality of their right-hand parts.

**Corollary 3** The pursuit problem for fractional Brownian particles of index \( H, X^{(i)}(t) = B^{(i)}_H(t), i = 0, \ldots, n, \) has the asymptotics

\[
\ln p_{T,n}(1)/[\ln T \ln n] = -1/d_H(1 + o(1)) \tag{1.13}
\]

with the hypothetical Li-Shao constant \( d_H, (1.5) \), provided that, \( c \ln \ln T < \ln n \leq C \ln T, c > 1 \) and \( T \to \infty \).

In addition, the result remains valid after replacing \( \{X^{(i)}(t), i = 1, \ldots, n\} \) with independent processes \( \{t^{H-H_i}B^{(i)}_H(t), i = 1, \ldots, n\}, \) where \( 0 < H, \min H_i, \max H_i < 1 \) and the number of different \( H_i \) is finite.

**Remark 4** Following Remark 2, we note that \( -1/d_H \) as the upper bound of the left part (1.13) can be obtained without additional restrictions on the growth of the \( (T, n) \) parameters.
Proof of Corollary 3. The Lamperti transform of the fractional Brownian motion $B_H(t)$ is the stationary process $\tilde{B}_H(\tau)$ with the covariance function

$$\tilde{r}_H(t) = [e^{t\tilde{H}} + e^{-t\tilde{H}} - (e^{t/2} - e^{-t/2})^{2\tilde{H}}]/2 > 0,$$

i.e. the Li-Shao constant $d_H = \int \tilde{r}_H(t)dt$ is related to the Lamperti transform of the process under consideration. Now the conditions of Theorem 1 are easily verified because $\tilde{r}_H(t) = 1 - |t|^{2\tilde{H}/2} + O(t^2), t \to 0$ and $\tilde{r}_H(t) = e^{-t\tilde{H}/2} + H e^{-t(1-H)} + O(e^{-t(2-H)}), t \to \infty$.

In addition, the spectral function of $\tilde{B}_H(\tau)$ is strictly positive:

$$\tilde{f}_H(\lambda) = c_{\tilde{H}} \cosh(\pi \lambda) |\Gamma(-H + i\lambda)|^2 > 0,$$

where $\Gamma(\cdot)$ is the Gamma-function. Therefore, $\ln \tilde{f}_H(\lambda) \in L_{1,loc}$; further, $\tilde{f}_H(\lambda) \leq \tilde{f}_H(0)$ because $\tilde{r}_H(t) > 0$.

Since the Lamperti transform of $t^{\tilde{H}-1} \tilde{B}_H(\tau)$ is equal to $\tilde{B}_H(\tau)$, the (a, b, c)-conditions of Theorem 1 follow from the asymptotic behavior of $\tilde{r}_H(\tau)$ at zero and at infinity. Finally, the condition (1.11) of Theorem 2 means nothing else but $\tilde{r}_H(t) \in L_1$ and $\int \tilde{r}_H(t)dt > 0$.

2 Proof of Theorem 1

Below, some non-essential, finite, and not necessarily identical constants can have the same notation. Equality $\equiv$ for stochastic objects means equality of their finite-dimensional distributions.

2.1 The Lower Bound of $P_{T,n}(0), 1 \ll n \leq CT$

We have to estimate $p_{T,n}(0) = P\{X^{(i)}(t) - X^{(0)}(t) \leq 0, 0 \leq t \leq T, 1 \leq i \leq n\}$ for independent stationary processes $\{X^{(i)}(t), i \geq 0\}$.

Let $M_T^{(i)} = \sup_{t \in [0,T]} |X^{(i)}(t)|, 0 \leq t \leq T\}.$

Since the processes $\{X^{(i)}(t), i = 0, 1, \ldots, n\}$ are independent and are stochastically symmetric, $\{X^{(0)}(t)\} \overset{d}{=} \{-X^{(0)}(t)\}$, we have for any $a > 0$

$$p_{T,n}(0) \geq P\{X^{(0)}(t) \leq -a, X^{(i)}(t) \leq a, 0 \leq t \leq T, 1 \leq i \leq n\}$$

$$= P(M_T^{(0)} \leq -a) \prod_{i=1}^n P\{M_T^{(i)} \leq a\}. \quad (2.1)$$

Estimate of $P(M_T^{(i)} \leq a)$.

Obviously, $P(M_T^{(i)} \leq a) \geq P(|X^{(i)}(t)| \leq a, t \in (0, T))$.

For any continuous centered Gaussian process, the Gaussian correlation inequality, [14, 23], implies

$$P(|X(t)| \leq a, 0 \leq t \leq T) \geq \prod_i P(|X(t)| \leq a, t \in \Delta_i), \cup \Delta_i = [0, T]$$

for any intervals $\Delta_i$.

Assuming $\Delta_i = [i\tau, (i+1)\tau], \Delta_i = [i\tau, (i+1)\tau], T = \tau \cdot m, m \in \mathbb{Z}_+$, and the stationarity of $X(t)$, one has $P(|X(t)| \leq a, 0 \leq t \leq T) \geq \prod [P(|X(t)| \leq a, 0 \leq t \leq \tau)]^m$.

Recall the concentration principle for the maximum of a continuous centered Gaussian process $X(t)$, [19]. Suppose $\mu_{\tau}$ is the median of the distribution of $M_{\tau} = \max(X(t), 0 \leq t \leq \tau)$.
where $\sigma^2(\tau) = \max_{0 \leq t \leq \tau} E[X(t)]^2$, and $\Phi(x)$ is the standard Gaussian distribution, then for any $x > 0$

$$P(M_\tau \leq \mu_\tau + x) \geq \Phi(x/\sigma(\tau)) \quad (2.2)$$

Note that $m_\tau = \inf(X(t), 0 \leq t \leq \tau) \overset{d}{=} -M_\tau$.

Therefore,

$$P(|X(t)| \leq a, 0 \leq t \leq \tau) \geq P(M_\tau \leq a) - P(m_\tau \leq -a) = P(M_\tau \leq a) - P(M_\tau \geq a) \geq 1 - 2\Psi((a - \mu_\tau)/\sigma(\tau)), \ a > \mu_\tau,$$

where $\Psi(x) = 1 - \Phi(x) = \Phi(-x)$.

In the stationary case, $\sigma^2(\tau) = \sigma^2(0)$.

Since $\Psi(x) \leq 0.5 \exp(-x^2/2), x > 0$ and $E[X^{(i)}(0)]^2 = 1$, we have

$$P(|X^{(i)}(t)| \leq a, 0 \leq t \leq \tau) \geq 1 - \exp(-a - (\mu^{(i)}_\tau)^2/2), \ a \geq \mu^{(i)}_\tau \quad (2.3)$$

Since the set of covariance functions is finite, $\mu_\tau = \sup_\xi \mu^{(i)}_\tau < \infty$.

Therefore, by setting

$$a = \sqrt{2 \ln n + \mu_\tau}, \ \tau > n/(n-1),$$

we have

$$\prod_1^n P(M^{(i)}_\tau < a) \geq (1 - \exp(-a - \mu_\tau^2/2))^{nm} \geq (1 - 1/n)^{nT/\tau} \geq (1 - 1/n)^{(n-1)T} \geq e^{-T}. \quad (2.5)$$

**Estimate of $P(M^{(i)}_\tau \leq -a$).**

Since $X^{(0)}(t)$ has a continuous spectral function $f_0(\lambda)$, $0, f_0^*(\delta) = \min\{f_0(\lambda), 0 \leq \lambda \leq \delta\} \rightarrow f_0(0) = (2\pi)^{-1} \int r_0(t)dt > 0, \ \delta \rightarrow 0$.

But then for small $\delta$, we can decompose the spectral function into two nonnegative terms: $f_0(\lambda) = f_\delta(\lambda) + f_c(\lambda)$ where $f_\delta(\lambda) = f_0^*(\delta) 1_{[\lambda < \delta]}$.

Hence, we can consider the Gaussian process $X^{(0)}(t)$ as sum $X^{(0)}(t) = \xi_\delta(t) + \xi_c(t)$ of two independent stationary processes with spectral functions $f_\delta(\lambda)$ and $f_c(\lambda)$ respectively.

Given the independence of the decomposition components, we have

$$P(M^{(i)}_\tau \leq -a) = P(\xi_\delta(t) + \xi_c(t) \leq -a, 0 \leq t \leq T) \geq P(M^{(i)}_\tau \leq -a - 1) P(M^{(c)}_\tau \leq 1) \quad (2.6)$$

where $M^{(i)}_\tau = \sup_\xi \xi_\delta(t), 0 \leq t \leq T, \alpha \in \{\delta, c\}$.

**Estimate of $P(M^{(c)}_\tau \leq 1$).**

By assumption (a),

$$E(X^{(0)}(t_1) - X^{(0)}(t_2))^2 = 2(1 - r_0(|t_1 - t_2|)) \leq c_0|t_1 - t_2|^{2H_0}, \ c_0 > 0.$$ 

Since $E(\Delta X^{(0)})^2 = E(\Delta \xi_\delta)^2 + E(\Delta \xi_c)^2$, we have $E(\Delta \xi_c)^2 \leq c_0|\Delta|^{2H_0}$.

Under such condition, the Talagrand’s theorem [24] guarantees that

$$P(\{\xi_c(t) \leq V, 0 \leq t \leq T\} \geq \exp(-KT/V^{1/H_0}) \quad (2.7)$$

with some $K > 0$.

In our case, $V = 1$.

**Estimate of $P(M^{(i)}_\tau \leq -a - 1$).**
The process $\xi_\delta(\tau)$ has finite spectrum and is not correlated at the points $\tau = \pi k/\delta$.

Therefore it admits the Kotelnikov-Shannon representation in terms of discrete white noise $\{\eta_n\}$:

$$
\xi_\delta(\tau) = \sigma_\delta \sum_{n \in \mathbb{Z}} \eta_n \frac{\sin(\delta \tau - \pi n)}{\delta \tau - \pi n} = \sigma_\delta S(\delta \tau / \pi),
$$

(2.8)

where $\sigma_\delta = \sqrt{2\delta \int_0^\pi (\delta \tau / \pi)^2}^{-1}$.

In terms of the random function $S(t)$ the probability under consideration is

$$
P(M_1^{(\delta)} \leq -a - 1) = P\{S(t) + a_\delta < 0, |t| \leq T_\delta/2\} := Q_{T_\delta},
$$

(2.9)

where $a_\delta = (a + 1)/\sigma_\delta$ and $T_\delta = T \delta / \pi$.

Following [3], consider for odd $N$

$$
g_N(t) = \sum_{|n| < N/2} \frac{\sin \pi (t - n)}{\pi (t - n)},
$$

then

$$
S(t) + a_\delta = S_N(t; a_\delta + 1) - 1 - (a_\delta + 1)(g_N(t) - 1) + R_N(t).
$$

(2.10)

Lemma 4 (see the Appendix for proof). There exists a constant $v_0 > 0$ such that

$$
|g_N(t) - 1| < v_0/(N - T_\delta) + e^{-\pi N/2} := \kappa(\delta, N)
$$

(2.11)

for $|t| \leq T_\delta/2 < N/2$.

Taking into account the independence of the terms in (2.10) and the inequality (2.11), we can estimate (2.9) as follows:

$$
Q_{T_\delta} \geq P\{|S_N(t; a_\delta + 1) < 1/2 - (a_\delta + 1)\kappa(\delta, N), |t| \leq T_\delta/2\} \times P\{R_N(t) < 1/2, |t| \leq T_\delta/2\} := Q_{T_\delta}^{(1)} \times Q_{T_\delta}^{(2)}.
$$

(2.12)

Suppose that

$$
(a_\delta + 1)\kappa(\delta, N) < 1/4.
$$

(2.13)

Then $Q_{T_\delta}^{(1)} \geq P\{|S_N(t; a_\delta + 1)| < 1/4, |t| \leq T_\delta/2\}$.

The function $S_N(t; a_\delta + 1)$ is obtained from $S_N(t; 0)$ by shifting the i.i.d. random variables $\{\eta_n, |n| < N/2\}$ by the constant $(a_\delta + 1)$. Therefore we can continue $Q_{T_\delta}^{(1)} \geq E_1 e^{\Theta_N} e^{-N(a_\delta + 1)^2/2}$, where $\Theta_N = \{|S_N(t; 0)| < 1/4, |t| \leq T_\delta/2\}$ and $\zeta = (a_\delta + 1) \sum_{|n| < N/2} \eta_n$.

Since $\eta_n = S_N(n; 0)$, the event $\Theta_N$ entails $\eta_n \geq -1/4, |n| \leq T_\delta/2$.

Therefore

$$
\zeta \geq -(a_\delta + 1)(T_\delta + 1)/4 + \zeta_\Delta, \zeta_\Delta = (a_\delta + 1) \sum_{T_\delta/2 < |n| < N/2} \eta_n.
$$

By the Cauchy–Schwartz inequality $E_1 e^{\zeta_\Delta/2} \cdot e^{-\zeta_\Delta/2} \leq E_1 e^{\zeta_\Delta} e^{-\zeta_\Delta} = E_1 e^{\zeta_\Delta} \exp((a_\delta + 1)^2(N - T_\delta)/2)$.  

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Hence,
\[
Q_{T_3}^{(1)} \geq E1\theta_{N}e^{\delta N}e^{-(a_\delta + 1)(T_\delta + 1)/4}e^{-N(a_\delta + 1)^2/2} \geq \left[ P(|S_N(t; 0)| < 1/4, |t| \leq T_\delta/2) \right]^2 \\
\times \exp[-(2N - T_\delta)(a_\delta + 1)^2/2 - (a_\delta + 1)(T_\delta + 1)/4]
\]
(2.14)

Since \(E|\Delta S_N(t; 0)|^2 \leq E|\Delta S(t)|^2 = \pi^{-1} \int_{0}^{\pi} (1 - \cos(\Delta \lambda)) d\lambda < 2|\Delta|^2\), the Talagrand’s theorem [24] implies
\[
P(|S_N(t; 0)| < 1/4, |t| \leq T_\delta/2) \geq e^{-K_1T_\delta} = e^{-K_1T_\delta}.
\]
(2.15)

According to [3] (Sect. 2.2.2),
\[
Q_{T_3}^{(2)} = P\{R_N(t) < 1/2, |t| \leq T_\delta/2\} \geq 1 - c_1/\sqrt{N - T_\delta}.
\]
(2.16)

Putting the above inequalities (2.5), (2.7), (2.14)–(2.16) together yields
\[
p_{T,n} \geq K \exp[-K_1T - 2K_2T_\delta - \{2(N - T_\delta) + T_\delta\}(a_\delta + 1)^2/2 - (a_\delta + 1)(T_\delta + 1)/4]
\]
(2.17)

for \(N - T_\delta \gg 1\).

To choose the appropriate \(\delta\), we recall all the constants that we used: \(T_\delta = T\delta/\pi, a_\delta = (a + 1)/\sigma_\delta, \sigma_\delta = \sqrt{2f_0(\delta)}, a = \sqrt{2\ln n + \mu_\rho}\), and our assumptions:
\[
(a_\delta + 1)[c/(N - T_\delta) + e^{-\pi\cdot N/2}] \leq 1/4 \text{ and } N - T_\delta \gg 1.
\]
(2.18)

Here \(T, n, \delta^{-1}\) as well as \(a\) and \(a_\delta\) are large numbers. It is not difficult to see that we can satisfy (2.18) by setting \(N - T_\delta = \kappa_0a_\delta\) with some \(\kappa_0 > 0\).

Then the argument of the exponent in (2.17) taken with the sign \((-)\) is
\[
O(T) + O(a_\delta^3) + T_\delta a_\delta^2/2 \cdot (1 + O(1/a_\delta))
\]
\[
= O(T) + O((\ln n/\delta)^{3/2}) + T\ln n \cdot (2\pi f_0(\delta))^{-1}(1 + O(\ln n/\delta)^{-1/2})
\]
(2.19)

Putting \(\delta^6 \ln n = 1\), we have \(p_{T,n}(0) \geq \exp(-c_1(T + (\ln n)^{-3/4}) - T \ln n/(2\pi f_0(\delta))^2)(1 + c_2(\ln n)^{-7/12}), c_1 > 0\).

Assume that, \(\ln n \leq CT\), then
\[
p_{T,n}(0) \geq \exp(-(T \ln n)(1 + c_1(T^{-1/4} + (\ln n)^{-7/12}))(2\pi f_0^*(\delta)),
\]
(2.20)
where \(f_0^*(\delta) - f_0(0) = o(1)\) since \(\delta = (\ln n)^{-1/6} \to 0\).

\[\square\]

### 2.2 The Upper Bound of \(p_{T,n}(0), c \ln T < \ln n, c > 1\)

The idea of proof goes back to the work [16], but it is not enough to analyze the general case under consideration. The proof relies on two lemmas:

**Lemma 5** [18] Let \(\{\xi_i, i = 1, \ldots, n\}\) be a centered Gaussian stationary sequence with a covariance function \(r(i)\) such that \(\max_{i \geq 1}|r(i)| = \bar{\rho} < r(0) = 1\).

Then for any subsequence \(\{\tilde{\xi}_{k(i)}, i = 1, \ldots, v \leq n\}\)
\[
|P[\tilde{\xi}_{k(i)} \leq a, i = 1, \ldots, v] - \Phi(a)v| \leq K(1 - \bar{\rho}^2)^{-1/2}v \sum_{i=1}^{n} |r(i)| \exp(-a^2/(1 + \bar{\rho}))
\]
(2.21)

where \(\Phi(x)\) is the standard Gaussian distribution.
Lemma 6 (see Appendix for proof). Let $X(t)$ be a centered Gaussian stationary process with a continuous spectral density $f_0(\lambda)$.

Assume that (a) $f_0(\lambda) \leq f_0(0) < \infty$, (b) $f_0(\lambda) \leq \tilde{f}_0(\lambda) \in L_1$, where $\tilde{f}_0(\lambda)$ is a monotone function, (c) $\ln f_0(\lambda) \in L_{1,\text{loc}}$.

Then for any real numbers $\{u_i\}$,

$$P[X(i\theta) > u_i, i = 1, \ldots, m] \leq P[\sigma_\theta \eta_i > u_i, i = 1, \ldots, m] \exp(A_\theta m/2) \quad (2.22)$$

where $\{\eta_i, i = 1, \ldots, m\}$ are i.i.d. standard Gaussian variables,

$$A_\theta = \sup_{0 \leq \lambda \leq \pi/\theta} \ln[2\pi f_0(0)/f_0(\lambda)] >_A +1, \theta \leq \theta_0, \quad (2.23)$$

$$< \psi(\lambda) >_A$$

is the mean of $\psi(\lambda)$ in the interval $(0, A)$, and

$$\theta \sigma_\theta^2 = 2\pi f_0(0) + o(1), \theta \to 0. \quad (2.24)$$

Proof of (1.8). We discretize the time in increments of $\theta$ so that $T = m\theta$ and $m$ is integer. Using notation $M_n(t) = \max[X(i)(t), i = 1, \ldots, n]$, we have

$$p_{T,n}(0) = P[M_n(t) < X^{(0)}(t), 0 \leq t \leq T] \leq P[M_n(k\theta) < X^{(0)}(k\theta), k = 0, \ldots, m]$$

$$= E\{P[M_n(k\theta) < X^{(0)}(k\theta), k = 0, \ldots, m|M_n(\cdot)]\}$$

(Here and below, $P(A|B)$ is the conditional probability of $A$ given $B$).

Lemma 6 allows replacing the process $X^{(0)}$ with a sequence $\{\sigma_\theta \eta_i, i = 1, \ldots, m\}$ of i.i.d. Gaussian variables, where $\theta \sigma_\theta^2 = 2\pi f_0(0) + o(1), \theta \to 0$.

For this reason,

$$p_{T,n}(0) \leq P[M_n(k\theta) < \sigma_\theta \eta_k, k = 0, \ldots, m] \exp(A_\theta m/2) := J_{n,m} \exp(A_\theta m/2), \quad (2.25)$$

where $A_\theta$ is given by (2.23).

Let $v_\alpha = \#[\sigma_\theta \eta_k > \alpha, k = 1, \ldots, m], \bar{v}_\alpha = m - v_\alpha$, and $\bar{\theta} = 1 - \theta$.

Then

$$J_{n,m} \leq P(v_\alpha > \bar{\theta} m) + P[M_n(k\theta) < \sigma_\theta \eta_k, k = 0, \ldots, m; \bar{v}_\alpha > \theta m] := J^{(1)}_{nm} + J^{(2)}_{nm}.$$

Since $v_\alpha$ is a binomial random variable with parameters $m$ and $p = \Psi(\alpha/\sigma_\theta)$, we have

$$J^{(1)}_{nm} \leq \sum_{k > \bar{\theta}} C_m^k p^k (1 - p)^{m-k} \leq C_m^{[\bar{\theta} m]} p^{[\bar{\theta} m]} (1 - p)^{-1} \leq c_2^m [\Psi(\alpha/\sigma_\theta)]^{[\bar{\theta} m]}, \quad (2.27)$$

$$J^{(1)}_{nm} \leq \sum_{k \geq \bar{\theta}} C_m^k p^k \geq p^{[\bar{\theta} m]} (1 - p)^{-1} > c_2^m [\Psi(\alpha/\sigma_\theta)]^{[\bar{\theta} m]}, \quad (2.28)$$

Note, that $\theta \sigma_\theta^2 = d_0 + o(1), \theta \to 0, d_0 = 2\pi f_0(0), T = \theta \cdot m$, and

$$< v < \Psi(u)/[u^{-1} e^{-u^2/2}] < V, u > u_0 > 0. \quad (2.26)$$

Here and below $\theta$ and the other parameters depend on $n$ and therefore the notation $o(1)$ means that $o(1) \to 0$ as $n \to \infty$.

For any $d^2_n = 2 \ln n(1 + o(1))$ we have

$$J^{(1)}_{nm} \leq K_1 \exp(-\bar{\theta} T \ln n/(d_0 + o(1)) + T \theta^{-1} \ln 2), K_1 > 0,$$

$$J^{(1)}_{nm} \geq k_1 \exp[-\bar{\theta} T \ln n/(d_0 + o(1)) - T \theta^{-1} \ln(c_1 \ln n)], k_1 > 0.$$
Moreover
\[ \ln[J_{nm}^{(1)} \exp(A_\theta T / 2\theta)] = -T \ln n/d_0(1 + o(1)) \text{ if } A_\theta/\theta = o(\ln n). \quad (2.28) \]

**Estimate of** \( J_{nm}^{(2)}. \)

Let \( I_a = \{ k_i, i = 1, \ldots, \bar{v}_a : \eta_k, \sigma_\theta \leq a_n \}. \)

Then
\[
J_{nm}^{(2)} = \mathcal{P}\{ M_a(k_\theta) < \sigma_\theta \eta_k, k = 0, \ldots, m; \bar{v}_a \geq \theta m \} \\
\leq \mathcal{P}\{ X^{(p)}(k_\theta) < a_n, p = 1, \ldots, n; k_i \in I_a, \bar{v}_a \geq \theta m \} \\
= E \prod_{p=1}^n \mathcal{P}\{ X^{(p)}(k_\theta) < a_n | \{ k_i \} = I_a, \bar{v}_a \geq \theta \cdot m \}.
\]

Due to the condition (b), we have \( \max_{k \geq 1} |r_i(k_\theta)| = \tilde{\rho}_{\theta,i} < 1. \)

Therefore, applying Lemma 5, we can continue
\[
\leq E \Pi_{i=1}^n \left[ \Phi^{\bar{v}_a}(a_n) + K(1 - \tilde{\rho}_{\theta,i}^2)^{-1/2} \bar{v}_a \theta^{-1} \right. \\
\times \exp(-a_n^2/(1 + \tilde{\rho}_{\theta,i}), i = 1, \ldots, \bar{v}_a | \bar{v}_a \geq \theta \cdot m = T). \]

According to the (c)-condition, \( \sum_{k \geq 1} |r_i(k_\theta)| \theta \leq \int_0^\infty \bar{r}_i(t) \, dt = I_i < \infty. \) Since the set of covariance functions is finite, we have \( \sup I_i < \text{const} \) and \( \tilde{\rho}_\theta = \sup_{i \geq 1} \tilde{\rho}_{\theta,i} < 1. \)

Therefore \( J_{nm}^{(2)} \leq \left[ \Phi^{T}(a_n) + K(1 - \tilde{\rho}_{\theta}^2)^{-1/2} T \theta^{-1} \exp(-a_n^2/(1 + \tilde{\rho}_{\theta})) \right]^n := [W_1 + W_2]^n. \)

Now we will refine \( a_n \) by finding it as the root of the equation \( \Psi(a_n) = n^{-1+\vartheta}, 0 < \vartheta < 1. \)

It’s easy to see that \( a_n^2 = (1 - \vartheta^2) n^2 - 2 \ln n n \cdot (1 + o(1)). \)

i.e. \( a_n^2 = 2 \ln n (1 + o(1)) \) if \( \vartheta = o(1). \)

In this case
\[ W_1^n = [\Phi^{T}(a_n)]^n = (1 - n^{-1+\vartheta})^n \leq \exp(-n^{\vartheta} T). \quad (2.29) \]

Now we estimate \([1 + W_2/W_1]^n. \) First of all,
\[
W_1 = (1 - n^{-1+\vartheta})^T \geq \exp(-T/(n^{1-\vartheta} - 1)) \\
\geq \exp(-2n^{\vartheta+q-1}), \text{ if } \ln T \leq q \ln n, 0 < \vartheta + q < 1, n > n_0; \\
W_2 = c(1 - \tilde{\rho}_{\theta}^2)^{-1/2} T \theta^{-1} \exp(-a_n^2/(1 + \tilde{\rho}_{\theta}) \).
\]

By the assumption, \( c \ln n \leq \ln T, c > 1. \) Therefore we can choose \( q = 1/c \) because \( \vartheta = o(1). \)

Let \( \rho_\theta = 1 - \tilde{\rho}_\theta \) and \( \phi(\theta) = \rho_\theta^{1/2} \), then
\[
W_2/W_1 \leq K_2 T/\phi(\theta) \cdot \exp[(-1 - \vartheta)/(1 - \rho_\theta/2) \ln n - 2 \ln \ln n(1 + o(1)) + 2n^{\vartheta+q-1}] \\
\leq n^{-1} K_2 T/\phi(\theta) \cdot \exp(-n n\cdot(\rho_\theta/2 - \vartheta)/(1 - \rho_\theta/2) + 1), n \geq n_0.
\]

By setting \( \vartheta = \rho_\theta/4, \) we get \([1 + W_2/W_1]^n \leq \exp(K_2 T/\phi(\theta) \cdot n^{-\rho_\theta/4}), K > 0. \)

Given (2.29), we have
\[ J_{nm}^{(2)} \leq [W_1 + W_2]^n \leq \exp[-T(n^{\rho_\theta/4} - \tilde{c} n^{-\rho_\theta/4} \rho_\theta^{1/2} \theta^{-1})]. \]

Using the condition (b) and finiteness of the covariance functions set, we can choose such constants, particularly \( h = \sup_{i \geq 1} h_i > 0, \) that for any \( i \geq 1 \)
\[ 1 - |r_i(t)| \geq c_\ast |t|^{2h}, |t| \leq \varepsilon. \]
Therefore, for small enough $\theta$, we have $\rho_\theta \geq c_\theta \theta^{2h}$. Assuming
\[ \theta^{-1} \leq \ln^\kappa n, \ 2h \kappa < 1 \] (2.31)
we will have $\rho_\theta \geq c_\theta \theta^{2h} \geq c_\theta (\ln n)^{-2h\kappa}$. It follows that
\[ n^{-\rho_\theta/4} \rho_\theta^{-1/2} \theta^{-1} \leq (c_\theta)^{-1/2} \exp(1) (\ln n)^{1-2h\kappa}(\ln n)^{(1+h)\kappa} = o(1) \]

Hence $J_{nm}^{(2)} \leq \exp[-T(n^{\rho_\theta/4} - o(1))] = o(J_{nm}^{(1)})$.

It remains only to fulfill the requirements (2.27) and (2.28), i.e., $1/\theta = o(\ln n/\ln \ln n)$, $A_\theta/\theta = o(\ln n)$. By definition (see 2.23), $A_\theta/\theta$ is unlimited strictly increasing function of $\theta^{-1}$. Therefore we can choose $\theta = \theta_n$ as the root of the equation $A_\theta/\theta = (\ln n)^\kappa$. As a result, we get $1/\theta \leq A_\theta/\theta = (\ln n)^\kappa$. The proof is complete.

3 Proof of Theorem 2

The argument of the proof is closely related to our work [20], but differs in some details due to the specifics of the problem under consideration.

3.1 The Lower Bound

Recall that $\{X^{(i)}(t)\}, X^{(i)}(0) = 0, i = 0, 1, \ldots, n\}$ are independent centered Gaussian continuous H-ss processes and $p_{\Delta,n}(u) = P\{X^{(i)}(t) - X^{(0)}(t) \leq u, t \in \Delta, 1 \leq i \leq n\}$.

Assume that $M_n(\Delta)$ is the maximum of the process $M_n(t) = \max[X^{(i)}(t) - X^{(0)}(t), 1 \leq i \leq n]$ on the interval $\Delta$ and $G_n(\Delta)$ is the far right position of $M_n(\Delta)$ on $\Delta$. Obviously, $p_{\Delta,n}(u) = P\{M_n(\Delta) \leq u\}$. Assume that
\[ \lim \inf_{T, n \to \infty} \ln p((1, T), n(0)/(\ln T \cdot \ln n) = \gamma_+ > -\infty \] (3.1)

where $(n, T)$ increase such that $\ln n < C \ln T$.

For any $ct, n > 0$
\[ P(M_n([1, T])) \leq 0 \leq P(G_n([0, T])) \leq 1 \leq P(G_n([0, 1])) \leq 0, M_n([0, 1]) \leq c T, n \}
\]
\[ + P(M_n([0, 1]) \geq c t, n) \leq P(M_n([0, T])) \leq c T, n + P(M_n([0, 1]) \geq c t, n). \] (3.2)

The H-ss property of the processes in question entails
\[ P(M_n([0, T])) \leq c T, n \}
\]
\[ = P(M_n([0, T])) \leq 1), T' = T c T, n \]

At the same time, $R_{T, n} := P(M_n([0, 1]) \geq c T, n) \leq \sum_{i=1}^{n} P(\max[\xi^{(i)}(t), 0 \leq t \leq 1] \geq c T, n)$ where $\xi^{(i)}(t) = X^{(i)}(t) - X^{(0)}(t)$.

Let $\mu_i$ be the median of the distribution of $M^{(i)} = \max[\xi^{(i)}(t), 0 \leq t \leq 1]$, and $\sigma_i^2 = \max_{0 \leq t \leq 1} E[\xi^{(i)}(t)]^2 = \max_{0 \leq t \leq 1} t^{2h} E[\xi^{(i)}(0)]^2 = E[\xi^{(i)}(0)]^2$.

We can assume that sup $\mu_i \leq \mu < \infty$ and sup $\sigma_i^2 \leq \sigma^2 < \infty$ since the set of the covariance functions of $\{X^{(i)}(t)\}$ is finite. Therefore, applying the concentration principle to $\{M^{(i)}\}, [19]$, we get for large $c n, T$
\[ R_{n, T} \leq \sum_{i=1}^{n} \Psi((c T, n - \mu_i)/\sigma_i) \leq n \Psi((c T, n - \mu)/\sigma) \] (3.4)

Put, $(c T, n - \mu)^2/2 = A \ln T \cdot \ln n, A > |\gamma_-|$. Then for large $T$ and $n, R_{n, T} \leq n \exp((c T, n - \mu)^2/2) = \exp[-A \ln T \cdot \ln n(1 + o(1))].$
Hence, by (3.1)

\[ R_{T,n} \leq o(P(M_n([1, T]) \leq 0)). \tag{3.5} \]

Collecting (3.2), (3.3) and (3.5) together we have

\[
\frac{\ln P(M_n([0, T']) \leq 1)}{\ln T' \cdot \ln n} \geq \frac{\ln P(M_n(1, T) \leq 0)(1 + o(1))}{\ln T \cdot \ln n} \times \frac{\ln T}{\ln T'}
\]

Assuming \( c < \ln n < C \ln T \) and taking \( T' = KT[\ln n \cdot \ln T]^{-1/2H} \) into account, we get

\[
\liminf_{T', n \to \infty} \frac{\ln p(0, T')}{\ln T' \cdot \ln n} \geq \gamma_-, \ln n < C' \ln T'
\]

### 3.2 The Upper Bound

Now we estimate \( p_{T,n}(1) = P(M_n([0, T]) \leq 1) \) from above using the upper bound \( \gamma_+ \) of \( \ln p_{1,T,n}(0)/\psi_{T,n} \), where \( \psi_{T,n} = \ln T \cdot \ln n \).

Let \( \mu(s) \geq 1, 1 \leq s \leq T \) be an element of the reproducing kernel Hilbert space \( H_0(T) \), associated with \( X^{(0)}(t), 0 \leq t \leq T \) (see [19]). We will need the following notation:

\[
X^{(0)}(t) = X^{(0)}(t) + \mu(t), \quad M_{n, \mu}(t)
\]

\[
\text{max} \{X^{(i)}(t) - X^{(0)}(t), 1 \leq i \leq n\}, \quad M_{n, \mu}(\Delta) = \sup_{t \in \Delta} M_{n, \mu}(t).
\]

Obviously, \( M_{n,0}(\Delta) = M_n(\Delta) \). Since \( 1 - \mu(t) \leq 0 \) for \( 1 \leq t \leq T \), we have

\[
P\{M_{n,0}([0, T]) \leq 1\} = P\{M_{n, \mu}(t) \leq 1 - \mu(t), 0 \leq t \leq T\}.
\tag{3.6}
\]

The function \( \mu(s) \) is the admissible shift of the Gaussian measure \( P^{(0)}(d\omega) \) related to the process \( X^{(0)}(t) \) on the interval \([0, T], [19]\). Therefore \( P(\Omega \mu) = E_1\Omega_0 \pi(\omega_T) \), where \( \pi(\omega_T) \) is the Radon–Nikodym derivative of two Gaussian measures corresponding to the processes \( X^{(0)} + \mu \) and \( X^{(0)} \) on \([0, T]\). By the Hölder inequality \( E_1\Omega_0 \pi(\omega_T) \leq (E_1\Omega_0)^{1-\varepsilon} (E e^{\xi_T/\varepsilon})^\varepsilon \)

where \( \xi_T = \ln \pi(\omega_T) \) is Gaussian variable with mean \( m_T = -\|\mu\|_H^2(0, T)/2 \) and variance \( \sigma_T^2 = \|\mu\|_H^2(0, T)^2 \cdot \|\psi_H(0, T)\|_2 \) is the norm in \( H_T(0, T) \), see [20]. Therefore,

\[
E_1\Omega_0 \pi(\omega_T) \leq (E_1\Omega_0)^{1-\varepsilon} \exp(m_T + \sigma_T^2/(2\varepsilon)).
\]

Choosing \( \varepsilon^2 = \|\mu\|_H^2(0, T)/\psi_{T,n} \), we can continue

\[
\leq (P\{M_{n,0}([1, T]) \leq 0\})^{1-\varepsilon} \exp(\varepsilon \psi_{T,n}/2).
\tag{3.7}
\]

Substituting (3.7) in (3.6) we have \( \ln P(M_{n,0}([0, T])) \leq 1)/\psi_{n,T} \leq (1 - \varepsilon) \ln P(M_{n,0}([1, T]) \leq 0)/\psi_{n,T} + \varepsilon/2 \)

It remains to find an element \( \mu(t) \) from the Hilbert space \( H_0(T) \) such that \( \mu(s) \geq 1 \) for \( 1 \leq s \leq T \) and \( \|\mu\|_H^2(0, T)/\ln T \ln n = \varepsilon^2 = o(1) \).

Let us consider the Lamperti transform, \( \tilde{X}^{(0)}(\tau) = X^{(0)}(e^\tau)e^{-\tau h} := LX^{(0)}(\tau) \), of the process \( X^{(0)}(t) \). Assume that \( \tilde{H}_0(\tilde{T}) \) is the Hilbert spaces with the reproducing kernels associated with the process \( \tilde{X}^{(0)}(\tau) \), \( \tau \in (-\infty, \tilde{T} = \ln T) \). Then the mapping \( \psi \to \tilde{\psi} = L\psi \) performs isometry of spaces \( H_0(T) \) and \( \tilde{H}_0(\tilde{T}) \).

Assume that \( dF_0(\lambda) \) is the spectral measure of the stationary process \( \tilde{X}^{(0)}(\tau) = LX^{(0)} \).

Then, as noted in [10], \( \tilde{\psi}(\tau) = 2 \int^{1/\tilde{T}}_0 \cos \tau \lambda dF_0(\lambda)/F_0((0, 1)) \geq 1, \tau \in [0, \tilde{T}] \) and

\[
\|\tilde{\psi}\|_{\tilde{H}}^2 = 2/F_0((0, 1)) \text{ where } \tilde{H} = \tilde{H}_0(\infty) \text{ and } H = H_0(\infty) \text{.Under the conditions}
\]
Let $\mu(t)$ be a stationary process with covariance function $r_0(t)$ and continuous spectral density $f_0(\lambda)$. Then the sequence \{X(k\theta)\} has spectral function
\[
f_0(\lambda) = \theta^{-1} f_0(\lambda/\theta) + 2 \sum_{k=1}^{\infty} \theta^{-1} f_0((\lambda + 2\pi k)/\theta), |\lambda| \leq \pi.
\] (4.1)

By the assumptions, $f_0(\lambda/\theta) \leq f_0(0)$ and $f_0(\lambda)$ is majorized by a monotone function $\tilde{f}_0(\lambda) \in L_1$. Therefore
\[
\sum_{k=1}^{\infty} f_0((\lambda + 2\pi k)/\theta) \leq f_0(\pi/\theta) + \sum_{k=1}^{\infty} f_0((2k + 1)\pi/\theta)
\] \[
\leq 2\pi^{-1} \theta \int_{\pi/(2\theta)}^{\pi/\theta} \tilde{f}_0(\lambda)d\lambda + (2\pi)^{-1} \theta \int_{\pi/(2\theta)}^{\pi/\theta} \tilde{f}_0(\lambda)d\lambda
\] \[
\leq 2\pi^{-1} \theta \int_{\pi/(2\theta)}^{\pi/\theta} \tilde{f}_0(\lambda)d\lambda
\]

Hence, putting $\sigma^2_0 = \sup_{0 \leq \lambda \leq \pi} 2\pi f_0(\lambda)$, we get
\[
\theta \sigma^2_0 = \sup_{0 \leq \lambda \leq \pi} \left[ 2\pi f_0(\lambda/\theta) + o(\theta) \right] = 2\pi f_0(0) + o(1), \theta \to 0.
\] (4.2)
Consider the $m \times m$ matrix $R_m = [r_0(i\theta - j\theta)]_{i,j=1-m}$. Because $\sigma^2_{\hat{\theta}} = \sup_{0 \leq \lambda \leq \pi} 2\pi f_0(\lambda)$, we have the following relation for the quadratic forms $(x, R_m x) = \int_{-\pi}^{\pi} \left| \sum_{i=1}^{m} x_k e^{i k \lambda} \right|^2 f_0(\lambda) d\lambda \leq \int_{-\pi}^{\pi} \left| \sum_{i=1}^{m} x_k e^{i k \lambda} \right|^2 \sigma^2_{\hat{\theta}} (2\pi) d\lambda = (x, x) \sigma^2_{\hat{\theta}}$.

We will see later that $R_m$ is non-degenerate. Assuming the existence $R_m^{-1}$, we have

$$\exp[-(x, R_m^{-1} x)/2] \leq \exp[ -(x, x)/2\sigma^2_{\hat{\theta}}].$$

The last one means that for any $\{u_i\}$ we have

$$P\{X(i\theta) > u_i, i = 1, \ldots, m\} \leq P(\sigma_{\hat{\theta}} u_i > u_i, i = 1, \ldots, m) Q_m$$

where $\{\eta_i, i = 1, \ldots, m\}$ are i.i.d. standard Gaussian variables, $Q_m = \sqrt{\sigma^2_{\hat{\theta}} D_m}$ and $D_m = \det R_m$.

According to the theory of the Toeplitz forms [11], $\delta^2_m = D_m/D_{m-1}$ is the mean-square error of the $X(0)$ prediction based on $\{X(i\theta), i = 1, \ldots, m, m-1\}$ data. Moreover, $\delta^2_m = D_m/D_{m-1}$ decreases and converges to the value $\delta^2_\infty = \exp((2\pi)^{-1} \int_{-\pi}^{\pi} \ln f_0(\lambda) d\lambda$.

Since $f_0(\lambda) \geq \theta^{-1} f_0(\lambda/\theta)$ we have $D_m = \delta^2_m \cdot \delta^2_{m-1} \cdots \delta^2_1 \geq (\delta^2_\infty)^m \geq \exp(m (\theta/\pi) \int_0^{\pi/\theta} \ln f_0(\lambda) d\lambda) \cdot \theta^{-m}$.

Therefore

$$Q_m = \sqrt{\sigma^2_{\hat{\theta}} D_m} \leq (\theta \sigma^2_{\hat{\theta}})^{m/2} \exp(-m/2 \cdot (\theta/\pi) \int_0^{\pi/\theta} \ln f_0(0)/f_0(\lambda) d\lambda + 1.$$ 

This estimate also proves the non-degeneracy of the matrix $R_m$. According to (4.2), $\theta \sigma^2_{\hat{\theta}} = 2\pi f_0(0) + o(1), \theta \to 0$.

Hence for small $\theta$ we have $Q_m \leq \exp(\tilde{A}_\theta m/2)$ with

$$\tilde{A}_\theta = \exp((\theta/\pi) \int_0^{\pi/\theta} \ln(2\pi f_0(0)/f_0(\lambda)) d\lambda + 1.$$ 

Obviously, $\tilde{A}_\theta$ can be replaced by a non-decreasing function of $\theta^{-1}$, namely

$$A_\theta = \sup_{0 \leq \Delta \leq \pi/\theta} < \ln[2\pi f_0(0)/f_0(\lambda)] > \Delta + 1, \theta \leq \theta_0$$

where $\langle \psi(\lambda) > \Delta$ is the mean of $\psi(\lambda)$ in the interval $(0, \Delta)$.

\[ \square \]

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