ABOUT THE BLOWUP OF QUASIMODES
ON RIEMANNIAN MANIFOLDS

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Abstract. On any compact Riemannian manifold \((M, g)\) of dimension \(n\), the \(L^2\)-normalized eigenfunctions \(\varphi_\lambda\) satisfy \(\|\varphi_\lambda\|_\infty \leq C\lambda^{n-1/2}\) where \(-\Delta \varphi_\lambda = \lambda^2 \varphi_\lambda\). The bound is sharp in the class of all \((M, g)\) since it is obtained by zonal spherical harmonics on the standard \(n\)-sphere \(S^n\). But of course, it is not sharp for many Riemannian manifolds, e.g., flat tori \(\mathbb{R}^n/\Gamma\). We say that \(S^n\), but not \(\mathbb{R}^n/\Gamma\), is a Riemannian manifold with maximal eigenfunction growth. The problem which motivates this paper is to determine the \((M, g)\) with maximal eigenfunction growth. In an earlier work, two of us showed that such an \((M, g)\) must have a point \(x\) where the set \(L_x\) of geodesic loops at \(x\) has positive measure in \(S^*_x M\). We strengthen this result here by showing that such a manifold must have a point where the set \(R_x\) of recurrent directions for the geodesic flow through \(x\) satisfies \(|R_x| > 0\). We also show that if there are no such points, \(L^2\)-normalized quasimodes have sup-norms that are \(o(\lambda^{n-1/2})\), and, in the other extreme, we show that if there is a point blow-down \(x\) at which the first return map for the flow is the identity, then there is a sequence of quasi-modes with \(L^\infty\)-norms that are \(\Omega(\lambda^{(n-1)/2})\).

1. Introduction

In a recent series of articles [SZ, TZ, TZ2, TZ3], the authors have been studying the relations between dynamics of the geodesic flow and \(L^p\) estimates of \(L^2\)-normalized eigenfunctions of the Laplacian on a compact Riemannian manifold \((M, g)\). The general aim is to understand how the behavior of geodesics modifies the universal estimates of \(L^\infty\) of Avakumovic-Levitan-Hörmander, and the general \(L^p\) norms obtained by Sogge [So1] (see also [KTZ] and [SS] for recent and more general results). In particular, we wish to characterize the global dynamical properties of the geodesic flow of \((M, g)\) which exhibit extremal behavior of eigenfunction growth. This problem is an example of global analysis of eigenfunctions as surveyed in [Z3].

This article continues the series. Its purpose is to sharpen the previous results on maximal eigenfunction growth and to prove they are sharp by giving converse results. To introduce our subject, we need some notation. Let \(\{-\lambda^2_\nu\}\) denote the eigenvalues of \(-\Delta\), where \(0 \leq \lambda^2_0 \leq \lambda^2_1 \leq \lambda^2_2 \leq \ldots \) are counted with multiplicity and let \(\{\varphi_{\lambda, \nu}(x)\}\) be an associated orthonormal basis of \(L^2\)-normalized eigenfunctions (modes). If \(\lambda^2\) is in the spectrum of \(-\Delta\), let \(V_\lambda = \{\varphi : \Delta \varphi = -\lambda^2 \varphi\}\) denote the corresponding eigenspace. We

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measure the growth rate of $L^\infty$-norms of modes by
\begin{equation}
L^\infty(\lambda, g) = \sup_{\varphi \in V_\lambda: ||\varphi||_{L^2} = 1} ||\varphi||_{L^\infty}.
\end{equation}

The general result of [Le, A] is that
\begin{equation}
L^\infty(\lambda, g) = O(\lambda^{n-1/2}).
\end{equation}

If this bound is achieved for some subsequence of eigenfunctions, i.e., $L^\infty(\lambda, g) = \Omega(\lambda^{n-1/2})$, we say that $(M, g)$ has maximal eigenfunction growth. The corresponding sequence of $L^2$-normalized eigenfunctions $\{\varphi_{\lambda_n}\}$ have $L^\infty$ norms which are comparable to those of zonal spherical harmonics on $S^n$. The main result of [SZ] is a necessary condition on maximal eigenfunction growth: there must then exist a point $z$ such that a positive measure of geodesics emanating from $z$ return to it at a fixed time $T$. In the case where all directions loop back, we will call $z$ a blow-down point since the natural projection $\pi : S^*M \to M$ has a blow-down singularity on $S^*_z M$. For lack of a standard term, in the general case of a positive measure of loops we call $z$ a partial blow-down point. Examples of blow-down points are poles of surfaces of revolution and umbilic points of two-dimensional tri-axial ellipsoids. In the case of surfaces of revolution, all geodesics emanating from poles smoothly close up while in the case of ellipsoids, the geodesics emanating from umbilic points loop back but with two exceptions do not close up smoothly. One can construct partial blow-down points by perturbing these metrics in small polar caps to obstruct some of the geodesics.

A comparison of surfaces of revolution and ellipsoids shows that the necessary condition on maximal eigenfunction growth in [SZ] is not sharp and focuses attention on the distinguishing dynamical invariant. It is easily seen that surfaces of revolution are of maximal eigenfunction growth (cf. e.g. [SZ]) and that zonal eigenfunctions achieve the sup norm bound (1.2) at the poles (see e.g. [So2]). However, ellipsoids are not of maximal eigenfunction growth. It is proved in [T1, T2] that obvious analogues of zonal eigenfunctions on an ellipsoid only have the growth rate $\frac{\lambda^{1/2}}{\log \lambda}$, and as a consequence of Theorem 2, it follows that $||\varphi_{\lambda_n}||_\infty = o(\lambda^{1/2})$ on such a surface. Indeed, although we will not prove it here, it is likely that $L^\infty(\lambda, g) = O\left(\frac{\lambda^{1/2}}{\log \lambda}\right)$ on the ellipsoid.

The obvious difference between the geodesics of surfaces of revolution and ellipsoids is in the nature of the first return map $\Phi_z$ on directions $\theta \in S^*_z M$. This map is simplest to define when $z$ is a blow-down point, i.e. if all directions loop back. The first return map is then the fixed time map of the geodesic flow $G^t$ acting on the sphere bundle:
\begin{equation}
\Phi_z = G^T_z : S^*_z M \to S^*_z M.
\end{equation}
Below we will define it in the case of a partial blow-down point.

In the case of a surface of revolution $G^T_z = id$ is the identity map on $S^*_z M$, while in the case of an ellipsoid it has just two fixed points, one attracting and one repelling, and all directions except for the repelling fixed point are in the basin of attraction of the attracting fixed point. This comparison motivates the first theme of the present article: to study of the relation of maximal eigenfunction growth and the dynamics of this first return map. The relevance of the first return map to problems in spectral theory was
already observed by Y. Safarov et al in studying clustering in the spectrum $[S, GS, SV]$. It seems reasonable to conjecture that maximal eigenfunction growth can only arise in the identity case, or at least when $\Phi_x$ has a positive measure of fixed points, i.e., if there exists a positive measure of smoothly closed geodesics through a point $z$ of $M$.

However, this necessary condition is not sufficient. A counterexample was constructed in $[SZ]$ of a surface with a positive measure of closed geodesics through a point $z$ but which does not have maximal eigenfunction growth. Further, we conjecture that even existence of blow-down points fails to be sufficient: for instance, every point is a blow-down point on a Zoll manifold, but we conjecture that generic Zoll manifolds fail to have maximal eigenfunction growth.

We can close the gap between necessary and sufficient conditions on eigenfunction growth by generalizing the problem of eigenfunction growth to include approximate eigenfunctions, or quasi-modes. As we shall see, the widest collection that one could hope to have pointwise $o(\lambda^{(n-1)/2})$ upperbounds are defined as follows:

**Definition 1.1.** A sequence $\{\psi_\lambda\}, \lambda = \lambda_j, j = 1, 2, \ldots$ is a sequence of admissible quasimodes if $\|\psi_\lambda\|_2 = 1$ and

$$
\|(\Delta + \lambda^2)\psi_\lambda\|_2 + \|S_{2\lambda}^\perp\psi_\lambda\|_\infty = o(\lambda).
$$

Here, $S_{\mu}^\perp$ denotes the projection onto the $[\mu, \infty)$ part of the spectrum of $\sqrt{-\Delta}$, and in what follows $S_{\mu} = I - S_{\mu}^\perp$, i.e., $S_{\mu}f = \sum_{\lambda_{j} < \mu} e_j(f)$, where $e_j(f)$ is the projection of $f$ onto the eigenspace with eigenvalue $\lambda_j$.

There are many notions of quasimodes in the literature, but the above one seems to be new. We shall describe why it seems to give the natural class of “approximate eigenfunctions” in theorems that say maximal pointwise blowup implies the existence of certain types of dynamics (see below). We should also point out that the technical condition in the definition that $\|S_{2\lambda}^\perp\psi_\lambda\|_\infty = o(\lambda)$ is typically not included in the definitions of quasimodes. We need it for some of our results, and we note that, by Sobolev, when the dimension is smaller than 4, it is a consequence of the main part of the definition, i.e., $\|(\Delta + \lambda^2)\psi_\lambda\|_2 = o(\lambda)$. A model case of functions satisfying (1.4) would be a sequence of $L^2$-normalized functions $\{\psi_{\lambda_j}\}$ whose $\sqrt{-\Delta}$ spectrum lies in intervals of the form $[\lambda_j - o(1), \lambda_j + o(1)]$ as $\lambda_j \to \infty$.

It is natural to consider quasi-modes because the methods of producing blowup apply in fact to quasi-modes. Results on modes are obtained only by specializing results on quasi-modes. In examples where one knows the eigenfunctions in detail, such as surfaces of revolution or ellipsoids, the reason is usually that the modes and quasi-modes are the same. We should also point out that while (1.4) is a natural condition for classes of “approximate eigenfunctions” satisfying $o(\lambda^{(n-1)/2})$ sup-norm upperbounds, it is not necessarily so for lowerbounds. Indeed, the quasimodes satisfying $\Omega(\lambda^{(n-1)/2})$ lowerbounds that we shall construct will satisfy $\|(\Delta + \lambda^2)\psi_\lambda\|_2 = o(1)$ (quasimodes of order zero), which is weaker than (1.4).
An important example of such a quasi-mode is a sequence of “shrinking spectral projections”, i.e. the $L^2$-normalized projection kernels

$$
\Phi^z_j(x) = \frac{\chi_{[\lambda_j, \lambda_j + \epsilon_j]}(x, z)}{\chi_{[\lambda_j, \lambda_j + \epsilon_j]}(z, z)}
$$

with second point frozen at a point $z \in M$ and with width $\epsilon_j \to 0$. Here, $\chi_{[\lambda_j, \lambda_j + \epsilon_j]}(x, z)$ is the orthogonal projection onto the sum of the eigenspaces $V_{\lambda}$ with $\lambda \in [\lambda_j, \lambda_j + \epsilon_j]$. The zonal eigenfunctions of a surface of revolution are examples of such shrinking spectral projections for a sufficiently small $\epsilon_j$, and when $z$ is a partial focus such $\Phi^z_j(x)$ are generalizations of zonal eigenfunctions. On a general Zoll manifold, shrinking spectral projections of widths $\epsilon_j = O(\lambda_j^{-1})$ are the direct analogues of zonal spherical harmonics, and they would satisfy the analog of (1.4) where $o(\lambda)$ is replaced by the much stronger $O(\lambda^{-1})$.

1.1. General results. Our first result is quite general. It shows that the main result of [SZ] extends to admissible quasi-modes and also gives a reasonable converse.

**Theorem 1.** Let $(M^n, g)$ be a compact Riemannian manifold with Laplacian $\Delta$. Then:

1. If there exists an admissible sequence of quasi-modes with $\|\psi_{\lambda_k}\|_{L^\infty} = \Omega(\lambda_k^{-n/2})$, then there exists a partial blow-down point $z \in M$ for the geodesic flow. If $(M, g)$ is real-analytic, then there exists a blow-down point.

2. Conversely, if there exists a blow-down point and if the first return map is the identity, $G^T_x = \text{id}$, then there exists a quasi-mode sequence $\{\psi_{\lambda_k}\}$ of order 0 with $\|\psi_{\lambda_k}\|_{L^\infty} = \Omega(\lambda_k^{(n-1)/2})$.

As we mentioned before, Part (1) of the Theorem was proved in [SZ] for modes. The improvement here is that there must be a partial blowdown point if a sequence of quasimodes has maximal sup-norm growth. We can make a further improvement and show that there must be a special type of partial blowdown point, a recurrent point for the geodesic flow.

Let us be more specific. Given $x \in M$, we let $\mathcal{L}_x$ the set of loop directions at $x$:

$$
\mathcal{L}_x = \{\xi \in S^*_{x}M : \exists T : \exp_x T\xi = x\}.
$$

Thus, $x$ is a partial blow-down point if $|\mathcal{L}_x| > 0$ where $|\cdot|_z$ denotes the surface measure on $S^*_x M$ determined by the metric $g_x$. We also let $T_x : S^*_x M \to \mathbb{R}_+ \cup \{\infty\}$ denote the return time function to $x$,

$$
T_x(\xi) = \begin{cases} 
\inf\{t > 0 : \exp_x t\xi = x\}, & \text{if } \xi \in \mathcal{L}_x; \\
+\infty, & \text{if no such } t \text{ exists.}
\end{cases}
$$

The first return map is thus

$$
G^{T_x}_x : \mathcal{L}_x \to S^*_z M.
$$
In the general case, \( \mathcal{L}_x \) is not necessarily invariant under \( G^T_x \). To obtain forward/backward invariant sets we put

\[
(1.6) \quad \mathcal{L}_{x}^{\pm \infty} = \bigcap_{\pm k \geq 0} (G^T_x)^k \mathcal{L}_x,
\]

and also put \( \mathcal{L}_{x}^{\infty} = \mathcal{L}_{x}^{+ \infty} \cap \mathcal{L}_{x}^{- \infty} \). Then \( (\mathcal{L}_x, G^T_x) \) defines a dynamical system. We equip it with the restriction of the surface measure \( | \cdot |_x \), but of course this measure is not generally invariant under \( G^T_x \). We further define the set of recurrent loop directions to be the subset

\[
\mathcal{R}_x = \{ \xi \in \mathcal{L}_x^{\infty} : \xi \in \omega(\xi) \},
\]

where \( \omega(\xi) \) denotes the \( \omega \)-limit set, i.e. the limit points of the orbit \( \{(G^T_x)^n \xi : n \in \mathbb{Z}_+ \} \).

Equivalently, \( \xi \in \mathcal{L}_x^{\infty} \) belongs to \( \mathcal{R}_x \) if infinitely many iterates, \( (G^T_x)^n \xi, n \in \mathbb{Z} \), belong to \( \Gamma \), whenever \( \Gamma \) is a neighborhood of \( \xi \) in \( S^*_x M \). Finally, we say that \( x \) is a recurrent point for the geodesic flow if \( |\mathcal{R}_x| > 0 \).

Our improvement of the first half of the preceding theorem will be based on the following result that will give upperbounds for admissible quasimodes under a natural dynamical assumption.

**Theorem 2.** Suppose that \( |\mathcal{R}_x| = 0 \) for every \( x \in M \). Then, given \( \varepsilon > 0 \), one can find \( \Lambda(\varepsilon) < \infty \) and \( \delta(\varepsilon) > 0 \) so that

\[
(1.7) \quad \| \chi_{[\lambda, \lambda + \delta(\varepsilon)]} f \|_{L^\infty(M)} \leq \varepsilon \lambda^{(n-1)/2} \| f \|_{L^2(M)}, \quad \lambda \geq \Lambda(\varepsilon).
\]

Under the stronger hypothesis that \( |\mathcal{L}_x| = 0 \) for every \( x \in M \) one has that for every \( \delta > 0 \) there is a \( \Lambda(\delta) \) so that

\[
(1.8) \quad \| \chi_{[\lambda, \lambda + \delta]} f \|_{L^\infty(M)} \leq C \delta^{1/2} \lambda^{(n-1)/2} \| f \|_{L^2(M)}, \quad \lambda \geq \Lambda(\delta),
\]

for some constant \( C = C(M, g) \) which is independent of \( \delta \) and \( \lambda \).

To show that Theorem 2 is indeed stronger than the result in [SZ], we note that it is well-known [11] that Liouville metrics on spheres (such as the triaxial ellipsoid) satisfy the condition \( |\mathcal{R}_x| = 0 \) for each \( x \in M \). However, \( |\mathcal{L}_x| = 1 \) when \( z \in M \) is an umbilic point for the metric and so, Theorem 2 applies to these examples as well and gives the \( L^\infty(\lambda, g) = o(\lambda^{(n-1)/2}) \)-bound.

Also, as noted in [SZ], if one uses an interpolation argument involving the estimates in [So1], then (1.7) implies that \( L^p \)-estimates are not saturated for \( p > 2(n+1)/(n-1) \) under the assumption that \( |\mathcal{R}_x| = 0 \) for all \( x \in M \). Recently, one of [So4] has formulated a sufficient condition for the non-saturation of \( L^p \)-estimates in dimension two for \( 2 < p < 6 \) that involves the concentration along geodesics. A condition for the endpoint case of \( p = 6 \) for dimension 2 or \( 2 < p < 2(n+1)/(n-1) \) or \( p = 2(n+1)/(n-1) \) in higher dimensions remains open.

A corollary of Theorem 2 will be given in Theorem 2.4 below which says that if there is a sequence of admissible quasimodes with maximal sup-norm blowup, then there must be a recurrent point for the geodesic flow. Equivalently, if there is no such point, then a sequence of admissible quasimodes must have sup-norms that are \( o(\lambda^{(n-1)/2}) \).

We can write the conclusion of Theorem 2 in the shorthand notation

\[
(1.9) \quad \| \chi_{[\lambda, \lambda + o(1)]} \|_{L^2(M) \to L^\infty(M)} = o(\lambda^{(n-1)/2}).
\]
This result is optimal in one sense because the well known sup-norm estimate
\[ \|\chi_{[\lambda, \lambda+1]}\|_{L^2(M) \to L^\infty(M)} = O(\lambda^{(n-1)/2}) \]
cannot be improved on any compact Riemannian manifold (see e.g., [So3]), and this together with (1.9) provides a motivation for Definition 1.1. On the other hand, it might be the case that (1.9) holds under the weaker hypothesis that \(|C_x| = 0\) for every \(x \in M\), if \(C_x \subset L_x\) denotes the set of periodic directions, i.e., initial directions for smoothly closed geodesics through \(x\). Also, because of the sharp Weyl formula, the bounds in (1.8) are clearly sharp in the sense that one cannot take a larger power of \(\delta\), and one also needs the hypothesis that \(\lambda\) is large depending on \(\delta\).

We should point out that Theorem 2 is related to the error estimates for the Weyl law of Duistermaat and Guillemin [DG] and Ivrii [Iv1] and the error estimates of Safarov [S] for a local Weyl law. Like Ivrii’s argument, ours are just based on exploiting the nature of the singularity of the wave kernel \(e^{it\sqrt{-\Delta}}\) at \(t = 0\). Unlike these other works, though, we can prove our main estimate, (1.7), without using Tauberian lemmas. Traditionally, sup-norm estimates like (1.7) were obtained by deducing the m from stronger asymptotic formulas, e.g., appropriate Weyl laws with remainder bounds. Two of us in [SZ] used this approach to prove the weaker variant of Theorem 2 where one deduces (1.7) under the stronger assumption that there are no partial blowdown points. In the present work, we are able to prove these stronger results using a simpler argument that yields the main estimate (1.7) directly but does not seem to yield a correspondingly strong local Weyl law under the assumption that \(|R_x| = 0\) for all \(x\).

1.2. Invariant tori and surfaces with maximal eigenfunction growth. An easy consequence of our results is the following:

**Theorem 3.** If a real analytic Riemannian \(n\)-manifold \((M, g)\) has maximal growth of eigenfunctions or admissible quasi-modes, then its geodesic flow has an invariant Lagrangian submanifold \(\Lambda \simeq S^1 \times S^{n-1} \subset S^*_g M\). Hence, a surface with ergodic geodesic never has maximal growth of eigenfunctions or admissible quasi-modes.

By [SZ], a real analytic surface with maximal eigenfunction growth must be a topological sphere, so the last result only adds new information when \(M \simeq S^2\). Real analytic ergodic metrics on \(S^2\) have been constructed by K. Burns - V. Donnay [BD] and by Donnay-Pugh [DP, DP2]. There even exist such surfaces embedded in \(\mathbb{R}^3\) (see [BD] for computer graphics of such surfaces). Note that such metrics must have conjugate points, so the logarithmic estimates of [Be] do not apply.

2. Recurrent points and upperbounds for quasimodes

We shall first prove Theorem 2. To do so, we first note that
\[ (2.1) \quad \|\chi_{[\lambda, \lambda+\delta]}\|_{L^2(M) \to L^\infty(M)}^2 = \sup_{x \in M} \sum_{\lambda_j \in [\lambda, \lambda+\delta]} |e_j(x)|^2, \]
if \(\{e_j\}\) is an orthonormal basis of eigenfunctions with eigenvalues \(\{\lambda_j\}\). By compactness, we conclude that the first inequality in Theorem 2 follows from the following local version
Proposition 2.1. Suppose that \( x_0 \in M \) satisfies \( |R_{x_0}| = 0 \). Then, given \( \varepsilon > 0 \) we can find a neighborhood \( N_{\varepsilon} \) of \( x_0 \), a \( \lambda_0 < \infty \) and a \( \delta(\varepsilon) > 0 \) so that

\[
\sum_{\lambda_j \in \lambda_0, \lambda + \delta(\varepsilon)} |e_j(x)|^2 \leq \varepsilon^2 \lambda^{(n-1)/2}, \quad x \in N_{\varepsilon}, \quad \lambda \geq \lambda_0.
\]

In [SZ] we exploited the lower semicontinuity of \( L(x, \xi) \) where \( L(x, \xi) \) equaled the shortest loop in the direction \( \xi \in S^*_x M \) if there was one and \( L(x, \xi) = +\infty \) if not.

To prove our improvement of the main result of [SZ], instead of watching all loops, we shall just watch all loops of a given length length \( \ell \) and initial direction \( \xi \) which have the property that \( \text{dist} \ (G_\ell^t(\xi), \xi) \leq \delta, \) with \( \text{dist} \ (\cdot, \cdot) \) being the distance induced by the metric, and, as before, \( G_\ell^t(\xi) \) being the terminal direction. So we let \( L_\delta(x, \xi) \) be the length of the shortest such loop fulfilling this requirement if it exists and \( +\infty \) otherwise. Then \( L_\delta(x, \xi) : S^*_x M \to (0, +\infty) \) is lower semicontinuous and \( 1/L_\delta(x, \xi) \) is upper semicontinuous. We then let \( R^\delta_\xi \) then is all \( \xi \) for which \( 1/L_\delta(x, \xi) \neq 0 \).

To exploit this, if \( x_0 \) is as in the proposition, we shall choose \( \delta \) large enough so that \( |R^\delta_{\xi_0}| < \varepsilon^2/2 \) and then take \( f(x, \xi) \) to be \( 1/L_\delta(x, \xi) \) in the following variant of Lemma 3.1 in [SZ]. We shall take the parameter \( \rho \) in the lemma to be \( 1/10T \) where \( T \) is much larger than \( 1/\delta^{(n-1)} \).

Lemma 2.2. Let \( f \) be a nonnegative upper semicontinuous function on \( O \times S^{n-1} \), where \( O \subset \mathbb{R}^n \) is open. Fix \( x_0 \in O \) and suppose that \( \{ \xi \in S^{n-1} : f(x_0, \xi) \neq 0 \} \) has measure \( \leq \varepsilon/2 \), with \( \varepsilon > 0 \) being fixed. Let \( \rho > 0 \) be given. Then there is a neighborhood \( N \) of \( x_0 \) an open set \( \Omega_b \subset S^{n-1} \) satisfying

\[
|f(x, \xi)| \leq \rho, \quad (x, \xi) \in N \times S^{n-1} \setminus \Omega_b
\]

\[
|\Omega_b| \leq \varepsilon.
\]

Furthermore, there is a \( b(\xi) \in C^\infty \) supported in \( \Omega_b \) satisfying \( 0 \leq b \leq 1 \), and having the property that if \( B(\xi) = 1 - b(\xi) \) then \( f(x, \xi) \leq \rho \) on \( N \times \text{supp} B \).

Proof: The proof is almost identical to Lemma 3.1 in [SZ].

By assumption the set \( E_\rho = \{ \xi \in S^{n-1} : f(x_0, \xi) \geq \rho \} \) satisfies \( |E_\rho| \leq \varepsilon/2 \). Let

\[
E_\rho(j) = \{ \xi \in S^{n-1} : f(x, \xi) \geq \rho, \text{some } x \in \overline{B}(x_0, 1/j) \},
\]

where \( \overline{B}(x_0, r) \) is the closed ball of radius \( r \) about \( x_0 \). Then clearly \( E_\rho(j + 1) \subset E_\rho(j) \). Also, if \( \xi \in \cap_{j \geq 1} E_\rho(j) \) then for all \( j \) one can find \( x_j \in \overline{B}(x_0, 1/j) \) such that \( f(x_j, \xi) \geq \rho \), which means that

\[
\rho \leq \limsup_{j \to \infty} f(x_j, \xi) \leq f(x_0, \xi),
\]

by the upper semicontinuity of \( f \). Thus,

\[
\cap_{j \geq 1} E_\rho(j) \subset E_\rho.
\]

Consequently, if \( j \) is large \( |E_\rho(j)| < \varepsilon \). Fix such a \( j = j_0 \) and choose an open set \( \Omega_b \) satisfying \( E_\rho(j) \subset \Omega_b \) and \( |\Omega_b| < \varepsilon \). Then clearly \( f(x, \xi) < \rho \) if \( (x, \xi) \in B(x_0, 1/j_0) \times S^{n-1} \setminus \Omega_b \).

For the last part, note that the argument we have just given will show that the sets \( E_\rho(j) \) are closed because of the upper semicontinuity property of \( f \). Thus, if \( E_\rho(j_0) \) and
Ω₀ are chosen as above, we need only apply the $C^∞$ Urysohn lemma to find a smooth function $b(ξ)$ supported in $Ω₀$ with range $[0, 1]$ and satisfying $b(ξ) = 1, ξ ∈ E₀(J₀)$, which then will clearly have the required properties.

To apply this lemma we first choose a coordinate patch $K$ with coordinates $κ(ξ)$ around $x₀$, which we identify with an open subset of $R^n$. Also, fix a number $δ > 0$ small enough so that $|R^δ_{x₀}| ≤ ε^2/2$. We then let $f(x, ξ)$ denote the image of $1/L_δ(x, ξ)$ in the induced coordinates for $\{(x, ξ) ∈ S^*M : x ∈ K\}$. Then, given a large number $T$ (to be specified later), we can find a function $b ∈ C^∞(S^{n-1})$ with range $[0, 1]$ so that

\[
\int_{S^{n-1}} b(ξ) dξ ≤ ε^2,
\]

and

\[
L_δ(x, ξ) ≥ 2T \quad \text{on} \quad N × \text{supp} B,
\]

where $N ⊂ υ_0$ is a neighborhood of $x₀$ and

\[
B(ξ) = 1 - b(ξ).
\]

Choose a function $ψ ∈ C^∞(R^n)$ with range $[0, 1]$ which vanishes outside of $N$ and equals one in a small ball centered at $κ(x₀)$. Using these functions we get zero-order pseudo-differential operators on $R^n$ by setting

\[
\hat{b}(x, D)f(x) = ψ(x)(2π)^{-n} \int \int e^{i(x-y)ξ} b(ξ/|ξ|)ψ(y)f(y) dydξ,
\]

and

\[
\hat{B}(x, D)f(x) = ψ(x)(2π)^{-n} \int \int e^{i(x-y)ξ} B(ξ/|ξ|)ψ(y)f(y) dydξ.
\]

Note that both variables of the kernels of these operators have support in $K$. If we let $b(x, D)$ and $B(x, D)$ in $Ψ^0(M)$ be the pullbacks of $\hat{b}$ and $\hat{B}$, respectively, then

\[
b(x, D) + B(x, D) = ψ^2(x).
\]

Since $ψ^2 χ_{[λ, λ+δ]} b(x, D) χ_{[λ, λ+δ]} + B(x, D) χ_{[λ, λ+δ]}$ it is clear that (2.2) would follow if we could show that there is a $T = T(ε) > S, λ(ε) < ∞$ and $δ(ε) > 0$ so that

\[
\|bχ_{[λ, λ+δ]}\|_{L^2 → L^∞} ≤ Cελ(λ-1)/2, \quad λ ≥ λ(ε), \quad (2.5)
\]

and

\[
\|Bχ_{[λ, λ+δ]}\|_{L^2 → L^∞} ≤ Cελ(λ-1)/2, \quad λ ≥ λ(ε), \quad (2.6)
\]

for some uniform constant $C$ which is independent of $ε$.

Note that

\[
\|bχ_{[λ, λ+δ]}\|_{L^2 → L^∞}^2 = \sup_x \sum_{λ_j∈[λ, λ+δ]} |be_j(x)|^2 \quad (2.7)
\]

\[
\|Bχ_{[λ, λ+δ]}\|_{L^2 → L^∞}^2 = \sup_x \sum_{λ_j∈[λ, λ+δ]} |Be_j(x)|^2 \quad (2.8)
\]

To exploit this we shall use a standard trick of dominating these truncated sums by smoothed-out versions in order to use the Fourier transform and the wave operator. To this end, we choose $ρ ∈ C^∞(R)$ which vanishes for $|t| > 1/2$ and satisfies $ρ ≥ 0$ and $ρ(0) = 1$. If we then take $T$ to be a fixed multiple of $1/δ(ε)$, we conclude from (2.7) and

\[
\]
Lemma 2.3. Let (2.8) that (2.5) and (2.6) would follow from showing that if $T = T(\varepsilon)$ and $\lambda(\varepsilon)$ are large, then
\begin{align}
(2.9) & \quad \sum_{j=1}^{\infty} (\hat{\rho}(T(\lambda - \lambda_j)))^2 |be_j(x)|^2 \leq C\varepsilon^2 \lambda^{n-1}, \quad \lambda \geq \lambda(\varepsilon) \\
(2.10) & \quad \sum_{j=1}^{\infty} (\hat{\rho}(T(\lambda - \lambda_j)))^2 |Be_j(x)|^2 \leq C\varepsilon^2 \lambda^{n-1}, \quad \lambda \geq \lambda(\varepsilon).
\end{align}

To prove these, we shall require the following standard result which is based on the singularity of the wave kernel restricted to the diagonal at $t = 0$. To state the notation, we let $U = e^{it\sqrt{-\Delta}}$ denote the wave group and $U(t, x, y)$ its kernel. Then we need the following result which follows from Proposition 2.2 in [SZ].

**Lemma 2.3.** Let $(M, g)$ have injectivity radius $> 10$ and let $A(x, D) \in \Psi^0(M)$ be a pseudo-differential operator of order 0. Let $\alpha \in C_0^\infty(\mathbb{R})$ vanishes for $|t| \geq 2$ and satisfies $\alpha(0) = 1$. Then, if $A_0(x, \xi)$ denotes the principal symbol of $A$,
\begin{align}
(2.11) & \quad (2\pi)^{-n} \int_{-\infty}^{\infty} \alpha(t)e^{-i\lambda t}(AUA^*)(t, x, x) \, dt \\
& \quad \quad \quad - (2\pi)^{-n} \lambda^{n-1} \int \sum g^{ik}(x)\delta_{\xi_k=1} |A_0(x, \xi)| \, d\sigma(\xi) = O(\lambda^{n-2}).
\end{align}

In what follows, we may assume without loss of generality that the hypothesis on the injectivity radius of $M$ is satisfied.

Note that we can rewrite the left side of (2.11) as
\begin{align}
(2.12) & \quad (2\pi)^{-n} \int_{-\infty}^{\infty} \alpha(t)e^{-i\lambda t}(AUA^*)(t, x, x) \, dt = \sum_j \hat{\lambda}(\lambda - \lambda_j)|Ae_j(x)|^2.
\end{align}

If we choose $\alpha$ as above so that $\hat{\lambda} \geq 0$, $\hat{\lambda}(0) = 1$, we conclude from (2.11) and (2.12) that
\begin{align}
(2.13) & \quad \sum_{|\lambda_j - \lambda| \leq 1} |Ae_j(x)|^2 \leq C\lambda^{n-1} \|A_0(x, \cdot)\|^2_{L^2(S^*_z M)} + C\lambda^{n-2},
\end{align}
where $C$ is independent of $A = A(x, D) \in \Psi^0(M)$. This will prove to be a useful estimate in what follows.

Using (2.13) we can get (2.9) if we assume, as we may, that $T > 1$. For then $(\hat{\rho}(T(\lambda - \lambda_j)))^2 \leq C_N(1 + |\lambda - \lambda_j|)^{-N}$ for any $N$, which yields (2.9) as $\|b(x, \cdot)\|^2_{L^2(S^*_z M)} \leq C\varepsilon^2$, by (2.8).

To finish the proof of (1.7) by proving (2.11), we first exploit (2.4) to see that we can construct a smooth partition of unity $1 = \sum_k \psi_k(\xi)$ of the unit sphere which consists of $O(\delta^{-n-1})$ terms each of which has range in $[0, 2]$ and is supported in a small spherical cap of diameter smaller than $\delta/10$. We then let $B_k(x, D)$ be the zero-order pseudo-differential operator whose symbol equals $B(x, \xi)\psi_k(\xi/|\xi|)$ in the coordinates used before. Since $\delta$ is fixed, we would have (2.11) if we could show that
\begin{align}
(2.14) & \quad \sum_{j=1}^{\infty} (\hat{\rho}(T(\lambda - \lambda_j)))^2 |B_k e_j(x)|^2 \leq CT^{-1} \lambda^{n-1} + C_{B_k, T} \lambda^{n-2}.
\end{align}
Indeed, if $T$ is chosen large enough so that $C\delta^{-(n-1)/2} \leq \varepsilon^2$, then, since $B = \sum B_k$, by applying the Cauchy-Schwarz inequality, we get (2.11) for large enough $\lambda$. As we shall see, the constant $C$ in (2.14) can be taken to be $O(1)$ as $\delta \to 0$; however, the reduction to estimates for each single $B_k$ contributes an additional factor $O(\delta^{-(n-1)})$ to the constant in (2.8).

To prove (2.14), we note that we can rewrite the left side as

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} T^{-1} (\rho * \rho)(t/T)(B_k UB_k^*)(t, x, x)e^{-it\lambda} dt.$$

To estimate this, we need to exploit the fact that our hypothesis (2.4) implies that $(t, x) \rightarrow (B_k UB_k^*)(t, x, x)$ is smooth when $0 < |t| \leq T$. Also, by construction, $(\rho * \rho)(t/T) = 0$ for $|t| > T$. To use these facts, we choose $\beta \in C^\infty_0(\mathbb{R})$ satisfying $\beta(t) = 1$, $|t| < 1$ and $\beta(t) = 0$, $|t| > 2$ and then split the left side of (2.14) as

$$\frac{1}{2\pi} \int \beta(t) T^{-1} (\rho * \rho)(t/T)(B_k UB_k^*)(t, x, x)e^{-it\lambda} dt + \frac{1}{2\pi} \int (1 - \beta(t)) T^{-1} (\rho * \rho)(t/T)(B_k UB_k^*)(t, x, x)e^{-it\lambda} dt = I + II.$$

If we integrate by parts we see that $II$ must be $O(\lambda^{-N})$ for any $N$, which means that we are left with showing that $I$ enjoys the bounds in (2.14). However, since we are assuming that $T > 1$, one can check that the inverse Fourier transform of $t \rightarrow \beta(t) T^{-1} (\rho * \rho)(t/T)$ must be $\leq C N T^{-1} (1 + |\tau|)^{-N}$ for any $N$ if $\tau$ is the variable dual to $t$. Thus, for every $N$,

$$I \leq C N T^{-1} \sum_{j=1}^{\infty} (1 + |\lambda - \lambda_j|)^{-N} |B_k e_j(x)|^2,$$

which means that our remaining estimate (2.14) also follows from (2.13).

One proves (1.8) by the above argument if one takes $\delta$ in the last step to be equal to 1.

2.1. Blowup rates for quasimodes: Proof of Theorem 1 (i). Next, we shall show that we can extend the blowup results of [SZ] for eigenfunctions to include the admissible quasimodes (defined in Definition 1.1) and also allow one to conclude that there must be points through which there is a positive measure of recurrent directions for the geodesic flow.

**Theorem 2.4.** Suppose that $\psi_\lambda$ is a sequence of admissible quasimodes satisfying

$$\|\psi_\lambda\|_\infty = \Omega(\lambda^{(n-1)/2}).$$

Then there must be a point $x \in M$ with $|R_x| > 0$.

Since $R_x \subset L_x$ this result is stronger than the first part of Theorem 1.

The proof of Theorem 2.4 is based on Theorem 2 and the following lemma.

**Lemma 2.5.** Fix $B > 0$ and suppose that for $\lambda = \lambda_j \to \infty$ we have

$$\|\psi_\lambda\|_\infty \geq B \lambda^{(n-1)/2}.$$
Then if \( 0 < \delta < 1 \) there exists \( \varepsilon > 0 \) so that if \( \lambda = \lambda_j \) and
\[
(\Delta + \lambda^2)\psi_\lambda \|_2 + \| S_{2\lambda}^1 \psi_\lambda \|_\infty \leq \varepsilon \lambda, \tag{2.16}
\]
then if \( \chi_{[\lambda-\delta,\lambda+\delta]}f = \sum_{\lambda_j \in [\lambda-\delta,\lambda+\delta]} e_j(f) \),
\[
\| \chi_{[\lambda-\delta,\lambda+\delta]} \psi_\lambda \|_\infty \geq \frac{B}{2} \lambda^{(n-1)/2}, \tag{2.17}
\]
for all sufficiently large \( \lambda = \lambda_j \).

Before proving Lemma 2.5 let us see why it and Theorem 2 implies Theorem 2.4.
To do this, let us suppose that we have a sequence of admissible quasimodes satisfying \( \| \psi_\lambda \|_\infty = \Omega(\lambda^{(n-1)/2}) \) If we apply Lemma 2.5 we conclude that there is a positive constant \( c > 0 \) so that for any \( 0 < \delta < 1 \) we have
\[
\| \chi_{[\lambda-\delta,\lambda+\delta]} \psi_\lambda \|_\infty \geq c\lambda^{(n-1)/2},
\]
for some sequence \( \lambda = \lambda_j \), if \( \lambda \) is large enough (depending on \( \delta \)).

Let \( \rho > 0 \). If there were no recurrent points, we could apply Theorem 2 to conclude that there is a \( \delta = \delta(\rho) \) so that for large enough \( \lambda \) (depending on \( \delta \))
\[
\| \chi_{[\lambda-\delta,\lambda+\delta]} \psi_\lambda \|_\infty \leq C\rho \lambda^{(n-1)/2},
\]
which leads to a contradiction if \( \rho \) is chosen small enough so that \( C\rho < c \). Thus, we conclude that there must be a recurrent point under the hypotheses of Theorem 2.4. \( \square \)

**Proof of Lemma 2.5** To simplify the notation, let us set \( \chi^\delta_\lambda = \chi_{[\lambda-\delta,\lambda+\delta]} \). We need to see under the hypotheses of Lemma 2.5 we have
\[
\| (I - \chi^\delta_\lambda) \psi_\lambda \|_\infty \leq \frac{B}{2} \lambda^{(n-1)/2}, \tag{2.18}
\]
if \( \lambda = \lambda_j \) is large enough.

This would follow from a couple of estimates. The first one says that there is a constant \( A \) which is independent of \( 0 < \delta < 1 \) and \( \lambda > 1 \) so that
\[
\| \chi^1_\lambda (I - \chi^\delta_\lambda) f \|_\infty \leq A\lambda^{(n-1)/2} (\lambda \delta)^{-1} \| (\Delta + \lambda^2) f \|_2, \tag{2.19}
\]
while the second one says that
\[
\| (I - \chi^1_\lambda) S_{2\lambda} f \|_\infty \leq C\lambda^{(n-1)/2} \lambda^{-1} \| (\Delta + \lambda^2) f \|_2. \tag{2.20}
\]

To see how these imply (2.17), we take \( f = \psi_\lambda \). Then since \( \delta < 1 \) we have
\[
(I - \chi^1_\lambda) \psi_\lambda = \chi^1_\lambda (I - \chi^\delta_\lambda) \psi_\lambda + (I - \chi^1_\lambda) S_{2\lambda} \psi_\lambda + S_{2\lambda}^1 \psi_\lambda. \tag{2.21}
\]

If \( n \geq 4 \) we estimate the last piece by the second part of our admissible quasimode hypothesis \( \| S_{2\lambda}^1 \psi_\lambda \|_\infty = o(\lambda) = o(\lambda^{(n-1)/2}) \). If \( n \leq 3 \) we use Sobolev to get that for a given \( 0 < \sigma < 1/2 \)
\[
\| S_{2\lambda}^1 \psi_\lambda \|_\infty \leq C \| (\sqrt{-\Delta})^{n/2+\sigma} S_{2\lambda}^1 \psi_\lambda \|_2 \leq C \| (\sqrt{-\Delta})^{n/2-\sigma/2} S_{2\lambda}^1 (\Delta + \lambda^2) \psi_\lambda \|_2 \leq C\lambda^{n/2-\sigma} \| (\Delta + \lambda^2) \psi_\lambda \|_2 \leq C\lambda^{n/2-\sigma} \lambda = o(\lambda^{(n-1)/2}),
\]
for all sufficiently large \( \lambda = \lambda_j \).
as desired since $\sigma < 1/2$.

Using (2.19), (2.19) and (2.20) we can estimate the remaining pieces in (2.21)

$$
\|\chi_\lambda^1(I - \chi_\lambda^1)S_{2\lambda}\psi_\lambda\|_\infty + \|(I - \chi_\lambda^1)S_{2\lambda}\psi_\lambda\|_\infty \leq (A + C)\lambda^{(n-1)/2}(\lambda\delta)^{-1}\varepsilon(\lambda/\log \lambda) \\
\leq 2(A + C)(\varepsilon/\delta)\lambda^{(n-1)/2},
$$

if $\lambda$ is large. Since this estimate and our earlier bounds for $S_{2\lambda}\psi_\lambda$ yield (2.18), we are left with proving (2.19) and (2.20).

The estimate (2.19) is easy. Using the fact that $\|\chi_\lambda^1\|_{L^2 - L^\infty} \leq A\lambda^{(n-1)/2}$, we get

$$
\|\chi_\lambda^1(I - \chi_\lambda^1)f\|_\infty \leq A\lambda^{(n-1)/2}\|(I - \chi_\lambda^1)f\|_2 \\
\leq A\lambda^{(n-1)/2}(\Delta + \lambda^2)^{-1}(I - \chi_\lambda^1)(\Delta + \lambda^2)f\|_2 \\
\leq A\lambda^{(n-1)/2}(\lambda\delta)^{-1}(\Delta + \lambda^2)f\|_2.
$$

To prove (2.20), let $\Pi_{(j,j+1)}$ denote the projection onto the $[j,j+1)$ part of the spectrum of $\sqrt{-\Delta}$. Then we can write

$$(I - \chi_\lambda^1)S_{2\lambda}f = \sum_{k=1}^\lambda \left( \Pi_{[\lambda+k,\lambda+k+1]}S_{2\lambda}f + \Pi_{[\lambda-k-1,\lambda-k]}S_{2\lambda}f \right).$$

Thus,

$$
\|(I - \chi_\lambda^1)S_{2\lambda}f\|_\infty \leq \sum_{k=1}^\lambda \left( \|\Pi_{[\lambda+k,\lambda+k+1]}S_{2\lambda}f\|_\infty + \|\Pi_{[\lambda-k-1,\lambda-k]}S_{2\lambda}f\|_\infty \right) \\
= I + II.
$$

We shall only estimate $I$ since the same argument will yield the same bounds for $II$.

To estimate $I$, we first note that for $1 \leq k \leq \lambda$

$$
\|\Pi_{[\lambda+k,\lambda+k+1]}g\|_\infty \leq C\lambda^{(n-1)/2}\|\Pi_{[\lambda+k,\lambda+k+1]}g\|_2 \\
= C\lambda^{(n-1)/2}(\Delta + \lambda^2)^{-1}\Pi_{[\lambda+k,\lambda+k+1]}(\Delta + \lambda^2)g\|_2 \\
\leq C\lambda^{(n-1)/2}(\lambda\delta)^{-1}(\Pi_{[\lambda+k,\lambda+k+1]}(\Delta + \lambda^2)g\|_2
$$

Therefore, by applying the Schwarz inequality, we get

$$
I = \sum_{k=1}^\lambda k^{-1}(k\|\Pi_{[\lambda+k,\lambda+k+1]}S_{2\lambda}f\|_\infty) \\
\leq C\lambda^{(n-1)/2}\lambda^{-1}\left(\sum_{k=1}^\lambda k\|\Pi_{[\lambda+k,\lambda+k+1]}(\Delta + \lambda^2)f\|_2^2\right)^{1/2} \\
\leq C\lambda^{(n-1)/2}\lambda^{-1}\|(\Delta + \lambda^2)f\|_2^2,
$$

as desired. Since, as we noted, the same argument works for $II$, we have completed the proof of Lemma 2.5. \[\square\]

Let us conclude this section by pointing out that the conclusion of the lemma is not valid for dimensions $n \geq 4$ if one just assumes $\|(\Delta + \lambda^2)^k\psi_{\lambda_k}\|_2 = o(\lambda_k)$ or even

(2.22) $\|(\Delta + \lambda^2)^k\psi_{\lambda_k}\|_2 = O(1)$
for the quasimode definition.

Let us first handle the case where \( n \geq 5 \) since that is slightly simpler than the \( n = 4 \) one. To handle this case, we fix a nonnegative function \( \eta \in C_0^\infty(\mathbb{R}) \) satisfying \( \eta(10) = 1 \) and \( \eta(s) = 0, s \notin [5, 20] \). We then set

\[
K = \lambda_k^s,
\]

where \( s > 1 \) is large and will be chosen later. Put

\[
\psi_{\lambda_k}(x) = K^n K^{-n} \eta(\sqrt{\Delta}/K)(x_0, x) = K^n K^{-n} \sum_{\lambda_j} \eta(\lambda_j/K) e_{\lambda_j}(x_0) e_{\lambda_j}(x),
\]

where \( x_0 \in M \) is fixed and \( \varepsilon > 0 \) is small.

We notice that the conclusion of the lemma is false for these functions since \( \chi_{[\lambda_k-\delta, \lambda_k+\delta]} \psi_{\lambda_k} \equiv 0 \) if \( s > 1 \) is fixed and if \( \lambda_k \) is large, due to the fact that the spectrum of \( \psi_{\lambda_k} \) is in \([5\lambda_k^s, 20\lambda_k^s]\) and \( \lambda_k \) does not lie in this interval for large \( k \). Also, it is not hard to verify that

\[
\psi_{\lambda_k}(x_0) \approx K^s = \lambda_k^s,
\]

and so by choosing \( s = \frac{n-1}{2} \), we have one of the assumptions of the lemma that \( \|\psi_{\lambda_k}\|_\infty = \Omega(\lambda_k^{(n-1)/2}) \). We also have \([2.22]\) if \( n \geq 5 \). For then

\[
\|\|\Delta + \lambda_k^2\| \psi_{\lambda_k} \|_2 \approx \|\|\Delta + 1\| \psi_{\lambda_k} \|_2 \approx \lambda_k^2 \|\psi_{\lambda_k}\|_2 \approx K^2 K^n K^{-n} K^{2n/2} = o(1),
\]

if, as we may, we choose \( \varepsilon < 1/2 \).

Minor modifications of this argument show that things break down for \( n = 4 \) as well if one just assumes \([2.22]\). Here one would take \( j_0 = 2\lambda_k^{n-1} \) so that \( \log j_0 = \lambda_k^{n-1} \), where \( \log \) is the base-2 \( \log \). Then, with the above notation, one sets

\[
\psi_{\lambda_k}(x) = (\log j_0)^{-1/2} \sum_{j \in [\log j_0, 2\log j_0]} 2^{-jn} \eta(\sqrt{\Delta}/2^j)(x_0, x).
\]

Then, one can see that

\[
\psi_{\lambda_k}(x_0) \approx (\log j_0)^{1/2} = \lambda_k^{(n-1)/2},
\]

\( \chi_{[\lambda_k-\delta, \lambda_k+\delta]} \psi_{\lambda_k} \equiv 0 \) if \( \lambda_k \) is large. Finally, if \( n = 4 \), \([2.22]\) is valid since

\[
\|\|\Delta + \lambda_k^2\| \psi_{\lambda_k} \|_2 \approx \|\|\Delta + 1\| \psi_{\lambda_k} \|_2 \approx (\log j_0)^{-1} \sum_{j \in [\log j_0, 2\log j_0]} 2^{4j} 2^{-8j} \|\eta(\sqrt{\Delta}/2^j)(x_0, \cdot)\|_2 \approx (\log j_0)^{-1} \sum_{j \in [\log j_0, 2\log j_0]} 1 \approx 1.
\]

These constructions will also show that when \( n \geq 4 \) one cannot use

\[
\|\|\Delta + \lambda_k^2\| \psi_{\lambda_k} \|_2 = O(\lambda_k^{-s})
\]

for any large \( s \) as the condition for quasimodes \( \{\psi_{\lambda_k}\} \) and have the conclusions of the lemma be valid.

### 2.2. Quasi-modes associated to blow-down points: Proof of Theorem 1 (2):

We now prove the converse result in Theorem 1 under the assumption that \( G_T^z = Id \). The method is to construct quasi-modes associated to the “blow-down” Lagrangian \( \Lambda_z \) in Definition 4 (see below). The analysis generalizes the one in [Z] in the Zoll case. The key point is the existence of an invariant 1/2-density on \( \Lambda_z \) for the geodesic flow. In this case, the invariant 1/2-density is \([du dt]\)^{1/2} where, \( du \) is Liouville measure on \( S_z^* M \).
2.2.1. The Blow-down Lagrangian. Since $z$ is a blow-down point, the geodesic flow induces a smooth first return map \[ (1.3) \]. Let $C^T_t$ denote the mapping cylinder of $G^T_t$, namely
\[ C^T_t = S^1_z \times [0, T] / \sim, \quad \text{where} \quad (\xi, T) \sim (\xi, T) \text{ for } G^T_t(z). \]
The $C^T_t$ is a smooth manifold. It naturally fibers over $S^1$ by the map
\[ \pi : C^T_t \to S^1, \quad \pi(\xi, t) = t \mod 2\pi. \]

**Proposition 4.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold, and assume that it possesses a blow down point $z$. Let $\iota_z : C^T_t \to T^*M$ be the map
\[ \iota_z(\xi, t) = G^t(z, \xi). \]

Then $\iota_z$ is a Lagrange embedding whose image is a geodesic-flow invariant Lagrangian manifold, $\Lambda_z$, diffeomorphic to $S^1 \times S^n \cong C^T_t$.

**Proof.** We let $\omega$ denote the canonical symplectic form on $T^*M$. Then, under the map \[ (2.24) \]
\[ \iota : S^1 \times S^*_z M \to T^*M, \quad \iota(t, x, \xi) \to G^t(x, \xi), \]
we have
\[ \iota^* \omega = \omega - dH \wedge dt, \quad H(x, \xi) = |\xi|^g. \]
The map $\iota_z$ is the restriction of $\iota$ to $R \times S^*_z M$. Since $dH = 0$ on $S^*M$ and $\omega = 0$ on $S^*_z M$, the right side equals zero.

Thus, $\iota_z$ is a Lagrange immersion. To see that it is an embedding, it suffices to prove that it is injective, but this is clear from the fact that $G^t$ has no fixed points.

Let $\alpha_A$ denote the action form $\alpha = \xi \cdot dx$ restricted to $\Lambda$. Also, let $m_A$ denote the Maslov class of $\Lambda$. A Lagrangian $\Lambda$ satisfies the Bohr-Sommerfeld quantization condition \[ [D] \] if
\[ (2.25) \]
\[ \frac{r_k}{2\pi} [\alpha_A] \equiv \frac{m_A}{4} \mod H^1(\Lambda, \mathbb{Z}), \]
where
\[ r_k = \frac{2\pi}{T}(k + \beta \frac{4}{4}), \]
with $\beta$ equal to the common Morse index of the geodesics $G^t(z, \xi), \xi \in S^*_z M$.

**Proposition 5.** $\Lambda_z$ satisfies the Bohr-Sommerfeld quantization condition.

**Proof.** We need to identify the action form and Maslov class.

**Lemma 6.** We have:
\begin{enumerate}
  \item $\iota_z^* \alpha_A = dt$.
  \item $\iota_z^* m_A = \frac{4}{T} [dt]$.
\end{enumerate}
Proof. (1) Let $\xi_H$ denote the Hamiltonian vector field of $H$. Since $(G^t)^*\alpha = \alpha$ for all $t$, we may restrict to $t = T$ and to $S^*_z M$ to obtain $(G^T)^*\alpha|_{S^*_z M} = \alpha|_{S^*_z M}$. But clearly, $\xi \cdot dx|_{S^*_z M} = 0$.

(2) We recall that $m_{\Lambda_z} \in H^1(\Lambda_z, \mathbb{Z})$ gives the oriented intersection class with the singular cycle $\Sigma \subset \Lambda_z$ of the projection $\pi : \Lambda_z \to M$. Given a closed curve $\alpha$ on $\Lambda_z$, we deform it to intersect $\Sigma$ transversally and then $\int_\alpha m_{\Lambda_z}$ is the oriented intersection number of the curve with $\Sigma$. Our claim is that $\int_\alpha m_{\Lambda_z} = \beta$ where $\beta$ is the common Morse index of the (not necessarily smoothly) closed geodesic loops $\gamma(t) = G^t(z, \xi), \xi \in S^*_z M$.

The inverse image of the singular cycle of $\Lambda_z$ under $\iota_z$ consists of the following components:

$$\iota_z^{-1} \Sigma = S^*_z M \cup \text{Conj}(z),$$

where $\text{Conj}(z) = \{(t, \xi) : 0 < t < T, \xi \in S^*_z, |\det d_z \exp_{t} t| = 0\}$ is the tangential conjugate locus of $z$. All of $S^*_z M$ consists of self-conjugate vectors at the time $T$.

If $\dim M \geq 3$, then $H^1(C_T, \mathbb{Z}) = \mathbb{Z}$ is generated by the homology class of a closed geodesic loop at $z$ and in this case $\int_\alpha m_{\Lambda_z} = \beta$ by definition of the Morse index. If $\dim M = 2$, then $H^1(C_T, \mathbb{Z})$ has two generators, that of a closed geodesic loop and that of $S^*_z M$. The value of $m_{\Lambda_z}$ on the former is the same as for $\dim M \geq 3$, so it suffices to determine $\int_{S^*_z M} m_{\Lambda_z}$. To calculate the intersection number, we deform $S^*_z M$ so that it intersects $\iota_z^{-1} \Sigma$ transversally. We can use $G^\epsilon S^*_z M$ as the small deformation, and observe that it has empty intersection with $\iota_z^{-1} \Sigma$ for small $\epsilon$ since the set of conjugate times and return times have non-zero lower bounds.

□

The Lemma immediately implies (2.25), completing the proof.

□

2.2.2. Construction of quasi-modes. We now ‘quantize’ $\Lambda_z$ as a space of oscillatory integrals.

**Lemma 7.** There exists $\Phi_k \in C^0(M, \Lambda_z, \{r_k\})$ with $i^* \sigma(\Phi_k) = e^{-ir_k t}\sqrt{dt} \otimes |d\mu|^{1/2}$, where $d\mu$ is Liouville measure on $S^*_z M$.

We will refer to $\Phi_k$ as quasi-modes associated to the blow down point $z \in M$.

**Examples**

1. In the case of $S^n$ and $z$ the north pole, $\Phi_k(z)$ is the zonal spherical harmonic of degree $k$. Equivalently, it equals, up to $L^2$-normalization, the orthogonal projection kernel $\Pi_k(\cdot, z)$ onto $k$th order with second variable fixed at $z$. In this case, it is an eigenfunction.

2. On a general Zoll manifold, with $z$ any point, the projection kernel onto the $k$th eigenvalue cluster is a quasi-mode of this type, see [Z]. In general, it is a
zeroth order quasi-mode, reflecting the width $k^{-1}$ of the $k$th cluster, and not an eigenfunction.

(3) On a surface of revolution diffeomorphic to $S^2$, the zonal eigenfunctions are oscillatory integrals of this type.

2.3. Proof of Theorem 3. By the results of [SZ], $(M, g)$ possesses a point $m$ such that all geodesics issuing from the point $m$ return to $m$ at some time $\ell$ (which with no loss of generality may be taken to be $2\pi$). By Proposition 3, the map $\iota$ of $(2.24)$ is a Lagrange immersion with image $\Lambda_m$.

If $\dim M = 2$, the image $\iota([0, 2\pi] \times S_M^* M)$ is a Lagrangian torus, the mapping torus of the first return map $G^{2\pi}|_{S_M^* M} : S_M^* M \to S_M^* M$. Obviously, $G^t(\Lambda) = \Lambda$ for all $t$, so $\Lambda$ is an invariant torus for the geodesic flow. Moreover, $M$ is diffeomorphic to $S^2$ or to $\mathbb{RP}^2$. Since $S_M^* M = \mathbb{RP}^3$ when $M = S^2$ (or in the case $\mathbb{RP}^2$ is a quotient by a $\mathbb{Z}_2$ action), we have $H^2(S^* M) = \{0\}$. Hence, $\Lambda = \partial\Omega$ where $\Omega \subset S^* M$ is a singular 3-chain. Since $\dim S^* M = 3$, $\Omega$ has a non-empty interior, so $\Lambda$ is the boundary of an open set. But $G^t \Omega \subset \Omega$. Hence, there exists an open invariant set, and $G^t$ cannot be ergodic.

In higher dimensions, we do not see how ergodicity rules out existence of invariant Lagrangian $S^1 \times S^{m-1}$ or blow down points. Hyperbolicity of the geodesic flow is inconsistent with existence of such Lagrangian submanifolds. But, as mentioned in the introduction, there are better estimates in the case of $(M, g)$ with Anosov (hyperbolic) geodesic flows. These never have conjugate points, and the generic sup norm estimate can be improved to $||\varphi_j||_{L^\infty} = O(\lambda_j^{\frac{1}{2}} \log \lambda_j)$ (3.5). But of course such flows do not exist for metrics on $S^2$, and the previous result provides new information for analytic metrics with ergodic geodesic flow on $S^2$.

2.3.1. Pointwise asymptotics of the quasimode $\Phi_k$. In the following we let $h \in \{r_k\}^{-1}; k = 1, 2, ..., \text{and let } B_j \subset M; j = 1, ..., N \text{ be small geodesically convex balls with } \pi(\Lambda_1) \subset \bigcup_{j=1}^N B_j. \text{Let } \chi_j \in C_0^\infty (B_j) \text{ be a partition of unity subordinate to this covering and } \chi_R(s) \in C_0^\infty (\mathbb{R}) \text{ be a cutoff equal to } 1 \text{ when } |s| < R \text{ with } R > 1 \text{ and zero when } |s| > 2R. \text{One then constructs the quasimode } \Phi_k(x) \text{ as a sum } \sum_{j=1}^N \chi_j \Phi_k^{(j)} \text{ where the } \Phi_k^{(j)} \in C^\infty (B_j) \text{ are local oscillatory integrals of the form}

$$
\Phi_k^{(j)}(x) = (2\pi h)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\varphi^{(j)}(x, \theta)/h} a^{(j)}(x, \theta; h) \chi_R(|\theta|) d\theta.
$$

Without loss of generality, we assume that $z \in B_1$ and let $x = (x_1, ..., x_n) \in B_1$ be geodesic normal coordinates with $x(z) = 0 \in \mathbb{R}^n$. Consider first

$$
\Phi_k^{(1)}(x) = (2\pi h)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\varphi^{(1)}(x, \theta)/h} a^{(1)}(x, \theta; h) \chi_R(|\theta|) d\theta.
$$

The $L^2$-normalized quasimode $\Phi_k$ is constructed to solve the equation $-\Delta_g \Phi_k - r_k^2 \Phi_k|_{L^2} = O(1)$ and for this, one needs to globally solve the eikonal equation and the first transport equation.

For the eikonal equation, we choose the phase $\varphi^{(1)} = \varphi^{(1)}(x, \theta)$ positive homogeneous of degree zero in the $\theta_j$-variables. Since $S^*_M \subset \Lambda_1 \cap \pi^{-1}(B_1)$ is non-characteristic for the geodesic flow, it follows that there exists a locally unique solution $\varphi^{(1)}(x, \theta)$ to the
initial value problem
\begin{align}
|\nabla_x \varphi^{(1)}(x, \theta)|_g^2 &= 1 \\
\varphi^{(1)}(0, \theta) &= 0,
\end{align}

with
\begin{equation}
\Lambda_z \cap \pi^{-1}(B_1) = \{(x, \partial_x \varphi^{(1)}(x, \theta)) \in B_1 \times \mathbb{R}^n; \partial_\theta \varphi^{(1)}(x, \theta) = 0\}.
\end{equation}

Consider the function
\begin{equation}
\varphi^{(1)}(x, \theta) = \langle x, \theta \rangle, \quad \theta \neq 0.
\end{equation}

By the Gauss lemma,
\begin{equation}
\sum_{j=1}^n g_{ij}(x) x_j = \sum_{j=1}^n g_{ij}(0) x_j = x_i,
\end{equation}
and so, \( \langle x, \theta \rangle_g = \sum_i x_i \theta_i \). Consequently, for \( \theta \neq 0 \) we have that
\begin{equation}
\varphi^{(1)}(x, \theta) = \langle x, \theta \rangle = \langle x, \theta \rangle_g.
\end{equation}

Then, from (2.29) it follows that \( \varphi^{(1)}(0, \theta) = 0 \) and
\begin{equation}
|\nabla_x \varphi^{(1)}(x, \theta)|_g^2 = \sum_{i,j=1}^n g^{ij}(x) \partial_i \varphi^{(1)} \partial_j \varphi^{(1)} = |\theta|_g^2 = 1.
\end{equation}

Thus, \( \varphi^{(1)}(x, \theta) = \langle x, \theta \rangle \) satisfies the initial value problem in (2.20) and (2.27). Moreover, a direct computation shows that
\begin{equation}
\{(x, \partial_x \varphi^{(1)}(x, \theta)) \in B_1 \times \mathbb{R}^n; \partial_\theta \varphi^{(1)}(x, \theta) = 0\} = \{(t\omega, \omega) \in \mathbb{R}^n \times S^{n-1}; |t| < \epsilon_0\}.
\end{equation}

Here, \( \epsilon_0 \) is the geodesic radius of the ball \( B_1 \). The latter set is just \( \Lambda_z \cap \pi^{-1}(B_1) \) written in normal coordinates.

The transport equation for \( a_0^{(1)}(x, \theta) \) is
\begin{equation}
g^{ij} \partial_x^i \varphi \cdot \partial_x^j a_0^{(1)} = g^{ij} \partial_x^i \partial_x^j \varphi \cdot a_0^{(1)} = g^{ij} \partial_x^i \partial_x^j (\langle x, \theta \rangle) \cdot a_0^{(1)} = 0,
\end{equation}
where, we impose the initial condition \( a_0^{(1)}(0, \theta) = 1 \). It follows that
\begin{equation}
a_0^{(1)}(x, \theta) = 1.
\end{equation}

\subsection*{2.3.2. \( L^2 \)-normalization.}
Consider first the local quasimode \( \Phi_k^{(1)} \) and choose \( \delta \in (1 - \frac{1}{n}, 1) \). Clearly,
\begin{equation}
\int_{|x| \leq h^\delta} |\Phi_k^{(1)}(x)|^2 dx = O(h^{1-(1-\delta)n}).
\end{equation}

In the annulus \( A_\delta(h) := \{x \in B_1; h^\delta < |x| < \epsilon_0\} \), we introduce polar coordinates and write
\begin{equation}
\Phi_k^{(1)}(x) = (2\pi h)^{\frac{1}{2}n} \int_{\mathbb{R}^n} e^{i \frac{|x|}{h}} \chi_R(|\theta|) d\theta
\end{equation}
This completes the proof of Theorem 1 (ii). □
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