On the stability of generalized mixed type quadratic and quartic functional equation in quasi-Banach spaces

M. Eshaghi Gordji
Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran
e-mail: madjid.eshaghi@gmail.com

S. Abbaszadeh
Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran
e-mail: s.abbaszadeh.math@gmail.com

Abstract. In this paper, we establish the general solution of the functional equation
\[ f(nx + y) + f(nx - y) = n^2 f(x + y) + n^2 f(x - y) + 2(n^2 - 1)f(x) \]
for fixed integers \( n \) with \( n \neq 0, \pm 1 \) and investigate the generalized Hyers-Ulam-Rassias stability of this equation in quasi-Banach spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [21] in 1940, concerning the stability of group homomorphisms. Let \((G_1, \cdot)\) be a group and let \((G_2, \ast)\) be a metric group with the metric \(d(\cdot, \cdot)\). Given \( \varepsilon > 0 \), does there exist a \( \delta > 0 \), such that if a mapping \( h : G_1 \rightarrow G_2 \) satisfies the inequality \( d(h(x.y), h(x) \ast h(y)) < \delta \) for all \( x, y \in G_1 \), then there exists a homomorphism \( H : G_1 \rightarrow G_2 \) with \( d(h(x), H(x)) < \varepsilon \) for all \( x \in G_1 \)? In the other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D.H. Hyers [10] gave a first affirmative answer to the question of Ulam for Banach spaces. Let \( f : E \rightarrow E' \) be a mapping between Banach spaces such that
\[ \|f(x + y) - f(x) - f(y)\| \leq \delta \]
for all \( x, y \in E \), and for some \( \delta > 0 \). Then there exists a unique additive mapping \( T : E \rightarrow E' \) such that
\[ \|f(x) - T(x)\| \leq \delta \]
for all \( x \in E \). Moreover, if \( f(tx) \) is continuous in \( t \) for each fixed \( x \in E \), then \( T \) is linear.

In 1978, Th. M. Rassias [17] provided a generalization of Hyers’ Theorem which allows the Cauchy difference to be unbounded. The functional equation
\[ f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad (1.1) \]
is related to symmetric bi-additive function\([1,2,11,13]\). It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function \( f \) between real vector

\[ 0 \] 2000 Mathematics Subject Classification: 39B82, 39B52.
\[ 0 \] Keywords: Hyers-Ulam-Rassias stability, Quartic function, Quadratic function.
In fact, they proved that a function \( f \) for all of (1.3) if and only if there exists a unique symmetric bi-additive function \( \| \) equation:

\[
\| x \|, \| y \| ≥ 0 \quad \text{for all } x ≠ 0.
\]

A Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.1) was proved by Skof for functions \( f : A → B \), where A is normed space and B Banach space (see [19]). Cholewa [4] noticed that the Theorem of Skof is still true if relevant domain A is replaced an abelian group. In the paper [6], Grabiec [9] has generalized these result mentioned above.

In [14], Won-Gil Prak and Jea Hyeong Bae, considered the following quartic functional equation:

\[
f(x + 2y) + f(x - 2y) = 4(f(x + y) + f(x - y) + 6f(y)) - 6f(x).
\]

In fact, they proved that a function \( f \) between two real vector spaces \( X \) and \( Y \) is a solution of (1.3) if and only if there exists a unique symmetric multi-additive function \( D : X × X × X × X → Y \) such that \( f(x) = D(x, x, x, x) \) for all \( x \). It is easy to show that the function \( f(x) = x^4 \) satisfies the functional equation (1.4), which is called a quartic functional equation (see also [5]).

In addition H. Kim [12], has obtained the generalized Hyers-Ulam-Rassias stability for the following mixed type of quartic and quadratic functional equation:

\[
\left\{ \begin{array}{l}
n ≥ 2 \quad \text{and} \quad \lambda ≥ 0 \quad \text{if } \lambda = 0.
\end{array} \right.
\]

for all \( n \)-variables \( x_1, x_2, ..., x_n \) in \( E_1 \), where \( n > 2 \) and \( f : E_1 → E_2 \) be a function between two real linear spaces \( E_1 \) and \( E_2 \).

Also A. Najati and G. Zamani Eskandani [16], have established the general solution and the generalized Hyers-Ulam-Rassias stability for a mixed type of cubic and additive functional equation, whenever \( f \) is a mapping between two quasi-Banach spaces.

Now, we introduce the following functional equation for fixed integers \( n \) with \( n ≠ 0, ±1: \)

\[
f(nx + y) + f(nx - y) = n^2 f(x + y) + n^2 f(x - y) + 2f(nx)
\]

\[
- 2n^2 f(x) - 2(n^2 - 1)f(y)
\]

in quasi Banach spaces. It is easy to see that the function \( f(x) = ax^4 + bx^2 \) is a solution of the functional equation (1.5). In the present paper we investigate the general solution of functional equation (1.5) when \( f \) is a function between vector spaces, and we establish the generalized Hyers-Ulam-Rassias stability of this functional equation whenever \( f \) is a function between two quasi-Banach spaces.

We recall some basic facts concerning quasi-Banach space and some preliminary results.

**Definition 1.1.** (See [3, 18].) Let \( X \) be a real linear space. A quasi-norm is a real-valued function on \( X \) satisfying the following:

1. \( \| x \| ≥ 0 \) for all \( x ∈ X \) and \( \| x \| = 0 \) if and only if \( x = 0 \).
2. \( \| λx \| = |λ| \| x \| \) for all \( λ ∈ \mathbb{R} \) and all \( x ∈ X \).
3. There is a constant \( K ≥ 1 \) such that \( \| x + y \| ≤ K(\| x \| + \| y \|) \) for all \( x, y ∈ X \).

It follows from condition (3) that

\[
\| \sum_{i=1}^{2m} x_i \| ≤ M^m \sum_{i=1}^{2m} \| x_i \|, \quad \| \sum_{i=1}^{2m+1} x_i \| ≤ M^{m+1} \sum_{i=1}^{2m+1} \| x_i \|
\]

for all \( m ≥ 1 \) and all \( x_1, x_2, ..., x_{2m+1} ∈ X \).
The pair \((X, \|\cdot\|)\) is called a quasi-normed space if \(\|\cdot\|\) is a quasi-norm on \(X\). The smallest possible \(M\) is called the modulus of concavity of \(\|\cdot\|\). A quasi-Banach space is a complete quasi-normed space.

A quasi-norm \(\|\cdot\|\) is called a p-norm \((0 < p \leq 1)\) if
\[
\|x + y\|^p \leq \|x\|^p + \|y\|^p
\]
for all \(x, y \in X\). In this case, a quasi-Banach space is called a p-Banach space.

Given a p-norm, the formula
\[
d(x, y) := \|x - y\|^p
\]
gives us a translation invariant metric on \(X\). By the Aoki-Rolewicz Theorem [18](see also [3]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms, henceforth we restrict our attention mainly to p-norms. In [20], J. Tabor has investigated a version of Hyers-Rassias-Gajda theorem (see [7, 17]) in quasi-Banach spaces.

2. General solution

Throughout this section, \(X\) and \(Y\) will be real vector spaces. We here present the general solution of (1.5).

**Lemma 2.1.** If a function \(f : X \rightarrow Y\) satisfies the functional equation (1.5), then \(f\) is a quadratic and quartic function.

**Proof.** By letting \(x = y = 0\) in (1.5), we get \(f(0) = 0\). Set \(x = 0\) in (1.5) to get \(f(y) = f(-y)\) for all \(y \in X\). So the function \(f\) is even. We substitute \(x = x + y\) in (1.5) and then \(x = x - y\) in (1.5) to obtain that
\[
f(nx + (n + 1)y) + f(nx + (n - 1)y) = n^2f(x + 2y) + n^2f(x) + 2f(nx + ny) - 2n^2f(x + y) - 2(n^2 - 1)f(y) \tag{2.1}
\]
and
\[
f(nx - (n - 1)y) + f(nx - (n + 1)y) = n^2f(x) + n^2f(x - 2y) + 2f(nx - ny) - 2n^2f(x - y) - 2(n^2 - 1)f(y) \tag{2.2}
\]
for all \(x, y \in X\). Interchanging \(x\) and \(y\) in (1.5) and using evenness of \(f\) to get the relation
\[
f(x + ny) + f(x - ny) = n^2f(x + y) + n^2f(x - y) + 2f(ny) - 2n^2f(y) - 2(n^2 - 1)f(x) \tag{2.3}
\]
for all \(x, y \in X\). Replacing \(y\) by \(ny\) in (1.5) and then using (2.3), we have
\[
f(nx + ny) + f(nx - ny) = n^2f(x + y) + n^2f(x - y) + 2f(ny) + 2f(nx) - 2n^4f(x) - 2n^4f(y) \tag{2.4}
\]
for all \(x, y \in X\). If we add (2.1) to (2.2) and use (2.4), we have
\[
f(nx + (n + 1)y) + f(nx - (n + 1)y) + f(nx + (n - 1)y) + f(nx - (n - 1)y) = n^2f(x + 2y) + n^2f(x - 2y) + 2n^2(n^2 - 1)f(x + y) + 2n^2(n^2 - 1)f(x - y) + 4f(ny) + 4f(nx) + (-4n^4 + 2n^2)f(x) + (-4n^4 - 4n^2 + 4)f(y) \tag{2.5}
\]
for all \(x, y \in X\). Substitute \(y = x + y\) in (1.5) and then \(y = x - y\) in (1.5) and using evenness of \(f\) to obtain that
\[
f((n+1)x+y)+f((n-1)x-y) = n^2f(2x+y) + n^2f(y) + 2f(nx) - 2n^2f(x) - 2(n^2-1)f(x+y)
\] (2.6)
and
\[
f((n+1)x-y)+f((n-1)x+y) = n^2f(2x-y) + n^2f(y) + 2f(nx) - 2n^2f(x) - 2(n^2-1)f(x-y)
\] (2.7)
for all \(x, y \in X\). Interchanging \(x\) with \(y\) in (2.6) and (2.7) and using evenness of \(f\), we get the relations
\[
f(x+(n+1)y)+f(x-(n-1)y) = n^2f(x+2y) + n^2f(x) + 2f(ny) - 2n^2f(y) - 2(n^2-1)f(x+y)
\] (2.8)
and
\[
f(x-(n+1)y)+f(x+(n-1)y) = n^2f(x-2y) + n^2f(x) + 2f(ny) - 2n^2f(y) - 2(n^2-1)f(x-y)
\] (2.9)
for all \(x, y \in X\). With the substitution \(y = (n+1)y\) in (1.5) and then \(y = (n-1)y\) in (1.5), we have
\[
f(nx+(n+1)y)+f(nx-(n+1)y) = n^2f(x+(n+1)y) + n^2f(x-(n+1)y) + 2f(nx) - 2n^2f(x) - 2(n^2-1)f((n+1)y)
\] (2.10)
and
\[
f(nx+(n-1)y)+f(nx-(n-1)y) = n^2f(x+(n-1)y) + n^2f(x-(n-1)y) + 2f(nx) - 2n^2f(x) - 2(n^2-1)f((n-1)y)
\] (2.11)
for all \(x, y \in X\). Replacing \(x\) by \(y\) in (1.5), we obtain
\[
f((n+1)y)+f((n-1)y) = n^2f(2y) - 2n^2f(y) + 2f(ny)
\] (2.12)
for all \(y \in X\). Adding (2.10) with (2.11) and using (2.8), (2.9) and (2.12), we lead to
\[
f(nx+(n+1)y)+f(nx-(n+1)y) + f(nx+(n-1)y) + f(nx-(n-1)y) =
\]
\[
n^4f(x+2y) + n^4f(x-2y) - 2n^2(n^2-1)f(x+y) - 2n^2(n^2-1)f(x-y) + 4f(ny) + 4f(nx) - 2n^2(n^2-1)f(2y) + (2n^4-4n^2)f(x)
\]
\[
+ (4n^4-12n^2+4)f(y)
\] (2.13)
for all \(x, y \in X\). By comparing (2.5) with (2.13), we arrive at
\[
f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) + 2f(2y) - 8f(y) - 6f(x)
\] (2.14)
for all \(x, y \in X\). Interchange \(x\) with \(y\) in (2.14) and use evenness of \(f\) to get the relation
\[
f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 2f(2x) - 8f(x) - 6f(y)
\] (2.15)
for all \(x, y \in X\).

We would show that (2.15) is a quadratic and quartic functional equation. To get this, we show that the functions \(g : X \rightarrow Y\) defined by \(g(x) = f(2x) - 16f(x)\) for all \(x \in X\) and \(h : X \rightarrow Y\) defined by \(h(x) = f(2x) - 4f(x)\) for all \(x \in X\), are quadratic and quartic, respectively.
Replacing $y$ by $2y$ in (2.15) and using evenness of $f$, we have
\[
f(2x + 2y) + f(2x - 2y) = 4f(2y + x) + 4f(2y - x) + 2f(2x) - 8f(x) - 6f(2y)
\]
for all $x, y \in X$. By interchanging $x$ with $y$ in (2.16) and then using (2.15), we obtain by evenness of $f$
\[
f(2x + 2y) + f(2x - 2y) = 4f(2x + y) + 4f(2x - y) + 2f(2y) - 8f(y) - 6f(2x)
\]
\[
= 16f(x + y) + 16f(x - y) + 2f(2x) + 2f(2y)
\]
\[
- 32f(x) - 32f(y)
\]
for all $x, y \in X$. By rearranging (2.17), we have
\[
[f(2x + 2y) - 16f(x + y)] + [f(2x - 2y) - 16f(x - y)] =
\]
\[
2[f(2x) - 16f(x)] + 2[f(2y) - 16f(y)]
\]
for all $x, y \in X$. This means that
\[
g(x + y) + g(x - y) = 2g(x) = 2g(y)
\]
for all $x, y \in X$. Therefore the function $g: X \rightarrow Y$ is quadratic.

To prove that $h: X \rightarrow Y$ is quartic, we have to show that
\[
h(2x + y) + h(2x - y) = 4h(x + y) + 4h(x - y) + 24h(x) - 6h(y)
\]
for all $x, y \in X$. Replacing $x$ and $y$ by $2x$ and $2y$ in (2.15), respectively, we get
\[
f(4x + 2y) + f(4x - 2y) = 4f(2x + 2y) + 4f(2x - 2y) + 2f(4x) - 8f(2x) - 6f(2y)
\]
for all $x, y \in X$. Since $g(2x) = 4g(x)$ for all $x \in X$ where $g: X \rightarrow Y$ is a quadratic function defined above, we have
\[
f(4x) = 20f(2x) - 64f(x)
\]
for all $x \in X$. Hence, it follows from (2.15), (2.18), and (2.19) that
\[
h(2x + y) + h(2x - y) = [f(4x + 2y) - 4f(2x + y)] + [f(4x - 2y) - 4f(2x - y)]
\]
\[
= 4[f(2x + 2y) - 4f(x + y)] + 4[f(2x - 2y) - 4f(x - y)]
\]
\[
+ 24[f(2x) - 4f(x)] - 64[f(2y) - 4f(y)]
\]
\[
= 4h(x + y) + 4h(x - y) + 24h(x) - 6h(y)
\]
for all $x, y \in X$. Therefore, $h: X \rightarrow Y$ is a quartic function. 

**Theorem 2.2.** A function $f: X \rightarrow Y$ satisfies (1.5) if and only if there exist a unique symmetric multi-additive function $D: X \times X \times X \times X \rightarrow Y$ and a unique symmetric bi-additive function $B: X \times X \rightarrow Y$ such that
\[
f(x) = D(x, x, x, x) + B(x, x)
\]
for all $x \in X$.

**Proof.** We first assume that the function $f: X \rightarrow Y$ satisfies (1.5). Let $g, h: X \rightarrow Y$ be functions defined by
\[
g(x) := f(2x) - 16f(x)
\]
\[
h(x) := f(2x) - 4f(x)
\]
for all $x \in X$. Hence, by Lemma (2.1), we achieve that the functions $g$ and $h$ are quadratic and quartic, respectively, and
\[
f(x) := \frac{1}{12}h(x) - \frac{1}{12}g(x)
\]
for all \( x \in X \). Therefore, there exist a unique symmetric multi-additive mapping \( D : X \times X \times X \times X \longrightarrow Y \) and a unique symmetric bi-additive mapping \( B : X \times X \longrightarrow Y \) such that 
\[
D(x, x, x, x) = \frac{1}{4} h(x) \quad \text{and} \quad B(x, x) = -\frac{1}{2} g(x)
\]
for all \( x \in X \) (see [1, 14]). So
\[
f(x) = D(x, x, x, x) + B(x, x)
\]
for all \( x \in X \).

Conversely assume that
\[
f(x) = D(x, x, x, x) + B(x, x)
\]
for all \( x \in X \), where the function \( D : X \times X \times X \times X \longrightarrow Y \) is symmetric multi-additive and \( B : X \times X \longrightarrow Y \) is bi-additive defined above. By a simple computation, one can show that the functions \( D \) and \( B \) satisfy the functional equation (1.5), so the function \( f \) satisfies (1.5).

3. Hyers-Ulam-Rassias stability of Eq. (1.5)

From now on, let \( X \) and \( Y \) be a quasi-Banach space with quasi-norm \( \| \|_X \) and a \( p \)-Banach space with \( p \)-norm \( \| \|_Y \), respectively. Let \( M \) be the modulus of concavity of \( \| \|_Y \). In this section using an idea of Gavruta [8] we prove the stability of Eq. (1.5) in the spirit of Hyers, Ulam and Rassias. For convenience we use the following abbreviation for a given function \( f : X \longrightarrow Y \):
\[
\Delta f(x, y) = f(nx + y) + f(nx - y) - n^2 f(x + y) - n^2 f(x - y) - 2f(nx) + 2n^2 f(x) + 2(n^2 - 1)f(y)
\]
for all \( x, y \in X \). We will use the following lemma in this section.

**Lemma 3.1.** (see [15,]) Let \( 0 < p \leq 1 \) and let \( x_1, x_2, \ldots, x_n \) be non-negative real numbers. Then
\[
\left( \sum_{i=1}^{n} x_i \right)^p \leq \sum_{i=1}^{n} x_i^p.
\]

**Theorem 3.2.** Let \( \varphi : X \times X \longrightarrow [0, \infty) \) be a function such that
\[
\lim_{m \to \infty} 4^m \varphi\left( \frac{x}{2m}, \frac{y}{2m} \right) = 0
\]
for all \( x, y \in X \) and
\[
\sum_{i=1}^{\infty} 4^i \varphi^p\left( \frac{x}{2^i}, \frac{y}{2^i} \right) < \infty
\]
for all \( x \in X \) and for all \( y \in \{ x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x \} \).
Suppose that a function \( f : X \longrightarrow Y \) with \( f(0) = 0 \) satisfies the inequality
\[
\| \Delta f(x, y) \|_Y \leq \varphi(x, y)
\]
for all \( x, y \in X \). Then the limit
\[
Q(x) := \lim_{m \to \infty} 4^m \left[ f\left( \frac{x}{2m-1} \right) - 16 f\left( \frac{x}{2m} \right) \right]
\]
exists for all \( x \in X \) and \( Q : X \longrightarrow Y \) is a unique quadratic function satisfying
\[
\| f(2x) - 16 f(x) - Q(x) \|_Y \leq \frac{M^{11}}{4} [\varphi(x)]^{\frac{1}{p}}
\]
for all $x \in X$, where

$$
\tilde{\varphi}_q(x) := \sum_{i=1}^{\infty} 4^p i \left\{ \frac{1}{n^{2p}(n^{2p} - 1)^p} \left[ \varphi_q^p \left( \frac{x}{2}, \frac{(n + 2)x}{2} \right) + \varphi_q^p \left( \frac{x}{2}, \frac{(n - 2)x}{2} \right) \right]
\right. \\
+ 4^p \varphi_q^p \left( \frac{x}{2}, \frac{(n + 1)x}{2} \right) + 4^p \varphi_q^p \left( \frac{x}{2}, \frac{(n - 1)x}{2} \right) + 4^p \varphi_q^p \left( \frac{x}{2}, \frac{nx}{2} \right) + \varphi_q^p \left( \frac{2x}{2}, \frac{2x}{2} \right) \\
+ 4^p \varphi_q^p \left( \frac{2x}{2}, \frac{x}{2} \right) + n^p \varphi_q^p \left( \frac{x}{2}, \frac{3x}{2} \right) + 2^p (3n^2 - 1)^p \varphi_q^p \left( \frac{x}{2}, \frac{2x}{2} \right) \\
+ (17n^2 - 8)^p \varphi_q^p \left( \frac{x}{2}, \frac{x}{2} \right) + \frac{n^{2p}}{(n^2 - 1)^p} \left[ \varphi_q^p (0, \frac{x(n + 1)x}{2}) + \varphi_q^p (0, \frac{(n - 3)x}{2}) \right] \\
+ 10^p \varphi_q^p (0, \frac{(n - 1)x}{2}) + 4^p \varphi_q^p (0, \frac{nx}{2}) + 4^p \varphi_q^p (0, \frac{(n - 2)x}{2}) + \frac{(n^4 + 1)^p}{(n^2 - 1)^p} \varphi_q^p (0, \frac{2x}{2}) \\
+ \frac{(2(3n^4 - n^2 + 2))^p}{(n^2 - 1)^p} \varphi_q^p (0, \frac{x}{2}) \right\}. \tag{3.6}
$$

Proof. Set $x = 0$ in (3.3) and then interchange $x$ with $y$ to get

$$
\| (n^2 - 1)f(x) - (n^2 - 1)f(-x) \| \leq \varphi_q(0, x) \tag{3.7}
$$

for all $x \in X$. Replacing $y$ by $x$, $2x$, $nx$, $(n + 1)x$ and $(n - 1)x$ in (3.3), respectively, we get

$$
\| f((n + 1)x) + f((n - 1)x) - n^2f(2x) - 2f(nx) + (4n^2 - 2)f(x) \| \leq \varphi_q(x, x) \tag{3.8}
$$

and

$$
\| f((n + 2)x) + f((n - 2)x) - n^2f(3x) - n^2f(-x) - 2f(nx) + 2n^2 f(x) \\
+ 2(n^2 - 1)f(2x) \| \leq \varphi_q(x, 2x) \tag{3.9}
$$

and

$$
\| f(2nx) - n^2f((n + 1)x) - n^2f((1 - n)x) + 2(n^2 - 2)f(nx) + 2n^2 f(x) \| \\
\leq \varphi_q(x, nx) \tag{3.10}
$$

and

$$
\| f((2n + 1)x) + f(-x) - n^2f((n + 2)x) - n^2f(-nx) - 2f(nx) + 2n^2 f(x) \\
+ 2(n^2 - 1)f((n + 1)x) \| \leq \varphi_q(x, (n + 1)x) \tag{3.11}
$$

and

$$
\| f((2n - 1)x) + f(x) - n^2f((2 - n)x) - (n^2 + 2)f(nx) + 2n^2 f(x) \\
+ 2(n^2 - 1)f((n - 1)x) \| \leq \varphi_q(x, (n - 1)x) \tag{3.12}
$$

and

$$
\| f((n + 1)x) + f(-2x) - n^2f((n + 3)x) - n^2f(-(n + 1)x) - 2f(nx) + 2n^2 f(x) \\
+ 2(n^2 - 1)f((n + 2)x) \| \leq \varphi_q(x, (n + 2)x) \tag{3.13}
$$

and

$$
\| f((2n - 1)x) + f(2x) - n^2f((n - 1)x) - n^2f(-(n - 3)x) - 2f(nx) + 2n^2 f(x) \\
+ 2(n^2 - 1)f((n - 2)x) \| \leq \varphi_q(x, (n - 2)x) \tag{3.14}
$$

and

$$
\| f((n + 3)x) + f((n - 3)x) - n^2f(4x) - n^2f(-2x) - 2f(nx) + 2n^2 f(x) \\
+ 2(n^2 - 1)f(3x) \| \leq \varphi_q(x, 3x) \tag{3.15}
$$
for all \( x \in X \). We combine (3.7) with (3.9), (3.10), (3.11), (3.12), (3.13), (3.14) and (3.15), respectively, to get the following inequalities:

\[
\|f((n+2)x) + f((n-2)x) - n^2 f(3x) - n^2 f(x) - 2f(nx) + 2n^2 f(x) + 2(n^2 - 1)f(2x)\| \leq \varphi_q(x, 2x) + \frac{n^2}{n^2 - 1} \varphi_q(0, x)
\]  

(3.16)

and

\[
\|f(2nx) - n^2 f((n+1)x) - n^2 f((n-1)x) + 2(n^2-2)f(nx) + 2n^2 f(x)\|
\leq \varphi_q(x, nx) + \frac{n^2}{n^2 - 1} \varphi_q(0, (n-1)x)
\]  

(3.17)

and

\[
\|f(2(n+1)x) + f(x) - n^2 f((n+2)x) - n^2 f(nx) - 2f(nx) + 2n^2 f(x) + 2(n^2 - 1)f((n+1)x)\| \leq \varphi_q(x, (n+1)x) + \frac{n^2}{n^2 - 1} \varphi_q(0, nx) + \frac{1}{n^2 - 1} \varphi_q(0, x)
\]  

(3.18)

and

\[
\|f(2(n-1)x) + f(x) - n^2 f((n-2)x) - (n^2 + 2)f(nx) + 2n^2 f(x) + 2(n^2 - 1)f((n-1)x)\| \leq \varphi_q(x, (n-1)x) + \frac{n^2}{n^2 - 1} \varphi_q(0, (n-2)x)
\]  

(3.19)

and

\[
\|f(2(n+1)x) + f(2x) - n^2 f((n+3)x) - n^2 f((n+1)x) - 2f(nx) + 2n^2 f(x) + 2(n^2 - 1)f((n+2)x)\| \leq \varphi_q(x, (n+2)x) + \frac{n^2}{n^2 - 1} \varphi_q(0, (n+1)x) + \varphi_q(0, 2x)
\]  

(3.20)

and

\[
\|f(2(n-1)x) + f(2x) - n^2 f((n-1)x) - n^2 f((n-3)x) - 2f(nx) + 2n^2 f(x) + 2(n^2 - 1)f((n-2)x)\| \leq \varphi_q(x, (n-2)x) + \frac{n^2}{n^2 - 1} \varphi_q(0, (n-3)x)
\]  

(3.21)

and

\[
\|f((n+3)x) + f((n-3)x) - n^2 f(4x) - n^2 f(2x) - 2f(nx) + 2n^2 f(x) + 2(n^2 - 1)f(3x)\| \leq \varphi_q(x, 3x) + \frac{n^2}{n^2 - 1} \varphi_q(0, 2x)
\]  

(3.22)

for all \( x \in X \). Replacing \( x \) and \( y \) by \( 2x \) and \( x \) in (3.3), respectively, we obtain

\[
\|f((2n+1)x) + f((2n-1)x) - n^2 f(3x) - 2f(2nx) + 2n^2 f(2x) + (n^2 - 2)f(x)\| \leq \varphi_q(2x, x)
\]  

(3.23)

for all \( x \in X \). Putting \( 2x \) and \( 2y \) instead of \( x \) and \( y \) in (3.3), respectively, we have

\[
\|f(2(n+1)x) + f(2(n-1)x) - n^2 f(4x) - 2f(2nx) + 2(2n^2 - 1)f(2x)\| \leq \varphi_q(2x, 2x)
\]  

(3.24)
On the stability of generalized mixed ...

for all $x \in X$. It follows from (3.8), (3.16), (3.17), (3.18), (3.19) and (3.23) that

$$
\|f(3x) - 6f(2x) + 15f(x)\| \leq \frac{M^5}{n^2(n^2 - 1)} [\varphi_q(x, (n + 1)x) + \varphi_q(x, (n - 1)x)
+ \varphi_q(2x, x) + 2\varphi_q(x, nx) + n^2\varphi_q(x, 2x) + (4n^2 - 2)\varphi_q(x, x)
+ \frac{n^2}{n^2 - 1}(2\varphi_q(0, (n - 1)x) + \varphi_q(0, nx) + \varphi_q(0, (n - 2)x))
+ \frac{n^4 + 1}{n^2 - 1}\varphi_q(0, x)]
$$

(3.25)

for all $x \in X$. Also, from (3.8), (3.16), (3.17), (3.20), (3.21), (3.22) and (3.24), we conclude

$$
\|f(4x) - 4f(3x) + 4f(2x) + 4f(x)\| \leq \frac{M^6}{n^2(n^2 - 1)} [\varphi_q(x, (n + 1)x) + \varphi_q(x, (n - 2)x)
+ 4\varphi_q(x, (n + 1)x) + 4\varphi_q(x, (n - 1)x) + 10\varphi_q(x, nx) + \varphi_q(2x, 2x)
+ 4\varphi_q(2x, x) + n^2\varphi_q(x, 3x) + 2(3n^2 - 1)\varphi_q(x, 2x) + (17n^2 - 8)\varphi_q(x, x)
+ \frac{n^2}{n^2 - 1}(\varphi_q(0, (n + 1)x) + \varphi_q(0, (n - 3)x) + 10\varphi_q(0, (n - 1)x) + 4\varphi_q(0, nx)
+ 4\varphi_q(0, (n - 2)x)) + \frac{n^4 + 1}{n^2 - 1}\varphi_q(0, 2x) + \frac{2(3n^3 - n^2 + 2)}{n^2 - 1}\varphi_q(0, x)]
$$

(3.26)

for all $x \in X$. Finally, combining (3.25) and (3.26) yields

$$
\|f(4x) - 24f(2x) + 64f(x)\| \leq \frac{M^6}{n^2(n^2 - 1)} [\varphi_q(x, (n + 1)x) + \varphi_q(x, (n - 2)x)
+ 4\varphi_q(x, (n + 1)x) + 4\varphi_q(x, (n - 1)x) + 10\varphi_q(x, nx) + \varphi_q(2x, 2x)
+ 4\varphi_q(2x, x) + n^2\varphi_q(x, 3x) + 2(3n^2 - 1)\varphi_q(x, 2x) + (17n^2 - 8)\varphi_q(x, x)
+ \frac{n^2}{n^2 - 1}(\varphi_q(0, (n + 1)x) + \varphi_q(0, (n - 3)x) + 10\varphi_q(0, (n - 1)x) + 4\varphi_q(0, nx)
+ 4\varphi_q(0, (n - 2)x)) + \frac{n^4 + 1}{n^2 - 1}\varphi_q(0, 2x) + \frac{2(3n^3 - n^2 + 2)}{n^2 - 1}\varphi_q(0, x)]
$$

(3.27)

for all $x \in X$. By substituting

$$
\psi_q(x) = \frac{1}{n^2(n^2 - 1)} [\varphi_q(x, (n + 2)x) + \varphi_q(x, (n - 2)x)
+ 4\varphi_q(x, (n + 1)x) + 4\varphi_q(x, (n - 1)x) + 10\varphi_q(x, nx) + \varphi_q(2x, 2x)
+ 4\varphi_q(2x, x) + n^2\varphi_q(x, 3x) + 2(3n^2 - 1)\varphi_q(x, 2x) + (17n^2 - 8)\varphi_q(x, x)
+ \frac{n^2}{n^2 - 1}(\varphi_q(0, (n + 1)x) + \varphi_q(0, (n - 3)x) + 10\varphi_q(0, (n - 1)x) + 4\varphi_q(0, nx)
+ 4\varphi_q(0, (n - 2)x)) + \frac{n^4 + 1}{n^2 - 1}\varphi_q(0, 2x) + \frac{2(3n^3 - n^2 + 2)}{n^2 - 1}\varphi_q(0, x)]
$$

(3.28)

(3.27) gives

$$
\|f(4x) - 20f(2x) + 64f(x)\| \leq M^6\psi_q(x)
$$

(3.29)

for all $x \in X$.

Let $g : X \to Y$ be a function defined by $g(x) := f(2x) - 16f(x)$ for all $x \in X$. From (3.29), we conclude that

$$
\|g(2x) - 4g(x)\| \leq M^6\psi_q(x)
$$

(3.30)
for all \( x \in X \). If we replace \( x \) in (3.30) by \( \frac{x}{2^{m+1}} \) and multiply both sides of (3.30) by \( 4^m \), we get
\[
\|4^{m+1} g\left(\frac{x}{2^{m+1}}\right) - 4^m g\left(\frac{x}{2^m}\right)\|_Y \leq M^8 4^m \psi_q\left(\frac{x}{2^{m+1}}\right)
\] (3.31)
for all \( x \in X \) and all non-negative integers \( m \). Since \( Y \) is a \( p \)-Banach space, then inequality (3.31) gives
\[
\|4^{m+1} g\left(\frac{x}{2^{m+1}}\right) - 4^k g\left(\frac{x}{2^k}\right)\|_Y \leq \sum_{i=k}^{m} \|4^{i+1} g\left(\frac{x}{2^{i+1}}\right) - 4^i g\left(\frac{x}{2^i}\right)\|_Y
\]
\[
\leq M^8 \sum_{i=k}^{m} 4^i \psi_q\left(\frac{x}{2^{i+1}}\right)
\] (3.32)
for all non-negative integers \( m \) and \( k \) with \( m \geq k \) and for all \( x \in X \). Since \( 0 < p \leq 1 \), then by Lemma 3.1, from (3.28), we conclude that
\[
\psi_q^p(x) \leq \frac{1}{n_2^p(n_2 - 1)^p} [f_q^p(x, (n + 2)x) + f_q^p(x, (n - 2)x) + 4^p f_q^p(x, (n + 1)x) + 4^p f_q^p(x, (n - 1)x) + 4^p f_q^p(2x, 2x)
\]
\[
+ 4^p f_q^p(2x, x) + n_2^p f_q^p(x, 3x) + 2^p (3n_2^2 - 1)^p f_q^p(x, 2x) + (17n_2^2 - 8)^p f_q^p(x, x)
\]
\[
+ \frac{n_2^p (n_2 - 1)^p [f_q^p(0, (n + 1)x) + f_q^p(0, (n - 3)x) + 10^p f_q^p(0, (n - 1)x) + 4^p f_q^p(0, nx) + 4^p f_q^p(0, (n - 2)x)]}{(n_2 - 1)^p}
\]
\[
\leq \frac{2(3n_2^2 - n_2^2 + 2)^p}{(n_2 - 1)^p} \psi_q^p(0, x)
\] (3.33)
for all \( x \in X \). Therefore, it follows from (3.2) and (3.33) that
\[
\sum_{i=1}^{\infty} 4^p \psi_q^p\left(\frac{x}{2^i}\right) < \infty
\] (3.34)
for all \( x \in X \). Thus, we conclude from (3.32) and (3.34) that the sequence \( \{4^m g\left(\frac{x}{2^m}\right)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, then, the sequence \( \{4^m g\left(\frac{x}{2^m}\right)\} \) converges for all \( x \in X \). So one can define the function \( Q : X \rightarrow Y \) by
\[
Q(x) = \lim_{m \rightarrow \infty} 4^m g\left(\frac{x}{2^m}\right)
\] (3.35)
for all \( x \in X \). Letting \( k = 0 \) and passing the limit \( m \rightarrow \infty \) in (3.32), we get
\[
\|g(x) - Q(x)\|_Y \leq M^8 \sum_{i=0}^{\infty} 4^i \psi_q^p\left(\frac{x}{2^{i+1}}\right) = \frac{M^8p}{4} \sum_{i=1}^{\infty} 4^i \psi_q^p\left(\frac{x}{2^i}\right)
\] (3.36)
for all \( x \in X \). Therefore, (3.5) follows from (3.2) and (3.36). Now we show that \( Q \) is quadratic. It follows from (3.1), (3.31) and (3.35) that
\[
\|Q(2x) - 4Q(x)\|_Y = \lim_{m \rightarrow \infty} \|4^m g\left(\frac{x}{2^{m-1}}\right) - 4^m g\left(\frac{x}{2^m}\right)\|_Y
\]
\[
= 4 \lim_{m \rightarrow \infty} \|4^{m-1} g\left(\frac{x}{2^{m-1}}\right) - 4^m g\left(\frac{x}{2^m}\right)\|_Y
\]
\[
\leq M^1 \lim_{m \rightarrow \infty} 4^m \psi_q\left(\frac{x}{2^m}\right) = 0
\]
for all \( x \in X \). So
\[
Q(2x) = 4Q(x)
\] (3.37)
for all \( x \in X \). On the other hand, it follows from (3.1), (3.3), (3.4) and (3.35) that

\[
\|\triangle Q(x,y)\|_Y = \lim_{m \to \infty} 4^m \|\triangle g(\frac{x}{2^m}, \frac{y}{2^m})\|_Y = \lim_{m \to \infty} 4^m \|\triangle \left(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}\right) - 16 \triangle f(\frac{x}{2^m}, \frac{y}{2^m})\|_Y
\]

\[
\leq M \lim_{m \to \infty} 4^m \left\{ \|\triangle f(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}})\|_Y + 16 \|\triangle g(\frac{x}{2^m}, \frac{y}{2^m})\|_Y \right\}
\]

\[
\leq M \lim_{m \to \infty} 4^m \left\{ \varphi(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}) + 16 \varphi(\frac{x}{2^m}, \frac{y}{2^m}) \right\} = 0
\]

for all \( x, y \in X \). Hence the function \( Q \) satisfies (1.5). By Lemma 2.1, the function \( x \mapsto Q(2x) - 4Q(x) \) is quadratic. Hence, (3.37) implies that the function \( Q \) is quadratic.

It remains to show that \( Q \) is unique. Suppose that there exists another quadratic function \( Q' : X \to Y \) which satisfies (1.5) and (3.5). Since \( Q' \left( \frac{x}{2^m} \right) = \frac{1}{4^m}Q' \left( x \right) \) and \( Q(\frac{x}{2^m}) = \frac{1}{4^m}Q(x) \) for all \( x \in X \), we conclude from (3.5) that

\[
\|Q(x) - Q'(x)\|_Y^p = \lim_{m \to \infty} 4^m \|g(\frac{x}{2^m}) - Q' \left( \frac{x}{2^m} \right)\|_Y^p \leq \frac{M^p}{4^p} \lim_{m \to \infty} 4^m \varphi(\frac{x}{2^m})
\]

(3.38)

for all \( x \in X \). On the other hand, since

\[
\lim_{m \to \infty} 4^m \sum_{i=1}^{\infty} 4^i \varphi(\frac{x}{2^{m+i}}, \frac{y}{2^{m+i}}) = \lim_{m \to \infty} \sum_{i=1}^{\infty} 4^i \varphi(\frac{x}{2^{m+i}}, \frac{y}{2^{m+i}}) = 0
\]

for all \( x \in X \) and for all \( y \in \{x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x\} \), therefore

\[
\lim_{m \to \infty} 4^m \varphi(\frac{x}{2^m}) = 0
\]

(3.39)

for all \( x \in X \). By using (3.39) in (3.38), we get \( Q = Q' \).

\[\square\]

**Theorem 3.3.** Let \( \varphi : X \times X \to [0, \infty) \) be a function such that

\[
\lim_{m \to \infty} \frac{1}{4^m} \varphi(2^m x, 2^m y) = 0
\]

for all \( x, y \in X \) and

\[
\sum_{i=0}^{\infty} \frac{1}{4^i} \varphi(2^i x, 2^i y) < \infty
\]

for all \( x \in X \) and for all \( y \in \{x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x\} \). Suppose that a function \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality

\[
\|\triangle f(x,y)\|_Y \leq \varphi(x,y)
\]

for all \( x, y \in X \). Then the limit

\[
Q(x) := \lim_{m \to \infty} \frac{1}{4^m} [f(2^{m+1} x) - 16f(2^m x)]
\]

exists for all \( x \in X \) and \( Q : X \to Y \) is a unique quadratic function satisfying

\[
\|f(2x) - 16f(x) - Q(x)\|_Y \leq \frac{M^8}{4} \|\varphi(x)\|^p
\]
for all $x \in X$, where

$$
\tilde{\psi}_q(x) := \sum_{i=0}^{\infty} \frac{1}{n^{2^p}(n^2 - 1)^p} \left[ \varphi^p_q(2^i x, 2^i(n + 2)x) + \varphi^p_q(2^i x, 2^i(n - 2)x) \\
+ 4^p \varphi^p_q(2^i x, 2^i(n + 1)x) + 4^p \varphi^p_q(2^i x, 2^i(n - 1)x) + 10^p \varphi^p_q(2^i x, 2^i n x) + \varphi^p_q(2^i x, 2^i 2x) \\
+ 4^p \varphi^p_q(2^i x, 2^i 2x) + n^{2p} \varphi^p_q(2^i x, 2^i 3x) + 2^p (3n^2 - 1)^p \varphi^p_q(2^i x, 2^i 2x) \\
+ (17n^2 - 8)^p \varphi^p_q(2^i x, 2^i x) + \frac{n^{2p}}{(n^2 - 1)^p} (\varphi^p_q(0, 2^i(n + 1)x) + \varphi^p_q(0, 2^i(n - 3)x) \\
+ 10^p \varphi^p_q(0, 2^i(n - 1)x) + 4^p \varphi^p_q(0, 2^i n x) + 4^p \varphi^p_q(0, 2^i(n - 2)x)) \\
+ \frac{(n^4 + 1)^p}{(n^2 - 1)^p} \varphi^p_q(0, 2^i 2x) + \frac{(2(3n^4 - n^2 + 2))^p}{(n^2 - 1)^p} \varphi^p_q(0, 2^i x) \right].
$$

Proof. The proof is similar to the proof of Theorem 3.2. \hfill \Box

Corollary 3.4. Let $\theta, r, s$ be non-negative real numbers such that $r, s > 2$ or $s < 2$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$
\|\Delta f(x, y)\| \leq \begin{cases} 
\theta, & r = s = 0; \\
\|x\|_X, & r > 0, s = 0; \\
\|y\|_X, & r = 0, s > 0; \\
\theta (\|x\|_X + \|y\|_X), & r, s > 0.
\end{cases}
$$

(3.40)

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying

$$
\|f(2x) - 16f(x) - Q(x)\| \leq \frac{M^8 \theta}{n^8(n^2 - 1)} \begin{cases} 
\delta_q, & r = s = 0; \\
\alpha_q(x), & r > 0, s = 0; \\
\beta_q(x), & r = 0, s > 0; \\
(\alpha_q(x) + \beta_q(x))^\frac{r}{s}, & r, s > 0.
\end{cases}
$$

for all $x \in X$, where

$$
\delta_q = \left\{ \frac{1}{4^p - 1(n^2 - 1)^p} \left[ (6n^2 - 2)^p(n^2 - 1)^p + (17n^2 - 8)^p(n^2 - 1)^p + (6n^4 - 2n^2 + 4)^p \\
+ n^{2p}(2 + 10^p + 2 \times 4^p) + (n^4 + 1)^p + n^{2p}(n^2 - 1)^p + 3 \times 4^p(n^2 - 1)^p + 10^p(n^2 - 1)^p \\
+ 3(n^2 - 1)^p \right] \right\}^\frac{r}{s},
$$

$$
\alpha_q(x) = \left\{ \frac{4^p(2 + 2^p) + 10^p + (6n^2 - 2)^p + (17n^2 - 8)^p + 2^p + n^{2p}}{|4^p - 2^p|} \right\}^\frac{r}{s} \|x\|_X
$$

and

$$
\beta_q(x) = \left\{ \frac{1}{(n^2 - 1)^p|4^p - 2^p|} \left[ 2^p(6n^2 - 2)^p(n^2 - 1)^p + (17n^2 - 8)^p(n^2 - 1)^p \\
+ (6n^4 - 2n^2 + 4)^p + n^{2p}(n + 1)^p + (n - 3)^p + 10^p(n - 1)^p \\
+ 4^p n^{2p} + 4^p(n - 2)^p + 2^p(n - 3)^p + 3^p n^{2p}(n^2 - 1)^p + 4^p(n^2 - 1)^p \\
+ (n + 2)^p(n^2 - 1)^p + (n - 2)^p(n^2 - 1)^p + 4^p(n + 1)^p(n^2 - 1)^p \\
+ 4^p(n - 1)^p(n^2 - 1)^p + 10^p n^{2p}(n^2 - 1)^p \right] \right\}^\frac{r}{s} \|x\|_X.
$$

Proof. In Theorem 3.2, putting $\varphi_q(x, y) := \theta(\|x\|_X + \|y\|_X)$ for all $x, y \in X$. \hfill \Box
Corollary 3.5. Let $\theta \geq 0$ and $r, s > 0$ be non-negative real numbers such that $\lambda := r + s \neq 2$.
Suppose that a function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality
\[
\|\Delta f(x, y)\|_Y \leq \theta \|x\|_X \|y\|_X,
\]
for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \rightarrow Y$ satisfying
\[
\|f(2x) - 16f(x) - Q(x)\|_Y \leq \frac{M^4 \theta}{n^2(n^2 - 1)^4} \left\{ \sum_{i=1}^{n^2} \frac{1}{|4^p - 2^p|^3} \left[ (n + 2)^{sp} + (n - 2)^{sp} + 4^p(n + 1)^{sp} + 4^p(n - 1)^{sp} + 10^n n^{sp} + 2^{(r+s)p} + 4^p 2^{rp} + n^{2p} \right] \right\} \|x\|_X^4
\]
for all $x \in X$.

Proof. In Theorem 3.2 putting $\phi(x, y) := \theta \|x\|_X^3 \|y\|_X^3$ for all $x, y \in X$. \qed

Theorem 3.6. Let $\varphi_1 : X \times X \rightarrow [0, \infty)$ be a function such that
\[
\lim_{m \rightarrow \infty} 16^m \varphi_1 \left( \frac{x}{2^m}, \frac{y}{2^m} \right) = 0
\]
for all $x, y \in X$ and
\[
\sum_{i=1}^{\infty} 16^i \varphi_1 \left( \frac{x}{2^i}, \frac{y}{2^i} \right) < \infty
\]
for all $x \in X$ and for all $y \in \{ x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x \}$. Suppose that a function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality
\[
\|\Delta f(x, y)\|_Y \leq \varphi(x, y)
\]
for all $x, y \in X$. Then the limit
\[
T(x) := \lim_{m \rightarrow \infty} 16^m [f(\frac{x}{2^{m-1}}) - 4f(\frac{x}{2^m})]
\]
exists for all $x \in X$ and $T : X \rightarrow Y$ is a unique quartic function satisfying
\[
\|f(2x) - 4f(x) - T(x)\|_Y \leq \frac{M^8}{16} \left[ \varphi_1(x) \right]^4
\]
for all $x \in X$, where
\[
\tilde{\varphi}_1(x) := \sum_{i=1}^{\infty} 16^i \left\{ \frac{1}{n^{2p}(n^2 - 1)^p} \left[ \varphi_1 \left( \frac{x}{2^i}, \frac{(n+2)x}{2^i} \right) + \varphi_1 \left( \frac{x}{2^i}, \frac{(n-2)x}{2^i} \right) \right] + 4^p \varphi_1 \left( \frac{x}{2^i}, \frac{(n+2)x}{2^i} \right) + 4^p \varphi_1 \left( \frac{x}{2^i}, \frac{(n-2)x}{2^i} \right) + 10^n \varphi_1 \left( \frac{x}{2^i}, \frac{nx}{2^i} \right) + \varphi_1 \left( \frac{2x}{2^i}, \frac{2x}{2^i} \right) + 4^p \varphi_1 \left( \frac{2x}{2^i}, \frac{2x}{2^i} \right) + n^{2p} \varphi_1 \left( \frac{x}{2^i}, \frac{3x}{2^i} \right) + 2^p (3n^2 - 1) \varphi_1 \left( \frac{x}{2^i}, \frac{2x}{2^i} \right) + (17n^2 - 8)^p \varphi_1 \left( \frac{x}{2^i}, \frac{2x}{2^i} \right) + \frac{n^{2p}}{(n^2 - 1)^p} \varphi_1 \left( \frac{x}{2^i}, \frac{nx}{2^i} \right) + \varphi_1 \left( 0, \frac{(n+1)x}{2^i} \right) + \varphi_1 \left( 0, \frac{(n-3)x}{2^i} \right) + \frac{10^n}{(n^2 - 1)^p} \varphi_1 \left( \frac{x}{2^i}, \frac{(n+1)x}{2^i} \right) + \frac{10^n}{(n^2 - 1)^p} \varphi_1 \left( \frac{x}{2^i}, \frac{(n-2)x}{2^i} \right) + 4^p \varphi_1 \left( 0, \frac{(n+1)x}{2^i} \right) + 4^p \varphi_1 \left( 0, \frac{(n-2)x}{2^i} \right) + \frac{(n^4 + 1)^p}{(n^2 - 1)^p} \varphi_1 \left( 0, \frac{2x}{2^i} \right) + \frac{2(3n^4 - n^2 + 2)^p}{(n^2 - 1)^p} \varphi_1 \left( 0, \frac{2x}{2^i} \right) \right\}.
\]
Proof. Similar to the proof Theorem 3.2, we have
\[ \|f(4x) - 20f(2x) + 64f(x)\| \leq M^8\psi_t(x), \] (3.48)
for all \( x \in X \), where
\[
\psi_t(x) = \frac{1}{n^2(n^2 - 1)} \left[ \varphi_t(x, (n + 2)x) + \varphi_t(x, (n - 2)x) \\
+ 4\varphi_t(x, (n + 1)x) + 4\varphi_t(x, (n - 1)x) + 10\varphi_t(x, nx) + \varphi_t(2x, 2x) \\
+ 4\varphi_t(2x, x) + n^2\varphi_t(x, 3x) + 2(3n^2 - 1)\varphi_t(x, 2x) + (17n^2 - 8)\varphi_t(x, x) \\
+ \frac{n^2}{n^2 - 1}(\varphi_t(0, (n + 1)x) + \varphi_t(0, (n - 3)x) + 10\varphi_t(0, (n - 1)x) + 4\varphi_t(0, nx) \\
+ 4\varphi_t(0, (n - 2)x)) + \frac{n^4 + 1}{n^2 - 1}\varphi_t(0, 2x) + \frac{2(3n^4 - n^2 + 2)}{n^2 - 1}\varphi_t(0, x) \right].
\] (3.49)

Let \( h : X \to Y \) be a function defined by \( h(x) := f(2x) - 4f(x) \). Then, we conclude that
\[ \|h(2x) - 16h(x)\| \leq M^8 \psi_t(x) \] (3.50)
for all \( x \in X \). If we replace \( x \) in (3.50) by \( \frac{x}{2^{m+1}} \), and multiply both sides of (3.50) by \( 16^m \), we get
\[ \|16^{m+1}h(\frac{x}{2^{m+1}}) - 16^mh(\frac{x}{2^m})\|_Y \leq M^8 16^m \psi_t(\frac{x}{2^{m+1}}) \] (3.51)
for all \( x \in X \) and all non-negative integers \( m \). Since \( Y \) is a p-Banach space, therefore, inequality (3.51) gives
\[ \|16^{m+1}h(\frac{x}{2^{m+1}}) - 16^kh(\frac{x}{2^k})\|_Y^p \leq \sum_{i=k}^{m} \|16^{i+1}h(\frac{x}{2^{i+1}}) - 16^i h(\frac{x}{2^{i}})\|_Y^p \leq M^9 \sum_{i=k}^{m} \psi_t^p(\frac{x}{2^{i+1}}) \] (3.52)
for all non-negative integers \( m \) and \( k \) with \( m \geq k \) and all \( x \in X \). Since \( 0 < p \leq 1 \), then by Lemma 3.1, we conclude from (3.49) that
\[ \psi_t^p(x) \leq \frac{1}{n^2p(n^2 - 1)^p} \left[ \varphi_t^p(x, (n + 2)x) + \varphi_t^p(x, (n - 2)x) \\
+ 4^p\varphi_t^p(x, (n + 1)x) + 4^p\varphi_t^p(x, (n - 1)x) + 10^p\varphi_t^p(x, nx) + \varphi_t^p(2x, 2x) \\
+ 4^p\varphi_t^p(2x, x) + n^2p\varphi_t^p(x, 3x) + 2^p(3n^2 - 1)^p\varphi_t^p(x, 2x) + (17n^2 - 8)^p\varphi_t^p(x, x) \\
+ \frac{n^2p^2}{n^2 - 1}(\varphi_t^p(0, (n + 1)x) + \varphi_t^p(0, (n - 3)x) + 10^p\varphi_t^p(0, (n - 1)x) + 4^p\varphi_t^p(0, nx) \\
+ 4^p\varphi_t^p(0, (n - 2)x)) + \frac{(n^4 + 1)p}{n^2 - 1}\varphi_t^p(0, 2x) + \frac{2(3n^4 - n^2 + 2)}{(n^2 - 1)p}\varphi_t^p(0, x) \right], \] (3.53)
for all \( x \in X \). Therefore, it follows from (3.42) and (3.52) that
\[ \sum_{i=1}^{\infty} \psi_t^p(\frac{x}{2^i}) < \infty \] (3.54)
for all \( x \in X \). Thus, we conclude from (3.52) and (3.54) that the sequence \( \{16^m h(\frac{x}{2^m})\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{16^m h(\frac{x}{2^m})\} \) converges
According to (3.42), (3.51) and (3.55), it follows that
\[ \| T_{x,y} \| \text{ for all } x, \]
for all \( x \). Letting \( k = 0 \) and passing the limit \( m \to \infty \) in (3.52), we get
\[ \| h(x) - T(x) \|_Y \leq M^{8p} \sum_{i=0}^{\infty} 16^i \psi^{p}\left( \frac{x}{2^{i+1}} \right) = M^{11p} \sum_{i=1}^{\infty} 16^i \psi^{p}\left( \frac{x}{2^i} \right) \]  
(3.56)
for all \( x \). Therefore (3.45) follows from (3.43) and (3.55). Now we show that \( T \) is quartic. According to (3.42), (3.51) and (3.55), it follows that
\[ \| T(2x) - 16T(x) \|_Y = \lim_{m \to \infty} 16^m \| h\left( \frac{x}{2^{m-1}} \right) - 16^m h\left( \frac{x}{2^m} \right) \|_Y \]
\[ = 16 \lim_{m \to \infty} 16^m \| h\left( \frac{x}{16^{m-1}} \right) - 16^m h\left( \frac{x}{2^m} \right) \|_Y \]
\[ \leq M^8 \lim_{m \to \infty} 16^m \psi\left( \frac{x}{2^m} \right) = 0 \]
for all \( x \). So
\[ T(2x) = 16T(x) \]  
(3.57)
for all \( x \). On the other hand, by (3.44), (3.54) and (3.55), we lead to
\[ \| \Delta T(x, y) \|_Y = \lim_{m \to \infty} 16^m \| \Delta h\left( \frac{x}{2^m} \cdot \frac{y}{2^m} \right) \|_Y = \lim_{m \to \infty} 16^m \| \Delta f\left( \frac{x}{2^m} \cdot \frac{y}{2^m} \right) - 4\Delta f\left( \frac{x}{2^m} \cdot \frac{y}{2^m} \right) \|_Y \]
\[ \leq M \lim_{m \to \infty} 16^m \left\{ \| \Delta f\left( \frac{x}{2^m} \cdot \frac{y}{2^m} \right) \|_Y + 4\| \Delta f\left( \frac{x}{2^m} \cdot \frac{y}{2^m} \right) \|_Y \right\} \]
\[ \leq M \lim_{m \to \infty} 16^m \left\{ \psi_i\left( \frac{x}{2^m} \cdot \frac{y}{2^m} \right) + 4\psi_i\left( \frac{x}{2^m} \cdot \frac{y}{2^m} \right) \right\} = 0 \]
for all \( x, y \in X \). Hence, the function \( T \) satisfies (1.5). By Lemma 2.1, the function \( x \to T(2x) - 16T(x) \) is quartic. Therefore (3.57) implies that the function \( T \) is quartic.

To prove the uniqueness property of \( T \), let \( T' : X \to Y \) be another quartic function satisfying (3.46). Since
\[ \lim_{m \to \infty} 16^m p \sum_{i=1}^{\infty} 16^i \psi_i^p\left( \frac{x}{2^{m+i}} \cdot \frac{x}{2^{m+i}} \right) = \lim_{m \to \infty} \sum_{i=m+1}^{\infty} 16^i \psi_i^p\left( \frac{x}{2^i} \cdot \frac{x}{2^i} \right) = 0 \]
for all \( x \in X \) and for all \( y \in \{ x, 2x, 3x, \ldots, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x \} \), then
\[ \lim_{m \to \infty} 16^m \psi_i\left( \frac{x}{2^m} \right) = 0 \]  
(3.58)
for all \( x \). It follows from (3.46) and (3.58) that
\[ \| T(x) - T'(x) \|_Y = \lim_{m \to \infty} 16^m \| h\left( \frac{x}{2^m} \right) - T'\left( \frac{x}{2^m} \right) \|_Y \leq M^{8p} \lim_{m \to \infty} 16^m \psi_i\left( \frac{x}{2^m} \right) = 0 \]
for all \( x \). So \( T = T' \).

\[ \square \]

**Theorem 3.7.** Let \( \psi : X \times X \to [0, \infty) \) be a function such that
\[ \lim_{m \to \infty} \frac{1}{16^m} \psi_i\left( \frac{2^n x}{2^m} \right) = 0 \]
for all \(x, y \in X\) and
\[
\sum_{i=0}^{\infty} \frac{1}{16^p} \varphi_i^p(2^i x, 2^i y) < \infty
\]
for all \(x \in X\) and for all \(y \in \{x, 2x, 3x, nx, (n + 1)x, (n - 1)x, (n + 2)x, (n - 2)x, (n - 3)x\}\). Suppose that a function \(f : X \to Y\) with \(f(0) = 0\) satisfies the inequality
\[
\|\Delta f(x, y)\|_Y \leq \delta_t(x, y)
\]
for all \(x, y \in X\). Then the limit
\[
T(x) := \lim_{m \to \infty} \frac{1}{16^m} [f(2^{m+1} x) - 4 f(2^m x)]
\]
exists for all \(x \in X\) and \(T : X \to Y\) is a unique quartic function satisfying
\[
\|f(2x) - 4 f(x) - T(x)\|_Y \leq \frac{M^8}{16} \|\psi_t(x)\|_Y
\]
for all \(x \in X\), where
\[
\psi_t(x) := \sum_{i=0}^{\infty} \frac{1}{16^i} \left\{ \frac{1}{n^2p(n^2 - 1)^p} \left[ \varphi_i^p(2^i x, 2^i (n+2)x) + \varphi_i^p(2^i x, 2^i (n-2)x) + 4^p \varphi_i^p(2^i x, 2^i (n+1)x) + 4^p \varphi_i^p(2^i x, 2^i (n-1)x) + 10^p \varphi_i^p(2^i x, 2^i nx) + \varphi_i^p(2^i x, 2^i 2x) + 4^p \varphi_i^p(2^i x, 2^i 2x) + n^{2p} \varphi_i^p(2^i x, 2^i 3x) + 2^p(3n^2 -1)^p \varphi_i^p(2^i x, 2^i 2x)
+ (17n^2 - 8)^p \varphi_i^p(2^i x, 2^i x) + \frac{n^{2p}}{(n^2 - 1)^p} \varphi_i^p(0, 2^i (n+1)x) + \varphi_i^p(0, 2^i (n-3)x) + 10^p \varphi_i^p(0, 2^i (n+1)x) + 4^p \varphi_i^p(0, 2^i 2x) + \varphi_i^p(0, 2^i 2x)
\right] \right}. \]

Proof. The proof is similar to the proof of Theorem 3.6.

\(\square\)

**Corollary 3.8.** Let \(\theta, r, s\) be non-negative real numbers such that \(r, s > 0\) or \(0 < r, s < 4\). Suppose that a function \(f : X \to Y\) with \(f(0) = 0\) satisfies the inequality (3.40) for all \(x, y \in X\). Then there exists a unique quartic function \(T : X \to Y\) satisfying
\[
\|f(2x) - 4 f(x) - T(x)\|_Y \leq \frac{M^8 \theta}{n^2} \left\{ \begin{array}{ll}
\delta_t, & r = s = 0; \\
\alpha_t(x), & r > 0, s = 0; \\
\beta_t(x), & r = 0, s > 0; \\
(\alpha_t^p(x) + \beta_t^p(x))^{1/2}, & r, s > 0.
\end{array} \right.
\]
for all \(x \in X\), where
\[
\delta_t = \left\{ \frac{1}{(16^p - 1)(n^2 - 1)^p} [(6n^2 - 2)^p(n^2 - 1)^p + (17n^2 - 8)^p(n^2 - 1)^p + (6n^2 - 2n^2 + 4)^p
+ n^{2p}(2 + 10^p + 2 * 4^p) + (n^2 + 1)^p + n^{2p}(n^2 - 1)^p + 3 * 4^p(n^2 - 1)^p + 10^p(n^2 - 1)^p
+ 3(n^2 - 1)^p] \right\}^{1/2},
\]
\[
\alpha_t(x) = \left\{ \frac{4^p(2 + 2^p) + 10^p + (6n^2 - 2)^p + (17n^2 - 8)^p + 2^p + n^{2p}}{|16^p - 2^p|} \right\}^{p} \|x\|
\]
On the stability of generalized mixed ... 17

and

\[ \beta_t(x) = \left\{ \frac{1}{(n^2 - 1)^p} \frac{1}{(16^p - 2^p)} \left[ 2^p (n^2 - 2)^p (n^2 - 1)^p + (17n^2 - 8)^p (n^2 - 1)^p \right. \right. \\
+ (6n^4 - 2n^2 + 4)^p + n^{2p} (n + 1)^p + (n - 3)^p + 10^p (n - 1)^p \right. \right. \\
+ 10^p n^{sp} + 4^p (n - 2)^p + 2^p (n^2 + 1)^p + 3^p n^{2p} (n^2 - 1)^p + 4^p (n^2 - 1)^p \\
+ (n + 2)^p (n^2 - 1)^p + (n - 2)^p (n^2 - 1)^p + 4^p (n + 1)^p (n^2 - 1)^p \\
+ 4^p (n - 1)^p (n^2 - 1)^p + 4^p n^{sp} (n^2 - 1)^p \left. \right\} \lambda \|x\|_X^p. \]

Proof. In Theorem 3.6, putting \( \varphi_t(x, y) := \theta \|x\|_X + \|y\|_X \) for all \( x, y \in X \). \( \square \)

Corollary 3.9. Let \( \theta \geq 0 \) and \( r, s > 0 \) be non-negative real numbers such that \( 0 \leq r + s \neq 4 \). Suppose that a function \( f : X \rightarrow Y \) with \( f(0) = 0 \) satisfies the inequality (3.41) for all \( x, y \in X \). Then there exists a unique quartic function \( T : X \rightarrow Y \) satisfying

\[ \|f(2x) - 4f(x) - T(x)\|_Y \leq \frac{\lambda M^2}{n^2(n^2 - 1)} \left\{ \frac{1}{(16^p - 2^p)} \left[ (n + 2)^p + (n - 2)^p + 4^p (n + 1)^p \right. \right. \\
+ 4^p (n - 1)^p + 10^p n^{sp} + 2^{(r+s)} \right. \right. \\
+ 4^p 2^{sp} + n^{sp} \left. \right\} \lambda \|x\|_X^p \]

for all \( x \in X \).

Proof. In Theorem 3.6, putting \( \varphi_t(x, y) := \theta \|x\|_X \|y\|_X \) for all \( x, y \in X \). \( \square \)

Theorem 3.10. Let \( \varphi : X \times X \rightarrow [0, \infty) \) be a function such that

\[ \lim_{m \to \infty} 4^m \varphi\left( \frac{x}{2^m}, \frac{y}{2^m} \right) = 0 = \lim_{m \to \infty} \frac{1}{4^m} \varphi(2^m x, 2^m y) \]

(3.59)

for all \( x, y \in X \) and

\[ \sum_{i=1}^{\infty} 4^m \varphi\left( \frac{x}{2^m}, \frac{y}{2^m} \right) < \infty \]

and

\[ \sum_{i=0}^{\infty} \frac{1}{16^m} \varphi\left( 2^m x, 2^m y \right) < \infty \]

for all \( x \in X \) and for all \( y \in \{ x, 2x, 3x, nx, (n + 1)x, (n - 1)x, (n + 2)x, (n - 2)x, (n - 3)x \} \). Suppose that a function \( f : X \rightarrow Y \) with \( f(0) = 0 \) satisfies the inequality

\[ ||\Delta f(x, y)||_Y \leq \varphi(x, y), \]

(3.60)

for all \( x, y \in X \). Then there exist a unique quadratic function \( Q : X \rightarrow Y \) and a unique quartic function \( T : X \rightarrow Y \) such that

\[ ||f(x) - Q(x) - T(x)||_Y \leq \frac{M^2}{192} \left( 4[\tilde{\psi}_q(x)]^2 + [\tilde{\psi}_t(x)]^2 \right) \]

(3.61)

for all \( x \in X \), where \( \tilde{\psi}_q(x) \) and \( \tilde{\psi}_t(x) \) have been defined in Theorems 3.2 and 3.7, respectively, for all \( x \in X \).
Proof. By Theorems 3.2 and 3.7, there exist a quadratic function \( Q_0 : X \to Y \) and a quartic function \( T_0 : X \to Y \) such that

\[
\|f(2x) - 16f(x) - Q_0(x)\|_Y \leq M^8 \frac{1}{4} (\tilde{v}_0(x))^\frac{1}{p}, \quad \|f(2x) - 4f(x) - T_0(x)\|_Y \leq M^8 \frac{1}{16} (\tilde{v}_1(x))^\frac{1}{p}
\]

for all \( x \in X \). Therefore, it follows from the last inequalities that

\[
\|f(x) + \frac{1}{12}Q_0(x) - \frac{1}{12}T_0(x)\|_Y \leq \frac{M^9}{192} (4[\tilde{v}_0(x)]^\frac{1}{p} + [\tilde{v}_1(x)]^\frac{1}{p})
\]

for all \( x \in X \). So we obtain (3.61) by letting \( Q(x) = -\frac{1}{12}Q_0(x) \) and \( T(x) = \frac{1}{12}T_0(x) \) for all \( x \in X \).

To prove the uniqueness property of \( Q \) and \( T \), we first show the uniqueness property for \( Q_0 \) and \( T_0 \) and then we conclude the uniqueness property of \( Q \) and \( T \). Let \( Q_1, T_1 : X \to Y \) be another quadratic and quartic functions satisfying (3.61) and let \( Q_2 = \frac{1}{12}Q_0, T_2 = \frac{1}{12}T_0 \), \( Q_3 = Q_2 - Q_1 \) and \( T_3 = T_2 - T_1 \). So

\[
\|Q_3(x) - T_3(x)\|_Y \leq M \left( \|f(x) - Q_2(x) - T_2(x)\|_Y + \|f(x) - Q_1(x) - T_1(x)\|_Y \right)
\]

\[
\leq \frac{M^9}{90} (4[\tilde{v}_0(x)]^\frac{1}{p} + [\tilde{v}_1(x)]^\frac{1}{p})
\]

(3.62) for all \( x \in X \). Since

\[
\lim_{m \to \infty} 4^m \tilde{v}_0 \left( \frac{x}{2^m} \right) = \lim_{m \to \infty} \frac{1}{16^m} \tilde{v}_1 (2^m x) = 0
\]

for all \( x \in X \), then (3.62) implies that \( \lim_{m \to \infty} \|4^m Q_3 (\frac{x}{2^m}) + \frac{1}{16^m} T_3 (2^m x)\|_Y = 0 \) for all \( x \in X \). Thus, \( T_3 = Q_3 \). But \( T_3 \) is only a quartic function and \( Q_3 \) is only a quadratic function. Therefore, we should have \( T_3 = Q_3 = 0 \) and this complete the uniqueness property of \( Q \) and \( T \). The other results proved similarly.

Corollary 3.11. Let \( \theta, r, s \) be non-negative real numbers such that \( r, s > 4 \) or \( 2 < r, s < 4 \) or \( 0 \leq r, s < 2 \). Suppose that a function \( f : X \to Y \) satisfies the inequality (3.40) for all \( x, y \in X \). Then there exist a unique quadratic function \( Q : X \to Y \) and a unique quartic function \( T : X \to Y \) such that

\[
\|f(x) - Q(x) - T(x)\|_Y \leq \frac{M^9 \theta}{12n^2 (n^2 - 1)} \left\{ \begin{array}{l}
\delta_\theta + \delta_t,
\alpha_\theta(x) + \alpha_t(x),
\beta_\theta(x) + \beta_t(x),
\left( \alpha_\theta(x) + \beta_\theta(x) \right)^\frac{1}{p} + \left( \alpha_\theta(x) + \beta_\theta(x) \right)^\frac{1}{q},
\end{array} \right. r = s = 0;
\right. r > 0, s = 0;
\right. r = 0, s > 0;
\right. r > 0, s > 0.
\]

for all \( x \in X \), where \( \delta_\theta, \delta_t, \alpha_\theta(x), \alpha_t(x), \beta_\theta(x) \) and \( \beta_t(x) \) are defined as in Corollaries 3.4 and 3.8.

Corollary 3.12. Let \( \theta \geq 0 \) and \( r, s > 0 \) be non-negative real numbers such that \( \lambda := r + s \in (0, 2) \cup (2, 4) \cup (4, \infty) \). Suppose that a function \( f : X \to Y \) satisfies the inequality (3.41) for all \( x, y \in X \). Then there exist a unique quadratic function \( Q : X \to Y \) and a unique quartic function \( T : X \to Y \) such that

\[
\|f(x) - Q(x) - T(x)\|_Y \leq \frac{M^9 \theta}{12n^2 (n^2 - 1)} \left\{ \begin{array}{l}
\frac{1}{|4p - 2sp|} [(n + 2)^{sp} + (n - 2)^{sp} + 4^p (n + 1)^{sp} + 4^p (n - 1)^{sp} + 10^p n^{sp} + 2^{(r+s)p} + 4^p 2^{sp} + n^{2sp} 3^{sp} + 2^p (6n^2 - 2)^p + (17n^2 - 8)^p] \right\}^\frac{1}{p} x \|_X
\end{array} \right.
\]

for all \( x \in X \).
On the stability of generalized mixed ...

REFERENCES

[1] J. Aczel, J. Dhombres, Functional Equations in Several Variables, *Cambridge Univ. Press*, 1989.

[2] D. Amir, Characterizations of inner product spaces. *Operator Theory: Advances and Applications*, 20. *Birkhäuser Verlag, Basel*, 1986. vi+200 pp. ISBN: 3-7643-1774-4

[3] Y. Benyamini, J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, vol. 1, Colloq. Publ. vol. 48, Amer. Math. Soc., Providence, RI, 2000.

[4] P. W. Cholewa, Remarks on the stability of functional equations, *Aequationes Math.* 27 (1984) 76-86.

[5] Jukang K. Chung and Prasanna K. Sahoo, On the general solution of a quartic functional equation, *Bull. Korean Math. Soc.* 40 (2003), no. 4, 565–576.

[6] S. Czerwik, On the stability of the quadratic mapping in normed spaces, *Abh. Math. Sem. Univ. Hamburg* 62 (1992) 59–64.

[7] Z. Gajda, On stability of additive mappings, *Internat. J. Math. Math. Sci.* 14(1991) 431-434.

[8] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* 184 (1994) 431-436.

[9] A. Grabiec, The generalized Hyers–Ulam stability of a class of functional equations, *Publ. Math. Debrecen* 48 (1996) 217-235.

[10] D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci.* 27 (1941) 222-224.

[11] P. Jordan, J. Von Neumann, on inner product in linear metric spaces, *Ann. of Math.* 36(1935) 719-723.

[12] H. M. Kim, On the stability problem for a mixed type of quartic and quadratic functional equation, *J. Math. Anal. Appl.* 324 (2006)358-372.

[13] Pl. Kannappan, Quadratic functional equation and inner product spaces, *Results Math.* 27 (1995) 368-372.

[14] Won-Gil Park, Jae-Hyeong Bae, On a bi-quadratic functional equation and its stability, *Nonlinear Anal.* 62 (2005), no. 4, 643–654.

[15] A. Najati, M. B. Moghimi, Stability of a functional equation deriving from quadratic and additive function in quasi-Banach spaces, *J. Math. Anal. Appl.* 337 (2008) 399-415.

[16] A. Najati, G. Zamani Eskandani, Stability of a mixed additive and cubic functional equation in quasi-Banach spaces, *J. Math. Anal. Appl.* 342 (2008) 13181331.

[17] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* 72 (1978) 297-307.

[18] S. Rolewicz, Metric Linear Spaces, *PWN-Polish Sci. Publ., Warszawa, Reidel, Dordrecht*, 1984. MR0802450 (88i:46004a)

[19] F. Skof, Propriet locali e approssimazione di operatori, *Rend. Sem. Mat. Fis. Milano*, 53 (1983), 113129.

[20] J. Tabor, stability of the Cauchy functional equation in quasi-Banach spaces, *Ann. Polon. Math.* 83(2004) 243-255.

[21] S. M. Ulam, Problems in Modern Mathematics, *Chapter VI, science ed.*, *Wiley, New York*, 1940.