Memory kernel approach to generalized Pauli channels: Markovian, semi-Markov, and beyond

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In this paper, we analyze the evolution of the generalized Pauli channels governed by the memory kernel master equation. We provide necessary and sufficient conditions for the memory kernel to give rise to the legitimate (completely positive and trace-preserving) quantum evolution. In particular, we analyze a class of kernels generating the quantum semi-Markov evolution, which is a natural generalization of the Markovian semigroup. Interestingly, the convex combination of Markovian semigroups goes beyond the semi-Markov case. Our analysis is illustrated with several examples.

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1. INTRODUCTION

In the theory of open quantum systems [1–3], the use of the Born-Markov approximation leads to the celebrated Markovian master equation,

$$\dot{\rho}_t = \mathcal{L}[\rho_t],$$

where $\mathcal{L}$ is the generator of the Markovian semigroup given by the well-known Gorini-Kossakowski-Sudarshan-Lindblad form [4, 5],

$$\mathcal{L}[\rho] = -i[H_{\text{eff}}, \rho] + \frac{1}{2} \sum_\alpha \gamma_\alpha \left( V_\alpha \rho V_\alpha^\dagger - \frac{1}{2} \{ V_\alpha^\dagger V_\alpha, \rho \} + \right),$$

where $H_{\text{eff}}$ is the effective Hamiltonian of the system, $V_\alpha$ denote the noise operators, and $\gamma_\alpha \geq 0$ are the decoherence rates. Equation (1) leads to the completely positive, trace-preserving (CPTP) dynamical map $\rho_0 \rightarrow \rho_t = \Lambda_t[\rho_0]$ satisfying the composition law,

$$\Lambda_t \Lambda_u = \Lambda_{t+u},$$

for all $t, u \geq 0$. The Born-Markov approximation assumes weak interactions and a separation of time scales between the system and its environment. Such approximation is usually valid in quantum optical systems. However, it is often violated in solid state physics. There are two natural generalizations of the above scheme. The first one introduces the time-local generator $\mathcal{L}_t$ which is of the form (2) but with time-dependent $V_\alpha(t)$ and $\gamma_\alpha(t)$. In the second approach, one takes into account non-local memory effects through the Nakajima-Zwanzig equation (4),

$$\dot{\rho}_t = \int_0^t K_{t-\tau} \rho_\tau \ d\tau,$$

with $K_t$ being the memory kernel. Indeed, contrary to (1), the rate $\dot{\rho}_t$ at the time $t$ depends on the whole history $\rho_\tau$ — starting from the initial time $\tau = 0$, up to the current time $\tau = t$. The Markovian semigroup (2) is recovered for $K_t = 2\delta(t)\mathcal{L}$.

The central problem with the memory kernel master equation (4) is to provide the necessary and sufficient conditions for the memory kernel super-operator $K_t$ which guarantee that the solution in the form of the dynamical map $\Lambda_t$ is CPTP. Such problem was originally posed by Barnett and Stenholm [6] for the memory kernel

$$K_t = k(t)\mathcal{L}$$

with the memory function $k(t)$ and the legitimate Markovian generator $\mathcal{L}$. Unfortunately, in general, such memory kernels may lead to unphysical results. This issue was further analyzed in [5, 10]. Shabani and Lidar [11] proposed the so-called post-Markovian master equation with

$$K_t = k(t)\mathcal{L}e^{\mathcal{L}t}.$$  

At this approach works for certain classes of Markovian generators $\mathcal{L}$ and memory functions $k(t)$. The authors succeeded in finding the necessary and sufficient conditions for this memory kernel to be legitimate. There is also the class of the qubit evolution [12] for which this kernel always produces physical results. Much attention was paid to finding the admissible memory kernels. It turned out that they can arise from the collisional model [12]. Another class was found for the random unitary qubit evolution [13]. The quantum analogue of the semi-Markov evolution was analyzed in [15, 17]. Interestingly, the proper definition of the quantum semi-Markov evolution was given only in [18], using the notion of legitimate pairs of quantum maps [14]. For recent papers discussing memory kernel approach see also [20, 22].

In this paper, we analyze the evolution of the generalized Pauli channels under the memory kernel master equation (4). We provide the necessary and sufficient conditions for the admissible memory kernel — that is, the kernel giving rise to the CPTP dynamical map $\Lambda_t$. A special class of memory kernels corresponds to the so-called semi-Markov quantum evolution, which is the quantum analogue of the classical semi-Markov process. We provide several examples of the semi-Markov evolution of the generalized Pauli channels. Interestingly, the convex combination of Markovian semigroups (which is also the generalized Pauli channel) is not semi-Markov.
2. GENERALIZED PAULI CHANNELS

The definition of the generalized Pauli channel involves the notion of mutually unbiased bases (MUBs). Two orthonormal bases |ψ_k⟩, |ϕ⟩ ∈ C^d are said to be mutually unbiased if and only if

|⟨ψ_k|ϕ⟩|^2 = \frac{1}{d}, \quad (5)

For d = p^r, where p is a prime number, the number of MUBs in C^d is maximal and equal to d + 1 [26, 27].

Take the d-dimensional Hilbert space for which one has d + 1 MUBs, { |ψ_0⟩, ..., |ψ_{d-1}⟩ }. The corresponding rank-1 projectors are given by P_i^(α) = |ψ_i^(α)⟩⟨ψ_i^(α)|. Now, let us define d + 1 unitary operators

U_α = \sum_{l=0}^{d-1} \omega^l P_l^(α), \quad (6)

where \omega = e^{2\pi i/d}, and the family of completely positive maps

U_α[ρ] = \sum_{k=1}^{d-1} U_k^α ρ U_k^α, \quad (7)

The evolution under the generalized Pauli channel is given by the following dynamical map [26, 27],

Λ_t = p_0(t) \mathbb{1} + \frac{1}{d - 1} \sum_{α=1}^{d+1} p_α(t) U_α, \quad (8)

where (p_0(t), p_1(t), ..., p_{d+1}(t)) denotes the probability vector such that p_0(0) = 1 and p_α(0) = 0 for α = 1, ..., d + 1. By the identity map \mathbb{1}, we understand \mathbb{1}[X] = X for any operator X. It is clear that this definition reproduces the Pauli channel for d = 2,

Λ_t = p_0(t) \mathbb{1} + \sum_{α=1}^{3} p_α(t) U_α, \quad (9)

with U_α[ρ] = σ_α ρ σ_α, and σ_α being the Pauli matrices.

One easily solves the eigenvalue problem for Λ_t,

Λ_t[U_k^α] = λ_α(t) U_k^α, \quad k = 1, ..., d - 1, \quad (10)

with the eigenvalues

λ_α(t) = p_0(t) + \frac{d}{d - 1} p_α(t) - \frac{1}{d - 1} \sum_{β=1}^{d+1} p_β(t), \quad (11)

and λ_0(t) = 1. All the eigenvalues are real, whereas λ_α(t) (α = 1, ..., d + 1) are (d - 1)-fold degenerated. The inverse relation reads

\frac{d}{d - 1} \mathbb{1} + \sum_{α=1}^{d+1} \lambda_α(t) = \frac{1}{d} \left[ 1 + (d - 1) \sum_{α=1}^{d+1} \lambda_α(t) \right], \quad (12)

The eigenvalue equation for L_t reads

L_t[U_k^α] = μ_α(t) U_k^α, \quad (19)

with μ_α(t) = γ_α(t) − γ(t) and γ(t) = \sum_{α=1}^{d+1} γ_α(t). Therefore, the time-dependent eigenvalues λ_α(t) of the dynamical map Λ_t are given by

λ_α(t) = \exp[Γ_α(t) − Γ(t)], \quad (20)

where Γ_α(t) = \int_0^t γ_α(τ) d τ and Γ(t) = \sum_{α=1}^{d+1} Γ_α(t).

3. MEMORY KERNEL APPROACH

In this paper, we analyze the evolution of the generalized Pauli channel Λ_t which is provided by the memory kernel equation

\dot{Λ}_t = \int_0^t K_{t-τ} Λ_τ d τ \quad (21)

with the following memory kernel,

K_t = \sum_{α=1}^{d+1} k_α(t) [Φ_α - \mathbb{1}], \quad (22)

Note that the eigenvalue equations of such a memory kernel are given by

K_t[U_k^α] = κ_α(t) U_k^α, \quad K_t[\mathbb{1}] = 0, \quad (23)
where
\[ \kappa_\alpha(t) = k_\alpha(t) - k(t), \]  
(24)
with \( k(t) = \sum_{\beta=1}^{d+1} k_\beta(t) \). Taking (23) and (10) into account, we can rewrite the relationship between the memory kernel \( K_t \) and the corresponding generalized Pauli channel \( \Lambda_t \) in terms of the corresponding eigenvalues,
\[ \dot{\lambda}_\alpha(t) = \int_0^t \kappa_\alpha(t - \tau) \lambda_\alpha(\tau) \, d\tau, \]
with \( \lambda_\alpha(0) = 1 \). In the Laplace transform (LT) domain, one finds the following relation,
\[ \tilde{\lambda}_\alpha(s) = \frac{1}{s - \tilde{\kappa}_\alpha(s)}, \]
(26)
where \( \tilde{f}(s) = \int_0^\infty f(t)e^{-st} \, dt \) stands for the Laplace transform of \( f(t) \). Let us parameterize the eigenvalues \( \lambda_\alpha(t) \) as follows,
\[ \lambda_\alpha(t) = 1 - \int_0^t \ell_\alpha(\tau) \, d\tau. \]
(27)
One arrives at the following theorem.

**Theorem 1.** The memory kernel \( K_t \) defined in (22) gives rise to a legitimate dynamical map \( \Lambda_t \) if and only if the corresponding eigenvalues \( \kappa_\alpha(t) \) are, in the LT domain, given by
\[ \tilde{\kappa}_\alpha(s) = -\frac{s\tilde{\ell}_\alpha(s)}{1 - \tilde{\ell}_\alpha(s)}, \]
(28)
and the functions \( \ell_\alpha(t) \) satisfy
\[ \int_0^t \ell_\alpha(\tau) \, d\tau \geq 0, \]
\[ \sum_{\alpha=1}^{d+1} \int_0^t \ell_\alpha(\tau) \, d\tau \leq \frac{d^2}{d-1}, \]
(29)
\[ \sum_{\alpha=1}^{d+1} \int_0^t \ell_\alpha(\tau) \, d\tau \geq d \int_0^t \ell_\beta(\tau) \, d\tau \]
for \( \beta = 1, \ldots, d+1 \).

The proof is evident since the above conditions reproduce \( \lambda_\alpha(t) \leq 1 \), \( p_0(t) \geq 0 \), and \( p_\beta(t) \geq 0 \) for \( \beta = 1, \ldots, d+1 \), respectively. The main problem is to find a reasonable class of functions \( \ell_\alpha(t) \) satisfying conditions (29).

**Proposition 1.** Consider \( \ell_\alpha(t) = \eta e^{-\xi_\alpha t} \) with \( \eta, \xi_\alpha > 0 \). If
\[ \eta \sum_{\alpha=1}^{d+1} \frac{1}{\xi_\alpha} \leq \frac{d^2}{d-1}, \]
(30)
\[ \sum_{\alpha=1}^{d+1} \frac{1}{\xi_\alpha} \geq \frac{d}{\xi_\beta}. \]
(31)
then \( \ell_\alpha(t) \)'s satisfy (29).

**Proof.** Let us start with showing that the first inequality in the Fujiwara-Algoet conditions (14),
\[ -\frac{1}{d-1} \leq \sum_{\beta=1}^{d+1} \lambda_\beta, \]
(32)
is equivalent to (30). For our choice of \( \ell_\alpha(t) \)'s, the eigenvalues of \( \Lambda_t \) are equal to
\[ \lambda_\alpha(t) = 1 - \frac{\eta}{\xi_\alpha} (1 - e^{-\xi_\alpha t}). \]
(33)
After inserting (33) into (32), one has
\[ -\frac{1}{d-1} \leq d + 1 - \sum_{\alpha=1}^{d+1} \frac{\eta}{\xi_\alpha} (1 - e^{-\xi_\alpha t}). \]
(34)
While this inequality holds for all \( t \geq 0 \), it is enough to check that it is true for \( t \to \infty \). Therefore, we arrive at
\[ -\frac{1}{d-1} \leq d + 1 - \sum_{\beta=1}^{d+1} \frac{\eta}{\xi_\beta}, \]
(35)
which is, indeed, equivalent to (30).

Now, we start from (31). Denote the minimal value of \( \xi_\alpha \) by \( \xi_{\min} = \min_{\alpha} \xi_\alpha \). If (31) holds for every \( \beta = 1, \ldots, d+1 \), then it is also true for \( \xi_{\min} \). Multiplying both sides of the inequality by the same (positive) coefficient, we get
\[ \sum_{\alpha=1}^{d+1} \frac{1}{\xi_\alpha} (1 - e^{-\xi_{\min} t}) \geq \frac{d}{\xi_{\min}} (1 - e^{-\xi_{\min} t}). \]
(36)
Observe that
\[ \sum_{\alpha=1}^{d+1} \frac{1}{\xi_\alpha} (1 - e^{-\xi_\alpha t}) \geq \sum_{\alpha=1}^{d+1} \frac{1}{\xi_\alpha} (1 - e^{-\xi_{\min} t}), \]
(37)
and therefore (36) reduces to
\[ \sum_{\alpha=1}^{d+1} \frac{1}{\xi_\alpha} (1 - e^{-\xi_\alpha t}) \geq \frac{d}{\xi_{\min}} (1 - e^{-\xi_{\min} t}). \]
(38)
Lastly, note that
\[ h(\xi, t) = \frac{d}{\xi} (1 - e^{-\xi t}) \]
(39)
is a function of \( \xi \) which decreases monotonically with the increasing value of \( \xi \) for each fixed \( t \geq 0 \). Hence, \( h(\xi_{\min}, t) = \max_\alpha h(\xi_\alpha, t) \), which means that (38) is equivalent to the second inequality in the Fujiwara-Algoet conditions (14). \( \square \)

For \( \ell_\alpha(t) = \eta e^{-\xi_\alpha t} \), one finds
\[ \kappa_\alpha(t) = -\eta \delta(t) + \eta(\xi_\alpha - \eta)e^{-\xi_\alpha t}, \]
(40)
and finally
\[ k_\alpha(t) = \frac{1}{d} \eta \delta(t) + \eta(\xi_\alpha - \eta)e^{-(\xi_\alpha - \eta)t} - \frac{1}{d} \sum_{\beta=1}^{d+1} \eta(\xi_\beta - \eta)e^{-(\xi_\beta - \eta)t}, \]  
which shows that a linear combination of simple exponential memory functions has to satisfy strong constraints \([10, 11]\). Observe that \(k_\alpha(t \to \infty) \to \infty\) for \(\xi_\alpha - \eta < 0\). Conditions \([10, 11]\) imply
\[ \xi_\alpha - \eta \geq -\frac{\eta}{d}, \]  
which means that we need an additional restriction for the choice of \(\eta\) and \(\xi_\alpha\)'s to obtain a physical memory kernel.

A special class of memory kernels is given by
\[ \ell_\alpha(t) = \frac{1}{a_\alpha} \ell(t). \]  
In this case, Theorem 1 implies the following.

**Proposition 2.** If the function \(\ell(t)\) and the collection of numbers \(\{a_1, \ldots, a_{d+1}\}\) satisfy
\[ \sum_{\alpha=1}^{d+1} \frac{1}{a_\alpha} \int_0^t \ell(\tau) \, d\tau \leq \frac{d^2}{d-1}, \]  
together with
\[ \sum_{\beta=1}^{d+1} \frac{1}{a_\beta} \geq \frac{d}{a_\alpha}, \]  
then \(K\), given by the following eigenvalues (in the LT domain),
\[ \tilde{\kappa}_\alpha(s) = \frac{s \tilde{t}(s)/a_\alpha}{1 - \ell(s)/a_\alpha}, \]  
defines the legitimate memory kernel for the evolution described by the generalized Pauli channel.

**Proposition 3.** Let \(\ell(t)\) be given by the following convolution,
\[ \ell(t) = e^{-z_1t} \ast \cdots \ast e^{-z_nt}, \]  
where \(z_k > 0\) and \(z_i \neq z_j\) for \(i \neq j\). If \(a_\alpha\) satisfy \([43]\) and
\[ \prod_{k=1}^n z_k \geq \frac{d-1}{d^2} \sum_{\alpha=1}^{d+1} \frac{1}{a_\alpha}, \]  
then \(\tilde{\kappa}_\alpha(s)\)'s given in \([43]\) define a legitimate memory kernel.

**Proof.** It is enough to verify that \(p_0(t \to \infty) \geq 0\), as the smallest value of \(p_0(t)\) corresponds to the asymptotic case where \(t \to \infty\). Using the following result,
\[ \int_0^\infty e^{-z_1t} \ast \cdots \ast e^{-z_nt} \, dt = \frac{1}{z_1 \cdots z_n}, \]  
with equation \([22]\), let us write out the explicit form of \(p_0(t \to \infty)\) for the chosen \(\ell(t)\),
\[ p_0(t \to \infty) = d^2 - (d-1) \sum_{\alpha=1}^{d+1} \frac{1}{a_\alpha} \frac{1}{\prod_{k=1}^n z_k}. \]  
This is always non-negative if \([48]\) is satisfied. \(\square\)

### 4. SEMI-MARKOV EVOLUTION

The quantum semi-Markov evolution is the quantum analogue of the classical concept of the stochastic semi-Markov process. Such process is defined in terms of the semi-Markov matrix \(q_{ij}(t) \geq 0\) \((t \geq 0)\), which determines the probability \(\int_0^t q_{ij}(\tau) \, d\tau\) of jump \(j \to i\) at \(\tau \in [0, t]\) if the system is in the state \(j\) at \(\tau = 0\). Using this matrix, one defines the waiting time distribution and the survival probability by
\[ f_j(t) = \sum_{i=1}^d q_{ij}(t), \quad g_j(t) = 1 - \int_0^t f_j(\tau) \, d\tau, \]  
respectively. The stochastic evolution of the probability vector \(p\),
\[ p(t) = T(t)p, \quad T(0) = \mathbb{I}, \]  
is provided by the stochastic map constructed as follows,
\[ T(t) = n(t) + (n * q)(t) + (n * q * q)(t) + \cdots, \]  
where \(n_{ij}(t) = g_j(t)\delta_{ij}\). It satisfies classical memory kernel master equation,
\[ \frac{d}{dt} T(t) = \int_0^t K(t - \tau) T(\tau) \, d\tau, \]  
with
\[ \tilde{K}(s) = s\mathbb{I} - [\mathbb{I} - \tilde{q}(s)]\tilde{\kappa}^{-1}(s). \]  

The quantum semi-Markov evolution \([18, 22]\) is defined in terms of the so-called quantum semi-Markov map, i.e. the completely positive map \(Q_t\) for which \(\int_0^t Q_t^\dagger[i] \, d\tau \leq \mathbb{I}\). By \(Q_t^\dagger\), we understand the map dual to \(Q_t\) in the sense that \(\text{Tr}(XQ_t[Y]) = \text{Tr}(Q_t^\dagger[X]Y)\). For the given semi-Markov map, one defines the waiting time operator \(F_t = Q_t^\dagger[i]\) and the survival operator
\[ G_t = \mathbb{I} - \int_0^t F_r \, d\tau, \]  

where \( G_t \geq 0, G_0 = \mathbb{I} \). In the quantum semi-Markov evolution, \( N_t \) is given by
\[
N_t[\rho] = \sqrt{G_t} \rho \sqrt{G_t}, \tag{57}
\]
and therefore it is fully determined by the choice of \( Q_t \). The dynamical map \( \Lambda_t \) is represented by the series of convolutions,
\[
\Lambda_t = N_t + N_t \ast Q_t + N_t \ast Q_t \ast Q_t + \ldots. \tag{58}
\]
This series is convergent if \( ||\tilde{Q}_s||_1 < 1 \), where \( ||X||_1 \) denotes the trace norm of \( X \). Such representation of the dynamical map allows us to construct the corresponding memory kernel via
\[
K_t = B_t - Z_t. \tag{59}
\]
The maps \( B_t \) and \( Z_t \) are defined, in the LT domain, by the following relations,
\[
\tilde{N}_s = [s \mathbb{I} + \tilde{Z}_s]^{-1}, \quad \tilde{Q}_s = \tilde{B}_s \tilde{N}_s, \tag{60}
\]
and give rise to the following formula for the memory kernel,
\[
\tilde{K}_s = s \mathbb{I} - [\mathbb{I} - \tilde{Q}_s] \tilde{N}_s^{-1}. \tag{61}
\]
Now, for the generalized Pauli channels, we take
\[
Q_t = \frac{1}{d} \sum_{a=1}^{d+1} f_\alpha(t) U_\alpha, \tag{62}
\]
with \( f_\alpha(t) \geq 0 \) and \( \int_0^\infty f(t) \, dt \leq 1 \), where
\[
f(t) = \sum_{a=1}^{d+1} f_\alpha(t). \tag{63}
\]
The quantum waiting time and the quantum survival time operators have simple forms,
\[
F_t = f(t) \mathbb{I}, \quad G_t = g(t) \mathbb{I}, \tag{64}
\]
with
\[
g(t) = 1 - \int_0^t f(\tau) \, d\tau. \tag{65}
\]
After some straightforward calculations, we obtain the following semi-Markov memory kernel,
\[
K_t = \sum_{\alpha=1}^{d+1} k_\alpha(t) [\Phi_\alpha - \mathbb{I}], \tag{66}
\]
where
\[
\tilde{k}_\alpha(s) = \frac{d}{d-1} \tilde{f}_\alpha(s) \frac{\tilde{f}(s)}{\tilde{g}(s)}. \tag{67}
\]
Finally, the generalized Pauli channel generated by \( \tilde{K}_t \) is determined by
\[
\tilde{\lambda}_\alpha(s) = -\frac{d-1}{s} \frac{\tilde{f}(s) - 1}{\tilde{f}(s) - df_\alpha(s) + d - 1}. \tag{68}
\]
It implies the following relations between \( \tilde{f}_\alpha(s) \) and \( \tilde{\ell}_\alpha(s) \):
\[
\tilde{\ell}_\alpha(s) = \frac{d (\tilde{f}(s) - \tilde{f}_\alpha(s))}{\tilde{f}(s) - df_\alpha(s) + d - 1}, \tag{69}
\]
\[
\tilde{f}_\alpha(s) = \sum_{\beta=1}^{d+1} \frac{1}{1 - \tilde{f}_\beta(s)} - \frac{d}{1 - \tilde{\ell}_\alpha(s)} - 1 \sum_{\beta=1}^{d+1} \frac{1}{1 - \tilde{f}_\beta(s)} + \frac{d}{d-1}. \tag{70}
\]
In the isotropic case – that is, when \( f_\alpha(t) = \chi(t) \) and
\[
\int_0^\infty \chi(t) \, dt \leq 1, \tag{71}
\]
one finds
\[
\tilde{\ell}_\alpha(s) = \tilde{\nu}(s) = \frac{d^2 \tilde{\chi}(s)}{\tilde{\chi}(s) + d - 1}, \tag{72}
\]
where the new kernel \( \tilde{K}_t \) is defined by
\[
\tilde{K}_s = s \tilde{N}_s \tilde{Q}_s \tilde{N}_s^{-1}. \tag{73}
\]
In particular, if \( \tilde{N}_s \) and \( \tilde{Q}_s \) commute, then \( \tilde{K}_s = s \tilde{Q}_s \) – or, equivalently, in the time domain, \( \tilde{K}_t = \tilde{Q}_t + Q_0 \delta(t) \).
In our case, it gives
\[
\tilde{K}_t = \sum_{\alpha=1}^{d+1} h_\alpha(t) U_\alpha, \tag{74}
\]
with \( h_\alpha(t) = f_\alpha(t) + f_\alpha(0) \delta(t) \), and hence eq. (72) provides the following inhomogeneous equation for the density operator \( \rho_t \) with the initial state \( \rho_0 \),
\[
\dot{\rho}_t = \int_0^t \sum_{\alpha=1}^{d+1} h_\alpha(t - \tau) U_\alpha [\rho_\tau] \, d\tau - f(t) \rho_0. \tag{75}
\]
**Example 1.** In the qubit case \( (d = 2) \), one finds
\[
\dot{\rho}_t = \int_0^t \sum_{\alpha=1}^{3} h_\alpha(t - \tau) \sigma_\alpha \rho_\tau \sigma_\alpha \, d\tau - f(t) \rho_0, \tag{76}
\]
or, introducing the Bloch vector \( x_\alpha(t) = \text{Tr} [\rho_\tau \sigma_\alpha] \),
\[
\dot{x}_\alpha(t) = \int_0^t [2h_\alpha(t - \tau) - h(t - \tau)] x_\alpha(\tau) \, d\tau - f(t) x_\alpha(0), \tag{77}
\]
with \( h(t) = \sum_{\alpha=1}^{3} h_\alpha(t) \).
5. DISCRETE WIGNER FUNCTIONS AND CLASSICAL SEMI-MARKOV EVOLUTION

The information encoded into the density operator $\rho$ can be translated into the following $d + 1$ probability distributions,

$$\pi_k^{(\alpha)} = \frac{1}{d} \text{Tr} \left( P_k^{(\alpha)} \rho \right). \quad (78)$$

The probability vectors $\left( \pi_0^{(\alpha)}, \ldots, \pi_{d+1}^{(\alpha)} \right)$ evolve according to the classical evolution equation

$$\pi_k^{(\alpha)}(t) = \sum_{i=0}^{d-1} T_{ki}^{(\alpha)}(t) \pi_i^{(\alpha)}(0) \quad (79)$$

with the stochastic (even doubly stochastic) map

$$T_{ij}^{(\alpha)}(t) = \text{Tr} \left( P_i^{(\alpha)} A_t \left[ P_j^{(\alpha)} \right] \right). \quad (80)$$

One easily finds

$$T^{(\alpha)}(t) = c_\alpha(t) \mathbb{I} + [1 - c_\alpha(t)] P, \quad (81)$$

where $P_{ij} = 1/d$ and

$$c_\alpha(t) = \frac{d}{d-1} \left[ p_0(t) + p_\alpha(t) - \frac{1}{d} \right]. \quad (82)$$

If $A_t$ is the solution of the quantum memory kernel master equation with $K_t$ as in Theorem 1, then the stochastic map takes the following form,

$$T^{(\alpha)}(t) = \left[ 1 - \int_0^t \ell_{\alpha}(\tau) d\tau \right] \mathbb{I} + \int_0^t \ell_{\alpha}(\tau) d\tau \mathbb{P}. \quad (83)$$

Observe that, knowing the probability distributions $\pi_k^{(\alpha)}$, one can express the discrete Wigner function $W_\alpha$ in terms of $\pi_k^{(\alpha)}$. Therefore, it is also possible to find the time-evolution evolution of $W_\alpha$. Recall the definition of the discrete Wigner function \[2\],

$$W_\alpha = \frac{1}{d} \text{Tr} \left( \rho A_\alpha \right), \quad (84)$$

where, after introducing $\alpha = (a_1, a_2)$, the operators $A_\alpha$ are given by

$$A_\alpha = \frac{1}{2} \left[ (-1)^{a_1} \sigma_3 + (-1)^{a_2} \sigma_1 + (-1)^{a_1 + a_2} \sigma_2 + \mathbb{I} \right]. \quad (85)$$

for $d = 2$ and

$$(A_\alpha)_{kl} = \delta_{2a_1, k+l} e^{2\pi i a_2 (k+l)/d}, \quad (86)$$

for a prime $d > 2$. Let us illustrate our claim in the following example.

**Example 2.** Calculate the discrete Wigner function for a qubit ($d = 2$). From definition, one has

$$W_{00}(t) = \frac{1}{4} \left( 1 + x_1(t) + x_2(t) + x_3(t) \right),$$

$$W_{01}(t) = \frac{1}{4} \left( 1 - x_1(t) - x_2(t) + x_3(t) \right),$$

$$W_{10}(t) = \frac{1}{4} \left( 1 + x_1(t) - x_2(t) - x_3(t) \right),$$

$$W_{11}(t) = \frac{1}{4} \left( 1 - x_1(t) + x_2(t) - x_3(t) \right), \quad (87)$$

where $x_\alpha(t) = \text{Tr}[\rho(t) \sigma_\alpha]$ is the Bloch vector. Observe that the Bloch vector is related to the probability distributions $\pi_k^{(\alpha)}(t)$ as follows,

$$\pi_k^{(\alpha)}(t) = \frac{1}{2} \left[ 1 - (-1)^k x_\alpha(t) \right]; \quad k = 1, 2. \quad (88)$$

Therefore, if $\pi_k^{(\alpha)}(t)$’s evolve according to the classical evolution equation (79) with the bistochastic map (80), then the corresponding discrete Wigner function satisfies the following evolution equation,

$$W(t) = S(t) W, \quad S(0) = \mathbb{I}, \quad (89)$$

with $W = (W_{00}, W_{01}, W_{10}, W_{11})$ and the bistochastic map $S(t)$. The map $S(t)$ has a simple structure,

$$S(t) = \frac{1}{4} \left[ s_0(t) s_3(t) s_1(t) s_2(t) \right. \left. s_3(t) s_0(t) s_2(t) s_1(t) \right. \left. s_1(t) s_2(t) s_0(t) s_3(t) \right. \left. s_2(t) s_1(t) s_3(t) s_0(t) \right], \quad (90)$$

where

$$s_0(t) = 4 - 3 \int_0^t \ell_{\beta}(\tau) d\tau,$$

$$s_\alpha(t) = \sum_{\beta=1}^{3} \int_0^t \ell_{\beta}(\tau) d\tau - 2 \int_0^t \ell_{\alpha}(\tau) d\tau. \quad (91)$$

Now, suppose that $A_t$ obeys the quantum semi-Markov evolution defined by the quantum semi-Markov map $Q_t = \frac{1}{d-1} \sum_{\alpha} f_\alpha(t) U_\alpha$. Using the representation (85),

$$A_t = N_t + N_{t} \ast Q_{t} + N_{t} \ast Q_{t} \ast Q_{t} + \ldots, \quad (92)$$

and the following property of $Q_t$,

$$Q_t[P_i^{(\alpha)}] = \sum_{j=0}^{d-1} q_{ij}^{(\alpha)}(t) P_j^{(\alpha)}, \quad (93)$$

with

$$q_{ij}^{(\alpha)}(t) = \text{Tr} \left( P_i^{(\alpha)} Q_t[P_j^{(\alpha)}] \right) = \delta_{ij} f_\alpha(t) + \frac{1 - \delta_{ij}}{d-1} [f(t) - f_\alpha(t)], \quad (94)$$

for $d > 2$. Let us illustrate our claim in the following example.
one finds the corresponding representation of the stochastic map $T^{(a)}(t)$,

$$T^{(a)} = n + n \ast q^{(a)} + n \ast q^{(a)} \ast q^{(a)} + \ldots,$$  \hspace{1cm} (94)

where

$$n_{ij}(t) = \text{Tr} \left( P_i^{(a)} N_j [P_j^{(a)}] \right) = g(t) \delta_{ij}. \hspace{1cm} (95)$$

Interestingly, the map $n(t)$ is universal – that is, it does not depend on ‘$a$’.

### 6. EXAMPLES

#### 6.1. Markovian semigroup

Let us consider the evolution provided by the following Gorini-Kossakowski-Sudarshan-Lindblad generator,

$$\mathcal{L} = \sum_{\alpha=1}^{d+1} \gamma_{\alpha} \mathcal{L}_{\alpha}, \hspace{1cm} (96)$$

where $\gamma_{\alpha} \geq 0$. Due to (20), one has

$$\lambda_{\alpha}(t) = e^{(\gamma_{\alpha} - \gamma)t}. \hspace{1cm} (97)$$

Now, the memory kernel equation (21), with $K_{t}$ satisfying the assumptions of Theorem 1, describes the dynamics of the Markovian semigroup if and only if

$$\ell_{\alpha}(t) = \frac{d(\gamma - \gamma_{\alpha})e^{-\frac{d+1}{d} \gamma t}}{(\gamma - d \gamma_{\alpha})e^{-\frac{d+1}{d} \gamma t} + d}. \hspace{1cm} (98)$$

Moreover, the Markovian semigroup is generated by the quantum semi-Markov map $Q_{t}$ (62) with

$$f_{\alpha}(t) = \frac{d-1}{d} \gamma_{\alpha} e^{-\frac{d+1}{d} \gamma t}. \hspace{1cm} (99)$$

#### 6.2. Oscillatory behaviour

Take the oscillating functions

$$\ell_{\alpha}(t) = \frac{\omega}{a_{\alpha}} \sin \omega t. \hspace{1cm} (100)$$

One finds the map (5) with the following probability vector,

$$p_{0}(t) = 1 - \frac{d-1}{d^2} (1 - \cos \omega t) \sum_{\beta=1}^{d+1} \frac{1}{a_{\beta}}, \hspace{1cm} (101)$$

$$p_{\alpha}(t) = \frac{d-1}{d^2} (1 - \cos \omega t) \left[ \sum_{\beta=1}^{d+1} \frac{1}{a_{\beta}} - \frac{d}{a_{\alpha}} \right]. \hspace{1cm} (102)$$

This corresponds to the legitimate generalized Pauli channel if and only if

$$\frac{d}{a_{\beta}} \leq \sum_{\alpha=1}^{d+1} \frac{1}{a_{\alpha}} \leq \frac{d^2}{2(d-1)}. \hspace{1cm} (103)$$

Note that $\ell_{\alpha}(t)$’s in (100) give rise to the memory kernel $K_{t}$ with the following eigenvalues,

$$\kappa_{\alpha}(t) = -\frac{\omega^2}{a_{\alpha}} \cos \left( \sqrt{1 - \frac{1}{a_{\alpha}} \omega t} \right). \hspace{1cm} (104)$$

Observe that (100) implies

$$a_{\alpha} \geq 2 \left( 1 - \frac{1}{d} \right), \hspace{1cm} (105)$$

which means that $a_{\alpha} \geq 1$ for $d \geq 2$, and hence $K_{t}$ is always well-defined.

#### 6.3. Convex combination of Markovian semigroups

Let us provide a simple generalization of the quantum channels considered in (22, 33),

$$\Lambda_{t} = \sum_{\alpha=1}^{d+1} x_{\alpha} e^{-dt} \mathcal{L}_{\alpha} = \frac{1}{d} \left[ (1 + [d-1]e^{-dt}) \mathbb{1} \right. \hspace{1cm} (106)$$

$$+ (1 - e^{-dt}) \sum_{\alpha=1}^{d+1} x_{\alpha} V_{\alpha} \left. \right],$$

where $x_{\alpha}$’s form the probability vector. Although the Kraus representation of $\Lambda_{t}$ is relatively complicated, each of its eigenvalues depends only on one $x_{\alpha}$,

$$\lambda_{\alpha}(t) = e^{-dt} + (1 - e^{-dt}) x_{\alpha}. \hspace{1cm} (107)$$

For (100), we can find the time-local generator $\mathcal{L}_{t}$ (10) with the following decoherence rates,

$$\gamma_{\alpha}(t) = \sum_{\beta=1}^{d+1} \frac{1 - x_{\beta}}{1 + (e^{dt} - 1) x_{\beta}} - \frac{1 - x_{\alpha}}{1 + (e^{dt} - 1) x_{\alpha}}. \hspace{1cm} (108)$$

This evolution belongs to the special class described by the memory kernels previously discussed in (13) with

$$\ell(t) = de^{-dt}, \hspace{1cm} a_{\alpha} = \frac{1}{1 - x_{\alpha}}. \hspace{1cm} (109)$$

Consider the qubit case ($d = 2$) and suppose that $x_{2} = x_{1} \equiv x$. For such a choice, it is possible to recover the semi-Markov map $Q_{t}$ (62) with

$$f_{1}(t) = f_{2}(t) = \frac{x}{\xi} e^{\frac{-3-2i}{2} \xi t} \left[ \xi \cosh \frac{\xi t}{2} - 3(1 - 2x) \sinh \frac{\xi t}{2} \right], \hspace{1cm} (110)$$
where $\xi := \sqrt{12x^2 - 4x + 1}$. Note that
\[
\int_0^\infty \left[ f_1(t) + f_2(t) + f_3(t) \right] dt = -\frac{3x^2 + 3x - 1}{x^2 + x - 1} \leq 1,
\]
and hence the evolution is semi-Markov if and only if $f_\alpha(t) \geq 0$ for all $t \geq 0$. This holds for the Markovian semigroup ($x = 0$) and for the maximally mixed probability vector, i.e. $x = 1/3$.

It turns out that this property carries over to higher dimensions. Indeed, for the probability vector $x_\alpha = \frac{1}{d+1}$, the generalized Pauli channel $\Lambda_t$ in (106) is generated by the quantum semi-Markov map $Q_t$ with
\[
f_\alpha(t) = \frac{d-1}{d+1} - \frac{d(d^2+1)-1}{d^2+1}.
\]
Note that this evolution is also Markovian, as (108) simplifies to
\[
\gamma_\alpha(t) = \frac{d}{d + e^{dt}}.
\]
Observe that, in this case, the semi-Markov evolution is a subclass of the Markovian evolution.

### 6.4. Eternally non-Markovian evolution

As another special case of the convex combination of the Markovian semigroups, we analyze the eternally non-Markovian evolution, where $x_\alpha = 1/d$ (for $\alpha = 1, \ldots, d$) and $x_{d+1} = 0$. This corresponds to the following choice of the decoherence rates,
\[
\gamma_\alpha(t) = 1, \quad \gamma_{d+1}(t) = -(d-1)\frac{e^{dt} - 1}{e^{dt} - 1 + d},
\]
with $\gamma_{d+1}(t) \leq 0$ for all $t \geq 0$. For $d = 2$, one recovers the well-known eternally non-Markovian evolution of the qubit [34],
\[
\gamma_1(t) = \gamma_2(t) = 1, \quad \gamma_3(t) = -\tanh t.
\]
To determine whether this evolution is semi-Markov, we find the map $Q_t$ [62] with
\[
f_\alpha(t) = (d-1)e^{-dt/2} \left[ \frac{1}{d} \cosh \left( \sqrt{\frac{d^3 - 4d + 4}{4d^2}}t \right) \right] - \frac{d-2}{\sqrt{d^3 - 4d^2 + 4d}} \sinh \left( \sqrt{\frac{d^3 - 4d + 4}{4d}}t \right).
\]
for $\alpha = 1, \ldots, d$, and
\[
f_{d+1}(t) = -\frac{2(d-1)^2}{\sqrt{d^4 - 4d^2 + 4d}} e^{-dt/2} \sinh \left( \frac{\sqrt{d^3 - 4d + 4}}{4d^2}t \right).
\]
Note that $f_\alpha(t) \geq 0$ for $\alpha = 1, \ldots, d$ but $f_{d+1}(t) < 0$ (for $t > 0$). Therefore, $Q_t$ is not completely positive, and hence the corresponding dynamical map $\Lambda_t$ is not semi-Markov. This example shows that, in general, the convex combination of Markovian semigroups goes beyond the semi-Markov evolution.

### 7. CONCLUSIONS

Using the memory kernel master equation, we analyzed the evolution of the special class of the dynamical maps, provided by the generalized Pauli channels. We found the necessary and sufficient conditions which guarantee that the corresponding solution defines the legitimate physical evolution (CPTP map). Moreover, we analyzed a special class of the kernels corresponding to the quantum semi-Markov evolution. Such evolution defines a generalization of the Markovian semigroup. Surprisingly, the convex combination of Markovian semigroups is not semi-Markov. Several examples illustrate the general approach.

It would be interesting to further analyze the memory kernels going beyond the semi-Markov case. The example of the eternally non-Markovian evolution shows that one can obtain the legitimate dynamical map
\[
\Lambda_t = N_t + N_t \ast Q_t + N_t \ast Q_t \ast Q_t + \ldots
\]
from not completely positive $Q_t$. Therefore, one would like to find weaker conditions for the maps $N_t, Q_t$ that still guarantee the complete positivity of $\Lambda_t$.

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