Abstract

Let $G$ be an inner form of a general linear group over a non-archimedean locally compact field of residue characteristic $p$, let $R$ be an algebraically closed field of characteristic different from $p$ and let $\mathcal{R}(G)$ be the category of smooth representations of $G$ over $R$. In this paper, we prove that a block (indecomposable summand) of $\mathcal{R}(G)$ is equivalent to a level-0 block (a block in which every object has non-zero invariant vectors for the pro-$p$-radical of a maximal compact open subgroup) of $\mathcal{R}(G')$, where $G'$ is a direct product of groups of the same type of $G$.

Keywords: Blocks, Type theory, Semisimple types, Equivalence of categories, Hecke algebras, Modular representations of $p$-adic reductive groups, Level-0 representations.

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Introduction

Let $F$ be a non-archimedean locally compact field of residue characteristic $p$ and let $D$ be a central division algebra of finite dimension over $F$ whose reduced degree is denoted by $d$. Given $m \in \mathbb{N}^*$, we consider the group $G = GL_m(D)$ which is an inner form of $GL_{md}(F)$. Let $R$ be an algebraically closed field of characteristic $\ell \neq p$ and let $\mathcal{R}(G)$ be the category of smooth representations of $G$ over $R$, that are called $\ell$-modular when $\ell$ is positive. In this paper, we are interested on the Bernstein decomposition of $\mathcal{R}(G)$ (see [SS16] or [Vig98] for $d = 1$) that is its decomposition as a direct sum of full indecomposable subcategories, called blocks. Actually a full understanding of blocks of $\mathcal{R}(G)$ is equivalent to a full understanding of the whole category.

The main purpose of this paper is to find an equivalence of categories between any block of $\mathcal{R}(G)$ and a level-0 block of $\mathcal{R}(G')$ where $G'$ is a suitable direct product of inner forms of general linear groups over finite extensions of $F$. We recall that a level-0 block of $\mathcal{R}(G')$ is a block in which every object has non-zero invariant vectors for the pro-$p$-radical of a maximal compact open subgroup of $G'$. This result is an important step in the attempt to describe blocks of $\mathcal{R}(G)$ because it reduces the problem to the description of level-0 blocks.

In the case of complex representations, Bernstein [Ber84] found a block decomposition of $\mathcal{R}(G)$ indexed by pairs $(M, \sigma)$ where $M$ is a Levi subgroup of $G$ and $\sigma$ is an irreducible cuspidal representation of $M$, up to a certain equivalence relation called inertial equivalence. In particular an irreducible representation $\pi$ of $G$ is in the block associated to the inertial class of $(M, \sigma)$ if its cuspidal support is in this class. In [BK98], Bushnell and Kutzko introduce a method to describe the blocks of $\mathcal{R}(C)(G)$: the theory of type. This method consists in associating
at every block of $\mathcal{R}_C(G)$ a pair $(J, \lambda)$, called type, where $J$ is a compact open subgroup of $G$ and $\lambda$ is an irreducible representation of $J$, such that the simple objects of the block are the irreducible subquotients of the compactly induced representation $\text{ind}_J^G(\lambda)$. In this case the block is equivalent to the category of modules over the $C$-algebra $\mathcal{H}_C(G, \lambda)$ of $G$-endomorphisms of $\text{ind}_J^G(\lambda)$. In [SS11] (see [BK99] for $d = 1$) Sécherre and Stevens describe explicitly this algebra as a tensor product of algebras of type $A$.

In the case of $\ell$-modular representations, in [SS16] Sécherre and Stevens (see [Vig98] for $d = 1$) found a block decomposition of $\mathcal{R}_R(G)$ indexed by inertial classes of pairs $(M, \sigma)$ where $M$ is a Levi subgroup of $G$ and $\sigma$ is an irreducible supercuspidal representation of $M$. In particular an irreducible representation $\pi$ of $G$ is in the block associated to the inertial class of $(M, \sigma)$ if its supercuspidal support is in this class. We recall that the notions of cuspidal and supercuspidal representation are not equivalent as in complex case; however, in [MS14a] Minguez and Sécherre prove the uniqueness of supercuspidal support, up to conjugation, for every irreducible representation of $G$. We remark that to obtain the block decomposition of $\mathcal{R}_R(G)$, Sécherre and Stevens do not use the same method as Bernstein, but they rely, like us in this paper, on the theory of semisimple types developed in [SS11] (see [BK99] for $d = 1$).

Actually, they associate at every block of $\mathcal{R}_R(G)$ a pair $(J, \lambda)$, called semisimple supertype. Unfortunately the construction of the equivalence, as in complex case, between the block and the category of modules over $\mathcal{H}_R(G, \lambda)$ does not hold and one of the problems that occurs is that the pro-order of $J$ can be divisible by $\ell$. Some partial results on descriptions of algebras which are Morita equivalent to blocks of $\mathcal{R}_R(GL_n(F))$ are given by Dat [Dat12], Helm [Hel16] and Guiraud [Gui13].

The idea of this paper is the following. We fix a block $\mathcal{R}(J, \lambda)$ of $\mathcal{R}_R(G)$ associated to the semisimple supertype $(J, \lambda)$ and, as in [SS16], we can associate to it a compact open subgroup $J_{\max}$ of $G$, its pro-$p$-radical $J_{\max}^1$ and an irreducible representation $\eta_{max}$ of $J_{\max}^1$. We remark that we can extend, not in a unique way, $\eta_{max}$ to an irreducible representation of $J_{\max}$. Thus, we denote $\mathcal{R}(G, \eta_{max})$ the direct sum of blocks of $\mathcal{R}_R(G)$ associated to $(J_{\max}^1, \eta_{max})$ and we consider the functor

$$M_{\eta_{max}} = \text{Hom}_G(\text{ind}_{J_{\max}^1}^G \eta_{max}, -) : \mathcal{R}(G, \eta_{max}) \rightarrow \text{Mod} - \mathcal{H}_R(G, \eta_{max})$$

where $\mathcal{H}_R(G, \eta_{max}) \cong \text{End}_G(\text{ind}_{J_{\max}^1}^G (\eta_{max}))$. Using the fact that $\eta_{max}$ is a projective representation, since $J_{\max}^1$ is a pro-$p$-group, we prove that $M_{\eta_{max}}$ is an equivalence of categories (theorem 5.10). This result generalizes corollary 3.3 of [Chi17] where $\eta_{max}$ is a trivial character. We can also associate to $(J, \lambda)$ a Levi subgroup $L$ of $G$ and a group $B_L^\times$, which is a direct product of inner forms of general linear groups over finite extensions of $F$ and which we have denoted $G'$ above. If $K_L$ is a maximal compact open subgroup of $B_L^\times$ and $K_1^L$ is its pro-$p$-radical then $K_L/K_1^L \cong J_{\max}/J_{\max}^1 = \mathcal{R}$ is a direct product of finite general linear groups. Actually, in [Chi17] is proved that the $K_1^L$-invariant functor $\text{inv}_{K_1^L}$ is an equivalence of categories between the level-0 subcategory $\mathcal{R}(B_L^\times, K_1^L)$ of $\mathcal{R}(B_L^\times)$, which is the direct sum of its level-0 blocks, and the category of modules over the algebra $\mathcal{H}_R(B_L^\times, K_1^L) \cong \text{End}_{B_L^\times}(\text{ind}^{B_L^\times}_{K_1^L} 1_{K_1^L})$. Now, thanks to the explicit presentation by generators and relations of $\mathcal{H}_R(B_L^\times, K_1^L)$, presented in [Chi17], in this paper we construct a homomorphism $\Theta_{r_{\eta_{max}}} : \mathcal{H}_R(B_L^\times, K_1^L) \rightarrow \mathcal{H}_R(G, \eta_{max})$, depending on the choice of the extension $\kappa_{max}$ of $\eta_{max}$ to $J_{\max}$ and on the choice of an intertwining element $\gamma$ of $\eta_{max}$, finding elements in $\mathcal{H}_R(G, \eta_{max})$ satisfying all relations defining $\mathcal{H}_R(B_L^\times, K_1^L)$. Moreover, using some properties of $\eta_{max}$, we prove that this homomorphism is actually an isomorphism. We remark that finding this isomorphism is one of the most difficult results obtained in this article and the proof in the case $L = G$ takes about half of the paper (section 3). This
also complete the results contained in the Phd thesis [Chi15] of the author because in it the construction of this isomorphism depends on a conjecture (see section 3.4 of [Chi15]). In this way we obtain an equivalence of categories $F_{\gamma, \kappa_{\text{max}}}: \mathcal{H}(G, \eta_{\text{max}}) \rightarrow \mathcal{H}(B^x_L, K^1_L)$ such that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{H}(G, \eta_{\text{max}}) & \xrightarrow{F_{\gamma, \kappa_{\text{max}}}} & \mathcal{H}(B^x_L, K^1_L) \\
\text{Mod} - \mathcal{H}(G, \eta_{\text{max}}) & \xrightarrow{\Theta_{\gamma, \kappa_{\text{max}}}} & \text{Mod} - \mathcal{H}(B^x_L, K^1_L).
\end{array}
$$

Then we obtain

$$
F_{\gamma, \kappa_{\text{max}}} (\pi, V) = M_{\eta_{\text{max}}} (\pi, V) \otimes \mathcal{H}(B^x_L, K^1_L) \text{mod}^{B^x_L}_{\kappa_{\text{max}}} (1_{K^1_L})
$$

for every $(\pi, V)$ in $\mathcal{H}(G, \eta_{\text{max}})$, where the action of $\mathcal{H}(B^x_L, K^1_L)$ on $M_{\eta_{\text{max}}} (\pi, V)$ depends on $\Theta_{\gamma, \kappa_{\text{max}}}$. Hence, $F_{\gamma, \kappa_{\text{max}}}$ induces an equivalence of categories between the block $\mathcal{H}(J, \lambda)$ and a level-0 block of $\mathcal{H}(B^x_L)$. To understand this correspondence we need to use the functor

$$
K_{\kappa_{\text{max}}}: \mathcal{H}(G, \eta_{\text{max}}) \rightarrow \mathcal{H}(J_{\text{max}}/J^1_{\text{max}}) = \mathcal{H}(\mathcal{G})
$$

where $J_{\text{max}}$ acts on $K_{\kappa_{\text{max}}} (\pi) = \text{Hom}_{j_{\text{max}}} (\eta_{\text{max}}, \pi)$ by $x \cdot \varphi = \pi(x) \circ \varphi \circ \kappa_{\text{max}}^{-1}$ for every representation $\pi$ of $G$, $\varphi \in \text{Hom}_{j_{\text{max}}} (\eta_{\text{max}}, \pi)$ and $x \in J_{\text{max}}$. This functor is strongly used in [SS16] to define $\mathcal{H}(J, \lambda)$ and to prove the Bernstein decomposition of $\mathcal{H}(G)$. We also consider the functor $K_{K_L}: \mathcal{H}(B^x_L, K^1_L) \rightarrow \mathcal{H}(K_L/K^1_L) = \mathcal{H}(\mathcal{G})$ given by $K_{K_L} (Z) = Z^{K^1_L}$ for every representation $(\varrho, Z)$ of $B^x_L$ where $x \in K_L$ acts on $Z \in K^1_L$ by $x \circ z = \varrho(x) z$. Then the functors $K_{K_L} \circ F_{\gamma, \kappa_{\text{max}}}$ and $K_{\kappa_{\text{max}}}$ are naturally isomorphic (proposition 5.14) and so $\mathcal{H}(J, \lambda)$ is equivalent to the level-0 block $\mathcal{B}$ of $\mathcal{H}(B^x_L)$ such that $\mathcal{K}_{\kappa_{\text{max}}} (\mathcal{H}(J, \lambda)) = \mathcal{K}_{K_L} (\mathcal{B})$. More precisely, if $J^1$ is the pro-p-radical of $J$, then $J/J^1 = \mathcal{M}$ is a Levi subgroup of $\mathcal{G}$ and the choice of $\kappa_{\text{max}}$ defines a decomposition $\lambda = \kappa \otimes \sigma$ where $\kappa$ is an irreducible representation of $J$ and $\sigma$ is a cuspidal representation of $\mathcal{M}$ viewed as an irreducible representation of $J$ trivial on $J^1$. If we can consider the pair $(\mathcal{M}, \sigma)$ up to the equivalence relation given in definition 1.14 of [SS16], then a representation $(\varrho, Z)$ of $B^x_L$ is in $\mathcal{B}$ if it is generated by the maximal subspace of $Z^{K^1_L}$ such that every irreducible subquotient has supercuspidal support in the class of $(\mathcal{M}, \sigma)$.

One question we do not address in this paper is the structure of level-0 blocks of $\mathcal{H}(B^x_L)$ when the characteristic of $R$ is positive. Thanks to results of [Chi17] we know that there is a correspondence between these blocks and the set $\mathcal{E}$ of primitive central idempotents of $\mathcal{H}(B^x_L, K^1_L)$, which are described in sections 2.5 and 2.6 of [Chi15]. Hence, one possibility for understanding level-0 blocks of $\mathcal{H}(B^x_L)$ is to describe the algebras $e \mathcal{H}(B^x_L, K^1_L)$ with $e \in \mathcal{E}$. On the other hand, we recall that in [Dat16] Dat proves that every level-0 block of $\mathcal{H}(GL_n(F))$ is equivalent to the unipotent block of $\mathcal{H}(G'')$ where $G''$ is a suitable product of general linear groups over non-archimedean locally compact fields. Hence, putting together the result of Dat and results of this article, we obtain a method to reduce the description of any block of $\mathcal{H}(GL_n(F))$ to that of an unipotent block. Unfortunately the description of the unipotent block of $\mathcal{H}(GL_n(F))$, or of $\mathcal{H}(G)$, is nowadays an hard question and it has no answer yet.

We now give a brief summary of the contents of each section of this paper. In section 1 we present general results on the convolution Hecke algebras $\mathcal{H}(G, \sigma)$ where $G$ is generic locally profinite group and $\sigma$ a representation of an open subgroup $H$ of $G$. We see that if $\sigma$ is finitely generated then $\mathcal{H}(G, \sigma)$ is isomorphic to the endomorphism algebra of $\text{ind}_H^G \sigma$. We also define
two subcategories of $\mathcal{R}(G)$ and we prove that, when they coincide, they are equivalent to the category of modules over $\mathcal{H}(G, \sigma)$. In section 2 we introduce the theory of maximal simple types; in particular we consider the Heisenberg representation $\eta_B$ that the algebras $\mathcal{R}(G, \eta_B)$ and $\mathcal{H}(G, \eta_B)$ are isomorphic. In section 4 we introduce the theory of semisimple types, we define the representation $\eta_{\text{max}}$ and the group $B_L^\times$ and we prove that the algebras $\mathcal{R}(B_L^\times, K_L^1)$ and $\mathcal{H}(G, \eta_{\text{max}})$ are isomorphic. Finally, in section 5 we prove that $M_{\eta_{\text{max}}}$ and $F_{\gamma, \kappa_{\text{max}}}$ are equivalences of categories, we describe the correspondence between blocks of $\mathcal{R}(G, \eta_{\text{max}})$ and of $\mathcal{H}(B_L^\times, K_L^1)$ and we investigate on the dependence of these results on the choice of the extension of $\eta_{\text{max}}$ to $J_{\text{max}}$.

1. Preliminaries

This section is written in much more generality than the remainder of this paper. We present general results for a generic locally profinite group.

Let $G$ be a locally profinite group (i.e. a locally compact and totally disconnected topological group) and let $R$ be a unitary commutative ring. We recall that a representation $(\pi, V)$ of $G$ over $R$ is smooth if for every $v \in V$ the stabilizer $\{g \in G \mid \pi(g)v = v\}$ is an open subgroup of $G$. We denote $\mathcal{H}(G)$ the (abelian) category of smooth representations of $G$ over $R$. From now on all representations are considered smooth.

1.1. Hecke algebras for a locally profinite group

In this paragraph we introduce an algebra associated to a representation $\sigma$ of a subgroup of $G$ and we prove that it is isomorphic to the endomorphism algebra of the compact induction of $\sigma$. This definition generalizes those in section 1 of [Chi17] that corresponds to the case in which $\sigma$ is trivial.

Let $H$ be an open subgroup of $G$ such that every $H$-double coset is a finite union of left $H$-cosets (or equivalently $H \cap ghg^{-1}$ is of finite index in $H$ for every $g \in G$) and let $(\sigma, V_\sigma)$ be a smooth representation of $H$ over $R$.

**Definition 1.1.** Let $\mathcal{H}(G, \sigma)$ be the $R$-algebra of functions $\Phi : G \to \text{End}_R(V_\sigma)$ such that $\Phi(hgh') = \sigma(h) \circ \Phi(g) \circ \sigma(h')$ for every $h, h' \in H$ and $g \in G$ and whose supports are a finite union of $H$-double cosets, endowed with convolution product

$$\tag{1.1} (\Phi_1 * \Phi_2)(g) = \sum_x \Phi_1(x)\Phi_2(x^{-1}g)$$

where $x$ describes a system of representatives of $G/H$ in $G$. This algebra is unitary and the identity element is $\sigma$ seen as a function on $G$ with support equal to $H$. To simplify the notation, from now on we denote $\Phi_1 \Phi_1 = \Phi_1 * \Phi_2$ for all $\Phi_1, \Phi_2 \in \mathcal{H}(G, \sigma)$.

We observe that the sum in (1.1) is finite since the support of $\Phi_1$ is a finite union of $H$-double cosets and by hypothesis, every $H$-double coset is a finite union of left $H$-cosets. Moreover, (1.1) is well-defined because for every $h \in H$ and $x, g \in G$ we have $\Phi_1(xh)\Phi_2((xh)^{-1}g) = \Phi_1(x)\circ \sigma(h) \circ \sigma(h^{-1}) \circ \Phi_2(x^{-1}g) = \Phi_1(x) \circ \Phi_2(x^{-1}g)$.

For every $g \in G$ we denote by $\mathcal{H}(G, \sigma)_{|\text{HgH}}$ the submodule of $\mathcal{H}(G, \sigma)$ of functions with support in $HgH$. If $g_1, g_2 \in G$, $\Phi_1 \in \mathcal{H}(G, \sigma)_{|Hg_1H}$ and $\Phi_2 \in \mathcal{H}(G, \sigma)_{|Hg_2H}$ then the support of $\Phi_1 \Phi_2$ is in $Hg_1Hg_2H$ and the support of $x \mapsto \Phi_1(x)\Phi_2(x^{-1}g)$ is in $Hg_1H \cap gHg_2^{\prime}$. $H$.

**Remark 1.2.** If $g_1$ or $g_2$ normalizes $H$ then the support of $\Phi_1 \Phi_2$ is in $Hg_1g_2H$ and the support of $x \mapsto \Phi_1(x)\Phi_2(x^{-1}g_1g_2)$ is in $g_1H$. Hence, we obtain $(\Phi_1 \Phi_2)(g_1g_2) = \Phi_1(g_1) \circ \Phi_2(g_2)$. 

4
For every \( g \in G \) we denote \( \mathbb{H}^g = g^{-1}Hg \) and \((\sigma^g, V_g)\) the representation of \( \mathbb{H}^g \) given by \( \sigma^g(x) = \sigma(gxg^{-1}) \) for every \( x \in \mathbb{H}^g \). We denote \( I_g(\sigma) \) the \( R \)-module \( \text{Hom}_{\mathbb{H}^g}(\sigma^g, \sigma^{g^2}) \) and \( I_g(\sigma) \) the set, called intertwining of \( \sigma \) in \( G \), of \( g \in G \) such that \( I_g(\sigma) \neq 0 \). For every \( g \in I_g(\sigma) \) the map \( \Phi \mapsto \Phi(g) \) is an isomorphism of \( R \)-modules between \( \mathbb{K}(G, \sigma) \) and \( I_g(\sigma) \) and so \( g \in G \) intertwines \( \sigma \) if and only if there exists an element \( \Phi \in \mathbb{K}(G, \sigma) \) such that \( \Phi(g) \neq 0 \).

Let \( \text{ind}_G^\mathbb{H}(\sigma) \) be the compact induced representation of \( \sigma \) to \( G \). It is the \( R \)-module of functions \( f : G \to V_\sigma \), compactly supported modulo \( \mathbb{H} \), such that \( f(hg) = \sigma(h)f(g) \) for every \( h \in \mathbb{H} \) and \( g \in G \) endowed with the action of \( G \) defined by \( x, f \mapsto f(gx) \) for every \( x, g \in G \) and \( f \in \text{ind}_G^\mathbb{H}(\sigma) \).

We remark that, since \( \mathbb{H} \) is open, by I.5.2(b) of [Vig96] it is a smooth representation of \( G \). For every \( v \in V_\sigma \) let \( i_v \in \text{ind}_G^\mathbb{H}(\sigma) \) with support in \( \mathbb{H} \) defined by \( i_v(h) = \sigma(h)v \) for every \( h \in \mathbb{H} \). Then for every \( x \in G \) the function \( x^{-1}i_v \) has support \( \mathbb{H}x \) and takes the value \( v \) on \( x \). Hence, for every \( f \in \text{ind}_G^\mathbb{H}(\sigma) \) we have

\[
 f = \sum_{x \in \mathbb{H} \backslash G} x^{-1}i_{f(x)} \tag{1.2}
\]

and so the image \( i_{V_\sigma} \) of \( v \mapsto i_v \) generates \( \text{ind}_G^\mathbb{H}(\sigma) \) as representation of \( G \).

Frobenius reciprocity (I.5.7 of [Vig96]) states that the map \( \text{Hom}_\mathbb{H}(\sigma, V) \to \text{Hom}_G(\text{ind}_G^\mathbb{H}(\sigma), V) \) given by \( \phi \mapsto \psi \) where \( \phi(v) = \psi(i_v) \) for every \( v \in V_\sigma \) is an isomorphism of \( R \)-modules.

**Lemma 1.3.** If \( V_\sigma \) is a finitely generated \( R \)-module, the map \( \xi : \mathbb{K}(G, \sigma) \to \text{End}_G(\text{ind}_G^\mathbb{H}(\sigma)) \) given by

\[
 \xi(\Phi)(f)(g) = (\Phi \ast f)(g) = \sum_{x \in G/\mathbb{H}} \Phi(x) f(x^{-1}g)
\]

for every \( \Phi \in \mathbb{K}(G, \sigma) \), \( f \in \text{ind}_G^\mathbb{H}(\sigma) \) and \( g \in G \) is an \( R \)-algebra isomorphism whose inverse is given by \( \xi^{-1}(\theta)(g)(v) = \theta(i_v)(g) \) for every \( \theta \in \text{End}_G(\text{ind}_G^\mathbb{H}(\sigma)) \), \( g \in G \) and \( v \in V_\sigma \).

**Proof.** See I.8.5-6 of [Vig96]. \( \square \)

**1.2. The categories \( \mathcal{B}_\sigma(\mathbb{G}) \) and \( \mathcal{B}(\mathbb{G}, \sigma) \)**

In this paragraph we associate to an irreducible projective representation of a compact open subgroup of \( \mathbb{G} \) two subcategories of \( \mathcal{B}(\mathbb{G}, \sigma) \).

Let \( K \) be a compact open subgroup of \( \mathbb{G} \) and \((\sigma, V_\sigma)\) be an irreducible projective representation of \( K \) such that \( V_\sigma \) is a finitely generated \( R \)-module. Then \( \rho = \text{ind}_K^K(\sigma) \) is a projective representation of \( \mathbb{G} \) by I.5.9(d) of [Vig96] and so the functor

\[
 M_\sigma = \text{Hom}_G(\rho, -) : \mathcal{B}(\mathbb{G}, \sigma) \to \text{Mod} - \mathcal{K}(\mathbb{G}, \sigma)
\]

is exact. We remark that for every representation \((\pi, V)\) of \( \mathbb{G} \) the right action of \( \Phi \in \mathcal{B}(\mathbb{G}, \sigma) \) on \( \varphi \in \text{Hom}_G(\rho, V) \) is given by \( \varphi, \Phi = \varphi \circ \xi(\Phi) \) where \( \xi \) is the isomorphism of lemma 1.3. Moreover if \( V_1 \) and \( V_2 \) are representations of \( \mathbb{G} \) and \( \epsilon \in \text{Hom}_G(V_1, V_2) \) then \( \rho_\sigma(\epsilon) \) maps \( \varphi \) to \( \varphi \circ \epsilon \) for every \( \varphi \in \text{Hom}_G(\rho, V_1) \).

**Definition 1.4.** Let \( \mathcal{B}_\sigma(\mathbb{G}) \) be the full subcategory of \( \mathcal{B}(\mathbb{G}, \sigma) \) whose objects are representations \( V \) such that \( M_\sigma(V') \neq 0 \) for every irreducible subquotient \( V' \) of \( V \).

For every representation \( V \) of \( \mathbb{G} \) we denote \( V^\sigma = \sum_{\varphi \in \text{Hom}_G(\rho, V)} \Phi(\sigma) \) which is a subrepresentation of the restriction of \( V \) to \( K \). We denote by \( V[\sigma] \) the representation of \( \mathbb{G} \) generated by \( V^\sigma \).

If \( \sigma \) is the trivial character of \( K \) then \( V^\sigma = V^K = \{ v \in V \mid \pi(k)v = v \text{ pour tout } k \in K \} \) is the set of \( K \)-invariant vectors of \( V \).
Proposition 1.5. For every representation $V$ of $G$ we have $V[\sigma] = \sum_{\psi \in M_{\sigma}(V)} \psi(\rho)$ and so $M_{\sigma}(V) = M_{\sigma}(V[\sigma])$. Moreover, if $W$ is a subrepresentation of $V$ then $M_{\sigma}(W) = M_{\sigma}(V)$ if and only if $W[\sigma] = V[\sigma]$.

Proof. By Frobenius reciprocity we have $\text{Hom}(\sigma, V) \cong M_{\sigma}(V)$ and so using (1.2) we obtain

$$V[\sigma] = \sum_{g \in G} \pi(g) \sum_{\psi \in M_{\sigma}(V)} \psi(i_{V, \rho}) = \sum_{\psi \in M_{\sigma}(V)} \psi \left( \sum_{g \in G} g \cdot i_{\rho} \right) = \sum_{\psi \in M_{\sigma}(V)} \psi(\rho)$$

that implies $M_{\sigma}(V) = M_{\sigma}(V[\sigma])$. Furthermore, if $W[\sigma] = V[\sigma]$ then $M_{\sigma}(W) = M_{\sigma}(V)$ and if $M_{\sigma}(W) = M_{\sigma}(V)$ then $W[\sigma] = \sum_{\psi \in M_{\sigma}(W)} \psi(\rho) = \sum_{\psi \in M_{\sigma}(V)} \psi(\rho) = V[\sigma]$. □

Definition 1.6. Let $\mathcal{R}(G, \sigma)$ be the full subcategory of $\mathcal{R}_R(G)$ whose objects are representations $V$ such that $V = V[\sigma]$. If $\sigma$ is the trivial character of $K$ we denote $\mathcal{R}(G, K)$ the subcategory of representations $V$ generated by $V^K$.

Proposition 1.7. Let $V$ be a representation of $G$. The following conditions are equivalent:

(i) for every irreducible subquotient $U$ of $V$ we have $M_{\sigma}(U) \neq 0$;
(ii) for every non-zero subquotient $W$ of $V$ we have $M_{\sigma}(W) \neq 0$;
(iii) for every subquotient $Z$ of $V$ we have $Z = Z[\sigma]$;
(iv) for every subrepresentation $Z$ of $V$ we have $Z = Z[\sigma]$.

Proof. (i) $\Rightarrow$ (ii): let $W$ be a non-zero subquotient of $V$ and $W_1 \subset W_2$ two subrepresentations of $W$ such that $U = W_2/W_1$ is irreducible. By (i) we have $M_{\sigma}(U) \neq 0$ which implies $M_{\sigma}(W_2) \neq 0$ and so $M_{\sigma}(W) \neq 0$. (ii) $\Rightarrow$ (iii): let $Z$ be a subquotient of $V$. By proposition 1.5 we have $M_{\sigma}(Z) = M_{\sigma}(Z[\sigma])$ and so $M_{\sigma}(Z/Z[\sigma]) = 0$. Hence, by (ii) we obtain $Z = Z[\sigma]$. (iv) $\Rightarrow$ (i): let $U$ be an irreducible subquotient of $V$ and $Z_1 \subset Z_2$ be two subrepresentations of $V$ such that $U = Z_2/Z_1$. By (iv) we have $Z_1[\sigma] = Z_1 \neq Z_2 = Z_2[\sigma]$ and by proposition 1.5 we have $M_{\sigma}(Z_1) \neq M_{\sigma}(Z_2)$. Hence, we obtain $M_{\sigma}(U) \neq 0$. □

Remark 1.8. Proposition 1.7 implies that $\mathcal{R}_{\sigma}(G)$ is contained in $\mathcal{R}(G, \sigma)$.

1.3. Equivalence of categories

In this paragraph we suppose that there exists a compact open subgroup $K_0$ of $G$ whose pro-order is invertible in $R^+$ and we consider the Haar measure $\text{dg}$ of $G$ with values in $R$ such that $\int_{K_0} \text{dg} = 1$ (see I.2 of [Vig96]). We prove that if the two categories introduced in paragraph 1.2 are equal then they are equivalent to the category of modules over the algebra introduced in paragraph 1.1.

The global Hecke algebra $\mathcal{H}_R(G)$ of $G$ is the $R$-algebra of locally constant and compactly supported functions $f : G \to R$ endowed with convolution product given by $(f_1 \ast f_2)(x) = \int_G f_1(g)f_2(g^{-1}x)d\text{g}$ for every $f_1, f_2 \in \mathcal{H}_R(G)$ and $x \in G$ (see ... of [Vig96]). In general $\mathcal{H}_R(G)$ is not unitary but it has enough idempotents by I.3.2 of [Vig96]. The categories $\mathcal{R}(G)$ and $\mathcal{R}(G) - \text{Mod}$ are equivalent by I.4.4 of [Vig96] and we have $\text{ind}_{K_0}^K(\tau) = \mathcal{H}_R(G) \otimes_{\mathcal{H}_R(K)} V_{\tau}$ for every representation $(\tau, V_{\tau})$ of an open subgroup $H$ of $G$ by I.5.2 of [Vig96].

Let $K$ be a compact open subgroup of $G$, let $(\sigma, V_{\sigma})$ be an irreducible projective representation of $K$ as in paragraph 1.2 and let $\rho = \text{ind}_{K_0}^K(\sigma)$. Since $V_{\sigma}$ is a simple projective module over the unitary algebra $\mathcal{H}_R(K)$, it is isomorphic to a direct summand of $\mathcal{H}_R(K)$ itself because any non-zero map $\mathcal{H}_R(K) \to V_{\sigma}$ is surjective and splits. Then it is isomorphic to a minimal ideal of $\mathcal{H}_R(K)$ and so there exists an idempotent $e$ of $\mathcal{H}_R(K)$ such that $V_{\sigma} = \mathcal{H}_R(K)e$. Hence, we obtain
\(\rho = \mathcal{H}_R(G)e\) because the map \(\sum_i (f_i \otimes h_i)e \mapsto (\sum_i f_i h_i)e\) is an isomorphism of \(\mathcal{H}_R(G)\)-modules between \(\mathcal{H}_R(G) \otimes_{\mathcal{H}_R(k)} \mathcal{H}_R(K)e\) and \(\mathcal{H}_R(G)e\) whose inverse is \(fe \mapsto fe \otimes e\).

The algebra \(\mathcal{H}_R(G, \sigma)\) is isomorphic to \(\text{End}_G(\rho) \cong \text{End}_{\mathcal{H}_R(G)}(\mathcal{H}_R(G)e)\) by lemma 1.3 and the map \(e.\mathcal{H}_R(G)e \rightarrow (\text{End}_{\mathcal{H}_R(G)}(\mathcal{H}_R(G)e))^{op}\) which maps \(efe \in e.\mathcal{H}_R(G)e\) to the endomorphism \(f'e \mapsto f'efe\) of \(\mathcal{H}_R(G)e\) is an algebra isomorphism whose inverse is \(\varphi \mapsto \varphi(e)\). Then we have \(\mathcal{H}_R(G, \sigma)^{op} \cong e.\mathcal{H}_R(G)e\) and so the categories \(\mathcal{H}_R(G)e-\text{Mod}\) and \(\text{Mod}-\mathcal{H}_R(G, \sigma)\) are equivalent.

**Theorem 1.9.** If \(\mathcal{R}_\sigma(G) = \mathcal{R}(G, \sigma)\) then \(V \mapsto M_\sigma(V)\) is an equivalence of categories between \(\mathcal{R}(G, \sigma)\) and \(\text{Mod}-\mathcal{H}_R(G, \sigma)\) whose quasi-inverse is \(W \mapsto W \otimes_{\mathcal{H}_R(G, \sigma)} \rho\).

**Proof.** We take \(A = \mathcal{H}_R(G)\) and \(\mathcal{H}_R(G)e = \rho\) in I.6.6 of [Vig96]. Since \(\mathcal{H}_R(G, \sigma)^{op} \cong e.\mathcal{H}_R(G)e\), left actions of \(e.\mathcal{H}_R(G)e\) become right actions of \(\mathcal{H}_R(G, \sigma)\). The functor \(V \mapsto eV\) of [Vig96] from \(\mathcal{H}_R(G) - \text{Mod}\) to \(e.\mathcal{H}_R(G)e - \text{Mod}\) becomes the functor \(V \mapsto \text{Hom}_{\mathcal{H}_R(G)}(\mathcal{H}_R(G)e, V)\) and so the functor \(M_\sigma\). Hypothesis of theorem "équivalence de catégories" in I.6.6 of [Vig96] are satisfied by the condition \(\mathcal{R}_\sigma(G) = \mathcal{R}(G, \sigma)\) and so we obtain the result. \(\square\)

2. Maximal simple types

In this section we introduce the theory of simple types of an inner form of a general linear group over a non-archimedean locally compact field in the case of modular representations. We refer to sections 2.1-5 of [MS14b] for more details.

Let \(p\) be a prime number. Let \(F\) be a non-archimedean locally compact field of residue characteristic \(p\) and let \(D\) be a central division algebra of finite dimension over \(F\) whose reduced degree is denoted by \(d\). Given a positive integer \(m\), we consider the ring \(A = M_m(D)\) and the group \(G = GL_m(D)\) which is an inner form of \(GL_{md}(F)\). Let \(R\) be an algebraically closed field of characteristic different from \(p\).

Let \(\Lambda\) be an \(O_D\)-lattice sequence of \(V = D^m\). It defines a hereditary \(O_F\)-order \(A = \mathfrak{A}(\Lambda)\) of \(A\) whose radical is denoted by \(\mathfrak{P}\), a compact open subgroup \(U(\Lambda) = U_0(\Lambda) = \mathfrak{A}(\Lambda)^{\times}\) of \(G\) and a filtration \(U_k(\Lambda) = 1 + \mathfrak{P}^k\) with \(k \geq 1\) of \(U(\Lambda)\) (see section 1 of [Séc04]). Let \([A, n, 0, \beta]\) be a simple stratum of \(A\) (see for instance section 1.6 of [SS08]). Then \(\beta \in A\) and the \(F\)-subalgebra \(F[\beta]\) of \(A\) generated by \(\beta\) is a field denoted by \(E\). The centralizer \(B\) of \(E\) in \(A\) is a simple central \(E\)-algebra and \(\mathfrak{B} = \mathfrak{A} \cap B\) is a hereditary \(O_F\)-order of \(B\) whose radical is \(\mathfrak{Q} = \mathfrak{P} \cap B\).

As in paragraphs 1.2 and 1.3 of [Séc05b] we can choose a simple right \(E \otimes_F D\)-module \(N\) such that the functor \(V \mapsto \text{Hom}_{E \otimes_P D}(N, V)\) defines a Morita equivalence between the category of modules over \(E \otimes_F D\) and the category of vector spaces over \(D' = \text{End}_{E \otimes_P D}(N)^{op}\) which is a central division algebra over \(E\). We denote \(A(E) = \text{End}_D(N)\) which is a central simple \(F\)-algebra. If \(d'\) is the reduced degree of \(D'\) over \(E\) and \(m'\) is the dimension of \(V' = \text{Hom}_{E \otimes_P D}(N, V)\) over \(D'\), then we have \(m'd' = md'[\text{E : F}]\). Fixing a basis of \(V'\) over \(D'\) we obtain, via the Morita equivalence above, an isomorphism \(N' \cong V\) of \(E \otimes_F D\)-modules. If for every \(i \in \{1, \ldots, m\}\) we denote by \(V^i\) the image of the \(i\)-th copy of \(N\) by this isomorphism, we obtain a decomposition \(V = V^1 \oplus \cdots \oplus V^{m'}\) into simple \(E \otimes_F D\)-submodules. By section 1.5 of [Séc05b] we can choose a basis \(\mathfrak{B}\) of \(V'\) over \(D'\) so that \(\Lambda\) decomposes into the direct sum of the \(\Lambda^i = \Lambda \cap V^i\) for \(i \in \{1, \ldots, m\}\). For every \(i \in \{1, \ldots, m'\}\), let \(e_i : V \rightarrow V^i\) be the projection on \(V^i\) with kernel \(\bigoplus_{j \neq i} V^j\). In accordance with paragraph 2.3.1 of [Séc04] (see also [BH96]) the family of idempotents \(e = (e_1, \ldots, e_{m'})\) is a decomposition conforms to \(\Lambda\) over \(E\).

By paragraphs 1.4.8 and 1.5.2 of [Séc05b] there exists a unique hereditary order \(\mathfrak{A}(E)\) normalized by \(E^{\times}\) in \(A(E)\) whose radical is denoted by \(\mathfrak{P}(E)\). For every \(i \in \{1, \ldots, m'\}\) we have an isomorphism \(\text{End}_E(V^i) \cong A(E)\) of \(F\)-algebras which induces an isomorphism of \(O_F\)-algebras between the hereditary orders \(\mathfrak{A}(\Lambda^i)\) and \(\mathfrak{A}(E)\). Moreover, to the choice of the basis \(\mathfrak{B}\) corresponds the isomorphisms \(M_m(D') \cong B\) of \(E\)-algebras and \(M_{m'}(A(E)) \cong A\) of \(F\)-algebras.
Remark 2.1. If $U(\Lambda) \cap B^\times$ is a maximal compact open subgroup of $B^\times$, these isomorphisms induce an isomorphism $\mathfrak{B} \cong M_{m'}(\mathcal{O}_D^\prime)$ of $\mathcal{O}_E$-algebras and, by lemma 1.6 of [Séc05a], two isomorphisms $\mathfrak{A} \cong M_{m'}(\mathfrak{A}(E))$ and $\mathfrak{P} \cong M_{m'}(\mathfrak{P}(E))$ of $\mathcal{O}_F$-algebras.

We can associate to $[\Lambda,n,0,\beta]$ two compact open subgroups $J = J(\beta,\Lambda)$, $H = H(\beta,\Lambda)$ of $U(\Lambda)$ (see 2.4 of [SS08]). For every integer $k \geq 1$ we denote $J^k = J^k(\beta,\Lambda) = J(\beta,\Lambda) \cap U_k(\Lambda)$ and $H^k = H^k(\beta,\Lambda) = H(\beta,\Lambda) \cap U_k(\Lambda)$ which are pro-$p$-groups. In particular $J^1$ and $H^1$ are normal pro-$p$-subgroups of $J$ and the quotient $J^1/H^1$ is a finite abelian $p$-group.

Remark 2.2. We have $J = (U(\Lambda) \cap B^\times)J^1$ and this induce a canonical group isomorphism $J/J^1 \cong (U(\Lambda) \cap B^\times)/(U_1(\Lambda) \cap B^\times)$ (see paragraph 2.3 of [MS14b]). It allows us to associate canonically and bijectively a representation of $J$ trivial on $J^1$ to a representation of $U(\Lambda) \cap B^\times$ trivial on $U_1(\Lambda) \cap B^\times$.

2.1. Simple characters, Heisenberg representation and $\beta$-extensions

Let $[\Lambda,n,0,\beta]$ be a simple stratum of $A$. We denote by $\mathcal{C}_R(\Lambda,0,\beta)$ the set of simple $R$-characters (see paragraph 2.2 of [MS14b] and [Séc04]) that is a finite set of simple characters of $H^1$ which depends on the choice of an additive $R$-character of $F$. If $\tilde{n} \in \mathbb{N}^*$ and $[\Lambda,\tilde{n},0,\tilde{\beta}]$ is a simple stratum of $M_{\tilde{n}}(D)$ such that there exists an isomorphism of $F$-algebras $\nu : F[\beta] \to F[\tilde{\beta}]$ with $\nu(\beta) = \tilde{\beta}$, then there exists a bijection $\mathcal{C}_R(\Lambda,0,\beta) \to \mathcal{C}_R(\Lambda,0,\tilde{\beta})$ canonically associated to $\nu$, called transfer map. There also exists an equivalence relation, called endo-equivalence, among simple characters in $\mathcal{C}_R(\Lambda,0,\beta)$ (see [BSS12]) whose equivalence classes are called endo-classes.

Let $\theta \in \mathcal{C}_R(\Lambda,0,\beta)$. By proposition 2.1 of [MS14b] there exists a finite dimensional irreducible representation $\eta$ of $J^1$, unique up to isomorphism, whose restriction to $H^1$ contains $\theta$. It is called Heisenberg representation associated to $\theta$. The intertwining of $\eta$ is $I_G(\eta) = J^1B^\times J^1 = JB^\times J$ and for every $y \in B^\times$ the $R$-vector space $I_y(\eta) = \text{Hom}_{R_{J^1}}(\eta,\eta^y)$ has dimension 1.

A $\beta$-extension of $\eta$ (or of $\theta$) is an irreducible representation $\kappa$ of $J$ extending $\eta$ such that $I_G(\kappa) = JB^\times J$. By proposition 2.4 of [MS14b], every simple character $\theta \in \mathcal{C}_R(\Lambda,0,\beta)$ admits a $\beta$-extension and by formula (2.2) of [MS14b] the set of $\beta$-extensions of $\theta$ is equal to

$$\mathcal{B}(\theta) = \{ \kappa \otimes (\chi \circ N_{B/E}) \mid \chi \text{ character of } \mathcal{O}_E^\times \text{ trivial on } 1 + \varphi E \}$$

where $N_{B/E}$ is the reduced norm of $B$ over $E$ and $\chi \circ N_{B/E}$ is seen as a character of $J$ trivial on $J^1$ thanks to remark 2.2. We observe that for every $\kappa \in \mathcal{B}(\theta)$ and every $y \in B^\times$, the $R$-vector space $I_y(\kappa)$ has dimension 1 because it is non-zero and it is contained in $I_y(\eta)$.

2.2. Maximal simple types

Let $[\Lambda,n,0,\beta]$ be a simple stratum of $A$ such that $U(\Lambda) \cap B^\times$ is a maximal compact open subgroup of $B^\times$. By remarks 2.1 and 2.2, there exists a group isomorphism $J/J^1 \cong GL_{m'}(\mathfrak{L}^\prime_D)$, which depends on the choice of $\beta$.

A maximal simple type of $G$ associated to $[\Lambda,n,0,\beta]$ is a pair $(J,\lambda)$ where $\lambda$ is an irreducible representation of $J$ of the form $\lambda = \kappa \otimes \sigma$ where $\kappa \in \mathcal{B}(\theta)$ with $\theta \in \mathcal{C}_R(\Lambda,0,\beta)$ and $\sigma$ is a cuspidal representation of $GL_{m'}(\mathfrak{L}^\prime_D)$ identified to an irreducible representation of $J$ trivial on $J^1$. If $\sigma$ is a supercuspidal representation of $GL_{m'}(\mathfrak{L}^\prime_D)$ then $(J,\lambda)$ is called maximal simple supertyp.

Remark 2.3. The choice of a $\beta$-extension $\kappa \in \mathcal{B}(\theta)$ determines the decomposition $\lambda = \kappa \otimes \sigma$.

If we choose another $\beta$-extension $\kappa' = \kappa \otimes (\chi \circ N_{B/E}) \in \mathcal{B}(\theta)$ we obtain the decomposition $\lambda = \kappa' \otimes \sigma'$ where $\sigma' = \sigma \otimes (\chi^{-1} \circ N_{B/E})$. 

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2.3. Covers

Let $\mathcal{M}$ be a Levi subgroup of $G$, $\mathcal{P}$ be a parabolic subgroup of $G$ with Levi component $\mathcal{M}$ and unipotent radical $\mathcal{U}$ and let $\mathcal{U}^{-}$ be the unipotent subgroup opposed to $\mathcal{U}$. We say that a compact open subgroup $K$ of $G$ is decomposed with respect to $(\mathcal{M}, \mathcal{P})$ if $K = (K \cap \mathcal{U}^{-})(K \cap \mathcal{M})(K \cap \mathcal{U})$ and every element $k \in K$ decomposes uniquely as $k = k_1k_2k_3$ with $k_1 \in K \cap \mathcal{U}^{-}$, $k_2 \in K \cap \mathcal{M}$ and $k_3 \in K \cap \mathcal{U}$. Furthermore, if $\pi$ is a representation of $K$ we say that the pair $(K, \pi)$ is decomposed with respect to $(\mathcal{M}, \mathcal{P})$ if $K$ is decomposed with respect to $(\mathcal{M}, \mathcal{P})$ and if $K \cap \mathcal{U}$ and $K \cap \mathcal{U}^{-}$ are in the kernel of $\pi$.

Let $\mathcal{M}$ be a Levi subgroup of $G$. Let $K$ and $K_M$ be two compact open subgroups of $G$ and $\mathcal{M}$ respectively and let $\varrho$ and $\varrho_M$ be two irreducible representations of $K$ and $K_M$ respectively. We say that the pair $(K, \varrho)$ is decomposed above $(K_M, \varrho_M)$ if $(K, \varrho)$ is decomposed with respect to $(\mathcal{M}, \mathcal{P})$ for every parabolic subgroup $\mathcal{P}$ with Levi component $\mathcal{M}$, if $K \cap \mathcal{M} = K_M$ and if the restriction of $\varrho$ to $K_M$ is equal to $\varrho_M$. A pair $(K, \varrho)$ is a cover of $(K_M, \varrho_M)$ if it is decomposed above $(K_M, \varrho_M)$ and it satisfies condition (0.3) of [Blo05]. For more details see [Blo05, Vig98].

3. The isomorphisms $\mathcal{H}_R(G, \eta) \cong \mathcal{H}_R(B^\times, U_1(\Lambda) \cap B^\times)$

Using notations of section 2, let $[\Lambda, n, 0, \beta]$ be a simple stratum of $A$ such that $U(\Lambda) \cap B^\times$ is a maximal compact open subgroup of $B^\times$. Let $\theta \in \mathcal{C}_R(\Lambda, 0, \beta)$ and let $\eta$ be the Heisenberg representation associated to $\theta$. In this section we want to prove that the algebras $\mathcal{H}_R(G, \eta)$ and $\mathcal{H}_R(B^\times, U_1(\Lambda) \cap B^\times)$ are isomorphic (theorem 3.46).

Thanks to section 2, from now on we identify $A$ with $M_{m'}(A(E))$, $G$ with $GL_{m'}(A(E))$, $U(\Lambda)$ with $GL_{m'}(\mathfrak{A}(E))$, $U_1(\Lambda)$ with $\mathfrak{I}_{m'} + M_{m'}(\mathfrak{P}(E))$, $B^\times$ with $GL_{m'}(D')$, $K_B = U(\Lambda) \cap B^\times$ with $GL_{m'}(\mathcal{O}_{D'})$ and $K_1^B = U_1(\Lambda) \cap B^\times$ with $\mathfrak{I}_{m'} + M_{m'}(\mathfrak{P}(D'))$. By section 2.4 of [Chi17] we know a presentation by generators and relations of the algebra $\mathcal{H}_R(B^\times, K_1^B) \cong \mathcal{H}_R(B^\times, K_1^B) \otimes_R \mathbb{Z}$. Using this presentation we want to find an isomorphism between $\mathcal{H}_R(B^\times, K_1^B)$ and $\mathcal{H}_R(G, \eta)$.

3.1. Root system of $GL_{m'}$

In this paragraph we recall some notations and results on the root system of $GL_{m'}$ contained in section 2.1 of [Chi17].

We denote by $\Phi = \{\alpha_{ij} \mid 1 \leq i \neq j \leq m'\}$ the set of roots of $GL_{m'}$ relative to torus of diagonal matrices. Let $\Phi^+ = \{\alpha_{ij} \mid 1 \leq i \leq j \leq m'\}$, $\Phi^- = -\Phi^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq m'\}$ and $\Sigma = \{\alpha_{i,i+1} \mid 1 \leq i \leq m' - 1\}$ be, respectively, the sets of positive, negative and simple roots relative to Borel subgroup of upper triangular matrices. For every $\alpha = \alpha_{i,i+1} \in \Sigma$ we write $s_\alpha$ or $s_i$ for the transposition $(i, i+1)$. Let $W$ be the group generated by the $s_i$ which is the group of permutations of $m'$ elements and so the Weyl group of $GL_{m'}$. Let $\ell : W \to \mathbb{N}$ be the length function of $W$ relative to $s_1, \ldots, s_{m'-1}$. The group $W$ acts on $\Phi$ by $w\alpha_{ij} = \alpha_{w(i)w(j)}$ and for every $w \in W$ and $\alpha \in \Sigma$ we have (see (2.2) of [Chi17])

$$\ell(ws_\alpha) = \begin{cases} \ell(w) + 1 & \text{if } wa \in \Phi^+ \\ \ell(w) - 1 & \text{if } wa \in \Phi^- \end{cases} \quad (3.1)$$

Remark 3.1. By proposition 2.2 of [Chi17] we have $\ell(w) = |\Phi^+ \cap w\Phi^-| = |\Phi^- \cap w\Phi^+|.$

For every $P \subset \Sigma$ we denote by $\Phi^+_P$ the set of positive roots generated by $P$, $\Phi^-_P = -\Phi^+_P$, $\Psi^+_P = \Phi^+ \setminus \Phi^+_P$ and $\Psi^-_P = -\Psi^+_P$. We denote by $W_P$ the subgroup of $W$ generated by the $s_\alpha$ with $\alpha \in P$ and by $\hat{P}$ the complement of $P$ in $\Sigma$. We abbreviate $\hat{\alpha} = \{\alpha\}.$

Example. If $\alpha = \alpha_{i,i+1}$ then $\hat{\alpha} = \{\alpha_{j,j+1} \in \Sigma \mid j \neq i\}$, $\Psi^+_\alpha = \{\alpha_{kk} \in \Phi^+ \mid 1 \leq h \leq i < k \leq m\}$ and $\Phi^+_\alpha = \{\alpha_{hh} \in \Phi^+ \mid 1 \leq h < k \leq i \text{ or } i+1 \leq h < k \leq m\}.$
Proposition 3.2. Let $P \subset \Sigma$ and $w$ be an element of minimal length in $wW_P \in W/W_P$. Then $w\alpha \in \Phi^+$ for every $\alpha \in \Phi^+_P$ and for every $w' \in W_P$ we have $\ell(ww') = \ell(w) + \ell(w')$.

Proof. Proposition 2.4 and lemma 2.5 of [Chi17].

Proposition 3.2 implies that in each class of $W/W_P$ with $P \subset \Sigma$, there exists a unique element of minimal length and the same holds in each class of $W_P \setminus W$.

If $\varpi$ is an uniformizer of $O_{\mathcal{D}'}$ we identify $\tau_i = \left(\frac{I_i}{0 \ \varpi^{0}_{m'_{-i}}}\right)$ with $i \in \{0, \ldots, m'\}$, defined in section 2.2 of [Chi17], to elements of $B^\times$ and then of $G$. For every $\alpha = \alpha_{i,i+1} \in \Sigma$ we write $\tau_\alpha = \tau_i$. Let $\Delta$ (resp. $\Delta'$) be the commutative monoid (resp. group) generated by $\tau_\alpha$ with $\alpha \in \Sigma$. Then we can write every element $\tau$ of $\Delta$ uniquely as $\tau = \prod_{\alpha \in \Sigma} \tau_\alpha^{a_\alpha}$ with $a_\alpha \in \mathbb{N}$ and uniquely as $\tau = \text{diag}(1, \varpi^{a_1}, \ldots, \varpi^{a_{m'-1}})$ with $0 \leq a_1 \leq \cdots \leq a_{m'-1}$. In this case we denote $P(\tau) = \{\alpha \in \Sigma | i_\alpha = 0\}$ and if $P \subset \{0, \ldots, m\}$ or if $P \subset \Sigma$ we write $\tau_P$ in place of $\prod_{x \in P} \tau_x$.

We remark that if $P(\tau_P) = \hat{P}$.

3.2. The representation $\eta_P$

Let $\mathcal{M} = A(E)^\times \times \cdots \times A(E)^\times$ (monomial representations) which is a Levi subgroup of $G$ and let $\mathcal{P}$ be the parabolic subgroup of $G$ of upper triangular matrices with Levi component $\mathcal{M}$ and unipotent radical $\mathcal{U}$. Let $\mathcal{P}^-$ be the opposite parabolic subgroup to $\mathcal{P}$ and $\mathcal{U}^+$ its unipotent radical.

We denote $U = K_B \cap \mathcal{U}$, $M = K_B \cap \mathcal{M}$ and $I_B = K_B^\perp M U$. Then $U$ is the group of unipotent upper triangular matrices with coefficients in $O_{\mathcal{D}'}$, $M$ is the group of diagonal matrices with coefficients in $O_{\mathcal{D}'}$, and $I_B$ is the standard Iwahori subgroup of $K_B$.

We denote by $\tilde{W}$ the group $W \times \hat{\Delta}$ of monomial matrices with coefficients in $\mathbb{Z}$ which is called the extended affine Weyl group of $B^\times$. We recall that $B^\times = I_B \tilde{W} I_B$ and actually it is the disjoint union of $I_B \tilde{w} I_B$ with $\tilde{w} \in \tilde{W}$.

Remark 3.3. By proposition 2.16 of [Sec05a], which works for every decomposition $e$ conforms to $\Lambda$ over $E$ and not necessarily subordinate to $\mathfrak{B}$, the groups $J^1$ and $H^1$ are decomposed with respect to $(\mathcal{M}, \mathcal{P})$. Moreover, if $M^t = \prod_{i=1}^r G\mathfrak{m}_i(A(E))$ with $\sum_{i=1}^r m'_i = m'$ is a standard Levi subgroup of $G$ containing $\mathcal{M}$ and $\mathcal{P}$, then $J^1 \subset M^t$ is the upper parabolic subgroup of $G$ with Levi component $M^t$, $J^1$ and $H^1$ are decomposed with respect to $(\mathcal{M}^t, \mathcal{P}^t)$.

Let $\mathfrak{J}_1 = \mathfrak{J}_1^1(\beta, \Lambda)$ and $\mathfrak{Y}_1 = \mathfrak{Y}_1^1(\beta, \Lambda)$ be the $O_{\mathcal{D}}$-lattices of $A$ such that $J^1 = 1 + \mathfrak{J}_1^1$ and $H^1 = 1 + \mathfrak{Y}_1^1$ (see section 3.3 of [Sec04] or chapter 3 of [BK93]). Then they are $(\mathfrak{B}, \mathfrak{B'})$-bimodules and we have $\varpi \mathfrak{J}_1^1 \subset \mathfrak{J}_1^1 \subset \mathfrak{J}_1^1 \subset M_{m'}(\mathfrak{B}(E))$.

Since $V^t \cong N$ for every $i \in \{1, \ldots, m\}$, we can identify every $\Lambda^t$ to a lattice sequence $\Lambda_0$ of $N$ with the same period of $\Lambda$, every $e^i \beta t$ to an element $\beta_0 \in A(E)$ and $\mathfrak{A}(\Lambda_0)$ to $\mathfrak{A}(E)$. By proposition 2.28 of [Sec04] the stratum $[\Lambda_0, n, 0, \beta_0]$ of $A(E)$ is simple and critical exponents $k_0(\beta, \Lambda)$ and $k_0(\beta_0, \Lambda_0)$ are equal (for a definition of critical exponent see section 2.1 of [Sec04]). This implies that $\beta$ is minimal (i.e. $-k_0(\beta, \Lambda) = n$) if and only if $\beta_0$ is minimal. We denote $\mathfrak{J}_0^1 = \mathfrak{J}_1(\beta_0, \Lambda_0)$, $\mathfrak{Y}_1^0 = \mathfrak{Y}_1(\beta_0, \Lambda_0)$, $J_0^1 = J^1(\beta_0, \Lambda_0) = 1 + \mathfrak{J}_0^1$, $H_0^1 = H^1(\beta_0, \Lambda_0) = 1 + \mathfrak{Y}_1^0$.

Proposition 3.4. We have $\mathfrak{J}_1 = M_{m'}(\mathfrak{J}_0^1)$ and $\mathfrak{Y}_1 = M_{m'}(\mathfrak{Y}_0^1)$.

Proof. We prove the result only for $\mathfrak{J}_1^1$ since the case of $\mathfrak{Y}_1^1$ is similar. We have to prove that for every $i, j \in \{1, \ldots, m\}$ we have $e^i \mathfrak{J}_1^1 e^j = \mathfrak{J}_0^1$. We need to recall the definition of $\mathfrak{J}_1(\beta, \Lambda) = \mathfrak{J}_0(\beta, \Lambda)$ and of $\mathfrak{J}_0^k(\beta, \Lambda)$ with $k \geq 1$. By proposition 3.42 of [Sec04] if we denote $q = -k_0(\beta, \Lambda)$ and $s = [(q + 1)/2]$ (where $[x]$ denotes the integer part of $x \in \mathbb{Q}$) we have $\mathfrak{J}(\beta, \Lambda) = \mathfrak{B} + \mathfrak{P}^s$ if $\beta$ is minimal and $\mathfrak{J}(\beta, \Lambda) = \mathfrak{B} + \mathfrak{J}_0^s(\gamma, \Lambda)$ if $[\Lambda, n, q, \gamma]$ is a simple stratum equivalent to $[\Lambda, n, q, \beta]$. Then, if $\beta$ is minimal, $\mathfrak{J}_0^k(\beta, \Lambda) = \mathfrak{J}_1(\beta, \Lambda) \cap \mathfrak{P}_k$ is equal to $\mathfrak{J}_1 = \mathfrak{J}_0^1 + \mathfrak{P}^s$ if $0 \leq k \leq s - 1$. [10]
and to $\mathfrak{P}^k$ if $k \geq s$. Otherwise, if $[\Lambda, n, q, \gamma]$ is a simple stratum equivalent to $[\Lambda, n, q, \beta]$, $3^k(\beta, \Lambda)$ is equal to $\mathfrak{Q}^k + 3^s(\gamma, \Lambda)$ if $0 \leq k \leq s - 1$ and to $3^k(\gamma, \Lambda)$ if $k \geq s$. Similarly we obtain that if $\beta_0$ is minimal then $3^k(\beta_0, \Lambda_0)$ is equal to $\psi_{\mathcal{D}}^k + \mathfrak{P}(E)^s$ if $0 \leq k \leq s - 1$ and to $\mathfrak{P}(E)^k$ if $k \geq s$. Otherwise, if $[\Lambda_0, n, q, \gamma_0]$ is a simple stratum equivalent to $[\Lambda_0, n, q, \beta_0]$, $3^k(\beta_0, \Lambda_0)$ is equal to $\psi_{\mathcal{D}}^k + 3^s(\gamma_0, \Lambda_0)$ if $k \leq s - 1$ and to $3^k(\gamma_0, \Lambda_0)$ if $k \geq s$. We prove that $e^i3^k(\beta, \Lambda)e^j = 3^k(\beta_0, \Lambda_0)$ for every $k \geq 0$ by induction on $q$. If $q = n$ and so if $\beta$ and $\beta_0$ are minimal, since $\Theta = M_m(\psi_{\mathcal{D}})$ and $\mathfrak{P} = M_m(\mathfrak{P}(E))$ we have $e^i\mathfrak{Q}^ke^j = \psi_{\mathcal{D}}^k$ and $e^i\mathfrak{P}^ke^j = \mathfrak{P}(E)^k$ for every $k$ and so $e^i3^k(\beta, \Lambda)e^j = 3^k(\beta_0, \Lambda_0)$ for every $k \geq 0$. Now if $q < n$ and so if $\beta$ and $\beta_0$ are not minimal, by proposition 1.20 of [SS08] (see also the proof of theorem 2.2 of [Séc05b]) we can choose a simple stratum $[\Lambda_0, n, q, \gamma_0]$ equivalent to $[\Lambda_0, n, q, \beta_0]$ such that if $\gamma$ is the image of $\gamma_0$ by the diagonal embedding $A(E) \to A$ then $[\Lambda, n, q, \gamma]$ is a simple stratum equivalent to $[\Lambda, n, q, \beta]$. By inductive hypothesis we have $e^i3^k(\gamma, \Lambda)e^j = 3^k(\gamma_0, \Lambda_0)$ for every $k \geq 0$ and then we obtain $e^i3^k(\beta, \Lambda)e^j = 3^k(\beta_0, \Lambda_0)$.

Let $\theta_0$ be the transfer of $\theta$ to $\mathcal{C}_R(\Lambda_0, 0, \beta)$. Since $H^1$ is a pro-$p$-group, proceeding as in proposition 2.16 of [Séc05a], the pair $(H^1, \theta)$ is decomposed with respect to $(\mathcal{M}, \mathcal{P})$ and the restriction of $\theta$ to $H^1 \cap \mathcal{M} = H^1_0 \times \cdots \times H^1_0$ is $\theta^\mathcal{M}_{\theta0}$. We remark that in general $(\mathcal{J}^1, \eta)$ is not decomposed with respect to $(\mathcal{M}, \mathcal{P})$. We denote by $\eta_0$ the Heisenberg representation of $\theta_0$ and we can consider the irreducible representation $\eta_{\mathcal{M}} = \eta_{\theta0}^{\mathcal{M}'}$ of $J^1_{\mathcal{M}} = J^1 \cap \mathcal{M} = J^1_0 \times \cdots \times J^1_0$.

We denote $J^1_P = (J^1 \cap \mathcal{P})H^1$ and $H^1_P = (J^1 \cap \mathcal{U})H^1$ which are subgroups of $J^1$. They are normal in $J^1$ because $H^1$ contains the derived group of $J^1$. Moreover, $J \cap \mathcal{P}$ normalizes $J^1_P$ because $H^1$ is normal in $J$ and $J^1 \cap \mathcal{P}$ is normal in $J \cap \mathcal{P}$. Then $J^1_P$ is normal in $J^1 \cap \mathcal{P}$.

**Remark 3.5.** Taking into account remark 5.7 of [SS08], proposition 5.3 of [SS08] states that $J^1_P$ and $H^1_P$ are decomposed with respect to $(\mathcal{M}, \mathcal{P})$ and so we have $J^1_P = (H^1 \cap \mathcal{U}^-)J^1_{\mathcal{M}}(J^1 \cap \mathcal{U})$ and $H^1_P = (H^1 \cap \mathcal{U}^+)(H^1 \cap \mathcal{M})(J^1 \cap \mathcal{U})$. Moreover, if if $M' = \prod_{i=1}^s GL_{m_i}(A(E))$ with $\sum_{i=1}^s m'_i = m'$ is a standard Levi subgroup of $G$ containing $\mathcal{M}$ and $\mathcal{P}'$ is the upper standard parabolic subgroup of $G$ with Levi component $M'$, then $J^1_P$ and $H^1_P$ are decomposed with respect to $(M', \mathcal{P}')$.

Let $\theta_\mathcal{P}$ be the character of $H^1_P$ defined by $\theta_\mathcal{P}(uh) = \theta(h)$ for every $u \in J^1 \cap \mathcal{U}$ and every $h \in H^1$. Since $J^1$ is a pro-$p$-group, proceeding as in Proposition 5.5 of [SS08] we can construct an irreducible representation $\eta_\mathcal{P}$ of $J^1_P$, unique up to isomorphism, whose restriction to $H^1_P$ contains $\theta_\mathcal{P}$. Actually it is the natural representation of $J^1_P$ on the $J^1 \cap \mathcal{U}$-invariants of $\eta$. Furthermore, $\text{ind}^{J^1_P}_{J^1}(\eta_\mathcal{P})$ is isomorphic to $\eta$, $\text{ind}^{J^1_P}_{J^1}(\eta_\mathcal{P}) = J^1_P \mathcal{B}J^1_P$ and for every $y \in B^X$ we have $\dim_R(J^1_P(\eta_\mathcal{P})) = 1$. We remark that $(J^1_P, \eta_\mathcal{P})$ is decomposed with respect to $(\mathcal{M}, \mathcal{P})$ and the restriction of $\eta_\mathcal{P}$ to $J^1_{\mathcal{M}}$ is $\eta_{\mathcal{M}}$. We denote by $V_{\mathcal{M}}$ the $R$-vector space of $\eta_{\mathcal{M}}$ and $\eta_\mathcal{P}$.

Since $\text{ind}^{J^1_P}_{J^1}(\eta_\mathcal{P})$ is isomorphic to $\eta$, we can identify the $R$-vector space $V_{\mathcal{P}}$ of $\eta$ with the vector space of function $\varphi : J^1 \to V_{\mathcal{M}}$ such that $\varphi(xj) = \eta_\mathcal{P}(x)\varphi(j)$ for every $x \in J^1_P$ and $j \in J^1$. In this case $\eta(\varphi) \varphi : x \mapsto \varphi(xj)$. By Mackey formula, $V_{\mathcal{M}}$ is a direct factor of $V_{\mathcal{P}}$ and we can identify the subspace of function $\varphi \in V_{\mathcal{P}}$ with support in $J^1_P$ with it. This identification is given by $\varphi \mapsto \varphi(1)$ whose inverse is $v \mapsto \varphi_v$ where the support of $\varphi_v$ is $J^1_P$ and $\varphi_v(1) = v$. Let $p : V_{\mathcal{P}} \to V_{\mathcal{M}}$ be the canonical projection, i.e. the restriction of a function in $V_{\mathcal{P}}$ to $J^1_P$, and let $\iota : V_{\mathcal{M}} \to V_{\mathcal{P}}$ be the inclusion.

**Remark 3.6.** In general we can not define a representation $\kappa_\mathcal{P}$ of $J^1 = (J \cap \mathcal{P})H^1$ as in section 2.3 of [Séc05a] or in section 5.5 of [SS08], because $e$ is a decomposition conforms to $\Lambda$ over $E$ but it is not subordinate to $\mathfrak{S}$. In our case ($\mathfrak{S}$ maximal) the only decomposition conforms to $\Lambda$ over $E$ and subordinate to $\mathfrak{S}$ is the trivial one.

**Lemma 3.7.**

1. For every $j \in J^1_P$ we have $\eta(j) \circ \iota = \iota \circ \eta_\mathcal{P}(j)$ and $p \circ \eta(j) = \eta_\mathcal{P}(j) \circ p$.  

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2. For every $j \in J^1$ we have $p \circ \eta(j) \circ \iota = \begin{cases} \eta P(j) & \text{if } j \in J^1_p \\ 0 & \text{otherwise} \end{cases}$

3. $\sum_{j \in J^1/J^1_p} \eta(j) \circ \iota \circ p \circ \eta(j^{-1})$ is the identity of $\text{End}_R(V_M)$.

Proof. To prove the first point, let $\varphi \in V_M$ and $\varphi \in V_n$. Then $\eta(j)(\iota(\varphi_n))(1) = \varphi_n(j) = \eta P(j)v$ and $p(\eta(j)(\varphi))(1) = \varphi(j) = \eta P(j)\varphi(1)$. To prove the second point we observe that if $j \in J^1_p$ then $p \circ \eta(j) \circ \iota = p \circ \iota \circ \eta(j) = \eta(j)$ while if $j \notin J^1_p$ the support of $\eta(j)(\iota(\varphi_n))$ is in $J^1_p j^{-1}$ for every $\varphi_n \in V_M$ and so $p \circ \eta(j) \circ \iota = 0$. Finally, to prove the third point we observe that for every $\varphi \in V_n$ the function $\varphi_j = (\eta(j) \circ \iota \circ p \circ \eta(j^{-1}))\varphi$ has support in $J^1_p j^{-1}$ and $\varphi_j(j^{-1}) = \varphi(j^{-1})$. □

We consider the surjective linear map $\mu : \text{End}_R(V_n) \rightarrow \text{End}_R(V_M)$ given by $f \mapsto p \circ f \circ \iota$.

**Lemma 3.8.** The map $\zeta : \mathcal{H}_R(G, \eta) \rightarrow \mathcal{H}_R(G, \eta P)$ defined by $\Phi \mapsto \mu \circ \Phi$ for every $\Phi \in \mathcal{H}_R(G, \eta)$ is an isomorphism of $R$-algebras. Moreover, if the support of $\Phi \in \mathcal{H}_R(G, \eta)$ is in $J^1 J^1$ with $x \in B^\times$ then the support of $\zeta(\Phi)$ is in $J^1_p x J^1_p$.

Proof. Let $\Phi \in \mathcal{H}_R(G, \eta)$. Then the support of $\mu \circ \Phi$ is included in the support of $\Phi$ which is compact. Moreover, for every $x_1, x_2 \in J^1$ and every $j \in J^1$ we have $\mu(\Phi(x_1 x_2)) = p \circ \eta(x_1) \circ \Phi(j) \circ \eta(x_2) \circ \iota$ that by lemma 3.7 is $\eta P(x_1) \circ \mu(\Phi(j)) \circ \eta P(x_2)$. Hence, $\zeta$ is well-defined and it is clearly an $R$-linear map. Let $\Phi_1, \Phi_2 \in \mathcal{H}_R(G, \eta)$. For every $g \in G$ we have

$$((\mu \circ \Phi_1) \ast (\mu \circ \Phi_2))(g) = \sum_{x \in G \backslash J^1_p} \sum_{y \in G \backslash J^1 J^1_p} \sum_{z \in J^1 \backslash J^1_p} p \circ \Phi_1(x) \circ \iota \circ p \circ \Phi_2(z^{-1} y^{-1} g) \circ \iota$$

and so $\zeta$ is a homomorphism of $R$-algebras. Let $\Phi \in \mathcal{H}_R(G, \eta)$ such that $p \circ \Phi(g) \circ \iota = 0$ for every $g \in G$. Then by lemma 3.7, for every $g' \in G$ we have

$$\Phi(g') = \sum_{j_1 \in J^1 \backslash J^1_p} \eta(j_1) \circ \iota \circ p \circ \eta(j_1^{-1}) \circ \Phi(g') \circ \sum_{j_2 \in J^1 J^1_p} \eta(j_2) \circ \iota \circ p \circ \eta(j_2^{-1})$$

and then $\zeta$ is injective. Now, we know that $\mathcal{H}_R(G, \eta) \cong \text{End}_G(\text{ind}^G_{J^1}(\eta))$, $\mathcal{H}_R(G, \eta P) \cong \text{End}_G(\text{ind}^G_{J^1_p}(\eta P))$ and $\text{ind}^G_{J^1_p}(\eta P) \cong \eta$. Then by transitivity of the induction we have $\mathcal{H}_R(G, \eta) \cong \mathcal{H}_R(G, \eta P)$ and then $\zeta$ must be bijective. Furthermore, if $\Phi \in \mathcal{H}_R(G, \eta)$ has support in $J^1 J^1$ with $x \in B^\times$ then the support of $\zeta(\Phi)$ is in $J^1 J^1 \cap I_G(\eta P) = J^1 J^1 \cap J^1_p B^\times J^1_p = J^1_p x J^1_p$. □

**Lemma 3.9.** Let $x_1, x_2 \in B^\times$ and let $\tilde{f}_i \in \mathcal{H}_R(G, \eta J^1 J^1)$ and $\tilde{f}_i = \zeta(\tilde{f}_i)$ for $i \in \{1, 2\}$.

1. If $x_1$ or $x_2$ normalizes $J^1_p$ then the support of $\tilde{f}_1 * \tilde{f}_2$ is in $J^1_p x_1 x_2 J^1_p$ and $(\tilde{f}_1 * \tilde{f}_2)(x_1 x_2) = \tilde{f}_1(x_1) \circ \tilde{f}_2(x_2)$. 

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2. If \( x_1 \) or \( x_2 \) normalizes \( J^1 \) then the support of \( \tilde{f}_1 \ast \tilde{f}_2 \) is in \( J_P^{} x_1 x_2 J_P^{} \) and
\[
(\tilde{f}_1 \ast \tilde{f}_2)(x_1 x_2) = p \circ \tilde{f}_1(x_1) \circ \tilde{f}_2(x_2) \circ t.
\]

**Proof.** First point follows by remark 1.2. If \( x_1 \) or \( x_2 \) normalizes \( J^1 \), by remark 1.2 the support of \( \tilde{f}_1 \ast \tilde{f}_2 \) is in \( J^1 x_1 x_2 J^1 \) and so the support of \( \tilde{f}_1 \ast \tilde{f}_2 = \zeta(\tilde{f}_1 \ast \tilde{f}_2) \) is in \( J^1 x_1 x_2 J^1 \cap I_G(\eta_P) = J_P^{} x_1 x_2 J_P^{} \), and moreover \( (\tilde{f}_1 \ast \tilde{f}_2)(x_1 x_2) = (\zeta(\tilde{f}_1 \ast \tilde{f}_2)(x_1 x_2) = p \circ \tilde{f}_1(x_1) \circ \tilde{f}_2(x_2) \circ t. \)

**Lemma 3.10.** For every \( x \in B^x \cap \mathcal{M} \) and every \( y \in I_G(\eta_P) \) which normalizes \( J_M^1 \) we have \( I_x(\eta_P) = I_x(\eta_M) \) and \( I_y(\eta_P) = I_y(\eta_M) \). Moreover, every non-zero element in \( I_z(\eta_P) \), with \( z \in I_G(\eta_P) \), is invertible.

**Proof.** For the first assertion, in both cases the \( R \)-vector spaces are 1-dimensional and so it suffices to prove an inclusion. Since \( \eta_M \) is the restriction of \( \eta_P \) to \( J_M^1 \), for every \( x' \in I_G(\eta_P) \) we have \( I_{x'}(\eta_P) \subseteq I_{x'}(\eta_M). \) For the second assertion, we observe that \( I_G(\eta_P) = J_P^{} B^x J_P^{} = J_P^{} I_B^{} W I_B^{} J_P^{} \). Now \( I_B^{} \) normalizes \( J_P^1 \) since it is contained in \( J^1(J \cap \mathcal{P}) \) while \( \tilde{W} \) normalizes \( J_M^1 \). Take \( z = z_1 z_2 z_3 \in I_G(\eta_P) \) with \( z_1 \in J_P^{} I_B^{}, z_2 \in \tilde{W} \) and \( z_3 \in I_B^{} J_P^1 \) and take a non-zero element \( \gamma \) in \( I_z(\eta_P) \). Let \( \gamma_1 \) and \( \gamma_3 \) two invertible elements in \( I_{z_1}(\eta_P) \) and in \( I_{z_3}(\eta_P) \) respectively. Then \( \gamma_1 \circ \gamma \circ \gamma_3 \) is a non-zero element in \( I_{z_2}(\eta_P) = I_{z_2}(\eta_M) \) and so it is invertible. \( \square \)

### 3.3. The isomorphism \( \mathcal{H}_R(J, \eta) \cong \mathcal{H}_R(K_B, K_B^1) \)

In this paragraph we want to prove that the subalgebra \( \mathcal{H}_R(K_B, K_B^1) \) of \( \mathcal{H}_R(B^x, K_B^1) \) is isomorphic to the subalgebra \( \mathcal{H}_R(J, \eta_P) \) of \( \mathcal{H}_R(G, \eta_P) \) and so to \( \mathcal{H}_R(J, \eta) \).

In accordance with chapter 2 of [Chi17], we denote by \( f_x \in \mathcal{H}_R(B^x, K_B^1) \) the characteristic function of \( K^1_B \times K^1_B \) for every \( x \in B^x \) and we write \( \Phi \Phi_1 = \Phi \ast \Phi_2 \) for every \( \Phi_1 \) and \( \Phi_2 \) in \( \mathcal{H}_R(B^x, K_B^1) \), in \( \mathcal{H}_R(G, \eta_P) \) or in \( \mathcal{H}_R(G, \eta_P) \).

We observe that every element in \( \mathcal{H}_R(J, \eta_P) \) has support in \( J \cap J_P^{} B^x J_P^{} = J_P^{} (J \cap B^x) J_P^{} = J_P^{} K_B J_P^{} \) and so its image by \( \zeta^{-1} \) has support in \( J^1 K_B J^1 \). This implies that \( \zeta \) induces an algebra isomorphism from \( \mathcal{H}_R(J, \eta_P) \) to \( \mathcal{H}_R(J, \eta_P) \). We also remark that \( \mathcal{H}_R(K_B, K_B^1) \) is isomorphic to the group algebra \( R[K_B, K_B^1] \cong R[J/J^1] \), then we can identify every \( \Phi \in \mathcal{H}_R(K_B, K_B^1) \) to a function \( \Phi \in \mathcal{H}_R(J, J^1) \).

From now on we fix a \( \beta \)-extension \( \kappa \) of \( \eta \). We recall that \( \text{res}_{J^1}^{} \kappa = \eta \), \( I_G(\eta) = I_G(\kappa) = J^1 B^x J^1 \) and for every \( y \in B^x \) we have \( I_y(\eta) = I_y(\kappa) \) which is an \( R \)-vector space of dimension 1. Then \( V_\eta \) is also the \( R \)-vector space of \( \kappa \) and \( \kappa(j) \in I_j(\eta) \) for every \( j \in J \).

**Lemma 3.11.** The map \( \Theta' : \mathcal{H}_R(K_B, K_B^1) \rightarrow \mathcal{H}_R(J, \eta) \) defined by \( \Phi \mapsto \Phi \circ \kappa \) for every \( \Phi \in \mathcal{H}_R(K_B, K_B^1) \) is an algebra isomorphism.

**Proof.** The map is well-defined since for every \( \Phi \in \mathcal{H}_R(K_B, K_B^1) \) we have \( \Phi \circ \kappa : J \rightarrow \text{End}_R(V_\eta) \) and \( (\Phi \circ \kappa)(j_1 j_2 j_1') \Phi(j_1 j_2 j_1') = \eta(j_1) \circ (\Phi(j_1) \kappa(j)) \circ \eta(j_1') \) for every \( j \in J \) and \( j_1, j_1' \in J^1 \).

It is clearly \( R \)-linear and
\[
\Theta'(\Phi_1 \ast \Phi_2)(j) = \sum_{x \in J / J^1} \Phi_1(x) \Phi_2(x^{-1} j) \kappa(j) = \sum_{x \in J / J^1} \Phi_1(x) \Phi_2(x^{-1} j) \kappa(x) \circ \kappa(x^{-1} j)
\]
\[
= \sum_{x \in J / J^1} (\Phi_1(x) \kappa(x)) \circ (\Phi_2(x^{-1} j) \kappa(x^{-1} j)) = (\Theta'(\Phi_1) \ast \Theta'(\Phi_2))(j)
\]
for every \( \Phi_1, \Phi_2 \in \mathcal{H}_R(K_B, K_B^1) \) and \( j \in J \). Hence, \( \Theta' \) is an \( R \)-algebra homomorphism. It is injective because \( \kappa(j) \in GL(V_\eta) \) for every \( j \in J \). Let \( \tilde{f} \in \mathcal{H}_R(J, \eta) \) and \( j \in J \). Since \( \tilde{f}(j) \in I_j(\eta) = \text{Hom}_{\mathcal{J}_j}(\eta, \eta') \), which is of dimension 1, we have \( \tilde{f}(j) \in R\kappa(j) \) and then we
can write $\tilde{f}(j) = \Phi(j)\kappa(j)$ with $\Phi : J \to R$. Since $\tilde{f} \in \mathcal{H}_R(J,\eta)$, for every $j_1 \in J^1$ we have $\Phi(j_1)\kappa_{\eta(j_1)} = \tilde{f}(j_1) = \eta(j_1)\tilde{f}(j) = \eta(j_1)\Phi(j)\kappa(j) = \Phi(j)\kappa_{\eta(j_1)}$ and so $\Phi \in \mathcal{H}_R(J, J^1)$.

We conclude that $\Theta'$ is surjective and then it is an algebra isomorphism. □

Composing the restriction of $\zeta$ to $\mathcal{H}_R(J,\eta)$ with $\Theta'$ we obtain an algebra isomorphism $\mathcal{H}_R(K_B, K_B^1) \to \mathcal{H}_R(J,\eta_P)$. For every $x \in K_B$ let $\tilde{f}_x = \Theta'(f_x) \in \mathcal{H}_R(J,\eta_P)$ which is given by $\tilde{f}_x(y) = \kappa(y)$ for every $y \in J^1 x J^1 = J^1 x$ and let $\tilde{f}_x = \zeta(\tilde{f}_x) \in \mathcal{H}_R(J,\eta_P)$ which is given by $\tilde{f}_x(z) = \Phi \circ \kappa(z) \circ \Theta$ for every $z \in J^1 x J^1_P$.

3.4. Generators and relations of $\mathcal{H}_R(B^\times, K_B^1)$

In this paragraph we introduce some notations and we recall the presentation by generators and relations of the algebra $\mathcal{H}_R(B^\times, K_B^1)$ presented in [Chi17].

We denote $\Omega = K_B \cup \{\tau_0, \tau_{0}^{-1}\} \cup \{\tau_\alpha : \alpha \in \Sigma\}$ and $\Omega = \{f_\omega : \omega \in \Omega\}$ which is a finite set. We define some subgroup of $G$, through its identification with $GL_m(A(E))$. For every $\alpha = \alpha_{ij} \in \Phi$ we denote by $\mathcal{U}_\alpha$ the subgroup of matrices $(a_{hk}) \in G$ with $a_{hk} = 1$ for every $h \in \{1, \ldots, m\}$, $\alpha_{ij} \in A(E)$ and $a_{hh} = 0$ if $h \neq k$ and $(h,k) \neq (i,j)$. For every $P \subset \Sigma$ we denote by $\mathcal{M}_P$ the standard Levi subgroup associated to $P$ and by $\mathcal{U}_P^\circ$ (resp. $\mathcal{U}_P^-$) the unipotent radical of upper (resp. lower) standard parabolic subgroups with Levi component $\mathcal{M}_P$. We remark that $\mathcal{M} = \mathcal{M}_\emptyset$, $\mathcal{U} = \mathcal{U}_\emptyset$ and $\mathcal{U}^- = \mathcal{U}_\emptyset^-$. Thus, we have $\mathcal{U}_P^\circ = \prod_{\alpha \in \Psi_P^+} \mathcal{U}_\alpha$ and $\mathcal{U}_P^- = \prod_{\alpha \in \Psi_P^-} \mathcal{U}_\alpha$.

Furthermore, if $P_1 \subset P_2 \subset \Sigma$ then $\mathcal{U}_{P_1}^\circ$ is a subgroup of $\mathcal{U}_{P_2}^\circ$ and $\mathcal{U}_{P_1}^-$ is a subgroup of $\mathcal{U}_{P_2}^-$. Remark 3.12. By proposition 3.4, if we take $\alpha = \alpha_{ij} \in \Phi$ (resp. $\alpha_{hk} \in \Omega_\alpha \cap J^1$) then $\alpha_{ij}$ in $\mathcal{U}_\emptyset$ (resp. $\mathcal{U}_\emptyset^\circ$).

Remark 3.13. In accordance with paragraph 2.2 of [Chi17] we denote $M_P = \mathcal{M}_P \cap K_B$, $U_P^\circ = \mathcal{U}_P^\circ \cap K_B$ and $U_P^- = \mathcal{U}_P^- \cap K_B$ for every $P \subset \Sigma$ and $U_\alpha = \mathcal{U}_\alpha \cap K_B$ for every $\alpha \in \Phi$.

As in paragraph 2.3 of [Chi17], for every $\alpha = \alpha_{i,i+1} \in \Sigma$ and $w \in W$ we consider the following sets: $A(w, \alpha) = \{w(j) | i + 1 \leq j \leq m\}$, $B(w, \alpha) = \{w(j) - 1 | i + 1 \leq j \leq m\}$, $P'(w, \alpha) = A(w, \alpha) \setminus B(w, \alpha)$, $P(w, \alpha) = \{\alpha_{i,i+1} \in \Sigma | i \in P'(w, \alpha)\}$ and $Q(w, \alpha) = B(w, \alpha) \setminus A(w, \alpha)$. We remark that $\tau_{P'(w,\alpha)} = \tau_{P(w,\alpha)} = \tau_{P(w,\alpha)}$ because $0 \notin P'(w,\alpha)$ and $\tau_m = \mathbb{I}_m$. Moreover, if $\alpha = \alpha_{i,i+1} \in \Sigma$, $w' \in W$ and $w$ is of minimal length in $w'W_\alpha \in W/W_\alpha$ then we have

$$w't_iw'^{-1} = w't_i^{-1}w = \prod_{h=i+1}^m w_i^h = \prod_{h=i+1}^m \tau_{w_i(h)-1}^\tau_{w_i(h)}^{-1} = \tau_{P(w,\alpha)}TQ(w,\alpha).$$

Lemma 3.14. The algebra $\mathcal{H}_R(B^\times, K_B^1)$ is the $R$-algebra generated by $\Omega$ subject to the following relations

1. $f_k = 1$ for every $k \in K^1$ and $f_{k_1}f_{k_2} = f_{k_1k_2}$ for every $k_1, k_2 \in K$;
2. $f_{\tau_0}f_{\tau_0}^{-1} = 1$ and $f_{\tau_0}^{-1}f_{\tau_0} = f_{\tau_0}^{-1}w_\alpha f_{\tau_0}^{-1}$ for every $\omega \in \Omega$;
3. $f_{\tau_0}f_x = f_{\tau_0}x f_{\tau_0}^{-1}$ for every $\alpha \in \Sigma$ and $x \in M_\alpha$;
4. $f_\alpha f_\alpha' = f_\alpha$ if $u \in U_{\alpha'}$ with $\alpha' \in \Psi_\alpha^+$, for every $\alpha \in \Sigma$;
5. $f_{\tau_0}f_\alpha = f_{\tau_0}$ if $u \in U_{\alpha'}$ with $\alpha' \in \Psi_\alpha^+$, for every $\alpha \in \Sigma$;
6. $f_{\tau_0}f_{\tau_0} = f_{\tau_0}f_{\tau_0}$ for every $\alpha, \alpha' \in \Sigma$;
7. $\left(\prod_{\alpha \in P(w,\alpha)} f_{\tau_0}^{-1}f_{\tau_0}^{w'}\right) f_{\tau_0}f_{\tau_0}^{-1} = q_{w}^{\tau}(\left(\prod_{\alpha' \in Q(w,\alpha)} f_{\tau_0}^{\alpha'}\right) \left(\sum_{u} f_{u}\right))$ for every $\alpha \in \Sigma$ and $w$ of minimal length in $wW_\alpha \in W/W_\alpha$ and where $u$ describes a system of representatives of $(U \cap wU^{-1} w^{-1})K_B^1/K_B$ in $U \cap wU^{-1} w^{-1}$. 

Proof. The only difference between this presentation and those in [Chi17] is the relation 3 which is equivalent to relations 3, 4 and 7 of definition 2.21 of [Chi17] because $M \cap K_B$, $U_{\alpha'}$ with $\alpha' \in \Phi^*_G$ and $W_\delta$ generate $M_G$.

Hence, to define an algebra homomorphism from $\mathcal{H}(B^*, K_B^*)$ to $\mathcal{H}(G, \eta \rho)$, it is sufficient to choose elements $f_\omega \in \mathcal{H}(G, \eta \rho)$ for every $\omega \in \Theta$ such that $f_\omega$ respect the relations of lemma 3.14. We remark that we can take $\hat{f}_\omega \in \mathcal{H}(G, \eta \rho)_{J_0^B, J_0^B}$ for every $\omega \in \Theta$ and we recall that in paragraph 3.3 we have just defined $\hat{f}_k$ for every $k \in K_B$ as the image of $f_\omega$ by $\odot \Theta'$.

3.5. Some decompositions of $J_0^B$-double cosets

In this paragraph we introduce some notations and some tools that we will use to construct elements in $\mathcal{H}(G, \eta \rho)_{J_0^B, J_0^B}$ with $i \in \{0, \ldots, m'-1\}$.

Lemma 3.15. Let $\tau \in \Delta$ and $P = P(\tau)$.

1. We have $J_0^P = (J_0^P \cap U_0^P)(J_0^P \cap M_P)(J_0^P \cap U_0^P) = (J_0^P \cap U_0^P)(J_0^P \cap M_P)(J_0^P \cap U_0^P)$.

2. We have $(J_0^P \cap U_0^P) \subset H^1 \cap U_0^P \subset J_0^P \cap U_0^P$, $(J_0^P \cap U_0^P)^{-1} \subset (J_0^1 \cap U_0^P)^{-1} \subset H^1 \cap U_0^P = (J_0^P \cap U_0^P)$ and $(J_0^P \cap M_P)^\tau = J_0^P \cap M_P$.

3. We have $(J_0^P \cap U_0^P)^{-1} \subset J_0^P \cap U_0^P$, $(J_0^P \cap U_0^P)^{-1} \subset J_0^P \cap U_0^P$ and $(J_0^P \cap U_0^P) = (J_0^P \cap U_0^P)^\tau = J_0^P \cap U_0^P$. 

Proof. The first point follows by remark 3.5. To prove the second point we observe that remark 3.12 implies that $(J_0^P \cap U_0^P)^\tau = (J_1^1 \cap \prod_{\alpha \in \Psi^*_G} U_0^\alpha)^\tau$ is contained in $(I_{m'} + \omega \Delta^1) \cap U_0^P$ which is in $H^1 \cap U_0^P \subset J_0^P \cap U_0^P$. Similarly we prove $(J_0^P \cap U_0^P)^{-1} \subset H^1 \cap U_0^P$. Moreover, since $\omega^{-1} 3_0^B \omega = 3_0^B$ and $\omega^{-1} 3_0^B \omega = 3_0^B$, we have $(J_0^P \cap M_P)^\tau = J_0^P \cap M_P$. To prove the third point, we observe that $(J_0^P \cap U_0^P) \subset ((J_0^P \cap M_P)(J_0^P \cap U_0^P))^{-1} \cap U_0^P$ which is in $(J_0^P \cap M_P)(J_0^P \cap U_0^P) \cap U_0^P = J_0^P \cap U_0^P$. Similarly we prove $(J_0^P \cap U_0^P)^{-1} \subset J_0^P \cap U_0^P$. Finally, since $\omega^{-1} 3_0^B \omega = 3_0^B$ we obtain $(J_0^P \cap U_0^P)^\tau = J_0^P \cap U_0^P$.

Lemma 3.16. If $\tau \in \Delta$ then $J_0^P \tau J_0^P = (J_0^P \cap U_0^P)^\tau J_0^P = J_0^P \tau (J_0^P \cap U_0^P)^\tau$ and $J_0^P \tau^{-1} J_0^P = (J_0^P \cap U_0^P)^{-1} J_0^P = J_0^P \tau^{-1} (J_0^P \cap U_0^P)^{-1}$.

Proof. Let $P = P(\tau)$. By lemma 3.15 we have $J_0^P = (J_0^P \cap U_0^P)(J_0^P \cap M_P)(J_0^P \cap U_0^P)$ and so we obtain $J_0^P \tau J_0^P = (J_0^P \cap U_0^P) \tau (J_0^P \cap M_P) \tau (J_0^P \cap U_0^P)^\tau J_0^P$ which is equal to $(J_0^P \cap U_0^P) \tau J_0^P$ by lemma 3.15. Similarly we prove other equalities.

Lemma 3.17. If $w \in W$ then $(J_0^P)^w J_0^P = (J^1 \cap U^w \cap U^\tau)^{-1} J_0^P$.

Proof. Since $(H^1 \cap U^w)^w \subset J_0^P$ and $(J_0^P)^w = J_0^P M_G$ we obtain $(J_0^P)^w J_0^P = (J^1 \cap U^w) J_0^P$. Moreover, we have $(J^1 \cap U^w \cap U^\tau) \subset J_0^P$ and so $(J_0^P)^w J_0^P = (J^1 \cap U^w \cap U^\tau) J_0^P$.

Lemma 3.18. We have $J_0^P U^{-1} J_0^P \cap U = J_0^P \cap U$ and $J_0^P U_0^P J_0^P \cap U = J_0^P \cap U$.

Proof. We have $J_0^P U^{-1} J_0^P \cap U = (J_0^P \cap U)(J_0^P \cap M)(J_0^P \cap U)^{-1} \cap U^{-1} \cap U = (J_0^P \cap U)(J_0^P \cap M)(J_0^P \cap U) = (J_0^P \cap U)^{-1} \cap U \cap (J_0^P \cap U) = J_0^P \cap U$ which is clearly contained in $J_0^P \cap U$. Similarly we prove the second statement.

Lemma 3.19. Let $\tau, \tau' \in \Delta$. Then $J_0^P \tau J_0^P \tau' J_0^P = J_0^P \tau' \tau J_0^P$ and $(J_0^P)^{\tau'} J_0^P \cap (J_0^P)^{-1} J_0^P = J_0^P$.

Proof. By lemma 3.16 we have $J_0^P \tau J_0^P \tau' J_0^P = J_0^P \tau (J_0^P \cap U_0^P)^{\tau'} J_0^P = J_0^P \tau' \tau J_0^P$.

By lemma 3.15 it is in $J_0^P \cap U_0^P \cap (J_0^P)^{-1} J_0^P$ and so $J_0^P \tau J_0^P \tau' J_0^P = J_0^P \tau' \tau J_0^P$. Now, by lemma 3.16, the set $(J_0^P)^{\tau'} J_0^P \cap (J_0^P)^{-1} J_0^P$ is contained in $(J_0^P \cap U_0^P)^{-1} J_0^P \cap (J_0^P \cap U_0^P)^{-1} J_0^P = (J_0^P \cap U_0^P)^{-1} J_0^P \cap (J_0^P \cap U_0^P)^{-1} J_0^P$ which is equal to $J_0^P$ by lemma 3.18.
Remark 3.20. We can prove similar results of lemmas 3.15, 3.16, 3.18 and 3.19 by replacing $J_1^1$ with $J_1^1$.

Lemma 3.21. Let $\alpha = \alpha_{i+1} \in \Sigma$, $w \in W$ and $P = P(w, \alpha)$. Then $\Psi_+^1 \cap w \Psi_+^- = \Phi^+ \cap w \Psi_+^- \cap \Psi_+^- \cap w \Psi_+^-$ and $\Psi_+^- \cap w \Psi_+^- = \Phi^- \cap w \Psi_+^-$. If in addition $w$ is of minimal length in $wW_\alpha \in W/W_\alpha$ then $\Phi^+ \cap w \Psi_+^- = \Phi^+ \cap w \Phi^-$ and $\Phi^- \cap w \Psi_+^- = \Phi^- \cap w \Phi^-$. 

Proof. It follows by lemma 2.19 of [Chi17]. □

From now on, we denote $\delta(\mathfrak{A}_0, \mathfrak{A}_1) = [\mathfrak{A}_0 : \mathfrak{A}_1]$ and $\delta(\mathfrak{S}_0, \mathfrak{S}_1) = [\mathfrak{S}_0, \mathfrak{S}_1] = [H^1 \cap U_{\alpha'} : (H^1 \cap U_{\alpha'})^\tau_{w}] = [H^1 \cap U_{\alpha'} : (H^1 \cap U_{\alpha'})^\tau_{w}]$ for every $\alpha \in \Phi$, $\alpha' \in \Phi^+$ and $\alpha'' \in \Phi^-$. In particular $\delta(\mathfrak{S}_0, \mathfrak{S}_1)$ and $\delta(\mathfrak{S}_0, \mathfrak{S}_1)$ are powers of $p$ and so they are invertible in $R$.

From now on we fix $1 \leq i \leq m - 1$ and we consider $\alpha = \alpha_{i+1}$, $w$ of minimal length in $wW_\alpha$, $P = P(w, \alpha)$ and $Q = Q(w, \alpha)$.

Remark 3.23. Lemma 3.21 implies that $wU_\alpha^w w^{-1} \cap U_\alpha^- = wU^- w^{-1} \cap U^+$ and $wU_\alpha^w w^{-1} \cap U_\alpha^- = wU^- w^{-1} \cap U^-$. Moreover, we have $\ell(w) = |\Psi_+^1 \cap w \Psi_+^-| = |\Psi_+^- \cap w \Psi_+^-|$ by remark 3.1.

We define

$$\mathcal{V}(w, \alpha) = (J_1^1 \cap wU_\alpha^w w^{-1} \cap U_\alpha^-) \cdot \tau_{w^{-1}} \cdot \tau_{w} w^{-1}$$

(3.2)

which is a pro-$p$-group. We remark that it is equal to $(J_1^1 \cap wU^- \cap U^-) \cdot \tau_{w^{-1}} \cdot \tau_{w} w^{-1}$ by remark 3.23 and to $(H^1 \cap wU_\alpha^w w^{-1} \cap U_\alpha^-) \cdot \tau_{w^{-1}} \cdot \tau_{w} w^{-1}$ since $J_1^1 \cap U_\alpha^- = H^1 \cap U_\alpha^-$. Then $\mathcal{V}(w, \alpha)$ is equal to

$$\prod_{a' \in \Psi_+^1 \cap \Psi_+^-} (H^1 \cap U_{\alpha'}) \cdot \tau_{w^{-1}} \cdot \tau_{w} = \prod_{a' \in \Psi_+^1 \cap \Psi_+^-} (H^1 \cap U_{\alpha'}) \cdot \tau_{w} w^{-1} = \prod_{a'' \in \Psi_+^1 \cap \Psi_+^-} (H^1 \cap U_{\alpha'}) \cdot \tau_{w} w^{-1}$$

which is $(\mathfrak{I}_m + \mathfrak{I}_m) \cap wU_\alpha^w w^{-1} \cap U_\alpha^-$. 

Lemma 3.24. The group $wU_\alpha^w w^{-1} \cap U_\alpha^- \subseteq \mathcal{V}(w, \alpha)$, it normalizes $\mathcal{V}(w, \alpha) \cap J_1^1$ and

$$\left( wU_\alpha^w w^{-1} \cap U_\alpha^- \right) \cap \mathcal{V}(w, \alpha) \cap J_1^1 = wU_\alpha^w w^{-1} \cap U_\alpha^- \cap K_B^1.$$

Proof. We recall that $wU_\alpha^w w^{-1} \cap U_\alpha^- = wU_\alpha^- \cap U_\alpha^- \cap K_B$ by remark 3.13. Since $U_{\alpha'} = \tau_{a}(K_B \cap U_{\alpha'}) \cdot \tau_{a}^{-1}$ for every $a' \in \Psi_+^1$ (see lemma 2.9 of [Chi17]), we have $wU_\alpha^w w^{-1} \cap U_\alpha^- = (K_B \cap wU_\alpha^- \cap U_\alpha^- \cap U_\alpha^-) \cdot \tau_{w^{-1}} \cdot \tau_{w} w^{-1}$ which is in $\mathcal{V}(w, \alpha)$. Moreover, the group $wU_\alpha^- \cap U_\alpha^- \cap K_B \text{ normalizes } \mathcal{V}(w, \alpha) \cap J_1^1 = \mathcal{V}(w, \alpha) \cap H^1$ because we have $wU_\alpha^w w^{-1} \cap U_\alpha^- \subset K_B$ and $K_B$ normalizes $H^1$. Finally, since $K_B \cap H^1 = K_B^1$, we have $wU_\alpha^w w^{-1} \cap U_\alpha^- \cap \mathcal{V}(w, \alpha) \cap J_1^1 = wU_\alpha^- \cap U_\alpha^- \cap K_B \cap H^1 = wU_\alpha^- \cap U_\alpha^- \cap K_B \cap H^1 = wU_\alpha^- \cap U_\alpha^- \cap K_B^1$. □

By lemma 3.24 the group $\mathcal{V}(w, \alpha) \cap J_1^1 = \mathcal{V}(w, \alpha) \cap J_1^1$ is a subgroup of $\mathcal{V}(w, \alpha)$. We set

$$d(w, \alpha) = |\mathcal{V}(w, \alpha) : \mathcal{V}(w, \alpha)| \in R$$

which is non-zero because it is a power of $p$.

Remark 3.25. We have $\mathcal{V}(w, \alpha) \cap J_1^1 = H^1 \cap wU_\alpha^- w^{-1} \cap U_\alpha^- = \prod_{a' \in \Psi_+^1 \cap \Psi_+^-} H^1 \cap U_{\alpha'}$. Hence, by remarks 3.22 and 3.23 we have $|\mathcal{V}(w, \alpha) : \mathcal{V}(w, \alpha) \cap J_1^1| = |\mathfrak{S}_0, \mathfrak{S}_1| = |\mathfrak{S}_0, \mathfrak{S}_1| = \delta(\mathfrak{S}_0, \mathfrak{S}_1)^{\ell(w)}$. On the other hand we have $|\mathcal{V}(w, \alpha) : \mathcal{V}(w, \alpha) \cap J_1^1| = d(w, \alpha) \cdot \mathcal{V}(w, \alpha) \cap J_1^1 = d(w, \alpha) \cdot \mathcal{V}(w, \alpha) \cap J_1^1$ which is equal to $d(w, \alpha) \cdot \mathcal{V}(w, \alpha) \cap J_1^1$ by remark 3.23 and so to $d(w, \alpha) |wU_\alpha^w w^{-1} \cap U^- = wU^- w^{-1} \cap U^- \cap K_B^1| = d(w, \alpha) q^{\ell(w)}$ where $q$ is the cardinality of $\mathcal{V}$. So, if we denote $\partial = \delta(\mathfrak{S}_0, \mathfrak{S}_1)/q \in \mathbb{R}^*$ then $d(w, \alpha) = \partial^{\ell(w)}$.  

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Lemma 3.26. We have \((J_P^1)^{\tau_P} J_P^1 \cap (J_P^1)^{w_{\tau w^{-1}}w^{-1}} J_P^1 = \mathcal{V}(w,\alpha) J_P^1\).

Proof. We have \((J_P^1)^{w_{\tau w^{-1}}w^{-1}} = (H^1 \cap w^{-1} \mathcal{U}w) w_{\tau w^{-1}}^{-1} (J_P^1)^{w_{\tau w^{-1}}w^{-1}} (J_P^1 \cap w^{-1} \mathcal{U}w) w_{\tau w^{-1}}^{-1}.\) Now consider the decompositions \(H^1 \cap w^{-1} \mathcal{U}w = (H^1 \cap w^{-1} \mathcal{U}w \cap \mathcal{U}^\#)(H^1 \cap \mathcal{U}w \cap \mathcal{U}^\#)\) and \(J_P^1 \cap w^{-1} \mathcal{U}w = (J_P^1 \cap \mathcal{U}w \cap \mathcal{U}^\#)(J_P^1 \cap w^{-1} \mathcal{U}w \cap \mathcal{U}^\#)\). By lemma 3.21 we have \(J_P^1 \cap w^{-1} \mathcal{U}w \cap \mathcal{U}^\# = J_P^1 \cap w^{-1} \mathcal{U}w \cap \mathcal{U}^\#\) and so by lemma 3.15 we obtain \((J_P^1 \cap w^{-1} \mathcal{U}w \cap \mathcal{U}^\#)^{w_{\tau w^{-1}}w^{-1}} \subset (J_P^1 \cap \mathcal{U}^\#)^{w_{\tau w^{-1}}w^{-1}} \subset (H^1 \cap \mathcal{U}^\#)^{w_{\tau w^{-1}}w^{-1}} \subset J_P^1\).

Then, since \((J_P^1)^{w_{\tau w^{-1}}w^{-1}} = J_P^1\) by lemma 3.15 and since \((H^1 \cap \mathcal{U}w \cap w^{-1})^{w_{\tau w^{-1}}w^{-1}} = \mathcal{V}(w,\alpha)\), we obtain \((J_P^1)^{w_{\tau w^{-1}}w^{-1}} \subset \mathcal{V}(w,\alpha) J_P^1 (J_P^1 \cap \mathcal{U}w \cap w^{-1})^{w_{\tau w^{-1}}w^{-1}}.\) By lemma 3.16 and by previous calculations we have

\[(J_P^1)^{\tau_P} J_P^1 \cap (J_P^1)^{w_{\tau w^{-1}}w^{-1}} J_P^1 = ((J_P^1 \cap \mathcal{U}^\#)^{\tau_P} \cap \mathcal{V}(w,\alpha)) J_P^1 (J_P^1 \cap \mathcal{U}w \cap w^{-1})^{w_{\tau w^{-1}}w^{-1}} J_P^1\]

Now, since \(w^{\tau w^{-1}}w^{-1} = \tau_Q^{-1} \tau_P\), the group \(\mathcal{V}(w,\alpha)\) is contained both in \((\mathcal{U}^\#)^{\tau_Q^{-1} \tau_P}\) and in \((J_P^1 \cap \mathcal{U}w \cap w^{-1})^{\tau_Q^{-1} \tau_P} \subset (J_P^1 \cap \mathcal{U}w \cap w^{-1})^{\tau_P} \subset (J_P^1)^{\tau_P}\) by lemma 3.15. This implies \(\mathcal{V}(w,\alpha) \subset (J_P^1 \cap \mathcal{U}^\#)^{\tau_P}\) and so \((J_P^1)^{\tau_P} J_P^1 \cap (J_P^1)^{w_{\tau w^{-1}}w^{-1}} J_P^1 = \mathcal{V}(w,\alpha) (J_P^1 \cap \mathcal{U}w \cap w^{-1})^{w_{\tau w^{-1}}w^{-1}} J_P^1\).

Now we have \((J_P^1 \cap \mathcal{U}^\#)^{\tau_P} \cap J_P^1 (J_P^1 \cap \mathcal{U}w \cap w^{-1})^{w_{\tau w^{-1}}w^{-1}} J_P^1 \subset \mathcal{U}w \cap J_P^1 \mathcal{U}^\# J_P^1\) that is in \(J_P^1\) by lemma 3.18.

3.6. The group \(\tilde{W}\)

In this paragraph we use a presentation by generators and relations of \(\tilde{W}\) to find a subgroup of \(\text{Aut}_R(V_M)\) isomorphic to a quotient of \(\tilde{W}\).

Remark 3.27. We know that the Iwahori-Hecke algebra (see section 14 of [Vig96]) is a deformation of the \(R\)-algebra \(\mathbb{R}^{[\tilde{W}]}\) and so it is not difficult to show that \(\tilde{W}\) is the group generated by \(s_1, \ldots, s_{m-1}\) and \(\tau_{m-1}\) subject to relations \(s_i s_j = s_j s_i\) for every \(i, j\) such that \(|i - j| > 1\), \(s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\) for every \(i \neq m' - 1\), \(s_i^2 = 1\) for every \(i\), \(\tau_{m'-1} s_i = s_i \tau_{m'-1}\) for every \(i\) \(\neq m' - 1\) and \(\tau_{m'-1} s_i = s_i \tau_{m'-1}\) for every \(i\) \(\neq m' - 1\) and \(\tau_{m'-1} s_i = s_i \tau_{m'-1}\).

Lemma 3.28. Let \(i \in \{1, \ldots, m' - 1\}, \alpha = \alpha_{i+1}, w \in W\) of minimal length in \(w W_{\alpha}\) and \(\Phi \in \mathcal{H}(G, \eta_P)_{J_P^1, J_P^1}\). Then the support of \(\hat{f}_w \Phi \hat{f}_{w^{-1}}\) in \(J_P^1 w T_i w^{-1} J_P^1\) and

\[(\hat{f}_w \Phi \hat{f}_{w^{-1}})(w T_i w^{-1}) = \delta(\delta_0, \delta_0) \hat{f}_w(w) \Phi(\tau_i) \circ \hat{f}_{w^{-1}}(w^{-1}).\]

Proof. Since \(w\) and \(w^{-1}\) normalize \(J_i^1\), by lemma 3.9 the support of \(\hat{f}_w \Phi \hat{f}_{w^{-1}}\) in \(J_P^1 w T_i w^{-1} J_P^1\). We recall that

\[(\hat{f}_w \Phi \hat{f}_{w^{-1}})(w T_i w^{-1}) = \sum_{x \in G \setminus J_P^1} (\hat{f}_w \Phi)(w T_i x) \hat{f}_{w^{-1}}(x^{-1} w^{-1}).\]

By lemma 3.17, the support of \(x \mapsto (\hat{f}_w \Phi)(w T_i x) \hat{f}_{w^{-1}}(x^{-1} w^{-1})\) is in \((J_P^1)^{w_T} J_P^1 \cap (J_P^1)^{w_T} J_P^1 = (J_P^1)^{w_T} J_P^1 \cap (J_P^1 \cap T_i w \cap T_i w^{-1}) J_P^1\). Once \(w\) is of minimal length in \(w W_{\alpha}\), by lemma 3.21 we have \(J_P^1 \cap T_i w \cap T_i w^{-1} = J_P^1 \cap T_i w \cap \mathcal{U}^\#\) which is included in \((J_P^1)^{w_T} w_T\) because \((J_P^1 \cap T_i w \cap \mathcal{U}^\#)^{w_{\tau w^{-1}}w^{-1}} = ((J_P^1 \cap \mathcal{U}^\#)^{w_{\tau w^{-1}}w^{-1}} \cap \mathcal{U}^\#)^{w_{\tau w^{-1}}w^{-1}}\) by lemma 3.15 is included in \((H^1 \cap \mathcal{U}^\#)^{w_{\tau w^{-1}}w^{-1}} \cap \mathcal{U}^\#\) and so in \(J_P^1\).

Hence, we obtain \((J_P^1)^{w_T} J_P^1 \cap (J_P^1)^{w_T} J_P^1 = (J_P^1 \cap T_i w \cap T_i w^{-1}) J_P^1\). Now since \((J_P^1 \cap T_i w \cap T_i w^{-1}) J_P^1\) is contained in \(J_P^1 \cap T_i w \cap \mathcal{U}^\#\) and so in the kernel of \(\eta_P\) and since we have \([J_P^1 \cap T_i w \cap T_i w^{-1}] J_P^1 = [J_P^1 \cap T_i w \cap T_i w^{-1}] H^1 \cap T_i w \cap T_i w^{-1} = 0\) we obtain \((\hat{f}_w \Phi \hat{f}_{w^{-1}})(w T_i w^{-1}) = \delta(\delta_0, \delta_0) \hat{f}_w(w) \Phi(\tau_i) \circ \hat{f}_{w^{-1}}(w^{-1})\).

To conclude we observe that by lemma 3.9 the support of \(\hat{f}_w \Phi\) is contained in \(J_P^1 w T_i w^{-1} J_P^1\) and by lemmas 3.16 and 3.17 the support of \(x \mapsto (\hat{f}_w(w x) \Phi(x^{-1} \tau_i)\) is in \((J_P^1)^{w_T} J_P^1 \cap (J_P^1)^{w_{\tau w^{-1}}w^{-1}} J_P^1 = (J_P^1 \cap T_i w \cap T_i w^{-1}) J_P^1 \cap (J_P^1 \cap T_i w \cap T_i w^{-1})^ {w_{\tau w^{-1}}w^{-1}} J_P^1\) that is included in \((T_i J_P^1 \cap T_i w) J_P^1 = J_P^1\) by lemma 3.18. Hence, \((\hat{f}_w \Phi)(w T_i = \hat{f}_w(w) \Phi(\tau_i).\)
Lemma 3.29. Let \( w \in W \) and \( \alpha \in \Sigma \). Then
\[
p \circ \kappa(w) \circ \iota \circ p \circ \kappa(s_\alpha) \circ \iota = \begin{cases} p \circ \kappa(ws_\alpha) \circ \iota & \text{if } \omega \alpha > 0 \\
\delta(\hat{\delta}(1), \hat{\delta}(1))^{-1}p \circ \kappa(ws_\alpha) \circ \iota & \text{if } \omega \alpha < 0. \end{cases}
\]

Proof. By lemma 3.11 we have \( \hat{f}_w \hat{f}_{s_\alpha} = \hat{f}_{ws_\alpha} \) and then \( (\hat{f}_w \hat{f}_{s_\alpha})(ws_\alpha) = p \circ \kappa(ws_\alpha) \circ \iota \). On the other hand we have
\[
(\hat{f}_w \hat{f}_{s_\alpha})(ws_\alpha) = \sum_{x \in G/J^p} (\hat{f}_w)(wx) \hat{f}_{s_\alpha}(x^{-1} s_\alpha).
\]
By lemma 3.17 the support of \( x \mapsto \hat{f}_w(wx) \hat{f}_{s_\alpha}(x^{-1} s_\alpha) \) is contained in \( (J^p)^w J^p \cap (J^p)^{s_\alpha} J^p = (J^p)^w J^p \cap (J^1 \cap U \cap U^{-1}) J^p = (J^p)^w J^p \cap (J^1 \cap U \cap U^{-1}) J^p \) which is equal to \( J^p \) if \( w(\alpha) < 0 \) and to \( (J^1 \cap U^{-1}) J^p \) if \( \omega(\alpha) > 0 \). Hence, if \( \omega(\alpha) > 0 \) we obtain \( (\hat{f}_w \hat{f}_{s_\alpha})(ws_\alpha) = p \circ \kappa(ws_\alpha) \circ \iota \) while if \( \omega(\alpha) < 0 \) since \( (J^1 \cap U^{-1}) J^p \) and \( (J^1 \cap U^{-1})^{s_\alpha} J^p \) are contained in \( J^1 \cap U \) and so in the kernel of \( \eta_p \) and since \( [(J^1 \cap U^{-1}) J^p : J^p] = [J^1 \cap U^{-1} : H^1 \cap U^{-1}] = \delta(\hat{\delta}(1), \hat{\delta}(1)) \) we obtain \( (\hat{f}_w \hat{f}_{s_\alpha})(ws_\alpha) = \delta(\hat{\delta}(1), \hat{\delta}(1)) p \circ \kappa(w) \circ \iota \circ \kappa(s_\alpha) \circ \iota. \)

From now on we fix a non-zero element \( \gamma \in I_{\tau_{m'-1}}(\eta_p) \) which is invertible by lemma 3.10 and we consider the function \( \hat{f}_{\tau_{m'-1}} \in \mathcal{M}(G, \eta_p)J^1_{\tau_{m'-1}J^p} \) defined by \( \hat{f}_{\tau_{m'-1}}(j_1 \tau_{m'-1} j_2) = \eta_p(j_1) \circ \gamma \circ \eta_p(j_2) \) for every \( j_1, j_2 \in J^p \).

From now on we fix a square root \( \delta(\hat{\delta}(1), \hat{\delta}(1))^{1/2} \) of \( \delta(\hat{\delta}(1), \hat{\delta}(1)) \) in \( R \). We consider the subgroup \( \hat{W} \) of \( \text{Aut}(V_M) \) generated by \( \gamma \) and by \( \delta(\hat{\delta}(1), \hat{\delta}(1))^{1/2} p \circ \kappa(s_i) \circ \iota \) with \( i \in \{1, \ldots, m'-1\} \).

Lemma 3.30. The function that maps \( s_i \) to \( \delta(\hat{\delta}(1), \hat{\delta}(1))^{1/2} p \circ \kappa(s_i) \circ \iota \) for every \( i \in \{1, \ldots, m'-1\} \) and \( \tau_{m'-1} \) to \( \gamma \) extends to a surjective group homomorphism \( \varepsilon : \hat{W} \to \hat{W} \).

Proof. Let \( \delta(\hat{\delta}(1), \hat{\delta}(1)) \). To prove that \( \varepsilon \) is a group homomorphism we use the presentation of \( \hat{W} \) given in remark 3.27. For every \( i, j \in \{1, \ldots, m'-1\} \) such that \( |i - j| > 1 \) we have \( \varepsilon(s_i) \varepsilon(s_j) = \delta \circ p \circ \kappa(s_i) \circ \iota \circ \kappa(s_j) \circ \iota \). By lemma 3.29 \( \varepsilon(s_i) \varepsilon(s_j) = \delta \circ \kappa(s_i) \circ \iota \circ \kappa(s_j) \circ \iota \). For every \( i \neq m'-1 \) we have \( \varepsilon(s_i) \varepsilon(s_{i+1}) \varepsilon(s_i) = \delta^{1/2} \circ \kappa(s_i) \circ \iota \circ \kappa(s_{i+1}) \circ \iota \circ \kappa(s_i) \circ \iota \). By lemma 3.29 \( \varepsilon(s_i) \varepsilon(s_{i+1}) \varepsilon(s_i) = \delta^{1/2} \circ \kappa(s_i) \circ \iota \circ \kappa(s_{i+1}) \circ \iota \). For every \( i \) we have \( \varepsilon(s_i)^2 = \delta \circ p \circ \kappa(s_i) \circ \iota \circ \kappa(s_i) \circ \iota \) that by lemma 3.29 is equal to \( p \circ \kappa(s_i) \circ \iota \) that is the identity of \( \text{Aut}(V_M) \). Let \( \tau = \tau_{m'-1} \) and \( \hat{f}_{\tau_{m'-1}} = \hat{f}_{\tau_{m'-1}} \). For every \( i \neq m'-1 \) we have \( \varepsilon(\tau) \varepsilon(s_i) = \delta^{1/2} \circ \kappa(s_i) \circ \iota \) that is equal to \( \delta^{1/2} \circ \kappa(s_i) \circ \iota \). Hence, by lemma 3.9 we have \( \varepsilon(\tau) \varepsilon(s_i) = \delta^{1/2} \circ p \circ \kappa(s_i) \circ \iota \circ \kappa(s_i) \circ \iota \). Since \( \varepsilon(\tau) \varepsilon(s_i) = \delta^{1/2} \circ p \circ \kappa(s_i) \circ \iota \circ \kappa(s_i) \circ \iota \) is equal to \( \delta^{1/2} \circ p \circ \kappa(s_i) \circ \iota \circ \gamma \circ \varepsilon(s_i) \circ \iota \) since the support of \( x \mapsto \hat{f}_s(x) \hat{f}_s(x^{-1} s_\alpha) \) is contained in \( (J^p)^w J^p \cap (J^p)^{s_\alpha} J^p = (J^p)^w J^p \cap (J^1 \cap U \cap U^{-1}) J^p \) which is equal to \( J^p \) if \( w(\alpha) < 0 \) and to \( (J^1 \cap U^{-1}) J^p \) if \( \omega(\alpha) > 0 \). Hence, if \( \omega(\alpha) > 0 \) we obtain \( (\hat{f}_w \hat{f}_{s_\alpha})(ws_\alpha) = p \circ \kappa(ws_\alpha) \circ \iota \) while if \( \omega(\alpha) < 0 \) since \( J^1 \cap U \cap U^{-1} \) \( J^p \) and \( J^1 \cap U \cap U^{-1} \) \( J^p \) are contained in \( J^1 \cap U \) and so in the kernel of \( \eta_p \) and since \( [(J^1 \cap U^{-1}) J^p : J^p] = [J^1 \cap U^{-1} : H^1 \cap U^{-1}] = \delta(\hat{\delta}(1), \hat{\delta}(1)) \) we obtain \( (\hat{f}_w \hat{f}_{s_\alpha})(ws_\alpha) = \delta(\hat{\delta}(1), \hat{\delta}(1)) p \circ \kappa(w) \circ \iota \circ \kappa(s_\alpha) \circ \iota. \)

The support of \( x \mapsto \hat{f}_w(x) \hat{f}_w(x^{-1} s_\alpha) \) is in \( (H^1 \cap U \cap U^{-1}) J^p \) with \( \alpha' = \alpha_{m', m'-1} \) by lemma 3.26. For every \( x \in (H^1 \cap U \cap U^{-1}) \) the elements \( x^{-1} \) and \( (x^{-1}) s_\alpha \) are in \( H^1 \cap U \) and so in the kernel
of $\eta_p$. Then $(\hat{f}_{\tau}, \hat{f}_{s}, \hat{f}_{\tau}, \hat{f}_{s})(\tau_{m'-2}) = (\hat{f}_{s}, \hat{f}_{\tau}, \hat{f}_{s}, \hat{f}_{\tau})(\tau_{m'-2})$ is equal to $\delta(\delta_0^1, \varpi \delta_0^1) \gamma \circ (\hat{f}_{s}, \hat{f}_{\tau}, \hat{f}_{s}, \hat{f}_{\tau})(s\tau s)$ and by lemma 3.28 it is also equal to $\delta(\delta_0^1, \varpi \delta_0^1) \varepsilon(\varepsilon)(s)(\varepsilon)(s)(\varepsilon)(s)$.

Now, if $\alpha'' = \alpha_{m'-2, m'-1}$ then $\alpha' \notin \Psi^+_{\alpha'} \cup \Psi^-_{\alpha'}$ and we have $(J_p^1)^sJ_p^1 \cap (J_p^1)^{m'-2}J_p^1 = J_p^1 = (J_p^1)^{m'-2}J_p^1 \cap (J_p^1)^sJ_p^1$.

Hence, $(\hat{f}_{s}, \hat{f}_{\tau}, \hat{f}_{s}, \hat{f}_{\tau})(s\tau s)$ is equal both to

$$\hat{f}_s(s)(\tau_{m'-2}) = \delta(\delta_0^1, \varpi \delta_0^1) \delta(\delta_0^1, \varpi \delta_0^1)^{-1/2} \varepsilon(\varepsilon)(s)(\varepsilon)(s)(\varepsilon)(s)$$

and also to

$$\hat{f}_s(s)(\tau_{m'-2}) = \delta(\delta_0^1, \varpi \delta_0^1) \delta(\delta_0^1, \varpi \delta_0^1)^{-1/2} \varepsilon(\varepsilon)(s)(\varepsilon)(s)(\varepsilon)(s)^2$$

This implies $\varepsilon(\varepsilon)(s)(\varepsilon)(s)(\varepsilon)(s) = (\varepsilon)(s)(\varepsilon)(s)(\varepsilon)(s)$ since both $\delta(\delta_0^1, \varpi \delta_0^1)$ and $\delta^{-1/2}$ are invertible in $R$. We conclude that $\varepsilon$ is a group homomorphism and it is clearly surjective.

Remark 3.31. For every $w \in W$ we have $\varepsilon(w) = \delta(\delta_0^1, \varpi \delta_0^1)^{((w)/2)} \circ \kappa(w) \circ \iota$.

Lemma 3.32. For every $\tilde{w} \in \tilde{W}$ we have $\varepsilon(\tilde{w}) \in I_{\tilde{w}}(\eta_p)$.

Proof. Since $\eta_{\tilde{w}}$ is the restriction of $\eta$ to the group $J_{\tilde{w}}^1$, we have $\varepsilon(w) = \delta(\delta_0^1, \varpi \delta_0^1)^{((w)/2)} \hat{f}_w(w) \in I_w(\eta_{\tilde{w}})$ for every $w \in W$ and $\varepsilon(\tilde{w}) \in I_{\tilde{w}}(\eta_{\tilde{w}})$. Then, since every $w \in W$ and $\tau_{m'-1}$ normalize $J_{\tilde{w}}$, we have $\varepsilon(\tilde{w}) \in I_{\tilde{w}}(\eta_{\tilde{w}})$ for every $\tilde{w} \in \tilde{W}$ and so $\varepsilon(\tilde{w}) \in I_{\tilde{w}}(\eta_p)$ by lemma 3.30.

Lemma 3.33. For every $\tau', \tau'' \in \Delta$, $\gamma' \in I_{\tau'}(\eta_p)$ and $\gamma'' \in I_{\tau''}(\eta_p)$ we have $\gamma' \circ \gamma'' = \gamma'' \circ \gamma'$.

Proof. We recall that $I_{\tau}(\eta_p)$ is 1-dimensional for every $\tau \in \Delta$ and so there exist $c', c'' \in R$ such that $\gamma' = c' \varepsilon(\tau')$ and $\gamma'' = c'' \varepsilon(\tau'')$. We obtain $\gamma' \circ \gamma'' = c' c'' \varepsilon(\tau') \circ \varepsilon(\tau'') = c' c'' \varepsilon(\tau' \tau'') = c' c'' \varepsilon(\tau'' \tau') = \gamma'' \circ \gamma'$.

3.7. The isomorphisms $\mathcal{H}_R(G, \eta_p) \cong \mathcal{H}_R(B^\times, K_B^1)$

In this paragraph we define elements $\tilde{f}_{\tau_i} \in \mathcal{H}_R(G, \eta_p)_{J_p^1, \tau_i, J_p^1}$ for every $i \in \{0, \ldots, m'-1\}$ and we prove that $\hat{f}_\omega$ with $\omega \in \Omega$ respect relations of lemma 3.34 obtaining an algebra homomorphism from $\mathcal{H}_R(B^\times, K_B^1)$ to $\mathcal{H}_R(G, \eta_p)$.

For every $i \in \{0, \ldots, m'-1\}$ we denote $\gamma_i = \partial(m'-i)(m'-i-1)/2 \varepsilon(\tau_i)$ where $\partial$ is the power of $p$ defined in remark 3.25. Then $\gamma_i$ is an invertible element in $I_{\tau_i}(\eta_p)$ and $\gamma_{m'-1} = 1$.

Lemma 3.34. We have $\gamma_{i-1} \circ \gamma_i = \partial^{m'-i-1} \varepsilon(\tau_{i-1} \tau_i^{-1})$ and $\gamma_i = \prod_{h=0}^{m'-i-1} \partial^{m'-h} \varepsilon(\tau_i)$ for every $i \in \{1, \ldots, m'-1\}$.

Proof. Since $((m' - (i - 1))(m' - (i - 1) - 1) - (m' - i)(m' - i - 1))/2 = m' - i$ we have $\gamma_{i-1} \circ \gamma_i = \partial^{m'-i-1} \varepsilon(\tau_{i-1} \tau_i^{-1}) = \partial^{m'-i} \varepsilon(\tau_{i-1} \tau_i^{-1})$. The second statement is true because $\sum_{h=0}^{m'-i-1} j = (m'-i)(m'-i-1)/2$.

For every $i \in \{0, \ldots, m'-1\}$ we consider the function $\tilde{f}_{\tau_i} \in \mathcal{H}_R(G, \eta_p)_{J_p^1, \tau_i, J_p^1}$ defined by $\tilde{f}_{\tau_i}(j_1 \tau_i j_2) = \eta_p(j_1) \circ \gamma_i \circ \eta_p(j_2)$ for every $j_1, j_2 \in J_p^1$. We remark that in general $\tilde{f}_{\tau_i}$ is not invertible but since $\tau_0$ normalizes $J_p^1$ the function $\tilde{f}_{\tau_0}$ is invertible in $\mathcal{H}_R(G, \eta_p)$ with inverse $\tilde{f}_{\tau_0}^{-1} : \tau_0^1 J_p^1 \to \text{End}_R(V_M)$ defined by $\tilde{f}_{\tau_0}(\tau_0^{-1} j) = \gamma_0^{-1} \circ \eta_p(j)$ for every $j \in J_p^1$.

Lemma 3.35. The map $\Theta'' : \Omega \to \mathcal{H}_R(G, \eta_p)$ given by $f_\omega \mapsto \hat{f}_\omega$ for every $f_\omega \in \Omega$ is well-defined.
Proof. The map is well-defined on \( f_k \) with \( k \in K_B \) because \( \Theta' \) is a homomorphism and it is well-defined on \( \tau_i \) with \( i \in \{0, \ldots, m' - 1\} \) because \( K_B^1 \cap K_B^1 = K_B^1 \cap K_B^1 \) implies \( i = j \).

**Lemma 3.36.** For every \( i, j \in \{0, \ldots, m' - 1\} \) the function \( \hat{f}_{\tau_i} \hat{f}_{\tau_j} \) is in \( \mathfrak{K}(G, \eta_P^1) \) and \( \hat{f}_{\tau_i} \hat{f}_{\tau_j}(\tau_i \tau_j) = \gamma_i \circ \gamma_j \).

Proof. If \( i \) or \( j \) is 0 then it follows by lemma 3.9 since \( \tau_0 \) normalizes \( J_P^1 \). Otherwise, by lemma 3.19 the support of \( \hat{f}_{\tau_i} \hat{f}_{\tau_j} \) is in \( J_P^1 \cap J_P^1 \) and the support of \( x \mapsto \hat{f}_{\tau_i}(\tau x) \hat{f}_{\tau_j}(x^{-1} \tau_j) \) is in \( (J_P^1)^x \cap (J_P^1)^{-1} \). We obtain \( \hat{f}_{\tau_i} \hat{f}_{\tau_j}(\tau_i \tau_j) = \sum_{x \in G/J_P^1} \hat{f}_{\tau_i}(\tau x) \hat{f}_{\tau_j}(x^{-1} \tau_j) = \hat{f}_{\tau_i}(\tau_i) \circ \hat{f}_{\tau_j}(\tau_j) = \gamma_i \circ \gamma_j \).

By lemmas 3.36 and 3.33 we obtain \( \hat{f}_{\tau_i} \hat{f}_{\tau_j} = \hat{f}_{\tau_i} \hat{f}_{\tau_j} \) for every \( i, j \in \{0, \ldots, m' - 1\} \). So, if \( P \subset \{0, \ldots, m' - 1\} \) we denote by \( \gamma_P \) the composition of \( \gamma_i \) with \( i \in P \), which is well-defined by lemma 3.33, and by \( \hat{f}_{\tau_P} \) the product of \( \hat{f}_{\tau_i} \) with \( i \in P \), which is well-defined because the \( \hat{f}_{\tau_i} \) commute. Furthermore, by lemma 3.19 we obtain that the support of \( \hat{f}_{\tau_P} \) is \( J_P^1 \tau_P J_P^1 \) and by lemma 3.36 we have \( \hat{f}_{\tau_P}(\tau_P) = \gamma_P \).

**Lemma 3.37.** We have \( \hat{f}_{\tau_i} \hat{f}_{\tau_x} = \hat{f}_{\tau_x \tau_i}^{-1} \hat{f}_{\tau_i} \) for every \( i \in \{0, \ldots, m' - 1\} \) and every \( x \in M_{\alpha_i, \alpha_{i+1}} = K_B \cap M_{\alpha_i, \alpha_{i+1}} \) if \( i \neq 0 \) or \( x \in K_B \) if \( i = 0 \).

Proof. Since \( x \) normalizes \( J^1 \) by lemma 3.9 the supports of \( \hat{f}_{\tau_x} \hat{f}_{\tau_i} \) and \( \hat{f}_{\tau_i} \tau_x \hat{f}_{\tau_x} \) are in \( J_P^1 \tau_x J_P^1 \) and \( (\hat{f}_{\tau_x} \hat{f}_{\tau_i})(\tau x) = \hat{f}_{\tau_i}(\tau x) \circ \kappa(x) \circ \iota \) is equal to \( \hat{f}_{\tau_x} \hat{f}_{\tau_i}(\tau x) \circ \kappa(x) \circ \iota = (\hat{f}_{\tau_i} \tau_x \hat{f}_{\tau_x} \tau_i)(\tau x) \) because \( \kappa(\hat{f}_{\tau_i}(\tau_i)) \in I_{\tau_i}(\kappa) \) and \( x \in J \cap J^1 \).

**Lemma 3.38.** Let \( i \in \{1, \ldots, m' - 1\} \) and \( \alpha \in \Psi_{\alpha_{i-1}}^+ \). Then for every \( u \in U_\alpha \) and \( u' \in U_{-\alpha} \) we have \( \hat{f}_u \hat{f}_{\tau_i} = \hat{f}_{\tau_i} \hat{f}_{\tau_i} u' = \hat{f}_{\tau_i} \).

Proof. The elements \( \tau_i^{-1} u \tau_i \) and \( \tau_i u' \tau_i^{-1} \) are in \( K_B^1 \subset K_B^1 \) and so, since \( u \) and \( u' \) normalize \( J^1 \), by lemma 3.9 the supports of \( \hat{f}_u \hat{f}_{\tau_i} \) and \( \hat{f}_{\tau_i} \hat{f}_{u'} \) are in \( J_P^1 u \tau_i J_P^1 = J_P^1 \tau_i u J_P^1 = J_P^1 \tau_i u' J_P^1 = J_P^1 \). Now since \( \zeta(\hat{f}_{\tau_i})(\tau_i) \in I_\tau(\kappa) = I_{\tau_i}(\kappa) \) and \( u \in J' \cap J' \), by lemma 3.9 we have \( (\hat{f}_u \hat{f}_{\tau_i})(u \tau_i) = \hat{f}_{\tau_i}(\tau_i) \circ \eta(\tau_i^{-1} u \tau_i) \circ i = \hat{f}_{\tau_i}(\tau_i) \circ \eta(\tau_i^{-1} u \tau_i) \circ i \). By lemma 3.7 we obtain \( (\hat{f}_u \hat{f}_{\tau_i})(u \tau_i) = \hat{f}_{\tau_i}(\tau_i) \circ \eta(\tau_i^{-1} u \tau_i) \circ i = \hat{f}_{\tau_i}(\tau_i) \circ \eta(\tau_i^{-1} u \tau_i) \circ i = \eta(\tau_i^{-1} u \tau_i) \circ i \). Similarly we have \( \hat{f}_{\tau_i}(u' \tau_i) = \hat{f}_{\tau_i}(\tau_i) \circ \eta(\tau_i u' \tau_i) \circ i = \hat{f}_{\tau_i}(\tau_i) \circ \eta(\tau_i u' \tau_i) \circ i = \eta(\tau_i u' \tau_i) \circ i \).

We introduce some subgroup of \( G \) through its identification with \( GL_m'(A(E)) \) in order to find the support of \( \hat{f}_{\tau_P} \hat{f}_u \hat{f}_{\tau_r} \hat{f}_{w^{-1}} \). We recall that \( A(E) \) is the unique hereditary order normalized by \( E^\times \) in \( A(E) \) and \( \Psi(E) \) is its radical.

- Let \( Z \) be the set of matrices \( (z_{ij}) \) such that \( z_{ii} = 1, z_{ij} \in \omega^{-1} \Psi(E) \) if \( i < j \) and \( z_{ij} = 0 \) if \( i > j \).
- Let \( V \) be the group \( (J^1 \cap wU^+_\alpha w^{-1} \cap U^{-1}_\beta)^{w_\alpha w^{-1}} = J^1 \cap wU^+_\beta \cap \phi^+_\alpha \subset Z \).

We remark that it is different from \( V(w, \alpha) \) defined by (3.2).

- Let \( \tilde{H}_1 \) be the group of matrices \( (m_{ij}) \) such that \( m_{ii} \in 1 + \Psi(E), m_{ij} \in A(E) \) if \( i < j \) and \( m_{ij} \in \Phi(E) \) if \( i > j \).
Let $W = W \ltimes M$ be the subgroup of $B^\times$ of monomial matrices with coefficients in $O_D^\times$. Then $B^\times$ is the disjoint union of $I_B(1)wI_B(1)$ with $w \in W$, where $I_B(1) = K^1U$ is the pro-$p$-Iwahori subgroup of $K_B$, i.e. the pro-$p$-radical of $I_B$.

Lemma 3.39. We have $J^1_{\tau P} \tau P_1 \tau w_1 w^{-1} J^1_{\tau P} = J^1_{\tau Q} \mathcal{V} J^1_{\tau P}$.

Proof. We proceed similarly to the beginning of proof of lemma 3.26: we can prove that $J^1_{\tau P} \tau w_1 w^{-1} J^1_{\tau P} = (J^1_{\tau P} \cap wU_q w^{-1}) \tau w_1 w^{-1} J^1_{\tau P}$. Now we consider the decomposition of the group $(J^1_{\tau P} \cap wU_q w^{-1} \cap U^{-})(J^1_{\tau P} \cap wU_q w^{-1} \cap U)$. By lemma 3.15 we have $(J^1_{\tau P} \cap wU_q w^{-1} \cap U^{-})^r \subset J^1_{\tau P}$ and by lemma 3.21 we have $J^1_{\tau P} \cap wU_q w^{-1} \cap U = J^1_{\tau P} \cap wU_q w^{-1} \cap U^+_P$. □

Lemma 3.40. Let $\tau \in \Delta$. If $z \in Z$ is such that $I^1_{\tau} \tau z \cap W \neq \emptyset$ then $I^1_{\tau} \tau z \cap W = \{\tau\}$.

Proof. For every $r \in \{1, \ldots, m^r\}$ we denote $\Delta_r, Z_r, \tilde{I}_r$ and $W_r$ the subsets of $GL_r(A(E))$ similar to those defined for $GL_{m^r}(A(E))$. We prove the statement of the lemma by induction on $r$. If $r = 1$ we have $\Delta_1 = \omega^2, Z_1 = \{1\}, \tilde{I}_1 = 1 + \Psi(E)$ and $W_1 = \omega^2$ and we have $(1 + \Psi(E)) \omega^a (1 + \Psi(E)) \cap \omega^a = \omega^a (1 + \Psi(E)) \cap \omega^a = \{\omega^a\}$ for every $a \in \mathbb{Z}$. Now we suppose the statement true for every $r < m^r$. Let $x, y \in \tilde{I}_r$ such that $x \tau y \in W$. We proceed by steps. First step. We consider the decomposition $\tilde{I}_1 = (\tilde{I}_1 \cap U^-)(\tilde{I}_1 \cap U)(\tilde{I}_1 \cap M)$ and we write $x = x_1 x_2 x_3$ with $x_1 \in \tilde{I}_1 \cap U^-, x_2 \in \tilde{I}_1 \cap U$ and $x_3 \in \tilde{I}_1 \cap M$. Then we have $x \tau y = x_1 \tau ((\tau^{-1} x_2 \tau)(\tau^{-1} x_3 \tau) z(\tau^{-1} x_3 \tau))((\tau^{-1} x_3 \tau)y).$

We observe that $\tau^{-1} x_3 \tau$ is a diagonal matrix with coefficients in $1 + \Psi(E)$ and the conjugate of $z$ by this element is in $Z$. Moreover, $\tau^{-1} x_2 \tau$ is in $\tilde{I}_1 \cap U$ and if we multiply it by an element of $Z$ we obtain another element of $Z$. If we set $z_1 = \tau^{-1} x_2 x_3 \tau z^{-1} x_3^{-1} \tau \in Z$ then $I^1_{\tau} \tau z \cap \tilde{I}_1 = I^1_{\tau} \tau z_1 \cap \tilde{I}_1$ and $(\tilde{I}_1 \cap U^-) \tau z_1 \cap \tilde{I}_1 \cap W \neq \emptyset$. Hence, we can suppose $x \in \tilde{I}_1 \cap U^-$. Second step. Let $a_1 \leq \cdots \leq a_{m^r} \in \mathbb{N}$ such that $\tau = \text{diag}(\omega^{a_1})$ and let $s \in \mathbb{N}^*$ such that $a_1 = \cdots = a_s$ and $a_1 < a_{s+1}$. We want to prove $z_{ij} \in \mathfrak{A}(E)$ for every $i \in \{1, \ldots, s\}$ so we assume the opposite and we look for a contradiction. Let $v$ be the valuation on $A(E)$ associated to $\mathfrak{P}(E)$ and let

$$b = \min \{v(\omega^{a_i} z_{ij}) \mid 1 \leq i \leq s, 1 \leq j \leq m^r\}$$

$$k = \min \{1 \leq j \leq m^r \mid \text{there exists } z_{ij} \text{ with } 1 \leq i \leq s \text{ such that } v(\omega^{a_i} z_{ij}) = b\}.$$

Let $1 \leq h \leq s$ be such that $v(\omega^{a_i} z_{hh}) = b$. By hypothesis the element $z_{hh}$ is not in $\mathfrak{A}(E)$ and so $h < k$ and

$$(a_1 - 1)v(\omega) < b < a_1 v(\omega). \quad (3.3)$$

We observe that for every $i \in \{1, \ldots, m^r\}$ and $j > i$ we have $v(\omega^{a_i} z_{ij}) \geq b$: if $i \leq s$ by definition of $b$ and if $i > s$ because $v(\omega^{a_i} z_{ij}) = a_1 v(\omega) + v(z_{ij}) > (a_1 - 1)v(\omega) \geq a_1 v(\omega) > b$. We consider the coefficient at position $(h, k)$ of $x \tau y$ which is equal to

$$\sum_{e=1}^{m^r} \sum_{f=1}^{m^r} x_{he} \omega^{a_e} z_{ef} y_{fk} = \sum_{e=1}^{m^r} \sum_{f=e}^{m^r} x_{he} \omega^{a_e} z_{ef} y_{fk},$$

since $x_{he} = 0$ if $e > h$ and $z_{ef} = 0$ if $f < e$. Now,

- if $e = h$ and $f = k$ then $v(x_{hh} \omega^{a_h} z_{hh} y_{kk}) = b$ because $x_{hh} = 1$, and $y_{kk} \in 1 + \mathfrak{P}(E)$;
- if $e = h$ and $f < k$ then $v(x_{hh} \omega^{a_h} z_{hf} y_{fk}) > b$ by definition of $k$;
We obtain an element of valuation \( b \). Then \( b \) must be a multiple of \( \nu(\varpi) \) because \( x\tau z y \in \mathbf{W} \) but this in contradiction with (3.3). Hence, \( z_{ij} \in \mathfrak{A}(E) \) for every \( i \in \{1, \ldots, s\} \). Now, we can write \( z = z'z'' \) with \( z'_{ii} = 1, z'_{ij} = z_{ij} \) if \( i \in \{s + 1, \ldots, m'\} \) and \( j > i \) and \( z''_{ij} = 0 \) otherwise and \( z''_{ii} = 1, z''_{ij} = z_{ij} \) if \( i \in \{1, \ldots, s\} \) and \( j > i \) and \( z''_{ii} = 0 \) otherwise. Then \( z'' \in 1^I \) and so \( 1^I \tau z'1^I = 1^I \tau z''1^I \) and \((1^I \cap U^-)\tau z'1^I \cap \mathbf{W} \neq \emptyset \). Then we can suppose \( z \) of the form \( (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) \) with \( \hat{z} \in Z(m'_{m-s}). \)

Third step. We write \( x = x'x'' \) with \( x'_{ii} = 1, x'_{ij} = x_{ij} \) if \( i \in \{s + 1, \ldots, m'\} \) and \( j < i \) and \( x'_{ij} = 0 \) otherwise and \( x''_{ii} = 1, x''_{ij} = x_{ij} \) if \( i \in \{1, \ldots, s\} \) and \( j < i \) and \( x''_{ij} = 0 \) otherwise. Then \( \tau^{-1}x'' \tau \in 1^I \) and it commutes with \( z \). Then we can suppose \( x \) of the form \( (\begin{smallmatrix} \frac{I^I_{m-s}}{1} & 0 \\ 0 & x'' \end{smallmatrix}) \) with \( x'' \in M(m'_{m-s})_{\times s}(\mathbf{P}(E)) \) and \( x'' \in 1^I(m'_{m-s}) \).

Fourth step. Let \( \tau = (\begin{smallmatrix} \varpi_{m-s} & 0 \\ 0 & \hat{\tau} \end{smallmatrix}) \) with \( \hat{\tau} \in \Delta(m'_{m-s}) \) and \( y = (\begin{smallmatrix} y_1 & y_2 \\ y_3 & y \end{smallmatrix}) \) with \( y_1 \in 1^I(s), y_2 \in M(s_{m-s})(\mathbf{A}(E)), y_3 \in M(1^I)_{\times x}(\mathbf{P}(E)) \) and \( y \in 1^I(m'_{m-s}) \). Then the product \( x\tau z y \) is

\[
\begin{pmatrix} I_s & 0 \\ x'' & \hat{\tau} \end{pmatrix} \begin{pmatrix} \varpi_{m-s} & 0 \\ 0 & \hat{\tau} \end{pmatrix} \begin{pmatrix} I_s & 0 \\ 0 & \hat{\tau} \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y \end{pmatrix} = \begin{pmatrix} \varpi_{m-s} & \varpi_{m-s} \\ 1 & \hat{\tau} \end{pmatrix} \begin{pmatrix} \varpi_{m-s} & \varpi_{m-s} \\ 1 & \hat{\tau} \end{pmatrix} \begin{pmatrix} x'' & y_2 \end{pmatrix} \begin{pmatrix} \varpi_{m-s} & \varpi_{m-s} \\ 1 & \hat{\tau} \end{pmatrix} \end{pmatrix}
\]

where \( t = x'' \varpi_{m-s} y_1 + \hat{\tau} \varpi_{m-s} y_3 \). Since \( x\tau z y \) is in \( \mathbf{W} \) and since \( y_1 \) is invertible then \( \varpi_{m-s} y_1 \) must be in \( \mathbf{W}(s) \) and so \( \varpi_{m-s} y_3 = t = 0 \). This implies \( y_1 \in \mathbf{I} \) and so \( x\tau z y = (\begin{smallmatrix} \varpi_{m-s} & 0 \\ 0 & \hat{\tau} \end{smallmatrix}) \) with \( \hat{\tau} \varpi_{m-s} \hat{\tau} \in \mathbf{W}(m'_{m-s}) \). Now, since \( 1^I(m'_{m-s}) \hat{\tau} \hat{\tau} 1^I(m'_{m-s}) \cap \mathbf{W}(m'_{m-s}) \neq \emptyset \), by inductive hypothesis we have \( \hat{\tau} \varpi_{m-s} \hat{\tau} \varpi_{m-s} = \tau \) and so \( x\tau z y = \tau \).

**Lemma 3.41.** We have \( J^I_p \tau P J^I_p \varpi w \alpha w^{-1} J^I_p \cap J^I_p B^{\times} J^I_p = J^I_p \tau Q(U \cap w U - w^{-1}) J^I_p \).

**Proof.** By lemma 3.39 we have \( J^I_p \tau P J^I_p \varpi w \alpha w^{-1} J^I_p = J^I_p \tau Q \mathbf{V} J^I_p \). Now, since \( \mathbf{I} \subset M(\mathbf{P}(E)) \) we have \( \mathbf{V} \subset \mathbf{Z} \) and \( J^I_p \subset 1^I \) and so we obtain

\[
J^I_p \tau P J^I_p \varpi w \alpha w^{-1} J^I_p \cap B^{\times} \subset 1^I \tau Q \mathbf{Z} 1^I \cap K^I_p U \mathbf{W} K^I_p = K^I_p U(1^I \tau Q \mathbf{Z} 1^I \cap \mathbf{W}) U K^I_p
\]

(lemma 3.40) \( = K^I_p B^I \tau Q K^I_B = K^I_p B^I \tau Q K^I_B \).

This implies \( J^I_p \tau P J^I_p \varpi w \alpha w^{-1} J^I_p \cap B^{\times} = J^I_p \tau Q \mathbf{V} J^I_p \cap K^I_p \tau Q K^I_B \). Let now \( v \in \mathbf{V} \) be such that \( J^I_p \tau Q \mathbf{V} J^I_p \cap K^I_p \tau Q K^I_B \neq \emptyset \). Then \( v \in \tau Q(1^I_p K^I_p \tau Q K^I_B \cap \mathbf{V}) \subset \tau Q(1^I_p K^I_p \tau Q K^I_B \cap \mathbf{U}) \). Now \( U = K^I_p \cap \mathbf{U} \subset J \cap P \) normalizes \( J^I_p \) and so \( v \in \tau Q^{-1}(1^I_p \tau Q J^I_p \cap \mathbf{U}) \) which is in \( \tau Q^{-1}(1^I_p \tau \mathbf{U} \cap \mathbf{U}) \). By lemma 3.16. Hence, by lemma 3.18 we obtain \( v \in U^I_p \cap \mathbf{V} \subset U J^I_p \cap \mathbf{V} \). By lemma 3.21 we have \( U \cap w U - w^{-1} = U^I_p \cap w U^I_p \) and proceeding similarly to proof of lemma 3.24 we can prove \( U^I_p \cap w U^I_p \subset \mathbf{V} \). We obtain

\[
U J^I \cap \mathbf{V} = (U \cap w U - w^{-1}) (U \cap w U - w^{-1}) (J^I \cap (U \cap w U - w^{-1}) \cap \mathbf{V}) = (U \cap w U - w^{-1}) (U J^I (U \cap w U - w^{-1}) \cap \mathbf{W}) \cap U^I_p \cap U^I_p \cap \mathbf{W} \subset \mathbf{V}
\]

By definition of \( \mathbf{V} \) we have \( \mathbf{W} = (J^I_p \cap w U^I_p \cap \mathbf{W}^I_p \cap \mathbf{W}^I_p) \cap U^I_p \cap U^I_p \cap \mathbf{W} \cap \mathbf{W} \) and then \( U J^I \cap \mathbf{V} \subset (U \cap w U - w^{-1}) (J^I \cap \mathbf{U} \cap \mathbf{U}) \cap U^I_p \cap U^I_p \cap \mathbf{W} \cap \mathbf{W} \) and then \( U J^I \cap \mathbf{V} \subset (U \cap w U - w^{-1}) (J^I \cap \mathbf{U} \cap \mathbf{U}) \cap U^I_p \cap U^I_p \cap \mathbf{W} \cap \mathbf{W} \). Hence, we have \( J^I_p \tau P J^I_p \varpi w \alpha w^{-1} J^I_p \cap J^I_p B^{\times} J^I_p = J^I_p \tau Q(U \cap w U - w^{-1}) J^I_p \).
Lemma 3.42. For every $u \in U \cap wU^{-1}$ we have

$$(\tilde{f}_p \tilde{f}_w \tilde{f}_{\tau_n} \tilde{f}_{w^{-1}})(\tau_Qu) = q^{\ell(w)}d(w, \alpha)\delta(\tilde{J}_0, \tilde{J}_0)\gamma_P \circ p \circ \kappa(w) \circ t \circ \gamma_i \circ p \circ \kappa(w^{-1}) \circ t \circ p \circ \kappa(u) \circ t.$$

Proof. By lemma 3.41 the support of $\tilde{f}_p \tilde{f}_w \tilde{f}_{\tau_n} \tilde{f}_{w^{-1}}$ is contained in $J_p^1 \cap U \cap U^{-1}$. By lemma 3.21 we have $U \cap wU^{-1} = U_1 \cap wU_1^{-1}$, by lemma 3.38 we have $\tilde{f}_{\tau_n} = \tilde{f}_w \tilde{f}_{w^{-1}}$ and by lemma 3.11 we have $\tilde{f}_{w^{-1}} \tilde{f}_w = \tilde{f}_{w^{-1}} \tilde{f}_w$. Since $u$ is in $U = K_{\beta} \cap U \cap J \cap P$, it normalizes $J_p$ and by lemma 3.9 we obtain $(\tilde{f}_p \tilde{f}_w \tilde{f}_{\tau_n} \tilde{f}_{w^{-1}})(\tau_Qu) = (\tilde{f}_p \tilde{f}_w \tilde{f}_{\tau_n} \tilde{f}_{w^{-1}})(\tau_Qu) \circ p \circ \kappa(u) \circ t$. It remains to calculate

$$(\tilde{f}_p \tilde{f}_w \tilde{f}_{\tau_n} \tilde{f}_{w^{-1}})(\tau_Qu) = \sum_{x \in \tilde{G}/\tilde{J}_p} \tilde{f}_{\tau_p}(\tilde{f}_p \tilde{f}_w \tilde{f}_{\tau_n} \tilde{f}_{w^{-1}})(\tau_Qu)(x^{-1}w\tau_Qu^{-1}).$$

By lemma 3.26 the support of function $x \mapsto (\tilde{f}_p \tilde{f}_w \tilde{f}_{\tau_n} \tilde{f}_{w^{-1}})(x^{-1}w\tau_Qu^{-1})$ is in $V(w, \alpha)J_p$. Now, since for every $x \in V(w, \alpha) \setminus \tau_1 H \tilde{J}_p \tilde{f}_p \tilde{f}_w \tilde{f}_{\tau_n} \tilde{f}_{w^{-1}}(w\tau_Qu^{-1})$ we have $(x^{-1}w\tau_Qu^{-1}) \in J_p^1 \cap \tilde{U}^{-1}$ and $x\tau_1^{-1} \in (J_p^1 \cap \tilde{U}^{-1}) \cap \tilde{J}_p^{-1} \in (J_p^1 \cap \tilde{U}^{-1}) \cap \tilde{J}_p^{-1}$ which is in $J_p^1 \cap \tilde{U}^{-1}$ by lemma 3.15, then $(x^{-1}w\tau_Qu^{-1})$ and $x\tau_1^{-1}$ are in the kernel of $\eta_P$. We obtain

$$(\tilde{f}_p \tilde{f}_w \tilde{f}_{\tau_n} \tilde{f}_{w^{-1}})(\tau_Qu) = [\kappa(w, \alpha) : \kappa(w, \alpha) \setminus H \tilde{J}_p \tilde{f}_p \tilde{f}_w \tilde{f}_{\tau_n} \tilde{f}_{w^{-1}}(w\tau_Qu^{-1})$$

(remark 3.25) $= d(w, \alpha)q^{\ell(w)}\tilde{f}_p(\tau_1 H \tilde{J}_p \tilde{f}_p \tilde{f}_w \tilde{f}_{\tau_n} \tilde{f}_{w^{-1}}(w\tau_Qu^{-1})^{-1})$

(lemma 3.28) $= d(w, \alpha)q^{\ell(w)}\delta(\tilde{J}_0, \tilde{J}_0)\gamma_P \circ p \circ \kappa(w) \circ t \circ \gamma_i \circ p \circ \kappa(w^{-1}) \circ t$

and so the result.

Lemma 3.43. We have $\gamma_Q = d(w, \alpha)\delta(\tilde{J}_0, \tilde{J}_0)\gamma_P \circ p \circ \kappa(w) \circ t \circ \gamma_i \circ p \circ \kappa(w^{-1}) \circ t$.

Proof. By definition of $P(w, \alpha)$ and $Q(w, \alpha)$ (see paragraph 3.4) we have $\tau_{w^{-1}} = \prod_{h=1}^{m'} \tau_{w(h)-1}$ and so

$$\gamma_{P^{-1}} \gamma_Q = \prod_{h=1}^{m'} \gamma_{w(h)}^{-1} \gamma_{w(h)-1}$$

(lemma 3.34) $= \prod_{h=1}^{m'} \partial^{\ell(w)} \varepsilon_{h-1} \varepsilon_{w(h)-1} = \left( \prod_{h=1}^{m'} \partial^{\ell(w)} \right) \varepsilon_{w(h)-1}$

(lemma 3.34) $= \left( \prod_{h=1}^{m'} \partial^{\ell(w)} \right) \left( \prod_{h=1}^{m'} \partial^{\ell(w)} \right) \varepsilon(w) \circ \gamma_i \circ \varepsilon(w^{-1})$

(remark 3.31) $= \left( \prod_{h=1}^{m'} \partial^{\ell(w)} \right) \delta(\tilde{J}_0, \tilde{J}_0) \varepsilon(w) \circ \gamma_i \circ \varepsilon(w^{-1})$

It remains to prove $d(w, \alpha) = \prod_{h=1}^{m'} \partial^{\ell(w)}$. Since by remark 3.25 we have $d(w, \alpha) = \partial^{\ell(w)}$, it is sufficient to prove $\sum_{h=i+1}^{m'} h - w(h) = \ell(w)$. We prove this statement by induction on $\ell(w)$. If $\ell(w) = 1$, since $w$ is of minimal length in $wW_0$, we have $w = s_\alpha = (i, i + 1)$ and $\sum_{h=i+1}^{m'} h - w(h) = i + 1 - w(i+1) + \sum_{h=i+2}^{m'} h - w(h) = i + 1 - i + 1 = 1$. Let now $w$ of
of hypothesis. Moreover, by definition of $\ell(w)$ for every $h \in \{i+1, \ldots, m\}'$ such that $j = w(h)$ and $j + 1 \neq w(h)$ for every $h \in \{i+1, \ldots, m\}'$ and then $w(h) = w'(h')$ for every $h \in \{i+1, \ldots, m\}'$ different from $\hat{h}$. We obtain $\sum_{h \neq \hat{h}} h - w(h) = \sum_{h \neq \hat{h}} h - w(h) + \hat{h} - w'(\hat{h}) + w'(\hat{h}) - w(\hat{h}) = \sum_{h \neq \hat{h}} w'(h) + \hat{h} - w'(\hat{h}) + (s(j) - j = \sum_{h \neq \hat{h}} h - w(h) + j + 1 - \ell(w') + 1 = \ell(w)$. 

Lemma 3.44. We have $\tilde{f}_{\tau p} \tilde{f}_{\tau w} \tilde{f}_{\tau n} \tilde{f}_{\tau -1} = q(\psi(\tau w)) \sum_{\tilde{f}_{\tau w}} u$ where $u$ describes a system of representatives of $(U \cap wU^{-1})K_1/K_1$ in $U \cap wU^{-1}$.

Proof. By lemma 3.41 the support of $\tilde{f}_{\tau p} \tilde{f}_{\tau w} \tilde{f}_{\tau n} \tilde{f}_{\tau -1}$ is contained in $J_{\tau p} \tau Q(U \cap wU^{-1})J_{\tau p}$. For every $u' \in U \cap wU^{-1}$, by lemmas 3.42 and 3.43 we have $(\tilde{f}_{\tau p} \tilde{f}_{\tau w} \tilde{f}_{\tau n} \tilde{f}_{\tau -1})(\tau Q u') = q(\psi(\tau w)) \delta(\tau \psi(\tau w)) \psi(p \circ \kappa(u')) \circ \gamma \circ \psi(p \circ \kappa(u')) \circ \tau = q(\psi(\tau w)) \psi(p \circ \kappa(u')) \circ \tau$. To conclude we observe that $(\tilde{f}_{\tau p} \sum_{\tilde{f}_{\tau w}} u)(\tau Q u') = (\tilde{f}_{\tau p} \tilde{f}_{\tau w}' \tilde{f}')(\tau Q u') = \gamma \circ \psi(p \circ \kappa(u')) \circ \tau$.

Proposition 3.45. The map $\Theta'$ of lemma 3.35 respect relations of lemma 3.14.

Proof. By lemma 3.11 the map $\Theta'$ respects relation 1. By lemma 3.37 it respects relation 3 and $\tilde{f}_{\tau p} \tilde{f}_{\tau w} \tilde{f}_{\tau n} \tilde{f}_{\tau -1}$ for every $k \in K_B$ and by lemmas 3.36 and 3.33 it respects relations 2 and 6. Moreover it respects relations 4 and 5 by lemma 3.38 and relation 7 by lemma 3.44.

Theorem 3.46. For every non-zero $\gamma \in I_{\tau w}^{-1}(\eta)$ and every $\beta$-extension $\kappa$ of $\eta$ there exists an algebra isomorphism $\Theta_{\gamma, \kappa}: \mathcal{H}_R(B^\times, K_B^{\times}) \rightarrow \mathcal{H}^B(G, \eta)$.

Proof. By proposition 3.45 and by lemma 3.8 there exists an algebra homomorphism from $\mathcal{H}_R(B^\times, K_B^{\times})$ to $\mathcal{H}_R(G, \eta)$ which depends on the choice of a $\beta$-extension of $\eta$ and of an element in $I_{\tau w}^{-1}(\eta)$ which is isomorphic to $I_{\tau w}^{-1}(\eta)$ by lemma 3.8. Let $\Xi$ be a set of representatives of $K_B^{\times}$-double cosets of $B^\times$. Then $\{f_x \mid x \in \Xi\}$ is a basis of $\mathcal{H}_R(B^\times, K_B^{\times})$ as $R$-vector space and, since $I_G(\eta) = J^1B^\times J^1$ and $\dim_R(I_B(\eta)) = 1$ for every $y \in I_G(\eta)$, the set $\{\Theta_{\gamma, \kappa}(f_x) \mid x \in \Xi\}$ is a set of generators of $\mathcal{H}_R(G, \eta)$ as $R$-vector space and so $\Theta_{\gamma, \kappa}$ is surjective. Moreover, the set $\{\Theta_{\gamma, \kappa}(f_x) \mid x \in \Xi\}$ is linearly independent and so $\Theta_{\gamma, \kappa}$ is also injective.

Remark 3.47. Let $\kappa$ and $\kappa'$ be two $\beta$-extensions of $\eta$. By paragraph 2.1 there exists a character $\chi$ of $\mathcal{O}_{E}^\times$ trivial on $1 + \mathcal{P} E$ such that $\kappa' = \kappa \otimes (\chi \circ N_B/E)$. If we denote $\tilde{\chi} = \inf_{\mathcal{O}_{E}^\times} \chi \circ N_B/E$, which is a character of $B^\times$, then $\Theta_{\gamma, \kappa}^{-1} \circ \Theta_{\gamma, \kappa'}$ maps $f_x$ to $\tilde{\chi}f_x = \tilde{\chi}(x)f_x$ for every $x \in B^\times$.

4. Semisimple types

Using notations of section 2, in this section we present the construction of semisimple types of $G$ with coefficients in $R$. We refer to sections 2.8-9 of [MS14b] for more details.

Let $r \in \mathbb{N}$ and let $(m_1, \ldots, m_r)$ be a family of strictly positive integers such that $\sum_{i=1}^r m_i = m$. For every $i \in \{1, \ldots, r\}$ we fix a maximal simple type $(J_i, \lambda_i)$ of $GL_{m_i}(D)$ and a simple stratum $[A_i, n_i, 0, \beta_i]$ of $A_i = M_{m_i}(D)$ such that $J_i = J(\beta_i, \Lambda_i)$. Then, the centralizer $B_i$ of $E_i = F[\beta_i]$ in $A_i$ is isomorphic to $M_{m_i}(D_i')$ for a suitable $E_i$-division algebra $D_i'$ of reduced degree $d_i'$ and a suitable $m_i' \in \mathbb{N}^*$. Moreover, $U(\Lambda_i) \cap B_i^\times$ is a maximal compact open subgroup of $B_i^\times$ that we identify with $GL_{m_i}(O_{D_i'})$.

Let $M$ be the standard Levi subgroup of $G$ of block diagonal matrices of sizes $m_1, \ldots, m_r$. The pair $(J_M, \lambda_M)$ with $J_M = \prod_{i=1}^r J_i$ and $\lambda_M = \bigotimes_{i=1}^r \lambda_i$ is called maximal simple type of $M$.

For every $i \in \{1, \ldots, r\}$ we fix a simple character $\theta_i \in \mathcal{O}_R(\Lambda_i, 0, \beta_i)$ contained in $\lambda_i$ and we observe that this choice does not depend on the choice of the $\beta$-extension $\kappa_i$ such that
\[ \lambda_i = \kappa_i \otimes \sigma_i. \] Grouping \( \theta_i \) according their endo-classes, we obtain a partition \( \{1, \ldots, r\} = \bigcup_{j=1}^r I_j \) with \( l \in \mathbb{N}^* \). Up to renumbering the \((J_i, \lambda_i)\) we can suppose that there exist integers \( 0 = a_0 < a_1 < \cdots < a_l = r \) such that we have \( I_j = \{ i \in \mathbb{N} | a_{j-1} < i \leq a_j \} \). For every \( j \in \{1, \ldots, l\} \) we denote \( m^j = \sum_{i \in I_j} m_i \) and \( m^j = \sum_{i \in I_j} m_i \) and we consider the standard Levi subgroup \( L \) of \( G \) containing \( M \) of block diagonal matrices of sizes \( m^1, \ldots, m^l \).

Let \( j \in \{1, \ldots, l\} \). We choose a simple stratum \([\lambda^j, n^j, 0, \beta^j]\) of \( M_{m}^{(D)}(D) \) as in paragraph 2.8 of [MS14b]. If we denote by \( B^j \) the centralizer of \( E^j = F[\beta^j] \) in \( M_{m}^{(D)}(D) \), there exist a \( E^j \)-division algebra \( D^j \) and an isomorphism that identifies \( B^j \) to \( M_{m}^{(D)}(D^j) \) and \( U(\lambda^j) \cap B^{j \times} \) to a standard parabolic subgroup of \( GL_{m^j}(O_{D^j}) \) associated to \( m_i^j \) with \( i \in I_j \). We denote by \( \beta^j \) the transfer of \( \beta_i \) with \( i \in I_j \) to \( \mathcal{E}\mathcal{R}(\lambda^j, 0, \beta^j) \) that does not depend on \( i \) and we fix a \( \beta \)-extension \( \kappa_j \) of \( \theta^j \). In section 2.8 of [MS14b] are defined two compact open subgroups \( J_j \subset J(\beta^j, \lambda^j) \) and \( J^j \subset J^1(\beta^j, \lambda^j) \) of \( G \) such that \( J_j / J^j \cong \prod_{i \in I_j} J_i / J^1_i \), and representations \( \kappa_j \) of \( J_j \) and \( \eta_j \) of \( J^j \) such that \( \text{ind}_{J^1}^{J_1}(\beta_j, \lambda^j) \eta_j \cong \text{res}_{J^1}^{J_1}(\beta_j, \lambda^j) \kappa_j \), \( \text{ind}_{J^1}^{J_1}(\beta_j, \lambda^j) \kappa_j \cong \eta_j \), \( J_j \cap M = \prod_{i \in I_j} J_i \) and \( \text{res}_{J^1}^{J_1}(\beta_j, \lambda^j) \eta_i \cong \text{res}_{J^1}^{J_1}(\beta_j, \lambda^j) \kappa_i \) viewed as a representation of \( J \) trivial on \( J^j \) and we set \( \lambda_j = \kappa_j \otimes \sigma_j \). Then \( (J_j, \lambda_j) \) is a cover of \( (\prod_{i \in I_j} J_i, \otimes_{i \in I_j} \lambda_i) \) (proposition 2.26 of [MS14b]), \( (J_j, \kappa_j) \) is decomposed above \( (\prod_{i \in I_j} J_i, \otimes_{i \in I_j} \kappa_i) \) and \( (J^j, \eta_j) \) is a cover of \( (\prod_{i \in I_j} J_i, \otimes_{i \in I_j} \eta_i) \) (proposition 2.27 of [MS14b]).

We set \( J^1 = \prod_{i=1}^r J_i, \lambda_M = \otimes_{i=1}^r \kappa_i, \eta_M = \otimes_{i=1}^r \eta_i, \lambda_L = \otimes_{i=1}^r \lambda_j, \kappa_L = \otimes_{i=1}^r \kappa_i, \eta_L = \otimes_{i=1}^r \eta_i, \text{ and } \sigma_L = \otimes_{i=1}^r \sigma_j \). By construction \( (J_L, \lambda_L) \) and \( (J^1_L, \eta_L) \) are covers of \( (J_M, \lambda_M) \) and \( (J^1_M, \eta_M) \) respectively and \( (J_L, \kappa_L) \) is decomposed above \( (J^1_M, \kappa_M) \).

Proposition 2.28 of [MS14b] defines a cover \( (J, \lambda) \) of \( (J_L, \lambda_L) \) and so of \( (J_M, \lambda_M) \), that we call semisimple type of \( G \). If the \( (J_i, \lambda_i) \) are maximal simple supertypes, we call \( (J, \lambda) \) semisimple supertype of \( G \). The semisimple type \( (J, \lambda) \) is associated to a stratum \([\lambda, n, 0, \beta]\) of \( A \), not necessarily simple (section 2.9 of [MS14b]). We denote by \( B \) the centralizer of \( \beta \) in \( A \), \( B^\times_L = B^\times \cap L = \prod_{j=1}^l B^\times_j \) and \( J^j = J \cap U_1(A) \). By propositions 2.30 and 2.31 of [MS14b] there exists a unique pair \( (J^1_j, \eta_L) \) decomposed above \( (J^j_j, \eta_L) \) and so above \( (J_M, \eta_M) \). Its intertwining set is \( I_G(\eta) = JB^\times_L J \) and for every \( y \in B^\times_L \) the \( \mathcal{R} \)-vector space \( I_y(\eta) \) is 1-dimensional. We also have the isomorphisms

\[
J / J^1 \cong J_L / J^1_L \cong \prod_{i=1}^r J_i / J^1_i \cong \prod_{i=1}^r GL_{m_i^j}(t_{D^j}).
\]

We can identify \( \sigma_L \) to an irreducible representation \( \sigma \) of \( J \) trivial on \( J^1 \). By proposition 2.33 of [MS14b] there exists a unique pair \( (J_L, \kappa_L) \) decomposed above \( (J_L, \kappa_L) \) and so above \( (J_M, \kappa_M) \). Moreover, we have \( \eta = \text{res}_{J^1}^{J_1}(\beta, \lambda) = \kappa \otimes \sigma \) and \( I_G(\kappa) = JB^\times_L J \). We denote \( \mathcal{A} \) the finite group \( \prod_{i=1}^r GL_{m_i^j}(t_{D^j}) \). Then we can identify \( \sigma \) to a cuspidal (supercuspidal if \( (J, \lambda) \) is a semisimple supertype) representation of \( \mathcal{A} \).

Remark 4.1. The choice of \( \beta \)-extensions \( \kappa_j \in B(\theta_i) \) for every \( j \in \{1, \ldots, l\} \) determines \( \kappa_i \in B(\theta_i) \) for every \( i \in \{1, \ldots, r\} \), \( \kappa_j \) for every \( j \in \{1, \ldots, l\} \), \( \kappa_L \) and \( \kappa \) and so the decompositions \( \lambda_i = \kappa_i \otimes \sigma \), \( \lambda_j = \kappa_j \otimes \sigma_j \) and \( \lambda = \kappa \otimes \sigma \).

4.1. The representation \( \eta_{\text{max}} \)

In this paragraph we associate to every semisimple supertype \( (J, \lambda) \) of \( G \) an irreducible projective representation \( \eta_{\text{max}} \) of a compact open subgroup of \( G \) and we prove that the alge-
Remark. Using proposition 4.3, the tensor product of these intertwining elements becomes an intertwining \( \beta \)-choices, for example the choice of the isomorphism \( \text{Corollary 4.4.} \)

Correspondence, we denote \( \kappa=J^{\max}(L) \) and \( J^{\max,j}=J^{\max}(L) \). We can also choose \( \theta^{(j)}\in\mathcal{H}(L,\lambda,0,\beta) \) such that its transfer to \( \mathcal{H}(\Lambda',0,\beta) \) is \( \theta' \). We fix a \( \beta \)-extension \( \kappa^{(j)} \) of \( \sigma^{(j)} \) and we denote \( \eta^{(j)} \) its restriction to \( J^{\max,j} \) By (5.2) of [SS16], there exists a unique \( \kappa^{j} \in \mathcal{B}(\theta') \) such that

\[
\text{ind}_{U}^{B}(\Lambda'),U(U(\Lambda'))=\text{ind}_{U}^{B}(\Lambda'),U(U(\Lambda'))\kappa^{j},
\]

and so by remark 4.1 the choice of \( \kappa^{(j)} \) determines \( \kappa^{j} \). We denote \( J^{\max}=\prod_{j=1}^{j} J^{\max,j} \), \( J^{\max}=\prod_{j=1}^{j} J^{\max,j}, \kappa^{\max}=\otimes_{j=1}^{j} \kappa^{\max,j}, \eta^{\max}=\otimes_{j=1}^{j} \eta^{\max,j}, K_{L}^{\max}=\prod_{j=1}^{j} U(L) \) and \( K_{L}^{\max}=\prod_{j=1}^{j} U(L) \). If we denote \( \mathcal{G} \) the finite group \( \prod_{j=1}^{j} GL_{m_{j}}(\mathbb{F}_{D'}^{j}) \), we obtain \( J_{\max}/J_{\max}^{\max}=K_{L}^{\max}/K_{L}^{\max} \). By sections II.6-8 of [Vig98] there exists an algebra isomorphism \( \mathcal{H}(\mathcal{G},U) \) is a supercuspidal pair of \( \mathcal{G} \).

As before in this section, by propositions 2.30, 2.31 and 2.33 of [MS14b] we can define two compact open subgroups \( J^{\max} \) and \( J^{\max,j} \) of \( G \) such that \( J_{\max}/J_{\max}^{\max}=J_{\max}^{\max}/J_{\max}^{\max} \) and \( J_{\max}^{\max} \) and \( J_{\max,j}^{\max} \) respectively. Then we have \( I_{G}(\kappa^{\max})=I_{G}(\eta^{\max})=J_{\max}B_{L}^{\Lambda}J_{\max} \) and the \( R \)-vector spaces \( I_{G}(\eta^{\max}) \) and \( I_{G}(\kappa^{\max}) \) have dimension 1 for every \( g \) in \( B_{L}^{\Lambda} \).

Remark 4.2. Since for every \( j \in \{1,\ldots,l\} \) the choice of \( \kappa^{(j)},j \in \mathcal{B}(\theta^{(j)} \) determine \( \kappa^{j} \), the choice of \( \kappa^{(j)} \) determine \( \kappa \) and so the decomposition \( \lambda=\kappa \otimes \sigma \). On the other hand the \( \eta^{(j)} \), the group \( \mathcal{G} \) and the conjugacy class of \( \mathcal{M} \) are uniquely determined by the semisimple supertyp (\( J,\lambda \)), independently by the choice of \( \kappa^{(j)} \) or \( \kappa^{j} \).

Proposition 4.3. The algebras \( \mathcal{H}(G,\eta^{\max}) \) and \( \otimes_{j=1}^{j} \mathcal{H}(G,\eta^{\max}) \) are isomorphic.

Proof. By lemma 2.4, proposition 2.5 of [Gui13] and lemma 1.3 there exists an algebra isomorphism \( \otimes_{j=1}^{j} \mathcal{H}(G,\eta^{\max}) \rightarrow \mathcal{H}(\mathcal{G},\delta^{\max})) \rightarrow \mathcal{H}(\mathcal{L},\delta^{\max}) \). Now, since \( I_{G}(\lambda^{\max}) \subset I_{G}(\lambda^{\max}) \mathcal{L}_{\lambda^{\max}} \mathcal{L}_{\lambda^{\max}} \), the subalgebra \( \mathcal{H}(\mathcal{L},\delta^{\max},\mathcal{L}_{\lambda^{\max}}) \) of \( \mathcal{H}(G,\eta^{\max}) \) of functions with support in \( \mathcal{L}_{\lambda^{\max}} \mathcal{L}_{\lambda^{\max}} \) is equal to \( \mathcal{H}(G,\eta^{\max}) \) and so by sections II.6-8 of [Vig98] there exists an algebra isomorphism \( \mathcal{H}(\mathcal{G},\delta^{\max}) \rightarrow \mathcal{H}(G,\eta^{\max}) \) which preserves the support.

Corollary 4.4. The \( R \)-algebras \( \mathcal{H}(B_{L}^{\Lambda},K_{L}^{\Lambda}) \) and \( \mathcal{H}(G,\eta^{\max}) \) are isomorphic.

Proof. By remark 1.5 of [Chil17] (see also theorem 6.3 of [Kri90]) we have \( \mathcal{H}(B_{L}^{\Lambda},K_{L}^{\Lambda}) \approx \bigotimes_{j=1}^{j} \mathcal{H}(B_{L}^{\Lambda},U(\Lambda^{\max,j} \cap B^{\Lambda,j})) \) and by theorem 3.46 we have \( \mathcal{H}(B_{L}^{\Lambda},U(\Lambda^{\max,j} \cap B^{\Lambda,j}) \approx \mathcal{H}(\mathcal{L},\delta^{\max},\mathcal{L}_{\lambda^{\max}}) \) for every \( j \in \{1,\ldots,l\} \).

Remark 4.5. By theorem 3.46 the isomorphism of corollary 4.4 depends on the choice of a \( \beta \)-extension \( \kappa^{\max,j} \) of \( \eta^{\max,j} \) and of an intertwining element of \( \eta^{\max,j} \) for every \( j \in \{1,\ldots,l\} \). Using proposition 4.3, the tensor product of these intertwining elements becomes an intertwining element of \( \eta^{\max} \).

Remark 4.6. The procedure that associates \( \eta^{\max} \) to \( (J,\lambda) \) depends on several non-canonical choices, for example the choice of the isomorphism \( B_{L}^{\Lambda} \rightarrow \prod_{j=1}^{j} GL_{m_{j}}(\mathbb{F}_{D}^{j}) \). To obtain a canonical correspondence, we denote \( \Theta_{i} \) the endo-class of \( \theta_{i} \) with \( i \in \{1,\ldots,r\} \) and we canonically associate to \( (J,\lambda) \) the formal sum \( \Theta(J,\lambda)=\sum_{i=1}^{r} \Theta_{i} \). Furthermore, the group \( \mathcal{G} \) and the \( \mathcal{G} \)-conjugacy class of \( \mathcal{M} \) depend only on \( (J,\lambda) \) and actually the group \( \mathcal{G} \) depends only on \( \Theta \) because \( m_{j}^{i}(\mathbb{F}_{D}^{j}) \neq 0 \) and actually the group \( \mathcal{G} \) depends only on \( \Theta \) because \( m_{j}^{i}(\mathbb{F}_{D}^{j}) = \sum_{i=1}^{m_{j}^{i}} \Theta_{i} \). We refer to paragraph 6.3 of [SS16] for more details.
5. The category equivalence $\mathcal{R}(G, \eta_{\text{max}}) \simeq \mathcal{R}(B_{L}^{\times}, K_{L}^{1})$

Using notations of section 4, in this section we prove that there exists an equivalence of categories between $\mathcal{R}(G, \eta_{\text{max}})$ and $\mathcal{R}(B_{L}^{\times}, K_{L}^{1})$. This allows to reduce the description of a positive-level block of $\mathcal{R}(G)$ to the description of a level-0 block of $\mathcal{R}(B_{L}^{\times})$.

5.1. The category $\mathcal{R}(J, \lambda)$

In this paragraph we associate to a semisimple supertype $(J, \lambda)$ of $G$ a subcategory of $\mathcal{R}(G)$. We refer to [SS16] for more details.

From now on we fix an extension $\kappa_{\text{max}}$ of $\eta_{\text{max}}$ to $J_{\text{max}}$, as in paragraph 4.1. This uniquely determines a decomposition $\lambda = \kappa \otimes \sigma$ where $\kappa$ is an irreducible representation of $J$ and $\sigma$ is a supercuspidal representation of $\mathcal{M}$ viewed as an irreducible representation of $J$ trivial on $J^{1}$. We consider the functor $K_{\kappa_{\text{max}}} : \mathcal{R}(G) \to \mathcal{R}(J_{\text{max}}/J_{\text{max}}) = \mathcal{R}(J)$ given by $K_{\kappa_{\text{max}}} (\pi) = \text{Hom}_{J_{\text{max}}} (\eta_{\text{max}}, \pi)$ for every representation $\pi$ of $G$ with $J_{\text{max}}$ that acts on $K_{\kappa_{\text{max}}} (\pi)$ by

$$x. \varphi = \pi(x) \circ \varphi \circ \kappa_{\text{max}}(x)^{-1}$$

for every $x \in J_{\text{max}}$. We denote $\pi(\kappa_{\text{max}})$ this representation of $J$. We remark that if $V_{1}$ and $V_{2}$ are representations of $G$ and $\phi \in \text{Hom}_{G}(V_{1}, V_{2})$ then $K_{\kappa_{\text{max}}} (\phi)$ maps $\varphi$ to $\phi \circ \varphi$ for every $\varphi \in \text{Hom}_{G}(\rho, V_{1})$. To more details on this functor see section 5 of [MS14b] and [SS16].

We recall that we have $\sigma = \bigotimes_{i=1}^{r} \sigma_{i}$, where $\sigma_{i}$ is a supercuspidal representation of $GL_{m_{i}} (t_{D_{i}})$. We denote $\Gamma_{\mathcal{M}} = \prod_{i=1}^{r} \text{Gal}(t_{D_{i}}/t_{D_{i}})[1]$. The equivalence class of $(\mathcal{M}, \sigma)$ (see definition 1.14 of [SS16]) is the set, denoted by $[\mathcal{M}, \sigma]$, of supercuspidal pairs $(\mathcal{M}', \sigma')$ of $J$ such that there exists $\epsilon \in \Gamma_{\mathcal{M}}$ such that $(\mathcal{M}', \sigma')$ is $\mathcal{J}$-conjugated to $(\mathcal{M}, \sigma')$.

Let $\Theta = \Theta(J, \lambda)$. For every representation $V$ of $G$ let $V[\Theta, \sigma]$ be the subrepresentation of $V$ generated by the maximal subspace of $K_{\kappa_{\text{max}}}(V)$ such that every irreducible subquotient has supercuspidal support in $[\mathcal{M}, \sigma]$ and let $V[\Theta]$ be the subrepresentation of $V$ generated by $K_{\kappa_{\text{max}}}(V)$ (see paragraph 9.1 of [SS16]).

**Definition 5.1.** Let $\mathcal{R}(J, \lambda)$ be the full subcategory of $\mathcal{R}(G)$ of representations $V$ such that $V = V[\Theta, \sigma]$. This does not depend on the choice of $\kappa_{\text{max}}$ (see paragraph 10.1 of [SS16]).

**Remark 5.2.** For every representation $V$ of $G$ we have $V[\Theta, \sigma][\Theta, \sigma] = V[\Theta, \sigma]$ (see lemma 9.1 of [SS16]) and so $V[\Theta, \sigma]$ is an object of $\mathcal{R}(J, \lambda)$.

We call **equivalence class of $(J, \lambda)$** the set $[J, \lambda]$ of semisimple supertypes $(\tilde{J}, \tilde{\lambda})$ of $G$ such that $\text{ind}_{J}^{\tilde{J}}(\tilde{\lambda}) \cong \text{ind}_{J}^{\tilde{J}}(\lambda)$.

**Theorem 5.3.** The category $\mathcal{R}(J, \lambda)$ depends only on the class $[J, \lambda]$ and it is a block of $\mathcal{R}(G)$.

**Proof.** It follows by propositions 10.2 and 10.5 and theorem 10.4 of [SS16].

**Remark 5.4.** The proof in [SS16] of theorem 5.3 use the notions of inertial class of a supercuspidal pair of $G$ and the notion of supercuspidal support (see 1.1.3, 2.1.2 and 2.1.3 of [MS14a]). These notions are very important in the study of representations of $GL_{m}(D)$ but in this article they are not used explicitly.
5.2. The category equivalence

Let \((J, \lambda)\) be a semisimple supertype of \(G\) and let \(\Theta = \Theta(J, \lambda)\) be the formal sum of endo-
classes associated to it. In general there exist several semisimple supertypes of \(G\) associated to \(\Theta\). We denote \(X = X_\Theta\) as \(\{(J', \lambda') | \Theta(J', \lambda') = \Theta\}\). In this paragraph we prove that the sum \(\bigoplus_{[J', \lambda'] \in X} \mathcal{R}(J', \lambda')\) is equivalent to the level-0 subcategory of \(\mathcal{R}(B_L^\times)\).

Let \(Y = Y_\Theta\) be the set of equivalence classes of supercuspidal pairs of \(J\), that is uniquely determined by \(\Theta\) by remark 4.6. Let \(K_{\text{max}}\) be a fixed extension of \(\eta_{\text{max}}\) to \(J_{\text{max}}\) as in paragraph 4.1 and let \(K = K_{K_{\text{max}}}\). By proposition 10.6 of [SS16] there exists a bijection

\[
\phi_{K_{\text{max}}} : X \to Y
\]  
(5.2)
given by \(\phi_{K_{\text{max}}}([J', \lambda']) = [\mathcal{M}, \sigma]\) if the supercuspidal supports of irreducible subquotients of \(K(V)\) are in \([\mathcal{M}, \sigma]\) for every (or equivalently for one) object \(V\) of \(\mathcal{R}(J', \lambda')\). This is equivalent to say that there exists \(\kappa\) as in section 4 (which depends on \(K_{\text{max}}\)) such that \(\lambda' = \kappa \otimes \sigma'\) with \((\mathcal{M}, \sigma') \in [\mathcal{M}, \sigma]\).

**Proposition 5.5** (Corollary 9.4 of [SS16]). For every representation \(V\) of \(G\) we have

\[
V[\Theta] = \bigoplus_{[\mathcal{M}, \sigma] \in Y} V[\Theta, \sigma'].
\]  
(5.3)

**Proposition 5.6** (Lemma 10.3 of [SS16]). If \([J', \lambda'] \in X\) and \(W\) is a simple object of \(\mathcal{R}(J', \lambda')\) then \(K(W) \neq 0\).

Since \(J_{\text{max}}\) has a pro-order invertible in \(R^\times\), the representation \(\eta_{\text{max}}\) is projective and so we can use notations and results of section 1.2. We have defined the functor

\[
M_{\eta_{\text{max}}} : \mathcal{R}(G, \eta_{\text{max}}) \to \text{Mod} - \mathcal{R}(G, \eta_{\text{max}})
\]

by \(M_{\eta_{\text{max}}}(V)_\mathcal{R}(G, \eta_{\text{max}}) = \text{Hom}_G(\text{ind}^G_{J_{\text{max}}}(\eta_{\text{max}}), V)\) and \(M_{\eta_{\text{max}}} : \phi \mapsto \phi \circ \phi\) for every representations \(V\) and \(V_1\) of \(G\), \(\phi \in \text{Hom}_G(V, V_1)\) and \(\varphi \in \text{Hom}_G(\text{ind}^G_{J_{\text{max}}}(\eta_{\text{max}}), V)\).

**Remark 5.7.** Frobenius reciprocity induces a natural isomorphism between the functor \(M_{\eta_{\text{max}}}\) composed with forget-functor \(\text{Mod} - \mathcal{R}(G, \eta_{\text{max}}) \to \text{Mod}_R\) and the functor \(K\) composed with the forget-functor \(\mathcal{R}(J) \to \text{Mod}_R\). This implies that for every representation \(V\) of \(G\) the subrepresentation \(V[\Theta]\) of \(V\) is the subrepresentation \(V[\eta_{\text{max}}]\) defined in paragraph 1.2.

We have also defined the full subcategories \(\mathcal{R}(\eta_{\text{max}})(G)\) and \(\mathcal{R}(G, \eta_{\text{max}})\) of \(\mathcal{R}(G)\). We recall that \(\mathcal{R}(G, \eta_{\text{max}})\) is the category of \(V\) such that \(V = V[\Theta]\) and \(\mathcal{R}(\eta_{\text{max}})(G)\) is the category of \(V\) such that \(M_{\eta_{\text{max}}}(V') \neq 0\) for every irreducible subquotient \(V'\) of \(V\).

**Lemma 5.8.** We have \(\mathcal{R}(G, \eta_{\text{max}}) = \mathcal{R}(\eta_{\text{max}})(G)\).

**Proof.** Thanks to remark 1.8 it is sufficient to prove \(\mathcal{R}(G, \eta_{\text{max}}) \subset \mathcal{R}(\eta_{\text{max}})(G)\). Let \(V\) be a representation in \(\mathcal{R}(G, \eta_{\text{max}})\). By proposition 5.5 we have \(V = \bigoplus_{\mathcal{M}, \sigma} V(\mathcal{M}, \sigma)\) and by remark 5.2 the representation \(V[\Theta, \sigma']\) is an object of \(\mathcal{R}(J', \lambda')\) where \([J', \lambda'] = \phi_{K_{\text{max}}}^{-1}(\mathcal{M}, \sigma') \in X\). Hence, we obtain the inclusion \(\mathcal{R}(G, \eta_{\text{max}}) \subset \bigoplus_{\mathcal{M}, \sigma} \mathcal{R}(J', \lambda')\). Let now \(W\) be an object of \(\bigoplus_{\mathcal{M}, \sigma} \mathcal{R}(J', \lambda')\) and \(W'\) an irreducible subquotient of \(W\). Then \(W'\) is an irreducible object of \(\mathcal{R}(J', \lambda')\) for a \([J', \lambda'] \in X\) and so by proposition 5.6 we have \(K_{K_{\text{max}}}(W') \neq 0\). Therefore, by remark 5.7 we have \(M_{\eta_{\text{max}}}(W') \neq 0\) which implies \(\bigoplus_{\mathcal{M}, \sigma} \mathcal{R}(J', \lambda') \subset \mathcal{R}(\eta_{\text{max}})(G)\).

**Remark 5.9.** We have proved that \(\mathcal{R}(G, \eta_{\text{max}}) = \mathcal{R}(\eta_{\text{max}})(G) = \bigoplus_{[J, \lambda] \in X} \mathcal{R}(J, \lambda)\). Moreover, by proposition 1.7, a representation \(V\) of \(G\) is in this category if and only if it verifies one of the following equivalent conditions: \(V = V[\Theta]\) for every subquotient \(Z\) of \(V\) we have \(Z = Z[\Theta]\), for every irreducible subquotient \(U\) of \(V\) we have \(M_{\eta_{\text{max}}}(U) \neq 0\) or for every non-zero subquotient \(W\) of \(V\) we have \(M_{\eta_{\text{max}}}(W) \neq 0\).
Theorem 5.10. The functor $\mathbb{M}_{\eta_{\text{max}}}$ is an equivalence of categories between $\mathbb{R}(G, \eta_{\text{max}})$ and $\text{Mod} - \mathcal{H}_R(G, \eta_{\text{max}})$.

Proof. We apply theorem 1.9 with $g = G$ and $\sigma = \eta_{\text{max}}$. □

Remark 5.11. We recall that a level-$0$ representation of $B_L^x$ is a representation generated by its $K_L^1$-invariant vectors. It is equivalent to say that all irreducible subquotients have non-zero $K_L^1$-invariant vectors (see section 3 of [Chi17]). The category $\mathbb{R}(B_L^x, K_L^1)$ is called level-$0$ subcategory of $\mathbb{R}(B_L^x)$.

By section 3 of [Chi17] and theorem 1.9, the $K_L^1$-invariant functor $\text{inv}_{K_L^1}$ induces an equivalence of categories between $\mathbb{R}(B_L^x, K_L^1)$ and $\text{Mod} - \mathcal{H}_R(B_L^x, K_L^1)$ whose quasi-inverse is $W \mapsto W \otimes \mathcal{H}_R(B_L^x, K_L^1) \otimes_{K_L^1} B_L^x(1)$. We recall that if $(\rho, Z)$ is a representation of $B_L^x$ then the action of $\Phi \in \mathcal{H}_R(B_L^x, K_L^1)$ on $z \in ZK_L^1$ is given by $z.\Phi = \sum_{x \in K_L^1} zB_L^x(1) \Phi(x) \rho(x^{-1})z$.

Corollary 5.12. There exists an equivalence of categories between $\mathbb{R}(G, \eta_{\text{max}})$ and $\mathbb{R}(B_L^x, K_L^1)$.

Proof. By corollary 4.4 the algebras $\mathcal{H}_R(B_L^x, K_L^1)$ and $\mathcal{H}_R(G, \eta_{\text{max}})$ are isomorphic. We obtain an equivalence of categories between $\text{Mod} - \mathcal{H}_R(G, \eta_{\text{max}})$ and $\text{Mod} - \mathcal{H}_R(B_L^x, K_L^1)$ and so between $\mathbb{R}(G, \eta_{\text{max}})$ and $\mathbb{R}(B_L^x, K_L^1)$ by theorem 5.10 and remark 5.11. □

Now we want to describe the functor that induces this equivalence of categories. We recall that we have fixed an isomorphism $B_L^x \cong \prod GL_{m_j}(D^{j})$ and an extension $\kappa_{\text{max}}$ of $\eta_{\text{max}}$. We also fix a non-zero intertwining element $\gamma$ of $\eta_{\text{max}}$ as in remark 4.5. By corollary 4.4 we have an isomorphism $\Theta_{\kappa_{\text{max}}}: \mathcal{H}_R(B_L^x, K_L^1) \rightarrow \mathcal{H}_R(G, \eta_{\text{max}})$ which induces an equivalence of categories $\Theta_{\gamma, \kappa_{\text{max}}}: \text{Mod} - \mathcal{H}_R(G, \eta_{\text{max}}) \rightarrow \text{Mod} - \mathcal{H}_R(B_L^x, K_L^1)$. We obtain the diagram

\[
\begin{array}{ccc}
\mathbb{R}(G, \eta_{\text{max}}) & \xrightarrow{\text{Corollary 5.12}} & \mathbb{R}(B_L^x, K_L^1) \\
\text{Mod} - \mathcal{H}_R(G, \eta_{\text{max}}) & \xrightarrow{\Theta_{\gamma, \kappa_{\text{max}}}} & \text{Mod} - \mathcal{H}_R(B_L^x, K_L^1).
\end{array}
\]

The functor $\mathbb{M}_{\eta_{\text{max}}}: \mathbb{R}(G, \eta_{\text{max}}) \rightarrow \text{Mod} - \mathcal{H}_R(G, \eta_{\text{max}})$ is an equivalence of categories by theorem 5.10. By lemma 1.3 the right action of $\mathcal{H}_R(G, \eta_{\text{max}})$ on $\mathbb{M}_{\eta_{\text{max}}}(V)$ is given by $(m, \Psi)(f) = m(\Psi * f)$ for every $m \in \mathbb{M}_{\eta_{\text{max}}}(V)$, $\Psi \in \mathcal{H}_R(G, \eta_{\text{max}})$ and $f \in \text{ind}_{\mathcal{H}_R(G, \eta_{\text{max}})}(\eta_{\text{max}})$. The right action of $\Phi \in \mathcal{H}_R(B_L^x, K_L^1)$ on a $\mathbb{H}(G, \eta_{\text{max}})$-module $N$ is given by $N.\Phi = N.\Theta_{\gamma, \kappa_{\text{max}}}(\Phi)$. By remark 5.11 the functor $W \mapsto W \otimes \mathcal{H}_R(B_L^x, K_L^1) \otimes_{K_L^1} B_L^x(1)$ is a category equivalence between $\text{Mod} - \mathcal{H}_R(B_L^x, K_L^1)$ and $\mathbb{R}(B_L^x, K_L^1)$ where, by lemma 1.3, the left action of $\Phi \in \mathcal{H}_R(B_L^x, K_L^1)$ on $f \in \text{ind}_{K_L^1} B_L^x(1)$ is given by $\Phi.f = \Phi * f$. Moreover, the left action of $x \in B_L^x$ on $w \otimes f \in W \otimes \mathcal{H}_R(B_L^x, K_L^1) \otimes_{K_L^1} B_L^x(1)$ is given by $(x.w \otimes f) = w \otimes (x.f)$.

Composing these three functors we obtain the equivalence of categories of corollary 5.12 which we denote $F_{\gamma, \kappa_{\text{max}}}$ and that is given by

\[
F_{\gamma, \kappa_{\text{max}}}((\pi, V)) = \mathbb{M}_{\eta_{\text{max}}}(\pi, V) \otimes \mathcal{H}_R(B_L^x, K_L^1) \otimes_{K_L^1} B_L^x(1)\]

(5.5)

for every $(\pi, V) \in \mathbb{R}(G, \eta_{\text{max}})$, where the right action of $\Phi \in \mathcal{H}_R(B_L^x, K_L^1)$ on $m \in \mathbb{M}_{\eta_{\text{max}}}(\pi, V)$ is given by $(m.\Phi)(f) = m(\Theta_{\gamma, \kappa_{\text{max}}}(\Phi) * f)$ for every $f \in \text{ind}_{\mathcal{H}_R(G, \eta_{\text{max}})}(\eta_{\text{max}})$. We remark that if $V_1$ and $V_2$ are in $\mathbb{R}(G, \eta_{\text{max}})$ and $\phi \in \text{Hom}_{G}(V_1, V_2)$ then $F_{\gamma, \kappa_{\text{max}}}((\phi))$ maps $m \otimes f$ to $(\phi \circ m) \otimes f$ for every $m \in \mathbb{M}_{\eta_{\text{max}}}(V_1)$ and $f \in \text{ind}_{K_L^1} B_L^x(1)$.
5.3. Correspondence between blocks

In this paragraph we discuss the correspondence among blocks of $\mathcal{A}(B^+_L, K^1_L)$ and those of $\mathcal{A}(G, \eta_{\max})$ induced by the equivalence of categories $F_{\gamma, \kappa_{\max}}$ defined in (5.5).

We consider the functor $K_{K_L}: \mathcal{A}(B^+_L, K^1_L) \to \mathcal{A}(R(L/K_L^1)) = \mathcal{A}(\varnothing)$ given by $K_{K_L}(Z) = \pi Z^L$ and $K_{K_L}(\phi) = \phi|_{Z^L}$ for every representations $(\varnothing, Z)$ and $(\varnothing, Z_1)$ of $B^+_L$ and every $\phi \in \text{Hom}_{B^+_L}(Z, Z_1)$, where $x \in K_L$ acts on $z \in \pi Z^L$ by $x.z = \varnothing(x)z$. It is the functor presented in paragraph 5.1 when we replace $G$ by $B^+_L$ and $\kappa_{\max}$ by trivial representation of $K_L$. We also consider the functor $H: \text{Mod} - \mathcal{H}(B^+_L, K^1_L) \to \mathcal{A}(R(K_L/K^1_L))$ given by $H(W) = (\varnothing, W)$ and $H(\phi) = \phi$ for every $\mathcal{H}(B^+_L, K^1_L)$-modules $W$ and $W_1$ and every $\phi \in \text{Hom}_{\mathcal{H}(B^+_L, K^1_L)}(W, W_1)$, where $\varnothing(k) = w = f_k^{-1} w$ for every $k \in K_L$ and $w \in W$.

Remark 5.13. The functor $K_{K_L}$ is the composition of $\text{inv}_{K_L}$ (see remark 5.11) and the functor $H$. Actually if $(\varnothing, Z)$ is an object of $\mathcal{A}(B^+_L, K^1_L)$ then $H(\text{inv}_{K_L}(Z)) = H(Z^{K_L^1}) = (\varnothing, Z^{K_L^1})$ where $\varnothing(z) = z.f_{k-1} = \sum z \in K_L \setminus B^+_L f_{k-1}(x)\varnothing(x)z = \varnothing(z)$ for every $z \in Z^{K_L^1}$ and $k \in K_L$.

We obtain the diagram

$$\begin{align*}
\mathcal{A}(G, \eta_{\max}) & \xrightarrow{\Theta_{\gamma, \kappa_{\max}} \circ M_{\eta_{\max}}} \mathcal{A}(B^+_L, K^1_L) \\
\text{Mod} - \mathcal{H}(B^+_L, K^1_L) & \xrightarrow{K_{K_L}} \mathcal{A}(R(\varnothing))
\end{align*}$$

(5.6)

Proposition 5.14. There exists a natural isomorphism between $K_{K_L} \circ F_{\gamma, \kappa_{\max}}$ and $K_{\kappa_{\max}}$.

Proof. By remark 5.13 we have $K_{K_L} \circ F_{\gamma, \kappa_{\max}} = H \circ \text{inv}_{K_L} \circ F_{\gamma, \kappa_{\max}}$ and by diagram (5.4) we have a natural isomorphism between $\text{inv}_{K_L} \circ F_{\gamma, \kappa_{\max}}$ and $\Theta_{\gamma, \kappa_{\max}} \circ M_{\eta_{\max}}$ so it is sufficient to find a natural isomorphism $3: H \circ \Theta_{\gamma, \kappa_{\max}} \circ M_{\eta_{\max}} \to K_{\kappa_{\max}}$. For every object $(\pi, V)$ of $\mathcal{A}(G, \eta_{\max})$, let $3_V: M_{\eta_{\max}}(V) \to K_{\kappa_{\max}}(V)$ be the isomorphism of $R$-modules given by remark 5.7. The action of $x \in K_L/K^1_L \cong \varnothing$ on $m \in M_{\eta_{\max}}(\pi, V)$ is given by $x.m = m.\Theta_{\gamma, \kappa_{\max}}(\varnothing_{x-1}) = m.\varnothing_{x-1}$ where $\varnothing_{x-1} \in \mathcal{H}(\pi, \eta_{\max})$ has support $x^{-1}J^1_{\max}Z$ and $\varnothing_{x-1}(x^{-1}) = \kappa_{\max}(x^{-1})$ while the action of $x \in J_{\max}/J_{\max} \cong \varnothing$ on $\varnothing \in K_{\kappa_{\max}}(V)$ is given by (5.1). We have to prove that $3_V(x.m) = x.3_V(m)$ for every $m \in M_{\eta_{\max}}(\pi, V)$ and $x \in \varnothing$.

We recall that in paragraph 1.1 we have defined elements $i_v: J^1_{\max} \to V_{\eta_{\max}}$ with $v \in V_{\eta_{\max}}$, which generate $\text{ind}^G_{\max}(\eta_{\max})$ as representation of $G$, such that $m(i_v) = 3_V(m)(v)$. Then for every $v \in V_{\eta_{\max}}$ we have $3_V(x.m)(v) = (x.m)(i_v) = (m.\varnothing_{x-1})(i_v) = m.\varnothing_{x-1}(x-1).i_v$. The support of $\varnothing_{x-1}(x-1)$ is $J^1_{\max}(x-1)$ and $\varnothing_{x-1}(x^{-1}) = \kappa_{\max}(x^{-1})$. Hence, we have $3_V(x.m)(v) = m(x.\kappa_{\max}(x^{-1})) = \pi(x)(m(\kappa_{\max}(x^{-1})) = \pi(x)(3_V(m)(\kappa_{\max}(x^{-1}))) = (x.3_V(m))(v)$. Now, let $V_1$ and $V_2$ be two objects of $\mathcal{A}(G, \eta_{\max})$ and let $\varnothing \in \text{Hom}_G(V_1, V_2)$. Then for every $m \in M_{\eta_{\max}}(V_1)$ and every $v \in V_{\eta_{\max}}$ we have $3_V(\varnothing(3_V(m)(v))) = 3_{V_2}(\varnothing(3_V(m)(v))) = 3_{V_2}(\varnothing(m(i_v))$ which is equal to $\varnothing(m(i_v))$ by Frobenius reciprocity. On the other hand we have $K_{\kappa_{\max}}(\varnothing)(3_{V_1}(m)(v)) = \varnothing(3_{V_1}(m)(v))$ which is equal to $\varnothing(m(i_v))$ by Frobenius reciprocity. This shows that $3_V(x.m) = x.3_V(m)$ is a natural isomorphism.

Now we look for a block decomposition of $\mathcal{A}(B^+_L, K^1_L)$. Let $[\mathcal{M}, \sigma] \in \mathcal{Y}$. Then $\mathcal{M} = \prod_{j=1}^l \mathcal{M}_j$ and $\sigma = \otimes_{j=1}^l \sigma_j$ where $\mathcal{M}_j \cong J_j/J_j^1$ and $[\mathcal{M}_j, \sigma_j]$ is class of supercuspidal pairs of
$GL_{inj}(\mathfrak{B})$. For every $j \in \{1, \ldots, l\}$, replacing $G$ by $B_{j}^{\times}$ and $\kappa_{\text{max}}$ by the trivial character of $U(\Lambda_{\text{max}, j}) \cap B_{j}^{\times}$ in definition 5.1, we obtain an abelian full subcategory $\mathcal{R}(U(\Lambda_{\text{max}, j}) \cap B_{j}^{\times}, \sigma_{j})$ of $\mathcal{R}(B_{j}^{\times})$ whose objects are representations $V_{j}$ of $B_{j}^{\times}$ generated by the maximal subspace of $V_{1}(\Lambda_{\text{max}, j}) \cap B_{j}^{\times}$ for which every irreducible subquotient has supercuspidal support in $[\mathcal{M}, \sigma]$. We obtain a full subcategory $\mathcal{R}(K_{L}, \sigma)$ of $\mathcal{R}(R_{L})$ (and of $\mathcal{R}(B_{L}^{\times}, K_{L}^{1})$) whose objects are representations $V$ of $B_{L}^{\times}$ generated by the maximal subspace of $V^{K_{L}^{1}}$ such that every irreducible subquotient has supercuspidal support in $[\mathcal{M}, \sigma]$. Theorem 5.3 and remark 5.9 give a block decomposition of $\mathcal{R}(B_{L}^{\times}, U_{1}(\Lambda_{\text{max}, j}) \cap B_{j}^{\times})$ for every $j \in \{1, \ldots, l\}$ and so we obtain a block decomposition
\[
\mathcal{R}(B_{L}^{\times}, K_{L}^{1}) = \bigoplus_{[\mathcal{M}, \sigma] \in \mathcal{Y}} \mathcal{R}(K_{L}, \sigma).
\]
We recall that we have a block decomposition $\mathcal{R}(G, \eta_{\text{max}}) = \bigoplus_{[J, \lambda] \in \mathcal{X}} \mathcal{R}(J, \lambda)$ by remark 5.9 and a bijection $\phi_{\kappa_{\text{max}}}: \mathcal{X} \to \mathcal{Y}$ defined in (5.2) which depends on the choice of $\kappa_{\text{max}}$.

**Theorem 5.15.** Let $[J, \lambda] \in \mathcal{X}$ and $[\mathcal{M}, \sigma] = \phi_{\kappa_{\text{max}}}(J, \lambda) \in \mathcal{Y}$. Then $F_{\gamma, \kappa_{\text{max}}}$ induces an equivalence of categories between the block $\mathcal{R}(J, \lambda)$ of $\mathcal{R}(G)$ and the block $\mathcal{R}(K_{L}, \sigma)$ of $\mathcal{R}(R_{L})$.

**Proof.** If $V$ is an object of $\mathcal{R}(J, \lambda)$, by proposition 5.14 there exists an isomorphism of representations of $\mathcal{G}$ between $K_{L}(F_{\gamma, \kappa_{\text{max}}} \cap B_{L}^{\times})$ and $K_{\kappa_{\text{max}}} \cap B_{L}^{\times}$. Then for every $1 \leq j \leq l$, we obtain an abelian full subcategory $\mathcal{R}(L_{j}^{\times}, K_{L}^{1})$ whose objects are representations $V_{j}$ of $L_{j}^{\times}$ generated by the maximal subspace of $V^{K_{L}^{1}}$ such that every irreducible subquotient has supercuspidal support in $[\mathcal{M}, \sigma]$. Theorem 5.3 and remark 5.9 give a block decomposition of $\mathcal{R}(L_{j}^{\times}, U_{1}(\Lambda_{\text{max}, j}) \cap B_{j}^{\times})$ for every $j \in \{1, \ldots, l\}$ and so we obtain a block decomposition
\[
\mathcal{R}(L_{j}^{\times}, K_{L}^{1}) = \bigoplus_{[\mathcal{M}, \sigma] \in \mathcal{Y}} \mathcal{R}(K_{L}, \sigma).
\]
We remark that this correspondence does not depend on the choice of the intertwining element $\gamma$ of $\eta_{\text{max}}$.

### 5.4. Dependence on the choice of $\kappa_{\text{max}}$

In this paragraph we discuss the dependence of results of paragraphs 5.1, 5.2 and 5.3 on the choice of the extension of $\eta_{\text{max}}$ to $J_{\text{max}}$.

Let $(J, \lambda)$ be a semisimple supertype of $G$. We have just seen in remark 4.6 that the group $\mathcal{G}$ depends only on $\Theta(J, \lambda)$ and by remark 4.6 and theorem 5.3 the $G$-conjugacy class of $\mathcal{M}$ and the category $\mathcal{R}(J, \lambda)$ do not depend on the choice of the extension of $\eta_{\text{max}}$ to $J_{\text{max}}$. Moreover, the sum (5.3) does not depend on this choice because a different one permutes the terms $V[\Theta, \sigma]$ in $V[\Theta]$. Then $V[\Theta]$, the equalities $\mathcal{R}(G, \eta_{\text{max}}) = \mathcal{R}(\eta_{\text{max}}(G)) = \bigoplus_{[J, \lambda] \in \mathcal{X}} \mathcal{R}(J, \lambda)$ and the equivalence of theorem 5.10 do not depend on the choice of the extension of $\eta_{\text{max}}$.

Let $\gamma$ be a fixed non-zero intertwining element of $\eta_{\text{max}}$ as in remark 4.5. Using notation of paragraph 4.1, let $\kappa_{\text{max}}$ and $\kappa'_{\text{max}}$ be two extensons of $\eta_{\text{max}}$ to $J_{\text{max}}$ and let $\kappa_{\text{max}, j} = \bigotimes_{j=1}^{l} \kappa_{\text{max}, j}$ and $\kappa'_{\text{max}, j}$ be the restrictions to $J_{\text{max}}$ of $\kappa_{\text{max}}$ and $\kappa'_{\text{max}}$, respectively. Then, for every $j \in \{1, \ldots, l\}$, $\kappa_{\text{max}, j}$ and $\kappa'_{\text{max}, j}$ are $\beta$-extensions of $\theta_{\text{max}, j}$ and so by paragraph 2.1 there exists a character $\chi_{j}$ of $\mathcal{O}_{E_{j}}^{\times}$ trivial on $1 + \varrho_{E_{j}}$ such that $\kappa'_{\text{max}, j} = \kappa_{\text{max}, j} \otimes (\chi_{j} \circ N_{B_{j}^{\times}/E_{j}})$. Let $\chi$ be the character $\bigotimes_{j=1}^{l} (\chi_{j} \circ N_{B_{j}^{\times}/E_{j}})$. Then $\chi$ and $\overline{\chi}$ be the character $\bigotimes_{j=1}^{l} (\frac{\kappa'_{\text{max}, j}}{\kappa_{\text{max}, j}} \otimes N_{B_{j}^{\times}/E_{j}})$ viewed as characters of $J_{\text{max}}$ and of $\mathcal{G}$ respectively and let $\tilde{\chi} = \bigotimes_{j=1}^{l} \left( (\text{infl}^{E_{j}}_{E_{j}})^{\times} \chi_{j} \circ N_{B_{j}^{\times}/E_{j}} \right)$ viewed as a character of $B_{L}^{\times}$.

We consider the functors $\tilde{\chi}: \mathcal{R}(B_{L}^{\times}, K_{L}^{1}) \to \mathcal{R}(B_{L}^{\times}, K_{L}^{1})$ and $\overline{\chi}: \mathcal{R}(\mathcal{G}) \to \mathcal{R}(\mathcal{G})$ given by $\tilde{\chi}(\varrho) = \varrho \otimes \overline{\chi}^{-1}$, $\overline{\chi}(\varrho) = \overline{\varrho}$, $\overline{\chi}(\tau) = \tau \otimes \overline{\chi}^{-1}$ and $\overline{\chi}(\varrho) = \overline{\varrho}$ for every $\varrho, \varrho_{1}$ in $\mathcal{R}(B_{L}^{\times}, K_{L}^{1})$, every $\overline{\varrho} \in \text{Hom}_{\mathcal{G}}(\varrho, \varrho_{1})$, every representations $\tau$ and $\tau_{1}$ of $\mathcal{G}$ and every $\overline{\varrho} \in \text{Hom}_{\mathcal{G}}(\tau, \tau_{1})$. We consider
Lemma 5.16. We have $K'_{\max} = \overline{\mathcal{X}} \circ K_{\max}$ and so for every representation $(\pi, V)$ in $\mathcal{D}(G, \eta_{max})$ we have $\pi(K'_{\max}) = \pi(K_{\max}) \otimes \overline{\chi}^{-1}$.

Proof. The space of $K'_{\max}$ and $\overline{\mathcal{X}}(K_{\max})$ is $\text{Hom}_{\mathcal{D}}(\pi_{\max}, V)$. Let $\varphi$ in this space and $x \in J_{\max}$. Let $Q$ be the standard parabolic subgroup of $G$ with Levi component $L$, let $N$ be the unipotent radical of $Q$ such that $Q = LN$ and let $N^-$ be the unipotent radical opposite to $N$. We choose $x_1 \in J_{\max} \cap N^-$, $x_2 \in J_{\max}$ and $x_3 \in J_{\max} \cap N$ such that $x = x_1 x_2 x_3$. Since $(\kappa_{\max}, J_{\max})$ and $(\kappa'_{\max}, J_{\max})$ are decomposed above $(\kappa_{\max}, J_{\max})$ and $(\kappa'_{\max}, J_{\max})$ respectively, we obtain $\pi(\kappa_{\max})(x)(\varphi) = \pi(x) \circ \varphi \circ \kappa_{\max}(x^{-1}) = \pi(x) \circ \varphi \circ \kappa'_{\max}(x_2^{-1}) = \pi(x) \circ \varphi \circ \kappa_{\max}(x_2^{-1}) \chi(x_2^{-1})^{-1} = \pi(\kappa'_{\max})(x)(\varphi) \chi(x_2^{-1})^{-1}$. Since $J_{\max} \cap N = J_{max}^1 \cap N$ and $J_{\max} \cap N^- = J_{max}^1 \cap N^-$ we obtain $\chi(x_2^{-1})^{-1} = \chi(x)^{-1}$. Now, let $V_1$ and $V_2$ be two objects of $\mathcal{D}(G, \eta_{max})$ and let $\varphi \in \text{Hom}_{\mathcal{D}}(V_1, V_2)$. Then for every $\varphi \in \text{Hom}_{\mathcal{D}}(\eta_{\max}, V_1)$ we have $K'_{\max}(\varphi)(\varphi) = \varphi \circ \varphi = \overline{\mathcal{D}}(K_{\max}(\varphi))(\varphi)$.

Lemma 5.17. We have $K_{KL} \circ \overline{x} = \overline{x} \circ K_{KL}$.

Proof. Let $(\rho, Z)$ be in $\mathcal{D}(B_L^\chi, K_L^1)$. The space of $K_{KL}(\overline{x}(Z))$ and of $\overline{\mathcal{D}}(K_{KL}(Z))$ is $Z K_L^1$. Let $x \in K_L$ and let $\overline{x}$ the projection of $x$ in $K_L / K_L^1 \cong \mathcal{D}$. For every $z \in Z K_L^1$, we have $K_{KL}(\overline{x}(\rho))(\overline{x}(z)) = \overline{\chi}(x^{-1}) g(x) v$ while $\overline{\mathcal{D}}(K_{KL}(\rho))(\overline{x}(z)) = \overline{\chi}(x^{-1}) g(x) v$. Now, let $Z_1$ and $Z_2$ be two objects of $\mathcal{D}(B_L^\chi, K_L^1)$ and let $\phi \in \text{Hom}_{\mathcal{D}}(Z_1, Z_2)$. Then we have $K_{KL}(\overline{x}(\phi)) = \phi |_{Z_1 K_L^1} = \overline{x}(K_{KL}(\phi))$.

We remark that by proposition 5.14, lemma 5.16 and lemma 5.17, the functor $K_{KL} \circ \mathcal{D}_{\gamma, k'_{\max}}$ is naturally isomorphic to $K'_{\max}$ which is equal to $\overline{x} \circ K_{\max}$ which is naturally isomorphic to $\overline{x} \circ K_{KL} \circ \mathcal{D}_{\gamma, k_{\max}}$.

Proposition 5.18. There exists a natural isomorphism between $\mathcal{D}_{\gamma, k_{\max}}$ and $\overline{x} \circ \mathcal{D}_{\gamma, k_{\max}}$.

Proof. For every object $(\pi, V)$ in $\mathcal{D}(G, \eta_{\max})$, the space of $\mathcal{D}_{\gamma, k_{\max}}(V)$ and of $\overline{x}(\mathcal{D}_{\gamma, k_{\max}}(V))$ is $M_{\eta_{\max}}(V) \otimes \mathcal{D}_{\gamma}(B_L^\chi, K_L^1) \text{Ind}^{B_L^\chi}_{K_L^1}(1 K_L^1)$. If $m \in M_{\eta_{\max}}(V)$ and $f \in \text{Ind}^{B_L^\chi}_{K_L^1}(1 K_L^1)$, in the first case the right action of $\Phi \in \mathcal{D}_{\gamma}(B_L^\chi, K_L^1)$ on $m$ and the left action of $x \in B_L^\chi$ on $m \otimes f$ are given by $m \ast' \Phi = m \ast \Theta_{\gamma, k_{\max}}(\Phi)$ and $x \ast (m \otimes f) = \overline{\chi}(x^{-1}) m \otimes x f$ while in the second case they are given by $m \ast \Phi = m \ast \Theta_{\gamma, k_{\max}}(\Phi)$ and $x \ast (m \otimes f) = \overline{\chi}(x^{-1}) m \otimes x f$. Let $3_V$ be the $R$-automorphism of $M_{\eta_{\max}}(V) \otimes \mathcal{D}_{\gamma}(B_L^\chi, K_L^1) \text{Ind}^{B_L^\chi}_{K_L^1}(1 K_L^1)$ that maps $m \otimes f$ to $m \otimes \overline{\chi} f$ for every $m \in M_{\eta_{\max}}(V)$ and $f \in \text{Ind}^{B_L^\chi}_{K_L^1}(1 K_L^1)$. By remark 3.47 we have $m \ast' \Phi = m \ast \overline{\chi} \Phi$ and then $3_V(m \ast' \Phi \otimes f) = (m \ast' \Phi \otimes \overline{\chi} f = m \ast \Phi \otimes \overline{\chi}(f) = m \otimes (\overline{\chi}(\Phi) \ast \overline{\chi}(f)) = m \otimes (\overline{\chi}(\Phi) \ast f) = 3_V(m \otimes (\Phi \ast f))$ and so $3_V$ is well-defined. Moreover, for every $x \in B_L^\chi$ we have $3_V(x \ast' (m \otimes f)) = \overline{\chi}(x^{-1}) m \otimes x f$ and so $3_V$ is an isomorphism of representations of $B_L^\chi$. Now, let $V_1$ and $V_2$ be two objects of $\mathcal{D}(G, \eta_{\max})$ and let $\phi \in \text{Hom}_{\mathcal{D}}(V_1, V_2)$. Then for every
\[ m \in M_{\eta_{\text{max}}}(V_1) \text{ and } f \in \text{ind}_{K_L}^J(1_{K_L}) \] we have
\[ \mathfrak{Z}(V_2(\mathbb{F},\eta_{\text{max}}(\phi)(m \otimes f)) = 3V_2((\phi \circ m) \otimes f) = (\phi \circ m) \otimes \tilde{x}f = \tilde{x}(\mathbb{F},\eta_{\text{max}})(\phi)(m \otimes \tilde{x}f) = \tilde{x}(\mathbb{F},\eta_{\text{max}}(\phi))(3V_1(m \otimes f)). \]

By remark 4.2, the representations \( \kappa_{\text{max}} \) and \( \kappa'_{\text{max}} \) determine two decompositions \( \lambda = \kappa \otimes \sigma \) and \( \lambda = \kappa' \otimes \sigma' \) where \( \sigma \) and \( \sigma' \) are supercuspidal representations of \( \mathcal{M} \) viewed as irreducible representations of \( J_L \) trivial on \( J_{K_L} \). Hence, the bijection \( \phi \circ \kappa_{\text{max}}^{-1} \) permutes the elements of \( \mathcal{Y} \) and it maps \( [\mathcal{M}, \sigma] \) to \( [\mathcal{M}, \sigma'] \). Let \( \kappa_L \) and \( \kappa'_L \) be the restrictions to \( J_L \) of \( \kappa \) and \( \kappa' \) respectively. By (4.1) and by (2.20) of [MS14b] for every \( j \in \{1, \ldots, l\} \) we have \( \kappa'_L = \kappa_L \otimes \chi \) and so \( \sigma' = \sigma \otimes \chi^{-1} \).

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