Tunnelling geometries I. Analyticity, unitarity and instantons in quantum cosmology

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Abstract

We present a theory of tunnelling geometries originating from the no-boundary quantum state of Hartle and Hawking, which describes in the language of analytic continuation the nucleation of the Lorentzian Universe from the Euclidean spacetime. We reformulate the no-boundary wavefunction in the manifestly unitary representation of true physical variables and calculate it in the one-loop approximation. For this purpose a special technique of complex extremals is developed, which reduces the formalism of complex tunnelling geometries to the real ones, and also the method of collective variables is applied, separating the macroscopic collective degrees of freedom from the microscopic modes that are treated perturbatively. The quantum distribution function of Lorentzian universes, defined on the space of such collective variables, incorporates the probability conservation and boils down to the partition function of quasi-DeSitter gravitational instantons weighted by their Euclidean effective action. These instantons represent closed compact manifolds obtained by the procedure of doubling the Euclidean spacetime which nucleates the Lorentzian universes. The over-Planckian behaviour of their distribution is determined by the anomalous scaling of the theory on such instantons, which serves as a criterion for the high-energy normalizability of the no-boundary wavefunction and the validity of the semiclassical expansion. Thus the presented formalism combines the covariance of the Euclidean effective action with unitarity and analyticity in the Lorentzian spacetime and can be regarded as a step towards the unification of Euclidean and Lorentzian versions of quantum gravity and its third quantization.

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1. Introduction

The history of quantum cosmology reveals in the last decade the whole cycle of the development characteristic of the theories standing at the advanced frontiers of physics and lacking a good experimental status. Being founded in the pioneering works of Dirac, Wheeler, DeWitt [1] and the others in late fifties and early sixties, this theory underwent a long period of a lull in order to recover in early eighties due to practically simultaneous invention of the inflationary cosmology and two, conceptually similar, proposals for its quantum state – no-boundary proposal of Hartle and Hawking [2, 3] and the tunnelling proposal of Vilenkin [4]. A productive application of these proposals in the theory of the inflationary Universe was followed by a great uprise of interest in the Euclidean quantum gravity – the cornerstone of the no-boundary wavefunction – in connection with the ideas of the topology change, baby universe production and the theory of the cosmological constant of Coleman [5, 6, 7], which could be regarded as a first visible step towards the third quantization of gravity. However, an extensive accumulation of applications in this field (see the bibliography in [8]), very useful from the viewpoint of the theory of the early Universe – the subject acquiring now a well established experimental status, could not resolve the principal difficulties of this theory. As a result, it has again entered a slugish stage, to an essential extent, caused by the fact that the fundamental ideas of gravitational tunnelling phenomena remained at the heuristic level and, actually, were never pushed beyond the tree-level approximation.

In short, the difficulties of quantum gravity, arising in the above context [1], can be formulated as a controversy between covariance and unitarity. In search for completely covariant formulation of quantum gravity, Hartle and Hawking postulated the path integral over Euclidean (positive-signature) four-geometries and matter fields, which under certain boundary conditions semiclassically generates the usual Lorentzian spacetimes and propagating quantum fields. It is obvious, that this construction does

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1We don’t discuss here the other fundamental difficulty of quantum gravity theory – its perturbative nonrenormalizability, which might be resolved in the framework of the fundamental string theory manifesting itself basically at the over-Planckian energy scale. We focus at the problems equally inherent to gravity theory both in the ultraviolet and infrared limits, arising, for example, in the theory of the cosmological constant of Coleman.
not guarantee the unitarity of the theory, the more so unitarity implies a well-defined notion of time and Lorentzian metric signature missing in such a Euclidean formalism. This feature of the covariant Euclidean quantum gravity makes it formulated at the half-heuristic level lacking the fundamental principles which close it as a self-contained physical theory: the selection of physical quantum states being incomplete, their physical inner product not defined, etc.

As a counterpart to this covariant approach there exists the quantization of the theory reduced to physical (ADM) variables, featuring particularly selected time, manifest unitarity and a Hilbert space of quantum states [9, 10, 11, 12, 13]. It is generally believed that this reduced phase-space quantization is not equivalent to the original Dirac-Wheeler-DeWitt scheme (or its Euclidean extension of Hartle and Hawking) – the statement usually relying on the obvious inequivalence of the ADM quantizations in different gauges with some ad hoc operator realizations of the physical Hamiltonian and observables. Ultimately, such statements might be true, because ADM quantization turns out to be intrinsically inconsistent in view of the problem of Gribov copies [13, 14, 15]. This problem underlies a purely mathematical motivation for the secondary quantization of a relativistic particle, which leads to a more fundamental concept of the quantum field theory, and apparently compells physicists for the third quantization of gravity [16, 3, 8, 4]. However, at the present stage of quantum gravity theory such statements do not give much insight into the nature of its Euclidean version and do not help to close it as a physical theory.

On the contrary, the extension of the ADM quantization to the Euclidean regime can yield the missing principles of constructing and interpreting the Euclidean quantum gravity and, also, clearly indicates the points where this method fails and what kind of new physical phenomena follow from this failure. This approach is very promising in the framework of the semiclassical expansion, for perturbatively there exist very strong theorems on the equivalence of covariant and reduced phase space quantizations [17, 18, 19], confirmed at the operatorial level in the one-loop approximation [14]. Despite its limitations, the asymptotic $\hbar$-expansion is universally applicable to theories of general type and, apparently, gives a key also to the phenomena which can be completely described only at the non-perturbative level. As a good analogy of this
approach, one can list a method of many-trajectory representation in quantum field theory [20], describing the former in terms of sets of first quantized relativistic particles, the first quantized string theory as an approximation to the full string field theory, etc.

With this paper we begin the series of publications devoted to a partial implementation of this strategy. Our starting point, partly formulated in [13, 21], will be a manifestly unitary gravity theory in Lorentzian spacetime, quantized in terms of physical variables. This theory produces by certain analytic continuation the Euclidean quantum gravity – the auxiliary mathematical tool which serves us to describe the classically forbidden states of the gravitational field [22], just as the concept of the imaginary time [23], Euclidean spacetime and instantons is usually used for the description of the decay and tunnelling phenomena in quantum mechanical and field models. Semiclassically, the necessity of such an analytic continuation originates from the fact that families of classical solutions may have caustics in the configuration space of the theory. In order to extend these solutions beyond caustics, into the "shadow" domains, one has to continue them analytically into the complex plane of time – the fact which follows (at least semiclassically) from the two-fold analyticity of the path integral. The gauge properties and the nature of the caustic surfaces for the Wheeler-DeWitt equations will be considered in the third paper of this series [25], while in the first two papers (this one and [24]) we develop elements of the general theory and the one-loop approximation for quantum systems penetrating beyond these caustics, which we shall call the tunnelling geometries.

A very simple picture of tunnelling geometry, demonstrating the purposes of this paper, is presented on Fig.1. In Einstein’s gravity theory with the positive cosmological constant $\Lambda = 3 H^2$, there exists a well known DeSitter solution with the Lorentzian (indefinite-signature) metric

$$ds_L^2 = -dt^2 + a_L^2(t) c_{ab} dx^a dx^b,$$

(1.1)

This paper demonstrates, in particular, that an important class of such caustics exactly corresponds to the occurrence of Gribov copies in the quantization of gravity as a gauge theory with local diffeomorphism invariance. In connection with the discussion of [13] this serves as a compelling argument that the quantum penetration through these caustics belongs to the phenomena of the universe creation in the third quantization of gravity theory, described in the language of the secondary quantization.
\[ a_L(t) = \frac{1}{H} \cosh (Ht) \]  

(1.2)

describing the expansion of a spatially homogeneous spherical hypersurface, having a round three-dimensional metric \( a^2_L(t) c_{ab} \) with the scale factor \( a_L(t) \). As a counterpart to this construction there exists the solution of Euclidean Einstein equations with the positive-signature DeSitter metric

\[ ds^2 = g^{DS}_{\mu \nu} dx^\mu dx^\nu = d\tau^2 + a^2(\tau) c_{ab} dx^a dx^b, \]

(1.3)

\[ a(\tau) = \frac{1}{H} \sin (H\tau), \]

(1.4)

which corresponds to the geometry of the four-dimensional sphere of radius \( R = 1/H \) with spherical three-dimensional sections labelled by the latitude angle \( \theta = H\tau \). The both metrics are related by the analytic continuation into the complex plane of the Euclidean “time” \( \tau \) \[26, 27\]

\[ \tau = \frac{\pi}{2H} + it, \quad a_L(t) = a(\pi/2H + it), \]

(1.5)

which is a Wick rotation with respect to the point \( \tau = \pi/2H \) in this plane. This analytic continuation can be interpreted as a quantum nucleation of the Lorentzian DeSitter spacetime from the Euclidean hemisphere and shown on Fig.1 as a matching of the two manifolds (1.1) - (1.4) across the equatorial section \( \tau = \pi/2H \) \((t = 0)\) – the bounce surface \( \Sigma_B \) of zero extrinsic curvature.

This quantum tunnelling from the classically forbidden Euclidean domain with \( a \leq 1/H \) served as a heuristic basis for the Hartle-Hawking no-boundary and Vilenkin tunnelling proposals, which recently constituted the whole epoch in the development of quantum cosmology. The two major difficulties with this simple picture are the following. First, this idea was never pushed beyond the tree-level approximation. In the best case, the quantum fields were considered on the classical background by using the well-known fact that the Wheeler-DeWitt equations generate the Schrodinger \[1, 28, 29\] (or the heat \[30\]) equation for quantized matter, when the collective variables of the tunnelling background are completely classical. Therefore, in this approach no quantum properties of this background can be analyzed and no question of unitarity in the sector of these variables can be posed.
The second difficulty is that, even in the tree-level approximation, the above simple picture is applicable only to a limited class of problems called real tunnelling geometries [31, 32]. In these problems the metric and matter fields are completely real both on the Lorentzian and Euclidean parts of spacetime and smoothly match across the nucleation surface which must necessarily have a vanishing extrinsic curvature and vanishing normal derivative of matter fields. This requirement of reality is very strong and in many physically interesting situations cannot be satisfied. The most important example is the Hawking model of chaotic inflation [3] driven by the effective Hubble constant $H(\varphi)$ which is generated by the inflaton scalar field $\varphi$. In this model the geometry of the spacetime is approximately described by the above equations (1.1) - (1.4) with $H = H(\varphi)$. As it follows from equations of motion, for large $H(\varphi)$ the scalar field is approximately conserved in time (which justifies the inflationary ansatz (1.2) with constant $H$), but its derivative never exactly vanishes for solutions satisfying appropriate boundary conditions. Therefore, no nucleation surface exists at which exactly real Euclidean solution can be smoothly matched to the real Lorentzian one [33].

Obviously, this is a general case of complex tunnelling in contrast to a rather narrow class of real tunnelling geometries. This situation is characteristic not only of quantum gravity, but also applicable to the theory of instantons in non-linear gauge theories. The curious fact is that not much attention has been paid in literature to the underbarrier penetration phenomena, described by complex-valued instantons, and filling in this omission in the body of quantum theory might lead to interesting physical applications extending beyond quantum gravity and cosmology [4].

Thus, in this paper we shall undertake several unassuming steps towards the resolution of the above difficulties and omissions in the theory of tunnelling geometries. To begin with, we present in Sect.2 the Euclidean quantum gravity as an analytic continuation from its Lorentzian counterpart and reformulate in this language the no-boundary proposal of Hartle and Hawking. In Sect.3 we discuss this analytic continuation for the wavefunction $\Psi(q, t)$ in the representation of true physical variables $q$. As it was discussed above, this representation is intrinsically non-covariant, but this obvious dis-

\footnote{One of the authors (A.O.B.) enjoiéd discussion of this point with S.Coleman.}
The advantage of the ADM wavefunction is justified by the transparency of its interpretation based on a simple inner product

\[ <\Psi_1 | \Psi_2> = \int dq \Psi_1^*(q, t) \Psi_2(q, t). \] (1.6)

The one-loop approximation for \( \Psi(q, t) \) also looks simple in this representation for it involves the functional determinant of the wave operator only on the subspace of physical modes. We use a reduction method of the second paper of this series \[24\] to convert this determinant to a special form which, together with (1.6), will be used for the proof of the conservation of the total probability – the cornerstone of unitarity.

Sect. 4 deals with the separation of the physical configuration space into collective macroscopic variables, describing the tunnelling spacetime background, and the microscopic modes treated perturbatively on this background – the only constructive way of handling the realistic models with the infinite amount of the degrees of freedom.

Sect. 5 contains the method generalizing the semiclassical expansion to the case of complex solutions of dynamical equations. Under good convexity properties of the Euclidean action this method boils down to the perturbation expansion in the imaginary part of the extremal, which can be used asymptotically in \( \hbar \to 0 \) even for large imaginary corrections. This method, in particular, explains the secondary role of complex instantons in quantum theory, which turn out to be exponentially suppressed in comparison with the real ones. The application of this method to the Hartle-Hawking wavefunction in Sect. 6 shows that the complex nature of the underlying tunnelling geometry does not prevent from interpreting this wavefunction as a special Euclidean vacuum of the microscopic physical modes.

Sect. 7 occupies the central place in the paper for it contains the derivation of the quantum distribution function for tunnelling Hartle-Hawking geometries on the space of collective variables. This distribution, on one hand, demonstrates unitarity and, on the other hand, reduces to the partition function of closed gravitational instantons of spherical topology weighted by their Euclidean effective action including all one-loop corrections. This algorithm, based on the reduction technique of the next paper in this series \[24\], establishes the link between the non-covariant but manifestly unitary Lorentzian theory with its covariant Euclidean counterpart – one of the main purposes of this paper.
Sect. 8 contains the first important application of this quantum distribution, briefly reported in \[34, 13\], – the over-Planckian behaviour of the partition function of the inflationary cosmologies in the Hawking model of the chaotic inflation \[35, 3\]. In contrast to the tree-level approximation, which always generates the Hartle-Hawking wavefunction unnormalizable at high energies \[36, 37\], the corresponding quantum distribution is determined asymptotically by the anomalous scaling of the Euclidean theory on the gravitational instanton of the above type. Thus, depending on the value of this scaling parameter determined by the particle content of the model, this distribution can either enhance the contribution of the over-Planckian energy scales, or suppress them and justify the use of the semiclassical expansion.

2. Euclidean quantum gravity as an analytic continuation from its Lorentzian counterpart

Let us denote the collection of three-dimensional metric \( g_{ab}(x) \) and matter fields \( \phi(x) \) in the canonical quantization of gravity by

\[
q = ( g_{ab}(x), \phi(x) ).
\]  (2.1)

Then, what is usually called the transition amplitude from the configuration \( q_- \) at spatial hypersurface \( \Sigma_- \) to the configuration \( q_+ \) at \( \Sigma_+ \) is given by the path integral over Lorentzian spacetime geometries and histories of matter fields \( g = ( g_{\mu\nu}(x,t), \phi(x,t) ) \) interpolating between the data on \( \Sigma_- \) and \( \Sigma_+ \)

\[
K(q_+, q_-) = \int D\mu[g] \, e^{\frac{i}{\hbar}S[g]}.
\]  (2.2)

Here \( S[g] \) is the gravitational action in spacetime domain sandwiched between the hypersurfaces \( \Sigma_- \) and \( \Sigma_+ \), which takes the form

\[
S[g] = \int_{t_-}^{t_+} dt \, \mathcal{L}(q, \dot{q}, N)
\]  (2.3)

of the time integral with the Lagrangian \( \mathcal{L}(q, \dot{q}, N) \) when spacetime is foliated by the \( t \)-parameter family of spacelike hypersurfaces, so that \( g = (q(t), N(t)) \) is decomposed into spatial 3-metric and matter fields \( q(t) \) (2.7) and lapse and shift functions
\( N(t) = (N^\perp(x, t), N^a(x, t)) \). In contrast to variables \( \mathbf{q}(t) \), which enter the gravitational Lagrangian together with their space and time derivatives, the time derivatives of lapse and shift functions \( N \) do not appear in (2.3) – a well-known fact accounting for the constrained nature of the gravitational theory.

The integration measure \( D\mu[\mathbf{g}] \) in (2.2) includes the Faddeev-Popov gauge fixing procedure for local diffeomorphism invariance of the theory with the corresponding ghost contribution and implies integration over the histories \( \mathbf{g} = (\mathbf{q}(t), N(t)) \) matching the fixed fields \( \mathbf{q}(t_\pm) = \mathbf{q}_\pm \) at \( \Sigma_\pm \). Lapse and shift functions \( N \), which have the nature of Lagrange multipliers generating the gravitational constraints, are integrated over at the boundary surfaces and, therefore, do not enter the arguments of the transition kernel (2.2) in accordance with the BRST symmetry truncated to the Dirac quantization of gravity \cite{38,13,14}. As a result this kernel satisfies with respect to both of its arguments the system of the Wheeler-DeWitt equations, which are just the operatorial version of the gravitational constraints, and does not depend on time coordinates \( t_\pm \) labeling the initial and final hypersurfaces.

### 2.1. Two-fold analyticity of the path integral

This mathematical construction allows to realize the qualitative ideas of tunnelling geometries, sketched above, in the following language of analytic continuation. Under the assumption that the field variables \( \mathbf{g} = (\mathbf{q}(t), N(t)) \) are analytic functions of time, one can deform the contour of integration over \( t \) in (2.3) into its complex plane without changing the value of the action. In view of the formal \( t_\pm \)-independence of the kernel (2.2) this can be done even without keeping the end points of this contour fixed: they can also be arbitrarily shifted into the complex plane of time. Such an assumption on the analyticity of the field variables in the integrand of the path integral is certainly of dubious nature, because generally the path integration goes in the class of non-smooth fields: the histories of \( \mathbf{q} \)-variables can be not differentiable, which fact is well known from the skeletonization of the path integral on the lattice, while the histories of Lagrange multipliers \( N(t) \) can be even discontinuous (at least in the class of unitary

\footnote{For the formulation of boundary conditions on the full set of integration variables and the boundary-value problems for their propagators in the Feynman diagrammatic technique corresponding to (2.2) see \cite{39}.}
nonrelativistic gauges of the Faddeev-Popov gauge fixing procedure \[13\]). However, the contribution of such non-smooth histories is inessential in the $\hbar$-expansion which we consider here.

As a consequence of such analytic continuation, the integration field variables generally become complex-valued, which means that the integral \( (2.2) \) must be taken over some complex contour $C$ in the configuration space of the theory. The possibility of doing it without changing the value of the transition kernel is again provided by the analyticity of the integrand, but this time, of the functional integral with respect to its functional argument $g$. Due to this two-fold analyticity in time and configuration-space variables the path integral representation \( (2.2) \) can be identically rewritten as

\[
K(q_+, q_-) = \int_C D\mu[\Phi] e^{-\frac{1}{\hbar} \mathcal{I}[\Phi(z)]}, \tag{2.4}
\]

where $z, \Phi(z)$ and $\mathcal{I}[\Phi(z)]$ are the results of the above two-fold analytic continuation of correspondingly the time, configuration-space fields and their gravitational action \( (2.3) \)

\[
t \to z, \quad g(t) \to \Phi(z) = (q(z), N_E(z)),
\]

\[
-iS[g] \to \mathcal{I}[\Phi(z)] = \int_C dz \mathcal{L}_E(q(z), dq(z)/dz, N_E(z)). \tag{2.5}
\]

Here the contour of integration $C$ can generally represent an arbitrary continuous curve in the complex plane of time variable, and the complex Lagrangian $\mathcal{L}_E(q, dq/d\tau, N_E)$ is related to the original gravitational Lagrangian by the equation

\[
\mathcal{L}_E(q, dq/d\tau, N_E) = -\mathcal{L}(q, idq/d\tau, N), \tag{2.6}
\]

\[
N \equiv (N^+, N^a) = (N^+_E, iN^a_E) \tag{2.7}
\]

for arbitrary functions $q = q(\tau)$ and $N_E = (N^+_E(\tau), N^a_E(\tau))$.

The notations in the eqs.\( (2.3)-(2.7) \) obviously imply that the quantities labelled by subscript $E$ denote the objects in the Euclidean spacetime having $\tau$ as a time variable. In particular, equation \( (2.4) \) means that in order to get the Euclidean Lagrangian from the Lorentzian one we have to perform the Wick rotation which basically boils down to multiplying the velocities and lapse functions by $i$. This procedure makes a formal transition from the Lorentzian metric

\[
ds^2 = -N^2dt^2 + g_{ab}(dx^a + N^a dt)(dx^b + N^b dt) \tag{2.8}
\]

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to the metric of Euclidean spacetime
\[ ds_E^2 = N^2 d\tau^2 + g_{ab} (dx^a + N^a_E d\tau) (dx^b + N^b_E d\tau), \quad (2.9) \]
foliated by the surfaces of constant \( \tau \).

Comparison of equations (2.7) and (2.3) shows that one can regard \( z \) as a variable in a complex plane of the Euclidean time \( \tau \) so that the real part of \( z \) can be identified with \( \tau \) itself, while its imaginary part coincides with \( t \). Correspondingly, in the integral (2.3) over arbitrary contour \( C \) the element of integration is \( dz = d\tau + idt \).

To clarify further the analytic continuation of the above type, let us consider the following two choices of such a contour \( C \). One choice is obvious and can be called the Lorentzian one:
\[ C_L : \{ z = it, \text{Im} t = 0, t_- \leq t \leq t_+ \}. \quad (2.10) \]
It generates exactly the Lorentzian gravitational action (2.3) provided the restriction of the fields \( \Phi(z) \) to this contour gives real-valued configuration space variables of the Lorentzian gravity theory \( g(t) \):
\[ -i S[g(t)] = \mathcal{I}[\Phi(z)]|_{C_L}, \quad \Phi(z)|_{C_L} = g(t). \quad (2.11) \]
With such a contour and the integration over real \( g(t) \) one gets from (2.4) the original Lorentzian path integral representation of the transition kernel.

Another choice is the Euclidean contour \( C_E \):
\[ C_E : \{ z = \tau, \text{Im} \tau = 0, \tau_- \leq \tau \leq \tau_+ \}, \quad (2.12) \]
which gives rise to the Euclidean action of the gravitational and matter variables \( g(\tau) \) corresponding to the metric (2.3) and generates the basic path integral of the Euclidean quantum gravity
\[ I[g(\tau)] = \mathcal{I}[\Phi(z)]|_{C_E}, \quad \Phi(z)|_{C_E} = g(\tau), \quad (2.13) \]
\[ K(q_+, q_-) = \int D\mu[g] e^{-\frac{i}{\hbar} I[g(\tau)]}. \quad (2.14) \]

\(^5\)Strictly speaking, Wick rotation implies multiplying by \( \pm i \) the time components of those matter fields \( q \) which have tensor indices, so that \( q \)'s in the left-hand side of (2.6) should be also labelled by \( E \), but for brevity we disregard this subtlety. Actually, in what follows we shall not encounter this problem because we shall work with the theory in the representation of true physical variables which usually comprise the subset of spatial tensor fields and do not include time components.
In view of the formal time independence of $K(q_+, q_-)$ and the analyticity assumptions, all three path integrals \((2.2), (2.4)\) and \((2.14)\) are equal, provided the configuration space integration contours in \((2.2)\) and \((2.14)\) either coincide or turn out to be continuously deformable to the contour $C$ in \((2.4)\) without crossing the domains of non-analyticity of the integrand. Actually the integral \((2.4)\) is parametrized only by the boundary fields $q\pm$ and by the contour $C$ (or, more precisely, by the equivalence class of the contours deformable in the above sense to each other). Recent studies of this property occupied one of the central places in the theory of quantum gravity and resulted in a major achievement of modern quantum cosmology – the no-boundary proposal for the quantum state of the Universe \([2, 3]\).

### 2.2. The no-boundary proposal

The formulation of the no-boundary proposal begins with the analysis of the $q$-dependence of the transition kernel \((2.2)\). It has two arguments $q\pm$ associated with two hypersurfaces $\Sigma\pm$ and describes the dynamical transition from $\Sigma-$ to $\Sigma+$ via the Lorentzian spacetime realizing the cobordism of these two spacelike surfaces. In contrast to transition amplitudes, the wavefunction has one argument associated with the spacelike hypersurface to which the quantum state is ascribed. The idea of Hartle and Hawking \([2, 3]\) to obtain the one-argument wavefunction from the two-argument transition kernel \((2.2)\) or \((2.14)\) consists in shrinking the spacetime section $\Sigma-$ to a point, demanding the regularity of all integration fields in its vicinity and integrating over all physical fields at this point compatible with the regularity condition. However this set of requirements cannot be implemented in spacetime with the Lorentzian signature: shrinking $\Sigma-$ to a point implies that $t$ plays the role of a radial coordinate in its neighbourhood, which is timelike in contrast to spacelike concentric spheres of constant $t$ around the origin $t = 0$. Such a 4-geometry is singular at the origin $t = 0$ which, thus, cannot be considered as a regular internal point treated on equal footing with the other points of spacetime. Therefore, the above transition to the one-argument path integral has to be performed with such integration 4-geometry which can be regular around the hypersurface tending to a point.

In the Hartle-Hawking no-boundary proposal the cosmological wavefunction is con-
constructed by integrating over Euclidean geometries and matter fields \( g \) on spacetime \( M \) which has a topology of a compact 4-dimensional ball \( \mathcal{B}^4 \) bounded by a 3-dimensional hypersurface \( \Sigma_+ \) with the boundary fields \( q_+ \):

\[
\Psi(q_+) = \int D\mu[g] e^{-\frac{1}{\hbar} I[g]}.
\] (2.15)

The Euclidean gravitational action \( I[g] \) in this equation can be regarded as a particular case of (2.13) corresponding to the Euclidean time contour \( C_E \) (2.12) with \( \tau^- = 0 \)

\[
I[g] = \int_{\tau^-}^{\tau^+} d\tau \mathcal{L}_E(q, dq/d\tau, N_E),
\] (2.16)

because the manifold \( M \) can be viewed as originating from the tube-like spacetime \( \Sigma \times [\tau^-, \tau^+] \) by the procedure of the above type: shrinking one of its boundaries \( \Sigma^- \) to a point and inhabiting it by a positive-signature metric and matter fields (see Fig.2).

In such a manifold the role of a radial coordinate is played by the Euclidean time \( \tau \) with the origin at \( \tau^- = 0 \) – point-like remnant of the boundary \( \Sigma^- \) of vanishing size.

Therefore the no-boundary path integral (2.13) is also a particular case of the kernel (2.4) with the no-boundary prescription replacing the specification of \( q^- \). This is the topological part of the definition of the Hartle-Hawking wave function. The rest part of this definition is the choice of the integration contour \( C \). Since the work [40], revealing the indefiniteness of the Euclidean gravitationa action in the sector of a conformal mode, it is known that this integration cannot run over real 4-geometries: to make the path integral formally convergent one should rotate the integration contour for this mode into the complex plane.

Unfortunately, at present, there is no theory which could have uniquely fixed this contour in the no-boundary proposal [41]. Its choice can be constrained by a number of compelling but disjoint arguments, including convergence of the path integral, the recovery of quantum field theory in a semiclassical curved spacetime, the enforcement of quantum constraints, etc. [41], but still has essential freedom, which was confirmed by considerations of several minisuperspace models [42]. The general conclusion claimed by the authors of [41] was that the program of finding the unique integration contour on the basis of quantizing the true physical degrees of freedom [43, 44] has not been successfully implemented in the theory of spatially closed cosmologies, in contrast to
asymptotically-flat gravitational systems subject to very powerful positive-energy and positive-action theorems [45, 46]. This conclusion can be, probably, understood by taking into account that the quantization of physical variables is intrinsically incomplete beyond the semiclassical expansion, for it suffers from the problem of Gribov copies [13, 14], which in canonical quantum gravity manifest themselves as different aspects of a notorious ”problem of time” [47, 48]. The presence of the Gribov problem serves as a compelling argument [13, 14] in favour of the third-order quantization of gravity which was intensively discussed recently in connection with the ideas of the wormhole physics, baby-universe production and Coleman’s ”big-fix” mechanism [5, 6, 7]. Then, the above ambiguity in integration contour of the path integral can be interpreted as corresponding to different choices of Green’s functions and vacua of the third-quantized gravity theory.

Here we shall not select the path integration contour from the variety of all possible equivalence classes of the above type. We suppose that this choice is already done by this or that selection rule, so that we have at our disposal one such class within which we can freely deform the contour. In particular, we suppose that we can pass it through the saddle point $g$ of the Euclidean gravitational action, which gives the following dominant contribution

$$\Psi(q_+) \sim e^{-\frac{1}{\hbar} I[g]}.$$  \hspace{1cm} (2.17)

to the path integral within the $\hbar$-expansion. Our purpose now will be to see how the complex nature of the time contour $C$ in the definition of the complexified action and configuration-space contour $C$ will show up in the calculation of the no-boundary wavefunction (2.13) and thus provide a mathematical ground for a simple picture of the tunnelling geometry given in Introduction.

\footnote{I am grateful to Bruce Campbell for this observation.}
2.3. Nucleation of the Lorentzian Universe from the Euclidean spacetime of the no-boundary type

The saddle point \( g = (q(\tau), N(\tau)) \) of the Euclidean action in (2.13) is a solution of the boundary-value problem

\[
\left. \frac{\delta I[g]}{\delta g} \right|_M = 0, 
\]

\[
q \left|_{\partial M} \equiv q(\tau_+) = q_+, \ g \left|_M \right. = \text{reg}, \]

which is a system of elliptic differential equations with the 3-metric and matter fields \( q_+ \) prescribed at the boundary and lapse and shift functions \( N \) determined by necessary gauge conditions fixing the coordinatization of \( M \). In a spherical coordinate system with \( \tau \) playing the role of geodetic radius in the vicinity of \( \tau_- = 0 \) the no-boundary regularity condition (2.19) reduces in the main to the following behaviour of the 4-metric

\[
ds^2 = d\tau^2 + \tau^2 c_{ab} dx^a dx^b + O(\tau^3), \ \tau \to 0, \]

and sufficiently smooth differentiability of matter fields.

When the argument \( q_+ \) of the wavefunction corresponds to a 3-geometry of "small size", that is close to the above asymptotic expression, one is granted, for obvious reasons, to have a real-valued solution of the classical Euclidean equations (2.18)-(2.19). For a pure-gravity theory with the cosmological constant \( \Lambda = 3H^2 \), considered in Introduction, in the case of the round 3-metric \( q_+ = a^2 c_{ab} \) with a scale factor \( a \leq 1/H \) this solution coincides with the 4-geometry (1.3) - (1.4). However, when the 3-geometry \( q_+ \) is big enough, such a real-valued solution with the positive-signature metric may not exist, as it happens in the above example for a scale factor \( a \geq 1/H \). This is a manifestation of the fact that, generally, the solutions of Einstein equations have caustics in superspace of \( q \) and cannot regularly be continued into its "shadow" domains. But such a solution, which we shall denote by \( \Phi(z) \), can exist when the segment of the Euclidean contour (2.12) is replaced, via the procedure of analytic continuation, with some contour in the complex plane of the Euclidean time \( z = \tau + it \), \( C_+ : \{ z = z(\sigma), 0 \leq \sigma \leq 1, z(0) = 0, z(1) = z_+ \} \), starting at zero value of the Euclidean "radius" and ending at some point \( z_+ = \tau_+ + it_+ \).
can represent an arbitrary curve in the complex z-plane, but for reasons of convenience and good physical interpretation it makes sense to break this curve into the union of two straight segments

\[ C_+ = C_E \cup C_L, \quad (2.21) \]

which are particular examples of the Euclidean (2.12) and Lorentzian (2.11) contours (see Fig.3):

\[ C_E : \{ z = \tau, \text{Im}\, \tau = 0, 0 \leq \tau \leq \tau_B \}, \quad (2.22) \]
\[ C_L : \{ z = \tau_B + it, \text{Im}\, t = 0, 0 \leq t \leq t_+ \}. \quad (2.23) \]

When the solution of classical equations on \( C_E \) and \( C_L \) is real-valued and represents respectively the real metrics (2.9) and (2.8) and correspondingly related matter fields, then the above two segments can be ascribed to real Euclidean and Lorentzian sections of one complex spacetime. These two sections are analytically matched together across the bounce surface \( \tau = \tau_B \) and form one spacetime manifold of a combined Euclidean-Lorentzian signature. At the ”moment” \( \tau_B \) the Euclidean solution suffers a bounce or tunnells into the Lorentzian regime and, thus, gives rise to the ”beginning of time” [32]. This is precisely the situation of the DeSitter Lorentzian spacetime (1.1) - (1.2) with the Hubble parameter \( H = \sqrt{\Lambda/3} \), nucleating from the Euclidean four-dimensional hemisphere (1.3) - (1.4) of radius \( R = 1/H \) at its equator – the spatial section at \( \tau_B = \pi/2H \) (see Fig.1).

The above case of the so-called real tunnelling geometries has been intensively studied since the invention of the no-boundary proposal. Semiclassically its interpretation goes as follows. Due to the complexity of the contour \( C_+ \) the exponential of (2.17) should be replaced by the complex functional \( \mathcal{I}[\Phi(z)] \) computed at the saddle point \( \Phi(z) \). In view of reality of \( \Phi(z) \) at both Euclidean and Lorentzian segments of the full contour \( C_+ \), this action functional has a complex structure

\[ \mathcal{I}[\Phi(z)] = I - iS \quad (2.24) \]

with the real and imaginary parts contributed respectively by the Euclidean and Lorentzian domains of the total spacetime. For the same reason these real fields satisfy in these domains correspondingly the Euclidean and Lorentzian equations of motion, so
that \( S \) turns out to be the classical Hamilton-Jacobi function of the system. This fact, when (2.24) is substituted into the expression (2.17), prompts to interpret the resulting wavefunction \( \Psi(q_+) \sim \exp\{-I/\hbar + iS/\hbar\} \) as describing the family of semiclassical Lorentzian universes weighted by the exponentiated action of the corresponding Euclidean domain responsible for their tunnelling or birth from "nothing" [2, 3]. The analyses of this weight [31] shows that under certain positive-energy assumptions for matter fields a real tunnelling solution can nucleate only a single connected Lorentzian spacetime with the most probable topology \( \mathbb{R} \times S^3 \) and the DeSitter metric. The rest of interpretation for the no-boundary wavefunction is usually based on incorporating the old method of deriving the quantum field theory of matter fields in curved spacetime from the semiclassically approximated Wheeler-DeWitt equations [1, 28, 29, 49]. This method shows that the quantum state of matter fields on the background of such a tunnelling geometry coincides with the Euclidean DeSitter invariant vacuum [49, 27, 32] which generates in the theory of the inflationary Universe the large scale cosmological structure compatible with observations [35, 51].

As was discussed in Introduction, there are basically two difficulties with the results of the above type: the limitations of the semiclassical interpretation and the restriction of the whole scheme to real tunnelling geometries and matter fields. The first difficulty invalidates the attempts to interpret beyond the tree level those modes of the gravitational field which are treated as classical in the \( \hbar \)-expansion of the Wheeler-DeWitt equation. In particular, for the Hawking model of chaotic inflation it does not allow to get the no-boundary quantum state as a normalizable wavefunction of the inflaton scalar field generating the effective cosmological constant. The second difficulty restricts the applicability of the above theory to the class of toy models with a special type of "inert" matter fields, because any realistic matter (including the chaotic inflation model of Hawking) generally gives tunnelling geometries which are complex both in the Euclidean (that is on \( C_E \)) and Lorentzian (on \( C_L \)) regimes.

As a remedy against the first difficulty (and as a starting point for generalizing the above theory to the case of complex tunnelling geometries), we shall consider the construction of the no-boundary wavefunction in the representation of true physical variables. For problems exploiting within the \( h \)-perturbation theory only local prop-
erties of the configuration space, this approach seems to be very promising \cite{13, 14}: under a proper operator realization it turns out to be equivalent to Dirac and BFV (BRST) quantization, allows to construct the conserved physical inner product in the space of physical states and, thus, provides the unitarity of the theory. Therefore, we shall use it here for the one-loop calculation of the no-boundary wavefunction.

3. No-boundary wavefunction in the representation of physical variables

3.1. ADM reduction and path-integral quantization

The perturbative construction of the no-boundary wavefunction of physical variables repeats with slight modifications the general formalism presented above. The starting point of this procedure is the ADM reduction of the gravitational theory to dynamically independent degrees of freedom \cite{9} adjusted to the case of spatially closed cosmology \cite{10, 11, 13}. It consists in imposing the gauge conditions on the original phase-space variables \( q \) and \( p \) (\( p \) forms the set of canonical momenta conjugated to \( q \)). These gauge conditions fix the local invariance of the theory with respect to general coordinate diffeomorphisms and allow one to disentangle the physical degrees of freedom as follows. The full system of the first-class gravitational constraints and imposed gauges can be solved for \((q, p)\) in terms of the dynamically independent canonical coordinates \( q = q^i \) and their conjugated momenta \( p = p_i \), which we shall label by a condensed index \( i \). The requirement of the conservation of gauge conditions in time yields also the lapse and shift functions \( N \) as functions of \((q, p)\) and thus specifies a concrete foliation of spacetime by spacelike hypersurfaces. Substituting \( q = q(q, p), \ p = p(q, p) \) into the canonical action of gravity theory produces its reduced phase space canonical action in terms of the unconstrained variables \((q, p)\). This action contains the nonvanishing, but generally time-dependent, Hamiltonian of these physical variables and by a standard procedure of the Legendre transform from \( p \) to \( \dot{q} = dq/dt \) generates the Lagrangian \( \mathcal{L}(q, dq/dt, t) \) and the corresponding Lagrangian action

\[
S[q(t)] = \int_{t_-}^{t_+} dt \mathcal{L}(q, dq/dt, t). \tag{3.1}
\]
According to the general theory of constrained systems \[17, 18\] the path integral \((2.2)\) over the full configuration space of fields \(g = (q, N)\) with the correct integration measure, including the gauge fixing and contribution of ghosts, can be identically rewritten as a path integral over reduced phase space variables \((q, p)\) of their exponentiated canonical action \[7\]. The ADM reduction to \((q, p)\) is very complicated due to the nonlinearity of the gravitational constraints. Therefore it generally leads to the physical Hamiltonian which is a non-polynomial function of \(p\), and the corresponding integration over momenta in this integral has a non-gaussian nature. This property complexifies the transition from the phase-space path integral to its Lagrangian version, but in the main it boils down to the expression \((2.2)\) with the covariant action \(S[g]\) replaced by its reduced version \(S[q]\) and the new integration measure \(D\mu[q]\). This measure accumulates the result of this nontrivial integration over momenta and can be calculated as a power series in \(\hbar\) beginning with the following one-loop contribution:

\[
D\mu[q] = \prod_t dq(t) \left[ \det a \right]^{1/2}(t) + O(\hbar), \quad dq = \prod_i dq^i,
\]

\[
\det a = \det a_{ik}, \quad a_{ik} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^k}.
\]

Here the determinant of the Hessian matrix \(a_{ik}\) is understood with respect to condensed indices \(i\) and \(k\). They include, depending on the representation of field variables, either continuous labels of spatial coordinates or discrete quantum numbers labelling some complete infinite set of functions on a spatial section of spacetime. Therefore the above determinant is functional, but its functional nature is restricted to a spatial slice of constant time \(t\). The product over time points of \(\det a(t)\) can be regarded as a determinant of higher functional dimensionality associated with the whole spacetime if we redefine \(a_{ik}\) as a time-ultralocal operator \(a = a_{ik}\delta(t - t')\). We shall denote such functional determinants for both ultralocal and differential operators in time by \(\text{Det}\). Therefore, in view of the ultralocality of \(a\), the contribution of the one-loop measure

\footnote{For a kernel \(K(q_+, q_-)\) of transition between the hypersurfaces \(\Sigma_{\pm}\) this relation holds up to certain operatorial factors associated with \(\Sigma_{\pm}\), which are responsible for the unitary map between the Dirac-Wheeler-DeWitt quantization and reduced phase space quantization. This property is discussed in much detail in the series of author’s papers \[12, 13, 14\] both at the level of operatorial and path-integral quantizations.}
becomes
\[
\prod_t \{ \det a \}^{1/2}(t) = \{ \text{Det } a \}^{1/2} = \exp \left\{ \frac{1}{2} \int_{t_-}^{t_+} dt \, \delta(0) \ln \det a(t) \right\}.
\] (3.4)

Since this expression contains the coincidence limit of the one-dimensional delta-function \( \delta(0) \), the local measure contributes to the path integral a pure power divergence. As was shown in [52] within the four-dimensional treatment of spacetime covariant differential operators, this contribution identically cancels the strongest (quartic) divergences of one-loop Feynman diagrams – the property which will be demonstrated in [24] within the canonical framework.

One more modification inherent to the reduced phase space quantization is that the transition kernel (2.2) and the Hartle-Hawking wavefunction (2.15) have in the representation of physical variables \( q \) the explicit dependence on time. This follows from the fact that in the ADM reduction the role of time is played by some functional combinations of the initial phase-space coordinates \( q \) (or momenta \( p \)), so that the arguments \( q_{\pm} \) of \( K(q_{+}, q_{-}) \) after the reduction give rise to time variables \( t_{\pm} \). Simultaneously with disentangling time from phase space of the theory, the ADM procedure recovers the nonvanishing physical Hamiltonian, and its operatorial version governs the Schrodinger evolution of the transition kernel and the wavefunction [11, 13, 14].

3.2. Analytic continuation technique

The reformulation of the analytic continuation technique in terms of physical variables is rather straightforward. Mainly it repeats the equations (2.2) - (2.6) and (2.10) - (2.24) with the original configuration variables \( g = (q, N) \), their Lagrangian \( \mathcal{L}(q, \dot{q}, N) \) and action \( S[g] \) being replaced by the corresponding physical space counterparts \( q = q^i, \mathcal{L}(q, dq/dt, t) \) and \( S[q(t)] \). In contrast to bold letters for the objects in the original formulation we shall use the usual letters for their analogues in the ADM quantization. In particular the physical transition kernel \( K(q_{+}, t_{+} | q_{-}, t_{-}) \) and the wavefunction \( \Psi(q_{+}, t_{+}) \) will replace \( K(q_{+}, q_{-}) \) and \( \Psi(q_{+}) \) [8].

[8]The precise relation between the unitary transition kernels and wavefunctions in the ADM and the Dirac-Wheeler-DeWitt quantization schemes, which establishes their local equivalence, is considered in [12, 13, 14].
The analytic continuation of both time $t$ and configuration space variables $q(t)$ into their complex planes $z$ and $\Phi(z)$ can be done similarly to eqs.(2.5) – the procedure that generates the complex action of physical variables

$$\mathcal{I}[\Phi(z)] = \int_C dz \mathcal{L}_E(\Phi(z), d\Phi(z)/dz, z)$$

with the Euclidean Lagrangian $\mathcal{L}_E(\Phi, d\Phi/dz, z)$. The latter is related to its Lorentzian version of the eq.(3.1) by the definition analogous to (2.6). There is now one subtlety in this definition originating from the explicit dependence of the physical Lagrangian $\mathcal{L}(q, dq/dt, t)$ on time. Since it is no longer translationally invariant in the complex plane of $z = \tau + it$, one should specify the point in this plane with respect to which the Wick rotation generates the Euclidean Lagrangian. In principle, the choice of this point is a part of the gauge fixing procedure to the same extent as the choice of time is a part of ADM reduction. Therefore one can expect that the physical results and the invariant unitary dynamics of the theory are independent of this choice. But there exists a physically distinguished definition of the Wick rotation associated with the no-boundary proposal, which looks as follows.

According to the Hartle-Hawking construction the contour of integration $C_+^{2.21}$ in the action functional runs from the point $z = 0$ to some complex point $z_+$. In the Dirac-Wheeler-DeWitt quantization this final point does not explicitly enter the wavefunction, being semiclassically determined by its argument $q_+$ from the solution of the boundary-value problem (2.18)-(2.19) in some coordinate gauge. On the contrary, in the ADM quantization the physical wavefunction explicitly depends on time, and according to the above method of analytic continuation real and imaginary ranges of this argument can be associated respectively with the classically allowed and forbidden transitions of the system. This interpretation urges us to identify the above final point of the integration contour $z_+ = \tau_B + it_+$ with the complex time argument of the physical wavefunction $\Psi(q_+, z_+)$.\footnote{This does not, certainly, mean that there is no gauge-fixing ambiguity in the determination of $z_+$ analogous to the coordinate gauge for the classical equations (2.18)-(2.19). The construction of time in the ADM reduction relies on some particular gauge which is implicitly equivalent to the coordinate gauge for these equations.} Breaking the contour $C_+$ into the union (2.21) of the Euclidean $C_E$ and Lorentzian $C_L$ segments heuristically implies that the no-boundary quantum state at the moment $t_+$ of the physical time is a result of the underbarrier
penetration along $C_E$ followed by the unitary evolution along the segment of real time $C_L$. An obvious choice of the center of Wick rotation, which follows from this picture, is the intersection point $z = \tau_B$ of these two segments. Under this choice the natural definition of the Euclidean Lagrangian of complexified physical variables $\Phi(z)$ looks like

$$\mathcal{L}_E(\Phi(z), d\Phi(z)/dz, z) = -\mathcal{L}(\Phi(z), id\Phi(z)/dz, (z - \tau_B)/i), \quad (3.6)$$

because it generates from the universal complex action functional (3.3) the Lorentzian action (3.1) on the contour $C_L$

$$iS[q(t)] = -\mathcal{I}[\Phi(z)]|_{C_L}, \quad q(t) = \Phi(z)|_{C_L} \quad (3.7)$$

and the Euclidean physical action on the contour $C_E$

$$I[\phi(\tau)] = \mathcal{I}[\Phi(z)]|_{C_E}, \quad \phi(\tau) = \Phi(z)|_{C_E} \quad (3.8)$$

$$I[\phi(\tau)] = \int_0^{\tau_B} d\tau \mathcal{L}_E(\phi, d\phi/d\tau, \tau). \quad (3.9)$$

Here we denoted the values of the complex physical field $\Phi(z)$ on the contours $C_L$ and $C_E$ respectively by $q(t)$ and $\phi(\tau)$, so that these functions are related by the following analytic continuation from the real axis of the Euclidean time $\tau$:

$$q(t) = \phi(\tau_B + it). \quad (3.10)$$

Now we can write the general expression for the no-boundary wavefunction of physical variables as a path integral

$$\Psi(q_+, z_+) = \int D\mu[\Phi(z)] e^{-\frac{1}{\hbar}\mathcal{I}[\Phi(z)]} \quad (3.11)$$

with the complex action (3.3) and a local measure (3.2) defined on the contour $C = C_+$ joining the points $z = 0$ and $z_+$ in the complex plane of time. The integration here goes over physical fields matching $q_+$ at the boundary $z = z_+$ of complex spacetime manifold $M = \Sigma \times C_+$ and satisfying the no-boundary regularity conditions in the center of it $z = 0$:

$$\Phi(z_+) = q_+, \quad \Phi(z)|_{z \to 0} = \text{reg}. \quad (3.12)$$

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The wavefunction (3.11) can be regarded as a result of the analytic continuation into the complex plane of \( \tau_+ \) of the Euclidean path integral

\[
\Psi(q_+, \tau_+) = \int D\mu [\phi(\tau)] e^{-\frac{1}{\hbar} I[\phi(\tau)]} \quad (3.13)
\]

which corresponds to \( z_+ = \tau_+ \) and the choice of the contour \( C_+ = C_E \) in (3.11). When \( z_+ = \tau_B + it_+ \) in \( \Psi(q_+, z_+) \) it makes sense to identify this function with the Lorentzian quantum state of the system \( \Psi_L(q_+, t_+) \) evolving in the real physical time \( t_+ \) and originating from \( \Psi(q_+, \tau_+) \) by this analytic continuation

\[
\Psi_L(q_+, t_+) = \Psi(q_+, \tau_B + it_+). \quad (3.14)
\]

So our further strategy will consist in the calculation of the Euclidean wavefunction (3.13) by the method of semiclassical expansion and its analytic continuation into the Lorentzian regime.

### 3.3. One-loop approximation

The semiclassical expansion of (3.13) is rather straightforward and begins in the one-loop approximation with the expression

\[
\Psi(q_+, \tau_+) = \left( \frac{\text{Det } F[\phi]}{\text{Det } a[\phi]} \right)^{-1/2} e^{-\frac{1}{\hbar} I[\phi]} \left[ 1 + O(\hbar) \right]. \quad (3.15)
\]

Here the dominant tree-level contribution originates from the classical extremal \( \phi \) satisfying the boundary-value problem for physical fields similar to (2.18) - (2.19)

\[
\frac{\delta I[\phi]}{\delta \phi(\tau)} \bigg|_M = 0, \quad (3.16)
\]

\[
\phi(\tau) \big|_M = \text{reg}, \quad \phi(\tau) \big|_{\partial M} \equiv \phi(\tau_+) = q_+, \quad (3.17)
\]

while the preexponential factor is a combination of the local measure (3.4) and contribution of the gaussian functional integration over quantum disturbances around this extremal. The latter is determined by the functional determinant of \( F \) – the kernel of the quadratic part of the action, which is a differential operator with respect to \( \tau \)

\[
F \equiv F(d/d\tau) \delta(\tau - \tau') = \frac{\delta^2 I[\phi]}{\delta \phi(\tau) \delta \phi(\tau')}, \quad (3.18)
\]
Because of the usual structure of the Lagrangian $L_E(\phi, d\phi/d\tau, \tau)$, containing at most first-order time derivatives of field variables, this matrix-valued differential operator $F(d/d\tau) = F_{ik}(d/d\tau)$ is of the second order and has the form

$$F(d/d\tau) = -\frac{d}{d\tau} a \frac{d}{d\tau} - \frac{d}{d\tau} b + b^T \frac{d}{d\tau} + c,$$

(3.19)

where the coefficients $a = a_{ik}$, $b = b_{ik}$ and $c = c_{ik}$ are the (functional) matrices acting in the space of field variables $\phi(\tau) = \phi^k(\tau)$, and the superscript $T$ denotes their (functional) transposition $(b^T)_{ik} \equiv b_{ki}$. These coefficients can be easily expressed as mixed second-order derivatives of the Euclidean Lagrangian with respect to $\phi^i$ and $\dot{\phi}^i = d\phi^i/d\tau$. In particular, the matrix of the second order derivatives $a_{ik}$ is given by the Euclidean version of the Hessian matrix ($3.3$): $a_{ik} = \partial^2 L_E/\partial \dot{\phi}^i \partial \phi^k$.

### 3.4. The choice of gauge and nature of physical variables

The general scheme of the above type bears inalienable ambiguity in the choice of gauge for physical variables $q^i$. This choice specifies the way these variables are disentangled from the initial phase space of $q$ and $p$ and fixes the physical internal time $t$ for their dynamics. This ambiguity shows up in the physical wavefunction and its analytic continuation $\Psi(q_+, z)$ into the complex plane of $t$, which is not gauge-invariant object in contrast to the Dirac-Wheeler-DeWitt wavefunction $\Psi(q)$. There exist several physically reasonable requirements which can restrict the excessive freedom in the choice of the gauge-fixing procedure. Note, first of all, that the Wick rotation ($3.6$) generally leads to a complex Euclidean Lagrangian in view of the explicit time dependence in $L(q, dq/dt, t)$. Therefore it makes sense to define such an ADM reduction that makes $\mathcal{L}_E(\phi, d\phi/d\tau, \tau)$ real valued at real values of the Euclidean time $\tau$ and $\phi(\tau)$.

Another property, which will be important for our further derivations, is the requirement of the boundedness from below for the Euclidean action of physical variables ($3.9$). It provides the formal (mode by mode) convergence of the Euclidean path integral ($3.13$) over real fields $\phi(\tau)$, establishes the normalizability of the wavefunction $\Psi(q, t)$ on the real section of the $q$-configuration space (and, therefore, the possibility to regard physical variables as Hermitian operators) and serves as a ground for a special technique of complex extremals which we develope in Sect.5. This property of the
Euclidean action can be a consequence of both a special reduction procedure and the properties of the original covariant gravitational action which, as is known, suffers from the problem of indefiniteness in the conformal sector that can invalidate the attempts to construct a good Euclidean action.

In this paper we shall not consider an ADM reduction which guarantees all the properties of the above type. We simply assume that the positivity of the quadratic form of the physical action near its extremum can be provided and, then, develop the consequences of this assumption. Actual reduction procedure and its application to the Hawking model of chaotic inflation will be considered in much detail in [25]. Just to give the idea of how this reduction works, we give here only its basic ingredients and formulate the no-boundary regularity conditions (3.17).

The basic approximation to the chaotic inflationary (spatially-closed) cosmology consists in the minisuperspace model described by the metric (1.1) - (1.2) with the scale factor $a_L(t)$ driven by the effective Hubble constant $H$ which is generated by the spatially homogeneous (inflaton) scalar field $\varphi$, $H = H(\varphi)$. All the other spatially inhomogeneous fields of all possible spins are treated as perturbations on this background and, therefore, the full set of the initial phase-space coordinates of the theory can be represented as $q = (a, \varphi, \phi(x), \psi(x), A_a(x), \psi_a(x), h_{ab}(x), ...)$.

A physically meaningful reduction from $\mathbf{q}$ to $\mathbf{q}$ goes separately in the minisuperspace sector of the full superspace $(a, \varphi)$ and the sector of spatially inhomogeneous modes. There is only one Hamiltonian constraint which is effectively imposed on spatially homogeneous modes $(a, \varphi)$, because the momentum constraints are identically satisfied in the minisuperspace model of Bianchi IX type with the round three-dimensional metric $c_{ab}$ [53]. Therefore, there can be only one physical degree of freedom among these two minisuperspace variables $(a, \varphi)$. The second of these variables has to be fixed by the gauge condition which simultaneously disentangles time (or, in other words, identifies the second variable with the internal time). It is useful to choose the inflaton scalar field $\varphi$ as this physical degree of freedom and interprete the approximate solution

\footnote{Time-components of vector $A_\mu$, gravitino $\psi_\mu$, graviton $h_{\mu\nu}$, etc. form in the canonical formalism the set of Lagrange multipliers (including lapse and shift functions) and, therefore, do not enter the set of phase-space coordinates.}
of classical equations of motion (1.2) as a gauge

\[ a(t) = \frac{1}{H(\varphi)} \cosh H(\varphi) t, \]  

(3.20)

which thus plays the role of the parametrization of the initial phase-space coordinates in terms of the physical ones in the minisuperspace sector of the theory.

This gauge is very convenient because it corresponds to the choice of cosmic time with the lapse \( N = 1 \) in classical solutions [25] and, what is most important for our purposes, provides the reality of the Euclidean Lagrangian (3.6). Indeed, choosing \( \tau_B = \pi/2H(\varphi) \) and analytically continuing the gauge (3.20) along the contour \( C_+ = C_E \cup C_L \) onto the real axis of the Euclidean time \( \tau \), one finds that the Euclidean scale factor remains real and coincides with the Euclidean solution (1.4). Therefore, the Lagrangian of physical variables on the background of such an Euclidean spacetime hemisphere \( 0 \leq \tau \leq \pi/2H(\varphi) \) is also real and even positive-definite in the sector of transverse-traceless graviton modes [21].

The ADM reduction in the sector of other fields can be performed in very many different ways. In the linearization approximation it mainly boils down to selecting the transverse \((T)\), transverse-traceless \((TT)\), etc. modes of spatial components of the corresponding tensor fields, so that the full set of physical variables can be written as

\[ q^i = (\varphi, \phi(x), \psi(x), A^T_a(x), \psi^T_a(x), h^{TT}_{ab}(x), ...). \]  

(3.21)

Here the index \( i \) is an element of condensed DeWitt notations which we shall intensively use throughout the paper. It includes discrete spin labels of field components and also continuous labels of spatial coordinates \( x \). The functions of spatial coordinates \( x \) in (3.21) can be decomposed as infinite series in the complete set of some spatial harmonics, which in view of the compactness of a spatial section is discrete and countable. Then the continuous label \( x \) in \( i \) will be replaced by the discrete quantum numbers enumerating these spatial harmonics. In both cases, however, the operations of respectively the integration over \( x \) and infinite summation over these numbers will be a part of a symbolic contraction of the condensed indices [1].

\[ ^{11} \text{It should be emphasized that, since we work in the noncovariant canonical formalism with the distinguished time variable, the condensed indices do not include time arguments of the fields. Correspondingly, these time arguments and such operations with them as integration will be, if necessary, explicitly written in all the formulae.} \]
3.5. The no-boundary regularity conditions for physical variables

The no-boundary regularity conditions (2.19)-(2.20) must be reformulated now in terms of physical variables (3.21). To begin with, note that the gauge (3.20) analytically continued to the Euclidean time, \( \tau = \pi / 2H(\phi) + it \), gives the scale factor \( a_E(\tau) = \tau + O(\tau^2) \) which automatically satisfies the asymptotic behaviour (2.20) of the metric (this gauge picks up the unit lapse \( N_E = 1 \)) and therefore guarantees that \( \tau \) measures the proper radial distance in the center of the Euclidean ball \( B^4 \) and does not produce the conical singularity for \( a_E(\tau) \sim \tau \) at \( \tau \to 0 \). Thus, it remains to check that all the physical fields (3.21) satisfy the condition of smooth regularity in the center of this Euclidean spacetime \( \tau = 0 \). For spatially homogeneous modes \( \phi = \phi(\tau) \) this condition implies that their radial derivative should vanish at this point \( (d\phi/d\tau)(0) = 0 \), while inhomogeneous modes should disappear themselves. This conclusion easily follows from the fact that the coordinates of a spatial section shrinking to a point \( \tau = 0 \) play the role of angular coordinates in a spherical coordinate system with the origin at this point. Therefore, the above properties of spatially homogeneous and inhomogeneous modes is a direct corollary of the direction independent limit of these modes or their derivatives when approaching the point \( \tau = 0 \).

4. The method of collective coordinates

In field-theoretical models the variables \( q \) represent the continuous infinitude of modes (3.21), so that their constructive treatment can be performed only in certain approximations. The idea of such approximations consists in disentangling from the set of \( q \) a certain finite subset which plays the most important role in the dynamics of the system and exactly or approximately decouple from the rest of degrees of freedom. Then these distinguished variables, which usually describe the collective behaviour of

\footnote{Note that we require the continuity of at most the first-order spacetime derivatives of fields. The explanation for this restricted notion of regularity in the no-boundary proposal can follow from the fact that the Lagrangian in the Euclidean action (3.9) contains at most the first-order derivatives of fields. Therefore the no-boundary proposal does not rule out the integration fields with discontinuous higher-order derivatives, because they do not give infinite contribution to the Euclidean action and are not automatically suppressed in the Euclidean path integral.}
the system as a whole, are treated exactly, while the rest of the modes are either supposed to be frozen out or considered perturbatively. A typical example of such an approach is a minisuperspace quantum cosmology with spatially homogeneous modes of the gravitational and matter fields as the only modes subject to dynamics and quantization [8].

Freezing out the dynamical degrees of freedom is not, however, a rigorous procedure beyond the tree-level approximation, because in the quantum domain all the modes have zero-point fluctuations and, therefore, can significantly contribute to the effective dynamics of the collective variables. That is why, we consider the approach, when neither of the modes are completely frozen out, but the variables complimentary to collective degrees of freedom are treated perturbatively. Such a method was developed in mid seventies in the context of matter field models [23] and in recent years was intensively applied in classical and quantum cosmology [49, 54, 55] for the purpose of studies in the theory of the early inflationary Universe and the formation of the large-scale cosmological structure. It consists in separating all the fields into the macroscopic collective variables \( \varphi \) (mainly it is the so-called inflaton scalar field) which drive the quasi-DeSitter dynamics of a spatially homogeneous cosmological background and a set of all inhomogeneous modes \( f \) describing the particle excitations. In [23] we shall consider in much detail the application of such a method to the Hawking model of chaotic inflation, while here we deal with the general theory of this method.

To begin with, consider the one-loop wavefunction (3.15) with the argument \( q^+ \) and make its splitting of the above type into the set of collective variables \( \varphi \) and the rest of degrees of freedom \( f \)

\[
q^+ = \begin{bmatrix} \varphi \\ f \end{bmatrix} = \begin{bmatrix} \varphi \\ 0 \end{bmatrix} + \eta, \quad \eta = \begin{bmatrix} 0 \\ f \end{bmatrix}.
\] (4.1)

Here, in general, \( \varphi \) can be regarded as subcolumn of \( q \) of finite dimensionality (like, for example, a finite set of physical variables responsible for the scale factor and anisotropy parameters in homogeneous Bianchi models). On the contrary, \( f \) is an infinite-dimensional vector (representing, in the same example, all spatially inhomogeneous field harmonics on a symmetric background).

Let us suppose that \( \phi(\tau) \) is a solution of the classical Euclidean equations (3.16) - (3.17)) corresponding to the unperturbed boundary condition at \( \tau^+ \), \( q^+ = (\varphi, 0) \) and
determined entirely by the collective variables $\varphi$. Similarly, we shall denote the solution of (3.16) - (3.17)) with the perturbed boundary conditions (4.1) as $\phi(\tau) + \eta(\tau)$, whence it follows that $\eta(\tau)$ satisfies up to quadratic terms the linearized Euclidean equations of motion subject to the condition of regularity at $\tau = 0$

$$F \frac{d}{d\tau} \eta(\tau) = O(\eta^2), \quad (4.2)$$

$$\eta(\tau_+) = \begin{bmatrix} 0 \\ f \end{bmatrix}, \quad \eta(0) = \text{reg.} \quad (4.3)$$

Thus the perturbation $\eta(\tau_+)$ of the boundary conditions $q = q(\tau_+)$ gives rise to to the perturbation $\eta(\tau)$ of the classical extremal $\phi(\tau)$ and the corresponding perturbation expansion of the Euclidean action:

$$I[\phi + \eta] = I[\phi] + \delta I + \frac{1}{2} \delta^2 I + O(\eta^3), \quad (4.4)$$

where the first-order variation of $I$, after integration by parts with respect to $\tau$, takes the following form

$$\delta I = \int_{0}^{\tau_+} d\tau \frac{\delta I}{\delta \phi(\tau)} \eta(\tau) + \left. \frac{\partial L_E}{\partial \dot{\phi}} \eta \right|_{\tau_+}. \quad (4.5)$$

This integration by parts yields the integral (volume) term containing the left-hand side of Euclidean equations of motion and the surface term – the contribution of the boundary at $\tau = \tau_+$ and, generally, the contribution of the lower integration limit $\tau_- = 0$. But in view of the no-boundary prescription the latter is vanishing because of the regularity conditions at the center of the Euclidean spacetime ball – the zero value of the radial coordinate $\tau_- = 0$. Depending on the type of the mode of the field $\phi(\tau)$ this takes place either because $\partial L_E/\partial \phi(\tau_-) = 0$ (in case of spatially homogeneous mode) or because of $\eta(\tau_-) = 0$ (for spatially inhomogeneous ones).

The second-order variation in (4.4)

$$\delta^2 I = \int_{0}^{\tau_+} d\tau \eta^T(F\eta) + \eta^T(W\eta) \bigg|_{\tau_+} \quad (4.6)$$

with $T$ denoting the transposition of the column $\eta = \eta^i$, follows from varying the equation (1.5) on account of the variational relations

$$\delta \frac{\delta I}{\delta \phi(\tau)} = F \left( \frac{d}{d\tau} \right) \eta(\tau), \quad (4.7)$$

$$\delta \frac{\partial L_E}{\partial \phi} = W \left( \frac{d}{d\tau} \right) \eta(\tau). \quad (4.8)$$
Here $F = F (d/d\tau)$ is, obviously, the differential operator of linearized Euclidean equations (3.18) - (3.19), while $W = W(d/d\tau)$ we shall call the Wronskian operator which enters the following relation

$$\varphi_1^T (F \varphi_2) - (F \varphi_1)^T \varphi_2 = -\frac{d}{d\tau} \left[ \varphi_1^T (W \varphi_2) - (W \varphi_1)^T \varphi_2 \right]$$

(4.9)

valid for arbitrary test functions $\varphi_1$ and $\varphi_2$ and usually used for the construction of the conserved inner product for linear modes of $F (d/d\tau)$. For $F (d/d\tau)$ of the form (3.19) the Wronskian operator equals

$$W(d/d\tau) = a \frac{d}{d\tau} + b.$$  

(4.10)

4.1. Basis functions of linearized field modes

The perturbation $\eta(\tau)$ satisfies the equations of motion (4.2) - (4.3) which can be solved by iterations in $\eta(\tau_+) = (0, f)$. In the linear approximation this solution can be represented in terms of regular basis functions of the Euclidean ”wave” operator $F (d/d\tau)$. They form the full set $u(\tau)$ of solutions of the homogeneous differential equation

$$F (d/d\tau) u(\tau) = 0, \quad u(0) = \text{reg},$$

(4.11)

which are regular in the Euclidean spacetime ball $0 \leq \tau \leq \tau_+$. But before discussing the regularity properties of these basis functions, let us consider their general spin-tensor structure and the form it takes in terms of condensed DeWitt notations.

In view of functional matrix nature of the operator $F (d/d\tau) = F_{ik} (d/d\tau)$ its basis functions also form a matrix $u(\tau) = u^k_A(\tau)$, in which the condensed upper index $k$ (acted upon by indices of the matrix operator) labels the components of a given basis function including its dependence on spatial coordinates, while the lower condensed index $A$ enumerates the basis functions themselves. The infinite ranges and the (discrete or continuous) nature of these indices $k$ and $A$ can be different depending on the parametrization of basic physical variables (3.21) and their possible decomposition in spatial harmonics. What is, however, in common to all field parametrizations is that there is a one to one map between the sets $\{k\}$ and $\{A\}$, so that the matrix
\( \mathbf{u}^i_A \) can be regarded as non-symmetric but quadratic and invertible matrix (parametrically depending on \( \tau \)) having with respect to its infinite-dimensional indices the inverse \( \mathbf{u}^{-1}(\tau) = (\mathbf{u}^{-1})^A_i \):

\[
\mathbf{u}^i_A (\mathbf{u}^{-1})^A_k = \delta^i_k. \tag{4.12}
\]

To illustrate the use of condensed DeWitt indices in the functional matrix \( \mathbf{u}(\tau) \) of the above type, let us consider a simple example of a scalar field \( q^i = \phi(x) \) in flat spacetime, when the condensed index does not contain any discrete labels and reduces to the set of continuous spatial coordinates \( i = x \). The linear equation of motion (4.11) in this case is the Euclidean Klein-Gordon equation which has, as a set of basis functions regular at past infinity \( \tau \to -\infty \), the following solutions

\[
\mathbf{u}^i_A(\tau) \equiv \mathbf{u}_k(x, \tau), \quad i = x, A = k, \tag{4.13}
\]

\[
\mathbf{u}_k(x, \tau) = e^{\omega(k) \tau + ikx}, \quad \omega(k) = \sqrt{k^2 + m^2}, \tag{4.14}
\]

enumerated by the continuous set of spatial momentum vectors \( k \). Every square-integrable function of spatial coordinates \( h^i = h(x) \) can be decomposed in plane waves of the above type in the form

\[
h(x) = \int d^3k e^{\omega(k) \tau + ikx} h_k, \tag{4.15}
\]

which can be rewritten in condensed notations as

\[
h^i = \mathbf{u}^i_A h^A, \quad h^A \equiv h_k \tag{4.16}
\]

(we omit for brevity the time argument here, for it enters this transformation only as a parameter). This means that this equation provides a linear one to one map between \( h^i \) and \( h^A \). The transformation inverse to (4.15)

\[
h_k = \frac{1}{(2\pi)^3} \int d^3x e^{-\omega(k) \tau - ikx} h(x), \tag{4.17}
\]

in condensed indices has a simple form

\[
h^A = (\mathbf{u}^{-1})_i^A h^i \tag{4.18}
\]

where \((\mathbf{u}^{-1})_i^A\) denotes the inverse of \( \mathbf{u}^i_A \) (4.12) with a kernel given by eq.(4.17).
It is also possible to decompose a scalar field in spherical-wave basis functions of the Klein-Gordon equation, enumerated instead of a spatial momentum vector by its continuous norm $k = |k|$ and discrete orbital $l = 0, 1, 2, \ldots$ and azimuthal $m$, $-l \leq m \leq l$, quantum numbers, in which case the condensed label $A = (k, l, m)$ will be of mixed continuous-discrete nature. In spatially closed cosmology the spatial section of spacetime is compact and, therefore, the corresponding set of spatial harmonics in the decomposition (4.16) is discrete and countable. For a general set of linearized physical fields (3.21), the condensed index $A$ of basis functions includes three discrete quantum numbers and the corresponding spin label $A = (n, l, m, \text{spin})$ which are again in one to one correspondence with $i = (x, aT, abTT, \ldots)$, and so on. But the above peculiarities of “fine” structure of various field models can always be encoded in universal relations (4.12), (4.16) and (4.18) written in DeWitt’s notations which we shall imply throughout the paper.

In view of the general decomposition (4.1) of $q^i$, the full set of basis functions $u(\tau)$ contains the modes of both the collective variable $\varphi$ and the rest of degrees of freedom $f$. Generally the collective variables of the system interact very nonlinearly with its microscopic modes. But physically the decomposition (4.1) makes sense when they decouple at least in the linearized approximation, which means that in the basis (4.1) of $\varphi$ and $f$ the differential operator $F(d/d\tau)$ has a block-diagonal structure

$$F(d/d\tau) = \begin{bmatrix} F_\varphi(d/d\tau) & 0 \\ 0 & F(d/d\tau) \end{bmatrix},$$

with $F_\varphi(d/d\tau)$ and $F(d/d\tau)$ acting respectively in subspaces of $\varphi$ and $f$. This structure of $F(d/d\tau)$ has a simple illustration in the case when collective variables $\varphi$ represent a spatially homogeneous background for inhomogeneous modes $f$. On such symmetric background the variables $f$ are decomposed into series of spatially inhomogeneous harmonics which are orthogonal to the homogeneous linear modes of $\varphi$: their bilinear combinations give zero in the cross $\varphi-f$ terms when integrated over a compact spatial section in the quadratic part of the Euclidean action (4.6) and, thus, provide the block-diagonal form (4.19).

The block-diagonal structure (4.19) implies a similar form of all the matrix coeffi-
coefficients of the operator $F \left( \frac{d}{d\tau} \right)$, its Wronskian operator \( W \left( \frac{d}{d\tau} \right) \) and also allows one to choose the matrix $u(\tau)$ in the block-diagonal form with the basis functions $u_\varphi(\tau)$ and $u(\tau)$ of the linearized modes of $\varphi$ and $f$ respectively

\[
W \left( \frac{d}{d\tau} \right) = \begin{bmatrix} W_\varphi \left( \frac{d}{d\tau} \right) & 0 \\ 0 & W \left( \frac{d}{d\tau} \right) \end{bmatrix} \quad \text{(4.20)}
\]

Finally let us consider the conditions which select the regular basis functions of the operator $F$. Due to the no-boundary nature of the underlying manifold $M$, its point of vanishing coordinate radius $\tau = \tau_- \equiv 0$ is a singular point of the radial part of $F \left( \frac{d}{d\tau} \right)$. Indeed, for physical fields (3.21) of all possible spins, $s = 0, 1/2, 1, 3/2, 2, \ldots$, the coefficient $a = a_{ik}$ in $F \left( \frac{d}{d\tau} \right)$ (the Euclidean version of eq.(3.3)) can be collectively written as

\[
a_{ik} = (4g)^{1/2} g^{\tau \tau} g^{a_1 a_2} ... g^{a_s a_2 s} \delta (x_i - x_k), \quad \text{(4.23)}
\]

\[i = (a_1, ... a_s, x_i), \quad k = (a_2, ... a_2 s, x_k), \quad \text{and in the regular metric (2.20) has the following behaviour}
\]

\[
a = a_0 \tau^k + O(\tau^{k+1}), \quad k = 3 - 2s, \quad \tau \to \tau_- = 0, \quad \text{(4.24)}
\]

where $a_0$ is defined by eq.(1.23) with respect to the round metric $c_{ab}$ on a 3-sphere of the unit radius and the unit lapse $g^{\tau \tau} = N^{-2} = 1^{13}$. Therefore the equations (1.11) for basis functions have the form

\[
\left( \frac{d^2}{d\tau^2} + f \frac{d}{d\tau} + g \right) u(\tau) = 0, \quad \text{(4.25)}
\]

with the coefficients $f$ and $g$ having the following asymptotic behaviour

\[
f = \frac{k}{\tau} I + O(\tau^0), \quad g = \frac{g_0}{\tau^2} + O(\tau^{-1}). \quad \text{(4.26)}
\]

\(^{13}\) Even though the expression (1.23) is formally valid only for integer-spin fields, this behaviour with the parameter $k = 3 - 2s$ also holds for half-integer spins, because every gamma matrix substituting the corresponding metric coefficient in (1.23) contributes one power of $\tau^{-1}$. 

33
Here the leading singularity in the potential term $g$ originates from the spatial Laplacian $g^{ab} \nabla_a \nabla_b$ entering the operator $F$, which scales in the metric (2.21) as $1/\tau^2$, and the leading term of $f$ is always a multiple of the unity matrix $I$ with the same parameter $k = 3 - 2s$ as in (1.24). In the representation of spatial harmonics, the eigenfunctions of a spatial Laplacian, the (functional) matrix $g_0$ can be also diagonalized, $g_0 = \text{diag}\{-\omega_i^2\}$, so that, without the loss of generality, the both singularities in (1.25) can be characterised by simple numbers $k$ and $\omega^2 = \omega_i^2$ for every component of $\mathbf{u} = \mathbf{u}^i$.

As it follows from the theory of differential equations with singular points [56], in this case there are two types of solutions $\mathbf{u}_-(\tau)$ and $\mathbf{u}_+(\tau)$ differing by their behaviour near $\tau_- = 0$:

$$F \mathbf{u}_\pm = 0,$$

$$\mathbf{u}_-(\tau) = U_- \tau^{\mu_-} + O(\tau^{1+\mu_-}),$$

$$\mathbf{u}_+(\tau) = V_+ \tau^{\mu_+} + O(\tau^{1+\mu_+}),$$

where $\mu_\pm$ are the roots of the quadratic equation involving only the coefficients of leading singularities $\mu^2 + (k - 1) \mu - \omega^2 = 0$. In view of non-negativity of $\omega^2$ (the eigenvalue of $-c^{ab} \nabla_a \nabla_b$) these roots are of opposite signs, $\mu_- \mu_+ = -\omega^2 \leq 0$, and we can choose $\mu_-$ to be non-negative in order to have $\mathbf{u}(\tau) = \mathbf{u}_-(\tau)$ as a set of regular basis functions at $\tau_- = 0$, the remaining part of them $\mathbf{u}_+(\tau)$ being singular. By our assumption the operator $F$ does not have zero eigenvalues on the Euclidean spacetime of the no-boundary type (otherwise, its functional determinant and the one-loop prefactor of the wavefunction are not defined). Therefore, there are no basis functions which are simultaneously regular at $\tau_- = 0$ and vanishing for positive $\tau \leq \tau_+$, and their matrix can be considered invertible everywhere in this range of $\tau$ except the origin $\tau_- = 0$.\footnote{This property holds until the first caustic of solutions of classical equations for physical variables at which $\mathbf{u}_-(\tau) = 0$ (because $\mathbf{u}_-(\tau)$ is a derivative of the parametric family of these solutions with respect to their parameter, this relation is just an equation for their enveloping curve). However, this is not the caustic of Einstein equations in superspace of the theory, responsible for the transition from the Euclidean to the Lorentzian regime. As it follows from the Hawking model of chaotic inflation [25], the latter is generated by zeros of the Faddeev-Popov determinant of the ADM reduction procedure, rather than by zeros of $\mathbf{u}_-(\tau)$. Degeneration of the Faddeev-Popov matrix indicates the presence of Gribov}

In what follows we shall denote the regular basis functions either by
$u(\tau)$ or by $u_-(\tau)$, when we prefer to emphasize their regularity in the ”center” of the Euclidean ball $\tau_\sim \equiv 0$.

4.2. Perturbation theory in microscopic variables and $\hbar$-expansion

The basis functions of the above type will serve us as a technical tool for two purposes: the perturbation theory in microscopic variables $f$ and the reduction method for the functional determinants in the one-loop prefactor. Let us here, first, develop this perturbation theory in powers of $\eta(\tau) = O(f)$ and show how it actually reduces to the expansion in $\hbar$.

To begin with, note that in virtue of the invertibility of $u(\tau)$ the linearized solution of the boundary-value problem (4.2) - (4.3) has the form:

$$\eta(\tau) = u(\tau) u^{-1}(\tau_+) \eta(\tau_+) + O(\eta^3),$$

Here we suppress the indices of functional matrices $u(\tau) = u^i_A(\tau), \ u^{-1}(\tau_+) = [u^{-1}(\tau_+)]^A_i$ and columns $\eta(\tau) = \eta^i(\tau)$ implying again the DeWitt rule of summation-integration over supercondensed labels. Substituting this solution into the linear and quadratic terms of the perturbed Euclidean action (4.4) one can see that the volume contributions vanish in the quadratic approximation due to the background $\delta I/\delta \phi(\tau) = 0$ and linearized (4.2) equations of motion. The remainig surface terms at the boundary $\tau = \tau_+$ give

$$I[\phi + \eta] = I[\phi] + \left[ \frac{\partial L_E}{\partial \dot{\phi}} \eta + \frac{1}{2} \eta^T (Wu) u^{-1} \eta \right]_{\tau_+} + O(\eta^3).$$

In view of the form of the boundary-value perturbation (4.3) only the $f$-component of the Euclidean momentum contributes to the right-hand side of this equation, but since it is computed at the field background $\phi(\tau) = (\varphi(\tau), 0)$ with identically vanishing $f(\tau)$ this momentum component $\partial L_E/\partial \dot{f}(\tau_+)$ also vanishes and only the quadratic form in $\eta$ survives in (4.31). This form in its turn can be completely rewritten in terms of copies in the quantization procedure and serves as a strong motivation for the third quantization of gravity \[13, 14, 15\], therefore, as it could have been expected on physical ground, the Euclidean-Lorentzian tunnelling phenomena are directly related to the physics of baby-universe production.
the quantities on the \( f \)-subspace due to (4.3) and the block-diagonal structure of basis functions and the Wronskian operator:

\[
I [ \phi + \eta ] = I [ \phi ] + \frac{1}{2} f^T D (\tau_+) f + O ( f^3 ),
\]

(4.32)

\[
D (\tau_+) = [ W (d/d\tau_+) u (\tau_+) ] u^{-1} (\tau_+).
\]

(4.33)

This expression can be used in the equation (3.15) for the wavefunction together with the preexponential factor \( \left( \text{Det} F / \text{Det} a \right)^{-1/2} \) calculated at the perturbed classical background \( \phi (\tau) + \eta (\tau) = O (f) \),

\[
\left( \frac{\text{Det} F [ \phi + \eta ]}{\text{Det} a [ \phi + \eta ]} \right)^{-1/2} = \left( \frac{\text{Det} F [ \phi ]}{\text{Det} a [ \phi ]} \right)^{-1/2} + O (f).
\]

(4.34)

Our purpose now will be to show that respectively cubic \( O (f^3) \) and linear \( O (f) \) corrections in eqs.(4.32) and (4.34) give the contributions to the wavefunction of one and the same order \( O (\hbar^{1/2}) \), belonging to the two-loop approximation of a semiclassical expansion. Substituting (4.32) and (4.34) into (3.13) and reexpanding the exponential in terms of \( O (f^3) \) we get

\[
\Psi (q_+, \tau_+) = \left( \frac{\text{Det} F [ \phi ]}{\text{Det} a [ \phi ]} \right)^{-1/2} \exp \left\{ -\frac{1}{\hbar} I [ \phi ] - \frac{1}{2\hbar} f^T D (\tau_+) f \right\}
\]

\[
\times \left[ 1 + O (f) + O (f^3/\hbar) \right].
\]

(4.35)

Thus, up to corrections of the above type the wavefunction is a gaussian state of microscopic variables \( f \). The gaussian exponent suppresses the states with large \( f \), because of the positive definiteness of the matrix \( D (\tau_+) \), which is a direct corollary of the boundedness of the Euclidean action from below. If this property is satisfied and the extremal \( \phi (\tau) \) realizes at least a local minimum of the action, then its quadratic perturbation (4.6) is positive-definite for arbitrary \( \eta (\tau) \), \( \delta^2 I > 0 \). On solutions of linearized equations of motion with the boundary data (4.3) specified by \( f \) it reduces to

\[
\delta^2 I = f^T D (\tau_+) f > 0
\]

(4.36)

and, thus, provides the negative definiteness of the quadratic form in the exponential of (4.35). Therefore one can use the asymptotic bound
$$e^{-\frac{1}{2\hbar} f^T D f} f^n = O (\hbar^{n/2}), \quad \hbar \to 0,$$

(4.37)

which can be rigorously proved under certain assumptions of uniform regularity of $D$ for a wide class of positive definite quadratic functionals on the space of functions $f$ of many variables [57]. In view of this bound the linear $O(f)$ and cubic $O(f^3/\hbar)$ corrections in (4.35) are both $O(\hbar^{1/2})$, and therefore go beyond the one-loop approximation considered in this paper. Actually, such terms of half integer power in $\hbar$ belong to the two-loop approximation $O(\hbar)$ because in quantum averages with the gaussian quantum state only even powers of $f$ will give a nonvanishing contribution and, therefore, the corrections $O(\hbar^{1/2})$ acquire at least one extra power of $\hbar^{1/2}$.

4.3. The basis functions algorithm for the one-loop preexponential factor

As is shown in [24] the regular basis functions $u_-(\tau)$ can be used for the calculation of the one-loop preexponential factor of the wavefunction (4.35). The nature of this procedure consists in the reduction which allows to obtain the functional determinant $\text{Det} F$ in terms of the quantity of the lower functional dimensionality – the determinant of the non-degenerate matrix of regular basis functions $u_A^i(\tau)$ taken with respect to its condensed indices. These basis functions have the behaviour (4.28) and are defined up to linear $\tau$-independent recombinations. The latter can be used to make the coefficient $U_-$ completely independent of the background fields $\phi$ on $M$ and, without loss of generality, equal the functional matrix unity $I$. Then this algorithm for a one-loop prefactor takes the form [24]

$$\left( \frac{\text{Det} F}{\text{Det} a} \right)^{-1/2} = \text{Const} \left[ \text{det} u_-(\tau_+) \right]^{-1/2},$$

(4.38)

$$u_-(\tau) = I \tau^{\mu_-} + O (\tau^{1+\mu_-}), \quad \tau \to 0.$$ 

(4.39)

15 This reduction method is actually a particular case of the Pauli-Van Vleck-Morette formula [58] for the one-loop kernel of the heat equation in the proper-time interval $[\tau_-, \tau_+]$, adjusted to the case of the no-boundary type, when $\tau_-$ is a singular point of the dynamical equations having a special behaviour (4.28) of the linearized modes [24]. The corresponding algorithm (1.38) was also obtained in the authors’ paper [21] by a special technique of the $\zeta$-functional regularization for operators with the explicitly unknown spectra on manifolds with a boundary.
Thus, combining eqs. (4.35) and (4.37) with this reduction algorithm, we get the one-loop Euclidean wavefunction \( \Psi (q^+, \tau^+) = \Psi (\varphi, f, \tau^+) \) of physical variables \( q^+ = (\varphi, f) \) in the form

\[
\Psi (\varphi, f, \tau^+) = \Psi (f, \tau^+) \bigg|_{\phi = \phi (\tau | \varphi, \tau^+)},
\]

\[
\Psi (f, \tau^+ \equiv \text{Const} \left( \det u_{\tau^+} [\phi] \right)^{-1/2} \exp \left\{ -\frac{1}{\hbar} I [\phi] - \frac{1}{2\hbar} f^T D (\tau^+) f \right\} \times \left[ 1 + O (\hbar^{1/2}) \right].
\]

In the equation (4.41) we clearly separated the dependence of the wavefunction on the collective variables from that on the microscopic ones. In contrast to a simple quadratic dependence on \( f \), the variables \( \varphi \) enter \( \Psi (\varphi, f, \tau^+) \) through the functional argument \( \phi (\tau) \) of \( \Psi (f, \tau^+) \), for they parametrize the extremal \( \phi (\tau) = \phi (\tau | \varphi, \tau^+) \) of the Euclidean equations of motion (3.16)-(3.17) with the boundary data \( q^+ = (\varphi, 0) \).

The argument \( \phi \), in its turn, enters the auxiliary functional (4.41) through the Euclidean action \( I [\phi] \), one-loop preexponential factor \( \left( \det u_{\tau^+} [\phi] \right)^{-1/2} \) (subscript \( [\phi] \) indicating the functional dependence on the field background) and the matrix of quantum dispersions \( D (\tau^+) = D (\tau^+ | \phi) \).

5. The method of complex extremals

According to the discussion of Sect.3 the "Lorentzian" wavefunction can be obtained from the one-loop expression (4.41) above by the analytic continuation (3.14) into the complex plane of the Euclidean time:

\[
\Psi_L (\varphi, f, t^+) = \Psi (f, z^+) \bigg|_{\Phi = \Phi (z | \varphi, z^+)},
\]

When this analytic continuation proceeds along the contour \( C^+ = C_E \cup C_L \) of Sect.2, the extremal field \( \Phi (z) \big|_{C^+} = \Phi (z | \varphi, z^+) \) on its Euclidean and Lorentzian segments will not generally represent real functions \( \phi (\tau) \) and \( q(t) \). The class of models in which all the fields can be real on the analytically matched Lorentzian and Euclidean sections of spacetime is very limited and comprises the so-called real tunnelling geometries [31].

In this section we present the semiclassical technique that allows to handle the general
case of complex extremals by reducing the formalism to real-valued solutions of both Euclidean and Lorentzian equations of motion.

5.1. Matching conditions between the Euclidean and Lorentzian spacetimes

To begin with, we shall reserve the notations \( q(t) \) and \( \phi(\tau) \) for real parts of the complex field \( \Phi(z) \) on respectively Lorentzian and Euclidean segments of the contour \( C_+ \), denote their imaginary parts by \( h(t) \) and \( \eta(\tau) \) and also introduce the notation \( Q(t) \) for the full complex field on \( C_L \):

\[
\Phi(\tau) = \phi(\tau) + i\eta(\tau), \quad Q(t) = q(t) + ih(t), \quad (5.2)
\]

\[
Q(t) \equiv \Phi(z) |_{C_L} = \Phi(\tau_B + it). \quad (5.3)
\]

Then the complex action \((3.5)\) on this contour takes the following form

\[
\mathcal{I}[\Phi(z)]|_{C_+} = I[\Phi(\tau)] - iS[Q(t)], \quad (5.4)
\]

where \( I[\Phi(\tau)] \) and \( S[Q(t)] \) are the Euclidean and Lorentzian actions \((3.9)\) and \((3.1)\) as functions of their complex functional arguments.

Let us now consider the variational principle for this Lorentzian-Euclidean action which gives the saddle point of the path integral \((3.11)\) for the no-boundary wavefunction. The first-order variation of \((5.4)\) can be obtained by using the equation \((4.5)\) and its analogue for the Lorentzian action

\[
\delta \mathcal{I} = \int_0^{t_+} dt \frac{\delta I}{\delta \Phi} \delta \Phi + \frac{\partial L_E}{\partial \dot{\Phi}} \delta \Phi \bigg|_{\tau_B} - i \int_0^{t_+} dt \frac{\delta S}{\delta Q} \delta Q + i \frac{\partial L}{\partial Q} \delta Q \bigg|_{t=0}. \quad (5.5)
\]

The fields and their variations satisfy the no-boundary regularity conditions at \( \tau = 0 \) and fixed boundary conditions \((3.12)\) at \( t = t_+ \), \( Q(t_+) = q_+ \). As a result, \( \delta Q(t_+) = 0 \) and the total-derivative term vanishes at the boundary of spacetime \( t = t_+ \) as well as in its regular center \( \tau = 0 \). Moreover, in view of the analytic continuation \((5.3)\), the Euclidean and Lorentzian fields satisfy the matching conditions

\[
\Phi(\tau_B) = Q(0), \quad (5.6)
\]
which means that $\delta \Phi(\tau_B) = \delta Q(0)$. Therefore, equating to zero separately the volume and surface terms of (5.5) in the variational equation $\delta I = 0$, one can get the system of Euclidean and Lorentzian equations of motion

$$\frac{\delta I}{\delta \Phi} = 0, \quad \frac{\delta S}{\delta Q} = 0$$

(5.7)

for fields subject to special matching conditions at the nucleation point $\tau = \tau_B (t = 0)$

$$\frac{\partial L_E}{\partial \Phi} \bigg|_{\tau_+} + i \frac{\partial L}{\partial Q} \bigg|_{t=0} = 0.$$  

(5.8)

These matching conditions show that the tunnelling geometries with real physical fields exist only in case when both the Euclidean $\partial L_E/\partial \Phi$ and Lorentzian $\partial L/\partial Q$ momenta separately vanish at the nucleation point. The covariant version of this statement in the gravitational sector of all fields sounds as a vanishing of the extrinsic curvature $K_{ab}$ of the nucleation surface [41, 31, 32]. This corresponds to the fact that the canonical gravitational momentum $p$ is linear in the second fundamental form of a hypersurface in the slicing of spacetime associated with the Hamiltonian formalism of the theory, and the above surface of the Euclidean-Lorentzian transition is supposed to be a member of such a slicing. In the example of the DeSitter Universe generated by the inert cosmological constant this surface coincides with the equator of the Euclidean four-dimensional sphere at which the time derivative of the scale factor (1.4) vanishes when approaching both from inside the Euclidean hemisphere and from the inflationary Lorentzian spacetime.

In the general case, when the momenta are nonzero at $\tau = \tau_B$, the fields $\Phi(\tau)$ and $Q(t)$ become complex, and the very notion of the Euclidean-Lorentzian transition becomes questionable, because complex physical fields generate complex-valued metric tensors which can hardly be ascribed to spacetimes of either Euclidean or Lorentzian signature. We shall show, however, that the Euclidean-Lorentzian decomposition of the full quantum dynamics still makes sense within the $\hbar$-expansion. To show this we shall develop the perturbation expansion in the imaginary parts $\varepsilon = (\eta, \hbar)$ of the complex fields (5.2) and demonstrate that it corresponds to the asymptotic expansion in $\hbar^{1/2}$.
5.2. Perturbation theory in the imaginary corrections and the $\hbar$-expansion

This perturbation expansion begins with substituting the expressions (5.2) into the classical equations and the matching conditions (5.6) - (5.8) and expanding the result in powers of $\varepsilon = (\eta, \hbar)$. The separation of the real and imaginary parts in (5.7) then immediately leads to the following system of equations for the real part of the Euclidean extremal and its imaginary part treated as a perturbation

$$\frac{\delta I}{\delta \phi (\tau)} = O (\eta^2), \quad F (d/d\tau) \eta (\tau) = O (\eta^3). \tag{5.9}$$

Similar equations hold for the Lorentzian fields

$$\frac{\delta S}{\delta q (t)} = O (h^2), \quad F_L (d/dt) h(t) = O (h^3), \tag{5.10}$$

where $F_L (d/dt)$ is the Lorentzian wave operator of linearized equations (at the background of $q(t)$), analogous to its Euclidean version (3.18)

$$F_L \equiv F_L (d/dt) \delta (t - t') = \frac{\delta^2 S [q]}{\delta q(t) \delta q(t')} \tag{5.11}.$$

The expansion of the matching condition (5.8) up to linear terms in $\varepsilon$ can be performed with the aid of the variational equation (4.8) and its Lorentzian equation

$$\delta \frac{\partial L_E}{\partial \dot{\phi}} \bigg|_{\tau_+} = W_L (d/dt) \delta q(t), \tag{5.12}$$

which serves as a definition of the Wronskian operator $W_L = W_L (d/dt)$ for (5.11).

The separation of the real and imaginary parts of eq.(5.8) then yields the system of matching conditions coupling the dynamics of Euclidean and Lorentzian variables

$$\delta \frac{\partial L_E}{\partial \phi} \bigg|_{\tau_+} = W_L \hbar \bigg|_{t=0} + O (\varepsilon^2), \tag{5.13}$$

$$\delta \frac{\partial L_L}{\partial \dot{q}} \bigg|_{t=0} = - W \eta \bigg|_{\tau_+} + O (\varepsilon^2). \tag{5.14}$$

Now we can calculate up to quadratic terms in $\varepsilon$ the complex action (5.4) at its complex extremal. Linear terms of the action follow from (5.5), while the quadratic terms can be obtained by using the equation (4.6) for the Euclidean action and its
obvious Lorentzian analogue involving the operators $F_L$ and $W_L$. In virtue of the equations (5.9) - (5.10) all the volume terms turn to be $O(\varepsilon^3)$, so that the contribution of imaginary corrections in the quadratic approximation reduces to the sum of surface terms at the nucleation point $\tau = \tau_+ (t = 0)$

$$\mathcal{I}[\Phi] = I[\phi] - i S[q] + i \left. \frac{\partial L_E}{\partial \dot{\phi}} \eta \right|_{\tau_+} - \left. \frac{\partial L_L}{\partial \dot{q}} h \right|_{t=0}$$

$$- \frac{1}{2} \eta^T(W\eta) \bigg|_{\tau_+} - \frac{i}{2} h^T(W_L h) \bigg|_{t=0} + O(\varepsilon^3).$$

(5.15)

Here we took into account the reality of $q_+(h(t_+) = 0)$ and the no-boundary regularity conditions leading to vanishing surface terms at $t = 0$ and $\tau = 0$. Now notice that in virtue of (5.6) $h(0) = \eta(\tau_+)$. Then the use of equations (5.13) - (5.14) allows to rewrite the above expression as

$$\mathcal{I}[\Phi] = I[\phi] - i S[q] + \frac{1}{2} \varepsilon^T(W\varepsilon) + O(\varepsilon^3),$$

(5.16)

where $\varepsilon^T(W\varepsilon)$ denotes the full resulting quadratic form in the variables $\varepsilon \equiv \text{Im} \Phi(z) = (h(t), \eta(\tau))$

$$\varepsilon^T(W\varepsilon) = \eta^T(W\eta) \bigg|_{\tau_+} + i h^T(W_L h) \bigg|_{t=0}.$$ (5.17)

The crucial point of our derivations in this section is that the net effect of the Lorentzian-Euclidean matching conditions (5.13)- (5.14) and linear terms in the expression (5.13) consists in changing the overall sign of the quadratic form in $\eta$ and $h$. This has a drastic consequence for the asymptotic $h$-expansion of the wavefunction (5.1) with the complex extremal $\Phi(z \mid \varphi, z_+)$. Indeed, substituting the expression (5.14) into the functional $\Psi[\Phi](f, z_+)$ given by (4.41) and reexpanding everything, except this exponentiated quadratic form, in powers of $\varepsilon$, one has

$$\Psi_L(\varphi, f, t_+) = e^{-\frac{1}{2h}\varepsilon^T(W\varepsilon)} \left\{ \Psi[\text{Re} \Phi](f, z_+) + O(\varepsilon) + O(\varepsilon^3/h) \right\},$$ (5.18)

Since the variables $\eta(\tau)$ satisfy up to higher order terms the linearized equations of motion (5.9) and no-boundary regularity conditions, the real part of this quadratic form coincides with the part of the Euclidean action quadratic in the field disturbancies $\eta$

$$\text{Re} \left[ \varepsilon^T(W\varepsilon) \right] = \delta^2_\eta I + O(\eta^4)$$ (5.19)

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and is positive definite by the assumption that the extremal realizes the local minimum of the Euclidean action.

Therefore, one can use the analogue of the asymptotic bound (4.37) to show that

$$e^{-\frac{1}{2\hbar}\varepsilon^T (\mathcal{W}\varepsilon)} \varepsilon^n = O\left(\bar{\hbar}^{n/2}\right), \: \bar{\hbar} \to 0,$$

whence it follows that all the perturbation corrections of the eq.(5.18) in powers of $\varepsilon$ actually belong to higher orders of a semiclassical expansion. In contrast to (4.37) the exponentiated quadratic form here has a kernel which is a differential operator $\mathcal{W}$ with respect to $t$ and $\tau$. This generalization, however, does not break the validity of the bound, because, according to [57], the statements like (4.37) or (5.20) are valid for quadratic functionals of integro-differential nature provided sufficient smoothness of their functional arguments. Another potential difficulty with the above bound is that the real part of our quadratic form involves only $\eta$ variables, and one would think that the powers of the Lorentzian field $h(t)$ are not exponentially suppressed in (5.20). This is not, however, the case because the fields $\eta(\tau)$ and $h(t)$ are not independent, for they satisfy the linearised differential equations (5.9)-(5.10) on the Euclidean and Lorentzian segments of $C^+$ with common boundary conditions at the nucleation point $\eta(\tau_B) = h(0)$. Therefore the Lorentzian imaginary disturbancies $h(t)$ can be parametrized in terms of $\eta(\tau)$ by certain integral operation, which provides the asymptotic bound (5.20) for the whole set of $\varepsilon$.

Thus, despite the complex nature of classical extremals $\Phi(z)$, the semiclassical expansion for tunnelling systems can still be performed on the real-valued background $\text{Re} \Phi(z) = (\phi(\tau), q(t))$, and with the corresponding elements of the Feynman diagrammatic technique – the inverse propagator, its basis functions and the matrix (4.33) of quantum dispersions for microscopic variables:

$$F = F[\text{Re} \Phi], \quad D_L(t) \equiv D(\tau_B + it)[\text{Re} \Phi].$$

16 This statement can be proved more rigorously by decomposing $\eta(\tau)$ and $h(t)$ in the basis functions of respectively Euclidean and Lorentzian wave operators subject to no-boundary regularity conditions at $\tau = 0$ and Dirichlet boundary conditions at $t = t_+$ (remember that $h(t_+) = 0$). Then the quadratic form and perturbative corrections become expressed entirely in terms of the independent set of variables $f \equiv \eta(\tau_B) = h(0)$, and one can apply the asymptotic bound (4.37) with some effective complex kernel $D$ having a positive definite real part.

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Imaginary corrections everywhere except the quadratic form of the action (5.16) can be treated by perturbations generating in higher orders additional set of Feynman diagrams.

One should emphasize a crucial role played by the boundedness from below of the Euclidean action, which provides the positivity of the form (5.19). In Einstein gravity theory this property is violated in the sector of the conformal mode which, as is widely believed, enters the set of physical variables in spatially closed quantum cosmology. The only known procedure of handling this mode consists in the rotation of its integration contour in the path integral to the complex plane so that to make the integration convergent due to a reversal of sign in the exponentiated quadratic form. In contrast to a widespread practice, this means that the same conformal rotation must be done in the argument of the wavefunction, which implies the complexification of the configuration-space point \( q + \), above, \( h (t_+) \neq 0 \), and the corresponding modification of the formalism of complex extremals. We shall not consider this modification here, and in what follows assume good properties of the Euclidean action. In this paper this will be justified by isolating the conformal mode into the sector of collective variables and considering (see Sect.8) only the high-energy behaviour of their quantum distribution. This behaviour is unaffected by the tree-level properties of the classical action and is determined by the quantum anomalous scaling of the theory (see discussion in Sect.9). As concerns the microscopic modes \( f \), for which the Euclidean action is supposed to have good positivity properties, we shall show the efficiency of the above technique for interpreting their quantum state in the Lorentzian world which nucleates from the Euclidean spacetime.

6. Euclidean vacuum via nucleation of the Lorentzian Universe from the Euclidean spacetime

Combining eqs. (4.41), (5.18) and (5.20) one can obtain the needed analytic continuation of the Euclidean wavefunction into the Lorentzian regime. Under this analytic

\[ ^{17} \text{In asymptotically flat spacetime this mode is unphysical due to the general coordinate invariance of the theory and the gravitational constraints – the basis of powerful positive-energy and positive-action theorems.} \]

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continuation the originally real Euclidean basis functions \( u_-(\tau) \) go over into complex functions \( u_-(z) \) on the contour \( C_+ \) \( (2.21) \). The complex nature of \( u_-(z) \equiv u_-(z) \mid _\Phi \) originates from the complexity of both their time argument \( z = \tau_B + it \) and the functional argument \( \Phi (z) \) – the complex classical background \( (5.2) \) at which their wave operator \( (3.18) \) is determined. Thus, if we introduce the following notation 

\[
\begin{align*}
\nu (t) &= \left( u_-(\tau_B + it) \mid _\Phi \right), \\
\nu^* (t) &= u_-(\tau_B + it) \mid _\Phi,
\end{align*}
\]

then the wavefunction of the Lorentzian Universe \( (5.1) \) takes the form 

\[
\Psi_L (\varphi, f, t_+) = \text{Const} \left[ \det \nu^* (t_+) \right]^{-1/2} \exp \left\{ -\frac{1}{2\hbar} f^T D_L (t_+) f \right\} 
\times e^{-\frac{\hbar}{2} \mathcal{I} \mid _\Phi} \left[ 1 + O \left( \hbar^{1/2} \right) \right], 
\]

where we have reabsorbed the quadratic form in \( \varepsilon \) into the full complex classical action \( (5.10) \).

In virtue of the relation \( (5.1) \) the functions \( (\nu (t), \nu^* (t)) \) satisfy the complex conjugated equations 

\[
\begin{align*}
F_{[\Phi]} (d/\!dt) \nu^* (t) &= 0, \quad \left[ F_{[\Phi]} (d/\!dt) \right]^* \nu (t) = 0, 
\end{align*}
\]

which are a direct corollary of the equation for \( u_-(z) \mid _\Phi \). However, the method of complex extremals allows us to treat the imaginary part \( \varepsilon \equiv \text{Im} \Phi (z) \) by perturbations and consider, instead of the complex operator \( F_{[\Phi]} (d/dz) \) acting on the contour \( C_+ = C_E \cup C_L \), the real Euclidean \( F (d/d\tau) \) and Lorentzian operators \( F_L (d/dt) \) acting on the corresponding segments \( C_E \) and \( C_L \) and related to \( F_{[\Phi]} (d/dz) \) by

\[
\begin{align*}
F_{[\Phi]} (d/dz) \bigg|_{C_E} &= F (d/d\tau) + O \left( \varepsilon \right), \\
F_{[\Phi]} (d/dz) \bigg|_{C_L} &= -F_L (d/dt) + O \left( \varepsilon \right).
\end{align*}
\]

Here the Euclidean operator is defined by eq.\( (5.21) \), while the Lorentzian operator

\[
F_L (d/dt) = -F_{[\text{Re} \Phi]} (d/\!dt) \bigg|_{C_L}.
\]

\[45\]
coincides with the hyperbolic wave operator (5.11) of the Lorentzian field theory, calculated at the real-valued background \( q(t) = \text{Re} Q(t) \), and has the form analogous to (3.19)

\[
F_L \left( \frac{d}{dt} \right) = -a \frac{d}{d\tau} - b_L + b_L^T \frac{d}{d\tau} - c.
\]  

(6.7)

Here the coefficients \( a \) and \( c \) are trivially related to their Euclidean versions (and, therefore, have the same notation), while the coefficient \( b_L \) represents a Wick rotation of its Euclidean counterpart: \( b_L = ib \).

Obviously, in view of eqs. (6.3)-(6.5) the functions (6.1) satisfy the inhomogeneous equations

\[
F_L v = O(\varepsilon), \quad F_L v^* = O(\varepsilon),
\]

but their right-hand sides \( O(\varepsilon) = O(\hbar^{1/2}) \) can be again discarded in the one-loop approximation due to the semiclassical technique of complex extremals. In what follows we shall work with this one-loop accuracy and, therefore, assume the following real-valued differential equation for the complex Lorentzian basis functions

\[
F_L v = 0, \quad F_L v^* = 0.
\]  

(6.8)

The Wronskian operator \( W_L (d/dt) \) corresponding to (6.7) generates the variational equation (5.12) for the Lorentzian canonical momentum and also enters the Lorentzian analogue of the Wronskian relation (4.9) valid for arbitrary test functions \( h_1(t) \) and \( h_2(t) \)

\[
h_1^T (F_L h_2) - (F_L h_1)^T h_2 = -\frac{d}{dt} \left[ h_1^T (W_L h_2) - (W_L h_1)^T h_2 \right],
\]  

(6.9)

\[
W_L (d/dt) = a \frac{d}{dt} + b_L, \quad W_L (d/dt) = iW_{[\text{Re} \Phi]} (d/idt) \mid _{C_L}.
\]  

(6.10)

A very important property of the above Lorentzian wave and Wronskian operators is their reality

\[
F_L^* (d/dt) = F_L (d/dt), \quad W_L^* (d/dt) = W_L (d/dt),
\]  

(6.11)

\[\text{18}\]

Alternatively, one can view \( v \) and \( v^* \) to be exact solutions of eq. (6.8) as well as \( u_-(\tau) \equiv u_-(\tau) \mid _{\text{Re} \Phi} \) to be the exact basis functions of the operator \( F(d/d\tau) \) (5.21). But these \( v \) and \( u_\ldots \) are not exactly related by (6.1) because the collection \( (F(d/d\tau), F_L (d/dt)) \) cannot be regarded as a smooth differential operator \( F(d/dz) \) on \( C_+ = C_E \cup C_L \). discarding \( \text{Im} \Phi \) in (5.21) and (6.4) means that the first-order derivatives of \( \text{Re} \Phi(z) \) and the operator coefficients \( a, b \) and \( c \) are discontinuous at \( z = \tau_B \) and result in the discontinuity of the second-order derivatives of \( v \) and \( u_\ldots \). Therefore, for such definition of \( v \) and \( u_\ldots \), one can at most demand matching their zeroth and first-order derivatives at \( z = \tau_B \), which implies the validity of (6.1) as well as the relation \( d u_- / d\tau = id v^* / dt \) only at \( z = \tau_B \) (\( t = 0 \)).
which is crucial for establishing the conventional complex structure on the space of classical solutions and gives rise in the wording of [32] to the "beginning of time". Let us emphasize here, that, in its turn, this property follows from the following two features of the above formalism. Firstly, it relies on the reality of both the Lorentzian and Euclidean Lagrangians of physical variables, related by the Wick rotation (3.6), when they are calculated at real-valued fields \( q(t), \phi(\tau) \) (cf. eqs. (3.7) - (3.9)). As it was discussed in Sect.3, this condition must be granted by a proper choice of gauge for physical variables, which should not introduce into the Lagrangian (3.6) the explicit time dependence generating its imaginary part on the complex contour \( C_+ \). And, secondly, it is based on the semiclassical method of Sect.5 which always reduces the calculations and the corresponding elements of the diagrammatic technique to those of the real field background \( \text{Re} \Phi(z) = (q(t), \phi(\tau)) \).

For the second-order differential equation \( F_L h = 0 \) with real coefficients, the complex linear space of solutions \( h(t) \) can be equipped with the indefinite inner product conserved in time

\[
< h_1, h_2 > = i \left[ h_1^\dagger (W_L h_2) - (W_L h_1)^\dagger h_2 \right],
\]

where \( h^\dagger \equiv (h^*)^T \) is a Hermitian conjugation involving both the transposition (of vectors and matrices) and the complex conjugation. When the functions \( h(t) \) are related by the analytic continuation to their Euclidean counterparts \( \varphi(\tau) \), this inner product can be expressed in terms of the Wronskian construction of the Euclidean operator \( F \)

\[
\varphi_1^T (W \varphi_2) - (W \varphi_1)^T \varphi_2 = -< h_1^*, h_2 >, \quad h_{1,2}(t) = \varphi_{1,2}(\tau_B + it).
\]

Now, if we take as \( \varphi_{1,2}(\tau) \) two regular basis functions of the Euclidean operator \( u_-(\tau) \), which have a vanishing Wronskian

\[
u_-(u_-(W u_-) - (W u_-)^T u_- = 0
\]

resulting from their regular behaviour at \( \tau = 0 \) (4.39), then it follows that the complex conjugated Lorentzian basis functions \( (5.1) \) satisfy the following orthogonality relation

\[
< v^*, v > = 0.
\]
On the other hand, Lorentzian basis functions of one "positive frequency" have the conserved matrix of inner products

\[
\Delta = \langle v, v \rangle, \quad \Delta \equiv \Delta_{AB},
\]

which can be calculated at the point of nucleation \( t = 0 \) \( (\tau = \tau_B) ) where the following matching conditions hold between the Lorentzian and Euclidean modes: \( u_-(\tau_B) = v(0) = v^*(0), \ (Wu_-(\tau_B) = iW_L v^*(0) \). In virtue of these matching conditions this matrix equals the following real symmetric matrix

\[
\Delta = 2 \ u^T(Wu_-) \mid_{\tau_B}, \quad \Delta = \Delta^T, \quad \Delta = \Delta^*,
\]

which coincides with the kernel of the positive definite quadratic part of the Euclidean action in field disturbances \( \delta \phi (\tau) = u(\tau) \eta \) satisfying linearized equations of motion (cf.eq.(4.31))

\[
\delta^2 I = \eta^T \left[ u^T(Wu_-) \right]_{\tau_B} \eta = \frac{1}{2} \eta^T \Delta \eta > 0.
\]

Therefore, \( \Delta \) is a real positive-definite symmetric matrix, which together with the orthogonality relations (6.13) implies that the Lorentzian modes \( v(t) \) and \( v^*(t) \) can be regarded as a set of positive and negative frequency modes of the Lorentzian wave operator.

Thus far we have considered the Lorentzian linearized modes of all physical variables \( q \). According to their decomposition (4.1) into collective variables \( \varphi \) and the microscopic variables \( f \) of Sect.4 and the corresponding block-diagonal structure of all the relevant Euclidean operators and modes (4.19) - (4.22), a similar decomposition properties hold for their Lorentzian counterparts

\[
F_L = \text{diag} \ (F_\varphi, F_L), \quad W_L = \text{diag} \ (W_\varphi, W_L), \quad v(t) = \text{diag} \ (v_\varphi(t), v(t)), \quad \Delta = \text{diag} \ (\Delta_\varphi, \Delta).
\]

Therefore all the Wronskian orthogonality relations of the above type hold separately in the sector of collective variables and the sector of microscopic ones.

19According to the discussion above, the Euclidean modes are real only up to negligible terms \( O(\varepsilon) = O(h^{1/2}) \) or, alternatively (see the previous footnote), can be regarded exactly real, in which case these matching conditions, replacing (6.1), serve for the continuation of the Euclidean modes into Lorentzian ones as exact solutions of the wave equations.
The operators $F_{\varphi L}$, $W_{\varphi L}$ and their modes $v_\varphi(t)$ determine the quantum properties of the collective variable $\varphi$ and of the corresponding field background $\Phi(z)$ in the definition of the wavefunction (6.2). Let us recall that, in contrast to the usual definitions of the classical background, our background has a quantum nature, for it is parametrized by the collective quantum variable $\varphi$, $\Phi(z) = \Phi(z | \varphi, z_+)$, as an extremal of equations of motion subject to the boundary conditions depending on $\varphi$ (cf. eq. (4.1)).

The quantum properties of the linearized microscopic modes $f$ in the wavefunctions (6.2) are determined by the matrix of quantum dispersions $D_L$ defined, according to (4.33) and (5.22), entirely in terms of $W_L(d/dt)$ and $v^*(t)$:

$$D_L(t) = -i \left[ W_L(d/dt) v^*(t) \right] \left[ v^*(t) \right]^{-1}.$$  

(6.20)

In virtue of eq. (6.15) this complex matrix is symmetric and has a positive-definite real part derivable from the corollary $W_L v^* = (v^1)^{-1} (W_L v)^T v^*$ of eq. (6.15):

$$D^T_L - D_L = (v^1)^{-1} < v, v^* > (v^*)^{-1} = 0, $$

(6.21)

$$D^*_L + D_L = (v^1)^{-1} \Delta v^{-1}. $$

(6.22)

These properties of $D_L$ finally allow us to show that the gaussian state (6.2) is a vacuum of linearized modes $f$ relative to the positive-negative frequency decomposition with respect to Lorentzian basis functions (5.1). Indeed, consider the Hermitian (in the quantum Hilbert space, but not in the space of complex matrices) Heisenberg operator of the linear quantum field $\hat{f}(t)$ decomposed into a set of $(v(t), v^*(t))$

$$\hat{f}(t) = v(t) \hat{a} + v^*(t) \hat{a}^* \equiv v_A(t) \hat{a}^A + v_A^*(t) \hat{a}^{*A} $$

(6.23)

with the operatorial Hermitian-conjugated coefficients $\hat{a} = \hat{a}^A$, and $\hat{a}^* = \hat{a}^{*A}$. The canonically conjugated momentum $\hat{p}(t)$ for this linearized field can be obtained in virtue of (5.12) by acting on $\hat{f}(t)$ with the Wronskian operator

$$\hat{p}(t) = W_L(d/dt) \hat{f}(t) = (W_L v)(t) \hat{a} + (W_L v^*)(t) \hat{a}^*. $$

(6.24)

As a corollary of the orthogonality (6.13) the following matrix relation holds

$$\begin{pmatrix} (W_L v)^T & -v^T \\ -v^T & W_L v \end{pmatrix} \begin{pmatrix} v & v^* \\ W_L v & W_L v^* \end{pmatrix} = \begin{pmatrix} i \Delta & 0 \\ 0 & -i \Delta \end{pmatrix} $$

(6.25)
which allows one immediately to solve the system of equations (6.23)-(6.24) for the operators (\(\hat{a}, \hat{a}^*\)):

\[
\hat{a} = i \Delta^{-1} v^\dagger \hat{p} - i \Delta^{-1} (W_L v)^\dagger \hat{f}, \quad \hat{a}^* = -i \Delta^{-1} v^T \hat{p} + i \Delta^{-1} (W_L v)^T \hat{f}.
\] (6.26)

In view of the standard equal-time canonical commutation relations for the phase-space operators [\(\hat{f}, \hat{p}^T\] = \(i \hbar I\) (\(I\) is a unit matrix in the \(f\)-sector of the full space of fields, \(I = \text{diag}(I_\phi, I)\), and all the other commutators vanish), \(\hat{a}\) and \(\hat{a}^*\) have the following only nonvanishing commutator

\[
[\hat{a}^A, \hat{a}^*^B] = \hbar (\Delta^{-1})^{AB}.
\] (6.27)

Since \(\Delta\) is a real positive definite matrix, it can be diagonalized by linear transformations of positive-frequency basis functions \(v(t)\) making their set orthonormal

\[
< v_A, v_B > = \Delta_{AB} = \delta_{AB},
\] (6.28)

so that \(\hat{a}\) and \(\hat{a}^*\) become respectively the usual annihilation and creation operators. In particular, the operator \(\hat{a}\) rewritten in the coordinate representation of the phase space operators, \(\hat{f} = f\), \(\hat{p} = \hbar \partial / i \partial f\),

\[
\hat{a} (f, \partial / \partial f) = \hbar v^\dagger \frac{\partial}{\partial f} - i \frac{(W_L v)^\dagger}{f}
\] (6.29)

annihilates the gaussian quantum state (6.2) of linearized quantum perturbations in the Lorentzian Universe

\[
\hat{a} (f, \partial / \partial f) \Psi_L (\phi, f, t_+) = 0,
\] (6.30)

which can be easily verified by using eq.(6.15).

Thus, in the no-boundary prescription of Hartle and Hawking the Lorentzian Universe nucleates from the Euclidean spacetime with the vacuum state of linearized physical modes, corresponding to the field decomposition in special positive and negative

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\(^{20}\)To avoid confusion with the matrix notations of this equation, we have written the indices of \(\hat{a}\) and \(\hat{a}^*\) explicitly. The correct matrix form of these commutation relation would be [\(\hat{a}, a^{*T}\] = \(\Delta\) reflecting the fact that [\(\hat{a}, a^{*T}\)] is a direct product of two vectors, but not the scalar contraction of their indices. For the same reason, the correct form of the commutation relations for \(\hat{f}\) and \(\hat{p}\) above also involves the transposed quantities: [\(\hat{f}, \hat{p}^T\] = \(-[\hat{p}, \hat{f}^T]\) = \(i \hbar I\).
frequency basis functions. They originate by the analytic continuation (6.1) from regular linear modes in the Euclidean ball. This decomposition and, therefore, the definition of the vacuum is unique, because the only admissible freedom in the choice of basis functions (6.1), which does not violate the regularity condition, consists in the unitary rotations of the positive-frequency subset of $v(t)$, preserving (6.28) and not mixing $v(t)$ and $v^*(t)$.

It is worth emphasizing here again a crucial role played by the boundedness of the Euclidean action from below. In view of the positivity of the quadratic form (6.18) it guarantees the positivity of $\Delta$ and of the real part of $D_L$. Therefore, it makes the gaussian state (6.2) normalizable and exponentially damping large quantum fluctuation. As we see, this is the same property which underlies the perturbation theory in the imaginary part of complex extremals and allows the construction of a special positive-negative frequency decomposition and vacuum state for complex tunnelling geometries.

The emergence of the vacuum state for quantum inhomogeneties from the Hartle-Hawking wavefunction has been observed by a number of authors [49, 50] in the context of different models with the real-valued classical DeSitter and quasi-DeSitter background and, in particular, has been most transparently demonstrated in the language of the path integral approach in [27]. In DeSitter models this special vacuum state coincides with the so-called Euclidean vacuum [61] which exhibits a number of remarkable properties [62] and has important applications in the theory of the inflationary Universe [51], because it provides the spectrum of density fluctuations responsible for the formation of the large scale structure of the observable Universe. Here we have extended these conclusions by showing that in the framework of $\hbar$-expansion a similar unique vacuum is selected by the no-boundary proposal even for complex-valued tunnelling geometries which certainly constitute the most general case.

\footnote{In particular, it is DeSitter invariant [62] and has a Hadamard singularity of two-point Green’s functions, thus providing the covariant renormalization of ultraviolet infinities with local counterterms [38, 39].}

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7. Quantum distribution of tunnelling universes

The Lorentzian wavefunction of the Universe (6.2) carries the information about all dynamical variables of the system. However, in practical applications, when only a part of variables are available to the observer at the experimental level, it suffices to have the corresponding density matrix which can be obtained from the wavefunction by tracing out the unmeasured degrees of freedom. This procedure of tracing out certain variables can drastically change the form of the initial full quantum state as a function of the needed observables and, therefore, represents a nontrivial step towards quantities having a direct physical interpretation. Usually the degrees of freedom which carry the most important information about the system are some macroscopic collective variables of the type considered in Sect.4, that is why the matter of primary interest within the above method of collective coordinates is their density matrix

\[
\hat{\rho}(t) = \text{Tr}_f |\Psi_L(t)\rangle\langle\Psi_L(t)|
\]  

obtained from the pure state density matrix by tracing out the microscopic degrees of freedom \(f\). This object encodes all the correlation functions of the collective variables \(\varphi\) generally nonlinearly interacting with one another and the rest of the physical modes. In particular, it includes the density of their probability distribution which is just the diagonal element \(\rho(\varphi, t) = \rho(\varphi, \varphi|t)\) of \(\hat{\rho}(t) = \rho(\varphi, \varphi'|t)\) in the coordinate representation of \(\varphi\). Since we work with the wavefunction of physical variables in the Hilbert space with the trivial inner product (1.6), this diagonal element equals

\[
\rho(\varphi, t) = \int df |\Psi_L(\varphi, f, t)|^2.
\]  

The knowledge of the wavefunction (6.2) easily allows us to calculate the full density matrix (7.1), but here we shall mainly concentrate on this quantity playing a very important role in quantum cosmology of tunnelling universes, for it determines their probability distribution in the space of such macroscopic variables as a Hubble constant, parameters of anisotropy, etc.

Substituting (6.2) into (7.2) and taking into account the equation (6.22) for the real part of the matrix of quantum dispersions, one immediately finds the following
answer for the gaussian integral over $f$

$$
\rho (\varphi, t) = \text{Const} \frac{(\det \Delta \varphi)^{1/2}}{|\det \varphi (t)|} (\det \Delta)^{-1/2} e^{-\frac{2}{\bar{h}} \text{Re} \mathcal{I} [\Phi]} \left[ 1 + O (\hbar^{1/2}) \right], \quad (7.3)
$$

where the real part of the complex action (5.16)

$$
\text{Re} \mathcal{I} [\Phi] = I [\phi] + \frac{1}{2} \eta^T (W \eta) (\tau_B) + O (\hbar^{3/2}), \quad (7.4)
$$

must include the positive definite real part of the quadratic form (5.17) damping the contribution of the imaginary part of the complex extremal $\varepsilon = O (\hbar^{1/2})$. Here we took into account the block-diagonal form (6.19) of the matrices $v$ and $\Delta$

$$
\det v (t) = \det v_\varphi (t) \det v (t), \quad \det \Delta = \det \Delta_\varphi \det \Delta, \quad (7.5)
$$

due to which the one-loop preexponential factor of (7.3) features the determinant of the full Wronskian matrix $\Delta$ and the determinant of the Lorentzian modes of the collective variables $v_\varphi (t)$ normalized by $(\det \Delta_\varphi)^{1/2}$ to unity in the inner product (6.12). The emergence of the full Wronskian matrix in this algorithm in conjunction with the reduction method for functional determinants on closed spacetimes of [24] make us to consider in the next section a special geometric interpretation of the result (7.3).

### 7.1. Doubling the Euclidean spacetime

Here we shall present a detailed geometrical construction of doubled Euclidean spacetime serving for both the interpretation and covariant calculation of the one-loop partition function (7.3), which was proposed for these purposes in [34] and also used in [31] for the general tree-level analysis of real tunnelling geometries, amounting to the so-called unique conception theorem.

To begin with, note that the Wronskian matrix $\Delta$ in (7.3) can be represented by the equation (6.17) in terms of the regular basis functions on the Euclidean spacetime ball $M \equiv M_- = B^4$ with the "center" at $\tau_-$. This spacetime carries a real Euclidean metric and matter fields characterized by the real physical fields $\phi (\tau) = \text{Re} \Phi (\tau)$ and has as a boundary the spatial hypersurface $\Sigma_B$ of constant $\tau = \tau_B$ at which a semiclassical nucleation of the Lorentzian Universe from the Euclidean one takes place. Now consider its orientation reversed copy $M_+$, which can be regarded as mirror image.
of $M_-$ with respect to this boundary. One can now construct the doubled manifold $2M$ by joining $M_-$ and $M_+$ across their common boundary $\Sigma_B$ (see Fig.4)

$$2M = M_- \cup M_+. \quad (7.6)$$

This doubled manifold is, obviously, closed, has a topology of a four-dimensional sphere and admits an isometry $\theta$ mapping its two halves $M_\pm$ onto one another

$$M_\pm = \theta M_\mp : \{ x \in M_\pm, \theta x \in M_\mp \}. \quad (7.7)$$

The foliation of $M_-$ by three-dimensional compact surfaces of constant $\tau$ can be continuously extended by this isometry to the whole of $2M$ with the parameter $\tau$ ranging between $\tau_-$ and $\tau_+$ – the value corresponding to the "center" of $M_+$ coinciding with the "north" pole of the sphere-like doubled manifold. This foliation can be represented by the continuous one-parameter family of surfaces $\Sigma(\tau)$ which expand from zero volume at the south cap of $2M$ and then again shrink to a point at the north cap after passing the equatorial section $\Sigma_B = \Sigma(\tau_B)$ at $\tau_B = (\tau_- + \tau_+)/2$

$$\tau (x) = \tau, \quad x (x) = x, \quad x \in M_-, \tau_- \leq \tau \leq \tau_B,$$

$$\tau (\theta x) = \tau_+ + \tau_- - \tau, \quad x (\theta x) = x, \quad \theta x \in M_+. \quad (7.8)$$

This foliation explicitly demonstrates the reversal of Euclidean time on $M_+$ in contrast to spatial coordinates $x$ identically related on surfaces $\Sigma$ and $\theta \Sigma$. Obviously, the coordinate $\tau$ parametrizing the whole of $2M$ plays the role of the latitude angle $\theta$ on the four-dimensional sphere homeomorphic to $2M$, ranging from 0 to $\pi$, while $x$ are the "angular" coordinates on quasispherical spatial sections $\Sigma$.

The four geometry and matter fields on $2M$ are also a subject of the isometry map (7.7), which means that on $M_+$ they are defined as a reflection image of those on the original spacetime $M_- = M$. In the foliation (7.8) this fact can be easily represented as a following definition of the physical variables $\phi(\tau)$ for $\tau_B \leq \tau \leq \tau_+$ in terms of those for $\tau_- \leq \tau \leq \tau_B$:

$$\phi(\tau) = \phi(\tau_+ + \tau_- - \tau). \quad (7.9)$$

Such fields are continuous but not generally analytic at the "equatorial" junction surface $\Sigma_B$ unless their normal derivative $d\phi/d\tau(\tau_B)$ vanishes there. In case of real tunnelling geometries considered in [31, 32] this condition is satisfied as a geometrically
invariant requirement of vanishing extrinsic curvature of $\Sigma_B$

\[ K_{ab} |_{\Sigma_B} = 0, \quad (7.10) \]

this fact providing the analytic matching of real Euclidean manifold $M_-$ with its double $M_+$ and with the nucleating real Lorentzian spacetime $M_L$. For complex tunnelling fields, we consider here, this condition is, however, generally violated, because the normal (time) derivative of $\phi(\tau)$ is proportional in view of the matching condition (5.13) to the imaginary part of the extremal: $\dot{\phi}(\tau_B) \sim \partial L_E / \partial \dot{\phi}(\tau_B) = O(\varepsilon)$.

The lack of smoothness of the background fields does not prevent, however, from extending the basis function $u_-(\tau)$ defined on $M_-$ to the whole of $2M$ as a solution of

\[ F u_- |_{2M} = 0 \quad (7.11) \]

with continuous zeroth and first order derivatives at $\Sigma_B$ (the second order derivatives will generally jump at $\Sigma_B$, because the coefficients of the differential operator $F$ are discontinuous at this surface). These basis functions are regular at the south pole $\tau_-$ of the doubled manifold, but singular at $\tau_+$, because we assume that the positive-definite Euclidean operator $F$ does not have zero modes on $2M$.

On the other hand, we can consider the set of basis functions $u_+$ on the doubled manifold which are the reflection image of $u_-$ defined in the foliation (7.8) by the relation

\[ u_+(\tau) = u_-(\tau_+ + \tau_- - \tau). \quad (7.12) \]

These basis functions are regular at $\tau_+$, singular at $\tau_-$ and satisfy the following matching conditions at the junction surface $\Sigma_B$

\[ u_-(\tau_B) = u_+(\tau_B), \quad W u_-(\tau_B) = -W u_+(\tau_B). \quad (7.13) \]

Therefore, the set of inner products of Lorentzian linear modes $\Delta$, given by eq. (6.17), can be rewritten in the form of the Wronskian matrix $\Delta^+_- = (\Delta_+_-)_{AB}$ of these two

22 Generally, this relation, as well as the equation (7.9), should be understood not for separate components of $u_+$ and $\phi$, but for their tensor objects as a whole, accounting for the orientation reversing map of local basis on $M_-$ and $M_+$. However, except for spinors, since the physical variables basically involve the spatial components of tensor fields, the relation (7.12) literally holds in a special coordinate system (7.8) which does not reverse the orientation of spatial sections. The more complicated case of spinors, accounting for the complex structure of the Dirac operator, was considered in [32] for Majorana, Dirac and Weyl spinor fields.
sets of Euclidean basis functions on the doubled manifold (independent of \( \tau \) in virtue of the relation (4.3))
\[
\begin{align*}
\Delta &= \Delta_{+-}, \\
\Delta_{+-} &= u_+^T (W u_-) - (W u_+)^T u_-.
\end{align*}
\] (7.14) (7.15)

This property serves as a ground for the following important observation. According to the reduction technique of [24] for functional determinants on a closed compact spacetime of spherical topology (which is just the case of \( 2M \)), the one-loop preexponential factor of the Euclidean quantum theory on such a spacetime can be generated by the determinant of the Wronskian matrix (7.15) of the two complete sets of basis functions \( u_{\pm}(\tau) \) which, in the \( \tau \)-foliation of the above type, are regular respectively at \( \tau_+ \) and \( \tau_- \) and have the asymptotic behaviour (cf. eq. (4.39))
\[
\begin{align*}
\left. u_-(\tau) = I (\tau - \tau_-)^{\mu-} + O \left[ (\tau - \tau_-)^{1+\mu_-} \right], \quad \tau \to \tau_- \right), \\
\left. u_+ (\tau) = I (\tau_+ - \tau)^{\mu+} + O \left[ (\tau_+ - \tau)^{1+\mu+} \right], \quad \tau \to \tau_+ \right).
\end{align*}
\] (7.16) (7.17)

with the field-independent coefficient – the matrix infinity \( I \). This reduction algorithm reads
\[
\begin{align*}
\left( \det \Delta_{+-} \right)^{-1/2} &= \text{Const} \left[ \frac{\det F}{\det a} \right]^{-1/2}.
\end{align*}
\] (7.18)

and implies that the one-loop preexponential factor of our partition function (7.3) in the main boils down to the contribution of functional determinants on \( 2M \). Such determinants are calculated on the representation space of \( F \) – the space of functions regular on closed compact manifold \( 2M \) and constitute the one-loop part of the effective action of the Euclidean theory on this spacetime.

### 7.2. Covariant distribution function: covariance versus unitarity

Using the above relation in (7.3) we arrive at the following fundamental algorithm for the one-loop partition function of tunnelling geometries
\[
\rho (\varphi, t) = \text{Const} \frac{(\det \Delta_\varphi)^{1/2}}{|\det v_\varphi (t)|} \exp \left\{ -\frac{1}{\hbar} \Gamma_{1-\text{loop}} [\phi] \right\} \\
\times e^{-\frac{1}{\hbar} \eta^T (W \eta) B \left[ 1 + O (\hbar^{1/2}) \right]}.
\] (7.19)
Here $v_\phi (t)$ is the set of linearized Lorentzian modes of collective variables $\phi$ which has a matrix of inner products (6.12)

$$F_{\phi L} (d/dt) v_\phi (t) = 0, \quad <v_\phi, v_\phi> = \Delta_\phi,$$  
(7.20)

$\eta^T (W_\eta)_B$ is a doubled quadratic form of the action (7.4) in the imaginary corrections to the classical extremal at the nucleation surface $\tau_B$ and

$$\Gamma_{1-\text{loop}} [\phi] = I_{2M} [\phi] + \frac{\hbar}{2} \text{Tr} \ln F - \frac{\hbar}{2} \text{Tr} \ln a$$  
(7.21)

is the one-loop effective action of the theory on the doubled Euclidean spacetime $2M$ with the real background field $\phi (\tau)$ – the real part of the exact complex extremal $\Phi (z) = \Phi (z|\phi, t)$, parametrized by the boundary data ($\phi$, $t$) which is the argument of the partition function (7.19):

$$\phi (\tau) = \text{Re} \Phi (\tau|\phi, t), \quad 0 \leq \tau \leq \tau_B.$$  
(7.22)

The effective action (7.21) includes the classical Euclidean action on $2M$

$$I_{2M} [\phi] = 2 I_M [\phi]$$  
(7.23)

and the one-loop contribution given by the logarithm of (7.18).

From the viewpoint of practical applications, it would seem that the new algorithm (7.19) does not have any advantages over the original expression (7.4), for the replacement of $\text{det} \Delta$ by its representation (7.18) with the determinant of higher functional dimensionality actually complicates the calculations. However, the new form of $\rho (\phi, t)$ has a very important property of covariance, because in contrast to $\text{det} \Delta$, involving the intrinsically non-covariant ADM reduction to physical variables, selection of the particular time, construction of basis functions, etc., there exist very powerful manifestly covariant methods for the calculation of $\Gamma_{1-\text{loop}} [\phi]$. Apart from this, the variety of methods for differential operators on compact spaces, which is just the case of $2M$, is much richer than on manifolds with boundaries. Altogether this allows one to perform a covariant regularization and renormalization of the generally divergent partition function and obtain its nontrivial high-energy behaviour, which will be considered in the last section of this paper where we shall briefly dwell on these methods.
As is known, the requirement of manifest general covariance served as one of the basic motivations for the creation of the Euclidean quantum gravity in the works of Hawking, Hartle and the others. Even the Euclideanization, as it is, was essentially aimed at getting rid of any signs of inequivalence between different coordinates on spacetime manifold. However, the usual price one pays for manifest covariance is the loss of manifest unitarity. The root of the difficulty is that in the covariant quantization the sector of physical fields and the physical inner product are deeply hidden in the full space of the theory involving ghosts, zero and negative norm states, etc., and usually it requires very subtle methods to recover unitarity from the manifestly covariant formalism or, vice versa, to render the unitary theory a manifestly covariant form \[56, 18, 52, 19, 13\]. A remarkable feature of the partition function (7.19) is that, being formulated in terms physical degrees of freedom with a standard inner product, it combines both of the desired properties: the covariance of radiative corrections accumulated in the Euclidean effective action (7.21) and its one-loop unitarity encoded in the preexponential factor. Let us first consider this unitarity property.

7.3. Unitarity and partition function of gravitational instantons

 Basically the unitarity in application to the density of the partition function means the conservation in the Lorentzian time \(t_+\) of the total probability

\[
\int d\varphi \rho(\varphi, t_+) = \text{Const.}
\]

(7.24)

To prove this statement, we shall make a number of observations. To begin with, note that a set of basis functions \(v_\varphi(t)\) featuring in (7.19) can be obtained from the family of classical extremals parametrized by certain variables (constants of integration of classical equations) by differentiating them with respect to these variables. The resulting functions satisfy the linearized equations of motion and the same regularity conditions as these classical extremals. In our case the extremals \(\Phi(z|\varphi, t_+)\) are parametrized by their boundary conditions \(\varphi\) at the final moment of Lorentzian time \(t_+)\), but for our purposes it will be more convenient to parametrize them by the value \(\phi_B\) of their real part at the moment \(\tau_B\) of the Lorentzian nucleation (we assume that \(\phi_B\) and \(\tau_B\) are in
one-to-one correspondence with \( \varphi \) and \( t_+ \):

\[
\Phi(z) = \Phi(z, \phi_B) = \phi(z, \phi_B) + i \eta(z, \phi_B). \quad \phi_B = \phi(\tau_B, \phi_B).
\]  

(7.25)

Thus, the Lorentzian basis functions can be regarded as

\[
v_\varphi^* (t) = \left. \frac{\partial \Phi(z, \phi_B)}{\partial \phi_B} \right|_{z=\tau_B+it}.
\]  

(7.26)

Since \( \Phi(z_+, \phi_B) = \varphi \), the matrix of the above basis functions is real at \( t_+ \), and its determinant coincides with the Jacobian of transformation from \( \phi_B \) to \( \varphi \)

\[
\varphi \rightarrow \phi_B, \quad \det v_\varphi (t_+) = \det \left( \frac{\partial \varphi}{\partial \phi_B} \right).
\]  

(7.27)

The further proof is based on the method of complex extremals according to which their imaginary part \( \varepsilon \) can be treated by perturbations in all the terms except the negative-definite quadratic form in the total exponential of \( (7.13) \). When combined with the classical action \( (7.23) \), this form gives rise to the (doubled) imaginary part of the full complex action \( (5.16) \), \( S[\Phi(z)] \equiv i I[\Phi(z)] = S(t_+, \varphi) + i I(t_+, \varphi) \) calculated at the contour \( C_+ \)

\[
I_{2M}[\phi] + \eta^T(W\eta)_B = 2 I(t_+, \varphi) + O(\eta^3),
\]  

(7.28)

which is a complex Hamilton-Jacobi function of the data \( (t_+, \varphi) \) at the end \( z_+ \) of the complex extremal, \( S[\Phi(z)] = S(t_+, \varphi) \). With respect to this data it satisfies the Hamilton-Jacobi equation with the physical Hamiltonian \( H(\varphi, p) \), generating the following equation for its imaginary part \( I = I(t_+, \varphi) \)

\[
\frac{\partial I}{\partial t_+} + \frac{\partial H}{\partial p} \bigg|_{p=\partial S(\varphi)} \frac{\partial I}{\partial \varphi} = O \left( \left( \frac{\partial I}{\partial \varphi} \right)^3 \right).
\]  

(7.29)

In view of the Euclidean-Lorentzian matching conditions of Sect.5, \( \partial I/\partial \varphi = O(\varepsilon) = O(h^{1/2}) \), so that the solution of \( (7.29) \), \( I(t, \varphi(t)) = I(0, \varphi(0)) + O(h^{3/2}) \), is practically a constant \( [4] \) along the real-valued trajectory \( \varphi(t) \) evolving according to

\[
\varphi = \left. \frac{\partial H(\varphi, p)}{\partial p} \right|_{p=\partial S(t, \varphi)/\partial \varphi}, \quad \varphi(t_+) = \varphi,
\]  

(7.30)
and differing from the real part of the exact complex extremal at most by $O(\varepsilon) = O(h^{1/2})$ terms, $\bar{\varphi}(t) = \phi(\tau_B + it, \phi_B) + O(\varepsilon)$.

Thus, the dependence on $t_+$ in (7.28) can be, with the needed one-loop accuracy, completely absorbed into the redefinition of the field variable, $\varphi \rightarrow \varphi_B \equiv \bar{\varphi}(0)$. Correspondingly, the tree-level part (\() can be regarded as the Euclidean action $I_{2M}[\bar{\phi}]$ on a new real classical background $\bar{\phi}(\tau)$ with the boundary condition $\varphi_B$ at $\tau_B$, the latter being determined as a function of $(t_+, \varphi)$ from the solution of (7.30):

$$\frac{\delta I[\bar{\phi}]}{\delta \bar{\phi}(\tau)} = 0, \quad \bar{\phi}(\tau_B) = \varphi_B(t_+, \varphi). \tag{7.31}$$

With the same accuracy the full one-loop exponential in (7.19) turns out to be the effective action on this new background $\bar{\phi}(\tau)$ or can be regarded as a function of its boundary data $\varphi_B = \phi_B + O(h^{1/2})$ at the nucleation surface

$$\Gamma_{1\text{-loop}}[\phi] + \eta^T(W\eta)_B = \Gamma_{1\text{-loop}}[\bar{\phi}] = \Gamma_{1\text{-loop}}(\varphi_B). \tag{7.32}$$

On the other hand, by differentiating the asymptotic bound (5.20) with respect to $\varphi_B$, one finds that not only $\varepsilon(\tau) = O(h^{1/2})$ but also $\partial \varepsilon / \partial \varphi_B = O(h^{1/2})$, whence one can rewrite (7.27) as a Jacobian of the following change of variables

$$\varphi \rightarrow \varphi_B, \quad \left[ \det v_\varphi(t_+) \right]^{-1} = \det (\partial \varphi_B / \partial \varphi) + O(h^{1/2}), \tag{7.33}$$

absorbing all the dependence of (7.19) on the Lorentzian time $t_+$.

Therefore, the quantum distribution function for tunneling geometries

$$\rho(\varphi, t_+) = \rho_{2M}(\varphi_B) \det \frac{\partial \varphi_B}{\partial \varphi} \bigg|_{\varphi_B = \varphi_B(\varphi, t_+)} \tag{7.34}$$

reduces to the following partition function of the gravitational instantons $2M$ with a special classical background field subject to the boundary conditions (7.31) at the junction surface $\Sigma_B$:

$$\rho_{2M}(\varphi_B) = \text{Const} \left[ \det \Delta_\phi(\varphi_B) \right]^{1/2} e^{-\frac{1}{\hbar} \Gamma_{1\text{-loop}}(\varphi_B)} \left[ 1 + O(h^{1/2}) \right]. \tag{7.35}$$

As a result one has

$$\int d\varphi \rho(\varphi, t_+) = \int d\varphi_B \rho_{2M}(\varphi_B) = \text{Const}, \tag{7.36}$$
which accomplishes the proof of unitarity for the distribution function \[\mathcal{F}\].

The algorithms (7.19) or (7.34) have a very good graphical illustration demonstrating their unitarity. The partition function, as an inner product of the wavefunction with itself, is shown on Fig.5 as a composition of the two spacetime manifolds combined of Euclidean and Lorentzian domains and associated respectively with \(\Psi_L(\phi, f, t)\) and \(\Psi^*_L(\phi, f, t)\). Due to unitarity, which makes sense only in physical Lorentzian spacetime, the Lorentzian "brims" of these two "hats" cancel, because this portion of the spacetime is described by the unitary evolution operator. What remains is, in the main, the doubled Euclidean manifold \(2M\) – the compact gravitational instanton of spherical topology serving as a support for the Euclidean action (7.21). This non-trivial remnant can be explained by the fact that the dynamical "evolution" on Euclidean spacetime is described by the non-unitary heat equation rather than the Schrödinger one.

The algorithms of analytic continuation from Euclidean spacetime (Wick rotation) for the matrix elements between different quantum states – IN and OUT asymptotic vacua – are well known in asymptotically flat case and usually serve as a calculational basis for conventional scattering theory. A similar technique for expectation values, that is for matrix elements of operators with respect to one and the same state, is much less popular because of the difficulties related to a manifest breakdown of analyticity. In contrast to the analyticity of the wavefunction, the corresponding expectation values can never be analytic for they involve both \(\Psi_L(q, t)\) and its complex conjugate \(\Psi^*_L(q, t)\). However, in the context of a special quantum state – the standard asymptotic IN-vacuum associated with the plane-wave decomposition of field operators – there exists a special technique relating the expectation values in Lorentzian spacetime to the Euclidean effective action [63]. Apparently, the algorithm (7.19) is the first analogue

\[23\text{Strictly speaking, both } \Gamma_1\text{-loop } (\varphi_B) \text{ and } \det \Delta_\varphi (\varphi_B) \text{ in (7.35) depend also on the value of } \tau_B \text{ which is a function of } t_+ \text{ (as a part of boundary conditions } (\varphi, t_+) \text{ on the complex extremal). However, as it follows from the analyses of these extremals [23], } d\tau_B/dt_+ = O(\varepsilon) = O(\hbar^{1/2}), \text{ so that this value is defined up to higher-order loop corrections by the location of caustic surfaces of the Euclidean classical histories in superspace of three-metrics and matter fields } q. \text{ According to the discussion of Sect.2, the necessity of introducing the complex time for tunnelling geometries originates from extending the classical extremals of the theory beyond these caustic surfaces, the gauge properties of which will be considered in [27].}\]
of this technique in the cosmological case of spatially closed spacetime and in the context of the no-boundary quantum state of Hartle and Hawking. Obviously, this state plays the role of the standard IN-vacuum of asymptotically flat worlds, its regular Euclidean modes being the counterparts of the positive energy plane waves which under the Wick rotation go over into the modes vanishing in the remote Euclidean "past". The same analogy also transpires in the realization of the reduction methods for functional determinants of \[24\] where the south and north poles of the compact sphere-like manifold were associated with the \(\pm\infty\) of the asymptotically flat spacetime, while the corresponding regular modes \(u_\pm\) were associated with the basis functions of the IN and OUT asymptotic vacua.

The above picture of unitarity on the Lorentzian portion of the tunnelling geometry takes the simplest form in the case of exactly real classical extremals, when the doubled manifold represents a completely smooth gravitational instanton \[4\]. In the general case of theories with the Euclidean action bounded from below, the exact extremals are complex, and in the one-loop approximation the effect of their complexity boils down to the gaussian factor in (7.19) damping the contribution of their large imaginary part. Apparently, this property explains the lack of interest in literature to complex instantons, the contribution of which is always exponentially suppressed in comparison with the real tunnelling solutions. However, as it was discussed in Introduction, there are physically interesting problems lacking the real solutions, in which cases the technique of the above type becomes indispensable. One of the fundamental examples is the Hawking model of chaotic inflation driven by a macroscopic collective variable – the inflaton scalar field. We shall consider this model in much detail in a forthcoming paper \[25\] and also use it in the next section to illustrate the last issue of the present work – the high energy behaviour and normalizability of the Hartle-Hawking wavefunction.

\[24\] Although, in this case the linearized modes of the collective variables get a zero norm with respect to the Wronskian inner product and, thus, acquire the status of zero modes on the Euclidean instanton, the discussion of which goes beyond the scope of this paper.
8. High-energy behaviour of the Hartle-Hawking wavefunction

The validity of a semiclassical loop expansion in quantum gravity essentially depends on the energy scale of the problem. At Planckian energies it is expected to break down because of the nonrenormalizability of the Einstein theory, related to the dimensional nature of its coupling constant. On the other hand, the energy scale of the problem is determined by the quantum state of the system and the location of maxima of the corresponding partition function for those variables which are supposed to play the major role in its dynamics. Therefore, the validity of the loop expansion in quantum gravity with the Hartle-Hawking state can, at least heuristically, follow from the behaviour of the partition function of collective variables constructed above.

This partition function includes the quantum contribution of the collective variables themselves and also of the infinite set of microscopic field modes. Therefore, it suffers from the ultraviolet divergences and requires regularization and renormalization. In principle, these procedures must be done at all stages of calculating the wavefunction and the corresponding partition function. Only in this case we would have the consistent and consequitive operatorial quantization. However, at the present state of art in high-energy physics, only in simple low dimensional field models such an approach has been realized and has a well-established status. In realistic field theories we still have to skip the operatorial stage of quantization at the unregularized level and make regularization and (if possible) renormalization only in the final algorithms for matrix elements, expectation values, etc., presented by loop Feynman diagrams. For this reason we shall discuss here the regularization in the final algorithm for a partition function rather than in the wavefunction itself.

Even apart from this liberty, there still remains a problem of whether the properly regularized infinities can be renormalized by physically sensible procedure. We shall not discuss here this issue, which is a subject of a vast literature on the over-Planckian structure of fundamental interactions. Instead, we shall simply assume that, whatever physical origin of this procedure is (either it is a fundamental finite string theory underlying its low-energy effective limit or the inclusion of the infinite set of counterterms), the correct procedure of renormalization consists in the subtraction of the covariantly
regularized ultraviolet infinities. It is difficult to perform a regularization, respecting
the four dimensional general covariance, in the manifestly non-covariant formalism of
the unitary ADM quantization. However, as it was shown in the previous section, our
partition function combines manifest unitarity with the Euclidean effective action ac-
cumulating the divergent quantum corrections, which can be rendered covariant form
and, therefore, covariantly renormalized. Here we shall briefly sketch this procedure
which eventually yields the high-energy behaviour of the partition function.

8.1. Covariant renormalization and anomalous scaling

The basic step of converting the effective action (7.21), calculated in terms of
reduced (Euclidean) physical variables $\phi$, into a covariant form consists in its transfor-
mation to the original set of gravitational four-metric and matter fields $g = (\phi, N_E)$
($\phi$ is a set of 3-metric coefficients and matter fields in spatial foliation of the Euclidean
spacetime, and $N_E$ are the corresponding lapse and shift functions) taken in some gauge
which has a form of local conditions on $g$ and its spacetime derivatives

$$\chi (g, g') = 0$$

(8.1)

(remember that, in the condensed canonical notations of our paper, only time deriva-
tives with respect to a selected foliation of spacetime are explicitly written, while the
spatial derivatives are encoded in the contraction of condensed indices). Such a trans-
formation is identical for the classical part of the effective action $I[\phi] = I[g]$, where
the boldfaced notation is used for the classical action in the initial variables (cf. Sect.2).
For its one-loop part it is given by the one-loop approximated Faddeev-Popov ansatz

$$\frac{\hbar}{2} \text{Tr} \ln F - \text{Tr} \ln a = \frac{\hbar}{2} \text{Tr} \ln \mathcal{F} - \hbar \text{Tr} \ln \mathcal{Q} + O (\delta I/\delta g).$$

(8.2)

It involves the wave operator of the full set of fields $g$, determined by the total action
$I_{\text{tot}}(g)$ which includes the gauge breaking term with the gauge of the above type

$$\mathcal{F} = \frac{\delta I_{\text{tot}}}{\delta g} \frac{\delta g}{\delta g}, \quad I_{\text{tot}}(g) = I[g] + \int d\tau \chi^2 (g, g')$$

(8.3)

and the corresponding ghost operator $\mathcal{Q}$ whose action on a small test function $f$ is
determined by the infinitesimal coordinate gauge transformation of gauge conditions
generated by this function, \( \mathcal{Q}f = \Delta f \chi^2 \).

On mass shell, that is on the solution of classical equations \( \delta I/\delta g = 0 \) for the background \( g \), which is just the case of eqs. (2.18) - (2.19) and (3.16) - (3.17), the expression (8.2) is gauge independent [54] and exactly generates the one-loop effective action in physical variables [26]. This freedom in the choice of \( \chi(g, \dot{g}) \) allows to choose them as belonging to the class of the background covariant gauge conditions [59, 60, 64], in which all the traces of the noncovariant (3+1)-splitting of the spacetime completely disappear and \( \mathcal{F} \) and \( \mathcal{Q} \) become local covariant differential operators of the second order. The functional determinants of such operators already admit the covariant regularization and have powerful calculational methods for their asymptotic scaling behaviour.

These methods, in the main, reduce to the combination of the Schwinger-DeWitt proper time technique [59, 60, 64] and the dimensional or \( \zeta \)-functional regularization. The latter has a technical advantage that it automatically subtracts the divergences, that is why we choose it here to demonstrate the covariant renormalization of the effective action. In the \( \zeta \)-functional regularization [65] one has for a second-order covariant differential operator \( \mathcal{F} \)

\[
\text{Tr} \ln \mathcal{F} = -\zeta'(0) - \zeta(0) \ln \mu^2, \tag{8.4}
\]

where the \( \zeta \)-function \( \zeta(p), p \to 0 \), is defined by the analytic continuation in \( p \) from the domain of convergence of the following functional trace (or, equivalently, the sum over the eigenvalues \( \lambda \) of \( \mathcal{F} \))

\[
\zeta(p) = \text{Tr} \mathcal{F}^{-p} = \sum_\lambda \lambda^{-p} \tag{8.5}
\]

and \( \mu^2 \) is the mass parameter reflecting the logarithmic renormalization ambiguity in

\[\text{We disregard in the right hand side of (8.4) the contribution of the local measure (the counterpart to } \text{Tr} \ln a \text{ in the left hand side) which is ultralocal and proportional to } \delta^4(0). \text{ In the covariant regularizations disregarding the strongest volume divergences (which are cancelled by this measure [52, 24]) it is either identically vanishing (dimensional regularization) or reduces to the logarithmic renormalization ambiguity (\( \zeta \)-functional regularization).} \]

\[\text{This is a particular case of the general statement on the gauge independence of the path integral in quantum gravity and local gauge theories, which is valid on shell beyond the one-loop approximation at least within the perturbative loop expansion [54, 17, 18, 26].}\]
The coefficient of $\ln \mu^2$ in (8.4) as a local functional of the background fields $g$

$$
\zeta(0) = \frac{1}{(4\pi)^2} A_2[g]
$$

(8.6)
determines the structure of (subtracted) ultraviolet divergences and is defined by the coefficient $A_2$ of the proper-time expansion for the functional trace of the heat kernel of the operator $\mathcal{F} = \mathcal{F}[g]$ [53, 60, 61]

$$
\text{Tr} e^{-s\mathcal{F}[g]} = \frac{1}{(4\pi s)^2} \sum_{n=0}^{\infty} A_{n/2}[g] s^{n/2}, \quad s \to 0.
$$

(8.7)

These coefficients are determined by volume and surface integrals of local invariants over the spacetime manifold $M$ and its boundary $\partial M$. Such invariants, in their turn, are constructed out of the coefficients of the operator $\mathcal{F}$, spacetime curvature and the extrinsic curvature of $\partial M$ and, therefore, are easily calculable for any spacetime and field background $g$.

The renormalization parameter $\mu^2$ introduces into the theory a new scale. The classical action has in the limit of small distances the asymptotic scaling invariance under the global conformal transformations of the 4-metric and matter fields $g = (g_{\mu\nu}(x), \phi(x))$

$$
\bar{g}_{\mu\nu}(x) = \Omega^2 g_{\mu\nu}(x), \quad \bar{\phi}(x) = \Omega^{c_\phi} \phi(x),
$$

(8.8)

$$
I[\bar{g}] \simeq I[g], \quad \Omega \to 0,
$$

(8.9)

where $c_\phi$ represents the set of conformal weights of fields $\phi(x)$ [7]. On the contrary, quantum corrections are asymptotically scale invariant only under the simultaneous rescaling of fields and the renormalization mass parameter $\mu^2 \to \bar{\mu}^2 = \Omega^{-2}\mu^2$, and, therefore, they have the anomalous scaling behaviour defined by the coefficient of $\ln \mu^2$ in (8.4)

$$
\int d^4x \left(2\bar{g}_{\mu\nu} \frac{\delta}{\delta \bar{g}_{\mu\nu}} + \sum_{\phi} c_\phi \phi \frac{\delta}{\delta \phi} \right) \frac{1}{2} \text{Tr} \ln \mathcal{F}[\bar{g}] \simeq -\zeta(0), \quad \Omega \to 0.
$$

(8.10)

This is a general case of the nonminimal coupling between the scalar curvature and the dilaton (or, in context of the inflationary Universe, inflaton) scalar field $\phi$ of the conformal weight $-1$, which generates the effective gravitational constant $k^2 \sim 1/\phi^2$, $\phi \to \infty$. For the minimal case different kinetic terms of the classical action scale differently, but this does not change essentially the scaling properties of quantum corrections considered below.
Thus, in this limit the scaling behaviour of the full effective action

\[ \Gamma_{1-\text{loop}}[g] \simeq -\zeta^{\text{tot}}(0) \ln \Omega + \Gamma_{1-\text{loop}}[g], \quad \Omega \to 0 \quad (8.11) \]

is determined by the total \( \zeta \)-function including the contributions of both the gauge field \( \mathcal{F} \) and ghost \( \mathcal{Q} \) operators in (8.2) \( ^{28} \).

### 8.2. Anomalous scaling on the gravitational instanton and the normalizability of the Hartle-Hawking wavefunction

Application of the last equation with due regard for the algorithm (7.19) is of crucial importance in the model of the quantum birth of the chaotic inflationary Universe. This model within a wide class of the field Lagrangians will be considered in \[ ^{25} \], where we shall demonstrate all the peculiarities of the general theory of the above type. However, there is a very important issue raised at the beginning of Sect.8 which can be resolved on the ground of equations (7.19) and (8.11) in the universal and model-independent way. This is the question of the high-energy behaviour of the partition function which serves as a criterion for the applicability of the WKB approximation and normalizability of the wavefunctions at over-Planckian scales.

As is well-known \[ ^{36, 37} \] the tree-level Hartle-Hawking wavefunction is not normalizable in this model. As a function of the only collective variable, the inflaton scalar field \( \varphi \), it goes to a constant for \( \varphi \to \infty \) and does not suppress the contribution of the over-Planckian energy scales. This can be clearly seen from the algorithm (7.34) with the tree-level partition function of gravitational instantons

\[ \rho_{\text{tree}}(\varphi_B) = \text{Const} e^{-\frac{1}{n} I_{2M}(\varphi_B)}. \quad (8.12) \]

Here the classical Euclidean action on the doubled manifold (7.23) is parametrized as in (7.32) in terms of the boundary conditions for an auxiliary classical field (??) at the nucleation surface \( \Sigma_B \) and has the following asymptotic behaviour at large \( \varphi_B \)

\[ I_{2M}(\varphi_B) = I_0 + I_1/\varphi_B^2 + O(1/\varphi_B^4), \quad \varphi_B \to \infty. \quad (8.13) \]

\( ^{28} \)For theories invariant under local Weyl transformations (8.8) with local parameter \( \Omega = \Omega(x) \) the relations (7.9) and (8.10) - (8.11) hold exactly, and the integrand on the left-hand side of (8.10) represents the well-known conformal anomaly given by the volume density of \( A_2 \) – the coincidence limit of the two-point DeWitt coefficient \( a_2(x) = a_2(x, x) \) \[ ^{59, 60} \].
This behaviour follows from the fact that in this model the Euclidean segment of a complex classical history \( \phi(\tau) = \Phi(\tau|\varphi, t), \ 0 \leq \tau \leq \tau_B \), giving the dominant contribution to the partition function, has in the large-\( \phi \) limit a simple form of a practically constant and real scalar field coinciding with its value at the nucleation point

\[
\phi(\tau) \simeq \text{Const} = \phi_B(\varphi, t), \quad \phi_B(\varphi, t) \simeq \varphi_B.
\]  

(8.14)

The value \( \phi_B(\varphi, t) \) is parametrized in accordance with the form of Lorentzian extremal by its final point \((\varphi, t)\) and always satisfies the inequality \( \phi_B > \varphi \) which follows from the fact that the scalar field slowly decreases with the growth of \( t \) during the Lorentzian inflationary stage. Therefore the limit \( \varphi \to \infty \) guarantees large values of \( \varphi_B \to \infty \). The corresponding constant scalar field (8.14) generates an effective cosmological constant \( \Lambda = 3H^2(\varphi_B) \). Its dependence on \( \varphi_B \) is determined by the form of the Lagrangian of the system, but for all Lagrangians viable from the viewpoint of the inflationary scenario it has the property of the monotonic growth \( H(\varphi_B) \to \infty \) for \( \varphi_B \to \infty \). The corresponding regular (in the no-boundary sense) solution of the Euclidean Einstein equations is the metric of the Euclidean DeSitter space (1.3) - (1.4) – the four-dimensional hemisphere of radius \( R = 1/H \), which generates the Lorentzian DeSitter Universe by the nucleation at \( \tau_B = \pi/2H \). The analysis of the matching conditions at the nucleation point \( [25] \) shows that both the deviation of the full Euclidean-Lorentzian extremal from the exactly DeSitter form and its imaginary corrections are vanishing for large \( H \) and, therefore, in this high-energy limit the doubled manifold \( 2M \) is given by a gravitational instanton – a four-dimensional sphere \( S^4 \) of vanishing radius \( R \), carrying the 4-geometry (1.3) - (1.4) and constant scalar field (8.14) which we shall denote by \( g_R = (g^{DS}_{\mu \nu}, \varphi_B) \)

\[
2M = S^4, \quad g_R = (g^{DS}_{\mu \nu}, \varphi_B), \quad R = 1/H(\varphi_B) \to 0.
\]  

(8.15)

The classical Euclidean action calculated on this instanton has a form (8.13) with the coefficients depending on the model of the Lagrangian for coupled gravitational and inflaton scalar field \([25] \ [29]\). Therefore, the tree-level partition function (8.12) of

\[\text{For example, in models with the minimal interaction of a scalar field the expansion (8.13) starts only with the subleading term, while for an inflaton field non-minimally coupled to a scalar curvature it has nonzero } I_0 \ [27].\]
the DeSitter gravitational instantons weighted by their action, does not suppress the contribution of over-Planckian energy scales $\phi_B \to \infty$ and yields unnormalizable Hartle-Hawking wavefunction. This basically means the inconsistency of the semiclassical approximation.

The situation drastically changes in the one-loop approximation, when the classical action must be replaced by the effective one (7.21). Since the scale of the DeSitter instanton is determined by the only dimensional quantity $H(\phi_B)$, the corresponding asymptotic behaviour of $\Gamma_{\text{1-loop}}$ follows from the equation (8.11) with the parameter $\Omega$ replaced by the dimensionless ratio $\Omega = \mu^2/H^2(\phi_B)$

$$
\Gamma_{\text{1-loop}}[\mathbf{g}_R] \simeq Z \ln \frac{H^2(\phi_B)}{\mu^2}, \quad H(\phi_B) \to \infty,
$$

(8.16)

where $Z$ is a total anomalous scaling (8.6) on the DeSitter instanton of vanishing size with the background metric and inflaton fields (8.13)

$$
Z = \left. \frac{1}{(4\pi)^2} A_{2}^{\text{tot}}[\mathbf{g}_R] \right|_{H(\phi_B) \to \infty}.
$$

(8.17)

Thus, the partition function $\rho(\phi_B)$ (7.35) of the DeSitter gravitational instantons with the effective Hubble constant $H(\phi_B)$ has the following high-energy behaviour

$$
\rho(\phi_B) \simeq \text{Const} [H(\phi_B)]^{-Z-1}, \quad H(\phi_B) \to \infty,
$$

(8.18)

where one extra negative power of $H(\phi_B)$ comes from the Wronskian normalization coefficient $(\Delta_\phi)^{1/2}$ for the Lorentzian inflaton mode $v_\phi(t) = [\partial \phi_B(\phi, t)/\partial \phi]^{-1}$ [25].

Therefore, depending on the value of the fundamental dimensionless quantity (8.17), this partition function either suppresses the contribution of the over-Planckian energy scales or infinitely enhances it and, thus, serves as a criterion of applicability of the semiclassical expansion. In particular, in generic theories with nonminimally coupled inflaton scalar field having quartic selfinteraction, for which $H(\phi_B) \sim \phi_B$, this partition function implies the high-energy normalizability of the no-boundary wavefunction

$$
\int_\infty d\phi_B \rho(\phi_B) < \infty,
$$

(8.19)

provided the anomalous scaling exponent satisfies the condition [34]

$$
Z > -1.
$$

(8.20)
The value of $Z$ is determined from (8.17) by the complete matter content of the Universe [67, 68] and, thus, the above condition can serve for a selection of physically viable particle models which have a physically consistent normalizable quantum state of the Universe justifying the use of a semiclassical $\hbar$-expansion.

These conclusions are, certainly, restricted to the one-loop approximation which was multiply used for the calculation of the path integral, for the justification of the perturbation theory in the imaginary part of the complex extremals and for the perturbation expansion in microscopic variables. But general principles of unitarity and covariance, which we have just verified by direct calculations in this approximation, allow us to conjecture that beyond one loop, and even non-perturbatively, the basic algorithm (7.34) – (7.35) will still be valid with $\Gamma_{1\text{-loop}}$ replaced by the full effective action $\Gamma$ calculated at the instanton solution of the exact effective equations. Therefore, the normalizability (that is, the quantum consistency) condition (8.20) will still hold with the exact nonperturbative anomalous scaling $Z$ replacing its simple one-loop counterpart (8.17).

As concerns the one-loop approximation, which definitely gives the possibility to improve the predictions of the intrinsically inconsistent tree-level theory, in the following paper of this series [25] we shall apply the presented technique to a generic model of the nonminimally coupled inflaton scalar field. In particular, we shall analyze the possible extrema of the obtained partition function which might be responsible for the most probable inflationary universes tunnelling according to the Hartle-Hawking-Vilenkin proposal.

9. Discussion

Thus we have considered some elements of the general theory for tunnelling geometries in the no-boundary quantum state of Hartle and Hawking. We have shown that within the $\hbar$-expansion this theory can be extended to complex tunnelling solutions which are exponentially suppressed as compared to real ones, but can contribute to interesting physical phenomena, especially, in problems lacking real solutions matching the classically forbidden Euclidean regime with the Lorentzian one. The nucleation of the latter from the former can be described perturbatively entirely in terms of the
Euclidean and Lorentzian spacetimes with real metrics and matter fields. In the calculation of the quantum distribution function for the collective physical variables, they naturally lead to the notion of the real gravitational instanton – topologically closed solution of covariant Euclidean equations of motion. Despite the underlying complexity, this distribution function features unitarity and, thus, demonstrates a subtle interplay between unitarity, analyticity and covariance, encoded in a partition function of gravitational instantons weighted by their Euclidean effective action.

The price one pays for such a real-field description of complex tunnelling phenomena is that the resulting real fields are not completely smooth: they suffer a jump of the first-order derivatives at the nucleation surface $\Sigma_B$ and, consequently, generate a sharp edge at the junction "equatorial" surface between $M_-$ and $M_+$ in the geometry of the instanton $2M$ (see Fig.4 and Fig.5). This means, that one cannot directly generalize, by using these reality properties, the conclusions of Gibbons and Hartle [31] and, in particular, their unique conception theorem about the nucleation of the topologically connected Lorentzian spacetime. The same property implies also that, from the viewpoint of physical implications, the framework of problems on the signature change in general relativity [22, 69] should be extended to account for the subtleties of the above analytic continuation. This framework might include the consideration of gravitational instantons even with the discontinuous metric and matter fields, which would naturally arise if one would consider the non-diagonal elements of the density matrix $\rho(\varphi, \varphi' | t)$ instead of the probability distribution $\rho(\varphi, t) = \rho(\varphi, \varphi | t)$.

The technique of this paper heavily relies on the boundedness from below of the Euclidean gravitational action of physical variables. In quantum cosmology of spatially closed worlds this presents an immediate difficulty in the conformal sector which gives a negative-definite contribution and, in contrast to asymptotically-flat spacetimes [13, 46] seems to be not ruled out from the sector of physical degrees of freedom. If there is no way to consistently declare the conformal mode the unphysical one, one is left with the only option briefly discussed in Sect.5 – to shift this variable into a complex plane both in the integration contour of the path integral [40] and in the argument of the wavefunction or its quantum distribution $\rho(\varphi, t)$. As discussed in [23], a first step of this program might consist in isolating this mode $\varphi^{\text{conf}}$ into the
sector of collective variables and modifying the corresponding perturbation theory in
the the imaginary part of the classical extremal for $\varphi^\text{conf}$. As is also shown in [25]
this difficulty is not very serious in the analysis of the over-Planckian, $H(\varphi^\text{conf}) \to \infty$,
behaviour of $\rho_{2M}(\varphi^\text{conf})$, because in this limit $\text{Im} \varphi^\text{conf} = O(1/H(\varphi^\text{conf}))$, and one
can use the usual perturbation theory in $\text{Im} \varphi^\text{conf}$ unrelated to the asymptotic bound
(5.20). This might be explained by the fact that the possible suppression of the over-
Planckian energy scales in $\rho_{2M}(\varphi^\text{conf})$ originates not from its good gaussian nature, but
from the anomalous scaling of $\Gamma_{\text{1-loop}}(\varphi^\text{conf})$, the calculation of which implies merely
the conformal rotation in the Euclidean path integral.

In [25] we shall consider in much detail the application of the general theory developed here and the peculiarities of the conformal mode for the model of chaotic inflation with the nonminimal self-interacting inflaton field. This model, in which the inflaton field coincides with the Brans-Dicke scalar, plays an important role in the theory of the early Universe, for it provides a very efficient resolution [70] of the known difficulties in the formation of the observable cosmological large-scale structure [25]. Similarly to the discussion of the previous section, the analysis of this model is most simple for large $H(\varphi_B)$, because in this limit a theory has a trivial behaviour of the caustic surfaces in superspace, responsible for the nucleation of the Lorentzian Universe from the Euclidean spacetime. On the contrary, the situation becomes extremely complicated for small effective $H(\varphi_B)$, when the inflationary ansatz is no longer efficient and the caustic surface disappears below some critical value of $H(\varphi_B)$ [71]. On the other hand, this infrared domain plays an important role in the theory of wormholes, baby universes and the cosmological constant [3, 6, 7] – the first step towards the third quantization of gravity. For this reason, we shall also have to consider in [25] some general properties of caustic surfaces for solutions of Einstein equations and, in particular, show their relevance to the problem of Gribov copies in the manifestly unitary quantization of gravity. According to the discussion in [13, 14, 15], the Gribov problem serves as a strong motivation for the third quantization of gravity, because due to this problem the secondary quantized gravity seems to be intrinsically inconsistent (just as a first quantized interacting relativistic particle is inconsistent outside of the secondary quantization framework). This gives a hope that such an analysis of superspace caustic
surfaces in gravity theory might pave a constructive perturbative approach to its third quantization.

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**Figure captions**

**Fig.1** Graphical representation of the Lorentzian spacetime $L$ nucleating at the bounce surface $\Sigma_B$ from the Euclidean manifold $E$ of the no-boundary type, having the topology of the four-dimensional ball.

**Fig.2** Euclidean spacetime of the no-boundary type originating from the tube-like manifold $\Sigma \times [\tau_-,\tau_+]$ by shrinking one of its boundaries $\Sigma_-$ to a point. It inherits the foliation with slices of constant Euclidean time $\tau$ in the form of quasi-spherical surfaces of ”radius” $\tau$ with the center at $\tau_- = 0$.

**Fig.3** The contour $C_+ = C_E \cup C_L$ of integration over complex time in the action, corresponding to the splitting of the whole spacetime into the combination of Euclidean ($C_E$) and Lorentzian ($C_L$) domains matched at the nucleation (bounce) point $\tau_B$ ($t = 0$).

**Fig.4** The doubling of the Euclidean manifold, which arises in the calculation of the quantum distribution function for Lorentzian universes. The Euclidean spacetime of the no-boundary type $M_-$ matched across the nucleation surface $\Sigma_B$ with its orientation reversed copy $M_+$ gives rise to a closed manifold $2M = M_- \cup M_+$ – the gravitational instanton of spherical topology. For generic complex tunnelling geometries the matching of $M_-$ with $M_+$ is not smooth, which is shown on the picture by the edge at $\Sigma_B$.

**Fig.5** The graphical representation of calculating the quantum distribution of tunnelling Lorentzian universes: a composition of the combined Euclidean-Lorentzian spacetime $M_- \cup L$ with its orientation reversed and complex conjugated copy $M_+ \cup L^*$ results in the doubled Euclidean manifold $2M$ – the gravitational instanton carrying the Euclidean effective action of the theory. The cancellation of the Lorentzian domains $L$ and $L^*$ reflects the unitarity of the theory in the physical spacetime of Lorentzian signature.
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