The ruin problem for Lévy-driven linear stochastic equations with applications to actuarial models with negative risk sums

Yuri KABANOV$^a$, Serguei PÉRAGMENTCHIKOV$^b$

$^a$Université Bourgogne Franche-Comté, Laboratoire de Mathématiques, 16 Route de Gray, 25030 Besançon cedex, France, and National Research University - Higher School of Economics, Moscow, Russia
Email: youri.kabanov@univ-fcomte.fr

$^b$Université de Rouen Normandie, Laboratoire de Mathématiques Raphaël Salem, Technopôle du Madrillet, 76801 Saint-Étienne-du-Rouvray, France, and National Research Tomsk State University, International Laboratory of Statistics of Stochastic Processes and Quantitative Finance, Tomsk, Russia
Email: Serge.Pergamenchtchikov@univ-rouen.fr

Abstract

We study the asymptotic of the ruin probability for a process which is the solution of linear SDE defined by a pair of independent Lévy processes. Our main interest is the model describing the evolution of the capital reserve of an insurance company selling annuities and investing in a risky asset. Let $\beta > 0$ be the root of the cumulant-generating function $H$ of the increment of the log price process $V$. We show that the ruin probability admits the exact asymptotic $Cu^{-\beta}$ as the initial capital $u \to \infty$ assuming only that the law of $V_T$ is non-arithmetic without any further assumptions on the price process.

Keywords: Ruin probabilities, Dual models, Price process, Distributional equation, Autoregression with random coefficients, Lévy process

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1. Introduction

The general ruin problem can be formulated as follows. We are given a family of scalar processes $X^u$ with the initial values $u > 0$. The object of interest is the exit probability of $X^u$ from the positive half-line as a function of $u$. More formally, let $\tau^u := \inf\{t : X^u_t \leq 0\}$. The question is to determine the
function $\Psi(u, T) := P(\tau^u \leq T)$ (the ruin probability on a finite interval $[0, T]$) or $\Psi(u) := P(\tau^u < \infty)$ (the ruin probability on $[0, \infty[$). The exact solution of the problem is available only in rare cases. For instance, for $X^u = u + W$ where $W$ is the Wiener process we have $\Psi(u, T) = P(\sup_{t \leq T} W_t \geq u)$ and it remains to recall that the explicit formula for the distribution of the supremum of the Wiener process was obtained already in the Louis Bachelier thesis of 1900 which is, probably, the first ever mathematical study on continuous stochastic processes. Another example is the well-known explicit formula for $\Psi(u)$ in the Lundberg model of the ruin of insurance company with exponential claims. Of course, for more complicated cases the explicit formulae are not available and only asymptotic results or bounds can be obtained as it is done, e.g., in the Lundberg–Cramér theory.

In this paper we consider the ruin problem for a rather general model, suggested by Paulsen in [27], in which $X^u$ is given as the solution of linear stochastic equation (sometimes called the generalized Ornstein–Uhlenbeck process)

$$X_t^u = u + P_t + \int_{[0,t]} X^u_s dR_s,$$

where $P$ and $R$ are independent Lévy processes with the Lévy triplets $(a, \sigma^2, \Pi)$ and $(a_P, \sigma^2_P, \Pi_P)$, respectively. We assume that $\Pi([-\infty, -1]) = 0$ (otherwise $\Psi(u) = 1$ for all $u > 0$) and $P$ is not a subordinator (otherwise $\Psi(u) = 0$ for all $u > 0$ since the process $X^u$ is strictly positive, see (3.2), (3.1)). Also we exclude the case $R = 0$ well studied in the literature, see [21].

There is a growing interest in models of this type because they describe the evolution of reserves of insurance companies investing in a risky asset with the price process $S$. In the actuarial context $R$ is interpreted as the relative price process with $dR_t = dS_t / S_t$, that is the price process $S$ is the stochastic (Doléans) exponential $\mathcal{E}(R)$. The log price process $V = \ln \mathcal{E}(R)$ is also a Lévy process with the triplet $(a_V, \sigma^2, \Pi_V)$. Recall that the behavior of the ruin probability in such models is radically different from that in the classical actuarial models. For instance, if the price of the risky asset follows a geometric Brownian motion, that is, $R_t = at + \sigma W_t$, and the risk process $P$ is as in the Lundberg model, then $\Psi(u) = O(u^{1-2a/\sigma^2})$, $u \to \infty$, if $2a/\sigma^2 > 1$, and $\Psi(u) \equiv 1$ otherwise, [11, 31, 18].

We are especially interested in the case where the process $P$ describing the “business part” of the model has only upward jumps (in other words,
$P$ is spectrally positive). In the classical actuarial literature such models are referred to as the annuity insurance models (or models with negative risk sums), [13, 33], while in modern sources they serve also to describe the capital reserve of a venture company investing in development of new technologies and selling innovations; sometimes they are referred to as the dual models, [1, 2, 3, 7], etc.

The mentioned specificity of models with negative risk sums leads to a continuous downcrossing of the zero level by the capital reserve process. This allows us to obtain the exact (up to a multiplicative constant) asymptotic of the ruin probability under weak assumptions on the price dynamics.

Let $H : q \mapsto \ln \mathbb{E} e^{-qV}$ be the cumulant-generating function of the increment of log price process $V$ on the interval $[0, 1]$. The function $H$ is convex and its effective domain $\text{dom} H$ is a convex subset of $\mathbb{R}$ containing zero.

It is well-known that the asymptotic of the ruin probability $\Psi(u)$ as $u \to \infty$ is determined by the strictly positive root $\beta$ of $H$, assumed existing and laying in the interior of $\text{dom} H$. Unfortunately, the existing results are overloaded by numerous integrability assumptions on processes $R$ and $P$ while the law $\mathcal{L}(V_T)$ of the random variable $V_T$ is required to contain an absolute continuous component where $T$ is independent random variable uniformly distributed on $[0, 1]$, see, e.g., Th. 3.2 in [29].

The aim of our study is to obtain the exact asymptotic of the exit probability under the weakest conditions. Our main result has the following easy to memorize formulation.

**Theorem 1.1.** Suppose that $H$ has a root $\beta > 0$ laying in $\text{int dom } H$ and $\int |x|^\beta I_{\{|x|>1\}} \Pi_P(dx) < \infty$. Then

$$0 < \liminf_{u \to \infty} u^\beta \Psi(u) \leq \limsup_{u \to \infty} u^\beta \Psi(u) < \infty. \quad (1.2)$$

If, moreover, $P$ has only upward jumps and the distribution $\mathcal{L}(V_1)$ is non-arithmetic, then $\Psi(u) \sim C_\infty u^{-\beta}$ where $C_\infty > 0$.

In our argument we are based on the theory of distributional equations as presented in the paper by Goldie, [12] and on the criterion by Guivarc'h and Le Page, [15], which simple proof can be found in the recent paper [5] by Buraczewski and Damek. This criterion gives a necessary and sufficient condition for the strict positivity of the constant in the Kesten–Goldie theorem determining the rate of decay of the tail of solution at infinity. Its obvious corollary allows us to simplify radically the proofs and get rid of additional
assumptions presented in the earlier papers, see [27, 28, 29, 25, 26, 19, 4] and references therein. Our technique involves only affine distributional equations and avoids more demanding Letac-type equations.

The structure of the paper is the following. In Section 2 we formulate the model and provide some prerequisites from Lévy processes. Section 3 contains a well-know reduction of the ruin problem to the study of asymptotic behavior of a stochastic integral. In Section 4 we prove moment inequalities for maximal functions of stochastic integrals needed to analysis of the limiting behavior of an exponential functional in Section 5. The latter section is concluded by the proof of the main result and some comments on its formulation. In Section 6 we establish Theorem 6.4 on the ruin with probability one using the technique suggested in [31]. This theorem implies, in particular, that in the classical model with negative risk sums and investments in the risky asset with price following a geometric Brownian motion the ruin is imminent if \( a \leq \sigma^2/2 \), [18]. In Section 7 we discuss examples. Our presentation is oriented towards the reader with preferences in the Lévy processes rather than in the theory of distributional equations (called also implicit renewal theory). That is why in Section 8 (Appendix) we provide a rather detailed information on the latter covering the arithmetic case. In particular, we give a proof of a version of Grincevičius theorem under slightly weaker conditions as in the original paper.

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2. Preliminaries from the theory of Lévy processes

Let \((a, \sigma^2, \Pi)\) and \((a_P, \sigma_P^2, \Pi_P)\) be the Lévy triplets of the processes \(R\) and \(P\) corresponding to the standard\(^1\) truncation function \(h(x) := x I_{\{|x| \leq 1\}}\). Putting \(\bar{h}(x) := x I_{\{|x| > 1\}}\) we can write the canonical decomposition of \(R\) in the form

\[
R_t = at + \sigma W_t + h \ast (\mu - \nu)_t + \bar{h} \ast \mu_t
\]  

(2.1)

where \(W\) is a standard Wiener process, the Poisson random measure \(\mu(dt, dx)\) is the jump measure of \(R\) having the (deterministic) compensator \(\nu(dt, dx) = \)

\(^1\)Other truncation functions are also used in the literature, see, e.g., [29]
\[ dt \Pi(dx). \] For notions and results see [17], Ch. 2 and also [9], Chs. 2 and 3.

As in [17], we use \(*\) for the standard notation of stochastic calculus for integrals with respect to random measures. For instance,

\[
h * (\mu - \nu)_t = \int_0^t \int h(x)(\mu - \nu)(ds, dx).
\]

We hope that the reader will be not confused that \( f(x) \) may denote the whole function \( f \) or its value at \( x \); the typical example is \( \ln(1 + x) \) explaining why such a flexibility is convenient. The symbols \( \Pi(f) \) or \( \Pi(f(x)) \) stands for the integral of \( f \) with respect to the measure \( \Pi \).

Recall that

\[
\Pi(|x|^2 \wedge 1) := \int (|x|^2 \wedge 1) \Pi(dx) < \infty
\]

and the condition \( \sigma = 0 \) and \( \Pi(|h|) < \infty \) is necessary and sufficient for \( R \) to have trajectories of (locally) finite variation, see Prop. 3.9 in [9].

The process \( P \) describing the actuarial (“business”) part of the model admits a similar representation:

\[
P_t = a_P t + \sigma_P W^P_t + h * (\mu^P - \nu^P)_t + \bar{h} * \mu^P_t. \tag{2.2}
\]

The Lévy processes \( R \) and \( P \) generate the filtration \( F^{R,P} = (F^{R,P}_t)_{t \geq 0} \).

**Standing assumption S.0** The Lévy measure \( \Pi \) is concentrated on the interval \( ] -1, \infty[ \); \( \sigma^2 \) and \( \Pi \) do not vanish simultaneously; the process \( P \) is not a subordinator.

Recall that subordinator is an increasing Lévy process. Accordingly to [9], Prop. 3.10, the process \( P \) is not a subordinator if and only if \( \sigma^2_P > 0 \), or one of the following three conditions hold:

1) \( \Pi_P(] - \infty, 0[) > 0 \),
2) \( \Pi_P(] - \infty, 0[) = 0, \Pi_P(x I_{x>0}) = \infty \),
3) \( \Pi_P(] - \infty, 0[) = 0, \Pi_P(x I_{x>0}) < \infty, \Pi_P(x I_{x>0}) - a_P > 0 \).

In the context of financial models the stochastic exponential

\[
\mathcal{E}_t(R) = e^{R_t - \frac{1}{2} \sigma^2_t + \sum_{s \leq t} (\ln(1+\Delta R_s) - \Delta R_s)}
\]

stands for the price of a risky asset (e.g., stock). The log price \( V := \ln \mathcal{E}(R) \) is a Lévy process and can be written in the form

\[
V_t = at - \frac{1}{2} \sigma^2 t + \sigma W_t + h * (\mu - \nu)_t + (\ln(1 + x) - h) * \mu_t. \tag{2.3}
\]
Its Lévy triplet is \((a, \sigma^2, \Pi_V)\) where
\[
a_V = a - \frac{\sigma^2}{2} + \Pi(h(\ln(1 + x)) - h)
\]
and \(\Pi_V = \Pi \varphi^{-1}, \varphi : x \mapsto \ln(1 + x)\).

The cumulate-generating function \(H : q \to \ln \mathbb{E} e^{-qV_1}\) of the random variable \(V_1\) admits an explicit expression. Namely,
\[
H(q) := -a_V q + \frac{\sigma^2}{2} q^2 + \Pi \left( e^{-q \ln(1 + x)} - 1 + qh(\ln(1 + x)) \right). \quad (2.4)
\]
Its effective domain \(\text{dom} H = \{q : H(q) < \infty\}\) is the set \(\{J(q) < \infty\}\) where
\[
J(q) := \Pi \left( I_{\{|\ln(1 + x)| > 1\}} e^{-q \ln(1 + x)} \right) = \Pi \left( I_{\{|\ln(1 + x)| > 1\}} (1 + x)^{-q} \right). \quad (2.5)
\]
Its interior is the open interval \([q, \bar{q}]\) with
\[
q := \inf\{q \leq 0 : J(q) < \infty\}, \quad \bar{q} := \sup\{q \geq 0 : J(q) < \infty\}.
\]
Being a convex function, \(H\) is continuous and admits finite right and left derivatives on \([q, \bar{q}]\). If \(\bar{q} > 0\), then the right derivative
\[
D^+ H(0) = -a_V - \Pi(h(\ln(1 + x))) < \infty,
\]
though it may be equal to \(-\infty\).

In formulations of our asymptotic results we shall always assume that \(\bar{q} > 0\) and the equation \(H(q) = 0\) has a root \(\beta \in ]0, \bar{q}].\) Since \(H\) is not constant, such a root is unique. Clearly, it exists if and only if \(D^+ H(0) < 0\) and \(\limsup_{q \uparrow \bar{q}} H(q)/q > 0\). In the case where \(q < 0\) the condition \(D^- H(0) > 0\) is necessary to ensure that \(H(q) < 0\) for sufficiently small in absolute value \(q < 0\).

If \(J(q) < \infty\), then the process \(m = (m_t(q))_{t \leq 1}\) with
\[
m_t(q) := e^{-qV_t - tH(q)} \quad (2.6)
\]
is a martingale and
\[
\mathbb{E} e^{-qV_t} = e^{tH(q)}, \quad t \in [0, 1]. \quad (2.7)
\]
In particular, we have that \(H(q) = \ln \mathbb{E} e^{-qV_1} = \ln \mathbb{E} M^q\). For the above properties see, e.g., Th. 25.17 in [32].
Note that
\[ E \sup_{t \leq 1} e^{-qVt} < \infty \quad \forall q \in ]q, \bar{q}[, \] (2.8)

Indeed, let \( q \in ]0, \bar{q}[, \). Take \( r \in ]1, \bar{q}/q[, \). Then \( E m_1^r(q) = e^{H(qr)-rH(q)} < \infty \). By virtue of the Doob inequality the maximal function \( m_1(q) := \sup_{t \leq 1} m_t(q) \) belongs to \( L^r \) and it remains to observe that \( e^{-qVt} \leq C q m_t(q) \) where the constant \( C_q = \sup_{t \leq 1} e^{tH(q)} \). Similar arguments work for \( q \in ]q, 0[, \).

3. Ruin problem: a reduction

Let us introduce the process
\[ Y_t := - \int_{[0,t]} \mathcal{E}^{-1}(R) dP_s = - \int_{[0,t]} e^{-V_s} dP_s. \] (3.1)

Due to independence of \( P \) and \( R \) the joint quadratic characteristic \([P, R]\) is zero, and the straightforward application of the product formula for semimartingales shows that the process
\[ X^u_t := \mathcal{E}_t(R)(u - Y_t) \] (3.2)
solves the non-homogeneous linear equation (1.1), i.e. the solution of the latter is given by this stochastic version of the Cauchy formula.

The positivity of \( \mathcal{E}(R) \) implies that \( \tau^u = \inf\{t \geq 0 : Y_t \geq u\} \).

The following lemma is due to Paulsen and Gjessing, see [30].

**Lemma 3.1.** If \( Y_t \to Y_\infty \) almost surely as \( t \to \infty \) where \( Y_\infty \) is a finite random variable unbounded from above, then for all \( u > 0 \)
\[ \bar{G}(u) \leq \Psi(u) = \frac{\bar{G}(u)}{\mathbb{E}(\bar{G}(X^u_{\tau^u}) | \tau^u < \infty)} \leq \frac{\bar{G}(u)}{\bar{G}(0)}, \] (3.3)

where \( \bar{G}(u) := \mathbb{P}(Y_\infty > u) \).

In particular, if \( \Pi_P([-\infty, 0]) = 0 \), then \( \Psi(u) = \bar{G}(u)/\bar{G}(0) \).

**Proof.** Let \( \tau \) be an arbitrary stopping time with respect to the filtration \((\mathcal{F}^P,R_t)\). As we assume that the finite limit \( Y_\infty \) exists, the random variable
\[ Y_{\tau,\infty} := \begin{cases} -\lim_{N \to \infty} \int_{[\tau, \tau+N]} e^{-(V_t-V_\tau)} dP_t, & \tau < \infty, \\ 0, & \tau = \infty, \end{cases} \]
is well defined. On the set \( \{ \tau < \infty \} \)
\[
Y_{\tau, \infty} = e^{V_{\tau}}(Y_{\infty} - Y_{\tau}) = X_{\tau} + e^{V_{\tau}}(Y_{\infty} - u). 
\tag{3.4}
\]
Let \( \xi \) be a \( \mathcal{F}_{\tau}^{P,R} \)-measurable random variable. Since the Lévy process \( Y \) starts afresh at \( \tau \), the conditional distribution of \( Y_{\tau, \infty} \) given \( (\tau, \xi) = (t, x) \in \mathbb{R}_+ \times \mathbb{R} \) is the same as the distribution of \( Y_{\infty} \). It follows that
\[
P(Y_{\tau, \infty} > \xi, \tau < \infty) = E \tilde{G}(\xi) 1_{\{\tau<\infty\}}.
\]
Thus, if \( P(\tau < \infty) > 0 \), then
\[
P(Y_{\tau, \infty} > \xi, \tau < \infty) = E(\tilde{G}(\xi) | \tau < \infty) \, P(\tau < \infty).
\]
Noting that \( \Psi(u) := P(\tau^u < \infty) \geq P(Y_{\infty} > u) > 0 \), we deduce from here using (3.4) that
\[
\tilde{G}(u) = P(Y_{\infty} > u, \tau^u < \infty) = P(Y_{\tau^{u, \infty}} > X_{\tau^u}, \tau^u < \infty)
= E(G(X_{\tau^u}) | \tau^u < \infty) \, P(\tau^u < \infty)
\]
implying the equality in (3.3). The result follows since \( X_{\tau^u} \leq 0 \) on the set \( \{ \tau^u < \infty \} \) and, in the case where \( \Pi_P([\infty, 0]) = 0 \), the process \( X^u \) crosses zero in a continuous way, i.e. \( X_{\tau^u} = 0 \) on this set. \( \square \)

In view of the above lemma the proof of Theorem 1.1 is reduced to establishing of the existence of finite limit \( Y_{\infty} \) and finding the asymptotic of the tail of its distribution.

4. Moments of the maximal function

In this section we prove a simple but important result on the existence of moments of the maximal function of the process \( Y \) on the interval \([0, 1]\), i.e. of the random variable \( Y^*_1 := \sup_{t \leq 1} |Y_t| \).

Before the formulation we recall the Novikov inequalities, [24], also referred to as the Bichteler–Jacod inequalities, see [8, 23], providing bounds for moments of the maximal function \( I^*_1 \) of stochastic integral \( I = g \ast (\mu^P - \nu^P) \) where \( g^2 \ast \nu^P < \infty \). In dependence of the parameter \( \alpha \in [1, 2] \) they have the following form:
\[
E I^*_1 \leq C_{p, \alpha} \begin{cases} 
E (|g|^\alpha \ast \nu^P)^{p/\alpha}, & \forall p \in [0, \alpha], \\
E (|g|^\alpha \ast \nu^P)^{p/\alpha} + E |g|^p \ast \nu^P, & \forall p \in [\alpha, \infty[.
\end{cases} \tag{4.1}
\]
Lemma 4.1. Let \( p > 0 \) be such that \( \Pi_P(|\bar{h}|^p) + \mathbb{E} \sup_{t \leq 1} e^{-pV_t} < \infty \). Then \( \mathbb{E} Y_1^p < \infty \).

Proof. We start with the case where \( p \in ]0, 1[ \). The elementary inequality \( (\sum x_k)^p \leq \sum x_k^p \) allows us to treat separately the integrals corresponding to each term in the representation

\[ P_t = a_P t + \sigma_P W_t^P + h \ast (\mu^P - \nu^P)_t + \bar{h} \ast \mu_1^P. \]

Recall that in the detailed notations \( f \ast \mu_1^P = \sum_{s \leq 1: \Delta P_s > 0} f(s, \Delta P_s) \) and \( V_\cdot = (V_{s\cdot}) \). Using the mentioned inequality we get that

\[ \mathbb{E} (e^{-V_\cdot - |\bar{h}|^p} \ast \mu_1^P)^p \leq \Pi_P(|\bar{h}|^p) \mathbb{E} \int_0^1 e^{-pV_t} dt \leq \Pi_P(|\bar{h}|^p) \mathbb{E} \sup_{t \leq 1} e^{-pV_t}. \]  

(4.2)

Note that

\[ \mathbb{E} \left( \int_0^1 e^{-V_t} dt \right)^p \leq \mathbb{E} \sup_{t \leq 1} e^{-pV_t}. \]  

(4.3)

By the Burkholder–Davis–Gundy inequality

\[ \mathbb{E} \sup_{t \leq 1} \left| \int_0^t e^{-V_s} dW_s^P \right|^p \leq C_p \mathbb{E} \left( \int_0^1 e^{-2V_t} ds \right)^{p/2} \leq C_p \mathbb{E} \sup_{t \leq 1} e^{-pV_t}. \]  

(4.4)

Using the Novikov inequality (with \( \alpha = 2 \)) we have

\[ \mathbb{E} \sup_{t \leq 1} \left| e^{-V_t - h \ast (\mu^P - \nu^P)_t} \right|^p \leq C_{p,2} \Pi(h^2)^{p/2} \mathbb{E} \left( \int_0^1 e^{-2V_t} dt \right)^{p/2} \]

\[ \leq C_{p,2} \Pi(h^2)^{p/2} \mathbb{E} \sup_{t \leq 1} e^{-pV_t}. \]  

(4.5)

From these estimates and the property (2.8) we have that \( \mathbb{E} Y_\ast^p < \infty \).

Let \( p \in ]1, 2[ \). By the Novikov inequality with \( \alpha = 1 \) and we have:

\[ \mathbb{E} \sup_{t \leq 1} \left| e^{-V_t - h \ast (\mu^P - \nu^P)_t} \right|^p \leq C_{p,1} \left( \mathbb{E} \left( e^{-V_t - h \ast \mu_1^P} \right)^p + \mathbb{E} e^{-pV_t} \ast \mu_1^P \right) \]

\[ \leq \tilde{C}_{p,1} \mathbb{E} \sup_{t \leq 1} e^{-pV_t} < \infty, \]
where \( \tilde{C}_{p,1} := C_{p,1} ((\Pi_P(\tilde{h}))^p + \Pi_P(\tilde{h}^p)) \). Using again the Novikov inequality but with \( \alpha = 2 \) we obtain that

\[
E \sup_{t \leq 1} |e^{-V - h} \ast (\mu^P - \nu^P)_t|^p \leq C_{p,2} E (e^{-2V - h^2} \ast \nu_t^p)^{p/2} \leq C_{p,2} (\Pi_P(h^2))^p E \sup_{t \leq 1} e^{-pV_t} < \infty.
\]

Estimates for the integrals with respect to \( dt \) and \( dW^P \) are of the same form as for the previous case. Using the inequality for the \( L^p \)-norm of the sum, we get that \( E Y_1^{*p} < \infty \).

Finally, let \( p \geq 2 \). Using the Novikov inequality with \( \alpha = 2 \), we have:

\[
E \sup_{t \leq 1} |g \ast (\mu^P - \nu^P)_t|^p \leq C_{p,2} (\Pi_P(|x|^2))^{p/2} E \left( \int_0^1 e^{-2V_t} dt \right)^{p/2} + C_{p,2} \Pi_P(|x|^p) E \int_0^1 e^{-pV_t} dt \leq C_{p,2} ((\Pi_P(|x|^2))^{p/2} + \Pi_P(|x|^p)) E \sup_{t \leq 1} e^{-pV_t} < \infty.
\]

Again the arguments for the integrals with respect to \( dt \) and \( dW^P \) remain valid. \( \square \)

5. Convergence of \( Y_t \)

Using Lemma 4.1 the convergence \( Y_t \) as \( t \to \infty \) can be easily established under very weak assumptions. Namely, we have the following:

**Proposition 5.1.** If there is \( p > 0 \) such that \( H(p) < 0 \), and \( \Pi_P(|\tilde{h}|^p) < \infty \), then \( Y_t \) converge a.s. to a finite random variable \( Y_\infty \) unbounded from above and solving the distributional equation

\[
Y_\infty \overset{d}{=} Y_1 + M_1 Y_\infty, \quad Y_\infty \text{ independent of } (M_1, Y_1), \quad (5.1)
\]

where \( M_1 := e^{-\nu_i} \).

**Proof.** Without loss of generality we assume that \( p < 1 \) and \( H(p+) \neq \infty \). For any integer \( j \geq 1 \)

\[
Y_j - Y_{j-1} = M_1 \ldots M_{j-1} Q_j, \quad .
\]

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where \((M_j, Q_j)\) are independent random variables,

\[
M_j := e^{-(V_j - V_{j-1})}, \quad Q_j := - \int_{[j-1,j]} e^{-(V_v - V_{j-1})} dP_v
\]

with distributions \(\mathcal{L}(M_j) = \mathcal{L}(M_1)\) and \(\mathcal{L}(Q_j) = \mathcal{L}(Y_1)\). By assumption, 
\[\rho := \mathbb{E}M_1^p = e^{H(p)} < 1\] and \(\mathbb{E}|Y_1|^p < \infty\) in virtue of (2.8) and Lemma 4.1. Since \(\mathbb{E}M_1\ldots M_{j-1}|Q_j| = \rho^j \mathbb{E}|Y_1|^p\) we have that \(\mathbb{E} \sum_{j \geq 1} |Y_j - Y_{j-1}|^p < \infty\) and, hence, \(\sum_{j \geq 1} |Y_j - Y_{j-1}|^p < \infty\) a.s. But then also \(\sum_{j \geq 1} |Y_j - Y_{j-1}| < \infty\) a.s. and, therefore, the sequence \(Y_n\) converges almost surely to some finite random variable \(Y_\infty\).

Put

\[
\Delta_n := \sup_{n-1 \leq v \leq n} \left| \int_{[n-1,v]} e^{-V_s} dP_s \right|, \quad n \geq 1.
\]

Note that

\[
\mathbb{E} \Delta_n^p = \mathbb{E} \prod_{j=1}^{n-1} M_j^p \sup_{n-1 \leq v \leq n} \left| \int_{[n-1,v]} e^{-(V_s - V_{n-1})} dP_s \right|^p = \rho^{n-1} \mathbb{E} Y_1^{sp} < \infty.
\]

For any \(\varepsilon > 0\) we get using the Chebyshev inequality that

\[
\sum_{n \geq 1} \mathbb{P}(\Delta_n > \varepsilon) \leq \varepsilon^{-p} \mathbb{E} Y_1^{sp} \sum_{n \geq 1} \rho^{n-1} < \infty.
\]

By the Borel–Cantelli lemma \(\Delta_n(\omega) \leq \varepsilon\) for all \(n \geq n_0(\omega)\) for each \(\omega \in \Omega\) except a null-set. This implies the convergence \(Y_t \to Y_\infty\) a.s., \(t \to \infty\).

Let us consider the sequence

\[
Y_{1,n} := Q_2 + M_2 Q_3 + \cdots + M_2 \ldots M_n Q_{n+1}
\]

converging almost surely to a random variable \(Y_{1,\infty}\) distributed as \(Y_\infty\). Passing to the limit in the obvious identity \(Y_n = Q_1 + M_1 Y_{1,n-1}\) we obtain that \(Y_\infty = Q_1 + M_1 Y_{1,\infty}\). For finite \(n\) the random variables \(Y_{1,n}\) and \((M_1, Q_1)\) are independent, \(\mathcal{L}(Y_{1,n}) = \mathcal{L}(Y_n)\). Hence, \(Y_{1,\infty}\) and \((M_1, Q_1)\) are independent, \(\mathcal{L}(Y_{1,\infty}) = \mathcal{L}(Y_\infty)\) and \(\mathcal{L}(Y_\infty) = \mathcal{L}(Q_1 + M_1 Y_{1,\infty})\). This is exactly the properties abbreviated by (5.1).

It remains to check that \(Y_\infty\) is unbounded from above. For this it is useful the following simple observation.

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Lemma 5.2. If the random variables $Q_1$ and $Q_1/M_1$ are unbounded from above, then $Y_\infty$ is also unbounded from above.

Proof. Since $Q_1/M_1$ is unbounded from above, we have, due to independence of $(Q_1/M_1)$ and $Y_1,\infty$, that $P(Y_1,\infty > 0) = P(Y_\infty > 0) > 0$. Take arbitrary $u > 0$. Then

$$P(Y_\infty > u) \geq P(Q_1 + M_1 Y_1,\infty > u, Y_1,\infty > 0) \geq P(Q_1 > u, Y_1,\infty > 0)$$

and the lemma is proven. \Box

Notation: $J_\theta := \int_0^1 e^{-\theta V_v} dv$, $Q_\theta := -\int_0^1 e^{-\theta V_v} dP_v$ where $\theta = 1$ or $-1$.

Lemma 5.3. $\mathcal{L}(Q_{-1}) = \mathcal{L}(Q_1/M_1)$.

Proof. We have:

$$\int_{[0,1]} \sum_{k=1}^n e^{V_{k/n}-I_{[(k-1)/n,k/n]}(v)} dP_v = \sum_{k=1}^n e^{V_{k/n}} (P_k/n - P_{(k-1)/n}),$$

$$e^{V_1} \int_{[0,1]} \sum_{k=1}^n e^{-V_{k/n}-I_{[(k-1)/n,k/n]}(v)} dP_v = \sum_{k=1}^n e^{V_1 - V_{k/n}} (P_k/n - P_{(k-1)/n}).$$

Note that $V$ and $P$ are independent, the increments $P_{k/n} - P_{(k-1)/n}$ are independent and identically distributed, and $\mathcal{L}(V_1 - V_{k/n}) = \mathcal{L}(V_{(k-1)/n})$. Thus, the right-hand sides of the above identities have the same distribution. The result follows because the left-hand sides tend in probability, respectively, to $-Q_{-1}$ and $-Q_1/M_1$. \Box

Thus, $Y_\infty$ is unbounded from above if so are the stochastic integrals $Q_\theta$. Lemma 5.4 below shows that $Q_\theta$ are unbounded from above if the ordinary integrals $J_\theta$ are unbounded from above. For the latter property we prove necessary and sufficient conditions in terms of defining characteristics (Lemma 5.7). The case where these conditions are not fulfilled we treat separately (Lemma 5.8).

Lemma 5.4. If $J_\theta$ is unbounded from above, so is $Q_\theta$.

Proof. We argue using the following observation: if $f(x, y)$ is a measurable function and $\xi$, $\eta$ are independent random variables with distributions $P_\xi$
and \( P_\eta \), then the distribution of \( f(\xi, \eta) \) is unbounded from below if the distribution of \( f(\xi, y) \) is unbounded from below on a set of \( y \) of positive measure \( P_\eta \).

In the case \( \sigma^2_P > 0 \), we use the representation

\[
Q_\theta = -\sigma_P \int_{[0,1]} e^{-\theta V_t} dW_t^P + \int_{[0,1]} e^{-\theta V_t} d(\sigma_P W_t - P_t).
\]

Applying the above observation (with \( \xi = W^P \) and \( \eta = (R, P - \sigma_P W^P) \)) and taking into account that the Wiener integral of a strictly positive deterministic function is a nonzero Gaussian random variable, we get that \( Q_\theta \) is unbounded.

Consider the case where \( \sigma^2_P = 0 \).

For \( \varepsilon > 0 \) we denote by \( \zeta^\varepsilon \) the locally square integrable martingale with

\[
\zeta^\varepsilon_t := e^{-\theta V_t} I_{\{|x| \leq \varepsilon\}} x * (\mu^P - \nu^P)_t. \tag{5.3}
\]

Since \( \langle \zeta^\varepsilon \rangle_1 = e^{-2\theta V} I_{\{|x| \leq \varepsilon\}} x^2 * \nu^1_P \to 0 \) as \( \varepsilon \to 0 \), we have that \( \sup_{t \leq 1} |\zeta^\varepsilon_t| \to 0 \) in probability.

Note that

\[
Q_\theta = (\Pi_P(x I_{\{|x| \leq 1\}}) - a_P) J_\theta - \zeta^\varepsilon_t - e^{-\theta V_t} I_{\{|x| < \varepsilon\}} x * \mu^1_P.
\]

Take \( N > 1 \). Since \( J_\theta \) is unbounded from above, there is \( N_1 > N + 1 \) such that the set \( \{N \leq J_\theta \leq N_1, \inf_{t \leq 1} e^{-V_t} \geq 1/N_1\} \) is non-null. Then

\[
\Gamma^\varepsilon := \{N \leq J_\theta \leq N_1, \inf_{t \leq 1} e^{-V_t} \geq 1/N_1, |\zeta^\varepsilon_t| \leq 1\}
\]

is also a non-null set for all sufficiently small \( \varepsilon > 0 \).

The process \( P \) is not a subordinator and, therefore, we have only three possible cases.

1) \( \Pi_P([-\infty, 0]) > 0 \). Then \( \Pi_P([-\infty, -\varepsilon_0]) > 0 \) for some \( \varepsilon_0 > 0 \). Due to independence, the intersection of \( \Gamma^\varepsilon \) with the set

\[
\{|I_{\{x < -\varepsilon\}} x * \mu^1_P| \geq N_1(a_P^+ N_1 + N), I_{\{x > \varepsilon\}} x * \mu^1_P = 0\}
\]

is non-null when \( \varepsilon \in ]0, \varepsilon_0[ \). On this intersection we have that

\[
Q_\theta \geq -a_P J_\theta - \zeta^\varepsilon_t - e^{-\theta V_t} I_{\{x < -\varepsilon\}} x * \mu^1_P \geq -a_P^+ N_1 - 1 + a_P^+ N_1 + N \geq N - 1.
\]
2) \( \Pi_P([-\infty, 0]) = 0, \Pi_P(h) = \infty. \) Diminishing in the need \( \varepsilon \) to ensure the inequality \( \Pi_P(xI_{x>\varepsilon}) \geq N_1(a_P^+N_1 + N) \), we have on the non-null set \( \Gamma^\varepsilon \cap \{I_{x>\varepsilon} \ast \mu_1^P = 0\} \) that
\[
Q_\theta = -a_P \mathcal{J}_\theta - \zeta_0 + e^{-\theta V_{I_{x>\varepsilon}}} I_{x>\varepsilon} \ast \nu_1 \geq -a_P^+N_1 - 1 + a_P^+N_1 + N \geq N - 1.
\]

3) \( \Pi_P([-\infty, 0]) = 0, \Pi_P(h) < \infty, \) and \( \Pi_P(h) - a_P > 0. \) Then on the non-null set \( \{\mathcal{J}_\theta \geq N\} \cap \{I_{x>0} \ast \mu_1 = 0\} \) we have that
\[
Q_\theta = (\Pi_P(h) - a_P) \mathcal{J}_\theta \geq (\Pi_P(h) - a_P)N.
\]

Since \( N \) is arbitrary, in all three cases \( Q_\theta \) is unbounded from above. \( \square \)

**Remark 5.5.** If \( \mathcal{J}_1I_{V_1<0} \) is unbounded from above, so is \( Q_1I_{V_1<0} \).

**Remark 5.6.** The proof above shows that in the case where \( \sigma_P = 0 \) there is a constant \( \kappa > 0 \) such that if the set \( \{\mathcal{J}_\theta > N\} \) is non-null, then \( Q_\theta > \kappa N \) on its \( \mathcal{F}_1^{R,P} \)-measurable non-null subset. The statement remains valid with obvious changes if the integration over the interval \([0, 1]\) is replaced by the integral over arbitrary finite interval \([0, T]\).

**Lemma 5.7.**

(i) The random variable \( \mathcal{J}_1 \) is unbounded from above if and only if \( \sigma^2 + \Pi([-1, 0]) > 0 \) or \( \Pi(xI_{|x|\leq 1}) = \infty. \)

(ii) The random variable \( \mathcal{J}_{-1} \) is unbounded from above if and only if \( \sigma^2 + \Pi([0, \infty]) > 0 \) or \( \Pi(xI_{x<0}) = -\infty. \)

**Proof.** In the case where \( \sigma^2 > 0 \) the “if” parts of the statements are obvious: \( W \) is independent of the jump part of \( V \) and the distribution of the random variable \( \int_0^1 e^{-\sigma W_t} g(v)dv \), where \( g > 0 \) is a deterministic function, has a support unbounded from above.

So, suppose that \( \sigma^2 = 0 \) and consider the “if” parts separately.

(i) Let \( \Pi([-1, 0]) > 0, \) i.e. \( \Pi([-1, -\varepsilon]) > 0 \) for some \( \varepsilon \in ]0, 1[. \) Then the process \( V \) given by (2.3) admits the decomposition
\[
V_t = at + h \ast (\mu - \nu)_t + (\ln(1 + x) - h) \ast \mu_t = (a - \Pi(xI_{[-1\leq x\leq -\varepsilon]}))t + V'_t + V''_t,
\]
where
\[
V'_t := I_{[-\varepsilon \leq x \leq 1\} \ast (\mu - \nu)_t + (\ln(1 + x) - x)I_{[-\varepsilon \leq x \leq 1\} \ast \mu_t
\]
\[
+ \ln(1 + x)I_{\{x>1\} \ast \mu_t,
\]
\[
V''_t := \ln(1 + x)I_{\{-1\leq x\leq -\varepsilon\} \ast \mu_t.
\]
The processes $V'$ and $V''$ are independent. The decreasing process $V''$ has jumps of the size not less than $|\ln(1 - \varepsilon)|$ and the number of jumps on the interval $[0, t]$ is a Poisson random variable with parameter $t\Pi([1, -\varepsilon]) > 0$. Hence, $V''_t$ is unbounded from below for any $t \in ]0, 1[$. In particular, for any $N > 0$, the set where $e^{-V''_t} \geq N$ on the interval $[1/2, 1]$ is non-null. The required property follows from these considerations.

Let $\Pi(h(x)I_{[x>0]}) = \infty$. We may assume without loss of generality that $\Pi([0, 0]) = 0$. In this case, the process $V$ has only positive jumps. Take arbitrary $N > 1$ and choose $\varepsilon > 0$ such that $\Pi(xI_{[\varepsilon<x\leq1]}) > 2N$ and $\Pi(I_{[0<x\leq\varepsilon]} \ln^2(1 + x)) \leq 1/(32N^2)$. We have the decomposition

$$V_t = ct + V^{(1)}_t + V^{(2)}_t + V^{(3)}_t,$$

where the processes

$$V^{(1)} := I_{[0<x\leq\varepsilon]} \ln(1 + x) \ast (\mu - \nu),$$
$$V^{(2)} := I_{[\varepsilon<x\leq1]} \ln(1 + x) \ast (\mu - \nu),$$
$$V^{(3)} := I_{[x>1]} \ln(1 + x) \ast \mu$$

are independent and the constant

$$c := a + \Pi((\ln(1 + x) - x)I_{[0<x\leq1]}) < \infty.$$ 

By the Doob inequality $P(\sup_{t\leq1} V^{(1)}_t < N/2) > 1/2$. The processes $V^{(2)}$ and $V^{(3)}$ have no jumps on $[0, 1]$ on a non-null set. In the absence of jumps the trajectory of $V^{(2)}$ is the linear function

$$y_t = -\Pi(xI_{[\varepsilon<x\leq1]})t \leq -2Nt.$$

It follows that $\sup_{1/2\leq t \leq 1} V_t \leq c - N/2$ on the set of positive probability. This implies that $\mathcal{J}_t$ is unbounded from above.

(ii) Let $\Pi([0, +\infty]) > 0$, i.e. $\Pi([0, \varepsilon]) > 0$ for some $\varepsilon \in ]0, 1[$. Then

$$V_t = at + h \ast (\mu - \nu)_t + (\ln(1 + x) - h) \ast \mu_t = (a - \Pi(hI_{[x>\varepsilon]})t + \tilde{V}'_t + \tilde{V}''_t,$$

where

$$\tilde{V}'_t := I_{[x\leq\varepsilon]} h \ast (\mu - \nu)_t + (\ln(1 + x) - h)I_{[x\leq\varepsilon]} \ast \mu_t,$$
$$\tilde{V}''_t := \ln(1 + x)I_{[x>\varepsilon]} \ast \mu_t.$$
The processes $\tilde{V}'$ and $\tilde{V}''$ are independent. The increasing process $\tilde{V}''$ has jumps of the size not less than $\ln(1 + \varepsilon)$ and the number of jumps on the interval $[0, t]$ is a Poisson random variable with parameter $t\Pi([\varepsilon, \infty[) > 0$. Hence, $V''_t$ is unbounded from above for any $t \in [0, 1]$. In particular, for any $N > 0$, the set where $e^{V''} \geq N$ on the interval $[1/2, 1]$ is non-null. These facts imply the required property.

Let $\Pi(xI_{\{x<0\}}) = -\infty$. We may assume without loss of generality that $\Pi([0, \infty[) = 0$. In this case, the process $V$ has only negative jumps. Take arbitrary $N > 1$ and choose $\varepsilon \in [0, 1/2[$ such that

$$-\Pi(\ln(1 + x)I_{\{-1/2 < x < -\varepsilon\}}) > 2N, \quad \Pi(I_{\{-\varepsilon < x < 0\}} \ln^2(1 + x)) \leq 1/(32N^2).$$

This time we use the representation

$$V_t = \tilde{c}t + \tilde{V}^{(1)}_t + \tilde{V}^{(2)}_t + \tilde{V}^{(3)}_t,$$

where the processes

$$\begin{align*}
\tilde{V}^{(1)} &:= I_{\{-\varepsilon < x < 0\}} \ln(1 + x) * (\mu - \nu), \\
\tilde{V}^{(2)} &:= I_{\{-1/2 < x \leq -\varepsilon\}} \ln(1 + x) * (\mu - \nu), \\
\tilde{V}^{(3)} &:= I_{\{-1 < x \leq -1/2\}} \ln(1 + x) * \mu
\end{align*}$$

are independent and the constant

$$\tilde{c} := a + \Pi(\ln(1 + x)I_{\{-1/2 < x < 0\}} - h).$$

By the Doob inequality $P(\sup_{t \leq 1} \tilde{V}^{(1)}_t < N/2) > 1/2$. The processes $\tilde{V}^{(2)}$ and $\tilde{V}^{(3)}$ have no jumps on $[0, 1]$ with strictly positive probability. In the absence of jumps the trajectory of $\tilde{V}^{(2)}$ is the linear function

$$y = -\Pi(\ln(1 + x)I_{\{-1/2 < x < -\varepsilon\}})t \geq 2Nt.$$ 

It follows that $\sup_{1/2 \leq t \leq 1} V_t \leq \tilde{c} + N/2$ on a non-null set. This implies that $J_{-1}$ is unbounded from above.

The “only if” parts of the lemma are obvious. ☐

Summarizing, we conclude that $Q_1$ and $Q_{-1}$ (hence, $Y_\infty$) are unbounded from above if $\sigma^2 > 0$, or $\sigma^2_p > 0$, or $P(|h|) = \infty$, or $\Pi([-1, 0[) > 0$ and $\Pi([0, \infty[) > 0$. The remaining cases are treated in the following:
Lemma 5.8. Suppose that $\sigma = 0$, $\Pi(\{|h|\} < \infty$, $\sigma_P = 0$. If $\Pi(\{1, 0\}) = 0$ or $\Pi(\{0, \infty\}) = 0$, then the random variable $Y_{\infty}$ is unbounded from above.

Proof. By our assumptions $V_t = ct + L_t$ where the constant $c := a - \Pi(h)$, $\Pi \neq 0$, and $L_t := \ln(1 + x) * \mu_t$. The assumption $\beta > 0$ implies that $V_t < 0$ with strictly positive probability and $V$ cannot be increasing or decreasing process. So, there are two cases which we consider separately:

(i) $c < 0$ and $\Pi(\{0, \infty\}) > 0$. Take any $T > 1$. Then the integral $\int_{[0, T]} e^{-V_t} dt \geq T/e$ on the non-null set $\{L_T \leq 1\}$. By virtue of Remark 5.6 on a non-null $\mathcal{F}_t^{R,P}$-measurable subset $\Gamma_T \subseteq \{L_T \leq 1\}$ we have the bound $\int_{[0, T]} e^{-V_t} dP_t \geq K_T$ where $K_T \to \infty$ as $T \to \infty$. For every $T > 1$

$P(\Gamma_T \cap \{L_T+1 - L_T \geq |c|(T + 1)\}) = P(\Gamma_T)P(L_T+1 - L_T \geq |c|(T + 1)) > 0$.

Let $\zeta^\varepsilon$ is the square integrable martingale $\zeta^\varepsilon$ defined by (5.3) with $\theta = 1$. Take $N > 1$ sufficiently large and $\varepsilon > 0$ sufficiently small to ensure that the set $\Gamma_T^{\varepsilon,N}$ defined as the intersection of sets $\Gamma_T \cap \{L_T+1 - L_T \geq |c|(T + 1)\}$, $\{\sup_{s \in [T,T+1]} e^{-V_s} \leq N$, $\inf_{s \in [T,T+1]} e^{-V_s} \geq 1/N\}$, and $\{\zeta_{T+1} - \zeta_T \leq 1\}$ is non-null.

Let us consider the representation

$$Y_{\infty} = -\int_{[0, T]} e^{-V_t} dP_t + a_P^{\varepsilon} \int_{[T,T+1]} e^{-V_t} dt - \zeta_{T+1}^\varepsilon + \zeta_T^\varepsilon - I_{[T, \infty]} e^{-V_t} I_{\{|x| > \varepsilon\}} * \mu_{T+1}^P + e^{-V_{T+1}} Y_{T+1}.$$

Take arbitrary $y > 0$ such that the set $\{Y_{T+1,\infty} > y\}$ is non-null.

Since the process $P$ is not a subordinator with $\sigma_P = 0$, it must satisfy one of the characterizing conditions 1), 2), 3) of Section 2. Let us consider them consecutively.

Suppose that $\Pi_P(\{-\infty, 0\}) > 0$. Then there exists $\varepsilon_0 > 0$ such that $\Pi_P(\{-\infty, -\varepsilon_0\}) > 0$. Due to the independence, the intersection of $\Gamma_T^{\varepsilon,N}$ with the set

$$\tilde{\Gamma}_T^{\varepsilon,N} := \{I_{[T, \infty]} I_{\{|x| < \varepsilon\}} * \mu_{T+1}^P \geq -(1/\varepsilon) N^2 a_P^{\varepsilon}, I_{[T, \infty]} I_{\{|x| > \varepsilon\}} * \mu_{T+1}^P = 0\}$$

is non-null when $\varepsilon \in [0, \varepsilon_0]$.

Due to independence, the intersection of $\Gamma_T^{\varepsilon,N} \cap \tilde{\Gamma}_T^{\varepsilon,N}$ and $\{Y_{T+1,\infty} > y\}$ also is a non-null set. But on this intersection we have inequality $Y_{\infty} \geq K_T - 1 + y$ implying that $Y_{\infty}$ is unbounded from above.
Suppose that $\Pi_P([-\infty, 0]) = 0$, $\Pi_P(h) = \infty$. Thus, for sufficiently small $\varepsilon > 0$ we have $a_P^\varepsilon > 0$. On the non-null set

$$\Gamma_T^c \cap \{I_{[T, \infty]}[I_{x > \varepsilon}] * \mu_{T+1}^P = 0\} \cap \{Y_{T+1, \infty} > y\}$$

the inequality $Y_\infty \geq K_T - 1 + y$ holds and we conclude as above.

Finally, suppose that $\Pi_P([-\infty, 0]) = 0$, $\Pi_P(h) < \infty$, and $\Pi_P(h) - a_P > 0$. In this case we can use the representation

$$Y_\infty = -\int_{[0,T]} e^{-V_i - dP_t + (\Pi_P(h) - a_P) \int_{[T, T+1]} e^{-V_i - dt} - I_{[T, \infty]} e^{-V_i - x I_{x > 0}} * \mu_{T+1}^P + e^{-V_{T+1}} Y_{T+1, \infty}.$$ 

On the non-null set $\Gamma_T^c \cap \{I_{[T, \infty]}[I_{x > 0}] * \mu_{T+1}^P = 0\} \cap \{Y_{T+1, \infty} > y\}$ we have that $Y_\infty \geq K_T + y$ implying that $Y_\infty$ is unbounded from above.

(ii) $c > 0$ and $\Pi([-1, 0]) > 0$. In this case there are $\gamma, \gamma_1 \in [0, 1]$, $\gamma < \gamma_1$, such that $\{I_{[\gamma-1, \gamma]} * \mu_1 = 0\}$, $\{I_{[\gamma-1, \gamma]} * \mu_{1/2} = I_{[\gamma-1, \gamma]} * \mu_1 = N\}$, and $\{\ln(1 + x) I_{[\gamma-1, 0]} * \mu_1 \geq -1\}$ are non-null sets. Due to independence, their intersection $A_N$ is also non-null.

On $A_N$ we have the bounds

$$c + N \ln(1 - \gamma) - 1 \leq V_1 \leq c + N \ln(1 - \gamma_1)$$

and

$$J_1 := \int_{[0,1]} e^{-V_i} d\gamma \geq e^{-c} \int_{[0,1/2]} e^{-\ln(1 + x) * \mu_1} d\gamma \geq \frac{1}{2} e^{-c} (1 - \gamma_1)^{-N}.$$

In virtue of Remark 5.6 there is a constant $\kappa_N$ an $\mathcal{F}_1^{R,P}$-measurable non-null subset $B_N$ of $A_N$ such that $Q_1 \geq \kappa_N$ on $B_N$ and $\kappa_N \to \infty$ as $N \to \infty$.

Take $T = T_N > 0$ such that $cT + N \ln(1 - \gamma) - 2 \geq 0$. The set $\{I_{[1, T]}[\ln(1 + x) * \mu_1 + T \geq -1]\}$ is non-null and its intersection with $B_N$ is also non-null. On this intersection $e^{-V_{T+1}} \leq 1$ and

$$c_1(N) \leq V_1 - \leq c_2(N)$$

where $c_1(N) := c + N \ln(1 - \gamma) - 2$, $c_2(N) := c(T + 1) + N \ln(1 - \gamma_1)$.

With this we accomplish the arguments by considering the cases corresponding to the properties 1), 2), and 3) with obvious modifications. □
With the above lemma the proof of Proposition 5.1 is complete. $\square$

**Proof of the main theorem.** In view of (2.8) and Lemma 4.1 we have that $E |Q_1|^\beta < \infty$. The hypothesis on $\beta$ and Proposition 5.1 allows us to use the results of the implicit renewal theory on the tail behavior of distribution of $Y_\infty$ resumed in Theorem 8.6 of Appendix. The reference to Lemma 3.1 completes the proof. $\square$

**Remark 5.9.** Note that the hypothesis $\beta \in \text{int dom } H$ can be replaced by the slightly weaker assumption $E e^{-\beta V_1} V_1^{-} < \infty$.

**Remark 5.10.** The hypothesis $\mathcal{L}(V_1)$ is non-arithmetic also can be replaced by a weaker one: one can assume that $\mathcal{L}(V_T)$ is non-arithmetic for some $T > 0$. Indeed, due to the identity $\ln E e^{-\beta V_T} = TH(\beta)$ the root $\beta$ does not depend on the choice of the time unit.

The following lemma shows that the condition on $\mathcal{L}(V_1)$ can be formulated in terms of the Lévy triplets.

**Lemma 5.11.** The (non-degenerate) distribution of $V_1$ is arithmetic if and only if $\sigma = 0$, $\Pi(\mathbb{R}) < \infty$, and there is $d > 0$ such that $\Pi_V$ is concentrated on the lattice $\Pi(h) - a + Zd$.

**Proof.** Recall that $\sigma_V = \sigma$ and $\Pi_V = \Pi V^{-1}$ where $\varphi : x \mapsto \ln(1 + x)$. So, we have $\Pi_V(\mathbb{R}) = \Pi(\mathbb{R})$. If $\sigma_V > 0$ or $\Pi_V(\mathbb{R}) = \infty$, the distribution of $V_1$ has a density, see Prop. 3.12 in [9]. If $\sigma = 0$ and $0 < \Pi_V(\mathbb{R}) < \infty$, then $V$ is a compound Poisson process with drift $c = a - \Pi(h)$ and distribution of jumps $F_V := \Pi_V^\beta / \Pi_V(\mathbb{R})$. In such a case $\mathcal{L}(V_1)$ is concentrated on the lattice $\mathbb{Z}d$ if and only if $\Pi_V$ is concentrated on the lattice $-c + \mathbb{Z}d$. $\square$

6. Ruin with probability one

In this section we give conditions under which the ruin is imminent whatever is the initial reserve.

Recall the following ergodic property of the autoregressive process $(X_n^u)_{n \geq 1}$ with random coefficients (see, [31], Prop. 7.1) which is defined recursively by the relations

$$X_n^u = A_n X_{n-1}^u + B_n, \quad n \geq 1, \quad X_0^u = u,$$

(6.1)

where $(A_n, B_n)_{n \geq 1}$ is a sequence of i.i.d. random variables in $\mathbb{R}^2$. 

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Lemma 6.1. Suppose that $E|A_n|^\delta < 1$ and $E|B_n|^\delta < \infty$ for some $\delta \in ]0, 1]$. Then for any $u \in \mathbb{R}$ the sequence $X_n^u$ converges in $L^\delta$ (hence, in probability) to the random variable

$$X_\infty^0 = \sum_{n=1}^{\infty} B_n \prod_{j=1}^{n-1} A_j$$

and for any bounded uniformly continuous function $f$

$$\frac{1}{N} \sum_{n=1}^{N} f(X_n^u) \to Ef(X_\infty^0) \quad \text{in probability as } N \to \infty. \quad (6.2)$$

Applying the lemma to the function $f(x) = I_{\{x<1\}} - x I_{\{-1\leq x<0\}}$ we get:

Corollary 6.2. Suppose that $E|A_n|^\delta < 1$ and $E|B_n|^\delta < \infty$ for some $\delta \in ]0, 1]$.

(i) If $P(X_\infty^0 < 0) > 0$, then $\inf_{n\geq1} X_n^u < 0$.

(ii) If $A_1 > 0$ and $B_1/A_1$ is unbounded from below, then $\inf_{n\geq1} X_n^u < 0$.

Proof. We get (i) by the straightforward application of (6.2) to the function $f(x) := I_{\{x<1\}} - x I_{\{-1\leq x<0\}}$. The statement (ii) follows from (i). Indeed, put $X_{\infty}^{0,1} := \sum_{n=2}^{\infty} B_n \prod_{j=2}^{n-1} A_j$. Then

$$X_\infty^0 = B_1 + A_1 X_\infty^{0,1} = A_1 (X_\infty^{0,1} + B_1/A_1).$$

Since $B_1/A_1$ and $X_\infty^{0,1}$ are independent and the random variable $B_1/A_1$ is unbounded from below, $P(X_\infty^0 < 0) > 0$. □

Let $M_j$ and $Q_j$ be the same as in (5.2).

Proposition 6.3. Suppose that $E|M_1^{-\delta} < 1$ and $E|Q_1|^{-\delta} < \infty$ for some $\delta \in [0, 1]$. If $Q_1$ is unbounded from above, then $\Psi(u) \equiv 1$.

Proof. The process $X^u$ solving the equation (1.1) and restricted to the integer values of the time scale admits the representation

$$X_n^u = e^{V_n V_{n-1} X_{n-1}^u} + e^{V_n \int_{[n-1, n]} e^{-V_t} dP_t}, \quad n \geq 1, \quad X_0^u = u.$$ 

That is, $X_n^u$ is given by (6.1) with $A_n = M_n^{-1}$ and $B_n = -M_n^{-1}Q_n$. The result follows from the statement (ii) of Corollary 6.2. □

Now we give more specific conditions of the ruin with probability one in terms of the triplets.

Theorem 6.4. Suppose that $0 \in \text{int dom } H$ and $\Pi_P([\hat{h}]^\varepsilon) < \infty$ for some $\varepsilon > 0$. If

$$a_V + \Pi(h(\ln(1 + x))) \leq 0,$$

then $\Psi(u) \equiv 1$.\n
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Proof. Note that $D^\ast H(0) = -a_V - \Pi(\bar{h}(\ln (1 + x)))$. If $D^\ast H(0) > 0$, then for all $q < 0$ sufficiently close to zero $H(q) < 0$, i.e. $\mathbb{E} M^q < 1$. By virtue of Lemma 5.3 $\mathcal{L}(M^{-1}_1 Q_1) = \mathcal{L}(Q_{-1})$. If $\Pi_P(|\bar{h}|^\varepsilon) < \infty$ for some $\varepsilon > 0$, then the same arguments as in the proof of the first part of the proof of Lemma 4.1 lead to the conclusion that $\mathbb{E} |Q_{-1}|^q < \infty$ for sufficiently small $q > 0$. To get the result we can use Proposition 6.3. Indeed, by virtue of Lemmata 5.4 and 5.7(i) the random variable $Q_1$ is unbounded from above except, eventually, the case where $\sigma^2 = 0$, $\sigma_P^2 = 0$, $\Pi(1) = 0$, $\Pi(xI_{0 < x < 1}) < \infty$, and $\Pi \neq 0$. But under such constrains on the characteristics the distribution of $X_0$ coincides with the distribution of the integral $\int_0^x e^{V_t} d\mathbb{P}_x$. Using the arguments similar to those in the proof of Lemma 5.8(i), it is easy to prove that the latter charges $]-\infty, 0[$ and we can apply Corollary 6.2(i).

In the case where $D^\ast H(0) = 0$ we consider, following [31], the discrete-time process $(\tilde{X}_n^u)_{n \in \mathbb{N}}$ where $\tilde{X}_n^u = X_{T_n}$ and the descending ladder times $T_n$ of the random walk $(V_n)_{n \in \mathbb{N}}$ which are defined as follows: $T_0 := 0,$

$$T_n := \inf\{k > T_{n-1} : V_k - V_{T_{n-1}} < 0\}.$$  

Since $J(q) = \Pi(I_{(|\ln(1+x)| > 1})(1 + x)^{-q}) < \infty$ for any $q \in [\bar{q}, \bar{q}[$, we have that $\Pi(\ln(1 + x)) < \infty$. It follows that the formula (2.3) can be written as

$$V_t = \left(a - \sigma^2/2 - \Pi(h)\right)t + \sigma W_t + \ln(1 + x) * \mu_t,$$

$\mathbb{E}V_t^2 < \infty$, and the condition $D^\ast H(0) = 0$ means that $\mathbb{E}V_1 = 0$.

Accordingly to Theorem 1a in Ch. XII.7 of Feller’s book [10] and the remark preceding the citing theorem, the above properties imply that there is a finite constant $c$ such that

$$\mathbb{P}(T_1 > n) \leq cn^{-1/2}. \quad (6.3)$$

It follows, in particular, that the differences $T_n - T_{n-1}$ are well-defined and form a sequence of finite independent random variables distributed as $T_1$. The discrete-time process $\tilde{X}_n^u = X_{T_n}^u$ has the representation

$$\tilde{X}_n^u = e^{V_{T_n} - V_{T_{n-1}}} \tilde{X}_{n-1}^u + e^{V_{T_n}} \int_{[T_{n-1}, T_n]} e^{-V_t} d\mathbb{P}_t, \quad n \geq 1, \quad \tilde{X}_0^u = u,$$

and solves the linear equation

$$\tilde{X}_n^u = \tilde{A}_n \tilde{X}_{n-1}^u + \tilde{B}_n, \quad n \geq 1, \quad X_0^u = u,$$

where

$$\tilde{A}_n := e^{V_{T_n} - V_{T_{n-1}}}, \quad \tilde{B}_n := e^{V_{T_n}} \int_{[T_{n-1}, T_n]} e^{-V_t} d\mathbb{P}_t,$$
and $\tilde{B}_1/\tilde{A}_1 = Y_{T_n}$ where $Y$ is given by (3.1).

By construction, $\tilde{A}_1^\delta < 1$ for any $\delta > 0$.

Using the definition of $Q_j$ given by (5.2) we have that

$$|\tilde{B}_1| \leq \sum_{j=1}^{T_1} e^{V_{T_1} - V_{j-1}} |Q_j| \leq \sum_{j=1}^{T_1} |Q_j|. $$

According to Lemma 4.1 $E|Q_j|^p < \infty$ for some $p \in ]0, 1]$. Then for $r \in ]0, p/5[$ and $l_n := [n^{4r}]$, we have, using the Chebyshev inequality and (6.3), that

$$E|\tilde{B}_1|^r \leq 1 + r \sum_{n \geq 1} n^{r-1} P \left( \sum_{j=1}^{T_1} |Q_j| > n \right)$$

$$\leq 1 + r \sum_{n \geq 1} n^{r-1} P \left( \sum_{j=1}^{l_n} |Q_j| > n \right) + r \sum_{n \geq 1} n^{r-1} P (T_1 > l_n)$$

$$\leq 1 + r E|Q_1|^p \sum_{n \geq 1} l_n n^{r-1-p} + rc \sum_{n \geq 1} n^{r-1} l_n^{1/2} < \infty.$$ 

To apply Corollary 6.2(ii) it remains to check that $Y_{T_1}$ is unbounded from above. Since $\{Q_1 > N, V_1 < 0\} \subseteq \{Y_{T_1} > N\}$, it is sufficient to check that the probability of the set in the left-hand side is strictly positive for all $N > 0$, or, by virtue of Remark 5.5, that

$$P(\mathcal{J}_1 > N, V_1 < 0) > 0 \quad \forall N > 0. \quad (6.4)$$

Let $\sigma^2 > 0$. Taking into account that the conditional distribution of the process $(W_s)_{s \leq 1}$ given $W_1 = x$ is the same as the (unconditional) distribution of the Brownian bridge $B^x = (B^x_s)_{s \leq 1}$ with $B^x_1 = W_s + s(x - W_1)$ we easily get that for any bounded positive function $g$ and any $y, M \in \mathbb{R}$ the probability

$$P \left( \int_0^1 e^{-\sigma W_s} g(v)dv > y, W_1 < M \right) > 0,$$

cf. with Lemma 4.2 in [18]. This implies (6.4).

Suppose that $\sigma^2 = 0$, but $\Pi([-1, 0]) > 0$, i.e. $\Pi([-1, -\varepsilon]) > 0$ for some $\varepsilon \in ]0, 1]$. In the decomposition $V = V^{(1)} + V^{(2)}$, where

$$V^{(1)}_t = I_{[-1 < x \leq -\varepsilon]} \ln(1 + x) * \mu_t,$$

$$V^{(2)}_t = (a - \Pi(hI_{[-1 < x \leq -\varepsilon]}))t + I_{[x > -\varepsilon]} h * (\mu - \nu)_t$$

$$+ I_{[x > -\varepsilon]} (\ln(1 + x) - h) * \mu_t,$$
the processes $V^{(1)}$ and $V^{(2)}$ are independent. The process $V^{(1)}$ is decreasing by negative jumps whose absolute value are larger or equal than $|\ln(1-\varepsilon)|$ and the number of jumps on the interval $[0,1/2]$ has the Poisson distribution with parameter $(1/2)\Pi(-1,-\varepsilon[)>0$. Thus, $P(V^{(1)}_{1/2}<-n)>0$ for any real $n$. It follows that

$$P(J_1 > N, V_1 < 0) \geq P \left( \int_0^1 e^{-V_1(t)} dt > N, V_1 < 0, V^{(1)}_{1/2} < -n \right) \geq P \left( e^n \int_{1/2}^1 e^{-V^{(2)}_1(t)} dt > N, V^{(2)}_1 < n, V^{(1)}_{1/2} < -n \right) = P \left( \int_{1/2}^1 e^{-V^{(2)}_1(t)} dt > Ne^{-n}, V^{(2)}_1 < n \right) P(V^{(1)}_{1/2} < -n).$$

The right-hand side is strictly positive for sufficiently large $n$ and (6.4) holds.

The case where $\Pi(xI_{0<x<1}) = \infty$ is treated similarly as in the last part of the proof of Lemma 5.7(i).

The exceptional case is treated by a reduction to Corollary 6.2(i). $\Box$

The above theorem implies that in the classical model with negative risk sums (where $\sigma = 0$, the jumps of $P$ are positive and form a compound Poisson process, $\Pi_P(|x|) < \infty$, trend is negative, i.e. $a_P - \Pi_P(x) < 0$) and investments into a risky asset with the price following a geometric Brownian motion (that is, $\Pi = 0$ and $\sigma \neq 0$), the ruin is imminent if $a_V = a - \sigma^2/2 \leq 0$.

7. Examples

Example 1. Let us consider the model with negative risk sums in which $\Pi_P(dx) = \lambda F_P(dx)$ where the constant $\lambda > 0$ and the probability distribution $F_P(dx)$ is concentrated on $]0,\infty[$, and

$$a^0_P := \lambda \int_{[0,1]} xF_P(dx) - a_P.$$

The process $P$ admits the representation as sum of an independent Wiener process with drift and a compound Poisson process:

$$P_t = -a^0_P t + \sigma_P W^P_t + \sum_{j=1}^{N^P_t} \xi_j, \quad (7.1)$$

where the Poisson process $N^P$ with intensity $\lambda_P$ is independent of the sequence $(\xi_j)_{j \geq 1}$ of positive i.i.d. random variables with common distribution $F_P$.
Suppose that the price process is a geometric Brownian motion
\[ \mathcal{E}_t(R) = e^{V_t} = e^{(a-\sigma^2/2)t+\sigma W_t}, \]
that is, \( \sigma \neq 0, \Pi = 0. \)

For this model \( q = -\infty, \bar{q} = \infty. \) The condition \( D^+H(0) < 0 \) is reduced to the inequality \( \sigma^2/2 < a \) and the function \( H(q) = (\sigma^2/2 - a + q\sigma^2/2)q \) has the root \( \beta = 2a/\sigma^2 - 1 > 0. \) Suppose that \( \sigma_p^2 + (a_p^0)^+ > 0. \) By Theorem 1.1 the exact asymptotic \( \Psi(u) \sim C_\infty u^{-\beta}, \) as \( u \to \infty, \) holds if \( E\xi_1^{\beta < \infty}. \) Since the exponential distribution has the above property, we recover, as a very particular case the asymptotic result of [18] where it was assumed that \( \sigma_p^2 = 0 \) and \( a_p^0 > 0. \)

If \( \sigma_p^2 + (a_p^0)^+ > 0, \sigma^2/2 \geq a, \) and \( E\xi_1^{1 < \infty} \) for some \( \epsilon > 0, \) then Theorem 6.4 implies that \( \Psi(u) \equiv 1. \)

**Example 2.** Let the process \( P \) be again given by (7.1) and suppose that the price process has a jump component, namely,
\[ \mathcal{E}_t(R) = \exp \left\{ (a - \sigma^2/2)t + \sigma W_t + \sum_{j=1}^{N_t} \ln(1 + \eta_j) \right\}, \]
where the Poisson process \( N \) with intensity \( \lambda > 0 \) is independent on the sequence \( (\eta_j)_{j \geq 1} \) of i.i.d. random variables with common distribution \( F \) not concentrated at zero and \( F([-\infty, -1]) = 0, \) see [22], Ch. 7. That is, the log price process is represented as
\[ V_t = (a - \sigma^2/2)t + \sigma W_t + \ln(1 + x) \ast \mu_t, \]
where \( \Pi(dx) = \lambda F(dx). \) The function \( H \) is given by the formula
\[ H(q) = (\sigma^2/2 - a + q\sigma^2/2)q + \lambda E(1 + \eta_1)^{-q} - 1. \]
Suppose that \( E(1 + \eta_1)^{-q} < \infty \) for all \( q > 0. \) Then \( \bar{q} = \infty. \)

Let \( \sigma \neq 0. \) Then \( \limsup_{q \to \infty} H(q)/q = \infty. \) If
\[ D^+H(0) = \sigma^2/2 - a - \lambda E \ln(1 + \eta_1) < 0, \quad (7.2) \]
then the root \( \beta > 0 \) of the equation \( H(q) = 0 \) does exist. Thus, if \( E\xi_1^{\beta < \infty}, \) then Theorem 1.1 can be applied to get that \( \Psi(u) \sim C_\infty u^{-\beta}, \) where \( C_\infty > 0. \)

If \( E(1 + \eta_1)^{-\beta_1} < 1 \) (resp., \( E(1 + \eta_1)^{-\beta_1} > 1 \)), the root \( \beta \) is smaller (resp., larger) than \( 2a/\sigma^2 - 1, \) the value of the root of \( H \) in model of the first example where the price process is continuous.

Let \( \sigma = 0. \) If
\[ D^+H(0) = -a - \lambda E \ln(1 + \eta_1) < 0, \]
and
\[ \limsup_{q \to \infty} q^{-1} E \left( (1 + \eta_1)^{-q} - 1 \right) > a/\lambda, \]
then the root \( \beta > 0 \) also exists. Theorem 1.1 can be applied when \( 0 < P(\eta_1 > 0) < 1 \) and the we have exact asymptotic if the distribution of \( \ln(1 + \eta_1) \) is non-arithmetic.

Suppose that \( E(1 + \eta_1)^{-q} < \infty \) for all \( q \in \mathbb{R} \). Then \( q = -\infty, \bar{q} = \infty \). If \( \sigma^2/2 - a - \lambda E \ln(1 + \eta_1) \geq 0, \sigma^2 + P(\eta_1 < 0) > 0, \) and \( E|\xi_1|^{\bar{\varepsilon}} < \infty \) for some \( \varepsilon > 0 \), then \( \Psi(u) \equiv 1 \) in virtue of Theorems 6.4.

8. Appendix: tails of distributions solving distributional equations

8.1. Kesten–Goldie theorem

Here we present a short account of needed results on distributional equations (random equations in the terminology of [12])

\[ Y_\infty \overset{d}{=} Q + M Y_\infty, \quad Y_\infty \text{ independent of } (M, Q), \tag{8.1} \]
where \((M, Q)\) is a given two-dimensional random variable with \( M > 0 \) and \( P(M \neq 1) > 0 \) and \( \overset{d}{=} \) is the equality in law. This is a symbolical notation which means that we are given in fact a two-dimensional distribution \( \mathcal{L} \) on \( \mathbb{R} \times \mathbb{R} \) not concentrated on \( \mathbb{R} \times \{1\} \) and the problem is to find a probability space with random variables \( Y_\infty \) and \((M, Q)\) on it such that \( Y_\infty \) and \((M, Q)\) are independent, \( \mathcal{L}(M, Q) = \mathcal{L} \), and \( \mathcal{L}(Y_\infty) = \mathcal{L}(Q + M Y_\infty) \). The uniqueness in this problem means the uniqueness of the distribution of \( Y_\infty \).

In the sequel \((M_j, Q_j)\) will be an i.i.d. sequence whose generic term \((M, Q)\) has the distribution \( \mathcal{L} \) and \( Z_j := M_1 \ldots M_j, Z_\infty^* := \sup_{j \leq n} Z_j \).

If there is \( p > 0 \) such that \( E M^p < 1 \) and \( E|Q|^p < \infty \), then the solution \( Y_\infty \) of (8.1) can be easily realized on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) where the sequence \((M_j, Q_j)\) is defined — just as the limit in \( L^p \) of the series \( \sum_{j \geq 1} Z_{j-1} Q_j \), see the beginning of the proof of Proposition 5.1.

The following classical result of the renewal theory is the Kesten–Goldie theorem, see Th. 4.1 in [12]:

**Theorem 8.1.** Suppose that \((Q, M)\) is such that the distribution of \( \ln M \) is non-arithmetic and, for some \( \beta > 0 \),

\[ E M^\beta = 1, \quad E M^{\beta (\ln M)^+} < \infty, \quad E |Q|^\beta < \infty. \tag{8.2} \]
Then
\[ \lim_{u \to \infty} u^\beta P(Y_\infty > u) = C_+ < \infty, \]
\[ \lim_{u \to \infty} u^\beta P(Y_\infty < -u) = C_- < \infty, \]
where \( C_+ + C_- > 0 \).

Theorem 8.1 left open the question when the constant \( C_+ \) is strictly positive. Recently, Guivarc’h and Le Page showed for the above case where the distribution of \( \ln M \) is non-arithmetic that \( C_+ > 0 \) if and only if \( Y_\infty \) is unbounded from above, see [15] and also the paper [5] for simpler arguments. The remaining part of the appendix deals mainly with the arithmetic case.

8.2. Grincevičius theorem

The theorem below is a simplified version of Th.2(b), [14], but with a slightly weaker assumption on \( Q \), namely, \( E|Q|^\beta < \infty \), used in our study. For the reader convenience we give its complete proof after recalling some concepts and facts from the renewal theory.

**Theorem 8.2.** Suppose that (8.2) holds and the distribution of \( \ln M \) is concentrated on the lattice \( \mathbb{Z}^d \) where \( d > 0 \). Then
\[ \limsup_{u \to \infty} u^\beta P(Y_\infty > u) < \infty. \] (8.3)

We consider the convolution-type linear operator which is well-defined for all positive as well as for (the Lebesgue) integrable functions by the formula
\[
\tilde{\psi}(x) = \int_{-\infty}^{x} e^{-(x-y)} \psi(y)dy.
\] (8.4)

Clearly, the functions \( \psi \) and \( \tilde{\psi} \) are integrable or not simultaneously and
\[
\int_{\mathbb{R}} \tilde{\psi}(x)dx = \int_{\mathbb{R}} \psi(x)dx.
\]

Suppose that \( \psi \geq 0 \) is integrable. Then \( \tilde{\psi}(x + \delta) \geq e^{-\delta} \tilde{\psi}(x) \) for any \( \delta > 0 \) and
\[
\delta \inf_{x \in [j\delta, (j+1)\delta]} \tilde{\psi}(x) \geq \delta e^{-\delta} \tilde{\psi}(j\delta) \geq e^{-2\delta} \int_{(j-1)\delta}^{j\delta} \tilde{\psi}(x)dx
\]

implying that
\[
U(\tilde{\psi}, \delta) := \delta \sum_{j \in \mathbb{Z}} \inf_{x \in [j\delta, (j+1)\delta]} \tilde{\psi}(x) \geq e^{-2\delta} \int_{\mathbb{R}} \tilde{\psi}(x)dx.
\]
Similarly,
\[
\tilde{U}(\tilde{\psi}, \delta) := \delta \sum_{j \in \mathbb{Z}} \sup_{x \in [j\delta, (j+1)\delta]} \tilde{\psi}(x) \leq e^{2\delta} \int_{\mathbb{R}} \tilde{\psi}(x) dx.
\]
Thus, \(\tilde{U}(\tilde{\psi}, \delta) < \infty\) and \(\tilde{U}(\tilde{\psi}, \delta) - U(\tilde{\psi}, \delta) \to 0\) as \(\delta \to \infty\). These two properties mean, by definition, that the function \(\tilde{\psi}\) is directly Riemann integrable. Arguing with the positive and negative parts, we obtain that if \(\psi\) is integrable, then \(\tilde{\psi}\) is directly Riemann integrable.

We shall use in the sequel the following renewal theorem for the random walk \(S_n := \sum_{i=1}^{n} \xi_i\) on a lattice, see Prop. 2.1, [16].

**Proposition 8.3.** Let \(\xi_i\) be i.i.d. random variables taking values in the lattice \(\mathbb{Z}_d, d > 0\), and having finite expectation \(m := E\xi_i > 0\). Let \(F : \mathbb{R} \to \mathbb{R}\) be a measurable function. If \(x \in \mathbb{R}\) is such that \(\sum_{j \in \mathbb{Z}} |F(x + jd)| < \infty\), then
\[
\lim_{n \to \infty} E\sum_{k \geq 0} F(x + nd - S_k) = \frac{d}{m} \sum_{j \in \mathbb{Z}} F(x + jd).
\]

**Proof of Theorem 8.2.** Let the solution of (8.1) be realized on some probability space \((\Omega, \mathcal{F}, P)\). We shall use the notation \((M, Q)\) instead of \((M_1, Q_1)\). Put \(G(u) := P(Y_\infty > u)\) and \(g(x) := e^{\beta x} G(e^x)\). Since \(Y_\infty\) and \(M\) are independent, \(P(MY_\infty > e^x) = E G(e^{x - \ln M})\). Defining the new probability measure \(\hat{P} := \hat{M}^\beta P\) and noting that
\[
e^{\beta x} P(MY_\infty > e^x) = EM^\beta e^{\beta(x - \ln M)} G(e^{x - \ln M}) = \hat{E} g(x - \ln M)
\]
we obtain the following identity (called renewal equation):
\[
g(x) = D(x) + \hat{E} g(x - \ln M),
\] (8.5)
where \(D(x) := e^{\beta x} P(Y_\infty > e^x) - P(MY_\infty > e^x)\). The Jensen inequality for the convex function \(x \mapsto x \ln x\) implies that \(\hat{E} \ln M = EM^\beta \ln M > 0\) and, hence, \(\hat{E} |\ln M| < \infty\).

Let us check that the function \(x \mapsto D(x)\) is integrable. To this aim, we note that for any random variables \(\xi, \eta\)
\[
|P(\xi > s) - P(\eta > s)| \leq P(\eta^+ \leq s < \xi^+) + P(\xi^+ \leq s < \eta^+).
\]
Using the Fubini theorem we obtain that
\[
\int_0^\infty P(\eta^+ \leq s < \xi^+) s^{\beta - 1} ds = E I_{\{\eta^+ < \xi^+\}} \int_{\eta^+}^{\xi^+} s^{\beta - 1} ds = \frac{1}{\beta} E ((\xi^+)^\beta - (\eta^+)^\beta)^+.
\]
Applying this bound with $\xi := Q + MY_\infty \overset{d}{=} Y_\infty$ and $\eta := MY_\infty$ we get that
\[
\int_\mathbb{R} |D(x)|dx = \int_0^\infty |\mathbf{P}(\xi > s) - \mathbf{P}(\eta > s)|s^{\beta - 1}ds \leq \frac{1}{\beta} \mathbf{E}|(\xi^+)^\beta - (\eta^+)^\beta|,
\]
and it remains to verify that
\[
\mathbf{E}|((Q + \eta^+)^\beta - (\eta^+)^\beta| < \infty \tag{8.6}
\]
when $\mathbf{E}|Q|^\beta < \infty$. But $|(Q + \eta^+)^\beta - (\eta^+)^\beta| = \zeta_1 + \zeta_2$ with positive summands
\[
\zeta_1 := I_{\{-Q < \eta \leq 0\}}(Q + \eta)^\beta + I_{\{0 < \eta \leq -Q\}}\eta^\beta \leq |Q|^\beta,
\]
\[
\zeta_2 := I_{\{Q + \eta > 0, \eta > 0\}}|(Q + \eta)^\beta - \eta^\beta|.
\]
If $\beta \leq 1$, then $\zeta_2$ is also dominated by $|Q|^\beta$. If $\beta > 1$, then the inequality $|x^\beta - y^\beta| \leq \beta|x - y|(|x \lor y|)^{\beta - 1}$ for $x, y \geq 0$ combined with the inequality $(|a| + |b|)^{\beta - 1} \leq 2^{(\beta - 2)^+}(|a|^\beta - |b|^\beta)$ leads to the estimate
\[
\zeta_2 \leq 2^{(\beta - 2)^+}\beta|Q|(|\eta|^{\beta - 1} + |Q|^{\beta - 1}).
\]
Using the independence of $(M, Q)$ and $Y_\infty$, the Hölder inequality, and taking into account that $\mathbf{E}M^\beta = 1$ and $\mathbf{E}|Y_\infty|^p < \infty$ for $p \in [0, \beta[$ we get that
\[
\mathbf{E}|Q||\eta|^{\beta - 1} = \mathbf{E}|Q|M^{\beta - 1} \mathbf{E}|Y_\infty|^{\beta - 1} \leq (\mathbf{E}|Q|^\beta)^{1/\beta} \mathbf{E}|Y_\infty|^{\beta - 1} < \infty.
\]
Thus, (8.6) holds.

The integrability of $D$ allows us to transform (8.5) into the equality
\[
\tilde{g}(x) = \tilde{D}(x) + \tilde{\mathbf{E}}\tilde{g}(x - \ln M).
\]
Iterating it, we obtain that
\[
\tilde{g}(x) = \sum_{n=0}^{N-1} \tilde{\mathbf{E}}\tilde{D}(x - S_n) + \tilde{\mathbf{E}}\tilde{g}(x - S_N), \tag{8.7}
\]
where $S_0 = 0$ and $S_n := \sum_{i=1}^n \xi_i$ for $n \geq 1$, $(\xi_i)$ is a sequence of independent random variables on $(\Omega, \mathcal{F}, \mathbf{P})$ independent on $Y_\infty$ such that the distribution $\mathcal{L}(\xi_i, \mathbf{P}) = \mathcal{L}(\ln M, \mathbf{P})$. In particular, $\tilde{\mathbf{E}}e^{-\beta \xi_i} = 1$.

By the strong law of large numbers $S_N/N \to \tilde{\mathbf{E}}\ln M > 0 \overset{\mathbf{P}}{-\text{a.s.}}, N \to \infty$, and, therefore, $y - S_N \to -\infty \overset{\mathbf{P}}{-\text{a.s.}}$ for every $y$. Since $\tilde{\mathbf{E}}e^{-\beta S_N} = 1$, we have by dominated convergence that
\[
\tilde{\mathbf{E}}\tilde{g}(y - S_N) = \tilde{\mathbf{E}}e^{\beta(y - S_N)}G(e^{y - S_N}) \to 0.
\]
It follows that the remainder term $\tilde{E}\tilde{g}(x - S_N)$ in (8.7) tends to zero, thus,

$$\tilde{g}(x) = \sum_{k \geq 0} \tilde{E} \tilde{D}(x - S_k). \tag{8.8}$$

Using Proposition 8.3 (with $F = \tilde{D}$) we obtain that for any $x > 0$

$$\lim_{n \to \infty} \tilde{g}(x + dn) = \frac{d}{\mathbb{E} \ln M} \sum_{j \in \mathbb{Z}} \tilde{D}(x + jd) \leq \tilde{U}(\tilde{D}, d) < \infty. \tag{8.9}$$

Replacing in the integrant the function $\tilde{G}(e^y)$ by its smallest value $G(e^x)$ we obtain that

$$\tilde{g}(x) := \int_{-\infty}^{x} e^{-(x-y)}e^{\beta y}G(e^y)dy \geq \frac{1}{\beta + 1}g(x)$$

and, therefore,

$$\limsup_{u \to \infty} u^\beta \tilde{P}(Y_\infty > u) = \limsup_{x \to \infty} g(x) \leq (\beta + 1) \limsup_{x \to \infty} \tilde{g}(x) < \infty.$$ 

Theorem 8.2 is proven. $\square$

8.3. Buraczewski–Damek approach

The following result, usually formulated in terms of the supremum of the random walk $S_n := \sum_{i=1}^{n} \ln M_i$, is well-known (see, e.g., Th. A, [20] for much more general setting).

**Proposition 8.4.** If $M$ satisfies (8.2), then

$$\liminf_{u \to \infty} u^\beta \tilde{P}(Z_\infty^* > u) > 0. \tag{8.10}$$

**Proof.** Let $F(x) := \tilde{P}(\ln M \leq x)$, $\bar{F}(x) := 1 - F(x)$, $S_n := \sum_{i=1}^{n} \xi_i$ where $\xi_i := \ln M_i$. The function $\bar{H}(x) := \tilde{P}(\sup_n S_n > x)$ admits the representation

$$\bar{H}(x) = \tilde{P}(\xi_1 > x) + \mathbb{E} I_{\{\xi_1 \leq x\}} \bar{H}(x - \xi_1) = \bar{F}(x) + \int_{-\infty}^{x} \bar{H}(x - t)d\bar{F}(t).$$

Putting $Z(x) := e^{\beta x} \bar{H}(x)$, $z(x) := e^{\beta x} \bar{F}(x)$, and $\tilde{P} := e^{\beta \xi_1} \tilde{P}$, we obtain from here that

$$Z(x) = z(x) + \tilde{E} Z(x - \xi_1) I_{\{\xi_1 \leq x\}}. \tag{8.11}$$

The same arguments as were used in deriving (8.7) lead to the representation

$$Z(x) = \tilde{E} \sum_{k \geq 0} z(x - S_k) I_{\{S_k \leq x\}}. \tag{8.12}$$
The function $\hat{z}(x) := z(x)I_{\{x \geq 0\}}$ is directly Riemann integrable. Indeed, for $j \geq 0$ we have that

$$
\sup_{x \in [j\delta,(j+1)\delta]} z(x) \leq e^{\beta(j+1)\delta} \tilde{F}(j\delta) \leq e^{2\beta\delta} \int_{(j-1)\delta}^{j\delta} e^{\beta v} \bar{F}(v) dv
$$

and, therefore,

$$
\bar{U}(\hat{z},\delta) = \delta z(0) + \delta \sum_{j \geq 0} \sup_{x \in [j\delta,(j+1)\delta]} z(x) \leq \delta z(0) + e^{2\beta\delta} \int_{-\delta}^{\infty} e^{\beta v} \bar{F}(v) dv.
$$

In the same spirit

$$
\inf_{x \in [j\delta,(j+1)\delta]} z(x) \geq e^{-\beta j\delta} \tilde{F}((j+1)\delta) \geq e^{-2\beta\delta} \int_{(j+1)\delta}^{(j+2)\delta} e^{\beta v} \bar{F}(v) dv
$$

and

$$
\underline{U}(\hat{z},\delta) = \delta \sum_{j \geq 0} \sup_{x \in [j\delta,(j+1)\delta]} z(x) \geq e^{-2\beta\delta} \int_{\delta}^{\infty} e^{\beta v} \bar{F}(v) dv.
$$

Taking into account that

$$
\int_{\mathbb{R}} e^{\beta v} \bar{F}(v) dv = \frac{1}{\beta} \mathbb{E} e^{\beta \xi_1} = \frac{1}{\beta} < \infty.
$$

We get from here that $\bar{U}(\hat{z},\delta) < \infty$ and $\bar{U}(\hat{z},\delta) - \underline{U}(\hat{z},\delta) \to 0$ as $\delta \to 0$.

Using the renewal theory, we obtain, if the law of $\xi$ is non-arithmetic, that

$$
\lim_{x \to \infty} e^{\beta x} \tilde{H}(x) = \frac{1}{\mathbb{E} \xi} \int_{0}^{\infty} z(v) dv,
$$

see, e.g., Ch. XI, 9, [10]. If the law of $\xi$ is arithmetic with the step $d > 0$, then, according to Proposition 8.3 for any $x > 0$

$$
\lim_{n \to \infty} e^{\beta(x+nd)} \tilde{H}(x + nd) = \frac{d}{\mathbb{E} \xi} \sum_{j \in \mathbb{Z}} z(x + jd) I_{\{x + jd \geq 0\}}.
$$

The equalities (8.13) and (8.14) implies the statement. □

The proof of the result below, formulated to cover our needs, follows the same line as in Lemma 2.6 of the Buraczewski–Damek paper [5] with minor changes to include also the arithmetic case.

**Theorem 8.5.** Suppose that (8.2) hold. If the support of distribution of $Y_\infty$ is unbounded from above then

$$
\liminf_{u \to \infty} w^\beta \mathbb{P}(Y_\infty > u) > 0.
$$

30
Proof. Let
\[
\tilde{Y}_n := - \sum_{j=1}^{n} Q_j Z_{j-1}, \quad Y_{n,\infty} := \sum_{j=n+1}^{\infty} Q_j \prod_{l=n+1}^{j-1} M_l
\]
and let \(Z_* := \sup_{j \leq n} Z_j\). Theorems 8.1, 8.2 imply that \(P(\tilde{Y}_\infty < -u) \leq C_1 u^{-\beta}\) with \(C_1 > 0\) for sufficiently large \(u\). On the other hand, by Proposition 8.4 \(P(Z_* > u) \geq C_2 u^{-\beta}\) with \(C_2 > 0\) and \(u \to \infty\).

Put \(U_n := \{Z_n > u, \tilde{Y}_n > -Cu\}\) where \(C_\beta := 4C_1/C_2\). The process \(\tilde{Y}\) decreases. Therefore, we have the inclusion \(\{Z_n > u\} \subseteq \{\tilde{Y}_\infty \leq -Cu\} \cup U_n\). It follows that for sufficiently large \(u > 0\)
\[
(3/4)C_2 u^{-\beta} \leq P(Z_* > u) = P(\cup_n \{Z_n > u\}) \leq P(\tilde{Y}_\infty \leq -Cu) + P(\cup_n U_n)
\]
\[
\leq 2C_1 C_\beta u^{-\beta} + P(\cup_n U_n)
\]
implying that \(P(\cup_n U_n) \geq (1/4)C_2 u^{-\beta}\).

Since \(Y_n + Z_n Y_{n,\infty} \leq Y_n + Z_n Y_{n,\infty} = Y_\infty\), we have that
\[
\{Y_{n,\infty} > C + 1\} \cap U_n \subseteq \{\tilde{Y}_n + Z_n Y_{n,\infty} > u\} \cap U_n \subseteq \{Y_\infty > u\} \cap U_n,
\]
Note that \(P(Y_\infty > C + 1) = P(Y_{n,\infty} > C + 1)\), because \(L(Y_{n,\infty}) = L(Y_\infty)\). Using the independence of \(Y_{n,\infty}\) and the sets \(W_n := U_n \cap (\cup_{k=1}^{u-1} U_k)\), forming a disjoint partition of \(\cup_n U_n\), we get that
\[
P(Y_\infty > C + 1)P(\cup_n W_n) = \sum_n P(\{Y_{n,\infty} > C + 1\} \cap W_n)
\]
\[
\leq \sum_n P(\{Y_\infty > u\} \cap W_n) \leq P(Y_\infty > u).
\]
Thus, \(P(Y_\infty > u) \geq (1/4)b C_2 u^{-\beta}\) where \(b := P(Y_\infty > C + 1) > 0\) by the assumption that the support of \(L(Y_\infty)\) is unbounded from above. The obtained asymptotic bound implies that \(C_+ > 0\). \(\square\)

Summarizing the above results we get for function \(\bar{G}(u) = P(Y_\infty > u)\) the following asymptotic properties when \(u \to \infty\):

**Theorem 8.6.** Suppose that (8.2) holds. Then \(\limsup u^\beta \bar{G}(u) < \infty\). If \(Y_\infty\) is unbounded from above, then \(\liminf u^\beta \bar{G}(u) > 0\) and in the case where \(L(M)\) is non-arithmetic \(\bar{G}(u) \sim C_+ u^{-\beta}\) where \(C_+ > 0\).

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