DIRAC SERIES OF $E_7(-25)$

YI-HAO DING AND CHAO-PING DONG

Abstract. By further sharpening the Helgason-Johnson bound in 1969, this paper classifies all the irreducible unitary representations with non-zero Dirac cohomology of the Hermitian symmetric real form $E_7(-25)$.

1. Introduction

As a sequel to [4] and [5], this article aims to classify the irreducible unitary representations with non-zero Dirac cohomology for the simple linear Lie group $E_7(-25)$, which is Hermitian symmetric.

We embark with a complex connected simple algebraic group $G_{\mathbb{C}}$ which has finite center. Let $\sigma : G_{\mathbb{C}} \to G_{\mathbb{C}}$ be a real form of $G_{\mathbb{C}}$. That is, $\sigma$ is an antiholomorphic Lie group automorphism and $\sigma^2 = \text{Id}$. Let $\theta : G_{\mathbb{C}} \to G_{\mathbb{C}}$ be the involutive algebraic automorphism of $G_{\mathbb{C}}$ corresponding to $\sigma$ via Cartan theorem (see Theorem 3.2 of the paper [1] by Adams, van Leeuwen, Trapa and Vogan). Put $G = G_{\mathbb{C}}^\theta$ as the group of real points. Note that $G$ must be in the Harish-Chandra class $[\mathcal{R}]$. Denote by $K_{\mathbb{C}} := G_{\mathbb{C}}^\theta$, and put $K := K_{\mathbb{C}}^\mathbb{R}$. Denote by $\mathfrak{g}_0$ the Lie algebra of $G$, and let

$$
\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0
$$

be the Cartan decomposition corresponding to $\theta$ on the Lie algebra level.

Let $H_f = T_f A_f$ be the $\theta$-stable fundamental Cartan subgroup for $G$. Then $T_f$ is a maximal torus of $K$, and on the Lie algebra level,

$$
\mathfrak{h}_{f,0} = \mathfrak{t}_{f,0} \oplus \mathfrak{a}_{f,0}
$$

is the unique $\theta$-stable fundamental Cartan subalgebra of $\mathfrak{g}_0$. As usual, the subscripts are dropped to stand for the complexified Lie algebras. For example, $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}, \mathfrak{h}_f = \mathfrak{h}_{f,0} \otimes \mathbb{C}$ and so on. We fix a non-degenerate invariant symmetric bilinear form $B(\cdot, \cdot)$ on $\mathfrak{g}$. Its restrictions to $\mathfrak{t}, \mathfrak{p},$ etc., will also be denoted by the same symbol.

Let $l$ be the rank of $\mathfrak{g}$. That is, $l = \dim_{\mathbb{C}} \mathfrak{h}_f$. Fix a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h}_f)$, and let $\{\zeta_1, \ldots, \zeta_l\}$ be the corresponding fundamental weights. Restricting the roots in $\Delta^+(\mathfrak{g}, \mathfrak{h}_f)$ to $\mathfrak{t}_f$, we have that

$$
\Delta^+(\mathfrak{g}, \mathfrak{t}_f) = \Delta^+(\mathfrak{t}, \mathfrak{t}_f) \cup \Delta^+(\mathfrak{p}, \mathfrak{t}_f).
$$

Let $\rho$ (resp., $\rho_c$) be the half sum of roots in $\Delta^+(\mathfrak{g}, \mathfrak{t}_f)$ (resp., $\Delta^+(\mathfrak{t}, \mathfrak{t}_f)$). Then $\rho := \rho - \rho_c$ is the half sum of roots in $\Delta^+(\mathfrak{p}, \mathfrak{t}_f)$. We will denote the Weyl groups corresponding to these root systems by $W(\mathfrak{g}, \mathfrak{h}_f), W(\mathfrak{g}, \mathfrak{t}_f), W(\mathfrak{t}, \mathfrak{t}_f)$. 

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Fix an orthonormal basis $Z_1, \ldots, Z_n$ of $p_0$ with respect to the inner product induced by the form $B(\cdot, \cdot)$. Let $U(g)$ be the universal enveloping algebra of $g$ and let $C(p)$ be the Clifford algebra of $p$ with respect to $B(\cdot, \cdot)$. In 1972, Parthasarathy introduced the Dirac operator as

$$D := \sum_{i=1}^{n} Z_i \otimes Z_i \in U(g) \otimes C(p).$$

It is easy to check that $D$ is independent of the choice of the orthonormal basis $Z_i$ and it is $K$-invariant under the diagonal action of $K$ given by adjoint actions on both factors. Moreover, we embed the Lie algebra $\mathfrak{k}$ diagonally into $U(g) \otimes C(p)$ in the following way:

$$X \mapsto X_\Delta := X \otimes 1 + 1 \otimes \sum_{i<j} \frac{1}{2} B(X, [Z_i, Z_j]) Z_i Z_j, \quad X \in \mathfrak{k}.$$ 

Denote the image of $\mathfrak{k}$ by $\mathfrak{k}_\Delta$. Let $\Omega_g$ (resp., $\Omega_{\mathfrak{k}_\Delta}$) be the Casimir element of $g$ (resp., $\mathfrak{k}_\Delta$). Then

$$D^2 = -\Omega_g \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + (\|\rho_c\|^2 - \|\rho\|^2) 1 \otimes 1.$$ 

Let $\pi$ be an irreducible $(g, K)$ module. Let $S_G$ be a spin module of $C(p)$. The Dirac operator acts on $\pi \otimes S_G$. When $\pi$ is unitary, the operator $D$ is self-adjoint with respect to a natural Hermitian inner product on $\pi \otimes S_G$, and $D^2$ has non-negative eigenvalue on any $\widetilde{K}$-type of $\pi \otimes S_G$. Here $\widetilde{K}$ is the subgroup of $K \times \text{Pin} p_0$ consisting of all pairs $(k, s)$ such that $\text{Ad}(k) = p(s)$, where $\text{Ad} : K \to O(p_0)$ is the adjoint action, and $p : \text{Pin} p_0 \to O(p_0)$ is the pin double covering map. Namely, $\widetilde{K}$ is constructed from the following diagram:

$$\begin{array}{ccc}
\widetilde{K} & \longrightarrow & \text{Pin} p_0 \\
\downarrow & & \downarrow p \\
K & \xrightarrow{\text{Ad}} & O(p_0)
\end{array}$$

Using (2), one can deduce that

$$\|\gamma + \rho_c\| \geq \|\Lambda\|,$$

where $\gamma$ is a highest weight of any $\widetilde{K}$-type occurring in $\pi \otimes S_G$. This is Parthasarathy’s Dirac operator inequality [26], which effectively detects non-unitarity. For instance, it is an important tool in the classification of irreducible unitary highest weight modules for Hermitian symmetric real forms by Enright, Howe and Wallach [12]. However, it is not sufficient for unitarity.

To sharpen the Dirac operator inequality, and to understand the unitary dual $\hat{G}$ better, Vogan formulated Dirac cohomology in 1997 [34] as the following $\widetilde{K}$-module:

$$H_D(\pi) = \text{Ker} D / (\text{Im} D \cap \text{Ker} D).$$

Here we note that $\widetilde{K}$ acts on $\pi$ through $K$ and on $S_G$ through the pin group $\text{Pin} p_0$. Moreover, since $\text{Ad}(k)(Z_1), \ldots, \text{Ad}(k)(Z_n)$ is still an orthonormal basis of $p_0$, it follows that $D$ is $\widetilde{K}$-invariant. Therefore, $\text{Ker} D$, $\text{Im} D$, and $H_D(X)$ are all $\widetilde{K}$-modules.
A fundamental result pertaining to Dirac cohomology is the Vogan conjecture proven by Huang and Pandžić in 2002. See Theorem 2.3 of [16]. Let us state its tiny extension to possibly disconnected groups. By setting the linear functionals on $t_f$ to be zero on $a_f$, we regard $t_f^*$ as a subspace of $h_f^*$. 

**Theorem 1.1.** (Theorem A of [10]) Let $\pi$ be an irreducible $(g, K)$-module with infinitesimal character $\Lambda$. Assume that the Dirac cohomology of $\pi$ is nonzero, and let $\gamma \in t_f^* \subset h_f^*$ be any highest weight of a $\tilde{K}$-type in $H_D(\pi)$. Then $\Lambda$ is conjugate to $\gamma + \rho_c$ under the action of the Weyl group $W(g, h_f)$. 

We care the most about the case that $\pi$ is unitary. Then $\text{Ker}D \cap \text{Im}D = 0$, and 

$$H_D(\pi) = \text{Ker} D = \text{Ker} D^2.$$ 

Moreover, by [17, Theorem 3.5.2], the inequality (3) becomes equality for certain $\tilde{K}$-types of $\pi \otimes S_G$ if and only if $H_D(\pi)$ is non-vanishing. 

Since Dirac cohomology is an invariant for Lie group representations, it is natural to ask: could we classify $\hat{G}^d$—the set consisting of all the members of $\hat{G}$ having non-vanishing Dirac cohomology? For convenience, we call members of $\hat{G}^d$ the **Dirac series** of $G$ (terminology suggested by J.-S. Huang). In view of the discussion after (5), the Dirac series of $G$ are exactly the members of $\hat{G}$ on which Dirac inequality becomes equality. 

The current paper aims to classify the Dirac series for $E_7(-25)$, by which we actually mean the connected simple real exceptional Lie group $E_7$ in Knapp [20], or by $E_7(-25)$ in other literature. Here atlas [37] is a software which computes various types of questions relevant to the representations of $G$. For instance, it detects whether $\pi$ is unitary or not based on the algorithm in [1]. See Section 2.2 for a very brief account of atlas. 

Let $\pi$ be any Dirac series representation of $G$ which has final atlas parameter $p = (x, \lambda, \nu)$. Then $\pi$ is a **FS-scattered** representation if the KGB element $x$ (see Section 2.2) is fully supported, i.e., if $\text{support}(x)$ equals $[0, 1, \ldots, l - 1]$. Otherwise, $\pi$ will be merged into a string of representations. Theorem A of [7] says that $\hat{G}^d$ consists of finitely many FS-scattered representations and finitely many strings of representations. One can pin down them explicitly without reference to the entire unitary dual $\hat{G}$. Indeed, it suffices to study the irreducible representations whose infinitesimal characters are relatively small. By further sharpening the Helason-Johnson bound [15] on the norm of $\nu$ for $E_7(-25)$ (see Proposition 4.1(c)), we are able to reduce the classification workload considerably. Besides the Dirac inequality (4), the **unitarily small** (u-small for short henceforth) convex hull introduced by Salamanca-Riba and Vogan [30] is also vital to Proposition 4.1(c). This concept and spin-lowest $K$-type will be recalled in Section 4.

Now we are able to state the main result. 

**Theorem 1.2.** The set $\hat{E}_{7(-25)}^d$ consists of 74 FS-scattered representations whose spin-lowest $K$-types are all u-small, and 878 strings of representations. Moreover, each spin lowest $K$-type of any Dirac series representation of $E_{7(-25)}$ occurs exactly once. 

The paper is organized as follows: Section 2 prepares necessary material on cohomological induction and the software atlas. Section 3 reviews the structure of $E_{7(-25)}$. Section
4 further improves the Helgason-Johnson bound. After these preparations, Section 5 pins down the Dirac series of $E_{7(-25)}$. Section 6 aims to look at certain Dirac series representations more carefully. Section 7 considers some special unipotent representations of $E_{7(-25)}$, while Section 8 lists all its non-trivial FS-scattered Dirac series representations.

2. Preliminaries

This section aims to collect necessary preliminaries.

2.1. Cohomological induction. Fix an element $H \in i_{f,0}$. Let $l$ be the zero eigenspace of $\text{ad}(H)$, and let $u$ be the sum of positive eigenspaces of $\text{ad}(H)$. Then $q = l \oplus u$ be a $\theta$-stable parabolic subalgebra of $g$ with Levi factor $L$ and nilpotent radical $u$. Set $L = N_G(q)$.

Let us arrange that $\Delta(u, h_f) \subseteq \Delta^+(g, h_f)$. Set $\Delta^+(l, h_f) = \Delta(l, h_f) \cap \Delta^+(g, h_f)$. Let $\rho^L$ denote the half sum of roots in $\Delta^+(l, h_f)$, and denote by $\rho(u)$ (resp., $\rho(u \cap p)$) the half sum of roots in $\Delta(u, h_f)$ (resp., $\Delta(u \cap p, h_f)$). Then

$$\rho = \rho^L + \rho(u).$$

Let $Z$ be an $(l, L \cap K)$-module. Cohomological induction functors attach to $Z$ certain $(g, K)$-modules $L_j(Z)$ and $R^j(Z)$, where $j$ is a nonnegative integer. Suppose that $Z$ has infinitesimal character $\lambda_L \in h_f^*$. After [22], we say that $Z$ is good or in good range if

$$\text{Re} \langle \lambda_L + \rho(u), \alpha^\vee \rangle > 0, \ \forall \alpha \in \Delta(u, h_f).$$

We say that $Z$ is weakly good if

$$\text{Re} \langle \lambda_L + \rho(u), \alpha^\vee \rangle \geq 0, \ \forall \alpha \in \Delta(u, h_f).$$

**Theorem 2.1.** [33] Theorems 1.2 and 1.3, or [22] Theorems 0.50 and 0.51) Suppose the admissible $(l, L \cap K)$-module $Z$ is weakly good. Then we have

(i) $L_j(Z) = R^j(Z) = 0$ for $j \neq S := \dim (u \cap \mathfrak{k})$.
(ii) $L_S(Z) \cong R^S(Z)$ as $(g, K)$-modules.
(iii) if $Z$ is irreducible, then $L_S(Z)$ is either zero or an irreducible $(g, K)$-module with infinitesimal character $\lambda_L + \rho(u)$.
(iv) if $Z$ is unitary, then $L_S(Z)$, if nonzero, is a unitary $(g, K)$-module.
(v) if $Z$ is in good range, then $L_S(Z)$ is nonzero, and it is unitary if and only if $Z$ is unitary.

In the special case that $Z$ is a one-dimensional unitary character $C_\lambda$, the module $L_S(Z)$ will be called an $A_q(\lambda)$ module. It is said to be fair if

$$\langle \lambda + \rho(u), \alpha \rangle > 0, \ \forall \alpha \in \Delta(u, h_f),$$

and to be weakly fair if

$$\langle \lambda + \rho(u), \alpha \rangle \geq 0, \ \forall \alpha \in \Delta(u, h_f);$$

Take $\Lambda \in h_f^*$ such that it is dominant for $\Delta^+(g, h_f)$. We say that $\Lambda$ is real if it belongs to $i t_{f,0} + a_{f,0}$, and $\Lambda$ is strongly regular if

$$\langle \Lambda - \rho, \alpha^\vee \rangle \geq 0, \ \forall \alpha \in \Delta^+(g, h_f).$$

The following result says that a large part of $\tilde{G}$ consists of $A_q(\lambda)$ modules.
**Theorem 2.2.** (Salamanca-Riba [29]) Let $\pi$ be an irreducible $(\mathfrak{g}, K)$-module with a strongly regular real infinitesimal character. If $\pi$ is unitary, then it is an $A_q(\lambda)$ module in the good range.

Example 6.5 suggests that $A_q(\lambda)$ modules should continue to play an important role in the unitary dual for singular infinitesimal characters.

2.2. The atlas software. Let us recall necessary notation from [1] regarding the Langlands parameters in the software atlas [37]. The recent seminar [35] is also a good reference. Let $H_C$ be a maximal torus of $G_C$. That is, $H_C$ is a maximal connected abelian subgroup of $G_C$ consisting of diagonalizable matrices. Note that $H_C$ is complex connected reductive algebraic. Its character lattice is the group of algebraic homomorphisms $X^* := \text{Hom}_{\text{alg}}(H_C, \mathbb{C}^\times)$.

Choose a Borel subgroup $B_C \supset H_C$. In atlas, an irreducible $(\mathfrak{g}, K)$-module $\pi$ is parameterized by a final parameter $p = (x, \lambda, \nu)$ via the Langlands classification [1], where $\lambda \in X^* + \rho$, $\nu \in (X^*)^\theta \otimes_{\mathbb{Z}} \mathbb{C}$, and $x$ is a KGB element. That is, $x$ is a $K_C$-orbit in the Borel variety $G_C/B_C$. The infinitesimal character of $\pi$ is represented by

$$
(12) \quad \frac{1}{2}(1 + \theta)\lambda + \nu.
$$

Note that the Cartan involution $\theta$ now becomes $\theta_x$—the involution of $x$, which is given by the command $\text{involution}(x)$ in atlas.

Among other things, Paul’s lecture [27] carefully explains how to do cohomological induction in atlas. In particular, the following canonical way is most important for us.

**Theorem 2.3.** (Vogan [33]) Let $p = (x, \lambda, \nu)$ be the atlas parameter of an irreducible $(\mathfrak{g}, K)$-module $\pi$. Let $S$ be the support of $x$, and $q(x)$ be the $\theta$-stable parabolic subalgebra given by the pair $(S, x)$, with Levi factor $L(x)$. Then $\pi$ is cohomologically induced, in the weakly good range, from an irreducible $(l, L \cap K)$-module $\pi_{L(x)}$ with parameter $p_L = (y, \lambda - \rho(u), \nu)$, where $y$ is the KGB element of $L(x)$ corresponding to $x$.

The atlas command for finding the $\pi_{L(x)}$ for any given $\pi$ is $\text{reduce good range}$. 

**Example 2.4.** Let us look at the irreducible representations of $SL(2, \mathbb{R})$ with infinitesimal character $\rho$. As we shall see, there are four representations in total, among which three are unitary.

```
G:SL(2,R)
rho(G)
Value: [ 1 ]/1
set all=all_parameters_gamma(G,[1])
#all
Value: 4
void: for p in all do prints(p, " ",is_unitary(p)) od
final parameter(x=2,lambda=[1]/1,nu=[1]/1) true
final parameter(x=2,lambda=[2]/1,nu=[1]/1) false
final parameter(x=1,lambda=[1]/1,nu=[0]/1) true
final parameter(x=0,lambda=[1]/1,nu=[0]/1) true
```
The first one is the trivial representation. Now let us look at the third representation in all. By Theorem 2.2, it is an $A_q(\lambda)$ module.

set $p=\text{all}[2]$

$\text{Value: final parameter}(x=1,\lambda=[1]/1,\nu=[0]/1)$

set $(Q,q)=\text{reduce_good_range}(p)$

$Q$

$\text{Value: (}[\text{KGB element #1}]$)

$\text{Levi}(Q)$

$\text{Value: compact connected quasisplit real group with Lie algebra 'u(1)'}$

$q$

$\text{Value: final parameter}(x=0,\lambda=[0]/1,\nu=[0]/1)$

dimension($q$)

$\text{Value: 1}$

goodness($q,G$)

$\text{Value: "Good"}$

$\rho_{u}(Q)+\text{infinitesimal_character}(q)=\text{infinitesimal_character}(p)$

$\text{Value: true}$

We see that $p$ is actually an $A_b(\lambda)$ module. Actually, it is a discrete series. $\square$

3. The structure of $E_7(-25)$

We will fix the group $G$ as $E_7(h)$ in atlas henceforth. This connected equal rank group has center $\mathbb{Z}/2\mathbb{Z}$. It is not simply connected. Note that $(G,K)$ is a Hermitian symmetric pair. The Lie algebra $\mathfrak{g}_0$ is labelled as EVII in [20, Appendix C]. We present the Vogan diagram for $\mathfrak{g}_0$ in Fig. 1 where $\alpha_1=\frac{1}{3}(1,-1,-1,-1,-1,-1,1)$, $\alpha_2=e_1+e_2$ and $\alpha_i=e_{i-1}-e_{i-2}$ for $3 \leq i \leq 7$. Let $\zeta_1,\ldots,\zeta_7 \in t_f^*$ be the corresponding fundamental weights for $\Delta^+(\mathfrak{g},t_f)$, where $t_f \subset \mathfrak{k}$ is the fundamental Cartan subalgebra of $\mathfrak{g}$. The dual space $t_f^*$ will be identified with $t_f$ under the form $B(\zeta,\eta)$. Put

\( (13) \quad \zeta := \zeta_7 = (0,0,0,0,0,1,-\frac{1}{2},-\frac{1}{2}) \).

Note that

\( \rho = (0,1,2,3,4,5,-\frac{17}{2},\frac{17}{2}) \).

We will use $\{\zeta_1,\ldots,\zeta_7\}$ as a basis to express the atlas parameters $\lambda, \nu$ and the infinitesimal character $\Lambda$. More precisely, in such cases, $[a,b,c,d,e,f,g]$ will stand for the vector $a\zeta_1 + \cdots + g\zeta_7$. For instance, the trivial representation of $E_7(-25)$ has infinitesimal character $\rho = [1,1,1,1,1,1,1]$. 
Denote by $\gamma_i = \alpha_i$ for $1 \leq i \leq 6$. Let $t_f^-$ be the real linear span of $\gamma_1, \ldots, \gamma_6$, which are the simple roots for $\Delta^+(\mathfrak{t}, t_f^-)$—the positive system for $\mathfrak{t}$ obtained from $\Delta^+(\mathfrak{g}, t_f^-)$ by restriction. We present the Dynkin diagram of $\Delta^+(\mathfrak{t}, t_f^-)$ in Fig. 2. Let $\varpi_1, \ldots, \varpi_6 \in (t_f^-)^*$ be the corresponding fundamental weights. Note that $\mathbb{R}\zeta$ is the one-dimensional center of $\mathfrak{k}$, that $t_f = t_f^- \oplus \mathbb{R}\zeta$, and that

$$\rho_c = (0, 1, 2, 3, 4, -4, -4, 4).$$

Moreover, we have the decomposition

$$p = p^+ \oplus p^-$$

as $\mathfrak{t}$-modules, where $p^+$ has highest weight

$$\beta := 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = (0, 0, 0, 0, 0, 0, -1, 1)$$

and $p^-$ has highest weight $-\alpha_7$. Both of them have dimension 27. Thus

$$-\dim \mathfrak{k} + \dim p = -79 + 54 = -25.$$  

This is how the number $-25$ enters the name $E_{7(-25)}$, see [14].

Let $E_\mu$ be the $\mathfrak{t}$-type with highest weight $\mu$. We will use $\{\varpi_1, \ldots, \varpi_6, \frac{1}{3}\zeta\}$ as a basis to express $\mu$. Namely, in such a case, $[a, b, c, d, e, f, g]$ stands for the vector $a\varpi_1 + b\varpi_2 + c\varpi_3 + d\varpi_4 + e\varpi_5 + f\varpi_6 + \frac{g}{3}\zeta$. For instance,

$$\beta = [1, 0, 0, 0, 0, 0, 2], \quad -\alpha_7 = [0, 0, 0, 0, 0, 1, -2],$$

and $\rho_c = [1, 1, 1, 1, 1, 0]$. The $\mathfrak{t}$-type $E_{[a,b,c,d,e,f,g]}$ has lowest weight $[-f, -b, -c, -d, -c, -a, g]$. Therefore, $E_{[f,b,e,d,c,a,-g]}$ is the contragredient $\mathfrak{t}$-type of $E_{[a,b,c,d,e,f,g]}$. 

**Figure 1.** The Vogan diagram for $E_{7(-25)}$ 

**Figure 2.** The Dynkin diagram for $\Delta^+(\mathfrak{t}, t_f^-)$
For $a, b, c, d, e, f \in \mathbb{Z}_{\geq 0}$ and $g \in \mathbb{Z}$, we have that $E_{[a,b,c,d,e,f,g]}$ is a $K$-type if and only if
\begin{equation}
-\frac{2a}{3} - b - \frac{4c}{3} - 2d - \frac{5e}{3} - \frac{4f}{3} + \frac{g}{3} \in \mathbb{Z}.
\end{equation}
Abusing the notation a bit, we may refer to a $\xi$-type (or $K$-type, $\widetilde{K}$-type) simply by its highest weight in terms of $\{\varpi_1, \ldots, \varpi_6, \frac{1}{3}\zeta\}$.

**Lemma 3.1.** Let $\Lambda = a\zeta_1 + b\zeta_2 + c\zeta_3 + d\zeta_4 + e\zeta_5 + f\zeta_6 + g\zeta_7$ be the infinitesimal character of any Dirac series representation $\pi$ of $E_7(-25)$ which is dominant with respect to $\Delta^+(g, t_f)$. Then $a, b, c, d, e, f, g$ must be non-negative integers such that
\begin{equation}
a + c > 0, b + d > 0, c + d > 0, d + e > 0, e + f > 0, f + g > 0,
\end{equation}
and that
\begin{equation}
a + b + e > 0, a + b + f > 0, a + b + g > 0, a + d + f > 0, a + d + g > 0,
\end{equation}
and that
\begin{equation}
a + e + g > 0, b + c + e > 0, b + c + f > 0, b + c + g > 0, c + e + g > 0.
\end{equation}

**Proof.** Since the group $E_7\mathfrak{h}$ is linear, it follows from Remark 4.1 of [9] that $a, b, c, d, e, f, g$ must be non-negative integers.

Now if $a + c = 0$, i.e., $a = c = 0$, a direct check says that for any $w \in W(g, t_f)^1$, at least one of the first six coordinates of $w\Lambda$ in terms of the basis $\{\varpi_1, \ldots, \varpi_6, \frac{1}{3}\}$ vanishes. Therefore,
\[
\{\mu - \rho_0^{(j)}\} + \rho_c = w\Lambda
\]
could not hold for any $K$-type $\mu$. This proves that $a + c > 0$. Other inequalities can be similarly deduced. \hfill \square

## 4. Sharpening the Helgason-Johnson Bound for $E_7(-25)$

Recall that $G$ is the group $E_7\mathfrak{h}$ in atlas. Moreover, we have fixed a Vogan diagram for $g_0$ in Fig. [1] By doing this, we have actually fixed a positive root system $\Delta^+(g, h_f)$. Recall that $h_f = t_f$. We start with $(\Delta^+)^{(0)}(g, t_f) := \Delta^+(g, h_f)$ which contains the $\Delta^+(\xi, t_f)$ corresponding to Fig. [2]. We fix this $\Delta^+(\xi, t_f)$ once for all. There are 56 ways of choosing positive root systems for $\Delta(g, t_f)$ containing the $\Delta^+(\xi, t_f)$. Note that $|W(g, t_f)/|W(\xi, t_f)| = 56$. We enumerate them as
\[
(\Delta^+)^{(j)}(g, t_f) = \Delta^+(\xi, t_f) \cup (\Delta^+)^{(j)}(p, t_f), \quad 0 \leq j \leq 55.
\]
Let us denote the half sum of roots of $(\Delta^+)^{(j)}(g, t_f)$ by $\rho^{(j)}$, and put $\rho_0^{(j)} := \rho^{(j)} - \rho_c$. Then $\rho^{(j)}_0$ is the half sum of roots in $(\Delta^+)^{(j)}(p, t_f)$. Let $w^{(j)}$ be the unique element in $W(g, h_f)$ such that $w^{(j)}\rho^{(0)} = \rho^{(j)}$, and collect them as the set $W(g, t_f)^1$. Then $w^{(0)} = e$, and by a result of Kostant [21], the multiplication map induces a bijection from $W(g, t_f)^1 \times W(\xi, t_f)$ onto $W(g, t_f)$. For any $0 \leq j \leq 55$, $w^{(j)}\alpha_1, \ldots, w^{(j)}\alpha_7$ are the simple roots of $(\Delta^+)^{(j)}(g, t_f)$, and $w^{(j)}\zeta_1, \ldots, w^{(j)}\zeta_7$ are the corresponding fundamental weights.
Let us denote the dominant Weyl chamber for \((\Delta^+(\mathfrak{g}, \mathfrak{t}_f))\) (resp., \(\Delta^+(\mathfrak{t}, \mathfrak{t}_f))\) by \(\mathcal{C}^{(j)}\) (resp., \(\mathcal{C}\)). Then
\[
(20) \quad \mathcal{C} = \bigcup_{j=0}^{55} \mathcal{C}^{(j)}.
\]
Note that \(w^{(j)}\mathcal{C}^{(0)} = \mathcal{C}^{(j)}\) for \(0 \leq j \leq 55\). The convex hull formed by the \(W(\mathfrak{t}, \mathfrak{t}_f)\) orbits of the points \(p^{(j)}_n\), \(0 \leq j \leq 55\), is called the \(u\)-small convex hull, and a \(K\)-type is said to be \(u\)-small if its highest weight lies in this polyhedron. See Salamanca-Riba and Vogan [30]. Otherwise, we will call this \(K\)-type \(u\)-large.

Consider an arbitrary \(K\)-type \(E_\mu\). Choose \(0 \leq j \leq 55\) such that \(\mu + 2\rho_c\) is dominant for \((\Delta^+(\mathfrak{g}, \mathfrak{t}_f))\). Put
\[
(21) \quad \lambda_a(\mu) := P(\mu + 2\rho_c - \rho^{(j)}).
\]
Here for any vector \(\eta \in \mathfrak{n}^*\), \(P(\eta)\) stands for the projection of \(\eta\) to the cone \(\mathcal{C}^{(j)}\). Namely, \(P(\eta)\) is the unique point in \(\mathcal{C}^{(j)}\) which is closest to \(\eta\). It turns out that the vector \(\lambda_a(\mu)\) is independent of the choice of an allowable \(j\), and the lambda norm of \(\mu\) is defined as
\[
(22) \quad \|\mu\|_{\lambda}\text{norm} := \|\lambda_a(\mu)\|.
\]
The above geometric way of describing the lambda norm in [32] is due to Carmona [3]. Let \(\pi\) be an irreducible \((\mathfrak{g}, \mathfrak{K})\) module. Then the \(K\)-type \(\mu\) is called a lowest \(K\)-type (LKT for short) of \(\pi\) if \(\mu\) shows up in \(\pi\) and its lambda norm attains the minimum among all the \(K\)-types of \(\pi\).

Given a LKT \(\mu\) of \(\pi\), the infinitesimal character \(\Lambda\) of \(\pi\) can be written as
\[
(23) \quad \Lambda = (\lambda_a(\mu), \nu) \in \mathfrak{h}^* = \mathfrak{t}^* + \mathfrak{a}^*.
\]
As in [32], \(G(\lambda_a(\mu))\) is the isotropy group at \(\lambda_a(\mu)\) for the \(G\) action; \(\mathfrak{h}\) is the complexified Lie algebra of a maximally split \(\theta\)-stable Cartan subgroup \(H = TA\) of \(G(\lambda_a(\mu))\). Note that \(\nu\) in (23) has the same norm as the \(\nu\)-part in the atlas parameter \(p = (x, \lambda, \nu)\) of \(\pi\). Abusing the notation a bit, we will not distinguish them.

Let \(S_G\) be the irreducible module of the Clifford algebra \(C(\mathfrak{p})\). As a special case of Lemma 9.3.2 of [36], we have the following decomposition
\[
(24) \quad S_G = \bigoplus_{j=0}^{55} E_{\rho_n^{(j)}}
\]
as \(\mathfrak{t}\)-modules. Now the spin norm introduced in [6] is
\[
(25) \quad \|\mu\|_{\text{spin}} := \min_{0 \leq j \leq 55} \|\{\mu - \rho_n^{(j)}\} + \rho_c\|.
\]
Here \(\{\mu - \rho_n^{(j)}\}\) is the unique dominant weight in the \(W(\mathfrak{t}, \mathfrak{t}_f)\) orbit of \(\mu - \rho_n^{(j)}\). We emphasize that \(\{\mu - \rho_n^{(j)}\}, 0 \leq j \leq 55\), are precisely all the PRV components [28] of the tensor product \(E_\mu \otimes S_G\) as \(\mathfrak{t}\)-modules. A \(K\)-type \(\delta\) of \(\pi\) is called a spin lowest \(K\)-type (spin LKT for short) of \(\pi\) if its spin norm attains the minimum among all the \(K\)-types of \(\pi\).
When $\pi$ is unitary, the case that we care the most, an equivalent way to formulate the Dirac inequality (3) is

$$\|\Lambda\| \leq \|\delta\|_{\text{spin}},$$

where $\delta$ is any $K$-type occurring in $\pi$. In view of the explanation after (5), $H_D(\pi)$ is non-zero if and only if (26) becomes equality on some $K$-types $\delta$ of $\pi$ (which then must be spin LKTs of $\pi$).

Now let us state the Helgason-Johnson bound and its improvements.

**Proposition 4.1.** In the above setting, further assume that $\pi$ is unitary, then we have that

(a) $\|\nu\| \leq \sqrt{\frac{399}{2}} = \|\rho\|$;

(b) $\|\nu\| \leq \sqrt{\frac{371}{2}}$;

(c) $\|\nu\| < \sqrt{94}$ if $\Lambda$ is integral and $\pi$ is not trivial or minimal.

Item (a) above is due to Helgason-Johnson [15] in 1969. Its slight extension (b) is given in [8] recently. In item (c), $\Lambda$ being integral means that it is an integer combination of $\zeta_1, \ldots, \zeta_7$.

**Example 4.2.** The trivial representation of $G$ has atlas parameter:

```
final parameter(x=3016,lambda=[1,1,1,1,1,1,1]/1,nu=[4,0,0,0,0,4,1]/1)
```

One computes that $\|\nu\| = \frac{371}{2}$, which attains the bound in item (b) of Proposition 4.1. Using atlas, one calculates that there are two irreducible representations with infinitesimal character [1,1,0,1,1,1] and GK dimension $(\rho, \beta^\vee) = 17$:

```
final parameter(x=2989,lambda=[3,2,2,-1,1,1,2]/1,nu=[8,5,5,-5,0,0,5]/2)
final parameter(x=2988,lambda=[3,2,2,-1,1,1,2]/1,nu=[8,5,5,-5,0,0,5]/2)
```

Both of them are unitary with $\|\nu\| = \sqrt{97}$. Thus they exhaust the minimal representations of $G$. Both of them are Dirac series, and show up in Table 10.

To deduce item (c) of Proposition 4.1, it suffices to prove the following

**Lemma 4.3.** Assume that $\pi$ is a unitary $(g,K)$ module with integral infinitesimal character $\Lambda$ which is given by (23). If $\|\nu\| \geq \sqrt{94}$, then $\pi$ must be a minimal representation or the trivial representation.

**Proof.** Let $\mu$ be a LKT of $\pi$. Since $\pi$ is unitary, the Dirac inequality (26) guarantees that

$$\|\Lambda\|^2 = \|\lambda_\mu(\mu)\|^2 + \|\nu\|^2 \leq \|\mu\|^2_{\text{spin}}.$$

Therefore,

$$\|\nu\|^2 \leq \|\mu\|^2_{\text{spin}} - \|\mu\|^2_{\text{lambda}}.$$

As computed in Section 6.2 of [8], we have that $\max\{A_j : 0 \leq j \leq 55\} = 79$. We refer the reader to Section 3 of [8] for the precise definition of $A_j$. It follows that for any u-large $K$-type $\mu$, one has that

$$\|\mu\|^2_{\text{spin}} - \|\mu\|^2_{\text{lambda}} \leq 79.$$

Since $\|\nu\| \geq \sqrt{94}$, we conclude from (27) that $\mu$ can not be u-large.
There are $21294$ u-small $K$-types in total. Among them, only $71$ have the property that

$$94 \leq \|\mu\|^2_{\text{spin}} - \|\mu\|^2_{\text{lambda}}.$$

Let us collect these $71$ u-small $K$-types as $\text{Certs}$. Its elements are listed as follow:

$$[0,0,0,0,0,0,0],$$
$$[0,2,0,0,0,0,0],$$
$$[0,1,0,0,0,0,3m] (-3 \leq m \leq 3),$$
$$[0,0,0,1,0,0,3m] (-1 \leq m \leq 1),$$
$$[1,0,0,0,0,1,3m] (-2 \leq m \leq 2),$$
$$[0,0,0,0,0,3m] (1 \leq m \leq 4),$$
$$[0,0,0,0,3,3m] (2 \leq m \leq 3),$$
$$[0,0,0,0,1,3m+1] (-4 \leq m \leq 2),$$
$$[0,0,0,0,2,3m-1] (-2 \leq m \leq 3),$$
$$[0,0,0,1,0,3m-1] (-2 \leq m \leq 2),$$
$$[0,1,0,0,0,1,3m+1] (-1 \leq m \leq 1).$$

Here we list a $K$-type and its contragredient in the same row. Moreover, we compute that

$$14 \leq \|\lambda_a(\mu)\|^2 \leq 49$$

for any $\mu \in \text{Certs}$. Therefore,

$$(28) \quad 14 + 94 \leq \|\Lambda\|^2 = \|\lambda_a(\mu)\|^2 + \|\nu\|^2 \leq 49 + \frac{371}{2}.$$

The right hand side above uses the proven Proposition $[4.1 b)$. There are $4676$ integral $\Lambda$s meeting the requirement $(28)$. We collect them as $\Omega$. Now a direct search using atlas says that there are $6474$ irreducible representations $\pi$ such that $\Lambda \in \Omega$ and that $\pi$ has a LKT which is a member of $\text{Certs}$. Furthermore, only three of them turn out to be unitary. These three representations are explicitly presented in Example $[4.2]$, they are either trivial or minimal. This finishes the proof. \hfill $\square$

5. Dirac series of $E_{7(-25)}$

5.1. FS-scattered representations of $E_{7(-25)}$. This subsection aims to sieve out all the FS-scattered Dirac series representations for $E_{7(-25)}$ using the algorithm in $[7]$.

To achieve the goal, it suffices to consider all the infinitesimal characters $\Lambda = [a, b, c, d, e, f, g]$ such that

- $a, b, c, d, e, f, g$ are non-negative integers such that $(17)$–$(19)$ hold;
- $\min\{a, b, c, d, e, f, g\} = 0$;
- there exists a fully supported KGB element $x$ such that $\|\frac{\Lambda - \theta_x(\Lambda)}{2}\| < \sqrt{94}$.

Let us collect them as $\Phi$. Note that the first item is guaranteed by Lemma $[3.1]$. The second item uses Theorem $[2.2]$. The third item needs some explanation. Let $\pi$ be an irreducible unitary $(g,K)$ module with infinitesimal character $\Lambda$ which is dominant integral for $\Delta_+^{\ast}(g, t_f)$. Let $p = (x, \lambda, \nu)$ be the atlas parameter for $\pi$. Then

$$\nu = \frac{\Lambda - \theta_x(\Lambda)}{2},$$

$$(29) \quad 14 + 94 \leq \|\Lambda\|^2 = \|\lambda_a(\mu)\|^2 + \|\nu\|^2 \leq 49 + \frac{371}{2}.$$
where \( \theta_t \) is involution(\( x \)) in atlas. Assume further that \( \pi \) is neither trivial nor minimal. Then by Proposition [11](c), we must have

\[
\| \nu \| < \sqrt{94}.
\]

The third item greatly reduces the cardinality of \( \Phi \), which turns out to be 178192.

Let us collect all the members of \( \Phi \) whose largest coordinate equals to \( i \) as \( \Phi_i \). Then \( \Phi \) is partitioned into \( \Phi_1, \ldots, \Phi_{13} \). Note that \( \Phi_1 \) has cardinality 23, and that

\[
\begin{array}{c|c|c|c|c|c|c}
\# \Phi_2 & \# \Phi_3 & \# \Phi_4 & \# \Phi_5 & \# \Phi_6 & \# \Phi_7 \\
921 & 7817 & 27246 & 42088 & 39685 & 28107 \\
\# \Phi_8 & \# \Phi_9 & \# \Phi_{10} & \# \Phi_{11} & \# \Phi_{12} & \# \Phi_{13} \\
17649 & 9042 & 4022 & 1359 & 220 & 13
\end{array}
\]

The elements of \( \Phi_1 \) are listed as follows:

\[
\begin{align*}
&[0, 0, 1, 1, 1, 1, 1], [0, 1, 1, 0, 1, 1, 1], [0, 1, 1, 1, 0, 1, 1], [0, 1, 1, 1, 1, 0, 1], \\
&[0, 1, 1, 1, 1, 1], [1, 0, 1, 1, 1, 1], [1, 0, 1, 1, 1, 0, 1], [1, 0, 1, 1, 1, 1, 1], \\
&[1, 0, 1, 1, 1, 1, 0], [1, 0, 1, 1, 1, 1, 1], [1, 1, 0, 1, 1, 1, 1], [1, 1, 0, 1, 1, 1, 0], \\
&[1, 1, 0, 1, 1, 1, 1], [1, 1, 1, 0, 1, 1, 1], [1, 1, 1, 0, 1, 1, 0], [1, 1, 1, 1, 1, 1, 0].
\end{align*}
\]

A careful study of the irreducible unitary representations under the above 23 infinitesimal characters leads us to Section 8.

For the elements \( \Lambda \) in \( \Phi_2, \Phi_3 \) up to \( \Phi_{13} \), we use Proposition [11](c) and atlas to verify that there is no fully supported irreducible unitary representations with infinitesimal character \( \Lambda \). Let us denote by \( \Pi^\Lambda_{FS}(\Lambda) \) the set of all the fully supported irreducible unitary representations (up to equivalence) with infinitesimal character \( \Lambda \).

**Example 5.1.** Let us consider the infinitesimal character \( \Lambda = [1, 0, 1, 1, 1, 0, 8] \) in \( \Phi_8 \).

```plaintext
G:E7_h
set all=all_parameters_gamma(G, [1, 0, 1, 1, 1, 0, 8])
#all
Value: 525
set allFS=## for p in all do if #support(p)=7 then [p] else [] fi od
#allFS
Value: 246
```

Therefore, there are 525 irreducible representations under \( \Lambda \), among which 246 are fully supported.

Now let us compare the original Helgason-Johnson bound and the sharpened one.

```plaintext
set HJ=94
set TgFWts=mat: [[0, 1, -1, 0, 0, 0, 0], [0, 1, 1, 0, 0, 0, 0], \\
[0, 1, 2, 0, 0, 0, 0], [0, 1, 1, 2, 2, 0, 0], [0, 1, 1, 2, 2, 2, 0], \\
[0, 1, 2, 2, 2, 2], [-2, -2, -3, -4, -3, -2, -1], [2, 2, 3, 4, 3, 2, 1]]
set oldHJ=##for p in allFS do if (nu(p)*TgFWts)*(nu(p)*TgFWts) <=4*399/2 then [p] else [] fi od
#oldHJ
```

Value: 218
set newHJ=##for p in allFS do if (nu(p)*TgFWts)*(nu(p)*TgFWts)
<4*HJ then [p] else [] fi od
#newHJ
Value: 29

Therefore, there are 218 fully supported irreducible representations satisfying $\|\nu\| \leq \sqrt{399}$,
while only 29 of them satisfy $\|\nu\| < \sqrt{94}$.

Finally, let us test the unitarity of the representations in newHJ.

void: for p in newHJ do if is_unitary(p) then prints(p) fi od

There is no output from atlas, meaning that no representation in newHJ is unitary. To sum up, we have that $\Pi^u_{FS}([1,0,1,1,0,8]) = \emptyset$. □

Remark 5.2. Since atlas takes some time to testing unitarity, adopting the sharpened Helason-Johnson bound (thus reducing the number of representations for which we need to test their unitarity) saves us a lot of time.

Example 5.3. Consider the infinitesimal character $\Lambda = [1,0,1,1,0,3,1]$ for $E_7$. It turns out that $\Pi^u_{FS}(\Lambda)$ consists of five representations, one of them is $p$=parameter(KGB(G,2969),[1,0,1,1,0,3,1],[0,0,0,0,0,4,0])

Theorem 1.1 guarantees that to study its Dirac cohomology, it suffices to look at its $K$-types up to the atlas height 248. The atlas height of a $K$-type $\mu$ is given by

$$\sum_{\alpha \in (\Delta^+)^{(j)}(g,t_f)} \langle \lambda_a(\mu), \alpha^\vee \rangle,$$

where the $0 \leq j \leq 55$ is chosen so that $\mu + 2\rho_c$ is dominant with respect to $(\Delta^+)^{(j)}(g,t_f)$, and $\langle \cdot, \cdot \rangle$ is the natural pairing between roots and co-roots. Then the atlas command

print_branch_irr_long(p,KGB(G,55), 248)

gives us 157 such $K$-types in total, and the minimum spin norm of them is $\sqrt{159}/2$, which is strictly larger than $\|\Lambda\| = \sqrt{78}$. Therefore, the unitary representation $p$ has zero Dirac cohomology in view of the discussion following (26).

The other four representations in $\Pi^u_{FS}(\Lambda)$ can be handled similarly. It turns out that they all have non-zero Dirac cohomology. We record them in Table 4 where the bolded 2949 under the column #x means that there is another representation having KGB element #2949, and having the same $\lambda$ and $\nu$ with that of #2950. □

All the other bolded entries under the columns #x in the tables of Section 8 are interpreted similarly. In particular, counting the number of KGB elements there (being bolded or not) gives that there are 73 non-trivial FS-scattered representations in Section 8.

5.2. Counting the strings in $E_7^{(-25)}$. Firstly, let us verify that Conjecture 2.6 of [9] and the binary condition holds for $E_7^{(-25)}$. 
Example 5.4. Consider the case that \( \text{support}(x) = [1, 2, 3, 4, 5, 6] \). There are 236 such KGB elements in total. We compute that there are 39002 infinitesimal characters \( \Lambda = [a, b, c, d, e, f, g] \) in total such that

- \( b, c, d, e, f, g \) are non-negative integers, \( a = 0 \) and that (17)–(19) hold;
- there exists a KGB element \( x \) with support \([1, 2, 3, 4, 5, 6]\) such that \( \| \Lambda - \theta \| \leq \sqrt{94} \).

We exhaust all the irreducible unitary representations under these infinitesimal characters with the above 236 KGB elements. It turns out that such representations occur only when \( b, c, d, e, f, g = 0 \) or 1. Then we check that each inducing module \( \pi_{L(x)} \) (see Theorem 2.3) is indeed unitary.

□

All the other non fully supported KGB elements are handled similarly. Thus Conjecture 2.6 of [9] and the binary condition hold for \( E_7(−25) \).

Now we use the formula in Section 5 of [9] to figure out the number of strings in \( \hat{E}_7(−25)^d \).

Recall that for any proper subset \( S \) of \([1, 2, 3, 4, 5, 6, 7]\), we use \( N(S) \) to denote the number of Dirac series representations with infinitesimal character \( \Lambda = \sum_{i=1}^{7} n_i \zeta_i \) such that \( n_i \) is either 0 or 1 for each \( i \in S \), and that \( n_i = 1 \) for each \( i \not\in S \). For example, \( N(\emptyset) \) counts the Dirac series representations with infinitesimal character \([1, 1, 1, 1, 1, 1, 1]\) and with KGB element \( x \) such that \( \text{support}(x) \) is empty. These representations are all tempered. It turns out that \( N(\emptyset) = 56 \). For each \( 0 \leq i \leq 6 \), we set

\[
N_i = \sum_{\#S=i} N(S).
\]

For instance, \( N_0 = N(\emptyset) = 56 \).

We compute that

\[
N([0, 1, 2, 4, 5, 6]) = 0, \quad N([0, 1, 2, 3, 5, 6]) = 4, \quad N([0, 1, 3, 4, 5, 6]) = 2, \\
N([0, 1, 2, 3, 6]) = 6, \quad N([0, 2, 3, 4, 5, 6]) = 34, \quad N([1, 2, 3, 4, 5, 6]) = 50, \\
N([0, 1, 2, 3, 4, 5]) = 62.
\]

In particular, it follows that \( N_6 = 158 \). We also compute that \( N_1 = 84, \quad N_2 = 102, \quad N_3 = 133, \quad N_4 = 164, \quad N_5 = 181 \).

Therefore, the total number of strings for \( E_7(−25) \) is equal to

\[
\sum_{i=0}^{6} N_i = 878.
\]

Some auxiliary files have been built up to facilitate the classification of the Dirac series of \( E_7(−25) \). They are available via the following link:

https://www.researchgate.net/publication/353352799_EVII-Files

6. Examples

Keeping the notation for \( E_7(−25) \) as in Section 3, we have a maximal \( \theta \)-stable parabolic subalgebra \( q := \mathfrak{k} + \mathfrak{p}^+ \) of \( g \). Let \( E_\mu \) be the \( K \)-type with highest weight \( \mu \in \mathfrak{t}'_f \). Form the
generalized Verma module

\[ N(\mu) := U(\mathfrak{g}) \otimes U(\mathfrak{q}) E_\mu. \]

Let \( L(\mu) \) denote the irreducible quotient of \( N(\mu) \). Those unitarizable \( L(\mu) \), called (irreducible) unitary highest weight modules in the literature, were classified in [12, 19], and were known to be Dirac series representations [18]. Indeed, by Proposition 3.7 of [18], the lowest \( K \)-type \( E_\mu \) of \( L(\mu) \) must contribute to \( H_D(L(\mu)) \). Thus \( E_\mu \) must be one of the spin LKTs of \( L(\mu) \). There is a similar story for irreducible unitary lowest weight modules. In the tables of Section 8, we have marked out all the LKTs if they are simultaneously spin LKTs. Therefore, whenever the term “LKT=” does not show up among the spin LKTs (see the #2960 representation in Table 2 for a specific instance), it means that the scattered representation is neither a highest weight module nor a lowest weight module. Thus like the \( E_6.h \) case [4], the Dirac series of \( E_7.h \) go beyond the highest/lowest weight modules.

Example 6.1. By Theorem 13.4 (c) of [12], one finds that \( L(z\zeta) \) is an irreducible unitary highest weight module of \( E_{7(-25)} \) if and only if \( z \) is an integer, and that \( z = 0, -4, -8 \) or \( z \in (-\infty, -9] \). We locate these representations in Table 1, where \( g \) runs over the non-negative integers.

If the representation is non-trivial and fully supported, the row “Table label” gives the label of the table where we can find the representation. For instance, the #2365 representations can be found in Table 9. The value \( z = 0 \) produces the trivial representation (see Example 4.2 for its atlas parameter), while the values \( z = -4 \) and \( -8 \) are the two Wallach modules whose Dirac cohomology will be studied in Example 6.2.

Whenever \( z \leq -12 \), the representation \( L(z\zeta) \) is not fully supported. Thus it will fit into a string, and does not occur in the tables of Section 8. The representation \( L(z\zeta) \) is tempered whenever \( z \leq -17 \).

| \( z \) | \(-11\) | \(-10\) | \(-9\) | \(-8\) | \(-4\) | \(0\) |
|---|---|---|---|---|---|---|
| \#x | #1224 | #1604 | #1975 | #2365 | #2988 | #3016 |
| Table label | 64 | 9 | 4 | 4 | 10 |
| \( z \) | \(-17 - g \) | \(-16\) | \(-15\) | \(-14\) | \(-13\) | \(-12\) |
| \#x | #52 | #66 | #180 | #301 | #443 | #807 |

Example 6.2. By Theorem 5.2 of [12], \( E_{7(-25)} \) has two Wallach modules. Namely, \( L(-4\zeta) \) and \( L(-8\zeta) \). See Example 6.1.

The module \( L(-4\zeta) \) has infinitesimal character \( \Lambda = [1, 1, 1, 0, 1, 1, 1] \). This is a minimal representation, and its \( K \)-types in terms of the basis \( \{\varpi_1, \ldots, \varpi_6, \frac{1}{2}\zeta\} \) are as follows:

\[ [0, 0, 0, 0, 0, 0, -12] + n[0, 0, 0, 0, 0, 1, -2], \]

where \( n \) runs over non-negative integers. It is easy to compute that the spin LKTs consist of those with \( 0 \leq n \leq 5 \), and they all have spin norm \( \sqrt{\frac{231}{2}} \) which equals to \( \|\Lambda\| \). Then we
compute that $H_D(L(-4\zeta))$ consists of the following twelve $\tilde{K}$-types without multiplicities:

$[1,0,0,0,0,1,1],[0,0,0,0,0,1,-1],[2,0,0,0,0,0,1],[0,0,0,0,0,2,-1],$
$[0,0,0,0,1,0,5],[0,0,0,0,0,0,5],[0,0,0,0,0,0,±15],[0,1,0,0,0,0,±9],[1,0,0,0,0,1,±3].$

On the other hand, the module $L(-8\zeta)$ has two spin LKTs:

$LKT = [0,0,0,0,0,-24], [1,0,0,0,0,-28],$

and infinitesimal character $\Lambda = [1,1,1,0,1,0,1]$. They have spin norm $\sqrt{\frac{150}{2}}$, which equals to $\|\Lambda\|$. Then we compute that $H_D(L(-8\zeta))$ consists of the $\tilde{K}$-types $\zeta$ and $-\zeta$ without multiplicities.

For each of the above Wallach modules, the $\tilde{K}$-types showing up in the Dirac chomology can be characterized as members of the following set

$\{w\Lambda - \rho_c \mid w \in W(g, t_f)^1\}$

which are dominant for $\Delta^+(t, t_f)$. □

**Remark 6.3.** The formulation (32) is inspired by Theorem 1.3 of [18], which has computed the Dirac cohomology of the Wallach modules for classical Hermitian symmetric real forms.

**Example 6.4.** Consider the #2950 representation in Table 4. It has infinitesimal character $\Lambda = [1,0,1,1,0,1,0]$, which is conjugate to $\rho_c$ under the action of $W(g, t_f)$. It has LKT $\mu = [0,0,0,0,0,0,3]$ and three spin LKTs:

$\mu_1 := [0,0,0,0,0,1,25], \mu_2 := [4,0,0,0,0,1,9], \mu_3 := [0,0,0,0,0,5,-7].$

Each $\mu_i$ contributes a trivial $\tilde{K}$-type to the Dirac cohomology. We compute that

$B(\mu_1 - \mu, \zeta) = 11, \quad B(\mu_2 - \mu, \zeta) = 3, \quad B(\mu_3 - \mu, \zeta) = 5.$

These integers are of the same parity. Thus by Theorem 6.2 of [11], there is no cancellation when passing from the Dirac cohomology to the Dirac index.

Therefore, up to a sign, the Dirac index of the representation is three copies of the trivial $\tilde{K}$-type. This can be checked directly by *atlas* as follows:

```
set p=parameter(KGB(G)[2950],[1,0,1,1,0,4,0],[0,0,0,0,0,4,0])
show_dirac_index(p)
```

The following example is another illustration of cohomological induction.

**Example 6.5.** Let us demonstrate that the fifth entry of Table 10 and the third entry of Table 6 are both $A_q(\lambda)$ modules.

G:E7_h
set x=KGB(G,1550)
support(x)
Value: [0,2,3,4,5,6]
set Q=Parabolic:(support(x),x)

...
set L=Levi(Q)
set t=trivial(L)
theta_induce_irreducible(parameter(x(t),lambda(t)-[0,2,0,0,0,0,0],nu(t)),G)
Value:
1*parameter(x=1769,lambda=[1,3,2,-2,1,2,1]/1,nu=[1,5,2,-5,0,2,1]/1) [257]
theta_induce_irreducible(parameter(x(t),lambda(t)-[0,3,0,0,0,0,0],nu(t)),G)
Value:
1*parameter(x=1923,lambda=[1,1,-1,3,-2,2,1]/1,nu=[1,0,-3,5,-5,2,1]/1) [208]

One can use the method in Example 6.5 to realize many FS-scattered representations as $A_q(\lambda)$ modules. But some FS-scattered representations of $E_7$ are not $A_q(\lambda)$ modules. To fill in this gap, we shall need special unipotent representations.

7. Special unipotent representations

The special unipotent representations for $E_7$ in the sense of [2] has been determined by Adams et al., see [38]. Whenever such a representation is non-trivial and is a Dirac series, we will mark it with a ♣ in Section 8. There are nine representations marked with ♣s in total, among which six are highest/lowest weight modules covered by Example 6.1. It remains to analyze the #2950/#2949 in Table 4 and the #2973 in Table 9.

G:E7
set kgp=KGP(G,[0,1,2,3,4,5])
set P=kgp[3]
set L=Levi(P)
L
Value: connected real group with Lie algebra 'e6(so(10).u(1)).u(1)'
set t=trivial(L)
set tm9=parameter(x(t),lambda(t)-[0,0,0,0,0,0,9],nu(t))
goodness(tm9,G)
Value: "Weakly fair"
theta_induce_irreducible(tm9,G)
Value:
1*parameter(x=2950,lambda=[1,0,1,1,0,4,0]/1,nu=[0,0,0,0,0,4,0]/1) [100]

We conclude that the #2950 representation in Table 4 is a weakly fair $A_q(\lambda)$ module. Then so is the #2949 representation. On the other hand, the #2973 in Table 9 is not a weakly fair $A_q(\lambda)$ module.

8. Appendix

This appendix presents all the 73 non-trivial FS-scattered Dirac series representations of $E_7(-25)$ according to their infinitesimal characters. The bolded KGB elements in the column #x' are explained in Example 5.3.
### Table 2. Infinitesimal character $[0, 1, 1, 0, 1, 1, 1]$

| $#x$ | $\lambda/\nu$ | Spin LKTs | $#x'$ |
|------|----------------|-----------|-------|
| 2960 | $[-2, 2, 4, -1, 1, 1, 2]$ | $[0, 0, 0, 0, 0, 1, 16], [0, 0, 0, 0, 0, 5, 2]$ | 2959 |
|      | $[-3, \frac{5}{2}, \frac{11}{2}, -\frac{5}{2}, 0, 0, \frac{5}{2}]$ | $[4, 0, 0, 0, 0, 1, 18]$ | |
| 915  | $[0, 2, 3, -2, 1, 1, 2]$ | $[3, 1, 0, 0, 0, 1, 16], [3, 0, 0, 0, 1, 20]$ | 914  |
|      | $[-2, \frac{5}{2}, 3, -3, 0, 0, \frac{5}{2}]$ | | |

### Table 3. Infinitesimal character $[1, 0, 1, 1, 1, 1]$  

| $#x$ | $\lambda/\nu$ | Spin LKTs | $#x'$ |
|------|----------------|-----------|-------|
| 2881 | $[2, -1, -3, 4, 1, 1, 2]$ | LKT $= [0, 0, 0, 0, 0, 1, 13], [0, 0, 0, 0, 0, 5, 5], [4, 0, 0, 0, 0, 1, 21]$ | 2880 |
|      | $[5, -\frac{5}{2}, -\frac{11}{2}, \frac{11}{2}, 0, 0, \frac{5}{2}]$ | | |

### Table 4. Infinitesimal character $[1, 0, 1, 1, 1, 1]$  

| $#x$ | $\lambda/\nu$ | Spin LKTs | $#x'$ |
|------|----------------|-----------|-------|
| 2950 | $[1, 0, 1, 1, 0, 4, 0]$ | $[0, 0, 0, 0, 0, 1, 25], [4, 0, 0, 0, 0, 1, 9]$ | 2949 |
|      | $[0, 0, 0, 0, 0, 4, 0]$ | $[0, 0, 0, 0, 0, 5, -7]$ | |
| 1977 | $[1, -2, 1, 3, -2, 3, 0]$ | LKT $= [0, 0, 0, 0, 0, 0, 27]$ | 1975 |
|      | $[0, -4, 0, 4, -4, 4, 0]$ | | |

### Table 5. Infinitesimal character $[1, 0, 1, 1, 1, 1]$  

| $#x$ | $\lambda/\nu$ | Spin LKTs | $#x'$ |
|------|----------------|-----------|-------|
| 2684 | $[1, -1, 1, 4, -3, 2, 1]$ | $[3, 2, 0, 0, 0, 0, 6], [0, 2, 0, 0, 0, 3, -6]$ | 2016 |
|      | $[0, -4, 2, 5, -5, 2, 0]$ | | |
| 2017 | $[1, -2, 1, 3, -2, 3, 1]$ | LKT $= [1, 0, 0, 0, 0, 0, 26]$ | |
|      | $[0, -\frac{9}{2}, 0, \frac{9}{2}, -\frac{9}{2}, 4, 1]$ | | |

### Table 6. Infinitesimal character $[1, 1, 0, 1, 1, 1]$  

| $#x$ | $\lambda/\nu$ | Spin LKTs | $#x'$ |
|------|----------------|-----------|-------|
| 2954 | $[1, 1, -1, 4, -2, 1, 3]$ | $[0, 0, 0, 0, 0, 1, 19], [0, 0, 0, 0, 0, 5, -1]$ | 2953 |
|      | $[0, 0, -2, 5, -3, 0, 3]$ | $[4, 0, 0, 0, 0, 1, 15]$ | |
| 2127 | $[4, 2, -2, 1, -1, 3, 1]$ | $[0, 3, 0, 0, 0, 0, 9], [0, 0, 0, 0, 3, 0, 9]$ | 2126 |
|      | $[5, \frac{3}{2}, -\frac{7}{2}, 0, -\frac{3}{2}, \frac{7}{2}, 0]$ | | |
| 1923 | $[1, 1, -1, 3, -2, 2, 1]$ | LKT $= [0, 0, 2, 0, 0, 0, 11]$ | 1922 |
|      | $[1, 0, -3, 5, -5, 2, 1]$ | $[0, 0, 2, 0, 0, 2, 7], [0, 0, 0, 2, 0, 0, 15]$ | |
| 1324 | $[3, 1, -2, 2, -1, 1, 4]$ | $[0, 0, 0, 0, 0, 1, 31]$ | 1323 |
|      | $[3, 0, -3, 2, -2, 0, 4]$ | | |
| 1226 | $[2, 1, -1, 2, -1, 1, 2]$ | LKT $= [0, 0, 0, 0, 0, 0, 33]$ | 1224 |
|      | $[3, 0, -3, 3, -3, 0, 3]$ | | |
### Table 7. Infinitesimal character \([1,1,0,1,1,0,1]\)

| \#x | \(\lambda/\nu\) | Spin LKTs | \#x' |
|-----|----------------|-----------|------|
| 1957 | \([4,2,-2,1,2,-2,3]\) | \([0,3,0,0,0,1,10], [0,1,0,0,2,1,8]\) | 1956 |
|      | \([5,\frac{3}{2},-\frac{7}{2},0,2,-\frac{7}{2},\frac{7}{2}]\) | | |
| 1524 | \([3,1,-2,1,2,-1,4]\) | \([1,0,0,0,0,1,30]\) | 1523 |
|      | \([\frac{7}{2},0,-\frac{2}{7},0,3,-3,4]\) | | |

### Table 8. Infinitesimal character \([1,1,0,1,1,1,1]\)

| \#x | \(\lambda/\nu\) | Spin LKTs | \#x' |
|-----|----------------|-----------|------|
| 2465 | \([4,1,-3,2,1,2,1]\) | LKT = \([0,4,0,0,0,0,0]\), \([1,4,0,0,0,0,2]\), \([0,4,0,0,0,1,-2]\) | |
|      | \([7,1,-7,2,0,2,0]\) | | |
| 1713 | \([2,1,-1,1,1,2,1]\) | LKT = \([3,0,0,0,0,0,30]\), \([2,0,1,0,0,32]\) | 1712 |
|      | \([\frac{9}{7},0,-\frac{9}{7},0,0,4,1]\) | | |
| $\#x$ | $\lambda / \nu$ | Spin LKTs | $\#x'$ |
|-----|----------------|-----------|-------|
| 2973 | $[1, 2, 1, -1, 3, -1, 4]$ | $[4, 0, 0, 0, 0, 1, 6]$, $[1, 0, 0, 0, 1, 0, 4, -6]$ | 2973 |
|      | $[0, 1, 0, -1, 4, -3, 4]$ | $[5, 0, 0, 0, 0, 1, 0, 10]$, $[0, 0, 0, 0, 0, 5, -10]$ | |
| 2958 | $[1, 2, 1, -1, 4, -2, 3]$ | $[0, 0, 0, 0, 0, 1, 22]$, $[0, 0, 0, 0, 0, 5, -4]$ | 2957 |
|      | $[0, 1, 0, -1, \frac{9}{7}, -\frac{7}{2}, \frac{7}{2}]$ | $[4, 0, 0, 0, 0, 1, 12]$ | |
| 2848 | $[1, 3, 1, -2, 5, -2, 1]$ | $[3, 1, 0, 0, 1, 4]$, $[1, 1, 0, 0, 3, -4]$ | 2848 |
|      | $[0, 4, 2, -4, 5, -3, 0]$ | $[4, 1, 0, 0, 0, 8]$, $[0, 1, 0, 0, 4, -8]$ | |
| 2366 | $[1, 4, 1, -3, 4, 0, 1]$ | LKT = $[0, 0, 0, 0, 0, 24]$, $[0, 0, 0, 0, 0, 1, 28]$ | 2366 |
|      | $[\frac{9}{7}, 0, -\frac{9}{7}, \frac{7}{2}, -\frac{7}{2}, \frac{7}{2}]$ | $[0, 0, 0, 0, 0, 1, 28]$ | |
| 2299 | $[1, 3, 1, 0, 1, -2, 3]$ | $[0, 0, 0, 0, 3, 0, 0]$, $[1, 0, 0, 0, 2, 1, 4]$ | 2299 |
|      | $[1, 4, 1, -1, 0, -4, 4]$ |  | |
| 2233 | $[1, 4, 1, -1, 2, -3, 5]$ | LKT = $[0, 0, 0, 0, 0, 1, 22]$, $[0, 0, 0, 0, 1, 0, 26]$ | 2233 |
|      | $[0, 4, 0, -1, 1, -4, 5]$ |  | |
| 2131 | $[2, 2, 3, -2, 2, 2, -2, 3]$ | $[0, 3, 0, 0, 0, 0, 12]$, $[0, 0, 0, 0, 3, 0, 6]$ | 2131 |
|      | $[\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}, 2, -\frac{3}{2}, \frac{3}{2}]$ |  | |
| 2081 | $[1, 2, 2, -2, 3, -2, 4]$ | LKT = $[0, 0, 0, 0, 1, 0, 20]$, $[0, 0, 0, 1, 0, 24]$ | 2081 |
|      | $[\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, 0, -\frac{3}{2}, \frac{3}{2}]$ | $[0, 0, 0, 0, 1, 0, 24]$ | |
| 1824 | $[1, 1, 4, -4, 5, -2, 3]$ | $[1, 0, 1, 0, 1, 1, 0]$, $[0, 0, 2, 0, 0, 2, 4]$ | 1824 |
|      | $[1, 0, 3, -4, 4, -3, 3]$ | $[2, 0, 0, 0, 2, 0, -4]$ | |
| 1741 | $[1, 1, 3, -2, 3, -2, 3]$ | $[0, 0, 1, 0, 1, 2, 2]$, $[1, 0, 0, 0, 2, 1, -2]$ | 1741 |
|      | $[1, 0, 3, -\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}, \frac{3}{2}]$ | $[0, 0, 0, 0, 1, 0, 28]$ | |
| 1669 | $[1, 1, 3, -2, 2, -1, 4]$ | LKT = $[0, 0, 0, 0, 0, 0, 30]$ | 1669 |
|      | $[0, 0, \frac{7}{2}, -\frac{7}{2}, \frac{7}{2}, -\frac{7}{2}]$ | $[0, 0, 0, 0, 0, 0, 30]$ | |
| 1606 | $[1, 1, 3, -2, 3, -2, 3]$ | LKT = $[0, 0, 0, 0, 0, 0, 30]$ | 1606 |
|      | $[0, 0, \frac{7}{2}, -\frac{7}{2}, \frac{7}{2}, -\frac{7}{2}]$ | $[0, 0, 0, 0, 0, 0, 30]$ | |
| 1580 | $[2, 4, 2, -3, 2, -1, 3]$ | LKT = $[0, 0, 2, 0, 0, 0, 14]$ | 1580 |
|      | $[1, 4, \frac{3}{2}, -4, \frac{3}{2}, -\frac{3}{2}, \frac{3}{2}]$ | $[1, 1, 1, 0, 0, 0, 18]$ | |
| 1025 | $[2, 2, 1, -1, 1, 0, 2]$ | $[0, 1, 0, 0, 0, 4, -2]$, $[0, 0, 0, 1, 0, 3, -6]$ | 1025 |
|      | $[\frac{5}{2}, \frac{5}{2}, 0, -\frac{5}{2}, 0, 0, \frac{5}{2}]$ | $[0, 1, 0, 0, 0, 4, -2]$, $[0, 0, 0, 1, 0, 3, -6]$ | |
| 959  | $[2, 2, 1, -2, 3, -1, 2]$ | LKT = $[3, 0, 0, 0, 1, 0, 8]$, $[3, 1, 0, 0, 0, 1, 10]$, $[3, 0, 0, 1, 0, 6]$ | 959 |
|      | $[2, 2, 0, -3, 3, -2, 1]$ | $[3, 0, 0, 0, 1, 0, 8]$, $[3, 1, 0, 0, 0, 1, 10]$, $[3, 0, 0, 1, 0, 6]$ | |
| #x   | λ/ν                  | Spin LKTs                  | #x'  |
|------|----------------------|----------------------------|------|
| 2989 | [3, 2, 2, −1, 1, 1, 2] | LKT = [0, 0, 0, 0, 0, 0, 12], [0, 0, 0, 0, 0, 0, 12] + nβ, 1 ≤ n ≤ 5 | 2988 |
| 2837 | [2, 2, 1, −2, 3, 1, 2] | LKT = [0, 0, 0, 0, 0, 2, 14], [1, 0, 0, 0, 0, 2, 14] | 2836 |
|      | [1, 0, 0, 0, 0, 2, 16], [0, 0, 0, 0, 0, 5, 8] | [2, 0, 0, 0, 0, 2, 18], [3, 0, 0, 0, 0, 2, 20] | 2579 |
| 2579 | [1, 1, 3, −1, 1, 1, 1] | LKT = [0, 3, 0, 0, 0, 4], [0, 3, 0, 0, 0, 2, 2, −4] | 1865 |
|      | [1, 0, 1, 0, 0, 2, 30], [0, 0, 2, 0, 0, 32] | [2, 3, 0, 0, 0, 36], [0, 0, 0, 1, 0, 38] | 1864 |
| 1769 | [1, 3, 2, −2, 1, 2, 1] | LKT = [0, 3, 0, 0, 0, 4], [0, 3, 0, 0, 0, 2, 2, −4] | 1768 |
|      | [0, 0, 0, 0, 1, 10], [0, 0, 2, 1, 0, 14] | [0, 0, 3, 0, 0, 0, 3], [0, 0, 3, 0, 0, 0, 3] | 1033 |
| 1033 | [2, 2, 1, −1, 1, 1, 2] | LKT = [0, 1, 0, 0, 0, 36], [0, 0, 0, 1, 0, 38] | 2768 |

| #x   | λ/ν                  | Spin LKTs                  | #x'  |
|------|----------------------|----------------------------|------|
| 1438 | [1, 1, 2, 1, −3, 4, 1] | [2, 0, 1, 0, 0, 3, 2], [3, 0, 0, 0, 1, 2, −2] | 1438 |
|      | [1, 0, 3, 0, −4, 4, −3] | 1438                        |      |

| #x   | λ/ν                  | Spin LKTs                  | #x'  |
|------|----------------------|----------------------------|------|
| 2768 | [2, 2, 1, 1, −2, 3, 2] | LKT = [0, 0, 0, 0, 0, 3, 15], [1, 0, 0, 0, 0, 3, 17], [2, 0, 0, 0, 0, 3, 19] | 2767 |
|      | [0, 0, 0, 0, 0, 5, 11], [2, 0, 0, 0, 0, 3, 19] | 2767                        |      |

| #x   | λ/ν                  | Spin LKTs                  | #x'  |
|------|----------------------|----------------------------|------|
| 2666 | [2, 1, 1, 1, −1, 3]   | LKT = [0, 0, 0, 0, 0, 4, 16], [0, 0, 0, 0, 0, 4, 16] | 2665 |
|      | [0, 0, 0, 0, 0, 5, 14], [1, 0, 0, 0, 0, 4, 18] | 2665                        |      |

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(Ding) School of Mathematical Sciences, Soochow University, Suzhou 215006, P. R. China
Email address: 435025738@qq.com

(Dong) School of Mathematical Sciences, Soochow University, Suzhou 215006, P. R. China
Email address: chaopindong@163.com