William I. Fine Theoretical Physics Institute  
University of Minnesota

Two-pion transitions in quarkonium revisited

M.B. Voloshin  
William I. Fine Theoretical Physics Institute, University of Minnesota,  
Minneapolis, MN 55455  
and  
Institute of Theoretical and Experimental Physics, Moscow, 117218

Abstract

Two-pion transitions in charmonium and bottomonium are considered with the leading relativistic effects taken into account. The contribution of the chromo-magnetic interaction of the charmed quarks to the amplitude of the decay $\psi(2S) \rightarrow \pi\pi J/\psi$ is estimated from the available data. This contribution is enhanced by a factor of three in the decay $\eta_c(2S) \rightarrow \pi\pi\eta_c$ and should produce a noticeable modification of the rate and the spectrum in the latter decay. It is argued that the peculiar observed spectrum in the decay $\Upsilon(3S) \rightarrow \pi\pi\Upsilon$ arises due to a dynamical suppression of the leading nonrelativistic quarkonium amplitude and thus enhanced prominence of the relativistic terms. Also discussed are the effects of the final state interaction between the pions.
1 Introduction

Hadronic transitions between states of the available heavy quarkonia, charmonium and bottomonium, present a very interesting case study in dynamics of both the heavy quarks and the light mesons emitted in the transitions. In particular, the observed properties of such processes with the emission of two pions are understood within the chiral low-energy dynamics of pions, starting with the earliest observation [1] of the decay $\psi(2S) \to \pi^+\pi^- J/\psi$ and the theoretical analyses of the data [2, 3]. Furthermore, the QCD picture of such decays is that the heavy quarkonium transition generates soft gluonic field which then produces the light mesons. The heavy quarkonium can be considered as a compact and nonrelativistic object in its interaction with a soft gluonic field, which justifies the use of the multipole expansion in QCD in analysing these processes [4]. In this approach the amplitude of the process factorizes into the heavy quarkonium part, i.e. the transition between the levels as a source of the field, and the light meson part, which describes the creation of the mesons by the field operators. The former part depends on the dynamics of the quarkonium, while the latter one can be understood in some detail by combining [5] the methods of chiral dynamics and the general low-energy relations in QCD, in particular those for the anomalies in the trace of the energy-momentum tensor and in the singlet axial current. Each of these two factors in the amplitudes of the discussed hadronic transitions can be used, in certain extent, independently. For instance, the decays $\psi(2S) \to \pi^+\pi^- J/\psi$ and $\psi(2S) \to \eta J/\psi$ (as well as $\Upsilon(2S) \to \pi^+\pi^- \Upsilon$ and $\Upsilon(2S) \to \eta \Upsilon$) have essentially the same heavy quarkonium part, so that the ratio of the decay rates is fully determined by the low-energy meson amplitudes [5, 6]. On the other hand the charmonium amplitude for the double interaction with the chromo-electric field (the chromo-polarizability), which dominates the quarkonium factor in the decay $\psi(2S) \to \pi^+\pi^- J/\psi$ can be used [7] for considering the low-energy limit of the conversion of $\psi(2S)$ into $J/\psi$ on a nucleon: $\psi(2S) + N \to J/\psi + N$, where the soft gluonic field is that inside the nucleon. The same charmonium transition amplitude can be used as a benchmark for the diagonal chromo-polarizability of $J/\psi$, which determines the low-energy elastic scattering of $J/\psi$ on a nucleon [8, 7]. For these reasons a more detailed understanding of both the heavy quarkonium amplitudes and of those determining the gluon conversion into light mesons is desirable.

Presently a quite detailed experimental data are available on the dominant hadronic transitions in charmonium [9]: $\psi(2S) \to \pi^+\pi^- J/\psi$, and in bottomonium [10, 11]: $\Upsilon(2S) \to \pi^+\pi^- \Upsilon$ and $\Upsilon(3S) \to \pi^+\pi^- \Upsilon(1S, 2S)$. Numerous other, less visible, transitions have been
observed, e.g. Υ(1D) → π⁺π⁻Υ. Thus it is not unreasonable to expect, given the capabilities of the modern e⁺e⁻ experiments at the charm and bottom thresholds, that a significantly improved data will become available. Such data may enable a study of finer effects in the amplitudes of the hadronic transitions beyond the leading ones thus far considered in the literature.

The purpose of the present paper is to analyze sub-leading relativistic effects in the amplitudes of the two-pion transitions in charmonium and bottomonium, and possible ways of studying such effects from the data. For the transitions between the 3S₁ states of quarkonium, dominated by the second order in the leading E₁ term of the multipole expansion, these corrections arise from the second order in the M₁ interaction. It will be argued that in the charmonium transition ψ(2S) → π⁺π⁻J/ψ the correction adds coherently to the leading term as an O(10%) term. Being of a rather moderate value in this particular decay, the correction is enhanced by a factor of 3 in the transition η₃(2S) → π⁺π⁻η₃ with a potentially very noticeable effect in the spectrum of the dipion invariant mass. Furthermore, this correction modifies the estimate of the charmonium chromo-polarizability with implications for charmonium scattering on nucleons. It will be further argued that, although generally the relativistic effects in bottomonium are quite small, in the particular case of the decay Υ(3S) → π⁺π⁻Υ it appears that the leading nonrelativistic term it strongly suppressed dynamically, and including the relativistic effects of the M₁ interaction and of the 3D₁ − 3S₁ mixing can possibly resolve the long standing puzzle of the peculiar observed spectrum of the dipion invariant masses in this decay, which spectrum by far does not conform with the predictions from the leading chromo-electric interaction.

Also some consideration will be given to the significance of the final state interaction (FSI) between the pions in the two-pion transitions. The FSI effects have been discussed ever since the earliest analyses, and are generally believed to be moderate in the kinematical region of the decays ψ(2S) → π⁺π⁻J/ψ and Υ(2S) → π⁺π⁻Υ. However once the amplitudes of the transitions are studied in finer detail, one will have to better quantify these effects. Although at present the FSI effects cannot be fully analyzed, their behavior at low invariant mass of the two pions can be estimated within the chiral expansion. For this purpose we give here a systematic derivation of the amplitudes for two pion production by gluonic operators in the leading chiral limit, including the terms with the pion mass, and consider the first sub-leading term due to iteration of the chiral amplitudes. At the invariant mass of the two pions near the threshold these corrections are quite small - only few percent. Their
continuation to higher values of the invariant mass can be studied experimentally in a rather straightforward way. Such study, in particular, would quantify the impact of the FSI effects on the estimates of the quarkonium amplitudes from the observed decay rates. It will be also argued that even with the already available data such impact can be estimated as quite moderate, as generally expected. \(^1\)

The material in the present paper is organized as follows. In Section 2 the chromo-electric and chromo-magnetic dipole interactions responsible for the two-pion transitions are discussed and the parameters determining the relevant quarkonium amplitudes to order \(v^2/c^2\) are introduced. The Section 3 contains a compilative derivation in the leading chiral order of the amplitudes for creation of the pion pair by gluonic operators, and in Section 4 the discussion of the two previous sections is combined in the expressions for the two-pion transition amplitudes with the leading relativistic terms included. The latter expressions are confronted with the available date on the transitions \(\psi(2S) \rightarrow \pi\pi J/\psi\) and \(\Upsilon(2S) \rightarrow \pi\pi\Upsilon\) in Section 5 and the significance of the chromo-magnetic interaction in transitions between charmonium resonances is evaluated. In Section 6 the FSI effects are considered using iteration of the chiral amplitudes and it is also argued on phenomenological grounds that these effects are only of a moderate magnitude in the observed two-pion transitions. The Section 7 addresses the long-standing puzzle of the unusual spectrum of the dipion invariant masses observed in the decay \(\Upsilon(3S) \rightarrow \pi\pi\Upsilon\) and it is suggested that this spectrum can be explained if the formally leading nonrelativistic quarkonium amplitude is dynamically suppressed in this transition, so that the relativistic terms provide a significant contribution. Finally, Section 8 contains a summary of the discussion in the present paper.

2 Multipole expansion for the two-pion transitions and relativistic terms

Considering quarkonium as a compact nonrelativistic object one can apply the multipole expansion for the quarkonium interaction with soft gluonic field \[\bar{E}a\]. The leading term in this expansion is the chromo-electric dipole interaction with the chromo-electric field \(\bar{E}a\)

\(^1\)In particular, the present data certainly exclude the recently claimed modification due to FSI of the charmonium amplitude by a factor of about 3.
described by the Hamiltonian

\[ H_{E1} = -\frac{1}{2} \xi^a \vec{r} \cdot \vec{E}^a(0) , \]

where \( \xi^a = t^1_1 - t^2_2 \) is the difference of the color generators acting on the quark and antiquark (e.g. \( t^a_1 = \lambda^a/2 \) with \( \lambda^a \) being the Gell-Mann matrices), and \( \vec{r} \) is the vector for relative position of the quark and the antiquark. The convention used throughout this paper is that the QCD coupling \( g \) is included in normalization of the gluon field operators, so that e.g. the gluon field Lagrangian is written as \(-F_{\mu\nu}^a)^2/(4g^2)\).

The amplitude of the two-pion transition between heavy quarkonium states \( \psi_2 \to \pi^+ \pi^- \psi_1 \) (with \( \psi_2 \) and \( \psi_1 \) used as a generic notation for the initial and the final quarkonium states) can thus be written as \(^2\)

\[ A(\psi_2 \to \pi^+ \pi^- \psi_1) = \frac{1}{2} \langle \pi^+ \pi^- | E^a_i E^a_j | 0 \rangle \alpha^{(12)}_{ij} , \]

where \( \alpha^{(12)}_{ij} \) can be termed, in complete analogy with the atomic properties in electric field, as the transitional chromo-polarizability of the quarkonium. In other words, the \( \psi_2 \to \psi_1 \) transition in the chromo-electric field is described by the effective Hamiltonian

\[ H_{\text{eff}} = -\frac{1}{2} \alpha^{(12)}_{ij} E^a_i E^a_j , \]

with the chromo-polarizability given by

\[ \alpha^{(12)}_{ij} = \frac{1}{16} \langle \psi_1 | \xi^a \vec{r}_i \mathcal{G} \vec{r}_j \xi^a | \psi_2 \rangle , \]

where \( \mathcal{G} \) is the Green’s function of the heavy quark pair in a color octet state. The latter function is not well understood presently, so that an \textit{ab initio} calculation of the chromo-polarizability would be at least highly model dependent.

Also a simple remark is in order that the discussed decays in quarkonia are governed by transitional polarizabilities, i.e. those linking different states of quarkonium. The diagonal chromo-polarizability of quarkonium states, in particular of the charmonium resonances \( \psi(2S) \) and \( J/\psi \) can also be measured \(^1\) and is relevant to the problem of scattering of these resonances on nuclei \(^2\).

Generally \( \alpha_{ij} \) is a symmetric tensor. Clearly, for transitions between pure \( S \) wave quarkonium states this tensor is necessarily proportional to \( \delta_{ij} \): \( \alpha_{ij} = \alpha_0 \delta_{ij} \). However for the \( ^3S_1 \)

\(^2\)For definiteness the processes with emission of a pair of charged pions, \( \pi^+ \pi^- \) are considered here, since the emission of neutral pions \( \pi^0 \pi^0 \) is trivially related by the isospin symmetry.
states a $^3D_1 - ^3S_1$ mixing generally takes place due to the relativistic effects in the order $v^2/c^2$ so that a traceless $D$ wave part of the chromo-polarizability is also present and is proportional to spin-2 combination of the polarizations, $\vec{\psi}_2$ and $\vec{\psi}_1$ of the initial and the final states:

$$\alpha_{ij} = \alpha_0 \delta_{ij} (\vec{\psi}_1 \cdot \vec{\psi}_2) + \alpha_2 \left[ \psi_{1i} \psi_{2j} + \psi_{1j} \psi_{2i} - (2/3) \delta_{ij} (\vec{\psi}_1 \cdot \vec{\psi}_2) \right].$$

(5)

Numerically, the $^3D_1 - ^3S_1$ mixing in charmonium appears to be comparable with the characteristic value of the relativistic parameter for this system, $v^2/c^2 \approx 0.2$. In particular, Rosner [16] using the $e^+e^-$ decay width of $\psi(3770)$ and considering this resonance as dominantly a $^3D_1$ state, estimates the angle $\theta$ of the $\psi(3770) - \psi(2S)$ mixing, i.e. the $^3D_1 - ^3S_1$ mixing for $\psi(2S)$, as $\theta \approx 0.2$. It can be noticed however that the mixing does not contribute to the spinless part $\alpha_0$ of the chromo-polarizability and that in the spin-averaged transition rate there is no interference between the $S$ and $D$ wave parts of $\alpha_{ij}$. Thus the effect of the $^3D_1 - ^3S_1$ mixing in the rate is of order $v^4/c^4$ and is thus generally very small.

It should be also mentioned that the effects of the recoil of the final quarkonium state in the decay also arise in the order $v^2/c^2$ and contribute terms quadratic in the momentum of the pion pair $\vec{q}$ to the spin-diagonal part of the tensor polarizability $\alpha_{ij}$. However such terms would also be of higher order in the chiral expansion in the pion momenta, which is used here. Therefore the recoil terms are suppressed by a product of two small parameters and can be neglected within the approximations adopted in this paper. 3

Besides the discussed $^3D_1 - ^3S_1$ mixing, the only other effect in the order $v^2/c^2$ in the amplitudes of two-pion transitions between quarkonium $S$ wave states arises through the second order in the M1 interaction with the chromo-magnetic field $\vec{B}^a$ described by the Hamiltonian

$$H_{M1} = -\frac{1}{2M} \xi^a (\vec{\Delta} \cdot \vec{B}^a),$$

(6)

where $\vec{\Delta} = \vec{s}_1 - \vec{s}_2$ is the difference of the spin operators acting on the quark and the antiquark, and $M$ is the heavy quark mass. The contribution of this term to the amplitude of two-pion transition between the $S$ states of quarkonium can be written as

$$A_M(\psi_2 \rightarrow \pi^+ \pi^- \psi_1) = \frac{1}{2} \langle \pi^+ \pi^- | B_i^a B_j^a | 0 \rangle \alpha^{(12)}_M \left[ \delta_{ij} (\vec{\psi}_1 \cdot \vec{\psi}_2) + \frac{3}{2} (\psi_{1i} \psi_{2j} + \psi_{1j} \psi_{2i} - (2/3) \delta_{ij} (\vec{\psi}_1 \cdot \vec{\psi}_2)) \right],$$

(7)

3The recoil effects in the quarkonium amplitude were considered in Ref. [17] and estimated to be quite small.
for the transitions between $^3S_1$ states, and

$$A_M(\eta_2 \rightarrow \pi^+\pi^- \eta_1) = \frac{3}{2} (\pi^+\pi^- | B_i^a B_i^a | 0) \alpha_M^{(12)} \quad (8)$$

for the transitions between $^1S_0$ states. (The notation $\eta_2$ and $\eta_1$ is used here for the states of quarkonium in order to emphasize that this relation is specific for the $^1S_0$ states.) The amplitude $\alpha_M^{(12)}$ is a chromo-magnetic analog of the chromo-electric term $\alpha_{ij}$ and is formally given by the formula

$$\alpha_M^{(12)} = \frac{1}{48 M^2} \langle \phi_1 | \xi^a G_\xi^a | \phi_2 \rangle \quad (9)$$

where $\phi_2$ and $\phi_1$ are the coordinate parts of the $S$-wave wave functions of the initial and the final states. It is taken into account here that the amplitudes generated by the M1 interaction are already suppressed in comparison with those described by Eq.(2) by the factor $v^2/c^2$. Therefore the wave functions of the quarkonium states can be taken in the form where the spin and coordinate parts are factorized.

The spin-2 part of the amplitude in Eq.(7) contributes to the decay rate only in the order $v^4/c^4$, similarly to the effect of the $^3D_1 - ^3S_1$ mixing. However, the spin-0 part in this amplitude as well as that in Eq.(8) (the only one present there) does interfere with the leading nonrelativistic amplitude in Eq.(2) proportional to $\alpha_0$ from Eq.(5). Therefore this part provides the first relativistic correction, proportional to $v^2/c^2$, to the transition rate. One can also notice that the effect for the $^1S_0$ states is three times bigger than for the $^3S_1$ states.

### 3 Two-pion creation by gluonic operator. Chiral limit.

As one can see from the previous section, a crucial role in the discussed approach to calculating the two-pion transition amplitudes is played by the amplitudes for production of two pions by operators quadratic in components of the gluon field strength tensor. Therefore in this section we consider the general amplitude of such type: $\langle \pi^+(p_1)\pi^-(p_2)|F^a_\mu F^a_\lambda | 0 \rangle$, describing the creation of two pions by the local gluonic operator $F^a_\mu F^a_\lambda$. In the leading chiral limit the momenta $p_1$ and $p_2$ of the pions as well as the pion mass $m$ are to be considered as small parameters, and the expression for the amplitude, quadratic in these parameters, can be written in the following general form

$$- \langle \pi^+(p_1)\pi^-(p_2)|F^a_\mu F^a_\lambda | 0 \rangle = \left[ X (p_1 \cdot p_2) + Y (p_1^2 + p_2^2 - m^2) \right] (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}) + Z t_{\mu\lambda\nu\sigma} \quad (10)$$
where the structure

\[
\begin{align*}
t_{\mu\nu\lambda\sigma} &= (p_1\mu p_2\lambda + p_1\lambda p_2\mu) g_{\nu\sigma} + (p_1\nu p_2\sigma + p_1\sigma p_2\nu) g_{\mu\lambda} \\
&\quad - (p_1\mu p_2\sigma + p_1\sigma p_2\mu) g_{\nu\lambda} - (p_1\nu p_2\lambda + p_1\lambda p_2\nu) g_{\mu\sigma} - (p_1 \cdot p_2) (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda})
\end{align*}
\]

(11)

has zero overall trace: \( t_{\mu\nu} = 0 \), and \( X \), \( Y \), and \( Z \) are yet undetermined coefficients. The form of the amplitude in Eq. (10) is uniquely determined by the symmetry (with respect to the indices) of the operator \( F_{\mu\nu} F_{\lambda\sigma} \) and by the Adler zero condition, which requires that the amplitude goes to zero when either one of the two pion momenta is set to zero and the other one is on the mass shell\(^4\).

The coefficients \( X \) and \( Y \) in Eq. (10) are in fact determined \(^5\) by the anomaly in the trace of the energy-momentum tensor \( \theta_{\mu\nu} \) in QCD. Indeed, in the low-energy limit in QCD, i.e. in QCD with three light quarks, one finds

\[
\theta_{\mu} = -\frac{b}{32\pi^2} F_{\mu\nu}^a F_{\nu}^{a\mu} + \sum_{q=u,d,s} m_q(\bar{q}q) ,
\]

(12)

where \( b = 9 \) is the first coefficient in the beta function for QCD with three quark flavors. The first term in Eq. (12) represents the conformal anomaly, while the quark mass term arises due to the explicit breaking of the scale invariance by quark masses. On the other hand, the matrix element of the energy-momentum tensor \( \theta_{\mu\nu} \) over the pions: \( \theta_{\mu\nu}(p_1, p_2) \equiv \langle \pi^+(p_1)\pi^-(p_2)|\theta_{\mu\nu}|0 \rangle \) is fully determined \(^5, 18, 19\) in the quadratic in \( p_1, p_2 \) and \( m \) order by the conditions of symmetry in \( \mu \) and \( \nu \), conservation on the mass shell \((p_1 + p_2)^\mu \theta_{\mu\nu}(p_1, p_2) = 0 \) at \( p_1^2 = p_2^2 = m^2 \), normalization \( (\theta_{\mu\nu}(p,-p) = 2 p_\mu p_\nu \) at \( p^2 = m^2 \), and the Adler zero condition \( (\theta_{\mu\nu}(p,0)|_{p^2=m^2} = 0) \):

\[
\theta_{\mu\nu}(p_1, p_2) = [(p_1 \cdot p_2) + p_1^2 + p_2^2 - m^2] g_{\mu\nu} - p_1\mu p_2\nu - p_1\nu p_2\mu .
\]

(13)

The equations (10) and (13) allow for the pion momenta to be off-shell in order to demonstrate the Adler zero. However in what follows only the amplitudes with pions on the mass shell will be considered, so that it will be henceforth implied that \( p_1^2 = p_2^2 = m^2 \). In particular one finds for the trace of the expression in Eq. (13)

\[
\theta^\mu(p_1, p_2) = 2 (p_1 \cdot p_2) + 4 m^2 .
\]

(14)

\(^4\)The proper index symmetry and the Adler zero condition also automatically ensure that the amplitude is \( C \) even, i.e. symmetric under the permutation of the pion momenta: \( p_1 \leftrightarrow p_2 \).
Furthermore, the quark mass term in Eq. (12), corresponding to the explicit breaking of the chiral symmetry in QCD corresponds to the same symmetry breaking by the pion mass term in the pion theory. Thus one finds to the quadratic order in \( m^2 \):

\[
\langle \pi^+ \pi^- | \sum_{q=u,d} m_q \bar{q} q | 0 \rangle = m^2, \tag{15}
\]

while the term with the strange quark makes no contribution to the discussed amplitude.

Combining the formula in Eq. (12) for \( \theta^\mu_\mu \) with the expressions (14) and (15) one finds the matrix element of the gluonic operator over the pions in the form

\[
- \langle \pi^+(p_1) \pi^-(p_2) | F^a_{\mu\nu} F^a_{\lambda\sigma} | 0 \rangle = \frac{32\pi^2}{b} \left[ 2 (p_1 \cdot p_2) + 3 m^2 \right], \tag{16}
\]

which thus allows to determine the coefficients \( X \) and \( Y \) in Eq. (10): \( X = 16\pi^2/(3b) \) and \( Y = 3X/2 = 8\pi^2/b \).

The coefficient \( Z \) of the traceless part in Eq. (10) cannot be found from the trace relation (12). Novikov and Shifman [18] estimated this coefficient by relating this part to the matrix element of the traceless (twist-two) energy-momentum tensor of the gluons only:

\[
\theta^G_{\mu\nu} = 4\pi\alpha_s \left(-F^a_{\mu\lambda} F^a_{\nu} + \frac{1}{4} g_{\mu\nu} F^a_{\lambda\sigma} F^a_{\lambda\sigma}\right),
\]

\[
Z t^\lambda_{\mu\nu} = 4\pi\alpha_s \langle \pi^+(p_1) \pi^-(p_2) | \theta^G_{\mu\nu} | 0 \rangle.
\tag{17}
\]

They then assume that the matrix element of the twist-two operator is proportional to the traceless part of the phenomenological energy momentum tensor of the pions,

\[
\langle \pi^+(p_1) \pi^-(p_2) | \theta^G_{\mu\nu} | 0 \rangle = \rho_G \left[ \frac{1}{2} (p_1 \cdot p_2) g_{\mu\nu} - p_{1\mu} p_{2\nu} - p_{1\nu} p_{2\mu} \right], \tag{18}
\]

with the proportionality coefficient interpreted, similarly to the deep inelastic scattering, as “the fraction of the pion momentum carried by gluons”. They further introduce a related parameter \( \kappa = b\alpha_s \rho_G/(6\pi) \) and conjecture that numerically \( \kappa \approx 0.15 - 0.20 \).

For the purpose of the present consideration the interpretation of the parameter \( \kappa \) as being related to the pion gluon structure function is not essential and we treat \( \kappa \) here as a phenomenological and measurable, and actually measured, parameter. Summarizing the results so far in this section one can write the expression for the general matrix element (10) for on-shell pions as

\[
- \langle \pi^+(p_1) \pi^-(p_2) | F^a_{\mu\nu} F^a_{\lambda\sigma} | 0 \rangle = \frac{8\pi^2}{3b} \left[ (q^2 + m^2) (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}) - \frac{9}{2} \kappa t_{\mu\nu} t_{\lambda\sigma} \right], \tag{19}
\]

It can be mentioned that this relation, taking into account the pion mass, was used in Ref. [20], and was also derived in a particular chiral model in Refs. [21, 22].
where \( q = p_1 + p_2 \) is the total four-momentum of the dipion.

Few remarks are due regarding effects of higher order in \( \alpha_s \). The trace term in Eq. (19) receives no renormalization, provided that the coefficient \( b \) is replaced by \( \beta(\alpha_s)/\alpha_s^2 \) with \( \beta(\alpha_s) = b \alpha_s^2 + O(\alpha_s^3) \) being the full beta function in QCD. This modification however only affects the overall normalization of the trace part, and can in fact be absorbed into the definition of the heavy quarkonium amplitudes. On the contrary, the relative coefficient of the traceless term in Eq. (19), i.e. the parameter \( \kappa \), does depend on the normalization scale, which scale is appropriate to be chosen as the characteristic size of the heavy quarkonium [18]. However, given other uncertainties in the analysis of the two-pion transitions, the logarithmic variation of \( \kappa \) is a small effect. In particular, this effect is likely to be smaller than the discussed in this paper relativistic effects in the amplitudes of the two-pion emission.

The matrix element in Eq. (19) describes the production of the two pions in two partial waves in their center of mass system: the \( S \) wave and the \( D \) wave. The two waves can be measured separately, and also any effects of the final state interaction between pions are different in these two orbital states. Therefore it is quite instructive for the subsequent discussion to explicitly separate the \( S \) and \( D \) waves in the matrix element, i.e. to rewrite the amplitude (19) in the form

\[
- \langle \pi^+(p_1)\pi^-(p_2)|F_\mu^a F_\lambda^a|0 \rangle = S_{\mu\nu\lambda\sigma} + D_{\mu\nu\lambda\sigma} . \tag{20}
\]

Clearly, the trace term in Eq. (19) corresponds to a pure \( S \) wave, while the traceless term proportional to \( \kappa \) contains both waves. In order to perform explicit partial wave separation in \( t_{\mu\nu\lambda\sigma} \), it is helpful to introduce [18] the four vector \( r = p_1 - p_2 \) describing the relative momentum of the two pions, which reduces to a purely spatial vector in the c.m. system of the pions ((\( r \cdot q \) = 0). Then the tensor

\[
\ell_{\mu\nu} = r_\mu r_\nu + \frac{1}{3} \left( 1 - \frac{4m^2}{q^2} \right) (q^2 g_{\mu\nu} - q_\mu q_\nu) \tag{21}
\]

is also purely spatial in the c.m. frame and corresponds to pure \( D \) wave. Using this tensor one can make the following series of replacements for the terms of the generic structure \( p_1 \alpha p_2 \beta \) in the tensor \( t_{\mu\nu\lambda\sigma} \):

\[
p_1 \alpha p_2 \beta \rightarrow \frac{1}{4} q_\alpha q_\beta - \frac{1}{4} r_\alpha r_\beta = \frac{1}{4} q_\alpha q_\beta + \frac{1}{12} \left( 1 - \frac{4m^2}{q^2} \right) (q^2 g_{\alpha\beta} - q_\alpha q_\beta) - \frac{1}{4} \ell_{\alpha\beta}
\]

\[
\rightarrow \frac{1}{6} \left( 1 + \frac{2m^2}{q^2} \right) q_\alpha q_\beta - \frac{1}{4} \ell_{\alpha\beta} . \tag{22}
\]
Here in the first replacement the cross terms between $r$ and $q$ are dropped since they cancel in $t_{\mu\nu\lambda\sigma}$ due to the C symmetry ($p_1 \leftrightarrow p_2$), while the $g_{\alpha\beta}$ term in the last transition is dropped, since such structure cancels in the traceless tensor $t$. Using Eq.(22) one readily finds from the formula (19) the expressions for the $S$ and $D$ wave amplitudes:

$$S_{\mu\nu\lambda\sigma} = \frac{8\pi^2}{3b} \left\{ (q^2 + m^2) (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}) - \frac{3}{2} \kappa \left( 1 + \frac{2m^2}{q^2} \right) \left[ q_\mu q_\lambda g_{\nu\sigma} + q_\nu q_\sigma g_{\mu\lambda} - q_\nu q_\lambda g_{\mu\sigma} - q_\mu q_\sigma g_{\nu\lambda} - \frac{1}{2} q^2 (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}) \right] \right\} ,$$

and

$$D_{\mu\nu\lambda\sigma} = \frac{8\pi^2}{3b} \frac{9\kappa}{4} \left( \ell_{\mu\lambda} g_{\nu\sigma} + \ell_{\nu\sigma} g_{\mu\lambda} - \ell_{\nu\lambda} g_{\mu\sigma} - \ell_{\mu\sigma} g_{\nu\lambda} \right) .$$

4 Two-pion transition amplitudes with the relativistic corrections

Using the formulas in the equations (2), (5) and (7) and the expressions (23) and (24) for the dipion production amplitudes in the chiral limit, one can readily find the amplitude of the transition $\psi_2 \rightarrow \pi^+\pi^- \psi_1$ between generic $3S_1$ states of a heavy quarkonium. After a straightforward calculation one finds the $S$ wave decay amplitude

$$S(\psi_2 \rightarrow \pi^+\pi^- \psi_1) =$$

$$-\frac{4\pi^2}{b} \alpha_0^{(12)} \left[ (1 - \chi_M) (q^2 + m^2) - (1 + \chi_M) \kappa \left( 1 + \frac{2m^2}{q^2} \right) \left( \frac{(q \cdot P)^2}{P^2} - \frac{1}{2} q^2 \right) \right] (\psi_1 \cdot \psi_2) ,$$

as well as three types of $D$ wave amplitude: one unrelated to the spins of the quarkonium states

$$D_1(\psi_2 \rightarrow \pi^+\pi^- \psi_1) = -\frac{4\pi^2}{b} \alpha_0^{(12)} (1 + \chi_M) \frac{3\kappa}{2} \frac{\ell_{\mu\nu} P^\mu P^\nu}{P^2} (\psi_1 \cdot \psi_2) ,$$

and two amplitudes with the correlation with the polarization of the initial and the final resonances

$$D_2(\psi_2 \rightarrow \pi^+\pi^- \psi_1) = \frac{4\pi^2}{b} \alpha_0^{(12)} \left( \chi_2 + \frac{3}{2} \chi_M \right) \frac{\kappa}{2} \left( 1 + \frac{2m^2}{q^2} \right) q_\mu q_\nu \psi^{\mu\nu}$$

and

$$D_3(\psi_2 \rightarrow \pi^+\pi^- \psi_1) = \frac{4\pi^2}{b} \alpha_0^{(12)} \left( \chi_2 + \frac{3}{2} \chi_M \right) \frac{3\kappa}{4} \ell_{\mu\nu} \psi^{\mu\nu} .$$
In these formulas the following notation is used: $P$ stands for the 4-momentum of the initial quarkonium resonance, $\psi_1^\mu$ and $\psi_2^\mu$ are the polarization 4-vectors for the $^3S_1$ states, and $\psi^{\mu\nu}$ is the spin-2 structure $\psi^{\mu\nu} = \psi_1^\mu \psi_2^\nu + \psi_1^\nu \psi_2^\mu - (2/3) (\psi_1 \cdot \psi_2) (P^\mu P^\nu / P^2 - g^{\mu\nu})$. Finally, $\chi_M$ and $\chi_2$ stand for the ratio

$$\chi_M = \frac{\alpha_M}{\alpha_0}, \quad \chi_2 = \frac{\alpha_2}{\alpha_0}$$

and encode the relative magnitude of the $O(v^2/c^2)$ relativistic effects due to respectively the chromo-magnetic interaction (Eq.(6)) and the $^3D_1 - ^3S_1$ mixing.

The three $D$ waves correspond to different angular correlations. The first one, $D_1$, given by Eq.(26) corresponds to a $D$-wave motion in the c.m. frame of two pions, which correlates with the motion of the c.m. system in the laboratory frame, i.e. with the direction of $\vec{q}$. This $D$ wave arises in the leading nonrelativistic approximation [15] and is in fact observed and measured experimentally [9] for the transition $\psi(2S) \rightarrow \pi^+ \pi^- J/\psi$. The second $D$-wave amplitude, $D_2$ in Eq.(27), corresponds to the two pions being in the $S$ wave in their c.m. system and describes the correlation of the spins of the initial and the final resonances with the $D$-wave motion of the two-pion system as a whole. Finally, the amplitude $D_3$ given by Eq.(28) corresponds to a $D$-wave motion of the pions in their c.m. frame, which $D$ wave is correlated with the spins of the quarkonium states. It can be noted that the two latter amplitudes are proportional to a product of two relatively small parameters $\kappa$ and $\alpha_2 + (3/2) \alpha_M \sim v^2/c^2$. Neither $D_2$ nor $D_3$ have yet been observed experimentally, although a study [23] of polarization effects in the decay $\Upsilon(2S) \rightarrow \pi^+ \pi^- \Upsilon$, utilizing a transversal polarization of the DORIS beams qualitatively confirms that these spin-dependent amplitudes are quite small. (A discussion can be found in the review [24].)

The transitions between $^1S_0$ states of quarkonium have not been observed yet. One may hope however that with a dedicated effort a two-pion transition from the recently found $\eta_c(2S)$ resonance: $\eta_c(2S) \rightarrow \pi^+ \pi^- \eta_c$ can be observed and studied. Within the approach discussed here such transition is closely related to the familiar decay $\psi(2S) \rightarrow \pi^+ \pi^- J/\psi$, and in fact can be used for a useful calibration of the total width of $\eta_c(2S)$ [25]. Clearly, on the theoretical side the transitions between $^1S_0$ states are simpler than those between the $^3S_1$ ones since no polarization effects are involved. On the other hand the effect of the $M_1$ interaction (Eq.(3)) is enhanced for the $^1S_0$ states (Eq.(7)) by a factor of 3, so that the relevant transition amplitudes of a generic $\eta_2 \rightarrow \pi^+ \pi^- \eta_1$ transition are given by

$$S(\eta_2 \rightarrow \pi^+ \pi^- \eta_1) =$$

(30)
\[ \frac{4\pi^2}{b} \alpha_0^{(12)} \left[ (1 - 3\chi_M)(q^2 + m^2) - (1 + 3\chi_M)\kappa \left( 1 + \frac{2m^2}{q^2} \right) \left( \frac{(q \cdot P)^2}{P^2} - \frac{1}{2} q^2 \right) \right], \]

and

\[ D_1(\eta_2 \rightarrow \pi^+\pi^-\eta_l) = \frac{4\pi^2}{b} \alpha_0^{(12)} (1 + 3\chi_M) \frac{3\kappa}{2} \frac{\ell_{\mu\nu} P^\mu P^\nu}{P^2}. \] (31)

The implications of the enhancement of the relativistic term for the spectrum of the dipion invariant masses is discussed in the next section.

5 Phenomenological analysis

The most experimentally studied so far transitions of the discussed type are \( \psi(2S) \rightarrow \pi^+\pi^- J/\psi \) and \( \Upsilon(2S) \rightarrow \pi^+\pi^- \Upsilon \). In particular, in the BES experiment \[9\] both the spectrum of the invariant masses of the two-pion system and the angular distribution described by the \( D \) wave amplitude of the type \( D_1 \) (Eq.(26) were analyzed, using the formulas equivalent to Eq.(25) and Eq.(26) with \( \chi_M = 0 \), and the parameter \( \kappa \) was determined from the fits. The fit to the mass spectrum resulted in the value \( \kappa = 0.186 \pm 0.003 \pm 0.006 \), while the fit to the ratio of the \( D_1 \) and \( S \) wave amplitude gave \( \kappa = 0.210 \pm 0.027 \pm 0.042 \). Clearly, the consistency of these two values implies that the discussed approach correctly predicts \[18\] the ratio of the \( D_1 \) wave in terms of the sub-dominant term proportional to \( \kappa \) in the \( S \) wave amplitude. Furthermore, considering the modification of the expressions for these amplitudes due to the relativistic parameter \( \chi_M \), it is clear that the fit parameter in the experimental analysis is not exactly \( \kappa \) but rather

\[ \kappa_{\text{eff}}^{(\psi'J/\psi)} = \frac{1 + \chi_M^{(\psi'J/\psi)}}{1 - \chi_M^{(\psi'J/\psi)}} \kappa \approx (1 + 2\chi_M^{(\psi'J/\psi)}) \kappa. \] (32)

In the estimates in this section we use the value with the smaller error: \( \kappa_{\text{eff}}^{(\psi'J/\psi)} = 0.186 \pm 0.003 \pm 0.006 \).

A similar analysis \[10\] of the decay \( \Upsilon(2S) \rightarrow \pi^+\pi^- \Upsilon \) was inconclusive regarding the \( D_1 \) wave amplitude, whereas the fit for the dipion invariant mass distribution resulted in \( \kappa_{\text{eff}}^{(\Upsilon'\Upsilon)} = 0.146 \pm 0.006 \). Thus the data display a statistically significant decrease of the parameter \( \kappa_{\text{eff}} \) in bottomonium in comparison with the similar parameter in charmonium: \( \frac{\kappa_{\text{eff}}^{(\psi'J/\psi)}}{\kappa_{\text{eff}}^{(\Upsilon'\Upsilon)}} - 1 = 0.27 \pm 0.07 \). Some decrease of this type of the parameter \( \kappa \) was in fact predicted \[18\] based on the different characteristic scale for normalization of the gluonic operator in Eq.(19) relevant for the transitions in the two systems. Indeed, the characteristic
size of $\Upsilon$, $r_\Upsilon \sim 1 \text{ GeV}^{-1}$, is somewhat smaller than that of $J/\psi$, $r_{J/\psi} \sim (0.7 \text{ GeV})^{-1}$. However this effect is rather difficult to quantify, given that at both these scales the applicability of the perturbation theory in QCD is marginal. Considering also that the difference in the size is not large enough, especially in the logarithmic scale, in order to explain the observed difference in the values of $\kappa_{\text{eff}}$, one may explore another approach to the observed difference by neglecting the variation of the parameter $\kappa$ altogether and ascribing the observed effect to different values of the relativistic term $\chi_M$ in these two quarkonium systems.

Indeed, the relativistic parameter $v^2/c^2$ in the lowest states of bottomonium is only about one third of that in charmonium, as can be inferred e.g. by comparing the excitation energies relative to the mass: $(M_{\psi(2S)} - M_{J/\psi})/(M_{\psi(2S)} + M_{J/\psi}) \approx 3 (M_{\Upsilon(2S)} - M_{\Upsilon})/(M_{\Upsilon(2S)} + M_{\Upsilon})$. Thus one can also expect $\chi_M^{(\psi' J/\psi)} \approx 3 \chi_M^{(\Upsilon \Upsilon')}$, and assuming that the value of $\kappa$ in Eq.(19) is the same for the transitions in both quarkonia, one can estimate the relativistic effect of the M1 interaction from

$$\frac{\kappa_{\text{eff}}^{(\psi' J/\psi)}}{\kappa_{\text{eff}}^{(\Upsilon \Upsilon')}} - 1 \approx 2 \chi_M^{(\psi' J/\psi)} - 2 \chi_M^{(\Upsilon \Upsilon')} \approx 0.27 \pm 0.07 .$$

(33)

Taken at face value, such estimate gives $\chi_M^{(\psi' J/\psi)} \approx 0.20 \pm 0.05$ if $\chi_M^{(\Upsilon \Upsilon')}$ is set equal to $\frac{1}{3} \chi_M^{(\psi' J/\psi)}$ and in $\chi_M^{(\psi' J/\psi)} \approx 0.14$ if $\chi_M^{(\Upsilon \Upsilon')}$ is neglected altogether. In either case the difference between the specific numerical estimates is within the experimental and theoretical uncertainty, while the estimated value compares well with the expected magnitude of the relativistic effects in charmonium.

A more definite determination of the parameter $\chi_M^{(\psi' J/\psi)}$ could be enabled by observing the distribution in the dipion mass in the decay $\eta_c(2S) \rightarrow \pi^+ \pi^- \eta_c$. Indeed, as previously mentioned, the effect of the chromo-magnetic M1 interaction is enhanced in this decay by a factor of 3. If the value of $\chi_M$ for the charmonium transitions is in the estimated range $0.15 - 0.2$, the effective parameter $\kappa_{\text{eff}}^{(\eta_c \eta_c)}$ for the latter decay can amount to $0.3 - 0.45$. Although the prospect of measuring the $D_1$ wave in the decay is likely very remote, the effect of such larger value of $\kappa_{\text{eff}}$ should be visible in the more experimentally accessible spectrum of the dipion mass, as illustrated in Fig.1. As can be seen from the plots the larger value of $\kappa_{\text{eff}}^{(\eta_c \eta_c)}$ results in a significant suppression of the spectrum at low invariant mass, due to the zero of the $S$ wave amplitude at a higher, than for $\psi(2S) \rightarrow \pi^+ \pi^- J/\psi$, value of $q^2$. Also the total rate contains an overall suppression due to the factor $(1 - 3 \chi_M)^2$ vs. the factor $(1 - \chi_M)^2$ for the transition between the $^3S_1$ states.
Figure 1: The spectrum of the invariant masses of the two-pion system described by the equations (25) and (26) for the decay $\psi(2S) \rightarrow \pi\pi J/\psi$ (solid line) and for the decay $\eta_c(2S) \rightarrow \pi\pi\eta_c$ with $\chi_M = 0.15$ (dashed) and $\chi_M = 0.2$ (dash-dot). The rates are normalized to the total rate $\Gamma_0$ of the decay $\psi(2S) \rightarrow \pi\pi J/\psi$.

6 Effects of the final state interaction between pions

So far the amplitudes of the two-pion transitions were considered here in the chiral limit. The formulas in Eq. (19) and in Eqs. (23) and (24) are exact in the leading chiral order, i.e. as far as the quadratic terms in the pion momenta and mass are concerned. The only dynamical modification of these expressions can arise from the previously mentioned QCD renormalization effects. In particular these expressions get no corrections due to the final state interaction (FSI) between the pions. The latter interaction however can give rise to the terms whose expansion starts with the fourth power of momenta and the pion mass, and generally can modify the amplitude at momenta of the pions relevant for actual transitions in quarkonium. The effects of FSI in chiral treatment of the two-pion transitions in quarkonium were a matter of concern ever since the earlier theoretical analyses [3]. The effect in the phases of the amplitudes is well known: these phases for the production amplitudes are equal to the two-pion scattering phases in the corresponding partial waves: $S = |S| \exp(i\delta_0)$, $D = |D| \exp(i\delta_2)$, where the $I = 0$ phases for the $S$ wave, $\delta_0$, and for the $D$ wave, $\delta_2$ are quite well
studied.\(^6\) It is also generally estimated both on theoretical and phenomenological grounds that the FSI corrections are not big (at most 20 - 25\%) in the transitions \(\psi(2S) \rightarrow \pi\pi J/\psi\) and \(\Upsilon(2S) \rightarrow \pi\pi \Upsilon\). (For a discussion see the review [24].) Some phenomenological arguments in favor of such estimate will also be discussed further towards the end of this section, and we start with a theoretical estimate of the onset of the higher term in the chiral expansion.

The interaction of pions at low energy in the \(D\) wave is quite weak, so that any modification by FSI of the \(D\) wave production amplitude of Eq.(24) can be safely neglected, and only the modification of the \(S\) wave amplitude (28), \(\delta S_{\mu\nu\lambda\sigma}\), is of interest for present phenomenology. The imaginary part of the correction at \(q^2 > 4m^2\) is found from the unitarity relation in terms of the isospin I=0 \(\pi\pi \rightarrow \pi\pi\) scattering amplitude \(T(q^2)\) in the \(S\) wave as

\[
\text{Im} (\delta S_{\mu\nu\lambda\sigma}) = \sqrt{1 - 4m^2 \over q^2} \frac{T(q^2)}{16\pi} S_{\mu\nu\lambda\sigma} . \tag{34}
\]

The amplitude \(T(q^2)\) is well known in the chiral limit, i.e. in the quadratic in \(q\) and \(m\) approximation, since the work of Weinberg [26]. In the normalization used here this amplitude has the form

\[
T(q^2) = \frac{2q^2 - m^2}{f^2_\pi} , \tag{35}
\]

where \(f_\pi \approx 130\) MeV is the \(\pi^+ \rightarrow \mu^+\nu\) decay constant. Clearly, the expression in Eq.(34) is of the fourth power in \(q\) and \(m\).

The real part of \(\delta S_{\mu\nu\lambda\sigma}\) can then be estimated from Eq.(34) using the dispersion relation in \(q^2\) for the amplitude \(S\). In doing so one should set the condition for the subtraction constants that this real part does not contain quadratic (and certainly also constant) terms in \(q\) and \(m\), since these are given by Eq.(23). After these subtractions the remaining dispersion integral is still logarithmically divergent and contains the well known ‘chiral logarithm’, depending on the ultraviolet cutoff \(\Lambda\), which is usually set at \(\Lambda \sim 1\) GeV, i.e. the scale where any chiral expansion certainly breaks down. Using the equations (23), (34) and (35) one can readily estimate the first FSI correction with a ‘logarithmic accuracy’. The expression for the full \(S\) wave production amplitude then takes the form

\[
S_{\mu\nu\lambda\sigma} = 8\pi^2 \over 3b \left( (q^2 + m^2) (1 + \xi_1) (g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}) - \right. \tag{36}
\]

\(^6\)It can be mentioned that the analysis [9] of the data on the \(J/\psi \rightarrow \pi^+\pi^- J/\psi\) decay does not take into account the relative phase between the \(S\) and \(D\) wave pion production amplitudes. Thus it would be interesting to know whether including the phase factor in the angular analysis, produces a significant impact on the results.  

15
\[
\frac{3}{4} \kappa \left(1 + \frac{2m^2}{q^2}\right) \left(1 + \xi_2\right) \left[q_\mu q_\lambda g_{\nu\sigma} + q_\nu q_\sigma g_{\mu\lambda} - q_\nu q_\sigma g_{\mu\lambda} - q_\mu q_\sigma g_{\nu\lambda} - \frac{1}{2} q^2 (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda})\right]
\]

where the correction terms \(\delta_1\) and \(\delta_2\) to Eq. (23) are given as

\[
\xi_1 = \frac{2 (q^2)^2 - 7 q^2 m^2 + 3 m^4}{16\pi^2 f_\pi^2 (q^2 + m^2)} \ln \frac{\Lambda^2}{m^2} + i \frac{2 q^2 - m^2}{16\pi f_\pi^2} \sqrt{1 - \frac{4m^2}{q^2}},
\]

(37)

and

\[
\xi_2 = \frac{2 (q^2)^2 - 9 q^2 m^2 + 8 m^4}{16\pi^2 f_\pi^2 (q^2 + 2m^2)} \ln \frac{\Lambda^2}{m^2} + i \frac{2 q^2 - m^2}{16\pi f_\pi^2} \sqrt{1 - \frac{4m^2}{q^2}},
\]

(38)

where the non-logarithmic imaginary part is retained for reference regarding the normalization. The lower limit under the logarithm is generally a function of both \(q^2\) and \(m^2\), however any difference of this function from the value \(m^2\) used in Eqs. (37) and (38) is a non-logarithmic term, i.e. beyond the accuracy of these equations. Since \(m^2\) is the smallest of the two parameters in the physical region of pion production, it can be expected that using this parameter provides a conservative estimate of the effect of FSI.

Estimating the corrections in Eq. (37) and Eq. (38), one finds that at the lower end of the physical phase space, i.e. near \(q^2 = 4m^2\), these terms do not exceed few percent. Thus the corrections only weakly modify the normalization of the pion production amplitude near the threshold. A theoretical extrapolation to higher values of \(q^2\) is problematic, and, most likely, one would have to resort to using actual data on the dipion spectra in order to judge upon the significance of deviation from the essentially linear in \(q^2\) behavior of the amplitude described by Eq. (25). A quantitative estimate of the deviation from this behavior has been attempted [24] using the data [23] on \(\Upsilon(2S) \to \pi^+\pi^-\Upsilon\) by parametrizing the deviation as a factor \((1 + q^2/M^2)\) in the amplitude with \(M\) being a parameter. The thus obtained lower limit on \(M\) is 1 GeV at 90% C.L.

Another phenomenological argument in favor of a relatively moderate FSI effect in the absolute value of the dominant \(S\) wave in the decay \(\psi(2S) \to \pi^+\pi^-J/\psi\) stems from the previously mentioned agreement of the observed [22] value of the ratio \(D/S\) with the parameter \(\kappa\) entering the expression for the \(S\) wave and extracted from the two-pion invariant mass spectrum. Clearly such an agreement would be ruined if there was a significant enhancement of the \(S\) wave by FSI.

Furthermore, the observed ratio of the rates of the transitions \(\psi(2S) \to \eta J/\psi\) and \(\psi(2S) \to \pi^+\pi^-J/\psi\) reasonably agrees with the calculation [5], which neglects any FSI effects in the latter decay, while it is clear that there is no such effects in the emission of
one-particle, i.e. of $\eta$. The expected accuracy of the theoretical calculation is mostly that of applying the nonrelativistic limit to charmonium, i.e. $O(20\%)$ in the amplitude. Thus the theoretical (and experimental) uncertainty may allow for a presence of the FSI effects in the two-pion system at such level, however large effects of this type, like those recently claimed in Ref. [13], are definitely excluded.

Finally, one can notice in connection with the equations (27) and (28) for the $D_2$ and $D_3$ waves involving the polarizations of the quarkonium resonances, that the unknown quarkonium matrix elements all cancel in the ratio $D_2/D_3$. The $D_2$ amplitude is determined by the $S$ wave of the pions in their c.m. system, and the $D_3$ contains the pions in the $D$ wave relative to each other. Thus if the ratio of actual amplitudes $D_2/D_3$ could be measured from angular distributions, e.g. in the decay $\psi(2S) \rightarrow \pi^+\pi^- J/\psi$, this would produce direct data on the modification by FSI of the $S$ wave relative to the $D$ wave.

7 Resolving the $\Upsilon(3S) \rightarrow \pi\pi \Upsilon$ puzzle?

The decay $\Upsilon(3S) \rightarrow \pi\pi \Upsilon$ is known to be quite different from the "well behaved" transitions $\psi(2S) \rightarrow \pi^+\pi^- J/\psi$ and $\Upsilon(2S) \rightarrow \pi^+\pi^- \Upsilon$ in that the spectrum [11] of the dipion invariant masses in this decay has two distinct maxima at low and high values of $q^2$. The proposed solutions to the puzzle presented by this behavior included presence of a hypothetical exotic resonance [27, 17], breakdown of the multipole expansion [28], and unusual FSI effects [29]. In this section I exercise an alternative, and somewhat more conventional explanation of the observed behavior of the spectrum in terms of the considered here terms of the multipole expansion. Namely, one can readily verify that the dipion mass spectrum in the decay $\Upsilon(3S) \rightarrow \pi\pi \Upsilon$ is reasonably reproduced by the equations (25) - (28) if the parameters $\chi_M$ and $\chi_2$ for this decay are rather large: of order one, which is probably a result of a dynamical suppression of the leading nonrelativistic quarkonium amplitude $\alpha_0$.

The overall amplitude of the decay $\Upsilon(3S) \rightarrow \pi\pi \Upsilon$ is arguably strongly suppressed. Indeed, the observed absolute rate of the transition $\Upsilon(3S) \rightarrow \pi^+\pi^- \Upsilon$ is only about 0.2 of the rate of $\Upsilon(2S) \rightarrow \pi^+\pi^- \Upsilon$ in spite of a significantly larger available phase space. The estimated suppression of the former decay would be even stronger, if its amplitude was "well behaved", i.e. given by leading E1 interaction and thus dominantly proportional to $q^2$. Furthermore, the total rate of another similar transition from the same $\Upsilon(3S)$ resonance, $\Upsilon(3S) \rightarrow \pi\pi \Upsilon(2S)$, is about 0.6 of the rate of $\Upsilon(3S) \rightarrow \pi\pi \Upsilon$, even though the energy released in the former
transition is only slightly above the two-pion threshold, and the transition is very strongly kinematically suppressed. Thus it appears quite reasonable to conclude that the pure $S$ wave part of the quarkonium amplitude in Eq. (11), $\alpha_0$ is small for the discussed transition. This can be a result of cancellation in the (appropriately weighed) overlap of the wave functions determining this amplitude due to the oscillations of the $3S$ wave function. Such cancellation however is not necessarily present in the overlap terms due to the $3D_1 - 3S_1$ mixing, $\alpha_2$ in Eq. (5), and due to the M1 interaction, $\alpha_M$ in Eq. (9). The latter terms are naturally expected to be of the order $O(v^2/c^2)$ in comparison with an unsuppressed leading amplitude, e.g. the amplitudes of the “well behaved” transitions: $\Upsilon(2S) \rightarrow \pi\pi\Upsilon$ or $\Upsilon(3S) \rightarrow \pi\pi\Upsilon(2S)$. In other words, the ratios $\chi_M$ and $\chi_2$ (Eq. (29)) can be large due to the small denominator $\alpha_0$. A fully quantitative analysis of the decay $\Upsilon(3S) \rightarrow \pi\pi\Upsilon$ is complicated by that the energy released in the transition $W = M_{\Upsilon(3S)} - M_{\Upsilon} \approx 895$ MeV is rather large for a straightforward application of the chiral-limit formulas for the pions. However, for at least a qualitative estimate, I neglect here this complication and apply the equations (25) - (28) to evaluate the dipion mass spectrum. The resulting behavior is illustrated in Fig. 2, where the parameters are set as $\chi_M = 0.7$, $\chi_2 = 1.0$, and $\kappa = 0.13$. Clearly, the evaluated dipion mass spectrum closely resembles the experimentally observed [11], although no attempt is made here at a quantitative fit to experimental data.

In order to assess whether the values of the relativistic parameters $\chi_M$ and $\chi_2$ used in the plots of Fig. 2 are of the order of the expected relativistic effects in bottomonium, it is instructive to compare the corresponding amplitudes $\alpha_{M}^{(\Upsilon''\Upsilon)}$ and $\alpha_{2}^{(\Upsilon''\Upsilon)}$ with the amplitude of the transition, that appears to be ‘normal’, namely $\Upsilon(3S) \rightarrow \pi\pi\Upsilon(2S)$. In the latter transition the released energy is only 332 MeV, and according to the discussed approach it should be absolutely dominated by the $S$ wave amplitude given by Eq. (25). Also any FSI effects in this decay should be very small due to the proximity of the pions to the threshold. To certain extent this approach can be tested by comparing the rates of the transitions with the charged pions and with the neutral, by using respectively the mass of the charged and neutral pion in considering each of these decay modes. The effect of the pion mass difference is quite essential due to the small available energy, so that after numerical integration of the rate calculated from Eq. (25) one finds, in place of the isotopic ratio 2, the estimate

$$\frac{\Gamma[\Upsilon(3S) \rightarrow \pi^+\pi^-\Upsilon(2S)]}{\Gamma[\Upsilon(3S) \rightarrow \pi^0\pi^0\Upsilon(2S)]} \approx 1.26,$$

(39)

which is in the agreement with the data [30]: $B[\Upsilon(3S) \rightarrow \pi^+\pi^-\Upsilon(2S)] = (2.8 \pm 0.6)\%$ and
Figure 2: The spectrum of the invariant masses of the two-pion system as given by the equations (25) - (28) for the decay $\Upsilon(3S) \rightarrow \pi\pi\Upsilon$. Shown in the plot are the total distribution (thick line), and the contribution of individual partial waves: S (thin solid line), $D_1$ (dashed), $D_2$ (dash-dot), and $D_3$ (dotted).

$\mathcal{B}[\Upsilon(3S) \rightarrow \pi^0\pi^0\Upsilon(2S)] = (2.00 \pm 0.32)\%$. The integral over the spectrum of either of the modes of transition $\Upsilon(3S) \rightarrow \pi\pi\Upsilon(2S)$ can then be compared with the result of the numerical integration over the spectrum of the decay $\Upsilon(3S) \rightarrow \pi^+\pi^-\Upsilon$ produced by the amplitudes in the equations (25) - (28) which gives the ratio of the rates of the two transitions from $\Upsilon(3S)$ in terms of the ratio of the squares of the corresponding amplitudes $\alpha_0$ and can be compared with the data. Performing this calculation with the values of the parameters $\chi^{(\Upsilon'')}_{M} = 0.7$ and $\chi^{(\Upsilon'')}_{2} = 1.0$ used in the plots of Fig.2, gives the estimate

$$
\frac{\alpha_0^{(\Upsilon'\Upsilon)}}{\alpha_0^{(\Upsilon''\Upsilon)}} \approx \frac{\alpha_2^{(\Upsilon'\Upsilon)}}{\alpha_0^{(\Upsilon''\Upsilon)}} \approx 0.06 , \quad \frac{\alpha_M^{(\Upsilon'\Upsilon)}}{\alpha_0^{(\Upsilon''\Upsilon)}} \approx 0.04 ,
$$

which certainly falls in the range of naturally expected magnitude of the relativistic terms in bottomonium: $v^2/c^2 \sim (M_{\Upsilon(2S)} - M_{\Upsilon})/(M_{\Upsilon(2S)} + M_{\Upsilon}) \approx 0.06$, and also quantifies the dynamical suppression of the leading nonrelativistic amplitude $\alpha_0$ in the transition $\Upsilon(3S) \rightarrow$.

---

7In this estimate the value of $\kappa_{eff}$ used is 0.146. Due to the small energy in the process, the estimate is quite insensitive to this particular value as long as $\kappa_{eff}$ is small.
However, given the large energy release in this transition it is rather difficult to quantify at present the FSI effect on the estimate [40]. In either event, the suggested explanation of the observed peculiarity of the decay $\Upsilon(3S) \to \pi\pi\Upsilon$, heavily relies on the relative prominence of the spin-dependent $D$ waves, $D_2$ and $D_3$, in this decay, as can be seen from the plots of Fig.2. The relative contribution of these two waves is the largest near the minimum in the invariant mass distribution at $m_{\pi\pi} \approx 0.65$ GeV, which is the main experimentally testable qualitative prediction of the suggested mechanism.

It should be mentioned in connection with the suggested presence of the spin-dependent $D$ waves that an experimental search for quarkonium polarization effects in $\Upsilon(3S) \to \pi\pi\Upsilon$ has been done [11]. It was found that the data were consistent at 65% confidence level “with the expectation that the daughter $\Upsilon(1S)$ retains the polarization of the parent $\Upsilon(3S)$ along the beam axis”. It thus appears that the test has not been sensitive enough and a more detailed study of the bottomonium polarization effects in this transition is needed.

8 Summary

In summary. Within the standard approach to two-pion transitions in heavy quarkonium, based on the multipole expansion in QCD and chiral dynamics of the pions, the first relativistic terms of order $v^2/c^2$ in the transition amplitudes arise from the effects of the $^3D_1 - ^3S_1$ mixing in the part determined by the E1 interaction and from the chromo-magnetic M1 interaction. These terms are parametrized by the quantities $\alpha_2$ (Eq.(5)) and $\alpha_M$ (Eq.(9)). The significance of the chromo-magnetic term in two-pions transitions in charmonium can be approximately estimated from the available data on the dipion invariant mass spectrum in the decays $\psi(2S) \to \pi\pi J/\psi$ and $\Upsilon(2S) \to \pi\pi\Upsilon$. The estimated value is 0.15 - 0.2 of the dominant nonrelativistic amplitude. The effect of the chromo-magnetic term is enhanced in the yet unobserved transition $\eta_c(2S) \to \pi\pi\eta_c$ and is expected to significantly distort the spectrum in the latter decay and also result in a suppression of its rate, as shown in Fig.1. The absolute determination of the leading nonrelativistic amplitude in charmonium, the chromo-polarizability, is of interest for other applications, e.g. the charmonium scattering on nuclei. Such determination generally suffers from FSI effects of the two-pion rescattering. These effects are estimated as a next term in the chiral expansion and amount to only few percent at a low invariant mass of the two-pion system near the threshold. An extrapolation to higher values of $q^2$ can be done using experimental data. With the presently available
data there is no indication of a large FSI effect. Furthermore, an agreement of the chiral-limit formulas with the data on the $D$ wave in the decay $\psi(2S) \rightarrow \pi\pi J/\psi$, as well as the agreement with the data of the theoretical prediction for the ratio of the rates of the decays $\psi(2S) \rightarrow \eta J/\psi$ and $\psi(2S) \rightarrow \pi\pi J/\psi$, suggest that the FSI effects may amount to at most a moderate fraction of the amplitude of the transition in charmonium. An experimental measurement of the ratio of the $D_2$ wave (Eq.(27)) and the $D_3$ wave (Eq.(28)) in $\psi(2S) \rightarrow \pi\pi J/\psi$ would provide a direct test of the FSI effect. The relativistic terms in the two-pion transitions may hold the clue to solving the puzzle of the unusual dipion mass spectrum observed in the transition $\Upsilon(3S) \rightarrow \pi\pi \Upsilon$, if the leading nonrelativistic quarkonium matrix element in this transition is strongly suppressed due to details of bottomonium wave functions. Although a detailed quantitative description of the latter decay is not yet attainable within the present knowledge, the suggested mechanism necessarily predicts a noticeable presence of the polarization-dependent $D_2$ and $D_3$ waves in the amplitude, which prediction can be tested experimentally.

Acknowledgements

I thank Misha Shifman and Arkady Vainshtein for illuminating discussions. This work is supported in part by the DOE grant DE-FG02-94ER40823.

References

[1] G. S. Abrams et. al., Phys. Rev. Lett. 34, 1181 (1975).

[2] M. B. Voloshin, JETP Lett. 21, 347 (1975) [Pisma Zh. Eksp. Teor. Fiz. 21, 733 (1975)].

[3] L. S. Brown and R. N. Cahn, Phys. Rev. Lett. 35, 1 (1975).

[4] K. Gottfried, Phys. Rev. Lett. 40, 598 (1978).

[5] M. B. Voloshin and V. I. Zakharov

[6] M. B. Voloshin, Phys. Lett. B 562, 68 (2003) [arXiv:hep-ph/0302261].

[7] A. Sibirtsev and M. B. Voloshin, Phys. Rev. D 71, 076005 (2005) [arXiv:hep-ph/0502068].
[8] A. B. Kaidalov and P. E. Volkovitsky, Phys. Rev. Lett. 69, 3155 (1992).

[9] J. Z. Bai et al. [BES Collaboration], Phys. Rev. D 62, 032002 (2000) \texttt{arXiv:hep-ex/9909038}.

[10] J. P. Alexander et al. [CLEO Collaboration], Phys. Rev. D 58, 052004 (1998) \texttt{arXiv:hep-ex/9802024}.

[11] F. Butler et al. [CLEO Collaboration], Phys. Rev. D 49, 40 (1994).

[12] S. E. Csorna et al. [CLEO Collaboration], Report CLEO CONF 02-06, \texttt{arXiv:hep-ex/0207060}.

[13] F. K. Guo, P. N. Shen and H. C. Chiang, \texttt{arXiv:hep-ph/0604252}.

[14] M. B. Voloshin, Nucl. Phys. B 154, 365 (1979).

[15] M. B. Voloshin, Mod. Phys. Lett. A 19, 665 (2004) \texttt{arXiv:hep-ph/0402011}.

[16] J. L. Rosner, Phys. Rev. D 64, 094002 (2001) \texttt{arXiv:hep-ph/0105327}.

[17] G. Belanger, T. A. Degrard and P. Moxhay, Phys. Rev. D 39, 257 (1989).

[18] V. A. Novikov and M. A. Shifman, Z. Phys. C 8, 43 (1981).

[19] M. B. Voloshin and A. D. Dolgov, Sov. J. Nucl. Phys. 35, 120 (1982) [Yad. Fiz. 35, 213 (1982)].

[20] M. B. Voloshin, Sov. J. Nucl. Phys. 44, 478 (1986) [Yad. Fiz. 44, 738 (1986)].

[21] R. S. Chivukula, A. G. Cohen, H. Georgi and A. V. Manohar, Phys. Lett. B 222, 258 (1989).

[22] R. S. Chivukula, A. G. Cohen, H. Georgi, B. Grinstein and A. V. Manohar, Annals Phys. 192 (1989) 93.

[23] H. Albrecht et al. [ARGUS Collaboration], Z. Phys. C 35, 283 (1987).

[24] M. B. Voloshin and Y. M. Zaitsev, Sov. Phys. Usp. 30, 553 (1987) [Usp. Fiz. Nauk 152, 361 (1987)].
[25] M. B. Voloshin, Mod. Phys. Lett. A 17, 1533 (2002) [arXiv:hep-ph/0206240].

[26] S. Weinberg, Phys. Rev. Lett. 17, 616 (1966).

[27] M. B. Voloshin, JETP Lett. 37, 69 (1983) [Pisma Zh. Eksp. Teor. Fiz. 37, 58 (1983)].

[28] P. Moxhay, Phys. Rev. D 39, 3497 (1989).

[29] S. Chakravarty, S. M. Kim and P. Ko, Phys. Rev. D 50, 389 (1994) [arXiv:hep-ph/9310376].

[30] S. Eidelman et al. [Particle Data Group], Phys. Lett. B 592, 1 (2004).