PARABOLIC MINKOWSKI CONVOLUTIONS OF VISCOSITY SOLUTIONS TO FULLY NONLINEAR EQUATIONS

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Abstract. This paper is concerned with the Minkowski convolution of viscosity solutions of fully nonlinear parabolic equations. We adopt this convolution to compare viscosity solutions of initial-boundary value problems in different domains. As a consequence, we can for instance obtain parabolic power concavity of solutions to a general class of parabolic equations. Our results apply to the Pucci operator, the normalized $q$-Laplacians with $1 < q \leq \infty$, the Finsler Laplacian and more general quasilinear operators.

1. Introduction

1.1. Background and motivation. This paper is connected to a general theory devised for the elliptic case in [44] and extended to the parabolic framework by two of the authors. In particular here we extend the results in [29] and [30] to a general class of fully nonlinear parabolic equations in the framework of viscosity solutions. In connection with the general theory of [44] and with the results and techniques of this paper, we also address the reader to the twin paper [22], where we consider spatial concavity properties as well as Brunn-Minkowski type inequalities for parabolic and elliptic problems.

Let us first describe the basic setting of our problem and introduce its background.

Let $m \geq 2$ and $n \geq 1$. For any $i = 1, 2, \ldots, m$, let $\Omega_i$ be a bounded smooth domain in $\mathbb{R}^n$. Let $\nu_i$ denote the inward unit normal vector to $\partial \Omega_i$. For any

$$
\lambda \in \Lambda_m = \left\{ (\lambda_1, \ldots, \lambda_m) \in (0, 1)^m : \sum_{i=1}^m \lambda_i = 1 \right\},
$$

let $\Omega_\lambda$ be the Minkowski combination of $\Omega_i$, defined by

$$
\Omega_\lambda = \sum_{i=1}^m \lambda_i \Omega_i = \left\{ \sum_{i=1}^m \lambda_i x_i : x_i \in \Omega_i, i = 1, 2, \ldots, m \right\}. \tag{1.1}
$$

It is easy to see that $\Omega_\lambda$ is bounded in $\mathbb{R}^n$. Please notice that when $\Omega_i = \Omega$ for $i = 1, \ldots, m$, we have of course $\Omega \subseteq \Omega_\lambda$, but the inclusion is in general strict unless $\Omega$ is convex. Hereafter for simplicity we set $Q_i = \Omega_i \times (0, \infty)$ and $\partial Q_i = (\partial \Omega_i \times (0, \infty)) \cup (\Omega_i \times \{0\})$ for $i = \lambda, 1, \ldots, m$. Our first aim is to connect the solution $u_\lambda$ of some Cauchy-Dirichlet problem in $\Omega_\lambda$ to the solutions $u_1, \ldots, u_m$ of similar (but not necessarily the same) Cauchy-Dirichlet problems in $\Omega_1, \ldots, \Omega_m$. 

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In particular, for \( i = \lambda \) and \( i = 1, 2, \ldots, m \), let us consider the following fully nonlinear Cauchy-Dirichlet problems:

\[
\begin{aligned}
\begin{cases}
\partial_t u + F_i(x, t, u, \nabla u, \nabla^2 u) = 0 & \text{in } Q_i, \\
u = 0 & \text{on } \partial Q_i,
\end{cases}
\end{aligned}
\tag{1.2}
\]

where \( F_i : \overline{Q}_i \times [0, \infty) \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n \to \mathbb{R} \) for \( i = \lambda, 1, 2, \ldots, m \) are given continuous elliptic operators, with \( F_\lambda \) suitably related to \( F_1, \ldots, F_m \). As we said, we are interested in finding some kind of relationships (which we will clarify later) between the solution of problem (1.2)–(1.3) with \( i = \lambda \) and the solutions with \( i = 1, \ldots, m \).

Let \( u_i \) be a positive solution of (1.2)–(1.3) in \( Q_i \) for every \( i = 1, 2, \ldots, m \). Let \( 1/2 \leq \alpha \leq 1 \) and \( p < 1 \) be two given parameters and define the \( \alpha \text{-parabolic Minkowski p-convolution} \) of \( \{u_i\}_{i=1}^m \) for any \( \lambda \in \Lambda_m \) as follows:

\[
U_{p, \lambda}(x, t) := \sup \left\{ M_p \left( (u_1(x_1, t_1), \ldots, u_m(x_m, t_m); \lambda) : (x_i, t_i) \in \overline{Q}_i, \right\},
\]

where, for given \( \lambda \in \Lambda_m \) and \( p \in [-\infty, +\infty] \), \( M_p(a_1, \ldots, a_m; \lambda) \) denotes the usual weighted \( p \)-means (with weight \( \lambda \) of \( a = (a_1, \ldots, a_m) \in [0, \infty]^m \), whose precise definition is given later in (2.1).

As shown in [30], when the equations are semilinear with \( F_i \) of the form

\[
F_i(x, t, r, \xi, X) = -\text{tr} X - f_i(x, t, r, \xi), \quad i = \lambda, 1, \ldots, m,
\tag{1.5}
\]

then, under suitable assumptions on the behavior of the \( u_i \)'s on \( \partial Q_i \)'s, \( U_{p, \lambda} \) is a subsolution of (1.2)–(1.3) with \( i = \lambda \), provided that \( f_\lambda \) and \( \{f_i\}_{i=1}^m \) satisfy

\[
g_\lambda \left( \sum_i \lambda_i x_i, \sum_i \lambda_i t_i, \sum_i \lambda r_i, \xi \right) \geq \sum_{i=1}^m \lambda_i g_i(x_i, t_i, r_i, \xi)
\tag{1.6}
\]

for any fixed \( \xi \in \mathbb{R}^n \) and any \( (x_i, t_i, r_i) \in Q_i \times (0, \infty) \), where

\[
g_i(x, t, r, \xi) = r^{\frac{p}{p-1}} f_i \left( x, t \frac{1}{p}, r \frac{1}{p}, \frac{1}{p} r \frac{1}{p-1} \xi \right), \quad i = \lambda, 1, \ldots, m.
\tag{1.7}
\]

This, coupled with a comparison principle for (1.2), results in a comparison between the solution of the problem in \( \Omega_\lambda \) with the solutions in the \( \Omega_i \)'s, \( i = 1, \ldots, m \), which consists in a sort of concavity principle for the solutions of the involved problems with respect to the Minkowski combination of the underlying domains. When the domains \( \Omega_1, \ldots, \Omega_m \) differ from each other, interesting applications are Brunn-Minkowski type inequalities for possibly connected functionals. For this, we refer to [44] and to the bibliography therein for the elliptic case, and to [30] for the parabolic case.

Notice that the condition (1.6) can be interpreted as a comparison relation between \( f_\lambda \) and a certain type of concave combination of the \( f_i \)'s \( (i = 1, 2, \ldots, m) \) under the transformation (1.7).

When all the \( \Omega_i \)'s coincide with a convex domain \( \Omega \) and all \( f_i \) are the same for \( i = \lambda, 1, \ldots, m \), all the problems clearly reduce to a single one. Then in this case the above
result, combined with a comparison principle for (1.2)–(1.3), immediately implies that the
unique solution \( u \) of such an equation is \( \alpha \)-parabolically \( p \)-concave in the sense that
\[
\begin{align*}
u \left( \sum_i \lambda_i x_i, M_\alpha(t_1, \ldots, t_m; \lambda) \right) & \geq M_p(u_1(x_1, t_1), \ldots, u_m(x_m, t_m); \lambda) .
\end{align*}
\] (1.8)

This type of concavity results was established in [29] and [30] (see also [27, 28]). Note that
(1.6) then turns into a concavity assumption for \( g_\lambda \).

When the \( \Omega_i \)'s truly differ from each other, then our result can be used to obtain Brunn-
Minkowski type inequalities for related functionals, as it will be more explicitly described
in [22] and has been already done in [30] in the parabolic framework and similarly, suitably
treating different specific cases, in [12, 9, 11, 10, 42, 39, 43, 7] in the elliptic case. Notice
that a general theory (for elliptic problems) is developed in [44], where however only classical
solutions and convex domains were considered, although all the results therein did not really
need convexity of the involved domains. And indeed non convex domains has been explicitly
treated in [30].

The purpose of this paper is to extend the results described above to a more general setting.
Our generalization lies at the following three aspects. First, we study the problem for a general
class of fully nonlinear parabolic equations, which certainly includes the known semilinear
case. We even allow the equations to bear mild singularity caused by vanishing gradient. By
“mild singularity”, we mean that for each \( i = \lambda, 1, \ldots, m \), there exists a continuous function
\( h_i : \overline{Q}_i \times [0, \infty) \rightarrow \mathbb{R} \) such that
\[
h_i(x, t, r) = (F_i)_s(x, t, r, 0, 0) = (F_i)^*(x, t, r, 0, 0) \quad \text{for} \quad (x, t, r) \in \overline{Q}_i \times [0, \infty) \quad \text{(1.9)}
\]
where \( (F_i)_s \) and \( (F_i)^* \) respectively stand for the lower and upper semicontinuous envelopes of
\( F_i \). Our results apply to several important types of nonlinear operators including the Pucci
operator, the normalized \( q \)-Laplacians (\( 1 < q \leq \infty \)) and more general quasilinear operators.

Secondly, in accordance to our generalization of the equations, another significant con-
tribution of this paper is that we use the weaker notion of viscosity solutions rather than
the classical solutions. We thus manage to reduce the \( C^2 \) regularity of the solutions in the
main theorems of [29, 30]. Let us emphasize that it is indeed possible to investigate spatial
convexity of solutions in the framework of viscosity theory; we refer to [17, 19, 1, 33, 40] for
viscosity techniques in different contexts and to [37, 38, 35, 6, 25, 26, 23, 24] etc for related
results for classical solutions. Our current work provides new results on parabolic power
concavity of viscosity solutions, which are not considered in the aforementioned references
(but let us point out that, right after completing this work, we have learnt also about [14],
where viscosity solutions, in the elliptic case, have been now considered).

Third, we allow more freedom to the parameters \( \alpha \) and \( p \), so that, depending on the
involved operators, we can consider \( \alpha \in [0, 1] \) and \( p \in (-\infty, 1] \). Notice that, although there
is no special difficulty, negative power concavity properties have not been explicitly treated
before to our knowledge.

Throughout this paper we assume the following fundamental well-posedness results for any
\( i = \lambda, 1, \ldots, m \).

- There exists a unique viscosity solution, locally Lipschitz in space, to (1.2)–(1.3).
- The comparison principle holds for (1.2)–(1.3), at least for \( i = \lambda \); that is, if \( u_\lambda \) and
  \( v_\lambda \) are respectively an upper semicontinuous subsolution and a lower semicontinuous
  supersolution satisfying \( u_\lambda \leq v_\lambda \) on \( \partial Q_\lambda \), then \( u_\lambda \leq v_\lambda \) in \( Q_\lambda \).
We refer to [13] and [18] for existence and uniqueness of viscosity solutions of (1.2)--(1.3). For the reader’s convenience, in Appendix (Section A.1) we list more precise structure assumptions on the \( F_i \) besides (1.9), which guarantee the comparison principle; see more details also in [13, Theorem 8.2] and [18, Theorem 3.6.1]. On the other hand, showing local Lipschitz regularity of the unique solution requires extra work and further assumptions on \( F_i \). We refer to the extensive literature on this subject in the context of viscosity solutions, for example [3, 45, 46, 36, 5, 4, 2] and the references therein.

1.2. Assumptions and main result. Our main result is based on a condition connecting \( F_i \) and \( F_i \) (i = 1, 2, . . . , m), which generalizes (1.6) in the fully nonlinear setting. In order to give a clear view of this condition, we introduce the following transformed operators with a parameter \( k \in \mathbb{R} \). Given \( p < 1 \) and \( \alpha \in (0, 1) \), let \( G_{i, k}^{p, \alpha} : \mathcal{Q}_i \times [0, \infty) \times (\mathbb{R}^n \setminus \{0\}) \times S^n \to \mathbb{R} \) be defined as follows for every \( i = \lambda, 1, \ldots, m \):

\[
G_{i, k}^{p, \alpha}(x,t,r,\xi,X) = r^k F_i \left( x, t^\frac{1}{p}, r^\frac{1}{p} \xi, r^{-1} X + \frac{1}{p^2 r^{1-2/p}} \xi \otimes \xi \right) \quad \text{if } p \neq 0,
\]

\[
G_{i, k}^{0, \alpha}(x,t,r,\xi,X) = e^{k r} F_i \left( x, t^\frac{1}{p}, e^r \xi, e^r (X + \xi \otimes \xi) \right) \quad \text{if } p = 0,
\]

for all \( x, t, r, \xi, X \in \mathcal{Q}_i \times (0, \infty) \times (\mathbb{R}^n \setminus \{0\}) \times S^n \).

To apply our method, we need to find \( k \in \mathbb{R} \) satisfying the following two key assumptions (H1), (H2), and (H2), \( p \);

(H1) \( p \), If \( p \neq 0 \), the parameter \( k \in \mathbb{R} \) satisfies

\[
either \frac{1}{p} - 1 + k \leq 0 \quad \text{or} \quad \alpha \left( \frac{1}{p} - 1 + k \right) \geq 1. \tag{1.11}
\]

(H2) \( p \), For any \( \lambda \in \Lambda \) and any \((x, t) \in Q_\lambda, r \geq 0, \xi \in \mathbb{R}^n \setminus \{0\}, Y \in S^n \),

\[
G_{\lambda, k}^{p, \alpha}(x,t,r,\xi,Y) \leq \sum_{i=1}^{m} \lambda_i G_{i, k}^{p, \alpha}(x_i, t_i, r_i, \xi, X_i) \tag{1.12}
\]

holds for \((x_i, t_i, r_i, X_i) \in Q_i \times (0, \infty) \times S^n (i = 1, 2, \ldots, m) \) satisfying

\[
\sum_{i} \lambda_i x_i = x, \quad \sum_{i} \lambda_i t_i = t, \quad \sum_{i} \lambda_i r_i = r \tag{1.13}
\]

and

\[
\text{sgn}^*(p) \begin{pmatrix}
\lambda_1 X_1 \\
\lambda_2 X_2 \\
\vdots \\
\lambda_m X_m
\end{pmatrix} \leq \text{sgn}^*(p) \begin{pmatrix}
\lambda_1^2 Y \\
\lambda_1 \lambda_2 Y \\
\vdots \\
\lambda_1 \lambda_m Y
\end{pmatrix},
\]

\[
\begin{pmatrix}
\lambda_2^2 Y \\
\lambda_2 \lambda_1 Y \\
\vdots \\
\lambda_2 \lambda_m Y
\end{pmatrix} \leq \begin{pmatrix}
\lambda_1^2 Y \\
\lambda_1 \lambda_2 Y \\
\vdots \\
\lambda_1 \lambda_m Y
\end{pmatrix},
\]

\[
\begin{pmatrix}
\lambda_3^2 Y \\
\lambda_3 \lambda_2 Y \\
\vdots \\
\lambda_3 \lambda_m Y
\end{pmatrix} \leq \begin{pmatrix}
\lambda_2^2 Y \\
\lambda_2 \lambda_1 Y \\
\vdots \\
\lambda_2 \lambda_m Y
\end{pmatrix},
\]

\[
\begin{pmatrix}
\lambda_m^2 Y \\
\lambda_m \lambda_2 Y \\
\vdots \\
\lambda_m \lambda_m Y
\end{pmatrix} \leq \begin{pmatrix}
\lambda_m^2 Y \\
\lambda_m \lambda_2 Y \\
\vdots \\
\lambda_m \lambda_m Y
\end{pmatrix},
\]

where

\[
\text{sgn}^*(p) = \begin{cases}
1 & \text{if } p \geq 0, \\
-1 & \text{if } p < 0.
\end{cases}
\]

We emphasize that when \( p = 0 \), condition (1.11) can be removed, i.e. we can take any \( k \in \mathbb{R} \). The condition (H1), \( p \) is equivalent to requiring the function \( g_k(r, t) = r^\frac{1}{p^2 - 1 - k^2 - 1 - \alpha} \) (when \( p \neq 0 \), while \( g_k(r, t) = e^{(k + 1)^2 r t^\frac{1}{p^2 - 1 - k^2 - 1 - \alpha}} \) when \( p = 0 \)) to be convex in \((0, \infty)^2\).
The reason for us to impose (H2)$_p$ in the form involving $G_{i,k}^{p,\alpha}$ rather than $F_i$ is that we will later transform our equation (1.2) into another form, which is more compatible with our convexity argument. The operator $G_{i,k}^{p,\alpha}$ appears in the new equation. The term $\text{sgn}^\ast(p)$ is needed in (1.14), since for the transformed equation we will consider subsolutions when $p \geq 0$ but supersolutions when $p < 0$.

Before stating our main result, we set

$$
\tilde{\nu}_i(x) := \begin{cases} 
\nu_i(x) & \text{if } x \in \partial \Omega_i, \\
0 & \text{if } x \in \Omega_i,
\end{cases} \quad \text{and} \quad \mu(t) := \begin{cases} 
1 & \text{if } t = 0, \\
0 & \text{if } t > 0.
\end{cases}
$$

(1.15)

**Theorem 1.1** (Subsolution property of Minkowski convolution). Fix $\lambda \in \Lambda_m$. Assume that $\Omega_i$ is a bounded smooth domain in $\mathbb{R}^n$ for any $i = 1, 2, \ldots, m$. Let $\Omega_{\lambda}$ be the Minkowski combination of $\{\Omega_i\}_{i=1}^m$ as defined in (1.1). Let $0 < \alpha \leq 1$ and $p < 1$. Suppose that there exists $k \in \mathbb{R}$ such that (H1)$_p$ and (H2)$_p$ hold. Let $u_i$ be the unique solution of (1.2)–(1.3) that is positive and locally Lipschitz in space in $Q_i$ for $i = 1, 2, \ldots, m$. Assume in addition that for any $i = 1, 2, \ldots, m$,

(i) $u_i$ is monotone in time, i.e.,

$$u_i(x, t) \geq u_i(x, s) \quad \text{for any } x \in \Omega_i \text{ and } t \geq s \geq 0; \quad (1.16)$$

(ii) if $0 < p \leq 1$, then

$$\frac{1}{p}u_i^p(x + \tilde{\nu}_i(x)\rho, t + \mu(t)p^{1/\alpha}) \to \infty \quad \text{as } \rho \to 0+ \quad \text{for any } (x, t) \in \partial Q_i.
$$

(1.17)

Then $U_{p,\lambda}$ as in (1.4) is a subsolution of (1.2)–(1.3) with $i = \lambda$.

We can use our general result to cover [30, Theorem 3.2]. Indeed, if $p \neq 0$, by taking $k = 3 - 1/p$ we get

$$G_{i,k}^{p,\alpha}(x, t, r, \xi, X) = -\frac{1}{p}r^2 \text{tr} X - \frac{1-p}{p^2}r - g_i(x, t, r, \xi)
$$

(1.18)

for all $(x, t, r, \xi, X) \in \Omega_{\lambda}$. We can verify the assumption (H2)$_p$ in Theorem 1.1 holds with the choice $k = 3 - 1/p$ and the condition (1.6). In the case $p = 0$, we can choose $k = 1$ to show that the same result holds under condition (1.6) but with

$$g_i(x, t, r, \xi) = e^r f_i \left(x, t, \mathbf{1}^r, e^r \xi \right), \quad i = \lambda, 1, \ldots, m.
$$

(1.19)

See more details in Section 5.1.

Compared to the key conditions (H1)$_p$ and (H2)$_p$, the additional assumptions (i)–(ii) are more technical. Notice however that assumption (1.17) is not needed for $p \leq 0$. Moreover, for $p \in (0, 1)$, even if in applications $F_i$ may not fulfill (i)–(ii), we can fix the issue by perturbing $F_i$ with a small $\varepsilon > 0$ as

$$F_{i,\varepsilon} = F_i - \varepsilon \quad (i = 1, 2, \ldots, m); \quad (1.20)
$$

in other words, we instead consider the equation

$$\partial_t u + F_{i,\varepsilon}(x, t, u, \nabla u, \nabla^2 u) = 0 \quad \text{in } Q_i.
$$

(1.21)

It turns out that such perturbation meet our needs in most of our applications. For $p \in (0, 1)$ we can prove (i) and (ii) for (1.21) with a larger class of parabolic operators $F_i$; see Appendix A.2 and Appendix A.3 for clarification. Such a perturbation causes no harm to the applications of our main results, since all of the other assumptions continue to hold.
in Theorem 1.1 with $F_i$ replaced by $F_i,\varepsilon$. We can still obtain the desired results by first considering the approximate problem (1.21) and then passing to the limit as $\varepsilon \to 0$ by standard stability theory. Let us finally notice that, although Theorem 1.1 holds the same also for $p = 1$, in this case it is very hard to get assumption (1.17), which would require $u_i$ to have vertical slope on the boundary (and indeed it is very hard to have concave solutions).

Our proof of Theorem 1.1 is based on the following two steps. We first take

$$v_i(x,t) = \begin{cases} u_i^p(x,t^{\frac{1}{p}}) & \text{if } p \neq 0 \\ \log u(x,t^{\frac{1}{p}}) & \text{if } p = 0 \end{cases}$$

(1.22)

for all $i = \lambda, 1, \ldots, m$. It is not difficult to verify, at least formally, that $v_i$ solves

$$\alpha v_i^{\frac{1}{p}} t^{1-\frac{1}{p}} \partial_t v_i + F_i \left( x, t^{\frac{1}{p}}, v_i^p, \frac{1}{p} v_i^{\frac{1}{p}-1} \nabla v_i, \frac{1}{p} v_i^{\frac{1}{p}-1} \nabla^2 v_i + \frac{1}{p^2} v_i^{\frac{1}{p}-2} \nabla v_i \otimes \nabla v_i \right) = 0$$

(1.23)

if $p \neq 0$ and

$$e^{v_i} t^{\frac{1}{p}} \partial_t v_i + F_i \left( x, t^{\frac{1}{p}}, e^{v_i}, e^{v_i} \nabla v_i, e^{v_i} \nabla^2 v_i + e^{v_i} \nabla v_i \otimes \nabla v_i \right) = 0$$

(1.24)

if $p = 0$, which are respectively equivalent to

$$v_i^{\frac{1}{p}} t^{1-\frac{1}{p}} \partial_t v_i(x,t) + \frac{1}{\alpha} G^{p,\alpha}_{i,k}(x,t,v_i(x,t),\nabla v_i(x,t),\nabla^2 v_i(x,t)) = 0$$

(1.25)

and

$$e^{(k+1)v_i} t^{\frac{1}{p}} \partial_t v_i(x,t) + \frac{1}{\alpha} G^{0,\alpha}_{i,k}(x,t,v_i(x,t),\nabla v_i(x,t),\nabla^2 v_i(x,t)) = 0$$

(1.26)

for any given parameter $k \in \mathbb{R}$. Here $G^{p,\alpha}_{i,k}$ is given by (1.10). In Section 3, we rigorously show that $u_i$ is a viscosity subsolution of (1.2) if and only if $v_i$ is a viscosity subsolution (resp., supersolution) of (1.25) when $p \geq 0$ (resp., $p < 0$).

After such a transformation, we next take the Minkowski convolution of $v_i$’s as follows:

$$V_{p,\lambda}(x,\tau) := \begin{cases} \sup \left\{ \sum_{i=1}^{m} \lambda_i v_i(x_i,\tau_i) : (x_i,\tau_i) \in \bar{Q}_i, \; x = \sum_i \lambda_i x_i, \; \tau = \sum_i \lambda_i \tau_i \right\} & \text{if } p \geq 0 \\
\inf \left\{ \sum_{i=1}^{m} \lambda_i v_i(x_i,\tau_i) : (x_i,\tau_i) \in \bar{Q}_i, \; x = \sum_i \lambda_i x_i, \; \tau = \sum_i \lambda_i \tau_i \right\} & \text{if } p < 0 \end{cases}$$

(1.27)

for every $(x,\tau) \in \bar{Q}_\lambda$. It is clear that

$$V_{p,\lambda}(x,\tau) = \begin{cases} U_{p,\lambda}(x,\tau^{\frac{1}{p}})^p & \text{if } p \neq 0 \\ \log U_{p,\lambda}(x,\tau^{\frac{1}{p}}) & \text{if } p = 0 \end{cases}$$

To prove Theorem 1.1, it thus suffices to prove that $V_{p,\lambda}$ is a subsolution if $p \geq 0$ or a supersolution if $p < 0$ of (1.25) with $i = \lambda$. The rest of the proof is inspired by [1], where the supersolution property is studied for the convex envelope of viscosity solutions to fully nonlinear elliptic equations with state constraint or Dirichlet boundary conditions. The key is to establish a relation between the semijets (weak derivatives) of $v_i$ and $V_{p,\lambda}$, which combined with (H1)$_p$–(H2)$_p$, leads to the desired conclusion.

1.3. Applications to parabolic power concavity. We can use Theorem 1.1 to study the parabolic power concavity of viscosity solutions to a general class of fully nonlinear parabolic equations. More precisely, when $F_i = F_\lambda$ and $\Omega_i = \Omega_\lambda$ for all $i = 1, 2, \ldots, m$ with $m = n + 2$,
assuming the convexity of $\Omega$, we can apply the above result to the unique solution $u$ of (1.2)–(1.3) with $i = \lambda$ to deduce that

$$u^*(x,t) := \sup \left\{ \left( \sum_{i=1}^{m} \lambda_i u^p(x_i, t_i) \right)^{\frac{1}{p}} : (x_i, t_i) \in \overline{Q}_i, x = \sum_i \lambda_i x_i, t = \left( \sum_i \lambda_i t_i^\alpha \right)^{\frac{1}{\alpha}} \right\}$$

(1.28)
is a subsolution of (1.2)–(1.3) with $i = \lambda$. Since $u \leq u^*$ by the definition and the comparison principle implies that $u \geq u^*$ in $\overline{Q}_i$, we obtain $u = u^*$, i.e. the parabolic power concavity of $u$ in the sense of (1.8). In this case, the assumption $(H2)_p$ becomes the following convexity assumption on the operator $G^{p,\alpha}_{\lambda,k}$ defined by (1.10):

$$(H2a)$$ For any $\lambda \in \Lambda$ and any $(x,t) \in Q, r \geq 0, \xi \in \mathbb{R}^n \setminus \{0\}, Y \in \mathbb{S}^n$,

$$G^{p,\alpha}_{\lambda,k}(x,t,r,\xi,Y) \leq \sum_{i}^{n+2} \lambda_i G^{p,\alpha}_{\lambda,k}(x_i, t_i, r_i, \xi_i, X_i)$$

holds whenever $(x_i, t_i) \in Q_i, r_i > 0$ and $X_i \in \mathbb{S}^n$ fulfilling (1.13) and (1.14) with $m = n + 2$.

**Theorem 1.2** (Parabolic power concavity). Assume that $\Omega \subset \mathbb{R}^n$ is a smooth bounded convex domain and that $F_\lambda$ satisfies (1.9) with $i = \lambda$. Let $u$ be the unique viscosity solution of (1.2)–(1.3) with $i = \lambda$ (that is positive and locally Lipschitz in space in $Q_\lambda = \Omega \times (0, \infty)$).

Let $k \in \mathbb{R}$, $0 < \alpha \leq 1$ and $p \leq 1$. Assume that $(H1)_p$ and $(H2a)$ hold, and, in addition, that

(i) $u$ is monotone in time, i.e.,

$$u(x,t) \geq u(x,s) \quad \text{for any } x \in \Omega \text{ and } t \geq s \geq 0;$$

(ii) if $p > 0$, then

$$\frac{1}{p} \left\{ \frac{1}{p} u^p \left( x + \tilde{v}_0(x)p, t + \mu(t)p^{1/\alpha} \right) \right\} \to \infty \quad \text{as } \rho \to 0+$$

for any $(x,t) \in \overline{Q}_i$.

Then $u$ is $\alpha$-parabolically $p$-concave in $Q_\lambda$ in the sense of (1.8).

It is worth remarking that $(H2a)$ is actually slightly weaker than the usual convexity of $(x,t,r,X) \mapsto G^{p,\alpha}_{\lambda,k}(x,t,r,\xi,X)$ combined with the ellipticity of $F_\lambda$, since (1.14) implies that

$$\text{sgn}^+(p) \sum_{i} \lambda_i X_i \leq \text{sgn}^+(p)Y.$$

As Theorem 1.1, Theorem 1.2 generalizes some previous results, precisely [29, Theorem 3] and [30, Theorem 4.2], which treat in the special case

$$F_i(x,t,r,\xi) = -\text{tr} X - f(x,t,r,\xi) \quad (i = \lambda, 1, 2, \ldots, m)$$

with $f \geq 0$ a given continuous function such that

$$f(x,t,r,\xi) \mapsto \begin{cases} r^{3-\frac{1}{p}} f \left( x, t^{\frac{1}{p}}, r^{\frac{1}{p}}, \frac{1}{p} r^{\frac{1}{p}+1} \xi \right) & p \neq 0 \\ e^{r} f(x, t^{1/\alpha}, e^r, e^r \xi) & p = 0 \end{cases}$$

(1.29)
is concave in $Q_\lambda \times (0, \infty)$ for any $\xi \in \mathbb{R}^n$.
For most applications of Theorem 1.1 and Theorem 1.2, we can take $k = 3 - 1/p$ for $p \neq 0$. It is clear that (H1)$_p$ is satisfied in this case. Denoting
\begin{equation}
G = G^{p,\alpha}_{\lambda} (p \neq 0),
\end{equation}
we see that the equation (1.25) with $i = \lambda$ reduces to
\begin{equation}
v^p t^{1 - \frac{p}{\alpha}} \partial_t v + \frac{p}{\alpha} G (x, t, v, \nabla v, \nabla^2 v) = 0.
\end{equation}
To meet the requirement (H2a) in Theorem 1.2, we only need to assume the following.

(H2b) For any $\lambda \in \Lambda$ and any $\xi \in \mathbb{R}^n \setminus \{0\}$,
\begin{equation}
\sum_i \lambda_i G(x_i, t_i, r_i, \xi, X_i) \geq G \left( \sum_i \lambda_i x_i, \sum_i \lambda_i t_i, \sum_i \lambda_i r_i, \xi, Y \right)
\end{equation}
for all $(x_i, t_i) \in Q_{\lambda}, r_i > 0$ and $X_i, Y \in S^n$ satisfying (1.14) with $m = n + 2$.

**Corollary 1.3** (A special case for parabolic power concavity). Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth convex domain. Assume that $F_{\lambda}$ satisfies (1.9) with $i = \lambda$. Let $0 < \alpha \leq 1$ and $0 \neq p \leq 1$. Assume that (H2b) holds. Let $u$ be a unique viscosity solution of (1.2)–(1.3) with $i = \lambda$ (that is positive and locally Lipschitz in space in $Q_{\lambda}$). Assume in addition that $u$ satisfies (i)(ii) in Theorem 1.2. Then $u$ is $\alpha$-parabolically $p$-concave in $Q_{\lambda}$.

We can verify that (H2b) holds when the operator $F_{\lambda}$ is in the form
\[ F_{\lambda}(x, t, r, \xi, X) = \mathcal{L}(\xi, X) - f(x, t, r, \xi), \]
where $\mathcal{L}$ is a degenerate elliptic operator satisfying proper assumptions (for instance $\mathcal{L}$ is 1-homogeneous with respect to $X$ and 0-homogeneous with respect to $\xi$) and $f \geq 0$ is a continuous function such that (1.29) is concave for any fixed $\xi \in \mathbb{R}^n$. Examples of $\mathcal{L}$ include the Laplacian, the Pucci operator, the normalized $q$-Laplacian ($1 < q \leq \infty$), the Finsler Laplacian, etc.; see details in Section 5.

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## 2. Preliminaries

### 2.1. Power means of nonnegative numbers.** For $a = (a_1, \ldots, a_m) \in (0, \infty)^m$, $\lambda \in \Lambda_m$, and $p \in [-\infty, +\infty]$, we set
\begin{equation}
M_p(a; \lambda) := \begin{cases} \\
\frac{[\lambda_1 a_1^p + \lambda_2 a_2^p + \cdots + \lambda_m a_m^p]^{1/p}}{\max\{a_1, \ldots, a_m\}} & \text{if } p \neq -\infty, 0, +\infty, \\
\max\{a_1, \ldots, a_m\} & \text{if } p = +\infty, \\
\lambda_1 a_1^{\lambda_1} \cdots a_m^{\lambda_m} & \text{if } p = 0, \\
\min\{a_1, a_2, \ldots, a_m\} & \text{if } p = -\infty.
\end{cases}
\end{equation}
which is the ($\lambda$-weighted) $p$-mean of $a$.

For $a = (a_1, \ldots, a_m) \in [0, \infty)^m$, we define $M_p(a; \lambda)$ as above if $p \geq 0$ and $M_p(a; \lambda) = 0$ if $p < 0$ and $\prod_{i=1}^m a_i = 0$.

Notice that $M_p(a; \lambda)$ is a continuous function of the argument $a$. Due to the Jensen inequality, we have

\[
M_p(a; \lambda) \leq M_q(a; \lambda) \quad \text{if} \quad -\infty \leq p \leq q \leq \infty,
\]

for any $a \in [0, \infty)^m$ and $\lambda \in \Lambda_m$. Moreover, it easily follows that

\[
\lim_{p \to +\infty} M_p(a; \lambda) = M_{+\infty}(a; \lambda), \quad \lim_{p \to 0} M_p(a; \lambda) = M_0(a; \lambda), \quad \lim_{p \to -\infty} M_p(a; \lambda) = M_{-\infty}(a; \lambda).
\]

For further details, see e.g. [21].

2.2. Definition of viscosity solutions. We recall the definition of viscosity solutions to (1.2), which can also be found in [13, 18]. In Appendix A we review more properties of viscosity solutions that are needed in this work.

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n$. Let $O$ denote an arbitrary open subset of $Q = \Omega \times (0, \infty)$. Consider a general parabolic equation

\[
\partial_t u + F(x, t, u, \nabla u, \nabla^2 u) = 0 \tag{2.3}
\]

in $Q$, where $F$ is a proper elliptic operator.

Here, by elliptic we mean that

\[
F(x, t, r, \xi, X_1) \leq F(x, t, r, \xi, X_2)
\]

for all $(x, t, r, \xi) \in \mathbb{R} \times [0, \infty) \times [0, \infty) \times (\mathbb{R}^n \setminus \{0\})$ and $X_1, X_2 \in \mathbb{S}^n$ satisfying $X_1 \geq X_2$. We also recall that $F$ is proper if there exists $c \in \mathbb{R}$ such that

\[
F(x, t, r_1, \xi, X) + cr_1 \leq F(x, t, r_2, \xi, X) + cr_2 \tag{2.5}
\]

for all $(x, t, \xi, X) \in \mathbb{R} \times [0, \infty) \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n$ and $r_1, r_2 \in [0, \infty)$ satisfying $r_1 \leq r_2$.

We further assume that $F$ satisfies (1.9) with the subindex $i$ omitted.

**Definition 2.1.** A locally bounded upper (resp., lower) semicontinuous function $u : O \to \mathbb{R}$ is said to be a subsolution (resp., supersolution) of (2.3) in $O$ if whenever there exist $(x_0, t_0) \in O$ and $\phi \in C^2(O)$ such that $u - \phi$ attains a maximum (resp., minimum) at $(x_0, t_0)$, we have

\[
\partial_t \phi(x_0, t_0) + F_u(x_0, t_0, u(x_0, t_0), \nabla \phi(x_0, t_0), \nabla^2 \phi(x_0, t_0)) \leq 0
\]

(resp., \[
\partial_t \phi(x_0, t_0) + F^*(x_0, t_0, u(x_0, t_0), \nabla \phi(x_0, t_0), \nabla^2 \phi(x_0, t_0)) \geq 0
\]).

A continuous function $u : O \to \mathbb{R}$ is called a solution of (2.3) in $O$ if it is both a subsolution and a supersolution in $O$.

It is clear that $F_u = F^* = F$ in $O \times (0, \infty) \times \mathbb{R}^n \times \mathbb{S}^n$ provided that $F$ is assumed to be continuous in $O \times (0, \infty) \times \mathbb{R}^n \times \mathbb{S}^n$.

**Remark 2.2.** It is standard in the theory of viscosity solutions to use the semijets to give an equivalent definition. More precisely, for any $(x_0, t_0) \in O$, setting $P^{2+}(x_0, t_0) \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ as

\[
P^{2+}(x_0, t_0) = \left\{ (\tau, \xi, X) : u(x, t) \leq u(x_0, t_0) + \tau(t - t_0) + \langle \xi, x - x_0 \rangle + \frac{1}{2} \langle X(x - x_0), (x - x_0) \rangle + o(|t - t_0| + |x - x_0|^2) \right\}
\]
and its “closure” as
\[
\overline{P}^{2+}u(x_0,t_0) = \left\{ (\tau, \xi, X) : \text{there exist } (x_j, t_j) \in \mathcal{O} \text{ and } (\tau_j, \xi_j, X_j) \in \overline{P}^{2+} u(x_j, t_j) \right\}
\]

such that \((x_j, t_j, \tau_j, \xi_j, X_j) \rightarrow (x_0, t_0, \tau, \xi, X) \text{ as } j \rightarrow \infty \},
\]
we then say \(u\) is a subsolution of (2.3) if
\[
\tau + F_\ast(x_0, t_0, u(x_0, t_0), \xi, X) \leq 0
\]
for every \((\tau, \xi, X) \in \overline{P}^{2+}u(x_0, t_0)\). The semijet \(P^{2-}u(x_0, t_0)\), its closure and supersolutions can be analogously defined in a symmetric way.

If \(F(x,t,r,\xi,X)\) is mildly singular at \(\xi = 0\), i.e. (1.9) holds, one can use the following equivalent definition, called \(F\)-solutions as in [18].

**Definition 2.3.** Suppose that there exists \(h \in C(\overline{\mathcal{O}} \times [0, \infty) \times [0, \infty))\) such that
\[
h(x, t, r, \xi, X) = F^\ast(x, t, r, 0, 0) = F_\ast(x, t, r, 0, 0)
\]
holds for all \((x, t, r, \xi, X) \in \mathcal{O} \times \mathbb{R}\). A locally bounded upper (resp., lower) semicontinuous function \(u : \mathcal{O} \rightarrow \mathbb{R}\) is said to be a subsolution (resp., supersolution) of (2.3) in \(\mathcal{O}\) if, whenever there exist \((x_0, t_0) \in \mathcal{O}\) and \(\phi \in C^2(\mathcal{O})\) such that \(u - \phi\) attains a maximum (resp., minimum) at \((x_0, t_0)\), we have
\[
\partial_t \phi(x_0, t_0) + F(x_0, t_0, u(x_0, t_0), \nabla \phi(x_0, t_0), \nabla^2 \phi(x_0, t_0)) \leq 0
\]
(resp., \(\partial_t \phi(x_0, t_0) + F(x_0, t_0, u(x_0, t_0), \nabla \phi(x_0, t_0), \nabla^2 \phi(x_0, t_0)) \geq 0\)) when \(\nabla \phi(x_0, t_0) \neq 0\) and
\[
\partial_t \phi(x_0, t_0) + h(x_0, t_0, u(x_0, t_0)) \leq 0
\]
(resp., \(\partial_t \phi(x_0, t_0) + h(x_0, t_0, u(x_0, t_0)) \geq 0\)) when \(\nabla \phi(x_0, t_0) = 0\) and \(\nabla^2 \phi(x_0, t_0) = 0\).

**Remark 2.4.** In the definition of subsolutions by semijets, these conditions are written as follows: for any \((\tau, \xi, X) \in \overline{P}^{2+}u(x_0, t_0)\), we require that
\[
\tau + F(x_0, t_0, u(x_0, t_0), \xi, X) \leq 0
\]
if \(\xi \neq 0\) and
\[
\tau + h(x_0, t_0, u(x_0, t_0)) \leq 0
\]
if \(\xi = \lambda\) and \(X = 0\).

### 3. A Useful Transformation of the Unknown Function

A straightforward way to study this problem is to directly turn the unknown function into a form that fits the desired parabolic power concavity.

If \(u_i\) is a smooth positive subsolution of (1.2) and \(F\) is not mildly singular, then by direct calculations we see that \(v_i\) defined in (1.22) is a smooth subsolution of (1.23) for all \(i = \lambda, 1, \ldots, m\). In fact, we have
\[
u_i(x,t) = v_i^\frac{1}{p+1}(x,t^\alpha),
\]
\[
\partial_t u_i(x,t) = \frac{\alpha}{p} v_i^{\frac{p}{p+1} - 1} t^{\alpha - 1} \partial_t v_i(x,t^\alpha),
\]
\[ \nabla u_i(x, t) = \frac{1}{p} v_i^{\frac{1}{p} - 1} \nabla v_i(x, t^\alpha), \]
\[ \nabla^2 u_i(x, t) = \frac{1}{p} v_i^{\frac{1}{p} - 1} \nabla^2 v_i(x, t^\alpha) + \frac{1 - p}{p^2} v_i^{\frac{1}{p} - 2} \nabla v_i(x, t^\alpha) \otimes \nabla v_i(x, t^\alpha). \]

Plugging these into (1.2), we easily obtain (1.23). It is clear that positive smooth solutions of (1.23) are equivalent to positive smooth solutions of (1.25), where \( G_{i,k}^{p,\alpha} \) is given by (1.10).

When \( u_i \) is not necessarily smooth, we can interpret such a result in the viscosity sense.

**Lemma 3.1** (Sub/supersolution properties under transformation). Fix \( i = \lambda, 1, \ldots, m \) arbitrarily. Assume that (1.9) holds. Let \( u_i \) be positive and upper semicontinuous in \( Q_i \). Let \( v_i \) be given by (1.22). Then

- if \( p \geq 0 \), \( u_i \) is a viscosity subsolution of (1.2) if and only if \( v_i \) is a viscosity subsolution of (1.25) in \( Q_i \);
- for \( p < 0 \), \( u_i \) is a viscosity subsolution of (1.2) if and only if \( v_i \) is a viscosity supersolution of (1.25) in \( Q_i \).

Moreover, a symmetric result holds also for supersolutions.

**Proof.** Let us give the proof in details for the case \( p > 0 \) and \( u_i \) is a subsolution of (1.2), then let us prove that this implies that \( v_i \) is a subsolution of (1.23). The converse implication can be similarly shown.

Assume that there exist \( (x_0, t_0) \in Q_i \) and \( \phi \in C^2(\overline{Q_i}) \) such that
\[ \max_{\overline{Q_i}} (v_i - \phi) = (v_i - \phi)(x_0, t_0) = 0. \]
In other words, we have
\[ v_i(x_0, t_0) = \phi(x_0, t_0), \quad v_i(x, t) \leq \phi(x, t) \quad \text{for all } (x, t) \in Q_i. \]
Since \( v_i > 0 \) in \( Q_i \), it follows that
\[ u_i(x_0, t_0^{1/\alpha}) = \phi^{1/p}(x_0, t_0), \quad u_i(x, t^{1/\alpha}) \leq \phi^{1/p}(x, t) \quad \text{for all } (x, t) \in Q_i. \]
This implies that \( u_i(x, t) - \psi(x, t) \) attains a maximum over \( Q_i \) at \( (x_0, t_0^{1/\alpha}) \), where
\[ \psi(x, t) = \phi^{\frac{1}{p}}(x, t^\alpha). \]

Suppose that \( \nabla \phi(x_0, t_0) \neq 0 \). Then \( \nabla \psi(x_0, t_0) \neq 0 \). Since \( u_i \) is a subsolution of (1.2), we see that
\[ \partial_t \psi + F_i \left( x_0, t_0^{1/\alpha}, u_i, \nabla \psi, \nabla^2 \psi \right) \leq 0 \quad \text{at } (x_0, t_0^{1/\alpha}). \]

By direct calculations, it follows that at \( (x_0, t_0) \)
\[ \frac{\alpha}{p} v_i^{\frac{1}{p} - 1} t^{1 - \frac{1}{\alpha}} \partial_t \phi + F_i \left( x_0, t_0, v_i, v_i^{\frac{1}{p} - 1} \nabla \phi, \frac{1}{p} v_i^{\frac{1}{p} - 1} \nabla^2 \phi + \frac{1 - p}{p^2} v_i^{\frac{1}{p} - 2} \nabla \phi \otimes \nabla \phi \right) \leq 0. \]

Multiplying (3.1) by \( pv_i(x_0, t_0)^k \), we obtain
\[ v_i(x_0, t_0)^{\frac{1}{p} - 1 + k} t^{1 - \frac{1}{\alpha}} \partial_t \phi(x_0, t_0) + \frac{p}{\alpha} G_{i,k}^{p,\alpha}(x_0, t_0, v_i(x_0, t_0), \nabla \phi(x_0, t_0), \nabla^2 v_i(x_0, t_0)) \leq 0. \]
If $\nabla \phi(x_0, t_0) = 0$, we have $\nabla \psi(x_0, t_0^{1/\alpha}) = 0$. Using Definition 2.3, we assume $\nabla^2 \phi(x_0, t_0) = 0$, which is equivalent to
\[
\nabla^2 \psi \left( x_0, t_0^{1/\alpha} \right) = 0.
\]
We thus can apply the definition of subsolution on $u_i$ to obtain
\[
\partial_t \psi(x_0, t_0^{1/\alpha}) + h_i \left( x_0, t_0^{1/\alpha}, u_i(x_0, t_0^{1/\alpha}) \right) \leq 0,
\]
which yields
\[
v_i(x_0, t_0) \frac{1}{p-1} t_0^{1-\frac{1}{\alpha}} \partial_t \phi(x_0, t_0) + \frac{p}{\alpha} h_i \left( x_0, t_0^{1/\alpha}, v_i(x_0, t_0) \right) \leq 0.
\]
The proof of the case $p > 0$ and $u_i$ is a subsolution is thus complete. The cases $p = 0$ and $p < 0$ can be treated similarly, and the same for the symmetric case when $u_i$ is a supersolution. □

If $F$ is mildly singular, it is not difficult to see that
\[
G_{i,k}^{p,\alpha}(x, t, r, \xi, X) \to r^k h_i \left( x, t^\frac{1}{\alpha}, r^\frac{1}{p} \right) \quad \text{as } \xi \to 0, \ X \to 0 \quad (3.2)
\]
locally uniformly for all $(x, t, r) \in \overline{Q}_i \times [0, \infty)$ and all $i = \lambda_1, \ldots, m$. In other words, the operator $G_{i,k}^{p,\alpha}$ satisfies the same properties as in (1.9). We are thus able to apply Definition 2.3 to define the sub- and supersolutions of (1.25). Let us denote
\[
\tilde{h}_i(x, t, r) = r^k h_i \left( x, t^\frac{1}{\alpha}, r^\frac{1}{p} \right) \quad (3.3)
\]
for all $(x, t, r) \in \overline{Q}_i \times [0, \infty)$ and $i = \lambda_1, \ldots, m$.

**Proposition 3.2.** Assume that (1.9) holds for each $i = \lambda_1, \ldots, m$. Suppose that there exists $k \in \mathbb{R}$ such that (H2)$_p$ holds. Then $\tilde{h}_i$ given by (3.3) satisfies
\[
\sum \lambda_i \tilde{h}_i(x_i, t_i, r_i) \geq \tilde{h}_0 \left( \sum \lambda_i x_i, \sum i \lambda_i t_i, \sum \lambda_i r_i \right) \quad (3.4)
\]
for any $\lambda \in \Lambda$, $(x_i, t_i) \in Q_i$, $r_i > 0$.

**Proof.** Since
\[
G_{\lambda,k}^{p,\alpha}(x, t, r, \xi, 0) \to \tilde{h}_0(x, t, r) \quad \text{locally uniformly as } \xi \to 0
\]
for any $\varepsilon > 0$, there exists $\xi_\varepsilon \in \mathbb{R}^n \setminus \{0\}$ such that
\[
G_{i,k}^{p,\alpha} \left( \sum \lambda_i x_i, \sum \lambda_i t_i, \sum \lambda_i r_i, \xi_\varepsilon, 0 \right) \geq \tilde{h}_0 \left( \sum \lambda_i x_i, \sum \lambda_i t_i, \sum \lambda_i r_i \right) - \varepsilon. \quad (3.5)
\]
Since (1.14) clearly holds with $Y = X_i = \lambda$ for all $i = 1, 2, \ldots, n + 2$. Then by (H2)$_p$, we get
\[
\sum \lambda_i G_{i,k}^{p,\alpha}(x_i, t_i, r_i, \xi_\varepsilon, 0) \geq G_{\lambda,k}^{p,\alpha} \left( \sum \lambda_i x_i, \sum \lambda_i t_i, \sum \lambda_i r_i, \xi_\varepsilon, 0 \right),
\]
which, by (3.5), yields
\[
\sum \lambda_i G_{i,k}^{p,\alpha}(x_i, t_i, r_i, \xi_\varepsilon, 0) \geq \tilde{h}_0 \left( \sum \lambda_i x_i, \sum \lambda_i t_i, \sum \lambda_i r_i \right) - \varepsilon
\]
Sending $\varepsilon \to 0$, we obtain (3.4) by (3.2) and (3.3). □
When $p > 0$, we easily see that $v_i$ satisfies the same initial and boundary conditions as $u_i$, we therefore can write the Cauchy-Dirichlet problem for $v_i$ ($i = \lambda, 1, \ldots, m$) as

$$\begin{cases} \frac{1}{p} \partial_t v_i^{1+k} + \frac{1}{\alpha} \partial_t v + \frac{p}{\alpha} G_{i,k}^{p,\alpha} (x, t, v, \nabla v, \nabla^2 v) = 0 & \text{in } Q_i, \\ v = 0 & \text{on } \partial Q_i, \end{cases} \quad (3.6)$$

Since we assume that a comparison principle holds for sub- and supersolutions of (1.2)–(1.3) that are positive in $\Omega \times (0, \infty)$, Lemma 3.1 implies that postive sub- and supersolutions of (3.6)–(3.7) also enjoy a comparison principle (which is what we truly need).

When $p \leq 0$, in place of (3.7) $v_i$ satisfies a blow-up boundary and initial condition (precisely $v_i \to -\infty$ for $p = 0$, while $v_i \to +\infty$ when $p < 0$ on $\partial Q_i$), which enter into the case of state constraints boundary conditions. Then we have to go back to $u_i$ and use the comparison principle for the problem satisfied by $u_{\lambda}$.

We conclude this section by pointing out the equivalence between (1.16) and the condition

$$v_i(x, t) \geq v_i(x, s) \quad \text{for any } x \in \Omega_i, \ t \geq s \geq 0 \ \text{and } i = 1, \ldots, m. \quad (3.8)$$

The monotonicity with respect to time will be used in the proof of Theorem 1.1.

4. The Minkowski Convolution

4.1. Achievability in the interior. For any given $\lambda \in \Lambda$ and $(x, t) = (\hat{x}, \hat{t})$, we show that the supremum in (1.4) can be attained at some $(x_i, t_i) \in Q_i$ for $i = 1, 2, \ldots, m$. Our proof is essentially the same of [30, Lemma 3.1]. We give the details for the sake of completeness.

**Lemma 4.1** (Interior maximizers for the envelope). Suppose that the assumptions of Theorem 1.1 hold. Then for any $(\hat{x}, \hat{t}) \in Q_{\lambda}$, there exist $(x_1, t_1) \in Q_1$, $(x_2, t_2) \in Q_2$, $\ldots, (x_m, t_m) \in Q_m$ such that

$$\hat{x} = \sum_{i=1}^{m} \lambda_i x_i, \quad (\hat{t})^\alpha = \sum_{i=1}^{m} \lambda_i t_i^\alpha \quad (4.1)$$

and

$$U_{p,\lambda} (\hat{x}, \hat{t}) = \left( \sum_{i=1}^{m} \lambda_i u_i^p (x_i, t_i) \right)^{\frac{1}{p}} \quad (4.2)$$

**Proof.** Let us only discuss the case $p > 0$, since the results with $p < 0$ or $p = 0$ clearly hold. In view of the compactness of the set

$$\left\{ (y_1, s_1, y_2, s_2, \ldots, y_m, s_m) \in \prod_{i=1}^{m} \overline{Q}_i : \hat{x} = \sum_{i=1}^{m} \lambda_i y_i, \ (\hat{t})^\alpha = \sum_{i=1}^{m} \lambda_i s_i^\alpha \right\}$$

and the continuity of

$$(y_1, s_1, y_2, s_2, \ldots, y_m, s_m) \mapsto \left( \sum_{i=1}^{m} \lambda_i u_i^p (x_i, t_i) \right)^{\frac{1}{p}},$$

we can find $(x_i, t_i) \in \overline{Q}_i$ for $i = 1, 2, \ldots, m$ such that (4.1) and (4.2) hold.

Let $\hat{\tau} = (\hat{t})^\alpha$ and $\tau_i = t_i^\alpha$ and recall that $v_i$ is given by (1.22). We have

$$\sum_{i=1}^{m} \lambda_i x_i = \hat{x}, \quad \sum_{i=1}^{m} \lambda_i \tau_i = \hat{\tau}, \quad (4.3)$$
\[ U_{p,\lambda}(\hat{x}, \hat{t})^p = V_{p,\lambda}(\hat{x}, \hat{t}) = \sum_{i=1}^{m} \lambda_i v(x_i, \tau_i). \tag{4.4} \]

It thus suffices to show that \((x_i, \tau_i) \in Q_i\) for all \(i = 1, 2, \ldots, m\).

Assume by contradiction that \((x_i, \tau_i) \in \partial Q_i\) for some \(i = 1, 2, \ldots, m\). We derive a contradiction in the following two cases.

**Case 1.** Suppose that \((x_i, \tau_i) \in \partial Q_i\) for all \(i = 1, 2, \ldots, m\), then by (1.3) and (4.2), we have \(U_{p,\lambda}(\hat{x}, \hat{t}) = 0\), which is a contradiction, since \(U_{p,\lambda}(\hat{x}, \hat{t}) > 0\) for every \((\hat{x}, \hat{t}) \in Q_\lambda\).

**Case 2.** Assume, without loss of generality, that \((x_1, \tau_1) \in \partial Q_1\) and \((x_2, \tau_2) \in Q_2\). Take \(\rho \in (0, 1)\) and put
\[
\tilde{x}_1 = x_1 + \frac{\rho}{\lambda_1} \tilde{v}_1, \quad \tilde{x}_2 = x_2 - \frac{\rho}{\lambda_2} \tilde{v}_1, \quad \tilde{x}_i = x_i, \quad (i = 3, 4, \ldots, m),
\]
\[
\tilde{\tau}_1 = \tau_1 + \mu(t_1) \frac{\rho}{\lambda_1}, \quad \tilde{\tau}_2 = \tau_2 - \mu(t_1) \frac{\rho}{\lambda_2}, \quad \tilde{\tau}_i = \tau_i, \quad (i = 3, 4, \ldots, m).
\]

Then it is clear that \(\sum_i \lambda_i \tilde{x}_i = \sum_i \lambda_i x_i = \tilde{x}, \sum_i \lambda_i \tilde{\tau}_i = \sum_i \lambda_i \tau_i = \tilde{\tau}\). By itaking \(\rho > 0\) small enough, we also have \((\tilde{x}_1, \tilde{\tau}_1) \in Q_1, (\tilde{x}_2, \tilde{\tau}_2) \in Q_2\).

Adopting the local Lipschitz regularity of \(u_2\), we get \(M > 0\) and \(\delta_1 > 0\) such that
\[
|\nabla v_2| + |\partial_t v_2| \leq M \quad \text{a.e. in } B_{\delta_1}(x_2) \times (\tau_2 - \delta_1, \tau_2 + \delta_1) \subset Q_2.
\]

It follows that
\[
\lambda_2 v_2(\tilde{x}_2, \tilde{\tau}_2) - \lambda_2 v_2(x_2, \tau_2) \geq -\lambda_2 M (|\tilde{x}_2 - x_2| + |\tilde{\tau}_2 - \tau_2|) \geq -2M \rho. \tag{4.5}
\]

On the other hand, the condition (1.17) implies that
\[
v_1(\tilde{x}_1, \tilde{\tau}_1) - v_1(x_1, \tau_1) = u_1 \left( x_1 + \frac{\rho}{\lambda_1} \tilde{v}_1(x_1), t_1 + \mu(t_1) \left( \frac{\rho}{\lambda_1} \right)^{\frac{1}{\alpha}} \right)^p \geq (2M + 1) \frac{\rho}{\lambda_1},
\]
which yields that
\[
\lambda_1 v_1(\tilde{x}_1, \tilde{\tau}_1) - \lambda_1 v_1(x_1, \tau_1) \geq (2M + 1) \rho \tag{4.6}
\]
when \(\rho\) is sufficiently small. By (4.5) and (4.6), we have
\[
\sum_i \lambda_i v_i(\tilde{x}_i, \tilde{\tau}_i) \geq \lambda_1 (v_1(\tilde{x}_1, \tilde{\tau}_1) - v_1(x_1, \tau_1)) + \lambda_2 (v_2(\tilde{x}_2, \tilde{\tau}_2) - v_2(x_2, \tau_2)) + \sum_i \lambda_i v_i(x_i, \tau_i)
\]
\[
> \sum_{i=1}^{m} \lambda_i v_i(x_i, \tau_i) = U_{p,\lambda}(\hat{x}, \hat{t})^p,
\]
which contradicts (4.4). \(\square\)

### 4.2. A key lemma

To show our main result, instead of using \(U_{p,\lambda}\) defined in (1.4), we consider the Minkowski convolution \(V_{p,\lambda}\) for \(v_i\) as given in (1.27).

It turns out that the following lemma plays a central role in the proof of Theorem 1.1.

**Lemma 4.2** (Minkowski convolution preserves subsolutions). Fix \(\lambda \in \Lambda_m\). Assume that \(\Omega_i\) is a bounded smooth domain in \(\mathbb{R}^n\) for any \(i = 1, 2, \ldots, m\). Let \(\Omega_\lambda\) be the Minkowski combination as defined in (1.1). Assume that \(F_i\) satisfies (1.9) for all \(i = \lambda, 1, \ldots, m\). Let \(0 < \alpha \leq 1\) and \(p < 1\). Suppose that there exists \(k \in \mathbb{R}\) such that \((H1)_p\) and \((H2)_p\) hold, where \(G_{i,k}^{p,\alpha}\) is given by (1.10). Then:
• Case 0 \leq p < 1.

Let \( v_i \) be a nondecreasing in time upper semicontinuous subsolution of (3.6) for every \( i = 1, 2, \ldots, m. \) Suppose that for any fixed \( (\hat{x}, \hat{t}) \in Q, \) the supremum in the definition of \( V_{p,\lambda} \) in (1.27) at \( (\hat{x}, \hat{\tau}) \) is attained at some \( (x_i, \tau_i) \in Q_i \) for \( i = 1, 2, \ldots, m, \) in other words, (4.3) and (4.4) hold. Then \( V_{p,\lambda} \) satisfies the subsolution property for (3.6) at \( (\hat{x}, \hat{t}). \)

• Case \( p < 0. \)

Let \( v_i \) be a nonincreasing in time lower semicontinuous supersolution of (3.6) for every \( i = 1, 2, \ldots, m. \) Suppose that for any fixed \( (\hat{x}, \hat{t}) \in Q, \) the infimum in the definition of \( V_{p,\lambda} \) in (1.27) at \( (\hat{x}, \hat{\tau}) \) is attained at some \( (x_i, \tau_i) \in Q_i \) for \( i = 1, 2, \ldots, m, \) in other words, (4.3) and (4.4) hold. Then \( V_{p,\lambda} \) satisfies the supersolution property for (3.6) at \( (\hat{x}, \hat{t}). \)

Using Lemma 4.2, we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Fix \( \lambda \in \Lambda \) arbitrarily. Let \( U_{p,\lambda}, v_i \) and \( V_{p,\lambda} \) be given respectively by (1.4), (1.22) and (1.27) \( (i = 1, 2, \ldots, m). \) Adopting Lemma 3.1, we can show that if \( p \geq 0 \) then \( v_i \) is a subsolution of (3.6), while for \( p < 0 \) it is a supersolution of (3.6) for any \( i = 1, 2, \ldots, m. \)

For any \( (\hat{x}, \hat{t}) \in Q, \) by Lemma 4.1, we see that the maximizers in the definition of \( U_{p,\lambda}(\hat{x}, \hat{t}) \) appear in \( Q_i \) for all \( i = 1, 2, \ldots, m. \) This enables us to apply Lemma 4.2 with \( \hat{\tau} = \hat{t}^{1/\alpha} \) to deduce that \( V_{p,\lambda} \) is a subsolution or a supersolution, according to the value of \( p, \) of (3.6) with \( i = \lambda. \) Adopting Lemma 3.1 again yields that \( U_{p,\lambda} \) is a subsolution of (1.2) with \( i = \lambda. \)

We next present a proof of of Lemma 4.2.

Proof of Lemma 4.2. Let us present the proof in details in the case \( p > 0. \) The cases \( p = 0 \) and \( p < 0 \) can be treated similarly.

Suppose that \( \phi \in C^2(\overline{Q}) \) is a test function of \( V_{p,\lambda} \) at \( (\hat{x}, \hat{\tau}) \in Q_\lambda, \) that is,
\[
(V_{p,\lambda} - \phi)(x, \tau) \leq (V_{p,\lambda} - \phi)(\hat{x}, \hat{\tau}) = 0
\]
for all \( (x, \tau) \in \overline{Q}_0. \) Due to the maximality of
\[
(y_1, s_1, \ldots, y_m, s_m) \mapsto \sum_{i=1}^{m} \lambda_i v_i(y_i, s_i) - V_{p,\lambda} \left( \sum_{i=1}^{m} \lambda_i y_i, \sum_{i=1}^{m} \lambda_i s_i \right) \tag{4.7}
\]
over \( \prod_{i=1}^{m} \overline{Q}_i \) at \( (x_1, \tau_1, \ldots, x_m, \tau_m) \in \prod_{i=1}^{m} Q_i, \) we see that
\[
(y_1, s_1, \ldots, y_m, s_m) \mapsto \sum_{i=1}^{m} \lambda_i v_i(x_i, s_i) - \phi \left( \sum_{i=1}^{m} \lambda_i y_i, \sum_{i=1}^{m} \lambda_i s_i \right) \tag{4.8}
\]
also attains a maximum over \( \prod_{i=1}^{m} \overline{Q}_i \) at \( (x_1, \tau_1, \ldots, x_m, \tau_m). \)

We next apply the Crandall-Ishii lemma [13]: for any \( \varepsilon > 0, \) there exist \( (\eta_i, \xi_i, A_i) \in \mathcal{P}^{2,+} v_i(x_i, \tau_i) \) \( (i = 1, 2, \ldots, m) \) such that
\[
\eta_i = \partial_\tau \phi(\hat{x}, \hat{\tau}) \tag{4.9}
\]
\[
\xi_i = \nabla \phi(\hat{x}, \hat{\tau}) \tag{4.10}
\]
Let us consider two different cases.

**Case 1.** Suppose that \( \nabla \phi(\hat{x}, \hat{\tau}) \neq 0 \). Then applying the definition of subsolutions of (3.6), we have

\[
v_i(x_i, \tau_i)^{\frac{1}{p} - 1 + k} \tau_i^{1 - 1/\alpha} \eta_i + \frac{p}{\alpha} G_{i,k}^{p,\alpha} (x_i, \tau_i, v_i(x_i, \tau_i), \xi_i, A_i) \leq 0, \tag{4.15}
\]

Multiplying (4.15) by \( \lambda_i \) and summing up the inequalities, we are led to

\[
\sum_i \lambda_i v_i(x_i, \tau_i)^{\frac{1}{p} - 1 + k} \tau_i^{1 - 1/\alpha} \eta_i + \frac{p}{\alpha} \sum_i \lambda_i G_{i,k}^{p,\alpha} (x_i, \tau_i, v_i(x_i, \tau_i), \xi_i, A_i) \leq 0,
\]

which by (4.9) yields that

\[
\left( \sum_i \lambda_i v_i(x_i, \tau_i)^{\frac{1}{p} - 1 + k} \tau_i^{1 - 1/\alpha} \right) \partial_t \phi(\hat{x}, \hat{\tau}) + \frac{p}{\alpha} \sum_i \lambda_i G_{i,k}^{p,\alpha} (x_i, \tau_i, v_i(x_i, \tau_i), \xi_i, A_i) \leq 0. \tag{4.16}
\]

By (H1), we can easily verify that

\[
(r, t) \mapsto r^{\frac{1}{p} - 1 - k} t^{1 - \frac{1}{\alpha}}
\]

is convex in \((0, \infty)^2\), which implies that

\[
\sum_i \lambda_i v_i(x_i, \tau_i)^{\frac{1}{p} - 1 + k} \tau_i^{1 - 1/\alpha} \geq \left( \sum_i \lambda_i v_i(x_i, \tau_i) \right)^{\frac{1}{p} - 1 + k} \left( \sum_i \lambda_i \tau_i \right)^{1 - \frac{1}{\alpha}} \tag{4.17}
\]

The last equality is due to (4.3) and (4.4). Using (4.14), (4.16) and (4.17), we thus obtain that

\[
V_{p, \lambda}(\hat{x}, \hat{\tau})^{\frac{1}{p} - 1 + k} \tau^{1 - \frac{1}{\alpha}} \partial_t \phi(\hat{x}, \hat{\tau}) + \frac{p}{\alpha} \sum_i \lambda_i G_{i,k}^{p,\alpha} (x_i, \tau_i, v_i(x_i, \tau_i), \xi_i, A_i) \leq 0. \tag{4.18}
\]
We next apply (H2)$_p$ with $X_i = \tilde{A}_i = A_i - C\varepsilon I$ and $Y = B$ to deduce that
\[ \sum_{i} \lambda_i G_{i,k}^{p,\alpha} (x_i, \tau_i, v_i(x_i, \tau_i), \xi_i, A_i - C\varepsilon I) \geq G_{\lambda,k}^{p,\alpha} (\hat{x}, \hat{\tau}, V_{p,\lambda}(\hat{x}, \hat{\tau}), \xi, B). \]
It follows from the continuity of $F_i$ (and therefore of $G_{i,k}^{p,\alpha}$) in $\overline{Q}_1 \times (0, \infty) \times (\mathbb{R}^n \setminus \{0\}) \times S^n$ that
\[ \sum_{i} \lambda_i G_{i,k}^{p,\alpha} (x_i, \tau_i, V(x_i, \tau_i), \xi_i, A_i) \geq G_{\lambda,k}^{p,\alpha} (\hat{x}, \hat{\tau}, V_{p,\lambda}(\hat{x}, \hat{\tau}), \xi, B) - \omega_F(\varepsilon) \tag{4.19} \]
where $\omega_F$ denotes a modulus of continuity describing the locally uniform continuity of $F_i$.

Plugging (4.19) into (4.18), we get
\[ V_{p,\lambda}(\hat{x}, \hat{\tau}) \frac{1}{p} \frac{1}{\hat{\tau}^{1 - \frac{1}{p}}} \left[ \frac{1}{\hat{\tau}^{1 - \frac{1}{p}}} \right] \partial_t \phi (\hat{x}, \hat{\tau}) + \frac{p}{\alpha} G_{\lambda,k}^{p,\alpha} (\hat{x}, \hat{\tau}, V_{p,\lambda}(\hat{x}, \hat{\tau}), \xi, B) \leq \frac{p}{\alpha} \omega_F(\varepsilon), \]
which yields, by letting $\varepsilon \to 0$, that
\[ V_{p,\lambda}(\hat{x}, \hat{\tau}) \frac{1}{p} \frac{1}{\hat{\tau}^{1 - \frac{1}{p}}} \left[ \frac{1}{\hat{\tau}^{1 - \frac{1}{p}}} \right] \partial_t \phi (\hat{x}, \hat{\tau}) + \frac{p}{\alpha} G_{\lambda,k}^{p,\alpha} (\hat{x}, \hat{\tau}, V_{p,\lambda}(\hat{x}, \hat{\tau}), \xi, B) \leq 0. \]

**Case 2.** Suppose that $\nabla^2 \phi(\hat{x}, \hat{t}) = 0$. We are able to apply Definition 2.3 for $V_{p,\lambda}$ by assuming $\nabla^2 \phi(\hat{x}, \hat{t}) = 0$, which by (4.11) further yields that $A_i \leq 0$ for all $i = 1, 2, \ldots, m$. Using Definition 2.3 for $v_i$ and the ellipticity of $G_{i,k}^{p,\alpha}$ with $i = 1, 2, \ldots, m$ along with (3.2) and (3.3), we then have
\[ v_i(x_i, \tau_i) \frac{1}{p} \frac{1}{\tau_i^{1 - \frac{1}{p}}} \eta_i + \frac{p}{\alpha} h_i(x_i, \tau_i) \leq 0. \]
Multiplying this inequality by $\lambda_i$ and summing up over $i = 1, 2, \ldots, n + 2$, we deduce that
\[ \left( \sum_{i} \lambda_i v_i(x_i, \tau_i) \frac{1}{p} \frac{1}{\tau_i^{1 - \frac{1}{p}}} \eta_i \right) \partial_t \phi (\hat{x}, \hat{\tau}) + \frac{p}{\alpha} \sum_{i} \lambda_i h_i(x_i, \tau_i, v(x_i, \tau_i)) \leq 0. \]
Thanks to (4.17) again and (3.4), we may use (4.3) and (4.4) to conclude that
\[ V_{p,\lambda}(\hat{x}, \hat{\tau}) \frac{1}{p} \frac{1}{\hat{\tau}^{1 - \frac{1}{p}}} \left[ \frac{1}{\hat{\tau}^{1 - \frac{1}{p}}} \right] \partial_t \phi (\hat{x}, \hat{\tau}) + \frac{p}{\alpha} h_0(\hat{x}, \hat{\tau}, V_{p,\lambda}(\hat{x}, \hat{\tau})) \leq 0. \]
The proof of the case $p > 0$ is now complete. As we mentioned at the beginning, the proof for the cases $p = 0$ and $p < 0$ can be done similarly and we leave the details to the reader. In the latter case, several inequalities need to be changed; for example, (4.14) should be reverted and (4.13) will become
\[ \begin{pmatrix} \lambda_1 \tilde{A}_1 \\ \lambda_2 \tilde{A}_2 \\ \vdots \\ \lambda_m \tilde{A}_m \end{pmatrix} \geq Z, \]
where $\tilde{A}_i = A_i + C\varepsilon I$ for $i = 1, 2, \ldots, n + 2$ this time.

\[ \square \]

5. Applications

Let us discuss applications of Theorem 1.1, Theorem 1.2 and Corollary 1.3 in this section. We will mainly verify (H1)$_p$ and (H2)$_p$ in Theorem 1.1 for various concrete examples of $F_i$. Most of our examples below satisfy the assumptions along with the conditions $1/2 \leq \alpha \leq 1$ and $p < 1$. 
5.1. The Laplacian. We are able to use Theorem 1.1 and Theorem 1.2 to recover the main results in [29, 30]. Let us first consider Theorem 1.1 when \( p \neq 0 \) and
\[
F_i(x, t, r, \xi, X) = -\text{tr} X - f_i(x, t, r, \xi),
\]
where we assume that \( f_i \geq 0 \) and (1.6) holds for \( g_i \) given in (1.7).

Taking \( k = 3 - 1/p \), we see that (H1) holds for any \( 1/2 \leq \alpha \leq 1 \) and \( 0 < p < 1 \). We can verify (H2) in this case with \( k = 3 - 1/p \). Since \( G_{i,k}^{p,\alpha} \) is given by (1.18) and (1.6) holds, it suffices to show that
\[
\sum_i \lambda_i H(r_i, X_i) \geq H \left( \sum_i \lambda_i r_i, Y \right),
\]
for \((x_i, t_i, r_i, X_i) \in Q_i \times (0, \infty) \times S^n \) (\( i = 1, 2, \ldots, m \)) satisfying (1.13) and (1.14), where
\[
H(r, X) = -\frac{1}{p} r^2 \text{tr} X,
\]
\((r, X) \in (0, \infty) \times S^n \).

In fact, multiplying both sides of (1.14) by \((r_1 \eta, \ldots, r_m \eta) \in \mathbb{R}^{mn} \) for an arbitrary \( \eta \in \mathbb{R}^n \) from left and right, we have
\[
\text{sgn}(p) \sum_i \lambda_i r_i^2 \langle X_i \eta, \eta \rangle \leq \text{sgn}(p) \left( \sum_i \lambda_i r_i \right)^2 \langle Y \eta, \eta \rangle;
\]
in other words,
\[
\text{sgn}(p) \sum_i \lambda_i r_i^2 X_i \leq \text{sgn}(p) \left( \sum_i \lambda_i r_i \right)^2 Y.
\]
Here \( \text{sgn}(p) \) denotes the sign of \( p \in \mathbb{R} \). This immediately implies that
\[
-\frac{1}{p} \sum_i \lambda_i r_i^2 \text{tr} X_i \geq -\frac{1}{p} \left( \sum_i \lambda_i r_i \right)^2 \text{tr} Y,
\]
which is equivalent to (5.1).

We can further use Corollary 1.3 to obtain the parabolic power concavity of the solution. Since the operator \( \overline{G} \) defined by (1.30) in this case is
\[
\overline{G}(x, t, r, \xi, X) = -\frac{r^2}{p} \text{tr} X - \left( \frac{1-p}{p^2} \right) r|\xi|^2 - r^{3-\frac{4}{p}} f \left( x, t, \frac{1}{\alpha}, r^\frac{1}{p}, \frac{1}{p} r^\frac{4}{p} - 1, \xi \right),
\]
the assumption (H2b) in Corollary 1.3 requires concavity of (1.29) in \( Q_\lambda \times (0, \infty) \).

We remark that, although the case of the Laplacian has been of course largely and deeply investigated, negative power concavity has never been considered before, to our knowledge.

We can treat the case \( p = 0 \) in an analogous way. When applying Theorem 1.1 in this case, noticing that (1.11) in (H1)_p is not required, we can choose \( k \in \mathbb{R} \) according to the given nonlinear terms \( f_i \) so that (H2)_p holds. We may take \( k = 1 \) provided that (1.6) holds with \( g_i \) given by (1.19). With such a choice, we can follow the argument in the case \( p > 0 \) to verify (5.1) under (1.13) and (1.14), where this time we take
\[
H(r, X) = -e^{2r} \text{tr} X, \quad (r, X) \in (0, \infty) \times S^n.
\]
5.2. **The normalized $q$-Laplacian.** We can apply our results to the normalized $q$-Laplacian operator with $1 < q < \infty$. Suppose that $F_i$ is given by

$$F_i(x, t, r, \xi, X) = -\text{tr} \left[ \left( I + (q - 2)\frac{\xi \otimes \xi}{|\xi|^2} \right) X \right] - f_i(x, t, r, \xi), \quad (5.2)$$

where $1 < q < \infty$ and $f_i \geq 0$ ($i = \lambda, 1, \ldots, m$). We take $k = 3 - 1/p$ and assume that (1.6) holds for $g_i$ in (1.7). Suppose that $1/2 \leq \alpha \leq 1$, $0 \neq p < 1$. Let us verify the assumption (H2)$_p$ with $p \neq 0$ again in this case.

Similar to the case $q = 2$ in Section 5.1, the key is to prove that for any fixed $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\sum_i \lambda_i H_q(r_i, \xi, X_i) \geq H_q \left( \sum_i \lambda_i r_i, \xi, Y \right) \quad (5.3)$$

holds for any $(x_i, t_i, r_i, X_i) \in Q_i \times (0, \infty) \times S^n$ ($i = 1, 2, \ldots, m$) satisfying (1.13) and (1.14), where

$$H_q(r, \xi, X) = \frac{1}{p} r^2 \text{tr} \left[ \left( I + (q - 2)\frac{\xi \otimes \xi}{|\xi|^2} \right) X \right], \quad (r, \xi, X) \in (0, \infty) \times (\mathbb{R}^n \setminus \{0\}) \times S^n.$$

To see this, we first notice that

$$M(\xi) := I + (q - 2)\frac{\xi \otimes \xi}{|\xi|^2}$$

is a positive semi-definite matrix in $S^n$. We thus can write

$$H_q(r, \xi, X) = -\frac{1}{p} r^2 \text{tr} \left( M^{1/2}(\xi) X M^{1/2}(\xi) \right),$$

where $M^{1/2}$ is the (nonnegative) square root of $M$. If (1.14) holds, then by multiplying (1.14) by $(r_1 M^{1/2}(\xi) \eta, r_2 M^{1/2}(\xi) \eta, \ldots, r_m M^{1/2}(\xi) \eta) \in \mathbb{R}^{nm}$ from both sides for any $\eta \in \mathbb{R}^n$, we can obtain

$$\text{sgn}(p) \sum_i \lambda_i r_i^2 \text{tr}(M(\xi) X_i) \leq \text{sgn}(p) \left( \sum_i \lambda_i r_i \right)^2 \text{tr}(M(\xi) Y),$$

which immediately yields the desired property (5.3) for $H_q$.

In this case we also have the parabolic power concavity result in Corollary 1.3 provided that (1.29) is concave in $Q_\lambda \times (0, \infty)$ for any $\xi \in \mathbb{R}^n$. The operator $\overline{G}$ in (1.30) is now given by

$$\overline{G}(x, t, r, \xi, X) = -\frac{r^2}{p} \text{tr} \left[ \left( I + (q - 2)\frac{\xi \otimes \xi}{|\xi|^2} \right) X \right] + \frac{r}{p^2} (1 - p)(1 - q)|\xi|^2$$

$$- r^{3-\frac{1}{p}} f \left( x, t^{1/p}, r^{1/p}, \frac{1}{p} r^{\frac{1}{p}-1} \xi \right). \quad (5.4)$$

To show that $\overline{G}$ verifies (H2b), we again need to assume the concavity of (1.29) in $Q_\lambda \times (0, \infty)$ for any $\xi \in \mathbb{R}^n$.

We omit the discussion for the case $p = 0$, since it can be handled analogously under appropriate assumptions on $f_i$. 
5.3. General quasilinear operators. We can further extend the situation in Section 5.2 to more general quasilinear operators in the form of
\[ F_i(x, t, r, \xi, X) = -\text{tr}(A_i(x, \xi)X) - f_i(x, t, r, \xi), \]
where \( f_i \geq 0 \) and \( A_i(x, \xi) \) a given nonnegative matrix for any \( x \in \Omega_i \) and \( \xi \in \mathbb{R}^n \setminus \{0\} \) for all \( i = \lambda, 1, \ldots, m \). Let \( 1/2 \leq \alpha \leq 1, \ p < 1 \). We assume that \( A_i(x, \xi) \) is uniformly continuous and bounded in \( \bar{\Omega}_i \times (\mathbb{R}^n \setminus \{0\}) \) for all \( i = 1, 2, \ldots, m \).

Let us again only consider the case \( p \neq 0 \). Besides the condition (1.6) with \( g_i \) in (1.7), the assumption (H2b) with \( k = 3 - 1/p \) requires that
\[ \sum_{i=1}^{m} \lambda_i H_{A_i}(x_i, r_i, \xi, X_i) \geq H_{A_\lambda} \left( \sum_{i=1}^{m} \lambda_i x_i, \sum_{i=1}^{m} \lambda_i r_i, \xi, Y \right) \] (5.5)
for \((x_i, t_i, r_i, X_i) \in Q_i \times (0, \infty) \times S^n \) \((i = 1, 2, \ldots, m)\) satisfying (1.13) and (1.14), where
\[ H_{A_i}(x, r, \xi, X) = -\frac{1}{p^2} \text{tr}(A_i(x, \xi)X) - \frac{1 - p}{p^2} r \langle A_i(x, \xi)\xi, \xi \rangle \] (5.6)
for \( i = \lambda, 1, \ldots, m \). This can be verified easily as in Section 5.2 when all \( A_i \) coincide and do not depend on the variable \( x \).

As for the application of Corollary 1.3, we see that \( \overline{G} \) in this case is given by
\[ \overline{G}(x, t, r, \xi, X) = -r^2 \text{tr}(A(x, \xi)X) - \frac{1 - p}{p^2} r \langle A(x, \xi)\xi, \xi \rangle - g(x, t, r, \xi), \]
where \( g \) is as in (1.29).

Since the first term on the right hand side can be handled analogously as in Section 5.2, we omit the details. Hence, a sufficient condition to guarantee the assumption (H2b) is the concavity of
\[ (x, t, r) \mapsto \frac{1 - p}{p^2} r \langle A(x, \xi)\xi, \xi \rangle + r^{3 - \frac{1}{p}} f \left( x, t, r, \frac{1}{r^{p-1}}, \frac{1}{p^{p-1}} \right) \]
in \( Q \times (0, \infty) \) for any fixed \( \xi \neq 0 \). In particular, if the coefficient matrix \( A \) does not depend on \( x \), i.e., \( A = A(\xi) \), then we require
\[ (x, t, r) \mapsto r^{3 - \frac{1}{p}} f \left( x, t, r, \frac{1}{r^{p-1}}, \frac{1}{p^{p-1}} \right) \]
is concave for any \( \xi \neq 0 \), as needed in the previous examples.

We remark that in addition to the normalized \( q \)-Laplacian discussed in Section 5.2, applicable quasilinear operators also include the so-called Finsler Laplacian as a special case. Recall that the Finsler-Laplace operator is defined by
\[ \mathcal{F}u = -\text{div}(J(\nabla u) \nabla J(\nabla u)), \]
where \( J : \mathbb{R}^n \to \mathbb{R} \) is a given nonnegative convex function of class \( C^2(\mathbb{R}^n \setminus \{0\}) \) which is positively homogeneous of degree 1, i.e.,
\[ J(k\xi) = |k|J(\xi), \quad \text{for all } k \in \mathbb{R}, \xi \in \mathbb{R}^n. \]
We can write
\[ \mathcal{F}u = -\text{tr} \left( A_J(\nabla u) \nabla^2 u \right), \]
where
\[ A_J(\xi) = \frac{1}{2} \nabla^2 J^2(\xi). \]
The homogeneity and regularity of the function \( J \) imply that the coefficient matrix \( A_J \) is bounded and continuous in \( \mathbb{R}^n \setminus \{0\} \).
It is now easily seen that Theorem 1.1 does apply to the equations with
\[ F_i(x,t,r,\xi,X) = -\text{tr} (A_j(\xi)X) - f_i(x,t,\xi), \quad i = \lambda, 1, \ldots, m. \]
Note that the boundedness and continuity of \( A_j \) in \( \mathbb{R}^n \setminus \{0\} \) enable us to apply the standard viscosity theory to equations involving \( F_i \); see basic structure assumptions (F1)–(F5) in Appendix A.1 for well-posedness.

Moreover, since in this case \( H_{A_i} \) in (5.6) is given by
\[ H_{A_i}(x,r,\xi,X) = -\frac{1}{p} r^2 \text{tr}(A_j(\xi)X) - \frac{1-p}{p^2} r \langle A_j(\xi)\xi,\xi \rangle, \]
for \( i = \lambda, 1, \ldots, m \), we can show that (5.5) holds for \( (x_i,t_i,r_i,X_i) \in Q_i \times (0,\infty) \times \mathbb{S}^n \) \( (i = 1,2,\ldots,m) \) satisfying (1.13) and (1.14), due to the convexity and nonnegativity of \( J \).

One can use a similar argument to justify the application of Corollary 1.3 to the Finsler Laplacian.

5.4. The Pucci operator. A typical example of fully nonlinear operators is the Pucci operator
\[ \mathcal{M}_{a,b}(X) = \inf_{\alpha I \leq A \leq \beta I} \text{tr}(AX) = a \sum_{e_i \geq 0} ae_i + b \sum_{e_i < 0} be_i, \]
where \( 0 < a \leq b \) are given and \( e_i = e_i(X) \) denotes the eigenvalues of any \( X \in \mathbb{S}^n \).

Consider
\[ F_i(x,t,r,\xi,X) = -\mathcal{M}_{a,b}(X) - f_i(x,t,r,\xi) \quad (5.7) \]
for \( (x,t) \in \overline{Q}, \ r \in [0,\infty), \ \xi \in \mathbb{R}^n \) and \( X \in \mathbb{S}^n \). As in the examples in Section 5.1 and Section 5.2, we again assume that \( f_i \) is nonnegative and satisfies the relation (1.6) with \( g_i \) defined in (1.7).

Assume that \( 1/2 \leq \alpha \leq 1, \ p < 1 \) and \( p \neq 0 \) so that (H1)_p holds with \( k = 3 - 1/p \). With such a choice of \( k \), we can also verify (H2)_p. In fact, the operator \( G_{i,3-1/p} \) in this case reads
\[ G_{i,3-1/p}(x,t,r,\xi,X) = \sup_{\alpha I \leq A \leq \beta I} H_A(r,\xi,X) - g_i(x,t,r,\xi), \]
where
\[ H_A(r,\xi,X) = -\frac{p^2}{p} \text{tr}(AX) - \frac{(1-p)r}{p^2} \langle A\xi,\xi \rangle. \]
As shown in Section 5.3, for any fixed \( A \in \mathbb{S}^n \) such that \( \alpha I \leq A \leq \beta I \) and \( \lambda \in \Lambda_m \), by (1.6) we have
\[ \sum_i \{ \lambda_i H_A(r_i,\xi,X_i) - \lambda_i g_i(x_i,t_i,r_i,\xi) \} \geq H_A \left( \sum_i \lambda_i r_i,\xi,Y \right) - g_0 \left( \sum_i \lambda_i x_i, \sum_i \lambda_i t_i, \sum_i \lambda_i r_i,\xi \right) \]
for any \( (x_i,t_i,r_i,X_i) \in Q_i \times (0,\infty) \times \mathbb{S}^n \) \( (i = 1,2,\ldots,m) \) satisfying (1.13)–(1.14). Maximizing both sides over \( \alpha I \leq A \leq \beta I \), we are led to
\[ \sum_i \left\{ \lambda_i \sup_{\alpha I \leq A \leq \beta I} H_A(r_i,\xi,X_i) - \lambda_i g_i(x_i,t_i,r_i,\xi) \right\} \geq \sup_{\alpha I \leq A \leq \beta I} H_A \left( \sum_i \lambda_i r_i,\xi,Y \right) - g_0 \left( \sum_i \lambda_i x_i, \sum_i \lambda_i t_i, \sum_i \lambda_i r_i,\xi \right), \]
which completes the verification of $(H2)_p$. Similar applications can be obtained in the case $p = 0$. One needs to fix $k \in \mathbb{R}$ in accordance with assumptions on $f_i (i = \lambda, 1, 2, \ldots, m)$.

We can therefore use Corollary 1.3 to give a corresponding parabolic power concavity result. Suppose that $f$ is a given nonnegative continuous function and (1.29) is concave with respect to $(x, t, r)$. Noticing that $G$ in (1.30) in this case is

$$G(x, t, r, \xi, X) = \sup_{aI \leq A \leq bI} H_A(r, \xi, X) - g(x, t, r, \xi),$$

we can show that it satisfies $(H2_b)$.

We remark that although the result of Theorem 1.1 holds for the operator in (5.7), in general it may not apply to the other type of Pucci operator, which reads

$$M_{a, b}^+(X) = \sup_{aI \leq A \leq bI} \text{tr}(AX) = a \sum_{e_i \leq 0} a e_i + b \sum_{e_i > 0} b e_i, \quad X \in \mathbb{S}^n.$$ 

Note that $-M_{a, b}^-(X)$ is convex in $X$ but $-M_{a, b}^+(X)$ is concave.

5.5. Porous medium equation. We also show an application of our concavity result to the porous medium equation. Suppose that the equation (1.2) reduces to

$$\partial_t u - \Delta (u^\sigma) = f_i(x, t, u, \nabla u) \quad \text{in } \Omega \times (0, \infty)$$

for a given $\sigma > 1$ and $f_i \geq 0$ satisfying assumptions to be specified later. In this case, the elliptic operator $F_i$ becomes

$$F_i(x, t, r, \xi, X) = -\sigma r^{\sigma-1} \text{tr} X - \sigma (\sigma - 1) r^{\sigma-2} |\xi|^2 - f_i(x, t, r, \xi).$$

The situation for this operator is different from the previous applications. In the case $p \neq 0$, we are actually not able to apply Theorem 1.1 with $k = 3 - 1/p$ or obtain a corresponding concavity result through Corollary 1.3, since our operators hardly satisfy $(H2)_p$ no matter what assumption is imposed on $f_i$. Instead, we use Theorem 1.1 with

$$k = \frac{\sigma}{p} - 3$$

so as to meet the requirement $(H2)_p$. Note that due to the choice of $k$ as in (5.9), we have

$$G_{i, k}^{p, \alpha}(x, t, r, \xi, X) = -\sigma r^{\sigma-2} \text{tr} X - \frac{\sigma(\sigma - p)}{p^2} r^\alpha |\xi|^2 - r^{\sigma-\frac{2}{p}} f_i \left( x, t^\frac{1}{p}, r^\frac{1}{p}, \frac{1}{p} r^{\frac{1}{p}-1} \xi \right),$$

where the major part (the first and second terms) is convex in the sense of $(H2)_p$. (They are in fact the same as those major terms in the previous examples.) This is precisely the reason why we chose $k$ as in (5.9).

Moreover, the assumption on $f_i$ is still as in (1.6) but with $g_i$ given by

$$g_i(x, t, r) = r^{3 - \frac{\sigma}{p}} f_i \left( x, t^\frac{1}{p}, r^\frac{1}{p}, \frac{1}{p} r^{\frac{1}{p}-1} \xi \right).$$

In order to meet the condition $(H1)_p$, we need to additionally assume that

$$\text{either } 2p - \sigma + 1 \leq 0 \quad \text{or } \frac{p}{2p - \sigma + 1} \leq \alpha \leq 1. \quad (5.10)$$

We again omit the details for the case $p = 0$. 

For the concavity result in Theorem 1.2, we can make the same choice of parameters $k, \alpha$ and $p$ as in (5.9) and (5.10), and assume that
\[(x, t, r) \mapsto r^{3-\frac{2}{p}} f \left( x, t^{\frac{1}{n}}, r^{\frac{1}{p}}, \frac{1}{p} r^{\frac{1}{p}-1} \xi \right) \tag{5.11} \]
is concave for any $\xi$.

**Remark 5.1.** The condition (5.10) means that, for any given $\sigma > 1$, in order to obtain parabolic power concavity with $p \in (-\infty, 1)$ satisfying $\sigma \geq 2p + 1$, we basically have no restrictions on $\alpha \in (0, 1)$ except for (5.11) when $p \neq 0$ (or a variant of (5.11) when $p = 0$). On the other hand, for $p \in (-\infty, 1)$ fulfilling $\sigma < 2p + 1$, (5.10) requires that $p \geq \sigma - 1$ so as to allow room for $\alpha$; there is no result for the range $(\sigma - 1)/2 < p < \sigma - 1$ no matter what $\alpha$ is taken.

Note that when $\sigma = 1$ the conditions (5.11) and (5.10) reduce to the assumptions for the Laplacian case as discussed in Section 5.1.

**Appendix A. Related issues on viscosity solutions**

Let us discuss several important properties of viscosity solutions, which are used in the previous sections. We first review well-posedness for general fully nonlinear parabolic equations and then give sufficient conditions for the time monotonicity of solutions and a Hopf-type property.

**A.1. Well-posedness.** Let $\Omega$ be a bounded smooth domain. We below consider the equation (2.3) in $Q = \Omega \times (0, \infty)$ with a homogeneous initial boundary condition, i.e.,
\[
\begin{align*}
&\left\{ \begin{array}{ll}
\partial_t u + F(x, t, u, \nabla u, \nabla^2 u) = 0 & \text{in } Q, \\
u = 0 & \text{on } \partial Q.
\end{array} \right. 
\end{align*}
\tag{A.1}
\]
We denote by $\nu$ the inward unit normal vector to $\partial \Omega$. Set $\tilde{\nu}(x) = \nu(x)$ if $x \in \partial \Omega$ and $\tilde{\nu}(x) = 0$ if $x \in \Omega$.

Let us impose the following basic structure assumptions on $F$:

(F1) $F : \overline{\Omega} \times [0, \infty) \times \mathbb{R} \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n \to \mathbb{R}$ is continuous.
(F2) $F$ is degenerate elliptic, i.e., (2.4) holds for all $x \in \overline{\Omega}$, $t \in [0, \infty)$, $r \in [0, \infty)$, $\xi \in \mathbb{R}^n \setminus \{0\}$ and $X_1, X_2 \in \mathbb{S}^n$ satisfying $X_1 \geq X_2$.
(F3) $F$ is monotone in the sense that there exists $c \in \mathbb{R}$ such that (2.5) holds for all $(x, t, \xi, X) \in \overline{\Omega} \times [0, \infty) \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n$ and $r_1, r_2 \in [0, \infty)$ satisfying $r_1 \leq r_2$.
(F4) For any $R > 0$ there exists a modulus of continuity $\omega_R$ such that
\[|F(x, t, r, \xi, X) - F(y, t, r, \xi, X)| \leq \omega_R(|x - y| + 1) \]
for all $x, y \in \overline{\Omega}$, $t \in [0, \infty)$, $|r| \leq R$, $\xi \in \mathbb{R}^n \setminus \{0\}$, $X \in \mathbb{S}^n$.
(F5) There exists a continuous function $h : \overline{Q} \times [0, \infty) \to \mathbb{R}$ such that
\[h(x, t, r) = (F)_u(x, t, r, 0, 0) = (F)_v(x, t, r, 0, 0) \quad \text{for } (x, t, r) \in Q \times (0, \infty) \tag{A.3}\]

Under these assumptions, viscosity solutions (sub- and supersolutions) of (A.1) are defined as in Section 2.2. It is known that the following comparison theorem holds.

**Theorem A.1** (Theorem 3.6.1 in [18]). Assume that $\Omega$ is bounded and (F1)–(F5) hold. Let $u$ and $v$ be respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution of (A.1). If $u \leq v$ on $\partial Q$, then $u \leq v$ in $\overline{Q}$.
Uniqueness of viscosity solutions to (A.1)–(A.2) is an immediate consequence of Theorem A.1. One can obtain existence of a positive viscosity solution by adopting Perron’s method for viscosity solutions if a positive subsolution exists; see precise arguments in [31] and [13, Section 4] for nonsingular equations and [18, Section 2.4] for singular one.

Moreover, a standard argument ([18, Theorem 2.2.1] for instance) yields that the unique solution $u$ is stable in the sup norm under uniform perturbation of the operator and initial boundary data.

As for the spatial Lipschitz regularity, which is needed in Theorem 1.1, we refer to relevant results in the literature. Lipschitz or Hölder regularity of viscosity solutions to fully nonlinear nonsingular parabolic equations is given in [3, 45, 46, 36, 5, 4] etc. We also consult local Lipschitz estimates for singular parabolic equations such as the normalized $q$-Laplace equations in [16, 41, 32] ($1 < q < ∞$) and in [34] ($q = ∞$).

A.2. Monotonicity in time. The next two subsections are devoted to discussion on the assumptions (i) and (ii) in Theorem 1.1 (and in Theorem 1.2) for $p ∈ (0,1)$. Since it is in general quite restrictive to assume (i) and (ii) on $F$, we consider the approximate equation (1.21). We will actually provide sufficient conditions to guarantee (i)(ii) for (1.21) instead of (1.2).

Let us first study the time monotonicity in (i). Suppose that
\begin{equation}
  h(x,0,0) = F_u(x,0,0,0,0) = F^*(x,0,0,0,0) \leq 0, \quad \text{(A.4)}
\end{equation}
and
\begin{equation}
  F(x,t,r,p,X) \leq F(x,s,r,p,X) \quad \text{for any } t \geq s \geq 0 \quad \text{(A.5)}
\end{equation}
\[\text{and } (x,r,p,X) \in \overline{\Omega} \times [0,∞) \times (\mathbb{R}^n \setminus \{0\}) \times S^n.\]

**Lemma A.2** (Monotonicity in time). Assume that $\Omega$ is bounded and $F$ satisfies (F1)–(F5). If (A.4) and (A.5) hold, then the unique solution $u$ of (1.2)–(1.3) satisfies (1.16).

**Proof.** The assumption (A.4) and (A.5) imply that the constant zero is a subsolution of (1.2)–(1.3). If follows that $u \geq 0$ in $\overline{Q}$ by the comparison principle. Fix $τ > 0$ arbitrarily and set $w_τ(x,t) = u(x,t + τ)$ for all $(x,t) \in \overline{Q}$. Then by (A.5) we can easily show that $w_τ$ is a supersolution of (1.2). Since $w_τ(\cdot,0) = u(\cdot,τ) \geq u(\cdot,0)$ in $\overline{\Omega}$, we can use the comparison principle again to prove that $w_τ \geq u$ in $\overline{Q}$, which immediately yields (1.16) due to the arbitrariness of $τ$. \[\Box\]

A more specific situation fulfilling (A.4), (A.5) and other assumptions needed in our main results is the case when
\[F(x,t,r,ξ,X) = L(ξ,X) - f(x,t,r)\]
for all $(x,t,r,ξ,X) \in \overline{Q} \times [0,∞) \times (\mathbb{R}^n \setminus \{0\}) \times S^n$, where $f$ is nonnegative in $\overline{Q} \times [0,∞)$ and nondecreasing in $t$, and $L(0,0) = L^*(0,0) = 0$. Concrete examples of $L$ include the Pucci operator, normalized $q$-Laplacian ($1 < q ≤ ∞$) and more general quasilinear operators as discussed in Section 5.

We next discuss the assumption (ii) in Theorem 1.1 and Theorem 1.2. Assume that $0 < p < 1$ for the rest of this section. Note that the condition (1.17) can be divided into two parts. One part is the following growth behavior near the initial moment:
\begin{equation}
  \frac{1}{ρ}u^p(x + \tilde{ρ}(x)ρ, ρ^{1/α}) \to ∞ \quad \text{as } ρ \to 0 \text{ for every } x \in \overline{Ω}. \quad \text{(A.6)}
\end{equation}
The other part can be expressed by
\[ \frac{1}{p} u^p (x + \nu(x) \rho, t) \to \infty \quad \text{as } \rho \to 0 \text{ for every } x \in \partial \Omega \text{ and } t > 0. \]  \hspace{1cm} (A.7)

We will later see that for \( p \in (0, 1) \) (A.7) is a consequence of the Hopf lemma; consult Section A.3.

In order to see that (A.6) holds, we need to strengthen the condition (A.4) in Lemma A.2.

**Lemma A.3** (Rapid initial growth). Let \( 0 < p < 1 \). Assume that \( \Omega \) is bounded and \( F \) satisfies \((F1)\)–\((F5)\). Assume that \((A.5)\) holds. Assume that

\[
\begin{cases}
\text{there exist } \beta, \beta', t_0 > 0 \text{ and } \psi_0 \in C^2(\overline{\Omega}) \text{ with } \psi_0 > 0 \text{ in } \Omega, \psi_0 = 0 \text{ on } \partial \Omega \text{ such that} \\
\frac{p\beta'}{\alpha} + \frac{p\beta}{\alpha} < 1, \quad \sup_{\Omega} \text{dist}(\cdot, \partial \Omega)^{\beta'} < \infty, \quad \text{and} \\
\beta \psi_0(x)t^{\beta-1} + F(x, t, \psi_0(x)t^\beta, \nabla \psi_0(x)t^\beta, \nabla^2 \psi_0(x)t^\beta) \leq 0 \quad \text{for all } (x, t) \in \Omega \times (0, t_0).
\end{cases}
\]  \hspace{1cm} (A.8)

Then the unique solution \( u \) of \((1.2)\)–\((1.3)\) satisfies \((A.6)\).

**Proof.** By the last inequality in \((A.8)\) we observe that

\[
(x, t) \mapsto \psi_0(x)t^\beta
\]

is a subsolution of \((1.2)\) restricted in \( \Omega \times (0, t_0) \). Noticing that \( \psi_0 = 0 \) on \( \partial \Omega \), we can use the comparison principle to obtain that

\[
u(x, t) \geq \psi_0(x)t^\beta,
\]

for \((x, t) \in \Omega \times (0, t_0) \). When \( x \in \Omega \), we easily deduce \((A.6)\), since \( \psi_0 > 0 \) in \( \Omega \) and \( \beta < \alpha/p \).

If \( x \in \partial \Omega \), noticing that \( \psi_0(x + \rho \nu(x)) \geq c \rho^{\beta'} \) in \( \Omega \) for some \( c > 0 \), we have

\[
u^p \left( x + \rho \nu(x), \rho^{1/\alpha} \right) \geq c^p \rho^{p\beta' + p\beta/\alpha},
\]

which implies \((A.6)\), due to the condition that \( p\beta' + p\beta/\alpha < 1 \). \( \square \)

Let us discuss how to apply Lemma A.3 in our applications under the assumption \( \alpha \geq p \).

Suppose that \( F_i \) (\( i = \lambda, 1, \ldots, m \)) satisfies \((A.4)\) and \((A.5)\), i.e.,

\[
h_i(x, 0, 0) = (F_i) \ast (x, 0, 0, 0, 0) = (F_i)^\ast(x, 0, 0, 0, 0) \leq 0,
\]

\[
F_i(x, t, r, p, X) \leq F_i(x, s, r, p, X) \quad \text{for any } t \geq s \geq 0
\]

and \((x, r, p, X) \in \overline{\Omega}_i \times [0, \infty) \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^n \).

As mentioned in the beginning of this section, these assumptions in general may not guarantee \((A.8)\) for \( F = F_i \). However, we can first turn to study the perturbed equation \((1.21)\) first and then let \( \varepsilon \to 0 \). In addition to the perturbation for the operators, we put \( p_\varepsilon = p - \varepsilon \) with \( \varepsilon > 0 \) small so that \( \alpha > p_\varepsilon \). We can show \((A.8)\) holds for \( p = p_\varepsilon \) and \( F = F_i, \varepsilon \) for any \( i = \lambda, 1, \ldots, m \). Indeed, we can choose

\[
1 < \beta < \frac{\alpha}{p_\varepsilon}, \quad 0 < \beta' < 1 - \frac{\beta p_\varepsilon}{\alpha},
\]

and \( \psi_0 \in C^2(\overline{\Omega}) \) such that \( \psi_0 = \text{dist}(\cdot, \partial \Omega)^{\beta'} \) near \( \partial \Omega \). Then we can verify the last inequality in \((A.8)\) with \( F = F_i, \varepsilon \) and \( p = p_\varepsilon \) provided that \( t_0 \) is sufficiently small.
A.3. The Hopf-type property. We finally discuss the property (A.7), which is used to derive the condition (ii) of Theorem 1.1. It is in fact related to the so-called Hopf-type property:

\((\text{HP})\) Fix any \(x_0 \in \partial \Omega \) and \(t_0 > 0\). Assume that there exist \(0 < \delta < t_0 \) and \(y_0 \in \Omega \) such that

\begin{itemize}
  \item \(B_\delta(y_0) \subset \Omega \) and \(\overline{B_\delta(y_0)} \cap \partial \Omega = \{x_0\}\);
  \item \(u\) is a supersolution of (A.1);
  \item \(u\) satisfies \(u(x, t) \geq u(x_0, t_0) = 0\) for any \((x, t) \in \overline{B_\delta(y_0)} \times [t_0 - \delta, t_0 + \delta]\).
\end{itemize}

Then

\[ \liminf_{\rho \to 0^+} \frac{1}{\rho} u \left( x_0 + \rho \frac{y_0 - x_0}{|y_0 - x_0|}, t_0 \right) > 0. \]

It is obvious that (A.7) is an immediate consequence of (HP). See [15, 20, 8] for sufficient conditions on \(F\) in order to obtain (HP).

For our own purpose in this work, following the same method described in Section (A.2), we can use (HP) for the approximate problem (1.21), where \(F_{i, \varepsilon}\) is the perturbed operator given in (1.20). It turns out that we still only need to assume (A.4) and (A.5) for \(F = F_i\) so as to get (HP) for (1.21).

Note that (A.4) and (A.5) applied to (1.21) yield that

\[ h_i(x, t, 0) = (F_i)_+(x, t, 0, 0, 0) = (F_i)^*(x, t, 0, 0, 0) \leq 0 \quad \text{for all} \quad (x, t) \in \overline{\Omega_i}. \quad (A.9) \]

Let \(u_{i, \varepsilon}\) be a supersolution of (1.21). Denote \(\zeta_0 = (y_0, t_0)\) and \(z = (x, t)\). By assumption, we have \(u_{i, \varepsilon} \geq u_{i, \varepsilon}(\zeta_0)\) in \(\overline{B_\delta(\zeta_0)}\). We take

\[ v_\gamma(z) = e^{-\gamma |z - \zeta_0|^2} - e^{-\gamma \delta^2} \]

with \(\gamma > 0\) large. Since

\[ \sup_{\overline{B_\delta(y_0)} \times [t_0 - \delta, t_0 + \delta]} \left( |v_\gamma| + |\partial_t v_\gamma| + |\nabla v_\gamma| + |\nabla^2 v_\gamma| \right) \to 0 \quad \text{as} \quad \gamma \to \infty, \]

it follows from (A.9) and (F5) that, when \(\gamma > 0\) is large,

\[ \partial_t v_\gamma(z) + (F_i, \varepsilon)^*(z, v_\gamma(z), \nabla v_\gamma(z), \nabla^2 v_\gamma(z)) \leq \partial_t v_\gamma(z) + (F_i)^*(z, v_\gamma(z), \nabla v_\gamma(z), \nabla^2 v_\gamma(z)) - \varepsilon \leq -\frac{\varepsilon}{2} \]

for any \(z \in B_\delta(y_0) \times (t_0 - \delta, t_0 + \delta)\).

We have shown that \(v_\gamma\) is a subsolution of (1.21) in \(B_\delta(y_0) \times (t_0 - \delta, t_0 + \delta)\). Noticing that

\[ u_{i, \varepsilon} \geq v_\gamma \quad \text{on} \quad \left( \overline{B_\delta(y_0)} \times \{t_0\} \right) \cup (\partial B_\delta(y_0) \times (t_0 - \delta, t_0 + \delta)), \]

by comparison principle we have

\[ u_{i, \varepsilon} \geq v_\gamma \quad \text{in} \quad \overline{B_\delta(y_0)} \times [t_0 - \delta, t_0 + \delta], \]

which implies that

\[ u_{i, \varepsilon} \left( x_0 + \rho \frac{y_0 - x_0}{|y_0 - x_0|}, t_0 \right) \geq v_\gamma \left( x_0 + \rho \frac{y_0 - x_0}{|y_0 - x_0|}, t_0 \right) \geq \rho \gamma |x_0 - y_0| e^{-\gamma |x_0 - y_0|^2} + o(\rho). \]

We thus complete the verification of (HP) for (1.21).
References

[1] O. Alvarez, J.-M. Lasry, and P.-L. Lions. Convex viscosity solutions and state constraints. J. Math. Pures Appl. (9), 76(3):265–288, 1997.

[2] A. Attouchi. Local regularity for quasi-linear parabolic equations in non-divergence form. preprint, 2018.

[3] G. Barles. A weak Bernstein method for fully nonlinear elliptic equations. Differential Integral Equations, 4(2):241–262, 1991.

[4] G. Barles. Local gradient estimates for second-order nonlinear elliptic and parabolic equations by the weak Bernstein’s method. preprint, 2017.

[5] G. Barles and P. E. Souganidis. Space-time periodic solutions and long-time behavior of solutions to quasi-linear parabolic equations. SIAM J. Math. Anal., 32(6):1311–1323, 2001.

[6] B. Bian and P. Guan. A microscopic convexity principle for nonlinear partial differential equations. Invent. Math., 177(2):307–335, 2009.

[7] C. Bianchini, P. Salani. Concavity properties for elliptic free boundary problems. Nonlinear Anal. 71 (2009), no. 10, 4461–4470.

[8] L. Caffarelli, Y. Li, and L. Nirenberg. Some remarks on singular solutions of nonlinear elliptic equations III: viscosity solutions including parabolic operators. Comm. Pure Appl. Math., 66(1):109–143, 2013.

[9] A. Colesanti. Brunn-Minkowski inequalities for variational functionals and related problems. Adv. Math.194 (2005), 105-140.

[10] A. Colesanti, P. Cuoghi. The Brunn-Minkowski inequality for the n-dimensional logarithmic capacity of convex bodies. Potential Anal. 22 (2005), no. 3, 289-304.

[11] A. Colesanti, P. Cuoghi, P. Salani. Brunn-Minkowski inequalities for two functionals involving the p-Laplace operator. Appl. Anal.85 (2006), 45-66.

[12] A. Colesanti, P. Salani. The Brunn-Minkowski inequality for p-capacity of convex bodies. Math. Ann. 327 (2003), 459-479.

[13] M. G. Crandall, H. Ishii, and P.-L. Lions. User’s guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.), 27(1):1–67, 1992.

[14] G. Crasta, I. Fraga, The Brunn-Minkowski inequality for the principal eigenvalue of fully nonlinear homogeneous elliptic operators preprint 2019.

[15] F. Da Lio. Remarks on the strong maximum principle for viscosity solutions to fully nonlinear parabolic equations. Commun. Pure Appl. Anal., 3(3):395–415, 2004.

[16] K. Does. An evolution equation involving the normalized p-Laplacian. Commun. Pure Appl. Anal., 10(1):361–396, 2011.

[17] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. I. J. Differential Geom., 33(3):635–681, 1991.

[18] Y. Giga. Surface evolution equations, volume 99 of Monographs in Mathematics. Birkhäuser Verlag, Basel, 2006. A level set approach.

[19] Y. Giga, S. Goto, H. Ishii, and M.-H. Sato. Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains. Indiana Univ. Math. J., 40(2):443–470, 1991.

[20] G. Gripenberg. On the strong maximum principle for degenerate parabolic equations. J. Differential Equations, 242(1):72–85, 2007.

[21] G. H. Hardy, J. E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, Cambridge, 1959.

[22] K. Ishige, Q. Liu and P. Salani. Concavity principles for solutions of elliptic and parabolic boundary value problems. In preparation.

[23] K. Ishige, K. Nakagawa and P. Salani. Power concavity in weakly coupled elliptic and parabolic systems. Nonlinear Analysis 131 (2016), 81-97.

[24] K. Ishige, K. Nakagawa and P. Salani. Spatial concavity of solutions to parabolic systems. Preprint 2018, to appear in Ann. SNS Pisa Cl. Sci.

[25] K. Ishige and P. Salani. Is quasi-concavity preserved by heat flow? Archiv der Mathematik, vol. 90 n.5 (2008), 450–460.

[26] K. Ishige and P. Salani. Convexity breaking of the free boundary for porous medium equations. Interfaces and Free Boundaries 12 (2010), 75-84.

[27] K. Ishige and P. Salani. On a new Kind of convexity for solutions of parabolic problems. Discrete and Continuous Dynamical Systems Series S, (DCDS-S) Vol. 4 n. 4 (2011), 851-864.

[28] K. Ishige and P. Salani. A note on parabolic power concavity. Kodai Math. J., 37(3):668–679, 2014.

[29] K. Ishige and P. Salani. Parabolic power concavity and parabolic boundary value problems. Math. Ann., 358(3-4):1091–1117, 2014.

[30] K. Ishige and P. Salani. Parabolic Minkowski convolutions of solutions to parabolic boundary value problems. Adv. Math., 287:640–673, 2016.
[31] H. Ishii. Perron’s method for Hamilton-Jacobi equations. Duke Math. J., 55(2):369–384, 1987.
[32] T. Jin and L. Silvestre. Hölder gradient estimates for parabolic homogeneous $p$-Laplacian equations. J. Math. Pures Appl. (9), 108(1):63–87, 2017.
[33] P. Juutinen. Concavity maximum principle for viscosity solutions of singular equations. NoDEA Nonlinear Differential Equations Appl., 17(5):601–618, 2010.
[34] P. Juutinen and B. Kawohl. On the evolution governed by the infinity Laplacian. Math. Ann., 335(4):819–851, 2006.
[35] B. Kawohl. Rearrangements and convexity of level sets in PDE, volume 1150 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1985.
[36] B. Kawohl and N. Kutev. Comparison principle and Lipschitz regularity for viscosity solutions of some classes of nonlinear partial differential equations. Funkcial. Ekvac., 43(2):241–253, 2000.
[37] A. U. Kennington. Power concavity and boundary value problems. Indiana Univ. Math. J., 34(3):687–704, 1985.
[38] N. J. Korevaar. Convex solutions to nonlinear elliptic and parabolic boundary value problems. Indiana Univ. Math. J., 32(4):603–614, 1983.
[39] P. Liu, X.-N. Ma, L. Xu. A Brunn–Minkowski inequality for the Hessian eigenvalue in three-dimensional convex domain. Adv. Math. 225 (2010) 1616–1633.
[40] Q. Liu, A. Schikorra, and X. Zhou. A game-theoretic proof of convexity preserving properties for motion by curvature. Indiana Univ. Math. J., 65:171–197, 2016.
[41] M. Parviainen and E. Ruosteenoja. Local regularity for time-dependent tug-of-war games with varying probabilities. J. Differential Equations, 261(2):1357–1398, 2016.
[42] P. Salani. A Brunn-Minkowski inequality for the Monge-Ampère eigenvalue. Adv. Math.194(2005), 67-86.
[43] P. Salani. Convexity of solutions and Brunn-Minkowski inequalities for Hessian equations in $R^3$. Adv. Math.229(2012), 1924-1948.
[44] P. Salani. Combination and mean width rearrangements of solutions of elliptic equations in convex sets. Ann. Inst. H. Poincaré Anal. Non Linéaire, 32(4):763–783, 2015.
[45] L. Wang. On the regularity theory of fully nonlinear parabolic equations. I. Comm. Pure Appl. Math., 45(1):27–76, 1992.
[46] L. Wang. On the regularity theory of fully nonlinear parabolic equations. II. Comm. Pure Appl. Math., 45(2):141–178, 1992.

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