2D REDUCTIONS OF THE EQUATION $u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$
AND THEIR NONLOCAL SYMMETRIES

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ABSTRACT. We consider the 3D equation $u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$ and its 2D reductions:
(1) $u_{yy} = (u_y + y)u_{xx} - u_x u_{xy} - 2$ (which is equivalent to the Gibbons-Tsarev equation) and
(2) $u_{yy} = (u_y + 2x)u_{xx} + (y - u_x)u_{xy} - u_x$. Using reduction of the known Lax pair for the
3D equation, we describe nonlocal symmetries of (1) and (2) and show that the Lie algebras
of these symmetries are isomorphic to the Witt algebra.

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INTRODUCTION

The equation
\[ u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy} \]  
belongs to the class of linearly degenerate integrable equations [6] and was, as far as we know, introduced first in [9] and independently in [10, 5]. In [1], we described all two-dimensional symmetry reductions of this equation. All these reductions are either linearizable or exactly solvable, except for the two ones:

\[ u_{yy} = (u_y + y)u_{xx} - u_x u_{xy} - 2 \]  
and

\[ u_{yy} = (u_y + 2x)u_{xx} + (y - u_x)u_{xy} - u_x. \]  

Equation (2) is reduced to the Gibbons-Tsarev equation [7] by the simple transformation \( u \mapsto u - y^2/2 \).

Equation (1) admits the Lax pair

\[ \begin{align*}
  w_t &= \left( \lambda^2 - \lambda u_x - u_y \right) w_x, \\
  w_y &= \left( \lambda - u_x \right) w_x
\end{align*} \]  

with the non-removable spectral parameter \( \lambda \). In [2], we, in particular, studied the behavior of this Lax pair under the symmetry reduction.

In this paper, we use system (4) to describe nonlocal symmetries of Equations (2) and (3) and prove that in both cases these symmetries form the Lie algebra isomorphic to the Witt algebra

\[ \mathfrak{w} = \{ e_i = z^{i+1} \frac{\partial}{\partial z} | i \in \mathbb{Z} \}. \]

In Section 1, we recall some necessary results obtained in the previous research and introduce the notions and constructions needed for the subsequent exposition. Section 2 is devoted to the proofs of the basic results. We discuss in detail Equation (2) and briefly repeat the main step for the second equation, because the reasoning is quite similar in both cases. The obtained results are discussed in Section 3 together with further perspectives.

1. Preliminaries

We recall here basic facts from the theory of nonlocal symmetries (see [3]) and previous results on Equation (1) and its reductions, see [1, 2].

1.1. Basics. Consider a differential equation

\[ F^\alpha \left( x, \ldots, \frac{\partial^{|\sigma|} u}{\partial x^\sigma}, \ldots \right) = 0, \quad \alpha = 1, \ldots, r, \]  
of order \( k \) in unknowns \( u = (u^1, \ldots, u^m) \), where \( u^j = u^j(x) \), \( x = (x^1, \ldots, x^n) \). To the infinite prolongation of (3) there corresponds a locus \( \mathcal{E} \subset J^\infty(\pi) \) in the space of infinite jets of the trivial bundle \( \pi: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), see [4]. Consider jet coordinates \( u^j_\sigma \) on \( J^\infty(\pi) \), \( j = 1, \ldots, m \), where \( \sigma = i_1 \ldots i_s \) is a symmetric multi-index, \( i_\alpha = 1, \ldots, n \), and choose a set \( I = \{ u^j_\sigma \} \) of internal variables on \( \mathcal{E} \). Denote by

\[ D_{x^i} = \frac{\partial}{\partial x^i} + \sum_I u_{\sigma i} \frac{\partial}{\partial u^j_\sigma}, \quad i = 1, \ldots, n, \]

the operators of total derivatives on \( \mathcal{E} \). For any function \( f \) on \( J^\infty(\pi) \) we denote by \( \ell_f \) the restriction of its linearization \( \sum \partial f / \partial u^j_\sigma D_\sigma \) to \( \mathcal{E} \), where \( D_\sigma = D_{x^1} \circ \cdots \circ D_{x^i} \). A (local) symmetry of \( \mathcal{E} \) is an evolutionary vector field

\[ E_\varphi = \sum_I D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma}, \]

\footnote{We do not distinguish between scalar and multi-component systems.}
where the generating function $\varphi = (\varphi^1, \ldots, \varphi^m)$ satisfies the equation $\ell_F(\varphi) = 0$ and $F = (F^1, \ldots, F^r)$ determines Equation (4). We identify symmetries with their generating functions and denote the Lie algebra of symmetries by $\text{sym} \, \mathcal{E}$.

A (differential) covering over $\mathcal{E}$ is a bundle $\tau: \tilde{\mathcal{E}} = \mathcal{E} \times \mathbb{R}^l \to \mathcal{E}$, $l = 1, 2, \ldots, \infty$, endowed with vector fields

$$\tilde{D}_{x^i} = D_{x^i} + \sum_\alpha X^\alpha_i \frac{\partial}{\partial w^\alpha}, \quad i = 1, \ldots, n,$$

that pair-wise commute, where $w^\alpha$ are coordinates in $\mathbb{R}^l$ called nonlocal variables. Equivalently, $\tilde{\mathcal{E}}$ may be regarded as an overdetermined system

$$w^\alpha_{x^i} = X^\alpha_i, \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, l,$$

whose compatibility conditions are consequences of $\mathcal{E}$. For a linear differential operator $\Delta = (\sum_\sigma a_{ij}^\sigma D_\sigma)$ on $\mathcal{E}$ we denote by $\tilde{\Delta} = (\sum_\sigma a_{ij}^\sigma \tilde{D}_\sigma)$ its lift to $\tilde{\mathcal{E}}$.

Coverings $\tau_1$ and $\tau_2$ are said to be equivalent if there exists a diffeomorphism $g: \tilde{\mathcal{E}}_1 \to \tilde{\mathcal{E}}_2$ such that $\tau_2 \circ g = \tau_1$ and which takes $\tilde{C}_1$ to $\tilde{C}_2$, where $\tilde{C}_i$ is the span of the fields $\tilde{D}_{x^i}$ on $\tilde{\mathcal{E}}_i$.

A nonlocal symmetry of $\tilde{\mathcal{E}}$ in the covering $\tau$ is a vector field

$$S_\phi = \sum_1^{+\infty} \tilde{D}_\sigma (\varphi^i) \frac{\partial}{\partial u^\sigma} + \sum_\alpha \psi^\alpha \frac{\partial}{\partial w^\alpha},$$

where $\varphi = (\varphi^1, \ldots, \varphi^m)$ and $\psi^\alpha$ are functions on $\tilde{\mathcal{E}}$ satisfying the system

$$\tilde{\ell}_F(\varphi) = 0,$$

$$\tilde{D}_{x^i}(\psi^\alpha) = \tilde{\ell}_{X^i_\sigma}(\varphi) + \sum_\beta \frac{\partial X^\alpha_i}{\partial w^\beta}$$

for all $i$ and $\alpha$. Any nonlocal symmetry is identified with the collection $\Phi = (\varphi, \psi^1, \ldots, \psi^\alpha, \ldots)$. A symmetry is called a lift of a local symmetry if the component $\varphi$ is a function on $\mathcal{E}$; it is called invisible if $\varphi = 0$. Nonlocal symmetries form a Lie algebra denoted by $\text{sym}_\tau \, \mathcal{E}$.

1.2. Coverings over Equation (1) and symmetries. Let us assume that $w = w(\lambda)$ in Equation (1) and consider the expansion

$$w = \sum_{i=\infty}^{+\infty} \lambda^{-i} w_i.$$

This leads to the infinite-dimensional covering

$$w_{i,x} = w_{i-1,y} + u_x w_{i-1,x},$$
$$w_{i,y} = w_{i-1,t} + u_y w_{i-1,x},$$

(6)

where $i \in \mathbb{Z}$.

Recall now that the space $\text{sym} \, \mathcal{E}$ for Equation (1) is spanned by the functions

$$\theta_1 = 2x - yu_x, \quad \theta_2 = 3u - 2xu_x - yu_y,$$

$$\theta_3(T) = Tu_y + T'(yu_x - x) - \frac{1}{2} T''y^2, \quad \theta_4(T) = Tu_x - T'y,$$

$$\theta_5(T) = Tu_t + T'(xu_x + uy_y - u) + \frac{1}{2} T''(y^2u_x - 2xy) - \frac{1}{6} T'''y^3, \quad \theta_6(T) = T,$$

where $T$ is a function in $t$ and ‘prime’ denotes the $t$-derivatives. In what follows, we shall need the following

**Proposition 1.** The symmetries $\theta_1$, $\theta_2$, and $\theta_5 = \theta_5(1) = u_t$ can be lifted to the covering (5).
Proof. Let us set

\[
\begin{align*}
\theta^i_1 &= -yw_{i,x} + (i + 2)w_{i-1}, \\
\theta^i_2 &= -2xw_{i,x} - yw_{i,y} + (i + 3)w_i, \\
\theta^i_5 &= w_{i,t}.
\end{align*}
\]

Then

\[\Theta_j = (\theta_j, \ldots, \theta^i_j, \ldots), \quad j = 1, 2, 5, \quad i \in \mathbb{Z},\]

are the desired lifts. \(\square\)

1.3. Reductions. Using Proposition 1, we now state the following

Proposition 2. Reduction of (1) with respect to the symmetry \(\Theta_5 + \Theta_1\) leads to the covering

\[
\begin{align*}
w_{i,x} &= w_{i-1,y} + u_x w_{i-1,x}, \\
w_{i,y} &= (u_y + y)w_{i-1,x} - (i + 1)w_{i-2}
\end{align*}
\]

over Equation (2), while reduction with respect to \(\Theta_5 + \Theta_2\) leads to the covering

\[
\begin{align*}
w_{i,x} &= w_{i-1,y} + u_x w_{i-1,x}, \\
w_{i,y} &= (u_y + 2x)w_{i-1,x} + yw_{i-1,y} - (i + 2)w_{i-1}
\end{align*}
\]

over Equation (3).

Let us finally describe the algebras of local symmetries for Equations (2) and (3). Direct computations show that sym \(E_1\) for the first equation is spanned by the functions

\[\varphi_{-4} = 1, \quad \varphi_{-3} = u_x, \quad \varphi_{-2} = -u_y - y, \quad \varphi_{-1} = -2x + yu_x, \quad \varphi_0 = 4u - 3xu_x - 2yu_y, \quad (9)\]

while in the second case we have the following generators of the symmetry algebra sym \(E_2\):

\[\gamma_{-3} = 1, \quad \gamma_{-2} = -y - \frac{1}{2}u_x, \quad \gamma_{-1} = y^2 - 2x + 2yu_x - 2u_y, \quad \gamma_0 = 3u - 2xu_x - yu_y.\]

It is also easily seen that

for Equation (2)

\[\left[ \varphi_i, \varphi_j \right] = \begin{cases} (j - i)\varphi_{i+j} & \text{if } i + j \geq -4, \\ 0 & \text{otherwise}, \end{cases}\]

for Equation (3)

\[\left[ \gamma_i, \gamma_j \right] = \begin{cases} (j - i)\gamma_{i+j} & \text{if } i + j \geq -3, \\ 0 & \text{otherwise}. \end{cases}\]

Denote by \(\mathfrak{M}_0 \subset \mathfrak{M}\) the subalgebra in the Witt algebra spanned by the fields \(e_i\) with \(i \leq 0\) and by \(\mathfrak{M}_k\) its ideal spanned by \(e_i, \ i \leq k < 0\).

Proposition 3. For Equation (2), one has sym \(E_1 = \mathfrak{M}_0/\mathfrak{M}_5\), while sym \(E_2 = \mathfrak{M}_0/\mathfrak{M}_4\) for Equation (3).
In a similar way, for Equation (3) and its covering (10) we have \( r_{i_1}^p = 0 \) for \( i < -3 \), \( r_{-3}^p = 1 \), and
\[
r_{-2}^p = -(p+3)y, \quad r_{-1}^p = -(p+3)x + \frac{1}{2}(p+3)^2y^2, \quad r_0^p = -(p+3)u + (p+3)^2xy - \frac{1}{6}(p+3)^3y^3,
\]
while
\[
\begin{align*}
    r_{i,x}^p &= r_{i-1,y}^p + u_x r_{i-1,x}^p, \\
    r_{i,y}^p &= (u_y + 2x)r_{i-1,x}^p + y r_{i-1,y}^p - (p + i + 5)r_{i}^p 
\end{align*}
\]
for \( i \geq 1 \). This covering will be denoted by \( \sigma^p \).

2. The main result

We prove the main result of the paper in this section, which states that in coverings \( \tau^p \), naturally associated with (7) and (8) the algebras of nonlocal symmetries for both Equations (2) and (3) are isomorphic to the Witt algebra \( \mathfrak{w} \). The case of Equation (2) is considered in detail, while for Equation (3) we provide a sketch of proofs only.

2.1. Equivalence of the coverings \( \tau^p \). The first step of the proof is to establish the equivalence of different \( \tau^p \) to each other.

**Proposition 4.** The covering \( \tau^p \) is equivalent to \( \tau^{-4} \) for any \( p \).

**Proof.** We distinguish the two cases: \( p \neq -3 \) and \( p = 3 \).

**The case \( p \neq -3 \).** Introduce the notation \( s_i = r_i^{-4} \) and consider the vector field
\[
\mathcal{X} = x \frac{\partial}{\partial y} + 2u \frac{\partial}{\partial x} + 3s_1 \frac{\partial}{\partial u} + \sum_{i \geq 1} (i + 3)s_{i+1} \frac{\partial}{\partial s_i}.
\]
Let us define the quantities \( Q_{k,j} \), \( k \geq 0 \), by
\[
Q_{k,0} = \frac{1}{(k+2)!} y^{k+2}, \quad Q_{k,j+1} = \frac{1}{j} \mathcal{X}(Q_{k,j})
\]
and assume \( Q_{k,j} = 0 \) for \( j < 0 \). Set
\[
d_i = \sum_{k \geq 0} (-1)^k \frac{(p + k + 4)!}{(p + 4)!} Q_{k,i-2k}.
\]
Then the transformation
\[
r_{i,x}^p = -(p+3)(s_i - (p+4)d_i), \quad i \geq 1,
\]
is the desired equivalence.

**The case \( p = -3 \).** The first three pairs of defining equations in this case are
\[
\begin{align*}
    r_{i,x} = 0, \quad r_{i,y} = 0, \quad i = -3, -2, -1,
\end{align*}
\]
while all the rest ones do not contain the variable \( r_{-3} \). Thus we that \( r_{-3} \) is a constant whose value does not influence the subsequent computations. So, we may set \( r_{-3} = 0 \) and this reduces the case under consideration to \( p > -3 \).

2.2. The Lie algebra of nonlocal symmetries. Let \( \Phi = (\varphi, \varphi^1, \ldots, \varphi^i, \ldots) \) be a nonlocal symmetry of Equation (2) in the covering (10). The defining equations for the components of \( \Phi \) are
\[
\begin{align*}
    \tilde{\ell}_F(\varphi) &\equiv \tilde{D}_y^2(\varphi) - (u_y + y)\tilde{D}_y^2(\varphi) - u_{xx}\tilde{D}_y(\varphi) + u_x\tilde{D}_x\tilde{D}_y(\varphi) + u_{xy}\tilde{D}_x(\varphi) = 0, \\
    \tilde{D}_x(\varphi^i) &\equiv \tilde{D}_y(\varphi^{i-1}) + u_x\tilde{D}_x(\varphi^{i-1}) + r_{i-1,x}^p \tilde{D}_x(\varphi), \\
    \tilde{D}_y(\varphi^i) &\equiv (u_y + y)\tilde{D}_x(\varphi^{i-1}) - (p + i + 1)\varphi^{i-2} + r_{i-1,x}^p \tilde{D}_y(\varphi).
\end{align*}
\]
We need a number of auxiliary results to describe the algebra \( \text{sym}_{p,q} \mathcal{E}_1 \). As a first step, it is convenient to assign weights to the internal coordinates in \( \tilde{\mathcal{E}}_1 \) in such a way that all polynomial objects become homogeneous with respect to these weights. Let us set
\[
|x| = 3, \quad |y| = 2, \quad |u| = 4.
\]
Then
\[
|u_x| = |u| - |x| = 1, \quad |u_y| = |u| - |y| = 2,
\]
etc., and
\[
|r_i^p| = i + 4, \quad i \geq 1.
\]
The weight of a monomial is the sum of weights of its factors and the weight of a vector field \( R \partial/\partial p \) is \( |R| - |p| \). In particular, one has \( |\varphi_k| = k, \ k = -4, \ldots, 0 \) for the local symmetries presented in \([9]\).

**Lemma 1.** The local symmetry \( \varphi_{-1} \) can be lifted to the covering \( \tau^{-4} \).

**Proof.** Let us set \( \Phi_{-1} = (\varphi_{-1}, \varphi_{-1}^1, \ldots, \varphi_{-1}^i, \ldots) \), where \( \varphi_{-1}^i = yr_{-1}^{-4} - (i + 2)r_{-1}^{-4} \). It is straightforward to check that Equations \([12]\) are satisfied for \( p = -4 \). \( \square \)

**Lemma 2.** The local symmetry \( \varphi_{-2} \) can be lifted to the covering \( \tau^{-5} \).

**Proof.** We set \( \Phi_{-2} = (\varphi_{-2}, -r_{-1}^{-5}, \ldots, -r_{-1}^{-5}, \ldots) \). Then \([12]\) are fulfilled in an obvious way for \( p = -5 \). \( \square \)

**Lemma 3.** There exists a nonlocal symmetry of weight 2 in the covering \( \tau^{-1} \).

**Proof.** Let us set \( \Phi_2 = (\varphi_2, \varphi_2^1, \ldots, \varphi_2^i, \ldots) \), where
\[
\varphi_2 = -\frac{1}{2} \left( 8r_{-1}^{-1} + 42y(3y^2 - 2u) + (80xy - 7r_{-1}^{-1})u_x + 2(7y^2 + 6u)u_y - 50x^2 - 96y^3 \right)
\]
and
\[
\varphi_2^i = -\frac{1}{2} \left( (80xy - 7r_{-1}^{-1})r_{i,x}^{-1} + 2(7y^2 + 6u)r_{i,y}^{-1} - 2(i + 8)r_{i+2}^{-1} - 8y(i + 6)r_i^{-1} - 10(i + 5)xr_{i-1}^{-1} \right)
\]
for all \( i \geq 1 \). \( \square \)

**Proposition 5.** The symmetries \( \Phi_{-2}, \Phi_{-1}, \) and \( \Phi_2 \) exist in any covering \( \tau^p \).

**Proof.** A direct consequence of Lemmas \([1-3]\) and Proposition \([4]\). \( \square \)

**Theorem 1.** The symmetries \( \Phi_{-2}, \Phi_{-1}, \) and \( \Phi_2 \) generate the entire Lie algebra \( \text{sym}_{p,q} \mathcal{E}_1 \) which is isomorphic to the Witt algebra \([20]\), i.e., there exists a basis \( \Phi_k, \ k \in \mathbb{Z} \), such that
\[
[\Phi_k, \Phi_l] = (l - k)\Phi_{k+l}
\]
for all \( k, l \in \mathbb{Z} \).

**Proof.** Let set
\[
\Phi_0 = 4[\Phi_{-2}, \Phi_2], \quad \Phi_1 = 3[\Phi_{-1}, \Phi_2]
\]
and by induction
\[
\Phi_{k+1} = (k - 1)[\Phi_1, \Phi_k], \quad \Phi_{k-1} = (1 - k)[\Phi_{-1}, \Phi_k]
\]
for all \( k \geq 2 \). \( \square \)

**Remark 1.** The symmetries \( \Phi_k \) are invisible for \( k < -4 \).

**Remark 2.** The symmetries \( \Phi_{-4}, \Phi_{-3}, \) and \( \Phi_0 \) are lifts of local symmetries \( \varphi_{-4}, \varphi_{-3}, \varphi_0 \), respectively.
2.3. Explicit formulas. To obtain explicit formulas for nonlocal symmetries in $\tau^p$ for arbitrary value of $p$ consider the fields

$$y_m = \sum_{i=m}^{\infty} (i - m + 1)r^p_{i-3} \frac{\partial}{\partial r^p_{i-4}}$$

on the space of $\tau^p$, where $m = 1 \ldots 4$, and the quantities $P^{\mu}_{i,j}$ defined by induction as follows:

$$P^{(m)}_{i,0} = \frac{1}{(i + 2)!} (r^p_{m-4})^{i+2}, \quad P^{(m)}_{i,j+1} = \frac{1}{j + 1} y_m(P^{(m)}_{i,j}),$$

where $i, j \geq 0$, $m = 1 \ldots 4$. We set $P^{(m)}_{i,j} = 0$ if at least one of the subscripts is $< 0$. In terms of these quantities, the lift of $\varphi_{-2}$ to $\tau^p$ acquires the form

$$\Phi^p_{-2} = (\varphi_{-2}^1, \varphi_{-2}^2, \varphi_{-2}^3, \ldots),$$

where

$$\varphi_{-2}^p = - r^p_{i,y} - (p + 5) \left( r^p_{i-2} + \sum_{j=1}^{\infty} \left( - \frac{1}{p + 3} \right)^j \prod_{l=0}^{j-1} ((p + 3)l - 2) \cdot P^{(2)}_{j-1,i-2j} \right).$$

The lift

$$\Phi^p_{-1} = (\varphi_{-1}^1, \varphi_{-1}^2, \varphi_{-1}^3, \ldots)$$

of $\varphi_{-1}$ is given by

$$\varphi_{-1}^p = yr^p_{i,x} - (i + 2)r^p_{i-1} + (p + 4) \sum_{j=1}^{\infty} \left( - \frac{1}{p + 3} \right)^j \prod_{l=0}^{j-1} ((p + 3)l - 1) \cdot P^{(2)}_{j-1,i-2j+1}.$$ 

The nonlocal symmetry $\Phi_2$, when passing to $\tau^p$, acquires the form

$$\Phi^p_2 = (\varphi^p_2, \varphi^p_2, \varphi^p_2, \ldots)$$

with

$$\varphi^p_2 = - \frac{1}{p + 3} \left( 8r^p_2 + 2(4p + 25)r^p_0 + (p + 3)(7p + 47)xy - 7r^p_1 \right) u_x$$

$$- \left( 7y^2 + 6y \right) u_y - (4p + 29)x^2 - \frac{1}{3} y^3 (8p^2 + 87p + 223)$$

and

$$\varphi^p_2 = - \frac{1}{p + 3} \left( (p + 3)(7p + 47)xy - 7r^p_1 \right) r^p_{i,x} - (7y^2 + 6u)r^p_{i,y}$$

$$+ (i + 8)r^p_{i+2} + 2y(p + 2i + 13)r^p_i + x(3p + 5i + 28)r^p_{i-1}$$

$$+ \frac{p + 1}{p + 3} \left( 2P^{(4)}_{i-2} + \sum_{j=1}^{\infty} \left( - \frac{1}{p + 3} \right)^j \prod_{l=0}^{j} ((l(p + 3) + 2) \cdot P^{(2)}_{j,i-2j+2} \right).$$

Explicit formulas for other nonlocal symmetries can be obtained using commutator relations from the proof of Theorem 11. For example, the symmetry $\Phi_1$ that generates the positive part of $\text{sym}_{np} \mathcal{E}_1$ is

$$\Phi_1 = (\varphi^p_1, \varphi^1_1, \varphi^1_1, \ldots)$$

with

$$\varphi^p_1 = 6 \frac{r^p}{p + 3} - \left( (4y^2 + 5u)u_x + 4xu_y - 2(3p + 16)xy \right)$$

and

$$\varphi^1_1 = - \left( (4y^2 + 5u)r^p_{i,x} + 4xr^p_{i,y} - (i + 6)r^p_{i+1} - y(2p + 3i + 16)r^p_{i-1} \right).$$

The case $p = -3$ is special, but, as it was indicated above, is reduced to the case $p = -2$.  

\[2\text{The case } p = -3 \text{ is special, but, as it was indicated above, is reduced to the case } p = -2.\]
2.4. Equation \((3)\). The results on this equation and their proofs are almost identical to those on Equation \((2)\). So, we confine ourselves with the final description of sym\(_2\), \(\mathcal{E}_2\). Introduce the weights
\[
|x| = 2, \quad |y| = 1, \quad |u| = 3.
\]
Consequently, the nonlocal variables in the covering \(\sigma^p\) acquire the weights \(|r_1^p| = i + 3\).

Then we have

**Theorem 2.** The symmetries \(\gamma_{-2}\) and \(\gamma_{-1}\) can be lifted to symmetries \(\Gamma_{-2}\) and \(\Gamma_{-1}\) in any covering \(\sigma^p\) over Equation \((3)\). In addition, there exists a nonlocal symmetry \(\Gamma_2\) of weight 2. These three symmetries generate the entire Lie algebra \(\text{sym}_{\sigma^p}\mathcal{E}_2\) which is isomorphic to the Witt algebra \(\mathfrak{W}\).

**Proof.** The symmetry \(\Gamma_{-2} = (\gamma_{-2}, \gamma_{-2}^1, \ldots, \gamma_{-2}^p, \ldots)\) is given by \(\gamma_{-2} = -y - u_x/2\) and
\[
\gamma_{-2}^p = -\frac{1}{2} \left( r_{i,x}^p + (p + 5) \left( r_{i-2}^p + \sum_{j=1}^\infty \left( -\frac{1}{p + 3} \right)^j \cdot \prod_{l=0}^{j-1} (l(p + 3) - 2) \cdot P_{j-1,i-j}^{(1)} \right) \right).
\]
For \(\Gamma_{-1} = (\gamma_{-1}, \gamma_{-1}^1, \ldots, \gamma_{-1}^p, \ldots)\) we have \(\gamma_{-1} = y^2 - 2x + 2yu_x - 2uy\) and
\[
\gamma_{-1}^p = 2 \left( r_{i,x}^p - r_{i,y}^p - (p + 4) \left( r_{i-1}^p + \sum_{j=1}^\infty \left( -\frac{1}{p + 3} \right)^j \cdot \prod_{l=0}^{j-1} (l(p + 3) - 1) \cdot P_{j-1,i-j+1}^{(1)} \right) \right).
\]
Finally, we have
\[
\gamma_2^p = 7r_2^p + (7p + 37)y - 6u_x r_1^p
\]
\[
+ (p + 3) \left( (6p + 31)uy + (3p + 19)x^2 - (3p^2 + 18p + 13)xy^2 + \frac{1}{12}(3p^3 + 27p^2 + 81p + 89)y^4 \right)
\]
\[
+ \left( 5u + 14xy + \frac{19}{3} y^3 \right) u_y - \frac{1}{2} \left( 7p^2 + 74p + 197 \right) y^2 u - \left( 7p + 37 \right) xu - 4(2p + 7)x^2 y
\]
\[
+ \frac{1}{3} \left( 7p^3 + 87p^2 + 333p + 397 \right) xy^3 - \frac{1}{30} \left( 7p^4 + 104p^3 + 558p^2 + 1296p + 1115 \right) y^5
\]
and
\[
\gamma_2^p = -(p + 3)((i + 7)r_{i+2}^p + (p + 3i + 19)yr_{i+1}^p) + Ar_{i,x}^p + Br_{i,y}^p + Cr_{i}^p + D
\]
for \(\Gamma_2 = (\gamma_2^p, \gamma_2^1, \ldots, \gamma_2^p, \ldots)\), where
\[
A = (p + 3) \left( (6p + 31)uy + (3p + 19)x^2 - (3p^2 + 18p + 13)xy^2 + \frac{1}{12}(3p^3 + 27p^2 + 81p + 89)y^4 \right)
\]
\[
- 6r_1^p,
\]
\[
B = (p + 3) \left( 5u + 14xy + \frac{19}{3} y^3 \right),
\]
\[
C = -(p + 3) \left( 2(p + 2i + 11)x + (p^2 + 9p + 5i + 33)y^2 \right),
\]
and
\[
D = -(p + 1) \left( 2P_{0,i-1}^{(3)} + \sum_{j=1}^\infty \left( -\frac{1}{p + 3} \right)^j \cdot \prod_{l=0}^{j-1} (l(p + 3) + 2) \cdot P_{j,i-j+3}^{(1)} \right)
\]
for \(i \geq 1\). \(\square\)
3. Discussion

We conclude with two remarks.

Remark 3. Consider the covering (10) with \( p = 2 \) and the generating series
\[
R = \sum_{i=1}^{\infty} \lambda^i r_i
\]
of the corresponding nonlocal variables. The function \( R \) satisfies the system
\[
R_x = \lambda(R_y + u_x R_x), \\
R_y = \lambda(u_y + y) R_x - (\lambda^3 R)_{\lambda},
\]
which is equivalent to
\[
\tilde{R}_x = \frac{\lambda^4 \tilde{R}_{\lambda}}{\lambda^2(u_y + y) + \lambda u_x - 1}, \\
\tilde{R}_y = \frac{\lambda^3(1 - \lambda u_x) \tilde{R}_{\lambda}}{\lambda^2(u_y + y) + \lambda u_x - 1},
\]
where \( \tilde{R} = \lambda^3 R \). Let \( \tilde{R} = \tilde{R}(x, y, \lambda) \) be a solution to (13) such that \( \tilde{R}_{\lambda} \neq 0 \). Then equation \( \tilde{R}(x, y, \lambda) = \text{const} \) defines \( \lambda \) as a function of \( x \) and \( y \), that is, \( \lambda = \psi(x, y) \), cf. [11]. Then
\[
\tilde{R}_{\lambda} \cdot \psi_x + \tilde{R}_x = 0, \\
\tilde{R}_{\lambda} \cdot \psi_y + \tilde{R}_y = 0
\]
and (13) transforms to
\[
\psi_x = \frac{\psi^4}{1 - u_x \psi - (u_y + y) \psi^2}, \\
\psi_y = \frac{\psi^3 (1 - u_x \psi)}{1 - u_x \psi - (u_y + y) \psi^2},
\]
which, by the gauge transformation \( \psi \mapsto \psi^{-1} \), is equivalent to
\[
\psi_x = \frac{1}{\psi^2 - u_x \psi - (u_y + y)}, \\
\psi_y = \frac{u_x - \psi}{\psi^2 - u_x \psi - (u_y + y)}.
\]
This is the covering obtained in [2] by the direct reduction and coincides with the known covering over the Gibbons-Tsarev equation, see [7].

Remark 4. In [1], two other equations,
\[
u_y u_{xy} - u_x u_{yy} = e^y u_{xx}
\]
and
\[
u_{yy} = (u_x + x) u_{xy} - u_y (u_{xx} + 2),
\]
were obtained as symmetry reductions of the universal hierarchy and the 3D rdDym equations, respectively, were obtained. We plan study these equations by the methods similar to the used above in the forthcoming publications.

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