The Adaptive Transient Hough method for long-duration gravitational wave transients

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This paper describes a new semi-coherent method to search for transient gravitational waves of intermediate duration (hours to days). We model the frequency evolution as a power law in order to search for newborn isolated neutron stars. The search uses short Fourier transforms from the output of ground-based gravitational wave detectors and applies a weighted Hough transform. We present the technical details for implementing the algorithm, its statistical properties, and a sensitivity estimate. A first example application of this method was in the search for GW170817 post-merger signals.

I. INTRODUCTION

The advanced gravitational wave (GW) detector era has provided us with multiple detections from binary compact objects including GW170817, the first binary neutron star coalescence. This detection motivated the development of the new search method presented in this paper, focusing on the possible birth of a rapidly rotating highly magnetized neutron star spinning down through some combination of GW and electromagnetic emission. For a very massive remnant, the collapse would occur in a short time scale (as explored in [3, 4]), but for low total mass and some equations of state, the emitted GW signal could have an intermediate duration on the order of hours to days [5, 6].

This regime of GW signal durations has long been mostly unexplored from the data analysis side, but other pre-existing or recently developed methods to search for intermediate-duration signals include the Stochastic Transient Analysis Multi-detector Pipeline (STAMP) [7], the Viterbi algorithm [8], and a generalization of the FrequencyHough method [9]. The first two are unmodeled searches, while the last is a modeled search similar to the one described in this paper. Together with those three other pipelines, our new Adaptive Transient Hough method has already contributed to the search for a long-duration transient signal from a putative neutron star remnant of GW170817 described in [10].

The Adaptive Transient Hough is a semi-coherent analysis adapted from the SkyHough search for continuous-wave signals [11, 12]. Whereas the original SkyHough assumes a constant intrinsic amplitude and slowly evolving frequency, the new method is designed to track transient GWs with rapid frequency and amplitude evolution following a power law. Such signals are possibly created by r-mode oscillations [14, 15] or quadrupolar deformations in a newborn neutron star, e.g. the product of a binary neutron star merger or a supernova event [16].

The paper is organized as follows: section II briefly describes the expected signal from a remnant neutron star. Section III summarizes the general strategy of a hierarchical search and its implementation, section IV studies its statistical properties, and section V introduces the threshold and vetoes required for a robust detection strategy. Finally section VI presents an estimate for the search sensitivity and section VII presents our conclusions.

II. THE SIGNAL MODEL

The output of a GW detector can be represented by

\[ x(t) = n(t) + h(t), \]

(1)

where \( n(t) \) is the detector noise at time \( t \), and \( h(t) \) is the strain induced by a GW signal:

\[ h(t) = F_+(n, \psi)h_+(t) + F_\times(n, \psi)h_\times(t), \]

(2)

where \( F_{+,\times} \) are the detector antenna patterns, which depend on a unit-vector \( \mathbf{n} \) corresponding to the sky location of the source and on the wave polarization angle \( \psi \). For ground-based detectors with perpendicular arms, the expressions for \( F_{+,\times} \) are [17]:

\[ F_+(t) = a(t) \cos 2\psi + b(t) \sin 2\psi, \]

(3)

\[ F_\times(t) = b(t) \cos 2\psi - a(t) \sin 2\psi, \]

(4)

where the functions \( a(t) \) and \( b(t) \) are independent of \( \psi \). Now the waveforms for the two polarizations \( h_{+,\times} \) are:

\[ h_+(t) = A_+(t) \cos \Phi(t), \]

(5)

\[ h_\times(t) = A_\times(t) \sin \Phi(t), \]

(6)

where \( \Phi(t) \) is the phase evolution of the signal and \( A_{+,\times} \) are amplitude parameters depending on the orientation \( \cos \iota \) of the source and on the strain amplitude evolution \( h_0(t) \) as follows:

\[ A_+(t) = \frac{1}{2} h_0(t)(1 + \cos^2 \iota), \]

(7)
\[ A_x(t) = h_0(t) \cos t \]. \tag{8}

The time evolution of the dimensionless strain amplitude depends on the emission mechanism; in this paper we focus on signals that can be written as

\[ h_0(t) = A_m(f_{gw}(t))^n \], \tag{9}

where \( n \) and \( A_m \) are constants defined by the emission mechanism. The frequency evolution covered by this method is described in [18, 19], and takes the following form:

\[ \dot{f}_{gw}(t) = \begin{cases} 
\frac{f_{gw0}}{\tau} \left( \frac{t - T_0}{\tau} + 1 \right)^{\frac{1}{n}} & \text{if } t \geq T_0 , \\
0 & \text{if } t < T_0 , 
\end{cases} \] \tag{10}

where \( n \) is an arbitrary braking index, \( \tau \) is associated to the spindown timescale, and \( f_{gw0} \) corresponds to the frequency at the start of the emission (\( t = T_0 \)); for simplicity let us set \( T_0 = 0 \). As in other semi-coherent searches, this method considers as negligible – and therefore ignores – relativistic corrections, and those due to the time delay between the detector and the solar-system barycenter (SSB). Therefore only the instantaneous signal frequency in the detector frame needs to be calculated:

\[ f_{gw}(t) = \dot{f}_{gw}(t) \left( 1 + \frac{v(t)}{c} \right), \tag{11} \]

where \( v(t) \) is the detector velocity with respect to the SSB frame. Note that now the time coordinate \( t \) corresponds to time at the detector.

The main difference between the signals we target with this method and those considered in continuous waves (CW) searches [20] are the larger characteristic spin-down values (derivatives of \( f_{gw} \)). These lead to signals of intermediate duration and at the same time to rapid amplitude evolution that makes the constant-amplitude approximation used e.g. in [13] unfeasible.

III. THE ADAPTIVE TRANSIENT HOUGH METHOD

This section discusses the implementation of the Adaptive Transient Hough (AtHough) method, a pipeline based on the semi-coherent SkyHough search for CWs described in [11, 13]. The common ground of both searches is the use of a weighted Hough transform on Short-time Fourier Transforms (SFTs) as the input data. The Hough transform is an algorithm widely used in pattern recognition; here the pattern is defined by the frequency evolution of the signal in the detector data. In both CW and transient cases, the weights take into account the amplitude modulation of the signal, caused by the antenna pattern, and the changing noise floor between SFTs. But as a difference to the CW SkyHough search, the new AtHough method also includes the source amplitude evolution in the weights.

![Image](https://example.com/image.png)

**FIG. 1.** Search setup: The maximum coherence length \( T_{coh} = \sqrt{(n-1)\tau}/\sqrt{2f_{gw0}} \) allowed for signals with fixed \( f_{gw0} = 2000 \) Hz and the other model parameters taking values in the intervals \( \tau \in [1000, 9640] \) s and \( n \in [2.5, 7] \).

A. Length of Short-duration Fourier Transforms

We first obtain a collection of SFTs by dividing the full observation time \( T_{obs} \) in \( N \) segments of length \( T_{coh} \). The maximum length of \( T_{coh} \) is calculated by imposing the 1/4-cycle criterion introduced in [17]: This leads to a requirement \( 2|df/dt| \leq T_{coh}^{-2} \). From eq. (11) the spin-down modulation is given by two effects, the spin-down of the source and the Doppler modulation resulting from the Earth’s motion. The constraint imposed by the spin-down of the source is:

\[ T_{coh} \leq \frac{\sqrt{(n-1)\tau}}{\sqrt{2f_{gw0}}}. \tag{12} \]

The range of maximum allowed \( T_{coh} \) for the parameter space covered in [10] is on the order of seconds, as shown in fig. [1]. On the other hand, the constraint imposed by Doppler modulation is on the order of hours, as discussed in [13]. Therefore we will consider only the spin-down of the source as the dominant threshold for \( T_{coh} \).

B. The peak-gram

The Hough transform requires a digitized spectrum as its input, with time-frequency bins categorized in two classes. The AtHough generates this by setting a threshold \( \rho_{th} \) on the normalized power spectrum \( \rho_i \) to conduct the bin selection:

\[ \rho_i \approx \frac{2|\hat{x}_i[f_k]|^2}{T_{SFT}S_n[f_k]}, \tag{13} \]

where \( [\cdot] \) indicates a discrete series and the index \( i \) corresponds to the \( i^{th} \) time step. That is, \( \hat{x}_i[f_k] \) is the value
obtained from the $i$th SFT on the $k$th frequency bin. Furthermore, $S_n$ is the single-sided Power Spectral Density (PSD) of the noise in the same bin. If $\rho_i \geq \rho_{th}$, then a value of 1 is assigned to that bin, and a 0 otherwise. The result of this process is known as the peak-gram.

C. Resolution in $\tau$ and $n$ space

The Hough transform is applied to find the statistical significance of each template in a bank over parameter space. A template is defined by the intrinsic parameters of the signal, $\xi = (f_{gw0}, n, \tau, t_0)$. To conduct a wide-parameter space search, we create a grid that ensures contiguous templates to deviate from each other by at most one frequency bin over a duration $T_{obs}$; this ensures the computation of at least all independent templates (by the 1/4-cycle criterion) between $t = 0 \, s$ and $t = T_{obs}$. The grid is constructed with the following step sizes:

$$\delta n = \left| \frac{\partial n}{\partial f_{gw}(t)} \right|_{t=T_{obs}} \delta f,$$  \hspace{1cm} (14)

$$\delta \tau = \left| \frac{\partial \tau}{\partial f_{gw}(t)} \right|_{t=T_{obs}} \delta f,$$ \hspace{1cm} (15)

where $\delta f = 1/T_{coh}$. Hence,

$$\delta n = \frac{(n-1)^2 (\frac{T_{obs}}{\tau} + 1)^{-\frac{1}{2(n-1)}}}{f_{gw0}T_{coh} \log (\frac{T_{obs}}{\tau} + 1)},$$ \hspace{1cm} (16)

$$\delta \tau = \frac{(n-1)\tau (\tau + T_{obs}) (\frac{T_{obs}}{\tau} + 1)^{-\frac{1}{2(n-1)}}}{f_{gw0}T_{coh}T_{obs}}.$$ \hspace{1cm} (17)

The two grid step sizes are inversely proportional to $f_{gw0}$. Fig. 2 represents the obtained $\delta \tau$ and $\delta n$ for a fixed $T_{coh}$, $T_{obs}$ and $f_{gw0}$ inside the $\tau$, $n$ ranges.

The practical implementation of the grid is defined by a nested loop; a pipeline diagram can be seen in Fig. 3. First, we select the minimum value of $\delta n$ over the $\tau$ range as shown in Fig. 4, given a set of $(T_{obs}, T_{coh}, n)$ and the maximum $f_{gw0}$; then we calculate $\delta \tau$ as in Fig. 5. We will recalculate $\delta n$ and $\delta \tau$ on each iteration of the $n$ and $\tau$ loops respectively.

In order to reduce the number of templates or grid points required by the search, we need to split the $\tau$ and $f_{gw0}$ ranges of the whole search space into smaller sub-domains. To do so, we will typically create bands for $\tau$ smaller than 10% of $T_{obs}$ and frequency bands between 50 and 100 Hz in width. Each sub-domain will be analyzed independently, making the computational load smaller. It is possible to make the domains larger, but the necessary refinement of the grid in certain areas will make the search less computationally efficient overall.

Fig. 6 shows the distribution and number of templates used for different $T_{obs}$ given a search that covers an analogous parameter space as 10. Here templates are calculated with the maximum integer coherence length allowed, and the minimum $T_{coh}$ considered for this figure and the search is 1 s.

IV. STATISTICAL PROPERTIES

A. The coherent statistic

For the following section we make the assumption of stationary Gaussian noise with zero mean in order to characterize the output of the detectors, for which the normalized power $2\rho_i$ in the presence of a signal $h$ follows a non-central $\chi^2$ distribution with 2 degrees of freedom.
and a non-centrality parameter
\[ \lambda_i = \frac{4|\tilde{h}_i[f_k]|^2}{T_{SFT}S_n[f_k]} , \]  
where $|\tilde{h}_i[f_k]|$ is the Fourier transform of the signal. Then the probability distribution for $\rho_i$ is:
\[ p(\rho_i|\lambda_i) = 2\lambda^2(2\rho_i|2, \lambda_i) = \exp(-\rho_i - \frac{\lambda_i}{2})I_0(\sqrt{2\lambda_i\rho_i}) , \]  
(19)

where $I_0$ is the zero-order modified Bessel function of the first kind.

The mean and variance for this distribution are respectively:
\[ \mathbb{E}[\rho_i] = 1 + \frac{\lambda_i}{2} , \]  
(20)
\[ \sigma^2[\rho_i] = 1 + \lambda_i . \]  
(21)

The false alarm and false dismissal probabilities for a frequency bin to be above the power spectrum threshold are:
\[ \alpha(\rho_{th}) = \int_{\rho_{th}}^{\infty} p(\rho|0)d\rho = \exp(-\rho_{th}) , \]  
(22)
\[ \beta_i(\rho_{th}) = \int_{0}^{\rho_{th}} p(\rho|\lambda_i)d\rho = 1 - \eta_i(\rho_{th}|\lambda_i) . \]  
(23)

The probability $\eta_i$ that a given frequency bin is selected is, in the small-signal approximation:
\[ \eta_i(\rho_{th}|\lambda_i) = \int_{\rho_{th}}^{\infty} p(\rho|\lambda_i)d\rho = \alpha(1 + \frac{\rho_{th}^2}{2\lambda_i} + O(\lambda_i^3)) . \]  
(24)
FIG. 6. The number of templates required for searches with four different $T_{\text{obs}}$. The total parameter-space covered is $n \in [2.5, 7]$, $f_{\text{gw}} \in [500, 2000]$ Hz, $\tau \in [10^3, 10^5]$ s and is evaluated in independently-processed sub-domains, each corresponding to a $\tau$ band of 100 s and a 100 Hz wide frequency band. In this figure, all panels show counts of templates after combining the $\tau$ bands. The top panel shows the number of templates for each frequency band when using the optimal $T_{\text{coh}}$ for each $T_{\text{obs}}$: it increases with $f_{\text{gw}}$ for each $T_{\text{obs}}$, and longer $T_{\text{obs}}$ require more templates at each frequency. The middle panel shows the total number of templates (summed over all frequency bands), for each $T_{\text{obs}}$, as a function of $T_{\text{coh}}$. The lower panel shows the total number of templates when again using the optimal $T_{\text{coh}}$ for each $T_{\text{obs}}$.

B. The incoherent number-count statistic

If a signal is present, the non-centrality parameter $\lambda_i$ will change for different SFTs. As pointed out previously, this can happen both because the noise may not be stationary and because the amplitude modulation of the signal changes over time. In other words, the observed signal power $|h|^2$ changes due to the non-uniform antenna pattern of the detector and due to the intrinsic spindown. Therefore, the detection probability $\eta$ changes across SFTs. This is taken into account by the non-demodulated weighted Hough approach mentioned before and covered in [11]; it is a similar strategy to the one applied in the StackSlide [21] and PowerFlux [22, 23] algorithms. The starting point is to generalize the integer number-count statistic, which we would obtain directly from the peak-map, to a non-integer weighted statistic

$$\nu = \sum_{i=1}^{N} w_i \nu_i ,$$

(25)

where $N$ is the number of SFTs, $\nu_i$ is the value assigned to the bin selected from the peak-gram in the $i^{\text{th}}$ time step for the current template, and for the weights $w_i$ we define

$$A := \sum_{i=1}^{N} w_i ,$$

$$||w||^2 := \sum_{i=1}^{N} w_i^2 .$$

(26)

(27)

This step is known as the incoherent sum; the templates in a search are then ranked based on their number count $\nu$.

If $\nu$ is considered as a continuous random variable, it follows a binomial distribution which can be approximated by a Gaussian with the right mean and variance:

$$p(\nu|\nu_0, \lambda) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\nu-\lambda\alpha)^2}{2\sigma^2}} .$$

(28)

The mean and variance in the absence of a signal are:

$$\langle \nu \rangle = A\alpha ,$$

$$\sigma^2_\nu = \langle \nu^2 \rangle - \langle \nu \rangle^2 = ||w||^2\alpha(1 - \alpha) .$$

(29)

(30)

Thus one can derive the number count threshold $\nu_{\text{th}}$ based on the incoherent false-alarm rate

$$\alpha_1 = \int_{\nu_{\text{th}}}^{\infty} p(\nu|\nu_0, 0) d\nu = \frac{1}{2} \text{erfc} \left( \frac{\nu_{\text{th}} - \langle \nu \rangle}{\sqrt{2}\sigma_\nu} \right) .$$

(31)

For a given set of weights and peak selection threshold, this equation decides what number count threshold must be used to obtain a desired $\alpha_1$. We can solve this as

$$\nu_{\text{th}} = \alpha A + \sqrt{2||w||^2\alpha(1 - \alpha)} \text{erfc}^{-1}(2\alpha_1) .$$

(32)
The false-dismissal rate requires the computation of the mean and variance of $\nu$ in the presence of a small signal:

$$\bar{\nu} = A\alpha + \alpha \sum_{i=1}^{N} w_i \lambda_i, \quad (33)$$

$$\sigma^2_{\nu} = \sum_{i=1}^{N} w_i^2 \eta_i (1 - \eta_i). \quad (34)$$

If the small signal approximation is applied, $\sigma^2_{\nu}$ can be expanded to first order in $\lambda_i$:

$$\sigma^2_{\nu} = ||w||^2 \alpha (1 - \alpha) \left(1 + \frac{\rho_{th}}{2||w||^2} \frac{1 - 2\alpha}{1 - \alpha} \sum_{i=1}^{N} w_i^2 \lambda_i\right). \quad (35)$$

We can then approximate the number count distribution $p(\nu|h)$ by a Gaussian distribution with the above mean and variance, yielding the false-dismissal rate as follows:

$$\beta_1 \approx \int_{-\infty}^{\nu_{th}} p(\nu|h) d\nu = \frac{1}{2} \text{erf}_c \left(\frac{\langle \nu \rangle - \nu_{th}}{\sqrt{2\sigma_{\nu}}}ight). \quad (36)$$

### C. Setting up the threshold

Considering the statistical significance in a template as $s := 1 - \alpha_1 - \beta_1$ and using the properties of the complementary error function, we can introduce a quantity

$$S = \text{erfc}^{-1}(2\alpha_1) + \text{erfc}^{-1}(2\beta_1). \quad (37)$$

This equation can be shown to reduce to $s$ when $S = 0$, and as it grows monotonically we can take it as a measure of the statistical significance of the search. By expanding to the first order in $\lambda_i$, we derive the following expression:

$$S = \sqrt{\frac{\alpha \rho_{th}^2}{8(1 - \alpha)}} \sum_{i=1}^{N} w_i \lambda_i
+ \frac{\rho_{th}}{4} \frac{1 - 2\alpha}{1 - \alpha} \sum_{i=1}^{N} w_i \lambda_i \text{erfc}^{-1}(2\alpha). \quad (38)$$

The first term on the right-hand side of this equation is proportional to $\sqrt{N}$, while the second term does not grow with $N$. For large values of $N$, the easiest way to see this is by taking $w_i \propto 1/N$. Thus the first term dominates, yielding

$$S = \sqrt{\frac{\alpha \rho_{th}^2}{8(1 - \alpha)}} \sum_{i=1}^{N} w_i \lambda_i. \quad (39)$$

The peak selection threshold is chosen to minimize $\beta_1$, or equivalently maximize $S$ for fixed $\alpha_1$:

$$\frac{d}{d\rho_{th}} \sqrt{\frac{\alpha \rho_{th}^2}{8(1 - \alpha)}} = 0. \quad (40)$$

As derived in [13], this is independent of the choice of weights; the solution to this equation is $\rho_{th} = 1.6$ which leads to $\alpha_1 = e^{-\rho_{th}} = 0.2$.

### D. Calibration of the weights

To define an appropriate set of weights, we start by considering the modulus square of the signal’s Fourier transform on the $i^{th}$ SFT:

$$|\tilde{h}_i[f_k]|^2 = \frac{A^2_{i,\nu} I_{i,\nu}^2 + A^2_{i,\nu} F_{w,\nu}^2}{4} \frac{\sin^2[p(f_i - f_k) T_{coh}]}{\pi(f_i - f_k)}. \quad (41)$$

The instantaneous frequency of the signal is $f_i$, and the selected central Fourier frequency is $f_k$. The $k^{th}$ frequency bin is selected so that $f_i \in (f_k - \delta f/2, f_k + \delta f/2)$. The average over that interval is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin^2[\pi x] (\pi x)^2 = 0.7737. \quad (42)$$

Now we can average over the pulsar orientation $\cos \iota$ and the polarization angle $\psi$ appearing in the antenna patterns and find the following relationships:

$$\langle |F_{+},i|^2 \rangle_{\iota,\psi} = \langle |F_{x},i|^2 \rangle_{\iota,\psi} = \frac{a_i^2 + b_i^2}{2}, \quad (43)$$

$$\langle (A_{+},i)^2 + (A_{x},i)^2 \rangle_{\iota,\psi} = \frac{4h_{0,0}^2}{5} \left(\frac{f_{gw,i}}{f_{gw0}}\right)^{2m}, \quad (44)$$

where $h_{0,0} = h_0(t = t_0)$ is the initial amplitude at $t_0$. Combining all these results:

$$\langle \lambda_i \rangle_{\iota,\psi} = 0.7737 \frac{2h_{0,0}^2 T_{coh} (a_i^2 + b_i^2)}{5 S_{n,i}} \left(\frac{f_{gw,i}}{f_{gw0}}\right)^{2m}, \quad (45)$$

and substituting this into eq. (39), the sensitivity is

$$S = \sqrt{\frac{\alpha \rho_{th}^2}{8(1 - \alpha)}} \sum_{i=1}^{N} \frac{2h_{0,0}^2 T_{coh} (a_i^2 + b_i^2)}{5 S_{n,i}} \left(\frac{f_{gw,i}}{f_{gw0}}\right)^{2m}. \quad (46)$$

We see that any rescaling of the weights ($\tilde{w}_i = k w_i$) has no impact on $S$, as for any constant $k$, the value of the detectable dimensionless strain amplitude $h_{0,0}$ at $t = 0$ s remains unchanged. The best sensitivity for a given template is obtained when the inner product $\mathbf{w} \cdot \mathbf{X}$ is maximal. This happens when the two vectors are proportional to each other:

$$w_i \propto X_i = \frac{(a_i^2 + b_i^2)}{S_{n,i}} \left(\frac{f_{gw,i}}{f_{gw0}}\right)^{2m}. \quad (47)$$

If the value $\rho_{th} = 1.6$ is substituted in eq. (46), the minimum theoretical value of the search for $h_0$ is:

$$h_{0,0} = 3.38 \sqrt{\frac{S_{1/2}^2}{T_{coh}} \langle ||w||^2 \rangle^{(1/2)} \langle \mathbf{w} \cdot \mathbf{X} \rangle} . \quad (48)$$
E. Critical Ratio $\Psi$

The critical ratio $\Psi$ is a new statistic that quantifies the significance of a given template. Based on the weighted number count \( \mathcal{N} \) and eqs\. \( 26, 27, 39 \), we define

$$\Psi = \frac{\nu - A\alpha}{\sigma} = \frac{\sum_{i=1}^{N}(w_i \nu_i) - \sum_{i=1}^{N}(w_i \alpha)}{\sqrt{\sum_{i=1}^{N}(w_i)^2 \alpha(1 - \alpha)}}. \quad (49)$$

As mentioned before, any normalization of the weights will not change the sensitivity of the search. It will also leave the significance or critical ratio in each template unchanged. Considering the previous equation as the single-detector case, the multi-detector critical ratio is defined as

$$\Psi_M = \frac{\sum_{k=1}^{M} \Psi_k \sqrt{\sum_{i=1}^{N_k}(w_{ik})^2}}{\sqrt{\sum_{k=1}^{M} \sum_{i=1}^{N_k}(w_{ik})^2}}, \quad (50)$$

where \( N_M \) is the number of detectors and \( N_k \) is the number of SFTs in detector \( k \), while \( w_{ik} \) and \( \nu_{ik} \) are the weights and number count assigned to the \( i^{th} \) SFT for that detector and a given template. We can also rewrite this as

$$\Psi_M = \sum_{k=1}^{M} \Psi_k \sqrt{\frac{\sum_{i=1}^{N_k}(w_{ik})^2}{\sum_{k=1}^{M} \sum_{i=1}^{N_k}(w_{ik})^2}}, \quad (51)$$

where \( \Psi_k \) is the critical ratio for each single detector \( k \).

In a multi-detector search, the duty factors (fraction of time a detector is recording usable data) and noise floors may differ between detectors. To quantify the contribution of each detector to the multi-detector critical ratio, the relative contribution ratio is defined as

$$r_j = \sqrt{\frac{\sum_{i=1}^{N}(w_{ij})^2}{\sum_{k=1}^{M} \sum_{i=1}^{N_k}(w_{ik})^2}}. \quad (52)$$

Using the previous equations, the critical ratio for a multi-detector search takes a very simple form:

$$\Psi_M = \sum_{k=1}^{M} \Psi_k r_k. \quad (53)$$

V. VETOES ON CRITICAL RATIO AND TIME CONSISTENCY

Candidates that appear significant by their critical ratio can be due to astrophysical sources, but also due to non-Gaussian noise artifacts in the data. To make the search robust against such artifacts, we introduce vetoes that test for each candidate (i) its consistency between detectors and (ii) the consistency of its transient behavior with the target astrophysical model.

A. The Critical ratio $\Psi$-veto

The threshold for a search is determined under the assumption of detector noise following a stationary zero-mean Gaussian distribution with a power spectral density \( S_n(f) \). A template is considered as a candidate when its \( \Psi \) exceeds a pre-specified threshold for which the probability of a false alarm due to noise alone is small. The overall false-alarm probability \( \alpha_S \) of the search can be approximated as the product of the number of trials (i.e., number of templates \( N_t \)) and the previously introduced false-alarm probability \( \alpha_I \). Now we can rewrite eq. \( 31 \) in terms of the critical-ratio threshold \( \Psi_{th} \):

$$\Psi_{th} = \sqrt{2} \text{erfc}^{-1}(2 \alpha_S / N_t), \quad (54)$$

If the critical ratio in a template exceeds the threshold, as a follow-up veto we can rephrase the question and consider each detector as an independent single trial, obtaining a threshold \( \Psi_{th}^D \) for each detector. This threshold will correspond to eq\. \( 54 \) with \( N_t = 1 \) and any given template that fails to satisfy it in either detector will be vetoed.

B. The time-inconsistency veto

To check that the transient behavior of the signal matches our model, we introduce an additional veto. Let us consider a candidate template \( \xi_C = (f_{gw0}, n, \tau, T_0 = T_{event}) \) and a time-shifted version \( \xi_F = (f_{gw0}, n, \tau, T_0 = T_{event} + T_F) \). These will be com-
FIG. 8. For the time-inconsistency veto, we consider time-shifted frequency tracks. The plot shows the frequency track in time domain for a candidate template $\xi_C = (f_{gw0} = 500 \text{ Hz}, n = 5, \tau = 10^4 \text{ s}, T_0 = 0 \text{ s})$ and a shifted template $\xi_F = (f_{gw0} = 500 \text{ Hz}, n = 5, \tau = 10^4 \text{ s}, T_0 = -T_{\text{obs}})$, showing that there is no overlap between the two tracks. Hence, the significance $\Psi_F$ of the shifted track can be used for a veto.

VI. SEARCH SENSITIVITY

In eq. (48) we have obtained an estimate for the sensitivity of a search as the smallest amplitude that would cross the number-count threshold for a given false-alarm rate $\alpha_1$ and false-dismissal rate $\beta_1$. As a specific astrophysical case, let us concentrate on the isolated non-axisymmetric magnetar scenario as considered in the GW170817 long-duration postmerger search [10]. In this model, the amplitude exponent $m$ in Eq. (9) takes a nominal value of 2 and the signal amplitude $h(t)$ is

$$h_0(t) = \frac{4\pi^2G}{c^4} \frac{I_{zz}c}{d} f_{gw}^2(t),$$

(55)

where $c$ is the speed of light, $I_{zz}$ is the z-z component of the star’s moment of inertia with the z-axis being its spin axis, $\epsilon := (I_{xx} - I_{yy}) / I_{zz}$ is the equatorial ellipticity of the star, and $d$ is the distance from Earth to the star. Other time shifts could be used for a veto as well, as long as the contribution of the candidate signal $\vec{\xi}$ is completely independent if $T_F = -T_{\text{obs}}$; see Fig. 8 for an example. Other time shifts could be used for a veto as well, as long as the contribution of the candidate signal $\xi_C$ to $\Psi_F$ of the shifted template $\xi_F$ is zero.

The obtained value $\Psi_F$ will indicate how much of the original candidate’s $\Psi_C$ seems to come from a stationary contribution instead. Stationary spectral line artifacts are common in the LIGO data [23] and hence this veto is important to remove non-astrophysical false candidates. In other words, we assign a probability to stationary lines to be the cause of the candidate. To estimate this probability we reuse eq. (54) for a single follow-up trial. If the resulting probability corresponds to more than 6 sigmas, we can safely reject the candidate.

FIG. 9. A comparison of analytically and empirically obtained sensitivity estimates for a GW170817 post-merger analysis with the AThrHough method. The analytic sensitivity estimate was done for aLIGO sensitivity $S_0$ during the GW170817 event (end of O2) and for $T_{\text{coh}} = 8 \text{ s}$. The empirical results correspond to the sensitive distance at 90% detectability, $d_{90\%}$, obtained for the $T_{\text{coh}} = 8 \text{ s}$ injection set in [10], using actual aLIGO data after GW170817 and NS parameters of $I_{zz} = 4.34 \times 10^{38}$ and $\cos \epsilon = 1$, as well as $f_{gw0}, \tau$ and $\epsilon$ as given in Fig. 10. See the appendix B of [10] for additional results at different $T_{\text{coh}}$.

FIG. 10. Parameters for the $T_{\text{coh}} = 8 \text{ s}$ injection set from [10], as also used for the comparison with the empirical sensitivity estimate in Fig. 9. Each set of values shown corresponds to the central value of an injection subset, with the parameters then further randomized in narrow ranges as described below.
Combining this amplitude with the sensitivity as given by eq. (48), the astrophysical range of the search is

\[ d = \frac{4\pi^2 GI_{zz} f_{gw0}^2}{c^4} \sqrt{\frac{\mathcal{J}_{coh}}{3.38S^2T^{1/2}}} \left( \frac{w \cdot X}{|w|} \right)^{(1/2)}. \] (56)

We now calculate an astrophysical range estimate for a search setup corresponding to the ATrHough analysis performed as one of four searches in [10]. We use the aLIGO O2 sensitivity \( S_n \) during the GW170817 event to calculate the weights, and for the remnant’s moment of inertia we use the same value as in [10], \( I_{zz} = 100M_\odot^2 \frac{G}{c^4} \approx 4.34 \times 10^{38} \text{ kg m}^2 \).

In Fig. 9 we compare the analytical estimate with the empirical recovery fraction for a set of injections. Those were originally performed for the sensitivity estimate in the GW170817 post-merger search [10]. The recovery criterion corresponds to \( \Psi_{th} = 9 \), or a 5\( \sigma \) significance. We have concentrated here on a braking index \( n = 5 \) that corresponds to pure gravitational-wave emission, and covered ranges of \( f_{gw0} \) and \( \tau \) as shown in Fig. 10. The procedure to obtain the experimental results consisted in selecting 10 Hz wide frequency bands, for each band injecting 1000 simulated signals into O2 data with amplitudes around the astrophysical range estimate. The purpose was to find the amplitude corresponding to 90\% recovery efficiency. The parameters \( \tau \) and \( f_{gw0} \) were randomized within 10 bins of their nominal value; i.e. the injection parameters are not perfectly aligned with the search grid, thus allowing for a realistic exploration of search mismatch in the recovery.

We do not expect an exact agreement between analytical prediction and sensitivity measured from injections, as the analytical estimate is based on a Gaussian noise approximation. But the results are sufficiently close to demonstrate that Eq. (48) is useful for the purpose of setting up future searches.

\[ \Psi_{th} = 9, \text{ or a 5}\sigma \text{ significance.} \]

\[ I_{zz} = 100M_\odot^2 \frac{G}{c^4} \approx 4.34 \times 10^{38} \text{ kg m}^2 \]

\[ d = \frac{4\pi^2 GI_{zz} f_{gw0}^2}{c^4} \sqrt{\frac{\mathcal{J}_{coh}}{3.38S^2T^{1/2}}} \left( \frac{w \cdot X}{|w|} \right)^{(1/2)}. \] (56)

VII. CONCLUSIONS

In this paper we have described a new semi-coherent search method using short incoherent steps of the order of seconds with the intention to track signals of intermediate durations (of the order of hours to days). We have concentrated on the model of power-law spin-down for a newborn neutron star. As applied in the GW170817 post-merger remnant search [10], the astrophysical range of this method at 90\% detection confidence is at \( \sim 1 \text{ Mpc} \) with LIGO sensitivity at the end of the second observing run. With future instruments like the Einstein Telescope [25,27], this range could increase by a factor of \( \sim 20 \).

One disadvantage of modeled semi-coherent methods like this one is the need to explicitly set a starting time for the signal model. On the other hand, it is a suitable method to perform fast and economic follow-ups of known merger events or for promising candidates identified by more generic searches, allowing to reliably set up a fixed false-alarm rate of the overall search.

The same strategy can also easily be translated to signals following other spin-down patterns than the power-law model we focused on so far, with the definition of weights and parameter space grids following the same procedure as introduced in this paper.

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