Admissible Permutations and an Algorithm of Frieze
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Section 2.1 Introduction.
In this paper, we define admissible permutations and use them to reduce the running time of a 3-phase algorithm of Frieze that obtains a hamilton cycle in an extremal random directed graph from \(O(n^{1.5})\) to \(O(n^{3-o(1)})\) by reducing the running time of Phase 1.

Section 2.2. Symbols, Definitions and Preliminary Theorems.
Let \(\tau\) be an n-cycle in \(S_n\), the symmetric group of permutations on the points, \(V_n = \{1,2,...,n\}\). Then \(\tau\) is a cyclic permutation. Define \(T_n\) to be the set of cyclic permutations of \(S_n\). Let \(L_{i,j}, i,j \in V_n\), be independent uniform \([0,1]\) random variables where the value of an arbitrary permutation \(\sigma\), is

\[
L_{\alpha} = \sum_{i \in V_n} L_{i,\alpha_i}
\]

Let \(\sigma^* = \sigma^*(L) \in S_n\) have the property that \(L_{\alpha} \leq L_{\sigma}\) for all \(\sigma\) in \(S_n\). Then \(\sigma^*\) is an optimal solution of the assignment problem whose table of values is given by \(L = \{L_{i,j}\}\) as \(\sigma\) runs through the elements of \(S_n\) and \(A(L) = L^\alpha\) denotes its value. On the other hand, let \(L_{\tau} \leq L_{\tau}\) for all \(\tau\) in \(T_n\) where \(\tau\) ranges through all elements of \(T_n\). Then \(\tau^* = \tau^*(L)\) is an optimal solution of the random asymmetric traveling salesman problem and \(T(L) = L_{\tau}\) denotes its value. Let \(L_{i,j}, i,j \in V_n\) be independent uniform \([0,1]\) random variables and consider the random variables \(A_n = A(L)\) and \(T_n = T(L)\). Karp [6] "patched together" the cycles of \(\sigma^*\) to obtain a tour, \(\tau\), of length \(T_{n,1}\) such that

\[
I \leq E(A_n) \leq E(T_n) \leq E(T_{n,1}) = E(A_n) + O((\log n)^{3/2} n^{0.24})
\]

where \(E(X)\) is the expected value of the random variable \(X\).

Karp and Steele [7] simplified this algorithm by constructing a simpler \(O(n^{3})\) algorithm which produced a tour \(\tau_2\) of length \(T_{n,2}\) where

\[
E(T_{n,2}) = E(A_n) + O(n^{-3/2}).
\]

In theorem 1.1 of [1], Dyer and Frieze improved Karp and Steele's estimate to

In [4], Frieze used patching procedures from Karp [5] to devise an algorithm that obtained a hamilton circuit in a random directed graph with probability approaching 1 as \(n \to \infty\). Its running time is \(O(n^{1.5})\). More precisely, let \(D_{n,m}\) denote the directed graph with set of vertices \(V = \{1,2,...,n\}\) and a set, \(E\), of \(m\) arcs chosen uniformly from \(K_n^2\), the complete directed graph on \(n\) vertices. Furthermore, let \(m^*\) be the smallest subset of \(E\) such that the directed graph, \(D_{n,m^*}\), has the property that \(\delta^*(D_{n,m^*}) \geq 1\) where \(\delta^*(D)\) denotes the minimum in-degree of the directed graph, \(D\). \(\delta^*(D)\) denotes the minimum in-degree of the directed graph, \(D\).
minimum out-degree of $D$. Frieze's algorithm consists of three steps. In the first one, (a), a small set of edges $E' \subseteq E_{m^*}$ is constructed which almost always contains a set of about $[\log n]$ vertex disjoint cycles which cover $V$. In the second stage, (b), patching algorithms are used which allow the cycles to be “patched” into larger ones by using 2-cycle exchanges. By the end of (b), there is almost always a cycle, $C'$, of length $n - o(n)$ plus at most $O(\log n)$ other cycles,

$$S = \{ C_i \mid i = 1, 2, ..., [c(\log n)] \}.$$  

In the last step, (c), the elements of $S$ are “added” to $C'$, one by one, by a process of double rotations, to obtain a hamilton circuit. Using an algorithm of Even and Tarjan in [4], the running time of Step 1 is $O(n^{1.5})$. The running times of steps (2) and (3) are $O(n(\log n)^{3/2})$ and $O(n^{3/2} + o(1))$, respectively. In this paper, the running time of Frieze’s algorithm is improved by replacing his Step 1 by another procedure which obtains a set of about $\log n$ disjoint cycles covering $V$ in $O(n(\log n)^{3/2})$ running time. Thus, the r. t. of his algorithm becomes $O(n^{3/2} + o(1))$ - that of Step 3 in [1]. Definitions given in the introduction apply to the remainder of the paper. From Frieze [4], we may assume in the construction of $D_{m^*}$ that $m^* \leq n(\log n) + kn$ where $k \to \infty$ as $n \to \infty$. More precisely, let

$$k_i, \text{ if } k \text{ is positive},$$

$$k = 0, \text{ if } k \text{ is zero},$$

$$-k_i, \text{ if } k \text{ is negative}.$$  

Furthermore, let

$$c = 1, \text{ if } k \text{ is zero},$$

$$c = e^{k_j}, \text{ if } k \text{ is positive},$$

$$c = e^{k_j}, \text{ if } k \text{ is negative}.$$  

Then $m^* \leq nl(\log cn)$. In general, if $s \in S_n$, then $s(a)$ represents the action of the permutation $s$ on the point $a$. If $s \in S_n$ moves every point in $P$, then it is a derangement. Thus, the $n$-cycle $h$ is a derangement. An arc not lying in $D_{m^*}$ is called a pseudo-arc associated with $D_{m^*}$. $D_i$ ($i = 1, 2, ...$) is a pseudo-derangement if it corresponds to a derangement, $d_i$, in $S_n$. A vertex of $D_i$ which is the initial vertex of a pseudo-arc of $D_i$ is a pseudo-arc vertex. All other vertices are arc vertices. Let $d_i$ be the derangement in $S_n$ corresponding to $D_i$. Then $s_i$ is admissible if $d_is_i$ is also a derangement. As an example, if $d_i = (1\ 2)(3\ 4)$ and $s_i = (2\ 3)$, $d_is_i = (1\ 2\ 3\ 4)$. Therefore, $d_is_i = d_{i+1}$ is a derangement and thus $s_i$ is admissible. We note that in order to obtain the pseudo-derangement, $D_{i+1}$, the permutation $s_i = (2\ 3)$ is replaced by the set of arcs

$$S = \{(2, \ d_i(3)), (3, \ d_i(2)) = \{(2, 4), (3, 1)\}$$
Lemma 2.1  \[ 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \approx \log n \]

**Proof.** Using an arbitrary partition of the interval from 1 through \( n \), say \((a_0, a_1, \ldots, a_n)\) with \( 1 \leq i \leq n \), the function \( \frac{1}{i} \) is a monotonically decreasing function in each subinterval \( a_{k-1} \leq i \leq a_k \). Using lower sums and upper sums, \( \frac{1}{i} \), it follows that

\[
\sum_{k=1}^{n} \frac{a_k - a_{k-1}}{a_k} \leq \log(n+1) \leq \sum_{k=1}^{n} \frac{a_k - a_{k-1}}{a_k}
\]

Therefore, if \( a_i = i + 1 \) for \( 0 \leq i \leq n \) with \( a_i \) in the interval \([1, n+1]\), we obtain

\[
\sum_{i=1}^{n+1} \frac{1}{i} \leq \log(n+1) \leq \sum_{i=1}^{n} \frac{1}{i}
\]

Since

\[
\log(n+1) - \log(n) = \log\left(\frac{n+1}{n}\right) = \log\left(1 + \frac{1}{n}\right) \to \log(1) = 0,
\]

as \( n \to \infty \), for large \( n \), \( \sum_{k=1}^{n+1} \frac{1}{k} \approx \log(n) \).

**Comment.** The best we can say about the above approximation is that it's less than 1.

**Theorem 2.1.** (Feller [2])

Let \( s \) be a permutation randomly chosen from \( S_n \). Then the number of disjoint cycles of \( s \) approaches \( \lfloor \log n \rfloor \) as \( n \to \infty \).

**Proof.** Let \( P = \{1, 2, \ldots, n, \ldots\} \) be the set of points of \( S_n \), the symmetric group on \( n \) points for large \( n \). Assume that we are randomly constructing a permutation \( s \) in \( S_n \) such that an identity element is considered a cycle of length 1. We can describe the cycles of a given permutation on the points of \( P \) by the inversions of their natural order. For instance, consider the following permutation:

\[
\begin{align*}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
7 & 4 & 2 & 10 & 6 & 3 & 8 & 1 & 5 & 9
\end{align*}
\]

The numbers in the top row may be thought of as ordinal numbers – in this case, the natural ordering of the numbers from 1 through 10. The numbers in the bottom row are the numbers being permuted. For instance, 1 is in the eighth place, 8 is in the seventh place, 7 is in the first place. Thus, \((1, 7, 8)\) is a disjoint cycle of the permutation. Next, 2 is in the third place, 3 is in the sixth place, 6 is in the fifth place, 5 is in the ninth place, 9 is in the tenth place, 10 is in the fourth place, 4 is in the second place. Thus, a second disjoint cycle is \((2, 3, 6, 5, 9, 10)\). The second (and last) disjoint cycle of the permutation. Now let \( s \) be a random permutation of the elements of \( P \). At the \( k-\text{th} \) step of the
construction of \( s \), let \( X_k = 1 \) if a cycle is completed at the end of the \( k \)-th step; otherwise, \( X_k = 0 \). In general,
\[
Pr(X_i = 1) = \frac{1}{n - i + 1}.
\]
It follows that the number of cycles in \( s \) is
\[
\sum_{i=1}^{n} = X_1 + X_2 + \ldots + X_n,
\]
while the average number of cycles in randomly constructing a permutation is
\[
m_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \approx \log n.
\]
From a result of Feller in [2], the number of cycles between \( \log n + \alpha(\log n)^{-1} \) and \( \log n + \beta(\log n)^{-1} \) is approximately given by \( n! \{ \Phi(\alpha) - \Phi(\beta) \} \). If we now assume that \( \alpha = -\log(\log n) \) while \( \beta = \log(\log n) \), then \( \Phi(\beta) - \Phi(\alpha) \to 1 \) as \( n \to \infty \). Since \( n^{\log n} > n + \log n \) for any positive number \( \epsilon \) and a correspondingly large value of \( n(\epsilon) = n \), it follows that, as \( n \to \infty \), the number of cycles in a randomly chosen permutation from \( S_n \) approaches \( \lceil \log n \rceil \).

**Section 2.3. The Algorithm.**

Before discussing the algorithm, we note that since Frieze's algorithm obtains a hamilton cycle, there can exist no cycles, \( C_i = (v_1, v_2, \ldots, v_i) \) (i = 2, 3, \ldots, m), \( m < n \) in \( D_n^* \), such that each vertex, \( v_j \), in \( C_i \) has \( d^-(v_j) = d^+(v_j) = 1 \): all arcs of such cycles must lie on every hamilton cycle of length \( n \) which is impossible. Also, the existence of such a cycle would mean that \( D_n^* \) is not a strongly connected digraph. We use \( D_m^* \) throughout. This allows us to include all arcs of \( D_m^* \), that must lie in *every* hamilton cycle of \( D_m^* \) in the set of cycles obtained while deleting only arcs than can lie on no hamilton cycle in \( D_m^* \). We construct \( D_m^* \) as a balanced, binary search tree.

(1) Before starting constructing permutations, let \( d^+(x) = 1 \), where \( (x, y) \) is the unique arc emanating from \( x \). As we construct \( D_m^* \), the contracted digraph of \( D_m^* \), \( (x, y) \) becomes the 2-vertex, \( xy \). If \( y \) also has out-degree 1, we construct a 3-vertex, say \( xyz \), etc.. Before continuing the algorithm, we delete all arcs that terminate in \( y \). Similarly, if \( d^-(v) = 1 \) where the arc \( (u, v) \) lies in \( D_m^* \), we construct the 2-vertex \( uv \) and then delete all arcs that terminate in \( v \). The set of vertices of \( D_m^* \) is \( V_C \).

(2) We next construct an \( n' \)-cycle, say \( d_0 \), in \( S_n \). (We will discuss the exact construction in section 2.3.) This \( n' \)-cycle generally includes 2-vertices, 3-vertices, \ldots, \( r \)-vertices implying that \( n' < n \). If \( d_0 = (a_1a_2 \ldots a_n) \), we construct a corresponding \( n' \)-cycle, \( D_0 \), consisting of arcs or pseudo-arcs of \( D_m^* \) that correspond to the arcs in \( d_0 \). i.e., if \( a_1a_2 \) is an "arc" of \( d_0 \), then \( (a_1, D_0(a_2)) \) is an arc or pseudo-arc of \( D_m^* \): If it lies in \( D_m^* \), then it is an arc; otherwise, it is a pseudo-arc. In general, \( D_0 \) is a pseudo derangement. Some (or
all) of these "arcs" may lie in $K^D_n - D^\ast_m$. If $(a_j,D_o(a_j))$ is an arc of $D^\ast_m$, then $a_j$ is an arc vertex; otherwise, it is a pseudo-arc vertex. To indicate an arc, note that we use $D_o(a_j)$ rather than $d_o(a_j)$. The object of the algorithm is to transform $D_o$ into a derangement in $D^\ast_m$, i.e., a disjoint set of cycles each of whose arcs lie in $D^\ast_m$, while the sum of the sets of vertices of the cycles is $V_c$. Our way of doing this is to construct a random permutation from the arcs in $D^\ast_m$. To understand the procedure, we give a simple example. Suppose that

$$S = \{(a_i,d_o(a_i)),(a_i,d_o(a_j)),(a_j,d_o(a_i))\}$$

is a set of arcs in $D^\ast_m$ where $a_i$, $a_j$, $a_j$ are pseudo-arc vertices. Then $s = (a,a,a_j)$ is a permutation in $S$, such that $d_i = d_o s$ corresponds to a new pseudo-derangement, $D_1$, of arcs where the elements of $S$ replace pseudo-arcs of $D_o$. Let $V_c = \{a_1,a_2,\ldots,a_n\}$. Assume that $a_i < a_j$ if $i < j$. Define $D_o = (a_1,a_2,\ldots,a_n)$. For simplicity, denote $a_i$ by $i$. It follows that $D_o = \{1,2,3,\ldots,n'\}$. The function $ORD$ represents the ordinal values of the elements of $D_o$. Thus, $ORD(1) = 1$, ..., $ORD(i) = i$, ..., $(1 \leq i \leq n')$. Analogously, $ORD^{-1}(a_i) = i$.

(3) We now construct the following balanced, binary search trees: $PSEUDO$, $ADD$, $DELETE$. Every pseudo-arc vertex of $D_o$ is placed in increasing order of magnitude on $PSEUDO$. $ORD[D_o]$ is a balanced, binary, search tree where each arc is on a branch headed by its initial vertex. $ORD^{-1}[D_o]$ is a balanced, binary search tree in which the entries are arranged in increasing order of magnitude of the domain values. Thus, we can find the $ORD^{-1}$ value of any vertex in $O(\log n)$. Let $a$ be an arbitrary element of $PSEUDO$. In the first step of the algorithm, we obtain an arc, say $[a,D_o(b)]$, lying in $D^\ast_m - D_o$. We then obtain $b$ in the following manner:

(i) $k \neq 1$.

If $ORD^{-1}(D_o(b)) = k$, then $ORD^{-1}(b) = k - 1$.

(ii) $k = 1$.

If $ORD^{-1}(D_o(b)) = 1$, then $ORD^{-1}(b) = n'$.

Finally, $ADD$ is represented by two ordered sets, $ADD(i)$ and $ADD(t)$.

In $ADD(i)$, the arcs of $ADD$ are placed in increasing order of magnitude of their initial vertices; in $ADD(t)$, in increasing order of magnitude of their terminal vertices.

(4) We randomly choose a vertex from $PSEUDO_o$, say $a$, and randomly choose an arc from $D^\ast_m - \{D_o \cup DELETE \cup ADD\}$, say $(a,D_o(b))$. (We define $DELETE$ in (5).)

Obtaining $ORD^{-1}(D_o(b)) = k$, if $k \neq 1$, we find $ORD^{-1}(b) = k - 1$; otherwise, $ORD^{-1}(b) = n'$. We then randomly choose arc $(b,D_o(c))$. Obtaining $ORD^{-1}(D_o(c)) = s$, let $c = ORD(s - 1)$. We then obtain $(c,D_o(d))$. We continue the construction of a permutation

$$s_o = (a,b,c,d\ldots,v\ldots)(a',b',c',d'\ldots,v'\ldots)(a'',b'',c''\ldots,v''\ldots)\ldots = \prod_{j=1}^{j=m} C_j,$$
placing each corresponding arc of \( D' \) in both \( ADD(i) \) and \( ADD(t) \). As we proceed, we continually check to see if the last arc chosen has an initial vertex in \( ADD(i) \) or a terminal vertex in \( ADD(t) \). Suppose that \((u, D_0(v))\) is randomly chosen from \( D' \). If an arc of form \((u, D_0(v'))\) lies in \( ADD(i) \), we delete it and place it in a balanced, binary search tree called \( DELETE \) whose elements are placed in increasing order of magnitude of the initial vertices of its elements. Similarly, suppose that \((c, D_0(v))\) lies in \( ADD(t) \). Again, we delete it, and place it in \( DELETE \). In the latter case, we continue the algorithm by choosing an arc emanating from \( c \). We now discuss how the structure of \( s_0 \) evolves as the algorithm progresses. Suppose \( v = d \), i.e., \((c, D_0(v)) = (c, D_0(d))\). If this occurs, we have two possibilities as shown in Figs. 2.1, 2.2:

(a) \((v, e, f, ... , u)\) is a cycle such that each of the arcs 
\((v, D_0(e)), (e, D_0(f)), ... , (u, D_0(v))\) lies in \( D'_m \). (Here \((u, D_0(v)) = (u, D_0(d))\). We are constructing a permutation, \( s_0 \), consisting of disjoint cycles. This can't be obtained if \((c, d), (u, d)\) are both in it. Thus, as mentioned earlier, we delete \((c, D_0(d))\) from \( ADD(i) \) and \( ADD(t) \), add it to \( DELETE \), and place \((u, D_0(v))\) in \( ADD(i) \) and \( ADD(t) \). As we proceed in this manner, each time we place an arc in \( ADD \), we check \( ADD(i) \) to see if the arc’s initial vertex is in \( PSEUDO \). If so, we delete it from \( PSEUDO \). Once we have placed \((c, D_0(d))\) in \( DELETE \), we continue the algorithm starting with the pseudo-arc \( c \).

(b) We have already obtained at least one cycle, say \( C_j \) \((j = 1, 2, ... )\).

In this case, \( v = d' \) may already belong to some cycle of \( s_0 \), say \( C_j = (d' e' f' ... , r') \). Therefore, we destroy \( C_j \) by deleting the arc \((r', D_0(d'))\) from \( ADD \), replacing it by \((u, D_0(v))\). Correspondingly, this yields the path \([a, b, c, d, ..., u, v, e', f', ..., r']\) used in the construction of \( s_0 \). (We note that \( r \) generally becomes a pseudo-arc vertex in this procedure. In that case, we would add it to \( PSEUDO \).) We then randomly choose an arc of 
\( D' = D'_m - \{ D_0 \cup DELETE \cup ADD \} \) emanating from \( c \).

Continuing in this manner, suppose the following occurs: There exist no arcs emanating from \( c \) in \( D' \). Then the heading of the “c” branch of \( D' \) is changed to \( cD \). We next randomly choose an arc, say \((c, D_0(z))\) from \( DELETE \).

Furthermore, if we delete an arc whose initial arc is \( c \), we place it on the \( cD \) branch of \( D' \). On the other hand, suppose that \( DELETE \) contains no arcs emanating from \( c \). We then change the \( cD \) heading of \( D' \) to \( c \) and commence randomly choosing arcs out of \( D' \), and, as we did initially, place arcs deleted from \( ADD \) in \( DELETE \). We continue the algorithm in this manner until \( PSEUDO \) contains no vertices, indicating that we have obtained a derangement all of whose arcs lie in \( D'_m \).

Suppose that the number of arcs of \( D' \) chosen is \( 2(1 + \alpha)n \log(cn) \). We next use the Poisson approximation to the solution of the classical occupancy problem given in Feller,
Fig. 2.1
Fig. 2.2
section V.2, [3], to obtain the probability that we will run through all of the vertices in $V_C$. Let $c \to \infty$ as $n \to \infty$. If $\lambda = ne^{\frac{(1+\alpha)n\log(cn)}{n}}$, the probability of success approaches $e^{-\lambda} = e^{-e^{\alpha n^{1/2}}} \to e^0 = 1$, the probability of $(1+\alpha)n\log(cn)$ random arcs passing through each vertex of $V_C$ approaches 1. Using theorem 2.1 in Feller [3], it follows that the probability of obtaining a derangement, $d$, in $D'$ containing $\log n$ cycles where only one arc of $d$ is a pseudo-arc approaches 1 as $n \to \infty$. Using another $(1+\alpha)n\log(cn)$ randomly chosen arcs, we now determine the probability of obtaining a derangement containing only arcs of $D'_{m^*}$. The probability that the last arc chosen in an admissible permutation, $s$, corresponds to an arc in $D'_{m^*}$ is at least $\frac{1}{n}$. Therefore, the probability of not obtaining an arc is at most $1 - \frac{1}{n}$ implying that the probability of not obtaining an arc in $(1+\alpha)n\log(cn)$ is at most $(1 - \frac{1}{n})^{(1+\alpha)n\log(cn)} < e^{-(1+\alpha)\log(cn)} = \frac{1}{(cn)^{1+\alpha}}$. It follows that the probability of obtaining an arc is at least $1 - \frac{1}{(cn)^{1+\alpha}} \to 1$ as $n \to \infty$ and $c \to \infty$, concluding the proof that the probability of success of the algorithm approaches 1 as $n \to \infty$. From lemma 2.1 and theorem 2.1, as $n \to \infty$, the probability that a derangement on $n'$ points has $\lceil \log n' \rceil$ cycles approaches 1.

The running time of the algorithm is $O(n(\log n)^2)$.

**Example 2.1** $D'_{m^*}$ is a digraph containing 27 vertices of which three are 2-vertices: 5-19, 7-25, 17-29.

$$D'_{m^*}$$

| 1: 5-19, 13, 15 | 15: 6, 12, 30 |
|----------------|--------------|
| 2: 9, 6        | 16: 30, 17-29, 21 |
| 3: 2, 14, 4    | 17-29: 26, 22, 18 |
| 4: 18, 9, 24   | 18: 7-25, 30, 13 |
| 5-19: 23, 27, 16 | 20: 13, 2, 27 |
| 6: 4, 24, 22   | 21: 20, 18, 28 |
| 7-25: 8, 3, 9  | 22: 3, 5-19, 2 |
| 8: 12, 20      | 23: 28, 1    |
| 9: 17-29, 26, 8 | 24: 14, 7-25, 3 |
10: 15, 16, 14
11: 1, 10
12: 10, 21, 11
13: 27, 4, 6
14: 16, 8, 10
26: 24, 11, 17
27: 11, 1, 9
28: 21, 23, 7-25
30: 22, 28, 20

$$D_o = (1 \ 2 \ 3 \ 4 \ 5-19 \ 6 \ 7-25 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ 16 \ 17-29 \ 18 \ 20 \ 21 \ 22 \ 23 \ 24 \ 26 \ 27 \ 28 \ 30)$$

The following arcs of $D_o$ lie in $D_{m^*}$: (3, 4), (7-25, 8), (16, 17-29), (17-29-18).
\[ ORD(1) = 1; \quad ORD^{-1}(1) = 1 \]
\[ ORD(2) = 2; \quad ORD^{-1}(2) = 2 \]
\[ ORD(3) = 3; \quad ORD^{-1}(3) = 3 \]
\[ ORD(4) = 4; \quad ORD^{-1}(4) = 4 \]
\[ ORD(5) = 5 - 19; \quad ORD^{-1}(5 - 19) = 5 \]
\[ ORD(6) = 6; \quad ORD^{-1}(6) = 6 \]
\[ ORD(7) = 7 - 25; \quad ORD^{-1}(7 - 25) = 7 \]
\[ ORD(8) = 8; \quad ORD^{-1}(8) = 8 \]
\[ ORD(9) = 9; \quad ORD^{-1}(9) = 9 \]
\[ ORD(10) = 10; \quad ORD^{-1}(10) = 10 \]
\[ ORD(11) = 11; \quad ORD^{-1}(11) = 11 \]
\[ ORD(12) = 12; \quad ORD^{-1}(12) = 12 \]
\[ ORD(13) = 13; \quad ORD^{-1}(13) = 13 \]
\[ ORD(14) = 14; \quad ORD^{-1}(14) = 14 \]
\[ ORD(15) = 15; \quad ORD^{-1}(15) = 15 \]
\[ ORD(16) = 16; \quad ORD^{-1}(16) = 16 \]
\[ ORD(17) = 17 - 29; \quad ORD^{-1}(17 - 29) = 17 \]
\[ ORD(18) = 18; \quad ORD^{-1}(18) = 18 \]
\[ ORD(19) = 20; \quad ORD^{-1}(20) = 19 \]
\[ ORD(20) = 21; \quad ORD^{-1}(21) = 20 \]
\[ ORD(21) = 22; \quad ORD^{-1}(22) = 21 \]
\[ ORD(22) = 23; \quad ORD^{-1}(23) = 22 \]
\[ ORD(23) = 24; \quad ORD^{-1}(24) = 23 \]
\[ ORD(24) = 26; \quad ORD^{-1}(26) = 24 \]
\[ ORD(25) = 27; \quad ORD^{-1}(27) = 25 \]
\[ ORD(26) = 28; \quad ORD^{-1}(28) = 26 \]
\[ ORD(27) = 30; \quad ORD^{-1}(30) = 27 \]

\[ D' = D_{m*} - \{ D_0 \cup \text{DELETE} \cup \text{ADD} \} \], \text{DELETE} = \{\}, \text{ADD} = \{\}, \text{PSEUDO} = \{1, 2, 4, 5-19, 6, 8, 9, 10, 11, 13, 14, 15, 18, 20, 21, 22, 23, 24, 26, 27, 28, 30\} \]

In order to save space, we will start each random choice of an arc, \((a, b)\), with the arc itself rather than saying “Choose \((a, b)\).”
(1, 5-19) 1 is not in \( ADD(i) \), 5-19 is not in \( ADD(t) \). Place (1, 5-19) is \( ADD(i) \), \( ADD(t) \).
Delete (1, 5-19) from \( D' \). Delete 1 from \( PSEUDO \).

\( ORD^{-1}(5-19) = 5, ORD(4) = 4. \)

(4, 9) 4 is not in \( ADD(i) \), 9 is not in \( ADD(t) \). Place (4, 9) in \( ADD(i) \), \( ADD(t) \).
Delete (4, 9) from \( D' \). Delete 4 from \( PSEUDO \).

\( ORD^{-1}(9) = 9, ORD(8) = 8. \)

(8, 12) 8 is not in \( ADD(i) \), 12 is not in \( ADD(t) \). Place (8, 12) in \( ADD(i) \), \( ADD(t) \).
Delete (8, 12) from \( D' \). Delete 8 from \( PSEUDO \).

\( ORD^{-1}(12) = 12, ORD(11) = 11. \)

(11, 10) 11 is not in \( ADD(i) \), 10 is not in \( ADD(t) \). Place (11, 10) in \( ADD(i) \), \( ADD(t) \).
Delete (11, 10) from \( D' \).

\( ORD^{-1}(10) = 10, ORD(9) = 9. \)

(9, 17-29) 9 is not in \( ADD(i) \), 17-29 is not in \( ADD(t) \). Place (9, 17-29) in \( ADD(i) \), \( ADD(t) \). Delete (9, 17-29) from \( D' \). Delete 9 from \( PSEUDO \).

\( PSEUDO = \{2, 5-19, 6, 10, 12, 13, 14, 15, 18, 20, 21, 22, 23, 24, 26, 27, 28, 30\} \)

\( ADD(i) = \{(1, 5-19), (4, 9), (8, 12), (9, 17, 29), (11, 10)\} \)

\( ADD(t) = \{(1, 5-19), (4, 9), (11, 10), (8, 12), (9, 17-29)\} \)

\( ARCS IN D_0 = \{(3, 4), (7-25, 8), (16, 17-29), (17-29, 18)\} \)

\( DELETE = \{ \} \)

\( ORD^{-1}(17-29) = 17, ORD(16) = 16. \)

(16, 21) 16 is not in \( ADD(i) \), 21 is not in \( ADD(t) \). Place (16, 21) in \( ADD(i) \), \( ADD(t) \).
Delete (16, 21) from \( D' \).

\( ORD^{-1}(21) = 20, ORD(19) = 20. \)

(20, 13) 20 is not in \( ADD(i) \), 13 is not in \( ADD(t) \). Place (20, 13) in \( ADD(i) \), \( ADD(t) \).
Delete (20, 13) from \( D' \). Delete 20 from \( PSEUDO \).

\( ORD^{-1}(13) = 13, ORD(12) = 12. \)

(12, 21) 12 is not in \( ADD(i) \), 21 is in \( ADD(t) \): (16, 21). Delete (16, 21) from \( ADD(i) \), \( ADD(t) \). Place (12, 21) in \( ADD(i) \), \( ADD(t) \).
Delete 12 from \( PSEUDO \).

(16, 30) Place (16, 21) in \( DELETE \). 16 is not in \( ADD(i) \), 30 is not in \( ADD(t) \). Place (16, 30) in \( ADD(i) \), \( ADD(t) \).
Delete (16, 30) from \( D' \).

\( ORD^{-1}(30) = 27, ORD(26) = 28. \)
(28, 7-25) 28 is not in $ADD(i)$, 7-25 is not in $ADD(t)$. Place (28, 7-25) in $ADD(i)$, $ADD(t)$. Delete (28, 7-25) from $D'$. Delete 28 from PSEUDO.

$ORD^{-1}(7-25)=7$, $ORD(6)=6$.

(6, 4) 6 is not in $ADD(i)$, 4 is not in $ADD(t)$. Place (6, 4) in $ADD(i)$, $ADD(t)$. Delete it from $D'$. Delete 6 from PSEUDO.

$PSEUDO = \{2, 5-19, 10, 13, 14, 15, 18, 21, 22, 23, 24, 26, 27, 30\}$

$ADD(i) = \{(1, 5-19), (4, 9), (6, 4), (8, 12), (9, 17-29), (11, 10), (12, 21), (16, 30), (20, 13), (28, 7-25)\}$

$ADD(t) = \{(6, 4), (1, 5-19), (28, 7-25), (4, 9), (11, 10), (8, 12), (20, 13), (9, 17-29), (12, 21), (16, 30)\}$

$DELETE = \{(6, 4), (4, 9), (16, 21)\}$

$ARCS in D_0 = \{(3, 4), (7-25, 8), (16, 17-29), (17-29, 18)\}$

$ORD^{-1}(4)=4$, $ORD(3)=3$.

(3, 14) 3 doesn’t lie in $ADD(i)$, 14 doesn’t lie in $ADD(t)$. Place (3, 14) in $ADD(i)$, $ADD(t)$. Delete (3, 14) from $D'$.

$ORD^{-1}(14)=14$, $ORD(13)=13$.

(13, 4) 13 is not in $ADD(i)$, 4 is in $ADD(t)$: (6, 4). Delete (6, 4) from $ADD(i)$, $ADD(t)$.

Place (13, 4) in $ADD(i)$, $ADD(t)$. Delete (13, 4) from $D'$.

(6, 24) Place (6, 4) in DELETE. 6 is not in $ADD(i)$, 24 is not in $ADD(t)$. Place (6, 24) in $ADD(i)$, $ADD(t)$. Delete it from $D'$.

$ORD^{-1}(24)=23$, $ORD(22)=23$.

(23, 28) 23 is not in $ADD(i)$, 28 is not in $ADD(t)$. Place (23, 28) in $ADD(i)$, $ADD(t)$. Delete it from $D'$. Delete 23 from PSEUDO.

$ORD^{-1}(28)=26$, $ORD(25)=27$.

(27, 29) 27 is not in $ADD(i)$, 9 is in $ADD(t)$: (4, 9). Delete (4, 9) from $ADD(i)$, $ADD(t)$.

Place (27, 9) in $ADD(i)$, $ADD(t)$. Delete it from $D'$. Delete 27 from PSEUDO.

$PSEUDO = \{2, 4, 5-19, 10, 14, 15, 18, 21, 22, 24, 26, 30\}$

$ADD(i) = \{(1, 5-19), (3, 14), (6, 24), (8, 12), (9, 17-29), (11, 10), (12, 21), (13, 4), (16, 30), (20, 13), (23, 28), (27, 9), (28, 7-25)\}$

$ADD(t) = \{(13, 4), (1, 5-19), (28, 7-25), (27, 9), (11, 10), (8, 12), (20, 13), (3, 14), (9, 17-29), (12, 21), (23, 28), (28, 7-25), (16, 30)\}$
\[
\text{DELETE} = \{(4, 9), (6, 4), (16, 21)\}
\]

\[
\text{ARCS in } D_0 = \{(3, 4), (7-25, 8), (16, 17-29), (17-29, 18)\}
\]

(4, 24) Place (4, 9) in DELETE. 4 is not in \textit{ADD}(i), 24 is in \textit{ADD}(t): (6, 24). Delete (6, 24) from \textit{ADD}(i), \textit{ADD}(t). Place (4, 24) in \textit{ADD}(i), \textit{ADD}(t). Delete (4, 24) from \textit{D}'. Delete 4 from \textit{PSEUDO}.

(6, 22) Place (6, 24) in DELETE. 6 is not in \textit{ADD}(i), 22 is not in \textit{ADD}(t). Place (6, 22) in \textit{ADD}(i), \textit{ADD}(t). Delete (6, 22) from \textit{D}'.

\[
\text{ORD}^{-1}(22) = 21, \text{ORD}(20) = 21.
\]

(21, 20) 21 is not in \textit{ADD}(i), 20 is not in \textit{ADD}(t). Place (21, 20) in \textit{ADD}(i), \textit{ADD}(t). Delete (21, 20) from \textit{D}'. Delete 21 from \textit{PSEUDO}.

\[
\text{ORD}^{-1}(20) = 19, \text{ORD}(18) = 18.
\]

(18, 30) 18 is not in \textit{ADD}(i), 30 is in \textit{ADD}(t): (16, 30). Delete (16, 30) from \textit{ADD}(i), \textit{ADD}(t). Place (18, 30) in \textit{ADD}(i), \textit{ADD}(t). Delete (18, 30) from \textit{D}'.

Delete 18 from \textit{PSEUDO}.

No arcs emanate from 16 in \textit{D'} Rename the heading of the branch 16 in \textit{D'}, 16D. Choose an arc emanating from 16 in DELETE.

(16, 21) 16 is not in \textit{ADD}(i), 21 is in \textit{ADD}(t): (12, 21). Delete (12, 21) from \textit{ADD}(i), \textit{ADD}(t). Place (16, 21) in \textit{ADD}(i), \textit{ADD}(t).

(12, 11) Place (12, 21) in DELETE. 12 is not in \textit{ADD}(i), 11 is not in \textit{ADD}(t). Place (12, 11) in \textit{ADD}(i), \textit{ADD}(t). Delete (12, 11) from \textit{D'}

\[
\text{ORD}^{-1}(11) = 11, \text{ORD}(10) = 10.
\]

\[
\text{PSEUDO} = \{2, 5-19, 10, 14, 15, 22, 24, 26, 30\}
\]

\[
\text{ADD}(i) = \{(1, 5-19), (3, 14), (4, 24), (6, 22), (8, 12), (9, 17-29), (11, 10), (12, 11), (13, 4), (16, 21), (18, 30), (20, 13), (21, 20), (23, 28), (27, 9), (28, 7-25)\}
\]

\[
\text{ADD}(t) = \{(13, 4), (1, 5-19), (28, 7-25), (27, 9), (11, 10), (12, 11), (8, 12), (20, 13), (3, 14), (9, 17-29), (21, 20), (16, 21), (6, 22), (4, 24), (23, 28), (18, 30)\}
\]

\[
\text{DELETE} = \{(3, 4), (4, 9), (6, 4), (6, 24), (12, 21), (16, 30)\}
\]

\[
\text{ARCS in } D_0 = \{(3, 4), (7-25, 8), (16, 17-29), (17-29, 18)\}
\]

(10, 16) 10 is not in \textit{ADD}(i), 16 is not in \textit{ADD}(t). Place (10, 16) in \textit{ADD}(i), \textit{ADD}(t). Delete (10, 16) from \textit{D}'. Delete 10 from \textit{PSEUDO}.

\[
\text{ORD}^{-1}(16) = 16, \text{ORD}(15) = 15.
\]

(15, 12) 15 is not in \textit{ADD}(i), 12 is in \textit{ADD}(t): (8, 12). Delete (8, 12) from \textit{ADD}(i), \textit{ADD}(t). Place (15, 12) in \textit{ADD}(i), \textit{ADD}(t). Delete (15, 12) from \textit{PSEUDO}.
$D'$. Delete 15 from $PSEUDO$.

(8, 20) 8 is not in $ADD(i)$, 20 is in $ADD(t)$: (21, 20). Delete (21, 20) from $ADD(i)$, $ADD(t)$. Place (8, 20) in $ADD(i)$, $ADD(t)$. Delete (8, 20) from $D'$.

(21, 18) Place (21, 20) in $DELETE$. 21 is not in $ADD(i)$, 18 is not in $ADD(t)$. Place (21, 18) in $ADD(i)$, $ADD(t)$. Delete (21, 18) from $D'$.

$ORD^{-1}(18) = 18$, $ORD(17) = 17 - 29$.

(17-29, 26) 17-29 is not in $ADD(i)$, 26 is not in $ADD(t)$. Place (17-29, 26) in $ADD(i)$, $ADD(t)$. Delete (17-29, 26) from $D'$.

$ORD^{-1}(26) = 24$, $ORD(23) = 24$.

$PSEUDO = \{2, 5-19, 14, 22, 24, 26, 30\}$

$ADD(i) = \{(1, 5-19), (3, 14), (4, 24), (6, 22), (8, 20), (9, 17-29), (10, 16), (11, 10), (12, 11), (15, 12), (16, 21), (17-29, 26), (18, 30), (20, 13), (21, 18), (23, 28), (27, 9), (28, 7-25)\}$

$ADD(t) = \{(13, 4), (1, 5-19), (28, 7-25), (27, 9), (11, 10), (12, 11), (15, 12), (20, 13), (3, 14), (10, 16), (9, 17-29), (21, 18), (8, 20), (16, 21), (6, 22), (4, 24), (17-29, 26), (23, 28), (18, 30)\}$

$DELETE = \{(4, 9), (6, 4), (6, 24), (8, 12), (16, 30), (21, 20)\}$

$ARCS$ in $D_0 = \{(3, 4), (7-25, 8), (16, 17-29), (17-29, 18)\}$

(24, 3) 24 is not in $ADD(i)$, 3 is not in $ADD(t)$. Place (24, 3) in $ADD(i)$, $ADD(t)$. Delete (24, 3) from $D'$. Delete 24 from $PSEUDO$.

$ORD^{-1}(3) = 3$, $ORD(2) = 2$.

(2, 6) 2 is not in $ADD(i)$, 6 is not in $ADD(t)$. Place (2, 6) in $ADD(i)$, $ADD(t)$. Delete (2, 6) from $D'$. Delete 2 from $PSEUDO$.

$ORD^{-1}(6) = 6$, $ORD(5) = 5 - 19$.

(5-19, 23) 5-19 is not in $ADD(i)$, 23 is not in $ADD(t)$. Place (5-19, 23) in $ADD(i)$, $ADD(t)$. Delete (5-19, 23) from $D'$. Delete 5-19 from $PSEUDO$.

$ORD^{-1}(23) = 22$, $ORD(21) = 22$.

(22, 2) 22 is not in $ADD(i)$, 2 is not in $ADD(t)$. Place (22, 2) in $ADD(i)$, $ADD(t)$. Delete (22, 2) from $D'$. Delete 22 from $PSEUDO$.

$ORD^{-1}(2) = 2$, $ORD(1) = 1$.

(1, 13) 1 is in $ADD(i)$: (1, 5-19), 13 is in $ADD(t)$: (20, 13). Delete (1, 5-19), (20, 13) from $ADD(i)$, $ADD(t)$. Place (1, 13) in $ADD(i)$, $ADD(t)$. Delete (1, 13) from $D'$.

(20, 27) Place (1, 5-19), (20, 13) in $DELETE$. 20 is not in $ADD(i)$, 27 is not in $ADD(t)$. Place (20, 27) in $ADD(i)$, $ADD(t)$. Delete (20, 27) from $D'$.

$ORD^{-1}(27) = 25$, $ORD(24) = 26$. 
\( PSEUDO = \{14, 26, 30\} \)

\( ADD(i) = \{(1, 13), (2, 6), (3, 14), (4, 24), (5-19, 23), (6, 22), (8, 20), (9, 17-29), (10, 16), (11, 10), (12, 11), (13, 4), (15, 12), (16, 21), (17-29, 26), (18, 30), (20, 27), (21, 18), (22, 2), (23, 28), (24, 3), (27, 9), (28, 7-25)\} \)

\( ADD(t) = \{(22, 2), (24, 3), (13, 4), (2, 6), (28, 7-25), (27, 9), (11, 10), (12, 11), (15, 12), (1, 13), (10, 16), (9, 17-29), (21, 18), (8, 20), (16, 21), (6, 22), (5-19, 23), (4, 24), (20, 27), (23, 28), (18, 30)\} \)

\( DELETE = \{(1, 5-19), (4, 9), (6, 4), (8, 12), (16, 30), (20, 13), (21, 20)\} \)

\( ARCS \text{ in } D_{o} = \{(3, 4), (7-25, 8), (16, 17-29), (17-29, 18)\} \)

(26, 24) 26 is not in \( ADD(i) \). 24 is in \( ADD(t) \): (4, 24). Delete (4, 24) from \( ADD(i), ADD(t) \). Place (26, 24) in \( ADD(i), ADD(t) \). Delete (26, 24) from \( D' \). Delete 26 from \( PSEUDO \).

(4, 18) Place (4, 24) in \( DELETE \). 4 is not in \( ADD(i) \), 18 is in \( ADD(t) \): (21, 18). Delete (21, 18) from \( ADD(i), ADD(t) \). Place (4, 18) in \( ADD(i), ADD(t) \). Delete (4, 18) from \( D' \).

(21, 28) Place (21, 18) in \( DELETE \). 21 is not in \( ADD(i) \), 28 is in \( ADD(t) \): (23, 28). Delete (23, 28) from \( ADD(i), ADD(t) \). Place (21, 28) in \( ADD(i), ADD(t) \). Delete (21, 28) from \( D' \).

(23, 1) Place (23, 28) in \( DELETE \). 23 is not in \( ADD(i) \), 1 is not in \( ADD(t) \). Place (23, 1) in \( ADD(i), ADD(t) \). Delete (23, 1) from \( D' \).

\( ORD^{-1}(1)=1, ORD(27)=30 \).

(30, 20) 30 is not in \( ADD(i) \), 20 is in \( ADD(t) \): (8, 20). Delete (8, 20) from \( ADD(i), ADD(t) \). Place (30, 20) in \( ADD(i), ADD(t) \). Delete (30, 20) from \( D' \). Delete 30 from \( PSEUDO \).

No arcs emanate out of 8 in \( D' \). Rename the heading 8 of \( D' \), 8D.

Choose (8, 12) from \( DELETE \). Delete (8, 12) from \( DELETE \). Place (8, 20) in \( D' \).

8 is not in \( ADD(i) \), 12 is in \( ADD(t) \): (15, 12). Delete (15, 12) from \( ADD(i), ADD(t) \). Place (8, 12) in \( ADD(i), ADD(t) \). Place 15 in \( PSEUDO \).

\( PSEUDO = \{14, 15\} \)

\( ADD(i) = \{(1, 13), (2, 6), (3, 14), (4, 18), (5-19, 23), (6, 22), (8, 12), (9, 17-29), (10, 16), (11, 10), (12, 11), (13, 4), (16, 21), (17-29, 26), (18, 30), (20, 27), (21, 28), (22, 2), (23, 1), (24, 3), (26, 24), (27, 9), (28, 7-25), (30, 20)\} \)

\( ADD(t) = \{(23, 1), (22, 2), (24, 3), (13, 4), (2, 6), (28, 7-25), (27, 9), (11, 10), (12, 11), (8, 12), (1, 13), (3, 14), (10, 16), (9, 17-29), (4, 18), (30, 20), (16, 21), (20, 13), (21, 20)\} \)


$\{6, 22\}, \{5-19, 23\}, \{26, 24\}, \{17-29, 26\}, \{20, 27\}, \{21, 28\}, \{18, 30\}\}

$\text{DELETE} = \{(1, 5-19), (4, 9), (4, 24), (6, 4), (6, 24), (8, 12), (16, 30), (20, 13), (21, 18)\}$

$\text{ARCS in } D_0 = \{(3, 4), (7-25, 8), (16, 17-29), (17-29, 18)\}$

(15, 6) Place (15, 12) in $\text{DELETE}. 15$ is not in $\text{ADD}(i)$. 6 is in $\text{ADD}(t): (2, 6)$. Delete (2, 6) from $\text{ADD}(i)$, $\text{ADD}(t)$. Place (15, 6) in $\text{ADD}(i)$, $\text{ADD}(t)$. Delete (15, 6) from $D'$. Delete 15 from $\text{PSEUDO}$.

(2, 9) Place (2, 6) in $\text{DELETE}. 2$ is not $\text{ADD}(i)$. 9 is in $\text{ADD}(t): (27, 9)$. Delete (27, 9) from $\text{ADD}(i)$, $\text{ADD}(t)$. Place (2, 9) in $\text{ADD}(i)$, $\text{ADD}(t)$. Delete (2, 9) from $D'$.

(27, 1) Place (27, 9) in $\text{DELETE}. 27$ is not in $\text{ADD}(i)$. 1 is in $\text{ADD}(t): (23, 1)$. Delete (23, 1) from $\text{ADD}(i)$, $\text{ADD}(t)$. Place (27, 1) in $\text{ADD}(i)$, $\text{ADD}(t)$. Delete (27, 1) from $D'$.

No arcs emanate from 23 in $D'$. Rename the heading 23 in $D'$, 23D. Choose (23, 28) from $\text{DELETE}. 23$ is not in $\text{ADD}(i)$. 28 is in $\text{ADD}(t): (21, 28)$. Delete (21, 28) from $\text{ADD}(i)$, $\text{ADD}(t)$. Place (23, 28) in $\text{ADD}(i)$, $\text{ADD}(t)$.

(21, 20) Place (21, 28) in $\text{DELETE}. 21$ is not in $\text{ADD}(i)$. 20 is in $\text{ADD}(t): (30, 20)$. Delete (30, 20) from $\text{ADD}(i)$, $\text{ADD}(t)$. Place (21, 20) in $\text{ADD}(i)$, $\text{ADD}(t)$.

Delete (21, 20) from $D'$. Place 30 in $\text{PSEUDO}$.

$\text{PSEUDO} = \{14, 30\}$

$\text{ADD}(i) = \{(1, 13), (2, 9), (3, 14), (4, 18), (5-19, 23), (6, 220, (8, 12), (9, 17-29), (10, 16), (11, 10), (12, 11), (13, 4), (15, 6), (16, 21), (17-29, 26), (18, 30), (20, 27), (21, 20), (22, 2), (23, 28), (24, 3), (26, 24), (27, 1), (28, 7-25)\}$

$\text{ADD}(t) = \{(27, 1), (22, 2), (24, 3), (13, 4), (15, 6), (28, 7-25), (2, 9), (11, 10), (12, 11), (8, 12), (1, 13), (3, 14), (10, 16), (9, 17-29), (4, 18), (21, 20), (16, 21), (6, 22), (5-19, 23), (26, 24), (17-29, 26), (20, 27), (23, 28), (18, 30)\}$

$\text{DELETE} = \{(1, 5-19), (2, 6), (4, 9), (4, 24), (6, 4), (6, 24), (16, 30), (20, 13), (21, 18), (21, 28), (23, 28)\}$

$\text{ARCS in } D_0 = \{(3, 4), (7-25, 8), (16, 17-29), (17-29, 18)\}$

(30, 22) Place (30, 20) in $\text{DELETE}. 30$ is not in $\text{ADD}(i)$. 22 is in $\text{ADD}(t): (6, 22)$. Delete (6, 22) from $\text{ADD}(i)$, $\text{ADD}(t)$. Place (30, 22) in $\text{ADD}(i)$, $\text{ADD}(t)$. Delete 30 from $\text{PSEUDO}$.

No arc emanates from 6 in $D'$. Rename 6, 6D in $D'$.

Choose (6, 4) from $\text{DELETE}. 6$ is in $\text{ADD}(i)$, $\text{ADD}(t)$. Place (6, 22) in $D'$. 

6 is not in $\text{ADD}(i)$, 4 is in $\text{ADD}(t)$: (13, 4). Delete (13, 4) from $\text{ADD}(i)$, $\text{ADD}(t)$. Place (6, 4) in $\text{ADD}(i)$, $\text{ADD}(t)$. Delete (6, 4) from $\text{DELETE}$.

(13, 6) Place (13, 4) in $\text{DELETE}$. 13 is not in $\text{ADD}(i)$, 6 is in $\text{ADD}(t)$: (15, 6). Delete (15, 6) from $\text{ADD}(i)$, $\text{ADD}(t)$. Place (13, 6) in $\text{ADD}(i)$, $\text{ADD}(t)$.

(15, 30) Place (15, 6) in $\text{DELETE}$. 15 is not in $\text{ADD}(i)$, 30 is in $\text{ADD}(t)$: (18, 30). Delete (18, 30) from $\text{ADD}(i)$, $\text{ADD}(t)$. Place (15, 30) in $\text{ADD}(i)$, $\text{ADD}(t)$.

(18, 13) Place (18, 30) in $\text{DELETE}$. 18 is not in $\text{ADD}(i)$, 13 is in $\text{ADD}(t)$: (1, 13). Delete (1, 13) from $\text{ADD}(i)$, $\text{ADD}(t)$. Place (18, 13) in $\text{ADD}(i)$, $\text{ADD}(t)$.

(1, 15) Place (1, 13) in $\text{DELETE}$. 1 is not in $\text{ADD}(i)$, 15 is not in $\text{ADD}(t)$. Place (1, 15) in $\text{ADD}(i)$, $\text{ADD}(t)$.

$\text{ORD}^{-1}(15) = 15$, $\text{ORD}(14) = 14$.

(14, 8) 14 is not in $\text{ADD}(i)$, 8 is not in $\text{ADD}(t)$. Place (14, 8) in $\text{ADD}(i)$, $\text{ADD}(t)$. Delete 14 from $\text{PSEUDO}$.

$\text{ORD}^{-1}(8) = 8$, $\text{ORD}(7) = 7 - 25$.

(7-25, 3) 7-25 is not in $\text{ADD}(i)$, 3 is in $\text{ADD}(t)$: (24, 3). Delete (24, 3) from $\text{ADD}(i)$, $\text{ADD}(t)$. Place (7-25, 3) in $\text{ADD}(i)$, $\text{ADD}(i)$. Place 24 in $\text{PSEUDO}$.

$\text{PSEUDO} = \{24\}$

$\text{ADD}(i) = \{(1, 15), (2, 9), (3, 14), (4, 18), (5-19, 23), (6, 4), (7-25, 3), (8, 12), (9, 17-29), (10, 16), (11, 10), (12, 11), (13, 6), (14, 8), (15, 30), (16, 21), (17-29, 26), (18, 13), (20, 27), (21, 20), (22, 2), (23, 28), (26, 24), (27, 1), (28, 7-25), (30, 22)\}$

$\text{ADD}(t) = \{(27, 1), (22, 2), (7-25, 3), (6, 4), (13, 6), (28, 7-25), (14, 8), (2, 9), (11, 10), (12, 11), (8, 12), (18, 13), (3, 14), (1, 15), (10, 16), (9, 17-29), (4, 18), (21, 20), (16, 21), (30, 22), (5-19, 23), (26, 24), (27, 1), (28, 7-25), (30, 22)\}$

$\text{DELETE} = \{(1, 5-19), (1, 13), (2, 6), (4, 9), (4, 24), (6, 4), (6, 24), (8, 12), (13, 4), (15, 6), (16, 30), (18, 30), (20, 13), (21, 18), (21, 28), (23, 28), (24, 3), (30, 20)\}$

$\text{ARCS in } D_0 = \{(3, 4), (7-25, 8), (16, 17-29), (17-29, 18)\}$

(24, 14) Place (24, 3) in $\text{DELETE}$. 24 is not in $\text{ADD}(i)$, 14 is in $\text{ADD}(t)$: (3, 14). Delete (3, 14) from $\text{ADD}(i)$, $\text{ADD}(t)$. Place (24, 14) in $\text{ADD}(i)$, $\text{ADD}(t)$. Delete 24 from $\text{PSEUDO}$.

(3, 2) Place (3, 14) in $\text{DELETE}$. 3 is not in $\text{ADD}(i)$, 2 is in $\text{ADD}(t)$: (22, 2).
Delete (22, 2) from \( ADD(i), ADD(t) \). Place (3, 2) in \( ADD(i), ADD(t) \).

(22, 5-19) Place (22, 2) in \( DELETE \). 22 is not in \( ADD(i) \), 5-19 is not in \( ADD(t) \).

We thus have no more vertices in \( PSEUDO \).

\[
PSEUDO = \{\}
\]

\[
ADD(i) = \{(1, 15), (2, 9), (3, 2), (4, 18), (5-19, 23), (6, 4), (7-25, 3), (8, 12), (9, 17-29), (10, 16), (11, 10), (12, 11), (13, 6), (14, 8), (15, 30), (16, 21), (17-29, 26), (18, 13), (20, 27), (21, 20), (22, 5-19), (23, 28), (24, 14), (26, 24), (27, 1), (28, 7-25), (30, 22)\}
\]

\[
D = (1 15 30 22 5-19 23 28 7-25 3 2 9 17-29 26 24 14 8 12 11 10 16 21 20 27) (4 18 13 6)
\]

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