Einstein-Podolsky-Rosen-steering using quantum correlations in non-Gaussian entangled states

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In view of the increasing importance of non-Gaussian entangled states in quantum information protocols like teleportation and violations of Bell inequalities, the steering of continuous variable non-Gaussian entangled states is investigated. The EPR steering for Gaussian states may be demonstrated through the violation of the Reid inequality involving products of the inferred variances of non-commuting observables. However, for arbitrary states the Reid inequality is not always necessary because of the higher order correlations in such states. One then needs to use the entropic steering inequality. We examine several classes of currently important non-Gaussian entangled states, such as the two-dimensional harmonic oscillator, the photon subtracted two mode squeezed vacuum, and the NOON state, in order to demonstrate the steering property of such states. A comparative study of the violation of the Bell-inequality for these states shows that the entanglement present is more easily revealed through steering compared to Bell-violation for several such states.

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I. INTRODUCTION

The pioneering work of Einstein, Podolsky and Rosen (EPR) [1] has over the years lead to the unfolding of several rich and arguably paradoxical features of quantum mechanics [2]. Considering a position-momentum correlated state of two particles, and assuming the notions of spatial separability, locality, and reality to hold true at the level of quantum particles, EPR argued that the quantum mechanical description of the state of a particle is not complete. An immediate consequence of correlations between spatially separated particles was then noted by Schrodinger [3] in that it allowed for the steering of the state on one side merely by the choice of the measurement basis on the other side without in any way having direct access to the affected particle. The word ‘entanglement’ was first coined by Schrodinger to describe the property of such spatially separated but correlated particles.

Inspired by the early works of EPR and Schrodinger, a formalism for quantifying the correlations in terms of joint measurements of observables corresponding to two spatially separated particles was first proposed by Bell [4] for the case of any general theory obeying the tenets of locality and realism. Bell’s inequality was shown to be violated in quantum mechanics, a fact that has since been empirically validated in several subsequent experiments [5]. From the practical point of view, quantum correlations have been used as resource in performing tasks that are unable to be achieved using classical means, leading to many interesting and important information theoretic applications, such as dense coding, teleportation, and secret key generation. Developments in quantum information theory for both discrete [6] as well as continuous variables [7] have brought about the realization of subtle differences in various categories of correlations, for example, the distinction of quantum entanglement from more general quantum correlations, viz., quantum discord [8] found in classes of separable states.

On the other hand, the understanding of the precise nature of correlations that lead to the EPR paradox had to wait for a number of years beyond Bell’s derivation of his inequality, and further advances in quantum information theory. In this direction, a testable formulation of the EPR paradox was proposed by Reid [9] in the realm of continuous variable systems using the position-momentum uncertainty relation, in terms of an inequality involving products of inferred variances of incompatible observables. This lead to the experimental realization of the EPR paradox by Ou et al. [10] for the case of two spatially separated and correlated light modes. Similar demonstrations of the EPR paradox using quadrature amplitudes of other radiation fields were performed [11]. Moreover, a much stronger violation of the Reid inequality for two mode squeezed vacuum states has been experimentally demonstrated recently [12]. The EPR criterion has been used to demonstrate entanglement in Bose-Einstein condensates, as well [13]. Other works have showed that the Reid inequality is effective in demonstrating the EPR paradox for systems in which correlations appear at the level of variances. However, in systems with correlations manifesting in higher than the second moment, the Reid formulation generally fails to show occurrence of the EPR paradox, even though Bell nonlocality may be exhibited [14, 15].

A more direct manifestation of EPR-type correlations has been proposed by the work of Wiseman et al. [16, 17], where steering is formulated in terms of an information theoretic task. Using similar formulations for entanglement as well as Bell nonlocality, a clear distinction between these three types of correlations is possible using joint probability distributions. Wiseman et al. [16, 17]
have further shown a hierarchy between the three types of correlations, with entanglement being the weakest, steering the intermediate, and Bell violation the strongest of the three. Bell nonlocal states constitute a strict subset of steerable states which, in turn, are a strict subset of entangled states. For the case of pure entangled states of two qubits the three classes overlap. An experimental demonstration of these differences has been demonstrated for mixed entangled states of two qubits [13]. A loophole free EPR-steering experiment has also been performed [19]. The case of continuous variable states however poses an additional difficulty, since there exist several pure entangled states which do not display steering through the Reid criterion based on variances of observables [9]. In order to exploit higher order correlation in such states, Walborn et al. [14] proposed a new steering condition which is derived using the entropic uncertainty principle [20]. Entropic functions by definition incorporate correlations up to all orders, and the Reid criterion can be seen to follow as a limiting case of the entropic steering relation [14]. Generalizations of entropic steering inequalities to the case of symmetric steering [21], loss-tolerant steering [22], as well as to the case of steering with quantum memories [23] has also been proposed recently.

EPR steering for Gaussian states has been studied extensively both theoretically and experimentally. It is realized though that Gaussian states are a rather special class of states, and there exist very common examples of states, such as the superposition of two oscillators in Fock states that are far from Gaussian in nature. The non-Gaussian states are usually generated by the process of photon subtraction and addition [24], and these states generally have higher degree of entanglement than the Gaussian states. Hence, non-Gaussian states have applications in tests of Bell inequalities, quantum teleportation and other quantum information protocols [25]. Extensions of the entanglement criteria for non-Gaussian states have been proposed recently [26]. Since the steering of correlated systems has started being studied only recently, it is important to understand the steering of systems with non-Gaussian correlations. A particular example of a non-Gaussian state was considered by Walborn et al. [14] revealing steering through the entropic inequality. Non-Gaussian entanglement and steering has also been recently studied in the context of Kerr-squeezed optical beams [27]. In the present work we consider several categories of non-Gaussian states with the motivation to investigate EPR-steering of such states. This should stimulate steering experiments using non-Gaussian states.

The plan of the paper is as follows. In the next section we present a brief review of the basic concepts involved in EPR steering. Here we first discuss the Reid criterion for demonstrating the EPR paradox, and recall its applicability for the case of the two mode squeezed vacuum state. We then discuss steering as an information processing task, and the entropic steering inequality for conjugate variable pairs. In Section III, several examples of non-Gaussian states are studied for their steering and nonlocality properties. Here we first consider entangled eigenstates of the two dimensional harmonic oscillator given by Laguerre-Gaussian wave functions that have been experimentally realized [28, 29], and may be capable of useful information processing due to their high available degrees of freedom. We show the inadequacy of the Reid criterion in revealing steering for such states. We then demonstrate steering using the entropic steering relation. Photon subtraction from light beams is useful for generating a variety of non-Gaussian states, and is thought to be of much practical use in quantum state engineering [30]. We next study the steering properties of photon subtracted squeezed vacuum states using the entropic steering inequality. Lastly, we study steering by N00N states [31] that are regarded to be of high utility in quantum metrology. In all the examples considered, we present a comparison of the magnitude of Bell violation with the strength of steering. Such an analysis also brings out the comparative efficiency of the steering framework in revealing quantum correlations in a given state compared to the Bell framework, that may be of practical relevance. A summary of our main results are presented in Section IV.

II. THE EPR PARADOX AND STEERING

The EPR paradox may be understood by considering a bipartite entangled state which may be expressed in two different ways, as

$$|\Psi\rangle = \sum_{n=1}^{\infty} c_n |\psi_n\rangle |u_n\rangle = \sum_{n=1}^{\infty} d_n |\phi_n\rangle |v_n\rangle$$ (1)

where \{ |u_n\rangle \} and \{ |v_n\rangle \} are two orthonormal bases for one of the parties (say, Alice). If Alice chose to measure in the \{ |u_n\rangle \} (\{ |v_n\rangle \}) basis, then she instantaneously projects Bob’s system into one of the states |ψ_n⟩ (|φ_n⟩). This ability of Alice to affect Bob’s state due to her choice of the measurement basis was dubbed as “steering” by Schrödinger [32]. Since there is no physical interaction between Alice and Bob, it is paradoxical that the ensemble of |ψ_n⟩s is different from the ensemble of |φ_n⟩s.

The EPR paradox stems from the correlations between two non-commuting observables of a sub-system with those of the other sub-system, i.e., \( x, p_y \neq 0 \), with \( x = 0 \Rightarrow y \neq 0 \) individually. In the original formulation of the paradox correlations between the measurement outcomes of positions and momenta for two separated particles was considered. Due to the presence of correlations, the measurement of the position of, say, the first particle leads one to infer the correlated value of the position for the second particle (say, \( x_{inf} \)). Now, if the momentum of the second particle is measured giving the outcome, say \( p \), the value of the product of uncertainties \( (\Delta x_{inf})^2(\Delta p_{inf})^2 \) may turn out to be less than that allowed by the uncertainty principle, viz. \( (\Delta x)^2(\Delta p)^2 \geq 1 \),
A. The Reid inequality and its violation for the two mode squeezed vacuum state

The possibility of demonstrating the EPR paradox in the context of continuous variable correlations was first proposed by Reid [9]. Such an idea has been experimentally realized [10] through quadrature phase measurements performed on the two output beams of a nondegenerate parametric amplifier. This technique of demonstrating the product of variances of the inferred values of correlated observables to be less than that allowed by the uncertainty principle, has since gained popularity [11], and has been employed recently for variables other than position and momentum, e.g., for correlations between optical and orbital angular momentum of light emitted through spontaneous parametric down-conversion [32].

Let us now consider the situation where the quadrature phase components of two correlated and spatially separated light fields are measured. The quadrature amplitudes associated with the fields \( E_\gamma = C[\gamma e^{-i\omega_{x}\lambda} + \gamma \, e^{i\omega_{x}\lambda}] \) (where, \( \gamma \in \{ a, b \} \), are the bosonic operators for two different modes, \( \omega_{\lambda} \) is the frequency, and \( C \) is a constant incorporating spatial factors taken to be equal for each mode) are given by

\[
\hat{X}_\gamma = \frac{\hat{a} e^{-i\theta} + \hat{a}^\dagger e^{i\theta}}{\sqrt{2}}, \quad \hat{Y}_\phi = \frac{\hat{b} e^{-i\phi} + \hat{b}^\dagger e^{i\phi}}{\sqrt{2}},
\]

where,

\[
\hat{a} = \frac{X + iP_x}{\sqrt{2}}, \quad \hat{a}^\dagger = \frac{X - iP_x}{\sqrt{2}},
\]

\[
\hat{b} = \frac{Y + iP_y}{\sqrt{2}}, \quad \hat{b}^\dagger = \frac{Y - iP_y}{\sqrt{2}},
\]

and the commutation relations of the bosonic operators are given by \([\hat{a}, \hat{a}^\dagger] = 1 = [\hat{b}, \hat{b}^\dagger] \). Now, using Eq.(3) the expression for the quadratures can be rewritten as

\[
\hat{X}_{\theta} = \cos[\theta] \hat{X} + \sin[\theta] \hat{P}_x, \quad \hat{Y}_{\phi} = \cos[\phi] \hat{Y} + \sin[\phi] \hat{P}_y
\]

The correlations between the quadrature amplitudes \( \hat{X}_\theta \) and \( \hat{Y}_\phi \) are captured by the correlation coefficient, \( C_{\theta,\phi} \) defined as [9][11]

\[
C_{\theta,\phi} = \frac{\langle \hat{X}_\theta \hat{Y}_\phi \rangle}{\sqrt{\langle \hat{X}^2 \rangle \langle \hat{Y}^2 \rangle}},
\]

where \( \langle \hat{X}_\theta \rangle = 0 = \langle \hat{Y}_\phi \rangle \). The correlation is perfect for some values of \( \theta \) and \( \phi \), if \( |C_{\theta,\phi}| = 1 \). Clearly \( |C_{\theta,\phi}| = 0 \) for uncorrelated variables.

Due to the presence of correlations, the quadrature amplitude \( \hat{X}_\theta \) can be inferred by measuring the corresponding amplitude \( \hat{Y}_\phi \). The EPR paradox arises due to the ability to infer an observable of one system from the result of measurement performed on a spatially separated second system. In realistic situations the correlations are not perfect because of the interaction with the environment as well as finite detector efficiency. Hence, the estimated amplitudes \( \hat{X}_{\theta 1} \) and \( \hat{X}_{\phi 2} \) with the help of \( \hat{Y}_{\phi 1} \) and \( \hat{Y}_{\phi 2} \), respectively, are subject to inference errors, and given by [10]

\[
\hat{X}_{\theta 1} = g_1 \hat{Y}_{\phi 1}, \quad \hat{X}_{\phi 2} = g_2 \hat{Y}_{\phi 2},
\]

where \( g_1 \) and \( g_2 \) are scaling parameters. Now, one may choose \( g_1, g_2, \phi_1, \) and \( \phi_2 \) in such a way that \( \hat{X}_{\theta 1} \) and \( \hat{X}_{\phi 2} \) are inferred with the highest possible accuracy. The errors given by the deviation of the estimated amplitudes from the true amplitudes \( \hat{X}_{\theta 1} \) and \( \hat{X}_{\phi 2} \) are captured by \( (\hat{X}_{\theta 1} - \hat{X}_{\theta 1}) \) and \( (\hat{X}_{\phi 2} - \hat{X}_{\phi 2}) \), respectively. The average errors of the inferences are given by

\[
(\Delta_{\text{inf}} \hat{X}_{\theta 1})^2 = \langle (\hat{X}_{\theta 1} - \hat{X}_{\theta 1})^2 \rangle = \langle (\hat{X}_{\theta 1} - g_1 \hat{Y}_{\phi 1})^2 \rangle,
\]

\[
(\Delta_{\text{inf}} \hat{X}_{\phi 2})^2 = \langle (\hat{X}_{\phi 2} - \hat{X}_{\phi 2})^2 \rangle = \langle (\hat{X}_{\phi 2} - g_2 \hat{Y}_{\phi 2})^2 \rangle.
\]

The values of the scaling parameters \( g_1 \) and \( g_2 \) are chosen such that

\[
\frac{\partial (\Delta_{\text{inf}} \hat{X}_{\theta 1})^2}{\partial g_1} = 0 = \frac{\partial (\Delta_{\text{inf}} \hat{X}_{\phi 2})^2}{\partial g_2},
\]

from which it follows that

\[
g_1 = \frac{\langle \hat{X}_{\theta 1} \hat{Y}_{\phi 1} \rangle}{\langle \hat{Y}_{\phi 1}^2 \rangle}, \quad g_2 = \frac{\langle \hat{X}_{\phi 2} \hat{Y}_{\phi 2} \rangle}{\langle \hat{Y}_{\phi 2}^2 \rangle}.
\]

The values of \( \phi_1 (\phi_2) \) are obtained by maximizing \( C_{\theta_1,\phi_1} \) \( (C_{\phi_2,\phi_2}) \). Now, due to the commutation relations \([\hat{X}, \hat{P}_x] = i; \quad [\hat{Y}, \hat{P}_y] = i \), it is required that the product of the variances of the above inferences \( (\Delta_{\text{inf}} \hat{X}_{\theta 1})^2 (\Delta_{\text{inf}} \hat{X}_{\phi 2})^2 \geq 1/4 \). Hence, the EPR paradox occurs if the correlations in the field quadratures lead to the condition

\[
EPR \equiv (\Delta_{\text{inf}} \hat{X}_{\theta 1})^2 (\Delta_{\text{inf}} \hat{X}_{\phi 2})^2 < \frac{1}{4}
\]

Let us consider a two mode squeezed vacuum (TMSV) state, the expression of which is given by [33]

\[
|\text{NOPA}\rangle = |\xi\rangle = S(\xi)|0,0\rangle
\]

\[
= \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n |n, n\rangle
\]

where, \( \lambda = \tanh(r) \in [0,1] \), the squeezing parameter \( r > 0 \) and \(|m, n\rangle = |m\rangle_A \otimes |n\rangle_B \) (where \(|m\rangle \) and \(|n\rangle \) are the usual Fock states), \( S(\xi) = e^{(\xi a_1^\dagger a_2^\dagger - \xi^* a_1 a_2)} \), where \( \xi = re^{i\phi} \) is the squeezing operator (unitary). \( A \) and \( B \) are the two involved modes for Alice and Bob respectively.

For the NOPA state given by Eq.(10), the inferred uncertainties is given by

\[
(\Delta_{\text{inf}} \hat{X}_{\theta 1})^2 = \frac{1}{2} \cosh[2r]
\]

\[
- \frac{1}{2} \tanh[2r] \sinh[2r] \cos^2[\theta + \phi],
\]
where the quadrature amplitude $X_\theta$ is inferred by measuring the corresponding amplitude $Y_\phi$. The minimum values for two different values of $\theta$ (i.e., $\theta_1 = 0$ and $\theta_2 = \pi/2$) of $(\Delta_{\inf} X_\theta)^2$ are

$$(\Delta_{\inf} X_{\theta_1})^2 = (\Delta_{\inf} X_{\theta_2})^2 = \frac{1}{2 \cosh[2r]},$$

which occur for $\phi_1 = 0$ and $\phi_2 = \pi/2$, respectively. The product of uncertainties is thus $\frac{1}{4 \cosh[2r]}$ which asymptotically reaches the value 0 for $r \to \infty$, and this shows that the Reid condition [9] for occurrence of the EPR paradox holds. Hence, the two mode squeezed vacuum state shows EPR steering for all values of $r$ except at $r = 0$. However, the Reid condition fails to demonstrate steering by more general non-Gaussian states, for example, the two-dimensional harmonic oscillator, as we will show in Section III.

### B. Steering and entropic inequalities

A modern formulation of EPR steering was presented by Wiseman et al. [16, 17] as an information processing task. They considered that one of two parties (say, Alice) prepares a bipartite quantum state and sends one of the particles to Bob. The procedure is repeated as many times as required. Bob’s particle is assumed to possess a definite state, even if it is unknown to him (local hidden state). No such assumption is made for Alice, and hence, this formulation of steering is an asymmetric task. Alice and Bob make measurements on their respective particles, and communicate classically. Alice’s task is to convince Bob that the state they share is entangled. If correlations between Bob’s measurement results and Alice’s declared results can be explained by a local hidden state (LHS) model for Bob, he is not convinced. This is because Alice could have drawn a pure state at random from some ensemble and sent it to Bob, and then chosen her result based on her knowledge of this LHS. Conversely, if the correlations cannot be so explained, then the state must be entangled. Alice will be successful in her task of steering if she can create genuinely different ensembles for Bob by steering Bob’s state. It may be noted that a similar formulation of Bell nonlocality as an information theoretic task is also possible [16], where the correlations between Alice and Bob may be described in terms of a local hidden variable model.

In the above situation, an EPR-steering inequality [34] may be derived involving an experimental situation for qubits with $n$ measurement settings for each side. Bob’s $k$-th measurement setting is taken to correspond with the observable $\hat{A}_k$, and Alice’s declared result is denoted by the random variable $A_k \to \{-1, 1\}$. Violation of the inequality

$$\frac{1}{n} \sum_{k=1}^{n} \langle A_k \sigma_k \rangle \leq C_n$$

reveals occurrence of steering, where $C_n = \max_{\|A_k\|} \left( \frac{n \max_{k=1}^{n} \sum_{k=1}^{n} A_k \sigma_k} {\lambda_{\text{max}}} \right)$ is the maximum value of the l.h.s. of (13) if Bob has a pre-existing state known to Alice, with $\lambda_{\text{max}}$ being the largest eigenvalue of the operator $\frac{1}{\lambda_{\text{max}}} \sum_{k=1}^{n} A_k \sigma_k$. Experimental demonstration of steering for mixed entangled states [18] that are Bell local has confirmed that steering is a weaker form of correlations compared to nonlocality.

For the case of continuous variable systems, the Reid criterion is an indicator for steering, as discussed above. However, there exist several pure entangled continuous variable states which do not reveal steering through the Reid criterion. An example of such a state is provided in Ref. [14], which we also discuss briefly below. Since entanglement is a weaker form of correlations compared to steering [16, 17], it is clear that for such states the steering correlations do not appear up to second order (variances) that may be checked by the Reid criterion. The Reid criterion itself is derived using the Heisenberg uncertainty relation involving product of variances of non-commuting observables. On the other hand, a more general form of the uncertainty relation containing correlations in all orders of, for example, the position and momentum distribution of a quantum system is provided by the entropic uncertainty relation [20]

$$h_Q(X) + h_Q(P) \geq \ln \pi e.$$  

(14)

Using the entropic uncertainty relation, Walborn et al. [14] have derived an entropic steering inequality. They considered a joint probability distribution of two parties corresponding to a non-steerable state for which there exists a local hidden state (LHS) description, given by

$$\mathcal{P}(r_A, r_B) = \sum_{\lambda} \mathcal{P}(\lambda) \mathcal{P}(r_A|\lambda) \mathcal{P}(r_B|\lambda),$$

(15)

where, $r_A$ and $r_B$ are the outcomes of measurements $R_A$ and $R_B$ respectively; $\lambda$ are hidden variables that specify an ensemble of states; $\mathcal{P}$ are general probability distributions; and $\mathcal{P}_Q$ are probability distributions corresponding to the quantum state specified by $\lambda$. Now, using a rule for conditional probabilities $P(a, b|c) = P(b|c)P(a|b)$ which holds when $\{b\} \in \{c\}$, i.e., there exists a local hidden state of Bob predetermined by Alice, it follows that the conditional probability $\mathcal{P}(r_B|r_A)$ is given by

$$\mathcal{P}(r_B|r_A) = \sum_{\lambda} \mathcal{P}(r_B, \lambda|r_A)$$

(16)

with $\mathcal{P}(r_B, \lambda|r_A) = P(\lambda|r_A)P(r_B|\lambda)$. Note that (15) and (16) are equivalent conditions for non-steerability. Next, considering the relative entropy (defined for two distributions $p(X)$ and $q(X)$ as $H(p(X)||q(X)) = \sum_x p_x \ln(p_x/q_x)$) between the probability distributions $\mathcal{P}(r_B, \lambda|r_A)$ and $\mathcal{P}(\lambda|r_A)\mathcal{P}(r_B|r_A)$, it follows from the positivity of relative entropy that

$$\sum_{\lambda} \int dr_B \mathcal{P}(r_B, \lambda|r_A) \ln \frac{\mathcal{P}(r_B, \lambda|r_A)}{\mathcal{P}(\lambda|r_A)\mathcal{P}(r_B|r_A)} \geq 0$$

(17)
Using the non-steering condition \( [16] \), the definition of the conditional entropy \( h(X|Y) = - \sum p(x,y) \ln p(x|y) \), and averaging over all measurement outcomes \( r_A \), it follows that the conditional entropy \( h(R_B|R_A) \) satisfies

\[
h(R_B|R_A) \geq \sum_{\lambda} \mathcal{P}(\lambda) h_Q(R_B|\lambda) \tag{18}\]

Considering a pair of variables \( S_A, S_B \) conjugate to \( R_A, R_B \), a similar bound on the conditional entropy may be written as

\[
h(S_B|S_A) \geq \sum_{\lambda} \mathcal{P}(\lambda) h_Q(S_B|\lambda) \tag{19}\]

For the LHS model for Bob, note that the entropic uncertainty relation \( [14] \) holds for each state marked by \( \lambda \). Averaging over all hidden variables, it follows that

\[
\sum_{\lambda} \mathcal{P}(\lambda) \left( h_Q(R_B|\lambda) + h_Q(S_B|\lambda) \right) \geq \ln \pi e \tag{20}\]

Now, using the bounds \( [18] \) and \( [19] \) in the relation \( [20] \) one gets the entropic steering inequality given by

\[
h(R_B|R_A) + h(S_B|S_A) \geq \ln \pi e. \tag{21}\]

Walborn et al. \( [14] \) presented an example of the state given by (up to a suitable normalization)

\[
\phi_n(x_A, x_B) = H_n \left( \frac{x_A + x_B}{\sqrt{2}\sigma_+} \right) e^{- \frac{(x_A - x_B)^2}{4\sigma_+^2}} e^{- \frac{(x_A - x_B)^2}{4\sigma_-^2}} \tag{22}\]

where \( H_n \) is the \( n \)-th order Hermite polynomial, which does not reveal steering using the Reid criterion when \( \sigma_+ / \sigma_- < 1 + 1.5\sqrt{n} \), whereas the entropic steering criterion \( [21] \) is able to show steering except when the state is separable, i.e., for \( n = 0 \), and \( \sigma_+ = \sigma_- \). Using the relation between information entropy and variance, it was further shown by Walborn et al. \( [14] \) that the Reid criterion follows in the limiting case of the entropic steering relation \( [21] \). In the following section we will use the entropic steering inequality for demonstrating steering by several continuous variable states.

### III. STEERING AND NONLOCALITY FOR NON-GAUSIAN STATES

In this section we study steering and nonlocality by several non-Gaussian states. Considering first entangled states constructed using the eigenstates of the two-dimensional harmonic oscillator, we study the steering and nonlocal properties of LG beams. We show that the Reid criterion is unable to reveal the steerability of LG modes. The entropic steering inequality shows that the strength of steering increases with angular momentum of the LG beams. We then discuss non-Gaussian states obtained by subtracting single and two photons from two-mode squeezed vacuum states. We show that the violation of Bell’s inequality for such states behaves differently with the increase of the squeezing parameter compared to the strength of steering. Finally, we investigate the nonlocal and steering properties of another class of non-Gaussian states, viz., N00N states.

#### A. Non-Gaussian entangled states of a two-dimensional harmonic oscillator

The importance of the two-dimensional harmonic oscillator cannot be overemphasized in the context of quantum mechanics. The historical development of radiation theory started with the correspondence with the two modes of the radiation field. The classic problem of the charged particle in the electromagnetic field leading to the existence of Landau levels was developed using the same machinery. The energy eigenfunctions of the two-dimensional harmonic oscillator may be expressed in terms of Hermite-Gaussian (HG) functions given by

\[
u_{nm}(x, y) = \sqrt{\frac{2}{\pi}} \left( \frac{1}{2n+m+2n!m!} \right)^{1/2} \times H_n \left( \frac{\sqrt{2}x}{w} \right) H_m \left( \frac{\sqrt{2}y}{w} \right) e^{- \frac{x^2+y^2}{2w^2}}, \tag{23}\]

\[
\int |u_{nm}(x, y)|^2 dxdy = 1 \tag{23}\]

Entangled states may be constructed from superpositions of HG wave functions \( [35] \)

\[
\Phi_{nm}(\rho, \theta) = \sum_{k=0}^{n+m} u_{n+m-k,k}(x, y) f_k^{(n,m)} \left( \sqrt{-1} \right)^k \times \sqrt{\frac{k!(n+m-k)!}{n!m!(2n+m)!}} \tag{24}\]

\[
f_k^{(n,m)} = \frac{\partial^k}{\partial t^k} ((1-t)^n(1+t)^m)|_{t=0}, \tag{25}\]

where \( \Phi_{nm}(\rho, \theta) \) are the well-known Laguerre-Gaussian functions that are physically realizable field configurations \( [25, 29] \) with interesting topological \( [36] \) and coherence \( [37, 38] \) properties, given by \( [39] \)

\[
\Phi_{nm}(\rho, \theta) = e^{i(n-m)\theta} e^{-\rho^2/w^2} (-1)^{\min(n,m)} \left( \frac{\rho \sqrt{2}}{w} \right)^{[n-m]} \tag{26}\]

\[
\times \sqrt{\frac{2}{\pi n! m! w^2}} L_{\min(n,m)}^{[n-m]} \left( \frac{2\rho^2}{w^2} \right) (\min(n,m))! \]

with \( \int |\Phi_{nm}(\rho, \theta)|^2 dxdy = 1 \), where \( w \) is the beam waist, and \( L_k(x) \) is the generalized Laguerre polynomial. The superposition \( [24] \) is like a Schmidt decomposition
The maximum correlation strength following Eqs. (7) and (8) it follows that
\[ C_{\text{max}} = \text{maximizing the correlation function} \]
where the strength of steering with the degree of nonlocality of relations present in the LG wave functions, first for the dimensions of quadratures \( \{X, P_X\} \) and \( \{Y, P_Y\} \), given by
\[ x(y) \rightarrow \frac{w}{\sqrt{2}} X(Y), \quad p_x(p_y) \rightarrow \sqrt{2}h \quad P_X(P_Y). \]
The canonical commutation relations are \( [\hat{X}, \hat{P}_X] = i; \quad [\hat{Y}, \hat{P}_Y] = i \), and the operator \( \hat{P}_X \) and \( \hat{P}_Y \) are given by
\[ \hat{P}_X = -i \frac{\partial}{\partial X} \quad \text{and} \quad \hat{P}_Y = -i \frac{\partial}{\partial Y}. \]
respectively. The Wigner function corresponding to the LG wave function in terms of the scaled variables is given by
\[ W_{nm}(X, P_X; Y, P_Y) = \frac{(-1)^{n+m}}{(\pi)^{2}} \left( L_{n}[4(Q_0 + Q_2)] \right) \left( L_{m}[4(Q_0 - Q_2)] \right) \exp(-4Q_0) \]
where
\[ Q_0 = \frac{1}{4} \left[ X^2 + Y^2 + P_X^2 + P_Y^2 \right], \]
\[ Q_2 = \frac{XP_Y - YP_X}{2}. \]
Let us now check how the Reid criterion applies to the case of LG wave functions. In order to do so we estimate the product of uncertainties of the values of inferred observables \( (\Delta_{\text{inf}}X_{\theta_1})^2(\Delta_{\text{inf}}X_{\theta_2})^2 \). This is performed by maximizing the correlation function \( C_{\theta_1, \theta_1}(C_{\theta_2, \theta_2}). \)
Using Eqs. (7) and (8), it follows that
\[ (\Delta_{\text{inf}}X_{\theta})^2 = \langle X^2 \rangle \left( 1 - (C_{\theta, \phi})^2 \right) \]
The maximum correlation strength \( |C_{\theta, \phi}| = \frac{1}{2} \) occurs for \( \phi - \theta = \frac{k\pi}{2} \) (where \( k \) is an odd integer). For arbitrary values of \( n, m \) it can be shown that the expression of the maximum correlation function is given by
\[ C_{\theta, \phi}^{\text{max}} = \frac{\langle XP_Y \rangle}{\sqrt{\langle X^2 \rangle \langle P_Y^2 \rangle}}, \quad C_{\theta, \phi}^{\text{max}} = \frac{\langle P_X Y \rangle}{\sqrt{\langle P_X^2 \rangle \langle Y^2 \rangle}} \]
In Figure 1 we plot the product of uncertainties \( (\Delta_{\text{inf}}X_{\theta_1})^2(\Delta_{\text{inf}}X_{\theta_2})^2 \) versus the angular momentum \( n \). It is seen that the Reid criterion given by Eq. (9) is not satisfied, since \( (\Delta_{\text{inf}}X_{\theta_1})^2(\Delta_{\text{inf}}X_{\theta_2})^2 \geq 1/4 \) for any value of \( n \). Hence, it is not possible to demonstrate steering by entangled LG modes using the Reid criterion.

We now apply the entropic steering criterion to the case of the LG wave functions. In the entropic steering inequality given by Eq. (21), the observables have to be chosen such that there exist correlations between \( R_A \) and \( R_B \) (\( S_A \) and \( S_B \)). For the case of the LG wave functions, we use the nonvanishing \( \langle XP_Y \rangle \) correlations, as evident from the Wigner function (31). Thus, in terms of the conjugate pairs of dimensionless quadratures, (21) becomes
\[ h(X'|P_Y) + h(P_X'|Y) \geq \ln \pi e, \]
where \( X, Y, P_X \) and \( P_Y \) are the outcomes of measurements \( X, Y, P_X \) and \( P_Y \) respectively. Here, the conditional entropies \( h(X'|P_Y) \) and \( h(P_X'|Y) \) are given by
\[ h(X'|P_Y) = h(X, P_Y) - h(P_Y), \]
\[ h(P_X'|Y) = h(P_X, Y) - h(Y), \]
with \( h(X, P_Y) = -\int_{-\infty}^{\infty} P(X, P_Y) \ln P(X, P_Y) \, dX \, dP_Y, \)
\( h(P_X|Y) = -\int_{-\infty}^{\infty} P(P_X|Y) \ln P(P_X|Y) \, dP_X, \) and similarly for \( h(P_X, Y) \) and \( h(Y) \). The marginal probability distributions are obtained using the Wigner function (31) for the LG wave function.

For \( n = 0 \) and \( m = 0 \), the LG wave function factorizes into a product state with the corresponding Wigner function given by
\[ W_{00}(X, P_X; Y, P_Y) = \frac{e^{-X^2 - Y^2 - P_X^2 - P_Y^2}}{\pi^2}. \]
In this case the relevant entropies turn out to be
\[ h(X'|P_Y) = h(P_X'|Y) = \ln \pi e \text{ and } h(Y) = h(P_Y) = \frac{1}{2} \ln \pi e, \]
and hence, the entropic steering inequality becomes saturated, i.e.,
\[ h(X'|P_Y) + h(P_X'|Y) = \ln \pi e. \]
For $n = 1$ and $m = 0$, the Wigner function has the form

$$W_{10}(X, P_X; Y, P_Y) = e^{-X^2 - Y^2 - P_X^2 - P_Y^2} \times \frac{(P_X - Y)^2 + (P_Y + X)^2 - 1}{\pi^2}$$  \hspace{1cm} (38)$$

and the relevant entropies are given by $h(X, P_Y) = h(P_X, Y) \approx 2.41509$, and $h(Y) = h(P_Y) \approx 1.38774$. Hence, the entropic steering relation in this case becomes

$$h(X|P_Y) + h(P_X|Y) \approx 2.05471 < \ln \pi e$$  \hspace{1cm} (39)$$

We thus see that steering is demonstrated. Note the non-Gaussian nature of the Wigner function for $n \geq 1$ which enables demonstration of steering through the entropic criterion. For higher values of angular momentum, we plot the l.h.s. of the entropic steering relation in Figure-2. We see that violation of the inequality becomes stronger for higher values of $n$.

![Entropic Steering Inequality](image1)

FIG. 2: (Coloronline) The figure shows that the violation of entropic steering inequality [34] for different values of $n$ (except $n = 0$) of the LG wave function keeping $m = 0$.

Now, for making a comparison between the strength of steering and the degree of nonlocality, we next study Bell violation by the LG wave function. In order to do so, we consider the Wigner transform $\Pi_{nm}(X, P_X; Y, P_Y) = \frac{\pi}{2} W_{nm}(X, P_X; Y, P_Y)$, where $W_{nm}(X, P_X; Y, P_Y)$ is given by Eq. (29) [10]. The Bell-CHSH inequality using Wigner transform is given by [42]

$$|BI| = |\Pi_{1,0}(X_1, P_{X_1}; Y_1, P_{Y_1}) + \Pi_{1,0}(X_2, P_{X_2}; Y_1, P_{Y_1}) + \Pi_{1,0}(X_1, P_{X_1}; Y_2, P_{Y_2}) - \Pi_{1,0}(X_2, P_{X_2}; Y_2, P_{Y_2})| < 2,$$  \hspace{1cm} (40)$$

In the following table, we make comparison among Bell violation and entropic EPR steering for different values of $n$ with $m = 0$.

| $n$ | $|B_{\text{max}}|$ | $\frac{(\ln \pi e)}{4n}$ | $4(\Delta_{\text{inf}}X_{\theta_1})^2(\Delta_{\text{inf}}X_{\theta_2})^2$ |
|-----|-----------------|-----------------|-----------------|
| 0   | 1               | 1               | 1               |
| 1   | 1.11934         | 1.04381         | 2.25            |
| 2   | 1.17437         | 1.0567          | 2.7777          |
| 3   | 1.20128         | 1.06256         | 3.0625          |
| 4   | 1.21738         | 1.06572         | 3.24            |
| 5   | 1.22813         | 1.06758         | 3.36111         |
| 6   | 1.23584         | 1.0687          | 3.44898         |
| 7   | 1.24165         | 1.06939         | 3.51563         |
| 8   | 1.24618         | 1.0698          | 3.5679          |
| 9   | 1.24982         | 1.07002         | 3.61            |
| 10  | 1.25281         | 1.07011         | 3.64463         |

Note here that $\frac{|B_{\text{max}}|}{2} > 1$ signifies Bell violation, and $\frac{(\ln \pi e)}{4n} > 1$ signifies steering by the entropic steering inequality. On the other hand the last column provides values of the products of inferred variances, showing that the Reid criterion is unable to identify steering for any value of $n$ in this case.

### B. Photon subtracted squeezed vacuum

Let us now consider non-Gaussian states derived from Gaussian states by the subtraction of photons. Consider the two mode squeezed vacuum state given by Eq. (10). The Wigner function associated with the state (10) is given by [24]

$$W_{\xi}(\alpha, \beta) = \frac{4}{\pi^2} \exp[-2|\alpha\cosh(r) - \beta^* \sinh(r) \exp[i\phi]|^2 - 2|\alpha^* \sinh(r) \exp[i\phi] + \beta \cosh(r)|^2],$$  \hspace{1cm} (41)$$

where $\alpha$ and $\beta$ represent complex phase space displacements and $\int \int W_{\xi}(\alpha, \beta) d^2\alpha d^2\beta = 1$, and $\{x, k_x\}$, $\{y, k_y\}$ are conjugate quadrature observables. In terms of the variables $X, P_X, Y$ and $P_Y$, the Wigner function (with the replacements $\alpha = \frac{X + iP_X}{\sqrt{2}}, \beta = \frac{Y + iP_Y}{\sqrt{2}}$, and $\phi = 0$) becomes

$$W_{\xi}(X, P_X; Y, P_Y) = \frac{1}{\pi^2} \exp[-2(P_X P_Y - XY) \sinh 2r - (X^2 + Y^2 + P_X^2 + P_Y^2) \cosh 2r].$$  \hspace{1cm} (42)$$

Bell violation by the NOPA state has been studied earlier [11]. In terms of the Wigner transform $W[\alpha, \beta] = \frac{\pi^2}{4} W_{\xi}(\alpha, \beta)$ the Bell sum is given by [11]

$$BI = \Pi[\alpha = 0, \beta = 0] + \Pi[\alpha = \sqrt{J}, \beta = 0] + \Pi[\alpha = 0, \beta = -\sqrt{J}] - \Pi[\alpha = \sqrt{J}, \beta = -\sqrt{J}]$$

$$= 1 + 2E \exp[-4J(\cosh^2(r) - 2 \cos(\phi) \cosh(r) \sinh(r) + \sinh^2(r))],$$  \hspace{1cm} (43)$$
where $J$ represents amount of displacement in the phase space. By choosing $\phi = 0$ and considering $r \to \infty$, the above expression becomes

$$BL(J,r) = 1 - E\exp[-4J e^{2r}] + 2E\exp[-J e^{2r}]$$  \hspace{1cm} (44)

The maximum value of $BI$ is 2.19055 \cite{11} (for the above choice of settings) which occurs for the constraints

$$JE\exp[2r] = \frac{1}{3} \ln 2,$$  

(45)

where $J << 1$. For example, $BI_{\text{max}} (= 2.19055)$ occurs for the choice of parameters $J = 0.0009467$ and $r = 3.9$. However, a more general choice of settings \cite{13 14}

$$BI = \Pi[\alpha, \beta_1] + \Pi[\alpha, \beta_2] + \Pi[\alpha, \beta_1] - \Pi[\alpha, \beta_2],$$  

(46)

leads to the maximum Bell violation $BI_{\text{max}} = 2.32449$ for the choice of parameters $\alpha_1 = 0.0036990$, $\alpha_2 = -0.0115244$, $\beta_1 = -0.0039127$, $\beta_2 = 0.0113108$, $r = 3.8853675$.

The subtraction of $n$ photons from the state $|\xi\rangle$ \cite{10} may be represented as

$$|\xi_n\rangle = (a \otimes I + (-1)^k I \otimes b)^n |\xi\rangle,$$  

(47)

where $k \in \{0, 1\}$, and it is assumed that one does not know from which mode the photon is subtracted. After normalization the state becomes $\sqrt{N_n}|\xi_n\rangle$, where the normalization constant $N_n$ is given by $(N_n)^{-1} = |\xi_n|\xi_n\rangle$. The Wigner function of the state $|\xi_n\rangle$ is related to the Wigner function of the state $|\xi_{n-1}\rangle$ by

$$W_n(\alpha, \beta) = \hat{A}(\alpha, \beta) W_{n-1}(\alpha, \beta),$$  

(48)

where the operator $\hat{A}(\alpha, \beta)$ is given by

$$\hat{A}(\alpha, \beta) = \left[ (\alpha^* + \frac{1}{2} \frac{\partial}{\partial \alpha}) (\alpha + \frac{1}{2} \frac{\partial}{\partial \alpha^*}) + (\alpha^* + \frac{1}{2} \frac{\partial}{\partial \alpha}) (\beta + \frac{1}{2} \frac{\partial}{\partial \beta^*}) + (\alpha + \frac{1}{2} \frac{\partial}{\partial \alpha^*}) (\beta^* + \frac{1}{2} \frac{\partial}{\partial \beta}) + (\beta + \frac{1}{2} \frac{\partial}{\partial \beta}) (\beta^* + \frac{1}{2} \frac{\partial}{\partial \beta^*}) \right].$$  

(49)

The Wigner function $W_n(\alpha, \beta)$ is obtained from $W(\alpha, \beta)$ given by Eq.\,(41) by applying $\hat{A}(\alpha, \beta)$ n times, i.e., $W_n(\alpha, \beta) = \hat{A}^n(\alpha, \beta) W(\alpha, \beta)$, and normalizing suitably ($\int W_n(\alpha, \beta) \, d\alpha \, d^2\beta = 1$). In terms of the $X, P_X, Y$ and $P_Y$, the Wigner function for the single photon subtracted squeezed vacuum state becomes

$$W_1(X, Y, P_X, P_Y) = \frac{1}{\pi^2} \exp[2 \sinh(2r)(XY - P_X P_Y - \cosh(2r)(X^2 + Y^2 + P_X^2 + P_Y^2) - 2P_X P_Y + (X - Y)^2] + \cosh(2r)(P_X^2 - 2P_X P_Y + P_Y^2) - 2P_X P_Y + P_Y^2 + (X - Y)^2 - 1)$$  

To evaluate the Bell violation, we use the Wigner transform $\Pi_n(\alpha, \beta) = \frac{1}{\pi^2} W_n(\alpha, \beta)$. The Bell sum using the above Wigner transform may be expressed as

$$BI_n = \Pi_n(\alpha_1, \beta_1) + \Pi_n(\alpha_2, \beta_2) + \Pi_n(\alpha_2, \beta_1) - \Pi_n(\alpha_1, \beta_2)$$  

(51)

Now, to obtain the maximum Bell violation, one maximizes $BI_n$ over $\alpha_1, \alpha_2, \beta_1, \beta_2, r$ for a given value of $n$.

Considering single photon subtraction from each mode, i.e., $a \otimes I + (-1)^k I \otimes b$, the state (47) becomes

$$|\xi^{1-}\rangle = \sqrt{1 - \lambda^2} \sum \lambda^n \sqrt{n} \{1, n\} + (-1)^k \{n, n - 1\}$$  

(52)

with normalization constant $N_1 = \frac{1}{2 \sinh(r)}$. The Wigner transform for the above state is given by

$$\Pi_1(\alpha, \beta) = \exp[2(\alpha^* \beta + \alpha^* \beta^* \sinh(2r) - 2 |\alpha|^2 + |\beta|^2 \cosh(2r)] (-2 \alpha^* \beta + 2 \alpha^* \beta^* + (\alpha^* \beta^*)^2) \cosh(2r) + 2 (\alpha^* \beta + (-1)^k \beta^*) \cosh(2r) - 1)$$  

(53)

The maximum Bell violation, i.e., $(BI_1)_{\text{max}} = -2.5444$ occurs for the choices $\alpha_1 = -0.0067, \alpha_2 = 0.0201, \beta_1 = 0.0067, \beta_2 = -0.0201, r = 3.0$ and $k = 1$. Now, comparing with the two-mode squeezed state where the Bell violation is $-2.3245$ \cite{12}, it is seen that by photon annihilation, the maximum Bell violation increases. For the case of two photon subtraction from each mode $((a \otimes I + (-1)^k I \otimes b)^2)$, we can similarly obtain the maximum Bell violation which turns out to be $(BI_2)_{\text{max}} = 2.6305$ for the choices $\alpha_1 = -0.1338, \alpha_2 = -0.1392, \beta_1 = -0.1365, \beta_2 = -0.1311, r = 4.4915$ and $k = 1$. We thus see that the maximum Bell violation increases further.

We have seen in the last section that the Reid criterion is able bring out the steering property of two mode squeezed vacuum state. Let us now see whether it is possible to demonstrate steering for single photon annihilated state \cite{52} using the Reid criterion. The uncertainty for the inferred observables is in this case given by

$$(\Delta_{\text{inf}} X_\theta)^2 = \cosh(2r) - \sinh(2r) \cosh(r) \cos(2\theta)$$  

(54)

$$- \frac{(\cosh(2r) \cos(\theta - \phi) - 2 \sinh(2r) \cos(\theta + \phi))}{4(\cosh(2r) - \sinh(r) \cosh(r) \cos(2\phi))}.$$  

Calculating the minimum value of $(\Delta_{\text{inf}} X_\theta)^2$ for two different values of $\theta$ (i.e., $\theta_1 = 0$ and $\theta_2 = \pi/2$), the product of uncertainties turns out to be

$$(\Delta_{\text{inf}} X_{\theta_1})^2(\Delta_{\text{inf}} X_{\theta_2})^2 = \frac{9}{2 (3 \cosh(4r) + 5)},$$  

(55)

which goes to 0 for $r \to \infty$. In the Figure 3a we compare the amount of violation of the Reid inequality by the
NOPA and the single photon annihilated NOPA states. We see that the Reid criterion fails in the latter case for smaller values of the squeezing parameter \( r \).

![Graphs showing entropic EPR steering criterion and Bell violation with entropic EPR steering for the NOPA and the single photon annihilated NOPA states.](Image)

**Fig. 3**: (Color online) a: The horizontal line represents the uncertainty bound below which steering is signified. The lower curve represents the product of inferred uncertainties for the two-mode squeezed vacuum state. Steering is demonstrated for all values of \( r \) through the Reid criterion. The upper curve represents the product of uncertainties for the photon subtracted state. Clearly, the Reid criterion fails to show steering for smaller values of \( r \) in the latter case.

b: The horizontal line represents the bound \( \ln \pi e \). The purple and blue curves represent the LHS of the steering inequality for the squeezed state and the single photon subtracted state, respectively.

We next demonstrate steering for the photon subtracted squeezed vacuum state through the entropic steering inequality. Considering the measurements corresponding to either position (\( r = x \)) or momentum (\( s = p \)), for the single photon subtracted squeezed vacuum state correlations exist between \( X \) and \( Y \), and \( P_X \) and \( P_Y \). \{\( X, P_X \)\} and \{\( Y, P_Y \)\} are conjugate pairs of dimensionless quadratures. So in terms of these variables, the steering inequality \(^{(21)}\) becomes

\[
h(Y|X) + h(P_Y|P_X) \geq \ln \pi e, \quad (56)
\]

where \( X, Y, P_X \), and \( P_Y \) are the outcomes of measurements \( X', Y', P_X \), and \( P_Y \) respectively. Here, the conditional entropies \( h(Y|X) \) and \( h(P_Y|P_X) \) are given by

\[
h(Y|X) = h(X', Y') - h(X'), \quad h(P_Y|P_X) = h(P_X, P_Y) - h(P_X), \quad (57)
\]

and calculated using the marginal probability distributions obtained from the Wigner function \(^{(50)}\). One can thus calculate the L.H.S. of the inequality \(^{(50)}\) for the single photon subtracted state for any value of the squeezing parameter \( r \). In Fig. 3b we plot the L.H.S. of the entropic steering inequality versus \( r \) for the squeezed vacuum state as well as the single photon subtracted state. The figure shows the violation of the steering inequality increases with \( r \) for each of these two states.

In the following table we show the comparison of Bell violation with entropic EPR steering for the NOPA state and the single photon annihilated NOPA state. Note here that \( \frac{B_{\text{max}}}{2} > \frac{\ln \pi e}{h(X(P_Y)) + h(P_Y|Y)} > 1 \) signifies Bell violation, and hence the magnitude of Bell violation reaches a maximum for a certain value of the squeezing parameter \( r \), and subsequently decreases gradually. The strength of steering increases monotonically with \( r \). Hence, it would be much easier to observe steering compared to Bell violation for higher values of \( r \).

| State \( \xi \) | \( r \) | Bell violation \( \frac{B_{\text{max}}}{2} \) | Entropic EPR steering criterion \( \frac{\ln \pi e}{h(X(P_Y)) + h(P_Y|Y)} \) |
|---|---|---|---|
| \( \xi \) | 0 | 1.0 | 1.0 |
| \( \xi \) | 0.2 | 1.040 | 1.038 |
| \( \xi \) | 0.4 | 1.091 | 1.157 |
| \( \xi \) | 0.6 | 1.125 | 1.383 |
| \( \xi \) | 0.8 | 1.144 | 1.790 |
| \( \xi \) | 1.0 | 1.153 | 2.616 |
| \( \xi \) | 1.2 | 1.159 | 4.991 |
| \( \xi \) | 1.4 | 1.160 | 62.737 |
| \( \xi \) | 0 | 1.120 | 1.044 |
| \( \xi \) | 0.2 | 1.189 | 1.061 |
| \( \xi \) | 0.4 | 1.229 | 1.124 |
| \( \xi \) | 0.6 | 1.252 | 1.264 |
| \( \xi \) | 0.8 | 1.263 | 1.529 |
| \( \xi \) | 1.0 | 1.267 | 2.027 |
| \( \xi \) | 1.2 | 1.271 | 3.132 |
| \( \xi \) | 1.4 | 1.271 | 7.531 |

C. N00N state

The maximally path-entangled number states have the form given by

\[
|\psi\rangle = \frac{1}{\sqrt{2}}(|N\rangle_a|0\rangle_b + e^{i\phi}|0\rangle_a|N\rangle_b). \quad (58)
\]

This is an example of a two-mode state such that \( N \) photons can be found either in the mode \( a \) or in the mode \( b \), and is referred to as ‘N00N’ states \(^{(31)}\). The utility of N00N states in making precise interferometric measurements is of much importance in quantum metrology. Such states have been recently experimentally realized up to \( N = 5 \) \(^{(45)}\). The entanglement of N00N states is obtained in terms of the logarithmic negativity, \( \nu_s = 1 \) \(^{(21)}\), a value that is independent of \( N \).

The Wigner distribution function for the N00N state is given by \(^{(46)}\)

\[
W(\alpha, \beta) = 2^{2N}e^{-2|\alpha|^2-2|\beta|^2}(-1)^N(L_N(4|\alpha|^2) + L_N(4|\beta|^2)) - \frac{2^{2N}}{N!}(\alpha^N \beta^N + \alpha^N \beta^N), \quad (59)
\]

where for simplicity we choose \( \phi = \pi \) and \( L_N(x) \) is the Laguerre polynomial. In terms of the dimensionless quadratures \{\( X, P_X \)\} and \{\( Y, P_Y \)\} the Wigner function...
becomes
\[ W(X, P_X, Y, P_Y) = \frac{1}{2\pi^{2N}} e^{-(X^2 + Y^2 + P_X^2 + P_Y^2)} \]
\[ \cdot [-2^N (X + iP_X)^N (Y - iP_Y)^N + (X - iP_X)^N (Y + iP_Y)^N + (-1)^N N! \{ L_N (2(X^2 + P_X^2)) + L_N (2(Y^2 + P_Y^2)) \}]. \] (60)

The Bell-CHSH inequality
\[ |BI| = \Pi(\alpha, \beta) + \Pi(\alpha', \beta) + \Pi(\alpha, \beta') - \Pi(\alpha', \beta') \leq 2 \]

is maximally violated with \( BI_{\text{max}} = -2.2387 \) which occurs for \( N = 1 \) and the corresponding settings are \( \alpha = -\beta = 0.0610285, \alpha' = -\beta' = -0.339053 \). States with larger \( N \) do not violate the inequality. However, there are some other Bell-type inequalities \([46]\) for six correlated events for which \( N00N \) states show the violation for any \( N \).

From the expression of the Wigner function \([60]\) for the \( N00N \) states the presence of correlations of the type \( \langle X,Y \rangle \) is clear. Using such correlations the entropic steering inequality for the \( N00N \) state may be written as
\[ h(\langle X \rangle |X) + h(\langle Y \rangle |P_X) \geq \ln \pi e, \] (62)
The conditional entropies \( h(\langle X \rangle |X) \) and \( h(\langle Y \rangle |P_X) \) can be calculated through the marginal probabilities obtained through the Wigner function \([60]\), using which the L.H.S. of the inequality \([62]\) may be obtained for different values of \( N \). It turns out that for \( N = 1 \), one gets \( h(\langle X \rangle |X) + h(\langle Y \rangle |P_X) \approx 2.05 < \ln \pi e \), thus violating the steering inequality. However, for \( N = 2 \), one gets \( h(\langle X \rangle |X) + h(\langle Y \rangle |P_X) \approx 2.25 > \ln \pi e \). Larger values of \( N \) lead to further higher values of \( h(\langle X \rangle |X) + h(\langle Y \rangle |P_X) \), and hence, no steering is possible for \( N > 1 \).

In Fig. 4, we plot the joint probability \( P(X, Y) \) for two different values of \( N \), viz., \( N = 1 \) and \( N = 4 \), respectively. The higher peak of the \( N = 1 \) curve indicates stronger \( \langle X,Y \rangle \) correlations responsible for steering in this case. The correlations weaken for larger values of \( N \) as is indicated by the lower peak value of the \( N = 4 \) curve, and are not sufficient for revealing steering through the entropic inequality. Thus, \( N00N \) states with \( N = 1 \) violate the entropic steering inequality, but for \( N \geq 1 \), these states are not steerable. This feature is similar to Bell violation for \( N00N \) states which is revealed for \( N = 1 \), but the violation of the standard Bell-CHSH inequality does not occur for \( N \geq 1 \).

IV. SUMMARY

In the present paper we have studied EPR steering by non-Gaussian continuous variable entangled states. Here we have considered several examples of such systems, i.e., the two-dimensional harmonic oscillator, the photon subtracted squeezed vacuum state, and the \( N00N \) state. Though such states are entangled pure states, we have shown that they fail to reveal steering through the Reid criterion for wide ranges of parameters. Steering with such states is demonstrated using the entropic steering inequality. We have computed the relevant conditional entropies using the Wigner function whose non-Gaussian nature plays an important role in demonstrating steering. For all the above examples we perform a quantitative study of the strength of steering (determined by the magnitude of violation of the entropic steering inequality) as a function of the state parameters. This leads to some interesting observations, especially in comparison with the magnitude of Bell nonlocality demonstrated by these states.

For the LG modes one sees that the steering strength increases with the increase of the angular momentum \( n \), a feature that is also common to the Bell violation. However, for both the two-mode squeezed vacuum state as well as the single photon subtracted state derived from it, we show that the behavior of the maximum Bell violation and steering strength versus the squeezing parameter are not similar. This is evident from the fact that though the maximum Bell violation peaks for a certain value of \( r \), the steering strength rises monotonically with increasing \( r \). This feature clearly establishes the fact that though Bell violation guarantees steerability, the two types of quantum correlations are distinct from each other. Moreover, the presence of quantum correlations in such class of states may be more easily detected through the violation of the entropic steering inequality compared to the violation of the Bell inequality for higher values of squeezing. Finally, we study steering by \( N00N \) states. Here, steering through the entropic steering condition is revealed only for \( N = 1 \), though the entanglement of such states remains constant with \( N \). This shows

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**FIG. 4:** (Coloronline) Correlations of the type \( \langle X,Y \rangle \) responsible for steering using the entropic steering inequality are revealed through the joint probability distributions \( P(X,Y) \). The figure shows that such correlations are sufficiently strong to admit steering for \( N = 1 \), but are significantly weakened for larger \( N \).
that entanglement is a different correlation compared to steering, as also it is different compared to Bell nonlocality. The above results should be useful for detecting and manipulating correlations in non-Gaussian states for practical purposes in different arenas such as information processing, quantum metrology, and Bose condensates.

Further work on the issue of the recently proposed symmetric steering framework [21] may be of interest using non-Gaussian resources.

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