IRREDUCIBILITY OF THE MONODROMY REPRESENTATION
OF LAURICELLA'S $F_C$

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Abstract. Let $E_C$ be the hypergeometric system of differential equations satisfied by Lauricella’s hypergeometric series $F_C$ of $m$ variables. We show that the monodromy representation of $E_C$ is irreducible under our assumption consisting of $2^m+1$ conditions for parameters. We also show that the monodromy representation is reducible if one of them is not satisfied.

1. Introduction

Lauricella’s hypergeometric series $F_C$ of $m$ variables $x_1, \ldots, x_m$ with complex parameters $a, b, c_1, \ldots, c_m$ is defined by

$$F_C(a, b, c; x) = \sum_{n_1, \ldots, n_m = 0}^{\infty} \frac{(a, n_1 + \cdots + n_m)(b, n_1 + \cdots + n_m)}{(c_1, n_1) \cdots (c_m, n_m)n_1! \cdots n_m!} x_1^{n_1} \cdots x_m^{n_m},$$

where $x = (x_1, \ldots, x_m)$, $c = (c_1, \ldots, c_m)$, $c_1, \ldots, c_m \notin \{0, -1, -2, \ldots\}$, and $(c_1, n_1) = \Gamma(c_1 + n_1)/\Gamma(c_1)$. This series converges in the domain

$$D_C = \left\{ (x_1, \ldots, x_m) \in \mathbb{C}^m \mid \sum_{k=1}^{m} \sqrt{|x_k|} < 1 \right\}.$$

It is shown in [3] that the hypergeometric system $E_C = E_C(a, b, c)$ of differential equations satisfied by $F_C(a, b, c; x)$ is a holonomic system of rank $2^m$ with the singular locus

$$S = \left\{ \prod_{k=1}^{m} x_k \cdot R(x) = 0 \right\} \subset \mathbb{C}^m,$$

$$R(x_1, \ldots, x_m) = \prod_{\varepsilon_1, \ldots, \varepsilon_m = \pm 1} \left( 1 + \sum_{k=1}^{m} \varepsilon_k \sqrt{|x_k|} \right),$$

and that the system $E_C$ is irreducible (in the sense of $D$-modules) if and only if

$$a - \sum_{k=1}^{m} i_k c_k, \quad b - \sum_{k=1}^{m} i_k c_k \notin \mathbb{Z}, \quad \forall I = (i_1, \ldots, i_m) \in \{0, 1\}^m. \quad (1)$$

It is classically known that there are $2^m$ solutions to $E_C(a, b, c)$ expressed in terms of $F_C$ with different parameters (see [3]). If parameters satisfy (1) and $c_1, \ldots, c_m \notin \{0, -1, -2, \ldots\}$,

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$\mathbb{Z}$ then they form a fundamental system of solutions to $E_C(a, b, c)$ in a simply connected domain in $D_C - S$.

Let $X$ be the complement of the singular locus $S$. The fundamental group of $X$ is generated by $m + 1$ loops $\rho_0, \rho_1, \ldots, \rho_m$ (see (2.2)). In [2], we express the circuit transformations $\mathcal{M}_i$ along $\rho_i$ ($i = 0, \ldots, m$) by using the $2^m$ solutions and the intersection form on twisted homology groups associated with Euler-type integrals of solutions to $E_C$. These expressions are independent of the choice of a basis of the twisted homology group. The circuit transformations $\mathcal{M}_i$ are also studied in [6] by the specification of the intersection form regarded as indeterminate.

In this paper, we show the following.

**Theorem 1.1 (Main theorem).** The monodromy representation

$$\mathcal{M} : \pi_1(X, \dot{x}) \rightarrow GL(\dot{\text{Sol}}_{\dot{x}})$$

is irreducible under the assumption (1), where $\dot{\text{Sol}}_{\dot{x}} = \text{Sol}_{\dot{x}}(a, b, c)$ is the local solution space to $E_C(a, b, c)$ around a point $\dot{x} \in D_C - S$.

We also show that the monodromy representation is reducible if one of the assumption (1) is not satisfied. We specify an invariant subspace in $\dot{\text{Sol}}_{\dot{x}}$ under the monodromy representation $\mathcal{M}$ in this case.

Note that under the assumption (1), for example, $c_k$ may be an integer. In such a case, the solutions (3) expressed by $F_C$ do not form a basis of the local solution space. We give a linear transformation of them so that the transformed solutions are valid even in cases where any of $c_k$'s are integers. We construct it inductively on $m$ using tensor products of matrices.

We remark that the irreducibility of the monodromy representation $\mathcal{M}$ is implied from that of the system $E_C$ under the assumption (1). We prove it explicitly by using properties of the circuit transformations in [2], not applying results of $D$-modules. We here briefly explain our idea of the proof of the main theorem. It is shown in [2] that the 1-eigenspace $V$ of $\mathcal{M}_0$ is $(2^m - 1)$-dimensional. Let $f_0 \in \dot{\text{Sol}}_{\dot{x}}$ (corresponding to $e_1, \ldots, 1$ in (2.2)) be a non-zero vector in its orthogonal complement with respect to the intersection form. It is quite easy to give a basis of the whole space $\dot{\text{Sol}}_{\dot{x}}$ by actions $\mathcal{M}_1, \ldots, \mathcal{M}_m$ on $f_0$. Let $W$ be an invariant subspace of $\dot{\text{Sol}}_{\dot{x}}$ under the monodromy representation $\mathcal{M}$. If $W \not\subset V$ then we can show $f_0 \notin W$, which yields that $W$ becomes the whole space $\dot{\text{Sol}}_{\dot{x}}$ by the previous fact. Otherwise, we can show that $W$ becomes the zero space by the perfectionness of the intersection form.

2. Preliminaries

Except in [5] we assume the conditions for parameters $a, b, c_1, \ldots, c_m$ in [1] (it is equivalent to [2] or [4] mentioned below).

In this section, we collect some facts about Lauricella’s $F_C$ mentioned in [1], [2], [3] and [5].

**Notation 2.1.** We put

$$\alpha = \exp(2\pi\sqrt{-1}a), \quad \beta = \exp(2\pi\sqrt{-1}b), \quad \gamma_k = \exp(2\pi\sqrt{-1}c_k) \ (k = 1, \ldots, m).$$

We often regard $\alpha$, $\beta$ and $\gamma_k$ as indeterminants, and consider the rational function field $\mathbb{C}(\alpha, \beta, \gamma) = \mathbb{C}(\alpha, \beta, \gamma_1, \ldots, \gamma_m)$. For a rational function $g(\alpha, \beta, \gamma_1, \ldots, \gamma_m) \in \mathbb{C}(\alpha, \beta, \gamma)$, we denote $g(\alpha, \beta, \gamma_1, \ldots, \gamma_m)^\nu = g(\alpha^{-1}, \beta^{-1}, \gamma_1^{-1}, \ldots, \gamma_m^{-1})$. 
Under these notations, the condition (1) is equivalent to

\[
\alpha - \prod_{k=1}^{m} \gamma_k^{i_k}, \quad \beta - \prod_{k=1}^{m} \gamma_k^{i_k} \neq 0, \quad \forall I = (i_1, \ldots, i_m) \in \{0, 1\}^m.
\]

For example, \(\gamma_k = 1\) or \(\alpha\beta - (-1)^{m-1} \prod_{k=1}^{m} \gamma_k = 0\) are allowed.

Note that though in [2] the indices \(I\) run the subsets of \(\{1, \ldots, m\}\), in this paper we use \(\{0, 1\}^m\) as a set of indices. The correspondence is given by

\[
\{1, \ldots, m\} \supset \{i_1, \ldots, i_r\} \longleftrightarrow e_{i_1} + \cdots + e_{i_r} \in \{0, 1\}^m,
\]

where \(e_k\) is the \(k\)-th unit vector of size \(m\). We put \(|I| = \sum_{k=1}^{m} i_k\).

2.1. System of differential equations. Let \(\partial_k (k = 1, \ldots, m)\) be the partial differential operator with respect to \(x_k\). We set \(\theta_k = x_k \partial_k, \theta = \sum_{k=1}^{m} \theta_k\). Lauricella’s \(F_C(a, b, c; x)\) satisfies differential equations

\[
[\theta_k (\theta_k + c_k - 1) - x_k (\theta + a)(\theta + b)] f(x) = 0, \quad k = 1, \ldots, m.
\]

The system generated by them is called Lauricella’s hypergeometric system \(E_C(a, b, c)\) of differential equations. The system \(E_C(a, b, c)\) is a holonomic system of rank \(2^m\) with the singular locus \(S\). It is shown in [3] that the system \(E_C(a, b, c)\) is irreducible, that is, the system \(E_C(a, b, c)\) defines a maximal ideal in the ring of differential operators with rational function coefficients, if and only if the parameters \(a, b, c_1, \ldots, c_m\) satisfy (1).

For an element \(I = (i_1, \ldots, i_m)\) of \(\{0, 1\}^m\), we set

\[
F_I(x) = \frac{\prod_{k=1}^{m} \Gamma((-1)^{i_k}(1-c_k))}{\Gamma(1-a^I)\Gamma(1-b^I)} \cdot \prod_{k=1}^{m} x_k^{i_k(1-c_k)} : F_C(a^I, b^I, c^I; x),
\]

where

\[
a^I = a + \sum_{k=1}^{m} i_k(1-c_k), \quad b^I = b + \sum_{k=1}^{m} i_k(1-c_k),
\]

\[
c^I = (c_1 + 2i_1(1-c_1), \ldots, c_m + 2i_m(1-c_m)).
\]

Note that the assumption (1) is equivalent to

\[
a^I, b^I \notin \mathbb{Z}, \quad \forall I = (i_1, \ldots, i_m) \in \{0, 1\}^m,
\]

and that

\[
c_k + 2i_k(1-c_k) = \begin{cases} 
c_k & \text{if } i_k = 0, \\
2 - c_k & \text{if } i_k = 1.
\end{cases}
\]
Example 2.2. We give examples for \( m = 1 \) (we put \( c_1 = c, x_1 = x \)):

\[
F_0(x) = \frac{\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)} F_C(a, b; c) = \frac{\Gamma(1-c)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} x^n,
\]

\[
F_1(x) = \frac{\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)} x^{1-c} F_C(a + 1 - c, b + 1 - c, 2 - c; x) = \frac{\Gamma(c-1)\Gamma(1-(c-1))}{\Gamma(c-a)\Gamma(c-b)\Gamma(1-c+a)\Gamma(1-c+b)} \sum_{n=0}^{\infty} \frac{\Gamma(a-c+1+n)\Gamma(b-c+1+n)}{\Gamma(2-c+n)\Gamma(1+n)} x^{n+1-c},
\]

where \( n' = n + 1 - c \) and \( \sum_{n'=1-c}^{\infty} \) means the sum of \( n' \) running over the set \( 1-c+N = \{1-c+n \mid n \in \mathbb{N} \} \).

The functions \( \{F_1(x)\}_{I \in \{0, 1\}^m} \) form a basis of the local solution space \( \text{Sol}_{\hat{x}} = \text{Sol}_{\hat{x}}(a, b, c) \) to the system \( E_C(a, b, c) \) around a point \( \hat{x} \) in \( D_C - S \) under conditions

\[(5) \quad c_1, \ldots, c_m \notin \mathbb{Z}. \]

We set a (row) vector valued function

\[
F(x) = (\ldots, F_I(x), \ldots),
\]

where \( I \in \{0, 1\}^m \) are aligned by the pure lexicographic order as

\((0, \ldots, 0), (1, 0, \ldots, 0), (0, 1, \ldots, 0), (1, 1, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, (1, 1, 1, \ldots, 1)\).

Note that its entries have factors

\[1, x_1^{1-c_1}, x_2^{1-c_2}, x_1^{1-c_1} x_2^{1-c_2}, x_3^{1-c_3}, \ldots, x_1^{1-c_1} x_2^{1-c_2} \cdots x_m^{1-c_m},\]

respectively.

2.2. Monodromy representation. Put \( \hat{x} = (\frac{1}{m_1}, \ldots, \frac{1}{m_m}) \in X \). For \( \rho \in \pi_1(X, \hat{x}) \) and \( g \in \text{Sol}_{\hat{x}} \), let \( \rho_\ast g \) be the analytic continuation of \( g \) along \( \rho \). Since \( \rho_\ast g \) is also a solution to \( E_C(a, b, c) \), the map \( \rho_\ast : \text{Sol}_{\hat{x}} \rightarrow \text{Sol}_{\hat{x}}; g \mapsto \rho_\ast g \) is a \( \mathbb{C} \)-linear automorphism which satisfies \( (\rho \cdot \rho')_\ast = \rho'_\ast \circ \rho_\ast \) for \( \rho, \rho' \in \pi_1(X, \hat{x}) \). Here, the composition \( \rho \cdot \rho' \) of loops \( \rho \) and \( \rho' \) is defined as the loop going first along \( \rho \), and then along \( \rho' \). We thus obtain a representation

\[\mathcal{M} : \pi_1(X, \hat{x}) \rightarrow GL(\text{Sol}_{\hat{x}})\]

of \( \pi_1(X, \hat{x}) \), where \( GL(V) \) is the general linear group on a \( \mathbb{C} \)-vector space \( V \). This representation \( \mathcal{M} \) is called the monodromy representation of \( E_C(a, b, c) \).

Let \( \rho_0, \rho_1, \ldots, \rho_m \) be loops in \( X \) so that

- \( \rho_0 \) turns the hypersurface \( (R(x) = 0) \) around the point \( (\frac{1}{m_1}, \ldots, \frac{1}{m_m}) \), positively,
- \( \rho_k \) \( (k = 1, \ldots, m) \) turns the hyperplane \( (x_k = 0) \), positively.
For explicit definitions of them, see [2].

**Fact 2.3** ([2]). The loops $\rho_0, \rho_1, \ldots, \rho_m$ generate the fundamental group $\pi_1(X, \dot{x})$. Moreover, if $m \geq 2$, then they satisfy the following relations:

$$
\rho_i \rho_j = \rho_j \rho_i \quad (i, j = 1, \ldots, m), \quad \rho_0 \rho_k = (\rho_k \rho_0)^2 \quad (k = 1, \ldots, m).
$$

In [2], $m + 1$ linear maps $M_i = M(\rho_i) \ (i = 0, \ldots, m)$ are investigated in terms of twisted homology groups and the intersection form. In this paper, we do not explain them. What we need is the following fact.

**Fact 2.4** ([1]). Suppose (1) and (5).

(i) By integration, the twisted homology group is isomorphic to the solution space $\text{Sol}_{\dot{x}}$.

(ii) We can construct twisted cycles $\{\Delta_I\}_I$ that correspond to $\{F_I\}_I$.

(iii) The intersection matrix $H = (H_{I,I'})_{I,I'}$ with respect to the basis $\{\Delta_I\}_I$ is diagonal, and its $(I, I')$-entry is

$$
H_{I,I'} = \prod_{k=1}^m \frac{(-1)^{i_k} \gamma_k^{i_k} - 1}{\gamma_k - 1} \cdot \frac{(\alpha - \prod_{k=1}^m \gamma_k^{i_k})(\beta - \prod_{k=1}^m \gamma_k^{i_k})}{(\alpha - \prod_{k=1}^m \gamma_k)(\beta - 1)}.
$$

By using this fact and properties of the intersection form, we can induce the intersection form on $\text{Sol}_{\dot{x}}$.

**Definition 2.5.** We assume (1) and (5). We define a bilinear form (called the intersection form)

$$
\mathcal{I}(\cdot, \cdot) : \text{Sol}_{\dot{x}} \times \text{Sol}_{\dot{x}} \to \mathbb{C}(\alpha, \beta, \gamma)
$$

as follows. For any $F(x), G(x) \in \text{Sol}_{\dot{x}}$, we express them as linear combinations of the basis $\{F_I\}_I$:

$$
F(x) = F(x) \cdot f, \quad G(x) = F(x) \cdot g, \quad f, g \in \mathbb{C}(\alpha, \beta, \gamma)^{2^m},
$$

and define

$$
\mathcal{I}(F(x), G(x)) = ^t f \cdot H \cdot g^\vee.
$$

**Remark 2.6.** Let $\mathcal{I}_H$ be the intersection form on $\text{Sol}_{\dot{x}}$ induced from that on the twisted homology group by the isomorphism in Fact 2.4 (i). The intersection form $\mathcal{I}$ coincides with $\mathcal{I}_H$ modulo a constant multiple which never vanishes under the conditions (1) and (5).

**Corollary 2.7.** Under the conditions (1) and (5), the intersection form $\mathcal{I}$ is a monodromy invariant form, that is, for any loops $\rho \in \pi_1(X, \dot{x})$, we have

$$
\mathcal{I}(\mathcal{M}(\rho)(F(x)), \mathcal{M}(\rho)(G(x))) = \mathcal{I}(F(x), G(x)).
$$

In other words, $H$ satisfies

$$
^t M_\rho \cdot H \cdot M_\rho^\vee = H,
$$

where $M_\rho$ is the representation matrix of $\mathcal{M}(\rho)$ with respect to the basis $\{F_I\}_I$.

Let $M_i$ be the representation matrix of $\mathcal{M}_i (i = 0, \ldots, m)$ with respect to the basis $\{F_I\}_I$. We give explicit expressions of them.
Fact 2.8 ([2]). We assume (7), (9) and $\lambda = (-1)^{m-1} \alpha^{-1} \beta^{-1} \prod_{k=1}^{m} \gamma_k \neq 1$. For $k = 1, \ldots, m$, the representation matrix $M_k$ is diagonal, and its $(I, I)$-entry is $\gamma_k^{-i_k}$.

The representation matrix $M_0$ is expressed as

$$M_0 = E_{2m} - \frac{1 - \lambda}{\mathbf{1} \cdot H \cdot \mathbf{1}} \cdot \mathbf{1} \cdot H = E_{2m} - \frac{(\beta - 1)(\alpha - \prod_{k=1}^{m} \gamma_k)}{\alpha \beta} \cdot \mathbf{1} \cdot \mathbf{1} \cdot H,$$

where $E_{2m}$ is the unit matrix of size $2^m$, $\mathbf{1}$ is the column vector of size $2^m$ with all entries $1$, and $H$ is the intersection matrix given in Fact 2.4.

Remark 2.9. These expressions are obtained from consideration to eigenvectors of each $M_i \in \text{GL}(\text{Sol}_x)$.

(i) $M_k$ ($k = 1, \ldots, m$): $F_I$ is an eigenvector of eigenvalue $\gamma_k^{-i_k}$ (resp. $1$) if $i_k = 1$ (resp. $i_k = 0$), where $I = (i_1, \ldots, i_m)$.

(ii) $M_0$: the eigenvalues of $M_0$ are $\lambda$ and $1$. The eigenspace of eigenvalue $\lambda$ is one-dimensional and spanned by

$$f_0 = \sum_{I \in \{0,1\}^m} F_I,$$

which corresponds to $\mathbf{1}$ when we take the basis $\{F_I\}_I$. The eigenspace of $M_0$ of eigenvalue $1$ is characterized as $\{g \in \text{Sol}_x \mid I(g, f_0) = 0\}$.

(iii) The first expression of $M_0$ is stable under the non-zero scalar multiple to $H$.

3. Another basis

In fact, $\{F_I\}$ does not form a basis of $\text{Sol}_x$ when $c_i$’s are integers. In this section, we introduce another basis $\{\tilde{F}_I\}_I$ which is a well-defined basis even if $c_i$’s are integers, and we give the circuit matrices with respect to this basis. Note that these do not coincide with solutions obtained by integrating the twisted cycles defined in [2], [6]

3.1. Basis of $\text{Sol}_x$. First, we construct a basis of $\text{Sol}_x$.

Lemma 3.1. Let $I = (i_1, i_2 \ldots, i_m)$ be any element of $\{0,1\}^m$ and $p = (p_1, p_2, \ldots, p_m)$ be any element of $\mathbb{Z}^m$. Then the limit function

$$\lim_{c \to p}(\gamma_1 - 1)(\gamma_2 - 1) \cdots (\gamma_m - 1)F_I(x)$$

is well-defined and not identically zero.

Proof. We have only to note that $F_I(x)$ has the factor

$$\prod_{k=1}^{m} (\Gamma(c_k - i_k)\Gamma(1 - (c_k - i_k))) = \prod_{k=1}^{m} \left(\frac{(-1)^{i_k} \pi}{\sin(\pi c_k)}\right),$$

which cancels out $(\gamma_1 - 1)(\gamma_2 - 1) \cdots (\gamma_m - 1)$.

\[\square\]

Lemma 3.2. Let $I = (i_1, \ldots, i_{k-th}, \ldots, i_m)$ and $I' = (i_1, \ldots, 1, \ldots, i_m)$ be elements of $\{0,1\}^m$ and $p_k$ be an integer. Then the limit function

$$\lim_{c_k \to p_k}(\gamma_m - 1) \cdots (\gamma_{k+1} - 1)(\gamma_k - 1 - 1) \cdots (\gamma_1 - 1)(F_I(x) + F_{I'}(x))$$

is well-defined and not identically zero.
Proof. We show the case $m = 1$ (we put $p_1 = p$). Firstly, we assume $p = 1$. By Example 2.2

$$F_0(x) = \frac{s_0}{\sin(\pi c)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} x^n,$$

$$F_1(x) = -\frac{s_1(c)}{\sin(\pi c)} \sum_{n' = 1-c}^{\infty} \frac{\Gamma(a+n')\Gamma(b+n')}{\Gamma(c+n')\Gamma(1+n')} x^{n'},$$

where $n' = n + 1 - c$, $s_0 = \frac{\sin(\pi a)\sin(\pi b)}{\pi}$, $s_1(c) = \frac{\sin(\pi(a-c))\sin(\pi(b-c))}{\pi}$. Both functions $\sin(\pi c)F_0(x)$ and $-\sin(\pi c)F_1(x)$ converge to

$$\sum_{n=0}^{\infty} s_0 \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} x^n = \sum_{n=0}^{\infty} A_n x^n$$

as $c \to p = 1$. Apply l'Hôpital's rule to the function

$$F_0(x) + F_1(x) = \frac{1}{\sin(\pi c)} \left[ \sum_{n=0}^{\infty} \left( s_0 \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} - A_n \right) x^n - \sum_{n' = 1-c}^{\infty} \left\{ (s_1(c) \frac{\Gamma(a+n')\Gamma(b+n')}{\Gamma(c+n')\Gamma(1+n')} - A_n) x^{n'} - A_n (x^n - x^{n'}) \right\} \right]$$

to verify that its limit as $c \to p = 1$ exists, where $n = n' + c - 1$ in the second sum.

Note that $\lim_{c \to 1} \frac{A_n (x^n - x^{n'})}{\sin(\pi c)}$ yields the factor $\log x$.

Secondly, we assume $p \geq 2$. In this case, the sum $\sum_{n' = 1-c}^{\infty}$ has negative terms for $n' = 1 - p, \ldots, -1$ as $c \to p$. Since $\lim_{c \to p} \frac{1}{\sin(\pi c)} \cdot \frac{1}{\Gamma(n' + 1)}$ converges to a non-zero value, these negative terms are well-defined. Thus $F_0(x) + F_1(x)$ consists of these finite terms and the infinite sum considered in the case $p = 1$.

Thirdly, we assume $p \leq 0$. By regarding $n$ as $n + p$ in the sums of $F_0(x)$ and $F_1(x)$, we can show that $F_0(x) + F_1(x)$ is well-defined and not identically zero as in the previous consideration.

For a general $m$, use a similar argument by regrading the variables except $x_k$ as constants. Note that the limit function has the factor $\log x_k$ coming from $\lim_{n_k \to n_k'} x_k^{n_k} - x_k^{n_k'}$. \hfill $\square$

We define the tensor product $A \otimes B$ of matrices $A = (b_{ij})_{1 \leq i \leq r}$ and $B = (b_{ij})_{1 \leq j \leq s}$ as

$$A \otimes B = \begin{pmatrix} A b_{11} & A b_{12} & \cdots & A b_{1s} \\ A b_{21} & A b_{22} & \cdots & A b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A b_{r1} & A b_{r2} & \cdots & A b_{rs} \end{pmatrix}.$$

We remark that this is different from the usual definition. We fix the number $m$ of variables. We set

$$G_k = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_k^{-1} \end{pmatrix}, \quad Q_k = \begin{pmatrix} 1 - \gamma_k & 0 \\ 0 & 1 \end{pmatrix},$$
for $k = 1, \ldots, m$ and

$$P_m = Q_1 \otimes Q_2 \otimes \cdots \otimes Q_m.$$  

By using these notations, the matrices $M_1, \ldots, M_m$ given in Fact 2.3 is expressed as

$$M_k = E_2 \otimes \cdots \otimes E_2 \otimes G_k \otimes E_2 \otimes \cdots \otimes E_2 \quad (k = 1, \ldots, m)$$

where $E_2$ is the unit matrix of size 2. For example, we have

$$M_1 = G_1 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_1^{-1} \end{pmatrix}, \quad P_1 = Q_1 = \begin{pmatrix} 1 - \gamma_1 & 1 \\ 0 & 1 \end{pmatrix}$$

in the case $m = 1$, and

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma_1^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_2^{-1} & 0 \\ 0 & 0 & 0 & \gamma_2^{-1} \end{pmatrix},$$

$$P_2 = Q_1 \otimes Q_2 = \begin{pmatrix} (1 - \gamma_1)(1 - \gamma_2) & 1 - \gamma_2 & 1 - \gamma_1 & 1 \\ 0 & 1 - \gamma_2 & 0 & 1 \\ 0 & 0 & 1 - \gamma_1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

in the case $m = 2$. Note that

$$P_m = \begin{pmatrix} P_{m-1}(1 - \gamma_m) & P_{m-1} \\ O & P_{m-1} \end{pmatrix},$$

where $O$ is the square zero matrix of size $2^{m-1}$. We have

$$\det(P_m) = \prod_{k=1}^{m} (1 - \gamma_k)^{2^{m-1}},$$

since $\det(P_1) = 1 - \gamma_1$ and

$$\det(P_m) = \det((1 - \gamma_m)P_{m-1})\det(P_{m-1}) = (1 - \gamma_m)^{2^{m-1}} \det(P_{m-1})^2.$$  

We use a new basis given by

$$\tilde{F}(x) = (\ldots, \tilde{F}_1(x), \ldots) = F(x) \cdot P_m.$$  

The vector-valued function $\tilde{F}(x)$ takes the form

$$\left((1 - \gamma_1)F_0(x), F_0(x) + F_1(x)\right)$$

for $m = 1$, and the form

$$\left((1 - \gamma_1)(1 - \gamma_2)F_{00}(x), (1 - \gamma_2)(F_{00}(x) + F_{10}(x)), \right.$$

$$\left. (1 - \gamma_1)(F_{00}(x) + F_{01}(x)), F_{00}(x) + F_{10}(x) + F_{01}(x) + F_{11}(x) \right)$$

for $m = 2$.

**Theorem 3.3.** The vector-valued function $\tilde{F}(x)$ gives a basis of the space $\text{Sol}_k$ of the local solutions to $E \gamma(a, b, c)$ around $\dot{x}$ even in cases $c_k \in \mathbb{Z} \quad (k = 1, \ldots, m)$.  

Proof. The entries of \( \tilde{F}(x) \) consist of

\[
\left[ \prod_{k=1}^{m} (1 - \gamma_k) \right] F_{0...0},
\]

\[
\left[ \prod_{1 \leq k \leq m} (1 - \gamma_k) \right] (F_{0...0} + F_{e_i}),
\]

\[
\left[ \prod_{1 \leq k \leq m} (1 - \gamma_k) \right] (F_{0...0} + F_{e_i} + F_{e_j} + F_{e_i + e_j}),
\]

\[
\vdots
\]

\[
(1 - \gamma_k) \sum_{I \in \{0, 1\}^m} (1 - i_k) F_I,
\]

\[
\sum_{I \in \{0, 1\}^m} F_I,
\]

where \( e_k \) is the \( k \)-th unit vector of size \( m \) and \( I = (i_1, \ldots, i_m) \). The functions in the first and second lines are well-defined by Lemmas 3.1 and 3.2. Since the functions

\[
\left[ \prod_{1 \leq k \leq m} (1 - \gamma_k) \right] [(F_{0...0} + F_{e_i}) + (F_{e_j} + F_{e_i + e_j})],
\]

\[
\left[ \prod_{1 \leq k \leq m} (1 - \gamma_k) \right] [(F_{0...0} + F_{e_j}) + (F_{e_i} + F_{e_i + e_j})]
\]

are well-defined by Lemma 3.2, the function

\[
\left[ \prod_{k \neq i, j} (1 - \gamma_k) \right] (F_{0...0} + F_{e_i} + F_{e_j} + F_{e_i + e_j})
\]

is also well-defined. In this way, we can show that the entries of \( \tilde{F}(x) \) are well-defined even in cases \( c_k \in \mathbb{Z} \). In the case \( c_k \in \mathbb{Z} \), the functions \( \tilde{F}_I(x) \) has the factor \( \log x_k \), if \( i_k = 1 \). This implies that \( \tilde{F}_I(x) \)'s are also linearly independent in such a case. \( \square \)

3.2. Representation matrices and the intersection matrix. Next, we consider the representation matrices of \( M_i \)'s and the intersection matrix with respect to the new basis \( \{ \tilde{F}_I \}_I \).

In the below discussion, we often use the following equality which is shown by a straightforward calculation:

\[
\sum_{J \leq I} (-1)^{|J|} \left( \alpha \beta \prod_{k=1}^{m} \gamma_k^{1-j_k} + \prod_{k=1}^{m} \gamma_k^{1+j_k} \right) = \left( \alpha \beta + (-1)^{|I|} \prod_{k=1}^{m} \gamma_k^{i_k} \right) \prod_{k=1}^{m} (\gamma_k^{1-i_k} (\gamma_k - 1)^{i_k}),
\]

(7)
where \( I = (i_1, \ldots, i_m) \), \( J = (j_1, \ldots, j_m) \) and we define a partial order \( \preceq \) on \( \{0, 1\}^m \) by

\[
J \preceq I \iff j_k \leq i_k, \quad k = 1, \ldots, m.
\]

**Corollary 3.4.** Let \( \tilde{M}_i \) be the representation matrix of \( M_i \) \((i = 0, \ldots, m)\) with respect to the basis \( \{\tilde{F}_1\}_I \). For \( k = 1, \ldots, m \), we have

\[
\tilde{M}_k = E_2 \otimes \cdots \otimes E_2 \otimes \tilde{G}_k \otimes E_2 \otimes \cdots \otimes E_2, \quad \tilde{G}_k = \begin{pmatrix} 1 & -\gamma_{k,1}^{-1} \\ 0 & \gamma_{k,1}^{-1} \end{pmatrix}.
\]

\( \tilde{M}_0 \) is written as

\[
\tilde{M}_0 = E_{2^m} - N_0, \quad N_0 = \begin{pmatrix} 0, \ldots, 0, v \end{pmatrix},
\]

where \( v \in \mathbb{C}^{2^m} \) is a column vector whose \( I \)-th entry is

\[
\begin{cases}
(-1)^m \frac{(\alpha - \prod_{k=1}^m \gamma_k)^{\alpha \beta}}{\alpha \beta} & (I = (0, \ldots, 0)), \\
(-1)^{m+i} \frac{(\alpha \beta + (-1)^i \prod_{k=1}^m \gamma_k^i) \prod_{k=1}^m \gamma_{k,1}^{i-k}}{\alpha \beta} & (I \neq (0, \ldots, 0)).
\end{cases}
\]

Under the condition \( m \) without assuming \( p \), \( \tilde{M}_0, \ldots, \tilde{M}_m \) are valid.

**Proof.** By the definition of \( \{\tilde{F}_1\}_I \), we have \( \tilde{M}_i = P_m^{-1} M_i P_m \). The first claim follows from \( Q_k^{-1} G_k Q_k = \tilde{G}_k \) and the expressions of \( M_k \) and \( P_k \) as tensor products.

We show the second claim. Note that since all of the entries of the \( 2^m \)-th column (we also say the \((1, \ldots, 1)\)-th column) of \( P_m \) are 1, we have \( P_m e_{1, \ldots, 1} = 1 \) and hence \( e_{1, \ldots, 1} \) is an eigenvector of \( \tilde{M}_0 \) of eigenvalue \( (-1)^{m-1} \alpha^{-1} \beta^{-1} \prod_k \gamma_k \), where \( e_{1, \ldots, 1} = (0, \ldots, 0, 1) \in \mathbb{C}^{2^m} \). By \( \tilde{M}_0 = P_m^{-1} M_0 P_m \) and Fact 2.8 \( \tilde{M}_0 - E_{2^m} \) should be

\[
\frac{(\beta - 1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha \beta} P_m^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot HP_m = \frac{(\beta - 1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha \beta} e_{1, \ldots, 1} \cdot \begin{pmatrix} \mu \end{pmatrix},
\]

\[
= \frac{(\beta - 1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha \beta} t(0, \ldots, 0, h) \cdot P_m,
\]

where \( h \in \mathbb{C}^{2^m} \) is a column vector whose \( I \)-th entry is \( H_{I,I} \). It is sufficient to show that

\[
\frac{(\beta - 1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha \beta} t h \cdot P_m = \begin{pmatrix} 0 \\ \alpha \beta \end{pmatrix}.
\]

The \( I \)-th entry of the left-hand side is equal to

\[
\frac{(\beta - 1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha \beta} \sum_{J \leq I} \left( H_{I,J} \cdot \prod_{k=1}^m (1 - \gamma_k)^{1-i_k} \right)
\]

\[
= \frac{(-1)^m}{\alpha \beta \prod_{k=1}^m (1 - \gamma_k)^{i_k}} \sum_{J \leq I} (-1)^{|J|} \prod_{k=1}^m \gamma_{k,j_k} - (\alpha - \prod_{k=1}^m \gamma_{k,j_k})(\beta - \prod_{k=1}^m \gamma_{k,j_k}) \begin{pmatrix} 0 \\ \alpha \beta \end{pmatrix}.
\]
If \( I = (0, \ldots, 0) \), then this is the \((0, \ldots, 0)\)-th entry of \( v \). If we assume \( I \neq (0, \ldots, 0) \), then it equals to

\[
\frac{(-1)^m}{\alpha^m \prod_{k=1}^m (1 - \gamma_k)^{i_k}} \sum_{J \subseteq I} (-1)^{|J|} \left( \alpha \prod_{k=1}^m \gamma_k^{1-j_k} + \prod_{k=1}^m \gamma_k^{1+j_k} \right)
\]

\[
= \frac{(-1)^m}{\alpha^m \prod_{k=1}^m (1 - \gamma_k)^{i_k}} \left( \alpha + (-1)^{|I|} \prod_{k=1}^m \gamma_k \right) \prod_{k=1}^m (\gamma_k^{1-i_k} (\gamma_k - 1)^{i_k})
\]

by (7), and this coincides with the \( I \)-th entry of \( v \). \( \square \)

**Lemma 3.5.** \( 2^m \) vectors

\[
\left( \prod_{k=1}^m \tilde{M}_k^{i_k} \right) \cdot e_{1, \ldots, 1} = \tilde{M}_1^{i_1} \tilde{M}_2^{i_2} \cdots \tilde{M}_m^{i_m} e_{1, \ldots, 1} \quad (I = (i_1, \ldots, i_m) \in \{0, 1\}^m)
\]

are linearly independent. In other words, actions \( M_1, \ldots, M_m \) on \( f_0 \) give a basis of the whole space \( \text{Sol}_2 \).

**Proof.** It is sufficient to show that the \( 2^m \times 2^m \) matrix

\[
\left( e_{1, \ldots, 1}, \tilde{M}_1 e_{1, \ldots, 1}, \tilde{M}_2 e_{1, \ldots, 1}, \tilde{M}_1 \tilde{M}_2 e_{1, \ldots, 1}, \tilde{M}_3 e_{1, \ldots, 1}, \ldots, \tilde{M}_1 \cdots \tilde{M}_m e_{1, \ldots, 1} \right)
\]

is invertible. We calculate its determinant. Because of \( \tilde{M}_k = P_m^{-1} M_k P_m \) and \( e_{1, \ldots, 1} = P_m^{-1} 1 \), this matrix equals to

\[
(8) \quad P_m^{-1} \cdot (1, M_1 1, M_2 1, M_1 M_2 1, M_3 1, \ldots, M_1 \cdots M_m 1).
\]

By the alignment of the indices set, the right side of this product is

\[
\left( \begin{array}{cc} 1 & 1 \\ 1 & \gamma_1^{-1} \end{array} \right) \otimes \left( \begin{array}{cc} 1 & 1 \\ 1 & \gamma_2^{-1} \end{array} \right) \otimes \cdots \otimes \left( \begin{array}{cc} 1 & 1 \\ 1 & \gamma_m^{-1} \end{array} \right),
\]

and its determinant is \( \prod_{k=1}^m (\gamma_k^{-1} - 1)^{2m-1} \). By (8), the determinant of (8) is equal to \( \prod_{k=1}^m \gamma_k^{-2m-1} \), which is not zero. \( \square \)

**Proposition 3.6.** Let \( \tilde{H} = {^t \! P_m H P_m^\gamma} \), which represents the intersection form \( I \) with respect to the basis \( \{ \tilde{F}_I \}_I \). Then \( \tilde{H} \) is well-defined, and its determinant is

\[
\det(\tilde{H}) = \frac{1}{(\alpha - \prod_{k=1}^m \gamma_k)^{2m} (\beta - 1)^{2m}} \cdot \prod_{I \in \{0, 1\}^m} \left( \alpha - \prod_{k=1}^m \gamma_k^{i_k} \right) \left( \beta - \prod_{k=1}^m \gamma_k^{i_k} \right).
\]

In particular, \( \tilde{H} \) is non-degenerate even in cases \( c_k \in \mathbb{Z} \) (\( k = 1, \ldots, m \)).

**Proof.** First, we show the well-definedness. For \( I = (i_1, \ldots, i_m) \), \( I' = (i_1', \ldots, i_m') \), we put

\[
I \cdot I' = (i_1 i_1', \ldots, i_m i_m') \in \{0, 1\}^m.
\]

Since \( H \) is diagonal, the \((I, I')\)-entry of \( \tilde{H} \) is

\[
\tilde{H}_{I, I'} = \prod_{k=1}^m (1 - \gamma_k)^{1-i_k} (1 - \gamma_k^{-1})^{1-i_k'} \sum_{J \subseteq I, J' \subseteq I'} \left( \prod_{k=1}^m \frac{(-1)^{j_k} \gamma_k^{1-j_k}}{\gamma_k - 1} \cdot \frac{(\alpha - \prod_{k=1}^m \gamma_k^{j_k}) (\beta - \prod_{k=1}^m \gamma_k^{j_k})}{(\alpha - \prod_{k=1}^m \gamma_k) (\beta - 1)} \right)
\]

\[
= \prod_{k=1}^m \frac{(-1)^{i_k} \gamma_k^{1-i_k} (1 - \gamma_k)^{1-i_k'}}{(\alpha - \prod_{k=1}^m \gamma_k) (\beta - 1)} \sum_{J \subseteq I, J' \subseteq I'} \left( -1 \right)^{|J|} \prod_{k=1}^m \gamma_k^{1-j_k} \cdot \left( \alpha - \prod_{k=1}^m \gamma_k^{j_k} \right) (\beta - \prod_{k=1}^m \gamma_k^{j_k}).
\]
If $I \cdot I' = (0, \ldots, 0)$, then $1 - i_k - i'_k \geq 0 \ (k = 1, \ldots, m)$, and hence

$$
\tilde{H}_{I, I'} = \prod_{k=1}^{m} (-\gamma_k)^{i_k} (1 - \gamma_k)^{1 - i_k - i'_k} \cdot \frac{\alpha - 1}{\alpha - \prod_{k=1}^{m} \gamma_k}
$$

is well-defined. If $I \cdot I' \neq (0, \ldots, 0)$, the same calculation as the proof of Corollary 3.4 shows

$$
\tilde{H}_{I, I'} = \prod_{k=1}^{m} (-1)^{i_k} (1 - \gamma_k)^{1 - i_k - i'_k} \left( \frac{\alpha - 1}{\alpha - \prod_{k=1}^{m} \gamma_k} \right)
$$

and we can see that its denominator does not vanish.

Next, we evaluate $\det(\tilde{H})$. Straightforward calculation and (6) show

$$
\det(\tilde{H}) = (-1)^{m2^{m-1}} \prod_{k=1}^{m} \gamma_k^{2^{m-1}} \cdot \prod_{I \cdot I' = 0} \left( \alpha - \prod_{k=1}^{m} \gamma_k^{i_k} \right) \cdot \prod_{I \cdot I' = 1} \left( \alpha - \prod_{k=1}^{m} \gamma_k^{i'_k} \right),
$$

$$
\det(P_0) = \prod_{k=1}^{m} (\gamma_k - 1)^{2^{m-1}}.
$$

We thus have

$$
\det(\tilde{H}) = \frac{1}{\prod_{k=1}^{m} \gamma_k^{2^{m-1}}} \cdot \prod_{I \cdot I' = 0} \left( \alpha - \prod_{k=1}^{m} \gamma_k^{i_k} \right) \cdot \prod_{I \cdot I' = 1} \left( \alpha - \prod_{k=1}^{m} \gamma_k^{i'_k} \right),
$$

and it is not zero under the condition (2).

By this proposition, we can relax the condition to define the intersection from on $\text{Sol}_2$.

**Corollary 3.7.** By using $\{\tilde{F}_I\}_I$ and $\tilde{H}$, the intersection form $I : \text{Sol}_2 \times \text{Sol}_2 \rightarrow \mathbb{C}(\alpha, \beta, \gamma)$ in Definition 2.2 can be extended even in cases $c_k \in \mathbb{Z}$ $(k = 1, \ldots, m)$.

**Lemma 3.8.** The eigenspace of $\tilde{M}_0$ with eigenvalue 1 is expressed as

$$
\ker N_0 = \ker \ i^* v.
$$

**Proof.** This is obvious because of the expression

$$
\tilde{M}_0 = E_{2^m} - N_0 = E_{2^m} - \ i^* (0, \ldots, 0, v).
$$

**Remark 3.9.** If the eigenvalue $(-1)^{m-1} \alpha^{-1} \beta^{-1} \prod_{k=1}^{m} \gamma_k$ of $\tilde{M}_0$ coincides with 1, then the $(1, \ldots, 1)$-th entry of $v$ is zero and $e_1, \ldots, 1$ also belongs to this eigenspace $\ker N_0$.

**Lemma 3.10.**

$$
\ker N_0 = \{ w \in \mathbb{C}^{2^m} \mid \ i^* \tilde{H} e_{1,\ldots,1} = 0 \}.
$$

**Proof.** This is also obvious because of the orthogonality of the eigenspaces (Remark 2.9 (ii)) and the definition of $\tilde{H}$. 

□
4. Irreducibility

We restate the main theorem and give its proof.

**Theorem 4.1.** The monodromy representation
\[ \mathcal{M} : \pi_1(X, \dot{x}) \to GL(Sol_{\mathbb{Z}}) \]
is irreducible under the condition [1].

**Proof.** By Theorem 3.3, it is sufficient to consider the matrix representation by $\tilde{M}_i$ under the isomorphism $Sol_{\mathbb{Z}} \simeq \mathbb{C}^{2^n}$. Let $W \subset \mathbb{C}^{2^n}$ be an invariant subspace.

(i) First, we suppose $W \not\subset \ker N_0$. We take $w \in W$ such that $N_0 w \neq 0$. By the definition of $N_0$, the image of $N_0$ is spanned by $e_1, \ldots, 1$. Thus $N_0 w \neq 0$ implies that there exists $\mu \neq 0$ such that $N_0 w = \mu e_1, \ldots, 1$. We obtain
\[ e_1, \ldots, 1 = \frac{1}{\mu} N_0 w = \frac{1}{\mu} (w - \tilde{M}_0 w) \in W. \]

By Lemma 3.5, the $2^m$ vectors
\[ \left( \prod_{k=1}^{m} \tilde{M}_k^i \right) \cdot e_1, \ldots, 1 \in W \quad (I = (i_1, \ldots, i_m) \in \{0, 1\}^m) \]
are linearly independent. This implies $W = \mathbb{C}^{2^n}$.

(ii) Next, we suppose $W \subset \ker N_0$. We fix an arbitrary $w \in W$. Since $W$ is an invariant subspace, we have
\[ (\tilde{M}_1^i)^{-1}(\tilde{M}_2^i)^{-1} \cdots (\tilde{M}_m^i)^{-1} w \in W \subset \ker N_0 \]
for any $I = (i_1, \ldots, i_m) \in \{0, 1\}^m$. By the monodromy invariance $t \tilde{M}_i \tilde{H} \tilde{M}_i^\dagger = \tilde{H}$ of the intersection matrix $\tilde{H}$, commutativity between $\tilde{M}_1, \tilde{M}_2, \ldots, \tilde{M}_m$, and Lemma 3.10, we obtain
\[
\begin{align*}
& t w \tilde{H}(\tilde{M}_1^i \tilde{M}_2^i \cdots \tilde{M}_m^i e_1, \ldots, 1)^\dagger \\
& = t w (\tilde{M}_1^i)^{-1}(\tilde{M}_2^i)^{-1} \cdots (\tilde{M}_m^i)^{-1} \tilde{H} e_1, \ldots, 1 \\
& = t (\tilde{M}_1^i)^{-1}(\tilde{M}_2^i)^{-1} \cdots (\tilde{M}_m^i)^{-1} w \tilde{H} e_1, \ldots, 1 = 0.
\end{align*}
\]

The linear independence of $\{\tilde{M}_1^i \tilde{M}_2^i \cdots \tilde{M}_m^i e_1, \ldots, 1\}_I$ and $\det(\tilde{H}) \neq 0$ (Proposition 3.4) means that $w = 0$. We thus have $W = 0$.

Therefore, the invariant subspaces should be the trivial ones. \( \square \)

5. Reducibility

Recall that our irreducibility assumption [1] consists of $2^{m+1}$ conditions for parameters. In this section, we show that if one of them is not satisfied then the monodromy representation $\mathcal{M}$ of $E_C(a, b, c)$ is reducible. More precisely, we have the following theorem.

**Theorem 5.1.** Suppose that there exists $I = (i_1, \ldots, i_m) \in \{0, 1\}^m$ such that $a^I \in \mathbb{Z}$, $b^I \notin \mathbb{Z}$ or $a^I \notin \mathbb{Z}$, $b^I \in \mathbb{Z}$. If $a^{I'}, b^{I'} \notin \mathbb{Z}$ for any $I' \in \{0, 1\}^m$ different from $I$, then the monodromy representation $\mathcal{M}$ of $E_C(a, b, c)$ is reducible, that is, there exists a non-trivial subspace in $Sol_{\mathbb{Z}}$ invariant under $\mathcal{M}$. 


Proof: We fix \( I = (i_1, \ldots, i_m) \in \{0, 1\}^m \) and assume that \( a^I \in \mathbb{Z} \), \( b^I \notin \mathbb{Z} \). If there exists \( j \) such that \( c_j \in \mathbb{Z} \) then
\[
\begin{cases}
  a^{I+c_j} = a^I + (1 - c_j) \in \mathbb{Z} & \text{if } i_j = 0, \\
  a^{I-c_j} = a^I - (1 - c_j) \in \mathbb{Z} & \text{if } i_j = 1,
\end{cases}
\]
which contradicts the assumption. Thus we have \( c_1, \ldots, c_m \notin \mathbb{Z} \) and the solutions \( F_I(x) \) in (3) for \( I' \neq I \) are valid. Note that these \( 2^m - 1 \) solutions are linearly independent.

Hereafter, we regard \( a^I \) as an indeterminant, and consider two cases:

(i) \( a^I \) approaches to a negative integer \(-L\);

(ii) \( a^I \) approaches to a non-negative integer \( L'\).

To prove the reducibility, we find a non-trivial invariant subspace in each case. 

(i) Note that the solution \( F_I(x) \) in (3) for this \( I \) is expressed as a non-zero constant multiple of
\[
\lim_{a^I \to -L} \sin(\pi a^I) \sum_{(n'_1, \ldots, n'_m)} \frac{\Gamma(a + n'_1 + \cdots + n'_m)\Gamma(b + n'_1 + \cdots + n'_m)}{\Gamma(c_1 + n'_1)\cdots \Gamma(c_m + n'_m)\Gamma(1 + n'_1)\cdots \Gamma(1 + n'_m)} x_{n'_1} \cdots x_{n'_m},
\]
where \( n'_k \) (\( 1 \leq k \leq m \)) runs over the set
\[
\begin{cases}
  \mathbb{N} = \{ n_k \mid n_k \in \mathbb{N} \} = \{ 0, 1, 2, \ldots \} & \text{if } k \notin I, \\
  1 - c_k + \mathbb{N} = \{ 1 - c_k + n_k \mid n_k \in \mathbb{N} \} & \text{if } k \in I.
\end{cases}
\]

Since
\[
a + n'_1 + \cdots + n'_m = a^I + n_1 + \cdots + n_m = -L + n_1 + \cdots + n_m \leq 0
\]
for \( n_1 + \cdots + n_m \leq L \) when \( a^I \to -L \), some finite terms of \( \Gamma(a + n'_1 + \cdots + n'_m) \) diverge. However, the poles of the Gamma function are simple, these poles are canceled by the limit of \( \sin(\pi a^I) \) as \( a^I \to -L \). Hence the solution \( F_I(x) \) is reduced to the sum of finite terms by the limit \( a^I \to -L \). Since \( F_I(x) \) is a polynomial times \( \prod_{k=1}^{m} x_k^{|c_k| - 1} \), the 1-dimensional span of \( F_I(x) \) in \( \text{Sol}_{b^I} \) is invariant under \( \mathcal{M} \). Therefore the monodromy representation \( \mathcal{M} \) is reducible in this case. Note that the representation matrices \( M_i \) (\( i = 0, 1, \ldots, m \)) in Fact 2.3 are valid under the limit and the \( I \)-th column of \( M_0 \) is the \( I \)-th unit column vector of size \( 2^m \). We can also see that the 1-dimensional span of \( F_I(x) \) is invariant under \( \mathcal{M} \) by these representation matrices.

(ii) If the parameter \( a^I \) goes to a non-negative integer \( L' \), then the solution \( F_I(x) \) in (3) for this \( I \) reduces to the identically zero. Thus we use the fundamental system \((\ldots, F_j(x), \ldots) = (\ldots, F_I, \ldots) H^{-1}\), where the diagonal matrix \( H \) is given in Fact 2.3. By the explicit form of \( H \), we can easily see that \( F'_I(x) \) for \( I' \neq I \) are valid under the limit \( a^I \to L' \). The limit \( \lim_{a^I \to L'} F'_I(x) \) is a non-zero constant multiple of
\[
\lim_{a^I \to L'} \sum_{(n'_1, \ldots, n'_m)} \frac{\Gamma(a + n'_1 + \cdots + n'_m)\Gamma(b + n'_1 + \cdots + n'_m)}{\Gamma(c_1 + n'_1)\cdots \Gamma(c_m + n'_m)\Gamma(1 + n'_1)\cdots \Gamma(1 + n'_m)} x_{n'_1} \cdots x_{n'_m},
\]
where \( n'_k \) (\( 1 \leq k \leq m \)) runs over the same set as (i). Since each term of this series converges as \( a^I \to L' \), this limit is a solution to \( E_C(a, b, c) \) with the factor \( \prod_{k=1}^{m} x_k^{|c_k| - 1} \). Thus the fundamental system \((\ldots, F_I(x), \ldots) \) is
valid under the limit \( a^I \to L' \). By this change of fundamental systems, the representation matrices \( M_i \) (\( i = 0, 1, \ldots, m \)) are transformed into
\[
M'_i = HM_iH^{-1}.
\]
For \( k = 1, \ldots, m \), since \( M_k \) and \( H \) are diagonal, we have \( M'_k = HM_kH^{-1} = M_k = {}^tM_k \). By Fact 2.8, \( M'_0 \) is given as
\[
M'_0 = H \left( E_{2m} - \frac{(\beta - 1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha \beta} \cdot {}^t1 \cdot H \right) H^{-1} = E_{2m} - \frac{(\beta - 1)(\alpha - \prod_{k=1}^m \gamma_k)}{\alpha \beta} \cdot {}^t1 = {}^tM_0.
\]
These representation matrices are valid under the limit \( a^I \to L' \). By this limit, the \( I \)-th row of \( M'_0 \) is the \( I \)-th unit row vector of size \( 2^m \). Hence the \((2^m - 1)\)-dimensional space spanned by \( F'_I \) (\( I' \neq I \)) is invariant under \( M \).
Therefore, we obtain non-trivial invariant subspaces, and complete the proof. \( \square \)

**Remark 5.2.** Even in the case of \( m = 1 \), we need detailed case analysis to give a fundamental system of solutions to \( E_C(a, b, c) \) in terms of the series \( \mathfrak{F} \) without the condition \( \mathfrak{H} \), refer to \( \mathfrak{I} \) and \( \mathfrak{J} \).

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