Dispersive representation of $K \to 3\pi$ amplitudes and cusps

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The NA48/2 collaboration has shown clear experimental evidence for a cusp in the data for $K \to \pi\pi^0\pi^0$. This effect can be used to extract information on the $\pi\pi$ scattering lengths. We address this issue using a two-loop dispersive construction of $\pi\pi \to \pi\pi$ and $K \to \pi\pi\pi$ amplitudes in the presence of isospin breaking.

1. Introduction

The observation of a cusp anomaly in the $\pi^0\pi^0$ invariant mass distribution in the data on the $K^+ \to \pi^+\pi^0\pi^0$ decay collected by the NA48/2 collaboration has triggered some theoretical activity. The basic explanation of the appearance of a unitarity cusp is very simple: the amplitude for $K^+ \to \pi^+\pi^0\pi^0$ has two basic contributions, one of which corresponds to the $\pi^+\pi^-$ intermediate state rescattering to $\pi^0\pi^0$. This intermediate state in the $s$-channel generates (at the one-loop level) a square root singularity, and the corresponding amplitude behaves at this level for $s \sim 4m_\pi^2$ ($m_\pi$ is the mass of $\pi^+$) as

$$A(s) = R(\sigma^2) + \pi S(\sigma^2) \begin{cases} i\sigma_+, & s > 4m_\pi^2 \\
-\bar{\sigma}_+, & s < 4m_\pi^2 \end{cases}$$

(1)

where $\sigma_P = \sqrt{1 - 4m_P^2/s}$, $\bar{\sigma}_P = \sqrt{4m_P^2/s - 1}$. The functions $R(\sigma^2)$ and $S(\sigma^2)$ can be expressed as convergent series in $s - 4m_\pi^2$ in the physical region of the $K^+ \to \pi^+\pi^0\pi^0$ decay. This singularity appears at $4m_\pi^2$, which is above the physical threshold $4m_0^2$ ($m_0$ is the mass of $\pi^0$), and the cusp is a result of the interference of the part containing the singularity and the rest without it. It is clear that the cusp appears only in the isospin breaking case and its strength is sensitive to the $\pi\pi$ scattering amplitude at the threshold. This is the reason why the investigation of the cusp effect can in principle serve as an independent method for the experimental determination of the $\pi\pi$ scattering lengths, provided a model independent description of the corresponding amplitude can be given.

In Cabibbo and Isidori have proposed to use the assumed simple analytical properties of the amplitude ($\pi\pi$ amplitudes are considered in the form (1) with $R(\sigma^2)$ and $S(\sigma^2)$ polynomial) and the unitarity of the scattering matrix in order to express the amplitude near the threshold as an expansion in scattering lengths $a_i$. This idea was further investigated in [4]. Another possibility how to address this issue is the framework of non-relativistic effective field theory as developed in [5]. This is done as a combined expansion again in $a_i$, and also in the pion momenta. Both of these different approaches compute contributions to the $K^+$ decay amplitude up to order $O(a^2_i)$. Each of these approaches has its own limitations that result from the assumptions made. Moreover, they give no connection between cusp effects and the traditional PDG parameterization of the $K$ decay amplitudes. We consider an alternative...
approach, which rests on general properties, unitarity, analyticity, crossing symmetry, relativistic invariance, and chiral power counting for partial wave amplitudes. This leads, through a two-step iterative procedure, to a two-loop representation of the $K \to \pi\pi\pi$ amplitudes.

In the following we illustrate the method on the simplest case of the $K_L \to 3\pi^0$ decay, mainly because the analytical expressions are less involved, but also because, for the time being, we want to avoid addressing some further issues appearing in the treatment of the processes involving charged pions. The cusp in this decay has been observed by KTeV [7], as well as by NA48/2 [8].

2. Reconstruction theorem

The approach we wish to implement proceeds in parallel with the construction of the two-loop representation for the $\pi\pi$ scattering amplitude achieved in [9][10]. We shall use the fact that it can be extended to other processes and that isospin symmetry is not an essential ingredient [11]. Here we are interested in $K\pi \to \pi\pi$, related to $K \to 3\pi$ by crossing symmetry. The essential ingredients required in order to implement this construction are the following. First, we need a decomposition of the amplitude of the type

$$A(s, t, u) = 16\pi(f_0(s) + 3f_1(s) \cos \theta) + A_{\ell \geq 2}$$

with the following chiral behaviour,

$$\text{Re} A_{\ell \geq 2} \sim O(p^4), \quad \text{Im} A_{\ell \geq 2} \sim O(p^5), \quad (3)$$

$$\text{Re} f_\ell \sim O(p^2), \quad \text{Im} f_\ell \sim O(p^3), \quad \ell = 0, 1. \quad (4)$$

Having this, one can reconstruct the amplitude $A(s,t,u)$ of a process $AB \to CD$ to $O(p^8)$ from the knowledge of the imaginary parts of $S$ and $P$ partial waves of all the crossed amplitudes:

$$A(s, t, u) = P(s, t; u) + \Phi_0(s)$$
$$+ \left[ s(t-u) + (m_A^2 - m_B^2)(m_0^2 - m_0^2) \right] \Phi_1(s)$$
$$+ \text{crossed channels} + O(p^8), \quad (5)$$

where $P(s,t,u)$ is a polynomial having the same $s, t, u$ symmetries as the amplitude $A(s, t; u)$ and of at most third order in the Mandelstam variables. $\Phi_0$ and $\Phi_1$ are the dispersive integrals of the partial waves of the $s$-channel amplitude

$$\Phi_0(s) = 16s^3 \int_{\text{thresh.}}^\infty dx \frac{\text{Im} f_0(x)}{x^3(x-s)}, \quad (6)$$

$$\Phi_1(s) = 48s^3 \int_{\text{thresh.}}^\infty dx \frac{\text{Im} f_1(x)}{x^3(x-s)\lambda_{AB}^{1/2}(x)\lambda_{CD}^{1/2}(x)}$$

and similar for the $t$- and $u$-crossed channel $[\lambda_{AB}(s) = (s - (m_A + m_B)^2)/(s - (m_A - m_B)^2)]$.

The imaginary parts that enter the above expressions are obtained from the unitarity relation,

$$\text{Im} f^{i-j}_\ell(s) = \sum_k \frac{1}{s} \lambda^{1/2}(s)$$
$$\times f^{i-k}_\ell(s) \left(f^{j-k}_\ell(s)\right)^* \theta(s - \text{thr}_k), \quad (7)$$

projected on the corresponding partial waves. The sum goes over all the possible intermediate states $k$ ($S$ is a symmetry factor, $S = 2$ for indistinguishable states and $S = 1$ otherwise). In the low-energy region, and up to two-loops, these are restricted to pairs of light pseudoscalar mesons.

To the extent that we are interested in the decay region only, we may further restrict them to intermediate $\pi\pi$ states. The contributions from other intermediate states, like e.g. $K\pi$, can be expanded in powers of the Mandelstam variables and absorbed into the polynomial $P(s, t, u)$.

This unitarity relation and the reconstruction theorem can be used iteratively, i.e. starting from the LO amplitudes, we obtain the NLO results. $S$ and $P$ partial wave projections thereof then allow to obtain the NNLO expressions. Details of the first step will be presented in the next section.

According to the reconstruction theorem and due to the crossing symmetry, the two-loop representation of the $K_L \to 3\pi^0$ amplitude looks like:

$$A_{L:00}(s, t, u) = P_{L:00} + \Phi^{L:00}_0(s) + \Phi^{L:00}_1(t)$$
$$+ \Phi^{L:00}_2(u) + O(p^8)$$

with the polynomial

$$P_{L:00} = C_F (A_{00}^L M^2_K + \{C_{L:00}(s-s_0^L)^2\}$$
$$+ E_{L:00}^3 [s - s_0^L]^3 + \{s \leftrightarrow t + \{s \leftrightarrow u\}), \quad (9)$$

where the centre of Dalitz plot was defined as $s_0^L = 1/3M^2_K + m_0^2$ and $C_F$ corresponds to the standard normalization, $C_F = -\frac{1}{2} V_{us}^* V_{ud} \Delta \bar{\chi}$. 
3. First iteration: one-loop expressions

As already stated, we need the leading order $\pi\pi$ and $K\pi \to \pi\pi$ scattering amplitudes. From the chiral perturbation theory we know that at $O(p^2)$ they are represented by polynomials of at most first order in the Mandelstam variables. Their particular choice (connected also with the particular choice of the polynomial of the reconstruction theorem) is important since different choices can possibly lead to different convergence properties of the chiral expansion and affect the stability of the fit to the data. The standard choice (believed to be stable) is the expansion in subthreshold parameters. For the $\pi\pi$ amplitude, this corresponds to

$$A_{LO}^{\pi\pi} = \frac{\beta_{\pm0}}{F_{\pi}^2} \left( s - \frac{2}{3}m_+^2 - \frac{2}{3}m_0^2 \right) - \frac{\alpha_{\pm0}m_0^2}{3F_{\pi}^2},$$

Another possibility would consist in choosing the scattering length and effective range parameter as independent coefficient. It is even possible to adjust the polynomial part $P(s,t;u)$ of the $\pi\pi$ amplitude so that these coefficients retain their physical interpretation up to two loops, just as in the non-relativistic approach [5]. We shall study this option elsewhere [12]. In the case of $K_L\pi^0 \to \pi^0\pi^0$ and $K_L\pi^0 \to \pi^+\pi^-$ amplitudes we have

$$A_{LO}^{K}\pi^0 = C_F A_{00}^L M_K^2,$$

$$A_{LO}^{K\pi^+\pi^-} = C_F \left[ B_{L-}^L - (s - s_{L-}^L) + A_{+}^L - M_K^2 \right],$$

where $s_{L-}^L = (M_K^2 + m_0^2 + 2m_0^2)/3$.

The $O(p^2)$ chiral perturbation theory result is reproduced by special values of the parameters $(\alpha_{00} = 1, \beta_{\pm0} = 1, \alpha_{\pm0} = (2m_+^2 - m_0^2)/m_0^2; \text{cf. [13]}$) and similarly for the $K\pi$ part (see [14]).

Using these amplitudes in the first iteration, we obtain the one-loop result for $\Phi_0^{LO}$ of [8] as

$$\Phi_0^{L,00}(s) = \frac{C_F}{2F_{\pi}^2} A_{00}^L M_K^2 \alpha_{00} m_0^2 \tilde{J}_0(s)$$

$$- \frac{C_F}{F_{\pi}^2} \left[ \beta_{\pm0} \left( s - \frac{2}{3}m_+^2 - \frac{2}{3}m_0^2 \right) - \frac{1}{3} \alpha_{\pm0} m_0^2 \right] (12)$$

$$\times [A_{+}^L - M_K^2 + B_{L-}^L - (s - s_{L-}^L)] \tilde{J}_\pm(s)$$

+ polynomial + $O(p^6)$.

4. Second iteration: some remarks

The second iteration leads to the two-loop expression of the $K \to 3\pi$ amplitude. However, a

![Figure 1. The partial decay rate $K_L \to 3\pi^0$ (in arbitrary units) as a function of the invariant mass of the $\pi^0\pi^0$ pair squared for one particular choice of parameters. Within the small frame the cusp region is zoomed.](image)
The contour $C(t_+, t_-)$ is defined in such a way that it avoids the intersection with the cuts attached to branch points corresponding to the normal threshold of the amplitude $A_{L,00}(s,t,u)$ in the $t$– and $u$– channel. The trajectory of $t_+(s)$ and the basic types of the contour $C(t_+, t_-)$ in the complex $t$–plane are depicted in [15]. In the case of the $K_L\pi^0 \rightarrow \pi^0\pi^0$ and $K_L\pi^0 \rightarrow \pi^+\pi^-$ scattering, the generalization of this prescription beyond the isospin limit is straightforward. However, this is not the case of e.g. the experimentally more interesting process $K^+\pi^- \rightarrow \pi^0\pi^0$, where the naive application of the prescription [15] shows some problems. Namely, for the reconstruction of the amplitude of this process beyond NLO we need to compute the partial waves of $K^+\pi^- \rightarrow \pi^+\pi^-$, where the trajectory of $t_-(s)$ crosses the $t$–channel cut of this amplitude instead of avoiding it, contrary to the isospin limit. In addition, the two-loop amplitude for $K^+\pi^- \rightarrow \pi^0\pi^0$ suffers from (complex) anomalous threshold stemming from the triangle Landau singularity. The consequence is that a careful analytic continuation of the normal dispersive integrals entering the representation of the amplitude by means of the reconstruction theorem has to be performed. These technical issues deserve a detailed further discussion which is however beyond the scope of this talk.

5. Conclusion

The cusp effect in $K \rightarrow 3\pi$ offers an interesting possibility to extract quantitative information on $\pi\pi$ scattering lengths from the experimental data. We have outlined a construction, in a fully relativistic framework, of the corresponding two-loop amplitudes, based on general properties, unitarity, analyticity, crossing, and chiral counting for the partial waves. Our analysis provides two parameterizations, one in terms of the subthreshold parameters and the other, as in existing analyses, directly in terms of scattering lengths.

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