On integrability of $D0$–brane equations on $AdS_4 \times \mathbb{CP}^3$ superbackground

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Abstract. Equations of motion for the $D0$–brane on $AdS_4 \times \mathbb{CP}^3$ superbackground are shown to be classically integrable by extending the argument previously elaborated for the massless superparticle model.

1. Introduction
Integrable structures have become nowadays the basic objects of study both in $AdS_5/CFT_4$ [1] and Aharony-Bergman-Jafferis-Maldacena (ABJM) [2] dualities. In the former case, classical integrability of the $AdS_5 \times S^5$ superstring equations was proved in the seminal work [3] based on the $AdS_5 \times S^5$ superstring description as a 2d sigma-model on the $PSU(2,2|4)/(SO(1,4) \times SO(5))$ supercoset manifold [4, 5]. In the ABJM case, theories conjectured to be dual share lower space-time (super)symmetry so that integrable structure is more difficult to unveil and explore. Gravity dual of the ABJM gauge theory just in the special sublimit of the 't Hooft limit reduces to the IIA superstring theory on $AdS_4 \times \mathbb{CP}^3$ superspace, that is not isomorphic to a supercoset manifold. Only its subspace of dimension $(10|24)$ can be described as the $OSp(4|6)/(SO(1,3) \times U(3))$ supermanifold. The $OSp(4|6)/(SO(1,3) \times U(3))$ sigma-model [6, 7] corresponds to gauging away 8 fermionic coordinates for the supersymmetries broken by the $AdS_4 \times \mathbb{CP}^3$ superbackground in the complete superstring action constructed in [9]. In analogy with the $AdS_5 \times S^5$ superstring case the $OSp(4|6)/(SO(1,3) \times U(3))$ sigma-model equations are classically integrable [6, 7]. But integrability of the full set of $AdS_4 \times \mathbb{CP}^3$ superstring equations, which depend non-trivially on those 8 fermions, is by no means obvious. However, in [10] and [11] it was verified perturbatively up to the second order in such fermionic coordinates that this is indeed the case.

In the present and companion papers [12, 13] we explore integrability of equations for the point-like dynamical objects in IIA superstring theory on $AdS_4 \times \mathbb{CP}^3$ superbackground. In [13] it was proved integrability of the equations of massless superparticle describing dynamics of the superstring zero modes. Here we extend the proof to the case of $D0$–brane, for which various formulations of the action and $\kappa$–symmetry gauge conditions were considered in [14].

2. Aspects of supergeometry of $AdS_4 \times \mathbb{CP}^3$ superspace
In this Section, to make the presentation self-contained, we sketch the construction of supervielbein and Ramond-Ramond (RR) 1-form on the $OSp(4|6)/(SO(1,3) \times U(3))$ supercoset manifold. An alternative approach to construction of the $OSp(4|6)/(SO(1,3) \times U(3))$ sigma-model action relies on the introduction of the pure spinor variables [8].
manifold and the $AdS_4 \times \mathbb{CP}^3$ superspace, that are 'building blocks' of the $D0$–brane action, with the emphasis on the relation to extended superconformal symmetry of the ABJM gauge theory [15]. Thorough treatment of the supergeometry of $AdS_4 \times \mathbb{CP}^3$ superspace is given in [9].

Using the isomorphism between $osp(4|6)$ superalgebra and $D = 3$ $\mathcal{N} = 6$ superconformal algebra, left-invariant $osp(4|6)$ Cartan forms can be expanded over the $D = 3$ $\mathcal{N} = 6$ superconformal generators [16]

$$\mathcal{C}(d) = \mathcal{G}^{-1} d\mathcal{G} = \Delta(d) D + \omega^m(d) P_m + c^m(d) K_m + G^mn(d) M_{mn}$$
$$+ \Omega_a(d) T^a + \Omega^a(d) T_a + \tilde{\Omega}_a^b(d) \tilde{V}_b^a + \tilde{\Omega}_b^a(d) \tilde{V}_a^b$$
$$+ \omega_\mu^a(d) \tilde{Q}_\mu^a + \tilde{\omega}^{\mu a}(d) \tilde{Q}_\mu a + \chi_{\mu a}(d) S^\mu a + \bar{\chi}_\mu^a(d) S_a^\mu.$$  

Bosonic subalgebra is spanned by $D = 3$ conformal generators $(D, P_m, K_m, M_{mn})$ and $su(4) \sim so(6)$ $R$–symmetry generators, that we divided into the $U(3)$ ones $\tilde{V}_b^a$ forming the stability algebra of $\mathbb{CP}^3 = SU(4)/U(3)$ manifold and the coset generators $(T_a, T^a)$. 24 fermionic generators correspond to $D = 3 \mathcal{N} = 6$ Poincare $(Q_\mu^a, \tilde{Q}_\mu a)$ and conformal $(S^\mu a, \bar{S}_a^\mu)$ supersymmetries. They are equipped with the $SL(2, \mathbb{R})$ spinor index $\mu = 1, 2$ and $SU(3)$ (anti)fundamental representation index $a = 1, 2, 3$ in accordance with the decomposition $6 = 3 \oplus 3$ of the $SO(6)$ vector on $SU(3)$ representations. Geometric constituents of the $(10|24)$–dimensional $OSp(4|6)/(SO(1, 3) \times U(3))$ supermanifold that enter the 2d sigma-model equations and Lax connection [6, 7] are identified with the Cartan forms associated with the generators $g(k)$, with $k = 0, ..., 3$, having definite eigenvalues w.r.t. the $\mathbb{Z}_4$ automorphism $\Upsilon$ of the $osp(4|6)$ superalgebra: $\Upsilon(g(k)) = i^k g(k)$. In particular, invariant eigenspace $g(0)$ is spanned by the $so(1, 3) \oplus u(3)$ stability algebra generators. So (1) can be presented in the manifestly $\mathbb{Z}_4$–graded form

$$\mathcal{C}(d) = \mathcal{C}(0)(d) + \mathcal{C}(2)(d) + \mathcal{C}(1)(d) + \mathcal{C}(3)(d),$$  

where

$$\mathcal{C}(0)(d) = 2G^{3m}(d) M_3 m + G^{mn}(d) M_{mn} + \tilde{\Omega}_a^b(d) \tilde{V}_b^a + \tilde{\Omega}_b^a(d) \tilde{V}_a^b \in g(0),$$
$$\mathcal{C}(2)(d) = 2G^{0m}(d) M_0 m + \Delta(d) D + \Omega_a(d) T^a + \Omega^a(d) T_a \in g(2),$$
$$\mathcal{C}(1)(d) = \omega_\mu^a(d) Q_\mu^a + \bar{\omega}^{\mu a}(d) \tilde{Q}_\mu a \in g(1),$$
$$\mathcal{C}(3)(d) = \omega_3^a(d) Q_3^a + \bar{\omega}_3^{\mu a}(d) \tilde{Q}_3 \mu a \in g(3).$$

The generators

$$M_0 m = \frac{1}{2}(P_m + K_m), \quad M_0 3 = \frac{1}{2} D, \quad M_{3m} = \frac{1}{2}(K_m - P_m), \quad M_{mn}$$

span the $so(2, 3)$ algebra and one can write

$$g(0) = (M_{3m}, M_{mn}, \tilde{V}_b^a), \quad g(2) = (M_0 m, D, T^a, T_a).$$

Fermionic generators from $g(1)$ and $g(3)$ eigenspaces are defined by the linear combinations of those of Poincare and conformal supersymmetries

$$Q_{(1, 3)}^a \mu = Q_\mu^a \pm i S_\mu^a, \quad \tilde{Q}_{(1, 3) \mu a} = \tilde{Q}_\mu a \pm i S_{a \mu}.$$  

As a result, bosonic and fermionic Cartan forms from $\mathcal{C}(1, 2, 3)$ eigenspaces

$$G^{0m}(d) = \frac{1}{2}(\omega^m(d) + c^m(d)), \quad \Delta(d), \quad \Omega_a(d), \quad \Omega^a(d)$$

2
and
\[ \omega_{(1,3),a}^\mu(d) = \frac{1}{2}(\omega_a^\mu(d) \pm i \chi_a^\mu(d)), \quad \bar{\omega}_{(1,3),a}^{\mu a}(d) = \frac{1}{2}(\bar{\omega}^{\mu a}(d) \mp i \bar{\chi}^{\mu a}(d)) \]

are identified with the \( OSp(4|6)/(SO(1,3) \times U(3)) \) supervielbein components, while remaining Cartan forms
\[ G^{3m}(d) = \frac{1}{2}(c^m(d) - \omega^m(d)), \quad G^{mn}(d), \quad \tilde{\Omega}_{a}^{b}(d) \]

describe the \( SO(1,3) \times U(3) \) connection.

Since the \( AdS_4 \times \mathbb{CP}^3 \) superspace cannot be realized as a supercoset manifold, the approach described above is not applicable directly to the construction of its geometric constituents, that depend not only on the coordinates of the \( OSp(4|6)/(SO(1,3) \times U(3)) \) subsuperspace but also on 8 fermionic coordinates for the broken supersymmetries. However, the \( AdS_4 \times \mathbb{CP}^3 \) background of IIA supergravity can be promoted to the maximally supersymmetric \( AdS_4 \times S^7 \) background of \( D = 11 \) supergravity thanks to the Hopf fibration realization of the 7-sphere \( S^7 = \mathbb{CP}^3 \times S^1 \) \([17],[18]\). The \( AdS_4 \times S^7 \) superspace is isomorphic to the \( OSp(4|8)/(SO(1,3) \times SO(7)) \) supercoset manifold and one can construct \( D = 11 \) supervielbein, connection and 4-form field strength out of the \( osp(4|8) \) Cartan forms similarly to the above discussion, albeit the \( \mathbb{Z}_2 \) automorphism of \( so(2,3) \oplus so(8) \) subalgebra consistent with the coset-space description of \( AdS_4 \times S^7 \) space-time cannot be extended to the \( \mathbb{Z}_4 \) automorphism of the full \( osp(4|8) \) superalgebra. Then performing dimensional reduction \([19],[20]\) yields supervielbein and other constituents of the \( AdS_4 \times \mathbb{CP}^3 \) supergeometry \([9]\).

Using the isomorphism between \( osp(4|8) \) superalgebra and \( D = 3 \mathcal{N} = 8 \) superconformal algebra, left-invariant \( osp(4|8) \) Cartan forms admit decomposition over the \( D = 3 \mathcal{N} = 8 \) superconformal generators \([21],[22]\)
\[ G^{-1}dG = \hat{\mathcal{G}}_{so(2,3)} + \hat{\mathcal{G}}_{so(8)} + \hat{\mathcal{G}}_{32suy}, \]

where
\[ \hat{\mathcal{G}}_{so(2,3)} = \Delta(d)D + \omega^m(d)P_m + \zeta^m(d)K_m + G^{mn}(d)M_{mn}, \]
\[ \hat{\mathcal{G}}_{so(8)} = \Omega_a(d)T^a + \Omega^a(d)T_a + \tilde{\Omega}_a^b(d)\tilde{T}_b^a + \tilde{\tilde{\Omega}}_a^b(d)\tilde{T}_a^b + h(d)H \]
\[ + \tilde{\Omega}_a(d)\tilde{T}^a + \tilde{\Omega}^a(d)\tilde{T}_a + \tilde{\Omega}_a^4(d)V_4^a + \tilde{\Omega}_a^4(d)V_4^a \]
and
\[ \hat{\mathcal{G}}_{32suy} = \omega_a^\mu(d)Q_\mu^a + \bar{\omega}^{\mu a}(d)\bar{Q}_\mu^a + \chi_{\mu a}(d)S^{\mu a} + \bar{\chi}_a^{\mu a}(d)\bar{S}_a^{\mu a} + \omega_4^\mu(d)Q_4^\mu + \bar{\omega}^{\mu 4}(d)\bar{Q}_4^\mu + \chi_{4 \mu}(d)S^{4 \mu} + \bar{\chi}_4^\mu(a)\bar{S}_4^\mu. \]

Eq. (11) contains \( so(2,3) \) Cartan forms in conformal basis and Eq. (12) introduces Cartan forms for the \( so(8) \) generators in a basis corresponding to the Hopf fibration realization of the 7-sphere \( S^7 = \mathbb{CP}^3 \times S^1 \). In this basis one explicitly singles out generators of the \( su(4) \oplus u(1) \) isometry algebra of \( \mathbb{CP}^3 \times S^1 \). The generators \( (\tilde{T}_a, \tilde{T}^a, V_4^a, V_4^a) \) span the \( su(4) \) algebra and commute with the \( u(1) \) generator \( H \). Remaining 12 generators \( (\tilde{T}_a, \tilde{T}^a, V_4^a, V_4^a) \) belong to the \( so(8)/(su(4) \times u(1)) \) coset. Commutation relations and transformation to conventional form of the \( so(8) \) generators are discussed in \([21],[22]\) and rely on a particular convenient realization \([23]\) for the Kähler 2-form on \( \mathbb{CP}^3 \) manifold.\(^2\) The advantage of that choice of the Kähler 2-form consists in diagonalization of two projectors \([17],[9]\) used to divide 32 fermionic generators of the \( osp(4|8) \) superalgebra (and associated coordinates) into 24 generators of the \( osp(4|6) \) superalgebra and 8 generators corresponding to the supersymmetries broken by the \( AdS_4 \times \mathbb{CP}^3 \) superbackground. Thus, the

\(^2\) For general consideration of the embedding of \( su(4) \) algebra into \( so(8) \) compatible with the Hopf fibration of the 7-sphere see \([9]\).
first line in (13) contains fermionic generators of $D = 3$ $\mathcal{N} = 6$ superconformal algebra while the second the generators of the broken supersymmetries.

$AdS_4 \times S^7$ supervielbein components tangent to Anti-de Sitter part of the background are equal to the $so(2,3)/so(1,3)$ Cartan forms

$$\hat{E}^{m'}(d) = \left( \frac{1}{2}(\tilde{\omega}^m(d) + \tilde{\varepsilon}^m(d)), -\Delta(d) \right), \quad m' = (m,3),$$

(14)

and those tangent to $S^7$ in the basis corresponding to the Hopf fibration realization of the 7-sphere are given by

$$E_a(d) = i(\Omega_a(d) + \tilde{\Omega}_a(d)), \quad E^a(d) = i(\Omega^a(d) + \tilde{\Omega}^a(d)), \quad \hat{E}^{11}(d) = h(d) + \tilde{\Omega}_a^a(d),$$

(15)

where the 11-th space-time dimension was identified with the $S^1$ fiber. Supervielbein fermionic components are identified with the odd Cartan forms in (13). Explicit form of the supervielbein depends on the choice of $OSp(4|8)/(SO(1,3) \times SO(7))$ representative $\hat{\mathcal{G}}$. To fulfil the reduction to 10 dimensions supervielbein components should match the Kaluza-Klein (KK) ansatz form [19, 20]. In the present case the reduction is performed along the $S^1$ parametrized by $y \in [0,2\pi)$ so that considering

$$\hat{\mathcal{G}} = \mathcal{G}e^{yH}\mathcal{G}_{br}(v),$$

(16)

where $\mathcal{G} \in OSp(4|6)/(SO(1,3) \times U(3))$ and $\mathcal{G}_{br}$ is a function of 8 Grassmann coordinates for the broken supersymmetries $\nu_\mu = (\theta_\mu, \tilde{\theta}_\mu, \eta_\mu, \tilde{\eta}_\mu)$, ensures that the $osp(4|8)$ Cartan forms do not depend on $y$. Underlining in (11), (13) and (14) was used to indicate those of the $osp(4|6)$ Cartan forms that acquire dependence on $dy$, as well as $v$ and $dv$ in addition to that on the coordinates of the $OSp(4|6)/(SO(1,3) \times U(3))$ supermanifold. In particular, the supervielbein components tangent to $AdS_4$ and $S^1$ acquire additive contributions proportional to $dy$

$$\hat{E}^{m'}(d) = G^{m'}(d) + C^{m'}_y dy, \quad \hat{E}^{11}(d) = \Phi dy + a(d),$$

(17)

where $G^{m'}_y$ and $\Phi$ are functions of $v$. Expression (17) deviates from the Kaluza-Klein ansatz [19], [20] so that the $SO(1,4)$ tangent-space Lorentz transformation $L$ should be applied to remove the term proportional to $dy$ in $E^{m'}(d)$

$$L \left( \begin{array}{c} \hat{E}^{m'}(d) \\ \hat{E}^{11}(d) \end{array} \right) = \left( \begin{array}{c} E^{m'}(d) \\ \Phi_L(dy + A_L(d)) \end{array} \right).$$

(18)

Then $E^{m'}(d)$ is identified with the tangent to $AdS_4$ components of the $D = 10$ supervielbein in KK frame, $\Phi_L = \sqrt{\Phi^2 + G^2_y} \left( G^2_y = G_{ym}G^{ym} = G_y \cdot G_y \right)$ is identified with $\exp(2\phi/3)$, where $\phi(v)$ is the $D = 10$ dilaton superfield, and $A_L$ is the Ramond-Ramond (RR) 1-form potential. Explicit form of the entries of the Lorentz rotation matrix is

$$L = \left( \begin{array}{cc} \varepsilon^{m'}_m + \frac{\Phi - \Phi_L}{\Phi_L G^2_y} C^{m'}_y G_{ym'} & -\Phi_L^{-1}G^{m'}_y \\ \Phi_L^{-1}C^{m'}_y & \Phi_L^{-1}\Phi \end{array} \right) \in SO(1,4).$$

(19)

$D = 11$ supervielbein bosonic components $E_a(d)$ and $E^a(d)$ in (15) do not depend on $dy$ and thus can be directly identified with the tangent to the $\mathbb{CP}^3$ components of $AdS_4 \times \mathbb{CP}^3$ supervielbein in the KK frame, hence they were not endowed with ‘hats’. This concludes characterization of

3 Associated Lorentz rotation acting on the supervielbein fermionic components is discussed in [9].
the $AdS_4 \times \mathbb{CP}^3$ supervielbein bosonic components and RR 1-form potential, i.e. the constituents of the $D0$--brane action.

For the analysis of $D0$--brane equations in analogy with those of the massless superparticle, it turns helpful to expand $G^{m'}(d)$, $E_a(d)$, $E^a(d)$ and $a(d)$ on the $\mathbb{Z}_4$--graded $osp(4/6)$ Cartan forms (2) and $dv$ [13]

\[
G^{m'}(d) = G^{0n}(d)M^{m'm'} + G^{3n}(d)N^{m'm'} + \Delta(d)L^{m'm'} + G^{kl}(d)K^{m'm'}
\]

\[+ q^{m'\mu}d\theta_\mu + \bar{q}^{m'\bar{\mu}}d\bar{\theta}_{\bar{\mu}} + s^{m'\mu}d\eta_\mu + \bar{s}^{m'\bar{\mu}}d\bar{\eta}_{\bar{\mu}},
\]

\[E_a(d) = i\Omega_a(d) + u_{(1)}^{\mu}\omega_{(1)\mu a}(d) + u_{(3)}^{\mu}\omega_{(3)\mu a}(d),
\]

\[E^a(d) = i\Omega^a(d) + \bar{u}_{(1)}^{\bar{\mu}}\bar{\omega}_{(1)\bar{\mu}a}(d) + \bar{u}_{(3)}^{\bar{\mu}}\bar{\omega}_{(3)\bar{\mu}a}(d)
\]

and

\[a(d) = \bar{\Xi}^{\alpha a} + G^{0m}(d)m_m + G^{3m}(d)n_m + \Delta(d)l + G^{mn}(d)k_{mn}
\]

\[+ h^{\mu}d\theta_\mu + \bar{h}^{\bar{\mu}}d\bar{\theta}_{\bar{\mu}} + p^{\mu}d\eta_\mu + \bar{p}^{\bar{\mu}}d\bar{\eta}_{\bar{\mu}}.
\]

Coefficients at the differentials of the 'broken' fermions $dv_\mu = (d\theta_\mu, d\bar{\theta}_{\bar{\mu}}, d\eta_\mu, d\bar{\eta}_{\bar{\mu}})$ represent corresponding $AdS_4 \times S^7$ supervielbein components, while those at the $osp(4/6)$ Cartan forms can be named 'previelbeins'. Due to the choice of the supercoset element (16) they are functions of $v$ only. To verify integrability of the $D0$--brane equations it is useful to have explicit expressions for them, as well as for $G^{m'}$ and $\Phi$, that can be derived by specifying $\vartheta_\mu$ (see [13] for further details).

3. $D0$--brane equations and their integrability

The $D0$--brane action

\[S = -m \int d\tau \Phi_1^{-1}\sqrt{-(E_{\tau m'}E^{m'}_{\tau} - E_{r\alpha}E^{\alpha}_{\tau})} + m \int A_L
\]

is the sum of kinetic and Wess-Zumino (WZ) terms defined by the world-line pull-back of the RR 1-form potential $A_L$ with $m$ measuring its flux. Introducing auxiliary 1$d$ metric, the action can be brought to the form without the square root

\[S = \frac{1}{2} \int d\tau \left[ e^{-2}\Phi_1^{-1}\left(E_{\tau m'}E^{m'}_{\tau} - E_{r\alpha}E^{\alpha}_{\tau}\right) - em^2 \right] + m \int A_L.
\]

Let us note arbitrariness in the definition of the Lagrange multiplier $e(\tau)$. The choice made in (23) gives conventional mass term $em^2$ independent of the superspace coordinates. It is also possible to 'absorb' $\Phi_1^{-1}$ factor into the definition of $e$ resulting in the action reducing in the $m \to 0$ limit to that of the massless superparticle. The variation of (23) on $e$ produces mass-shell constraint that is irrelevant for establishing integrability of other (dynamical) equations of motion and hence in the subsequent discussion we set $e = 1/2$, i.e. concentrate on the '1$d$ sigma-model' case.

Considering the $osp(4/6)/(so(1,3) \times u(3))$ Cartan forms (2) as independent variation parameters allows to derive the set of 10 bosonic and 24 fermionic equations of motion. They can be written as a system of the first order ordinary differential equations with the coefficients given by the world-line pull-backs of the $osp(4/6)$ Cartan forms to facilitate derivation of the Lax
representation

\[ -\frac{\delta S}{\delta G^m_{\alpha}(\delta)} = \hat{a} \omega_n - 4f G_{\tau n} - 4 \phi G_{\tau n} + 4 f G_{\tau n} - 2 \Delta \cdot \phi \]

\[ - 4i (\omega_1 \tau \epsilon (1)) - \epsilon (1) \delta^{\alpha} \omega (1)^{\alpha} = 0, \]

\[ \frac{1}{2} \frac{\delta S}{\delta \Delta (\delta)} = \hat{f} G_{\tau n} - 2 \omega (1)^{\alpha} = 0, \]

\[ -\frac{\delta S}{\delta \Delta (\delta)} = \hat{g}^{\b} + i \epsilon^{\b} (\bar{\omega} \tau \beta - \bar{g} \tau \beta) - 4 \omega \tau \epsilon a \]

\[ + 4 \epsilon_{abc} \left( \omega (1)^{\alpha} \epsilon (1) + \omega (3)^{\alpha} \epsilon (3) \epsilon (3) \right) = 0 \]

and

\[ -\frac{\delta S}{\delta \epsilon (\beta)} = \hat{g}^{\b} + i \epsilon^{\b} (\bar{\omega} \tau \beta - \bar{g} \tau \beta) \]

\[ + 4 \epsilon_{abc} \left( \omega (1)^{\alpha} \epsilon (1) + \omega (3)^{\alpha} \epsilon (3) \epsilon (3) \right) = 0 \]

and c.c. equations. In Eqs. (24), (25) the following quantities have been introduced

\[ \begin{pmatrix} \hat{a} \\ \hat{f} \\ \hat{g} \end{pmatrix} = 2 \Phi L^{2} \begin{pmatrix} M^{n} \\ L^{n} \\ N^{n} \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{f} \\ \hat{g} \end{pmatrix} = 2 \Phi L^{2} \begin{pmatrix} M^{n} \\ L^{n} \\ N^{n} \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{f} \\ \hat{g} \end{pmatrix} \]

\[ + \Phi L^{2} \begin{pmatrix} m_n \\ l_n \\ m_n \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{f} \\ \hat{g} \end{pmatrix} = 2 \Phi L^{2} (G^2 a - G^2 G + m \Phi), \]

\[ y^a = -i \Phi L^{-2} E_{\tau a}, \quad y_a = -i \Phi L^{-2} E_{\tau a}, \quad w = \frac{1}{2} \Phi L^{-4} (G^2 a - G^2 G + m \Phi), \]

\[ \epsilon (1)^{\alpha} = -\frac{1}{2} \Phi L^{-2} E_{\tau a} \epsilon (1)^{\alpha}, \quad \epsilon (3)^{\alpha} = \frac{1}{4} \Phi L^{-2} E_{\tau a} \epsilon (1)^{\alpha}, \]

\[ \bar{\epsilon} (1)^{\alpha} = \frac{1}{4} \Phi L^{-2} E_{\tau a} \bar{\epsilon} (1)^{\alpha}, \quad \bar{\epsilon} (3)^{\alpha} = -\frac{1}{4} \Phi L^{-2} E_{\tau a} \bar{\epsilon} (1)^{\alpha}. \]

Compared to the massless superparticle case, the above introduced quantities take into account the WZ term contribution and the overall factor of $\Phi L^{-2}$ in the kinetic term. Finally there are 8 equations for the ‘broken’ fermions $v$ that can be brought to the following form

\[ - 2 \frac{\delta}{\delta f} \left[ \Phi L^{-2} (G_{\tau a} + \epsilon (1)^{\alpha} G_{\tau a} + \epsilon (3)^{\alpha} G_{\tau a}) \right] \frac{\delta G_{\tau a}}{\delta \phi_a} = 0, \]

\[ + \Phi L^{-2} \left( G_{\tau a} + \epsilon (1)^{\alpha} G_{\tau a} + \epsilon (3)^{\alpha} G_{\tau a} \right) \frac{\delta G_{\tau a}}{\delta \phi_a} = 0, \]

\[ + \Phi L^{-2} \left( G_{\tau a} + \epsilon (1)^{\alpha} G_{\tau a} + \epsilon (3)^{\alpha} G_{\tau a} \right) \frac{\delta G_{\tau a}}{\delta \phi_a} = 0, \]

\[ + \left[ m + 2 \Phi L^{-2} (G_{\tau a} + \epsilon (1)^{\alpha} G_{\tau a} + \epsilon (3)^{\alpha} G_{\tau a}) \right] \frac{\delta G_{\tau a}}{\delta \phi_a} = 0, \]
where the following shorthand notation was introduced: \( \frac{\partial G_{\nu_1}^{\mu_1}}{\partial \nu_\mu} = (q^{\mu_1}, \tilde{q}^{\mu_1}, s^{\mu_1}, s^{\mu_1}) \),

\[ \frac{\partial G_{\nu_1}^{\mu_1}}{\partial \nu_\mu} = \left( \begin{array}{c}
G_r \nu_1^m \frac{\partial M_{\nu_1}^{\mu_1}}{\partial \nu_\mu} + G_r \nu_1^m \frac{\partial N_{\nu_1}^{\mu_1}}{\partial \nu_\mu} + \Delta_r \frac{\partial M_{\nu_1}^{\mu_1}}{\partial \nu_\mu} + G_r \nu_1^m \frac{\partial K_{\nu_1}^{\mu_1}}{\partial \nu_\mu} \\
\theta_\nu \frac{\partial q_{\nu_1}^{\mu_1}}{\partial \nu_\mu} - \frac{\partial q_{\nu_1}^{\mu_1}}{\partial \nu_\mu} - \frac{\partial q_{\nu_1}^{\mu_1}}{\partial \nu_\mu} - \frac{\partial q_{\nu_1}^{\mu_1}}{\partial \nu_\mu}
\end{array} \right) \]

(30)

\[-\frac{\partial E_{\nu a}}{\partial \nu_\mu} = \omega_{(1) \tau a} \frac{\partial u_{(1)}^{\nu a}}{\partial \nu_\mu} + \omega_{(3) \tau a} \frac{\partial u_{(3)}^{\nu a}}{\partial \nu_\mu}, \quad -\frac{\partial E_{\tau a}}{\partial \nu_\mu} = \tilde{\omega}_{(1) \tau a} \frac{\partial \tilde{u}_{(1)}^{\nu a}}{\partial \nu_\mu} + \tilde{\omega}_{(3) \tau a} \frac{\partial \tilde{u}_{(3)}^{\nu a}}{\partial \nu_\mu}. \]

(31)

Analogously are defined \( \frac{\partial G_{\nu_1}^{\mu_1}}{\partial \nu_\mu} \) and \( \frac{\partial G_{\nu_1}^{\mu_1}}{\partial \nu_\mu} \).

Equations of motion (24), (25) and (29) are equivalent to the Lax equation

\[ \frac{d\mathcal{L}}{d\tau} + [\mathcal{M}, \mathcal{L}] = 0 \]

with the Lax component \( \mathcal{M} \) given by the world-line projection of the \( osp(4|6) \) Cartan forms (2) and the component \( \mathcal{L} \) that can be presented as a sum

\[ \mathcal{L} = \mathcal{L}_{so(2,3)} + \mathcal{L}_{su(4)} + \mathcal{L}_{24susys} \in osp(4|6), \]

(33)

where each summand is given by the linear combination of the quantities introduced in (26)-(28)

\[ \mathcal{L}_{so(2,3)} = a_0^\nu a_0^\mu M_0 a_0^m + f D + b_3 n M_3 m + f m n M_{mn}, \]

\[ \mathcal{L}_{su(4)} = y^a T_a + \bar{y}_a T^a + 4 w \bar{V}_a, \]

\[ \mathcal{L}_{24susys} = \varepsilon_{(1) \mu}^0 Q_{(1) \mu} + \varepsilon_{(1) \mu}^a \bar{Q}_{(1) \mu a} + \varepsilon_{(3) \mu}^a Q_{(3) \mu a} + \varepsilon_{(3) \mu}^a \bar{Q}_{(3) \mu a}. \]

\( \mathcal{L} \) takes the same form as for the massless superparticle modulo the definition of coefficients (26)-(28). Analogously to the superparticle case the Lax component \( \mathcal{L} \) can be presented in the form of \( osp(4|6) \)-valued differential operator acting on the \( D0 \)-brane action (23)

\[ \mathcal{L} = \left( \begin{array}{c}
M_0 a_0^m \frac{\partial}{\partial \nu_\mu} + \frac{1}{4} D \frac{\partial}{\partial \nu_\mu} - M_{mn} \frac{\partial}{\partial \nu_\mu} - M_{3m} \frac{\partial}{\partial \nu_\mu} \\
T_a \frac{\partial}{\partial \nu_\mu} + T^a \frac{\partial}{\partial \nu_\mu} + \bar{V}_a \frac{\partial}{\partial \nu_\mu} \\
- \frac{1}{4} Q_{(1) \mu} a_0 \frac{\partial}{\partial \nu_\mu} + \frac{1}{4} Q_{(1) \mu a} \frac{\partial}{\partial \nu_\mu} + \frac{1}{4} Q_{(3) \mu} a_0 \frac{\partial}{\partial \nu_\mu} - \frac{1}{4} Q_{(3) \mu a} \frac{\partial}{\partial \nu_\mu}
\end{array} \right) S. \]

(35)

4. Conclusion

In this note we focused on the proof of the classical integrability of equations of the \( D0 \)-brane on \( AdS_4 \times CP^3 \) superbackground. They can be written as the system of the first order ordinary differential equations that admit Lax representation with the Lax pair components taking value in the \( osp(4|6) \) isometry superalgebra of \( AdS_4 \times CP^3 \) superbackground. This generalizes the proof of integrability of the massless superparticle equations given in [13]. However, the most important problem is to find a generalization to the superstring case, i.e. to find a zero curvature representation for the \( AdS_4 \times CP^3 \) superstring equations. As follows from the discussion in [12] on the relation between the superparticle’s Lax pair and the Lax connection of the superstring, the component \( \mathcal{L} \) of the Lax pair captures the form of the contribution linear in \( \ell_3 \) to the \( 2D \) Hodge dualized part of the Lax connection up to insertions of the functions \( \ell_1, \ell_3, \ell_4 \) of the spectral parameter that is convenient to identify with \( \ell_3 \). It is thus interesting to find out whether the form of the remaining part of the Lax connection is also determined by the constituents of \( AdS_4 \times S^7 \) supervielbein bosonic components (17) and ‘previelbeins’ analogously to \( \mathcal{L} \).
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