Global well-posedness of the Cauchy problem of two-dimensional compressible Navier-Stokes equations in weighted spaces

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Abstract: In this paper, we study the global well-posedness of classical solution to 2D Cauchy problem of the compressible Navier-Stokes equations with large initial data and vacuum. It is proved that if the shear viscosity \( \mu \) is a positive constant and the bulk viscosity \( \lambda \) is the power function of the density, that is, \( \lambda(\rho) = \rho^\beta \) with \( \beta > 3 \), then the 2D Cauchy problem of the compressible Navier-Stokes equations on the whole space \( \mathbb{R}^2 \) admit a unique global classical solution \((\rho, u)\) which may contain vacuums in an open set of \( \mathbb{R}^2 \). Note that the initial data can be arbitrarily large to contain vacuum states. Various weighted estimates of the density and velocity are obtained in this paper and these self-contained estimates reflect the fact that the weighted density and weighted velocity propagate along with the flow.

Key Words: compressible Navier-Stokes equations, density-dependent viscosity, global well-posedness, vacuum, weighted estimates

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1 Introduction and main results

In this paper, we consider the following compressible and isentropic Navier-Stokes equations with density-dependent viscosities

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P(\rho) &= \mu \Delta u + \nabla ((\mu + \lambda(\rho))\text{div} u), \quad x \in \mathbb{R}^2, t > 0,
\end{align*}
\]

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where $\rho(t, x) \geq 0$, $u(t, x) = (u_1, u_2)(t, x)$ represent the density and the velocity of the fluid, respectively. And $x = (x_1, x_2) \in \mathbb{R}^2$ and $t \in [0, T]$ for any fixed $T > 0$. We denote the right hand side of (1.1) by

$$L_{\rho}u = \mu \Delta u + \nabla((\mu + \lambda(\rho)) \text{div} u).$$

Here, it is assumed that

$$\mu = \text{const.} > 0, \quad \lambda(\rho) = \rho^\beta, \quad \beta > 3,$$

such that the operator $L_{\rho}$ is strictly elliptic.

For simplicity, we assume that the pressure function is given by

$$P(\rho) = A \rho^\gamma,$$

where $\gamma > 1$ denotes the adiabatic exponent and $A = \text{const.}$. Without loss of generality, $A$ is normalized to be 1. The initial values are imposed as

$$(\rho, u)(t = 0, x) = (\rho_0, u_0)(x).$$

The system (1.1)-(1.2) was first proposed and studied by Vaigant-Kazhikhov in [50] in which the global well-posedness of the classical solution to (1.1)-(1.3) with general data satisfying periodic boundary conditions was obtained provided that the initial density is uniformly away from vacuum. To authors’s knowledge, this is the first result of the global well-posedness to the multi-dimensional compressible Navier-Stokes equations with large initial data in the absence of vacuum. Then Perepelitsa [45] studied the global existence and large time behavior of weak solution to (1.1)-(1.2) with 2D periodic boundary conditions. Recently, Jiu-Wang-Xin [31] improved the result in [50] and obtained the global well-posedness of the classical solution to the periodic problem with general initial data permitting vacuum. Later on, based on [50], [45] and [31], Huang-Li relaxed the index $\beta$ to be $\beta > \frac{4}{3}$ and studied the large time behavior of the solutions in [21]. However, all these results are concerned with the 2D periodic problems. In the present paper, we are interested in the global existence and uniqueness of classical solution to 2D Cauchy problem with large data and vacuum.

There are extensive studies on global well-posedness of the compressible Navier-Stokes equations when both the shear and bulk viscosities are positive constants. In particular, the one-dimensional theory is rather satisfactory, see [20, 37, 33, 34] and the references therein. In multi-dimensional case, the local well-posedness theory of classical solutions was established in the absence of vacuum (see [43, 25] and [49]) and the global well-posedness theory of classical solutions was obtained for initial data close to a non-vacuum steady state (see [40, 19, 11, 8] and references therein). For the large initial data which may contain vacuums, the global existence of weak solutions was obtained when $\gamma > \frac{N}{2}, (N = 2, 3)$ in general case and $\gamma > 1$ if assuming space symmetry (see [36, 13, 28]). However, the uniqueness of such weak solutions remain completely open in general. By the weak-strong uniqueness of [16], this is equivalent to the problem of global (in time) well-posedness of strong solution in the presence of vacuum. It should be noted that if the solutions contain possible vacuums, the regularity and uniqueness become difficult and subtle issues. In 1998, Xin showed [51] that if the initial density has compact support, any smooth solution in $C^1([0, T]; H^s(R^N))$ with $s \geq [N/2] + 2$ to the Cauchy problem of the CNS without heat conduction blows up in finite time for any $N \geq 1$. Then Rozanova [46] generalized the results in [51] to the case the data with highly decreasing at infinity. Very recently, Xin-Yan [52] improves the blow-up results in [51] by removing the assumptions that the initial density has compact support and the smooth solution has finite energy. On the other
hand, the short time well-posedness of either strong or classical solutions containing vacuum was studied recently by Cho-Kim [9] and Luo [39] in 3D and 2D case, respectively. A natural compatibility condition was imposed in [9] to guarantee the local well-posedness of the classical solution for the isentropic CNS with general nonnegative initial density. More recently, Huang-Li-Xin [22] proved the global well-posedness of classical solutions with small energy but large oscillations which can contain vacuums to 3D isentropic compressible Navier-Stokes equations.

The case that the viscosity coefficients depend on the density has received a lot attention recently, see [3, 4, 5, 11, 12, 17, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 41, 42, 43, 44, 45, 48, 53, 54, 55] and the references therein. When deriving by Chapman-Enskog expansions from the Boltzmann equation, the viscosity of the compressible Navier-Stokes equations depends on the temperature and thus on the density for isentropic flows (see [38]). On the other hand, in the geophysical flow, the viscous Saint-Venant system for the shallow water corresponds exactly to a kind of compressible Navier-Stokes equations with density-dependent viscosities (see [15]). Similar to the case of constant viscosities, the well-posedness theory to the one-dimensional problem with viscosity coefficients depending on the density has been well-understood. However, the progress is very limited for multi-dimensional problems. Even the short time well-posedness of strong or classical solutions has not been established in the presence of vacuum. Also, the global existence of weak solutions to the compressible Navier-Stokes equations with density-dependent viscosities remains open, except assuming some space symmetry [17]. One can refer to [5], [18], and references therein for recent developments along this line. In this paper, we are concerned with the global well-posedness of the classical solution to the 2D Cauchy problem (1.1)-(1.4) with general data permitting vacuum. Compared with [50] and references therein. When deriving by Chapman-Enskog expansions from the Boltzmann equation, the viscosity of the compressible Navier-Stokes equations depends on the temperature and thus on the density for isentropic flows (see [38]). On the other hand, in the geophysical flow, the viscous Saint-Venant system for the shallow water corresponds exactly to a kind of compressible Navier-Stokes equations with density-dependent viscosities (see [15]). Similar to the case of constant viscosities, the well-posedness theory to the one-dimensional problem with viscosity coefficients depending on the density has been well-understood. However, the progress is very limited for multi-dimensional problems. Even the short time well-posedness of strong or classical solutions has not been established in the presence of vacuum. Also, the global existence of weak solutions to the compressible Navier-Stokes equations with density-dependent viscosities remains open, except assuming some space symmetry [17]. One can refer to [5], [18], and references therein for recent developments along this line.

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the upper bound of the density, and then get our main results.

It is also interesting to obtain various weighted estimates of the density and velocity itself in $L^p(1 < p < \infty)$ spaces. These self-contained estimates reflect the fact that the weighted density and weighted velocity propagate along with the flow. Moreover, the weighted estimates will provide an appropriate approach to deal with the two-dimensional Cauchy problem of other fluid models having similar structure. As an example, it is possible that the methods here can be applied to 2D Cauchy problem of MHD systems as in [2]. Very recently, we just learned that Huang-Li [24] studied the Cauchy problem (1.1)-(1.4) independently and obtained the global well-posedness of strong and classical solution in quite different weighted spaces and by using different approaches, in which the index $\beta > \frac{4}{3}$.

The main results of the present paper can be stated in the following.

**Theorem 1.1** Suppose that the initial values $(\rho_0, u_0)(x)$ satisfy
\[
0 \leq (\rho_0(x), P(\rho_0)(x)) \in W^{2,q}(\mathbb{R}^2) \times W^{2,q}(\mathbb{R}^2), \quad u_0(x) \in D^1 \cap D^2(\mathbb{R}^2),
\]
\[
\rho_0(1 + |x|^{\alpha_1}) \in L^1(\mathbb{R}^2), \quad \sqrt{\rho_0u_0}(1 + |x|^{\frac{\alpha_1}{2}}) \in L^2(\mathbb{R}^2), \quad \nabla u_0|x|^{\frac{\alpha_1}{2}} \in L^2(\mathbb{R}^2),
\]
for some $q > 2$ and the weights $0 < \alpha < 2\sqrt{2} - 1$, $\alpha < \alpha_1$, and the compatibility condition
\[
\mathcal{L}_{\rho_0}u_0 - \nabla P(\rho_0) = \sqrt{\rho_0}g(x)
\]
with some $g$ satisfying $g(1 + |x|^{\frac{\alpha_1}{2}}) \in L^2(\mathbb{R}^2)$. If one of the following restrictions holds:
1) $1 < \alpha < 2\sqrt{2} - 1$, $\beta > 3$, $\gamma > 1$,
2) $0 < \alpha \leq 1$, $\beta > 3$, $1 < \gamma \leq 2\beta$,
then there exists a unique global classical solution $(\rho, u)(t, x)$ to the Cauchy problem (1.1)-(1.4) with
\[
0 \leq \rho \leq C, \quad (\rho, P(\rho))(t, x) \in C([0, T]; W^{2,q}(\mathbb{R}^2)), \quad \rho(1 + |x|^{\alpha_1}) \in C([0, T]; L^1(\mathbb{R}^2)),
\]
\[
\sqrt{\rho_0u}(1 + |x|^{\frac{\alpha_1}{2}}), \sqrt{\rho_0u}(1 + |x|^{\frac{\alpha_1}{2}}), \nabla u|x|^{\frac{\alpha_1}{2}} \in C([0, T]; L^2(\mathbb{R}^2)),
\]
\[
\rho \in C([0, T]; L^2(\mathbb{R}^2) \cap L^2(0, T; L^2(\mathbb{R}^2)), \quad \sqrt{tu} \in L^\infty(0, T; D^3(\mathbb{R}^2)),
\]
\[
tu \in L^\infty(0, T; D^3(\mathbb{R}^2)), \quad ut \in L^2(0, T; L^2(\mathbb{R}^2) \cap D^1(\mathbb{R}^2)),
\]
\[
\sqrt{tu} \in L^2(0, T; D^2(\mathbb{R}^2)) \cap L^\infty(0, T; L^2(\mathbb{R}^2)), \quad tu \in L^\infty(0, T; D^2(\mathbb{R}^2)),
\]
\[
\sqrt{t^2u} \in L^2(0, T; L^2(\mathbb{R}^2)), \quad t\sqrt{tu} \in L^\infty(0, T; L^2(\mathbb{R}^2)), \quad t\nabla u \in L^2(0, T; L^2(\mathbb{R}^2)),
\]
where $\dot{u}$ is the material derivative of $u$ defined in (2.3).

**Remark 1.1** From the regularity of the solution $(\rho, u)(t, x)$, it can be shown that $(\rho, u)$ is a classical solution of the system (1.1) in $[0, T] \times \mathbb{R}^2$ (see the details in Section 5).

**Remark 1.2** If the initial data contains vacuum, then the compatibility condition (1.6) is necessary for the existence of the classical solution, just as the case of constant viscosity coefficients in [3].
If the initial values are more regular, based on Theorem 1.1, we can prove

**Theorem 1.2** Under assumptions of (1.5)-(1.8), assume further that

\[ 0 \leq (\rho_0(x), P(\rho_0(x))) \in H^3(\mathbb{R}^2) \times H^3(\mathbb{R}^2), \quad u_0(x) \in D^1 \cap D^3(\mathbb{R}^2) \] (1.10)

and the compatibility condition (1.6), then there exists a unique global classical solution \((\rho, u)(t, x)\) to the Cauchy problem (1.1)-(1.4) satisfying all the properties listed in (1.9) in Theorem 1.1 with any \(2 < q < \infty\). Furthermore, it holds that

\[
\begin{align*}
& \quad u \in L^2(0, T; D^4(\mathbb{R}^2)), \quad (\rho, P(\rho)) \in C([0, T]; H^3(\mathbb{R}^2)), \\
& \quad \rho u \in C([0, T]; D^1 \cap D^3(\mathbb{R}^2)), \quad \sqrt{\rho} \nabla^3 u \in C([0, T]; L^2(\mathbb{R}^2)).
\end{align*}
\] (1.11)

**Remark 1.3** In fact, the conditions on the initial velocity \(u_0\) in (1.10) can be weakened to \(u_0 \in D^1 \cap D^2(\mathbb{R}^2)\) and \(\sqrt{\rho} \nabla^3 u_0 \in L^2(\mathbb{R}^2)\) to get (1.11).

**Remark 1.4** In Theorem 1.2, it is not clear whether or not \(u \in C([0, T]; D^3(\mathbb{R}^2))\) even though one has \(\rho u \in C([0, T]; D^3(\mathbb{R}^2))\).

The plan of the paper is as follows. In Section 2, we present the reformulations of the system, some elementary facts and inequalities. In Sections 3-4, we derive a priori estimates which are needed to extend the local solution to global one. Finally, in Section 5, we prove our main results.

**Notations.** Throughout this paper, positive generic constants are denoted by \(c\) and \(C\), which are independent of \(m\) and \(t \in [0, T]\), without confusion, and \(C(\cdot)\) stands for some generic constant(s) depending only on the quantity listed in the parenthesis. For functional spaces, \(L^p(\mathbb{R}^2), 1 \leq p \leq \infty\), denote the usual Lebesgue spaces on \(\mathbb{R}^2\) and \(||\cdot||_p\) denotes its \(L^p\) norm. \(W^{k,p}(\mathbb{R}^2)\) denotes the standard \(k\)-th order Sobolev space and \(H^k(\mathbb{R}^2) := W^{k,2}(\mathbb{R}^2)\). For \(1 < p < \infty\), the homogenous Sobolev space \(D^{k,p}(\mathbb{R}^2)\) is defined by \(D^{k,p}(\mathbb{R}^2) = \{u \in L^1_{\text{loc}}(\mathbb{R}^2) \, | \, \nabla^k u \in L^p(\mathbb{R}^2)\} \) with \(\|u\|_{D^{k,p}} := \|\nabla^k u\|_p\) and \(D^k(\mathbb{R}^2) := D^{k,2}(\mathbb{R}^2)\).

## 2 Preliminaries

As in [50], we introduce the following variables. First denote the effective viscous flux by

\[
F = (2\mu + \lambda(\rho)) \text{div} u - P(\rho),
\] (2.1)

and the vorticity by

\[
\omega = \partial_{x_1} u_2 - \partial_{x_2} u_1.
\]

Also, we define that

\[
H = \frac{1}{\rho} (\mu \omega_{x_1} + F_{x_2}), \quad L = \frac{1}{\rho} (-\mu \omega_{x_2} + F_{x_1}).
\]

Then the momentum equation (1.1) can be rewritten as

\[
\begin{aligned}
\dot{u}_1 &= u_{1t} + u \cdot \nabla u_1 = \frac{1}{\rho} (-\mu \omega_{x_2} + F_{x_1}) = L, \\
\dot{u}_2 &= u_{2t} + u \cdot \nabla u_2 = \frac{1}{\rho} (\mu \omega_{x_1} + F_{x_2}) = H,
\end{aligned}
\] (2.2)
that is,
\[ \dot{u} = (\ddot{u}_1, \ddot{u}_2)^t = (L, H)^t. \] (2.3)

Then the effective viscous flux \( F \) and the vorticity \( \omega \) solve the following system:
\[
\begin{aligned}
\omega_t + u \cdot \nabla \omega + \omega \text{div} u &= H_{x_1} - L_{x_2}, \\
\frac{F + P(\rho)}{2\mu + \lambda(\rho)}_t + u \cdot \nabla \left( \frac{F + P(\rho)}{2\mu + \lambda(\rho)} \right) + (u_{x_1})^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2 &= H_{x_2} + L_{x_1}.
\end{aligned}
\] (2.4)

Due to the continuity equation (1.1), we have
\[
\begin{aligned}
\{&\omega_t + u \cdot \nabla \omega + \omega \text{div} u = H_{x_1} - L_{x_2}, \\
&F_t + u \cdot \nabla F - \rho(2\mu + \lambda(\rho))[F(\frac{1}{2\mu + \lambda(\rho)})'] + (\frac{P(\rho)}{2\mu + \lambda(\rho)})']\text{div} u \\
&+(2\mu + \lambda(\rho))(u_{x_1})^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2 = (2\mu + \lambda(\rho))(H_{x_2} + L_{x_1}).
\end{aligned}
\] (2.5)

Furthermore, the system for \((H, L)\) can be derived as
\[
\begin{aligned}
\rho H_t + pu \cdot \nabla H - \rho H\text{div} u + u_{x_2} \cdot \nabla F + \mu u_{x_1} \cdot \nabla \omega + \mu(\omega \text{div} u)_{x_1} &= \\
-\{&(\rho(2\mu + \lambda(\rho))[F(\frac{1}{2\mu + \lambda(\rho)})'] + (\frac{P(\rho)}{2\mu + \lambda(\rho)})']\text{div} u\}_{x_2} \\
+&(2\mu + \lambda(\rho))(u_{x_1})^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2\}_{x_2} \\
= &[(2\mu + \lambda(\rho))(H_{x_2} + L_{x_1})]_{x_2} + \mu(H_{x_1} - L_{x_2})_{x_1},
\end{aligned}
\] (2.6)

In the following, we will utilize the above systems in different steps. Note that these systems are equivalent to each other for the smooth solution to the original system (1.1). We first state the local existence and uniqueness of classical solution when the initial data may contain vacuum.

**Lemma 2.1** Under assumptions of Theorem 1.1, there exists a \( T_s > 0 \) and a unique classical solution \((\rho, u)\) to the Cauchy problem (1.1)-(1.4) satisfying (1.9) with \( T \) replaced by \( T_s \).

**Lemma 2.1** can be proved in a similar way as in [9] and [39], by using the linearization method, Schauder fixed point theorem and borrowing a priori estimates in Sections 3-4 of this paper. We omit the details here.

Several elementary Lemmas are needed later. The first one is the various Gagliardo-Nirenberg inequalities.

**Lemma 2.2** (1) \( \forall h \in W^{1,m}(\mathbb{R}^2) \cap L^r(\mathbb{R}^2), \) it holds that
\[
\|h\|_q \leq C \|\nabla h\|_m^\theta \|h\|_r^{1-\theta},
\] (2.7)

where \( \theta = (\frac{1}{r} - \frac{1}{q})(\frac{1}{r} - 1 + \frac{1}{q})^{-1}, \) and if \( m < 2, \) then \( q \) is between \( r \) and \( \frac{2m}{2-m}, \) that is, \( q \in [r, \frac{2m}{2-m}] \) if \( r < \frac{2m}{2-m}, \) \( q \in [\frac{2m}{2-m}, r] \) if \( r \geq \frac{2m}{2-m}, \) if \( m = 2, \) then \( q \in [r, +\infty), \) if \( m > 2, \) then \( q \in [r, +\infty]. \)
(2) (Best constant for Gagliardo-Nirenberg inequality)
\[ \forall h \in D^m(\mathbb{R}^2) = \left\{ h \in L^{m+1}(\mathbb{R}^2) \big| \nabla h \in L^2(\mathbb{R}^2), h \in L^{2m}(\mathbb{R}^2) \right\} \text{ with } m > 1, \text{ it holds that} \]
\[ \|h\|_{2m} \leq A_m \|\nabla h\|_{L_2}^\theta \|h\|_{m+1}^{1-\theta}, \quad (2.8) \]
where \( \theta = \frac{1}{2} - \frac{1}{2m} \) and
\[ A_m = \left( \frac{m+1}{2\pi} \right)^\frac{\theta}{2} \left( \frac{2}{m+1} \right)^\frac{1}{2m} \leq C m^{\frac{1}{2}} \]
with the positive constant \( C \) independent of \( m \), and \( A_m \) is the optimal constant.

(3) \( \forall h \in W^{1,m}(\mathbb{R}^2) \) with \( 1 \leq m < 2 \), then
\[ \|h\|_{2m} \leq C(2 - m)^{-\frac{1}{2}} \|\nabla h\|_m, \quad (2.9) \]
where the positive constant \( C \) is independent of \( m \).

(4) \( \forall h \in W^{1,\frac{2m}{m+\eta}}(\mathbb{R}^2) \) with \( m \geq 2 \) and \( 0 < \eta \leq 1 \), we have
\[ \|h\|_{2m} \leq C m^{\frac{1}{2}} \|h\|_{L_2}^{\frac{s}{2(1-\varepsilon)}} \|\nabla h\|_{\frac{m+\eta}{2(1-\varepsilon)}}, \quad (2.10) \]
where \( \varepsilon \in [0, \frac{1}{2}] \), \( s = \frac{(1-\varepsilon)(1-\eta)}{m-\eta(1-\varepsilon)} \) and the positive constant \( C \) is independent of \( m \).

**Proof:** The proof of (1) can be found in [44], while the proof of (2) can be found in [10]. The proof of (3) can be found in [13] and the proof of (4) is a direct consequence of (2) and the interpolation inequality.

The following Lemma is about the Caffarelli-Kohn-Nirenberg inequalities, which are crucial to the weighted estimates in 2D Cauchy problem.

**Lemma 2.3**

(1) \( \forall h \in C_0^\infty(\mathbb{R}^2) \), it holds that
\[ \| |x|^\alpha h\|_r \leq C \| |x|^\alpha \nabla h\|_{L_p}^\theta \| |x|^\beta h\|_q^{1-\theta} \quad (2.11) \]
where \( 1 \leq p, q < \infty, 0 < r < \infty, 0 \leq \theta \leq 1, \frac{1}{p} + \frac{\kappa}{2} > 0, \frac{1}{q} + \frac{\kappa}{2} > 0, 0 \leq \theta \leq 1 \) and satisfying
\[ \frac{1}{r} + \frac{\kappa}{2} = \theta \left( \frac{1}{p} + \frac{\alpha-1}{2} \right) + (1-\theta) \left( \frac{1}{q} + \frac{\beta}{2} \right), \quad (2.12) \]
and
\[ \kappa = \theta \sigma + (1-\theta) \beta, \]
with \( 0 \leq \alpha - \sigma \) if \( \theta > 0 \) and \( 0 \leq \alpha - \sigma \leq 1 \) if \( \theta > 0 \) and \( \frac{1}{p} + \frac{\alpha-1}{2} = \frac{1}{r} + \frac{\kappa}{2} \).

(2) (Best constant for Caffarelli-Kohn-Nirenberg inequality)
\[ \forall h \in C_0^\infty(\mathbb{R}^2), \text{ it holds that} \]
\[ \| |x|^b h\|_p \leq C_{a,b} \| |x|^a \nabla h\|_2 \quad (2.13) \]
where \( a > 0, a-1 \leq b \leq a \) and \( p = \frac{2}{a-b} \). If \( b = a-1 \), then \( p = 2 \) and the best constant in the inequality (2.13) is
\[ C_{a,b} = C_{a,a-1} = a. \]
Proof: The proof of (1) can be found in [6] while the proof of (2) can be found in [7]. □

Lemma 2.4  (1) It holds that for $1 < p < \infty$ and $u \in C^\infty_0(\mathbb{R}^2)$,

$$
\|\nabla u\|_p \leq C(\|\text{div} u\|_p + \|\omega\|_p); \quad (2.14)
$$

(2) It holds that for $1 < p < \infty$, $-2 < \alpha < 2(p-1)$ and $u \in C^\infty_0(\mathbb{R}^2)$,

$$
\|\left|x\right|^\alpha \nabla u\|_p \leq C\left(\|\left|x\right|^\alpha \text{div} u\|_p + \|\left|x\right|^\alpha \omega\|_p\right). \quad (2.15)
$$

Proof: (1) Since $\Delta u = \nabla(\text{div} u) - \nabla \times \nabla \times u = \nabla(\text{div} u) - \nabla \times \omega$,

where $\nabla \times$ denotes the 3-dimensional curl operator, and

$$
\nabla \times \omega = (\partial_{x_2} \omega, -\partial_{x_1} \omega, 0)
$$
is regarded as the 2-dimensional vector $(\partial_{x_2} \omega, -\partial_{x_1} \omega)^t$, then it holds that

$$
\nabla u = \nabla \Delta^{-1} \nabla(\text{div} u) - \nabla \Delta^{-1} \nabla \times \omega := T_1(\text{div} u) + T_2 \omega, \quad (2.16)
$$

where $T_1 = \nabla \Delta^{-1} \nabla$ and $T_2 = -\nabla \Delta^{-1} \nabla \times$ both are the singular integral operators of the convolution type which are bounded in $L^p(\mathbb{R}^2)$. Thus Lemma 2.4 (1) is proved.

(2) If $-2 < \alpha < 2(p-1)$, then $\left|x\right|^{\alpha}$ is in the class $A_p$ (cf. p. 194 in [47]), that is,

$$
\frac{1}{|B|} \int_B \left|x\right|^{\alpha} dx \cdot \left[ \frac{1}{|B|} \int_B \left(\left|x\right|^{\alpha}\right)^{\frac{p'}{p}} dx \right]^{\frac{p}{p'}} < \infty,
$$

for all balls $B$ in $\mathbb{R}^2$, where $p'$ is the dual to $p$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Then by the Corollary in p. 205 of [47], there exist positive constants $C_1, C_2$ such that for $u \in C^\infty_0(\mathbb{R}^2)$,

$$
\int_{\mathbb{R}^2} |T_1(\text{div} u)|^p |\left|x\right|^{\alpha} dx \leq C_1 \int_{\mathbb{R}^2} |\text{div} u|^p |\left|x\right|^{\alpha} dx,
$$

and

$$
\int_{\mathbb{R}^2} |T_2 \omega|^p |\left|x\right|^{\alpha} dx \leq C_2 \int_{\mathbb{R}^2} |\omega|^p |\left|x\right|^{\alpha} dx.
$$

Therefore, it follows from (2.16) that

$$
\left|x\right|^\alpha \nabla u\|_p \leq \int_{\mathbb{R}^2} |\nabla u|^p |\left|x\right|^{\alpha} dx \leq C_p \left[ \int_{\mathbb{R}^2} |T_1(\text{div} u)|^p |\left|x\right|^{\alpha} dx + \int_{\mathbb{R}^2} |T_2 \omega|^p |\left|x\right|^{\alpha} dx \right] \leq C_p \left[ \int_{\mathbb{R}^2} |\text{div} u|^p |\left|x\right|^{\alpha} dx + \int_{\mathbb{R}^2} |\omega|^p |\left|x\right|^{\alpha} dx \right] = C_p \left[ \left|x\right|^\alpha \text{div} u\|_p^p + \left|x\right|^\alpha \omega\|_p^p \right].
$$

Thus the proof of Lemma 2.4 (2) is completed. □
3 A priori estimates (I)

In this section, we obtain various a priori estimates and weighted estimates on the classical solution \((\rho, u)\) on the time interval \([0, T]\). Denote

\[
M = \| (\rho_0, P(\rho_0)) \|_{W^{2,q}} + \| \rho_0 (1 + |x|^{\alpha_1}) \|_1 + \| u_0 \|_{D^{1,q} \cap \Omega_2} + \| x^{\frac{\gamma}{2}} (\sqrt{\rho_0} u_0, \nabla u_0) \|_2 + \| g(1 + |x|^{\beta_1}) \|_2.
\]

(3.1)

Step 1. Elementary energy estimates:

**Lemma 3.1** There exists a positive constant \(C\) only depending on \(M\), such that

\[
\sup_{t \in [0, T]} \left( \| \sqrt{\rho} u \|_2^2 + \| \rho \|_{\gamma}^\gamma + \| \rho \|_1 \right) + \int_0^T \left( \| \nabla u \|_2^2 + \| \omega \|_2^2 + \| (2\mu + \lambda(\rho))^{\frac{1}{2}} \text{div} u \|_2^2 \right) dt \leq C(M).
\]

**Proof:** Multiplying the equation (1.1) by \(u\) and the continuity equation (1.1) by \(\gamma \rho \gamma \rho \gamma - 1\), then summing the resulting equations, we have

\[
\left( \rho \frac{|u|^2}{2} + \frac{\rho^\gamma}{\gamma - 1} \right)_t + \text{div}(\rho \frac{|u|^2}{2} + \frac{\rho^\gamma u}{\gamma - 1}) = \text{div} \left[ \mu \nabla |u|^2 + (\mu + \lambda(\rho)) (\text{div} u)^2 \right] - \mu |\nabla u|^2 - (\mu + \lambda(\rho)) (\text{div} u)^2.
\]

(3.2)

Integrating the above equality over \([0, t] \times \mathbb{R}^2\) with respect to \(t\) and \(x\) yields that

\[
\int_0^t \left( \frac{1}{2} \rho |u|^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) dt + \int_0^t \int \left[ \mu |\nabla u|^2 + (\mu + \lambda(\rho)) (\text{div} u)^2 \right] dx dt = \int_0^t \left( \frac{1}{2} \rho_0 |u_0|^2 + \frac{1}{\gamma - 1} \rho_0^\gamma \right) dx dt \leq C.
\]

(3.3)

Note that

\[
\int |\mu |\nabla u|^2 + (\mu + \lambda(\rho)) (\text{div} u)^2 \right| dx = \int \left[ \mu \omega^2 + (2\mu + \lambda(\rho)) (\text{div} u)^2 \right] dx.
\]

(3.4)

Integrating the continuity equation (1.1) with respect to \(t, x\) over \([0, t] \times \mathbb{R}^2\) yields that

\[
\int \rho(t, x) dx = \int \rho_0(x) dx.
\]

Thus the proof of Lemma 3.1 is completed. \(\square\)

Step 2. Weighted energy estimates:

The following weighted energy estimates is fundamental and crucial in our paper.

**Lemma 3.2** If one of the following restrictions holds:

1) \(1 < \alpha < 2\sqrt{2} - 1, \beta > 0, \gamma > 1\),

2) \(0 < \alpha \leq 1, \beta > \frac{1}{2}, 1 < \gamma \leq 2\beta\),

(3.5)

(3.6)
then it holds that for sufficiently large $m > 1$ and $\forall t \in [0,T],$

$$
\int_{\mathbb{R}^2} |x|^\alpha (\rho |u|^2 + \rho^\gamma)(t, x) dx + \int_0^t \left[ ||x|^\frac{\gamma}{2} \nabla u^2 ||_2^2 (s) + \| ||x|^\frac{\gamma}{2} \text{div} u \|_2^2 (s) + \| ||x|^\frac{\gamma}{2} \sqrt{\lambda(\rho)} \text{div} u \|_2^2 (s) \right] ds

\leq C_\alpha(M) \left[ 1 + \int_0^t (\| \rho \|_{2m+1}^2 (s) + 1)(\| \nabla u \|_2^2 (s) + 1) ds \right],
$$

(3.7)

where the positive constant $C_\alpha(M)$ depend on $\alpha$ and $M$ but is independent of $m.$

**Proof:** Multiplying the equality (3.2) by $|x|^\alpha$ yields that

$$
\left[ |x|^\alpha (\rho |u|^2 + \rho^\gamma) \right]_t + \left[ \mu |\nabla u|^2 + (\mu + \lambda(\rho)) (\text{div} u)^2 \right] |x|^\alpha 

= - \text{div} \left[ |x|^\alpha (\rho u |u|^2 + \frac{\gamma \rho^\gamma u}{\gamma - 1}) \right] + \text{div} \left[ (\mu |\nabla u|^2 + (\mu + \lambda(\rho)) (\text{div} u)) |x|^\alpha \right] 

+ (\rho \frac{|u|^2}{2} + \frac{\gamma \rho^\gamma}{\gamma - 1}) u \cdot \nabla (|x|^\alpha) - \left[ \mu \nabla \frac{|u|^2}{2} + (\mu + \lambda(\rho)) (\text{div} u) u \right] \cdot \nabla (|x|^\alpha).
$$

(3.8)

Integrating the above equation (3.8) with respect to $x$ over $\mathbb{R}^2$ yields that

$$
\frac{d}{dt} \int |x|^\alpha (\rho \frac{|u|^2}{2} + \frac{\rho^\gamma}{\gamma - 1}) (t, x) dx + \mu \| |x|^\frac{\gamma}{2} \nabla u \|_2^2 (t) + \mu \| ||x|^\frac{\gamma}{2} \text{div} u \|_2^2 (t) + \| ||x|^\frac{\gamma}{2} \sqrt{\lambda(\rho)} \text{div} u \|_2^2 (t)

= \int (\rho \frac{|u|^2}{2} + \frac{\gamma \rho^\gamma}{\gamma - 1}) u \cdot \nabla (|x|^\alpha) dx - \int \left[ \mu \nabla \frac{|u|^2}{2} + (\mu + \lambda(\rho)) (\text{div} u) u \right] \cdot \nabla (|x|^\alpha) dx.
$$

(3.9)

Note that the conservation terms in (3.8) is vanished, which can be proved rigourously by multiplying a smooth cutting-off function $\phi_R(x) = \phi(\frac{x}{R})$ on both sides of the equation (3.8), where

$$
\phi(x) = \phi(|x|) = \begin{cases} 
1, & |x| \leq 1, \\
0, & |x| \geq 2,
\end{cases}
$$

satisfying $|D\phi(x)| \leq 2$ and then taking the limit $R \to +\infty.$

Now we estimate the terms on the right hand side of (3.9). First, it holds that

$$
\left| \int \rho \frac{|u|^2}{2} u \cdot \nabla (|x|^\alpha) dx \right| = \alpha \int \rho \frac{|u|^2}{2} |x|^\alpha - 2 u \cdot x dx

\leq \frac{\alpha}{2} \int \rho |u|^3 |x|^\alpha - 1 dx \leq \frac{\alpha}{2} \| \sqrt{\rho} u \|_2 \| \sqrt{\rho} |x|^\alpha - 1 \|_2 \| u \|_{q_1}

\leq C \| \sqrt{\rho} \|_{p_1} \| |x|^\frac{\alpha - 1}{2} u \|_{2q_1} \leq C \| \sqrt{\rho} \|_{p_1} \| \nabla u \|_{2q_1}^2 \| |x|^\alpha u \|_{q_1}^{2(1 - \theta_1)}

\leq C \| \rho \|_{p_1}^2 \| \nabla u \|_{2q_1}^{2\theta_1} \| |x|^\frac{\gamma}{2} \nabla u \|_2^{2(1 - \theta_1)} \leq \sigma \| \rho \|_{p_1} \frac{\gamma}{2} \| \nabla u \|_2^2 + C_\sigma \| \rho \|_{p_1} \frac{\gamma}{2} \| \nabla u \|_2^2,
$$

(3.10)

where and in the sequel $\sigma > 0$ is a small constant to be determined, $C_\sigma$ is a positive constant depending on $\sigma.$ By the Hölder inequality and the Caffarelli-Kohn-Nirenberg inequality in Lemma 2.3 (1), the positive constants $p_1 > 2, q_1 > 2, q_1^* > 1, \beta_1 > 0, \theta_1 \in (0, 1)$ in the above inequality (3.10) satisfying

$$
\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2},

\frac{1}{2q_1} + \frac{\alpha - 1}{2} = \theta_1 \left( \frac{1}{2} + \frac{0 - 1}{2} \right) + (1 - \theta_1) \left( \frac{1}{q_1} + \frac{\beta_1}{2} \right),
$$
\[
\frac{1}{q_1} + \frac{\beta_1}{2} = \frac{1}{2} + \frac{q - 1}{2}.
\]

The combination of the above three equalities yields that
\[
p_1 = \frac{2}{\alpha \theta_1}.
\]

(3.11)

Note that one should choose the parameters \(\alpha > 0\) and \(0 < \theta_1 < 1\) such that \(p_1 > 2\) in (3.11). Now choose \(m > 1\) sufficiently large such that \(2m\beta + 1 > \frac{q_1}{2}\). Therefore, by the interpolation inequality, it holds that
\[
\|\rho\|_\frac{2q_1}{2} \leq \|\rho\|_1^{\frac{1}{q_1}} \|\rho\|_\frac{2q_1}{2(2m\beta + 1)},
\]

(3.12)

with \(a_1 \in (0, 1)\) satisfying
\[
a_1 + \frac{1 - a_1}{2m\beta + 1} = \frac{2}{p_1} = \theta_1 \alpha,
\]

which implies that
\[
a_1 = \theta_1 \alpha \left(1 + \frac{1}{2m\beta}\right) + \frac{1}{2m\beta}.
\]

To close the estimates in Lemma 3.5, the following restriction should be imposed to (3.12)
\[
\frac{1 - a_1}{2\theta_1} \leq \beta, \quad \text{i.e.} \quad \theta_1 \geq \frac{1 - a_1}{2\beta},
\]

For definiteness, we can choose \(\theta_1 = \frac{1}{2}\) and then \(a_1 = \frac{2}{2} - \frac{2 - \alpha}{4m\beta} \in (0, 1)\). Obviously, the restriction \(\theta_1 = \frac{1}{2} \geq \frac{1}{2}\left(1 - a_1\right) = \frac{1}{2}\left(1 - \frac{2}{2} + \frac{2 - \alpha}{4m\beta}\right)\) is satisfied if \(m \gg 1\). Then it follows from (3.12) that
\[
\|\rho\|_\frac{2q_1}{2} \leq C(\|\rho\|_\frac{2q_1}{2(2m\beta + 1)} + 1),
\]

(3.13)

with the positive constant \(C\) independent of \(m\).

Then it holds that
\[
\left| \int \frac{\gamma \rho^{\frac{\gamma}{\gamma - 1}} u \cdot \nabla (|x|^\alpha) dx \right| = \frac{\gamma \alpha}{\gamma - 1} \int \rho^\gamma |x|^\alpha u \cdot dx \leq \frac{\gamma \alpha}{\gamma - 1} \|\rho\|_{p_2} \|\nabla u\|_{p_2}^\beta \|\nabla u\|_{q_2}^{1 - \theta_2} \leq C \|\rho\|_{p_2} \|\nabla u\|_{p_2} \|\nabla u\|_{q_2} \]
\[
\leq \sigma \|x\|^{\frac{\gamma}{\gamma - 1}} \|\nabla u\|_{q_2}^2 + C \|\rho\|_{p_2} \|\nabla u\|_{p_2} \|\nabla u\|_{q_2} \]
\[
\leq \sigma \|x\|^{\frac{\gamma}{\gamma - 1}} \|\nabla u\|_{q_2}^2 + C \|\rho\|_{p_2} \|\nabla u\|_{p_2} \|\nabla u\|_{q_2} \]
\[
\leq \sigma \|x\|^{\frac{\gamma}{\gamma - 1}} \|\nabla u\|_{q_2}^2 + C \|\rho\|_{p_2} \|\nabla u\|_{p_2} \|\nabla u\|_{q_2} \]
\]

By the Hölder inequality and the Caffarelli-Kohn-Nirenberg inequality in Lemma 2.3 (1), the positive constants \(p_2 > 1, q_2 > 1, \bar{q}_2 > 1, \beta_2 > 0, \theta_2 \in (0, 1)\) in the above inequality (3.14) satisfying
\[
\frac{1}{p_2} + \frac{1}{q_2} = 1,
\]

\[
\frac{1}{q_2} + \frac{\alpha - 1}{2} = \theta_2 (\frac{1}{2} + \frac{0 - 1}{2}) + (1 - \theta_2) (\frac{1}{q_2} + \frac{\beta_2}{2}),
\]

and
\[
\frac{1}{q_2} + \frac{\beta_2}{2} = \frac{1}{2} + \frac{\alpha - 1}{2}.
\]
The combination of the above three equalities yields that
\[ p_2 = \frac{4}{2 + \alpha(1 + \theta_2)}. \] (3.15)

Note that one should choose the parameters \( \alpha > 0 \) and \( 0 < \theta_2 < 1 \) such that \( p_2 > 1 \) in (3.15). Now choose \( m > 1 \) sufficiently large such that \( 2m\beta + 1 > p_2\gamma \). Therefore, by the interpolation inequality, it holds that
\[ \|\rho\|_{p_2\gamma}^{2\gamma} \leq \|\rho\|_{\gamma}^{2\gamma} \|\rho\|_{2m\beta+1}^{2\gamma(1-a_2)}, \] (3.16)
with \( a_2 \in (0, 1) \) satisfying
\[ \frac{a_2}{\gamma} + \frac{1-a_2}{2m\beta+1} = \frac{1}{p_2\gamma} = \frac{2 + \alpha(1 + \theta_2)}{4\gamma}, \]
which implies that
\[ a_2 \rightarrow \frac{2 + \alpha(1 + \theta_2)}{4}, \text{ as } m \rightarrow +\infty. \]

The following restriction should be imposed to (3.16)
\[ \frac{2\gamma}{1 + \theta_2}(1-a_2) \leq \beta, \text{ i.e. } 1 + \theta_2 \geq \frac{2\gamma}{\beta}(1-a_2). \] (3.17)

For \( m \gg 1 \) large enough, it is sufficient to have the following restriction
\[ 1 + \theta_2 > \frac{\gamma}{\beta}(1 - \frac{\alpha(1 + \theta_2)}{2}), \]
That is
\[ (1 + \theta_2)(\frac{\beta}{\gamma} + \frac{\alpha}{2}) > 1. \] (3.18)

Consequently, if
\[ 1 < \alpha \leq 2, \quad \beta > 0, \quad \gamma > 1, \] (3.19)
we can choose \( 0 \leq \frac{2}{\alpha} - 1 < \theta_2 < 1 \) such that (3.18) and hence (3.17) hold true for \( m \gg 1 \). If
\[ 0 < \alpha \leq 1, \quad \beta > \frac{1}{2}, \quad 1 < \gamma \leq 2\beta, \] (3.20)
we can choose \( \max\{\frac{\gamma}{\beta} - 1, 0\} < \theta_2 < 1 \) such that (3.18) and hence (3.17) hold true for large \( m \gg 1 \).

Then it follows from (3.16) that
\[ \|\rho\|_{p_2\gamma}^{2\gamma} \leq C(||\rho||_{2m\beta+1}^{\beta} + 1) \] (3.21)
with the positive constant \( C \) independent of \( m \).

Now one can compute that
\[ | - \int \mu \nabla \frac{|u|^2}{2} \cdot \nabla (|x|^\alpha) dx | = \mu \alpha \int u \cdot \nabla u \cdot x |x|^{\alpha-2} dx | \leq \mu \alpha \||x|^\frac{\alpha}{2} \nabla u\|_2 \||x|^{\alpha-2} u\|_2 \leq \frac{\mu \alpha^2}{2} \||x|^\frac{\alpha}{2} \nabla u\|_2^2, \] (3.22)
where in the last inequality one has used the best constant $\frac{\alpha}{2}$ for the Caffarelli-Kohn-Nirenberg inequality in Lemma 2.3 (2). Similarly, it holds that
\[
| - \int \mu(\text{div} u) \cdot \nabla(|x|^\alpha) dx | = \mu \alpha | \int (\text{div} u)|x|^{\alpha - 2} u \cdot x dx |
\leq \mu \alpha \| |x|^\frac{\alpha}{2} \text{div} u \|_2 \| |x|^\frac{\alpha}{2} - 1 u \|_2 \leq \frac{\mu \alpha^2}{2} \| |x|^\frac{\alpha}{2} \text{div} u \|_2 \| |x|^\frac{\alpha}{2} \nabla u \|_2.
\] (3.23)

Then it follows that
\[
| - \int \lambda(\rho)(\text{div} u) u \cdot \nabla(|x|^\alpha) dx | = \alpha | \int \lambda(\rho)(\text{div} u)|x|^{\alpha - 2} u \cdot x dx |
\leq \alpha \| \sqrt{\lambda(\rho)}|x|^\frac{\alpha}{2} \text{div} u \|_2 \| \sqrt{\lambda(\rho)}||x|^\frac{\alpha}{2} - 1 u \|_q_3
\leq C \| \sqrt{\lambda(\rho)}|x|^\frac{\alpha}{2} \text{div} u \|_2 \| \rho^\frac{\beta}{\beta_3} \|_{\beta_3} \| \nabla u \|^\theta_3 \| |x|^\theta_3 u \|^{1-\theta_3}_3
\leq C \| \sqrt{\lambda(\rho)}|x|^\frac{\alpha}{2} \text{div} u \|_2 \| \rho^\frac{\beta}{\beta_3} \|_{\beta_3} \| \nabla u \|^\theta_3 \| |x|^\theta_3 \nabla u \|^{1-\theta_3}_2
\leq \sigma \| \sqrt{\lambda(\rho)}|x|^\frac{\alpha}{2} \text{div} u \|_2^2 + C_\sigma \| \rho^\frac{\beta}{\beta_3} \|_{\beta_3} \| \nabla u \|^{2\theta_3}_2 \| |x|^\frac{\alpha}{2} \nabla u \|^{2(1-\theta_3)}_2
\leq \sigma \| \sqrt{\lambda(\rho)}|x|^\frac{\alpha}{2} \text{div} u \|_2^2 + \| |x|^\frac{\alpha}{2} \nabla u \|_2^2 + C_\sigma \| \rho^\frac{\beta}{\beta_3} \|_{\beta_3} \| \nabla u \|_2^2,
\] (3.24)

By the Hölder inequality and the Caffarelli-Kohn-Nirenberg inequality in Lemma 2.3 (1), the positive constants $p_3 > 2, q_3 > 2, \bar{q}_3 > 1, \beta_3 > 0, \theta_3 \in (0,1)$ in the above inequality (3.24) satisfying
\[
\frac{1}{p_3} + \frac{1}{q_3} = \frac{1}{2},
\]
\[
\frac{1}{q_3} + \frac{\alpha}{2} - 1 = \theta_3 (\frac{1}{2} + \frac{0-1}{2}) + (1-\theta_3)(\frac{1}{q_3} + \frac{\beta_3}{2}),
\]
and
\[
\frac{1}{q_3} + \frac{\beta_3}{2} = \frac{1}{2} + \frac{\alpha}{2} - 1.
\]

The combination of the above three equalities yields that
\[
p_3 = \frac{4}{\alpha \theta_3}.
\] (3.25)

Note that one should choose the parameters $\alpha, \theta_3 \in (0,1)$ such that $p_3 > 2$ in (3.25). By the interpolation inequality, it holds that
\[
\| \rho \|_{\frac{\beta}{\beta_3}} \leq \| \rho \|_{1}^{\frac{\alpha}{p_3} a_3} \| \rho \|_{\frac{\beta}{2m \beta + 1}}^{\frac{\beta}{2m \beta + 1}(1-a_3)},
\] (3.26)

with $a_3 \in (0,1)$ satisfying
\[
\frac{a_3}{1} + \frac{1-a_3}{2m \beta + 1} = \frac{2}{p_3 \beta} = \frac{\alpha \theta_3}{2 \beta},
\]
which implies that
\[
a_3 = \frac{\alpha \theta_3}{2 \beta} + \frac{\alpha \theta_3}{2 \beta} - 1.
\]

The following restriction should be imposed to (3.26)
\[
\frac{\beta}{\theta_3} (1-a_3) \leq \beta, \text{ i.e. } \theta_3 \geq (1-a_3).
\]
For definiteness, one can choose $\theta_3 \in (0, 1)$

$$\theta_3 \geq 1 - \frac{\alpha \theta_3}{2\beta}$$

if $m$ is sufficiently large. Then it follows from (3.16) that

$$\|\rho\|_{L^3}^3 \leq C(\|\rho\|_{L^{2m+1}}^3 + 1) \tag{3.27}$$

with the positive constant $C$ independent of $m$.

Substituting (3.10), (3.13), (3.14), (3.21), (3.22), (3.23), (3.24) and (3.27) into (3.9) yields

$$\int |x|^{\alpha} \left( \rho \left[ \frac{|\nabla u|^2}{2} + \frac{\rho}{\gamma - 1} \right](t, x) + \|x\|^{\frac{\gamma}{2}} \sqrt{\lambda(\rho)} \|\text{div} u\|_2^2(t) \right)$$

$$+ \mu \left( 1 - \frac{\alpha^2}{2} \right) \|x\|^{\frac{\gamma}{2}} \|\nabla u\|_2^2 - \frac{\alpha^2}{2} \|x\|^{\frac{\gamma}{2}} \text{div} u \|_2 + \|x\|^{\frac{\gamma}{2}} \text{div} u \|_2^2 \right)$$

$$\leq \sigma \left[ \|\sqrt{\lambda(\rho)} \|^{\frac{\gamma}{2}} \|\text{div} u\|_2^2 + 3 \|x\|^{\frac{\gamma}{2}} \|\nabla u\|_2^2 \right] + C_{\sigma}(\|\rho\|_{L^{2m+1}}^3 + 1)(\|\nabla u\|_2^2 + 1).$$

The determinant of the quadratic term in the second line of (3.28) can be calculated by

$$\Delta = \alpha^4 - 4(1 - \frac{\alpha^2}{2}) = \frac{1}{4}(\alpha^4 + 8\alpha^2 - 16).$$

Therefore, if the weight $\alpha$ satisfies

$$0 < \alpha^2 < 4(\sqrt{2} - 1), \tag{3.29}$$

then the determinant $\Delta < 0$, and thus there exists a positive constant $C_\alpha$ such that

$$(1 - \frac{\alpha^2}{2}) \|x\|^{\frac{\gamma}{2}} \|\nabla u\|_2^2 - \frac{\alpha^2}{2} \|x\|^{\frac{\gamma}{2}} \|\nabla u\|_2^2 + \|x\|^{\frac{\gamma}{2}} \|\text{div} u\|_2^2 \geq C_{\alpha}^{-1} \left[ \|x\|^{\frac{\gamma}{2}} \|\nabla u\|_2^2 + \|x\|^{\frac{\gamma}{2}} \|\text{div} u\|_2^2 \right]. \tag{3.30}$$

Substituting (3.30) into (3.28), choosing $\sigma$ suitably small in (3.28) and noting that

$$\int \rho_0^\gamma |x|^{\alpha} dx \leq \|\rho_0|x|^{\alpha}\|_1 \|\rho_0^{\gamma-1}\|_\infty \leq C\|\rho_0(1 + |x|^{\alpha})\|_1 \|\rho_0|^{\gamma-1}W_{2,q}(\mathbb{R}^2) \leq C,$$

yield the estimate (3.7) in Lemma 3.2. The restrictions of $\alpha$ (3.5) and (3.6) follow from (3.19), (3.20) and (3.29).

**Step 4. Density estimates:**

Applying the operator $\text{div}$ to the momentum equation (1.1), it holds that

$$[\text{div}(\rho u)]_t + \text{div}[\text{div}(\rho u \otimes u)] = \Delta F. \tag{3.31}$$

Consider the following two elliptic problems on the whole space $\mathbb{R}^2$:

$$\Delta \xi = \text{div}(\rho u), \tag{3.32}$$

$$\Delta \eta = \text{div}[\text{div}(\rho u \otimes u)], \tag{3.33}$$

both with the boundary conditions $\xi, \eta \to 0$ as $|x| \to \infty$. By the elliptic estimates and Hölder inequality, it holds that
Lemma 3.3  
(1) $\| \nabla \xi \|_{2m} \leq C m \| \rho \|_{2m} \| u \|_{2mk}$, for any $k > 1, m \geq 1$;

(2) $\| \nabla \xi \|_{2-r} \leq C \sqrt{r} \| u \|_{2} \| \rho \|_{2-r} \leq C \| \rho \|_{2-r}$, for any $0 < r < 1$;

(3) $\| \eta \|_{2m} \leq C m \| \rho \|_{2m} \| u \|_{4mk}$, for any $k > 1, m \geq 1$;

where $C$ are positive constants independent of $m, k$ and $r$.

**Proof:** (1) By the elliptic estimates to the equation (3.32) and then using the Hölder inequality, one has for any $k > 1, m \geq 1$,

$$\| \nabla \xi \|_{2m} \leq C m \| \rho u \|_{2m} \leq C m \| \rho \|_{2m} \| u \|_{2mk}.$$

Similarly, the statements (2) and (3) can be proved.

Based on Lemmas 2.2-2.4 and Lemma 3.3, it holds that

Lemma 3.4  
(1) $\| \xi \|_{2m} \leq C m^{\frac{1}{2}} \| \nabla \xi \|_{2m} \leq C m^{\frac{1}{2}} \| \rho \|_{m+1}^{\frac{1}{2}}$, for any $m \geq 2$;

(2) $\| u \|_{2m} \leq C m^{\frac{1}{2}} \| \nabla u \|_{2}^{\frac{1}{2}} \| x |^{\frac{1}{2}} \nabla u \|_{2}^{\frac{1}{2}}$, for any $m + 1 \geq \frac{1}{4}$;

(3) $\| \nabla \xi \|_{2m} \leq C m^{\frac{3}{2}} k^{\frac{1}{2}} \| \rho \|_{2mk} \| \nabla u \|_{2}^{\frac{1}{2}} \| x |^{\frac{3}{2}} \nabla u \|_{2}^{\frac{1}{2}}$, for any $k > 1, m + 1 \geq \frac{1}{4}$;

(4) $\| \eta \|_{2m} \leq C m^{2} k \| \rho \|_{2mk} \| \nabla u \|_{2}^{\frac{2}{m+1}} \| x |^{\frac{3}{2}} \nabla u \|_{2}^{\frac{2}{m+1}}$, for any $k > 1, m + 1 \geq \frac{1}{4}$;

where $C$ are positive constants independent of $m, k$.

**Proof:** (1) By Lemma 2.3 and Lemma 3.3 (2), it holds that

$$\| \xi \|_{2m} \leq C m^{\frac{1}{2}} \| \nabla \xi \|_{2m} \leq C m^{\frac{1}{2}} \| \rho \|_{m+1}^{\frac{1}{2}} \leq C m^{\frac{1}{2}} \| \rho \|_{m}^{\frac{1}{2}},$$

where in the last inequality one has used the elementary energy estimates (3.3).

(2) If $m + 1 > \frac{1}{2}$, then by interpolation inequality and Caffarelli-Kohn-Nirenberg inequality, it holds that

$$\| u \|_{m+1} \leq \| u \|_{2m}^{\theta} \| u \|_{2m}^{1-\theta} \leq C \| u \|_{2m}^{\theta} \| x |^{\frac{1}{2}} \nabla u \|_{2}^{1-\theta}$$

(3.34)

where

$$\theta = \frac{1}{2m} - \frac{\alpha}{4}.$$

Then it follows from Lemma 2.2 (2) and (3.31) that

$$\| u \|_{2m} \leq C m^{\frac{1}{2}} \| \nabla u \|_{2}^{\frac{1}{2}} \| x |^{\frac{1}{2}} \nabla u \|_{2}^{\frac{1}{2}} \| u \|_{2m}^{\frac{1}{2}} \| u \|_{2m}^{\frac{1}{2}} \| x |^{\frac{1}{2}} \nabla u \|_{2}^{\frac{1}{2}} \| x |^{\frac{1}{2}} \nabla u \|_{2}^{\frac{1}{2}} \| u \|_{2m}^{\theta} \| u \|_{2m}^{1-\theta},$$

which implies Lemma 3.4 (2) immediately.

The assertions (3) and (4) in Lemma 3.4 are the direct consequences of Lemma 3.4 (2) and Lemma 3.3 (1), (3), respectively. Thus the proof of Lemma 3.4 is completed.

Substituting (3.32) and (3.33) into (3.31) yields that

$$\Delta (\xi + \eta - F) = 0,$$
which implies that  
\[ \xi_t + \eta - F = 0. \]

It follows from the definition (2.1) of the effective viscous flux \( F \) that 
\[ \xi_t - (2\mu + \lambda(\rho)) \text{div} u + P(\rho) + \eta = 0. \]

Then the continuity equation (1.1) yields that 
\[ \xi_t + \frac{2\mu + \lambda(\rho)}{\rho}(\rho_t + u \cdot \nabla \rho) + P(\rho) + \eta = 0. \]

Define
\[ \nu(\rho) = \int_1^\rho \frac{2\mu + \lambda(s)}{s} ds = 2\mu \ln \rho + \frac{1}{\beta}(\rho^\beta - 1). \]  

(3.35) Then we obtain a new transport equation 
\[ (\xi + \nu(\rho))_t + u \cdot \nabla (\xi + \nu(\rho)) + P(\rho) + \eta - u \cdot \nabla \xi = 0, \] 
which is crucial in the following Lemma for the higher integrability of the density function.

**Lemma 3.5** For any \( k \geq 1 \), it holds that 
\[ \sup_{t \in [0, T]} \|\rho(t, \cdot)\|_k \leq C(M) \ k^{\frac{2}{\beta - 1}}. \]  

(3.37)

**Proof:** Multiplying the equation (3.36) by \( \rho[(\xi + \nu(\rho))_+^{2m-1}] \) with \( m \) sufficiently large integer, here and in what follows, the notation \((\cdot \cdot \cdot)_+\) denotes the positive part of \((\cdot \cdot \cdot)\), one can get that 
\[ \frac{1}{2m} \frac{d}{dt} \int\rho[(\xi + \nu(\rho))_+^{2m-1}] dx + \int \rho P(\rho)[(\xi + \nu(\rho))_+^{2m-1}] dx = -\int \rho \eta[(\xi + \nu(\rho))_+^{2m-1}] dx + \int pu \cdot \nabla \xi[(\xi + \nu(\rho))_+^{2m-1}] dx. \]  

(3.38)

Denote 
\[ f(t) = \left\{ \int \rho[(\xi + \nu(\rho))_+^{2m}] dx \right\}^{\frac{1}{2m}}, \quad t \in [0, T]. \]

Now we estimate the two terms on the right hand side of (3.38). First, it holds that 
\[ | - \int \rho \eta[(\xi + \nu(\rho))_+^{2m-1}] dx | \leq \int \rho \frac{1}{2m} \left\| \eta \right\|_{2m+1} \right\| \rho[(\xi + \nu(\rho))_+^{2m-1}] \right\|_{2m} \]
\[ \leq C \|\rho\|_{2m+1} \|\eta\|_{2m+1} \|\nabla u\|_{2m+1} \int f(t)^{2m-1}. \]  

(3.39)
where in the last inequality we have taken \( k = \frac{\beta}{2m+1} \). Next, for \( \frac{1}{2m+1} + \frac{1}{p} + \frac{1}{q} = 1 \) with \( p, q \geq 1 \), one has

\[
\left| \int \rho u \cdot \nabla \xi((\xi + \nu(\rho))_{+})^{2m-1} \, dx \right| \leq \int \rho \frac{1}{2m+1} \|u\|_{2mp} \|\nabla \xi\|_{2m} \|\rho(\xi + \nu(\rho))_{+}\|_{2m}^{2m-1} \frac{2m-1}{2m} \, dx \\
\leq \|\rho\|_{2m+1} \|u\|_{2mp} \|\nabla \xi\|_{2m} \|\rho(\xi + \nu(\rho))_{+}\|_{1}^{2m-1} \\
\leq C \|\rho\|_{2m+1} \frac{1}{2m+1} (mp)^{\frac{1}{2}} \|\nabla u\|_{2}^{-\frac{2}{mp+1}} \|\xi\|_{mp}^{\frac{2}{mp+1}} \|\nabla u\|_{2}^{\frac{2}{mp+1}} f(t)^{2m-1} \\
\leq C m^{2} \|\rho\|^{\frac{1}{2m+1}} \|\nabla u\|_{2}^{-\frac{4(\beta-1)}{\alpha(2m+1)+1}} \|\nabla u\|_{2}^{\frac{4(\beta-1)}{\alpha(2m+1)+1}} f(t)^{2m-1},
\]

(3.40)

where in the third inequality one has chosen \( p = q = \frac{2m+1}{mp} \) and \( k = \frac{\beta}{2m+1} \). Substituting (3.39) and (3.40) into (3.38) yields that

\[
\frac{1}{2m} \frac{d}{dt} (f_{2m}(t)) + \int \rho P(\rho)((\xi + \nu(\rho))_{+})^{2m-1} \, dx \\
\leq C m^{2} \|\rho\|_{2m+1} \|\nabla u\|_{2}^{-\frac{4(\beta-1)}{\alpha(2m+1)+1}} \|\nabla u\|_{2}^{\frac{4(\beta-1)}{\alpha(2m+1)+1}} f(t)^{2m-1}.
\]

Then it holds that

\[
\frac{d}{dt} f(t) \leq f(0) + C m^{2} \int_{0}^{t} \|\rho\|^{\frac{1}{2m+1}} \|\nabla u\|_{2}^{-\frac{4(\beta-1)}{\alpha(2m+1)+1}} \|\nabla u\|_{2}^{\frac{4(\beta-1)}{\alpha(2m+1)+1}} d\tau.
\]

(3.41)

Integrating the above inequality over \([0, t]\) gives that

\[
f(t) \leq f(0) + C m^{2} \left[ 1 + m^{2} \int_{0}^{t} \|\rho\|^{\frac{1}{2m+1}} \|\nabla u\|_{2}^{-\frac{4(\beta-1)}{\alpha(2m+1)+1}} \|\nabla u\|_{2}^{\frac{4(\beta-1)}{\alpha(2m+1)+1}} d\tau \right],
\]

(3.42)

and

\[
\|\rho\|_{2m+1}^{\beta} \leq C \left[ 1 + f(t) + m^{2} \|\rho\|_{2m+1}^{\frac{1}{2m+1}}(t) \right] \\
\leq \frac{1}{2} \|\rho\|_{2m+1}^{\beta} + C \left[ 1 + f(t) + m^{2} \|\rho\|_{2m+1}^{\frac{1}{2m+1}} \right].
\]

(3.43)

It follows from (3.42), (3.43) and Lemma 3.2 that

\[
\|\rho\|_{2m+1}^{\beta} \leq C \left[ f(t) + m^{2} \|\rho\|_{2m+1}^{\frac{1}{2m+1}} \right] \\
\leq C \left[ m^{2} \int_{0}^{t} \|\rho\|^{\frac{1}{2m+1}} \|\nabla u\|_{2}^{-\frac{4(\beta-1)}{\alpha(2m+1)+1}} \|\nabla u\|_{2}^{\frac{4(\beta-1)}{\alpha(2m+1)+1}} d\tau \right] \\
\leq C \left[ m^{2} \int_{0}^{t} \|\nabla u\|_{2}^{\frac{4(\beta-1)}{\alpha(2m+1)+1}} \|\nabla u\|_{2}^{\frac{4(\beta-1)}{\alpha(2m+1)+1}} d\tau \right] \\
\leq C \left[ m^{2} \int_{0}^{t} \left( \|\rho\|_{2m+1}^{\frac{1}{2m+1}}(\tau) + 1 \right) \|\nabla u\|_{2}^{\frac{1}{2}}(\tau) + 1 d\tau \right] \\
+ m^{2} \int_{0}^{t} \|\rho\|_{2m+1}^{\frac{1}{2m+1}}(\tau) \|\nabla u\|_{2}^{\frac{1}{2}}(\tau) d\tau.
\]
Applying Gronwall’s inequality to the above inequality yields that

\[
\|\rho\|_{2m,\beta+1}(t) \leq C \left[ m\frac{\beta}{2-\beta} + m^2 \int_0^t \|\rho\|_{2m,\beta+1}^{(1+\frac{\beta}{2m+1})/(1-\frac{2}{2m,\beta+1})} \|\nabla u\|_2^2(\tau) d\tau \right].
\]

Denote

\[
y(t) = m^{-\frac{\beta}{2-\beta}}\|\rho\|_{2m,\beta+1}(t).
\]

Then it holds that

\[
y^2(t) \leq C \left[ m\frac{\beta(1-3\beta)}{2(2\beta-1)} + \int_0^t y(\tau)^{(1+\frac{\beta}{2m})/(1+\frac{2(\beta-1)}{2m,\beta+1})} \|\nabla u\|_2^2(\tau) d\tau \right]
\]

\[\leq C \left[ 1 + \int_0^t (y^2(\tau) + 1) \|\nabla u\|_2^2(\tau) d\tau \right].\]

So applying the Gronwall’s inequality the above inequality yields that

\[y(t) \leq C, \quad \forall t \in [0, T],\]

that is,

\[\|\rho\|_{2m,\beta+1}(t) \leq C m^{\frac{2}{2-\beta}}, \quad \forall t \in [0, T].\]

Equivalently, (3.37) holds. Thus Lemma 3.5 is proved.

\[\square\]

Step 4: First-order derivative estimates of the velocity.

**Lemma 3.6** There exists a positive constant \( C = C(M) \), such that

\[
\sup_{t \in [0, T]} \int (\mu \omega^2 + \frac{F^2}{2\mu + \lambda(\rho)}) dx + \int_0^T \int \rho(H^2 + L^2) dx dt \leq C.
\]

**Proof:** The proof of the lemma is similar to that of Lemma 3.5 in [31] besides the weighted estimates. Multiplying the equation (2.31) by \( \mu \omega \), the equation (2.32) by \( \frac{F}{2\mu + \lambda(\rho)} \), respectively, and then summing and integrating the resulted equations with respect to \( x \in \mathbb{R}^2 \), one has

\[
\frac{1}{2} \frac{d}{dt} \int (\mu \omega^2 + \frac{F^2}{2\mu + \lambda(\rho)}) dx + \int \rho(H^2 + L^2) dx
\]

\[
= \mu \int \omega^2 \text{div} u dx + \frac{1}{2} \int F^2(\text{div} u) \left[ \rho \left( \frac{P(\rho)}{2\mu + \lambda(\rho)} \right) - \frac{1}{2\mu + \lambda(\rho)} \right] dx
\]

\[+ \int F(\text{div} u) \left[ \rho \left( \frac{P(\rho)}{2\mu + \lambda(\rho)} \right) - \frac{P(\rho)}{2\mu + \lambda(\rho)} \right] dx - \int 2F(u_{1x_2}u_{2x_1} - u_{1x_1}u_{2x_2}) dx.\] (3.44)

Set

\[Z^2(t) = \int (\mu \omega^2 + \frac{F^2}{2\mu + \lambda(\rho)}) dx,\]

\[\varphi^2(t) = \int \rho(H^2 + L^2) dx = \frac{1}{\rho} \left[ (\mu \omega_{x_1} + F_{x_2})^2 + (-\mu \omega_{x_2} + F_{x_1})^2 \right].\]

Then similar to the proof of Lemma 3.5 in [29], it yields that

\[
\frac{1}{2} \frac{d}{dt} Z^2(t) + \varphi(t)^2 \leq \sigma \varphi(t)^2 + C_\sigma (Z(t)^2 + 1)^{2+\frac{\mu}{2\mu + \lambda(\rho)}} \int_0^1 \int \left[ \frac{|F|^3}{2\mu + \lambda(\rho)} + \int |F| \nabla u|^2 \right] dx dt.
\] (3.45)
Now it remains to estimate the terms \( \int \frac{|F|^3}{2\mu + \lambda(\rho)} dx \) and \( \int |F| |\nabla u|^2 dx \) on the right hand side of (3.45). By Lemma 2.4, for \( \varepsilon \in [0, \frac{1}{2}] \) and \( \eta = \varepsilon \), it holds that

\[
\|F\|_{2m} \leq C m^{\frac{1}{2}} \|\nabla F\|_{2m}^{\frac{1-s}{2m}} \|F\|_{2(1-\varepsilon)}^{s},
\]

(3.46)

where \( s = \frac{(1-\varepsilon)^2}{m - \varepsilon (1-\varepsilon)} \) and the positive constant \( C \) is independent of \( m \) and \( \varepsilon \).

Note that if \( \varepsilon \) is sufficiently small, then it holds that

\[
\|F\|_{2(1-\varepsilon)} = (2\mu + \lambda(\rho))\text{div}v - P(\rho)\|_{2(1-\varepsilon)} \\
\leq 2\mu\|\text{div}u\|_{2(1-\varepsilon)} + \|\lambda(\rho)\text{div}u\|_{2(1-\varepsilon)} + \|P(\rho)\|_{2(1-\varepsilon)} \\
\leq 2\mu\| (1 + |x|^2)\text{div}u_2 \|_{1(1-\varepsilon)} + \|\lambda(\rho)\text{div}u\|_{2(1-\varepsilon)} + C \\
\leq C \left[ \left( \| (1 + |x|^2)\text{div}u_2 \|_{1(1-\varepsilon)} + \|\lambda(\rho)\text{div}u\|_{2(1-\varepsilon)} \right) \right] + 1 \]

(3.47)

\[
\leq C \left[ \left( \| (1 + |x|^2)\text{div}u_2 \|_{1(1-\varepsilon)} + \|\lambda(\rho)\text{div}u\|_{2(1-\varepsilon)} \right) \right] + 1 \]

(3.48)

where in the third inequality we have used the fact that

\[
\| (1 + |x|^2)\|_{1(1-\varepsilon)} \leq +\infty,
\]

provided that \( \varepsilon \) is sufficiently small. By (3.46), (3.47) and setting \( \varepsilon = 2^{-m} \) with \( m \) sufficiently large, it holds that

\[
\|F\|_{2m} \leq C m^{\frac{1}{2}} \left( \frac{m + \varepsilon}{m} \right)^{1-s} \|\nabla F\|_{2m}^{\frac{1-s}{2m}} \|F\|_{2(1-\varepsilon)}^{s} \\
\leq C m^{\frac{1}{2}} \left( \frac{m + \varepsilon}{m} \right)^{1-s} \|\nabla F\|_{2m}^{\frac{1-s}{2m}} \|F\|_{2(1-\varepsilon)}^{s} \\
\leq C m^{\frac{1}{2}} \left( \frac{m + \varepsilon}{m} \right)^{1-s} \|\nabla F\|_{2m}^{\frac{1-s}{2m}} \|F\|_{2(1-\varepsilon)}^{s} \\
\leq C m^{\frac{1}{2}} \left( \frac{m + \varepsilon}{m} \right)^{1-s} \|\nabla F\|_{2m}^{\frac{1-s}{2m}} \|F\|_{2(1-\varepsilon)}^{s}
\]

(3.49)
where one has used the fact that $ms = \frac{m(1-\varepsilon)^2}{m-\varepsilon(1-\varepsilon)} \to 1$ with $\varepsilon = 2^{-m}$ as $m \to +\infty$.

Furthermore, it holds that

$$
\int |F| \|\nabla u\|^2 \, dx \leq \|F\|_{2m} \|\nabla u\|^2 \frac{2m(2m-3)}{2m^2-1} \leq C \|F\|_{2m} \left( \|\text{div} u\|^2 + \|\omega\|^2 \right)
$$

Note that

$$
\frac{F}{2\mu + \lambda(\rho)}\left( \frac{2m(2m-3)}{2m^2-1} \right) \leq \|F\|_{2m} \left( \frac{2m^2-1}{2m^2-1} \right)
$$

and from Lemma 2.2 (1), one has

$$
\|\omega\|^2 \leq C \|\omega\|^2 \left[ \|\nabla u\|^2 \frac{1}{2m^2-1} \leq CZ(t)^2 \left( \frac{1}{m(1-\varepsilon)} \right) \right]
$$

Substituting (3.48) into (3.51), then substituting the resulted (3.51) and (3.52) into (3.50) give that

$$
\sigma \leq \varphi(t)^2 + C_\sigma \left\{ \left[ |x|^2 \text{div} u^2 \right] \left( \frac{2m^2-1}{m^2+1} \right) \sigma \left( \frac{1}{m+1} \right) \right\}^2
$$

(3.53)
Substituting (3.49) and (3.53) into (3.45) and choosing \( \sigma \) sufficiently small yield that

\[
\frac{1}{2} \frac{d}{dt} (Z^2(t)) + \frac{1}{2} \varphi(t)^2 \leq C \varepsilon^{\frac{2}{\sigma-1}} (Z(t)^2 + 1)^{2+\frac{2}{\sigma-1}}
\]

\[
+ C m \left( \frac{m}{\varepsilon} \right)^{\frac{2}{\sigma-1}} \left[ \left( \|x\|^{\frac{2}{\sigma}} \text{div} u \right)^2 + 1 \right] (Z(t)^2 + 1) + (1 + Z(t)^2)^{2+\frac{1}{m} - \frac{1}{m+1} (2m+1) \varepsilon} \right]
\]

\[
+ (1 + Z(t)^2)^{2+\frac{1}{m} - \frac{1}{m+1} \varepsilon} + (1 + Z(t)^2)^{2+\frac{1}{m+1} \varepsilon} + (Z(t)^2 + 1)^{2+\frac{1}{m} - \frac{1}{m+1} \varepsilon} \right].
\]  

(3.54)

Note that \( \lim_{m \to +\infty} [2^m (1 - ms)] = 2 \), and so \( 1 - ms \sim 2\varepsilon \) as \( m \to +\infty \). Thus for \( m \) sufficiently large, one has

\[
\frac{1 - ms + (2ms - 1)\varepsilon}{m(1 + s)(1 - 2\varepsilon) - 1 + \varepsilon} \sim \frac{2\varepsilon + \varepsilon(1 - 4\varepsilon)}{m(1 - 2\varepsilon)(1 - 2\varepsilon) - 1 + \varepsilon} \leq 3\varepsilon,
\]

\[
\frac{1 - ms - \varepsilon}{m(1 - 2\varepsilon) - 1 + \varepsilon} \sim \frac{\varepsilon}{m(1 - 2\varepsilon) - 1 + \varepsilon} \leq \varepsilon,
\]

\[
\frac{1 - ms}{m(s + 1) - 2} \sim \frac{2\varepsilon}{1 - 2\varepsilon + m - 2} = \frac{2\varepsilon}{m - 1 - 2\varepsilon} \leq 2\varepsilon,
\]

and

\[
\frac{1 - ms}{m - 2} \sim \frac{2\varepsilon}{m - 2} \leq 2\varepsilon.
\]

Then (3.54) yields the following inequality for suitably large \( m \),

\[
\frac{1}{2} \frac{d}{dt} (Z^2(t)) + \frac{1}{2} \varphi(t)^2 \leq C m \left( \frac{m}{\varepsilon} \right)^{\frac{2}{\sigma-1}} (1 + Z(t)^2)^{2+3\varepsilon}.
\]  

(3.55)

Note that

\[
Z^2(t) = \int (\mu \omega^2 + \frac{F^2}{2\mu + \lambda(\rho)}) dx
\]

\[
\leq C \int \left[ \mu \omega^2 + (2\mu + \lambda(\rho))(\text{div} u)^2 + \frac{P^2(\rho)}{2\mu + \lambda(\rho)} \right] dx
\]

\[
\leq C (\phi(t)) + \int P^2(\rho) dx \in L^1(0, T),
\]

where \( \phi(t) \) is defined as in (3.4).

Applying the Gronwall’s inequality to (3.55) and using (3.56) show that

\[
\frac{1}{(1 + Z^2(t))^{3\varepsilon}} - \frac{1}{(1 + Z^2(0))^{3\varepsilon}} + C m \varepsilon \left( \frac{m}{\varepsilon} \right)^{\frac{2}{\sigma-1}} \geq 0.
\]

Then we have the inequality

\[
\frac{1}{(1 + Z^2(t))^{3\varepsilon}} \geq \frac{1}{2(1 + Z^2(0))^{3\varepsilon}},
\]  

(3.57)

provided that

\[
C m \varepsilon \left( \frac{m}{\varepsilon} \right)^{\frac{2}{\sigma-1}} \leq \frac{1}{2(1 + Z^2(0))^{3\varepsilon}}.
\]  

(3.58)

This condition, i. e., (3.58), is satisfied if

\[
C m^{1+\frac{2}{\sigma-1}} 2^{-m(1-\frac{2}{\sigma-1})} \leq \frac{1}{2},
\]  

(3.59)
since
\[ Z^2(0) = \int \left[ \mu(\omega_0)^2 + \frac{(F_0)^2}{2\mu + \lambda(\rho_0)} \right] dx \leq C \left[ \| \nabla^2 u_0 \|_2^2 + \| \rho_0 \|_{W^{2,s}(\mathbb{R}^2)}^2 \| \nabla^2 u_0 \|_2^2 + \| \rho_0 \|_{W^{2,s}(\mathbb{R}^2)}^2 \right] \leq C. \]

Now if \( \beta > 3 \), that is, \( 1 - \frac{2}{\beta - 1} > 0 \), then we can choose sufficiently large \( m > 2 \) to guarantee the condition (3.59). Consequently, the inequality (3.57) is satisfied with \( \beta > 3 \) and sufficiently large \( m > 2 \). Then
\[ Z^2(t) \leq 2^{m-1} (1 + Z^2(0)) - 1 \leq C, \]
and
\[ \int_0^T \varphi(t) dt \leq C. \]

Thus the proof of Lemma 3.6 is completed. \( \square \)

Step 5: Weighted estimates for the density:
The following Lemma 3.7 is used in estimating the nonlinear terms (3.95) and (3.96).

**Lemma 3.7** It holds that for \( \alpha_1 > \alpha \) with \( \alpha \) being the weight in Lemma 3.2
\[ \int \rho |x|^{\alpha_1} dx \leq C. \]

**Proof:** Multiplying the continuity equation (1.1) by \( |x|^{\alpha_1} \) yields that
\[ (\rho |x|^{\alpha_1})_t + \text{div}(\rho u |x|^{\alpha_1}) - \rho u \cdot \nabla(|x|^{\alpha_1}) = 0. \]

Integrating the above equation with respect to \( t, x \) over \([0, t] \times \mathbb{R}^2 \) gives that
\[
\begin{align*}
\int \rho(t, x)|x|^{\alpha_1} dx &= \int \rho_0(x)|x|^{\alpha_1} dx + \int_0^t \int \rho u \cdot \nabla(|x|^{\alpha_1}) dx dt \\
&\leq M + \alpha_1 \int_0^T \int \rho |u| |x|^{\alpha_1-1} dx dt \\
&\leq M + \alpha_1 \int_0^T \| |u| |x|^{\alpha_1-1} \|_{p_1} \| \rho \|_{p_1} dt \\
&\leq M + C \int_0^T \| |x|^{\frac{\alpha}{2}} \nabla u \|_2 dt \leq C(M),
\end{align*}
\]

where in the second and third inequalities we have used the Hölder inequality and Caffarelli-Kohn-Nirenberg inequality Lemma 2.3 (1), respectively, such that the positive constants \( p, p_1, \alpha_1, \alpha \) satisfying the relations
\[
\frac{1}{p} + \frac{1}{p_1} = 1, \quad \frac{1}{p} + \frac{\alpha_1 - 1}{2} = \frac{1}{2} + \frac{\frac{\alpha}{2} - 1}{2} = \frac{\alpha}{4},
\]
which implies that
\[ \alpha_1 = \alpha \frac{2}{2} + 1 - \frac{2}{p} > \alpha, \text{ if we choose } p > \frac{4}{2 - \alpha}. \]
Thus the proof of Lemma 3.7 is completed. \( \square \)

Step 6: Second order derivative estimates for the velocity:
Lemma 3.8 There exists a positive constant \( C = C(M) \), such that
\[
\sup_{t \in [0,T]} \| (1 + |x|^2) \sqrt{\rho(H, L)} \|_2^2(t) + \int_0^T \| (1 + |x|^2) \nabla(H, L) \|_2^2 dt \leq C.
\]

Proof: Multiplying the equations (2.6) and (2.6) by \( H \) and \( L \), respectively, summing the resulted equations together and then integrating with respect to \( x \) over \( \mathbb{R}^2 \) yields that
\[
\frac{1}{2} \frac{d}{dt} \int \rho(H^2 + L^2) dx + \int \mu(H_1 - L_2)^2 + (2\mu + \lambda(\rho))(H_2 + L_1)^2 dx
\]
\[
= \int \rho(H^2 + L^2) \text{div} u dx - \int \mu \text{div} (L_2 - H_1) dx
\]
\[
- \int \rho(2\mu + \lambda(\rho)) \left[ F(\frac{1}{2\mu + \lambda(\rho)})^\prime + \left( \frac{P(\rho)}{2\mu + \lambda(\rho)} \right)^\prime \right] \text{div} u \left( H_2 + L_1 \right) dx
\]
\[
- \int \left[ H(u_{x_2} \cdot \nabla F + \mu u_{x_1} \cdot \nabla \omega) + L(u_{x_1} \cdot \nabla F - \mu u_{x_2} \cdot \nabla \omega) \right] dx
\]
\[
+ \int (2\mu + \lambda(\rho))(u_{x_1})^2 + 2u_{1x_2} u_{2x_1} + (u_{x_2})^2 (H_2 + L_1) dx.
\]

Set
\[
Y(t) = \left( \int \rho(H^2 + L^2) dx \right)^{\frac{1}{2}},
\]
and
\[
\psi(t) = \left( \int \mu(H_1 - L_2)^2 + (2\mu + \lambda(\rho))(H_2 + L_1)^2 dx \right)^{\frac{1}{2}}.
\]

Note that
\[
\int (|\nabla H|^2 + |\nabla L|^2) dx = \int (H_{x_1}^2 + H_{x_2}^2 + L_{x_1}^2 + L_{x_2}^2) dx
\]
\[
= \int ((H_1 - L_2)^2 + (H_2 + L_1)^2) dx \leq \frac{1}{\mu} \psi^2(t).
\]

Then it follows from the elliptic system
\[
\mu \omega_{x_1} + F_{x_2} = \rho H, \quad -\mu \omega_{x_2} + F_{x_1} = \rho L,
\]
that
\[
\| \nabla (F, \omega) \|_p \leq C \| \rho(H, L) \|_p, \quad \forall 1 < p < +\infty.
\]

Now we estimate the right hand side of (3.60) term by term. First, by the Hölder inequality, Caffarelli-Kohn-Nirenberg inequality in Lemma 2.3 (1) and the density estimate (3.37), it holds that
\[
\| F \|_{\frac{1}{2}} \leq C \| \rho(H, L) \|_{\frac{1}{2}}(1 + 1)
\]
\[
\| \nabla F \|_{\frac{1}{2}} \leq C \| \rho(H, L) \|_{\frac{1}{2}}(1 + 1)
\]
\[
\| \nabla (F, \omega) \|_{\frac{1}{2}} \leq C \| \rho(H, L) \|_{\frac{1}{2}}(1 + 1)
\]
\[
\| \nabla (L, H) \|_{\frac{1}{2}} \leq C \| \rho(H, L) \|_{\frac{1}{2}}(1 + 1)
\]
\[
\| \nabla (H, L) \|_{\frac{1}{2}} \leq C \| \rho(H, L) \|_{\frac{1}{2}}(1 + 1)
\]
\[
\leq \sigma_1 \| x \|^{\frac{1}{2}} \| \nabla (L, H) \|^{\frac{1}{2}} + C \sigma_1 (Y(t))^4 + 1
\]
\[
(3.63)
\]
where $\sigma_1 > 0$ is a small constant to be determined and $C_{\sigma_1}$ is a positive constant depending on $\sigma_1$. Second, direct estimates give

$$\left| - \int \mu \omega \text{div}(L_{x_2} - H_{x_1})dx \right| \leq \mu \left( \int (L_{x_2} - H_{x_1})^2 dx \right)^{\frac{1}{2}} \left( \int \omega^2 (\text{div}u)^2 dx \right)^{\frac{1}{2}}$$

$$\leq \sigma \psi^2(t) + C_\sigma \int \omega^2 (\text{div}u)^2 dx \leq \sigma \psi^2(t) + C_\sigma \|\omega\|_2^2 \|F + P(\rho)\|_{2\mu + \lambda(\rho)}^2$$

$$\leq \sigma \psi^2(t) + C_\sigma \|\omega\|_2^2 (1 + \|F\|_2^2) \leq \sigma \psi^2(t) + C_\sigma \|\nabla \omega\|_{2/3}^2 (\|\nabla F\|_{2/3}^2 + 1)$$

$$\leq \sigma \psi^2(t) + C_\sigma (\|\rho(H, L)\|_{4/3}^4 + 1) \leq \sigma \psi^2(t) + C_\sigma (\|\nabla F\|_{4/3}^4 + 1)$$

$$\leq \sigma \psi^2(t) + C_\sigma (\|\rho(H, L)\|_{4/3}^4 + 1) \leq \sigma \psi^2(t) + C_\sigma (\|\nabla F\|_{4/3}^4 + 1)$$

Similarly, one has

$$\left| - \int \rho(2\mu + \lambda(\rho)) \left[ F \left( \frac{1}{2\mu + \lambda(\rho)} \right) \right]' \left( \frac{P(\rho)}{2\mu + \lambda(\rho)} \right)' \text{div}(H_{x_2} + L_{x_1})dx \right|$$

$$\leq \sigma \int (2\mu + \lambda(\rho))(H_{x_2} + L_{x_1})^2 dx$$

$$+ C_\sigma \int \rho^2(2\mu + \lambda(\rho)) \left[ F \left( \frac{1}{2\mu + \lambda(\rho)} \right) \right]' \left( \frac{P(\rho)}{2\mu + \lambda(\rho)} \right)'^2 (\text{div}u)^2 dx$$

$$\leq \sigma \psi^2(t) + C_\sigma \int \rho^2(2\mu + \lambda(\rho)) \left[ F \left( \frac{1}{2\mu + \lambda(\rho)} \right) \right]' \left( \frac{P(\rho)}{2\mu + \lambda(\rho)} \right)'^2 |F|^2 + P(\rho) \frac{dx}{2\mu + \lambda(\rho)}$$

$$\leq \sigma \psi^2(t) + C_\sigma \|F\|_{4/3}^4 + 1 \leq \sigma \psi^2(t) + C_\sigma (\|\nabla F\|_{4/3}^4 + 1)$$

$$\leq \sigma \psi^2(t) + C_\sigma (\|\rho(H, L)\|_{4/3}^4 + 1) \leq \sigma \psi^2(t) + C_\sigma (\|\nabla F\|_{4/3}^4 + 1).$$

Next,

$$\left| - \int [H(u_{x_2} \cdot \nabla F + \mu u_{x_1} \cdot \nabla \omega) + L(u_{x_1} \cdot \nabla F - \mu u_{x_2} \cdot \nabla \omega)] dx \right|$$

$$\leq C \int \|H, L\| \|\nabla u\| \|\nabla (F, \omega)\| dx \leq C \|\nabla u\|_2 \|H, L\|_{4/3} \|\nabla (F, \omega)\|_{4/3 - \alpha}$$

$$\leq C \|\nabla u\|_2 \|\nabla (H, L)\|_{2} \|\rho(H, L)\|_{4/3 - \alpha},$$

where one has used the fact that

$$\|\nabla u\|_2 \leq C (\|\text{div}u\|_2 + \|\omega\|_2) \leq C (\|F + P(\rho)\|_{2\mu + \lambda(\rho)} + \|\omega\|_2) \leq C.$$

Note that

$$\|\rho(H, L)\|_{4/3 - \alpha}^4 = \left( \int \rho^{2\alpha} |(H, L)|^{2\alpha} dx \right)^{\frac{2\alpha}{4}} = \left( \int \sqrt{\rho} |(H, L)|^{\frac{2\alpha}{\alpha} + \frac{6\alpha}{2(2\alpha) - \alpha}} dx \right)^{\frac{2\alpha}{4}}$$

$$\leq \|\sqrt{\rho}(H, L)\|_2^{\frac{2\alpha}{4}} \|\rho(H, L)\|_{\frac{2\alpha}{\alpha} + \frac{6\alpha}{2(2\alpha) - \alpha}} \leq C \|\sqrt{\rho}(H, L)\|_2^{\frac{2\alpha}{4}} \left[ \|\rho(H, L)\|_{\frac{2\alpha}{\alpha} + \frac{6\alpha}{2(2\alpha) - \alpha}} + \|\nabla (H, L)\|_{\frac{2\alpha}{\alpha}} \right]$$

$$\leq C \|\sqrt{\rho}(H, L)\|_2^{\frac{2\alpha}{4}} \|\rho(H, L)\|_{\frac{2\alpha}{\alpha}} \|\nabla (H, L)\|_{\frac{2\alpha}{\alpha}},$$
In order to close the estimates in Lemma 3.8, we need to carry out the following weighted estimates. Substituting the estimates (3.63)-(3.65), (3.68) and (3.69) into (3.60), one can arrive at
\[
\sigma \leq \sigma \|
\nabla(h, L)\|_2^2 + \sigma_1 \|x|\nabla(h, L)\|_2^2 + C_{\sigma_1} \gamma(1 + \gamma^2). \tag{3.68}
\]

Moreover,
\[
\bigg| \int (2\mu + \lambda(\rho))(H x_1)^2 + 2u x_1 u_1 x_1 + (u_2 x_2)^2 (H x_2 + L x_1) dx \bigg| \leq \sigma_1 \int (2\mu + \lambda(\rho)) \|\nabla u\|^4 dx + C_\sigma \int (2\mu + \lambda(\rho)) \|\nabla u\|^4 dx \leq \sigma \psi(t)^2 + C_\sigma \int (2\mu + \lambda(\rho)) \|\nabla u\|^4 dx \leq \sigma \psi(t)^2 + C_\sigma \|\nabla(H, L)\|^4_3 + \|\nabla(H, L)\|^4_5 + 1 \leq \sigma \psi(t)^2 + C_\sigma \gamma(t)^4 + 1. \tag{3.69}
\]

Substituting the estimates (3.63), (3.65), (3.68) and (3.69) into (3.60), one can arrive at
\[
\frac{1}{2} \int \frac{d}{dt} \int (h^2(t) + \psi^2(t)) \leq 4\sigma \psi^2(t) + 2\sigma_1 \|x|\nabla(h, L)\|^2_2 + C_{\sigma_1} (1 + \gamma^2(t))^2.
\]

Choosing \(4\sigma = \frac{1}{2}\), noting that \(\gamma^2(t) = \varphi^2(t) \in L^1(0, T)\), and then using Gronwall’s inequality yield that
\[
\gamma^2(t) + \int_0^t \psi^2(t) dt \leq \gamma^2(0) + 2\sigma_1 \int_0^t \|x|\nabla(h, L)\|^2_2 d\tau + C_{\sigma_1}. \tag{3.70}
\]

By the compatibility condition (1.6), one has
\[
\gamma^2(0) = \|\sqrt{\rho_0} (h_0, L_0)\|_2^2 = \|g\|^2_2 \leq C.
\]

This, together with (3.70), shows that
\[
\gamma^2(t) + \int_0^t \psi^2(\tau) d\tau \leq 2\sigma_1 \int_0^t \|x|\nabla(h, L)\|^2_2 d\tau + C_{\sigma_1}. \tag{3.71}
\]

In order to close the estimates in Lemma 3.8, we need to carry out the following weighted estimates to \(\sqrt{p}(h, L)\). Note that
\[
\mu(h x_1 - L x_2)^2 + (2\mu + \lambda(\rho))(H x_2 + L x_1)^2 - \text{div} \left\{ (2\mu + \lambda(\rho))(H x_2 + L x_1)(L, H)^t \right\} \]
\[- \left[ \mu(H h x_1 - L x_2) \right] x_1 + \left[ \mu L(h x_1 - L x_2) \right] x_2 = \mu \nabla(h, L)^2 + (\mu + \lambda(\rho))(H x_2 + L x_1)^2 - \text{div} \left[ \mu \nabla \left( \frac{H^2 + L^2}{2} \right) + (\mu + \lambda(\rho))(H x_2 + L x_1)(L, H)^t \right].
\]
Similar to (3.60), it follows from (2.6) that

\[
\frac{d}{dt} \int \frac{1}{2} \rho(H^2 + L^2)|x|^\alpha \, dx + J(t) = \int \frac{1}{2} (H^2 + L^2) \rho u \cdot \nabla(|x|^\alpha) \, dx + \int \rho(H^2 + L^2)(\nabla u \cdot |x|^\alpha) \, dx \\
+ \int \mu \omega (\nabla u \cdot |x|^\alpha \cdot x_1 - L(|x|^\alpha \cdot x_2) \, dx - \int \mu \omega (\nabla u \cdot (L_{x_2} - H_{x_1})) \, dx \\
- \int [H(u_{x_2} \cdot \nabla F + \mu u_{x_1} \cdot \nabla \omega) + L(u_{x_1} \cdot \nabla F - \mu u_{x_2} \cdot \nabla \omega)] \, dx \\
- \int \rho(2\mu + \lambda(\rho))(\nabla u \cdot |x|^\alpha) \, dx \\
+ \int \rho(2\mu + \lambda(\rho))(\nabla u \cdot |x|^\alpha) \, dx \\
- \int \rho(P(\rho)) [(\nabla u \cdot |x|^\alpha) \, dx \\
- \frac{\mu}{2\mu + \lambda(\rho)} (\nabla u \cdot |x|^\alpha) \, dx \\
+ \int \rho(2\mu + \lambda(\rho))(\nabla u \cdot |x|^\alpha) \, dx \\
+ \int (2\mu + \lambda(\rho))(|u_{x_1}|^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2) (L, H)^t \cdot \nabla(|x|^\alpha) \, dx \\
+ \int (2\mu + \lambda(\rho))(|u_{x_1}|^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2) (H_{x_2} + L_{x_1}) |x|^\alpha \, dx := \sum_{i=1}^{9} I_i,
\]

where

\[
J(t) = \int \left\{ |x|^\alpha \left[ \mu |\nabla(H, L)|^2 + (\mu + \lambda(\rho))(H_{x_2} + L_{x_1})^2 \right] \\
- \left[ \mu \nabla \left( \frac{H^2 + L^2}{2} \right) + (\mu + \lambda(\rho))(H_{x_2} + L_{x_1})(L, H)^t \cdot \nabla(|x|^\alpha) \right] \right\} \, dx.
\]

First, \( J(t) \) in (3.73) can be estimated similarly as in (3.22), (3.23), (3.24) and (3.30) if replacing \( u \) by \((L, H)^t = \dot{u}\). Therefore, if \( \alpha^2 < 4(\sqrt{2} - 1) \), then there exists a positive constant \( C_\alpha \) such that

\[
J(t) \geq C_\alpha^{-1} \left( |||x|^{\frac{2}{\alpha}} \nabla(H, L)||^2 + |||x|^{\frac{2}{\alpha}} (H_{x_2} + L_{x_1})||^2 + ||\sqrt{\lambda(\rho)}|x|^{\frac{2}{\alpha}} (H_{x_2} + L_{x_1})||^2 \right) - C_\alpha ||\nabla(H, L)||^2.
\]

Then the terms \( I_i \) \((i = 1, 2, \ldots, 9)\) on the right hand side of (3.72) will be estimated as follows. By the Hölder inequality, Caffarelli-Kohn-Nirenberg inequality in Lemma 2.3 (1) and Young inequality, it holds that

\[
|I_1| \leq \frac{\alpha}{2} \int \rho(H, L)^2 |u||x|^{\alpha - 1} \, dx \leq \frac{\alpha}{2} \int \sqrt{\rho(H, L)} |||x|^{\frac{2}{\alpha}} \nabla(H, L)||^2 |||x|^{\frac{2}{\alpha}} (H_{x_2} + L_{x_1})||^2 |||x|^{\frac{2}{\alpha}} \nabla u||^2 \, dx
\]

where the positive constants in the above inequality (3.75) satisfying that

\[
\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{2}, \quad \frac{1}{q} + \frac{\beta_1}{2} = \frac{\alpha}{4}(1 - \theta), \quad \frac{1}{r} + \frac{\alpha - \beta_1 - 1}{2} = \frac{\alpha}{4}.
\]

which implies that

\[
p = \frac{4}{\alpha \theta}, \quad \text{with} \quad \theta \in (0, 1).
\]

Then Hölder inequality gives that

\[
|I_2| \leq \sqrt{\rho(H, L)} |||x|^{\frac{2}{\alpha}} \nabla(H, L)||^2 |||x|^{\frac{2}{\alpha}} \nabla u||^2 |||x|^{\frac{2}{\alpha}} \nabla \nabla u||^2 |||x|^{\frac{2}{\alpha}} \nabla \nabla u||^2 \, dx.
\]

(3.76)
where \( \beta_1 > 0 \) is to be determined and \( p, q > 2 \) satisfying that

\[
\frac{1}{2\gamma} + \frac{1}{p} + \frac{1}{q} = \frac{1}{2}.
\]

Note that it follows from Lemma 3.2 and Lemma 3.5 that if \( 2\gamma\beta_1 = \alpha \), then

\[
\left\| \sqrt{\rho} |x|^{\beta_1} \right\|_{2\gamma} = \left( \int \rho^\gamma |x|^{2\gamma\beta_1} \, dx \right)^{\frac{1}{2\gamma}} \leq C.
\]

Thus, if \( \beta_1 = \frac{\alpha}{2\gamma} \), and

\[
\frac{1}{q} + \frac{\beta_1}{2} = \frac{1}{4}(1 - \theta), \quad \text{with} \quad \theta \in (0, 1),
\]

then by Caffarelli-Kohn-Nirenberg inequality and Sobolev inequality, it follows from (3.79) that

\[
|I_2| \leq C|\sqrt{\rho}(H, L)| |x|^\frac{2}{p} \left\| \frac{F + P(\rho)}{2\mu + \lambda(\rho)} \right\|_p \| |x|^{\frac{\alpha}{2} - \frac{\alpha}{2\gamma}} \|_{2\mu + \lambda(\rho)} \| |x|^{\beta_1} \|_{q} \|
\]

\[
\leq C\left| \sqrt{\rho}(H, L) \right| |x|^\frac{2}{p} \left( \| F \|_p + \| P(\rho) \|_p \right) \| \nabla (H, L) \|_2 \| |x|^{\frac{\alpha}{2}} \|_{q} \|
\]

\[
\leq C\left| \sqrt{\rho}(H, L) \right| |x|^\frac{2}{p} \left( \| F \|_p + \| P(\rho) \|_p \right) \| \nabla (H, L) \|_2 \| |x|^{\frac{\alpha}{2}} \|_{q} \|
\]

\[
\leq C\left| \sqrt{\rho}(H, L) \right| |x|^\frac{2}{p} \left( \| F \|_p + \| P(\rho) \|_p \right) \| \nabla (H, L) \|_2 \| |x|^{\frac{\alpha}{2}} \|_{q} \|
\]

\[
\leq \sigma \left( \| \nabla (H, L) \|_2 \right)^2 + C\| \sqrt{\rho}(H, L) \| |x|^\frac{\alpha}{2} \|_{q} \|
\]

Similarly, it follows that

\[
|I_3| \leq C\int |\omega||(H, L)||\text{div} u||x|^{\alpha - 1} \, dx \leq C\| \text{div} u \|_2 \| |x|^{\beta_1} \omega \|_p \| |x|^{\alpha - \beta_1 - 1} \|_{q} \|
\]

\[
\leq \sigma \left( \| \nabla (H, L) \|_2 \right)^2 + C\| \sqrt{\rho}(H, L) \| |x|^\frac{\alpha}{2} \|_{q} \|
\]

if one has \( \theta \in (0, 1), p, p_1, q > 2, \beta_1 > 0 \) and

\[
\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad \frac{1}{p} + \frac{\beta_1}{2} = \frac{1}{p_1} + \frac{1}{2}, \quad \frac{1}{q} + \frac{\alpha - \beta_1 - 1}{2} = \frac{1}{4}(1 - \theta),
\]

which implies that

\[
p_1 = \frac{4}{2 + \alpha(1 + \theta)} < 2.
\]

Thus one can obtain from (3.78) that

\[
|I_3| \leq C\| \rho(H, L) \|_{p_1} \left( \| 1 + |x|^\frac{\alpha}{2} \| \| \nabla (H, L) \|_2 \right)^2 \leq C\| \sqrt{\rho}(H, L) \| \| 1 + |x|^\frac{\alpha}{2} \| \| \nabla (H, L) \|_2 \right)^2
\]

\[
\leq \sigma \left( \| \nabla (H, L) \|_2 \right)^2 + C\| \sqrt{\rho}(H, L) \| |x|^\frac{\alpha}{2} \|_{q} \|
\]

Then for \( \beta_1 > 0 \) to be determined and \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \), it holds that

\[
|I_4| \leq C\| |x|^\frac{\alpha}{2} \| \| 1 + |x|^\frac{\alpha}{2} \| \| \nabla (H, L) \| \| x |^{\beta_1} \omega \|_{q} \|
\]

\[
\leq C\| |x|^\frac{\alpha}{2} \| \| 1 + |x|^\frac{\alpha}{2} \| \| \nabla (H, L) \| \| x |^{\beta_1} \omega \|_{q} \|
\]

\[
\leq C\| |x|^\frac{\alpha}{2} \| \| 1 + |x|^\frac{\alpha}{2} \| \| \nabla (H, L) \| \| x |^{\beta_1} \omega \|_{q} \|
\]

(3.80)
where we have chosen $\beta_1 = \frac{\alpha}{q_1}$ and by Caffarelli-Kohn-Nirenberg inequality, $p_1$ and $q_1$ satisfy $1 < p_i < 2$ ($i = 1, 2$) and

$$
\frac{1}{p} + \frac{\beta_1}{2} = \frac{1}{p_1} - \frac{1}{2}, \quad \frac{1}{q} + \frac{\sigma - \beta}{2} = \frac{1}{q_1} - \frac{1}{2}.
$$

Thus one can get from (3.80) that

$$
|I_4| \leq C||x|^{\frac{\alpha}{q}} \nabla (H, L)||_2 \left[ \|\rho (H, L)\|_{p_1} + 1 \right] \|\rho (H, L)\|_{q_1}
$$

$$
\leq C||x|^{\frac{\alpha - \beta_1}{2}} (H, L)||\nabla u||_2 \|x|^\beta_1 \nabla (F, \omega)|| dx
$$

$$
\leq C \|\nabla u||_2 \||x|^{\alpha - \beta_1} (H, L)||_{p_1} \||x|^{\beta_1} \nabla (F, \omega)||_q
$$

(3.81)

$$
\leq \sigma \||x|^{\frac{\alpha - \beta_1}{2}} (H, L)||_{p_1} \|\rho (H, L)|| \|x|^\beta_1 ||_q.
$$

Now if we also choose

$$
\frac{\alpha}{2} < \beta_1 < \alpha,
$$

then

$$
\||x|^{\alpha - \beta_1} (H, L)||_{p_1} \leq C \|\nabla (H, L)||_{2}^{\beta_1} \|x|^{\frac{\alpha}{2}} \nabla (H, L)||_{2}^{1 - \theta_1} \leq C \||1 + |x|^{\frac{\alpha}{2}} \nabla (H, L)||_{2},
$$

(3.85)

where

$$
\frac{1}{p} + \frac{\alpha - \beta_1}{2} = \frac{\alpha}{4} (1 - \theta_1), \quad \text{that is,} \quad p = \frac{4}{2 \beta_1 - \alpha (1 + \theta_1)}.
$$

(3.86)

For $\beta_2 > 0$ to be determined and for $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}$, it holds that

$$
\|\rho (H, L)|x|^{\beta_1}||^2 = \int \rho^{|q|} (H, L)^{|q|} |x|^{\beta_1 q} dx
$$

$$
\leq \||\sqrt{\rho} (H, L)||^{\frac{\alpha}{q}} \|x|^{\frac{\alpha}{2}} \nabla (H, L)||^{q - 1} \|x|^{\beta_1 q - \frac{\alpha}{2}} - \beta_2\|_{p_1} \|\rho^{\frac{q}{2}}|x|^{\beta_2}||_q.
$$

If we choose $\beta_2 = \frac{\alpha}{q_1 \gamma}$, then it holds that

$$
\|\rho^{\frac{q}{2}}|x|^{\beta_2}||_q = \left( \int \rho^{(q - \frac{\alpha}{2}) q_1} |x|^{\beta_2 q_1} dx \right)^{\frac{1}{q_1}}
$$

$$
= \left( \int \rho^{(q - \frac{\alpha}{2}) q_1} |x|^{\frac{\alpha}{q_1}} dx \right)^{\frac{1}{q_1}} \leq C \||\rho| |x|^{\frac{\alpha}{q}} \|^{\frac{\alpha}{q}} \|\rho^{(q - \frac{\alpha}{2}) q_1 - 1}||^{\frac{q_1}{q_1}} \leq C,
$$

$$
\int \rho^{(q - \frac{\alpha}{2}) q_1} |x|^{\frac{\alpha}{q_1}} dx \leq C.
$$
and thus
\[
\|\rho(H,L)|x|^{\beta_1}\|_q \leq C\sqrt{\rho(H,L)|x|^{\frac{1}{2}}}\|_2\|\rho(H,L)|^{q-1}|x|^{\beta_1q-\frac{q}{2}-\beta_2}\|_{p_1}
\]
\[
\leq C\sqrt{\rho(H,L)|x|^{\frac{1}{2}}}\|_2\|\rho(H,L)|^{q-1}|x|^{\beta_1q-\frac{q}{2}-\beta_2}\|_{p_1(q-1)}
\]
\[
\leq C\sqrt{\rho(H,L)|x|^{\frac{1}{2}}}\|\nabla(H,L)\|_2^{\rho_2(q-1)}\|\nabla(H,L)\|_2^{(1-\theta_2)(q-1)}
\]
\[
\leq C\sqrt{\rho(H,L)|x|^{\frac{1}{2}}}\|\nabla(H,L)\|_2^{q-1}.
\]

where
\[
\frac{1}{p_1(q-1)} + \frac{\beta_1q-\frac{q}{2}-\beta_2}{2(q-1)} = \frac{\alpha}{4}(1-\theta_2)
\]

(3.87)

It follows from (3.86) and (3.88) that
\[
q_1 = (1 + \frac{\alpha}{2\gamma})\left[\frac{q - 1}{2} + \frac{\alpha}{4}(\theta_1q + \theta_2(q-1))\right]^{-1} > 2, \text{ if } q \rightarrow 2+, \theta_i \rightarrow 0+, \text{ (i = 1, 2}).
\]

Note that \(p\) is sufficiently large when \(q \rightarrow 2+\) and thus the above restrictions (3.83) and (3.84) on \(\beta_1\) when estimating \(I_5\) could be satisfied. Substituting (3.85) and (3.87) into (3.82) yields that
\[
|I_5| \leq C||(1 + |x|^{\frac{1}{2}})\nabla(H,L)||_2\left[\|\sqrt{\rho(H,L)|x|^{\frac{1}{2}}}((1 + |x|^{\frac{1}{2}})\nabla(H,L)||_2^{1-\gamma} + \|\sqrt{\rho(H,L)}\right]
\]
\[
\leq \sigma||(1 + |x|^{\frac{1}{2}})\nabla(H,L)||_2 + C\sigma\|\sqrt{\rho(H,L)}(1 + |x|^{\frac{1}{2}})||_2^{2}.
\]

(3.89)

Then, for \(\beta_1 > 0\) to be determined and for \(\frac{1}{p} + \frac{1}{q} = \frac{1}{2}\), it holds that
\[
|I_6| \leq C\|\text{divu}\|_2\|\rho(H,L)|^{\beta_1}\|_p\left[\|\rho(2\mu + \lambda(\rho))F(\frac{1}{2\mu + \lambda(\rho)})^\gamma|x|^{\alpha-\beta_1-1}\|_q
\]
\[
+ \|\rho(2\mu + \lambda(\rho))(\frac{P(\rho)}{2\mu + \lambda(\rho)})^\gamma|x|^{\alpha-\beta_1-1}\|_q\right]
\]
\[
\leq C\|\nabla(H,L)\|_2\left[\|F\|_q|x|^{\alpha-\beta_1-1}\|_q + \|P(\rho)\|_q|x|^{\alpha-\beta_1-1}\|_q\right],
\]

where
\[
\frac{1}{p} + \frac{\beta_1}{2} = \frac{\alpha}{4}.
\]

Furthermore, for \(1 < q_1 < 2\) and
\[
\frac{1}{q} + \frac{\alpha - \beta_1 - 1}{2} = \frac{1}{q_1} - \frac{1}{2},
\]

it follows that
\[
\|F\|_q|x|^{\alpha-\beta_1-1}\|_q \leq C\|\nabla F\|_{q_1} \leq C\|\rho(H,L)\|_{q_1} \leq C\|\sqrt{\rho(H,L)}\|_2,
\]

and
\[
\|P(\rho)|x|^{\alpha-\beta_1-1}\|_q = \left(\int \rho^{\gamma q}|x|^{(\alpha-\beta_1-1)q}dx\right)^{\frac{1}{q}} \leq C\|\rho|x|^{(\alpha-\beta_1-1)q}\|_q \|\rho^{\gamma q-1}\|_\gamma \leq C,
\]

provided \(\beta_1\) is chosen such that
\[
(\alpha - \beta_1 - 1)q\gamma \leq \alpha.
\]
Therefore, from (3.90), it holds that

$$|I_6| \leq C\|x\|^\frac{2}{\alpha} \div (H, L)\|_2 \left[ \|\sqrt{\rho}(H, L)\|_2^2 + 1 \right] \leq \sigma \|x\|^\frac{2}{\alpha} \div (H, L)\|_2^2 + C_{\sigma} \left[ \|\sqrt{\rho}(H, L)\|_2^2 + 1 \right].$$

(3.91)

Then

$$|I_7| \leq C\|(1 + \lambda(\rho))(H_{x_2} + L_{x_1})\|\|x\|^\frac{2}{\alpha} \div (F) + P(\rho)\| \div u\|_2^2$$

$$\leq C\|(1 + \lambda(\rho))(H_{x_2} + L_{x_1})\|\|x\|^\frac{2}{\alpha} \div (F) + P(\rho)\| \div u\|_2^2$$

(3.92)

Next, for $p = \frac{2}{1+\alpha}$ and $q = \frac{4}{2+\alpha}$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, one has

$$|I_8| \leq C\|(H, L)\| \|x\|^{\alpha-1} \|\|(2\mu + \lambda(\rho))\|\|\div u\|_q^2$$

$$\leq C\|\div (H, L)\| \|x\|^\frac{2}{\alpha} \|\|\div u\|_q^2 + \|\lambda(\rho)\|\|\div u\|_q^2$$

$$\leq C\|\div (H, L)\| \|x\|^\frac{2}{\alpha} \|\|\div u\|_q^2 + 1 \leq C\|\div (H, L)\| \|x\|^\frac{2}{\alpha} \left[ \|\omega\|_{\alpha q}^2 + \|\div u\|_{\alpha q}^2 + 1 \right]$$

(3.93)

$$\leq C\|\div (H, L)\| \|x\|^\frac{2}{\alpha} \left[ \|\div (F, \omega)\|_{\alpha q}^2 + 1 \right] \leq C\|\div (H, L)\| \|x\|^\frac{2}{\alpha} \left[ \|\div (F, \omega)\|_{\frac{4}{\alpha q} + 1}^2 + 1 \right]$$

$$\leq C\|\div (H, L)\| \|x\|^\frac{2}{\alpha} \left[ \|\sqrt{\rho}(H, L)\|_{\alpha q}^2 + 1 \right] \leq C\|\div (H, L)\| \|x\|^\frac{2}{\alpha} + C_{\sigma} \left[ \|\sqrt{\rho}(H, L)\|_{\alpha q}^2 + 1 \right].$$

Finally, it holds that

$$|I_9| \leq \sigma \int (1 + \lambda(\rho))(H_{x_2} + L_{x_1})^2 \|x\|^\alpha \|\div u\|_4^4 dx + C_{\sigma} \int (1 + \lambda(\rho)) \|x\|^\alpha \|\div u\|_4^4 dx.$$

(3.94)

At the same time, it follows from Lemma 2.4 (2) and Lemma 3.7 that

$$\int \|x\|^\alpha \|\div u\|_4^4 dx = \|x\|^\frac{2}{\alpha} \|\div u\|_4^2 \leq C\left[ \|x\|^\frac{2}{\alpha} \|\div u\|_4^2 + \|x\|^\frac{2}{\alpha} \|\sqrt{\rho}(H, L)\| \|x\|^\frac{2}{\alpha} \|\omega\|_4^2 \right]$$

$$\leq C\left[ \|x\|^\frac{2}{\alpha} \|F\|_4^2 + \|x\|^\frac{2}{\alpha} \|P(\rho)\|_4^2 + \|x\|^\frac{2}{\alpha} \|\delta\|_4^2 \right]$$

$$\leq C\left[ \|\div (F, \omega)\|_{\frac{4}{\alpha q} + 1}^4 + \|\rho x\|_{\alpha q}^4 \|\omega\|_{\alpha q}^4 \right]$$

$$\leq C\left[ \|\rho x\|_{\alpha q}^4 \|\omega\|_{\alpha q}^4 + \|\rho x\|_{\alpha q}^2 \|\omega\|_{\alpha q}^2 \right] \leq C\left[ \|\sqrt{\rho}(H, L)\|_{\alpha q}^2 + 1 \right].$$

(3.95)

and for $p > q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} + \frac{\alpha}{\alpha_1} = 1$, one has

$$\int \lambda(\rho) \|x\|^\alpha \|\div u\|_4^4 dx \leq \|\rho x\|_{\alpha q}^4 \|\omega\|_{\alpha q}^4 \|\div u\|_q^4 = \|\rho x\|_{\alpha q}^2 \|\omega\|_{\alpha q}^2 \|\div u\|_q^4 \leq C\left[ \|\div u\|_{\alpha q}^4 + \|\omega\|_{\alpha q}^4 \right] \leq C\left[ \|\div u\|_{\alpha q}^4 + \|\omega\|_{\alpha q}^4 + 1 \right]$$

$$\leq C\left[ \|\rho x\|_{\alpha q}^4 \|\omega\|_{\alpha q}^4 + 1 \right] \leq C\left[ \|\rho(\rho, H, L)\|_{\alpha q}^4 + 1 \right].$$

(3.96)

Substituting (3.95) and (3.96) into (3.94) yields that

$$|I_9| \leq \sigma \int (1 + \lambda(\rho))(H_{x_2} + L_{x_1})^2 \|x\|^\alpha dx + C_{\sigma} \left[ \|\sqrt{\rho}(H, L)\|_{\alpha q}^2 + 1 \right].$$

(3.97)

Substituting the estimates (3.71), (3.75), (3.77), (3.79), (3.81), (3.89), (3.91), (3.92), (3.93) and (3.97) into (3.72), then integrating the resulted inequality with respect to $t$ over $[0, t]$, and noting that

$$\|\sqrt{\rho_0}(H_0, L_0)\|_{\alpha q}^2 = \|g\|_{\alpha q}^2,$$
it holds that
\[ \| \sqrt{\rho} (H, L) |x|^\frac{2}{p} \|_2^2 (t) + \int_0^t \| |x|^\frac{2}{p} \nabla (H, L) \|_2^2 dt \leq C \int_0^t \| \nabla (H, L) \|_2^2 dt + C, \quad (3.98) \]
which, together with the estimate \[ \text{(3.71)} \] and choosing \( \sigma, \sigma_1 \) suitably small, completes the proof of Lemma \[ \text{3.8} \]. \[ \square \]

**Step 7. Upper bound of the density:** We are now ready to derive the upper bound for the density in the super-norm. First, one has

**Lemma 3.9** It holds that
\[ \int_0^T \| (F, \omega) \|_\infty^2 dt \leq C(M). \]

**Proof:** By \[ (3.62) \] with \( p = 3 \), one has for \( \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2} \),
\[ \int_0^T \| \nabla (F, \omega) \|_3^2 dt \leq C \int_0^T \| \rho (H, L) \|_3^2 dt = C \int_0^T \int \rho |(H, L)|^3 dx dt \]
\[ = C \int_0^T \int \sqrt{\rho} (H, L) |(H, L)|^2 \| \nabla (H, L) \|_2^2 \rho^\frac{2}{2} dx dt \]
\[ \leq C \int_0^T \| \sqrt{\rho} (H, L) \|_2 \| (H, L) \|_2^2 \| \rho \|_\frac{2}{3}^{\frac{2}{2}} dt. \quad (3.99) \]

By Caffarelli-Kohn-Nirenberg inequality in Lemma \[ 2.3 \] (1), it holds that
\[ \| (H, L) \|_2 p \leq C \| \nabla (H, L) \|_p^\theta \| |x|^\frac{2}{p_1} (H, L) \|_p^{1-\theta} \]
\[ \leq C \| \nabla (H, L) \|_2^\theta \| |x|^\frac{2}{p_1} \nabla (H, L) \|_2^{1-\theta} \leq C \| (1 + |x|^\frac{2}{p_1}) \nabla (H, L) \|_2, \quad (3.100) \]
with \( p > 2 \) and \( \theta \in (0, 1) \) satisfying
\[ \frac{1}{2p} = (1 - \theta) \left( \frac{1}{p_1} + \frac{\beta_1}{2} \right) = (1 - \theta) \left( \frac{1}{2} + \frac{\frac{2}{2} - 1}{2} \right) = \frac{\alpha(1 - \theta)}{4}. \]

Substituting \[ (3.100) \] into \[ (3.99) \] yields that
\[ \int_0^T \| \nabla (F, \omega) \|_3^2 dt \leq C \int_0^T \| (1 + |x|^\frac{2}{p_1}) \nabla (H, L) \|_2^2 dt \leq C, \]
which, combined with the estimate
\[ \int_0^T \| (F, \omega) \|_3^2 dt \leq C \int_0^T \| \nabla (F, \omega) \|_\frac{2}{5}^2 dt \leq C \int_0^T \| \rho (H, L) \|_\frac{2}{5}^2 dt \]
\[ \leq C \int_0^T \| \sqrt{\rho} (H, L) \|_2^\frac{2}{5} \| \sqrt{\rho} \|_3^2 dt \]
\[ \leq C \sup_{t \in [0, T]} \| \sqrt{\rho} (H, L) \|_2^\frac{2}{5} \int_0^T (\| \sqrt{\rho} (H, L) \|_2^2 + 1) dt \leq C, \]
yields that
\[ \int_0^T \| (F, \omega) \|_\infty^2 dt \leq \int_0^T \| (F, \omega) \|_{W^{1, 3}(\mathbb{R}^2)}^2 dt \leq C. \quad (3.101) \]

The proof of Lemma \[ 3.9 \] is finished. \[ \square \]

With Lemma \[ 3.9 \] in hand, we can obtain the uniform upper bound for the density.
Lemma 3.10  It holds that
\[ \rho(t, x) \leq C(M), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^2. \]

Proof: From the continuity equation (1.1), we have
\[ \nu(\rho)_t + u \cdot \nabla \nu(\rho) + P(\rho) + F = 0, \]
where \( \nu(\rho) \) is defined in (3.35).

Along the particle path \( \vec{X}(\tau; t, x) \) through the point \((t, x) \in [0, T] \times \mathbb{R}^2 \) defined by
\[
\begin{cases}
\frac{d\vec{X}(\tau; t, x)}{d\tau} = u(\tau, \vec{X}(\tau; t, x)), \\
\vec{X}(\tau; t, x)|_{\tau=t} = x,
\end{cases}
\]
there holds the following ODE
\[ \frac{d}{d\tau} \nu(\rho)(\tau, \vec{X}(\tau; t, x)) = -P(\rho)(\tau, \vec{X}(\tau; t, x)) - F(\tau, \vec{X}(\tau; t, x)), \]
which is integrated over \([0, t]\) to yield that
\[ \nu(\rho)(t, x) - \nu(\rho_0)(\vec{X}_0) = -\int_0^t (P(\rho) + F)(\tau, \vec{X}(\tau; t, x))d\tau, \tag{3.102} \]
with \( \vec{X}_0 = \vec{X}(\tau; t, x)|_{\tau=0} \).

It follows from (3.102) that
\[ 2\mu \ln \frac{\rho(t, x)}{\rho_0(\vec{X}_0)} + \frac{1}{\beta} \rho^\beta(t, x) + \int_0^t P(\rho)(\tau, \vec{X}(\tau; t, x))d\tau = \frac{1}{\beta} \rho_0(\vec{X}_0)^\beta - \int_0^t F(\tau, \vec{X}(\tau; t, x))d\tau. \]
So
\[ 2\mu \ln \frac{\rho(t, x)}{\rho_0(\vec{X}_0)} \leq \frac{1}{\beta} \|\rho_0\|_\infty^\beta + \int_0^t \|F(\tau, \cdot)\|_\infty d\tau \leq C, \]
which implies that
\[ \frac{\rho(t, x)}{\rho_0(\vec{X}_0)} \leq C. \]
Therefore, we have
\[ \rho(t, x) \leq C, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^2. \]
Hence Lemma 3.10 is proved. \( \square \)

As an immediate consequence of the upper bound of the density, one has

Lemma 3.11  It holds that for any \( 1 < p < \infty \),
\[ \int_0^T \left( \|\text{div} u\|_\infty^3 + \|\nabla (F, \omega)\|_p^2 \right) dt \leq C(M). \tag{3.103} \]
Integrating the above equation with respect to $x$

Then for $1 < p \leq 2$, it follows that

For $\frac{4}{\alpha} \leq p < \infty$,

Thus Lemma 3.11 is proved. \qed

### 4 Higher order estimates

Based on the basic estimates and bound of the density obtained in Section 3, we can derive some uniform estimates on their higher order derivatives. We start with estimates on first order derivatives.

**Lemma 4.1** It holds that for any $1 < p < +\infty$,

\[
\sup_{t \in [0,T]} \| (\nabla \rho,\nabla P(\rho))(t,\cdot) \|_p + \int_0^T \| \nabla u \|_\infty^2 (t) dt \leq C(M).
\]

**Proof:** Applying the operator $\nabla$ to the continuity equation (4.1), one has

\[
(\nabla \rho)_t + \nabla u \cdot \nabla \rho + u \cdot \nabla (\nabla \rho) + (\nabla \rho) \text{div} u + \rho \nabla (\text{div} u) = 0. \tag{4.1}
\]

Multiplying the equation (4.1) by $p|\nabla \rho|^{p-2}\nabla \rho$ with $p \geq 2$ implies that

\[
(|\nabla \rho|^p)_t + \text{div}(u|\nabla \rho|^p) + (p-1)|\nabla \rho|^p \text{div} u + p|\nabla \rho|^{p-2} \nabla \rho \cdot (\nabla u \cdot \nabla \rho) + p\rho |\nabla \rho|^{p-2} \nabla \rho \cdot \nabla (\text{div} u) = 0.
\]

Integrating the above equation with respect to $x$ over $\mathbb{R}^2$ gives that

\[
\frac{d}{dt}\| \nabla \rho \|_p \leq C \left[ \| \nabla u \|_\infty \| \nabla \rho \|_p + \| \nabla \text{div} u \|_p \right] \leq C \left[ \| \nabla u \|_\infty \| \nabla \rho \|_p + \| \nabla (\frac{F + P(\rho)}{2\mu + \lambda(\rho)}) \|_p \right] \tag{4.2}
\]

By (4.3), one has

\[
L_\rho u = \nabla P(\rho) + \rho\dot{u} = \nabla P(\rho) + \rho(L, H)'. \tag{4.3}
\]

Thus the elliptic estimates yields that for any $\frac{4}{\alpha} \leq p < \infty$,

\[
\| \nabla^2 u \|_p \leq C \left[ \| \nabla P(\rho) \|_p + \| \rho(L, H) \|_p \right] \leq C \left[ \| \nabla \rho \|_p + \| (L, H) \|_p \right] \leq C \left[ \| \nabla \rho \|_p + \| (1 + |x|^{\frac{2}{\alpha}}) \nabla (L, H) \|_2 \right]. \tag{4.4}
\]
By Beal-Kato-Majda type inequality (see [1], [23] and [50]) and (4.4), it holds that
\[
\|\nabla u\|_\infty \leq C (\|\text{div} u\|_\infty + \|\omega\|_\infty) \ln(e + \|\nabla^2 u\|_\alpha^\alpha) \\
\leq C (\|\text{div} u\|_\infty + \|\omega\|_\infty) \left[ \ln(e + \|\nabla\|_\alpha^\alpha) + \ln(e + \|1 + |x|^{\frac{\alpha}{2}}\nabla(H, L)\|_2) \right].
\] (4.5)

The combination of (4.2) with \( p = \frac{4}{\alpha} \) and (4.5) yields that
\[
\frac{d}{dt} \|\nabla\rho\|_\alpha^\frac{4}{\alpha} \leq C \|\nabla F\|_\alpha^\frac{4}{\alpha} \\
+ C \left[ (\|\text{div} u\|_\infty + \|\omega\|_\infty) \ln(e + \|\nabla(H, L)\|_2) + \|F\|_\infty + 1 \right] \|\nabla\rho\|_\alpha^\frac{4}{\alpha} \ln(e + \|\nabla\|_\alpha^\alpha).
\]

By the estimates (3.101), (3.104), (3.103) and the Gronwall’s inequality, it holds that
\[
\sup_{t \in [0, T]} \|\nabla\rho\|_p \leq C (\|\nabla\rho_0\|_p + 1) \leq C, \quad \forall p \in (1, +\infty).
\]

Therefore, by (4.6), Lemma 3.9, Lemma 3.11 and Gronwall inequality, one can derive from (4.2) that
\[
\sup_{t \in [0, T]} \|\nabla\rho\|_p \leq C (\|\nabla\rho_0\|_p + 1) \leq C, \quad \forall p \in (1, +\infty).
\]

Thus the proof of Lemma 4.1 is completed. \( \square \)

**Lemma 4.2** It holds that
\[
\sup_{t \in [0, T]} \left[ \|u(t, \cdot)\|_\alpha^\frac{4}{\alpha} + \|x|^{\frac{2}{\alpha}} \nabla u(t, \cdot)\|_2 \right] + \int_0^T \|x|^{\frac{8}{\alpha}} \sqrt{\rho\dot{u}}\|_2^2 dt \leq C(M).
\]

**Proof:** The momentum equation (1.1) can be rewritten as
\[
\rho \dot{u} + \nabla P(\rho) = \mu \Delta u + \nabla(\mu + \lambda(\rho)\text{div} u).
\]

Multiplying the above equation by \( \dot{u}|x|^\alpha \) with \( \alpha \) being the weight in Lemma 3.2 and integrating the resulted equations with respect to \( x \) over \( \mathbb{R}^2 \) give that
\[
\frac{d}{dt} \int \left[ \mu \frac{|\nabla u|^2}{2} + (\mu + \lambda(\rho))\frac{(\text{div} u)^2}{2} - P(\rho)\text{div} u \right]|x|^\alpha dx + \int \rho |\dot{u}|^2 |x|^\alpha dx = \int \left[ (\dot{u} - u)(\text{div} u) \right] |x|^\alpha dx \\
+ \mu \int \frac{|\nabla u|^2}{2} - \mu \nabla u \cdot \dot{u} + (\mu + \lambda(\rho))u \frac{(\text{div} u)^2}{2} - (\mu + \lambda(\rho))(\text{div} u)\dot{u} \cdot \nabla(|x|^\alpha) dx \\
- \int \left[ \mu \sum_{i,j,k=1}^2 \partial_{x_i} u_j \partial_{x_k} u_k \partial_{x_k} u_j + \mu \frac{|\nabla u|^2}{2} \text{div} u - (\mu + \lambda(\rho))(\text{div} u) \sum_{i,j=1}^2 \partial_{x_i} u_j \partial_{x_j} u_i \\
- \rho \lambda'(\rho) \frac{(\text{div} u)^3}{2} \right] |x|^\alpha dx + \int \left[ P(\rho) \sum_{i,j=1}^2 \partial_{x_i} u_j \partial_{x_j} u_i + (\gamma - 1)P(\rho)(\text{div} u)^2 \right] |x|^\alpha dx := \sum_{i=1}^3 K_i.
\] (4.7)
First, $K_1$ can be estimated as

$$
|K_1| \leq C \int \left[ P(\rho)|\dot{u}| + P(\rho)|u|\text{div}u + |u||\nabla u|^2 + |\nabla u||\dot{u}|
\right.

\begin{align*}
&\quad + (1 + \lambda(\rho))(|u|(|\nabla u|^2 + |\text{div}u||\dot{u}|)) \left]|x|^{-\alpha}dx\right.
\end{align*}

\begin{align*}
&\quad \leq C \left[\|P(\rho)x^\frac{\alpha}{2}\|_2||\dot{u}||x^\frac{\alpha}{2} - 1\|_2 + \|P(\rho)\|_\infty(\|\nabla u||x^\frac{\alpha}{2}\|_2|u|x^\frac{\alpha}{2} - 1\|_2
\right.
\end{align*}

\begin{align*}
&\quad + (1 + \lambda(\rho))\|x^\frac{\alpha}{2}\|_2 + ||\nabla u||_\infty\|\nabla u||x^\frac{\alpha}{2}\|_2 + \|\nabla u||_\infty\|\nabla u||x^\frac{\alpha}{2}\|_2 + ||\nabla u||_\infty\|\nabla u||x^\frac{\alpha}{2}\|_2 + ||\nabla u||_\infty\|\nabla u||x^\frac{\alpha}{2}\|_2\right]

&\quad \leq C \left[\|\nabla u||x^\frac{\alpha}{2}\|_2 + ||\text{div}u||x^\frac{\alpha}{2}\|_2\|\nabla u||x^\frac{\alpha}{2}\|_2 + ||\nabla u||_\infty\|\nabla u||x^\frac{\alpha}{2}\|_2 + ||\nabla u||_\infty\|\nabla u||x^\frac{\alpha}{2}\|_2\right]

\begin{align*}
&\quad \leq C \left[\|\nabla u\|_\infty + 1\|\nabla u||x^\frac{\alpha}{2}\|_2 + \|\nabla(H, L)||x^\frac{\alpha}{2}\|_2^2 + 1\right].
\end{align*}

(4.8)

Then, it follows that

$$
|K_2| \leq C \int (1 + \lambda(\rho))|\nabla u|^3|x|^{\alpha}dx \leq C\|\nabla u\|_\infty\|\nabla u||x^\frac{\alpha}{2}\|_2^2,
$$

(4.9)

and

$$
|K_3| \leq C \int P(\rho)|\nabla u|^2|x|^{\alpha}dx \leq C\|\nabla u||x^\frac{\alpha}{2}\|_2^3.
$$

(4.10)

Note that for sufficiently small constant $\sigma > 0$, it holds that

$$
\int \left[\frac{1}{2}|\nabla u|^2 + (\mu + \lambda(\rho))\frac{(|\text{div}u|)^2}{2} - P(\rho)|\nabla u||\text{div}u| \right]|x|^{\alpha}dx
$$

\begin{align*}
&\quad \geq \int \left[\frac{1}{2}|\nabla u|^2 + (\mu + \lambda(\rho))\frac{(|\text{div}u|)^2}{2}\right]|x|^{\alpha}dx - \sigma \int (\text{div}u)^2|x|^{\alpha}dx - C_{\sigma} \int P^2(\rho)|x|^{\alpha}dx
\end{align*}

(4.11)

if we choose $\sigma = \frac{\mu}{4}$.

Substituting (4.8), (4.9) and (4.11) into (4.7), and integrating the resulted equation with respect to $t$ over $[0, t]$, and then using (4.11) and Gronwall inequality, it holds that

$$
\sup_{t \in [0, T]} \|\nabla u||x|^{\frac{\alpha}{2}}\|_2^2 + \int_0^T \|\sqrt{\rho}||x|^{\frac{\alpha}{2}}\|_2^2(t)dt \leq C,
$$

which, together with the Caffarelli-Kohn-Nirenberg inequality, completes the proof of Lemma 4.2.

Lemma 4.3 It holds that for any $2 \leq p < +\infty$,

$$
\sup_{t \in [0, T]} \left[\|u(t, \cdot)||_\infty + \|\nabla u||_p + \|\rho(t, P_t)||_p + \|\nabla^2 \rho, \nabla^2 P(t, \nabla^2 u)||_2\right] + \int_0^T \|\nabla^3 u||_2^2dt \leq C.
$$

Proof: By $L^2$–estimates to the elliptic system (4.3), one has

$$
\sup_{t \in [0, T]} \|\nabla^2 u||_2 \leq C \sup_{t \in [0, T]} \left(\|\nabla P(\rho)||_2 + \|\rho(H, L)||_2\right)
\leq C \sup_{t \in [0, T]} \left(\|\nabla P(\rho)||_2 + \|\sqrt{\rho}(H, L)||_2\right) \leq C.
$$

(4.12)
It follows from the interpolation theorem, Lemma 4.2 and (4.12) that
\[
\sup_{t \in [0,T]} \|u(t, \cdot)\|_\infty \leq C \sup_{t \in [0,T]} \|u(t, \cdot)\|_{L^\frac{2}{\alpha}} \|\nabla^2 u\|_{L^\frac{2}{\alpha}} \leq C. \tag{4.13}
\]
For any \( p \in [2, \infty) \), by Sobolev embedding theorem, Lemma 3.6 and (4.12), it holds that
\[
\sup_{t \in [0,T]} \|\nabla u\|_p \leq C \sup_{t \in [0,T]} \|\nabla u\|_{H^1} \leq C. \tag{4.14}
\]
Due to (1.1), one can get \( \rho_t = -\nabla \rho - \rho \div \nu \) and \( P_t = -\nabla P - \rho P'(\rho) \div \nu \), which, together with the uniform upper bound of the density and the estimates in Lemma 4.1 and (4.13)-(4.14), yields that
\[
\sup_{t \in [0,T]} \|\rho_t, P_t\|_p \leq C, \quad \forall p \in [2, +\infty).
\]
Applying \( \nabla^2 \) to the continuity equation (1.1), then multiplying the resulted equation by \( \nabla^2 \rho \), and then integrating with respect to \( x \) over \( \mathbb{R}^2 \), one can get that
\[
\frac{d}{dt} \|\nabla^2 \rho\|_2^2 \leq C \left[ \|\nabla u\|_\infty \|\nabla^2 \rho\|_2^2 + \|\nabla \rho\|_4 \|\nabla^2 \rho\|_2 \|\nabla^2 u\|_4 + \|\rho\|_\infty \|\nabla^2 \rho\|_2 \|\nabla^3 u\|_2 \right] 
\leq C \left[ (\|\nabla u\|_\infty + 1) \|\nabla^2 \rho\|_2^2 + \|\nabla^3 u\|_2 + 1 \right], \tag{4.15}
\]
where one has used the fact that
\[
\|\nabla^2 u\|_4 \leq C \|\nabla^2 u\|_2^\frac{1}{2} \|\nabla^3 u\|_2^\frac{1}{2} \leq C \|\nabla^3 u\|_2^\frac{1}{2}. \tag{4.16}
\]
Similarly,
\[
\frac{d}{dt} \|\nabla^2 P(\rho)\|_2^2 \leq C \left[ (\|\nabla u\|_\infty + 1) \|\nabla^2 P(\rho)\|_2^2 + \|\nabla^3 u\|_2^\frac{1}{2} + 1 \right]. \tag{4.17}
\]
Note that (1.3) implies that
\[
\mathcal{L}(\nabla u) = \nabla^2 P(\rho) + \nabla [\rho \nabla \lambda(H, L)] + \nabla (\nabla \lambda(\rho) \div \nu) := \Phi. \tag{4.18}
\]
Then the standard elliptic estimates and the estimate (4.16) give that
\[
\|\nabla^3 u\|_2 \leq C \|\Phi\|_2 \leq C \left[ \|\nabla^2 P(\rho)\|_2 + \|\rho\|_\infty \|\nabla (H, L)\|_2 + \|\nabla \rho\|_4 \|\nabla (H, L)\|_2 \right] 
+ (\|\nabla^2 \rho\|_2 + \|\nabla \rho\|_4) \|\div \nu\|_\infty + \|\nabla \rho\|_4 \|\nabla^2 u\|_4 \right] 
\leq C \left[ \|\nabla^2 P(\rho)\|_2 (\|\nabla u\|_\infty + 1) + (1 + |x|^{\frac{\alpha}{2}}) \|\nabla (H, L)\|_2 + \|\nabla^3 u\|_2^{\frac{1}{2}} \right] 
\tag{4.19}
\]
Consequently,
\[
\|\nabla^3 u\|_2 \leq C \left[ \|\nabla^2 P(\rho, \nabla^2 \rho)\|_2 (\|\nabla u\|_\infty + 1) + (1 + |x|^{\frac{\alpha}{2}}) \|\nabla (H, L)\|_2 + 1 \right]. \tag{4.19}
\]
Substituting (4.19) into (4.15) and (4.17) yields that
\[
\frac{d}{dt} \|\nabla^2 \rho, \nabla^2 P(\rho)\|_2^2 \leq C \left[ \|\nabla u\|_\infty^2 + 1 \right] \|\nabla^2 P(\rho)\|_2^2 + (1 + |x|^{\frac{\alpha}{2}}) \|\nabla (H, L)\|_2^2 + 1 \right].
\]
Then the Gronwall’s inequality yields that
\[
\|\nabla^2 \rho, \nabla^2 P(\rho)\|_2^2(t) \leq \left[ \|\nabla^2 P_0, \nabla^2 P_0\|_2^2 
+ C \int_0^T \left( (1 + |x|^{\frac{\alpha}{2}}) \|\nabla (H, L)\|_2^2 + 1 \right) \dt \right] C \int_0^T \|\nabla u\|_\infty^2 + 1 \dt \leq C,
\]
which also implies that
\[
\sup_{t \in [0,T]} \| (\nabla^2 \rho, \nabla^2 P(\rho)) \|_2(t) + \int_0^T \| \nabla^3 u \|_2^2 dt \leq C.
\]
The proof of Lemma 4.3 is completed. \( \square \)

**Lemma 4.4** It holds that for \( \frac{4}{\alpha} \leq p < \infty \),
\[
\sup_{t \in [0,T]} \| \sqrt{\rho} u_t (1 + |x|^{\frac{2}{\alpha}}) \|_2^2(t) + \int_0^T (\| \nabla u_t \|_2^2 + \| u_t \|_{p}^2) dt \leq C(M).
\]

**Proof:** First, it holds that
\[
\sup_{t \in [0,T]} \| \sqrt{\rho} u_t (1 + |x|^{\frac{2}{\alpha}}) \|_2^2(t) \leq \sup_{t \in [0,T]} \left[ \| \sqrt{\rho} u_t (1 + |x|^{\frac{2}{\alpha}}) \|_2^2(t) + \| \sqrt{\rho} \cdot \nabla u (1 + |x|^{\frac{2}{\alpha}}) \|_2^2(t) \right]
\]
\[
\leq \sup_{t \in [0,T]} \left[ \| \sqrt{\rho} (L, H)^t (1 + |x|^{\frac{2}{\alpha}}) \|_2^2(t) + \| \sqrt{\rho} \|_\infty^2 \| \nabla u (1 + |x|^{\frac{2}{\alpha}}) \|_2^2(t) \right] \leq C.
\]
Then, one can arrive at
\[
\int_0^T \| \nabla u_t \|_2^2 dt \leq \int_0^T \left[ \| \nabla u \|_2^2 + \| u \|_{\infty}^2 \| \nabla^2 u \|_2^2 + \| \nabla u \|_2^2 \right] dt \leq C.
\]
For any \( \frac{4}{\alpha} \leq p < +\infty \),
\[
\int_0^T \| u_t \|_p^2 dt \leq \int_0^T \left( \| (H, L) \|_p^2 + \| u \|_{\infty}^2 \| \nabla u \|_p^2 \right) dt
\]
\[
\leq \int_0^T \left( \| (1 + |x|^{\frac{2}{\alpha}}) \nabla (H, L) \|_2^2 + \| u \|_{\infty}^2 \| \nabla u \|_p^2 \right) dt \leq C.
\]
Thus the proof of Lemma 4.4 is completed. \( \square \)

**Lemma 4.5** It holds that
\[
\sup_{t \in [0,T]} \| (\rho_t, P(\rho)_t, \lambda(\rho)_t) \|_{H^1}(t) + \int_0^T \| (\rho_{tt}, P(\rho)_{tt}, \lambda(\rho)_{tt}) \|_2^2 dt \leq C.
\]

**Proof:** From the continuity equation, it holds that \( \rho_t = -u \nabla \rho - \rho \text{div} u \) and \( \rho_{tt} = -u_t \cdot \nabla \rho - u \cdot \nabla \rho_t - \rho_{t} \text{div} u_t + \rho_{tt} \text{div} u \), and thus
\[
\sup_{t \in [0,T]} \| \nabla \rho_t \|_2^2(t) \leq \sup_{t \in [0,T]} \left[ \| \nabla \rho \|_4^2 \| \nabla u \|_4 + \| u \|_\infty^2 \| \nabla^2 \rho \|_2 + \| \rho \|_\infty \| \nabla^2 u \|_2 \right] \leq C.
\]
and
\[
\| \rho_{tt} \|_2^2 \leq \left[ \| u_t \|_4^2 \| \nabla \rho \|_2^2 \| \nabla u \|_4 + \| u \|_{\infty}^2 \| \nabla \rho_t \|_2^2 + \| \rho_t \|_4^2 \| \nabla^2 u \|_2^2 + \| \rho \|_{\infty}^2 \| \nabla u_t \|_2^2 \right]
\]
\[
\leq C(\| \nabla u_t \|_2^2 + \| u_t \|_4^2 + 1).
\]
Therefore, it holds that
\[
\int_0^T \| \rho_{tt} \|_2^2 dt \leq C \int_0^T (\| \nabla u_t \|_2^2 + \| u_t \|_4^2 + 1) dt \leq C.
\]
Similarly, one has
\[
\sup_{t \in [0, T]} \left\| \nabla (P(\rho)_t, \lambda(\rho)_t) \right\|_2^2 + \int_0^T \left\| (P(\rho)_{tt}, \lambda(\rho)_{tt}) \right\|_2^2 dt \leq C.
\]
Thus the proof of Lemma 4.5 is completed. \[\square\]

**Lemma 4.6** It holds that
\[
\sup_{t \in [0, T]} \left[ t\left\| \nabla u_t \right\|_2^2 + \left\| (\rho, P(\rho)) \right\|_{W^{2,q}(\mathbb{R}^2)} + \left\| (\nabla \rho, \nabla P(\rho)) \right\|_{\infty} \right]
+ \int_0^T t \left[ \left\| \nabla \rho u_{tt} \right\|_2^2(t) + \left\| \nabla^2 u_t \right\|_2^2(t) \right] dt \leq C,
\]
where \( q > 2 \) is given in Theorem 1.1.

**Proof:** The estimates are similar to Lemma 4.5 in [31] except noting that, by (2.3),
\[
\nabla \dot{u} = \nabla (L, H)_t = \nabla u_t - \nabla (u \cdot \nabla u).
\]
Consequently, it holds that
\[
\sup_{t \in [0, T]} \left[ t\left\| \nabla \dot{u} \right\|_2^2(t) \right] \leq \sup_{t \in [0, T]} \left[ t\left\| \nabla u_t \right\|_2^2(t) + t\left\| u \right\|_{\infty}^2 \left\| \nabla^2 u \right\|_2^2 + t\left\| \nabla u \right\|_{\infty}^2 \right] \leq C. \tag{4.20}
\]
We omit the details and the proof of Lemma 4.6 is completed. \[\square\]

Based on the estimates obtained so far, similar to Lemma 4.6 in [31], one has

**Lemma 4.7** It holds that for any \( 0 < \tau \leq T \),
\[
\sup_{t \in [0, T]} \left[ t^2 \left\| \nabla u_{tt} \right\|_2^2(t) \right] + \int_0^T t^2 \left\| \nabla u_{tt} \right\|_2^2(t) dt \leq C.
\]

Finally, we have the following

**Lemma 4.8** It holds that
\[
\sup_{t \in [0, T]} \left[ t\left| \nabla \dot{u} \cdot x \right|_2 \left| |x|_2 \right|_2 + t\left\| (u_t, \dot{u}_t) \right\|_2^2 + t\left\| \nabla^2 u_t \right\|_2^2 + t^2 \left\| \nabla^3 u \right\|_2^2 \right]
+ \int_0^T t\left| \nabla \rho u_{tt} \cdot x \right|_2^2(t) dt \leq C.
\]

**Proof:** Applying the operator \( \partial_t + \text{div}(u \cdot) \) to the equation (???), \( i = 1, 2 \) gives that
\[
\rho \dot{u}_{tt} + \rho u \cdot \nabla \dot{u}_i = \mu \Delta \dot{u}_i + \mu \text{div}(u \Delta u_i)
+ \partial_{x_i}(\mu + \lambda(\rho)) \text{div} u + \text{div} \left[ u \partial_{x_i}(\mu + \lambda(\rho)) \text{div} u \right] - \partial_{x_i} P(\rho) - \text{div}(u \partial_{x_i} P(\rho)). \tag{4.21}
\]
First, it holds that

\[ Q_1 = | - \int \rho u \cdot \nabla \hat{u} \cdot u_t| x^\frac{\alpha}{2} dx | \leq \| \sqrt{\rho} u_t | x^\frac{\alpha}{2} \|_2 \| \sqrt{\rho} u | x^\frac{\alpha}{2} \|_2 \]
\[ \leq C \| \sqrt{\rho} u_t | x^\frac{\alpha}{2} \|_2 \| \nabla \hat{u} (1 + | x^\frac{\alpha}{2} \|_2 \| + C_\sigma \| \nabla \hat{u} (1 + | x^\frac{\alpha}{2} \|_2. \]  

Then, one can obtain

\[ Q_2 = - \frac{d}{dt} \left[ \int \mu \cdot u_t \partial_{x} x^\frac{\alpha}{2} dx + \int (\mu + \lambda) (\text{div} \hat{u}) \mu \cdot \nabla | x^\frac{\alpha}{2} dx \right] \]
\[ + \int \mu \cdot u_t \partial_{x} x^\frac{\alpha}{2} dx + \int (\mu + \lambda) (\text{div} \hat{u}) \mu \cdot \nabla | x^\frac{\alpha}{2} dx + \int (\mu + \lambda) \text{div} \hat{u} \mu \cdot \nabla | x^\frac{\alpha}{2} dx \]
\[ \leq - \frac{d}{dt} \left[ \int \mu \cdot u_t \partial_{x} x^\frac{\alpha}{2} dx + \int (\mu + \lambda) (\text{div} \hat{u}) \mu \cdot \nabla | x^\frac{\alpha}{2} dx \right] \]
\[ + C \| \hat{u} | x^\frac{\alpha}{2} \|_2 \| \nabla \hat{u}_t \|_2 + C | | \lambda(\rho) | \|_\infty \| \hat{u} | x^\frac{\alpha}{2} \|_2 \| \nabla \hat{u}_t \|_2 \]
\[ \leq - \frac{d}{dt} \left[ \int \mu \cdot u_t \partial_{x} x^\frac{\alpha}{2} dx + \int (\mu + \lambda) (\text{div} \hat{u}) \mu \cdot \nabla | x^\frac{\alpha}{2} dx \right] \]
\[ + C \| \nabla \hat{u} | x^\frac{\alpha}{2} \|_2 \| \nabla \hat{u}_t \|_2 + \| \text{div} \hat{u} \|_\infty (\| \text{div} \hat{u}_t \|_2 + \| \nabla \hat{u}_t \|_2 + \| \nabla \hat{u} \|_2 + \| \text{div} \hat{u} \|_\infty \| \nabla^2 u \|_2 + \| \hat{u}_t \|_\infty \| \nabla^2 u \|_2. \]

Obviously,

\[ | Q_5 | \leq C \| \lambda(\rho) | \|_\infty \| (\text{div} \hat{u}) | x^\frac{\alpha}{2} \|_2 \]
\[ \leq C (1 + \| \text{div} u \|_\infty) \| \nabla \hat{u} | x^\frac{\alpha}{2} \|_2. \]  

Now we estimate \( Q_4 \), which contains six integrals. For simplicity, only the first and the last terms in \( Q_4 \), denoted by \( Q_4^1 \) and \( Q_4^6 \), respectively, will be computed as follows. The others terms
in $Q_4$ can be done similarly.

\[
Q_4^1 = \frac{d}{dt} \int \partial_{x_j} u \cdot \nabla u \cdot (\dot{u} | x^{\frac{3}{2}}) dx + \int \partial_{x_j} u_t \cdot \nabla u \cdot (\dot{u} | x^{\frac{3}{2}}) dx + \int \partial_{x_j} u \cdot \nabla u_t \cdot (\dot{u} | x^{\frac{3}{2}}) dx
\]

\[
\leq \frac{d}{dt} \int \partial_{x_j} u \cdot \nabla u \cdot (\dot{u} | x^{\frac{3}{2}}) dx + \|\nabla u\|_\infty \|\nabla u_t\|_2 \left[ \|\nabla \dot{u} \cdot x^{\frac{3}{2}}\|_2 + \|\dot{u} | x^{\frac{3}{2}}\|^{-1}_2 \right]
\]

\[
\leq \frac{d}{dt} \int \partial_{x_j} u \cdot \nabla u \cdot (\dot{u} | x^{\frac{3}{2}}) dx + C \|\nabla u\|_\infty \|\nabla u_t\|_2 \|\dot{u} | x^{\frac{3}{2}}\|_2
\]

\[
\leq \frac{d}{dt} \int \partial_{x_j} u \cdot \nabla u \cdot (\dot{u} | x^{\frac{3}{2}}) dx + C \left[ \|\nabla \dot{u} | x^{\frac{3}{2}}\|_2 + \|\nabla u\|_\infty^2 \|\nabla u_t\|_2^2 \right],
\]

(4.26)

and

\[
Q_4^6 = \frac{d}{dt} \int (\mu + \lambda(\rho))(\partial_{x_j} u_k \partial_{x_k} u_j) \text{div}(\dot{u} | x^{\frac{3}{2}}) dx
\]

\[
+ \int (\mu + \lambda(\rho)) (\partial_{x_j} u_k \partial_{x_k} u_j) \text{div}(\dot{u} | x^{\frac{3}{2}}) dx + \int \lambda(\rho) \partial_{x_j} u_k \partial_{x_k} u_j \text{div}(\dot{u} | x^{\frac{3}{2}}) dx
\]

\[
\leq \frac{d}{dt} \int (\mu + \lambda(\rho))(\partial_{x_j} u_k \partial_{x_k} u_j) \text{div}(\dot{u} | x^{\frac{3}{2}}) dx
\]

\[
+ \left( \|\nabla u\|_\infty \|\nabla u_t\|_2 + \|\lambda(\rho)\| \|\nabla \dot{u}\|_2 \right) \left[ \|\nabla \dot{u} | x^{\frac{3}{2}}\|_2 + \|\dot{u} | x^{\frac{3}{2}}\|^{-1}_2 \right]
\]

\[
\leq \frac{d}{dt} \int \partial_{x_j} u \cdot \nabla u \cdot (\dot{u} | x^{\frac{3}{2}}) dx + C \left[ \|\nabla \dot{u} | x^{\frac{3}{2}}\|_2 + \|\nabla u\|_\infty^2 (1 + \|\nabla u_t\|_2^2) \right].
\]

It follows that

\[
Q_5 = -\frac{d}{dt} \left[ \int P(\rho) u_{x_j} \cdot \nabla (\dot{u} | x^{\frac{3}{2}}) + \int (\gamma - 1) P(\rho) \text{div}(u) \text{div}(\dot{u} | x^{\frac{3}{2}}) dx \right]
\]

\[
+ \int P(\rho) u_{x_j} \cdot \nabla (\dot{u} | x^{\frac{3}{2}}) + \int (\gamma - 1) P(\rho) \text{div}(u) \text{div}(\dot{u} | x^{\frac{3}{2}}) dx
\]

\[
+ \int P(\rho) u_{x_j} \cdot \nabla (\dot{u} | x^{\frac{3}{2}}) + \int (\gamma - 1) P(\rho) \text{div}(u) \text{div}(\dot{u} | x^{\frac{3}{2}}) dx
\]

\[
\leq -\frac{d}{dt} \left[ \int P(\rho) u_{x_j} \cdot \nabla (\dot{u} | x^{\frac{3}{2}}) + \int (\gamma - 1) P(\rho) \text{div}(u) \text{div}(\dot{u} | x^{\frac{3}{2}}) dx \right]
\]

\[
+ C |P(\rho)| \|\nabla u\|_2 \|\nabla \dot{u} | x^{\frac{3}{2}}\|_2 + C \|P(\rho)\| \|\nabla u_t\|_2 ||\nabla \dot{u} | x^{\frac{3}{2}}\|_2
\]

\[
\leq -\frac{d}{dt} \left[ \int P(\rho) u_{x_j} \cdot \nabla (\dot{u} | x^{\frac{3}{2}}) + \int (\gamma - 1) P(\rho) \text{div}(u) \text{div}(\dot{u} | x^{\frac{3}{2}}) dx \right]
\]

\[
+ C \left[ \|\nabla \dot{u} | x^{\frac{3}{2}}\|_2 + \|\nabla u\|_\infty^2 + \|\nabla u_t\|_2^2 \right].
\]

(4.28)

Substituting (4.23), (4.24), (4.25), (4.26), (4.27) and (4.28) into (4.22) and choosing $\sigma = \frac{1}{2}$ gives that

\[
\frac{d}{dt} R(t) + \frac{1}{2} \|\sqrt{\rho} \dot{u} | x^{\frac{3}{2}}\|_2^2 \leq C \left[ \left( \|\nabla u\|_\infty^2 + 1 \right) \left( 1 + \|\nabla u_t\|_2^2 + \|\nabla \dot{u}\|_2^2 + \|\nabla \dot{u} | x^{\frac{3}{2}}\|_2 \right) + \left( 1 + \|\text{div} u\|_\infty \right) \left( \|\nabla \dot{u} | x^{\frac{3}{2}}\|_2 + \|\nabla u_t\|_2 + \|\nabla^2 u_t\|_2^2 + \|u_t\|_{\frac{3}{2}} \right) \right]
\]

\[
\leq C \left[ \left( \|\nabla u\|_\infty^2 + 1 \right) \left( 1 + \|\nabla u_t\|_2^2 + \|\nabla \dot{u}\|_2^2 + \|\nabla \dot{u} | x^{\frac{3}{2}}\|_2 + \left( 1 + \|\text{div} u\|_\infty \right) \|\nabla \dot{u} | x^{\frac{3}{2}}\|_2 \right)
\]

\[
+ \|\nabla \dot{u} | x^{\frac{3}{2}}\|_2 \|\nabla u_t\|_2 + \|\nabla^2 u_t\|_2^2 + \|u_t\|_{\frac{3}{2}} \|\nabla^2 u_t\|_2^2 \right],
\]

(4.29)
where

\[ R(t) = \frac{1}{2} \left[ \mu \| \nabla \dot{u} \|_2^2 + \mu \| (\text{div} \dot{u}) \|_2^2 + \| \lambda(\rho) \| \| (\text{div} \dot{u}) \|_2^2 \right] + \int (\mu + \lambda(\rho)) (\text{div} \dot{u}) \cdot \nabla (|x|^{\frac{2}{r}} \dx - \int \partial_{x_j} u \cdot \nabla u \cdot (\dot{u} |x|^{\frac{2}{r}})_{x_j} \dx 
- \int \partial_{x_j} u \cdot \nabla u \cdot (\dot{u} |x|^{\frac{2}{r}})_{x_j} \dx + \int (\text{div} \partial_{x_j} u \cdot (\dot{u} |x|^{\frac{2}{r}})_{x_j} \dx - \int u_{x_j} \cdot \nabla (\dot{u} |x|^{\frac{2}{r}}) \cdot u_{x_j} \dx 
+ \int (\mu + \lambda(\rho) + \rho \lambda'(\rho)) (\text{div} u)^2 \text{div}(\dot{u} |x|^{\frac{2}{r}}) \dx - \int (\mu + \lambda(\rho)) (\text{div} u) u_{x_j} \cdot \nabla (\dot{u} |x|^{\frac{2}{r}}) \dx 
+ \int P(\rho) u_{x_j} \cdot \nabla (\dot{u} |x|^{\frac{2}{r}}) + \int (\gamma - 1)P(\rho) (\text{div} u) \text{div}(\dot{u} |x|^{\frac{2}{r}}) dx. \]

(4.30)

It can be computed that

\[ \int_0^T R(t) dt \leq C \int_0^T \left[ \| \nabla \dot{u} (1 + |x|^{\frac{2}{r}}) \|_2^2 + \| \nabla u \|_4^2 + \| \nabla u \|_2^2 \right] dt \leq C, \]

and

\[ \left| \int \mu \dot{u} \cdot \partial_{x_j} (|x|^{\frac{2}{r}}) dx \right| \leq \frac{\mu \alpha}{2} \| \nabla \dot{u} |x|^{\frac{2}{r}} \|_2^2 \| \dot{u} |x|^{\frac{2}{r}} \|_2 \leq \frac{\mu \alpha^2}{8} \| \nabla \dot{u} |x|^{\frac{2}{r}} \|_2^2, \]

(4.32)

For \( r > 2 \) and close to 2 and satisfying \( \frac{1}{r} + \frac{1}{2} = \frac{1}{2} \) and \( \frac{1}{r} + \frac{2}{2} = \frac{2}{8} \theta \) with \( \theta \in (0, 1) \), it holds that

\[ \left| \int (\mu + \lambda(\rho)) (\text{div} \dot{u}) \dot{u} \cdot \nabla (|x|^{\frac{2}{r}}) \right| \leq \frac{\mu \alpha}{2} \left[ \| (\text{div} \dot{u}) |x|^{\frac{2}{r}} \|_2 \| \dot{u} |x|^{\frac{2}{r}} \|_2 + \| \sqrt{\lambda(\rho)} (\text{div} \dot{u}) \|_2 \| \dot{u} |x|^{\frac{2}{r}} \|_2 \right] \]
\[ \leq \frac{\mu \alpha^2}{8} \| (\text{div} \dot{u}) |x|^{\frac{2}{r}} \|_2 \| \dot{u} |x|^{\frac{2}{r}} \|_2 + C \| \sqrt{\lambda(\rho)} (\text{div} \dot{u}) \|_2 \| \dot{u} |x|^{\frac{2}{r}} \|_2 \]
\[ \leq \frac{\mu \alpha^2}{8} \| (\text{div} \dot{u}) |x|^{\frac{2}{r}} \|_2 \| \dot{u} |x|^{\frac{2}{r}} \|_2 + C \| \sqrt{\lambda(\rho)} (\text{div} \dot{u}) \|_2 \| \dot{u} |x|^{\frac{2}{r}} \|_2 \]
\[ + C \| \nabla \dot{u} \|_2 + C \| \nabla u |x|^{\frac{2}{r}} \|_2. \]

By (4.18), it holds that

\[ \| \nabla^3 u \|_2 \leq C \| \Phi \|_2 \leq C \left[ \| \nabla^2 P(\rho) \|_2 + \| \rho \|_{\infty} \| \nabla \dot{u} \|_2 + \| \nabla \rho \|_{\frac{5}{4}} \| \dot{u} \|_{\frac{5}{4} \theta} + \left( \| \nabla^2 \rho \|_{1} + \| \rho \|_{\frac{3}{8}} \right) \| \text{div} u \|_4 + \| \nabla \rho \|_{\infty} \| \nabla^2 u \|_2 \right], \]

(4.34)

Then the other terms in (4.30) can be estimated as

\[ - \int \partial_{x_j} u \cdot \nabla u \cdot (\dot{u} |x|^{\frac{2}{r}})_{x_j} \dx + \int \partial_{x_j} u \cdot \nabla u \cdot (\dot{u} |x|^{\frac{2}{r}})_{x_j} \dx 
+ \int (\text{div} u) \partial_{x_j} u \cdot (\dot{u} |x|^{\frac{2}{r}})_{x_j} \dx - \int u_{x_j} \cdot \nabla (\dot{u} |x|^{\frac{2}{r}}) \cdot u_{x_j} \dx 
+ \int (\mu + \lambda(\rho) + \rho \lambda'(\rho)) (\text{div} u)^2 \text{div}(\dot{u} |x|^{\frac{2}{r}}) \dx - \int (\mu + \lambda(\rho)) (\text{div} u) u_{x_j} \cdot \nabla (\dot{u} |x|^{\frac{2}{r}}) \dx 
+ \int P(\rho) u_{x_j} \cdot \nabla (\dot{u} |x|^{\frac{2}{r}}) + \int (\gamma - 1)P(\rho) (\text{div} u) \text{div}(\dot{u} |x|^{\frac{2}{r}}) dx \]

(4.35)

\[ \leq C \| \nabla \dot{u} |x|^{\frac{2}{r}} \|_2 \left[ \| \nabla u |x|^{\frac{2}{r}} \|_2 + \| \dot{u} |x|^{\frac{2}{r}} \|_2 \right] \left( \| \nabla u \|_{\infty} + 1 \right) \]
\[ \leq C \| \nabla \dot{u} |x|^{\frac{2}{r}} \|_2 \| \nabla u |x|^{\frac{2}{r}} \|_2 \left( \| \nabla u \|_{\infty} + 1 \right) \]
\[ \leq C \| \nabla \dot{u} |x|^{\frac{2}{r}} \|_2 \| \text{div} u (1 + |x|^{\frac{2}{r}}) \|_2 \left( \| \nabla u \|_{\frac{3}{2}} + \| \nabla \dot{u} |x|^{\frac{2}{r}} \|_2 + 1 \right) \leq C \| \nabla \dot{u} |x|^{\frac{2}{r}} \|_2 \| \nabla u \|_{\frac{3}{2}} + 1 \]
\[ \leq C \| \nabla \dot{u} |x|^{\frac{2}{r}} \|_2 \left( \| \nabla u \|_{\frac{3}{2}} + \| \nabla \dot{u} |x|^{\frac{2}{r}} \|_2 + 1 \right) \leq \| \nabla \dot{u} |x|^{\frac{2}{r}} \|_2 + C \| \nabla \dot{u} \|_{\frac{3}{2} + 1}. \]
Thus it holds that

\[ R(t) \geq \frac{\mu}{2} \left[ \left( 1 - \frac{\alpha^2}{4} \right) \|\nabla \dot{u} \|_2^2 + \|\text{div} \dot{u} \|_2^2 - \frac{\alpha^2}{4} (\|\text{div} \dot{u} \|_2^2) \right] + \left( \frac{1}{2} - \sigma \right) \|\sqrt{\lambda(\rho)}(\text{div} \dot{u}) \|_2^2 - 2 \sigma \|\nabla \dot{u} \|_2^2 - C_\sigma (\|\nabla \dot{u} \|_2^2 + 1) \]

(4.36)

where in the last inequality one has used the fact that the quadratic term in the square brackets is positively definite when \( \alpha^2 < 4(\sqrt{2} - 1) \) and one has chosen \( \sigma \) suitably small.

Multiplying the inequality (4.29) by \( t \) and then integrating the resulting inequality with respect to \( t \) over \([\tau, t_1] \) with both \( \tau, t_1 \in [0, T] \) give that

\[
t_1 R(t_1) + \frac{1}{2} \int_\tau^{t_1} t \|\sqrt{\rho} \dot{u}_t \|_2^2 dt \leq \tau R(\tau) + C \int_\tau^{t_1} \left[ (\|\nabla u \|_\infty^2 + 1)(1 + \|u_t \|_2 + \|\dot{u} \|_2^2) + t \|\nabla u \|_2^2 + \|u_t \|_2^2 + \|\dot{u} \|_2^2 + \|\nabla u \|_2^2 + \|u_t \|_2^2 + \|\dot{u} \|_2^2 + R(t) \right] dt
\]

(4.37)

By (4.31), there exists a subsequence \( \tau_k \) such that

\[
\tau_k \to 0, \quad \tau_k R(\tau_k) \to 0, \quad \text{as} \quad k \to +\infty.
\]

Take \( \tau = \tau_k \) in (4.37), then \( k \to +\infty \) and using Gronwall inequality, one can obtain

\[
\sup_{t \in [0, T]} \left[ t R(t) \right] + \int_0^T \|\sqrt{\rho} \dot{u}_t \|_2^2 dt \leq C,
\]

which, combined with (4.36) and Lemma 4.6 yields that

\[
\sup_{t \in [0, T]} \left[ t \|\nabla \ddot{u} \|_2^2 + t \|u_t \|_2^2 + \|\dot{u} \|_2^2 \right] + \int_0^T \|\sqrt{\rho} \dot{u}_t \|_2^2 dt \leq C.
\]

Finally, by (4.18), it holds that

\[
\|\nabla^3 u \|_2^2 \leq C \left[ (\|\nabla^2 \rho, \nabla^2 \rho \|_2^2 (\|\nabla u \|_\infty + 1) + \|\nabla \dot{u} \|_2 + \|\nabla \dot{u} \|_2 + \|\dot{u} \|_2 + \|\nabla \rho \|_\infty \|\nabla^2 u \|_2 \right]
\]

\[
\leq C \left[ [\|\nabla u \|_2^2 + \|\nabla^3 u \|_2^2] + \|\nabla \dot{u} \|_2 + \|\dot{u} \|_2 + 1 \right] \leq C \left[ \|\nabla^3 u \|_2^2 + \|\nabla \dot{u} \|_2 + \|\dot{u} \|_2 + 1 \right],
\]

which implies that

\[
\sup_{t \in [0, T]} \left[ t \|\nabla^3 u \|_2^2 \right] \leq C \sup_{t \in [0, T]} \left[ t \|\nabla \ddot{u} \|_2^2 + t \|u_t \|_2^2 \right] \leq C.
\]

Then it holds that

\[
\sup_{t \in [0, T]} \left[ t^2 \|\nabla u \|_2^2 \right] \leq C \sup_{t \in [0, T]} \left[ t^2 \|\nabla u \|_2^2 + t^2 \|u_t \|_2^2 \right] \leq C.
\]

and one can obtain

\[
\sup_{t \in [0, T]} \left[ t^2 \|\nabla u \|_2^2 \right] \leq C \sup_{t \in [0, T]} \left[ t^2 \|\nabla u \|_2^2 + t^2 \|\dot{u} \|_2^2 \right] \leq C.
\]

So the proof of Lemma 4.8 is completed. \( \square \)
5 The proof of main results

In this section, we give the proof of our main results.

The proof of Theorem 1.1. We first show that \((\rho, u)\) is a classical solution to (1.1) if \((\rho, u)\) satisfies (1.9). Since \(u \in L^2(0; T; L^2 \cap D^3(\mathbb{R}^2))\) and \(u_t \in L^2(0; T; L^{\frac{2}{3}} \cap D^1(\mathbb{R}^2))\), so the Sobolev’s embedding theorem implies that

\[
u \in C([0, T]; L^2 \cap D^2(\mathbb{R}^2)) \hookrightarrow C([0, T] \times \mathbb{R}^2).
\]

Then it follows from \((\rho, P(\rho)) \in L^\infty(0, T; W^{2,q}(\mathbb{R}^2))\) and \((\rho, P(\rho))_t \in L^\infty(0, T; H^1(\mathbb{R}^2))\) that \((\rho, P(\rho)) \in C([0, T]; W^{1,q}(\mathbb{R}^2)) \cap C([0, T]; W^{2,q}(\mathbb{R}^2) - \text{weak})\). This and Lemma 2.6 then imply that

\[
(\rho, P(\rho)) \in C([0, T]; W^{2,q}(\mathbb{R}^2)).
\]

Since for any \(\tau \in (0, T)\),

\[
(\nabla u, \nabla^2 u) \in L^\infty(\tau, T; W^{1,q}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)), \quad (\nabla u_t, \nabla^2 u_t) \in L^\infty(\tau, T; L^2(\mathbb{R}^2)).
\]

Therefore,

\[
(\nabla u, \nabla^2 u) \in C([\tau, T] \times \mathbb{R}^2),
\]

Due to the fact that

\[
\nabla(\rho, P(\rho)) \in C([0, T]; W^{1,q}(\mathbb{R}^2)) \hookrightarrow C([0, T] \times \mathbb{R}^2)
\]

and the continuity equation (1.1) \(_1\), it holds that

\[
\rho_t = u \cdot \nabla \rho + \rho \text{div} u \in C([\tau, T] \times \mathbb{R}^2).
\]

It follows from the momentum equation (1.1) \(_2\) that

\[
(\rho u)_t = \mathcal{L}_p u - \text{div}(\rho u \otimes u) - \nabla P(\rho)
= \mu \Delta u + (\mu + \lambda(\rho))\nabla(\text{div} u) + (\text{div} u)\nabla \lambda(\rho) + \rho u \cdot \nabla u + \rho \text{div} u + (u \cdot \nabla \rho)u - \nabla P(\rho)
\in C([\tau, T] \times \mathbb{R}^2).
\]

Then the proof of Theorem 1.1 follows from Lemma 2.1 which is about the local well-posedness of the classical solution and the global (in time) a priori estimates in Sections 3-4. In fact, by Lemma 2.1 there exists a local classical solution \((\rho, u)\) on the time interval \((0, T_\ast)\) with \(T_\ast > 0\). Now let \(T^\ast\) be the maximal existing time of the classical solution \((\rho, u)\) in Lemma 2.1. Then obviously one has \(T^\ast \geq T_\ast\). Now we claim that \(T^\ast > T\) with \(T > 0\) being any fixed positive constant given in Theorem 1.1. Otherwise, if \(T^\ast \leq T\), then all the a priori estimates in Sections 3-4 hold with \(T\) being replaced by \(T^\ast\). In particular, from the inequality (3.98), it holds that

\[
(1 + |x|^\frac{1}{p})\sqrt{\rho \mu} \in C([0, T^\ast]; L^2(\mathbb{R}^2)).
\]

Therefore, it follows from a priori estimates in Sections 3-4 that \((\rho, u)(x, T^\ast)\) satisfy (1.5) and the compatibility condition (1.6) at time \(t = T^\ast\). By using Lemma 2.1 again, there exists a \(T_1^\ast > 0\) such that the classical solution \((\rho, u)\) in Lemma 2.1 exist on \((0, T^\ast + T_1^\ast)\), which contradicts with \(T^\ast\) being the maximal existing time of the classical solution \((\rho, u)\). Thus it holds that \(T^\ast > T\), and the proof of Theorem 1.1 is completed. \(\square\)
The proof of Theorem 1.2. Based on Theorem 1.1, one can prove Theorem 1.2 easily as follows. Since

$$\rho_0 \in H^3(\mathbb{R}^2) \hookrightarrow W^{2,q}(\mathbb{R}^2)$$

for any $2 < q < +\infty$, it follows that under the conditions of Theorem 1.2, Theorem 1.1 holds for any $2 < q < +\infty$. Thus, we need only to prove the higher order regularity presented in Theorem 1.2.

□

Lemma 5.1 It holds that

$$\sup_{t \in [0,T]} \left[ \| \sqrt{\rho} \nabla^3 u(t) \|_2 + \| (\rho, P(\rho), \lambda(\rho)) \|_{H^3}(t) \right] + \int_0^T \| \nabla^4 u \|_2^2 dt \leq C.$$

The proof of Lemma 5.1 is completely similar to Lemma 6.1 in [31]. We omit the details here for simplicity and the proof Theorem 1.2 is complete.

□

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