LEVEL REPULSION FOR SCHRÖDINGER
OPERATORS WITH SINGULAR CONTINUOUS
SPECTRUM

JONATHAN BREUER AND DANIEL WEISSMAN

Abstract. We describe a family of half-line continuum Schrödinger operators with purely singular continuous spectrum on \((0, \infty)\), exhibiting asymptotic strong level repulsion (known as clock behavior). This follows from the convergence of the renormalized continuum Christoffel-Darboux kernel to the sine kernel.

1. Introduction

The problem of understanding the asymptotics of finite-volume level spacings for Schrödinger operators has been receiving a considerable amount of attention in recent years ([1, 2, 3, 5, 6, 10, 11, 14, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 32, 33] is a small subset of relevant references). While results in the multidimensional setting were obtained predominantly for random operators, in the one-dimensional case both deterministic and random operators were studied.

Particularly interesting in this context is the problem of understanding how the asymptotics of level spacings are connected to continuity properties of the spectral measures. Known results indicate a certain rough correspondence between asymptotic repulsion and continuity of the spectral measures. Perhaps the most studied case is that of the localized regime in the Anderson model, where it has been shown under various conditions that the rescaled finite-volume eigenvalue process converges to a Poisson process on the line (see, e.g., [14, 25, 26]). Results for one-dimensional operators with random decaying potentials [6, 16, 17, 27] seem to indicate a pattern whereby greater continuity of the spectral measure corresponds to greater repulsion. Deterministic results establishing asymptotic regular spacing for absolutely continuous measures [2, 10, 19, 20, 21, 22, 23, 24, 29, 30, 32] work in the same vein.

Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel. E-mail: jbreuer@math.huji.ac.il; daniel.weissman@mail.huji.ac.il. Supported in part by The Israel Science Foundation (Grant No. 1105/10).
Our aim in the present paper is to show that the situation is more subtle than what may be thought in light of the discussion above. We shall present a family of half-line Schrödinger operators with purely singular continuous spectrum on the positive half-line, whose finite-volume eigenvalues display clock asymptotic behavior (see Definition 1.1 below), which is a very strong form of repulsion. This shows that strong repulsion should not be associated only with absolutely continuous spectrum.

For discrete Schrödinger operators, an analogous result was obtained by one of us [5] in the context of studying conditions on measures guaranteeing universal behavior of the associated Christoffel-Darboux (CD) kernel. In particular, in [5] clock spacing is a consequence of the convergence of the CD kernel to the sine kernel. We shall obtain our result here by exploiting the analogy between the discrete and continuous case.

To fix terminology and notation, a Schrödinger operator acting on \( \mathbb{R}^+ \) is an operator of the form

\[
H = \Delta + V
\]

where \( \Delta = -\frac{d^2}{dx^2} \) denotes the Laplacian, and the operator defined by multiplication by \( V(x) \) is the potential (we shall soon specify conditions on \( V \)). We will assume Neumann boundary conditions throughout, i.e.

\[
u (0) = 1 \text{ and } \frac{d}{dx}u(x)|_{x=0} = 0.
\]

By the spectral theorem there exists a unique measure, \( \mu \), for which \( H \) is unitarily equivalent to the operator of multiplication by \( x \) on \( L^2(\mathbb{R},d\mu(x)) \). We call \( \mu \) the spectral measure associated with the operator \( H \).

Given \( H \) as above, we can restrict it to intervals [0, \( L \)] and consider the operator

\[
H^L = \Delta + V|_{[0,L]}
\]

acting on \( L^2(0,L) \), this time with Neumann boundary conditions at \( L \) as well as at 0, i.e., with the additional condition

\[
u' (L) = 0.
\]

The spectrum of \( H^L \) is a discrete set of eigenvalues \( \{ \xi^L_j \} \). We are interested in the connection between properties of \( \mu \) and asymptotic properties (as \( L \to \infty \)) of the spacings between the \( \xi^L_j \)'s. The asymptotic density of these eigenvalues is measured by the \textit{density of states measure}, \( \nu \), defined as the weak limit (if it exists) of the normalized
Let $I \subseteq \mathbb{R}^+$ be a closed interval, and let $\xi^* \in I$. For each $L > 0$, reenumerate the eigenvalues \( \{ \xi^L_j \} \) around $\xi^*$ as follows:

$$
\ldots < \xi^L_{n+1} (\xi^*) < \xi^* \leq \xi^L_{n} (\xi^*) < \xi^L_1 (\xi^*) < \ldots
$$

Following [2], we say there is strong clock behavior at $\xi^*$, if the density of states exists and for each fixed $n$,\n
$$
\lim_{L \to \infty} L \left( \xi^L_{n+1} (\xi^*) - \xi^L_{n} (\xi^*) \right) \rho (\xi^*) = 1 \quad (1.3)
$$

We say there is uniform clock behavior on the interval $I \subseteq \mathbb{R}^+$, if the above limit is uniform on $I$ for each fixed $n$.

**Remark 1.1.** Originally, the concept was defined for operators arising in the theory of orthogonal polynomials on the unit circle [29]. On the unit circle, the equally-spaced eigenvalues appear as marks on a clock – hence the name.

**Definition 1.2.** If $u (\xi, x)$ is the unique solution of the equation

$$
- \frac{d^2}{dx^2} u (\xi, x) + V (x) u (\xi, x) = \xi \cdot u (\xi, x) \quad (1.4)
$$

with Neumann boundary conditions, the continuous Christoffel-Darboux (CD) kernel at $L$ is

$$
S_L (\xi, \zeta) = \int_0^L u (\xi, r) u (\zeta, r) \, dr \quad (1.5)
$$

This object, introduced in [23], is the continuous analog of the classical CD kernel. The CD kernel arises naturally in the study of orthogonal polynomials and has a wide range of applications (see [31] for a survey). In particular, the phenomenon of universality in random matrix theory is intimately connected with the fact that in many cases the rescaled CD kernel converges to the sine kernel ([9]).

The significance of this, in our case, lies in the fact that if $u' (\xi, L) = 0$, then $u' (\zeta, L) = 0$ iff $S_L (\xi, \zeta) = 0$. Since the zeros of $u' (\cdot, L)$ are the...
eigenvalues of $H^L$, this means there is a connection between the asymptotic properties of $S_L(\xi + \frac{a}{L}, \xi + \frac{b}{L})$, and the small-scale behavior of the $\xi_j^L$'s around $\xi$, as $L \to \infty$.

We shall prove uniform clock behavior for the operators under consideration by showing that the associated CD kernel satisfies

$$\frac{S_L(\xi + \frac{a}{L}, \xi + \frac{b}{L})}{S_L(\xi, \xi)} \xrightarrow{L \to \infty} \frac{\sin (\pi \rho(\xi) (b - a))}{\pi \rho(\xi) (b - a)}$$

(1.6)

uniformly for $\xi$ in compact subsets of $\mathbb{R}^+$ and $a, b$ in compact subsets of the strip $|\text{Im}z| \leq 1$. The fact that this, together with existence of the density of states (which needs to be established separately), implies uniform clock behavior in the discrete case is known as the Freud-Levin Theorem [12, 20, 31]. The proof given in [31, Section 23] translates directly to the continuum case and so

**Proposition 1.3.** For an operator $H = \Delta + V$ on $L^2(\mathbb{R}^+)$, assume that (1.6) holds uniformly for $\xi$ in compact subsets of $\mathbb{R}^+$ and $a, b$ in compact subsets of the strip $|\text{Im}z| \leq 1$. Assume moreover that the density of states for $H$ exists and is absolutely continuous with respect to Lebesgue measure. Then uniform clock behavior follows on any compact interval $I \subseteq (0, \infty)$.

In the case of discrete CD kernels (1.6) has been proven under a wide range of conditions. However, except for [5] (and this work), it was always assumed that the spectral measure was absolutely continuous in a neighborhood of the point under consideration. That (1.6) implies clock behavior in the discrete case was discovered by Freud [12], and rediscovered by Levin and Lubinsky [20]. Additional results were obtained by Lubinsky in [21, 22], and in greater generality, by Simon, [30], Findley [10], and Totik [32]. Avila, Last and Simon, in [2], proved (1.6) and clock behavior for ergodic Jacobi matrices with a.c. spectrum.

To the best of our knowledge, Maltsev’s paper [23] is the only prior work dealing with (1.6) in the continuous case, where it was used to prove clock asymptotics for eigenvalues of perturbed periodic Schrödinger operators.

We are now ready to take a closer look at the potentials we consider in this work. Given a sequence of real numbers, $\{\lambda_n\}_{n=1}^{\infty}$ such that $\lambda_n \to 0$, a sequence of positive numbers $\{N_n\}_{n=1}^{\infty}$ such that $\frac{N_n}{N_{n+1}} \to 0$, and a bounded, non-negative, compactly supported function $W(x)$ (which we think of as a recurring perturbation of varying amplitude, where the amplitudes are determined by the $\lambda_n$), we define a so-called
Pearson potential (see [15]),

\[ V(x) = \sum_{n=1}^{\infty} \lambda_n W(x - N_n) \quad (1.7) \]

Thus, if we fix boundary conditions as above, there is an associated self-adjoint Schrödinger operator,

\[ H = \Delta + V. \quad (1.8) \]

Sparse potentials were first introduced by Pearson in [28] in 1978, in the construction of the first explicit examples of Schrödinger operators exhibiting purely singular continuous spectrum. The following theorem of Kiselev, Last and Simon extends the original Pearson result:

**Theorem 1.4 (Theorem 1.6 of [15]).** Let \( V \) be a Pearson potential. If \( \sum_{n=1}^{\infty} \lambda_n^2 < \infty \) (resp. \( \sum_{n=1}^{\infty} \lambda_n^2 = \infty \)), the spectrum of the operator \( \Delta + V \) is purely absolutely continuous on \( \mathbb{R}^+ \) (resp. purely singular continuous), for any boundary condition at 0.

We are now ready to state our main result.

**Theorem 1.5.** Let \( W(x) \) be a smooth, non-negative function which is supported on \([0, 1]\), and let \( \{\lambda_n\}_{n=1}^{\infty} \) be a sequence such that \( \lambda_n \to 0 \). If the sequence \( \{N_n\}_{n=1}^{\infty} \) is sufficiently sparse, then for the operator \( H = \Delta + V \), with \( V \) as defined in (1.7), and with Neumann boundary conditions, (1.6) holds uniformly for \( \xi \) in compact subsets of \( \mathbb{R}^+ \) and \( a, b \) in compact subsets of the strip \( |\text{Im} z| \leq 1 \).

**Remark 1.2.** By ‘\( \{N_n\}_{n=1}^{\infty} \) is sufficiently sparse’ we mean that \( N_{k+1} \) has to be chosen sufficiently large, as a function of \( N_1, N_2, \ldots, N_k \). In other words, for every \( k \geq 1 \), there exists a function \( \tilde{N}_k (N_1, N_2, \ldots, N_k) \), such that \( N_{k+1} \geq \tilde{N}_k (N_1, N_2, \ldots, N_k) \). The functions \( \tilde{N}_k \) will depend on \( \{\lambda_n\}_{n=1}^{\infty} \). In particular, we require that \( \frac{N_k}{N_{k+1}} \to 0 \).

**Corollary 1.6.** There exist operators with purely singular continuous spectrum on \((0, \infty)\), exhibiting uniform clock behavior on any compact interval \( I \subseteq (0, \infty) \).

**Proof.** Since \( V(x) \to 0 \) as \( x \to \infty \), it is not difficult to show that the density of states for the Pearson operators considered here exists and is equal to the free density of states (e.g., by imitating the proof of [13]). This, together with Proposition 1.3 implies the conclusion. \( \square \)

In rough terms, the strategy of our proof is as follows. Given \( H \), we define a sequence of operators by truncating the potential at increasing
points. We refer to them as the truncated operators. Like the original operator, $H$, they act on $L^2(\mathbb{R}^+)$. Let
\[ H^{(\ell)} = \Delta + V^{(\ell)} = \Delta + \sum_{n=1}^{\ell} \lambda_n W(x - N_n). \] (1.9)
These operators are defined using the same boundary conditions as $H$ (and should not be confused with the restricted operators, $H^L$). Denote by $u^{(\ell)}(\xi, x)$ the unique solution of the associated eigenfunction equation
\[ (\Delta + V^{(\ell)}) u^{(\ell)}(\xi, x) = \xi u^{(\ell)}(\xi, x). \] (1.10)
The associated CD kernel is
\[ S^{(\ell)}_L(\xi, \zeta) = \int_0^L u^{(\ell)}(\xi, r) u^{(\ell)}(\zeta, r) dr. \] (1.11)

Our strategy will be to show that since (1.6) holds for any $H^{(\ell)}$, if we place the perturbations sparsely enough (i.e. the sequence $\{N_n\}_{n=1}^\infty$), then for sufficiently small $\lambda$ the renormalized CD kernel remains close enough to its sine kernel limit. Since we want to construct the sequence $\{N_n\}_{n=1}^\infty$ such that (1.6) holds for all $\xi \in \mathbb{R}^+$ (and not only in a compact interval), the main challenge will be to control the constants as we increase the size of the interval that we study. After obtaining some preliminary results in Section 2, we prove Theorem 1.5 in Section 3.

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2. Preliminaries

We denote by $\Phi(\xi, x)$ the solution to
\[ -\frac{d^2}{dx^2}u(\xi, x) = \xi \cdot u(\xi, x) \] (2.1)
with Dirichlet boundary conditions, namely $\Phi(\xi, 0) = 0$ and $\Phi'(\xi, 0) = 1$. Similarly, we denote by $\Psi(\xi, x)$ the Neumann solution, $\Psi(\xi, 0) = 1$ and $\Psi'(\xi, 0) = 0$. We define the transfer matrix by
\[ T^{(0)}_{x, 0}(\xi) = \begin{bmatrix} \Psi(\xi, x) & \Phi(\xi, x) \\ \Psi'(\xi, x) & \Phi'(\xi, x) \end{bmatrix} = \begin{bmatrix} \cos(\sqrt{\xi}x) & \frac{1}{\sqrt{\xi}}\sin(\sqrt{\xi}x) \\ -\sqrt{\xi}\sin(\sqrt{\xi}x) & \cos(\sqrt{\xi}x) \end{bmatrix} \] (2.2)
so that if $u^{(0)}(\xi, x)$ is some solution to (2.1) then
\[ \begin{bmatrix} u^{(0)}(\xi, x) \\ u^{(0),'}(\xi, x) \end{bmatrix} = T^{(0)}_{x, 0}(\xi) \begin{bmatrix} u^{(0)}(\xi, 0) \\ u^{(0),'}(\xi, 0) \end{bmatrix}. \]
For $0 \leq a \leq b$ we further define $T_{b,a}^{(0)} = T_{b,0}^{(0)} T_{a,0}^{(0)}$. Note that, for any $0 \leq a \leq b$,

$$\det \left( T_{b,a}^{(0)} (\xi) \right) = 1. \quad (2.3)$$

Given a Pearson potential as in (1.7) with $W$ smooth, nonnegative, and with support $\subseteq [0, 1]$, recall that $u(\xi, x)$ is the unique solution of (1.4) with Neumann boundary conditions. Using the Dirichlet solution, we may define analogously the transfer matrix for $H$ to get

$$\begin{bmatrix} u(\xi, x) \\ u'(\xi, x) \end{bmatrix} = T_{x,0}^{(0)} (\xi) \begin{bmatrix} u(0) \\ u'(0) \end{bmatrix}. \quad (2.4)$$

Our analysis will rely on variation of parameters. Namely we define the functions $A_1(\xi, x)$ and $A_2(\xi, x)$ through

$$\begin{bmatrix} A_1(\xi, x) \\ A_2(\xi, x) \end{bmatrix} = T_{x,0}^{(0)} (\xi)^{-1} \begin{bmatrix} u(\xi, x) \\ u'(\xi, x) \end{bmatrix},$$

i.e.

$$u(\xi, x) = A_1(\xi, x) \Phi(\xi, x) + A_2(\xi, x) \Psi(\xi, x)$$

$$u'(\xi, x) = A_1(\xi, x) \Phi'(\xi, x) + A_2(\xi, x) \Psi'(\xi, x).$$

Analogously, for the truncated operators, $H^{(\ell)}$, and the associated generalized eigenfunctions, $u^{(\ell)}$, defined in (1.9) and (1.10), we define the associated transfer matrix $T_{x,0}^{(\ell)} (\xi)$, which satisfies

$$\begin{bmatrix} u^{(\ell)}(\xi, x) \\ u'^{(\ell)}(\xi, x) \end{bmatrix} = T_{x,0}^{(\ell)} (\xi) \begin{bmatrix} u^{(\ell)}(0) \\ u'^{(\ell)}(0) \end{bmatrix}.$$

The functions $A_1^{(\ell)}$ and $A_2^{(\ell)}$ are also defined through

$$\begin{bmatrix} A_1^{(\ell)}(\xi, x) \\ A_2^{(\ell)}(\xi, x) \end{bmatrix} = T_{x,0}^{(\ell)} (\xi)^{-1} \begin{bmatrix} u^{(\ell)}(\xi, x) \\ u'^{(\ell)}(\xi, x) \end{bmatrix}. \quad (2.5)$$

Note that for $x \geq N_\ell + 1$, the functions $A_1^{(\ell)}, A_2^{(\ell)}$ are constant in $x$. For later reference we note that, by [8, Chapter 1, Theorem 8.4], for any $x$ the functions $u(\xi, x), u^{(\ell)}(\xi, x)$ and therefore also $A_1(\xi, x), A_2(\xi, x), A_1^{(\ell)}(\xi, x), A_2^{(\ell)}(\xi, x)$ are entire functions of $\xi$.

Let $I_m = \left[ \frac{1}{m}, m \right]$. The following is clear:

**Lemma 2.1.** There is a constant $\hat{C} \in \mathbb{R}$, such that for any $\xi \in [a, b] \subseteq \mathbb{R}^+$, for any $x \in \mathbb{R}$

$$\| T_{x,0}^{(0)} (\xi) \|, \| T_{x,0}^{(0)}^{-1} (\xi) \| \leq \hat{C} \cdot \max \left\{ \sqrt{b}, \frac{1}{\sqrt{a}} \right\}.$$
Thus, for any interval $I_m$

$$\left|T_{x,0}^{(0)}(\xi)\right|, \left|T_{x,0}^{(0) -1}(\xi)\right| \leq \hat{C}\sqrt{m}.$$  

We need to extend this bound slightly to the complex plane, so that it holds on strips around $\mathbb{R}^+$. 

**Lemma 2.2.** For any closed interval $I = [a, b] \subseteq \mathbb{R}^+$, there exists a constant $M_I \in \mathbb{R}$, such that for any $\xi \in I$, $0 < x$ and $t \in \mathbb{R}$ with $|t| \leq 1$,

$$\left|T_{x,0}^{(0)}\left(\frac{\xi + it}{x}\right)\right|, \left|T_{x,0}^{(0) -1}\left(\frac{\xi + it}{x}\right)\right| \leq M_I$$

**Remark 2.1.** We consider the principal branch of the square root defined on $\mathbb{C} \setminus (-\infty, 0]$. This is not a problem since we only consider $\xi > 0$.

**Proof.** We need to uniformly bound the entries of the matrix

$$T_{x,0}^{(0)}\left(\frac{\xi + it}{x}\right) = \begin{bmatrix} \cos\left(\sqrt{\xi + \frac{it}{x}}\right) & \frac{1}{\sqrt{\xi + \frac{it}{x}}} \sin\left(\sqrt{\xi + \frac{it}{x}}\right) \\ -\frac{1}{\sqrt{\xi + \frac{it}{x}}} \sin\left(\sqrt{\xi + \frac{it}{x}}\right) & \cos\left(\sqrt{\xi + \frac{it}{x}}\right) \end{bmatrix}$$

By developing the square root in a Taylor series around $\xi$, it is not hard to see that the entries are bounded at $x \to \infty$. As for the limit $x \to 0$, we only need to estimate $\sqrt{\xi + \frac{it}{x}} \sin\left(\sqrt{\xi + \frac{it}{x}}\right)$ (bounds for the other entries are obvious by continuity and the fact that the argument goes to zero). Writing series around $x = 0$,

$$\sin\left(\sqrt{\xi + \frac{it}{x}}\right) = \sqrt{\xi + \frac{it}{x}} - \frac{1}{3!} \left(\sqrt{\xi + \frac{it}{x}}\right)^3 + o\left(x^2\right)$$

we see that

$$\sqrt{\xi + \frac{it}{x}} \sin\left(\sqrt{\xi + \frac{it}{x}}\right) = x \left(\xi + \frac{it}{x}\right) - \frac{1}{3!} x^3 \left(\xi + \frac{it}{x}\right)^2 + o(x)$$

and we are done. \hfill \Box

For the intervals $I_m$, we denote the constants above simply as $M_m = M_{I_m}$. From (2.5) we conclude that for any $\xi \in I_m$, and for any $t \in [-1, 1]$

$$\left\|\begin{bmatrix} A_1^{(t)}(\xi + \frac{it}{x}, x) \\ A_2^{(t)}(\xi + \frac{it}{x}, x) \end{bmatrix}\right\| \leq M_m \left\|\begin{bmatrix} u^{(t)}(\xi + \frac{it}{x}, x) \\ u^{(t)}(\xi + \frac{it}{x}, x) \end{bmatrix}\right\|$$
\[ \left\| \frac{u^{(\ell)}(\xi + \frac{\iota}{2}, x)}{u^{(\ell)'}(\xi + \frac{\iota}{2}, x)} \right\| \leq M_m \left\| \frac{A_1^{(\ell)}(\xi + \frac{\iota}{2}, x)}{A_2^{(\ell)}(\xi + \frac{\iota}{2}, x)} \right\| \]  

(2.6)

We assume from now on also that, for any \( m, M_m \geq 1 \).

The following is known as the continuous Christoffel-Darboux formula.

**Lemma 2.3.** If \( \xi \neq \zeta \),

\[ S_L(\xi, \zeta) = \frac{u(\xi, L) u'(\zeta, L) - u(\zeta, L) u'(\xi, L)}{\xi - \zeta} \]  

(2.7)

and for the diagonal case,

\[ S_L(\xi, \xi) = u'(\xi, L) \frac{d}{d\xi} u(\xi, L) - \frac{d}{d\xi} u'(\xi, L) u(\xi, L) \]  

(2.8)

**Proof.** This is easily proved using integration by parts; see Lemma 3.4 of [23].

Recall the CD kernels, \( S_x^{(\ell)}(\xi, \zeta) \), associated with \( H^{(\ell)} \) defined in (1.11). Let

\[ \widetilde{A}_1^{(\ell)}(\xi, x) = \frac{A_1^{(\ell)}(\xi, x)}{\sqrt{\xi}} \]

\[ \widetilde{A}_2^{(\ell)}(\xi, x) = A_2^{(\ell)}(\xi, x) \]  

(2.9)

**Lemma 2.4.** For any \( \ell \in \mathbb{N} \) and \( \xi \in \mathbb{R}^+ \):

\[ \lim_{x \to \infty} \frac{S_x^{(\ell)}(\xi + \frac{a}{2}, \xi + \frac{b}{2})}{x \left( \frac{\widetilde{A}_1^{(\ell)}(\xi, x)^2 + \widetilde{A}_2^{(\ell)}(\xi, x)^2}{2} \right)} = \frac{\sin(\pi \cdot \rho(\xi) (a - b))}{\pi \cdot \rho(\xi) (a - b)} \]

where \( \rho(\xi) = \frac{1}{2\pi \sqrt{\xi}} \) is the derivative of the density of states of \( \Delta \). Moreover, for any \( m \) and \( C > 0 \), the convergence is uniform in \( \xi \in I_m \) and in \( |a|, |b| \leq C \). That is, for any \( m, C > 0 \) and any \( \varepsilon > 0 \), there exists \( N(\varepsilon, m) \) so that for any \( x \geq N(\varepsilon, m) \), any \( \xi \in I_m \), and any \( |a|, |b| \leq C \)

\[ \left| \frac{S_x^{(\ell)}(\xi + \frac{a}{2}, \xi + \frac{b}{2})}{x \left( \frac{\widetilde{A}_1^{(\ell)}(\xi, x)^2 + \widetilde{A}_2^{(\ell)}(\xi, x)^2}{2} \right)} - \frac{\sin(\pi \cdot \rho(\xi) (a - b))}{\pi \cdot \rho(\xi) (a - b)} \right| < \varepsilon. \]

**Proof.** We first prove uniform convergence for \( |a|, |b| \leq C \), \( |a - b| \geq \delta \) for any \( \delta > 0 \). For any \( \xi' \) and \( j = 1, 2 \) we let

\[ \widetilde{A}_j(\xi') = A_j^{(\ell)}(\xi') = \lim_{x \to \infty} \widetilde{A}_j^{(\ell)}(\xi', x) \]
(since $\tilde{A}_j^{(t)}(\xi', x)$ is constant in $x$ for $x \geq N_{t+1}$ the limit clearly exists), and we note that for any $a$

$$\lim_{x \to \infty} \tilde{A}_j^{(t)}(\xi + \frac{a}{x}, x) = \lim_{x \to \infty} \tilde{A}_j^{(t)}(\xi + \frac{a}{x}, N_{t+1}) = \tilde{A}_j(\xi). \quad (2.10)$$

Fix $\xi \in (0, \infty)$. In order to streamline the computations below, let

$$\alpha_x = \xi + \frac{a}{x}$$
$$\beta_x = \xi + \frac{b}{x}$$

Now apply Lemma 2.3 to $S_x^{(t)}(\xi + \frac{a}{x}, \xi + \frac{b}{x})$ and use (2.5) to get

$$\frac{a-b}{x} S_x^{(t)}(\alpha_x, \beta_x) = A_1^{(t)}(\alpha_x, x) A_1^{(t)}(\beta_x, x)$$
$$\cdot \left( \frac{\sin \left( \sqrt{\alpha_x} x \right) \cos \left( \sqrt{\beta_x} x \right)}{\sqrt{\alpha_x}} - \frac{\sin \left( \sqrt{\beta_x} x \right) \cos \left( \sqrt{\alpha_x} x \right)}{\sqrt{\beta_x}} \right)$$
$$+ A_2^{(t)}(\alpha_x, x) A_2^{(t)}(\beta_x, x)$$
$$\cdot \left( \frac{\sin \left( \sqrt{\alpha_x} x \right) \cos \left( \sqrt{\beta_x} x \right)}{\sqrt{\alpha_x}} - \frac{\sin \left( \sqrt{\beta_x} x \right) \cos \left( \sqrt{\alpha_x} x \right)}{\sqrt{\beta_x}} \right)^{-1}$$
$$+ \cos \left( \sqrt{\beta_x} x \right) \cos \left( \sqrt{\alpha_x} x \right)$$
$$\cdot \left( A_1^{(t)}(\beta_x, x) A_1^{(t)}(\alpha_x, x) - A_1^{(t)}(\alpha_x, x) A_1^{(t)}(\beta_x, x) \right)$$
$$+ \sin \left( \sqrt{\beta_x} x \right) \sin \left( \sqrt{\alpha_x} x \right)$$
$$\cdot \left( \frac{A_1^{(t)}(\beta_x, x) A_2^{(t)}(\alpha_x, x)}{\sqrt{\beta_x}} - \frac{A_1^{(t)}(\alpha_x, x) A_2^{(t)}(\beta_x, x)}{\sqrt{\alpha_x}} \right)$$
$$= \tilde{A}_1^{(t)}(\alpha_x, x) \tilde{A}_1^{(t)}(\beta_x, x) \sqrt{\alpha_x} \sqrt{\beta_x}$$
$$\cdot \left( \frac{\sin \left( \sqrt{\alpha_x} x \right) \cos \left( \sqrt{\beta_x} x \right)}{\sqrt{\alpha_x}} - \frac{\sin \left( \sqrt{\beta_x} x \right) \cos \left( \sqrt{\alpha_x} x \right)}{\sqrt{\beta_x}} \right)$$
$$+ \tilde{A}_2^{(t)}(\alpha_x, x) \tilde{A}_2^{(t)}(\beta_x, x)$$
$$\cdot \left( \frac{\sin \left( \sqrt{\alpha_x} x \right) \cos \left( \sqrt{\beta_x} x \right)}{\sqrt{\alpha_x}} - \frac{\sin \left( \sqrt{\beta_x} x \right) \cos \left( \sqrt{\alpha_x} x \right)}{\sqrt{\beta_x}} \right) + o(1)$$

since $\lim_{x \to \infty} \alpha_x = \lim_{x \to \infty} \beta_x = \xi$. 
For the first term, write
\[\widetilde{A}_1^{(\ell)} (\alpha_x, x) \widetilde{A}_1^{(\ell)} (\beta_x, x) \sqrt{\alpha_x} \sqrt{\beta_x} \]
\[\cdot \left( \frac{\sin (\sqrt{\alpha_x} x)}{\sqrt{\alpha_x}} \cos (\sqrt{\beta_x} x) - \frac{\sin (\sqrt{\beta_x} x)}{\sqrt{\beta_x}} \cos (\sqrt{\alpha_x} x) \right) \]
\[= \widetilde{A}_1^{(\ell)} (\alpha_x, x) \widetilde{A}_1^{(\ell)} (\beta_x, x) \sqrt{\alpha_x} \sqrt{\beta_x} \sin \left( (\sqrt{\alpha_x} - \sqrt{\beta_x}) x \right) \]
\[\cdot \left( \frac{1}{2 \sqrt{\alpha_x}} + \frac{1}{2 \sqrt{\beta_x}} \right) \]
\[+ \widetilde{A}_1^{(\ell)} (\alpha_x, x) \widetilde{A}_1^{(\ell)} (\beta_x, x) \sqrt{\alpha_x} \sqrt{\beta_x} \sin \left( (\sqrt{\alpha_x} + \sqrt{\beta_x}) x \right) \]
\[\cdot \left( \frac{1}{2 \sqrt{\alpha_x}} - \frac{1}{2 \sqrt{\beta_x}} \right) \]
\[= \widetilde{A}_1^{(\ell)} (\alpha_x, x) \widetilde{A}_1^{(\ell)} (\beta_x, x) \sqrt{\alpha_x} \sqrt{\beta_x} \sin \left( (\sqrt{\alpha_x} - \sqrt{\beta_x}) x \right) \]
\[\cdot \left( \frac{1}{2 \sqrt{\alpha_x}} + \frac{1}{2 \sqrt{\beta_x}} \right) + o(1). \]

Using
\[\lim_{x \to \infty} \left( \sqrt{\xi + \frac{a}{x}} - \sqrt{\xi + \frac{b}{x}} \right) x = \frac{a - b}{2 \sqrt{\xi}} \]
it now follows that
\[\lim_{x \to \infty} \widetilde{A}_1^{(\ell)} (\alpha_x, x) \widetilde{A}_1^{(\ell)} (\beta_x, x) \sqrt{\alpha_x} \sqrt{\beta_x} \]
\[\cdot \left( \frac{\sin (\sqrt{\alpha_x} x)}{\sqrt{\alpha_x}} \cos (\sqrt{\beta_x} x) - \frac{\sin (\sqrt{\beta_x} x)}{\sqrt{\beta_x}} \cos (\sqrt{\alpha_x} x) \right) \]
\[= \widetilde{A}_1^{(\ell)} (\xi)^2 \sqrt{\xi} \sin \left( \frac{a - b}{2 \sqrt{\xi}} \right). \]

A similar computation shows that
\[\lim_{x \to \infty} A_2^{(\ell)} (\alpha_x, x) A_2^{(\ell)} (\beta_x, x) \]
\[\cdot \left( \frac{\sin (\sqrt{\alpha_x} x)}{\sqrt{\alpha_x}} \cos (\sqrt{\beta_x} x) - \frac{\sin (\sqrt{\beta_x} x)}{\sqrt{\beta_x}} \cos (\sqrt{\alpha_x} x) \right) \]
\[= \widetilde{A}_2^{(\ell)} (\xi)^2 \sqrt{\xi} \sin \left( \frac{a - b}{2 \sqrt{\xi}} \right). \]
Recalling that \( \widetilde{A}_j^{(\ell)}(\xi, x) = \widehat{A}_j^{(\ell)}(\xi) \) for any \( x \geq N_{\ell+1} \), and that \( \pi\rho(\xi) = \frac{1}{2\sqrt{\xi}} \), it follows that we have uniform convergence for \(|a|, |b| \leq C\) and \(|a - b| \geq \delta\) for any \( \delta > 0 \).

Now, since \( u^{(\ell)}(\xi, x) \) is entire in \( \xi \), \( f^{(\ell)}(x) = S^{(\ell)}_\xi(\xi + a, \xi + b) \), for fixed \( a \), is an analytic function of \( b \). Clearly, \( f(b) = \sin(\pi \cdot \rho(\xi)(a - b)) \) is also an analytic function of \( b \). Therefore, by using the Cauchy formula, uniform convergence of \( f^{(\ell)}(b) \) to \( f(b) \) in the annulus \( \delta \leq |a - b| \leq 1 \), implies uniform convergence on the disk \( |a - b| < \delta \). This then implies uniform convergence on the disk \( |a - b| \leq 1 \). Thus we have uniform convergence for \(|a|, |b| \leq C\) and we are done.

\[ \square \]

The lemma above shows that for any \( H^{(\ell)} \) the CD kernel is close to the desired limit for sufficiently large \( x \). As a first step towards understanding \( H \) we want to understand the effect of the addition of \( \lambda^{(\ell+1)}W(x - N_{\ell+1}) \) to the potential of \( H^{(\ell)} \) (i.e., going to \( H^{(\ell+1)} \)). The lemma below serves this purpose.

**Lemma 2.5.** Let \( A, B : \mathbb{R} \rightarrow M_2(\mathbb{C}) \) be the solutions to the following matrix initial value problems

\[
A'(x) = \begin{bmatrix} 0 & 1 \\ -\xi & 0 \end{bmatrix} A(x) \tag{2.11}
\]

\[
B'(x) = \begin{bmatrix} 0 & 1 \\ \lambda W(x) - \xi & 0 \end{bmatrix} B(x) \tag{2.12}
\]

with the initial condition

\[
A(0) = B(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

where \( W(x) \) is a smooth, nonnegative function with \( \text{supp}W \subseteq [0, 1] \), and \( \xi \in \mathbb{R} \). Then

\[
||A(x) - B(x)|| \leq C(\xi, W) |\lambda| \tag{2.13}
\]

where \( C(\xi, W) \) is a constant depending on \( \xi \) and \( W(x) \).

**Proof.** Recall the free transfer matrix, \( T_{x,0}(\xi) \), defined in (2.2). It is straightforward to verify that

\[
\frac{d}{dx} T_{x,0}^{(0)}(\xi) = \begin{bmatrix} 0 & 1 \\ -\xi & 0 \end{bmatrix} T_{x,0}^{(0)}(\xi). \tag{2.14}
\]
Since $T_{0,0}(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, it follows that $A(x) = T^{(0)}_{x,0}(\xi)$. Thus, $A(x)$ is bounded in $x$ and invertible by (2.3). Denote

$$S(x) = A(x)^{-1} B(x), \quad S(0) = \text{Id}.$$ 

By differentiating $B(x)$ and by (2.12),

$$B'(x) = A'(x) S(x) + A(x) S'(x) = \begin{bmatrix} 0 & 1 \\ \lambda W(x) - \xi & 0 \end{bmatrix} A(x) S(x)$$

So that by (2.11),

$$\begin{bmatrix} 0 & 1 \\ -\xi & 0 \end{bmatrix} A(x) S(x) + A(x) S'(x) = \begin{bmatrix} 0 & 1 \\ \lambda W(x) - \xi & 0 \end{bmatrix} A(x) S(x)$$

and upon rearranging,

$$A(x) S'(x) = \begin{bmatrix} 0 & 0 \\ \lambda W(x) & 0 \end{bmatrix} A(x) S(x)$$

$$S'(x) = \lambda W(x) A^{-1}(x) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} A(x) S(x).$$

This is a linear ODE so clearly (e.g. by the method of successive approximations [7]), for any $x \in [0, 1]$

$$||S(x) - I|| \leq |\lambda| \tilde{C}(\xi, W)x \leq |\lambda| \tilde{C}(\xi, W)$$

(2.15)

where $\tilde{C}$ depends on $\sup_x \|T^{(0)}_{x,0}(\xi)\|$ and on $\|W\|_\infty$. Thus writing

$$\|A(x) - B(x)\| = \|A(x)(I - S(x))\|$$

we immediately see that for $x \in [0, 1]$

$$\|A(x) - B(x)\| \leq \sup_x \|T^{(0)}_{x,0}(\xi)\| \tilde{C}(\xi, W)|\lambda|.$$

For $x > 1$ we have that

$$(A(x) - B(x))' = \begin{bmatrix} 0 & 1 \\ -\xi & 0 \end{bmatrix} (A(x) - B(x))$$

since $\text{supp} W \subseteq [0, 1]$ and so, from (2.14) we see that

$$A(x) - B(x) = T^{(0)}_{x,1}(\xi) (A(1) - B(1))$$

so that for $x > 1$

$$\|A(x) - B(x)\| \leq \sup_x \|T^{(0)}_{x,1}(\xi)\| \|T^{(0)}_{x,0}(\xi)\| \tilde{C}(\xi, W)|\lambda|,$$

and we are done. \qed

For an interval $I_m = [1/m, m]$, we extend the definition of $M_m$ so that for all $\xi \in I_m$, $C(\xi, W) \leq M_m$. 

Lemma 2.6. Fix $m \in \mathbb{N}$. For any $\xi \in I_m$ and for large enough values of $\ell$,
\[
\left\| \begin{bmatrix} u^{(\ell+1)}(\xi, x) \\ u^{(\ell+1)'}(\xi, x) \end{bmatrix} - \begin{bmatrix} u^{(\ell)}(\xi, x) \\ u^{(\ell)'}(\xi, x) \end{bmatrix} \right\| \leq C \cdot M_m \cdot |\lambda_{\ell+1}| \cdot \left\| \begin{bmatrix} u^{(\ell)}(\xi, x) \\ u^{(\ell)'}(\xi, x) \end{bmatrix} \right\| \leq \tilde{C} \cdot M_m \cdot |\lambda_{\ell+1}| \left\| \begin{bmatrix} u^{(\ell+1)}(\xi, x) \\ u^{(\ell+1)'}(\xi, x) \end{bmatrix} \right\|
\]
where the constants $C, \tilde{C}$ depend only on the function $W$. Similarly,
\[
\left\| \begin{bmatrix} A_1^{(\ell+1)}(\xi, x) \\ A_2^{(\ell+1)}(\xi, x) \end{bmatrix} - \begin{bmatrix} A_1^{(\ell)}(\xi, x) \\ A_2^{(\ell)}(\xi, x) \end{bmatrix} \right\| \leq C \cdot M_m^2 \cdot |\lambda_{\ell+1}| \cdot \left\| \begin{bmatrix} u^{(\ell)}(\xi, x) \\ u^{(\ell)'}(\xi, x) \end{bmatrix} \right\| \leq \tilde{C} \cdot M_m^2 \cdot |\lambda_{\ell+1}| \left\| \begin{bmatrix} u^{(\ell+1)}(\xi, x) \\ u^{(\ell+1)'}(\xi, x) \end{bmatrix} \right\|
\]
and
\[
\left\| \begin{bmatrix} \tilde{A}_1^{(\ell+1)}(\xi, x) \\ \tilde{A}_2^{(\ell+1)}(\xi, x) \end{bmatrix} - \begin{bmatrix} \tilde{A}_1^{(\ell)}(\xi, x) \\ \tilde{A}_2^{(\ell)}(\xi, x) \end{bmatrix} \right\| \leq C \cdot \tilde{M}_m^2 \cdot |\lambda_{\ell+1}| \cdot \left\| \begin{bmatrix} u^{(\ell)}(\xi, x) \\ u^{(\ell)'}(\xi, x) \end{bmatrix} \right\| \leq \tilde{C} \cdot \tilde{M}_m^2 \cdot |\lambda_{\ell+1}| \left\| \begin{bmatrix} u^{(\ell+1)}(\xi, x) \\ u^{(\ell+1)'}(\xi, x) \end{bmatrix} \right\|
\]
where $\tilde{M}_m = M_m \cdot \max_{\xi \in I_m} \left(\sqrt{\xi}, \sqrt{\xi^{-1}}\right) = \sqrt{m}M_m$.

Proof. Fix $N_\ell + 1 \leq x_0 \leq N_{\ell+1}$ and let $T^{(\ell)}_{x,x_0}(\xi)$, $T^{(\ell+1)}_{x,x_0}(\xi)$ be the associated transfer matrices from $x_0$ to $x$. It is a simple computation to check that for $x_0 < x$
\[
T^{(\ell+1)}_{x,x_0}(\xi) = \begin{bmatrix} 0 & 1 \\ -\xi & 0 \end{bmatrix} T^{(\ell)}_{x,x_0}(\xi),
\]
\[
T^{(\ell+1)'}_{x,x_0}(\xi) = \begin{bmatrix} 0 & 1 \\ \lambda_{\ell+1}W(x-N_{\ell+1}) - \xi & 0 \end{bmatrix} T^{(\ell+1)}_{x,x_0}(\xi).
\]
Since
\[
\left\| \begin{bmatrix} u^{(\ell+1)}(\xi, x) \\ u^{(\ell+1)'}(\xi, x) \end{bmatrix} - \begin{bmatrix} u^{(\ell)}(\xi, x) \\ u^{(\ell)'}(\xi, x) \end{bmatrix} \right\| = \left\| \begin{bmatrix} T^{(\ell)}_{x,x_0} - T^{(\ell+1)}_{x,x_0} \end{bmatrix} \begin{bmatrix} u^{(\ell)}(\xi, x_0) \\ u^{(\ell)'}(\xi, x_0) \end{bmatrix} \right\|
\]
Lemma 2.5 implies that
\[
\left\| \begin{bmatrix} u^{(\ell+1)}(\xi, x) \\ u^{(\ell+1)'}(\xi, x) \end{bmatrix} - \begin{bmatrix} u^{(\ell)}(\xi, x) \\ u^{(\ell)'}(\xi, x) \end{bmatrix} \right\| \leq C' \cdot M_m |\lambda_{\ell+1}| \cdot \left\| \begin{bmatrix} u^{(\ell)}(\xi, x_0) \\ u^{(\ell)'}(\xi, x_0) \end{bmatrix} \right\|
\]
for some $C' > 0$. But
\[
\left\| \begin{bmatrix} u^{(\ell)}(\xi, x_0) \\ u^{(\ell)'}(\xi, x_0) \end{bmatrix} \right\| \leq \|T_{x,x_0}^{-1}\| \left\| \begin{bmatrix} u^{(\ell)}(\xi, x) \\ u^{(\ell)'}(\xi, x) \end{bmatrix} \right\|
\]
so the first inequality follows with $C = \sup_x \|T_{x,x_0}^{-1}\|C'$.

The second inequality follows from the fact that $\lambda_n \to 0$, so for large enough values of $\ell$, $CM_m |\lambda_{\ell+1}| < \frac{1}{2}$. By the triangle inequality,
\[
\left\| \begin{bmatrix} u^{(\ell)}(\xi, x) \\ u^{(\ell)'}(\xi, x) \end{bmatrix} \right\| - \left\| \begin{bmatrix} u^{(\ell+1)}(\xi, x) \\ u^{(\ell+1)'}(\xi, x) \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} u^{(\ell+1)}(\xi, x) - u^{(\ell)}(\xi, x) \\ u^{(\ell+1)'}(\xi, x) - u^{(\ell)'}(\xi, x) \end{bmatrix} \right\| \leq C \cdot M_m \cdot |\lambda_{\ell+1}| \cdot \left\| \begin{bmatrix} u^{(\ell)}(\xi, x) \\ u^{(\ell)'}(\xi, x) \end{bmatrix} \right\|
\]
so for such values of $\ell$
\[
\frac{1}{2} \left\| \begin{bmatrix} u^{(\ell)}(\xi, x) \\ u^{(\ell)'}(\xi, x) \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} u^{(\ell+1)}(\xi, x) \\ u^{(\ell+1)'}(\xi, x) \end{bmatrix} \right\| \quad (2.19)
\]
and we can take $\tilde{C} = 2C$. Finally, (2.17) follows by (2.5) and (2.18) is immediate from the definition of $A_j^{(\ell)}$. \qed 

3. Proof of Theorem 1.5

Proof of Theorem 1.5. Given $\{\lambda_n\}_{n=1}^\infty$, let $\{m_n\}_{n=1}^\infty$ satisfy $m_n \to \infty$ monotonically, and also
\[
\lim_{n \to \infty} |\lambda_n| \overline{M_{m_n}}^6 \to 0 \quad (3.1)
\]
where we recall that $\overline{M}_m = \sqrt{m} M_m$ and $M_m = M_{l_m}$. Such a subsequence exists, since $|\lambda_n| \to 0$ as $n \to \infty$, so for any natural $r$, there exists an $N_r$ such that if $N_r < n$, then $|\lambda_n| \leq r^{-1} \overline{M}_r^{-6}$. Thus, we can choose $m_1 = m_2 = \cdots = m_{N_2} = 1$ and $m_{N_2+1} = m_{N_2+2} = \cdots = m_{N_3} = 2$ etc.

Assume we’ve fixed $\{N_j\}_{j=1}^\ell$. Let $\tilde{I}_\ell = I_{m_{\ell+1}^{1/\ell}}$ (a closed interval in the interior of $I_{m_{\ell+1}^{1/\ell}}$). By Lemma 2.4 there exists $\tilde{N}_\ell$ such that for any $x \geq \tilde{N}_\ell$
\[
\left| \frac{S_x^{(\ell)}(\xi + \frac{a}{\ell}, \xi + \frac{b}{\ell})}{x} - \frac{\sin (\pi \cdot \rho(\xi)(a-b))}{\pi \cdot \rho(\xi)(a-b)} \right| < 1/\ell \quad (3.2)
\]
for any $\xi \in \tilde{I}_\ell$, and $a, b \in \mathbb{C}$ with $|a|, |b| \leq \ell$. From now on, we shall only consider $a, b \in \mathbb{C}$ with $\text{Im}a, \text{Im}b \leq 1$. By taking $\tilde{N}_\ell$ large enough
we may also assume that \( \text{Re} \left( \xi + \frac{a}{x} \right) \), \( \text{Re} \left( \xi + \frac{b}{x} \right) \) \( \in I_{m+1} \) for \( \xi \in \tilde{I}_\ell \), \( x \geq \tilde{N}_\ell \). We may also assume that for such values of the parameters

\[
\frac{1}{2} \leq \frac{\left| \frac{\tilde{A}_1^{(\ell)}(\xi + \frac{a}{x}, x)}{\tilde{A}_1^{(\ell)}(\xi, x)} \right|^2 + \left| \frac{\tilde{A}_2^{(\ell)}(\xi + \frac{a}{x}, x)}{\tilde{A}_2^{(\ell)}(\xi, x)} \right|^2}{2} \leq 2. \quad (3.3)
\]

We will show that, as long as we pick \( N_{\ell+1} \geq \tilde{N}_\ell \) inductively,

\[
\frac{S_x(\xi + \frac{a}{x}, \xi + \frac{b}{x})}{S_x(\xi, \xi)} \xrightarrow{x \to \infty} \frac{\sin(\pi \cdot \rho(\xi)(b - a))}{\pi \cdot \rho(\xi)(b - a)}, \quad (3.4)
\]

uniformly for \( \xi \) in closed intervals of \( \mathbb{R}^+ \) and \( a, b \) in compact subsets of the strip \( |\text{Im}z| \leq 1 \). Our strategy will be to first prove that

\[
\lim_{x \to \infty} \frac{S_x(\xi + \frac{a}{x}, \xi + \frac{b}{x})}{x \left( \frac{1}{A_1(\xi, x)^2 + A_2(\xi, x)^2} \right)} = \frac{\sin(\pi \cdot \rho(\xi)(a - b))}{\pi \cdot \rho(\xi)(a - b)}, \quad (3.5)
\]

uniformly for complex \( |a|, |b| \leq C \) with \( |\text{Im}a|, |\text{Im}b| \leq 1 \) and \( |a - b| > \delta > 0 \) and then deduce (3.4) from analyticity.

Note that for \( N_{\ell+1} \leq x \leq N_{\ell+2} \) \( u(\xi, x) = u^{(\ell+1)}(\xi, x) \) and \( A_\ell(\xi, x) = A^{(\ell+1)}_\ell(\xi, x) \) \( i = 1, 2 \). We claim that it is enough to show that

\[
\max_{N_{\ell+1} \leq x \leq N_{\ell+2}} \left| \frac{S_x^{(\ell)}(\xi + \frac{a}{x}, \xi + \frac{b}{x})}{x \left( \frac{\tilde{A}_1^{(\ell)}(\xi, x)^2 + \tilde{A}_2^{(\ell)}(\xi, x)^2}{2} \right)} - \frac{S_x^{(\ell+1)}(\xi + \frac{a}{x}, \xi + \frac{b}{x})}{x \left( \frac{A^{(\ell+1)}_1(\xi, x)^2 + A^{(\ell+1)}_2(\xi, x)^2}{2} \right)} \right| \xrightarrow{\ell \to \infty} 0 \quad (3.6)
\]

as \( \ell \to \infty \). This is because, assuming (3.6), given \( \varepsilon > 0 \) we may choose \( L \) so that for any \( \ell > L \) both the quantity in (3.6) and the left hand side of (3.2) are smaller than \( \varepsilon/2 \). Now, for any \( x > N_{\ell+1} \) it follows that \( N_{\ell+1} \leq x \leq N_{\ell+2} \) for some \( \ell > L \) so that

\[
\frac{S_x(\xi + \frac{a}{x}, \xi + \frac{b}{x})}{x \left( \frac{A_1(\xi, x)^2 + A_2(\xi, x)^2}{2} \right)} = \frac{S_x^{(\ell+1)}(\xi + \frac{a}{x}, \xi + \frac{b}{x})}{x \left( \frac{A^{(\ell+1)}_1(\xi, x)^2 + A^{(\ell+1)}_2(\xi, x)^2}{2} \right)}.
\]

The \( \varepsilon/2 \) bound on (3.6) and on (3.2) then combine to show that

\[
\lim_{x \to \infty} \left| \frac{S_x(\xi + \frac{a}{x}, \xi + \frac{b}{x})}{x \left( \frac{A_1(\xi, x)^2 + A_2(\xi, x)^2}{2} \right)} - \frac{\sin(\pi \cdot \rho(\xi)(a - b))}{\pi \cdot \rho(\xi)(a - b)} \right| < \varepsilon.
\]
We now proceed to prove (3.6). Denoting \( \kappa^{(\ell)}_x = \frac{\tilde{A}_1^{(\ell)}(\xi,x)^2 + \tilde{A}_2^{(\ell)}(\xi,x)^2}{2} \), we estimate

\[
\begin{align*}
\left| \frac{S^{(\ell)}_x (\xi + \frac{a}{x}, \xi + \frac{b}{x})}{x \kappa^{(\ell)}_x} - \frac{S^{(\ell+1)}_x (\xi + \frac{b}{x}, \xi + \frac{b}{x})}{x \kappa^{(\ell+1)}_x} \right| & \leq \left| \frac{S^{(\ell)}_x (\xi + \frac{a}{x}, \xi + \frac{b}{x})}{x \kappa^{(\ell)}_x} - \frac{S^{(\ell+1)}_x (\xi + \frac{b}{x}, \xi + \frac{b}{x})}{x \kappa^{(\ell+1)}_x} \right| \\
& \quad + \left| \frac{S^{(\ell)}_x (\xi + \frac{a}{x}, \xi + \frac{b}{x})}{x \kappa^{(\ell)}_x} - \frac{S^{(\ell+1)}_x (\xi + \frac{b}{x}, \xi + \frac{b}{x})}{x \kappa^{(\ell+1)}_x} \right| \\
& \quad + \left| \frac{S^{(\ell)}_x (\xi + \frac{a}{x}, \xi + \frac{b}{x})}{x \kappa^{(\ell)}_x} - \frac{S^{(\ell+1)}_x (\xi + \frac{b}{x}, \xi + \frac{b}{x})}{x \kappa^{(\ell+1)}_x} \right| \cdot \left( \frac{\kappa^{(\ell+1)}_x - \kappa^{(\ell)}_x}{\kappa^{(\ell+1)}_x} \right) \end{align*}
\]

As in the proof of Lemma 2.4, we introduce the notation \( \alpha_x = \xi + \frac{a}{x} \) and \( \beta_x = \xi + \frac{b}{x} \); also, in the following calculation, we will omit the second variable, which will always be \( x \). By (2.7) and by introducing the notation

\[
\Delta u^{(\ell+1)}(\alpha_x) = u^{(\ell+1)}(\alpha_x) - u^{(\ell)}(\alpha_x)
\]

we evaluate

\[
\begin{align*}
\left| \frac{S^{(\ell)}_x (\xi + \frac{a}{x}, \xi + \frac{b}{x})}{x \kappa^{(\ell)}_x} - \frac{S^{(\ell+1)}_x (\xi + \frac{b}{x}, \xi + \frac{b}{x})}{x \kappa^{(\ell+1)}_x} \right| & = \frac{2}{a - b} \left[ \tilde{A}_1^{(\ell+1)}(\xi,x)^2 + \tilde{A}_2^{(\ell+1)}(\xi,x)^2 \right]^{-1} \times \\
& \left( u^{(\ell)}(\alpha_x) u^{(\ell)}(\beta_x) - u^{(\ell)}(\beta_x) u^{(\ell)}(\alpha_x) \\
\quad - u^{(\ell+1)}(\alpha_x) u^{(\ell+1)}(\beta_x) + u^{(\ell+1)}(\beta_x) u^{(\ell+1)}(\alpha_x) \right) \\
& = \frac{2}{a - b} \left[ \tilde{A}_1^{(\ell+1)}(\xi,x)^2 + \tilde{A}_2^{(\ell+1)}(\xi,x)^2 \right] \times \\
& \left( u^{(\ell)}(\beta_x) \Delta u^{(\ell+1)}(\alpha_x) + \Delta u^{(\ell+1)}(\beta_x) u^{(\ell)}(\alpha_x) \\
\quad - u^{(\ell)}(\alpha_x) \Delta u^{(\ell+1)}(\beta_x) - \Delta u^{(\ell+1)}(\alpha_x) u^{(\ell)}(\beta_x) \\
\quad + \Delta u^{(\ell+1)}(\beta_x) \Delta u^{(\ell+1)}(\alpha_x) \\
\quad - \Delta u^{(\ell+1)}(\alpha_x) \Delta u^{(\ell+1)}(\beta_x) \right) .
\end{align*}
\]
By (2.16) and (2.6) (recall that $\tilde{M}_m \leq M_m \max_{\xi \in \mathcal{E}_m} \max \left( \sqrt{\xi}, \sqrt{\xi^{-1}} \right)$)

\[
\left\| \begin{bmatrix} u^{(\ell+1)}(\alpha_x) \\ u^{(\ell+1)'}(\alpha_x) \end{bmatrix} - \begin{bmatrix} u^{(\ell)}(\alpha_x) \\ u^{(\ell)'}(\alpha_x) \end{bmatrix} \right\| \leq \tilde{C} M_{m_\ell} |\lambda_{\ell+1}| \cdot \left\| \begin{bmatrix} u^{(\ell+1)}(\alpha_x) \\ u^{(\ell+1)'}(\alpha_x) \end{bmatrix} \right\| 
\]

\[
\leq \tilde{C} M_{m_\ell}^{-2} |\lambda_{\ell+1}| \left\| \begin{bmatrix} \tilde{A}_1^{(\ell+1)}(\alpha_x, x) \\ \tilde{A}_2^{(\ell+1)}(\alpha_x, x) \end{bmatrix} \right\| 
\]

\[
\leq 2\tilde{C} M_{m_\ell}^{-2} |\lambda_{\ell+1}| \left\| \begin{bmatrix} \tilde{A}_1^{(\ell+1)}(\xi, x) \\ \tilde{A}_2^{(\ell+1)}(\xi, x) \end{bmatrix} \right\| 
\]

where the last inequality follows from (3.3). A similar estimate holds for $\beta_x$. Therefore

\[
\frac{S_x^{(\ell)}(\xi + \frac{a}{2}, \xi + \frac{b}{2})}{x \kappa_x^{(\ell+1)}} - \frac{S_x^{(\ell+1)}(\xi + \frac{a}{2}, \xi + \frac{b}{2})}{x \kappa_x^{(\ell+1)}} 
\]

\[
= \frac{2}{a-b} \left| \tilde{A}_1^{(\ell+1)}(\xi, x) \right|^{-2} \times \left| u^{(\ell)}(\beta_x) \Delta u^{(\ell+1)'}(\alpha_x) 
+ \Delta u^{(\ell+1)}(\beta_x) u^{(\ell)'}(\alpha_x) 
- u^{(\ell)}(\alpha_x) \Delta u^{(\ell+1)'}(\beta_x) 
- \Delta u^{(\ell+1)}(\alpha_x) u^{(\ell)'}(\beta_x) 
+ \Delta u^{(\ell+1)}(\beta_x) \Delta u^{(\ell+1)'}(\alpha_x) 
- \Delta u^{(\ell+1)}(\alpha_x) \Delta u^{(\ell+1)'}(\beta_x) \right| 
\]

\[
\leq \frac{4}{a-b} \tilde{C} M_{m_\ell}^{-2} |\lambda_{\ell+1}| \left| \tilde{A}_1^{(\ell+1)}(\xi, x) \right|^{-1} \times 
\left( 2|u^{(\ell)}(\alpha_x)| + 2|u^{(\ell+1)}(\beta_x)| + |u^{(\ell)'}(\alpha_x)| + |u^{(\ell)'}(\beta_x)| 
+ |u^{(\ell+1)}(\alpha_x)| + |u^{(\ell+1)}(\beta_x)| \right). 
\]

We are almost done with the first term. the only real issue is that the numerator is evaluated at $\alpha_x$ and $\beta_x$, while the denominator is evaluated at $\xi$. To fix this, use (2.16) to replace $u^{(\ell+1)}$ with $u^{(\ell)}$ (for sufficiently large $\ell$) and then (2.6) to replace $u^{(\ell)}$ with $\tilde{A}^{(\ell)}$ (paying a price with some fixed constants and powers of $\tilde{M}_{m_\ell}$). Now apply
(3.3) to replace the $\alpha_x$’s and $\beta_x$’s with $\xi$ and use (2.18) and (2.6) again to show that the numerator can be bounded by (a constant times) $\|\tilde{A}^{(\ell+1)}\|$. Since $\tilde{M}_m > 1$ we conclude that for $\ell$ sufficiently large

$$\left| \frac{S_x^{(\ell)}(\xi + \frac{x}{\ell}, \xi + \frac{x}{\ell} \pm 1)}{\kappa^{(\ell+1)}(x \kappa^{(\ell+1)}(x \kappa^{(\ell+1)})} \right| \leq \frac{D}{a - b} C |\lambda_{\ell+1}| \tilde{M}_m^6$$

(where $D$ is some universal constant) so that by (3.1), we are done with the first term.

As for the second term, $\left| \frac{S_x^{(\ell)}(\xi + \xi + \frac{x}{\ell})}{x \kappa^{(\ell+1)}(x \kappa^{(\ell+1)})} \right|$, the left factor $\left| \frac{S_x^{(\ell)}(\xi + \xi + \frac{x}{\ell})}{x \kappa^{(\ell+1)}(x \kappa^{(\ell+1)})} \right|$ is bounded since, by Lemma 2.4, it converges (uniformly on $I_m$) to the sine kernel. Regarding the right factor,

$$\frac{\kappa^{(\ell+1)}(x \kappa^{(\ell+1)}) - \kappa^{(\ell)}(x \kappa^{(\ell+1)})}{\kappa^{(\ell+1)}(x \kappa^{(\ell+1)})} = \frac{\tilde{A}^{(\ell+1)}(\xi, x)^2 + \tilde{A}^{(\ell+1)}(\xi, x)^2 - \tilde{A}^{(\ell)}(\xi, x)^2 - \tilde{A}^{(\ell)}(\xi, x)^2}{\tilde{A}^{(\ell+1)}(\xi, x)^2 + \tilde{A}^{(\ell+1)}(\xi, x)^2}$$

$$= \frac{\left(\tilde{A}^{(\ell+1)} - \tilde{A}^{(\ell)}\right) \left(\tilde{A}^{(\ell+1)} + \tilde{A}^{(\ell)}\right)}{\left(\tilde{A}^{(\ell+1)} + \tilde{A}^{(\ell)}\right)^2 + \left(\tilde{A}^{(\ell+1)} + \tilde{A}^{(\ell)}\right)^2}$$

By (2.18), the definition of $\tilde{A}^{(\ell)}$, and (2.6)

$$\left| \tilde{A}^{(\ell+1)} - \tilde{A}^{(\ell)} \right| \leq \tilde{C} \tilde{M}_m^3 |\lambda_{\ell+1}| \left| \tilde{A}^{(\ell+1)} - \tilde{A}^{(\ell)} \right|$$

and the same bound applies to $\left| \tilde{A}^{(\ell+1)} - \tilde{A}^{(\ell)} \right|$. Therefore,

$$\frac{\kappa^{(\ell+1)}(x \kappa^{(\ell+1)}) - \kappa^{(\ell)}(x \kappa^{(\ell+1)})}{\kappa^{(\ell+1)}(x \kappa^{(\ell+1)})} \leq \tilde{C} \tilde{M}_m^3 |\lambda_{\ell+1}| \left| \tilde{A}^{(\ell+1)} - \tilde{A}^{(\ell)} \right| + \tilde{A}^{(\ell+1)} + \tilde{A}^{(\ell)} + \tilde{A}^{(\ell)}$$

$$\leq 8 \tilde{C} \tilde{M}_m^5 |\lambda_{\ell+1}|$$

for $\ell$ sufficiently large ((2.19) and (2.5) are used). By (3.1) we are done with the second term as well.

We have shown (3.5) uniformly for $\xi$ in compact subsets of $\mathbb{R}^+$ and $a, b$ complex with $|\text{Im}a|, |\text{Im}b| \leq 1, |a|, |b| \leq C$ for any $C > 0$, and $|a - b| > \delta$ for some $\delta > 0$. We now repeat the argument at the end of the proof of Lemma 2.4. Since for fixed $a$, all functions involved are analytic in $b$ (note that $\kappa^\ell_x > 0$ for all $x, \xi$), the Cauchy formula implies
uniform convergence also for $|a - b| \leq \delta$. This implies (3.5) uniformly without the restriction $|a - b| > \delta$. Since the limit for $a = b = 0$ is 1, this implies (3.4) uniformly for $\xi$ in compact subsets of $\mathbb{R}^+$ and $a, b$ in compact subsets of $\mathbb{R}$ and finishes the proof.

□

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