GEOMETRIC SECOND DERIVATIVE ESTIMATES IN CARNOT GROUPS
AND CONVEXITY

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Abstract. We prove some new a priori estimates for $H^2$-convex functions which are zero on the boundary of a bounded smooth domain $\Omega$ in a Carnot group $G$. Such estimates are global and are geometric in nature as they involve the horizontal mean curvature $H$ of $\partial\Omega$. As a consequence of our bounds we show that if $G$ has step two, then for any smooth $H^2$-convex function in $\Omega \subset G$ vanishing on $\partial\Omega$ one has

$$\sum_{i,j=1}^m \int_\Omega ([X_i, X_j]u)^2 \, dg \leq \frac{4}{3} \int_{\partial\Omega} H |\nabla_H u|^2 \, d\sigma_H.$$ 

1. Introduction and statement of the results

In this paper we study some a priori estimates of geometric type for functions vanishing on the boundary of a smooth bounded open set in a Carnot group. The estimates that we obtain are of interest in connection with the study of the geometric notion of convexity in Carnot groups recently introduced in [DGNT], see also [LMS], [GMI], [CT], [DGNT], [Wa1], [Wa2], [Ma]. They also play a central role in establishing a priori bounds in $L^2$ for the (horizontal) second derivatives of solutions of non-variational operators with rough coefficients. In this respect, a global version of our results (with sharp constant) for compactly supported functions in the Heisenberg group $\mathbb{H}^n$ has been recently established in the very interesting recent work of Domokos and Manfredi, see Lemma 1.1 in [DM], using the deeper spectral decomposition of Strichartz [Str]. An alternative proof of the estimates in [DM], which however does not produce the best constant, would be to use the subelliptic estimates, see [Ko] and [H].

Our results are intimately connected to those in [DGNT], except that the approach in that paper was based on monotonicity formulas for a certain fully nonlinear subelliptic operator, rather than geometric inequalities as those in the present paper. One interesting and novel aspect here is that the relevant estimates depend in an explicit way on a new geometric object, the so called horizontal mean curvature of the boundary of the ground domain. Whether our method is capable of producing estimates with sharp constants or not presently remains an open question. Our approach is based on some delicate (but otherwise fairly elementary) integration by parts formulas which are combined with a sub-Riemannian Bochner type identity. The latter is inspired by the classical one from Riemannian geometry which states that for a Riemannian manifold $M$ with Levi-Civita connection $\nabla$, one has for $u \in C^0(M)$

$$\Delta (|\nabla u|^2) = 2 |\nabla^2 u|^2 + 2 \langle \nabla u, \nabla (\Delta u) \rangle + 2 Ric(\nabla u, \nabla u),$$

where $Ric(\cdot, \cdot)$ represents the Ricci tensor. A beautiful generalization to CR manifolds of (1.1) was found by Greenleaf in [Gre]. Another tool that we use in the proof of Theorem 1.8 below is a sub-Riemannian Rellich identity discovered in [GV].

In the classical case a basic a priori estimate of the elliptic theory reads

$$||u||_{W^{2,2}(\Omega)} \leq C \left( ||Lu||_{L^2(\Omega)} + ||u||_{L^2(\Omega)} \right),$$

where $Lu$ denotes the horizontal second derivatives of $u$. This inequality is a consequence of the classical elliptic estimates$^{1}$.

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where $\Omega \subset \mathbb{R}^n$, and $L$ is a second order uniformly elliptic operator. One of the main tools in the obtainment of \(1.2\) is the following geometric a priori inequality

\[
\int_{\Omega} ||\nabla^2 u||^2 \, dx \leq C \left( \int_{\Omega} (\Delta u)^2 \, dx + \int_{\Omega} u^2 \, dx \right),
\]

valid for any $u \in C^2(\Omega)$, with $u = 0$ on $\partial \Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded open set which is piecewise $C^1$ and whose principal curvatures are bounded, see [LU], Lemma 8.1 on p.175. Here $\nabla^2 u$ denotes the Hessian matrix of $u$, and $||\nabla^2 u||$ indicates its Hilbert-Schmidt norm. The constant $C > 0$ depends on various parameters, among which appropriate bounds on the principal curvatures of $\partial \Omega$. The prototype of estimates such as \(1.2\) and \(1.3\) first appeared in two dimension in the pioneering works of S. N. Bernstein [Be1], [Be2]. Several years later Kadlec [Ka] first obtained a higher dimensional version of \(1.3\) for convex domains. One should also see the works of Ladyzenskaya and Uraltseva [LU], Talenti [Ta], Grisvard [Gr], Lewis [Le]. We consider a Carnot group $G$ with Lie algebra $\mathfrak{g}$ (for the relevant definitions and properties see Section 2). We assume throughout that $G$ is endowed with a left-invariant Riemannian metric with respect to which the Lie algebra generating left-invariant vector fields $X_1, \ldots, X_m$ defined in \(2.5\) below constitute an orthonormal basis of the horizontal subbundle $H \! G$. If $\Omega \subset G$ is an open set and $k \in \mathbb{N}$, we denote by $\Gamma_k(\Omega)$ the Folland-Stein class of functions $u \in C(\Omega)$ such that for every $1 \leq s \leq k$ one has $X_{j_1} \ldots X_{j_s} u \in C(\Omega)$, where $j_i \in \{1, \ldots, m\}$ for $1 \leq \ell \leq s$. Given a function $u \in \Gamma^2(\Omega)$ we let $\nabla H u = \sum_{i=1}^m X_i u X_i$ denote the horizontal gradient of $u$. If $\zeta = \sum_{i=1}^m \zeta_i X_i \in \Gamma^1(G, H \! G)$, then we let $\text{div}_H \zeta = \sum_{i=1}^m X_i \zeta_i$. The horizontal Laplacian of $u \in \Gamma^2(\Omega)$ is then given by

$$
\Delta_H u = \text{div}_H \nabla H u = \sum_{i=1}^m X_i^2 u.
$$

The symmetrized Hessian of $u$ is the $m \times m$ matrix defined by

$$
\nabla^2 H u = [u_{ij}], \quad \text{where} \quad u_{ij} = \frac{X_i X_j u + X_j X_i u}{2},
$$

and clearly $\Delta_H u = \text{trace} \nabla^2 H u$. The matrix $\nabla^2 H u$ plays a central role in the study of convexity in Carnot groups. It was in fact proved in [DGN1], [LMS] that a function $u \in \Gamma^2$ is $H$-convex (see definition \(1.17\) below) if and only if $\nabla^2 H u \geq 0$.

Our first result is an integral identity which connects an interesting fully nonlinear subelliptic operator to the geometry of the ground domain through the $H$-mean curvature $\mathcal{H}$ of its boundary (for the latter notion, see Definition \(2.2\) below). In what follows we denote by $d\sigma_H$ the $H$-perimeter measure on $\partial \Omega$, see Definition \(2.5\) and also [DGN2].

**Theorem 1.1.** Let $G$ be a Carnot group and consider a $C^2$ bounded open set $\Omega \subset G$. Let $u \in \Gamma^3(\Omega) \cap C^2(\Omega)$ with $u \leq 0$ in $\Omega$ and $u = 0$ on $\partial \Omega$. One has

\[
\int_{\Omega} \left\{ (\Delta_H u)^2 - ||\nabla^2 H u||^2 \right\} \, dg + \frac{3}{4} \sum_{i,j=1}^m \int_{\Omega} ([X_i, X_j] u)^2 \, dg
\]

\[+ \sum_{i,j=1}^m \int_{\Omega} X_i u \, [X_i, X_j] X_j u \, dg = \int_{\partial \Omega} \mathcal{H} \, |\nabla H u|^2 d\sigma_H.
\]


When the step of $\mathbb{G}$ is two, then the third integral in the left-hand side of (1.4) vanishes. If instead $u \in C_0^\infty(\Omega)$, then one obtains

$$
(1.5) \quad \int_{\Omega} \|\nabla_H^2 u\|^2 \, dg = \int_{\Omega} (\Delta_H u)^2 \, dg + \frac{3}{4} \sum_{i,j=1}^{m} \int_{\Omega} ([X_i, X_j] u)^2 \, dg
$$

$$
+ \sum_{i,j=1}^{m} \int_{\Omega} X_i u ([X_i, X_j], X_j) u \, dg .
$$

It is interesting to state explicitly Theorem 1.1 in the special, yet important situation, when $\mathbb{G} = \mathbb{H}^n$, the Heisenberg group, with group law

$$
g \circ g' = (x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(x \cdot y' - y' \cdot x)) ,
$$

and the left-invariant basis for the Lie algebra

$$
(1.6) \quad X_j = \frac{\partial}{\partial x_j}, \quad X_{n+j} = \frac{\partial}{\partial y_j} + \frac{x_j}{2} \frac{\partial}{\partial t}, \quad j = 1, \ldots, n, \quad T = \frac{\partial}{\partial t} .
$$

We note that the vector fields $X_1, \ldots, X_{2n}$ satisfy the commutation relations

$$
[X_j, X_{n+k}] = \delta_{jk} T, \quad j, k = 1, \ldots, n ,
$$

and therefore they generate the Lie algebra of $\mathbb{H}^n$.

**Corollary 1.2.** Consider a $C^2$ bounded open set $\Omega \subset \mathbb{H}^n$. Let $u \in \Gamma^3(\Omega) \cap C^2(\Omega)$ with $u \leq 0$ in $\Omega$ and $u = 0$ on $\partial \Omega$. One has

$$
(1.7) \quad \int_{\Omega} (\Delta_H u)^2 \, dg - \int_{\Omega} \|\nabla_H^2 u\|^2 \, dg + \frac{3}{2} n \int_{\Omega} (T u)^2 \, dg
$$

$$
= \int_{\partial \Omega} \mathcal{H} |\nabla_H u|^2 \, d\sigma_H .
$$

As a consequence, if $\partial \Omega$ has $H$-mean curvature $\mathcal{H} \geq 0$, then

$$
(1.8) \quad \int_{\Omega} \|\nabla_H^2 u\|^2 \, dg \leq \int_{\Omega} (\Delta_H u)^2 \, dg + \frac{3}{2} n \int_{\Omega} (T u)^2 \, dg .
$$

If instead $u \in C_0^\infty(\Omega)$, then regardless of the sign of $\mathcal{H}$ one obtains

$$
(1.9) \quad \int_{\Omega} \|\nabla_H^2 u\|^2 \, dg = \int_{\Omega} (\Delta_H u)^2 \, dg + \frac{3}{2} n \int_{\Omega} (T u)^2 \, dg .
$$

When $n = 1$, then (1.7) becomes

$$
(1.10) \quad \int_{\Omega} \det(\nabla_H^2 u) \, dg + \frac{3}{4} \int_{\Omega} (T u)^2 = \frac{1}{2} \int_{\partial \Omega} \mathcal{H} |\nabla_H u|^2 \, d\sigma_H .
$$

With some appropriate modifications, Theorems 1.1 and Corollary 1.2 are still valid if one removes the assumption that $u \leq 0$ in $\Omega$. The following result easily follows by keeping track of the various terms appearing in the proof of Theorem 1.1

**Theorem 1.3.** Let $\mathbb{G}$ be a Carnot group of step $r = 2$, and consider a $C^2$ bounded open set $\Omega \subset \mathbb{G}$. Let $u \in \Gamma^3(\Omega) \cap C^2(\Omega)$ with $u = 0$ on $\partial \Omega$. One has

$$
(1.11) \quad \left| \int_{\Omega} (\Delta_H u)^2 \, dg - \int_{\Omega} \|\nabla_H^2 u\|^2 \, dg + \frac{3}{4} \sum_{i,j=1}^{m} \int_{\Omega} ([X_i, X_j] u)^2 \, dg \right|
$$

$$
\leq \int_{\partial \Omega} |\mathcal{H}| \|\nabla_H u\|^2 \, d\sigma_H .
$$
We next obtain some basic consequences of Theorem 1.1 when the function $u$ is $H_2$-convex. With this hypothesis we are able to bound the $L^2$ norm of the commutators $[X_i, X_j]u$ in terms of a weighted integral of the $H$-mean curvature of $\partial \Omega$.

To introduce the relevant notions we recall that for $r = 1, \ldots, m$, the $r$-th elementary symmetric function is defined by

$$S_r(x) = \sum_{i_1 < \ldots < i_r} x_{i_1} \ldots x_{i_r}, \quad 1 \leq r \leq m.$$  

When $r > 1$ we can use such functions to form the fully nonlinear differential operators

$$F_r[u] = S_r(\lambda_1(u), \ldots, \lambda_m(u)),$$

where $\lambda_1(u), \ldots, \lambda_m(u)$ denote the eigenvalues of the symmetrized Hessian of $u$. One easily recognizes that

$$F_1[u] = S_1(\lambda) = \text{trace}(\nabla^2_H u) = \Delta_H u = \sum_{i=1}^m u_{ii} \quad \text{(horizontal Laplacian)},$$

$$F_2[u] = S_2(\lambda) = \sum_{i<j} (u_{ii} u_{jj} - u_{ij}^2) = \frac{1}{2} \{ (\Delta_H u)^2 - ||\nabla^2_H u||^2 \} ,$$

$$F_m[u] = S_m(\lambda) = \det \nabla^2_H(u) \quad \text{(horizontal Monge-Ampère)}.$$  

**Definition 1.4.** For $r = 1, \ldots, m$, a function $u \in \Gamma^2(\mathbb{G})$ is called $H_r$-convex, if $F_k(u) \geq 0$ for $k = 1, \ldots, r$.

For these notions and for related results we refer the reader to the paper [DGNT].

**Remark 1.5.** We observe that $H_1$-convex functions correspond to subharmonic functions, i.e., $\Delta_H u \geq 0$, whereas a function $u$ is $H_2$-convex if $\Delta_H u \geq 0$ and $(\Delta_H u)^2 - ||\nabla^2_H u||^2 \geq 0$. We recall the following geometric notion of convexity introduced in [DGN1]. One should also see [LMS] where a notion of convexity in the viscosity sense was introduced. These two notions have been recently shown to be equivalent, see [Wa1, Wa2, Ri]. A function $u : \mathbb{G} \to \mathbb{R}$ is called $H$-convex if given any point $g \in \mathbb{G}$ and $0 \leq \lambda \leq 1$, the following inequality holds

$$u(g\delta_\lambda(g^{-1}g')) \leq (1 - \lambda)u(g) + \lambda u(g') , \quad \text{for every} \quad g' \in H_g ,$$

where $H_g$ indicates the horizontal plane through $g \in \mathbb{G}$. In (1.17) we have denoted by $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ the anisotropic dilations on $\mathbb{G}$. The point $g\delta_\lambda(g^{-1}g')$ denotes the twisted convex combination of $g$ and $g'$ based at $g$. According to Theorem 5.12 in [DGN1], a function $u$ is $H_m$-convex according to Definition 1.4 if and only if $u$ is $H$-convex.

**Theorem 1.6.** Let $\mathbb{G}$ be a Carnot group of step $r = 2$, and consider a $C^2$ bounded open set $\Omega \subset \mathbb{G}$. Let $u \in \Gamma^3(\bar{\Omega}) \cap C^2(\bar{\Omega})$ be $H_2$-convex in $\Omega$ with $u = 0$ on $\partial \Omega$. One has

$$\sum_{i,j=1}^m \int_{\Omega} [(X_i, X_j)u]^2 \, dg \leq \frac{4}{3} \int_{\partial \Omega} \mathcal{H} \, |\nabla_H u|^2 \, d\sigma_H .$$

In particular, if $\mathbb{G} = \mathbb{H}^n$, one obtains

$$\int_{\Omega} (Tu)^2 \leq \frac{2}{3n} \int_{\partial \Omega} \mathcal{H} \, |\nabla_H u|^2 \, d\sigma_H .$$
We note that, according to Lemma 2.4, under the hypothesis of Theorem 1.6 we must have $H \geq 0$ on $\partial \Omega \setminus \Sigma$, where $\Sigma$ denotes the characteristic set of $\partial \Omega$, see definition (2.7) below. Since thanks to results of Balogh [Ba] and Magnani [Ma] we know that $\sigma_H(\Sigma) = 0$, we conclude that $H \geq 0$ $\sigma_H$-a.e. on $\partial \Omega$.

One should compare the sharp geometric bounds in Theorem 1.6 with the following non-geometric local a priori bound established in [DGNT], see also [GM1], [GT] and [GM2].

**Theorem 1.7.** Consider a bounded open set $\Omega$ in a group of step two $\mathbb{G}$. Let $u \in \Gamma^3(\Omega)$ be a $H^2$-convex function. For any $D \subset \subset D' \subset \subset \Omega$ we have for some constant $C > 0$ depending on $\mathbb{G}, \Omega, D', D$

$$\sum_{i,j=1}^m \int_D ([X_i, X_j]u)^2 \, dg \leq C \left( \text{osc}_{D'} u \right)^2.$$ 

The next theorem provides a basic global a priori bound for the $L^2$ norms of the commutators of an $H^2$-convex function vanishing on the boundary under the assumption that the ground domain be starlike (in a weak sense) and that the horizontal mean curvature of its boundary be bounded. In the starlikeness assumption in (1.20) below, the vector field $Z$ indicates the infinitesimal generator of the non-isotropic group dilations. For its definition see (4.9) below.

**Theorem 1.8.** Let $\mathbb{G}$ be a Carnot group of step $r = 2$, and let $\Omega \subset \mathbb{G}$ be a $C^2$ bounded open set such that for some $M, \alpha > 0$,

(1.19) $\sup_{\partial \Omega} |\mathcal{H}| \leq M$ ,

and, with $W$ as in (2.6) below, suppose that

(1.20) $\inf_{\partial \Omega} < Z, \nu > \geq \alpha W$ .

There exists a constant $C(\mathbb{G}, \Omega, M, \alpha) > 0$ such that for $u \in \Gamma^3(\Omega) \cap C^2(\Omega)$ which is $H^2$-convex in $\Omega$, and satisfies $u = 0$ on $\partial \Omega$, one has

(1.21) $\sum_{i,j=1}^m \int_\Omega ([X_i, X_j]u)^2 \, dg \leq C \int_\Omega (\Delta_H u)^2 \, dg$ .

The proof of Theorem 1.8 is accomplished by combining Theorem 1.6 with a sub-Riemannian Rellich identity discovered in [GV], see Theorem 4.2 and Corollaries 4.3 and 4.4. These results allow to establish Lemma 4.5 which is instrumental in controlling the commutator term in (1.21) in Theorem 1.8 solely in terms of the $L^2$ norm of the horizontal Laplacian. We notice explicitly that, since by (2.7) at the characteristic points of $\partial \Omega$ the angle function $W$ vanishes, condition (1.20) is a weak starlikeness assumption with respect to the non-isotropic group dilations (2.1) below. We also emphasize that in a Carnot group of arbitrary step a basic family of domains satisfying the hypothesis (1.19) and (1.20) in Theorem 1.8 is represented by the gauge pseudoballs centered at the group identity, see Proposition 4.6.

We mention at this point that after this paper was submitted we learnt from J. Manfredi of his interesting preprint [CM] joint with Chanillo in which, using the above mentioned Bochner identity due to Greenleaf [Gre], the authors obtain a priori estimates connected to those in Theorem 1.1 but for strictly pseudo-convex CR hypersurfaces $M^{2n+1} \subset \mathbb{C}^{n+1}$. There are however two essential differences between our work and [CM], and neither of these papers is contained in the other. On the one hand there is the obvious fact that there exists in nature a plentiful supply of Carnot groups which are not CR hypersurfaces. The second distinction has to do with the different goals of the papers. To explain this point we mention that, as it is well-known, see e.g. [S], the Heisenberg group $\mathbb{H}^n$ with its flat CR structure, via its identification with the boundary of the Siegel upper half-space, is the basic prototype of a CR hypersurface. However, even in this specialized context the results in [CM] do not contain ours since we are
primarily concerned with global geometric estimates connecting the fully nonlinear operators introduced in [DGNT] to the horizontal mean curvature of a relatively compact sub-domain of the group itself. On the other hand, in [CM] in the non-compact case the authors work exclusively with $C_0^\infty(M^{2n+1})$ functions and the geometry of the boundary plays no role for them. It appears that the ultimate goal in [CM] is achieving the sharp constant in their commutator estimates since this allows them to generalize to the CR setting the above cited Cordes type results in [DM].

In closing we briefly describe the organization of the paper. Section 2 is devoted to recalling some basic facts about Carnot groups and the notion of horizontal (or sub-Riemannian) mean curvature of a hypersurface in such a group. In section 3 we prove the Bochner identity in Proposition 3.3. In section 4 we prove our main results: Theorems 1.1, 1.6 and 1.8. Finally, in Proposition 4.6 we show that in any Carnot group the gauge pseudo-balls satisfy the hypothesis of Theorem 1.8.

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2. Preliminaries

Consider a Carnot group $G$ of step $r$. This is a simply connected Lie group whose Lie algebra $\mathfrak{g}$ is graded and $r$-nilpotent. This means that there exists vector sub-spaces $V_1,\ldots,V_r \subset \mathfrak{g}$ such that $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$, with $[V_i,V_j] = V_{i+j}$, $i = 1,\ldots,r-1$, $[V_1,V_r] = \{0\}$. A natural family of non-isotropic dilations on $\mathfrak{g}$ associated with this grading is given by $\Delta_\lambda(\xi) = \lambda \xi_1 + \cdots + \lambda^r \xi_r$, if $\xi = \xi_1 + \cdots + \xi_r \in \mathfrak{g}$. Using the global diffeomorphism $\exp: \mathfrak{g} \to G$, one then lifts these dilations to the one-parameter family of group automorphisms

$$\delta_\lambda(g) = \exp \circ \Delta_\lambda \circ \exp^{-1}(g), \quad \lambda > 0.$$  

The homogeneous dimension associated with the dilations $\{\delta_\lambda\}_{\lambda > 0}$ is given by

$$Q = \sum_{j=1}^r j \dim V_j.$$

Such number often replaces the topological dimension $N = \sum_{j=1}^r \dim V_j$ in the analysis of $G$, see for instance Corollary 1.4 below. Its relevance is expressed by the fact that, if $dg$ denotes the bi-invariant Haar measure on $G$ obtained by pushing forward through the exponential mapping the Lebesgue measure on $\mathfrak{g}$, then $d(\delta_\lambda(g)) = \lambda^Q dg$. Here, bi-invariant means with respect to the operators of left- and right-translation $L_g(g') = gg'$, $R_g(g') = g'g$, on $G$.

The gauge pseudo-distance $\rho(g,g')$ is defined as follows. Let $|\cdot|$ denote the Euclidean distance to the origin on $\mathfrak{g}$. For $\xi = \xi_1 + \cdots + \xi_r \in \mathfrak{g}$, $\xi_i \in V_i$, one lets

$$|\xi|_\mathfrak{g} = \left( \sum_{i=1}^r |\xi_i|^{2r_i/i} \right)^{2r_i/i}, \quad |g|_G = |\exp^{-1} g|_\mathfrak{g}, \quad g \in G.$$

The pseudo-distance on $G$ associated to $|\cdot|_G$ is given by

$$\rho(g,g') = |g^{-1} g'|_G.$$

With a slight abuse of notation when we write $\rho(g)$ we indicate $\rho(g,e)$, where $e$ is the group identity. Since the function $g \to \rho(g)$ is homogeneous of degree one with respect to the non-isotropic dilations $\delta_\lambda$, we have for the gauge pseudo-ball,

$$B(g,R) = \{g' \in G \mid \rho(g',g) < R\},$$
We denote by $\nabla \vert \Omega$ the Riemannian manifold $\Omega$ in a Carnot group $G$. The vectors $\{\epsilon_j, s\}$ constitute an orthonormal basis of $G$. Because of the special role played by the first two layers $V_1, V_2$ in the grading of $g$, it is convenient to have a simpler notation for the elements of their basis. We thus set henceforth $m = \text{dim } V_1$, $k = \text{dim } V_2$, and will indicate with $\{e_1, ..., e_m\}$, $\{\epsilon_1, ..., \epsilon_k\}$ the corresponding basis of $V_1, V_2$. Denoting with $(L_g)_*$ the differential of left-translations, we define a family of left-invariant vector fields on $G$ by letting

$$
\begin{align*}
X_i(g) &= (L_g)_*(e_i), \ i = 1, ..., m, \ T_s(g) = (L_g)_*(\epsilon_s), \ s = 1, ..., k, \\
X_{j,s}(g) &= (L_g)_*(\epsilon_{j,s}), \ j = 3, ..., k, \ s = 1, ..., m_j.
\end{align*}
$$

Hereafter, we assume that $G$ is endowed with a left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ with respect to which the vector fields $X_1, ..., X_m, T_1, ..., T_k, X_{j,s}, j = 3, ..., r, s = 1, ..., m_j$, are orthonormal. In view of the grading assumption on $g$, it is clear that the vector fields $X_1, ..., X_m$ generate the Lie algebra of all left-invariant vector fields on $G$. They generate a sub-bundle $H_G$ of the tangent bundle $T_G$ which is usually called the horizontal bundle.

We next recall some basic concepts from the sub-Riemannian geometry of an hypersurface in a Carnot group $G$. For a detailed account we refer the reader to [DG2]. We consider the Riemannian manifold $M = G$ with the left-invariant metric tensor with respect to which $X_1, ..., X_m, ..., X_{m,n}$ is an orthonormal basis, the corresponding Levi-Civita connection $\nabla$ on $G$, and the horizontal Levi-Civita connection $\nabla^H$. Let $\Omega \subset G$ be a bounded $C^k$ domain, with $k \geq 2$. We denote by $\nu$ the Riemannian outer normal to $\partial \Omega$, and define the so-called angle function on $\partial \Omega$ as follows

$$
W = |N^H| = \sqrt{\sum_{j=1}^m \langle \nu, X_j \rangle^2}.
$$

The characteristic set of $\Omega$, hereafter denoted by $\Sigma$, is the compact subset of $\partial \Omega$ where the continuous function $W$ vanishes

$$
\Sigma = \{g \in \partial \Omega \mid W(g) = 0\}.
$$

The next definition plays a basic role in sub-Riemannian geometry.

**Definition 2.1.** We define the outer horizontal normal on $\partial \Omega$ as follows

$$
N^H = \sum_{j=1}^m \langle \nu, X_j \rangle X_j,
$$

so that $W = |N^H|$. The horizontal Gauss map $\nu^H$ on $\partial \Omega$ is defined by

$$
\nu^H = \frac{N^H}{|N^H|}, \quad \text{on } \partial \Omega \setminus \Sigma.
$$

We note that $N^H$ is the projection of the Riemannian Gauss map on $\partial \Omega$ onto the horizontal subbundle $H_G \subset T_G$. Such projection vanishes only at characteristic points, and this is why the horizontal Gauss map is not defined on $\Sigma$. The following definition is taken from [DG2], but the reader should also see [HP] for a related notion in the more general setting of vertically rigid spaces.

**Definition 2.2.** The horizontal or $H$-mean curvature of $\partial \Omega$ at a point $g_0 \in \partial \Omega \setminus \Sigma$ is defined as

$$
H = \sum_{i=1}^{m-1} \langle \nabla^H_{e_i} e_i, \nu^H \rangle,
$$
where \( \{e_1, \ldots, e_{m-1}\} \) denotes an orthonormal basis of the horizontal tangent bundle \( T_H \partial \Omega \overset{\text{def}}{=} T\partial \Omega \cap H \mathbb{G} \) on \( \partial \Omega \). If instead \( g_0 \in \Sigma \), then we define \( \mathcal{H}(g_0) = \lim_{g \to g_0} \mathcal{H}(g) \), provided that the limit exists and is finite.

We next consider the following nonlinear operator

\[
\Delta_{H, \infty} u \overset{\text{def}}{=} \sum_{i,j=1}^m u_{ij} X_i u X_j u ,
\]

which by analogy with its by now classical Euclidean ancestor we call the horizontal \( \infty \)-Laplacian. This operator has been recently studied by various people, see e.g. [Bi], [BiC], [Wa3], [GT]. The reason for introducing the operator \( \Delta_{H, \infty} \) is in the following proposition which is often useful in computing the \( H \)-mean curvature. To state it we recall the notion of a defining function for \( \Omega \).

We consider a \( C^2 \) bounded open set \( \Omega \subset \mathbb{G} \) and we assume for convenience that there exists a globally defined \( \phi \in C^2(\mathbb{G}) \) (a defining function) such that

\[
\Omega = \{ g \in \mathbb{G} \mid \phi(g) < 0 \},
\]

and for which \( |\nabla \phi| \geq \alpha > 0 \) in an open neighborhood \( O \) of \( \partial \Omega \), where \( \nabla \phi \) denotes the Riemannian gradient of \( \phi \). The Riemannian outer unit normal to \( \partial \Omega \) is presently given by \( \nu = \nabla \phi / |\nabla \phi| \).

We observe that

\[
|N^H| = \frac{|\nabla^H \phi|}{|\nabla \phi|},
\]

and that on \( \partial \Omega \setminus \Sigma \) one has

\[
\nu^H = \frac{\nabla^H \phi}{|\nabla^H \phi|}.
\]

The next result is Proposition 9.12 in [DGN2].

**Proposition 2.3.** At every point of \( \partial \Omega \setminus \Sigma \) one has in terms of a local defining function \( \phi \) of \( S \)

\[
|\nabla^H \phi|^3 \mathcal{H} = |\nabla^H \phi|^2 \Delta_H \phi - \Delta_{H, \infty} \phi .
\]

We will need the following lemma.

**Lemma 2.4.** Let \( u \in C^2(\mathbb{G}) \) be \( H_2 \)-convex, then for every \( s \in \mathbb{R} \) such that the level set

\[
\Omega_s = \{ g \in \mathbb{G} \mid u(g) < s \}
\]

is a \( C^2 \) domain, the \( H \)-mean curvature of \( \mathcal{E}_s = \partial \Omega_s \) (wherever it is defined) is nonnegative.

**Proof.** Recall that the hypothesis that \( u \) be \( H_2 \)-convex means that \( \Delta_H u \geq 0 \), and that moreover

\[
(\Delta_H u)^2 - ||\nabla^2_H u||^2 \geq 0.
\]

According to Proposition 2.3 it suffices to show that on \( \mathcal{E}_s \) one has \( |\nabla^H u|^2 \Delta_H u - \Delta_{\infty} u \geq 0 \). On the other hand, Schwarz inequality gives

\[
\Delta_{H, \infty} u = \sum_{i,j=1}^m u_{ij} X_i u X_j u \leq ||\nabla^2_H u|| |\nabla^H u|^2 \leq \Delta_H u |\nabla^H u|^2 ,
\]

where in the last inequality we have used (2.14).

\[ \square \]

Given an open set \( \Omega \subset \mathbb{G} \) denote by

\[
\mathcal{F}(\Omega) = \{ \phi = \sum_{j=1}^m \phi_j X_j \in C^1_0(\Omega, H \mathbb{G}) \mid ||\phi||_\infty = \sup_{g \in \Omega} (\sum_{j=1}^m \phi_j^2)^{1/2} \leq 1 \} .
\]
The $H$-perimeter of a measurable set $E \subset \mathbb{G}$ with respect to $\Omega$ was defined in [CDG] as
\[
P_H(E; \Omega) = \sup_{\phi \in \mathcal{F}(\Omega)} \int_{E \cap \Omega} \text{div}_H \phi \, dg .
\]

If $E$ is a bounded open set of class $C^1$, then the divergence theorem gives
\[
P_H(E; \Omega) = \sup_{\phi \in \mathcal{F}(\Omega)} \int_{\partial E \cap \Omega} \nu \cdot \nabla \phi \, d\sigma = \int_{\partial E \cap \Omega} \left| \mathcal{N}_H \right| \sigma ,
\]
where $d\sigma$ is the Riemannian surface measure on $\partial E$. It is clear from this formula that the measure on $\partial E$, defined by $\sigma_H(\partial E \cap \Omega) \overset{\text{def}}{=} P_H(E; \Omega)$ on the open sets of $\partial E$, is absolutely continuous with respect to $\sigma$, and its density is represented by the angle function $W$ of $\partial E$. We formalize this observation in the following definition.

**Definition 2.5.** Given a bounded domain $E \subset \mathbb{G}$ of class $C^1$, with angle function $W$ as in (2.7), we will denote by
\[(2.15) \quad d\sigma_H = \left| \mathcal{N}_H \right| d\sigma = W d\sigma ,
\]
the $H$-perimeter measure supported on $\partial E$.

### 3. A Bochner type identity

In this section we establish a sub-Riemannian version of the classical Bochner identity (1.1) for the sub-Laplacian of the square of the length of the horizontal gradient of a function on a Carnot group $\mathbb{G}$, see Proposition 3.3. A deeper CR version of such formula first appeared in the beautiful paper by A. Greenleaf [Gre]. We begin by finding the formula which expresses the connection between the Hilbert-Schmidt norm of the symmetrized, and that of the un-symmetrized horizontal Hessian.

**Lemma 3.1.** Let $u \in \Gamma^2(\mathbb{G})$, then one has
\[
\sum_{i,j=1}^{m} (X_i X_j u)^2 = \| \nabla_H^2 u \|^2 + \frac{1}{4} \sum_{i,j=1}^{m} ([X_i, X_j] u)^2 .
\]

**Proof.** Notice that the un-symmetrized and the symmetrized second derivatives are connected by the formula
\[
(3.1) \quad X_i X_j u = u_{,ij} + \frac{1}{2} [X_i, X_j] u .
\]

We obtain from (3.1)
\[
\sum_{i,j=1}^{m} (X_i X_j u)^2 = \| \nabla_H^2 u \|^2 + \frac{1}{4} \sum_{i,j=1}^{m} ([X_i, X_j] u)^2 + \sum_{i,j=1}^{m} u_{,ij} [X_i, X_j] u .
\]

To reach the conclusion, it is now enough to observe that, thanks to the skew-symmetry of the matrix $\{ [X_i, X_j] \}$, we have
\[
\sum_{i,j=1}^{m} u_{,ij} [X_i, X_j] u = \sum_{i<j} u_{,ij} [X_i, X_j] u + \sum_{i>j} u_{,ij} [X_i, X_j] u = 0 .
\]

\[\Box\]

**Lemma 3.2.** For $u \in \Gamma^2(\mathbb{G})$ one has
\[
\sum_{i,j=1}^{m} X_i X_j u [X_i, X_j] u = \frac{1}{2} \sum_{i,j=1}^{m} ([X_i, X_j] u)^2 .
\]
Proof. To check this formula we proceed as follows

\begin{equation}
\frac{1}{2} \Delta_H (|\nabla^H u|^2) = < \nabla^H u, \nabla^H (\Delta_H u) > + ||\nabla^2_H u||^2 + \frac{1}{4} \sum_{i,j=1}^m ([X_i, X_j] u)^2 \\
+ 2 \sum_{i,j=1}^m X_j u [X_i, X_j] X_i u + \sum_{i,j=1}^m X_j u [X_i, [X_i, X_j]] u.
\end{equation}

Proposition 3.3. Let $G$ be a Carnot group, $u \in \Gamma^3(G)$, then the following sub-Riemannian Bochner formula holds

\begin{equation}
\frac{1}{2} \Delta_H (|\nabla^H u|^2) = < \nabla^H u, \nabla^H (\Delta_H u) > + ||\nabla^2_H u||^2 + \frac{3}{2} n (Tu)^2 \\
+ 2 \sum_{i,j=1}^m X_j u [X_i, X_j] X_i u.
\end{equation}

Proof. We observe that for any function $F$ we have

\begin{equation}
\Delta_H (F^2) = 2 F \Delta_H F + 2 |\nabla^H F|^2.
\end{equation}

Applying (3.4) to $F = X_j u$ we obtain

\begin{equation}
\frac{1}{2} \Delta_H (|\nabla^H u|^2) = \frac{1}{2} \sum_{j=1}^m \Delta_H ([X_j u]^2) = \sum_{j=1}^m X_j u \Delta_H (X_j u) + \sum_{i,j=1}^m (X_i X_j u)^2.
\end{equation}

We next compute $\Delta_H (X_j u)$. One has

\begin{equation}
\Delta_H (X_j u) = \sum_{i=1}^m X_i X_i X_j u = \sum_{i=1}^m X_i ([X_j u] + [X_i, X_j] u) \\
= \sum_{i=1}^m (X_j X_i + [X_i, X_j]) X_i u + \sum_{i=1}^m X_i [X_i, X_j] u \\
= X_j (\Delta_H u) + \sum_{i=1}^m [X_i, X_j] X_i u + 2 \sum_{i=1}^m [X_i, X_j] X_i u.
\end{equation}

On the other hand, Lemma 3.1 gives

\begin{equation}
\sum_{i,j=1}^m (X_i X_j u)^2 = ||\nabla^2_H u||^2 + \frac{1}{4} \sum_{i,j=1}^m ([X_i, X_j] u)^2.
\end{equation}
Substituting (3.6), (3.7) in (3.5) we find
\[
\frac{1}{2} \Delta_H(|\nabla^H u|^2) = <\nabla^H u, \nabla^H (\Delta_H)> + ||\nabla^H u||^2 + \frac{1}{4} \sum_{i,j=1}^m ([X_i, X_j]u)^2 \\
+ 2 \sum_{i,j=1}^m X_j u [X_i, X_j]u + m \sum_{i,j=1}^m X_j u [X_i, [X_i, X_j]]u ,
\]

which gives the desired conclusion.

\[ \square \]

4. Geometric second derivative estimates

In this section using the horizontal Bochner identity in Proposition 3.3 we prove the various results stated in the introduction.

**Proof of Theorem 1.1.** We begin by observing that, if we denote by \( \nu \) the outer unit Riemannian normal on \( \partial \Omega \), then the assumptions \( u \leq 0 \) in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \) imply
\[
(4.1) \quad \nabla u = |\nabla u| \nu , \quad \text{on} \quad \partial \Omega .
\]

Next, we rewrite the identity in Proposition 3.3 as follows
\[
\frac{1}{2} \Delta_H(|\nabla^H u|^2) = <\nabla^H u, \nabla^H (\Delta_H)> + ||\nabla^H u||^2 + \frac{1}{4} \sum_{i,j=1}^m ([X_i, X_j]u)^2 \\
+ 2 \sum_{i,j=1}^m X_j u [X_i, X_j]u + m \sum_{i,j=1}^m X_j u [X_i, [X_i, X_j]]u
\]

This gives
\[
(4.2) \quad \frac{1}{2} \Delta_H(|\nabla^H u|^2) = <\nabla^H u, \nabla^H (\Delta_H)> + ||\nabla^H u||^2 + \frac{1}{4} \sum_{i,j=1}^m ([X_i, X_j]u)^2 \\
+ 2 \sum_{i,j=1}^m X_j u [X_i, X_j]u + m \sum_{i,j=1}^m X_j u [X_i, [X_i, X_j]]u.
\]

We now integrate the identity (4.2) on \( \Omega \)
\[
(4.3) \quad \frac{1}{2} \int_{\Omega} \Delta_H(|\nabla^H u|^2) \, dg = \int_{\Omega} <\nabla^H u, \nabla^H (\Delta_H u)> \, dg \\
+ \int_{\Omega} ||\nabla^H u||^2 \, dg + \frac{1}{4} \sum_{i,j=1}^m \int_{\Omega} ([X_i, X_j]u)^2 \, dg \\
+ 2 \sum_{i,j=1}^m \int_{\Omega} X_j u [X_i, X_j]u \, dg + \sum_{i,j=1}^m \int_{\Omega} X_j u [X_i, [X_i, X_j]]u \, dg .
\]

Using (4.1) we have from the divergence theorem
\[
(4.4) \quad \frac{1}{2} \int_{\Omega} \Delta_H(|\nabla^H u|^2) \, dg = \frac{1}{2} \int_{\partial \Omega} \frac{<\nabla^H (|\nabla^H u|^2), \nabla^H u>}{|\nabla u|} \, d\sigma .
\]
Also, again from (4.1), we find

\[(4.5) \quad \int_{\Omega} < \nabla^H u, \nabla^H (\Delta^H u) > dg = \int_{\partial \Omega} \frac{\nabla^H u \cdot \nabla^H (\Delta^H u)}{|\nabla u|} d\sigma - \int_{\Omega} (\Delta^H u)^2 dg .\]

Finally, we have

\[(4.6) \quad 2 \sum_{i,j=1}^{m} \int_{\Omega} X_i u X_i [X_i, X_j] u dg = 2 \sum_{i,j=1}^{m} \int_{\partial \Omega} \frac{[X_i, X_j] u X_i u X_j u}{|\nabla u|} d\sigma \]

\[- 2 \sum_{i,j=1}^{m} \int_{\Omega} [X_i, X_j] u X_i X_j u dg = - 2 \sum_{i,j=1}^{m} \int_{\Omega} [X_i, X_j] u X_i X_j u d\sigma , \]

where to eliminate the boundary integral we have used the skew-symmetry of the matrix \([X_i, X_j] u \}_{i,j=1,...,m} \). Substituting (4.4)-(4.6) into (4.3) we conclude

\[\int_{\Omega} \left\{ (\Delta^H u)^2 - ||\nabla^2^H u||^2 \right\} dg + \frac{3}{4} \sum_{i,j=1}^{m} \int_{\Omega} ([X_i, X_j] u)^2 dg + \sum_{i,j=1}^{m} \int_{\Omega} X_i u ([X_i, X_j], X_j] u d\sigma \]

\[= \int_{\partial \Omega} \mathcal{H} |\nabla^H u|^2 d\sigma_H , \]

where in the last equality we have used Proposition 2.3 and Definition 2.5. This gives the desired conclusion.

\[\square\]

**Corollary 4.1.** Let \( G \) be a Carnot group of step \( r = 2 \), and consider a \( C^2 \) bounded open set \( \Omega \subset G \). Let \( u \in \Gamma^2(\Omega) \) with \( u \leq 0 \) in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \). One has

\[(4.7) \quad \int_{\Omega} ||\nabla^2^H u||^2 dg = \int_{\Omega} (\Delta^H u)^2 dg + \frac{3}{4} \sum_{i,j=1}^{m} \int_{\Omega} ([X_i, X_j] u)^2 dg \]

\[\quad - \int_{\partial \Omega} \mathcal{H} |\nabla^H u|^2 d\sigma_H . \]

If instead \( u \in \Gamma^2_0(\Omega) \), then one obtains

\[(4.8) \quad \int_{\Omega} ||\nabla^2^H u||^2 dg = \int_{\Omega} (\Delta^H u)^2 dg + \frac{3}{4} \sum_{i,j=1}^{m} \int_{\Omega} ([X_i, X_j] u)^2 dg . \]

We can now present the

**Proof of Theorem 1.6.** We notice that, since \( u \) is \( H_2 \)-convex, then \( \Delta^H u \geq 0 \), and \( (\Delta^H u)^2 - ||\nabla^2^H u||^2 \geq 0 \) in \( \Omega \). In particular, since by assumption \( u = 0 \) on \( \partial \Omega \), from Bony’s weak maximum principle \[Bo\] we infer that \( u \leq 0 \) in \( \Omega \), and therefore we can apply Theorem 1.1. The desired conclusion now follows from (1.4) in Theorem 1.1.

\[\square\]

We next want to control the commutator term in the right-hand side of (4.7) in Corollary 4.1. To reach this goal we will make use of a sub-Riemannian Rellich identity discovered in \[GV\]. In the following results, \( \Omega \) will indicate a piecewise \( C^1 \) bounded open subset of a Carnot group \( G \) with outer unit normal \( \nu \) and surface measure \( \sigma \).
**Theorem 4.2.** For \( u \in \Gamma^2(\Omega) \) one has
\[
2 \int_{\partial \Omega} \zeta u < \nabla^H u, \mathbf{N}^H > d\sigma + \int_{\Omega} \text{div}_G \zeta |\nabla^H u|^2 \, dg \\
- 2 \sum_{i=1}^{m} \int_{\Omega} X_i u [X_i, \zeta] u \, dg - 2 \int_{\Omega} \zeta u \Delta^H u \, dg \\
= \int_{\partial \Omega} |\nabla^H u|^2 < \zeta, \nu > d\sigma,
\]
where \( \zeta \) is a \( C^1 \) vector field on \( G \).

**Corollary 4.3.** Let \( u \in \Gamma^2(\Omega) \) and assume, in addition, that \( u = 0 \) on \( \partial \Omega \). One has
\[
\int_{\partial \Omega} |\nabla^H u|^2 < \zeta, \nu > d\sigma + \int_{\Omega} \text{div}_G \zeta |\nabla^H u|^2 \, dg \\
- 2 \sum_{i=1}^{m} \int_{\Omega} X_i u [X_i, \zeta] u \, dg - 2 \int_{\Omega} \zeta u \Delta^H u \, dg = 0.
\]

In what follows we indicate with \( Z \) the infinitesimal generator of the non-isotropic dilations (2.1). We note that in the exponential coordinates it is given by
\[
Z = \sum_{i=1}^{m} x_i(g) X_i + 2 \sum_{s=1}^{k} t_s(g) T_s + \sum_{j=3}^{r} \sum_{s=1}^{m_j} x_{j,s}(g) X_{j,s}.
\]

When the step of the group is \( r = 2 \) the third sum in the right-hand side of (4.9) does not appear.

\[
[X_i, Z] = X_i, \quad i = 1, \ldots, m, \quad \text{div}_G Z = Q.
\]

For a proof of the first identity in (4.10) see Lemma 2.1 in [DG]. The second identity follows by using the expression (4.9) of \( Z \) in the exponential coordinates. Choosing \( \eta = Z \) in Corollary 4.3, and using (4.10) we easily obtain.

**Corollary 4.4.** Let \( u \in \Gamma^2(\Omega) \) and assume, in addition, that \( u = 0 \) on \( \partial \Omega \). One has
\[
\int_{\partial \Omega} |\nabla^H u|^2 < Z, \nu > d\sigma + (Q - 2) \int_{\Omega} |\nabla^H u|^2 \, dg = 2 \int_{\Omega} Zu \Delta^H u \, dg.
\]

Using Corollary 4.4 we can now prove the following useful estimate.

**Lemma 4.5.** Suppose that \( G \) be a Carnot group of step \( r = 2 \). Under the hypothesis of Corollary 4.4 on the function \( u \) there exists a constant \( C = C(G, \Omega) > 0 \) such that for any \( \epsilon > 0 \) one has
\[
\int_{\partial \Omega} |\nabla^H u|^2 < Z, \nu > d\sigma + \frac{(Q - 2)}{2} \int_{\Omega} |\nabla^H u|^2 \, dg \\
\leq C \left\{ \left( 1 + \frac{1}{\epsilon} \right) \int_{\Omega} (\Delta^H u)^2 \, dg + \epsilon \sum_{i,j=1}^{m} \int_{\Omega} ([X_i, X_j]u)^2 \, dg \right\}.
\]

**Proof.** Since \( G \) has step \( r = 2 \), from the bracket generating assumption for every \( s = 1, \ldots, k \) there exist \( \alpha^s_{i,j} \in \mathbb{R}, \ i, j = 1, \ldots, m \), such that
\[
T_s = \sum_{i,j=1}^{m} \alpha^s_{i,j} [X_i, X_j].
\]
Therefore, it is possible to find \( \beta_s > 0 \) such that

\[
|T_s u| \leq \beta_s \left( \sum_{i,j=1}^{m} ([X_i, X_j]u)^2 \right)^{1/2}.
\]

From this estimate, from (4.9) and from the boundedness of \( \Omega \) we conclude that there exists \( C = C(G, \Omega) > 0 \) such that one has

\[
|Z u \Delta_H u| \leq C \left( |\nabla H u| + \left( \sum_{i,j=1}^{m} ([X_i, X_j]u)^2 \right)^{1/2} \right) |\Delta_H u|, \quad \text{in} \ \Omega.
\]

Inserting this estimate in the identity of Corollary 4.4, for every \( \delta, \epsilon > 0 \) we find

\[
\int_{\partial \Omega} |\nabla H u|^2 < Z, \nu > d\sigma + (Q - 2) \int_{\Omega} |\nabla H u|^2 \ dg \leq 2 \int_{\Omega} |Z u \Delta_H u| \ dg
\]

\[
\leq C \delta \int_{\Omega} |\nabla H u|^2 \ dg + \frac{C}{\delta} \int_{\Omega} (\Delta_H u)^2 \ dg
\]

\[
+ C \epsilon \sum_{i,j=1}^{m} \int_{\Omega} ([X_i, X_j]u)^2 \ dg + \frac{C}{\epsilon} \int_{\Omega} (\Delta_H u)^2 \ dg.
\]

Choosing now \( \delta > 0 \) such that \( C\delta = \frac{Q-2}{2} \) we obtain the desired conclusion (with a possibly different constant \( C = C(G, \Omega) > 0 \)).

\[\square\]

We can now provide the

Proof of Theorem 1.8. We start with the inequality (1.18) in Theorem 1.6, which gives

\[
3 \left( \frac{m}{4} \right) \sum_{i,j=1}^{m} \int_{\Omega} ([X_i, X_j]u)^2 \ dg \leq \int_{\partial \Omega} H |\nabla H u|^2 d\sigma_H \leq M \int_{\partial \Omega} |\nabla H u|^2 d\sigma_H,
\]

where we have used the hypothesis (1.19). According to Lemma 4.5 if \( \Omega \) satisfies the hypothesis (1.20) we obtain for any \( \epsilon > 0 \)

\[
\alpha \int_{\partial \Omega} |\nabla H u|^2 W \ d\sigma + \frac{(Q - 2)}{2} \int_{\Omega} |\nabla H u|^2 \ dg
\]

\[
\leq \int_{\partial \Omega} |\nabla H u|^2 < Z, \nu > d\sigma + \frac{(Q - 2)}{2} \int_{\Omega} |\nabla H u|^2 \ dg
\]

\[
\leq C \left\{ \left( 1 + \frac{1}{\epsilon} \right) \int_{\Omega} (\Delta_H u)^2 \ dg + \epsilon \sum_{i,j=1}^{m} \int_{\Omega} ([X_i, X_j]u)^2 \ dg \right\}.
\]

Keeping (2.15) in mind, we have proved that for any \( \epsilon > 0 \)

\[
\int_{\partial \Omega} |\nabla H u|^2 d\sigma_H \leq C(G, \alpha) \left\{ \left( 1 + \frac{1}{\epsilon} \right) \int_{\Omega} (\Delta_H u)^2 \ dg + \epsilon \sum_{i,j=1}^{m} \int_{\Omega} ([X_i, X_j]u)^2 \ dg \right\}.
\]

Combining this estimate with (4.11) we obtain

\[
\sum_{i,j=1}^{m} \int_{\Omega} ([X_i, X_j]u)^2 \ dg \leq C(G, M, \alpha) \left\{ \left( 1 + \frac{1}{\epsilon} \right) \int_{\Omega} (\Delta_H u)^2 \ dg + \epsilon \sum_{i,j=1}^{m} \int_{\Omega} ([X_i, X_j]u)^2 \ dg \right\}.
\]
Choosing $\epsilon > 0$ in (4.12) such that $\epsilon C(G, M, \alpha) < 1$ we finally reach the desired conclusion.

We close with a proposition which provides a significant class of domains which satisfy the two geometric hypothesis in Theorem 1.8.

**Proposition 4.6.** In a Carnot group $G$ of arbitrary step consider a gauge pseudo-ball $B_R = \{ g \in G \mid \rho(g) < R \}$, where $\rho$ is the Folland-Stein gauge (2.2), (2.3). There exists $C = C(G) > 0$, $\alpha = \alpha(G) > 0$ such that

\begin{equation}
\sup_{\partial B_R} |\mathcal{H}| \leq \frac{C}{R},
\end{equation}

and

\begin{equation}
\inf_{\partial B_R} < Z, \nu > \geq \alpha R W.
\end{equation}

**Proof.** The outer unit normal to $B_R$ at a point of its boundary is given by $\nu = \nabla\rho / \|
abla\rho\|$. Since the function $\rho$ is homogeneous of degree one with respect to the non-isotropic group dilations, from the Euler type formula for Carnot groups we obtain on $\partial B_R$

\begin{equation}
< Z, \nu > = < Z, \frac{\nabla\rho}{\|
abla\rho\|} > = \frac{Z \rho}{\|
abla\rho\|} = \frac{\rho}{\|
abla\rho\|} = \frac{R}{\|
abla\rho\|}.
\end{equation}

On the other hand, since $\rho \in C^\infty(G \setminus \{e\})$, and since $|\nabla^H \rho|$ is homogeneous of degree zero, we have for every $g \neq e$

\begin{equation}
W(g) = \frac{|\nabla^H \rho(g)|}{|\nabla \rho(g)|} \leq \frac{\sup_{\rho(g')=1} |\nabla^H \rho(g')|}{|\nabla \rho(g)|} = \frac{C(G)}{|\nabla \rho(g)|}.
\end{equation}

We thus obtain on $\partial B_R$

\begin{equation}
< Z, \nu > \geq C(G)^{-1} R W = \alpha R W.
\end{equation}

This proves (4.14). To prove the qualitative estimate (4.13) we again employ homogeneity considerations. According to Proposition 2.3 we have on $\partial B_R$

\begin{equation}
|\nabla^H \rho|^3 \mathcal{H} = |\nabla^H \rho|^2 \Delta_H \rho - \Delta_{H, \infty} \rho.
\end{equation}

Now, $\Delta_H \rho$ and $\Delta_{H, \infty} \rho$ both have homogeneity $-1$, and hence so does $\mathcal{H}$. We thus find on $\partial B_R$

\begin{equation}
|\mathcal{H}(g)| \leq \frac{1}{\rho(g)} \sup_{\rho(g')=1} |\mathcal{H}(g')| = \frac{C(G)}{R},
\end{equation}

which establishes (4.13).

\[\square\]
