What is an atom laser?

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An atom laser is a hypothetical device which would produce an atomic field analogous to the electromagnetic field of a photon laser. Here I argue that for this analogy to be meaningful it is necessary to have a precise definition of a laser which applies equally to photon or atom lasers. The definition I propose is based upon the principle that the output of a laser is well-approximated by a classical wave of fixed intensity and phase. This principle yields four quantitative conditions which the output of a device must satisfy in order for that device to be considered a laser. While explaining these requirements, I analyse the similarities and differences between atom and photon lasers. I show how these conditions are satisfied first by an idealized photon laser model, and then by a more generic model which can apply to atom lasers also. Lastly, I briefly discuss the current proposals for atom lasers and whether they could be true lasers.

I. INTRODUCTION

The field of quantum optics exists courtesy of the invention of the laser, now almost four decades ago [1]. Not surprisingly, the mechanism of a laser is treated in most modern quantum optics textbooks [2,3] although there are exceptions [4]. However, in none of these books is it to be found an actual definition of the term “laser”. The expansion of the acronym itself — light amplification by the stimulated emission of radiation — falls well short of explaining what is special about laser light. For example, stimulated emission may dominate spontaneous emission even in a laser below threshold, and yet at least the quantum optics community believes that operation above threshold is essential for a device to be considered a true laser [4,5]. The precise definition of a laser would clearly be a difficult task, so the approach by the authors of these books — to present an analysis of the dynamics of some more or less idealized model of a laser, and then to look at the consequences of these dynamics on the properties of the light — is quite understandable.

As well as being difficult, constructing a definition of a laser might be thought futile. One reason is that different groups of people (whether in theory, experiment, or applications) require quite different properties from a laser, and hence would differ in their own intuitive understanding of what makes a laser special. Another reason is that the same physics often underpins the operation of lasers in vastly different regimes, so it would seem unnatural to exclude some regimes by saying “this is no longer a true laser”.

These are powerful arguments against attempting a definition of the term “laser”. However, a new field of physics has recently arisen which seems to need such a definition if its future direction is to be clear. This is the field of atom optics and, more particularly, atomic Bose condensation. Atom optics is the study of atoms under conditions where their de Broglie wave nature becomes important, suggesting an analogy with photons [6]. The recent experimental achievement of dilute gases of atoms which are highly Bose-degenerate [7,8] has now deepened the analogy between photons and Bose atoms by demonstrating that their quantum statistics are identical. It is generally recognized that the next logical step would be to try to create an atom laser, “the equivalent of a laser for atoms and . . . a source of coherent matter waves” [9]. But in what sense is an atom laser to be equivalent to a photon laser, and what are the properties which make matter waves coherent?

Recently there have been a number of proposals for building a device under various names similar to that of ‘atom laser’ [10–12]. Very recently Mewes et al. [13] have actually built an output coupler for a Bose condensate which they say “can be regarded as a pulsed ‘atom laser’”. Although their experimental results are an important milestone, their cautious language makes it clear that the output of their device is still a long way short of the coherent matter waves one would envisage for a true atom laser. When a true atom laser is built, perhaps along the lines of one of the theoretical proposals mentioned above, it may be that it has as many and varied applications as a photon laser. In that case, the users of atom lasers will define what they mean by that term. But at present the applications for an atom laser are still speculative. For this reason it seems to me that if experimentalists wish to build a true atom laser, there must be a fixed goal at which they can aim. That is to say, what is required is a definition of exactly what would constitute an atom laser. In order to construct such a definition, it is first necessary to answer the question of what constitutes a photon laser. If the term ‘laser’ is to have any sensible meaning, its defining features should not depend on the ‘substrate’ of the laser (atoms or photons). In particular, the acronym referring to the amplification of radiation is of no use.

In this paper I present a definition of a laser which applies equally as well to photon lasers as to atom lasers. In its most concise form, the definition is that the output of

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a laser should be well approximated by a classical wave of fixed amplitude and phase. This definition is clearly based on fundamental considerations rather than practical applications. A consequence of this is that some devices which are usually called lasers (such as pulsed lasers) would not be so termed under my definition. Of course I do not think that the community at large should change its vocabulary. However, if the creation of an atom laser is to represent a really significant advance beyond the creation of a Bose condensate of atoms, then I think that the workers attempting to do this must accept some definition of a laser similar to that presented here.

In the atom laser models which have been published so far most authors [14,16,17] have not attempted to define what they mean by a laser. Rather, they have followed the course of the text-books mentioned above, in presenting the model for their atom laser, and analysing its dynamics. However, they generally fail to go on to analyse the properties of the output field. This seems to me to miss the most important point, because ultimately it is the output field of a laser which is actually used. The internal state of a laser is not particularly interesting in isolation: it comes quickly to a steady-state which is a roughly Poissonian mixture of number states (be it number of photons or atoms). Such a state has already been created in the atomic Bose condensates, so there is no need to mimic the complex dynamics of a laser in order to reproduce this state. Moreover, some of these analyses [14,15,19] begin with a rate equation approach. While this is sensible for obtaining preliminary results, it precludes any possibility of calculating the first-order coherence of the laser output.

In Ref. [18], Holland et al. do claim to define a laser in terms of the “essential properties of [its] output field”. However, the properties listed refer not only to the output field but also to the internal state of the laser, as they include the statement that the linewidth should be inversely proportional to the intracavity boson number. It seems to me that the exact meaning of the statement is unclear: what is held fixed when the linewidth changes inversely with the change in the intracavity photon number? If the bare cavity linewidth is held fixed, then it is true that the laser linewidth for an ideal laser will vary inversely with the mean intracavity boson number, as will be shown in Sec. IV. Unfortunately, the atom laser model presented in Ref. [18] has other contributions to its linewidth apart from this fundamental one [17]. Thus their model explicitly fails to satisfy their criterion, as would most optical lasers.

By contrast, the definition to be presented here is unambiguous and is based purely on the properties of the laser output beam; the laser is treated as a black box with one relevant output. The only condition on the inputs is that they cannot be laser beams themselves, at least not of the same substrate (photons or atoms) as that beam which the device itself is supposed to produce. This concentration on the laser output does not imply that there are not analogies between the dynamics of typical photon lasers and those of suggested atom lasers, nor that these analogies are not worth examining (in fact I will examine them in this paper). However, as will become apparent, the nature of the output field of the laser is sufficient to define what we mean by a laser. Some parts of this definition have been published before [14,17], but it has not been presented as a coherent whole.

This paper is organized as follows. The larger part is Secs. II and III, in which I propose a set of four criteria for deciding whether a device can be considered a true laser. Sec. II contains the elementary definitions (which can be understood from the single-particle viewpoint) and Sec. III contains the more interesting definitions which require quantum statistics to formulate. At the same time as giving this definition I will discuss their consequences for atom and photon lasers. In Sec. IV, I present an extremely idealized model of a photon laser to show how the required properties can be satisfied in principle. In Sec. V, I present a more realistic model, which could apply to either a photon laser or an atom laser. This enables an identification of the typical dynamics which enable a laser to produce its characteristic output beam. I discuss how this model applies to photon lasers and the proposed atom lasers. Sec. VI summarizes the most important points.

II. ELEMENTARY DEFINITIONS

A. Directionality

As stated in the introduction, the premise of this paper is that a laser should be defined solely in terms of its output beam. By stating the premise in this way, I have already made an assumption, namely that the output of a laser is a beam. That is to say, the first condition on a laser is that

1. The output is highly directional.

This assumption allows us to separate out a longitudinal direction (the direction of propagation) and two transverse directions (the directions of diffraction). It is not implicit in this condition that the output of the laser must propagate in free space. Even for optical lasers, it is common now for the output to propagate in an optical waveguide such as a fibre. For an atom laser, a waveguide such as proposed in Refs. [21,22] may be essential. This is because, unlike photons, atoms do not naturally travel at the speed of light. In fact, the ‘natural’ speed of atoms at the temperatures of the achieved Bose condensates is a few cm/s. At these speeds, the effect of gravity on the atomic motion is far from negligible over a macroscopic distance. As well as supporting the atoms against gravity, a waveguide could prevent spreading of the atomic
beam due to diffraction. In addition, it would limit the number of transverse modes in the beam. In the ideal case, there would only be one transverse mode. While this is not an essential requirement for a laser, it is a desirable one, so I will list it as

\( (1^+) \) Ideally, the output has a single transverse mode.

A single transverse mode also implies a single polarization state for photons and a single electronic state for atoms.

### B. Monochromaticity

A second property of a laser output which can be stated in single-particle terms is that of monochromaticity:

(2) The longitudinal spatial frequency of the output beam has a small spread in the sense that \( \delta k \ll \bar{k} \).

The uncertainty in the spatial frequency is the reciprocal of the characteristic coherence length \( \ell_{\text{coh}} = 1/\delta k \). Thus the condition (2) can be expressed as

\[
\ell_{\text{coh}} \gg \lambda, \tag{2.1}
\]

where \( \lambda = 2\pi/\bar{k} \) is the mean wavelength. Because atoms travel nonrelativistically, for this condition to be meaningful it is necessary to state the reference frame in which the spatial frequency \( k \) is defined. This is because in a Galileian transformation to a frame moving at fixed speed \( u \) the spatial frequency of a de Broglie wave is transformed to \( k' = k - Mu/\hbar \), where \( M \) is the mass of the atom. The obvious reference frame to choose is that in which the laser itself is at rest.

For photons, there is no difficulty in reformulating condition (2) in terms of the frequency spread or spectral width \( \delta \omega = \epsilon \delta k \) being small compared to the central frequency \( \bar{\omega} = \epsilon \bar{k} \). For atoms we have to be careful about what frequency we are referring to. This is because unlike a photon field (the electromagnetic field), the atom field is unobservable, as will be discussed later. In particular, its global phase has no physical meaning, so its frequency is arbitrary. For instance, we could choose to include the rest mass \( M \) in the energy which would give \( \hbar \omega = Mc^2 + p^2/(2M) \), or we could choose to leave it out. The most physically sensible definition is to use only the kinetic energy, so that

\[
\bar{\omega} = \frac{\hbar \bar{k}^2}{2M}, \quad \frac{\delta \omega}{\bar{\omega}} = 2\frac{\delta k}{\bar{k}}. \tag{2.2}
\]

This allows the monochromaticity condition to be stated as \( \delta \omega \ll \bar{\omega} \) for the atomic case also.

### 1. Dispersion

As stated above, an important difference between photons and atoms is that the latter do not travel at a fixed speed. That is to say, a beam of atoms will disperse even in a vacuum [23]. It is important not to confuse dispersion with diffraction. The former is a longitudinal spreading of an atomic wavepacket, the latter a transverse spreading which is present also for photons. Although dispersion of an atomic beam will occur, it will be negligible over distances much shorter than a characteristic length which I will call the dispersion length. The dispersion length can be defined to be the propagation length over which the spread in an atomic wavepacket due to dispersion becomes comparable to the original spread in the wavepacket due to the uncertainty principle. One would expect that for a laser this dispersion length would be much larger than the wavelength of the output. For photons this is trivially satisfied, so below we need consider only atoms (or massive particles in general).

Let us assume that the momentum distribution in the output is Gaussian with width \( \hbar \delta k \). Then we can imagine that each individual atom has a wavepacket which begins as a real Gaussian wavepacket in \( x \) with \( \delta x = 1/(2\delta k) \). Then for Gaussian wavepackets it is easy to show that the dispersion time (the time for \( x \)-variance of the wavepacket to double) is given by

\[
\tau_{\text{disp}} = \frac{2M(\delta x)^2}{\hbar} = \frac{M}{2\hbar(\delta k)^2} = \frac{\bar{\omega}}{(\delta \omega)^2}. \tag{2.3}
\]

The dispersion length is found by multiplying by the mean velocity

\[
\ell_{\text{disp}} = \tau_{\text{disp}} \frac{\hbar k}{M} = \frac{\bar{k}}{2(\delta k)^2}. \tag{2.4}
\]

Since the wavelength of the output is given by \( 2\pi/\bar{k} \), the dispersion length will be much greater than a wavelength as long as \( \delta k \ll \bar{k} \). This is already guaranteed by the monochromaticity condition (2) above. Also, the dispersion length will be much greater than the coherence length. This can be summarized as follows:

\[
\ell_{\text{disp}} \sim \frac{\bar{k}}{(\delta k)^2} \gg \ell_{\text{coh}} \sim \frac{1}{\delta k} \gg \lambda \sim \frac{1}{\bar{k}}. \tag{2.5}
\]

### 2. Longitudinal acceleration

In the absence of relativistic effects, the frequency of a beam of photons is fixed. The same is not true of a beam of atoms. For example, a wave guide for an atomic beam could be directed so as to gradually lower the atoms a distance \( d \). This causes the atoms to gain kinetic energy. Thus we have \( \bar{\omega}' = \bar{\omega} + Mgd/\hbar \), but \( \delta \omega' = \delta \omega \) so the
emerging beam is more monochromatic than it was at the laser output. If we define $\zeta^2 = \tilde{\omega}/\tilde{\omega}$, then the effect of this longitudinal acceleration on the spatial frequency is

$$\tilde{k}' = \zeta \tilde{k}, \quad \delta k' = \delta k/\zeta \quad (2.6)$$

From this we find that the coherence length has increased both relative to the (decreased) wavelength and absolutely

$$\ell'_{\text{coh}}/\lambda' = \zeta^2 \ell_{\text{coh}}/\lambda, \quad \ell'_{\text{coh}} = \zeta^2 \ell_{\text{coh}} \quad (2.7)$$

and the dispersion length has increased both absolutely and relative to the (increased) coherence length:

$$\ell'_{\text{disp}} = \zeta^3 \ell_{\text{disp}}, \quad \ell'_{\text{disp}}/\ell_{\text{coh}} = \zeta^2 \ell_{\text{disp}}/\ell_{\text{coh}}. \quad (2.8)$$

Despite the plasticity of these features of an atom laser, it is still necessary for the monochromaticity condition (2) to hold at the output of the laser. This is because we have assumed that the dispersion is still negligible after the longitudinal acceleration, which would be possible only if the original beam were approximately monochromatic.

### III. QUANTUM STATISTICAL DEFINITIONS

#### A. Quantum field theory

In this section we introduce definitions which require a many-body description of the output beam. The best description is that using quantum field theory, which I will now briefly outline.

1. **Photon field**

   Photons are the quanta of the (first) quantized electromagnetic field (see for example Ref. [24]). Under conditions (1), (1'), and (2) we need consider only longitudinal modes of the electromagnetic field propagating away from the laser with spatial frequency $k$ in the region of $\tilde{k}$. If we denote the annihilation operator for each longitudinal mode $a(k)$, then the canonical commutation relations are

   $$[a(k), a^\dagger(k')] = \delta(k - k'), \quad (3.1)$$

   and the Hamiltonian for these field modes is

   $$H = \int_0^\infty dk \ c k a^\dagger(k) a(k), \quad (3.2)$$

   where $c$ is the speed of light. This is simply interpreted if $a^\dagger(k) a(k)$ is recognized as the operator for the density of photons in $k$ space. The evolution of these operators in the Heisenberg picture is simply $a_t(k) = e^{-ik\hat{t}} a_0(k)$.

   The field modes labelled by $k$ are completely delocalized. For the definitions which follow it is more convenient to have localized field operators, which we can define as

   $$\tilde{a}(z) = \int_{\tilde{k} - \Delta k}^{\tilde{k} + \Delta k} dk \ a(k) e^{ikz}, \quad (3.3)$$

   where $\delta k \ll \Delta k \ll \tilde{k}$. These operators obey the approximate commutation relations

   $$[\tilde{a}(z), \tilde{a}^\dagger(z')] \approx \delta(z - z'). \quad (3.4)$$

   The actual width of this $\delta$ function is of order $(2\Delta k)^{-1}$ because $2\Delta k$ is the range of the integration. This spatial resolution is at least of order a wavelength $2\pi/\tilde{k}$, because $\Delta k \ll \tilde{k}$. This commutation relation indicates that the operator $a^\dagger(z) a(z)$ can be thought of as the density of photons in real (one-dimensional) space, provided one does not try to probe that space on a scale of order a wavelength or finer.

   The evolution of these operators in the Heisenberg picture is simply

   $$\tilde{a}_t(z) = \tilde{a}_0(z - ct). \quad (3.5)$$

   Thus we could just as well parameterize this local field operator in time rather than space by defining

   $$b(t) = \sqrt{c} a_0(z_1) = \sqrt{c} \tilde{a}_0(z_1 - ct), \quad (3.6)$$

   where $z_1$ is some convenient point in the output. The prefactor $\sqrt{c}$ is so that these operators obey the commutation relation

   $$[b(t), b^\dagger(t')] = \delta(t - t'), \quad (3.7)$$

   where again the width of this $\delta$ function must be at least of order an optical period. With this proviso in mind, the operator

   $$I(t) = b^\dagger(t) b(t) \quad (3.8)$$

   represents the density of photons in time, or the photon flux at the position $z_1$.

2. **Atom field**

   Atoms are composed of elementary fermions: quarks and electrons. An atom which comprises an even number of elementary fermions (which is to say an atom containing an even number of neutrons) will, under dilute conditions, behave like a particle which is a boson. That is to say, a dilute gas of such atoms has the same statistics as a gas of photons. However, there are important differences between these two gases. Photons are gauge bosons which carry the electromagnetic force, and arise
from quantization of the electromagnetic field. There is no analogous force for which atoms are the carrier, and hence there is no fundamental atom field which when quantized yields atoms. Rather, atoms are particles, and the fundamental description for a system with many indistinguishable atoms is the appropriately symmetrized many-particle wavefunction. However, this formalism is very unwieldy and inefficient. For this reason we introduce a quantum field description of the many atoms, by a process commonly called second quantization [24]. Thus the atom field is secondary to the atoms, whereas the electromagnetic field is primary to the photons.

The end result of the second-quantization procedure is that the single particle Schrödinger wavefunction is transformed into a set of operators \( \psi(z) \) parameterized by position, such that \( \psi^\dagger(z)\psi(z) \) is the operator for the (linear) density of particles at position \( z \). In this case, the commutation relations for Bose particles
\[
[\psi(z), \psi^\dagger(z')] = \delta(z - z') \quad (3.9)
\]
is exact. However for atoms it must be remembered that they can only be treated as bosons if the interatomic separation is large compared to an atomic radius, but this has already been assumed [23]. The field \( \psi(z) \), although analogous to the electromagnetic field for photons, is different in that it does not carry any force, as explained above. This is a consequence of the conservation laws for elementary fermions, which imply that no Hamiltonian of the form \( \delta \) can be linear in a matter field such as \( \psi \). Thus the field \( \psi \) is strictly unobservable, although a bilinear combination such as the density \( \psi^\dagger \psi \) is certainly observable.

Now consider an atom laser, and in particular a region of its output around some point \( z_1 \) where there is no longer any longitudinal acceleration, and where dispersion is still negligible. The field operator \( \psi(z) \) may be decomposed using the single-particle energy eigenfunctions \( u_k(z) \) as
\[
\psi(z) = \int_{-\infty}^{\infty} dk \, a(k)u_k(z), \quad (3.10)
\]
where \( k \) is the spatial frequency at the point \( z_1 \). The modes \( u_k(z) \) will not necessarily be of the form \( e^{ikz} \) because we have allowed for the possibility of longitudinal acceleration prior to the point \( z_1 \). Nevertheless, a decomposition is possible in principle. It is therefore also possible to recombine these delocalized operators into a new \( \tilde{a}(z) \) which contains only those \( a(k) \) in the region of \( k \) (the mean spatial frequency of the output at the point \( z_1 \)), as was done for the photon case (3.3).

Because of the possibility of longitudinal acceleration, we cannot write down as simple an expression as (3.3). Nevertheless the assumed negligibility of dispersion up to \( z_1 \) means that we can write
\[
\tilde{a}_t(z_1) = \tilde{a}_{t_0}(z_1 - \xi(t - t_1)), \quad (3.11)
\]
for some function \( \xi(t) \) which satisfies \( \xi(0) = 0 \) and \( \xi(t) > 0 \). This again means that we can parameterize the output using the time variable by
\[
b(t) = \sqrt{\delta a_t(z_1)}. \quad (3.12)
\]
Here \( c = \hbar k/M = \xi(0) \) is the mean speed of the atoms at \( z_1 \). We can then calculate the commutation relations
\[
[b(t), b^\dagger(t')] = c\tilde{a}_t(z_1), \quad c\tilde{a}_t(z_1) = c\delta(t - t')/\xi(0) = \delta(t - t') \quad (3.13)
\]
to be the same as for the photon case (3.7), providing the width of this \( \delta \) function is again taken to be large compared to \( \hbar \) divided by the mean kinetic energy. Thus \( I(t) = b^\dagger(t)b(t) \) can be interpreted as the approximate atom-flux operator at \( z_1 \).

**B. The fundamental principle**

Having established that the output field of a laser can be represented by a field operator \( b(t) \) satisfying approximate \( \delta \)-function commutation relations, I can now state the fundamental principle behind the criteria for deciding whether a device can be considered a laser. That principle is that

(\( \pi \)) The output is well-approximated by a classical wave of fixed intensity and phase.

In slightly more mathematical terms, we require
\[
b(t) \approx \beta(t) \equiv \beta e^{-i\omega t}, \quad (3.14)
\]
where \( \beta \) is a c-number and the meaning of the \( \approx \) sign will become clear in the remainder of Sec. III. The fact that \( \beta(t) \) has only trivial time-dependence indicates that I am here assuming that the output of the laser must be a stationary process. This excludes pulsed lasers from the class of true lasers. This is necessary if there is to be a distinction between projecting an atomic Bose-condensed gas across the laboratory and constructing a true atom laser.

The zeroth order approximation \( b(t) = \beta(t) \) is used in quantum optics whenever the driving of a system (such as an atom or a cavity) by a laser is treated as classical driving. Assumption (\( \pi \)) is also necessary for the first order (linearized) approximation of the intensity fluctuations of a laser by
\[
\beta^*(t)\delta b(t) + \beta(t)\delta b^\dagger(t), \quad (3.15)
\]
where \( \delta b = b - \beta \). Thus the linearization of the fluctuation in an atomic travelling-wave field is only permissible if that field is the output of an atom laser.
1. Mean amplitude?

The first obvious requirement for \( b(t) \) to be approximated by \( \beta(t) \) would seem to be

\[
\langle b(t) \rangle = \beta(t). \tag{3.16}
\]

Unfortunately, this cannot be true for an atom laser. As explained above, atoms are particles of matter, rather than gauge bosons like photons. Thus, the atom field is not an observable like the photon field. Only bilinear combinations of the atom field are observable. All single-atom quantities, such as the atomic dipole, or the atomic momentum, correspond to field observables which are bilinear in the atom field. Similarly, no Hamiltonian is linear in the atom field, because this would imply the creation of atoms out of nothing. Of course, it is possible to create atoms in a particular electronic state from atoms in a different electronic state. At higher energies, it is possible to create atoms by splitting up molecules. At still higher energies, it is possible to create atoms out of pure energy by simultaneously creating an anti-atom. But all of these processes involve other matter fields, and the fundamental Hamiltonian will always be bilinear in the total matter field which includes all elementary particles in the universe. Thus no interaction can produce a quantum state in which the mean value of a matter field is different from its initial value of zero. We can therefore conclude that

\[
\langle b(t) \rangle = 0 \tag{3.17}
\]

for an atom laser.

In contrast to an atom field, the electric field is an observable, and it is linear in \( b(t) \), so in principle an optical laser can have a mean field. However, in an optical laser starting from the vacuum, the phase of the field produced cannot be predicted unless it is seeded with another laser. Thus in a strict sense the expected value of the mean field of an optical laser is zero also, because the average of \( e^{i\phi} \) over all phases \( \phi \) is zero. One could argue that the actual value of \( \phi \) is found by making a phase-sensitive measurement of the output of the laser. The problem is that at optical frequencies, such measurements are almost always made relative to the phase of another laser beam, so they do nothing to establish the absolute phase of the electromagnetic field. Nevertheless it is possible in principle to determine this phase absolutely, for example by shooting an electron through a specific point in an electromagnetic standing wave at a particular time, and seeing in which direction it is deflected due to the Lorentz force. This would require timing to better than \( 10^{-16} \) seconds, which is why in practice only relative phase is measured at optical frequencies. For a more technical discussion of the differences and similarities between photon and matter fields, see the Appendix.

In summary, the zero mean field result \( \langle b(t) \rangle = 0 \) is true for both photon and atom lasers. For a photon laser we can believe that the output really is in a state with a well-defined amplitude and phase, but we don’t know what that phase is. In this view, the result \( \langle b(t) \rangle = 0 \) is to be understood as a classical average over all possible phases. For an atom laser this is the wrong physical picture because a state with a mean field amplitude is physically impossible and the absolute phase of an atom field is absolutely unobservable. Nevertheless, this picture does not lead to contradictions, and may even be useful. Indeed, the imagining of the existence of a well-defined phase for a matter field is the essential element in the spontaneous symmetry breaking hypothesis in superfluidity and superconductivity which has served the condensed matter physics community well for many decades. However, it is not necessary to use this non-physical hypothesis in order to define a laser, and therefore I will not use it.

C. Well-defined intensity

Even if the mean amplitude of a laser is zero, its mean intensity certainly is not. Thus the first requirement for approximating \( b(t) \) by \( \beta(t) \) is that

\[
\langle I(t) \rangle = \langle b(t)^\dagger b(t) \rangle = |\beta|^2. \tag{3.18}
\]

However, in order to justify the \( \approx \) sign in Eq. \( \langle b(t)^\dagger b(t) \rangle \rightarrow +\infty \), we require more than an equation of mean values. We also require that the fluctuations in the intensity be small in some sense. One’s first thought might be to require that the standard deviation in \( I(t) \) be small compared to its mean. Unfortunately, \( \langle I(t)^2 \rangle \rightarrow +\infty \), because of the \( \delta \)-function commutation relations. For this reason it is necessary instead to consider the two-time correlation function for \( I(t) \), and split off the singular part:

\[
\langle I(t+\tau)I(t) \rangle = \langle b(t+\tau)^\dagger b(t+\tau) b(t)^\dagger b(t) \rangle = |\beta|^2 + \langle b(t+\tau)^\dagger b(t+\tau) b(t)^\dagger b(t) \rangle
\]

where the \( \langle \cdot \rangle \) denotes normal ordering as usual. The final term in Eq. \( \langle b(t)^\dagger b(t) \rangle \rightarrow +\infty \) is proportional to \( \delta(\tau) \), the shot noise. It is present because the field is really composed of discrete quanta (photons) or particles (atoms). Although this is an irreducible quantum fluctuation in the field intensity, I will ignore it for now and concentrate on the normally-ordered fluctuations. As will be seen below, the shot-noise is (perhaps surprisingly) more conveniently analysed when considering phase fluctuations. The normally-ordered fluctuations are analogous to fluctuations in a classical field but their probability distribution function, the Glauber-Sudarshan P-function, may be non-positive. Consideration
of these fluctuations leads to the third requirement on a laser that

\[(3)\] The output intensity fluctuations are small in the sense that \(\forall \tau \ |\{\langle I(t + \tau), I(t) \rangle\}| \ll \langle I \rangle^2.\]

Here I am using the notation

\[\langle : I(t + \tau), I(t) : \rangle = \langle : I(t + \tau)I(t) : \rangle - \langle I \rangle^2. \quad (3.20)\]

1. Second-order coherence

The above condition can be recast in terms of Glauber’s normalized second-order coherence function \[g^{(2)}(\tau) = \langle : I(t + \tau)I(t) : \rangle / \langle I \rangle^2 \quad (3.21)\] as

\[|g^{(2)}(\tau) - 1| \ll 1. \quad (3.22)\]

For light which is second-order coherent, \(g^{(2)}(\tau) = 1\). Thus, the condition (3) can be restated as: the output of a laser is approximately second-order coherent. This can be contrasted with the light from a thermal source, for which \(g^{(2)}(0) = 2\).

The interpretation of \(g^{(2)}(\tau)\) is that it gives the relative change in the likelihood that a boson (photon or atom) will be observed a time \(\tau\) later than a boson-detection. Explicitly, the probability for observing a boson in the interval \((t + \tau, t + \tau + dt)\) given that one was observed at time \(t\) is \(g^{(2)}(\tau)\langle I \rangle dt\). In an approximately second-order coherent beam, the boson arrival times are approximately independent. In an exactly second-order coherent beam the arrival times would be Poissonian. Thermal light, with \(g^{(2)}(0) = 2\) is super-Poissonian. This bunching has recently been confirmed for a thermal source of Bose atoms as well.

It might be thought that the definition adopted here would also exclude sub-Poissonian light with \(g^{(2)}(0) - 1 < 0\), as produced by a regularly-pumped laser for example. However, this is not the case because the arrival of a photon in this case has only a very small negative influence on the arrival likelihood multiplier \(g^{(2)}(\tau)\). Specifically, if \(n \gg 1\) is the mean number of photons in the laser cavity then \(g^{(2)}(\tau) - 1 > -1/(2n)\).

2. The noise spectrum

Another common way that intensity noise is quantified is by the intensity noise spectrum

\[S(\omega) = \langle I \rangle^{-1} \int d\tau e^{i\omega \tau} \langle I(t + \tau), I(t) \rangle. \quad (3.23)\]

The normalization here is chosen so that the shot noise contribution is equal to one:

\[S(\omega) = 1 + \langle I \rangle \int d\tau e^{i\omega \tau} [g^{(2)}(\tau) - 1]. \quad (3.24)\]

For an exactly second-order coherent beam the shot noise is the only contribution. As will be seen, the intensity noise spectrum shows an enormous difference between a laser and a thermal source with the same intensity and spectral width.

D. Well-defined phase

If the laser output has small intensity fluctuations then the only significant variation in its complex amplitude is that due to phase fluctuations. This implies that the first order coherence function

\[G^{(1)}(\tau) = \langle b^\dagger(t + \tau)b(t) \rangle, \quad (3.25)\]

or its normalized form

\[g^{(1)}(\tau) = \langle b^\dagger(t + \tau)b(t) \rangle / \langle b^\dagger b \rangle, \quad (3.26)\]

is a useful measure of phase fluctuations. It also has the advantage that it is sensitive to phase fluctuations without requiring the existence of a mean field having a well-defined phase. Unlike the field itself \(b(t)\), the bilinear combination in Eq. (3.25) is measurable even for an atom field. For \(\tau = 0\), it is simply the mean intensity \(\langle I \rangle\), but as \(\tau\) increases \(\langle G^{(1)}(\tau) \rangle\) decreases as the phase of the field gradually becomes decorrelated from its value at time \(t\). As \(\tau \to \infty\), \(\langle G^{(1)}(\tau) \rangle \to 0\). By contrast, an approximately first-order coherent source would satisfy \(\langle g^{(1)}(\tau) \rangle \approx 1 \forall \tau\). This clearly cannot be imposed as a condition on any finite laser.

If we cannot expect \(\langle G^{(1)}(\tau) \rangle\) to remain constant for all time, we can enquire how long it takes to decay. The characteristic time for this decay is known as the coherence time, and can be simply defined as

\[\tau_{\text{coh}} = \int_0^\infty |g^{(1)}(\tau)| d\tau. \quad (3.27)\]

The coherence time is the time scale over which the phase of the field remains roughly constant. If we wish to say that the phase of the field is well-defined, then we should be able to measure its value and verify that it does remain constant over this time scale. Of course we cannot measure the absolute phase of the atom field, and we very rarely measure the absolute phase of an optical-frequency electromagnetic field. However we can measure the phase relative to another source (a laser), which we know to have a very long coherence time, by doing an interference experiment. Alternatively, we can measure the two-time correlation function \(g^{(1)}(\tau)\) by putting a time-delay in one arm of an interferometer.
If the field were classical then there would be no difficulty with making an arbitrarily precise phase measurement in an arbitrarily short time. But we are dealing with quantum fields, and the accuracy of a phase measurement (via interferometry or otherwise) is limited by the graininess of the field. This is the problem of shot noise which was mentioned above. In order to obtain a precise result from a phase measurement, we need to measure a portion of the output field containing a large mean number $\bar{n}$ of bosons. That is because for a phase measurement on any state with a mean number $\bar{n}$ of bosons, the minimum uncertainty scales as \[ \delta \phi \gtrsim \bar{n}^{-1}, \] while for states with a Poisson distribution of mean $\bar{n} \gg 1$ (which is more relevant to a laser) \[ \delta \phi \gtrsim \bar{n}^{-1/2}. \] In either case we clearly require $\bar{n} \gg 1$ to obtain a good measurement of phase. Now if the phase measurement lasts a time $T$, then the mean number of bosons available is $\bar{n} = \langle I \rangle T$. But we also require $T$ to be much less than the coherence time $\tau_{coh}$, otherwise the phase will diffuse over the course of the measurement. Thus we have the following condition on the laser output: \[ \langle I \rangle \tau_{coh} \gg 1. \] (3.30) This can be stated very concisely using $G^{(1)}(\tau)$ as \[ (4) \] The output phase fluctuations are small in the sense that \( \int dt |G^{(1)}(\tau)| \gg 1. \) To my knowledge a condition such as this has never been proposed before, so in that sense this final condition is the most important of the four presented here. This condition is easily satisfied by all optical lasers, as will be explained below. It is only with lasers which have been deliberately filtered that the shot noise becomes a serious limitation to whether an interference pattern can be observed \[ (10) \] This filtering has been done to prove that spatial interference exists even when there is at most one photon in the apparatus at any given time \[ (3.32) \] or temporal interference when there is at most one photon arrival per beat period \[ (3.33) \].

1. Power Spectrum

The coherence time introduced here is basically the reciprocal of the spectral width $\delta \omega$. The latter is usually defined as the full-width at half-maximum height of the power spectrum \[ P(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \, e^{i\omega \tau} G^{(1)}(\tau), \] which is defined so that $\int_{-\infty}^{\infty} d\omega P(\omega) = \langle I \rangle$. For many sources the power spectrum is Lorentzian:

\[ P(\omega) = (2\pi)^{-1} \frac{\langle I \rangle \delta \omega}{(\omega - \bar{\omega})^2 + (\delta \omega/2)^2}. \] (3.32)

This gives an alternative statement of the fourth condition, namely that the resonant spectral intensity $P(\bar{\omega})$ be much greater than unity.

From these considerations we see that the spectral width of a laser must be small in two senses. From condition (2) we require $\delta \omega \ll \omega$ and from condition (4) we require $\delta \omega \ll \langle I \rangle$. For single-mode optical lasers with an output power of greater than about one milliwatt, $\langle I \rangle > \bar{\omega}$, so condition (4) is actually weaker than condition (2), and is always satisfied. Atom lasers are a different matter; as will be discussed in Sec. V they may have difficulty satisfying condition (4). Furthermore, the quantity on the left side of Eq. (3.30) cannot be changed by longitudinal acceleration of the laser beam. Although such acceleration can increase the monochromaticity of the beam, it does so by increasing $\bar{\omega}$ but leaving $\delta \omega$ unchanged, as we saw in Sec. III B 2. Since it cannot change the flux $\langle I \rangle$ it cannot change the ratio $\delta \omega : \langle I \rangle$.

2. Bose statistics

Although I have derived condition (4) from considerations of phase measurements, it could also have been derived from considerations of another fundamental property of a laser output: its constituents must be bosons. None of the first three conditions require this but the fourth does as can be seen as follows. Consider a section of the laser output of length $Z$ much longer than the coherence length $l_{coh} \sim \epsilon_{coh}$, where $c$ is the mean speed of the bosons. Being of finite length, this section can be described by discrete longitudinal modes in $K$ space with separation $2\pi/Z$. If the uncertainty in the spatial frequency is of order $\delta k \sim \delta \omega/c$, then the number of these modes which are significantly populated is of order $Z\delta k \sim Z\delta\omega/c$. Now the temporal duration of this section is $T = Z/c$, so the mean number of bosons present in the section is $\langle I \rangle Z/c$. Thus the mean number of bosons per mode is of order $\langle I \rangle / \delta \omega$, which is independent of $Z$ and must be much greater than unity according to condition (4). That is, condition (4) is equivalent to the condition that the output field be highly Bose degenerate.

This requirement can be understood from the fundamental principle ($\pi$). For a quantum field to approach a classical field it must be highly excited, with many quanta per mode. This is necessary for the quantum fluctuations (caused by the discreteness of the quanta) to be comparatively small. A fermion field (such as describes all of the known fundamental constituents of matter) cannot have a classical limit because there can be at most one
particle per mode (the Pauli exclusion principle). Thus while condition (3) serves to limit the size of the ‘classical’ fluctuations in the field, condition (4) does likewise for the quantum fluctuations.

Yet another way to interpret condition (4) is as follows. The time \( (I)^{-1} \) is the mean time between one boson and the next in the output of the laser. Writing condition (4) as \( (I)^{-1} \ll \tau_{\text{coh}} \), this implies that the output field is still coherent from one boson to the next. If instead \( (I)^{-1} \gtrsim \tau_{\text{coh}} \), then the “phase” of one boson emitted by the laser would be essentially independent of the “phase” of the subsequent ones (I use quote marks here because one boson individually can have no phase). Stated in this way, condition (4) seems to come closest to the intuitive conviction that that an atom laser output should be “coherent”. Without condition (4), quantum statistics would play no significant role, and there would be no reason to treat the beam as a quantum field at all.

**IV. ADDING BOSONS, ONE BY ONE**

In this section I will present an idealized model of a device which satisfies the conditions (1)–(4) and hence can be classified as a laser. Consider an optical model consisting of a cavity supporting a single mode which is damped from one end mirror. This automatically produces an output beam with the desired directional properties. The laser gain mechanism required to maintain a stationary operation is a scheme which simply adds photons to the laser mode, one at a time, at Poisson-distributed times. Part of the motivation for presenting this model is to dispel any belief that the laser gain mechanism has to be “coherent” in some special way.

**A. The laser model**

The laser gain is achieved by the passage of two-level atoms through a cavity. The atoms are resonant with the field, and are initially excited. Each atom is put through the cavity repeatedly until it is known to have given up its quantum of energy to the field. The interaction Hamiltonian is

\[
H = i\Omega (\sigma a^\dagger - \sigma^\dagger a),
\]

where \( a \) is the annihilation operator for the cavity mode, \( \sigma = |g\rangle\langle e| \) is the lowering operator for the atom, and \( \Omega \) is the one-photon Rabi frequency. Let the interaction time \( \tau \) be such that \( \epsilon \sqrt{\bar{n}} \ll 1 \), where \( \epsilon = \Omega \tau \) and \( \bar{n} \) is the mean intracavity photon number. Then the unitary operator \( \exp(-iH\tau) \) acting on the initially factorized state \( \rho \otimes |e\rangle\langle e| \) can be expanded to second order in \( \epsilon \) to give the entangled state

\[
R = \rho \otimes |e\rangle\langle e| + \epsilon (a^\dagger \rho \otimes |g\rangle\langle e| + \text{H.c.})
+ \epsilon^2 (a^\dagger \rho a \otimes |g\rangle\langle e| - \frac{1}{2} \{aa^\dagger, \rho \} \otimes |e\rangle\langle e|).
\]  

(4.2)

If the outgoing atom is detected in the excited state, then the conditioned state of the field (the norm of which represents the probability of this detection result) is, to first order in \( \epsilon^2 \),

\[
\hat{\rho}_e = \langle e| R |e\rangle = (1 - \epsilon^2 A[a^\dagger]) \rho
+ \exp(-\epsilon^2 aa^\dagger/2)\rho\exp(-\epsilon^2 aa^\dagger/2),
\]

(4.3)

where the superoperator \( A[c] \) is defined for an arbitrary operator \( c \) by

\[
A[c] \rho = \frac{1}{2} (c^\dagger c, \rho).
\]

(4.4)

If the atom is detected in the ground state (which happens rarely), the state is

\[
\hat{\rho}_g = \langle g| R |g\rangle = \epsilon^2 J[a^\dagger] \rho,
\]

(4.5)

where the superoperator \( J[c] \) is defined by

\[
J[c] \rho = c^\dagger c \rho.
\]

(4.6)

If the atom is detected in the ground state, then the field has gained a photon and the process can stop. If it is detected in the excited state, one must try again with the same atom (or, more realistically, another excited atom). Say \( K \) atoms are required before the \((K + 1)\)th is detected in the ground state. The unnormalized state after the \((K + 1)\)th atom is

\[
\hat{\rho}_K = \epsilon^2 J[a^\dagger] \exp[-K \epsilon^2 aa^\dagger/2] \rho \exp[-K \epsilon^2 aa^\dagger/2].
\]

(4.7)

The norm of this density operator is equal to the probability that this many atoms are needed. Thus, the average density operator, given that an atom is finally detected in the ground state, is

\[
\rho' = \sum_{K=0}^{\infty} \hat{\rho}_K.
\]

(4.8)

Using the fact that \( \epsilon^2 \) is small, the sum can be converted to an integral by setting \( u = \epsilon^2 K \):

\[
\rho' = J[a^\dagger] \int_{0}^{\infty} \exp(-uaa^\dagger/2) \rho \exp(-uaa^\dagger/2) du.
\]

(4.9)

This can be formally evaluated as

\[
\rho' = J[a^\dagger] A[a^\dagger]^{-1} \rho.
\]

(4.10)

The superoperator \( A[a^\dagger]^{-1} \) is well-defined because \( aa^\dagger \) is a strictly positive operator [41].

The action of the superoperator \( J[a^\dagger] A[a^\dagger]^{-1} \) is to add a photon to the system irrespective of its initial state. That is to say, it shifts the photon number distribution
upwards by one. If this addition of a photon is assumed to occur at Poisson-distributed times, then we can obtain a Markovian master equation for the field. If we also include the linear damping at rate \( \kappa \), and let the gain (the rate of photon addition) be \( \kappa \mu \), then we get

\[
\kappa^{-1} \dot{\rho} = \mu (\mathcal{J}[a^\dagger]A[a^\dagger]^{-1} - 1) \rho + \mathcal{D}[a] \rho = \mu \mathcal{D}[a^\dagger]A[a^\dagger]^{-1} \rho + \mathcal{D}[a] \rho. \tag{4.11}
\]

Here I am using another superoperator \( \mathcal{D}[c] \) defined for an arbitrary operator \( c \) by

\[
\mathcal{D}[c] \equiv \mathcal{J}[c] - \mathcal{A}[c]. \tag{4.12}
\]

It can be verified that the ideal laser master equation (4.11), first derived in Ref. [42], is of the required Lindblad form \( \mathcal{L} \).

### B. The laser statistics

This master equation is conveniently solved using the Glauber-Sudarshan \( P(\alpha, \alpha^*) \) representation \( | \alpha \rangle \langle \alpha | \). Defining number and phase variables in terms of the complex amplitudes

\[
n = |\alpha|^2; \quad \varphi = \text{Im} \log \alpha, \tag{4.13}
\]

and using the operator correspondences for the \( P \) function, one can demonstrate the following superoperator correspondences:

\[
\mathcal{A}[a^\dagger] \rho \rightarrow n (1 - \partial_n) P(n, \varphi), \\
\mathcal{D}[a^\dagger] \rho \rightarrow -\partial_n n (1 - \partial_n) + \frac{1}{4n} \partial^2 \varphi P(n, \varphi), \\
\mathcal{D}[a] \rho \rightarrow \partial_n n P(n, \varphi). \tag{4.14}
\]

Putting these into Eq. (4.11), and dropping terms of order \( 1/n^2 \) and smaller gives the Fokker-Planck equation

\[
\kappa^{-1} \dot{P} = \left( -\partial_n [\mu - n] + \frac{\mu}{4n^2} \partial^2 \varphi \right) P. \tag{4.15}
\]

The first notable feature of this equation is that there is no diffusion in the intensity \( n \). Thus \( n \) relaxes exponentially at rate \( \kappa \) to its stationary value \( \mu \). From the definition of the \( P \) function, this means that the stationary state of this laser is a mixture of coherent states of mean photon number \( \mu \). Thus the intracavity photon number statistics are Poissonian with mean \( \mu \). Furthermore, the second-order coherence function is defined in terms of the \( P \) function variables by

\[
g^{(2)}(\tau) = \langle n(t + \tau) n(t) \rangle / \langle n \rangle^2; \tag{4.16}
\]

so for this ideal model we have the stationary result

\[
g^{(2)}(\tau) = 1 \ \forall \tau. \quad \text{That is to say, the output is exactly second-order coherent, satisfying condition (3).}
\]

The mean intensity of the output is given by \( \langle I \rangle = \kappa \langle n \rangle = \kappa \mu \), which is equal to the gain (as it must be at steady-state).

The stationary value \( n = \mu \) also allows us to replace \( n \) in the diffusion term of (4.15) by \( \mu \). Then the dynamics of the laser is completely determined by the phase diffusion

\[
\dot{P}(\varphi) = \frac{\Gamma}{2} \partial^2 \varphi P(\varphi), \tag{4.17}
\]

where \( \Gamma \) is the fundamental rate of phase diffusion of the laser, given by

\[
\Gamma = \frac{\kappa}{2\mu}. \tag{4.18}
\]

This expression is well-known and an heuristic derivation is given by Loudon [2].

Equation (4.17) implies that \( \varphi(t) \) is a Wiener process, so that \( \langle |\varphi(t + \tau) - \varphi(t)|^2 \rangle = \Gamma \tau \) [6]. We can use this fact to prove that the coherence time \( \tau_{\text{coh}} \) is equal to \( 2\Gamma^{-1} \), since in steady state

\[
|g^{(1)}(\tau)| = \langle \sqrt{n(t + \tau)} e^{-i\varphi(t)} \sqrt{n(t)} e^{i\varphi(t)} \rangle / \langle n \rangle = \langle e^{-i|\varphi(t + \tau) - \varphi(t)|} \rangle = \exp(-\Gamma \tau / 2), \tag{4.19}
\]

Thus the power spectrum of the laser is Lorentzian as in Eq. (3.32), with a width \( \delta \omega = \Gamma \). Since we must have \( \kappa \ll \omega \) in order to use the Markovian description of damping, it is clear that the condition (2) is satisfied with \( \delta \omega \ll \omega \). Also we have

\[
\tau_{\text{coh}} \langle I \rangle \sim \langle I \rangle / \Gamma = 2\mu^2, \tag{4.20}
\]

so providing that the mean intracavity photon number \( \mu \) is much greater than one [as was assumed above in obtaining the Fokker-Planck equation (4.17)], condition (4) is easily satisfied.

### C. Comparison with linear amplifier

The notable feature of the laser gain mechanism presented above is that it adds photons one by one, independent of the state. This is to be contrasted with a linear amplifier, in which the rate of addition of photons is proportional to the number of photons plus one, as usual for stimulated emission. The master equation for a linear amplifier with the same damping term and the same mean photon number as the above laser is

\[
\kappa^{-1} \dot{\rho} = \mu \mathcal{D}[a^\dagger]((\mu + 1)^{-1} \rho + \mathcal{D}[a] \rho . \tag{4.21}
\]

This is best converted into a Fokker-Planck equation for the \( P \) function in terms of \( \alpha \) and \( \alpha^* \)

\[
\dot{P} = \frac{\kappa}{\mu + 1} \left[ \frac{1}{2} \partial_\alpha \alpha + \frac{1}{2} \partial_{\alpha^*} \alpha^* + \mu \partial_{\alpha} \partial_{\alpha^*} \right] P. \tag{4.22}
\]
From this it is easy to show that the stationary first order correlation function is
\[
g^{(1)}(\tau) = \frac{\langle \alpha^* (t + \tau) \alpha(t) \rangle}{\langle \alpha \rangle^2} = \exp \left( -\frac{\kappa \tau}{\mu + 1} \right). \tag{4.23}
\]
For \(\mu \gg 1\), the spectral width is \(\delta \omega = \kappa/(\mu + 1) \approx \kappa/\mu\), which is only twice that of a laser \[4.1\] with the same intensity \(\langle I \rangle = \kappa \mu\). Clearly conditions (2) and (4) are still satisfied for \(\mu \gg 1\).

The factor of two difference in the linewidth is because this source has amplitude fluctuations as well as phase fluctuations contributing to the decorrelation of the field. It is these amplitude fluctuations which exclude the linear amplifier from the class of lasers. We find in this case
\[
g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2 = 1 + \exp[-\kappa \tau/(\mu + 1)], \tag{4.24}
\]
which clearly clearly violates condition (3). The difference in the intensity fluctuations is even more striking in the low-frequency intensity noise spectrum \[3.24\]. For the laser model above we have
\[
S_{\text{laser}}(0) = 1, \tag{4.25}
\]
representing shot noise only. For the linear amplifier,
\[
S_{\text{lin,amp.}}(0) = 1 + 2\mu(\mu + 1), \tag{4.26}
\]
which is enormously far above the shot-noise level for \(\mu \gg 1\).

The result obtained here for a linear amplifier is identical to what would be obtained by passively filtering a beam emitted by a source of bosons in thermal equilibrium. The source would have to be extremely Bose degenerate, with \(\mu\) bosons per mode, and the filter would have to block all except a narrow range of energies \(\hbar \delta \omega = \hbar \kappa/(\mu + 1)\). As well as failing to satisfy condition (3), this method of producing an intense monochromatic beam is very inefficient, in that it filters out almost all of the input. As pointed out by Holland et al. \[11\], this is in contrast to a laser in which Bose statistics enhance the transfer of bosons into the laser mode.

D. Is stimulated emission necessary?

Since the “stimulated emission of radiation” is part of the acronym for laser, it might be thought that stimulated emission is essential to produce a laser. While this is true for a typical laser (as will be examined in Sec. V), the fact that the model of Sec. IV A adds photons one by one suggests that it is not true in general. I will now show that stimulated emission is indeed not necessary for laser action.

Stimulated emission is a simple consequence of the linear coupling of the laser field to its source, as in Eq. \[4.1\]. That is to say, the Hamiltonian \[4.1\] is linear in the annihilation operator \(a\) which, for classical fields, can be replaced by the \(c\)-number \(\sqrt{n} e^{i\phi}\). Whenever a rate is calculated in quantum theory it depends on the square of the Hamiltonian. Hence the fundamental gain rate from a linear coupling will vary as \(n\), which is the so-called stimulated emission or Bose-enhancement factor. A fully quantum calculation of course gives spontaneous emission as well, and hence a gain rate proportional to \(n + 1\).

Since stimulated emission can be traced to the presence of \(a\) in the coupling Hamiltonian, the only way to remove it is to substitute for \(a\) a different lowering operator, one which does not increase with \(n\). That is to say, in Eq. \[4.1\], we replace
\[
a = \sum_{n=1}^{\infty} \sqrt{n} |n - 1\rangle \langle n| \tag{4.27}
\]
by the Susskind-Glogower \[44\] \(\hat{e} \equiv e^{i\phi}\) operator
\[
e = a(a^\dagger a)^{-1/2} = \sum_{n=1}^{\infty} |n - 1\rangle \langle n|. \tag{4.28}
\]
The new Hamiltonian would be extremely nonlinear if expressed in terms of \(a\) and \(a^\dagger\), and would be quite impossible to realize physically, but it cannot be denied that it will not exhibit any stimulated emission. The above derivation will follow through, and indeed will be simplified by the fact that \(\hat{e} \hat{e}^\dagger = 1\). Thus in place of Eq. \[4.11\] we will have
\[
\kappa^{-1} \dot{\rho} = \mu D[\hat{e}^\dagger] \rho + D[a] \rho. \tag{4.29}
\]
It can be shown that this master equation produces exactly the same photon number statistics as does Eq. \[4.11\]. Moreover, for \(\mu \gg 1\), the linewidth of the laser described by Eq. \[4.29\] will be half that of the laser described by Eq. \[4.11\]. Thus, all of the conditions for the device to be considered a laser can be satisfied without there being any stimulated emission at all. However, as stressed above, this is a completely artificial model and in real lasers (as will be analysed in the next section), the Bose enhancement factor is essential for their operation above threshold.

V. A GENERIC LASER MODEL

The laser model presented in the preceding section proves that it is possible to construct a device satisfying the criteria \(1\)–\(4\) simply by adding bosons one at a time to a damped mode. However, it has two shortcomings: first, its properties are too idealized; second, its dynamics are not typical of a laser. For these reasons I will present in this section a more realistic model of a laser which will illustrate the typical features of laser dynamics. I suggest that the following list contains the most important of such features:
1. a cavity supporting the laser mode of the boson field and allowing an output beam to form.
2. a source of bosons which is coupled to the laser mode.
3. an irreversibility which favours the transfer of bosons from source to laser mode.
4. a sink which also takes bosons from the source.
5. a pump to replenish the source.

As we will see, a crucial requirement for the device to operate as a laser is that the transfer of bosons from source to laser mode be at least comparable to the transfer from source to sink.

A. The dynamical model

Before discussing the realization of the above laser dynamics in specific physical systems, I wish to present a generic model which can be analyzed in these terms. For simplicity I will model the laser mode and the source as two different modes of the boson field with annihilation operators \(a\) and \(c\) respectively. In addition there is another mode \(b\) (not necessarily of the same field) which will cause the irreversibility in the coupling between \(a\) and \(c\). The dynamics of the total system is then governed by the following master equation

\[
\dot{W} = \sum_{i=1}^{5} L_i W,
\]

where the five Liouville superoperators corresponding to the five features listed above are

\[
\begin{align*}
L_1 W &= \kappa D[a] W \quad (5.2) \\
L_2 W &= -i [g(c^\dagger a + ca^\dagger b^\dagger), W] \quad (5.3) \\
L_3 W &= \lambda D[b] W \quad (5.4) \\
L_4 W &= \gamma (N + 1) D[c] W \quad (5.5) \\
L_5 W &= \gamma N D[c^\dagger] W \quad (5.6)
\end{align*}
\]

The first superoperator \(L_1\) describes damping of the laser mode at rate \(\kappa\), as in the model of the preceding section. The second \(L_2\) is a Hamiltonian (reversible) coupling of the source mode \(c\) to the laser mode \(a\) and another mode \(b\). The third \(L_3\) damps this other mode \(b\), and \(\lambda\) will be assumed very large so that the coupling \(L_2\) is effectively made unidirectional. The fourth and fifth together \(L_4 + L_5\) describe the coupling of the source to a broad-band reservoir with \(N\) bosons per mode. The pump term \(L_3\) is the usual linear amplifier while the sink term \(L_4\) is the usual linear loss.

We wish to simplify this three-mode master equation into a single-mode master equation by eliminating first mode \(b\) and then mode \(c\). Under the assumption that \(\lambda \gg g\) we can follow the adiabatic elimination method of Refs. [17] and [12] to derive a master equation for \(R\), the state matrix for modes \(a\) and \(c\) alone:

\[
\dot{R} = [L_1 + L_c + L_4 + L_5] R, \quad (5.7)
\]

where the new superoperator replacing \(L_2\) and \(L_3\) is

\[
L_c R = \eta D[a^\dagger] R, \quad (5.8)
\]

where \(\eta = g^2 / \lambda\). This is the (now irreversible) coupling between the source and laser mode. In order to adiabatically eliminate mode \(c\) as well we assume that \(N \ll 1\). This is a natural assumption to make as Bose-degenerate beams \((N \gtrsim 1)\) are difficult to come by unless one already has a laser. Under this assumption we can expand the state matrix \(R\) as

\[
R = \rho_0 \otimes |0\rangle \langle 0| + \rho_1 \otimes |1\rangle \langle 1|, \quad (5.9)
\]

where the \(\rho_0\) and \(\rho_1\) are density operators for the laser mode \(a\), and the number states are those for the source mode \(c\). We will see that in steady-state \(\rho_1 \sim N \ll 1\) so that \(\rho \simeq \rho_0\).

Substituting the expansion (5.9) into the master equation (5.7) and using \(N \ll 1\) yields

\[
\dot{\rho}_0 = (L_1 - \gamma N) \rho_0 + (\gamma + \eta A[a^\dagger]) \rho_1 \quad (5.10)
\]
\[
\dot{\rho}_1 = (L_1 - \eta A[a^\dagger] - \gamma) \rho_1 + \gamma N \rho_0 \quad (5.11)
\]

Assuming that \(\gamma \gg \kappa\), we can neglect the \(L_1\) term in the equation for \(\rho_1\) and, since \(\rho_0 \simeq \rho\) varies on the time scale \(\kappa\) (as will be shown), we can assume that \(\rho_0\) is constant on the time scale that \(\rho_1\) relaxes. Thus we can slave \(\rho_1\) to \(\rho_0\) by calculating the stationary state of Eq. (5.11):

\[
\rho_1 \simeq \gamma N (\gamma + \eta A[a^\dagger])^{-1} \rho_0, \quad (5.12)
\]

which is of order \(N\) as promised. Substituting this back into Eq. (5.10) yields

\[
\dot{\rho}_0 = \left[ L_1 + \gamma N (\gamma + \eta J[a^\dagger]) (\gamma + \eta A[a^\dagger])^{-1} - \gamma N \right] \rho_0. \quad (5.13)
\]

Since \(\rho \simeq \rho_0\) we can rewrite this as the following master equation for the laser mode alone:

\[
\kappa^{-1} \dot{\rho} = D[a] \rho + \nu D[a^\dagger] (n_s + A[a^\dagger])^{-1} \rho, \quad (5.14)
\]

where \(\nu = \gamma N / \kappa\) is a dimensionless gain parameter and \(n_s = \gamma / \eta\) is a saturation boson number (also dimensionless). Both are typically much greater than unity, and this will be assumed in all that follows. As promised, the characteristic rate of evolution of \(\rho\) is \(\kappa\). It can also be verified that this master equation (5.14) is of the Lindblad form.
B. The laser threshold

The most important aspect in which Eq. (5.14) is a more realistic laser master equation than Eq. (4.11) is that it exhibits a threshold. If we define a threshold parameter

\[ \theta = \nu/n_s, \]

then the threshold is at \( \theta = 1 \). However, for \( n_s \) finite this threshold is not infinitely sharp, as I explain in this section.

1. Above threshold

For the laser to be above threshold we require \( \theta \) not just greater than one, but finitely greater than 1. By this I mean that \( \theta - 1 \) must be a positive number not much less than one. Using the \( P(n, \varphi) \) function as in Sec. IV, it can then be shown that the mean intracavity boson number is, to a very good approximation, given by \( \bar{n} = \nu - n_s = n_s(\theta - 1) \), and the variance by \( \nu = n_s \theta \).

The second-order coherence function is given by

\[ g^{(2)}(\tau) = 1 + \frac{\exp[-\kappa(1 - \theta^{-1})\tau]}{n_s(n_s(1 - \theta^{-2}))}. \]

(5.16)

From this it is apparent that the condition \( \theta \) finitely \( > 1 \) is necessary to satisfy condition (3). In fact we require

\[ \theta - 1 \gg n_s^{-1/2} \]

(5.17)

As for condition (4), it can be shown that the rate of phase diffusion is given by \( \Gamma = \kappa/(2\bar{n}) \) so that

\[ \langle I \rangle/\Gamma = 2\bar{n}^2 = n_s^2(\theta - 1)^2. \]

(5.18)

Thus condition (4) that \( \langle I \rangle \gg \Gamma \) is still satisfied without difficulty providing \( \theta \) finitely \( > 1 \) and \( n_s \gg 1 \). In fact, it is satisfied providing \( \theta - 1 \gg n_s^{-1} \), which is a weaker condition than (5.17) required to satisfy (3).

Very far above threshold, \( \theta \gg 1 \) with \( \nu = n_s \theta \) constant, and Eqs. (5.16) and (5.18) go over to the ideal limits derived in Sec. IV with \( \mu = \nu \). This limit can be obtained directly from the master equation (5.14), for if \( n_s \ll \bar{n} \) then \( n_s \) can be ignored compared to the superoperator \( \mathcal{A}[a^\dagger] \) which is of order \( a a^\dagger \). Then Eq. (5.14) goes over to the idealized Eq. (4.11) with \( \mu = \nu \). This is the limit in which the stimulated emission is so strong that the gain becomes independent of the number of bosons in the laser mode. This is not a paradox, because the state of the source depends on the number of bosons in the laser mode. When the laser boson number is sufficiently large, the source is so depleted that the \( n + 1 \) Bose-enhancement factor is neutralized and the absolutely constant gain of the model of Sec. IV is reproduced.

2. Below threshold

The below threshold regime is, unsurprisingly, \( \theta \) finitely \( < 1 \). In this regime the source is, to a first approximation, undepleted, and it can be shown that the mean boson number is \( \bar{n} \approx \theta/(1 - \theta) \). Therefore, for \( \theta > 1/2 \), \( \bar{n} \) will be greater than unity, implying that stimulated emission will dominate spontaneous emission. But this does not mean that \( \theta > 1/2 \) puts the laser above threshold. The laser will remain below threshold as long as \( \bar{n} \ll n_s \). This translates as the condition

\[ 1 - \theta > n_s^{-1}. \]

(5.19)

Under this condition the superoperator \( \mathcal{A}[a^\dagger] \) (which scales as \( \bar{n} + 1 \)) will also be negligible compared to \( n_s \). Then Eq. (5.14) can be approximated by

\[ \kappa^{-1}\dot{\rho} = \mathcal{D}[a]\rho + \theta \mathcal{D}[a^\dagger]\rho, \]

(5.20)

from which the result \( \bar{n} = \theta/(1 - \theta) \) is trivial to derive. This master equation is of the same form as the linear amplifier (5.21), with \( \kappa \) the linear loss and \( \theta \kappa \) the linear gain. However unlike Eq. (5.22) we have \( \theta \) finitely \( < 1 \) whereas \( \mu/(\mu + 1) \) approaches 1 for \( \mu \gg 1 \).

A laser below threshold fails condition (3), since \( g^{(2)}(0) = 2 \) as in Eq. (4.24). It will also fail condition (4), as \( \langle I \rangle = \kappa\theta(1 - \theta)^{-1} \) and the linewidth is \( \Gamma = \kappa(1 - \theta) \), so that

\[ \int_0^\infty d\tau G^{(1)}(\tau) \sim \langle I \rangle/\Gamma = \frac{\theta}{(1 - \theta)^2}. \]

(5.21)

is only of order unity for \( \theta \) finitely \( < 1 \). The laser above threshold satisfies condition (4) not only because the intensity \( \langle I \rangle \) is much greater, but because the linewidth \( \Gamma \) is much narrower.

3. The significance of the threshold

The increase in intensity and linewidth-narrowing of the laser above threshold are simple consequences of the large mean mean photon number. However the same cannot be said for the change in the intensity fluctuations. It is therefore instructive to enquire what has changed in the laser dynamics to make such a difference to these fluctuations. The requirement for the laser to be above threshold is, as we have seen above, that \( \theta - 1 \) is positive and not very small. This can be restated as: the mean boson number \( \bar{n} \) must be not very small compared to the saturation boson number \( n_s \). Returning to the above derivation, this is saying that in Eq. (5.11), the damping of \( \rho_1 \) proportional to \( \Gamma\mathcal{A}[a^\dagger] \) should not be negligible compared to the damping due to \( \gamma \). In more physical terms, it means that the rate of the transfer of bosons from the source to the laser mode should be comparable to the rate of transfer from the source to the sink.
In other words, the laser mode itself must significantly deplete the source, giving a sort of dynamical negative feedback, in order for the laser to go above threshold.

In practical terms, this characterization of the threshold transition is equivalent to the usual one, that a laser goes above threshold when the linear gain exceeds the linear loss. The exponential rise in boson number implied by linear gain exceeding linear loss must eventually deplete the source, causing the gain to change from being linear to being nonlinear. Very far above threshold, where Eq. (4.11) holds, the gain is so nonlinear that it is a constant.

C. Photon lasers

The starting master equation (5.1) used above could represent a photon laser based on parametric down conversion. To my knowledge, no such laser has been built, but it could be in principle. In this laser, the fundamental coupling would be a $\chi^{(2)}$ nonlinearity between the three electromagnetic field modes $a, b, c$, with $\omega_c = \omega_a + \omega_b$. It would be possible to have $\omega_a = \omega_b$, but only if the down-converted modes $a$ and $b$ were non-degenerate through their polarization for example. The high frequency mode $c$ would be pumped by a thermal light source (driving by another laser is, under the strict conditions established above, prohibited if the device is to qualify as a laser itself). The cavity for mode $b$ would be low quality (i.e. would have a short lifetime), which is easy to achieve. What would be difficult to achieve would be to make the cavity for mode $a$ sufficiently high quality that a very large number of photons could build up inside it while still satisfying the inequalities assumed above in deriving the final master equation (5.14).

In most photon lasers, matter plays a more important role in the gain mechanism than simply supplying a nonlinear coupling between optical modes. Probably the simplest systems to model are lasers using a dilute gas (see for example Ref. [24]). In this case the source system is the gas of atoms (or molecules). A particular transition of the atoms is coupled to the laser mode by a resonant electric-dipole interaction of the form

$$H = g \left( a^\dagger |1\rangle \langle 2| + a |2\rangle \langle 1| \right).$$  \hfill (5.22)

The atoms are pumped by being excited (perhaps by a thermal light source such as a flashlamp) into a state from which they can make a spontaneous transition into the upper level $|2\rangle$. This by itself is insufficient, because the interaction (5.22) is reversible. What is needed is the analogue of mode $b$, which makes the interaction irreversible and favours the transfer of energy to the photon field. This role is played by the continuum of electromagnetic field modes which cause the level $|1\rangle$ to spontaneously decay. Providing this spontaneous decay rate $\gamma_1$ is rapid enough, any atom which gives up its photon to the laser field is immediately transferred to another state from which it cannot reabsorb that photon. This gives rise to what is called in gas lasers a situation of population-inversion, but it is apparent from the generic model of Sec. V A that the term “population-inversion” is not appropriate for describing laser action in general.

Assuming that the atoms can be adiabatically eliminated, the rate for transferring quanta from the atoms to the field is equal to $N_2 g^2 a^\dagger a / \gamma_1$, where $N_2$ is the number of excited state atoms, and $aa^\dagger = a^\dagger a + 1$ is the number of photons in the laser mode plus one. The condition for the laser to be above threshold is that the number of photons in the laser mode is so large that this rate be at least comparable with the rate for other processes de-exciting the upper level $|2\rangle$. These other processes include spontaneous emission at rate $\gamma_2$, and collisional de-excitation.

D. Atom lasers

Being material particles, atoms cannot be created out of energy (as can photons and other gauge bosons). This means that the source for atoms must be a mode (or set of modes) of the atom field. Typically these modes would exist in a trap for atoms and the laser mode itself would be the lowest mode of that trap (although this is not necessarily so). The mechanism which have been proposed for transferring atoms from the source to the laser mode fall into two classes: optical cooling [13–15,19] and evaporative cooling [16–18]. These will be discussed separately below. In both cases atoms can be added to the source by pumping some reservoir, either external to the trap or in the high-energy modes of the trap. The removal of the atoms from the laser mode to form a coherent output beam is a considerable technical challenge, and would depend on the nature of the trap. As stated in Sec. II, it would be useful for the output to be in the form of a guided atomic wave.

1. Optical cooling

Optical cooling is the reduction in the kinetic energy of atoms by the transfer of energy and momentum between the atoms, optical laser beams, and spontaneously emitted photons. This is the mechanism for transferring atoms from the source modes of the trap to the laser mode of the trap in the proposed atom lasers of Refs. [13–15,19]. It is clear that the irreversibility in this transfer comes from the spontaneously emitted photons. Thus, as in a photon laser with a gaseous gain medium, the role of mode $b$ is played by the continuum of electromagnetic field modes coupled to a particular atomic transition. One problem with this cooling mechanism is
that the emitted photons may be reabsorbed by other atoms in the laser mode. However this problem could be minimized for a laser mode which is linear in aspect (that is, long and narrow). This might be achieved by siting the laser cavity within an atom waveguide, which would also produce the output in a desirable form.

The condition for threshold in an atom laser of this sort is that the depletion of the source due to optical cooling into the laser mode (which will of course be proportional to the number of laser mode atoms plus one) be at least comparable to the depletion of the source due to losses from the trap or laser-induced transitions to other trap modes. Provided that photon reabsorption is not too great a problem, this threshold condition can be achieved, so that the intensity fluctuations in the laser output can satisfy condition (3). To satisfy condition (4) it is necessary to calculate the linewidth of the the laser. So far this has been done only for the model of Ref. [13]. Because of unwanted processes, the linewidth found in Ref. [13] far exceeded the ideal limit of Eq. (4.13), but was still small enough to satisfy condition (4). However, this calculation ignored many complicating factors such as inter-atomic collisions, so it remains to be seen whether it really would be possible to build an atom laser based on this or any other method of optical cooling.

2. Evaporative cooling

Evaporative cooling at first sight appears a more promising route to an atom laser because unlike optical cooling it has already produced Bose condensation of atoms. The basic mechanism for evaporative cooling is removal of ‘hot’ atoms (evaporation) accompanied by thermalization via collisions. For a cold dilute gas the dominant form of collisions are binary ones resulting from s-wave scattering, so the coupling Hamiltonian is of the form

\[ H = \sum_{i \leq j, k \leq l} g_{ijkl} a_i^\dagger a_j^\dagger a_k a_l, \]

(5.23)

where \(g_{ijkl} = g_{kij}\) and \(a_i\) is the annihilation operator for the atom field in the \(i\)th trap mode. To transfer atoms from the source to the laser mode requires collisions between two atoms in the source modes \((k, l)\), putting one atom into the laser mode (say \(j = 0\)) and the second atom into some other mode \((i)\) with high energy. This process will be irreversible, as desired, if the atom in mode \(i\) has sufficient energy to escape the trap (so that \(i\) is really a continuum mode). Thus in this case \(a_i\) plays the role of \(b\) and \(a_k a_l\) that of the source \(c\). For the regime of weak pumping \((N \ll 1)\) the fact that the coupling of the source to the laser mode is quadratic in the former causes fairly trivial differences from the linear coupling used above [18,17]. However, in the other limit \((N \gg 1)\), some differences do arise [17]; in particular the intensity fluctuations are not necessarily Poissonian far above threshold. Nevertheless it seems that condition (3) could be satisfied for this sort of atom laser providing a suitable output coupler is found.

What is unclear about this sort of laser is whether it could satisfy condition (4). Using the simple model proposed by Holland et al. [18], it was shown in Ref. [17] that the linewidth of an atom laser based on evaporative cooling would typically be much larger than the output flux, so that condition (4) would not be satisfied. The dominant linewidth broadening is due to the fluctuations in the self-interaction energy of the laser mode coming from the Hamiltonian term \(g_{0000} a_0^\dagger a_0^\dagger a_0^2\). This is linked to the dynamics of the laser because it is the same s-wave scattering Hamiltonian [5,23] which causes both the undesired self-interaction energy of the laser mode and the desired coupling between the source and the laser modes. Thus it may be impossible to build an atom laser using evaporative cooling. There are various ways that this conclusion could be avoided. First, it may be that by using dipole-dipole collisions, as suggested in Ref. [16], the cross-section for self-interactions of the laser mode could be made negligible compared to the cross-section for source-laser coupling, and so overcome the linewidth problem. This might also be achievable by using sympathetic cooling [46], in which another species of atom (which could have a large collisional cross-section) is cooled evaporatively and takes the desired species (which could have a small cross-section) into the Bose degenerate regime through thermalization. A third possibility is that a more elaborate model of standard evaporative cooling based on s-wave scattering might lead to a different conclusion. All of these suggestions deserve further investigation.

VI. SUMMARY

In this paper I have argued that before attempting to build an atom laser (an analogue of the familiar photon laser), it is necessary to have a rigorous definition of what constitutes a laser. This definition should apply equally to photon lasers and to atom lasers. The fundamental principle underlying the definition which I propose is that a laser is a device which produces an output field which is well-approximated by a classical wave of fixed intensity and phase.

Thus, the laser itself is treated as a black box; we are only interested in its output. The inputs are arbitrary except that they cannot themselves be laser fields of the same substrate (atoms or photons for an atom or photon laser respectively).

This fundamental principle can be quantified by requiring that the output of a laser satisfy four conditions.
The first two conditions are elementary conditions that can be understood using single-boson concepts:

1. The output is highly directional, and ideally has a single transverse mode.

2. The longitudinal spatial frequency of the output beam has a small spread in the sense that \( \delta k \ll k \).

The second two conditions require many-boson concepts and are most easily stated using quantum field theory. Under the first two conditions the output field of a laser may be described by a field operator \( b(t) \) satisfying \( [b(t), b(t')] = \delta(t - t') \). The intensity operator \( I(t) = b^\dagger(t)b(t) \) measures the output boson flux in units of bosons per second. Using these quantities the final two conditions can be stated as

3. The output intensity fluctuations are small in the sense that \( \forall \tau \neq 0 \left| \langle I(t + \tau), I(t) \rangle \right| \ll \langle I \rangle^2 \).

4. The output phase fluctuations are small in the sense that \( \int d\tau \left| b^\dagger(t + \tau)b(t) \right| \gg 1 \).

The fourth condition has, to my knowledge, not been proposed by any previous authors and so in some sense is the most important contribution of this paper.

As well as defining a laser, I introduced a generic laser model and listed five characteristic features which it possesses:

1. a cavity supporting the laser mode of the boson field and allowing an output beam to form.
2. a source of bosons which is coupled to the laser mode.
3. an irreversibility which favours the transfer of bosons from source to laser mode.
4. a sink which also takes bosons from the source.
5. a pump to replenish the source.

The presence of a sink implies that this laser can operate either above or below threshold. Only when operating above threshold will it satisfy the final two conditions above. For the laser to be above threshold requires that the transfer of bosons from source to laser mode be at least comparable to the transfer from source to sink. In this regime there is a negative correlation between the number of source excitations and the number of bosons in the laser mode. This provides the self-regulation of intensity which enables condition (3) to be satisfied. By contrast, condition (4) may be satisfied for a laser above threshold simply because the intracavity boson number is very large.

Finally, I presented a very brief analysis of some of the atom laser models which have been proposed, in terms of the above conditions and features. These atom laser schemes fall into two broad classes: those based on optical cooling of atoms and those based on evaporative cooling. The former is closer in operation to the generic laser model introduced here, but would be hard to implement because of the technical difficulties involved in cooling atoms to extremely low temperatures in the presence of optical laser beams. The latter appears a more promising route because it has already been used to achieve Bose condensation. However, it appears that atom lasers based on evaporative cooling would have difficulty satisfying condition (4). The identification of this difficulty highlights the importance of having a definition for an atom laser.

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APPENDIX: SYMMETRIES AND CONSERVATION LAWS

The difference between matter fields and the electromagnetic field can be seen from the fundamental coupling Hamiltonian for quantum electrodynamics (QED), which can be written as

\[
H_{\text{QED}} = \sum_{ijk} u_{ijk} \Psi_i^\dagger (a_j + a_j^\dagger) \Psi_k. \tag{A1}
\]

Here \( \Psi_i \) represents the modes of an electron field, \( a_j \) represents the annihilation operator for the modes of the photon field, and \( u_{ijk} \) is a set of coupling constants. This Hamiltonian is invariant under the global gauge transformation

\[
\Psi_i \rightarrow \Psi_i e^{if(t)}, \tag{A2}
\]

which shows that the absolute phase of a matter field is completely unobservable, and thus that it is impossible to create a matter field with a well-defined absolute phase. This is implicit in the discussion in Sec. II B about the fact that the frequency of a matter field is undefined up to an arbitrary additional constant. Only the relative phase (and the relative frequency) of one mode of a matter field to another mode is physically meaningful. The same is not true of the photon field because both \( a \) and \( a^\dagger \) appear linearly in Eq. (A1) so it is not invariant under the transformation

\[
a_j \rightarrow a_j e^{ig(t)}. \tag{A3}
\]

The invariance (A2) leads to a conservation law for the total number of electrons \( \sum \Psi_i^\dagger \Psi_i \).
\[ N_{\text{electrons}} = \sum_i \Psi_i^\dagger \Psi_i, \quad (A4) \]

while the non-invariance \(^{(A3)}\) explains why there is no such conservation law for photon number.

Although there is no absolute conservation law for photon number, there is an approximate conservation law at visible frequencies. At frequencies \(\omega \sim 10^{16}\text{s}^{-1}\), the characteristic size \(d\) of the material oscillators (that is, atoms) with these frequencies is typically much smaller than the wavelength of the resonant Maxwell field. That is, \(kd \ll 1\), where \(k = c/\omega\). Under this condition the coupling between atom and the vacuum electromagnetic field can be treated using the dipole approximation. This gives a quality factor for the oscillator of \([25]\).

\( \frac{\gamma}{\omega} = \frac{4}{3} \alpha (kd)^2, \quad (A5) \)

where \(\alpha \approx 1/137\) is the QED fine structure constant. Because \(kd\) is typically of order \(10^{-3}\), the ratio \(\gamma/\omega\) is very small so we can also make the so-called rotating wave approximation (RWA). This amounts to ignoring non-energy-conserving terms in the Hamiltonian \(^{(A1)}\).

Let us assume for simplicity that each electron is bound in a two-level atom. Then the field mode index \(i\) can be replaced by one letter, \(u\) or \(l\), corresponding to the electron being in the upper or lower electronic level, plus another index \(\alpha\) for the motional mode of the atom as a whole. Then the RWA is effected by replacing the fundamental QED Hamiltonian density \(^{(A1)}\) by the approximate Hamiltonian

\[ H_{\text{RWA}} = \sum_{\alpha j \beta} v_{\alpha j \beta} \left[ \Psi_{u,\alpha}^\dagger a_j \Psi_{l,\beta} + \Psi_{l,\alpha}^\dagger a_j^\dagger \Psi_{u,\beta} \right], \quad (A6) \]

where \(v\) is a set of coupling constants related to the set \(u\). This Hamiltonian is invariant under the transformation \(^{(A3)}\), providing it is combined with

\[ \Psi_{u,\alpha} \rightarrow \Psi_{u,\alpha} e^{ig(t)} \quad (A7) \]

for the same function \(g(t)\). This leads to a conservation law for excitation number, that is, the total number of photons and excited state atoms:

\[ N_{\text{excitations}} = \sum_{\alpha j} \left[ \Psi_{u,\alpha}^\dagger \Psi_{u,\alpha} + a_j^\dagger a_j \right]. \quad (A8) \]

This conservation law can be generalized for more complicated atoms. It implies that under usual optical conditions it is neither possible to measure the absolute phase of the electromagnetic field, nor is it possible to create a field with a well-defined phase. One could thus conclude that the mean field of an optical laser is as much a myth as the non-zero mean value of a matter field, and this is the view taken in Ref. \([17]\), I prefer to stress that there is an in-principle difference between matter fields and the electromagnetic field. After all, the RWA is only an approximation and in particular it does not apply for the gedankenexperiment described in the text, in which the Lorentz force on a free electron is used to measure the temporal phase of an optical standing wave. Similarly, as noted in Ref. \([17]\), a free-electron laser is a source which may produce visible light with a temporal phase which could, in principle, be predicted.

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