STABILITY OF LINE SOLITONS FOR THE KP-II EQUATION IN $\mathbb{R}^2$

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Abstract. We prove nonlinear stability of line soliton solutions of the KP-II equation with respect to transverse perturbations that are exponentially localized as $x \to \infty$. We find that the amplitude of the line soliton converges to that of the line soliton at initial time whereas jumps of the local phase shift of the crest propagate in a finite speed toward $y = \pm \infty$. The local amplitude and the phase shift of the crest of the line solitons are described by a system of 1D wave equations with diffraction terms.

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1. Introduction

The KP-II equation

\[ \partial_x(\partial_t u + \partial_x u + 3\partial_x(u^2)) + 3\partial_y^2 u = 0 \quad \text{for } t > 0 \text{ and } (x, y) \in \mathbb{R}^2, \]

is a generalization to two spatial dimensions of the KdV equation

\[ \partial_t u + \partial_x^3 u + 3\partial_x(u^2) = 0, \]

and has been derived as a model in the study of the transverse stability of solitary wave solutions to the KdV equation with respect to two dimensional perturbation when the surface tension is weak or absent. See [17] for the derivation of (1.1). Note that every solution of the KdV equation (1.2) is a planar solution of the KP-II equation (1.1).

The global well-posedness of (1.1) in \( H^s(\mathbb{R}^2) \) \( (s \geq 0) \) on the background of line solitons has been studied by Molinet, Saut and Tzvetkov [29] whose proof is based on the work of Bourgain [8]. For the other contributions on the Cauchy problem of the KP-II equation, see e.g. [12, 13, 15, 16, 35, 36, 37, 38] and the references therein.

Let

\[ \varphi_c(x) = c \sech^2 \left( \sqrt{\frac{c}{2}} x \right). \]

It is well known that the 1-soliton solution \( \varphi_c(x - 2ct) \) of the KdV equation (1.2) is orbitally stable because \( \varphi_c \) is a minimizer of the Hamiltonian of (1.2) restricted on the manifold \( \{ u \in H^1(\mathbb{R}) \mid \|u\|_{L^2} = \|\varphi_c\|_{L^2} \} \). See [2, 4] and [5, 11, 40] for stability of solitary wave solutions of Hamiltonian systems. We remark that the KP-II equation does not fit into those standard argument because the first two terms of the Hamiltonian of the KP-II equation

\[ \int \left( u_x^2(t, x, y) - 3(\partial_x^{-1}\partial_y u(t, x, y))^2 - 2u^3(t, x, y) \right) dxdy, \]

have the opposite sign. Recently, Mizumachi and Tzvetkov [26] have proved that \( \varphi_c(x - 2ct) \) is orbitally stable as a solution of the KP-II equation in \( L^2(\mathbb{R}_x \times T_y) \). They used the Bäcklund transformation to prove that \( L^2(\mathbb{R}_x \times T_y) \)-stability follows from the \( L^2 \)-stability of the 0-solution, which is an immediate consequence of the conservation law of the \( L^2(\mathbb{R}_x \times T_y) \)-norm.

Unlike the perturbations which are periodic in the transverse directions, the perturbations in \( L^2(\mathbb{R}^2) \) does not allow phase shifts of line solitons that are uniform in the transverse direction. This is because the difference of any translated line solitons and itself has infinite \( L^2(\mathbb{R}^2) \)-mass whereas the well-posedness result [29] tells us the perturbation to the line soliton stay in \( L^2(\mathbb{R}^2) \) for all the time. In order to analyze modulation of line solitons, we express solutions around the line soliton as

\[ u(t, x, y) = \varphi_{c(t, y)}(x) - \psi_{c(t, y)}(x - x(t, y) + 4t) + v(t, x - x(t, y), y), \]

where \( c(t, y) \) and \( x(t, y) \) are the local amplitude and the local phase shift of the modulating line soliton, \( v \) is a remainder part which is expected to behave like an oscillating tail and \( \psi_{c(t, y)} \) is an auxiliary function so that

\[ \int v(t, x, y) dx = \int v(0, x, y) dx \quad \text{for any } t > 0, \]
Eq. (1.4) means that if the line soliton is locally amplified, then small waves are emitted from the rear of the line soliton. By introducing the auxiliary function \( \psi_{c}(t, y) \), we have \( v(t) \in L^{2}(\mathbb{R}^{2}) \) for every \( t \geq 0 \) and we are able to show that the \( L^{2} \)-norm of \( v \) is almost conserved. We find that local modulations of the amplitude and phase shift can be described by a system of 1-dimensional wave equations with diffraction (viscous damping) terms, that a modulating line soliton converges to a line soliton with the same height as the original soliton on any compact subset of \( \mathbb{R}^{2} \) (Theorem 1.1) and that “jumps” of the phase shift of the modulating line soliton propagate toward \( y = \pm \infty \) along the crest of line solitons, which makes the set of all line soliton solutions unstable (Theorem 1.2).

Using geometric optics, Pedersen (39) heuristically explained that the amplitude and the orientation of the crest are described by a system of the Burgers equation. Since both the KP-II equation and the Boussinesq equations are long wave models for the 3D shallow water waves, it is natural to expect the same phenomena for KP-II. We find that the first order asymptotics of \( \partial_{y}x(t, y) \) and \( c(t, y) \) around \( y = \pm (8c_{0})^{1/4}t + O(\sqrt{t}) \) are given by self-similar solutions of the Burgers equations as \( t \to \infty \) (Theorem 1.3).

Now let us introduce our results. The first result is the stability of line soliton solutions for exponentially localized perturbations.

**Theorem 1.1.** Let \( c_{0} > 0 \) and \( a \in (0, \sqrt{c_{0}/2}) \). Then there exist positive constants \( \varepsilon_{0} \) and \( C \) satisfying the following: if \( u(0, x) = \varphi_{c_{0}}(x - x_{0}) + v_{0}(x) \) and \( \varepsilon := \|e^{ax}v_{0}\|_{L^{2}(\mathbb{R}^{2})} + \|e^{ax}\varphi_{0}\|_{L^{2}(\mathbb{R}^{2})} + \|v_{0}\|_{L^{2}(\mathbb{R}^{2})} < \varepsilon_{0} \), then there exist \( C^{1} \)-functions \( c(t, y) \) and \( x(t, y) \) such that for \( t \geq 0 \),

\[
\begin{align*}
(1.5) \quad & \|u(t, x, y) - \varphi_{c(t,y)}(x - x(t, y))\|_{L^{2}(\mathbb{R}^{2})} \leq C\varepsilon, \\
(1.6) \quad & \sup_{y \in \mathbb{R}}|c(t, y) - c_{0}| + |x_{y}(t, y)| \leq C\varepsilon(1 + t)^{-1/2}, \\
(1.7) \quad & \|x_{t}(t, \cdot) - 2c(t, \cdot)\|_{L^{2}} \leq C\varepsilon(1 + t)^{-3/4}, \\
(1.8) \quad & \|e^{ax}(u(t, x + x(t, y), y) - \varphi_{c(t,y)}(x))\|_{L^{2}} \leq C\varepsilon(1 + t)^{-3/4}.
\end{align*}
\]

**Remark 1.1.** The KP-II equation has no localized solitary waves (see [6]). On the other hand, the KP-I equation has stable localized solitary waves (see [7] [22]) and line solitons of the KP-I equation are unstable (33 [34] [42]).

The KP-II equation (1.1) is invariant under a change of variables

\[
(1.9) \quad x \mapsto x + ky - 3k^{2}t + \gamma \quad \text{and} \quad y \mapsto y - 6kt \quad \text{for any} \ k, \gamma \in \mathbb{R},
\]

and has a 3-parameter family of line soliton solutions

\[
\mathcal{A} = \{ \varphi_{c}(x + ky - (2c + 3k^{2})t + \gamma) \mid c > 0, k, \gamma \in \mathbb{R} \}.
\]

The set of all 1-soliton solutions of KdV or line soliton solutions of KP-II under the \( y \)-periodic boundary conditions are known to be stable in \( L^{2}(\mathbb{R}^{2}) \) (see [24] [26]). However the set \( \mathcal{A} \) is not large enough to be stable for the flow generated by KP-II in \( L^{2}(\mathbb{R}^{2}) \).

**Theorem 1.2.** Let \( c_{0} > 0 \). There exists a positive constant \( C \) such that for any \( \varepsilon > 0 \), there exists a solution of (1.1) such that \( \|u(0, x, y) - \varphi_{c_{0}}(x)\|_{L^{2}} < \varepsilon \) and

\[
\lim_{t \to \infty} t^{-1/4}\|u(t, x, y) - \varphi_{c_{0}}(x)\|_{L^{2}(\mathbb{R}^{2})} \geq C\varepsilon.
\]
Remark 1.2. If \((c, \gamma) \neq (c_0, 0)\), then \(\|u(t, x, y) - \varphi_c(x - \gamma)\|_{L^2(\mathbb{R}^2)} = \infty\) thanks to the well-posedness result (29). Thus the orbital instability

\[
\liminf_{t \to \infty} t^{-1/4} \inf_{v \in A} \|u(t, \cdot) - v\|_{L^2(\mathbb{R}^2)} \geq C\varepsilon
\]

follows immediately from Theorem 1.2.

Orbital instability is a consequence of finite speed propagations of local phase shifts along the crest of the modulating line soliton. We find that \(c(t, y)\) and \(\partial_y x(t, y)\) behave like a self-similar solution of the Burgers equation around \(y = \pm \sqrt{8c_0 t}\).

**Theorem 1.3.** Let \(c_0 = 2\) and let \(\nu_0\) and \(\varepsilon\) be the same as in Theorem 1.1. Then for any \(R > 0\),

\[
\left\| \begin{pmatrix} u_B(t, y) \\ y_B(t, y) \end{pmatrix} \right\|_{L^2([-R/4, R/4] \times \mathbb{R}^2)} = o(t^{-1/4})
\]

as \(t \to \infty\), where \(u_B^\pm\) are self similar solutions of the Burgers equation

\[\partial_t u = 2\partial_y^2 u + 4\partial_y(u^2)\]

such that

\[
u_B^\pm(t, y) = \frac{\pm m_{\pm} H_{21}(y)}{2(1 + m_{\pm} \int_0^\pi H_{21}(y_1) dy_1)}, \quad H_i(y) = (4\pi t)^{-1/2} e^{-y^2/4t},
\]

and that \(m_{\pm}\) are constants satisfying

\[
\int_{\mathbb{R}^2} u_B^\pm(t, y) dy = \frac{1}{4} \int_{\mathbb{R}^2} c(0, y) dy + O(\varepsilon^2).
\]

Now we recall known results on stability of planar traveling wave solutions. Stability of planar traveling waves in \(L^2(\mathbb{R}^n)\) \((n \geq 2)\) were studied for reaction diffusion equations by Xin (41), Levermore and Xin (21) and Kapitula (18). Stability of kink solutions of Hamiltonian systems has been studied for 3-dimensional \(\phi^4\)-model by Cuccagna (9).

The difficulty of those problems is that the spectrum of the linearized operator \(\mathcal{L}\) around planar traveling waves has continuous spectrum converging to 0 whereas in the case where \(n = 1\), we see that 0 is an isolated eigenvalue of the linearized operator around the traveling wave solution and all the rest of the spectrum is in the left half plane and away from the imaginary axis. When \(n \geq 2\), the paper (41) tells us that the semigroup generated by the linearized operator decays to zero like \(t^{-(n-1)/4}\). This corresponds to the relation between our results and the asymptotic stability result for the KdV equation by Pego and Weinstein (32) where the spectrum of the linearized operator in \(L^2(\mathbb{R}; e^{2ax}dx)\) consists of the isolated eigenvalue 0 and \(\sigma_c\) satisfying \(\sigma_c \subset \{ \lambda \in \mathbb{C} \mid \Re \lambda < -b \}\) for some \(b > 0\). By measuring the size of perturbations with an exponentially weighted norm biased in the direction of motion, one obtains that exponential decay of the oscillating tail of the solution for both KdV and KP-II and that leads to exponential stability of the KdV 1-soliton. However, thanks to the transverse direction, the linearized operator around a line soliton of the KP-II equation has two branches of continuous spectrum all the way up to 0 in \(L^2(\mathbb{R}; e^{ax}dy)\) with \(a > 0\). We remark that those resonant modes are exponentially growing as \(x \to -\infty\) (see Lemma 2.1) and that
the corresponding continuous spectrum does not show up when we consider $L^2(\mathbb{R}^2)$-linear stability of the line soliton. We refer the readers e.g. [14] for linear stability of solitary waves and cnoidal waves to transverse perturbations.

Since the transverse direction is 1-dimensional, the rate of decay of $\|\partial_y^k c(t, \cdot)\|_{L^2}$ and $\|\partial_y^{k+1} x(t, \cdot)\|_{L^2}$ is at most $t^{-\frac{3}{2}(k+1)/4}$ and the nonlinearity of the modulation equations is quadratic, it was fortunate that they have the similar structure as the Burgers equations. Indeed, there are 1D-heat equations with quadratic nonlinearity whose solutions may not exist global in time (10).

Our plan of the present paper is as follows. In Section 2 we obtain explicit formula of resonant modes of $\mathcal{L}$ and $\mathcal{L}^*$, where $\mathcal{L}$ is a linearized operator of the KP-II equation around the line soliton $\varphi(x - 4t)$ by using the linearized Miura transformations. As is well known, the Miura transformations connect line solitons and the null solution of the KP-II equation with kink solutions of the modified KP-II equation and all the slowly decaying eigenmodes of the linearized equation $\partial_t u = \mathcal{L} u$ can be found by investigating the kernel and the cokernel of the linearized Miura transformation. We find two branches of (resonant) eigenmodes $\{g(x, \pm \eta)e^{iy\eta}\}$ of $\mathcal{L}$ such that $g(x, \eta) \in L^2(\mathbb{R}; e^{2\alpha x} dx)$ for a $\alpha > 0$ and $\eta \in (-\eta_*, \eta_*)$, where $\eta_*$ is a positive number depending on $\alpha$. In Section 3 we prove that solutions which are orthogonal to resonant modes of $\mathcal{L}^*$ decay exponentially in $L^2(\mathbb{R}; e^{2\alpha x} dx dy)$ like solutions of the linearized KP-II equation around the null solution by using a bijection composed of the linearized Miura transformations by using an idea of [25].

In Section 4, we collect linear estimates of 1D damped wave equations which shall be used to analyze modulation equations of line solitons. In Section 5 we fix the decomposition (1.3) by imposing that $v(t)$ is orthogonal to secular resonant modes of $\mathcal{L}^*$. In Section 6 we derive modulation equations on $c(t, y)$ and $x(t, y)$ from the non-secular conditions introduced in Section 5. Since the KP equations are anisotropic in $x$ and $y$, the resonant eigenfunctions cannot be written in the form $\{g(x)e^{iy\eta} \mid \eta \in \mathbb{R}\}$ as in the case for reaction diffusion equations ([18, 41]) or the $\phi^4$ model ([15, 21]). Moreover, the resonant eigenfunctions grow like $g(x, \eta) \sim e^{\eta^2|x|/2}$ as $x \to -\infty$. For this reason, we work on exponentially weighted space $X$ and impose the non-secular conditions only for small $\eta$. To rewrite modulation equations of $c(t, y)$ and $x(t, y)$ in a PDE form, we compute the time derivative of the non-secular condition, take the inverse Fourier transform of the resulting equation. Although the modulation equations are non-local due to the $\eta$-dependence of the resonant modes $g(x, \eta)$, the dominant part of the modulation equations are damped wave equations. Indeed, the modulation equations for the line soliton $\varphi_{c_0}(x - 2c_0t)$ with $c_0 = 2$ are approximately

\begin{equation}
(1.10) \quad \left( \begin{array}{c}
\frac{b_t}{\tilde{x}_t} \\
\frac{b}{\tilde{x}}
\end{array} \right) \simeq \left( \begin{array}{cc}
\frac{3\partial_y^2}{2} & \frac{8\partial_y^2}{\partial_y^2 y} \\
2 - \mu_3 \partial_y^2 & \partial_y^2 y
\end{array} \right) \left( \begin{array}{c}
\frac{b}{\tilde{x}} \\
\frac{6(bx_y) y}{3(\tilde{x}_y)^2 - 4b^2}
\end{array} \right),
\end{equation}

where $\mu_3 = 1/2 + \pi^2/24$ and $b(t, y) = 4/3\{(c(t, y)/2)^{3/2} - 1\}$ (see (6,12) for the precise definition) and $\tilde{x}(t, y) = x(t, y) - 4t$. We remark that $\partial_t x(t, y) \simeq 2c(t, y)$ and $b(t, y) \simeq c(t, y) - 2 \to 0$ as $t \to \infty$. If we translate (1.10) into a system of $b(t, y)$ and $\partial_t x(t, y)$ and diagonalize the resulting equation, then we obtain a coupled Burgers equations.

In Section 7 we obtain $\mathcal{F}^{-1}L^\infty - L^2$ decay estimates on $b(t, y)$ and $\partial_t x(t, y)$ assuming a decay estimate on $v(t)$ in $X := L^2(\mathbb{R}; e^{2\alpha x} dx dy)$ and the $L^2$-bound of $v(t)$. In Section 8 we prove the $L^2$-estimate of $v$ assuming the decay estimate on
v(t) in X. In Section 5 we estimate the low-frequency part of v in $L^2(\mathbb{R}^2; e^{2ax}dx)dy$ by using the semigroup estimates obtained in Section 3. We estimate the high frequency part separately in Section 10 by using the virial type estimate to avoid the derivative loss. We remark that the potential term produced by the linearization around the line soliton is negligible to obtain time-global virial type estimates for the high frequency part. In Sections 11 and 12 we prove Theorems 1.1 and 1.2. In Section 13 we prove Theorem 1.3 by using a rescaling argument by Karch (20).

Using the inverse scattering method, Villarroel and Ablowitz (39) studied the Cauchy problem and stability of line solitons of the KP-II equation. However, it is not clear from their result how modulations to line solitons evolve because they did not explain in which sense line solitons are stable. Moreover, our method does not rely on integrability of the equation except for the linear estimate and can possibly be applied to bidirectional models such as the Benney-Luke equation (33 31). Our result is a first step toward $H^1$-asymptotic stability of 1D solitary waves for gKdV.

Finally, let us introduce several notations. For Banach spaces V and W, let $B(V, W)$ be the space of all linear continuous operators from V to W and let $\|T\|_{B(V, W)} = \sup_{\|v\|_V = 1} \|Tu\|_W$ for $T \in B(V, W)$. We abbreviate $B(V, V)$ as $B(V)$.

For $f \in \mathcal{S}(\mathbb{R}^n)$ and $m \in \mathcal{S}'(\mathbb{R}^n)$, let

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-ix\xi}dx,$$

$$\mathcal{F}^{-1}f(x) = \hat{f}(x) = \hat{f}(-x), \quad (m(D_x)f)(x) = (2\pi)^{-n/2}(\hat{m} * \hat{f})(x).$$

We use $a \lesssim b$ and $a = O(b)$ to mean that there exists a positive constant such that $a \leq Cb$. Various constants will be simply denoted by $C$ and $C_i (i \in \mathbb{N})$ in the course of the calculations. We denote $\langle x \rangle = \sqrt{1 + x^2}$ for $x \in \mathbb{R}$.

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2. THE MIURA TRANSFORMATION AND RESONANT MODES OF THE LINEARIZED OPERATOR

In this section, we will find resonant eigenmodes of the linearized operator around line solitons and prove exponential stability of non-resonant modes in an exponentially weighted space by using the linearized Miura transformations.

For $p \in [1, \infty]$ and $k \in \mathbb{N}$, let $L^p_0(\mathbb{R}) = \{v \mid e^{ax}v \in L^p(\mathbb{R})\}$ and $H^k_0(\mathbb{R}) = \{v \mid e^{ax}v \in H^k(\mathbb{R})\}$ whose norms are given by

$$\|v\|_{L^p_0(\mathbb{R})} = \|e^{ax}v\|_{L^p(\mathbb{R})}, \quad \|v\|_{H^k_0(\mathbb{R})} = \left(\sum_{j=0}^{k} \|\partial_x^j v\|_{L^2_0(\mathbb{R})}^2\right)^{1/2}.$$
For any $a > 0$, we define the anti-derivative operator $\partial_x^{-1}$ on $L^2_{\pm a}(\mathbb{R})$ by

$$
(\partial_x^{-1} u)(x) = -\int_x^\infty u(x_1) \, dx_1 \quad \text{for} \; u \in L^2_{\pm a}(\mathbb{R}),
$$

$$
(\partial_x^{-1} u)(x) = \int_x^-\infty u(x_1) \, dx_1 \quad \text{for} \; u \in L^2_{\pm a}(\mathbb{R}).
$$

The operator $\partial_x^{-1}$ is bounded on $L^2_{\pm a}(\mathbb{R})$. Indeed, it follows from Young’s inequality that $\|\partial_x^{-1}\|_{B(L^2(\mathbb{R}))} = \|\partial_x^{-1}\|_{B(L^2,\mathbb{R})} = 1/a$ for $a > 0$.

We interpret (1.1) in the “integrated” form

$$
(2.1) \quad \partial_t u + \partial^3_x u + 3\partial_x^{-1}\partial^2_y u + 3\partial_x u^2 = 0,
$$

where $\partial_x^{-1}\partial_y^2 u(x, y) = -\int_x^\infty \partial^2_y u(x, y_1) \, dx_1$ in the sense of distribution, that is,

$$
(2.2) \quad \langle \partial_x^{-1}\partial_y^2 u, \psi \rangle = -\int_{\mathbb{R}^2} \left( \int_x^\infty u(x_1, y) \, dx_1 \right) \partial_y^2 \psi(x, y) \, dx \, dy \quad \text{for} \; \psi \in C^\infty_0(\mathbb{R}^2).
$$

If $u$ is smooth and $u, \partial_y u \in X := L^2(\mathbb{R}^2; e^{2xy} \, dx \, dy)$, then

$$
\langle \partial_x^{-1}\partial_y^2 u, \psi \rangle = \int_{\mathbb{R}^2} \left( \int_x^\infty \partial_y u(x_1, y) \, dx_1 \right) \partial_y \psi(x, y) \, dx \, dy.
$$

Eq. (2.2) follows from the standard definition $\partial_x^{-1}\partial_y u = \mathcal{F}^{-1}(\frac{1}{\xi^2} \mathcal{F}(\xi, \eta))$ when a solution is exponentially localized in the $x$-direction. Indeed, we have $\langle \partial_x^{-1} u(x, y) = (\mathcal{F}^{-1}(1/\xi^2)\mathcal{F}(\xi, \eta)) \rangle$ for $u \in \{ f \in C^\infty_0(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} f(x, y) \, dx = 0 \; \forall y \in \mathbb{R} \}$ = $\mathcal{B}$. Since $\mathcal{B}$ is a dense subset of $X \cap L^2(\mathbb{R}^2)$, we have (2.2) for $u \in X \cap L^2(\mathbb{R}^2)$ by taking a limit.

We remark that (2.1) has a solution in the class

$$
\varphi(x - 2ct) \in L^\infty_0(\mathbb{R} ; \|X \cap C(\mathbb{R} ; L^2(\mathbb{R}^2))
$$

for initial data $u(0) \in X \cap L^2(\mathbb{R}^2)$ (see Appendix E).

### 2.1. Resonant modes

Let $\varphi = \varphi_2, u(t, x, y) = \varphi(x - 4t) + U(t, x - 4t, y)$. Linearizing (2.1) around $U = 0$, we have

$$
(2.3) \quad \partial_t U = \mathcal{L} U, \quad \mathcal{L} U = -\partial^4_x U + 4\partial_x U - 3\partial_x^{-1}\partial^2_y U - 6\partial_x(\varphi U).
$$

Let $\mathcal{L}(\eta) := -\partial^4_x U + 4\partial_x U + 3\eta^2 \partial_x^{-1} U - 6\partial_x(\varphi U)$ be the operator on $L^2_{\pm a}(\mathbb{R})$ with its domain $D(\mathcal{L}(\eta)) = H^4_{\pm a}(\mathbb{R})$. Since the potential of $\mathcal{L}$ does not depend on $y$, we have $\mathcal{L}(u(x)e^{+i\eta y}) = e^{+i\eta y}\mathcal{L}(\eta)u(x)$. We will look for resonant modes $\{g(x, \eta)e^{i\eta y}\}$ such that $g(\cdot, \eta) \in L^2_{\pm a}(\mathbb{R})$ is a solution of $\mathcal{L}(\eta)u = \lambda u$.

**Lemma 2.1.** Let $\eta \in \mathbb{R} \setminus \{0\}$, $\beta(\eta) = \sqrt{1 + \eta^2}$, $\lambda(\eta) = 4\eta \beta(\eta)$ and

$$
g(x, \eta) = \frac{-i}{2\eta \beta(\eta)} \partial_x^2 e^{-\beta(\eta)x} \mathrm{sech} x, \quad g^*(x, \eta) = \partial_x(e^{\beta(\eta)x} \mathrm{sech} x).
$$

Then

$$
(2.4) \quad \mathcal{L}(\eta)g(x, \pm \eta) = \lambda(\pm \eta)g(x, \pm \eta),
$$

$$
(2.5) \quad \mathcal{L}(\eta)g^*(x, \pm \eta) = \lambda(\mp \eta)g^*(x, \pm \eta),
$$

$$
(2.6) \quad \int_{\mathbb{R}} g(x, \eta)g^*(x, \eta) \, dx = 1, \quad \int_{\mathbb{R}} g(x, \eta)g^*(x, -\eta) \, dx = 0.
$$
Lemma 2.1 will be proved in Subsection 2.2.

To resolve the singularity of \( g(x, \eta) \) and the degeneracy of \( g_\nu(x, \eta) \) at \( \eta = 0 \), we decompose resonant modes and adjoint resonant modes into their real parts and imaginary parts. Let

\[
\begin{align*}
  g_1(x, \eta) &= g(x, \eta) + g(x, -\eta), \quad g_2(x, \eta) = i\eta \{ g(x, \eta) - g(x, -\eta) \}, \\
  g_1^*(x, \eta) &= \frac{1}{2} \{ g^*(x, \eta) + g^*(x, -\eta) \}, \quad g_2^*(x, \eta) = \frac{i}{2\eta} \{ g^*(x, \eta) - g^*(x, -\eta) \}.
\end{align*}
\]

Then we have the following.

**Lemma 2.2.**

\[
\int_{\mathbb{R}} g_j(x, \eta) \overline{g_k^*(x, \eta)} \, dx = \delta_{jk} \quad \text{for } j, k = 1, 2.
\]

\[
\begin{align*}
  \mathcal{L}(\eta) g_1(x, \eta) &= \Re(\lambda(\eta)) g_1(x, \eta) + \frac{3\lambda(\eta)}{\eta} g_2(x, \eta), \\
  \mathcal{L}(\eta) g_2(x, \eta) &= -\eta \Im(\lambda(\eta)) g_1(x, \eta) + \Re(\lambda(\eta)) g_2(x, \eta), \\
  \mathcal{L}(\eta)^* g_1^*(x, \eta) &= \Re(\lambda(\eta)) g_1^*(x, \eta) - \eta \Im(\lambda(\eta)) g_2^*(x, \eta), \\
  \mathcal{L}(\eta)^* g_2^*(x, \eta) &= \frac{3\lambda(\eta)}{\eta} g_1^*(x, \eta) + \Re(\lambda(\eta)) g_2^*(x, \eta).
\end{align*}
\]

**Proof.** Lemma 2.2 follows immediately from Lemma 2.1 since \( \overline{\lambda(\eta)} = \lambda(-\eta) \).

We remark that \( \mathcal{L}(0) \) coincides with the linearized operator of the KdV equation around the 1-soliton \( \varphi(x - 4t) \) and that \( g_k(x, 0) \in \ker_g(\mathcal{L}(0)) \), \( g_k^*(x, 0) \in \ker_g(\mathcal{L}(0)^*) \) for \( k = 1 \) and 2.

**Claim 2.1.** Let \( a \in (0, 2) \) and \( \nu(\eta) = \Re(\beta(\eta)) - 1 \). Let \( \eta_0 \) be a positive number such that \( \nu_0 := \nu(\eta_0) < a \). Then for \( \eta \in [-\eta_0, \eta_0] \),

\[
\begin{align*}
  g_1(x, \eta) &= \frac{1}{4} \phi' + \frac{x}{4} \phi' + \frac{1}{2} \phi + O(\eta^2), \quad g_2(x, \eta) = -\frac{1}{2} \phi' + O(\eta^2) \quad \text{in } L^2_a(\mathbb{R}), \\
  g_1^*(x, \eta) &= \frac{1}{2} \phi + O(\eta^2), \quad g_2^*(x, \eta) = \int_{-\infty}^{x} \partial_x \varphi dx + O(\eta^2) \quad \text{in } L^2_{-a}(\mathbb{R}),
\end{align*}
\]

where \( \partial_x \varphi = \partial_{c\varphi} |_{c=2} \).

**Proof.** Since \( g_1(x, \eta) \) and \( g_2(x, \eta) \) are even in \( \eta \),

\[
\begin{align*}
  g_1(x, \eta) &= \frac{1}{2i\eta} \partial_x^2 \left\{ \left( e^{-\beta(\eta)x} \frac{e^{-\beta(-\eta)x}}{\beta(\eta) - \beta(-\eta)} \right) \sech x \right\} \\
  &= \partial_x \left( e^{-\sqrt{s} x} \frac{\sech x}{\sqrt{s}} \right) \bigg|_{s=1} + O(\eta^2) \\
  &= \frac{x}{4} \varphi' + \frac{1}{2} \varphi + O(\eta^2),
\end{align*}
\]

and

\[
  g_2(x, \eta) = (e^{-x} \sech x)_{xx} + O(\eta^2) = -\frac{1}{2} \varphi' + O(\eta^2).
\]

We can compute \( g_1^*(x, \eta) \) and \( g_2^*(x, \eta) \) in the same way. \( \square \)
2.2. Linearized Miura transformation. Now we recall the Miura transformation of the KP-II equation. Let
\[ M_\pm^c(v) = \pm \partial_x v + \partial_x^{-1} \partial_y v - v^2 + \frac{c}{2}. \]
The transformations \( M_+^c \) relate the KP-II equation to the mKP-II equation (mKP-II) which reads
\[
\partial_t v + \partial_y^2 v + 3\partial_x^{-1} \partial_y^2 v - 6u^2 \partial_x v + 6\partial_x v \partial_y^{-1} \partial_y v = 0.
\]
Formally, if \( v(t, x, y) \) is a solution of (2.7) and \( c > 0 \), then \( M_+^c(v)(t, x, c) \) are solutions of the KP-II equation (1.1). A line soliton solution \( \varphi_c(x - ct) \) of the KP-II equation is related to a kink solution \( Q_c(x) \) of (2.7), where \( Q_c(x) = \sqrt{\frac{c}{2}} \text{tanh} \left( \sqrt{\frac{c}{2}} x \right) \). Indeed, we have
\[
M_+^c(Q_c) = \varphi_c, \quad M_-^c(Q_c) = 0.
\]
From now on, let \( c = 2 \), \( Q = Q_2 \) and \( M_\pm = M_\pm^2 \). Let \( v(t, x, y) = Q(x + 2t) + V(t, x + 2t, y) \) and linearize (2.7) around \( V = 0 \). Then
\[
\partial_t V = L_M V,
\]
\[
L_M V := -\partial_y^3 V - 2\partial_x V - 3\partial_x^{-1} \partial_y^2 V + 6\partial_x (Q^2 V) - 6Q' \partial_x^{-1} \partial_y V
\]
\[
- \partial_y^3 V + 4\partial_x V - 3\partial_x^{-1} \partial_y^2 V - 6\partial_x (Q' V) - 6Q' \partial_x^{-1} \partial_y V.
\]
In the last line, we use \( Q' = 1 - Q^2 \). Let \( X_M \) be the Banach space equipped with the norm \( \|v\|_{X_M} := (\|v\|_X^2 + \|\partial_x v\|_X^2 + \|\partial_x^{-1} \partial_y v\|_X^2)^{1/2} \). Thanks to the smoothing effect of \( L_0 \) in \( X \) (see Lemma 3.4 in Section 3), the initial value problem
\[
\partial_t v = L_M v, \quad v(0) = v_0
\]
has a unique solution in the class \( C([0, \infty); X_M) \).

Solutions of (2.3) are related to those of (2.9) by the linearized Miura transformation
\[
\partial_t v = \nabla \nabla_\pm(Q) v = \partial_x v + \partial_x^{-1} \partial_y v - 2Qv.
\]
Another linearized Miura transformation
\[
\partial_t u + \partial_y^3 u - 4\partial_x u + 3\partial_x^{-1} \partial_y^2 u = 0.
\]
Lemma 2.3. Suppose that \( v \) is a solution to (2.9). Then \( u = \nabla M_\pm(Q)v \) satisfies (2.3) and \( u = \nabla M_\pm(Q)v \) satisfies (2.12).

Proof of Lemma 2.3. By a straightforward computation, we find that
\[
L_0 \nabla M_\pm(Q) = \nabla M_\pm(Q)L_M, \quad L_0 \nabla M_-^c(Q) = \nabla M_-^c(Q)L_M.
\]
Let \( u_\pm = \nabla M_\pm(Q)v \). Then it follows from (2.13) that
\[
\partial_t u_\pm - L_u_\pm = \nabla M_\pm(Q)(\partial_t v - L_M v),
\]
\[
\partial_t u_- - L_u_- = \nabla M_-^c(Q)(\partial_t v - L_M v).
\]
Therefore \( u_\pm \) and \( u_- \) are solutions of (2.3) and (2.12), respectively, if \( v \) is a solution to (2.9). Thus we complete the proof.

Lemma 2.4. Let \( a > 0 \) and \( v(t) \in C([0, \infty); X_M) \) be a solution to (2.9).
(1) Suppose that \( u(t) \in C([0, \infty); X) \) is a solution to (2.13) satisfying \( u(0) = \nabla M_+(Q)v(0) \). Then \( u(t) = \nabla M_+(Q)v(t) \) holds for every \( t \geq 0 \).

(2) Suppose that \( u(t) \in C([0, \infty); X) \) is a solution to (2.12) satisfying \( u(0) = \nabla M_-(Q)v(0) \). Then \( u(t) = \nabla M_-(Q)v(t) \) holds for every \( t \geq 0 \).

**Proof.** Let \( \tilde{u}(t) = \nabla M_+(Q)v(t) \). Then \( \tilde{u}(t) \in C([0, \infty); X) \). Moreover, Lemma 2.7 implies that \( \tilde{u}(t) \) is a solution of (2.13). Since \( u(0) = \tilde{u}(0) \) and both \( u(t) \) and \( \tilde{u}(t) \) are solutions of (2.13) in the class \( C([0, \infty); X) \), we have \( u(t) = \tilde{u}(t) \). Thus we prove (i). We can prove (ii) in exactly the same way. This completes the proof of Lemma 2.4.

Let

\[ L_M(\eta)v := -\partial_x^2 v + 4\partial_x v + 3\eta^2 \partial_x^{-1} v - 6\partial_x(Q'v) - 6i\eta Q' \partial_x^{-1} v. \]

Then \( L_M(v(x)e^{i\eta y}) = e^{i\eta y} L_M(\eta)v(x) \) and \( L_M(\eta) \) has the following resonant modes.

**Lemma 2.5.** Let \( \eta \in \mathbb{R} \) and

\[ g_M(x, \eta) = -\frac{1}{2\beta(\eta)} \partial_x(e^{-\beta(\eta)x} \text{sech } x), \quad g^*_M(x, \eta) = e^{\beta(-\eta)x} \text{sech } x. \]

Then

\begin{align*}
(2.14) & \quad L_M(\eta)g_M(x, \eta) = \lambda(\eta)g_M(x, \eta), \\
(2.15) & \quad L_M(\eta)^* g^*_M(x, \eta) = \lambda(-\eta)g^*_M(x, \eta), \\
(2.16) & \quad \int_\mathbb{R} g_M(x, \eta)g^*_M(x, \eta) \, dx = 1.
\end{align*}

The eigenvalue problem \( Lu = \lambda u \) is related to the eigenvalue problem \( L_Mv = \lambda v \) via (2.13). Before we prove Lemmas 2.4 and 2.5, we will investigate the kernel and the cokernel of bounded operators \( M_\pm(\eta) : H^1_\alpha(\mathbb{R}) \to L^2_\alpha(\mathbb{R}) \) defined by

\[ M_\pm(\eta)g(x) := \pm g'(x) - i\eta \int_x^\infty g(t) \, dt - 2Q(x)g(x). \]

**Lemma 2.6.** Let \( a \in (0, 2) \) and \( \eta_0 \) be a positive number satisfying \( a > \eta_0 \). Then \( \ker(M_+(\eta)) = \text{span}\{g^*(:, \eta)\} \) and \( \text{Range}(M_+(\eta)) = L^2_\alpha(\mathbb{R}) \). Moreover, for any \( \eta \in [-\eta_0, \eta_0] \) and \( f \in L^2_\alpha(\mathbb{R}) \), there exists a unique solution \( v \in H^1_\alpha(\mathbb{R}) \) of

\[ M_+(\eta)v = f, \]

that satisfies \( \int_\mathbb{R} v(x)g^*_M(x, \eta) \, dx = 0 \). Moreover,

\[ ||v||_{H^1_\alpha(\mathbb{R})} + ||\eta||\partial_x^{-1}v||_{L^2_\alpha(\mathbb{R})} \leq \frac{C}{a - \nu(\eta_0)} ||f||_{L^2_\alpha(\mathbb{R})}, \]

where \( C \) is a constant depending only on \( a \).

**Lemma 2.7.** Let \( a \in (0, 2) \) and \( \eta_0 \) be a positive number satisfying \( a > \eta_0 \). If \( \eta \in [-\eta_0, \eta_0] \), then \( \ker(M_-(\eta)) = \{0\} \) and \( \text{Range}(M_+(\eta)) = \text{span}\{g^*(x, -, \eta)\} \). Moreover, for any \( f \in L^2_\alpha(\mathbb{R}) \) satisfying \( \int_\mathbb{R} f(x)g^*(x, -\eta) \, dx = 0 \), there exists a unique solution \( v \in H^1_\alpha(\mathbb{R}) \) of

\[ M_-(\eta)v = f, \]

satisfying

\[ ||v||_{H^1_\alpha(\mathbb{R})} + ||\eta||\partial_x^{-1}v||_{L^2_\alpha(\mathbb{R})} \leq C ||f||_{L^2_\alpha(\mathbb{R})}, \]
where $C$ is a constant depending only on $\alpha$. If $f$ satisfies $\int_{\mathbb{R}} f(x) g^*(x, \eta) \, dx = 0$ in addition, then $\int_{\mathbb{R}} v(x) g_M^*(x, \eta) \, dx = 0$.

Proof of Lemma 2.20: Suppose $v \in \text{ker}(\mathcal{M}_-(\eta))$. Then $v \in H^1_{L}(\mathbb{R})$ and
\begin{equation}
(2.20)\quad -v'' + i\nu v - 2(Qv)' = 0.
\end{equation}

Eq. (2.20) has a fundamental system $\{\tilde{g}_1, \tilde{g}_2\}$, where
\begin{equation*}
\tilde{g}_1(x) = \left(e^{-\beta(x)} \text{sech} x \right)_x, \quad \tilde{g}_2(x) = \left(e^{\beta(x)} \text{sech} x \right)_x.
\end{equation*}

Since $1 \leq \Re \beta(\nu) \leq \Re \beta(\eta_0)$ and
\begin{equation}
(2.21)\quad \tilde{g}_1(x) \sim e^{-(\beta(\eta) \pm 1)x} \quad \text{and} \quad \tilde{g}_2(x) \sim e^{(\beta(\eta) \mp 1)x} \quad \text{as} x \to \pm \infty,
\end{equation}

it follows that $\tilde{g}_1 \in H^1_{L}(\mathbb{R})$ and $\tilde{g}_2 \not\in H^1_{L}(\mathbb{R})$ and that $v(x) = \alpha \tilde{g}_1(x)$ for an $\alpha$. Thus we prove $\text{ker}(\mathcal{M}_-(\eta)) = \text{span}\{g_M(\cdot, \eta)\}$.

Suppose $v \in H^1_{L}(\mathbb{R})$ is a solution of (2.17). Then $v$ satisfies an ODE
\begin{equation}
(2.22)\quad -v'' + i\nu v - 2(Qv)' = f'.
\end{equation}

By the variation of the constants formula,
\begin{equation*}
v(x) = \tilde{g}_1(x) \int_x^\infty \frac{\tilde{g}_2(t)'(t)}{W(t)} \, dt - \tilde{g}_2(x) \int_x^\infty \frac{\tilde{g}_1(t)'(t)}{W(t)} \, dt = \tilde{g}_1(x) \int_x^\infty k_1'(t) f(t) \, dt + \tilde{g}_2(x) \int_x^\infty k_2'(t) f(t) \, dt,
\end{equation*}

where $W(t) = \tilde{g}_1(t) \tilde{g}_2(t) - \tilde{g}_1(t) \tilde{g}_2(t) = -2i\eta \beta(\eta) \text{sech}^2 t$,
\begin{align*}
k_1(t) &= -\frac{\tilde{g}_2(t)}{W(t)} = \frac{e^{\beta(t)}(\beta(t) \cosh t - \sinh t)}{2i\eta \beta(\eta)}, \\
k_2(t) &= \frac{\tilde{g}_1(t)}{W(t)} = \frac{e^{-\beta(t)}(\beta(t) \cosh t + \sinh t)}{2i\eta \beta(\eta)},
\end{align*}

$k_1'(t) = (2\beta(\eta))^{-1} e^{\beta(t)} \cosh t$ and $k_2'(t) = -(2\beta(\eta))^{-1} e^{-\beta(t)} \cosh t$. Now let
\begin{equation}
(2.23)\quad v(x) = \alpha \tilde{g}_1(x) + T_1(f) + T_2(f), \\
\quad T_1(f) = -\tilde{g}_1(x) \int_x^\infty k_1'(t) f(t) \, dt, \quad T_2(f) = -\tilde{g}_2(x) \int_x^\infty k_2'(t) f(t) \, dt,
\end{equation}

where $\alpha$ is a constant to be chosen later. Since $\text{sech} x \cosh t \leq e^{t-x}$ for $t \in [x, \infty)$ and $\nu(\eta) \leq \nu_0$ for $\eta \in [-\eta_0, \eta_0]$,
\begin{equation*}
|\tilde{g}_1(x) k_1'(t)| \lesssim e^{\nu_0(t-x)} \quad \text{if} \quad t \geq x.
\end{equation*}

Using Young’s inequality and the above, we have
\begin{equation*}
\|T_1(f)\|_{L^2(\mathbb{R})} \lesssim \left\| \int_x^\infty e^{\nu(t-x)} |f(t)| \, dt \right\|_{L^2(\mathbb{R})} \lesssim \|e^{-(a-\nu_0)t}\|_{L^1(0, \infty)} \|f\|_{L^2(\mathbb{R})} \leq \frac{C_0}{a - \nu_0} \|f\|_{L^2(\mathbb{R})},
\end{equation*}
where $C_0$ is a constant independent of $\eta_0$ and $f \in L^2_\alpha(\mathbb{R})$. Using the fact that $0 \leq \cosh t \sech x \leq e^{t-x}$ if $x \leq t$ and that $\nu(\eta) \geq 0$, we have

$$\|T_2(f)\|_{L^2(\mathbb{R})} \lesssim \left\| \int_0^\infty e^{\nu(\eta)(x-t)} |f(t)| \, dt \right\|_{L^2_\alpha(\mathbb{R})} \lesssim e^{-(\alpha + \nu(\eta))t} \|f\|_{L^2(0, \infty)} \leq C_1 \|f\|_{L^2_\alpha(\mathbb{R})},$$

where $C_1$ is a constant independent of $\eta_0$ and $f \in L^2_\alpha(\mathbb{R})$. Since

$$\int_\mathbb{R} \tilde{g}_1(x)\overline{g}_M(x, \eta) \, dx = - \int_\mathbb{R} \sech^2 x (\beta(\eta) - \tanh x) \, dx = -2\beta(\eta) \neq 0,$$

there exists a unique $\alpha$ such that $\int v(x) g^*(x, \eta) \, dx = 0$. Since $L^2_\alpha(\mathbb{R}) \ni f \mapsto T_1(f)$, $T_2(f) \in L^2_\alpha(\mathbb{R})$ are continuous, $\alpha = \alpha(f)$ is also continuous in $f$. Thus we prove that there exists a constant $C_2$ such that

$$\|v\|_{L^2_\alpha(\mathbb{R})} \leq C_2 \|f\|_{L^2_\alpha(\mathbb{R})} \tag{2.24}$$

for every $\eta \in [-\eta_0, \eta_0] \setminus \{0\}$ and $f \in L^2_\alpha(\mathbb{R})$.

Differentiating (2.23) with respect to $x$, we have

$$v'(x) = \alpha \tilde{g}_1'(x) - f(x) - \tilde{g}_1'(x) \int_x^\infty k_1'(t) f(t) \, dt - \tilde{g}_2'(x) \int_x^\infty k_2'(t) f(t) \, dt.$$

We can prove

$$\|v'(x)\|_{L^2_\alpha(\mathbb{R})} \leq \frac{C_3}{\alpha - \nu_0} \|f\|_{L^2_\alpha(\mathbb{R})}, \tag{2.25}$$

in the same way as (2.24), where $C_3$ is a positive constant independent of $\eta_0$ and $f \in L^2_\alpha(\mathbb{R})$. Combining (2.24) and (2.25) with (2.17), we have

$$|\eta| \|\partial_x^{-1} v\|_{L^2_\alpha(\mathbb{R})} \leq \|v'(x)\|_{L^2_\alpha(\mathbb{R})} + 2 \|Q v\|_{L^2_\alpha(\mathbb{R})} + \|f\|_{L^2_\alpha(\mathbb{R})} \leq \frac{C_4}{\alpha - \nu_0} \|f\|_{L^2_\alpha(\mathbb{R})},$$

where $C_4$ is a positive constant independent of $\eta_0$ and $f \in L^2_\alpha(\mathbb{R})$. Thus we complete the proof.

**Proof of Lemma 2.24** First, we will show that $\ker (\mathcal{M}_+(\eta)^*) = \mathrm{span}\{\tilde{g}_2(x)\}$. Since $\mathcal{M}_-(\eta)$ is formally an adjoint of $\mathcal{M}_+(\eta)$, we easily see that $h \in \ker(\mathcal{M}_+(\eta)) \subset L^2_{-\alpha}(\mathbb{R})$ is a solution of (2.22) and that $h(x) = \alpha \tilde{g}_2(x) = \alpha g^*(x, -\eta)$ for an $\alpha \in \mathbb{C}$. Since $\ker(\mathcal{M}_+(\eta)^*) = \mathrm{span}\{\tilde{g}_2(x)\}$, we have $\mathrm{Range}(\mathcal{M}_+(\eta)) \subset \mathrm{span}\{g^*(x, -\eta)\}$.

Next we will show that $\ker((\mathcal{M}_+(\eta)) = \{0\}$. Suppose $\mathcal{M}_+(\eta) h = 0$. Then

$$h'' - 2(Q h)' + i \eta h = 0. \tag{2.26}$$

Eq. (2.26) has a fundamental system $\{h_1(x), h_2(x)\}$, where

$$h_1(x) = e^{\beta(-\eta)x} \cosh x, \quad h_2(x) = e^{-\beta(-\eta)x} \sinh x.$$

Since

$$h_1(x) \sim e^{(\beta(-\eta)\pm 1)x}, \quad h_2(x) \sim e^{-(\beta(-\eta)\pm 1)x} \quad \text{as} \ x \to \pm \infty,$$

it is clear that $h \in H^1_\alpha(\mathbb{R})$ if and only if $h = 0$. Thus we prove $\ker(\mathcal{M}_+(\eta)) = \{0\}$.

Secondly, we will show that $\mathrm{Range}(\mathcal{M}_+(\eta)) = \mathrm{span}\{g^*(x, -\eta)\}$. Suppose that $v \in H^1_\alpha(\mathbb{R})$ is a solution of (2.18). Then

$$v'' - 2(Q v)' + i \eta v = f'. \tag{2.28}$$
By the variation of constants formula, we can find the following solution of (2.28).

\begin{equation}
(2.29) \quad v(x) = T_3(f) + T_4(f),
\end{equation}

\begin{align*}
T_3(f) &:= \frac{e^{\beta(-\eta)x} \cosh x}{2\beta(-\eta)} \int_x^\infty \left(e^{-\beta(-\eta)t} \text{sech} t\right)_t f(t) \, dt, \\
T_4(f) &:= \frac{e^{-\beta(-\eta)x} \cosh x}{2\beta(-\eta)} \int_x^\infty \left(e^{\beta(-\eta)t} \text{sech} t\right)_t f(t) \, dt.
\end{align*}

Since \( \cosh x \text{sech} t \leq e^{|x-t|} \), we have

\begin{align*}
\|T_3(f)\|_{L^2_t(R)} &\leq \left\| \int_x^\infty e^{\nu(x-t)} |f(t)| \, dt \right\|_{L^2_t(R)} \\
&\leq C_1 \|f\|_{L^2_t(R)},
\end{align*}

where \( C_1 \) is a constant depending only on \( a \). If \( \int_R f(x)g^*(x,-\eta) \, dx = 0 \), then \( T_4(f) \) can be rewritten as

\begin{equation}
(2.31) \quad T_4(f) = \frac{e^{-\beta(-\eta)x} \cosh x}{2\beta(-\eta)} \int_x^\infty \left(e^{\beta(-\eta)t} \text{sech} t\right)_t f(t) \, dt.
\end{equation}

Using (2.29) for \( x \geq 0 \) and (2.31) for \( x \leq 0 \) and the fact that \( \cosh x \text{sech} t \leq 2e^{-|x-t|} \) for \( t \) satisfying \( |t| \geq |x| \), we have

\begin{align*}
\|T_4(f)\|_{L^2_t(R)} &\leq \left\| \int_x^\infty e^{(\nu(x-t)) |f(t)|} \, dt \right\|_{L^1(0,\infty)} \\
&\quad + \left\| \int_{-\infty}^x e^{(\nu(x-t)) - 2(x-t) |f(t)|} \, dt \right\|_{L^1(-\infty,0)} \\
&\leq C_2 \|f\|_{L^2_t(R)},
\end{align*}

where \( C_2 \) is a constant depending only on \( a \). Thus we prove that (2.18) has a unique solution \( v \in L^2_t(R) \). We can prove (2.19) in the same way as Lemma 2.6.

Suppose \( f \) satisfies \( \int_R f(x)g^*(x,\pm\eta) \, dx = 0 \), then it follows from (2.29) and (2.18) that

\begin{align*}
2i\nu \int_R v(x)g^*_M(x,\eta) \, dx &= -\int_R \mathcal{M}_+(\eta)v(x)g^*(x,\eta) \, dx \\
&= -\int_R f(x)g^*(x,\eta) = 0.
\end{align*}

Thus we have \( \int_R v(x)g^*_M(x,\eta) \, dx = 0 \) for \( \eta \in [-\eta_0,\eta_0] \setminus \{0\} \). This completes the proof of Lemma 2.7. \( \square \)

Now we are in position to prove Lemmas 2.1 and 2.5.

**Proof of Lemmas 2.1 and 2.5.** First, we will show that \( \nabla \mathcal{M}_+(Q)g_M(x,\eta)e^{i\eta y} \) are the resonant eigenmodes of \( \mathcal{L} \) and that \( \nabla \mathcal{M}_-(Q)(g_M(x,\eta)e^{i\eta y}) = 0 \) by using (2.13).

Let \( \mathcal{L}_0 u := -\partial_x^2 u + 4\partial_x u + 3\eta^2 \partial_x^{-1} u \) be the operator on \( L^2_\alpha(R) \) with its domain \( D(\mathcal{L}_0(\eta)) = H^3_\alpha(R) \). By the definition of \( \mathcal{L}(\eta) \) and \( \mathcal{M}_\pm(\eta) \), we have \( \mathcal{L}_0(u(x)e^{\pm i\eta y}) = e^{\pm i\eta y}\mathcal{L}_0(\eta)u(x) \) and

\[ \nabla \mathcal{M}_\pm(Q)(g(x)e^{i\eta y}) = (\mathcal{M}_\pm(\eta)g)(x)e^{i\eta y}. \]
In view of (2.13),

\begin{equation}
\mathcal{L}(\eta)\mathcal{M}_+(\eta) = \mathcal{M}_+(\eta)\mathcal{L}(\eta),
\end{equation}

\begin{equation}
\mathcal{L}(\eta)\mathcal{M}_-(\eta) = \mathcal{M}_-(\eta)\mathcal{L}(\eta).
\end{equation}

By a simple computation, we find

\begin{equation}
\mathcal{M}_+(\eta)g_M(x, \eta) = -2i\eta g(x, \eta), \quad \mathcal{M}_-(\eta)g_M(x, \eta) = 0.
\end{equation}

Combining (2.33) and (2.34), we have

\begin{equation}
\mathcal{M}_-(\eta)\mathcal{L}(\eta)g_M(x, \eta) = \mathcal{L}(\eta)\mathcal{M}_-(\eta)g_M(x, \eta) = 0.
\end{equation}

Since \(g_M(x, \eta) \in H^1_c(\mathbb{R})\) for an \(a \in (\nu(\eta), 2)\), we have \(\mathcal{L}(\eta)g_M(x, \eta) \in H^1_c(\mathbb{R})\) and \(\mathcal{L}(\eta)g_M(x, \eta) \in \ker \mathcal{M}_-(\eta).\) Lemma 2.6 implies that there exists a \(\lambda(\eta) \in \mathbb{C}\) such that \(\mathcal{L}(\eta)g_M(x, \eta) = \lambda(\eta)g_M(x, \eta)\). Since \(g_M(x, \eta) \sim e^{-(1+\beta(\eta))x}\) as \(x \to \infty\), we see that

\[\lambda(\eta) = (1 + \beta(\eta))^3 - 4(1 + \beta(\eta)) - \frac{3\eta^2}{1 + \beta(\eta)} = 4i\eta\beta(\eta).\]

Thus we prove (2.14). It follows from (2.14), (2.32) and (2.34) that

\[\mathcal{L}(\eta)g(x, \eta) = \frac{i}{2\eta}\mathcal{M}_+(\eta)\mathcal{L}(\eta)g_M(x, \eta)\]

\[= i(1 + \beta(\eta))^3 - 4(1 + \beta(\eta)) - \frac{3\eta^2}{1 + \beta(\eta)} = 4i\eta\beta(\eta),\]

and \(\mathcal{L}(\eta)g(x, -\eta) = \mathcal{L}(\eta)g(x, \eta) = \lambda(-\eta)g(x, -\eta).\)

Using the fact that \(\partial_x \mathcal{L}(\eta)^* = -\mathcal{L}(\eta)\partial_x\) (formally) and \(\varphi\) is even, we can easily confirm (2.20). Since \(g_M(x, \eta)\) is a solution of (2.21) and

\[g^*(x, \eta) = -2\beta(-\eta)g_M(-x, -\eta), \quad Q(-x) = -Q(x),\]

we have \(\partial_x^2 g^*(x, \eta) + 2\partial_x(Q(x)g^*(x, \eta)) + i\eta g^*(x, \eta) = 0\). Combining the above with \(g^*(x, \eta) = \partial_x g^*_M(x, \eta),\) we have

\begin{equation}
\mathcal{M}_+(\eta)^*g^*(x, \eta) = -\partial_x g^*(x, \eta) + i\eta \int_{-\infty}^x g^*(t, \eta) dt - 2Q(x)g^*(x, \eta)
\end{equation}

\[= 2i\eta g^*_M(x, \eta).
\]

By (2.32),

\begin{equation}
M_+(\eta)^*\mathcal{L}(\eta)^* = \mathcal{L}(\eta)^*\mathcal{M}_+(\eta)^*.
\end{equation}

Eq. (2.14) follows immediately from (2.25), (2.35) and (2.36).

Next, we will prove (2.16). By integration by parts,

\[\int g_M(x, \eta)g_M^*(x, \eta) dx = \frac{1}{2\beta(\eta)} \int ((\beta(\eta) + \tanh x) \sech^2 x) dx = 1.\]
Finally, we will prove (2.6). By (2.34) and (2.35),
\[
2\nu \int_{\mathbb{R}} g(x, \eta) g^*(x, \eta) \, dx = -\int_{\mathbb{R}} \mathcal{M}_+(\eta) g_M(x, \eta) g^*(x, \eta) \, dx
= -\int_{\mathbb{R}} g_M(x, \eta) \mathcal{M}_+ g^*(x, \eta) \, dx
= 2\nu \int_{\mathbb{R}} g_M(x, \eta) g_M^*(x, \eta) \, dx = 2\nu.
\]

Thus we prove (2.6) for \( \eta \neq 0 \).

If \( \eta \) is large, the operators \( \mathcal{M}_\pm(\eta) : H^1_{\eta}(\mathbb{R}) \rightarrow L^2_{\eta}(\mathbb{R}) \) have bounded inverse.

**Lemma 2.8.** Suppose \( a \in (0, 2) \), \( \eta > 0 \) and \( \nu(\eta) > a \).

1. For every \( f \in L^2_\eta(\mathbb{R}) \), there exists a unique solution \( v_+ \) of (2.17) satisfying
\[
\|v_+\|_{H^1_{\eta}(\mathbb{R})} + |\nu(\eta)| \|\partial_x^{-1} v_+\|_{L^2_{\eta}(\mathbb{R})} \leq C \frac{|\nu(\eta)| - a}{\|\nu(\eta) - a\|} \|f\|_{L^2_{\eta}(\mathbb{R})},
\]
where \( C \) is a constant depending only on \( a \).
2. For every \( f \in L^2_\eta(\mathbb{R}) \), there exists a unique solution \( v_- \) of (2.18) satisfying
\[
\|v_-\|_{H^1_{\eta}(\mathbb{R})} + |\nu(\eta)| \|\partial_x^{-1} v_-\|_{L^2_{\eta}(\mathbb{R})} \leq C \frac{|\nu(\eta)| - a}{\|\nu(\eta) - a\|} \|f\|_{L^2_{\eta}(\mathbb{R})},
\]
where \( C \) is a constant depending only on \( a \).

**Proof of Lemma 2.8.** If \( \nu(\eta) > a > 0 \), then (2.21) and (2.27) imply \( \ker(\mathcal{M}_\pm(\eta)) = \{0\} \) and that (2.17) and (2.18) have at most one solution.

First we prove (1). Let
\[
v_+(x) = g_1(x) \int_{-\infty}^{x} k_1(t) f(t) \, dt + g_2(x) \int_{x}^{\infty} k_2(t) \, dt.
\]

Then \( v(x) \) is a solution of (2.17). Since \( |g_1(x)k_1(t)| + |g_2(x)k_2(t)| \leq e^{-\nu(\eta)|x-t|} \) if \( x > t \) and \( |g_2(x)k_2(t)| \leq e^{\nu(\eta)|x-t|} \) if \( x < t \), we have
\[
|v_+(x)| + |\partial_x v_+(x)| \lesssim \int_{\mathbb{R}} e^{-\nu(\eta)|x-t|} |f(t)| \, dt.
\]

Using Young’s inequality, we have
\[
\|v_+\|_{L^2_{\eta}(\mathbb{R})} + \|\partial_x v\|_{L^2_{\eta}(\mathbb{R})} \lesssim \|e^{-\nu(\eta)|x|} \|_{L^1(\mathbb{R})} \|f\|_{L^2_{\eta}(\mathbb{R})} \lesssim (\nu(\eta) - a)^{-1} \|f\|_{L^2_{\eta}(\mathbb{R})}
\]

Thus we can prove (2.37) in the same way as the proof of Lemma 2.6.

Now we prove (2). Let \( v_- = T_3(f) + T_4(f) \). Obviously, \( v_- \) is a solution of (2.18) satisfying
\[
|v_-(x)| + |\partial_x v_-(x)| \lesssim \int_{\mathbb{R}} e^{-\nu(\eta)|x-t|} |f(t)| \, dt.
\]

Thus we can prove (2.38) in the same way as (2.37). This completes the proof of Lemma 2.8.

Using Lemmas 2.6, 2.7, and 2.8, we will investigate the spectrum \( \sigma(\mathcal{L}(\eta)) \) of \( \mathcal{L}(\eta) \).

**Lemma 2.9.** Let \( a \in (0, 2) \) and \( \eta_* \) be a positive number satisfying \( \nu(\eta_*) = a \).

1. If \( \eta \in (-\eta_, \eta_*) \), then \( \mathcal{L}(\eta) \) has no eigenvalue other than \( \lambda(\pm \eta) \) and
\[
\sigma(\mathcal{L}(\eta)) = \{\lambda(\pm \eta_*)\} \cup \{i\rho(\xi + ia, \eta) : \xi \in \mathbb{R}\}.
\]
2. If \( \eta \in \mathbb{R} \setminus [-\eta_, \eta_*] \), then \( \sigma(\mathcal{L}(\eta)) = \{i\rho(\xi + ia, \eta) : \xi \in \mathbb{R}\} \).
Proof of Lemma 2.9. The equation $\nu(\eta) = a$ has a unique positive root $\eta_*$ because $\nu(\eta)$ is monotone increasing for $\eta \geq 0$, $\nu(0) = 0$ and $\nu(\infty) = \infty$.

Since $\lambda - L(\eta)$ and $\lambda - L_0(\eta)$ are invertible for large $\lambda > 0$ and $(\lambda - L(\eta))^{-1} - (\lambda - L_0(\eta))^{-1}$ is compact, it follows from the Weyl essential spectrum theorem that

$$\sigma(L(\eta)) \setminus \sigma_p(L(\eta)) = \{ ip(\xi + ia) \mid \xi \in \mathbb{R} \}.$$ 

Suppose that $\eta \in (-\eta_*, \eta_*)$ and that $L(\eta)u = \nu u$ for some $u \in H^4_\alpha(\mathbb{R})$ and $\lambda \in \mathbb{C} \setminus \{ \lambda(\pm \eta) \}$. Then

$$\int_{\mathbb{R}} u(x) g^*(x, \pm \eta) \, dx = 0. \tag{2.39}$$

Indeed, it follows from Lemma 2.1 that

$$(\lambda - \lambda(\pm \eta)) \int_{\mathbb{R}} u(x) g^*(x, \pm \eta) \, dx = \int_{\mathbb{R}} \{ \mu u(x) - (L(\eta)u(x)) \} g^*(x, \pm \eta) \, dx = 0.$$ 

Lemma 2.7 implies that there exists a solution $v \in H^4_\alpha(\mathbb{R})$ of $u = M_+(\eta)v$ satisfying $\int_{\mathbb{R}} v(x) g^*_M(x, \eta) \, dx = 0$. By (2.32),

$$M_+(\eta)(L_M(\eta)v - \lambda v) = (L(\eta) - \nu)M_+(\eta)v = L(\eta)u - \lambda u = 0.$$ 

Since $\ker(M_+(\eta)) = \{ 0 \}$, it follows that $L(\eta)v = \lambda v$. Using (2.33), we have

$$(L_0(\eta) - \lambda)M_-(\eta)v = M_-(\eta)(L_M(\eta)v - \lambda v) = 0, \tag{2.40}$$

whence $M_-(\eta)v = 0$ because (2.40) implies that the support of $F_x(M_-(\eta)v)(\xi)$ is contained in $\{ \xi \in \mathbb{R} \mid \xi^4 + 4\xi^2 + i\lambda \xi - \eta^2 = 0 \}$. Lemma 2.6 implies there exists an $\alpha \in \mathbb{C}$ such that $v(x) = \alpha g_M(x, \eta)$ and hence it follows from (2.34) that

$$u(x) = M_+(\eta)v = -2i\alpha g(x, \eta).$$

By Lemma 2.1 and (2.39)

$$\int_{\mathbb{R}} u(x) g^*(x, \eta) \, dx = -2i\alpha = 0,$$

whence $u = 0$. Thus we prove (1).

Suppose $\eta \in \mathbb{R} \setminus [-\epsilon a_*, \eta_*]$ and that $\mathcal{L}u = \nu u$ for some $u \in H^4_\alpha(\mathbb{R})$ and $\lambda \in \mathbb{C}$. Lemma 2.8 implies that there exists $v \in H^4_\alpha(\mathbb{R})$ satisfying $u = M_+(\eta)v$ and we can prove that $M_-(\eta)v = 0$ in the same way as the proof of (1). Since $M_-(\eta)$ has the bounded inverse, it follows that $v = 0$ and $u = M_+(\eta)v = 0$. Thus we complete the proof.

3. Semigroup estimates for the linearized KP-II equation

In this section, we will prove exponential decay estimates of solutions to (2.3). To begin with, we define a spectral projection to low frequency resonant modes. Let $P_0(\eta_0)$ be an operator defined by

$$P_0(\eta_0)f(x, y) = \frac{1}{2\pi} \sum_{k=1, 2} \int_{-\eta_0}^{\eta_0} a_k(\eta)g_k(x, \eta)e^{i\nu \eta} \, d\eta,$$
\[ a_k(\eta) = \int_{\mathbb{R}} \lim_{M \to \infty} \left( \int_{-M}^{M} f(x_1, y_1) e^{-iy_1 \eta} dy_1 \right) g_k^*_{\eta}(x_1, \eta) \, dx_1 \]
\[ = \sqrt{2\pi} \int_{\mathbb{R}} \mathcal{F}_y f(x, \eta) g_k^*_{\eta}(x, \eta) \, dx. \]

We will show that \( P_0(\eta_0) \) is a spectral projection on \( X = L^2(\mathbb{R}^2; e^{2ax} dx dy) \).

**Lemma 3.1.** Let \( a \in (0, 2) \) and \( \eta_1 \) be a positive constant satisfying \( \nu(\eta_1) < a \). If \( \eta_0 \in [-\eta_1, \eta_1] \), then

1. \( \|P_0(\eta_0) f\|_X + \|P_0(\eta_0) \partial_y f\|_X \leq C \|f\|_X \) for any \( f \in X \), where \( C \) is a positive constant depending only on \( a \) and \( \eta_1 \),
2. \( \|P_0(\eta_0) f\|_X + \|P_0(\eta_0) \partial_y f\|_X \leq C \|e^{ax} f\|_{L^1_t L^2_x} \) for any \( e^{ax} f \in L^1_t L^2_x \), where \( C \) is a positive constant depending only on \( a \) and \( \eta_1 \),
3. \( \mathcal{L} P_0(\eta_0) f = \mathcal{P}(\eta_0) \mathcal{L} f \) for any \( f \in D(\mathcal{L}) = \{ u \mid u, \partial_x^3 u, \partial_x \partial_y u \in X \} \),
4. \( P_0(\eta_0) = P_0(\eta_0) e^{\mathcal{L}} \) on \( X \),
5. \( e^{\mathcal{L}} P_0(\eta_0) = P_0(\eta_0) e^{\mathcal{L}} \) on \( X \).

**Proof.** First, we show (1). Since \( C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \) is dense in \( X \), we may assume \( f \in C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \). Let

\[ f_k(x, y) = \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} a_k(\eta) g_k(x, \eta) e^{i\eta y} \, d\eta \quad \text{for} \quad k = 1, 2. \]

By Plancherel’s theorem,

\[ \|f_k(x, y)\|_{L^2_y} = \frac{1}{\sqrt{2\pi}} \left( \int_{-\eta_0}^{\eta_0} |a_k(\eta)| g_k(x, \eta)|^2 \, d\eta \right)^{1/2} \]
\[ \leq \frac{1}{\sqrt{2\pi}} \sup_{\eta \in [-\eta_0, \eta_0]} |g_k(x, \eta)| \left( \int_{-\eta_0}^{\eta_0} |a_k(\eta)|^2 \, d\eta \right)^{1/2}. \]

If \( \nu(\eta_1) < a \), then it follows from the definition of \( g_k \) and \( g^*_k \) that there exists a positive constant \( C' \) such that for \( \eta \in [-\eta_1, \eta_1] \) and \( x \in \mathbb{R} \),

\[ |g_1(x, \eta)| \leq C'(x) e^{-2x + e^{\nu(\eta_1)x}}, \quad |g_2(x, \eta)| \leq C'e^{-2x + e^{\nu(\eta_1)x}}, \]
\[ |g^*_1(x, \eta)| \leq C'e^{\nu(\eta)x + e^{-2x}}, \quad |g^*_2(x, \eta)| \leq C'(x)e^{\nu(\eta)x} e^{-2x}, \]

where \( x_{\pm} = \max(\pm x, 0) \) and \( C' \) is a constant depending only \( \eta_1 \). Hence it follows from (3.1) and (3.2) that

\[ \|P_0(\eta_0) f\|_X \leq \sum_{k=1,2} \|\|f_k\|_{L^2_y}\|_{L^2_x} \]
\[ \leq C_1 \left( \int_{-\eta_0}^{\eta_0} (|a_1(\eta)|^2 + |a_2(\eta)|^2) \, d\eta \right)^{1/2}, \]

where \( C_1 \) is a constant depending only on \( a \) and \( \eta_1 \). Using the Schwarz inequality and (3.2), we have for \( \eta \in [-\eta_0, \eta_0] \),

\[ |a_k(\eta)| \leq \sqrt{2\pi} \|\mathcal{F}_y f(x, \eta)\|_{L^2_x} \|g^*_k(x, \eta)\|_{L^2_y} \leq C_2 \|\mathcal{F}_y f(x, \eta)\|_{L^2_x}, \]
where $C_2$ is a constant depending only on $a$ and $\eta_1$. Hence it follows that for any $f \in C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$,

$$\|P_0(\eta)f\|_X \leq C_1 C_2 \left( \int_{-\eta_0}^{\eta_0} \|\mathcal{F}_y f(x, \eta)\|_{L_x^2(\mathbb{R}_x)}^2 \, d\eta \right)^{1/2} = C_1 C_2 \|f\|_X.$$  

We can prove $\|P_0(\eta)\partial_x f\|_X \lesssim \|f\|_X$ in exactly the same way.  

Next we will prove (2). Using Minkowski’s inequality and applying (3.2) and Plancherel’s theorem to the resulting equation, we have

$$\|a_k\|_{L^2(-\eta_0, \eta_0)} \leq \sqrt{2\pi} \sup_{x \in \mathbb{R}, \eta \in [-\eta_0, \eta_0]} |e^{-ax}g_k^*(x, \eta)| \int_{\mathbb{R}} e^{ax}\|\mathcal{F}f(x, \cdot)\|_{L^2(-\eta_0, \eta_0)} \, dx$$

$$\lesssim \|e^{ax}f\|_{L^1 L^2_x}.$$ 

Substituting the above into (3.3), we have $\|P_0(\eta)f\|_X \lesssim \|e^{ax}f\|_{L^1 L^2_x}$. We can prove $\|P_0(\eta)\partial_x f\|_X \lesssim \|e^{ax}f\|_{L^1 L^2_x}$ in exactly the same way. 

Since the potential of $\mathcal{L}$ is independent of $y$, it suffices to show (3) for $f \in D(\mathcal{L}) \cap X$, where $X = \{f \in X \mid (\mathcal{F}_y f)(\cdot, \eta) = 0 \text{ a.e. } \eta \notin [-\eta_0, \eta_0]\}$. Since $\lambda(\pm \eta)$ are isolated eigenvalue of $\mathcal{L}(\eta)$ by Lemma 2.9, it follows from Lemmas 2.1 and 2.2 that

$$P_0(\eta_0)f = \frac{1}{(2\pi)^{3/2}} \int_{-\eta_0}^{\eta_0} \int_{\Gamma} (\lambda - \mathcal{L}(\eta))^{-1}(\mathcal{F}_y f)(\cdot, \eta) e^{i\eta y} \, d\lambda \, d\eta$$

$$= \frac{1}{2\pi i} \int_{\Gamma} (\lambda - \mathcal{L})^{-1} f \, d\lambda,$$

where $\Gamma$ is the boundary of a domain $D \supset \{\lambda(\pm \eta) \mid \eta \in [-\eta_0, \eta_0]\}$ satisfying $D \cap \{p(\eta + ia) \mid \eta \in \mathbb{R}\} = \emptyset$. Thus $P_0(\eta_0)$ equals to a spectral projection of $\mathcal{L}|_X$ defined by the Dunford integral and (3.1)–(3.3) can be obtained by a standard argument. We remark that $e^{t\mathcal{L}}$ is a $C^0$-semigroup on $X$ because $\mathcal{L}_0 := -\partial_x^2 + 4\partial_x - 3\partial_x^{-1}\partial_y^2$ is $m$-dissipative on $X$ and $\mathcal{L} - \mathcal{L}_0$ is infinitesimally small with respect to $\mathcal{L}_0$. Thus we complete the proof of Lemma 3.1.  

Let $0 < \eta_1 \leq \eta_2 \leq \infty$ and $P_1(\eta_1, \eta_2)$ and $P_2(\eta_1, \eta_2)$ be projections defined by

$$P_1(\eta_1, \eta_2)u(x, y) = \frac{1}{2\pi} \int_{y_1 \leq \eta_1} \int_{\mathbb{R}} u(x, y_1) e^{i\eta(y-y_1)} \, dy_1 \, d\eta_1,$$

$$P_2(\eta_1, \eta_2) = P_1(0, \eta_2) - P_0(\eta_1).$$

We remark that $P_2(\eta_1, \eta_2)$ is a projection onto non-resonant low frequency modes and that $\|P_2(\eta_1, \eta_2)e^{t\mathcal{L}}\|_{\mathcal{B}(X)}$ decays exponentially as $t \to \infty$.  

**Proposition 3.2.** Let $a \in (0, 2)$ and $\eta_1$ be a positive number satisfying $\nu(\eta_1) < a$. Then there exist positive constants $K$ and $b$ such that for any $\eta_0 \in (0, \eta_1]$, $f \in X$ and $t \geq 0$,

$$\|e^{t\mathcal{L}}P_2(\eta_0, \infty)f\|_X \leq K(\eta_0^{-1} e^{\beta\lambda(\eta_0)t} + e^{-bt}) \|f\|_X.$$
Corollary 3.3. Let a and \( \eta_0 \) be as in Lemma 3.2. Then there exist positive constants \( K_1 \) and \( b \) such that for every \( M \geq \eta_0 \) and \( f \in X \) and \( t > 0 \),

\[
\| e^{t\mathcal{L}_2}(\eta_0, M) \partial_x f \|_X \leq K_1(1 + \eta_0^{-1} + t^{-1/2})e^{-bt} \| f \|_X ,
\]

\[
\| e^{t\mathcal{L}_2}(\eta_0, M) \partial_x f \|_X \leq K_1(1 + \eta_0^{-1} + t^{-3/4})e^{-bt} \| e^{ax} f \|_{L^1_xL^2_y} .
\]

To prove Proposition 3.2, we need decay estimates for the free semigroup \( e^{t\mathcal{L}_0} \).

Lemma 3.4. Let \( a \in (0, 2) \). Then there exists a positive constant \( C \) such that for every \( f \in C_0^{\infty}(\mathbb{R}^2) \) and \( t > 0 \),

\[
\| e^{t\mathcal{L}_0} f \|_X \leq Ce^{-a(4-a^2)t} \| f \|_X ,
\]

\[
\| e^{t\mathcal{L}_0} \partial_x f \|_X \leq C(1 + t^{-1/2})e^{-a(4-a^2)t} \| f \|_X ,
\]

\[
\| e^{t\mathcal{L}_0} \partial_x f \|_X \leq Ct^{-1/2}e^{-a(4-a^2)t} \| f \|_X ,
\]

\[
\| e^{t\mathcal{L}_0} \partial_x f \|_X \leq C(1 + t^{-2/3})e^{-a(4-a^2)t} \| e^{ax} f \|_{L^2_xL^2_y} ,
\]

\[
\| e^{t\mathcal{L}_0} f \|_X \leq C(t^{-1/2} + t^{-3/4})e^{-a(4-a^2)t} \| e^{ax} f \|_{L^1_x} .
\]

Proof. Let \( u(t) \) be a solution to (2.12) satisfying the initial condition \( u(0) = f \). Then

\[
\hat{u}(t, \xi, \eta) = e^{ip(\xi, \eta) t} \hat{f}(\xi, \eta), \quad p(\xi, \eta) = \xi^3 + 4\xi - \frac{3\eta^2}{\xi} .
\]

It follows from Plancherel’s theorem that for every \( g \in X \),

\[
\|g\|^2_X = \int_{\mathbb{R}^2} e^{2ax} g(x, y)^2 dxdy = \int_{\mathbb{R}^2} |\hat{g}(\xi + ia, \eta)|^2 d\xi d\eta .
\]

Making use of (3.11) and the fact that

\[
\exists p(\xi + ia, \eta) = a(4 - a^2) + 3a\xi^2 + 3a\eta^2/(\xi^2 + a^2) ,
\]

we have for \( j \geq 0 \),

\[
\| (\partial_x^j e^{t\mathcal{L}_0} f \|_X \lesssim \| (\xi + ia)^j e^{-i3p(\xi + ia, \eta) t} \hat{f}(\xi + ia, \eta) \|_{L^2_x} \lesssim e^{a(4-a^2)t} (\sup_\xi (|\xi| + a)^j e^{-3at||\xi||^2}) \| \hat{f}(\cdot + ia, \cdot) \|_{L^2(\mathbb{R}^2)} \lesssim e^{-a(4-a^2)t} (1 + t^{-j/2}) \| f \|_X ,
\]

and

\[
\| (\partial_x^j e^{t\mathcal{L}_0} f \|_X \lesssim \| (\eta^j / (\xi^2 + a^2)^{j/2}) e^{-i3p(\xi + ia, \eta) t} \hat{f}(\xi + ia, \eta) \|_{L^2_x} \lesssim e^{a(4-a^2)t} \| \hat{f}(\cdot + ia, \cdot) \|_{L^2(\mathbb{R}^2)} \frac{\eta^j}{(\xi^2 + a^2)^{j/2}} e^{-3at/(\xi^2 + a^2)} \lesssim e^{a(4-a^2)t} t^{-j/2} \| f \|_X .
\]
Similarly,
\[ \| \partial_x e^{tL_0} f \|_X \lesssim \left\| \left( |\xi| + |a| \right)e^{-t^{3/2}(|\xi| + |a|)} \hat{f}(\xi + ia, \eta) \right\|_{L^2_\xi} \]
\[ \lesssim e^{-a(t-2)/4} \left\| \left( |\xi| + |a| \right)e^{-3at}\xi^2 \right\|_{L^2_\xi} \| \hat{f}(\cdot + ia, \cdot) \|_{L^2_\xi L^\infty_\eta} \]
\[ \lesssim e^{-a(t-2)/4} \left( 1 + t^{-2(2j+1)/4} \right) \| e^{ax} f \|_{L^2_\xi L^1_\eta} , \]
and
\[ \| e^{tL_0} f \|_X \lesssim \left\| e^{-t^{3/2}(|\xi| + |a|)} \right\|_{L^2_\xi} \left\| \hat{f}(\xi + ia, \eta) \right\|_{L^2_\xi L^\infty_\eta} \]
\[ \lesssim e^{-a(t-1/2)/4} \left\| e^{-3at}\xi^2/((a^2 + \xi^2)} \right\|_{L^2_\xi} \| \hat{f}(\cdot + ia, \cdot) \|_{L^2_\xi L^\infty_\eta} \]
\[ \lesssim e^{-a(t-1/2)}(t^{-1/2} + t^{-3/4}) \| e^{ax} f \|_{L^1(\mathbb{R}^2)} . \]
This completes the proof of Lemma 3.4.

Combining properties of the linearized Miura transformation and Lemma 3.4, we will prove linear decay estimates for non-resonant modes.

**Lemma 3.5.** Let \( a \) and \( \eta_* \) be as in Lemma 2.4 and let \( \eta_1 \in (0, \eta_*) \). Then there exists a positive constant \( K \) such that for every \( t \geq 0 \), \( \eta_0 \in [-\eta_1, \eta_1] \) and \( f \in C^\infty_0(\mathbb{R}^2) \),
\[
\| e^{tL} P_2(\eta_0, \eta_0) f \|_X \leq Ke^{-a(t-1/2)}\| f \|_X .
\]

**Proof.** Since \( C^\infty_0(\mathbb{R}) \) is dense in \( X \), it suffices to prove (3.13) for \( f \in C^\infty_0(\mathbb{R}) \).

Let \( u(t) = e^{tL} P_2(\eta_0, \eta_0) f \). Since \( P_0(\eta) \) is a spectral projection associated with \( L \) (Lemma 3.4.5), we have \( P_0(\eta_0) u(t) = 0 \) for every \( t \geq 0 \). Let \( u_\eta(t,x) = (\mathcal{F}_y u)(t,x,\eta) \). Then \( u_\eta(t,\cdot) \in L^2_\eta(\mathbb{R}) \) and \( \int_{\mathbb{R}} u_\eta(t,x) g(x,\pm \eta) \, dx = 0 \) for a.e. \( \eta \in [-\eta_0, \eta_0] \). It follows from Lemma 2.4 that there exists \( v_\eta(t,x) \in H^1_\eta(\mathbb{R}) \) such that for \( t \geq 0 \) and a.e. \( \eta \in [-\eta_0, \eta_0] \),
\[
\begin{align*}
\int_{\mathbb{R}} v_\eta(t,x) g_M(x,\eta) \, dx & = 0 , \\
(C_1 \| u_\eta(t) \|_{L^2(\mathbb{R})})^2 & \leq \| v_\eta(t) \|_{H^1(\mathbb{R})}^2 + (|\eta| \| \partial_\xi^{-1} v_\eta(t) \|_{L^2(\mathbb{R})})^2 \leq (C_2 \| u_\eta(t) \|_{L^2(\mathbb{R})})^2 ,
\end{align*}
\]
where \( C_1 \) and \( C_2 \) are positive constants depending only on \( a \) and \( \eta_1 \). Moreover,
\[
v(t) = \frac{1}{\sqrt{2\pi}} \int_{-\eta_0}^{\eta_0} v_\eta(t,x,\eta) e^{i\eta \eta} \, d\eta
\]
satisfies \( u(t) = \nabla M_+(Q)v(t) \) for every \( t \geq 0 \). Hence it follows from Lemma 2.4 that \( v(t) \) is a solution of (2.3) satisfying. Moreover, we have for \( t \geq 0 \),
\[
\int_{\mathbb{R}} (\mathcal{F}_y v)(t,x,\eta) g_M^*(x,\eta) e^{-i\eta \eta} \, dx = 0 .
\]
Integrating (3.14) over \([-\eta_0, \eta_0]\) and using Plancherel’s theorem, we obtain
\[
C_1 \| u(t) \|_X \leq \| v(t) \|_X \leq C_2 \| u(t) \|_X .
\]
Let \( \tilde{u}(t) = \nabla M_{\gamma}(Q)v(t) \) and \( \tilde{u}_y(t, x) = (\mathcal{F}_y \tilde{u})(t, x) \). Then \( \tilde{u}_y(t) = M_{\gamma}(\eta)v_y(t) \) and it follows from Lemma 2.3 that \( \tilde{u}(t) \) is a solution to (2.12). Using Lemma 2.6 we can prove that for \( t \geq 0 \),

\[
(3.17) \quad C_1' \|\tilde{u}(t)\|_X \leq \|v(t)\|_{X_M} \leq C_2' \|\tilde{u}(t)\|_X,
\]

in the same way as (3.10). Here \( C_1' \) and \( C_2' \) are positive constants depending only on \( \eta_1 \) and \( \eta_2 \). By Lemma 3.4

\[
(3.18) \quad \|\tilde{u}(t)\|_X \leq C\|\tilde{u}(0)\|_X e^{-\alpha(4-a^2)t}.
\]

Combining (3.16), (3.17) and (3.18), we obtain (3.13). Thus we complete the proof.

Lemma 3.6. Let \( a \) and \( \eta_* \) be as in Lemma 2.7 and let \( \eta_2 > \eta_* \). Then there exists a positive constant \( K \) such that for every \( t \geq 0 \) and \( f \in C_{0}^{\infty}(\mathbb{R}^2) \),

\[
(3.19) \quad \|e^{t\mathcal{L}}P_1(\eta_2, \infty)f\|_X \leq Ke^{-\alpha(4-a^2)t}\|f\|_X.
\]

Using Lemma 2.8 instead of Lemmas 2.6 and 2.7, we can prove Lemma 3.6 in exactly the same way as Lemma 3.5. Thus we omit the proof.

Middle frequency resonant modes are exponentially stable. We can obtain decay estimates of these modes by a direct computation.

Lemma 3.7. Let \( a \) and \( \eta_* \) be as in Lemma 2.7. Let \( \eta_0 \) and \( \eta_1 \) be positive numbers satisfying \( 0 < \eta_0 < \eta_1 < \eta_* \). Then for every \( f \in X \),

\[
\|e^{t\mathcal{L}}(P_0(\eta_1) - P_0(\eta_0))f\|_X \leq C(1 + \eta_0^{-1})e^{2\Re(\eta_0)t}\|f\|_X,
\]

where \( C \) is a constant depending only on \( a \) and \( \eta_1 \).

Proof. Let \( a_k(t, \eta) = \int_{\mathbb{R}}(\mathcal{F}_y u)(t, x, \eta)g_k^1(x, \eta)e^{-iy\eta}dx \) for \( k = 1, 2 \) and let

\[
E_a(t, \eta_0, \eta_1) = \int_{|\eta| \leq \eta_1} (|a_1(t, \eta)|^2 + \eta^2|a_2(t, \eta)|^2)d\eta.
\]

Since \( u(t) \) is a solution of (2.15), it follows from Lemma 2.2 that

\[
(3.20) \quad \partial_t a_1(t, \eta) = \int_{\mathbb{R}}\mathcal{L}(\eta)(\mathcal{F}_y u)(t, x, \eta)g_1^1(x, \eta)dx
= \Re(\lambda) a_1 - \eta \Im(\lambda) a_2,
\]

\[
(3.21) \quad \partial_t a_2(t, \eta) = \int_{\mathbb{R}}\mathcal{L}(\eta)(\mathcal{F}_y u)(t, x, \eta)g_2^1(x, \eta)dx
= \eta^{-1} \Im(\lambda) a_1 + \Re(\lambda) a_2,
\]

Using (3.20), (3.21) and the fact that \( \Re(\eta) \) is even and monotone decreasing for \( \eta \geq 0 \),

\[
\partial_t E_a(t, \eta_0, \eta_1) = 2\int_{\eta_0 \leq |\eta| \leq \eta_1} \Re(\lambda)\{(|a_1(t, \eta)|^2 + \eta^2|a_2(t, \eta)|^2)d\eta
\leq 2\Re(\eta_0) E_a(t, \eta_0, \eta_1).
\]

Thus we have for \( t \geq 0 \),

\[
(3.22) \quad E_a(t, \eta_0, \eta_1) \leq E_a(0, \eta_0, \eta_1) e^{2\Re(\eta_0)t}.
\]

As in the proof of Lemma 3.1 we have

\[
(3.23) \quad \|e^{t\mathcal{L}}(P_0(\eta_1) - P_0(\eta_0))f\|_X \leq C_1(1 + \eta_0^{-1}) E_a(t, \eta_0, \eta_1)^{1/2},
\]
Combining (3.23)–(3.25), we obtain Lemma 3.7. Thus we complete the proof. \(\square\)

Now we are in position to prove Proposition 3.2.

**Proof of Proposition 3.2.** Let \(a_1, a_2, \eta_1 \) and \(\eta_2\) be positive numbers satisfying \(a_1 < \nu(\eta_1) < a < \nu(\eta_2) < a_2\). Note that \(\eta_0 < \eta_1 < \eta_s < \eta_2\), where \(\eta_s\) is a root of \(\nu(\eta) = a\). Since

\[
P_2(\eta_0, \infty) = P_2(\eta_1, 1) + P_0(\eta_1) + P_1(\eta_1, 1) + P_1(\eta_2, \infty),
\]

it follows from Lemmas 3.5, 3.6 and 3.7 that for \(\nu(\eta) < a_2\), it follows from (3.26)–(3.28) that

\[
\|e^{t\mathcal{L}}P_2(\eta_0, \infty) - P_1(\eta_1, \eta_2)f\|_X \lesssim \|e^{a_1(4-a^2)t} + e^{a_2(4-a^2)t}\|_{L^2(\mathbb{R}^2)}^2,
\]

\[
\|e^{a_1x}e^{t\mathcal{L}}P_2(\eta_1, \eta_2)f\|_{L^2(\mathbb{R}^2)} \lesssim \|e^{a_1x}f\|_{L^2(\mathbb{R}^2)},
\]

\[
\|e^{a_2x}e^{t\mathcal{L}}P_1(\eta_1, \eta_2)f\|_{L^2(\mathbb{R}^2)} \lesssim \|e^{a_2x}f\|_{L^2(\mathbb{R}^2)}.
\]

On the other hand, Lemma 3.7 implies that

\[
\|e^{a_1x}e^{t\mathcal{L}}P_1(\eta_1, \eta_2)f\|_{L^2(\mathbb{R}^2)} \lesssim \|e^{a_1x}f\|_{L^2(\mathbb{R}^2)}.
\]

Hence it follows from the complex interpolation theorem that

\[
\|e^{t\mathcal{L}}P_1(\eta_1, \eta_2)f\|_X \lesssim \left\{ e^{-a_1(4-a^2)t} + e^{-a_2(4-a^2)t} + (1 + \eta_1^{-1})e^{\mathbb{R}(\mathcal{L}(\eta_1))t} \right\} \|f\|_X.
\]

By (3.20) and (3.24), we obtain

\[
\|e^{t\mathcal{L}}P_2(\eta_0, \infty)f\|_X \lesssim \left\{ e^{-a_1(4-a^2)t} + (1 + \eta_0^{-1})e^{\mathbb{R}(\mathcal{L}(\eta_0))t} \right\} \|f\|_X + \left\{ e^{-a_1(4-a^2)t} + e^{-a_2(4-a^2)t} + (1 + \eta_1^{-1})e^{\mathbb{R}(\mathcal{L}(\eta_1))t} \right\} \|f\|_X.
\]

Thus we complete the proof of Proposition 3.2. \(\square\)

**Proof of Corollary 3.3.** Without loss of generality, we may assume that \(M = \infty\). By the variation of constants formula, we have for any \(f \in X\),

\[
e^{t\mathcal{L}}P_2(\eta_0, \infty)\partial_x f = e^{t\mathcal{L}_0}P_2(\eta_0, \infty)\partial_x f
\]

\[
-6 \int_0^t \partial_x \left( e^{(t-s)\mathcal{L}_0} \right) \left( \varphi e^{s\mathcal{L}}P_2(\eta_0, \infty)\partial_x f \right) ds.
\]

(3.28)
Let $t \in (0, 2]$. Applying Proposition 3.2 and Lemmas 3.1 [44] to (3.28),
\[
\| e^{t} P_{2}(\eta, \infty) \partial_{x} f \|_{X} \leq \| e^{t} e \| P_{2}(\eta, \infty) \partial_{x} f \|_{X} + 6 \int_{0}^{t} \| \partial_{x} e^{(t-s)E} \left( S_{x} e^{t} P_{2}(\eta, \infty) \partial_{x} f \right) \|_{X} \leq (1 + t^{-1/2}) \| f \|_{X} + \int_{0}^{t} (t-s)^{-1/2} \| e^{s} P_{2}(\eta, \infty) \partial_{x} f \|_{L^{2}}.
\]
By Gronwall’s inequality, we have
\[
\| e^{t} P_{2}(\eta, \infty) \partial_{x} f \|_{X} \leq C t^{-1/2} \| f \|_{X}
\]
for $t \in (0, 2]$, where $C$ is a constant independent of $t \in (0, 2]$ and $f \in X$.

Let $t \geq 2$. Eq. (3.29) implies that $e^{t} P_{2}(\eta, \infty) \partial_{x} f$ is bounded on $X$. Applying Proposition 3.2 to $e^{t} P_{2}(\eta, \infty) \partial_{x} = e^{t-1} e^{t} P_{2}(\eta, \infty) e^{t} P_{2}(\eta, \infty) \partial_{x}$, we have for $t \geq 2$,
\[
\| e^{t} P_{2}(\eta, \infty) \partial_{x} f \|_{X} \leq e^{-bt} \| f \|_{X}.
\]
Combining the above with (3.29), we obtain (3.4).

Using (3.7) and (3.9) and Lemma 3.2 we can prove (3.5) in the same way as (3.4).

\[\Box\]

4. Preliminaries

To begin with, we will introduce notation of Banach spaces which shall be used to analyze modulation equations. For an $\eta_{0} > 0$, let $Y$ and $Z$ be closed subspaces of $L^{2}(\mathbb{R})$ defined by
\[
Y = F^{\eta}_{\eta}^{-1} Z \quad \text{and} \quad Z = \{ f \in L^{2}(\mathbb{R}) \mid \text{supp } f \subset [-\eta_{0}, \eta_{0}] \},
\]
and let $Y_{1} = F^{\eta}_{\eta}^{-1} Z_{1}$ and $Z_{1} = \{ f \in Z \mid \| f \|_{Z_{1}} := \| f \|_{L^{\infty}} < \infty \}$.

Remark 4.1. We have
\[
\| f \|_{H^{s}} \leq (1 + \eta_{0}^{2})^{s/2} \| f \|_{L^{2}} \quad \text{for any } s \geq 0 \text{ and } f \in Y,
\]
since $f$ is 0 outside of $[-\eta_{0}, \eta_{0}]$. Especially, we have $\| f \|_{L^{\infty}} \leq \| f \|_{L^{2}}$ for any $f \in Y$.

Let $P_{1}$ be a projection defined by $P_{1} f = F_{\eta}^{\eta^{-1}} 1_{[-\eta_{0}, \eta_{0}]}, f$, where $1_{[-\eta_{0}, \eta_{0}]}(\eta) = 1$ for $\eta \in [-\eta_{0}, \eta_{0}]$ and $1_{[-\eta_{0}, \eta_{0}]}(\eta) = 0$ for $\eta \not\in [-\eta_{0}, \eta_{0}]$. Then $\| P_{1} f \|_{Y_{1}} \leq (2\pi)^{-1/2} \| f \|_{L^{1}(\mathbb{R})}$ for any $f \in L^{1}(\mathbb{R})$. In particular, for any $f, g \in Y$,
\[
\| P_{1} (f g) \|_{Y_{1}} \leq (2\pi)^{-1/2} \| f \|_{L^{1}} \leq (2\pi)^{-1/2} \| f \|_{Y} \| g \|_{Y}.
\]

In order to estimate modulation parameters $c(t, y)$ and $x(t, y)$, we will use a linear estimate for solutions to
\[
\frac{\partial u}{\partial t} = A(t) u,
\]
where $A(t) = A_{0}(D_{y}) + A_{1}(t, D_{y}), u(t, y) = \{ u_{1}(t, y), u_{2}(t, y) \}$,
\[
A_{0}(D_{y}) = \begin{pmatrix} a_{11}(D_{y}) & a_{12}(D_{y}) \ a_{21}(D_{y}) & a_{22}(D_{y}) \end{pmatrix}, \quad A_{1}(t, D_{y}) = \begin{pmatrix} b_{11}(t, D_{y}) & b_{12}(t, D_{y}) \\ b_{21}(t, D_{y}) & b_{22}(t, D_{y}) \end{pmatrix},
\]
and $a_{ij}(\eta)$ and $b_{ij}(t, \eta)$ are continuous in $\eta \in [-\eta_{0}, \eta_{0}]$ and $t \geq 0$. We denote by $U(t, s) f$ a solution of (1.3) satisfying $u(s, y) = f(y)$. Then we have the following.
Lemma 4.1. Let $k \in \mathbb{Z}_{>0}$, $\mu > 1/8$. Let $\delta_1$, $\delta_2$, $\kappa$ be positive constants. Suppose that $a_{ij}(\eta)$, $b_{ij}(t, \eta)$ ($i, j = 1, 2$) satisfy
\[
\begin{align*}
|a_{11}(\eta) + 3\eta^2| &\leq \delta_1 \eta^2, \\
|a_{12}(\eta) - 8| &\leq \delta_1 |\eta|,
\end{align*}
\]
(H)
\[
\begin{align*}
|a_{21}(\eta) - (2 + \mu \eta^2)| &\leq \delta_1 \eta^2, \\
|a_{22}(\eta) + \eta^2| &\leq \delta_1 \eta^2,
\end{align*}
\]
\[
|b_{ij}(t, \eta)| \leq \delta_2 e^{-\kappa t} \quad \text{for } j = 1, 2.
\]
If $\delta_1$ is sufficiently small, then for every $t \geq s \geq 0$ and $f \in Y$,
\[
\begin{align*}
\|\partial_t^k U(t, s)f\|_Y &\leq C(1 + t - s)^{-k/2}\|f\|_Y, \\
\|\partial_t^k U(t, s)f\|_Y &\leq C(1 + t - s)^{-(2k+1)/4}\|f\|_Y,
\end{align*}
\]
where $C = C(\eta_0)$ is a constant satisfying $\limsup_{\eta_0 \downarrow 0} C(\eta_0) < \infty$.

Proof. We will prove Lemma 4.1 by the energy method. Let
\[
\omega(\eta) = \sqrt{16 + (8\mu - 1)\eta^2},
\]
\[
A_\eta(\eta) = \begin{pmatrix} -3\eta^2 & 8i \eta \\ 8i \eta & -\eta^2 \end{pmatrix}, \quad \Pi_\eta(\eta) = \begin{pmatrix} 8i \\ \eta + i\omega(\eta) \end{pmatrix}.
\]
The matrix $A_\eta(\eta)$ has eigenvalues $\lambda_{\pm}(\eta) = -2\eta^2 \pm i\eta \omega$ and $\Pi_\eta(\eta)^{-1} A_\eta(\eta) \Pi_\eta(\eta) = \text{diag}(\lambda_{+}(\eta), \lambda_{-}(\eta))$. By the assumption, there exist eigenvalues $\lambda_{\pm}(\eta)$ and an eigensystem $\Pi(\eta)$ of $A_0(\eta)$ satisfying for $\eta \in [-\eta_0, \eta_0]$,
\[
|\lambda_{\pm}(\eta) - \lambda_{\pm}(\eta_0)| \lesssim \delta_1 \eta^2, \quad |\Pi(\eta) - \Pi(\eta_0)| \lesssim \delta_1.
\]
Let $\Lambda(\eta) = \text{diag}(\lambda_{+}(\eta), \lambda_{-}(\eta))$, $B(t, \eta) = \Pi(\eta)^{-1} A(t, \eta) \Pi(\eta)$ and
\[
\mathbf{e}(t, \eta) = \begin{pmatrix} e_+(t, \eta) \\ e_-(t, \eta) \end{pmatrix} = \Pi(\eta)^{-1} \mathbf{F}_u(t, \eta).
\]
Then (4.3) can be rewritten as
\[
\partial_t \mathbf{e}(t, \eta) = (\Lambda(\eta) + B(t, \eta)) \mathbf{e}(t, \eta).
\]
Differentiating the energy function $e(t, \eta) := |e_+(t, \eta)|^2 + |e_-(t, \eta)|^2$ with respect to $t$, we have
\[
\begin{align*}
\partial_t e(t, \eta) = &\sum_{\pm} 2 \Re \lambda_{\pm}(\eta) |e_{\pm}(t, \eta)|^2 + 2 \Re \langle B(t, \eta) \mathbf{e}(t, \eta), \mathbf{e}(t, \eta) \rangle_{\mathbb{C}^2} \\
\leq &(-4 + O(\delta_1))\eta^2 e(t, \eta) + C\delta_2 e^{-\kappa t} e(t, \eta),
\end{align*}
\]
where $C$ is a positive constant and $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ is the standard inner product on $\mathbb{C}^2$. By Gronwall’s inequality, there exists a positive constant $c_3$ such that
\[
e(t, \eta) \leq c_3 e(s, \eta)e^{(-4 + O(\delta_1))\eta^2(t - s)} \quad \text{for } t \geq s \geq 0.
\]
Since $\|\partial_t^k u(t)\|_Y^2 \simeq \int_{|\eta| \leq \eta_0} \eta^{2k} e(t, \eta) d\eta$,
\[
\begin{align*}
\|\partial_t^k u(t)\|_Y^2 \lesssim &\left( \sup_{|\eta| \leq \eta_0} \eta^{2k} e^{-4\eta^2(t - s)} \right) \int_{|\eta| \leq \eta_0} e(s, \eta) d\eta \\
\lesssim & (t - s)^{-k} \|u(s)\|_Y^2 \quad \text{for } t \geq s \geq 0.
\end{align*}
\]
Similarly, we have
\[
\| \partial_y^k u(t) \|^2 \lesssim \sup_{|\eta| \leq \eta_0} e(s, \eta) \int_{|\eta| \leq \eta_0} \eta^{2k} e(-4O(\delta_1)) \eta^2 (t-s) d\eta
\]
\[
\lesssim (t-s)^{-(2k+1)/2} \| u(s) \|^2_{Y^1}, \quad \text{for } t \geq s \geq 0.
\]
Thus we complete the proof. □

Let \( A_\ast = \mathcal{F}^{-1} A_\ast(\eta) \mathcal{F} \), where \( A_\ast(\eta) \) is a matrix defined by (4.6). For a specific choice of \( \mu \), we can express the semigroup \( e^{tA_\ast} \) by using the kernel \( H_t(y) = (4\pi t)^{-1/2} e^{-y^2/4t} \).

**Lemma 4.2.** Let \( \mu = 1/8 \) and \((f_1, f_2) \in Y \times Y\). Then

\[
e^{tA_\ast} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} k_{11}(t, \cdot) * f_1 + k_{12}(t, \cdot) * f_2 \\ k_{21}(t, \cdot) * f_1 + k_{22}(t, \cdot) * f_2 \end{pmatrix},
\]

where
\[
k_{11}(t, y) = \left( \frac{1}{2} + \frac{1}{8} \partial_y \right) H_{2t}(y + 4t) + \left( \frac{1}{2} - \frac{1}{8} \partial_y \right) H_{2t}(y - 4t),
\]
\[
k_{12}(t, y) = H_{2t}(y + 4t) - H_{2t}(y - 4t),
\]
\[
k_{21}(t, y) = \left( \frac{1}{4} - \frac{1}{64} \partial_y^2 \right) \left( H_{2t}(y + 4t) - H_{2t}(y - 4t) \right),
\]
\[
k_{22}(t, y) = \left( \frac{1}{2} - \frac{1}{8} \partial_y \right) H_{2t}(y + 4t) + \left( \frac{1}{2} + \frac{1}{8} \partial_y \right) H_{2t}(y - 4t).
\]
Moreover, for every \( k \in \mathbb{Z}_{\geq 0} \), there exists a positive constant \( C \) such that
\[
\| \partial_y^k e^{tA_\ast} \|_{B(Y,Y)} \leq C(t)^{-k/2}, \quad \| \partial_y^k e^{tA_\ast} \|_{B(Y_1,Y)} \leq C(t)^{-2k+1/4}.
\]

**Proof.** In view of the proof of Lemma 4.1, we have
\[
e^{tA_\ast}(\eta) = e^{-2t\eta^2} \begin{pmatrix} \cos 4t\eta - \frac{\eta}{8} \sin 4t\eta & 2i \sin 4t\eta \\ \left( \frac{t^2}{2\pi} + \frac{\eta}{2} \right) \sin 4t\eta & \cos 4t\eta + \frac{\eta}{8} \sin 4t\eta \end{pmatrix},
\]
provided \( \mu = 1/8 \). Taking the inverse Fourier transform of the above, we obtain (4.9). The decay estimates follow immediately from (4.9) and the fact that \( \| \partial_y^k H_t \|_{B(Y,Y)} \lesssim (t)^{-k/2} \) and \( \| \partial_y^k H_t \|_{B(Y_1,Y)} \lesssim (t)^{-2k+1/4} \). Thus we complete the proof. □

Using (4.9) and the fact that \( \| \partial_y^k H_{2t} * f \|_Y \lesssim (t)^{-(2k+1)/4} \| f \|_{Y_1} \) for \( t \geq 0 \) and \( k \in \mathbb{Z}_{\geq 0} \), we can obtain the first order asymptotics of \( e^{tA_\ast}(f_1, f_2) \) as \( t \to \infty \).

**Corollary 4.3.** Let \( \mu \) and \( A_\ast \) be as in Lemma 4.2. Then there exists a positive constant \( C \) such that for every \((f_1, f_2) \in Y_1 \times Y_1\),
\[
\left\| e^{tA_\ast} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - e^{4t\partial_y} H_{2t} * \begin{pmatrix} 2f_+ \\ f_+ \end{pmatrix} - e^{-4t\partial_y} H_{2t} * \begin{pmatrix} 2f_- \\ -f_- \end{pmatrix} \right\|_Y \leq C(t)^{-3/4} \sum_{i=1,2} \| f_i \|_{Y_1},
\]
where \( f_+ = \frac{1}{4} f_1 + \frac{1}{2} f_2 \) and \( f_- = \frac{1}{4} f_1 - \frac{1}{2} f_2 \).

To estimate inhomogeneous terms of modulation equations, we will use the following.
More precisely, at the time $t$ the phase shift of the modulating line soliton $\phi(5.1)$

exists a positive constant $C$ such that

$$\int_0^t (t-s)^{-\alpha}(s)^{-\beta} ds \leq C(t)^{-\gamma}.$$ 

If $\alpha < 1$ and $\beta = 1$ or $\alpha = 1$ and $\beta < 1$, then there exists a $C > 0$ such that

$$\int_0^t (t-s)^{-\alpha}(s)^{-\beta} ds \leq C(t)^{-\min(\alpha, \beta) \log(t)}.$$ 

5. Decomposition of the perturbed line soliton

In this section, we will decompose a solution around a line soliton solution $\varphi(x - 4t)$ into a sum of a modulating line soliton and a non-resonant dispersive part plus a small wave which is caused by amplitude changes of the line soliton:

$$(5.1) \quad u(t, x, y) = \varphi_{c(t,y)}(z) - \psi_{c(t,y), L}(z + 4t) + v(t, z, y), \quad z = x - x(t, y).$$

The modulation parameters $c(t_0, y_0)$ and $x(t_0, y_0)$ denote the maximum height and the phase shift of the modulating line soliton $\varphi_{c(t,y)}(x - x(t, y))$ along the line $y = y_0$ at the time $t = t_0$, and $\psi_{c,L}$ is an auxiliary function such that

$$(5.2) \quad \int_{\mathbb{R}} \psi_{c,L}(x) dx = \int_{\mathbb{R}} (\varphi_c(x) - \varphi(x)) dx.$$ 

More precisely,

$$\psi_{c,L}(x) = 2(\sqrt{2c} - 2)\psi(x + L),$$

where $L > 0$ is a large constant to be fixed later and $\psi(x)$ is a nonnegative function such that $\psi(x) = 0$ if $|x| \geq 1$ and that $\int_{\mathbb{R}} \psi(x) dx = 1$. Since a localized solution to KP-type equations satisfies $\int_{\mathbb{R}} u(t, x, y) dx = 0$ for any $y \in \mathbb{R}$ and $t > 0$ (see [27]), it is natural to expect small perturbations appear in the rear of the solitary wave if the solitary wave is amplified.

To fix the decomposition (5.1), we impose that $v(t, z, y)$ is symplectically orthogonal to low frequency resonant modes. More precisely, we impose the constraint that for $k = 1, 2$,

$$(5.3) \quad \lim_{M \to \infty} \int_{-M}^M \int_{\mathbb{R}} v(t, z, y) g^*_k(z, \eta, c(t, y)) e^{-i\varphi} dz dy = 0 \quad \text{in } L^2(-\eta_0, \eta_0),$$

where $g^*_1(x, \eta, c) = cg^*_1(\sqrt{c/2}x, \eta)$ and $g^*_2(x, \eta, c) = \frac{\xi}{c} g^*_2(\sqrt{c/2}x, \eta)$.

We will show that the decomposition (5.1) with (5.3) is well defined if $u$ is close to a modulating line soliton in the exponentially weighted space $X$. It is expected that $\|c(t, \cdot) - 2\|_{L^\infty}$ remains small as long as (5.1) persists.

Now let us introduce functional to prove the existence of the representation (5.1) that satisfies the orthogonality condition (5.3). For $v \in X$ and $\gamma, \tilde{c} \in Y$ and $L \geq 0$, let $c(y) = 2 + \tilde{c}(y)$ and

$$F_k[u, \tilde{c}, \gamma, L](\eta) := 1_{(-\eta_0, \eta_0)}(\eta) \lim_{M \to \infty} \int_{-M}^M \int_{\mathbb{R}} \left\{ u(x, y) + \varphi(x) - \varphi_{c(y)}(x - \gamma(y)) + \psi_{c(y), L}(x - \gamma(y)) \right\} g^*_k(x - \gamma(y), \eta, \tilde{c}(y)) e^{-i\varphi} dz dy.$$
To begin with, we will show that \( F = (F_1, F_2) \) is a mapping from \( X \times Y \times \mathbb{R} \) into \( Z \times Z \).

**Lemma 5.1.** Let \( a \in (0, 2) \), \( u \in X = L^2(\mathbb{R}^2; e^{2az} \, dx \, dy), \tilde{c}, \gamma \in Y \) and \( L \geq 0 \). Then there exists a \( \delta > 0 \) such that if \( \| \tilde{c} \|_Y + \| \gamma \|_Y \leq \delta \), then \( F_k[u, \tilde{c}, \gamma, L] \in Z \) for \( k = 1, 2 \). Moreover, if \( u \in X_1 := L^1(\mathbb{R}_y; L^2(\mathbb{R}_x)) \) and \( \tilde{c}, \gamma \in Y_1 \), then \( F_k[u, \tilde{c}, \gamma, L] \in Z_1 \) for \( k = 1, 2 \).

**Proof.** Let \( u \in C_0^\infty(\mathbb{R}^2) \) and

\[
\Phi_1(x, y) = \varphi(c(y))(x - \gamma(y)) - \varphi(x) - \psi_{c(y), L}(x - \gamma(y)), \\
\Phi_{1, 0}(x, y) = \partial_x \varphi(x) \tilde{c}(y) - \varphi'(x) \gamma(y) - \psi_{c(y), L}(x), \\
\Phi_2(x, y) = \Phi_1(x, y) - \Phi_{1, 0}(x, y), \quad \Psi(x, y) = g_k^2(x - \gamma(y), \eta, c(y)) - g_k^2(x, \eta).
\]

Then

\[
\int_{\mathbb{R}^2} \{u(x, y) - \Phi_1(x, y)\} g_k^2(x - \gamma(y), \eta, c(y)) e^{-i\eta \gamma} \, dx \, dy = \sum_{j=1}^4 I_j(\eta).
\]

where

\[
I_1 = \int_{\mathbb{R}^2} u(x, y) g_k^2(x, \eta) e^{-i\eta \gamma} \, dx \, dy, \\
I_2 = -\int_{\mathbb{R}^2} \Phi_{1, 0}(x, y) \tilde{g}_k(x, \eta) e^{-i\eta \gamma} \, dx \, dy, \\
I_3 = -\int_{\mathbb{R}^2} \Phi_2(x, y) \tilde{g}_k(x, \eta) e^{-i\eta \gamma} \, dx \, dy, \\
I_4 = \int_{\mathbb{R}^2} \{u(x, y) - \Phi_1(x, y)\} \Psi(x, y) e^{-i\eta \gamma} \, dx \, dy.
\]

By Claim [2.1],

\[
(5.4) \quad \sup_{c \in [2-\delta, 2+\delta]} \sup_{\eta \in [-m_0, 0]} \| \partial_x^j \partial_y^k \tilde{g}_k(\cdot, \eta, c) \|_{L^2_x(\mathbb{R})} < \infty \quad \text{for } j, k \geq 0,
\]

and it follows from Plancherel’s theorem and (5.4) that

\[
\int_{-m_0}^{m_0} |I_1(\eta)|^2 \, d\eta \lesssim \|u\|_X^2, \quad \int_{-m_0}^{m_0} |I_2(\eta)|^2 \, d\eta \lesssim \|\tilde{c}\|_Y^2 + \|\gamma\|_Y^2.
\]

Since \( \sup_y (|\tilde{c}(y)| + |\gamma(y)|) \lesssim \|\tilde{c}\|_Y + \|\gamma\|_Y \), we have

\[
\|\Phi_1\|_X + \|\Phi_{1, 0}\|_X \leq C(\|\tilde{c}\|_Y + \|\gamma\|_Y), \\
\|e^{2ax} \Phi_2(x, y)\|_{L^1(\mathbb{R}^2)} \leq C(\|\tilde{c}\|_Y + \|\gamma\|_Y)^2, \\
\|e^{-2ax} \Psi(x, y)\|_{L^2(\mathbb{R}^2)} \leq C(\|\tilde{c}\|_Y + \|\gamma\|_Y),
\]

where \( C \) is a positive constant depending only on \( \delta \).

Combining the above, we obtain

\[
\sup_{-m_0 \leq \eta \leq m_0} (|I_3(\eta)| + |I_4(\eta)|) \lesssim \|u\|_X (\|\tilde{c}\|_Y + \|\gamma\|_Y) + (\|\tilde{c}\|_Y + \|\gamma\|_Y)^2.
\]

Since \( C_0^\infty(\mathbb{R}^2) \) is dense in \( X \), it follows that for any \( u \in X \),

\[
1_{[-m_0, m_0]}(I_1 + I_2) \in Z, \quad 1_{[-m_0, m_0]}(I_3 + I_4) \in Z_1 \subset Z.
\]
Suppose \( u \in X_1 \) and \( \tilde{c}, \gamma \in Y_1 \). Noting that \( \sqrt{c} - 2 = \tilde{c}/2 + O(\tilde{c}^2) \in Y_1 \), we have
\[
\sup_{[-\eta_0, \eta_0]} |I_1(\eta)| \lesssim \|u\|_{X_1}, \quad \sup_{[-\eta_0, \eta_0]} |I_2(\eta)| \lesssim \|\tilde{c}\|_{Y_1} + \|\gamma\|_{Y_1},
\]
and \( 1_{[-\eta_0, \eta_0]} \sum_{1 \leq i \leq 4} I_i \in Z_1 \). Thus we complete the proof. \( \square \)

**Lemma 5.2.** Let \( a \in (0, 2) \). There exist positive constants \( \delta_0, \delta_1, L_0 \) and \( C \) such that if \( \|u\|_X < \delta_0 \) and \( L \geq L_0 \), then there exists a unique \((\tilde{c}, \gamma)\) with \( c = 2 + \tilde{c} \) satisfying
\begin{align}
&\|\tilde{c}\|_Y + \|\gamma\|_Y < \delta_1, \label{5.5} \\
&F_1[u, \tilde{c}, \gamma, L] = F_2[u, \tilde{c}, \gamma, L] = 0. \label{5.6}
\end{align}
Moreover, the mapping \( \{u \in X \mid \|u\|_X < \delta_0\} \ni u \mapsto (\tilde{c}, \gamma) =: \Phi(u) \) is \( C^1 \).

**Proof.** Clearly, we have \((F_1, F_2) \in C^1(X \times Y \times Y \times \mathbb{R}; Z \times Z)\) and for \( \tilde{c}, \gamma \in Y \),
\[
D_{(\tilde{c}, \gamma)}(F_1, F_2)(0, 0, 0, L) \left( \begin{array}{c}
\tilde{c} \\
\gamma
\end{array} \right) = \sqrt{2\pi} \left( \begin{array}{cc}
f_{11} & f_{12} \\
f_{21} & f_{22}
\end{array} \right) \left( \begin{array}{c}
\tilde{c} \\
\gamma
\end{array} \right),
\]
where
\[
\begin{align*}
f_{11} &= -\int_R (\partial_c \varphi(x) - \psi(x + L))g_1^*(x, \eta) \, dx, \\
f_{12} &= \int_R \varphi'(x)g_1^*(x, \eta) \, dx, \\
f_{21} &= -\int_R (\partial_c \varphi(x) - \psi(x + L))g_2^*(x, \eta) \, dx, \\
f_{22} &= \int_R \varphi'(x)g_2^*(x, \eta) \, dx.
\end{align*}
\]
By Claims 2.1 and A.1 in Appendix A,
\[
\begin{align*}
f_{11} &= -1 + O(\eta_0^2) + O(e^{-aL}), \\
f_{12} &= O(\eta_0^2), \\
f_{21} &= -\frac{1}{2} + O(\eta_0^2) + O(e^{-aL}), \\
f_{22} &= -2 + O(\eta_0^2),
\end{align*}
\]
and \( D_{(\tilde{c}, \gamma)}(F_1, F_2)(0, 0, 0, L) \in B(Y \times Y, Z \times Z) \) has a bounded inverse if \( \delta_0 \) and \( e^{-aL} \) are sufficiently small. Hence it follows from the implicit function theorem that for any \( u \) satisfying \( \|u\|_X < \delta_0 \), there exists a unique \((c, \gamma) \in Y \times Y\) satisfying (5.5) and (5.6). Moreover, the mapping \((\tilde{c}, \gamma) = \Phi(u)\) is \( C^1 \).

**Remark 5.1.** In Lemma 5.2, we can replace \( X \) by a Banach space \( X_2 \) whose norm is
\[
\|u\|_{X_2} = \left( \int_{-\eta_0}^{\eta_0} \int_R \frac{|\hat{u}(\xi + ia, \eta)|^2}{(1 + \xi^2)^3} \, d\xi d\eta \right)^{1/2}.
\]
Suppose \( u(t, x, y) \) is a solution of (2.1) satisfying \( u(0, x, y) = \varphi(x) + v_0(x, y) \) and \( v_0 \in X \cap H^1(\mathbb{R}) \). Then for any \( T > 0 \),
\[
\tilde{v}(t, x, y) := u(t, x + 4t, y) - \varphi(x) \in C([0, \infty); X), \label{5.7}
\]
(see Proposition 5.1 in Appendix 5). Combining (5.7) and the fact that \( \tilde{v}(t) \) is a solution of (5.1), we have \( \partial_t P_1(0, \eta_0) u \in C([0, \infty); X_2) \) and
\[
P_1(0, \eta_0) \tilde{v}(t) \in C^1([0, \infty); X_2). \label{5.8}
\]
If \( \sup_{t \in [0, T]} \|\tilde{v}(t)\|_{X_2} \) is sufficiently small for a \( T > 0 \), then there exists \((\tilde{c}(t), \tilde{x}(t)) := \Phi(\tilde{v}(t))\) satisfying (5.3) for \( t \in [0, T] \), where \( c(t, y) = \tilde{c}(t, y) + 2 \) and \( x(t, y) = 4t + \tilde{x}(t, y) \) and \( v \) and \( z \) are defined by (2.1). That is, the decomposition (5.1)
satisfying (5.3) exists on $[0, T]$ if $\|v_0\|_{X_2} \lesssim \|v_0\|_X$ is sufficiently small. Since $X_2 \ni u \mapsto \Phi(u) \in Y \times Y$ is $C^1$, it follows from (5.3) that

$$(\hat{c}(t), \hat{x}(t)) = \Phi(\tilde{v}(t)) \in C([0, T]; Y \times Y) \cap C^1([0, T]; Y \times Y).$$

We use the following lemma to decompose initial data around the line soliton.

**Lemma 5.3.** Let $a \in (0, 2)$. There exist positive constants $\delta_2$, $\delta_3$ and $L'_0$ such that if $\|u\|_{X_1} < \delta_2$ and $L \geq L'_0$, then there exists a unique $(\hat{c}, \gamma)$ with $c = 2 + \hat{c}$ satisfying $\|\hat{c}\|_{Y_1} + \|\gamma\|_{Y_1} < \delta_3$ and (5.6).

Lemma 5.3 can be proved in exactly the same as Lemma 5.2.

We provide a continuation principle that ensures the existence of (5.1) as long as $\|v(t)\|_X$ and $\|\hat{c}(t)\|_Y$ remain small.

**Proposition 5.4.** Let $a$, $\delta_0$ and $L$ be the same as in Lemma 5.3 and let $u(t)$ be a solution of (2.1) such that $u(t, x, y) - \varphi(x - 4t) \in C([0, \infty); X \cap L^2(\mathbb{R}^2))$. Then there exists a constant $\delta_2 > 0$ such that if (5.1) and (5.3) hold for $t \in [0, T)$ and $v(t, z, y), \hat{c}(t, y) := c(t, y) - 2$ and $\hat{x}(t, y) := x(t, y) - 4t$ satisfy

$$(\hat{c}, \hat{x}) \in C([0, T); Y \times Y) \cap C^1((0, T); Y \times Y),$$

$$(\hat{c}(t), \hat{x}(t)) = \Phi(\tilde{v}(t)) \in C([0, T]; Y \times Y) \cap C^1([0, T]; Y \times Y)$$

then either $T = \infty$ or $T$ is not the maximal time of the decomposition (5.1) satisfying (5.3), (5.9) and (5.10).

**Proof.** Suppose $T < \infty$. Let $\tau \in (0, T)$, $\tilde{x}(t, y) = x(t, y)$ for $t \in [0, T - \tau]$ and $\tilde{x}(t, y) = x(T - \tau, y) + 4(t + \tau - T)$ for $t \geq T - \tau$. Let $u_1(t, x, y) = u(t, x + \tilde{x}(t, y), y - \varphi(x))$. Then

$$ \sup_{t \in [0, T - \tau]} \|u_1(t)\|_X \leq \sup_{t \in [0, T]} (\|v(t)\|_X + \|\varphi(t, y) - \varphi\|_X + \|\varphi(t, y)\|_X)$$

$$ \leq \frac{\delta_0}{2} + C_1 \sup_{t \in [0, T]} \|\hat{c}(t)\|_Y \leq \frac{\delta_0}{2} + C_1 \delta_2,$$

where $\tilde{\psi}_c(x) = \psi_c(x + 4t)$ and $C_1$ is a constant that does not depend on $\tau$. Since $Y \subset L^\infty(\mathbb{R})$, it follows from the assumption that $C_2 := \sup_{\tau \in (0, T)} \sup_y e^{-a\tilde{\varphi}(\tau, y)} < \infty$. Thus for $t \in [T - \tau, T)$,

$$ \|u_1(t)\|_X \leq \|u_1(T - \tau)\|_X + \left\|e^{-a\tilde{\varphi}(T - \tau, y)} \{\tilde{v}(t) - \tilde{v}(T - \tau)\}\right\|_X$$

$$ \leq \frac{\delta_0}{2} + C_1 \delta_2 + C_2 \|\hat{v}(t) - \hat{v}(T - \tau)\|_X.$$ 

Now we choose $\delta_2$ and $\tau$ so that

$$ \delta_2 < \min \{\delta_1, \delta_0/(4C_1)\}, \quad \sup_{t_1, t_2 \in [T - \tau, T + \tau]} \|\tilde{v}(t_2) - \tilde{v}(t_1)\|_X < \delta_0/(4C_2).$$

Then we have $\sup_{t \in [0, T + \tau]} \|u_1(t)\|_X < \delta_0$ and it follows from Lemma 5.2 and Remark 5.1 that there exists a unique

$$(\hat{c}_1(t), \hat{x}_1(t)) \in C([T - \tau, T + \tau]; Y \times Y) \cap C^1((T - \tau, T + \tau); Y \times Y)$$

and $\hat{c}_1(t) \in C([T - \tau, T + \tau]; Y \times Y) \cap C^1((T - \tau, T + \tau); Y \times Y)$. 


satisfying $\sup_{t \in (T-\tau, T+\tau)} (||\hat{c}_1(t)||_Y + ||\hat{x}_1(t)||_Y) < \delta_1$ and
\begin{equation}
(5.11) \quad u(t, x + \hat{x}(t, y), y) = \varphi_{c_1(t, y)}(z_1) - \tilde{\psi}_{c_1(t, y)}(z_1) + v_1(t, z_1, y),
\end{equation}
\begin{equation}
(5.12) \quad \lim_{M \to \infty} \int_{-M}^M \int_{\mathbb{R}} v_1(t, z_1, y) g^*_k(z_1, \eta, c_1(t, y)) e^{-i\eta \eta} d\eta \ dy = 0 \quad \text{in} \ L^2(\eta_0, \eta_0)
\end{equation}
for $k = 1$ and 2, where $c_1(t, y) = 2 + \hat{c}_1(t, y)$ and $z_1 = x - x_1(t, y)$. By the local uniqueness of the decomposition, we have for $t \in [T-\tau, T]$,
\begin{equation}
(5.13) \quad \hat{c}(t) = \hat{c}_1(t), \quad \hat{x}(t) = \hat{x}(T-\tau) + \hat{x}_1(t), \quad v(t, x, y) = v_1(t, x, y).
\end{equation}
Let us define $\hat{c}(t)$ and $\hat{x}(t)$ by (5.13) and $v(t)$ by (5.1) for $t \in [T, T + \tau]$. Then $(\hat{c}, \hat{x}) \in C([0, T + \tau]; Y \times Y) \cap C^1((0, T + \tau); Y \times Y)$ and (5.11) and (5.12) imply that $v(t)$ satisfies (5.5) for $t \in [0, T + \tau]$. Thus we prove that $T$ is not maximal. This completes the proof of Proposition 5.4.

6. Modulation equations

In this section, we will derive a system of PDEs which describe the motion of $c(t, y)$ and $x(t, y)$. Substituting the ansatz (5.1) into (2.1), we obtain
\begin{equation}
(6.1) \quad \partial_t v = \mathcal{L}_{c(t, y)} v + \ell + \partial_x(N_1 + N_2) + N_3,
\end{equation}
where $\mathcal{L}_c v = -\partial_x(\partial^2_x - 2c + 6\varphi_c)v - 3\partial_{y}^{-1} \partial_y^3 v$, $\ell = \ell_1 + \ell_2$, $\ell_k = \ell_{k1} + \ell_{k2} + \ell_{k3}$ ($k = 1, 2$) and
\begin{align*}
\ell_1 & = (\ell_1 - 2c - 3(3y^2)\varphi'^c_c - (c_x - 6c_y y) \partial_c \varphi_c), & \ell_2 & = 3x_{yy} \varphi_c, \\
\ell_3 & = 3c_y \int_y^x \partial_c \varphi_c(z_1) d z_1 + 3(c_y)^2 \int_y^x \partial^2_c \varphi_c(z_1) d z_1, \\
\ell_4 & = (c_x - 6c_y y) \partial_c \tilde{\psi}_c - (c_y - 4 - 3(3y^2)\tilde{\psi}'_c), \\
\ell_5 & = 3\tilde{\psi}'_c - 3\partial_x(\partial^2_x \tilde{\psi}_c) + 6 \partial_x(c \tilde{\psi}_c) - 3x_{yy} \tilde{\psi}_c, \\
\ell_6 & = -3c_y \int_y^x \partial_c \tilde{\psi}_c(z_1) d z_1 - 3(c_y)^2 \int_y^x \partial^2_c \tilde{\psi}_c(z_1) d z_1,
\end{align*}
$N_1 = -3v^2$, \quad $N_2 = \{x_1 - 2c - 3(3y^2)^2\}v + 6\tilde{\psi}_c v$, \quad $N_3 = 6x_y \partial_y v + 3x_{yy} v = 6 \partial_y(x_y v) - 3x_{yy} v$.

Here we abbreviate $c(t, y)$ as $c$ and $x(t, y)$ as $x$.

First, we will derive modulation equations of $c(t, y)$ and $x(t, y)$ from the orthogonality condition (5.3) assuming that $v_0 \in X \cap H^3(\mathbb{R}^2)$ and $\partial_x^{-1} v_0 \in H^2(\mathbb{R}^2)$. If $v_0 \in H^3(\mathbb{R}^2)$ and $\partial_x^{-1} v_0 \in H^2(\mathbb{R}^2)$, then it follows from (2.9) that $\hat{v}(t) \in H^3(\mathbb{R}^2)$ is $C(\mathbb{R}; H^3(\mathbb{R}^2))$ and $\partial_x^{-1} \hat{v}(t) \in C(\mathbb{R}; H^2(\mathbb{R}^2))$. Moreover, Proposition 5.1 implies that $\hat{v}(t) \in C([0, \infty); X)$. If $M_1(T)$ and $M_2(T)$ are sufficiently small, then the decomposition of (6.1) satisfying (5.3) and (5.4) exists for $t \in [0, T]$ by Lemma 5.2. Remark 5.3 and Proposition 5.4. Since $Y \subset H^4(\mathbb{R})$,
\begin{equation}
(6.2) \quad v(t, z, y) - \hat{v}(t, z + \hat{x}(t, y), y) = \varphi(z + x(t, y)) - \varphi_{c(t, y)}(z) + \tilde{\psi}_{c(t, y)}(z) \in H^3(\mathbb{R}^2),
\end{equation}
and we easily see that $v(t) \in C([0, T]; X \cap H^3(\mathbb{R}^2))$. Using
\begin{equation}
\int_{\mathbb{R}} (v(t, z, y) - \hat{v}(t, z + \hat{x}(t, y), y)) d z = 0,
\end{equation}
for $k = 1$ and 2, where $c_1(t, y) = 2 + \hat{c}_1(t, y)$ and $z_1 = x - x_1(t, y)$. By the local uniqueness of the decomposition, we have for $t \in [T-\tau, T]$,
by \([5.2]\) and its integrand decays exponentially as \(z \to \pm \infty\), we have

\[
(\partial_z^{-1} v)(t, z, y) = - \int_z^\infty v(t, z_1, y) \, dz_1 \in X \cap H^2(\mathbb{R}^2).
\]

By Proposition 5.4 and Remark 5.1 the mapping

\[
t \mapsto \int_{\mathbb{R}^2} v(t, z, y) g_k^\ast(z, \eta, c(t, y)) e^{-iy\eta} \, dz \, dy \in Z
\]

is \(C^1\) for \(t \in [0, T]\) if we have \([5.9]\) and \([5.10]\). Differentiating \([6.1]\) with respect to \(t\) and substituting \([6.1]\) into the resulting equation, we have in \(L^2(-\eta_0, \eta_0)\)

\[
\frac{d}{dt} \int_{\mathbb{R}^2} v(t, z, y) g_k^\ast(z, \eta, c(t, y)) e^{-iy\eta} \, dz \, dy
\]

\[
= \int_{\mathbb{R}^2} (t g_k^\ast(z, \eta, c(t, y)) e^{-iy\eta} \, dz \, dy + \sum_{j=1}^5 II_j^k(t, \eta) = 0,
\]

where

\[
II_1^k = \int_{\mathbb{R}^2} v(t, z, y) \mathcal{L}_{c(t, y)} g_k^\ast(z, c(t, y)) e^{iy\eta} \, dz \, dy,
\]

\[
II_2^k = - \int_{\mathbb{R}^2} N_1 \partial_z g_k^\ast(z, \eta, c(t, y)) e^{-iy\eta} \, dz \, dy,
\]

\[
II_3^k = \int_{\mathbb{R}^2} N_3 g_k^\ast(z, \eta, c(t, y)) e^{-iy\eta} \, dz \, dy
\]

\[
+ 6 \int_{\mathbb{R}^2} v(t, z, y) c_y(t, y) x_y(t, y) \partial_z g_k^\ast(z, \eta, c(t, y)) e^{-iy\eta} \, dz \, dy,
\]

\[
II_4^k = \int_{\mathbb{R}^2} v(t, z, y) (c_t - 6c_y x_y(t, y)) \partial_z g_k^\ast(z, \eta, c(t, y)) e^{-iy\eta} \, dz \, dy,
\]

\[
II_5^k = - \int_{\mathbb{R}^2} N_2 \partial_z g_k^\ast(z, \eta, c(t, y)) e^{-iy\eta} \, dz \, dy.
\]

Next, we will show the second equation of \([6.2]\) for \(t \in [0, T]\) and \(v(t) \in C([0, T]; L^2(\mathbb{R}^2)) \cap L^\infty([0, T]; X)\) assuming that \(M_1(T)\) and \(M_2(T)\) are sufficiently small. Let \(\{v_{0n}\}_{n=1}^\infty\) be a sequence such that

\[
v_{0n} \in H^2(\mathbb{R}^2) \cap X, \quad \partial_z^{-1} v_{0n} \in H^2(\mathbb{R}^2), \quad \lim_{n \to \infty} \left( \|v_{0n} - v_0\|_X + \|v_{0n} - v_0\|_{L^2(\mathbb{R}^2)} \right) = 0,
\]

and let \(u_n(t)\) be a solution of \([2.1]\) satisfying \(u_n(0, x, y) = \varphi(x) + v_{0n}(x, y)\) and \(\tilde{v}_n(t, x, y) = u_n(t, x, y) - \varphi(x)\). Since \(\sup_{t \in [0, T]} \|\tilde{v}_n(t)\|_{L^2(\mathbb{R}^2)} \to 0\) as \(n \to \infty\) by \([29]\) and \(\sup_{n} \sup_{t \in [0, T]} \|\tilde{v}_n(t)\|_X < \infty\) by \([E.4]\) in Appendix E we have \(\lim_{n \to \infty} \sup_{t \in [0, T]} \|\tilde{v}_n(t) - \tilde{v}(t)\|_{L^2(\mathbb{R}^2)} = 0\) for any \(t \geq 0\). If \(\eta_0\) is so small that \(a/2 > \eta_0\), we can replace the weight function \(e^{2az}\) by \(e^{az}\) in Lemma 5.2, Remark 5.1 and Proposition 5.4 and see that there exist \(v_n(t), c_n(t)\) and \(x_n(t)\) satisfying for
Using the asymptotic formula of \( g(z) \) in Lemma 6.1.

Thus we can obtain the second equation of (6.2) on \([0, T]\) passing to the limit \( n \to \infty \).

The proof is given in Appendix A. We remark that (6.2) is the inverse Fourier transform of (6.1) in the \( \eta \)-variable. The leading term of

\[
\int_{\mathbb{R}^2} \ell_1 g_k^*(z, \eta, c(t, y)) e^{-i \eta \gamma} dz dy
\]

is \( \sqrt{2\pi F_y G_k(t, \eta)} \), where

(6.3)

\[
G_k(t, y) = \int_{\mathbb{R}} \ell_1 g_k^*(z, 0, c(t, y)) dz .
\]

Using the asymptotic formula of \( g_k^*(z, \eta) \), we can see that \( G_k \) has the following expression.

**Lemma 6.1.** Let \( \mu_1 = \frac{1}{2} - \frac{\pi^2}{12} \) and \( \mu_2 = \frac{\pi^2}{32} - \frac{3}{16} \). Then

\[
G_1 = 16x_{yy}(c_f^2)^{3/2} - 2(c_t - 6c_y x_y) \left( \frac{c}{2} \right)^{1/2} + 6c_y y_y - \frac{3}{c} (c_y)^2 ,
\]

\[
G_2 = -2(x_t - 2c - 3(x_y)^2) \left( \frac{c}{2} \right)^{3/2} + 6x_{yy} \left( \frac{c}{2} \right)^{3/2} - \frac{1}{2} (c_t - 6c_y x_y) \left( \frac{c}{2} \right)^{1/2} + \mu_1 c_{yy} + \mu_2 (c_y)^2 \left( \frac{c}{2} \right)^{-1} .
\]

The proof is given in Appendix A. We remark that \((G_1, G_2)\) are the dominant part of diffusion wave equations for \(c\) and \(x\).

Next, we will expand

\[
\int_{\mathbb{R}} \ell_1 \left( g_k^*(z, \eta, c(t, y)) - g_k^*(z, 0, c(t, y)) \right) e^{-i \eta \gamma} dz dy
\]

in \(c(t, y)\) and \(x(t, y)\) up to the second order. In order to express the coefficients of \(c_t\), \(x_t\), \(c_{yy}\) and \(x_{yy}\), let us introduce the operators \(S_k^j (j = k, 1, 2)\). For \(q_c(z) = \varphi_c(z), \varphi_c'(z), \partial_c \varphi_c(z)\) and \(\partial_z^{-1} \partial_c q_c(z) = -\int_\infty^z \partial_c q_c(z) dz_1 (m \geq 1)\), let

\[
S_k^1[q_c](f)(t, y) = \frac{1}{2\pi} \int_{-\gamma_0}^{\gamma_0} \int_{\mathbb{R}^2} f(y_1) g_2(z) g_{k1}^*(z, \eta, 2) e^{i(y_1 \eta)} dy_1 dz dy_1 dz dy ,
\]

\[
S_k^2[q_c](f)(t, y) = \frac{1}{2\pi} \int_{-\gamma_0}^{\gamma_0} \int_{\mathbb{R}^2} f(y_1) \partial_c(t, y) g_{k2}^*(z, \eta, c(t, y)) e^{i(y_1 \eta)} dy_1 dz dy_1 dz dy ,
\]

where

\[
g_{k1}^*(z, \eta, c) = g_k^*(z, \eta, c) - g_k^*(z, 0, c) , \quad \delta q_c(z) = \frac{q_c(z) - q_2(z)}{c - 2} ,
\]

\[
g_{k2}^*(z, \eta, c) = g_{k1}^*(z, \eta, 2) \delta q_c(z) + \frac{g_{k1}^*(z, \eta, c) - g_{k1}^*(z, \eta, 2)}{c - 2} q_c(z) .
\]
We have $S_1^k \in B(Y)$ and $S_1^k$ are independent of $c(t, y)$ whereas $\|S_1^k\|_{B(Y, Y)} \leq \|\tilde{c}\|_Y$. See Claims B.3 and B.2 in Appendix B. Using $S_1^k$ ($j, k = 1, 2$), we have

\[(6.4)\]

\[
\frac{1}{\sqrt{2\pi}} \tilde{P}_1 \mathcal{F}_{\eta}^{-1} \left( \int_{\mathbb{R}} \ell_1 \left( g_k^0(z, \eta, c(t, y)) - g_k^2(z, 0, c(t, y)) \right) e^{-i\eta y} dz dy \right) = -\sum_{j=1,2} \partial_\eta^2 \left( S_1^j [\psi c](x_t - 2c - 3(x_y)^2) - S_1^j [\partial_\psi \varphi c](c_t - 6c_y x_y) \right) - \partial_\eta^2 (R_1^k + R_2^k),
\]

\[
R_1^k = 3S_1^k [\varphi c](x_{yy}) - 3S_1^k [\partial_\varphi \varphi c](c_{yy}) - S_1^j [\partial_\psi \varphi c](c_{yy}) - 3 \sum_{j=1,2} S_1^j [\partial_\psi \varphi c](c_{yy}).
\]

We rewrite the linear term $R_1^k$ as

\[
\begin{pmatrix} R_1^k \\ R_2^k \end{pmatrix} = \tilde{S}_0 \begin{pmatrix} c_{yy} \\ x_{yy} \end{pmatrix}, \quad \tilde{S}_0 = 3 \begin{pmatrix} -S_1^1 [\partial_\psi \varphi c] & S_1^1 [\varphi c] \\ -S_2^1 [\partial_\psi \varphi c] & S_2^1 [\varphi c] \end{pmatrix}.
\]

Next we deal with $\int_{\mathbb{R}} \ell_2 g_k^0(z, \eta, c(t, y)) e^{-i\eta y} dz dy$. Let $S_3^k[p]$ and $S_4^k[p]$ be operators defined by

\[
S_3^k[p](f)(t, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} f(y_1) p(z + 4t + L) g_k^0(z, \eta) e^{i(y_1 - y) \eta} dy_1 dz d\eta,
\]

\[
S_4^k[p](f)(t, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} f(y_1) \tilde{c}(y_1) p(z + 4t + L) g_k^0(z, \eta) e^{i(y_1 - y) \eta} dy_1 dz d\eta,
\]

where

\[
g_k^*(z, \eta, c) = \frac{g_k^0(z, \eta, c) - g_k^2(z, \eta)}{c - 2}.
\]

By the definition of $\tilde{\psi}_c$,

\[(6.5)\]

\[
\begin{align*}
&\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \ell_2 g_k^0(z, \eta, c(t, y)) e^{-i\eta y} dz dy \\
&= \mathcal{F}_y \left\{ (S_3^k[\psi] + S_4^k[\psi]) \left( \sqrt{2/c} c_t - 6c_y x_y \right) \right\}(t, \eta) \\
&\quad - 2\sqrt{2} \mathcal{F}_y \left\{ (S_3^k[\psi^*] + S_4^k[\psi^*]) \left( \sqrt{c - \sqrt{2}} (x_t - 4 - 3(x_y)^2) \right) \right\}(t, \eta).
\end{align*}
\]

The operator norms of $S_3^k[\psi]$, $S_4^k[\psi^*]$ ($j = 3, 4, k = 1, 2$) decay exponentially because $g_k^0(z, \eta)$ and $g_k^0(z, \eta, c)$ are exponentially localized as $z \to -\infty$ and $\psi \in C_0^\infty(\mathbb{R})$. See Claims B.3 and B.4 in Appendix B.

Next we decompose $(2\pi)^{-1} \int_{\mathbb{R}^2} (\ell_{21} + \ell_{23}) g_k^0(z, \eta, c(t, y)) e^{-i\eta y} dz dy$ into a linear part and a nonlinear part with respect to $\tilde{c}$ and $\tilde{x}$. The linear part can be written as

\[(6.6)\]

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \ell_{2,lin}(t, z, y_1) g_k^0(z, \eta) e^{i(y_1 - y) \eta} dy_1 dz d\eta =: \tilde{a}_k(t, D_y) \tilde{c},
\]

where

\[
\ell_{2,lin}(t, z, y) = \tilde{c}(t, y) \partial_x \left\{ \partial_x^2 + 6c(\varphi(z)) \right\} \psi(z + 4t + L) - 3c_{yy}(t, y) \int_{-\infty}^{\infty} \psi(z_1 + 4t + L) dz_1,
\]
\[
\tilde{a}_k(t, \eta) = \left[ \int_{\mathbb{R}} \{\psi'''(z + 4t + L) + 6(\varphi(z)\psi(z + 4t + L))_z\} \overline{g_k(z, \eta)} dz \right. \\
+ \left. 3\eta^2 \int_{\mathbb{R}} \left( \int_{-\infty}^{\infty} \psi(z_1 + 4t + L) dz_1 \right) \overline{g_k(z, \eta)} dz \right] 1_{[-\eta_0, \eta_0]}(\eta),
\]

(6.7)

and the nonlinear part is

\[
R_k^3(t, y) := \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \left( \ell_{22} + \ell_{23} \right) g_k^2(z, \eta, c(t, y_1)) e^{i(y - y_1)\eta} dz dy_1 d\eta \\
- \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \ell_{2,11} g_k(z, \eta) e^{i(y - y_1)\eta} dz dy_1 d\eta.
\]

(6.8)

Next, we deal with II_k^j (j = 1, \cdots, 5) in (6.2). Let

\[
II_k^1 = -3 \int_{\mathbb{R}^2} v(t, z, y) x_y(t, y) g_k^2(z, \eta, c(t, y)) e^{-i\eta \gamma} dz dy,
\]

\[
II_k^2 = 6 \int_{\mathbb{R}^2} v(t, z, y) x_y(t, y) g_k^2(z, \eta, c(t, y)) e^{-i\eta \gamma} dz dy
\]

so that II_k^j = II_k^3 + i\eta II_k^2. Let

\[
R_k^1(t, y) = \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \left\{ II_k^2(t, \eta) + II_k^3(t, \eta) + II_k^1(t, \eta) \right\} e^{i\eta \gamma} d\eta,
\]

(6.9)

\[
R_k^2(t, y) = \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} II_k^1(t, \eta) e^{i\eta \gamma} d\eta.
\]

Let \( S_k^5 \) and \( S_k^6 \) be operators defined by

\[
S_k^5 f(t, y) = \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} v(t, z, y_1) f(y_1) \partial_z g_k^2(z, \eta, c(t, y_1)) e^{i(y - y_1)\eta} dz dy_1 d\eta,
\]

\[
S_k^6 f(t, y) = \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} v(t, z, y_1) f(y_1) \partial_y g_k^2(z, \eta, c(t, y_1)) e^{i(y - y_1)\eta} dz dy_1 d\eta
\]

so that

\[
1_{[-\eta_0, \eta_0]}(\eta) II_k^1(t, \eta) = \sqrt{2\pi} F_y (S_k^5(c_t - 6cy x_y)),
\]

\[
1_{[-\eta_0, \eta_0]}(\eta) II_k^2(t, \eta) = \sqrt{2\pi} F_y \left\{ S_k^6 \left( x_t - 2c - 3(x_y)^2 \right) + R_k^6 \right\},
\]

where

\[
R_k^6 = -\frac{3}{\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} v_c(t, y_1, L)(z + 4t)v(t, z, y_1) \partial_z g_k^2(z, \eta, c(t, y_1)) e^{i(y - y_1)\eta} dz dy_1 d\eta.
\]

Now we are in position to translate (6.2) into a PDE form. Using (6.3)–(6.5) and (6.6)–(6.10), we can translate (6.2) into

\[
\tilde{P}_1 \left( \begin{array}{c} G_1 \\ G_2 \end{array} \right) - \left( \partial_y^2 (S_1 + \tilde{S}_2) - S_3 - \tilde{S}_4 - \tilde{S}_5 \right) \left( \begin{array}{c} c_t - 6cy x_y \\ x_t - 2c - 3(x_y)^2 \end{array} \right)
\]

\[
+ \tilde{A}_1(t) \left( \begin{array}{c} \tilde{C} \\ \tilde{x} \end{array} \right) - \partial_y^2 R_1 + \tilde{R}^1 + \partial_y \tilde{R}^2 = 0,
\]

(6.11)
where
\[
\bar{S}_j = \begin{pmatrix} -S_1'[\varphi_c] & S_1'[\varphi_c] \\ -S_2'[\varphi_c] & S_2'[\varphi_c] \end{pmatrix} \text{ for } j = 1, 2, \quad \bar{S}_3 = \begin{pmatrix} S_3'[\varphi] \\ 0 \end{pmatrix},
\]
\[
\bar{S}_4 = \begin{pmatrix} S_1'[\varphi]((\sqrt{2/c} - 1) \cdot) + S_1'[\varphi]((\sqrt{2/c} - 1) \cdot) & -2(S_1'[\varphi'] + S_1'[\varphi']((\sqrt{2/c} - 1) \cdot)) \\ S_2'[\varphi]((\sqrt{2/c} - 1) \cdot) & -2(S_2'[\varphi'] + S_2'[\varphi']((\sqrt{2/c} - 1) \cdot)) \end{pmatrix},
\]
\[
\bar{S}_5 = \begin{pmatrix} S_5' \\ S_6' \\ S_5' \\ S_6' \end{pmatrix}, \quad \bar{A}_1(t) = \begin{pmatrix} \bar{a}_1(t, D_y) \\ \bar{a}_2(t, D_y) \end{pmatrix},
\]
\[
\bar{R}^3 = R^3 + R^4 + R^5 - \bar{S}_4 \left( \frac{3}{2 \bar{c}} \right), \quad \bar{R}^2 = R^5 - \partial_y R^2,
\]
and \( R^j = (R_1^j, R_2^j) \) for \( 1 \leq j \leq 6 \). In \( G_1 \), the nonlinear terms \((c/2)^{1/2}c_y x_y \) and \( 16x_{yy}((c/2)^{3/2} - 1) \) are critical because they are expected to decay like \( t^{-1} \) as \( t \to \infty \). To translate these nonlinearity into a divergence form, we will make use of the following change of variables. Let
\[
\bar{x}(t, \cdot) = x(t, \cdot) - 4t, \quad b(t, \cdot) = \frac{1}{3} \bar{P}_1 \left\{ \sqrt{2c(t, \cdot)^{3/2} - 4} \right\},
\]
\[
C_1 = \frac{1}{2} \bar{P}_1 \left\{ c(t, \cdot)^2 - 4 \right\} \bar{P}_1,
\]
\[
\bar{C}_1 = \begin{pmatrix} 0 & 0 \\ 0 & C_1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 6 & 16 \\ \mu_1 & 6 \end{pmatrix}.
\]
We remark that \( b \simeq \bar{c} = c - 2 \) if \( c \) is close to 2 (see Claim [D.6]). By (6.12), we have \( \bar{b}_t = \bar{P}_1 (c/2)^{1/2}c_t, b_y = \bar{P}_1 (c/2)^{1/2}c_y \) and it follows from Lemma 6.4 that
\[
(6.13) \quad \bar{P}_1 \left( \frac{G_1}{G_2} \right) = -(B_1 + \bar{C}_1) \bar{P}_1 \left\{ \frac{b_t - 6(bx_{yy})_y}{x_t - 2c - 3(x_{yy})^2} \right\} + B_2 \left\{ b_{yy} \right\} + \bar{P}_1 R^7,
\]
where \( R^7 = t(R_1^7, R_2^7) \) and
\[
R_1^7 = \begin{pmatrix} 4 \sqrt{2} c^{3/2} - 16 - 12b \\ x_{yy} - 6(b_{yy} - c_{yy}) \\ -6(2b_y - (2c)^{1/2}c_y)_y - 3c^{-1}(c_y)^2 \end{pmatrix},
\]
\[
R_2^7 = 6 \left( \frac{c}{2} \right)^{3/2} - 1 \left\{ x_{yy} + 3 \left( \frac{c}{2} \right)^{1/2} c_y x_y - 3(b_{xy})_y \\ -\mu_1(b_{yy} - c_{yy}) + \mu_2 \frac{2}{c}(c_{yy})^2 \right\}.
\]
Let \( C_2 = \bar{P}_1 \left\{ \left( \frac{c(t, \cdot)}{2} \right)^{1/2} - 1 \right\} \bar{P}_1, \quad \bar{C}_2 = \begin{pmatrix} C_2 \\ 0 \end{pmatrix}, \quad \bar{S}_j = \bar{S}_j(I + \bar{C}_2)^{-1} \text{ for } 1 \leq j \leq 5 \)
and
\[
(6.15) \quad B_3 = B_1 + \bar{C}_1 + \partial_y^2(\bar{S}_1 + \bar{S}_2) - \bar{S}_3 - \bar{S}_4 - \bar{S}_5.
\]
Note that \( I + \bar{C}_2 \) is invertible as long as \( c(t, \cdot) \) remains small in \( Y \) and that \( B_3 \) is a bounded operator on \( Y \times Y \) depending on \( \bar{c} \) and \( v \). Substituting (6.13) into (6.11), we have
\[
B_3 \bar{P}_1 \left\{ \frac{b_t - 6(bx_{yy})_y}{x_t - 2c - 3(x_{yy})^2} \right\} = \left\{ (B_2 - \partial_y^2 \bar{S}_0) \partial_y^2 + \bar{A}_1(t) \right\} \left\{ \frac{b}{x} \right\} + \bar{P}_1 R^7 + \bar{R}^1 + \partial_y \bar{R}^2 + \bar{R}^3,
\]
Let

\[
\begin{align*}
R^8 &= 6(B_3 - B_1 - \tilde{C}_1) \left( (I + C_2)(c_yx_y) - (bx_y)_y \right), \\
R^9 &= \partial_y^2 \tilde{S}_0 \left( b_{yy} - c_{yy} \right), \\
R^{10} &= \bar{A}_1(t) \left( \tilde{c} - b \right).
\end{align*}
\]

Thus we have the following.

**Proposition 6.2.** There exists a \( \delta_3 > 0 \) such that if \( M_1(T) + M_2(T) + \eta_0 + e^{-aL} < \delta_3 \) for a \( T \geq 0 \), then

\[
(6.16) \quad \left( \begin{array}{l} b_l \\ \tilde{x}_t \end{array} \right) = A(t) \left( \begin{array}{l} b_l \\ \tilde{x}_t \end{array} \right) + \sum_{i=1}^{8} N_i
\]

where \( B_4 = B_1 + \partial_y^2 \tilde{S}_1 - \tilde{S}_3 = B_3|_{\varepsilon=0, v=0} \),

\[
A(t) = B_4^{-1} \left\{ (B_2 - \partial_y^2 \tilde{S}_0) \tilde{c} + \bar{A}_1(t) \right\} + \left( \begin{array}{ll} 0 & 0 \\ 2 & 0 \end{array} \right),
\]

\[
N_1 = \bar{P}_1 \left( \begin{array}{l} 6(b_{\tilde{x}_y}y) \\ 2(\tilde{c} - b) + 3(\tilde{x}_y)^2 \end{array} \right), \\
N_2 = B_3^{-1} \bar{P}_1 \tilde{R}^7, \\
N_3 = B_3^{-1} \bar{R}^1, \\
N_4 = B_3^{-1} \partial_y \tilde{R}^2, \\
N_5 = B_3^{-1} \bar{R}^3, \\
N_6 = (B_3^{-1} - B_4^{-1}) \bar{A}_1(t) \left( \begin{array}{l} b_{yy} \\ x_{yy} \end{array} \right), \\
N_7 = (B_3^{-1} - B_4^{-1})(B_2 - \partial_y^2 \tilde{S}_0) \left( \begin{array}{l} b_{yy} \\ 0 \end{array} \right), \\
N_8 = (B_3^{-1} - B_4^{-1})(B_2 - \partial_y^2 \tilde{S}_0) \left( \begin{array}{l} 0 \\ x_{yy} \end{array} \right).
\]

**Proof.** Proposition \( [6.4] \) implies that the \( (5.3) \) persists on \([0, T]\) if \( \delta_3 \) is sufficiently small. Moreover Claims \( 6.1 \) below imply that \( B_3, B_1 \) and \( I + \tilde{C}_k \) are invertible if \( \| \tilde{c}(t) \|_Y, \| v(t) \|_X, \eta_0 \) and \( e^{-aL} \) are sufficiently small. Thus we have \( (6.16) \). \( \Box \)

**Claim 6.1.** There exist positive constants \( \delta \) and \( C \) such that if \( M_1(T) \leq \delta \), then for \( s \in [0, T] \) and \( k = 1, 2 \),

\[
(6.17) \quad \| \tilde{C}_k \|_{B(Y)} \leq C M_1(T) \| s \|_T^{-1/2},
\]

\[
(6.18) \quad \| \tilde{C}_k \|_{B(Y,Y_1)} \leq C M_1(T) \| s \|_T^{-1/4},
\]

\[
\| (I + \tilde{C}_k)^{-1} \|_{B(Y)} + \| (I + \tilde{C}_k)^{-1} \|_{B(Y_1)} \leq C.
\]

Claim \( 6.1 \) immediately follows from Claim \( 3.6 \) in Appendix \( B \) and the fact that \( Y_1 \subset Y \) and \( \| \tilde{C}_k \|_{B(Y,Y_1)} \approx \| \tilde{C}_k \|_{B(Y,Y_1)} \).

**Claim 6.2.** There exist positive constants \( \delta \) and \( C \) such that if \( M_1(T) + M_2(T) + \eta_0 + e^{-aL} \leq \delta \), then

\[
\| B_3^{-1} \|_{B(Y)} \leq C \quad \text{and} \quad \| B_3^{-1} \|_{B(Y_1)} \leq C \quad \text{for} \ s \in [0, T].
\]

\[
\begin{align*}
\tilde{A}^3 &= R^8 + R^9 + R^{10} \\
R^8 &= 6(B_3 - B_1 - \tilde{C}_1) \left( (I + C_2)(c_yx_y) - (bx_y)_y \right), \\
R^9 &= \partial_y^2 \tilde{S}_0 \left( b_{yy} - c_{yy} \right), \\
R^{10} &= \bar{A}_1(t) \left( \tilde{c} - b \right).
\end{align*}
\]
Claim 6.3. There exist positive constants $C$ and $\delta$ such that if $\eta_0^2 + e^{-aL} \leq \delta$, then

$$
\|B_1^{-1}\|_{B(Y)} + \|B_1^{-1}\|_{B(Y')} \leq C.
$$

The proof of Claims 6.2 and 6.3 will be given in Appendix C.

7. A PRIORI ESTIMATES FOR $c(t, y)$ AND $x_y(t, y)$

In this section, we will estimate $M_1(T)$ assuming that $M_1(T)$ $(1 \leq i \leq 3)$, $\eta_0$ and $e^{-aL}$ are sufficiently small.

Lemma 7.1. There exist positive constants $\delta_1$ and $C$ such that if $M_1(T) + M_2(T) + \eta_0 + e^{-aL} \leq \delta_1$, then

$$
M_1(T) \leq C\|v_0\|_{X_1} + C(M_1(T) + M_2(T))^2.
$$

To prove Lemma 7.1, we need the following.

Claim 7.1. There exist positive constants $\eta_1$, $\delta$ and $C$ such that if $\eta_0 \in (0, \eta_1]$ and $M_1(T) \leq \delta$, then

$$
\|[\partial_y, B_3]\|_{B(Y,Y')} \leq C(M_1(T) + M_2(T))(s)^{-3/4} \text{ for } s \in [0, T].
$$

The proof is given in Appendix C.

Claim 7.2. There exist positive constants $\eta_1$, $\delta$ and $C$ such that if $\eta_0 \in (0, \eta_1]$ and $M_1(T) \leq \delta$, then for $t \in [0, T]$,

$$
\|S_1 - \tilde{S}_1\|_{B(Y,Y')} \leq CM_1(T)(t)^{-1/4},
$$

$$
\|S_3 - \tilde{S}_3\|_{B(Y,Y')} \leq CM_1(T)(t)^{-1/4}e^{-a(4t+L)}.
$$

Claim 7.2 follows immediately from (C.4), (C.8) and Claim 6.2.

Proof of Lemma 7.1. To apply Lemma 4.1 we will transform (6.16) into a system of $b$

where

$$
M \eta = \text{diag}(1, \eta_x). A(t) \text{ diag}(1, \eta_x^{-1}),
$$

$$
A_0 = \text{diag}(1, \partial_y) \begin{pmatrix} B_0^{-1}B_2 - \partial_y^2 \tilde{S}_1 \partial_y^2 + \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \end{pmatrix} \text{ diag}(1, \partial_y)^{-1},
$$

$$
A_1(t) = \text{diag}(1, \partial_y)(B_4^{-1} - B_1^{-1})(B_4 - \partial_y^2 \tilde{S}_0) \text{ diag}(\partial_y^2, \partial_y) + \text{ diag}(1, \partial_y)B_4^{-1}\tilde{A}_1(t),
$$

where $\partial_y^{-1} = J^{-1}(\eta_0)^{-1}J$. Then $A(t) = A_0 + A_1(t)$. Note that $\tilde{A}_1(t) = \tilde{A}_1(t) \text{ diag}(1, \partial_y^{-1})$. Multiplying (6.16) by $\text{ diag}(1, \partial_y)$ from the left, we can transform (6.16) into

$$
\partial_t \begin{pmatrix} b \\ x_y \end{pmatrix} = A(t) \begin{pmatrix} b \\ x_y \end{pmatrix} + \sum_{i=1}^{8} \text{ diag}(1, \partial_y)\mathcal{N}_i.
$$

Now we will show that $A(t)$ satisfies the hypothesis $\mathcal{H}$ of Lemma 4.1. Let $A_0(\eta)$ be the Fourier transform of the operator $A_0$. Then

$$
A_0(\eta) = \begin{pmatrix} 1 & 0 \\ 0 & i\eta \end{pmatrix} (B_0^{-1} + O(\eta^2))(B_2 + O(\eta^2)) \begin{pmatrix} -\eta^2 & 0 \\ 0 & i\eta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2i\eta & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} -3\eta^2 & 8i\eta \\ i\eta(2 + \mu_3\eta^2) & -\eta^2 \end{pmatrix} + \begin{pmatrix} O(\eta^3) & O(\eta^3) \\ O(\eta^3) & O(\eta^3) \end{pmatrix},
$$

where $\mu_3 = \frac{-2\mu_2}{t} + \frac{1}{t} = \frac{1}{t} + \frac{\mu_2}{2t} > 1/8$. Claim D.4 in Appendix D implies $\|A_1(t)\|_{B(Y)} \leq e^{-a(4t+L)}$. Thus we prove that $A(t) = A_0 + A_1(t)$ satisfies $\mathcal{H}$.  


Let $U(t, s)$ be the semigroup generated by $A(t)$. By the variation of the constant formula,
\[
\begin{pmatrix}
b(t) \\
x_y(t)
\end{pmatrix} = U(t, 0) \begin{pmatrix}
b(0) \\
x_y(0)
\end{pmatrix} + \sum_{i=1}^{8} \int_0^t U(t, s) \text{diag}(1, \partial_y) \mathcal{N}_i(s) ds.
\]
By Lemma 5.3 and Claim D.6,
\[
\|b(0)\|_{Y_1} + \|x_y(0)\|_{Y_1} \lesssim \|\hat{c}(0)\|_{Y_1} + \eta_0 \|\hat{x}(0)\|_{Y_1} \lesssim \|v_0\|_{X_1}.
\]
Applying Lemma 4.1 to the first term of the right hand side, we have for $k \geq 0$,
\[
\begin{aligned}
\|\partial_y^k b(t)\|_Y + \|\partial_y^{k+1} x(t)\|_Y & \lesssim (1 + t)^{-(2k+1)/4} \|v_0\|_{X_1} + \mathcal{N}_{1}^k + \mathcal{N}_{2}^k, \\
\mathcal{N}_{1}^k &= \int_0^t \left\| \partial_y^k U(t, s) \text{diag}(1, \partial_y) \mathcal{N}_1(s) \right\|_Y ds, \\
\mathcal{N}_{2}^k &= \int_0^t \left\| \partial_y^k U(t, s) \sum_{i=2}^{8} \text{diag}(1, \partial_y) \mathcal{N}_i(s) \right\|_Y ds.
\end{aligned}
\]
Now we will estimate $\mathcal{N}_i^k$ ($i = 1, 2$, $k = 0, 1, 2$). First, we estimate $\mathcal{N}_1^k$. Let $n_1 = 6bx_y$ and $n_2 = 2(\hat{c} - b) + 3(x_y)^2$. Then $\text{diag}(1, \partial_y) \mathcal{N}_1 = \partial_y \tilde{P}_1(t, s)^t (n_1, n_2)$. Since $[\partial_y, U(t, s)] = 0$,
\[
\partial_y^k U(t, s) \text{diag}(1, \partial_y) \mathcal{N}_1 = \partial_y^{k+1} U(t, s)^t \tilde{P}_1(t, s)^t (n_1(s), n_2(s)).
\]
By (4.2), Claim D.6 and the fact that $[\partial_y, \tilde{P}_1] = 0$,
\[
\|\tilde{P}_1 n_1\|_{Y_1} + \|\tilde{P}_1 n_2\|_{Y_1} \lesssim \|b\|_Y \|x_y\|_Y + \|b\|_Y \|x_y\|_Y + \|b - \hat{c}\|_{Y_1} \\
\lesssim (1 + \|\hat{c}\|_{L^{\infty}}) \|\hat{c}\|_Y \|x_y\|_Y + \|b\|_Y \|x_y\|_Y + \|\hat{c}\|_Y \\
\lesssim M_1(T)^2 \langle s \rangle^{-1/2} \quad \text{for } s \in [0, T],
\]
\[
\|\partial_y \tilde{P}_1 n_1\|_{Y_1} + \|\partial_y \tilde{P}_1 n_2\|_{Y_1} \lesssim \|b\|_Y \|x_y\|_Y + \|b\|_Y \|x_y\|_Y + \|b - c_y\|_{Y_1} \\
\lesssim M_1(T)^2 \langle s \rangle^{-1} \quad \text{for } s \in [0, T],
\]
\[
\|\partial_y^2 \tilde{P}_1 n_1(s)\|_{Y_1} + \|\partial_y^2 \tilde{P}_1 n_2(s)\|_{Y_1} \lesssim \|(b x_y)_{yy}\|_{L^1} + \|b_{yy} - \hat{c}_{yy}\|_{Y_1} + \|((x_y)^2)_{yy}\|_{L^1} \\
\lesssim \|\hat{c}\|_Y \|x_{yy}\|_Y + \|c_y\|_Y \|x_{yy}\|_Y + \|c_y\|_Y \|x_{yy}\|_Y + \|\hat{c}\|_Y \|c_{yy}\|_Y + \|c_y\|_Y \\
+ \|x_y\|_Y \|x_{yy}\|_Y + \|x_{yy}\|_Y^2 \\
\lesssim M_1(T)^2 \langle s \rangle^{-5/4} \quad \text{for any } s \in [0, T].
\]
Using Lemma 4.1, 7.5 with $k = 0$ and (7.6), we obtain
\[
\mathcal{N}_1^0 \lesssim M_1(T)^2 \int_0^t \langle t - s \rangle^{-3/4} \langle s \rangle^{-1/2} ds \lesssim M_1(T)^2 \langle t \rangle^{-1/4} \quad \text{for } t \in [0, T].
\]
Indeed, Lemma 4.1 implies that for $t \in [0, t/2]$ and (7.7) for $s \in [t/2, t]$, we obtain

\[
\mathcal{N}_1^k \lesssim \int_0^{t/2} \|\partial_y^2 U(t, s)\|_{B(Y_1, Y_1)}(\|\tilde{P}_1 n_1(s)\|_{Y_1} + \|\tilde{P}_1 n_2(s)\|_{Y_1}) \, ds \\
+ \int_{t/2}^t \|\partial_y U(t, s)\|_{B(Y_1, Y_1)}(\|\partial_y \tilde{P}_1 n_1(s)\|_{Y_1} + \|\partial_y \tilde{P}_1 n_2(s)\|_{Y_1}) \, ds \\
\lesssim \mathcal{M}_1(T)^2 \left( \int_0^{t/2} \langle t-s \rangle^{-5/4} \langle s \rangle^{-1/2} \, ds + \int_{t/2}^t \langle t-s \rangle^{-3/4} \langle s \rangle^{-1} \, ds \right) \\
\lesssim \mathcal{M}_1(T)^2 \langle t \rangle^{-3/4} \quad \text{for } t \in [0, T].
\]

Similarly, we have

\[
\mathcal{N}_2^k \lesssim \int_0^{t/2} \|\partial_y^2 U(t, s)\|_{B(Y_1, Y_1)}(\|\tilde{P}_1 n_1(s)\|_{Y_1} + \|\tilde{P}_1 n_2(s)\|_{Y_1}) \, ds \\
+ \int_{t/2}^t \|\partial_y U(t, s)\|_{B(Y_1, Y_1)}(\|\partial_y \tilde{P}_1 n_1(s)\|_{Y_1} + \|\partial_y \tilde{P}_1 n_2(s)\|_{Y_1}) \, ds \\
\lesssim \mathcal{M}_1(T)^2 \left( \int_0^{t/2} \langle t-s \rangle^{-7/4} \langle s \rangle^{-1/2} \, ds + \int_{t/2}^t \langle t-s \rangle^{-3/4} \langle s \rangle^{-5/4} \, ds \right) \\
\lesssim \mathcal{M}_1(T)^2 \langle t \rangle^{-1} \quad \text{for } t \in [0, T].
\]

The rest of nonlinear terms $\sum_{i=2}^8 \text{diag}(1, \partial_y)\mathcal{N}_i$ can be rewritten as a sum of $\mathcal{N}'(t)$ and $\partial_y \mathcal{N}''(t)$ satisfying

\[
\mathcal{N}'(t) \lesssim (\mathcal{M}_1(T) + \mathcal{M}_2(T))^2 \langle t \rangle^{-5/4} \quad \text{for } t \in [0, T], \\
\mathcal{N}''(t) \lesssim (\mathcal{M}_1(T) + \mathcal{M}_2(T))^2 \langle t \rangle^{-1} \quad \text{for } t \in [0, T].
\]

First, we prove decay estimates of $\mathcal{N}_2^k$ $(k = 0, 1, 2)$ presuming that (7.12) is true. Then for $t \in [0, T]$ and $0 \leq k \leq 2,$

\[
\mathcal{N}_2^k \lesssim (\mathcal{M}_1(T) + \mathcal{M}_2(T))^2 \langle t \rangle^{-\min\{1, (2k+1)/4\}}.
\]

Indeed, Lemma 4.1 implies that for $t \in [0, T],$

\[
\int_0^t \|\partial_y^k U(t, s)\mathcal{N}'\|_Y \, ds \lesssim \int_0^t \|\partial_y^k U(t, s)\|_{B(Y_1, Y_1)} \|\mathcal{N}'(s)\|_{Y_1} \, ds \\
\lesssim (\mathcal{M}_1(T) + \mathcal{M}_2(T))^2 \int_0^t \langle t-s \rangle^{-(2k+1)/4} \langle s \rangle^{-5/4} \, ds,
\]

\[
\int_0^t \|\partial_y^k U(t, s)\partial_y \mathcal{N}''\|_Y \, ds \lesssim \int_0^t \|\partial_y^{k+1} U(t, s)\|_{B(Y_1, Y_1)} \|\mathcal{N}''(s)\|_{Y_1} \, ds \\
\lesssim \mathcal{M}_1(T)(\mathcal{M}_1(T) + \mathcal{M}_2(T)) \int_0^t \langle t-s \rangle^{-(2k+3)/4} \langle s \rangle^{-1} \, ds.
\]

By Claim 4.1

\[
\int_0^t \langle t-s \rangle^{-(2k+1)/4} \langle s \rangle^{-5/4} \, ds \lesssim \langle t \rangle^{-(2k+1)/4} \quad \text{for } k = 0, 1, 2,
\]
Combining (7.14), (7.16) and (7.17), we have
\[
\| \frac{\partial}{\partial y} (\sqrt{c} - \sqrt{\tilde{c}}) R_2^T \|_{Y_1} \lesssim \sum_{j=3,4, k=1,2} \| S_k^j [\psi^c] (\sqrt{c} - \sqrt{\tilde{c}}) R_2^T \|_{Y_1} \]
\[
\lesssim \left( \| S_k^1 [\psi^c] \|_{B(1, \infty)} ((\sqrt{c} - \sqrt{\tilde{c}}) R_2^T \|_{L^1} + \| S_k^2 [\psi^c] \|_{B(1, \infty)} ((\sqrt{c} - \sqrt{\tilde{c}}) R_2^T \|_{L^2} \right) \]
\[
\lesssim M_1 (T)^3 (s)^{-3/2},
\]
\[
\| \tilde{S}_4^2 E_2 R_2^T \|_{Y_1} \lesssim \sum_{j=3,4, k=1,2} \| S_k^j [\psi^c] (\sqrt{c} - \sqrt{\tilde{c}}) R_2^T \|_{Y_1} \]
\[
\lesssim \left( \| S_k^1 [\psi^c] \|_{B(1, \infty)} \| v(t, \cdot) \|_{X_1} \right) \lesssim M_1 (T)^3 (s)^{-3/2}.
\]
Combining the above, we have
\[
\| \mathcal{N}_{22} \|_{Y_1} \lesssim M_1 (T)^3 (s)^{-5/4} \quad \text{for } s \in [0, T].
\]
By Claim D.7 and the fact that diag(1, \partial_y) \mathcal{N}_{23} = \frac{1}{2} \partial_y (0, R_2^T),
\[
\| \text{diag}(1, \partial_y) \mathcal{N}_{23} \|_{Y_1} \lesssim \| \partial_y R_2^T \|_{Y_1} \lesssim M_1 (T)^2 \| s \|^{-5/4} \quad \text{for } s \in [0, T].
\]
Combining (7.14), (7.16) and (7.17), we have
\[
\| \text{diag}(1, \partial_y) \mathcal{N}_{2} \|_{Y_1} \lesssim M_1 (T)^2 \| s \|^{-5/4} \quad \text{for } s \in [0, T].
\]
Next, we will estimate \mathcal{N}_3. Claims D.2 and D.5 imply that for s \in [0, T],
\[
\| R^T \|_{Y_1} + \| R^T \|_{Y_1} + \| \tilde{S}_4^2 R_2^T \|_{Y_1} \lesssim (M_1 (T) + M_2 (T))^2 \| s \|^{-3/2}.
\]
Using Claims \textbf{B.3} \textbf{B.4} and \textbf{1.2}, we can show that for $s \in [0, T]$, 
\[
\left\| \bar{S}_4 \left( \frac{0}{\varepsilon} \right) \right\|_{Y_1} \lesssim \sum_{j=3,4} \| S_j^3 \psi' (\sqrt{2} - \sqrt{3}) \|_{Y_1} \lesssim M_1(T)^2 e^{-a(4t+L)}(s)^{-1/2}
\]
in the same way as \textbf{(7.15)}. Thus for $s \in [0, T]$, 
\[
\|N_3(s)\|_{Y_1} \lesssim \|B_3^{-1}\|_{B(Y_1)} \|\bar{R}_3\|_{Y_1} \lesssim (M_1(T) + M_2(T))^2(s)^{-3/2}.
\]

Next we will estimate $N_4^2$. Let $N_{41} = B_3^{-1} \bar{R}_2$ and $N_{42} = [B_3^{-1}, \partial_y] \bar{R}_2$. Then $N_4 = \partial_y N_{41} + N_{42}$. By Claims \textbf{D.1} \textbf{D.5} and \textbf{C.6}, 
\[
\|\bar{R}_2\|_{Y_1} \lesssim M_1(T)(M_1(T) + M_2(T))(s)^{-1} \quad \text{for } s \in [0, T].
\]
By Claims \textbf{6.2} and \textbf{7.11} 
\[
\|B_3^{-1}, \partial_y\|_{B(Y, Y_1)} \lesssim (M_1(T) + M_2(T))(s)^{-3/4} \quad \text{for } s \in [0, T].
\]
Combining \textbf{(7.20)}, \textbf{(7.21)} and Claim \textbf{6.2} we have for $s \in [0, T]$, 
\[
\|N_{41}(s)\|_{Y_1} \lesssim M_1(T)(M_1(T) + M_2(T))(s)^{-1},
\]
\[
\|N_{42}(s)\|_{Y_1} \lesssim M_1(T)(M_1(T) + M_2(T))^2(s)^{-7/4}.
\]

Next we will estimate $N_5$. Let $r_8 = \bar{P}_1 \{(I + C_2)(c_y x_y) - (bx_y y)\}$. Then 
\[
\|r_8\|_{Y} \lesssim \|r_8\|_{Y_1} \lesssim M_1(T)^2(s)^{-1}, \quad \|\partial_y r_8\|_{Y_1} \lesssim M_1(T)^2(s)^{-5/4}.
\]
Here we use Claims \textbf{6.1} and \textbf{B.7}. By \textbf{(6.15)} and \textbf{C.6}, 
\[
\|R^8\|_{Y_1} \lesssim \bar{S}_1 \|B(Y_1)\|_{\partial_y} \|r_8\|_{Y_1} + \left( \|\partial_y, \bar{S}_1\|_{B(Y, Y_1)} + \sum_{j=2,4,5} \|\bar{S}_j\|_{B(Y, Y_1)} \right) \|r_8\|_{Y} + \|\bar{S}_5\|_{B(Y_1)} \|r_8\|_{Y_1}.
\]
Combining the above with \textbf{(C.1) - (C.5)} and \textbf{(C.10)} in Appendix \textbf{C} we have 
\[
\|R^8\|_{Y_1} \lesssim M_1(T)^2(1 + M_1(T) + M_2(T))(s)^{-5/4} \quad \text{for } s \in [0, T].
\]
Since $\|\partial_y \bar{S}_0\|_{B(Y, Y)} \lesssim \sigma_0^2$ by Claim \textbf{B.1} and \textbf{C.6}, it follows from Claim \textbf{D.6} that 
\[
\|R^{10}\|_{Y_1} \lesssim \|b_{yy} - c_{yy}\|_{Y_1} \lesssim M_1(T)^2(s)^{-5/4} \quad \text{for } s \in [0, T].
\]
By Claims \textbf{D.3} and \textbf{D.6} 
\[
\|R^{10}\|_{Y_1} \lesssim e^{-a(4s+L)}\|b - \bar{c}\|_{Y_1} \lesssim M_1(T)^2(s)^{-1/2} e^{-a(4s+L)} \quad \text{for } s \in [0, T].
\]
Thus we have 
\[
\|N_5\|_{Y_1} \lesssim \|\bar{R}_3\|_{Y_1} \lesssim M_1(T)^2(s)^{-5/4} \quad \text{for } s \in [0, T].
\]

Next we will estimate $N_6^2$. Since the second column of $\tilde{A}_1(t)$ is 0, 
\[
N_6 = (B_3^{-1} - B_4^{-1}) \tilde{A}_1(t) \begin{pmatrix} b \\ 0 \end{pmatrix}.
\]
By the definitions of $B_3$ and $B_4$, 
\[
B_3 - B_4 = \tilde{c}_1 + \partial_y^2 (\bar{S}_1 - \bar{S}_1) + \partial_y^2 \bar{S}_2 - (\bar{S}_3 - \bar{S}_3) - \bar{S}_4 - \bar{S}_5.
\]
Hence it follows from Claims 6.1, 7.2, (C.2) and (C.4)–(C.6) that
\[ \|B_1 - B_3\|_{B(Y,Y_1)} \leq ||\mathcal{C}_1||_{B(Y,Y_1)} + \sum_{j=1}^{3} ||\partial_y^3 (\bar{S}_j - \bar{S}_j)\|_{B(Y,Y_1)} + \sum_{j=2,4,5} ||\partial_y^2 \bar{S}_j\|_{B(Y,Y_1)} \]
\[ \lesssim (M_1(T) + M_2(T)) \langle s \rangle^{-1/4} \quad \text{for } s \in [0,T]. \]

In view of Claims 6.2 and 6.3 and the above, we have for \( s \in [0,T] \),
\[ \|B_3^{-1} - B_4^{-1}\|_{B(Y,Y_1)} \leq \|B_4^{-1}\|_{B(Y_1)} \|B_4 - B_3\|_{B(Y,Y_1)} \|B_3^{-1}\|_{B(Y)} \]
\[ \lesssim (M_1(T) + M_2(T)) \langle s \rangle^{-1/4}. \]

Combining Claim D.3 and (7.25), we have for \( s \in [0,T] \),
\[ ||N_0||_Y \lesssim \|B_3^{-1} - B_4^{-1}\|_{B(Y,Y_1)} \|\bar{A}_1(t)\|_{B(Y)} \|b\|_Y \]
\[ \lesssim (M_1(T) + M_2(T))(M_1(T)) \langle s \rangle^{-1/2} e^{-a(4s+L)}. \]

Since \( ||\bar{S}_0||_{B(Y)} \lesssim 1 \) by Claim B.1, it follows from C.6 and (7.25) that for \( s \in [0,T], \)
\[ ||N_1||_Y \lesssim \|B_3^{-1} - B_4^{-1}\|_{B(Y,Y_1)} \|b_{yy}\|_Y \lesssim (M_1(T) + M_2(T)) \langle s \rangle^{-5/4}. \]

Finally, we will estimate \( N_8 \). Let
\[ N_{s1} = (B_4^{-1} - B_3^{-1}) \partial_y^3 \bar{S}_0 \left( \begin{array}{c} 0 \\ x_{yy} \end{array} \right), \quad \tilde{R}^4 = B_4^{-1} B_2 \left( \begin{array}{c} 0 \\ x_{yy} \end{array} \right), \]
\[ N_{s2} = B_3^{-1}(B_4 - B_3 + \bar{C}_1) \tilde{R}^4, \quad N_{s3} = B_1^{-1}(B_3 - B_1 - \bar{C}_1) B_3^{-1} \bar{C}_1 \tilde{R}^4, \]
\[ N_{s4} = -B_1^{-1} \bar{C}_1(I - B_3^{-1} \bar{C}_1) \tilde{R}^4. \]

Then \( N_8 = \sum_{1 \leq j \leq 4} N_{sj} \). Since \( [\partial_y, \bar{S}_0] = 0 \), we have
\[ \left\| \partial_y \bar{S}_0 \left( \begin{array}{c} 0 \\ x_{yy} \end{array} \right) \right\|_Y \lesssim ||x_{yyy}||_Y. \]

Combining the above with C.6 and (7.25), we see that for \( s \in [0,T], \)
\[ ||N_{s1}||_Y \lesssim \|b_{yy}\|(M_1(T) + M_2(T)) \langle s \rangle^{-1/4} ||x_{yyy}||_Y \]
\[ \lesssim (M_1(T) + M_2(T))(M_1(T)) \langle s \rangle^{-5/4}. \]

Next we will estimate \( N_{s2} \). Let
\[ n_3 = \partial_y(\bar{S}_1 - \bar{S}_1 + \bar{S}_2) \partial_y \tilde{R}^4, \quad n_4 = \partial_y [\partial_y, \bar{S}_1 + \bar{S}_2] \tilde{R}^4, \quad n_5 = (\bar{S}_3 - \bar{S}_3 + \bar{S}_4 + \bar{S}_5) \tilde{R}_4. \]

Then \( N_{s2} = B_3^{-1}(n_5 - n_3 - n_4) \). Claim 6.3 and the fact that \( [\partial_y, B_4] = 0 \) imply that for \( s \in [0,T], \)
\[ \|\tilde{R}^4\|_Y \lesssim ||x_{yyy}||_Y \lesssim M_1(T) \langle s \rangle^{-3/4}, \]
\[ \|\partial_y \tilde{R}^4\|_Y \lesssim ||x_{yyy}||_Y \lesssim M_1(T) \langle s \rangle^{-1}. \]

We see that \( ||n_3||_Y \lesssim M_1(T)^2 \langle s \rangle^{-5/4} \) follows from Claim 7.2, C.2, C.6 and (7.29) and that \( ||n_4||_Y \lesssim M_1(T)^2 \langle s \rangle^{-3/2} \) follows from C.10, (7.11), C.6 and (7.29). By Claim 7.2, C.4, C.5 and (7.29),
\[ ||n_5||_Y \lesssim M_1(T) \langle s \rangle^{-3/4}(M_1(T)e^{-a(4s+L)} \langle s \rangle^{-1/4} + M_2(T) \langle s \rangle^{-3/4}). \]

Thus we have
\[ ||N_{s2}||_Y \lesssim (M_1(T)(M_1(T) + M_2(T)) \langle s \rangle^{-5/4} \quad \text{for } s \in [0,T]. \]
Next we will estimate \( N_{83} \). Let
\[
\begin{align*}
n_6 &= [\partial_y, \tilde{S}_1 + \tilde{S}_2] B_3^{-1} \tilde{C}_1 R^4, \\
n_7 &= (\tilde{S}_1 + \tilde{S}_2) \partial_y, B_3^{-1} \tilde{C}_1 R^4, \\
n_8 &= (\tilde{S}_1 + \tilde{S}_2) B_3^{-1} \partial_y \tilde{C}_1 R^4, \\
n_9 &= (\tilde{S}_3 + \tilde{S}_4 + \tilde{S}_5) B_3^{-1} \tilde{C}_1 R^4.
\end{align*}
\]
Then \( N_{83} = B_3^{-1} \partial_y (n_6 + n_7 + n_8) - B_3^{-1} n_9 \). By Claims 6.2, B.6, B.7 and 7.1 with (7.29), we obtain
\[
\|n_6\|_{Y_1} \lesssim \sum_{j=1,2} \|\partial_y, \tilde{S}_j\|_B(Y, Y_1) \|B_3^{-1}\|_{B(Y)} \|\tilde{C}_1\|_{B(Y)} \|R^4\|_{Y Y_1} \lesssim M_1(T)^2(s)^{-2} \quad \text{for} \quad s \in [0, T].
\]
Using Claims 6.2, B.6, B.7 and (7.29), we can obtain
\[
\|n_7\|_{Y_1} \lesssim M_1(T)^2(M_1(T) + M_2(T))\langle s \rangle^{-2}, \quad \|n_8\|_{Y_1} \lesssim M_1(T)^2(s)^{-5/4}, \quad \|n_9\|_{Y_1} \lesssim M_1(T)^2(s)^{-1}(\varepsilon_0 + M_2(T)\langle s \rangle^{-3/4}).
\]
Combining the above, we have
\[
(7.31) \quad \|N_{83}\|_{Y_1} \lesssim M_1(T)^2(s)^{-5/4}.
\]
Since \( \text{diag}(\partial_y) B_3^{-1} \tilde{C}_1 = \frac{1}{2} \partial_y \tilde{C}_1 \),
\[
2 \text{diag}(1, \partial_y) N_{84} = \left\{ [\partial_y, \tilde{C}_1] B_3^{-1} + \tilde{C}_1 [\partial_y, B_3^{-1}] + (\tilde{C}_1 B_3^{-1} - I) \partial_y \right\} \tilde{C}_1 R^4.
\]
Combining Claims 0.2, B.6, B.7 and (7.29) with (7.29), we obtain
\[
(7.32) \quad \|\text{diag}(1, \partial_y) N_{84}\|_{Y_1} \lesssim M_1(T)^2(s)^{-5/4} \quad \text{for} \quad s \in [0, T].
\]
Thus for \( s \in [0, T], \)
\[
(7.33) \quad \sum_{1 \leq j \leq 4} \|\text{diag}(1, \partial_y) N_{8j}\|_{Y_1} \lesssim M_1(T)(M_1(T) + M_2(T))\langle s \rangle^{-5/4}
\]
follows from (7.28), (7.30), (7.31) and (7.32).

Now let \( N' = \text{diag}(1, \partial_y) \left( \sum_{i \neq j} N_i + N_{42} \right) \) and \( N'' = \text{diag}(1, \partial_y) N_{41} \). Then
\[
(7.34) \quad \|\partial_y^k c(t)\|_{Y} + \|\partial_y^{k+1} x(t)\|_{Y} \lesssim \langle t \rangle^{-(2k+1)/4} \|v_0\|_X + \Theta_1^k + \Theta_2^k + M_1(T)^2\langle t \rangle^{-\min\{2k+3/4,3/2\}}.
\]
for \( 0 \leq k \leq 2 \) and \( t \in [0, T] \). Combining (7.30)–(7.31) and (7.32) with (7.33), we obtain (7.41). This completes the proof of Lemma 7.1. \( \square \)

8. \( L^2 \) bound on \( v(t, z, y) \)

In this section, we will estimate \( M_3(T) \) assuming that \( M_1(T) \) and \( M_2(T) \) are small. First, we will show a variant of the \( L^2 \) conservation law on \( v \).

**Lemma 8.1.** Let \( a \in (0, 2) \) and \( T > 0 \). Suppose \( v(t) \in C([0, T], X \cap L^2(\mathbb{R}^2)) \) is a solution of (6.1) and that \( v(t), c(t) \) and \( x(t) \) satisfy (5.3), (5.9) and (5.10). Then
\[
Q(t, v) := \int_{\mathbb{R}^2} \left\{ v(t, z, y)^2 - 2\psi_{c(t, y), L}(z + 4t)v(t, z, y) \right\} \, dz \, dy
\]
satisfies for \( t \in [0, T] \),

\[
Q(t, v) = Q(0, v) + 2 \int_0^t \int_{\mathbb{R}^2} \left( \ell_{11} + \ell_{12} + 6 \varphi'_c(s, y)(z) \tilde{\varphi}_c(s, y)(z) \right) v(s, z, y) \, dzdyds
- 2 \int_0^t \int_{\mathbb{R}^2} \ell \tilde{\varphi}_c(t, y, L) \tilde{T}(z + 4t) \, dzdyds - 6 \int_{\mathbb{R}^2} \varphi'_c(t, y)(z) v(t, z, y)^2 \, dzdy
- 6 \int_{\mathbb{R}^2} \nabla \varphi_c(t, y)(z_1) \, dz_1
+ c_p(t, y)^2 \int_{-\infty}^z \partial^2 \varphi_c(t, y)(z_1) \, dz_1 \} \, dzdy.
\]

**Proof.** Suppose \( v_0 \in H^3(\mathbb{R}^2) \) and \( \partial^{-1}_z v_0 \in H^2(\mathbb{R}^2) \). Then as in Section 6, we have \( v(t) \in C([0, T]; H^3(\mathbb{R}^2)) \) and \( \partial^{-1}_z \partial_y v(t) \in C([0, T]; H^3(\mathbb{R}^2)) \). Using (6.1), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \psi_c(t, y) v(t, x, y)^2 \, dx \, dy = 2 \int v(\mathcal{L}_c(t, y)) v + \ell + \partial_z(N_1 + N_2) + N_3 \, dzdy
= 2 \int \ell v \, dzdy + 6 \int (\tilde{\psi}_c' - \varphi'_c) v^2 \, dzdy,
\]

and

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \tilde{\psi}_c(t, y, L) \tilde{T}(z + 4t) v(t, z, y) \, dzdy = \int \left( c_p \partial_z \tilde{\psi}_c + 4 \tilde{\psi}_c' \right) v
+ \int \tilde{\psi}_c \{ \mathcal{L}_c(t, y) v + \ell + \partial_z(N_1 + N_2) + N_3 \} \, dzdy.
\]

Since \( \partial^{-1}_z \partial^2_y v(z, y) = - \int_{-\infty}^z \partial^2_y v(z_1, y) \, dz_1 \), we have

\[
\int_{\mathbb{R}^2} \tilde{\psi}_c \mathcal{L}_c \, v \, dzdy = - \int \tilde{\psi}_c \partial_z (\tilde{\psi}_c^2 - 2c + 6 \varphi_c) v \, dzdy - 3 \int \tilde{\psi}_c \partial^{-1}_z \partial_y^2 v \, dzdy
= \int v(\tilde{\psi}_c^m - 2c \tilde{\psi}_c' + 6 \varphi_c \tilde{\psi}_c') \, dzdy
+ 3 \int_{\mathbb{R}^2} v \left\{ c_{yy} \int_{-\infty}^z \partial_z \tilde{\psi}_c + (c_y)^2 \int_{-\infty}^z \partial^2_z \tilde{\psi}_c \right\} \, dzdy,
\]

where \( \partial^k \tilde{\psi}_c = \partial^k \tilde{\psi}_c(t, y, L)(x + 4t) \) for \( k \geq 0 \). By integration by parts, we have

\[
\int_{\mathbb{R}^2} \tilde{\psi}_c \partial_z N_1 \, dzdy = 3 \int \tilde{\psi}_c' \, dzdy,
\]

and

\[
\int (\partial_z N_2 + N_3) \tilde{\psi}_c \, dzdy
= - \int v \left\{ (x_t - 2c - 3(x_y)^2) \tilde{\psi}_c' + 3x_{yy} \tilde{\psi}_c + 6c_y x_y \partial_z \tilde{\psi}_c + 6 \tilde{\psi}_c' \tilde{\psi}_c \right\} \, dzdy.
\]
Combining the above, we have
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \left\{ v(t, z, y)^2 - 2\psi_{c(t,y),L}(z + 4t)v(t, z, y) \right\} \, dz \, dy \\
= 2\int_{\mathbb{R}^2} \left\{ \left( \ell_{11} + \ell_{12} + 6\varphi_{c(t,y)}(z)\psi_{c(t,y)}(z) \right) v(t, z, y) - \ell\psi_{c(t,y),L}(z + 4t) \right\} \, dz \, dy \\
+ 6\int_{\mathbb{R}^2} v(t, z, y) \left\{ c_{yy}(t, y) \int_{\mathbb{R}} \left( \partial_z \varphi_{c(t,y)}(z_1) - \partial_z \psi_{c(t,y),L}(z_1 + 4t) \right) \, dz_1 \right\} \, dz \, dy \\
+ 6\int_{\mathbb{R}^2} v(t, z, y) \left\{ c_{y}(t, y)^2 \int_{\mathbb{R}} \left( \partial_z^2 \varphi_{c(t,y)}(z_1) - \partial_z^2 \psi_{c(t,y),L}(z_1 + 4t) \right) \, dz_1 \right\} \, dz \, dy \\
- 6\int_{\mathbb{R}^2} \ell_{13}^* v(t, z, y) \, dz \, dy - 6\int_{\mathbb{R}^2} \varphi_{c(t,y)}(z) v(t, z, y)^2 \, dz \, dy,
\]
where
\[
\ell_{13}^* = c_{yy}(t, y) \int_{-\infty}^{z} \partial_z \varphi_{c(t,y)}(z_1) \, dz_1 + c_y(t, y)^2 \int_{-\infty}^{z} \partial_z^2 \varphi_{c(t,y)}(z_1) \, dz_1.
\]
Since \( \int_{\mathbb{R}} \partial_z \varphi_{c}(x) \, dx = \int_{\mathbb{R}} \partial_x \psi_{c,L}(x) \, dx \) for \( k \geq 0 \) by (5.2), we see that Lemma 8.1 holds provided \( v_0 \) and \( \partial_x^{-1} \partial_y v_0 \) are smooth.

For general \( v_0 \in X \cap L^2(\mathbb{R}^2) \), we can prove Lemma 8.1 by a standard limiting argument. The mapping
\[
L^2(\mathbb{R}^2) \ni v_0 \mapsto \tilde{v}(t) \in C([0, T]; L^2(\mathbb{R}^2))
\]
is continuous for any \( T > 0 \) by [29]. On the other hand, it follows from (E.1) that a solution \( \tilde{v}(t) \) of (E.1) satisfies \( \sup_{t \in [0, T]} \| \tilde{v}(t) \|_X \leq C \), where \( C \) is a constant depending only on \( T \), \( \| \tilde{v}(0) \|_{L^2(\mathbb{R}^2)} \) and \( \| \tilde{v}(0) \|_X \). Thus the mapping
\[
X \cap L^2(\mathbb{R}^2) \ni v_0 \mapsto \tilde{v}(t) \in C([0, T]; L^2(\mathbb{R}^2; e^{ax} \, dx \, dy))
\]
is continuous since \( \| u \|_{L^2(\mathbb{R}^2; e^{ax} \, dx \, dy)} \lesssim \| u \|_{L^2(\mathbb{R}^2)}^{1/2} \| u \|_X^{1/2} \) for every \( u \in X \cap L^2(\mathbb{R}^2) \). If \( \eta_0 \) is sufficiently small, it is clear from Lemma 5.2 and Remark 5.4 that \( (\tilde{v}(t), \tilde{x}(t)) \) is \( Y \times Y \) as well as its time derivate depends continuously on \( v(t) \in L^2(\mathbb{R}^2; e^{ax} \, dx \, dy) \). This completes the proof of Lemma 8.1.

Using Lemma 8.1, we will estimate the upper bound of \( \| v(t) \|_{L^2} \).

**Lemma 8.2.** Let \( a \in (0, 1) \) and \( \delta_4 \) be as in Lemma 7.1. Then there exists a positive constant \( C \) such that if \( M_1(T) + M_2(T) + \eta_0 + e^{-\delta_4 z} \leq \delta_4 \), then
\[
M_3(T) \leq C(\| v_0 \|_{L^2(\mathbb{R}^2)} + M_1(T) + M_2(T))
\]

**Proof.** Remark 5.4 and Proposition 5.4 tell us that we can apply Lemma 8.1 for \( t \in [0, T] \) if \( M_3(T) \) and \( M_2(T) \) are sufficiently small.

Since we have for \( j, k \geq 0 \) and \( z \in \mathbb{R} \),
\[
\partial_z^j \partial_x^k \varphi_c(z) \lesssim e^{-2a|z|}, \quad \int_{-\infty}^{z} \partial_z^j \varphi_c(z_1) \, dz_1 \lesssim \min(1, e^{2az}),
\]
it follows that
\[
\int_{\mathbb{R}^2} \left( \ell_{11} + \ell_{12} \right)^* v \, dz \, dy \lesssim \| c_t - 6c_yx_y \|_{L^2} + \| x_t - 2c - 3(x_y)^2 \|_{L^2} + \| x_{yy} \|_{L^2} + \| c_{yy} \|_{L^2} + \| c_y \|_{L^2} \| v \|_X,
\]
Following the proof of Lemma 7.1, we see that
\[
\left| \int_{\mathbb{R}^2} \varphi_c'(t,y)(z)v(t,z,y)^2 \, dz \, dy \right| \lesssim \|v\|_{L^2_X}^2,
\]
Similarly,
\[
\left| \int_{\mathbb{R}^2} \varphi_c'(t,y)(z)\tilde{\psi}_c(t,y)(z)v(t,z,y) \, dz \, dy \right| \lesssim \|e^{a_x} \tilde{\psi}_c(t,y)\|_{L^2_{t,z}} \|v\|_{X} e^{-a(4t+L)}.
\]
Next, we will estimate \( \int_{\mathbb{R}^2} \ell \tilde{\psi}_c(t,y) \). In view of (8.3), we see that \( \ell_{11} + \ell_{12} \) is exponentially localized and that
\[
\left| \int_{\mathbb{R}^2} (\ell_{11} + \ell_{12})\tilde{\psi}_c(t,y)(t,z,y) \, dz \, dy \right| \leq \|e^{-a_k} (\ell_{11} + \ell_{12})\|_{L^2_{t,z}} \|e^{a_k} \tilde{\psi}_c(t,y)\|_{L^2_{t,z}}
\]
\[
\lesssim \left( \|c_t - 6c_y x_y\|_{L^2} + \|x_t - 2c - 3(x_y)^2\|_{L^2} + \|x_{yy}\|_{L^2} \right) \|e^{a_k} \tilde{\psi}_c(t,y)\|_{L^2_{t,z}}.
\]
By integration by parts and the fact that \( \|\partial_z \tilde{\psi}_c(t,y)(z)\|_{L^\infty_{t,z}} \lesssim 1 \),
\[
\int_{\mathbb{R}^2} \ell_1 \tilde{\psi}_c(t,y)(t,z,y) \, dz \, dy = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \psi_c(t,y)(z)^2 \, dz \, dy + O(\|c_y x_y\|_{L^2}\|\tilde{c}\|_{L^2}).
\]
Similarly,
\[
\left| \int_{\mathbb{R}^2} \ell_2 \tilde{\psi}_c(t,y) \, dz \, dy \right| = 3 \left| \int_{\mathbb{R}^2} \left\{ \varphi_c'(t,y)(z) - x_{yy}(t,y) \right\} \tilde{\psi}_c(t,y)(z)^2 \, dz \, dy \right|
\]
\[
\lesssim \|e^{a_k} \tilde{\psi}_c\|_{L^2_{t,z}}^2 + \|x_{yy}\|_{L^\infty} \|\tilde{\psi}_c(t,y)\|_{L^2_{t,z}}.
\]
Since \( \|\ell_{13}\|_{L^\infty_{t} L^2_{y}} + \|\ell_{23}\|_{L^\infty_{t} L^2_{y}} \lesssim \|c_{yy}\|_{L^2} + \|c_y\|_{L^4}^2 \) and \( \|\psi_c(t,y)\|_{L^1_{t} L^2_{y}} = O(\|\tilde{c}\|_{L^2}) \),
\[
\sum_{j=1,2} \left| \int_{\mathbb{R}^2} \ell_3 \tilde{\psi}_c(t,y) \, dz \, dy \right| \lesssim \|\tilde{c}\|_{L^2} (\|c_{yy}\|_{L^2} + \|c_y\|_{L^4}^2) .
\]
In view of the definition of \( \tilde{\psi} \),
\[
\|\tilde{\psi}_c(t,y)\|_{X} \lesssim \|\tilde{c}\|_{L^2} e^{-a(4t+L)},
\]
\[
\|\tilde{\psi}_c(t,y)\|_{L^2_{t}(\mathbb{R}^2)} = 2\sqrt{2} \|\sqrt{\psi} - \sqrt{2} \|_{L^2(R)} \|\tilde{\psi}\|_{L^2(R)} \lesssim \|\tilde{c}\|_{L^2}.
\]
Claims [D.3] [D.6] and (6.10) imply that for \( t \in [0,T] \),
\[
\|c_t\|_{Y} + \|x_t - 2c - 3(x_y)^2\|_{L^2} \lesssim \|b_t\|_{Y} + \|x_t - 2c - 3(x_y)^2\|_{L^2}
\]
\[
\lesssim \|c_{yy}\|_{Y} + \|x_{yy}\|_{Y} + \|A_1(t)\|_{B(Y)} \|b\|_{Y} + \|Ax_{yy}\|_{Y} + \sum_{i=2}^{8} \|N_i\|_{Y}
\]
\[
\lesssim M_1(T)(t)^{-3/4} + \sum_{i=2}^{8} \|N_i\|_{Y} .
\]
Following the proof of Lemma [7.1], we see that
\[
\sum_{2 \leq i \leq 8} \|N_i\|_{Y} \lesssim (M_1(T) + M_2(T))^2(t)^{-1}.
\]
Thus we have
\[
\|c_t\|_{Y} + \|x_t - 2c - 3(x_y)^2\|_{L^2} \lesssim M_1(T)(t)^{-3/4} + (M_1(T) + M_2(T))^2(t)^{-1}.
\]
Combining (8.1)–(8.10), (8.11) and (8.12) with Lemma 8.1 we see that for \( t \in (0, T) \),

\[
\left[ Q(s, v) + 8\|v\|_2^2 + \sqrt{c(s)} - \sqrt{\mathcal{E}} \right]_{s=0}^t \leq (\mathcal{E}_1(T) + \mathcal{E}_2(T))^2 \int_0^t \langle s \rangle^{-5/4} \, ds \leq (\mathcal{E}_1(T) + \mathcal{E}_2(T))^2. \tag{8.13}
\]

Since \( \|\sqrt{c(0)} - \sqrt{\mathcal{E}}\|_{L^2} \lesssim \|\tilde{c}(0)\|_{Y} \lesssim \|v_0\|_{X_1} \) and

\[
Q(t, v) = \|v(t)\|^2_{L^2(\mathbb{R}^2)} + O(\|\tilde{c}(t)\|_{Y}\|v(t)\|_{L^2(\mathbb{R}^2)}),
\]

Lemma 8.2 follows immediately from (8.13). Thus we complete the proof. \( \square \)

9. Low frequencies bound of \( v(t, x, y) \) in \( y \)

Let \( v_1(t) = P_1(0, 2M)v(t) \). Since \( v_1(t) \) does not include high frequency modes in the \( y \) variable, we can estimate \( v_1(t) \) in the similar manner as generalized KdV equations (32) by using the semigroup estimates obtained in Section 3. In this section, we will estimate \( v_1(t) \) in the exponentially weighted space \( X \).

**Lemma 9.1.** Let \( c_0, a \) and \( M \) be positive constants satisfying \( c_0 < a < 2 \) and \( a(2M) > a \). Suppose that \( v(t) \) is a solution of (6.1). Then there exist positive constants \( b_1, \delta_5 \) and \( C \) such that if \( \mathcal{E}_1(T) + \mathcal{E}_2(T) < \delta_5 \), then for \( t \in [0, T] \),

\[
\|v_1(t, \cdot)\|_{X} \leq Ce^{-2b_1T^2} \|v(0, \cdot)\|_{X} + \left\{ \mathcal{E}_1(T) + \mathcal{E}_2(T) \sum_{i=1}^{3} \mathcal{E}_i(T) \right\} \langle t \rangle^{-3/4}.
\]

Let \( \chi(\eta) \) be a nonnegative smooth function such that \( \chi(\eta) = 1 \) if \( |\eta| \leq 1 \) and \( \chi(\eta) = 0 \) if \( |\eta| \geq 2 \). Let \( \chi_M(\eta) = \chi(\eta/M) \) and

\[
P_{\leq M} u := \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi_M(\eta) \hat{u}(\xi, \eta) e^{i(x\xi+y\eta)} \, d\xi d\eta, \quad P_2 = I - P_{\leq M}.
\]

To estimate \( v_1(t) \), we need the following.

**Claim 9.1.** There exists a positive constant \( C \) such that

\[
\|P_{\leq M} u\|_{L^1_x L^2_y} \leq C\sqrt{\mathcal{E}}\|u\|_{L^1(\mathbb{R}^2)}.
\]

**Proof.** Applying Young’s inequality to

\[
(P_{\leq M} u)(x, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F^{-1}(\chi_M)(y - y_1) u(x, y_1) \, dy_1,
\]

we have

\[
\|P_{\leq M} u\|_{L^2_y} \leq \|F^{-1}(\chi_M)\|_{L^2(\mathbb{R})}\|u(x, \cdot)\|_{L^1(\mathbb{R})} \lesssim \sqrt{\mathcal{E}}\|u(x, \cdot)\|_{L^1(\mathbb{R})}.
\]

Integrating the above over \( \mathbb{R} \) in \( x \), we obtain (9.11). \( \square \)

**Proof of Lemma 9.1.** Let \( v_2(t) = P_2(\eta_0, M)v(t) \). Then

\[
\partial_t v_2 = \mathcal{L} v_2 + P_2(\eta_0, 2M)\{\ell + \partial_x (N_1 + N_2 + N_3) + N_3\},
\]

...
where \( N'_1 = 2\tilde{c}(t, y)v(t, z, y) + 6(\varphi(z) - \varphi_c(t, y)(z))v(t, z, y) \). Hereafter we abbreviate \( P_2(y_0, 2M) \) as \( P_2 \). By Proposition 3.2 and Corollary 3.3 we may assume that

\[
\|e^{t\mathcal{L}}P_2f\|_X \leq Ke^{-2b_1\eta^2 t}\|f\|_X ,
\]

\[
\|e^{t\mathcal{L}}P_2\partial_z f\|_X \leq K(1 + t^{-1/2})e^{-2b_1\eta^2 t}\|f\|_X ,
\]

\[
\|e^{t\mathcal{L}}P_2\partial_x f\|_X \leq K(1 + t^{-3/4})e^{-2b_1\eta^2 t}\|e^{az}f\|_{L_1^2 L_2^2} ,
\]

where \( K \) and \( b_1 \) are positive constants independent of \( f \in X \) and \( t > 0 \). Applying the semigroup estimates Lemma 3.2 and Corollary 3.3 to (9.3), we have

\[
\|v_2(t)\|_X \lesssim e^{-2b_1\eta^2 t}\|v_2(0)\|_X + \int_0^t e^{-2b_1\eta^2 (t-s)}(t-s)^{-3/4}\|e^{az}N_1(s)\|_{L_1^2 L_2^2} ds + \int_0^t e^{-2b_1\eta^2 (t-s)}(t-s)^{-1/2}(\|N_2(s)\|_X + \|N'_2(s)\|_X) ds + \int_0^t e^{-2b_1\eta^2 (t-s)}(\|\ell(s)\|_X + \|N_3(s)\|_X) ds .
\]

By Claim 5.1

\[
\|e^{az}P_2N_1\|_{L_1^2 L_2^2} \lesssim \sqrt{M}\|v\|_{L_2^2}\|v\|_X \lesssim \sqrt{M}M_2(T)M_3(T)(t)^{-3/4} \quad \text{for } t \in [0, T].
\]

By (5.12), we have for \( t \in [0, T] \),

\[
\|\ell_1\|_X \lesssim \|x_t - 2c - 3(x_y)^2\|_{L_2^2} + \|c_t - 6c_y x_y\|_{L_2^2} + \|x_{yy}\|_{L_2^2} + \|c_{yy}\|_{L_2^2} + \|c_y\|_{L_4^2} \lesssim (M_1(T) + M_2(T)^2)(t)^{-3/4} ,
\]

\[
\|\ell_2\|_X \lesssim e^{-a(4t + L)}(\|c_t - 6c_y x_y\|_{Y} + \|x_t - 2c - 3(x_y)^2\|_{Y} + \|\tilde{c}\|_{Y} + \|x_{yy}\|_{Y} + \|c_{yy}\|_{Y} + \|c_y\|_{L_4^2}) \lesssim e^{-a(4t + L)}(M_1(T) + M_2(T)^2)(t)^{-1/4} ,
\]

and

\[
\|N_2\|_X + \|N'_2\|_X \lesssim (\|x_t - 2c - 3(x_y)^2\|_{L_\infty} + \|\tilde{c}\|_{L_\infty})\|v\|_X \lesssim (M_1(T) + M_2(T))M_3(T)(t)^{-5/4} .
\]

Here we use \( \sup_{y,z}(|\varphi_c(t,y)(z) - \varphi(z)| + |\tilde{\varphi}_c(t,y)|) \lesssim \|\tilde{c}(t)\|_{L_\infty} \). Since \( \|\partial_y P_2\|_{B(X)} \lesssim M \),

\[
\|P_2N_3\|_X \lesssim M(\|x_y\|_{L_\infty} + \|x_{yy}\|_{L_\infty})\|v\|_X \lesssim MM_1(T)M_2(T)(t)^{-5/4} \quad \text{for } t \in [0, T].
\]

As long as \( v(t) \) satisfies the orthogonality condition (5.3) and \( \tilde{c}(t, y) \) remains small, we have

\[
\|v_1(t) - v_2(t)\|_X \lesssim \sup_{y}|\tilde{c}(t, y)||v_1(t)\|_X ,
\]
and \( \frac{1}{2} \| v_2(t) \|_X \leq \| v_1(t) \|_X \leq 2 \| v_2(t) \|_X \). Thus we have for \( t \in [0, T] \),

\[
\| v_1(t) \|_X \leq e^{-2b_1\delta^2 t} \| v(0) \|_X + M_4(T) \int_0^t e^{-2b_1\delta^2 (t-s)} \langle s \rangle^{-3/4} ds \\
+ M_2(T) \sum_{i=1}^3 M_i(T) \int_0^t e^{-2b_1\delta^2 (t-s)} \left\{ 1 + (t-s)^{-3/4} \right\} \langle s \rangle^{-3/4} ds \\
\leq e^{-2b_1\delta^2 t} \| v(0) \|_X + \left\{ M_1(T) + M_2(T) \sum_{i=1}^3 M_i(T) \right\} \langle t \rangle^{-3/4}.
\]

Thus we complete the proof. \( \square \)

10. Virial estimates

If we apply the argument in Section 9 to \( P_{\geq M} v(t) \), it requires boundedness of \( \| v(t) \|_{L^p(R^2)} \) with \( p > 2 \), which remains unknown even for small solutions around 0. Instead of the semigroup estimate in Section 3, we will make use of a virial estimates of \( v \) in the exponentially weighted space. We remark that the virial estimate for \( L^2 \)-solutions to the KP-II equation \((2.1)\) was shown in \([8]\).

**Lemma 10.1.** Let \( \alpha \in (0, 2) \) and \( v \) be a solution to \((6.1)\). There exist positive constants \( \delta_6, M \) and \( C \) such that if \( \sum_{i=1}^3 M_i(T) < \delta_6 \), then

\[
\| v(t) \|_X^2 \leq e^{-2 \alpha t} \| v(0) \|_X^2 + C \int_0^t e^{-2 \alpha (t-s)} \left( \| \ell(s) \|_X^2 + \| P_{\leq M} v(s) \|_X^2 \right) ds.
\]

To prove Lemma 10.1, we use the following.

**Claim 10.1.** Let \( \alpha > 0 \) and \( p_n(x) = e^{2\alpha n}(1 + \tanh \alpha(x - n)) \). There exists a \( C > 0 \) such that for every \( n \in \mathbb{N} \)

\[
(\int_{\mathbb{R}^2} p_n(x) u^2(x, y) \, dx \, dy)^{1/2} \leq C \int_{\mathbb{R}^2} p_n'(x) \left( (\partial_x u)^2 + (\partial_y u)^2 + u^2 \right) (x, y) \, dx \, dy.
\]

Claim 10.1 follows in exactly the same way as \([28\text{ Lemma 2]} \) and \([26\text{ Claim 5.1]} \). So we omit the proof.

**Proof of Lemma 10.1.** Let \( p_n \) be as in Claim 10.1. Then \( p_n(z) \uparrow e^{2az} \) and \( p_n'(z) \uparrow 2ae^{2az} \) as \( n \to \infty \) and \( 0 < p_n''(z) \leq ap_n(z), \ |p_n'''(z)| \leq 4a^2 p_n'(z) \) and \( ap_n(z)^2 = e^{2az}p_n'(z) \) for \( z \in \mathbb{R} \).

First, we will derive a virial identity for \( v(t) \) assuming \( v_0 \in H^3(\mathbb{R}^2) \) and \( \partial^{-1} v_0 \in H^2(\mathbb{R}^2) \) so that \( v(t) \in C([0, T]; H^3(\mathbb{R}^2)) \) and \( \partial^{-1} v(t) \in C([0, T]; H^2(\mathbb{R}^2)) \). Multiplying \((6.1)\) by \( 2e^{2at} p_n(z) v(t, z, y) \) and integrating the resulting equation by part, we have for \( t \in [0, T] \),

\[
\frac{d}{dt} \left( e^{2at} \int_{\mathbb{R}^2} p_n(z) v(t, z, y)^2 \, dz \, dy \right) + e^{2at} \int_{\mathbb{R}^2} p_n'(z) \left( E(v) - 4v^3 \right) (t, z, y) \, dz \, dy \\
= e^{2at} \left\{ \int_{\mathbb{R}^2} 2ap_n(z) v(t, z, y)^2 \, dz \, dy + \sum_{k=1}^3 III_k(t) \right\},
\]

where \( \sum_{k=1}^3 III_k(t) = \sum_{k=1}^3 \int_{\mathbb{R}^2} \left( \sum_{i,j,l=1}^3 \partial_i \partial_j \partial_l v(t, z, y) \right)^2 \, dz \, dy \).
By the Schwarz inequality, 

\[ III_1 = 2 \int_{R^2} p_n(z) \xi(t, z, y) dz dy ds, \]

\[ III_2 = - \int_{R^2} p'_n(z) \left( (\hat{x}_t(t, y) - 3x_y(t, y))^2 \right) v(t, z, y)^2 dz dy, \]

\[ III_3 = \int_{R^2} \left\{ p''_n(z) + 6[\partial_z, p_n(z)] \left( \psi_c(t, z) - \psi_c(t, y), L(z + 4t) \right) \right\} v(t, z, y)^2 dz dy. \]

Integrating (10.2) over \([0, t]\), we have

\[ e^{2at} \int_{R^2} p_n(z) v(t, z, y)^2 dz dy + \int_0^t e^{2as} \int_{R^2} p'_n(z) (E(v) - 4v^2) (s, z, y) dz dy ds \]

\[ = \int_{R^2} p_n(z) v(0, z, y)^2 dz dy + \int_0^t e^{2as} \int_{R^2} 2ap_n(z) v(s, z, y)^2 dz dy ds \]

\[ + \int_0^t e^{2as} \int_{R^2} \{ III_1(s) + III_2(s) + III_3(s) \} ds, \]

We can prove (10.3) for any \( v(t) \in C([0, T]; L^2(R^2)) \cap L^\infty([0, T]; X) \) satisfying \( \sum_{i=1}^3 M_i(T) < \delta \) in the same way as the proof of Lemma 8.1.

By the Schwarz inequality and Claim 10.1,

\[ \left| \int_{R^2} p'_n(z) v(t, z, y)^3 dz dy \right| \leq \| v(t) \|_{L^2} \left( \int_{R^2} p'_n(z)^2 v(t, z, y)^4 dz dy \right)^{1/2} \]

\[ \leq \| v(t) \|_{L^2} \int_{R^2} p'_n(z) E(v(t, z, y)) dz dy. \]

By the Schwarz inequality,

\[ |III_1| \leq \int_{R^2} p'_n(z)^2 v^2 dz dy + \int_{R^2} \frac{p_n(z)^2}{p'_n(z)} v^2 dz dy. \]

Since \( Y \subset H^1(R) \), we have \( \sup_{t \in [0, T], y \in R} |\hat{x}_t(t, y) - 3x_y(t, y)|^2 \leq M_1(T) \) from (8.12) and

\[ |III_2| \leq \| M_1(T) \int_{R^2} p_n(z) v(t, z, y)^2 dz dy. \]

Let

\[ M = \sup_{n, z} \left| p''_n(z) \right| + 6 \sup_{n, t, y, z} \frac{\left| [\partial_z, p_n(z)] \left( \psi_c(t, z) - \psi_c(t, y), L(z + 4t) \right) \right|}{p'_n(z)}. \]

Then

\[ |III_3| \leq M \int_{R^2} p_n(z) v(t, z, y)^2 dz dy. \]

Let \( v_\prec = P_{\leq M} v \) and \( v_\succ = P_{\geq M} v \). For \( y \)-high frequencies, the potential term can be absorbed into the left hand side. Indeed it follows from Plancherel’s theorem.
and the Schwarz inequality that
\[
\int_{\mathbb{R}^2} p_n'(z) \left( \left( \partial_z v > \right)^2 + \left( \partial_y^{-1} \partial_y v > \right)^2 \right) (t, z, y) \, dz \, dy = \int_{\mathbb{R}^2} p_n'(z) \left( |\partial_z F_y (v >)|^2 + \eta^2 |\partial_y^{-1} F_y (v >)|^2 \right) (t, z, \eta) \, dz \, d\eta \geq 2M \int_{\mathbb{R}^2} p_n'(z)v_>(t, z, y)^2 \, dz \, dy.
\]
Combining the above, we have for \( t \) and the Schwarz inequality that
\[
\int_{\mathbb{R}^2} p_n'(z) \left( \left( \partial_z v > \right)^2 + \left( \partial_y^{-1} \partial_y v > \right)^2 \right) (t, z, y) \, dz \, dy = \int_{\mathbb{R}^2} p_n'(z) \left( \left( \partial_z F_y (v >) \right)^2 + \eta^2 |\partial_y^{-1} F_y (v >)|^2 \right) (t, z, \eta) \, dz \, d\eta \geq 2M \int_{\mathbb{R}^2} p_n'(z)v_>(t, z, y)^2 \, dz \, dy.
\]
Combining the above, we have for \( t \in [0, T] \),
\[
e^{2at} \int_{\mathbb{R}} p_n(z) v(t, z, y)^2 \, dz \, dy \leq \int_{\mathbb{R}} p_n(z) v(0, z, y)^2 \, dz \, dy + \int_{0}^{t} e^{2as} \frac{p_n(z)^2}{\rho_n(z)} \ell(s)^2 \, dz \, dy + M \int_{0}^{t} e^{2as} p_n'(z) v < (s, z, y)^2 \, dz \, dy \, ds
\]
if \( \delta_6 \) is sufficiently small. By passing to the limit as \( n \to \infty \), we obtain Lemma 10.1. Thus we complete the proof. □

Combining Lemmas 9.1 and 10.1 we obtain the following.

**Lemma 10.2.** Let \( a \) and \( M \) be as in Lemmas 9.1 and 10.1. There exist positive constants \( \delta_7 \) and \( C \) such that if \( \sum_{i=1}^{3} M_i(T) \leq \delta_7 \), then
\[
M_2(T) \leq C(\|v_0\|_X + M_1(T)).
\]

**Proof.** Since \( \chi_M(\eta) = 0 \) for \( \eta \in \mathbb{R} \setminus [-2M, 2M] \), we have \( \|P_{\leq M} v(t)\|_X \leq \|v_1(t)\|_X \). Combining Lemma 10.1 with Lemma 9.1, (9.4) and (9.5), we have for \( t \in [0, T] \),
\[
\|v(t)\|_X \leq e^{-bt} \|v(0)\|_X + \{M_1(t) + M_2(T) (M_1(T) + M_2(T) + M_3(T))\} \langle t \rangle^{-\frac{3}{4}}.\]
Since \( \|v(0)\|_X \lesssim \|v_0\|_X \) by Lemma 5.2, we obtain (10.5) if \( \delta_7 \) is sufficiently small. Thus we complete the proof. □

11. **Proof of Theorem 1.1**

Now we are in position to complete the proof of Theorem 1.1.

**Proof.** Since the KP-II equation has the scaling invariance, we may assume that \( c_0 = 2 \) without loss of generality. Let \( \delta_5 = \min_{0 \leq 1 \leq 7} \delta_5 / 2 \).

Since \( v_0 \in H^1(\mathbb{R}^2) \cap X \),
\[
\tilde{v}(t, x, y) = u(t, x + 4t, y) - \varphi(x) \in C([0, \infty); X \cap H^1(\mathbb{R}^2))
\]
(see 29) and Proposition 11.1. If \( \|v_0\|_X + \|v_0\|_{L^2} \) is sufficiently small, Lemma 5.2 and Remark 5.1 imply that there exists \( T > 0 \) and \((c(t), x(t))\) satisfying (5.1), (5.3), (5.9) and
\[
\|\tilde{v}(t)\|_Y + \|\tilde{u}(t)\|_Y \lesssim \|\tilde{v}(t)\|_X \quad \text{for } t \in [0, T],
\]
and it follows that \( v(t) \in C([0, T]; X \cap L^2(\mathbb{R}^2)) \) and
\[
M_{tot}(T) := M_1(T) + M_2(T) + M_3(T) \leq \delta_4 / 2.
\]
By Proposition 5.4 we can extend the decomposition (5.1) satisfying (5.3) beyond \( t = T \). Let \( T_1 \in (0, \infty) \) be the maximal time such that the decomposition (5.1) with
From Lemma 7.2, 8.2 and 10.2 we have
\begin{equation}
M_{\text{tot}}(T_1) \lesssim \|v_0\|_{X_1} + \|v_0\|_{L^2(\mathbb{R}^2)} + M_{\text{tot}}(T_1)^2.
\end{equation}
If \(\|v_0\|_{X_1} + \|v_0\|_{L^2(\mathbb{R}^2)}\) is sufficiently small, then \(M_{\text{tot}}(T_1) \leq \delta_s/2\) follows from (11.2), which contradicts to the definition of \(T_1\). Thus we prove \(T_1 = \infty\) and
\begin{equation}
M_{\text{tot}}(\infty) \lesssim \|v_0\|_{X_1} + \|v_0\|_{L^2}.
\end{equation}
Now we will prove (1.5) and (1.8). By (5.1), (8.11) and (11.3),
\begin{align*}
\|u(t, x, y) - \varphi_{c(t,y)}(x - x(t, y))\|_{L^2(\mathbb{R}^2)} &\leq \|v(t)\|_{L^2(\mathbb{R}^2)} + \|\hat{\varphi}_{c(t,y)}\|_{L^2(\mathbb{R}^2)} \\
&\lesssim M_2(\infty) + M_1(\infty),
\end{align*}
where
\begin{align*}
\|e^{ax}(u(t, x + x(t, y), y) - \varphi_{c(t,y)}(x))\|_{L^2} &\leq \|v(t)\|_{L^2(\mathbb{R}^2)} + \|\hat{\varphi}_{c(t,y)}\|_{L^2(\mathbb{R}^2)} \\
&\lesssim M_2(\infty)\langle t \rangle^{-3/4} + M_1(\infty)e^{-a(4t + L)}\langle t \rangle^{-1/4}.
\end{align*}
Since \(\|f\|_{L^\infty} \lesssim \|f\|_{Y_1}^{1/2}\|\partial_y f\|_{Y_1}^{1/2}\) for any \(f \in Y\), we see that (1.6) and (1.7) follow immediately from (11.3) and (8.12). Thus we complete the proof of Theorem 1.1.

12. PROOF OF THEOREM 1.2.

In this section, we will prove orbital instability of line solitons. For the purpose, we will utilize that \(b, x_y\) is a solution to the diffusion wave equation (7.2) and its profile can be approximated by the heat kernel in some region.

Proof of Theorem 1.2. First we remark that if \(\|u(t, x) - \varphi_{c_0}(x - x_0)\|_{L^2(\mathbb{R}^2)} < \infty\), then \(x_0 = 2c_0t\). Indeed, it follow from [29] that \(u(t, x, y) - \varphi_{c_0}(x - 2c_0 t) \in L^2(\mathbb{R}^2)\) for every \(t \geq 0\) and
\begin{align*}
\|\varphi_{c_0}(x - 2c_0 t) - \varphi_{c_0}(x - x_0)\|_{L^2(\mathbb{R}^2)} \\
&\leq \|u(t, x, y) - \varphi_{c_0}(x - x_0)\|_{L^2(\mathbb{R}^2)} + \|u(t, x, y) - \varphi_{c_0}(x - 2c_0 t)\|_{L^2(\mathbb{R}^2)} < \infty,
\end{align*}
whereas \(\|\varphi_{c_0}(\cdot - 2c_0 t) - \varphi(\cdot - x_0)\|_{L^2(\mathbb{R}^2)} = \infty\) if \(x_0 \neq 2c_0 t\).

On the other hand, Theorem 1.1 implies that
\begin{align*}
\|u(t, \cdot) - \varphi_{c_0}(x - 2c_0 t)\|_{L^2(\mathbb{R}^2)} \\
&\geq \|\varphi_{c_0}(x - x(t, y)) - \varphi_{c_0}(x - 2c_0 t)\|_{L^2(\mathbb{R}^2)} - \|u(t, x, y) - \varphi_{c(t,y)}(x - x(t, y))\|_{L^2(\mathbb{R}^2)} \\
&\quad - \|\varphi_{c(t,y)}(x - x(t, y)) - \varphi_{c_0}(x - x(t, y))\|_{L^2(\mathbb{R}^2)} \\
&\gtrsim \|x(t, \cdot) - 2c_0 t\|_{L^2(\mathbb{R})} - O(\varepsilon).
\end{align*}
Thus to prove orbital instability of line solitons, it suffices to show that \(\|x(t, \cdot)\|_{L^2(\mathbb{R})}\) grows up as \(t \to \infty\).

Now we will construct a solution satisfying \(\|x(t, \cdot)\|_{Y} \gtrsim t^{1/4}\) as \(t \to \infty\). We may assume that \(c_0 = 2\) without loss of generality. If \(b(0)\) and \(x_y(0)\) are sufficiently small and \(\int_{\mathbb{R}} b(0) dy\) is nonzero, then \(e^{tA_0}(b(0), x_y(0))\) is expected to be the main part of the solution \((b(t), x_y(t))\). To investigate the behavior of \(e^{tA_0}(b(0), x_y(0))\), we represent the semigroup \(e^{tA_0}\) by using the heat kernel \(H_t(y)\). Let
\begin{align*}
A_{0,1}(\eta) &= \left(\begin{array}{cc}
-3\eta^2 & 8i\eta \\
\eta(2 + \frac{1}{\eta^2}) & -\eta^2
\end{array}\right), & A_{0,2}(\eta) &= \frac{1}{\eta^3} (A_0(\eta) - A_{0,1}(\eta)).
\end{align*}
Note that $A_{0,1}$ is equal to $A_*$ in Lemma 4.2 and that $A_{0,2} = \begin{pmatrix} O(\eta) & O(1) \\ O(1) & O(\eta) \end{pmatrix}$.

Let $U_1(t, s)$ be the $2 \times 2$ matrix such that
\[
\partial_t U_1(t) = A_1(t, 0)U_1(t), \quad \lim_{t \to \infty} U_1(t) = I.
\]

Since $|A_1(t, 0)| \lesssim e^{-a(4t + L)}$ for $t \geq 0$, we have $\sup_{\tau \geq t} |U_1(\tau) - I| \lesssim e^{-a(4t + L)}$. Now let
\[
\begin{pmatrix} b(t) \\ x_y(t) \end{pmatrix} = U_1(t) \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}.
\]

Then
\[
(12.1) \quad \partial_t \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} = (A_0(D_y) + D_y A_2(t, D_y)) \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} + \sum_{i=1}^{8} U_1(t)^{-1} \operatorname{diag}(1, \partial_y) N_i,
\]
where $A_2(t, \eta) = A_{21}(t, \eta) + A_{22}(t, \eta)$ and
\[
A_{21}(t, \eta) = \frac{U_1(t)^{-1} A_0(\eta) U_1(t) - A_0(\eta)}{\eta},
\]
\[
A_{22}(t, \eta) = \frac{U_1(t)^{-1} A_1(t, \eta) - A_1(t, 0)}{\eta} U_1(t).
\]

Clearly, we have $\|A_2(t, D_y)\|_{B(Y)} \lesssim e^{-a(4t + L)}$. By the variation of constants formula,
\[
\begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} = e^{tA_{0,1}} \begin{pmatrix} b_1(0) \\ b_2(0) \end{pmatrix} + IV_1 + IV_2 + IV_3,
\]
where
\[
IV_1 = \int_0^t e^{(t-s)A_{0,1}} D_y^3 A_{0,2} \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix} \, ds,
\]
\[
IV_2 = \int_0^t e^{(t-s)A_{0,1}} D_y A_2(s, D_y) \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix} \, ds,
\]
\[
IV_3 = \sum_{i=1}^{8} \int_0^t e^{(t-s)A_{0,1}} U_1(s)^{-1} \operatorname{diag}(1, \partial_y) N_i(s) \, ds.
\]

Let $h \in C_0^\infty(-\eta, \eta)$ such that $h(0) = 1$ and let
\[
u(0, x, y) = \phi_{2+c_1}(y) \psi_{2+c_2}(x), \quad c_1(y) = 2 + \varepsilon (f - h)(y).
\]

Then it follows from Lemma 5.2 that $\hat{c}(0, y) = c_1(y)$, $x(0, y) = 0$ and $v(0, \cdot) = 0$. Since $\|h(0) - \hat{c}(0)\|_{Y_1} \lesssim \|\hat{c}(0)\|_{L_2}$ by Claim 12.6 and $\|U_1(t) - I\|_{B(Y)} \lesssim e^{-a(4t + L)}$,
\[
(12.2) \quad \left\| \begin{pmatrix} b_1(0) \\ b_2(0) \end{pmatrix} - \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \right\|_{Y_1} \lesssim \left\| U_1(0)^{-1} \begin{pmatrix} b(0) \\ 0 \end{pmatrix} - \begin{pmatrix} \hat{c}(0) \\ 0 \end{pmatrix} \right\|_{Y_1} \lesssim \varepsilon (\varepsilon + e^{-aL}).
\]

Since $\|e^{tA_{0,1}}\|_{B(Y_1, Y)} \lesssim (1 + t)^{-1/4}$ by Lemma 4.2 it follows from 12.2 that
\[
\left\| e^{tA_{0,1}} \begin{pmatrix} b_1(0) \\ b_2(0) \end{pmatrix} - e^{tA_{0,1}} \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \right\|_Y \lesssim \varepsilon (\varepsilon + e^{-aL}) (1 + t)^{-1/4}.
\]

By Corollary 4.3
\[
\left\| e^{tA_{0,1}} \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \right\|_Y \lesssim \frac{1}{4} \left( 2 \left( e^{4t\partial_y(H_{2t} * c_1)} + e^{-4t\partial_y(H_{2t} * c_1)} \right) \right) \lesssim \varepsilon (t)^{-3/4}.
\]
Using Lemma 4.2, we can prove in exactly the same way as the proof of Lemma 7.1. Combining (12.3)–(12.6), we have

\[ \left\| \mathcal{P}_I \left( e^{\pm 4t \partial_y} (H_{2t} * c_\varepsilon) - e^{\pm 4t \partial_y} H_{2t} \right) \right\|_Y = \varepsilon \left\| e^{-2t \eta Y} (h(\eta) - h(0)) \right\|_{L^2(-\eta_0, \eta_0)} \lesssim \varepsilon \sup_{\eta} |h'(\eta)| \| e^{-2t \eta Y} \|_{L^2(-\eta_0, \eta_0)} \lesssim \varepsilon \langle t \rangle^{-3/4}. \]

Thus we have

\[ \varepsilon \langle t \rangle^{-3/4} \lesssim \varepsilon (\varepsilon + e^{-aL}) \langle t \rangle^{-1/4} + \varepsilon \langle t \rangle^{-3/4}. \]

In view of the proof of Theorem 1.1, we have for \( k = 0 \) and 1,

\[ \sup_{t \geq 0} \langle t \rangle^{(2k+1)/4} (\| b_1(t) \|_Y + \| b_2(t) \|_Y) \lesssim M_1(\infty) \lesssim \varepsilon. \]

By Lemma 4.2 and Claim 4.3, we have

\[ \| IV_1 \|_Y \lesssim \int_0^t \| \partial_y e^{(t-s)\partial_y} \|_{B(Y)} \left( \| \partial_y b_1(s) \|_Y + \| \partial_y b_2(s) \|_Y \right) ds \lesssim M_1(t) \int_0^t \langle t-s \rangle^{-1} \| s \|^{-3/4} ds \lesssim \varepsilon \langle t \rangle^{-3/4} \log \langle t \rangle, \]

and

\[ \| IV_2 \|_Y \lesssim \int_0^t \| \partial_y e^{(t-s)\partial_y} \|_{B(Y)} \left( \| b_2(s) \|_Y \right) ds \lesssim M_1(t) \int_0^t \langle t-s \rangle^{-1/2} e^{-a(4s+L)} \| s \|^{-1/4} ds \lesssim \varepsilon \langle t \rangle^{-1/2}. \]

Using Lemma 4.2, we can prove

\[ \| IV_3 \|_Y \lesssim \langle t \rangle^{-1/4} (M_1(\infty)^2 + M_2(\infty)^2) \lesssim \varepsilon^2 \langle t \rangle^{-1/4} \]

in exactly the same way as the proof of Lemma 7.1. Combining (12.3)–(12.6), we have

\[ \left\| \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} - \frac{\varepsilon}{4} \mathcal{P}_I \left( \begin{pmatrix} 2 (e^{4t \partial_y} H_{2t} + e^{-4t \partial_y} H_{2t}) \\ e^{4t \partial_y} H_{2t} - e^{-4t \partial_y} H_{2t} \end{pmatrix} \right) \right\|_Y \lesssim \varepsilon (\varepsilon + e^{-aL}) \langle t \rangle^{-1/4} + \varepsilon \langle t \rangle^{-1/2}. \]

Since \( |U_1(t) - I| \lesssim e^{-a(4t+L)} \) and \( \| (I - \mathcal{P}_I) H_{2t} \|_{L^2} \lesssim e^{-2t \eta_0^2} \),

\[ \langle x_y(t) - \frac{\varepsilon}{4} (e^{4t \partial_y} H_{2t} - e^{-4t \partial_y} H_{2t}) \rangle_{L^2} \lesssim \varepsilon (\varepsilon + e^{-aL}) \langle t \rangle^{-1/4} + \varepsilon \langle t \rangle^{-1/2}. \]

Now let \( d_1 \) and \( d_2 \) be constants satisfying \( d_2 > d_1 > 1 \) and let \( y_1 \in [-4t + 4(d_1 - 1)\sqrt{t}, -4t + 4d_1 \sqrt{t}] \), \( y_2 \in [-4t + 4d_2 \sqrt{t}, -4t + 4(d_2 + 1)\sqrt{t}] \) for \( t \geq 0 \). By (12.7),
there exist positive constants $C_1$, $C_2$ and $t_1$ such that

$$x(t, y_2) - x(t, y_1) = \int_{y_1}^{y_2} x_y(t, \bar{y}) \, d\bar{y}$$

\[
\geq \varepsilon \int_{y_1}^{y_2} (H_{2t}(\bar{y} + 4t) - H_{2t}(\bar{y} - 4t)) \, d\bar{y} \\
- (y_2 - y_1)^{1/2} \left\| x_y(t) - \frac{\varepsilon}{4} \left( e^{4t \partial_y} H_{2t} - e^{-4t \partial_y} H_{2t} \right) \right\|_{L^2} \\
\geq \varepsilon \int_{d_1 \sqrt{t}}^{d_2 \sqrt{t}} H_{2t}(y) \, dy - C_1 \varepsilon \langle \varepsilon + e^{-aL} + \langle t \rangle^{-1/4} \rangle \\
= \varepsilon \left( \text{erf}(\sqrt{2}d_2) - \text{erf}(\sqrt{2}d_1) \right) - C_1 \varepsilon \langle \varepsilon + e^{-aL} + \langle t \rangle^{-1/4} \rangle, \\
\geq C_2 \varepsilon \quad \text{for } t \geq t_1,
\]

if $\varepsilon$ and $e^{-aL}$ are sufficiently small. Recall that $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} \, ds$. Since $L$ is an auxiliary parameter introduced in Section 3 which can be chosen arbitrary large, we see that $|x(t, y)| \geq C_2 \varepsilon/2$ either on $[-4t + 4(d_1 - 1)\sqrt{t}, -4t + 4d_1 \sqrt{t}]$ or on $[-4t + 4d_2 \sqrt{t}, -4t + 4(d_2 + 1)\sqrt{t}]$. Therefore $\|x(t)\|_Y \geq \varepsilon \langle t \rangle^{1/4}$. Thus we complete the proof. \qed

13. Proof of Theorem 1.3

In order to prove Theorem 1.3 we will show that the first order asymptotics of solutions to (12.1) around $y = \pm 4t + O(\sqrt{t})$ is given by a sum of self-similar solutions to the Burgers equations. We apply the scaling argument by Karch [20] to obtain the asymptotics of (12.1) and use a virial type estimate to show that interaction between $b_1(t, y)$ and $b_2(t, y)$ tends to 0 around $y = \pm 4t + O(\sqrt{t})$ as $t \to \infty$. Since $\sup_{t>0} ^{1/4} (\|b_1(t)\|_{L^2(\mathbb{R})} + \|b_2(t)\|_{L^2(\mathbb{R})}) \ll 1$, we have the uniqueness of the limiting profile.

Roughly speaking, a solution of (12.1) can be decomposed into two parts that move to the opposite direction. Now we recenter each component of solutions to (12.1) and diagonalize the equations. Let $A_\ast(\eta)$, $\Pi_\ast(\eta)$ and $\omega(\eta)$ be as (4.6) with $\mu = \mu_3$. By the change of variables

$$b(t, y) = \begin{pmatrix} b_1(t, y) \\ b_2(t, y) \end{pmatrix}, \quad \Pi_1(t, \eta) = \frac{1}{4i} \Pi_\ast(\eta) \text{diag}(e^{4it\eta}, e^{-4it\eta}),$$

$$d(t, y) = \mathcal{F}_\eta^{-1}\Pi_1(t, \eta)^{-1}(\mathcal{F}_\eta b)(t, \eta),$$

we have

(13.1) $\partial_t d = \{2\partial_t^2 I + \partial_y (A_3(t, D_y) + A_4(t, D_y))\} d + A_5(t, D_y) \sum_{i=1}^8 \text{diag}(1, \partial_y) \mathcal{N}_i,$

where

$$A_3(t, \eta) = (\omega(\eta) - 4) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i\eta^{-1}\Pi_1(t, \eta)^{-1}(A_0(\eta) - A_\ast(\eta)),$$

$$A_4(t, \eta) = -i\Pi_1(t, \eta)^{-1} A_2(t, \eta) \Pi_1(t, \eta), \quad A_5(t, \eta) = \Pi_1(t, \eta)^{-1} \mathcal{U}_1(t)^{-1}.$$
To detect the dominant part of the equation, let us consider the rescaled solution \( d_\lambda(t, y) = \lambda d(\lambda^2 t, \lambda y) \). Our aim is to find a self-similar profile \( d_\infty(t, y) \) such that
\[
\lambda d_\infty(\lambda^2 t, \lambda y) = d_\infty(t, y),
\]
and that for any \( t_1 \) and \( t_2 \) satisfying \( 0 < t_1 \leq t_2 < \infty \) and any \( R > 0 \),
\[
\lim_{\lambda \to \infty} \sup_{t \in [t_1, t_2]} \| d_\lambda(t, y) - d_\infty(t, y) \|_{L^2(|y| < R)} = 0.
\]
If (13.2) and (13.3) hold, then letting \( \lambda = t^{1/2} \to \infty \), we have
\[
t^{1/4} \| d(t, \cdot) - d_\infty(t, \cdot) \|_{L^2(|y| < R)} = \lambda^{1/2} \| d(\lambda^2, \cdot) - d_\infty(\lambda^2, \cdot) \|_{L^2(|y| < \lambda R)} = \| d_\lambda(1, \cdot) - d_\infty(1, \cdot) \|_{L^2(|y| < R)} \to 0.
\]
To prove (13.3), we need the upper bounds of \( d_\lambda(t) \) for \( k = 0, 1 \) that do not depend on \( \lambda \geq 1 \).

**Lemma 13.1.** Let \( \varepsilon \) be as in Theorem [17]. Then there exists a positive constant \( C \) such that for any \( \lambda \geq 1 \) and \( t \in (0, \infty) \),
\[
\sum_{k=0,1} \| \partial_y^k d_\lambda(t, \cdot) \|_{L^2} \leq C \varepsilon t^{-(2k+1)/4}, \quad \| \partial_y^2 d_\lambda(t, \cdot) \|_{L^2} \leq C \varepsilon \lambda^{1/2} t^{-1},
\]
\[
\| \partial_t d_\lambda(t, \cdot) \|_{H^{-2}} \leq C(t^{-1/4} + t^{-3/2}) \varepsilon.
\]

**Proof.** Since \( M_1(\infty) \leq \varepsilon \) by (11.3), we have
\[
\sum_{k=0,1} \sup_{t \geq 0} \| \partial_y^k d(t) \|_Y + \| \partial_y^2 d(t) \|_Y \leq \varepsilon.
\]
Thus we have
\[
\| \partial_y^k d_\lambda(t, \cdot) \|_{L^2} = \lambda^{2k+1/2} \| \partial_y^k d(\lambda^2 t, \cdot) \|_Y \leq \lambda^{2k+1/2} (1 + \lambda^2 t)^{-(2k+1)/4} \leq t^{-(2k+1)/4} \varepsilon \quad \text{for } k = 0, 1,
\]
\[
\| \partial_y^2 d_\lambda(t, \cdot) \|_{L^2} \leq \lambda^{3/2} \| \partial_y^2 d(\lambda^2 t, \cdot) \|_Y \leq \lambda^{3/2} (1 + \lambda^2 t)^{-1} \varepsilon \lesssim \lambda^{1/2} t^{-1} \varepsilon.
\]
Thus we prove (13.5).

Next we will show (13.6). Let \( N'(t, y) + \partial_y N''(t, y) = \text{diag}(1, \partial_y) \sum_{i=2}^8 N_i(t, y), \ N_3(t, y) = \lambda^3 N'(\lambda^2 t, \lambda y), \ N'_3(t, y) = \lambda^3 N''(\lambda^2 t, \lambda y), \ N_\lambda(t, y) = \tilde{P}_1 \left( n_2(t, y) \right), \ N_\lambda(t, y) = \lambda^2 \tilde{N}(\lambda^2 t, \lambda y). \)

Then (13.1) can be rewritten as
\[
\partial_t d_\lambda = \{ 2 \partial_y^2 I + \lambda \partial_y (A_3(\lambda^2 t, \lambda y) + A_4(\lambda^2 t, \lambda y)) \} d_\lambda + A_5(\lambda^2 t, \lambda y) \partial_y (\tilde{N}_\lambda + N'_3) + N'_3.
\]
By (13.7),
\[
\| \partial_t d_\lambda \|_{H^{-2}} \leq 2 \| d_\lambda \|_{L^2} + \lambda \| A_3(\lambda^2 t, \lambda y) d_\lambda \|_{H^{-1}} + \lambda \| A_4(\lambda^2 t, \lambda y) d_\lambda \|_{L^2} + \| A_5(\lambda^2 t, \lambda y) \tilde{N}_\lambda + N'_3 \|_{L^2} + \| A_5(\lambda^2 t, \lambda y) N'_3 \|_{L^2}
\]
Now we will estimate each term of the right hand side. By (13.3) and the fact that \( \omega(\eta) = 4 + O(\eta^2) \), we have
\[
\| A_3(\lambda^2 t, \lambda y) \| \lesssim \lambda^{-2} \eta^2.
\]
Thus by (13.3) and Plancherel’s theorem,
\begin{equation}
\lambda \| A_3(\lambda^2 t, \lambda^{-1} D_y) d_{\lambda} \|_{H^{-1}} \lesssim \lambda^{-1} \| \langle \eta \rangle^{-1} \eta^2 (F_y d_{\lambda})(t, \eta) \|_{L^2} \lesssim \lambda^{-1} \| \partial_y d_{\lambda}(t, \cdot) \|_{L^2} \lesssim \lambda^{-1} t^{-3/4} \varepsilon.
\end{equation}
(13.10)

Since \( \| A_4(t, D_y) \|_{B(Y)} \lesssim \| A_2(t, D_y) \|_{B(Y)} \lesssim e^{-a(\lambda^2 t + L)} \), it follows from (13.5) and the scaling argument that
\begin{equation}
\lambda \| A_4(\lambda^2 t, \lambda^{-1} D_y) d_{\lambda}(t, \cdot) \|_{L^2} = \lambda^{3/2} \| A_4(\lambda^2 t, D_y) d(\lambda^{2} t, \cdot) \|_{Y} \lesssim \lambda^{3/2} e^{-a(\lambda^2 t + L)} \| d(\lambda^2 t, \cdot) \|_{Y} \lesssim \lambda^{3/2} t^{-2} e^{-a(\lambda^2 t + L)} \lesssim \lambda^{-1/4} t^{-7/8} \varepsilon.
\end{equation}
(13.11)

Following the proof of Lemma 7.1, we have for \( t \geq 0 \),
\begin{equation}
\| \tilde{N} \|_{Y} \lesssim \langle t \rangle^{-3/4} \varepsilon, \quad \| \tilde{N}' \|_{Y} \lesssim \langle t \rangle^{-3/2} \varepsilon, \quad \| \tilde{N}'' \|_{Y} \lesssim \langle t \rangle^{-5/4} \varepsilon.
\end{equation}
(13.12)

Nonlinear terms decay \( t^{-1/4} \) times faster in (13.12) than those in (7.6) and (7.12) because \( Y \) and \( Y_1 \) have the same scaling as \( L^2(\mathbb{R}) \) and \( L^4(\mathbb{R}) \), respectively. By (13.12),
\begin{align*}
\| \tilde{N}_\lambda \|_{L^2} &= \lambda^{3/2} \| \tilde{N}(\lambda^2 t, \cdot) \|_{Y} \lesssim \lambda^{3/2} (1 + \lambda^2 t)^{-3/4} \varepsilon \lesssim t^{-3/4} \varepsilon, \\
\| \tilde{N}'_\lambda \|_{L^2} &= \lambda^{5/2} \| \tilde{N}'(\lambda^2 t, \cdot) \|_{Y} \lesssim \lambda^{5/2} (1 + \lambda^2 t)^{-3/2} \varepsilon \lesssim \lambda^{-1/2} t^{-3/2} \varepsilon, \\
\| \tilde{N}''_\lambda \|_{L^2} &= \lambda^{3/2} \| \tilde{N}''(\lambda^2 t, \cdot) \|_{Y} \lesssim \lambda^{3/2} (1 + \lambda^2 t)^{-5/4} \varepsilon \lesssim \lambda^{-1/4} t^{-7/8} \varepsilon.
\end{align*}
(13.13)–(13.15)

Since \( \sup_{\lambda \geq 1} \| A_5(\lambda^2 t, \lambda^{-1} D_y) \|_{B(L^2)} \lesssim 1 \), we have
\begin{align}
\| A_5(\lambda^2 t, \lambda^{-1} D_y) \tilde{N}_\lambda \|_{L^2} &\lesssim t^{-3/4} \varepsilon, \\
\| A_5(\lambda^2 t, \lambda^{-1} D_y) \tilde{N}'_\lambda \|_{L^2} &\lesssim \lambda^{-1/4} t^{-7/8} \varepsilon, \\
\| A_5(\lambda^2 t, \lambda^{-1} D_y) \tilde{N}''_\lambda \|_{L^2} &\lesssim \lambda^{-1/2} t^{-3/2} \varepsilon.
\end{align}
(13.13)–(13.15)

Combining (13.3) and (13.8)–(13.11), (13.13)–(13.15), we obtain (13.6). \( \square \)

By Lemma 13.1 and the Arzelà-Ascoli theorem, we have the following.

**Corollary 13.2.** There exist a sequence \( \{ \lambda_n \}_{n \geq 1} \) satisfying \( \lim_{n \to \infty} \lambda_n = \infty \) and \( d_\infty(t, y) \) such that
\begin{align*}
d_{\lambda_n}(t, \cdot) &\to d_\infty(t, \cdot) \quad \text{weakly star in } L^\infty_{\text{loc}}((0, \infty); H^1(\mathbb{R})), \\
\partial_t d_{\lambda_n}(t, \cdot) &\to \partial_t d_\infty(t, \cdot) \quad \text{weakly star in } L^\infty_{\text{loc}}((0, \infty); H^{-2}(\mathbb{R})),
\end{align*}
\begin{equation}
\sup_{t > 0} \| d_{\lambda_n}(t, \cdot) \|_{L^2} \leq C \varepsilon,
\end{equation}
where \( C \) is a constant given in Lemma 7.1. Moreover, for any \( R > 0 \) and \( t_1, t_2 \) with \( 0 < t_1 \leq t_2 < \infty \),
\begin{equation}
\lim_{n \to \infty} \sup_{t \in [t_1, t_2]} \| d_{\lambda_n}(t, \cdot) - d_\infty(t, \cdot) \|_{L^2(\{ |y| \leq R \})} = 0.
\end{equation}

Next we will show that \( d_\infty(t) \) is a self-similar solution to a system of Burgers equations. To begin with, we will prove the following.

**Lemma 13.3.** Let \( d_\infty(t) = \xi(d_+(t, y), d_-(t, y)) \). Then for \( t > 0 \) and \( y \in \mathbb{R} \),
\begin{equation}
\begin{cases}
\partial_t d_+ = 2 \partial_y^2 d_+ + 4 \partial_y (d_+^2), \\
\partial_t d_- = 2 \partial_y^2 d_- - 4 \partial_y (d_-^2),
\end{cases}
\end{equation}
(13.16)
and
\begin{equation}
\lim_{t \downarrow 0} \int_{\mathbb{R}} d_{\infty}(t, y) h(y) \, dy = \sqrt{2\pi} (F_y \tilde{d})(0, 0) h(0) \quad \text{for any } h \in H^2(\mathbb{R}),
\end{equation}
where
\begin{equation}
\tilde{d}(t, y) = d(t, y) + \int_{t}^{\infty} A_{5}(s, D_{y}) \mathcal{N}'(s, y) \, ds.
\end{equation}

Proof. Let \( \tilde{d}_{\lambda}(t, y) = \lambda \tilde{d}(\lambda t, \lambda^2 y) \). The limiting profile of \( d_{\lambda}(t) \) and \( \tilde{d}_{\lambda}(t) \) as \( \lambda \to \infty \) are the same for every \( t > 0 \). Indeed, it follows from (13.15) that
\begin{equation}
\| \tilde{d}_{\lambda}(t, \cdot) - d_{\lambda}(t, \cdot) \|_{L^2} \lesssim \int_{t}^{\infty} \| A_{5}(\lambda^2 s, \lambda^{-1} D_{y}) \mathcal{N}'(s, y) \|_{L^2} \, ds \lesssim \lambda^{-1/2} \int_{t}^{\infty} \tau^{-3/2} \, d\tau \lesssim \lambda^{-1/2} t^{-1/2}.
\end{equation}
By (13.7),
\begin{equation}
\partial_{t} \tilde{d}_{\lambda} = 2 \partial_{y}^2 d_{\lambda} + \lambda \partial_{y} \left\{ (A_{3}(\lambda^2 t, \lambda^{-1} D_{y}) + A_{4}(\lambda^2 t, \lambda^{-1} D_{y})) d_{\lambda} + \partial_{y} A_{5}(\lambda^2 t, \lambda^{-1} D_{y}) (\tilde{N}_{\lambda} + \mathcal{N}'_{\lambda}) \right\},
\end{equation}
and we have \( \sup_{\lambda > 0} \| \partial_{t} \tilde{d}_{\lambda}(t, \cdot) \|_{H^{-2}} \lesssim t^{-1/4} + t^{-7/8} \) from (13.5), (13.10), (13.11), (13.12), (13.13) and (13.14). Thus for \( t > s > 0 \) and \( h \in H^2(\mathbb{R}) \),
\begin{equation}
\left| \int_{\mathbb{R}} \tilde{d}_{\lambda}(t, y) h(y) \, dy - \int_{\mathbb{R}} \tilde{d}_{\lambda}(s, y) h(y) \, dy \right| \leq C \{(t - s)^{3/4} + (t - s)^{1/8} \},
\end{equation}
where \( C \) is a constant independent of \( \lambda \). Passing to the limit as \( s \downarrow 0 \) in the above, we obtain for \( t > 0 \),
\begin{equation}
\left| \int_{\mathbb{R}} \tilde{d}_{\lambda}(t, y) h(y) \, dy - \int_{\mathbb{R}} \tilde{d}_{\lambda}(0, y) h(y) \, dy \right| \leq C (t^{3/4} + t^{1/8}).
\end{equation}
Since \( d(0, \cdot) \in Y_{1} \) by the definition and \( \| \mathcal{N}'(\cdot, \cdot) \|_{Y_{1}} \lesssim (\tau)^{-5/4} \), we have
\begin{equation}
\tilde{d}_{1}(0, y) = d(0, y) + \int_{0}^{\infty} A_{5}(\tau, D_{y}) \mathcal{N}'(\tau, y) \, d\tau \in Y_{1},
\end{equation}
and it follows from Lebesgue’s dominated convergence theorem that
\begin{equation}
\int_{\mathbb{R}} \tilde{d}_{\lambda}(0, y) h(y) \, dy = \int_{\mathbb{R}} (F_{y} \tilde{d}_{1})(0, \lambda^{-1} \eta) (F_{y}^{-1} h)(\eta) \, d\eta \to \sqrt{2\pi} (F_{y} \tilde{d}_{1})(0, 0) h(0)
\end{equation}
as \( \lambda \to \infty \) for any \( h \in H^1(\mathbb{R}) \). Letting \( \lambda = \lambda_{n} \) and passing to the limit as \( n \to \infty \) in (13.20), we see that (13.17) follows from Corollary 13.2.

Next, we will show (13.16). By Corollary 13.2 and (13.18),
\begin{equation}
\partial_{t} \tilde{d}_{\lambda_{n}} - 2 \partial_{y}^2 d_{\lambda_{n}} \to \partial_{t} d_{\infty} - 2 \partial_{y}^2 d_{\infty}.
\end{equation}
By (13.10), (13.11) and (13.14),
\begin{equation}
\lambda \{ (A_{3}(\lambda_{n}^2 t, \lambda_{n}^{-1} D_{y}) + A_{4}(\lambda_{n}^2 t, \lambda_{n}^{-1} D_{y})) d_{\lambda_{n}} + A_{5}(\lambda_{n}^2 t, \lambda_{n}^{-1} D_{y}) \mathcal{N}'_{\lambda_{n}} \} \to 0,
\end{equation}
as \( n \to \infty \) in \( \mathcal{D}'((0, \infty) \times \mathbb{R}) \).

Now we investigate the limit of \( \tilde{N}_{\lambda} \). Let
\begin{equation}
d_{\lambda}(t, y) = \begin{pmatrix} d_{+, \lambda}(t, y) \\ d_{-, \lambda}(t, y) \end{pmatrix}.
\end{equation}
By the definition of $d(t)$,
\[
\begin{pmatrix}
(b(t,·) \\
x_y(t,·)
\end{pmatrix}
= U_1(t) \Pi_1(t, D_y) d(t,·).
\]
Since $|U_1(t) - 1| \lesssim e^{-a(4t+L)}$ and
\[
\begin{align*}
(4t)^{-1} \Pi_+(\lambda^{-1} \eta) - \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \lesssim \lambda^{-1} \eta \quad & \text{for } \lambda \geq 1,
\end{align*}
\]
we have
\[
(13.23)
\begin{align*}
\left\| \begin{pmatrix} \lambda & (b(\lambda^2 t, \lambda)) \\ x_y(\lambda^2 t, \lambda) \end{pmatrix} + \begin{pmatrix} 8 & 8 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} e^{4\lambda t\partial_y} d_{+,\lambda}(t,·) \\ e^{-4\lambda t\partial_y} d_{-,\lambda}(t,·) \end{pmatrix} \right\|_{L^2} 
\lesssim (\lambda^{-1} + e^{-4a\lambda^2 t}) \| d(t) \|_{H^1} \lesssim (\lambda^{-1} + e^{-4a\lambda^2 t})(t^{-1/4} + t^{-3/4}).
\end{align*}
\]
Recall that $(n_1, n_2) = (6bx_y, 2(\bar{c} - b) + 3(x_y)^2)$. Since $\|n_1(t)\|_{L^1} + \|n_2(t)\|_{L^1} \lesssim t^{-1/2}$,
\[
\begin{align*}
\| \bar{\Lambda}_\lambda - \lambda^2 (n_1(n_2) (\lambda^2 t, \lambda)) \|_{H^{-1}} 
\lesssim \lambda^{1/2} (\|n_1(\lambda^2 t,·)\|_{L^1} + \|n_2(\lambda^2 t,·)\|_{L^1}) \lesssim (\lambda t)^{-1/2}.
\end{align*}
\]
Combining (13.25) with (13.24), we have
\[
\begin{align*}
\left\| \lambda^2 (\bar{P}_1 n_1)(\lambda^2 t, \lambda) - 12 \{ (e^{4\lambda t\partial_y} d_{+,\lambda})^2 - (e^{-4\lambda t\partial_y} d_{-,\lambda})^2 \} \right\|_{H^{-1}} 
\lesssim C(t)(\lambda^{-1/2} + e^{-4a\lambda^2 t}),
\end{align*}
\]
where $C(t)$ is a monotone decreasing function of $t$. Claim (1.6) implies
\[
\begin{align*}
\left\| b - \bar{c} - \frac{1}{8} \bar{P}_1 (b^2) \right\|_{L^2} \lesssim \|b\|_{L^2} \lesssim \|d\|_{L^2} \lesssim \|\partial_y d\|_{L^2},
\end{align*}
\]
whence
\[
\begin{align*}
\lambda^2 \left\| b - \bar{c} - \frac{1}{8} \bar{P}_1 (b^2) \right\|_{Y} \lesssim \lambda^{-1} \|d\|_{L^2} \|\partial_y d\|_{L^2} \lesssim \lambda^{-1} t^{-5/4}.
\end{align*}
\]
We can obtain $\lambda^2 \left\| (I - \bar{P}_1) b^2(\lambda^2 t, \lambda) \right\|_{H^{-1}} \lesssim (\lambda t)^{-1/2}$ in the same way as (13.25).
Combining the above with (13.24) and (13.25), we have
\[
\begin{align*}
\left\| \lambda^2 (\bar{P}_1 n_2)(\lambda^2 t, \lambda) - 2 \{ (e^{4\lambda t\partial_y} d_{+,\lambda})^2 - 4(e^{4\lambda t\partial_y} d_{+,\lambda})(e^{-4\lambda t\partial_y} d_{-,\lambda}) + (e^{-4\lambda t\partial_y} d_{-,\lambda})^2 \} \right\|_{H^{-1}} \lesssim C'(t)(\lambda^{-1/2} + e^{-4a\lambda^2 t}),
\end{align*}
\]
where $C'(t)$ is a monotone decreasing function of $t$. Since
\[
\begin{align*}
\left\| 4\lambda \Pi_+(\lambda^{-1} \eta)^{-1} - \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \right\|_{L^2} \lesssim \lambda^{-1} \eta \quad & \text{for } \lambda \geq 1,
\end{align*}
\]
it follows from (13.26) and (13.27) that
\[
\begin{align*}
\left\| A_5(\lambda^2 t, \lambda^{-1} D_y) \bar{\Lambda}_\lambda - 2 \begin{pmatrix} 2d_y^2 \lambda - 2d_{+,\lambda}(e^{-8\lambda t\partial_y} d_{-,\lambda}) - (e^{-8\lambda t\partial_y} d_{-,\lambda})^2 \\ 2d_y^2 \lambda - 2d_{+,\lambda}(e^{8\lambda t\partial_y} d_{+,\lambda}) - (e^{8\lambda t\partial_y} d_{+,\lambda})^2 \end{pmatrix} \right\|_{H^{-2}} 
\lesssim C(t)(\lambda^{-1/2} + e^{-4a\lambda^2 t}),
\end{align*}
\]
where $C(t)$ is a monotone decreasing function of $t$. 
Next, we will show that \( e^{\pm 4\lambda t} \partial_y d_{+} \) locally tends to 0 around \( y = \pm 4\lambda t \). Let \( \alpha > 0 \) and \( \zeta_{\pm}(y) = 1 \pm \tanh \alpha y \). Then we have \( 0 \leq \zeta_{\pm}(y) \leq 2, 0 < \pm \zeta_{\pm}'(y) \leq \alpha \) and \( 0 \leq |\zeta_{\pm}''(y)/\zeta_{\pm}'(y)| \leq 2\alpha \) for \( y \in \mathbb{R} \). By (13.7),
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \zeta_{+}(y - 8\lambda t) d_{+}^{2}(t, y) \, dy + 4\lambda \int_{\mathbb{R}} \zeta_{+}'(y - 8\lambda t) d_{+}^{2}(t, y) \, dy \\
\leq \int_{\mathbb{R}} \zeta_{+}'(y - 8\lambda t) (d_{+}^{2}(t, y) \, dy - 2 \int_{\mathbb{R}} \zeta_{+}(y - 8\lambda t) (\partial_y d_{+}(t, y))^{2}(t, y) \, dy + V,
\]
where
\[
V = (2 + \alpha) \lambda \| A_{3}(\lambda^{2} t, \lambda^{-1} D_y) d_{\lambda}(t) \|_{L^{2}} \| d_{\lambda}(t) \|_{H^{1}} \\
+ (2 + \alpha) \lambda \| A_{1}(\lambda^{2} t, \lambda^{-1} D_y) d_{\lambda}(t) \|_{L^{2}} \| d_{\lambda}(t) \|_{H^{1}} \\
+ (2 + \alpha) \| d_{\lambda}(t) \|_{H^{1}} \| A_{3}(\lambda^{2} t, \lambda^{-1} D_y) (N_{\lambda}(t) + N_{\lambda}''(t)) \|_{L^{2}} \\
+ 2 \| d_{\lambda}(t) \|_{L^{2}} \| A_{3}(\lambda^{2} t, \lambda^{-1} D_y) N_{\lambda}''(t) \|_{L^{2}}.
\]
Using Lemma 13.4 and (13.9), we have
(13.30) \( \lambda \| A_{3}(\lambda^{2} t, \lambda^{-1} D_y) d_{\lambda}(t) \|_{L^{2}} \lesssim \lambda^{-1/2} t^{-1} \).

By Lemma 13.1, (13.11), (13.13)–(13.15) and (13.30),
\[
V \lesssim \lambda^{-1/4}(t^{-9/8} + t^{-9/4}) + t^{-1} + t^{-3/2}.
\]
If \( \alpha \) is sufficiently small, it follows that
(13.31) \[
\frac{d}{dt} \int_{\mathbb{R}} \zeta_{+}(y - 8\lambda t) d_{+}^{2}(t, y) \, dy + 4\lambda \int_{\mathbb{R}} \zeta_{+}'(y - 8\lambda t) d_{+}^{2}(t, y) \, dy \\
\leq C \lambda^{-1/4}(t^{-9/8} + t^{-9/4}) + C(t^{-1} + t^{-3/2}),
\]
where \( C \) is a positive constant independent of \( t > 0 \) and \( \lambda \geq 1 \). Let \( 0 < t_1 < t_2 < \infty \). Integrating (13.31) over \([t_1, t_2]\), we obtain
(13.32) \[
0 < \int_{t_1}^{t_2} \int_{\mathbb{R}} \zeta_{+}'(y - 8\lambda t) d_{+}^{2}(t, y) \, dy \, dt \leq C(t_1, t_2) \lambda^{-1},
\]
where \( C(t_1, t_2) \) is a constant independent of \( \lambda \geq 1 \). We can prove
(13.33) \[
0 < - \int_{t_1}^{t_2} \int_{\mathbb{R}} \zeta_{+}'(y + 8\lambda t) d_{-}^{2}(t, y) \, dy \, dt \leq C(t_1, t_2) \lambda^{-1},
\]
in exactly the same way. By (13.32) and (13.33),
\[
\lim_{\lambda \to \infty} \| d_{+} \|_{L^{2}([t_1, t_2] \times B_{R}^{+})} = 0
\]
for any \( R > 0 \), where \( B_{R}^{+} = \{ y \in \mathbb{R} \mid |y + 8\lambda t| \leq R \} \). Combining the above with Corollary 13.2, 13.7, 13.21, 13.22 and 13.29, we see that \( d_{\lambda} \) satisfies (13.10).

Now we are in position to prove Theorem 13.8.

**Proof of Theorem 13.8.** By Corollary 13.2,
\[
\| \partial_y d_{2}^{2}(t, \cdot) \|_{H^{-2}} \lesssim \| d_{\lambda}(t, \cdot) \|_{L^{2}}^{2} \lesssim t^{-1/2},
\]
whence $\partial_y(d_\pm(t)^2) \in L^1_{\text{loc}}((0, \infty); H^{-2}(\mathbb{R}))$. Combining the above with (13.16) and (13.17), we have $d_\pm(t) \in C([0, \infty); H^{-2}(\mathbb{R}))$ and

$$d_\pm(t) = c_\pm H_{2t} \pm 4 \int_0^t e^{2(t-s)} \partial_y^2(d_\pm(s)^2) \, ds,$$

where $(c_+, c_-) = \sqrt{2\pi}(F_y d_1)(0,0)$. If we choose $m_\pm \in (-2\sqrt{2}, 2\sqrt{2})$ so that

$$\int_R u^\pm_B(t, y) \, dy = \frac{1}{2} \log \left( \frac{2\sqrt{2} + m_\pm}{2\sqrt{2} + m_\pm} \right) = c_\pm,$$

then $u^\pm_B(t)$ is also a solution to (13.33) satisfying $\sup_{t \geq 0} t^{1/4} \|u^\pm_B(t)\|_{L^2} \lesssim \varepsilon$. Let $\|\cdot\|_W = \sup_{t \geq 0} t^{1/4} \|u(t, \cdot)\|_{L^2(\mathbb{R})}$. Since $\|\partial_y e^{2t\partial_y^2}\|_{B(L^1; L^2)} \lesssim t^{-3/4}$,

$$d_\pm - u^\pm_B \|W \leq 4 \sup_{t \geq 0} t^{1/4} \int_0^t \|\partial_y e^{2(t-s)\partial_y^2}\|_{B(L^1; L^2)} \|d_\pm(s)^2 - u^\pm_B(s)^2\|_{L^1} \, ds \lesssim (\|d_\pm\|_W + \|u^\pm_B\|_W) \|d_\pm - u^\pm_B\|_W t^{1/4} \int_0^t (t-s)^{-3/4} s^{-1/2} \, ds \lesssim \varepsilon \|d_\pm - u^\pm_B\|_W.$$

Thus we have $d_\pm(t, y) = u^\pm_B(t, y)$ for small $\varepsilon$ and (13.33) follows from the uniqueness of the limiting profile $d_\infty(t, y) = (d_+(t, y), d_-(t, y))$. Obviously $d_\infty(t, y) = (u^+_B(t, y), u^-_B(t, y))$ satisfies (13.2). Now Theorem 1.3 follows immediately from (13.3), (13.24) and the definition of $b(t, y)$. Thus we complete the proof. □

**APPENDIX A. PROOF OF LEMMA 6.1**

To prove Lemma 6.1, we need the following.

**Claim A.1.** Let $\varphi_c(x) = \text{csech}^2(\sqrt{c/2} x)$, $\varphi = \varphi_2$ and $\partial_c^k \varphi = \partial_c^k \varphi_c |_{c=2}$ for $k \in \mathbb{N}$. Then

(A.1) \( \int_R \varphi(x) \, dx = 4 \), \( \int_R \varphi(x)^2 \, dx = \frac{16}{3} \),

(A.2) \( \int_R \varphi(x) \partial_x \varphi(x) \, dx = -\int_R \varphi'(x) \left( \int_{-\infty}^x \partial_x \varphi(z) \, dz \right) = 2 \),

(A.3) \( \int_R \varphi(x) \left( \int_{-\infty}^x \partial_x \varphi(z) \, dz \right) \, dx = \int_R \varphi(x) \left( \int_{-\infty}^x \partial_x \varphi(z) \, dz \right) \, dx = 2 \),

(A.4) \( \int_R \varphi(x) \left( \int_{-\infty}^x \partial_x^2 \varphi(z) \, dz \right) \, dx = -\frac{1}{2} \), \( \int_R \partial_x \varphi(x) \left( \int_{-\infty}^x \partial_x \varphi(z) \, dz \right) \, dx = \frac{1}{2} \),

(A.5) \( \int_R \left( \int_{-\infty}^x \partial_x \varphi(z) \, dz \right) \left( \int_{-\infty}^x \partial_x \varphi(z) \, dz \right) \, dx = \frac{1}{6} - \frac{\pi^2}{36} \),

(A.6) \( \int_R \left( \int_{-\infty}^x \partial_x^2 \varphi(z) \, dz \right) \left( \int_{-\infty}^x \partial_x \varphi(z) \, dz \right) \, dx = \frac{\pi^2}{96} - \frac{1}{16} \).

**Proof.** Eq. (A.1) can be obtained by using the change of variable $s = \tanh x$. Since

(A.7) \( \varphi_c(x) = \frac{c}{2} \varphi(\sqrt{c/2} x) \),

\( \int \varphi \partial_x \varphi \, dx = \frac{1}{2} \frac{d}{dc} \left( \frac{c}{2} \right)^{3/2} \Bigg|_{c=2} \int \varphi^2 \, dx = 2 \).
Using (A.7), we have

\[ \partial_c \varphi(x) = \frac{x}{4} \varphi'(x) + \frac{1}{2} \varphi(x) = \frac{1}{4} (x\varphi)' + \frac{1}{4} \varphi, \]

(A.8)  \[ \partial_c^2 \varphi(x) = \frac{x^2}{16} \varphi''(x) + \frac{3x}{16} \varphi'(x) = \frac{1}{16} (x^2 \varphi' + x\varphi)' - \frac{1}{16} \varphi, \]

\[ \int_{-\infty}^{\infty} \partial_c \varphi = \frac{x^2}{4} + \frac{\tanh x + 1}{2}, \quad \int_{-\infty}^{x} \partial_c^2 \varphi = -\frac{x^2 \varphi' + x\varphi}{16} + \frac{\tanh x - 1}{8}. \]

By (A.8) and the fact that \( \varphi \) is even,

\[ \int \varphi \int_{-\infty}^{x} \partial_c \varphi = \frac{1}{2} \int \varphi = 2, \]

\[ \int \varphi \int_{-\infty}^{\infty} \partial_c^2 \varphi = -\frac{1}{8} \int \varphi = -\frac{1}{2}, \]

\[ \int_{\mathbb{R}} \partial_c \varphi \int_{-\infty}^{x} \partial_c \varphi = \frac{1}{8} \int \varphi = \frac{1}{2}. \]

By (A.8),

\[ \int_{\mathbb{R}} \left( \int_{-\infty}^{x} \partial_c \varphi(z) \right) \left( \int_{-\infty}^{\infty} \partial_c \varphi(z) \right) dx = \int_{\mathbb{R}} \left\{ \frac{1}{4} - \left( \frac{x}{4} \varphi + \frac{1}{2} \tanh x \right)^2 \right\} dx \]

\[ = \frac{1}{4} \int \sec^2 x - \frac{1}{2} \int x \sec^3 x \sinh x - \frac{1}{4} \int x^2 \sec^4 x dx \]

\[ = -\frac{1}{4} \int x^2 \sec^4 x dx = -\frac{\pi^2 - 6}{36}. \]

Here use the fact that \( \int_{0}^{\infty} \frac{x}{e^{x} + 1} dx = \frac{\pi^2}{12}. \) We have (A.6) in the same way. \( \square \)

Now we are in position to prove Lemma 6.1.

**Proof of Lemma 6.1.** By Claims A.1 and 2.1

\[ G_1 = \int_{\mathbb{R}} \ell_1 \varphi_c(z) dz \]

\[ = 3x_{yy} \int \varphi_c^2 - (c_t - 6c_y x_y) \int \varphi_c \partial_c \varphi_c \]

\[ + 3c_{yy} \int \varphi_c \int_{-\infty}^{\infty} \partial_c \varphi_c + 3(c_y)^2 \int \varphi_c \int_{-\infty}^{\infty} \partial_c^2 \varphi_c \]

\[ = 16x_{yy} \left( \frac{c}{2} \right)^{3/2} - 2(c_t - 6c_y x_y) \left( \frac{c}{2} \right)^{1/2} + 6c_{yy} - \frac{3}{2} \left( c_y \right)^2 \left( \frac{2}{c} \right), \]
and
\[
\left(\frac{c}{2}\right)^{-3/2} G_2 = \int_{\mathbb{R}} \ell_1 \left( \int_{-\infty}^{x} \partial_z \varphi_c(z) \, dz \right) \, dz \nonumber
\]
\[
= (x_t - 2c - 3(x_y)^2) \left( \int_{-\infty}^{x} \partial_z \varphi_c \right) \nonumber
\]
\[
+ 3x_{yy} \int \varphi_c \left( \int_{-\infty}^{\infty} \partial_z \varphi_c \right) - (c_t - 6c_y x_y) \int \partial_z \varphi_c \left( \int_{-\infty}^{x} \partial_z \varphi_c \right) \nonumber
\]
\[
+ 3c_{yy} \left( \int_{-\infty}^{\infty} \partial_z \varphi_c \right) - (c_t - 6c_y x_y) \left( \int_{-\infty}^{\infty} \partial_z \varphi_c \right) \nonumber
\]
\[
= -2(x_t - 2c - 3(x_y)^2) \left( \frac{c}{2} \right)^{1/2} + 6x_{yy} \left( \frac{c}{2} \right)^{1/2} + \mu_1 c_{yy} \left( \frac{c}{2} \right)^{-3/2} \nonumber
\]
\[
+ \mu_2 (c_y)^2 \left( \frac{c}{2} \right)^{-5/2}. \nonumber
\]

□

APPENDIX B. OPERATOR NORMS OF \( S_k^j \) AND \( \tilde{C}_k \)

Claim B.1. There exist positive constants \( \eta_1 \) and \( C \) such that for \( \eta \in (0, \eta_1) \), \( j \in \mathbb{Z}_{\geq 0} \), \( k = 1, 2 \) and \( f \in L^2(\mathbb{R}) \),

(B.1) \[ \| \partial_y \tilde{S}_k^j [q_c](f)(t, \cdot) \|_{Y_1} \leq C \| e^{az} q_2 \|_{L^2(\mathbb{R})} \| \partial_y^3 \tilde{P}_1 f \| \| Y_1 \|, \]

(B.2) \[ \| \partial_y \tilde{S}_k^j [q_c](f)(t, \cdot) \|_{Y_1} \leq C \| e^{az} q_2 \|_{L^2(\mathbb{R})} \| \partial_y^3 \tilde{P}_1 f \| \| Y_1 \|, \]

(B.3) \[ \| \partial_y, S_k^j [q_c] \| = 0. \]

Proof. Since the Fourier transform of \( S_k^j f \) can be rewritten as

(B.4) \[ \mathcal{F}_y(S_k^j f)(t, \eta) = 1_{[-\eta_0, \eta_0]}(\eta) \hat{f}(\eta) \int dz q_2(z) \tilde{g}_{k1}^{*}(z, \eta, 2), \]

we have \( [\partial_y, S_k^j] = i \mathcal{F}^{-1} \left[ \eta_1, \mathcal{F}_y(S_k^j f)(t, \eta) \right] = 0 \). Since

\[ \sup_{\eta \in [-\eta_0, \eta_0]} \left| \int dz q_2(z) \tilde{g}_{k1}^{*}(z, \eta, 2) \right| \leq \| e^{az} q_2(z) \|_{L^2(\mathbb{R})} \]

by Claim 2.1 we see that (B.1) and (B.2) follow immediately from (B.4) and (B.3).

□

Claim B.2. There exist positive constants \( \eta_1 \), \( \delta \) and \( C \) such that if \( \eta \in (0, \eta_1) \) and \( M_1(T) \leq \delta \), then for \( k = 1, 2 \), \( t \in [0, T] \) and \( f \in L^2(\mathbb{R}) \),

(B.5) \[ \| S_k^2 [q_c](f)(t, \cdot) \|_{Y_1} \leq C \sup_{c \in [2-\delta, 2+\delta]} \left( \| e^{az} q_c \|_{L^2(\mathbb{R})} + \| e^{az} \partial_c q_c \|_{L^2(\mathbb{R})} \right) \| \tilde{c} \|_{Y} \| f \|_{L^2(\mathbb{R})}, \]

(B.6) \[ \| \partial_y S_k^2 [q_c](f)(t, \cdot) \|_{Y_1} \leq C \sum_{i=1,2} \sup_{c \in [2-\delta, 2+\delta]} \| e^{az} \partial_y^i q_c \|_{L^2(\mathbb{R})} \| \tilde{c} \|_{Y} \| f \|_{L^2(\mathbb{R})}, \]

(B.7) \[ \| S_k^2 [q_c](f)(t, \cdot) \|_{Y_1} \leq C \sum_{0 \leq i \leq 2} \sup_{c \in [2-\delta, 2+\delta]} \| e^{az} \partial_y^i q_c \|_{L^2(\mathbb{R})} \| \tilde{c} \|_{Y} \| f \|_{L^2(\mathbb{R})}, \]

(B.8) \[ \| [\partial_y, S_k^2 [q_c]](f, t, \cdot) \|_{Y_1} \leq C \sum_{0 \leq i \leq 3} \sup_{c \in [2-\delta, 2+\delta]} \| e^{az} \partial_y^i q_c \|_{L^2(\mathbb{R})} \| \tilde{c} \|_{Y} \| f \|_{L^2(\mathbb{R})}. \]
Proof. By the definition of $g^*_k$:

$$\sup_{c \in [2^{-\delta}, 2^{\delta}]} \sup_{|\eta| \leq \eta_0} \left| \int_{\mathbb{R}} g^*_{k2}(z, \eta, c)dz \right| \lesssim \sup_{c \in [2^{-\delta}, 2^{\delta}]} \left( \|e^{ax}q_c\|_{L^2_z} + \|e^{ax}\partial_c q_c\|_{L^2_z} \right).$$

Since

$$\mathcal{F}_y(S^2_k[q_c]f)(t, \eta) = \int dy e^{-iyn} f(y) \overline{\hat{c}(t, y)} \frac{1-\eta_0}{\sqrt{2\pi}} \int dz g^*_{k2}(z, \eta, c(t, y)),$$

we have

$$\|S^2_k[q_c](f)(t, \cdot)\|_{y_1} = \|\mathcal{F}_y(S^2_k[q_c](f))(t, \eta)\|_{L^\infty[-\eta_0, \eta_0]} \lesssim \sup_{c \in [2^{-\delta}, 2^{\delta}]} \left( \|e^{ax}q_c\|_{L^2_z} + \|e^{ax}\partial_c q_c\|_{L^2_z} \right) \int |f(y)\overline{\hat{c}(t, y)}| dy \lesssim \sup_{c \in [2^{-\delta}, 2^{\delta}]} \left( \|e^{ax}q_c\|_{L^2_z} + \|e^{ax}\partial_c q_c\|_{L^2_z} \right) \|f\|_{L^2} \|\hat{c}\|_{Y}.$$

Next, we will prove (B.7). Let

$$S^2_k[q_c](f)(t, y) = \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} f(y_1) \hat{c}(t, y_1) \overline{g^*_{k2}(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dy_1 dz d\eta,$$

$$S^2_k[q_c](f)(t, y) = \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} f(y_1) \hat{c}(t, y_1) \overline{g^*_{k4}(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dy_1 dz d\eta,$$

where

$$g^*_{k4}(z, \eta, c) = \frac{g^*_{k2}(z, \eta, c) - g^*_{k2}(z, \eta, 2)}{c - 2}.$$

Then $S^2_k[q_c] = S^2_k[q_c] + S^2_k[q_c]$ and we can prove

$$\|S^2_k[q_c]f(t, \cdot)\|_{Y} \lesssim \sum_{0 \leq i \leq 2} \sup_{c \in [2^{-\delta}, 2^{\delta}]} \|e^{ax}\partial^i_c q_c\|_{L^2_z} \|\hat{c}\|_{Y} \|f\|_{L^2},$$

(B.9)

$$\|S^2_k[q_c]f(t, \cdot)\|_{y_1} \lesssim \sum_{0 \leq i \leq 2} \sup_{c \in [2^{-\delta}, 2^{\delta}]} \|e^{ax}\partial^i_c q_c\|_{L^2_z} \|\hat{c}\|_{L^2} \|f\|_{L^2},$$

in exactly the same way as (B.1) and (B.5). Since

$$\|S^2_k[q_c]f(t, \cdot)\|_{y} \lesssim \|S^2_k[q_c]f(t, \cdot)\|_{Y} + \|S^2_k[q_c]f(t, \cdot)\|_{y_1},$$

(B.7) follows from (B.9).

Now we will show (B.8). Noting that

$$\mathcal{F}_y([\partial_y, S^2_k[q_c]] f)(t, \eta) = \frac{1-\eta_0}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) c_y(t, y) e^{-iyn} dy \int_{\mathbb{R}} \overline{g^*_{k2}(z, \eta, 2)} dz,$$

$$\mathcal{F}_y([\partial_y, S^2_k[q_c]] f)(t, \eta) = \frac{1-\eta_0}{\sqrt{2\pi}} \int_{\mathbb{R}^2} f(y) \partial_y \left( \hat{c}(t, y)^2 \overline{g^*_{k4}(z, \eta, c(t, y))} \right) \times e^{-iyn} dz dy,$$

we can prove (B.8) in the same way as (B.3). Eq. (B.6) immediately follows from (B.5) and (B.8). Thus we complete the proof. □

Next we will estimate the operator norm of $S^3[p](f)$. 


Claim B.3. There exist positive constants $\eta_1$ and $C$ such that for $\eta_0 \in (0, \eta_1]$, $k = 1, 2, t \geq 0$ and $f \in L^2(\mathbb{R})$,
\begin{align}
\|S_k^4[p](f)(t, \cdot)\|_{Y_1} &\leq Ce^{-a(4t + L)}\|e^{az}p\|_{L^2}\|\tilde{F}_yf\|_{Y_1}, \\
\|S_k^4[p](f)(t, \cdot)\|_{Y_1} &\leq Ce^{-a(4t + L)}\|e^{az}p\|_{L^2}\|\tilde{F}_yf\|_{Y_1}.
\end{align}
Moreover,
\begin{equation}
[\partial_y, S_k^4[p]] = 0.
\end{equation}

**Proof.** The Fourier transform of $S_k^4f$ is
\begin{equation}
\mathcal{F}_y(S_k^4f)(t, \eta) = 1_{[\eta_0, \eta_0]}(\eta)\hat{f}(\eta) \int_\mathbb{R} p(z + 4t + L)\overline{g_k^*(z, \eta)} dz.
\end{equation}
By Claim 2.1,
\begin{equation}
\left| \int p(z + 4t + L)\overline{g_k^*(z, \eta)} dz \right| \leq e^{-a(4t + L)}\|e^{az}p\|_{L^2} \sup_{|\eta| \leq \eta_0} \|e^{-az}g_k^*(z, \eta)\|_{L^2}
\end{equation}
\begin{equation}
\lesssim e^{-a(4t + L)}\|e^{az}p\|_{L^2}.
\end{equation}
Combining (B.13) and (B.14), we immediately have (B.10) and (B.11). Eq. (B.12) clearly follows from the definition of $S_k^4$. Thus we complete the proof. \hfill $\square$

Claim B.4. There exist positive constants $\eta_1$, $\delta$ and $C$ such that if $\eta_0 \in (0, \eta_1]$ and $M_1(T) \leq \delta$, then for $k = 1, 2, t \in [0, T]$ and $f \in L^2$,
\begin{align}
\|S_k^4[p](f)(t, \cdot)\|_{Y_1} &\leq Ce^{-a(4t + L)}\|e^{az}p\|_{L^2}\|\tilde{F}_yf\|_{L^2}, \\
\|[\partial_y, S_k^4[p]](f)(t, \cdot)\|_{Y_1} &\leq Ce^{-a(4t + L)}\|e^{az}p\|_{L^2}\|\tilde{F}_yf\|_{L^2}.
\end{align}

**Proof.** Since
\begin{equation}
\mathcal{F}_y(S_k^4[p](f))(t, \eta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^2} f(y)\overline{\tilde{c}(t, y)p(z + 4t + L)\overline{g_k^*(z, \eta, c(t, y))e^{-iy\eta}}} dz dy,
\end{equation}
we have
\begin{equation}
\|S_k^4[p](f)(t, \cdot)\|_{Y_1} \lesssim \|f\|_{L^2}\|\tilde{c}\|_{L^2}e^{-a(4t + L)}\|e^{az}p\|_{L^2} \sup_{c \in [2-\delta, 2+\delta]} \sup_{\eta \in [-\eta_0, \eta_0]} \|e^{-az}g_k^*(z, \eta, c)\|_{L^2}
\end{equation}
\begin{equation}
\lesssim e^{-a(4t + L)}\|f\|_{L^2}\|\tilde{c}(t)\|_{Y_1}.
\end{equation}
Noting that
\begin{equation}
[\partial_y, S_k^4[p]](f) = \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} f(y_1)p(z + 4t + L)\partial_{y_1} \left\{ e^{iy_1\eta}g_k^*(z, \eta, c(t, y_1)) \right\} \times e^{iy\eta} dz dy_1 d\eta,
\end{equation}
we can prove (B.16) in the same way as (B.15). Thus we complete the proof. \hfill $\square$

Claim B.5. There exist positive constants $\eta_1$, $\delta$ and $C$ such that if $\eta_0 \in (0, \eta_1]$ and $M_1(T) \leq \delta$, then for $k = 1, 2$ and $t \in [0, T]$,
\begin{equation}
\|S_k^4f\|_{Y_1} + \|S_k^4f\|_{Y_1} \leq C\|\tilde{v}(t, \cdot)\|\|f\|_{L^2}.
\end{equation}
Proof. Using the Schwarz inequality, we have

\[ \| S_k^6 f \|_{Y_1} = \sup_{|\nu| \leq \eta_0} \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} v(t, z, y) \overline{f(y) \partial_z g_k^6(z, \eta, \nu)} e^{-i\nu t} \, dzdy \right| \]

\[ \lesssim \sup_{c \in [2-\delta, 2+\delta], \eta \in [-\eta_0, \eta_0]} \| e^{-a_2 \partial_z g_k^6(z, \eta, \nu)} \|_{L^2} \| v(t) \|_{X} \| f \|_{L^2}. \]

Since \( \| e^{-a_2 \partial_z g_k^6(z, \eta, \nu)} \|_{L^2} \) is bounded for \( c \in (1, 3) \) and \( \eta \in [-\eta_0, \eta_0] \), we have

\[ \| S_k^6 f \|_{Y_1} \lesssim \| v(t, \cdot) \|_{X} \| f \|_{L^2}. \]

We can estimate \( S_k^6 \) in exactly the same way. Thus we complete the proof. \( \square \)

Next, we will estimate operator norms of \( \tilde{C}_k \) (\( k = 1, 2 \)).

Claim B.6. There exist positive constants \( \delta \) and \( C \) such that if \( \sup_{t \in [0, T]} \| \tilde{c}(t) \|_Y \leq \delta \), then for \( k = 1, 2 \) and \( t \in [0, T] \),

\[ \| \mathcal{C}_k f \|_Y \leq C \| \tilde{c} \|_{L^\infty} \| \tilde{P}_1 f \|_Y, \]

\[ \| \mathcal{C}_k f \|_{Y_1} \leq C \| \tilde{c} \|_Y \mathcal{P}_1 f \|_Y. \]

Proof. Since \( \| \tilde{c} \|_{L^\infty} \lesssim \| \tilde{c} \|_Y \) by Remark [14], it follows that \( |c^2 - 4| \leq (2 + O(\delta))|\tilde{c}| \).

Thus we have

\[ \| \mathcal{C}_1 f \|_Y \leq \frac{1}{2} \| c^2 - 4 \|_{L^\infty} \| \tilde{P}_1 f \|_{L^2} \lesssim \| \tilde{c} \|_{L^\infty} \| \tilde{P}_1 f \|_Y, \]

\[ \| \mathcal{C}_1 f \|_{Y_1} = \frac{1}{2} \left\| \mathcal{F} (c^2 - 4) \ast \mathcal{F}(\tilde{P}_1 f) \right\|_{L^\infty([-\eta_0, \eta_0])} \lesssim \| c^2 - 4 \|_{L^2} \| \tilde{P}_1 f \|_{L^2} \lesssim \| \tilde{c} \|_Y \| \tilde{P}_1 f \|_Y. \]

We can estimate \( \mathcal{C}_2 \) in exactly the same way. Thus we complete the proof. \( \square \)

Claim B.7. There exist positive constants \( \delta \) and \( C \) such that if \( \sup_{t \in [0, T]} \| \tilde{c}(t) \|_Y \leq \delta \), then for \( k = 1, 2 \) and \( t \in [0, T] \),

\[ \| [\partial_y, \mathcal{C}_k] f \|_Y \leq C \| c_y \|_{L^\infty} \| f \|_{L^2}, \]

\[ \| [\partial_y, \mathcal{C}_k] f \|_{Y_1} \leq C \| c_y \|_Y \| f \|_{L^2}. \]

Proof. Since \( [\partial_y, \mathcal{C}_1] = \tilde{P}_1 c_y \tilde{P}_1 \),

\[ \| [\partial_y, \mathcal{C}_1] f \|_Y \lesssim \| c_y \|_{L^\infty} \| f \|_{L^2}, \]

\[ \| [\partial_y, \mathcal{C}_1] f \|_{Y_1} = \left\| \mathcal{F} (c c_y) \ast \mathcal{F}(\tilde{P}_1 f) \right\|_{L^\infty([-\eta_0, \eta_0])} \lesssim \| c_y \|_Y \| \tilde{P}_1 f \|_Y. \]

We can prove the estimate for \( [\partial_y, \mathcal{C}_2] \) in the same way. Thus we complete the proof. \( \square \)
APPENDIX C. PROOF OF CLAIMS 6.1, 6.2 AND 6.3

Proof of Claims 6.2 and 6.3

Claims 6.1, 6.2 and 6.1 imply that for \( s \in [0, T] \),

\[
\| \tilde{S}_1 \|_{B(Y)} \lesssim \| S_1 \|_{B(Y)} (1 + \tilde{C}_2)^{-1} \| B(Y) \|
\]

\[
(C.1)
\]

\[
\| \tilde{S}_2 \|_{B(Y,Y_1)} \lesssim \| S_2 \|_{B(Y,Y_1)} (1 + \tilde{C}_2)^{-1} \| B(Y) \|
\]

\[
(C.2)
\]

\[
\| \tilde{S}_3 \|_{B(Y)} \lesssim \| S_3 \|_{B(Y)} \lesssim e^{-a(4s + L)},
\]

\[
(C.3)
\]

By Claims 6.1 and 6.1

\[
\| \tilde{S}_4 \|_{B(Y,Y_1)} \lesssim \sum_{k=1,2} \left( \| S_k^3 \|_{B(Y)} + \| S_k^2 \|_{B(Y)} \right) \lesssim M_1(T) \| s \|^{-1/4} e^{-a(4s + L)}.
\]

\[
(C.4)
\]

By Claims 6.1 and 6.1

\[
\| \tilde{S}_5 \|_{B(Y,Y_1)} \lesssim \sum_{k=1,2} \left( \| S_k^6 \|_{B(Y,Y_1)} \right) \lesssim M_2(T) \| s \|^{-3/4}.
\]

\[
(C.5)
\]

Obviously,

\[
\| \theta_y \|_{B(Y)} + \| \partial_y \|_{B(Y)} \lesssim \eta_0.
\]

By 6.15, 6.17, 6.21 and 6.6 and the fact that \( Y_1 \subset Y \),

\[
\| B_3 - B_1 \|_{B(Y)} \lesssim \| \tilde{C}_1 \|_{B(Y)} + \eta_0^2 \sum_{j=1,2} \| \tilde{S}_j \|_{B(Y)} + \sum_{j=3,4,5} \| \tilde{S}_j \|_{B(Y)}
\]

\[
\lesssim M_1(T) + M_2(T) + \eta_0^2 + e^{-aL}.
\]

Since \( B_1 \) is invertible, we see that \( \| B_3^{-1} \|_{B(Y)} \) is bounded for \( t \in [0, T] \) if \( \| B_3 - B_1 \|_{B(Y)} \) remains small on \( [0, T] \). We can prove the boundedness of \( \| B_3^{-1} \|_{B(Y)} \) in the same way. This completes the proof of Claim 6.2.
Using Claims B.1 and B.3 we can prove

\[(C.7) \quad \| \tilde{S}_1 \|_{B(Y)} + \| \tilde{S}_1 \|_{B(Y_1)} \lesssim 1,\]

\[(C.8) \quad \| \tilde{S}_3 \|_{B(Y)} + \| \tilde{S}_3 \|_{B(Y_1)} \lesssim e^{-\alpha(4t+L)} \quad \text{for} \ t \geq 0,\]

in the same way as (C.1) and (C.3). Claim 6.3 immediately follows from (C.6), (C.7) and (C.8). Thus we complete the proof.

Proof of Claim 7.1 In view of (6.15),

\[\sum_{j=1,2} \partial_y^2[\partial_y, \tilde{S}_j] = \sum_{j=3,4,5} \partial_y \delta_y, \tilde{S}_j,\]

Now we will estimate each term of the right hand side. By Claim B.7 and the definition of \( \tilde{C}_k \),

\[(C.9) \quad \| [\partial_y, \tilde{C}_k] \|_{B(Y, Y_1)} \lesssim M_4(T)(s)^{-3/4} \quad \text{for} \ k = 1, 2 \ \text{and} \ s \in [0, T].\]

Since \([\partial_y, \tilde{S}_1] = 0\) by B.3, we have \([\partial_y, \tilde{S}_1] = \tilde{S}_1 [\tilde{C}_2, \partial_y](1 + \tilde{C}_2)^{-1} \). Thus by Claim 6.1 (C.1) and (C.9),

\[(C.10) \quad \| [\partial_y, \tilde{S}_1] \|_{B(Y, Y_1)} \lesssim \| \tilde{S}_1 \|_{B(Y_1)} \| [\tilde{C}_2, \partial_y] \|_{B(Y)} \| (1 + \tilde{C}_2)^{-1} \|_{B(Y)} \lesssim M_4(T)(s)^{-3/4} \quad \text{for} \ s \in [0, T].\]

Applying Claims B.1 B.2 B.7 and (C.2) to \([\partial_y, \tilde{S}_2] = \{ [\partial_y, \tilde{S}_2] + \tilde{S}_2 [\tilde{C}_2, \partial_y] \}(I + \tilde{C}_2)^{-1}\), we obtain

\[(C.11) \quad \| [\partial_y, \tilde{S}_2] \|_{B(Y, Y_1)} \lesssim \| [\partial_y, \tilde{S}_2] \|_{B(Y, Y_1)} + \| \tilde{S}_2 \|_{B(Y, Y_1)} \| [\partial_y, \tilde{C}_2] \|_{B(Y)} \lesssim \| \tilde{S}_2 \|_{B(Y, Y_1)} \| [\partial_y, \tilde{C}_2] \|_{B(Y)} \lesssim e^s \| \tilde{C}_2 \|_{Y, Y_1} \| c_\gamma \|_{L^\infty} \lesssim M_4(T)(s)^{-3/4} \quad \text{for} \ s \in [0, T].\]

Since \([\partial_y, \tilde{S}_3] = 0\) by B.12, we have \([\partial_y, \tilde{S}_3] = \tilde{S}_3 [\tilde{C}_2, \partial_y](I + \tilde{C}_2)^{-1}\). Hence it follows from Claims B.1 B.3 and (C.9) that

\[(C.12) \quad \| [\partial_y, \tilde{S}_3] \|_{B(Y, Y_1)} \lesssim \| \tilde{S}_3 \|_{B(Y_1)} \| c_\gamma \|_{Y} \lesssim M_4(T)(s)^{-3/4} e^{-\alpha(4s+L)}.\]

By (C.4), (C.5) and (C.6), we have for \( s \in [0, T] \),

\[(C.13) \quad \| [\partial_y, \tilde{S}_4] \|_{B(Y, Y_1)} \lesssim \eta M_1(T)(s)^{-1/4} e^{-\alpha(4s+L)},\]

\[(C.14) \quad \| [\partial_y, \tilde{S}_5] \|_{B(Y, Y_1)} \lesssim \eta M_2(T)(s)^{-3/4}.\]

Combining (C.9) C.14, we obtain Claim 7.1 Thus we complete the proof.

**Appendix D. Estimates of \( R^k \)**

**Claim D.1.** There exist positive constants \( \delta \) and \( C \) such that if \( M_4(T) \leq \delta \), then for \( t \in [0, T] \),

\[\| R_k^2(t, \cdot) \|_{Y_1} \leq C M_1(T)^2(t)^{-1}, \quad \| \partial_y R_k^2(t, \cdot) \|_{Y_1} \leq C M_1(T)^2(t)^{-5/4}.\]
Proof. By Claims [B.1] [B.2] and [B.3],
\[
\|R_k^2\|_{Y_1} \lesssim \|\hat{c}\|_{Y}(\|x_{yy}\|_{Y} + \|c_{yy}\|_{Y}) + \|c_y\|^2_{Y}(1 + \|\hat{c}\|_{L^\infty}) \\
\lesssim M_1(T) (T)^2 (s)^{-1}.
\]
We can estimate \(\|\partial_y R_k^2\|_{Y_1}\) in the same way. Thus we complete the proof. \(\square\)

Claim D.2. There exist positive constants \(\delta\) and \(C\) such that if \(M_1(T) \leq \delta\), then for \(t \in [0, T]\), \(\|R_k^3(t, \cdot)\|_{Y_1} \leq C(t)^{-1/2} e^{-\alpha(4+L)M_1(T)^2}\).

Proof. We decompose \(R_k^3\) into three terms. Let
\[
R_k^1 = \frac{3}{2} \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \left\{ \partial_z \psi^2(z, t, y) + x_{yy} \psi(z, t, y) + 3c_y(t, y)^2 \int_{z}^\infty \partial_z^2 \psi(z, t, y) \right\} dx dy dt,
\]
and
\[
R_k^2 = \frac{3}{2} \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \left( \partial_z \psi(z, t, y) - 2 \partial \psi(z, t, y) \right) dx dy dt,
\]
where \(g_{k5}(z, \eta, c)^* = (\varphi(z) \partial_z g_{k5}(z, \eta, c) - \varphi(z) \partial_z g_{k5}(z, \eta, c))/\hat{c}\). Then \(R_k^3 = \sum_{i=1}^3 R_k^{3i}\).

Let us estimate \(R_k^1\) by using Claims [B.3] and [B.4]. Since \(\psi(z) = (2\sqrt{2c-2}) \psi(z + 4t + L)\), we have
\[
R_k^1 = S_k^3[\psi'''](2\sqrt{2c-2} - \hat{c}) + S_k^3[\psi'''](2\sqrt{2c-2}) + 3S_k^3[\partial_z^{-1}\psi(\{(2/c)^{1/2} - 1\} c_{yy}) + 3S_k^3[\partial_z^{-1}\psi(\{(2/c)^{1/2} c_{yy})
\]
\[
R_k^2 = -24(S_k^3 + S_k^4)(\psi^2')((\sqrt{c} - \sqrt{2}c) - 6\sqrt{2}(S_k^3 + S_k^4)[\psi(\{(\sqrt{c} - \sqrt{2})x_{yy})
\]
\[
- \frac{3\sqrt{2}}{2}(S_k^3 + S_k^4)[\partial_z^{-1}\psi(c^{-3/2}(c_{yy})^2).
\]
Since \(2\sqrt{2c-2} - \hat{c} + O(c^2)\) and \((2/c)^{1/2} - 1 = O(\hat{c})\) and \(\hat{P}L^1 \subset Y_1\), it follows from Claim [B.3] that
\[
\|S_k^3[\psi'''](2\sqrt{2c-2} - \hat{c})\|_{Y_1} \lesssim e^{-\alpha(4t + L)}\|\hat{c}\|_{Y}^2,
\]
and
\[
\|\sqrt{2}S_k^3[\partial_z^{-1}\psi(\{(2/c)^{1/2} - 1\} c_{yy})\|_{Y_1} \lesssim e^{-\alpha(4t + L)}\|\hat{c}\|_{Y} c_{yy}\|_{Y}.
\]
By Claim [B.4]
\[
\|S_k^3[\psi'''](\sqrt{c} - \sqrt{2})\|_{Y_1} + \|S_k^3[\partial_z^{-1}\psi(c^{-1/2} c_{yy})]\|_{Y_1} \lesssim e^{-\alpha(4t + L)}\|\hat{c}\|_{Y} (\|\hat{c}\|_{Y} + \|c_{yy}\|_{Y}).
\]
Thus we prove $\|R_{k1}^3\|_{Y_1} \lesssim e^{-a(4t+L)}\|\hat{c}\|_Y (\|\hat{c}\|_Y + \|c_{yy}\|_Y)$. Similarly, we have $\|R_{k2}^3\|_{Y_1} \lesssim e^{-a(4t+L)}(\|\hat{c}\|_Y^2 + \|\hat{c}\|_Y|x_{yy}\|_Y + \|c_{yy}\|_Y^2)$. $\|R_{k3}^3\|_{Y_1} \lesssim e^{-a(4t+L)}\|\hat{c}\|_Y^2$.

Thus we complete the proof. □

Claim D.3. There exists a positive constant $C$ such that

$$\|\hat{A}_1(t)\|_{B(Y)} + \|\hat{A}_1(t)\|_{B(Y_1)} \leq Ce^{-a(4t+L)} \quad \text{for every } t \geq 0 \text{ and } L \geq 0.$$  

Proof. In view of (6.7),

$$\hat{a}_k(t, D_y)\hat{c} = S_k^2(\psi')(\hat{c}) + 3S_k^3(\partial_z^{-1}\psi)(c_{yy}) - 6F_{\eta}^{-1}\left\{ \int \varphi(z)(z + 4t + L)\partial_z g_{\eta}^k(z, \eta)dz(F_{\eta}\hat{c})(t, \eta) \right\}.$$  

Hence it follows from Claim B.3 and (B.14) that

$$\|\hat{a}_k(t, D_y)\|_{B(Y)} + \|\hat{a}_k(t, D_y)\|_{B(Y_1)} \lesssim e^{-a(4t+L)}.$$  

Thus we complete the proof of Claim D.3. □

Claim D.4. There exist positive constants $C$ and $L_0$ such that if $L \geq L_0$, then

$$\|A_1(t)\|_{B(Y)} \leq Ce^{-a(4t+L)} \quad \text{for every } t \geq 0.$$  

Proof. Since $B_1$ is invertible and $\|\hat{S}_3\|_{B(Y)} \lesssim \sum_{k=1,2} \|S_k^2(\psi)\|_{B(Y)} \lesssim e^{-a(4t+L)}$, we have Claim D.4. □

Claim D.5. Suppose $a \in (0,1)$ and $M_1(T) \leq \delta$. If $\delta$ is sufficiently small, then there exists a positive constant $C$ such that

$$(D.1) \quad \|R_{k1}^1(t)\|_{Y_1} \leq C(M_1(T) + M_2(T)M_2(T))(t)^{-3/2},$$  

$$(D.2) \quad \|R_{k2}^1(t)\|_{Y_1} \leq CM_1(T)M_2(T)(t)^{-1},$$  

$$(D.3) \quad \|R_{k3}^1(t)\|_{Y_1} \leq Ce^{-a(4t+L)}(t)^{-1}M_1(T)M_2(T),$$  

$$(D.4) \quad \|R_{k4}^1(t)\|_{Y} \leq CM_1(T)M_2(T)(t)^{-5/4}.$$  

Proof. By Lemma [2.2] and [5.3], we can rewrite $II_1^k$ as $II_1^k = i\eta II_1^{k1} + II_1^{k2} + II_1^{k3}$, where

$$II_1^{k1}(t, \eta) = -6 \int_{\mathbb{R}} c_y(t, y) h_{1k}(t, y, \eta)e^{-iy\eta}dy,$$  

$$II_1^{k2}(t, \eta) = 3 \int_{\mathbb{R}} c_{yy}(t, y) h_{1k}(t, y, \eta)e^{-iy\eta}dy,$$  

$$II_1^{k3}(t, \eta) = 3 \int_{\mathbb{R}} (c_y(t, y))^2 h_{2k}(t, y, \eta)e^{-iy\eta}dy,$$  

$$h_{jk}(t, y, \eta) = \int_{\mathbb{R}} v(t, z, y) \left( \int_{\mathbb{R}} \partial_z g_{\eta}^k(z_1, \eta, c(t, y))dz_1 \right)dz \quad \text{for } j = 1, 2.$$  

First, we will estimate $II_1^{k1}(t, \cdot)$. Since

$$\sup_{-\nu_0 \leq \eta \leq \nu_0, 2-\delta \leq \epsilon \leq 2+\delta} \left\| e^{-az} \int_{-\infty}^{\epsilon} g_{\eta}^k(z, \eta, c)dz \right\|_{L^2_z} < \infty,$$

there exists a positive constant $C$ such that

$$\sup_{-\nu_0 \leq \eta \leq \nu_0} |h_{jk}(t, y, \eta)| \leq C\|e^{az}v(t, z, y)\|_{L^2_z} \quad \text{for any } y \in \mathbb{R} \text{ and } t \geq 0.$$
Thus by the Schwarz inequality,

\[ \|II_{k1}^1(t, \cdot)\|_{L^\infty([-\eta_0, \eta_0])} \leq \|c_y(t)\|_{Y} \sup_{\eta \in [-\eta_0, \eta_0]} \left( \int_{\mathbb{R}} |h_{1k}(t, y, \eta)|^2 dy \right)^{1/2} \lesssim \|c_y(t)\|_{Y} \|v(t)\|_{X}. \]  

We can prove

\[ \|II_{k2}^1(t, \eta)\|_{L^\infty([-\eta_0, \eta_0])} \lesssim \|c_{yy}\|_{Y} \|v(t, \cdot)\|_{X}, \]  

\[ \|II_{k3}^1(t, \eta)\|_{L^\infty([-\eta_0, \eta_0])} \lesssim \|c_y\|_{\tilde{L}^2(\mathbb{R})} \|v(t, \cdot)\|_{X}, \]  

in exactly the same way.

Next, we will estimate \( II_{k}^2 \) and \( II_{k}^3 \). Since

\[ \sup_{c \in [2^{-\delta} \eta, \eta_0], \eta \in [-\eta_0, \eta_0]} \|e^{-2az} g_k^c(z, \eta, c)\|_{L_x^\infty} < \infty, \]

\[ \sup_{c \in [2^{-\delta} \eta, \eta_0], \eta \in [-\eta_0, \eta_0]} \left( \|e^{-2az} g_k^c\|_{L^2} + \|e^{-az} \partial_z g_k^c\|_{L^2} + \|e^{-az} \partial_c g_k^c\|_{L^2} \right) < \infty, \]

we have

\[ \|II_{k}^2\|_{L^\infty([-\eta_0, \eta_0])} = 3 \sup_{\eta \in [-\eta_0, \eta_0]} \left( \int_{\mathbb{R}^2} v(t, z, y)^2 \partial_z g_k^c(z, \eta, c(t, y)) e^{-iy\eta} dz dy \right) \lesssim \|v\|_{X} \sup_{c \in [2^{-\delta} \eta, \eta_0], \eta \in [-\eta_0, \eta_0]} \|e^{-2az} g_k^c(z, \eta, c)\|_{L_x^\infty} \lesssim \|v\|_{X}^2, \]

and

\[ \|II_{k1}^3(t, \eta)\|_{L^\infty([-\eta_0, \eta_0])} \lesssim \|v(t)\|_{X} \|x_{yy}\|_{Y}, \quad \|R_{k}^2\|_{Y_1} \lesssim \|II_{k2}^3\|_{L^\infty([-\eta_0, \eta_0])} \lesssim \|v(t)\|_{X} \|x_{yy}\|_{Y}. \]

Combining the above, we have

\[ \|R_{k}^1(t)\|_{Y_1} \lesssim \sup_{-\eta_0 \leq \eta \leq \eta_0} (||II_{k1}^1(t, \eta)|| + ||II_{k2}^1(t, \eta)|| + ||II_{k3}^1(t, \eta)||) \lesssim \|v(t, \cdot)\|_{X} (\|c_y(t)\|_{Y} + \|c_{yy}\|_{Y}) + \|c_y(t)\|_{\tilde{L}^2}^2 + \|x_{yy}\|_{Y} + \|v(t, \cdot)\|_{X}^2, \]

which implies \( (D.1) \).

By the Schwarz inequality and \( (S.11) \),

\[ \|R_{k}^1\|_{Y_1} \lesssim \sup_{|\eta| \leq \eta_0} \int_{\mathbb{R}^2} v(t, x, y) \partial_{t,c} g_k^c(z, \eta, c(t, y)) e^{-iy\eta} dz dy \lesssim \|v(t)\|_{X} \sup_{c \in [2^{-\delta} \eta, \eta_0], \eta \in [-\eta_0, \eta_0]} \|e^{-2az} g_k^c(z, \eta, c)\|_{L_x^\infty} \lesssim e^{-a(4t + L)} \|\tilde{c}(t)\|_{L^2(\mathbb{R})} \|v(t)\|_{X}. \]

Finally, we will estimate \( \|R_{k}^2\|_{Y} \). Let

\[ II_{k21} = 6 \int_{\mathbb{R}^2} v(t, z, y) x_y(t, y) g_k^c(z, \eta) e^{-iy\eta} dz dy \]

\[ = 6\sqrt{2\pi} \int_{\mathbb{R}} F_y (x_y(t, \cdot) v(t, z, \cdot)) (\eta) g_k^c(z, \eta) dz, \]

\[ II_{k22} = 6 \int_{\mathbb{R}^2} v(t, z, y) x_y(t, y) \tilde{c}(t, y) g_k^c(z, \eta, c(t, y)) e^{-iy\eta} dz dy. \]
Then $II_{k2}^3 = II_{k21}^3 + II_{k22}^3$. By the Schwarz inequality,
\[
\|II_{k21}^3\| \lesssim \|e^{-az}g_k^*(z, \eta)\|_{L^2} \left( \int_{\mathbb{R}} |\mathcal{F}_y(x_y(t, \cdot) v(t, z, \cdot))(\eta)|^2 \, dz \right)^{1/2}.
\]
By (D.9) and Plancherel's theorem,
\[
\|II_{k21}^3\|_{L^2[-y_0, y_0]} \lesssim \left( \int_{-y_0}^{y_0} \int_{\mathbb{R}} |\mathcal{F}_y(x_y(t, \cdot) e^{-az} v(t, z, \cdot))(\eta)|^2 \, dz \, d\eta \right)^{1/2}
\lesssim \|x_y(t)v(t)\|_X.
\]
By the Schwarz inequality,
\[
\|II_{k22}^3\|_{L^2[-y_0, y_0]} \lesssim \left[ \sup_{y \in [-y_0, y_0], c \in [2-\delta, 2+\delta]} \|e^{-az}g_k^*(z, \eta, c)\|_{L^2} \right] \lesssim \|v(t)\|_X \|x_y(t)\|_Y \lesssim \mathcal{M}_1(T)^2 \mathcal{M}_2(T)(t)^{-3/2}.
\]
Combining the above, we have for $t \in [0, T]$,
\[
\|R_k^3\|_Y \lesssim \|II_{k21}^3\|_{L^2[-y_0, y_0]} + \|II_{k22}^3\|_{L^2[-y_0, y_0]} \lesssim \mathcal{M}_1(T)^2 \mathcal{M}_2(T)(t)^{-5/4}.
\]
Thus we complete the proof. \(\square\)

To estimate $R_k^4$, we need the following.

Claim D.6. There exist positive constants $\delta$ and $C$ such that if $\sup_{t \in [0, T]} \|\tilde{v}(t)\|_Y \leq \delta$, then for $t \in [0, T]$,
\[
\begin{align*}
\|b - \tilde{c}\|_Y &\leq C\|\tilde{c}\|_{L^\infty} \|\tilde{c}\|_Y, \\
\|b - \tilde{c}\|_{L^1} &\leq C\|\tilde{c}\|^2, \\
\|b_y - c_y\|_Y &\leq C\|\tilde{c}\|_Y \|c_y\|_Y, \\
\|b_y - c_y\|_{L^1} &\leq C\|\tilde{c}\|_Y \|c_y\|_Y, \\
\|b_{yy} - c_{yy}\|_Y &\leq C\|\tilde{c}\|_Y \|c_{yy}\|_Y + \|c_y\|_{L^\infty} \|c_y\|_Y, \\
\|b_{yy} - c_{yy}\|_{L^1} &\leq C\|\tilde{c}\|_Y \|c_{yy}\|_Y + \|c_{yy}\|^2, \\
\left\| \frac{(c/2)^{3/2}}{2} - \frac{1}{4} b \right\|_{L^2} &\leq C\|\tilde{c}\|_{L^\infty} \|\tilde{c}\|_Y, \\
\|b - \tilde{c} - \frac{1}{8} \tilde{P}_1 \tilde{c}^2\|_Y &\leq C\|\tilde{c}\|^2, \\
\end{align*}
\]

Proof. By (D.12),
\[
b - \tilde{c} = \frac{4}{3} \tilde{P}_1 \left\{ \left( \frac{c}{2} \right)^{3/2} - 1 - \frac{3}{4} \tilde{c} \right\},
\]
\[
\begin{align*}
b_y - c_y &\approx \tilde{P}_1 \left\{ (c/2)^{1/2} - 1 \right\} c_y, \\
b_{yy} - c_{yy} &\approx \tilde{P}_1 \left\{ (c/2)^{1/2} - 1 \right\} c_{yy} + \frac{1}{4} \tilde{P}_1 (c/2)^{-1/2} (c_y)^2.
\end{align*}
\]

Using the fact that $(c/2)^{3/2} - 1 - 3\tilde{c}/4 - 3c^2/32 = O(\tilde{c}^3)$, we can prove (D.10) – (D.14) and (D.16) in the same way as the proof of Claim B.6.
Finally, we will show (D.15). Let $\tilde{P}_2 = I - \tilde{P}_1$. Since $\tilde{P}_2 \tilde{c} = 0$ and
\[
\left( \frac{c}{2} \right)^{3/2} - 1 - \frac{3b}{4} = \tilde{P}_2 \left\{ \left( \frac{c}{2} \right)^{3/2} - 1 \right\},
\]
we have
\[
\left\| \left( \frac{c}{2} \right)^{3/2} - 1 - \frac{3b}{4} \right\|_{L^2} = \left\| \tilde{P}_2 \left\{ \left( \frac{c}{2} \right)^{3/2} - 1 - \frac{3b}{4} \right\} \right\|_{L^2} \lesssim \| \tilde{c} \|^2_{L^4}.
\]
Thus we complete the proof. \(\square\)

**Claim D.7.** There exist positive constants $\delta$ and $C$ such that if $\sup_{t \in [0, T]} \| \tilde{c}(t) \|_Y \leq \delta$, then for $t \in [0, T]$,
\[
(D.17) \quad \| \tilde{P}_1 R_1^1(s) \|_{Y_1} \leq CM_1(T)^2(s)^{-5/4}, \quad \| \tilde{P}_1 R_2^1(s) \|_{Y_1} \leq CM_1(T)^2(s)^{-1},
\]
\[
(D.18) \quad \| \tilde{P}_1 R_1^3(s) \|_Y \leq CM_1(T)^2(s)^{-3/2}, \quad \| \tilde{P}_1 R_2^3(s) \|_Y \leq CM_1(T)^2(s)^{-5/4},
\]
\[
(D.19) \quad \| \tilde{P}_1 \partial_y R_2^3(s) \|_{Y_1} \leq CM_1(T)^2(s)^{-5/4}, \quad \| \tilde{P}_1 \partial_y R_2^3(s) \|_Y \leq CM_1(T)^2(s)^{-3/2}.
\]

**Proof.** In view of (6.14) and (4.2),
\[
\| \tilde{P}_1 R_1^2 \|_{Y_1} \lesssim \| x_{yy} \|_Y \left( \left( \frac{c}{2} \right)^{3/2} - 1 - \frac{3b}{4} \right)_{L^2} + \| b_{yy} - c_{yy} \|_{Y_1} + \| b_y - c_y \|_Y + \| \tilde{c} \|_{L^\infty} \| c_y \|_Y \| x_y \|_Y + \| c_y \|_Y^2,
\]
\[
\| \tilde{P}_1 R_2^2 \|_{Y_1} \lesssim \| \tilde{c} \|_Y \| x_{yy} \|_Y + \| c_y \|_Y \| x_y \|_Y + \| b_{yy} - c_{yy} \|_{Y_1} + \| c_y \|_Y^2.
\]
Combining the above with Claim D.6, we have (D.17). We can obtain (D.18) and (D.19) in the same way. Thus we complete the proof. \(\square\)

**APPENDIX E. LOCAL WELL-POSEDNESS IN EXponentially WEIGHTED SPACE**

The $L^2$ well-posedness of the KP-II equation around line solitons has been proved by Mollinet, Saut and Tzvetkov (29) by using Bourgain’s norm. In this section, we will explain well-posedness for exponentially localized initial data around a line soliton.

Let $u(t, x, y) = \varphi(x - 4t) + \tilde{v}(t, x - 4t, y)$ be a solution to (2.1). Then
\[
(E.1) \quad \partial_t \tilde{v} = \mathcal{L} \tilde{v} - 3\partial_x (\tilde{v}^2).
\]

**Proposition E.1.** Suppose $a > 0$ and $\nu_0 \in X \cap L^2(\mathbb{R}^2)$. If $\tilde{v}(0) = \nu_0$, then there exists a unique solution of (E.1) such that for any $T > 0$,
\[
(E.2) \quad \tilde{v} \in L^\infty(0, T; X) \cap X_T,
\]
where $X_T$ is the auxiliary Banach space used in Theorem 1.1 of (29). If $\nu_0 \in H^1(\mathbb{R}^2)$ in addition, then $\tilde{v}(t) \in C([0, \infty); X)$.

**Remark E.1.** The Banach space $X_T$ is continuously imbedded into $C([0, T]; L^2(\mathbb{R}^2))$. Moreover (29) Lemma 4.1] tells us that
\[
(E.3) \quad \tilde{v}(t) \in C([0, T]; L^2(\mathbb{R}^2)) \cap L^4(0, T; L^4(\mathbb{R}^2))
\]
and that $\tilde{v}(t) \in C([0, T]; H^s(\mathbb{R}^2))$ if $\tilde{v}(0) \in H^s(\mathbb{R}^2)$ for an $s \geq 0$. 

Proof of Proposition [E.7]. To prove $\tilde{v}(t) \in L^\infty(0, T; X)$ for any $T > 0$, we will use the virial identity for the KP-II equation (E.1). Let $a > 0$, $p(x) = 1 + \tanh ax$ and $p_n(x) = e^{2\pi n} p(x - n)$ for $n \in \mathbb{N}$. Suppose $\nu_0 \in H^3(\mathbb{R}^2) \cap X$ and $\partial_x^{-1} \nu_0 \in H^2(\mathbb{R}^2)$. Then by [29], we have $\tilde{v}(t) \in C([0, \infty); H^3)$ and $\partial_x^{-1} \nu_0 \in C([0, \infty); H^2)$. Multiplying (E.1) by $2p_n(x) \nu(t, x, y)$ and integrating the resulting equation over $\mathbb{R}^2$, we have

$$
\frac{d}{dt} \int_{\mathbb{R}^2} p_n(x) \tilde{v}(t, x, y)^2 dxdy + \int_{\mathbb{R}^2} p_n'(x) \left\{ 3(\partial_x \tilde{v})^2 + 3(\partial_x^{-1} \partial_y \tilde{v})^2 - 4\tilde{v}^3 \right\} dxdy
$$

$$
= 3 \int_{\mathbb{R}^2} \{ p_n'(x) \varphi(x) - p_n(x) \varphi'(x) \} \tilde{v}(t, x, y)^2 dxdy .
$$

By Claim 5.1 in [29],

$$
\left| \int_{\mathbb{R}^2} \tilde{p}_n(x) \tilde{v}^3 dxdy \right| \leq C_1 \left( \int_{\mathbb{R}^2} \tilde{v}^2 dxdy \right)^{1/2} \left( \int_{\mathbb{R}^2} p_n'(x) E(\tilde{v}) dxdy \right)^{1/2},
$$

where $C_1$ is a constant independent of $n \in \mathbb{N}$. Hence there exists a positive constant $C$ such that for every $n \in \mathbb{N},$

$$
\int_{\mathbb{R}^2} p_n(x) \tilde{v}(t, x, y)^2 dxdy + 2 \int_{0}^{t} \int_{\mathbb{R}^2} p_n'(x) \left\{ (\partial_x \tilde{v})^2 + (\partial_x^{-1} \partial_y \tilde{v})^2 \right\} (s, x, y) dxdys
$$

$$
\leq \int_{\mathbb{R}^2} p_n(x) \nu_0(x, y)^2 dxdy + C \int_{0}^{t} ||\tilde{v}(s)||_{L^2}^2 ds .
$$

By approximating a solution $\tilde{v}(t)$ of (E.1) with $\tilde{v}(0) = \nu_0 \in X \cap L^2(\mathbb{R}^2)$ by a sequence solutions $\{ \tilde{v}_k(t) \}$ of (E.1) satisfying

$$
\tilde{v}_k(0) \in H^3(\mathbb{R}^2), \quad \partial_x^{-1} \tilde{v}_k(0) \in H^2(\mathbb{R}^2), \quad \lim_{k \to \infty} ||\tilde{v}(0) - \nu_0||_{L^2(\mathbb{R}^2)} = 0 ,
$$

we have for any $\nu_0 \in L^2(\mathbb{R}^2),$

$$
\int_{\mathbb{R}^2} p_n(x) \tilde{v}(t, x, y)^2 dxdy \leq \int_{\mathbb{R}^2} p_n(x) \nu_0(x, y)^2 dxdy + C \int_{0}^{t} ||\tilde{v}(s)||_{L^2}^2 ds .
$$

Passing to the limit $n \to \infty$, we obtain

$$
||\tilde{v}(t)||_X^2 \leq ||\nu_0||_X^2 + C \int_{0}^{t} ||\tilde{v}(s)||_{L^2}^2 ds .
$$

Since $\sup_{t \in [0, T]} ||\tilde{v}(t)||_{L^2(\mathbb{R}^2)} < \infty$ for any $T > 0$, we have (E.2).

Suppose $\nu_0 \in H^1(\mathbb{R}^2) \cap X$. Then we have (E.2) and $\tilde{v}(t) \in C(\mathbb{R}; H^1(\mathbb{R}^2))$. By the variation of constants formula,

$$
\tilde{v}(t) = e^{t\mathcal{L}_0} \nu_0 - \int_{0}^{t} e^{(t-s)\mathcal{L}_0} \partial_x (\varphi(\tilde{v}(s))) ds - \int_{0}^{t} e^{(t-s)\mathcal{L}_0} \tilde{v}(s) \partial_x \tilde{v}(s) ds .
$$

Since $\|e^{ax\varphi(\tilde{v}(s))}\|_{L^1} \leq ||\tilde{v}(s)||_X ||\tilde{v}(s)||_{H^1(\mathbb{R}^2)}$, we have $\tilde{v}(t) \in C([0, \infty); X)$ by using (3.6) and (3.10) in Lemma 3.4. Thus we complete the proof.  

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