A SPECTRAL DEFERRED CORRECTION METHOD FOR INCOMPRESSIBLE FLOW WITH VARIABLE VISCOSITY

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Abstract. This paper presents a semi-implicit spectral deferred correction (SDC) method for incompressible Navier-Stokes problems with variable viscosity and time-dependent boundary conditions. The proposed method integrates elements of velocity- and pressure-correction schemes, which yields a simpler pressure handling in comparison to the SDPC method of Minion & Saye (J. Comput. Phys. 375:797–822, 2018). Combined with the discontinuous Galerkin spectral-element method for spatial discretization it can in theory reach arbitrary order of accuracy in time and space. Numerical experiments in three space dimensions demonstrate up to order 12 in time and 17 in space for constant, spatiotemporally varying as well as solution-dependent viscosity. The phenomenon of order reduction also reported by Minion & Saye is observed in the case of time-dependent boundary conditions, where it manifests in terms of a slower convergence of the correction sweeps.

1. Introduction

High-order discretization methods are gaining interest in fluid mechanics [15, 56, 76, 84], solid mechanics [30, 85], electrodynamics [21, 22] and other areas of computational science that are governed by partial differential equations [65]. This development is driven by the expectation of achieving a superior algorithmic efficiency which enables high-fidelity simulations at a scale beyond the reach of low-order methods. In simulations of processes evolving in space-time, the accuracy of spatial and temporal approximations must be tuned to each other. The natural and only scalable way for achieving this is to match the convergence rates in space and time. However, examining recent work on high-order methods in computational fluid dynamics (CFD) reveals that the spatial order ranges typically from 4 to 16, whereas the temporal order rarely exceeds 3. This discrepancy constitutes the principal motivation of the present work which adopts the spectral deferred correction (SDC) method to reach arbitrary temporal convergence rates for incompressible flows with variable viscosity.

Before reviewing the state of the art in high-order time integration the implications of high-order space discretization shall be briefly recapitulated. In CFD, element based Galerkin methods with piecewise polynomial expansions represent the prevalent approach apart from spectral, finite-difference and isogeometric methods [16, 54, 82, 87]. The numerical properties of these methods have been thoroughly studied at the hand of convection, diffusion and wave problems as well as combinations thereof. Ainsworth and Wajid [2] and Gassner and Kopriva [40] showed that high-order spectral element approximations achieve far lower dispersion and dissipation errors than second-order finite-element or finite-volume methods using a comparable mesh spacing. This property yields a tremendous advantage in marginally or underresolved simulations.

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Key words and phrases. Incompressible flow, Variable viscosity, High-order time integration, Discontinuous Galerkin method.
of turbulent flows \cite{10}. On the other hand, the condition of the discrete operators worsens when increasing the degree of the expansion basis. Denoting the element size with $h$ and the polynomial degree with $P$, the largest eigenvalues grow asymptotically as $\lambda_c \sim P^2/h$ for the convection operator and $\lambda_d \sim P^4/h^2$ for the diffusion operator, see \cite{17, Ch. 7.3}. For explicit time integration schemes these estimates imply stability restrictions of the form $\Delta t_c \sim h/P^2$ and $\Delta t_d \sim h^2/P^4$, respectively. This corresponds to a reduction of the admissible time step by a factor of $1/P$ for convection and $1/P^2$ for diffusion in comparison to low-order finite-element or finite-volume methods. These stability issues lead to a preference of implicit time integration methods, especially for diffusion. With convection, however, nonlinearity may impair implicit methods and render semi-implicit or even fully explicit approaches attractive.

Time integration methods applied in computational fluid dynamics cover a wide range of approaches, including multistep, Runge-Kutta (RK), Rosenbrock, extrapolation, deferred correction and variational methods, see e.g. \cite{28, 42, 45, 46, 86}. Early work on high-order space discretization for incompressible flows advocated semi-implicit multistep methods \cite{55, 73} which are based on the projection method introduced by Chorin \cite{23}. These methods were investigated and generalized in numerous follow-up studies, and gained considerable popularity, mainly because of their simplicity and low cost per time step \cite{35, 37, 43, 44, 58, 63}. Their advantage is offset, however, by aggravating stability restrictions of the explicit part and the loss of A-stability of the implicit part for convergence orders greater than two (second Dahlquist barrier \cite{45}). Accordingly, the vast majority of studies based on multistep methods uses order two in time, whereas the spatial order ranges from 4 to well above 10. As a notable exception Klein et al. \cite{57} used a fully implicit SIMPLE method based on backward differentiation formulas (BDF) up to order 4.

In contrast to linear multistep methods, implicit Runge-Kutta methods can be constructed to reach high convergence orders along with excellent stability properties. For flow problems, however, convection or solution dependent viscosity yield nonlinear equations which need to be linearized and solved on every stage. In order to constrain complexity, all studies known to the author used only diagonally implicit Runge-Kutta (DIRK) methods. Uranga et al. \cite{57} applied a third order DIRK for large-eddy simulations of transitional flow wings in conjunction with Newton’s method and preconditioned conjugate gradients for solving the nonlinear equation systems on each stage. Rosenbrock-type methods are build on the Jacobian of the right-hand side (RHS) and, thus, achieve linearization more directly. John et al. \cite{52} compared Rosenbrock methods of order 3 for incompressible Navier-Stokes problems with fractional step methods and showed their competitiveness, especially in terms of accuracy and robustness. More recently, Bassi et al. \cite{9} and Noventa et al. \cite{72} applied Rosenbrock methods of orders up to 6 combined with discontinuous Galerkin methods in space to compressible and incompressible flows past airfoils and other configurations. In \cite{72} the authors mention the possibility of order reduction with time-dependent boundary conditions, but did not include corresponding convergence studies. As an alternative to the fully implicit approach, implicit-explicit (IMEX) methods combine implicit RK for the stiff (and often linear) part of the RHS with explicit RK schemes for the nonlinear part \cite{8, 13, 20, 56}. So far, however, IMEX RK methods appear to be rarely used in flow simulations featuring high-order spatial approximations \cite{39, 48, 74}. Moreover, the increasing number of matching conditions \cite{56} complicates the construction of higher order IMEX RK and methods of order higher than five are not available to the knowledge of the author. A further issue arising in the construction of high-order RK methods is the order reduction phenomenon, which can be triggered, e.g., by stiff source terms or time-dependent boundary conditions \cite{18, 45, 50}. Although several approaches have been proposed to mitigate
this issue, there exists no general solution for complex equations such as Navier-Stokes problems \[3, 6, 75\]. One possibility to mitigate order reduction is to use RK methods possessing a high stage order \[14, 45\]. Unfortunately, DIRK and Rosenbrock methods are limited to stage order two by construction.

Extrapolation and deferred correction methods employ low-order time-integration schemes within an iterative framework to achieve convergence of (in principle) arbitrary high order \[29, 33, 59, 68\]. In recent years, the spectral deferred correction (SDC) method gained attention \[4, 19, 24, 27, 66, 79, 80, 88\]. It was proposed by Dutt et al. \[33\] for solving the Cauchy problem for ordinary differential equations and extended by Kress and Gustafsson \[59\] to initial boundary value problems. The basic idea is to convert the differential evolution problem into a Picard integral equation which is solved by a deferred correction procedure, driven by a lower order marching scheme. In this procedure, the lower order scheme sweeps repeatedly through subintervals defined by a set of collocation points, which also serve for Lagrange interpolation and integration. Choosing these points from a Gauss-type quadrature yields an SDC method that converges toward the solution of the corresponding implicit Gauss collocation method \[45\]. Applying a first order corrector such as implicit Euler, each sweep ideally elevates the order by one, until reaching the maximum depending on the chosen set of collocation points. Like other high-order methods SDC is also susceptible to order reduction. However, in contrast to RK methods, the phenomenon manifests rather in a slower convergence toward the collocation solution, whereas the converged solution still attains the optimal order. Although several approaches have been proposed for accelerating convergence, the general solution to this problem remains an open issue \[24, 25, 60, 61, 88\]. In spite of its capability to reach arbitrary high orders and straightforward extension to parallel-in-time methods boosting the efficiency on high-performance computers \[12, 70\], SDC has been rarely applied to fluid dynamics problems. Moreover, most studies are confined to simple configurations with constant properties and periodic boundaries \[4, 67, 69, 79\]. Only recently, Minion and Saye \[66\] proposed an SDC method for computing incompressible flows with time-dependent BC.

Variational methods resemble SDC in harnessing piecewise polynomial expansions in the time direction. Unlike the latter, they achieve discretization by application of a variational principle, predominantly the discontinuous Galerkin (DG) method. Recent applications of DG in time include incompressible flows \[1, 84\], elasticity \[85\] and hyperbolic conservation laws \[38\]. Although inherently implicit, the method allows to incorporate semi-implicit strategies similar to SDC as proposed e.g. in \[84\]. Like SDC, DG methods can be based on Lagrange polynomials constructed from Gauss points. However, assuming that \(Q+1\) points are used, DG methods generally converge with order \(Q+1\), whereas SDC methods reach order \(2Q\) with Gauss-Lobatto-Legendre (GLL) and \(2Q+2\) with Gauss-Legendre points \[19\].

The goal of this study is to develop an SDC method for incompressible Navier-Stokes problems which, in combination with the DG spectral element method for spatial discretization, is capable to reach arbitrary high order in time and space. As the backbone of the new method, a semi-implicit correction scheme is devised which yields a simpler and more robust pressure handling than the SDPC method proposed by Minion and Saye \[66\]. Moreover, the present approach allows for variable, solution-dependent viscosity as well as time-dependent boundary conditions.

The remainder of the paper is organized as follows: Section 2 summarizes the incompressible Navier-Stokes equations with variable viscosity. Section 3 reviews the spectral deferred correction method and extends it to the flow problem. Section 4 presents the spatial discretization
followed by a compilation and discussion of numerical results in section 5. Section 6 concludes the paper.

2. The Navier-Stokes equations with variable viscosity

This paper considers incompressible flows with constant density and variable viscosity in a simply connected domain $\Omega \in \mathbb{R}^3$. The velocity $v(x, t)$ satisfies the momentum (Navier-Stokes) and continuity equations

$$\partial_t v + \nabla \cdot vv + \nabla p = \nabla \cdot \tau + f,$$

$$\nabla \cdot v = 0$$

in $\Omega$, where $p$ represents the pressure and

$$\tau = \nu[\nabla v + (\nabla v)^T]$$

the viscous stress tensor, both divided by density; $\nu(x, t, v)$ is the kinematic viscosity and $f(x, t)$ an explicitly defined forcing term. The flow problem is closed by stating initial and boundary conditions

$$v(x, 0) = v_0(x) \quad x \in \Omega,$$

$$v(x, t) = v_b(x, t) \quad x \in \partial \Omega.$$ (4)

For continuity, $v_0$ must be divergence free and $v_b$ satisfy the compatibility condition

$$\int_{\partial \Omega} n \cdot v_0 d\Gamma = 0.$$ (6)

Assuming a constant viscosity and using the identity $\nabla^2 v + \nabla \times \nabla \times v = \nabla \nabla \cdot v$ the viscous term in the momentum equation (1) can be written in the following forms

$$\nabla \cdot \tau = \nu[\nabla^2 v + \nabla \nabla \cdot v] \quad \text{(native)}$$ (7a)

$$= \nu \nabla^2 v \quad \text{(laplacian)}$$ (7b)

$$= -\nu \nabla \times \nabla \times v \quad \text{(rotational)}.$$ (7c)

For variable $\nu$ they can be generalized to

$$\nabla \cdot \tau = \nabla \cdot \nu[\nabla v + \nabla v^T - \chi I \nabla \cdot v],$$ (8)

where $\chi = 0$ corresponds to the native, $\chi = 1$ the laplacian, and $\chi = 2$ the rotational form, respectively. These forms are equivalent when applied to solenoidal vector fields, but not with approximate solutions that are not divergence free. This needs to be considered in the discrete case.

3. Spectral deferred correction method

3.1. General approach. This section briefly reviews the SDC method based on the model problem

$$\frac{dv}{dt} = F(t, v(t))$$ (9)

with $t \in (t_0, T]$, $T = t_0 + \Delta t$ and initial condition $v(t_0) = v_0$. 
3.1.1. Preliminaries. As a prerequisite for developing the method, the time interval is divided into subintervals $\{(t_{i-1}, t_i)\}_{i=1}^M$ such that $t_0 = t^n$ and $t_M = t^{n+1}$. Further, let $\Delta t_i = t_i - t_{i-1}$ denote the length of the $i$-th subinterval, $v_i \approx v(t_i)$ the discrete solution at time $t_i$ and $v_i^k$ the $k$-th approximation to $v_i$. In addition to this, the intermediate times $t_i$ serve as collocation points for Lagrange interpolation and as quadrature points for numerical integration. Depending on the underlying quadrature rule, one or both endpoints may be dropped, see e.g. [19, 62]. The following description is based on the Gauss-Lobatto-Legendre (GLL) rule and, hence, includes both endpoints for interpolation and integration. Accordingly, $\{t_i\}_{i=0}^M$ represent the GLL points scaled to $[t_0, T]$, $v = [v_i]$ the discrete solution vector, $v^k = [v_i^k]$, and $Iv^k(t)$ the corresponding Langrange interpolant at time $t$.

3.1.2. Predictor. The initial approximation $v^0$ is obtained by performing a predictor sweep of the form

$$v_i^0 = v_{i-1}^0 + H_i(v^0), \quad i = 1, \ldots, M,$$

(10)

where $v_i^0 = v_0$ and $H_i(v)$ is an approximation to $\int_{t_{i-1}}^{t_i} F dt$. Using, for example, a combination of forward and backward Euler rules based on the decomposition $F = F^{IM} + F^{EX}$ yields an IMEX Euler predictor with

$$H_i(v) = \Delta t_i [F^{IM}(t_i, v_i) + F^{EX}(t_{i-1}, v_{i-1})].$$

(11)

Alternatively, the predictor can be constructed from higher order time integration schemes such as RK or multistep methods [60]. This approach may give an advantage by providing more accurate start values, but will not be investigated in frame of the present study.

3.1.3. Corrector. The goal of the corrector is to remove the error from a given approximation $v^k$. For deriving the correction equation, the error function is defined as

$$\delta^k(t) = v(t) - Iv^k(t).$$

(12)

Further, the residual function is introduced by

$$\varepsilon^k(t) = \int_{t_0}^t F(\tau, Iv^k(\tau)) d\tau - Iv^k(t).$$

(13)

Differentiating and subtracting [12] and [13] yields

$$d_t(\delta^k - \varepsilon^k) = d_t v(t) - F(t, Iv^k(t)).$$

(14)

This equation can be rearranged using [9] and [12] to give the error equation

$$d_t(\delta^k - \varepsilon^k) = F(t, Iv^k(t) + \delta^k(t)) - F(t, Iv^k(t)),$$

(15)

which is supplemented with the initial condition $\delta^k(t_0) = 0$.

The error equation is solved numerically by means of a time integration scheme sweeping through the subintervals. For example, application of IMEX Euler yields

$$\delta_i^k = \delta_{i-1}^k + \varepsilon^k(t_i) - \varepsilon^k(t_{i-1}) + H_i(v^k + \delta^k) - H_i(v^k)$$

(16)

for $i = 1, \ldots, M$, where $\delta_i^k$ represents the approximation to $\delta^k(t_i)$. Substituting [13] for $\varepsilon^k(t_i)$, eliminating $\delta_i^k$ by means of [12] and defining the new approximate solution by $v_{i+1}^k = v_i^k + \delta_i^k$ finally gives the update equation

$$v_i^{k+1} = v_i^k + H_i(v^{k+1}) - H_i(v^k) + \int_{t_{i-1}}^{t_i} F(\tau, Iv^k(\tau)) d\tau.$$
The last term in (17) is usually approximated by replacing the integrand by its Lagrange interpolant, i.e.

\[ \int_{t_{i-1}}^{t_i} F(\tau, I_k(\tau)) \, d\tau \approx \int_{t_{i-1}}^{t_i} Ik F(\tau) \, d\tau =: S^k_i, \tag{18} \]

with \( F^k = [F(t_i, v^k_i)] \). The approximate integral can be expressed in terms of a quadrature formula,

\[ S^k_i = \Delta t \sum_{j=0}^{M} w^s_{i,j} F(t_j, v^k_j), \tag{19} \]

where \( w^s_{i,j} \) equals the integral of the \( j \)-th interpolation polynomial over subinterval \((t_{i-1}, t_i)\), normalized with \( \Delta t \). As a consequence, the sum \( \sum_i w^s_{i,j} \) recovers the weights of the underlying quadrature rule and, hence, \( v^k \) converges to the solution of the related collocation method.

The sketched SDC method attains order \( 2M + 1 \) for a single interval and \( 2M \) when applied to a sequence of multiple intervals. Using a first order corrector as sketched above, every sweep, ideally, elevates the order by one, until reaching the maximum order \([33]\). However, stiff terms and boundary conditions may affect convergence such that more iterations are required to attain the optimal order. For details and possible remedies the reader is referred to \([49, 66, 88]\).

3.2. Application to incompressible flow.

3.2.1. Considerations. The generalization of the SDC approach to the incompressible Navier-Stokes problem (1–5) follows a similar approach as outlined above. Starting from an identical partitioning of a given time interval, the semi-discrete solution is denoted by \( v_i(x) \approx v(x, t_i) \) for the velocity and \( p_i(x) \approx p(x, t_i) \) for the pressure. Before proceeding it is important to note the following differences between the flow problem and the model problem (9): Although the momentum balance (1) resembles an evolution equation for the velocity, it involves an additional variable in terms of the pressure. Moreover, the velocity is required to satisfy the continuity equation (2), which lacks a time derivative and, hence, looks like an algebraic constraint from perspective of time integration. Finally, the flow equations are subject to boundary conditions (5) that may depend on time themselves.

3.2.2. Predictor. The complex nature of the flow problem complicates the construction of the predictor (10). Instead of defining the operator \( H_i \) directly it is more appropriate to derive its structure from a single time step across some subinterval \((t_{i-1}, t_i)\). In analogy to the model problem, on could use the IMEX Euler method for incompressible flow, i.e.

\[ \frac{v_i - v_{i-1}}{\Delta t_i} + \nabla \cdot (vv)_{i-1} + \nabla p_i = \nabla \cdot \tau_i + f_i, \tag{20} \]

\[ \nabla \cdot v_i = 0. \tag{21} \]

While this scheme looks reasonably simple and elegant, it yields a coupled system for \( v_i \) and \( p_i \), which renders the solution costly, especially in view of the pertinent stability restrictions and low accuracy. Therefore, it seems attractive turning to projection schemes that decouple continuity from the momentum balance. These schemes employ some approximation of pressure \( p_i \) in the (incomplete) momentum step and achieve continuity by performing a separate projection step \([41, 43, 66]\). Moreover, the latter yields a Poisson equation for correcting the pressure.

Depending on the order of the substeps two classes of projection schemes can be distinguished: pressure-correction and velocity-correction methods. Pressure-correction methods were introduced by Chorin \([23]\). They first solve an implicit diffusion problem for each velocity component,
including approximations for convection and pressure terms, and then project the provisional velocity to a divergence-free field. In comparison to IMEX Euler, the splitting leads to an additional error caused by the violation of tangential boundary conditions in the projection step. However, several approaches exist for controlling the splitting error and retaining first order convergence [43]. Minion and Saye [66] investigated different variants of the pressure-correction scheme as a basis for their SDPC method.

Velocity-correction methods were introduced by Orszag et al. [73] and further extended, e.g. in [44, 55]. As a common feature, these methods perform the projection step before solving the diffusion problem. With constant viscosity this approach preserves continuity. However, similar to the pressure-correction method, it introduces a splitting error due to inaccurate pressure boundary conditions in the projection step. Using the rotational form of the velocity-correction method mitigates this error and recovers optimal convergence [44].

Several authors adapted the pressure-correction method to simulate flows with variable viscosity [31, 34, 71], whereas the author is not aware of corresponding extensions of the velocity-correction method. On the other hand, the pressure-correction method implies a rather complicated handling of the pressure when applied as a base method for SDC [66]. The method proposed in the following combines the advantages of both approaches. It starts with a velocity-correction step and concludes with a projection like the pressure-correction method.

For stating the base time-integration method, the momentum equation is rewritten in the form

\[
\partial_t v = F(x, t, v, p),
\]

where

\[
F = F_c(v) + F_d(\nu, v) + F_p(p) + f(x, t)
\]

with

\[
F_c = -\nabla \cdot vv
\]

\[
F_d = F_{d1} + F_{d2} + F_{d3}
\]

\[
F_{d1} = \nabla \cdot \nu \nabla v
\]

\[
F_{d2} = \nabla \cdot \nu (\nabla v)^T
\]

\[
F_{d3} = -\chi \nabla (\nu \nabla \cdot v)
\]

\[
F_p = -\nabla p.
\]

The predictor is then defined as follows:

\[
\frac{v_i^0 - v_i^{0-1}}{\Delta t_i} = (F_c + F_d)^{i-1} + f_i,
\]

\[
\frac{v_i^0 - v_i^{0-1}}{\Delta t_i} = -\nabla p_i^{0-1}, \quad \nabla \cdot v_i^{0-1} = 0, \quad n \cdot v_i^{0-1} |_{\partial \Omega} = n \cdot v_b(t_i),
\]

\[
\frac{v_i^{0-1} - v_i^{0-0}}{\Delta t_i} = \nabla \cdot (\nu_{i-1}^{0-1} \nabla v_i^{0-0} - (F_{d1} + F_{d3})^{i-1}, \quad v_i^{0-0} |_{\partial \Omega} = v_b(t_i),
\]

\[
\frac{v_i^{0-0} - v_i^{0-0}}{\Delta t_i} = -\nabla (p_i^{0-0} - p_i^{0-0}), \quad \nabla \cdot v_i^{0-0} = 0, \quad n \cdot v_i^{0-0} |_{\partial \Omega} = n \cdot v_b(t_i)
\]

for \(i = 1, \ldots, M\). The first three substeps comprise a velocity-correction method. In particular, (30) represents an incomplete Euler step using the forward rule for convection and diffusion, backward rule for the forcing term and skipping the pressure part. It is followed by the first
projection \((31)\) and the viscous correction \((32)\). The latter computes a new semi-implicit approximation of the diffusion term \(\nabla \cdot \nu \nabla v\), while the explicit guess \(F_{d,i-1}^{0}\) and the viscous divergence contribution \(F_{d,i-1}^{0}\) introduced in \((30)\) are dropped. For constant viscosity \((30 - 32)\) reproduce with \(\chi = 1\) the standard and with \(\chi = 2\) the rotational velocity-correction method as defined in \([14]\). In the case of variable viscosity, the diffusion step produces a divergence error of the order \(O(\Delta t_i)\). This error is removed by the final projection step \((33)\). Alternatively, it can be tolerated as a part of the overall discretization error, which will be considered as an option in the numerical experiments.

In contrast to pressure-correction, the proposed method requires no initial approximation of the pressure. The intermediate pressure \(p_{i,i-1}^{0}\) and the final pressure \(p_{i,i-1}^{0}\) are obtained each by solving a Poisson problem which follows from the corresponding projection step. For example, taking the divergence and, respectively, the normal projection of the first equation in \((31)\) leads to

\[
\nabla^2 p_{i,i-1}^{0} = \frac{1}{\Delta t_i} \nabla \cdot v_{i}^{0}, \quad n \cdot \nabla p_{i,i-1}^{0}|_{\partial \Omega} = \frac{1}{\Delta t_i} n \cdot (v_{i}^{0}|_{\partial \Omega} - v_{b}(t_i)). \tag{34}
\]

Similarly, \((33)\) yields a Poisson equation for \(p_{i,i-1}^{0} - p_{i,i-1}^{0}\) with homogeneous Neumann conditions.

### 3.2.3. Corrector

Apart from the additional low- and high-order contributions, the corrector resembles the predictor. The substeps for sweep \(k\) read

\[
\frac{v_{i}^{k} - v_{i}^{k-1}}{\Delta t_i} = F_{c,i-1}^{k} + \nabla \cdot v_{i-1}^{k} \nabla v_{i}^{k-1} + F_{d2,i-1}^{k} + F_{d3,i-1}^{k} - \frac{H_{i}^{k-1}}{\Delta t_i} + S_{i}^{k-1}, \tag{35}
\]

\[
\frac{v_{i}^{i} - v_{i}^{k}}{\Delta t_i} = -\nabla p_{i}^{k}, \quad \nabla \cdot v_{i}^{i} = 0, \quad n \cdot v_{i}^{i}|_{\partial \Omega} = n \cdot v_{b}(t_i), \tag{36}
\]

\[
\frac{v_{i}^{m} - v_{i}^{k}}{\Delta t_i} = \nabla \cdot v_{i-1}^{i} \nabla (v_{i}^{m} - v_{i}^{k-1}) - F_{d3,i-1}^{k}, \quad v_{i}^{m}|_{\partial \Omega} = v_{b}(t_i), \tag{37}
\]

\[
\frac{v_{i}^{k} - v_{i}^{m}}{\Delta t_i} = -\nabla (p_{i}^{k} - p_{i}^{m}), \quad \nabla \cdot v_{i}^{k} = 0, \quad n \cdot v_{i}^{k}|_{\partial \Omega} = n \cdot v_{b}(t_i), \tag{38}
\]

where

\[
S_{i}^{k-1} = \Delta t \sum_{j=0}^{M} w_{i,j}^{s} \left[ F_{c,i}^{k-1} + F_{d,i}^{k-1} + f_{i} \right] - \nabla P_{i}^{k-1} \tag{39}
\]

represents the subinterval integral similar to \([19]\) with the pressure part \(P_{i}^{k-1}\) yet to be defined. In contrast to the predictor, the corrector exploits the previous approximation of \(v_{i}\) to provide a better starting value for the implicit diffusion term in the extrapolation step \((35)\). The pressures computed in the projection steps \((36)\) and \((38)\), respectively, are generally no approximations of \(p_{i}\) and, hence, marked by a tilde. Adding and rearranging the equations for \(v_{i}^{k}, v_{i}^{m}, v_{i}^{m} k\) and \(v_{i}^{k}\) gives

\[
v_{i}^{k} - v_{i}^{k-1} = \Delta t_i \left[ F_{c,i-1}^{k} + \nabla \cdot v_{i-1}^{k} \nabla v_{i}^{k-1} + F_{d2,i-1}^{k} \right] - H_{i}^{k-1} + S_{i}^{k-1}, \tag{40}
\]

and leads to the following ansatz for the low-order contribution

\[
H_{i}^{k-1} = \Delta t_i \left[ F_{c,i-1}^{k} + \nabla \cdot v_{i-1}^{k} \nabla v_{i}^{k-1} + F_{d2,i-1}^{k} \right] - H_{i}^{k-1} \tag{41}
\]
Substituting the high- and low-order contributions in (40) and considering the converged case yields

\[ v_i - v_{i-1} = \Delta t \sum_{j=0}^{M} w_{i,j}^s [F_{c,i} + F_{d,i} + f_i] - \nabla [P_i + \Delta t \bar{p}_i]. \]  

(42)

Comparing this result to the corresponding collocation formulation implies

\[ P_i + \Delta t \bar{p}_i = \Delta t \sum_{j=0}^{M} w_{i,j}^s p_i \approx \int_{t_{i-1}}^{t_i} p \, dt. \]  

(43)

As a consequence it may be conjectured that the choice of \( p_i^h \) and \( P_i \) has no influence on the corrector and, hence, does not affect the convergence of the SDC method. This conjecture is confirmed by preliminary studies exploring several approaches, including the evaluation of \( P_i^k = \Delta t \sum_j w_{i,j}^s p_i^k \) using the recomputed pressure obtained from

\[ \nabla^2 p_i^k = \nabla \cdot (F_c + F_d + f)_i^k, \]  

(44)

\[ \partial_n p_i^k = n \cdot (F_c + F_d + f)_i^k - \partial_t v_{b,i}|_{\partial \Omega}. \]  

(45)

Consequently, all studies in this work were performed with the simplest choice, \( p_i^h = 0 \) and \( P_i = 0 \). Finally it is noted that the high-order contribution includes the viscous divergence contribution \( F_{i3} \). While this term vanishes for the exact solution, it improves stability and accuracy with non-solenoidal approximations.

4. Spatial discretization

The semi-discrete SDC formulation developed in the previous section is discretized in space using the discontinuous Galerkin spectral element method (DG-SEM) with nodal base functions. The method is based largely on the approach of Fehn et al., but also includes measures to attain pressure robustness proposed in.

Note that the following description is constrained to cuboidal domains. This restriction serves only for convenience and can be lifted easily without affecting the proposed SDC method.

4.1. Preliminaries. First, the computational domain \( \Omega \) is decomposed into \( N_e \) rectangular hexahedral elements to obtain the discrete domain

\[ \Omega_h = \bigcup_{e=1}^{N_e} \Omega^e. \]  

(46)

Let \( \Gamma_h^i \) denote the set of all interior (including periodic) faces in \( \Omega_h \) and \( \Gamma_h^p \) the set of boundary faces. The union of these sets defines the skeleton \( \Gamma_h \). For any interior face \( \Gamma_i^f \in \Gamma_h^i \) there exist two adjoining elements \( \Omega^- \) and \( \Omega^+ \) with unit normal vectors \( n^- \) and \( n^+ \), respectively. The standard average and jump operators for element-wise continuous functions \( \phi \) of any dimension are defined as

\[ \{ \phi \}_f = \frac{1}{2} (\phi^- + \phi^+), \]  

(47)

\[ [ \phi ]_f = n^- \phi^- + n^+ \phi^+, \]  

(48)

where \( \phi^- \) are the traces of the function from within \( \Omega^- \). A further jump operator, involving the inner product with the normal vectors, is introduced for vector or higher rank tensor functions:

\[ \langle \phi \rangle_f = n^- \cdot \phi^- + n^+ \cdot \phi^+. \]  

(49)
These definitions are extended to boundary faces by assuming \( n^- = n = -n^+ \) and providing exterior values \( \phi^+ = \phi^0 \) on \( \Gamma_h^0 \) depending on boundary conditions \([17]\). Hereafter, the index \( f \) is dropped to indicate a quantity that is defined on any face or a set of faces.

Let \( Q_P(\Omega^e) \) denote the tensor-product space of all polynomials on \( \Omega^e \) with degree less or equal \( P \) in each direction. Glueing all element spaces together yields the global space of element-wise polynomial, discontinuous functions

\[
Q_P = \bigoplus_{\Omega^e \in \Omega_h} Q_P(\Omega^e). \tag{50}
\]

This allows to define the ansatz spaces

\[
V = [Q_P]^3 \quad \text{and} \quad P = Q_P
\]

for velocity and pressure, respectively. Note that \( P_v > P_p \) is required for inf-sup stability, see e.g. \([11, 51]\). In the following the degree is set to \( P_v = P \) for velocity and \( P_p = P - 1 \) for pressure.

### 4.2. DG-SEM formulation.

#### 4.2.1. Gradient and divergence functionals. Consider the test functions \( q_h \in P \) and \( w_h \in V \). The gradient functional of a scalar \( p_h \in P \) is given by

\[
G_h(p_h, w_h) = -\int_{\Omega_h} (\nabla \cdot w_h)p_h \, d\Omega + \int_{\Gamma_h} \langle w_h \rangle \cdot \langle p_h \rangle \, d\Gamma. \tag{52}
\]

Further,

\[
D_h(v_h, q_h) = -\int_{\Omega_h} \nabla q_h \cdot v_h \, d\Omega + \int_{\Gamma_h} [q_h] \cdot \langle v_h \rangle \, d\Gamma, \tag{53}
\]

\[
D_h(\sigma_h, w_h) = -\int_{\Omega_h} (\nabla w_h) : \sigma_h \, d\Omega + \int_{\Gamma_h} [w_h] \cdot \langle \sigma_h \rangle \, d\Gamma \tag{54}
\]

define the divergence functionals for any vector \( v_h \in V \) and second rank tensor \( \sigma_h \in V \otimes V \), respectively. Boundary conditions are considered by providing proper exterior values, as will be detailed below. Based on these functionals the discrete gradient \( \nabla_h p_h \) and divergence \( \nabla_h \cdot v_h \) are introduced such that

\[
\int_{\Omega_h} w_h \cdot \nabla_h p_h \, d\Omega = G_h(p_h, w_h) \quad \forall q_h \in P, \tag{55}
\]

\[
\int_{\Omega_h} q_h \nabla_h \cdot v_h \, d\Omega = D_h(v_h, q_h) \quad \forall w_h \in V. \tag{56}
\]

#### 4.2.2. Time derivative. The time derivative comprises the discrete counterparts of the convection, diffusion, pressure and forcing terms introduced in \([23, 25]\). For the convection term \( F_{h,c} \) application of the local Lax-Friedrichs flux leads to

\[
\int_{\Omega_h} w_h \cdot F_{h,c} \, d\Omega = -\int_{\Omega_h} \nabla w_h : v_h v_h \, d\Omega
\]

\[
+ \int_{\Gamma_h} [w_h] : (\langle v_h v_h \rangle + \hat{v}_n [v_h] ) \, d\Gamma \quad \forall w_h \in V, \tag{57}
\]
where \( \hat{v}_n = \max (|n \cdot v_h^+|, |n \cdot v_h^-|) \). The diffusive and pressure terms are based on the divergence and gradient functionals, i.e.

\[
\int_{\Omega_h} w_h \cdot F_{h,d1} \, d\Omega = D_h(v_h \nabla v_h, w_h),
\]

(58)

\[
\int_{\Omega_h} w_h \cdot F_{h,d2} \, d\Omega = D_h(v_h(\nabla v_h)^T, w_h),
\]

(59)

\[
\int_{\Omega_h} w_h \cdot F_{h,d3} \, d\Omega = -\chi \mathcal{G}_h(v_h \nabla \cdot v_h, w_h),
\]

(60)

\[
\int_{\Omega_h} w_h \cdot F_{h,p} \, d\Omega = \mathcal{G}_h(p_h, w_h)
\]

(61)

for all \( w_h \in V \). Finally, the forcing term follows from

\[
\int_{\Omega_h} w_h \cdot f_h \, d\Omega = \int_{\Omega_h} w_h \cdot f \, d\Omega \quad \forall w_h \in V,
\]

(62)

which is equivalent to element-wise \( L^2 \) projection.

4.2.3. Laplace and viscous diffusion operators. The Laplacian occurring in the pressure equations such as (34) and the viscous diffusion operator in (32) and (37) are discretized using the symmetric interior penalty (SIP) method \[7\].

Application to the pressure Laplacian \( \nabla^2 p \) yields

\[
L_h(p_h, q_h) = -\int_{\Omega_h} \nabla q_h \cdot \nabla p_h \, d\Omega + \int_{\Omega_h} ([q_h] : [[\nabla p_h]] + [\nabla q_h] \cdot [p_h]) \, d\Gamma
\]

\[
-\int_{\Gamma_h} \mu_p [q_h] \cdot [p_h] \, d\Gamma,
\]

(63)

for \( p_h, q_h \in \mathbb{P} \). The penalty parameter is defined as \( \mu_p = \mu(P_p) \) with

\[
\mu(P) = \mu_* \frac{P(P+1)}{2} \left( \frac{1}{\Delta x_n} \right).
\]

(64)

where \( \Delta x_n \) is the mesh spacing normal to the face and \( \mu_* > 1 \) a constant parameter \[82\]. On Neumann boundaries the face averages and jumps are given by

\[
n \cdot \{\{\nabla p\}\} = (\partial_n p)_b, \quad [p_h] = 0.
\]

(65)

Note that the latter implies \( p^+ = p^- \).

For the diffusion operator \( \lambda v - \nabla \cdot \nu \nabla v \) with constant \( \lambda \) the SIP method gives

\[
\mathcal{V}_h(\lambda, v_h; v, w_h) = \int_{\Omega_h} \lambda w_h \cdot v_h + \int_{\Omega_h} \nu_h \nabla w_h : \nabla v_h \, d\Omega
\]

\[
-\int_{\Omega_h} ([w_h] : \{\nu_h \nabla v_h\} + \{\nu_h \nabla w_h\} : [v_h]) \, d\Gamma
\]

\[
+\int_{\Gamma_h} \mu \hat{v} [w_h] : [v_h] \, d\Gamma
\]

(66)

where \( \mu_v = \mu_* (P_v) \) and \( \hat{v} = \max(\nu_h^-, \nu_h^+) \). Dirichlet boundary conditions are weakly imposed by setting

\[
\{\{v\}\} = v_b, \quad \{\nu_h \nabla v_h\} = (\nu_h \nabla v_h)^-.
\]

(67)
The first of these relations is equivalent to $v^+ = 2v_h - v^-$ and thus also defines the jump $[v_h]$. Remarkably, $\mathcal{V}_h$ does not couple across $v_h$ such that the corresponding diffusion problems can be solved component by component as long as $\nu_h$ and the RHS do not depend on the solution.

4.2.4. Divergence/mass-flux stabilization. The discrete projection steps are augmented with the penalty functional

$$J_h(v_h, w_h) = \int_{\Omega_h} \tau_d (\nabla \cdot w_h)(\nabla \cdot v_h) \, d\Omega + \int_{\Gamma_h} \tau_j (\nu_h)(\nu_h) \, d\Gamma.$$  \hfill (68)

This functional was introduced by Joshi et al. \cite{53} in frame of a post-processing technique for stabilizing pressure-correction methods for incompressible inviscid flow. It has no counterpart in the differential formulation, but vanishes for continuous, element-wise divergence-free vector fields $v_h$. In the general case, the first part of $J_h$ penalizes the divergence of $v_h$ within elements and the second part jumps of the normal flux across faces. Akbas et al. \cite{3} recently proved that both parts are required for pressure robustness.

The divergence penalty functional (68) has been applied with projection methods as well as coupled methods \cite{3, 31, 53, 58}. In these studies, various expressions have been proposed for the stabilization parameters $\tau_d$ and $\tau_j$. The present work follows \cite{3} by setting

$$\tau_d = \tau_* \nu_{ref} \quad \text{and} \quad \tau_j = \frac{\tau_*}{\Delta x_n} \nu_{ref},$$  \hfill (69)

where $\tau_*$ is a positive constant, $\Delta x_n$ the mesh spacing in the normal direction and $\nu_{ref}$ a reference value of viscosity.

4.2.5. Predictor. Starting from $v_h^0$ at time $t_0$ the predictor sweeps through all subintervals $i$, performing the following steps:

1. Extrapolation:

$$v_{h,i}^0 = v_{h,i-1} + \Delta t_i [F_{h,c}(v_{h,i-1}) + F_{h,d}(\nu_{h,i-1}, v_{h,i-1}) + f_{h,i}].$$  \hfill (70)

2. First projection: Determine pressure $p_{h,i}'' \in \mathbb{P}$ by solving

$$\mathcal{L}_h(p_{h,i}'', q_h) = -\frac{1}{\Delta t_i} \mathcal{D}_h(v_{h,i}', q_h) \quad \forall q_h \in \mathbb{P}$$  \hfill (71)

with Neumann boundary conditions

$$n \cdot \nabla p_{h,i}'' |_{\partial \Omega_h} = \frac{1}{\Delta t_i} n \cdot (v_{h,i}'|_{\partial \Omega_h} - v_h(t_i)).$$  \hfill (72)

Subsequently compute the intermediate velocity $v_{h,i}''' \in \mathbb{V}$ such that

$$\int_{\Omega_h} w_h \cdot \frac{v_{h,i}''' - v_{h,i}^0}{\Delta t_i} \, d\Omega + J_h(v_{h,i}'', w_h) + G_h(p_{h,i}'', w_h) = 0 \quad \forall w_h \in \mathbb{V}$$  \hfill (73)

with homogeneous Neumann conditions (constant extrapolation) on $\partial \Omega_h$. 

(3) **Diffusion:** Find $v_{h,i}^{m0} \in V$ such that for all $w_h \in V$

$$
\nabla_h(\Delta t_i^{-1} \nu_{h,i-1}; v_{h,i}^{m0}, w_h) = \int_{\Omega_h} w_h \cdot \frac{v_{h,i}^{m0}}{\Delta t_i} \, d\Omega
- \int_{\Omega_h} w_h \cdot F_{h,d1}(\nu_{h,i-1}, v_{h,i-1}^{0}) \, d\Omega
- \int_{\Omega_h} w_h \cdot F_{h,d3}(\nu_{h,i-1}, v_{h,i-1}^{0}) \, d\Omega
$$

with Dirichlet conditions

$$
v_{h,i}^{m0} |_{\partial \Omega_h} = v_b(t_i).$$

(4) **Final projection:** Solve

$$
\mathcal{L}_h(p_{h,i}, q_h) = -\frac{1}{\Delta t_i} \mathcal{D}_h(v_{h,i}^{m0}, q_h) \quad \forall q_h \in \mathbb{P}
$$

for $p_{h,i}^{0} \in \mathbb{P}$ and, subsequently,

$$
\int_{\Omega_h} w_h \cdot \frac{v_{h,i}^{0} - v_{h,i}^{m0}}{\Delta t_i} \, d\Omega + \mathcal{J}_h(v_{h,i}^{0}, w_h) + \mathcal{G}_h(p_{h,i}^{0} - p_{h,i}^{0}, w_h) = 0 \quad \forall w_h \in V
$$

to obtain the velocity $v_{h,i}^{0} \in V$. For both problems, (76) as well as (77), homogeneous Neumann conditions are imposed.

4.2.6. **Corrector.** The DG-SEM formulation of the corrector resembles that of the predictor. It is summarized below, skipping specifications of function spaces and homogeneous boundary conditions for brevity.

For $k = 1, \ldots, K$ sweep through all subintervals $i$ and perform the following steps:

(0) **Low- and high-order contributions:**

$$
H_{h,i}^{k-1} = \Delta t_i \left[ F_{h,c}(v_{h,i-1}^{k-1}) + F_{h,d1}(\nu_{h,i-1}^{k-1}, v_{h,i-1}^{m,k-1}) + F_{h,d2}(\nu_{h,i-1}^{k-1}, v_{h,i-1}^{k-1}) \right],
$$

$$
S_{h,i}^{k-1} = \Delta t_i \sum_{j=0}^{M} w_{i,j}^{s} \left[ F_{c}(v_{h,i}^{k-1}) + F_{h,d}(\nu_{h,i}^{k-1}, v_{h,i}^{k-1}) + f_{h,i} \right].
$$

(1) **Extrapolation:**

$$
v_{h,i}^{k} = v_{h,i-1}^{k} + \Delta t_i F_{h,c}(v_{h,i-1}^{k-1})
+ \Delta t_i \left[ F_{h,d1}(\nu_{h,i-1}^{k-1}, v_{h,i-1}^{k-1}) + F_{h,d2}(\nu_{h,i-1}^{k-1}, v_{h,i-1}^{k-1}) + F_{h,d3}(\nu_{h,i-1}^{k-1}, v_{h,i-1}^{k-1}) \right]
- H_{h,i}^{k-1} + S_{h,i}^{k-1}.
$$

(2) **First projection:**

$$
\mathcal{L}_h(\tilde{p}_{h,i}^{k}, q_h) = -\frac{1}{\Delta t_i} \mathcal{D}_h(v_{h,i}^{k}, q_h),
$$

$$
n \cdot \nabla \tilde{p}_{h,i}^{k} |_{\partial \Omega_h} = \frac{1}{\Delta t_i} n \cdot (v_{h,i}^{k} |_{\partial \Omega_h} - v_b(t_i)),
$$

$$
\int_{\Omega_h} w_h \cdot \frac{v_{h,i}^{k} - v_{h,i}^{k}}{\Delta t_i} \, d\Omega + \mathcal{J}_h(v_{h,i}^{k}, w_h) + \mathcal{G}_h(\tilde{p}_{h,i}^{k}, w_h) = 0.
$$
Diffusion:

$$V_h(\Delta t^{-1}, \nu_h^{k-1}, \nu_h^k, \mathbf{w}_h) = \int_{\Omega_h} \mathbf{w}_h \cdot \frac{\nu_h^k}{\Delta t} \, d\Omega$$

$$- \int_{\Omega_h} \mathbf{w}_h \cdot \mathbf{F}_{h, d1}(\nu_h^{k-1}, \nu_h^k) \, d\Omega$$

$$- \int_{\Omega_h} \mathbf{w}_h \cdot \mathbf{F}_{h, d2}(\nu_h^{k-1}, \nu_h^k) \, d\Omega$$

(84)

with Dirichlet conditions $\nu_h^k|_{\partial \Omega_h} = \mathbf{v}_b(t_i)$.

Final projection:

$$L_h(p_h^k, \mathbf{v}_h) = -\frac{1}{\Delta t} \mathcal{D}_h(\mathbf{v}_h^k, \mathbf{q}_h),$$

$$\int_{\Omega_h} \mathbf{w}_h \cdot \frac{\nu_h^k}{\Delta t} \, d\Omega + \mathcal{J}_h(\nu_h^k, \mathbf{w}_h) + \mathcal{G}_h(p_h^k, \mathbf{v}_h) = 0.$$ 

(86)

4.2.7. Base functions and numerical quadrature. The discrete solution is approximated by means of tensor-product Lagrange bases constructed from GLL points of degree $P_p$ for the pressure and $P_v$ for the velocity and all remaining variables. Integrals are evaluated numerically with GLL quadrature on the collocation points except for the convection term (57) and the functional (52), which are integrated using $\lceil 3P_v^2 \rceil + 1$ and $P_v + 1$ points, respectively. This choice avoids aliasing errors and preserves the equivalence between the gradient and divergence functionals, i.e.

$$\mathcal{G}_h(p_h, \mathbf{v}_h) = -\mathcal{D}_h(\mathbf{v}_h, p_h).$$

(87)

According to Maday and Rønquist [64] optimal convergence with variable viscosity requires $P_v + 2$ Lobatto points for integrating the diffusion term (66), as opposed to $P_v + 1$ which are actually used. However, elevating the quadrature order would also increase the cost of solving the diffusion problems and is therefore postponed to future studies.

4.3. Solution methods and implementation. The pressure equations (71, 76, 81) and (85) are solved by means of a Krylov-accelerated polynomial multigrid technique using an element-based overlapping Schwarz method for smoothing [82, 83]. To cope with variable coefficients in diffusion problems (74, 84) the Schwarz smoother was extended by adopting the linearization strategy developed in [81] for continuous spectral elements. Details will be presented in a forthcoming paper. The projection steps (73, 77, 83) and (86) are solved with a diagonally preconditioned conjugate gradient method [78]. The SDC method and all examples presented in this paper are implemented in the high-order spectral-element techniques library HiSPEET, which is under development at the Institute of Fluid Mechanics at TU Dresden (fusionforge.zih.tu-dresden.de/projects/hispeet). All solver components are parallelized with MPI and exploit SIMD techniques for accelerating the element operators on CPUs [50].

5. Numerical experiments

In the following, numerical results are presented for various test cases including those with constant viscosity as well as several scenarios with variable viscosity. For the pressure and diffusion problems the multigrid solver is used with a relative tolerance of $10^{-12}$ and an absolute tolerance of $10^{-14}$. The preconditioned conjugate gradient method in the projection steps is terminated after 10 iterations or reaching a relative tolerance of $10^{-10}$. The velocity error $\varepsilon_v$ is
computed as the RMS value over all collocation points at the end time $T$. Correspondingly, $\varepsilon_p$ denotes the pressure error and $\varepsilon_{\text{div}}$ the divergence error, i.e., the RMS value of $\nabla_h \cdot \mathbf{v}_h$.

5.1. Traveling Taylor-Green vortex with constant viscosity. The first test problem is adopted from Minion and Saye [66]. It is a Taylor-Green vortex traveling through the two-dimensional domain $\Omega^{2D} = [-1/2, 1/2]^2$. The exact solution is given by

$$
v_x(x, y, t) = 1 + \sin(2\pi(x - t)) \cos(2\pi(y - \frac{1}{8} - t)) \exp(-8\pi^2\nu t), \quad (88)
$$

$$
v_y(x, y, t) = 1 - \cos(2\pi(x - t)) \sin(2\pi(y - \frac{1}{8} - t)) \exp(-8\pi^2\nu t), \quad (89)
$$

$$
p^{\text{ex}}(x, y, t) = \frac{1}{4} \left[ \cos(4\pi(x - t)) + \cos(4\pi(y - \frac{1}{8} - t)) \right] \exp(-16\pi^2\nu t). \quad (90)
$$

There are no external sources, i.e., $f = 0$. Based on this problem Minion and Saye [66] defined two test cases: Example 1 with $\nu = 0.02$ and periodic conditions in both directions, and Example 2 with $\nu = 0.01$, periodic conditions in $x$-direction and Dirichlet conditions in $y$-direction. The final time is $T = 0.25$ in both cases. For the present study these cases are extended into three dimensions by assuming $v_x = 0$ and periodicity in the $z$-direction. Following [66] the domain is discretized with $8 \times 8$ elements of degree $P = 10$ in the $x$-$y$ plane and one element layer in the $z$-direction. Time integration is performed with the SDC method using $M = 3$ subintervals. The corresponding numerical studies are labeled PPP for the periodic case and PDP for the case with Dirichlet conditions in the $y$-direction.

Before considering the studies in detail, a comparison of different SDC variants is given in Figure 1. It shows the error $\varepsilon_v$ after $K = 5$ correction sweeps depending on the time step size $\Delta t$. The SDC method was applied either with the standard velocity correction scheme ($\chi = 1$) or the rotational scheme ($\chi = 2$). Further, the final projection step (FP) was omitted in selected runs. Indeed it seems redundant with the rotational scheme, whereas the standard scheme requires the FP to reach the same accuracy. Apparently, the rotational scheme is more effective in reducing the splitting error. The convergence behavior follows a similar pattern for all variants. For time steps $\Delta t > 3 \times 10^{-3}$ they attain only a convergence rate of 3, whereas 6 is expected for $M = 6$. A comparable order reduction was observed with SDPC in the 2D study of Minion and Saye [66]. However, with smaller steps convergence accelerates and reaches almost the optimal rate of 6, except for the standard scheme with no FP. This remarkable feature is not discernible in the SDPC results reported in [66].

Figure 2 presents the results for cases PPP and PDP obtained with the rotational method for a different number of correction sweeps, ranging from $K = 0$ to 9. For PPP the convergence rate grows by one with each correction until the maximum of $2M$ is reached (Fig. 2a). Hence, the method shows the optimal convergence behavior in the periodic case. This can be explained by the lack of a splitting error and was also observed in [66]. In case PDP the imposition of Dirichlet conditions causes an order reduction which manifests in a larger error and a flatter slope for identical $K$ in comparison to PPP (Fig. 2b). Nevertheless, the SDC method regains the optimal convergence rate when increasing the number of sweeps, i.e. $K = 9$ in this example.

5.2. Traveling 3D vortex with variable viscosity.

5.2.1. Test cases. The suitability for Navier-Stokes problems with variable viscosity is examined on the basis of a manufactured solution proposed by [71]. The exact velocity and pressure are
Figure 1. Velocity error of SDC with $M = 3$ subintervals and $K = 5$ correction sweeps for the Taylor-Green vortex with Dirichlet conditions in the $y$-direction. The choices $\chi = 1$ and $\chi = 2$ refer to the method based on standard and rotational velocity correction, respectively, and FP indicates the application of the final projection step.

Figure 2. Velocity error for the Taylor-Green vortex with $M = 3$ subintervals, $\chi = 2$ and no final projection. Labels indicate the number of correction sweeps and lines the approximate slope.

given by

$$v^e_x = (\sin 2\pi (x + t) + \cos 2\pi (y + t)) \sin 2\pi (z + t),$$  \hspace{1cm} (91) 
$$v^e_y = (\cos 2\pi (x + t) + \sin 2\pi (y + t)) \sin 2\pi (z + t),$$  \hspace{1cm} (92) 
$$v^e_z = (\cos 2\pi (x + t) + \cos 2\pi (y + t)) \cos 2\pi (z + t),$$  \hspace{1cm} (93) 
$$p^e = \sin 2\pi (x + t) \sin 2\pi (y + t) \sin 2\pi (z + t).$$  \hspace{1cm} (94)
Equations (91–93) define a periodic vortex array with wave length \( l = 1 \) and velocity magnitude \( v^\text{ex}_{\max} = 2 \), traveling with a phase velocity of 1 in each direction, separately. The exact solution is supplemented with a spatially and temporally varying viscosity of the form \( \nu = \nu_0 + \nu(t, v) \).

Three different scenarios are considered for the fluctuation \( \nu_t \):

\[
\begin{align*}
\nu_t^{(1)}(x) &= \nu_1 \sin^2(2\pi x) \sin^2(2\pi y) \sin^2(2\pi z) \\
\nu_t^{(2)}(x, t) &= \nu_1 \sin^2(2\pi(x - t)) \sin^2(2\pi(y - t)) \sin^2(2\pi(z - t)) \\
\nu_t^{(3)}(v) &= \nu_1 \frac{v^2_{\text{max}}}{v^\text{ex}^2_{\max}}
\end{align*}
\]

Note that the expressions are normalized such that \( \max|\nu_t| = \nu_1 \) provided that \( v = v^\text{ex} \). The spatially varying fluctuation \( \nu_t^{(1)} \) was already given in (71). Complementing it with a unit phase velocity which is opposed to that of \( v^\text{ex} \) leads to \( \nu_t^{(2)} \). Finally, \( \nu_t^{(3)} \) depends on the approximate velocity and, thus, renders the viscous term genuinely nonlinear. Based on the above specifications, the RHS of the Navier-Stokes problem is computed as

\[
f(x, t) = \partial_t v^\text{ex} + \nabla \cdot v^\text{ex} v^\text{ex} - \nabla \cdot \left[ \nu(x, t, v^\text{ex}) \left( \nabla v^\text{ex} + (\nabla v^\text{ex})^T \right) \right] + \nabla p^\text{ex}
\]

Omitting the convection term in (98) yields the RHS of the corresponding Stokes problem.

In all studies reported below time-dependent Dirichlet conditions are imposed such that \( v_\text{b} = v^\text{ex} \) on \( \partial\Omega \).

5.2.2. Influence of the base time-integration scheme. To investigate the role of the parameter \( \chi \) and the final projection in the predictor and corrector, a preliminary study was conducted for a solution-dependent viscosity \( \nu^{(3)} \) with coefficients \( \nu_0 = \nu_1 = 10^{-2} \). This choice corresponds to Reynolds number of \( Re = l v_{\max}/\nu_{\text{ref}} \approx 133 \), where \( \nu_{\text{ref}} = \nu_0 + \frac{1}{2} \nu_1 \). The numerical tests were computed in the domain \( \Omega = [-1/2, 1/2]^3 \) for \( t \leq 0.25 \). Spatial discretization is based on a uniform mesh comprising \( 2^3 \) cubic elements of degree \( P = 16 \). For time integration the SDC method was applied with \( M = 6 \) and \( K = 11 \) correction sweeps. Figure 3 shows the resulting velocity error for different predictor/corrector variants. As in case of the Taylor-Green vortex, the variants with FP achieve the best results and virtually coincide regardless of the choice for \( \chi \). They reach a convergence rate of approximately 7.5, which corresponds to a reduction of 4.5 or 37.5 percent of the expected order of 12. The standard scheme (\( \chi = 1 \)) without FP attains almost the same accuracy, whereas the rotational scheme (\( \chi = 2 \)) converges at a rate of only about 4.8. This contradicts the results obtained with the Taylor-Green vortex, for which the rotational scheme surpassed the standard one. These observations indicate that the final projection eliminates a substantial part of the splitting error, while the choice of \( \chi \) is of minor importance. All of the following studies are based on the rotational scheme with FP.

5.2.3. Temporal convergence. Three different scenarios were chosen for assessing the robustness of the SDC method against variable viscosity: 1) spatially varying, \( \nu(x) = \nu^{(1)} \), 2) spatiotemporally varying, \( \nu(x, t) = \nu^{(2)} \), and 3) solution-dependent, \( \nu(v) = \nu^{(3)} \), with coefficients \( \nu_0 = \nu_1 = 10^{-2} \). Additionally, the case with constant \( \nu = 0.015 \) is considered for reference. The computational domain, spatial discretization and final time are identical to the previous study. Figure 4 shows the velocity error obtained with \( M = 6 \) subintervals and a different number of correction sweeps, ranging from \( K = 0 \) up to 30. All investigated scenarios exhibit a similar behavior and achieve convergence rates comparable to the reference case. Using 30 corrections yields a rate of 12, which equals the theoretical order of the underlying collocation method.
Variable viscosity $\nu^{(3)}$ – SDC(6,11)

Figure 3. Velocity error of SDC with $M = 6$ subintervals and $K = 11$ correction sweeps for the traveling 3D vortex with solution-dependent viscosity. The choices $\chi = 1$ and $\chi = 2$ refer to the method based on standard and rotational velocity correction, respectively, and FP indicates the application of the final projection step.

Runs with a lower number of sweeps suffer an order reduction. The extent of this reduction is similar for all scenarios, which indicates that the presence of a variable viscosity is not the primary cause. It further noted that Fig. 4b and 4c lack the errors for the two largest time steps with $K = 30$. This is because $\Delta t$ exceeds the long term stability threshold, which is considered in more detail below.

As discussed in Sec. 3.2.3, the SDC method does not provide the discrete pressure. It can be computed, however, by solving the discrete version of the consistent pressure equation (44). This yields the pressure with an accuracy comparable to that of velocity, see, e.g., Fig. 5a for the case of solution-dependent viscosity. Figure 5 depicts the corresponding divergence error. Except for larger time steps with $K = 1$ it shows roughly the same behavior as the velocity and pressure errors. Additional studies revealed that an even stronger divergence penalization fails to reduce $\varepsilon_{\text{div}}$ significantly. This observation is somewhat surprising. It can be explained with the splitting error in the final projection step, which incurs a violation of the tangential velocity boundary conditions and causes a growth of divergence near the edges of the computational domain.

5.2.4. Stability. The stability of the SDC method was investigated for the limiting cases of convection-dominated flow and Stokes flow. For this study the domain $\Omega = [-1/2, 1/2]^3$ was discretized in three ways: 1) $6^3$ elements of degree $P = 6$, 2) $3^3$ elements of degree 11 and 3) $2^3$ elements of degree 16. The numerical tests were run until reaching the final time $T = 10$, which corresponds to 10 convective units in terms of the phase velocity. A test was considered unstable when exceeding a velocity magnitude of $v_{\text{max}}^{10}/2$ or detecting a NaN in the numerical solution. Based on this criterion the time step was adapted via bisection until a reaching sufficiently accurate approximation of the critical time step $\Delta t^*$. 
In the convection-dominated case the viscosity $\nu^{(1)}$ is adopted with $\nu_0 = \nu_1 = 10^{-3}$. This corresponds to a Reynolds number of approximately 1333, based on $v_{\text{max}}^\text{ex}$ and $\nu_\text{ref} = \nu_0 + \frac{1}{2}\nu_1$. The resulting convective stability threshold is converted into dimensionless form by introducing the critical Courant-Friedrichs-Lewy (CFL) number

$$CFL^* = \frac{\Delta t^* v_{\text{max}}}{\delta},$$

(99)

where $\delta$ is a length scale characterizing the mesh spacing. For high-order element methods different length scales have been proposed, in particular, $\delta_1 = h/(P + 1)$ and $\delta_2 = h/P^2$, see...
The corresponding CFL numbers are denoted as $CFL_1^*$ and $CFL_2^*$, respectively. Alternatively, the length scale can be defined as the inverse maximum modulus of the eigenvalues of the one-dimensional element convection operator, i.e., $\delta_\lambda = |\lambda|_{\text{max}}^{-1}$. In the present case, the eigenvalues result from the GLL collocation differentiation operator of degree $P$ combined with one-sided Dirichlet conditions. For details see Canuto et al. [17, Sec. 6.3.3]. The resulting CFL number is denoted as $CFL_\lambda^*$. Tabular 1 compiles the critical CFL numbers determined in this study. Note that the first three lines correspond to the case with only one subinterval. Starting with the split semi-implicit Euler method ($K=0$), the admissible time step grows by factor of 3 with one correction and even a factor of 5.3 with two. Elevating the number of subintervals and, proportionally, the number correction sweeps yields a further increase of the stability threshold.

A comparison of the stability limits for equal $M$ further reveals a strong dependence of $CFL_1^*$ and $CFL_2^*$ on $P$, whereas $CFL_\lambda^*$ is virtually independent of the polynomial degree. Figure 6 confirms this observation and, moreover, illustrates that the stability threshold grows linearly when increasing the number of subintervals. This behavior was expected, since the predictor and the corrector perform substeps with the size scaling as $\Delta t/M$. Given the influence of the flow configuration, the Reynolds number, the stability and dissipativity of spatial discretization and the termination criterium, these results cannot be compared directly to other studies. Yet it is worth noting that the observed $CFL_2^*$ reside in the same range as those reported by Fehn et al. [35], who use a similar space discretization in conjunction with a second-order projection method.

A similar study was conducted for the Stokes flow with solution-dependent viscosity $\nu^{(3)}$ featuring $\nu_0 = \nu_1 = 1$. Notwithstanding the semi-implicit treatment of the viscous term all test runs remained stable up to the maximal time step size of $\Delta t = 5$, which is about $10^4$ times the diffusive fluctuation time scale $\tau_d = h^2/\nu_1 P^4$. This allows the conclusion that, for moderate viscosity fluctuations, the semi-implicit approach does not affect the stability of the SDC method.
Table 1. Critical CFL numbers for $Re \approx 1333$ and SDC with a different number of subintervals $M$ and corrections sweeps $K$.

| $M$ | $K$ | $P = 6$ | $P = 11$ | $P = 16$ |
|-----|-----|---------|----------|---------|
|     |     | $CFL^*_1$ | $CFL^*_2$ | $CFL^*_\lambda$ | $CFL^*_1$ | $CFL^*_2$ | $CFL^*_\lambda$ | $CFL^*_1$ | $CFL^*_2$ | $CFL^*_\lambda$ |
| 1   | 0   | .174    | .895     | .222     | .142    | 1.43     | .277     | .118    | 1.78     | .317     |
| 1   | 1   | .522    | 2.68     | .667     | .368    | 3.72     | .720     | .275    | 4.14     | .737     |
| 1   | 2   | .922    | 4.74     | 1.18     | .639    | 6.45     | 1.25     | .476    | 7.17     | 1.28     |
| 2   | 4   | 1.16    | 5.97     | 1.48     | .781    | 7.87     | 1.53     | .551    | 8.29     | 1.48     |
| 3   | 6   | 1.41    | 7.25     | 1.80     | .925    | 9.32     | 1.81     | .673    | 10.1     | 1.80     |
| 4   | 8   | 1.75    | 9.02     | 2.24     | 1.14    | 11.6     | 2.24     | .818    | 12.3     | 2.19     |
| 5   | 10  | 2.09    | 10.7     | 2.67     | 1.37    | 13.8     | 2.68     | .966    | 14.5     | 2.59     |
| 6   | 12  | 2.38    | 12.3     | 3.04     | 1.54    | 15.6     | 3.02     | 1.09    | 16.4     | 2.91     |
| 7   | 14  | 2.64    | 13.6     | 3.38     | 1.71    | 17.2     | 3.34     | 1.21    | 18.2     | 3.25     |
| 8   | 16  | 2.92    | 15.0     | 3.74     | 1.89    | 19.1     | 3.70     | 1.34    | 20.1     | 3.58     |

Figure 6. Critical CFL number for a varying number of subintervals $M$ and $K = 2M$ correction sweeps.

5.2.5. Spatial convergence. The final study serves for examining the spatial convergence with variable viscosity. It is based on the on the highly-nonlinear, solution-dependent viscosity $\nu^{(3)}$ with coefficients $\nu_0 = \nu_1 = 10^{-2}$. The numerical tests were computed in the domain $\Omega = [-2, 2]^3$ for polynomial degrees up to $P = 16$. Starting from one, the number of elements per direction was gradually increased to at least 8. Time integration was performed using the SDC method with $M = 6$ and $K = 11$ until reaching $T = 0.25$. The time step size was confined to $\Delta t = 2^{-8}$ such that the temporal discretization error is negligible. Figure 7 shows the velocity error for polynomial degrees $P = 6, 11$ and 16. In all three cases the method converges approximately with $h^{P+1}$. This result is surprising, since the viscous terms are integrated with just $P + 1$ GLL points, for which only order $P$ is expected [64]. Possibly, the higher convergence rate
6. Conclusions

This paper presents a high-order, semi-implicit time-integration strategy for incompressible Navier-Stokes problems with variable viscosity based on the spectral deferred correction (SDC) method. Combining SDC with the discontinuous Galerkin spectral-element method for spatial discretization yields a powerful approach for targeting arbitrary order in space and time.

The key ingredients of the method, the predictor and the corrector, are derived from a first-order velocity-correction method, which is augmented by an additional projection step to remove divergence errors caused by variable viscosity. In contrast to the SDPC method of Minion and Saye \cite{MinionSaye}, the pressure occurs only as an auxiliary variable in the substeps and needs not to be stored. Furthermore, mixed-order \((P/P-1)\) polynomial approximations are used for inf-sup stability and combined with divergence/mass-flux stabilization for pressure robustness \cite{Falk}.\]

The performance of the SDC method was assessed at the example of a Taylor-Green (TG) vortex and a manufactured 3D vortex array, both traveling with a prescribed phase velocity. For the TG vortex with constant viscosity and periodic boundaries, each correction sweep elevates the order by 1, whereas the imposition of time-dependent Dirichlet reduces this improvement to approximately 0.8. The manufactured 3D example involves a variable viscosity \(\nu\) that can be chosen to depend on 1) space, 2) space and time, or 3) the velocity \(v\). While this example incurred a slightly stronger order reduction, it also proved the robustness of the proposed SDC method against variable viscosity, even in the nonlinear case, \(\nu(v)\). Regardless of the viscosity variation, the method was capable to reach the order of the underlying collocation scheme, e.g. order 12 with 13 Lobatto points and 30 correction sweeps. Further studies showed that the critical time step grows linearly with the number of subintervals in the convection-dominated case. In the Stokes case with moderate viscosity fluctuations, the method remained stable for steps up 4 order of magnitude above the viscous time scale \(h^2/P^4\nu\), where \(h\) is the element size and \(P\) the polynomial degree of the discrete velocity.
Future work should be directed to improve the efficiency of the method and, ultimately, render it competitive to common approaches such as multistep and Runge-Kutta schemes. While recent studies give little hope for developing higher-order correctors [19], the use of higher-order schemes for the predictor could accelerate the method by providing a better initial approximation [60]. Additionally, the number of iterations could be reduced by adopting a space-time multilevel strategy as proposed with the MLSDC method in [79].

Acknowledgements

Funding by German Research Foundation (DFG) in frame of the project STI 57/8-1 is gratefully acknowledged. The author would like to thank ZIH for providing computational resources and Karl Schoppmann for his assistance in devising and implementing the manufactured solution for the variable viscosity test case.

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