INFORMATION THEORY AND MODULI OF Riemann SURFACES

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ABSTRACT. One interpretation of Torelli’s Theorem, which asserts that a compact Riemann Surface $X$ of genus $g > 1$ is determined by the $g(g+1)/2$ entries of the period matrix, is that the period matrix is a message about $X$. Since this message depends on only $3g - 3$ moduli, it is sparse, or at least approximately so, in the sense of information theory. Thus, methods from information theory may be useful in reconstructing the period matrix, and hence the Riemann surface, from a small subset of the periods. The results here show that, with high probability, any set of $3g - 3$ periods form moduli for the surface.

1. Introduction

The connection between Information Theory and Algebraic Geometry, once unthinkable, was established by the rise of applications like Goppa codes [13] and elliptic- and hyperelliptic curve cryptography [1].

Both coding and cryptography are branches of information theory as originally formulated by Shannon [10]. A message is a sequence of symbols from a known alphabet with known probabilities (to be concrete, imagine a stream of bits selected from \{0, 1\} with known probability of choosing each). The information content of such a message is the smallest number of bits required to send it; this is related to its Kolmogorov complexity. In coding, the goal is to adjoin bits, that is, redundancy, to make the message more likely to understand; in cryptography, one of the goals is to hide redundancy.

Goppa codes are constructed from the line bundles and points of an algebraic curve defined over a finite field; the field elements constitute the alphabet. The parameters of the code are determined using the Riemann–Roch theorem.

The most common cryptographic constructions use the Jacobian of an algebraic curve, most often an elliptic or hyperelliptic curve; the cryptographic strength comes from the difficulty of solving the discrete logarithm problem in these groups.

One generally discusses cryptographic protocols using a sender Alyce sending a message to a receiver Bob that may be intercepted by an eavesdropper Eve. Now suppose that Alyce wants to describe a compact Riemann surface with genus $g > 1$ to Bob. By Torelli’s theorem she can, in principle, send Bob a period matrix of the curve (although in practice it can be difficult to tell much about the curve from its period matrix). This message consists of $g(g + 1)/2$ complex numbers, or their approximations. However, the dimension of the moduli space of compact Riemann surfaces of genus $g > 1$ is only $3g - 3$, so in principle, or locally, the information content of Alyce’s message is much smaller than its length.
In the language of information theory, then, the message is compressible. This concept is familiar to computer users who use utilities like gzip to compress files and save disk space.

Another perspective on this situation is that the message describing the Riemann surface is sparse, or approximately so. To illustrate what sparsity means in this context, imagine some finite energy signal (say, a voice reading aloud) and its Fourier transform. In many cases drawn from nature (see [2]) the set of coefficients of this expansion (or other expansion, such as wavelets, or even non-orthonormal expansions [3]) contains only a few large numbers, while the rest of the coefficients are zero or small; this is the meaning of sparse in this context. Put differently, the spectrum of the signal is concentrated in a few bands.

The essence of many compression and de-noising schemes is to look at the spectrum of a signal and approximate it using only the terms with large coefficients. Compressed Sensing [3] takes a completely different approach; rather than collecting the whole signal before compressing, compressed sensing collects a smaller number of samples, in effect compressing at the source. One way to do this might be take a random set of Fourier coefficients. For example, given $x$, the discrete Fourier transform of some signal, let $A$ be a random $m \times n$ matrix with entries in $\{0, 1\}$, and let $y = Ax$; $y$ is the observation. In theory, the problem of reconstructing $x$ from $y$ is not well-defined, but in practice when the signal is sparse excellent approximations and even exact reconstructions are possible.

It seems natural that at least the ideas of compressed sensing might apply to Alyce’s description of a Riemann Surface.

Furthermore, when Eve hears Alyce’s list of $g(g + 1)/2$ complex numbers, it is natural for her to ask whether they are the periods of a compact Riemann Surface. This is just a restatement of the classical Schottky problem, asking how to distinguish period matrices from ordinary elements of the Siegel upper half space. Call this version the Information-Theoretic Schottky Problem.

Eve might also wonder about the properties of the Riemann Surface represented by Alyce’s message; for example, she might ask whether the surface in question is hyperelliptic, or trigonal, or whether it has a non-trivial automorphism group, etc. Call this the Information-Theoretic Torelli Problem. In this case, numerical results [15] (and unpublished analytical results) indicate that one can distinguish the period matrices of hyperelliptic surfaces among arbitrary period matrices by examining the statistics of the periods: the periods are band-limited in the sense that their frequencies (arguments) are tightly clustered.

2. Period Matrices

Begin by fixing notation; consult [6] as a general reference. Let $X$ be a compact Riemann Surface of genus $g > 1$; equivalently, $X$ is a nonsingular complex algebraic curve. Choose a basis $\omega_1, \ldots, \omega_g$ for the space $H^{1,0}(X)$ of holomorphic differentials on $X$, and a symplectic basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ for the singular homology $H_1(X, \mathbb{Z})$, normalized so $\int_{\alpha_i} \omega_j = \delta_{ij}$, the Dirac delta. The matrix $\Omega_{ij} := \int_{\beta_j} \omega_i$ is the period matrix; Riemann proved that it is symmetric with positive definite imaginary part. The torus $\mathbb{C}^g/[\mathbb{Z}]^g$ is the Jacobian of $X$. Torelli’s Theorem asserts that the Jacobian determines all of the properties of $X$. In practice deciding which properties apply is seldom successful, and any success (like [14]) depends on deep tools like Riemann’s theta–function.
The period matrix is symmetric with positive-definite imaginary part, and the space of such matrices forms the Siegel upper half-space $\mathcal{H}_g$. Its dimension is $g(g + 1)/2$, while the dimension of the moduli space of curves of degree $g$ is $3g - 3$. Distinguishing the period matrices from arbitrary elements of $\mathcal{H}_g$ is the Schottky Problem.

It would be unusual – and noteworthy – were a problem as old as the Schottky problem to remain unsolved. Grushevsky’s survey [4] contains more detail, but previous solutions have involved $\theta$-function identities, the geometry of the $\Theta$ divisor, Kummer varieties, and solitons; the wide range of techniques indicates that the Schottky problem is a central problem in mathematics. The information-theoretic approach deepens this idea.

The aim now is to study and then exploit the information-theoretic properties of period matrices. One interpretation of Torelli’s Theorem is that the period matrix is a message or signal about $X$. This message is highly compressible, while a random element of $\mathcal{H}_g$ is not.

The result here can also be interpreted as a new result in Compressed Sensing (described below). Typically, compressed sensing works with a linear relationship between the signal and the measurement, but in the Schottky problem the relationship between the signal – that is, the period matrix – and the measurement – that is, explicit moduli – is nonlinear. Nonetheless, a compressed sensing manipulation of a precursor to the signal (here, the holomorphic differentials) still leads to a compressed sensing result, at least with high probability.

Nonlinear compressed sensing is an emerging field; the few results to appear so far include [5] and [8].

## 3. Compressive Sensing

Compressed (or Compressive) Sensing [3] is the study of taking small representations or samples of sparse (or nearly sparse) signals without any loss of information. A familiar example of compression is the Joint Photographic Experts Group (JPEG) standard for storing and transmitting images, which can encode an image comprising millions of pixels into a few kilobytes with minimal loss of quality. Following [3] and [11], in compressed sensing the compression happens at the sensor rather than in post-processing. “Sparse” has many meanings; one of the simpler ones is that the signal can be constructed from some basis (e.g., Fourier series, impulses, wavelets, ...) with only a few non-zero coefficients.

Many compression methods have the following outline: transform the signal with respect to some basis functions $\Psi_i$, remove most of the “unimportant” terms (e.g., those with the smallest coefficients), and finally perform the inverse transform. For example one can do this with Fourier coefficients, and the Shannon-Nyquist Sampling Theorem [7] determines which coefficients one needs to retain for perfect or lossless compression. The same general outline applies to wavelets or other basis functions (which need not be orthonormal). Another concept of sparsity is “lying on a submanifold (or subvariety) of small dimension.” [3] discusses other definitions.

Abstractly, this corresponds to a signal vector $\tilde{x}$ and a linear set of observations $\tilde{y}$ formed using a sensing matrix $M$, i.e., $\tilde{y} = M \tilde{x}$. In many applications it suffices to use a random matrix $M$ in which each entry is the result of a Bernoulli process of fixed probability [2].
Compression works when the signal under consideration is sparse, or approximately so. This is the case with a period matrix, which, by Torelli’s Theorem, represents a compact Riemann surface completely: while the matrix has $g(g+1)/2$ distinct entries, the Riemann surface it encodes depends on $3g-3$ moduli. In this work, the period matrix is treated as a signal $\Pi$ of length $g(g+1)/2$ which depends on $3g-3$ unknown underlying parameters. This is a list of complex numbers; the matrix structure is in some sense irrelevant because their placement in the matrix depends on choices of bases for $H^0(K)$ and $H_1(X,\mathbb{Z})$. Instead of multiplying $\Pi$ by a random measurement matrix, as would be done in compressed sensing, a random change–of–basis is applied to $H^0(K)$. With high probability, the first three rows of the period matrix constructed from the new basis constitute a set of moduli for $X$, in a sense explained below.

4. Period Matrices and Moduli

The primary tool relating period matrices to moduli is the following theorem of Rauch [9]. Let $K$ denote the canonical divisor on $X$.

**Theorem 4.1.** [Rauch] Let $\{\zeta_1, \ldots, \zeta_g\}$ be a normalized basis for $H^{(1,0)}(X)$ of a non-hyperelliptic Riemann surface $X$, and suppose that $\{\zeta_i\zeta_j : (i, j) \in (I, J)\}$ form a basis for the quadratic differentials $H^0(X, 2K)$. If another Riemann surface $X'$ has the same entries as $X$ in the $(I, J)$ positions of its period matrix then $X$ and $X'$ are holomorphically equivalent.

In other words, some sets of $3g-3$ periods form local coordinates on the moduli space of compact Riemann Surfaces. However, except in special cases like smooth plane curves (see [10]), there are no results indicating which sets of $3g-3$ entries constitute moduli. (In the case of plane curves, one can use the explicit basis for $H^0(K)$ to determine sets of moduli.)

5. Numerical Experiments

Numerical experiments with period matrices of large genus show that the distribution of the squared absolute values of the periods is consistent with a compressible signal. These distributions appear to follow a power law (the $n$th entry is bounded by $1/n^p$ for some power $p$).

For example, consider the periods of the Fermat Curve of degree 11, given in homogeneous by $X^{11} + Y^{11} + Z^{11} = 0$. Using Maple, the squared moduli of the periods show the characteristic shape of a compressible signal.
Other numerical experiments have exhibited unusual phenomena in the distribution of the periods of special classes of curves; in particular, the periods of hyperelliptic curves appear to be “band–limited”. See [15].

6. Main Theorem

This work takes a probabilistic approach to the problem of determining which sets of \(3g - 3\) entries form moduli. The theorem of Rauch above reduces this to finding a set of \(3g - 3\) quadratic differentials that form a basis of \(H^0(2K)\).

**Theorem 6.1.** Let \(X\) be a compact Riemann surface of genus \(g\) and choose a symplectic homology basis for \(H^0(K)\). Let \(M\) be a \(g \times g\) matrix whose entries lie in \(\{0, 1\}\), and are determined independently by a Bernoulli process. Use \(M\) as a change-of-basis matrix for \(H^0(K)\). Then after the change–of–basis the first three rows of the period matrix for \(X\) form moduli for \(X\), with high probability.

**Proof.** Using the notation from Theorem 4.1, let \(\{\zeta_1, \ldots, \zeta_g\}\) be a normalized basis for \(H^{(1,0)}(X)\) and suppose that \(\{\zeta_i\zeta_j : (i,j) \in (I,J)\}\) form a basis for \(H^0(2K)\). The elements of \((I, J)\) are unknown.

Now, let \(\omega_i = \sum b_{ij} \zeta_j\), where \(b_{ij}\) is a random binary determined by a Bernoulli process. Determining the probability that \(\Omega = \{\omega_i\}\) forms a basis, which is the probability that the matrix \([b_{ij}]\) is nonsingular, is rather subtle. Tao and Vu [12], conjecture the probability to be

\[
1 - \left(\frac{1}{2} + o(1)\right)^p,
\]

and prove it greater than

\[
1 - \left(\frac{3}{7} + o(1)\right)^p.
\]

In any case, it is very likely that \(\Omega\) forms a basis for \(H^0(K)\).
Now, assuming that \( \Omega \) forms a basis for \( H^0(K) \), the question becomes determining the probability that \( \{ \omega_1, \omega_2, \ldots, \omega_g, \rho_1, \rho_2, \ldots, \rho_g, \gamma_1, \ldots, \gamma_g \} \) forms a basis for \( H^0(2K) \). Notice that this set contains \( 3g - 3 \) elements because of the symmetry of the period matrix.

Again, because the coefficients are binary, and, in particular, positive, it suffices for each \( \zeta_i \zeta_j \) for \((i, j) \in (I, J)\) to have a non-zero coefficient in one expansion of the products of the \( \omega \)s.

To find the probability that every coefficient of \( \zeta_i \zeta_j \) vanishes in the expansion of \( \omega_\ell \omega_m \), notice that all four coefficients \( b_{i,\ell}, b_{j,\ell}, b_{i,m} \) and \( b_{j,m} \) must vanish, so the probability is \( 1/16 \). For this to happen for all coefficients, then, the probability is \( (1/16)^{3g-3} \). And, since this must happen independently for all \( \zeta_i \zeta_j \), the probability of a basis for \( H^0(2K) \) is

\[
\frac{3g - 3}{16^{3g-3}}.
\]

\[\square\]

7. Complements

The proof of the main theorem is deceptively simple, because it depends on Rauch’s deep theorem.

Compressed Sensing usually depends on linear measurement, and there are few results about nonlinear measurement processes [11]. Here, the “measurement” is linear at the level of the holomorphic differentials, but the resulting transformation of the period matrix is non-linear.

A direct relationship between the moduli space of compact Riemann Surfaces and the period matrix is a long-standing difficult problem. One possibility here is to look at a discrete version of Rauch’s Theorem, whose main construction is the minimal energy element of the homotopy class of differentiable maps between compact Riemann surfaces.

In practice, as D. Litt pointed out in a conversation, it may not be possible to transmit periods in a finite message, although many complex numbers do have compact descriptions (e.g., Gaussian rationals, surds). In other cases it may only be possible to transmit an approximation of the periods, in which case the conclusion is that the curve in question is close (in an analytic sense) to the Schottky locus, which is already significant.

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