NEARNESS RINGS ON NEARNESS APPROXIMATION SPACES*

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Abstract. In this paper, we consider the problem of how to establish algebraic structures on nearness approximation spaces. Essentially, our approach is to define the nearness ring, nearness ideal and nearness ring of all weak cosets by considering new operations on the set of all weak cosets. Afterwards, our aim is to study nearness homomorphism on nearness approximation spaces, and to investigate some properties of nearness rings and ideals.

1. Introduction

Nearness approximation spaces and near sets were introduced in 2007 as a generalization of rough set theory [12, 10]. More recent work considers generalized approach theory in the study of the nearness of non-empty sets that resemble each other [14] and a topological framework for the study of nearness and apartness of sets [8]. An algebraic approach of rough sets has been given by Iwinski [4]. Afterwards, rough subgroups were introduced by Biswas and Nanda [1]. In 2004 Davvaz investigated the concept of roughness of rings [3] (and other algebraic approaches of rough sets in [17, 16]).

Near set theory begins with the selection of probe functions that provide a basis for describing and discerning affinities between objects in distinct perceptual granules. A probe function is a real-valued function representing a feature of physical objects such as images or collections of artificial organisms, e.g., robot societies.

In the concept of ordinary algebraic structures, such a structure that consists of a nonempty set of abstract points with one or more binary operations, which are required to satisfy certain axioms. For example, a groupoid is an algebraic structure \((A, \circ)\) consisting of a nonempty set \(A\) and a binary operation \(\circ\) defined on \(A\) [2]. In a groupoid, the binary operation \(\circ\) must be only closed in \(A\), i.e., for all \(a, b \in A\), the result of the operation \(a \circ b\) is also in \(A\). As for the nearness approximation space, the sets are composed of perceptual objects (non-abstract points) instead of abstract points. Perceptual objects are points that have features. And these points describable with feature vectors in nearness approximation spaces [10]. Upper approximation of a nonempty set is obtained by using the set of objects composed by the nearness approximation space together with matching objects. In the algebraic structures constructed on nearness approximation spaces, the basic tool is consideration of upper approximations of the subsets of perceptual objects.

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In a groupoid $A$ on nearness approximation space, the binary operation “$\circ$” may be closed in upper approximation of $A$, i.e., for all $a, b$ in $A$, $a \circ b$ is in upper approximation of $A$.

There are two important differences between ordinary algebraic structures and nearness algebraic structures. The first one is working with non-abstract points while the second one is considering of upper approximations of the subsets of perceptual objects for the closeness of binary operations.

In 2012, E. İnan and M. A. Öztürk [5, 6] investigated the concept of near groups on nearness approximation spaces. Moreover, in 2013, M. A. Öztürk at all [9] introduced near group of weak cosets on nearness approximation spaces. And in 2015, E. İnan and M. A. Öztürk [7] investigated the nearness semigroups. In this paper, we consider the problem of how to establish and improve algebraic structures of nearness approximation spaces. Essentially, our aim is to obtain algebraic structures such as nearness rings using sets and operations that ordinary are not being algebraic structures. Moreover, we define the nearness ring of all weak cosets by considering operations on the set of all weak cosets. To define this quotient structure we don’t need to consider ideals.

2. Preliminaries

2.1. Nearness Approximation Spaces [10]. Perceptual objects are points that describable with feature vectors. Let $O$ be a set of perceptual objects. An object description is defined by means of a tuple of function values $\Phi (x)$ associated with an object $x \in X \subseteq O$. The important thing to notice is the choice of functions $\varphi_i \in B$ used to describe any object of interest. Assume that $B \subseteq F$ is a given set of functions representing features of sample objects $X \subseteq O$. Let $\varphi_i \in B$, where $\varphi_i : O \rightarrow \mathbb{R}$. In combination, the functions representing object features provide a basis for an object description $\Phi : O \rightarrow \mathbb{R}^L$, a vector containing measurements (returned values) associated with each functional value $\varphi_i (x)$, where the description length is $|\Phi| = L$.

Object Description: $\Phi (x) = (\varphi_1 (x), \varphi_2 (x), \varphi_3 (x), ..., \varphi_i (x), ..., \varphi_L (x))$.

Sample objects $X \subseteq O$ are near to each other if and only if the objects have similar descriptions. Recall that each $\varphi$ defines a description of an object. Then let $\Delta_{\varphi_i}$ denote $\Delta_{\varphi_i} = |\varphi_i (x') - \varphi_i (x)|$, where $x, x' \in O$. The difference $\Delta_{\varphi}$ leads to a definition of the indiscernibility relation “$\sim_B$”.

Let $x, x' \in O, B \subseteq F$.

$$\sim_B = \{(x, x') \in O \times O | \forall \varphi_i \in B, \Delta_{\varphi_i} = 0\}$$

is called the indiscernibility relation on $O$, where description length $i \leq |\Phi|$. 

Interpretation

Partition

Indiscernibility relation defined using \( N_B \) on \( \nu \) in the set.

problems arising from disjoint sets that resemble each other [10, 12].

sets were introduced as a means of solving classification and pattern recognition

collection of partitions \( B \) associated with a feature of an object \( x \)." "

lap function \( \nu \) denote a set of features for objects in a set \( F \).

...into non-empty, pairwise disjoint subsets that are equivalence classes denoted

by \( [x]_{B_r} \), where \( [x]_{B_r} = \{ x' \in \mathcal{O} \mid x \sim_{B_r} x' \} \). These classes form a new set called

the quotient set \( \mathcal{O}/\sim_{B_r} \), where \( \mathcal{O}/\sim_{B_r} = \{ [x]_{B_r} \mid x \in \mathcal{O} \} \). In effect, each choice

of probe functions \( B_r \) defines a partition \( \xi_{\mathcal{O},B_r} \) on a set of objects \( \mathcal{O} \), namely,

\( \xi_{\mathcal{O},B_r} = \mathcal{O}/\sim_{B_r} \). Every choice of the set \( B_r \) leads to a new partition of \( \mathcal{O} \). Let \( F \) denote a set of features for objects in a set \( X \), where each \( \phi_i \in F \) that maps \( X \) to some value set \( V_{\phi_i} \) (range of \( \phi_i \)). The value of \( \phi_i (x) \) is a measurement associated with a feature of an object \( x \in X \). The overlap function \( \nu_N \) is defined by \( \nu_N : \phi(O) \times \phi(O) \rightarrow [0,1] \), where \( \phi(O) \) is the powerset of \( \mathcal{O} \). The overlap function \( \nu_N \) maps a pair of sets in \( [0,1] \) representing the degree of overlap between sets of objects with their features defined by probe functions \( B_r \subseteq B \) [12]. For each subset \( B_r \subseteq B \) of probe functions, define the binary relation \( \sim_{B_r} = \{ (x, x') \in \mathcal{O} \times \mathcal{O} \mid \forall \phi_i \in B_r, \phi_i (x) = \phi_i (x') \} \). Since each \( \sim_{B_r} \) is, in fact, the usual indiscernibility relation, for \( B_r \subseteq B \) and \( x \in \mathcal{O} \), let \( [x]_{B_r} \) denote the equivalence class containing \( x \). If \( (x, x') \in \sim_{B_r} \), then \( x \) and \( x' \) are said to be \( B \)-indiscernible with respect to all feature probe functions in \( B_r \). Then define a collection of partitions \( N_r (B) \), where \( N_r (B) = \{ \xi_{\mathcal{O},B_r} \mid B_r \subseteq B \} \).

2.2. Descriptively Near Sets. We need the notion of nearness between sets, and so we consider the concept of the descriptively near sets. In 2007, descriptively near sets were introduced as a means of solving classification and pattern recognition problems arising from disjoint sets that resemble each other [10, 12].

A set of objects \( A \subseteq \mathcal{O} \) is characterized by the unique description of each object in the set.

| Symbol | Interpretation |
|--------|----------------|
| \( B \) | \( B \subseteq F \), |
| \( r \) | \( \{\{x_i\}\} \), i.e., \( |B| \) probe functions \( \phi_i \in B \) taken \( r \) at a time, |
| \( B_r \) | \( r \leq |B| \) probe functions in \( B \), |
| \( \sim_{B_r} \) | Indiscernibility relation defined using \( B_r \), |
| \( [x]_{B_r} \) | \( \{x' \in O \mid x \sim_{B_r} x' \} \), equivalence (nearness) class, |
| \( \mathcal{O}/\sim_{B_r} \) | \( \mathcal{O}/\sim_{B_r} = \{ [x]_{B_r} \mid x \in \mathcal{O} \} \), quotient set, |
| \( \xi_{\mathcal{O},B_r} \) | Partition \( \xi_{\mathcal{O},B_r} = \mathcal{O}/\sim_{B_r} \), |
| \( N_r (B) \) | \( N_r (B) = \{ \xi_{\mathcal{O},B_r} \mid B_r \subseteq B \} \), set of partitions, |
| \( \nu_N \) | \( \nu_N : \phi(O) \times \phi(O) \rightarrow [0,1] \), overlap function, |
| \( N_r (B)_x X \) | \( N_r (B)_x X = \bigcup_{x \in x} [x]_{B_r} \subseteq X \), lower approximation, |
| \( N_r (B)^* X \) | \( N_r (B)^* X = \bigcup_{x \in x} [x]_{B_r} \cap X \not= \emptyset \subseteq X \), upper approximation, |
| \( Bnd_{N_r (B)} (x) \) | \( N_r (B)^* X \setminus N_r (B)_x X = \{ x \in N_r (B)^* X \mid x \notin N_r (B)_x X \} \). |

Table 1 : Nearness Approximation Space Symbols
Set Description: Let $O$ be a set of perceptual objects, $\Phi$ an object description and $A \subseteq O$. Then the set description of $A$ is defined as

$$Q(A) = \{\Phi(a) \mid a \in A\}.$$ 

Descriptive Set Intersection: Let $O$ be a set of perceptual objects, $A$ and $B$ any two subsets of $O$. Then the descriptive (set) intersection of $A$ and $B$ is defined as

$$A \cap B = \{x \in A \cup B \mid \Phi(x) \in Q(A) \text{ and } \Phi(x) \in Q(B)\}.$$ 

If $Q(A) \cap Q(B) \neq \emptyset$, then $A$ is called descriptively near $B$ and denoted by $A \delta_\Phi B$.

Descriptive Nearness Collections: Let $\Phi$ be an object description, $A$ any subset of $O$ and $\xi_\Phi (A)$ a descriptive nearness collections. Then $A \in \xi_\Phi (A)$.

2.3. Some Algebraic Structures on NAS. A binary operation on a set $G$ is a mapping of $G \times G$ into $G$, where $G \times G$ is the set of all ordered pairs of elements of $G$. A groupoid is a system $G (\cdot)$ consisting of a nonempty set $G$ together with a binary operation “$\cdot$” on $G$.

Let $NAS = (O, \mathcal{F}, \sim_B, N_r (B), \nu_{N_r})$ be a nearness approximation space (NAS) and let “$\cdot$” a binary operation defined on $O$. A subset $G$ of the set of perceptual objects $O$ is called a near group on NAS if the following properties are satisfied:

- $(NG_1)$ For all $x, y \in G$, $x \cdot y \in N_r (B)^* G$.
- $(NG_2)$ For all $x, y, z \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $N_r (B)^* G$.
- $(NG_3)$ There exists $e \in N_r (B)^* G$ such that $x \cdot e = e \cdot x = x$ for all $x \in G$ ($e$ is called the near identity element of $G$).
- $(NG_4)$ There exists $y \in G$ such that $x \cdot y = y \cdot x = e$ for all $x \in G$ ($y$ is called the near inverse of $x$ in $G$ and denoted as $x^{-1}$).

If in addition, for all $x, y, z \in G$, $x \cdot y = y \cdot x$ property holds in $N_r (B)^* G$, then $G$ is said to be an abelian near group on NAS.

Also, a nonempty subset $S \subseteq O$ is called a near semigroup on NAS if $x \cdot y \in N_r (B)^* S$ for all $x, y \in S$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in S$ property holds in $N_r (B)^* (S)$.

**Theorem 1.** Let $G$ be a near group on NAS.

(i) There exists a unique near identity element $e \in N_r (B)^* G$ such that $x \cdot e = e \cdot x = x$ for all $x \in G$.

(ii) For all $x \in G$, there exists a unique $y \in G$ such that $x \cdot y = e = y \cdot x$.

**Theorem 2.** Let $G$ be a near group on NAS.

(i) $(x^{-1})^{-1} = x$ for all $x \in G$.

(ii) If $x \cdot y \in G$, then $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$ for all $x, y \in G$.

(iii) If either $x \cdot z = y \cdot z$ or $z \cdot x = z \cdot y$, then $x = y$ for all $x, y, z \in G$.

$H$ is called a subnear group of near group $G$ if $H$ is a near group relative to the operation in $G$. There is only one guaranteed trivial subnear group of near group $G$, i.e., $G$ itself. Moreover, $[e]$ is a trivial subnear group of near group $G$ if and only if $e \in G$. 
Theorem 3. [6] Let $G$ be a near group on nearness approximation space, $H$ be a nonempty subset of $G$ and $N_r (B)^+ H$ be a groupoid. $H \subseteq G$ is a subnear group of $G$ if and only if $x^{-1} \in H$ for all $x \in H$.

Let $H_1$ and $H_2$ be two near subgroups of the near group $G$ and $N_r (B)^+ H_1$, $N_r (B)^+ H_2$ groupoids. If $(N_r (B)^+ H_1) \cap (N_r (B)^+ H_2) = N_r (B)^+ (H_1 \cap H_2)$, then $H_1 \cap H_2$ is a near subgroup of near group $G$ [6].

Let $G \subseteq O$ be a near group and $H$ be a subnear group of $G$. The left weak equivalence relation (compatible relation) “$\sim_L$” defined as

$$a \sim_L b : \iff a^{-1} \cdot b \in H \cup \{e\}.$$ 

A weak class defined by relation “$\sim_L$” is called left weak coset. The left weak coset that contains the element $a$ is denoted by $\bar{a}_L$, i.e.

$$\bar{a}_L = \{a \cdot h \mid h \in H, a \in G, a \cdot h \in G\} \cup \{a\} = aH.$$

Let $(O_1, F_1, \sim_{B_1}, N_r (B)_1, \nu_{N_r_1})$ and $(O_2, F_2, \sim_{B_2}, N_r (B)_2, \nu_{N_r_2})$ be two nearness approximation spaces and “$\sim_{B_1}$”, “$\sim_{B_2}$” binary operations over $O_1$ and $O_2$, respectively.

Let $G_1 \subseteq O_1$, $G_2 \subseteq O_2$ be two near groups and $\sigma$ a mapping from $N_{r_1} (B)^+ G_1$ onto $N_{r_2} (B)^+ G_2$. If $\sigma (x \cdot y) = \sigma (x) \circ \sigma (y)$ for all $x, y \in G_1$, then $\sigma$ is called a near homomorphism and also, $G_1$ is called near homomorphic to $G_2$.

Let $G_1 \subseteq O_1$, $G_2 \subseteq O_2$ be near homomorphic groups, $H_1$ a near subgroup and $N_{r_1} (B)^+ H_1$ a groupoid. If $\sigma (N_{r_1} (B)^+ H_1) = N_{r_2} (B)^+ \sigma (H_1)$, then $\sigma (H_1)$ is a near subgroup of $G_2$ [6].

The kernel of $\sigma$ is defined to be the set $\text{Ker} \sigma = \{x \in G_1 \mid \sigma (x) = e'\}$, where $e'$ is the near identity element of $G_2$.

Theorem 4. [6] Let $G_1 \subseteq O_1, G_2 \subseteq O_2$ be near groups that are near homomorphic, $\text{Ker} \sigma = N$ be near homomorphism kernel and $N_r (B)^+ N$ be a groupoid. Then $N$ is a near normal subgroup of $G_1$.

Definition 1. [9] Let $O$ be a set of perceptual objects, $G \subseteq O$ a near group and $H$ a subnear group of $G$. Let $G / \sim_L$ be a set of all left weak cosets of $G$ by $H$, $\xi_{\Phi} (A)$ a descriptive nearness collections and $A \in \mathcal{P} (O)$. Then

$$N_r (B)^+ (G / \sim_L) = \bigcup_{\xi \Phi (A) \subseteq G / \sim_L \neq \emptyset} \xi \Phi (A)$$

is called upper approximation of $G / \sim_L$.

Theorem 5. [9] Let $G$ be a near group, $H$ a subnear group of $G$ and $G / \sim_L$ a set of all left weak cosets of $G$ by $H$. If $(N_r (B)^+ G) / \sim_L \subseteq N_r (B)^+ (G / \sim_L)$, then $G / \sim_L$ is a near group under the operation given by $aH \circ bH = (a \cdot b)H$ for all $a, b \in G$.

Let $G$ be a near group and $H$ a subnear group of $G$. The near group $G / \sim_L$ is called a near group of all left weak cosets of $G$ by $H$ and denoted by $G / wH$ [9].

3. Nearness Rings on Nearness Approximation Spaces

Definition 2. Let $NAS = (O, F, \sim_{B_r} , N_r (B), \nu_{N_r})$ be a nearness approximation space and “$+$” and “$\cdot$” binary operations defined on $O$. A subset $R$ of the set of perceptual objects $O$ is called a nearness ring on NAS if the following properties are satisfied:
(NR₁) $R$ is an abelian near group on $NAS$ with binary operation “+”,

(NR₂) $R$ is a near semigroup on $NAS$ with binary operation “·”,

(NR₃) For all $x, y, z \in R$,

\[ x \cdot (y + z) = (x \cdot y) + (x \cdot z) \quad \text{and} \quad (x + y) \cdot z = (x \cdot z) + (y \cdot z) \]

properties hold in $N_r(B)^* R$.

If in addition:

(NR₄) $x \cdot y = y \cdot x$ for all $x, y \in R$,

then $R$ is said to be a commutative nearness ring.

(NR₅) If $N_r(B)^* R$ contains an element $1_R$ such that $1_R \cdot x = x \cdot 1_R = x$ for all $x \in R$,

then $R$ is said to be a nearness ring with identity.

These properties have to hold in $N_r(B)^* R$. Sometimes they may be hold in $O/N_r(B)^* R$, then $R$ is not a nearness ring on $NAS$. Multiplying or sum of finite number of elements in $R$ may not always belongs to $N_r(B)^* R$. Therefore always we can not say that $x^n \in N_r(B)^* R$ or $nx \in N_r(B)^* R$, for all $x \in R$ and some positive integer $n$. If $(N_r(B)^* R, R_{\cdot})$ and $(N_r(B)^* R, R_{\cdot})$ are groupoids, then we can say that $x^n \in N_r(B)^* R$ for all positive integer $n$ or $nx \in N_r(B)^* R$ all integer $n$, for all $x \in R$.

An element $x$ in nearness ring $R$ with identity is said to be left (resp. right) invertible if there exists $y \in N_r(B)^* R$ (resp. $z \in N_r(B)^* R$) such that $y \cdot x = 1_R$ (resp. $x \cdot z = 1_R$). The element $y$ (resp. $z$) is called a left (resp. right) inverse of $x$. If $x \in R$ is both left and right invertible, then $x$ is said to be nearness invertible or nearness unit. The set of nearness units in a nearness ring $R$ with identity forms is a near group on $NAS$ with multiplication.

A nearness ring $R$ is a nearness division ring iff $(R \setminus \{0\}, \cdot)$ is a near group on $NAS$, i.e., every nonzero elements in $R$ is a nearness unit. Similarly, a nearness ring $R$ is a nearness field iff $(R \setminus \{0\}, \cdot)$ is a commutative near group on $NAS$.

Some elementary properties of elements in nearness rings are not always provided as in ordinary rings. If we consider $N_r(B)^* R$ as a ordinary ring, then elementary properties of elements in nearness ring are provided.

**Lemma 1.** Every ordinary rings on $NAS$ are nearness rings on $NAS$.

**Example 1.** Let $O = \{o, p, r, s, t, v, w, x\}$ be a set of perceptual objects and $B = \{\varphi_1, \varphi_2, \varphi_3\} \subseteq F$ a set of probe functions. Values of the probe functions

\[
\varphi_1 : O \rightarrow V_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \\
\varphi_2 : O \rightarrow V_2 = \{\beta_1, \beta_2, \beta_3\},
\]

are given in Table 2.

|    | o  | p  | r  | s  | t  | v  | w  | x  |
|----|----|----|----|----|----|----|----|----|
| $\varphi_1$ | $\alpha_4$ | $\alpha_2$ | $\alpha_1$ | $\alpha_2$ | $\alpha_1$ | $\alpha_3$ | $\alpha_4$ | $\alpha_3$ |
| $\varphi_2$ | $\beta_1$ | $\beta_3$ | $\beta_2$ | $\beta_3$ | $\beta_2$ | $\beta_3$ | $\beta_1$ | $\beta_3$ |

Table 2.

Let “+” and “·” be binary operations of perceptual objects on $O$ as in Tables 3 and 4.
Since \( r + (s + s) \neq (r + s) + s \), \((\mathcal{O}, +)\) is not a group, i.e., \((\mathcal{O}, +, \cdot)\) is not a ring.

Let \( R = \{r, t, w\} \) be a subset of perceptual objects. Let \( \cdot \) and \( + \) be operations of perceptual objects on \( R \subseteq \mathcal{O} \) as in Tables 5 and 6.

| \( \cdot \) | \( o \) | \( p \) | \( r \) | \( s \) | \( t \) | \( v \) | \( w \) | \( x \) |
|---|---|---|---|---|---|---|---|---|
| \( o \) | \( o \) | \( o \) | \( o \) | \( o \) | \( o \) | \( o \) | \( o \) | \( o \) |
| \( p \) | \( p \) | \( o \) | \( o \) | \( p \) | \( s \) | \( t \) | \( v \) | \( w \) |
| \( r \) | \( r \) | \( r \) | \( o \) | \( p \) | \( r \) | \( t \) | \( w \) | \( r \) |
| \( s \) | \( s \) | \( t \) | \( v \) | \( w \) | \( o \) | \( p \) | \( r \) | \( s \) |
| \( t \) | \( t \) | \( w \) | \( v \) | \( x \) | \( o \) | \( p \) | \( r \) | \( s \) |
| \( v \) | \( v \) | \( w \) | \( x \) | \( p \) | \( r \) | \( s \) | \( t \) | \( v \) |
| \( w \) | \( w \) | \( x \) | \( o \) | \( p \) | \( r \) | \( s \) | \( t \) | \( v \) |
| \( x \) | \( x \) | \( o \) | \( p \) | \( r \) | \( s \) | \( t \) | \( v \) | \( w \) |

\begin{align*}
\{o\}_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(o) = \alpha_4\} = \{o, w\}, \\
\{p\}_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(p) = \alpha_2\} \\
&= \{p, s\} \\
&= \{s\}_{\varphi_1}, \\
\{r\}_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(r) = \alpha_1\} \\
&= \{r, t\} \\
&= \{t\}_{\varphi_1}, \\
\{v\}_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(v) = \alpha_3\} \\
&= \{v, x\} \\
&= \{x\}_{\varphi_1}.
\end{align*}

Hence we have that \( \xi_{\varphi_1} = \{\{o\}_{\varphi_1}, \{r\}_{\varphi_1}, \{v\}_{\varphi_1}, \{w\}_{\varphi_1}\} \).

| \( + \) | \( r \) | \( t \) | \( w \) |
|---|---|---|---|
| \( r \) | \( t \) | \( w \) | \( o \) |
| \( t \) | \( w \) | \( o \) | \( r \) |
| \( w \) | \( o \) | \( r \) | \( t \) |

Table 5.

| \( \cdot \) | \( r \) | \( t \) | \( w \) |
|---|---|---|---|
| \( r \) | \( t \) | \( o \) |
| \( t \) | \( o \) | \( o \) | \( o \) |

Table 6.
Proposition 1. Let $x, y$ then for all "

Definition 4. Let $R$ be a near group on NAS with binary operation "•",

Theorem 6. For all $x, y, z$ hold, $R$ is a near ring on NAS.

Proposition 1. Let $R$ be a nearness ring on NAS and $0 \in R$. If $0 \cdot x, x \cdot 0 \in R$, then for all $x, y \in R$

(i) $x \cdot 0 = 0 \cdot x = 0$,

(ii) $x \cdot (-y) = (-x) \cdot y = -(x \cdot y)$,

(iii) $(-x) \cdot (-y) = x \cdot y$.

Definition 3. Let $R$ be a nearness ring on NAS and $S$ a nonempty subset of $R$. $S$ is called subnearness ring of $R$, if $S$ is a nearness ring with binary operations “+” and “•” on nearness ring $R$.

Definition 4. Let we consider nearness field $R$ and a nonempty subset $S$ of $R$. $S$ is called subnearness field of $R$, if $S$ is a nearness field.

Theorem 6. Let $R$ be a nearness ring on NAS and $(N_r (B)^* S, +)$, $(N_r (B)^* S, \cdot)$ groupoids. A nonempty subset $S$ of nearness ring $R$ is a subnearness ring of $R$ if $-x \in S$ for all $x \in S$.

Proof. Suppose that $S$ is a subnearness ring of $R$. Then $S$ is a nearness ring and $-x \in S$ for all $x \in S$. Conversely, suppose $-x \in S$ for all $x \in S$. Then since $(N_r (B)^* S, +)$ is a groupoid, from Theorem 3 $(S, +)$ is a commutative near group on NAS. By the hypothesis, since $(N_r (B)^* S, \cdot)$ is a groupoid and $S \subseteq R$, then associative property holds in $N_r (B)^* S$. Hence $(S, \cdot)$ is a near semigroup on NAS. For all $x, y, z \in S \subseteq R$, $y + z \in N_r (B)^* S$ and $x \cdot (y + z) \in N_r (B)^* S$. Also $x \cdot y + x \cdot z \in N_r (B)^* S$. Since $R$ is a nearness ring, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ property holds in $N_r (B)^* S$. Similarly we can show that $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ property holds in $N_r (B)^* S$. Therefore $S$ is a subnearness ring of nearness ring $R$.

Example 2. From Example 4 let we consider the nearness ring $R = \{r, t, w\}$ on NAS. Let $S = \{r, w\}$ be a subset of nearness ring $R$. Then, “+” and “•” are binary operations of perceptual objects on $S \subseteq R$ as in Tables 7 and 8.
We know from Example 7 for \( r = 1 \), a classification of \( O \) is \( N_1(B) = \{\xi_{(\varphi_1)}, \xi_{(\varphi_2)}\} \).

Then, we can obtain \( N_1(B)^* S = \{o, r, t, w\} \).

Hence we can observe that \((N_r(B)^* S, +), (N_r(B)^* S, \cdot)\) are groupoids and \(-r = w, -w = r \in N_r(B)^* S\). Therefore from Theorem 6 \( S \) is a subnearness ring of nearness ring \( R \).

**Theorem 7.** Let \( R \) be a nearness ring on NAS, \( S_1 \) and \( S_2 \) two subnearness rings of \( R \) and \( N_r(B)^* S_1, N_r(B)^* S_2 \) groupoids with the binary operations \( + \) and \( \cdot \). If

\[
(N_r(B)^* S_1) \cap (N_r(B)^* S_2) = N_r(B)^* (S_1 \cap S_2),
\]

then \( S_1 \cap S_2 \) is a subnearness ring of \( R \).

**Corollary 1.** Let \( R \) be a nearness ring on NAS, \( \{S_i : i \in \Delta\} \) a nonempty family of subnearness rings of \( R \) and \( N_r(B)^* S_i \) groupoids. If

\[
\bigcap_{i \in \Delta} (N_r(B)^* S_i) = N_r(B)^* \left( \bigcap_{i \in \Delta} S_i \right),
\]

then \( \bigcap_{i \in \Delta} S_i \) is a subnearness ring of \( R \).

**Definition 5.** Let \( R \) be a nearness ring on NAS and \( I \) be a nonempty subset of \( R \). \( I \) is a left (right) nearness ideal of \( R \) provided for all \( x, y \in I \) and for all \( r \in R \), \( x - y \in N_r(B)^* I, r \cdot x \in N_r(B)^* I \) (\( x - y \in N_r(B)^* I, x \cdot r \in N_r(B)^* I \)).

A nonempty set \( I \) of a nearness ring \( R \) is called a nearness ideal of \( R \) if \( I \) is both a left and a right nearness ideal of \( R \).

There is only one guaranteed trivial nearness ideal of nearness ring \( R \), i.e., \( R \) itself. Furthermore, \( \{0\} \) is a trivial nearness ideal of nearness ring \( R \) iff \( 0 \in R \).

**Lemma 2.** Every nearness ideal is a subnearness ring of nearness ring \( R \).

**Example 3.** From Example 7 and 8, let us consider the nearness ring \( R = \{r, t, w\} \) on NAS and subnearness ring \( S = \{r, w\} \) of \( R \). We can observe that \( x - y \in N_r(B)^* S, r \cdot x \in N_r(B)^* S \) and \( x \cdot r \in N_r(B)^* S \) for all \( x, y \in S \) and for all \( r \in R \). Hence, from Definition 5 \( S \) is a nearness ideal of \( R \).

**Theorem 8.** Let \( R \) be a nearness ring on NAS, \( I_1 \) and \( I_2 \) two nearness ideals of \( R \) and \( N_r(B)^* I_1, N_r(B)^* I_2 \) groupoids with the binary operations \( + \) and \( \cdot \). If

\[
(N_r(B)^* I_1) \cap (N_r(B)^* I_2) = N_r(B)^* (I_1 \cap I_2),
\]

then \( I_1 \cap I_2 \) is a nearness ideal of \( R \).

**Proof.** Suppose \( I_1 \) and \( I_2 \) be two nearness ideals of the nearness ring \( R \). It is obvious that \( I_1 \cap I_2 \subset R \). Consider \( x, y \in I_1 \cap I_2 \). Since \( I_1 \) and \( I_2 \) are nearness ideals, we have \( x - y, r \cdot x \in N_r(B)^* I_1 \) and \( x - y, r \cdot x \in N_r(B)^* I_2 \), i.e., \( x - y, r \cdot x \in (N_r(B)^* I_1) \cap (N_r(B)^* I_2) \) for all \( x, y \in I_1, I_2 \) and \( r \in R \). Assuming \((N_r(B)^* I_1) \cap (N_r(B)^* I_2) = N_r(B)^* (I_1 \cap I_2)\), we have \( x - y, r \cdot x \in N_r(B)^* (I_1 \cap I_2) \). From Definition 5 \( I_1 \cap I_2 \) is a nearness ideal of \( R \). 

\[\square\]
Corollary 2. Let $R$ be a nearness ring on $\text{NAS}$, $\{I_i : i \in \Delta\}$ a nonempty family of nearness ideals of $R$ and $N_r(B)^* I_i$ groupoids with the binary operations “+” and “.”. If

$$\bigcap_{i \in \Delta} (N_r(B)^* I_i) = N_r(B)^* \left( \bigcap_{i \in \Delta} I_i \right),$$

then $\bigcap_{i \in \Delta} I_i$ is a nearness ideal of $R$.

Let $R$ be a nearness ring and $S$ a subnearness ring of $R$. The left weak equivalence relation (compatible relation) “$\sim_L$” defined as

$$x \sim_L y :\iff -x + y \in S \cup \{e\}.$$ 

A weak class defined by relation “$\sim_L$” is called left weak coset. The left weak coset that contains the element $x \in R$ is denoted by $\tilde{x}_L$, i.e.,

$$\tilde{x}_L = \{x + s \mid s \in S, \ x \in R, \ x + s \in R\} \cup \{x\}.$$ 

Similarly we can define the right weak coset that contains the element $x \in R$ is denoted by $\tilde{x}_R$, i.e.,

$$\tilde{x}_R = \{s + x \mid s \in S, \ x \in R, \ s + x \in R\} \cup \{x\}.$$ 

We can easily show that $\tilde{x}_L = x + S$ and $\tilde{x}_R = S + x$. Since $(R, +)$ is a abelian near group on $\text{NAS}$, $\tilde{x}_L = \tilde{x}_R$ and so we use only notation $\tilde{x}$. Then

$$R/\sim = \{x + S \mid x \in R\}$$

is a set of all weak cosets of $R$ by $S$. In this case, if we consider $N_r(B)^* R$ instead of nearness ring $R$

$$(N_r(B)^* R) /\sim = \{x + S \mid x \in N_r(B)^* R\}.$$ 

Definition 6. Let $R$ be a nearness ring and $S$ be a subnearness ring of $R$. For $x, y \in R$, let $x + S$ and $y + S$ be two weak cosets that determined the elements $x$ and $y$, respectively. Then sum of two weak cosets that determined by $x + y \in N_r(B)^* R$ can be defined as

$$(x + y) + S = \{(x + y) + s \mid s \in S, \ x + y \in N_r(B)^* R, \ (x + y) + s \in R\} \cup \{x + y\}$$

and denoted by

$$(x + S) \oplus (y + S) = (x + y) + S.$$ 

Definition 7. Let $R$ be a nearness ring and $S$ be a subnearness ring of $R$. For $x, y \in R$, let $x + S$ and $y + S$ be two weak cosets that determined the elements $x$ and $y$, respectively. Then product of two weak cosets that determined by $x \cdot y \in N_r(B)^* R$ can be defined as

$$(x \cdot y) + S = \{(x \cdot y) + s \mid s \in S, \ x \cdot y \in N_r(B)^* R, \ (x \cdot y) + s \in R\} \cup \{x \cdot y\}$$

and denoted by

$$(x + S) \odot (y + S) = (x \cdot y) + S.$$ 

Definition 8. Let $R/\sim$ be a set of all weak cosets of $R$ by $S$, $\xi\Phi(A)$ a descriptive nearness collections and $A \in \mathcal{P}(O)$. Then

$$N_r(B)^* (R/\sim) = \bigcup_{\xi\Phi(A) \cap R/\sim \neq \emptyset} \xi\Phi(A)$$

is called upper approximation of $R/\sim$. 

Theorem 9. Let $R$ be a nearness ring, $S$ a subnearness ring of $R$ and $R/\sim$ be a set of all weak cosets of $R$ by $S$. If $\{N_r (B)^* R\} / \sim \subseteq N_r (B)^* R / \sim$, then $R / \sim$ is a nearness ring under the operations given by $(x + S) \oplus (y + S) = (x + y) + S$ and $(x + S) \circ (y + S) = (x \cdot y) + S$ for all $x, y \in R$.

Proof. (NR1) Let $\{N_r (B)^* R\} / \sim \subseteq N_r (B)^* R / \sim$. Since $R$ is a nearness ring from Theorem 5 $\{R / \sim, \oplus\}$ is a abelian near group of all weak cosets of $R$ by $S$.

(NR2) Since $(R, \cdot, +, \cdot)$ is a near semigroup;

\begin{align*}
\{N_r (B)^* R\} / \sim & \subseteq \{N_r (B)^* R / \sim\}.
\end{align*}

From the hypothesis, $(x + S) \circ (y + S) = (x \cdot y) + S \in \{N_r (B)^* R\} / \sim$ for all $(x + S), (y + S) \in R / \sim$. From the hypothesis, $(x + S) + (y + S) = (x + y) + S \in \{N_r (B)^* R\} / \sim$ for all $(x + S), (y + S) \in R / \sim$.

(NR3) Since $R$ is a nearness ring, left distributive law holds in $\{N_r (B)^* R\} / \sim$. For all $(x + S), (y + S), (z + S) \in R / \sim$,

\begin{align*}
(x + S) \circ ((y + S) \oplus (z + S)) &= (x + S) \circ ((y + z) + S) \\
&= ((x + y) + S) \oplus ((x + z) + S) \\
&= ((x + y) + S) \oplus (x + S \circ (y + S)) = (x + S) \circ (y + S) \oplus (z + S).
\end{align*}

Hence left distributive law holds in $\{N_r (B)^* R\} / \sim$. Similarly we can show that right distributive law holds in $\{N_r (B)^* R\} / \sim$.

\begin{align*}
(x + S) \oplus (y + S) \oplus (z + S) &= ((x + S) \circ (z + S)) \oplus ((x + S) \circ (y + S)) \oplus ((x + S) \circ (z + S))
\end{align*}

for all $(x + S), (y + S), (z + S) \in R / \sim$.

Thus we have that $R / \sim S = \{r + S, t + S, w + S\}$.

Example 4. Let $S = \{r, w\}$ be a subset of $R = \{r, t, w\}$. From Example 2 $S$ is a subnearness ring of nearness ring $R$.

Now, we can compute the all weak cosets of $R$ by $S$. By using the definition of weak coset,

\begin{align*}
r + S &= \{r\} \cup \{r\} = \{r\} \\
t + S &= \{w, r\} \cup \{t\} = \{w, r, t\} \\
w + S &= \{t\} \cup \{w\} = \{t, w\}.
\end{align*}

Thus we have that $R / _w S = \{r + S, t + S, w + S\}$.

Since $N_1 (B)^* R = \{o, r, t, w\}$, we can write the all weak cosets of $N_1 (B)^* R$ by $S$.

In this case

\begin{align*}
o + S &= \{r, w\} \cup \{o\} = \{r, w, o\}.
\end{align*}
Then \((N_1(B)^* R) / \sim = \{o + S, r + S, t + S, w + S\} \subset \mathcal{P}(O)\).

Let “\(\oplus\)” and “\(\ominus\)” be operations on \(R/wS\), by using the Definition 8 and 9 as in Tables 9 and 10.

| \(\oplus\) | \(r + S\) | \(t + S\) | \(w + S\) |
|---|---|---|---|
| \(r + S\) | \(t + S\) | \(w + S\) | \(o + S\) |
| \(t + S\) | \(w + S\) | \(o + S\) | \(r + S\) |
| \(w + S\) | \(r + S\) | \(t + S\) |

| \(\ominus\) | \(r + S\) | \(t + S\) | \(w + S\) |
|---|---|---|---|
| \(r + S\) | \(t + S\) | \(o + S\) | \(t + S\) |
| \(t + S\) | \(o + S\) | \(o + S\) | \(o + S\) |
| \(w + S\) | \(t + S\) | \(o + S\) | \(t + S\) |

**Table 9.**

It is enough to show that every element of \((N_1(B)^* R) / \sim\) is also an element of \(N_1(B)^* (R/wS)\) in order to ensure \((N_r(B)^* R) / \sim \subseteq N_r(B)^* (R/wS)\).

\[
\mathcal{Q}(R/wS) = \{\Phi(A) \mid A \in R/wS\} = \{\Phi(r + S), \Phi(t + S), \Phi(w + S)\}
\]

Then \(\{\Phi(r), \Phi(w), \Phi(o)\} = \{(\alpha_1, \beta_2), (\alpha_4, \beta_1), (\alpha_4, \beta_1)\}\).

Since \(\mathcal{Q}(r + S) \cap \mathcal{Q}(o + S) = \{(\alpha_1, \beta_2)\} \neq \emptyset\), it follows that \(o + S \in \mathcal{E}_\Phi(r + S)\).

Hence \(\mathcal{Q}(r + S) \cap R/wS \neq \emptyset\) and \(r + S, o + S \in N_1(B)^* (R/wS)\) by Definition 8.

For \(r + S \in R/wS\), \(w + S\) we get that

\[
\mathcal{Q}(r + S) = \{\Phi(r)\} = \{(\alpha_1, \beta_2)\},
\]

\[
\mathcal{Q}(o + S) = \{\Phi(o)\} = \{(\alpha_1, \beta_2), (\alpha_4, \beta_1), (\alpha_4, \beta_1)\}.
\]

Thus, the Theorem 22 \(R/wS\) is a nearness ring of all weak cosets of \(R\) by \(S\) with the operations given by Tables 9 and 10.

**Definition 10.** Let \(R_1, R_2 \subset O\) be two nearness rings and \(\eta\) a mapping from \(N_r(B)^* R_1\) onto \(N_r(B)^* R_2\). If \(\eta(x + y) = \eta(x) + \eta(y)\) and \(\eta(x \cdot y) = \eta(x) \cdot \eta(y)\) for all \(x, y \in R_1\), then \(\eta\) is called a nearness ring homomorphism and also, \(R_1\) is called near homomorphic to \(R_2\), denoted by \(R_1 \simeq_n R_2\).

A nearness ring homomorphism \(\eta\) of \(N_r(B)^* R_1\) into \(N_r(B)^* R_2\) is called

(i) a nearness monomorphism if \(\eta\) is one-one,

(ii) a nearness epimorphism if \(\eta\) is onto \(N_r(B)^* R_2\) and

(iii) a nearness isomorphism if \(\eta\) is one-one and maps \(N_r(B)^* R_1\) onto \(N_r(B)^* R_2\).

**Theorem 10.** Let \(R_1, R_2\) be two nearness rings and \(\eta\) a nearness homomorphism of \(N_r(B)^* R_1\) into \(N_r(B)^* R_2\). Then the following properties hold.

(i) \(\eta(0_{R_1}) = 0_{R_2}\), where \(0_{R_2} \in N_r(B)^* R_2\) is the nearness zero of \(R_2\).
Proof. (i) Since $\eta$ is a nearness homomorphism, $\eta(0_{R_1}) \cdot \eta(0_{R_1}) = \eta(0_{R_1} \cdot 0_{R_1}) = \eta(0_{R_1})$. Thus we have that $\eta(0_{R_1}) = 0_{R_2}$ by Theorem 2(iii).

(ii) Let $x \in R_1$. Then $\eta(x) \cdot \eta(-x) = \eta(x - x) = \eta(0_{R_1}) = 0_{R_2}$. Similarly we can obtain that $\eta(-x) \cdot \eta(x) = 0_{R_2}$ for all $x \in R_1$. From Theorem 2(ii), since $\eta(x)$ has a unique inverse, $\eta(-x) = -\eta(x)$ for all $x \in R_1$. □

Theorem 11. Let $R_1, R_2$ be two nearness rings and $\eta$ a nearness homomorphism of $N_r(B)^+ R_1$ into $N_r(B)^+ R_2$ and $N_r(B)^+ S$ a groupoid. Then the following properties hold.

(i) If $S$ is a subnearness ring of nearness ring $R_1$ and $\eta(N_r(B)^+ S) = N_r(B)^+ \eta(S)$, then $\eta(S) = \langle \eta(x) : x \in S \rangle$ is a subnearness ring of $R_2$.

(ii) If $S$ is a commutative subnearness ring $R_1$ and $\eta(N_r(B)^+ S) = N_r(B)^+ \eta(S)$, then $\eta(S)$ is a commutative subnearness ring of $R_2$.

Proof. (i) Let $S$ be a subnearness ring of nearness ring $R_1$. Then $0_S \in N_r(B)^+ S$ and by Theorem 10(i), $\eta(0_S) = 0_{R_2}$, where $0_{R_2} \in N_r(B)^+ R_2$. Thus, $0_{R_2} = \eta(0_S) \in \eta(N_r(B)^+ S) = N_r(B)^+ \eta(S)$. This means that $\eta(S) \neq \emptyset$. Let $\eta(x) \notin \eta(S)$, where $x \in S$. Since $S$ is a subnearness ring of $R_1$, $-x \in N_r(B)^+ S$ for all $x \in S$. Thus $-\eta(x) = \eta(-x) \in \eta(N_r(B)^+ S) = N_r(B)^+ \eta(S)$ for all $\eta(x) \in \eta(S)$. Hence by Theorem 1(i), $\eta(S)$ is subnearness ring of $R_2$.

(ii) Let $S$ be a commutative subnearness ring and $\eta(x), \eta(y) \in \eta(S)$. We have that $\eta(S)$ is a subnearness ring of $R_2$ by (i), i.e., $\eta(S)$ is a nearness ring. Then $\eta(x) \cdot \eta(y) = \eta(x \cdot y) = \eta(y \cdot x) = \eta(x) \cdot \eta(y)$ for all $\eta(x), \eta(y) \in \eta(R_1)$. Hence $\eta(S)$ is commutative subnearness ring of $R_2$. □

Definition 11. Let $R_1, R_2$ be two nearness rings and $\eta$ be a nearness homomorphism of $N_r(B)^+ R_1$ into $N_r(B)^+ R_2$. The kernel of $\eta$, denoted by $\text{Ker}_\eta$, is defined to be the set

$$\text{Ker}_\eta = \{ x \in R_1 : \eta(x) = 0_{R_2} \}.$$ 

Theorem 12. Let $R_1, R_2$ be two nearness rings, $\eta$ a nearness homomorphism of $N_r(B)^+ R_1$ into $N_r(B)^+ R_2$ and $N_r(B)^+ \text{Ker}_\eta$ a groupoid with binary operations “+” and “.”. Then $\emptyset \neq \text{Ker}_\eta$ is a nearness ideal of $R_1$.

Proof. Let $x, y \in \text{Ker}_\eta$. Then $f(x - y) = f(x) - f(y) = 0_{R_2} - 0_{R_2} = 0_{R_2} \in N_r(B)^+ R_2$ and so $x - y \in N_r(B)^+ (\text{Ker}_\eta)$. Let $r \in R_1$. Then $f(r \cdot x) = f(r) \cdot f(x) = f(r) \cdot 0_{R_2} = 0_{R_2} \in N_r(B)^+ R_2$ and so $r \cdot x \in N_r(B)^+ (\text{Ker}_\eta)$. Similarly, $x \cdot r \in N_r(B)^+ (\text{Ker}_\eta)$. Hence, from Definition 14 $\text{Ker}_\eta$ is a nearness ideal of $R_1$. □

Theorem 13. Let $R$ be a nearness ring and $S$ a subnearness ring of $R$. Then the mapping $\Pi : N_r(B)^+ R \to N_r(B)^+ (R/\omega S)$ defined by $\Pi(x) = x + S$ for all $x \in N_r(B)^+ R$ is a nearness homomorphism.

Proof. From the definition of $\Pi$, $\Pi$ is a mapping from $N_r(B)^+ R$ into $N_r(B)^+ (R/\omega S)$. By using the Definition 7

$$\Pi(x + y) = (x + y) + S = (x + S) \oplus (y + S) = \Pi(x) \oplus \Pi(y),$$

$$\Pi(x \cdot y) = (x \cdot y) + S = (x + S) \odot (y + S) = \Pi(x) \odot \Pi(y)$$

for all $x, y \in R$. Thus $\Pi$ is a nearness homomorphism from Definition 10. □
Definition 12. The near homomorphism \( \Pi \) is called a nearness natural homomorphism from \( N_r (B)^* R \) into \( N_r (B)^* (R/wS) \).

Example 5. From Example 4, we consider the nearness ring of all weak cosets \( R/wS \). Define \( \Pi : N_r (B)^* R \to N_r (B)^* (R/wS) \)

\[
x \mapsto \Pi (x) = x + S
\]

for all \( x \in N_r (B)^* R \). By using the Definitions 6 and 7, we have that

\[
\Pi (x + y) = (x + y) + S = (x + S) \oplus (y + S) = \Pi (x) \oplus \Pi (y),
\]

\[
\Pi (x \cdot y) = (x \cdot y) + S = (x + S) \odot (y + S) = \Pi (x) \odot \Pi (y)
\]

for all \( x, y \in R \). Hence, \( \Pi \) is a nearness natural homomorphism from \( N_r (B)^* R \) into \( N_r (B)^* (R/wS) \).

Definition 13. Let \( R_1, R_2 \) be two nearness rings and \( S \) be a non-empty subset of \( R_1 \). Let \( \chi : N_r (B)^* R_1 \to N_r (B)^* R_2 \)

be a mapping and

\[
\chi_S = \chi \mid_S : S \to N_r (B)^* R_2
\]

a restricted mapping. If \( \chi(x + y) = \chi_S(x + y) = \chi_S(x) + \chi_S(y) = \chi(x) + \chi(y) \) and \( \chi(x \cdot y) = \chi_S(x \cdot y) = \chi_S(x) \cdot \chi_S(y) = \chi(x) \cdot \chi(y) \) for all \( x, y \in S \), then \( \chi \) is called a restricted nearness homomorphism and also, \( R_1 \) is called restrict nearness homomorphic to \( R_2 \), denoted by \( R_1 \simeq_{\text{rn}} R_2 \).

Theorem 14. Let \( R_1, R_2 \) be two nearness rings and \( \chi \) be a nearness homomorphism from \( N_r (B)^* R_1 \) into \( N_r (B)^* R_2 \). Let \( (N_r (B)^* \text{Ker} \chi, +) \) and \( (N_r (B)^* \text{Ker} \chi, -) \) be groupoids and \( (N_r (B)^* R_1)/\sim \) be a set of all weak cosets of \( N_r (B)^* R_1 \) by \( \text{Ker} \chi \). If \( (N_r (B)^* R_1)/\sim \subseteq N_r (B)^* (R_1/wKer \chi) \) and \( N_r (B)^* \chi (R_1) = \chi (N_r (B)^* R_1) \), then

\[
R_1/wKer \chi \simeq_{\text{rn}} \chi (R_1).
\]

Proof. Since \( (N_r (B)^* \text{Ker} \chi, +) \) and \( (N_r (B)^* \text{Ker} \chi, -) \) are groupoids, from Theorem 12 \( \text{Ker} \chi \) is a subnearness ring of \( R_1 \). Since \( \text{Ker} \chi \) is a subnearness ring of \( R_1 \) and \( (N_r (B)^* R_1)/\sim \subseteq N_r (B)^* (R_1/wKer \chi) \), then \( R_1/wKer \chi \) is a nearness ring of all weak cosets of \( R_1 \) by \( \text{Ker} \chi \) from Theorem 9. Since \( N_r (B)^* \chi (R_1) = \chi (N_r (B)^* R_1) \), \( \chi (R_1) \) is a subnearness ring of \( R_2 \). Define

\[
\eta : N_r (B)^* (R_1/wKer \chi) \to N_r (B)^* \chi (R_1)
\]

\[
A \mapsto \eta (A) = \begin{cases} \eta_{R_1/wKer \chi} (A) & , A \in (N_r (B)^* R_1)/\sim \\ \epsilon_{\chi (R_1)} & , A \notin (N_r (B)^* R_1)/\sim \end{cases}
\]

where

\[
\eta_{R_1/wKer \chi} : N_r (B)^* (R_1/wKer \chi) \to N_r (B)^* \chi (R_1)
\]

\[
x + Ker \chi \mapsto \eta_{R_1/wKer \chi} (x + Ker \chi) = \chi (x)
\]

for all \( x + Ker \chi \in R_1/wKer \chi \).

Since

\[
x + Ker \chi = \{ x + k \mid k \in Ker \chi, x + k \in R_1 \} \cup \{ x \},
\]

\[
y + Ker \chi = \{ y + k' \mid k' \in Ker \chi, y + k' \in R_1 \} \cup \{ y \},
\]
and the mapping $\chi$ is a nearness homomorphism,

$$x + \text{Ker} \chi = y + \text{Ker} \chi$$

$$\Rightarrow x \in y + \text{Ker} \chi$$

$$\Rightarrow x \in \{ y + k' \mid k' \in \text{Ker} \chi, y + k' \in R_1 \} \text{ or } x \notin \{y\}$$

$$\Rightarrow x = y + k', k' \in \text{Ker} \chi, y + k' \in R_1 \text{ or } x = y$$

$$\Rightarrow y + x = (-y + y) + k', k' \in \text{Ker} \chi \text{ or } \chi(x) = \chi(y)$$

$$\Rightarrow y + x = k', k' \in \text{Ker} \chi$$

$$\Rightarrow y + x \in \text{Ker} \chi$$

$$\Rightarrow \chi(-y + x) = e_{\chi(R_1)}$$

$$\Rightarrow \chi(-y) + \chi(x) = e_{\chi(R_1)}$$

$$\Rightarrow -\chi(y) + \chi(x) = e_{\chi(R_1)}$$

$$\Rightarrow \chi(x) = \chi(y)$$

$$\Rightarrow \eta_{R_1/\text{Ker} \chi}(x + \text{Ker} \chi) = \eta_{R_1/\text{Ker} \chi}(y + \text{Ker} \chi)$$

Therefore $\eta_{R_1/\text{Ker} \chi}$ is well defined.

For $A, B \in N_r(B)^* (R_1/\text{Ker} \chi)$, we suppose that $A = B$. Since the mapping $\eta_{R_1/\text{Ker} \chi}$ is well defined,

$$\eta(A) = \begin{cases} 
\eta_{R_1/\text{Ker} \chi}(A), & A \in (N_r(B)^* R_1) / \sim, \\
e_{\chi(R_1)}, & A \notin (N_r(B)^* R_1) / \sim
\end{cases}$$

$$= \begin{cases} 
\eta_{R_1/\text{Ker} \chi}(B), & B \in (N_r(B)^* R_1) / \sim, \\
e_{\chi(R_1)}, & B \notin (N_r(B)^* R_1) / \sim
\end{cases}$$

$$= \eta(B).$$

Consequently $\eta$ is well defined.

For all $x + \text{Ker} \chi, y + \text{Ker} \chi \in R_1/\text{Ker} \chi \subset N_r(B)^*(R_1/\text{Ker} \chi)$,

$$\eta((x + \text{Ker} \chi) \oplus (y + \text{Ker} \chi))$$

$$= \eta_{R_1/\text{Ker} \chi}((x + \text{Ker} \chi) \oplus (y + \text{Ker} \chi))$$

$$= \eta_{R_1/\text{Ker} \chi}((x + y) + \text{Ker} \chi)$$

$$= \chi(x + y)$$

$$= \chi(x) + \chi(y)$$

$$= \eta_{R_1/\text{Ker} \chi}(x + \text{Ker} \chi) + \eta_{R_1/\text{Ker} \chi}(y + \text{Ker} \chi)$$

$$= \eta(x + \text{Ker} \chi) + \eta(y + \text{Ker} \chi).$$

and

$$\eta((x + \text{Ker} \chi) \circ (y + \text{Ker} \chi))$$

$$= \eta_{R_1/\text{Ker} \chi}((x + \text{Ker} \chi) \circ (y + \text{Ker} \chi))$$

$$= \chi_{R_1/\text{Ker} \chi}((x \cdot y) + \text{Ker} \chi)$$

$$= \chi(x \cdot y)$$

$$= \chi(x) \cdot \chi(y)$$

$$= \eta_{R_1/\text{Ker} \chi}(x + \text{Ker} \chi) \cdot \eta_{R_1/\text{Ker} \chi}(y + \text{Ker} \chi)$$

$$= \eta(x + \text{Ker} \chi) \cdot \eta(y + \text{Ker} \chi).$$

Therefore $\eta$ is a restricted nearness homomorphism by Definition. Hence, $R_{1/\text{Ker} \chi} \simeq_{\text{rn}} \chi(R_1)$. □
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