Less is More: Born’s Rule from Quantum Frequentism

Lionel Brits
lbrits@mun.ca
(Dated: November 26, 2019)

Accounting for Born’s rule from first principles remains an open problem among the foundational issues in quantum mechanics. Proponents of the many-worlds interpretation have argued that Born’s rule is observed simply because those histories that violate the rule have vanishing norms, and so must be unphysical. This argument has only been made explicit for contrived situations involving measurements on infinitely many identically-prepared systems. We prove a more general result, namely that for systems containing infinitely many degrees of freedom in arbitrarily-prepared states the universal wavefunction contains no histories that violate Born’s rule.

I. INTRODUCTION

Despite the predictive success of the Copenhagen formulation of quantum mechanics it cannot be considered a complete description of nature as it relies on a distinguished observer in order to make sense of the measurement process. This can be seen either as an inconsistency – that physical interactions are unitary or non-unitary depending on whether they are observed [1] – or merely a nuisance – that only a single observer is actually necessary to bootstrap the measurement process, one that can be pushed to the very edges of a system and then be forgotten [2]. This issue has, in part, lead to the development of the many worlds interpretation (MWI) [3, 4], which aims to explain the role of the observer within the unitary framework of the theory itself. In this interpretation, the apparent collapse of the wavefunction is understood as a loss of coherence between environmental states corresponding to different measurement outcomes, so that local degrees of freedom seem to evolve irreversibly. However, the MWI suffers from its own minimalism – having thrown out everything that is discontinuous and non-unitary, it has yet to give a wholly satisfactory explanation for the origin of probability, and in particular, Born’s rule. This paper aims to shed light on the problem, while keeping as much as possible of the spirit of the MWI intact.

A. The Problem with (Too) Many Worlds

For definiteness, we will summarize the key ideas behind the MWI [3, 4]. According to this interpretation, the state of any isolated system is at all times described by a state vector $|\Psi\rangle$ evolving unitarily according to the Schrödinger equation. In particular, the MWI deduces that if the universe is an isolated system, then it too must be described in this way. The key strength of the MWI is that it explains quite clearly how the stochastic nature of quantum mechanics comes about. Suppose that we divide our universe into a microscopic system $S$ being measured and an apparatus $A$ that is performing the measurement (the observer may be considered as part of the apparatus subsystem). Then a factorizable state $|S_0\rangle \otimes |A_0\rangle$ of the composite system will evolve into a superposition $\sum_i c_i |S_i\rangle \otimes |A_i\rangle$. One may say that the universe in which the system and apparatus was in state $|S_0\rangle \otimes |A_0\rangle$ has branched into a (possibly dense) set of universes each in the definite state $|S_i\rangle \otimes |A_i\rangle$. Within each universe, the observer sees a different measurement outcome (represented by $A_i$), despite having identical initial conditions, and since any particular observer has no way of knowing which branch they will find themselves in, their measurement outcome is completely unpredictable. We clarify that the universes described so far are merely arbitrary orthogonal decompositions of the universal wavefunction, and it is the task of the decoherence program to find among these decompositions emergent classical realities [5, 6], which we will not do here.

Despite the appeal of dealing with the observer as an integral part of the system, the MWI has been criticized for failing to account for the appearance of Born’s rule in the measurement process. Since linearity puts every universe on an equal footing, it would suggest that they are all equally likely, including so-called maverick worlds in which Born’s rule is grossly violated. This is of course in contrast to what we actually observe, i.e., that likelihoods are proportional to the absolute squares of the coefficients $c_i$.

To see how this comes about, let us construct a minimal model of measurement in which Born’s rule can be verified in a completely transparent way. We imagine a number of spin-$\frac{1}{2}$ particles are prepared in identical states equal to $|\psi\rangle = \alpha |\downarrow\rangle + \beta |\uparrow\rangle = \alpha |1\rangle + \beta |0\rangle$ after which the number of particles in one of the states, $|\downarrow\rangle$ say, is recorded by a particle counter. This may be represented by a quantum circuit that counts the number of 1s in a set of identically prepared qubits (i.e., its Hamming weight) and records the result in a binary register consisting of some other previously initialized qubits. For the sake of clarity we will construct it from a sequence of unoptimized controlled-[ADD 1] gates. This gate increments the target register if the control qubit is set to 1, and leaves it unaltered otherwise. The full circuit is shown in Fig. 1. An implementation for $N = 3$ is also reproduced in Fig. 2.

From this simple model we see that the act of registering the frequency of certain events can be done
in a manifestly unitary way. The circuit therefore exemplifies a sort of robotic “Wigners’s friend” [1, 3], one in which no proposed mechanisms for collapse, such as consciousness, can hide. Consequently, at the end of the registration process the circuit remains in a superposition of states in which every possible value of \( n \) has been registered. Except in the case in which \(|\alpha| = |\beta|\), the majority of these states correspond to frequencies that violate Born’s rule, which is at odds with what we actually observe, i.e., that \( n/N \approx |\alpha|^2 \). At this point the role of the experimenter seems indispensable for obtaining the correct result, but this can only be true if he or she is to be endowed with some non-unitary quality [3]. While this possibility has yet to be ruled out experimentally, recent demonstrations of quantum superposition on both mesoscopic and macroscopic [7, 8] scales make this argument hard to accept. The alternative conclusion, originally put forth by Everett [3] and others, is that maverick worlds are nonphysical, since they are assumed to have vanishing norm. Perhaps the most promising result is that of Graham and DeWitt [4], and independently, Hartle [9], who showed that when measurements are performed on an ensemble of \( N \) identically prepared systems, histories that deviate from Born’s rule have zero norm in the limit that the \( N \) is taken to infinity. In the next section we will review this result before moving on to its generalization.

### B. Quantum Frequentism

Prior to measurement, our circuit may be considered to be in the state \( |\Psi_N\rangle = (|\alpha\rangle + |\beta\rangle)^N |0\rangle \), with the rightmost \(|0\rangle\) representing the state of the counter. After \( N \) measurements the system will be in the state

\[
|\Psi_N\rangle \rightarrow \sum_{n=1}^{N} \alpha^n \beta^{N-n} \sum_{h} |s_{n,h}\rangle |n\rangle,
\]

where \( \{|s_{n,h}\rangle\} \) is the set of subsequent \( N \)-particle states corresponding to the \( \binom{N}{n} \) initial states which contain \( n \) downward spins. Making use of the fact that the states \( |s_{n,h}\rangle \) are orthogonal, one finds,

\[
|\langle n|\Psi_N\rangle| = \sqrt{\alpha^{2n}\beta^{2(N-n)} \binom{N}{n}}.
\]

We recognize the term \( \alpha^{2n}\beta^{2(N-n)} \binom{N}{n} \) as the binomial distribution for the number of successes in a sequence of \( N \) independent Bernoulli trials, each with success probability \( p = |\alpha|^2 \). This function has relative width \( \frac{\mu}{\sqrt{N}} = \frac{p(1-p)}{N} \) and mean \( \mu = |\alpha|^2 N \), so that after a large number of measurements, the state of the system will be narrowly peaked around the expected frequency \( |\alpha|^2 \), decaying rapidly to zero elsewhere. Since the counter has so far only served a bookkeeping purpose, organizing the initial multi-particle state \( |\Psi_N\rangle = (|\alpha\rangle + |\beta\rangle)^N |0\rangle \) into a superposition of states with definite numbers of downward spins, we shall suppress the counter and rewrite the final state in terms of the computational basis, i.e.,

\[
|\Psi_N\rangle = \sum_{x_1,x_2,...,x_N} c_{x_1,x_2,...,x_N} |x_1\rangle |x_2\rangle \cdots |x_N\rangle,
\]

where \( x_i \in \{0,1\} \) and \( c_{x_1,x_2,...,x_N} = \alpha^n \beta^{N-n} \) with \( n = \sum x_i \). Each binary string \( (x_1, x_2, \ldots, x_N) \) and its associated counter value \( n \) then defines a world in which a particular set of outcomes will be measured. Note that as we let \( N \rightarrow \infty \), the vector space \( \mathcal{H} = \mathbb{C}^N \) spanned by the basis vectors \( |x_1\rangle |x_2\rangle \cdots \) becomes non-separable, so that care must be taken to maintain a well-behaved inner product structure [10]. Let us define a maverick world to be one in which the empirical error \( \bar{\epsilon} \) differs from the expected frequency \( E[\bar{\epsilon}] = |\alpha|^2 \) by some finite positive error \( \epsilon \), i.e., \( |\bar{\epsilon} - |\alpha|^2| > \epsilon \) (here and elsewhere expectation values will always mean those computed according to Born’s rule). We can then decompose \( |\Psi_N\rangle \) into the projection \( |\Psi_N\rangle_{M(\text{maverick})} \) containing all maverick worlds, as well as the projection \( |\Psi_N\rangle_{B(\text{born})} \) containing all regular, or Born worlds, so that
|Ψ_N⟩ = |Ψ_N⟩_B + |Ψ_N⟩_M. To find ∥|Ψ_N⟩_M∥, we would need to evaluate

\[ ∥|Ψ_N⟩_M∥^2 = \sum_{|\bar{x} - |a||^2 > \epsilon} |c_{x_1x_2...x_N}|^2. \] (4)

We can however place an upper bound on this quantity. Because \( n \) is the sum of independent random variables taking the values \{0, 1\} we use Hoeffding’s inequality \([11]\) to find

\[ ∥|Ψ_N⟩_M∥^2 \leq 2 e^{-2\epsilon^2 N}. \] (5)

It follows that \( \lim_{N \to \infty} ∥|Ψ_N⟩_M∥^2 = 0 \) for any finite \( \epsilon \), and therefore that \( |Ψ_\infty⟩ \) differs from \( |Ψ_\infty⟩_B \) by a quantity of zero norm. Using the Cauchy–Schwarz inequality,

\[ |⟨a|b⟩| \leq ⟨a|a⟩ ⟨b|b⟩, \quad \forall ⟨a⟩, ⟨b⟩ \in \mathcal{H}, \] (6)

we see that \( |Ψ_\infty⟩_M \) is orthogonal to (and decoupled from) all other states. Such vectors are not proper elements of the Hilbert space, and must be removed in order to maintain a positive definite inner product. We therefore identify those elements of \( \mathcal{H} \) that differ by elements of \( \mathcal{H}_0 \), the subspace of zero norm states. Our Hilbert space is then the quotient space denoted by \( \mathcal{H}/\mathcal{H}_0 \), so that we may consider \( |Ψ_\infty⟩ \) and \( |Ψ_\infty⟩_B \) to represent the same physical state, i.e., \( |Ψ_\infty⟩ = |Ψ_\infty⟩_B \). We conclude that in the \( N \to \infty \) limit the universal wavefunction contains only those worlds in which Born’s rule is observed.

Since the counter plays no role in this result, it strictly unnecessary that the spin of every particle be measured, only that infinitely many identically prepared particles be available to measure. However, a serious criticism of the frequentist program is that we do not perform measurements on systems containing infinitely many identically prepared particles \([12]\). In the finite case, all frequencies of events have non-zero amplitudes, and consequently non-zero probabilities of occurring, so that it seems hardly even possible to define maverick worlds in this case. Everett and others have ultimately argued that the only way to produce the correct Born probabilities in this case is to assign a probability measure on the universal Hilbert space, so that maverick worlds are never observed simply because one is very unlikely to find oneself in such a world. However, by getting rid of one postulate at the cost of gaining another, this apparent resolution stands at odds with the position of the wavefunction as a complete description of the system. If the theory is to be self consistent, then the structure of the wavefunction itself must account for the appearance of Born’s rule.

II. MAVERICK WORLDS IN THE THERMODYNAMIC LIMIT

A way out of this problem is to realize that any experiment performed on a finite multi-particle state such as \(|ψ⟩|ψ⟩ ... |ψ⟩ \) must take place inside a Hilbert space containing also all the particles in the environment. Aguirre and Tegmark \([13]\) have argued that in an infinite, statistically uniform cosmological model the laboratory state \(|ψ⟩ \) must be replicated infinitely many times throughout the universe, thereby realizing the “fictitious” infinite ensemble needed to derive Born’s rule. (More specifically, the authors of \([13]\) impose the stronger condition that both system plus experimenter be replicated infinitely many times, although this does not seem necessary.) However, this argument rests on some knowledge of the distribution of states that make up the universal wavefunction, and does not account for states that come arbitrarily close to, but never equal \(|ψ⟩ \). As we will show, the replica condition is actually unnecessary, so that we need only consider states of the form

\[ |Ω⟩ = ... |φ_{−2}⟩ |φ_{−1}⟩ (|ψ⟩ |ψ⟩ ... |ψ⟩ ) |φ_1⟩ |φ_2⟩ ..., \] (7)

where \( \{|φ_i⟩\} \) now represent arbitrary environmental component states. Since there is no real distinction between the system and the environment, it is convenient to treat all degrees of freedom on an equal footing by absorbing the multi-particle state \(|ψ⟩ |ψ⟩ ... |ψ⟩ \) into the environmental degrees of freedom, letting

\[ |Ω_N⟩ = |φ_1⟩ |φ_2⟩ ... |φ_N⟩. \] (8)

Let us now find a suitable definition of a maverick world in this case. In terms of the computational basis, this state may again be written as

\[ |Ω_N⟩ = \sum_{x_1x_2...x_N} c_{x_1x_2...x_N} |x_1⟩ |x_2⟩ ... |x_N⟩. \] (9)

As we have argued, there can be no condition placed on the outcome of any finite subset of measurement outcomes \( \{x_i\} \) (provided that \( p_i \) is neither 0 or 1). Instead, we will take an information theoretic approach. First, given a joint probability distribution \( p(x_1, x_2, ..., x_N) \) obtained from \( |Ω_N⟩ \), we define a typical sequence to be a sequence \( (x_i) \) such that

\[ \left| - \frac{1}{N} \log p(x_1, x_2, ..., x_N) - H(x) \right| \leq \epsilon, \] (10)

for some value \( \epsilon > 0 \), where

\[ H(x) = \frac{1}{N} E [- \log p(x_1, x_2, ..., x_N)], \] (11)

is the Shannon entropy rate of the distribution \( p(x_1, x_2, ..., x_N) \) \([14]\). This definition is motivated by the asymptotic equipartition property (AEP) \([14, 15]\), which states that

\[ \lim_{N \to \infty} \Pr \left[ \left| - \frac{1}{N} \log p(x_1, x_2, ..., x_N) - H(x) \right| > \epsilon \right] = 0. \] (12)

That is, as \( N \) becomes large, the empirical entropy rate of a sequence chosen at random from the distribution
$p(x_1, x_2, \ldots, x_N)$ tends towards $H(x)$, which follows directly from the weak law of large numbers applied to the quantity $-\log_2 p(x_1, x_2, \ldots, x_N)$. We can therefore partition the set of all possible sequences $(x_i)$ into two sets: a typical set, in which every element has probability $p(x_1, x_2, \ldots, x_N) \approx 2^{-N H(x)}$, and a non-typical set, containing all other sequences. The AEP tells us that, in the $N \to \infty$ limit, the probability of randomly selecting an element that belongs to the non-typical set is zero.

Having established the expected behaviour of the classical sequences $\{(x_i)\}$, we identify typical sequences with Born worlds and the remaining (non-typical) sequences with maverick worlds. To see that this identification agrees with our intuition, note that if we have $N$ repetitions of the state $\sum_k c_k |k\rangle$ then the inequality in equation 10 will contain the term

$$\sum_i \log_2 |c_i|^2 - N \sum_k |c_k|^2 \log_2 |c_k|^2,$$

which attains a minimum when $\log_2 |c_k|^2$ occurs roughly $N |c_k|^2$ times in the first sum, or equivalently, when there are roughly $N |c_k|^2$ particles in the state $|k\rangle$. It remains to be shown that the state $|\Omega_N\rangle = |\Omega_N\rangle_B + |\Omega_N\rangle_M$ contains no non-typical sequences in the $N \to \infty$ limit, i.e., that $|\Omega_N\rangle_M = 0$. But since $||\Omega_N\rangle_M||^2$ is precisely the quantity $\Pr\left\{ -\frac{1}{\log_2} \log_2 p(x_1, x_2, \ldots, x_N) - H(x) > \epsilon \right\}$, by the AEP, we may conclude that $\lim_{N \to \infty} ||\Omega_N\rangle_M||^2 = 0$ and that $|\Omega_N\rangle = |\Omega_N\rangle_B$ in general. The fact that we observe Born’s rule therefore stems from a rather surprising place: From a practical point of view, choosing a sequence $(x_1, x_2, \ldots)$ at random from the distribution $p(x_1, x_2, \ldots)$ is indistinguishable from choosing among the set of typical sequences with a uniform probability distribution. Since $|\Omega_N\rangle$ contains only such sequences, our observations must appear to be governed by the distribution $p(x_1, x_2, \ldots)$ as obtained from Born’s rule.

Although it seems that we needed to make a detour into the classical world to derive this result, we stress that the AEP and the weak law of large numbers from which it is derived are purely algebraic inequalities which may be applied to the quantities $|c_{x_1, x_2, \ldots, x_N}|^2$ without any mention of probabilities. Since our argument hinges on this point, it is worth doing so explicitly. We start by recalling an important inequality (see appendix):

**Theorem II.1 (Chebyshev’s Inequality)** Given an arbitrary state $|\Psi\rangle$ and a Hermitian operator $\hat{Y}$, let $\hat{P}_{|Y - \mu_Y| > \epsilon}$ be the projection operator that preserves states for which $|Y - \langle \Psi | \hat{Y} | \Psi \rangle| > \epsilon$ for some $\epsilon > 0$. Then

$$\left\| \hat{P}_{|Y - \mu_Y| > \epsilon} |\Psi\rangle \right\|^2 \leq \frac{\text{Var}(Y)}{\epsilon^2},$$

where $\text{Var}(Y) = \langle \Psi | (\hat{Y} - \langle \Psi | \hat{Y} | \Psi \rangle)^2 | \Psi \rangle$.

Phrased in this form, Chebyshev’s inequality is independent of any probabilistic interpretation, and may be used to derive the AEP by applying it to $|\Omega_N\rangle$, letting $Y(x_1, x_2, \ldots, x_N) = -\frac{1}{\log_2} \log_2 |c_{x_1, x_2, \ldots, x_N}|^2$. Then

$$\left\| \hat{P}_{|Y - \mu_Y| > \epsilon} |\Omega_N\rangle \right\|^2 \leq \frac{\text{Var}(\log |c_{x_1, x_2, \ldots, x_N}|^2)}{N^2 \epsilon^2}.$$  

(15)

Since $|\Omega_N\rangle = |\phi_1 \rangle |\phi_2 \rangle \ldots |\phi_N \rangle$ the quantity $|c_{x_1, x_2, \ldots, x_N}|^2$ can be factorized into the form $|c_1(x_1)|^2 \ldots |c_N(x_N)|^2$ so that

$$\left\| \hat{P}_{|Y - \mu_Y| > \epsilon} |\Omega_N\rangle \right\|^2 \leq \frac{\sum_i \text{Var}(\log |c_i(x_i)|^2)}{N^2 \epsilon^2}.$$  

(16)

Then, provided that $\text{Var}(\log |c_i(x_i)|^2) < M$ for all $i$, we may write

$$\left\| \hat{P}_{|Y - \mu_Y| > \epsilon} |\Omega_N\rangle \right\|^2 \leq \frac{M}{N \epsilon^2}.$$  

(17)

It follows then that $\lim_{N \to \infty} ||\Omega_N\rangle_M||^2 = 0$ and therefore that $|\Omega_N\rangle = |\Omega_N\rangle_B$ as claimed.

### III. CONCLUDING REMARKS

The importance of environmental degrees of freedom in obtaining Born’s rule was already recognized by Zurek [5] in the context of environmentally induced decoherence. However, in our result the environment plays a non-dynamical role, serving to define, and then get rid of, maverick worlds, a sort of Mach’s principle for quantum states. Therefore, while system and environmental degrees of freedom are a priori independent, they must be considered together as parts of a single quantum state. Our result shows that if one allows for systems with infinitely many degrees of freedom to exist, then Born’s rule arises from the theory quite automatically. However, we do not impose any ad hoc measure on the Hilbert space in order to achieve this result. Instead, we note that in order for the inner product to be positive definite, all vectors of zero norm must be identified with the zero vector. Thus the physical Hilbert space is not $\mathcal{H}$ but $\mathcal{H}/\mathcal{H}_0$, in which $|\Omega_\infty\rangle = |\Omega_\infty\rangle_B$ is identically zero.

It is also worth noting that the argument presented in this paper works strictly in the limit $N \to \infty$, and fails for any finite value of $N$, where maverick states are in the majority. Thus the statistical behaviour of the finite system does not approach that of the infinite system continuously. To fully account for Born’s rule, we must assume that our universe contains infinitely many degrees of freedom (rather than some arbitrarily large number, as is normally done). While this does not seem to be an overly objectionable assumption to us, we can turn this reasoning around and view Born’s rule instead as an experimental validation of this possibility. That is, the absence of maverick states supports the idea that the universe is an infinite system.
IV. ACKNOWLEDGMENTS

The author wishes to thank Conor Stokes for helpful discussion.

V. APPENDIX

Lemma V.1 (Markov’s Inequality) Given an arbitrary state \( |\Psi\rangle \) and a non-negative Hermitian operator \( \hat{f} \), let \( \hat{P}_{f>a} \) and \( \hat{P}_{f\leq a} \) be projection operators that separate those states for which \( f > a \) from those for which \( f \leq a \) for some \( a > 0 \). Then

\[
\left\| \hat{P}_{f>a} |\Psi\rangle \right\|^2 \leq \frac{1}{a} \langle \Psi | \hat{f} |\Psi\rangle. \tag{18}
\]

Proof Without loss of generality, we may consider a basis that diagonalizes \( \hat{f} \), i.e., let

\[
|\Psi\rangle = \sum_{x_1x_2...x_N} \Psi_{x_1x_2...x_N} |x_1\rangle |x_2\rangle ... |x_N\rangle, \tag{19}
\]

such that

\[
\hat{f} |x_1\rangle |x_2\rangle ... |x_N\rangle = f(x_1, x_2, ..., x_N) |x_1\rangle |x_2\rangle ... |x_N\rangle. \tag{20}
\]

Then

\[
\langle \Psi | \hat{f} |\Psi\rangle = \langle \Psi | \hat{P}_{f\leq a} \hat{f} |\Psi\rangle + \langle \Psi | \hat{P}_{f>a} \hat{f} |\Psi\rangle,
\]

\[
\geq \langle \Psi | \hat{P}_{f>a} \hat{f} |\Psi\rangle,
\]

\[
\geq a \langle \Psi | \hat{P}_{f>a} |\Psi\rangle.
\]

Since \( a > 0 \),

\[
\left\| \hat{P}_{f>a} |\Psi\rangle \right\|^2 \leq \frac{1}{a} \langle \Psi | \hat{f} |\Psi\rangle. \tag{21}
\]

Proof of theorem II.1 (Chebyshev’s Inequality)

The result follows directly by taking \( \hat{f} = (\hat{Y} - \mu_Y)^2 \) where \( \mu_Y = \langle \Psi | \hat{Y} |\Psi\rangle \) and \( a = \epsilon^2 \).