A new three-dimensional autonomous chaotic oscillation system

Chong-Xin Liu\textsuperscript{1}, Ling Liu \textsuperscript{2}

1. Institute of Electrical Engineering, Xi’an Jiaotong University, Xi’an 710049, P.R. China.
2. State Key Laboratory of Electrical Insulation and Power Equipment, Xi’an 710049, P.R. China

E-mail: liucx@mail.xjtu.edu.cn

Abstract. In this paper, a new chaotic system of three-dimensional quadratic autonomous differential equations is reported. Some basic dynamical properties, such as Lyapunov exponents, fractal dimension, Poincaré mapping, the continuous spectrum and chaotic behavior of this new transverse butterfly attractor are investigated. Furthermore, the complex chaotic dynamical behavior of the system has been also proofed by experimental simulation of a designed electronic oscillator circuit based on Electronics Work Bench 5.12 (EWB).

1. Introduction
In this letter a new chaotic system is proposed. It is a three-dimensional autonomous system which relies on one multiplier and one quadratic term to introduce the non-linearity necessary for folding trajectories. The chaotic attractor obtained from this new system according to the detailed numerical as well as theoretical analysis is also the butterfly-shaped attractor, but it is a transverse butterfly-shaped attractor, which exhibiting complex and abundant chaotic dynamics.

This chaotic system is a new similar to Lorenz chaotic attractor, but not equivalent chaotic attractor in the topological structure [1-6]. Meanwhile, its reverse structure system is also proposed. This new butterfly attractor is also a compound structure obtained by merging together two simple attractors after performing one mirror operation that is proved by detailed numerical and theoretical analysis [7-10]. In addition, the complex chaotic dynamical behavior of this system has been also proofed by experimental simulation of an electronic oscillator circuit to be designed based on EWB in this paper [11].

2. New Chaotic System
The new system is given by the following equations

\textsuperscript{1} To whom any correspondence should be addressed.
\[
\begin{align*}
\dot{x} &= a(z-x) + z^2 \\
\dot{y} &= by - hxz \\
\dot{z} &= ky - gz
\end{align*}
\]

(1)

Here \(a, b, g, k\) and \(h\) are all positive real constants with \(a = 0.5\), \(b = 2.5\), \(k = 1\), \(g = 4\), \(h = 1\), this nonlinear system (1) is chaotic. Apparently, it is also a three-dimensional autonomous system with one multiplier and one quadratic term that its topological structure is different from Lorenz system, Chen system and Lü system.

The system has three equilibria, which are respectively described as follows

\[O(0, 0, 0), E_1(x_1, y_1, z_1), E_2(x_2, y_2, z_2)\]

Here

\[
E_1 = \left( \frac{bg}{hk}, \frac{-a - \sqrt{a^2 + 4abg}}{hk}, \frac{-a + \sqrt{a^2 + 4abg}}{hk} \right),
\]

\[
E_2 = \left( \frac{bg}{hk}, \frac{-a + \sqrt{a^2 + 4abg}}{hk}, \frac{-a - \sqrt{a^2 + 4abg}}{hk} \right).
\]

We operate these nonlinear algebraic equations \((a(z-x) + z^2 = 0, by - hxz = 0, ky - gz = 0)\) and obtained \(O(0,0,0), E_1(10, -10, -2.5) E_2(10, 8, 2)\). For equilibrium \(O(0,0,0)\), the Jacobian matrix is defined as

\[
J_0 = \begin{bmatrix}
-a & 0 & a + 2z \\
-hz & b & -hx \\
0 & k & -g
\end{bmatrix} = \begin{bmatrix}
-0.5 & 0 & 0.5 \\
0 & 2.5 & 0 \\
0 & 1 & -4
\end{bmatrix}
\]

To gained its eigenvalues, we let

\[|\lambda I - J_0| = 0\]

These eigenvalues that corresponding to equilibrium \(O(0,0,0)\) are respectively obtained as follows

\[
\lambda_1 = -0.5, \quad \lambda_2 = -4, \quad \lambda_3 = 2.5
\]

Here \(\lambda_3\) is a positive real number, \(\lambda_1\) and \(\lambda_2\) are two negative real numbers. Therefore, the equilibrium \(O(0,0,0)\) is a saddle point; this equilibrium \(O(0,0,0)\) is unstable.

Next, linearizing the system (1) about the nonzero equilibria \(E_1\) and \(E_2\) yields the following characteristic operation.

For equilibrium point \(E_1\), has a Jacobian matrix equal to

\[
J_1 = \begin{bmatrix}
-a & 0 & a + 2z \\
-hz & b & -hx \\
0 & k & -g
\end{bmatrix} = \begin{bmatrix}
-0.5 & 0 & -4.5 \\
2.5 & 2.5 & -10 \\
0 & 1 & -4
\end{bmatrix}
\]

We let \[|\lambda I - J_1| = 0\] these eigenvalues corresponding to the equilibrium point \(E_1(x_1, y_1, z_1)\) are
\[ \lambda_1 = -3, \lambda_2 = 0.5 + j1.8708, \lambda_3 = 0.5 - j1.8708 \]

Here \( \lambda_1 \) is a negative real root, \( \lambda_2 \) and \( \lambda_3 \) become a pair of complex conjugate eigenvalues with positive real parts. So, the equilibrium point \( E_1 \) is saddle-focus point; apparently, this equilibrium point is unstable.

For equilibrium point \( E_2(x_2, y_2, z_2) \), eigenvalues are

\[ \lambda_1 = -2.847, \lambda_2 = 0.4235 + j1.7268, \lambda_3 = 0.4235 - j1.7268 \]

here \( \lambda_1 \) is a negative real root, \( \lambda_2 \) and \( \lambda_3 \) is pair of complex conjugate eigenvalues with positive real parts. The equilibrium point \( E_2 \) is also a saddle-focus point; this equilibrium point is similarly unstable.

The above brief analyses show that the three equilibria of this system (1) are all saddle focus-nodes.

The initial values of the system (1) are selected as \((0.04, 0.2, 0)\). Using MATLAB program, the numerical simulation have been completed. This nonlinear system exhibits the complex and abundant of the chaotic dynamics behaviors, the strange attractors are shown in Figure 1. Obviously, the strange attractors in this nonlinear-system are similar to Lorenz chaos attractors, but it is a transverse butterfly-shaped attractor to rightward; the waveforms of \( x(t) \) in time domain are shown in Figure 2; the waveforms of \( x(t) \) is non-periodic. Its spectrum is continuous as shown in Figure 3. Poincare mapping are these points of the confusion as shown in Figure 4.

\[ t \quad x \quad y \quad z \]

(a) Three-dimensional view          (b) \( x-z \) phase plane strange attractors

Figure 1. Strange attractors of the chaotic system (1)

Figure 2. Waveform of \( x(t) \)
Apparently, the evolution of the chaos trajectories is very sensitive to initial conditions in this nonlinear-system (1). If the appointed initial values be changed, the chaotic dynamical behavior disappears soon. For dynamical system (1), it is noticed that

\[
\nabla V = \frac{\partial \tilde{x}}{\partial x} + \frac{\partial \tilde{y}}{\partial y} + \frac{\partial \tilde{z}}{\partial z} = -a + b - g = p = -2
\]

where \( V = (a(z - x) + z^2), by - hxy, ky - gz \), \( p \) is a negative, dynamical system described by (1) is a forced dissipative system, an exponential contraction of the system (1) is \( \frac{dV}{dt} = e^p = e^{-2t} \). In dynamical system (1), a volume element \( V_0 \) is apparently contracted by the flow into a volume element \( V_0 e^{pt} = V_0 e^{-2t} \) in time \( t \). This means that each volume containing the trajectory of this dynamical system shrinks to zero as \( t \to \infty \) at an exponential rate \( p \). So, all this dynamical system orbits are eventually confined to a specific subset that have zero volume, the asymptotic motion settles onto an attractor of the system (1).

As is well known, the Lyapunov exponents measure the exponential rates of divergence or convergence of nearby trajectories in phase space, according to the detailed numerical and as well as theoretical analysis, the largest value of positive Lyapunov exponent of this chaotic system (1) is obtained as \( \lambda_{+1} = 0.3028 \). It is related to the expanding nature of different direction in phase space.

Another one Lyapunov exponent is \( \lambda_{-2} = 0 \). Obviously, it is related to the critical nature between the expanding and the contracting nature of different direction in phase space.

While negative Lyapunov exponent is \( \lambda_{-3} = -2.3028 \). It is related to the contracting nature of different direction in phase space [12].

The Liapunov dimension of chaos attractors in this nonlinear-system (1) is of fraction dimension, it is described as

\[
D_L = j + \frac{1}{|\lambda_j|} \sum_{i=1}^{j} \lambda_i = 2 + \frac{(\lambda_{+1} + \lambda_{-2})}{|\lambda_3|} = 2 + \frac{0.3028 + 0}{-2.3028} \approx 2.13
\]

The fractal nature of an attractor does not merely imply non-periodic orbits; it also causes nearby trajectories to diverge. As all strange attractors, orbits that are initiated from different initial conditions soon reach the attracting set, but two nearby orbits do not stay close to each other, they soon diverge and follow totally different paths in the attractor. Therefore, the system (1) is really a new chaotic system.
The forming mechanism of its compound structure obtained by merging together single upper scroll attractor and other lower scroll attractor after performing one mirror operation has also been investigated by detailed numerical as well as theoretical analysis.

3. Reverse structure of the chaotic system (1)

Here, another chaotic system (2) is also proposed; this new chaotic system (2) is reverse to chaos system (1) in the topological structure.

The autonomy differential equations that describe the system are expressed as

\[
\begin{align*}
\dot{x} &= a(z - x) - z^2 \\
\dot{y} &= by + hxz \\
\dot{z} &= -ky - cz
\end{align*}
\]

(2)

here, let \(a = 0.5, b = 2.5, k = 1, g = 4, h = 1\). The sign of quadratic term is negative and the sign of multiplier is positive in this system (2), but the sign of quadratic term is positive and the sign of multiplier is negative in the system (1), this is only difference of between the system (2) and the system (1).

The initial values of this system (2) are still selected as \((0.04, 0.2, 0)\). This system (2) is also chaotic. Its chaos attractor is also a transverse butterfly-shaped attractor and it is to leftward. So, this nonlinear system (2) is reverse to chaos system (1) in the topological structure.

4. Oscillator circuitry design and simulation of the system based on EWB

In this section, an electronic oscillator circuit is designed to realize the new chaotic system (1) based on EWB. Electronics Workbench (EWB) is an electronics lab in the computer and it is modeled on a real electronics technique workbench. So, EWB is very useful in analysis and design verifications of electronic circuits. The designed chaotic oscillator circuit diagram is shown in Figure 5. Linear resistors, capacitors, operational amplifiers and analog multiplier structured it. The three state variables, \(x, y\) and \(z\) are respectively obtained from the terminal outputs of \(u_{c1}, u_{c2}\) and \(u_{c3}\) in this electronic circuit. The experimental simulation phase portraits of the new transverse butterfly-shaped attractor are shown in Figure 6 based on EWB. Results confirm the validity of the designed chaotic oscillator circuitry. We hope that the designed chaos oscillator as shown in Figure 5 will be very useful in electronic technique and communication engineering. The chaotic system (2) can be similarly realized with an electronic oscillator circuit.

Op-amps: LM741, Multiplier: AD633.

Figure 5. Chaotic oscillator circuit diagram
5. Conclusion
This article has presented a new chaotic system and its reverse structure system. This new attractor is
different from the Lorenz attractor; but it is a new butterfly-shaped chaotic attractor of Lorenz –like
system. This new attractors proposed can be also realized with an electronic oscillator circuit and have
great potential for electronic technique and communication engineering. These new attractors and their
forming mechanism need further to study and explore. Their topological structure should be
completely and thoroughly investigated. It is expecting that more detailed theory analysis and
simulation investigation will be provided in the near future.

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