A new approach to bounds on mixing

Flavien Léger

Courant Institute of Mathematical Science
NYU
Outline

1 Introduction
   - Mixing
   - Bressan’s conjecture

2 Main result
   - New perspective
   - Proof
   - A remark

3 Summary
   - Perspective
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Setting

Incompressible passive scalar advection

\[
\begin{aligned}
\partial_t \theta + \text{div}(u\theta) &= 0 \\
\text{div}(u) &= 0 \\
\theta(0, \cdot) &= \theta_0
\end{aligned}
\]
Natural question: how well can we mix?

▶ Need an energetic constraint on the flow.

Also need to quantify mixing.

Our problem

How much can an incompressible flow mix, under energetic constraint?

Figure credits: A. Bressan, 2003
Mixing

Natural question: how well can we mix?

Figure credits: A. Bressan, 2003
Natural question: how well can we mix?

- Need an energetic constraint on the flow.
  Also need to quantify mixing.

Our problem

How much can an incompressible flow mix, under energetic constraint?
Energetic constraint

Cost of stirring $\theta(0, \cdot)$ to $\theta(T, \cdot)$ is

$$\int_0^T \|\nabla u(t, \cdot)\|_{L^p} \, dt$$

For $p = 2$ it is the energy transferred to a Stokes flow

$$\begin{cases} -\Delta u = f - \nabla p \\ \text{div } u = 0 \end{cases}$$

since:

$$\int f \cdot u \, dx = \int |\nabla u|^2 \, dx$$
Measuring mixing

Definition (Mixing measure)

\[ \varepsilon(t) := \| \theta(t, \cdot) \|_{H^{-s}}^2 = \int |\xi|^{-2s} |\hat{\theta}(t, \xi)|^2 \, d\xi \]

\( s = 1 : \frac{\| \theta(t, \cdot) \|_{H^{-1}}}{\| \theta(t, \cdot) \|_{L^2}} \) scales as length.
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Bressan’s conjecture

Conjecture (Bressan, 2006)
Let $\varepsilon(t) = \text{mixing measure of } \theta(t, \cdot)$. Then

$$\varepsilon(t) \geq C^{-1} \exp \left( -C \int_0^t \| \nabla u(t', \cdot) \|_1 \, dt' \right)$$
Bressan’s conjecture

Conjecture (Bressan, 2006)

Let \( \varepsilon(t) = \text{mixing measure of } \theta(t, \cdot) \). Then

\[
\varepsilon(t) \geq C^{-1} \exp \left( -C \int_0^t \| \nabla u(t', \cdot) \|_1 \, dt' \right)
\]

\( L^1 \): not known

\( L^p \): solved (\( p > 1 \)) (Crippa – De Lellis, 2008)
Previous results

- Crippa – De Lellis (2008)
  - Binary mixtures, Lagrangian coord
  - log of derivative in physical space
  - Use of maximal function

- Seis (2013)
  - Binary mixtures
  - Optimal transportation distance
  - Also use of maximal function
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Consider

\[ \mathcal{V}(\theta) = \int_{\mathbb{R}^d} \log |\xi| |\hat{\theta}(\xi)|^2 \, d\xi \]

Then

\[ \mathcal{V}(\theta(t, \cdot)) - \mathcal{V}(\theta_0) \leq C \|\theta_0\|_\infty \|\theta_0\|_{p'} \int_{t_0}^{t} \|\nabla u(t', \cdot)\|_p \, dt' \quad (p > 1) \]

Implies a restriction on mixing

\[ \|\theta(t, \cdot)\|_{\dot{H}^{-1}} \geq C^{-1} \exp(-C \int_{t_0}^{t} \|\nabla u(t', \cdot)\|_p \, dt') \]
Our approach

Consider

\[ V(\theta) = \int_{\mathbb{R}^d} \log |\xi| |\hat{\theta}(\xi)|^2 \, d\xi \]

Then

\[ V(\theta(t, \cdot)) - V(\theta_0) \leq C \|\theta_0\|_\infty \|\theta_0\|_{p'} \int_0^t \|\nabla u(t', \cdot)\|_p \, dt' \]

\((p > 1)\)
Our approach

Consider
\[ \mathcal{V} (\theta) = \int_{\mathbb{R}^d} \log |\xi| |\hat{\theta}(\xi)|^2 \, d\xi \]

Then
\[ \mathcal{V} (\theta(t, \cdot)) - \mathcal{V} (\theta_0) \leq C \| \theta_0 \|_\infty \| \theta_0 \|_{p'} \int_0^t \| \nabla u(t', \cdot) \|_p \, dt' \]

\((p > 1)\)

Implies a restriction on mixing
\[ \| \theta(t, \cdot) \|_{\dot{H}^{-1}} \geq C^{-1} \exp \left( -C \int_0^t \| \nabla u(t', \cdot) \|_p \, dt' \right) \]
Features

- Control of the log of the derivative
- Stronger than $\dot{H}^{-1}$ norm
- Technique is different
Decay of mixing measures

Simple convexity inequality: if \( \| \theta_0 \|_{L^2} = 1 \),

\[
\| \theta(t, \cdot) \|_{\dot{H}^{-1}} \geq \exp \left( - \mathcal{V}(\theta(t, \cdot)) \right)
\]

- Mixing measure decay at most exponentially.

Recall main theorem:

\[
\mathcal{V}(\theta(t, \cdot)) - \mathcal{V}(\theta_0) \leq C \| \theta_0 \|_\infty \| \theta_0 \|_{p'} \int_0^t \| \nabla u(t', \cdot) \|_p \, dt'
\]
Stronger than $\dot{H}^{-1}$

Compare

$$\int_0^T \|\nabla u(t, \cdot)\|_p \, dt \leq M \Rightarrow \int |\xi|^{-2} |\hat{\theta}(T, \xi)|^2 \, d\xi \gtrsim \exp(-M)$$
Stronger than $\dot{H}^{-1}$

Compare

$$\int_0^T \| \nabla u(t, \cdot) \|_p \, dt \leq M \implies \int |\xi|^{-2} |\hat{\theta}(T, \xi)|^2 \, d\xi \gtrsim \exp(-M)$$

and

$$\int_0^T \| \nabla u(t, \cdot) \|_p \, dt \leq M \implies \int \log|\xi| \, |\hat{\theta}(T, \xi)|^2 \, d\xi \lesssim M$$
Corollary: on the blowup of Sobolev norms

If $\|\nabla u(t, \cdot)\|_{L^2} \leq C$:

- $\int |\hat{\theta}(t, \xi)|^2 d\xi = C$
- $\int \log|\xi||\hat{\theta}(t, \xi)|^2 d\xi \leq C (1 + t)$
- $\int (\log|\xi|)^2 |\hat{\theta}(t, \xi)|^2 d\xi \leq C (1 + t)^2$
- $\int |\xi|^{2s} |\hat{\theta}(t, \xi)|^2 d\xi$ can blow up, for any $s > 0$
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Outline of proof

1. Write $\mathcal{V}(f) = \int \log|\xi| |\hat{f}(\xi)|^2 d\xi$ in physical space. Morally
   \[ \mathcal{V}(f) = \iint \frac{f(x)f(y)}{|x-y|^d} \, dx \, dy \]

2. Time derivative is
   \[ \frac{d}{dt} \mathcal{V}(\theta) = \iint \theta(x) \theta(y) \frac{u(x) - u(y)}{|x-y|} \cdot \frac{x-y}{|x-y|^{d+1}} \, dx \, dy \]

3. Incompressibility implies
   \[ \frac{d}{dt} \mathcal{V}(\theta) = \iint \theta(x) \theta(y) (m_{xy} \nabla u) : K(x-y) \, dx \, dy \]

4. Hölder-type bounds. Mixing result follows by elementary arguments.
More details

1. Recall

\[ \mathcal{V}(f) = \int_{\mathbb{R}^d} \log |\xi| |\hat{f}(\xi)|^2 \, d\xi \]

In physical space:

\[ \mathcal{V}(f) = \alpha_d \left( \frac{1}{2} \iint_{|x-y| \leq 1} \frac{|f(x) - f(y)|^2}{|x-y|^d} \, dx \, dy \right) \\
- \iint_{|x-y| > 1} \frac{f(x)f(y)}{|x-y|^d} \, dx \, dy \right) + \beta_d \|f\|_{L^2}^2 \]
Time-derivative of $V$ along the flow $\partial_t \theta + \text{div}(u\theta) = 0$:

$$\frac{d}{dt} V(\theta(t, \cdot)) = c_d \text{PV} \int \int \theta(t, x) \theta(t, y) \frac{u(t, x) - u(t, y)}{|x - y|} \cdot \frac{x - y}{|x - y|^{d+1}} dx \, dy$$

Incompressibility constraint $\text{div} \, u = 0 \rightarrow$ multilinear singular integral can be written as

$$\frac{u(t, x) - u(t, y)}{|x - y|} \cdot \frac{x - y}{|x - y|^{d+1}} = (m_{xy} \nabla u(t, \cdot)) : K(x - y)$$

with $K$ a Calderón–Zygmund kernel.
Multilinear singular integral bounds (Seeger, Smart, Street):

$$\text{PV} \int \int \theta(t, x)\theta(t, y)(m_{xy} \nabla u(t, \cdot)) : K(x - y) \, dx \, dy \leq$$

$$C \|	heta(t, \cdot)\|_{\infty} \|	heta(t, \cdot)\|_{p'} \|
abla u(t, \cdot)\|_{p}$$

Hence

$$\left| \frac{d}{dt} \mathcal{V}(\theta(t, \cdot)) \right| \leq C \|	heta_{0}\|_{\infty} \|	heta_{0}\|_{p'} \|
abla u(t, \cdot)\|_{p}$$
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A word on the harmonic analysis

Trilinear form

$$\Lambda(a, \theta, \phi) = \text{PV} \int \int K(x - y) \left( m_{xy} a \right) \theta(x) \phi(y) \, dx \, dy$$

$m_{xy} a = \text{average of } a \text{ on } [x, y]$

- Christ-Journé (1987): $C(a, \theta, \phi) \lesssim \| \theta \|_q \| \phi \|_{q'} \| a \|_\infty$
- Seeger, Smart, Street (2015): $\| a \|_p \ (p > 1)$
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Summary

New approach to

\[ \varepsilon(t) \geq C^{-1} \exp \left( -C \int_0^t \| \nabla u(t') \|_p \, dt' \right) \]

\( (p > 1) \). Use of

\[ \mathcal{V}(\theta(t, \cdot)) = \int \log |\xi| \| \hat{\theta}(t, \xi) \|^2 \, d\xi \]

Still open

Can we get Bressan’s conjecture: unlikely.
Easier version than Bressan? \( L \log L \) ok.
Note: no \( L^1 \) result available yet.
Thank you for your attention!