A polynomial time algorithm to compute the connected tree-width of a series-parallel graph

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Abstract

It is well known that the treewidth of a graph $G$ corresponds to the node search number where a team of cops is pursuing a robber that is lazy, visible and has the ability to move at infinite speed via unguarded path. In recent papers, connected node search strategies have been considered. A search strategy is connected if at each step the set of vertices that is or has been occupied by the team of cops, induced a connected subgraph of $G$. It has been shown that the connected search number of a graph $G$ can be expressed as the connected treewidth, denoted $\text{ctw}(G)$, that is defined as the minimum width of a rooted tree-decomposition $(X, T, r)$ such that the union of the bags corresponding to the nodes of a path of $T$ containing the root $r$ is connected. Clearly we have that $\text{tw}(G) \leq \text{ctw}(G)$. It is paper, we initiate the algorithmic study of connected treewidth. We design a $O(n^2 \cdot \log n)$-time dynamic programming algorithm to compute the connected treewidth of a biconnected series-parallel graphs. At the price of an extra $n$ factor in the running time, our algorithm generalizes to graphs of treewidth at most 2.

Keywords: Series-parallel graphs, treewidth, connected treewidth, dynamic programming.

1 Introduction

Since its introduction [8, 24, 34], the concept of treewidth of a graph has led to major advancements in graph theory. The treewidth parameter, denoted by $\text{tw}$, is central to the design of efficient graph algorithms (see [2] for a survey on early works in this direction and [10] for a recent survey). According to the theorem of Courcelle [13], properties expressible in MSO$_2$ logic can be tested in parameterized linear time on graph of bounded treewidth. This result established treewidth as an important structural parameter in the context of parameterized complexity [18]. Treewidth can be defined in several equivalent ways, while the standard definition is by means of a tree-decomposition (see Section 2). During the last two decades, a number of width-parameters have been introduced as combinatorial variants or alternatives to treewidth (see [25] for a survey on width parameters). This paper deals with the connected treewidth parameter, a new variant of treewidth that is motivated by the study of connected search games in graphs.
A node search game is a game opposing a group of searchers and a robber occupying the vertices of a graph. In a search strategy, one searcher may either be placed to or removed from a vertex and a winning search strategy is a sequence of moves of the searchers that eventually leads to the capture of the robber. The capture of the robber happens when a searcher lands on the vertex occupied by the robber while the robber cannot escape along a searcher-free path. The cost of a search strategy is the maximum number of searchers simultaneously occupying vertices of the graph. The clean territory during some step of the game is the set of vertices from which the robber has been, so far, expelled. A strategy is monotone if it guarantees that the robber will not be able to visit an already cleaned territory. Also, a strategy is connected if it guarantees, that at any step, the clean territory induced a connected graph. The robber can be lazy or agile. Being a lazy means that the robber is staying at its position as long as a searcher is not moving on that position. An agile robber may move at any time regardless of the move of the searchers. Also a robber can be visible or invisible in the sense that the searchers strategy may or may not take into account the current position of the robber. The node search number of a node search number of a graph is the minimum cost of a winning search strategy (for some of the above variants) is the minimum cost of a winning search strategy.

It is well-known that the search number against an invisible and lazy robber is equal to treewidth plus one, while the same quantity is also equal to the search number against an visible and agile robber. On the other hand, if the robber is invisible and agile, then the corresponding search number is equal to the parameter of pathwidth plus one. Moreover all the aforementioned versions of the game are monotone in the sense that for every search strategy there is a monotone one with the same cost. Interestingly, the motivating story of one of the seminal papers on graph searching was inspired by an earlier article of the Breisch in Southwestern Cavers Journal. This papers concerned speleotopological explorations, which are, by essence, connected explorations as the searchers cannot “teleport to a vertex that is away from the current clean territory. The connectivity constraint was considered for the first time in, where the price of connectivity, defined as the ratio between monotone connected node search number and the node search number, was first considered. In, it was proven that the price of connectivity for node search against a visible and lazy robber (or, equivalently, an invisible and lazy one) is Θ(log n). Dereniowsky introduced the notion of connected pathwidth of a graph, denoted cpw(G) and showed its equivalence with the node search number against an visible and agile robber. He proved that, for every graph G, cpw(G) ≤ 2 · pw(G) + O(1), henceforth resolving the question of the price of connectivity for a node search against an agile robber. Extending the work of, Adler et al. recently introduced a definition of connected treewidth, denoted hereafter ctw(G). They proved that, as for treewidth, the connected treewidth parameter is equivalent to the connected node search number against an visible and lazy robber and, as in the non-connected setting, the same holds for the invisible and agile case. Also they proved that connected treewidth is equivalent to a connected variant of the vertex separation number (see Section 2).

In this paper, we are interested in the problem of computing the connected treewidth of a graph. So far, very little is known on the algorithmic aspects of the connected treewidth and connected pathwidth parameters. First of all, checking, given a graph G and an integer k, whether the connected treewidth (or the connected pathwidth) is at most k is an NP-complete problem (see Theorem 4). This means that, in general, we may not expect any polynomial
algorithm for computing connected treewidth for these two parameters. To the best of our
knowledge, the only existent result is a recent \(n^{O(k^2)}\)-time algorithm to check whether\(\text{cpw}(G) \leq k\) [10]. An explanation for this lack of algorithmic results certainly relies on the fact that, unlike
pathwidth and treewidth, connected treewidth and connected pathwidth parameters do not
enjoy nice combinatorial properties such as closeness under taking of minors. But this is not
correct for connected pathwidth/treewidth. Interestingly both parameters are closed under
contractions [1]. However there is no analogue of the algorithmic machinery developed in the
context of graph minors for graph contractions.

This motivated us to initiate the study of computing the connected treewidth on the class
of series-parallel graphs. First introduced by Macmahon in 1992 [29] (see also [30, 33]), series-
parallel graphs are essentially graphs of treewidth at most two [19]. More precisely, a graph
has treewidth at most two if and only if each of its biconnected components induces a series-parallel
graph. The recursive construction, by means of series and parallel composition (see Section 2),
of series-parallel graphs allows to solve a large number of \(\text{NP}\)-hard problem in polynomial (or
even linear) time, see for example [6, 7]. It follows that the class of series-parallel graphs,
among others, forms a natural test bed for the existence of efficient graph algorithms [11]. In
this paper, we design a \(O(n^2 \cdot \log n)\)-time algorithm to compute the connected treewidth of a
biconnected series-parallel graph. The algorithm is extended to a \(O(n^3 \cdot \log n)\)-algorithm for
graphs of treewidth at most 2. This result constitutes a first step toward the computation of
connected treewidth parameterized by treewidth (see Section 4 for a discussion).

2 Preliminaries

We use standard notations for graphs, as for example in [17]. We consider undirected and simple
graphs. We let \(G = (V, E)\) denote a graph on \(n\) vertices, with \(V = V(G)\) its vertex set and
\(E = E(G)\) its edge set. A vertex \(x\) is a neighbor of \(y\) if \(xy\) is an edge of \(E\). If \(S\) is a subset of
\(V\), then \(G[S]\) is the subgraph of \(G\) induced by \(S\). A path \(P\) between vertex \(x\) and \(y\) is called an
\((x, y)\)-path and the vertices of \(P\) distinct from \(x\) and \(y\) are the internal vertices of \(P\). A vertex
\(x \in V\) is a cut vertex if \(G \setminus x\) has strictly more connected components than \(G\). A graph is
biconnected if it is connected and does not contains a cut vertex. Given an integer \(q\), we use \([q]\)
as a shortcut for the set \(\{1, \ldots, q\}\).

A layout \(\sigma\) of a graph \(G = (V, E)\) is a total ordering of its vertices, in other words \(\sigma\) is a
bijection from \(V\) to \([n]\). For two vertices \(x\) and \(y\), we write \(x <_\sigma y\) if \(\sigma(x) < \sigma(y)\). We define
\(\sigma_{<i} = \{x \in V \mid \sigma(x) < i\}\) (the sets \(\sigma_{>i}\), \(\sigma_{\leq i}\) and \(\sigma_{\geq i}\) are similarly defined). If \(S\) is a subset of
\(V\), then \(\sigma[S]\) is the layout of \(G[S]\) such that for every \(x, y \in S, \sigma(x) < \sigma(y)\) if and only
if \(\sigma[S](x) < \sigma[S](y)\). Let \(\sigma_1\) and \(\sigma_2\) be two layouts on disjoint vertex sets \(V_1\) and \(V_2\). Then
\(\sigma = \sigma_1 \circ \sigma_2\), the concatenation of \(\sigma_1\) and \(\sigma_2\), satisfies for every \(x_1 \in V_1\) and every \(x_2 \in V_2\),
\(\sigma(x_1) < \sigma(x_2)\), \(\sigma[V_1] = \sigma_1\) and \(\sigma[V_2] = \sigma_2\).

2.1 Series-parallel graphs

A 2-terminal graph is a pair \(G = (G, (x, y))\) where \(G = (V, E)\) is a graph and \((x, y)\) is a
pair of distinguished vertices, hereafter called the terminals. Consider two 2-terminal graphs
\((G_1, (x_1, y_1))\) and \((G_2, (x_2, y_2))\), where \(G_1 = (V_1, E_1)\) and \(G_2 = (V_2, E_2)\). Then:

- the series-composition, denoted \((G_1, (x_1, y_1)) \otimes (G_2, (x_2, y_2))\), yields the 2-terminal graph
(G, (x_1, y_2)) with G being the graph obtained by identifying the terminal y_1 with x_2;

- the parallel-composition, denoted (G_1, (x_1, y_1)) \oplus (G_2, (x_2, y_2)), yields the 2-terminal graph (G, (x_1, y_1)) with G being the graph obtained by identifying the terminal x_1 with x_2 and the terminal y_1 with y_2.

**Definition 1.** A 2-terminal graph (G, (x, y)) is series-parallel if either it is the single edge graph with \{x, y\} as vertex set, or if it results from the series or the parallel composition of two 2-terminal series-parallel graphs. A graph G = (V, E) is a series-parallel graph if for some pair of vertices x, y \in V, (G, (x, y)) is a 2-terminal series-parallel graph.

Observe that from the definition above, we may generate multi-graphs. However, in this paper we only consider simple graphs. When clear from the context, G will denote the 2-terminal series-parallel graph (G, (x, y)). Observe that a series-parallel graph G can be represented by a so-called SP-tree T(G). The leaves of T(G) are labelled by the edges of G. Every internal node of T(G) is labelled by a composition operation (\oplus or \otimes) and a pair of terminal vertices. For an internal node t of T(G), we let T_t denote the subtree of T(G) rooted at t and V_t the subset of vertices incident to an edge labelling a leaf of T_t. Suppose that t is labelled (\otimes, (x_t, y_t)), then the node u represents the 2-terminal graph (G[V_t], (x_t, y_t)).

**Theorem 1.** [20] If a graph G = (V, E) is a biconnected series-parallel graph, then for any edge xy \in E, (G, (x, y)) is a 2-terminal series-parallel graph.

Theorem 1 can be rephrased as follows: if xy is an edge of a biconnected series-parallel graph, then G = (G, (x, y)) is a 2-terminal series-parallel graph such that G = (G_1, (x, y)) \oplus (G_2, (x, y)) where G_1 = (\{x, y\}, \{xy\}) and G_2 = (V, E \setminus \{xy\}).

**Theorem 2.** [36] The problem of testing whether a graph G is series-parallel is linear time can be solved in linear time. Moreover if G is a biconnected series-parallel graph, then a series-parallel tree of (G, (x, y)), where xy is an edge, can be build in linear time.

### 2.2 Connected tree-decomposition and connected layouts

A tree-decomposition of a graph G = (V, E) is pair (T, \mathcal{F}) where T = (V_T, E_T) is a tree and \mathcal{F} = \{X_t \subseteq V \mid t \in V_T\} such that

1. \bigcup_{t \in V_T} X_t = V;
2. for every edge e \in E, there exists a node t \in V_T such that e \subseteq X_t;
3. for every vertex x \in V, the set \{t \in V_T \mid x \in X_t\} induces a connected subgraph of T.

We refer to V_T as the set of nodes of T and the sets of \mathcal{F} as the bags of (T, \mathcal{F}). The width of a tree-decomposition (T, \mathcal{F}) is\[\text{width}(T, \mathcal{F}) = \max\{|X| - 1 \mid X \in \mathcal{F}\}\] and the tree-width of a graph G is\[\text{tw}(G) = \min\{\text{width}(T, \mathcal{F}) \mid (T, \mathcal{F}) \text{ is a tree-decomposition of } G\}\]

For two nodes u and v of V_T, we define P_{u,v} the unique path between u and v in T and the set V_{u,v} \subseteq V as \bigcup_{t \in P_{u,v}} X_t. A tree-decomposition (T, \mathcal{F}) is connected if there exists a node
\( r \in V_T \) such that for every node \( u \in V_T \), the subgraph \( G[V_{r,u}] \) is connected. Then the connected tree-width of a graph \( G \) is

\[
\text{ctw}(G) = \min \{ \text{width}(T, \mathcal{F}) \mid (T, \mathcal{F}) \text{ is a connected tree-decomposition of } G \}.
\]

A layout \( \sigma \) of \( G \) is connected if for every \( i \in [n] \), the subgraph \( G[\sigma_{\leq i}] \) is connected. We let \( L^c(G) \) be the set of connected layouts of a graph \( G \). For every \( v \in V \), we define the supporting set of \( x \) as

\[
S_\sigma(v) = \{ x \in V(G) \mid \sigma(x) < \sigma(v) \text{ and there exists a } (x,v)\text{-path } P \text{ such that every internal vertex } y \text{ of } P \text{ satisfies } \sigma(y) > \sigma(v) \}.
\]

We set \( \text{cost}(G, \sigma) = \max \{|S_\sigma(v)| \mid v \in V \} \). The vertex separation number of a graph is defined as

\[
\text{vs}(G) = \min \{ \text{cost}(G, \sigma) \mid \sigma \in \mathcal{L}(G) \}.
\]

When restricting to the set of connected layouts, we obtain the connected vertex separation number

\[
\text{cvs}(G) = \min \{ \text{cost}(G, \sigma) \mid \sigma \in L^c(G) \}.
\]

**Theorem 3** ([1]). For every graph \( G = (V,E) \), we have \( \text{ctw}(G) = \text{cvs}(G) \).

Notice that if in the above definitions we drop the connectivity demand from the considered layouts, we have an alternative layout-definition of the parameter of treewidth, as observed in [14]. It is known that deciding whether \( \text{tw}(G) \leq k \) is an NP-complete problem [3]. An easy reduction of treewidth to connected treewidth is the following: consider a graph \( G \), add a vertex \( v_{\text{new}} \), and make \( v_{\text{new}} \) adjacent to all the vertices of \( G \). We call the new graph \( G^+ \). It follows, as a direct consequence of the layout definitions, that \( \text{ctw}(G^+) = \text{tw}(G) + 1 \). We conclude to the following.

**Theorem 4.** The problem of deciding, given a graph \( G \) and an integer \( k \), whether \( \text{ctw}(G) \leq k \) is NP-complete.

### 2.3 Rooted graphs and extended graphs

**Rooted graphs.** A rooted graph is a pair \( (G,R) \) where \( G = (V,E) \) is a graph and \( R \subseteq V \) is a subset of vertices, hereafter called roots. Observe that every graph \( G \) can be considered as the rooted graph \( (G,\emptyset) \). The definition of rooted graph naturally extends to two-terminal graphs. If \( G = (G,(x,y)) \) is a series-parallel graph and \( R \subseteq V \) a set of roots, then the corresponding rooted two-terminal graph will be denoted by \( (G,R) \). Observe that the set of roots \( R \) may be different from the terminal pair \( (x,y) \).

A rooted graph \( (G,R) \) is connected if and only if every connected component of \( G \) contains a root from \( R \). A rooted layout of \( (G,R) \) is a layout \( \sigma \) of \( G \) such that \( \sigma_{\leq |R|} = R \). Based on this, the notion of connected layout naturally extends to rooted graphs and rooted layouts as follows. A rooted layout \( \sigma \) of \( (G,R) \) is connected if for every \( i, |R| < i \leq n \), \( (G[\sigma_{\leq i}], R) \) is a connected rooted graph.
Extended graphs. Let \( G = (G, (x, y)) \) be a 2-terminal graph where \( G = (V, E) \). Suppose that \( F \subseteq \left\{ \frac{1}{2} \right\} \setminus E(G) \), i.e., \( F \) is a subset of edges distinct from \( E \). We define the extended graph \( G^+F \) as the 2-terminal graph \( (G^+F, (x, y)) \) where \( G^+F = (V, E \cup F) \). The edges in \( E \) are called solid edges, while the edges of \( F \) are called fictive edges. Hereafter the extended graph \( G \) (and the graph \( G \) respectively) is called the solid graph of \( G^+F \) (and \( G^+F \) respectively).

The purpose of introducing fictive edges is not to augment the connectivity of the solid graph but to keep track of cumulative cost in the recursive calls of the dynamic programming algorithm while computing the connected treewidth of a series-parallel graph. Indeed, the connected components of the extended graph \( G^+F \) are the connected component of its solid graph \( G \). In particular, \( G^+F \) is connected if and only if \( G \) is connected.

This connectivity definition of an extended graph also transfers to (rooted) layouts. More precisely, if \( R \subseteq V \) is a set of roots and \( G^+F \) an extended graph, then \( (G^+F, R) \) is a rooted extended graph. A layout \( \sigma \) of \( (G^+F, R) \) is connected if and only if it is a connected layout of \( (G, R) \). Observe that if \( G^+F \) is not connected, then a connected layout of \( G^+F \) exists if and only if every connected component of \( G^+F \) contains a root from \( R \).

An extended path of \( G^+F \) is a path that may contains a fictive edge of \( F \), while a solid path in \( G^+F \) is a path of \( G \), that is, a path that only contain solid edges. The extended cost of a layout \( \sigma \) of the extended graph \( G^+F \) as follows:

\[
S_{\sigma}^{(e)}(v) = \{ x \in V \mid \sigma(x) < \sigma(v) \text{ and there exists an extended } (x, v)\text{-path } P \text{ such that every internal vertex } y \text{ of } P \text{ belongs to } \sigma_{>1}\}
\]

The definitions of the extended cost \( \text{ecost}(G^+F, \sigma) \) and of the extended connected vertex separation number \( \text{ecvs}(G^+F) \) follow accordingly:

\[
\text{ecvs}(G^+F) = \min\{\text{ecost}(G^+F, \sigma) \mid \sigma \in \mathcal{L}(G^+F)\}
\]

where \( \text{ecost}(G^+F, \sigma) = \max\{|S_{\sigma}^{(e)}(v)| \mid v \in V\} \).

3 A dynamic programming algorithm

We require the following result about the maximum value of connected treewidth on a graph with bounded treewidth.

**Theorem 5 ([21][22][23]).** Every graph \( G \) on \( n \) vertices satisfies \( \text{ctw}(G) \leq tw(G) \cdot (\log n + 1) \).

As series-parallel graphs have treewidth at most two, Theorem 5 implies that the connected treewidth of a series-parallel graph on \( n \) vertices is at most \( c_{sp} := \lceil 2(\log n + 1) \rceil \). This bound allows us to optimize the size of the table in our dynamic programming algorithm. Moreover this bound is tight \([22][23]\), even on series-parallel graphs as shown by a construction of minimal obstructions for connected treewidth \([1]\).

3.1 Biconnected series-parallel graphs

In this section, let \( G = (G, (x, y)) \) be a series-parallel graph such that \( G = (V, E) \). We suppose that \( G \) results from the series or the parallel composition of \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \). When dealing with the 2-terminal graphs \( G_1 \) and \( G_2 \), the terminal pairs will be clear from the context.
3.1.1 Parallel composition

**Lemma 1.** Let \((G^+, R)\) be a rooted extended graph such that \(R = \{x, y\}\) and \(G = G_1 \oplus G_2\) with \(G_1 = (G_1, (x, y))\) and \(G_2 = (G_2, (x, y))\). Then

\[
\text{ecvs}(G^+, R) = \max \left\{ \text{ecvs}(G_1^+, R), \text{ecvs}(G_2^+, R) \right\}.
\]

**Proof.** Let \(\sigma^* \in \mathcal{L}^c(G^+, R)\) be a connected layout of minimum cost. From \(\sigma^*\), we define a layout \(\sigma\) of \((G^+, R)\) as follows: \(\sigma = \sigma^*|V_1 \cup \sigma^*|V_2 \setminus \{x, y\}\) (see Figure 1).

![Figure 1: Rearranging a layout of minimum cost. Black vertices belongs to \(V_1 \setminus \{x, y\}\) and white vertices belong to \(V_2 \setminus \{x, y\}\).](image)

Observe that as \(\{x, y\}\) separates \(V_1\) from \(V_2\) and that \(\sigma^* \in \mathcal{L}^c(G^+, R)\) imply that \(\sigma \in \mathcal{L}^c(G^+, R)\), \(\sigma_1 = \sigma|V_1 \in \mathcal{L}^c(G^+, R)\) and \(\sigma_2 = \sigma|V_2 \in \mathcal{L}^c(G^+, R)\). Moreover, \(\{x, y\}\) separating \(V_1\) from \(V_2\) implies that for every vertex \(v_1 \in V_1 \setminus \{x, y\}\), we have \(S^{(e)}(v_1) = S^{(e)}(v_1) \subseteq V_1\) and that for every vertex \(v_2 \in V_2 \setminus \{x, y\}\), we have \(S^{(e)}(v_2) = S^{(e)}(v_2) \subseteq V_2\). It follows that for every vertex \(v \in V \setminus \{x, y\}\), \(S^{(e)}(v) = S^{(e)}(v)\), implying that \(\text{ecost}(G^+, \sigma) = \text{ecost}(G^+, \sigma^*)\).

![Figure 2: Decomposition of an extended graph resulting from a parallel composition.](image)

To conclude, we observe that if \(\tau_1 \in \mathcal{L}^c(G_1^+, R)\) and \(\tau_2 \in \mathcal{L}^c(G_2^+, R)\), then \(\tau = \tau_1 \circ \tau_2|V_2 \setminus \{x, y\}\) belongs to \(\mathcal{L}^c(G^+, R)\) and that \(\text{ecost}(G^+, \tau) = \max \{\text{ecost}(G_1^+, \tau_1), \text{ecost}(G_2^+, \tau_2)\}\). So the optimality of \(\sigma^*\) and \(\sigma\) imply that \(\text{ecvs}(G^+, R) = \max \left\{ \text{ecvs}(G_1^+, R), \text{ecvs}(G_2^+, R) \right\}\). \(\square\)

**Lemma 2.** Let \((G^+, R)\) be a rooted extended graph such that \(R = \{x, r_1 \ldots r_k\}\) (with \(k > 0\)), \(F = \{y r_i | i \in [k]\}\) and \((G[V \setminus \{r_1 \ldots r_k\}], (x, y)) = G_1 \oplus G_2\) with \(G_1 = (G_1, (x, y))\) and \(G_2 = (G_2, (x, y))\). Then

\[
\text{ecvs}(G^+, R) = \min \left\{ \begin{array}{c}
\max \left\{ \text{ecvs}(G|V_1 \cup R + F \cup \{xy\}, R), \text{ecvs}(G_2^+, \{x, y\}) \right\} \\
\max \left\{ \text{ecvs}(G|V_2 \cup R + F \cup \{xy\}, R), \text{ecvs}(G_1^+, \{x, y\}) \right\}
\end{array} \right\}.
\]
Proof. We observe that $G^+F$ contains $k$ isolated vertices, the root vertices $r_1, \ldots, r_k$, and a connected component resulting from a parallel composition. Let $\sigma^* \in L^c(G^+F, R)$ be a connected layout of minimum cost. Consider the neighbor $v$ of $y$ such that $\sigma^*(v)$ is minimum. By the connectivity of $\sigma^*$, $\sigma^*(v) < \sigma^*(y)$. Suppose without loss of generality that $v \in V_1$. The case $v \in V_2$ is symmetric. (Observe that $v$ can be the vertex $x$, in which case it can arbitrarily be considered as a vertex of $V_1$ and $V_2$). From $\sigma^*$, we define a layout $\sigma$ of $(G^+F, R)$ as follows: $\sigma = \{r_1, \ldots, r_k\} \cup \sigma^*[V_1] \cup \sigma^*[V_2 \setminus \{x, y\}]$.

Claim 1. $\sigma \in L^c(G^+F, R)$.

Let $v_1$ be a vertex of $V_1$ distinct from $x$ and $y$. Observe that every neighbor of $v_1$ belongs to $V_1$. By the assumption above, we know that $y$ has a neighbor $v$ prior to it in $\sigma^*$. As the relative ordering between vertices of $V_1$ is left unchanged, every vertex of $V_1 \setminus \{x\}$ has a neighbor prior to it in $\sigma$ as well. Consider now a vertex $v_2 \in V_2$ distinct from $x$ and $y$. As in the previous case, every neighbor of $v_2$ belongs to $V_2$. As the relative ordering between vertices of $V_2$ has only been modified by moving $y$ ahead, $v_2$ has a neighbor prior to it in $\sigma$. It follows that $\sigma \in L^c(G^+F, R)$.

Claim 2. $\text{ecost}((G^+F, R), \sigma) = \text{ecost}((G^+F, R), \sigma^*)$.

We first consider a vertex $v_1 \in V_1$. By construction, we have $\sigma^*[V_1] = \sigma[V_1]$ and for every vertex $v_2 \in V_2 \setminus \{x, y\}$, $\sigma(v_1) \leq \sigma(v_2)$. It follows that $S^c_\sigma(v_1) \subseteq V_1 \cup R$. Suppose that $\sigma(v_1) > \sigma(y)$. As $\{x, y\}$ separates $v_1$ from the vertices of $V_2 \cup \{r_1, \ldots, r_k\}$, we have $S^c_\sigma(v_1) = S^c_\sigma(v_1)$. Suppose that $\sigma(v_1) \leq \sigma(y)$. We observe that if for $i \in [k]$, $r_i \in S^c_\sigma(v_1)$, then there exists a $(v_1, y)$-path $P$ in $G_1$ such that every vertex $v \in P$ satisfies $\sigma(v_1) \leq \sigma(v)$. Similarly, if $v \in V_1$ belongs to $S^c_\sigma(v_1)$, then there exists a $(v_1, v')$-path $P'$ in $G_1$ such that every vertex $v' \in P'$ distinct from $v$ satisfies $\sigma(v_1) \leq \sigma(v')$. The existence of the paths $P$ and $P'$ is a consequence of the fact that $\{x, y\}$ separates the vertices of $V_1 \setminus \{x, y\}$ from the rest of the graph. As $\sigma^*[V_1 \cup R] = \sigma[V_1 \cup R]$, we have $\sigma^*(v_1) \leq \sigma^*(v)$ and $\sigma^*(v_1) \leq \sigma^*(v')$. It follows that $r_i, v' \in S^c_\sigma(v_1)$ implying that $|S^c_\sigma(v_1)| \leq |S^c_\sigma(v_1)|$.

Figure 3: Decomposition of an extended graph with several isolated roots and resulting from a parallel composition.

Let $v_2$ be a vertex of $V_2 \setminus \{x, y\}$. Observe that, as $\{x, y\}$ separates the vertices of $V_2 \setminus \{x, y\}$ from the vertices of $R \cup V_1 \setminus \{x, y\}$ and as $\sigma(x) < \sigma(y) < \sigma(v_2)$, we have $S^c_\sigma(v_2) \subseteq V_2$. As $\sigma[V_2 \setminus \{y\}] = \sigma^*[V_2 \setminus \{y\}]$, every vertex $v \in V_2 \setminus \{y\}$ that belongs to $S^c_\sigma(v_2)$ also belongs to $S^c_\sigma(v_2)$. Suppose that $y \in S^c_\sigma(v_2)$. Then there exists a $(y, v_2)$-path $P$ such that every internal vertex $v$ of $P$ satisfies $\sigma(v_2) < \sigma(v)$. As $\sigma[V_2 \setminus \{y\}] = \sigma^*[V_2 \setminus \{y\}]$, we also have $\sigma^*(v_2) < \sigma^*(v)$.
\begin{itemize}
\item If \( \sigma^*(y) < \sigma^*(v_2) \), then \( y \in S^{(e)}(v_2) \) implying that \( S^{(e)}(v_2) \subseteq S^{(e)}(v_2) \).
\item Otherwise, \( \sigma^*(y) > \sigma^*(v_2) \) and then \( y \notin S^{(e)}(v_2) \). But in that case, let us recall that by assumption the first neighbor \( v \) of \( y \) in \( \sigma^* \) belongs to \( V_1 \). It follows that \( v \in S^{(e)}(v_2) \). As we argue that \( S^{(e)}(v_2) \subseteq V_2 \), \( v \in S^{(e)}(v_2) \setminus S^{(e)}(v_2) \). So \( v \) is a replacement vertex for \( y \) in \( S^{(e)}(v_2) \), implying that \( |S^{(e)}(v_2)| = |S^{(e)}(v_2)| \).
\end{itemize}

So we proved that \( \text{ecost}((G^{+F}, R), \sigma) \leq \text{ecost}((G^{+F}, R), \sigma^*) \). As by Claim 1 \( \sigma \in \mathcal{L}^c(G^{+F}, R) \), the optimality of \( \sigma^* \) implies that \( \text{ecost}((G^{+F}, R), \sigma) = \text{ecost}((G^{+F}, R), \sigma^*) \).

**Claim 3.** \( \text{ecvs}(G^{+F}, R) = \max \left\{ \text{ecvs}(G[V_1 \cup R]^{+F \cup \{xy\}}, R), \text{ecvs}(G_2^{+\emptyset}, \{x, y\}) \right\} \). (see Figure 3)

From the proof of Claim 1 we deduce that \( \sigma_1 = \sigma[V_1 \cup R] \in \mathcal{L}^c(G[V_1 \cup R]^{+F \cup \{xy\}}, R) \). Observe that in the extended graph \( G[V_1 \cup R]^{+F \cup \{xy\}} \), the edge \( xy \) simulates every \( (x, y) \)-path of \( G \) whose internal vertices belong to \( V_2 \). It follows that for every vertex \( v_1 \in V_1 \cup R, S^{(e)}_1(v_1) = S^{(e)}_2(v_1) \). Likewise, from the proof of Claim 2 we deduce that \( \sigma_2 = \sigma[V_2] \in \mathcal{L}^c(G_2, \{x, y\}) \). As \( \{x, y\} \) separates the vertices of \( V_2 \) from the other vertices, for every \( v_2 \in V_2 \setminus \{x, y\} \), we have \( S^{(e)}_1(v_2) = S^{(e)}_2(v_2) \), proving the claim.

**Lemma 3.** Let \( (G^{+F \cup \{xy\}}, R) \) be a rooted extended graph such that \( R = \{x, r_1 \ldots r_k\} \) (with \( k > 0 \)), \( F = \{yr_i | i \in [k]\} \) and \( (G[V \setminus \{r_1, \ldots, r_k\}], (x, y)) = G_1 \oplus G_2 \) with \( G_1 = (G_1, (x, y)) \) and \( G_2 = (G_2, (x, y)) \). Then

\[ \text{ecvs}(G^{+F \cup \{xy\}}, R) = \text{ecvs}(G^{+F}, R). \]

**Proof.** Let \( \sigma^* \in \mathcal{L}^c(G^{+F \cup \{xy\}}, R) \) be a connected layout of minimum cost. Consider the neighbor \( v \) of \( y \) such that \( \sigma^*(v) \) is minimum. By the connectivity of \( \sigma^* \), \( \sigma^*(v) < \sigma^*(y) \). Suppose without loss of generality that \( v \in V_1 \). The case \( v \in V_2 \) is symmetric. We construct a layout \( \sigma \) of \( (G^{+F \cup \{xy\}}, R) \) as in the proof of Lemma 2 that is: \( \sigma = (r_1, \ldots, r_k) \circ \sigma^*[V_1] \circ \sigma^*[V_2 \setminus \{x, y\}] \). We first observe that Claim 1 and Claim 2 apply to \( \sigma \). Moreover the proof of Claim 3 shows that \( \text{ecvs}(G^{+F \cup \{xy\}}, R) = \max \left\{ \text{ecvs}(G[V_1 \cup R]^{+F \cup \{xy\}}, R), \text{ecvs}(G_2^{+\emptyset}, \{x, y\}) \right\} \). It follows that the fictive edge \( xy \) is irrelevant (see Figure 3) and that \( \text{ecvs}(G^{+F \cup \{xy\}}, R) = \text{ecvs}(G^{+F}, R) \) as claimed.

\[ \square \]
3.1.2 Series composition

Lemma 4. Let \((G^{+0}, R)\) be a rooted extended graph with \(R = \{x, y\}\) and such that \(G = G_1 \otimes G_2\) with \(G_1 = (G_1, (x, z))\) and \(G_2 = (G_2, (z, y))\). Then

\[
ecvs(G^{+0}, R) = \min \left\{ \max \left\{ ecvs(G_1^{+0}, R), ecvs(G_2^{+0}, R) \right\} \right. \\
\left. \max \left\{ ecvs(G_1^{+0}, R), ecvs(G_2^{+0}, R) \right\} \right. \\
\]

where \(G_1\) (resp. \(G_2\)) is obtained from \(G_1\) (resp. \(G_2\)) by adding \(y\) (resp. \(x\)) as an isolated vertex, and where \(R_1 = \{z, y\}, R_2 = \{x, z\}\).

Proof. Let \(\sigma^* \in L^e(G, R)\) be a connected layout of minimum cost. Consider the neighbor \(v\) of \(z\) such that \(\sigma^*(v)\) is minimum. By the connectivity of \(\sigma^*, \sigma^*(v) < \sigma^*(z)\). Suppose without loss of generality that \(v \in V_1\). The case \(v \in V_2\) is symmetric. From \(\sigma^*\), we define a layout \(\sigma\) of \((G^{+F}, R)\) as follows: \(\sigma = (x, y) \odot \sigma^*[V_1 \setminus \{x\}] \odot \sigma^*[V_2 \setminus \{y, z\}]\) (see Figure 5).

![Figure 5: Rearranging a layout \(\sigma^*\) of minimum cost into \(\sigma = (x, y) \odot \sigma^*[V_1 \setminus \{x\}] \odot \sigma^*[V_2 \setminus \{y, z\}]\). Red diamond vertices belong to \(V_1\) \(\setminus \{x\}\), blue diamond vertices belong to \(V_2\) \(\setminus \{y, z\}\) and red square vertices are the roots. Observe that the path \(v, z, v_2\) certifies that \(v \in S_{\sigma}^{(e)}(v_2)\). But as \(\sigma(z) < \sigma(v_2), v \notin S_{\sigma}^{(e)}(v_2)\). Instead we have that \(z \in S_{\sigma}^{(e)}(v_2)\).

Claim 4. \(\sigma \in L^e(G^{+0}, R)\).

Let \(v_1\) be a vertex of \(V_1\) distinct from \(x\) and \(z\). Every neighbor of \(v_1\) belongs to \(V_1\). As the relative ordering between vertices of \(V_1\) is left unchanged, vertex \(v_1\) has a neighbor prior to it in \(\sigma\). Suppose that \(v_2\) is a vertex of \(V_2\) distinct from \(y\) and \(z\). Then as in the previous case, every neighbor of \(v_2\) belongs to \(V_2\). As the relative ordering between vertices of \(V_2\) has only been modified by moving \(z\) ahead, \(v_2\) has a neighbor prior to it in \(\sigma\). We are left with vertex \(z\) : by assumption, \(z\) has a neighbor in \(V_1\) such that \(\sigma^*(v) < \sigma^*(z)\), implying that \(\sigma(v) < \sigma(z)\). As every vertex has a neighbor prior to it is \(\sigma\), the layout \(\sigma\) is connected.

Claim 5. \(ecost((G^{+0}, R), \sigma) = ecost((G^{+0}, R), \sigma^*)\).

We first consider a vertex \(v_1 \in V_1\). By construction, we have \(\sigma^*[V_1] = \sigma[V_1]\) and for every vertex \(v_2 \in V_2 \setminus \{x, y\}\), \(\sigma(v_1) \leq \sigma(v_2)\). It follows that \(S_{\sigma}^{(e)}(v_1) \subseteq V_1 \cup R\). Suppose that \(\sigma(v_1) > \sigma(z)\). As \(z\) separates \(v_1\) from the vertices of \(V_2\), we have that \(S_{\sigma}^{(e)}(v_1) = S_{\sigma}^{(e)}(v_1)\). Suppose now that \(\sigma(v_1) \leq \sigma(z)\) and let \(v \in V_1\) a vertex that belongs to \(S_{\sigma}^{(e)}(v_1)\). Then, as \(z\) is a cut vertex of \(G^{+0}\), there exists a \((v, v_1)\)-path \(P\) in \(G_1\) such that every vertex \(v' \in P\) distinct from \(v\) satisfies \(\sigma(v_1) \leq \sigma(v')\). As \(\sigma^*[V_1] = \sigma[V_1]\), the path \(P\) certifies that \(v \in S_{\sigma}^{(e)}(v_1)\), implying that \(|S_{\sigma}^{(e)}(v_1)| \leq |S_{\sigma}^{(e)}(v_1)|\).
Let us now consider a vertex \( v_2 \in V_2 \setminus \{ y, z \} \). Observe that, as \( y \) is a cut vertex, \( \sigma^*(y) < \sigma^*(v_2) \) implies that \( S^{(e)}_\sigma(v_2) \subseteq V_2 \). As \( \sigma[V_2 \setminus \{ z \}] = \sigma^*[V_2 \setminus \{ z \}] \), every vertex \( v \in V_2 \setminus \{ z \} \) that belongs to \( S^{(e)}_\sigma(v_2) \) also belongs to \( S^{(e)}_{\sigma^*}(v_2) \). Suppose that \( z \in S^{(e)}_\sigma(v_2) \). Then there exists a \((z,v_2)\)-path \( P \) such that every internal vertex \( v \) of \( P \) satisfies \( \sigma(v_2) < \sigma(v) \). As \( \sigma[V_2 \setminus \{ z \}] = \sigma^*[V_2 \setminus \{ z \}] \), we also have \( \sigma^*(v_2) < \sigma^*(v) \). Let us distinguish two cases:

- If \( \sigma^*(z) < \sigma^*(v_2) \), then \( z \in S^{(e)}_{\sigma^*}(v_2) \) implying that \( S^{(e)}_\sigma(v_2) \subseteq S^{(e)}_{\sigma^*}(v_2) \).

- Otherwise, \( \sigma^*(z) > \sigma^*(v_2) \) and then \( z \notin S^{(e)}_{\sigma^*}(v_2) \). But in that case, let us recall that by assumption the first neighbor \( v \) of \( z \) in \( \sigma^* \) belongs to \( V_1 \). It follows that \( v \in S^{(e)}_{\sigma^*}(v_2) \). As we argue that \( S^{(e)}_{\sigma^*}(v_2) \subseteq V_2 \), \( v \in S^{(e)}_\sigma(v_2) \setminus S^{(e)}_{\sigma^*}(v_2) \). So \( v \) is a replacement vertex for \( z \) in \( S^{(e)}_{\sigma^*}(v_2) \), implying that \( |S^{(e)}_\sigma(v_2)| \leq |S^{(e)}_{\sigma^*}(v_2)| \).

So we proved that \( \text{ecost}(L(G_1^+, R), \sigma) \leq \text{ecost}(L(G_2^+, R), \sigma^*) \). As by Claim 4, \( \sigma \in L^c(G_1^+, R) \), the optimality of \( \sigma^* \) implies that \( \text{ecost}(L(G_1^+, R), \sigma) = \text{ecost}(L(G_1^+, R), \sigma^*) \).

**Claim 6.** \( \text{ecvs}(G_1^+, R) = \max \left\{ \text{ecvs}(G_1^+(xy), R), \text{ecvs}(G_2^+, R_2) \right\} \). (See Figure 7)

![Figure 6: Decomposition of an extended graph resulting from a series composition.](image)

From the proof of Claim 4, we deduce that \( \sigma_1 = \sigma[V_1 \cup R] \in L^c(G_1^+(xy), R) \). Observe that in the extended graph \( G_1^+(xy) \), the fictive edge \( xy \) simulates every \((z,y)\)-path of \( G \) whose internal vertices belong to \( V_2 \). It follows that for every vertex \( v_1 \in V_1 \cup R \), \( S_\sigma^+(v_1) = S_\sigma(v_1) \). Likewise, from the proof of Claim 4, we deduce that \( \sigma_2 = \sigma[V_2] \in L^c(G_2^+, \{y,z\}) \). As \( z \) separates the vertices of \( V_2 \) from the other vertices, for every \( v_2 \in V_2 \setminus \{ z \} \), we have \( S^{(e)}_\sigma(v_2) = S^{(e)}_{\sigma^*}(v_2) \), proving the claim.

**Lemma 5.** Let \((G^+, R)\) be a rooted extended graph such that \( R = \{ x, r_1 \ldots r_k \} \) (with \( k > 0 \)), \( F = \{ y_i \mid i \in [k] \} \) and \((G[V \setminus \{ r_1, \ldots, r_k \}, (x,y)] = G_1 \otimes G_2 \) with \( G_1 = (G_1(x,z)) \) and \( G_2 = (G_2(z,y)) \). Then

\[
\text{ecvs}(G^+, R) = \max \left\{ \text{ecvs}(G_1 \cup R)^+ F', R), \text{ecvs}(G_2 \cup R')^+, R' \right\},
\]

where \( F' = \{ z_i \mid i \in [k] \} \) and \( R' = \{ z, r_1, \ldots, r_k \} \).

**Proof.** Let \( \sigma^* \in L^c(G^+, R) \) be a connected layout of minimum cost. From \( \sigma^* \), we define a layout \( \sigma \) of \((G^+, R)\) as follows: \( \sigma = \langle r_1, \ldots, r_k \rangle \circ \sigma^*[V_1] \circ \sigma^*[V_2 \setminus \{ z \}] \) (see Figure 7). We observe that in this case, as \( y \) is not a root, \( \sigma^*(z) \leq \sigma^*(v_2) \) for every vertex \( v_2 \in V_2 \).
Suppose that \( \sigma \) is a connected layout of \( G^+ \) with \( \sigma^* \) as its minimum cost. Let \( v \in V_1 \setminus \{x\} \) and let \( z \) be a neighbor of \( v \) in \( V \). Then as in the previous case, every neighbor of \( v \) belongs to \( V_2 \). As the relative ordering between vertices of \( V_2 \) is left unchanged, vertex \( v \) has a neighbor prior to it in \( \sigma \). It follows that the layout \( \sigma \) is connected.

Let us now conclude the proof of the lemma. Claim 7 and the optimality of \( \sigma^* \) imply that \( \sigma \) is a connected layout of \( G^+ \) of minimum cost. From the proof of Claim 7 we deduce that \( \sigma = \sigma_1 \in L^c(G[V_1 \cup R]^{+F'}, R) \). Observe that in the extended graph \( G[V_1 \cup R]^{+F'} \), for \( i \in [k] \), the fictive edge \( zr_i \in F' \) simulates every simple extended \((z, r_i)\)-path in \( G^+ \). It follows that for every vertex \( v \in V_1 \cup R \), \( \sigma^*(v) \) simulates every simple extended \((z, r_i)\)-path in \( G^+ \). Likewise, from the
proof of Claim 7, we deduce that \( \sigma_2 = \sigma[V_2 \cup R'] \in \mathcal{L}^c(G[V_2 \cup R'] + F', R') \). As noticed before, \( z \) separates the vertices of \( V_2 \) from vertices of \( V_1 \). It follows for every \( v_2 \in V_2 \setminus \{z\} \), we have \( S_{\sigma}^{(e)}(v_2) = S_{\sigma_2}^{(e)}(v_2) \), completing the proof. 

**Lemma 6.** Let \((G^{+F \cup \{xy\}}, R)\) be a rooted extended graph such that \( R = \{z, r_1 \ldots r_k\} \) (with \( k > 0 \)), \( F = \{yr_i \mid i \in [k]\} \) and \((G[V \setminus \{r_1, \ldots, r_k\}], (x, y)) = G_1 \otimes G_2 \) with \( G_1 = (G_1, (x, z)) \) and \( G_2 = (G_2, (z, y)) \). Then

\[
\text{ecvs}(G^{+F \cup \{xy\}}, R) = \max \left\{ \text{ecvs}(G[V_1 \cup R] + F' \cup \{xz\}, R), \text{ecvs}(G[V_2 \cup R'] + F \cup \{xy\}, R') \right\},
\]

where \( R' = \{z, r_1, \ldots, r_k, x\} \) and \( F' = \{zr_i \mid i \in [k]\} \).

**Proof.** We proceed as in the proof of Lemma 5. We transform a layout \( \sigma \in \mathcal{L}^c(G^{+F \cup \{xy\}}, R) \) of minimum cost into the layout \( \sigma = \langle r_1, \ldots, r_k \rangle \otimes \sigma^*[V_1] \otimes \sigma^*[V_2 \setminus \{z\}] \) (see Figure 7). Observe that since the solid graph \( G \) is the same as in Lemma 5 Claim 7 applies and thereby \( \sigma \in \mathcal{L}^c(G^{+F \cup \{xy\}}, R) \).

![Figure 9: Decomposition of an extended graph resulting from a series composition.](image)

**Claim 9.** \( \text{ecost}((G^{+F \cup \{xy\}}, R), \sigma) = \text{ecost}((G^{+F \cup \{xy\}}, R), \sigma^*) \).

The existence of the fictive edge \( xy \) does not change the arguments used in the proof of Claim 5. Let us consider \( v_1 \in V_1 \) and \( v_2 \in V_2 \setminus \{z\} \). First as \( \sigma^*(x) < \sigma^*(z) < \sigma^*(v_2) \) and \( z \) separates vertices of \( V_2 \) from vertices of \( V_1 \), we obtain that \( S_{\sigma}^{(e)}(v_1) \cap (V_2 \setminus \{z\}) = \emptyset \) and \( S_{\sigma}^{(e)}(v_2) \cap (V_1 \setminus \{x, z\}) = \emptyset \). Moreover \( \sigma^*[V_1 \cup R] = \sigma[V_1 \cup R] \) implies \( S_{\sigma}^{(e)}(v_1) = S_{\sigma}^{(e)}(v_1) \) and \( \sigma^*[V_2 \cup R] = \sigma[V_2 \cup R] \) implies \( S_{\sigma}^{(e)}(v_2) = S_{\sigma}^{(e)}(v_2) \), proving the claim. In other words, \( \sigma \) is a connected layout of \( G^{+F \cup \{xy\}} \) of minimum cost.

Let us now conclude the proof of the lemma. From the proof of Claim 7 we deduce that \( \sigma_1 = \sigma[V_1 \cup R] \in \mathcal{L}^c(G[V_1 \cup R] + F' \cup \{xy\}, R) \). Observe that in the extended graph \( G[V_1 \cup R] + F' \), for \( i \in [k] \), the fictive edge \( zr_i \in F' \) (for \( i \in [k] \)) simulates every simple extended \((z, r_i)\)-path in \( G^{+F \cup \{xy\}} \). Similarly the fictive edge \( zx \) aims at representing extended \((z, x)\)-paths avoiding \( V_1 \setminus \{x\} \) in \( G^{+F \cup \{xy\}} \). It follows that for every vertex \( v_1 \in V_1 \cup R \), \( S_{\sigma_1}^{(e)}(v_1) = S_{\sigma}^{(e)}(v_1) \). Likewise, from the proof of Claim 7 we deduce that \( \sigma_2 = \sigma[V_2 \cup R'] \in \mathcal{L}^c(G[V_2 \cup R'] + F \cup \{xz\}, R') \). As noticed before, \( z \) separates the vertices of \( V_2 \) from vertices of \( V_1 \), and thereby for every \( v_2 \in V_2 \),
$V_1 \setminus \{z\} \cap S^{(c)}(v_2) = \emptyset$. Observe that despite the fact that $x \notin V_2$, it is preserved as a (pendant) root in $R'$. It follows for every $v_2 \in V_2 \setminus \{z\}$, we have $S^{(c)}(v_2) = S^{(c)}(v_2)$, completing the proof.

3.2 The dynamic programming algorithm

Let us first focus on the case of biconnected series-parallel graph. Recall that $c_{sp} := [2(\log n + 1)]$, as defined in the beginning of Section 3.

Proposition 1. Let $G$ be a biconnected series-parallel graph on $n$ vertices. Then computing $ctw(G)$ can be done $O(n^2 \cdot \log n)$-time.

Proof. Let $G = (G, (x, y))$, with $G = (V, E)$, be a biconnected 2-terminal graph such that $xy \in E$. By Theorem 1, we have $G = G_1 \oplus G_2$ where $G_1 = (G_1, (x, y))$ with $G_1 = (\{x, y\}, \{xy\})$ and $G_2 = (G_2, (x, y))$ with $G_2 = (V, E \setminus \{xy\})$. By Theorem 2, in linear time, we can compute $T(G)$, the series-parallel tree of $G$. Recall that the root of $T(G)$ corresponds to the parallel composition $G_1 \oplus G_2$. We let $G_t = (G_t, (x_t, y_t))$ with $G_t = (V_t, E_t)$ denote the subgraph represented by node $t$ of $T(G)$. We let denote $G_t$ be the graph $G_t$ augmented with $k$ isolated vertices $r_1, \ldots, r_k$. In order to apply the rules described in Lemmas 1–3 the table $DP_t[\cdot]$ stored at every node $t$ contains the following values:

- $DP_t[0] = \text{ecvs}(G_t^+, \{x_t, y_t\});$
- for $k \in [c_{sp}, \]$, $DP_t[k, x_t] = \text{ecvs}(G_t[V_t \cup R]^+, F)$ where $R = \{x_t, r_1, \ldots, r_k\}$ and $F = \{y_tr_i \mid i \in [k]\}$;
- for $k \in [c_{sp}, \]$, $DP_t[k, y_t] = \text{ecvs}(G_t[V_t \cup R]^+, F)$ where $R = \{y_t, r_1, \ldots, r_k\}$ and $F = \{x_tr_i \mid i \in [k]\}$;
- for $k \in [c_{sp} - 1, \]$, $DP_t[k, x_t, x_t y_t] = \text{ecvs}(G_t[V_t \cup R]^+, F)$ where $R = \{x_t, r_1, \ldots, r_k\}$ and $F = \{x_t y_t\} \cup \{y_tr_i \mid i \in [k]\}$;
- for $k \in [c_{sp} - 1, \]$, $DP_t[k, y_t, x_t y_t] = \text{ecvs}(G_t[V_t \cup R]^+, F)$ where $R = \{y_t, r_1, \ldots, r_k\}$ and $F = \{x_t y_t\} \cup \{x_tr_i \mid i \in [k]\}$.

The bounds on the integer $k$ determining the number of entries in the table $DP[t]$ of a node $t$ is fixed by the upper-bound $c_{sp}$ implied by Theorem 3. The initialization of the table for leaf nodes (see below) guarantees this bound is respected.

We observe that for every node $t$, every entry of $DP[t]$ corresponds to an extended rooted two-terminal graphs $(H^+, F)$ such that: $R$ contains at least two vertices; at least one vertex of $R$ is a terminal vertex; and every root vertex that is not a terminal vertex is an isolated vertex. These properties implies that every connected component of $H^+$ contains a root vertex, and thereby it guarantees the existence of a connected layout of $(H^+, F)$.

Suppose that $t$ represents a parallel composition $G_t = G'_t \oplus G''_t$. The children $t_1$ and $t_2$ of $t$ respectively represent the 2-terminal graphs $G'_t = (G'_t, (x_t, y_t))$ and $G''_t = (G''_t, (x_t, y_t))$. Then $DP_t$ is computed as follows:

- By Lemma 1, $DP_t[0] = \max \{DP(t_1)[0], DP(t_2)[0]\}$.
• By Lemma \(2\) we have for \(k \in [c_{sp} - 1]\):

\[
DP_t[k, x_t] = \min \left\{ \max\{DP(t_1)[k, x_t, x_t y_t], DP(t_2)[0]\}, \max\{DP(t_2)[k, x_t, x_t y_t], DP(t_1)[0]\}\right\}
\]

and

\[
DP_t[k, y_t] = \min \left\{ \max\{DP(t_1)[k, y_t, x_t y_t], DP(t_2)[0]\}, \max\{DP(t_2)[k, y_t, x_t y_t], DP(t_1)[0]\}\right\}.
\]

• By Lemma \(3\) we have for \(k \in [c_{sp} - 1]\): \(DP_t[k, x_t, x_t y_t] = DP_t[k, x_t]\) and \(DP_t[k, y_t, x_t y_t] = DP_t[k, y_t]\).

Suppose that \(t\) represents a series composition \(G_t = G'_t \otimes G''_2\). The children \(t_1\) and \(t_2\) of \(t\) respectively represent the 2-terminal graphs \(G'_t = (G'_1, (x_t, z))\) and \(G''_2 = (G'_2, (z, y_t))\). Then \(DP_t\) is computed as follows:

• By Lemma \(4\) we have for \(k \in [c_{sp} - 1]\):

\[
DP_t[0] = \min \left\{ \max\{DP(t_1)[0], DP(t_2)[0]\}, \max\{DP(t_2)[0], DP(t_1)[0]\}\right\}.
\]

• By Lemma \(5\) we have for \(k \in [c_{sp} - 1]\):

\[
DP_t[k, x_t] = \max\{DP(t_1)[k, x_t], DP(t_2)[k, z]\} \quad \text{and} \quad DP_t[k, y_t] = \max\{DP(t_1)[k, y_t], DP(t_2)[k, z]\}.
\]

• By Lemma \(6\) we have for \(k \in [c_{sp} - 2]\):

\[
DP_t[k, x_t, x_t y_t] = \max\{DP(t_1)[k, x_t, x_t z], DP(t_2)[k + 1, z]\} \quad \text{and} \quad DP_t[k, y_t, x_t y_t] = \max\{DP(t_1)[k, y_t, x_t z], DP(t_2)[k + 1, z]\}.
\]

For every non-leaf node \(t\) of \(T(G)\), the entries of \(DP_t\) are initialized to some dummy value \(\bot\). Every leaf node \(t\) of \(T(G)\) represents the single edge graph, that is \(G_t = (V_t, E_t)\) with \(V = \{x_t, y_t\}\) and \(E = \{x_t y_t\}\). We let denote \(G_t\) the graph \(G_t\) augmented with \(k\) isolated vertices \(r_1, \ldots, r_k\). Then we can initialize the values associated to a leaf node \(t\) as follows:

• \(DP_t[0] = ecvs(G_t^{+\theta}, \{x_t, y_t\}) = 1\)

• for \(k \in [c_{sp}-1]\), \(DP_t[k, x_t] = ecvs(G_t^{+F}, \{x_t, r_1, \ldots, r_k\}) = k + 1\) where \(F = \{y_t r_i \mid i \in [k]\}\).

• for \(k \in [c_{sp}-1]\), \(DP_t[k, y_t] = ecvs(G_t^{+F}, \{y_t, r_1, \ldots, r_k\}) = k + 1\) where \(F = \{x_t r_i \mid i \in [k]\}\).

• for \(k \in [c_{sp}-2]\), \(DP_t[k, x_t, x_t y_t] = ecvs(G_t^{+F}, \{x_t, r_1, \ldots, r_k\}) = k + 1\) where \(F = \{y_t r_i \mid i \in [k]\} \cup \{x_t y_t\}\).

• for \(k \in [c_{sp}-2]\), \(DP_t[k, y_t, x_t y_t] = ecvs(G_t^{+F}, \{y_t, r_1, \ldots, r_k\}) = k + 1\) where \(F = \{y_t r_i \mid i \in [k]\} \cup \{x_t y_t\}\).

As the series-parallel tree \(T(G)\) contains \(O(n)\) nodes, filling the table \(DP_t\) for every node \(t\), is achieved in \(O(n \cdot \log n)\)-time. By Theorem \(8\) we obtain that \(ctw(G) = cvs(G) = \min\{ecvs(G^{+\theta}, \{x, y\}) \mid xy \in E\}\). This implies that the whole algorithm runs in \(O(n^2 \cdot \log n)\)-time. \(\Box\)
3.3 Generalization to graph of treewidth at most two

Recall that a graph $G$ has treewidth at most two if and only if every biconnected component of $G$ is a series-parallel graph. So we need a lemma to deal with cut-vertices.

**Lemma 7.** Let $G = (V, E)$ be a graph containing a cut vertex $x$ and let $G_1 = [C_1 \cup \{x\}]$, $G_k = G[C_k \cup \{x\}]$ be the induced subgraphs where $C_1, \ldots, C_k$ denote the connected components of $G - x$. Then

$$
\text{cvs}(G) = \min_{i \in [k]} \left\{ \max \left\{ \text{cvs}(G_i), \max \{ \text{cvs}(G_j, \{x\}) \mid j \in [k], j \neq i \} \right\} \right\} \text{ (see Figure 10)}.
$$

![Figure 10: Decomposition of a graph with a cut vertex. If an optimal connected layout $\sigma^*$ starts at an arbitrary vertex of $G_1$, then $\sigma^*[V_2]$ and $\sigma^*[V_3]$ start at $x$, which becomes a root of $G_2$ and of $G_3$.](image)

**Proof.** Let us consider $\sigma \in \mathcal{L}(c)(G)$ and suppose that the first vertex of $\sigma$ belongs to $V_1$. Then observe that $\sigma$ can be rearranged into $\tau = \sigma[V_1] \circ \sigma[V_2 \setminus \{x\}] \cdots \circ \sigma[V_k \setminus \{x\}]$ and that since $x$ is a cut-vertex then $\tau \in \mathcal{L}(c)(G)$ as well. The statement follows.

**Theorem 6.** Computing the connected treewidth of a graph of treewidth at most 2 requires $O(n^3 \cdot \log n)$-time.

**Proof.** Let $G$ be a graph of treewidth at most 2. The algorithm first computes the biconnected tree decomposition of $G$. This can be done in linear time. Following Lemma 7, we select a biconnected component $C_1$ in which the connected layout will start. This generates for every biconnected component $C_k$ distinct from $C_1$ a root vertex $r_k$. Then using Proposition 11 in $O(n^2 \cdot \log n)$ we can compute $\text{cvs}(G[C_1])$ and $\text{cvs}(G[C_k], \{r_k\})$ for each $k \neq 1$. This leads to an $O(n^3 \cdot \log n)$-time algorithm.

4 Discussion and open problems

We obtained a polynomial time algorithm to compute the connected treewidth for the class of treewidth at most two graphs. This result naturally leads to the problem of determining the algorithmic complexity of computing the connected treewidth for the class of bounded treewidth graphs. To discuss this, we present the problem as a decision our problem:
**Connected Treewidth**

*Input:* A graph $G$ and an integer $k$.

*Question:* $\text{ctw}(G) \leq k$?

Our result implies that Connected Treewidth can be solved in $O(n^3 \cdot k)$-time for graphs of treewidth at most 2. We conjecture the following:

**Conjecture 1.** Connected Treewidth can be solved by an $O(n^{f(tw(G))})$-time algorithm.

**Conjecture 2.** Connected Treewidth can be solved by an $O(f(k, tw(G)) \cdot n)$-time algorithm.

**Conjecture 3.** Connected Treewidth can be solved by an $O(n^{f(k)})$-time algorithm.

Our result can be seen as a special case of Conjecture 1 (when $tw(G) \leq 2$). A general resolution of Conjecture 1 would require an vast extension of our dynamic programming approach. In our approach we essentially solve a slightly modified problem where the input is pair $(G, e)$, where $e \in E(G)$, and we return the minimum cost of a layout that begins with the endpoints of $e$. Then we reduce the computation of connected treewidth to this problem by paying an overhead of $O(n^2)$. An interesting question is whether and how a similar approach might work for the general case. Of course, one may try to reduce the $O(n^3 \cdot \log n)$-time complexity of our algorithm by avoiding such reductions and directly build a dynamic programming scheme for Connected Treewidth on graphs of treewidth $\leq k$. We believe that this is possible and can reduce the time complexity to $O(n^c)$ for some $1 < c \leq 3$.

For Conjecture 2 one may attempt to use tools related to Courcelle’s theorem. This would require to express the question $\text{ctw}(G) \leq k$ in using a formula $\phi_k$ in Monadic Second Order Logic (MSOL) which is far from obvious. A possible direction would be to consider the contraction-obstruction set $\mathcal{Z}_k$ of the class $\mathcal{G}_k = \{G \mid \text{ctw}(G) \leq k\}$, i.e., the contraction minimal graphs not in $\mathcal{G}_k$. If this set is finite (as it is the case for treewidth and pathwidth) then MSOL expressibility follows. However, it turns out that $\mathcal{Z}_k$ is infinite for every $k \geq 2$, as observed in [1]. Another promising direction is to directly prove that the question $\text{ctw}(G) \leq k$ has finite index avoiding Courcelles theorem and devising an ad-hoc dynamic programming algorithm.

A proof of Conjecture 3 would follow if we devise an algorithm to check whether $\text{ctw}(G) \leq k$ in $n^{f(k, tw(G))}$ time. This follows directly from the fact that yes-instances of Connected Treewidth have always treewidth at most $k$. Such a result would be analogous to the one of [16] for connected pathwidth and is perhaps the first (and easier) to be attacked among the three above conjectures. Certainly the first two conjectures have their counterparts on connected pathwidth and remain open.

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