A new discrepancy principle ∗†

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1. Introduction
The aim of this note is to prove a new discrepancy principle. The advantage of the new discrepancy principle compared with the known one consists of solving a minimization problem approximately, rather than exactly, and in the proof of a stability result. To explain this in more detail, let us recall the usual discrepancy principle, which can be stated as follows. Consider an operator equation

\[ Au = f, \]

where \( A : H \to H \) is a bounded linear operator on a Hilbert space \( H \), and assume that the range \( R(A) \) is not closed, so that problem (1) is ill-posed. Assume that \( f = Ay \) where \( y \) is the minimal-norm solution to (1), and that noisy data \( f_\delta \) are given, such that \( ||f_\delta - f|| \leq \delta \). One wants to construct a stable approximation to \( y \), given \( f_\delta \).

The variational regularization method for solving this problem consists of solving the minimization problem

\[ F(u) := ||Au - f_\delta||^2 + \epsilon ||u||^2 = \min. \]

(2)

It is well known that problem (2) has a solution and this solution is unique (see e.g. [1]). Let \( u_{\delta,\epsilon} \) solve (2). Consider the equation for finding \( \epsilon = \epsilon(\delta) \):

\[ ||Au_{\delta,\epsilon} - f_\delta|| = C\delta, \]

(3)

where \( C = \text{const} > 1 \). Equation (3) is the usual discrepancy principle. One can prove that equation (3) determines \( \epsilon = \epsilon(\delta) \) uniquely, \( \epsilon(\delta) \to 0 \) as \( \delta \to 0 \), and \( u_\delta := u_{\delta,\epsilon(\delta)} \to y \) as \( \delta \to 0 \). This justifies the usual discrepancy principle for choosing the regularization parameter (see [1] and [2] for various justifications of this principle, [3] for the dynamical systems method for stable solution of equation (1), and [4] for a method of solving nonlinear ill-posed problems).

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The drawback of this principle consists of the necessity to solve problem (2) exactly. The other drawback is the lack of information concerning stability of the solution to (3): if one solves (2) approximately in some sense, will the element \( u_{\delta,\epsilon(\delta)} \) (with \( \epsilon(\delta) \) being an approximate solution to (3)) converge to \( y \)?

Our aim is to formulate and justify a new discrepancy principle which deals with the both issues mentioned above.

Our basic result is:

**Theorem 1.** Assume:

i) \( A \) is a bounded linear operator in a Hilbert space \( H \),

ii) equation \( Au = f \) is solvable, and \( y \) is its minimal-norm solution,

iii) \( ||f_\delta - f|| \leq \delta, \ ||f_\delta|| > C\delta \), where \( C > 1 \) is a constant.

Then:

j) equation (3) is solvable for \( \epsilon \) for any fixed \( \delta > 0 \), where \( u_{\delta,\epsilon} \) is any element satisfying inequality \( F(u_{\delta,\epsilon}) \leq m + (C^2 - 1 - b)\delta^2 \), \( F(u) := ||A(u) - f_\delta||^2 + \epsilon||u||^2, \ m = m(\delta, \epsilon) := \inf_u F(u), \ b = \text{const} > 0, \) and \( C^2 > 1 + b \), and

jj) if \( \epsilon = \epsilon(\delta) \) solves (3), and \( u_\delta := u_{\delta,\epsilon(\delta)} \), then \( \lim_{\delta \to 0} ||u_\delta - y|| = 0. \)

In Section 2 proof of Theorem 1 is given.

2. **Proof**

**Proof of Theorem 1.** Let us first prove the existence of a solution to (3). We claim that the function \( h(\delta, \epsilon) := ||Au_{\delta,\epsilon} - f_\delta|| \) is greater than \( C\delta \) for sufficiently large \( \epsilon \), and smaller than \( C\delta \) for sufficiently small \( \epsilon \). If this is proved, then the continuity of \( h(\delta, \epsilon) \) with respect to \( \epsilon \) on \( (0, \infty) \) implies that the equation \( h(\delta, \epsilon) = C\delta \) has a solution.

Let us prove the claim. As \( \epsilon \to \infty \), we use the inequality:

\[
\epsilon||u_{\delta,\epsilon}||^2 \leq F(u_{\delta,\epsilon}) \leq m + (C^2 - 1 - b)\delta^2 \leq F(0) + (C^2 - 1 - b)\delta^2,
\]

and, as \( \epsilon \to 0 \), we use another inequality:

\[
||Au_{\delta,\epsilon} - f_\delta||^2 < F(u_{\delta,\epsilon}) \leq m + (C^2 - 1 - b)\delta^2 \leq F(y) + (C^2 - 1 - b)\delta^2 = \epsilon||y||^2 + (C^2 - b)\delta^2.
\]

This inequality implies

\[
h^2(\delta, \epsilon) < \epsilon||y||^2 + (C^2 - b)\delta^2.
\]

As \( \epsilon \to \infty \), one gets \( ||u_{\delta,\epsilon}|| \leq \frac{c}{\epsilon} \to 0 \), where \( c > 0 \) is a constant depending on \( \delta \). Thus, by the continuity of \( A \), one obtains

\[
\lim_{\epsilon \to \infty} h(\delta, \epsilon) = ||A(0) - f_\delta|| = ||f_\delta|| > C\delta.
\]

As \( \epsilon \to 0 \), one gets

\[
\lim_{\epsilon \to 0} h(\delta, \epsilon) = \lim_{\epsilon \to 0} \inf \epsilon||y||^2 + (C^2 - b)\delta^2)^{1/2} < C\delta.
\]

Therefore equation \( h(\delta, \epsilon) = C\delta \) has a solution \( \epsilon = \epsilon(\delta) > 0. \)
Let us now prove that if \( u_\delta := u_{\delta, \epsilon(\delta)} \), then
\[
\lim_{\delta \to 0} ||u_\delta - y|| = 0. \tag{4}
\]
From the estimate
\[
F(u_\delta = ||Au_\delta - f_\delta||^2 + \epsilon ||u_\delta||^2 \leq C^2 \delta^2 + \epsilon |y|^2,
\]
and from (3), it follows that
\[
||u_\delta|| \leq ||w||, \tag{5}
\]
where \( w \) is any solution to (1). We will use (5) with \( w = y \) and \( w = U \), where \( U \) is a solution to (1) constructed below, and \( y \) is a minimal-norm solution to (1).

Thus, one may assume that \( u_\delta \to U \), and from (3) it follows that \( Au_\delta \to f \) as \( \delta \to 0 \). This implies, as we prove below (see (8)), that
\[
AU = f. \tag{6}
\]
We also prove below that from (5) it follows that
\[
\lim_{\delta \to 0} ||u_\delta - U|| = 0, \quad ||U|| \leq ||y||. \tag{7}
\]
The minimal norm solution to equation (1) is unique. Consequently, (7) implies \( U = y \). Thus, (4) holds.

Let us now prove (6). We have
\[
(f, v) = \lim_{\delta \to 0} (Au_\delta, v) = \lim_{\delta \to 0} (u_\delta, A^*v) = (U, A^*v) = (AU, v) \quad \forall v \in H. \tag{8}
\]
Since \( v \) is arbitrary, (8) implies (6).

Finally, we prove (7). We have \( u_\delta \to U \). Thus, \( ||U|| \leq \liminf_{\delta \to 0} ||u_\delta|| \). Inequality (5) implies \( \limsup_{\delta \to 0} ||u_\delta|| \leq ||U|| \). Consequently, \( ||u_\delta|| \to ||U|| \). It is well known that the weak convergence together with convergence of the norms imply in a Hilbert space strong convergence. Therefore \( \lim_{\delta \to 0} ||u_\delta - U|| = 0 \). Taking \( w = y \) in (5), and then passing to the limit \( \delta \to 0 \) in (5), yields inequality (7). Thus, both parts of (7) are established.

Since \( U \) solves equation (1), and \( ||U|| \leq ||y|| \), it follows that \( U = y \), and (4) holds.

Theorem 1 is proved □.

References

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