New Approaches to Pythagorean Fuzzy Averaging Aggregation Operators

Khaista Rahman¹, Muhammad Sajjad Ali Khan¹, Murad Ullah²

¹Department of Mathematics, Hazara University, Mansehra, Pakistan
²Department of Mathematics, Islamia College University Peshawar, Pakistan

Email address: Khaista355@yahoo.com (K. Rahman), sajjadalimath@yahoo.com (M. S. A. Khan), muradullah90@yahoo.com (M. Ullah),

To cite this article:
Khaista Rahman, Muhammad Sajjad Ali Khan, Murad Ullah. New Approaches to Pythagorean Fuzzy Averaging Aggregation Operators. Mathematics Letters. Vol. 3, No. 2, 2017, pp. 29-36. doi: 10.11648/j.ml.20170302.12

Received: April 13, 2017; Accepted: April 21, 2017; Published: May 22, 2017

Abstract: In this paper, we present two Pythagorean fuzzy averaging aggregation operators such as, Pythagorean fuzzy weighted averaging (PFWA) operator, Pythagorean fuzzy ordered weighted averaging (PFOWA) operator and also introduce some of their basic properties.

Keywords: Pythagorean Fuzzy Sets, PFWA Operator, PFOWA Operator

1. Introduction

Atanassov [1] introduced the concept of IFS characterized by a membership function and a non-membership function. It is more suitable for dealing with fuzziness and uncertainty than the ordinary fuzzy set developed by Zadeh [2] characterized by one membership function. Gau and Buehrer [3] proposed the notion of vague set. Chen and Tan [4] and Hong and Choi [5] presented some techniques for handling multi-criteria fuzzy decision-making problems based on vague sets. Bustine and Burillo [6] showed that the vague set is equivalent to IFS. In 1986, many scholars [7, 8, 9, 10, 11, 12] have done works in the field of AIFS and its applications. Particularly, information aggregation is a very crucial research area in IFS theory that has been receiving more and more focus. Xu [13] developed some basic arithmetic aggregation operators, including IFWA operator, IFOWA operator and IFHA operator and applied them to group decision making. Xu and Yager [14] defined some basic geometric aggregation operators such as, IFWG operator, IFOWG operator and IFHG operator, and applied them to multiple attribute decision making (MADM) based on intuitionistic fuzzy information. Wang and Liu [15] introduced the notion of IFEWG operator and geometric IFEOWG operator and applied them to group decision making. In [16] Wang and Liu also introduced the concept of IFEWA operator and IFEOWA operator. Zhao and Wei [17] introduced the notion of two new types of hybrid aggregation operators such as, IFEHA operator and IFEHG operator. But there are many cases where the decision maker may provide the degree of membership and nonmembership of a particular attribute in such a way that their sum is greater than one. Therefore, Yager [18] introduced the concept of PFS. PFS is more powerful tool to solve uncertain problems. In 2013, Yager and Abbasov [19] introduced the notion of two new Pythagorean fuzzy aggregation operators such as, PFWA operator and PFOWA operator. In [20, 21, 22, 23] K. Rahman et al. introduced the concept of PFHA operator, PFWG operator, PFOWG operator and PFH operator.

Thus keeping the advantage of the above mention aggregation operators, in this paper we introduce two new types Pythagorean fuzzy aggregation operators for Pythagorean fuzzy values such as, PFWA operator, PFOWA operator.
2. Preliminaries

Definition 1: [18] Let \( Q \) be a universal set, then a PFS, \( E \) can be defined as:
\[
E = \{ (q, \Delta_E(q), \nabla_E(q)) \mid q \in Q \},
\]
where \( \Delta_E(q) \) and \( \nabla_E(q) \) are mappings from \( E \) to \([0,1]\), with some conditions such as, \( 0 \leq \Delta_E(q) \leq 1, 0 \leq \nabla_E(q) \leq 1, 0 \leq \Delta_E^2(q) + \nabla_E^2(q) \leq 1, \forall q \in Q \).

In this paper, we consider the interval \( [\Delta_E(q), 1-\Delta_E(q)] \) is a PFS, and replace equation (1) with
\[
E = \{ (q, [\Delta_E(q), 1-\Delta_E(q)]) \mid q \in Q \},
\]
correspondingly. Here the PFS \([\Delta_E(q), 1-\Delta_E(q)]\) show that the fixed degree of the membership \( \Delta_E(q) \) is not defined. However it can lie between, \( \Delta_E(q) \leq \Delta(E(q)) \leq 1 - \Delta_E(q) \), where \( \Delta_E(q) + \Delta_E^2(q) \leq 1 \).

Definition 2: [13] Let \( \varphi_j = [\Delta_{\varphi_j}, 1-\varphi_j] \) be a collection of IFVs, then IFWA operator can be defined as:
\[
IFWA_1 (\varphi_1, \varphi_2, \ldots, \varphi_n) = I_1 \varphi_1 \boxplus I_2 \varphi_2 \boxplus \cdots \boxplus I_n \varphi_n,
\]
where \( I = (I_1, I_2, I_3, \ldots, I_n)^T \) be the weighted vector of \( \varphi_j \) with \( I_j \in [0,1] \) and \( \sum_{j=1}^n I_j = 1 \).

Definition 3: [13] Let \( \varphi_j = [\epsilon_j, 1-\varphi_j] \) be a collection of IFVs, then IFWA operator can be defined as:
\[
IFWA_1 (\varphi_1, \varphi_2, \ldots, \varphi_n) = I_1 \varphi_1 \boxplus I_2 \varphi_2 \boxplus \cdots \boxplus I_n \varphi_n.
\]

3. Operational Laws and Relations

Definition 4. Let \( \mathcal{R}_1 = [h_{\mathcal{R}_1}, 1-\partial_{\mathcal{R}_1}] \), \( \mathcal{R}_2 = [h_{\mathcal{R}_2}, 1-\partial_{\mathcal{R}_2}] \) be the two PFVs, then \( S(\mathcal{R}_1) = h_{\mathcal{R}_1} - \partial_{\mathcal{R}_1} \) and \( S(\mathcal{R}_2) = h_{\mathcal{R}_2} - \partial_{\mathcal{R}_2} \) be the scores of \( \mathcal{R}_1 \), \( \mathcal{R}_2 \) and \( H(\mathcal{R}_1) = h_{\mathcal{R}_1} + \partial_{\mathcal{R}_1} \), \( H(\mathcal{R}_2) = h_{\mathcal{R}_2} + \partial_{\mathcal{R}_2} \) be the accuracy degrees of \( \mathcal{R}_1 \), \( \mathcal{R}_2 \) respectively, then

(a) If \( S(\mathcal{R}_1) < S(\mathcal{R}_2) \), then \( \mathcal{R}_1 < \mathcal{R}_2 \),
(b) If \( S(\mathcal{R}_1) > S(\mathcal{R}_2) \), then \( \mathcal{R}_1 > \mathcal{R}_2 \),
(c) If \( H(\mathcal{R}_1) < H(\mathcal{R}_2) \) then \( \mathcal{R}_1 < \mathcal{R}_2 \),
(d) If \( H(\mathcal{R}_1) > H(\mathcal{R}_2) \) then \( \mathcal{R}_1 > \mathcal{R}_2 \).

Theorem 1: Let \( \mathcal{R}_1 = [h_{\mathcal{R}_1}, 1-\partial_{\mathcal{R}_1}] \), \( \mathcal{R}_2 = [h_{\mathcal{R}_2}, 1-\partial_{\mathcal{R}_2}] \) be the collection of two PFVs, if \( h_{\mathcal{R}_1} \leq h_{\mathcal{R}_2} \) and \( \partial_{\mathcal{R}_1} \geq \partial_{\mathcal{R}_2} \), then \( \mathcal{R}_1 \leq \mathcal{R}_2 \).

Proof: Since \( S(\mathcal{R}_1) = h_{\mathcal{R}_1} - \partial_{\mathcal{R}_1} \) and \( S(\mathcal{R}_2) = h_{\mathcal{R}_2} - \partial_{\mathcal{R}_2} \).
Now \( h_{\mathcal{R}_1} \leq h_{\mathcal{R}_2} \), \( \partial_{\mathcal{R}_1} \geq \partial_{\mathcal{R}_2} \), then
\[
S(\mathcal{R}_1) - S(\mathcal{R}_2) = (h_{\mathcal{R}_1} - \partial_{\mathcal{R}_1}) - (h_{\mathcal{R}_2} - \partial_{\mathcal{R}_2}) = h_{\mathcal{R}_1} - h_{\mathcal{R}_2} + \partial_{\mathcal{R}_1} - \partial_{\mathcal{R}_2} = (h_{\mathcal{R}_1} - h_{\mathcal{R}_2}) + (\partial_{\mathcal{R}_1} - \partial_{\mathcal{R}_2}).
\]
If \( h_{\mathcal{R}_1} = h_{\mathcal{R}_2} \), \( h_{\mathcal{R}_1} + (\partial_{\mathcal{R}_1} - \partial_{\mathcal{R}_2}) = h_{\mathcal{R}_1} - \partial_{\mathcal{R}_2} \).
Then \( \mathcal{R}_1 = \mathcal{R}_2 \).
(5)

And
\[
\partial_{\mathcal{R}_1} - \partial_{\mathcal{R}_2} = \partial_{\mathcal{R}_1} - \partial_{\mathcal{R}_2} = \partial_{\mathcal{R}_1} - \partial_{\mathcal{R}_2}.
\]
(6)

From equation (5) and equation (6), we have
\[
\mathcal{R}_1 = \mathcal{R}_2.
\]
(7)

Otherwise we have \( S(\mathcal{R}_1) - S(\mathcal{R}_2) < 0 \) i.e., \( S(\mathcal{R}_1) < S(\mathcal{R}_2) \).
Thus \( \mathcal{R}_1 < \mathcal{R}_2 \).
(8)

From equation (7) and equation (8), we have \( \mathcal{R}_1 \leq \mathcal{R}_2 \).

Definition 5. Let \( \mathcal{R}_1 = [h_{\mathcal{R}_1}, 1-\partial_{\mathcal{R}_1}] \), \( \mathcal{R}_2 = [h_{\mathcal{R}_2}, 1-\partial_{\mathcal{R}_2}] \) and \( \mathcal{R}_3 = [h_{\mathcal{R}_3}, 1-\partial_{\mathcal{R}_3}] \) be the collection of three PFVs, and \( \lambda > 0 \), then
(1) \( \mathcal{R}_1 \oplus \mathcal{R}_2 = \left[ \sqrt{h_{\mathcal{R}_1}^2 + h_{\mathcal{R}_2}^2} - h_{\mathcal{R}_1}^2, 1-\partial_{\mathcal{R}_1} \partial_{\mathcal{R}_2} \right] \)
(2) \( \lambda \mathcal{R} = \left[ \lambda \left(1 - h_{\mathcal{R}}^2 \right)^{2}, 1-(\partial_{\mathcal{R}})^t \right] \)

Let \( S_{\mathcal{R}_1} (h_{\mathcal{R}_1}, h_{\mathcal{R}_2}) = \sqrt{h_{\mathcal{R}_1}^2 + h_{\mathcal{R}_2}^2} - h_{\mathcal{R}_1} h_{\mathcal{R}_2} \) and \( T_{\mathcal{R}_1} (\partial_{\mathcal{R}_1}, \partial_{\mathcal{R}_2}) = \partial_{\mathcal{R}_1} \partial_{\mathcal{R}_2} \), then the operational law (1) can be rewritten as follows:
\[
\mathcal{R}_1 \oplus \mathcal{R}_2 = \left[ S_{\mathcal{R}_1} (h_{\mathcal{R}_1}, h_{\mathcal{R}_2}), 1-T_{\mathcal{R}_1} (\partial_{\mathcal{R}_1}, \partial_{\mathcal{R}_2}) \right].
\]
(9)

where \( S_{\mathcal{R}_1} (h_{\mathcal{R}_1}, h_{\mathcal{R}_2}) = \sqrt{h_{\mathcal{R}_1}^2 + h_{\mathcal{R}_2}^2} - h_{\mathcal{R}_1} h_{\mathcal{R}_2} \) is called \( t \)-conorm.

Especially, if \( h_{\mathcal{R}_1} = 1-\partial_{\mathcal{R}_1} \) and \( h_{\mathcal{R}_2} = 1-\partial_{\mathcal{R}_2} \), then both
$\mathcal{R}_1 = [h_{\mathcal{R}_1}, 1 - \partial_{\mathcal{R}_1}]$ and $\mathcal{R}_2 = [h_{\mathcal{R}_2}, 1 - \partial_{\mathcal{R}_2}]$ are converted to $h_{\mathcal{R}_1}$ and $h_{\mathcal{R}_2}$, respectively. Here the above mention laws are converted into the following positions:

1. $\mathcal{R}_1 \oplus \mathcal{R}_2 = S_P(h_{\mathcal{R}_1}, h_{\mathcal{R}_2})$

2. $\lambda \mathcal{R} = \sqrt{1 - (1 - h_{\mathcal{R}}^2)\lambda}, \lambda > 0$

Let $\delta(\lambda, \mathcal{R}) = \sqrt{1 - (1 - h_{\mathcal{R}}^2)\lambda}, \lambda > 0$, then expression (2) is a unit interval monotone increasing function on $\lambda$ and $\mathcal{R}$ having the following good properties.

1. $0 \leq \lambda(\lambda, 0) \leq 1$. Especially, $\lambda(\lambda, 0) = 0, \lambda(1, \mathcal{R}) = 1$ and $\lambda(1, \mathcal{R}) = \mathcal{R}$.

2. If $\lambda \rightarrow 0$ and $0 < \mathcal{R} < 1$, then $\lambda(\lambda, \mathcal{R}) \rightarrow 0$.

3. If $\lambda \rightarrow +\infty$ and $0 < \lambda > 1$, then $\lambda(\lambda, \mathcal{R}) \rightarrow 1$.

4. If $\partial_{\mathcal{R}_1} > \partial_{\mathcal{R}_2}$, then $\lambda(\lambda, \mathcal{R}_1) > \lambda(\lambda, \mathcal{R}_2)$.

5. If $\mathcal{R}_1 > \mathcal{R}_2$, then $\lambda(\lambda, \mathcal{R}_1) > \lambda(\lambda, \mathcal{R}_2)$.

These desirable properties provide a theoretic basis for the application of the operational law (2) to the aggregation of PFVs.

**Theorem 2:** Let $\mathcal{R}_1 = [h_{\mathcal{R}_1}, 1 - \partial_{\mathcal{R}_1}]$, $\mathcal{R}_2 = [h_{\mathcal{R}_2}, 1 - \partial_{\mathcal{R}_2}]$ and $\mathcal{R}_3 = [h_{\mathcal{R}_3}, 1 - \partial_{\mathcal{R}_3}]$ be the collection of three PFVs and let $\beta_1 = \mathcal{R}_1 \oplus \mathcal{R}_2$ and $\beta_2 = \lambda \mathcal{R}(\lambda > 0)$, then both $\beta_1$ and $\beta_2$ are also PFVs.

**Proof:** Since: Let $\mathcal{R}_1 = [h_{\mathcal{R}_1}, 1 - \partial_{\mathcal{R}_1}]$, $\mathcal{R}_2 = [h_{\mathcal{R}_2}, 1 - \partial_{\mathcal{R}_2}]$ be two PFVs, then $h_{\mathcal{R}_1} + \partial_{\mathcal{R}_1} \leq 1, h_{\mathcal{R}_2} + \partial_{\mathcal{R}_2} \leq 1$. Then we have

$$\sqrt{h_{\mathcal{R}_1}^2 + h_{\mathcal{R}_2}^2 - h_{\mathcal{R}_1}^2 - h_{\mathcal{R}_2}^2} \leq 1 - (1 - \partial_{\mathcal{R}_1} \partial_{\mathcal{R}_2})$$

Therefore, $\beta_1$ is a PFV. Now $\sqrt{1 - (1 - h_{\mathcal{R}}^2)\lambda} \geq 0$ and $(\partial_{\mathcal{R}})^\lambda \geq 0$, then

$$\sqrt{1 - (1 - h_{\mathcal{R}}^2)\lambda} \geq 0$$

Thus $\beta_2$ is a PFV.

Now we are define some basic cases as follows:

1. If $\mathcal{R} = [h_{\mathcal{R}}, 1 - \partial_{\mathcal{R}}] = [1, 1]$, i.e., $h_{\mathcal{R}} = 1 - \partial_{\mathcal{R}} = 1$, then

$$\lambda \mathcal{R} = \sqrt{1 - (1 - h_{\mathcal{R}}^2)\lambda}, \lambda > 0$$

$$= [1, 1].$$

2. If $\mathcal{R} = [h_{\mathcal{R}}, 1 - \partial_{\mathcal{R}}] = [0, 0]$, i.e., $h_{\mathcal{R}} = 0, 1 - \partial_{\mathcal{R}} = 0$, then

$$\lambda \mathcal{R} = \sqrt{1 - (1 - h_{\mathcal{R}}^2)\lambda}, \lambda > 0$$

$$= [0, 0].$$

3. If $\mathcal{R} = [h_{\mathcal{R}}, 1 - \partial_{\mathcal{R}}] = [0, 1]$, i.e., $h_{\mathcal{R}} = 0, 1 - \partial_{\mathcal{R}} = 1$, then

$$\lambda \mathcal{R} = \sqrt{1 - (1 - h_{\mathcal{R}}^2)\lambda}, \lambda > 0$$

$$= [0, 1].$$

4. If $\lambda \rightarrow 0$ and $0 < h_{\mathcal{R}}, \partial_{\mathcal{R}} < 1$, then

$$\lambda \mathcal{R} = \sqrt{1 - (1 - h_{\mathcal{R}}^2)\lambda}, \lambda > 0$$

$$= [0, 0].$$

5. If $\lambda \rightarrow +\infty$ and $0 < h_{\mathcal{R}}, \partial_{\mathcal{R}} < 1$, then

$$\lambda \mathcal{R} = \sqrt{1 - (1 - h_{\mathcal{R}}^2)\lambda}, \lambda > 0$$

$$= [1, 1].$$

6. If $\lambda = 1$, then

$$\lambda \mathcal{R} = \sqrt{1 - (1 - h_{\mathcal{R}}^2)\lambda}, \lambda > 0$$

$$= \mathcal{R}.$$
\[ \mathcal{R}_1 \oplus \mathcal{R}_2 = \begin{bmatrix} h_{R_1}^2 + h_{R_2}^2 - h_{R_1} h_{R_2}, 1 - \partial \mathcal{R}_1, \partial \mathcal{R}_1, \\
\end{bmatrix} \]
\[ = \begin{bmatrix} h_{R_1} + h_{R_2} - h_{R_1} h_{R_2}, 1 - \partial \mathcal{R}_1, \partial \mathcal{R}_1, \\
\end{bmatrix} \]
\[ = [\mathcal{R}_2 \oplus \mathcal{R}_1]. \]

(2) Since
\[ \mathcal{R}_1 \oplus \mathcal{R}_2 = \begin{bmatrix} h_{R_1}^2 + h_{R_2}^2 - h_{R_1} h_{R_2}, 1 - \partial \mathcal{R}_1, \partial \mathcal{R}_1, \\
\end{bmatrix} \]
Then
\[ \lambda (\mathcal{R}_1 \oplus \mathcal{R}_2) = \begin{bmatrix} \left(1 - \left(1 - \left(h_{R_1}^2 + h_{R_2}^2 - h_{R_1} h_{R_2}^2\right)^2\right) \right)^{\lambda} \\
1 - \left(\partial \mathcal{R}_1, \partial \mathcal{R}_1, \right)^{\lambda} \\
\end{bmatrix} \]
\[ = \begin{bmatrix} \left(1 - \left(1 - \left(h_{R_1}^2 + h_{R_2}^2 - h_{R_1} h_{R_2}^2\right)^2\right) \right)^{\lambda} \\
1 - \left(\partial \mathcal{R}_1, \partial \mathcal{R}_1, \right)^{\lambda} \\
\end{bmatrix} \]
\[ \lambda \mathcal{R}_1 \oplus \lambda \mathcal{R}_2 = \begin{bmatrix} \left(1 - \left(1 - \left(h_{R_1}^2 + h_{R_2}^2 - h_{R_1} h_{R_2}^2\right)^2\right) \right)^{\lambda} \\
1 - \left(\partial \mathcal{R}_1, \partial \mathcal{R}_1, \right)^{\lambda} \\
\end{bmatrix} \]
\[ = \begin{bmatrix} \left(1 - \left(1 - \left(h_{R_1}^2 + h_{R_2}^2 - h_{R_1} h_{R_2}^2\right)^2\right) \right)^{\lambda} \\
1 - \left(\partial \mathcal{R}_1, \partial \mathcal{R}_1, \right)^{\lambda} \\
\end{bmatrix} \]

4. Pythagorean Fuzzy Weighted Averaging Aggregation Operators

In this section we introduced the notion of two new types of aggregation operators such as, Pythagorean fuzzy weighted averaging aggregation operator and Pythagorean fuzzy ordered weighted averaging aggregation operator and also discuss some of their basic properties.

4.1. Pythagorean Fuzzy Weighted Averaging Operator

**Definition 6:** Let \( \mathbf{K}_j = [h_{\mathcal{R}_j}, 1 - \partial \mathcal{R}_j] \) \((j = 1, 2, 3, \ldots, n)\) be a collection of PFVs, and let \( \text{PFWA} : \mathcal{U}^n \rightarrow \mathcal{U}, \) if
\[ \text{PFWA}_\mathbf{K} (\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_n) = \left[ \begin{array}{c}
\left(1 - \prod_{j=1}^{n} (1 - h_{\mathcal{R}_j})^{\frac{1}{\varpi_j}}\right), \left(1 - \prod_{j=1}^{n} (\partial \mathcal{R}_j)^{\frac{1}{\varpi_j}}\right)
\end{array} \right], \]
then \( \text{PFWA} \) is called Pythagorean fuzzy weighted averaging \((\text{PFWA})\) operator of dimension \( n. \) Especially, if \( \mathbf{\sigma} = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T, \) then the PFWA operator is reduced to a \( \text{PFA} \) operator of dimension \( n, \) which can be defined as follows:
\[ \text{PFA} (\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_n) = \frac{1}{n} \left( \mathbf{K}_1 \oplus \mathbf{K}_2 \oplus \ldots \oplus \mathbf{K}_n \right). \]

**Theorem 4:** Let \( \mathbf{K}_j = [h_{\mathcal{R}_j}, 1 - \partial \mathcal{R}_j] \) \((j = 1, 2, 3, \ldots, n)\) be a collection of PFVs, then their aggregated value by using the PFWA operator is also a PFV.

Proof: Straightforward.

**Example 1:** Let
\[ \mathcal{R}_1 = [0.8, 0.5], \mathcal{R}_2 = [0.7, 0.6], \mathcal{R}_3 = [0.6, 0.7], \]
\[ \mathcal{R}_4 = [0.5, 0.8], \mathcal{R}_5 = [0.4, 0.9], \]
be the PFVs and let \( \mathbf{\sigma} = \left(0.1, 0.2, 0.2, 0.2, 0.3\right)^T \) be the weight vector of \( \mathcal{R}_j \) \((j = 1, 2, 3, 4, 5)\), then
Then for all \( j \),

\[
1 - \partial_{\mathcal{R}_j} = 0.5 \Rightarrow \partial_{\mathcal{R}_j} = 0.5, 1 - \partial_{\mathcal{R}_2} = 0.6 \Rightarrow \partial_{\mathcal{R}_2} = 0.4 \\
1 - \partial_{\mathcal{R}_3} = 0.7 \Rightarrow \partial_{\mathcal{R}_3} = 0.3, 1 - \partial_{\mathcal{R}_4} = 0.8 \Rightarrow \partial_{\mathcal{R}_4} = 0.2 \\
1 - \partial_{\mathcal{R}_5} = 0.9 \Rightarrow \partial_{\mathcal{R}_5} = 0.1
\]

Thus

\[
\text{PFWA}_\sigma (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5) = \left[ \sqrt{1 - \prod_{j=1}^{5} \left( 1 - h_{\mathcal{R}_j} \right) \sigma_j}, 1 - \prod_{j=1}^{5} \left( \partial_{\mathcal{R}_j} \right)^{\sigma_j} \right]
\]

\[
= \left[ \sqrt{1 - 0.643}, 1 - 0.221 \right] = (0.597, 0.778)
\]

Theorem 5. Let \( \mathcal{R}_j = (3_{\mathcal{R}_j}, 1 - \varphi_{\mathcal{R}_j}) \) \( (j = 1, 2, 3, ..., n) \) be a collection of PFVs and \( \chi = (\chi_1, \chi_2, \chi_3, ..., \chi_n)^T \) is the weighted vector of \( \mathcal{R}_j \) with \( \chi_j \in [0, 1] \) and \( \sum_{j=1}^{n} \chi_j = 1 \), then the following conditions hold.

1. (Idempotency): If \( \mathcal{R}_j = \mathcal{R}_j \) \( \forall j \), then

\[
\text{PFWA}_\chi (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, ..., \mathcal{R}_n) = \mathcal{R}_j.
\]

2. (Boundary):

\[
\mathcal{R}^- \leq \text{PFWA}_\chi (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, ..., \mathcal{R}_n) \leq \mathcal{R}^+, \text{ for all } \chi.
\]

Where

\[
\mathcal{R}^- = \left[ \min \{3_{\mathcal{R}_j}\}, 1 - \max \{\varphi_{\mathcal{R}_j}\} \right],
\]

\[
\mathcal{R}^+ = \left[ \max \{3_{\mathcal{R}_j}\}, 1 - \min \{\varphi_{\mathcal{R}_j}\} \right].
\]

3. (Monotonicity): If \( h_{\mathcal{R}_j} \leq h_{\mathcal{R}_j'} \) \( \partial_{\mathcal{R}_j} \geq \partial_{\mathcal{R}_j'} \) for all \( j \), then

\[
\text{PFWA}_\sigma (\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_n) \leq \text{PFWA}_\sigma (\mathcal{R}_1', \mathcal{R}_2', ..., \mathcal{R}_n').
\]

Proof: (1) As we know that

\[
\text{PFWA}_\chi (\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_n) = \chi_1 \mathcal{R}_1 \oplus \chi_2 \mathcal{R}_2 \oplus ... \oplus \chi_n \mathcal{R}_n
\]

Let \( \mathcal{R}_j = \mathcal{R}_j \). Then

\[
\text{PFWA}_\chi (\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_n) = \chi_j \mathcal{R}_j \oplus \chi_2 \mathcal{R}_2 \oplus ... \oplus \chi_n \mathcal{R}_n
\]

(2) (Boundary): From equation (16) we have

\[
\Rightarrow \sqrt{\min \{3_{\mathcal{R}_j}\}} \leq \sqrt{\max \{3_{\mathcal{R}_j}\}}
\]

\[
\Rightarrow \prod_{j=1}^{n} \left[ 1 - \max \{3_{\mathcal{R}_j}\} \right] \leq \prod_{j=1}^{n} \left[ 1 - \min \{3_{\mathcal{R}_j}\} \right]
\]

\[
\Rightarrow \left[ 1 - \prod_{j=1}^{n} \left[ 1 - 3_{\mathcal{R}_j}\right] \right] \leq \left[ 1 - \prod_{j=1}^{n} \left[ 1 - 3_{\mathcal{R}_j}\right] \right]
\]

From equation (17) we have

\[
\Rightarrow \min \{\varphi_{\mathcal{R}_j}\} \leq \max \{\varphi_{\mathcal{R}_j}\}
\]

\[
\Rightarrow \sum_{j=1}^{n} \varphi_{\mathcal{R}_j} \leq \max \{\varphi_{\mathcal{R}_j}\}
\]

(19)

(20)

Let

\[
\text{PFWA}_\chi (\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_n) = \mathcal{R} = [3_{\mathcal{R}_j}, 1 - \varphi_{\mathcal{R}_j}]
\]

Then

\[
S(\mathcal{R}) = 3_{\mathcal{R}_j}^2 - \varphi_{\mathcal{R}_j}^2
\]

\[
\leq \left[ \max \{3_{\mathcal{R}_j}\} \right]^2 - \left[ \min \{\varphi_{\mathcal{R}_j}\} \right]^2
\]

(22)

Again
\( S(\mathfrak{R}) = 3^2 - \phi_k^2 \)
\[ \geq \left[ \min_j \left( 3_{\mathfrak{R}_j} \right) \right]^2 - \left[ \max_j \left( \phi_k \right) \right]^2 = S(\mathfrak{R}^{-}). \]  

(23)

If
\( S(\mathfrak{R}) \prec S(\mathfrak{R}^+). \)  

(24)

And
\( S(\mathfrak{R}) \succ S(\mathfrak{R}^-). \)  

(25)

Thus from equations (24) and (25), we have
\( \mathfrak{R}^{-} < \text{PFWA}_\chi (\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, ..., \mathfrak{R}_n) < \mathfrak{R}^+. \)  

(26)

If
\( S(\mathfrak{R}) = S(\mathfrak{R}^+). \)  

(27)

Then
\[ \begin{align*}
\Rightarrow 3_k^2 - \phi_k^2 & = \left[ \max_j \left( 3_{\mathfrak{R}_j} \right) \right]^2 - \left[ \min_j \left( \phi_k \right) \right]^2 = S(\mathfrak{R}^+). \\
\Rightarrow 3_k^2 & = \left[ \max_j \left( 3_{\mathfrak{R}_j} \right) \right]^2 - \phi_k^2 = \left[ \min_j \left( \phi_k \right) \right]^2 \\
\Rightarrow 3_k & = \max_j \left( 3_{\mathfrak{R}_j} \right), \phi_k = \min_j \left( \phi_k \right)
\end{align*} \]

(28)

Since
\[ \begin{align*}
H(\mathfrak{R}) & = 3^2 + \phi_k^2 \\
& = \left[ \max_j \left( 3_{\mathfrak{R}_j} \right) \right]^2 + \left[ \min_j \left( \phi_k \right) \right]^2 = H(\mathfrak{R}^+).
\end{align*} \]

(31)

Thus from equation (31), we have
\[ \text{PFWA}_\chi (\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, ..., \mathfrak{R}_n) = \mathfrak{R}^+. \]  

(32)

Then from equations (26), (29), and (32), we have (15) always holds.

3. (Monotonicity): Follows the above proof.

4.2. Pythagorean Fuzzy Ordered Weighted Averaging

Definition 7. Let \( \varphi_j = 3_{\mathfrak{R}_j} \cdot 1 - \varnothing_{\mathfrak{R}_j} \) \((j = 1, ..., n)\), then a PFOWA can be define as:
\[ \text{PFOWA}_\chi (\varphi_1, \varphi_2, ..., \varphi_n) = i_1\varphi_{\tau(1)} + i_2\varphi_{\tau(2)} + ... + i_n\varphi_{\tau(n)}, \]  

(33)

where \((\tau(1), \tau(2), \tau(3), ..., \tau(n))\) is a permutation of \((1, 2, ..., n)\) with condition \(\varphi_{\tau(j-1)} \geq \varphi_{\tau(j)}\) for all \(j\), and \(i = (i_1, i_2, i_3, ..., i_n)^T\) be the weighted vector of \(\varphi_j\).

Theorem 6. Let \( \mathfrak{R}_j = 3_{\mathfrak{R}_j} \cdot 1 - \varnothing_{\mathfrak{R}_j} \) \((j = 1, 2, ..., n)\) be a collection of PFVs, by applying PFOWA operator the result is also a PFV.

Proof: The proof is similar to the Theorem 4.

Example 2. Let \( \mathfrak{R}_1 = [0.4, 0.7], \mathfrak{R}_2 = [0.5, 0.6], \mathfrak{R}_3 = [0.6, 0.7], \mathfrak{R}_4 = [0.8, 0.5], \mathfrak{R}_5 = [0.8, 0.4], \)
and \( \chi = (0.1, 0.2, 0.2, 0.4)^T \), then
\[ h_{\mathfrak{R}_1} = 0.4, h_{\mathfrak{R}_2} = 0.5, h_{\mathfrak{R}_3} = 0.6, h_{\mathfrak{R}_4} = 0.8, h_{\mathfrak{R}_5} = 0.8, \]
and
\[ 1 - \partial_{\mathfrak{R}_1} = 0.7 \leftrightarrow \partial_{\mathfrak{R}_1} = 1 - 0.7 = 0.3 \]
\[ 1 - \partial_{\mathfrak{R}_2} = 0.6 \leftrightarrow \partial_{\mathfrak{R}_2} = 1 - 0.6 = 0.4 \]
\[ 1 - \partial_{\mathfrak{R}_3} = 0.7 \leftrightarrow \partial_{\mathfrak{R}_3} = 1 - 0.7 = 0.3 \]
\[ 1 - \partial_{\mathfrak{R}_4} = 0.5 \leftrightarrow \partial_{\mathfrak{R}_4} = 1 - 0.5 = 0.5 \]
\[ 1 - \partial_{\mathfrak{R}_5} = 0.4 \leftrightarrow \partial_{\mathfrak{R}_5} = 1 - 0.4 = 0.6 \]

Now we can find the score function \( \mathfrak{R}_j \) \((j = 1, 2, 3, 4, 5)\).
\[ S(\mathcal{R}_j) = (0.4)^2 - (0.3)^2 = 0.07 \]
\[ S(\mathcal{R}_j) = (0.5)^2 - (0.4)^2 = 0.09 \]
\[ S(\mathcal{R}_3) = (0.6)^2 - (0.3)^2 = 0.27 \]
\[ S(\mathcal{R}_j) = (0.8)^2 - (0.5)^2 = 0.39 \]
\[ S(\mathcal{R}_3) = (0.8)^2 - (0.6)^2 = 0.28 \]

Hence \( S(\mathcal{R}_4) > S(\mathcal{R}_3) > S(\mathcal{R}_3) > S(\mathcal{R}_2) > S(\mathcal{R}_1) \)

Thus \( PFWA_\ell(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5) \)

\[
= \left[ 1 - \sum_{j=1}^{5} \left( 1 - h^{\mathcal{R}_j}_{(j)} \right) \chi_j \right] \left( 1 - \sum_{j=1}^{5} \left( \partial h^{\mathcal{R}_j}_{(j)} \right) \chi_j \right)
\]

\[
= \left[ 1 - \left( 1 - 0.64 \right)^{0.1} \left( 1 - 0.64 \right)^{0.1} \left( 1 - 0.36 \right)^{0.2} \right] \left[ 1 - \left( 1 - 0.25 \right)^{0.2} \left( 1 - 0.16 \right)^{0.4} \right]
\]

\[
= \left[ 0.588, 1 - 0.643 \right]
\]

**Theorem 7.** Let \( \mathcal{R}_j = \left[ h_{\mathcal{R}_j}, 1 - \partial h_{\mathcal{R}_j} \right] \) \((j = 1, 2, 3, ..., n)\) be a collection of PFVs and \( \sigma = (\sigma_1, \sigma_2, \sigma_3, ..., \sigma_n)^T \) is the weighted vector of \( \mathcal{R}_j \) with \( \sigma_j \in [0, 1] \) and \( \sum_{j=1}^{n} \sigma_j = 1 \). Then we have the following.

1. (Idempotency): If \( \mathcal{R}_j = \mathcal{R}_j \) for all \( j \), then

\[
PFWA_\sigma(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, ..., \mathcal{R}_n) = \mathcal{R}_j. \quad (34)
\]

2. (Boundary):

\[
\mathcal{R}_j^- \leq PFWA_\sigma(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, ..., \mathcal{R}_n) \leq \mathcal{R}_j^+, \quad (35)
\]

\[
\mathcal{R}_j^- = \left[ \min_j \left( h_{\mathcal{R}_j} \right), 1 - \max_j \left( \partial h_{\mathcal{R}_j} \right) \right]
\]

\[
\mathcal{R}_j^+ = \left[ \max_j \left( h_{\mathcal{R}_j} \right), 1 - \min_j \left( \partial h_{\mathcal{R}_j} \right) \right]
\]

3. (Monotonicity): If \( h_{\mathcal{R}_j} \leq h_{\mathcal{R}_j^{'}} \) and \( \partial h_{\mathcal{R}_j} \geq \partial h_{\mathcal{R}_j^{'}} \) for all \( j \), then

\[
PFWA_\sigma(\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_n) \leq PFWA_\sigma(\mathcal{R}_1^{'}, \mathcal{R}_2^{'}, ..., \mathcal{R}_n^{'}). \quad (36)
\]

Proof: The proof is similar to the Theorem 5.

**Theorem 8.** Let \( \mathcal{R}_j = \left[ 3_{\mathcal{R}_j}, 1 - \varphi_{\mathcal{R}_j} \right] \) \((j = 1, 2, ..., n)\) be a group of PFVs and \( \ell = (\ell_1, \ell_2, \ell_3, ..., \ell_n)^T \) is the weighted vector of \( \mathcal{R}_j \), such that \( \ell_j \in [0, 1] \) and \( \sum_{j=1}^{n} \ell_j = 1 \). Then

1. If \( \ell = (1, 0, 0, ..., 0)^T \), then

\[
PFWA_\ell(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, ..., \mathcal{R}_n) = \max_j \left\{ \mathcal{R}_j \right\},
\]

2. If \( \ell = (0, 0, 0, ..., 1)^T \), then

\[
PFWA_\ell(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, ..., \mathcal{R}_n) = \min_j \left\{ \mathcal{R}_j \right\},
\]

3. If \( \ell_j = 1 \) and \( \ell_j = 0 (i \neq j) \), then

\[
PFWA_\ell(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, ..., \mathcal{R}_n) = \mathcal{R}_j,
\]

where \( \mathcal{R}_j \) is the jth greatest of \( \mathcal{R}_j \).

Proof: Straightforward.

**5. Conclusion**

In this work we have familiarized the idea of PFOWA operator and PFWA operator and also discussed some of their basic properties.

**References**

1. K. Atanassov, Intuitionistic fuzzy sets. Fuzzy Sets Syst. 1986, 85, 87-96.
2. L. A. Zadeh, Fuzzy sets, Inf Control, (1965), 338-353.
3. H. Bustine and P. Burillo, Vague sets are intuitionistic fuzzy sets. Fuzzy sets and systems, (1996), 79 (3), 403–405.
4. C. H. Tan and X. H. Chen, Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making, Expert Syst App, (2010), 149.157.
5. D. H. Hong and C.H. Choi, Multicriteria fuzzy decision-making problems based on vague set theory. Fuzzy sets and systems, (2000) 114 (1), 103–113.
6. H. Bustine and P. Burillo, Vague sets are intuitionistic fuzzy sets. Fuzzy sets and systems, (1996) 79 (3), 403–405.
7. K. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy Sets Syst, (1994), 137-142.
8. K. Atanassov, Remarks on the intuitionistic fuzzy sets. III, Fuzzy Sets Syst, (1995), 401-402.
9. K. Atanassov, equality between intuitionistic fuzzy sets, Fuzzy Sets Syst, (1996), 257-258.
10. K. Atanassov, Intuitionistic fuzzy sets: theory and applications, Heidelberg, Germany: Physica-Verlag (1999).
11. M. Xia and Z. S. Xu, Generalized point operators for aggregating intuitionistic fuzzy information, Int J Intell Syst, (2010), 1061-1080.
12. S. K. De, R. Biswas and A. R. Roy, Some operations on intuitionistic fuzzy sets, Fuzzy Set Syst, (2000), 477-484.
[13] Z. S. Xu, Intuitionistic fuzzy aggregation operators. IEEE Trans Fuzzy Syst, (2007), 1179-1187.

[14] Z. S. Xu, R. R. Yager. Some geometric aggregation operators based on intuitionistic fuzzy sets, Int J Gen Syst (2006), 417-433.

[15] W. Wang and X. Liu, Intuitionistic Fuzzy Geometric Aggregation Operators Based on Einstein Operations, international journal of intelligent systems, (2011), 1049-1075.

[16] Weize Wang, Xinwang Liu, Intuitionistic Fuzzy Information Aggregation Using Einstein Operations, IEEE Trans. Fuzzy Systems, (2012) 923-938.

[17] X. Zhao and G. Wei, Some intuitionistic fuzzy Einstein hybrid aggregation operators And their application to multiple attribute decision making, Knowledge-Based Systems, (2013). 472-479.

[18] R. R. Yager, Pythagorean fuzzy subsets, In Proc. Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada (2013), 57-61.

[19] R. R. Yager, A. M. Abbasov, Pythagorean membership grades, complex numbers and decision making. Int J Intell Syst (2013), 28.436:452.

[20] K. Rahman, S. Abdullah, M. S. Ali Khan, A. Ali and F. Amin, Pythagorean fuzzy hybrid averaging aggregation operator and its application to multiple attribute decision making. Accepted.

[21] K. Rahman, M. S. Ali. Khan, Murad Ullah and A. Fahmi, Multiple attribute group decision making for plant location selection with Pythagorean fuzzy weighted geometric aggregation operator, The Nucleus (2017), 54, 66-74.

[22] K. Rahman, S. Abdullah, F. Husain M. S. Ali Khan, M. Shakeel, Pythagorean fuzzy ordered weighted geometric aggregation operator and their application to multiple attribute group decision making, J. Appl. Environ. Biol. Sci., (2017), 7(4) 67-83.

[23] K. Rahman, S. Abdullah, M. S. Ali Khan and M. Shakeel, Pythagorean fuzzy hybrid geometric aggregation operator and their applications to multiple attribute decision making, International Journal of Computer Science and Information Security (IJCISIS), (2016), 14, No. 6, 837-854.

[24] H. Garg, A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making, international journal of intelligent systems, (2016), 1-35.