Systematic classical continuum limits of integrable spin chains and emerging novel dualities

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Abstract

We examine certain classical continuum long wave-length limits of prototype integrable quantum spin chains. We define the corresponding construction of classical continuum Lax operators. Our discussion starts with the XXX chain, the anisotropic Heisenberg model and their generalizations and extends to the generic isotropic and anisotropic $gl_n$ magnets. Certain classical and quantum integrable models emerging from special “dualities” of quantum spin chains, parametrized by $c$-number matrices, are also presented.
1 Introduction

Locally interacting discrete integrable spin chains have been the subject of much interest since they cropped up in string theory in the study of the AdS/CFT correspondence [1]. Their classical, long wavelength limit provides a connection to continuous $\sigma$-models describing particular dynamics of the string (references on this subject can be found in e.g. [2, 3]).

This has lead us to tackle here the problem of formulating the classical continuum long wavelength limit of the (simpler) quantum integrable closed spin chains in a way that directly preserves integrability. Accordingly we will describe the classical Lax-matrix formulation,
including the associated classical $r$-matrix structure, which consistently yields the classical, long-wavelength limit, derived for integrable closed quantum spin chain models (see e.g. [2, 3]).

We shall first describe and implement in detail the general Hamiltonian procedure. Then we will tackle a number of specific examples, and explicitly compare with already known results from alternative derivations. These identifications will establish the validity of our approach. We shall in particular consider the paradigmatic example of the long wavelength limit of the $XXX$ spin chain, followed by the anisotropic Heisenberg model and the $gl_n$ classical magnet. We finally consider some more complicated cases where the original quantum $R$-matrix used to build the spin chain by coproduct is “twisted” by a scalar solution of the exchange algebra. The corresponding Hamiltonians will be discussed in general, realizing interesting formal connections between different classical integrable models. We shall also briefly touch upon the inhomogeneous case where the specific twist matrix will be site-dependent. Some technical derivations will be exposed in the Appendices.

Our motivation for this work is to develop a Hamiltonian approach different in its principle from the usual Lagrangian formulation of the long wavelength limit, in order to use in cases where the latter cannot be applied. In our approach we start from the Hamiltonian integrability formulation (quantum $R$-matrix and Lax matrix) guaranteeing a priori Liouville integrability of the classical continuous models through a Lax matrix-classical $r$-matrix formulation, provided that some consistency checks be made. On all known specific examples it will be checked that it yields the same results as the Lagrangian approach. It is indeed a key result that the Poisson structure is the same, in all cases when comparison is available, as the canonical structure derived from the long wavelength classical Lagrangian. This thereby validates the procedure and allows to use it in more general situations where the Lagrangian approach may not be used, in particular as a systematic way to build more general types of classical continuous integrable models by exploiting the richness of the algebraic approach.

2 The general procedure

In this section we outline the general procedure for obtaining a classical Lax formulation from the classical limits of the $R$ and monodromy matrices.
2.1 Classical limit for the $R$-matrix and the monodromy matrix

A quantum $c$-number non-dynamical $R$-matrix obeys the quantum Yang–Baxter (YB) equation [4]

\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} , \]  

(2.1)

where the labels $i = 1, 2, 3$ may include dependence on a complex spectral parameter $\lambda_i$. The auxiliary spaces are in this case loop-spaces $V_i \otimes \mathbb{C}(\lambda_i)$, where $V_i$ are (isomorphic) finite-dimensional vector spaces.

Assuming that $R$ admits an expansion (“semiclassical”) in positive power series of a parameter (usually denoted $\hbar$) as

\[ R_{12} = 1 \otimes 1 + \hbar r_{12} + \mathcal{O}(\hbar^2) , \]  

(2.2)

the first non-trivial term arising when we substitute this in (2.1) is order of order two and yields the classical YB equation

\[ [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 . \]  

(2.3)

This is the canonically known “classical Yang–Baxter equation”. It is not in general the sufficient associativity condition for a classical linear Poisson bracket, except when $r$ is non-dynamical and skew-symmetric (see e.g. [5]). We shall hereafter limit ourselves to such situations.\(^{1}\)

A quantum monodromy matrix $T$ is generically built as a tensor product over “quantum spaces” and algebraic product over “auxiliary space” of representations of the YB algebra associated to $R$. Namely, one assumes a collection operators assembled in matrices $L_{1i}$, acting on “quantum” Hilbert spaces labeled by $i$ and encapsulated in a matrix “acting” on the auxiliary space $V_1$. For any quantum space $q$ they obey the quadratic exchange algebra [9, 10, 11]

\[ R_{12} L_{1q} L_{2q} = L_{2q} L_{1q} R_{12} , \]  

(2.4)

where operators acting on different quantum spaces commute. The form of the monodromy matrix $T$ is then deduced from the co-module structure of the YB algebra

\[ T_a \equiv L_{a1} L_{a2} \ldots L_{aN} \]  

(2.5)

\(^{1}\)The dynamical YB equation is related to Drinfel’s deformations of quantum groups, whereas the non-skew symmetric equation is associated to reflection algebras (see e.g. [6, 7, 8]) and hence to open spin chains which we do not consider here.
and thus naturally obeys the same quadratic exchange algebra \((2.4)\). In particular one can pick \(L = R\), the operators now acting on the second auxiliary space identified as “quantum space”. This way, one builds closed inhomogeneous spin chains with general spins at each lattice site (labeled by \((i)\)) belonging to locally chosen representations of some Lie algebra (labeled by \(i\)).

We now establish that \(T\) has a classical limit by considering in addition the classical counterpart of \(L\), labeled by \(L^c\) which then satisfies the quadratic Poisson algebra, emerging directly as a semi-classical limit of \((2.4)\), after setting \(\frac{1}{\hbar}[A, B] \rightarrow \{A, B\}\). It reads

\[
\{L^c_\alpha(\lambda_1), L^c_\beta(\lambda_2)\} = [r_{ab}(\lambda_1 - \lambda_2), L^c_\alpha(\lambda_1) L^c_\beta(\lambda_2)].
\]

(2.6)

The quantum monodromy matrix has also a classical limit given by (see also \([12, 13]\))

\[
T^c_{a,\{i\}} = L^c_{a_1} \cdots L^c_{a_N}.
\]

(2.7)

The exchange algebra for \(T^c\) takes the form

\[
\{T^c_a, T^c_b\} = [r_{ab}, T^c_a T^c_b].
\]

(2.8)

This quadratic Poisson structure implies that the traces of powers of the monodromy matrix \(tr(T^c)\) generate Poisson-commuting quantities identified as classically integrable Hamiltonians. In particular, when \(T^c\) depends on a spectral parameter, the auxiliary space is a loop space \(V \otimes \mathbb{C}(\lambda)\). Performing the trace over the finite vector space yields a generating function \(tr(T^c(\lambda))\) for classically integrable Hamiltonians obtained by series expansion in \(\lambda\).

### 2.2 The long wavelength limit

The usual presentation of the long wavelength limit, such as can be found in \([2, 3]\), is a Lagrangian one where the Poisson structure is obtained from the standard derivation of canonical variables using a Lagrangian density. Instead, we will present here a purely Hamiltonian version of this limit by defining the long wavelength limit of a hierarchy of integrable quantum Hamiltonians based on some affine Lie algebra \(\hat{G}\). We shall define a priori the Poisson structure of the classical variables by imposing classical integrability of the long wavelength limit of the Hamiltonian through its associated classical Lax matrix. We consider a \(N\)-site closed spin chain Hamiltonian \(H\), initially assumed to be governed by nearest-neighbour interaction that takes the form

\[
H \equiv \sum_{l=1}^{N} H_{ll+1}.
\]

(2.9)
The classical, long wavelength limit, is obtained by first defining local quantum states as linear combinations of the base quantum states, parametrized by a complete set of \( k \) continuous variables. The number \( k \) depends on the choice of \( \hat{G} \) and essentially \( k = \text{dim}(G) \). These variables, which can be identified as Euler angles in the simplest case of \( sl(2) \), become the classical dynamical variables once a suitable Poisson structure is imposed. The bras and kets are denoted respectively by \( \langle n(l, \theta_k) \rangle \) and \( |n(l, \theta_k)\rangle \), where \( l \) denotes the site index and \( \theta_k \) denote the set of \( k \) angular variables. The condition of “closed” spin chain, essentially formulated as \( N + l \equiv l \), imposes periodicity or quasi-periodicity conditions on the \( \theta_k \)'s. We note that we assume that the base quantum states different only by the fact that they are defined in distinct sites, hence the frequently used notation below \( |n_l\rangle \), instead of \( |n(l, \theta_k)\rangle \), should not be confusing.

If one considers nearest-neighbor interactions (local) then one defines the classical, but still defined in the lattice, Hamiltonian as

\[
    \mathcal{H} \equiv \sum_{l=1}^{N} \mathcal{H}_l(t) , \quad \mathcal{H}_l(x, t) = \langle n_l \rangle \otimes \langle n_{l+1} \rangle \ H_{l_{l+1}} \ |n_l\rangle \otimes |n_{l+1}\rangle . \tag{2.10}
\]

For integrable models, we may similarly define the continuum limit of the full set of commuting Hamiltonians. In these cases the generic Hamiltonians \( H^{(n)} \) of the integrable hierarchy are obtained directly from the analytic series expansion around some value \( \lambda_0 \) of the spectral parameter of the trace of the monodromy matrix (transfer matrix) as

\[
    \text{tr} T(\lambda) \equiv \sum_{n=1}^{\infty} (\lambda - \lambda_0)^n H^{(n)} . \tag{2.11}
\]

By extension, we define in this case the classical Hamiltonians as the expectation value, over the \( N \) site lattice quantum state, of \( H^{(n)} \)

\[
    \mathcal{H}^{(n)}(x, t) = \otimes_1^N \ldots \langle n_l \rangle \otimes \langle n_{l+1} \rangle \ldots \ H^{(n)} \ldots |n_l\rangle \otimes |n_{l+1}\rangle \ldots . \tag{2.12}
\]

We next define a continuous limit and take simultaneously the thermodynamical limit in which \( N \to \infty \). Accordingly, this is achieved by identifying the lattice spacing \( \delta \) as being of order \( 1/N \) and subsequently consider only slow-varying spin configurations (the long wavelength limit proper) for which

\[
    l_i \to l(x) , \quad l_{i+1} \to l(x + \delta) . \tag{2.13}
\]

In this limit, the finite “site differences” turn into derivatives.
Given that (2.12) is applied to Hamiltonians of the integrable hierarchy obtained directly from the series expansion of the trace of the monodromy matrix, it is immediate that the expectation value procedure goes straightforwardly to the full monodromy matrix \(T\) (and thence to its trace over the auxiliary space which is altogether decoupled from the quantum expectation value procedure). Accordingly, we define first a lattice expectation value

\[
T_a = \ldots \langle n_l | \otimes \langle n_{l+1} | \ldots (L_{a_1} L_{a_2} \ldots L_{a_N}) \ldots | n_l \rangle \otimes | n_{l+1} \rangle \ldots ,
\]

(2.14)

which nicely factors out as

\[
T_a = \prod_{i=1}^{N} \langle n_i | L_{ai} | n_i \rangle .
\]

(2.15)

Assuming now that \(L\) admits an expansion in powers of \(\delta\) as

\[
L_{ai} = 1 + \delta l_{ai} + O(\delta^2) ,
\]

(2.16)

we consider the product (setting \(\langle n_i | l_{ai} | n_i \rangle = l_a(x_i)\))

\[
T_a = \prod_{i=1}^{N} (1 + \delta l_{ai} + \sum_{n=2}^{\infty} \delta^n l_{ai}^{(n)}) .
\]

(2.17)

Expanding this expression in powers of \(\delta\), we get

\[
T_a = 1 + \delta \sum_i l_{ai} + \delta^2 \sum_{i<j} l_{ai} l_{aj} + \delta^2 \sum_i l_{ai}^{(2)} + \ldots .
\]

(2.18)

These, multiple in general, infinite series of the products of local terms, are characterized by two indices: the overall power \(n\) of \(\delta\), and the number \(m\) of the set of indices \(i\) (that is the number of distinct summation indices) over which the series is summed. Note that, in the \(T\) expansion one always has \(n \geq m\). The continuum limit soon to be defined more precisely, will entail the limit \(\delta \to 0\) with \(O(N) = O(1/\delta)\). We now formulate the following power-counting rule, that is terms of the form (for notational convenience \(l_{ai} = l_{ai}^{(1)}\) below)

\[
\delta^n \sum_{i_1 < i_2 < \ldots < i_m} l_{ai_1}^{(m_1)} \ldots l_{ai_m}^{(m_m)} , \quad \sum_{j=1}^{m} n_j = n ,
\]

(2.19)

with \(n > m\) are omitted in the continuum limit. The latter is defined by

\[
\delta \sum_i l_{ai} \to \int_{0}^{A} dx \ l_a(x)
\]

(2.20)

and similarly for multiple integrals. Here \(A\) is the length of the continuous interval defined as the limit of \(N\delta\). In other words, contributions to the continuum limit may only come
from the terms with \( n = m \) for which the power \( \delta^n \) can be exactly matched by the “scale” factor \( N^m \) of the \( m \)-multiple sum over \( m \) indices \( i \). In particular, only terms of order one in the \( \delta \) expansion of local classical matrices \( L_{ai} \equiv \langle n_i | L_{a} | n_i \rangle \) will contribute to the continuum limit. Any other contribution acquires a scale factor \( \delta^{n-m} \to 0 \), when the continuum limit is taken. This argument is of course valid term by term in the double expansion. Being only a weak limit argument, it always has to be checked for consistency.

Let’s remark that if \( L \) is taken to be \( R \), one naturally identifies \( \delta \) with the small parameter \( \hbar \), thus identifying in some sense the classical and the continuum limits. However, this is not required in general. It is clear to characterize separately both notions in our discussion as

\[
\text{classical limit: } \quad R = 1 + \hbar r , \\
\text{continuum limit: } \quad L = 1 + \delta l .
\]  

(2.21)

Recalling (2.13), the continuous limit of \( T \), hereafter denoted \( T \), is then immediately identified from (2.15), as the path-ordered exponential from \( x = 0 \) to \( x = A \)

\[
T = P \exp \left( \int_0^A dx \ l(x) \right) ,
\]

(2.22)

where suitable (quasi) periodicity conditions on the continuous variables \( \theta_k(x) \) of the classical matrix \( l(x) \), acting on the auxiliary space \( V \otimes C(\lambda) \), are assumed. Of course the definition of a continuous limit requires that the \( L \)-matrices are not too inhomogeneous (e.g. \( L \)-matrices at neighbor sites should not be too different. This is in fact assured by the long wavelength limit assumption.

### 2.3 The Lax matrix and \( r \)-matrix formulation

The above identification of \( T \) also defines it as the monodromy matrix of the first order differential operator \( d/dx + l(x) \). In addition, it has been built so that to straightforwardly generate the classical continuous limit of the Hamiltonians in (2.12) from the analytic expansion

\[
\text{tr}(T(\lambda)) = \sum_{n=1}^{\infty} (\lambda - \lambda_0)^n \mathcal{H}^{(n)} .
\]

(2.23)

We thus characterize \( l(x) \) as a local Lax matrix yielding the hierarchy of continuous Hamiltonians \( \mathcal{H}^{(n)} \). In order for this statement to agree with the key assumption of preservation
of integrability we are now lead to require a Poisson structure for \( l \) (inducing one for the continuous dynamical variables \( \theta_k(x) \)) compatible with the demand of classical integrability of the continuous Hamiltonians. Indeed, such a structure is deduced from (2.6), as the ultra-local Poisson bracket

\[
\{ l_1(x, \lambda_1), l_2(y, \lambda_2) \} = [r_{12}(\lambda_1 - \lambda_2), l_1(x, \lambda_1) + l_2(y, \lambda_2)]\delta(x - y),
\]  

(2.24)

where \( r \) is the classical limit (2.2) of the \( R \)-matrix characterizing the exchange algebra of the \( L \)-operators. More specifically, recalling that \( L_{ai} = 1 + \delta l_{ai} + \mathcal{O}(\delta^2) \), plugging it into (2.6) and assuming ultra-locality of Poisson brackets one gets

\[
\{ l_{ai}, l_{bj} \} = [r_{ab}, l_{ai} + l_{bj}]\frac{\delta_{ij}}{\delta}. \tag{2.25}
\]

One then identifies, in the continuum limit \( \delta \to 0 \), the factor \( \delta_{ij}/\delta \) with \( \delta(x - y) \). Reciprocally, it is a well known result (see, for instance L.D. Faddeev’s Les Houches Lectures in 1982) that if \( l(x) \) has a such an ultra-local linear Poisson bracket (2.24) the full monodromy matrix between 0 and \( A \) of \( d/dx + l(x) \) has the quadratic Poisson bracket structure (2.8), thereby guaranteeing Poisson commutation of the Hamiltonians.

We thus obtain a hierarchy of classically integrable, mutually Poisson commuting Hamiltonians from the explicit computation of the monodromy matrix \( t(\lambda) \) of the Lax operator \( d/dx + l(x) \) as \( H^{(n)} = \frac{d^n}{dx^n} t(\lambda)|_{\lambda = \lambda_0} \). Such Hamiltonians are however generally highly non-local and not necessarily very relevant as physical models. We shall thus extend our discussion to local Hamiltonians.

### 2.4 The case of local spin chains

Local spin chain Hamiltonians are more interesting, physically meaningful and easier to manipulate. In particular, they are the most relevant objects in connection with string theory and the AdS/CFT duality [1]. Their construction generically requires the determination of a so-called “regular value” \( \lambda_0 \) of the spectral parameter such that \( L_{ai}(\lambda_0) \propto \mathcal{P}_{ai} \), where \( \mathcal{P} \) is the permutation operator. In this sense the expansion of \( L \) can be expressed up to an appropriate normalization factor as (see also Appendix C)

\[
L(\lambda) = f(\lambda)(1 + \delta l + \mathcal{O}(\delta^2)) \tag{2.26}
\]

Of course only when the auxiliary space \( a \) and quantum space \( i \) are isomorphic has this “regular value” any relevance. One then defines the local Hamiltonians as (denoting as
usual $t(\lambda) = \text{tr}_a T_a(\lambda))$

$$H^{(n)} = \frac{d^n}{d\lambda^n} \ln(t(\lambda))|_{\lambda=\lambda_0}, \quad (2.27)$$

implying that they are no more such Hamiltonians expressed as linear combinations of higher derivatives of $t(\lambda)$. Their long wavelength limit (e.g. (2.10)) is not obviously derivable from a straightforward “diagonal” expectation value of the $T$-matrix contrary to (2.12), since in general $\langle F(A) \rangle \neq F(\langle A \rangle)$, for any functional of a set of operators $A$. However, we show below that this is indeed the case due to locality properties. Let us first focus for simplicity (but, as we shall see, without loss of generality) on the first local Hamiltonian

$$H^{(1)} = t(\lambda_0)^{-1} \frac{d}{d\lambda} t(\lambda)|_{\lambda=\lambda_0}, \quad (2.28)$$

where, $t^{-1}(\lambda_0) = \mathcal{P}_{12}\mathcal{P}_{23} \ldots \mathcal{P}_{N-1N}$. This operator acts exactly as a one-site shift on tensorized states, identifying of course site labels according to the assumed periodicity, i.e. $N + 1 = 1$. (Normalization issues will be discussed in Appendix C). Computing the expectation value of $H^{(1)}$ we obtain

$$\langle H^{(1)} \rangle = \langle n_1 | \otimes \ldots \otimes | n_N \rangle t^{-1}(\lambda_0) \frac{d}{d\lambda} \left( f^N(\lambda) \text{tr}_a \prod_{i=1}^N (1 + \delta l_{ai} + \mathcal{O}(\delta^2)) \right) | n_1 \rangle \otimes \ldots \otimes | n_N \rangle. \quad (2.29)$$

One has

$$\langle n_1 | \otimes \langle n_2 | \otimes \ldots \otimes | n_N \rangle t^{-1}(\lambda_0) = \langle n_2 | \otimes \langle n_3 | \otimes \ldots \otimes | n_1 \rangle \quad (2.30)$$

and of course $N + 1 \equiv 1$.

Taking into account the power-counting rule described in section 2.2 we obtain (see also Appendix C) that

$$\langle H^{(1)} \rangle = \prod_{i=1}^N \langle n_{i+1} | n_i \rangle \frac{d}{d\lambda} \left( f^N(\lambda) \text{tr}_a \prod_{i=1}^N (1 + \delta \langle l_{ai} \rangle + \mathcal{O}(\delta^2)) \right). \quad (2.31)$$

We then easily establish that in the continuum limit, using the power counting rule and the factorized form of both the state vector as $\langle n_1 | \otimes \ldots \otimes | n_N \rangle$ that the operator to be valued over it $t^{-1}(\lambda_0) = \mathcal{P}_{12}\mathcal{P}_{23} \ldots \mathcal{P}_{N-1N}$, $\langle t^{-1}(\lambda_0) \rangle = (t(\lambda_0))^{-1}$. We finally obtain that in the continuum limit

$$\langle H^{(1)} \rangle = \langle t^{-1}(\lambda_0) \frac{d}{d\lambda} t(\lambda)|_{\lambda=\lambda_0} = \langle t(\lambda_0) \rangle^{-1} \frac{d}{d\lambda} \langle t(\lambda) \rangle|_{\lambda=\lambda_0} = \frac{d}{d\lambda} (\ln(t(\lambda)))|_{\lambda=\lambda_0}. \quad (2.32)$$
The computation may be easily generalized along the same lines for any higher Hamiltonian. Higher local Hamiltonians are indeed obtained from (2.27), admitting thus an expansion as

\[ H^{(n)} = t^{-1}(\lambda_0) \frac{d^n}{d\lambda^n} t(\lambda) \big|_{\lambda_0} + \text{polynomials}, \]  

(2.33)

depending only on lower order local Hamiltonians. When computing the expectation value of such higher Hamiltonians one gets the expectation value of \( t^{-1}(\lambda_0) \frac{d^n}{d\lambda^n} t(\lambda) \big|_{\lambda_0} \) which in the continuum classical limit yields

\[ \langle t^{-1}(\lambda_0) \frac{d^n}{d\lambda^n} t(\lambda) \big|_{\lambda_0} \rangle = \langle t(\lambda_0) \rangle^{-1} \frac{d^n}{d\lambda^n} \langle t(\lambda) \rangle \big|_{\lambda_0}, \]  

(2.34)

using the same arguments as in the \( n = 1 \) case. In addition, one obtains expectation values of the polynomials of order \( k \) in the local Hamiltonians. In this case expectation values by tensor product of local vectors \( \langle n_1 \rangle \ldots \langle n_N \rangle \) are exactly factorized over products of \( k \) local monomials \( h_{i_1} \ldots h_{i_k} \), except if indices \( i \) coincide (or at least overlap for multiple indices). Locality of the lower Hamiltonians plays here a crucial role. It is clear that such families of terms with coinciding or overlapping indices correspond to a second “label” \( M = k - 1 \) and therefore their contribution will necessarily be suppressed in the continuum limit, with respect to the contribution of the generic terms (non-coinciding indices) with \( M = k \) by the power-counting argument. Hence, it is consistent to conclude that in the continuum limit

\[ \langle \text{Polynomial in } (H^{(i)}) \rangle = \text{Polynomial in } (\langle H^{(i)} \rangle) \]  

(2.35)

and therefore

\[ \langle H^{(n)} \rangle = \langle \frac{d^n}{d\lambda^n} \ln(t(\lambda)) \big|_{\lambda=\lambda_0} \rangle = \frac{d^n}{d\lambda^n} \ln(\langle t(\lambda) \rangle) \big|_{\lambda=\lambda_0}. \]  

(2.36)

This is the final, key result in systematically establishing the classical continuum limit of integrable spin chains. We may now apply this general procedure to all sorts of examples, starting with the simpler applications.

### 3 The XXX chain

The XXX model Hamiltonian describing first neighbor spin-spin interactions is given by

\[ H = \frac{1}{2} \sum_{j=1}^{N} \left( \sigma_{j}^{x} \sigma_{j+1}^{x} + \sigma_{j}^{y} \sigma_{j+1}^{y} + \sigma_{j}^{z} \sigma_{j+1}^{z} \right). \]  

(3.1)
It is well known that when one considers the long wavelength limit one obtains a classical \( \sigma \)-model [2, 3]. We shall briefly review how this process works. The coherent spin state is parametrized by the parameters \( x, t \) via the fields \( \theta, \phi \) as

\[
|n(x, t)\rangle = \cos \theta(x, t) \ e^{i\varphi(x, t)} \ |+\rangle + \sin \theta(x, t) \ e^{-i\varphi(x, t)} \ |-\rangle ,
\]

(3.2)

where the ranges of variables is \( \theta \in (0, \pi/2) \) and \( \varphi \in (0, \pi) \). One can verify the completeness relation

\[
\int d\mu(n) \langle n|\langle n = 1 ,
\]

(3.3)

where the integration measure is given by

\[
d\mu(n) = \frac{4}{\pi} \sin \theta \ \cos \theta \ d\theta \ d\varphi .
\]

(3.4)

Then as was described in [2, 3] and in subsection 2.1, one obtains a classical Hamiltonian via the expectation value procedure by employing (2.10). The appropriate XXX 2-site Hamiltonian is

\[
H_{ll+1} \propto (\mathcal{P}_{ll+1} - \mathbb{I}) ,
\]

(3.5)

where \( \mathcal{P} \) is the permutation operator acting as \( \mathcal{P}(a \otimes b) = b \otimes a \) for \( a, b \) vectors in \( V \). From the definition of \( \mathcal{H} \) we are led to compute quantities of the type

\[
\langle a| \otimes \langle b| \mathcal{P} |a| \otimes |b\rangle = \langle a|b\rangle \otimes \langle b|a\rangle = |\langle a|b\rangle|^2 .
\]

(3.6)

They are expressed in terms of scalar products of the form

\[
\langle n|n\rangle = \cos(\theta - \tilde{\theta}) \ \cos(\varphi - \tilde{\varphi}) + i \cos(\theta + \tilde{\theta}) \ \sin(\varphi - \tilde{\varphi}) .
\]

(3.7)

In the long wavelength limit, \( |n\rangle \rightarrow |\delta n\rangle \), \( \tilde{\theta}(x) = \theta(x + \delta) \) and \( \tilde{\varphi}(x) = \varphi(x + \delta) \). We conclude that

\[
H \propto \int dx \ (\theta'^2 + \sin^2(2\theta) \ \varphi'^2) .
\]

(3.8)

We shall now derive the Lax representation yielding (3.8) following section 2. The \( R \)-matrix for the XXX model is [14]

\[
R(\lambda) = \lambda + i\hbar \mathcal{P} ,
\]

(3.9)
where for normalization issues we refer to Appendix C. This $R$-matrix is a solution of the quantum YB equation \[4\]. It has a consistent normalized classical limit defined as
\[
r(\lambda) = \frac{1}{\lambda} P ,
\]
which satisfies the classical YB equation. Alternatively, the classical $r$-matrix may be written as
\[
r(\lambda) = \frac{1}{\lambda} \left( \begin{array}{cc}
\frac{i}{2}(\sigma^z + 1) & \sigma^- \\
\sigma^+ & \frac{i}{2}(-\sigma^z + 1)
\end{array} \right) .
\]
Set first
\[
L_{an}(\lambda) = R_{an}(\lambda - \frac{i\hbar}{2}) \quad (3.12)
\]
and demand that $L$ satisfies the fundamental algebraic relation
\[
R_{ab}(\lambda_1 - \lambda_2) \quad L_{an}(\lambda_1) \quad L_{bn}(\lambda_2) = L_{bn}(\lambda_2) \quad L_{an}(\lambda_1) \quad R_{ab}(\lambda_1 - \lambda_2) ,
\]
where as usual in the spin chain framework we call $n$ the quantum space and $a$ the auxiliary space. Following the general derivation of section 2 and going directly to the continuous limit we disregard higher powers in $\delta = \hbar$ (in this case the two small parameters are naturally identified). We next define a “local Lax matrix” as a mean value of $L$ on the same coherent spin state, taken solely over the quantum space
\[
\langle n | L_{an}(\lambda) | n \rangle = 1 + i\hbar l(x, \lambda) ,
\]
where
\[
l = \left( \begin{array}{cc}
\frac{1}{2}\langle n | \sigma^z | n \rangle & \langle n | \sigma^+ | n \rangle \\
\langle n | \sigma^+ | n \rangle & -\frac{1}{2}\langle n | \sigma^z | n \rangle
\end{array} \right) = \frac{1}{2} \left( \begin{array}{cc}
\cos 2\theta(x) & \sin 2\theta(x) e^{-2i\varphi(x)} \\
\sin 2\theta(x) e^{+2i\varphi(x)} & -\cos 2\theta(x)
\end{array} \right) ,
\]
where we have used the form of the coherent states to compute the matrix elements explicitly. Then $l$ satisfies the classical fundamental algebraic relation
\[
\{ l_1(x, \lambda_1), \quad l_2(y, \lambda_2) \} = [r_{12}(\lambda_1 - \lambda_2), \quad l_1(\lambda_1) + l_2(\lambda_2)] \delta(x - y) .
\]
Setting $l(x, \lambda) = \Pi/\lambda$ and taking into account the above algebraic relations we get
\[
\{ \Pi_1, \quad \Pi_2 \} = P_{12}(\Pi_2 - \Pi_1) \delta(x - y) .
\]
The parametrization in terms of the continuum parameters \( \theta(x), \phi(x) \) gives rise to the classical version of \( sl_2 \). Indeed, parametrizing the generators of the classical current algebra as

\[
S^z = \cos 2\theta, \quad S^\pm = \frac{1}{2} \sin 2\theta \ e^{\mp i\phi}.
\]  

(3.18)

we obtain from the fundamental relation that

\[
\{S^+, S^-\} = S^z \delta(x - y), \quad \{S^z, S^\pm\} = \pm 2S^\pm \delta(x - y).
\]  

(3.19)

The continuum parameters \( \theta(x) \) and \( \phi(x) \) can also be expressed in terms of canonical variables \( p \) and \( q \) as

\[
\cos 2\theta(x) = p(x), \quad \varphi(x) = q(x) \quad \text{and} \quad \{q(x), p(y)\} = i\delta(x - y).
\]  

(3.20)

The \( l \)-matrix in (3.15) coincides obviously with the potential term in the Lax matrix of the classical Heisenberg model. Precisely, one recalls that one must consider as classical Lax operator a la Zakharov–Shabat

\[
L = \frac{d}{dx} + l(x).
\]

The monodromy matrix for \( L \) is well known now to yield the classical Hamiltonians including the first non trivial one (see [12])

\[
H \propto \int dx \left( \left( \frac{dS^z}{dx} \right)^2 + \left( \frac{dS^x}{dx} \right)^2 + \left( \frac{dS^y}{dx} \right)^2 \right).
\]  

(3.21)

Recalling the expressions (3.18) and substituting in the expression above we obtain the Hamiltonian (3.8), hence the process above works consistently.

Having exemplified the general construction of Section 2 to a simple system and checked the consistency of the approach we now turn to more complicated systems by first moving to trigonometric and elliptic \( sl(2) \) \( R \)-matrices, corresponding to the XXZ and XYZ spin chains.

### 4 The anisotropic Heisenberg model

Consider the generic anisotropic XYZ model with Hamiltonian

\[
H = \frac{1}{2} \sum_{j=1}^{N} \left( J_x \sigma^x_j \sigma^x_{j+1} + J_y \sigma^y_j \sigma^y_{j+1} + J_z \sigma^z_j \sigma^z_{j+1} \right).
\]  

(4.1)

For the following computations it is convenient to set

\[
J_\xi = 1 - \delta^2 a_\xi, \quad \xi \in \{x, y, z\}.
\]  

(4.2)
The Hamiltonian is written as
\[ H = \sum_{j=1}^{N} P_{jj+1} - \frac{N}{2} - \frac{\delta^2 a_x}{2} \sum_{j=1}^{N} \sigma_x^j \sigma_x^{j+1} - \frac{\delta^2 a_y}{2} \sum_{j=1}^{N} \sigma_y^j \sigma_y^{j+1} - \frac{\delta^2 a_z}{2} \sum_{j=1}^{N} \sigma_z^j \sigma_z^{j+1}. \] (4.3)

The additive constant may be omitted here. Taking into account equations (3.5)–(3.8), (4.3) and keeping terms of order \( \delta^2 \) we get
\[ H \propto \int dx \left( \theta'^2 + \sin^2(2\theta) \phi'^2 + a_x \sin^2(2\theta) \cos^2(2\phi) + a_y \sin^2(2\theta) \sin^2(2\phi) + a_z \cos^2(2\theta) \right). \] (4.4)

This may be seen as a “deformation” of the classical Heisenberg Hamiltonian. The last three terms are essentially potential-like terms. In the special case of the XXZ model the terms with coupling constant \( a_x, a_y \) are zero, whereas in the XXX case all potential terms vanish and one recovers the Hamiltonian (3.8). If we now recall the parametrization (3.18), then the expression above reduces to the Hamiltonian of the Landau-Lifshitz model or the anisotropic classical magnet [12]
\[ H \propto \int dx \left( \left( \frac{dS_x}{dx} \right)^2 + \left( \frac{dS_y}{dx} \right)^2 + \left( \frac{dS_z}{dx} \right)^2 + a_x S_x^2 + a_y S_y^2 + a_z S_z^2 \right). \] (4.5)

We now derive the classical \( l \)-matrix for the anisotropic cases. We focus in more detail on the XXZ \( R \)-matrix
\[ R(\lambda) = \begin{pmatrix} \sinh(\lambda + \frac{i\mu}{2} \sigma^z + \frac{i\mu}{2}) & \sinh(i\mu) \sigma^- \\ \sinh(i\mu) \sigma^+ & \sinh(\lambda - \frac{i\mu}{2} \sigma^z + \frac{i\mu}{2}) \end{pmatrix}. \] (4.6)

The classical limit of the XXZ \( R \)-matrix, after appropriate normalization, is given as (we divide with the constant factor \( \sinh \lambda \))
\[ R(\lambda) = 1 + i \mu \ r(\lambda) + O(\mu^2), \] (4.7)

where
\[ r(\lambda) = \frac{1}{\sinh \lambda} \begin{pmatrix} \cos \lambda + \frac{1}{2} \sigma^+ & \sigma^- \\ \sigma^+ & -\cos \lambda + \frac{1}{2} \sigma^- \end{pmatrix}. \] (4.8)

The associated classical Lax operator is again obtained from \( L(\lambda) = R(\lambda - \frac{i\mu}{2}) \) as (once again moving immediately to the continuous limit)
\[ \langle n|L(\lambda)|n \rangle = 1 + i \mu \ l(x, \lambda) + O(\mu^2), \] (4.9)
where

\[
\ell(\lambda) = \frac{1}{\sinh \lambda} \begin{pmatrix}
\langle n|\sigma^z|n\rangle \cosh \lambda & \langle n|\sigma^-|n\rangle \\
\langle n|\sigma^+|n\rangle & -\langle n|\sigma^z|n\rangle \cosh \lambda
\end{pmatrix}
\]

\[
= \frac{1}{\sinh \lambda} \begin{pmatrix}
\frac{1}{2}S^z \cosh \lambda & S^-
\\
S^+ & -\frac{1}{2}S^z \cosh \lambda
\end{pmatrix},
\]

(4.10)

where \(S^Z, S^\pm\) are the classical generators of the current \(sl(2)\) algebra realized in terms of the angular variables in (3.18). The continuous variables \(x, y\) were omitted here for simplicity and will be from now on whenever there is no ambiguity.

Let us also briefly characterize the classical algebra underlying the model. We set

\[
l_i(\lambda) = \frac{\cosh \lambda}{\sinh \lambda} D_i + \frac{1}{\sinh(\lambda)} A_i, \quad r_{12}(\lambda) = \frac{\cosh \lambda}{\sinh \lambda} D_{12} + \frac{1}{\sinh(\lambda)} A_{12}.
\]

(4.11)

Substituting this expressions to (3.16) and taking into account that

\[
[A_{12}, A_1] = -[D_{12}, A_2],
\]

(4.12)

we end up with the following set of Poisson structures

\[
\{D_1, D_2\} = 0, \quad \{D_1, A_2\} = [D_{12}, A_2]\delta(x - y), \quad \{A_1, A_2\} = -[A_{12}, D_1]\delta(x - y)
\]

(4.13)

which give rise to the \(sl_2\) Poisson algebra (3.19).

The full XYZ classical \(r\)-matrix also yields, through this process, the classical Lax operator of the fully anisotropic classical Heisenberg model, satisfying also the fundamental linear algebraic relation (3.16) (see also [12]). A detailed presentation of this derivation is omitted here for the sake of brevity.

5 The \(gl_n\) classical magnet

In this section we further extend our analysis to the case of higher rank algebras. In particular, we study the classical limit of isotropic and anisotropic \(gl_n\) type magnets.

5.1 The isotropic case

First consider the generic situation of the isotropic \(gl_n\) quantum spin chain. The \(R\)-matrix is given by the general form (3.9), where the permutation operator is of the form

\[
\mathcal{P} = \sum_{i,j=1}^n e_{ij} \otimes e_{ji}, \quad (e_{ij})_{kl} = \delta_{ik}\delta_{jl}.
\]

(5.1)
The coherent state is parametrized by $n$ continuum parameters as

$$|n(x, t)\rangle = \sum_{i=1}^{n} \alpha_i(x, t) |e_i\rangle ,$$  
(5.2)

where $|e_i\rangle$ is the $n$ column vector with one at position $i$ and zero elsewhere. In addition

$$\langle n|n \rangle = 1 \Rightarrow \sum_{i=1}^{n} |\alpha_i|^2 = 1 .$$  
(5.3)

Following the process described in the previous sections we end up with the classical $r$ and $l$ operators defined as (in here $L(\lambda) = R(\lambda)$, instead of (3.12))

$$r(\lambda) = \frac{1}{\lambda} P , \quad l(\lambda) = \frac{1}{\lambda} \sum_{i,j=1}^{n} e_{ij} \otimes \langle n|e_{ji}|n \rangle = \frac{1}{\lambda} \sum_{i,j=1}^{n} e_{ij} l_{ij} .$$  
(5.4)

The $l$-matrix satisfies the linear algebraic relation (3.16), which clearly gives rise to the classical current-$gl_n$ exchange relations among the elements $l_{ij}(x)$. These are given by

$$\{l_{ij}(x), l_{kl}(y)\} = (\delta_{il} l_{jk} - \delta_{jk} l_{il}) \delta(x - y) .$$  
(5.5)

We compute next the first local classical integral of motion starting from the spin chain Hamiltonian

$$H^{(0)} \propto \sum_{j=1}^{N} \mathcal{P}_{jj+1} ,$$  
(5.6)

where we have dropped from the beginning the constant compared to (3.5). Then, defining first the Hamiltonian density as

$$\mathcal{H}^{(0)}(x) = \langle n| \otimes \langle \tilde{n} | \mathcal{P} | n \rangle \otimes |\tilde{n}\rangle = \sum_{i,j=1}^{n} l_{ij}(x) l_{ji}(x + \delta) .$$  
(5.7)

Expanding appropriately this, we conclude that

$$\mathcal{H}^{(0)}(x) = \sum_{i,j=1}^{n} l_{ij}(x) l_{ji}(x) - \frac{1}{2} \delta^2 \sum_{i,j=1}^{n} \frac{dl_{ij}(x)}{dx} \frac{dl_{ji}(x)}{dx} ,$$  
(5.8)

where we have dropped boundary terms by imposing appropriate boundary conditions. The first term above is the quadratic Casimir and can be dropped. The second term, proportional to $\delta^2$, provides, upon integration, the classical Hamiltonian

$$H^{(0)} \propto \int dx \sum_{i,j=1}^{n} \frac{dl_{ij}(x)}{dx} \frac{dl_{ji}(x)}{dx} .$$  
(5.9)
The classical integrals of motion on the other hand are obtained from the monodromy matrix of \( l(x) \) for the generic case along the lines described in the Appendix A (see also [12]). Comparing with (5.8) they are seen to coincide.

The direct computation from the classical \( l(x) \) matrix is actually presented in the Appendix for another model, but it goes along the same lines for the generalized Heisenberg model, and is omitted here for brevity.

### 5.2 The anisotropic case

Consider now the anisotropic case. Recall the classical \( r \)-matrix associated to \( A_{n-1}^{(1)} \) [15]

\[
r(\lambda) = \frac{\cosh(\lambda)}{\sinh(\lambda)} \sum_{i \neq j} e_{ii} \otimes e_{jj} + \frac{1}{\sinh(\lambda)} \sum_{i \neq j} e^{(\text{sgn}(i-j)-(i-j)\frac{2}{n+1})\lambda} e_{ij} \otimes e_{ji}.
\]  

(5.10)

The associated classical \( l \)-matrix will be of the form

\[
l(\lambda) = \frac{\cosh(\lambda)}{\sinh(\lambda)} \sum_{i \neq j} l_{ij}(x)e_{ii} + \frac{1}{\sinh(\lambda)} \sum_{i \neq j} e^{(\text{sgn}(i-j)-(i-j)\frac{2}{n+1})\lambda} l_{ji}(x)e_{ij}
\]  

(5.11)

and satisfies the linear algebraic relation (3.16). Take now the Hamiltonian of the deformed spin chain (see e.g. [16])

\[
H \propto \sum_j U_{jj+1},
\]  

(5.12)

where the matrix \( U \) is a representation of the Hecke algebra expressed as (\( q = e^\mu \))

\[
U = \sum_{i \neq j=1}^{n} \left( e_{ij} \otimes e_{ji} - q^{-\text{sgn}(i-j)} e_{ii} \otimes e_{jj} \right).
\]  

(5.13)

It is convenient to rewrite it as

\[
U = \mathcal{P} - \mathbb{I} + \sum_{i \neq j=1}^{n} (1 - q^{-\text{sgn}(i-j)}) e_{ii} \otimes e_{jj}
\]  

(5.14)

and also set

\[
\mu = \delta \alpha, \quad q^{-\text{sgn}(i-j)} \sim 1 - \text{sgn}(i - j)\delta \alpha + \frac{\delta^2 \alpha^2}{2}.
\]  

(5.15)

The Hamiltonian in this case is basically a “deformation” of the isotropic case and again the first non-trivial terms arises at order \( \delta^2 \). We get that

\[
H \propto \int dx \left( \sum_{i,j=1}^{n} \frac{dl_{ij}(x)}{dx} \frac{dl_{ji}(x)}{dx} + \alpha^2 \sum_{i \neq j=1}^{n} l_{ji}(x)l_{ij}(x) + 2a \sum_{i < j=1}^{n} (l_{ii}(x)l'_{jj}(x) - l_{jj}(x)l'_{ii}(x)) \right)
\]  

(5.16)
It clearly provides a generalization of the Landau–Lifshitz model\(^2\). Note that the last term, proportional to \(a\), disappears in the case \(n = 2\), given that in this case \(e_1 + e_2 = \mathbb{I}\). A generalization starting from the elliptic classical \(r\)-matrix can be also obtained, but is omitted here for brevity. Similarly, this classical Hamiltonian may be again directly obtained from the classical \(l\) operator (5.11) as is described e.g. in [12].

6 Novel “dualities” of integrable models

In this section we shall investigate certain integrable “duals” of the XXX spin chain and its higher rank generalizations, and we shall derive their classical counterparts. They will be based on the coproduct structure applied this time to \(c\)-number matrices.

Given an initial quantum \(R\)-matrix, the \(c\)-number YB equation reads

\[
[R_{12}(\lambda), \ U_{1}U_{2}] = 0 ,
\]

(6.1)

where \(U\) is a particular scalar \(n \times n\) matrix. Considering, for instance, \(R\) to be the Yangian \(R\)-matrix, the latter equation is valid for any \(n \times n\) matrix. In the case of the XXX matrix one may take for example the Pauli matrices to write

\[
\sigma_1^\xi \sigma_2^\xi R_{12}(\lambda) \sigma_1^\xi \sigma_2^\xi = R_{12}(\lambda) .
\]

(6.2)

Given the relation (6.1) we may always define a new \(L\)-operator \(\tilde{L}_{12} = U_1L_{12}\), which obviously satisfies (3.13) as long as \(L\) also satisfies it.

Before we proceed with the classical limit of the models, let us first examine the quantum local Hamiltonian arising from the \(\tilde{L}\)-matrices. For simplicity we choose \(\lambda = 0\) to be the regular value and as usual we define this Hamiltonian as

\[
H \propto \frac{d}{d\lambda} \log t(\lambda)|_{\lambda = 0} ,
\]

(6.3)

where we now have

\[
t(\lambda) = \text{tr}_0[\tilde{L}_{0N}(\lambda) \ldots \tilde{L}_{01}(\lambda)] = \text{tr}_0[U_0R_{0N}(\lambda) \ldots U_0R_{01}(\lambda)] .
\]

(6.4)

\(^2\)Note that for \(q = e^{i\mu}\) one obtains a non-Hermitian Hamiltonian. This is not so surprising given that higher rank spin chain as well as higher \(A_{n-1}^{(1)}\) affine Toda field theories with imaginary coupling are non-unitary models. Nevertheless, the relevant physical quantities, such as spectrum excitations, exact \(S\) matrices etc. have been extensively studied.
The transfer matrix at $\lambda = 0$ becomes

$$t(0) \propto \text{tr}_0[U_0 P_0 \ldots U_0 P_{01}] = \ldots = U_N \ldots U_{i+1} \Pi U_{i-1} \ldots U_1 U_N,$$

$$\Pi = \mathcal{P}_{12} \mathcal{P}_{23} \ldots \mathcal{P}_{N-1N}.$$  

(6.5)

Taking the derivative of the transfer matrix we find that

$$\left. \frac{dt(\lambda)}{d\lambda} \right|_{\lambda=0} \propto U_N \ldots U_{i+1} \left. \frac{d\tilde{R}_{ii+1}(\lambda)}{d\lambda} \right|_{\lambda=0} \Pi U_{i-1} \ldots U_1 U_N,$$

(6.6)

where $\tilde{R} = \mathcal{P} R$ and we consider here the XXX $R$-matrix. Gathering the information above we conclude that the Hamiltonian is again local and reads

$$H \propto \sum_{i=1}^{N} U_{i+1} \left. \frac{d\tilde{R}_{ii+1}(\lambda)}{d\lambda} \right|_{\lambda=0} U_{i-1}^{-1}.$$  

(6.7)

Note that more general “regularity conditions” of the form $L_{ab}(\lambda_0) = U_a \mathcal{P}_{ab}$, will similarly guarantee locality of the Hamiltonians derived from (6.3).

An inhomogeneous generalization of this construction is obviously available by using distinct solutions to (6.1) at each site of the chain. One starts from a set of solutions

$$[R_{12}(\lambda), U_1^{(i)} U_2^{(i)}] = 0, \quad i = 1, 2, \ldots, N.$$  

(6.8)

The quantum Hamiltonians are derived from the monodromy matrix

$$t(\lambda) = \text{tr}_0[\tilde{L}_{0N}(\lambda) \ldots \tilde{L}_{01}(\lambda)] = \text{tr}_0[U_0^{(N)} R_{0N}(\lambda) \ldots U_0^{(i)} R_{0i}(\lambda)],$$

(6.9)

by applying the co-module structure to generate an inhomogeneous transfer matrix by successive tensoring by $U_0^{(i)} R_{0i}(\lambda)$. Note that the same procedure (albeit with shifts over the spectral parameter, or distinct choices of representations of the quantum space $i$, instead of twists by a $U^i$ c-number constant matrix) is used to get inhomogeneous spin chains in many examples.

Local Hamiltonians are again obtained via (6.3) and following the exact steps of the proof of the homogeneous Hamiltonian above. We end up with the generic final expression

$$H \propto \sum_{i=1}^{N} U_{i+1}^{(i)} \left. \frac{d\tilde{R}_{ii+1}(\lambda)}{d\lambda} \right|_{\lambda=0} (U_{i+1}^{(i)})^{-1}.$$  

(6.10)

Such Hamiltonians may be interpreted as describing spin chains in inhomogeneous backgrounds modifying the nearest-neighbor couplings in a site-dependant pattern. Possibly
interesting types of inhomogeneities include a local defect (one single \( U_i \) distinct from 1); a domain-like defect (\( U_i = U \) for \( i \leq i_0 \) and \( U_i = 1 \) elsewhere) or a “double-chain” effect (\( U_i = U \) for even \( i \) and 1 for odd \( i \)). We shall not discuss the general construction of the classical limit for such chain Hamiltonians, postponing it for future studies.

We now move back to the homogeneous case. A first remark is in order here. It follows from (6.6), (6.10) that one expects to have conditions on the choice of \( U_i \) and the parametrization of \(|n_i\rangle\) in order to be able to define classical continuum limits. To illustrate this point let us discuss in detail the computation of \( \langle t^{-1}(0) \rangle \). From (6.6) one has

\[
\langle t^{-1}(0) \rangle = \otimes_i \langle n_i | (\prod_i U_i)^{-1} \otimes_i | n_i \rangle = \prod_i \langle n_{i+1} | (U_i)^{-1} | n_i \rangle .
\]  

(6.12)

It follows that if \( U_i \) and \(|n_i\rangle\) are such that

\[
U_i |n_i\rangle = |n_i\rangle + \delta |v_i\rangle + \mathcal{O}(\delta^2) ,
\]  

(6.13)

where \( \delta \) is the same scaling parameter as for the continuous limit of \( L \) and \(|v_i\rangle\) is some vector, which can be chosen to be orthogonal to \(|n_i\rangle\) without loss of generality. If this condition is fulfilled the expectation value then has the following form

\[
\langle t^{-1}(0) \rangle = \prod_{i=1}^N (1 - \delta n_i |n_i\rangle - \delta \langle n_i |v_i\rangle + \mathcal{O}(\delta^2) ,
\]  

(6.14)

where the key identifications hold up to order \( \delta^2 \) in the discrete case. Hence (due to the power-counting argument) exactly in the continuous limit

\[
\otimes_i \langle n_i | (\prod_i U_i)^{-1} \otimes_i | n_i \rangle = \prod_i \langle n_i | U_i | n_i \rangle^{-1} .
\]  

(6.15)

Hence, the technical derivation of section 2.4 will hold, yielding again \( \langle t^{-1}(0) \rangle = \langle t(0) \rangle^{-1} \).

It is to be expected that similar conditions will arise when considering the derivative term.

We next examine the most general situation associated to the XXX model. Consider the generic \( 2 \times 2 \) matrix \( U \) and its inverse

\[
U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad U^{-1} = \frac{1}{D} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} , \quad D = ad - bc .
\]  

(6.16)

Take also into account the form of \( U^\xi U^{-1} \) we conclude that the most general 2-site
Hamiltonian after applying the $U$ transformation to XXX is given by the following form

$$\begin{align*}
H &= \frac{1}{2D} \sigma^z \otimes \left((ad + bc)\sigma^z - 2ab\sigma^+ + 2cd\sigma^-ight) \\
& \quad + \frac{1}{D} \sigma^+ \otimes \left(bd\sigma^z - b^2\sigma^+ + d^2\sigma^-\right) \\
& \quad + \frac{1}{D} \sigma^- \otimes \left(-ac\sigma^z + a^2\sigma^+ - c^2\sigma^-\right).
\end{align*}$$

(6.17)

To gain further insight we focus on particular examples. Taking, for instance, the XXX chain and setting $U = \sigma^z$, the local Hamiltonian becomes

$$H \propto \sum_{j=1}^{N} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \sigma_j^z \sigma_{j+1}^z\right),$$

(6.18)

with the characteristic flip of signs in front of the $\sigma^z \otimes \sigma^z$ term. Similarly, for $U = \sigma^x, \sigma^y$ a minus sign in front of the $\sigma^x \otimes \sigma^x$ and $\sigma^y \otimes \sigma^y$ terms, respectively, is attached.

A classical limit can be defined for these modified Lax matrix (recall $L(\lambda) = R(\lambda - \frac{i\hbar}{2})$) set also $\tilde{L}_{12} = U_1 \ L_{12}$. More precisely, for $U = \sigma^z$ we have, after acting from the left and right with the coherent state:

$$\langle n|\tilde{L}(\lambda)|n\rangle = \begin{pmatrix} \lambda + \frac{i\hbar}{2} \cos 2\theta & \frac{i\hbar}{2} \sin 2\theta e^{2i\varphi} \\ -\frac{i\hbar}{2} \sin 2\theta e^{-2i\varphi} & -\lambda + \frac{i\hbar}{2} \cos 2\theta \end{pmatrix}. \quad (6.19)$$

Now consider the rescaling $\theta \to \hbar \theta$, in the small $\hbar$ limit and also appropriately rescale $\lambda \to \hbar^2 \lambda$. This is precisely a realization of the condition $[6.13)$ on $U$ and $|n\rangle$ discussed above. The linear limit of the $L$ operator above becomes after setting $i\theta e^{2i\varphi} = \psi, -i\theta e^{-2i\varphi} = \bar{\psi}$ ($\tilde{L}(\lambda) \sim 1 + \hbar \tilde{l}(\lambda)$)

$$\langle n|\tilde{L}_a(\lambda)|n\rangle = 1 + i\hbar \tilde{l}(\lambda), \quad \tilde{l}(\lambda) = -2 \begin{pmatrix} \lambda & \psi \\ \bar{\psi} & -\lambda \end{pmatrix}. \quad (6.20)$$

The $l$-matrix above is nothing else than the classical NLS Lax operator (see also [17] for lattice versions NLS). The new spectral parameter is here defined as $\tilde{\lambda} = \frac{\lambda + i\hbar}{\hbar^2}$ hence the critical value for $\tilde{\lambda}$ becomes infinite in the continuum limit (in agreement with the computations in [12] and Appendix A).

Consider next $U = \sigma^x$ and recall the parametrization (3.20). Taking a similar limit we get

$$\begin{align*}
\cos 2\theta & \to \hbar p, \quad \sin 2\theta \to 1, \quad e^{\pm 2i\varphi} \to 1 \pm 2i\hbar q, \quad \lambda \to \hbar^2 \lambda
\end{align*}$$

(6.21)

and keep only lowest order terms. The classical Lax operator takes the following form

$$\tilde{l}(\lambda) = 2 \begin{pmatrix} -q & -\lambda + \frac{i\hbar}{2} \\ -\lambda - \frac{i\hbar}{2} & q \end{pmatrix}. \quad (6.22)$$
The latter is just the classical Lax operator for the harmonic oscillator. Note that the
classical $l$-matrices presented above are of the form: $\tilde{l}(\lambda) = \lambda A + B$. Taking into account
that $\tilde{l}$ satisfies the linear algebraic relation (3.16) we end up with the Poisson relations
\[
\{B_1, B_2\} = \mathcal{P}(A_1 - A_2)\delta(x - y),
\]
which lead to the following expected canonical exchange relations
\[
\text{NLS model : } \{\psi(x), \bar{\psi}(y)\} = \delta(x - y),
\]
\[
\text{Harmonic oscillator : } \{q(x), p(y)\} = i\delta(x - y).
\]
We next discuss the classical limits of these “dual” quantum Hamiltonians. First, consider
the quantum Hamiltonian which corresponds to the NLS model (i.e. $U = \sigma^z$) and set
\[
\mathcal{H}(x) = \langle a| \otimes \langle b| (\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y - \sigma^z \otimes \sigma^z) |a\rangle \otimes |b\rangle,
\]
where $\sin 2\theta \to 2\hbar\theta$, $\cos 2\theta \to 1$. The corresponding Hamiltonian becomes
\[
H = \int dx \theta^2(x) = \int dx \psi(x)\bar{\psi}(x),
\]
which is just the first integral of motion of the NLS model (see [12] for details on the
computation of the classical integrals of motion), that is the number of particles. Note that
higher integrals of motion may be obtained from the higher quantum Hamiltonian (higher
derivatives of $\log(t(\lambda))$) (see also Appendix B, where higher integrals of motion are also
computed for NLS). Consider now the situation where $U = \sigma^x$
\[
\mathcal{H}(x) \propto \langle a| \otimes \langle b| (-\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \sigma^z \otimes \sigma^z) |a\rangle \otimes |b\rangle,
\]
and taking into account (6.21), the Hamiltonian density becomes
\[
\mathcal{H} \propto p^2(x) + \frac{q^2(x)}{4} \Rightarrow H \propto \int dx \left( q^2(x) + \frac{p^2(x)}{4} \right),
\]
which is simply a classical harmonic oscillator type Hamiltonian, and coincides with the
Hamiltonian obtained directly from the classical continuum model (see Appendix A). A
similar Hamiltonian is obtained in the case where $U = \sigma^y$. The only difference is a relative
minus sign between the $p$ and $q$ terms. The considerations above may be generalized to the
$gl_n$ case. Choose for instance
\[
U = \sum_{i=1}^{n-1} e_{ii} - e_{nn} \quad \text{also set}
\]
\[
l_{ii} \to 1, \quad l_{nn} \to -1, \quad l_{in} \to -\hbar \psi_i, \quad l_{ni} \to \hbar \bar{\psi}_i, \quad l_{ij} \to 0, \quad \lambda \to \hbar^2 \lambda,
\]
\[
i, j \neq n.
\]
Notice that above we keep only first order terms, recall also that $l_{ij}$ are the generators of the classical $\mathfrak{gl}_n$ (see section 4). Once again we have implemented condition (6.13).

From the now standard construction

$$
\langle n | \tilde{L}_a(\lambda) | n \rangle = 1 + i \hbar \tilde{l}_a(x, \lambda) ,
$$

we conclude that the linear Lax operator in this case takes the form

$$
\tilde{l}(\lambda) = -i \sum_{i=1}^{n-1} \left( \psi_i e_{ni} + \bar{\psi}_i e_{in} \right) - \lambda \left( \sum_{i=1}^{n-1} e_{ii} - e_{nn} \right) ,
$$

which is just the generalized NLS Lax operator (see e.g [18] and references therein). It is clear from (3.16) that

$$
\{ \psi_i(x), \bar{\psi}_j(y) \} = \delta_{ij} \delta(x - y) .
$$

By choosing $U$ given in (6.29) we conclude that the relevant 2-site Hamiltonian is

$$
H_2 = \sum_{k,l \neq n} e_{kl} \otimes e_{lk} - \sum_{l \neq n} e_{nl} \otimes e_{ln} - \sum_{l \neq n} e_{ln} \otimes e_{nl} + e_{nn} \otimes e_{nn} ,
$$

acting non-trivially on the sites $j, j + 1$ of the spin chain. Computing now the Hamiltonian density in the usual procedure, while recalling the expansion defined in (6.29) one has

$$
\mathcal{H}(x) = \langle a | \otimes \langle b | H_2 | a \rangle \otimes | b \rangle = \ldots \propto \sum_{i=1}^{n-1} \psi_i(x) \bar{\psi}_i(x) .
$$

The later provides the total number of particles of the model (see also [18])

$$
N = \int dx \sum_i \psi_i(x) \bar{\psi}(x) .
$$

Similar transformations may be found in anisotropic models, however the whole analysis is quite subtle in this situation, and will be left for future investigations.

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A Classical local Hamiltonians

We compute the classical integrals of motion for the classical harmonic oscillator respectively, starting from the associated classical Lax operators. Consider the monodromy matrix

\[ T(x, y) = P \exp \left( \int_y^x dx' \ l(x', t, \lambda) \right), \]  

(A.1)
satisfying the first order differential equation

\[ \frac{dT(x, y)}{dx} = l(x) \ T(x, y). \]  

(A.2)

It may be expressed in the following form [12]

\[ T(x, y) = (1 + W(x)) \ e^{Z(x, y)} (1 + W(y))^{-1}, \]  

(A.3)

where \( W \) is anti-diagonal, \( Z \) diagonal, and both are expanded at \( \lambda \to \infty \)

\[ W(x) = \sum_{m=0}^{\infty} \frac{W^{(m)}(x)}{\lambda^m}, Z(x) = \sum_{m=-1}^{\infty} \frac{Z^{(m)}(x)}{\lambda^m}. \]  

(A.4)

Our purpose is to identify the various \( W^{(m)} \) and \( Z^{(m)} \) and hence the associated integrals of motion.

By substituting the monodromy matrix as in (A.3), and setting \( l(\lambda) = D + A \) (\( D, A \) being the diagonal and anti-diagonal part of the Lax operator) we obtain

\[ \frac{dW}{dx} + 2 \ D \ W + W \ A \ W - A = 0, \]

\[ \frac{dZ}{dx} = D + A \ W. \]  

(A.5)

Substituting expressions (A.4) in (A.5) we find

\[ W^{(0)} = \sigma^x, \]

\[ W^{(1)} = (+iq + \frac{p}{2})\sigma^y, \]  

(A.6)

\[ W^{(2)} = -\frac{1}{2}(iq + \frac{p}{2})^2\sigma^x, \ldots \]

and

\[ \frac{dZ^{(-1)}}{dx} = -2I, \]

\[ \frac{dZ^{(0)}}{dx} = 0, \]  

(A.7)

\[ \frac{dZ^{(1)}}{dx} = -(q^2 + \frac{p^2}{4})I \ldots, \]
where $\mathbb{I}$ is the $2 \times 2$ unit-matrix. The first non-trivial integral of motion is obtained essentially from the $\text{tr} \int dx Z^{(1)}$

$$I^{(1)} \propto \int dx (q^2(x) + \frac{p^2(x)}{4}) . \quad (A.8)$$

## B Higher Hamiltonians

We focus on the computation of higher charges in the NLS context starting from the corresponding quantum model examined in section 6. We shall show that the quantity emerging from the quantum higher charge is identical with the higher classical charge, that is the momentum. This gives an illustration of the statement in Section 2, eq. 2.36 that the second quantum local Hamiltonian derived from the quantum $R$ matrix formulation of the spin chain, also becomes in the continuous classical limit the second conserved quantity obtained from the monodromy matrix derived from the classical Lax matrix $l(x)$. Hence the construction is consistent.

Let us compute the quantum higher charge starting from the quantum NLS Hamiltonian

$$H = \sum_j h_{jj+1} , \quad h_{jj+1} = \sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1} - \sigma^z_j \sigma^z_{j+1} \sigma^z_j \sigma^y_{j+1} . \quad (B.1)$$

Define now the so called boost operator as (see e.g. [19] and references therein)

$$\mathcal{B} = \sum_j j h_{jj+1} . \quad (B.2)$$

All higher charges in involution may be obtained via the boost operator $\mathcal{B}$ as follows

$$H^{(n+1)} = [\mathcal{B}, H^{(n)}] . \quad (B.3)$$

So the next charge one obtains via (B.3) is of the form

$$H^{(2)} \propto \sum_j [h_{jj+1}, h_{j+1j+2}] . \quad (B.4)$$

The three site quantum higher Hamiltonian is then

$$h^{(2)} = [h \otimes \mathbb{I}, \mathbb{I} \otimes h] , \quad h = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y - \sigma^z \otimes \sigma^z . \quad (B.5)$$

It is now straightforward to show that

$$h^{(2)} = \sigma^x \otimes (\sigma^z \otimes \sigma^y + \sigma^y \otimes \sigma^z) - \sigma^y \otimes (\sigma^x \otimes \sigma^z + \sigma^z \otimes \sigma^x) - \sigma^z \otimes (\sigma^y \otimes \sigma^z - \sigma^x \otimes \sigma^y) . \quad (B.6)$$
Define now the Hamiltonian density as

\[ H^{(2)}(x) = \langle n | \otimes \langle n | \otimes \langle n | h^{(2)} | n \rangle \otimes | n \rangle \otimes | n \rangle. \]  

(B.7)

Recalling the identifications: \( \sin 2\theta \to 2\hbar \theta, \cos 2\theta \to 1 \), we have

\[ \langle n | \sigma^x | n \rangle \to 2 \hbar \theta \sin 2\varphi, \quad \langle n | \sigma^y | n \rangle \to 2 \hbar \theta \cos 2\varphi, \quad \langle n | \sigma^z | n \rangle \to 1 \]  

(B.8)

and taking into account the first non trivial contribution (\( \mathcal{O}(\delta) \)), after expanding the difference operators between neighbor sites \( \theta_{i+1} - \theta_i \to \theta(x+\delta) - \theta(x) \) (same for \( \varphi_i \)), we conclude that ( recall also \( i\theta e^{2i\varphi} = \psi, -i\theta e^{-2i\varphi} = \bar{\psi} \))

\[ H^{(2)} = 4\theta^2 \varphi' \propto \psi(x)\bar{\psi}'(x) - \psi'(x)\bar{\psi}(x). \]  

(B.9)

And indeed the second conserved quantity is the momentum of NLS i.e.

\[ H^{(2)} \propto \int dx \left( \psi(x)\bar{\psi}'(x) - \psi'(x)\bar{\psi}(x) \right). \]  

(B.10)

It is clear that similar computations can be done for higher charges and for other models, but these are beyond the intended scope of the present work. We simply focus here on a simple example the NLS case to further illustrate the consistency of our approach.

C Local spin chains: normalization factor

A delicate normalization issue arises in the considerations of section 2.4. Superficially the assumption of “regular limit” of the \( L \) matrix \( L_{ai}(\lambda_0) \equiv \mathcal{P}_{ai} \) clashes with the assumption of “semiclassical limit” \( L_{ai} = 1 \otimes 1 + \delta l_{ai} \). In fact one is considering two different normalizations of the same initial \( R \) matrix, yielding respectively the semiclassical \( L_{cl} \) matrix and the regular \( L_r \) matrix. They will differ in the simplest case by an overall c-number factor as \( L_{cl} = \delta (\lambda - \lambda_0)^{-1} L_r \).

The transfer matrix \( t \) yielding \( \frac{d}{d\lambda} \log t^{(\lambda)} \bigg|_{\lambda=\lambda_0} \), with \( t^{-1}(\lambda_0) = \mathcal{P}_{12}\mathcal{P}_{23} \cdots \mathcal{P}_{N-1N} \) is obtained from application of the co-module structure to \( L_r \). Whenever an overall normalization factor \( f(\lambda) \) is applied to \( L \) the “new” \( T \) matrix acquires an overall factor \( f(\lambda)^N \) and the Hamiltonian \( H^{(1)} \) is shifted by a trivial identity operator \( N \frac{d}{d\lambda} \log f(\lambda) \bigg|_{\lambda=\lambda_0} \). One can then regularize the whole construction by suitable substraction of this infinite identity operator whilst keeping quantum integrability.
The remaining problem in this case is the issue of having to consider the continuous limit of the product $\prod_1^N (1 \otimes 1 + \delta l_{1i})$ around the regular value $\lambda_0$ since it would appear from the definition of $L_{cl} = (\lambda - \lambda_0)^{-1} L_r$ that the generic term of the infinite product is then singular. But in fact one is always dealing with formal series expansion around this value, or equivalently integrals on an arbitrarily small contour around but not touching this value. It is understood that evaluations of derivatives of the ln of the transfer matrix at point $\lambda = \lambda_0$ generically mean extraction of the leading term in formal series expansion. The $\delta \to 0$ limit is thus always defined, and the continuous limit thus computed will generate, as shown in the various examples studied, the correct Hamiltonians, providing a definite check of consistency of the procedure.

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