Global cosmic string networks as a function of tension

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Abstract. We investigate the properties of global cosmic string networks as a function of the ratio of string tension to Goldstone-field coupling, and as a function of the Hubble damping strength. Our results show unambiguously that the string density is sensitive to this ratio. We also find that existing semi-analytical (one-scale) models must be missing some important aspect of the network dynamics. Our results point the way towards improving such models.

Keywords: Cosmic strings, domain walls, monopoles, axions

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1 Introduction

Cosmic strings are an interesting consequence of any beyond-Standard-Model theory in which a U(1) symmetry spontaneously breaks. They could play a role in modern cosmology — modern constraints show that they play a negligible role in cosmic structure formation [1–5], but they could contribute to the gravitational radiation background [6, 7]. Perhaps most interesting, axionic cosmic strings may play a pivotal role in establishing the density of axions in the Universe [8–18], and therefore the relation between the axion dark matter abundance and the axion mass [19]. And they determine the level of short-scale inhomogeneities in the axion dark-matter background, which could lead to small, very overdense axionic dark matter features (axion miniclusters or axion stars) [20–27].

Most research to date has focused on local cosmic strings, meaning strings where the spontaneously broken U(1) symmetry couples to a U(1) gauge field [28] or otherwise leads to no massless degrees of freedom, and therefore no long-range interactions between strings. Large-scale numerical simulations [29–34] generally treat this case, and analytical models designed to describe string networks [35–38] are generally fitted to these simulations. It appears that such analytical models now do quite a good job in describing the core features of these string networks.

But axions would arise from global cosmic string networks, meaning strings arising from a U(1) symmetry breaking where no gauge bosons receive a mass [39, 40]. Instead, these networks couple to massless Goldstone bosons, which can be radiated from the network and which can communicate long-range interactions between strings. Effective theories governing these networks are known [41], but existing large-scale numerical simulations [15, 17, 27, 42] face resolution issues, which limit their reliability. Specifically, the string evolution is a two-scale problem: there is an infrared scale, set by the light-crossing distance of the spacetime, which is of order the Hubble scale $H^{-1}$. The inter-string separation is of order this scale. And there is a microphysical scale, of order the inverse mass of the heavy fields which must exist in the model giving rise to the string. For instance, if the string arises from a microphysical scalar theory with a Mexican-hat potential, this is the inverse of the mass scale for radial excitations $m_r$. This scale sets the size of the string’s core, $r_{\text{core}} \sim 1/m_r$. The string’s gradient energy density scales with distance from the string core as $1/r^2$, and therefore the string tension is logarithmically dependent on this scale hierarchy: the tension $T \sim$
\[ \pi f^2 \int_0^{1/m_r} r \, dr/r^2 \simeq \pi f^2 \ln(m_r/H), \quad \text{with } f \text{ the symmetry-breaking scale and } \ln(m_r/H) \equiv \kappa \] the logarithm of this scale ratio. Note however that the strength of the string’s interactions with long-range Goldstone boson modes is proportional to \( f^2 \) but without this factor of \( \kappa \). Therefore \( f \) cancels out in establishing the network dynamics, but \( \kappa \) does not, and the network dynamics could show logarithmic sensitivity to this ratio of scales.

Some numerical results observe such a sensitivity [17, 27, 42–44], while others, including recent results, do not [45]. This discrepancy needs to be resolved. We also need simulations which achieve much larger scale hierarchies than straightforward scalar-field simulations can reach. For instance, for axions at the QCD epoch we have \( m_r \sim f_a \sim 10^{11} \text{ GeV} \) and \( H \sim T^2/m_{\text{pl}} \sim (0.3 \text{ GeV})^2/10^{19} \text{ GeV} \), and therefore \( \kappa \sim 70 \); whereas in numerical studies \( m_r \) must be smaller than the inverse lattice spacing \( 1/a_x \) and \( H \) must be larger than the inverse box size, constraining (roughly) \( \kappa \leq 7 \). Multigrid methods [46] might be able to reach spacing-to-separation ratios of order \( 10^6 \), giving \( \kappa \sim 14 \), but this is still severely insufficient.

We see a need to improve this situation with better simulations of global string networks. These would allow us to confront analytical models with data to see where they come up short. Recently, a similar effort aimed at improving simulations of domain wall networks showed significant shortcomings in the existing analytical models [47].

In this work we will address this problem with two sorts of simulations. First, we will perform simulations where the ratio of the Hubble scale to the string core size is held fixed. These are unphysical but they have the virtue that a true scaling solution exists, and lattice-spacing artifacts and initial-condition transients rapidly disappear. Using these simulations, we will demonstrate robustly, within scalar-only simulations, that the string network really does depend significantly on \( \kappa \).

Next, we will use our recently introduced numerical method for introducing a large string tension into global string networks, to see how network properties depend on tension at large tension values. These studies rapidly run into resolution problems if we stick with the radiation-dominated FRW metric. But by considering matter-dominated FRW metrics and metrics for universes with a slightly negative-pressure equation of state, we can explore networks with much stronger Hubble drag. This Hubble drag gives an additional variable for comparison with analytical models. It also destroys kinks and other small-scale structures on strings, which makes it easier to achieve scaling in the string network evolution. Furthermore, as we will discuss, existing one-scale models [35, 36] predict that the behavior of global and local networks will rapidly become more similar as the Hubble drag is increased — a prediction we will test.

The next section sets the stage by reviewing our expectations for global string networks, based on analytical “one-scale” models. Then section 3 will explore networks in the radiation era with a small but fixed \( \kappa \) value, as a function of \( \kappa \). We show robustly that the network density is sensitive to this measure of the scale hierarchy. Then we present our methodology and results for studying higher tension networks with varying Hubble drag, looking at the string velocity and the kinkiness of strings as well as the network density. We also present results for the rate of string loop production. Section 5 discusses what we have learned, and in particular, the implications of these results for existing analytical models. Our most notable findings are that global string networks have larger average string velocity than local networks, opposite to the predictions of the one-scale model [35, 36], and that the string density does not rapidly approach that of a local network as one increases the Hubble drag. We believe that some work is needed on the modeling side, probably by incorporating the effects of long-range inter-string interactions.
2 Analytical expectations

This section reviews the analytical one-scale model of string networks, originally introduced by Martins and Shellard [35] and later extended for global string networks [36].

First we quickly review the cosmological background spacetimes we will consider. We assume an FRW universe with equation of state \( P = w \varepsilon \). The physically most interesting cases are \( w = 0 \) (matter domination) and \( w = 1/3 \) (radiation domination), but we will allow all values \( w \in [-1/3, 1] \). Values \( w > 1 \) violate the dominant energy condition and do not appear to arise from any state of a renormalizable field theory. Values \( w < -1/3 \) do not allow for string network scaling solutions. But by considering values \( w \in [-1/3, 0] \), we will be able to examine string networks with particularly tractable dynamics, which challenge one-scale models.

The two Einstein equations describing the evolution of the energy density \( \varepsilon \) and scale factor \( a \) as a function of \( t \) for such a metric are

\[
\frac{1}{\varepsilon} \frac{d\varepsilon}{dt} = -3(1 + w)H, \quad H \equiv \frac{da}{adt} = \frac{\sqrt{\varepsilon}}{M_{\text{pl}}},
\]

with \( M_{\text{pl}} \) the reduced Planck mass. We also introduce the conformal time

\[
d\tau = \frac{1}{a} dt
\]

in terms of which the metric is \( g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu} \). After some straightforward algebra we find

\[
\frac{a}{a_0} = \left( \frac{t}{t_0} \right)^{\frac{2}{1+w}}, \quad H = \frac{2}{3(1+w)}t^{-1}, \quad \text{and} \quad \tau \propto a^{\frac{4+3w}{1+w}}.
\]

For future reference, we can work in comoving coordinates and conformal time, in which case the metric scales with conformal time as

\[
g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu} = \tau^n \eta_{\mu\nu}, \quad n \equiv \frac{4}{1+3w}.
\]

We will see that \( n \) parameterizes how strongly Hubble drag acts to slow down cosmic strings.\(^1\) By considering values of \( w \) from 1 to \(-1/3\) we can arrange for \( n \) to take values between 1 and \( \infty \), though values \( n > 4 \) require values \( w < 0 \) which are difficult to achieve with physically sensible cosmological fluids.

We now return to the issue of describing a string network. One-scale models [35, 36] postulate that a string network is described by the mean inter-string separation \( L \) and the mean string velocity \( v \). We should understand \( L \) in terms of the string length per unit volume, \( \ell_{\text{tot},\gamma}/V = 1/L^2 \); here \( \ell_{\text{tot},\gamma} \) is the total invariant length of string in a large volume \( V \), and invariant length means that we weight a string’s length with a factor of \( \gamma \equiv 1/\sqrt{1-v^2} \) so that string energy, rather than geometrical string length, is the relevant quantity; \( \ell_{\text{tot},\gamma} = \int \gamma d\sigma \) where the integral is over all string and \( d\sigma \) is the differential geometrical length of string. The velocity \( v \) should be understood as the RMS string velocity of the network, weighted by string energy and excluding small loops. Because strings are curved, string tension tends to accelerate strings and increase \( v \) provided that \( v^2 < 1/2 \); this effect gets stronger the smaller

\(^1\)For comparison with other literature: some authors [48] work in terms of \( m \equiv Ht \), which is related to \( n \) via \( m = n/(2+n) \) or \( n = 2m/(1-m) \).
$L$ is. Hubble drag slows down the network in a way which depends only on the value of $H$. Martins and Shellard argue for velocity evolution obeying

$$\frac{dv}{dt} = (1 - v^2) \left( \frac{k(v)}{L} - 2Hv \right),$$

where $k(v)$ is a velocity-dependent coefficient reflecting how much the string curvature accelerates the string motion. Analytic estimates give $k = 2\sqrt{2}/\pi$, but it is better to treat it as a velocity-dependent function to be fit to network evolution data. Recently Correia and Martins [48] have extended the model by considering $k(v)$ to be a more general function of velocity. Their fit to the abelian Higgs model at a range of $n$ values from $n = 2$ to $n = 38$ obtains

$$k(v) = k_0 \frac{1 - (qv^2)^{\beta}}{1 + (qv^2)^{\beta}}, \quad k_0 \simeq 1.37, \quad q \simeq 2.3, \quad \beta \simeq 1.5.$$  \hspace{1cm} (2.6)

We will use this form in what follows.

For a true scaling solution, we expect $dv/dt = 0$, in which case eq. (2.5) reduces to $L = k(v)/2Hv$. As $k(v)/v$ is a strictly decreasing function of $v$, this predicts that smaller velocities correspond with larger $L$ values and therefore lower-density networks. It is also useful to introduce the conformal inter-string separation $\xi \equiv L/a$; in terms of the scale factor, time $t$, and conformal time $\tau$, we have

$$a\tau = \frac{n + 2}{n + 2}, \quad Ht = \frac{n}{n + 2} \Rightarrow \frac{L}{a\tau} = \frac{\xi}{\tau} = \frac{1}{n} \frac{k(v)}{v}.$$ \hspace{1cm} (2.7)

This version is useful for identifying the network density in conformal time, which is typically used in lattice simulations. To convert to regular time, use $\xi/\tau = (L/t) \times 2/(n + 2)$.

The other network evolution equation tells how fast the amount of string changes. Note that $\ell_{\text{tot},\gamma} \propto 1/L^2$, and therefore $(1/\ell_{\text{tot},\gamma})d\ell_{\text{tot},\gamma}/dt = -(2/L)dL/dt$; so something which reduces the amount of string increases $L$. Therefore the rate at which $L$ changes is [35, 36]

$$2\frac{dL}{dt} = 2HL(1 + v^2) + F(v), \quad F(v) = c_{\text{loop}}v + d(k_0 - k(v))^r + \frac{s_{\text{global}}v^6}{\kappa}.$$ \hspace{1cm} (2.8)

Here $2HL(1 + v^2)$ is the loss of string density due to expansion, with 1 arising from dilution of the string network and $v^2$ from loss of string energy due to Hubble drag. The term $F(v)$ represents all mechanisms for string energy to convert into some other type of energy, such as the production of string loops $c_{\text{loop}}v$, radiation of heavy modes represented phenomenologically as $d(k_0 - k(v))^r$, and radiation of Goldstone modes $s_{\text{global}}v^6/\kappa$. We will again follow [48] and choose

$$c_{\text{loop}} = 0.34, \quad d = 0.22, \quad r = 1.8, \quad s_{\text{global}} = 70.$$ \hspace{1cm} (2.9)

The value $s_{\text{global}} = 70$ is from ref. [38], since this parameter was not considered in ref. [48]. The inverse dependence on $\kappa$ arises because energy loss through Goldstone boson radiation is proportional to $f^2$, but the string tension is proportional to $f^2\kappa$; therefore, the larger the $\kappa$ value is, the less importance Goldstone radiation has to the string network’s energy budget. The $v^6$ behavior is a generic prediction for emission of massless, relativistic waves from a nonrelativistic source [36]. It is also conceivable that some coefficients, particularly $c_{\text{loop}}$, are different for global than local networks; and $c_{\text{loop}}$ may also have velocity dependence which is not reflected above.
Table 1. Expected network velocity-squared $v^2$ and string density $\zeta$ for local networks and global networks at three values of the tension parameter $\kappa$. The values $n = 4, 8, 16$ correspond to equations of state with $w = 0, -\frac{1}{6}, -\frac{1}{4}$ respectively.

| $n$ | $v^2$ | $\kappa \to \infty$ | $\kappa = 32$ | $\kappa = 16$ | $\kappa = 6$ |
|-----|-------|-----------------|---------------|---------------|---------------|
| 4   | $0.238$ | $0.233$ | $0.229$ | $0.217$ |
| $w = 0$ | $\zeta$ | $11.4$ | $10.4$ | $9.7$ | $8.1$ |
| 8   | $0.159$ | $0.158$ | $0.156$ | $0.151$ |
| $w = -\frac{1}{6}$ | $\zeta$ | $13.4$ | $13.0$ | $12.7$ | $11.9$ |
| 16  | $0.092$ | $0.092$ | $0.092$ | $0.091$ |
| $w = -\frac{1}{4}$ | $\zeta$ | $18.7$ | $18.6$ | $18.5$ | $18.2$ |

Returning to eq. (2.8), the scaling expectation is $L \propto t$, and therefore $dL/dt = L/t$ would represent a scaling solution. Again using $Ht = n/(n+2)$, rearranging, and converting to conformal units, we find

$$(2 - nv^2)\frac{\xi}{\tau} = F(v)$$

or

$$\frac{\xi}{\tau} = \frac{F(v)}{2 - nv^2}.$$  (2.10)

We can determine the scaling solutions for $v$ and $\xi/\tau$ by plotting eq. (2.7) and eq. (2.10) in the $v, \xi/\tau$ plane and finding the intersection point. But we can already see that the presence of $s_{\text{global}}$ increases the value of $L/t$ and therefore decreases $v$. Therefore, two robust predictions of the one-scale model are that global networks have smaller values of $v^2$ and larger values of $L/t$ (lower network densities). And for $n > 4$ the prefactor on $\xi/\tau$ in eq. (2.10) vanishes for $v^2 = 2/n$, which is therefore an upper bound on the string mean-squared velocity. For values of $w \approx -1/3$ the network velocity becomes small, which makes the friction term $\propto v^6$ irrelevant. Therefore the one-scale model as formulated above predicts that global and local networks should behave almost identically in this regime.

To test this one-scale model, we should consider both different values of $\kappa$ and different values of $w$ or equivalently $n$, including $0 > w > -1/3$ or $n > 4$. Another advantage of considering $w < 0$ is that, in this regime, strong Hubble drag tends to round off kinks and damp cusps, making strings smoother and more easily resolved by lattice studies. In preparation for numerical results on these quantities, we show in table 1 what the one-scale model predictions are for local and global string-network scaling densities

$$\zeta \equiv \left(\frac{\xi}{\tau}\right)^{-2}$$  (2.11)

and mean-squared velocities $v^2$ at a few $\kappa$ values and for the three $n$ values which we will consider further in later sections. Note that we express the network density in comoving coordinates and conformal time, to make it easiest to compare to lattice results.

The table shows smaller network densities and slower strings for the global cases, especially for the smallest value of $\kappa$. However, as the Hubble expansion rate is increased such that the mean network velocity becomes small, radiation of Goldstone modes becomes inefficient and the results become almost indistinguishable. This appears to be a robust prediction of the current one-scale models. We will see how these predictions hold up in subsequent sections.
3 Networks with fixed, small scale hierarchy

In discussing scaling solutions so far, we have ignored the complication that $\kappa$ is not a constant. The scale hierarchy at play is $\kappa \equiv \ln(m_r/H)$ with $m_r$ a microscopic scale which is fixed in physical units, and $H$ the Hubble scale which grows with time. This leads to an evolution of the “scaling” solution (which is then not strictly a scaling but a tracking solution). However, in a numerical simulation one always starts away from the tracking solution. Therefore previous global string simulations all contain simultaneous evolution towards the tracking solution, and evolution of the tracking solution. Separating these two effects is nontrivial. Recently the authors of ref. [45] have claimed that previous studies are mistaken, and the whole evolution of network densities in numerical simulations can be ascribed to the approach to tracking solutions. Their results are consistent with no $\kappa$ dependence in the actual tracking or scaling network solution. In this section we will show how to perform numerical simulations of global strings, such that $\kappa$ actually remains a constant and a scaling solution unambiguously exists. We can consider networks at different $\kappa$ values, and we will show that they unambiguously have different scaling solutions.

Consider cosmic strings arising from a theory of a complex scalar field $\phi = \phi_0 + i\phi_1/\sqrt{2}$ with a symmetry-breaking potential. The Lagrangian is

$$-L = g^{\mu
u} \partial_\mu \phi^* \partial_\nu \phi + \frac{m_r^2}{8f^2} (f^2 - 2\phi^* \phi)^2. \tag{3.1}$$

Here $m_r$ is the mass of the radial field excitation (Higgs particle) and $f$ is the vacuum expectation value of the scalar field. In a radiation dominated FRW universe and using conformal time $\tau$, the metric is $g_{\mu\nu} = \tau^2 \eta_{\mu\nu}$, $g^{\mu\nu} = \tau^{-2} \eta^{\mu\nu}$, and $\sqrt{-g} = \tau^4$. Therefore, we can rewrite the Lagrangian as

$$-L = \tau^2 \left( \eta^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + \frac{m_r^2 \tau^2}{8f^2} (f^2 - 2\phi^* \phi)^2 \right). \tag{3.2}$$

Note that the potential term scales as $\tau^2$ relative to the gradient term. Therefore, the string core’s radius $r_{\text{core}} \sim 1/m_r$, when expressed in terms of comoving coordinates, scales as $1/(m_r \tau)$, while the mean string separation scales as $R \sim \tau$. Therefore the ratio of these scales behaves as $R/r_{\text{core}} \propto \tau^2$. This is a problem for simulations, because $r_{\text{core}}$ must be kept larger than the lattice spacing at all times, and the simulation must end before $R = L_{\text{box}}$ the box size.

Press Ryden and Spergel [49] (henceforward PRS) proposed an alternative numerical approach, in which the $\tau^2$ scaling is simply dropped from the potential term, such that the string core stays the same size in comoving coordinates and therefore in lattice units. While unphysical, this approach gives the maximum dynamic range over which the simulation describes an evolving network, and therefore provides a better chance for the network to approach its tracking behavior. Some but not all studies take this approach.

But in either of these approaches, the Lagrangian does not display conformal scaling, and so there is no rigorous argument that the dynamics will either. The ratio of string separation to core size changes with time, which we might expect to lead to an evolution in the scaled network density. But at the same time, the initial conditions never have a string

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$^2$We use the $[-+++]$ metric convention and natural units throughout.
Figure 1. The plot shows different ways the log of a comoving scale (vertical axis) can depend on the log of conformal time (horizontal axis). The Hubble scale is a straight line of slope 1. The lattice spacing and lattice length scales are horizontal lines. The physical case, where the string core is of fixed size in physical units, is a downward-sloping line, while the PRS or “fat string” case is a straight line. We propose to follow a straight line until a given $m_r/H$ ratio is achieved, and then to follow a slope-1 line, keeping $\ln(m_r/H)$ fixed — conformal scaling, allowing a true attractor solution. The logarithm of the scale hierarchy is given by $\kappa = \ln(\sqrt{\lambda})$. We can then investigate network dependence on the core-to-separation hierarchy by performing simulations with a range of $\lambda$ values.

Figure 1 provides a hopefully helpful cartoon of these three possibilities. In a log-log plot of the comoving length against the conformal time, the lattice spacing $a_x$ and the lattice extent or length $L_{box}$ are flat lines, while the Hubble scale $H^{-1}$ is a diagonal line of slope 1. The physical case has a string core size which falls (in comoving units) as $\tau^{-1}$. The network only really comes into existence once $m_r/H \gg 1$, so the dynamic range available (the interval of time during which the curve is below the Hubble curve but above the lattice-spacing curve) is compressed. The PRS approach keeps the core size fixed in lattice units, which provides the maximum possible dynamic range. Our proposal initially follows the PRS choice, but
then switches to a slope-1 line, that is, \( m_r \propto \tau^{-1} \), so the ratio \( m_r/H \) remains fixed, the theory becomes conformal, and a true scaling solution exists.

In carrying out a numerical study we make the following modifications.\(^3\) First, at early times eq. (3.3) calls for a mass which is large compared to the inverse lattice spacing. Instead, we start with \( m_r = 1.5/a_x \) (\( a_x \) the lattice spacing) and we switch over to \( m_r = \sqrt{\lambda}/\tau \) once this quantity falls below \( 1.5/a_x \). Second, we increase the Hubble damping strength by a factor of \( 8 \) (strongly damped early evolution) until \( \tau = 24a_x \) so that the initial string density will be higher, close to the scaling density. In treating the resulting data we will only make use of that part of the simulation which occurs well after these modifications have been switched off.

We performed simulations with \( m_r\tau = 20, 50, 100, 200, \) and \( 500; \) the first three on a \( 1024^3 \) lattice to \( \tau = 512a_x \) and the latter two on a \( 2048^3 \) lattice to \( \tau = 1024a_x \) (where \( a_x \) is the lattice spacing). The final time is chosen so that causality still ensures the results to be equivalent to the infinite-volume limit. For the smaller volumes we gather 32 independent evolutions; for the larger volumes we gather 8. In each simulation we evaluate the length of string by counting plaquettes pierced by a string and then multiplying by \( 2/3 \) to correct for the direction-average of the number of plaquettes pierced per unit length of string (Manhattan effect \([49, 50]\)). We also estimate the gamma-factor of the string, using the method of Fleury and Moore\(^4\)\(^\text{[17]}\). We then report the total string length, scaled by factors of the time and the lattice volume such that it should scale, both with and without a factor of \( \gamma \):

\[
\zeta_{\text{len}} \equiv \frac{\ell_{\text{tot,}2\tau}^2}{V} = \frac{2n_{\text{pierced-plaq}}(\tau/a_x)^2}{N^3}, \quad \zeta_{\text{inv}} \equiv \frac{\ell_{\text{tot,}2\tau}^2}{V} = \frac{2}{3}(\tau/a_x)^2 \sum_{\text{pierced-plaq}} \gamma. \tag{3.4}
\]

Note that we are reporting the network density in comoving coordinates and conformal time, as we do throughout this paper. The factor \( \gamma \) is conventionally included when finding the network density because it accounts for the energy content, rather than comoving-frame length, of the string.

Figure 2 shows the raw and averaged data for each \( m_r\tau \) value we consider. We see very clearly that each evolution rapidly approaches a plateau, and that the plateau value shows clear dependence on \( m_r\tau \), with larger \( m_r\tau \) (smaller core size) giving rise to a denser string network. The figure also displays our fit to the data. We use the time range during which \( m_r a_x < 0.78 \), and we fit including coefficients for \( \tau^{-1} \) and \( \tau^{-2} \) early-time transients, which we weakly constrain with priors that they are initially of order 20\% and 4\% of the total network density. The resulting best-fit network densities are summarized in table 2. The table also reports the mean-squared string velocity \( \int \gamma v^2 dl / \int \gamma dl \). Here we use the Fleury-Moore estimate for velocity and evaluate at the time when \( ma_x = 0.5 \), such that the lattice-spacing errors are comparable for each case. We indicate the statistical errors only; by considering twice smaller values of \( ma_x \) (twice-later times) we find the errors in \( \langle v^2 \rangle \) from lattice spacing and transient effects are \( \sim 0.005 \). The systematic errors from, e.g., breather modes, string curvature, and exactly what we mean by the velocity of a string with finite core thickness are difficult to estimate but are the most severe for the smallest \( m_r\tau \) values.

\(^3\)In other aspects our numerics are rather standard, e.g., cubic \( N \times N \times N \) box with periodic boundary conditions, Leapfrog update algorithm, \( \phi \) initialized to lie on the vacuum manifold with independent phase at each point in space. We approximate the \( \nabla^2 \) term in the equations of motion with an improved next-nearest neighbor approximation (13-point rather than 7-point stencil).

\(^4\)The method uses the value of \( |\partial_x \varphi|^2 \) at the points around each plaquette which the string pierces. For the known profile of a straight string, this can be used to determine \( v^2/(1-v^2) \) at this point on the string. The method receives errors when \( (m_r a_x) \) is not small enough. These errors shrink over the course of the simulation.
Figure 2. Raw data (left) and average over simulations (right) with 1σ statistical error bars, for the scaled string length at several $m_r\tau$ values. The top figures use the geometrical string length, the bottom figures include the string gamma-factor in computing the string length. Right-hand figures also show the best-fit lines.

How do we expect this data to be related to the cases of the PRS network or the “physical” network? Call the density of the conformal network with a given $\kappa = \ln(m_r/H)$ value $\zeta_c(\kappa)$. Intuitively, at a given moment when $\kappa(\tau) = \ln(m_r(\tau)/H(\tau))$ takes a given value, the string network density $\zeta(\tau)$ should evolve towards the attractor value $\zeta_c(\kappa(\tau))$; for small differences, we expect

$$
\tau \frac{d\zeta(\tau)}{d\tau} = C(\zeta_c(\kappa(\tau)) - \zeta(\tau))
$$

(3.5)

It is also systematically wrong on highly curved parts of string, parts with large “breather” mode fluctuations, or near string intersections. The importance of these issues is expected to diminish with increasing $m_r\tau$. 

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| $m_r t$ | $\kappa$ | $\zeta_{\text{len}}$ | $\zeta_{\text{inv}}$ | $\langle v^2 \rangle$ |
|--------|--------|-------------------|-------------------|----------------|
| 20     | 3.0    | 1.74(3)           | 2.64(3)           | 0.4919(3)     |
| 50     | 3.9    | 2.42(2)           | 3.49(3)           | 0.4731(5)     |
| 100    | 4.6    | 3.06(4)           | 4.34(5)           | 0.446(1)      |
| 200    | 5.3    | 3.65(10)          | 5.08(12)          | 0.423(3)      |
| 500    | 6.2    | 4.18(11)          | 5.65(14)          | 0.419(7)      |

Table 2. String density, with and without $\gamma$ factor, as a function of the (constant) $m_r t$ value during the simulation, after extrapolation to late time; and mean squared string velocity.

![Scaling analysis of Network density](image)

Figure 3. Network density $\zeta_{\text{len}}$ against $\kappa$ for conformal networks (green points), PRS networks (upper error bars, labeled as “fat”) and physical-string networks (lower error bars). The PRS and physical-string network densities lag the conformal-network value as expected. Blue and red fits are explained in the main text.

with $C$ some order-1 constant, telling how fast a network approaches the scaling behavior. But for PRS or “physical” networks, we have $d\kappa/d\ln(\tau) = 1$ (PRS) or 2 (“physical”). Therefore, $\zeta(\tau)$ is evolving towards a moving target. If $\zeta_c(\kappa)$ depends approximately linearly on $\kappa$, $d\zeta_c/d\kappa = \zeta'$, then the “tracking” solution to eq. (3.5) is $\zeta(\tau) = \zeta_c(\kappa(\tau)) - \zeta' \times (1 \text{ or } 2)$.

We can check this picture by simulating PRS networks and physical networks and plotting the resulting $\zeta(\kappa(\tau))$ on the same plot as $\zeta_c(\kappa)$. Our results are shown in figure 3, which shows the network density $\zeta_{\text{len}}$ as a function of $\kappa = \ln(m_r/H)$ for PRS and physical networks, carried out on $3072^3$ boxes up to time $\tau = 1536a_x$ with $m_r = 1/a_x$ in the PRS.
case and \( m_r = \tau / \tau_{\text{final}} \) in the physical case. In each case the field is initialized to a random independent value at each point and then evolved under radiation-era damping without an initial high-damping stage. The black error bars are the scaled string density \( \zeta_{\text{len}} \) averaged over a few independent runs, with 1-sigma statistical errors. The statistical errors are highly correlated for time differences less than about a factor of 2, or about 1 unit along the horizontal axis. We have made no attempt here to estimate systematic lattice-spacing effects, but we expect based on previous studies \([17]\) they are a few percent.

It is common in the literature to assume that \( \zeta_{\text{len}} \) should scale linearly with \( \kappa \). Also, initial conditions lead to an initial transient as the network approaches the tracking solution; this effect should decay with time. With this in mind, we include in the figure a fit to the assumed functional form

\[
\zeta_{\text{len}} = \frac{a}{\tau} + b + c \kappa, \tag{3.6}
\]

which is shown as the blue dotted line. Such a fit appears to work well for each full data series. Another assumption in the literature, due to Hindmarsh et al. \([45]\), is that \( \zeta_{\text{len}} \) should approach a constant, and that early time transients can be accommodated by assuming that the mean inter-string separation \( \xi \) evolves with time as

\[
\xi = A + B \tau, \quad \zeta_{\text{len}} = \frac{\tau^2}{\xi^2}. \tag{3.7}
\]

This is shown as the red dotted curve; it does not provide a good description of the data over the full range shown. The figure indeed illustrates that the PRS network is shifted down (or rightward) with respect to the conformal scaling solutions, and the “physical” network is shifted by a larger coefficient, which within errors and initial transients is consistent with a factor of 2.

In summary, our data shows robustly that the network density is sensitive to the core-to-separation ratio of the network. Both lattice spacing effects and initial conditions are thoroughly under control in this determination. The standard PRS and “physical” networks appear to evolve towards tracking solutions which trail behind the scaling solutions we find in the conformal network case.

It would be straightforward to extend this treatment to larger damping strengths \( n > 2 \). But instead we will consider a wider range of string tensions at high damping, in the next section.

4 Networks with large tension

Realistic networks have \( \kappa \geq 70 \) as noted before, so simulations of U(1) symmetric scalar field theory are quite far from the regime where they reproduce the physically relevant network dynamics. Therefore we need some method to study networks with much larger \( \kappa \) values.

In \([51]\) we present a way to do so, based on effective field theory ideas from \([41]\). Consider a network with an extremely large scale hierarchy, \( 1/m_r \ll t \) or \( 1/H \). Then we can ask what the network looks like, after “fuzzing” details smaller than an intermediate scale \( r_{\text{cut}} \ll 1/H \) but \( r_{\text{cut}} \gg 1/m_r \). On this scale strings are nearly straight and have negligible core thickness. The string then looks like an object with tension \( T = \pi f^2 \ln(r_{\text{cut}} m_r) \), interacting with long-range Goldstone modes via a Kalb-Ramond interaction \([41]\). If we find some other — any other — model which also behaves, at the scale \( r_{\text{cut}} \), like strings with tension \( T \) interacting with Goldstone modes via a Kalb-Ramond interaction, then it will describe the evolution
of the same combination of string network plus Goldstone fields. We refer the reader to our previous paper [51] for further details. Suffice it to say that we can introduce additional, heavy, degrees of freedom, at a scale \( m_e \). These contribute to the string core’s mass, but away from the string core they have a negligible effect because they are so heavy. Choosing \( r_{\text{cut}} \) to be large compared to the scale \( 1/m_e \), we will have strings interacting with Goldstone bosons through a Kalb-Ramond interaction, only now the tension associated with the short-distance physics is \( \ln(r_{\text{cut}} m_r) \rightarrow \ln(r_{\text{cut}} m_e) + \Delta \kappa \), where \( \Delta \kappa \) is an additional tension associated with the string core energy due to the added degrees of freedom. It turns out that the most easily achieved values for \( \Delta \kappa \) are of form \( \Delta \kappa = 2(i^2 + (i + 1)^2) \) with \( i \) an integer, for instance 10 \( (i = 1) \), 26 \( (i = 2) \), 50 \( (i = 3) \), and so forth.

It is still necessary to keep the introduced scale \( m_e \) light compared to the inverse lattice spacing, \( m_e a_x \leq 1 \), so that the string core with new degrees of freedom is properly resolved by the lattice. And we need \( 1/m_e \ll r_{\text{cut}} \ll \tau < L_{\text{box}} \) so that the core dynamics is well separated from the inter-string dynamics. Therefore only the later time range on our lattice has sufficient scale separation for the effective description to work. We will work on \( 2048^3 \) lattices and will use the time range \( \tau \in [512a_x, 1024a_x] \). Therefore the total value of \( \kappa \) is approximately \( \kappa = \ln(m_e \tau) + \Delta \kappa \simeq 6 + \Delta \kappa \). We will perform simulations of the pure complex scalar theory, the modified theory with \( \Delta \kappa = 10 \) and \( \Delta \kappa = 26 \), and the abelian Higgs model. In what follows we will refer to these cases as \( \kappa = 6 \) (simulations with complex scalars only), \( \kappa = 16 \) (simulations with two scalars and a gauge field, with charges 2, 1), \( \kappa = 32 \) (simulations with two scalars and a gauge field, with charges 3, 2), and \( \kappa \rightarrow \infty \) (abelian-Higgs model). We perform a few simulations for each combination, using the fluctuations between simulations — and between a larger number of simulations on \( 1024^3 \) lattices — to estimate statistical errors. Because we use an improved action, we choose the mass scale for all heavy excitations to be \( m_e a_x = 1 \) for each case. Previous results show that making the lattice spacing finer makes little difference in the results [17, 51].

In this section we will only consider \( n \geq 4 \), that is, strong Hubble damping corresponding to an equation of state with \( w \leq 0 \). In particular we study \( n = 4 \), \( n = 8 \), and \( n = 16 \). This parameter enters in the dynamics, in conformal coordinates, through an overall coefficient on the Lagrangian; eq. (3.3) is replaced with

\[
\mathcal{L} = \tau^n \left( \eta^{\mu\nu} D_\mu \varphi^* D_\nu \varphi + \frac{m^2}{8 f^2} (f^2 - 2 \varphi^* \varphi)^2 + \frac{1}{4 e^2} \eta^{\mu\nu} \eta^{\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \right),
\]

(4.1)

where for the pure scalar case \( D_\mu \rightarrow \partial_\mu \) and for the two-scalar case (to get intermediate string tensions) the scalar term should be duplicated with two fields with different charges as described in [51]. Note that we use the Press-Ryden-Spergel method for both the scalar potential and the gauge fields, such that both gauge and scalar masses remain fixed (and equal) in comoving coordinates. When deriving equations of motion, the factor \( \tau^n \) leads to

\[
\partial_\tau^2 \varphi \rightarrow \partial_\tau^2 \varphi + \frac{n}{\tau} \partial_\tau \varphi, \quad \partial_\tau E_i \rightarrow \partial_\tau E_i + \frac{n}{\tau} E_i,
\]

(4.2)

such that \( n \) controls the strength of dissipative forces in the field evolution and therefore in the network’s evolution.

We already explained the rationale for studying disparate \( n \) values, in terms of probing a broader range of network behaviors. But there is a second reason, which is that numerical simulations become better under control as \( n \) gets larger. This is because Hubble damping tends to smooth the strings and remove the locations with the largest velocities;
short-distance structures and fast-moving (highly Lorentz contracted) strings are two things which simulations handle badly, so the simulations become more faithful as we move in this direction. Simulations tend to approach the scaling network density from below, and we accelerate this approach by starting our simulations with a much larger Hubble damping rate, so that the network initially evolves slowly and remains dense; this high-damping regime is turned off around $\tau = 128a_x$, and the strength of early damping is tuned such that the network density at $\tau = 256a_x$ and $\tau = 512a_x$ are the same. We will see that the network is nevertheless not quite in the scaling limit; the network density increases somewhat between $\tau = 512a_x$ and the end of the simulation at $\tau = 1024a_x$, leaving some non-scaling corrections in our results. But the difference between different network types will be much larger than these non-scaling corrections, so we feel that the simulations still provide interesting results.

We start by examining whether the networks really have reduced small-scale structure as $n$ is increased. We do so by examining the string-tangent autocorrelation function as a function of separation along a string. That is, at each point on the string we can define the string unit-tangent direction $\hat{x}'(\sigma)$ with $\sigma$ the affine parameter indicating the position along the string. The autocorrelation function is

$$D(\Delta \sigma) \equiv \frac{1}{\ell_{\text{tot}}} \int_0^{\ell_{\text{tot}}} \hat{x}'(\sigma) \cdot \hat{x}'(\sigma + \Delta \sigma) \, d\sigma.$$  \hspace{1cm} (4.3)

This tells how quickly the string’s direction changes as one moves along the string. Details of how we determine the string’s location are provided in appendix A. The autocorrelator is shown in figure 4. As with all studies of string properties, the figure is based on an average over all string in a simulation, both long strings and short loops; however the long strings dominate the network, so removing loops before taking averages would lead to modest changes.

The most pertinent feature seen in the figure is that, as the damping gets stronger, the behavior of the autocorrelator near zero goes from being approximately linear to being approximately quadratic. A linear behavior reflects a string with kinks — points of abrupt change in the unit tangent — while a quadratic behavior indicates a string which is locally smooth with finite radius of curvature. We see that this difference applies equally well for local strings — the damping rounds off the kinks. Kinkless strings are expected to be well described in field-theoretical simulations. Therefore, while field theory simulations are probably not reliable at reasonable lattice size and spacing for radiation-era ($n = 2$) simulations, they may be reliable for $n \geq 8$. The figure also shows that small $\kappa$ corresponds to more rounded strings and longer-distance directional correlations along the string. The exception is that, for $\kappa = 6$, while the direction is highly correlated over short distances, it becomes less correlated at larger distances, falling below some other curves. We do not yet understand this effect.

Next we examine the string’s mean squared velocity. We determine this velocity in two ways. The first is from the rate of variation of the fields at the string’s core, as described in [17, 51]. The second is geometrical. We identify the set of points the string goes through by interpolating the crossing-point within each plaquette penetrated by the string, as explained in appendix A. The string is taken as the set of straight line segments connecting these plaquette-crossing points. Then we compare all points on the string network at time $t + \Delta t$ to the network at time $\tau$, finding the closest point on the network at time $\tau$ and taking the velocity to be distance over time. We rescale values $v > 0.95$ back to $v = 0.95$ to prevent
superluminal motion (which can occur close to cusps or string intersections). We have used \( \Delta \tau = 2a_x \), but we check that the answers are almost the same using \( \Delta \tau = 4a_x \), 6\( a_x \), and 8\( a_x \).

Our results are presented in figure 5. The figure shows that the mean squared velocity gets smaller as we increase the strength of Hubble damping, as expected. There is some discrepancy between the two estimates of the string velocity, but it is clear from both methods that the scalar-only simulation provides the largest string velocity, with higher-tension networks displaying a smaller mean velocity.

Next we turn to the network density. We define as before the invariant network density

\[
\zeta_{\text{conf}} \equiv \frac{\tau^2}{\xi^2} = \frac{\tau^2 \int \gamma dl_{\text{comoving}}}{V_{\text{comoving}}}, \tag{4.4}
\]

We have again included a factor of the local \( \gamma \)-factor in the relation between the string length and the network density. We average this quantity between simulations, once at time \( \tau = 512a_x \) and again at time \( \tau = 1024a_x \), to find any residual corrections from the approach to scaling. The results are shown in figure 6. We see immediately that the network density

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**Figure 4.** The string-tangent autocorrelation function for strings with four different tensions \( \kappa \), at three Hubble drag strengths \( n = 4 \) \((w = 0)\), \( n = 8 \) \((w = -1/6)\), and \( n = 16 \) \((w = -1/4)\).
Figure 5. Mean-squared string velocity, measured two ways: using the geometrical definition (measured distance between strings at different time steps); and using the field-derivative method [17]. The string velocity decreases as we increase the Hubble damping strength, and it is higher for low-tension scalar-only networks than for networks with larger string tension.

increases as we increase the damping strength. We also find that the network density is systematically larger as we consider higher-tension networks — or perhaps more accurately, systematically smaller as we consider networks with stronger interactions with the Goldstone modes. This persists at large Hubble damping, in strong contrast to the expectations from one-scale models. Considering abelian-Higgs networks, we find a somewhat larger network density than [48], probably because our boxes are a factor of 4 larger than theirs, so finite core-to-separation ratio effects are less under control in their simulations. This might imply that our results would change further if we had access to larger boxes; indeed we see a difference between $\tau = 512a_x$ and $\tau = 1024a_x$ which indicates that this would be the case.

Finally, we attempt a direct determination of the loop chopping rate. This first involves writing code to identify points along the string and to connect them together into loops (in a periodic box all string comes in loops). We then identify short loops, defined as loops whose total (invariant) length is smaller than $\xi = \sqrt{V/\ell_{\text{tot}}}$. We then compare to the strings output at the previous timestep, finding the string which is closest to the loop in question. If the string in the previous time step is not a loop of nearly the same size, but is instead part of a loop which stretches well beyond the loop in question, then a loop-creation event has occurred, and we count the length of the loop to be an amount of cut-off loop at this time.
Figure 6. Network density for global and local string networks with various tensions at three Hubble-damping strengths. In each case the lower/upper data point is the network density at $\tau = 512a_x/\tau = 1024a_x$, with the lower/upper extending error bar the statistical error for that point.

Note that we do not consider it “loop creation” when an existing loop of length slightly larger than $L$ shrinks, and $L$ increases, such that the loop now qualifies as small. We only consider it loop chopping when a loop at time $\tau + \Delta \tau$ corresponds to part of a larger loop at time $\tau$. Also note that a small loop occasionally reconnects onto the network. We do not count this as negative loop production; since it is rare and has almost no effect. Finally, we average the amount of loop production per unit volume over the last half of each evolution and average over evolutions to estimate the statistical error bars.

The results of this study are shown in figure 7. We find that loop production is more common when the Hubble damping is weaker than when it is stronger. We also find that it is much more common for global networks, especially those with low tension, than for local networks. Indeed, in our matter-dominated simulations, the chopping efficiency $c_{\text{loop}} \sim 0.6$ for scalar-only simulations, but it is $\sim 0.013$ for abelian Higgs networks.

All of our results are summarized in table 3.

5 Discussion: lessons for global string models

Interactions with massless Goldstone modes play a role in the evolution of global string networks, but are absent in local network evolution, where only massive modes exist off the
network. We can learn more about the detailed physical impact of the interactions with Goldstone bosons by examining network evolution at a range of network tensions and at a range of expansion rates (which in conformal coordinates means, a range of Hubble damping parameters). We have done so in this paper.

First, we showed that, within radiation-era scalar-only simulations, there is robustly a difference between network evolution when the “core-to-separation ratio” is small and when it is large. In our language, this is comparing network evolution for radiation domination, $n = 8$, with that when the core-to-separation ratio is order 400, has about 1/3 the network density of a scalar-only network. And the mean-squared velocity of the global network is larger. This is despite the fact that the mean network velocity is very small, small enough that radiation of Goldstone modes is predicted to play almost no role in the network dynamics.

We believe that this difference arises because inter-string forces can play a role in accelerating the strings. This effect does not disappear for slowly-moving strings. Acceleration of a string due to Goldstone-mediated inter-string forces would add another term in eq. (2.5), effectively modifying $k(v) → k(v) + g/κ$ with $g$ a (possibly velocity-dependent) coefficient representing the effects of inter-string forces and $1/κ$ representing that these forces become less important as the string core’s tension becomes larger.

To examine this a little more, we can adopt the philosophy of Correia and Martins [48] and use our results for $ζ$ and $v$, together with eq. (2.7) (rewritten as $k(v) = n v ξ/τ$) and

### Table 3. Measured properties of different network types. The $ζ_{\text{conf}}$ value is obtained by using eq. (4.4), $⟨v^2⟩_\text{fb}$ is the field-based velocity squared, explained in [17], and $⟨v^2⟩_g$ the geometrical velocity squared explained in appendix A. The last parameter is the measured loop chopping coefficient $c_{\text{loop}}$. 

| Network | $ζ_{\text{conf}}$ | $⟨v^2⟩_\text{fb}$ | $⟨v^2⟩_g$ | $c_{\text{loop}}$ |
|---------|------------------|-----------------|-------------|-----------------|
| $κ = 6$ | $n = 4$ | $5.16(33) → 6.03(81)$ | $0.2896(19)$ | $0.2715(16)$ | $0.658(39)$ |
| $n = 8$ | $6.33(17) → 6.47(96)$ | $0.2074(12)$ | $0.2027(12)$ | $0.231(24)$ |
| $n = 16$ | $8.760(224) → 10.01(89)$ | $0.1181(15)$ | $0.1103(6)$ | $0.0395(77)$ |
| $κ = 16$ | $n = 4$ | $12.36(49) → 12.61(105)$ | $0.2212(10)$ | $0.2280(10)$ | $0.0622(75)$ |
| $n = 8$ | $13.97(31) → 15.15(88)$ | $0.1506(9)$ | $0.1594(12)$ | $0.0251(41)$ |
| $n = 16$ | $18.45(29) → 20.53(121)$ | $0.0866(4)$ | $0.0959(12)$ | $0.0212(62)$ |
| $κ = 32$ | $n = 4$ | $16.23(38) → 17.09(137)$ | $0.2200(5)$ | $0.2308(10)$ | $0.0310(43)$ |
| $n = 8$ | $17.45(32) → 18.70(78)$ | $0.1524(6)$ | $0.1651(8)$ | $0.0128(39)$ |
| $n = 16$ | $22.68(29) → 25.22(129)$ | $0.0866(5)$ | $0.0987(9)$ | $0.0206(69)$ |
| $κ = ∞$ | $n = 4$ | $19.25(43) → 20.78(85)$ | $0.1982(7)$ | $0.237(1)$ | $0.0129(30)$ |
| $n = 8$ | $19.47(43) → 20.61(122)$ | $0.1409(3)$ | $0.1734(7)$ | $0.0105(34)$ |
| $n = 16$ | $26.10(56) → 29.10(137)$ | $0.0795(3)$ | $0.1020(8)$ | $0.0113(29)$ |
eq. (2.10), to determine the “acceleration” function $k(v)$ and the energy-loss or drag function $F(v)$. Adopting our geometrically determined network velocities $\langle v^2 \rangle_g$ from table 3, we find the results displayed in figure 8. Note that the results for $F(v)$ the drag are rather sensitive to the network velocity. The difference between our geometrical and field-based velocity estimator shifts around the points in detail within the right plot, and should be viewed as a systematic uncertainty. Therefore we have no convincing information on whether $F(v)$, the efficiency with which the network radiates energy, increases or decreases as we pass from local to global networks. But on the contrary, there is a large and robust difference in the acceleration function $k(v)$. Because $nv \xi / \tau$ has only a weak dependence on the velocity $v$, the acceleration-function $k(v)$ is rather robustly predicted despite the differences in our string velocity estimate. And it is dramatically different for the scalar-only simulations than for the simulations with enhanced string tension. A global network, especially one with a modest separation-to-core ratio (a small $\kappa$), shows markedly more acceleration than a global network. This almost certainly arises because the Goldstone modes induce inter-string forces, and it appears to be the dominant feature leading to differences in the network evolution, at least for the rather strongly Hubble-damped networks we consider here. This effect should be taken into account in future models which seek to describe global axion networks.
Figure 8. Observed values of the network-acceleration function $k(v)$ and the network energy-loss function $F(v)$, as determined using eq. (2.7), eq. (2.10), and our data. We use the geometrically determined network velocity. In each figure, the four points at lowest/mid/highest velocity are $n = 16/8/4$.

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A String-finding details

Here we present a few details of how we identify the location of the string. Our working definition of the location of the string in pure scalar field theory is the line of points where $\varphi$ vanishes. The same definition works in the abelian Higgs theory. In the two-Higgs theory which we use to describe global strings with enhanced tension [51], we can identify the string as the points where the higher-charge field $\varphi_1$ vanishes.

This location has to be interpolated from the field values on the actual lattice points. We do this in three steps. First, we identify the plaquettes which the string penetrates. Second, we interpolate the $\varphi = 0$ point inside each plaquette. Third, we connect together these plaquette-penetration points with straight-line segments. The remainder of this appendix will give more detail on each of these steps.

First consider the pure scalar theory. Label the four corners of a plaquette $x_1 = (n_x, n_y), x_2 = (n_x + 1, n_y), x_3 = (n_x + 1, n_y + 1), \text{ and } x_4 = (n_x, n_y + 1)$, with scalar values $\varphi(x_1)\ldots\varphi(x_4)$. We define the angle between two points as

$$\theta_{i,i-1} = \text{Arg} \varphi(x_i)\varphi^*(x_{i-1})$$

(A.1)
Figure 9. Illustration of where the four points $\varphi(x_1), \ldots, \varphi(x_4)$ reside in the complex-$\varphi$ plane. The fact that the quadrangle formed by the four $\varphi$ points encloses the origin indicates that the string penetrates the plaquette. The quadrangle is not in general square, because the string may be moving and may not penetrate the plaquette at its normal. The four areas $A_1, \ldots, A_4$ can be used to interpolate the location of the string-crossing within the plaquette; the smaller $A_2$ is relative to $A_4$, the closer the string lies to the line (in real-space) running between $x_2$ and $x_3$ and the farther it is from the line (in real-space) between $x_4$ and $x_1$.

where Arg means, as usual, the phase in the complex plane, taken between $-\pi$ and $\pi$. These phases are summed as one goes around the plaquette, $\theta_{\text{tot}} = \theta_{21} + \theta_{32} + \theta_{43} + \theta_{14}$, and there is a string of positive/negative sense if this total is $\pm 2\pi$, rather than zero. Reference [17] presents an equivalent but more numerically efficient way to implement this condition.

For the abelian Higgs theory, we need to make a slight modification. To make the answer gauge invariant, we define

$$\theta_{i,i-1} \equiv \text{Arg} \left( e^{-iA} \varphi(x_i) \varphi^*(x_{i-1}) \right),$$

with $A$ the gauge field value on the link between the points $x_i, x_{i-1}$, oriented such that $\varphi(x_i)$ and $e^{iA} \varphi(x_{i-1})$ have the same gauge transformation properties. The sum of the $\theta_{i,i-1}$ around the circle now equals $-\sum A$, which is minus the magnetic field which penetrates the plaquette, plus or minus $2\pi$ in the case that the string penetrates the plaquette. This algorithm originates with Kajantie et al. [52].

Having identified the plaquettes which contain a string, we must next interpolate $\varphi(x)$ into the interior of the plaquette to find the location where the $\varphi$ field is zero. We start with the pure scalar field. Consider the illustration in figure 9, which shows where the points $\varphi(x_1), \ldots, \varphi(x_4)$ might appear as points in the complex plane. It is elementary to evaluate the four indicated areas as, e.g., $A_1 = \text{Im} \varphi^*(x_1) \varphi(x_2)/2$. We then interpolate the $x$ value of the string as $x_{\text{str}} = n_x + A_4/(A_2 + A_4)$ and $y_{\text{str}} = n_y + A_1/(A_1 + A_3)$. That is, $x_{\text{str}}$ gets closer to

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\footnote{We use the noncompact formulation of U(1) gauge theory.}
as the area of the triangle with corners at 0, \( \varphi(n_x, n_y) \), \( \varphi(n_x, n_y + 1) \) gets smaller; and it gets closer to \( n_x + 1 \) as the area of the triangle with corners at 0, \( \varphi(n_x + 1, n_y + 1) \), \( \varphi(n_x + 1, n_y) \) gets smaller; and similarly in the \( y \) direction. For the case of the abelian Higgs theory or the two-scalar theory, we use the same procedure, working in the gauge where each link \( A \)-field equals 1/4 of the magnetic flux, that is, the gauge which minimizes the sum of squares of \( A \)-fields around the plaquette.

It occasionally occurs that one of the areas \( A_i \) has the opposite sign of the other three, leading to an estimated interpolated zero-point which is outside of the plaquette. In this case we place the zero-point 0.001 lattice units in from the edge of the plaquette.

The next step is to attach these plaquette-penetrating points together with straight line segments. We do this as follows. Every point where a string goes through a plaquette carries the string from one lattice cell (a \( 1 \times 1 \times 1 \) box with lattice points as corners and plaquettes as faces) into another; the orientation is determined by whether the angles sum to 2\( \pi \) or \(-2\pi\) (or whether the areas are positive or negative). Each cell has a total oriented number of strings entering equal to zero. If one string enters and one exits, we connect the entry/exit points with a straight segment and continue to follow the string from the exit point. If two strings enter a cell and two exit, we pick the pairing which leads to the shortest total length of string within the box.

One check of our procedure is to find the total length of string, based on the lengths of the straight segments found by our algorithm, and to compare it to the estimate of 2/3 of the sum of penetrated plaquettes.\(^6\) We find good agreement, whereas if we always interpolate the string to go through the center of the plaquette we underestimate the string length by \(<5\%\).

However, the interpolation of the location within a plaquette where the string penetrates is not perfect, and this leads to random fluctuations in the direction of the straight segments. This “renormalizes” the dot product of the string tangent vectors by a few percent. In making figure 4, we have applied a multiplicative rescaling of the direction autocorrelation function such that it approaches 1 at short distances.

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\(^6\)The factor 2/3 results by assuming that the string is locally straight and averaging over the directions which the string can point; a string pointing in the \( \vec{n} \) direction penetrates \( |n_x| + |n_y| + |n_z| \) plaquettes per unit length, which averages over directions to 3/2.
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