Uncertainty principles for the windowed Opdam–Cherednik transform

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In this paper, we study a few versions of the uncertainty principle for the windowed Opdam–Cherednik transform. In particular, we establish the uncertainty principle for orthonormal sequences, Donoho–Stark's uncertainty principle, Benedicks-type uncertainty principle, Heisenberg-type uncertainty principle, and local uncertainty inequality for this transform. We also obtain the Heisenberg-type uncertainty inequality using the $k$-entropy of the windowed Opdam–Cherednik transform.

KEYWORDS
Benedicks-type uncertainty principle, Donoho–Stark's uncertainty principle, Heisenberg-type uncertainty inequality, Opdam–Cherednik transform, windowed Opdam–Cherednik transform

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1 | INTRODUCTION

Uncertainty principles play a fundamental role in the field of mathematics, physics, in addition to some engineering areas such as signal processing, image processing, quantum theory, optics, and many other well-known areas [1–5]. In quantum physics, it says that a particle's speed and position cannot both be measured with infinite precision. The classical Heisenberg uncertainty principle was established in the Schrödinger space (the square-integrable function space). It states that a non-zero function and its Fourier transform cannot be both sharply localized, that is, it is impossible for a non-zero function and its Fourier transform to be simultaneously small. This phenomenon has been under intensive study for almost a century now and extensively investigated in different settings. One can formulate different forms of the uncertainty principle depending on various ways of measuring the localization of a function. Uncertainty principles can be divided into two categories: quantitative and qualitative uncertainty principles. Quantitative uncertainty principles are some special types of inequalities that tell us how a function and its Fourier transform relate. For example, Benedicks [6], Donoho and Stark [3], and Slepian and Pollak [7] gave quantitative uncertainty principles for the Fourier transforms. On the other hand, qualitative uncertainty principles imply the vanishing of a function under some strong conditions on the function. In particular, Hardy [8], Morgan [9], Cowling and Price [10], and Beurling [11] theorems are the examples of qualitative uncertainty principles. For a more detailed study on the history of the uncertainty principle, and for many other generalizations and variations of the uncertainty principle, we refer to the book of Havin and Jöricke [12], and the excellent survey of Folland and Sitaram [13].

Considerable attention has been devoted to discovering generalizations to new contexts for quantitative uncertainty principles. For example, quantitative uncertainty principles were studied in Mejjaoli [14] for the generalized Fourier transform. Over the years, discovering new mathematical formulations of the uncertainty principle for the windowed Fourier transform have drawn significant attention among many researchers; see, for example, previous works.
The motivation to prove these types of quantitative uncertainty principles for the windowed Opdam–Cherednik transform in the framework of the Opdam–Cherednik transform arises from the classical uncertainty principles for the windowed Fourier transform and the remarkable contribution of the Opdam–Cherednik transform in harmonic analysis (see previous works [30–33]). Another important motivation to study the Jacobi–Cherednik operators arises from their relevance in the algebraic description of exactly solvable quantum many-body systems of Calogero–Moser–Sutherland type (see the literature [34, 35]) and they provide a useful tool in the study of special functions with root systems (see previous works [36, 37]). These describe algebraically integrable systems in one dimension and have gained considerable interest in mathematical physics. Other motivation for the investigation of the Jacobi–Cherednik operator and the windowed Opdam–Cherednik transform is to generalize the previous subjects which are bound with the physics. For a more detailed discussion, one can see Mejjaoli and Trimèche [38].

Uncertainty inequalities are some special class of uncertainty principles that give us information about how a function and its Fourier transform relate and have got considerable importance in signal analysis, physics, optics, and many other well-known areas [1, 3–5, 39]. A well-known example of uncertainty inequality is the Heisenberg inequality. Considerable attention has been devoted to discovering generalizations to new contexts for uncertainty inequalities for various generalized transforms by several researchers. For instance, these uncertainty inequalities were obtained in Johansen [24] for the Heckman–Opdam transform, and in Mondal and Poria [40] for the Opdam–Cherednik transform. Some recent work on the windowed Opdam–Cherednik transform [29, 41] motivates us to study a few quantitative uncertainty principles for this transform. The main aim of this paper is to study a few uncertainty principles related to the windowed Opdam–Cherednik transform. More precisely, we prove the uncertainty principle for orthonormal sequences, Donoho–Stark’s uncertainty principle, Benedicks-type uncertainty principle, Heisenberg-type uncertainty principle and local uncertainty inequality for the windowed Opdam–Cherednik transform. We study the version of Donoho–Stark’s uncertainty principle and show that the windowed Opdam–Cherednik transform cannot be concentrated in any small set. Also, we obtain an estimate for the size of the essential support of this transform under the condition that the windowed Opdam–Cherednik transform of a non-zero function is time-frequency concentrated on a measurable set. Then, we investigate the Benedicks-type uncertainty principle and show that the windowed Opdam–Cherednik transform cannot be concentrated inside a set of measures arbitrarily small. Further, we study the Heisenberg-type uncertainty inequality for a general magnitude and provide the result related to the $L^2(\mathbb{R}, A_{\alpha, \beta})$-mass of the windowed Opdam–Cherednik transform outside sets of finite measure. Finally, we obtain the Heisenberg-type uncertainty inequality using the $k$-entropy and study the localization of the $k$-entropy of the windowed Opdam–Cherednik transform.

The presentation of this manuscript is divided into four sections apart from the introduction. In Section 2, we present some preliminaries related to the Opdam–Cherednik transform. In Section 3, we recall some essential properties related to the windowed Opdam–Cherednik transform. Some different types of uncertainty principles associated with the windowed Opdam–Cherednik transform are provided in Section 4. In particular, we prove the uncertainty principle for orthonormal sequences, Donoho–Stark’s uncertainty principle, Benedicks-type uncertainty principle, Heisenberg-type uncertainty principle and local uncertainty inequality for the windowed Opdam–Cherednik transform. We conclude the paper with the Heisenberg-type uncertainty inequality using the $k$-entropy of this transform.
2 | HARMONIC ANALYSIS AND THE OPDAM–CHEREDNIK TRANSFORM

In this section, we recall some necessary definitions and results from the harmonic analysis related to the Opdam–Cherednik transform, which will be used frequently. A complete account of harmonic analysis related to this transform can be found in the literature [27, 29–33, 42, 43]. However, we will use the notations given in Poria [27].

Let \( T_{\alpha,\beta} \) denote the Jacobi–Cherednik differential–difference operator (also called the Dunkl–Cherednik operator)

\[
T_{\alpha,\beta} f(x) = \frac{d}{dx} f(x) + [(2\alpha + 1) \coth x + (2\beta + 1) \tanh x] \frac{f(x) - f(-x)}{2} - \rho f(-x),
\]

where \( \alpha, \beta \) are two parameters satisfying \( \alpha \geq \beta \geq -\frac{1}{2} \) and \( \alpha > -\frac{1}{2} \), and \( \rho = \alpha + \beta + 1 \). Let \( \lambda \in \mathbb{C} \). The Opdam hypergeometric functions \( G_{\alpha,\beta}^\rho \) on \( \mathbb{R} \) are eigenfunctions \( T_{\alpha,\beta} G_{\alpha,\beta}^\rho(x) = i\lambda G_{\alpha,\beta}^\rho(x) \) of \( T_{\alpha,\beta} \) that are normalized such that \( G_{\alpha,\beta}^\rho(0) = 1 \). The eigenfunction \( G_{\alpha,\beta}^\rho \) is given by

\[
G_{\alpha,\beta}^\rho(x) = \phi_{\alpha,\beta}^\rho(x) - \frac{1}{\rho - i\lambda} \frac{d}{dx} \phi_{\alpha,\beta}^\rho(x) = \phi_{\alpha,\beta}^\rho(x) + \frac{\rho + i\lambda}{4(\alpha + 1)} \sinh 2x \phi_{\alpha,\beta}^{\alpha+1,\beta+1}(x),
\]

where \( \phi_{\alpha,\beta}^\rho(x) = {}_2F_1 \left( \frac{\rho + i\lambda}{2}, \frac{\rho - i\lambda}{2}; \alpha + 1; -\sinh^2 x \right) \) is the classical Jacobi function.

For every \( \lambda \in \mathbb{C} \) and \( x \in \mathbb{R} \), the eigenfunction \( G_{\alpha,\beta}^\rho \) satisfy

\[
\left| G_{\alpha,\beta}^\rho(x) \right| \leq C e^{-\rho |x|} e^{\text{Im}(\lambda)|x|},
\]

where \( C \) is a positive constant. Since \( \rho > 0 \), we have

\[
\left| G_{\alpha,\beta}^\rho(x) \right| \leq C e^{\text{Im}(\lambda)|x|}. \tag{2.1}
\]

Let us denote by \( C_c(\mathbb{R}) \) the space of continuous functions on \( \mathbb{R} \) with compact support. The Opdam–Cherednik transform is the Fourier transform in the trigonometric Dunkl setting, and it is defined as follows.

**Definition 2.1.** Let \( \alpha \geq \beta \geq -\frac{1}{2} \) with \( \alpha > -\frac{1}{2} \). The Opdam–Cherednik transform \( \mathcal{H}_{\alpha,\beta}(f) \) of a function \( f \in C_c(\mathbb{R}) \) is defined by

\[
\mathcal{H}_{\alpha,\beta}(f)(\lambda) = \int_{\mathbb{R}} f(x) G_{\alpha,\beta}^\rho(-x) A_{\alpha,\beta}(x) dx \quad \text{for all} \ \lambda \in \mathbb{C},
\]

where \( A_{\alpha,\beta}(x) = (\sinh |x|)^{2\alpha+1} (\cosh |x|)^{2\beta+1} \). The inverse Opdam–Cherednik transform for a suitable function \( g \) on \( \mathbb{R} \) is given by

\[
\mathcal{H}^{-1}_{\alpha,\beta}(g)(x) = \int_{\mathbb{R}} g(\lambda) G_{\alpha,\beta}^\rho(x) d\sigma_{\alpha,\beta}(\lambda) \quad \text{for all} \ x \in \mathbb{R},
\]

where

\[
d\sigma_{\alpha,\beta}(\lambda) = \left( 1 - \frac{\rho}{i\lambda} \right) \frac{d\lambda}{8\pi |C_{\alpha,\beta}(\lambda)|^2}
\]

and

\[
C_{\alpha,\beta}(\lambda) = \frac{2^{\rho-i\lambda} \Gamma(\alpha+1) \Gamma(i\lambda)}{\Gamma(\frac{\rho+i\lambda}{2}) \Gamma(\frac{\alpha-\beta+1+i\lambda}{2})}, \ \lambda \in \mathbb{C}\setminus\mathbb{N}.
\]

The Plancherel formula is given by

\[
\int_{\mathbb{R}} |f(x)|^2 A_{\alpha,\beta}(x) dx = \int_{\mathbb{R}} \mathcal{H}_{\alpha,\beta}(f)(\lambda) \overline{\mathcal{H}_{\alpha,\beta}(f)(-\lambda)} d\sigma_{\alpha,\beta}(\lambda), \tag{2.2}
\]

where \( \tilde{f}(x) := f(-x) \).
Let $L^p(\mathbb{R}, A_{\alpha, \beta})$ (resp. $L^p(\mathbb{R}, \sigma_{\alpha, \beta})$), $p \in [1, \infty]$, denote the $L^p$-spaces corresponding to the measure $A_{\alpha, \beta}(x)dx$ (resp. $d|\sigma_{\alpha, \beta}(x)$).

The generalized translation operator associated with the Opdam–Cherednik transform is defined by [44]

$$t_x^{(\alpha, \beta)} f(y) = \int_{\mathbb{R}} f(z) d\mu_x^{(\alpha, \beta)}(z),$$

(2.3)

where $d\mu_x^{(\alpha, \beta)}$ is given by

$$d\mu_x^{(\alpha, \beta)}(z) = \begin{cases} \mathcal{K}_{\alpha, \beta}(x, y, z) A_{\alpha, \beta}(z) dz & \text{if } xy \neq 0 \\ d\delta_x(z) & \text{if } y = 0 \\ d\delta_y(z) & \text{if } x = 0 \end{cases}$$

(2.4)

and

$$\mathcal{K}_{\alpha, \beta}(x, y, z) = M_{\alpha, \beta} \sinh x \cdot \sinh y \cdot \sinh z|^{-2\alpha} \int_0^\infty g(x, y, z, \chi)^{\alpha-\beta-1}$$

$$\times \left[1 - \sigma_{x,y,z}^x + \sigma_{x,y,z}^y + \sigma_{x,y,z}^z + \frac{\rho}{\beta + \frac{1}{2}} \coth x \cdot \coth y \cdot \coth z \sinh^2(\sin \chi)^2 \right] \times (\sin \chi)^{2\beta} d\chi$$

if $x, y, z \in \mathbb{R}\setminus\{0\}$ satisfy the triangular inequality $||x| - |y|| < |z| < |x| + |y|$, and $\mathcal{K}_{\alpha, \beta}(x, y, z) = 0$ otherwise. Here

$$\sigma_{x,y,z}^x = \begin{cases} \frac{\cosh x \cosh y - \cosh z \cos \chi}{\sinh x \sinh y} & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$$

for $x, y, z \in \mathbb{R}$, $\chi \in [0, \pi]$.

$$g(x, y, z, \chi) = 1 - \cosh^2 x - \cosh^2 y - \cosh^2 z + 2 \cosh x \cdot \cosh y \cdot \cosh z \cdot \cos \chi,$$

and

$$g_+ = \begin{cases} g & \text{if } g > 0 \\ 0 & \text{if } g \leq 0. \end{cases}$$

The kernel $\mathcal{K}_{\alpha, \beta}(x, y, z)$ satisfies the following symmetry properties:

$$\mathcal{K}_{\alpha, \beta}(x, y, z) = \mathcal{K}_{\alpha, \beta}(y, x, z), \quad \mathcal{K}_{\alpha, \beta}(x, y, z) = \mathcal{K}_{\alpha, \beta}(-z, y, -x), \quad \mathcal{K}_{\alpha, \beta}(x, y, z) = \mathcal{K}_{\alpha, \beta}(x, -z, -y).$$

For every $x, y \in \mathbb{R}$, we have

$$t_x^{(\alpha, \beta)} f(y) = t_y^{(\alpha, \beta)} f(x),$$

(2.5)

and

$$\mathcal{H}_{\alpha, \beta} \left( t_x^{(\alpha, \beta)} f \right) (\lambda) = G_{\lambda}^{(\alpha, \beta)} (x) \mathcal{H}_{\alpha, \beta}(f)(\lambda),$$

(2.6)

for $f \in C_c(\mathbb{R})$.

If $f \in L^1(\mathbb{R}, A_{\alpha, \beta})$, then

$$\int_{\mathbb{R}} t_x^{(\alpha, \beta)} f(y) A_{\alpha, \beta}(y) dy = \int_{\mathbb{R}} f(y) A_{\alpha, \beta}(y) dy.$$

(2.7)

For every $f \in L^p(\mathbb{R}, A_{\alpha, \beta})$ and every $x \in \mathbb{R}$, the function $t_x^{(\alpha, \beta)} f$ belongs to the space $L^p(\mathbb{R}, A_{\alpha, \beta})$ and

$$\left\| t_x^{(\alpha, \beta)} f \right\|_{L^p(\mathbb{R}, A_{\alpha, \beta})} \leq C_{\alpha, \beta} \| f \|_{L^p(\mathbb{R}, A_{\alpha, \beta})},$$

(2.8)

where $C_{\alpha, \beta}$ is a positive constant.
The convolution product associated with the Opdam–Cherednik transform is defined for two suitable functions $f$ and $g$ by [44]

$$(f \ast_{a,\beta} g)(x) = \int_{\mathbb{R}} \xi^{(a,\beta)} f(-y) g(y) A_{a,\beta}(y) \, dy$$

and

$$\mathbb{H}_{a,\beta}(f \ast_{a,\beta} g) = \mathbb{H}_{a,\beta}(f) \mathbb{H}_{a,\beta}(g).$$

(2.9)

### 3 | THE WINDOWED OPDAM–CHEREDNIK TRANSFORM

In this section, we collect the necessary definitions and results from the harmonic analysis related to the windowed Opdam–Cherednik transform. For a detailed discussion on this transform, we refer to Poria [29].

Let $g \in L^2(\mathbb{R}, A_{a,\beta})$ and $\xi \in \mathbb{R}$, the modulation operator of $g$ associated with the Opdam–Cherednik transform is defined by

$$M_{\xi}^{(a,\beta)} g = \mathbb{H}_{a,\beta}^{-1} \left( \sqrt{\xi^{(a,\beta)} g} \right).$$

(3.1)

Then, for every $g \in L^2(\mathbb{R}, A_{a,\beta})$ and $\xi \in \mathbb{R}$, by using the Plancherel formula (2.2) and the translation invariance of the Plancherel measure $d\sigma_{a,\beta}$, we obtain

$$\left\| M_{\xi}^{(a,\beta)} g \right\|_{L^2(\mathbb{R}, A_{a,\beta})} = \|g\|_{L^2(\mathbb{R}, A_{a,\beta})}. $$

(3.2)

Now, for a non-zero window function $g \in L^2(\mathbb{R}, A_{a,\beta})$ and $(x, \xi) \in \mathbb{R}^2$, we consider the function $g_{x,\xi}^{(a,\beta)}$ defined by

$$g_{x,\xi}^{(a,\beta)} = \xi^{(a,\beta)} M_{\xi}^{(a,\beta)} g.$$  

(3.3)

For any function $f \in L^2(\mathbb{R}, A_{a,\beta})$, we define the windowed Opdam–Cherednik transform by

$$\mathcal{W}_g^{(a,\beta)}(f)(x, \xi) = \int_{\mathbb{R}} f(s) g_{x,\xi}^{(a,\beta)}(-s) A_{a,\beta}(s) \, ds, \quad (x, \xi) \in \mathbb{R}^2.$$ 

(3.4)

which can be also written in the form

$$\mathcal{W}_g^{(a,\beta)}(f)(x, \xi) = \left( f \ast_{a,\beta} M_{\xi}^{(a,\beta)} g \right)(x).$$ 

(3.5)

We define the measure $A_{a,\beta} \otimes \sigma_{a,\beta}$ on $\mathbb{R}^2$ by

$$d(A_{a,\beta} \otimes \sigma_{a,\beta})(x, \xi) = A_{a,\beta}(x) dx d\sigma_{a,\beta}(\xi).$$ 

(3.6)

The windowed Opdam–Cherednik transform satisfies the following properties.

**Proposition 3.1** (Poria [29]). Let $g \in L^2(\mathbb{R}, A_{a,\beta})$ be a non-zero window function. Then we have

1. (Plancherel’s formula) For every $f \in L^2(\mathbb{R}, A_{a,\beta})$,

$$\left\| \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta})} = \|f\|_{L^2(\mathbb{R}, A_{a,\beta})} \left\| g \right\|_{L^2(\mathbb{R}, A_{a,\beta})}.$$ 

(3.7)

2. (Orthogonality relation) For every $f, h \in L^2(\mathbb{R}, A_{a,\beta})$, we have

$$\int_{\mathbb{R}^2} \mathcal{W}_g^{(a,\beta)}(f)(x, \xi) \mathcal{W}_g^{(a,\beta)}(h)(x, \xi) d(A_{a,\beta} \otimes \sigma_{a,\beta})(x, \xi) = \|g\|_{L^2(\mathbb{R}, A_{a,\beta})}^2 \int_{\mathbb{R}} f(s) \overline{h(s)} A_{a,\beta}(s) \, ds.$$ 

(3.8)
4.1 Uncertainty principle for orthonormal sequences

In this subsection, we establish the uncertainty principle for orthonormal sequences associated with the windowed Opdam–Cherednik transform. First, we consider the following orthogonal projections:

1. Let $P_g$ be the orthogonal projection from $L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta})$ onto $\mathcal{W}_g^{(a,\beta)}(L^2(\mathbb{R}, A_{a,\beta}))$ and $\operatorname{Im} P_g$ denotes the range of $P_g$.

2. Let $P_\Sigma$ be the orthogonal projection on $L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta})$ defined by

$$P_\Sigma F = \chi_\Sigma F, \quad F \in L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta}),$$

where $\Sigma \subset \mathbb{R}^2$ and $\operatorname{Im} P_\Sigma$ is the range of $P_\Sigma$.

Also, we define

$$\|P_\Sigma P_g\| = \sup \left\{ \|P_\Sigma P_g(F)\|_{L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta})} : F \in L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta}), \|F\|_{L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta})} = 1 \right\}.$$

We first need the following result.

**Theorem 4.1.** Let $g \in L^2(\mathbb{R}, A_{a,\beta})$ be a non-zero window function. Then for any $\Sigma \subset \mathbb{R}^2$ of finite measure $A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma) < \infty$, the operator $P_\Sigma P_g$ is a Hilbert–Schmidt operator. Moreover, we have the following estimation:

$$\|P_\Sigma P_g\|^2 \leq A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma).$$
**Proof.** Since $P_g$ is a projection onto a reproducing kernel Hilbert space, for any function $F \in L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta})$, the orthogonal projection $P_g$ can be expressed as

$$P_g(F)(x, \xi) = \iint_{\mathbb{R}^2} F(x', \xi') K_g((x', \xi') ; (x, \xi)) \, d(A_{a,\beta} \otimes \sigma_{a,\beta})(x', \xi'),$$

where $K_g((x', \xi') ; (x, \xi))$ is given by (3.9). Using the relation (4.1), we obtain

$$P_2P_g(F)(x, \xi) = \iint_{\mathbb{R}^2} \mathcal{X}_2(x, \xi) F(x', \xi') K_g((x', \xi') ; (x, \xi)) \, d(A_{a,\beta} \otimes \sigma_{a,\beta})(x', \xi').$$

This shows that the operator $P_2P_g$ is an integral operator with kernel $K((x', \xi') ; (x, \xi)) = \mathcal{X}_2(x, \xi) K_g((x', \xi') ; (x, \xi))$. Using the relation (3.9), Plancherel's formula (3.7), and Fubini's theorem, we have

$$\|P_2P_g\|_{\text{HS}}^2 = \iint_{\mathbb{R}^2} \iint_{\mathbb{R}^2} |K((x', \xi') ; (x, \xi))|^2 \, d(A_{a,\beta} \otimes \sigma_{a,\beta})(x', \xi') \, d(A_{a,\beta} \otimes \sigma_{a,\beta})(x, \xi)$$

$$= \iint_{\mathbb{R}^2} \iint_{\mathbb{R}^2} |\mathcal{X}_2(x, \xi)|^2 K_g((x', \xi') ; (x, \xi)) \, d(A_{a,\beta} \otimes \sigma_{a,\beta})(x', \xi') \, d(A_{a,\beta} \otimes \sigma_{a,\beta})(x, \xi)$$

$$= \frac{1}{\|g\|_{L^1(\mathbb{R}, A_{a,\beta})}^4} \iint_{\Sigma} \left( \iint_{\mathbb{R}^2} W_g^{(a,\beta)}(x, \xi) \, d(A_{a,\beta} \otimes \sigma_{a,\beta})(x', \xi') \right)^2 \, d(A_{a,\beta} \otimes \sigma_{a,\beta})(x, \xi)$$

$$\leq \frac{\|g\|_{L^1(\mathbb{R}, A_{a,\beta})}^4}{\|g\|_{L^1(\mathbb{R}, A_{a,\beta})}^4} A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma) = A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma).$$

Thus, the operator $P_2P_g$ is a Hilbert–Schmidt operator. Now, the proof follows from the fact that $\|P_2P_g\| \leq \|P_2P_g\|_{\text{HS}}$. 

In the following, we obtain the uncertainty principle for orthonormal sequences associated with the windowed Opdam–Cherednik transform.

**Theorem 4.2.** Let $g \in L^2(\mathbb{R}, A_{a,\beta})$ be a non-zero window function and $\{\phi_n\}_{n \in \mathbb{N}}$ be an orthonormal sequence in $L^2(\mathbb{R}, A_{a,\beta})$. Then for any $\Sigma \subset \mathbb{R}^2$ of finite measure $A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma) < \infty$, we have

$$\sum_{n=1}^N \left( 1 - \|\mathcal{X}_2 W_g^{(a,\beta)}(\phi_n)\|_{L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta})} \right) \leq A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma),$$

for every $N \in \mathbb{N}$.

**Proof.** Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for $L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta})$. Since $P_2P_g$ is a Hilbert–Schmidt operator, from Theorem 4.1, we get

$$\text{tr} \left( P_g P_2P_g \right) = \sum_{n \in \mathbb{N}} \langle P_g P_2P_g e_n, e_n \rangle_{L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta})} = \|P_2P_g\|_{\text{HS}}^2 \leq A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma),$$

where $\text{tr} \left( P_g P_2P_g \right)$ denotes the trace of the operator $P_g P_2P_g$ and $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta})}$ denotes the inner product of $L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta})$. Since $\{\phi_n\}_{n \in \mathbb{N}}$ be an orthonormal sequence in $L^2(\mathbb{R}, A_{a,\beta})$, from the orthogonality relation (3.8), we obtain that $\left\{ \mathcal{X}_2 W_g^{(a,\beta)}(\phi_n) \right\}_{n \in \mathbb{N}}$ is also an orthonormal sequence in $L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta})$. Therefore,
Here, we study the version of Donoho–Stark’s uncertainty principle for the windowed Opdam–Cherednik transform. In particular, we investigate the case where \( f \) and \( \mathcal{W}^{(a,\beta)}_\Sigma(f) \) are close to zero outside measurable sets. We start with the following result.

**Theorem 4.3.** Let \( g \in L^2(\mathbb{R}, A_{a,\beta}) \) be a non-zero window function and \( f \in L^2(\mathbb{R}, A_{a,\beta}) \) such that \( f \neq 0 \). Then for any \( \Sigma \subset \mathbb{R}^2 \) and \( \varepsilon \geq 0 \) such that

\[
\int_\Sigma \left| \mathcal{W}^{(a,\beta)}_\Sigma(f)(x,\xi) \right|^2 \, d(A_{a,\beta} \otimes \sigma_{a,\beta})(x,\xi) \geq (1 - \varepsilon) \| f \|^2_{L^2(\mathbb{R}, A_{a,\beta})} \| g \|^2_{L^2(\mathbb{R}, A_{a,\beta})},
\]

we have

\[
A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma) \geq 1 - \varepsilon.
\]

**Proof.** Using the relation (3.11), we get

\[
(1 - \varepsilon) \| f \|^2_{L^2(\mathbb{R}, A_{a,\beta})} \| g \|^2_{L^2(\mathbb{R}, A_{a,\beta})} \leq \int_\Sigma \left| \mathcal{W}^{(a,\beta)}_\Sigma(f)(x,\xi) \right|^2 \, d(A_{a,\beta} \otimes \sigma_{a,\beta})(x,\xi)
\leq \left| \mathcal{W}^{(a,\beta)}_\Sigma(f) \right|^2_{L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta})} A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma)
\leq A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma) \| f \|^2_{L^2(\mathbb{R}, A_{a,\beta})} \| g \|^2_{L^2(\mathbb{R}, A_{a,\beta})},
\]

Therefore, \( A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma) \geq 1 - \varepsilon \). □
The following proposition shows that the windowed Opdam–Cherednik transform cannot be concentrated in any small set.

**Proposition 4.4.** Let $g \in L^2(\mathbb{R}, A_{a,\beta})$ be a non-zero window function. Then for any function $f \in L^2(\mathbb{R}, A_{a,\beta})$ and for any $\Sigma \subset \mathbb{R}^2$ such that $A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma) < 1$, we have

$$\left\| \chi_\Sigma \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})} \geq \sqrt{1 - A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma)} \left\| f \right\|_{L^2(\mathbb{R}, A_{a,\beta})} \left\| g \right\|_{L^2(\mathbb{R}, A_{a,\beta})}. $$

**Proof.** For any function $f \in L^2(\mathbb{R}, A_{a,\beta})$, using the relation (3.11), we get

$$\left\| \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 = \left\| \chi_\Sigma \mathcal{W}_g^{(a,\beta)}(f) + \chi_\Sigma \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2

= \left\| \chi_\Sigma \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 + \left\| \chi_\Sigma \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2

\leq A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma) \left\| \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 + \left\| \chi_\Sigma \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2

\leq A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma) \left\| f \right\|_{L^2(\mathbb{R}, A_{a,\beta})}^2 \left\| g \right\|_{L^2(\mathbb{R}, A_{a,\beta})}^2 + \left\| \chi_\Sigma \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2.

Thus, using Plancherel’s formula (3.7), we obtain

$$\left\| \chi_\Sigma \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})} \geq \sqrt{1 - A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma)} \left\| f \right\|_{L^2(\mathbb{R}, A_{a,\beta})} \left\| g \right\|_{L^2(\mathbb{R}, A_{a,\beta})}. $$

**Definition 4.5.** Let $E$ be a measurable subset of $\mathbb{R}$ and $0 \leq \epsilon_E < 1$. Then we say that a function $f \in L^p(\mathbb{R}, A_{a,\beta}), 1 \leq p \leq 2$, is $\epsilon_E$-concentrated on $E$ in $L^p(\mathbb{R}, A_{a,\beta})$-norm, if

$$\left\| \chi_E f \right\|_{L^p(\mathbb{R}, A_{a,\beta})} \leq \epsilon_E \left\| f \right\|_{L^p(\mathbb{R}, A_{a,\beta})}. $$

If $\epsilon_E = 0$, then $E$ contains the support of $f$.

**Definition 4.6.** Let $\Sigma$ be a measurable subset of $\mathbb{R}^2$ and $0 \leq \epsilon_\Sigma < 1$. Let $f, g \in L^2(\mathbb{R}, A_{a,\beta})$ be two non-zero functions. We say that $\mathcal{W}_g^{(a,\beta)}(f)$ is $\epsilon_\Sigma$-time-frequency concentrated on $\Sigma$, if

$$\left\| \chi_\Sigma \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})} \leq \epsilon_\Sigma \left\| \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}.$$

If $\mathcal{W}_g^{(a,\beta)}(f)$ is $\epsilon_\Sigma$-time-frequency concentrated on $\Sigma$, then in the following, we obtain an estimate for the size of the essential support of the windowed Opdam–Cherednik transform.

**Theorem 4.7.** Let $g \in L^2(\mathbb{R}, A_{a,\beta})$ be a non-zero window function and $f \in L^2(\mathbb{R}, A_{a,\beta})$ such that $f \neq 0$. Let $\Sigma \subset \mathbb{R}^2$ such that $A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma) < \infty$ and $\epsilon_\Sigma \geq 0$. If $\mathcal{W}_g^{(a,\beta)}(f)$ is $\epsilon_\Sigma$-time-frequency concentrated on $\Sigma$, then

$$A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma) \geq 1 - \epsilon_\Sigma^2.$$

**Proof.** Since $\mathcal{W}_g^{(a,\beta)}(f)$ is $\epsilon_\Sigma$-time-frequency concentrated on $\Sigma$, using Plancherel's formula (3.7), we deduce that

$$\left\| f \right\|_{L^2(\mathbb{R}, A_{a,\beta})}^2 \left\| g \right\|_{L^2(\mathbb{R}, A_{a,\beta})}^2 = \left\| \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2

= \left\| \chi_\Sigma \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 + \left\| \chi_\Sigma \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2

\leq \epsilon_\Sigma^2 \left\| \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 + \left\| \chi_\Sigma \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2.$$
Hence, using the relation (3.11), we obtain

\[
(1 - \epsilon_2^2) \|f\|_{L^2(\mathbb{R}^2, \Sigma)}^2 \|g\|_{L^2(\mathbb{R}^2, \Sigma)}^2 \leq \left\| \chi_S \mathcal{W}_g^{(\alpha, \beta)}(f) \right\|_{L^2(\mathbb{R}^2, \mathbb{R}^2, \sigma_{\alpha, \beta})}^2 \\
\leq \left\| \mathcal{W}_g^{(\alpha, \beta)}(f) \right\|_{L^2(\mathbb{R}^2, \mathbb{R}^2, \sigma_{\alpha, \beta})}^2 \left\| A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta}(\Sigma) \right\|_r^2 \\
\leq \|f\|_{L^2(\mathbb{R}^2, \Sigma)}^2 \|g\|_{L^2(\mathbb{R}^2, \Sigma)}^2 \left\| A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta}(\Sigma) \right\|_r^2,
\]

which completes the proof. \[\square\]

**Theorem 4.8.** Let \( \Sigma \subset \mathbb{R}^2 \) such that \( A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta}(\Sigma) < \infty, \epsilon_2 \geq 0, g \in L^2(\mathbb{R}, A_{\alpha, \beta}) \) be a non-zero window function, and \( f \in L^2(\mathbb{R}, A_{\alpha, \beta}) \) such that \( f \neq 0 \). If \( \mathcal{W}_g^{(\alpha, \beta)}(f) \) is \( \epsilon_2 \)-time-frequency concentrated on \( \Sigma \), then for every \( p > 2 \), we have

\[
A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta}(\Sigma) \geq \left( 1 - \epsilon_2^2 \right)^{\frac{r}{p-2}}.
\]

**Proof.** Since \( \mathcal{W}_g^{(\alpha, \beta)}(f) \) is \( \epsilon_2 \)-time-frequency concentrated on \( \Sigma \), from (4.2), we have

\[
(1 - \epsilon_2^2) \|f\|_{L^2(\mathbb{R}^2, \Sigma)}^2 \|g\|_{L^2(\mathbb{R}^2, \Sigma)}^2 \leq \left\| \chi_S \mathcal{W}_g^{(\alpha, \beta)}(f) \right\|_{L^2(\mathbb{R}^2, \Sigma)}^2.
\]

Again, applying Hölder's inequality for the conjugate exponents \( \frac{p}{2} \) and \( \frac{p}{p-2} \), we get

\[
\left\| \chi_S \mathcal{W}_g^{(\alpha, \beta)}(f) \right\|_{L^2(\mathbb{R}^2, \Sigma)}^2 \leq \left\| \mathcal{W}_g^{(\alpha, \beta)}(f) \right\|_{L^2(\mathbb{R}^2, \Sigma)}^2 \left\| A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta}(\Sigma) \right\|_r^{\frac{r}{p-2}}.
\]

Now, using the relation (3.12), we obtain

\[
\left\| \chi_S \mathcal{W}_g^{(\alpha, \beta)}(f) \right\|_{L^2(\mathbb{R}^2, \Sigma)}^2 \leq \|f\|_{L^2(\mathbb{R}^2, \Sigma)}^2 \|g\|_{L^2(\mathbb{R}^2, \Sigma)}^2 \left\| A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta}(\Sigma) \right\|_r^{\frac{r}{p-2}}.
\]

Hence,

\[
A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta}(\Sigma) \geq \left( 1 - \epsilon_2^2 \right)^{\frac{r}{p-2}}.
\]

\[\square\]

**Theorem 4.9.** Let \( \Sigma \subset \mathbb{R}^2 \) such that \( A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta}(\Sigma) < \infty, g \in L^2(\mathbb{R}, A_{\alpha, \beta}) \) be a non-zero window function, and \( f \in L^1(\mathbb{R}, A_{\alpha, \beta}) \cap L^2(\mathbb{R}, A_{\alpha, \beta}) \) such that \( \| \mathcal{W}_g^{(\alpha, \beta)}(f) \|_{L^2(\mathbb{R}^2, \sigma_{\alpha, \beta})} = 1 \). Let \( E \subset \mathbb{R} \) such that \( A_{\alpha, \beta}(E) < \infty \). If \( f \) is \( \epsilon_2 \)-concentrated on \( E \) in \( L^1(\mathbb{R}, A_{\alpha, \beta}) \)-norm and \( \mathcal{W}_g^{(\alpha, \beta)}(f) \) is \( \epsilon_2 \)-time-frequency concentrated on \( \Sigma \), then

\[
A_{\alpha, \beta}(E) \geq (1 - \epsilon_2^2) \|f\|_{L^1(\mathbb{R}, A_{\alpha, \beta})}^2 \|g\|_{L^2(\mathbb{R}, A_{\alpha, \beta})}^2,
\]

and

\[
\|f\|_{L^1(\mathbb{R}, A_{\alpha, \beta})} \|g\|_{L^2(\mathbb{R}, A_{\alpha, \beta})} A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta}(\Sigma) \geq (1 - \epsilon_2^2).
\]

In particular,

\[
A_{\alpha, \beta}(E) A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta}(\Sigma) \|f\|_{L^1(\mathbb{R}, A_{\alpha, \beta})}^2 \geq (1 - \epsilon_2^2) \|f\|_{L^1(\mathbb{R}, A_{\alpha, \beta})}^2 (1 - \epsilon_2^2) \|f\|_{L^1(\mathbb{R}, A_{\alpha, \beta})}^2.
\]

**Proof.** Since \( \mathcal{W}_g^{(\alpha, \beta)}(f) \) is \( \epsilon_2 \)-time-frequency concentrated on \( \Sigma \), from (4.2), we have

\[
(1 - \epsilon_2^2) \|f\|_{L^2(\mathbb{R}, A_{\alpha, \beta})}^2 \|g\|_{L^2(\mathbb{R}, A_{\alpha, \beta})}^2 \leq \left\| \chi_S \mathcal{W}_g^{(\alpha, \beta)}(f) \right\|_{L^2(\mathbb{R}^2, \Sigma)}^2.
\]
Since \( \| \mathcal{W}_g^{(a,\beta)}(f) \|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})} = 1 \), using the relation (3.11) and Plancherel’s formula (3.7), we obtain

\[
(1 - \epsilon_E^2) \leq \left\| \mathcal{X}\mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 \\
\leq \left\| \mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 \cdot A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma) \\
\leq \| f \|_{L^2(\mathbb{R}; A_{a,\beta})}^2 \| g \|_{L^2(\mathbb{R}; A_{a,\beta})}^2 \cdot A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma).
\]

(4.3)

Similarly, since \( f \) is \( \epsilon_E \)-concentrated on \( E \) in \( L^1(\mathbb{R}, A_{a,\beta}) \)-norm, using the Cauchy–Schwarz inequality and the fact that \( \| f \|_{L^2(\mathbb{R}; A_{a,\beta})} \| g \|_{L^2(\mathbb{R}; A_{a,\beta})} = 1 \), we get

\[
(1 - \epsilon_E) \| f \|_{L^2(\mathbb{R}; A_{a,\beta})} \leq \| \mathcal{X}_E f \|_{L^1(\mathbb{R}; A_{a,\beta})} \leq \| f \|_{L^2(\mathbb{R}; A_{a,\beta})} A_{a,\beta}(E) \frac{1}{2} = \frac{A_{a,\beta}(E)^{\frac{1}{2}}}{\| g \|_{L^2(\mathbb{R}; A_{a,\beta})}}.
\]

(4.4)

Finally, from (4.3) and (4.4), we have

\[
A_{a,\beta}(E) A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma) \| f \|_{L^2(\mathbb{R}; A_{a,\beta})}^2 \geq (1 - \epsilon_E)^2 \left( 1 - \epsilon_E^2 \right) \| f \|_{L^2(\mathbb{R}; A_{a,\beta})}^2.
\]

This completes the proof of the theorem. \( \square \)

### 4.3 Benedicks-type uncertainty principle for the windowed Opdam–Cherednik transform

In this subsection, we study Benedicks-type uncertainty principle for the windowed Opdam–Cherednik transform. The following proposition shows that this transform cannot be concentrated inside a set of measure arbitrarily small.

**Proposition 4.10.** Let \( g \in L^2(\mathbb{R}, A_{a,\beta}) \) be a non-zero window function and \( \Sigma \subset \mathbb{R}^2 \). If \( \| P_\Sigma P_g \| < 1 \), then there exists a constant \( c(\Sigma, g) > 0 \) such that for every \( f \in L^2(\mathbb{R}, A_{a,\beta}) \), we have

\[
\| f \|_{L^2(\mathbb{R}; A_{a,\beta})} \| g \|_{L^2(\mathbb{R}; A_{a,\beta})} \leq c(\Sigma, g) \left\| \mathcal{X}\mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}.
\]

**Proof.** Since \( P_\Sigma \) is an orthogonal projection on \( L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta}) \), for any \( F \in L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta}) \), we get

\[
\| P_\Sigma P_g(F) \|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})} = \| P_\Sigma P_g(F) \|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})} = \| P_\Sigma P_g(F) \|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}.
\]

Again, using the identity \( P_\Sigma P_g(F) = P_\Sigma P_g \cdot P_g(F) \), we get

\[
\| P_\Sigma P_g(F) \|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})} \leq \| P_\Sigma P_g \|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})} \| P_g(F) \|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}.
\]

Hence,

\[
\| P_g(F) \|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})} \leq \frac{1}{1 - \| P_\Sigma P_g \|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2}.
\]

(4.5)

Since \( P_g \) is an orthogonal projection from \( L^2(\mathbb{R}^2, A_{a,\beta} \otimes \sigma_{a,\beta}) \) onto \( \mathcal{W}_g^{(a,\beta)} \left( L^2(\mathbb{R}, A_{a,\beta}) \right) \), for any \( f \in L^2(\mathbb{R}, A_{a,\beta}) \), using the relation (4.5) and Plancherel’s formula (3.7), we get

\[
\| f \|_{L^2(\mathbb{R}; A_{a,\beta})} \| g \|_{L^2(\mathbb{R}; A_{a,\beta})} \leq \frac{1}{\sqrt{1 - \| P_\Sigma P_g \|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2}} \left\| \mathcal{X}\mathcal{W}_g^{(a,\beta)}(f) \right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}.
\]

The desired result follows by choosing the constant \( c(\Sigma, g) = \frac{1}{\sqrt{1 - \| P_\Sigma P_g \|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2}} \). \( \square \)

Next, we obtain Benedicks-type uncertainty principle for the windowed Opdam–Cherednik transform.
Theorem 4.11. Let \( g \in L^2(\mathbb{R}, A_{\alpha,\beta}) \) be a non-zero window function such that
\[
\sigma_{\alpha,\beta} \left( \left\{ \mathbb{H}_{\alpha,\beta}(g) \neq 0 \right\} \right) < \infty.
\]
Then for any \( \Sigma \subset \mathbb{R}^2 \) such that for almost every \( \xi \in \mathbb{R} \), \( \int_{\mathbb{R}} \chi_{\Sigma}(x, \xi) A_{\alpha,\beta}(x) \, dx < \infty \), we have
\[
\mathcal{W}_{g}^{(\alpha,\beta)} \left( L^2 (\mathbb{R}, A_{\alpha,\beta}) \right) \cap \text{Im} P_{\Sigma} = \{ 0 \}.
\]

Proof. Let \( F \in \mathcal{W}_{g}^{(\alpha,\beta)} \left( L^2 (\mathbb{R}, A_{\alpha,\beta}) \right) \) \cap \text{Im} P_{\Sigma} be a non-trivial function. Then there exists a function \( f \in L^2(\mathbb{R}, A_{\alpha,\beta}) \) such that \( F = \mathcal{W}_{g}^{(\alpha,\beta)}(f) \) and \( \text{Supp} F \subset \Sigma \). For any \( \xi \in \mathbb{R} \), we consider the function
\[
F_{\xi}(x) = \mathcal{W}_{g}^{(\alpha,\beta)}(f)(x, \xi), \quad x \in \mathbb{R}.
\]
Then, we get \( \text{Supp} F_{\xi} \subset \{ x \in \mathbb{R} : (x, \xi) \in \Sigma \} \). Since for almost every \( \xi \in \mathbb{R} \), \( \int_{\mathbb{R}} \chi_{\Sigma}(x, \xi) A_{\alpha,\beta}(x) \, dx < \infty \), we have \( A_{\alpha,\beta}(\text{Supp} F_{\xi}) < \infty \). Using the relation (3.5), we get
\[
\mathbb{H}_{\alpha,\beta}(F_{\xi}) = \mathbb{H}_{\alpha,\beta}(f) \mathbb{H}_{\alpha,\beta} \left( \mathcal{M}_{\xi}^{(\alpha,\beta)} g \right) \ \text{a.e.}
\]
Hence,
\[
\text{Supp} \mathbb{H}_{\alpha,\beta}(F_{\xi}) \subset \text{Supp} \mathcal{M}_{\xi}^{(\alpha,\beta)}(\mathbb{H}_{\alpha,\beta}(g))^2.
\]
and from the hypothesis, we get \( \sigma_{\alpha,\beta} \left( \left\{ \mathbb{H}_{\alpha,\beta}(F_{\xi}) \neq 0 \right\} \right) < \infty \). From Benedicks-type result for the Opdam–Cherednik transform [28], we conclude that for every \( \xi \in \mathbb{R}, F(\cdot, \xi) = 0 \), which eventually implies that \( F = 0 \).

Remark 4.12. Let \( g \in L^2(\mathbb{R}, A_{\alpha,\beta}) \) be a non-zero window function such that
\[
\sigma_{\alpha,\beta} \left( \left\{ \mathbb{H}_{\alpha,\beta}(g) \neq 0 \right\} \right) < \infty.
\]
Then, for any non-zero function \( f \in L^2(\mathbb{R}, A_{\alpha,\beta}) \), we have \( A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta}(\text{Supp} \mathcal{W}_{g}^{(\alpha,\beta)}(f)) = \infty \), that is, the support of \( \mathcal{W}_{g}^{(\alpha,\beta)}(f) \) cannot be of finite measure.

Proposition 4.13. Let \( g \in L^2(\mathbb{R}, A_{\alpha,\beta}) \) be a non-zero window function such that
\[
\sigma_{\alpha,\beta} \left( \left\{ \mathbb{H}_{\alpha,\beta}(g) \neq 0 \right\} \right) < \infty.
\]
Let \( \Sigma \subset \mathbb{R}^2 \) such that \( A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta}(\Sigma) < \infty \), then there exists a constant \( c(\Sigma, g) > 0 \) such that
\[
\| f \|_{L^2 (\mathbb{R}, A_{\alpha,\beta})} \leq c(\Sigma, g) \| \chi_{\Sigma} \mathcal{W}_{g}^{(\alpha,\beta)}(f) \|_{L^2 (\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})}.
\]

Proof. Assume that
\[
\| P_{\Sigma} P_{g} (F) \|_{L^2 (\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})} = \| F \|_{L^2 (\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})}, \quad F \in L^2 (\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta}).
\]
Since \( P_{\Sigma} \) and \( P_{g} \) are orthogonal projections, we obtain \( P_{\Sigma}(F) = P_{g}(F) = F \). Again, since \( A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta}(\Sigma) < \infty \), for almost every \( \xi \in \mathbb{R}, \int_{\mathbb{R}} \chi_{\Sigma}(x, \xi) A_{\alpha,\beta}(x) \, dx < \infty \) and from Theorem 4.11, we get \( F = 0 \). Hence, for \( F \neq 0 \), we obtain
\[
\| P_{\Sigma} P_{g}(F) \|_{L^2 (\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})} < \| F \|_{L^2 (\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})}.
\]
Since \( P_{\Sigma} P_{g} \) is a Hilbert–Schmidt operator, we obtain that the largest eigenvalue \( | \lambda | \) of the operator \( P_{\Sigma} P_{g} \) satisfy \( | \lambda | < 1 \) and \( \| P_{\Sigma} P_{g} \| = | \lambda | < 1 \). Finally, using Proposition 4.10, we get the desired result. \( \square \)
4.4 Heisenberg-type uncertainty principle for the windowed Opdam–Cherednik transform

This subsection is devoted to study Heisenberg-type uncertainty inequality for the windowed Opdam–Cherednik transform for general magnitude \(s > 0\). Indeed, we have the following result.

**Theorem 4.14.** Let \(s > 0\). Then there exists a constant \(c(s, a, \beta) > 0\) such that for all \(f, g \in L^2(\mathbb{R}, A_{a,\beta})\), we have

\[
\left\|\mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 + \left\|e^{i\xi \cdot x} \mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 \geq c(s, a, \beta) \left\|f\right\|_{L^2(\mathbb{R}, A_{a,\beta})}^2 \left\|g\right\|_{L^2(\mathbb{R}, A_{a,\beta})}^2.
\]

**Proof.** Let \(r > 0\) and \(B_r = \{(x, \xi) \in \mathbb{R}^2 : |(x, \xi)| < r\}\) be the ball with center at origin and radius \(r\) in \(\mathbb{R}^2\). Fix \(\varepsilon_0 \leq 1\) small enough such that \(A_{a,\beta} \otimes \sigma_{a,\beta}(B_{\varepsilon_0}) < 1\). From Proposition 4.4, we have

\[
\left\|f\right\|_{L^2(\mathbb{R}, A_{a,\beta})}^2 \left\|g\right\|_{L^2(\mathbb{R}, A_{a,\beta})}^2 \leq \frac{1}{1 - A_{a,\beta} \otimes \sigma_{a,\beta}(B_{\varepsilon_0})} \left\|\mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 + \frac{1}{1 - A_{a,\beta} \otimes \sigma_{a,\beta}(B_{\varepsilon_0})} \left\|e^{i\xi \cdot x} \mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 \geq \varepsilon_0 \left\|\mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 \left\|f\right\|_{L^2(\mathbb{R}, A_{a,\beta})}^2 \left\|g\right\|_{L^2(\mathbb{R}, A_{a,\beta})}^2.
\]

Consequently,

\[
\left\|[x, \xi] \mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 \geq \varepsilon_0 \left\|\mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 \left\|f\right\|_{L^2(\mathbb{R}, A_{a,\beta})}^2 \left\|g\right\|_{L^2(\mathbb{R}, A_{a,\beta})}^2.
\]

Finally, using the fact that \(|a + b|^s \leq 2^s (|a|^s + |b|^s)\) and from (4.6), we get

\[
\varepsilon_0 \left\|\mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 \leq \left\|\mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 + 2 \varepsilon_0 \left\|e^{i\xi \cdot x} \mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2.
\]

Hence, we get the desired result with \(c(s, a, \beta) = \varepsilon_0 \left\|\mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}^2 \left\|f\right\|_{L^2(\mathbb{R}, A_{a,\beta})}^2 \left\|g\right\|_{L^2(\mathbb{R}, A_{a,\beta})}^2\). \(\square\)

4.5 Local uncertainty inequality for the windowed Opdam–Cherednik transform

Here, we discuss results related to the \(L^2(\mathbb{R}, A_{a,\beta})\)-mass of the windowed Opdam–Cherednik transform outside sets of finite measure. Indeed, we establish the following result.

**Theorem 4.15.** Let \(s > 0\). Then there exists a constant \(c(s, a, \beta) > 0\) such that for any \(f, g \in L^2(\mathbb{R}, A_{a,\beta})\) and any \(\Sigma \subset \mathbb{R}^2\) such that \(A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma) < \infty\), we have

\[
\left\|\mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\Sigma; A_{a,\beta} \otimes \sigma_{a,\beta})} \leq c(s, a, \beta)\left[A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma)\right]^{\frac{1}{2}} \left\|[x, \xi] \mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})}.
\]

**Proof.** Since \(\left\|\mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\Sigma; A_{a,\beta} \otimes \sigma_{a,\beta})} \leq \left\|\mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\mathbb{R}^2; A_{a,\beta} \otimes \sigma_{a,\beta})} \left[A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma)\right]^{\frac{1}{2}}\), using the relation (3.11), we get

\[
\left\|\mathcal{W}_g^{(a,\beta)}(f)\right\|_{L^2(\Sigma; A_{a,\beta} \otimes \sigma_{a,\beta})} \leq \left[A_{a,\beta} \otimes \sigma_{a,\beta}(\Sigma)\right]^{\frac{1}{2}} \left\|f\right\|_{L^2(\mathbb{R}, A_{a,\beta})} \left\|g\right\|_{L^2(\mathbb{R}, A_{a,\beta})}.
\]
Thus, using Heisenberg’s inequality (4.6), we get
\[
\left\| \mathcal{W}_g^{(\alpha,\beta)}(f) \right\|_{L^2(\Sigma_{A,\beta})} \leq \frac{[A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta}(\Sigma)]^{\frac{1}{2}}}{\varepsilon_0^4} \left\| (x, \xi) \mathcal{W}_g^{(\alpha,\beta)}(f) \right\|_{L^1(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})}.
\]

This completes the proof of the theorem with \( c(s, \alpha, \beta) = \varepsilon_0^{-4} \left( 1 - A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta}(B_{\varepsilon_0}) \right)^{-\frac{1}{2}} \).

The following result shows that Theorem 4.15 gives a general form of Heisenberg-type inequality with a different constant.

**Corollary 4.16.** Let \( s > 0 \). Then there exists a constant \( c_{s,\alpha,\beta} > 0 \) such that for all \( f, g \in L^2(\mathbb{R}, A_{\alpha,\beta}) \), we have
\[
\left\| (x, \xi) \mathcal{W}_g^{(\alpha,\beta)}(f) \right\|_{L^1(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})} \geq c_{s,\alpha,\beta} \| f \|_{L^1(\mathbb{R}, A_{\alpha,\beta})} \| g \|_{L^1(\mathbb{R}, A_{\alpha,\beta})}.
\]

In particular,
\[
\left\| x^2 \mathcal{W}_g^{(\alpha,\beta)}(f) \right\|_{L^2(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})} + \left\| \xi^2 \mathcal{W}_g^{(\alpha,\beta)}(f) \right\|_{L^2(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})} \geq c_{s,\alpha,\beta} \| f \|_{L^2(\mathbb{R}, A_{\alpha,\beta})} \| g \|_{L^2(\mathbb{R}, A_{\alpha,\beta})}.
\]

**Proof.** Let \( r > 0 \) and \( B_r = \{(x, \xi) \in \mathbb{R}^2 : |(x, \xi)| < r \} \) be the ball with center at origin and radius \( r \) in \( \mathbb{R}^2 \). Using Plancherel’s formula (3.7) and Theorem 4.15, we obtain
\[
\| f \|_{L^2(\mathbb{R}, A_{\alpha,\beta})} \| g \|_{L^2(\mathbb{R}, A_{\alpha,\beta})} = \left\| \mathcal{W}_g^{(\alpha,\beta)}(f) \right\|_{L^2(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})}^2 = \left\| \mathcal{W}_g^{(\alpha,\beta)}(f) \right\|_{L^2(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})}^2 + \left\| \chi_{B_r} \mathcal{W}_g^{(\alpha,\beta)}(f) \right\|_{L^2(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})}^2 \geq c_{s,\alpha,\beta} \| f \|_{L^2(\mathbb{R}, A_{\alpha,\beta})} \| g \|_{L^2(\mathbb{R}, A_{\alpha,\beta})}.
\]

We get the inequality (4.7) by minimizing the right-hand side of the above inequality over \( r > 0 \). Now, proceeding similarly as in the proof of Theorem 4.14, we obtain
\[
2^s \| x^2 \mathcal{W}_g^{(\alpha,\beta)}(f) \|_{L^2(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})} + 2^s \| \xi^2 \mathcal{W}_g^{(\alpha,\beta)}(f) \|_{L^2(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})} \geq c_{s,\alpha,\beta} \| f \|_{L^2(\mathbb{R}, A_{\alpha,\beta})} \| g \|_{L^2(\mathbb{R}, A_{\alpha,\beta})}.
\]
and this completes the proof.

### 4.6 Heisenberg-type uncertainty inequality via the \( k \)-entropy

In this subsection, we study the localization of the \( k \)-entropy of the windowed Opdam–Cherednik transform over the space \( \mathbb{R}^2 \). Before going to prove the main result, we first need the following definition.

**Definition 4.17.**

1. A probability density function \( \rho \) on \( \mathbb{R}^2 \) is a non-negative measurable function on \( \mathbb{R}^2 \) satisfying
\[
\int_{\mathbb{R}^2} \rho(x, \xi) d(A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})(x, \xi) = 1.
\]
2. Let $\rho$ be a probability density function on $\mathbb{R}^2$. Then the $k$-entropy of $\rho$ is defined by

$$E_k(\rho) := -\int_{\mathbb{R}^2} \ln(\rho(x, \xi)) \rho(x, \xi) d(A_{\alpha, \beta} \otimes \sigma_{\gamma, \delta})(x, \xi),$$

whenever the integral on the right-hand side is well defined.

In the following, we prove the main result of this subsection.

**Theorem 4.18.** Let $g \in L^2(\mathbb{R}, A_{\alpha, \beta})$ be a non-zero window function and $f \in L^2(\mathbb{R}, A_{\alpha, \beta})$ such that $f \neq 0$. Then, we have

$$E_k \left( \left| \mathcal{W}^{(\alpha, \beta)}_g(f) \right|^2 \right) \geq -2 \ln \left( \|f\|_{L^2(\mathbb{R}, A_{\alpha, \beta})} \|g\|_{L^2(\mathbb{R}, A_{\alpha, \beta})} \right) \|f\|^2_{L^2(\mathbb{R}, A_{\alpha, \beta})} \|g\|^2_{L^2(\mathbb{R}, A_{\alpha, \beta})}.$$  \hspace{1cm} (4.8)

**Proof.** First, we assume that $\|f\|_{L^2(\mathbb{R}, A_{\alpha, \beta})} = \|g\|_{L^2(\mathbb{R}, A_{\alpha, \beta})} = 1$. For any $(x, \xi) \in \mathbb{R}^2$, using the relation (3.11), we get

$$\left| \mathcal{W}^{(\alpha, \beta)}_g(f)(x, \xi) \right| \leq \|f\|_{L^2(\mathbb{R}, A_{\alpha, \beta})} \|g\|_{L^2(\mathbb{R}, A_{\alpha, \beta})} = 1.$$

Consequently, $\ln \left( \left| \mathcal{W}^{(\alpha, \beta)}_g(f) \right|^2 \right) \leq 0$ and therefore $E_k \left( \left| \mathcal{W}^{(\alpha, \beta)}_g(f) \right|^2 \right) \geq 0$. The desired inequality (4.8) holds trivially if the entropy $E_k \left( \left| \mathcal{W}^{(\alpha, \beta)}_g(f) \right|^2 \right)$ is infinite. Now, suppose that the entropy $E_k \left( \left| \mathcal{W}^{(\alpha, \beta)}_g(f) \right|^2 \right)$ is finite. Let $f$ and $g$ be two non-zero functions in $L^2(\mathbb{R}, A_{\alpha, \beta})$. We define

$$\phi = \frac{f}{\|f\|_{L^2(\mathbb{R}, A_{\alpha, \beta})}} \quad \text{and} \quad \psi = \frac{g}{\|g\|_{L^2(\mathbb{R}, A_{\alpha, \beta})}}.$$ 

Then $\phi, \psi \in L^2(\mathbb{R}, A_{\alpha, \beta})$ with $\|\phi\|_{L^2(\mathbb{R}, A_{\alpha, \beta})} = \|\psi\|_{L^2(\mathbb{R}, A_{\alpha, \beta})} = 1$ and consequently

$$E_k \left( \left| \mathcal{W}^{(\alpha, \beta)}_\psi(f) \right|^2 \right) \geq 0.$$  \hspace{1cm} (4.9)

Since $\mathcal{W}^{(\alpha, \beta)}_\psi(f) = \frac{1}{\|f\|_{L^2(\mathbb{R}, A_{\alpha, \beta})}} \mathcal{W}^{(\alpha, \beta)}_g(f)$, we have

$$E_k \left( \left| \mathcal{W}^{(\alpha, \beta)}_\psi(f) \right|^2 \right) = -\int_{\mathbb{R}^2} \ln \left( \left| \mathcal{W}^{(\alpha, \beta)}_\psi(f)(x, \xi) \right|^2 \right) \left| \mathcal{W}^{(\alpha, \beta)}_\psi(f)(x, \xi) \right|^2 d(A_{\alpha, \beta} \otimes \sigma_{\gamma, \delta})(x, \xi)
$$

$$= \frac{1}{\|f\|^2_{L^2(\mathbb{R}, A_{\alpha, \beta})} \|g\|^2_{L^2(\mathbb{R}, A_{\alpha, \beta})}} E_k \left( \left| \mathcal{W}^{(\alpha, \beta)}_g(f) \right|^2 \right)
$$

$$+ 2 \ln \left( \|f\|_{L^2(\mathbb{R}, A_{\alpha, \beta})} \|g\|_{L^2(\mathbb{R}, A_{\alpha, \beta})} \right).$$

Finally, from (4.9), we obtain

$$E_k \left( \left| \mathcal{W}^{(\alpha, \beta)}_g(f) \right|^2 \right) \geq -2 \ln \left( \|f\|_{L^2(\mathbb{R}, A_{\alpha, \beta})} \|g\|_{L^2(\mathbb{R}, A_{\alpha, \beta})} \right) \|f\|^2_{L^2(\mathbb{R}, A_{\alpha, \beta})} \|g\|^2_{L^2(\mathbb{R}, A_{\alpha, \beta})}.$$ 

This completes the proof of the theorem.

4.7 | Application in signal processing

Here, we present an application of these uncertainty principles in compressive sensing. Mainly, we show that uncertainty principles can be used for the separation of signals. The signal separation problem is an extremely ill-defined signal processing problem, which is also important in many engineering problems. It consists in splitting a signal $f$ into a sum
of components $f_k$ of different nature: $f = f_1 + f_2 + \cdots + f_n$. Since this notion of different nature often makes sense in applied domains, it is generally extremely difficult to formalize mathematically. Sparsity offers a convenient framework for approaching such a notion. More precisely, signals of different natures can be sparsely represented in different wave-form systems. Given a union of several frames $U^{(1)}, U^{(2)}, \ldots, U^{(n)}$ in a Hilbert space $\mathbb{H}$, the separation problem can be given various formulations, among which the analysis and synthesis formulations (see Ricaud and Torrsani [5]).

Let $U^{(k)}$ denotes the analysis operator of frame $k$. In the case of $n$ frames, applying these uncertainty principles and using the similar method as in Ricaud and Torrsani [5], it can be proven that if one frame gives a splitting $f = f_1 + f_2 + \cdots + f_n$, obtained via any algorithm, if $\|U^{(1)}f_1\| + \|U^{(2)}f_2\| + \cdots + \|U^{(n)}f_n\|$ is small enough, then this splitting is necessarily optimal. More precisely, we have the following result. Let $U^{(1)}, U^{(2)}, \ldots, U^{(n)}$ denote $n$ frames in $\mathbb{H}$. For any $f \in \mathbb{H}$, let $f = f_1 + f_2 + \cdots + f_n$ denote a splitting such that

$$\|U^{(1)}f_1\| + \|U^{(2)}f_2\| + \cdots + \|U^{(n)}f_n\| < \frac{1}{\mu_*},$$

where $\mu_*$ is the generalized coherence function (see Ricaud and Torrsani [5]). Then, using these uncertainty principles for the windowed Opdam–Cherednik transform, we obtain that this splitting minimizes $\|U^{(1)}f_1\| + \|U^{(2)}f_2\| + \cdots + \|U^{(n)}f_n\|$. Similarly, using these uncertainty principles one can study sparsity-based algorithms for window optimization in time-frequency analysis. Many other applications can be given using uncertainty principles for the windowed Opdam–Cherednik transform.

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CONFLICT OF INTEREST STATEMENT

No potential conflict of interest was reported by the authors.

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