Large time behavior for a compressible two-fluid model with algebraic pressure closure and large initial data

Yang Li\textsuperscript{1}, Yongzhong Sun\textsuperscript{2} and Ewelina Zatorska\textsuperscript{3,4}

\textsuperscript{1} School of Mathematical Sciences, Anhui University, Hefei 230601, People’s Republic of China
\textsuperscript{2} Department of Mathematics, Nanjing University, Nanjing 210093, People’s Republic of China
\textsuperscript{3} Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, United Kingdom

E-mail: lynjum@163.com, sunyz@nju.edu.cn and e.zatorska@ucl.ac.uk

Received 15 December 2019
Accepted for publication 16 March 2020
Published 2 July 2020

Abstract

In this paper, we consider a compressible two-fluid system with a common velocity field and algebraic pressure closure in dimension one. Existence, uniqueness and stability of global weak solutions to this system are obtained with arbitrarily large initial data. Making use of the uniform-in-time bounds for the densities from above and below, exponential decay of weak solution to the unique steady state is obtained without any smallness restriction to the size of the initial data. In particular, our results show that degeneration to single-fluid motion will not occur as long as in the initial distribution both components are present at every point.

Keywords: compressible two-fluid model, weak solutions, large time behavior

Mathematics Subject Classification numbers: 76T10, 35D30, 35B40, 74G25.

1. Introduction

In multi-component flows the presence of topologically complex interphase separating the components is a great difficulty from physical as well as mathematical point of view. However, in most of engineering applications precise description of motion of each component of interphase are not rarely needed and only the averaged macroscopic description is important. We will focus on the averaged two-component model derived in the monograph of Ishii and Hibiki in its inviscid form \cite{Ishii}. We refer the interested reader to \cite{Danchin} for concise overview of...
various modelling and mathematical aspects related to such models. In the present paper we immediately assume that the two components of the flow share a common velocity field and that their pressures are equal (algebraic pressure closure). We obtain the following one-dimensional system of partial differential equations:

\[ \partial_t(\alpha \pm \varrho \pm u) + \partial_x(\alpha \pm \varrho \pm u) = 0, \]  
\[ \partial_t[(\alpha + \varrho + \alpha - \varrho)u] + \partial_x[(\alpha + \varrho + \alpha - \varrho)u^2] + \partial_x p = \mu \partial_{xx} u, \]
\[ \alpha + \alpha = 1, \quad \alpha \geq 0, \]
\[ p = p_+ = p_-. \]

with the space variable \( x \in \Omega = (0, 1) \) and the time variable \( t \in (0, T) \). Here, \( \alpha_+ \) and \( \alpha_- \) are the volumetric rates of the two fluids; \( \varrho_+ \) and \( \varrho_- \) are the two mass densities; \( u \) is the common velocity field, and \( \mu > 0 \) is the viscosity coefficient. The two internal pressures are given by

\[ p_+ = \varrho_+^{\gamma_+}, \quad p_- = \varrho_-^{\gamma_-}, \]

for adiabatic exponents \( \gamma_+ > 1 \). The algebraic closure (1.4) is only one of possible choices, but it seems to be well accepted in the physics community. For more thorough discussion on this and other choices we refer to [1, 2].

We restrict ourselves to the case of Dirichlet boundary conditions for the velocity:

\[ u|_{x=0,1} = 0. \]

Following [3], we introduce the notation

\[ R = \alpha_+ \varrho_+, \quad Q = \alpha_- \varrho_-, \quad Z = \varrho_+, \]

and reformulate (1.1)–(1.5) to

\[ \partial_t R + \partial_x(Ru) = 0, \]
\[ \partial_t Q + \partial_x(Qu) = 0, \]
\[ \partial_t [(R + Q)u] + \partial_x[(R + Q)u^2] + \partial_x Z^{\gamma} = \mu \partial_{xx} u, \]

again supplemented with the Dirichlet boundary conditions (1.6) for the velocity and the initial conditions:

\[ (R, Q, u)|_{t=0} = (R_0, Q_0, u_0). \]

Without loss of generality, we may assume that

\[ \int_\Omega (R_0 + Q_0) d\xi = 1. \]

Due to the algebraic closure (1.4), \( Z \) is an implicit function of \( R \) and \( Q \) interrelated by

\[ Q = \left(1 - \frac{R}{Z}\right) Z^\gamma, \quad \gamma := \frac{\gamma_+}{\gamma_-}. \]
The same model, but in semi-stationary Stokes regime, has been recently investigated by Bresch, Mucha and the third author in the three-dimensional setting. They proved the global-in-time existence of weak solutions without any restriction on the initial data. Similar result for the general Navier–Stokes system, with generalized equation of state was later obtained by Novotný and Pokorný [4]. Earlier results in this spirit concern existence of weak solutions to very particular two-component models including the fluid model of atmospheric flow with transport of potential temperature [5], and the hydrodynamic limit of Vlasov–Fokker–Planck system modelling suspension of the particles in the compressible fluid [6]. For other related results in case of one-dimensional two-fluid models, including density dependent coefficients or the so-called drift-flux model, we refer to [7], to the works of Evje et al [8–12] and to the recent overview paper [13].

The present paper is, as far as we know, the first attempt to provide some more information about quantitative properties of weak solutions to this system. In order to investigate the large time behavior of solutions, we furthermore rewrite (1.8)–(1.10) in Lagrangian coordinates. To do this, we make the change of variables

\[ y := \int_0^x (R + Q)(\xi, t) d\xi, \quad s := t. \]

Observing that \( y \) is nondecreasing with respect to \( x \) and, in light of (1.8), (1.9) and (1.11),

\[ \int_{\Omega} (R + Q)(\xi, t) d\xi = \int_{\Omega} (R_0 + Q_0)(\xi) d\xi = 1, \]

we find that \( y \in [0, 1] \) when \( x \in [0, 1] \). Based on this property, we deduce from (1.6) that the velocity obeys the Dirichlet boundary conditions in the Lagrangian coordinates as well.

As a consequence,

\[ \partial_t \tau = \partial_y u, \]

\[ \partial_y (Q \tau) = 0, \]

\[ \partial_y u = \partial_y \left( \mu \frac{\partial_y u}{\tau} - Z(R, Q)^{\gamma+} \right), \]

where \( y \in \Omega, t \in (0, T) \) and

\[ \tau := \frac{1}{R + Q}. \]

It follows readily that \( R \) satisfies

\[ \partial_y (R \tau) = 0. \]

By introducing the two time-independent quantities

\[ c_+(y) := \frac{R_0}{R_0 + Q_0}, \quad c_-(y) := \frac{Q_0}{R_0 + Q_0}, \]
which are called concentrations of the components, we realize then the pressure takes the form

\[ p = p(y, \tau) = Z^+ \left( \frac{c_+(y)}{\tau}, \frac{c_-(y)}{\tau} \right). \]

Hence, we may reformulate (1.14)–(1.16) in a more convenient form

\[
\begin{align*}
\partial_t \tau &= \partial_y u, \\
\partial_t u &= \partial_{yy} \left[ \frac{\partial_y u}{\tau} - Z^+ \left( \frac{c_+(y)}{\tau}, \frac{c_-(y)}{\tau} \right) \right].
\end{align*}
\]

(1.17)

The equation (1.17) are reminiscent of 1D viscous, barotropic, compressible Navier–Stokes system with pressure depending on both the specific volume and the Lagrangian mass coordinate. In [14] Zlotnik showed the existence, uniqueness and Lipschitz continuous dependence on the initial data of weak solutions to 1D barotropic (or heat-conductive) flow in Lagrangian mass coordinate; regularity of weak solutions was also considered therein. We refer to [15] for asymptotic behavior of weak (or strong) solutions to the two-scale equations of a viscous compressible barotropic medium. The main ideas, in particular the asymptotic analysis, of the aforementioned papers play a crucial role in our model system.

The equation (1.17) are supplemented with the initial and boundary conditions as follows:

\[
(\tau, u)|_{t=0} = (\tau_0, u_0),
\]

(1.18)

\[
u |_{y=0,1} = 0,
\]

(1.19)

and we denote

\[
\tau_0 := \frac{1}{R_0 + Q_0}.
\]

(1.20)

This paper is mainly devoted to the large time behavior of weak solutions to (1.17)–(1.19) with large initial data. Existence, uniqueness and stability of weak solutions are obtained by making full use of the specific structure of the equations. Unlike in the three-dimensional regime [3, 4], we prove the existence of weak solutions by approximation based on the strong solutions. Then the stability of weak solutions is verified by adapting the arguments for single-fluid equations [16, 17]. The key step in the asymptotic analysis is to show uniform-in-time bounds on the densities from above and below. Due to the complicated form of the pressure, classical methods used in [18–22] cannot be applied here. However, thanks to the structure of the pressure, we are able to adapt the argument from [23] so as to obtain the two-sided bounds; see lemma 4.1. Based on these bounds, we show the exponential decay of weak solution by choosing suitable test functions in the momentum equation and making another use of the structure of the pressure.

**Notation.** We denote \( \Omega_t := \Omega \times (0, t); L^p \) stands for the Lebesgue space \( L^p(\Omega) \) with the norm \( \| \cdot \|_{L^p} \); \( H^1 \) denotes the Sobolev space \( W^{1,2}(\Omega) \). \( H^1_0 \) is the subspace of \( H^1 \) with vanishing trace. Further we denote

\[
L^\infty(0, T; L^2) := \left\{ f : (0, T) \to L^2 \text{ strongly measurable : } \sup_{t \in (0, T)} \| f(\cdot, t) \|_{L^2} < \infty \right\}.
\]

For \( f = f(y, t) \), we define

\[
\| f \|_{V_2(G_T)} := \| f \|_{L^\infty(0,T,L^2)} + \| \partial_y f \|_{L^2(G_T)}.
\]
Before stating our main results, we specify the meaning of weak solutions.

**Definition 1.1.** Let $R_0, Q_0, u_0$ satisfy
\begin{align}
0 < R_0 &\leq R_0 \leq R_0 < \infty, \\
0 < Q_0 &\leq Q_0 \leq Q_0 < \infty, \\
u_0 &\in L^2.
\end{align}

A triple $(R, Q, u)$ is said to be a weak solution of (1.17)–(1.19) on $\Omega_T$ provided that

- $(R, Q, u)$ belongs to the spaces
  \[0 < R(y, t), Q(y, t), \text{ a.e. in } \Omega_T,\]
  \[(\frac{R}{Q}, 1) \in L^\infty(\Omega_T), \quad (\partial_r R, \partial_r Q) \in L^2(\Omega_T), \quad u \in V_2(\Omega_T),\]

- The initial-boundary conditions
  \[(R, Q, u)|_{t=0} = (R_0, Q_0, u_0), \quad u|_{y=0,1} = 0,\]
  are satisfied in the sense of traces,

- The equation
  \[\partial_t \tau = \partial_t u,\]
  makes sense in $L^2(\Omega_T)$ and the equalities
  \[R = \frac{c_+(y)}{\tau}, \quad Q = \frac{c_-(y)}{\tau}, \quad \tau = \frac{1}{R + Q},\]
  hold true a.e. in $\Omega_T$,

- The momentum equation is understood in the sense of distributions, i.e.,
  \[\int_{\Omega_T} \left\{ u \partial_t \phi - \left[ \mu \frac{\partial u}{\tau} - Z^+ \left( \frac{c_+(y)}{\tau}, \frac{c_-(y)}{\tau} \right) \partial_y \phi \right] \right\} dydx = 0,\]
  for any $\phi \in C_0^\infty(\Omega_T)$.

**Remark 1.1.** Given $(R_0, Q_0)$ in (1.21) and (1.22), we tacitly assume that $Z_0$ satisfies
\[\begin{cases}
Q_0 = \left( 1 - \frac{R_0}{Z_0} \right) Z_0^+, \\
R_0 \leq Z_0.
\end{cases}\]  

Clearly, the positive lower bound of $Z_0$ follows from (1.24) and moreover $R_0 < Z_0$ a.e. in $\Omega$ in accordance with (1.24). To get the upper bound of $Z_0$ we again make use of (1.24). Indeed, suppose on the contrary that $Z_0 > \max \left\{ 2R_0, (2Q_0)^{1/\gamma} \right\}$, then we would have
\[\overline{Q_0} > Q_0 = \left( 1 - \frac{R_0}{Z_0} \right) Z_0^+ \geq \frac{1}{2} Z_0^+ > \overline{Q_0},\]
which is a contradiction. Therefore $Z_0$ must be bounded from above, more precisely

$$Z_0 \leq \max \left\{ 2R_0, (2Q_0)^{1/\gamma} \right\}.$$

The main results of this paper are the following two theorems. The first one is concerned with Lipschitz continuous dependence on the initial data of weak solutions.

**Theorem 1.1.** Let $\gamma_\pm > 1$ and (1.21)–(1.23) be satisfied. Then there exists a unique global-in-time weak solution to (1.17)–(1.19). Moreover, if $(R, Q, u)$ and $(\tilde{R}, \tilde{Q}, \tilde{u})$ are two weak solutions on $\Omega_T$ corresponding to the initial data $(R_0, Q_0, u_0)$ and $(\tilde{R}_0, \tilde{Q}_0, \tilde{u}_0)$, respectively, then

$$\left( \| R - \tilde{R} \|_{L^\infty(\Omega_T)} + \| Q - \tilde{Q} \|_{L^\infty(\Omega_T)} + \| u - \tilde{u} \|_{V^2(\Omega_T)} \right) \leq C \left( \| R_0 - \tilde{R}_0 \|_{L^\infty} + \| Q_0 - \tilde{Q}_0 \|_{L^\infty} + \| u_0 - \tilde{u}_0 \|_{L^2} \right),$$

(1.25)

where $C$ is a positive constant depending on the lower and upper bounds of $(R_0, Q_0, \tilde{R}_0, \tilde{Q}_0)$, the $L^2$-norms of $(u_0, \tilde{u}_0)$, $\mu$, $\gamma_\pm$, and $T$.

The second theorem gives the large time behavior of weak solutions. More precisely, we show the asymptotic decay of weak solutions to $(R_\infty, Q_\infty, u_\infty)$—the unique steady state for problem (1.17)–(1.19) given implicitly by

$$\begin{align*}
Q_\infty &= c_+ \tau_\infty^{-1}, \quad R_\infty = c_+ \tau_\infty^{-1},
\tau_\infty &= (R_\infty + Q_\infty)^{-1},
u_\infty = 0, \quad Z_\infty^+ = C_*,
Q_\infty &= \left( 1 - \frac{R_\infty}{Z_\infty} \right) Z_\infty^-, \quad R_\infty \leq Z_\infty,
\int_\Omega \tau_\infty \, dy &= \int_\Omega \tilde{\tau}_0 \, dy.
\end{align*}$$

(1.26)

Here, $C_*$ is the positive constant uniquely determined by $R_0, Q_0, \gamma_\pm$ and the conservation of mass (1.26).

**Theorem 1.2.** Let $(R, Q, u)$ be the unique weak solution to (1.17)–(1.19) provided by theorem 1.1. Then, for any $t \geq 0$, it holds

$$\left\| (R - R_\infty, Q - Q_\infty, u - u_\infty) \right\|_{L^2} \leq C_1 \exp(-C_2 t).$$

(1.27)

Here, $C_1$ and $C_2$ are positive constants depending on the initial data, $\mu, \gamma_\pm$, but independent of time.

**Remark 1.2.** Given suitably regular initial data, i.e.,

$$(R_0, Q_0) \in L^\infty, \quad (\partial_t R_0, \partial_t Q_0) \in L^\infty, \quad u_0 \in H^1_0,$$

$$0 < R_0(y), \quad Q_0(y), \quad \text{for any } y \in \Omega,$$

it can be shown, adapting the arguments from [23, 24], that

$$\left\| (R - R_\infty, Q - Q_\infty, u - u_\infty) \right\|_{H^1} \leq C_1 \exp(-C_2 t).$$
Remark 1.3. Theorems 1.1 and 1.2 are obtained based on the hypothesis that the two components of flows share the same velocity field. However, it is not straightforward how to generalise these results to system with two different velocities. First of all, it is not clear how to transform such system from Eulerian to Lagrangian coordinates. Secondly, the pressure term in the two-velocity system would not be in the conservative part, causing that the energy estimates and all the good properties of the pressure would be much harder to obtain.

The rest of this paper is structured as follows. In section 2.1 we show global existence and uniqueness of strong solutions to (1.17)–(1.19). In section 2.2 we prove the existence of global weak solutions via approximation based on regular solutions corresponding to regularized initial data and the weak convergence method. In section 3 we verify the stability of weak solutions. In section 4, we obtain the exponential decay of weak solution to the unique steady state in $L^2$-norm with large initial data.

2. Global existence of weak solutions

2.1. Global well-posedness to (1.17)–(1.19)

In this subsection, we prove global existence and uniqueness of strong solutions to (1.17)–(1.19) with large data. This will be useful in construction of weak solutions. Throughout this section, we denote by $C$ a generic positive constant depending on the initial data, $\mu$, $\gamma_\pm$ and $T$.

Proposition 2.1. Let (1.21) and (1.22) be satisfied. Assume that

$(R_0, Q_0) \in H^1, \quad u_0 \in H^1_0$.

Then there exists a unique global strong solution $(R, Q, u)$ to (1.17)–(1.19). Furthermore, for any $0 < T < \infty$, it holds that

$C^{-1} \leq R(y, t), Q(y, t), Z(y, t) \leq C$, for any $(y, t) \in \Omega_T$, (2.2)

$\|u\|_{L^\infty(0,T;L^2)} + \|\partial_y u\|_{L^2(\Omega_T)} \leq C$, (2.3)

$\|\partial_y R\|_{L^2(\Omega_T)} + \|\partial_y Q\|_{L^2(\Omega_T)} \leq C$. (2.4)

Local-in-time existence and uniqueness of strong solutions to (1.17)–(1.19) is proved by the classical method based on the linearization of the problem and Banach fixed point theorem. We refer to [25, 26] for similar calculations. Therefore, it remains to derive sufficient global a priori estimates so as to extend the local solution globally.

We start by giving the conservation of mass and the elementary energy inequality. To simplify the expression, we define

$\alpha := \frac{R}{Z}$. (2.5)

Lemma 2.1. Let $(R, Q, u)$ be a smooth solution to (1.17)–(1.19) on $\Omega_T$ with regular initial data (2.1), then we have

$\int_\Omega \tau(y, t) \, dy = \int_\Omega \tau_0(y) \, dy$, for any $t \in [0, T]$. (2.6)
\[
\sup_{0 \leq t \leq T} \int \left( \frac{1}{2} u^2 + \frac{(c_+)^{\gamma+}}{\gamma+ - 1} (\alpha \tau)^{\gamma+ + 1} + \frac{(c_-)^{\gamma-}}{\gamma- - 1} [(1 - \alpha) \tau]^{\gamma- + 1} \right) dy + \mu \int_{\Omega_T} (\partial_t u)^2 \, dy ds \leq C.
\] (2.7)

**Proof.** (2.6) follows from (1.17), immediately after integration over \( \Omega \). To show (2.7), we adopt the Lagrangian coordinates and the technique of pressure decomposition in [3]. One deduces from (1.4), (1.7), and (2.5) that
\[
\left( \frac{R}{\alpha} \right)^{\gamma^+} = \left( \frac{Q}{1 - \alpha} \right)^{\gamma^-}.
\] (2.8)

Thus we can decompose the pressure as
\[
Z^{\gamma^+} = \alpha \left( \frac{R}{\alpha} \right)^{\gamma^+} + (1 - \alpha) \left( \frac{Q}{1 - \alpha} \right)^{\gamma^-}.
\] (2.9)

Multiplying (1.17) by \( u \) and integrating by parts yields
\[
0 = \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dy - \int_{\Omega} Z^{\gamma^+} \partial_t u \, dy + \mu \int_{\Omega} \frac{(\partial_t u)^2}{\tau} \, dy
\]
\[
= \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dy - \int_{\Omega} Z^{\gamma^+} \partial_t \tau \, dy + \mu \int_{\Omega} \frac{(\partial_t u)^2}{\tau} \, dy.
\] (2.10)

The second term on the right-hand side of (2.10) can be computed, with the help of (2.9), through
\[
- \int_{\Omega} Z^{\gamma^+} \partial_t \tau \, dy = \int_{\Omega} \alpha \left( \frac{R}{\alpha} \right)^{\gamma^+} \partial_t \tau \, dy - \int_{\Omega} (1 - \alpha) \left( \frac{Q}{1 - \alpha} \right)^{\gamma^-} \partial_t \tau \, dy
\]
\[
= - \int_{\Omega} \left( \frac{c_+}{\alpha \tau} \right)^{\gamma^+} \alpha \partial_t \tau \, dy - \int_{\Omega} \left( \frac{c_-}{(1 - \alpha) \tau} \right)^{\gamma^-} (1 - \alpha) \partial_t \tau \, dy
\]
\[
= - \int_{\Omega} \left( \frac{c_+}{\alpha \tau} \right)^{\gamma^+} \partial_t (\alpha \tau) \, dy + \int_{\Omega} \left( \frac{c_+}{\alpha \tau} \right)^{\gamma^+} \tau \, \partial_t \alpha \, dy
\]
\[
- \int_{\Omega} \left( \frac{c_-}{(1 - \alpha) \tau} \right)^{\gamma^-} \partial_t [(1 - \alpha) \tau] \, dy + \int_{\Omega} \left( \frac{c_-}{(1 - \alpha) \tau} \right)^{\gamma^-} \tau \, \partial_t [(1 - \alpha) \tau] \, dy
\]
\[
= - \frac{d}{dt} \int_{\Omega} \left( \frac{c_+}{\alpha \tau} \right)^{\gamma^+} \partial_t (\alpha \tau) \, dy + \int_{\Omega} \left( \frac{c_-}{(1 - \alpha) \tau} \right)^{\gamma^-} \partial_t [(1 - \alpha) \tau] \, dy
\]
\[
= \frac{d}{dt} \int_{\Omega} (c_+)^{\gamma^+} (\alpha \tau)^{\gamma^+ + 1} \, dy + \frac{d}{dt} \int_{\Omega} (c_-)^{\gamma^-} [(1 - \alpha) \tau]^{\gamma^- + 1} \, dy.
\] (2.11)

where we have used (2.8) in the fourth equality. Thus, combining (2.10) and (2.11) gives rise to (2.7). This finishes the proof of lemma 2.1.

By virtue of lemma 2.1 and the specific mathematical structure of the equations, we are able to show the upper and lower bounds for \( R \) and \( Q \). This plays a crucial role in the proof of proposition 2.1.
Lemma 2.2. Let the assumptions of lemma 2.1 be satisfied. Then
\[ C^{-1} \leq R(y, t), Q(y, t), Z(y, t) \leq C, \quad \text{for any } (y, t) \in \overline{\Omega_T}. \] (2.12)

Proof. We observe first that the positiveness of \( R \) and \( Q \) follow from the method of characteristics and the regular initial data. Given positive \( R \) and \( Q \), there exists a unique \( Z \) satisfying (1.12) and (1.13). This fact can be justified easily and we refer to lemma 2.1 in [3] for the details. That is, \( Z \) can be regarded as a function of \( R \) and \( Q \). Furthermore, due to \( R = \frac{\xi}{\tau} \) and \( Q = \frac{\eta}{\tau} \), \( Z \) can be also seen as a function of \( y \) and \( \tau \), i.e., \( Z = Z \left( \frac{c(y)}{\tau}, \frac{c(0)}{\tau} \right) \).

Rewriting (1.12) as
\[ c_+ \tau^{-1} = \left( 1 - \frac{c_+ \tau^{-1}}{Z} \right) Z. \] (2.13)
Differentiating both sides of (2.13) with respect to \( \tau \) leads to
\[ -c_- \tau^{-2} = \gamma Z^{-1} \partial_{\tau} Z - (c_+ \tau^{-1} (\gamma - 1) Z^{-2} \partial_{\tau} Z - c_+ \tau^{-2} Z^{\gamma - 1}), \]
or equivalently,
\[ \partial_{\tau} Z = -\frac{c_- \tau^{-2} + c_+ \tau^{-2} Z^{\gamma - 1}}{\gamma Z^{-1} - c_+ \tau^{-1} (\gamma - 1) Z^{-2}}. \] (2.14)
The denominator is positive as we have
\[ \gamma Z^{-1} - c_+ \tau^{-1} (\gamma - 1) Z^{-2} = Z^{-2} [\gamma Z - (\gamma - 1) R] \]
\[ = Z^{-2} [\gamma (Z - R) + R] \geq Z^{-2} R > 0, \] (2.15)
due to (1.13), assumption about the smoothness of the solution, and remark 1.1. Therefore,
\[ \partial_{\tau} (Z^+) = \gamma Z^{\gamma - 1} \partial_{\tau} Z < 0, \] (2.16)
which means that the pressure is decreasing with respect to \( \tau \). Based on this crucial observation, the two-sided bounds of \( \tau \) are proved by the same method as Zlotnik [14] (see also Antontsev et al [27]). The details are omitted here. The two-sided bounds of \( Z \) follows in the same manner as remark 1.1.

With lemma 2.2 at hand, high-order energy estimates are obtained in a routine manner. Thus, the local-in-time solution can be extended globally. Uniqueness of solutions is proved by the classical energy method. This finishes the proof of proposition 2.1.

2.2. Existence of weak solutions

The main task of this subsection is to construct global-in-time weak solutions to (1.17)–(1.19) using approximation based on regular solutions. We start from regularizing the initial data \((R_0, Q_0, u_0)\) in such a way that \( \{(R_0^\varepsilon, Q_0^\varepsilon, u_0^\varepsilon)\}_{\varepsilon > 0} \) satisfy
\[ (R_0^\varepsilon, Q_0^\varepsilon) \in C^2(\overline{\Omega}), \quad C^{-1} \leq R_0^\varepsilon, Q_0^\varepsilon \leq C, \quad u_0^\varepsilon \in C^2(\Omega), \]
\[ (R_0^\varepsilon, Q_0^\varepsilon, u_0^\varepsilon) \to (R_0, Q_0, u_0) \text{ strongly in } L^2 \text{ as } \varepsilon \to 0. \]
Moreover, we define
\[ \tau_0^\varepsilon := (R_0^\varepsilon + Q_0^\varepsilon)^{-1}, \quad \tau_\varepsilon := (R_\varepsilon + Q_\varepsilon)^{-1}. \]
Therefore, it follows from proposition 2.1 that there exists a unique global strong solution \((R, Q, u_c)\) to (1.17)--(1.19) with initial data \((R_0, Q_0, u_0)\). Furthermore, from proposition 2.1 we conclude the following uniform-in-\(\varepsilon\) estimates:
\[
C^{-1} \leq R(\gamma, t), Q(\gamma, t) \leq C, \quad \text{for any } (\gamma, t) \in \Omega_T, \tag{2.17}
\]
\[
\|u_c\|_{L^\infty(0; T; L^1)} + \|\partial_t u_c\|_{L^2(\Omega_T)} \leq C, \tag{2.18}
\]
\[
\|\partial_t R\|_{L^2(\Omega_T)} + \|\partial_t Q\|_{L^2(\Omega_T)} \leq C. \tag{2.19}
\]
From (2.17)--(2.19) it follows that there exists a subsequence of \(\{(R, Q, u_c)\}_{\varepsilon > 0}\), not relabeling, such that as \(\varepsilon \to 0\),
\[
(R, Q) \to (R, Q) \text{ weakly } - * \text{ in } L^\infty(\Omega_T), \tag{2.20}
\]
\[
u_c \to u \text{ weakly } - * \text{ in } L^\infty(0; T; L^2), \tag{2.21}
\]
\[
(\partial_t R, \partial_t Q, \partial_t u) \to (\partial_t R, \partial_t Q, \partial_t u) \text{ weakly in } L^2(\Omega_T), \tag{2.22}
\]
for some limit triple \((R, Q, u)\), and we infer from (2.17)--(2.22) that
\[
C^{-1} \leq R(\gamma, t), Q(\gamma, t) \leq C, \quad \text{for a.e. } (\gamma, t) \in \Omega_T, \tag{2.23}
\]
\[
\|u\|_{L^\infty(0; T; L^1)} + \|\partial_t u\|_{L^2(\Omega_T)} \leq C, \tag{2.24}
\]
\[
\|\partial_t R\|_{L^2(\Omega_T)} + \|\partial_t Q\|_{L^2(\Omega_T)} \leq C. \tag{2.25}
\]

The weak convergence results (2.20)--(2.22) are not sufficient to pass to the limit in (1.17)--(1.19), in particular, in the strongly nonlinear pressure function. For the moment we only know that \(Z_c\) is the unique solution of
\[
Q_c = \left(1 - \frac{R_c}{Z_c}\right) \cdot Z_c, \quad R_c \leq Z_c.
\]
To identify the pressure term, it suffices to verify that the pointwise limit of \(\{Z_c\}_{\varepsilon > 0}\) is the unique solution of
\[
Q = \left(1 - \frac{R}{Z}\right) \cdot Z, \quad R \leq Z,
\]
for which we need the strong convergence of the sequence \(\{Z_c\}_{\varepsilon > 0}\). In fact, since \(Q_c = Q_0\tau_0^{-1} Z_c^{-1}, R_c = R_0\tau_0^{-1} Z_c^{-1}\), \(Z_c\) can also be regarded as \(Z_c = Z_s(Q_0, R_0, \tau_0, \tau_c)\). Therefore the strong convergence of \(\{Z_c\}_{\varepsilon > 0}\) will follow from that of \(\{\tau_c\}_{\varepsilon > 0}\). The necessary compactness property in space is provided by the following lemma.

**Lemma 2.3.** For any \(0 < h < 1\), there holds
\[
\|\Delta_h \tau|\|_{L^\infty(0; T; L^2)} \leq C(\|\Delta_h R_0\|_{L^2(\Omega_T)} + r\|\Delta_h Q_0\|_{L^2(\Omega_T)} + h), \tag{2.26}
\]
where \(\Delta_h F(y) := F(y + h) - F(y)\) is the translation in spatial variable with the step \(h\).

**Proof.** By setting
\[
\sigma_c := \mu \frac{\partial_t u_c}{\tau_c} = Z_c^{\varepsilon},
\]
it follows that
\[\tau_c = D_c \left( \tau_0 + \frac{1}{\mu} \int_0^t D_z^{-1}(y, s) \left( \tau_c Z_c^{-1} \right)(y, s) ds \right),\] (2.27)

where
\[D_c(y, t) := \exp \left( \frac{1}{\mu} \int_0^t \sigma_c(y, s) ds \right).\]

By definition it holds that
\[\Delta_h \tau_c(y, t) = \Delta_h D_c(y, t) \left( \tau_0(y + h) + \frac{1}{\mu} \int_0^t D_z^{-1} \left( \tau_c Z_c^{-1} \right)(y + h, s) ds \right)
+ D_c(y, t) \left( \Delta_h \tau_0(y) + \frac{1}{\mu} \int_0^t D_z^{-1} \left( \tau_c Z_c^{-1} \right)(y + h, s) \Delta_h \left( \tau_c Z_c^{-1} \right)(y, s) ds \right)
- D_c(y, t) \left( \frac{1}{\mu} \int_0^t D_z^{-1} \left( \tau_c Z_c^{-1} \right)(y, s) \Delta_h D_c(y, s) ds \right).\] (2.28)

Thanks to (2.17), we have
\[C^{-1} \leq Z_c(y, t), \quad D_c(y, t) \leq C, \quad \text{for any } (y, t) \in \Omega_T.\] (2.29)

The delicate issue is to compute \(\Delta_h \left( \tau_c Z_c^{-1} \right).\) In fact,
\[\Delta_h \left( \tau_c Z_c^{-1} \right) = Z_c^{-1} \Delta_h \tau_c + \tau_c(y + h, t) \Delta_h Z_c^{-1}.\] (2.30)

Furthermore,
\[\Delta_h Z_c = (\partial_{\Omega_0} Z_c) \Delta_h Q_0^{-1} + (\partial_{\rho_0} Z_c) \Delta_h R_0^{-1} + (\partial_{\tau_0} Z_c) \Delta_h \tau_0^{-1} + (\partial_{\tau_1} Z_c) \Delta_h \tau_1.\]

Subsequent differentiations of (2.13) with respect to \(Q_0^{-1}, R_0^{-1}, \tau_0^{-1},\) and \(\tau_1\) give rise to
\[\partial_{\Omega_0} Z_c = \frac{\tau_0^{-1}\tau_1^{-1}}{\gamma Z_c^{-1} - R_0 \tau_0^{-1}\tau_1^{-1}(\gamma - 1)Z_c^{-1}Z_c^{-1}},\]
\[\partial_{\rho_0} Z_c = \frac{\tau_0^{-1}\tau_1^{-1}Z_c^{-1}}{\gamma Z_c^{-1} - R_0 \tau_0^{-1}\tau_1^{-1}(\gamma - 1)Z_c^{-1}Z_c^{-1}},\]
\[\partial_{\tau_0} Z_c = \frac{Q_0^{-1}\tau_1^{-1} + R_0 \tau_0^{-1}\tau_1^{-1}(\gamma - 1)Z_c^{-1}Z_c^{-1}}{\gamma Z_c^{-1} - R_0 \tau_0^{-1}\tau_1^{-1}(\gamma - 1)Z_c^{-1}Z_c^{-1}},\]
\[\partial_{\tau_1} Z_c = \frac{Q_0^{-1}\tau_1^{-2} + R_0 \tau_0^{-1}\tau_1^{-2}Z_c^{-1}}{\gamma Z_c^{-1} - R_0 \tau_0^{-1}\tau_1^{-2}(\gamma - 1)Z_c^{-1}Z_c^{-1}}.\]

In view of (2.15), (2.17) and (2.29), it follows that
\[\left\| \left( \partial_{\Omega_0} Z_c, \partial_{\rho_0} Z_c, \partial_{\tau_0} Z_c, \partial_{\tau_1} Z_c \right) \right\|_{L^\infty(\Omega_T)} \leq C.\] (2.31)

Consequently, we conclude from (2.17) and (2.18) and (2.28)–(2.31) that
Recalling that $\|\Delta_h \tau \|_{L^2} \leq C (\|\Delta_h D_t(y, t)\|_{L^2} + \|\Delta_h \tau_0\|_{L^2} + \int_0^t (\|\Delta_h D_s(y, s)\|_{L^2} + \|\Delta_h \tau_s\|_{L^2} + \|\Delta_h Q_s\|_{L^2} + \|\Delta_h R_s\|_{L^2}) ds) \leq C (h \|u - u_0\|_{L^\infty(0,T;L^2)} + \|\Delta_h Q_0\|_{L^2} + \|\Delta_h R_0\|_{L^2} + \int_0^t \|\Delta_h \tau_s\|_{L^2} ds) \leq C \left( h + \|\Delta_h Q_0\|_{L^2} + \|\Delta_h R_0\|_{L^2} + \int_0^t \|\Delta_h \tau_s\|_{L^2} ds \right)$. This particularly implies the strong convergence of $\{\tau_s\} \to \tau$ in $L^2(\Omega_T)$ and furthermore in $L^p(\Omega_T)$ for any $1 \leq p < \infty$. Consequently, it holds that

$$Q \to c_- \tau^{-1}, \quad R \to c_+ \tau^{-1}, \quad \text{a.e. in } \Omega_T,$$

which yields

$$Q = c_- \tau^{-1}, \quad R = c_+ \tau^{-1}, \quad \text{a.e. in } \Omega_T. \quad (2.33)$$

Recalling that $Z = Z(0, T - \tau, \tau, \tau)$, we find $Z$ converges to some limit function $Z$ almost everywhere. Upon passing to the limit in the relations

$$Q = \left( 1 - \frac{R}{Z} \right) Z, \quad R \leq Z,$$

we conclude from (2.29) that $\{Z\} \to Z$ strongly in $L^p(\Omega_T)$ for any $1 \leq p < \infty$ and $Z$ solves exactly

$$Q = \left( 1 - \frac{R}{Z} \right) Z, \quad R \leq Z. \quad (2.34)$$

This finishes the proof of existence of a weak solution.

### 3. Stability of weak solutions

In the present section, we show Lipschitz continuous dependence on the initial data of weak solutions, i.e., we prove our first main theorem 1.1. We remark that the proof relies on the structure of the equations. As a preliminary step, we state the following lemma, the proof of which is omitted as it is similar to relevant results from [17, 24]. Throughout this section, various positive constants are denoted by the same symbol $C$ depending on the initial data, $\mu, \gamma, h$ and $T$.

**Lemma 3.1.** Let $(R, Q, u)$ be a weak solution to (1.17)–(1.19). Then

$$\tau(y, t) = \exp \left( \frac{1}{\mu} \int_0^t \sigma(y, s) ds \right)$$
Lemma 3.2. Let the assumptions of theorem 1.1 be fulfilled, then we have

\[ \| \Delta \tau \|_{L^{\infty}([0,T])} \leq C \left( \| \Delta R_0 \|_{L^\infty} + \| \Delta Q_0 \|_{L^\infty} + \| \Delta u_0 \|_{L^2} + \| \Delta u \|_{L^\infty} + \| \Delta u \|_{L^2} \right) \]

for any \( t \in (0,T) \).

Proof. It follows from (3.1) that

\[ \Delta \tau = D \left\{ \Delta \tau_0 + \frac{1}{\mu} \int_0^t \exp \left( -\frac{1}{\mu} \int_0^s \sigma(y,s)ds \right) (\tau Z^+) \ (y, \xi) d\xi \right\} \]

and

\[ \int_0^t \sigma(y,s)ds = (I(u - u_0))(y, t) + \int_0^t (\sigma, s)ds, \]

where

\[ \sigma(y,t) := \left( \frac{\partial \mu}{\tau} - Z^+ \right)(y,t), \]

\[ I(f) := \int_0^t f(y)dy - \left\langle \int_0^t f(y)dy \right\rangle, \quad \langle f \rangle := \int_\Omega f(y)dy. \]

To verify the stability estimate (1.25) from theorem 1.1, we follow the arguments in [16, 24]. Let us start from introducing the following notation:

\[
\begin{cases}
(\Delta \tau, \Delta R, \Delta Q, \Delta u) = (\tau - \tilde{\tau}, R - \tilde{R}, Q - \tilde{Q}, u - \tilde{u}), \\
(\Delta \tau_0, \Delta R_0, \Delta Q_0, \Delta u_0) = (\tau_0 - \tilde{\tau}_0, R_0 - \tilde{R}_0, Q_0 - \tilde{Q}_0, u_0 - \tilde{u}_0), \\
\Delta \sigma := \sigma - \tilde{\sigma}, \quad \tilde{\sigma} := \mu \frac{\partial \mu}{\tau} - (\tilde{Z})^+, \\
D := \exp \left( \frac{1}{\mu} \int_0^t \sigma(y,s)ds \right), \quad \tilde{D} := \exp \left( \frac{1}{\mu} \int_0^t \tilde{\sigma}(y,s)ds \right), \\
\tilde{\varrho} := \tilde{R} + \tilde{Q}, \quad \Delta \tilde{\varrho} := \varrho - \tilde{\varrho}.
\end{cases}
\]

Recalling that \( Q = c_+ \tau^{-1}, R = c_+ \tau^{-1} \), one has in light of uniform bounds for \( R, Q \) from below and above, i.e., (2.23), that

\[
\begin{align*}
|\Delta R| &\leq C \left( |\Delta R_0| + |\Delta \tau_0| + |\Delta \tau| \right) \\
|\Delta Q| &\leq C \left( |\Delta Q_0| + |\Delta \tau_0| + |\Delta \tau| \right)
\end{align*}
\]

Consequently, in order to estimate \( L^\infty(\Omega_T) \)-norm of \( \Delta R \) and \( \Delta Q \), it suffices to control \( \| \Delta \tau \|_{L^\infty(\Omega_T)} \). This is the key step in proving stability of weak solutions. We follow the idea in [16, 17] to accomplish this goal.

Lemma 3.2. Let the assumptions of theorem 1.1 be fulfilled, then we have

\[ \| \Delta \tau \|_{L^{\infty}([0,T])} \leq C \left( \| \Delta R_0 \|_{L^\infty} + \| \Delta Q_0 \|_{L^\infty} + \| \Delta u_0 \|_{L^2} + \| \Delta u \|_{L^\infty} + \| \Delta u \|_{L^2} \right) \]

for any \( t \in (0,T) \).
\[ + \left( D - \tilde{D} \right) \left( \tilde{\sigma}_0 + \frac{1}{\mu} \int_0^t \frac{\tilde{\tau}(\tilde{Z})^+}{D} \, ds \right). \quad (3.5) \]

Similarly to (2.29), it holds that
\[ C^{-1} \leq \left( Z, \tilde{Z}, D, \tilde{D} \right) (y, t) \leq C, \quad \text{for a.e. } (y, t) \in \Omega_F. \quad (3.6) \]

Indeed, the upper bound of \( Z \) and \( \tilde{Z} \) is derived by the same argument as (2.29). Based on (2.23) and (3.6), we observe that
\[
\left| \tau Z^+ - \tilde{\tau}(\tilde{Z})^+ \right| \leq C \left( \| \partial_t Z \|_{L^\infty(\Omega_T)} \| \Delta \tau \| + \| \partial_\nu Z \|_{L^\infty(\Omega_T)} \| \Delta Q_0 \| \\
+ \| \partial_\nu Z \|_{L^\infty(\Omega_T)} \| \Delta R_0 \| + \| \partial_\nu Z \|_{L^\infty(\Omega_T)} \| \Delta \tau_0 \| \right)
\leq C \left( \| \Delta \tau \| + \| \Delta R_0 \| + \| \Delta Q_0 \| \right), \quad (3.7)
\]

where we used a version of (2.31) for the limit functions. Therefore, we deduce from (3.5)–(3.7) that
\[
\| \Delta \tau \| \leq C \left( \| \Delta R_0 \| + \| \Delta Q_0 \| + \int_0^t \left( \left\| \int_0^s \Delta \sigma \, ds \right\| + \| \Delta \tau \| \right) \, ds \right) + C \left| \int_0^t \Delta \sigma \, ds \right|;
\]
whence
\[
\| \Delta \tau (\cdot, t) \|_{L^\infty} \leq C \left( \| \Delta R_0 \|_{L^\infty} + \| \Delta Q_0 \|_{L^\infty} + \int_0^t \| \Delta \sigma \|_{L^\infty(\Omega_t)} + \int_0^t \| \Delta \tau (\cdot, s) \|_{L^\infty} \, ds \right). \quad (3.8)
\]

The rest of the proof follows the same lines as [24], and we write down the details only for the convenience of the reader. First, from the identity (3.2) and Hölder’s inequality we obtain
\[
\left\| \int_0^t \Delta \sigma \, ds \right\|_{L^\infty(\Omega_t)} \leq \left( \| \mathcal{I}(\Delta u_0) \|_{L^\infty} + \| \mathcal{I}(\Delta u) \|_{L^\infty(\Omega_t)} + \| \Delta \sigma \|_{L^2(\Omega_t)} \right). \quad (3.9)
\]

It remains to bound \( \Delta \sigma \). Notice that we have
\[
\Delta \sigma = \mu \frac{\partial_t (\Delta u)(\cdot, t)}{\tau} + \mu (\Delta \phi) \partial_\nu \tilde{u} - \left( Z^+ - (\tilde{Z})^+ \right),
\]
and so, as for (3.7) we obtain
\[
\| \Delta \sigma \|_{L^2(\Omega_t)} \leq C \left( \| \partial_t (\Delta u) \|_{L^2(\Omega_t)} + \| \Delta \tau (\| \partial_\nu \tilde{u} \| + 1) \| \right).
\]

It follows that
\[
\| \Delta \sigma \|_{L^2(\Omega_t)} \leq C \left( \| \partial_t (\Delta u) \|_{L^2(\Omega_t)} + \| \Delta R_0 \|_{L^\infty} + \| \Delta Q_0 \|_{L^\infty} \\
+ \int_0^t \left( \| \partial_\nu \tilde{u} \|_{L^2} + 1 \right) \| \Delta \tau (\cdot, s) \|_{L^\infty} \, ds \right). \quad (3.10)
\]

Since from (2.24) we deduce that \( \int_0^T \| \partial_\nu \tilde{u} \|_{L^2} \, ds \leq C \), and therefore we can put together (3.9) and (3.10), and apply Gronwall’s inequality to (3.8) to deduce (3.4). The proof of lemma 3.2 is thus finished. \( \square \)
In order to use (3.4) to conclude (1.25), we need the estimates for \( \Delta u \). In fact, standard energy estimate for parabolic equation [28] gives

**Lemma 3.3.** For any \( t \in (0, T] \), it holds that

\[
\| \Delta u \|_{L^\infty(0, t; L^2)} + \| \partial_y (\Delta u) \|_{L^2(0, t; L^2)} \leq C \left( \| \Delta u_0 \|_{L^2} + \| \Delta R_0 \|_{L^\infty} + \| \Delta Q_0 \|_{L^\infty} + \| \left( \| (\partial_y \tilde{u})(\cdot, s) \|_{L^2} + 1 \right) \| \Delta \tau(\cdot, s) \|_{L^\infty(0, t; L^2)} \right) .
\] (3.11)

Having this, (1.25) follows by suitable combination of lemmas 3.2 and 3.3. For the sake of brevity, we omit the details and refer the reader to [24] for similar steps. Clearly, (1.25) implies the uniqueness of weak solutions and so the proof of theorem 1.1 is complete.

4. Large time behavior of weak solution

In this section, we show the exponential decay of weak solution in \( L^2 \)-norm. The classical methods to handle the large time behavior of the one-dimensional single-phase Navier–Stokes equations [20–22] are not readily applicable to our two-fluid model system. In [23], the author developed a new technique to treat one-dimensional viscous barotropic gas with nonmonotone pressure. Of great importance in [23] is to obtain the uniform-in-time bounds of the density from above and below. It turns out that the idea can be adapted to our two-fluid model. As a matter of fact, it has already been successfully adapted before to the case of one-dimensional nonresistive magnetohydrodynamic equations [24].

4.1. Two-sided bounds for \( R \) and \( Q \)

To begin with, we notice that the estimates in lemma 2.1 are uniform-in-time. Then we have the following lemma, which is essential for the proof of theorem 1.2. Throughout this section we use \( C \) and \( C_i \) to denote generic positive constants depending on the initial data, \( \mu, \gamma_{\pm} \), while independent of time.

**Lemma 4.1.** Let \( (R, Q, u) \) be the unique weak solution to (1.17)–(1.19) ensured by theorem 1.1. Then

\[
C^{-1} \leq R(y, t), Q(y, t) \leq C, \text{ for a.e. } (y, t) \in \Omega_\infty.
\] (4.1)

**Proof.** From (2.33) and the assumptions on the initial data (1.21) and (1.22) one sees that verification of (4.1) requires only to show the two-sided bounds for \( \tau \). By adapting the arguments in [23] (see also [24]), this follows from lemma 2.1 and the three items below.

- \( 0 < C_1 \leq \int_{\Omega} \tau \tilde{Z}^+ \, dy \leq C_2 < \infty \),
- \( \tilde{Z}^+ \) is sufficiently large if \( \tau \) is sufficiently small,
- \( \tilde{Z}^+ \) is sufficiently small if \( \tau \) is sufficiently large.

As a consequence, it remains to check that the three items above are satisfied. By the identity of pressure decomposition (2.9) and (2.33), it holds that

\[
\int_{\Omega} \tau \tilde{Z}^+ \, dy = \int_{\Omega} \left( \tau \alpha \left( \frac{R}{\alpha} \right)^{\gamma^+} + \tau (1 - \alpha) \left( \frac{Q}{1 - \alpha} \right)^{\gamma^-} \right) \, dy
\]
\[
\int_{\Omega} \left( (c_+)^{\gamma+} + (c_-)^{\gamma+} - (1 - \alpha) \tau \right)^{\gamma+} \, dy \leq C_2,
\]

where we have used the energy estimate (2.7). Clearly, we conclude from the definition of \(\alpha\), i.e., (2.5), and Jensen’s inequality that

\[
\int_{\Omega} \tau Z^{\gamma+} \, dy = \int_{\Omega} \left( \tau \alpha \left( \frac{R}{\alpha} \right)^{\gamma+} + \tau (1 - \alpha) \left( \frac{Q}{1 - \alpha} \right)^{\gamma-} \right) \, dy
\]
\[
\geq \int_{\Omega} \alpha^{\gamma-} + \tau \gamma+ \left( c_+ \right)^{\gamma+} \, dy
\]
\[
\geq C \int_{\Omega} \tau^{\gamma-} \, dy
\]
\[
\geq C \left( \int_{\Omega} \tau \, dy \right)^{\gamma+}.
\]

Suppose now that \(\tau\) is small, i.e., \(R + Q\) is large and we consider two possible cases. If \(R\) is large, then \(Z^{\gamma+}\) is also large due to \(R \leq Z\). If, on the other hand, \(Q\) is large, then also \(Z\) is large. Indeed, otherwise, we would arrive at a contradiction in the relation

\[
Q = \left( 1 - \frac{R}{Z} \right) Z^\gamma.
\]

The third item is verified by using similar observation as above. We refer to [23] and lemma 5.3 in [24] for the remaining details. \(\square\)

**Remark 4.1.** The key observations in lemma 4.1 are as follows. Firstly, the pressure term is a function with variables \(y\) and \(\tau\) by virtue of (1.12) and (1.13), tending to infinity as \(\tau\) goes to zero and tending to zero as \(\tau\) goes to infinity. Secondly, the two internal pressures satisfy \(\gamma\)-laws. This leads to a positive lower bound of the integral \(\int_{\Omega} \tau Z^{\gamma+} \, dy\); while the upper bound is obtained by the energy inequality. In this way, the arguments in [23, 24] are naturally adapted.

### 4.2. Exponential decay

In this subsection, we prove the exponential decay of weak solution in \(L^2\)-norm by adapting the ideas from [23, 24]. It should be emphasized that the structure of pressure function is crucial for a modification of these arguments to work.

**Step 1.** Let \((R_\infty, Q_\infty, u_\infty)\) be the unique steady state for problem (1.17)–(1.19) given by (1.26). Thanks to (1.26), we rewrite the momentum equation (1.17) as

\[
\partial_t u + \partial_y \left( Z^{\gamma+} - Z_\infty^{\gamma+} \right) = \mu \partial_y \left( \frac{\partial_y u}{\tau} \right).
\]

(4.2)

Since \(Z\) is a function of \(y\) and \(\tau\), it follows that \(Z_\infty = Z(y, \tau_\infty)\). Therefore, testing (4.2) by \(u\) and integrating by parts yields

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dy + \int_{\Omega} (Z^{\gamma+}(y, \tau) - Z^{\gamma+}(y, \tau_\infty)) \partial_y u \, dy + \mu \int_{\Omega} \frac{(\partial_y u)^2}{\tau} \, dy = 0.
\]
Using the continuity equation (1.17), one has
\[
\int \Omega \left( Z^+(y, \tau) - Z^+(y, \tau_\infty) \right) \partial_\tau u dy = \int \Omega \left( Z^+(y, \tau) - Z^+(y, \tau) \right) \partial_\tau \tau dy \\
= \frac{d}{d\tau} \int \Omega G(y, \tau, \tau_\infty) dy,
\]

where we denoted
\[
G(y, \tau, \tau_\infty) := \int_{\tau_\infty}^\tau \left( Z^+(y, \tau) - Z^+(y, \tau) \right) d\xi.
\]

Thus we obtain
\[
\frac{d}{d\tau} \int \Omega \left( \frac{u^2}{2} + G(y, \tau, \tau_\infty) \right) dy + \mu \int \Omega \frac{(\partial_\tau u)^2}{\tau} dy = 0. 
\]

(4.3)

Step 2. The key step in obtaining the exponential decay is to show that
\[
C^{-1}(\tau - \tau_\infty)^2 \leq G(y, \tau, \tau_\infty) \leq C(\tau - \tau_\infty)^2. \tag{4.4}
\]

The main observation is as follows. By setting
\[
F(\tau) := - \int_{\tau_\infty}^\tau Z^+(y, \xi) d\xi,
\]

\[
G(y, \tau, \tau_\infty) \text{ is reformulated as }
\]
\[
G(y, \tau, \tau_\infty) = F(\tau) - F(\tau_\infty) - F'(\tau_\infty)(\tau - \tau_\infty). \tag{4.5}
\]

Therefore, in order to deduce (4.4) it is enough to estimate the second derivative of \( F(\tau) \). To this purpose we use the expression for \( \partial_\tau \tau \) from (2.14) to get
\[
\partial_\tau(Z^+) = -\gamma Z^+ \frac{-c_+ \tau^{-2} + c_+ \tau^{-2} Z^{-1}}{\gamma Z^{-1} - c_+ \tau^{-1}(\gamma - 1)Z^{1-\gamma}}. \tag{4.6}
\]

As in (2.15) we first observe that the denominator is strictly positive. Moreover, in spirit of remark 1.1, we infer from (4.1) and the relation \( Q = (1 - \frac{2}{\gamma}) Z^+ \) that
\[
C^{-1} \leq Z(y, t) \leq C, \quad \text{for a.e. } (y, t) \in \Omega_\infty, \tag{4.7}
\]

which together with lower and upper bound for \( \tau \) implies boundedness of the numerator of (4.6).

The remaining arguments follow largely the ones from [23, 24]. We incorporate the detailed proof for the sake of completeness.

Step 3. Let 0 < \( \varepsilon < 1 \) and
\[
K(y, t) := \int_0^t (\tau(\xi, t) - \tau_\infty(\xi)) d\xi.
\]

Testing (4.2) by \( \varepsilon K \) gives rise to
\[
\frac{d}{d\tau} \int \Omega \varepsilon u K dy - \varepsilon \int \Omega \left( Z^+(y, \tau) - Z^+(y, \tau_\infty) \right) (\tau - \tau_\infty) dy
\]
From (4.3) and (4.8) we obtain

\[
\begin{align*}
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} u^2 + G(y, \tau, \tau_c) + \varepsilon uK \right) \, dy \\
+ \mu \int_{\Omega} \left( \frac{\partial u}{\tau} \right)^2 \, dy - \varepsilon \int_{\Omega} (Z^+ - Z^+(y, \tau_c)) (\tau - \tau_c) \, dy \\
= \varepsilon \int_{\Omega} u^2 \, dy - \varepsilon \int_{\Omega} \frac{\partial u}{\tau} (\tau - \tau_c) \, dy.
\end{align*}
\]  
(4.9)

It follows from (4.1), (4.6) and (4.7) that

\[
\int_{\Omega} (Z^+ - Z^+(y, \tau_c)) (\tau - \tau_c) \, dy \geq C_1 \int_{\Omega} (\tau - \tau_c)^2 \, dy.
\]  
(4.10)

With the help of Cauchy–Schwarz’s inequality and (4.1), we find

\[
\left| \int_{\Omega} \frac{\partial u}{\tau} (\tau - \tau_c) \, dy \right| \leq \frac{C_2}{C_1} \mu \int_{\Omega} \left( \frac{\partial u}{\tau} \right)^2 \, dy + \frac{C_1}{2} \int_{\Omega} (\tau - \tau_c)^2 \, dy; \quad (4.11)
\]

\[
\int_{\Omega} u^2 \, dy \leq C_3 \mu \int_{\Omega} \left( \frac{\partial u}{\tau} \right)^2 \, dy. \quad (4.12)
\]

Using (4.10)–(4.12), (4.9) implies

\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} u^2 + G(y, \tau, \tau_c) + \varepsilon uK \right) \, dy \\
+ \mu \int_{\Omega} \left( \frac{\partial u}{\tau} \right)^2 \, dy - \varepsilon \int_{\Omega} (Z^+ - Z^+(y, \tau_c)) (\tau - \tau_c) \, dy \\
\leq 0. \quad (4.13)
\]

Step 4. Due to the definition of $K$, it holds that

\[
\left| \int_{\Omega} \varepsilon uK \, dy \right| \leq \frac{\varepsilon}{2} \int_{\Omega} u^2 \, dy + \frac{\varepsilon}{2} \int_{\Omega} (\tau - \tau_c)^2 \, dy. \quad (4.14)
\]

Based on (4.14), after choosing $\varepsilon$ suitably small, we see

\[
C^{-1}(\|\tau - \tau_c\|_{L^2}^2 + \|u\|_{L^2}^2) \leq \int_{\Omega} \left( \frac{1}{2} u^2 + G(y, \tau, \tau_c) + \varepsilon uK \right) \, dy \\
\leq C(\|\tau - \tau_c\|_{L^2}^2 + \|u\|_{L^2}^2),
\]

where we essentially used the property (4.4) from step 2. Combining the above with (4.13) leads to

\[
\|\tau - \tau_c\|_{L^2}^2 + \|u\|_{L^2}^2 \leq C \exp(-Ct), \quad (4.15)
\]

for any $t \geq 0$. 

4092
Step 5. Finally, the exponential decay of $\|R - R_\infty\|_{L^2}$ and $\|Q - Q_\infty\|_{L^2}$ is a direct consequence of (4.15) and the relations

$$Q = c_- \tau^{-1}, \quad R = c_+ \tau^{-1};$$

$$Q_\infty = c_- \tau_\infty^{-1}, \quad R_\infty = c_+ \tau_\infty^{-1}.$$

The proof of theorem 1.2 is complete. □

**Remark 4.2.** We observe that the exponential decay of $Z$ follows from that of $\tau$. Indeed,

$$\|Z(y, \tau) - Z(y, \tau_\infty)\|_{L^2} \leq \| \partial_\tau Z \|_{L^\infty} \| \tau - \tau_\infty \|_{L^2} \leq C \exp(-Ct),$$

in light of (4.6), (4.7) and (4.15).

**Remark 4.3.** The strategy adopted in this paper is strong enough to show existence, stability and exponential decay of global weak solution to two-fluid models with more general form of pressure considered for example in [4]. In particular, the two-fluid model with pressure satisfying $\gamma$-laws could be included.

**Acknowledgments**

The research of Y Sun is supported by the NSF of China under Grant Numbers 11571167, 11771395, 11771206 and PAPD of Jiangsu Higher Education Institutions. E Zatorska acknowledges the support from the Polish Ministry of Science and Higher Education ‘Juventus Plus’ under Grant Number 0888/IP3/2016/74. The authors would like to thank the referees for carefully reading of the manuscript and providing many helpful comments and suggestions.

**References**

[1] Ishii M and Hibiki T 2006 *Thermo-Fluid Dynamics of Two-phase Flow* (Berlin: Springer)

[2] Bresch D, Desjardins B, Ghidaglia J M, Grenier E and Hilliairet M 2018 Multi fluid models including compressible fluids *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids* ed Y Giga et al and A Novotný (Berlin: Springer) p 52

[3] Bresch D, Mucha P B and Zatorska E 2019 Finite-energy solutions for compressible two-fluid Stokes system *Arch. Ration. Mech. Anal.* 232 987–1029

[4] Novotný A and Pokorný M 2020 Weak solutions for some compressible multicomponent fluid models *Arch. Ration. Mech. Anal.* 235 355–403

[5] Maltese D, Michálek M, Mucha P B, Novotný A, Pokorný M and Zatorska E 2016 Existence of weak solutions for compressible Navier–Stokes equations with entropy transport *J. Differ. Equ.* 261 4448–85

[6] Vasseur A, Wen H and Yu C 2019 Global weak solution to the viscous two-fluid model with finite energy *J. Pure Appl. Math.* 125 247–82

[7] Bresch D, Huang X and Li J 2012 Global weak solutions to one-dimensional non-conservative viscous compressible two-phase system *Commun. Math. Phys.* 309 737–55

[8] Evje S and Wen H 2013 Weak solutions of a gas-liquid drift-flux model with general slip law for wellbore operators *Discrete Continuous Dyn. Syst.* 33 4497–530

[9] Evje S and Wen H 2015 Weak solutions of a two-phase Navier–Stokes model with a general slip law *J. Funct. Anal.* 268 93–139

[10] Evje S and Wen H 2015 On the large time behavior of the compressible gas–liquid drift-flux model with slip *Math. Model Methods Appl. Sci.* 25 2175–215
[11] Evje S, Wen H and Yao L 2016 Global solutions to a one-dimensional non-conservative two-phase model Discrete Continuous Dyn. Syst. 36 1927–55
[12] Evje S, Wen H and Zhu C 2017 On global solutions to the viscous liquid–gas model with unconstrained transition to single-phase flow Math. Model Methods Appl. Sci. 27 323–46
[13] Wen H, Yao L and Zhu C 2018 Review on mathematical analysis of some two-phase flow models Acta Math. Sci. 38 1617–36
[14] Zlotnik A A 2017 Well-posedness of the IBVPs for the 1D viscous gas equations Handbook of Mathematical Analysis in Mechanics of Viscous Fluids ed Y Giga, A Novotný and des. (Cham: Springer) ch 3 pp 1–73
[15] Zlotnik A A 2002 Stabilization of solutions to the two-scale equations of a viscous compressible barotropic medium Dokl. Math. 65 103–7
[16] Amosov A A and Zlotnik A A 1994 Uniqueness and stability of generalized solutions for a class of quasilinear systems of composite type equations Math. Notes 55 555–67
[17] Zlotnik A A and Amosov A A 1988 Global generalized solutions of one-dimensional motion equations for a viscous barotropic gas Dokl. Akad. Nauk SSSR 299 1303–7
[18] Beirão da Veiga H 1989 Long time behavior for one-dimensional motion of a general barotropic fluid Arch. Ration. Mech. Anal. 108 141–60
[19] Kanel Y 1968 On a model system of equations for the one-dimensional motion of a gas Differ. Uravn. 4 721–34
[20] Kazhikhov A V 1979 Stabilization of solutions of the initial-boundary value problem for barotropic viscous fluid equations Differ. Uravn. 15 662–7
[21] Matsumura A and Yanagi S 1996 Uniform boundedness of the solutions for a one-dimensional isentropic model system of compressible viscous gas Commun. Math. Phys. 175 259–74
[22] Straškraba I and Valli A 1988 Asymptotic behavior of the density for one-dimensional Navier–Stokes equations Manuscripta Math. 62 401–16
[23] Zlotnik A A 1992 On equations for one-dimensional motion of a viscous barotropic gas in the presence of a body force Siberian Math. J. 33 798–815
[24] Li Y and Sun Y 2019 Global weak solutions and long time behavior for 1D compressible MHD equations without resistivity J. Math. Phys. 60 071511
[25] Evje S and Karlsen K H 2008 Global existence of weak solutions for a viscous two-phase model J. Differ. Equ. 245 2660–703
[26] Nash J 1962 Le problème de Cauchy pour les équations différentielles d’un fluide général Bull. Soc. Math. Fr. 90 487–97
[27] Antonie S N, Kazhikhov A V and Monakhov V N 1990 Boundary Value Problems in Mechanics of Nonhomogeneous Fluids (Amsterdam: North-Holland)
[28] Evans L C 2010 Partial Differential Equations 2nd edn (Graduate Studies in Mathematics vol 19) (Providence, RI: American Mathematical Society)