A NOTE ON A BRILL-NOETHER LOCUS
OVER A NON-HYPERELLIPTIC CURVE OF GENUS 4

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Abstract. We prove that a certain Brill-Noether locus over a non-hyperelliptic curve C of genus 4, is isomorphic to the Donagi-Izadi cubic threefold in the case when the pencils of the two trigonal line bundles of C coincide.

1. Introduction

Let C be a non-hyperelliptic curve of genus 4 over C and then C is embedded into \( \mathbb{P}_3 \) by the canonical embedding and there exists a unique quadric surface \( Q \subset \mathbb{P}_3 \) containing C. If we let \( g_3^1 \) and \( h_3^1 \) be the two trigonal line bundles such that \( g_3^1 \otimes h_3^1 = \mathcal{O}_C(K_C) \), the canonical line bundle, then Q is singular if and only if the two pencils \( |g_3^1| \) and \( |h_3^1| \) coincide.

Let \( SU_C(2, K_C) \) be the moduli space of semi-stable bundles of rank 2 on C with the canonical determinant and \( W^r \) be the Brill-Noether locus defined as the closure of the set of stable bundles \( E \) with \( h^0(E) \geq r + 1 \). In [3], \( W^2 \) was proven to be isomorphic to the Donagi-Izadi cubic threefold. In [1], we gave a different proof of this when \( Q \) is smooth, using the fact that the moduli space of stable sheaves of rank 2 on \( Q \) with the Chern classes \( c_1 = \mathcal{O}_Q(1,1) \) and \( c_2 = 2 \), is isomorphic to \( \mathbb{P}_3 \).

In this article, we use the same trick of [1] to the Hirzebruch surface \( F_2 \) and derive the same result on \( W^2 \) when \( Q \) is a quadric cone in \( \mathbb{P}_3 \). Unlike the situation in [1], the determinant of sheaves that we choose over \( F_2 \) is not ample, which prevents us from using the definition of stability. Instead, we use a parametrization \( P \) of the vector bundles on \( F_2 \) admitting a certain exact sequence. We show that \( P \) is isomorphic to \( \mathbb{P}_3 \), the original ambient space into which \( C \) is embedded by the canonical embedding. From the investigation of this parametrization, we show that the restriction map from \( P \) to \( W^2 \) is given by the complete linear system \( |I_C(3)| \), implying that \( W^2 \) is isomorphic to the Donagi-Izadi cubic threefold.

2. Main Theorem

Let \( F_2 = \mathbb{P}(\mathbb{F}) \) be the Hirzebruch surface with a section \( \sigma \) whose self-intersection is \( -2 \), where \( F \simeq \mathcal{O}_{F_1} \oplus \mathcal{O}_{F_1}(2) \). Recall that \( F_2 \) is the minimal

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resolution of the quadric cone $Q \subset \mathbb{P}_3$ at the vertex point $P_0$. The section $\sigma$ is the exceptional curve of the resolution. Let $f$ be a fibre of the ruling $\pi : \mathbb{F}_2 \rightarrow \mathbb{P}_1$ and then $\text{Pic}(\mathbb{F}_2)$ is freely generated by $\sigma$ and $f$. We will denote the line bundle $\mathcal{O}_{\mathbb{F}_2}(a \sigma + bf)$ by $\mathcal{O}(a, b)$ and $E \otimes \mathcal{O}(a, b)$ by $E(a, b)$ for a coherent sheaf $E$ on $\mathbb{F}_2$. Note that the canonical line bundle is $\mathcal{O}(-2, -4)$. Then the resolution $\varphi : \mathbb{F}_2 \rightarrow Q$ is given by the linear system $|O(1, 2)|$. Here, the line bundle $H = O(1, 2)$ is the tautological line bundle $\mathcal{O}_{\mathbb{F}_2}(1)$ on $\mathbb{F}_2$, which is nef but not ample.

**Lemma 2.1.** We have

$$H^i(\mathcal{O}_{\mathbb{F}_2}(aH + bf)) = \begin{cases} 0, & \text{if } a = -1; \\ H^i(\mathbb{P}_1, S^a(F) \otimes \mathcal{O}_{\mathbb{F}_1}(b)), & \text{if } a \geq 0; \\ H^{2-i}(\mathbb{P}_1, S^{-2-a}(F) \otimes \mathcal{O}_{\mathbb{F}_1}(-b)), & \text{if } a \leq -2. \end{cases}$$

**Proof.** From the Leray spectral sequence,

$$H^i(\mathbb{P}_1, R^j \pi_* \mathcal{O}_{\mathbb{F}_2}(aH + bf)) \Rightarrow H^{i+j}(\mathbb{F}_2, \mathcal{O}_{\mathbb{F}_2}(aH + bf))$$

and $R^i \pi_* \mathcal{O}_{\mathbb{F}_2}(aH + bf) = R^i \pi_* \mathcal{O}_{\mathbb{F}_2}(aH) \otimes \mathcal{O}_{\mathbb{F}_1}(b) = 0$ for $i > 0$ and $a \geq -1,$ we have

$$H^i(\mathbb{F}_2, \mathcal{O}_{\mathbb{F}_2}(aH + bf)) = H^i(\mathbb{P}_1, \pi_* \mathcal{O}_{\mathbb{F}_2}(aH + bf)),$$

for all $a \geq -1.$ Since $\pi_* \mathcal{O}_{\mathbb{F}_2}(aH + bf) = S^a(F) \otimes \mathcal{O}_{\mathbb{F}_1}(b)$ if $a \geq 0$ and 0 otherwise, we get the first and second assertions. The last case can be derived from the second case, using the Serre duality. □

Let $\mathcal{P}$ be the set of non-trivial sheaves of rank 2 on $\mathbb{F}_2$ with the Chern classes $c_1 = O(1, 2)$ and $c_2 = 2$, which are fitted into the following exact sequence,

$$0 \rightarrow \mathcal{O}(1, 0) \rightarrow E \rightarrow \mathcal{O}(0, 2) \rightarrow 0.$$ 

Note that $h^0(E)$ is 3 or 4 from the sequence (1). If we let $O(a, b)$ be a sub-bundle of $E$ from a section $s \in H^0(E)$, we have $(a, b) = (0, 0), (0, 1), (0, 2), (1, 0)$ or $(1, 1)$, since $c_2(E) = 2$. In the case of $(a, b) = (0, 2)$, $E$ turns out to be isomorphic to $O(1, 0) \oplus O(0, 2)$, which is excluded. In the case of $(1, 1)$, $E$ is fitted into

$$0 \rightarrow \mathcal{O}(1, 1) \rightarrow E \rightarrow I_p(0, 1) \rightarrow 0,$$

where $p$ is a point on $\mathbb{F}_2$, implying that $h^0(E(-1, 0)) = 2$. But this is not true since $h^0(E(-1, 0)) = 1$ from (1). Thus we obtain only $O(0, 0), O(1, 0)$ or $O(0, 1)$ as sub-bundles of $E \in \mathcal{P}$ from sections of $E$.

Let $\mathcal{D}$ be the set of sheaves $E \in \mathcal{P}$, fitted into the following exact sequence,

$$0 \rightarrow \mathcal{O}(0, 1) \rightarrow E \rightarrow I_p(1, 1) \rightarrow 0,$$

where $p$ is a point on $\mathbb{F}_2$. Since the dimension of $\text{Ext}^1(I_p(1, 1), \mathcal{O}(0, 1))$ is 1, so we have the unique non-trivial extension of (3) to each point $p \in \mathbb{F}_2$. (It can be easily checked that the trivial extension does not lie in $\mathcal{P}$). Since $h^0(E(-1, 0)) = 1$, we have $E \in \mathcal{P}$ and thus $\mathcal{D} \subset \mathcal{P}$. We also have
h^0(E(0, -1)) = 1 + h^0(I_p(1, 0)) = 2 if \( p \in \sigma \) and 1 otherwise. Similarly, 
\( h^0(E) = 4 \) if \( p \in \sigma \) and 3 otherwise.

Let \((a, b) = (0, 0)\) and so we have the exact sequence,

\[
0 \to \mathcal{O} \to E \to I_Z(1, 2) \to 0,
\]

where \( Z \) is a 0-cycle on \( \mathbb{F}_2 \) with length 2. Let us denote the extension classes of type (4) by \( \mathbb{P}(Z) := \mathbb{P} \text{Ext}^1(I_Z(1, 2), \mathcal{O}) \), then \( \mathbb{P}(Z) \) is isomorphic to \( \mathbb{P}H^0(\mathcal{O}_Z)^* \simeq \mathbb{P}_1 \). Since \( 1 = H^0(E(-1, 0)) = H^0(I_Z(0, 2)) \), we have two fibres \( f_1 \) and \( f_2 \) of \( \pi \), each containing a point of \( Z \). In fact, in the case when \( Z \) is contained in a fibre \( f \), we have the extension (2).

**Proposition 2.2.** We have the following descriptions on \( \mathcal{P} \):

1. \( \mathcal{P} \) is isomorphic to \( \mathbb{P}_3 \).
2. \( \mathcal{D} \subset \mathcal{P} \) is a quadric cone \( Q' \).
3. The vertex \( P_0' \) of \( \mathcal{D} \) corresponds to the unique vector bundle \( E_0 \in \mathcal{P} \) such that \( h^0(E_0) = 4 \).

**Proof.** The assertion (1) is clear since \( h^0(E(-1, 0)) = 1 \) for all \( E \in \mathcal{P} \) and \( \mathbb{P} \text{Ext}^1(\mathcal{O}(0, 2), \mathcal{O}(1, 0)) \) is isomorphic to \( \mathbb{P}H^1(\mathcal{O}(1, -2)) \simeq \mathbb{P}_3 \). Now there exists a universal extension

\[
0 \to q^*\mathcal{O}(1, 0) \to \mathcal{E} \to q^*\mathcal{O}(0, 2) \to 0,
\]
on \( \mathcal{P} \times \mathbb{F}_2 \) (\( q \) is the projection to \( \mathbb{F}_2 \)) such that \( \mathcal{E}|_{\{p\} \times \mathbb{F}_2} \) is isomorphic to an extension corresponding to \( p \in \mathcal{P} \). Let \( \mathcal{E}' \) be an extension of \( I_\triangle \otimes q^*\mathcal{O}(1, 1) \) by \( q^*\mathcal{O}(0, 1) \) over \( \mathbb{F}_2 \times \mathbb{F}_2 \) such that the restriction of \( \mathcal{E}' \) to \( \{p\} \times \mathbb{F}_2 \) is the unique non-trivial extension of \( I_p(1, 1) \) by \( \mathcal{O}(0, 1) \). Here, \( \triangle \) is the diagonal of \( \mathbb{F}_2 \times \mathbb{F}_2 \). The existence of such \( \mathcal{E}' \) is guaranteed because

\[
\begin{align*}
H^2(I_\triangle \otimes p^*\mathcal{O}(-1, -4) \otimes q^*\mathcal{O}(-2, -4)) \\
\simeq H^2(\mathcal{O}_\triangle \otimes p^*\mathcal{O}(-1, -4) \otimes q^*\mathcal{O}(-2, -4)) \\
\simeq H^2(\mathbb{F}_2, \mathcal{O}(-1, -4) \otimes p_\circ q^*\mathcal{O}(-2, -4)) \\
\simeq H^2(\mathcal{O}(-3, -8)) \simeq H^0(\mathcal{O}(1, 4))
\end{align*}
\]
is not zero. Since each restriction to \( \{p\} \times \mathbb{F}_2 \) is contained in \( \mathcal{P} \), we have a morphism \( \chi \) from \( \mathbb{F}_2 \) to \( \mathcal{P} \) and the image of \( \chi \) is \( \mathcal{D} \). Now assume that \( h^0(E) = 4 \) for \( E \in \mathcal{P} \). It can be easily checked that there exists a section of \( E \) for which \( E \) is fitted into (4). Thus, \( 4 = h^0(E) = 1 + h^0(I_Z(1, 2)) \), i.e. \( h^0(I_Z(1, 2)) = 3 \). This implies that \( Z \) is contained in \( \sigma \). From (4), we also have \( h^0(E(0, -1)) > 0 \). In particular, \( E \) is also fitted into (6) with \( p \in \sigma \).

Let \( s_1, s_2 \) be two sections of \( E(-1, 0) \) such that \( p_1 \) is the only zero of \( s_1 \) and \( s_2 \). If \( s_1 \) and \( s_2 \) are different, we can find \( p_2 \neq p_1 \) such that \( as_1 + bs_2 \) is zero at \( p_2 \) for some \( a, b \neq 0 \), which is absurd because \( p_1 \) is also the unique zero of \( as_1 + bs_2 \). Thus for all \( p_1 \in \sigma \), we have the unique \( E \) such that \( h^0(E) = 4 \). In particular, the map \( \chi \) contracts \( \sigma \) to a point in \( \mathcal{P} \). Let \( E \in \mathcal{D} \). If \( p \not\in \sigma \),
we have $h^0(E(0, -1)) = 1$ so that we can assign a different $E$ for each $p \not\in \sigma$.
Thus $\chi$ is the minimal resolution of a quadric cone $Q' \subset P$ at the vertex point $P_0'$ corresponding to the sheaf $E_0$ admitting (3) with $p \in \sigma$. \hfill \Box

Remark 2.3.

(1) Let us consider the definition of stability on the sheaves of rank 2 on $P_2$ with the Chern classes $c_1 = O(1, 2)$ and $c_2 = 2$ with respect to the nef divisor $H = O(1, 2)$. It can be checked that such sheaves admits an exact sequence (1). Since all the sheaves in $\mathcal{D}$ contains $O(0, 1)$ as sub-bundle, it contradicts to the stability condition. So the space of stable sheaves in this sense, is isomorphic to $P_3 \backslash Q$ and in particular, it is not projective.

(2) Let us assume that a non-trivial bundle $E$ with the extension (3), $p \in P_2 \backslash \sigma$, admits an extension (4) with $Z \in P_2^{[2]}$, where $P_2^{[2]}$ be the Hilbert scheme of 2-cycles of length 2 on $P_2$. In these two extensions, $O$ is a sub-bundle of $O(0, 1)$, otherwise, $E$ contains $O \oplus O(0, 1)$ as a sub-bundle, which is absurd. Thus we have a surjection from $I_Z(1, 2)$ to $I_p(1, 1)$. In particular, $\text{Hom}(I_Z(1, 2), I_p(1, 1))$ non-trivial. As a result, if we take $\text{Hom}(\cdot, I_p(1, 1))$ to the exact sequence,

$$0 \to I_Z(1, 2) \to O(1, 2) \to O_Z \to 0,$$

then we know that $\text{Ext}^1(O_Z, I_p)$ is non-trivial, which implies that $p \in Z$. Let us denote this $p$ by $p_E$.

For $E \in \mathcal{P}$, we consider the determinat map

$$\lambda_E : \wedge^2 H^0(E) \to H^0(O(1, 2)).$$

Recall that the dimension of $\wedge^2 H^0(E)$ is 3 if $E \neq E_0$.

Lemma 2.4. If $E \in \mathcal{P} \backslash \mathcal{D}$, then $\lambda_E$ is injective.

Proof. Let $s_1$ and $s_2$ be two sections of $H^0(E)$ for which $s_1 \wedge s_2$ is a non-trivial element in $\ker(\lambda_E)$. It would generate a subsheaf $F$ of $E$ such that $h^0(F) \geq 2$. Since $c_2(E) = 2$, it can be easily checked that the only possibility for $F$ is $O(0, 1)$ or $I_p(1, 1)$, where $p$ is a point on $P_2$ with the following exact sequence,

$$0 \to I_p(1, 1) \to E \to O(0, 1) \to 0.$$

Let us assume that $E \not\in \mathcal{D}$ and in particular, $I_p(1, 1)$ is the only possibility for $F$. From the previous result, $E$ is locally free. Since $O(0, 1)$ is torsion-free, so $I_p(1, 1)$ must be a line bundle, which is absurd. \hfill \Box

Let us denote by $p_E \in P_3$, the point corresponding to the dual of the cokernel of $\lambda_E$. As vector subspaces of $H^0(O(1, 2))$, we see that $H^0(I_Z(1, 2))$ is contained in the image of $\wedge^2 H^0(E)$ and so $p_E$ is contained in $H^0(O_Z)^*$ as a vector subspace of $H^0(O(1, 2))^*$. It implies that $p_E$ is a point in $P_3$, contained in all secant lines of $\varphi(Z)$ for which $E$ admits an extension (4).

This argument with the remark 2.3 gives us a map from $\eta : \mathcal{P} \to P_3$ sending
$E$ to $p_E$ for $E \in \mathcal{P}\setminus\{E_0\}$. Clearly, this map extends to $E_0$ by assigning the vertex $P_0 \in Q$ because in the extension $\tilde{\mathcal{E}}$ of $E_0$, the support of $Z$ should lie on $\sigma$ due to the fact that $h^0(E) = 4$ and so $h^0(I_Z(1,1)) = 2$. Note that, for $E \in \mathcal{P}\setminus\mathcal{D}$, $p_E$ lies outside $Q$ and so we get the following statement.

**Proposition 2.5.** The map $\eta : \mathcal{P} \to \mathbb{P}H^0(\mathcal{O}(1,2))^*$ is an isomorphism. Moreover, the restriction of $\eta$ to $Q'$ is an isomorphism to $Q$ sending an extension of type $\mathcal{Z}$ to $\varphi(p) \in Q$.

**Remark 2.6.** Let $p \in \mathbb{P}_3 \setminus Q$ and $\varphi'$ be the restriction of the projection from $\mathbb{P}_3$ to $\mathbb{P}_2$ at $p$, to $Q$. For the cotangent bundle of $\mathbb{P}_2$, twisted by $\mathcal{O}_{\mathbb{P}_2}(2)$, admits the following exact sequence,

$$0 \to \mathcal{O}_{\mathbb{P}_2} \to \Omega_{\mathbb{P}_2}(2) \to I_{\mathbb{P}_2}(1) \to 0,$$

where $p$ is a point on $\mathbb{P}_2$, not the point corresponding to the line passing through $p$ and the vertex point $P_0$. If we pull back the sequence via $\varphi' \circ \varphi$, then we get a vector bundle $E$ admitting an exact sequence (4), where $Z$ is $\varphi^{-1} \circ \varphi'^{-1}(p)$. This defines a map from $\mathbb{P}_3 \to \mathcal{P}$ and in fact, it extends to the inverse morphism of $\eta$.

Let $C$ be a non-hyperelliptic curve of genus 4 with the two trigonal line bundles $g_3^1$ and $h_3^1$ such that $|g_3^1| = |h_3^1|$. In particular, $C$ is embedded into $\mathbb{P}_3$ by the canonical embedding and there exists a unique quadric cone $Q \subset \mathbb{P}_3$ containing $C$. Let $P_0$ be the vertex point of $Q$. Recall that $\mathbb{F}_2$ is the minimal resolution of $Q$ at $P_0$. Let $C'$ be the proper transform of $C$ in $\mathbb{F}_2$. Note that $C$ and $C'$ are isomorphic, so we will use $C$ instead of $C'$ if there is no confusion. Let us assume that the divisor type of $C \subset \mathbb{F}_2$ is $(a,b)$. From the adjunction formula, we have

$$6 = 2g(C) - 2 = C.(C + K) = (a,b).(a - 2, b - 4).$$

Since $C$ does meet the vertex $P_0$, we have $C.\sigma = 0$. Hence $(a,b) = (3,6)$. If we tensor the following exact sequence

$$0 \to \mathcal{O}(-3,-6) \to \mathcal{O} \to \mathcal{O}_C \to 0,
$$

with a bundle $E \in \mathcal{P}$ and take the long exact sequence of cohomology, we have $h^0(E|\mathcal{C}) = h^0(E) = 3$ since $h^1(E(-3,-6)) = h^1(E) = 0$. By the adjunction formula, we have

$$\mathcal{O}_C(K) = \mathcal{O}(K_{\mathbb{F}_2}) \otimes \mathcal{O}(3,6) \otimes \mathcal{O}_C = \mathcal{O}(1,2) \otimes \mathcal{O}_C,$$

i.e. the determinant of $E|\mathcal{C}$ is $\mathcal{O}_C(K_C)$.

**Lemma 2.7.** The restriction map

$$\Phi : \mathcal{P} \longrightarrow \mathcal{W},$$

sending $E$ to $E|\mathcal{C}$, is well-defined.

**Proof.** It is enough to prove that $E|\mathcal{C}$ is stable. Let us assume that there exists a sub-bundle $\mathcal{O}_C(D)$ with $d = \deg(D) \geq 3$. Since the degree of $K_C - D$ is less than 4, we have $h^0(\mathcal{O}_C(D)) > 0$ due to the Clifford theorem [?] and
$h^0(E|C) = 3$. Thus we can assume that $D$ is effective. Since $H^0(E) \simeq H^0(E|C)$, $D$ can be considered as the zero section of $H^0(E)$ with $C$. For a section in $H^0(E)$, let us consider an exact sequence,

$$0 \to \mathcal{O}(a,b) \to E \to I_Z(1-a,2-b) \to 0,$$

where $a, b \geq 0$. From the numeric invariants of $E$ and the fact that $h^0(E) = 3$, we have $(a,b) = (0,0), (1,0)$ or $(0,1)$. For a general vector bundle $E \in \mathcal{P}$, the case of $(a,b) = (0,1)$ cannot happen. Indeed, it happens only when $E \in \mathcal{D}$. Since the length of $Z$ is at most 2 in each case, $d$ must be less than 3. Hence, $E|C$ is stable. 

Let $g^1_3$ be the trigonal line bundle on $C$ and we have $\mathcal{O}(0,1)|C = g^1_3$. If $E \in \mathcal{D}$, we have an exact sequence \([3]\). If $p \not\in C$, we obtain the following exact sequence

$$0 \to g^1_3 \to E|C \to g^1_3 \to 0,$$

after tensoring with $\mathcal{O}_C$. In particular, $E|C$ is in the same equivalent class of $g^1_3 \oplus g^1_3$. If $p \in C$, then we obtain an exact sequence,

$$0 \to g^1_3 \otimes \mathcal{O}_C(p) \to E|C \to g^1_3 \otimes \mathcal{O}_C(-p) \to 0,$$

which implies that $E|C$ is not semi-stable. Thus we have the following assertion.

**Proposition 2.8.** The restriction map $\Phi : \mathcal{P} \longrightarrow \mathcal{W}^2$ is defined by the complete linear system $|I_C(3)|$. In particular, $\mathcal{W}^2$ is isomorphic to the Donagi-Izadi cubic threefold.

**Proof.** The proof is similar with the one in \([1]\). If we choose a general hyperplane section $H \subset \mathcal{P}$, then the restriction of $\Phi$ to $H$ is not defined on 6 intersection points of $C$ with $H$. Since this indeterminacy locus lie on a conic on $H$, the blow-up of $H$ at these points is a singular cubic surface in $\mathbb{P}_3$. In particular, the degree of $\Phi$ is 3.

Let $E$ be a general vector bundle in $\mathcal{W}^2$ with $h^0(E) = 3$. It can be checked as in \([2,4]\) that the determinant map from $\wedge^2 H^0(E)$ to $H^0(\mathcal{O}_C(K_C))$ is injective and so we can assign a point $p_E \in \mathbb{P}_3$ corresponding to the dual of the cokernel of the determinant map. This defines a map $\rho$ from $\mathcal{W}^2$ to $\mathbb{P}_3$ and $\eta^{-1} \circ \rho \circ \Phi$ is the identity on $\mathcal{P}$. In particular, the dimension of $\mathcal{W}^2$ is at least 3. Conversely, the dimension of $\mathcal{W}^2$ can be shown to be at most 3 as follows: Let us assume that $E$ is the extension of $\mathcal{O}_C(K_C - D)$ by $\mathcal{O}_C(D)$, where $D$ is a divisor of $C$ with the degree $d$. Because of the stability of $E$ and the result of \([2]\), we can assume that $d = 2$ and so $h^0(\mathcal{O}_C(D)) = 1$. In particular, we can assume that $D$ is effective. In the extension space $\mathbb{P} \text{Ext}^1(\mathcal{O}_C(K_C - D), \mathcal{O}_C) \simeq \mathbb{P}_1$, there exists $\mathbb{P}_1$-parametrization corresponding to the vector bundles $E$ with $h^0(E) \geq 3$ \([3]\). Thus, we have a dominant map from a $\mathbb{P}_1$-bundle over $\text{Sec}^2(C)$ to $\mathcal{W}^2$ and so the dimension of $\mathcal{W}^2$ is at most 3. Now, we know that $h^0(I_C(3)) = 5$ and so $\Phi$ is given by the complete linear system $|I_C(3)|$ and the image is exactly $\mathcal{W}^2$. \qed
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