STABILITY OF SOLUTIONS TO DAMPED EQUATIONS WITH NEGATIVE STIFFNESS

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ABSTRACT. This article concerns the stability of a model for mass-spring systems with positive damping and negative stiffness. It is well known that when the coefficients are frozen in time the system is unstable. Here we find conditions on the variable coefficients to prove stability. In particular, we disprove the belief that if the eigenvalues of the system change slowly in time the system remains unstable. We extend some of our results for nonlinear systems.

1. INTRODUCTION

In this article, we present conditions for the stability of solutions to the differential equation

\[ u''(t) + b(t)u'(t) + k(t)u(t) = 0, \]
\[ u(t_0) = u_0, \quad u'(t_0) = u_1, \]

where the coefficient \( k \) may have negative values. This equation has been used for modelling mass-spring systems, where the mass is one unit, the coefficient \( b \) produces a damping effect proportional to velocity, and the coefficient \( k \) is the stiffness coefficient. Physical examples of systems with negative stiffness can be found in [11, 12].

Equation (1.1) is also written in matrix form as

\[
\begin{pmatrix}
  u \\
  u'
\end{pmatrix}' = A(t) \begin{pmatrix}
  u \\
  u'
\end{pmatrix}, \quad A(t) = \begin{pmatrix}
  0 & 1 \\
  -k(t) & -b(t)
\end{pmatrix}.
\]

Note that the roots of the auxiliary equation of (1.1) and the eigenvalues of the matrix \( A \) are \( \lambda_k = (-b \pm \sqrt{b^2 - 4k})/2 \). For constant coefficients \( b, k \) with \( k < 0 \), one eigenvalue is negative and one is positive. This makes the point \( u = 0, \ u' = 0 \) a saddle point and the zero solution unstable (see definition below).

The literature for this differential equations with time-varying coefficient has several results about instability with negative stiffness, but none about stability. In an attempt to extend the stability results to time-varying coefficients, the so called frozen coefficient method has been developed. In this technique the coefficients are frozen in time and the system is analyzed as a system of constant coefficients [11, Sec. 10.7]. Thus arises a belief that if the eigenvalues corresponding to time-varying coefficients change slowly with respect to time, then the instability obtained for constant coefficients remains valid. However, we did not find a precise statement of how small should be the rate of change of the eigenvalues. In this article, we
disprove the believed instability by showing that for each positive number, there exist coefficients \(b\) and \(k\) for which (1.1) is stable and the rate of change in the eigenvalues does not exceed the given number. See Remark 3.6.

The main objective of this article is to find conditions on \(b(t)\) and \(k(t)\) for stability in the negative stiffness case. More precisely, we find conditions for the transition from instability (at the frozen state) to stability of systems with variable coefficients. To this end we use Lyapunov functionals in sections 2, and a fixed point argument in section 3. Also we show that for every stiffness coefficient, there is a damping coefficient that makes (1.1) stable. Similarly, for every non-negative damping coefficient, we find a stiffness coefficient so that (1.1) is stable if and only if an integral condition on \(b\) is satisfied. Then as an application, we extend the stability results to nonlinear systems. We conclude this article by presenting some instability results that complement those in the literature.

In this article, we assume that \(b(t)\) and \(k(t)\) are continuous functions so that standard arguments in differential equations guarantee the existence and uniqueness of a solution \(u(t) = u(t, u_0, u_1)\).

**Definition.** The zero solution is stable if for each \(\epsilon > 0\), there exists a corresponding \(\delta(\epsilon, t_0) > 0\) such that \(u(t_0)^2 + u'(t_0)^2 < \delta^2\) implies \(u(t)^2 + u'(t)^2 < \epsilon^2\) for all \(t \geq t_0\). Equivalently, \(\max\{|u(t_0)|, |u'(t_0)|\} < \delta\) implies \(\max\{|u(t)|, |u'(t)|\} < \epsilon\).

The zero solution is asymptotically stable if for some \(\delta > 0\), the condition \(u(t_0)^2 + u'(t_0)^2 < \delta^2\) implies \(\lim_{t \to -\infty} u(t)^2 + u'(t)^2 = 0\). Equivalently, \(\max\{|u(t_0)|, |u'(t_0)|\} < \delta\) implies \(\lim_{t \to -\infty} |u(t)| = \lim_{t \to -\infty} |u'(t)| = 0\).

The zero solution is strictly stable if it is stable and asymptotically stable. A solution that is not stable is called unstable. For linear systems, the stability of one solution implies the stability of all solutions, in which case the system is called stable.

2. Stability using Lyapunov functionals

Stability for (1.1), with positive stiffness, has been established when \(b(t)\) and \(k(t)\) are bounded above and below by positive constants in [10]. Assuming that \(|b|, \ |k|, \ |k'\) are bounded above, Ignatiev [5] proved uniform asymptotic stability, under the assumption that \(k\) and \(k'/(2k) + b\) are bounded below by two positive constants.

Stability for systems of the form \(x' = A(t)x\) and \(x' = A(t,x)x\) has been studied by several authors [3, 4, 7, 8, 9]. However, their restrictions on the matrix \(A(t)\) do not allow for negative stiffness. In [4], the eigenvalues have negative real part, and the matrix satisfies some growth conditions. In [9] the average of the real part of the eigenvalues is negative and the matrix satisfies some growth conditions.

Our first stability result reads as follows.

**Theorem 2.1.** The zero solution of (1.1) is stable if, for all \(t \geq t_0\), the following conditions are satisfied:

\[
b(t) > 0, \quad \frac{1}{b(t)} + k(t) \geq M \quad \text{for a constant } M \quad \text{(which may be negative)}.
\]

\[
\frac{d}{dt} e^{\frac{1}{2}+k} u^2 \leq -\left(e^{\frac{1}{2}+k} - k\right)^2/(2b).
\]

**Proof.** First, we define the Lyapunov functional

\[
E(t) = e^{\frac{1}{2}+k} u^2 + (u')^2
\]
and compute its derivative along the solutions of (1.1).

\[ E'(t) = \frac{d}{dt}[e^{\frac{t}{b} + k}]u^2 + 2(e^{\frac{t}{b} + k} - k)uu' - 2b(u')^2. \]

Then, factoring \(-2b\) in the last two terms, and completing the square, we have

\[ E'(t) = \left[ \frac{1}{2b}(e^{\frac{t}{b} + k} - k)^2 + \frac{d}{dt}[e^{\frac{t}{b} + k}]u^2 - 2b\left( \frac{1}{2b}(e^{\frac{t}{b} + k} - k)u - u' \right) \right] - 2b(u')^2. \]

(2.3)

By (2.2), the coefficient of \(u^2\) is non-positive, and because \(b > 0\), \(E'(t) \leq 0\); therefore, \(E(t) \leq E(t_0)\) for all \(t \geq t_0\).

Now, we show that the zero solution is stable. For \(\epsilon > 0\), we select \(\delta > 0\) such that

\[ \delta^2 < \min\{e^{M}, 1\} \epsilon^2 / \max\{1, \exp\left(\frac{1}{b(t_0)} + k(t_0)\right)\}. \]

Note that \(u(t_0)^2 + u'(t_0)^2 < \delta^2\) implies

\[ \min\{e^{M}, 1\} \epsilon^2 > \max\{1, \exp\left(\frac{1}{b(t_0)} + k(t_0)\right)\}\delta^2 \geq E(t_0). \]

Also note that

\[ E(t_0) \geq E(t) \geq e^M u(t)^2 + u'(t)^2 \geq \min\{e^M, 1\}(u(t)^2 + u'(t)^2). \]

The stability of the zero solution follows from the two inequalities above. This completes the proof. \(\square\)

Uniform stability is obtained under the additional assumption that \(1/b + k\) is bounded above; because, the delta in the proof can be chosen independent of \(t_0\).

**Remark 2.2.** The above theorem makes the transition from instability (at the frozen state) to stability take place. However, when \(k(t)\) is non-positive and non-decreasing, Conditions (2.1) and (2.2) imply \(b(t)\) growing exponentially, which is very restrictive. To prove this remark, note that for non-decreasing \(k\), we have \((1/b)' \leq (1/b + k)\). Then by (2.1), \(e^M \leq e^{t/k} + k \geq e^M\). Then by (2.2), \((1/b)'e^M \leq -\frac{1}{2b}e^M\). This implies \(b' \geq b/2\), which in turn implies \(b(t) \geq b(0)e^{t/2}\).

**Example.** Among the equations with exponential damping, there are stable equations that satisfy and some others that do not satisfy (2.1)-(2.2). For instance, if \(k = -1\) and \(b = e^{nt}\), then (1.1) has solutions of the form

\[ \exp\left(\frac{n^2 t - e^{nt}}{2n}\right) \left(c_1 \left[ I\left( -\frac{1}{2} + \frac{1}{n}, z \right) + I\left( \frac{1}{2} + \frac{1}{n}, z \right) \right] + c_2 \left[ K\left( -\frac{1}{2} + \frac{1}{n}, z \right) - K\left( \frac{1}{2} + \frac{1}{n}, z \right) \right] \right). \]

where \(c_1\) and \(c_2\) are the integration constants, \(z \equiv e^{nt}/2n\), and \(I(a, z)\) and \(B(a, z)\) are the modified Bessel functions of first and second kind, respectively. When \(n = 1\) this solution converges uniformly to \(c_1\), when \(n > 1\) it converges to \(2c_1 \sqrt{n/\pi}\), as \(t \to \infty\). In both cases this implies stability. The zero solution is also stable for \(n \leq 1/2\). For instance, for \(n = 1/2\) the solution converges to \(32c_1\), and for \(n = 1/4\) it does to \(781250c_1\). However, (2.1)-(2.2) are satisfied only for \(n \geq 2\).

In an attempt to weaken the growth restrictions on \(b\), we consider now the case where the damping is a positive constant. Note that the assumptions below restrict the stiffness to remain negative. Also note that the larger (smaller) the constant damping is, the slower (faster) the negative stiffness is needed to be compensated.
Theorem 2.3. The zero solution of (1.1) is stable if \( b(t) \) is a positive constant and for all \( t \geq t_0 \):

There exists a positive constant \( \alpha \) such that \(-k(t) \geq \alpha\), \( -k' + 2k^2/b \leq 0 \). \hspace{1cm} (2.4)

\( -k' + 2k^2/b \leq 0 \). \hspace{1cm} (2.5)

Proof. We define the Lyapunov functional

\[ E(t) = -k(t)u(t)^2 + u'(t)^2, \]

whose derivative along solutions of (1.1) is

\[ E'(t) = -k'u^2 - 2kuv' + 2v'u'' \]

\[ = -k'u^2 - 4kuv' - 2v'(u')^2 \]

\[ = [-k' + 2\frac{k^2}{b}]u^2 - 2b\frac{k}{b}u + u'. \]

By (2.5), the coefficient of \( u^2 \) is non-negative. Since \( b > 0 \), \( E'(t) \leq 0 \) so that \( E(t) \leq E(t_0) \) for all \( t \geq t_0 \). To show stability of the zero solution, for each positive \( \epsilon \), we select \( \delta > 0 \) such that

\[ \max\{1, -k(t_0)\}\delta^2 < \min\{1, \alpha\}\epsilon^2. \]

With this delta, we can show that the definition of stability is satisfied, and hence the proof is complete. \( \square \)

Note that (2.5) implies \( k' \geq 2k^2/b \) which yields a lower bound for the rate of change in \( k \). Since \( b \) is constant, this inequality provides bounds for the rate of change in the eigenvalues of \( A(t) \):

\[ \frac{d\lambda_+}{dt} \leq -\frac{2k^2}{b\sqrt{b^2 - 4k}} \leq 0 \quad \text{and} \quad \frac{d\lambda_-}{dt} \geq \frac{2k^2}{b\sqrt{b^2 - 4k}} \geq 0. \]

So that the transition to stability happens when both eigenvalues approach zero sufficiently fast.

3. Stability using a fixed point theorem

In this section we eliminate the restriction that \( b \) must grow exponentially, by using a fixed point argument similar to those used in [2]. We start by stating a condition that is necessary (but not sufficient) for stability.

Lemma 3.1. Assume \( k(t) \leq 0 \) for \( t \geq t_0 \). Then the condition

\[ \int_{t_0}^{\infty} e^{-\int_{t_0}^{s} b} ds < \infty \] \hspace{1cm} (3.1)

is necessary for stability of (1.1).

Proof. First, using the integrating factor \( \exp \left( \int_{t_0}^{s} b(s) \right) ds \) we transform (1.1) into the equivalent equation

\[ (e^{f b} u')' + ke^{f b} u = 0 \] \hspace{1cm} (3.2)

which is used for setting up a contrapositive argument. Let the initial values \( u(t_0) \) and \( u'(t_0) \) be positive. Then by the continuity of the solution there is a non-empty maximal interval \([t_0, t_1]\) where \( u(t) \geq 0 \). On this interval,

\[ (e^{f b} u')' = -ke^{f b} u \geq 0. \]
So that \( e^{\int b} u' \) is non-decreasing; hence,
\[
u'(t) \geq u'(t_0) e^{-\int_{t_0}^t b} > 0.
\]
Therefore, \( u(t) \) is increasing and the maximal interval can be extended to \([t_0, \infty)\).
Integration on the above inequality yields
\[
u(t) \geq u(t_0) + u'(t_0) \int_{t_0}^t e^{-\int_{t_0}^s b} ds, \quad \forall t \geq t_0.
\]
Note that when (3.1) is not satisfied, the solution \( u(t) \) is unbounded which implies (1.1) being unstable. This completes the proof. \( \square \)

**Remark 3.2.** For each damping coefficient \( b(t) \), there is a stiffness coefficient \( k \) so that (1.1) is stable if and only if (3.1) is satisfied. In fact,
\[
k(t) = -\exp \left( -2 \int_{t_0}^t b(s) ds \right)
\]
leads to \( \exp(\pm \int_{t_0}^t e^{-\int_{t_0}^s b} ds \) being solutions of (1.1). To check stability, we use that all solutions can be written as \( u(t) = c_1 e^r + c_2 e^{-r} \) with \( r(t) = \int_{t_0}^t e^{-\int_{t_0}^s b} ds, \)
and that by (3.1), \( r(t) \) and \( r'(t) \) are bounded. On the other hand if (3.1) is not satisfied, one of the two solutions is unbounded which leads to instability.

To check that the two functions above are solutions, let \( \dot{\phi}(t) = e^{r(t)} \). Then \( \dot{\phi}' = r e^{r} \) and \( \phi'' = (r')^2 e^{r} + r'' e^{r} \). Since \( r' = e^{-f(b)} \), \( (r')^2 = e^{-2f(b)} = -k \), and \( r'' = -be^{-f(b)} \), it follows that \( k\dot{\phi} + (r')^2 e^{r} = 0 \) and \( b\dot{\phi}' + r'' e^{r} = 0 \). Therefore, \( \phi \) is a solution of (1.1).

As an illustration of the remark above, we have the following two examples: Firstly, when \( b(t) = 2/(t+1) \), (3.1) is satisfied and the equation
\[
u''(t) + \frac{2}{t+1} u'(t) - \frac{1}{(t+1)^2} u(t) = 0
\]
has solutions of the form \( u(t) = c_1 e^{1/(t+1)} + c_2 e^{-1/(t+1)} \). Since both exponential functions are bounded on \([0, \infty)\), we can prove stability. Secondly, when \( b(t) = 1/(t+1) \), (3.1) is not satisfied and the equation
\[
u''(t) + \frac{1}{t+1} u'(t) - \frac{1}{(t+1)^2} u(t) = 0
\]
has solutions of the form \( u(t) = c_1(t+1) + c_2/(t+1) \). Since the first function is unbounded on \([0, \infty)\), we have instability.

Now we set up a mapping whose fixed points are solutions of (1.1). From (3.2), it follows that the solution \( u(t) \) satisfies
\[
u'(t) = u'(t_0) e^{-\int_{t_0}^t b} - e^{-\int_{t_0}^t b} \int_{t_0}^t k(\tau) e^{\int_{t_0}^\tau b} u(\tau) d\tau,
\]
and
\[
u(t) = u(t_0) + u'(t_0) \int_{t_0}^t e^{-\int_{t_0}^s b} ds - \int_{t_0}^t e^{-\int_{t_0}^s b} \int_{t_0}^s k(\tau) e^{\int_{t_0}^\tau b} u(\tau) d\tau ds := F[u(t)]
\]
(3.3)
For $\epsilon > 0$, we define the following convex subset of the space of continuous differentiable functions. Let
\[
B_\epsilon = \{ u : |u(t_0)| \leq \epsilon/4, \ |u'(t_0)| \leq \min\{\epsilon/4, \epsilon/(4 \int_{t_0}^\infty e^{-f(t)b} d\tau)\}, \ |u(t)| \leq \epsilon, \ |u'(t)| \leq \epsilon \forall t \geq t_0 \}.
\]
Using the supremum norm $\|u\|_\infty = \sup_{t \geq t_0} |u(t)|$, we can show that this set is closed under the norm $\|u\|_\infty + \|u'\|_\infty$. Note that under assumption 3.1, this set is not empty; at least, there are constant functions in this set.

For the next result, we define the hypotheses:
\[
e^{-f_{t_0}^s b} < 2 \quad \forall s \geq t_0, \tag{3.4}
\]
\[
\int_{t_0}^\infty e^{-f_{t_0}^s b} \int_{t_0}^s |k(\tau)| e^{f_{t_0}^\tau b} d\tau d\tau < \frac{1}{2}, \tag{3.5}
\]
\[
e^{-f_{t_0}^s b} \int_{t_0}^s |k(\tau)| e^{f_{t_0}^\tau b} d\tau < \frac{1}{2} \quad \forall s \geq t_0. \tag{3.6}
\]

**Lemma 3.3.** Under assumptions 3.1 and 3.4–3.6, the transformation $F$ maps $B_\epsilon$ into $B_\epsilon$ and has a fixed point.

**Proof.** Let $u$ be a function in $B_\epsilon$. Then by 3.2 and 3.6,
\[
|F[u'](t)| \leq |u'(t_0)| e^{-f_{t_0}^s b} + \|u\|_\infty e^{-f_{t_0}^s b} \int_{t_0}^t |k(\tau)| e^{f_{t_0}^\tau b} d\tau < \epsilon/2 + \epsilon/2 = \epsilon.
\]
By 3.5,
\[
|F[u](t)| \leq |u(t_0)| + |u'(t_0)| \int_{t_0}^t e^{-f_{t_0}^s b} + \|u\|_\infty \int_{t_0}^t e^{-f_{t_0}^s b} \int_{t_0}^s |k(\tau)| e^{f_{t_0}^\tau b} d\tau d\tau < \epsilon/4 + \epsilon/4 + \epsilon/2 = \epsilon.
\]
Therefore, $F$ maps $B_\epsilon$ into itself. To find a fixed point for $F$, we define an iterative process that can start at any function $w_0$ in $B_\epsilon$. For $n = 1, 2, \ldots$, define $w_n = F[w_{n-1}]$. Note that the values $w_n(t_0)$ and $w_n'(t_0)$ remain unchanged in these iterations. Also note that by 3.3 and 3.6,
\[
|(w_{n+1} - w_n)(t)| \leq \int_{t_0}^t e^{-f_{t_0}^s b} \int_{t_0}^s |k(\tau)| e^{f_{t_0}^\tau b} \|w_n - w_{n-1}\|_\infty d\tau d\tau < \frac{1}{2} \|w_n - w_{n-1}\|_\infty
\]
and
\[
|(w_{n+1}' - w_n')(t)| \leq e^{-f_{t_0}^s b} \int_{t_0}^t |k(\tau)| e^{f_{t_0}^\tau b} \|w_n - w_{n-1}\|_\infty d\tau d\tau < \frac{1}{2} \|w_n - w_{n-1}\|_\infty.
\]
Therefore, $F$ is a contraction and $\{w_n\}$ converges to a fixed point of $F$ in $B_\epsilon$; hence the solution of (3.2) is in the set $B_\epsilon$ and satisfies the conditions for stability. This completes the proof. \hfill \Box

We are ready to present the main result of this section.

**Theorem 3.4.** Under assumptions 3.1 and 3.4–3.6, the zero solution of (1.1) is stable.

**Proof.** For each $\epsilon > 0$, we define $B_\epsilon$ as above and set $\delta = \min\{\epsilon/4, \epsilon/(4 \int_{t_0}^\infty e^{-f(t)b})\}$. For initial conditions $|u(t_0)| < \delta$ and $|u'(t_0)| < \delta$, the solution is obtained as a fixed point of $F$; therefore, $|u(t)| < \epsilon$ and $|u'(t)| < \epsilon$ which implies stability of (1.1). \hfill \Box
Remark 3.5. For each stiffness coefficient $k(t)$, there exists a damping coefficient $b(t)$ that makes (1.1) stable. In fact, for a constant $\alpha > 1$, we let

$$b(t) \geq 2|k(t)|(t + \alpha)^2 + 2/(t + \alpha),$$

(3.7)

so that the conditions in Theorem 3.4 are satisfied with $t_0 = 0$. Condition (3.4) follows from $b(t) \geq 0$. Note that $b \geq 2/(t + \alpha)$ and $\int_0^t b \geq \int_0^t 2/(t + \alpha) = 2 \ln(t + \alpha)$. So that

$$\exp(-\int_0^t b) \leq \exp(-2 \ln(t + 1\alpha)) = (t + \alpha)^{-2}.$$ 

Condition (3.1) follows from integrating in the inequality above. Note that from (3.7), $|k| \leq (t + \alpha)^{-2}b/2 - (t + \alpha)^{-3}$ which implies

$$|k|e^{\int b} \leq \frac{1}{2}(t + \alpha)^{-2}be^{\int b} - (t + \alpha)^{-3}e^{\int b}$$

Integrating on $[0, t]$, we have

$$\int_0^t |k|e^{\int b}d\tau \leq \frac{1}{2}(t + \alpha)^{-2}e^{\int b} - \frac{\alpha}{2},$$

Then

$$e^{-\int b} \int_0^t |k|e^{\int b}d\tau < \frac{1}{2}(t + \alpha)^{-2} \leq \frac{1}{2},$$

which is (3.6). Integrating on $[0, \infty)$, we have

$$\int_0^\infty e^{-\int b} \int_0^t |k|e^{\int b}d\tau dt < \frac{1}{2} \int_0^\infty (t + \alpha)^{-2}dt = \frac{1}{2\alpha^2} \leq \frac{1}{2},$$

which is (3.5). Therefore, (1.1) is stable with this choice of $b(t)$.

Remark 3.6. The believed instability is disproved as follows: For each $\epsilon > 0$, we find $b(t)$ and $k(t)$ such that (1.1) is stable and the rate of change in the eigenvalues of $A(t)$ is less than $\epsilon$, in absolute value.

Let $b(t) = 4/(t + \alpha)$ and $k(t) = -1/(t + \alpha)^3$, where $\alpha = \max\{1, \sqrt{5}/\epsilon\}$. Note that (3.7) is satisfied, and hence the conditions for Theorem 3.4 are satisfied; so that (1.1) is stable.

The rate of change in the eigenvalues of $A(t)$ is

$$\frac{d\lambda_+}{dt} = \frac{1}{2} \left( -1 \pm \frac{b}{\sqrt{b^2 - 4k}} \right) \frac{db}{dt} \mp \frac{1}{\sqrt{b^2 - 4k}} \frac{dk}{dt}.$$ 

The absolute value of the coefficient of $db/dt$ is bounded by 1, while $|db/dt| = 4/(t + \alpha)^2 \leq 4/\alpha^2$. The coefficient of $dk/dt$ is bounded as follows

$$\left| \frac{1}{\sqrt{b^2 - 4k}} \right| = \left| \frac{1}{b^2 - 4k} \right|^{1/2} \leq \frac{1}{b} = \frac{(t + \alpha)}{4}.$$ 

Since $dk/dt = 3/(t + \alpha)^4$, and $\alpha \geq 1$,

$$\left| \frac{d\lambda_+}{dt} \right| \leq \frac{4}{\alpha^2} + \frac{3}{4(t + \alpha)^3} \leq \frac{4}{\alpha^2} + \frac{3}{4\alpha^3} \leq \frac{19}{4\alpha^2} < \epsilon.$$ 

Which proves the claim of this remark.

We conclude this section with a stability result for the non-linear case.
Remark 3.7. As an applications to non-linear equations, we consider the differential equation
\[ u''(t) = f(t, u', u), \]  
where \( f(t, 0, 0) = 0 \) and \( f \) is differentiable at \( (t, 0, 0) \). Then \( u(t) \equiv 0 \) is a solution of (3.8). The stability of the zero solution is studied by considering the linearized version
\[ u''(t) = f_2(t, 0, 0)u' + f_3(t, 0, 0)u, \]  
where \( f_2(t, x, y) = \partial_u f(t, x, y) \) and \( f_3(t, x, y) = \partial_{uu} f(t, x, y) \). Note that this linear approximation is valid only for small values of \( u \) and of \( u' \). The stability of (3.9) is studied by setting \( b(t) = -f_2(t, 0, 0), \quad k(t) = -f_3(t, 0, 0) \) and applying results from this section, without any further modifications. However, the instability results in the next section may not hold because the linear approximation is valid only for values \( u, u' \) close to zero.

4. Instability Using Lyapunov and Chetaev Functionals

Instability of (1.1), with negative stiffness, was obtained by Ignatiev [5], assuming that \( |b|, |k|, |k'| \) are bounded above, and that \(-k\) and \( \frac{1}{b} + b \) are bounded below by two positive constants. In the same article, instability is proved when \( \frac{1}{t}b^2 + k \leq 0 \), and when \( \frac{1}{t}b^2 + k > 0 \) with some additional assumptions.

Our next result states that the zero solution is unstable, for non-positive stiffness and non-positive damping.

Theorem 4.1. If \( b(t) \leq 0 \) and \( k(t) \leq 0 \) for all \( t \geq t_0 \), then the zero solution is unstable.

Proof. For each pair of initial values \( u(t_0) > 0 \) and \( u'(t_0) > 0 \), by continuity of the solution and its derivative, there exists an interval where \( u(t) \geq 0 \) and \( u'(t) \geq 0 \). Since \( b \leq 0 \) and \( k \leq 0 \), from (1.1), it follows that \( u''(t) \geq 0 \); i.e., \( u \) is concave up and \( u'(t) \geq u'(t_0) \) on this interval. Let \( t_1 \) be the largest value such that \( u''(t) \geq 0 \) \( t_1 \leq t \leq t_1 \). If \( t_1 < +\infty \), the graph of \( u \) is concave up on \([t_0, t_1]\), \( u(t_1) \geq u(t_0) > 0 \) and \( u'(t_1) \geq u'(t_0) > 0 \). By continuity of the solution and its derivative, there exists \( t_2 > t_1 \), such that \( u(t) \geq 0 \) and \( u'(t) \geq 0 \) on \([t_1, t_2]\). Since \( b \leq 0 \) and \( k \leq 0 \), from (1.1), it follows that \( u''(t) \geq 0 \) on \([t_1, t_2]\). This contradicts \( t_1 \) being maximal; therefore, \( t_1 = +\infty \).

Because \( t_1 = +\infty \), the graph of \( u \) is concave up and \( u'(t_0) \geq u'(t_0) > 0 \) for all \( t \geq t_0 \). Therefore, \( \lim_{t \to +\infty} u(t) = +\infty \) for all arbitrarily small and positive initial values. This implies instability of the zero solution and completes the proof. \( \square \)

Example. The equation
\[ u''(t) - \frac{1}{t}u'(t) - \frac{1}{t}u(t) = 0 \]  
(4.1)

satisfies the conditions of Theorem 4.1 and has solutions of the form
\[ u(t) = c_1 t \mathcal{I}(2, 2\sqrt{t}) + c_2 t \mathcal{K}(2, 2\sqrt{t}). \]

For large \( t \) the dominant terms in each branch of this solution are
\[ \frac{c_1}{2\sqrt{\pi}} \frac{t^2}{4} e^{2\sqrt{t}} + \frac{c_2}{2} \frac{t^4}{4} e^{-2\sqrt{t}}. \]
which diverges as \( t \to \infty \), which implies instability.

**Theorem 4.2.** The zero solution of (1.1) is unstable under the following conditions: For all \( t \geq t_0 \),

\[
\text{there exists a positive constant } \alpha \text{ such that } -k(t) \geq \alpha, \quad (4.2)
\]

\[
k'(t) + 2b(t)k(t) \geq 0. \quad (4.3)
\]

**Proof.** We construct a functional similar to the one in Chetaev’s theorem [6]. However, the proof for the variable coefficient case is not the same as the constant case. Let

\[
V(t) = u(t)^2 + \frac{1}{k(t)}u'(t)^2,
\]

whose derivative along the solutions of (1.1) is

\[
V'(t) = 2uu' - \frac{k'}{k^2}(u')^2 + \frac{2}{k}u''u' = -\frac{k'}{k^2} + \frac{2kb}{k^2}(u')^2. \quad (4.4)
\]

Note that \( V(t) \) can be positive or negative and that we can select \( u(t_0) \) and \( u'(t_0) \) so that \( V(t_0) < 0 \). Then by (4.3), \( V'(t) \leq 0 \) and \( V(t) \) is non-increasing, which allows only two possible cases:

- **Case 1.** When \( \lim_{t \to \infty} V(t) = -\infty \), since \( V \geq \frac{1}{k}(u')^2 \geq -\frac{1}{\alpha}(u')^2 \), we have \( \lim_{t \to \infty} u'(t)^2 = +\infty \). Therefore, \( u' \) is unbounded and the zero solution is unstable.
- **Case 2.** When \( V(t) \) is bounded below, being non-increasing, it converges to some negative number \( -L^2 \). Then there exists \( t_1 \) such that \( V(t) \leq -L^2/4 \) for \( t \geq t_1 \). By (4.2),

\[
-\frac{1}{\alpha}(u')^2 \leq \frac{1}{k(t)}(u')^2 \leq V(t) \leq -L^2/4 \quad \text{for } t \geq t_1,
\]

which implies \( |u'(t)| \geq L\sqrt{\alpha}/2 > 0 \). If \( u' \) is positive, then it is bounded below by a positive constant for \( t \geq t_1 \). Therefore, \( \lim_{t \to \infty} u(t) = +\infty \) and the zero solution is unstable. If \( u' \) is negative, then it is bounded above by a negative constant for \( t \geq t_1 \). Therefore, \( \lim_{t \to \infty} u(t) = -\infty \) and the zero solution is unstable. This completes the proof. \( \square \)

**Theorem 4.3.** The zero solution of (1.1) is unstable under the following conditions: For all \( t \geq t_0 \),

\[
\text{there exists a positive constant } \alpha \text{ such that } -k(t) \geq \alpha, \quad (4.5)
\]

\[
k'(t) + 2b(t)k(t) \leq 0, \quad (4.6)
\]

\[
\text{there exist positive constants } \alpha_3 \text{ and } t_3 \text{ such that}
\]

\[
-k'(t) - 2b(t)k(t) \geq \frac{1}{\sqrt{\alpha_3}}b(t)^2k(t)^2, \quad \text{for } t \geq t_3. \quad (4.7)
\]

**Proof.** We define the Chetaev’s functional and compute its derivative as in the proof Theorem [1.2]. Next we select \( u(t_0) \) and \( u'(t_0) \) so that \( V(t_0) > 0 \). Then by (4.4), \( V'(t) \geq 0 \) and \( V(t) \) is non-decreasing, which allows only two possible cases:

- **Case 1.** When \( \lim_{t \to \infty} V(t) = +\infty \), since \( V(t) \leq u(t)^2 \), we have \( \lim_{t \to \infty} u(t) = +\infty \) and the zero solution is unstable.
- **Case 2.** When \( V(t) \) is bounded above, being nondecreasing, it converges to some positive number \( L^2 \). Therefore, \( \int_{t_0}^{\infty} V'(t) \, dt \) converges. Since the integral \( \int_{t_0}^{\infty} 1/t \, dt \)
diverges, a limit comparison yields \( \lim_{t \to -\infty} tV'(t) = 0 \). From [4.4], [1.5], and [1.7], we have
\[
0 = \lim_{t \to -\infty} tV'(t) \geq \lim_{t \to -\infty} a \beta b(t)^2(u'(t))^2 \geq 0.
\]
Then \( \lim_{t \to -\infty} |bu'| = 0 \); therefore, there exists a time \( t_1 \) such that \( |b(t)u'(t)| \leq L\alpha/4 \) for all \( t \geq t_1 \).

Since \( V(t) \leq u(t)^2 \) and \( \lim_{t \to -\infty} V(t) = L^2 \), there exists a time \( t_2 \) such that \( |u(t)| \geq 3L/4 \) for all \( t \geq t_2 \). There are two possible cases for \( t \geq \max\{t_1, t_2\} \):

Case 1: \( u(t) \geq 3L/4 \). From [1.1] and [4.5],
\[
u''(t) = -bu' - ku \geq -\frac{L\alpha}{4} - \frac{k3L}{4} \geq -\frac{L\alpha}{4} + \frac{3\alpha L}{4} = \frac{L\alpha}{2} > 0.
\]
The solution \( u \), being bounded below and having concavity greater than a positive constant, must have \( \lim_{t \to -\infty} u(t) = +\infty \). This implies instability of the zero solution.

Case 2: \( u(t) \leq -3L/4 \). From [1.1] and [4.5],
\[
u''(t) = -bu' - ku \leq \frac{L\alpha}{4} + \frac{k3L}{4} \leq \frac{L\alpha}{4} - \frac{3\alpha L}{4} = -\frac{L\alpha}{2} < 0.
\]
The solution \( u \), being bounded above and having concavity less than a negative constant, must have \( \lim_{t \to -\infty} u(t) = -\infty \). This implies instability of the zero solution.

Since the above reasoning applies to arbitrarily small positive initial conditions such that \( -k(t_0)u(t_0)^2 \geq u'(t_0)^2 \), the zero solution is unstable and the proof is complete.

\(\square\)

Acknowledgements. C.A.T.-E. wants to thank, for the kind hospitality, the MCTP at the University of Michigan, Ann Arbor, where part of this work was done.

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