THE SPECTRUM OF GROTHENDIECK MONOID: A NEW APPROACH TO CLASSIFY SERRE SUBCATEGORIES

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Abstract. The Grothendieck monoid of an exact category is a monoid version of the Grothendieck group. We use it to classify Serre subcategories of an exact category and to reconstruct the topology of a noetherian scheme. We first construct bijections between (i) the set of Serre subcategories of an exact category, (ii) the set of faces of its Grothendieck monoid, and (iii) the monoid spectrum of its Grothendieck monoid. By using (ii), we classify Serre subcategories of exact categories related to a finite dimensional algebra and a smooth projective curve. In particular, we determine the Grothendieck monoid of the category of coherent sheaves on a smooth projective curve. By using (iii), we introduce a topology on the set of Serre subcategories. As a consequence, we recover the topology of a noetherian scheme from the Grothendieck monoid.

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1. INTRODUCTION

1.1. Background. Classifying nice subcategories of an abelian category or a triangulated category is quite an active subject which has been studied in various areas of mathematics such as representation theory and algebraic geometry. A typical example is the following result given by Gabriel:

Fact 1.1 ([Gab62, Proposition VI.2.4]). Let $X$ be a noetherian scheme. There is an inclusion-preserving bijection between the following sets:

- The set of Serre subcategories of the category $\text{coh} X$ of coherent sheaves on $X$.
- The set of specialization-closed subsets of $X$

In [Gab62], Gabriel also proved that any noetherian scheme $X$ can be reconstructed from the category $\text{Qcoh} X$ of quasi-coherent sheaves on $X$. These results has been generalized by several ways. See [Zie84, Her97, Kan12] for classifications of Serre subcategories and [Ros98, BKS07, GP08, Bra16] for...
reconstruction theorems. The present paper sheds new light on these results by using Grothendieck monoids.

The Grothendieck monoid $M(E)$ is a monoid version of the Grothendieck group, which is defined for each exact category $E$. Several authors studied the Grothendieck monoid and extract information the Grothendieck group does not contain. See [Bro97, Bro98, Bro03] for module categories, [Eno22] for exact categories related to finite dimensional algebras and [BG16] for an application to Ringel-Hall algebras. This paper aims to study Serre subcategories of an exact category via its Grothendieck monoid.

1.2. Main results. Let us introduce the main subject in this paper. A subcategory $S$ of an exact category $E$ is Serre if for any conflation $0 \to X \to Y \to Z \to 0$ in $E$, we have that $Y \in S$ if and only if both $X \in S$ and $Z \in S$. The reader unfamiliar with the language of exact categories can think of exact categories and conflations as abelian categories and short exact sequences, respectively. The following result is the starting point of this paper.

**Theorem A** (Proposition 5.10). For an exact category $E$, there are bijections between the following sets:

1. The set $\text{Serre}(E)$ of Serre subcategories of $E$.
2. The set $\text{Face}(M(E))$ of faces of the Grothendieck monoid $M(E)$.
3. The set $\text{MSpec}(M(E))$ of prime ideals of the Grothendieck monoid $M(E)$.

Let us explain the terminologies used above. Let $M$ be a commutative monoid. A subset $F$ of $M$ is a face if for all $x, y \in M$, we have that $x + y \in F$ if and only if both $x \in F$ and $y \in F$. A subset $p$ of $M$ is a prime ideal if $p^c := M \setminus p$ is a face of $M$.

The second set $\text{Face}(M(E))$ can be computed purely algebraically, and its computation is much easier than examining the whole structure of the exact category $E$. The third set $\text{MSpec}(M(E))$ has a topology, which is a natural analogue of the Zariski topology on the spectrum $\text{Spec} R$ of a commutative ring $R$. These lead us in two directions.

The one direction is a classification of Serre subcategories by using faces of the Grothendieck monoid. We propose the following strategy to classify Serre subcategories of an exact category $E$.

1. Relate the Grothendieck monoid $M(E)$ with an abstract monoid $M$.
2. Classify faces of the abstract monoid $M$.
3. Classify Serre subcategories of $E$ by using (1) and (2).

Following this strategy, we classify Serre subcategories of some explicit exact categories. We present a primitive example to illustrate this idea.

**Example 1.2.** Consider the category $\text{vect} k$ of finite dimensional vector spaces over a field $k$. We classify extension-closed subcategories of $\text{vect} k$ by using a monoid. Here a subcategory $X$ of $\text{vect} k$ is extension-closed if it contains a zero object and for any exact sequence $0 \to U \to V \to W \to 0$, the condition $U, W \in X$ implies $V \in X$. Let $[\text{vect} k]$ be the set of isomorphism classes of objects of $\text{vect} k$. The assignment $V \mapsto \text{dim}_k V$, where $\text{dim}_k V$ denotes the dimension of $V$ over $k$, gives a bijection $[\text{vect} k] \cong \mathbb{N}$. Here $\mathbb{N}$ is the set of non-negative integers. We identify subcategories of $\text{vect} k$ (closed under isomorphisms) with subsets of $\mathbb{N}$ by this bijection. For vector spaces $U, V$ and $W$, there is an exact sequence $0 \to U \to V \to W \to 0$ if and only if $\text{dim}_k V = \text{dim}_k U + \text{dim}_k W$. Thus a subcategory of $\text{vect} k$ is extension-closed if and only if the corresponding subset of $\mathbb{N}$ is a submonoid. Therefore, classifying extension-closed subcategories of $\text{vect} k$ and submonoids of $\mathbb{N}$ are equivalent. This example shows why and how monoids are used to classify subcategories of an exact category.

In general, computation of the Grothendieck monoid $M(E)$ is difficult even if $E$ is an extension-closed subcategory of $\text{mod} \Lambda$ for a finite dimensional algebra $\Lambda$ (cf. [Eno22, Section 7]). However, we determine the Grothendieck monoids of the following exact category related to a smooth projective curve $C$:

- The category $\text{coh} C$ of coherent sheaves on $C$,
- The category $\text{vect} C$ of vector bundles over $C$,
- The category $\text{tor} C$ of coherent torsion sheaves on $C$.

**Theorem B** (Proposition 4.4, 4.9 and 4.12). Let $C$ be a smooth projective curve over an algebraically closed field $k$.

1. $M(\text{tor} C) \cong \text{Div}^+ C$ holds, where $\text{Div}^+ C$ is the monoid of effective divisors on $C$.
2. $M(\text{vect} C) \cong (\text{Pic} C \times \mathbb{N}^+) \cup \{(\mathcal{O}_C, 0)\} \subseteq \text{Pic} C \times \mathbb{Z}$ holds, where $\text{Pic}(C)$ is the Picard group of $C$ and $\mathbb{N}^+$ is the semigroup of strictly positive integers.
(3) We can regard $M(\text{tor } C)$ and $M(\text{vect } C)$ as submonoids of $M(\text{coh } C)$. Then $M(\text{coh } C)$ is the disjoint union of $M(\text{tor } C)$ and $M(\text{vect } C)^+ := M(\text{vect } C) \setminus \{0\}$. See Corollary 4.17 for the complete description of $M(\text{coh } C)$ as a monoid.

See Example 4.14 for the comparison of the Grothendieck group $K_0(\text{coh } C)$ and the Grothendieck monoid $M(\text{coh } C)$. As a corollary of this theorem, we have the following:

**Theorem C** (Corollary 4.10). Let $C$ be a smooth projective curve over an algebraically closed $k$. Then $\text{vect } C$ has no nontrivial Serre subcategories.

The other direction is a study of the space $\text{Serre}(E)$ whose topology induced by the Zariski topology on $\text{MSpec } M(E)$ (see §5.1). For a noetherian scheme $X$, we can construct an immersion $X \hookrightarrow \text{Serre}(\text{coh } X)$ of topological spaces. We obtain the following result by using this observation.

**Theorem D** (Corollary 5.23). Consider the following conditions for noetherian schemes $X$ and $Y$.

1. $X \cong Y$ as schemes.
2. $M(\text{coh } X) \cong M(\text{coh } Y)$ as monoids.
3. $\text{MSpec } M(\text{coh } X) \cong \text{MSpec } M(\text{coh } Y)$ as topological spaces.
4. $X \cong Y$ as topological spaces.

Then the implications “(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4)” hold.

It is surprisingly enough since the Grothendieck monoid $M(\text{coh } X)$ loses a lot of information and the Grothendieck group $K_0(\text{coh } X)$ never recover the topology of $X$.

**Organization.** This paper is organized as follows.

In Section 2, we first review some definitions and properties of commutative monoids, exact categories and its Grothendieck monoid. Then we establish a bijection between Serre subcategories of an exact category and faces of its Grothendieck monoid.

In Section 3, we study the Grothendieck monoid of an exact category with finiteness conditions and classify its Serre subcategories. In particular, we give an explicit example of classifying Serre subcategories of a complicated exact category related to a finite dimensional algebra by using its Grothendieck monoid (Example 3.18).

In Section 4, we determine the Grothendieck monoids of exact categories related to a smooth projective curve and classify Serre subcategories of them.

In Section 5, we first review the spectrum of a monoid and monoidal spaces, which are natural analogies of the spectrum of a commutative ring and ringed spaces, respectively. Next, we introduce a topology on the set of Serre subcategories and study relations with the spectrum of the Grothendieck monoid. Finally, we recover the topology of a noetherian scheme $X$ from the Grothendieck monoid $M(\text{coh } X)$.

**Conventions.** For a category $\mathcal{C}$, we denote $[\mathcal{C}]$ by the class of all isomorphism classes of objects and $\text{Hom}_\mathcal{C}(X, Y)$ by the set of morphisms between objects $X$ and $Y$ in $\mathcal{C}$. The isomorphism class containing $X \in \mathcal{C}$ is denoted by $[X]$. For a map $f : [\mathcal{C}] \to S$, we abbreviate $f([X])$ to $f(X)$. In this paper, we suppose that all categories are *skeletonally small*, that is, the class $[\mathcal{C}]$ forms a set. Also, we suppose that all subcategories are full subcategories closed under isomorphisms. We often identify subcategories of $\mathcal{C}$ and subsets of $[\mathcal{C}]$.

For a noetherian ring $\Lambda$ (with identity and associative multiplication), we denote by $\text{mod } \Lambda$ the category of finitely generated (right) $\Lambda$-modules. We set $\text{Hom}_\Lambda(X, Y) := \text{Hom}_{\text{mod } \Lambda}(X, Y)$.

A *variety* over a field $k$ means a separated integral scheme of finite type over $k$. A variety of dimension 1 is called a *curve*.

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2. **Classifying Serre subcategories via Grothendieck monoid**

In this section, we establish a bijection between the set of Serre subcategories of an exact category and the set of faces of its Grothendieck monoid (§2.3). In the first half of this section, we review some definitions and properties of commutative monoids (§2.1), exact categories and its Grothendieck monoid (§2.2).
2.1. Preliminaries: commutative monoids. We collect minimal definitions and properties of commutative monoids to describe our first results. The main reference of this subsection is [Ogu18].

A monoid is a semigroup with a unit. In this paper, every monoid is assumed to be commutative. Thus we use the additive notation, that is, the binary operation is denoted by +, and the unit is denoted by 0. A monoid homomorphism is a map \( f: M \to N \) satisfying \( f(x+y) = f(x) + f(y) \) and \( f(0_M) = 0_N \). We denote \( \text{Mon} \) by the category of (commutative) monoids and monoid homomorphisms. The category \( \text{Mon} \) has arbitrary small limits and colimits (see [Ogu18, Section I.1.1]). Note that the forgetful functor \( \text{Mon} \to \text{Set} \) preserves small limits, where \( \text{Set} \) denotes the category of sets and maps. We can define products \( \prod_{i \in I} M_i \) and direct sums (= coproducts) \( \bigoplus_{i \in I} M_i \) of monoids in a similar manner to vector spaces. In particular, finite products and finite direct sums coincide.

A basic example of monoids is the set \( N \) of non-negative integers with the arithmetic addition. A monoid \( M \) is said to be free if it is isomorphic to \( \mathbb{N}^{\oplus I} \) for some index set \( I \). The rank and a basis of a free monoid \( M \) are defined by similar ways to vector spaces.

Remark 2.1. For a monoid homomorphism \( f: M \to N \), define a submonoid of \( M \) by
\[
\ker(f) := \{ x \in M \mid f(x) = 0 \}.
\]
A caution is that the condition \( \ker(f) = 0 \) does not imply \( f \) is injective. Indeed, the map
\[
\mathbb{N}^{\oplus 2} \to \mathbb{N}, \quad (x, y) \mapsto x + y
\]
is a monoid homomorphism which is not injective and satisfies \( \ker(f) = 0 \).

The notion of quotients of monoids slightly differs from that of vector spaces. We introduce a class of equivalence relations \( \sim \) on a monoid \( M \) to guarantee that the quotient set \( M/\sim \) has the natural monoid structure.

Definition 2.2. The equivalence relation \( \sim \) on a monoid \( M \) is called a congruence if \( x \sim y \) implies \( a + x \sim a + y \) for every \( a, x, y \in M \).

We can check easily that the quotient set \( M/\sim \) of a monoid \( M \) by a congruence \( \sim \) has a unique monoid structure such that the quotient map \( M \to M/\sim \) is a monoid homomorphism. A submonoid \( N \) of \( M \) defines a congruence on \( M \) as follows:
\[
x \sim_N y : \iff \text{there exist } n, n' \in N \text{ such that } x + n = y + n'.
\]
Then the monoid \( M/N := M/\sim_N \) is called the quotient monoid of \( M \) by \( N \). We write \( x \equiv y \pmod{N} \) if \( x \sim_N y \). The equivalence class containing \( x \in M \) is denoted by \( x \mod{N} \). We can see immediately that the quotient monoids have the following universal property.

Proposition 2.3. Let \( N \) be a submonoid of \( M \) and let \( \pi: M \to M/N \) be the quotient homomorphism. Then for any monoid homomorphism \( f: M \to X \) of monoids such that \( f(N) = 0 \), there exists a unique monoid homomorphism \( \overline{f}: M/N \to X \) satisfying \( \overline{f} \pi = f \). This means that the diagram
\[
\begin{array}{ccc}
M & \xrightarrow{f} & X \\
\downarrow \pi & & \downarrow \overline{f} \\
M/N & \xrightarrow{f} & M/N
\end{array}
\]
is a coequalizer diagram in \( \text{Mon} \).

Recall that a binary relation on a set \( X \) is a subset \( R \subseteq X \times X \). It is clear that for any family of \( \{ R_i \subseteq M \times M \}_{i \in I} \) of congruences on a monoid \( M \), the intersection \( \bigcap_{i \in I} R_i \subseteq M \times M \) is also a congruence on \( M \). Thus for any binary relation on \( M \), there is the smallest subset of \( M \times M \) containing \( R \) which is a congruence on \( M \). We call it the congruence generated by a binary relation \( R \) on a monoid \( M \).

Next, we introduce a class of submonoids which corresponds to Serre subcategories in \( \mathcal{S} \).

Definition 2.4. Let \( M \) be a monoid.

1. A submonoid \( N \) of \( M \) is called a face if for all \( x, y \in M \), we have that \( x + y \in N \) if and only if both \( x \in M \) and \( y \in M \).
2. \( \text{Face}(M) \) denotes the set of faces of \( M \).

We regard \( \text{Face}(M) \) as a poset by the inclusion-order.

Remark 2.5. Let \( M \) be a monoid. An element \( x \in M \) is a unit if there exists \( y \in M \) such that \( x + y = 0 \). We denote by \( M^\times \) the set of units of \( M \). Then \( M^\times \) is the smallest face. On the other hand, \( M \) itself is the largest face of \( M \). Thus \( \text{Face}(M) \) has the maximum and minimum elements. We see that \( \text{Face}(M) \) has only one point if and only if \( M \) is a group.
Example 2.6. Let $M$ be a free monoid of rank 2 with a basis $e_1$ and $e_2$.

1. $\mathbb{N}(e_1 + e_2)$ is a submonoid of $M$ but not a face.
2. A face of $M$ is one of the following: $M$ itself, $Ne_1$, $Ne_2$ or 0.

We list properties of monoids which we will use.

**Definition 2.7.** Let $M$ be a monoid.

1. $M$ is **sharp** (or reduced) if $a + b = 0$ implies $a = b = 0$ for any $a, b \in M$.
2. $M$ is **cancellative** (or integral) if $a + x = a + y$ implies $x = y$ for any $a, x, y \in M$.

Finally, we discuss the relationship between monoids and groups.

**Definition 2.8.** The group completion $\text{gp}M = (\text{gp}M, \rho: M \to \text{gp}M)$ of a monoid $M$ is defined by the following universal property:

- For every monoid homomorphism $f: M \to G$ into a group $G$, there exists a unique group homomorphism $\overline{f}: \text{gp}M \to G$ such that $f = \rho \overline{f}$.

The group completion $M$ exists for any monoid $M$ and it is constructed as the localization of $M$ with respect to $M$ itself (see §5.1). The group completion has the following properties by the construction.

**Proposition 2.9** (cf. §5.1). Let $M$ be a monoid and $(\text{gp}M, \rho: M \to \text{gp}M)$ its group completion.

1. $\text{gp}M$ is an abelian group.
2. For any $x, y \in M$, the equality $\rho(x) = \rho(y)$ holds in $\text{gp}M$ if and only if $x + s = y + s$ in $M$ for some $s \in M$.

The cancellation property is related to the group completion as follows.

**Proposition 2.10.** The following are equivalent.

1. $M$ is cancellative.
2. The localization homomorphism $\rho: M \to \text{gp}M$ is injective.
3. There is an injective monoid homomorphism from $M$ to some group.

**Proof.** It follows immediately from Proposition 2.9 and the fact that any submonoid of an abelian group is cancellative. \qed

2.2. Preliminaries: Grothendieck monoids of exact categories. In this subsection, we first give a brief review of an exact category and then define the Grothendieck monoid of an exact category, which we study mainly in this paper.

Let $\mathcal{A}$ be an abelian category. An additive subcategory $\mathcal{E} \subseteq \mathcal{A}$ is said to be **extension-closed** if for any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, the condition $X, Z \in \mathcal{E}$ implies $Y \in \mathcal{E}$. We also say that $\mathcal{E}$ is an exact category when we omit the ambient abelian category $\mathcal{A}$. A **conflation** in $\mathcal{E}$ is an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ with $X, Z \in \mathcal{E}$. Note that $Y$ also belongs to $\mathcal{E}$ automatically. For a conflation $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{E}$, the morphism $f$ (resp. $g$) is called an **inflation** (resp. deflation). In this case, we say that $Z$ (resp. $X$) is the cokernel of the inflation $f$ (resp. the kernel of the deflation of $g$). An inflation (resp. a deflation) is sometimes denoted by $X \xrightarrow{f} Y$ (resp. $Y \xrightarrow{g} Z$).

**Example 2.11.** Consider the category $\text{mod}Z$ of finitely generated modules over the ring $Z$ of integers. The category $\text{proj}Z$ of finitely generated free (=projective) $Z$-modules is an extension-closed subcategory of the abelian category $\text{mod}Z$. Then the sequence

$$0 \to Z \xrightarrow{[1]} Z \oplus Z \xrightarrow{[1][1]} Z \to 0$$

is a conflation in $\text{proj}Z$, while the sequence

$$0 \to Z \xrightarrow{2} Z \to Z/2Z \to 0$$

is not a conflation in $\text{proj}Z$. Thus the monomorphism $Z \xrightarrow{[1]} Z \oplus Z$ is an inflation in $\text{proj}Z$ but the monomorphism $Z \xrightarrow{2} Z$ is not.

Let $\mathcal{E}$ be an exact category. An additive subcategory $\mathcal{F} \subseteq \mathcal{E}$ is said to be **conflation-closed** if for any conflation $0 \to X \to Y \to Z \to 0$ in $\mathcal{E}$, the condition $X, Z \in \mathcal{F}$ implies $Y \in \mathcal{F}$. For an abelian category, conflation-closed subcategories are equivalent to extension-closed subcategories. A conflation-closed subcategory $\mathcal{F} \subseteq \mathcal{E}$ is also an exact category which has the same ambient abelian category as that of $\mathcal{E}$.
Remark 2.12. There is an axiomatic definition of exact categories which does not depend on the embedding \( \mathcal{E} \hookrightarrow \mathcal{A} \) to an abelian category. See [Büh10] for sophisticated and standard treatments of exact categories. We do not follow this formulation for accessibility. Our treatment of exact categories is justified by Thomason’s embedding theorem (cf. [Büh10, Appendix A]). Even if the reader thinks of exact categories as the axiomatic ones, there is no problem at all in this paper.

The Grothendieck monoid \( M(\mathcal{E}) \) of an exact category \( \mathcal{E} \) is a monoid constructed as follows:

- Define the operation + on the set \([\mathcal{E}]\) of isomorphism classes of objects by \([X] + [Y] := [X \oplus Y]\). A pair \( M^{\oplus}(\mathcal{E}) := ([\mathcal{E}], +) \) is clearly a commutative monoid with an unit \([0]\), which we call the split Grothendieck monoid of \( \mathcal{E} \). Note that it only depends on the underlying additive category of \( \mathcal{E} \).
- Define a congruence \( \sim \sim \) on \( M^{\oplus}(\mathcal{E}) \) which is generated by the following relations:
  \[ [Y] \sim [X] + [Z] \text{ for every conflation } 0 \to X \to Y \to Z \to 0 \text{ in } \mathcal{E}. \]

The monoid \( M(\mathcal{E}) := M^{\oplus}(\mathcal{E})/\sim \) is called the Grothendieck monoid of \( \mathcal{E} \). An equivalence class containing \([X] \in [\mathcal{E}]\) with respect to \( \sim \) is also denoted by \([X]\). We only use \([X] = [Y]\) as an equality in \( M(\mathcal{E}) \) and it does not imply \( X \cong Y \) to avoid ambiguity.

Let us explain a universal property of the Grothendieck monoid \( M(\mathcal{E}) \). An additive function on an exact category \( \mathcal{E} \) with values in a monoid \( M \) is a map \( f : [\mathcal{E}] \to M \) satisfying the following conditions:

1. \( f(0) = 0 \) holds.
2. For any conflation \( 0 \to X \to Y \to Z \to 0 \) in \( \mathcal{E} \), we have that \( f(Y) = f(X) + f(Z) \) in \( M \).

Fact 2.13 ([Eno22, Definition 3.2, Proposition 3.3]). The Grothendieck monoid \( M(\mathcal{E}) \) of an exact category \( \mathcal{E} \) and the canonical map \( \pi : [\mathcal{E}] \to M(\mathcal{E}) \) have the following universal property:

1. \( \pi \) is an additive function on \( \mathcal{E} \).
2. For any additive function \( f : [\mathcal{E}] \to M \) with values in a monoid \( M \), there exists a unique monoid homomorphism \( \overline{f} : M(\mathcal{E}) \to M \) such that \( f = \overline{f}\pi \).

We introduce some terminologies to give a more direct characterization whether \([X] = [Y]\) in \( M(\mathcal{E}) \). Let \( X \) be an object of an exact category \( \mathcal{E} \). Two inflations \( Y \twoheadrightarrow X \) and \( Z \twoheadrightarrow X \) are equivalent if there is an isomorphism \( Y \cong Z \) such that the following diagram commutes:

\[
\begin{array}{ccc}
Y & \cong & X \\
\downarrow & & \downarrow \\
Z & \twoheadrightarrow & X
\end{array}
\]

An admissible subobject of \( X \) is the equivalence class \([Y \twoheadrightarrow X]\) of an inflation \( Y \twoheadrightarrow X \). We often say that \( Y \) is an admissible subobject of \( X \) and denote the cokernel of \( Y \twoheadrightarrow X \) by \( X/Y \). If \( \mathcal{E} \) is an abelian category, we omit the adjective admissible. For two admissible subobjects \( Y \) and \( Z \) of \( X \), we write \( Y \leq Z \) if there exists an inflation \( Y \twoheadrightarrow Z \) such that the following diagram commutes:

\[
\begin{array}{ccc}
Y & \twoheadrightarrow & X \\
\downarrow & & \downarrow \\
Z & \twoheadrightarrow & X
\end{array}
\]

This binary relation \( \leq \) yields a partial order on the set of admissible subobjects of \( X \). See [Eno22, Section 2] for a detailed study of the poset of admissible subobjects.

An admissible subobject series of \( X \) is a finite sequence \( 0 = X_0 \leq X_1 \leq \cdots \leq X_n = X \) of admissible subobjects of \( X \). Two admissible subobject series \( 0 = X_0 \leq X_1 \leq \cdots \leq X_n = X \) and \( 0 = Y_0 \leq Y_1 \leq \cdots \leq Y_m = Y \) are isomorphic if \( n = m \) and there exists a permutation \( \sigma \in \Sigma_n \) such that \( X_i/X_{i-1} \cong Y_{\sigma(i)}/Y_{\sigma(i)-1} \) for all \( 1 \leq i \leq n \). In this case, we say that \( X \) and \( Y \) are \( S \)-equivalent \(^1\) and

\(^1\)The terminology \( S \)-equivalence comes from [Kin94], which is abbreviation of strongly equivalence introduced in [Ses67]. The original notion of S-equivalence is used to study the moduli space of representations of a finite dimensional algebra.
denote it by \( X \sim_S Y \). We have the following equality in \( M(\mathcal{E}) \):
\[
[X] = [X_0] + [X_2/X_1] + \cdots + [X_n/X_{n-1}] = [Y_1] + [Y_2/Y_1] + \cdots + [Y_n/Y_{n-1}] = [Y].
\]
Conversely, the following holds.

**Fact 2.14** ([Eno22, Proposition 3.4]). For two objects \( X \) and \( Y \) in \( \mathcal{E} \), the equality \( [X] = [Y] \) holds in \( M(\mathcal{E}) \) if and only if there exists a sequence of objects \( X = X_0, X_1, \ldots, X_m = Y \) in \( \mathcal{E} \) such that \( X_i \sim_S X_{i-1} \) for each \( i \).

If \( \mathcal{E} \) is an abelian category, we have a stronger version of this fact.

**Fact 2.15** ([Bro98, Proposition 3.3]). Let \( \mathcal{A} \) be an abelian category. For any two objects \( X \) and \( Y \) in \( \mathcal{A} \), the equality \( [X] = [Y] \) holds in \( M(\mathcal{A}) \) if and only if \( X \) and \( Y \) are \( S \)-equivalent.

Using this description, we can obtain the following properties of the Grothendieck monoid.

**Fact 2.16** ([Eno22, Proposition 3.5]). Let \( \mathcal{E} \) be an exact category.

1. For an object \( X \in \mathcal{E} \), we have that \( [X] = 0 \) in \( M(\mathcal{E}) \) if and only if \( X \cong 0 \).
2. \( M(\mathcal{E}) \) is sharp (see Definition 2.7).

Next, we investigate a functorial property of Grothendieck monoids. An additive functor \( F : D \to \mathcal{E} \) between exact categories is said to be exact if for any conflation \( 0 \to X \to Y \to Z \to 0 \) in \( D \), we have that \( 0 \to FX \to FY \to FZ \to 0 \) is also a conflation in \( \mathcal{E} \). An exact equivalence is an exact functor which has an exact quasi-inverse. Note that a fully faithful and essentially surjective exact functor is not necessarily an exact equivalence.

Let \( F : D \to \mathcal{E} \) be an exact functor between exact categories. Define an additive function \([D] \to M(\mathcal{E})\) by \([X] \mapsto [FX]\). It gives rise to a monoid homomorphism \( M(F) : M(D) \to M(\mathcal{E}) \) by Fact 2.13. For two exact functors \( F : \mathcal{E}_1 \to \mathcal{E}_2 \) and \( G : \mathcal{E}_2 \to \mathcal{E}_3 \), we have \( M(GF) = M(G)M(F) \). It is easy to check that for any exact equivalence \( F : D \xrightarrow{\sim} \mathcal{E} \), the monoid homomorphism \( M(F) : M(D) \to M(\mathcal{E}) \) is an isomorphism.

Finally, we compare the Grothendieck monoid \( M(\mathcal{E}) \) with the Grothendieck group \( K_0(\mathcal{E}) \).

**Remark 2.17.** Let \( \mathcal{E} \) be an exact category.

1. Recall that the Grothendieck group \( K_0(\mathcal{E}) \) of \( \mathcal{E} \) is defined by
\[
K_0(\mathcal{E}) := \bigoplus_{[X] \in \mathcal{E}} \mathbb{Z}[X]/([A] - [B] + [C] \mid 0 \to A \to B \to C \to 0 \text{ is a conflation}).
\]
Then there is a natural monoid homomorphism
\[
\rho : M(\mathcal{E}) \to K_0(\mathcal{E}), \quad [X] \mapsto [X].
\]
We can easily check that \( (K_0(\mathcal{E}), \rho) : (M(\mathcal{E}) \to K_0(\mathcal{E})) \) is the group completion.
2. The natural map \( \rho \) is injective if and only if \( M(\mathcal{E}) \) is cancellative by Proposition 2.10. In this case, the Grothendieck monoid \( M(\mathcal{E}) \) can be identified with the positive part
\[
K_0^+(\mathcal{E}) := \{[X] \in K_0(\mathcal{E}) \mid X \in \mathcal{E}\}
\]
of the Grothendieck group. Thus if \( M(\mathcal{E}) \) is cancellative, the computation of \( M(\mathcal{E}) \) becomes much easier. However, not much is known about the conditions for an exact category \( \mathcal{E} \) under which \( M(\mathcal{E}) \) becomes cancellative. Nevertheless, we prove that the Grothendieck monoid \( M(\text{vect}\,C) \) of the category of vector bundles over a smooth projective curve is cancellative in §4.2.
3. An element of \( M(\mathcal{E}) \) can be expressed by \([X]\) for some single object \( X \in \mathcal{E} \), while an element of \( K_0(\mathcal{E}) \) can only be expressed by \([X] - [Y]\) for some objects \( X, Y \in \mathcal{E} \) in general. It is an advantage of the Grothendieck monoid.
4. Grothendieck monoids are more rigid than Grothendieck groups since they are not invariant under triangulated equivalences of the bounded derived categories \( D^b(\mathcal{E}) \) but only invariant under exact equivalences of exact categories (see Example 4.14).

### 2.3. Serre subcategories and faces

In this subsection, \( \mathcal{E} \) is an exact category. Let us give a correspondence between subcategories of \( \mathcal{E} \) and subsets of \( M(\mathcal{E}) \). For a subcategory \( D \) of \( \mathcal{E} \), we define a subset of \( M(\mathcal{E}) \) by
\[
M_D := \{[X] \in M(\mathcal{E}) \mid X \in D\}.
\]
For a subset \( N \subseteq M(\mathcal{E}) \), we define a subcategory of \( \mathcal{E} \) by
\[
D_N := \{X \in \mathcal{E} \mid [X] \in N\}.
\]
We will show that these assignments give a bijection between certain subcategories of \( \mathcal{E} \) and certain subsets of \( \text{M}(\mathcal{E}) \).

**Definition 2.18.** A subcategory \( \mathcal{D} \) of \( \mathcal{E} \) is said to be closed under \( S \)-equivalences if \( X \sim_S Y \) and \( X \in \mathcal{D} \) implies \( Y \in \mathcal{D} \) for any \( X, Y \in \mathcal{E} \). In this case, we also say that \( \mathcal{D} \) is an \( S \)-closed subcategory for short.

The relation between the categories \( \mathcal{D}_N \) defined as above and \( S \)-closed subcategories is as follows.

**Proposition 2.19.** The following hold.

1. For a subset \( N \subseteq \text{M}(\mathcal{E}) \), the subcategory \( \mathcal{D}_N \) is closed under \( S \)-equivalences.
2. The maps \( \Phi: \mathcal{D} \to \text{M}_\mathcal{D} \) and \( \Psi: \mathcal{N} \to \mathcal{D}_N \) are mutually inverse bijections between the set of \( S \)-closed subcategories and the power set of \( \text{M}(\mathcal{E}) \).

**Proof.** (1) Let \( X \in \mathcal{D}_N \) and \( Y \in \mathcal{E} \) satisfying \( X \sim_S Y \). Then we have \([X] \in N \) and \([X] = [Y] \) in \( \text{M}(\mathcal{E}) \), and hence \([Y] \in N \). Thus \( Y \) also belongs to \( \mathcal{D}_N \).

(2) We first prove that \( \text{M}_{\mathcal{D}_N} = N \). Clearly, we have \( \text{M}_{\mathcal{D}_N} \supseteq N \). Take \([X] \in \text{M}_{\mathcal{D}_N} \). Then there is an object \( Y \in \mathcal{D}_N \) such that \([X] = [Y] \). Because \( \mathcal{D}_N \) is closed under \( S \)-equivalences by (1), the object \( Y \) also belongs to \( \mathcal{D}_N \), and thus \([X] \in N \). It follows that \( \text{M}_{\mathcal{D}_N} = N \). Next, we prove that \( \mathcal{D}_{\text{M}_\mathcal{D}} = \mathcal{D} \) for an \( S \)-closed subcategory \( \mathcal{D} \) of \( \mathcal{E} \). It is obvious that \( \mathcal{D}_{\text{M}_\mathcal{D}} \supseteq \mathcal{D} \). Take \( X \in \mathcal{D}_{\text{M}_\mathcal{D}} \). We have \([X] \in \text{M}_\mathcal{D} \) and then there exists an object \( Y \in \mathcal{D} \) such that \([X] = [Y] \). Since \( \mathcal{D} \) is closed under \( S \)-equivalences, we obtain that \( X \in \mathcal{D} \), which proves \( \mathcal{D}_{\text{M}_\mathcal{D}} = \mathcal{D} \). Therefore \( \Phi \) and \( \Psi \) are mutually inverse bijections. \( \square \)

We recall that properties of a subcategory \( \mathcal{D} \) of an exact category \( \mathcal{E} \).

- \( \mathcal{D} \) is closed under direct sums of two objects if for any objects \( X, Y \in \mathcal{D} \), their direct sum \( X \oplus Y \) in \( \mathcal{C} \) belongs to \( \mathcal{D} \).
- \( \mathcal{D} \) is closed under direct summands if \( X \oplus Y \in \mathcal{D} \) implies that both \( X \) and \( Y \) belong to \( \mathcal{D} \) for any objects \( X, Y \in \mathcal{E} \).
- \( \mathcal{D} \) is a Serre subcategory if it is nonempty and for any conflation \( 0 \to X \to Y \to Z \to 0 \) in \( \mathcal{E} \), we have that \( X, Z \in \mathcal{D} \) if and only if \( Y \in \mathcal{D} \).

In our terminology, an additive subcategory is a subcategory closed under direct sums of two objects and containing a zero object of \( \mathcal{C} \).

In what follows, we translate properties of subcategories of \( \mathcal{E} \) into those of subsets of the Grothendieck monoid \( \text{M}(\mathcal{E}) \). In particular, we give a bijection between Serre subcategories of \( \mathcal{E} \) and faces of \( \text{M}(\mathcal{E}) \).

**Lemma 2.20 (cf. [Emo22, Proposition 3.7]).** A Serre subcategory \( \mathcal{D} \) of \( \mathcal{E} \) is an additive subcategory closed under direct summands and \( S \)-equivalences.

**Proof.** The Serre subcategory \( \mathcal{D} \) is an additive subcategory closed under direct summands since there is an conflation \( 0 \to X \to X \oplus Y \to Y \to 0 \) for any \( X, Y \in \mathcal{E} \). We prove that \( \mathcal{D} \) is closed under \( S \)-equivalences. Let \( X \in \mathcal{D} \) and \( Y \in \mathcal{E} \) satisfying \( X \sim_S Y \). There are admissible subobject series \( 0 = X_0 \leq X_1 \leq \cdots \leq X_n = X \) and \( 0 = Y_0 \leq Y_1 \leq \cdots \leq Y_n = Y \), and a permutation \( \sigma \in \mathfrak{S}_n \) such that \( X_i/X_{i-1} \cong Y_{\sigma(i)}/Y_{\sigma(i)-1} \) for all \( i \). Since \( \mathcal{D} \) is Serre and \( X \in \mathcal{D} \), we have that \( X_i/X_{i-1} \in \mathcal{D} \), which implies \( Y_{\sigma(i)}/Y_{\sigma(i)-1} \in \mathcal{D} \) for all \( i \). Then we conclude that \( Y \) belongs to \( \mathcal{D} \) because it is conflation-closed. \( \square \)

**Lemma 2.21.** Let \( \mathcal{D} \) be a subcategory of \( \mathcal{E} \).

1. If \( \mathcal{D} \) is closed under direct sums of two objects, then \( \text{M}_\mathcal{D} \) is a subsemigroup of \( \text{M}(\mathcal{E}) \).
2. If \( \mathcal{D} \) is additive, then \( \text{M}_\mathcal{D} \) is a submonoid of \( \text{M}(\mathcal{E}) \).
3. If \( \mathcal{D} \) is Serre, then \( \text{M}_\mathcal{D} \) is a face of \( \text{M}(\mathcal{E}) \).

**Proof.** (1) and (2) are immediate. We only show that \( \text{M}_\mathcal{D} \) is a face of \( \text{M}(\mathcal{E}) \) for a Serre subcategory \( \mathcal{D} \) of \( \mathcal{E} \). Note that \( \text{M}_\mathcal{D} \) is a submonoid of \( \text{M}(\mathcal{E}) \) by (2) and Lemma 2.20. Suppose that \([X] + [Y] \in \text{M}_\mathcal{D} \) for some objects \( X, Y \in \mathcal{E} \). Then there exists \( Z \in \mathcal{D} \) such that \([Z] = [X] + [Y] = [X \oplus Y] \). This yields \( X \oplus Y \in \mathcal{D} \) because \( \mathcal{D} \) is \( S \)-closed. Since \( \mathcal{D} \) is closed under direct summands, both \( X \in \mathcal{D} \) and \( Y \in \mathcal{D} \), and hence we have that both \([X] \in \text{M}_\mathcal{D} \) and \([Y] \in \text{M}_\mathcal{D} \). This proves \( \text{M}_\mathcal{D} \) is a face of \( \text{M}(\mathcal{E}) \). \( \square \)

**Corollary 2.22.** The bijection in Proposition 2.19 induces inclusion-preserving bijections between the following sets:

1. The set of \( S \)-closed subcategories of \( \mathcal{E} \) and the power set of \( \text{M}(\mathcal{E}) \).
2. The set of \( S \)-closed subcategories of \( \mathcal{E} \) closed under direct sums of two objects and the set of subsemigroups of \( \text{M}(\mathcal{E}) \).
3. The set of \( S \)-closed additive subcategories of \( \mathcal{E} \) and the set of submonoids of \( \text{M}(\mathcal{E}) \).
4. The set \( \text{Serre}(\mathcal{E}) \) of Serre subcategories of \( \mathcal{E} \) and the set \( \text{Face}(\text{M}(\mathcal{E})) \) of faces of \( \text{M}(\mathcal{E}) \).
Proof. It follows form Proposition 2.19, Lemma 2.21 and direct consideration. \qed

We think of classifying Serre categories of an exact category as a special case of classifying faces of a monoid by this corollary. Following this philosophy, we classify faces of an abstract monoid and apply it to classify Serre subcategories in §3 and §4.

Finally, we compare \( M_\mathcal{D} \) with \( M(\mathcal{D}) \) for a Serre subcategory \( \mathcal{D} \).

Proposition 2.23. Let \( \mathcal{D} \) be a Serre subcategory of \( \mathcal{E} \) and \( i: \mathcal{D} \hookrightarrow \mathcal{E} \) be the natural inclusion functor. Then \( M(i): M(\mathcal{D}) \to M(\mathcal{E}) \) is an injective monoid homomorphism whose image is \( M_\mathcal{D} \subseteq M(\mathcal{E}) \).

Proof. It is clear that the image of \( M(i) \) is \( M_\mathcal{D} \). Thus we only show that \( M(i) \) is injective. By Lemma 2.20 and Fact 2.14, it is enough to show that if \( X \) and \( Y \) are \( S \)-equivalent in \( \mathcal{E} \), then they are also \( S \)-equivalent in \( \mathcal{D} \) for any \( X, Y \in \mathcal{D} \). The same proof of Lemma 2.20 also works to prove this. \( \square \)

For a Serre subcategory \( \mathcal{D} \) of \( \mathcal{E} \), we identify \( M(\mathcal{D}) \) with the submonoid \( M_\mathcal{D} \) of \( M(\mathcal{E}) \) by this proposition.

Remark 2.24. The bijection of Corollary 2.22 (3) is a not completely new result, which was originally mentioned in [Bro97, Proposition 16.8] for the category \( \mathsf{Mod}_\Lambda \) of all modules over a \( (not \ necessarily \ noetherian) \) ring \( \Lambda \). Brookfield used this bijection to study the Grothendieck monoid \( M(\mathsf{noeth}_\Lambda) \) of the category \( \mathsf{noeth}_\Lambda \) of noetherian \( \Lambda \)-modules by identifying it with \( M(\mathsf{Mod}_\Lambda) \) in our terminology.

In the present paper, we treat this result more seriously to classify Serre subcategories of an exact category.

3. The case of exact categories related to finite dimensional algebras

In this section, we give concrete examples of classifying Serre subcategories of an exact category \( \mathcal{E} \) by using its Grothendieck monoid \( M(\mathcal{E}) \). Our strategy is the following:

1. Relate the Grothendieck monoid \( M(\mathcal{E}) \) with an abstract monoid \( M \).
2. Classify faces of the abstract monoid \( M \).
3. Classify Serre subcategories of \( \mathcal{E} \) by using (1) and (2).

Although we think that some results in this section are well-known to experts, we give the proofs from the viewpoint of Grothendieck monoids.

In §3.1, we classify faces of some abstract monoid generated by a subset. In §3.2, we classify Serre subcategories of exact category with finiteness conditions by using the result of §3.1. In particular, we give an explicit example of classifying Serre subcategories of a complicated exact category related to a finite dimensional algebra (Example 3.18).

3.1. More properties of faces. Let us study more properties of faces to classify faces of an abstract monoid. Hereafter \( M \) is a monoid. We first give a description of the face generated by a subset. This explicit description is useful to study faces.

Fact 3.1 ([Ogu18, Proposition I.1.4.2]). Let \( S \) be a subset of \( M \).

1. The smallest submonoid of \( M \) containing \( S \) is equal to
   \[
   (S)_M := \langle x \mid x \in S \rangle_M := \left\{ \sum_{i=1}^{m} n_i x_i \mid m, n_i \in \mathbb{N}, x_i \in S \right\}.
   \]
   We call it the submonoid of \( M \) generated by \( S \).

2. The smallest face of \( M \) containing \( S \) is equal to
   \[
   (S)_\text{face} := \langle x \mid x \in S \rangle_{\text{face}} := \{ x \in M \mid there \ exists \ y \in M \ such \ that \ x + y \in (S)_M \}.
   \]
   We call it the face of \( M \) generated by \( S \).

We study the relation between faces of \( M \) and those of its quotient monoid \( M/N \). Unlike the case of vector spaces, submonoids of \( M/N \) do not correspond to submonoids of \( M \) containing \( N \).

Example 3.2. Let \( M := \mathbb{N}^{\mathbb{N}^{\mathbb{N}}} \) and \( N := \mathbb{N}(1,0) + \mathbb{N}(1,1) \subseteq M \). Then we have \( M/N = 0 \) but \( M \) and \( N \) themselves are distinct submonoids of \( M \) containing \( N \).

However, we have a bijection for faces.

Proposition 3.3. Let \( N \) be a submonoid of \( M \), and let \( \pi: M \to M/N \) be the quotient homomorphism.

1. If \( X \) is a face of \( M \) containing \( N \), then \( X/N := \pi(X) \) is also a face of \( M/N \).
2. If \( X' \) is a face of \( M/N \), then \( \pi^{-1}(X') \) is also a face of \( M \) containing \( N \).
(3) The assignments given in (1) and (2) give inclusion-preserving bijections between the set of faces of $M$ containing $N$ and that of $M/N$.

Proof. We only prove (1) and (3) since the proof of (2) is straightforward. The symbol $\overline{\pi}$ denotes the equivalence class of $x$ in $M/N$.

(1) It is clear that $X/N$ is a submonoid of $M/N$. Let $a, b \in M$ such that $\overline{a + b} \in X/N$. Then there exist $x \in X$ and $a, b' \in N$ such that $x + a = \overline{a + b} + b'$. Since $x + a \in X$ and $X$ is a face of $M$, we have that $a, b \in X$. Thus both $\overline{a}$ and $\overline{b}$ belong to $X/N$, which proves $X/N$ is a face.

(3) We have that $\pi^{-1}(X')/N = \pi(\pi^{-1}(X')) = X'$ since $\pi$ is surjective. It is easy to check that $\pi^{-1}(X/N) \subseteq X$. Let $a \in \pi^{-1}(X/N)$. Then we have $\overline{a} \in X/N$. There exist $x \in X$ and $a, b' \in N$ such that $x + a = a + b'$. Since $x + a \in X$ and $X$ is a face, we have $a \in X$, which proves $\pi^{-1}(X/N) \subseteq X$. \qed

Corollary 3.4. For a submonoid $N$ of $M$, there is an inclusion-preserving bijection between $\text{Face}(M/N)$ and $\text{Face}(M/\langle N \rangle_{\text{face}})$. In particular, we have an inclusion-preserving bijection $\text{Face}(M) \cong \text{Face}(M/M^\times)$, where $M^\times$ is the set of units of $M$.

Proof. It follows from Remark 2.5 and Proposition 3.3. \qed

Let $f : M \to N$ be a monoid homomorphism. For any face $X$ of $N$, the inverse image $f^{-1}(X)$ is also a face of $M$. Thus we have an inclusion-preserving bijection $\text{Face}(f) : \text{Face}(N) \to \text{Face}(M)$. The following lemma is obvious but useful.

Lemma 3.5. The map $\text{Face}(f) : \text{Face}(N) \to \text{Face}(M)$ is injective for a surjective monoid homomorphism $f : M \to N$.

Let us consider a finiteness condition on a monoid and classify faces of a monoid satisfying it.

Definition 3.6. Let $M$ be a monoid.

(1) $M$ is finitely generated if $M = \langle S \rangle_{\overline{N}}$ for a finite subset $S$ of $M$.

(2) A face $N$ of $M$ is finitely generated if $N = \langle S \rangle_{\text{face}}$ for a finite subset $S$ of $N$.

Remark 3.7. Let $M$ be a monoid.

(1) If $M$ is finitely generated, then it is finitely generated as a face.

(2) A face $N$ of $M$ is finitely generated if and only if $M = \langle x \rangle_{\text{face}}$ for some element $x \in M$. Indeed, if $M = \langle S \rangle_{\text{face}}$ for a finite subset $S$ of $M$, then we can easily see that $M = \langle \sum_{s \in S} s \rangle_{\text{face}}$.

Lemma 3.8. If $M$ is generated by a (not necessarily finite) subset $S \subseteq M$, then the map $\langle - \rangle_{\text{face}} : P(S) \to \text{Face}(M), \ A \mapsto \langle A \rangle_{\text{face}}$ is an inclusion-preserving surjection, where $P(S)$ is the power set of $S$.

Proof. Let $N$ be a face of $M$ and set $S_N := \{ x \in S \mid x \in N \}$. We show that $N = \langle S_N \rangle_{\text{face}}$. Clearly $N \supseteq \langle S_N \rangle_{\text{face}}$. Take $x \in N$. Then $x = \sum_{i=1}^n n_i x_i$ for some $x_i \in S$ and $0 \neq n_i \in N$ by Fact 3.1. Since $N$ is a face, we obtain that $S_i \subseteq N$ for all $i$, which shows $x \in \langle S_N \rangle_{\text{face}}$. Thus we conclude that $N = \langle S_N \rangle_{\text{face}}$ and $\langle - \rangle_{\text{face}} : P(S) \to \text{Face}(M)$ is surjective. Note that we actually prove $N = \langle S_N \rangle_{\overline{N}}$. \qed

Corollary 3.9. If $M$ is finitely generated, then $\text{Face}(M)$ is a finite set.

Proof. There is a finite subset $S$ of $M$ such that $M = \langle S \rangle_{\overline{N}}$ because $M$ is finitely generated. Then $P(S)$ is also a finite set, and we conclude that $\text{Face}(M)$ is a finite set by Lemma 3.8. \qed

Example 3.10. Let $M$ be a free monoid with a basis $\{ e_i \mid i \in I \}$. Then it is clear that the map $\langle - \rangle_{\text{face}} : P(\{ e_i \mid i \in I \}) \to \text{Face}(M)$ is bijective.

3.2. Serre subcategories of length exact categories. In this subsection, we study the Grothendieck monoid of an exact category with finiteness conditions and apply it to classify Serre subcategories of exact categories related to finite dimensional algebras.

We briefly review terminologies related to composition series to introduce finiteness conditions of exact categories. Let $\mathcal{E}$ be an exact category. An admissible subobject series $0 = X_0 \leq X_1 \leq \cdots \leq X_n = X$ of $X \in \mathcal{E}$ is proper if $X_{i+1}/X_i \neq 0$ for all $i$. In this case, we say that this proper admissible subobject series has length $n$. A nonzero object $X \in \mathcal{E}$ is said to be simple if it has no admissible subobject except 0 and $X$ itself. We denote by $\text{simp} \mathcal{E}$ the set of isomorphism classes of simple objects of $\mathcal{E}$. An admissible subobject series $0 = X_0 \leq X_1 \leq \cdots \leq X_n = X$ of $X \in \mathcal{E}$ is a composition series if $X_{i+1}/X_i$ is simple for all $i$. 10
Definition 3.11. Let $\mathcal{E}$ be an exact category.

1. An object $X$ of $\mathcal{E}$ is of finite length if the lengths of proper admissible subobject series of $X$ have an upper bound.
2. $\mathcal{E}$ is said to be length if every object in $\mathcal{E}$ is of finite length.
3. A length exact category $\mathcal{E}$ satisfies the Jordan-Hölder property if, for every $X \in \mathcal{E}$, all composition series of $X$ are isomorphic to each other.

Note that any object of finite length has a composition series since proper admissible subobject series of a maximal length are composition series (see [Eno22, Proposition 2.5]).

Example 3.12. Let $\mathcal{E}$ be an exact category.

1. A length-like function is an additive function $\ell : [\mathcal{E}] \to \mathbb{N}$ such that $\ell(X) = 0$ implies $X \cong 0$. If $\mathcal{E}$ has a length-like function, then $\mathcal{E}$ is a length exact category (see [Eno22, Lemma 4.3]).
2. Let $\Lambda$ be a finite dimensional algebra over a field $k$. Then $\text{mod} \, \Lambda$ is length abelian category since the dimension as vector spaces gives rise to a length-like function $\text{dim}_k : [\text{mod} \, \Lambda] \to \mathbb{N}$. An extension-closed subcategory of $\text{mod} \, \Lambda$ is also a length exact category.
3. A length abelian category satisfies the Jordan-Hölder property (see [Ste75, p.92, Examples 2]).

The following facts are basics to study the Grothendieck monoid of a length exact category.

Fact 3.13 ([Eno22, Proposition 4.8]). Let $\mathcal{E}$ be a length exact category. Then $M(\mathcal{E})$ is generated by the set $\{[S] \mid S \in \sim \mathcal{E}\}$. Moreover, $M(\mathcal{E})$ is finitely generated if and only if $\sim \mathcal{E}$ is a finite set.

Fact 3.14 ([Eno22, Theorem 4.12]). The following are equivalent for an exact category $\mathcal{E}$.

1. $\mathcal{E}$ satisfies the Jordan-Hölder property.
2. $M(\mathcal{E})$ is a free monoid with a basis $\{[S] \mid S \in \sim \mathcal{E}\}$.

In particular, if $A$ is a length abelian category, then $M(A)$ is a free monoid with a basis $\{[S] \mid S \in \sim A\}$.

We will now begin to classify Serre subcategories of a length exact category. For a subcategory $\mathcal{X}$ of an exact category $\mathcal{E}$, the Serre subcategory generated by $\mathcal{X}$ is the smallest Serre subcategory $\langle \mathcal{X} \rangle_{\text{Serre}}$ containing $\mathcal{X}$. We give a concrete description of $\langle \mathcal{X} \rangle_{\text{Serre}}$ by using the Grothendieck monoid:

$$\langle \mathcal{X} \rangle_{\text{Serre}} = \{ A \in \mathcal{E} \mid \text{there exist } B \in \mathcal{E} \text{ and } X \in \mathcal{X} \text{ such that } [A] + [B] = [X] \}$$

for an additive subcategory $\mathcal{X}$. It follows from Corollary 2.22 and Fact 3.1. In particular, we have

$$\langle \mathcal{X} \rangle_{\text{Serre}} = \{ A \in \mathcal{E} \mid \text{there exist } B \in \mathcal{E} \text{ and } n \in \mathbb{N} \text{ such that } [A] + [B] = [X^{\oplus n}] \}$$

for an object $X \in \mathcal{E}$. A Serre subcategory of the form $\langle \mathcal{X} \rangle_{\text{Serre}}$ is said to be finitely generated.

Proposition 3.15. Let $\mathcal{E}$ be a length exact category. Then we have an inclusion-preserving surjection $(-)_{\text{Serre}} : \mathcal{P}(\sim \mathcal{E}) \to \text{Serre}(\mathcal{E}), \mathcal{X} \mapsto \langle \mathcal{X} \rangle_{\text{Serre}}$.

Proof. It follows from Corollary 2.22, Lemma 3.8 and Fact 3.13.

As a corollary, we obtain a classification of Serre subcategories of an exact category satisfying the Jordan-Hölder property.

Corollary 3.16. Let $\mathcal{E}$ be an exact category satisfying the Jordan-Hölder property. Then we have an inclusion-preserving bijection $(-)_{\text{Serre}} : \mathcal{P}(\sim \mathcal{E}) \to \text{Serre}(\mathcal{E}), \mathcal{X} \mapsto \langle \mathcal{X} \rangle_{\text{Serre}}$.

Proof. It follows from Example 3.10, Fact 3.14 and Proposition 3.15.

We give a nontrivial example of classifying Serre subcategories of a length exact category which does not satisfy the Jordan-Hölder property. We first introduce the Cayley quiver, which is a monoid version of the Cayley graph of a group.

Definition 3.17 ([Eno22, Definition 7.5]). Let $M$ be a monoid generated by $A \subseteq M$. Then the Cayley quiver of $M$ with respect to $A$ is a quiver defined as follows:

- The vertex set is $M$.
- For each $a \in A$ and $m \in M$, we draw a (labeled) arrow $m \xrightarrow{a} m + a$.

For a length exact category $\mathcal{E}$, the natural choice of $A$ above is $\{[S] \mid S \in \sim \mathcal{E}\}$. 
Example 3.18 (cf. [Eno22, Section 7.2]). Let \( \Lambda \) be the path algebra of the quiver \( 1 \leftarrow 2 \) over a field \( k \). Then \( \text{mod} \Lambda \) is a length abelian category whose indecomposable objects are exactly two simple modules \( S_1, S_2 \) and one projective injective module \( P \). Thus \( \text{M} (\text{mod} \Lambda) = \mathbb{N} [S_1] \oplus \mathbb{N} [S_2] \cong \mathbb{N} [P] \) by Fact 3.14. We identify \( \text{M} (\text{mod} \Lambda) \) with \( \mathbb{N} [P] \) via this isomorphism. Let \( N := \text{N}(m,n) \) be the submonoid of \( \text{M} (\text{mod} \Lambda) \) generated by \( (m,n) \in \mathbb{N} [P] \) such that \( (m,n) \neq (0,0) \). Consider the extension-closed subcategory \( \mathcal{D}_N \) of \( \text{mod} \Lambda \) corresponding to \( N \) (see \$2.3$). Then \( \mathcal{D}_N \) is a length exact category by Example 3.12. The structure of \( \text{M} (\mathcal{D}_N) \) is determined by Enomoto [Eno22, Proposition 7.6] as follows:

1. \( \mathcal{D}_N \) has exactly \( l + 1 \) distinct simple objects \( A_0, \ldots, A_l \), where \( l := \min \{ m, n \} \) and

\[
A_i := P^{\oplus i} \oplus S_1^{\oplus (m-i)} \oplus S_2^{\oplus (n-i)}.
\]

Thus \( \text{M} (\mathcal{D}_N) \) is generated by \( [A_0], \ldots, [A_l] \).

2. Set \( a_i := [A_i] \) for \( 0 \leq i \leq l \). Then the Cayley quiver of \( \text{M} (\mathcal{D}_N) \) with respect to \( \{ a_i \mid 0 \leq i \leq l \} \) is determined as follows, where \( a_{0-k} \) denotes \( k + 1 \) arrows \( a_0, \ldots, a_k \) for \( 0 \leq k \leq l \).

(Case 1) The case \( m \neq n \):

\[
\begin{array}{ccccccc}
0 & \rightarrow & a_0 & \rightarrow & a_1 & \rightarrow & a_2 & \rightarrow & \ldots \\
& & a_0 & \rightarrow & a_1 & \rightarrow & a_2 & \rightarrow & \ldots \\
& & a_0 & \rightarrow & a_1 & \rightarrow & a_2 & \rightarrow & \ldots \\
& & a_0 & \rightarrow & a_1 & \rightarrow & a_2 & \rightarrow & \ldots
\end{array}
\]

In particular, \( \text{M} (\mathcal{C}) \) is free if and only if either \( m = 0 \) or \( n = 0 \).

(Case 2) The case \( m = n \):

\[
\begin{array}{ccccccc}
0 & \rightarrow & a_0 & \rightarrow & a_1 & \rightarrow & a_2 & \rightarrow & \ldots \\
& & a_0 & \rightarrow & a_1 & \rightarrow & a_2 & \rightarrow & \ldots \\
& & a_0 & \rightarrow & a_1 & \rightarrow & a_2 & \rightarrow & \ldots \\
& & a_0 & \rightarrow & a_1 & \rightarrow & a_2 & \rightarrow & \ldots
\end{array}
\]

Now we determine the faces of \( \text{M} (\mathcal{D}_N) \) to classify the Serre subcategories of \( \mathcal{D}_N \):

(Case 1) Any face \( F \) of \( \text{M} (\mathcal{D}_N) \) is of the form \( (a_i \mid i \in I) \text{face} \) for some \( I \subseteq \{0, \ldots, l\} \) by Lemma 3.8. If \( I \) is not empty, then \( F \) contains \( 2a_0 \). Thus all \( a_i \) belong to \( F \) since \( F \) is a face, and then \( F = \text{M} (\mathcal{D}_N) \). Therefore \( \mathcal{D}_N \) has no nontrivial Serre subcategories.

(Case 2) Let \( F = (a_i \mid i \in I) \text{face} \) be a face of \( \text{M} (\mathcal{D}_N) \) for some \( I \subseteq \{0, \ldots, n\} \). If \( i \in I \) for \( 0 \leq i \leq n-1 \), then \( 2a_0 \in F \), and thus \( F = \text{M} (\mathcal{D}_N) \). Unlike the case \( m \neq n \), \( \text{M} (\mathcal{D}_N) \) has a nontrivial face \( F = (a_i) \text{face} \). Hence \( \mathcal{D}_N \) has exactly three Serre subcategories \( 0, \mathcal{D}_N \), and \( (P^{\oplus n})_{\text{Serre}} \).

4. The case of exact categories related to a smooth projective curve

Hereafter \( C \) is a smooth projective curve over an algebraically closed field \( k \). There are three exact categories related to \( C \):

- The category \( \text{coh}(C) \) of coherent sheaves on \( C \).
- The category \( \text{vect}(C) \) of vector bundles over \( C \).
- The category \( \text{tor}(C) \) of coherent torsion sheaves on \( C \).

We determine the Grothendieck monoids of them and classify Serre subcategories of them.

For the basics of algebraic geometry, we refer to [Har77, GW20]. We fix notation on schemes. Let \( X \) be a noetherian scheme with structure sheaf \( \mathcal{O}_X \). A point of \( X \) is not necessarily assumed to be closed. For a point \( x \in X \), we denote by \( \mathfrak{m}_x \) the maximal ideal of \( \mathcal{O}_{X,x} \) and \( \kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x \) the residue field of \( x \). We denote by \( \text{coh} X \) the category of coherent sheaves on \( X \). We set \( \text{Hom}_{\text{coh}}(\mathcal{F}, \mathcal{G}) := \text{Hom}_{\text{coh}}(\mathcal{F}, \mathcal{G}) \). The tensor product of \( \mathcal{F} \) and \( \mathcal{G} \) over \( \mathcal{O}_X \) is denoted by \( \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \). The sheaf of homomorphisms from \( \mathcal{F} \) to \( \mathcal{G} \) is denoted by \( \text{Hom}_{\text{coh}}(\mathcal{F}, \mathcal{G}) \). If no ambiguity can arise, we will often omit the subscript \( \mathcal{O}_X \). The support of \( \mathcal{F} \) is a closed subset of \( X \) defined by \( \text{Supp} \mathcal{F} := \{ x \in X \mid \mathcal{F}_x \neq 0 \} \). For a noetherian commutative ring \( R \), we identify \( \text{mod} R \) with \( \text{coh}(\text{Spec } R) \).

For a morphism \( f: X \to Y \) of noetherian schemes, we denote by \( f_* \mathcal{F} \) the direct image of \( \mathcal{F} \in \text{coh } X \) and \( f^* \mathcal{G} \) the pull-back of \( \mathcal{G} \in \text{coh } Y \). We always have a functor \( f^*: \text{coh } Y \to \text{coh } X \), while we have a functor \( f_*: \text{coh } X \to \text{coh } Y \) when \( f \) is proper (e.g., \( f \) is a closed immersion).
We will review the categorical properties of \( \text{coh} \, C \) in each of the following subsections. Those are well-known and easy to prove but we write down the proofs since we do not find a suitable reference. The results of this section are new, except for the lemmas.

4.1. The case of coherent torsion sheaves. We first review a categorical characterization of coherent torsion sheaves. Let \( i : Z \rightarrow X \) be a closed immersion into a noetherian scheme \( X \), and let \( I \) be the quasi-coherent ideal sheaf corresponding to \( Z \). Then the functor \( i_* : \text{coh} \, Z \rightarrow \text{coh} \, X \) is a fully faithful exact functor whose essential image \( \text{Im} \, i_* \) is the subcategory consisting of coherent sheaves \( F \) such that \( I_* F = 0 \) (see [SP, Tag 01QX]). It follows immediately that \( \text{Im} \, i_* \) is closed under subobjects in \( \text{coh} \, X \). This means that there is no difference between subobjects of \( F \in \text{coh} \, Z \) and subobjects of \( i_* F \in \text{coh} \, X \). For a closed point \( x \in X \), consider the natural closed immersion \( i : \text{Spec} \, \kappa(x) \rightarrow X \). Then \( O_x := i_* O_{\text{Spec} \, \kappa(x)} \) is a simple object of \( \text{coh} \, X \) by the above discussion.

**Lemma 4.1.** The following are equivalent for a coherent sheaf \( F \) on a noetherian scheme \( X 

1. \( F \) is a simple object in \( \text{coh} \, X \).
2. \( F \cong O_x \) for some closed point \( x \in X \).

**Proof.** It is enough to show that (1) implies (2). Suppose that \( F \) is a simple object in \( \text{coh} \, X \). Note that \( \text{Supp} \, F \neq \emptyset \) since \( F \neq 0 \). There is a closed point \( x \in \text{Supp} \, F \) because \( X \) is noetherian (cf. [GW20, Lemma 1.25, Exercise 3.13]). Let \( i : \text{Spec} \, \kappa(x) \rightarrow X \) be the natural closed immersion. Then \( F(x) := i_* F = F_x / m_x \in \text{mod} \, \kappa(x) \) is nonzero by Nakayama’s lemma. Because the unit morphism \( F \rightarrow i_* i^* F = i_* F(x) \) is surjective and \( F \) is simple, we have that \( F \cong i_* F(x) \). Then \( F(x) \) is also a simple object in \( \text{mod} \, \kappa(x) \). This means \( F(x) \cong \kappa(x) \), and we obtain the desired result.

**Lemma 4.2.** The following are equivalent for a coherent sheaf \( F \) on a noetherian scheme \( X 

1. \( F \) is an object of finite length in \( \text{coh} \, X \).
2. \( \text{Supp}(F) \) has only finitely many points.

In this case, the following hold:

1. \( F_x \) is an \( O_{X,x} \)-module of finite length for any \( x \in X \).
2. The natural morphism \( F \rightarrow \bigoplus_{x \in \text{Supp} \, F} i_* F_x \) is an isomorphism, where \( i_* \) is the natural morphism \( \text{Spec} \, O_{X,x} \rightarrow X \).

**Proof.** (1) \( \Rightarrow \) (2): There is a composition series \( 0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = F \) of finite length. Then \( F_{i+1} / F_i \cong O_{x_i} \) for some closed point \( x_i \in X \) by Lemma 4.1. Thus we have \( \text{Supp} \, F = \bigcup_{i=1}^{n-1} \text{Supp} \, O_{x_i} = \{ x_i \mid 1 \leq i \leq n - 1 \} \).

(2) \( \Rightarrow \) (1): We regard \( Z := \text{Supp} \, F \) as a closed subscheme of \( X \) which corresponds to the annihilator \( \text{Ann}(F) \) of \( F \) (cf. [GW20, Subsection 7.17]). Note that \( O_{Z,x} = O_{X,x} / \text{Ann}_{O_{X,x}}(F_x) \) as rings. Then the natural morphism \( \prod_{x \in \text{Supp} \, F} \text{Spec} \, O_{Z,x} \rightarrow Z \) is an isomorphism and \( O_{Z,x} \) is an artinian local ring by (2) and [GW20, Proposition 5.11]. Since \( F_x \) is finitely generated over the artinian ring \( O_{Z,x} \), it is of finite length as an \( O_{Z,x} \)-module, and hence (i) holds. Let \( j : Z \rightarrow X \) and \( j_x : \text{Spec} \, O_{Z,x} \rightarrow Z \) be the natural closed immersions. Then we have

\[
j^* F \cong \bigoplus_{x \in \text{Supp} \, F} j_x^* (j^* F)_x \cong \bigoplus_{x \in \text{Supp} \, F} j_x^* (F_x / \text{Ann}_{O_{X,x}}(F_x)) = \bigoplus_{x \in \text{Supp} \, F} j_x F_x.
\]

Thus we obtain an isomorphism

\[
F \cong j_* j^* F \cong j_* \left( \bigoplus_{x \in \text{Supp} \, F} j_x F_x \right) \cong \bigoplus_{x \in \text{Supp} \, F} (j j_x)_* F_x.
\]

Since \( j j_x \) is a closed immersion and \( F_x \) is of finite length, we conclude that \( F \) is also of finite length in \( \text{coh} \, X \). It is easily seen that (ii) holds by considering the following diagram:

\[
\begin{array}{ccc}
Z & \xrightarrow{j} & X \\
\downarrow{j_x} & & \downarrow{i_x} \\
\text{Spec} \, O_{Z,x} & \xrightarrow{i} & \text{Spec} \, O_{X,x}.
\end{array}
\]

Let us characterize coherent sheaves of finite length on a smooth projective curve \( C \). For any closed point \( x \in C \), we set \( O_n \times := i_* (O_{C,x} / m_n \times) \), where \( i \) is the natural morphism \( \text{Spec} \, O_{C,x} \rightarrow C \).
Lemma 4.3. The following are equivalent for a coherent sheaf $F$ on $C$:

1. $F$ is of finite length in $\text{coh} C$.
2. $\text{Supp}(F)$ has only finitely many points.
3. $F_{\eta} = 0$ holds, where $\eta$ is the generic point of $C$.

In this case, the following hold:

1. $F_x$ is a torsion $\text{O}_{C,x}$-module for any $x \in C$.
2. For any $x \in \text{Supp} F$, there is some positive integer $n_x > 0$ such that $F \cong \bigoplus_{x \in \text{Supp} F} \text{O}_{n_x}$.

Proof. It is obvious that (2) is equivalent to (3) since $C$ is an integral scheme of dimension 1 and $\text{Supp} F$ is closed. A module of finite length over the discrete valuation ring $\text{O}_{C,x}$ is of the form $\text{O}_{C,x}/m_{n_x}$ for some integer $n_x > 0$. Thus (i) and (ii) follow.

A coherent sheaf $F$ is said to be torsion if it satisfies the conditions of Corollary 4.3. We denote by $\text{tor} C$ the category of coherent torsion sheaves. It is immediate that $\text{tor} C$ is a Serre subcategory of an abelian category $\text{coh} C$. It is also clear that $\text{tor} C$ is a length abelian category.

We will calculate the Grothendieck monoid $\text{M}(\text{coh} C)$ and classify Serre subcategories of $\text{tor} C$. For this we recall divisors on $C$. Let $C(k)$ be the set of closed points of $C$. $\text{Div}(C)$ denotes the free abelian group generated by $C(k)$. An element $D = \sum_{i=1}^{n} m_i x_i$ of $\text{Div}(C)$ is called a divisor on $C$. The integer $\deg D := \sum_{i=1}^{n} m_i$ is called the degree of $D$. A divisor $D = \sum_{i=1}^{n} m_i x_i$ is said to be effective if $m_i \geq 0$ for all $i$. $\text{Div}^+(C)$ denotes the set of effective divisors on $C$. It is a submonoid of $\text{Div}(C)$.

Proposition 4.4. The following hold.

1. $\text{Sim}(\text{tor} C) = \{ O_x \mid x \in C(k) \}$ holds.
2. There is a monoid isomorphism

$$\text{Div}^+(C) \cong \text{M}(\text{tor} C), \quad \sum_{i=1}^{n} m_i x_i \mapsto \sum_{i=1}^{n} m_i [O_{x_i}].$$

Proof. It follows from Lemma 4.1 and Fact 3.14.

Corollary 4.5. There is an inclusion-preserving bijection

$$\text{P}(C(k)) \cong \text{Serre}(\text{tor} C), \quad A \mapsto \langle O_x \mid x \in A \rangle_{\text{Serre}}.$$

Note that $\text{P}(C(k))$ is exactly the set of specialization-closed subset except $C$ itself.

4.2. The case of vector bundles. We begin with a review on vector bundles on a noetherian scheme $X$. A locally free sheaf of rank $n$ on $X$ is a coherent sheaf $F$ such that $F_x \cong \mathcal{O}_{X,x}^n$ for all $x \in X$ (cf. [GW20, Proposition 7.41]). We call a locally free sheaf of finite rank on $X$ a vector bundle over $X$. We denote by vect $X$ the category of vector bundles over $X$. Then vect $X$ is an extension-closed subcategory of $\text{coh} X$. Indeed, for any exact sequence $0 \to F \to G \to H \to 0$ in $\text{coh} X$ with $F, G, H \in \text{vect} X$ and any $x \in X$, the exact sequence $0 \to F_x \to G_x \to H_x \to 0$ splits since $H_x$ is a free $\mathcal{O}_{X,x}$-module. Thus $G_x \cong F_x \oplus H_x$ is also a free $\mathcal{O}_{X,x}$-module for any $x \in X$. This implies $G \in \text{vect} X$, and hence vect $X$ is extension-closed. Then vect $X$ is a length exact category because the ranks of vector bundles gives rise to a length-like function $\text{rk}: \text{vect} X \to \mathbb{N}$. An admissible subobject in vect $X$ is called a subbundle.

Before studying the Grothendieck monoid $\text{M}(\text{vect} C)$, we recall the structure of the Grothendieck group $K_0(\text{vect} C)$. For this, we will introduce the Picard group of a noetherian scheme $X$. A line bundle $\mathcal{L}$ is a vector bundle of rank 1. It gives rise to an exact equivalence $- \otimes \mathcal{L}: \text{coh} X \xrightarrow{\sim} \text{coh} X$, which restricts to an exact equivalence vect $X \xrightarrow{\sim} \text{vect} X$. It is clear that $\text{rk}(U \otimes V) = \text{rk}(U) \cdot \text{rk}(V)$ for any vector bundles $U$ and $V$. In particular, we have that $\text{rk}(\mathcal{L} \otimes V) = \text{rk}(V)$ if $\mathcal{L}$ is a line bundle. The set Pic $X$ of isomorphism classes of line bundles over $X$ becomes a group whose operation is the tensor product $\otimes$ and unit is $\mathcal{O}_X$. The inverse of $\mathcal{L}$ in Pic($X$) is given by the dual $\mathcal{L}^\vee := \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ of $\mathcal{L}$. The group Pic$X$ is called the Picard group of $X$. We can assign a vector bundle $V$ of rank $r \geq 1$ with a line bundle $\det V := \mathcal{L}_r^\vee$, which is called the determinant bundle of $V$. We define the determinant bundle of zero sheaf 0 by det(0) := $\mathcal{O}_X$. It gives rise to an additive function $\text{det}: \text{vect} X \to \text{Pic} X$.

Fact 4.6 ([LeP97, Section 2.6]). The following holds for a smooth projective curve $C$.

1. The inclusion functor vect $C \hookrightarrow \text{coh} C$ induces a group isomorphism $K_0(\text{vect} C) \cong K_0(\text{coh} C)$.
2. There is a group isomorphism

$$K_0(\text{vect} C) \cong \text{Pic}(C) \times \mathbb{Z}, \quad [\mathcal{V}] \mapsto (\text{det} \mathcal{V}, \text{rk} \mathcal{V}).$$
We will determine the Grothendieck monoid $\mathbb{M}(\text{vect} C)$ in Proposition 4.9 below. Let us give a few preliminaries for Proposition 4.9. A coherent sheaf $\mathcal{F}$ on a noetherian scheme $X$ is globally generated if there exists a surjective morphism $\mathcal{O}_X^{\oplus n} \to \mathcal{F}$. We do not give the definition of very ample line bundles which appear in the following fact. See [Har77, Section II.5, page 120] for the definition. We only note that any projective variety has a very ample line bundle.

**Fact 4.7** (Serre [Ser55, Theorem 66.2], cf. [Har77, Theorem II.5.17]). Let $X$ be a projective variety over $k$, and let $\mathcal{O}(1)$ be a very ample line bundle on $X$. Then for any coherent sheaf $\mathcal{F}$ on $X$, there is an integer $n_0$ such that $\mathcal{F} \otimes \mathcal{O}(1)^{\oplus n}$ is globally generated for all $n \geq n_0$.

**Fact 4.8** (Atiyah [Ati57, Theorem 2]). Let $X$ be a smooth projective variety of dimension $d$ over $k$, and let $\mathcal{V}$ be a globally generated vector bundle of rank $r$ over $X$. If $r > d$, then $\mathcal{V}$ contains a trivial subbundle of rank $r - d$, that is, there is an inflation $\mathcal{O}_X^{\oplus (r-d)} \to \mathcal{V}$ in $\text{vect} X$.

We prepare notations to use the following proof. Let $\mathcal{O}(1)$ be a very ample line bundle on a smooth projective curve $C$. We set $\mathcal{O}(n) := \mathcal{O}(1)^{\oplus n}$ when $n \geq 0$ and $\mathcal{O}(n) := (\mathcal{O}(1)^{\vee})^{\oplus n}$ when $n < 0$. For a coherent sheaf $\mathcal{F}$ on $C$, we set $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}(n)$. Then $\mathcal{F}(n) \otimes \mathcal{O}(m) \cong \mathcal{F}(n+m)$ holds for any integers $n$ and $m$.

**Proposition 4.9.** The following holds.

1. A vector bundle is simple in $\text{vect} C$ if and only if it is a line bundle.
2. $\mathbb{M}(\text{vect} C)$ is a cancellative monoid, that is, the natural monoid homomorphism $\mathbb{M}(\text{vect} C) \to K_0(\text{vect} C)$ is injective (see Definition 2.7 and Remark 2.17).
3. There is a monoid isomorphism $\mathbb{M}(\text{vect} C) \overset{\sim}{\to} (\text{Pic} C \times \mathbb{N}^+) \cup \{(\mathcal{O}_C, 0)\} \subseteq \text{Pic} C \times \mathbb{Z}$, $[\mathcal{V}] \mapsto (\det \mathcal{V}, \text{rk} \mathcal{V})$, where $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ is the semigroup of strictly positive integers.

**Proof.**

1. Let $\mathcal{V}$ be a vector bundle of rank $r$. Then there is some integer $n$ such that $\mathcal{V}(n)$ is globally generated by Fact 4.7. If $r > 1$, then there is an inflation $\mathcal{O}_C^{\oplus (r-1)} \to \mathcal{V}(n)$ in $\text{vect}(C)$ by Fact 4.8. Since the functor $- \otimes \mathcal{O}(-n) : \text{vect} C \to \text{vect} C$ is exact, we have an inflation $\mathcal{O}(-n)^{\oplus (r-1)} \to \mathcal{V}$. Thus a simple object in $\text{vect} C$ has to be a line bundle. Conversely, a line bundle is a simple object in $\text{vect} C$ because $\text{rk} : \{\text{[vect} C]\} \to \mathbb{N}$ is a length-like function.

2. Define a monoid homomorphism by $\Phi := (\det, \text{rk}) : \mathbb{M}(\text{vect} C) \to \text{Pic} C \times \mathbb{N}$. Considering the commutative diagram

$$
\begin{array}{ccc}
\mathbb{M}(\text{vect} C) & \longrightarrow & K_0(\text{vect} C) \\
\downarrow & & \downarrow \Psi \\
\text{Pic} C \times \mathbb{N} & \longrightarrow & \text{Pic} C \times \mathbb{Z},
\end{array}
$$

it is enough to show that $\Phi$ is injective. Take vector bundles $\mathcal{U}$ and $\mathcal{V}$ such that $\Phi(\mathcal{U}) = \Phi(\mathcal{V})$. That is, they satisfy $\det \mathcal{U} \cong \det \mathcal{V}$ and $r := \text{rk} \mathcal{U} = \text{rk} \mathcal{V}$. It follows from Fact 4.7 that $\mathcal{U}(n)$ and $\mathcal{V}(n)$ are globally generated for the same integer $n$. Then there are conflations

$$
0 \to \mathcal{O}(-n)^{\oplus (r-1)} \to \mathcal{U} \to \mathcal{L} \to 0 \quad \text{and} \quad 0 \to \mathcal{O}(-n)^{\oplus (r-1)} \to \mathcal{V} \to \mathcal{M} \to 0
$$

in $\text{vect} C$ by Fact 4.8. Here $\mathcal{L}$ and $\mathcal{M}$ are line bundles. Then we have

$$
\mathcal{L} = \det \mathcal{L} \cong \det \mathcal{U} \otimes \det \left(\mathcal{O}(-n)^{\oplus (r-1)}\right)^{\vee} \cong \det \mathcal{V} \otimes \det \left(\mathcal{O}(-n)^{\oplus (r-1)}\right)^{\vee} \cong \det \mathcal{M} = \mathcal{M}.
$$

Hence we obtain $[\mathcal{U}] = [\mathcal{L}] + (r-1)[\mathcal{O}(-n)] = [\mathcal{M}] + (r-1)[\mathcal{O}(-n)] = [\mathcal{V}]$ in $\mathbb{M}(\text{vect} C)$. This proves $\Phi$ is injective.

3. It follows from $\text{Im} \Phi = (\text{Pic} C \times \mathbb{N}^+) \cup \{(\mathcal{O}_C, 0)\}$.

**Corollary 4.10.** The exact category $\text{vect} C$ has no nontrivial Serre subcategories.

**Proof.** We need to classify faces of the monoid $M := (\text{Pic} C \times \mathbb{N}^+) \cup \{(\mathcal{O}_C, 0)\} \subseteq \text{Pic} C \times \mathbb{Z}$ by Corollary 2.22 (4) and Proposition 4.9 (3). Let $F$ be a nonzero face of $M$. There is a nonzero element $(\mathcal{L}, r) \in F$ such that $(\mathcal{L}, r) \not= (\mathcal{O}_C, 0)$. Then we have $(\mathcal{O}_C, 1) \in F$ since $2(\mathcal{L}, r) = (\mathcal{L}^{\oplus 2}, 2r-1) + (\mathcal{O}_C, 1)$ in $M$ and $F$ is a face. For any non-zero element $(\mathcal{M}, s) \in M$, we obtain $(\mathcal{M}, s) + (\mathcal{M}', s) = (\mathcal{O}_C, 2s) = 2s(\mathcal{O}_C, 1) \in F$, and thus $(\mathcal{M}, s) \in F$. Therefore $M$ has no nontrivial faces.
4.3. The case of coherent sheaves. We finally deal with the case of the category $\text{coh}C$ of coherent sheaves. We begin with the relationship between $\text{tor}C$, $\text{vect}C$ and $\text{coh}C$.

Lemma 4.11. The following hold.

1. $\text{Hom}_{\text{O}_C}(T, V) = 0$ holds for all $T \in \text{tor}C$ and $V \in \text{vect}C$.
2. For every coherent sheaf $F$ on $C$, there exists an exact sequence

\[ 0 \to F_{\text{tor}} \to F \to F_{\text{vect}} \to 0 \]

in $\text{coh}C$ with $F_{\text{tor}} \in \text{tor}C$ and $F_{\text{vect}} \in \text{vect}C$.

In particular, $(\text{tor}C, \text{vect}C)$ is a torsion pair in $\text{coh}C$ (see [Ste75, Section VI.2] for the definition).

Proof. (1) Let $f: T \to V$ be a morphism from a coherent torsion sheaf to a vector bundle in $\text{coh}C$. Then $T_x$ is a torsion module and $V_x$ is a free module for any $x \in C$ by Corollary 4.3 and the definition of vector bundles. Hence $f_x: T_x \to V_x$ is equal to zero for all $x \in C$. This implies $f = 0$.

(2) Let $\eta$ be the generic point of $C$ and $K(C) := \mathcal{O}_{C,\eta}$ the function field of $C$. Consider the natural morphism $j: \text{Spec}K(C) \to C$. Define a coherent sheaf $F_{\text{tor}}$ by the kernel of the unit morphism $F \to j_*j^*F = j_*F_\eta$. Note that $j_*F_\eta$ is a constant sheaf on $C$ with value $F_\eta$. Thus we have $F_{\text{tor}}(U) = \{s \in F(U) \mid s_x = 0\}$ for every open subset $U$ of $C$. Then it is clear that $F_{\text{tor},\eta} = 0$, and thus $F_{\text{tor}}$ is a coherent torsion sheaf. Set $F_{\text{vect}} := F/F_{\text{tor}} \in \text{coh}C$. Then $F_{\text{vect}}$ is a subsheaf of the constant sheaf $j_*F_\eta$. Hence $F_{\text{vect},x}$ is an $\mathcal{O}_{C,x}$-submodule of $F_\eta$ for every point $x \in C$. This implies $F_{\text{vect},x}$ is a torsion free $\mathcal{O}_{C,x}$-module, and thus it is a free $\mathcal{O}_{C,x}$-module since $\mathcal{O}_{C,x}$ is a discrete valuation ring. We conclude that $F_{\text{vect}}$ is a vector bundle. 

We will determine the structure of the Grothendieck monoid $M(\text{coh}C)$ in Proposition 4.12 below. For this, we recall the relation between divisors and line bundles. We can attach to a divisor $D$ a line bundle $\mathcal{O}_C(D)$. It gives rise to a group homomorphism

\[ \text{Div}C \to \text{Pic}C, \quad D \mapsto \mathcal{O}_C(D). \]

For any effective divisor $D = \sum_{i=1}^n n_i x_i$ on $C$, we set $\mathcal{O}_D := \bigoplus_{i=1}^n \mathcal{O}_{n_ix_i}$. Then there is the following exact sequence in $\text{coh}C$:

\[ 0 \to \mathcal{O}_C(-D) \to \mathcal{O}_C \to \mathcal{O}_D \to 0. \]

(4.1)

Note that the abelian category $\text{coh}C$ is not length since there is an infinite subobject series of $\mathcal{O}_C$:

\[ \cdots \subseteq \mathcal{O}_C(-3x) \subseteq \mathcal{O}_C(-2x) \subseteq \mathcal{O}_C(-x) \subseteq \mathcal{O}_C, \]

where $x$ is a closed point of $C$. Thus we cannot use the results in §3.2.

Proposition 4.12. The following hold.

1. The inclusion functors $\text{tor}C \hookrightarrow \text{coh}C$ and $\text{vect}C \hookrightarrow \text{coh}C$ induce injective monoid homomorphisms $M(\text{tor}C) \hookrightarrow M(\text{coh}C)$ and $M(\text{vect}C) \hookrightarrow M(\text{coh}C)$, respectively.
2. For any line bundle $\mathcal{L}$ and any effective divisor $D$, we have $[\mathcal{L}] + [\mathcal{O}_D] = [\mathcal{L} \otimes \mathcal{O}_C(D)]$.
3. $M(\text{coh}C)$ is the disjoint union of $M_{\text{tor}C}$ and $M_{\text{vect}C} := M_{\text{vect}C} \setminus \{0\}$ as a set.

Proof. (1) The natural monoid homomorphism $M(\text{tor}C) \to M(\text{coh}C)$ is injective by Proposition 2.23. We prove that the natural monoid homomorphism $\iota: M(\text{vect}C) \to M(\text{coh}C)$ is injective. Recall that $M(\text{vect}C)$ is cancellative and the natural homomorphism $K_0(\text{vect}C) \to K_0(\text{coh}C)$ is an isomorphism by Proposition 4.9 and Fact 4.6. It follows that $\iota$ is injective by the following commutative diagram:

\[ \begin{array}{ccc}
M(\text{vect}C) & \xrightarrow{\iota} & M(\text{coh}C) \\
\downarrow & & \downarrow \\
K_0(\text{vect}C) & \xrightarrow{=} & K_0(\text{coh}C).
\end{array} \]

(2) We first note that $\mathcal{T} \otimes \mathcal{L} \cong \mathcal{T}$ for any coherent torsion sheaf $\mathcal{T}$. Applying the exact functor $- \otimes (\mathcal{L} \otimes \mathcal{O}_C(D)) : \text{coh}C \cong \text{coh}C$ to the exact sequence (4.1), we get an exact sequence

\[ 0 \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_C(D) \to \mathcal{O}_D \to 0. \]

Hence we have the equality $[\mathcal{L}] + [\mathcal{O}_D] = [\mathcal{L} \otimes \mathcal{O}_C(D)]$. 

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(3) For any coherent sheaf \( F \), there exists a coherent torsion sheaf \( T \) and a vector bundle \( V \) such that \( [F] = [T] + [V] \) by Lemma 4.11. Then there is a effective divisor \( D \) such that \( T \cong \mathcal{O}_D \). We can write
\[
[V] = \sum_{i=1}^r [L_i]
\]
for some line bundles \( L_i \) by Proposition 4.9. If \( V \) is a nonzero vector bundle, we have
\[
[F] = [\mathcal{O}_D] + \sum_{i=1}^r [L_i] = [L_1 \otimes \mathcal{O}_C(D)] + \sum_{i=2}^r [L_i] = \left( L_1 \otimes \mathcal{O}_C(D) \right) \oplus \bigoplus_{i=2}^r L_i \in \text{M}_{\text{vect}} C.
\]
This proves the desired conclusion. \( \square \)

As a corollary of Proposition 4.12, we recover Fact 1.1 for smooth projective curves. We will define a specialization-closed subset of a topological space in \( \S 5.1 \). We only note here that a specialization-closed subset of \( C \) is either a set of closed points or \( C \) itself.

**Corollary 4.13** (cf. [Gab62, Proposition VI.2.4]). There is an inclusion-preserving bijection between the following sets:

- The set of Serre subcategories of \( \text{coh} C \).
- The set of specialization-closed subset of \( C \).

**Proof.** It is enough to classify faces of \( \text{M}(\text{coh} C) \) by Corollary 2.22. Let \( F \) be a face of \( \text{M}(\text{coh} C) \). If \( [V] \in F \) for some nonzero vector bundle \( V \), it contains \( \text{M}_{\text{vect}} C \) by Corollary 4.10. Then \( F \) must coincide with \( \text{M}(\text{coh} C) \) by the exact sequence (4.1). Thus if \( F \neq \text{M}(\text{coh} C) \), it is contained in \( \text{M}_{\text{tor}} C \). Faces of \( \text{M}(\text{tor} C) \) bijectively corresponds to subsets of the set \( \text{C}(k) \) of closed points by Corollary 4.5. Extending this bijection by assigning \( \text{M}(\text{coh} C) \) with \( C \), we obtain the desired bijection. \( \square \)

Now we compare the Grothendieck monoid \( \text{M}(\text{coh} C) \) with the Grothendieck group \( K_0(\text{coh} C) \). There are unique group homomorphisms \( \text{deg}, \text{rk} : K_0(\text{coh} C) \to \mathbb{Z} \) satisfying the following conditions (see [LeP97, Sectin 2.6]):

- \( \text{rk}(F) = \text{rk}(F_{\text{vect}}) \) for any coherent sheaf \( F \) on \( C \).
- \( \text{deg}(\mathcal{O}_C(D)) = \text{deg} D \) for any divisor \( D \) on \( C \).
- \( \text{deg}(\mathcal{O}_D) = \text{deg} D \) for any effective divisor \( D \) on \( C \).

The image of the map \( (\text{rk}, \text{deg}) : K_0(\text{coh} C) \to \mathbb{Z}^{\oplus 2} \) is illustrated as follows:

Here the gray region corresponds to the Grothendieck monoid \( \text{M}(\text{coh} C) \). Let \( \rho : \text{M}(\text{coh} C) \to K_0(\text{coh} C) \) be the natural map. The map \( \rho \) is injective on \( \text{M}_{\text{vect}} C \) by Proposition 4.9 and 4.12. Whereas, the map \( \rho \) loses a lot of information on \( \text{M}_{\text{tor}} C \). Indeed, for two effective divisors \( D \) and \( E \), the equality \( [\mathcal{O}_D] = [\mathcal{O}_E] \) holds in \( K_0(\text{coh} C) \) if and only if \( \mathcal{O}_C(D) = \mathcal{O}_C(E) \) in \( \text{Pic} C \).

**Example 4.14.** Let \( \mathbb{P}^1 \) be the projective line. Then \( \text{deg} : \text{Pic} C \to \mathbb{Z} \) is a group isomorphism. In particular, the map \( (\text{rk}, \text{deg}) : K_0(\text{coh} \mathbb{P}^1) \to \mathbb{Z}^{\oplus 2} \) is a group isomorphism. For two effective divisors \( D \) and \( E \) on \( \mathbb{P}^1 \), the equality \( [\mathcal{O}_D] = [\mathcal{O}_E] \) holds in \( K_0(\text{coh} C) \) if and only if \( \text{deg} D = \text{deg} E \). Thus the map
\(\rho\) loses all information except the degrees for torsion sheaves. In particular, the equality \([O_x] = [O_y]\) holds in \(K_0(\text{coh} \mathcal{C})\) for any closed points \(x, y \in \mathbb{P}^1(k)\). Thus the Grothendieck group \(K_0(\text{coh} \mathbb{P}^1)\) has no information about closed points of \(\mathbb{P}^1\). In contrast, the Grothendieck monoid \(M(\text{coh} \mathbb{P}^1)\) remembers all closed points of \(\mathbb{P}^1\) because \(M(\text{coh} \mathcal{C}) \supseteq M_{\text{tor}} \mathcal{C} = \bigoplus_{x \in \mathbb{P}^1(k)} \mathbb{N}[O_x]\).

This example has another consequence. Let \(kQ\) be the path algebra of Kronecker quiver. It is well-known that the bounded derived categories \(D^b(\text{coh} \mathbb{P}^1)\) and \(D^b(\text{mod} kQ)\) are triangulated equivalent. However, we have a monoid isomorphism \(M(\text{coh} \mathbb{P}^1) \cong \mathbb{N}^{\mathbb{P}^1}\) by Fact 3.14. Thus \(M(\text{coh} \mathbb{P}^1)\) and \(M(\text{mod} kQ)\) are not isomorphic as monoids. This implies the Grothendieck monoids are not derived invariants.

Finally, we will introduce the notion of the twisted disjoint union to describe the structure of \(M(\text{coh} \mathcal{C})\) in terms of purely monoid-theoretic language. The rest of this section does not affect the other sections and can be skipped. We first recall the notion of a monoid action. Let \(M\) be a monoid. An \(M\)-action on a set \(X\) is a monoid homomorphism \(\sigma : M \to \text{End}_\mathbb{Z}(X) := \text{Hom}_\mathbb{Z}(X, X)\). The pair \(X = (X, \sigma)\) is called an \(M\)-set. Set \(\sigma_m := \sigma(m)\) and \(m \cdot x := \sigma_m(x)\) for all \(m \in M\) and \(x \in X\). A map \(f : X \to Y\) between \(M\)-sets is \(M\)-equivariant if \(f(m \cdot x) = m \cdot f(x)\) for all \(m \in M\) and \(x \in X\).

Let \(X, Y\) and \(Z\) be \(M\)-sets. A map \(\alpha : X \times Y \to Z\) is an \(M\)-bimorphism if it satisfies \(m \cdot \alpha(x, y) = \alpha(m \cdot x, y) = \alpha(x, m \cdot y)\) holds for all \(m \in M\), \(x \in X\), and \(y \in Y\). An \(M\)-semigroup is an \(M\)-set \(S\) with an \(M\)-bimorphism \(\alpha : S \times S \to S\) satisfying associativity and commutativity. In other words, it is a (commutative) semigroup \(S\) with an \(M\)-action satisfying \(m \cdot (x + y) = m \cdot x + y = x + m \cdot y\) for all \(m \in M\), \(x, y \in S\). An \(M\)-semigroup homomorphism is an \(M\)-equivariant map \(f : S \to T\) satisfying \(f(x + y) = f(x) + f(y)\) for all \(x, y \in S\). We denote by \(\text{SemiGrp}_M\) the category of \(M\)-semigroups and \(M\)-semigroup homomorphisms.

**Example 4.15.** Let \(\phi : M \to X\) be a monoid homomorphism. Then \(\phi\) defines an action of \(M\) on \(X\) by \(m \cdot x := \phi(m) + x\) for all \(m \in M\) and \(x \in X\). We can easily check that \(X\) is an \(M\)-semigroup by this action.

The twisted disjoint union \(M \sqcup_\sigma S\) of a monoid \(M\) and an \(M\)-semigroup \(S\) whose action is given by \(\sigma : M \to \text{End}_\mathbb{Z}(S)\) is the set-theoretic disjoint union \(M \sqcup S\) with a binary operation given by

\[
x + y := \begin{cases} 
x +_M y & \text{if both } x \in M \text{ and } y \in M, 
x +_S y & \text{if both } x \in S \text{ and } y \in S, 
x +_\sigma \sigma_m(y) & \text{if } x \in M \text{ and } y \in S, 
x +_\sigma \sigma_n(x) & \text{if } x \in S \text{ and } y \in M,
\end{cases}
\]

where \(+_M\) (resp. \(+_S\)) denotes the binary operation on \(M\) (resp. \(S\)). We can check easily that \(M \sqcup_\sigma S\) is a (commutative) monoid. The natural inclusion \(i : M \to M \sqcup_\sigma S\) is a monoid homomorphism. Hence we can think of \(M \sqcup_\sigma S\) as an \(M\)-semigroup by Example 4.15. Then the natural inclusion \(j : S \to M \sqcup_\sigma S\) is an \(M\)-semigroup homomorphism.

We describe a universal property of the twisted disjoint union. We denote by \(\text{Mon}_M\) the slice category of \(\text{Mon}\) under a monoid \(M\). That is, its objects are monoid homomorphisms \(M \to X\), and morphisms between \(\phi : M \to X\) and \(\psi : M \to Y\) are monoid homomorphisms \(f : X \to Y\) satisfying \(f \phi = \psi\).

**Proposition 4.16.** Let \(\phi : M \to X\) be a monoid homomorphism. We regard \(X\) as an \(M\)-semigroup. Let \(S\) be an \(M\)-semigroup, and let \(i : M \to M \sqcup_\sigma S\) and \(j : S \to M \sqcup_\sigma S\) be the natural inclusions. Then there is a natural isomorphism

\[
\text{Hom}_{\text{Mon}_M}(M \sqcup_\sigma S, X) \cong \text{Hom}_{\text{SemiGrp}_M}(S, X), \quad h \mapsto hj.
\]

**Proof.** We omit the proof since it is straightforward. \(\square\)

Consider the Grothendieck monoid \(M(\text{coh} \mathcal{C})\). Then \(M(\text{coh} \mathcal{C})\) is an \(M_{\text{tor}} \mathcal{C}\)-semigroup by the inclusion homomorphism \(M_{\text{tor}} \mathcal{C} \to M(\text{coh} \mathcal{C})\). The subsemigroup \(M_{\text{vec}} \mathcal{C}\) is also an \(M_{\text{tor}} \mathcal{C}\)-semigroup whose \(M_{\text{tor}} \mathcal{C}\)-action is given by \(\sigma_{[O_\mathcal{C}]}([Y]) := [\mathcal{O}_\mathcal{D}] + [Y]\). Then the natural inclusion map \(M_{\text{vec}} \mathcal{C} \to M(\text{coh} \mathcal{C})\) is an \(M_{\text{tor}} \mathcal{C}\)-semigroup homomorphism. It induces a monoid homomorphism \(h : M_{\text{tor}} \mathcal{C} \sqcup_\sigma M_{\text{vec}} \mathcal{C} \to M(\text{coh} \mathcal{C})\) by Proposition 4.16. It is clear that \(h\) is an isomorphism. Thus the following statement follows.

**Corollary 4.17.** There is a monoid isomorphism

\[
\text{Div}^+(C) \sqcup_\sigma (\text{Pic} C \times \mathbb{N}^+) \cong M(\text{coh} \mathcal{C}),
\]

where the \(\text{Div}^+(C)\)-action on \(\text{Pic} C \times \mathbb{N}^+\) is defined by \(\sigma_{\mathcal{D}}(L, r) := (L \otimes \mathcal{O}_C(D), r)\).

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5. The spectrum of Grothendieck monoid

In this section, we study the monoid spectrum of the Grothendieck monoid \( M(\mathcal{E}) \) and a topology on the set \( \text{Serre}(\mathcal{E}) \) of Serre subcategories.

In §5.1, we review the spectrum of a monoid and monoidal spaces, which is a natural analogy of the spectrum of a commutative ring and ringed spaces, respectively. In §5.2, we introduce a topology on \( \text{Serre}(\mathcal{E}) \) for an exact category \( \mathcal{E} \). We first reveal the relation between the topologies on \( \text{Serre}(\mathcal{E}) \) and \( \text{MSpec} M(\mathcal{E}) \). Next, we construct a sheaf \( \mathcal{M}_A \) of monoids on \( \text{Serre}(A) \) for an abelian category \( A \), which has a property that the stalk \( \mathcal{M}_{A,S} \) is isomorphic to \( M(A/S) \) for any point \( S \in \text{Serre}(A) \). Here \( A/S \) is the abelian quotient category. We compare \( \text{MSpec} M(A) \) and \( (\text{Serre}(A), \mathcal{M}_A) \) as monoidal spaces. In §5.3, we recover the topology of a noetherian scheme \( X \) from the Grothendieck monoid \( M(\text{coh} X) \).

5.1. Preliminaries: the spectrum of commutative monoid. In this subsection, we review the spectrum of a monoid. The main reference is [Ogu18]. We often refer to [GW20], a textbook of scheme theory since many constructions are analogies of the spectrum of a commutative ring. Throughout this subsection, \( M \) is a monoid.

**Definition 5.1.**

1. A subset \( I \) of \( M \) is called an ideal if for all \( x \in I \) and \( a \in M \), we have \( x + a \in I \).
2. An ideal \( p \) of \( M \) is said to be prime if it satisfies (i) \( p \neq M \) and (ii) \( x + y \in p \) implies \( x \in p \) or \( y \in p \) for all \( x, y \in M \).
3. \( \text{MSpec} M \) denotes the set of prime ideals of \( M \) and is called the monoid spectrum of \( M \).

For a subset \( S \) of \( M \), let

\[
\langle S \rangle_{\text{ideal}} := \{ x + a \mid x \in S, a \in M \}.
\]

It is obviously the smallest ideal containing \( S \). We call it the ideal generated by \( S \).

**Remark 5.2.**

1. The set \( M^+ := M \setminus M^\times \) of non-units is the unique maximal ideal of \( M \). It is also a prime ideal of \( M \).
2. The empty set \( \emptyset \) is the unique minimal ideal of \( M \). It is also a prime ideal of \( M \).
3. The spectrum \( \text{MSpec} M \) is never empty and has the maximum and minimum element with respect to inclusion by (1) and (2). \( \text{MSpec} M \) has only one point if and only if \( M \) is a group.

For a subset \( S \) of \( M \), we set

\[
V(S) := \{ p \in \text{MSpec} M \mid p \supseteq S \}.
\]

Note that \( V(S) = V(\langle S \rangle_{\text{ideal}}) \) holds. They satisfy the following equalities(cf. [GW20, Lemma 2.1]):

- \( V(M) = \emptyset \) and \( V(\emptyset) = \text{MSpec} M \).
- \( \bigcap_{\alpha \in A} V(S_\alpha) = V(\bigcup_{\alpha \in A} S_\alpha) \) for a family \( \{ S_\alpha \}_{\alpha \in A} \) of subsets of \( M \).
- \( V(I) \cup V(J) = V(I \cap J) \) for ideals \( I, J \) of \( M \).

We define the Zariski topology on \( \text{MSpec} M \) by taking the subsets of the form \( V(S) \) to be the closed subsets. Note that \( \text{MSpec} M \) has a unique closed point \( M^+ \) and a unique generic point \( \emptyset \) by Remark 5.2. In particular, \( \text{MSpec} M \) is an irreducible topological space.

Let

\[
D(f) := \{ p \in \text{MSpec} M \mid f \notin p \}
\]

for each element \( f \in M \). They are open in \( \text{MSpec} M \) since \( D(f) = \text{MSpec} M \setminus V(f) \). They satisfy

\[
D(f) \cap D(g) = D(f + g)
\]

for any \( f, g \in M \). Open subsets of \( \text{MSpec} M \) of this form are called principal open subsets of \( \text{MSpec} M \).

The set of principal open subsets \( D(f) \) forms a basis of the Zariski topology on \( \text{MSpec} M \). (cf. [GW20, Proposition 2.5]).

We define a preorder on a topological space \( X \). For two points \( x, y \in X \), we say that \( x \) is a specialization of \( y \) or that \( y \) is a generalization of \( x \) if \( x \) belongs to the topological closure \( \overline{\{y\}} \) of \( \{y\} \) in \( X \). Define a preorder \( \preceq \) on \( X \) by

\[
x \preceq y :\Leftrightarrow x \text{ is a specialization of } y.
\]

We call it the specialization order on \( X \). When we regard \( X \) as a poset by the specialization order, it is denoted by \( X_{\text{spcl}} := (X, \preceq) \). A subset \( A \) of \( X \) is specialization-closed (resp. generalization-closed) if for any \( x \in A \) and every its specialization (resp. generalization) \( x' \in X \), we have \( x' \in A \). Any closed subset (resp. open subset) is specialization-closed (resp. generalization-closed).

The specialization order on \( \text{MSpec} M \) recovers the inclusion-order on prime ideals.
Proposition 5.3. The following hold.

1. \{p\} = V(p) for any prime ideal p ⊆ M.
2. MSpec M_{spec} is isomorphic to MSpec M ordered by reverse inclusion as posets.

Proof. We omit the proof since it is straightforward. □

A relation between faces and prime ideals are the following.

Fact 5.4 ([Ogu18, Section I.1.4]).

1. \( p^c := M \setminus p \) is a face of M for any prime ideal p of M.
2. \( F^c := M \setminus F \) is a prime ideal of M for any face F of M.
3. The assignments given in (1) and (2) give inclusion-reversing bijections between Face(M) and MSpec M.

Let us recall localization of monoids, which is a monoid version of localization of commutative rings, to introduce the structure sheaf on the monoid spectrum MSpec M. We recommend that the reader skips the remaining part of this subsection in the first reading.

Definition 5.5. Let S be a subset of M. The localization of M with respect to S is a monoid M_S together with a monoid homomorphism \( \rho: M \to M_S \), which is called the localization homomorphism, satisfying the following universal property:

1. (i) \( \rho(s) \) is invertible for each s ∈ S.
2. (ii) For any monoid homomorphism \( \phi: M \to X \) such that \( \phi(s) \) is invertible for each s ∈ S, there is a unique monoid homomorphism \( \overline{\phi}: M_S \to X \) satisfying \( \phi = \overline{\phi} \rho \).

The localization of M with respect to a subset \( S \subseteq M \) always exists. It is constructed as follows:

Let \( (S)_N \) be the submonoid of M generated by S. Define an equivalence relation on \( M \times (S)_N \) by

\[(x, s) \sim (y, t) :\iff \exists \text{ there exist } u \in (S)_N \text{ such that } x + t + u = y + s + u \in M.\]

Then the set \( M_S := M \times (S)_N / \sim \) of equivalence classes has a natural monoid structure given by

\[[x, s] + [y, t] := [x + y, s + t],\]

where \([x, s]\) denotes the equivalence class containing \((x, s) \in M \times (S)_N\). We can think of \([x, s]\) as \(\sim x - s\). Then we can check easily that the monoid \( M_S \) with a monoid homomorphism \( \rho: M \to M_S \) defined by \( \rho(m) = [m, 0] \) satisfies the universal property of the localization.

We introduce a sheaf \( O_M \) of monoids on MSpec M, which is called the structure sheaf of MSpec M.

Fact 5.6 ([Ogu18, Section II.1.2]). There is a sheaf \( O_M \) of monoids on MSpec M satisfying the following.

1. For any element \( f \in M \), we have \( O_M(D(f)) = M_f \).
2. In particular, we have \( O_M(\text{MSpec } M(A)) = M \).
3. For any point \( p \in \text{MSpec } M \), the stalk \( O_{\text{MSpec } M, p} \) of \( O_M \) is isomorphic to \( M_p \), where \( p^c := M \setminus p \).

A monoidal space is a pair \((X, \mathcal{M})\) of a topological space X and a sheaf \( \mathcal{M} \) of monoids on X. A morphism \((f, f') : (X, \mathcal{M}) \to (Y, \mathcal{N})\) of monoidal spaces is a pair of continuous map \( f : X \to Y \) and a morphism \( f'^\#: \mathcal{N}^{-1}(\mathcal{N}) \to \mathcal{M} \) of sheaves of monoids such that the map on the stalks \( \mathcal{N}_{f(x)} \to \mathcal{M}_x \) are local monoid homomorphism for all \( x \in X \). Here a monoid morphism \( \phi: M \to N \) is local if \( \phi^{-1}(N^c) = M^c \).

An affine monoid scheme is a monoidal space isomorphic to \((\text{MSpec } M, \mathcal{O}_M)\) for some monoid M.

Remark 5.7. An affine monoid scheme \((\text{MSpec } M, \mathcal{O}_M)\) was first introduced by Kato [Kat94] to study toric singularities. Deitmar [Dei05] used it to construct a theory of “schemes over the field \( \mathbb{F}_1 \) with one element”. See [LP11] for more information.

The following lemmas will be used in the proof of Proposition 5.19.

Lemma 5.8. Let S be a subset of M. Then the natural monoid homomorphism

\[ M_S \to M_{(S)_{face}}, \quad [x, s] \mapsto [x, s] \]

is an isomorphism.

Proof. Let \( \rho: M \to M_S \) be the localization homomorphism. We have that \( \rho^{-1}(M_S^c) = (S)_\text{face} \) by [Ogu18, The text following Proposition 1.4.4]. Then the conclusion follows immediately from the universal property. □

Lemma 5.9. Let S be a face of M.
2.22 5.4

We can easily check that Proposition 5.11.

Thus we can define a topology on Serre(\mathcal{E})

\text{Proposition 5.10.}

We conclude that \([x, s] = [y, t]\) mod \(M^\xi\)

\text{Proposition 5.12.}

Therefore \(M/S\) is sharp.

\(\text{(2) The quotient homomorphism} \ M \rightarrow M/S\)

\(\text{is an isomorphism.}\)

\textbf{Proof.} (1) Let \(x, y \in M\) such that \(x + y \equiv 0\) mod \(S\). There are elements \(s, t \in S\) such that \(x + y + s = t\)

\(\text{in} \ M\). Since \(S\) is a face, both \(x\) and \(y\) belong to \(S\), which imply both \(x \equiv 0\) mod \(S\)

and \(y \equiv 0\) mod \(S\). Therefore \(M/S\) is sharp.

(2) The quotient homomorphism \(M \rightarrow M/S\) induces a monoid homomorphism \(\phi': M_S \rightarrow M/S\)

by the universal property of \(M_S\). Then \(\phi'(M^\xi_S) = 0\) since \(M/S\) is sharp. Thus \(\phi'\)

\(\text{induces a monoid homomorphism} \ M_S/M^\xi_S \rightarrow M/S\)

by the universal property of \(M_S/M^\xi_S\). The homomorphism \(\phi\)

is clearly surjective. We prove that \(\phi\) is injective. Let \([x, s], [y, t] \in M_S\) such that \(x \equiv y\) mod \(S\) in \(M\).

Then there are \(n, n' \in S\) such that \(x + n = y + n'\) in \(M\). Hence we have the following equalities in \(M_S\):

\([x, s] + [s + n, 0] = [x + s + n, s] = [x + n, 0] = [y + n', 0] = [y, t] + [t + n', 0]\).

We conclude that \([x, s] \equiv [y, t]\) mod \(M^\xi_S\) because \([s + n, 0], [t + n', 0] \in M^\xi_S\).

Therefore \(\phi\) is injective. \(\square\)

5.2. The spectrum of Grothendieck monoid. In this subsection, \(\mathcal{E}\) is an exact category. We first introduce a topology on the set \(\text{Serre}(\mathcal{E})\) of Serre subcategories and study the relation between the topologies on \(\text{Serre}(\mathcal{E})\) and \(\text{MSpec}(M(\mathcal{E}))\). Next, we classify finitely generated Serre subcategories by using this topology. Finally, we introduce a sheaf \(\mathcal{M}\) of monoids on \(\text{Serre}(\mathcal{A})\) for an abelian category \(\mathcal{A}\), which is related to the quotient abelian category \(\mathcal{A}/S\), and compare it with the structure sheaf \(\mathcal{O}_{M(\mathcal{A})}\)

of \(\text{MSpec}(M(\mathcal{A}))\).

Let us begin with the bijections which follow from Corollary 2.22 and Fact 5.4.

\textbf{Proposition 5.10.} \(\text{There are bijections between the following sets:}\)

\(\text{(1) The set} \ \text{Serre}(\mathcal{E}) \ \text{of Serre subcategories of} \ \mathcal{E}.\)

\(\text{(2) The set} \ \text{Face} M(\mathcal{E}) \ \text{of faces of} \ M(\mathcal{E}).\)

\(\text{(3) The set} \ \text{MSpec} M(\mathcal{E}) \ \text{of prime ideals of} \ M(\mathcal{E}).\)

Moreover, the bijection between (1) and (2) is inclusion-preserving while the one between (2) and (3) is inclusion-reversing.

The bijection between (1) and (3) induces a topology on \(\text{Serre}(\mathcal{E})\) from \(\text{MSpec} M(\mathcal{E}).\) In the following, we describe this topology explicitly. For a subcategory \(\mathcal{X}\) of \(\mathcal{E}\), we set

\(V(\mathcal{X}) := \{\mathcal{S} \in \text{Serre}(\mathcal{X}) \mid \mathcal{S} \cap \mathcal{X} = \emptyset\}.\)

We can easily check that the following equalities hold:

- \(V(\mathcal{E}) = \emptyset\) and \(V(\emptyset) = \text{Face}(\mathcal{E}).\)
- \(\bigcap_{\mathcal{X} \in \mathcal{A}} V(\mathcal{X}_a) = V(\bigcup_{\mathcal{X} \in \mathcal{A}} \mathcal{X}_a)\) for a family \(\{\mathcal{X}_a\}_{a \in \mathcal{A}}\) of subcategories of \(\mathcal{X}\).
- \(V(\mathcal{X}) \cap V(\mathcal{Y}) = V(\mathcal{X} \oplus \mathcal{Y})\) for subcategories \(\mathcal{X}, \mathcal{Y}\) of \(\mathcal{E}\), where \(\mathcal{X} \oplus \mathcal{Y} := \{X \oplus Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}\).

Thus we can define a topology on \(\text{Serre}(\mathcal{E})\), which is called the Zariski topology, by taking the subsets of the form \(V(\mathcal{X})\) to be the closed subsets.

For an object \(X \in \mathcal{E}\), we put

\(U_X := \{\mathcal{S} \in \text{Serre}(\mathcal{E}) \mid X \in \mathcal{S}\}.\)

We can easily check that \(U_X \cap U_Y = U_{X \oplus Y}\) for any \(X, Y \in \mathcal{E}\).

\textbf{Proposition 5.11.} The following hold.

\(\text{(1) The bijection} \ \Phi: \text{Serre}(\mathcal{E}) \xrightarrow{\sim} \text{MSpec} M(\mathcal{E}) \ \text{in} \ \text{Proposition 5.10} \ \text{is a homeomorphism.}\)

\(\text{(2) The set of subsets of the form} \ U_X \ \text{forms an open basis of} \ \text{Serre}(\mathcal{E}).\)

\(\text{(3) \text{Serre}(\mathcal{E})_{\text{spec}} \cong (\text{Serre}(\mathcal{E}), \subseteq) \ as \ posets.}\)

\textbf{Proof.} We first note that \(\Phi(\mathcal{S}) = M^\xi_S := M(\mathcal{E}) \setminus M_S\) for any \(\mathcal{S} \in \text{Serre}(\mathcal{E}).\)

(1) Let \(\mathcal{X}\) be a subcategory of \(\mathcal{E}\) and \(\mathcal{S}\) a Serre subcategory of \(\mathcal{E}\). Then \(\mathcal{S} \cap \mathcal{X} = \emptyset\) if and only if \(M_S \cap M_X = \emptyset\) since \(\mathcal{S}\) is \(S\)-closed by Lemma 2.20. It is equivalent to \(\Phi(\mathcal{S}) = M^\xi_S \supseteq M_X\). Thus we obtain \(\Phi(V(\mathcal{X})) = V(M_X)\), which implies \(\Phi\) is a homeomorphism since \(M_X\) runs through all subsets of \(M(\mathcal{E})\) by Proposition 2.19.

(2) It is clear since \(\Phi(U_X) = D([X])\) for any \(X \in \mathcal{E}\).

(3) It follows from Proposition 5.3 and the fact that \(\Phi\) is inclusion-reversing. \(\square\)
Remark 5.12. The topology on $\text{Serre}(\mathcal{E})$ is a natural analogy of the topology on the set of thick subcategories of a triangulate category, which is introduced by Balmer [Bal05] (see also [MT20, Mat21]).

Next, we give a topological characterization of $U_{A}$ to classify finitely generated $\text{Serre}$ subcategories. Recall that a $\text{Serre}$ subcategory $\mathcal{S}$ of $\mathcal{E}$ is finitely generated if $\mathcal{S} = \langle M \rangle_{\text{Serre}}$ for some object $M \in \mathcal{E}$.

Definition 5.13. A topological space $X$ is strongly quasi-compact if for every open covering $\{U_{i}\}_{i \in I}$ of $X$, there exists $i \in I$ such that $X = U_{i}$.

Lemma 5.14. Let $M$ be a monoid. An open subset $U$ of $\text{MSpec } M$ is strongly quasi-compact if and only if $U = D(f)$ for some $f \in M$.

Proof. We first show that $D(f)$ is strongly quasi-compact. The subset $D(f)$ has the maximum element $(f)_{\text{face}}$ with respect to inclusions. Indeed, for any $p \in D(f)$, we have $f \not\subseteq p$. Since $p_{\text{face}}$ is a face and $f \in p_{\text{face}}$, we obtain $(f)_{\text{face}} \subseteq p_{\text{face}}$. Thus we conclude that $(f)_{\text{face}} \subseteq p$. Let $\{U_{i}\}_{i \in I}$ be an open covering of $D(f)$. Then there exists $i \in I$ such that $(f)_{\text{face}} \subseteq U_{i}$, which implies $D(f) = U_{i}$ because $U_{i}$ is generalization-closed. This proves $D(f)$ is strongly quasi-compact.

We construct $U$ as a strongly quasi-compact open subset of $\text{MSpec } M$. Since the principal open subsets are a basis of Zariski topology, the open subset $U$ is covered by them. Thus $U = D(f)$ for some $f \in M$ because $U$ is strongly quasi-compact.

We rephrase this lemma in terms of $\text{Serre}$ subcategories.

Corollary 5.15. An open subset $U$ of $\text{Serre}(\mathcal{E})$ is strongly quasi-compact if and only if $U = U_{A}$ for some $A \in \mathcal{E}$.

We classify finitely generated $\text{Serre}$ subcategories of $\mathcal{E}$ via its monoid spectrum $\text{MSpec } M(\mathcal{E})$.

Corollary 5.16. There are bijections between the following sets:

1. The set of finitely generated $\text{Serre}$ subcategories of $\mathcal{E}$.
2. The set of strongly quasi-compact open subsets of $\text{Serre}(\mathcal{E})$.
3. The set of strongly quasi-compact open subsets of $\text{MSpec } M(\mathcal{E})$.

The bijection from (1) to (2) is given by $X = \langle X \rangle_{\text{Serre}} \mapsto U_{X}$.

Proof. Note that the assignment $X = \langle X \rangle_{\text{Serre}} \mapsto U_{X}$ is well-defined since $U_{X} = U_{Y}$ if $\langle X \rangle_{\text{Serre}} = \langle Y \rangle_{\text{Serre}}$. This assignment is surjective by Corollary 5.15. We prove the injectivity. Take $X, Y \in \mathcal{E}$ such that $U_{X} = U_{Y}$. Then $\langle X \rangle_{\text{Serre}} \subseteq U_{X} = U_{Y}$, which implies $Y \in \langle X \rangle_{\text{Serre}}$. We also have $X \in \langle Y \rangle_{\text{Serre}}$. Thus we have $\langle X \rangle_{\text{Serre}} = \langle Y \rangle_{\text{Serre}}$.

Finally, we construct a sheaf of monoids on $\text{Serre}(A)$ for an abelian category $A$, which is related to the quotient abelian category $\mathcal{A}/\mathcal{S}$. There is no application of this sheaf at the moment. However, it may be interesting from the viewpoint of geometry over the field $\mathbb{F}_{1}$ with one element. Even if the reader skips the rest of this subsection, there is no problem to read the next subsection.

We begin with a review of the notion of abelian quotient categories. See [Pop73, Section 4.3] for details. For a $\text{Serre}$ subcategory $\mathcal{S}$ of $\mathcal{A}$, there are an abelian category $\mathcal{A}/\mathcal{S}$ and an exact functor $Q: \mathcal{A} \to \mathcal{A}/\mathcal{S}$ which satisfy the following universal property:

- For any exact functor $F: \mathcal{A} \to \mathcal{C}$ of abelian categories such that $F(\mathcal{S}) = 0$, there exists a unique exact functor $\overline{F}: \mathcal{A}/\mathcal{S} \to \mathcal{C}$ satisfying $F = \overline{F}Q$.

We call $\mathcal{A}/\mathcal{S}$ the abelian quotient category of $\mathcal{A}$ with respect to $\mathcal{S}$, and $Q: \mathcal{A} \to \mathcal{A}/\mathcal{S}$ the quotient functor. The following facts are useful to study the abelian quotient category $\mathcal{A}/\mathcal{S}$.

Fact 5.17 ([Pop73, Lemma 4.3.4, 4.3.7, 4.3.9]). Let $\mathcal{S}$ be a $\text{Serre}$ subcategory of an abelian category $\mathcal{A}$, and let $Q: \mathcal{A} \to \mathcal{A}/\mathcal{S}$ be the quotient functor.

1. $\mathcal{S} = \{X \in \mathcal{A} \mid Q(X) = 0\}$ holds.
2. For any morphism $f: X \to Y$ in $\mathcal{A}$, the morphism $Q(f)$ is an isomorphism if and only if both $\ker f \in \mathcal{S}$ and $\cok f \in \mathcal{S}$ hold.
3. Any morphism of $\mathcal{A}/\mathcal{S}$ can be written by $Q(s)^{-1}Q(f)Q(t)^{-1}$ for some morphisms $s, t, f$ in $\mathcal{A}$.

Let us construct a sheaf of monoids on $\text{Serre}(A)$. Let $B$ be the set of strongly quasi-compact open subsets of $\text{Serre}(A)$. Explicitly, we have $B = \{U_{X} \mid X \in \mathcal{A}\}$ by Corollary 5.15. Then $B$ is an open basis of $\text{Serre}(A)$ by Proposition 5.11. We have that $U_{X} \supseteq U_{Y}$ if and only if $\langle X \rangle_{\text{Serre}} \subseteq \langle Y \rangle_{\text{Serre}}$ for any $X, Y \in \mathcal{A}$. In this case, there is an exact functor $F_{X,Y}: \mathcal{A}/\langle X \rangle_{\text{Serre}} \to \mathcal{A}/\langle Y \rangle_{\text{Serre}}$ induced by
the universal property of the abelian quotient category \( A / (X)_{\text{Serre}} \). In particular, we obtain a monoid homomorphism \( r_{X,Y} := M(F_{X,Y}) : M(A / (X)_{\text{Serre}}) \to M(A / (Y)_{\text{Serre}}) \). Thus the assignment
\[
U_{X} \mapsto \mathcal{M}_{A}(U_{X}) := M(A / (X)_{\text{Serre}})
\]
defines a presheaf of monoids on \( B \). Define a presheaf \( \mathcal{M}_{A} \) on \( \text{Serre}(A) \) by
\[
V \mapsto \mathcal{M}_{A}(V) := \varinjlim_{U} \mathcal{M}_{A}(U),
\]
where \( U \) runs through the set of \( U \in B \) with \( U \subset V \). Then it satisfies the condition of [GW20, Proposition 2.20] since \( U \in B \) is strongly quasi-compact. Thus \( \mathcal{M}_{A} \) is a sheaf on \( \text{Serre}(A) \).

**Proposition 5.18.** Let \( A \) be an abelian category and \( \mathcal{M}_{A} \) a sheaf on \( \text{Serre}(A) \) constructed as above.

1. For any \( X \in A \), we have \( \mathcal{M}_{A}(U_{X}) = M(A / (X)_{\text{Serre}}) \).
2. In particular, we have \( \mathcal{M}_{A}(\text{Serre}(A)) = M(A) \).
3. For any point \( s \in \text{Serre}(A) \), the stalk \( \mathcal{M}_{A,s} \) of \( \mathcal{M}_{A} \) is isomorphic to \( M(A/S) \).

**Proof.** We only prove (3) because (1) and (2) are obvious by the definition of \( \mathcal{M}_{A} \). Let \( \mathcal{S} \) be a Serre subcategory of \( A \). For any \( X \in A \) with \( \mathcal{S} \in U_{X} \), we have the natural exact functor \( A / (X)_{\text{Serre}} \to A / \mathcal{S} \).

They induce a monoid homomorphism
\[
\phi : \mathcal{M}_{A,S} = \colim_{\mathcal{V} \in \mathcal{S}} \mathcal{M}_{A}(U_{X}) = \colim_{\mathcal{V} \in \mathcal{S}} M(A / (X)_{\text{Serre}}) \to M(A / \mathcal{S}).
\]
It is clear that \( \phi \) is surjective. We now prove \( \phi \) is injective. We first note that the natural map \( M(A) \to \mathcal{M}_{A,S} \) is surjective, and denote by \( [X]_{\mathcal{S}} \) the element of \( \mathcal{M}_{A,S} \) represented by \( X \in A \). Suppose that \( \phi ([X]_{\mathcal{S}}) = \phi ([Y]_{\mathcal{S}}) \) for some \( X, Y \in A \). Then \( X \) and \( Y \) are \( \mathcal{S} \)-equivalent in \( A / \mathcal{S} \) by Fact 2.15. Hence there are admissible subobject series \( 0 = X_{0} \leq X_{1} \leq \cdots \leq X_{n} = X \) and \( 0 = Y_{0} \leq Y_{1} \leq \cdots \leq Y_{n} = Y \) such that \( X_{i}/X_{i-1} \cong Y_{\sigma(i)}/Y_{\sigma(i)-1} \) in \( A / \mathcal{S} \) for some permutation \( \sigma \in S_{n} \). The isomorphism \( X_{i}/X_{i-1} \cong Y_{\sigma(i)}/Y_{\sigma(i)-1} \) can be written \( Q(s_{i})^{-1}Q(f_{i})Q(t_{i})^{-1} \) for some morphisms \( s_{i}, t_{i}, f_{i}, 1 \leq i \leq n \) in \( A \) by Fact 5.17. The monomorphisms \( X_{i-1} \to X_{i} \) and \( Y_{i-1} \to Y_{i} \) in \( A / \mathcal{S} \) are also written by \( Q(s_{i})^{-1}Q(f_{i})Q(t_{i})^{-1} \) for \( n + 1 \leq i \leq 2n \) and \( 2n + 1 \leq i \leq 3n \), respectively. We set
\[
M := \bigoplus_{i=1}^{3n}(\text{Ker}(s_{i}) \oplus \text{Cok}(s_{i}) \oplus \text{Ker}(t_{i}) \oplus \text{Cok}(t_{i}) \oplus \text{Ker}(f_{i})) \oplus \bigoplus_{i=1}^{n} \text{Cok}(f_{i}).
\]
Then \( M \in \mathcal{S} \) by Fact 5.17, and \( X \) and \( Y \) are still \( \mathcal{S} \)-equivalent in \( A / (M)_{\text{Serre}} \). Thus \( [X] = [Y] \) in \( M(A / (M)_{\text{Serre}}) \) and \( \mathcal{S} \in U_{M} \). This proves \( [X]_{\mathcal{S}} = [Y]_{\mathcal{S}} \) in \( \mathcal{M}_{A,S} \).

We compare the monoidal space \( (\text{Serre}(A), \mathcal{M}_{A}) \) with the affine monoid scheme \( (\text{MSpec}(M(A), O_{M(A)})). \)

Define a sheaf \( \overline{O}_{M(A)} \) on \( \text{MSpec}(M(A)) \) by the sheafification of presheaf
\[
U \mapsto O_{M(A)}(U)/O_{M(A)}(U)^{\times}.
\]
For any object \( X \in A \), we have an isomorphism
\[
O_{M(A)}(D([X]))/O_{M(A)}(D([X]))^{\times} \cong M(A)_{[X]}/M(A)_{[X]}^{\times} \overset{\cong}{\to} M(A) / ([X])_{\text{face}}
\]
by Lemma 5.8, 5.9. In [ES], the author and Enomoto prove that the natural monoid homomorphism \( M(A) / M_{S} \to M(A / \mathcal{S}) \) is an isomorphism for any Serre subcategory \( \mathcal{S} \). In particular, we have an isomorphism
\[
M(A) / ([X])_{\text{face}} \overset{\cong}{\to} M(A / (X)_{\text{Serre}}) = \mathcal{M}_{A}(U_{X}).
\]
Let \( \Phi : \text{Serre}(\mathcal{S}) \overset{\cong}{\to} \text{MSpec}(M(A)) \) be the homeomorphism in Proposition 5.11. Combining the isomorphisms (5.1) and (5.2), we obtain an isomorphism \( \Phi^{-1}(\overline{\mathcal{O}}_{M(A)}) \to \mathcal{M}_{A} \) of sheaves of monoids. Thus we have the following proposition.

**Proposition 5.19.** The bijection in Proposition 5.10 induces an isomorphism of monoidal spaces
\[
(\text{Serre}(A), \mathcal{M}_{A}) \cong (\text{MSpec}(M(A), \overline{\mathcal{O}}_{M(A)}).\)
5.3. Reconstructing the topology of a noetherian scheme. In this subsection, we recover the topology of a noetherian scheme $X$ from the Grothendieck monoid $\text{M}(\text{coh} X)$. Hereafter $X$ is a noetherian scheme.

We first construct an immersion from $X$ to $\text{Serre}(\text{coh} X)$ as topological spaces. For any point $x \in X$, define a subcategory of $\text{coh} X$ by

$$\text{coh}^x X := \{ \mathcal{F} \in \text{coh} X \mid \mathcal{F}_x = 0 \}.$$

It is clear that $\text{coh}^x X$ is a Serre subcategory of $\text{coh} X$. Let $j : X \to \text{Serre}(\text{coh} X)$ be a map defined by $j(x) := \text{coh}^x X$.

**Lemma 5.20.** The map $j : X \to \text{Serre}(\text{coh} X)$ is an immersion of topological spaces. That is, it is a homeomorphism onto a Serre subcategory of $\text{Serre}(\text{coh} X)$.

**Proof.**

Let $x, y \in X$ be distinct points. Then $\mathcal{O}_x \in \text{coh}^y X$ but $\mathcal{O}_y \not\in \text{coh}^y X$, where $\mathcal{O}_x$ is the skyscraper sheaf supported at $x$ (see §4.1). Hence $j$ is injective. For a coherent sheaf $\mathcal{F}$ on $X$, we have

$$j^{-1}(U_{\mathcal{F}}) = \{ x \in X \mid \mathcal{F} \in \text{coh}^x X \} = \{ x \in X \mid \mathcal{F}_x = 0 \} = X \setminus \text{Supp} \mathcal{F}.$$

Thus $j$ is continuous. Let $Z$ be a closed subset of $X$, and let $\mathcal{I}$ be the quasi-coherent ideal sheaf of $\mathcal{O}_X$ corresponding to $Z$ with the reduced subscheme structure. It is straightforward that $\text{coh}^x X \in V(\mathcal{O}_X/\mathcal{I})$ if and only if $x \in \text{Supp}(\mathcal{O}_X/\mathcal{I}) = Z$ for any $x \in X$. Thus we have $j(Z) = j(X) \cap V(\mathcal{O}_X/\mathcal{I})$, and hence $j(Z)$ is a closed subset of $j(X)$. Therefore $j$ is a homeomorphism onto the subspace $j(X)$ of $\text{Serre}(\text{coh} X)$. \hfill \Box

Next, we determine the image of the immersion $j : X \hookrightarrow \text{Serre}(\text{coh} X)$. A Serre subcategory $\mathcal{S}$ of an abelian category $\mathcal{A}$ is meet-irreducible if $X \cap Y \subseteq \mathcal{S}$ implies $X \subseteq \mathcal{S}$ or $Y \subseteq \mathcal{S}$ for any $X, Y \in \text{Serre}(\mathcal{A})$. We have

$$j(X) = \{ \mathcal{S} \in \text{Serre}(\text{coh} X) \mid \mathcal{S} \text{ is meet-irreducible} \}$$

by the following fact.

**Fact 5.21 ([BKS07, Proposition 3.7, 9.1]).** For a Serre subcategory $\mathcal{S}$ of $\text{coh} X$, it is meet-irreducible if and only if $\mathcal{S} = \text{coh}^x X$ for some point $x \in X$.

**Lemma 5.22.** Let $X$ and $Y$ be noetherian schemes. If there is a homeomorphism $\text{Serre}(\text{coh} X) \xrightarrow{\cong} \text{Serre}(\text{coh} Y)$, then it restricts to a homeomorphism $j(X) \xrightarrow{\cong} j(Y)$.

**Proof.** Let $\Psi : \text{Serre}(\text{coh} X) \xrightarrow{\cong} \text{Serre}(\text{coh} Y)$ be a homeomorphism, and let $\mathcal{S}$ be a meet-irreducible Serre subcategory of $\text{coh} X$. Since $\Psi$ is also an isomorphism of posets by Proposition 5.11, we see that $\Psi(\mathcal{S})$ is also meet-irreducible in $\text{Serre}(\text{coh} Y)$. This proves $\Psi(j(X)) \subseteq j(Y)$. \hfill \Box

**Corollary 5.23.** Consider the following conditions for noetherian schemes $X$ and $Y$.

1. $X \cong Y$ as schemes.
2. $\text{M}(\text{coh} X) \cong \text{M}(\text{coh} Y)$ as monoids.
3. $\text{MSpec} \text{M}(\text{coh} X) \cong \text{MSpec} \text{M}(\text{coh} Y)$ as topological spaces.
4. $X \cong Y$ as topological spaces.

Then the implications "(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4)" hold.

**Proof.** The implications "(1) $\Rightarrow$ (2) $\Rightarrow$ (3)" is obvious. The implication "(3) $\Rightarrow$ (4)" follows from Lemma 5.20 and 5.22. \hfill \Box

**Remark 5.24.** There is another proof of the implication "(3) $\Rightarrow$ (4)". Let us give a sketch of the proof.

We begin with a review of the spectrum of a frame. See [BKS07] and [PP12] for detailed explanations. A frame is a poset $L = (L, \leq)$ satisfying the following conditions:

- The poset $L$ is a complete lattice, that is, any subset $A$ of $L$ admits a supremum $\text{sup} A = \bigvee_{a \in A} a$ and an infimum $\text{inf} A = \bigwedge_{a \in A} a$. We denote by $a \vee b := \text{sup} \{ a, b \}$ and $a \wedge b := \text{inf} \{ a, b \}$.
- The complete lattice $L$ satisfies the distributed law:

$$\bigvee_{a \in A} a \wedge b = \bigvee_{a \in A} (a \wedge b)$$

for any subset $A \subseteq L$ and any element $b \in L$.  

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An element $p$ of a frame $L$ is meet-irreducible if $x \land y \leq p$ implies $x \leq p$ or $y \leq p$ for any $x, y \in L$. The set $\text{LSpec} L$ of meet-irreducible elements of a frame $L$ is called the lattice spectrum of $L$. The set $\text{LSpec} L$ has a topology whose closed subsets are of the form $V(a) := \{ p \in \text{LSpec} L \mid a \leq p \}$, $a \in L$.

For a noetherian scheme $X$, we can endow the underlying set $\text{LSpec}(\text{Serre}(\text{coh} X))$ with a new topology by taking subsets of the following form to be the open subsets:

$$Y = \bigcup_{i \in I} Y_i$$

such that $X \setminus Y_i$ is a quasi-compact open in $\text{LSpec}(\text{Serre}(\text{coh} X))$ for all $i \in I$.

We denote this new space by $\text{LSpec}^\ast(\text{Serre}(\text{coh} X))$. Buan, Krause and Solberg proved that $X$ and $\text{LSpec}^\ast(\text{Serre}(\text{coh} X))$ are homeomorphic for any noetherian scheme $X$ in [BKS07, Section 9].

We now come back to the proof of the implication “(3) $\Rightarrow$ (4)”. It follows that $\text{Serre}(\text{coh} X) \cong \text{Serre}(\text{coh} Y)$ as posets by the condition (3) and Proposition 5.11. Thus we have homeomorphisms

$$X \cong \text{LSpec}^\ast(\text{Serre}(\text{coh} X)) \cong \text{LSpec}^\ast(\text{Serre}(\text{coh} Y)) \cong Y$$

by the discussion above.

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