Research Article

A Note on Tsuji’s Criterion for Numerical Triviality

Shigetaka Fukuda

Faculty of Education, Gifu Shotoku Gakuen University, Gifu, Japan

Correspondence should be addressed to Shigetaka Fukuda; fukuda@gifu.shotoku.ac.jp

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Abstract

In this study, we give an alternative and elementary proof to Tsuji’s criterion for a Cartier divisor to be numerically trivial.

1. Introduction

In this article, every algebraic variety is proper over the field of complex numbers \( \mathbb{C} \).

In 1970s, Iitaka [1] initiated the classification theory of higher dimensional algebraic varieties by using the pluricanonical systems. In 1980s, Mori [2] deepened the Iitaka theory by cutting off the subvarieties of elliptic type. In [3], Tsuji gave an interesting and useful criterion for a Cartier divisor to be numerically trivial.

Theorem 1 (Tsuji [[3], Lemma 5.1], cf. Bauer et al. [[4], Theorem 2.4]). Let \( f: M \rightarrow B \) be a surjective morphism between complete varieties. Let \( L \) be a nef Cartier divisor on \( M \) and \( W \) some subvarieties of \( M \), such that \( f(W) = B \) and \( B_0 \) a subset of \( B \) which is a union of countably many proper Zariski-closed subsets. Assume that

1. For some \( b \in B \), \( (L, C) = 0 \) for every curve \( C \) on \( f^{-1}(b) \)
2. \( (L, W) = 0 \) for some irreducible curve \( W \) on \( M \), such that \( f(W) \neq B_0 \)

Then, \( L \) is numerically trivial.

Remark 1. In the statement of Lemma 1, condition (1) immediately implies that \( L \) is numerically trivial on every general fiber of the morphism \( f \) by considering the flattening. By the normalization, the Stein factorization, and the desingularization, the article ([4], Proposition 2.5), for an algebraically fibered surface, implies the assertion of Lemma 1.

2. Elementary Proof of Main Theorem 1

Proof. We prove the assertion by induction on \( (\dim M, \dim B) \).

First, we take a commutative diagram as shown in Figure 1 with the following properties:

1. \( M' \) and \( B' \) are nonsingular projective varieties
2. \( \mu \) is a birational morphism
3. \( \nu \) is a generically finite morphism
(4) $g$ is a morphism with only connected fibers.

There exists some irreducible component $W'$ of $\mu^{-1}(W)$, such that $g(W') = B'$. We set $L' := \mu' L$.

The locus $g(\cup [C']_i) \subset M'$ is an irreducible curve on $M'$, $g(C')$ is a point, and the intersection number $(L', C') > 0$ is included in a union of at most countably many proper Zariski-closed subsets of $B'$ (Proposition 1). Thus, we obtain a union $B' \supseteq \mu^{-1}(B_0)$ of countably many proper Zariski-closed subsets of $B'$ with the following two properties:

1. $L'$ is numerically trivial on every fiber of $g$ over $B' \setminus B_0$

2. $(L', C') = 0$ for every irreducible curve $C_i$ on $M'$, such that $g(C_i) \notin B_0$

It suffices to prove that $(L', C') = 0$ for every irreducible curve $C'$ on $M'$. We fix an irreducible curve $C'$ on $M'$.

**Case 1.** $g(C') \notin B_0'$, Case this divides into Subcases 1 and 2.

**Subcase 1.** $g(C') \notin B_0'$ and $g(C')$ is a point.

We have $(L', C') = 0$ from (1).

**Subcase 2.** $g(C') \notin B_0'$ and $g(C')$ is a curve. This subcase divides into Subcases 3 and 7.

**Subcase 3.** $g(C') \notin B_0'$, $g(C')$ is a curve and $\dim B = 1$. This subcase divides into Subcases 4, 5, and 6.

We note that $g(W_1) = g(C') = B'$ for some irreducible curve $W_1$ on $W'$.

**Subcase 4.** $g(C') \notin B_0'$, $g(C')$ is a curve, $\dim B = 1$, and $\dim M = 1$.

$M' = W' = W_1 = C'$. Thus, $(L', C') = 0$.

**Subcase 5.** $g(C') \notin B_0'$, $g(C')$ is a curve, $\dim B = 1$, and $\dim M = 2$.

Because $(L', W_1) = 0$ from (2), Lemma 1 implies that $L'$ is numerically trivial, and thus, $(L', C') = 0$.

**Subcase 6.** $g(C') \notin B_0'$, $g(C')$ is a curve, $\dim B = 1$, and $\dim M \geq 3$.

Because the codimension of $\dim (W_1 \cup C', M') \geq 2$, we have an irreducible hyperplane section $H$ of $M'$ that includes $W_1$ and $C'$ (Proposition 2). Then, $L'_{\mid H}$ is numerically trivial from the induction hypothesis. Consequently, $(L', C') = 0$.

**Subcase 7 (cf. [4], 2.1.2]).** $g(C') \notin B_0'$, $g(C')$ is a curve, and $\dim B \geq 2$.

Let $S := \{S_i\}$ be the set of irreducible components of $g^{-1}(g(C'))$. We note that $g(W_1) = g(C')$ for some irreducible curve $W_1$ on $W'$. Thus, $L'_{\mid S_i}$ is numerically trivial for some $S_i \in S$, such that $S_i \supseteq W_1$ from the property (2) and from the induction hypothesis.

If $g((\cup_{m \neq 1}S_m)) = g(C')$, then $g(S_i \cap (\cup_{m \neq 1}S_m)) = g(C')$ from the connectedness of fibers of $g$, and therefore, $g(S_i \cap S_j) = g(C')$ for some $S_j \in S$. Thus, $\dim g((\cup_{m \neq 1}S_m)) \subseteq 0$ or $g(S_i \cap S_j) = g(C')$.

Therefore, $((S_i \cup S_j) \cap (\cup_{m \neq 1}S_m)) = g(C')$, then $(S_i \cup S_j) \cap (\cup_{m \neq 1}S_m) = g(C')$ from the connectedness of fibers of $g$, and therefore, $g(S_i \cup S_j) \cap S_j = g(C')$ for some $S_j \in S$. From this argument, we obtain the following properties:

1. $g(S_i) = g(S_i) = \cdots = g(S_i) = g(C')$

2. $g((S_i \cup S_j \cup \cdots \cup S_k)) = g(C')$ for all $i$ with $1 \leq i \leq k$

3. $\dim g((\cup_{m \neq 1}S_m)) \subseteq 0$

The fact that $L'_{\mid S_j}$ is numerically trivial and that $g(S_i \cap S_j) = g(C')$ implies that $L'_{\mid S_j}$ is numerically trivial from the induction hypothesis. The fact that $L'_{\mid S_j \cup S_j}$ is numerically trivial and that $g((S_i \cup S_j) \cap S_j) = g(C')$ implies that $L'_{\mid S_j}$ is numerically trivial from the induction hypothesis. This argument implies that $L'_{\mid S_j \cup S_j \cup \cdots \cup S_k}$ is numerically trivial. Because $\dim g((\cup_{m \neq 1}S_m)) \subseteq 0$, we have that $C' \subseteq S_1 \cup S_2 \cup \cdots \cup S_k$. Consequently, $(L', C') = 0$.

**Case 2.** $g(C') \subseteq B_0'$, This case divides into Subcases 8 and 11.

**Subcase 8.** $g(C') \subseteq B_0'$ and $\dim M = 2$. This subcase divides into Subcases 9 and 10.

**Subcase 9.** $g(C') \subseteq B_0'$, $\dim M = 2$, and $\dim B = 1$.

Lemma 1 implies that $L'$ is numerically trivial, and thus, $(L', C') = 0$.

**Subcase 10.** $g(C') \subseteq B_0'$, $\dim M = 2$, and $\dim B = 2$.

Because $W' = M'$, we have that $(L', H) = 0$ for an irreducible hyperplane section $H$ of $M'$ from the property (2) of the divisor $L'$. The Hodge index theorem implies that $L'$ is numerically trivial. Thus, $(L', C') = 0$.

**Subcase 11.** $g(C') \subseteq B_0'$ and $\dim M \geq 3$.

Because the codimension of $\dim (C', M') \geq 2$, there exists an irreducible hyperplane section $H$ of $M'$ that includes $C'$ (Proposition 2). We may assume that $g(H) \notin B_0'$. Note that, from Case 1, $(L', C') = 0$ for every irreducible curve $C''$ on $H$, such that $g(C'') \notin B_0'$. Thus, $L'_{\mid H}$ is numerically trivial from the induction hypothesis. Consequently, $(L', C') = 0$.

**3. Appendix**

In this appendix, we state two elementary propositions and their proofs, which are well known to the experts, for the readers’ convenience.
Proposition 1. Let \( f : M \rightarrow B \) be a surjective morphism between projective varieties and \( L \) a nef Cartier divisor on \( M \).

We assume that for some \( b \in B \), the intersection number \((L, C) = 0\) for every irreducible curve \( C \) on \( f^{-1}(b) \).

Then, the locus \( f(\cup |C|) \) is an irreducible curve on \( M \), \((f(C) \) is a point, and the intersection number \((L, C) > 0\) \) is included in a union of at most countably many proper Zariski-closed subsets of \( B \).

Proof. There exists some ample divisor \( A \) on \( B \). Assume that \( C \) is an irreducible curve on \( M \), such that \((f^* A, C) = 0\) (i.e., \( f(C) \) is a point) and that \((L, C) > 0\). There exists some irreducible component \( W \) of the universal scheme for the Hilbert scheme \( Hilb(M) \) of \( M \), such that \( W \) includes \( p_1^{-1}([C]) \), where \([C] \) is the point \((\in Hilb(M)) \), which represents the subscheme \( C \) of \( M \), and Figure 2 shows the projections \( p_1 \) and \( p_2 \) and the property that \( dim W = dim p_2(W) + 1 \). We set \( T := p_2(W) \).

First, we consider the normalization \( n_1: W_n \rightarrow W \), \( n_2: T_n \rightarrow T \), and \( n_3: W_n \rightarrow T_n \) of the morphism \( p_2|_{W_n} : W_n \rightarrow T \). Next, consider the Stein factorization \( W_n \rightarrow T' \rightarrow T \) of the morphism \( n_3: W_n \rightarrow T_n \).

Last, consider the flattening \( f_1: W'' \rightarrow W_n \), \( f_2: T'' \rightarrow T' \), and \( f_3: W'' \rightarrow T'' \) of the morphism \( s_1: W_n \rightarrow T' \), where the morphism \( f_2 \) is birational and the variety \( T'' \) is nonsingular. We note that the morphism \( f_3: W'' \rightarrow T'' \) is flat and with only connected fibers.

We put \( h = n_i f_i \).

Thus, we have the commutative diagram, as shown in Figure 3.

From the flatness of the morphism \( f_3: W'' \rightarrow T'' \), the intersection number \( (h' p_1^* f^* A, F_{T''}) > 0 \) for every fiber \( F_{T''} \) of the morphism \( f_3: W'' \rightarrow T'' \) because \((p_1^{-1} f^* A, p_2^{-1}([C])) = 0 \). Thus, for every fiber \( F_{T''} \) of the morphism \( f_3: W'' \rightarrow T'' \), the morphism \( f_3 \) contracts \( p_1 h(F') \) to one point from the connectedness of \( F' \). In other words, \( p_1 h(F') \) is included in some fiber of \( f \).

There exists some ample divisor \( A' \) on \( W \). Of course, \((A', F) > 0 \) for every curve \( F \) on \( W \). Because the morphism \( h' \) is birational, we have that \( h(F') \) is not a point (i.e., \((h' A', F') > 0 \) for some fiber \( F' \) of \( f_3: W'' \rightarrow T'' \)). From the flatness, \((h' A', F') > 0 \) for every fiber \( F' \) of \( f_3: W'' \rightarrow T'' \). Thus, every fiber \( F' \) of \( f_3: W'' \rightarrow T'' \) cannot be contracted to a point by the morphism \( h' \).

There exists some fiber \( F'' \) of \( f_3: W'' \rightarrow T'' \), such that \( h^{-1}(p_1^{-1}([C])) \cap F'' \neq \emptyset \). Then, \( F'' \subset h^{-1}(p_1^{-1}([C])) \) because the morphism \( p_1 h \) maps \( F'' \) to a point \([C] \in T \). Consequently, \( h(F''') = p_2^{-1}([C]) \) because \( F'' \) does not contract to a point by the morphism \( h' \). Thus, \((h' p_1, F''') > 0 \). From the flatness of the morphism \( f_3: W'' \rightarrow T'' \), the intersection number \((h' p_1 L, F''') > 0 \) for every fiber \( F' \) of the morphism \( f_3: W'' \rightarrow T'' \).

We note that every fiber of \( f_3: W'' \rightarrow T'' \) is mapped in some fiber of \( p_2|_{W_n}: W \rightarrow T \). In other words, every fiber of \( p_2|_{W_n} \) is swept out by fibers of \( f_3 \).

So, for every fiber \( F \) of \( p_2|_{W_n}: W \rightarrow T \), the locus \( p_1(F) \) is swept out by connected curves \( C' \), such that \( f(C') \) is one point and that the intersection number \((L, C') > 0 \). We note that we consider \( p_1 h(F') \) as \( C' \) and that \( C' = p_1 h(F') = Supp((p_1 h, F')) \) from the connectedness of \( F' \). Thus, \( b \notin f(p_1(F)) \). Consequently, \( b \notin f(p_1(W)) \). In other words, \( p_1(W) \) is disjoint with \( f^{-1}(b) \).

The countability of the irreducible components of the Hilbert scheme \( Hilb(M) \) of \( M \) implies the assertion. \( \square \)

Proposition 2. Let \( M \) be a nonsingular projective variety and \( C \) a Zariski-closed subset with codimension \( codim(C, M) \geq 2 \). Then, there exists some irreducible hyperplane section \( H \), such that \( H^C \).

Proof. We take some ample divisor \( A \) on \( M \). We have a birational morphism \( f: M' \rightarrow M \), such that \( M' \) is a nonsingular projective variety, that \( f^{-1}(C) \) is divisorial with only simple normal crossings and that there exists an effective divisor \( C_0 \) with the property that \( Supp(C_0) = f^{-1}(C) \) and \( -C_0 \) is \( f \)-ample. Then, \( mf^* A - C_0 \) is ample for a sufficiently large integer \( m \). For a sufficiently large and divisible integer \( l \), the divisor \( lim A \) is very ample, and there exists a member \( H_0 \in l(mf^* A - C_0) \) which is very ample and irreducible. We put \( H = f_*(H_0 + lC_0) \). Then, \( H \in lim A \).

The locus \( f^{-1}(H) \) coincides with \( Supp(f^* H) = Supp(H_0 + lC_0) \). Thus, \( f^{-1}(H)^C \).

Data Availability

No data were used to support this study.

Disclosure

The updated version of the manuscript is presented in arXiv: 2109.02034v1 ([8]).
Conflicts of Interest
The author declares that there are no conflicts of interest.

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