From Kepler’s laws to Newton’s law: a didactical proof

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Abstract
An elementary derivation of the Newton “inverse square law” from the three Kepler laws is proposed. Our proof, thought essentially for first-year undergraduates, basically rests on Euclidean geometry. It could then be offered even to high-school students possessing only the first basics of Calculus.
What makes planets go around the sun? At the time of Kepler some people answered this problem by saying that there were angels behind them beating their wings and pushing the planets around an orbit. As you will see, the answer is not very far from the truth. The only difference is that the angels sit in a different direction and their wings push inwards.

(Richard P. Feynman [1])
1 Introduction

It is generally easier to find mathematical derivations of Kepler’s laws from Newton’s inverse square law (Newton’s law henceforth) rather than the opposite, the probably most famous one being that given by Richard Feynman in his celebrated “Lost Lecture” [2]. However, the importance of the fact that Newton’s law is a logical consequence of Kepler’s laws was considerably emphasized by Max Born in his beautiful textbook on cause and chance [3]. Born poses such a logical consequence “... as the basis on which my (his) whole conception of causality in physics rests,” [3, p. 129], and furnished a full rigorous proof by using the most natural coordinate system for dealing with problems involving central forces, namely the polar one [3, Appendix II]. In 1993 a compact and interesting mathematical derivation of Newton’s law from Kepler within Cartesian realm has been published in [4] and, to use the author own words, “... without need of the ‘clever tricks’ that are often used when polar coordinates are employed.”

The aim of the present paper is purely pedagogical. In fact, although Kepler’s laws and Newton’s law are central topics in any first-year undergraduate physics course, the mathematical background and knowledge of the audience is still too far from being acceptable for a complete presentation to be adequately grasped. As far as my teaching experience is concerned, this implies that a rigorous justification of Newton’s law is carried out only for the simplest case of circular orbits while it is left unsolved for elliptical orbits. Such an unsatisfactory state of fact pushed me to conceive a proof to be offered also to first-year undergraduates, or even to high-school students possessing only the first basics of Calculus.

To help teachers, the present work is organized in the form of a self-contained didactical unit, which can be provided in three steps. In fact, since our proof is ultimately based on two geometrical properties of ellipses, which could not necessarily be known to students, a couple of appendixes have been added to the paper. Each appendix could then constitute the subject of a practice session.

2 The proof

2.1 Kepler’s laws

Here the Kepler laws are listed for reader’s convenience:

I. Each planet moves along an ellipse with the Sun at one of the two foci;

II. The segment joining the Sun and the planet sweeps out equal areas in equal times;

III. The square of the orbital period divided by the cube of the elliptical orbit major axis is the same for all planets.

The proof then follows within the following three steps.
2.2 Step 1: use of first Kepler’s law

We start from the Kepler law I and the geometry is depicted in Fig. 1: the planet is represented by the point $P$ which is supposed to move counterclockwise with velocity $v$ along the elliptical trajectory whose foci are $F_1$ and $F_2$. The Sun is at $F_1$. We have

$$PF_1 + PF_2 = 2a,$$

where $2a$ denotes the ellipse major axis. Needless to say, in the following of the paper the symbol $a$ should not be confused with the modulus of the acceleration $a$, which will then be denoted by $|a|$. Moreover, the symbol $2f$ stands for the focal distance $F_1 F_2$, in such a way the orbit eccentricity, say $\epsilon$, is given by the ratio $f/a$ and the minor half-axis, say $b$, turns out to be

$$b = \sqrt{a^2 - f^2} = a \sqrt{1 - \epsilon^2}.$$  

Moreover, a Cartesian reference frame has also been introduced in such a way

the ellipse representation reads

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$  

The first of the above quoted properties of the ellipse we are going to use within our proof is the following:

$P1$: the normal to the ellipse at $P$ bisects the angle $\angle F_1 PF_2$

A clever, purely geometrical proof of $P1$ can be found, for instance, in the above quoted Feynman “Lost Lecture” [2, pp. 150-151]. A less clever, although straightforward proof, based on the Cartesian representation (3), is outlined in Appendix A and could be used, as it was said above, as the topic of a preparatory practice session.
2.3 Step 2: use of second Kepler’s law

In the following it will be assumed that all students know that the areal speed, which will be denoted $\dot{A}$, can be mathematically defined in terms of the cross product between the position vector $\vec{F_1P}$ and the point velocity $v$ as

$$\dot{A} = \frac{1}{2} \vec{F_1P} \times v.$$  \hfill (4)

Moreover, since the motion is planar, in the following the symbol $\times$ has to be meant as the sole cross product component along the direction perpendicular to the motion plane. In other terms, it is thought of as a *scalar* quantity. Kepler’s law II then implies that the areal speed $\dot{A}$ is a constant of the motion. This, in turns, has two consequences.

The first of them is that the acceleration of $P$ points toward $F_1$. The original geometrical proof provided by Newton has been summarized again in [2, pp. 153-157]. An alternative way is to take the time derivative of both sides of Eq. (4), and on taking into account that, since the areal speed is constant, it turns out that $\ddot{A} = 0$, so that

$$2\ddot{A} = \vec{F_1P} \times \dot{v} + \vec{F_1P} \times \ddot{v} = \vec{F_1P} \times \ddot{v} = 0,$$  \hfill (5)

where use has been made of the fact that $\vec{F_1P} \times \dot{v}$ and that $\ddot{v} = a$. Equation (4) then implies that $a \parallel \vec{F_1P}$.

The second consequence of Kepler’s law II represents the central point of our proof. With reference to Fig. 2 write the areal speed by using the geometrical definition of cross product, i.e.,

$$2\dot{A} = \vec{F_1P} v \sin \alpha = \vec{F_1P} v \cos \theta,$$  \hfill (6)
where, due to the property P1, θ = ∠F_1PQ = ∠F_2PQ. On applying cosine’s law to both triangles F_1PQ and F_2PQ we have
\[
\begin{align*}
F_1Q^2 &= F_1P^2 + PQ^2 - 2 F_1P PQ \cos \theta, \\
F_2Q^2 &= F_2P^2 + PQ^2 - 2 F_2P PQ \cos \theta,
\end{align*}
\] (7)
which, on taking Eq. (16) into account, after simple algebra gives
\[
\cos \theta = \frac{b}{(F_1P F_2P)^{1/2}}. 
\] (8)
On substituting from Eq. (8) into Eq. (6), the following relationship between the point speed $v$ and the areal speed $\dot{A}$ is then obtained:
\[
v = \frac{2}{b} \dot{A} \sqrt{\frac{F_2P}{F_1P}}. 
\] (9)
Moreover, the constant value of the areal speed, say $K$, is obtained simply by dividing the ellipse area, $\pi ab$, by the orbital period, say $T$, i.e.,
\[
\dot{A} = K = \frac{\pi ab}{T}. 
\] (10)

### 2.4 Step 3: use of third Kepler’s law and finalization of the proof

We are now ready to finalize the proof, i.e., to show that the modulus of the acceleration is proportional to the inverse square of $F_1P$. Since $a$ is parallel to $\vec{F_1P}$, the normal component of the acceleration, say $a_\nu$, is given by $a_\nu = |a| \cos \theta$, with $\theta = \angle F_1PQ$ (see again Fig. 2). Then, on introducing the curvature radius, say $\rho_P$, of the ellipse at point $P$, we have
\[
a_\nu = \frac{v^2}{\rho_P} \implies |a| = \frac{v^2}{\rho_P \cos \theta}. 
\] (11)
In order to continue, the explicit expression of the curvature radius $\rho_P$ is needed. This is just the second property of ellipses mentioned at the beginning of the paper, which states:

**P2: the radius of curvature of the ellipse in $P$ is**
\[
\rho_P = \frac{(F_1P F_2P)^{3/2}}{ab} 
\] (12)
An elementary proof of P2 is outlined in Appendix B and could constitute the subject of another preliminary practice session.
On substituting from Eqs. (8), (9), (10), and (12) into Eq. (11), we thus have
\[ |a| = \frac{4}{\sqrt{b^2}} \frac{\pi^2 a^2 b^2}{T^2} \frac{K}{F_1 P} \frac{a \beta}{F_2 P} \frac{T_1 P^3/2}{F_2 P^3/2} \frac{T_2 K^2}{F_2 P^{1/2}} = \]
\[ = 4\pi \left( \frac{a^3}{T^2} \right) \frac{1}{F_1 P^2}, \quad (13) \]
and, finally, the last step: Kepler’s law III asserts that the ratio \( a^3/T^2 \) must be a constant independent of the planet. On denoting \( C \) such a constant, Eq. (13) finally gives
\[ |a| = \frac{4\pi C}{F_1 P^2}, \quad (14) \]

Quod Erat Demonstrandum.

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**A Proof of P1**

The starting point is Eq. (3), where both coordinates should be thought of as functions of time \( t \) in order to describe an hypothetical motion of \( P \) on the ellipse. Then, on deriving both sides with respect \( t \) we have
\[ \frac{x \dot{x}}{a^2} + \frac{y \dot{y}}{b^2} = 0, \quad (15) \]
whose left side can be interpreted as the scalar product between the velocity of the point, whose Cartesian representation is \((\dot{x}, \dot{y})\), and the vector \((x/a^2, y/b^2)\). Since such scalar product is null, the latter vector must be directed along the ellipse normal at \( P \).

On denoting \( Q \) the intersection of such normal with the \( x \)-axis (see again Fig. 2), and on taking into account that \( F_1 P = y_i \), it is not difficult to show that \( Q \equiv (x \epsilon^2, 0) \). Moreover, on taking into account that \( F_1 P = a + x \epsilon \) and that \( F_2 P = a - x \epsilon \), it follows at once that:
\[ F_1 Q = \epsilon F_1 P, \quad F_2 Q = \epsilon F_2 P, \quad (16) \]
Finally, on applying the sine theorem to the triangles \( PQF_1 \) and \( PQF_2 \), property P1 then follows.

\[ ^1 \text{Proving these relationships could be left to students as a useful geometry problem.} \]
B Proof of P2

To evaluate the radius of curvature, imagine a point moving across the ellipse according to the following law of motion:

\[
\begin{align*}
  x(t) &= a \cos t, \\
  y(t) &= b \sin t,
\end{align*}
\]

with \( t \in [0, 2\pi] \) in suitable units. On deriving both \( x \) and \( y \), Cartesian representations for both velocity and acceleration reads

\[
\begin{align*}
  \mathbf{v} &= \begin{cases} \\
  \dot{x}(t) &= -a \sin t, \\
  \dot{y}(t) &= b \cos t,
\end{cases} \\
\end{align*}
\]

and

\[
\begin{align*}
  \mathbf{a} &= \begin{cases} \\
  \ddot{x}(t) &= -a \cos t, \\
  \ddot{y}(t) &= -b \sin t,
\end{cases}
\end{align*}
\]

respectively. Then, the radius of curvature stems from the well known intrinsic expression,

\[
\rho_P = \frac{v^3}{|\mathbf{v} \times \mathbf{a}|},
\]

which, on taking Eqs. (19) and (20) into account, after simple algebra gives at once

\[
\rho_P = \frac{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}{ab} = \frac{(a^2 - (a^2 - b^2) \cos^2 t)^{3/2}}{ab} = \frac{(a^2 - a^2 (1 - b^2/a^2) \cos^2 t)^{3/2}}{ab} = \frac{(a^2 - x^2 \epsilon^2)^{3/2}}{ab} = \frac{(a + x \epsilon)^{3/2} (a + x \epsilon)^{3/2}}{ab} = \frac{F_1 F_2^{3/2} F_3^{3/2}}{ab}.
\]
References

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