The strong interaction limit of continuous-time weakly self-avoiding walk

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Dedicated to Erwin Bolthausen and Jürgen Gärtner on the occasion of their 65th and 60th birthday celebration

Abstract The strong interaction limit of the discrete-time weakly self-avoiding walk (or Domb–Joyce model) is trivially seen to be the usual strictly self-avoiding walk. For the continuous-time weakly self-avoiding walk, the situation is more delicate, and is clarified in this paper. The strong interaction limit in the continuous-time setting depends on how the fugacity is scaled, and in one extreme leads to the strictly self-avoiding walk, in another to simple random walk. These two extremes are interpolated by a new model of a self-repelling walk that we call the “quick step” model. We study the limit both for walks taking a fixed number of steps, and for the two-point function.

1 Domb–Joyce model: discrete time

The discrete-time weakly self-avoiding walk, or Domb–Joyce model [6], is a useful adaptation of the strictly self-avoiding walk that continues to be actively studied [1]. It is defined as follows. For simplicity, we restrict attention to the nearest-neighbour model on $\mathbb{Z}^d$, although a more general formulation is easy to obtain.

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Let $d \geq 1$ and $n \geq 0$ be integers, and let $\mathcal{W}_n$ denote the set of nearest-neighbour walks in $\mathbb{Z}^d$, of length $n$, which start from the origin. In other words, $\mathcal{W}_n$ consists of sequences $Y = (Y_0, Y_1, \ldots, Y_n)$ with $Y_i \in \mathbb{Z}^d$, $Y_0 = 0$, $|Y_{i+1} - Y_i| = 1$ (Euclidean distance). Let $\mathcal{S}_n$ denote the set of nearest-neighbour self-avoiding walks in $\mathcal{W}_n$; these are the walks with $Y_i \neq Y_j$ for all $i \neq j$. Let $c_n$ denote the cardinality of $\mathcal{S}_n$. For $Y \in \mathcal{W}_n$ and $x \in \mathbb{Z}^d$, let $n_x = n_x(Y) = \sum_{i=0}^{n} 1_{Y_i=x}$ denote the number of visits to $x$ by $Y$. The Domb–Joyce model is the measure on $\mathcal{W}_n$ which assigns to a walk $Y \in \mathcal{W}_n$ the probability

$$P_{g,n}^{\text{DJ}}(Y) = \frac{1}{c_n^{\text{DJ}}(g)} e^{-g \sum_{i \in \mathbb{Z}^d} n_i(Y)(n_i(Y)-1)},$$

where $g$ is a positive parameter and

$$c_n^{\text{DJ}}(g) = \sum_{Y \in \mathcal{W}_n} e^{-g \sum_{i \in \mathbb{Z}^d} n_i(Y)(n_i(Y)-1)}.$$  

The Domb–Joyce model interpolates between simple random walk and self-avoiding walk. Indeed, the case $g = 0$ corresponds to simple random walk by definition, and also

$$\lim_{g \to \infty} e^{-g \sum_{i \in \mathbb{Z}^d} n_i(Y)(n_i(Y)-1)} = \mathbb{1}_{Y \in \mathcal{S}_n}$$

and hence

$$\lim_{g \to \infty} P_{g,n}^{\text{DJ}}(Y) = \frac{1}{c_n} \mathbb{1}_{Y \in \mathcal{S}_n}.$$  

This shows that the strong interaction limit of the Domb–Joyce model is the uniform measure on $\mathcal{S}_n$. (For an analogous result for weakly self-avoiding lattice trees, which is more subtle than for self-avoiding walks, see [2].)

A standard subadditivity argument (see, e.g., [10, Lemma 1.2.2]) implies that the limits

$$\mu(g) = \lim_{n \to \infty} c_n^{\text{DJ}}(g)^{1/n}, \quad \mu = \lim_{n \to \infty} c_n^{1/n}$$

exist and obey $c_n^{\text{DJ}}(g) \geq \mu(g)^n$ and $c_n \geq \mu^n$ for all $n$. The number of walks that take steps only in the positive coordinate directions is $d^n$, and such walks are self-avoiding, so $c_n \geq d^n$. Also, it follows from [2] that if $0 \leq g < g_0$ then $(2d)^n \geq c_n^{\text{DJ}}(g) \geq c_n^{\text{DJ}}(g_0) \geq c_n \geq d^n$, and hence $2d \geq \mu(g) \geq \mu(g_0) \geq \mu \geq d$. In particular, by monotonicity, $\lim_{g \to \infty} \mu(g)$ exists in $[\mu, 2d]$. If we take the limit $g \to \infty$ in the inequality $c_n^{\text{DJ}}(g) \geq \mu(g)^n \geq \mu^n$, we obtain $c_n \geq (\lim_{g \to \infty} \mu(g))^n \geq \mu^n$. Taking $n^{th}$ roots and then the limit $n \to \infty$ then gives

$$\lim_{g \to \infty} \mu(g) = \mu.$$  

Let $\mathcal{W}_n(x)$ denote the subset of $\mathcal{W}_n$ consisting of walks that end at $x \in \mathbb{Z}^d$. Let $\mathcal{S}_n(x) = \mathcal{S}_n \cap \mathcal{W}_n(x)$, and let $c_n(x)$ denote the cardinality of $\mathcal{S}_n(x)$. Let

$$c_{n,g}^{\text{DJ}}(x) = \sum_{Y \in \mathcal{W}_n(x)} e^{-g \sum_{i \in \mathbb{Z}^d} n_i(Y)(n_i(Y)-1)}.$$  

Let $z \geq 0$. The two-point functions of the Domb–Joyce and self-avoiding walk models are defined as follows:

$$G_{g, z}^{DJ}(x) = \sum_{n=0}^{\infty} c_{n, g}^{DJ}(x) z^n, \quad G_{z}(x) = \sum_{n=0}^{\infty} c_n(x) z^n. \quad (8)$$

These series converge for $z < \mu(g)^{-1}$ and $z < \mu^{-1}$ respectively. Presumably they converge also for $z = \mu(g)^{-1}$ and $z = \mu^{-1}$ but this is a delicate question that is unproven except in high dimensions (in fact, the decay of the two-point function with $z = \mu^{-1}$ is known in some cases [4, 8, 9]). The following proposition shows that the strong interaction limit of $G_{g, z}^{DJ}(x)$ is $G_{z}(x)$.

**Proposition 1.** For $z \in [0, \mu^{-1})$ and $x \in \mathbb{Z}^d$,

$$\lim_{g \to \infty} G_{g, z}^{DJ}(x) = G_{z}(x). \quad (9)$$

**Proof.** Fix $z \in [0, \mu^{-1})$. By (6), if $g_0$ is sufficiently large then $z < \mu(g_0)^{-1}$. Thus, since $c_{n, g}^{DJ}(x)$ is nonincreasing in $g$, there are $r < 1$ and $C > 0$ such that $c_{n, g}(x) z^n \leq c_{n, g_0}(x) z^n \leq Cr^n$ for all $n$, uniformly in $g \geq g_0$. Thus, for all $g \geq g_0$,

$$G_{g, z}^{DJ}(x) \leq \sum_{x \in \mathbb{Z}^d} G_{g, z}^{DJ}(x) = \sum_{n=0}^{\infty} c_{n, g}^{DJ}(x) z^n \leq C \frac{1}{1-r} < \infty. \quad (10)$$

By (3), $\lim_{g \to \infty} c_{n, g}^{DJ}(x) = c_n(x)$, and the desired result then follows by dominated convergence. $\square$

## 2 The continuous-time weakly self-avoiding walk

Our goal is to study the analogues of (4) and Proposition 1 for the continuous-time weakly self-avoiding walk. The continuous-time model is a lattice version of the Edwards model [7]. It has been useful in particular due to its representation in terms of functional integrals [5] that have been employed in renormalisation group analyses.

### 2.1 Fixed-length walks

We first consider the case of fixed-length walks, in which a fixed number $n$ of steps is taken by the walk. We will find that the strong interaction limit depends on how an auxiliary parameter $\rho$ is scaled, where $e^{\rho}$ plays the role of a fugacity. The scaling is parametrized by $a \in [-\infty, \infty]$. The case $a = \infty$ leads to the strictly self-avoiding walk, the case $a = -\infty$ leads to simple random walk, and the interpolating cases,
Let $X$ denote the continuous-time Markov process with state space $\mathbb{Z}^d$, in which uniformly random nearest-neighbour steps are taken after independent $\text{Exp}(1)$ holding times. Let $\mathbb{E}$ denote expectation for this process started at 0. We distinguish between the continuous-time walk $X$ and the sequence of sites visited during its first $n$ steps, which we typically denote by $Y \in \mathcal{W}_n$. Conditioning on the first $n$ steps of $X$ to be $Y$ is denoted by $\mathbb{E}(\cdot \mid Y)$.

For fixed-length walks, the continuous-time weakly self-avoiding walk is the measure $Q_{g,\rho,n}$ on $\mathcal{W}_n$ defined as follows. Here $\rho$ is a real parameter at our disposal, which we allow to depend on $g > 0$. Let $T_n$ denote the time of the $(n+1)^{\text{st}}$ jump of $X$, and let $L_{x,n}(X) = \int_0^{T_n} 1_{X(s) = x} ds$ denote the local time at $x$ up to time $T_n$. By definition, $\sum_{x \in \mathbb{Z}^d} L_{x,n} = T_n$. For $Y \in \mathcal{W}_n$, let

$$Q_{g,\rho,n}(Y) = \frac{1}{Z_n(g,\rho)} \mathbb{E}\left( e^{-g \sum_{x} L_{x,n}^2 + \rho \sum_{x} L_{x,n}} \mid Y \right),$$

(11)

where

$$Z_n(g,\rho) = \sum_{Y \in \mathcal{W}_n} \mathbb{E}\left( e^{-g \sum_{x} L_{x,n}^2 + \rho \sum_{x} L_{x,n}} \mid Y \right).$$

(12)

For $a \in \mathbb{R}$ and $m \in \mathbb{N}$, let

$$I_m(a) = \int_{-a}^{a} \frac{(u + a)^{m-1}}{(m-1)!} e^{-u^2} du.$$ 

(13)

**Proposition 2.** Let $\alpha = \alpha(g,\rho) = \frac{1}{2} g^{-1/2} (\rho - 1)$, and let $\rho = \rho(g)$ be chosen in such a way that $a = \lim_{g \to \infty} \alpha(g,\rho(g))$ exists in $[-\infty, \infty]$. Let $n \geq 1$ and $Y \in \mathcal{W}_n$. Then

$$\lim_{g \to \infty} Q_{g,\rho(g),n}(Y) = \begin{cases} \frac{1}{2} \prod_{x \in Y} e^{a^2 I_{n_x}(Y)(a)} & \text{if } a \in (-\infty, \infty), \\ \frac{1}{e^{a^2}} \mathbb{I}_{Y \in \mathcal{W}_n} & \text{if } a = \infty, \\ \frac{1}{(2\pi)^d} & \text{if } a = -\infty, \end{cases}$$

(14)

where $Z_a$ is a normalisation constant, and the product over $x$ is over the distinct vertices visited by $Y$.

**Proof.** As before, we write $n_x = n_x(Y)$ for the number of times that $x$ is visited by $Y$. Thus $\sum_x n_x = n + 1$ is the number of vertices visited by $Y$ (with multiplicity). Since the sum of $m$ independent $\text{Exp}(1)$ random variables has a Gamma$(m, 1)$ distribution, we have

$$\mathbb{E}\left( e^{-g \sum_{x} L_{x,n}^2 + \rho \sum_{x} L_{x,n}} \mid Y \right) = \prod_{x \in Y} \int_0^{\infty} \frac{s_x^{n_x-1}}{(n_x-1)!} e^{-s_x} e^{-g s_x^2 + \rho s_x^2} ds_x,$$

(15)

where the product is over the distinct vertices visited by $Y$. We make the changes of variables $t_x = g^{1/2} s_x$ and then $u_x = t_x - \alpha$. After completing the square, this leads to
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\[ E \left( e^{-g \sum L_n^2 + \rho \sum L_n} \mid Y \right) = g^{-(n+1)/2} \prod_{x \in Y} e^{\alpha^2} l_n(\alpha). \]  (16)

**Case a \(\in (-\infty, \infty)\): the quick step model.** Suppose that \(\alpha \to a \in (-\infty, \infty)\) as \(g \to \infty\).

In this case, by the continuity of \(l_m(a)\) in \(a\),

\[ E \left( e^{-g \sum L_n^2 + \rho \sum L_n} \mid Y \right) \sim g^{-(n+1)/2} \prod_{x \in Y} e^{\alpha^2} l_n(a), \]  (17)

and thus

\[ \lim_{g \to \infty} Q_{\rho, Y_n}(Y) = \frac{1}{Z_{\alpha}} \prod_{x \in Y} e^{\alpha^2} l_n(Y_{\alpha}) \quad (\alpha \to a \in (-\infty, \infty)). \]  (18)

**Case \(a = \infty\): limit is uniform on \(Y_n\).** Suppose that \(\alpha \to \infty\) as \(g \to \infty\). In this case, since \(\alpha\) is nonzero we can use (16) to write

\[ E \left( e^{-g \sum L_n^2 + \rho \sum L_n} \mid Y \right) \]

\[ = (g^{1/2} e^{\alpha^2})^{n+1} (\alpha e^{-\alpha^2})^{n+1-|Y|} \prod_{x \in Y} \int_{-\alpha}^{\infty} \frac{(1 + u_x/\alpha)^{n_x-1}}{(n_x-1)!} e^{-u_x^2} du_x, \]  (19)

where \(|Y|\) denotes the number of distinct vertices visited by \(Y\). Since the factor \((\alpha e^{-\alpha^2})^{n+1-|Y|}\) goes to zero unless \(Y\) is self-avoiding, in which case the factor is equal to 1 and \(n_x = 1\) for the vertices visited by \(Y\), and since also

\[ \lim_{\alpha \to \infty} \int_{-\alpha}^{\infty} e^{-u_x^2} du_x = \sqrt{\pi}, \]  (20)

this gives

\[ E \left( e^{-g \sum L_n^2 + \rho \sum L_n} \mid Y \right) \sim (g^{1/2} e^{\alpha^2})^{n+1} I_{Y \in Y_n}. \]  (21)

When we take the normalisation into account we find that

\[ \lim_{g \to \infty} Q_{\rho, Y_n}(Y) = \frac{1}{Z_{\alpha}} I_{Y \in Y_n} \quad (\alpha \to \infty). \]  (22)

**Case \(a = -\infty\): limit is uniform on \(Y_n\).** Suppose that \(\alpha \to -\infty\) as \(g \to \infty\). We will show that, for \(m \geq 1,

\[ e^{\alpha^2} l_m(\alpha) \sim (-2\alpha)^{-m} \quad \text{as } \alpha \to -\infty. \]  (23)

With (16), this claim implies that

\[ E \left( e^{-g \sum L_n^2 + \rho \sum L_n} \mid Y \right) \sim g^{-(n+1)/2} \prod_{x \in Y} (-2\alpha)^{-n_x} = (-2\alpha g^{-1/2})^{n+1}. \]  (24)
Since the right-hand side is independent of $Y$, this proves that the limiting measure is uniform on $W_n$, as required. Finally, to prove (23), we set $b = -\alpha$ and obtain

\[
(2b)^m e^{b^2} I_m(-b) = (2b)^m e^{b^2} \int_b^\infty \frac{(-b+u)^{m-1}}{(m-1)!} e^{-u^2} \, du = \int_0^\infty \frac{u^{m-1}}{(m-1)!} e^{-u^2} \left(1 + \frac{u(2b)^2}{2} + \int_0^\infty \frac{u(2b)^2}{2} \left(1 + \frac{u(2b)^2}{2} + \ldots \right) \, du \right) = (2b)^m e^{b^2} \int_0^\infty \frac{u^{m-1}}{(m-1)!} e^{-u^2} \, du.
\]  

(25)

By dominated convergence, as $b \to \infty$, the integral on the right-hand side approaches 1 because it becomes the integral over the $\Gamma(m, 1)$ probability density function. 

Proposition 2 shows that the case $\alpha \to \infty$ leads to the uniform measure on self-avoiding walks, whereas $\alpha \to -\infty$ leads to simple random walk. These two extremes are interpolated by the quick step walk, for $\alpha \to a \in (-\infty, \infty)$ (e.g., $a = 0$ if $|\rho| = o(g^{1/2})$ or $a = c$ if $\rho \sim 2c g^{1/2}$). The name “quick step walk” is intended to reflect that idea that the large $g$ limit of the continuous-time walk should be dominated by quickly moving continuous-time walks. In fact, when $\rho = 2ag^{1/2}$, by completing the square the weight $e^{-\sum x(Y) (2a g^{1/2} x - n)}$ can be rewritten as $e^{\sum x(Y) (2a g^{1/2} x - n)}$. Thus walks with smaller $L_{\alpha,n}$ receive larger weight, and this effect grows in importance as $g \to \infty$.

The particular case of Proposition 2 for the choice $\rho(g) = (2g \log(g/\pi))^{1/2}$, which corresponds to $a = \infty$, was proved previously in [3].

For the case $a = 0$, evaluation of $I(n, Y) \rho(Y)$ in (13) gives

\[
\lim_{g \to \infty} Q_g \rho(Y) = \frac{1}{Z_0} \prod_{x \in Y} \frac{\Gamma(n_x(Y)/2)}{2\Gamma(n_x(Y))} (\alpha \to 0).
\]  

(27)

Large values of $n_x$ are penalised under this limiting probability, so this is a model of a self-repelling walk. It is an interesting question whether the quick step walk is in the same universality class as the self-avoiding walk, for $a \in (-\infty, \infty)$. We do not have an answer to this question.

2.2 Two-point function

Now we show that when $\rho$ is chosen carefully, depending on $g$, the two-point function for the continuous-time weakly self-avoiding walk converges, as $g \to \infty$, to the two-point function of the strictly self-avoiding walk. The two-point function of the continuous-time weakly self-avoiding walk can be written in two equivalent ways. This is discussed in a self-contained manner in [5], and we summarise the situation as follows.

The version of the two-point function that we will work with is written in terms of a modified Markov process $X = X(t)$, whose definition depends on a choice of
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\( \delta \in (0, 1) \). The state space is \( \mathbb{Z}^d \cup \{ \partial \} \), where \( \partial \) is an absorbing state called the cemetery. When \( X \) arrives at state \( x \) it waits for an \( \text{Exp}(1) \) holding time and then jumps to a neighbour of \( x \) with probability \( (2d)^{-1}(1 - \delta) \) and jumps to the cemetery with probability \( \delta \). The holding times are independent of each other and of the jumps. The two-point function is defined, for \( x \in \mathbb{Z}^d \), to be

\[
G_{g,\rho}^{\text{CT}}(x) = \frac{1}{\delta^n \mathbb{E}(\delta)} \left( e^{-\sum_{i \in \mathbb{Z}^d} \frac{L_i^2}{2} + \rho \xi \mathbb{1}_{X(\xi) = x}} \right),
\]

where we leave implicit the dependence of \( G^{\text{CT}} \) on \( \delta \), where \( \mathbb{E}(\delta) \) denotes expectation with respect to the modified process, and where \( \rho \) is any real number for which the expectation is finite. The random number of steps taken by \( X \) before jumping to the cemetery is denoted \( \eta \), and the independent sequence of holding times will be denoted \( \sigma_0, \sigma_1, \ldots, \sigma_\eta \).

A special case of the conclusions of [5, Section 3.2] (there with \( d_c = 1 \) and \( \pi_{x,\delta} = \delta \) for all \( x \), and restricted to finite state space) is the equivalent formula

\[
G_{g,\rho}^{\text{CT}}(x) = \int_0^\infty \mathbb{E} \left( e^{-\sum_{i \in \mathbb{Z}^d} \frac{L_i^2}{2} \mathbb{1}_{X(\xi) = x}} \right) e^{(\rho - \delta)T} dT,
\]

where now \( X \) is the original continuous-time Markov process \( X \) without cemetery state, and \( \mathbb{E} \) denotes its expectation when started from the origin of \( \mathbb{Z}^d \). Here \( L_v T = \int_0^T \mathbb{1}_{X(s)=v} ds \) is the local time of \( X \) at \( v \in \mathbb{Z}^d \) up to time \( T \). We will work with \( (28) \) rather than \( (29) \).

As in Proposition 2 we write \( \alpha = \alpha(g, \rho) = \frac{1}{g} \rho^{-1/2} (\rho - 1) \). Throughout this section, we mainly choose \( \rho = \rho(g) \) in such a way that

\[
\lim_{g \to \infty} g^{-1/2} e^{\alpha^2(g, \rho)} = p \in [0, \infty)
\]

(30)

For example, \( (30) \) holds for \( p > 0 \) when \( \rho(g) = 2 [g \log(p \sqrt{g})]^{1/2} \), which is a choice closely related to that in \( (26) \). Note that \( \lim_{g \to \infty} \rho(g) = \infty \) when \( p > 0 \). It is natural to consider \( \rho \to \infty \), because if \( \rho \) is fixed to a value such that \( G_{g_0,\rho}^{\text{CT}}(x) < \infty \) for some \( g_0 > 0 \), then by dominated convergence \( \lim_{g \to \infty} G_{g,\rho}^{\text{CT}}(x) = 0 \). The conclusion of Proposition 3 shows that this trivial behaviour persists even when \( \rho(g) \to \infty \) in such a way that \( p = 0 \).

Given \( p \in [0, \infty) \), let

\[
z = (2d)^{-1}(1 - \delta)p\sqrt{\pi}.
\]

(31)

The following proposition shows that, under the scaling \( (30) \), the strong interaction limit of the continuous-time weakly self-avoiding walk two-point function is the two-point function of the strictly self-avoiding walk defined in \( (8) \).

**Proposition 3.** Let \( \delta \in (0, 1), z \in [0, \mu^{-1}) \), and \( x \in \mathbb{Z}^d \). Suppose that \( (30) \) holds with the value of \( p \in [0, \infty) \) specified by \( z \) via \( (31) \). Then

\[
\lim_{g \to \infty} G_{g,\rho(g)}^{\text{CT}}(x) = p\sqrt{\pi} G_z(x).
\]

(32)
The proof of Proposition 3 uses three lemmas, and we discuss these next. For $m \in \mathbb{N}$ and $\alpha > 0$, let
\begin{equation}
J_m(\alpha) = \int_{-\alpha}^{\infty} \frac{(1 + u/\alpha)^{m-1}}{(m-1)!} e^{-u^2} du.
\end{equation}

**Lemma 1.** Given any $\epsilon > 0$ there exists $A_0 > 0$ such that for all $\alpha \geq A \geq A_0$ and $m \geq 1$,
\begin{equation}
J_m(\alpha) \leq (1 + \epsilon)J_m(A).
\end{equation}

**Proof.** For $m \geq 2$, $J_m(\alpha)$ is a non-increasing function of $\alpha \in (0, \infty)$ because
\begin{align*}
\frac{dJ_m(\alpha)}{d\alpha} &= -\frac{1}{(m-2)!} \int_{-\alpha}^{\infty} \frac{u}{\alpha^2} (1 + u/\alpha)^{m-2} e^{-u^2} du \\
&= -\frac{1}{(m-2)!} \left[ \int_{-\alpha}^{\infty} \frac{u}{\alpha^2} (1 + u/\alpha)^{m-2} e^{-u^2} du \\
&+ \int_{0}^{\alpha} \frac{u}{\alpha^2} [(1 + u/\alpha)^{m-2} - (1 - u/\alpha)^{m-2}] e^{-u^2} du \right] \\
&\leq 0
\end{align*}
(note that in the first line the contribution from differentiating the limit of integration vanishes), and thus (34) holds even with $\epsilon = 0$. For the remaining case $m = 1$, since $J_1$ is increasing and $\lim_{\alpha \to \infty} J_1(\alpha) = \sqrt{\pi}$ (see (20)), given any $\epsilon > 0$ there exists $A_0 > 0$ such that if $\alpha \geq A \geq A_0$ then $1 \leq J_1(\alpha)/J_1(A) \leq 1 + \epsilon$. □

Recall that $\eta$ is the random number of steps taken by $X$ before jumping to the cemetery state. For $x \in \mathbb{Z}^d$, let
\begin{align*}
w_n(g; x) &= \frac{1}{\delta} \mathbb{E}(\delta)[e^{-\delta \sum l_{z_i}^2 + \rho \zeta_{\delta}(x) - x} \mathbbm{1}_{\eta=n}], \\
w_n(g, \rho) &= \frac{1}{\delta} \mathbb{E}(\delta)[e^{-\delta \sum l_{z_i}^2 + \rho \zeta_{\delta}} \mathbbm{1}_{\eta=n}].
\end{align*}

Let $w_n(g; x) = w_n(g, \rho(g); x)$ and $w_n(g) = w_n(g, \rho(g))$ with $\rho(g)$ chosen according to (30).

**Lemma 2.** Suppose that (30) holds with $p > 0$, and let $z$ be given by (31). Then for $n \geq 0$ and $x \in \mathbb{Z}^d$,
\begin{equation}
\lim_{g \to \infty} w_n(g; x) = p \sqrt{\pi c_n(x)} z^n.
\end{equation}

**Proof.** Given that $\eta = n$, let $Y \in \mathcal{W}_n(x)$ denote the sequence of jumps made by $X$ before landing in the cemetery, and let $|Y|$ denote the cardinality of the range of $Y$. By conditioning on $Y$ and using (19), we see that, as $g \to \infty$, ...
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\[ w_n(g; x) = [(2d)^{-1} (1 - \delta)]^n (g^{-1/2} e^{\alpha \delta^2})^{n+1} \sum_{Y \in \mathcal{W}_n(x)} (\alpha e^{-\alpha^2})^{n+1-|Y|} \prod_{v \in Y} J_{n_1}(\alpha) \]

\[ \sim [(2d)^{-1} (1 - \delta)]^n p^{n+1} \sum_{Y \in \mathcal{W}_n(x)} (\alpha e^{-\alpha^2})^{n+1-|Y|} \prod_{v \in Y} J_{n_1}(\alpha), \tag{39} \]

where the product is over the distinct vertices visited by \( Y \) and \( |Y| \) denotes the number of such vertices. It suffices to show that, for any \( Y \in \mathcal{W}_n(x) \),

\[ \lim_{g \to \infty} (\alpha e^{-\alpha^2})^{n+1-|Y|} \prod_{v \in Y} J_{n_1}(\alpha) = 1_{Y \in \mathcal{S}_n} \pi^{(n+1)/2}. \tag{40} \]

Since \( p > 0 \), we have \( \alpha \to \infty \), and so \( \alpha e^{-\alpha^2} \to 0 \). Therefore, the above limit is zero unless \( n + 1 = |Y| \), which corresponds to \( Y \in \mathcal{S}_n \); the product over \( v \) remains bounded as \( \alpha \to \infty \) and poses no difficulty. Since \( J_1(\alpha) \to \sqrt{\pi} \) as in (20), the result follows. \( \Box \)

**Lemma 3.** Suppose that (30) holds with \( p \in (0, \infty) \), and let \( z \) be specified by (31). Let

\[ \mu(g, \rho) = \limsup_{n \to \infty} w_n(g, \rho)^{1/n}. \tag{41} \]

Then

\[ \limsup_{g \to \infty} \mu(g, \rho(g)) \leq z \mu. \tag{42} \]

**Proof.** Let \( L_{x|[y]} = \sum_{k \in \mathbb{Z}_+} \sigma_k \mathbb{1}_{Y_k = x} \), where the \( \sigma_k \) are the exponential holding times. Let \( E_n^{(y)} \) denote the expectation for the process started in state \( y \) instead of state 0. For integers \( n \geq 1 \) and \( m \geq 1 \), an elementary argument using the strong Markov property leads to

\[ w_{n+m}(g, \rho) \leq \frac{1}{\delta} E_n^{(y)} \left[ e^{-\delta \sum_{k \in \mathbb{Z}_+} L_k^{\Sigma_{k}} \mathbb{1}_{Y_k = x}} e^{-\delta \sum_{k \in \mathbb{Z}_+} L_k^{\Sigma_{k+1} \mathbb{1}_{Y_k = x}}} \right] \]

\[ = \sum_{y} E_n^{(y)} \left[ e^{-\delta \sum_{k \in \mathbb{Z}_+} L_k^{\Sigma_{k}} \mathbb{1}_{Y_k = x}} e^{-\delta \sum_{k \in \mathbb{Z}_+} L_k^{\Sigma_{k+1} \mathbb{1}_{Y_k = x}}} \prod_{\eta = m+1} \right] \]

\[ \leq w_n(g, \rho) w_{m-1}(g, \rho), \tag{43} \]

It is straightforward to adapt the proof of [10] Lemma 1.2.2 to obtain from this approximate subadditivity the equality

\[ \mu(g, \rho) = \inf_{n \geq 1} w_n(g, \rho)^{1/(n+1)}. \tag{44} \]

Then we have

\[ w_n(g, \rho)^{1/(n+1)} \geq \mu(g, \rho). \tag{45} \]
We let \( g \to \infty \) in the above inequality, with \( \rho(g) \) chosen as in (30); note that \( \alpha \to \infty \) since \( p > 0 \). By Lemma 2, for \( n \geq 0 \),
\[
\lim_{g \to \infty} w_n(g) = p \sqrt{\pi} c_n z^n. \tag{46}
\]
By (45), this gives
\[
(p \sqrt{\pi} c_n)^{1/(n+1)} z^n/(n+1) \geq \limsup_{g \to \infty} \mu(g, \rho(g)). \tag{47}
\]
Now we take \( n \to \infty \) to get
\[
\mu z \geq \limsup_{g \to \infty} \mu(g, \rho(g)), \tag{48}
\]
as required. \( \square \)

**Proof of Proposition 3.** We consider separately the cases \( p > 0 \) and \( p = 0 \).

**Case** \( p > 0 \). We write \( \rho = \rho(g) \). By (28), and by (36) with \( \rho = \rho(g) \),
\[
G_{\mu \rho}^\mu(x) = \sum_{n=0}^{\infty} w_n(g; x). \tag{49}
\]
By Lemma 2 the result of taking the limit \( g \to \infty \) under the summation gives the desired result
\[
p \sqrt{\pi} \sum_{n=0}^{\infty} c_n(x) z^n, \tag{50}
\]
and it suffices to justify the interchange of limit and summation. For this, we will use dominated convergence. Since \( w_n(g; x) \leq w_n(g) \), it suffices to find a \( g_0 > 0 \) and a summable sequence \( B_n \) such that, for \( g \geq g_0 \) and \( n \in \mathbb{N}_0 \),
\[
w_n(g; x) \leq B_n. \tag{51}
\]
This will follow if we show the stronger statement that for large \( g \)
\[
w_n(g) \leq B_n. \tag{52}
\]
Since \( z \mu < 1 \), there exists \( \varepsilon > 0 \) such that \( c = (1 + \varepsilon)^2 (\mu z + \varepsilon) < 1 \). Since \( g^{-1/2} e^{\alpha^2} \to p > 0 \), there is a (large) \( g_0 \) such that if \( g \geq g_0 \) then \( g^{-1/2} e^{\alpha^2} \leq g_0^{-1/2} e^{\alpha_0^2} (1 + \varepsilon) \), where \( \alpha_0 \) is the value of \( \alpha \) corresponding to \( g = g_0 \); also \( \alpha e^{-\alpha^2} \leq \alpha_0 e^{-\alpha_0^2} \). Therefore, by (39), and by Lemma 1 (increasing \( g_0 \) if necessary),
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\[ w_n(g) = [(2d)^{-1}(1 - \delta)]^n (g^{-1/2}e^{\alpha_2})^{n+1} \sum_{Y \in \mathcal{W}_n} (\alpha e^{-\alpha^2})^{n+1-|Y|} \prod_{v \in Y} J_n(\alpha) \]

\[ \leq [(2d)^{-1}(1 - \delta)]^n (g_0^{-1/2}e^{\alpha_0^2}(1 + \delta)^2)^{n+1} \sum_{Y \in \mathcal{W}_n} (\alpha_0 e^{-\alpha_0^2})^{n+1-|Y|} \prod_{v \in Y} J_n(\alpha_0) \]

\[ = (1 + \delta)^{2(n+1)} w_n(g_0). \quad (53) \]

We set \( B_n = (1 + \delta)^{2(n+1)} w_n(g_0) \). Then

\[ \limsup_{n \to \infty} B_n^{1/n} = (1 + \delta)^2 \mu(g_0, \rho(g_0)) \leq (1 + \delta)^2 (\mu + \epsilon) < 1, \quad (54) \]

by taking \( g_0 \) larger if necessary and applying Lemma 3. Therefore \( \sum B_n \) converges, and the proof is complete for the case \( p > 0 \).

\textit{Case } \( p = 0 \). We will prove that

\[ \lim_{g \to \infty} \sum_{n=0}^{\infty} w_n(g) = 0. \quad (55) \]

By (49), this is more than sufficient. We again write \( \rho = \rho(g) \). By conditioning on \( Y \) and using (16), for \( n \geq 0 \) we have

\[ w_n(g) = [(2d)^{-1}(1 - \delta)]^n \sum_{Y \in \mathcal{W}_n} \prod_{x \in Y} e^{-m/2}e^{\alpha_2} I_n(\alpha). \quad (56) \]

The change of variables \( s = a + u \) in (13) gives, for \( m \geq 1 \),

\[ e^{\alpha_2} I_m(\alpha) = e^{\alpha_2} \int_0^\infty \frac{s^{m-1}}{(m-1)!} e^{-(s-\alpha)^2} ds \]

\[ \leq e^{\alpha_2} \int_0^\infty \frac{s^{m-1}}{(m-1)!} e^{-s} \left( \sup_{s \in \mathbb{R}} e^{s^2} \right) ds = e^{\alpha_2 + \alpha + 1/4}. \quad (57) \]

Let \( \epsilon > 0 \). Since \( g^{-1/2}e^{\alpha_2} \to p = 0 \), we can find \( g(\epsilon) \) such that for \( g \geq g(\epsilon) \) and

\[ g^{-1/2}e^{\alpha_2} \sqrt{\pi} \leq \epsilon, \quad g^{-m/2}e^{\alpha_2 + \alpha + 1/4} \leq \epsilon^m. \quad (58) \]

Henceforth we assume that \( g \geq g(\epsilon) \). By (57),

\[ g^{-m/2}e^{\alpha_2} I_m(\alpha) \leq \epsilon^m \quad \text{for } m \geq 2. \quad (59) \]

For \( m = 1 \), we obtain an upper bound by extending the range of the integral in the first line of (57) to the entire real line, whereupon it evaluates to \( \sqrt{\pi} \). Thus, by (58), \( g^{-1/2}e^{\alpha_2} I_1(\alpha) \leq \epsilon \). By (56) and the fact that the number of walks in \( \mathcal{W}_n \) is \( (2d)^n \), for \( n \geq 0 \) we then have

\[ w_n(g) \leq [(2d)^{-1}(1 - \delta)]^n \sum_{Y \in \mathcal{W}_n} \prod_{v \in Y} e^{nv} = (1 - \delta)^n e^{n+1}. \quad (60) \]
The case $n = 0$ corresponds to $m = 1$ because the number of visits to state 0 is $n_0 = 1$. Therefore $\limsup_{g \to \infty} \sum_{n=0}^{\infty} w_n(g) = O(\varepsilon)$. Since $\varepsilon$ is arbitrary, this proves (55), and the proof is complete. □

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