Can Renormalization Group Flow End in a Big Mess?

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ABSTRACT

The field theoretical renormalization group equations have many common features with the equations of dynamical systems. In particular, the manner how Callan-Symanzik equation ensures the independence of a theory from its subtraction point is reminiscent of self-similarity in autonomous flows towards attractors. Motivated by such analogies we propose that besides isolated fixed points, the couplings in a renormalizable field theory may also flow towards more general, even fractal attractors. This could lead to Big Mess scenarios in applications to multiphase systems, from spin-glasses and neural networks to fundamental string (M?) theory. We consider various general aspects of such chaotic flows. We argue that they pose no obvious contradictions with the known properties of effective actions, the existence of dissipative Lyapunov functions, and even the strong version of the $c$-theorem. We also explain the difficulties encountered when constructing effective actions with chaotic renormalization group flows and observe that they have many common virtues with realistic field theory effective actions. We conclude that if chaotic renormalization group flows are to be excluded, conceptually novel no-go theorems must be developed.

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1. Introduction

The concept of a renormalization group (RG) flow is a basic notion both in quantum field theory and string theory [1]-[15]. In the Wilsonian approach, the RG equation describes a flow in the space of operators towards a subspace of relevant and marginal operators. This subspace can be viewed as a functional attractor for the flow. The behaviour within the functional attractor is then determined by the β-functions for the relevant couplings. Conventionally one expects that these couplings flow towards attractors which consist of a finite number of isolated fixed points. This ensures that the limit of the RG flow yields a definite quantum field theory with destined values for its couplings.

However, already in [2] Wilson noted that the RG flow of the couplings could approach attractors which are more elaborate than plain isolated fixed points. In particular, he suggested that the end of a RG coupling flow could be a limit cycle. Some evidence in support of this conjecture has been recently provided in [16]-[18], where a coupling flow with a (periodic) blow-up is interpreted in terms of a limit cycle.

Here we shall inquire whether a RG flow could indeed tend towards a non-trivial attractor, with even a fractal structure corresponding to a chaotic flow in the space of couplings. Such chaotic flows could lead to interesting Big Mess scenarios in various applications of quantum field and string theories. Indeed, we suggest that the commonly accepted dogma that RG flows can only approach plain, discreet fixed points is in an apparent contradiction with the existence of multiphase systems described by spin glasses and neural networks, which are expected to emerge as the IR endpoints of RG flows from simple microscopic Hamiltonians. Furthermore, it should be only natural to speculate that the IR finality of the full (but yet to be discovered) string theory possesses complex multiphase structures [19] with a variety of quantum field theories, strings and brane models, emerging at the ends of some as yet unidentified chaotic RG evolutions. These flows are expected originate from a relatively simple (fundamental
microscopic) system, for example from the celebrated $M$-theory [20, 21]. We propose that such chaotic RG scenarios are realizable, very much like condensed matter physics with its highly complex long-distance structures emerges from a simple microscopic Hamiltonian. Indeed, if complex systems are to appear as asymptotic IR limits of some RG flows it should be clear that these flows can not be towards a finite set of plain isolated fixed points but rather towards more general, even strange attractors with self-similar structures over several orders of magnitude.

We emphasize that we are not considering the possibility of chaos in an underlying field theory. For example, in the classical Yang-Mills theory chaotic behaviour has already been well established [22]. Consequently such chaotic behaviour will not be considered here. Obviously, a chaotic RG flow also necessitates the consideration of field (string) theories with at least three couplings.

In the present article we shall be interested in the possibility of chaotic RG flows in the IR limits of quantum field and string theories. While we are not in a position to present concrete examples of theories which exhibit a chaotic RG flow, we do have a number of plausibility arguments which support their existence. Furthermore, we can understand and explain the difficulties encountered in their constructions. At least, our work should motivate the derivation of no-go theorems. But at the moment we do not see any immediate contradictions between the existence of chaotic RG flows and known properties of field and string theories. Indeed, we believe that the potential existence of a chaotic RG flow is an important question, and either examples should be searched for or then conceptually novel no-go theorems forbidding chaotic RG behaviour should be established.

Exact RG group equations are operative everywhere in the space $M$ of couplings $\{t^i\}$. Occasionally, and in particular in the context of string theories, these couplings are identified with the moduli space of the theory under consideration. The Wilsonian approach to the exact RG is based on an effective action,

$$\mathcal{F}_\mathcal{A}(t|\phi) \sim \log Z_\mathcal{A}(t|\phi),$$
The integration extends over a functional space $\mathcal{A}$ of fields $\phi$. The effective action $F$ depends both on the background fields $\varphi$ and on a functional form which is parameterized by the couplings $t^i$. Consequently it is a section of a line bundle over background fields $\varphi$ and the manifold $\mathcal{M}$ of couplings $t^i$. The Wilsonian RG flow (in the sequel we consider mainly flows from the UV to the IR) describes the change of $F$ under the change of $\mathcal{A}$, when some of the background fields are integrated out. In the simplest case these background fields are Fourier components with momenta exceeding some normalization scale $\mu^2$. The boundary $\partial\mathcal{A}$ can have a generic shape. It is parametrized by Whitham times $T_\alpha$, and the Wilsonian RG equations can be understood as evolution equations in all possible Whitham directions [12, 13]. This leads to a relation between the RG flow and the concept of self-similarity (or functional similarity in the terminology of [9]) between effective actions evaluated at different Whitham times. The present version of renormalizability then makes exact RG flows a part of the general theory of dynamical systems [23].

The specific feature of a RG flow when viewed as a dynamical system is that it involves the effective action. As a section in a line bundle the effective action is in general a multivalued function of the background fields and couplings. Consequently there is an element of local integrability, that may be eventually lost due to global obstructions. Furthermore, since the effective action is defined by a (functional) integral it possesses extensive symmetries relating to changes of the integration variables (quantum fields) $\phi$. These symmetries are usually expressed in terms of Ward identities (or Picard-Fucks equations) [24]-[26]. For an exact RG, the number of independent couplings $t^i$ must also be large - in fact infinite - to ensure that the operators involved indeed form a complete basis of functionals. Furthermore, when we ignore the background fields $\varphi$, the exact RG equations [3] for the effective action acquire a Callan-Symanzik form, which
is linear in the derivatives w.r.t. couplings,

\[ \dot{Z}(t) = \frac{\partial Z(t)}{\partial s} = \sum_i \beta^i(t) \frac{\partial Z(t)}{\partial t^i}, \]

\[ \dot{F}(t) = \sum_i \beta^i(t) \frac{\partial F(t)}{\partial t^i}, \]

\[ \beta^i(t) = i^i = -\mu \frac{\partial t^i}{\partial \mu}, \quad (s = -\log \mu) \quad (2) \]

This functional form emerges for a strongly complete basis of operators \( O_i(\phi) \) (for definitions see [15]) which includes all linearly independent generators. (For a weakly complete basis which includes all algebraically independent operators, the RG equations contain higher derivatives of \( Z(t) \) [15].)

In the next section we shall first consider certain general aspects of the coupling flow. We then review some of the assumptions that underlie RG flows, in particular the \( c \)-theorem(s). In section 3 we first relate a few field theory \( \beta \)-functions to the \( c \)-theorems. We then argue by considering the Lorenz model, that the properties of these coupling flows are not inconsistent with chaotic RG flows. In section 4 we consider limit cycles from the point of view of RG flows, and inspect vorticity as a RG scheme independent tool for describing multicoupling flows. In section 5 we study model effective actions as toy models for reproducing realistic scaling properties of field theories. In section 6 we explain how to construct model effective actions from the \( \beta \)-function flows. In particular, we explain how the construction fails in case of chaotic flows and suggests this parallels the problems encountered in constructing actual field theory effective actions. This also explains why it is very hard to construct actual field theory models with chaotic RG flow. In section 7 we consider possible extensions and scenarios for realizing chaotic RG flows, including spectral flow in general and in particular in stringy context. We conclude with some suggestions on Early Universe models.
2. The $\epsilon$-theorems

The idea of a chaotic attractor is actually not too distant from a quantum field theoretical RG equation. This can be seen already by inspecting the functional form of familiar one-loop single coupling $\beta$-functions. For example, in $d = 2 + \epsilon$ dimensions the one-loop $\beta$ function of the $O(N)$ nonlinear $\sigma$-model has the functional form

$$\dot{g} = \epsilon ag(1 - g)$$

which coincides with the form of the Verhulst equation of population growth [27]. In its discretized version, this clearly relates to the logistic equation

$$x_{n+1} = cx_n(1 - x_n)$$

which is the classic example of an iterative equation with chaos.

But for chaos in a continuous RG flow, we need at least three independent couplings: In a renormalizable field theory with several couplings $t^i$, the RG flow is described by the following autonomous linear system (recall that when considering flows towards the IR limit we have $s = -\ln \mu$ which means that the ensuing (IR) $\beta$-functions are positive in asymptotically free models)

$$-\frac{\partial t^i}{\partial \mu} = \frac{\partial t^i}{\partial s} = \beta^i(t)$$

Conventionally, one assumes that in a field theory this IR flow is asymptotically approaching an isolated fixed point which is hyperbolic. For classification purposes we may then evoke the Hartman-Grobman theorem [28] which allows us to consider $\beta$-functions which are linear in the couplings,

$$\beta^i(t) \approx B^i_j t^j + \mathcal{O}(t^2)$$

($B^i_j$ constant). We note that this may not be attainable by conventional changes in the renormalization scheme, which are analytic diffeomorphisms on the space of couplings of the form

$$t^i \rightarrow \tilde{t}^i(t) = A^i_j t^j + \mathcal{O}(t^2)$$
where $A^i_j$ is a constant nonsingular matrix with positive eigenvalues. Obviously, the first term in (6) describes only integrable, linear flow along the eigendirections of the matrix $B^i_j$. But it is widely respected that even in the presence of apparently very simple non-linear corrections, such as in the case of the Rössler system [29]

$$
\begin{align*}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= bx + cz + xz
\end{align*}
$$

(8)

the equations (5) can possess a multitude of asymptotic behaviours as the flow-time $s$ goes to infinity. Besides isolated fixed points and limit cycles, the trajectories can also approach chaotic strange attractors with complex geometries and fractal dimensions.

Notice that the equation (5) is renormalization scheme covariant, i.e. form invariant under the diffeomorphisms (7). Indeed, under a general coordinate transformation $t^i \rightarrow \tilde{t}^i(t)$ the $\beta^i$ transform as components of a vector field

$$
\tilde{\beta}^i(\tilde{t}) = \frac{\partial \tilde{t}^i}{\partial t^j} \beta^j(t),
$$

Infinitesimally, for $\tilde{t}^i = t^i + \epsilon^i(t)$ we then have

$$
\delta \beta^i(t) = \tilde{\beta}^i(t) - \beta^i(t) = \beta^j \frac{\partial \epsilon^i}{\partial t^j} - \epsilon^j \frac{\partial \beta^i}{\partial t^j} = - (\mathcal{L}_\epsilon \beta)^i
$$

which is the Lie derivative of $\beta$ along $\epsilon$. Consequently we can interpret the flow (5) geometrically in a renormalization scheme independent manner, with $\beta^i$ a vector field in the tangent bundle of $\mathcal{M}$.

In formal quantum field theory investigations one traditionally assumes that the flow (5) can only tend towards isolated fixed points. For this, the vector field $\beta^i$ is subjected to a variety of conditions. A pivotal requirement is that the renormalization group flow must be irreversible. According to Zamolodchikov’s

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\[1\] Notice that this representation of the Rössler equations differs from the conventional one [29] where the last equation reads $\dot{z} = \tilde{b} + z(x + \tilde{c})$, by a linear transformation of variables: $x \rightarrow x + ab$, $y \rightarrow y - b$, $z \rightarrow z + b$ so that $\tilde{b} = -bc$ and $\tilde{c} = c - ab$. 

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(weak) $c$-theorem [30, 31] this irreversibility is ensured by the existence of a Liapunov function $c(t)$ which is monotonically decreasing along the RG flow towards the IR

$$\frac{dc(t)}{ds} = \beta^i(t) \frac{\partial c(t)}{\partial t^i} < 0$$

(9)

Occasionally one also assumes that the Liapunov function is positive semidefinite $c(t) \geq 0$. It may then be related to the number of degrees of freedom in the theory.

A strong version of the $c$-theorem states that the vector field $\beta^i$ is a gradient,

$$\beta^i(t) = -G^{ij}(t) \frac{\partial \sigma(t)}{\partial t^j}$$

(10)

with a symmetric metric $G_{ij}(t) = G_{ji}(t)$. Furthermore, if $G_{ij}(t)$ is positive-definite as is usually assumed in the strong $c$-theorem, the generating function $\sigma(t)$ is a Lyapunov function:

$$\dot{\sigma} = \frac{d\sigma}{ds} = -\sum_{ij} G^{ij} \frac{\partial \sigma}{\partial t^i} \frac{\partial \sigma}{\partial t^j} < 0$$

(11)

Furthermore, in the case of two-dimensional field theories it can be argued that $\sigma(t)$ is a (perfect) Morse function [32] with its (isolated) critical points corresponding to conformal field theories. In that case (11) becomes a gradient flow between the critical points of $c$, i.e. between different conformal field theories.

We note that these statements on the RG flow are renormalization scheme independent, and intrinsically geometric.
3. Some Examples

We shall now argue, with examples, that many of the widely accepted properties of the RG flow do not exclude strange attractors from appearing in the IR limit of the flow. We start by considering examples of field theoretical RG coupling flow equations from the perspective of the $c$-theorems.

An important example of a RG coupling flow with three independent couplings (the minimal number required for a chaotic flow) is the $U(1) \times SU(2) \times SU(3)$ standard model in four dimensions. At the two-loop level, and ignoring the contribution from the Higgs sector, the three gauge couplings $t^i$ flow according to [33]

$$ t^i = g^{ij} \partial_j F $$

where the metric $g^{ij}$ has the form

$$ g^{ij} = -\delta^{ij} + \sum_k b^{i}_k (t^k)^2 $$

with

$$ 16 \pi^2 b^1 = \frac{4}{3} n + \frac{1}{10} $$
$$ 16 \pi^2 b^2 = \frac{22}{3} - \frac{4}{3} n - \frac{1}{6} $$
$$ 16 \pi^2 b^3 = 11 - \frac{4}{3} n $$

$$ b^{i}_k = \frac{1}{(16 \pi^2)^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{14}{15} & 0 \\ 0 & 0 & 102 \end{pmatrix} - \frac{n}{(16 \pi^2)^2} \begin{pmatrix} -19/15 & -1/5 & -11/30 \\ -4/15 & -3/5 & -7/10 \\ -4/15 & -2/5 & 7/3 \end{pmatrix} $$

where $n$ is the number of generations and $F$ is a (degenerate) Morse function for the critical point at $t^i = 0$,

$$ F(t) = \frac{1}{4} \left[ (t^1)^4 - (t^2)^4 - (t^3)^4 \right] $$

Here the indefiniteness of $F(t)$ (which is usually excluded by $c$-theorems due to its naive contradiction with unitarity) reflects UV asymptotic freedom of the non-abelian components in the model.
The first \( (b^i) \) term in (13) is the one-loop contribution, the second \( (b^i k) \) term is the two-loop correction. Note that the entire two-loop contribution can be viewed as a correction to the one-loop metric. We also note that depending on \( n \), the metric can have different signatures. Furthermore, depending on \( t^i \) the signature of the two-loop metric can differ from the signature of the one-loop metric. While these observations on the qualitative attributes of the metric as such can hardly have much relevance to the Physics of Standard Model they are still instructive in revealing the variety of properties that coupling flows in quantum field theories share.

As another example, where already at the one-loop level the signature of the metric depends on the relative (small) values of the couplings, we consider the standard Yukawa coupling between a pseudoscalar meson and a Dirac fermion. There are now two couplings, the Yukawa coupling \( g \) and the quartic self-coupling \( \lambda \) of the pseudoscalar. At the one-loop level the flow equations are (in \( d = 4 + \epsilon \) dimensions, with \( (2\pi)^d N_d \) the area of the unit sphere in \( d \) dimensions) [34]

\[
\begin{align*}
\dot{g} &= \frac{\epsilon}{2} g + \frac{5}{2} N_d g^3 + \ldots \\
\dot{\lambda} &= \epsilon \lambda + \frac{3}{2} N_d \left( \lambda^2 + \frac{8}{3} \lambda g^2 - 16 g^4 \right) + \ldots 
\end{align*}
\] (16)

The origin \( g = \lambda = 0 \) is a critical point with its stability depending on the sign of \( \epsilon = d - 4 \). If we introduce the non-degenerate Morse function

\[
F = \frac{1}{2} \left( g^2 + \lambda^2 \right)
\] (17)

these equations can be written in the gradient-flow form

\[
\begin{align*}
\dot{g} &= G^{g g} \partial_g F = \frac{1}{2} (\epsilon + 5 N_d g^2) \partial_g F \\
\dot{\lambda} &= G^{\lambda \lambda} \partial_\lambda F = (\epsilon + 4 N_d g^2 - 24 N_d g^4 + \frac{3}{2} N_d \lambda) \partial_\lambda F
\end{align*}
\] (18)

Obviously the metric can be either positive definite, negative definite or indefinite depending on the parameters, and the relative strength of the couplings. To some extent this can be compensated, by adjusting the relative signs of the two
terms in $F$. But since the metric can also change its signature at non-vanishing, even small values of $g$ and $\lambda$ it cannot be made positive everywhere. This reflects the fixed point structure of the theory on the $(g, \lambda)$ plane.

As a third example we consider the model which has been studied in [17] as a candidate for limit cycle behaviour in the coupling flow. This is the $su(2)$ level $k = 1$ Wess-Zumino model, at the one-loop level its couplings $g$ and $h$ flow according to

$$
\dot{g} = -h^2 = -h \partial_g(hg)
$$

$$
\dot{h} = -gh = -h \partial_h(hg)
$$

Consequently in these coordinates the RG equations have the gradient flow form but the $c$-function is not a non-degenerate Morse function. Furthermore, unless $h > 0$ the metric is not positive definite. In the present model the exact $\beta$-functions are also known. The ensuing RG flow equations are [35], [17]

$$
\dot{g} = -h \frac{h}{2 - g} \partial_g(hg)
$$

$$
\dot{h} = -h \frac{h}{2 - g} \partial_h(hg)
$$

In these (isothermal) coordinates we then have a metric tensor which is singular, and a $c$-function which is not a nondegenerate Morse function. Notice in particular that the $c$-function does not receive corrections beyond the one-loop, all higher order corrections lead only to a modification of the metric. We note that the ensuing coupling flow on the $(g, h)$ plane has a blow-up at finite value of the flow parameter $s$. It has been suggested [16], [17] that this could be interpreted as a flow towards a limit cycle (see below).

From these examples it is clear that the $\beta$-functions that appear in quantum field theories are not always of the form suggested by the various versions of the $c$-theorem. In particular, while the equations do admit the gradient flow form (10) with (trivially) symmetric metric, the $c$-functions are not necessarily nondegenerate Morse functions nor are the metric tensors necessarily positive definite or even regular everywhere in the space of couplings. While in some
models such deviations from the $c$-theorems could be attributed to limitations of perturbation theory and removed by higher order corrections, the example (20) shows that this is not always the case. Moreover, even though the flows admit a gradient representation, they do not in general describe simple, structureless laminar flows towards isolated fixed points along gradients of the $c$-functions: If we consider the vector fields which appear on the r.h.s. of the flow equations, in all of the three examples, we note that each of the vector fields carries a non-trivial vorticity two-form (defined as the exterior differential of the covector of the $\beta$-functions)

$$\omega_{ij} = \partial_i \beta_j - \partial_j \beta_i$$  \hspace{1cm} (21)

Curiously, for the standard model the vorticity vanishes at the one-loop level and appears only at the two-loop level (and beyond). This suggests that higher loop corrections have a qualitative effect on the theory. Both in the pseudoscalar model and WZW model vorticity is present and regular already at the one-loop level. But from (20) we find that the vorticity in the full theory can have a singularity at finite coupling

$$\omega = \epsilon_{ij} \partial_i \beta_j = \frac{g(g - 2) - h^2}{(g - 2)^2}$$

reflecting the blow-up at finite flow time. We note that since vorticity is a closed two-form it can be made constant in a neighborhood of any regular point by diffeomorphisms. But if it is non-vanishing in one coordinate system it remains non-vanishing in all coordinate systems. Consequently vorticity is a renormalization scheme independent characteristic of the flow.

Maybe somewhat surprisingly several qualitative aspects of our examples, as well as many general assumptions in the $c$-theorems, can also be realized in chaotic systems. As an example, we consider the three dimensional Lorenz system [36],

$$\dot{x} = -\sigma x + \sigma y = \beta^x$$

$$\dot{y} = rx - y - xz = \beta^y$$

$$\dot{z} = xy - bz = \beta^z$$  \hspace{1cm} (22)
\(\sigma, r, b\) are positive constants. We start by introducing a (Lyapunov) \(c\)-function, an arbitrary positive semidefinite function \(\rho \geq 0\) in \(R^3\) which we advect along the Lorenz flow. This is described by the conservation of the current

\[
\mathbf{j}^\mu = (\rho, \rho \beta^i) \\
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \beta) = 0
\]

which reduces to

\[
\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \beta \cdot \nabla \rho = -(\nabla \cdot \beta) \rho = -(\sigma + b + 1) \rho \leq 0 \tag{23}
\]

Consequently any positive function on \(R^3\) which is advected by the Lorenz flow satisfies the irreversibility requirement of the \(c\)-theorem.

Consistent with the \(c\)-theorems, the Lorenz system can also be presented in the form of a gradient flow (10) with a symmetric metric. For example, if we introduce a non-degenerate Morse function for the critical point at the origin

\[
F(x, y, z) = x^2 + \sigma(y^2 + z^2) \tag{24}
\]

we can write the Lorenz system as

\[
\begin{align*}
\dot{x} &= g^{11} \partial_x F = \frac{\sigma y - x}{2x} \partial_x F \\
\dot{y} &= g^{22} \partial_y F = \frac{rx - y - xz}{2\sigma y} \partial_y F \\
\dot{z} &= g^{33} \partial_z F = \frac{xy - bz}{2\sigma z} \partial_z F
\end{align*} \tag{25}
\]

As in the previous three field theory examples, the positive definiteness of the metric tensor \(g_{ab}\) depends on the relative values of \(x, y\) and \(z\). Like the metric in (20), this metric is also non-singular except for the critical points of the flow (and the \(x, y, z = 0\) lines) while \(F\) is both a Morse function for the critical point at origin, and a global Liapunov function when \(r < 1\) since

\[
\frac{dF}{dt} = -2\sigma \left( x - \frac{1}{2}(1 + r)y \right)^2 + \frac{1}{4}(1 - r)(3 + r)y^2 + 2\sigma bz^2 < 0 \quad (x, y, z \neq 0)
\]

Consequently, at this level of analysis we do not see much difference in the qualitative properties of the (chaotic) Lorenz model and the coupling flows in
our three field theory examples. In particular, nothing appears to prevent chaos from occurring in RG flows as well.
4. Limit Cycles and Vorticity

The existence of a positive definite metric with a $c$-function which decreases along the orbits can also be satisfied by flows without simple fixed points. For example, if we take

$$c(t, \bar{t}) = (t\bar{t} - a^2)^2$$

the entire cycle $t = ae^{i\theta}$ forms an attractor. In order to induce a motion along this cycle we then consider ($\eta > 0$)

$$\dot{x} = -\eta y = -\beta x$$
$$\dot{y} = \eta x = -\beta y$$

(26)

This flow is consistent with the strong $c$-theorem, with a positive-definite metric

$$G^{ij} = \eta \delta^{ij} (x^2 + y^2)$$

and a (multivalued) Liapunov $c$-function which is monotonically decreasing along the flow,

$$c(x, y) = \arctan(y/x)$$

The ensuing flow has constant vorticity,

$$\omega = \epsilon_{ij} \partial_i \beta_j = 2\eta$$

Such flows may actually be realistic in models where the coupling (moduli) space has nontrivial topology, with non-vanishing $\pi_1(M)$ (cf. [37]). Indeed, since the $c$-function is highly nonlinear the flow could be viewed as a non-perturbative one. We are not aware of any apparent reason why this kind of flow should in general be excluded by the $c$-theorem.

There is a simple generalization of (26) to non-constant vorticity,

$$\dot{x} = -y^k,$$
$$\dot{y} = x^k$$

(27)
with $k$ an integer. This flow is of the form of strong $c$-theorem, with $$c(x, y) = y^{1-k} - x^{1-k}$$

The corresponding metric tensor is

$$g^{ij} = \frac{\delta^{ij}}{k-1} (xy)^k$$

For $k = 2n$ this metric is positive on the $(x, y)$ plane, and the $c$-function has the form of a (degenerate) Morse function as expected by the strong $c$-theorem. But for $k = 2n + 1$ i.e. odd, the metric is in general not positive. It is instructive to consider in more detail the reasons for the failure of strong $c$-theorem when $k$ odd. For this we note that the flow possesses a conserved quantity, $x^{k+1} + y^{k+1} = \text{const}$. This suggests us to change variables

$$x = R \cos^{1/(n+1)} \theta$$
$$y = R \sin^{1/(n+1)} \theta$$

so that

$$\dot{R} = 0, \quad \dot{\theta} = \frac{\partial c(\theta)}{\partial \theta}$$

(29)

where $c(\theta)$ is $R$-independent. Consequently, in these coordinates it appears that the flow is consistent with the strong version of the $c$-theorem with a positive definite metric

$$dR^2 + R^2 d\theta^2$$

But when we transform back to the cartesian $x, y$ coordinates,

$$\left( R^2 \left( \frac{\partial X}{\partial R} \right)^2 + \left( \frac{\partial X}{\partial \theta} \right)^2 \right) dX^2 + 2 \left( R^2 \frac{\partial X}{\partial R} \frac{\partial Y}{\partial R} + \frac{\partial X}{\partial \theta} \frac{\partial Y}{\partial \theta} \right) +$$
$$+ \left( R^2 \left( \frac{\partial Y}{\partial R} \right)^2 + \left( \frac{\partial Y}{\partial \theta} \right)^2 \right) dY^2$$

(30)

We find that due to the fractional powers of trigonometric functions in (28) the ensuing metric is not positive definite, for example when $k = 1$ we have

$$G^{XX} = \left( \frac{R \sin \theta}{2 \sqrt{\cos \theta}} \right)^2 + \left( \frac{R \sqrt{\cos \theta}}{2} \right)^2 = \frac{R^2}{4 \cos \theta}$$

(31)
which is not positive definite.

Notice that besides a gradient flow, the flows (26), (27) can also be interpreted in terms of a symplectic (Hamiltonian) flow of the form

$$\beta^i(t) = \sum_j \omega_{ij} \frac{\partial H}{\partial t^j}$$

(32)

with a closed symplectic two-form $\omega = \omega_{ij} dt^i \wedge dt^j$ and a Hamiltonian $H$. In particular, (26) is the harmonic oscillator.

We note that the presence of such limit cycles in RG flows is quite important from the point of view of (possible) chaotic flow. Obviously, a limit cycle behaviour is much easier to analyse than a chaotic behaviour. But in addition, limit cycles can also provide a period doubling (Feigenbaum) route to chaos (for flows with at least three couplings) [38]. Indeed, if for some value of external control parameters a system exhibits a stable limit cycle, its stability can be lost by period doubling when the control parameter changes. When this happens, the attractive cycle becomes repelling and instead there is a new attractive limit cycle which exhibits period doubling and links around the previously stable cycle. When the control parameter is varied further, there will be additional period doublings and eventually these bifurcate into an infinitely long limit cycle, with a fractal structure. The Rössler system (8) is the simplest three dimensional example which exhibits this Feigenbaum route to chaos by consecutive period-doublings in its limit cycles, while in the Lorenz system (22) the transition to chaos is by intermittency [27].

Unfortunately, it appears to be quite difficult to find RG limit cycles. While periodic dependency on the coupling constants in the $\beta$-functions has been well established, for example in the context of topological charge renormalization [39], this is not sufficient for obtaining a limit cycle. Until now, only one example of limit cycle behaviour in RG equations has been constructed [16]-[18], but unfortunately this example can not be considered fully satisfactory. Essentially, the flow is of the form

$$\dot{g} = h^2 + g^2$$
\[ \dot{h} = 0, \]

which exhibits a blow-up, rather than smooth cyclic behaviour. We note that for small values of \( g \) these equations are quite similar to the conventional asymptotically free RG equations (with \( h = 0 \)), the only difference is that whenever the second coupling constant \( h \neq 0 \) the flow becomes accelerated. The ensuing flow

\[ g(\mu) = h \cot(h \log \mu) \]

is formally periodic in the sense that

\[ g(e^{\pi/h} \mu) = g(\mu) \]

but a discontinuous jump from \( g = +\infty \) to \( g = -\infty \) is necessary in order to close the cycle. Unfortunately, concrete examples with continuous limit cycle behaviour in realistic field theory models have not yet been found.

The previous examples underline the importance of developing general classification schemes for \( \beta \)-function flows in multidimensional cases. As we have proposed, a local approach could be based on the Hartman-Grobman theorem [28] on hyperbolic fixed points. This suggests that for classification purposes we consider flow equations with \( \beta \)-functions which are at most bilinear in the couplings, the bilinearities representing either corrections to, or deviations from hyperbolic behaviour. This subclass of flows is particularly interesting in three dimensions, since bilinear nonlinearities are sufficient for a chaotic flow to occur. For this we consider three dimensional flows of the form

\[ \dot{x}^i = \beta^i \quad (i = 1, 2, 3) \quad (33) \]

The three dimensional (co)vector \( \beta_i \) can be presented using a Glebsch decomposition

\[ \beta_i = \mu \partial_i \rho + \partial_i \gamma \]

with three functions \( \mu, \rho, \gamma \). When \( \mu = 0 \) the vorticity

\[ \omega_i = \epsilon_{ijk} \partial_j \beta_k = \epsilon_{ijk} \partial_j \mu \partial_k \rho \]

18
vanishes, limit cycles are absent and the ensuing advection of the couplings is laminar, non-chaotic gradient flow. But whenever $\mu$ is non-vanishing the vorticity is non-vanishing and either constant or linear in the couplings. We now argue that both for constant and linear vorticity, the advection can be chaotic. (For a flow with vanishing vorticity, chaotic advection is hardly possible.) Recall that since vorticity is a closed two-form, we can introduce a diffeomorphism which maps it into a constant in a neighborhood of a regular point. But it can not be made to vanish in that neighborhood by diffeomorphisms.

Consider first the Lorenz equations (22). The $\beta$-functions can be Glebsch-decomposed according to

$$
\beta_i = xy^2 \partial_i \left( \frac{z + \sigma - r}{y} \right) + \partial_i \left( \sigma xy - \frac{1}{2} (\sigma x^2 + y^2 + b z^2) \right)
$$

and for the vorticity we find

$$
\omega = -z dx \wedge dy + (2x - \sigma) dy \wedge dz - y dz \wedge dx
$$

Notice that on the surface $x^2 + y^2 + z^2 = \text{const}$ this involves a term which represents $H^2(S^2)$, the (unique) volume element of $S^2$. Consequently the vorticity (34) of the Lorenz system is a representative of the monopole bundle in $R^3$.

For the Rössler equation (8) we find the vorticity (here we use the original form of these equations, see footnote (1) in connection of equation (8).)

$$
\omega = -2 dx \wedge dy + (1 + z) dx \wedge dz = -2 dx \wedge dy + \frac{1}{2} dx \wedge d(1 + z)^2
$$

Consequently the vorticity can be made constant by a quadratic diffeomorphism, but at the expense of loosing the quadratic nature of the equations.

A chaotic system with simultaneously quadratic nonlinearities and constant vorticity can also be constructed, for example by combining the behaviour of the $\beta$-functions in (19) and (26) into the following three dimensional flow

$$
\begin{align*}
\beta_x &= \frac{1}{2} z + y - 1 + px + \frac{1}{2} y^2 \\
\beta_y &= -x + xy - y - 1 \\
\beta_z &= -\frac{1}{2} x + az
\end{align*}
$$

19
This can be shown to be chaotic for appropriate values of the parameters $p, a$ [40]. Vorticity is linear

$$\omega = -2dx \wedge dy + dz \wedge dx$$

In $D$ dimensions we can introduce a generalization of the Glebsch decomposition

$$\beta_i = \sum_{k=1}^{N} \mu_k \partial_i \rho_k + \partial_i \gamma$$

Where $2N = D$ for $D$ even, and $2N = D - 1$ for $D$ odd, and $\gamma$ is present only for odd $D$. Presumably a necessary condition for chaotic advection in $D \geq 3$ is that the ensuing vorticity

$$\omega_{ij} = \sum_{k=1}^{2N} (\partial_i \mu_k \partial_j \rho_k - \partial_j \mu_k \partial_i \rho_k)$$

is non-vanishing. Since a non-trivial vorticity appears to be generic for $\beta$-functions in quantum field theories, additional restrictions are then needed to exclude chaotic advection of the couplings.
5. RG Flows And Model Effective Actions

It appears that nontrivial vorticity is generic for the coupling flows, to the extent that even the one-coupling $\beta$-functions can be related to flows with vorticity. For this, we consider the familiar one-loop formula

$$\frac{\partial}{\partial s} \left( \frac{1}{g^2} \right) = -2b,$$

(37)

which appears e.g. at one-loop four dimensional Yang-Mills theories, cf. (12), (13). We then introduce a Coleman-Weinberg type model effective action (for strong fields). This is an ordinary function of a single real variable $F$,

$$\mathcal{F}(g, h|F) = \frac{1}{g^2} F^2 + h F^2 \log F,$$

(38)

It turns out that many properties of RG flows can be understood by inspecting the scaling properties of such ordinary functions. Indeed, since the $\beta$-functions in (5) are independent of spacetime coordinates, we can expect that the essential aspects of RG flows are independent of spacetime coordinates and can be studied in terms of such ordinary functions (i.e. model effective actions) in lieu of the actual quantum effective actions. This is certainly the case in theories where we can have an effective potential, such as Higgs models.

When we introduce the scaling $F \rightarrow \lambda^2 F$ and $\mathcal{F} \rightarrow \lambda^4 \mathcal{F}$ in (38), we get

$$\frac{\partial}{\partial s} \left( \frac{1}{g^2} \right) = -\frac{2\dot{g}}{g^3} = h,$$

$$\dot{h} = 0$$

(39)

The solution

$$h = \text{const} = -2b$$

$$g^{-2} = 2b \log \mu$$

of these equations leads to the $\beta$-function (37). Nevertheless, the system (39) has a non-trivial vorticity

$$\omega = dh \wedge dg^{-2} \neq 0$$

(40)
More generally, if we consider model effective actions of the form
\[ F(h|F) = \sum_{k=0} h_k F^2 \log^k F \] (41)
the scaling \( F \to \lambda^2 F \) leads to the evolution equations
\[ \dot{h}_k = (k+1)h_{k+1}, \] (42)
or, for \( g_k = h_k/h_0 \),
\[ \dot{g}_k = (k+1)g_{k+1} - g_kg. \] (43)
If all \( h_k \) with \( k > N \) vanish, (42) is solved by a polynomial \( P_N(s) \) of degree \( N \) for \( h_0 \), and its \( k \)-th derivative for \( h_k \). Thus solutions \( g_k \) of (43) are rational functions of the form \( g_k = \partial s^k P_N(s)/P_N(s) \). The scaling properties of (41) and (42) can then be employed to derive realistic multiloop \( \beta \)-functions of the form
\[ \beta(g) = \sum_j b_j g^{2j+1} \]
generalizing our one-loop result (38). For this, we substitute for \( h_k \) in (41)
\[ h_0(g^2) = \frac{1}{g^2}, \]
\[ h_1(g^2) = h_0 = -\frac{2\dot{g}}{g^3} = -\frac{2\beta(g)}{g^3} = -2 \sum_j b_j g^{2(j-1)}, \]
\[ h_2(g^2) = \dot{h}_1 = 2g\beta(g)h_1(g^2) = -4 \sum_{j,k} (j-1)b_j b_k g^{2(j+k-1)}, \]
\[ \ldots \] (44)
If \( b_0 = 0 \), then
\[ h_1 = -2(b_1 + b_2 g^2 + \ldots), \]
\[ h_k = -2^k b_1^{k-1} b_2 k! g^{2k} + \ldots \text{ for } k > 1. \] (45)

The scaling properties of the model effective actions (38), (41) lead to the Callan-Symanzik type equations
\[ \frac{\partial F(t|\varphi)}{\partial \varphi} + \sum_i \beta_i(t) \frac{\partial F(t|\varphi)}{\partial t_i} + kF(t|\varphi) = 0 \] (46)
where the model background field $\varphi$ in (46) is related to $F$ in (41) by $F = e^{-\varphi}$. The equation (46) can then provide relations between field theoretical RG equations and dynamical systems. In particular, we shall now propose that issues concerning chaoticity and integrability of the coupling flow equations can be directly related to the construction of solutions to the equation (46).
6. Model Effective Actions And Chaos in RG Flows

In quantum field theories, it is known that the effective actions are usually highly complicated, multivalued functions of the couplings. Indeed, in general the effective actions are not ordinary function(al)s but highly nontrivial sections of line bundles which are defined over fibrations of the background fields $\varphi$ over the spaces $\mathcal{M}$ of couplings $t_i$. We shall now propose that these complexities in their functional form could be viewed as an indication of chaotic behaviour in the ensuing flows of the couplings. This turns out to be an issue that can be addressed at the level of model effective actions i.e. ordinary functions which model the scaling properties of the actual field theoretical effective actions. These model effective actions can often be constructed by explicit integrations of the ensuing RG flow equations:

Consider the model RG equation (46), which we now solve using the method of characteristics. For this, we first need a solution to the extended system (5),

$$\dot{t}_i = \beta_i(t),$$
$$\dot{\varphi} = 1$$

(47)

These solutions yield the flows $t_i(s, a_{i-1})$, with $a_{i-1}$ as the integrations constants. We then invert these relations, to express $s$ and the integration constants $a_{i-1}$ as functions of the $t_i$ and $\varphi$. The general solution to the model RG equation (46) is now obtained by introducing an arbitrary function $f[a_{i-1}]$ of $a_{i-1}(t|\varphi)$ and setting

$$\mathcal{F}(t|\varphi) = f[a_{i-1}(t|\varphi)] + ks(t|\varphi).$$

(48)

Obviously, complex behaviour such as bifurcations in the original equations (47) relate to singularities in these functional inversions.

Clearly, if the equation (5) describes flow towards a strange attractor i.e. the flow is chaotic, the solution (48) will also lead to a function which reflects the structure of the attractor. Asymptotically, the model effective action (48)
then flows towards a generalized function or rather a (Lebesque) measure with support on the strange attractor. Obviously this means that we need to extend the concept of model effective actions to include measures with support on fractal structures. In particular, when the flow (47) is chaotic the construction of its solutions can only be symbolic, which then translates to the impossibility to construct the model effective action (48) by quadratures. Presumably, this could be developed into a criteria for identifying actual field theory models where chaotic RG flows are present.

For this, we first consider explicit constructions of model effective actions for the flows that we have analysed previously. We start by noting that the last equation in (47) is clearly oversimplified and easily solved, \( \varphi = s + a_s \), so that \( \varphi \) will actually enters (48) only through a single integration constant \( a_s \). Moreover, it will always appear in the combination \( a_s = \varphi - s(t) \), where \( s(t) \) is obtained, together with the integration constants \( a_{i-1} \), as the functional inverse of the flows, derived from (5). For the same reason the last term at the r.h.s. of (48) does not depend on \( \varphi \) and can be written as \( ks(t) \).

We first consider the flow of a single coupling,

\[
\dot{t} = \beta(t)
\]

(49)

The generic solution to (46) is

\[
\mathcal{F}(t|\varphi) = f[s(t) - \varphi] = \tilde{f} \left[ \gamma(t) e^{-\varphi} \right],
\]

(50)

where \( f[x] = \tilde{f}[e^x] \) is arbitrary function of a single variable, and \( \gamma'(t)/\gamma(t) = 1/\beta(t) \), i.e.

\[
\gamma(t) = e^{\epsilon(t)} = \exp \left( \int^{t} \frac{dx}{\beta(x)} \right)
\]

(51)

Similarly, if the \( i \)-th \( \beta \)-function depends on \( t_i \) only, \( \beta_i(t) = \beta_i(t_i) \),

\[
\mathcal{F}(t|\varphi) = \tilde{f} \left[ \gamma_i(t_i)e^{-\varphi} \right],
\]

(52)

where \( \tilde{f}[x] \) is arbitrary function of its variables and

\[
\gamma_i(t_i) = \exp \left( \int^{t_i} \frac{dx}{\beta_i(x)} \right)
\]

(53)
We then consider
\[ \dot{t}_i = A_{ij} t_j, \]  
(54)
with \( t \)-independent matrix \( A_{ij} \). According to the Hartman-Grobman theorem \cite{28}, a generic flow is homeomorphic to this near its hyperbolic fixed points. But we note that this can also include models with limit cycles, such as (26). We diagonalize \( A_{ij} \) so that the system is transformed into \( \dot{\xi}_i = \lambda_i \xi_i \) with
\[ \xi_i = \sum_j U_{ij} t_j \]
\[ (UAU^{-1})_{ij} = \lambda_i \delta_{ij} \]
The generic solution to (46) then becomes
\[ F(t|\varphi) = f \left[ \frac{\xi_i^{1/\lambda_i}}{\xi_i^{1/\lambda_1}}, \varphi - \frac{1}{\lambda_1} \log \xi_1 \right] = \]
\[ = \tilde{f} \left[ \xi_i e^{-\lambda_i \varphi} \right] = \tilde{f} \left[ e^{-\lambda_i \varphi} \sum_j U_{ij} t_j \right] \]
(55)
with arbitrary functions \( f[x_i] \) and \( \tilde{f}[x_i] \).

Finally, we consider the most general linear flow
\[ \dot{t}_i = A_{ij} t_j + B_i, \]  
(56)
with \( t \)-independent \( A_{ij} \) and \( B_i \). We get
\[ F(t|\varphi) = \tilde{f} \left[ e^{-\lambda_i \varphi} \sum_j U_{ij} t_j + \frac{1 - e^{-\lambda_i \varphi}}{\lambda_i} \sum_j U_{ij} B_j \right]. \]  
(57)
Note that we have defined the arguments of \( \tilde{f}[x_i] \) so that they make sense even if \( A_{ij} \) is degenerate i.e. some of the eigenvalues \( \lambda_i \) vanish.

A particular example of the previous construction is given by the limit cycle flow (26),
\[ \dot{x} = -\eta y, \]
\[ \dot{y} = \eta x \]  
(58)
we have $x = -a \sin \eta s$, $y = a \cos \eta s$, and the generic solution is
\[
F(x, y | \varphi) = f \left[ \sqrt{x^2 + y^2}, \ \eta \varphi + \arctan(x/y) \right] =
\begin{align*}
\tilde{f}[x \cos \eta \varphi - y \sin \eta \varphi, & \ x \sin \eta \varphi + y \cos \eta \varphi],
\end{align*}
\tag{59}
\]
where $f[x, y]$ and $\tilde{f}[x, y]$ are arbitrary functions of two variables.

As a further, nonlinear example we consider the model effective action for the $\beta$-functions (19), (20). For this we consider (19), in the form
\[
\begin{align*}
\dot{x} &= xy, \\
\dot{y} &= x^2
\end{align*}
\tag{60}
\]
Indeed, this is the system considered in [16, 17], with
\[
x(s) = \frac{Q}{\sin Qs} \quad \& \quad y(s) = Q \cot Qs
\]
The generic solution of (46) is
\[
F(x, y | \varphi) = f \left[ Q, \ Q \varphi - \arccos \left( -\frac{y}{x} \right) \right] = \tilde{f} \left[ Q, \ Q \sin Q \varphi - y \cos Q \varphi, \ x \right],
\tag{61}
\]
where $f[x, y]$ and $\tilde{f}[x, y]$ are arbitrary functions of two variables, and
\[
Q(x, y) = \sqrt{x^2 - y^2}
\]
It is clear that since the method of characteristics is based on solving the system (5), in the case of chaotic flows the method can not be explicitly implemented, even in principle. Consequently model effective actions for chaotic flows can not be explicitly constructed, which may also be an explanation why concrete examples are not known in the literature. Indeed, this is an obvious conceptual issue that explains why it is very difficult (may be even impossible?) to construct even simplistic model effective actions with chaotic renormalization group flow in the first place. To exemplify the problems encountered, we shall now consider the construction of the model effective action for the Lorenz equations (22), by employing a perturbative expansion. A natural perturbative parameter is $r$, which corresponds to the ratio of Rayleigh number to its critical
value in the underlying hydrodynamical model. This suggests that we search for a perturbative construction of the model effective action by first expanding (22) around the (chaotic) large-$r$ limit. (Recall that with the canonical values $\sigma = 10$ and $b = 8/3$ the model becomes chaotic only when $r > 24.74$ ... ) The expansion parameter we use is $\epsilon = 1/\sqrt{r}$, and setting $x \to \epsilon x$, $y \to \epsilon^2 \sigma y$, $z \to \sigma(\epsilon^2 z - 1)$ and $t \to t/\epsilon$ we get for the Lorenz equations

$$
\dot{x} = y - \epsilon \sigma x \\
\dot{y} = -xz - \epsilon y \\
\dot{z} = xy - \epsilon b(z + \sigma)
$$

We then expand

$$
x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots \\
y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots \\
z = z_0 + \epsilon z_1 + \epsilon^2 z_2 + \ldots
$$

To the leading order we get

$$
\dot{x}_0 = y_0 \\
\dot{y}_0 = -x_0 z_0 \\
\dot{z}_0 = x_0 y_0
$$

These equations can be integrated by quadratures, in terms of the Jacobi elliptic functions. When we substitute the solution to the higher order equations for $(x_n, y_n, z_n)$ ($n \geq 1$) we find at each order a set of equations which are linear in their unknown variables. For example at $O(\epsilon)$ we get the linear equations

$$
\dot{x}_1 = y_1 - \sigma x_0 \\
\dot{y}_1 = -x_0 z_1 - x_1 z_0 - y_0 \\
\dot{z}_1 = x_0 y_1 + x_1 y_0 - b(z_0 + \sigma)
$$

and so forth at higher orders in $\epsilon$. Consequently these higher order equations can in principle also be solved by quadratures. Thus the Lorenz equations can be
solved by quadratures, at least formally in a perturbation expansion to an arbitrary order in $\epsilon$. The ensuing model effective action can then also be constructed by employing the method of characteristics, order by order in a perturbative expansion in $\epsilon$. However, it turns out that this perturbative solution of the Lorenz equations is at most asymptotic. It does not converge towards a solution of the original Lorenz system. Instead, numerical investigations indicate that it leads to an diverging asymptotic expansion which describes the chaotic solutions of the Lorenz equations only for (relatively) small values of the flowtime $s$.

Clearly, it must be a general feature of chaotic systems that model effective actions can not be constructed by quadratures. Not even in principle, as this would amount to solving the original chaotic equations. For the same reason any perturbative approach can only lead to an asymptotic expansion, which approximates the exact model effective action at most during a limited period of the flow. Indeed, since a chaotic flow approaches an attractor which is a fractal, the ensuing model effective actions must also approach generalized functions (measures) with support on a fractal set of points. We note that this is in a very curious resemblance with the familiar complex behaviour of actual effective field theory actions. These effective actions are usually highly complicated and multivalued function(al)s (rather section(al)s) for which any perturbative expansion yields at most an asymptotic series.
7. Further Developments

It was proposed already in [2] that (at least) cyclic behaviour of asymptotic RG flows cannot be excluded. As we have argued in the present article even chaotic flow does not appear to conflict (most of) the assumptions in $c$-theorems. In fact, cyclicity of the flow seems to be quite natural in many field theory problems with spectral flow, with an adiabatic evolution of the energy eigenvalues. In terms of spectral flow, a cycle can arise whenever the energy spectrum is mapped onto itself by an adiabatic process, but with a rearrangement: The level $E_n$ becomes shifted into some level $E_{n-k}$. From the point of view of an effective action, this rearrangement of the energy levels is clearly a cycle.

One example of nontrivial spectral flow can be developed by considering a stringy spectrum of the form

$$E_n = nm^2(g) + \alpha(g)$$

A cycle arises whenever

$$m^2(g') = m^2(g) \quad \text{and} \quad \alpha(g') = \alpha(g) - km^2(g), \quad k \in \mathbb{Z}. \quad (64)$$

The intercept $\alpha(g)$ is now identified as the dissipative Lyapunov $c$-function. Note that in simple string models the intercept is exactly (proportional to) the central charge of the underlying 2$d$ conformal field theory (effective space-time dimension). Clearly, this example is very much in the spirit of the original introduction of the $c$-theorem in the context of conformal field theories [30].

In our examples, we have inspected the $c$-theorem in RG flows without background fields. To some extent this can be justified, by arguing that the background fields can be amalgamated with couplings. However, there are also differences. For this, we consider a hypothetical field theory model (for example a Higgs model), with background fields which describe a vacuum state of the effective theory. If the parameters in the effective action change e.g. as a consequence of coupling flow, then the functional form of the effective action will also
change. But as the effective action changes, so does the vacuum. Moreover, in the case of first-order phase transitions there are discontinuous changes in the ensuing order parameters i.e. background fields. Even if only second-order phase transitions occur in the course of RG evolution and the location of vacuum is changing continuously, its evolution is still important for determining the effective action. The physical partition function which emerges after accounting for the change in the vacuum obeys RG equations somewhat different from the original theory. For this, we consider a model effective action

\[ \mathcal{F}(\varphi) = \sum_k t_k \varphi^k \]

which flows as

\[ \dot{\mathcal{F}}(\varphi) = B(\varphi) = \sum_k \beta_k \varphi^k. \]  \hspace{1cm} (65)

Define a vacuum order parameter \( \varphi_0 \) as an extremum of \( \mathcal{F}(\varphi) \):

\[ \mathcal{F}'(\varphi_0) = 0. \]  \hspace{1cm} (66)

Often we are interested in effective potential

\[ \mathcal{F}_0(\varphi) \equiv \mathcal{F}(\varphi_0 + \varphi) - \mathcal{F}(\varphi_0), \]  \hspace{1cm} (67)

which describes fluctuations around the vacuum. Then, since

\[ \dot{\varphi}_0 = -B(\varphi_0)/\mathcal{F}''(\varphi_0) \]

we get

\[ \dot{\mathcal{F}}_0(\varphi) = B_0(\varphi), \]

and

\[ B_0(\varphi) = B(\varphi_0 + \varphi) - B(\varphi_0) - \left[ \mathcal{F}'(\varphi_0 + \varphi) - \mathcal{F}'(\varphi_0) \right] \frac{B(\varphi_0)}{\mathcal{F}''(\varphi_0)} \]  \hspace{1cm} (68)

Now, even if \( B(\varphi) \) is subject to a gradient flow according to the strong c-theorem, there is no apparent reason why this should be the case with \( B_0(\varphi) \).
Once we account for the RG flow of the vacuum, new phenomena become possible. For example, consider the potential

$$V(\phi) = -\frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4$$

The vacuum $\phi_0 = \sqrt{m^2/\lambda}$ changes smoothly when $m^2$ and $\lambda$ are renormalized according to conventional RG equations. Now suppose the field $\phi$ is further coupled to another field $\chi$, for example by

$$\chi^4 \cos(\omega \phi)$$

Then the effective coupling $g_4$ in the $\chi^4$ vertex is actually equal to

$$g_4 = \cos(\omega \phi_0) = \cos(\omega \sqrt{m^2/\lambda})$$

and, for sufficiently large $\omega$, this $g_4$ can be oscillating along the RG flow with ensuing changes for $m^2$ and $\lambda$. This model could then indicate how to construct realistic field theory models with RG limit cycles.

An alternative is to consider

$$V(\phi) = g_1 \cos \omega_1 \phi + g_2 \cos \omega_2 \phi$$

with $\omega_1/\omega_2$ irrational. The potential has infinitely many irregular local minima, which change when $g_1$ and $g_2$ are smoothly evolving along any RG flow (which may be subject to the strong $c$-theorem). The lowest energy minimum is a complex function of these couplings, and the ensuing v.e.v. $\phi_0$ can at least a priori exhibit chaotic (irregular) behaviour. Such irregular behaviour of $\phi_0$ can then give rise to similarly irregular behaviour of some of the couplings.

Similar phenomena can also be realized in the context of finite temperature field theories, with various applications including in particular Very Early Universe and Cosmology. In the presence of a finite temperature, the role of flow-time $s$ is taken by the inverse temperature $s = 1/T$. As temperature decreases, the number of states that contribute to the partition function decreases - the flow is contracting, irreversible. Again, we can illustrate the phenomena...
with a model partition function: We select a somewhat unconventional set of harmonic oscillators, with potentials

$$\tilde{\alpha}_i + \frac{1}{2}\omega_i^2 q_i^2$$

The ensuing spectrum consists of oscillators $\alpha_i + \omega_i k_i$ with integer non-negative $k_i$, with the ground state energy absorbed into the intercept $\alpha_i = \tilde{\alpha}_i + \frac{1}{2}\omega_i$. The partition function is

$$Z(s) = \exp(\mathcal{F}(s)) = \sum_i e^{-\alpha_i s}$$

If $\alpha_i \gg \omega_i$, this can be approximated by

$$Z_{app}(s) = \frac{1}{s} \sum \omega e^{-\alpha(\omega)s}$$

In the presence of only two frequencies $\omega_1 \ll \omega_2$, with $\alpha_1 \gg \alpha_2$, we then have

$$sZ_{app}(s) = \frac{e^{-\alpha_1 s}}{\omega_1} + \frac{e^{-\alpha_2 s}}{\omega_2}$$

with the first term dominating at $s \ll 1/\alpha_1$ and the second one at $1/\alpha_1 \ll s \ll 1/\alpha_2$. For infinitely many chaotically distributed $\omega_i$ and $\alpha_i$, the behaviour of $Z(s)$ also exhibits irregular (chaotic) features. As an extreme, one can consider a somewhat peculiar distribution of oscillators with $\omega_n \sim \sqrt{n}$ and $\alpha_n = \alpha \log n$. Then

$$sZ_{app}(s) = \zeta \left( \frac{1}{2} + \alpha s \right)$$

is the Riemann $\zeta$-function, which is suspected to have a relation with a chaotic dynamical system.
8. Conclusions

In conclusion, we feel that there is a clear need to investigate the relations between RG flows from the perspective of chaotic dynamical systems. We have analyzed the restrictions that can be imposed on the RG flows by employing various aspects of c-theorems and local integrability conditions, stemming from the fact that RG flows reflect the scaling properties of effective actions. These effective actions are in general not functions but sections in a line bundle over background fields and couplings. In fact, we have argued that the involved structure of field theory effective actions, in particular their multivaluedness in the couplings, can be naturally understood in terms of the impossibility to explicitely construct model effective actions for chaotic dynamical systems. In particular, both admit perturbative expansions which fail to converge except in an asymptotic sense.

We have also proposed that many of the familiar properties imposed on RG flows are not in any kind of apparent contradiction with the existence of chaotic behaviour. In fact, several RG flows do reflect features such as nontrivial vorticity which are necessary for a chaotic flow to occur. Furthermore, we have suggested that a self-similar, chaotic RG flow in the IR limit could actually be desirable in many applications, e.g. to spin-glasses and neural networks. Perhaps even more importantly to the Early Universe Cosmology, and maybe even M-theory with the possibility that various field theory, string and brane models emerge at the end of its chaotic RG trajectories. In a sense, we are then proposing that chaotic RG orbits are as natural as the emergence of condensed matter physics with its highly elaborate and largely self-similar structure which emanates from the relatively simple microscopic quantum electrodynamics.

While we feel that the idea of self-similar chaotic orbits in field theory RG flows is a very natural one, we can not exclude their absence. But for this, novel no-go theorems are needed. Such theorems should most likely be based on conceptually new physical principles. Whatever the necessary restrictions are,
they are bound to shed important light to studies of hidden integrable structures
of effective actions (with complex relations between different RG flows taken
into account). They will also lead to a deeper understanding of singularities of
generalized $\tau$-functions, and in a wide sense to the general structure of phase
transitions with applications ranging from condensed matter to early universe
and fundamental string theories.

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