REGULAR FAMILIES OF KERNELS FOR NONLINEAR APPROXIMATION

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Abstract. This article studies sufficient conditions on families of approximating kernels which provide $N$–term approximation errors from an associated nonlinear approximation space which match the best known orders of $N$–term wavelet expansion. These conditions provide a framework which encompasses some notable approximation kernels including splines, so-called cardinal functions, and many radial basis functions such as the Gaussians and general multiquadrics. Examples of such kernels are given to justify the criteria, and some computational experiments are done to demonstrate the theoretical results. Additionally, the techniques involved allow for some new results on $N$–term interpolation of Sobolev functions via radial basis functions.

1. Introduction

Among the many notions of smoothness spaces of functions on $\mathbb{R}^d$ are spaces of fractional smoothness which may be defined by, or intimately related to, a wavelet expansion. The primary examples to be considered here are the Besov and Triebel–Lizorkin spaces. For an overview of the basic facts about these spaces, the reader is invited to consult [18] and the many references therein. DeVore, Jawerth, and Popov [10] studied approximation orders for best $N$–term approximation of functions from these spaces with respect to a given wavelet system; in particular, it was shown that if $f$ is in the Triebel–Lizorkin space $F^s_{\tau,q}$ (where $s \in \mathbb{R}^+$ is the smoothness) then the error of the best $N$–term approximation in $L^p$ (for $p = (1/\tau - s/d)^{-1}$) via many wavelet systems is $O(N^{-s/d})$.

This article studies $N$–term approximation of functions in these smoothness spaces from nonlinear spaces associated with different kernels, which are typically related to a spline system or a radial basis function. The motivation for this analysis is the work of DeVore and Ron [11], which studies approximation from both linear and nonlinear spaces using kernels that arise as the solution of certain elliptic differential operators. This technique encompasses some growing kernels, but does not allow the use of some notable examples, such as multiquadrics. Additionally, Hangelbroek and Ron [17] analyzed best $N$–term approximations via Gaussians. The analyticity and rapid decay of the Gaussian kernel allows for very fast approximation in theory; however, their analysis does not directly extend to allow kernels of finite smoothness, and certainly not growing kernels. With these beginnings in mind, we aim to provide sufficient conditions on a family of kernels which yield $O(N^{-s/d})$ approximation orders from a related nonlinear space.

More specifically, suppose we have a family of continuous kernels depending upon some parameter, $(\phi_\alpha)_{\alpha \in A}$ for some unbounded $A \subset (0, \infty)$. Considering the following nonlinear approximation space:

\[
\Phi_N := \left\{ \sum_{j=1}^{N} a_j \phi_{\alpha_j} (\cdot - x_j) : (a_j) \subset \mathbb{C}, (\alpha_j) \subset A, (x_j) \subset \mathbb{R}^d \right\},
\]

we seek to answer the following problem.

Problem 1.1. Find sufficient conditions on the family $(\phi_\alpha)$ such that for $f \in F^s_{\tau,q}$, there exists an $S_{f,N} \in \Phi_N$ such that

\[
\|f - S_{f,N}\|_{L^p} \leq C f N^{-s/d}.
\]

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Our main results give sufficient conditions on finite-smoothness kernels (both decaying and growing ones) which allow for such approximation orders. Some notable implications of this are that other growing kernels such as the multiquadrics of all orders may be used, and also our methods allow for the use of so-called cardinal functions, which gives rise to some new nonlinear methods involving interpolation. The use of cardinal functions in turn allows for error estimates for Sobolev spaces, which cannot be achieved directly from the Besov and Triebel–Lizorkin space estimates.

The primary concern of this work is to demonstrate that the approximation orders of the nonlinear spaces $\Phi_N$ associated with a large variety of kernels match the known rates for nonlinear wavelet systems, which are determined by the smoothness of the target functions. However, in the course of the proof, we exhibit a concrete approximant to a given function $f$ which attains the error bound of $O(N^{-s/d})$. As an intermediate step in the procedure, we use the fact that such $f \in F_{\tau,q}$ admits a wavelet expansion of a certain type, though the choice of the wavelet is not overly important.

Specifically, we employ a two-stage approximation process as follows:

Step 1: Form a linear approximation space which provides extremely good error bounds for approximating mother wavelets which are bandlimited Schwartz functions. Then truncate the approximant. If $\psi$ is the wavelet, let $T_N\psi$ be its truncated approximant from the linear space, which is also an $N$–term approximant in $\Phi_N$.

Step 2: Given a wavelet expansion $f = \sum_I f_I \psi_I$, where $I$ are dyadic cubes, and $\psi$ is as in Step 1, approximate $S_{f,N} = \sum_I f_I T_N \psi_I$, for some cost distribution ($N_I$) with $\sum_I N_I \leq N$.

For a particular cost distribution (i.e. choice of ($N_I$) based on the wavelet coefficients ($f_I$)), this $S_{f,N}$ obtains the desired error bounds for the given $f$. In general terms, we now state the main theorems contained below (Theorems 4.5 and 7.6):

Theorem. Suppose that $(\phi_\alpha)_{\alpha \in A}$ is a family of kernels satisfying conditions (A1)–(A6) or (B1)–(B4) below. Then for the appropriate choices of parameters $s, \tau, q,$ and $p$, every $f \in F_{\tau,q}$ has an $N$–term approximant $S_{f,N} \in \Phi_N$ such that

$$\|f - S_{f,N}\|_{L^p} \leq CN^{-s/d}\|f\|_{F_{\tau,q}}.$$
We denote by $F_{r,q}^s$ the Triebel–Lizorkin space (defined in Section 4 where $s \in \mathbb{R}^+$ essentially determines the smoothness of the functions in the space). For $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, let $W_p^k(\Omega)$ be the Sobolev space of functions in $L_p(\Omega)$ whose weak derivatives of order up to $k$ are also in $L_p(\Omega)$. The norm and seminorm on the Sobolev space may be defined, respectively, via
\[
\|f\|_{W_p^k(\Omega)} := \|f\|_{L_p(\Omega)} + \|f\|_{W_p^k(\Omega)}, \quad \text{and} \quad |f|_{W_p^k(\Omega)} = \max_{|\beta|=k} \|D^\beta f\|_{L_p(\Omega)}.
\]

Let $PW_k$ be the Paley–Wiener space of bandlimited $L_2$ functions whose Fourier transform is supported on $S$. If $B(0,R)$ is the Euclidean ball of radius $R$ centered about the origin, define the space $H_B := \mathcal{F} \cap PW_{B(0,R)}$; in the sequel, $R$ will be determined from the context. For a set $S$, let $\chi_S$ be the function which takes value 1 on $S$ and 0 elsewhere, and let $|S|$ denote the Lebesgue measure of the set $S$. We use $C$ to denote constants, and where appropriate add subscripts to emphasize the parameters they may depend on.

3. Regularity Criteria and Approximation in $H_B$

We begin with Step 1 described above, which is to give criteria on kernels which allows for rapid approximation of bandlimited Schwartz functions from a linear space involving the kernels. For now, we resolve our problem for decaying kernels, and turn to growing ones in later sections. Let $A$ be an infinite subset of $(0, \infty)$, and $(\phi_\alpha)_{\alpha \in A}$ be a family of continuous kernels depending on the parameter $\alpha$, and let $\Phi_N$ be as defined in \[4\].

Letting $R > 0$ be fixed, but arbitrary, and $B := B(0,R)$, we introduce the following regularity conditions and assume that they hold from here on unless otherwise noted.

(A1) $\Phi_N$ is closed under translation and dilation.
(A2) Each $\phi_\alpha$ is continuous, and $\sup_{\alpha \in A} \|\phi_\alpha\|_{L_\infty} \leq C$.
(A3) Let $A_{\alpha,h,j} := \frac{\phi_\alpha(\cdot + 2\pi j/h)}{\phi_\alpha}$. Suppose that $\sum_{j \neq 0} \|A_{\alpha,h,j}\|_{L_\infty(B)} \leq g_\alpha(h)$ for some function $g_\alpha(h)$.

Then for every $k \in \mathbb{N}_0$, there exists an $\alpha_k \in A$ such that for every $\alpha \geq \alpha_k$, $g_\alpha(h) \leq Ch^k$ for some absolute constant $C$.

(A4) For every $k \in \mathbb{N}$, there exists an $\alpha'_k \in A$ such that for every $\alpha \geq \alpha'_k$, $D^\gamma(1/\hat{\phi}_\alpha) \in L_2(B)$ for every $|\gamma| \leq k$.

(A5) For every $k \in \mathbb{N}$, there exists an $\alpha''_k \in A$ such that for every $\alpha \geq \alpha''_k$, $|\phi_\alpha(x)| = O(|x|^{-2k})$, $|x| \to \infty$.

(A6) For every $\alpha \in A$, $|\phi_\alpha(x)| + |\hat{\phi}_\alpha(x)| \leq C(1 + |x|)^{-d-\varepsilon}$ for some $C, \varepsilon > 0$.

The analysis in \[17\] is aided by the fact that the Gaussian has rapid decay – a requirement we drop in order to consider a family of general inverse multiquadrics which serves as the typical example of a collection of kernels which satisfies (A1)–(A6). These conditions allow us to provide norm and pointwise estimates for functions in $H_B$. The requirement (A1) is to ensure the ease of approximation of the wavelet basis, while (A6) allows one to use the Poisson summation and Fourier inversion formulas (of course (A6) may be replaced by another condition which allows these to hold). The utility of the remaining criteria will reveal itself in the sequel. Presently, we define an approximation suited to our purposes.

For $f \in H_B$ and $\phi \in (\phi_\alpha)$, we first define $f_\phi$ to be the function such that $\hat{f_\phi} = \hat{f} \hat{\phi}$. The main obstruction is that $f_\phi$ need not be smooth. Nevertheless, following \[17\], we define
\[
T^\phi_h f(x) := \sum_{j \in \mathbb{Z}^d} f_\phi(hj)\phi(x - hj).
\]

**Proposition 3.1** (c.f. Proposition 3.1, \[17\]). If $\alpha \geq \alpha_0$ and $h < \pi/R$, then for every $f \in H_B$,
\[
\|f - h^d T^\phi_h f\|_{L_\infty} \leq C\|\hat{f}\|_{L_1} g_\alpha(h),
\]
where $C > 0$ is independent of $f$ and $h$. 


Proof. By the inversion formula and the Poisson summation formula (both are justified by (A6)),
\[ h^d T_h^2 f(x) = \frac{h^d}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) \left[ \frac{1}{\phi(\xi)} \sum_{j \in \mathbb{Z}^d} \phi(x - hj) e^{i(\xi, hj)} \right] d\xi \]
\[ = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) \left[ \frac{e^{i(\xi, x)}}{\phi(\xi)} \sum_{j \in \mathbb{Z}^d} \hat{\phi}(\xi + 2\pi j / h) e^{i(x, 2\pi j / h)} \right] d\xi. \]
The \( j = 0 \) term is nothing but \( f(x) \), hence
\[ f(x) - h^d T_h^2 f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) \left[ \frac{e^{i(\xi, x)}}{\phi(\xi)} \sum_{j \neq 0} \hat{\phi}(\xi + 2\pi j / h) e^{i(x, 2\pi j / h)} \right] d\xi. \]
Recalling that \( \text{supp}(\hat{f}) \subset B \), we have the straightforward estimate
\[ \|f - h^d T_h^2 f\|_{L^\infty} \leq \frac{1}{(2\pi)^d} \sum_{j \neq 0} \|A_{\alpha, h, j}\|_{L^\infty(B)} \|\hat{f}\|_{L_1}, \]
which is at most \( C \|\hat{f}\|_{L_1} g_{\alpha}(h) \) by (A3).

Note that by the restriction \( \alpha \geq \alpha_k \) above, we mean that for any \( \phi \in (\phi_\alpha)_{\alpha \geq \alpha_k} \), the conclusion of the proposition holds.

We move now to truncate our approximant to put it in the space \( \Phi_N \). Define \( T_h^0 f \) via
\[ T_h^0 f(x) := \sum_{j \in \mathbb{Z}^d \cap B_h} f_\phi(hj) \phi(x - hj), \]
where \( B_h \) is the ball of radius \( h^{-2} \) centered at the origin. This is a finite sum of \( N \sim h^{-2d} \) terms, hence \( T_h^0 f \in \Phi_N \). The next pair of propositions show that this truncated approximant performs quite well both in \( L^\infty \) and pointwise.

**Proposition 3.2** (cf. Lemma 3.2, [17]). Let \( k \in \mathbb{N} \) and \( f \in H_B \). For \( h < \pi / R \), there is a constant \( C_{f, k} > 0 \), independent of \( h \), so that for all \( \alpha \geq \max\{\alpha_k, \alpha_{k+d}\} \),
\[ \|f - h^d T_h^0 f\|_{L^\infty} \leq C_{f, k} h^k. \]

**Proof.** Note that the norm in question is at most \( C \|\hat{f}\|_{L_1} g_{\alpha}(h) + h^d \|T_h^0 f - T_h^0 f\|_{L^\infty} \) by the triangle inequality and Proposition 3.1, but the first term is at most \( C_j h^k \) by (A3) and the assumption \( \alpha \geq \alpha_k \). Thus, it remains to show that \( \|T_h^0 f - T_h^0 f\|_{L^\infty} \leq C h^{k-d} \). By (A2), we have
\[ (2) \quad \|T_h^0 f - T_h^0 f\|_{L^\infty} \leq C \sum_{|\beta| \geq h^{-2}} |f_\phi(hj)|. \]

Now if \( \alpha \geq \alpha_{k+d} \), then for all multi-indices \( |\gamma| \leq k + d \), \( D^\gamma (1/\hat{\phi}) \in L_2(B) \). Consequently, by standard Fourier transform techniques,
\[ |f_\phi(x)| \leq \left( \sum_{|\gamma| \leq k+d} \|D^\gamma (\hat{f}/\hat{\phi})\|_{L_1} \right) (1 + |x|)^{-k-d}, \]
which is, by Leibniz’s rule, the Cauchy–Schwarz inequality, the fact that \( f \in \mathcal{S} \), and (A4), majorized by
\[ \sum_{|\gamma| \leq k+d} \sum_{\beta \leq \gamma} \|D^\beta \hat{f}\|_{L_2} \left\| D^{\gamma-\beta} \left( \frac{1}{\hat{\phi}} \right) \right\|_{L_2} (1 + |x|)^{-k-d} \leq C_{f, k} (1 + |x|)^{-k-d}. \]
Thus the series on the right side of (2) is at most
\[ C_{f,k}h^{k-d} \sum_{|j|>h^{-2}} \frac{1}{|j|^{k+d}} \leq C_{f,k}h^{k-d} \int_{h^{-2}}^{\infty} r^{d-1}r^{-k-d}dr = C_{f,k}h^{k-d}, \]
concluding the proof. \[ \square \]

**Proposition 3.3** (cf. Lemma 3.3, [17]). Let \( k \in \mathbb{N} \) and \( f \in H_B \). Then for all \( \alpha \geq \max\{\alpha_{2k}, \alpha'_{2k+d}, \alpha''_k\} \) and sufficiently small \( h \), there is a constant \( C \), independent of \( h \), such that
\[ |f(x) - h^d T^\alpha f(x)| \leq Ch^{k(1 + |x|)^{-k}}. \]

**Proof.** First notice that if \( |x| \leq 2/h \), then \((1 + |x|)^{-k} \geq (h/3)^k\), and so the desired inequality arises from Proposition 3.2 (note this requires \( \alpha \geq \max\{\alpha_{2k}, \alpha'_{2k+d}\} \)).

If \( |x| > 2/h \), \( h^k(1 + |x|)^{-k} \geq C|x/2|^{-2k} \), and so it suffices to show that \( |f(x) - h^d T^\alpha f(x)| \leq C(1 + |x|)^{-2k} \) in this range. Note that \( |f(x)| \leq C(1 + |x|)^{-2k} \) since it is in \( \mathscr{S} \). On the other hand,
\[ h^d |T^\alpha f(x)| \leq h^d \sum_{j \in \mathbb{Z}^d} |f_\phi(hj)| \sup_{|j| \leq h^{-2}} |\phi(x - hj)|. \]

Using the same bound as in the proof of Proposition 3.2, we see that the series above is bounded as long as \( \alpha \geq \alpha'_{d+1} \). Moreover, in this case, the series is bounded by a constant times \( h \) to a positive power, in which case, we may say \( h^d \sum_{j \in \mathbb{Z}^d} |f_\phi(hj)| \leq C \) for some constant independent of \( h \). Next, notice that \( |x - hj| \geq |x|/2 \), and so if \( \alpha \geq \alpha''_k \), (A5) implies that \( |\phi(x - hj)| = O(|x - hj|^{-2k}) = O(|x|^{-2k}) \) for sufficiently small \( h \). Therefore, \( |h^d T^\alpha f(x)| \leq C(1 + |x|)^{-2k} \) as required. \[ \square \]

Now to make the dependence on \( N \) more explicit, we can replace each occurrence of \( h \) above with \( N^{-1/(2d)} \) and rewrite \( T^\alpha f \) as
\[ T_N f(x) := N^{-1/2} T^\alpha_{N^{-1/(2d)}} f(x). \]
We denote by \( N_0 \) the smallest such \( N \) that satisfies the requirements of Proposition 3.3.

4. Approximation in \( L_p \), \( 1 \leq p < \infty \)

To continue our study in the manner described in Step 2 above, we will make use of the wavelet expansion of a given function and approximate the wavelet as in Section 3. For this we need a wavelet system whose generators are in \( H_B \) where \( B = B(0, R) \) contains the support of the Fourier transform of the mother wavelet. Fortunately, such wavelet systems are well known; for example the Meyer wavelet (see [13, 21]) forms one such system. So that the proofs extend easily to arbitrarily high dimensions (see especially Section 7), we restrict to considering multivariate wavelet systems which are tensor products of univariate ones. Since we use the same methodology as [17], Section 3, we only list the relevant details adapted to our set up. Using the notation we find there, \( I = c(I) + [0, \ell(I)]^d \) is a cube with corner \( c(I) \in \mathbb{R}^d \) and side length \( \ell(I) > 0 \). Encumbered upon the wavelet structure, we make use of dyadic cubes, i.e.
\[ I \in \mathcal{D} := \{2^m(n + [0, 1]^d) : m \in \mathbb{Z}, n \in \mathbb{Z}^d \}. \]
Let \( \mathcal{D}_j \) be the subset of cubes with common edge-length \( 2^j \). We will denote by \( \psi_I \) the the natural affine change of variables:
\[ \psi_I(x) := \psi((x - c(I))/\ell(I)). \]
If \( \psi \in H_B \), then since (A1) is satisfied, we have the following theorem as a direct result of Proposition 3.3.

**Theorem 4.1.** Let \( k \in \mathbb{N}, \psi \in H_B, N > N_0 \) and let \( I \in \mathcal{D} \). There exists \( \alpha_K \in A \) such that for all \( \alpha \geq \alpha_K \), there is a constant \( C \), independent of \( N \), such that
\[ |\psi_I(x) - (T_N \psi)_I(x)| \leq C N^{-k/d} \left(1 + \frac{|x - c(I)|}{\ell(I)}\right)^{-2k}. \]
In particular, Theorem 4.1 says that to approximate any element of the wavelet system, it suffices to consider the approximant of the mother wavelet under the same affine change of variables.

We will assume throughout that we have a wavelet system \( \Psi \) formed from a mother wavelet \( \psi \) as described above. This allows us to use the results of the previous section because we have (for smooth enough \( f \))

\[
f = \sum_{I \in \mathcal{D}} f_I \psi_I.
\]

To approximate \( f \) with \( N \) terms, we follow the methodology of \cite{17}. To begin, we assign to each cube \( I \) a cost \( c_I \), and subsequently a budget

\[
N_I := \begin{cases} |c_I|, & c_I \geq N_0, \\ 0, & \text{otherwise}, \end{cases}
\]

where \( N_0 \) is the same as in Theorem 4.1 and depends only on the wavelet system. Requiring that \( \sum I N_I \leq N \), we then simply approximate \( f_I \psi_I \) with \( N_I \) terms, and set

\[
f_{f,N} := \sum_{I \in \mathcal{D}} f_I T_{N_I} \psi_I
\]

to be our \( N \)-term approximant in \( \Phi_N \) which will exhibit the desired rate of convergence. The cost distribution depends on several auxiliary quantities, which we define below.

**Definition 4.2.** Given \( s,q > 0 \), define the maximal function \( M_{s,q}f \) via

\[
M_{s,q}f(x) := \left( \sum_{I \in \mathcal{D}} |I|^{-sq/d} |f_I|^q \chi_I(x) \right)^{1/q}.
\]

For a fixed dyadic cube \( I \), we define a partial maximal function by

\[
M_{s,q,I}f(x) := \left( \sum_{I' \subset I \in \mathcal{D}} |I'|^{-sq/d} |f_{I'}|^q \chi_{I'}(x) \right)^{1/q}.
\]

Now given \( \tau, s, q > 0 \), we define the Triebel–Lizorkin space \( F_{\tau,q}^s \) via

\[
F_{\tau,q}^s := \{ f : |f|_{F_{\tau,q}^s} := \| M_{s,q}f \|_{L^s(\mathbb{R}^d)} < \infty \},
\]

where \( |f|_{F_{\tau,q}^s} \) is a quasi-seminorm.

For the reader unfamiliar with the Triebel–Lizorkin spaces, it should be mentioned that they may be viewed from many different equivalent vantage points. While the definition above is in terms of mixed sequence and function space norms of a maximal function, they may also be defined via a Littlewood–Paley decomposition, or additionally in terms of a wavelet system; for more details, consult Section 3 of \cite{18}. Returning to the problem at hand, we are now in position to define our cost distribution.

**Definition 4.3.** Let \( s > 0 \) and \( 1 \leq p < \infty \). Define \( \tau \) and \( q \) by \( 1/\tau := 1/p + s/d \) and \( 1/q := 1 + s/d \). Let \( f \in F_{\tau,q}^s \), with the wavelet expansion \( \sum_{I \in \mathcal{D}} f_I \psi_I \). The cost of a dyadic cube \( I \in \mathcal{D} \) is

\[
c_I := |f|_{F_{\tau,q}^s} m_{s,q,I}^{-q} |f_I|^q [I]^q N,
\]

where \( m_{s,q,I} := \sup_{x \in \mathbb{R}^d} M_{s,q,I} f(x) \).

As shown in \cite{17}, the sum of all costs is bounded by \( N \). Once we have our budget apportioned, we can use Theorem 4.1 to see that we are estimating \( f_I \psi_I \) with \( N_I \) translates of the kernel \( \phi_a \). Additionally, we can see that for \( \alpha \geq \alpha_N \):

\[
|\psi_I(x)| \leq C_{k,d} \min \{1, N^{-k/d} \} \left( 1 + \frac{\text{dist}(x,I)}{\ell(I)} \right)^{-2k} := R_I(x);
\]

\[
|R_I(x)| \leq C_{k,d} \min \{1, c_I^{-k/d} \} \left( 1 + \frac{\text{dist}(x,I)}{\ell(I)} \right)^{-2k}.
\]

The analogue of \cite{17} Theorem 9 relies on the following lemma.
Lemma 4.4 (c.f. Lemma 8, [17]). Let $1 \leq p < \infty$ and $\alpha \geq \alpha_K$, then
\[
\left\| \sum_{I \in \mathcal{D}} |f_I| R_I \right\|_{L_p} \leq C_{k,d} \left\| \sum_{I \in \mathcal{D}} \min \left\{ 1, c_I^{-k/d} \right\} |f_I| \chi_I \right\|_{L_p}.
\]

Theorem 4.5 (c.f. Theorem 9, [17]). Given $s > 0$ and $1 \leq p < \infty$, there exists a constant $C_{s,p,d} > 0$ such that for $f \in F_{s,q}$, with $1/\tau = 1/p + s/d$ and $1/q = 1 + s/d$, there is a function $S_{f,N} \in \Phi_N$ so that
\[
\|f - S_{f,N}\|_{L_p} \leq C_{s,p,d} N^{-s/d} |f|_{F_{s,q}}.
\]
In particular, we may take $S_{f,N}$ as in (5).

Proof. The proof is the same as the one given in [17], mutatis mutandis. □

5. APPROXIMATION IN $L_\infty$

It is noted in [17] (and the situation is no different here) that the proof of Lemma 4.4 relies upon the Fefferman–Stein inequality, which is not valid in $L_\infty$. Thus, for $L_\infty$ convergence phenomena, we restrict our attentions to Besov spaces, which are alternative (slightly smaller) smoothness spaces than the Triebel–Lizorkin spaces which do not rely on maximal functions in their definitions.

Here, we list the relevant results suited to our set up. It closely matches that of [17, Section 4], thus the proofs are omitted.

Definition 5.1. For $\tau = d/s \in (0, \infty)$ and $q \in (0, \infty)$, the Besov space $B^s_{\tau,q}$ is defined by the finiteness of the quasi-seminorm
\[
|f|_{B^s_{\tau,q}} := \left\| k \mapsto \left( \sum_{I \in \mathcal{D}} |f_I|^\tau \right)^{1/\tau} \right\|_{\ell_q},
\]
that is
\[
B^s_{\tau,q} := \left\{ f \in L_\tau : |f|_{B^s_{\tau,q}} < \infty \right\}.
\]

Lemma 5.2. Let $k > d$ and $\alpha > \alpha_K$. Suppose that $(a_I)_{I \in \mathcal{D}_j}$ is a finitely supported sequence of coefficients; then
\[
\left\| \sum_{I \in \mathcal{D}_j} a_I \psi_I - \sum_{I \in \mathcal{D}_j} a_I [T_{N_i} \psi_I] \right\|_{L_\infty} \leq C_{k,d} \sup_{I \in \mathcal{D}_j} \left| a_I N_i^{-k/d} \right|.
\]

Theorem 5.3. Given $s > 0$, there exists a constant $C_{s,d} > 0$ such that for $f \in B^s_{\tau,q}$, with $1/\tau = s/d$ and $1/q = 1 + s/d$, there is a function $S_{f,N} \in \Phi_N$ so that
\[
\|f - S_{f,N}\|_{L_\infty} \leq C_{s,d} N^{-s/d} |f|_{B^s_{\tau,q}}.
\]
For the cost distribution for the Besov space case, see [17, Section 4.1].

6. EXAMPLES

In this section, we provide numerous examples of kernels which satisfy the conditions prescribed above whose associated $N$–term approximation spaces $\Phi_N$ exhibit optimal rates for nonlinear approximation of smooth functions.
6.1. Gaussians. This example is due to Hangelbroek and Ron and can be found in [17]. It served as the inspiration for our analysis.

Here we will let $\phi_\alpha$ be the Gaussian:

$$\phi_\alpha(x) := e^{-|x/\alpha|^2},$$

whose Fourier transform is given by

$$\hat{\phi_\alpha}(\xi) = (\alpha \sqrt{\pi})^d \phi_{2/\alpha}(\xi).$$

We check conditions (A1)–(A6) below, which are straightforward since the Gaussian is nice.

Condition (A1) is clear from the definition of $\Phi_N$ and the fact that $\phi_\alpha(x/\beta) = \phi_{\alpha \beta}(x)$. The constant in (A2) can be taken to be $\phi_\alpha(0) = 1$. The quantity in (A3) is calculated in [17] as follows:

$$\sum_{j \neq 0} \left| \frac{\hat{\phi_\alpha}(\xi + 2\pi j/h)}{\hat{\phi_\alpha}(\xi)} \right| \leq C \hat{\phi_\alpha}(2(\pi/h-R)) \leq C \alpha R e^{c/h^2},$$

where the constant $c > 0$ depends on $\alpha$ and $R$. We may take our parameter $\alpha_k = 0$.

Condition (A4) is checked easily by noting that $1/\hat{\phi_\alpha} = e^{2\pi i \xi x/\alpha}$, which is in $L^1_{\text{loc}}$ since it is continuous. As with (A3), we may take $\alpha_k' = 0$. Likewise, conditions (A5) and (A6) are both clearly satisfied (with $\alpha''_k = 0$) because of the exponential decay of the Gaussian.

6.2. Multiquadrics I. We will now consider the general (inverse) multiquadric,

$$\phi_{\alpha,c}(x) := (|x|^2 + c^2)^{-\alpha},$$

where $| \cdot |$ denotes the Euclidean distance on $\mathbb{R}^d$; $c > 0$ is called the shape parameter, and we take $\alpha \in A = (d+1/2, \infty)$. Consider

$$\Phi_N = \left\{ \sum_{j=1}^N a_j \phi_{\alpha_j,c_j}(:-x_j:) : (a_j) \subset \mathbb{C}, (\alpha_j) \subset A, (c_j) \subset \mathbb{R}^+, (x_j) \subset \mathbb{R}^d \right\}.$$ 

Since $\Phi_N$ is manufactured to be closed under translations, we check only dilation. If $\delta > 0$, we have $\phi_{\alpha,c}(\delta \cdot) = \delta^{2\alpha} \phi_{\alpha,c/\delta}$; hence $\Phi_N$ is dilation invariant, and (A1) is satisfied.

Condition (A2) is straightforward by noting that $\phi_{\alpha,c}$ is decreasing radially when $|x| \in (0, \infty)$; hence $|\phi_{\alpha,c}(x)| \leq e^{-2\alpha}$, which is bounded by $e^{-2d+1}$ for all $\alpha \in A$. Condition (A3) requires the use of the Fourier transform of $\phi_{\alpha,c}$, which is given by (see, for example, [39 Theorem 8.15]):

$$\hat{\phi_{\alpha,c}}(\xi) = (2\pi)^\frac{d}{2} \frac{2^{1-\alpha}}{\Gamma(\alpha)} \left( \frac{c}{|\xi|} \right)^{\frac{d}{2} - \alpha} K_{\frac{d}{2} - \alpha}(c|\xi|), \quad \xi \in \mathbb{R}^d \setminus \{0\},$$

where

$$K_\nu(r) := \int_0^\infty e^{-r \cosh t} \cosh(\nu t) dt, \quad r > 0, \nu \in \mathbb{R}.$$ 

The function $K_\nu$ is called the modified Bessel function of the second kind (see [21 p.376]). We note that $\phi_{\alpha,c}$, and consequently its Fourier transform, are radial functions (i.e. $\phi_{\alpha,c}(x) = \phi_{\alpha,c(|x|)}$). It is also clear from (7) that $K$ is symmetric in its order; that is, $K_{-\nu} = K_\nu$ for any $\nu \in \mathbb{R}$. Additionally, the decay of the Bessel function governs the decay of $\hat{\phi_{\alpha,c}}$, which is exponential away from the origin.

Using the estimates from [15], we may estimate $A_{\alpha,h,j}$ as follows:

$$|A_{\alpha,h,j}(\xi)| \leq \left( \frac{R}{|\xi| + 2\pi j/h} \right)^{d/2-\alpha} e^{cR-c(|\xi|+2\pi j/h)} \leq C_{\alpha,R} e^{cR-c(|\xi|+2\pi j/h)} e^{-\pi c|j|/h}.$$ 

where we have estimated the polynomial term with an exponential. Thus the series obtained by summing over $j \neq 0$ is bounded by $C_{\alpha,R} e^{cR-c|\xi|/h}$. Consequently, we may take $\alpha_k > d+1/2$ to satisfy the condition.

Estimates from [15] show that $1/\hat{\phi_{\alpha,c}}$ is only $(2a-d)$-times differentiable near the origin, hence we must have $\alpha_k' > k/2 + d/2$. Additionally, each of these derivatives are bounded, hence $1/\hat{\phi_{\alpha,c}} \in L^1_{\text{loc}}$ as desired.
Condition (A5) is obvious by taking $\alpha_k'' > 2k$. Finally, (A6) follows from the decay of $\phi_{\alpha,c}$ when $\alpha \in A$ and the exponential decay of $\hat{\phi}_{\alpha,c}$.

6.3. Matérn Kernels. The Matérn kernels are simply the Fourier transform of the inverse multiquadrics, i.e. $\rho_{\alpha,c}(x) := \phi_{\alpha,c}(x)$, where $\phi_{\alpha,c}$ is given by (6). Consequently, $\tilde{\rho}_{\alpha,c}(\xi) = (\xi^2 + c^2)^{-\alpha}$. The approximation space associated with these kernels is closed under dilations since $\rho_{\alpha,c}(d\xi) = \delta^{2\alpha-d}\rho_{\alpha,\delta}(x)$. It is a simple exercise to verify that condition (A3) holds if $\alpha > d/2$, whilst the other conditions are also readily checked in a fashion similar to the multiquadrics.

6.4. Cardinal Functions. Associated with many kernels are so-called cardinal functions which exhibit the interpolatory condition on the integer lattice that the cardinal sine function does, i.e. they are functions $L$ such that $L(j) = \delta_{\alpha,j}$, $j \in \mathbb{Z}^d$. Given a kernel $\phi$, formally define

$$\hat{L}_{\phi}(\xi) := \sum_{j \in \mathbb{Z}^d} \hat{\phi}(\xi - 2\pi j), \quad \xi \in \mathbb{R}^d.$$ 

Then as long as the Fourier inversion formula holds and $\hat{\phi}$ decays suitably, one may show that $L_{\phi}$ given by the inverse Fourier transform is a cardinal function. For sufficient conditions on the kernel $\phi$ for $L_{\phi}$ to be a cardinal function, see [19]. In particular, we assume from here on that $\hat{\phi} > 0$ on $\mathbb{R}^d$.

Cardinal functions associated with radial basis functions have been studied rather extensively [3][6][14][16][19][20][22][25][27]. Specifically, the cardinal functions associated with the Gaussian and general multiquadrics are known to have nice decay (exponential in the former case and polynomial based on the exponent $\alpha$ in the latter).

In the examples given below, we also have $L_{\phi}(x) = \sum_{j \in \mathbb{Z}^d} c_j \phi(x - j)$, where convergence is at least uniform on compact subsets of $\mathbb{R}^d$, but in many cases, the convergence is uniform and in $L_2(\mathbb{R}^d)$. We note that the impetus for considering such cardinal functions may be found in the ubiquitous literature on spline interpolation (particularly cardinal splines), see for example [2][26] and references therein. For some works involving splines used in sampling, see [2][28][29].

**Lemma 6.1.** Let $(\phi_\alpha)_{\alpha \in A}$ satisfy (A3), (denote the quantity there by $A^{\phi}_{\alpha,h,j}$). Then the associated cardinal functions $(L_\alpha)_{\alpha \in A}$ satisfy (A3) (denote the associated functions in the condition by $A^{L}_{\alpha,h,j}$), and moreover

$$\sum_{j \neq 0} \| A^{L}_{\alpha,h,j} \|_{L_\infty(B)} \leq C \sum_{j \neq 0} \| A^{\phi}_{\alpha,h,j} \|_{L_\infty(B)} \left( 1 + \sum_{j \neq 0} \| A^{\phi}_{\alpha,1,j} \|_{L_\infty(B)} \right).$$

**Proof.** Note that

$$A^{L}_{\alpha,h,j}(\xi) = \frac{\hat{L}_\alpha(\xi + 2\pi j / h)}{\hat{L}_\alpha(\xi)} = \frac{\hat{\phi}_\alpha(\xi + 2\pi j / h)}{\hat{\phi}_\alpha(\xi)} \sum_{\ell \in \mathbb{Z}^d} \hat{\phi}_\alpha(\xi - 2\pi \ell).$$

The series in the denominator on the right is at least $\hat{\phi}_\alpha(\xi)$ by the assumption of positivity. Thus

$$A^{L}_{\alpha,h,j}(\xi) \leq A^{\phi}_{\alpha,h,j}(\xi) \left( 1 + \sum_{\ell \neq 0} A^{\phi}_{\alpha,1,\ell}(\xi) \right),$$

and the required inequality follows from summing over $j \neq 0$. \hfill $\square$

It follows immediately from this that $g^L_\alpha(h) \leq C g^\phi_\alpha(h)$, where the functions $g_\alpha$ are the rates assumed in (A3).

For a given set of cardinal functions $(L_\alpha)$, define the $N$-term approximation space via

$$L_N := \left\{ \sum_{j=1}^N a_j L_\alpha \left( \frac{x_j - x}{c_j} \right) : (a_j) \subset C, (\alpha_j) \subset A, (x_j) \subset \mathbb{R}^d, (c_j) \subset \mathbb{R} \right\}.$$
Note that these spaces are defined to be closed under translation and dilation, so (A1) is automatically satisfied; this is done because the cardinal functions themselves are not generally preserved under these operations. However, defining the approximation space this way is not unnatural since often in such methods, one considers interpolation using cardinal functions for different lattices. Therefore, the use of shifted and dilated cardinal functions will correspond to interpolation of the wavelet at differently scaled lattices to form the approximant. We note also that (A2) is satisfied for all cardinal functions since \( |L_\alpha(x)| \leq 1 \) for all \( x \in \mathbb{R}^d \) for the examples listed in the sequel.

6.4.1. Gaussian Cardinal Function. The cardinal function associated with the Gaussian kernel \( \phi_\alpha \) described in section 6.1 was studied extensively by Riemenschneider and Sivakumar [22, 23, 27]. In particular, they showed that the cardinal function decays exponentially away from the origin, as does its Fourier transform, which implies (A5) with \( \alpha''_\ell = 0 \) and (A6). Additionally, for every \( \alpha \), \( |L_\alpha(x)| \leq 1 \) for every \( x \in \mathbb{R}^d \). Finally, (A3) follows from Lemma 6.1, while (A4) is a simple exercise based on the exponential decay of the Gaussian.

6.4.2. Multiquadric Cardinal Functions. Details on the behavior of the cardinal functions associated with the general multiquadrics may be found in [14, 15] for a broad range of exponents \( \alpha \), whereas the particular cases of \( \alpha = \pm 1/2 \), \( 1 \), \( \alpha > 1/2 \), \( \alpha = 1 \), \( \alpha = 1/2 \), \( \alpha = 1/2 \), and \( \alpha = 1/2 \) have well-defined cardinal functions which also satisfy (A1)–(A6) [14, 15].

In fact, checking the condition (A3) was done in Section 7 of [14] for the univariate cardinal function, while a different estimate in [15] demonstrates (A3) in higher dimensions. (A4) and (A5) also follow from those estimates (even with the same parameters \( \alpha_k, \alpha'_k, \alpha''_k \) as in Section 6.2). For any \( \alpha > d/2 \), the associated multiquadric cardinal function satisfies (A1)–(A6).

It should also be noted that the growing multiquadrics \( \phi_{\beta,c} := (|\cdot|^2 + c^2)^{\beta} \) for \( \beta \geq 1/2 \) have well-defined cardinal functions which also satisfy (A1)–(A6) [14, 15].

7. Regularity Criteria for Growing Kernels

In this section we investigate the possibilities for growing kernels, which requires a reworking of conditions (A1)–(A6). To begin, given \( N \geq N_0 \), we seek an \( M := M_N \) term approximation space

\[
\Phi_M := \left\{ \sum_{j=1}^{M} a_j \phi_{\alpha_j} (\cdot - x_j) : (a_j) \subset C, (\alpha_j) \subset A, (x_j) \subset \mathbb{R}^d \right\}
\]

which is closed under translation and dilation. Relying on the analysis above, we seek functions \( \phi_\alpha \) such that a finite linear combination of their translates decay. For the moment, consider the univariate case, where we could hope to use the divided difference to obtain (A2) and perhaps (A5). In principle and practice, condition (A3) is the hardest to check. Notice that a growing kernel leads to a singularity at the origin in the Fourier domain; however, the function \( \sum_{j \in \mathbb{Z}} A_{\alpha,h,j} \) has no such singularity. With this and the formulation in Proposition 6.1 in mind, we define the bivariate kernel \( \tilde{k}_{\phi,h} \) whose Fourier transform in the first variable is given by

\[
(8) \quad \tilde{k}_{\phi,h}(\xi, x) := \sum_{j \in \mathbb{Z}} \frac{\tilde{\phi}(\xi + 2\pi j/h)e^{2\pi ijx/h}}{\phi(\xi)} \chi_B(\xi).
\]

By (A3), this kernel is well defined for all \( x \), since

\[
\| \tilde{k}_{\phi,h}(\cdot, x) \|_{L_1} \leq C \left( 1 + \sum_{j \neq 0} \| A_{\alpha,h,j} \|_{L_\infty(B)} \right).
\]

Thus we make our approximation of \( f \in H_B \) of the form

\[
(9) \quad \tilde{T}_h f(x) := [f * k_h(\cdot, x)](x)
\]

(i.e. the convolution is taken in the first variable) and note that when everything is smooth and decays well enough, the Poisson summation formula holds and we recover our previous work; that is, \( \tilde{T}_h f = T_h f \).
The advantage of this technique is that we can consider growing kernels from the outset. For illustration purposes, consider the family of univariate multiquadrics

\[ \tau_\alpha(x) := (x^2 + c^2)^{\alpha - 1/2}, \quad \alpha \in \mathbb{N}. \]

We know that by taking sufficiently many divided differences, we will be able to obtain any polynomial of degree \( \alpha \). Then, we define

\[ \tilde{f} \tau_\alpha(x) := \frac{1}{(2\pi)^n} \sum_{j=-n}^{n} (-1)^{j+n} (2n)^j f(j_n) \]

so that if we choose \( h = 1/N \), we get

\[ A^\phi_{\alpha,h,j}(\xi) = \frac{1 - \cos(\xi + 2\pi j/h)}{(1 - \cos(\xi))^\alpha} \tilde{\tau}_\alpha(\xi) \]

so that if we choose \( h = 1/N \), we get

\[ A^\phi_{\alpha,1/N,j}(\xi) = \frac{1 - \cos(\xi + 2\pi j/N)}{(1 - \cos(\xi))^\alpha} \tilde{\tau}_\alpha(\xi) = A^\tau_{\alpha,1/N,j}(\xi). \]

The interplay between the divided difference and the choice of \( h = 1/N \) is vital and allows us to proceed. To do so, we need to truncate our approximation, so we set \( k^\phi_{\alpha,1/N}(\cdot, x) \) to be the function whose Fourier transform is given by

\[ \tilde{k}^\phi_{\alpha,1/N}(\xi, x) := \sum_{|j| \leq N^2} \tilde{\phi}(\xi + 2\pi j/N) e^{2\pi ijN} \]

then we define

\[ \tilde{T}_{1/N}^\phi f(x) := [f * k^\phi_{\alpha,1/N}(\cdot, x)](x). \]

Hereafter, we make the following assumptions:

- (B1) \( \Phi_M \) is closed under translation and dilation.
- (B2) \( \phi_\alpha = [\tau_\alpha]_{n_\alpha} \) for some \( (n_\alpha) \subset \mathbb{N} \); that is, the kernels are built out of divided differences.
- (B3) (A3) holds for each \( \phi_\alpha \) and \( h = 1/N \).
- (B4) For every \( k \in \mathbb{N} \) and \( j \neq 0 \), there exists \( \tilde{\alpha}_k \in A \) and \( C \) independent of \( N \) such that for every \( \alpha \geq \tilde{\alpha}_k \) and \( 0 \leq l \leq k \), \( D^l A^\tau_{\alpha,1/N,j} \in L_\infty(B) \), and

\[ \sum_{j \neq 0} \|D^l A^\tau_{\alpha,1/N,j}\|_{L_\infty(B)} \leq C. \]

We now use these hypotheses to prove the results analogous to those found in Section 3; note that here we have \( B = [-R, R] \).

Lemma 7.1. There exists \( \alpha_0 \in A \) such that for all \( \alpha \geq \alpha_0 \), \( N > R/\pi \), and \( f \in H_B \),

\[ |(1/N)\tilde{T}_{1/N}^\phi f(x)| \leq \|\tilde{f}\|_{L_1} (1 + g_\alpha(1/N)). \]

Proof. Using (9) and the inversion formula, we have

\[ |(1/N)\tilde{T}_{1/N}^\phi f(x)| \leq \left\| \int \tilde{f} k_{\phi,1/N}(\cdot, x) \right\|_{L_1} \leq \|\tilde{f}\|_{L_1} (1 + g_\alpha(1/N)), \]

the last inequality coming from (B3).

Proposition 7.2. There exists \( \alpha_0 \in A \) such that for all \( \alpha \geq \alpha_0 \), \( N > R/\pi \), and \( f \in H_B \),

\[ \|f - (1/N)\tilde{T}_{1/N}^\phi f\|_{L_\infty} \leq C \|\tilde{f}\|_{L_1} g_\alpha(1/N), \]

where \( C > 0 \) is independent of \( f \) and \( N \).

Proof. We have, from (B3),

\[ \left\| f(x) - (1/N)\tilde{T}_{1/N}^\phi f(x) \right\|_{L_1(B)} \leq \left\| \tilde{f} \sum_{j \neq 0} A^\phi_{\alpha,1/N,j} \right\|_{L_1(B)} \leq C \|\tilde{f}\|_{L_1} g_\alpha(1/N). \]


Lemma 7.3. Let $k \in \mathbb{N}$ and $f \in H_B$. There exists $\alpha_k \in A$, such that for sufficiently large $N$, there is a constant $C_{f,k} > 0$ independent of $N$, so that for all $\alpha \geq \alpha_k$, 
\[ \|f - \tilde{T}^\alpha_{1/N} f\|_{L_\infty} \leq C_{f,k} N^{-k}. \]

Proof. On account of (B3) and Proposition 7.2, we have 
\[ \left| f(x) - (1/N)\tilde{T}^\alpha_{1/N} f(x) \right| \leq \left| f(x) - (1/N)\tilde{T}^\alpha_{1/N} f(x) \right| + \left| (1/N)\tilde{T}^\alpha_{1/N} f(x) - (1/N)\tilde{T}^\alpha_{1/N} f(x) \right| \]
\[ \leq C\|\tilde{f}\|_{L_1(B)} g_\alpha(1/N) + (1/N)\|\tilde{f}\|_{L_1(B)} \sum_{|j| > N^2} \|A_{\alpha,1/N,j}^\phi\|_{L_\infty(B)} \]
\[ \leq C_{f,k} N^{-k}. \]
\[ \square \]

Proposition 7.4. Let $k \in \mathbb{N}_0$ and $f \in H_B$. There exists $\alpha'_k \in A$ so that for all $\alpha \geq \alpha'_k$ and sufficiently large $N$, there is a constant $C$, independent of $N$ such that 
\[ \left| f(x) - (1/N)\tilde{T}^\alpha_{1/N} f(x) \right| \leq C N^{-k} (1 + |x|)^{-k}. \]

Proof. Similar to Proposition 7.3, we need only check that if $|x| > 2/h$, then 
\[ \left| f(x) - (1/N)\tilde{T}^\alpha_{1/N} f(x) \right| \leq C_{k,f} (1 + |x|)^{-2k}. \]

Note that $f \in \mathcal{S}$, $|f(x)| \leq C(1 + |x|)^{-2k}$. From (B4), when $\alpha \geq 2\alpha_k$, we have 
\[ \left| (1 + |x|)^{2k} (1/N)\tilde{T}^\alpha_{1/N} f(x) \right| \leq C_{f,k} \sum_{|j| > N^2} 2k \|A_{\alpha,1/N,j}^\phi\|_{L_\infty(B)} \]
\[ \leq C_{f,k}. \]
\[ \square \]

For a given $k$, we set $\alpha_K := \max\{\alpha_0, \alpha'_k, \alpha_k\}$. Thus the conclusions of the results above hold for every $\alpha \geq \alpha_K$.

Theorem 7.5. Let $\psi \in H_B$ be given. Suppose $k \in \mathbb{N}_0$, $N \geq N_0$, and let $I$ be a cube. Then there exists a constant $C > 0$, independent of $N$ and $I$, such that for all $N$ large enough and suitably large $\alpha$
\[ \left| \psi_I(x) - \tilde{T}^\alpha_{1/N} \psi_I(x) \right| \leq C N^{-k} \left( 1 + \frac{|x - c(I)|}{\ell(I)} \right)^{-2k}. \]

We need only double $\alpha_K$ to achieve the stated bound.

In order to obtain results in the multivariate case we use tensor products, which is sufficient due to the fact that the wavelets considered here are tensor products themselves. The univariate estimates above lead us to prove the main results of this section. Note that $\Phi^{\otimes d}_M$ is the space of $d$–fold tensor products of functions in $\Phi_M$, i.e. $h \in \Phi^{\otimes 2}_M$ has the form $h(x,y) = f(x)g(y)$, for $f, g \in \Phi_M$.

Theorem 7.6. Suppose that $s > 0$, $N \geq N_0$, and $1 \leq p < \infty$. There is a constant $C_{p,s,d} > 0$, $M_{s,N} \in \mathbb{N}$ so that for $f \in F^s_{\tau,q}$, with $1/\tau = 1/p + s/d$ and $1/q = 1 + s/d$, there is $S_f \in \Phi^{\otimes d}_M$ so that 
\[ \|f - S_f\|_{L_p} \leq C_{p,s,d} N^{-s/d} \|f\|_{F^s_{\tau,q}}. \]

Proof. We begin with the wavelet expansion of $f$, given by 
\[ f = \sum f_I \psi_I, \]
and define $S_f$ in terms of this expansion
\[ S_f := \sum f_I \tilde{T}^\alpha_{1/N_1} \psi_I. \]
Recall that $\tilde{T}_{1/N}^{\alpha} \psi_{I} = \sum_{j=1}^{N_{I}} a_{I,j} \phi (\cdot - c(I)/\ell(I))$, so choose $\phi = \phi_{\alpha}$ with $\alpha = \alpha_{s}$ large enough so that Theorem 7.5 holds. Notice also that we have used no more than $\sum M_{s,N_{I}} \leq M_{s,N}$ total centers. If everything is smooth, we have $M_{s,N} = N$, while if we use divided differences, then we have $M_{s,N} = (2\alpha_{s} + 1)^{d}(N^{2} + 1)^{d}$ terms involving the original kernel.

Since the bound in Theorem 7.5 justifies Lemma 4.4 with no further restriction, we use this to obtain

$$\|f - S_{f}\|_{L_{p}} \leq C_{k,d} \left( \sum_{I} \min\{1, c_{I}^{-k/d}\} |f|_{I} \chi_{I} \right)^{1/p}.$$  

This estimate also makes use of the tensor product structure of the wavelet and approximant $S_{f}$. To complete the proof, we must have $k > s$, which provides the restriction on $\alpha_{s}$. The rest of the proof follows the same reasoning given in the proof of Theorem 9 in [17].

Unfortunately, the conclusion of Theorem 7.6 does not completely resolve the problem of approximation from the space $\Phi_{M}$ as the construction therein relies quite heavily on taking tensor products for approximation in dimension larger than 1. It remains of interest to determine an approximation method from the space $\Phi_{M}$ in 1 dimension that does not require taking divided differences of the growing kernel and which can be extended in a natural way to higher dimensions.

8. Examples of Growing Kernels

8.1. Multiquadrics II. We will show that the family of divided differences of multiquadrics mentioned above satisfies properties (B1)–(B4). To wit, consider

$$\phi_{\alpha} = [\tau_{\alpha}]_{2\alpha},$$

where $\tau_{\alpha}(x) = (x^{2} + \alpha^{2})^{\alpha - 1/2}$, $\alpha \in \mathbb{N}$, $x \in \mathbb{R}$.

Property (B1) is satisfied by using the same reasoning as that for (A1) in Section 8.2. Property (B2) is obvious, and note the fact that we want the divided difference of order $2\alpha$ of $\tau_{\alpha}$. To see that (B3) holds, note that

$$A_{\alpha,1/N,j}^{\phi}(\xi) = \frac{\tilde{\tau}_{\alpha}(\xi + 2\pi j N)}{\tau_{\alpha}(\xi)} = A_{\alpha,1/N,j}^{\tau}(\xi),$$

so using estimates from Lemma 1 in [14], we have

$$||A_{\alpha,1/N,j}^{\tau}(\xi)||_{L_{\infty}(B)} \leq e^{2\epsilon R} e^{-2\pi c |j| N (2N) - 1} - \alpha,$$

hence (A3) is satisfied. Estimates from [15] show that $A_{\alpha,1/N,j}^{\tau}$ is $(2\alpha - 1)$–times differentiable and Theorem 5.1 there may be adapted to our needs, showing that the sum in (B4) can be bounded independent of $h$.

8.2. Power kernels. Our next example is $\phi_{\alpha} = [\tau_{\alpha}]_{2\alpha}$, where $\tau_{\alpha}(x) = |x|^\alpha$ for $\alpha \in \mathbb{R}^{+} \setminus 2\mathbb{N}$. We note that

$$\phi_{\alpha}(x) = O \left( \frac{1}{x} \right), \quad |x| \to \infty.$$  

We find the Fourier transform in Section 8.3 of [20]:

$$\tilde{\tau}_{\alpha}(\xi) = (2\pi)^{\frac{d}{2}} 2^{\alpha + \frac{d}{2}} \Gamma \left( \frac{\alpha + 1}{2} \right) \Gamma \left( - \frac{\alpha}{2} \right) |\xi|^{\alpha - 1},$$

which allows us to move forward with our computations. Conditions (B1) and (B2) are evident, so we begin with (B3):

$$A_{\alpha,1/N,j}^{\phi}(\xi) = \frac{|\xi|^{\alpha + 1}}{|\xi + 2\pi j N|^{\alpha + 1}},$$

thus

$$||A_{\alpha,1/N,j}^{\phi}(\xi)||_{L_{\infty}(B)} \leq \frac{R^{\alpha + 1} | N^{\alpha + 1} (2N)^{\alpha + 1} | (2j - 1)^{-\alpha - 1}},$$

hence (B3) is satisfied. To see (B4), we apply the quotient rule repeatedly and find that

$$|D_{\alpha}^{j} A_{\alpha,1/N,j}^{\phi}(\xi)| \leq C_{\alpha,l} N^{-l} (2N - 1)^{-\alpha},$$

hence as long as $\alpha > 1$, (B4) is satisfied.
9. Cost Distribution

9.1. Examples. In this section, we analyze what the cost distribution looks like for some particular types of functions. Suppose that $\psi$ is the mother wavelet, and that $\psi_I$ is as defined previously for a dyadic cube $I$. Then suppose that $\{I_j\}_{j=1}^M$ are disjointly supported dyadic cubes, and we consider the cost distribution for the function

$$f = \sum_{j=1}^M a_j \psi_{I_j}.$$  

We recall the relation of the parameters: for a fixed $p$ and $s$, we have $1/\tau = 1/p + s/d$, and $1/q = 1 + s/d$. Then

$$M_{s,q}f(x) = \left( \sum_{j=1}^M |I_j|^{-\frac{s}{p}} |a_j|^q \chi_{I_j}(x) \right)^{\frac{1}{q}}.$$  

Therefore, since the $I_j$ are disjoint,

$$|f|_{F_{p,q}} = \left( \int_{\mathbb{R}} \left( \sum_{j=1}^M |I_j|^{-\frac{s}{p}} |a_j|^q \chi_{I_j}(x) \right)^{\frac{1}{q}} \right)^{\frac{1}{p}}$$

$$= \left( \sum_{j=1}^M |I_j|^{-\frac{s}{q}} |a_j|^\tau dx \right)^{\frac{1}{q}}$$

$$= \left( \sum_{j=1}^M |a_j|^\tau |I_j|^{\frac{p}{q}} \right)^{\frac{1}{q}},$$

the final step coming from the observation that $1 - s\tau/d = \tau/p$.

Next, recall that $m_{s,q,I_j} := M_{s,q,I_j} = |I_j|^{-\frac{s}{q}} |a_j|$, and by definition $f_{I_j} = a_j$. Then the cost of the cube $I_j$ is

$$c_{I_j} = \frac{1}{\left( \sum_{m=1}^M |a_m|^\tau |I_m|^{\frac{p}{q}} \right)^{\frac{1}{q}}} |I_j|^{-\frac{s}{q}(\tau-q)} |a_j|^{\tau-q} |a_j|^q |I_j|^q N$$

$$= \frac{|a_j|^{\tau} |I_j|^{\frac{p}{q}}}{\left( \sum_{m=1}^M |a_m|^\tau |I_m|^{\frac{p}{q}} \right)^{\frac{1}{q}}} N. \quad (11)$$

Recalling that $N_{I_j} = \lfloor c_{I_j} \rfloor$ if the right-hand side is at least $N_0$, and $N_{I_j} = 0$ otherwise, we have that

$$S_f = \sum_{j=1}^M a_j T_{N_{I_j}} \psi_{I_j}.$$  

9.2. Limitations. Here, we present an inherent limit to this scheme. It should not be surprising based on the discussion above, but the cost distribution scheme can fail to capture enough information on a signal whose wavelet expansion is very spread out. Let us suppose that $N$ is fixed, and arbitrarily large. We will exhibit a signal whose $N$–term approximant $S_{f,N}$ is identically 0 because the cost of each dyadic cube is 0.

Let $M > N^\tau$, and let $(I_j)_{j=1}^M$ be disjointly supported cubes of unit volume, and define $f = \sum_{j=1}^M \psi_{I_j}$. By $(11)$, we find that for each $1 \leq j \leq M$,

$$c_{I_j} = M^{-\tau} N < 1,$$

whereby, $N_{I_j} = 0$ for every $j$.  

REGULAR FAMILIES OF KERNELS FOR NONLINEAR APPROXIMATION

10. Sobolev Interpolation Using Cardinal Functions

Let us now look more closely at the example of using cardinal functions as the family of kernels. Due to some existing theory, the use of cardinal functions allows us to obtain error estimates for the classical Sobolev smoothness spaces which are not included in the framework above due to the restrictions on the smoothness $s$ of the Triebel–Lizorkin space. The method here is one of interpolation, whereby more details may be found in the cardinal interpolation literature mentioned previously.

To begin, formally define an alternate approximant of $f$ via

$$I_h^k f(x) := \sum_{j \in \mathbb{Z}^d} f(hj) L_{\alpha, \tau(h)} \left( \frac{x}{h} - j \right),$$

where $L_{\alpha, \tau(h)}$ is one of the cardinal functions discussed in Section 6.4 and $\tau(h) = h^2$ in the case of the Gaussian, and $h^{-1}$ for the multiquadrics. Note that this is different from $T_h^k f$ due to the fact that we use the samples of $f$ at the lattice $h\mathbb{Z}^d$ in the approximant rather than the values of $f_{L_n}$ as defined previously. This object has been studied in various instances before [5,15,16], and is actually an interpolant of $f$. By definition of the cardinal functions, it is easy to see that $I_h^k f(hk) = f(hk), k \in \mathbb{Z}^d$.

It is known that for certain classes of Sobolev functions, these interpolants exhibit nice approximation rates. For example, the following holds.

**Theorem 10.1** [10, Theorem 2.1 and 15, Theorem 3.1]. Let $L_\alpha$ be the cardinal function associated with the Gaussian or the general multiquadrics. If $1 < p < \infty$, and $k > d/p$, then there exists a constant $C$, independent of $h$, such that for every $f \in W^k_p(\mathbb{R}^d)$,

$$\|I_h^k f - f\|_{L_p} \leq Ch^k \|f\|_{W^k_p}.$$

If $p = 1, \infty$, then the bound changes to $C(1 + |\ln h|) h^k \|f\|_{W^k_1}$.

With theorem 10.1 in mind, one would naturally desire some estimate for a related interpolant which makes use of only finitely many samples of the function. While there are many feasible ways to do this, we focus here on a method similar to the preceding analysis; to wit, let

$$I_N f(x) := \sum_{j \in \mathbb{Z}^d \cap B_h} f(hj) L_{\alpha, \tau(h)} \left( \frac{x}{h} - j \right),$$

where $B_h$ is the ball of radius $h^{-2}$ centered about the origin. Similarly, let $I_N f$ be as in [3] where $N \sim h^{-2d}$.

For arbitrary Sobolev functions, it is difficult to ascertain the behavior of the truncated interpolant; however, if one assumes a certain asymptotic decay rate on the function itself, then something may be said. Define $\mathcal{K}_\rho$ to be the class of functions on $\mathbb{R}^d$ with decay $O(|x|^{-\rho}), |x| \to \infty$. Then the following holds.

**Theorem 10.2.** Let $1 < p < \infty$, $\kappa > d$, and $k > d/p$. There exists a constant $C$, independent of $h$, such that for every $f \in W^k_p(\mathbb{R}^d) \cap \mathcal{K}_\rho$,

$$\|I_h^k f - f\|_{L_p} \leq Ch^\rho \|f\|_{W^k_p},$$

where $\rho = \min\{k, \kappa - 2d + d/p\}$ for multiquadric interpolation, and $\rho = \min\{k, \kappa - d\}$ for Gaussian interpolation.

Before supplying the complete proof, we collect some useful lemmas.

**Lemma 10.3.** If $f \in \mathcal{K}_\rho$, then

$$\sum_{|j| > h^{-2}} |f(hj)| \leq C h^{\kappa - 2d}.$$

**Proof.** By the assumption on the decay of $f$ and the same estimate as in the proof of Lemma 3.2 we find that

$$\sum_{|j| > h^{-2}} |f(hj)| \leq C \sum_{|j| > h^{-2}} \frac{1}{|j|^\kappa} h^\kappa \leq C_d h^{-\kappa} \int_{h^{-2}}^\infty r^{d-1-\kappa} dr = C_d \kappa h^{-\kappa} h^{-2d+2\kappa},$$

which is at most $C h^{\kappa - 2d}$ as required. \(\square\)
Lemma 10.4. Let $1 < p < \infty$. If $L_{\alpha, h^2}$ is the cardinal function associated with the general multiquadric, then
\[
\|L_{\alpha, h^2}(\cdot/h)\|_{L^p} \leq C h^d/p,
\]
whereas if $L_{\alpha,h^2}$ is that associated with the Gaussian, then
\[
\|L_{\alpha,h^2}(\cdot/h)\|_{L^p} \leq C h^d.
\]

Proof. Let us begin with the multiquadric case. From [15, Corollary 5.3], we see that $|L_{\alpha, h^2}(x/h)| \leq C_{\alpha,d} \min \{1, h^d|x|^{-d}\}$ (we also note that the estimate may be obtained in a straightforward manner from Section 4 of [12]). Therefore, the $p$-th power of the $L_p$ norm in question is bounded by
\[
\int_{B(0,h)} dx + \int_{\mathbb{R}^d \setminus B(0,h)} \frac{h^{dp}}{|x|^{dp}} dx \leq C h^d + C h^d \int_h^\infty r^{-dp}r^{-1} dr,
\]
which is at most $C_{\alpha,p,d} h^d$.

For the Gaussian case, one need only notice that the multivariate Gaussian cardinal function is nothing but the $d$-fold tensor product of the univariate version. Consequently, we may use the bound of [16, Eq. 4.4], which says that for the univariate Gaussian cardinal function, $L_{\alpha,h^2}(x/h) \leq C \min \{h, h|x|^{-1}\}$. Consequently,
\[
\|L_{\alpha,h^2}(\cdot/h)\|_{L^p(\mathbb{R})}^p \leq C h^p \int_{-h}^h dx + C h^p \int_h^\infty \frac{1}{|x|^p} dx \leq C h^p.
\]
Thus the multivariate estimate is $C h^{dp}$, whence taking $p$-th roots gives the desired inequality.

With these ingredients in hand, we are now ready to supply the proof of the theorem.

Proof of Theorem 10.2. First note that $\|I_{\alpha} f - f\|_{L^p} \leq \|I_{\alpha} f - f\|_{L^p} + \|I_{\alpha} f - I_{\alpha} f\|_{L^p}$, and the first term is majorized by $C h^k$ by Theorem 10.1. Now to estimate the second term, it follows from Minkowski’s integral inequality [12, Theorem 6.19, p. 194] that
\[
\|I_{\alpha} f - I_{\alpha} f\|_{L^p} = \left( \int_{\mathbb{R}^d} \left| \sum_{|j| > h} f(hj)L_{\alpha,\tau(h)} \left( \frac{x}{h} - j \right) \right|^p dx \right)^{1/p} \leq \sum_{|j| > h} \left( \int_{\mathbb{R}^d} |f(hj)|^p \left| L_{\alpha,\tau(h)} \left( \frac{x}{h} - j \right) \right|^p dx \right)^{1/p} = \|f(hj)\|_{L^p(\mathbb{R})} \left| L_{\alpha,\tau(h)}(\cdot/h)\right|_{L^p}.
\]
From Lemmas 10.3 and 10.4, we see that for the multiquadric, $\|I_{\alpha} f - I_{\alpha} f\|_{L^p} \leq C h^{n-2d+d/p}$, while the bound for the Gaussian is $C h^{n-d}$, thus completing the proof.

Corollary 10.5. With the parameters as in Theorem 10.3 if $N \sim h^{-2d}$, then for every $f \in W^k_p(\mathbb{R}^d) \cap \mathcal{S}$,$$
\|I_N f - f\|_{L^p} \leq C N^{-p/(2d)} \|f\|_{W^k_p}. $$
We conclude this section by noting that the $N$–term interpolants considered here work well for Sobolev functions which decay away from the origin. Of course, if a given function peaks far away from the origin but still decays away from the peak, one should translate the peak to the origin and then interpolate, in which case the same estimate as in Corollary 10.5 holds for $I_N (f) - f$.

It follows easily from the lemmas above that one can still estimate the difference of the full and truncated interpolants in $L_1$ and $L_\infty$, and get analogous bounds in terms of $h$. However, the method of proof of Theorem 10.2 is not sufficient to estimate approximation orders of $I_{\alpha} f - f$ in these spaces due to the logarithmic term found in Theorem 10.1. Moreover, it has proven elusive to estimate the truncated interpolant via other means.
As mentioned above, it would be interesting for future work to consider other $N$-term interpolation methods involving cardinal functions than the one here. Of particular note would be the so-called greedy interpolant of $f$, which for a given lattice would be formed by keeping only the $N$ largest (in absolute value) samples $f(hj)$. That is, $G_N f(x) = \sum_{j\in \Lambda} f(hj) L_{\alpha,\tau(h)}(x/h-j)$, where $|f(hj)| \geq |f(hk)|$, $j \in \Lambda$, $k \notin \Lambda$, and $|\Lambda| = N$.

11. Computational Examples

To conclude our discussion, we present some numerical examples of the implementation of the approximation method discussed in Section 4. The first task is to estimate how well the given nonlinear approximation methods discussed above perform when approximating a given mother wavelet in the class $H_B$. To wit, we consider the classical Meyer wavelet \[21\], whose Fourier transform may be given by the following [8, p. 117]

\[
\hat{\psi}(\xi) := \begin{cases} 
\sin \left( \frac{3}{2} \nu \left( \frac{3\xi}{\pi} - 1 \right) \right) e^{i\frac{\xi}{3}}, & \frac{2\pi}{3} \leq |\xi| \leq \frac{4\pi}{3}, \\
\cos \left( \frac{2}{3} \nu \left( \frac{3\xi}{4\pi} - 1 \right) \right) e^{i\frac{\xi}{2}}, & \frac{2\pi}{3} \leq |\xi| \leq \frac{8\pi}{3}, \\
0, & \text{otherwise,}
\end{cases}
\]

where $\nu$ is any function (of finite or infinite smoothness) satisfying $\nu(x) = 0$ for $x \leq 0$ and $\nu(x) = 1$ for $x > 1$ with $\nu(x) + \nu(1-x) = 1$ for all $x \in \mathbb{R}$. The smoothness of the mother wavelet $\psi$ is determined by the smoothness of $\nu$.

For practical purposes of experimentation, we choose the simple case $\nu(x) = x$ for $x \in [0, 1]$, and defined as above outside that range. However we note that in Matlab’s implementation, stemming from the discussion in [8, p. 119], the function used is $\nu(x) = x^4(35 - 84x + 70x^2 - 20x^3)$. For our choice of $\nu$, a tedious, but nonetheless straightforward calculus exercise yields the following closed-form expression for the mother wavelet in the time domain which was used in the implementation:

\[
\psi(x) = \frac{\sin \left( \frac{2\pi}{3} (x + \frac{1}{3}) \right) - \sin \left( \frac{2\pi}{3} (x + \frac{1}{3}) \right)}{2\pi(x + \frac{1}{3})} + \frac{\sin \left( \frac{2\pi}{3} (x - \frac{1}{3}) \right) - \sin \left( \frac{2\pi}{3} (x - \frac{1}{3}) \right)}{2\pi(x - \frac{1}{3})} + \frac{\cos \left( \frac{2\pi}{3} (x + \frac{1}{3}) \right) - \cos \left( \frac{2\pi}{3} (x + \frac{1}{3}) \right)}{2\pi(x + \frac{1}{3})} + \frac{\cos \left( \frac{2\pi}{3} (x + \frac{1}{3}) \right) - \cos \left( \frac{2\pi}{3} (x + \frac{1}{3}) \right)}{2\pi(x + \frac{1}{3})}.
\]

11.1. Approximation of the Meyer wavelet with decaying kernels. To estimate the base performance of the approximation algorithm, many experiments were performed for a variety of kernels with different parameters with two aims: to demonstrate the theoretical approximation results as in Proposition 5.1 and to give some evidence for what parameters to choose, though in general the latter is a rather difficult task. As an example, Figure 11 shows an example of the $N$-term Gaussian approximation (with $\alpha = 0.2$) of the Meyer mother wavelet for small and large budgets $N = 10$ and 50, respectively.

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$N$ & $\alpha = .5$ & $\alpha = .4$ & $\alpha = .3$ & $\alpha = .2$ & $\alpha = .5$ & $\alpha = .4$ & $\alpha = .3$ & $\alpha = .2$ \\
\hline
10 & .25987 & .10465 & .11326 & .30130 & .18277 & .07657 & .04877 & .20808 \\
20 & .06592 & .03001 & .02259 & .0427 & .05831 & .01885 & .01408 & .01408 \\
30 & .06410 & .0142 & .00469 & .00397 & .05932 & .01863 & .00855 & .00511 \\
40 & .00587 & .00072 & .6824-10^{-6} & .2915-10^{-6} & .01575 & .00331 & .00056 & 1.870-10^{-5} \\
50 & 2.268-10^{-5} & 3.248-10^{-6} & 1.960-10^{-6} & 1.886-10^{-6} & 9.671-10^{-7} & 2.569-10^{-6} & 1.954-10^{-6} & 2.232-10^{-6} \\
\hline
\end{tabular}
\caption{L_1 (left) and L_\infty (right) error for Gaussian approximation of $\psi$ for different $\alpha$ and $N$.}
\end{table}
Figure 1. Gaussian approximant of $\psi$ for $N = 10$ (left) and $N = 50$ (right) with parameter $\alpha = 0.2$.

Table 2. $L_1$ (left) and $L_\infty$ (right) error for multiquadric approximation of $\psi$ for different $\alpha$ and $N$.

| $N$ | $\alpha = -1.5$ | $\alpha = -3.5$ | $\alpha = -5.5$ | $\alpha = -7.5$ | $\alpha = -1.5$ | $\alpha = -3.5$ | $\alpha = -5.5$ | $\alpha = -7.5$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 10  | 2.9961          | .29455          | .13485          | .11067          | 1.0390          | .18572          | .09012          | .06405          |
| 20  | .66997          | .07469          | .03662          | .02795          | .33928          | .05469          | .02334          | .01460          |
| 30  | .70073          | .06446          | .01999          | .00979          | .34522          | .05479          | .02306          | .01426          |
| 40  | .19996          | .00949          | .00176          | .00055          | .14652          | .01594          | .00510          | .00228          |
| 50  | .06267          | .00062          | $2.827 \cdot 10^{-5}$ | 4.869$\cdot 10^{-6}$ | .03918          | .00108          | 7.377$\cdot 10^{-5}$ | 7.232$\cdot 10^{-6}$ |

Figure 2 shows the Gaussian approximation error for a fixed budget of $N = 50$ centers and varying parameters $\alpha$. In theory, one expects to see the error diminish as the value of $\alpha$ gets smaller; however, likely due to floating point calculation error, for $\alpha$ smaller than approximately 0.2, the error begins to increase.

Figure 2. Log of $L_1$ error for Gaussian approximant of $\psi$ for $N = 50$ with different $\alpha$ values.

One observes a similar phenomenon for inverse multiquadric approximation whenever the exponent $\alpha$ is sufficiently negative. In theory, as $\alpha$ becomes more negative, the error should decrease for a fixed budget.
It appears that in the range $\alpha \in (-20, -8)$, the error levels out rather than decreasing, and beyond $-20$, the error begins to increase, likely due to rounding error.

11.2. Approximation of disjoint wavelet expansions. As a second experiment, we consider the performance and approximation order for some functions which have a disjoint wavelet expansion. First, we consider a fixed 7–term wavelet, $f = a_1 \psi(2^{j_1} (\cdot + 3)) + a_2 \psi(2^{j_2} (\cdot + 2)) + \cdots + a_7 \psi(2^{j_7} (\cdot - 3))$ where the coefficients $a_1, \ldots, a_7$ were randomly generated from the uniform distribution on $(0, 10)$, and the dilations were randomly generated between 0 and 4. The coefficients produced were $(9.45, 4.91, 4.89, 3.38, 9.00, 3.69, 1.11)$ and the powers of the dilations were $(3, 1, 1, 2, 0, 0, 4)$. The number of terms is of course arbitrary, but we consider the approximation of $f$ based on the cost distribution set out in (11) (noting that the intervals are disjoint). Figure 3 shows the $L_\infty$ error of the $N$–term approximation via Gaussians with $\alpha = .2$ for this fixed $f$.

![Figure 3. $L_\infty$ error vs. Budget for Gaussian approximation with $\alpha = .2$ of fixed $f$ with 7–term sparse wavelet expansion.](image)

Additionally, Table 3 shows the error as the budget increases as well as showing the cost distribution.

| $N$ | $L_\infty$ Error | Cost $(c_1, \ldots, c_7)$ |
|-----|------------------|-----------------------------|
| 100 | 6.660            | (54, 7, 7, 10, 6, 3, 13)    |
| 200 | 3.215            | (109, 14, 14, 19, 13, 5, 26)|
| 300 | 1.020            | (180, 23, 23, 32, 21, 9, 11)|
| 400 | .4470            | (240, 31, 31, 43, 29, 12, 14)|

Table 3. $L_\infty$ Error for Gaussian approximant with $\alpha = .2$ of fixed $f$ with sparse wavelet representation.

Note that the error does decrease as expected as $N$ increases, however the rate is rather slow (indeed, the slope on a semi-log scale is approximately -.009). One also notices that the large coefficient and large dilation on the first term cause it to dominate the cost distribution, which is to be expected.

Finally, we would like to see what happens on average as the budget increases for sparse wavelet expansions such as the function $f$ just considered. Thus, we end with an experiment of approximating 5–term wavelet expansions whose shifts are randomly generated between 1 and 20, and whose coefficients and dilations are generated randomly as in the previous experiment. Figure 4 shows the average $L_\infty$ error of $N$–term Gaussian approximation ($\alpha = .2$); the error bars capture the standard error based on 10 trials for each $N$. The error does seem to decrease (as does the variance); however, the rate is again slow when the coefficients are allowed to vary.
11.3. Conclusion. While the theoretical error rates for approximating the mother wavelet are demonstrated with these experiments, there are evidently drawbacks for the proposed reconstruction. Nonetheless, the purpose of this work was primarily to demonstrate the theoretical approximation rates for the nonlinear $N$-term approximation spaces related to a variety of kernels considered above, and to support in some way the theory with computational experiments. For the future, it would be interesting to consider other approximations from the spaces $\Phi_N$ above which may obtain the optimal recovery rates and which are more readily implemented numerically.

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