New application of mesh-free particle method to geotechnical engineering

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ABSTRACT

The smoothed particle hydrodynamics SPH method, which is based on the mesh-free Lagrangian scheme, is commonly used to solve large deformation problems in geotechnical engineering. The method can be used to solve large deformation problems without distortion of the mesh. Moreover, it can handle the governing equations and constitutive models for geomaterials based on continuum mechanics. However, numerical inaccuracies are a problem with the conventional SPH method. In this paper, we reported an improved method, Symmetric SPH (SSPH). Validations of simulations of simple shear problems in elastic and elasto-plastic materials were carried out, and the SSPH method demonstrated significant improvements in numerical accuracy compared with the SPH method. Furthermore, excavation problems with loose to dense sand were analyzed to examine the applications of the SSPH method.

Keywords: mesh-free method, constitutive model, excavation

1. INTRODUCTION

Recently, the frequency of geo-disasters, including embankment failures and slope failures due to the heavy rainfall or earthquakes, has increased. Many of these disasters exhibit the large deformation behavior. The smoothed particle hydrodynamics (SPH) method has been used to solve large deformation problems of geotechnical structures. The SPH method is a particle method based on the mesh-free Lagrangian scheme, which can be used to solve large deformation problems without distortion of the mesh. Moreover, it can handle governing equations and existing constitutive models for geomaterials based on continuum mechanics; however, it is generally said that inaccuracies in analytical method are a problem with this conventional method.

Here, we describe a comparison between the symmetric SPH (SSPH) method (Batra & Zhang, 2008) and the conventional SPH method in terms of numerical accuracy. Simple shear problems of elastic and elasto-plastic materials were simulated using both methods. The Green-Nagdhi stress rate was used as the objectivity. To implement the elasto-plastic analysis, the SYS Cam-clay model (Asaoka et al., 2002) was used in both mesh-free methods. Excavation problems with loose to dense sand were analyzed to discuss the application of the SSPH method.

2 NUMERICAL METHOD

2.1 Conventional SPH

With the SPH method, an object is described as an assembly of particles. If the motion of each particle is solved individually, however, the deformation behavior of the continuum cannot be accurately represented using this technique. To treat an object as a continuum, an interpolation theory is required. The interpolation theory includes a kernel approximation and a particle approximation. The first step is the kernel approximation of the field functions, which is based on neighboring particles $\beta$ located at points $x^\beta$ within the support domain $k^\alpha h$ of a smoothing function $W$ for a reference particle $\alpha$ located at point $x^\alpha$. In the first step of the interpolation, we define the smoothed physical quantity $<f(x^\alpha)>$ to describe the physical quantity $f(x^\alpha)$ at the reference particle $\alpha$ as follows:

$$<f(x^\alpha)> = \int_{\Omega} f(x^\beta) W(r, h) dx^\beta$$

(1)

where $r = |x^\alpha - x^\beta|$, $h$ is the radius of the influence domain, and $\Omega$ is the volume of the integral that contains $x^\alpha$ and $x^\beta$.

In the second step of the interpolation, the physical quantity $<f(x^\alpha)>$ for reference particle $\alpha$ is expressed as the summation of the distributions of the assumed physical quantities $f(x^\beta)$ for each particle. Thus, the physical quantity can be expressed in terms of $N$
discrete points, i.e.,

\[
\langle f(x^s) \rangle = \sum_{\beta}^{N} \frac{m^\beta}{\rho^\beta} f(x^\beta) W_{\alpha \beta}^{\alpha \beta}
\]

(2)

where \( m^\beta \) is the mass, and \( \rho^\beta \) is the density of neighboring particles \( \beta \). \( N \) is the number of neighboring particles in the support domain, and \( W_{\alpha \beta}^{\alpha \beta} \) is the smoothing function used to express the contribution from the neighboring particles \( \beta \) to the reference particle \( \alpha \).

Equation (2) can be used to evaluate a physical quantity using the SPH method. It is also possible to approximate the spatial gradient of a physical quantity in a similar manner, using the spatial derivative of the smoothing function. The spatial derivative of Eq. (2) can be expressed as

\[
\frac{\partial}{\partial x_i} \langle f(x^s) \rangle = \sum_{\beta}^{N} \frac{m^\beta}{\rho^\beta} \frac{\partial f(x^\beta)}{\partial x_i} W_{\alpha \beta}^{\alpha \beta}
\]

(3)

Based on the two-step interpolation procedure, it is possible to calculate any physical quantity as well as the special derivative thereof.

2.2 SSPH method

The SSPH method was proposed by Batra and Zhang (2008). This method is formulated taking into account the higher-order terms based on a Taylor series expansion. The spatial gradient of the physical quantity is obtained via the SSPH method as follows:

\[
Q = K^{-1} T
\]

(4)

where \( Q \) is given by

\[
Q = \begin{pmatrix}
\frac{\partial f(x^s)}{\partial x_i}
\frac{\partial^2 f(x^s)}{\partial x_i \partial x_j}
\frac{\partial^2 f(x^s)}{\partial x_j \partial x_k}
\end{pmatrix}
\]

(5)

\( K \) is given by

\[
K = \begin{pmatrix}
X_1^3 \tilde{W} & X_1^1 X_3 \tilde{W} & X_1^2 X_3 \tilde{W} & X_1^3 \tilde{X}_3 \tilde{W} \\
X_1 X_3 \tilde{W} & X_2 X_3 \tilde{W} & X_2^2 \tilde{W} & X_2 X_3 \tilde{W} \\
X_1 X_2 \tilde{W} & X_3 X_2 \tilde{W} & X_3^2 \tilde{W} & X_3 X_2 \tilde{W} \\
X_1 X_2 \tilde{W} & X_1 X_2 \tilde{W} & X_1 X_3 \tilde{W} & X_1 X_3 \tilde{W}
\end{pmatrix}
\]

(6)

and \( T \) is given by

\[
T = \begin{pmatrix}
(f(x^s) - f(x^\beta)) X_i \tilde{W}
(f(x^s) - f(x^\beta)) X_i \tilde{W}
(f(x^s) - f(x^\beta)) X_i \tilde{W}
(f(x^s) - f(x^\beta)) X_i \tilde{W}
\end{pmatrix}
\]

(7)

and where \( \tilde{W} \), \( X_1 \) and \( X_2 \) are given by

\[
\tilde{W} = \sum_{\beta=1}^{N} \frac{m^\beta}{\rho^\beta} W_{\alpha \beta}^{\alpha \beta}
\]

(8)

3 COMPARISON WITH THE SPH METHOD

3.1 Simple shear simulations of an elastic material

To validate the numerical scheme introduced to describe the Green-Naghdi stress rate, we simulated a simple shear test of an elastic material, with a shear modulus of \( G=10.0 \text{ Pa} \), Poisson's ratio of \( \nu=0.30 \) under plane strain conditions using the SPH and the SSPH methods. The stress strain relations obtained were compared with the theoretical solution at the center of specimen, as well as the distribution of deformation gradient tensor \( F_{11} \).

Fig. 1 shows an illustration of the numerical model. The specimen was a square object (10 cm by 10 cm), and the calculation points located on the periphery of the specimen were deformed using a constant displacement to represent simple shear conditions.

Fig. 2 shows the resulting shear stress- strain relation at the center of the specimen, and Fig. 3 shows the distribution of deformation gradient tensor \( F_{11} \) for various shear strains. In this simple shear problem of elastic material, the component of the deformation gradient tensor is already known, for example, we have \( F_{11} = 1.0 \).

With the SPH method, the simulated result was in poor agreement with theoretical result at shear strains greater than 50%. As shown in Fig. 3, it was not possible to accurately calculate the deformation in the vicinity of the boundary. In contrast, however, with the SSPH method, the simulated result was in good agreement with the theoretical solution for strains of up to 200%. Additionally, a uniform distribution of the deformation gradient tensor was found.
increased, the deviator stress \( q \) was not consistent with the theoretical solution particularly with cases 2 and 3. This is because of limitations in the accuracy of the calculations at the boundaries. The simulated results obtained using the SSPH method were in good agreement with the theoretical solutions. Furthermore, the shear stress-strain relation, the anisotropy-strain relation, the \( R \)-strain relation and the \( R^* \)-strain relation at the center of the specimen were in good agreement with the theoretical solution using the both methods. Although constant-volume conditions were assumed, it was possible to simulate the shear behaviors of loose and dense sand using the SSPH method, shown in the previous research (Nakai 2005).

Fig. 4 and 5 show the (a) stress paths, (b) \( p \)-specific volume relation, (c) shear stress-strain relation, (d) anisotropy-strain relation, (e) \( R \)-strain relation and (f) \( R^* \)-strain relation using both methods. The simulated results and the theoretical solution are compared in these figures. With the SPH method, as the strain is increased, the deviator stress \( q \) was not consistent with the theoretical solution particularly with cases 2 and 3. This is because of limitations in the accuracy of the calculations at the boundaries. The simulated results obtained using the SSPH method were in good agreement with the theoretical solutions. Furthermore, the shear stress-strain relation, the anisotropy-strain relation, the \( R \)-strain relation and the \( R^* \)-strain relation at the center of the specimen were in good agreement with the theoretical solution using the both methods. Although constant-volume conditions were assumed, it was possible to simulate the shear behaviors of loose and dense sand using the SSPH method, shown in the previous research (Nakai 2005).

### 3.2 Simple shear simulations of an elasto-plastic material

We simulated a simple shear test under plane strain conditions is carried out using the SYS Cam-clay model (Asaoka et al., 2002). The stress-strain relation, and stress paths were obtained and compared with the theoretical solution at the center of specimen. The parameters of the sand under constant mean stress and three different values of the initial degree of overconsolidation, structure, anisotropy and the initial specific volume were used, as listed in Tables 1 and 2. Three different cases were investigated: [1] loose sand, [2] medium dense sand, [3] very dense sand.

![Distribution of deformation gradient tensor](image)

**Fig. 3.** Distribution of deformation gradient tensor \( F_{11} \) with different shear strains.

| Table 1. The material parameters used in the simulations. |
|-----------------------------------------------|
| **Elasto-plastic parameters**                  |
| Compression index  \( \lambda \)  0.050         |
| Swelling index  \( k \)  0.012                  |
| Critical state constant \( M \)  1.0            |
| NCL intercept \( N \)  1.98                    |
| Poisson’s ratio \( \nu \)  0.3                  |
| **Evolution parameters**                       |
| Degradation index of overconsolidation \( m \) 0.06 |
| Degradation index of structure \( a \)  2.2     |
| Degradation index of structure \( b_c \)  1.0   |
| Evolution index of rotational hardening \( b \) 3.5 |
| Limit of rotational hardening \( m_b \)  0.7   |

| Table 2. The initial values for the simulations. |
|-----------------------------------------------|
| Case | 1 | 2 | 3 |
| Initial degree of overconsolidation \( 1/R_0 \) 1.25 | 4.23 | 6.58 |
| Initial degree of structure \( 1/R_0^* \) 73.73 | 2.85 | 2.01 |
| Initial specific volume \( v_0 \) 2.08 | 1.91 | 1.88 |
| Initial mean stress \( p_0 \) 294.3             |
| Initial degree of anisotropy \( \gamma_0 \) 0.01 | 0.23 | 0.30 |

Fig. 4 and 5 show the (a) stress paths, (b) \( p \)-specific volume relation, (c) shear stress-strain relation, (d) anisotropy-strain relation, (e) \( R \)-strain relation and (f) \( R^* \)-strain relation using both methods. The simulated results and the theoretical solution are compared in these figures. With the SPH method, as the strain is increased, the deviator stress \( q \) was not consistent with the theoretical solution particularly with cases 2 and 3. This is because of limitations in the accuracy of the calculations at the boundaries. The simulated results obtained using the SSPH method were in good agreement with the theoretical solutions. Furthermore, the shear stress-strain relation, the anisotropy-strain relation, the \( R \)-strain relation and the \( R^* \)-strain relation at the center of the specimen were in good agreement with the theoretical solution using the both methods. Although constant-volume conditions were assumed, it was possible to simulate the shear behaviors of loose and dense sand using the SSPH method, shown in the previous research (Nakai 2005).

### 4 EXCAVATION ANALYSIS

An excavation analysis was carried out for loose to dense sand using the SSPH method and considering the soil skeleton structure of the soil. The material parameters and initial values of the evolution rules determined by the simple shear simulation in the previous section were used. The initial depth of the initial was 9 m, and the width was 20 m. Following excavation, the slope height was 4.5 m, and the slope angle was 1H: 2V, as shown in Fig. 6. The time histories of \( R \) and \( R^* \) at point A were calculated. The
ground was initially homogeneous. The overconsolidation ratio was calculated from the structure, anisotropy, and specific volumes. The initial values of the evolution rules used in this simulation are listed in Table 3. The isotropic stress corresponding to static earth pressure was used as the initial stress. For the boundary conditions, the horizontal direction at the side wall of the ground is fixed, and the vertical direction is free. The horizontal and vertical directions at the bottom of the ground are fixed. Fixed boundary particles are used to describe the walls. A static analysis was carried out until the ground was stable, and then particles were removed instantly in the excavation area as shown in Fig. 6.

Fig. 6. Numerical models.

Table 3. The initial values used in the simulation shown in Fig. 6.

| Case | 1 | 2 | 3 |
|------|---|---|---|
| Initial degree of overconsolidation $1/R_e$ | 5.30 | 17.92 | 167.71 |
| Initial degree of structure $1/R_e^*$ | 73.62 | 2.84 | 1.13 |
| Initial specific volume $v_0$ | 2.08 | 1.91 | 1.79 |

Fig. 7 shows the distribution of the maximum shear strain at different times for cases 1, 2 and 3. Fig. 8 shows the time histories of $R^*$ and $R$ at point A following the excavation. The time of failure was different in each case as shown in Fig. 7. With case 1, slope failure occurred immediately following excavation. With case 2, a slope failure occurred after some delay following the excavation. With case 3, slope failure did not occur.

Loose sand ($R^* \to 1.0$) can decay more easily that dense sand. Furthermore, the ground was highly structured soil as shown in Fig. 7(a), and loss of overconsolidation ($R \to 1.0$) occurred following the decay of the structure ($R^* \to 1.0$), as shown in Fig. 8(b). Slope failure occurred because the state of the ground changed from overconsolidation to a normal consolidated state. With case 1, the result of focusing on the time history of $R$ at the failure point (Point A), the ground reached the normal consolidated state. With case 2, the time of failure was later than with case 1 because the structure was less likely to decay than loose sand would be. As with case 1, the ground reached the normal consolidated state at the failure point (Point A). With case 3, because the decay of the structure did not occur following excavation, the ground remained in the overconsolidation state, and slope failure did not occur.

Fig. 7. Distribution of the accumulated maximum shear strain at different times. The boundary particles used at the walls are not shown.

Fig. 8. Time history of (a) $R^*$ and (b) $R$ at point A.

5 CONCLUSIONS

We have used the SSPH method to develop numerical simulation to describe the large deformation behavior of geomaterial, and the results were compared with simulations using the SPH method. The main findings can be summarized as follows.

1) Simulated results using the SSPH method were in good agreement with the theoretical solution. In contrast, the accuracy in the vicinity of the boundary that calculation point was insufficient is confirmed in the SPH method.

2) We simulated an excavation, describing the deformation behavior for loose to dense sand. We found that the differences in the behavior of the geomaterials were due to the differences in the development of the structure and overconsolidation.

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