The Casimir Effect in the Presence of Compactified Universal Extra Dimensions

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The Casimir force in a system consisting of two parallel conducting plates in the presence of compactified universal extra dimensions (UXD) is analyzed. The Casimir force with UXDs differs from the force obtained without extra dimensions. A new power law for the Casimir force is derived. By comparison to experimental data the size $R$ of the universal extra dimensions can be restricted to $R \leq 10 \text{ nm}$ for one extra dimension.

The idea that our world has more than three spatial dimensions has been discussed for more than 80 years. Already Kaluza and Klein [1, 2] postulated an additional compactified dimension to unify gravity and classical electrodynamics. Today, there is a large variety of promising models and theories which suggest the existence of more than three spatial dimensions. Most notably, string theory [3, 4] suggests the existence of seven additional spatial dimensions. In string theory (and also in older approaches) one expects that the compactification scale of the extra dimensions is of order $M_{\text{Planck}} \sim 10^{19}$ GeV. Thus, observable effects are shifted into an energy domain out of reach of today’s and near future experimental possibilities.

However, recently models with compactification radii up to the mm-scale and with a lowered Planck Mass (TeV region) have been introduced [5, 6, 7, 8, 9, 10, 11, 12]. In the ADD [6] and RS [10] type of models, the hierarchy problem is solved or reformulated, resp. in a geometric language. Here, the existence of $d$ compactified large extra dimensions (LXDs), because the size of the additional dimensions can be as large as $100-1000 \mu\text{m}$ in which only the gravitons can propagate is assumed. The standard-model particles however, are bound to our 4-dimensional submanifold, often called our 3-brane.

This obvious asymmetry between standard model fields and gravity has given rise to the introduction of universal extra dimensions (UXDs) in which all particle species are allowed to propagate [5, 6, 12]. In this model the present limit on the size of the extra dimensions is $R \leq (300 \text{ GeV})^{-1} \approx 10^{-9} \text{ nm}$ due to the non-observation of Kaluza-Klein excitations at Tevatron [12, 13, 14, 15].

Especially the Casimir effect has received great attention and has been extensively studied in a wide variety of topics in those and related scenarios:

- The question how vacuum fluctuations affect the stability of extra dimensions has been explored in [16, 17, 18, 19, 20, 21]. Especially the detailed studies in the Randall-Sundrum model have shown the major contribution of the Casimir effect to stabilise the radion [22, 23, 24, 25].

- Cosmological aspects like the cosmological constant as a manifestation of the Casimir energy or effects of Casimir energy during the primordial cosmic inflation have been analyzed [26, 27, 28, 29, 30, 31, 32, 33, 34].

- The Casimir effect in the context of string theory has been investigated in [35, 36, 37, 38].

The commonly known and experimentally accessible Casimir effect [39] has recently gained intense attention. Experimentally [40] the precision of the measurements has been greatly enhanced, while on the theoretical side major progress has been reported (for a review, the reader is referred to [41]). In fact, the Casimir effect has been suggested as an experimentally powerful tool for the investigation of new physics beyond the standard model [42].

In this letter, we scrutinize the Casimir force between two parallel plates to probe the possible existence and size of additional Universal Extra Dimensions (UXDs) [12]. In general and independently from the considered field, zero-point fluctuations of any quantum field give rise to observable Casimir forces if boundaries are present [39]. In this study, the zeta function method as suggested in [43, 44, 45, 46, 47, 48, 49] is applied to renormalize the Casimir energy. For simplicity, we limit ourselves to the investigation of new physics beyond the standard model [42].
UXDs, we also need the boundary condition $k_N = \pi N/a$, where $k_N$ is the wave vector in the directions restricted by the plates, $N$ an integer and $a$ the distance of the plates.

In the case of one extra dimension we find the frequency of the vacuum fluctuations to be

$$\omega_{nN} = \sqrt{k_+^2 + \frac{n^2}{R^2} + \left(\frac{\pi N}{a}\right)^2}, \quad (1)$$

with $k_+ = \sqrt{(k_1^2 + k_2^2)}$. $k_1$ and $k_2$ are the wave vectors in direction of the unbound space coordinates. Therefore the Casimir energy per unit plate area reads

$$\varepsilon_{nr} = 2 \cdot \frac{\hbar c}{4\pi\Gamma(-\frac{3}{2})} \int_0^{\infty} \frac{dx}{x} x^{-3/2} \left[ \sum_{n,N=0}^{+\infty'} p \cdot \omega_{nN} - \sum_{n=0}^{+\infty'} \omega_n \right] \cdot (2)$$

Here, the prime indicates that the term with $n = N = 0$ has to be skipped. The factor 2 arises from the volume of the orbifold, and the factor $p$ from the possible polarizations of the photon ($p = 3$ for one UXD). The modes polarized in the direction of our brane, $n = 0$, cause the additional term which has to be subtracted.

Using the Schwinger representation [50] of the square root and the Gauss integral in 2 dimensions one obtains

$$\varepsilon_{nr} = \frac{\hbar c}{4\pi\Gamma(-\frac{3}{2})} \Gamma\left(-\frac{3}{2}\right) \left[ \sum_{n,N=0}^{+\infty'} p \cdot \omega_{nN} - \sum_{n=0}^{+\infty'} \omega_n \right] \cdot (3)$$

With the Gamma-function [51] this yields

$$\varepsilon_{nr} = \frac{\hbar c}{4\pi\Gamma(-\frac{3}{2})} \left[ \sum_{n,N=0}^{+\infty'} \left( n^2 + \left(\frac{\pi N}{a}\right)^2 \right)^{3/2} - \sum_{n=0}^{+\infty'} \left( \frac{n^2}{R^2} \right)^{3/2} \right] \cdot (4)$$

To use the Epstein zeta function [52] renormalization, the sums are re-written with indices running from $-\infty$ to $+\infty$. For the double sum, one has to take care of the modes where one index equals zero. The resulting energy density reads in terms of the Epstein zeta-function $Z$:

$$\varepsilon_{nr} = \frac{\hbar c}{4\pi\Gamma(-\frac{3}{2})} \left[ \sum_{n,N=0}^{+\infty'} \left( n^2 + \left(\frac{\pi N}{a}\right)^2 \right)^{3/2} - \sum_{n=0}^{+\infty'} \left( \frac{n^2}{R^2} \right)^{3/2} \right] \cdot (5)$$

The second and the third term are due to the modes with one index equal to zero. According to [52], the reflexion relation of the Epstein zeta function yields the renormalized energy density:

$$\varepsilon_{ren} = -\frac{\hbar c}{16\pi^3} \left[ \frac{3}{8\pi} p \cdot R a Z_2 \left( R, \frac{a}{\pi}, 5 \right) \right] + \frac{p^2}{R^3} \zeta(4) + \frac{p}{a^3} \zeta(4) \cdot (6)$$

Note that this quantity is regularized with respect to $3 + 1$-dimensional Minkowski space. To obtain the total energy in the space between the plates, one has to multiply by the surface $A$ of the plates: $E(R, a) = \varepsilon_{ren} \cdot A$.

In an analogous calculation we find the renormalized vacuum energy in the volume between the plates, but plates absent to be

$$E_{UXD}(R) = -\frac{\hbar c}{16\pi^5} \frac{3}{4\pi} p \cdot a \frac{1}{R^3} \zeta(5) \cdot A \cdot (7)$$

The quantity of interest for the Casimir effect in the present setting is the energy difference between the $3 + 1$-dimensional space with plates and $3 + 1$-dimensional space without plates. Thus, in the setting with one compactified universal extra dimension the Casimir energy is

$$E_{cas}(R, a) = E^{\text{Plate}}_{UXD}(R, a) - E_{UXD}(R, a) \cdot (8)$$

The Casimir force is now given by the derivative of the

![Graph showing Casimir forces with different plate separations](image)
Casimir energy $E_{\text{cas}}$ with respect to the plate distance $a$:

$$F_{\text{cas}} = -\frac{\partial E_{\text{cas}}}{\partial a} = \frac{\hbar c}{16\pi^3} p A \left[ \frac{3}{8\pi} R \cdot Z_2 \left( R, \frac{a}{\pi}, 5 \right) \right. \right. - 15 \left. \left. \frac{a^2 R}{8\pi^3} \sum_{n,N=-\infty}^{+\infty'} N^2 \left[ n^2 R^2 + \frac{a^2}{\pi^2} N^2 \right]^{-1} \right] \right] - 3\pi^3\zeta(4) \frac{1}{a}\left[ \frac{3}{4\pi^2} \zeta(5) \frac{1}{R^4} \right]$$

(9)

Now we compare the Casimir force in this modified space-time to data and the normal Casimir force $F$ between parallel plates without extra dimensions, given by

$$F = -\frac{\hbar c\pi^2}{240} \cdot \frac{A}{a^4}.$$  

(10)

It should be noted that the measurement of Casimir forces between parallel plates is experimentally difficult because exact parallelity cannot be obtained easily\(^1\). In spite of this problem, one experiment with relatively high accuracy was done by Sparnaay\(^5\) with chromium plates\(^2\).

Figure 1 depicts the dependence of the Casimir force between two parallel plates on the radius of the extra dimension (dashed and dotted lines) and the distance of the parallel plates. One clearly observes that the data can be reproduced either by a calculation without UXDs (full line) or by a calculation with one UXD of small size. In the present setting with one universal extra dimension, good agreement with the data can only be obtained if the radius of the UXD is smaller than $\sim 10\text{ nm}$. Similar results are obtained for more than one Universal Extra Dimension.

In conclusion, the Casimir force between two conducting plates in the presence of UXDs is studied. For UXD sizes previously discussed in the literature\(^3, 4, 12, 13, 14\), the Casimir force is in line with the measured data. The present study of the Casimir force with one UXD yields an upper limit of $R \leq 10\text{ nm}$ on the extension of UXDs.

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The present experimental results should be handled with care.

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\(^1\) Unfortunately, Sparnaay’s measurement with other metals showed partly repulsive instead of attractive forces, so the present experimental results should be handled with care.

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