Position and momentum operators for a moving particle in bulk

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Received: 21 June 2022 / Accepted: 21 October 2022 / Published online: 1 November 2022
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Abstract In this paper we explore how to describe a bulk moving particle in the dual conformal field theories (CFTs). One aspect of this problem is to construct the dual state of the moving particle. On the other hand one should find the corresponding operators associated with the particle. The dynamics of the particle, i.e., the geodesic equation, can be formulated as a Hamiltonian system with canonical variables. The achievements of our paper are to construct the dual CFT states and the operators corresponding to the canonical variables. The expectation values of the operators give the expected solutions of the geodesic line, and the quantum commutators reduce to the classical Poisson brackets to leading order in the bulk gravitational coupling. Our work provides a framework to understand the geodesic equation, that is gravitational attraction, in the dual CFTs.

1 Introduction

To understand gravity one should not only know the curved spacetime but also how matter moves in the spacetime. The AdS/CFT correspondence provides us a framework to explore both aspects of gravity in the conformal field theory (CFT) on the boundary of the asymptotically AdS spacetime [1–3].

In the context of AdS/CFT, there are a lot of studies on the first aspect, that is emergence of spacetime from the non-gravitational degrees of freedom. The concepts from the quantum information theories are found to be useful. Many quantities, such as entanglement entropy, complexity, are expected to be related to the bulk geometry [4–8]. One could refer to the recent review [9] for more references on these studies.

On the other aspect it is also significant to understand how to describe the bulk moving particle in the CFTs, that is the CFT duality of the geodesic line. The geodesic approximation for correlation functions in AdS/CFT has a long history. Certain two point functions of primary operators in the boundary CFT can be associated with the geodesic line connecting the location of the two operators. More precisely, for a bulk free scalar field $\phi$ with large mass $m$ ($1/\ell_{ads} \ll m \ll 1/\ell_p$ where $\ell_{ads}$ is the radius of AdS), the dual boundary operator $O$ has conformal dimension $\Delta \simeq m\ell_{ads}$. The two point functions $\langle \psi | O(x) O(y) | \psi \rangle$ can be approximated by

$$\langle \psi | O(x) O(y) | \psi \rangle \propto e^{-mL(x, y)},$$

where $L(x, y)$ is the length of geodesic line between $x$ and $y$ in the bulk geometry dual to the state $| \psi \rangle$. The geodesic approximation has many interesting applications to explore the bulk and boundary physics, see, e.g., [10–17].

In this paper will also study the geodesic line. But our motivation is different from the geodesic approximation of correlation functions. We will consider a particle moves in the bulk, which follows the timelike geodesic line. According to AdS/CFT, the information of the bulk particle should be encoded on the boundary CFTs. We can take this as a dynamical system and use the canonical variables such as position and momentum to describe the information of the bulk particle. More precisely, our goal is to construct the dual CFT operators associated with the canonical variables. To achieve this, one should firstly construct the dual state of the moving bulk particle. Besides that, it is also necessary to construct the corresponding operators that are associated with the particle.

In this paper we will only focus on the vacuum AdS$_3$ spacetime. The map between the bulk isometries and boundary conformal symmetry would help us to construct the operators dual to the canonical variables. The constructions should follow some general rules:

1. The background geometry $g_{\mu\nu}$ can be effectively described by a CFT state $| g \rangle$. 

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2. A bulk moving particle can be seen as excited state of the bulk, denoted by $|\psi\rangle$, which is also a state in the Hilbert space of the dual CFT. The energy of the particle should be given by $\langle \psi | \hat{H}_{cfft} | \psi \rangle - \langle g | \hat{H}_{cfft} | g \rangle$, where $\hat{H}_{cfft}$ is the Hamiltonian operator of the CFT.

3. There exists Hermitian operators corresponding to the canonical variables of the classical particle. The expectation values of these operators in the state $|\psi\rangle$ satisfy the equation of motion of the particle at the leading order of $G$.

4. The quantum commutators of the constructed operators should reduce to the classical Poisson brackets in the semiclassical limit $G \to 0$.

In the paper we only focus on the vacuum state of AdS$_3$. With the observation on the geodesic solution, we assume that the position and momentum operators can be constructed by stress energy tensor $T$ and $\tilde{T}$, or equally the Virasoro generators. Actually, we will show below only global Virasoro generators are needed.

We expect the operators would be state-dependent [18]. In the Hamiltonian formulation the canonical variables would be associated with the metric $g_{\mu\nu}$. According to the point 3 of the general rules, we would like to construct the operators associated with the canonical variables. It is nature to consider the operators should depend on the background geometry. However, they should not depend on the dual state of the particle, which is related to the initial conditions of the bulk particle.

The organization of this paper is as follows. In Sect. 2 we will briefly discuss the so-called geometric state. One of the feature is that the correlators of stress energy tensor will satisfy the factorization property, which is important for our constructions. In Sect. 3, the solution of the geodesic line in the global coordinate is shown. In Sect. 4 the radial moving particle is discussed. The dual CFT state is assumed to be associated with the local bulk state with suitable regularization. We show how to construct the radial position and momentum operator in this case. In Sect. 5 we discuss another example. The boundary locally excited state can be taken as particle starting from the AdS boundary. We also construct a new state that are expected to be dual to particle with angular momentum. The angular momentum of the particle has a dictionary with the rapidity of a boost in the CFT. The final section is conclusion and discussion. We discuss three interesting problems that are worthy to explore in the near future.

2 Geometric state and factorization

In the introduction we have mentioned that the background geometry is expected to be dual to a CFT state $|g\rangle$. We will call these kinds of CFT states geometric states. The states $|\Psi\rangle$ that are expected to be dual to the moving particle should also be geometric states. It is expected that the particle should have backreaction on the background geometry. If the backreaction can be neglected in the semiclassical limit $G \to 0$, it is not expected the observables in the CFT could detect the difference from the reference state $|g\rangle$. For example, the energy is same in both states in the limit $G \to 0$. The construction would be meaningless. Therefore, in the following examples the mass of the particle will be taken to be $O(1/G)$.

We construct the position and momentum operators by using stress energy tensor $T$ and $\tilde{T}$. One of the feature of the geometric states is the factorization property. For a given geometric state $|g\rangle$, the expectation value of $T$ is of order $c$ or $1/G$ [19], for 2-point correlator

$$\langle g|T(z_1)T(z_2)|g\rangle \sim \langle g|T(z_1)|g\rangle\langle g|T(z_2)|g\rangle \sim O(c).$$

Or we can define the scaled operator $u := T/c$, the expectation value of which is of order $c^0$. The above condition becomes

$$\lim_{c\to\infty} \left( \langle g|u(z_1)u(z_2)|g\rangle \right) \sim \langle g|u(z_1)|g\rangle\langle g|u(z_2)|g\rangle = 0. \quad (3)$$

This means the operators $u$ satisfy the factorization property in the limit $c \to \infty$. Actually, for n-point correlator we also have the factorization property. In [20] we have shown the factorization condition is associated with the geometric state by the scaling behavior of holographic Rényi entropy.

If the operators are functions of stress energy tensor, they also satisfy the factorization property. For an arbitrary operator $\tilde{X}$ as a function of $T$ and $\tilde{T}$, if the expectation value of it is finite in the semiclassical limit $c \to \infty$, we will call it classical operator. Two arbitrary classical operators $\tilde{X}$ and $\tilde{Y}$, we expect the factorization

$$\lim_{c\to\infty} \left( \langle g|\tilde{X}\tilde{Y}|g\rangle \right) \sim \langle g|\tilde{X}|g\rangle\langle g|\tilde{Y}|g\rangle = 0. \quad (4)$$

For given geometric states and classical operators, such as the ones that we construct in the following, one could check the above statement by direct calculation. Actually, the factorization property is general for quantum system which has a well defined classical limit [21].

The factorization property is very useful for our calculations. For example, consider $\langle \tilde{X} \rangle_g := \langle g|\tilde{X}|g\rangle$, $\langle \tilde{Y} \rangle_g := \langle g|\tilde{Y}|g\rangle \sim O(c^0)$. By the factorization (4) we have $\langle \tilde{X}\tilde{Y} \rangle_g \sim O(c^0)$. Hence, we expect the commutator

$$\langle [\tilde{X}, \tilde{Y}] \rangle_g \sim O(c^{-1}). \quad (5)$$

Now the commutator

$$\langle [\tilde{X}^2, \tilde{Y}] \rangle_g = \langle (\tilde{X}[\tilde{X}, \tilde{Y}] + [\tilde{X}, \tilde{Y}]\tilde{X}) \rangle_g = 2\langle \tilde{X}\tilde{X}\tilde{Y}\rangle_g + O(c^{-2}). \quad (6)$$
More generally, one could check
\[ ([\hat{X}^n, \hat{Y}])_g = -n\langle \hat{X}^{n-1}[\hat{X}, \hat{Y}] \rangle_g + O(c^{-2}). \]  
(7)
The above results will be used in the following sections.

### 3 Geodesic line

We will focus on the global coordinate. In the following we will take the radius \( \ell_{ads} = 1 \). The metric is
\[ ds^2 = -\cosh^2(\rho)dt^2 + d\rho^2 + \sinh^2(\rho)d\phi^2, \]
where we take the radius of AdS to be 1.

Consider a particle starting from \((\rho_0, \phi_0)\) with velocity \( \left. \frac{d\phi}{dt} \right|_{t=0} = v_\phi \) and \( \left. \frac{d\rho}{dt} \right|_{t=0} = 0 \). The action of the particle with mass \( m \) is
\[ S = \int dt L(t), \]
where the Lagrangian is
\[ L(t) := -m \sqrt{\cosh^2(\rho) - \rho^2 - \sinh^2(\rho)\phi^2}. \]
(10)
The canonical momentum associated with the coordinates \( \rho \) and \( \phi \) is
\[ P_\rho := \frac{\partial L}{\partial \dot{\rho}} = \frac{m \dot{\rho}}{\sqrt{\cosh^2(\rho) - \rho^2 - \sinh^2(\rho)\phi^2}}, \]
\[ P_\phi := \frac{\partial L}{\partial \dot{\phi}} = \frac{m \sinh^2(\rho)\phi}{\sqrt{\cosh^2(\rho) - \rho^2 - \sinh^2(\rho)\phi^2}}. \]
(11)

Using these we obtain the Hamiltonian
\[ H = P_\rho \dot{\rho} + P_\phi \dot{\phi} - L = \frac{m \cosh^2(\rho)}{\sqrt{\cosh^2(\rho) - \rho^2 - \sinh^2(\rho)\phi^2}}. \]
(12)
The geodesic line of the particle can be obtained by solving the Hamiltonian equations associated with the canonical variables \( \{\rho, \phi, P_\rho, P_\phi\} \). The result is
\[ \tanh \rho(t) = \tanh(\rho_0) \sqrt{1 - v_\phi^2} \cos^2(t) + v_\phi^2 \]
\[ \tan \phi(t) = v_\phi \tan(t) + \phi_0. \]
(13)
We can get the momentum \( P_\rho(t) \) and \( P_\phi(t) \) by taking the solutions (13) into (11)
\[ P_\rho(t) = \frac{m \left( 1 - v_\phi^2 \right) \sinh(\rho_0) \sin(t)}{\sqrt{1 + v_\phi^2 \tan^2(t)} \sqrt{1 - v_\phi^2 \tan^2(\rho_0)}}, \]
\[ P_\phi(t) = \frac{mv_\phi \sinh (\rho_0) \tan (\rho_0)}{\sqrt{1 - v_\phi^2 \tan^2(\rho_0)}}. \]
(14)
The Hamiltonian of the particle is conserved. Thus, the energy of the particle is constant, that is given by
\[ E = \frac{m \cosh(\rho_0)}{\sqrt{1 - v_\phi^2 \tan^2(\rho_0)}}. \]
(15)

Another constant of motion is the angular momentum \( P_\phi \), which is independent with \( t \) as we can see from (14).

### 4 Radial moving particle

Firstly, let us consider the radial moving particle, that is the velocity \( v_\phi = 0 \). We would like to show the dual CFT state of the radial moving particle. Then we will construct the position and momentum operators corresponding to the canonical variables \( \{\rho, P_\rho\} \).

#### 4.1 State dual to radial moving particle

The first step is to construct the dual CFT states of the bulk radial moving particle. The particle can be seen as excitation of CFTs in the bulk. It is expected to be related to the bulk local states, which have been explored in many literatures. The Hamilton–Kabat–Lifshytz–Lowe (HKLL) construction is a well known method to express the bulk local operator as CFT operators \([22–24]\). A different view on the construction is proposed in \([25]\), for which the symmetry of AdS and CFT play an important role \([26]\). One could also refer to \([27–29]\) for the reconstructions of bulk operator in more general background. Therefore, the bulk local states can be written as superposition of states in CFT. We will briefly review the methods and show the bulk local states with suitable regularization can be taken as the dual state of the radial moving particle in Appendix A.

Actually, it is expected an object with large mass \( m \) at rest in AdS3 is dual to primary state \( |O_a\rangle := |O_a(0)\rangle \) with the conformal dimension of \( \Delta_a \simeq m \) \([30,31]\). Here we still assume \( \Delta_a := \epsilon_a c \), where \( \epsilon_a \) satisfies \( \epsilon_a \ll 1 \). In Appendix A we constructed the state \( |\Psi_a\rangle \Lambda \) (91) where \( \Lambda \) is a regulator. Taking the limit \( \Lambda \to \infty, |\Psi_a\rangle \Lambda \) would approach to \( |O_a\rangle \). However, even taking \( \Lambda \sim O(c^0) \) we find the bulk metric still corresponds to the backreacted geometry with the stationary massive particle at \( \rho = 0 \). The bulk metric is not sensitive to the cut-off parameter \( \Lambda \). For our purpose we will take \( |O_a\rangle \) as the stationary massive particle at \( \rho = 0 \).

According to \([27]\) the states located at \((\rho_0, \phi_0)\) can be associated with \( |O_a\rangle \) by a unitary operator. The state of a particle located at \( \rho = \rho_0, \phi = \phi_0 \) is expected to be given by
\[ |\Psi_a(\rho_0, \phi_0)\rangle = g(\rho_0, \phi_0) |O_a\rangle, \]
(16)
where
\[ g(\rho, \phi) = e^{i(L_0 - \bar{L}_0)\phi} e^{-\frac{2}{L_1 - L_{-1} + L_1 - L_{-1}}}. \] (17)

One could refer to [27] or Appendix A for more details.

In the radial moving case one could always to fix the angular coordinate \( \phi_0 = 0 \). Consider its time evolution we have the state
\[ |\Psi_\alpha(\rho_0, 0, t)\rangle := U_\rho(t)|\Psi_\alpha(\rho_0, 0)\rangle, \] (18)
where \( U_\rho(t) := e^{itH_\rho} \) is the unitary evolution operator. In the following we would like to show this state is dual to a radial moving particle in the bulk by directly constructing the associated position and momentum operators.

4.2 Position and momentum operator

Let’s calculate the expectation value of the Hamiltonian \( H_\rho \) in the (18). It is obvious that the energy
\[ \langle \rho_0, 0 | H_\rho | \Psi_\alpha(\rho_0, 0, t) \rangle \] is independent with \( t \). To evaluate it we need the formula
\[ g^{-1}(\rho_0, 0) L_0 g(\rho_0, 0) = L_0 \cosh(\rho_0) + \frac{L_1 + L_{-1}}{2} \sinh(\rho_0). \] (19)

By using (92) we have
\[ \langle \Psi_\alpha(\rho_0, 0, t) | H_\rho | \Psi_\alpha(\rho_0, 0, t) \rangle = \Delta_\alpha \cosh(\rho_0) - \frac{c}{12}. \] (20)
The first term is same with the classical particle energy (15) with \( v_\phi = 0 \) by taking \( \Delta_\alpha \simeq m \). The second term is the Casimir energy in the vacuum. The expectation value of the operator \( \hat{H} = H_\rho + \frac{c}{12} \) gives energy of the particle. This suggests \( \hat{H} \) can be taken as the operator dual to the Hamiltonian (12).

One could also check the expectation value of the momentum operator \( \hat{P}_\rho := L_0 - \bar{L}_0 \) is zero by using the fact \( h_\alpha = \bar{h}_\alpha \). This is consistent with the result that \( \hat{P}_\rho = 0 \) for \( v_\phi = 0 \).

Now we move on to the construction of position operator \( \hat{\rho}_r \) and momentum operator \( \hat{\rho}_\rho \) of the radial moving particle. To simplify the notations we will denote the expectation value \( \langle \Psi_\alpha(\rho_0, 0, t) | \hat{X} | \Psi_\alpha(\rho_0, 0, t) \rangle \) as \( \langle \hat{X} \rangle_{\Psi_\alpha(t)} \).

The basic requirement for the CFT operators \( \hat{\rho} \) and \( \hat{\rho}_\rho \) is that
\[ \langle \hat{\rho}_\rho \rangle_{\Psi_\alpha(t)} = \rho(t), \]
\[ \langle \hat{\rho}_\rho \rangle_{\Psi_\alpha(t)} = P_\rho(t), \] (21)
where \( \rho(t) \) and \( \hat{\rho}_\rho(t) \) are given by (11) with \( v_\phi = 0 \). As we have discussed above the Hamiltonian \( \hat{H} \) and momentum operator \( \hat{P}_\rho \) can be associated with energy and angular momentum of the bulk moving particle. They are constructed by the Virasoro generators \( L_n \) and \( \bar{L}_n \). Motivated by this we can try to build \( \hat{\rho}_r \) and \( \hat{\rho}_\rho \) by the same way. Actually, we only need the generators associated with global conformal symmetry, that is \( \{ L_{-1}, L_0, \bar{L}_0 \} \) and \( \{ \bar{L}_{-1}, L_0, \bar{L}_0 \} \).

Firstly, let’s show the following formulas that are useful for the constructions,
\[ g^{-1}(\rho_0, 0) U_\rho^{-1}(t) L_0 U_\rho(t) g(\rho_0, 0) = g^{-1}(\rho_0, 0) L_0 g(\rho_0, 0), \]
\[ g^{-1}(\rho_0, 0) U_\rho^{-1}(t) \bar{L}_0 U_\rho(t) g(\rho_0, 0) = \bar{L}_0 g(\rho_0, 0) \]
\[ \frac{L_0 \sinh(\rho_0) e^{it}}{2} (L_1 + L_{-1}) + \frac{L_1 - L_{-1}}{2} e^{-it}, \]
\[ \frac{L_0 \sinh(\rho_0) e^{-it}}{2} (L_1 + L_{-1}) - \frac{L_1 - L_{-1}}{2} e^{it}. \] (22)

It can be shown that
\[ \left\langle \frac{L_1 - L_{-1}}{2i} \right\rangle_{\Psi_\alpha(t)} = \Delta_\alpha \langle \rho_0 | \rho_0 \rangle \sin(t) = h_\alpha \sinh(\rho_0) \sin(t) + O(e^0). \] (23)

where we have used (92). Our proposal of the radial momentum operator is
\[ \hat{\rho}_\rho = \frac{L_1 - L_{-1} + \bar{L}_1 - \bar{L}_{-1}}{2i}, \] (24)
which gives the expected relation
\[ \langle \hat{\rho}_\rho \rangle_{\Psi_\alpha(t)} = \Delta_\alpha \sinh(\rho_0) \sin(t) \simeq m \sinh(\rho_0) \sin(t). \] (25)

Actually, we can take \( \hat{\rho}_\rho \) as the generator of the radial transformation \( g(\rho, 0) \) since \( g(\rho, 0) = e^{-i\hat{\rho}_\rho}. \)

We can construct the position operator \( \hat{\rho}_r \) from the classical Hamiltonian of the particle. By using (11) and (12) the Hamiltonian with \( \phi = 0 \) is
\[ H = \cosh(\rho) \sqrt{m^2 + \hat{P}_\rho^2}. \] (26)
Taking the Hamiltonian operator \( \hat{H} = L_0 + \bar{L}_0 \) and radial momentum operator \( \hat{\rho}_\rho \) (24) into the above equation, one could obtain the operator \( \hat{\rho}_r \) by solving the operator equation. This suggests the position operator \( \hat{\rho}_r \) can be constructed as
\[ \hat{\rho} := \text{arccosh} \left( \tilde{A} \right), \quad \tilde{A} := \hat{H}(\Delta_\alpha^2 + \hat{P}_\rho^2)^{-1/2}. \] (27)

In the above expression \( \text{arccosh}(\tilde{A}) \) is defined as \( \sum_{n=1}^{\infty} a_n \tilde{A}^n \) where \( a_n \) are Taylor coefficients of the function \( \text{arccosh}(x) \). To make \( \text{arccosh}(A) \) to be a well defined bounded operator the series expansion should be convergent in the sense of operator algebra. In this section we only focus on the expectation value of the operators in the state \( |\Psi_\alpha(\rho_0, 0, t)\rangle \). For our purpose we would take the operator \( \text{arccosh}(\tilde{A}) \) to be a well defined operator if the expansion \( \sum_{n=0}^{\infty} a_n (\hat{\tilde{A}}^n)_{\Psi_\alpha(t)} \) is finite.
4.3 Check of our proposal

The operator \( \hat{A} \) are polynomials in \( \hat{H} \) and \( \hat{P}_{\rho} \), which are associated with the energy momentum operator \( T \) and \( \bar{T} \). Roughly, the relation is \( \hat{H}, \hat{P}_{\rho} \sim \int f \hat{T} \hat{f} \bar{T} \), where \( f \) and \( \hat{f} \) are some functions. The state \( |\psi_{\alpha}(\rho_0, 0, t)\rangle \) is explained as a moving bulk particle state, which obviously should be a geometric state. Therefore, using the factorization property for the operators \( \hat{H} \) and \( \hat{P}_{\rho} \), we obtain

\[
\langle \hat{A} \rangle_{\psi_{\alpha}(t)} = \langle \hat{H} \rangle_{\psi_{\alpha}(t)} (\Delta_a^2 + \langle \hat{P}_{\rho} \rangle_{\psi_{\alpha}(t)}^2)^{-1/2} + O(c^{-1}). \tag{28}
\]

Similarly, the operator \( \hat{A} \) also satisfies the factorization property

\[
\langle \hat{A}^n \rangle_{\psi_{\alpha}(t)} = \langle \hat{A} \rangle_{\psi_{\alpha}(t)}^n + O(c^{-1}). \tag{29}
\]

One could show this by direct calculations for a given \( n \).

Taking (20) and (25) into (28), we have

\[
\langle \hat{A} \rangle_{\psi_{\alpha}(t)} = \frac{1}{\sqrt{1 - \tanh^2(\rho_0) \cos^2(t)}} + O(c^{-1}). \tag{30}
\]

Using the above result and (29) we have the expected relation

\[
\langle \hat{A}^n \rangle_{\psi_{\alpha}(t)} = \langle \hat{A} \rangle_{\psi_{\alpha}(t)}^n + O(c^{-1}). \tag{31}
\]

The expectation values of the operator \( \hat{r} \) and \( \hat{p} \) in the state \( |\psi_{\alpha}(\rho_0, 0, t)\rangle \) give the classical results (13) and (14) at the leading order of \( c \).

It is convenient to introduce the scaled momentum operator \( \hat{P}_{\rho} := \hat{P}_{\rho} / c \). They can be taken as the operators related to the particle with mass \( m/c \). \( \hat{P}_{\rho} \) are classical operators, since their expectation values in the state \( |\psi_{\alpha}(\rho_0, 0, t)\rangle \) are finite in the limit \( c \to \infty \). We can also define more general operators \( \hat{X}(\hat{r}, \hat{p}), \hat{Y}(\hat{r}, \hat{p}) \), which are functions of \( \hat{r} \) and \( \hat{p} \). We also have the factorization property

\[
\langle \hat{X}(\hat{r}, \hat{p}), \hat{Y}(\hat{r}, \hat{p}) \rangle_{\psi_{\alpha}(t)} = \langle \hat{X}(\hat{r}, \hat{p}) \hat{Y}(\hat{r}, \hat{p}) \rangle_{\psi_{\alpha}(t)} + O(c^{-1}), \tag{32}
\]

at the leading order of \( c \), where \( \rho(t), p(t) := \rho_{\rho}(t) / c \) are given by (21). The proof is similar as (29). Therefore, the classical operators behave as \( c \)-number in the state \( |\psi_{\alpha}(\rho_0, 0, t)\rangle \).

The Newton constant \( G \) or \( 1/c \) plays the role as the parameter \( \hbar \) in quantum mechanics.

The commutator \([\hat{X}, \hat{Y}]\) would also have a correspondence to the Poisson bracket \([X, Y]\). For the radial moving particle the phase space is 2-dimensional, for which \( \rho \) and \( p_{\rho} \) are canonical variables. The classical Poisson brackets of two functions \( X(\rho, p_{\rho}) \) and \( Y(\rho, p_{\rho}) \) are defined as

\[
[X, Y] := \frac{\partial X}{\partial \rho} \frac{\partial Y}{\partial p_{\rho}} - \frac{\partial X}{\partial p_{\rho}} \frac{\partial Y}{\partial \rho}. \tag{33}
\]

One special case is the fundamental Poisson bracket \([\rho, p_{\rho}] = 1 \). Since we have constructed the position and momentum operators, their commutators can be evaluated by the Virasoro algebra. Our task is to show how to obtain the classical Poisson brackets from the quantum commutators. This is similar as the process that the quantum commutators reduce to classical Poisson brackets in the limit \( \hbar \to 0 \).

Let’s begin with the fundamental bracket \([\rho, p_{\rho}] = 1 \). To evaluate the corresponding quantum commutator \([\hat{\rho}, \hat{p}_{\rho}] \), we need \([\hat{A}^n, \hat{P}_{\rho}] \).

\[
[\hat{A}, \hat{P}_{\rho}] = [\hat{H}, \hat{P}_{\rho}](m^2 + \hat{P}_{\rho}^2)^{-1/2} = -i \hat{Q}_{\rho}(\Delta_a^2 + \hat{P}_{\rho}^2)^{-1/2}, \tag{34}
\]

with

\[
\hat{Q}_{\rho} := \frac{L_1 + L_{-1} + L_1 + L_{-1}}{2} \tag{35}
\]

For general \( n \) it is not easy to write down the results. However, if we consider the commutators in the state \( |\psi_{\alpha}(\rho_0, 0, t)\rangle \) the expression would be very simple at the leading order of \( c \).

By using the factorization property of operators, we have

\[
\langle [\hat{A}^n, \hat{P}_{\rho}] \rangle_{\psi_{\alpha}(t)} = \sum_n a_n \langle [\hat{A}^n, \hat{P}_{\rho}] \rangle_{\psi_{\alpha}(t)} = -i \tanh(\rho_0) \cos(t) + O(c^{-1}) \tag{36}
\]

In the above calculation we only keep the leading order \( c \) results. In the last step we use (31) and the fact

\[
\langle \hat{Q}_{\rho} \hat{H}^{-1} \rangle_{\psi_{\alpha}(t)} = \frac{\sinh(\rho_0) \cos(t)}{\cosh(\rho_0)} + O(c^{-1}) \tag{37}
\]

Finally, we have the result

\[
\langle [\hat{\rho}, \hat{p}_{\rho}] \rangle_{\psi_{\alpha}(t)} = \sum_n a_n \langle [\hat{A}^n, \hat{P}_{\rho}] \rangle_{\psi_{\alpha}(t)} \tag{38}
\]

The quantum commutator reduces to classical Poisson brackets as

\[
\lim_{c \to \infty} i c \langle [\hat{\rho}, \hat{p}_{\rho}] \rangle_{\psi_{\alpha}(t)} = [\rho, p_{\rho}] = 1. \tag{39}
\]

The above result is consistent with the factorization property. The expectation values of \( \hat{\rho} \) and \( \hat{p}_{\rho} \) are of order \( O(c^0) \).
According to the factorization property (32) the commutator $\{\hat{p}_\rho, \hat{p}_\rho\}u,\gamma\}$ should be vanishing at the order $O(e^0)$. The result (39) shows that the commutator is of $O(1/e)$. With the fundamental brackets one could derive the general Poisson brackets (33).

5 Locally excited state in CFT

According to the extrapolate dictionary of AdS/CFT, the bulk operator $\phi_\alpha(\rho, x)$ and the dual boundary CFT operator $O_\alpha(x)$ are related by

$$O_\alpha(x) = \lim_{\rho \to \infty} e^{\rho A} \phi_\alpha(\rho, x),$$

in the global coordinate. We expect the bulk state $\phi_\alpha(\rho, x)|0\rangle$ should reduce to the locally excited state $O_\alpha(x)|0\rangle$ in CFT near the AdS boundary. To regularize this state $O_\alpha(x)$ one could introduce a cut-off $\epsilon$ and define

$$|\psi(x)\rangle_{\epsilon} := \mathcal{N} e^{-\epsilon H} O_\alpha(x)|0\rangle,$$

where $\mathcal{N}$ is the normalization constant. We can take $\epsilon$ as the UV cut-off of theory with $\epsilon \ll 1$. The locally excited states has been studied in many literatures on the dynamics behavior of entanglement entropy, see [32–41] and references therein.

In the following we would like to focus on such state, which is expected to be described by a point particle with the initial location near the AdS boundary [42].

Denote the boundary coordinate as $x^\pm = \phi \pm t$. By a Wick-rotation we have the Euclidean coordinate $\omega := \phi - i t$ and $\tilde{w} := \phi + i t$. With a conformal mapping $z = e^{\omega}$, the cylinder is mapped to $z$-plane. The state will be defined on the $z$-plane. The local state (41) inserted at $w_0 = \tilde{w}_0 = 0$ is given by

$$|\psi_\alpha(z_0, \tilde{z}_0)\rangle_{\epsilon} = \mathcal{N}(z_0, \tilde{z}_0)O_\alpha(z_0, \tilde{z}_0)|0\rangle,$$

where $z_0 = e^{i\omega_0} = e^\epsilon, \tilde{z} = e^{-i\tilde{w}_0} = e^{-\epsilon}$, and normalization constant $\mathcal{N}(z_0, \tilde{z}_0) = (z_0z_0^* - 1)^{h_\alpha}(\tilde{z}_0\tilde{z}_0^* - 1)^{\tilde{h}_\alpha}$. We can also write the above state as

$$|\psi_\alpha(z_0, \tilde{z}_0)\rangle_{\epsilon} = \mathcal{N}(z_0, \tilde{z}_0)e^{i\omega z_{\text{L}} - 1 + \tilde{z}_{\text{L}} - 1}|O_\alpha\rangle,$$

with the primary state $|O_\alpha\rangle := \lim_{z \to 0} O\alpha|0\rangle$. For $h_\alpha \sim O(e)$, the energy of this state is

$$\epsilon \langle \psi_\alpha(z_0, \tilde{z}_0)|\hat{H}|\psi_\alpha(z_0, \tilde{z}_0)\rangle_{\epsilon} = \frac{h_\alpha(z_0z_0^* + 1)}{z_0z_0^* - 1} + \frac{\tilde{h}_\alpha(\tilde{z}_0\tilde{z}_0^* + 1)}{\tilde{z}_0\tilde{z}_0^* - 1},$$

where $\hat{H} = L_0 + \tilde{L}_0$. If $\epsilon \ll 1$ we have

$$\epsilon \langle \psi_\alpha(z_0, \tilde{z}_0)|\hat{H}|\psi_\alpha(z_0, \tilde{z}_0)\rangle_{\epsilon} = \frac{\Delta_\alpha}{\epsilon}.$$  

For the static particle located at the AdS boundary $\rho_0 \gg 1$, the energy is given by (15) with $\nu_\phi = 0$, that is

$$E = m \cosh(\rho_0) \simeq \frac{1}{2} \Delta_\alpha e^{\rho_0}.$$  

Comparing with (45) we obtain the relation $\frac{1}{\epsilon} \simeq \frac{1}{2} e^{\rho_0}$, which provides an interpretation of $\log(\frac{1}{\epsilon})$ as the initial location of the bulk particle. This is also consistent with the UV/IR relation in the context of AdS/CFT [43].

One could check the expectation value of momentum operator $P_\phi = L_0 - \tilde{L}_0$ in the state (42) is vanishing. Therefore, we can interpret this state is dual to a particle moving in the radial direction in the bulk, that is $\nu_\phi = 0$.

5.1 State with angular momentum

It is more interesting to construct the state with non-vanishing $\nu_\phi$. From (14) and (12) with $\rho_0 \gg 1$ we have

$$P_\phi \simeq \frac{me^{\rho_0} v_\phi}{2 \sqrt{1 - v_\phi^2}},$$

$$E \simeq \frac{m e^{\rho_0}}{2 \sqrt{1 - v_\phi^2}}.$$  

This motivates us to construct the state with non-vanishing $\nu_\phi$ by a boost with velocity $v_\phi$. The coordinates transform as

$$x^+ = e^{-\lambda} x^+, \quad x^- = e^{\lambda} x^-,$$

where $\lambda$ is the rapidity with $\nu_\phi = \tanh \lambda$.

We propose that the dual state of a moving bulk particle with initial position $(\rho, \phi) = (\rho_0, 0)$ ($\rho_0 \gg 1$) and velocity $v_\phi$ is given by the locally excited state $O_\alpha(w_0 e^{-\lambda}, \tilde{w}_0 e^{\lambda})|0\rangle$. On the $w$-plane the state is defined as

$$|\psi_\alpha(z_\lambda, \tilde{z}_\lambda)\rangle_{\epsilon} := \mathcal{N}(z_\lambda, \tilde{z}_\lambda)e^{i\omega z_{\text{L}} - 1 + \tilde{z}_{\text{L}} - 1}|O_\alpha\rangle,$$

where $z_\lambda = e^{i\omega_0} e^{i\lambda} = e^{i\lambda} \tilde{z}_\lambda = e^{-i\tilde{w}_0} e^{i\lambda} = e^{i\lambda}$, and normalization constant $\mathcal{N}(z_\lambda, \tilde{z}_\lambda) = (z_\lambda z_\lambda^* - 1)^{h_\alpha}(\tilde{z}_\lambda \tilde{z}_\lambda^* - 1)^{\tilde{h}_\alpha}$. The energy can be obtained by the replacement $z_0 \to z_\lambda, \tilde{z}_0 \to \tilde{z}_\lambda$ in (44). Keeping the leading order of $\epsilon$ we have

$$E = \epsilon \langle \psi_\alpha(z_\lambda, \tilde{z}_\lambda)|((L_0 + \tilde{L}_0)|\psi_\alpha(z_\lambda, \tilde{z}_\lambda)\rangle_{\epsilon} = \frac{\Delta_\alpha}{\epsilon} \cosh \lambda.$$  

Similarly, the angular momentum is given by

$$P_\phi = \epsilon \langle \psi_\alpha(z_\lambda, \tilde{z}_\lambda)|((L_0 - \tilde{L}_0)|\psi_\alpha(z_\lambda, \tilde{z}_\lambda)\rangle_{\epsilon} = \frac{\Delta_\alpha}{\epsilon} \sinh \lambda.$$  

These are consistent with the results (47).
Consider the time evolution and define the time-dependent state
\[ |\psi_0(t)\rangle_e := U_R(t) |\psi_a(z_\lambda, \bar{z}_\lambda)\rangle_e, \tag{52} \]
where \( U_R = e^{i H_R t} \). It is obvious that the energy and angular momentum are independent with \( t \).

### 5.2 Angular coordinate operator

We can construct the position and momentum operators as we have done for the radial moving case. We will show they can be expressed as operator functions of the global Virasoro generators.

To simplify the notation the expectation value of operator \( \hat{X} \) in the state \( |\psi_\alpha(t)\rangle_e \) is denoted by \( \langle \hat{X} \rangle_{\psi_\alpha(t)} \). The following formulas are useful for our construction,
\[
\begin{align*}
\langle L_1 \rangle_{\psi_\alpha(t)} &= \frac{2h_a^2 \bar{z}_\lambda e^{it}}{z_\lambda \bar{z}_\lambda - 1} \approx \frac{h_a e^{\lambda + it}}{e}, \\
\langle \bar{L}_1 \rangle_{\psi_\alpha(t)} &= \frac{2\bar{h}_a \bar{z}_\lambda e^{it}}{z_\lambda \bar{z}_\lambda - 1} \approx \frac{\bar{h}_a e^{-\lambda + it}}{e}, \\
\langle L_{-1} \rangle_{\psi_\alpha(t)} &= \frac{2h_a \bar{z}_\lambda e^{-it}}{z_\lambda \bar{z}_\lambda - 1} \approx \frac{h_a e^{-\lambda - it}}{e}, \\
\langle \bar{L}_{-1} \rangle_{\psi_\alpha(t)} &= \frac{2\bar{h}_a \bar{z}_\lambda e^{-it}}{z_\lambda \bar{z}_\lambda - 1} \approx \frac{\bar{h}_a e^{\lambda - it}}{e}.
\end{align*}
\tag{53} \]

In previous section we have defined two Hermitian operators \( \hat{P}_\rho \) and \( \hat{Q}_\rho \) which are linear combinations of the global Virasoro generators. Let’s define two more independent operators
\[
\begin{align*}
\hat{S}_\phi &:= \frac{L_1 - L_{-1} - (\bar{L}_1 - \bar{L}_{-1})}{2i}, \\
\hat{T}_\phi &:= \frac{L_1 + L_{-1} - (\bar{L}_1 + \bar{L}_{-1})}{2}.
\end{align*}
\tag{54} \]

By using (53) it is straightforward to evaluate the expectation values of the four Hermitian operators. The results are
\[
\begin{align*}
\langle \hat{P}_\rho \rangle_{\psi_\alpha(t)} &= \frac{\Delta_a}{\epsilon} \cosh \lambda \sin t, \\
\langle \hat{Q}_\rho \rangle_{\psi_\alpha(t)} &= \frac{\Delta_a}{\epsilon} \cosh \lambda \cos t, \\
\langle \hat{S}_\phi \rangle_{\psi_\alpha(t)} &= \frac{\Delta_a}{\epsilon} \sinh \lambda \sin t, \\
\langle \hat{T}_\phi \rangle_{\psi_\alpha(t)} &= \frac{\Delta_a}{\epsilon} \sinh \lambda \cos t.
\end{align*}
\tag{55} \]

The above results are consistent with the radial moving case \( \lambda = 0 \).

Since the state (49) is a geometric state, the global Virasoro generators also satisfy the factorization property.

Now we move to construct the operator \( \hat{\phi} \). The expectation \( \langle \hat{\phi} \rangle_{\psi_\alpha(t)} \) is expected to give the classical solution (13).

Rewrite (13) as
\[
\phi = \text{arccos} \left[ \frac{1}{\sqrt{1 + (\tanh \lambda \tan t)^2}} \right] = \text{arccos} \left[ \frac{\cosh \lambda \cos t}{\sqrt{\cosh^2 \lambda \cos^2 t + \sinh^2 \lambda \sin^2 t}} \right]. \tag{56} \]

The angular coordinate operator \( \hat{\phi} \) is suggested to be
\[
\hat{\phi} = \text{arccos} \hat{B}, \quad \hat{B} = \hat{Q}_\rho (\hat{Q}_\rho^2 + \hat{S}_\phi^2)^{-1/2}, \tag{57} \]

where \( \text{arccos} \hat{B} := \sum_n b_n \hat{B}^n \), \( b_n \) are Taylor coefficients of the function \( \text{arccos}(x) \). Using (55) we have the result
\[
\langle \hat{B} \rangle_{\psi_\alpha(t)} = \frac{1}{\sqrt{1 + (\tanh \lambda \tan t)^2}} + O(e^{-1}). \tag{58} \]

One could check that \( \langle \hat{\phi} \rangle_{\psi_\alpha(t)} = \phi(t) \). For the radial moving particle we have \( \langle \hat{\phi} \rangle_{\psi_\alpha(t)} = 0 \).

### 5.3 Radial momentum and coordinate operator

For \( \rho_0 \gg 1 \) the radial momentum is
\[
P_\rho(t) \approx \frac{\Delta_a}{\epsilon} \sqrt{1 - v_{\phi}^2 \sin^2(t)}, \tag{59} \]

which is different from \( \langle \hat{P}_\rho \rangle_{\psi_\alpha(t)} \) (55). We should include more terms to produce the above expected result. By using (55) we suggest the following radial momentum operator
\[
\hat{P}_\rho := \hat{P}_\rho \cos \phi - \hat{T}_\rho \sin \phi. \tag{60} \]

By the definition of \( \phi \) we have \( \sin \phi = \sqrt{1 - \cos^2 \phi} = \sqrt{1 - \hat{B}^2} \), which can be written as
\[
\sin \phi = \hat{S}_\phi (\hat{Q}_\rho^2 + \hat{S}_\phi^2)^{-1/2}. \tag{61} \]

The expectation value of \( \hat{P}_\rho \) is given by
\[
\langle \hat{P}_\rho \rangle_{\psi_\alpha(t)} = \frac{\Delta_a}{\epsilon} \frac{\sin t}{\cosh \lambda \sqrt{1 + \tanh^2 \lambda \tan^2 t}} + O(e^{0}, e^{0}), \tag{62} \]

which is equal to (59). For the special radial moving case \( \langle \hat{\phi} \rangle_{\psi_\alpha(t)} = 0 \), we can effectively take \( \hat{P}_\rho \) as the radial momentum operator.

The radial position operator \( \hat{\rho} \) can be constructed by using the relation
\[
H = \coth \rho \sqrt{\hat{P}_\rho^2 + (m^2 + \hat{P}_\rho^2) \sinh^2 \rho}. \tag{63} \]

The solution of the above equation for \( \rho \) actually gives an ansatz of the radial position operator \( \hat{\rho} \). We have constructed the Hamiltonian operator \( \hat{H} \), the angular momentum operator...
\( \hat{P}_\phi \) and the radial momentum operator \( \hat{P}_\phi \). Taking them into the solution we obtain

\[
\hat{r} = \frac{1}{2} \text{arccosh}(\hat{r}),
\]

\[
\hat{r} := \left\{ (\hat{H} + \hat{P}_\phi)(\hat{H} - \hat{P}_\phi) + [(\hat{H} + \hat{P}_\phi)^2 - \Delta_\alpha - \hat{P}^2_\phi]^{1/2} \times [(\hat{H} - \hat{P}_\phi)^2 - \Delta_\alpha - \hat{P}^2_\phi]^{1/2} \right\}(\Delta^2_\alpha + \hat{P}^2_\phi)^{-1}.
\]  

(64)

One could check the above expression will become (27) for \( \hat{P}_\phi = 0 \). With some calculations we can find the expected relation

\[
\langle \hat{\rho} \rangle_{\psi_\alpha(t)} = \text{arctanh} \left[ \sqrt{\left(1 - v^2_\phi \right) \cos^2 t + v^2_\phi} \right] + O(c^{-1}, \epsilon).
\]

(65)

5.4 Poisson brackets

As a check of our proposals we will show how to get the Poisson bracket from the position and momentum operators. The phase space of the bulk moving particle is 4-dimensional with the canonical variables \( \{ \phi, \rho, \hat{P}_\phi, \hat{P}_\rho \} \). We will focus on the fundamental Poisson brackets.

Firstly, consider the commutator \([\hat{\phi}, \hat{\rho}_\phi]\). With the definitions we have the following commutation relations,

\[
[\hat{Q}_\rho, \hat{P}_\phi] = i \hat{S}_\phi, \quad [\hat{S}_\phi, \hat{P}_\phi] = -i \hat{Q}_\rho,
\]

\[
[\hat{P}_\rho, \hat{P}_\phi] = -i \hat{T}_\phi, \quad [\hat{T}_\phi, \hat{P}_\phi] = i \hat{P}_\rho,
\]

\[
[\hat{P}_\rho, \hat{Q}_\phi] = -i \hat{H}, \quad [\hat{P}_\rho, \hat{S}_\phi] = 0,
\]

\[
[\hat{N}_\rho, \hat{\phi}] = -i \hat{\rho}_\rho, \quad [\hat{N}_\phi, \hat{\rho}] = 0,
\]

\[
[\hat{N}_\rho, \hat{H}] = -i \hat{\rho}_\rho, \quad [\hat{N}_\phi, \hat{H}] = i \hat{\phi},
\]

\[
[\hat{N}_\rho, \hat{\phi}] = i \hat{\rho}_\rho, \quad [\hat{N}_\phi, \hat{\phi}] = -i \hat{\phi}.
\]

(66)

With these and the definition of \( \hat{B} (57) \) we can obtain

\[
\langle [\hat{B}, \hat{\rho}_\phi] \rangle_{\psi_\alpha(t)} = \langle [\hat{B}, \hat{\rho}_\phi] \rangle_{\psi_\alpha(t)} = \langle \{[\hat{Q}_\rho, \hat{P}_\phi], \{\hat{Q}_\rho, \hat{S}_\phi\}^{-1/2} + \hat{Q}_\rho\{\hat{Q}_\rho, \hat{S}_\phi\}^{-1/2}, \hat{P}_\phi\} \rangle_{\psi_\alpha(t)} = \langle \{\hat{Q}_\rho, \hat{P}_\phi\} \rangle_{\psi_\alpha(t)},
\]

(67)

where in the second step we have used

\[
\langle ([\hat{Q}_\rho, \hat{P}_\phi])_{\psi_\alpha(t)} \rangle_{\psi_\alpha(t)} = -\langle ([\hat{Q}_\rho, \hat{P}_\phi])_{\psi_\alpha(t)} \rangle_{\psi_\alpha(t)} = \langle \{\hat{Q}_\rho, \hat{P}_\phi\} \rangle_{\psi_\alpha(t)} + O(c^{-1})
\]

\[
= \frac{1}{2} \langle ([\hat{Q}_\rho, \hat{P}_\phi])_{\psi_\alpha(t)} \rangle_{\psi_\alpha(t)} = \langle \{\hat{Q}_\rho, \hat{P}_\phi\} \rangle_{\psi_\alpha(t)} + O(c^{-1})
\]

\[
= \frac{1}{2} \langle \{\hat{Q}_\rho, \hat{P}_\phi\} \rangle_{\psi_\alpha(t)} = \langle \{\hat{Q}_\rho, \hat{P}_\phi\} \rangle_{\psi_\alpha(t)} + O(c^{-1})
\]

\[
= 0 + O(c^{-1}).
\]

(68)

where the last step follows from (66). In the above evaluation we have used the factorization property. Therefore, the equality is established only in the leading order of \( c \).

Now we can evaluate the commutator

\[
\langle \{\hat{\phi}, \hat{\rho}_\phi\} \rangle_{\psi_\alpha(t)} = \sum_n b_n \langle \{\hat{\phi}, \hat{\rho}_\phi\} \rangle_{\psi_\alpha(t)} + \sum_n nb_n \langle \{\hat{\phi}, \hat{\rho}_\phi\} \rangle_{\psi_\alpha(t)} + O(c^{-1})
\]

\[
= i \sqrt{1 - \langle \{\hat{\phi}, \hat{\rho}_\phi\} \rangle_{\psi_\alpha(t)}^2} = -i \sqrt{1 - \langle \{\hat{\phi}, \hat{\rho}_\phi\} \rangle_{\psi_\alpha(t)}^2} + O(c^{-1})
\]

(69)

In the last step we have used (55) and (58).

Let’s introduce the scaled operator \( \hat{\rho}_\phi := \hat{\rho}_\phi/c \). The quantum commutator reduces to Poisson brackets as

\[
\lim_{c \to \infty} \text{ic} \langle \{\hat{\phi}, \hat{\rho}_\phi\} \rangle_{\psi_\alpha(t)} = 1.
\]

(70)

Using the above result we can evaluate the more general commutators such as

\[
\lim_{c \to \infty} \text{ic} \langle \{\hat{F}(\hat{\phi}), \hat{\rho}_\phi\} \rangle_{\psi_\alpha(t)} = \left( \frac{\partial F(\hat{\phi})}{\partial \phi} \right)_{\psi_\alpha(t)},
\]

(71)

where \( F(\hat{\phi}) \) is arbitrary functions of \( \hat{\phi} \). One could derive the above expression by taking \( \hat{\rho}_\phi \) as \( \frac{\partial}{\partial \phi} \) when evaluating the commutators.

Now let’s consider the commutator \([\hat{\rho}_\phi, \hat{\rho}_\rho]\), where we define the scaled radial momentum operator \( \hat{\rho}_\rho := \hat{\rho}_\rho/c \). Using the definition of \( \hat{\rho}_\rho \) (60) and commutation relations (66) and (71), we have

\[
\lim_{c \to \infty} \text{ic} \langle \{\hat{\rho}_\rho, \hat{\rho}_\phi\} \rangle_{\psi_\alpha(t)} = \lim_{c \to \infty} \text{ic} \langle \{\hat{\rho}_\rho, \hat{\rho}_\phi\} \rangle_{\psi_\alpha(t)} = \lim_{c \to \infty} \text{ic} \langle \{\hat{\rho}_\rho, \hat{\rho}_\phi\} \rangle_{\psi_\alpha(t)} = 0.
\]

(72)

With some calculations one can also get the following commutators,

\[
\lim_{c \to \infty} \text{ic} \langle \{\hat{\rho}_\rho, \hat{\phi}\} \rangle_{\psi_\alpha(t)} = 0,
\]

\[
\lim_{c \to \infty} \text{ic} \langle \{\hat{\rho}_\phi, \hat{\rho}_\rho\} \rangle_{\psi_\alpha(t)} = 0,
\]

\[
\lim_{c \to \infty} \text{ic} \langle \{\hat{\rho}_\rho, \hat{\rho}_\phi\} \rangle_{\psi_\alpha(t)} = 1.
\]

(73)

We show the details of the calculation in Appendix B
5.5 Equation of motion

With the Poisson brackets one could easily obtain the equation of motion of the particle. For a classical system with canonical variable \( \{q_i, p_i\} \) the Hamiltonian equation can be expressed as

\[
\frac{dq_i}{dt} = \{q_i, H\}, \quad \frac{dp_i}{dt} = \{p_i, H\},
\]

where \( H \) is the Hamiltonian of the system. In the previous section we have constructed the position and momentum operators and shown the quantum commutators can reduce to the classical Poisson brackets in the limit \( c \to \infty \). Now we would like to find the classical equation in the same limit. Take \( \dot{\phi} \) as an example. Let’s define the scaled Hamiltonian operator \( \hat{H} := \hat{H}/c \). Using the result (104) and \( \dot{\phi} := \arccos \hat{B} \) we have

\[
\lim_{c \to \infty} i c \langle \phi, \hat{H} \rangle_{\psi_\alpha(t)} = -\lim_{c \to \infty} i (1 - \langle \hat{B} \rangle_{\phi(t)})^{-1/2} \langle \left[ \hat{B}, \hat{H} \right] \rangle_{\phi(t)} = \lim_{c \to \infty} \langle \left( \hat{P}_\rho \hat{S}_\phi + \hat{Q}_\rho \hat{T}_\phi \right) (\hat{Q}_\rho^2 + \hat{S}_\phi^2)^{-1} \rangle_{\phi(t)} = \frac{\sinh \lambda \cosh \lambda}{(\sinh \lambda \sin \tau)^2 + (\cosh \lambda \cos \tau)^2},
\]

which in the last step we have used (55). The final result is same as \( \frac{d\langle \phi \rangle_{\psi_\alpha(t)}}{dt} \). Therefore, we find the equation of motion

\[
\frac{d\langle \phi \rangle_{\psi_\alpha(t)}}{dt} = \lim_{c \to \infty} i c \langle \left[ \phi, \hat{H} \right] \rangle_{\psi_\alpha(t)},
\]

which can be taken as the classical limit of the Heisenberg equation.

Another example is the operator \( \hat{P}_\rho \). Using (103) and (55) one could check the equation of motion

\[
\frac{d\langle \hat{P}_\rho \rangle_{\psi_\alpha(t)}}{dt} = \lim_{c \to \infty} i c \langle \left[ \hat{P}_\rho, \hat{H} \right] \rangle_{\psi_\alpha(t)}.
\]

The interested reader can check the other two equations associated with \( \hat{P}_\rho \) and \( \hat{P}_\phi \).

6 Conclusion and discussion

The main result of our paper is to explore the CFT dual of a bulk moving particle. As mentioned in the introduction the problem has two aspects.

Firstly, we construct the state that are expected to be dual to the moving particle. Two examples are shown. For the radial moving particle starting from arbitrary position \( \rho_0 \) we find the CFT state can be described by the regularized bulk local states that are discussed in previous paper [27]. The other one is the boundary locally excited state. This state can be explained as particle starting from the AdS boundary. In this case we find the dual state of the particle with angular momentum can be related to a boost. The rapidity of the boost is associated with the velocity in the \( \phi \) direction. As far as we know the state with a boost hasn’t been discussed in other papers.

The other aspect of the problem is to construct the position and momentum operators associated with the particle. We should also note that the operators in the radial moving example are special case of \( \{\rho, \hat{P}_\rho, \hat{P}_\phi\} \). However, we haven’t successfully constructed the state dual to arbitrary moving particles in the bulk. It would be interesting to find such states and check whether the constructed operators could give the correct results.

Generally, we have no systematical method to find the operators. Therefore, we should discuss case by case at present. Of course, there are some basic constraints on the constructions.

Let us summarize some important clues. The energy and angular momentum of the particle should be related to the Hamiltonian and momentum operators of the dual CFTs. The particle can be taken as excited state of the bulk. Hence, the energy and angular momentum of the particle should be equal to the difference between the excited state and the background state. In the CFTs they should be related to the expectation value of the Hamiltonian and momentum operators.

The basic requirement is that the expectation value of the constructed operators in the dual CFT states should give the classical solution at the leading order of \( G \). Actually, this is the important guidance to the constructions. For the examples that are shown in our paper are simple, since we could find the exact classical solution. We find the operators can be constructed only by using the stress energy tensor, in fact only by the global Virasoro generators. The reason is that the dual state can be obtained by global symmetry. For the background state beyond the vacuum the symmetry is broken. We don’t expect these operators should be universe, that is independent with the background geometry. But the stress energy tensor should be the building block of the position and momentum operators.

Another requirement is the quantum commutators of the constructed operators should reduce to the classical Poisson brackets in the semiclassical limit \( c \to \infty \). This can also be seen as a check of our constructions. In both examples in our paper we show the correspondence between the quantum and classical brackets.

There are many important and interesting problems that we haven’t touch in this paper. In the following we will briefly discuss three such problems that are worthy to study in the future.
6.1 Bulk isometries and the dual operators

In this paper we only focus on the pure AdS3 spacetime. There exists well established duality between the bulk isometries and boundary conformal symmetry, see, e.g., [27]. This could be used as a guidance to construct the operators associated with bulk moving particle. In [25] the arbitrary bulk locally excited states are constructed by using the isometries. Here the states dual to bulk operator located at \((\rho_0, \phi_0)\) \((16)\) are also constructed by similar way. In Sect. 4 the momentum operator \(\hat{P}_\rho\) \((24)\) actually can be understood as generator of translation in the radial direction \(\rho\).

The isometries also play an important role for the construction in the Sect. 5. In fact all the geodesic lines in AdS3 are equivalent up to isometries. That is a geodesic lines can be associated with the particle at rest. This suggests a systematical method to build the dual states and operators for general case. It would be interesting to explore how to construct the dual operators with the help of isometries. However, the background geometry may not has isometries. In this case, the constructions would be more difficult.

6.2 Other examples

We only focus on the vacuum AdS3 in the global coordinate. It is easy to generalize to other situations, such as the Poincaré coordinate, AdS Rindler. One can use the similar methods that we have used. It is an interesting exercise to work out the results in different coordinate and compare with our results. In particular, the AdS Rindler will help us to understand more on the entanglement wedge construction or subregiona/subregion duality [44–46]. For example, the Poincaré coordinate of pure AdS3 is

\[ ds^2 = \frac{dy^2 - dt_p^2 + dx^2}{y^2}. \]  

(78)

The state \(|\Psi_\alpha\rangle\) in this case corresponds to the bulk local excitation at point \((t_p, y, x) = (0, 1, 0)\). The bulk local state at point \((t_p, y, x) = (0, y, x)\) is given by

\[ |\Psi_\alpha(y, x)\rangle = g(y, x)|\Psi_\alpha\rangle, \]  

(79)

where

\[ g(y, x) := e^{i(L_0 + \frac{L_1 + L_{-1}}{2})x^+} e^{-i(L_0 + \frac{L_1 + L_{-1}}{2})x^-} \times y^{\frac{L_1-L_{-1}+L_{-1}}{2}}, \]  

(80)

where \(x^\pm := x \pm t_p\). The time evolution is controlled by the operator \(U_p := e^{(L_0 + L_{-1})t_p}\), where \(L_0 := -L_{-1} + \frac{L_1 + L_{-1}}{2}\) and \(L_{-1} := -L_0 + \frac{L_1 + L_{-1}}{2}\) are the generators in the hyperbolic basis as shown in [27]. By similar method we expect one could obtain the position and momentum operators in the Poincaré coordinate. It would be interesting to see the difference and relation with the global coordinate.

More interesting case is BTZ black hole as the background geometry. One could use the thermofield double states to work out the results. In this case there is a horizon in the bulk. It is interesting to explore how to construct the corresponding operators once the particle is inside the horizon.

The generalization to higher dimension is not so straightforward, but we expect one could have a similar construction as the 3D AdS. In the vacuum case the operators are only associated with the global symmetry. In higher dimension the dual CFT also has global conformal symmetry. But in higher dimension the situation will be more complicated, thus more interesting phenomena are expected to appear.

6.3 Coordinate dependence

In the last section we discuss the generalization to other coordinate such as the Poincaré AdS. The vacuums in different coordinate are not same. The particle states are actually different in these coordinate.

It is obvious that our constructions of position and momentum operators depend on the coordinate. Even in the global coordinate, one could choose different coordinates. The operators should depend on the canonical variables that one choose.

For example, in the radial moving case one could choose \(D := \arccosh \rho\) as the position coordinate of the particle. The Lagrangian is

\[ L(t) = -m \sqrt{D^2 - \frac{\dot{D}^2}{D^2 - 1} - (D^2 - 1)\dot{\phi}^2}. \]  

(81)

Hence, the canonical momentum is given by

\[ P_D := \frac{\partial L}{\partial \dot{D}} = \frac{m\dot{D}}{(D^2 - 1)\sqrt{D^2 - \frac{\dot{D}^2}{D^2 - 1} - (D^2 - 1)\dot{\phi}^2}}. \]  

(82)

The Hamiltonian of the classical particle with \(\dot{\phi} = 0\) is given by

\[ H = D\sqrt{m^2 + (D^2 - 1)P_D}. \]  

(83)

Thanks to the factorization property of the geometric state, we could guess the position operator should be

\[ \hat{D} := \cos \hat{\rho} = \hat{A} = \hat{\Delta}_a^2 + \hat{\Delta}^2_{-\rho} = 1/2. \]  

(84)

The Eq. \((83)\) gives the expression of the canonical momentum operator

\[ \hat{P}_D = \hat{P}_\rho(\Delta_a^2 + \Delta_{-\rho}^2)^{1/2}(\Delta_a^2 - \Delta_{-\rho}^2)^{-1/2}. \]  

(85)
One could check the above operators by comparing the expectation value of them with the classical geodesic solution.

The coordinate transformation is a special case of the canonical transformation of the phase space of the classical particle. One could choose any set of the canonical variables. And the corresponding position and momentum operators can be constructed by the same methods as above.

6.4 How to understand gravity in the CFTs?

Our results provide a framework to explore the explanation of gravity in the CFTs. In general relativity the dynamics of the particle is given by the geodesic equation. The geodesic equation is equal to equation of motion for the system with Lagrangian \( L = -m \int d\tau \), where \( \tau \) is the proper time of the particle.

In our approach we assume the existence of the CFT operators that are dual to the canonical variables of the particle. Moreover, we construct the operators, the expectation values of which give the particle’s position and momentum. A remarkable fact is that all these operators are constructed by the Hermitian operators \( \hat{P}_\rho, \hat{Q}_\rho, \hat{S}_\rho, \hat{T}_\rho, \hat{H} \) and \( \hat{P}_\phi \), which are independent linear combinations of global Virasoro generators.

We also show how to obtain the classical Poisson brackets from the quantum commutators of the constructed operators. The Hamiltonian equations, i.e., the geodesic equations, can be expressed by the Poisson brackets as we have shown in Sect. 5.5. Hence, the dynamics of the bulk particle is determined by the commutation relations algebra (86). Further, these relations can be derived from the stress energy tensor commutators, i.e., \([T_{\mu\nu}, T_{\rho\sigma}]\). The geodesic equation should have a correspondence to the stress energy tensor commutator, at least in our special cases. However, we have no evidence to conclude that the correspondence is also true for general background geometry. It is worth to study more general examples and make the correspondence more explicit.

Recently, Susskind proposes the size-momentum correspondence, that gives a connection between radial momentum of a bulk particle, operator size and complexity [47]. The size of the operator is expected to be proportional to the radial momentum of a bulk moving particle. One could refer to [47–49] for the definition of these concepts. The operator size can be evaluated in SYK models [50,51]. For general theories, such as holographic field theory, it is expected the operator size may be associated with a Hermitian operator [52,53]. Follow the notation of [52], the size of operator \( O \) that act on reference state \( |\Psi\rangle \) is given by

\[
S_{\langle \Psi |O|\Psi \rangle} := \frac{\langle \Psi |O^\dagger \hat{S}_{\langle \Psi |O|\Psi \rangle} |\Psi \rangle}{\langle \Psi |O^\dagger O|\Psi \rangle},
\]

where \( \hat{S}_{\langle \Psi |O|\Psi \rangle} \) is expected to be a semi-definite, Hermitian operator.

Actually, in our approach the radial momentum of the bulk moving particle is also given by the expectation value of the operator \( \hat{P}_\rho \). It is very interesting to check whether the operator size operator \( \hat{S}_{\langle \Psi |O|\Psi \rangle} \) has some connection with the radial momentum operator \( \hat{P}_\rho \) in our paper.

Acknowledgements I would like to thank Qin Qin for useful discussions. I also would like to thank the anonymous referee on her/his comments on the significance of the isometries in the bulk for the constructions of the dual states and operators. I am supported by the National Natural Science Foundation of China under Grant no. 12005070 and the Fundamental Research Funds for the Central Universities under Grants no. 2020kfyXJS041.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: This is a paper on theoretical physics. No datasets were generated or analysed in this work.]

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Appendix A: Bulk locally excited states

The bulk scalar operator \( \hat{\phi}_\alpha(X^\mu) \) satisfies the equation of motion on the background geometry \( g_{\mu\nu}(X^\mu) \),

\[
(\Box^2_{\mu\nu} + m^2)\hat{\phi}_\alpha(X^\mu) = 0,
\]

where \( m \) is the mass of scalar field. Suppose the metric \( g_{\mu\nu} \) can be associated with a geometric state |\( \Psi(g_{\mu\nu}) \rangle \). The bulk local state is defined as \( |\phi_\alpha(X^\mu)\rangle = \hat{\phi}_\alpha(X^\mu)|\Psi(g_{\mu\nu})\rangle \), where \( X^\mu \) is the coordinate of the local operator. It is expected the bulk operator \( \hat{\phi}_\alpha(X^\mu) \) can be expanded by the CFT operators. Thus the bulk local state \( |\phi_\alpha(X^\mu)\rangle \) can be taken as states in Hilbert space of the CFT.

We only focus on the vacuum state |0\rangle. Consider the global coordinate, the state located in the origin of AdS \( \rho = 0 \), denoted by \( |\Psi_\alpha\rangle \), can be expanded as the superposition of Ishibashi states [25]

\[
|\Psi_\alpha\rangle = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\Delta_\alpha)}{\Gamma(k+1)\Gamma(\Delta_\alpha+k)} L^k_{-1} \hat{L}^k_{-1} |\alpha\rangle,
\]

where \( L^k_{-1} |\alpha\rangle \) are the bulk local states for the CFT.
where $\Delta_a = h_a + \hat{h}_a$ is the conformal dimension of primary operator $O_a$, the primary state $|O_a \rangle := \lim_{\varepsilon \to 0} O_a |0 \rangle$. The standard AdS/CFT dictionary gives the relation $m = \sqrt{\Delta_a (\Delta_a - 2)}$.

The bulk local states at point $(\rho, \phi)$ can be associated with $|\Psi_\alpha \rangle$ by a unitary transformation $g(\rho, \phi)$. The bulk local state at point $(\rho, \phi)$ is given by

$$|\Psi_\alpha(\rho, \phi) \rangle = g(\rho, \phi)|\Psi_\alpha \rangle,$$

with the unitary operator

$$g(\rho, \phi) = e^{i(L_0 - \bar{L}_0)\phi} e^{-\frac{\phi}{2}(L_1 - L_{-1} + \bar{L}_1 - \bar{L}_{-1})}.$$ (90)

The state $|\Psi_\alpha \rangle$ is unnormalized since the local operator $\hat{\phi}_\alpha$ is unbounded operator. We can introduce a regulator $\Lambda$ and define the state

$$|\Psi_\alpha \rangle_\Lambda := \mathcal{N}(\Lambda) e^{-\Lambda \hat{H}} |\Psi_\alpha \rangle,$$ (91)

where $\hat{H} := L_0 + \bar{L}_0$, the normalization constant $\mathcal{N}(\Lambda) = e^{-\Lambda \Delta_a \sqrt{1 - e^{-4\Lambda}}}$. It is straightforward to obtain the following one-point functions of $L_n$

$$\Lambda \langle \Psi_\alpha | L_0 | \Psi_\alpha \rangle_\Lambda = h_a + \frac{1}{e^{4\Lambda} - 1}$$ (92)

and $\Lambda \langle \Psi_\alpha | L_n | \Psi_\alpha \rangle_\Lambda = 0$ for $n \neq 0$. It is also useful to evaluate the two-point functions

$$\Lambda \langle \Psi_\alpha | L_0^2 | \Psi_\alpha \rangle_\Lambda = h_a^2 + \frac{2h_a}{e^{4\Lambda} - 1} + \frac{e^{4\Lambda} + 1}{(e^{4\Lambda} - 1)^2},$$

$$\Lambda \langle \Psi_\alpha | L_1 L_{-1} | \Psi_\alpha \rangle_\Lambda = \frac{2e^{4\Lambda} h_a}{e^{4\Lambda} - 1} + \frac{2e^{4\Lambda}}{(e^{4\Lambda} - 1)^2},$$

$$\Lambda \langle \Psi_\alpha | L_1^2 | \Psi_\alpha \rangle_\Lambda = \Lambda \langle \Psi_\alpha | L_1^2 | \Psi_\alpha \rangle_\Lambda = 0.$$ (93)

More generally, we have

$$\Lambda \langle \Psi_\alpha | L_n^m | \Psi_\alpha \rangle_\Lambda = h_a^m + \sum_{m-k} h_a^{m-k} C_k \sum_{n=0}^{\infty} n^k e^{-4\Lambda n} (1 - e^{-4\Lambda}),$$ (94)

where $C_k := \frac{m!}{k!(m-k)!}$. As we have argued in the beginning of Sect. 2 we are interested in the case $\Delta_a \sim O(c)$ in the holographic CFTs with $c \gg 1$. Define the operator $l_n := L_n/c$, which can be taken as the classical operator. In the regime of $\Lambda \gg 1$, we would have the following clustering property for $l_n$,

$$\Lambda \langle \Psi_\alpha | l_n^m | \Psi_\alpha \rangle_\Lambda = \Lambda \langle \Psi_\alpha | l_n | \Psi_\alpha \rangle_\Lambda^m + O(c^{-1}),$$ (95)

for $n = -1, 0, +1$. This is a necessary condition for the geometric states as we have discussed in the introduction. Here we would like to explain the state $|\Psi_\alpha \rangle_\Lambda$ with $\Delta_a \sim O(c)$ and $\Lambda \gg 1$ to be dual to a particle with mass $m$ at rest in the center of AdS in the global coordinate (8). We have the parameter relation $\Delta_a \approx m$.

The energy in the state $|\Psi_\alpha \rangle_\Lambda$ is given by

$$E = \Lambda \langle \Psi_\alpha | H_g | \Psi_\alpha \rangle_\Lambda = \Delta_a - \frac{c}{12} + O(c^0),$$ (96)

where $H_g := L_0 + \bar{L}_0 - \frac{c}{12}$ is the Hamiltonian of the boundary CFT in the global coordinate. The constant $-\frac{c}{12}$ is the Casimir energy in the vacuum of global coordinate. This is consistent with the holographic result of a stationary particle with mass $m$ at $\rho = 0$ by using the fact $\Delta_a \approx m$ at leading order of $c$.

In fact, one could take the limit $\Lambda \to \infty$. We have $|\Psi_\alpha \rangle_\Lambda \to |O_a \rangle_0$, i.e., the primary state $|O_a \rangle$. This result is consistent with the discussion in [30].

**Appendix B: Commutators**

In this section we will show the details of the calculation of commutators (73).

Consider the commutator $[\hat{P} \rho, \hat{\phi}]_{\psi_\alpha(t)}$. By definition we have

$$[\hat{P} \rho, \hat{\phi}]_{\psi_\alpha(t)} = [\hat{P} \rho \cos \phi - \hat{\tau} \rho \sin \phi, \hat{\phi}]_{\psi_\alpha(t)} = [\hat{P} \rho \cos \phi - [\hat{\tau} \rho, \hat{\phi}] \sin \phi]_{\psi_\alpha(t)}.$$ (96)

It is useful to evaluate $[\hat{P} \rho, \hat{\phi}]_{\psi_\alpha(t)}$. By using (66) we have

$$[\hat{P} \rho, \hat{\phi}]_{\psi_\alpha(t)} = [\hat{P} \rho, \hat{\phi}]_{\psi_\alpha(0)} = [\hat{P} \rho \cos \phi - [\hat{\tau} \rho, \hat{\phi}] \sin \phi]_{\psi_\alpha(0)}.$$ (97)

Similarly, we have

$$[\hat{\tau} \rho, \hat{\phi}]_{\psi_\alpha(t)} = [\hat{\tau} \rho, \hat{\phi}]_{\psi_\alpha(0)} = [\hat{\tau} \rho \cos \phi - [\hat{\tau} \rho, \hat{\phi}] \sin \phi]_{\psi_\alpha(0)} = 0.$$ (98)
Using the above results we can obtain (97), the result is

\[
\begin{align*}
&\langle [\hat{P}_\rho, \hat{\phi}] \rangle_{\psi_a(t)} \\
&= \sum_n b_n \left\{ \langle [\hat{P}_\rho, \hat{B}^n] \right\} \cos \phi - [\hat{T}_\rho, \hat{B}^n] \sin \phi \} \right\}_{\psi_a(t)} \\
&= \sum_n b_n \left\{ \langle [\hat{P}_\rho, \hat{B}] \hat{B}^{n-1} \rangle \cos \phi - [\hat{T}_\rho, \hat{B}] \hat{B}^{n-1} \sin \phi \} \right\}_{\psi_a(t)} \\
&\quad + O(c^{-1}) \\
&= \sum_n b_n \left\{ \langle \hat{B}^{n-1} \right\} \hat{H} \hat{S}_\phi \left( \hat{Q}_\rho^2 + \hat{S}_\phi^2 \right)^{-3/2} \right\}_{\psi_a(t)} \\
&\quad \times \left\{ -i \hat{S}_\phi \cos \phi + i \hat{Q}_\rho \sin \phi \} \right\}_{\psi_a(t)} + O(c^{-1}) \\
&= 0 + O(c^{-1}),
\end{align*}
\]

(99)

where in the last step we use the fact \(\langle \hat{S}_\phi \cos \phi - \hat{Q}_\rho \sin \phi \} \rangle_{\psi_a(t)} = O(c^{-1}).

Therefore, we have the classical Poisson bracket

\[
\lim_{c \to \infty} ic \langle [\hat{P}_\rho, \hat{\phi}] \rangle_{\psi_a(t)} = 0.
\]

(100)

Now let’s turn to the commutator \([\hat{P}_\rho, \hat{\phi}]\). We should calculate \([\hat{F}, \hat{p}_\rho]\), where \(\hat{F}\) is defined as (64). With some tedious but straightforward calculations we find \([\hat{F}, \hat{p}_\rho] \propto \langle [\hat{P}_\rho, \hat{\phi}] \rangle_{\psi_a(t)}\). We have shown that \(\lim_{c \to \infty} ic \langle [\hat{P}_\rho, \hat{\phi}] \rangle_{\psi_a(t)} = 0\). One could conclude that

\[
\lim_{c \to \infty} ic \langle [\hat{P}_\rho, \hat{\phi}] \rangle_{\psi_a(t)} = 0.
\]

(101)

We leave it as an exercise for interested reader to prove \(\lim_{c \to \infty} ic \langle [\hat{P}_\rho, \hat{\phi}] \rangle_{\psi_a(t)} = 0\).

Finally, let’s show the last commutator in (73), \(\hat{P}_\rho\) is a function of \(\hat{H}, \hat{P}_\rho\) and \(\hat{\phi}\). We will need the commutator \([\hat{P}_\rho, \hat{H}]\), which can be written as

\[
\begin{align*}
[\hat{P}_\rho, \hat{H}] &= [\hat{P}_\rho \cos \phi - \hat{T}_\rho \sin \phi, \hat{H}] \\
&= [\hat{P}_\rho, \hat{B} - \hat{T}_\rho (1 - \hat{B}^2)^{1/2}, \hat{H}] \\
&= [\hat{P}_\rho, \hat{H}] \hat{B} + \hat{P}_\rho [\hat{B}, \hat{H}] - [\hat{T}_\rho, \hat{H}] (1 - \hat{B}^2)^{1/2} \hat{B} \\
&\quad - \hat{T}_\rho [1 - (1 - \hat{B}^2)^{1/2}, \hat{H}] \\
&= -i \hat{Q}_\rho \hat{B} + \hat{P}_\rho [\hat{B}, \hat{H}] - i \hat{S}_\phi (1 - \hat{B}^2)^{1/2} \hat{B} \\
&\quad - \hat{T}_\rho [1 - (1 - \hat{B}^2)^{1/2}, \hat{H}].
\end{align*}
\]

(102)

The expectation value of \([\hat{P}_\rho, \hat{H}]\)

\[
\begin{align*}
\langle [\hat{P}_\rho, \hat{H}] \rangle_{\psi_a(t)} &= -i \left( \langle \hat{Q}_\rho^2 + \hat{S}_\phi^2 \rangle (2\hat{Q}_\rho^2 - \hat{T}_\rho^2) - 2\hat{Q}_\rho \hat{P}_\rho \hat{S}_\phi \hat{T}_\rho + \hat{S}_\phi^2 (\hat{Q}_\rho^2 - \hat{T}_\rho^2) \right) \\
&\quad \times \left( \hat{Q}_\rho^2 + \hat{S}_\phi^2 \right)^{-3/2} \psi_a(t) + O(c^0).
\end{align*}
\]

(103)

where we have used

\[
\begin{align*}
\langle [\hat{B}, \hat{H}] \rangle_{\psi_a(t)} &= i \langle [\hat{P}_\rho, \hat{S}_\phi + \hat{Q}_\rho \hat{T}_\rho] \rangle_{\psi_a(t)} O(c^{-1}).
\end{align*}
\]

(104)

Now recall the definition of \(\hat{r}\) (64), we obtain the commutator

\[
\begin{align*}
\langle [\hat{r}, \hat{P}_\rho] \rangle_{\psi_a(t)} &= -2i \left\{ \hat{H} (\hat{P}_\rho, \hat{T}_\rho) \right\} (\hat{H} - \hat{P}_\rho^2 - \hat{Q}_\rho^2 - \hat{T}_\rho^2) (\hat{P}_\rho^2 + \hat{Q}_\rho^2)^{-1} \\
&\quad \times \left\{ \left( \hat{H}^2 - \hat{P}_\rho^2 - \hat{Q}_\rho^2 + \hat{T}_\rho^2 - 4 \hat{P}_\rho^2 \hat{Q}_\rho^2 + \hat{Q}_\rho^2 \right)^{-1/2} \right\}_{\psi_a(t)} \\
&\quad + O(c^{-1}).
\end{align*}
\]

(105)

Thus we have

\[
\langle [\hat{r}, \hat{P}_\rho] \rangle_{\psi_a(t)} = \frac{1}{\sqrt{\langle r^2 \rangle_{\psi_a(t)}} - 1}
\]

(106)

Taking (105) into the above expression and replacing the operator by their expectation, we finally obtain the expected result

\[
\langle [\hat{r}, \hat{P}_\rho] \rangle_{\psi_a(t)} = -i + O(c^{-1}).
\]

(107)

Therefore, the classical Poisson bracket is given by

\[
\lim_{c \to \infty} ic \langle [\hat{r}, \hat{P}_\rho] \rangle_{\psi_a(t)} = 1.
\]

(108)

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