Prime Coloring of Crossing Number Zero Graphs

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ABSTRACT

In this paper, prime coloring and its chromatic number of some crossing number zero graphs are depicted and its results are validated with few theorems. Prime Coloring is defined as G be a loop less and without multiple edges with n distinct Vertices on Color class C={c1, c2, c3, ..., cn} a bijection ψ: V(G)→ Z∗−{0}, if for each edge e = ci, cj, i≠j, gcd {ψ (ci), ψ (cj)}=1, ψ (ci) and ψ (cj) receive distinct colors. The Chromatic number of Prime coloring is minimum cardinality taken by all the Prime colors. It is denoted by η(G).

1. Introduction

We use the terminology as Bondy.J.A. and Murty U.S.R (1976). Let G= (V, E) be simple graph with crossing number zero i.e (Cr(G)=0) and here order of V be |V| = n. In this paper, we discuss Prime graphs and Coloring. The Prime labeling emanate with Entringer et al. 1980. Be the graph G = (V, E) with vertex set V is said to have a Prime labeling if there exists a bijection f: V(G)→Z∗−{0}, Z∗ is the set of positive integers such that for each edge xy ∈ E(G),gcd |f(x),f(y)|=1. It begin to be an interesting problem to investigate the Prime coloring as possible in a graph with n vertices. Moreover, the general graph G is proper colorable but not prime graph forever. The Coloring of a graph is explained as pair wise adjacent vertices receive conflict colors.

2. Prime Coloring of Crossing Number Zero Graphs

In this entire paper, we consider ψ is the bijective function and ψ′ is the Chromatic number of G as well as η is also chromatic number of prime graph.

Theorem 2.1
Every path Pn graph is Prime if it is Prime colorable graph.

Proof: Let the path Pn with n distinct vertices, the bijection ψ: V(Pn)→ {c1, c2, c3, ..., cn} graph if for each edge e = cici, gcd {ψ (ci), ψ (cj)}=1, i≠j,
ψ (ci) = ci
ψ (cj) = cj receive distinct colors. Then the path Pn Graph is Prime colorable graph.

Corollary 2.1.1: If path Pn graph is prime then η(Pn) = 2.

Example 2.1: Pn is Prime colorable graph

Cr(G)=0, gcd {ψ (ci), ψ (cj)}=1, η(Pn) = 2.

Figure 1

Theorem 2.2: Every Cycle Cn, n≥4 if it is Prime colorable graph.

Proof: Assume that Cycle Cn on n≥4 vertices with the bijection ψ: V(Cn)→ {c1, c2, c3, ..., cn} graph if for each edge e = cici, gcd {ψ (ci), ψ (cj)}=1,i≠j,
ψ (ci) = ci
ψ (cj) = cj receive distinct colors on cycle Cn on n≥4.

Corollary 2.2.1: If Cycle graph Cn, n≥4 is Prime,
then \( \eta(C_n) = \begin{cases} 
2, & \text{if } n \text{ is even} \\
3, & \text{if } n \text{ is odd} 
\end{cases} \)

**Example 2.2:** Cycle \( C_4 \)

\[
\text{Cr}(G)=0, \gcd \{\psi(c_i), \psi(c_j)\}=1, \eta(C_4) = 2
\]

**Figure 2**

**Example 2.3**

The complete graph \( K_4 \) is Prime Cloring graph.

\[
\text{Cr}(G)=0, \gcd \{\psi(c_i), \psi(c_j)\}=1, \eta(K_4) = 3.
\]

**Figure 4**

Theorem 2.3 If a Complete graph \( K_n \), \( n \leq 3 \) is Prime graph, then \( K_n \) is Prime colorable graph with \( n \leq 3 \).

**Proof**

Proof by induction \( n=1 \) obviously holds.

\( n=2 \) then the graph is a path by the theorem (2.1) holds.

\( n=3 \) then the graph is triangle every \( \gcd \{\psi(c_i), \psi(c_j)\} = 1, i \neq j \) then the graph is prime and also receive distinct colors.

\( n=4 \) then the vertices assigns distinct colors but \( \gcd \{\psi(c_i), \psi(c_j)\} \neq 1 \). But consider the general \( K_4 \) is proper colorable with Chromatic number \( \psi'(K_4) = 4 \), but \( \eta(K_4) \) not defined.

**Complete graph \( K_4 \)**

\[
\text{Cr}(G)\neq0, \gcd \{\psi(c_i), \psi(c_j)\}=1, \gcd \{\psi(c_i), \psi(c_j)\}=1, \gcd \{\psi(c_i), \psi(c_j)\}=1, \text{But } \gcd \{\psi(c_i), \psi(c_j)\} \neq 1.
\]

**Figure 5**

Consider \( K_5 \) is complete graph pair wise adjacent vertices are connected by the definition of graph coloring it has to assign distinct colors, but the labeled graph \( K_n \) with \( \{1,2,\ldots,n\} , \gcd (1,n) \text{ not equal to 1 for } n \geq 3 \), \( K_n \) is not Prime graph. Thus \( K_n \) is not a Prime colorable.

**Corollary 2.3.1** Every Complete graph \( K_n \) is Proper Colorable but not Prime colorable.

**Figure 6**

Here, \( \text{Cr}(G)\neq0, \gcd \{\psi(c_i), \psi(c_j)\}=1, \gcd \{\psi(c_i), \psi(c_j)\}=1, \gcd \{\psi(c_i), \psi(c_j)\}=1, \text{But } \gcd \{\psi(c_i), \psi(c_j)\} \neq 1. \)
\[
gcd \{ \psi(c_i), \psi(c_j) \} = 1,\\
gcd \{ \psi(c_i), \psi(c_j) \} = 1,\\
gcd \{ \psi(c_i), \psi(c_j) \} = 1,\\
gcd \{ \psi(c_i), \psi(c_j) \} = 1.
\]

But \( \gcd \{ \psi(c_i), \psi(c_j) \} \neq 1 \). Consider the General \( K_2 \) is proper colorable with chromatic number \( \psi'(K_2) = 5 \). But \( \eta(K_2) \) is not defined.

**Theorem 2.4** If Star graphs \( K_{1,n} \) is Prime graph then \( \eta(K_{1,n}) = 2 \).

**Proof**
Let \( V = \{ c_1, c_2, c_3, \ldots, c_{n+1} \} \) vertices of Star graph \( K_{1,n} \). The graph labeled as well as colored by the bijection \( \psi : V(K_{1,n}) \to \{ c_1, c_2, c_3, \ldots, c_{n+1} \} \) graph if for each edge \( e = c_i c_j, \gcd \{ \psi(c_i), \psi(c_j) \} = 1, i \neq j \) \( \psi(c_i) \) and \( \psi(c_j) \) receive distinct colors. Therefore the coloring pairs of are the distinct. The coloring pairs are at most 2 and also every \( \gcd (i_j) = 1 \) is a prime graph.

Thus,
\[
\psi(c_i) = c_i, i \leq i_j \leq n + 1 \text{ then every } i \neq j \text{ is Prime colorable. Therefore minimum coloring needed for the Star graph is } \eta(K_{1,n}) = 2.
\]

**Example 2.4:** Star graph \( K_{1,n} \)

\[
\psi(c_i) = c_i, i \leq i_j \leq n + 1 \text{ then every } i \neq j \text{ is Prime colorable. Therefore minimum coloring needed for the Star graph is } \eta(K_{1,n}) = 2.
\]

**Theorem 2.5** If Wheel graph \( W_n \), \( n \geq 4 \) is Prime graph, then \( \eta(W_n) = 2 \) when \( n \) is even.

**Proof**
Let \( V = \{ c_1, c_2, c_3, \ldots, c_{n+1} \} \) vertices of Wheel graph \( W_n \). The graph labeled as well as colored by the bijection \( \psi : V(W_n) \to \{ c_1, c_2, c_3, \ldots, c_{n+1} \} \) graph if for each edge \( e = c_i c_j, \gcd \{ \psi(c_i), \psi(c_j) \} = 1, i \neq j \) \( \psi(c_i) \) and \( \psi(c_j) \) receive distinct colors. Therefore the coloring pairs of are the distinct. Therefore the coloring pairs are at most 4 and also \( \gcd (i_j) = 1 \) is a prime graph.

Thus, \( \psi(c_i) = c_i \) for \( 1 \leq i_j \leq n+1 \).

Then every \( i \neq j \) is prime colorable. Therefore minimum coloring needed for the Wheel graph is \( \eta(W_n) = 2 \).

**Example 2.5** Wheel graph \( W_5 \)

\[
\eta(W_n) = 2.
\]

**3. Main Results**

**Lemma 3.1:** \( G \) is Prime graph \( \iff \) \( G \) is Prime colorable graph.

**Proof**
Let us assume \( G \) is prime label with the bijection \( \psi : V(G) \to \{ c_1, c_2, c_3, \ldots, c_n \} \) graph if for each edge \( w = c_i c_j, \gcd \{ \psi(c_i), \psi(c_j) \} = 1, i \neq j \) \( \psi(c_i) \) and \( \psi(c_j) \) receive distinct colors. Now coloring the vertices is proper coloring. Then the resulting graph with different colors. Conversely \( G \) is Prime coloring by the definition of graph coloring holds it is a prime graph.

**Corollary 3.1.1** If \( G \) is Prime Graph then \( \psi'(G) = \eta(G) \)

**Corollary 3.1.2** Every spanning sub graph of prime graph is prime colorable graph.

Seoud and youse et.al (1999) proved every spanning sub graph is prime graph. Now coloring proper by the graph become an prime colorable \( \gcd \{ \psi(c_i), \psi(c_j) \} = 1, i \neq j \) \( \psi(c_i) \) and \( \psi(c_j) \) receive distinct colors. We observe that every spanning sub graph is also prime colorable.

**Corollary 3.1.3:** Every tree is prime colorable graph.

Every tree is prime proved by Entringer \( n \leq 50 \) vertices. The tree have been properly colored as the color class \( C = \{ c_1, c_2, c_3, \ldots, c_n \} \). Tree is a spanning sub graph of the general graph \( G \) by the above corollary (3.1.1) holds, there for every tree is prime colorable.

**Corollary 3.1.4:** Every prime graph is proper colorable graph.

**Corollary 3.1.5:** Every graph \( G \) is proper Colorable but not prime color.

**Lemma 3.2**
If \( \psi'(P_n) = \eta(P_n) = 2 \), then the following are equivalent
i. $P_n$ is Path.
ii. $P_n$ is Prime graph.
iii. $P_n$ is Prime colorable graph.

Proof

(i) ⇒ (ii) let the path $P_n$ with $n$ distinct vertices then the chromatic number of the graph $\psi(P_n) = 2$ we have to prove $P_n$ is a prime graph with $\gcd\{\psi(c_i),\psi(c_j)\} = 1$, $i \neq j$ then $P_n$ is a prime graph.

(ii) ⇒ (iii)

By the theorem (2.1) refers to every prime graph $P_n$ which assigned the distinct colors and by the definition of a prime color then $\eta(P_n) = 2$.

(iii) ⇒ (i) from the proof given prime colorable graph $\psi(P_n) = \eta(P_n) = 2$ Then $n$ distinct vertices are connected as a path.

Lemma 3.3

If $\psi(C_n) = \eta(C_n) = \begin{cases} 2, & \text{n is even} \\ 3, & \text{n is odd} \end{cases}$, then the following are equivalent

i. $C_n$ is Cycle
ii. $C_n$ is Prime graph.
iii. $C_n$ is Prime colorable graph.

Proof

Case (i) n is odd:

(i) ⇒ (ii)

Consider $C_n$ be the cycle with distinct $n$ vertices. If Cycle $C_n$, $n>3$ is an odd cycle with $2n-1$ vertices (Samir K. Vaidya, Udayan M. Prajapati (2012)) et al., by the Fusion, switching, duplication of the vertices than the graph is the prime graph.

(ii) ⇒ (iii)

Consider $C_n$ is prime graph when $n$ is odd, be the fusion, switching, duplication is allowed then the graph is prime by the definition of prime coloring $C_n$ is prime colorable graph.

(iii) ⇒ (i)

By the definition prime coloring , all the vertices are distinct when $n$ is odd i.e $c_i = c_{2n-1}$, $n>3$ and holds $\psi(C_n) = \eta(C_n) = 3$.

Case (ii) n is even

(i) ⇒ (ii) Clearly the even number of vertices on the cycle is prime and $\gcd\{\psi(c_i),\psi(c_j)\} = 1$, $i \neq j$

(ii) ⇒ (iii)

By the theorem (2.2) the graph $\gcd\{\psi(c_i),\psi(c_j)\} = 1$, $i \neq j$, then $\psi(C_n) = \eta(C_n) = 2$ pair wise adjacent vertices receive distinct colors.

(iii) ⇒ (i)

$\psi(C_n) = \eta(C_n) = 2$ with the Cycle $C_n$, $n>3$ we get $c_i = c_{2n-2}$, $i \neq j$ vertices are distinct and even vertices of $C_n$ is cycle.

Lemma 3.4 If G is the prime graph and $\Delta$ is the maximum degree of the graph then $\eta(G) \leq \Delta + 1$

Conclusion

This article discussed & absorbed the upshot of graph coloring technique which associated with the prime labeled graph. We elucidated the coloring techniques of some crossing number zero graphs and also evinced the some interesting results obtained by Prime Coloring graphs.

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