REMARK ON ARITHMETIC TOPOLOGY

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Abstract. We formalize the arithmetic topology, i.e. a relationship between knots and primes. Namely, using the notion of a cluster \( C^* \)-algebra we construct a functor from the category of 3-dimensional manifolds \( \mathcal{M} \) to a category of algebraic number fields \( K \), such that the prime ideals (ideals, resp.) in the ring of integers of \( K \) correspond to knots (links, resp.) in \( \mathcal{M} \). It is proved that the functor realizes all axioms of the arithmetic topology conjectured in the 1960’s by Manin, Mazur and Mumford.

1. Introduction

The famous Weil’s Conjectures expose an amazing link between topology and number theory [Weil 1949] [12]. Likewise, the arithmetic topology studies an analogy between knots and primes; we refer the reader to the book [Morishita 2012] [7] for an excellent introduction. To give an idea of the analogy, we quote [Mazur 1964] [6]:

“Guided by the results of Artin and Tate applied to the calculation of the Grothendieck Cohomology Groups of the schemes:

\[ \text{Spec } (\mathbb{Z}/p\mathbb{Z}) \subset \text{Spec } \mathbb{Z} \]

Mumford has suggested a most elegant model as a geometric interpretation of the above situation: \( \text{Spec } (\mathbb{Z}/p\mathbb{Z}) \) is like a one-dimensional knot in \( \text{Spec } \mathbb{Z} \) which is like a simply connected three-manifold.”

Roughly speaking, the idea is this. The 3-dimensional sphere \( \mathbb{R}^3 \) corresponds to the field of rational numbers \( \mathbb{Q} \). The prime ideals \( p \mathbb{Z} \) in the ring of integers \( \mathbb{Z} \) correspond to the knots \( \mathcal{K} \subset \mathbb{R}^3 \) and the ideals in \( \mathbb{Z} \) correspond to the links \( \mathcal{L} \subset \mathbb{R}^3 \). In general, a 3-dimensional manifold \( \mathcal{M} \) corresponds to an algebraic number field \( K \). The prime ideals in the ring of integers \( \mathcal{O}_K \) of the field \( K \) correspond to the knots \( \mathcal{K} \subset \mathcal{M} \) and the ideals in \( \mathcal{O}_K \) correspond to the links \( \mathcal{L} \subset \mathcal{M} \).

The aim of our note is a functor from the category of closed 3-dimensional manifolds to a category of algebraic number fields realizing all axioms of the arithmetic topology. The construction of such a functor is based on a representation of the braid group into a cluster \( C^* \)-algebra [9].

Namely, denote by \( S_{g,n} \) a Riemann surface of genus \( g \) with \( n \) cusps. Recall that a cluster algebra \( \mathcal{A}(x, S_{g,n}) \) of the \( S_{g,n} \) is a subring of the ring of the Laurent polynomials \( \mathbb{Z}[x^{\pm 1}] \) with integer coefficients and variables \( x := (x_1, \ldots, x_{6g-6+2n}) \). The algebra \( \mathcal{A}(x, S_{g,n}) \) is a coordinate ring of the Teichmüller space \( T_{g,n} \) of surface

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$S_{g,n}$ [Williams 2014, Section 3] [13]. The $\mathcal{A}(x, S_{g,n})$ is a commutative algebra with an additive abelian semigroup consisting of the Laurent polynomials with positive coefficients. In particular, the $\mathcal{A}(x, S_{g,n})$ is an abelian group with order satisfying the Riesz interpolation property, i.e. a dimension group [Effros 1981, Theorem 3.1] [2]. By a cluster $C^*$-algebra $\mathcal{A}(x, S_{g,n})$ we understand an Approximately Finite $C^*$-algebra (AF-algebra) such that $K_0(\mathcal{A}(x, S_{g,n})) \cong \mathcal{A}(x, S_{g,n})$, where $\cong$ is an isomorphism of the dimension groups. We refer the reader to [9] for the details and examples. The algebra $\mathcal{A}(x, S_{g,n})$ is a non-commutative coordinate ring of the Teichmüller space $T_{g,n}$. This observation and the Birman-Hilden theorem imply a representation

$$\rho : B_{2g+n} \to \mathcal{A}(x, S_{g,n}), \quad n \in \{0; 1\},$$

(1.2)

where $B_{2g+n} := \{\sigma_1, \ldots, \sigma_{2g+n-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2\}$ is the braid group, the map $\rho$ acts by the formula $\sigma_i \mapsto e_i + 1$ and $e_i$ are projections of the algebra $\mathcal{A}(x, S_{g,n})$ [10].

Let $b \in B_{2g+n}$ be a braid. Denote by $\mathcal{L}_b$ a link obtained by the closure of $b$ and let $\pi_1(\mathcal{L}_b)$ be the fundamental group of $\mathcal{L}_b$. Recall that

$$\pi_1(\mathcal{L}_b) \cong \langle x_1, \ldots, x_{2g+n} | x_i = r(b)x_i, \quad 1 \leq i \leq 2g+n \rangle,$$

(1.3)

where $x_i$ are generators of the free group $F^{2g+n}$ and $r : B_{2g+n} \to Aut(F^{2g+n})$ is the Artin representation of $B_{2g+n}$ [Artin 1925] [1, Theorem 6]. Let $I_b$ be a two-sided ideal in the algebra $\mathcal{A}(x, S_{g,n})$ generated by relations (1.3). In particular, the ideal $I_b$ is self-adjoint and representation (1.2) induces a representation

$$R : \pi_1(\mathcal{L}_b) \to \mathcal{A}(x, S_{g,n}) / I_b$$

(1.4)

on the quotient $\mathcal{A}(x, S_{g,n}) / I_b := \mathcal{A}_b$, see lemma 3.1. The $\mathcal{A}_b$ is a stationary AF-algebra of rank $6g - 6 + 2n$; we refer the reader to [Effros 1981] [2, Chapter 5] or Section 2.2 for the definitions. It is known that the group $K_0(\mathcal{A}_b) \cong O_K$, where $K$ is a number field of degree $6g - 6 + 2n$ over $\mathbb{Q}$ [Effros 1981] [2, Chapter 5].

Thus we obtain a map $F : \mathcal{L} \to \mathcal{O}$, where $\mathcal{L}$ is a category of all links $\mathcal{L}$ modulo a homotopy equivalence and $\mathcal{O}$ is a category of rings of the algebraic integers $O_K$ modulo an isomorphism. The map $F$ acts by the formula:

$$\mathcal{L} \xrightarrow{\pi_1(\mathcal{L}_b)} \mathcal{A}_b \xrightarrow{K_0} O_K,$$

(1.5)

where $R$ is given by (1.4).

**Remark 1.1.** Using the Lickorish-Wallace Theorem [Lickorish 1962] [5], one can extend the map $F$ to a category of 3-dimensional manifolds. Indeed, recall that if $\mathcal{M}$ is a closed, orientable, connected 3-dimensional manifold, then there exists a link $\mathcal{L} \subset \mathcal{I}^3$ such that the Dehn surgery of $\mathcal{L}$ with the $\pm 1$ coefficients is homeomorphic to $\mathcal{M}$. (Notice that such a link is not unique, but using the Kirby calculus one can define a canonical link $\mathcal{L}$ attached to $\mathcal{M}$.) Thus we get a map $\mathcal{M} \mapsto \mathcal{L}$.

**Theorem 1.2.** The map $F$ is a functor, such that:

(i) $F(\mathcal{I}^3) = \mathbb{Z}$;

(ii) each ideal $I \subseteq O_K = F(\mathcal{M})$ corresponds to a link $\mathcal{L} \subset \mathcal{M}$;

(iii) each prime ideal $I \subseteq O_K = F(\mathcal{M})$ corresponds to a knot $\mathcal{K} \subset \mathcal{M}$.

The article is organized as follows. Section 2 contains a brief review of braids, links and cluster $C^*$-algebras. Theorem 1.2 is proved in Section 3. An illustration of theorem 1.2 can be found in Section 4.
2. Preliminaries

A brief review of braids, links, AF-algebras and cluster $C^*$-algebras is given below. We refer the reader to [Artin 1925] [1], [Effros 1981] [2], [Morishita 2012] [7], [Williams 2014] [13] and [9] for a detailed account.

2.1. Braids, links and Galois covering. By an $n$-string braid $b_n$ one understands two parallel copies of the plane $\mathbb{R}^2$ in $\mathbb{R}^3$ with $n$ distinguished points taken together with $n$ disjoint smooth paths (“strings”) joining pairwise the distinguished points of the planes; the tangent vector to each string is never parallel to the planes. The braids $b$ are endowed with a natural equivalence relation: two braids $b$ and $b'$ are equivalent if $b$ can be deformed into $b'$ without intersection of the strings and so that at each moment of the deformation $b$ remains a braid. By an $n$-string braid group $B_n$ one understands the set of all $n$-string braids $b$ endowed with a multiplication operation of the concatenation of $b \in B_n$ and $b' \in B_n$, i.e. the identification of the bottom of $b$ with the top of $b'$. The group is non-commutative and the identity is given by the trivial braid. The $B_n$ is isomorphic to a group on generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ satisfying the relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| \geq 2$. By the Artin representation we understand an injective homomorphism $r : B_n \rightarrow \text{Aut} (\mathbb{F}^n)$ into the group of automorphisms of the free group on generators $x_1, \ldots, x_n$ given by the formula $\sigma_i : x_i \mapsto x_{i+1} x_i x_i^{-1}$, $\sigma_{i+1} : x_{i+1} \mapsto x_i$ and $\sigma_k = \text{Id}$ if $k \neq i$ or $k \neq i + 1$.

A closure of the braid $b$ is a link $\mathcal{L}_b \subset \mathbb{R}^3$ obtained by glueing the endpoints of strings at the top of the braid with such at the bottom of the braid. The closure of two braids $b \in B_n$ and $b' \in B_m$ give the same link $\mathcal{L}_b \subset \mathbb{R}^3$ if and only if $b$ and $b'$ can be connected by a sequence of the Markov moves of type I: $b \mapsto aba^{-1}$ for a braid $a \in B_n$ and type II: $b \mapsto ba^\pm 1 \in B_{n+1}$, where $\sigma \in B_{n+1}$.

**Theorem 2.1.** ([1, Theorem 6]) $\pi_1(\mathcal{L}_b) \cong \langle x_1, \ldots, x_n \mid x_1 = r(b) x_1, \ldots, x_n = r(b) x_n, $ where $x_i$ are generators of the free group $\mathbb{F}^n$ and $r : B_n \rightarrow \text{Aut} (\mathbb{F}^n)$ is the Artin representation of group $B_n$.

Let $X$ be a topological space. A covering space of $X$ is a topological space $X'$ and a continuous surjective map $p : X' \rightarrow X$ such that for an open neighborhood $U$ of every point $x \in X$ the set $p^{-1}(U)$ is a union of disjoint open sets in $X$. A deck transformation of the covering space $X'$ is a homeomorphism $f : X' \rightarrow X'$ such that $p \circ f = p$. The set of all deck transformations is a group under composition denoted by $\text{Aut} (X')$. The covering $p : X' \rightarrow X$ is called Galois (or regular) if the group $\text{Aut} (X')$ acts transitively on each fiber $p^{-1}(x)$, i.e. for any points $y_1, y_2 \in p^{-1}(x)$ there exists $g \in \text{Aut} (X')$ such that $y_2 = g(y_1)$. The covering $p : X' \rightarrow X$ is Galois if and only if the group $G := p_*(\pi_1(X'))$ is a normal subgroup of the fundamental group $\pi_1(X)$. In what follows we consider the Galois coverings of the space $X$ such that $|\pi_1(X)/G| < \infty$, i.e. the quotient $\pi_1(X)/G$ is a finite group.

2.2. AF-algebras. A $C^*$-algebra is an algebra $A$ over $\mathbb{C}$ with a norm $a \mapsto ||a||$ and an involution $a \mapsto a^*$ such that it is complete with respect to the norm and $||ab|| \leq ||a|| \cdot ||b||$ and $||a^*a|| = ||a^2||$ for all $a, b \in A$. Any commutative $C^*$-algebra is isomorphic to the algebra $C_0(X)$ of continuous complex-valued functions on some locally compact Hausdorff space $X$; otherwise, $A$ represents a noncommutative topological space. For a unital $C^*$-algebra $A$, let $V(A)$ be the union (over $n$) of projections in the $n \times n$ matrix $C^*$-algebra with entries in $A$; projections $p, q \in V(A)$
are Murray - von Neumann equivalent if there exists a partial isometry \( u \) such that 
\( p = u^*u \) and \( q = uu^* \). The equivalence class of projection \( p \) is denoted by \([p]\); 
the equivalence classes of orthogonal projections can be made to a semigroup by 
putting \([p] + [q] = [p + q] \). The Grothendieck completion of this semigroup to an 
abelian group is called the \( K_0 \)-group of the algebra \( A \). The functor \( A \to K_0(A) \) 
maps the category of unital \( C^* \)-algebras into the category of abelian groups, so that 
projections in the algebra \( A \) correspond to a positive cone \( K_0^+ \subset K_0(A) \) and the unit 
element \( 1 \in A \) corresponds to an order unit \( u \in K_0(A) \). The ordered abelian group 
\((K_0, K_0^+, u)\) with an order unit is called a \textit{dimension group}; an order-isomorphism 
class of the latter we denote by \((G, G^+)\).

An \textit{AF-algebra} \( \mathcal{A} \) (Approximately Finite \textit{C}*-algebra) is defined to be the norm 
closure of an ascending sequence of finite dimensional \( C^* \)-algebras \( M_n \), where \( M_n \) is 
the \( C^* \)-algebra of the \( n \times n \) matrices with entries in \( \mathbb{C} \). Each embedding 
\( M_n \to M_{n+1} \) is given by an integer non-negative matrix \( A_n \). An infinite graph 
given by the incidence matrices \( A_n \) is called a \textit{Bratteli diagram} of the AF-algebra. 
The AF-algebra is defined by the Bratteli diagram. If \( A_n = \text{Const} \) for all \( n \) the 
corresponding AF-algebra is called \textit{stationary}. The rank of a stationary AF-algebra 
is defined as the rank of matrix \( A_n \). The dimension group \((K_0(\mathcal{A}), K_0^+(\mathcal{A}), u)\) is 
a complete isomorphism invariant of the algebra \( \mathcal{A} \). The order-isomorphism class 
\((K_0(\mathcal{A}), K_0^+(\mathcal{A}))\) is an invariant of the Morita equivalence of algebra \( \mathcal{A} \), i.e. 
an isomorphism class in the category of finitely generated projective modules over \( \mathcal{A} \). 
The dimension group of any stationary AF-algebra has the form \((O_K, O_K^+, 1)\), where 
\( O_K \) is the ring of integers of the number field \( K \) generated by the Perron-Frobenius 
eigenvalue of matrix \( A_n \), \( O_K^+ \) consists of the positive elements of \( O_K \) and \( 1 \) is the 
rational unit. The degree of \( O_K \) over \( \mathbb{Q} \) is equal to the rank of the corresponding 
stationary AF-algebra.

2.3. \textbf{Cluster \textit{C}*-algebras.} \textit{Cluster algebra} \( \mathcal{A}(\mathbf{x}, B) \) of rank \( n \) is a subring of the 
field of rational functions in \( n \) variables depending on a cluster of variables \( \mathbf{x} = (x_1, \ldots, x_n) \) and a skew-symmetric matrix \( B = (b_{ij}) \in M_n(\mathbb{Z}) \); the pair \((\mathbf{x}, B)\) is 
called a seed. A new cluster \( \mathbf{x}' = (x_1', \ldots, x_n') \) and a new skew-symmetric matrix 
\( B' = (b_{ij}') \) is obtained from \((\mathbf{x}, B)\) by the exchange relations:

\[
\begin{align*}
  x_k x'_k &= \prod_{i=1}^n x_i \max(b_{ik}, 0) + \prod_{i=1}^n x_i \max(-b_{ik}, 0), \\
  b'_{ij} &= \begin{cases} 
    -b_{ij} & \text{if } i = k \text{ or } j = k \\
    b_{ij} + \frac{|b_{ik}|b_{kj} + b_{jk}|b_{ik}|}{2} & \text{otherwise.}
  \end{cases}
\end{align*}
\] (2.1)

The seed \((\mathbf{x}', B')\) is said to be a mutation of \((\mathbf{x}, B)\) in direction \( k \), where \( 1 \leq k \leq n \); 
the algebra \( \mathcal{A}(\mathbf{x}, B) \) is generated by cluster variables \( \{x_i\}_{i=1}^n \) obtained from 
the initial seed \((\mathbf{x}, B)\) by the iteration of mutations in all possible directions \( k \). The 
Laurent phenomenon says that \( \mathcal{A}(\mathbf{x}, B) \subset \mathbb{Z}[\mathbf{x}^\pm 1] \) is the ring of the 
Laurent polynomials in variables \( \mathbf{x} = (x_1, \ldots, x_n) \) depending on an initial seed 
\((\mathbf{x}, B)\); in other words, each generator \( x_i \) of algebra \( \mathcal{A}(\mathbf{x}, B) \) can be written as a 
Laurent polynomial in \( n \) variables with the integer coefficients.

Let \( S_{g,n} \) be a Riemann surface of genus \( g \) with \( n \) cusps, such that \( 2g - 2 + n > 0 \). 
Denote by \( T_{g,n} \cong \mathbb{R}^{6g-6+2n} \) the (decorated) Teichmüller space of \( S_{g,n} \), i.e. a 
collection of all Riemann surfaces of genus \( g \) with \( n \) cusps endowed with the natural 
topology [Penner 1987] [11]. In what follows, we focus on the cluster algebras
Indeed, the generating relation $\mathcal{A}(x, B)$ comes from an ideal triangulation of the surface $S_{g,n}$ [Fomin, Shapiro & Thurston 2008] [3]. We denote by $\mathcal{A}(x, S_{g,n})$ the corresponding cluster algebra of rank $6g - 6 + 3n$. The $\mathcal{A}(x, S_{g,n})$ is a coordinate ring of the space $T_{g,n}$ [Williams 2014] [13].

The $\mathcal{A}(x, S_{g,n})$ is a commutative algebra with an additive abelian semigroup consisting of the Laurent polynomials with positive coefficients. Thus the algebra $\mathcal{A}(x, S_{g,n})$ is a countable abelian group with an order satisfying the Riesz interpolation property, i.e. a dimension group [Effros 1981, Theorem 3.1] [2]. A cluster $C^*$-algebra $\mathcal{A}(x, S_{g,n})$ is an AF-algebra, such that $K_0(\mathcal{A}(x, S_{g,n})) \cong \mathcal{A}(x, S_{g,n})$, where $\cong$ is an isomorphism of the dimension groups [9]. An element $e$ in a $C^*$-algebra is called a projection if $e^* = e = e^2$.

**Theorem 2.2.** ([10]) The formula $\sigma_i \mapsto e_i + 1$ defines a representation $\rho : B_{2g+n} \to \mathcal{A}(x, S_{g,n})$, where $e_i$ are projections in the algebra $\mathcal{A}(x, S_{g,n})$ and $n \in \{0; 1\}$.

### 3. Proof of Theorem 1.2

We shall split the proof in a series of lemmas.

**Lemma 3.1.** There exists a faithful representation $R : \pi_1(\mathcal{L}_b) \to \mathcal{A}_b$, where $\mathcal{A}_b$ is a stationary AF-algebra of rank $6g - 6 + 2n$.

**Proof.** (i) Let us construct a representation

$$R : \pi_1(\mathcal{L}_b) \to \mathcal{A}(x, S_{g,n})/\mathcal{I}_b := \mathcal{A}_b.$$ (3.1)

Suppose that $r : B_{2g+n} \to \text{Aut}(F_{2g+n})$ is the Artin representation of the braid group $B_{2g+n}$, see Section 2.1. If $x_i$ is a generator of the free group $F_{2g+n}$, one can think of $x_i$ as an element of the group $\text{Aut}(F_{2g+n})$ representing an automorphism of the left multiplication $F_{2g+n} \to x_iF_{2g+n}$. In view of Theorem 2.1, we get an embedding $\pi_1(\mathcal{L}_b) \hookrightarrow \text{Aut}(F_{2g+n})$, where $r(b) = Id$ is a trivial automorphism.

Recall that the braid relations $\Sigma_i = \{x_ix_{i+1}x_i^{-1}x_{i+1}^{-1}x_i^{-1}x_{i+1} = Id, x_ix_j\sigma^{-1}_i\sigma_j^{-1} = Id \text{ if } |i-j| \geq 2\}$ correspond to the trivial automorphisms of the group $F_{2g+n}$. Thus

$$\pi_1(\mathcal{L}_b) \cong \langle x_1, \ldots, x_{2g+n} \mid r(b) = \Sigma_i = Id, 1 \leq i \leq 2g + n - 1 \rangle.$$ (3.2)

Let now $\rho : B_{2g+n} \to \mathcal{A}(x, S_{g,n})$ be the representation constructed in Theorem 2.2. Because $B_{2g+n} \cong \langle x_i \mid \Sigma_i = Id, 1 \leq i \leq 2g + n - 1 \rangle$ so that the ideal $\mathcal{I}_b \subset \mathcal{A}(x, S_{g,n})$ is generated by the relation $r(b) = Id$, one gets from (3.2) a faithful representation

$$R : \pi_1(\mathcal{L}_b) \to \mathcal{A}(x, S_{g,n})/\mathcal{I}_b.$$ (3.3)

(ii) Let us show that the quotient $\mathcal{A}_b = \mathcal{A}(x, S_{g,n})/\mathcal{I}_b$ is a stationary AF-algebra of rank $6g - 6 + 2n$.

First, let us show that the ideal $\mathcal{I}_b \subset \mathcal{A}(x, S_{g,n})$ is self-adjoint, i.e. $\mathcal{I}_b^* \cong \mathcal{I}_b$. Indeed, the generating relation $r(b) = Id$ for such an ideal is invariant under $*$-involutions. To prove the claim, we follow the argument and notation of [10, Remark 4]. Namely, from Theorem 2.2 the relation $r(b) = Id$ has the form $(e_1 + 1)^{k_1} \ldots (e_{n-1} + 1)^{k_{n-1}} = 1$, where $k_i \in \mathbb{Z}$ and $e_i$ are projections. Using the braid relations, one can write the product at the LHS in the form $\sum_{i=1}^{\infty} a_i e_i$, where $a_i \in \mathbb{Z}$ and $e_i$ are elements of a finite multiplicatively closed set $\mathcal{E}$. Moreover, each $e_i$ is (the Murray-von Neumann equivalent to) a projection. In other words,
\[ \varepsilon^* = \varepsilon \text{ and } \left( \sum \varepsilon_i \right)^* = \sum \varepsilon_i. \] We conclude that the relation \( r(b) = Id \) is invariant under \(*\)-involution. Therefore, we have \( I_{\varepsilon} \cong I_b \).

Recall that the quotient of the cluster \( C^* \)-algebra \( \mathbb{A}(x, S_{g,n}) \) by a self-adjoint (primitive) ideal is a simple \( \text{AF} \)-algebra of rank \( 6g - 6 + 2n \) [9, Theorem 2]. Let us show that the quotient \( \mathbb{A}(x, S_{g,n}) / I_{\varepsilon} \) is a stationary \( \text{AF} \)-algebra.

Consider an inner automorphism \( \varphi_b \) of the group \( B_{2g + n} \) given by the formula \( x \mapsto b^{-1}xb \). Using the representation \( \rho \) of Theorem 2.2, one can extend the \( \varphi_b \) to an automorphism of the algebra \( \mathbb{A}(x, S_{g,n}) \). Since the braid \( b \) is a fixed point of \( \varphi_b \), we conclude that \( \varphi_b \) induces a non-trivial automorphism of the \( \text{AF} \)-algebra \( \mathbb{A}_b := \mathbb{A}(x, S_{g,n}) / I_{\varepsilon} \). But each simple \( \text{AF} \)-algebra with a non-trivial group of automorphisms must be a stationary \( \text{AF} \)-algebra [Effros 1981] [2, Chapter 5]. Lemma 3.1 follows.

**Lemma 3.2.** There is a one-to-one correspondence between normal subgroups of the group \( \pi_1(\mathcal{L}_b) \) and \( \text{AF} \)-subalgebras of the algebra \( \mathbb{A}_b \).

**Proof.** Consider a representation \( R : \pi_1(\mathcal{L}_b) \to \mathbb{A}_b \) constructed in lemma 3.1. The algebra \( \mathbb{A}_b \) is a closure in the norm topology of a self-adjoint representation of the group ring \( C[\pi_1(\mathcal{L}_b)] \) by bounded linear operators acting on a Hilbert space. Namely, such a representation is given by the formula \( x_i \mapsto e_i + 1 \), where \( x_i \) is a generator of the group \( \pi_1(\mathcal{L}_b) \) and \( e_i \) is a projection in the algebra \( \mathbb{A}_b \).

Let \( G \) be a subgroup of \( \pi_1(\mathcal{L}_b) \). The \( C[G] \) is a subring of the group ring \( C[\pi_1(\mathcal{L}_b)] \). Taking the closure of a self-adjoint representation of \( C[G] \), one gets a \( \text{AF} \)-subalgebra \( \mathbb{A}_G \) of the algebra \( \mathbb{A}_b \).

Let us show that if \( G \) is a normal subgroup, then the \( \mathbb{A}_G \) is an \( \text{AF} \)-algebra. Indeed, if \( G \) is a normal subgroup one gets an exact sequence:

\[ 0 \to \mathbb{A}_G \to \mathbb{A}_b \to \pi_1(\mathcal{L}_b) / G \to 0, \] (3.4)

where \( \pi_1(\mathcal{L}_b) / G \) is a finite group. Let \( \{ M_k(C) \}_{k=1}^\infty \) be an ascending sequence of the finite-dimensional \( C^* \)-algebras, such that \( \mathbb{A}_b = \lim_{k \to \infty} M_k(C) \). Let \( M_k'(C) = M_k(C) \cap \mathbb{A}_G \). From (3.4) we obtain an exact sequence:

\[ 0 \to M_k'(C) \to M_k(C) \to \pi_1(\mathcal{L}_b) / G \to 0. \] (3.5)

Since \( |\pi_1(\mathcal{L}_b) / G| < \infty \), the \( M_k'(C) \) is a finite-dimensional \( C^* \)-algebra. Thus \( \mathbb{A}_G = \lim_{k \to \infty} M_k'(C) \), i.e. the \( \mathbb{A}_G \) is an \( \text{AF} \)-algebra. Lemma 3.2 follows.

**Remark 3.3.** The \( \mathbb{A}_G \) is a stationary \( \text{AF} \)-algebra, since it is an \( \text{AF} \)-subalgebra of a stationary \( \text{AF} \)-algebra [Effros 1981] [2, Chapter 5].

**Corollary 3.4.** There is a one-to-one correspondence between normal subgroups of the group \( \pi_1(\mathcal{L}_b) \) and ideals in the ring of integers \( O_K \cong K_0(\mathbb{A}_b) \) of a number field \( K \), where \( \text{deg } (K|Q) = 6g - 6 + 2n \).

**Proof.** A one-to-one correspondence between stationary \( \text{AF} \)-algebras and the rings of integers in number fields has been established by [Handelman 1981] [4]. Namely, the dimension groups of stationary \( \text{AF} \)-algebras are one-to-one with the triples \((\Lambda, [I], i)\), where \( \Lambda \subset O_K \) is an order (i.e. a ring with the unit) in the number field \( K \), \([I]\) is the equivalence class of ideals corresponding to \( \Lambda \) and \( i \) is the embedding class of the field \( K \). The degree of \( K \) over \( Q \) is equal to the rank of stationary \( \text{AF} \)-algebra, i.e. \( \text{deg } (K|Q) = 6g - 6 + 2n \) by lemma 3.1.

Assume for simplicity that \( \Lambda \cong O_K \), i.e. that \( \Lambda \) is the maximal order in the field \( K \). From (3.4) we get an inclusion \( K_0(\mathbb{A}_G) \subset K_0(\mathbb{A}_b) \cong O_K \). By remark
3.3, the $K_0(\mathcal{A})$ is an order in $O_K$. Since by (3.4) the $K_0(\mathcal{A})$ is the kernel of a homomorphism, we conclude that it is an ideal in $O_K$. The rest of the proof follows from lemma 3.2. Corollary 3.4 is proved.

**Lemma 3.5.** Let $\mathcal{M}$ be a 3-dimensional manifold, such that $\pi_1(\mathcal{M}) \cong \pi_1(\mathcal{L})$ and let $O_K \cong K_0(\mathcal{L})$. There is a one-to-one correspondence between the Galois coverings of $\mathcal{M}$ ramified over a link $\mathcal{L} \subset \mathcal{M}$ (a knot $\mathcal{K} \subset \mathcal{M}$, resp.) and the ideals (the prime ideals, resp.) of the ring $O_K$. In other words, each link $\mathcal{L} \rightarrow \mathcal{M}$ (each knot $\mathcal{K} \rightarrow \mathcal{M}$, resp.) corresponds to an ideal (a prime ideal, resp.) of the ring $O_K$.

**Proof.** Let

$$\mathcal{L} \cong S^1 \cup S^1 \cup \cdots \cup S^1$$

(3.6)

and let $\mathcal{L} \rightarrow \mathcal{M}$ be an embedding of link $\mathcal{L}$ into a 3-dimensional manifold $\mathcal{M}$. Let $\mathcal{M}_1$ be the Galois covering of $\mathcal{M}$ such that the deck transformations fix the remaining components of $\mathcal{L}$. Let

$$\mathcal{L}_1 \cong S^1 \cup S^1 \cup \cdots \cup S^1$$

(3.7)

and let $\mathcal{L}_1 \rightarrow \mathcal{M}_1$ be an embedding of link $\mathcal{L}_1$ into $\mathcal{M}_1$. Denote by $G_1$ a normal subgroup of $\pi_1(\mathcal{M})$ corresponding to the Galois covering $\mathcal{M}_1$.

Let $\mathcal{M}_2$ be the Galois covering of $\mathcal{M}_1$ such that the remaining components are fixed by the corresponding deck transformations. We denote by $G_2 \leq G_1$ a normal subgroup of $G_1$ corresponding to the Galois covering $\mathcal{M}_2$.

Proceeding by the induction, one gets the following lattice of the normal subgroups:

$$G_k \leq G_{k-1} \leq \cdots \leq G_1 \leq \pi_1(\mathcal{M}).$$

(3.8)

By corollary 3.4, the normal subgroups (3.8) correspond to a chain of ideals of the ring $O_K$:

$$I_k \subset I_{k-1} \subset \cdots \subset I_1 \subset O_K.$$  

(3.9)

(i) Suppose that $k \geq 2$. In this case the link $\mathcal{L}$ has at least two components, and i.e. $\mathcal{L}$ is distinct from a knot. In view of (3.9), the ideal $I_k$ cannot be a maximal ideal of the ring $O_K$, since $I_k \subset I_{k-1} \subset O_K$.

(ii) Suppose that $k = 1$. In this case $\mathcal{L} \cong \mathcal{K}$ is a knot. From (3.9) the ideal $I_k$ is the maximal ideal of the ring $O_K$. Since $O_K$ is the Dedekind domain, we conclude that $I_k$ is a prime ideal. This argument finishes the proof of lemma 3.5. □

Items (ii) and (iii) of theorem 1.2 follow from lemma 3.5.

To prove item (i) of theorem 1.2, one needs to show that $\mathcal{M} \cong \mathcal{S}^3$ implies $O_K \cong \mathbb{Z}$. Indeed, consider the Riemann surface $S_{0,1}$, i.e. sphere with a cusp. Since the fundamental group $\pi_1(S_{0,1})$ is trivial, the 3-dimensional sphere $\mathcal{S}^3$ is homeomorphic to the mapping torus $\mathcal{M}_\phi$ of surface $S_{0,1}$ by an automorphism $\phi : S_{0,1} \to S_{0,1}$, i.e. $\mathcal{S}^3 \cong \mathcal{M}_\phi$.

On the other hand, the $S_{0,1}$ is homeomorphic to the interior of a planar $d$-gon. Thus the cluster algebra $\mathcal{A}(x, S_{0,1})$ is isomorphic to the algebra $\mathcal{A}_{d-3}$ of the
triangulated $d$-gon for $d \geq 3$, see [Williams 2014] [13, Example 2.2]. It is known that the $\mathcal{A}_d$ is a cluster algebra of finite type, i.e. has finitely many seeds, \textit{ibid.} Therefore the corresponding cluster $C^*$-algebra must be finite-dimensional, i.e. $\mathcal{A}(x, S_{0,1}) \cong M_n(\mathbb{C})$. But $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$ [Effros 1981] [2]. We conclude therefore that the homeomorphism $\mathcal{A} \cong \mathbb{R}^3$ implies an isomorphism $O_K \cong \mathbb{Z}$. This argument finishes the proof of item (i) of theorem 1.2.

Theorem 1.2 follows.

4. Example

Let $g = n = 1$, i.e. surface $S_{g,n}$ is homeomorphic to the torus with a cusp. The matrix $B$ associated to an ideal triangulation of surface $S_{1,1}$ has the form:

$$B = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$

[Fomin, Shapiro & Thurston 2008] [3, Example 4.6]. The cluster $C^*$-algebra $\mathcal{A}(x, S_{1,1})$ is given by the Bratteli diagram shown in Figure 1. Theorem 2.2 says that there exists a faithful representation of the braid group $B_3$:

$$\rho : B_3 \rightarrow \mathcal{A}(x, S_{1,1}).$$

For every $b \in B_3$ the inner automorphism $\varphi_b : x \mapsto b^{-1}xb$ of $B_3$ defines a unique automorphism of the algebra $\mathcal{A}(x, S_{1,1})$. Since the algebra $\mathcal{A}(x, S_{1,1})$ is a coordinate ring of the Teichmüller space $T_{1,1}$ and $\text{Aut} (T_{1,1}) \cong \text{SL}_2(\mathbb{Z})$, we conclude that $\varphi_b$ corresponds to an element of the modular group $\text{SL}_2(\mathbb{Z})$.

An explicit formula for the correspondence $b \mapsto \varphi_b$ is well known. Namely, if $\sigma_1$ and $\sigma_2$ are the standard generators of the braid group $B_3 \cong \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ then the formula

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

defines a surjective homomorphism $B_3 \rightarrow \text{SL}_2(\mathbb{Z})$.

Example 4.1. Let $L_b$ be a link given by the closure of a braid of the form $b = \sigma_1^p \sigma_2^{-q}$, where $p \geq 1$ and $q \geq 1$. In this case

$$\varphi_b = \begin{pmatrix} pq + 1 & p \\ q & 1 \end{pmatrix}.$$
The algebra $\mathbb{A}_b = \mathbb{A}(x, S_{1,1})/I_b$ is a stationary AF-algebra of rank 2 given by the Bratteli diagram shown in Figure 2. (The numbers in Figure 2 show the multiplicity of the corresponding edge of the graph.) The Perron-Frobenius eigenvalue of the matrix $\varphi_b$ is equal to

$$\lambda_{\varphi_b} = \frac{pq + 2 + \sqrt{pq(pq + 4)}}{2}.$$  

Therefore $K_0(\mathbb{A}_b) \cong \mathbb{Z} + \mathbb{Z}\sqrt{D}$, where $D = pq(pq + 4)$. The number field $K$ corresponding to the link $\mathcal{L}_b$ is a real quadratic field of the form:

$$K \cong \mathbb{Q}\left(\sqrt{pq(pq + 4)}\right).$$

Denote by $\mathcal{M}_{p,q}$ a 3-dimensional manifold, such that $\pi_1(\mathcal{M}_{p,q}) \cong \pi_1(\mathcal{L}_{\sigma_p^p\sigma_q^{-q}})$. The manifolds $\mathcal{M}_{p,q}$ corresponding to the quadratic fields with a small square-free discriminant $D$ are recorded below.

| Manifold $\mathcal{M}_{p,q}$ | Number field $K = F(\mathcal{M}_{p,q})$ |
|-------------------------------|-----------------------------------------|
| $\mathcal{M}_{1,1}$           | $\mathbb{Q}(\sqrt{5})$                 |
| $\mathcal{M}_{1,3}$           | $\mathbb{Q}(\sqrt{21})$                |
| $\mathcal{M}_{1,7}$           | $\mathbb{Q}(\sqrt{77})$                |
| $\mathcal{M}_{1,11}$          | $\mathbb{Q}(\sqrt{165})$               |
| $\mathcal{M}_{1,13}$          | $\mathbb{Q}(\sqrt{221})$               |
| $\mathcal{M}_{3,5}$           | $\mathbb{Q}(\sqrt{285})$               |
| $\mathcal{M}_{3,7}$           | $\mathbb{Q}(\sqrt{525})$               |
| $\mathcal{M}_{3,11}$          | $\mathbb{Q}(\sqrt{1221})$              |

Remark 4.2. The manifold $\mathcal{M}_{p,q}$ can be realized as a torus bundle over the circle with the monodromy given by matrix (4.4). The arithmetic invariants of surface bundles over the circle were studied in [8].

References

1. E. Artin, *Theorie der Zöpfen*, Abhandlungen aus dem Mathematischen Seminar der Hamburgerschen Universität 4 (1925), 47-72.
2. E. G. Effros, *Dimensions and C*-Algebras*, in: Conf. Board of the Math. Sciences, Regional conference series in Math., No.46, AMS, 1981.
3. S. Fomin, M. Shapiro and D. Thurston, *Cluster algebras and triangulated surfaces, I. Cluster complexes*, Acta Math. 201 (2008), 83-146.
4. D. Handelman, *Positive matrices and dimension groups affiliated to C*-algebras and topological Markov chains*, J. Operator Theory 6 (1981), 55-74.
5. W. B. R. Lickorish, *A representation of orientable combinatorial 3-manifolds*, Annals of Math. **76** (1962), 531-540.

6. B. Mazur, *Remarks on the Alexander polynomial*, unpublished notes, c. 1964; available at http://www.math.harvard.edu/~mazur/papers/alexander_polynomial.pdf

7. M. Morishita, *Knots and Primes, An Introduction to Arithmetic Topology*, Springer Universitext, London, Dordrecht, Heidelberg, New York, 2012.

8. I. Nikolaev, *Operator algebras and conjugacy problem for the pseudo-Anosov automorphisms of a surface*, Pacific J. Math. **261** (2013) 445-462.

9. I. Nikolaev, *On cluster C*-algebras*, J. Funct. Spaces **2016**, Art. ID 9639875, 8 pp.

10. I. Nikolaev, *Cluster C*-algebras and knot polynomials*, arXiv:1603.01180

11. R. C. Penner, *The decorated Teichmüller space of punctured surfaces*, Comm. Math. Phys. **113** (1987), 299-339.

12. A. Weil, *Numbers of solutions of equations in finite fields*, Bull. Amer. Math. Soc. **55** (1949), 497-508.

13. L. K. Williams, *Cluster algebras: an introduction*, Bull. Amer. Math. Soc. **51** (2014), 1-26.

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