Malliavin regularity of solutions to mixed stochastic differential equations

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Abstract

For a mixed stochastic differential driven by independent fractional Brownian motions and Wiener processes, the existence and integrability of the Malliavin derivative of its solution are established. It is also proved that the solution possesses exponential moments.

Keywords: Mixed stochastic differential equation, fractional Brownian motion, Wiener process, Malliavin regularity

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1. Introduction

This paper is devoted to the following mixed stochastic differential equation (SDE) in $\mathbb{R}^d$:

$$X_t = X_0 + \int_0^t a(X_s) \, ds + \int_0^t b(X_s) \, dW_s + \int_0^t c(X_s) \, dB_s,$$

where $W = \{W_t, t \geq 0\}$ is an $m$-dimensional standard Wiener process, $B = \{B_t, t \geq 0\}$ is an $l$-dimensional fractional Brownian motion; the coefficients $a: \mathbb{R}^d \to \mathbb{R}^d$, $b: \mathbb{R}^d \to \mathbb{R}^{d \times m}$, $c: \mathbb{R}^d \to \mathbb{R}^{d \times l}$ are continuous, $X_0 \in \mathbb{R}^d$ is non-random. (See Section 2 for precise definitions of all objects.) Unique solvability of such equations was established under different sets of conditions in \cite{2, 4, 9, 8}. Article \cite{9} also contains an important result that the solution to (1) is a limit of solutions to Itô SDEs, which gives a tool to transfer some elements of a well developed theory for the Itô SDEs to equation (1). In \cite{5}, it is used to obtain stochastic viability and comparison results for equation (1).

Over last decades, the Malliavin calculus of variations has become one of the most important tools in stochastic analysis. While originally it was developed by Malliavin to study existence and regularity of densities of solutions to SDEs, now it has numerous applications in mathematical finance, statistics, optimal control, etc. For this reason, questions of Malliavin regularity generate considerable scientific interest. There is a huge amount of articles devoted to the Malliavin calculus for Itô SDEs, see \cite{11} and references therein. For SDEs driven by fBm, the questions of Malliavin regularity were studied in \cite{1, 4, 10, 12}. For mixed equations, the Malliavin regularity was already studied in \cite{7} in the one-dimensional case. Equation (1) can be treated with the help of the rough path theory, however, it requires rather high regularity of coefficients (usually they are assumed to be infinitely differentiable with all bounded derivatives). In this paper we will use another techniques, namely, the ad hoc approach developed in \cite{8, 9, 13}, to study equation (1), which enables us to prove the Malliavin regularity under less restrictive assumptions about the coefficients. One of the key ingredients of the

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proof is the above-mentioned approximation by solutions of Itô SDEs. As a side result of independent interest, we prove exponential integrability of solutions to mixed SDEs with bounded coefficients.

The paper is organized as follows. Section 2 contains a precise definition of the main object. It also provides a brief summary on the pathwise integration and the Malliavin calculus of variations for fractional Brownian motion. In Section 3 we prove the main results of the article: exponential integrability of solution to (1) and existence and integrability of Malliavin derivatives. Proofs of auxiliary results are given in Appendix.

2. Preliminaries

2.1. Main object, notation and assumptions

On a complete filtered probability space \((\Omega, \mathcal{F}, \mathbb{P} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\), let \(W = \{W_t = (W^1_t, \ldots, W^m_t), t \geq 0\}\) be a standard \(\mathcal{F}\)-Wiener process in \(\mathbb{R}^m\), \(B = \{B_t = (B^{H,1}_t, \ldots, B^{H,l}_t), t \geq 0\}\) be an independent of \(W\) \(\mathcal{F}\)-adapted fractional Brownian motion (fBm) in \(\mathbb{R}^l\), i.e. a collection of independent fBms \(B^{H,k}\) with Hurst index \(H \in (1/2, 1)\). We recall that an fBm with Hurst index \(H \in (0, 1)\) is a centered Gaussian process \(B^H = \{B^H_t, t \geq 0\}\) with the covariance function

\[
R_H(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H});
\]

in the case \(H \in (1/2, 1)\) considered here, \(B^H\) has the property of long-range dependence. It is known that fBm has a continuous modification (even Hölder continuous of any order up to \(H\)), and in what follows we will assume that the process \(B^H\) is continuous.

The main object of the article, equation (1), is understood precisely as a system of equations on \([0, T]\)

\[
X^i_t = X^i_0 + \int_0^t a_i(X_s) \, ds + \sum_{j=1}^m \int_0^t b_{i,j}(X_s) \, dW^j_s + \sum_{k=1}^l \int_0^t c_{i,k}(X_s) \, dB^{H,k}_s, \quad i = 1, 2, \ldots, d, \quad (2)
\]

where for \(i = 1, \ldots, d\), \(j = 1, \ldots, m\), \(k = 1, \ldots, l\) the functions \(a_i, b_{i,j}, c_{i,k} : \mathbb{R}^d \to \mathbb{R}\) are continuous; the integrals w.r.t. \(W^j\) are understood in the Itô sense, whereas those w.r.t. \(B^{H,k}\), in the pathwise sense, as defined in 2.2.

Throughout the paper, we will use the following notation: \(|\cdot|\) will denote the absolute value of a number, the Euclidean norm of a vector, and the operator norm: \(|Ax| = \sup_{|x|=1} |Ax|\). The inner product in \(\mathbb{R}^d\) will be denoted by \(\langle \cdot, \cdot \rangle\); \(\|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|\) is the supremum norm of a function \(f\), either real-valued, vector-valued, or operator-valued. The symbol \(C\) will be used for a generic constant, whose value is not important and may change from one line to another.

We will impose the following assumptions on the coefficients \(a = (a_1, \ldots, a_d), b = (b_{i,j})_{i=1,d,j=1,m}\) and \(c = (c_{i,j})_{i=1,d,j=1,m}\) of (2):

(A1) \(a, b, c\) are bounded and have bounded continuous derivatives,

(A2) \(c\) is twice differentiable and \(c''\) is bounded.

2.2. Integration with respect to fractional Brownian motion

The integral with respect to fBm will be understood in the pathwise (Young) sense. Specifically, for a \(\mu\)-Hölder continuous function \(f\) and a \(\nu\)-Hölder continuous function \(g\) with \(\mu + \nu > 1\) the integral \(\int_a^b f(x) \, dg(x)\) exists as a limit of integral sums, moreover, the inequality

\[
\left| \int_a^b f(s) \, dg(s) \right| \leq K_{\mu, \nu} \|g\|_{a, b, \mu} \left( \|f\|_{a, b, \infty} (b - a)^{\mu} + \|f\|_{a, b, \nu} (b - a)^{\mu + \nu} \right) \quad (3)
\]
holds, where \( \|f\|_{a,b,\infty} = \sup_{x \in [a, b]} |f(x)| \) is the supremum norm on \([a, b]\), and for \( \gamma \in (0, 1) \)
\[
\|f\|_{a,b,\gamma} = \sup_{a \leq s < t \leq b} \frac{|f(t) - f(s)|}{|t - s|^\gamma}
\]
is the Hölder seminorm on \([a, b]\); \( K_{\mu,\nu} \) is a universal constant. Thus, since \( \text{fBm} \) is Hölder continuous of any order less than \( H \), the integral \( \int_a^b f(s) dB^H_s \) is well defined provided \( f \) is \( \beta \)-Hölder continuous on \([a, b]\) with \( \beta > 1 - H \). We will use inequality (3) also in a multidimensional case; for simplicity we will write it with the same constant \( K_{\mu,\nu} \).

2.3. Malliavin calculus of variations

Here we give only basics of Malliavin calculus of variations with respect to \( \text{fBm} \), see [11] for a deeper exposition. Let \( S[0, T] \) denote the set of step functions of the form \( f(t) = \sum_{k=1}^n c_k 1_{[a_k, b_k)}(t) \) defined on \([0, T]\). For functions \( f, g \in S[0, T] \) define the scalar product
\[
\langle f, g \rangle_H = H(2H - 1) \int_0^T \int_0^T f(t) g(s) |t - s|^{2H-2} \, dt \, ds.
\]
Let \( L^2_H[0, T] \) denote the closure of \( S[0, T] \) w.r.t. this scalar product. It is a separable Hilbert space, which contains not only classical functions, but also some distributions. Then the product
\[
\mathfrak{H} = \left( L^2_H[0, T] \right)^l \times \left( L^2_H[0, T] \right)^m
\]
is a separable Hilbert space with a scalar product \( \langle \cdot, \cdot \rangle_{\mathfrak{H}} \). The map
\[
\mathcal{I} : (1_{[0,t_1]), 1_{[0,t_2)}, \ldots, 1_{[0,t_l)}, 1_{[0,s_1], 1_{[0,s_2]}, \ldots, 1_{[0,s_m)}} \mapsto (B^H_{t_1}, B^H_{t_2}, \ldots, B^H_{t_l}, W^1_{s_1}, W^2_{s_2}, \ldots, W^m_{s_m})
\]
can be extended by linearity to \( L^2(\Omega; \mathbb{R}^{l+m}) \). It appears that for \( f, g \in S[0, T] \)
\[
\mathbb{E} \left[ \langle \mathcal{I}(f), \mathcal{I}(g) \rangle \right] = \langle f, g \rangle_{\mathfrak{H}},
\]
so \( \mathcal{I} \) can be extended to an isometry between \( \mathfrak{H} \) and a subspace of \( L^2(\Omega; \mathbb{R}^{l+m}) \).

For a smooth cylindrical variable of the form \( \xi = F(\mathcal{I}(f_1), \ldots, \mathcal{I}(f_n)) \), where \( f_1, \ldots, f_n \in \mathfrak{H} \), \( F : \mathbb{R}^n \to \mathbb{R} \) is a continuously differentiable finitely supported function, define the Malliavin derivative
\[
D\xi = \sum_{i=1}^n \partial_i F(\mathcal{I}(f_1), \ldots, \mathcal{I}(f_n)) f_i
\]
as an element of \( \mathfrak{H} \). For \( p \geq 1 \) denote by \( D^{1,p} \) the closure of the space of smooth cylindrical random variables with respect to the norm
\[
\|\xi\|_{D^{1,p}} = \mathbb{E} \left[ |\xi|^p + \|D\xi\|_{\mathfrak{H}}^p \right]^{1/p},
\]
\( D \) is closable in this space and its closure will be denoted likewise. Finally, the Malliavin derivative is a (possibly, generalized) function from \([0, T]\) to \( \mathbb{R}^{l+m} \), so we can introduce the notation
\[
D\xi = \left\{ D_t \xi = \left( D_t^{H,1}_t \xi, \ldots, D_t^{H,l}_t \xi, D_t^{W,1}_t \xi, \ldots, D_t^{W,m}_t \xi \right), \ t \in [0, T] \right\}.
\]
3. Existence of moments and Malliavin regularity

**Theorem 1.** The solution \( X \) of (2) satisfies \( E \left[ \exp \left\{ z \| X \|_{0,T,\infty}^{\alpha} \right\} \right] < \infty \) for any \( \alpha \in (0, 4H/(2H + 1)) \), \( z > 0 \).

**Proof.** The inequality \( \alpha < 4H/(2H + 1) \) is equivalent to \( (2H)^{-1} < 2\alpha^{-1} - 1 \), therefore, it is possible to choose some \( \nu \in (1/2, H) \) such that \( (2\nu)^{-1} < 2\alpha^{-1} - 1 \). Take also arbitrary \( \beta \in ((2\nu)^{-1}, 2\alpha^{-1} - 1) \) so that \( \alpha(1 + \beta) < 2 \).

Now if \( \kappa \in (1 - \nu, 1/2) \) is sufficiently close to 1/2, it follows from Lemma 2 that

\[
\|X\|_{0,T,\infty} \leq C\left(1 + \|B\|_{0,T,\nu}^{1+\beta} + J_{X,\kappa}(T)\left(1 + \|B\|_{0,T,\nu}^{\beta}\right)\right).
\]

By the Young inequality,

\[
J_{X,\kappa}(T)\|B\|_{0,T,\nu}^{\beta} \leq \frac{J_{X,\kappa}(T)^{1+\beta}}{1+\beta} + \beta\|B\|_{0,T,\nu}^{1+\beta}.
\]

Since \( \|B\|_{0,T,\nu} \) is an almost surely finite supremum of a centered Gaussian family, and \( \alpha(1 + \beta) < 2 \), for any \( y > 0 \) we have \( E \left[ \exp \left\{ y\|B\|_{0,T,\nu}^{(1+\beta)} \right\} \right] < \infty \) thanks to Fernique’s theorem. Further, it follows from Lemma 1 that for any \( y > 0 \) \( E \left[ \exp \left\{ yJ_{X,\kappa}(T)^{\alpha(1+\beta)} \right\} \right] < \infty \) and \( E \left[ \exp \left\{ yJ_{X,\kappa}(T)^{\alpha} \right\} \right] < \infty \). Thus, writing

\[
\|X\|_{0,T,\infty}^{\alpha} \leq C\alpha \left(1 + \|B\|_{0,T,\nu}^{(1+\beta)} + J_{X,\kappa}(T)^{\alpha} + J_{X,\kappa}(T)^{\alpha(1+\beta)}\right),
\]

we get the required statement with the help of the Hölder inequality. \( \square \)

**Theorem 2.** Let \( X \) be the solution of (2). Then for all \( t > 0 \) \( X_t \in \bigcap_{p \geq 1} L^{1,p} \).

**Proof.** Consider the sequence \( \left\{ Z^n_t = n \int_{(t-1/n)^0}^t B_s \, ds, n \geq 1 \right\} \) of processes approximating \( B \). It can be easily checked (see e.g. [3]) that \( \|Z^n - B\|_{0,T,\mu} \to 0, n \to \infty \) a.s. Processes \( Z^n \) are absolutely continuous: \( Z^n_t = \int_0^t Z^n_s \, ds \) with \( Z^n_t = n(B_t - B_{(t-1/n)^0}) \).

Now define \( X^n = \{X^n_t, t \in [0, T]\} = \{(X_t^n, \ldots, X_t^{n,d}), t \in [0, T]\}_n \) as the solution to the SDE

\[
dX_t^n = \left(a(X_t^n) + c(X_t^n)\dot{Z}_t^n\right) \, dt + b(X_t^n) \, dW_t;
\]

coordinatewise, for \( i = 1, \ldots, d \)

\[
X_t^{n,i} = \int_0^t \left( a_i(X_u^n) + \sum_{j=1}^l c_{i,j}(X_u^n)\dot{Z}_{u,j}^n \right) \, du + \sum_{k=1}^m \int_0^t b_{i,k}(X_u^n) \, dW_{u,k},
\]

where \( \dot{Z}_{u,j}^n = n(B_{H,j}^H - B_{(t-1/n)^0}) \). By [3, Theorem 4.1], \( X_t^n \to X_t, n \to \infty \), uniformly on \([0, T]\) in probability.

First note that all moments of \( X^n \) are bounded uniformly in \( n \). Indeed, from the almost sure convergence \( \|Z^n - Z\|_{0,T,\mu} \to 0, n \to \infty \), we have \( \zeta = \sup_{n \geq 1} \|Z^n\|_{0,T,\mu} < \infty \) a.s. But \( \zeta \) is a supremum of some centered Gaussian family, so by Fernique’s theorem, \( E \left[ \exp \left\{ \zeta \right\} \right] < \infty \) for all \( z > 0, \alpha \in (0, 2) \). A fortiori, \( \sup_{n \geq 1} E \left[ \exp \left\{ z \|Z^n\|_{0,T,\mu}^\alpha \right\} \right] < \infty \). Arguing as in the proof of Theorem 1 we get \( \sup_{n \geq 1} E \left[ \exp \left\{ z \|X^n\|_{0,T,\infty}^\alpha \right\} \right] < \infty \) for any \( \alpha \in (0, 4H/(2H + 1)) \), \( z > 0 \). The uniform boundedness of moments clearly follows.
Further, the solution of (I) is Malliavin differentiable w.r.t. \( W \) (see e.g. [11]) as a solution to an Itô SDE. It is Malliavin differentiable w.r.t. \( B \) as a solution of an Itô SDE with a parameter. The equations satisfied by the derivatives w.r.t. \( B \) and \( W \) are similar, so we will study those for \( B \), as they are slightly more involved.

Fix \( s \in [0,T] \) and \( q = 1, \ldots, l \). Define \( D^{n,i}_t = D^{H,q}_{s,i} X^{n,i}_t, \ i = 1, \ldots, d \). Then \( D^n \) satisfies

\[
D^{n,i}_t = D^{n,i}_s + \int_s^t \langle \text{grad } a_i(X^n_u), D^n_u \rangle du + \sum_{j=1}^m \int_s^t \langle \text{grad } b_{i,j}(X^n_u), D^n_u \rangle dW^j_u + \sum_{j=1}^l \int_s^t \langle \text{grad } c_{i,j}(X^n_u), D^n_u \rangle dZ^{j,n}_u,
\]

where

\[
D^{n,i}_{s,t} = \sum_{j=1}^l \int_0^t c_{i,j}(X^n_u) D^{H,q}_{s,i} Z^{n,j}_u du = n \int_s^t \left\langle c_{i,q}(X^n_u) \right\rangle du.
\]

Due to linearity, the solution to equation (A.3) can be written as

\[
D^n_t = n \int_s^{(s+1/n)t} R^n_t(z) dz,
\]

where for \( z \in (s, s + 1/n) \) the process \( \{ R^n_t(z) = (R^{n,1}_t(z), \ldots, R^{n,d}_t(z)), t \geq z \} \) solves

\[
R^{n,i}_t(z) = c_{i,q}(X_z) + \int_z^t \langle \text{grad } a_i(X^n_u), R^n_u(z) \rangle du + \sum_{j=1}^m \int_z^t \langle \text{grad } b_{i,j}(X^n_u), R^n_u(z) \rangle dW^j_u + \sum_{j=1}^l \int_z^t \langle \text{grad } c_{i,j}(X^n_u), R^n_u(z) \rangle dZ^{j,n}_u,
\]

Therefore,

\[
\|D^n\|_{s,T,\infty} \leq n \int_s^{(s+1/n)} \|R^n(z)\|_{z,T,\infty} dz,
\]

whence for any \( p \geq 1 \) by Jensen’s inequality,

\[
E \left[ \|D^n\|_{s,T,\infty}^p \right] \leq n \int_s^{(s+1/n)} E \left[ \|R^n(z)\|_{z,T,\infty}^p \right] dz.
\]

From Lemma 3 we have

\[
E \left[ \|R^n(z)\|_{z,T,\infty}^p \right] \leq K_p \left( E \left[ \exp \left\{ K_p \|Z^n\|_{0,T,\mu}^\alpha \right\} \right] \right)^{1/4}.
\]

As it was shown above, \( \sup_{n \geq 1} E \left[ \exp \left\{ K_p \|Z^n\|_{0,T,\mu}^\alpha \right\} \right] < \infty \), thus, we obtain that \( E \left[ \|D^n\|_{s,T,\infty}^p \right] \) is bounded by a constant independent of \( n \) and of \( s \). So we have for any \( p \geq 1 \),

\[
\sup_{n \geq 1} \sup_{s,t \in [0,T]} E \left[ \|D^{H,q}_{s,i} X^{n,i}_t\|^p \right] < \infty, \quad q = 1, \ldots, l.
\]

Similarly,

\[
\sup_{n \geq 1} \sup_{s,t \in [0,T]} E \left[ \|D^{W,j}_{s,i} X^{n,i}_t\|^p \right] < \infty, \quad j = 1, \ldots, m.
\]

Hence it is easy to deduce that \( \sup_{n \geq 1} E \left[ \|D X^n\|^p \right] < \infty \); and, taking into account that all moments of \( X^n \) are bounded uniformly in \( n \), we get \( \sup_{n \geq 1} E \left[ \|X^n\|^p_{D_t,p} \right] < \infty \) for any \( p \geq 1 \). From here the Malliavin differentiability of \( X \) is deduced by a standard argument.

Remark 1. Using the same techniques, it is possible to generalize the results of the paper to the case where the driving fBm’s have different Hurst exponents.
Appendix A. Technical lemmas

Lemma 1. Let $A > 0$, $\kappa \in (0, 1/2)$, $\alpha \in (0, 2)$, $z > 0$, $t > 0$. There exists a constant $K_{A,\kappa,\alpha,z,t}$ such that if an $\mathbb{F}$-adapted process $\{\xi_s, s \in [0,t]\}$ satisfies $|\xi_s| \leq A$ for almost all $s \in [0,t]$ and $\omega \in \Omega$, and $V$ is an $\mathbb{F}$-Wiener process, then $\mathbb{E} \left[ \exp \left\{ z \|\int_0^t \xi_s dW_s\|_{0,t,\kappa}^\alpha \right\} \right] \leq K_{A,\kappa,\alpha,z,t}$.

Proof. Using the Garsia–Rodemich–Rumsey inequality, we can write for $p > (1/2 - \kappa)^{-1}$

$$
\mu := \mathbb{E} \left[ \|Z\|_{0,t,\kappa} \right] \leq C_{p,\kappa,\alpha} \mathbb{E} \left[ \left( \int_0^t \int_0^t \frac{|\xi_s dV_r|^p}{|s-u|^p + 2} \, du \, ds \right)^{1/p} \right] \leq C_{p,\kappa,\alpha} \left( \int_0^t \int_0^t \mathbb{E} \left[ \frac{|\xi_s dV_r|^p}{|s-u|^p + 2} \right] \, du \, ds \right)^{1/p} \leq C_{p,\kappa,\alpha} A \left( \int_0^t \int_0^t |s-u|^{p(1/2 - \kappa)^{-2}} \, du \, ds \right)^{1/p} \leq C_{p,\kappa,\alpha} A.
$$

Further, let $D_v$ denote the Malliavin derivative with respect to $W$, and $Z_{u,s} = \int_u^s \xi_r dV_r (s-u)^{-\kappa}$, $(u, s) \in T := \{(a, b) \mid 0 \leq a < b \leq t\}$. Then

$$
\int_0^t |D_v Z_{u,s}|^2 \, dr = (u-s)^{-2\kappa} \int_u^s |\xi_r|^2 \, dr \leq A^2 (u-s)^{1-2\kappa} \leq A^2 t^{1-2\kappa}
$$

almost surely. Therefore, it follows from [14, Theorem 3.6] that for any $x > 0$

$$
\Pr \left( \left\| \int_0^t \xi_s dV_s \right\|_{0,t,\kappa} > \mu + x \right) = \Pr \left( \sup_{(u,s) \in T} Z_{u,s} > \mu + x \right) \leq 4 \exp \left\{ -\frac{x^2}{2A^2 t^{1-2\kappa}} \right\},
$$

which provides the required statement.

Further we estimate the solution of a slightly more general version of equation (2):

$$
Y_t^i = Y_0^i + \int_0^t a_i(Y_s) \, ds + \sum_{j=1}^m \int_0^t b_{i,j}(Y_s) \, dW_s^j + \sum_{k=1}^l \int_0^t c_{i,k}(Y_s) \, d\gamma_s^k, \quad i = 1, \ldots, d, \tag{A.1}
$$

where $\gamma = \{(\gamma_1^t, \ldots, \gamma_l^t), t \geq 0\}$ is a process in $\mathbb{R}^d$ with $\mu$-Hölder continuous paths, $\mu > 1/2$; the integral $\int_0^t c_{i,k}(X_s) \, d\gamma_s^k$ is understood in the Young sense.

Fix some $\theta \in (1 - \mu, 1/2)$ and define

$$
J_{Y,\theta}(t) = \sum_{i=1}^d \sum_{j=1}^m \left\| \int_0^t b_{i,j}(Y_s) \, dW_s^j \right\|_{0,t,\theta}.
$$

The following result establishes pathwise estimates of the solution to (A.1), which are better than those in [13, Lemma 4.1], but require stronger assumptions. To prove it, we modify the approach of [3].

Lemma 2. The solution $Y$ of (A.1) satisfies

$$
\|Y\|_{0,t,\infty} \leq |Y_0| + 2 \left( \|a\|_\infty + J_{Y,\theta}(t) + K_{\theta,\mu} \|\gamma\|_{0,t,\mu} \|c\|_\infty \right) \left( t^\theta + t \left( 2K_{\theta,\mu} \|\gamma\|_{0,t,\mu} \|c^\gamma\|_\infty + 1 \right)^{(1-\theta)/\mu} \right).
$$
Applying (3), we can write for $u, s$
Proof.
\[ |Y_s - Y_u| \leq \left| \int_u^s a(Y_v) \, dv \right| + \left| \int_u^s b(Y_v) \, dW_v \right| + \int_u^s c(Y_v) \, d\gamma_v \]
\[ \leq \|a\|_\infty (s - u) + J_{Y,0}(s - u) + K_{\theta,\mu} \|Y\|_{|0,t,\mu} \left( \|c\|_\infty (s - u) + \|c(Y)\|_{u,s,\theta} (s - u)^{\theta + \mu} \right) \]
\[ \leq \left( \|a\|_\infty + J_{Y,0}(t) \right) (s - u)^{\theta + \mu} + K_{\theta,\mu} \|Y\|_{|0,t,\mu} \|c\|_\infty (s - u)^{\theta + \mu} \]
It follows that
\[ \|Y\|_{u,s,\theta} \leq \|a\|_\infty + J_{Y,0}(t) + K_{\theta,\mu} \|c\|_\infty \|c\|_\infty \|Y\|_{u,s,\theta} (s - u)^{\theta} \]
Now put $\Delta = \left( 2K_{\theta,\mu} \|Y\|_{|0,t,\mu} \|c\|_\infty + 1 \right)^{-1/\mu}$. With this choice, for $(s - u) \leq \Delta$
\[ \|Y\|_{u,s,\theta} \leq 2 \left( \|a\|_\infty + J_{Y,0}(t) + K_{\theta,\mu} \|Y\|_{|0,t,\mu} \|c\|_\infty \right) (s - u)^{\theta} \]
Therefore, for $(s - u) \leq \Delta$
\[ \|Y\|_{u,s,\infty} \leq |Y_u| + \|Y\|_{u,s,\infty} (s - u)^{\theta} \leq |Y_u| + 2 \left( \|a\|_\infty + J_{Y,0} + K_{\theta,\mu} \|Y\|_{|0,T,\mu} \|c\|_\infty \right) (s - u)^{\theta} \]
If $\Delta \geq t$, we set $u = 0, s = t$ and obtain
\[ \|Y\|_{0,t,\infty} \leq \|Y\|_{0,u,\infty} + 2 \left( \|a\|_\infty + J_{Y,0}(t) + K_{\theta,\mu} \|Y\|_{|0,t,\mu} \|c\|_\infty \right) t^{\theta} \]
as needed. In the case where $\Delta < t$, write
\[ \|Y\|_{u,s,\infty} \leq \|Y\|_{0,u,\infty} + 2 \left( \|a\|_\infty + J_{Y,0}(t) + K_{\theta,\mu} \|Y\|_{|0,t,\mu} \|c\|_\infty \right) \Delta^{\theta} \]
Hence, dividing the interval $[0, t]$ into $[t/\Delta] + 1$ subintervals of length at most $\Delta$, we obtain by induction
\[ \|Y\|_{0,t,\infty} \leq \|Y\|_{0,u,\infty} + 2 \left( \|a\|_\infty + J_{Y,0}(t) + K_{\theta,\mu} \|Y\|_{|0,t,\mu} \|c\|_\infty \right) (1 + [t/\Delta]) \Delta^{\theta} \]
\[ \leq |Y_0| + 2 \left( \|a\|_\infty + J_{Y,0}(t) + K_{\theta,\mu} \|Y\|_{|0,t,\mu} \|c\|_\infty \right) (t^{\theta} + t\Delta^{\theta-1}) \]
which implies the required statement. \qed
For a fixed $s \in [0, T]$ and a fixed almost surely bounded $\mathcal{F}_s$-measurable random vector $R_s = (R_s^1, \ldots, R_s^d) \in \mathbb{R}^d$, let $\{R_t, t \in [s, T]\} = \{(R_t^1, \ldots, R_t^d), t \in [s, T]\}$ be the solution of
\[ R_t^i = R_s^i + \int_s^t \langle \text{grad} a_i(X_u^n), R_u \rangle \, du + \sum_{j=1}^m \int_s^t \langle \text{grad} b_i,j(X_u^n), R_u \rangle \, dW_u^j + \sum_{k=1}^l \int_s^t \langle \text{grad} c_i,k(X_u^n), R_u \rangle \, d\gamma_u^k, \]
\[ i = 1, \ldots, d, t \in [s, T]. \]
We will write equation (A.3) shortly as
\[ R_t = R_s + \int_s^t a'(Y_u) \, du + \int_s^t [b'(Y_u), R_u] \, dW_u + \int_s^t [c'(Y_u), R_u] \, d\gamma_u. \]
Using the same methods as in [8, 9], it can be shown that this equation has a unique solution such that $\|R\|_{s,T,\theta} < \infty$ a.s.
Lemma 3. For any $p > 0$

$$E \left[ \|R\|_{s,T,\infty}^{2p} \right] \leq K_p \left( E \left[ \exp \left\{ K_p \|\gamma\|_{0,T,\mu}^p \right\} \right] \right)^{1/4},$$

where $\alpha = \max \{1/\mu, 2/(\mu + \theta)\}$, the constant $K_p$ depends only on $p, T, \theta, \mu$, ess sup $|R_s|$, $\|a\|_{\infty}$, $\|a'\|_{\infty}$, $\|b\|_{\infty}$, $\|b'\|_{\infty}$, $\|c\|_{\infty}$, $\|c'\|_{\infty}$, $\|c''\|_{\infty}$. 

Proof. In this proof the symbol $C$ will denote a generic constant which depends only on the parameters mentioned in the statement.

Fix some $N \geq 1, M \geq 1$ and define for $t \in [s, T] A_t = \left\{ \|\gamma\|_{0,t,\mu} + J_{Y,\theta}(t) \leq N, \|R\|_{s,t,\infty} \leq M \right\}$,

$\mathbb{I}_t = 1_{A_t}, Z_t = \int_{s}^{t} [b'(Y_s), R_s] dW_s$. Let also $\Delta = \min \{\Delta_1, \Delta_2, \Delta_3\}$, where

\begin{align*}
\Delta_1 &= (9K_{\theta,\mu} \|c''\|_{\infty} N + 1)^{-1/\mu}, \\
\Delta_2 &= (9 \|a''\|_{\infty} + 1)^{-1}, \\
\Delta_3 &= (9K_{\theta,\mu} \|c''\|_{\infty} (\|a\|_{\infty} + K_{\theta,\mu} \|c\|_{\infty} + 1) N^2 + 1)^{-1/(\mu + \theta)}.
\end{align*}

Take some $u, t \in [s, T]$ such that $u < t$ and estimate

\begin{align*}
\int_{u}^{t} a'(Y_r) R_r d\gamma_r &\leq \|a'\|_{\infty} \|R\|_{s,t,\infty} (t - u), \\
\int_{u}^{t} c'(Y_r) R_r d\gamma_r &\leq K_{\theta,\mu} \|\gamma\|_{0,t,\mu} \left( \|c'(Y)\|_{s,t,\infty} (t - u) + \|c'(Y)\|_{u,t,\theta} (t - u)^{\theta + \mu} \right) \\
&\leq K_{\theta,\mu} \|\gamma\|_{0,t,\mu} \left( \|c''\|_{\infty} \|R\|_{s,t,\infty} (t - u) \right. \\
&\quad \left. + \|c''\|_{\infty} \|Y\|_{s,t,\theta} (t - u)^{\theta + \mu} \right) \|R\|_{s,t,\infty}.
\end{align*}

Therefore,

\begin{align*}
\|R_t - R_u\| &\leq K_{\theta,\mu} \|\gamma\|_{0,t,\mu} \|c''\|_{\infty} (t - u)^{\theta + \mu} \|R\|_{s,t,\theta} + \|Z_t - Z_u\| \\
&\quad + \left( \|a'\|_{\infty} (t - u) + K_{\theta,\mu} \|\gamma\|_{0,t,\mu} \left( \|c''\|_{s,t,\infty} (t - u)^{\theta + \mu} \right. \\
&\quad \left. \|c''\|_{\infty} \|Y\|_{s,t,\theta} (t - u)^{\theta + \mu} \right) \right) \|R\|_{s,t,\infty}.
\end{align*}

Now let $(t - u) \leq \Delta$ and $\omega \in A_t$. Then

\begin{align*}
\|R\|_{s,t,\theta} &\leq K_{\theta,\mu} \|c''\|_{\infty} N \Delta^{\mu} \|R\|_{s,t,\theta} + \|Z\|_{s,t,\theta} \\
&\quad + \left( \|a'\|_{\infty} \Delta^{1 - \theta} + K_{\theta,\mu} N \left( \|c''\|_{\infty} \Delta^{\mu - \theta} + \|c''\|_{\infty} \|Y\|_{s,t,\theta} \Delta^{\mu} \right) \right) ||R||_{s,t,\infty} \\
&\leq \frac{1}{9} \|R\|_{s,t,\theta} + \|Z\|_{s,t,\theta} + \left( \frac{1}{9} \Delta^{1 - \theta} + \frac{1}{9} \Delta^{1 - \theta} + K_{\theta,\mu} \|c''\|_{\infty} \|Y\|_{s,t,\theta} \Delta^{\mu} \right) ||R||_{s,t,\infty}.
\end{align*}

Applying the estimate (A.2) and the definition of $\Delta_3$, we get $K_{\theta,\mu} \|c''\|_{\infty} \|Y\|_{s,t,\theta} N \Delta^{\mu + \theta} \leq 2/9$, which leads to

$$\|R\|_{s,t,\theta} \leq \frac{1}{2\Delta^{\theta}} \|R\|_{s,t,\infty} + \frac{9}{8} \|Z\|_{s,t,\theta}.$$ 

Obviously, $\|R\|_{s,t,\infty} \leq |R_u| + \|R\|_{s,t,\theta} \Delta^{\theta} \leq \|R\|_{s,t,\infty} + \|R\|_{s,t,\infty} / 2 + 9 \Delta^{\theta} \|Z\|_{s,t,\theta} / 8$, whence

$$\|R\|_{s,t,\infty} \leq 2 \|R\|_{s,t,\infty} + \frac{9\Delta^{\theta}}{4} \|Z\|_{s,t,\theta}$$

for $\omega \in A_t$. Therefore,

$$E \left[ \|R\|_{s,t,\infty}^{2p} \mathbb{I}_t \right] \leq C \left(E \left[ \|R\|_{s,t,\infty}^{2p} \mathbb{I}_t \right] + \Delta^{2p\theta} E \left[ \|Z\|_{s,t,\theta}^{2p} \mathbb{I}_t \right] \right) \leq C \left(E \left[ \|R\|_{s,t,\infty}^{2p} \mathbb{I}_s \right] + E \left[ \|Z\|_{s,t,\theta}^{2p} \mathbb{I}_t \right] \right),$$

(A.4)
We can assume without loss of generality that \( p > 2(1 - 2\theta)^{-1} \). Then, by the Garsia–Rodemich–Rumsey inequality,

\[
E \left[ \left\| Z \right\|_{u,t,\theta}^{2p} \right] \leq C \int_u^t \int_u^t E \left[ \left| Z_x - Z_y \right|^{2p} \right] \, dx \, dy \leq C \int_u^t \int_u^t E \left[ \left| f'_x b'(Y,r) \right|^{2p} \right] \, dx \, dy \leq C \int_u^t \int_u^t E \left[ \left| R^t \right|^{2p} \right] \, dx \, dy \leq C \int_u^t \int_u^t \left| x - y \right|^{p(1 - 2\theta)^{-3}} \, dx \, dy \leq C \int_u^t \int_u^t \left| x - y \right|^{p(1 - 2\theta)^{-3}} \, dx \, dy
\]

Plugging this estimate into (A.4) we get with the help of the Gronwall lemma that

\[
E \left[ \left\| R \right\|_{s,\theta}^{2p} \right] \leq C \left( E \left[ \left\| R \right\|_{s,u,\infty}^{2p} \right] \right) e^{C\Delta} \leq C \left( E \left[ \left\| R \right\|_{s,u,\infty}^{2p} \right] \right) 
\]

whenever \((t - u) < \Delta\). By induction, we get

\[
E \left[ \left\| R \right\|_{s,T,\infty}^{2p} \right] \leq E \left[ \left\| R \right\|_{s,T,\infty}^{2p} \right] e^{C(T - s)/\Delta} \leq C e^{CT/\Delta} \leq K e^{KN^\alpha},
\]

where we use the symbol \( K \) for the constant, as it will be fixed from now on. Denote \( \zeta = \left\| \gamma \right\|_{0,T,\mu} + J_{\gamma}(T) \). Observe that in (A.5) the right-hand side is independent of \( M \), so letting \( M \rightarrow \infty \), we get

\[
E \left[ \left\| R \right\|_{s,T,\infty}^{2p} 1_{\zeta \leq N} \right] \leq K \exp \{ KN^\alpha \} \text{ for any } N \geq 1.
\]

Now write

\[
E \left[ \left\| R \right\|_{s,T,\infty}^{2p} e^{-2K\zeta^\alpha} \right] = \sum_{n=1}^\infty E \left[ \left\| R \right\|_{s,T,\infty}^{2p} e^{-2K\zeta^\alpha} 1_{\zeta \in [n-1,n]} \right] 
\]

By the Cauchy–Schwarz inequality,

\[
E \left[ \left\| R \right\|_{s,T,\infty}^{2p} e^{-2K\zeta^\alpha} \right] \leq \left( E \left[ \left\| R \right\|_{s,T,\infty}^{2p} e^{-2K\zeta^\alpha} \right] E \left[ e^{2K\zeta^\alpha} \right] \right)^{1/2} \leq \left( K' E \left[ e^{2K\zeta^\alpha} \right] \right)^{1/2}.
\]

Finally,

\[
E \left[ e^{2K\zeta^\alpha} \right] \leq E \left[ \exp \left\{ 2^\alpha K \left( \left\| \gamma \right\|_{0,T,\mu} + J_{\gamma}(T)^\alpha \right) \right\} \right] 
\]

Applying Lemma 1 to the last term, we get the required statement. \( \square \)

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