Cobordism independence of Grassmann manifolds

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MS received 11 April 2003; revised 9 October 2003

Abstract. This note proves that, for \( F = \mathbb{R}, \mathbb{C}, \) or \( \mathbb{H}, \) the bordism classes of all non-bounding Grassmannian manifolds \( G_k(F^{n+k}) \), with \( k < n \) and having real dimension \( d \), constitute a linearly independent set in the unoriented bordism group \( \Omega_d \) regarded as a \( \mathbb{Z}_2 \)-vector space.

Keywords. Grassmannians; bordism; Stiefel–Whitney class.

1. Introduction

This paper is a continuation of the ongoing study of cobordism of Grassmann manifolds. Let \( F \) denote one of the division rings \( \mathbb{R} \) of reals, \( \mathbb{C} \) of complex numbers, or \( \mathbb{H} \) of quaternions. Let \( t = \dim_\mathbb{R} F \). Then the Grassmannian manifold \( G_k(F^{n+k}) \) is defined to be the set of all \( k \)-dimensional (left) subspaces of \( F^{n+k} \). \( G_k(F^{n+k}) \) is a closed manifold of real dimension \( nkt \). Using the orthogonal complement of a subspace one identifies \( G_k(F^{n+k}) \) with \( G_n(F^{n+k}) \).

In [8], Sankaran has proved that, for \( F = \mathbb{R}, \mathbb{C}, \) or \( \mathbb{H}, \) the Grassmannian manifold \( G_k(F^{n+k}) \) bounds if and only if \( \nu(n+k) > \nu(k) \), where, given a positive integer \( m \), \( \nu(m) \) denotes the largest integer such that \( 2^\nu(m) \) divides \( m \).

Given a positive integer \( d \), let \( \mathcal{G}(d) \) denote the set of bordism classes of all non-bounding Grassmannian manifolds \( G_k(F^{n+k}) \) having real dimension \( d \) such that \( k < n \). The restriction \( k < n \) is imposed because \( G_k(F^{n+k}) \approx G_n(F^{n+k}) \) and, for \( k = n, G_k(F^{n+k}) \) bounds. Thus, \( \mathcal{G}(d) = \{ [G_k(F^{n+k})] \in \Omega_n \mid nkt = d, k < n, \text{ and } \nu(n+k) \leq \nu(k) \} \subset \Omega_d \).

The purpose of this paper is to prove the following:

**Theorem 1.1.** \( \mathcal{G}(d) \) is a linearly independent set in the \( \mathbb{Z}_2 \)-vector space \( \Omega_d \).

Similar results for Dold and Milnor manifolds can be found in [6] and [1] respectively.

2. The real Grassmannians — a Brief review

The real Grassmannian manifold \( G_k(\mathbb{R}^{n+k}) \) is an \( nk \)-dimensional closed manifold of \( k \)-planes in \( \mathbb{R}^{n+k} \). It is well-known (see [5]) that the mod-2 cohomology of \( G_k(\mathbb{R}^{n+k}) \) is given by

\[
H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \ldots, w_k, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n]/\{w, \bar{w} = 1\},
\]
where \( w = 1 + w_1 + w_2 + \cdots + w_k \) and \( \tilde{w} = 1 + \tilde{w}_1 + \tilde{w}_2 + \cdots + \tilde{w}_n \) are the total Stiefel–Whitney classes of the universal \( k \)-plane bundle \( \gamma_k \) and the corresponding complementary bundle \( \gamma_k^c \), both over \( G_k(\mathbb{R}^{n+k}) \), respectively.

For computational convenience in this cohomology one uses the flag manifold \( \text{Flag}(\mathbb{R}^{n+k}) \) consisting of all ordered \((n+k)\)-tuples \((V_1, V_2, \ldots, V_{n+k})\) of mutually orthogonal one-dimensional subspaces of \( \mathbb{R}^{n+k} \) with respect to the ‘standard’ inner product on \( \mathbb{R}^{n+k} \). It is standard (see [4]) that the mod-2 cohomology of \( \text{Flag}(\mathbb{R}^{n+k}) \) is given by

\[
H^*(\text{Flag}(\mathbb{R}^{n+k}); \mathbb{Z}_2) \cong \mathbb{Z}_2[e_1, e_2, \ldots, e_{n+k}] / \left\{ \prod_{i=1}^{n+k} (1 + e_i) = 1 \right\},
\]

where \( e_1, e_2, \ldots, e_{n+k} \) are one-dimensional classes. In fact each \( e_i \) is the first Stiefel–Whitney class of the line bundle \( \lambda_i \) over \( \text{Flag}(\mathbb{R}^{n+k}) \) whose total space consists of pairs, a flag \((V_1, V_2, \ldots, V_{n+k})\) and a vector in \( V_i \).

There is a map \( \pi_{n+k} : \text{Flag}(\mathbb{R}^{n+k}) \rightarrow G_k(\mathbb{R}^{n+k}) \) which assigns to \((V_1, V_2, \ldots, V_{n+k})\), the \( k \)-dimensional subspace \( V_1 \oplus V_2 \oplus \cdots \oplus V_k \). In the cohomology, \( \pi_{n+k}^* : H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2) \rightarrow H^*(\text{Flag}(\mathbb{R}^{n+k}); \mathbb{Z}_2) \) is injective and is described by

\[
\pi_{n+k}^*(w) = \prod_{i=1}^{k} (1 + e_i), \quad \pi_{n+k}^*(\tilde{w}) = \prod_{i=k+1}^{n+k} (1 + e_i).
\]

In [2], Stong has observed, among others, the following facts:

**Fact 2.1.** The value of the class \( u \in H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2) \) on the fundamental class of \( G_k(\mathbb{R}^{n+k}) \) is the same as the value of

\[
\pi_{n+k}^*(u) e_1^{k-1} e_2^{k-2} \cdots e_k^{k-1} e_{k+1}^{n-k+1} e_{k+2}^{n-k+1} \cdots e_{n+k-1}^{n-k+1}
\]

on the fundamental class of \( \text{Flag}(\mathbb{R}^{n+k}) \).

**Fact 2.2.** In \( H^*(\text{Flag}(\mathbb{R}^{n+k}); \mathbb{Z}_2) \) one has

\[
e_{i_1}^{n+k-(r-1)} e_{i_2}^{n+k-(r-1)} \cdots e_{i_r}^{n+k-(r-1)} = 0
\]

if \( 1 \leq r \leq n+k \) and the set \( \{i_1, i_2, \ldots, i_r\} \subset \{1, 2, \ldots, n+k\} \). In particular \( e_i^{n+k} = 0 \) for each \( i, 1 \leq i \leq n+k \).

**Fact 2.3.** In the top dimensional cohomology of \( \text{Flag}(\mathbb{R}^{n+k}) \), a monomial \( e_{i_1}^{i_1} e_{i_2}^{i_2} \cdots e_{i_{n+k}}^{i_{n+k}} \) represents the non-zero class if and only if the set \( \{i_1, i_2, \ldots, i_{n+k}\} = \{0, 1, \ldots, n+k-1\} \).

The tangent bundle \( \tau \) over \( G_k(\mathbb{R}^{n+k}) \) is given (see [5]) by

\[
\tau \oplus \gamma_k \oplus \gamma_k \cong (n+k) \gamma_k.
\]

In particular, the total Stiefel–Whitney class \( W(G_k(\mathbb{R}^{n+k})) \) of the tangent bundle over \( G_k(\mathbb{R}^{n+k}) \) maps under \( \pi_{n+k}^* \) to

\[
\prod_{i=1}^{k} (1 + e_i)^{n+k} \cdot \prod_{1 \leq i < j \leq k} (1 + e_i + e_j)^{-2}.
\]
Choosing a positive integer $\alpha$ such that $2^\alpha \geq n+k$, we have, using Fact 2.2
\[
\pi_{n+k}^*(W(G_k(\mathbb{R}^{n+k}))) = \prod_{1 \leq i \leq k} (1+e_i)^{n+k} \cdot \prod_{1 \leq i < j \leq k} (1+e_i+e_j)^{2^\alpha-2}.
\]
Thus, the $m$th Stiefel–Whitney class $W_m = W_m(G_k(\mathbb{R}^{n+k}))$ maps under $\pi_{n+k}^*$ to the $m$th elementary symmetric polynomial in $e_i$, $1 \leq i \leq k$, each with multiplicity $n+k$, and $e_i+e_j$, $1 \leq i < j \leq k$, each with multiplicity $2^\alpha-2$. Therefore, if $S_p(\sigma_1, \sigma_2, \ldots, \sigma_p)$ denotes the expression of the power sum $\sum_{m=1}^q y_m^p$ as a polynomial in elementary symmetric polynomials $\sigma_m$’s in $q$ ‘unknowns’ $y_1, y_2, \ldots, y_q$, $q \geq p$, we have (see \cite{3})
\[
S_p(\pi_{n+k}^*(W_1), \pi_{n+k}^*(W_2), \ldots, \pi_{n+k}^*(W_p)) = \sum_{1 \leq i \leq k} (n+k)e_i^p.
\]
Thus we have a polynomial
\[
S_p(G_k(\mathbb{R}^{n+k})) = S_p(W_1, W_2, \ldots, W_p) \in H^p(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2)
\]
of Stiefel–Whitney classes of $G_k(\mathbb{R}^{n+k})$ such that
\[
\pi_{n+k}^*(S_p(G_k(\mathbb{R}^{n+k}))) = \begin{cases} 
\sum_{1 \leq i \leq k} e_i^p, & \text{if } n+k \text{ is odd and } p < n+k \\
0, & \text{otherwise}.
\end{cases} \tag{2.4}
\]

3. Proof of Theorem 1.1

It is shown in \cite{2} that
\[
[G_{2k}(\mathbb{R}^{2n+2k})] = [G_k(\mathbb{R}^{n+k})]^4 \quad \text{in } \mathfrak{M}_{4nk}.
\]
From this, we have, in particular,
\[
[G_k(F^{n+k})] = [G_k(\mathbb{R}^{n+k})]^\nu \quad \text{in } \mathfrak{M}_{nk}.
\]
For this one has to simply observe that the mod-2 cohomology of the $F$-Grassmannian is isomorphic as ring to that of the corresponding real Grassmannian by an obvious isomorphism that multiples the degree by $t$. On the other hand, since $\mathfrak{M}_*$ is a polynomial ring over the field $\mathbb{Z}_2$, we have the following:

Remark 3.1. A set $\{[M_1], [M_2], \ldots, [M_m]\}$ is linearly independent in $\mathfrak{M}_d$ if and only if the set $\{[M_1]^{2\beta}, [M_2]^{2\beta}, \ldots, [M_m]^{2\beta}\}$ is linearly independent in $\mathfrak{M}_{d,2\beta}$, $\beta \geq 0$.

Therefore, noting that $t = 1, 2, 4$, it is enough to prove Theorem 1.1 for real Grassmannians only. Thus, from now onwards, we shall take
\[
\mathcal{G}(d) = \{[G_k(\mathbb{R}^{n+k})] \mid nk = d, k < n, \text{ and } \nu(n+k) \leq \nu(k)\}.
\]

If $G_k(\mathbb{R}^{n+k})$ is an odd-dimensional real Grassmannian manifold then both $n$ and $k$ must be odd, and so $\nu(n+k) > \nu(k)$. This means that $G_k(\mathbb{R}^{n+k})$ bounds and so it follows that $\mathcal{G}(d) = \emptyset$ if $d$ is odd. Therefore we assume that $d$ is even.
Lemma 3.2. In $H^*(\text{Flag}(\mathbb{R}^{n+k}); \mathbb{Z}_2)$ one has, for $1 \leq j \leq k$,

$$\left( \sum_{1 \leq i \leq k} e_{i}^{n+k-(2j-1)} \right) e_{1}^{j-1} e_{2}^{j-2} \ldots e_{j-1}^{j-1} e_{k-(j-1)}^{n+k-(j-1)} \ldots e_{k}^{n+k-1} = e_{1}^{j-1} e_{2}^{j-2} \ldots e_{k-(j-1)}^{j-1} e_{k-(j-2)}^{n+k-(j-1)} \ldots e_{k}^{n+k-1}.$$

**Proof.** Note that

(a) if $i \neq k - (j - 1)$ then the exponent of $e_i$ in the product

$$e_{1}^{j-1} e_{2}^{j-2} \ldots e_{k-(j-1)}^{j-1} e_{k-(j-2)}^{n+k-(j-1)} \ldots e_{k}^{n+k-1}$$

is greater than or equal to $j$, and

(b) $\{n+k-(2j-1)\} + j = n+k-(j-1)$.

Therefore, invoking Fact 2.2, the lemma follows.

**PROPOSITION 3.3.**

Let $\mathcal{O}(d) = \{[G_k(\mathbb{R}^{n+k})] \in \mathcal{G}(d) \mid n+k \text{ is odd } \}$. Then $\mathcal{O}(d)$ is linearly independent in $\mathcal{R}_d$.

**Proof.** Arrange the members of $\mathcal{O}(d)$ in descending order of the values of $n+k$, so that

$$\mathcal{O}(d) = \{[G_{k_1}(\mathbb{R}^{n_1+k_1})], [G_{k_2}(\mathbb{R}^{n_2+k_2})], \ldots, [G_{k_s}(\mathbb{R}^{n_s+k_s})]\},$$

where $n_1+k_1 > n_2+k_2 > \cdots > n_s+k_s$. Note that $n_1 = d$ and $k_1 = 1$.

For a $d$-dimensional Grassmannian manifold $G_k(\mathbb{R}^{n+k})$, consider the polynomials

$$f_\ell(G_k(\mathbb{R}^{n+k})) = \prod_{1 \leq i \leq k} S_{n_i+k_i-(2j-1)}(G_k(\mathbb{R}^{n+k})) \in H^d(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2)$$

of Stiefel–Whitney classes of $G_k(\mathbb{R}^{n+k})$, where $1 \leq \ell \leq s$. Then, for each $\ell$, $1 \leq \ell \leq s$, we have, using 2.4,

$$\pi_{n_i+k_i}(f_\ell(G_k(\mathbb{R}^{n+k}))) e_1^{k_1-1} e_2^{k_2-2} \ldots e_{k_{\ell-1}}^{n_{\ell-1}-1} e_{k_{\ell+1}}^{n_{\ell+1}-2} \ldots e_{n+k_{\ell}}^{n_{s}-1}$$

$$= \left( \prod_{1 \leq i \leq k} \left( \sum_{1 \leq \ell \leq k} e_i^{n_i+k_i-(2j-1)} \right) \right) e_1^{k_1-1} e_2^{k_2-2} \ldots e_{k_{\ell}}^{n_i+k_{i}-1} e_{k_{\ell+1}}^{n_{i}-1} e_{k_{\ell+2}}^{n_{i}-2} \ldots e_{n+k_{\ell}}^{n_{i}-1},$$

applying Lemma 3.2 repeatedly for successive values of $j$.

Thus, in view of Facts 2.1 and 2.3 the Stiefel–Whitney number

$$\langle f_\ell(G_k(\mathbb{R}^{n+k})), [G_{k_\ell}(\mathbb{R}^{n+k})] \rangle \neq 0$$

for each $\ell$, $1 \leq \ell \leq s$. On the other hand, using 2.4, it is clear that

$$\langle f_\ell(G_k(\mathbb{R}^{n+k})), [G_{k_\ell}(\mathbb{R}^{n+k})] \rangle = 0$$
for each $h > \ell$, since $n_\ell + k_{\ell} - 1 \geq n_h + k_h$. Therefore, it follows that the $s \times s$ matrix
\[
\begin{bmatrix}
[f_s(G_k(\mathbb{R}^{n_k+k_h})), [G_k(\mathbb{R}^{n_k+k_h})]] \\
1 \leq \ell \leq s, 1 \leq k \leq s
\end{bmatrix}
\]
is non-singular; being lower triangular with 1’s in the diagonal. This completes the proof.

Now we shall complete the proof of Theorem 1.1 using induction on $d$. First note that
\[
\mathcal{G}(2) = \{ [G_1(\mathbb{R}^{2+1})] \} = \{ [\mathbb{R}^2] \},
\]
\[
\mathcal{G}(4) = \{ [G_1(\mathbb{R}^{4+1})] \} = \{ [\mathbb{R}^4] \},
\]
and so both are linearly independent in $\mathcal{M}_2$, $\mathcal{M}_4$ respectively. Assume that the theorem holds for all dimensions less than $d$.

We have $\mathcal{G}(d) = \mathcal{E}(d) \cup \mathcal{O}(d)$, where
\[
\mathcal{E}(d) = \{ [G_k(\mathbb{R}^{n+k})] \in \mathcal{G}(d) | n+k \text{ is even} \}
\]
and
\[
\mathcal{O}(d) = \{ [G_k(\mathbb{R}^{n+k})] \in \mathcal{G}(d) | n+k \text{ is odd} \}.
\]

Observe that if $[G_k(\mathbb{R}^{n+k})] \in \mathcal{E}(d)$ then both $n$ and $k$ are even with $v(k) \neq v(n)$. On the other hand, $[G_k(\mathbb{R}^{\frac{d}{2}+\frac{d}{2}})] \in \mathcal{E}(d)$ if $d \equiv 0 \pmod{8}$. Thus, $\mathcal{E}(d) \neq \emptyset$ if and only if $d \equiv 0 \pmod{8}$.

In view of Proposition 3.3, we may assume without any loss that $\mathcal{E}(d) \neq \emptyset$. Then, by the above observation and by Theorem 2.2 of [3], every member of $\mathcal{E}(d)$ is of the form $[G_k(\mathbb{R}^{\frac{d}{2}+\frac{d}{2}})]^d$, where $[G_k(\mathbb{R}^{\frac{d}{2}+\frac{d}{2}})] \in \mathcal{G}(\frac{d}{2})$. By induction hypothesis, $\mathcal{G}(\frac{d}{2})$ is linearly independent in $\mathcal{M}_d$.

So, by Remark 3.1
\[
\mathcal{E}(d) \text{ is linearly independent in } \mathcal{M}_d. \tag{3.4}
\]

Again note that if $[G_k(\mathbb{R}^{n+k})] \in \mathcal{E}(d)$, then, by 2.4, the polynomial $S_p(G_k(\mathbb{R}^{n+k})) = 0$, $\forall p \geq 1$. So, for each of the polynomials $f_s$, $1 \leq \ell \leq s$, considered in Proposition 3.3 we have
\[
\langle f_s(G_k(\mathbb{R}^{n+k})), [G_k(\mathbb{R}^{n+k})] \rangle = 0.
\]

Therefore, writing
\[
\mathcal{E}(d) = \{ [G_{k+1}(\mathbb{R}^{n_{k+1}+k_{k+1}})], [G_{k+2}(\mathbb{R}^{n_{k+2}+k_{k+2}})], \ldots, [G_{k+s+q}(\mathbb{R}^{n_{k+s+q}+k_{k+s+q}})] \},
\]
where $n_{s+1} + k_{s+1} > n_{s+2} + k_{s+2} > \cdots > n_{s+q} + k_{s+q}$, we see that the $s \times (s+q)$ matrix
\[
\langle f_s(G_k(\mathbb{R}^{n_k+k_h})), [G_k(\mathbb{R}^{n_k+k_h})] \rangle, 1 \leq \ell \leq s, 1 \leq k \leq s+q
\]
is of the form
\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\star & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\star & \star & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\star & \star & \star & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\star & \star & \star & \cdots & \star & \cdots & \cdots & \cdots & \cdots \\
\ell(d) & \ell(d)
\end{bmatrix} \tag{3.5}
\]
Thus, no non-trivial linear combination of members of $\mathcal{O}(d)$ can be expressed as a linear combination of the members of $\mathcal{E}(d)$. This, together with (3.4) and Proposition 3.3, proves that the set $\mathcal{G}(d) = \mathcal{E}(d) \cup \mathcal{O}(d)$ is linearly independent in $\mathcal{N}_d$. Hence, by induction, Theorem 1.1 is completely proved.

**Remark 3.6.** Using the decomposition of the members of $\mathcal{E}(d)$, and the polynomials $f_i$, in the lower dimensions together with the doubling homomorphism defined by Milnor [7], one can obtain a set of polynomials of Stiefel–Whitney classes which yield, as in Proposition 3.3, a lower triangular matrix for $\mathcal{E}(d)$ with 1’s in the diagonal. Thus using (3.5) we have a lower triangular matrix, with 1’s in the diagonal, for the whole set $\mathcal{G}(d)$.

**Acknowledgement**

Part of this work was done under a DST project

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