Information inefficiency in a random linear economy model

JOÃO PEDRO JERICÓ¹ and RENATO VICENTE²

¹ Dep. de Física Geral, Instituto de Física, Universidade de São Paulo, CP 66319, 05315-970, São Paulo-SP, Brazil
² Dept. of Applied Mathematics, Instituto de Matemática e Estatística, Universidade de São Paulo, 05508-090, São Paulo-SP, Brazil

PACS 89.75.–k – Complex systems
PACS 89.65.Gh – Economics; econophysics, financial markets, business and management

Abstract – We study the effects of introducing information inefficiency in a model for a random linear economy with a representative consumer. This is done by considering statistical, instead of classical, economic general equilibria. Employing two different approaches we show that inefficiency increases the consumption set of a consumer but decreases her expected utility. In this scenario economic activity grows while welfare shrinks, what is contrary to the behavior obtained by considering a rational consumer.

Introduction. – Standard microeconomic theory states that an agent faced with a choice of goods or services to purchase will maximize an expected utility function given her constraints [1]. It also predicts an equilibrium state resulting from these agents simultaneously and independently trying to optimize their own choices.

Over the last decade these assumptions have been under intense scrutiny [2, 3]. Among other factors, the 2007-2008 mortgage crisis, for which the predictions made by mainstream theories were ineffective in preventing, and the growing empirical body of evidence being gathered in the field of Behavioral Economics have shown that perfect rationality may be a very poor proxy for the observed behavior of economic agents.

In fact, it has been shown that modelling agents to have zero intelligence also produces realistic results [4,5] due to general statistical properties that may emerge in interacting systems. As argued by the authors in [6], treating economic agents as having full rationality and perfect knowledge is clearly a significant assumption, so why not model the other end of the spectrum? It has been shown in [6] that, for an economy in which trades occur in a random fashion, the equilibrium wealth distribution is a Gibbs distribution. This is very close to empirical observations for several countries, at least in the bulk of the distribution. By adding a rate of return on capital, power law distributions are obtained [7], what is again consistent with empirical data for top earners.

Statistical mechanics provides a robust framework, based on maximum entropy principles, for identifying and calculating macroscopic quantities relevant for the description of interacting systems. Within this framework, the temperature of a system represents how large observed deviations from the ground state are likely to be. Clearly, a zero temperature system is always at the configuration which minimizes its energy function, while a system at infinite temperature will be at any configuration allowed by its constraints with equal probability.

These considerations suggest a way to model consumers that may turn to be more realistic: the degree of rationality may be represented as a parameter within a statistical mechanics model with energy given by the negative utility. A similar approach has already appeared in the contexts of game theory [8,9] and general equilibrium [10,11].

Although deviations from maximum utility are here named, in accordance with economics literature, “suboptimal” or “inefficient”, it is common knowledge in statistical inference that a “softmax” decision rule can actually be beneficial if the agent has incomplete knowledge about the universe of available options.

In this paper we discuss two approaches for considering consumer irrationality and show that they qualitatively lead to the same behavior for important macroeconomic quantities.

The Random Linear Economy Model. – In a random linear economy [12] there are M goods, N technologies and a single representative consumer. Let n = N/M represent the multiplicity of technologies per good.

A consumer is endowed by an initial bundle of goods $x_0 = (x_0^1, \ldots, x_0^M)$. We will be interested in an ensemble...
of economies with a given technology multiplicity. We suppose an endowment \( x_0^\mu \geq 0 \) is drawn independently from an exponential distribution. Consumer preferences are ordered according to a separable utility function

\[ U(x) = \sum_{\mu=1}^{M} u(x^\mu). \]

The consumer optimizes its utility subject to a budget constraint \( x \cdot p \leq x_0 \cdot p \), where \( p \) represents a vector of prices.

In a closed economy, markets clear at equilibrium so that firms inputs have to come from consumer’s initial endowment. The \( N \) dimensional production scale vector \( s \) satisfies

\[ x = x_0 + \sum_{i=1}^{N} s_i \xi_i. \tag{1} \]

Every firm in the market is equipped with a random technology \( \xi_i = (\xi_1^i, \ldots, \xi_M^i) \), where \( \xi_0^i < 0 \) represents an input and \( \xi_M^i > 0 \) represents an output. Thus, a firm represents a transformation in the space of goods given by the \( M \) dimensional vector \( s_i \xi_i \). The firm has to decide at what scale \( s_i \) it will operate its technology so that profits \( s_i p_i \cdot \xi_i \) are maximized. We consider normally distributed elements \( \xi_1^i \) with variance \( 1/M \) constrained by

\[ \sum_{i=1}^{M} \xi_i^\mu = -\epsilon. \]

For \( \epsilon > 0 \) every technology is inefficient and it is impossible to couple two or more of them in order to produce an infinite amount of a good.

The market clearing condition implies that the utilitities and profits can be maximized simultaneously. Optimal consumer bundle \( x^* = \arg \max_s U(x) \) and production scales \( s^*_i = \arg \max_p s_i p_i \cdot \xi_i \) are attained when prices are set to satisfy the first order condition for the consumer utility maximization problem \( p_\mu = \frac{\partial U}{\partial p_\mu} \). At the equilibrium, firms production and the market prices can be set to satisfy consumer’s desired demand, and no actor in the market has an incentive to deviate from this state.

Market clearing, however, carries a strong restriction. If we multiply both sides of the equation (1) by \( p \), we get

\[ p \cdot (x - x_0) = \sum_{i=1}^{N} s_i p_i \cdot \xi_i. \tag{2} \]

The budget condition implies a non-positive left side, while the right hand side of this equation is necessarily non-negative, as the scale of production can be made \( s_i = 0 \) to avoid losses. The equation is, therefore, only satisfied when agents spend their budget to the limit, to say, when \( p \cdot x = p \cdot x_0 \), and when firm profits vanish, \( p \cdot \xi_i = 0 \), or firms leave the market, and \( s_i = 0 \).

Equation (2) also implies that not more than \( M \) different technologies can be simultaneously active at equilibrium. This can be seen by observing that, for \( n > 1 \), \( p \cdot \xi = 0 \) represents an overdetermined linear system with \( N \) equations and \( M \) variables.

This model has some very interesting properties which are described at length in [12] and more recently explored in [13]. In particular, given an ensemble of economies, we can calculate the probability distributions of \( x \) and \( s \) (and therefore of \( p \)) and find that all macroscopic quantities derived from these two quantities depend on the technology multiplicity per good \( n = N/M \). When \( n < 2 \), the market is competitive and the fraction of active firms \( \phi = \sum_{i} I(s_i > 0)/N \) is around \( \phi = 0.5 \). As each firm has on average half the goods as inputs and half as outputs, when \( n < 2 \), the number of technologies is not sufficient to produce every combination of \( M \) goods demanded. The space of goods is not spanned by the technologies available.

When \( n > 2 \), however, there are so many technologies to choose from that it is possible to find \( M \) linear independent firms that span the whole space of goods. The market thus becomes monopolistic and \( \phi \) vanishes asymptotically with \( n \) as the maximum number of active technologies is \( M \).

In Economics the Gross Domestic Product (GDP) is a measure of the total market value of final goods produced in a given period. In the random linear economy model, the consequence of the market clearing condition (2) is that no net market value is created by the transformation process. An alternative measure of economic activity within the model is provided by

\[ Y = \frac{\sum_{\mu=1}^{M} |x_\mu - x_0^\mu| p_\mu}{2 \sum_{\mu=0}^{M} p_\mu}. \tag{3} \]

In [12] it is shown that in the competitive regime \( n < 2 \), a new technology has a positive effect on economic activity \( Y \), while in the monopolistic regime \( n > 2 \), new technologies have negligible impact on the economic activity. Our goals in this paper is revisiting this result in a setting where the representative consumer is an inefficient optimizer.

In what follows we do that in two different ways. In the first approach we employ a maximum entropy argument for replacing the idea that the consumer is a hyperrational maximizer. In the second approach we suppose that the consumer actually optimizes a noisy version of her utility.

The Irrational Consumer. — We now turn to the problem of treating the consumer as an agent that makes suboptimal choices with as little extra assumptions for her behavior as possible.

An answer for this type of question is provided by statistical mechanics in the form of the following inference problem: find the probability distribution for \( x \) that has maximum entropy given the constraints that \( P(x) \) is a probability distribution and the average utility \( \bar{U} \) is observed [15]. Using the simplifying notation that \( \int dx = \int_0^\infty \cdots \int_0^\infty \prod_{\mu} dx_\mu \), we want to find the function \( P(x) \) that maximizes the functional
where $\alpha$ and $\beta$ are Lagrange multipliers for the constraints. The well known solution for this entropy maximization problem is the Gibbs distribution:

$$P(x) = \frac{1}{Z(\beta)} e^{\beta U(x)}, \quad (5)$$

where $Z = \int_0^\infty dx e^{\beta U(x)}$ is the normalization term and $\beta$ is a free parameter that controls how much we allow for deviations from the maximum. It is clear that in the limit $\beta \to \infty$ the distribution collapses to a Dirac delta in its maximum and the inference reduces to the classical consumer problem. We, however, have found in $\beta$ a way to introduce deviations from the maximum. Additionally, the maximum entropy argument assures we are not introducing unjustified assumptions about consumer’s behavior.

Because market clearing still holds, we can plug equation (1) directly into the Gibbs distribution. The probability of the consumer choosing a bundle $x$ becomes the probability of firms setting a production vector $s$, given by:

$$P(s|\xi, x_0) = \frac{1}{Z(\beta)} e^{\beta U(x_0 + \sum_i s_i \xi_i)} \quad (6)$$

Although this distribution is intractable, except for $\beta \to \infty$, we still can find the expected values of macroeconomic variables for a random linear economy by Metropolis sampling averaged over the disorder $(\xi, x_0)$. We proceed by first finding the average fraction of active firms $\langle \phi \rangle_{\xi, x_0}$ at the equilibrium as a function of $\beta$. The results are shown in Figure 1. We see that the inefficiency parameter $\beta$ has a strong influence on the “two regimes” observed in the efficient case, which is shown by making $\beta > 10^5$. For lower values, the monopolistic regime vanishes, and after a certain point the fraction of active firms actually increases with $n$.

More interesting, however, are the behavior of other macroeconomic variables of the model. By definition, a rational consumer increases her utility as $n$ increases. If the supply of trades is expanded by the introduction of new firms operating in a market, the agent can always choose to keep her current exchange or replace it for a better one.

However, when agents are irrational that is not always the case. In this case we can compute numerically the average utility per good $\langle u \rangle$ for the consumer as a function of $n$:

$$\langle u \rangle = \frac{1}{M} \langle U \rangle_{\xi, x_0} = \frac{1}{M} \int dx \int d\xi dx_0 U(x) \frac{1}{Z} e^{\beta U(x)} \quad (7)$$

As can be seen in Figure 2, for large $\beta$, $\langle u \rangle$ is an increasing function of $n$, as expected. However, when inefficiency is increased, the average utility of the consumer becomes a decreasing function. This result corroborates a number of empirical observations in Behavioral Economics claiming that consuming options can turn to be excessive [16, 17]: as the number of options available to a consumer without perfect knowledge and full rationality increases the probability of choice mistakes also increases.

As a proxy for economic activity, we also calculate the absolute amount of goods traded, that is given by

$$\Lambda = \frac{1}{M} \sum_{\mu=1}^M |x_\mu - x_\mu^n| \quad (8)$$

We observe in the simulations depicted in the lower panel of Figure 2 that the economic activity always increases as a function of the technological multiplicity $n$, but the rate of increase in $\Lambda$ also increases with the inefficiency, as opposed to the utility. This is our main observation: irrational choices enhance the amount of goods traded while worsening the quality of what is acquired, decreasing consumer’s utility.

A difficulty with our analysis still remains: whereas originally the prices were uniquely defined by the first order condition of the consumer’s optimization problem, in this new framework there is no such restraint. Given a probability distribution $P(x|\xi, x_0)$, it is an ill-posed problem to define the price vector from it. In fact, any price vector $p$ satisfying

$$p \cdot \xi_i = 0, \text{ for all } i \text{ if } s_i > 0 \quad (9)$$

would be an equally good choice.

Since we have $N_0$ equations of this type, one for each active firm, and $M$ unknown variables $p_\mu$, the above equa-
However, she is actually optimizing another utility. The curve $\phi = 1/n$ is plotted (dashed line) in Figure 3. The model only stays, for all $n$, in the region where global prices are possible when very close to rationality ($\beta$ large). When there is a level of inefficiency, there is always a threshold $n$, above which no global prices can exist.

This does not mean that no market transactions will take place, instead it means that in this approach prices must be defined for each of the $N$ suppliers.

**Noisy Utility.** As an alternative along the lines of [14], we can modify the classical setup so that the equilibrium is still reached from a maximization problem solved by the consumer. We suppose now that the representative consumer thinks she is rationally optimizing an utility $U$. However, she is actually optimizing another utility $U_T$, a noisy version of $U$.

Call $U_T$ the “true” utility function and write it as

$$U_T(x) = U(x) + h \cdot x,$$

where $h$ is a $M$ dimensional vector with all entries sampled independently from an exponential distribution with scale $\lambda$. This extra term represents a part of the utility unknown by the consumer herself. We point out that the true utility function $U_T$ is still a concave function and therefore the equilibrium is still unique and well behaved.

It can be shown [14] that the resulting probability distribution for $x$, when the consumer is maximizing the utility distribution $U_T(x)$, is the Gibbs distribution for $U(x)$. This allows us to reformulate the problem in terms of a consumer that does not choose the “optimal” utility, keeping a well defined price vector which results from the usual first order conditions. For example, for the utility $U(x) = \sum p \log(x)$ the price of a good would be given by $p_n = \partial_n U_t(x) = 1/n + h_n$.

The results for the fraction of active firms $\phi$ are now the same of the rational agent case and we still have a competitive regime for $n < 2$ and a monopolistic regime for $n > 2$. However, the behavior of the macroeconomic quantities reproduces that of the inefficient consumer. Figure 3 depicts solutions of the maximization process of $U_T$ for various values of the scale $\lambda$ for the exponential distribution of $h_n$.

As in the previous case, above a threshold $\lambda$, the observed utility decreases with $n$, while the GDP increases.

In this approach we had to introduce additional assumptions about consumer’s utility. As it was noticed by the authors of [14], the Gibbs distribution for the observed depends on very general conditions for the unknown part, therefore, the results we have obtained are supposed to be robust to different choices in the perturbed utility.

**Conclusion.** We have shown that the introduction of irrationality in the behavior a representative consumer implies in macroeconomic behavior very distinct of what is the derived from classical rationality axioms.

Provided the coupling between demand and supply imposed by a market clearing scenario, as inefficiency is introduced to the consumer behavior, the supply side is exposed to less selection and the consumer is flooded by an overabundance of low utility goods. What corroborates results from Behavioral Economics [16][17].

With less selection the supply side produces more and GDP grows. Thus a larger number of consumption options may result in poorer choices while increasing economic activity.

Our conclusion may be intriguing: for a whole country’s economy it is seems desirable that its consumers are less rational agents as less efficiency in choice leads to enhanced GDP and tax revenues. If we thinks of country development as an evolutionary process, there would be incentives for less rational consumers, penalizing the lower economic activity of more a efficient demand side. Additionally, if the average utility can be considered a proxy for

\[
\text{Average utility of the consumer as a function of } n, \text{ for different values of } \beta. \text{ Bottom. Average economic activity for the economy as a function of } n.
\]
for individual fitness, then a more rational consumer would thrive in a country with low choice efficiency, suggesting that an interesting competition process may take place.

This is certainly too deep a conclusion to be arrived from just this simple model. However, the results may be robust for other equilibrium scenarios, and we hope this paper sets the stage for future research on the topic.

** REFERENCES **

[1] Microeconomic theory - Mas-Colell, Andreu and Whinston, Michael Dennis and Green, Jerry R and others, (Oxford university press New York) 1995

[2] Bouchaud, Jean-Philippe, Nature, 455 (2008)

---

J.P.J. is thankful for the grant conceded by FAPESP, process number 2012/19521-8. We also thank Matteo Marsili for his contributions to this paper.