The Work of Vladimir Voevodsky

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Vladimir Voevodsky was born in 1966. He studied at Moscow State University and Harvard university. He is now Professor at the Institute for Advanced Study in Princeton.

Among his main achievements are the following: he defined and developed motivic cohomology and the $\mathbb{A}^1$-homotopy theory of algebraic varieties; he proved the Milnor conjectures on the $K$-theory of fields.

Let us state the first Milnor conjecture. Let $F$ be a field and $n$ a positive integer. The Milnor $K$-group of $F$ is the abelian group $K^M_n(F)$ defined by the following generators and relations. The generators are sequences $\{a_1, \ldots, a_n\}$ of $n$ units $a_i \in F^*$. The relations are

$$\{a_1, \ldots, a_{k-1}, xy, a_{k+1}, \ldots, a_n\} = \{a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_n\} + \{a_1, \ldots, a_{k-1}, y, a_{k+1}, \ldots, a_n\}$$

for all $a_i, x, y \in F^*$, $1 \leq k \leq n$, and the Steinberg relation

$$\{a_1, \ldots, x, \ldots, 1-x, \ldots, a_n\} = 0$$

for all $a_i \in F^*$ and $x \in F - \{0,1\}$.

On the other hand, let $\overline{F}$ be an algebraic closure of $F$ and $G = \text{Gal}(\overline{F}/F)$ the absolute Galois group of $F$, with its profinite topology. The Galois cohomology of $F$ with $\mathbb{Z}/2$ coefficients is, by definition,

$$H^n(F, \mathbb{Z}/2) = H^n_{\text{continuous}}(G, \mathbb{Z}/2).$$

**Theorem 1.** (Voevodsky 1996) Assume $1/2 \in F$ and $n \geq 1$. The Galois symbol

$$h_n : K^M_n(F)/2K^M_n(F) \to H^n(F, \mathbb{Z}/2)$$

is an isomorphism.

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This was conjectured by Milnor in 1970 \[\[1\]. When \(n = 2\), Theorem 1 was proved by Merkurjev in 1983. The case \(n = 3\) was then solved independently by Merkurjev-Suslin and Rost.

There exists also a Galois symbol on \(K_n^M(F)/pK_n^M(F)\) for any prime \(p\) invertible in \(F\). When \(n = 2\) and \(F\) is a number field, Tate proved that it is an isomorphism. In 1983 Merkurjev and Suslin proved that it is an isomorphism when \(n = 2\) and \(F\) is any field. Both Voevodsky and Rost have made a lot of progress towards proving that, for any \(F\), any \(n > 0\) and any \(p\) invertible in \(F\), the Galois symbol is an isomorphism.

The map \(h_n\) in Theorem 1 is defined as follows. When \(n = 1\), we have \(K_1^M(F) = F^*\) and \(H^1(F;\mathbb{Z}/2) = \text{Hom}(G,\mathbb{Z}/2)\). The map

\[
  h_1 : (F^* / (F^*)^2) \to \text{Hom}(G,\mathbb{Z}/2)
\]

maps \(a \in F^*\) to the quadratic character \(\chi_a\) defined by

\[
  \chi_a(g) = g(\sqrt{a})/\sqrt{a} = \pm 1
\]

for any \(g \in G\) and any square root \(\sqrt{a}\) of \(a\) in \(F\). That \(h_1\) is bijective is a special case of Kummer theory. When \(n \geq 2\), we just need to define \(h_n\) on the generators \(\{a_1, \ldots, a_n\}\) of \(K_n^M(F)\). It is given by a cup-product:

\[
  h_n(\{a_1, \ldots, a_n\}) = \chi_{a_1} \cup \cdots \cup \chi_{a_n}.
\]

The fact that \(h_n\) is compatible with the Steinberg relation was first noticed by Bass and Tate.

Theorem 1 says that \(H^n(F;\mathbb{Z}/2)\) has a very explicit description. In particular, an immediate consequence of Theorem 1 and the definition of \(h_n\) is the following

**Corollary 1.** The graded \(\mathbb{Z}/2\)-algebra \(\bigoplus_{n \geq 0} H^n(F;\mathbb{Z}/2)\) is spanned by elements of degree one.

This means that absolute Galois groups are very special groups. Indeed, it is seldom seen that the cohomology of a group or a topological space is spanned in degree one.

**Corollary 2.** (Bloch) Let \(X\) be a complex algebraic variety and \(\alpha \in H^n(X(\mathbb{C}),\mathbb{Z})\) a class in its singular cohomology. Assume that \(2\alpha = 0\). Then, there exists a nonempty Zariski open subset \(U \subset X\) such that the restriction of \(\alpha\) to \(U\) vanishes.

If Theorem 1 was extended to \(K_n^M(F)/pK_n^M(F)\) for all \(n\) and \(p\), Corollary 2 would say that any torsion class in the integral singular cohomology of \(X\) is supported on some hypersurface. (Hodge seems to have believed that such a torsion class should be Poincaré dual to an analytic cycle, but this is not always true.)
With Orlov and Vishik, Voevodsky proved a second conjecture of Milnor relating the Witt group of quadratic forms over $F$ to its Milnor $K$-theory [3].

A very serious difficulty that Voevodsky had to overcome to prove Theorem 1 was that, when $n = 2$, Merkurjev made use of the algebraic $K$-theory of conics over $F$, but, when $n \geq 2$, one needed to study special quadric hypersurfaces of dimension $2^{n-1} - 1$. And it is quite hard to compute the algebraic $K$-theory of varieties of such a high dimension. Although Rost had obtained crucial information about the $K$-theory of these quadrics, this was not enough to conclude the proof when $n > 3$. Instead of algebraic $K$-theory, Voevodsky used motivic cohomology, which turned out to be more computable.

Given an algebraic variety $X$ over $F$ and two integers $p, q \in \mathbb{Z}$, Voevodsky defined an abelian group $H^{p,q}(X, \mathbb{Z})$, called motivic cohomology. These groups are analogs of the singular cohomology of CW-complexes. They satisfy a long list of properties, which had been anticipated by Beilinson and Lichtenbaum. For example, when $n$ is a positive integer and $X$ is smooth, the group

$$H^{2n,n}(X, \mathbb{Z}) = \text{CH}^n(X)$$

is the Chow group of codimension $n$ algebraic cycles on $X$ modulo linear equivalence. And when $X$ is a point we have

$$H^{n,n}(\text{point}) = K^n_M(F).$$

It is also possible to compute Quillen’s algebraic $K$-theory from motivic cohomology. Earlier constructions of motivic cohomology are due to Bloch (at the end of the seventies) and, later, to Suslin. The way Suslin modified Bloch’s definition was crucial to Voevodsky’s approach and, as a matter of fact, several important papers on this topic were written jointly by Suslin and Voevodsky [4, 5]. There exist also two very different definitions of $H^{p,q}(X, \mathbb{Z})$, due to Levine and Hanamura; according to the experts they lead to the same groups. But it seems fair to say that Voevodsky’s approach to motivic cohomology is the most complete and satisfactory one.

A larger context in which Voevodsky developed motivic cohomology is the $\mathbb{A}^1$-homotopy of algebraic manifolds [6], which is a theory of “algebraic varieties up to deformations”, developed jointly with Morel [2]. Starting with the category of smooth manifolds (over a fixed field $F$), they first embed this category into the category of Nisnevich sheaves, by sending a given manifold to the sheaf it represents. A Nisnevich sheaf is a sheaf of sets on the category of smooth manifolds for the Nisnevich topology, a topology which is finer (resp. coarser) than the Zariski (resp. étale) topology. Then Morel and Voevodsky define a homotopy theory of Nisnevich sheaves in much the same way the homotopy theory of CW-complexes is defined. The parameter space of deformations is the affine line $\mathbb{A}^1$ instead of the real unit interval $[0, 1]$. Note that, in this theory there are two circles (corresponding to the two degrees $p$ and $q$ for motivic cohomology)! The first circle is the sheaf represented by the smooth manifold $\mathbb{A}^1 - \{0\}$ (indeed, $\mathbb{C} - \{0\}$ has the homotopy type of a circle). The second circle is $\mathbb{A}^1/\{0, 1\}$ (note that $\mathbb{R}/\{0, 1\}$ is a loop). The latter is not represented by a smooth manifold. But, if we identify 0 and 1 in the sheaf
of sets represented by \( \mathbb{A}^1 \) we get a presheaf of sets, and \( \mathbb{A}^1/\{0, 1\} \) can be defined as the sheaf attached to this presheaf. This example shows why it was useful to embed the category of algebraic manifolds into a category of sheaves.

It is quite extraordinary that such a homotopy theory of algebraic manifolds exists at all. In the fifties and sixties, interesting invariants of differentiable manifolds were introduced using algebraic topology. But very few mathematicians anticipated that these “soft” methods would ever be successful for algebraic manifolds. It seems now that any notion in algebraic topology will find a partner in algebraic geometry. This has long been the case with Quillen’s algebraic \( K \)-theory, which is precisely analogous to topological \( K \)-theory. We mentioned that motivic cohomology is an algebraic analog of singular cohomology. Voevodsky also computed the algebraic analog of the Steenrod algebra, i.e. cohomological operations on motivic cohomology (this played a decisive role in the proof of Theorem 1). Morel and Voevodsky developed the (stable) \( \mathbb{A}^1 \)-homotopy theory of algebraic manifolds. Voevodsky defined *algebraic cobordism* as homotopy classes of maps from the suspension of an algebraic manifold to the classifying space \( \operatorname{MGL} \). There is also a direct geometric definition of algebraic cobordism, due to Levine and Morel (see Levine’s talk in these proceedings), which should compare well with Voevodsky’s definition. And the list is growing: Morava \( K \)-theories, stable homotopy groups of spheres, etc.

Vladimir Voevodsky is an amazing mathematician. He has demonstrated an exceptional talent for creating new abstract theories, about which he proved highly nontrivial theorems. He was able to use these theories to solve several of the main long standing problems in algebraic \( K \)-theory. The field is completely different after his work. He opened large new avenues and, to use the same word as Laumon, he is leading us closer to the world of *motives* that Grothendieck was dreaming about in the sixties.

**References**

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Vladimir Voevodsky

June 4, 1966

| Year    | Event                                                                 |
|---------|----------------------------------------------------------------------|
| 1989    | B.S. in Mathematics, Moscow University                               |
| 1992    | Ph.D. in Mathematics, Harvard University                            |
| 1992–1993| Institute for Advanced Study, Member                                |
| 1993–1996| Harvard University, Junior Fellow of Harvard Society of Fellows     |
| 1996–1997| Harvard University, Visiting Scholar                                |
| 1996–1997| Max-Planck Institute, Visiting Scholar                              |
| 1996–1999| Northwestern University, Associate Professor                       |
| 1998–2001| Institute for Advanced Study, Member                                |
| 2002    | Institute for Advanced Study, Professor                            |

V. Voevodsky (right) and C. Soulé