Changeover phenomenon in randomly colored Potts model

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A hybrid Potts model where a random concentration $p$ of the spins assume $q_0$ states and a random concentration $1 - p$ of the spins assume $q > q_0$ states is introduced. It is known that when the system is homogeneous, with an integer spin number $q_0$ or $q$, it undergoes a second or a first order transition, respectively. It is argued that there is a concentration $p^*$ such that the transition nature of the model is changed at $p^*$. This idea is demonstrated analytically and by simulations for two different types of interaction: the usual square lattice nearest neighboring and the mean field all-to-all interaction. Exact expressions for the second order critical line in concentration-temperature parameter space of the mean field model together with some other related critical properties, are derived.

I. INTRODUCTION

The Potts model [1, 2] has been extensively studied in the past few decades and the amount of related literature is enormous. Though the Potts model has some experimental realizations [3–5], it is mostly successful as a testing laboratory to analytical concepts and simulation methods [6–20]. In the context of temperature-driven phase transitions, the Potts model can be thought of as a generalization to the celebrated Ising model with the typical “up-down” symmetry, as it has a multi-fold symmetry that is spontaneously broken.

An interesting problem that has been first studied by Baxter [6] is the dependence of the transition behavior (first versus second order) on the spin number $q$. Baxter considered the square lattice model with nearest neighbors interaction (hereafter referred to as the standard model) and obtained an exact expression for the latent heat. That expression has remained finite for $q > 4$ and has vanished for $q = 4$. Using an equivalence of the Potts model to a six-vertex model [21] Baxter has shown that the free energy of the latter near the critical temperature was negligibly small. Based on these findings, Baxter conjectured that the model in subject exhibits a continuous transition for $q \leq q_c$ and discontinuous transition for $q > q_c$ with $q_c = 4$ being the changeover integer or the maximal integer for which the transition is continuous. Recently, Duminil-Copin et al [22, 23] have rigorously proven Baxter’s conjecture using the random cluster or the Fortuin-Kasteleyn (FK) [24] representation of the Potts model. Though the changeover integer $q_c = 4$ has been found for the square lattice, it is believed to be lattice independent [7].

The general convention is that two dimensional systems accompanying the spontaneous breaking of the $q$-fold Potts symmetry, maintain the changeover behavior of the standard model. There are, however, a few counterexamples [25–31]. For instance, in [25] it has been shown that for a mean-field-like finite but long range interaction, the system undergoes a first order transition for $q = 3$. The same result has been rigorously established in [26] for a system with an exponentially decaying infinite range interaction.

In [30, 31] the authors introduced the $(q, r)$ Potts model defined by a Hamiltonian with $q$ “visible” colors (Potts states) as in the standard model and additional $r$ “invisible” colors that control the entropy but do not affect the energy. The model has a growing interest in recent years and problems such as its marginal dimensions [32] or its behavior in scale-free networks [33] have been addressed.

It has been demonstrated [30] using a mean field solution of the BEG model [34] together with simulations, and also rigorously proven [35, 36] that the $(q, r)$ model exhibits a first order transition for any $q$ and $r$ sufficiently large. A non-trivial outcome of the latter result is the occurrence of a first order transition for $q \leq 4$. The reason is that, typically, in the continuous transition cases large fractal-like configurations buffer between order and disorder, and, even though the number of such fractals grows exponentially with their size, they do not compensate the entropic cost of their construction when the total number of microstates is large enough.

We present a hybrid Potts (HP) model that captures the concept of manipulating the transition order by controlling the number of microstates, keeping the interaction parameters fixed, in a different way. Consider a Potts Hamiltonian $H\{\sigma\}$ where $\{\sigma\}$ is a configuration of $N$ spins, each can select one out of $\{1, 2, ..., Q_i\}$ colors where

$$Q_i = -X_i(q - q_0) + q, \quad i = 1, ..., N,$$

(1)

and $X_i \sim Ber(p)$ are i.i.d. random variables. In other words, each spin can be colored by $q_0 \leq q_c$ “strong” colors with probability $p$ and by $q > q_c$ colors (containing $q - q_0$ additional “weak” colors), with probability $1 - p$. We say that spins chosen with probability $p$ or $1 - p$ belong to strong or weak regions, respectively.

As opposed to the $(q, r)$ model, in the present model the weak colors contribute both to the energy and the entropy of the system. Thus, the HP model cannot be mapped onto a $(q_0, 1)$ [30] model where spins in the single invisible state couple to an effective external field. Still, when the strong regions are large enough, it is entropically disadvantageous for the system to construct large clusters with $q$ colors. In that case one can neglect the interactions induced by weak colors. Thus, weak regions
typically dominate the entropy production and not the
energy saving. Consequently, an effective \((q_0, 1)\) description
may be reasonable then.

Since the HP model assumes different color densities lying between
limiting homogeneous instances where the transition is either continuous or discon-  

Since the HP model assumes different color densities lying between limiting homogeneous instances where the transition is either continuous or discontinuous, one might expect that for some concentration, \(0 < p^* < 1\), the model displays a \textit{changeover phenomenon}, that is, switches from one type of transition to another for every combination of \(q_0 \leq q_c\) strong colors and a total number of \(q > q_c\) colors.

The rest of the paper is organized as follows. In section II we find \(p^*\) for the standard bond-interaction model. We also present the results of Monte-Carlo (MC) simulations. A detailed analysis of the mean field (MF) model equipped with the HP prescription (1) is reported in section III. Finally, our conclusions are drawn in section IV.

II. THE STANDARD MODEL

In the present section we derive \(p^*\) using a simple
energy-entropy first principle argument. We start with a
system of \(N\) spins situated on the vertices of the square
lattice and interacting via the usual Hamiltonian

\[
\mathcal{H}_B = -J \sum_{\langle i,j \rangle} \delta_{\sigma_i, \sigma_j},
\]

where \(J > 0\) is the ferromagnetic coupling constant (we
take hereafter \(J = k_B = 1\) for convenience) and the
summation is taken over nearest neighboring spins.

We call a large cluster with \(l\) bonds \textit{simple} if its number of
sites per bond is minimal, i.e., \(1/2 + o(1)\). We term a
\textit{snake}, a fractal with a maximal number of \(1 + o(1)\) sites
per bond. Any fractal therefore grows no faster than a
snake and has \(1/2 + \delta + o(1)\) sites per bond with \(\delta \leq 1/2\).

Consider a (fractal) contour made of \(l = O(N)\) bonds. The contour has energy \(-l\). Let \(#(k, n)\) be the number of
fractals with \(n\) sites such that \(k\) of them are positioned in
strong regions. The expected number of such fractals is

\[
\langle #(k, n) \rangle = \mu^k \binom{n}{k} p^k (1 - p)^{n-k},
\]

where \(\sum_k \#(k, n) \sim \mu^n\) for some \(1 < \mu \leq \lambda\) depending on \(\delta\) [38] and \(\lambda\) is the growth constant of all
the fractals with \(n\) sites. The expected change in the number of states is
given by \(\langle #(k, n) \rangle q_0^{-k} q^{-\langle n-k \rangle}\) so that, assuming \(#(k, n)\) is narrowly distributed around its mean and using (3), the free energy per site can be written

\[
-\beta f_{\text{frac}} = \frac{2\beta}{1 + 2\delta} + \ln \mu - \ln q + \sum_k \left( \frac{1}{n} \ln \binom{n}{k} p^k (1 - p)^{n-k} \right) + \frac{k}{n} \ln \left( \frac{q}{q_0} \right).
\]

In the large \(n\) limit, the sum in (4) can be replaced with
the maximum of the summand obtained at \(k\) satisfying

\[
\kappa = \frac{pq}{pq + (1 - p)q_0},
\]

with \(\kappa = k/n\). This brings (4) into the form

\[
-\beta f_{\text{frac}} = \frac{2\beta}{1 + 2\delta} + \ln \mu - \ln q - \kappa \ln \kappa - (1 - \kappa) \ln (1 - \kappa) + \kappa \ln p + (1 - \kappa) \ln (1 - p) + \kappa \ln \left( \frac{q}{q_0} \right).
\]

Unlike fractals, large simple clusters grow exponentially with their size and therefore randomly distributed on the lattice. Accordingly, the free energy per site of simple clusters with the same energy \(-l\) is given by

\[
-\beta f_{\text{sim}} = 2\beta - p \ln q_0 - (1 - p) \ln q.
\]

Note that in the homogeneous first order limit of (6), i.e.,
\(p, \delta\) tend to 0 and \(\mu\) tends to 1, that equation reduces to
(7).

It may be more constructive to form large monochromatic simple clusters rather than fractals. Strictly speaking, if at \(\beta\) solving \(f_{\text{sim}} = 0\) we have \(f_{\text{frac}} \geq 0\), then it
is entropically disadvantageous for the system to possess
large fractals at that temperature. Instead, large simple
monochromatic clusters are formed and the system
undergoes a first order transition [38] at

\[
\beta_c \approx \frac{1}{2} (p \ln q_0 + (1 - p) \ln q).
\]

The marginal concentration \(p^*\) below which large simple clusters are entropically more favorable than fractals can be estimated by taking \(\kappa^*\) to satisfy (5) at \(p^*\), plugging it into (6) together with the RHS of (8) and solving

\[
\sup_{\delta} \left( p^* \ln q_0 + (1 - p^*) \ln q + (1 + 2\delta) \left( \ln \mu - \ln q - \kappa^* \ln \kappa^* - (1 - \kappa^*) \ln (1 - \kappa^*) \right) + \kappa^* \ln p^* + (1 - \kappa^*) \ln (1 - p^*) + \kappa^* \ln (q/q_0) \right) = 0
\]

for \(p^*\).

For small concentrations of strong regions, a first order transition occurs when the total number of colors is large. The reason is that, when \(p\) is small, the construction of monochromatic exponential families of fractals is entropically too wasteful for \(q\) sufficiently large. Thus, the critical behavior is determined by homogeneously distributed simple clusters and the transition is of first order. For finding the large \(q\) behavior of \(p^*\) we expand the LHS of (9) up to \(o(p^*)\) and explicitly solve for \(p^*\). This yields

\[
p^* \sim \frac{\ln q}{q}.
\]
Our next goal is to numerically support the theory presented up to this point. For that purpose, we have performed extensive MC simulations using the Wang-Landau (WL) method [19, 20]. The WL method is specially designed to approximate the entropy \( S(E) \propto \ln \Omega(E) \) where \( \Omega(E) \) is the number of states with energy \( E \), at any desired accuracy. This has been exploited to calculate the energy probability-distribution-function (PDF) for samples of various sizes \( L \) at pseudo-criticality, e.g., at a temperature where the specific heat is maximal [38, 39]. While for a second order transition the pseudo-critical (PC) energy PDF is expected to display a single peak centred around the energies of the coexisting ordered and disordered states is conventional [40].

In Fig. 1 we plot the PC energy PDF for a standard model with \( q = 50 \) and different concentrations. The sample size is \( L = 40 \). While the PDF for \( p = 0.1 \) is broad and its shape may smear out a weak first order transition, two pronounced peaks are observed for concentrations lower than \( p = 0.1 \) and a clear single peak is present for \( p = 0.15 \). The marginal concentration for a first order transition is therefore expected to be \( 0.05 \leq p^* < 0.15 \) in reasonable agreement with (10) yielding \( p^* \sim \frac{\ln 50}{\ln 40} \approx 0.08 \).

![FIG. 1 PC energy PDF for the bond-interaction model on the square lattice. The “strong” Potts variable is \( q_0 = 2 \) and the total number of colors is \( q = 50 \). The lattice linear size is \( L = 40 \).](image)

**III. MEAN FIELD MODEL**

In the present section we adopt the Bragg-Williams approach to solve the MF HP model. The MF Hamiltonian is given by

\[
\mathcal{H}_{MF} = -\frac{1}{N} \sum_{i<j} \delta_{\sigma_i, \sigma_j} ,
\]

where the normalization factor \( N^{-1} \) assures that the total energy is extensive. Let \( \xi_i \) and \( \eta_j \) be the fraction of spins at state \( i \in \{0, 1, ..., q_0 - 1, ..., q - 1\} \), in strong and weak regions, respectively. The number of states with energy \( -\frac{1}{2}N \sum_i (p\xi_i + (1 - p)\eta_i)^2 \) is given by

\[
\left(p\xi_i \cdots p\xi_i \cdots \right)^{(1-p)N} \left((1-p)\eta_j \cdots (1-p)\eta_j \cdots \right) .
\]

Thus, introducing the Lagrange multipliers \( a, b \), the free energy per site takes the form

\[
\beta f = \sum_i \left( p\xi_i \ln \xi_i + (1 - p)\eta_i \ln \eta_i \right) - \frac{1}{2} \beta \left( p\xi_i + (1 - p)\eta_i \right)^2 + a \left( \sum_i \xi_i - 1 \right) + b \left( \sum_i \eta_i - 1 \right) .
\]

Setting \( \xi_j = 0, \ j = q_0,...,q - 1 \), differentiating (12) with respect to \( \xi_j, \eta_j \) and equating to zero gives

\[
\ln \xi_j - \beta \left( p\xi_j + (1 - p)\eta_j \right) + a/p + 1 = 0 , j = 0, ..., q_0 - 1 ,
\]

\[
\ln \eta_j - \beta \left( p\xi_j + (1 - p)\eta_j \right) + b/(1 - p) + 1 = 0 , j = 0, ..., q - 1 .
\]

The fractions \( \xi_j, \eta_j \) therefore satisfy the coupled equations

\[
\xi_j = \begin{cases} 
\frac{e^{\beta \left( p\xi_j + (1 - p)\eta_j \right)}}{Z_0} , & j = 0, ..., q_0 - 1 , \\
0 , & j = q_0, ..., q - 1 ,
\end{cases}
\]

and

\[
\eta_j = \begin{cases} 
\frac{e^{\beta \left( p\xi_j + (1 - p)\eta_j \right)}}{Z} , & j = 0, ..., q_0 - 1 , \\
\frac{e^{\beta (1 - p)\eta_j}}{Z} , & j = q_0, ..., q - 1 ,
\end{cases}
\]

where

\[
Z_0 = e^{a/p+1} = \sum_{i=0}^{q_0-1} e^{\beta (p\xi_i + (1-p)\eta_i)}
\]

and

\[
Z = e^{b/(1-p)+1} = Z_0 + \sum_{i=q_0}^{q-1} e^{\beta (1-p)\eta_i} .
\]

The quantities in (14),(15) can take the form

\[
\xi_j = \begin{cases} 
\frac{1}{q_0} \left( 1 + (q_0 - 1)m_0 \right) , & j = 0 , \\
\frac{1}{q_0} \left( 1 - m_0 \right) , & j = 1, ..., q_0 - 1 , \\
0 , & j = q_0, ..., q - 1 ,
\end{cases}
\]

and

\[
\eta_j = \begin{cases} 
\frac{1}{q} \left( 1 + (q_0 - 1)m_0 \right) \times (1 + (q/q_0 - 1)m_1) , & j = 0 , \\
\frac{1}{q} \left( 1 - m_0 \right) (1 + (q/q_0 - 1)m_1) , & j = 1, ..., q_0 - 1 , \\
\frac{1}{q} \left( 1 - m_1 \right) , & j = q_0, ..., q - 1 ,
\end{cases}
\]

(17)
where \( m_0, m_1 \) are the components of a two-fold magnetization \( m \). Indeed, there are favored fractions \( \xi_0 \geq \xi_j \) and \( \eta_0 \geq \eta_j \) for all \( 1 \leq j \leq q - 1 \), assuring the existence of long range order. Furthermore, (16),(17) capture the idea that the favored \( j = 0 \) color and the other \( q_0 - 1 \) strong colors are evenly weighted in strong and weak regions.

Above the critical point, at the disordered state, strong colors are uniformly distributed so that (16),(17) imply

\[
m_0 = 0.
\]

In the case that the transition is continuous, when plugging (16),(17) into (12), the magnetization at the critical point, \( m^* \), must satisfy \( \nabla f = 0 \) such that (18) holds. It is known [2, 41] that for the homogeneous model \( q_c = 2 \). Thus, in the following we set \( q_0 = 2 \). The gradient components then read

\[
g_0 = -\frac{1}{2q^2} \left( (1-p)q (2 + m_1(q-2)) \ln \left( \frac{1-m_0}{1+m_0} \right) + \beta m_0(m_1(1-p)(q-2) + p(q-2) + 2)^2 \right) + p \tanh^{-1}(m_0),
\]

\[
g_1 = -\frac{(1-p)(q-2)}{2q^2} \left( \beta \left( 1 + m_0^2 \right) q(m_1(1-p) + p) + 2\beta m_0^2(1-m_1)(1-p) + q \left( m_0 \ln \left( \frac{1-m_0}{1+m_0} \right) \right) \right)
- 2 \ln \left( \frac{2 + m_1(q-2)}{2q} \right) - \ln \left( 1 - m_0^2 \right).
\]

Substituting (18) in (19), \( g_0 \) indeed vanish. The critical temperature is obtained by the further condition that the Hessian matrix \( H(m, \beta, p, q) \) given by

\[
\begin{pmatrix}
\frac{(m_1(1-p)(q-2)-p(q-2)-2)(\beta(1-m_0^2)(q(p-2p+m_1(1-p)(q-2)+2)-2q)}{2(1-m_0^2)q^2} \\
\frac{(1-p)(q-2)(2\beta m_0(q-p+2p-m_1(1-p)(q-2)-2)+q \ln \left( \frac{1-m_0}{1+m_0} \right))}{2q^2} \\
\end{pmatrix}
\]

computed at \( m^* \) satisfying \( m_0^* = 0 \), \( g_1(m^*, \beta_c, p, q) = 0 \), has a vanishing eigenvalue, or, alternatively, that

\[
det H(m^*(\beta_c, p, q), \beta_c, p, q) = 0.
\]

Fixing \( q \), (22) implicitly determines the second order critical line in concentration-temperature plane by

\[
\frac{1}{2} \exp \left( \frac{\beta_c - q}{q - 2} \right) = \frac{\beta_c - 2}{(q - 2)(2 - \beta_c p)}.
\]

The critical magnetization reads

\[
m^* = \left( 0, \frac{q(2 - \beta_c p)}{\beta_c(1-p)(q-2) - 2(q-2)} \right)
\]

Pluggin (24) into the energy term of (12) yields the following expression for the critical energy

\[
\varepsilon_c = -\frac{(\beta_c - 4)\beta_c + 2q}{2\beta_c^2(q-2)}.
\]

It is known [2] that the inverse critical temperature of a homogeneous system with Ising-like spins is equal to 2.

We shall now provide a heuristic argument to support the existence of \( p^* \) for the MF model. First, note that \( p = 0, \beta_c = q \) simultaneously solve (23). Then, assuming that the phase portrait is continuous, in particular at \( p = 0 \), we have that it must converge to \( q \) in the limit \( p \to 0 \). On the other hand, in a homogeneous system with \( q > 2 \) colors, the first order inverse critical temperature is given by [2] \( \beta_c^0 = \frac{2(q-1)\ln(q-1)}{q-2} \). Of course, \( \beta_c^0 < q \), hence, \( \beta_c(0, q) = q \) is not a physical solution, which means that the critical line satisfying (23) must be truncated at some finite concentration, \( p^* \).

A first order transition is associated with a discontinuity of the magnetization at the critical point. In other words, there are two points \( m^* \) and \( m^{**} \) such that both
simultaneously minimize the free energy, that is, simultaneously solve $\nabla f = 0$, $f(m^*) = f(m^{**})$.

The nature of the transition and, consequently, the presence of $p^*$, can be phenomenologically detected by fixing the concentration and examining the behavior of $m$ that numerically minimizes (12) when the temperature is varied. In Fig. 2, the “strong” component $m_0$ is plotted against temperature, for $q_0 = 2, q = 6$ and different concentrations. Apparently, for $p \leq 0.2$ the magnetization has a discontinuity in the vicinity of $T_0$ (taken to be the point where the numerical derivative $\Delta m$ is “large”), indicating the presence of $m_0 = 0, m_0^{**} > 0$ [42], whereas for $p \geq 0.4$ it is continuous at $T_0$. It is therefore expected that $0.2 < p^* \leq 0.4$ for these spin numbers.

We next discuss the results of Metropolis [43] MC simulations performed to capture numerically the concentration-dependent changeover phenomenon for the MF model. We consider the simulated magnetization (SM)

$$\text{SM} = p\frac{q_0 z_0 - 1}{q_0 - 1} + (1 - p)\frac{q z - 1}{q - 1},$$

where $z_0$ and $z$ are the maximal fractions of monochromatic spins in strong and weak locations, respectively. A sample of $N = 1000$ spins is used in the simulations where the SM together with the energy given by (11) are observed. In Fig. 3(a) it is seen that the SM hops between two values of ordered and disordered states, as expected from a system undergoing a first order transition. On the other hand, in Fig. 3(b) the system exhibits a typical second order transition by means of fluctuations around a single value.

**FIG. 2 Strong magnetization component $m_0$ as a function of temperature, for different concentrations.** The strong and weak spin numbers are $q_0 = 2$ and $q = 6$, respectively. It should be noted that the symbols composing the graphs for $p \geq 0.4$ are more dense than for $p \leq 0.2$, in some neighborhood of $T_0$.

**FIG. 3 MC time variation of the simulated magnetization given by (27).** The simulated system has $N = 1000$ spins and spin numbers $q_0 = 2, q = 6$. Two different concentrations corresponding to a first or a second order transition together with temperatures in the vicinity of $T_0$ (to be precise $T = T_0 + \frac{10}{N} = T_0 + 0.01$, are chosen. (a) $p = 0.1$ and $T = 0.2661$, (b) $p = 0.6$ and $T = 0.4199$. Energy, $\varepsilon$, against MC time is plotted in the inset. The energy fluctuates around $\langle \varepsilon \rangle = -0.178(9)$ which is in good agreement with $\varepsilon_c = -0.172(0)$ due to (25).

We conclude this section by delving into the continuous regime of Fig. 2, first by solving (23) taking $q = 6$ and $p = 0.6$, to give $T_c = \beta_c^{-1} = 0.409(8)$. Next, from Fig. 4, where a plot of $m$ for $q_0 = 2, q = 6$ and $p = 0.6$ against temperature is presented, it can be deduced that since the value of $T_0$ agrees well with $T_c$, $m$ is continuous at $T_c$. It should be emphasized at this point that, unlike $m_0$, $m_1$ is actually not a proper order parameter as it does not vanish at any finite temperature. It has, however, a computational value. For instance, it can be verified from Fig. 4 that $m_1$ is not differentiable at $T_c$ which makes $T_c$ a unique temperature [44]. Furthermore, substituting $q = 6$, $p = 0.6$, $\beta_c = 2.440(1)$ in (24), gives $m_1^* = 0.323(5)$ which is in excellent agreement with $m_1$ computed at $T_0$.

**IV. CONCLUDING REMARKS**

It is demonstrated on the standard Potts model that in a mixed system of spins randomly colored by $q_0$ or $q > q_0$ colors, the transition order depends on the concentration $p$ of $q_0$-type spins. For that system there is a critical concentration $0 < p^* < 1$ such that the tran-
The pseudo-critical temperature is a function of temperature, for $q_0 = 2, q = 6$ and $p = 0.6$. The pseudo-critical temperature is $T_0 = 0.4099$. The pseudo-critical magnetization at $T_0$ is $m_1 = 0.3235$.

The Bragg-Williams approximation, applied to the MF model, uncovers the presence of the marginal concentration for that model, where the strong regions are occupied by Ising-like spins. Unlike in the standard model where we use an argument that explains the occurrence of a first order transition for concentrations below $p^*$, in the MF case we show exactly that, given a spin number $q$, the vanishing order parameter is unstable at concentration-dependent temperatures, on some interval of concentrations, $I_p$. Thus, the model undergoes a second order transition on $I_p$, making any temperature in that interval a critical point depending both on $p$ and $q$. From continuity of the phase portrait it follows that the continuous line is truncated at some finite concentration, defining the left boundary of $I_p$. It is therefore reasonable to assume that the phase portrait contains a first order branch, terminated at $\frac{1}{\beta_0} = \frac{q-2}{2(q-1)\log(q-2)}$, on the interval $[0, p^*)$.

It should be noted that the simulations produce observables that have a different qualitative behavior for different concentrations. In the standard model case the PC energy PDF changes its shape from dual to single peaked, with increasing values of $p$ and in the MF case the order parameter has a (noisy) typical single versus dual valued temporal behavior, for different values of $p$. Thus, in both cases, the essence of $p^*$ as a changeover point is detected.

Our model provides a rather general framework of governing the transition order of well known Potts systems, without explicitly interfering with the interaction content of such systems when no heterogeneity in the number of colors is introduced, in the sense that a changeover phenomenon is observed for any setup of strong and weak colors. We believe that the HP machinery can be applied to other Potts systems and it is expected that a concentration-dependent changeover phenomenon is detected in those systems.

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