A combinatorial Yamabe problem on two and three dimensional manifolds

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Abstract

In this paper, we introduce a new combinatorial curvature on two and three dimensional triangulated manifolds, which transforms in the same way as that of the smooth scalar curvature under scaling of the metric and could be used to approximate the Gauss curvature on two dimensional manifolds. Then we use the flow method to study the corresponding constant curvature problem, which is called combinatorial Yamabe problem.

In two dimensional case, we introduce a combinatorial Ricci flow and a combinatorial Calabi flow to study the combinatorial Yamabe problem. Interestingly, the evolutions of the curvature along the two combinatorial curvature flows have the same form as that of the scalar curvature along the smooth curvature flows respectively. By a discrete maximum principle, we prove that the existence of constant curvature metric is equivalent to a combinatorial and topological condition in case of nonpositive Euler characteristic, which is similar to Thurston’s criterion for classical discrete curvature. We further prove the equivalence between the existence of constant curvature metric, the convergence of the combinatorial curvature flows and the combinatorial and topological condition for surfaces with nonpositive Euler characteristic, in which case the constant curvature metric is unique. For surfaces with positive Euler characteristic, we find counterexamples which show that the properties no longer hold. So we introduce combinatorial $\alpha$-curvature, combinatorial $\alpha$-Ricci flow and $\alpha$-Calabi flow and get similar properties. Specially, in case of $\alpha = 0$, the $\alpha$-curvature and $\alpha$-flows are just the classical discrete Gauss curvature and the corresponding curvature flows studied in [13, 18] respectively. We also study the prescribing curvature problems and get some results similar to Kazdan and Warner’s theorems.

In three dimensional case, we find that the constant curvature metrics are critical points of a combinatorial Yamabe functional, which has almost the same form as the classical Yamabe functional. Then we introduce a combinatorial Yamabe flow to study the combinatorial Yamabe problem. Importantly, this flow is a natural discretization of the classical Yamabe flow and the curvature evolves according to a reaction diffusion equation which has almost the same form as that of the scalar curvature along the
classical Yamabe flow. Then we use the combinatorial Yamabe flow to give some almost equivalent conditions for the existence of constant curvature metric.

**Keywords:** Circle packing metric; Combinatorial Yamabe problem; Combinatorial Yamabe flow; Combinatorial Ricci flow; Combinatorial Calabi flow

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1 **Introduction**

One of the central topics in differential geometry is the existence of canonical metrics on a given manifold, especially metrics with constant curvatures. This is inspired by the uniformization theorem in two dimension, which states that any two dimension surface admits a constant curvature metric conformal to the original metric on the manifold. To find the canonical metrics, lots of theories were developed and many tools were introduced. As the achievement is fruitful, we just mention some results related to the topic of this
manuscript in the following. To find constant scalar curvature metric on a closed manifold, Yamabe [48] proposed a generalization of the uniformization theorem to higher dimensional manifolds, which is now called the Yamabe problem. The problem was solved by Trudinger [47], Aubin [1] and Schoen [44] using variational method. Hamilton [29, 30] once proposed a flow method to solve this problem, i.e. the Yamabe flow. We refer to [49, 4, 5] for related results. Ricci flow was introduced by Hamilton [28] to study the low dimension topology. Ricci flow is a powerful tool and has lots of applications. Specially, it has been used to prove the Poincaré conjecture [38, 39, 40]. Interestingly, it is proved [29, 12, 11] that the Ricci flow can be used to give a new proof of the uniformization theorem for Riemann surfaces. Given a compact complex manifold admitting at least one Kähler metric, to find the extreme metric which minimizing the $L^2$ norm of the curvature tensor in a given principal cohomology class, Calabi [6] proposed a flow, which is now called the Calabi flow. It is proved [15, 10] that the Calabi flow on closed surfaces exists for all time and converges to a constant Gaussian curvature metric. It is interesting that the Calabi flow could also be used to give a proof of the uniformization theorem on closed surfaces [9].

In this paper, we consider piecewise flat manifolds, which could be taken as discretization and approximation of Riemannian manifolds. There is also a notion of curvature on such manifolds. The curvature on such spaces was first proposed by Regge [42], which coincides with the discrete Gauss curvature in two dimensional case. For two dimensional triangulated manifolds, the discrete Gauss curvature is defined as the angle deficit at a vertex. There has been much work on this curvature, among which we mention the following results. Chow and Luo [13] introduced a combinatorial Ricci flow, an analogue of the smooth Ricci flow on surfaces, for triangulated surfaces with circle packing metrics and gave a complete description of the behavior of the flow. This work is the cornerstone of the application of surface Ricci flow in engineering up to now. A discrete form of Ricci flow for higher dimensional piecewise flat simplicial geometry is also established in [37]. To extend the results in [13], Luo [34] introduced a combinatorial Yamabe flow to study the constant curvature problem for triangulated surfaces with piecewise flat metrics. The first author [18] and the authors [19] also introduced a combinatorial Calabi flow to study the constant curvature problem on triangulated surfaces with circle packing metrics and gave a characterization of the existence of constant curvature metric. Recently, it is proved [26, 27] that uniformization theorem of the discrete Gauss curvature is valid for triangulated surfaces.

For three dimensional triangulated manifold, Cooper and Rivin [17] introduced a notion of combinatorial scalar curvature, which is defined as the angle deficit of solid angles at a vertex. Cooper and Rivin’s definition of combinatorial scalar curvature is motivated by the fact that, in smooth case, the scalar curvature at a point $p$ locally measures the difference of the volume growth rate of the geodesic ball with center $p$ to the Euclidean
ball [33, 2]. Glickenstein [22, 23] introduced a combinatorial Yamabe flow to study the corresponding constant curvature problem. He found that the combinatorial scalar curvature evolves according to a heat type equation and showed that the solution converges to a constant curvature metric under some additional conditions. Luo [35] once introduced a combinatorial curvature flow to deform metrics of triangulated three-dimensional manifolds with boundary. However, the definition of curvature he used is different from that of Cooper and Rivin’s definition. The authors [20] once modified Cooper and Rivin’s definition of combinatorial scalar curvature and introduced a combinatorial Yamabe flow to study the corresponding constant curvature problem on 3-dimensional triangulated manifolds. For piecewise flat manifolds, Cheeger, Müller and Schrader [8] defined analogous curvatures of Lipschitz-Killing curvatures for smooth Riemannian manifolds and proved that the curvatures converge to scalar curvature measure.

However, both the discrete Gauss curvature and Cooper and Rivin’s combinatorial scalar curvature are scaling invariant, which is different from that of the smooth case. To mend this flaw, we introduce a new definition of curvature for two and three dimensional triangulated manifolds with sphere packing metrics in Euclidean background geometry in this paper. The authors also consider the corresponding problem with hyperbolic background geometry in [21]. In what follows, we first state our main results briefly in this section. The full version of the results related to 2-dimensional triangulated surfaces are stated and proved in Section 2 and the results for 3-dimensional triangulated manifolds appear in Section 3.

For a weighted triangulated surface \((M, T, \Phi)\) with Thurston’s circle packing metric \(r \in \mathbb{R}_{>0}^N\), we define the new combinatorial curvature as \(R_i = K_i/r_i^2\), where \(K_i\) is the classical discrete Gauss curvature. We sometimes refer to this curvature as \(R\)-curvature in the following. Here the denominator \(r_i^2\) may be thought as an “area element” attached to vertex \(i\). For this new curvature, we study the constant curvature problem and prescribing curvature problem by introducing combinatorial Ricci flow and Calabi flow. The combinatorial Ricci flow, which is the same as combinatorial Yamabe flow in two dimension, is defined as

\[
\frac{dg_i}{dt} = (R_{av} - R_i)g_i,
\]

where \(R_{av} = 2\pi\chi(M)/\|r\|_2^2\) and \(g_i = r_i^2\) is a discrete version of Riemann metric tensor. This flow exhibits similar properties to the smooth Ricci flow on surfaces. The curvature \(R_i\) evolves according to a heat-type equation, which has almost the same form as the evolution of the scalar curvature along the classical Ricci flow and permits a discrete maximum principle. Using the discrete maximum principle, we have the following existence theorem of constant combinatorial \(R\)-curvature metric.

**Theorem 1.1.** Suppose \((M, T, \Phi)\) is a weighted triangulated surface with circle packing
metric \( r \) satisfying \( R_i < 0 \) for all \( i \in V \), then there exists a negative constant curvature metric \( r^* \). Furthermore, the solution \( r(t) \) to the combinatorial Ricci flow (1.1) exists for all time and converges exponentially fast to \( r^* \) as \( t \to +\infty \).

Notice that, without Thurston’s topological and combinatorial condition (2.45), we can still derive the existence of constant curvature metric under the existence of an initial metric with negative curvature.

By introducing a new discrete Laplace operator
\[
\Delta f_i = \frac{1}{r_i^2} \sum_{j \sim i} (-\frac{\partial K_i}{\partial u_j})(f_j - f_i),
\]
where \( f \) is a function defined on all vertices and \( u_j = \ln g_j \) is a coordinate transformation, we can define the combinatorial Calabi flow as
\[
\frac{dg_i}{dt} = \Delta R_i \cdot g_i, \tag{1.2}
\]
which is similar to the smooth Calabi flow. By considering a discrete quadratic energy functional \( \tilde{C}(r) = \sum_{i=1}^{N} \varphi_i^2 \), where \( \varphi_i = K_i - R_{av}r_i^2 \), we can define a modified Calabi flow as
\[
\dot{u} = -\frac{1}{2} \nabla u \tilde{C}. \tag{1.3}
\]

For any proper subset \( I \subset V \), let \( F_I \) be the subcomplex whose vertices are in \( I \) and let \( Lk(I) \) be the set of pairs \((e,v)\) of an edge \( e \) and a vertex \( v \) such the following three conditions: (1) the end points of \( e \) are not in \( I \); (2) \( v \) is in \( I \); (3) \( e \) and \( v \) form a triangle. Denote
\[
\mathcal{Y}_I \triangleq \{ x \in \mathbb{R}^N | \sum_{i \in I} x_i > -\sum_{(e,v) \in Lk(I)} (\pi - \Phi(e)) + 2\pi \chi(F_I) \},
\]
and
\[
\mathcal{X}_{GB} \triangleq \{ x \in \mathbb{R}^N | \sum_{i=1}^{N} x_i = 2\pi \chi(X) \}.
\]
For classical combinatorial Gauss curvature \( K \), by the work of Thurston [46], Marden and Rodin [36], Chow and Luo [13], the space of all admissible \( K \)-curvature is \( \{ K = K(r) | r \in \mathbb{R}_{>0}^N \} = \mathcal{X}_{GB} \cap (\cap_{\phi \neq I \subset V} \mathcal{Y}_I) \). Combining Corollary 2.17, Theorem 2.23, Theorem 2.27, Theorem 2.32 and Theorem 2.34, we have the following main theorem.

**Theorem 1.2.** Suppose \((M,T,\Phi)\) is a weighted triangulated surface with \( \chi(M) \leq 0 \). Then the existence of constant curvature circle packing metric, the convergence of combinatorial Ricci flow (1.1), the convergence of combinatorial Calabi flow (1.2) and the convergence of modified Calabi flow (1.3) are mutually equivalent. They are all equivalent to the following combinatorial and topological condition
\[
\mathcal{X}_{GB} \cap (\cap_{\phi \neq I \subset V} \mathcal{Y}_I) \cap \mathbb{R}^N_{<0} \neq \emptyset \tag{1.4}
\]
when \( \chi(M) < 0 \), and to \( 0 \in \mathcal{X}_{GB} \cap (\cap_{\phi \neq I \subset V} \mathcal{Y}_I) \) when \( \chi(M) = 0 \). Furthermore, the constant curvature metric is unique (if exists) up to scaling.
When $\chi(M) > 0$, the conclusions in Theorem 1.2 are not true. We have two interesting counterexamples. Example 2, the tetrahedron triangulation of the sphere, gives a simple counterexample for the convergence of the combinatorial Ricci flow. While Example 3 shows that there are more than one constant curvature metric on the sphere with tetrahedron triangulation. These counterexamples reveal the difference between smooth curvature flows and combinatorial curvature flows. The behavior of combinatorial flows are deeply constrained by the combinatorial structure of the triangulation.

However, for surface with $\chi(M) > 0$, it's very interesting that we could evolve flow (1.1) very well just by substituting the “area element” $r_i^2$ by $r_i^{-2}$, or $r_i^\alpha$ with $\alpha \leq 0$ more generally. In fact, define $\alpha$-curvature as $R_{\alpha,i} = K_i/r_i^\alpha$, and introduce the corresponding $\alpha$-flows, we have the following result for constant $\alpha$-curvature problem, which is a combination of Theorem 2.30, Theorem 2.33 and Corollary 2.36.

**Theorem 1.3.** Suppose $(M, T, \Phi)$ is a weighted triangulated surface with $\alpha \chi(M) \leq 0$. Then the existence of constant $\alpha$-curvature metric, the convergence of $\alpha$-Ricci flow (2.39), the convergence of $\alpha$-Calabi flow (2.42) and the convergence of modified $\alpha$-Calabi flow (2.43) are mutually equivalent. When $\alpha > 0$ and $\chi(M) < 0$, they are equivalent to $K_{GB} \cap (\cap_{\emptyset \neq I \subseteq V} B_I) \cap R^N_{\geq 0} \neq \emptyset$; When $\alpha < 0$ and $\chi(M) > 0$, they are equivalent to $K_{GB} \cap (\cap_{\emptyset \neq I \subseteq V} B_I) \cap R^N_{> 0} \neq \emptyset$; When $\chi(M) = 0$, they are equivalent to $0 \in K_{GB} \cap (\cap_{\emptyset \neq I \subseteq V} B_I)$. Furthermore, the constant $\alpha$-curvature metric is unique (if exists) up to scaling.

By the maximum principle, we have the following existence result for non-negative constant $\alpha$-curvature metric.

**Theorem 1.4.** Suppose $(M, T, \Phi)$ is a weighted triangulated surface. If there is a metric $r$ satisfying $R_{\alpha,i} \geq 0$ for all $i \in V$, and

$$
- \sum_{(e,v) \in Lk(I)} (\pi - \Phi(e)) + 2\pi \chi(F_I) < 0, \quad \forall I : \emptyset \subseteq I \subseteq V,
$$

then there exists a non-negative constant curvature metric $r^\ast$. 

For triangulated 3-manifolds with sphere packing metrics, we also have a new definition of combinatorial scalar curvature. Different from Cooper and Rivin’s scalar curvature $K_i$, the new combinatorial scalar curvature is defined as $R_i = K_i/r_i^{-2}$, which is formally the same as the new combinatorial Gauss curvature. Similar to the smooth case, we find that sphere packing metrics with constant combinatorial scalar curvature are isolated and are exactly the critical points of a normalized Einstein-Hilbert-Regge functional $\sum R_i r_i^3/(\sum r_i^3)^{1/3}$. Hence we propose to study a combinatorial Yamabe problem with respect to the new combinatorial scalar curvature $R_i$. 

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Combinatorial Yamabe Problem. Given a 3-dimensional manifold $M$ with triangulation $\mathcal{T}$, find a sphere packing metric with constant combinatorial scalar curvature in the combinatorial conformal class $\mathfrak{M}_\mathcal{T}$.

Here $\mathfrak{M}_\mathcal{T}$ is the space of admissible sphere packing metrics determined by $\mathcal{T}$.

Set $R_{av} = \sum R_i r_i^3 / \sum r_i^3$, we introduce the following combinatorial Yamabe flow

$$\frac{dg_i}{dt} = (R_{av} - R_i)g_i$$

(1.5)

to study the combinatorial Yamabe problem. Nonpositive constant curvature metrics are local attractors of the flow (1.5) and hence we have

**Theorem 1.5.** Suppose $r^*$ is a sphere packing metric on $(M, \mathcal{T})$ with nonpositive constant combinatorial scalar curvature. If $||R(0) - R^*||^2$ is small enough, the solution of the normalized combinatorial Yamabe flow (3.14) exists for all time and converges to $r^*$.

The paper is organized as follows. In Section 2, we introduce the new definition of discrete Gauss curvature for triangulated surfaces with circle packing metrics and then give the proof of Theorem 1.1, 1.2, 1.3 and 1.4. In Section 2, we also give a solution of the prescribing curvature problem using the curvature flows we introduced. In Section 3, we introduce the new definition of combinatorial scalar curvature and give the proof of Theorem 1.5. In Section 4, we list some unsolved problems which are closely related to the paper.

## 2 2-dimensional combinatorial Yamabe problem

### 2.1 The definition of combinatorial Gauss curvature

Suppose $M$ is a connected closed surface with triangulation $\mathcal{T} = \{V, E, F\}$, where $V, E, F$ represent the sets of vertices, edges and faces respectively. A weight for the triangulation is a function $\Phi : E \rightarrow [0, \frac{\pi}{2}]$ defined on the edges. The triple $(M, \mathcal{T}, \Phi)$ will be referred as a weighted triangulation of $M$ in the following. All the vertices are supposed to be ordered one by one, marked by $v_1, \cdots, v_N$, where $N = V^2$ is the number of vertices. We often write $i$ instead of $v_i$ and use $i \sim j$ to denote that the vertices $v_i$ and $v_j$ are adjacent. Throughout this paper, all functions $f : V \rightarrow \mathbb{R}$ will be regarded as column vectors and $f_i$ is the value of $f$ at $i$. And we use $C(V)$ to denote the sets of functions defined in this way.

The circle packing metric is defined to be a map $r : V \rightarrow (0, +\infty)$. Given $(M, \mathcal{T}, \Phi)$, we can attach each edge $\{ij\}$ a length

$$l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_ir_j \cos \Phi_{ij}}.$$  

(2.1)
It is proved [46] that the lengths \( \{l_{ij}, l_{jk}, l_{ik}\} \) satisfies the triangle inequality for \( \Phi \in [0, \pi/2] \), which ensures that the face \( \triangle v_iv_jv_k \) could be realized as a Euclidean triangle with lengths \( \{l_{ij}, l_{jk}, l_{ik}\} \). In this sense, the triangulated surface \((M, T, \Phi)\) could be taken as gluing many Euclidean triangles coherently. This case is called as the Euclidean background geometry. However, this length structure produces singularities at the vertices. To describe the singularity, the concept of curvature is introduced. Suppose \( \theta_{ijk}^i \) is the inner angle of triangle \( \triangle v_iv_jv_k \) at the vertex \( v_i \), the classical discrete Gauss curvature at the vertex \( v_i \) is defined as

\[
K_i = 2\pi - \sum_{\triangle v_iv_jv_k \in F} \theta_{ijk}^i,
\]

where the sum is taken over all the triangles with \( v_i \) as one of its vertices. This curvature locally measures the difference of the triangulated surface from the Euclidean plane at the vertex. Similar to the classical Gauss-Bonnet Theorem for smooth surfaces

\[
\int_M K dA_g = 2\pi\chi(M),
\]

the curvature \( K_i \) satisfies the following Gauss-Bonnet identity

\[
\sum_{i \in V} K_i = 2\pi\chi(M) \tag{2.3}
\]

in this case [13]. The curvature \( K_i \) has been widely studied in discrete geometry, we refer to [13, 26, 27] for recent progress.

However, there are two disadvantages of the classical definition of \( K_i \). For one thing, classical discrete Gauss curvature does not perform so perfectly in that it is scaling invariant, i.e. if \( \tilde{r}_i = \lambda r_i \) for some positive constant \( \lambda \), then \( \tilde{K}_i = K_i \), which is different from the transformation of scalar curvature \( R_{\lambda g} = \lambda^{-1} R_g \) in the smooth case. For another, classical discrete Gauss curvature can’t be used directly to approximate smooth Gauss curvature when the triangulation is finer and finer. As the triangulation of a fixed surface is finer and finer, we can get a sequence of polyhedrons which approximate the surface. However, the classical discrete Gauss curvature at each vertex \( i \) tends to zero, for all triangles surrounding the vertex \( i \) will finally run into the tangent plane and the triangulation of the fixed surface locally becomes a triangulation of the tangent plane at vertex \( i \). To approximate the smooth Gauss curvature, dividing \( K_i \) by an “area elment” \( A_i \) is necessary. Since we are considering circle packing metrics, we may choose the area of the disk packed at \( i \), i.e. \( A_i = \pi r_i^2 \), as the “area elment” attached to vertex \( i \). Omitting the coefficient \( \pi \), we introduce the following definition of discrete curvature on triangulated surfaces.

**Definition 2.1.** Given a weighted triangulated surface \((M, T, \Phi)\) with circle packing metric \( r : V \to (0, +\infty) \), the modified discrete Gauss curvature at the vertex \( i \) is defined to
be

\[ R_i = \frac{K_i}{r_i^2}, \]

where \( K_i \) is the classical discrete Gauss curvature defined as the angle deficit at \( i \) by (2.2).

In the following of this subsection, we will give some evidence from which we see that the Definition 2.1 is a good candidate to avoid the two disadvantages we mentioned in the above paragraph. Firstly, let us have a look at the following example.

**Example 1.** Consider the standard sphere \( S^2 \) imbedded in \( \mathbb{R}^3 \). The smooth Gauss curvature of \( S^2 \) is +1 everywhere. Consider a local triangulation of \( S^2 \) at the north pole, which is denoted as a vertex \( i \). \( O \) is the origin of \( \mathbb{R}^3 \). \( h \) is a point lying in \( Oi \), see Figure 1. The intersection between the horizontal plane passing through \( h \) and \( S^2 \) is a circle. Divide this circle into \( n \) equal portions. Denote any two conjoint points by \( j \) and \( k \), thus \( i, j, k \) forms a triangle. Denote \( x = \angle iOj \), by taking \( r_i = r_j = r_k = \frac{ij}{2} = \sin \frac{x}{2}, \Phi_{ij} = \Phi_{ik} = 0 \), and suitably choosing \( \Phi_{jk} \), we can always get a circle packing metric locally. Moreover, we can get an infinitesimal triangulation at vertex \( i \) by letting \( n \to +\infty \) and \( x \to 0 \). By a trivial calculation we can get \( \theta_{jk}^{ij} = \angle kij = 2 \arcsin(\sin \frac{x}{n} \cos \frac{x}{2}) \), hence the classical Gauss curvature is \( K_i = 2\pi - 2n \arcsin(\sin \frac{x}{n} \cos \frac{x}{2}) \). It’s obviously that

\[
\lim_{n \to \infty, x \to 0} K_i = 0.
\]

However, the modified combinatorial Gauss curvature approaches the smooth Gauss curvature from the fact that

\[
\frac{R_i}{\pi} = \frac{K_i}{\pi r_i^2} = \frac{2\pi - 2n \arcsin(\sin \frac{x}{n} \cos \frac{x}{2})}{\pi \sin^2 \frac{x}{2}} \to +1.
\]

Secondly, let’s analyse from the view point of the scaling law of Riemann tensor and Gauss curvature. Recall that, for a \( C^1 \) curve \( \gamma : [a, b] \to M \) in a Riemannian manifold \( (M, g) \), the length of the curve is defined to be \( L(\gamma, g) = \int_a^b \sqrt{g(\gamma(t), \gamma'(t))} dt \), where \( \gamma'(t) \) is the tangent vector of \( \gamma(t) \) in \( M \). So we would have \( L(\gamma, \bar{g}) = \lambda^{1/2} L(\gamma, g) \) if \( \bar{g} = \lambda g \) for
some positive constant $\lambda$. Note that, for a weighted triangulated surface $(M, T, \Phi)$ with circle packing metric $r : V \to (0, +\infty)$, the length $l_{ij}$ of the edge $e_{ij}$ is given by (2.1). So the quantity corresponding to the Riemannian metric $g$ on weighted triangulated surface $(M, T, \Phi)$ with circle packing metric $r$ should be quadratic in $r$, among which $r^2_i$ is the simplest. Furthermore, according to the definition $R_i = \frac{K_i}{r_i^2}$, we have $R_i(\lambda r_i^2) = \lambda^{-1} R_i(r_i^2)$, which has the same form as the transformation for the smooth scalar curvature.

Finally, the Definition [2.1] is geometrically reasonable. Let us recall the original definition of Gauss curvature. We just give a sketch of the definition here and the readers could refer to [15] for details. Suppose $M$ is a surface embedded in $\mathbb{R}^3$ and $\nu$ is the unit normal vector of $M$ in $\mathbb{R}^3$. $\nu$ defines the well-known Gauss map $\nu : M \to S^2 \subset \mathbb{R}^3$. Then the Gauss curvature at $p \in M$ is defined as

$$K(p) = \lim_{A \to p} \frac{\text{Area} \nu(A)}{\text{Area} A},$$

where the limit is taken as the region $A$ around $p$ becomes smaller and smaller. For the weighted triangulated surface $(M, T, \Phi)$ with circle packing metric $r$, assuming it is embedded in $\mathbb{R}^3$ and taking the normal vector as a set-value function, then the definition of discrete Gauss curvature $R_i = \frac{K_i}{r_i^2}$ is an approximation of the original Gauss curvature up to a uniform constant $\pi$, as the numerator $K_i$ is the measure of the set-valued function $\nu$ and the denominator $r_i^2$ is the area of the disk with radius $r_i$ up to a uniform constant $\pi$.

Note that the classical discrete Gauss curvature satisfies the Gauss-Bonnet identity (2.3). If we define a discrete measure $\mu$ on the vertices by $\mu_i = r_i^2$, then we have the following Gauss-Bonnet identity

$$\int_M R d\mu = \sum_i R_i d\mu_i = 2\pi \chi(M). \quad (2.4)$$

In this sense, the average curvature is

$$R_{av} = \frac{\int_M R d\mu}{\int_M d\mu} = \frac{2\pi \chi(M)}{||r||^2}, \quad (2.5)$$

where $||r||^2 = \sum_{i=1}^{N} r_i^2$ is the total measure of $M$ with respect to $\mu$.

We give the following table for the corresponding quantities for smooth surfaces and triangulated surfaces.

For the curvature $R_i$ defined by (2.1), it is natural to consider the corresponding combinatorial Yamabe problem, i.e. does there exist any circle packing metric $r$ with constant curvature $R_i$? Inspired by [13, 18, 22, 34], we study the combinatorial Yamabe problem by introducing combinatorial curvature flows in the following.
Table 1: Corresponding between smooth surfaces and triangulated surfaces

| Smooth surfaces | Weighted triangulated surfaces |
|-----------------|--------------------------------|
| Metric          | $g = g_{ij} dx^i dx^j$         | $g_i = r_i^2$ |
| Length          | $L(\gamma, g) = \int_a^b \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt$ | $l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_ir_j \cos \Phi_{ij}}$ |
| Measure         | $d\mu = \sqrt{\det g_{ij}} \, dx$ | $d\mu_i = r_i^2$ |
| Curvature       | Gauss curvature $K$            | $R_i = \frac{K_i}{r_i^2}$ |
| Total curvature | $\int_M K \, d\mu = 2\pi\chi(M)$ | $\sum_i R_i d\mu_i = 2\pi\chi(M)$ |

2.2 The 2-dimensional combinatorial Ricci flow

2.2.1 Definition of 2-dimensional combinatorial Ricci flow

Ricci flow was introduced by Hamilton [28] to study the low dimension topology. For a closed Riemannian manifold $(M^n, g_{ij})$, the Ricci flow is defined as

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$  \hspace{1cm} (2.6)

with normalization

$$\frac{\partial}{\partial t} g_{ij} = \frac{2}{n} r g_{ij} - 2R_{ij},$$  \hspace{1cm} (2.7)

where $r$ is the average of the scalar curvature. Ricci flow is a powerful tool and has lots of applications. Specially, it has been used to prove the Poincaré conjecture. For a closed surface $(M^2, g_{ij})$, the Ricci flow equation (2.6) is reduced to

$$\frac{\partial}{\partial t} g_{ij} = -R g_{ij}$$  \hspace{1cm} (2.8)

with normalization

$$\frac{\partial}{\partial t} g_{ij} = (r - R) g_{ij},$$  \hspace{1cm} (2.9)

where $R$ is the scalar curvature. It is proved [29, 12] that, for any closed surface with any initial Riemannian metric, the solution of the normalized Ricci flow (2.9) on surface exists for all time and converges to a constant curvature metric conformal to the initial metric as time goes to infinity. It is further pointed out by Chen, Lu and Tian [11] that the Ricci flow can be used to give a new proof of the famous uniformization theorem of Riemann surfaces. The discrete version of Ricci flow on surfaces was first introduced by Chow and Luo in [13], in which they give another proof of Thurston’s existence of circle packing theorem. In this subsection, we will introduce a modified combinatorial Ricci flow on surfaces to study the constant curvature problem of $R_i$. 

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\textbf{Definition 2.2.} For a weighted triangulated surface \((M, \mathcal{T}, \Phi)\) with circle packing metric \(r\), the modified combinatorial Ricci flow is defined as

\[
\frac{dg_i}{dt} = -R_i g_i,
\]  

(2.10)

where \(g_i = r_i^2\).

Following Hamilton’s approach, we introduce the following normalization of the flow (2.10)

\[
\frac{dg_i}{dt} = (R_{av} - R_i) g_i,
\]  

(2.11)

where \(R_{av}\) is the average curvature of \(R_i\) defined by (2.5). It is easy to check that the total measure \(\mu(M) = ||r||^2\) of \(M\) with respect to \(\mu\) is invariant along the normalized flow (2.11), from which we know that the average curvature \(R_{av}\) is invariant along (2.11). As our goal is to study the existence of constant combinatorial curvature metric, we will focus on the properties of the flow (2.11) in the following. We assume \(r(0) \in S^{N-1}\) and then \(r(t) \in S^{N-1}\) along the flow in the following.

The flows (2.10) and (2.11) differ only by a change of scale in space and a change of parametrization in time. Let \(t, r, R\) denote the variables for the unnormalized flow (2.10), and \(\tilde{t}, \tilde{r}, \tilde{R}\) for the normalized flow (2.11). Suppose \(r(t), t \in [0, T]\), is a solution of (2.10). Set \(\tilde{r}(\tilde{t}) = \varphi(t)^{1/2} r(t)\), where \(\varphi(t) = ||r||^{-2}\) and \(\tilde{t} = \int_0^t \varphi(\tau) d\tau\). Then we have

\[
||\tilde{r}||^2 = 1, \quad \tilde{R}_i = \varphi(t)^{-1} R_i, \quad \tilde{R}_{av} = \varphi(t)^{-1} R_{av} = 2\pi \chi(M).
\]

This gives

\[
\frac{d\tilde{g}_i}{d\tilde{t}} = \frac{dr_i^2}{dt} \frac{dt}{d\tilde{t}} = (\varphi' r_i^2 + \varphi \frac{dr_i^2}{dt}) \varphi^{-1} = (\varphi' \varphi^{-1} r_i^2 - \varphi R_i r_i^2) \varphi^{-1} = (\tilde{R}_i - \tilde{R}_i) \tilde{g}_i,
\]

where, in the last step, we use

\[
\varphi' = \frac{d}{dt} ||r||^{-2} = -||r||^{-4} \frac{d}{dt} ||r||^2 = 2\pi \chi(M) \varphi^2.
\]

Conversely, if \(\tilde{r}(\tilde{t}), \tilde{t} \in [0, \tilde{T})\), is a solution of (2.11), set \(r(t) = \varphi(\tilde{t})^{1/2} \tilde{r}(\tilde{t})\), where

\[
\varphi(\tilde{t}) = e^{-\tilde{R}_{av}\tilde{t}}, \quad t = \int_0^{\tilde{t}} \varphi(\tau) d\tau.
\]

Then it is easy to check that \(\frac{d\varphi}{d\tilde{t}} = -R_i \varphi_i\).

For simplicity, we will set \(u_i = \ln g_i\) in the following. Then the flows (2.10) and (2.11) could be written as

\[
\dot{u} = -R
\]  

(2.12)

and

\[
\dot{u} = R_{av} 1 - R
\]  

(2.13)

respectively, where \(u = (u_1, \ldots, u_N)^T\), \(1 = (1, \ldots, 1)^T\) and \(R = (R_1, \ldots, R_N)^T\).
2.2.2 Discrete Laplace operator

Since \( r_i^2 \) is an analogy of \( d\mu \), we can define an inner product \( \langle \cdot, \cdot \rangle \) on \((M, \mathcal{T}, \Phi)\) with circle packing metric \( r \) by

\[
\langle f, h \rangle = \sum_{i=1}^{N} f_i h_i r_i^2 = h^T \Sigma f
\]

for real functions \( f, h \in C(V) \), where

\[
\Sigma \triangleq \text{diag}\{r_1^2, \ldots, r_N^2\}.
\]

A combinatorial operator \( S : C(V) \rightarrow C(V) \) is said to be self-adjoint if

\[
\langle Sf, h \rangle = \langle f, Sh \rangle
\]

for any \( f, h \in C(V) \).

The classical discrete Laplace operator \([16]\) \( \Delta \) is often written in the following form

\[
\Delta f_i = \sum_{j \sim i} \omega_{ij} (f_j - f_i),
\]

where the weight \( \omega_{ij} \) can be arbitrarily selected for different purpose. Bennett Chow and Feng Luo \([13]\) first gave a special weight which comes from the dual structure of circle patterns, while Glickenstein \([22]\) gave a similar weight in three dimension. The curvature \( K_i \) is determined by the circle packing metric \( r \). Consider the curvature as a function of \( u \), where \( u_i = \ln g_i = 2 \ln r_i \) are coordinate transformations, then Bennett Chow and Feng Luo’s discrete Laplacian \( \Delta_{CL} \) can be interpreted \([18]\) as the Jacobian of curvature map, i.e. \( \Delta_{CL} = -2L \), where

\[
L = \frac{\partial (K_1, \ldots, K_N)}{\partial (u_1, \ldots, u_N)}.
\]

It is noticeable that symbol \( L \) above is different from that in \([18]\) by a factor 2, which comes from the coordinate transformations.

Using the discrete measure \( \mu_i = r_i^2 \), we can give a new definition of discrete Laplace operator, which is slightly different from that in \([13, 18]\).

**Definition 2.3.** For a weighted triangulated surface \((M, \mathcal{T}, \Phi)\) with circle packing metric \( r : V \rightarrow (0, +\infty) \), the discrete Laplace operator \( \Delta : C(V) \rightarrow C(V) \) is defined as

\[
\Delta f_i = \frac{1}{r_i^2} \sum_{j \sim i} (-\frac{\partial K_i}{\partial u_j})(f_j - f_i)
\]

for \( f \in C(V) \).

The following property of \( L \) will be used frequently in the following of the paper.
Lemma 2.4. ([13], Proposition 3.9.) $L$ is positive semi-definite with rank $N-1$ and kernel \( \{ t1 | t \in \mathbb{R} \} \). Moreover, \( \frac{\partial K_i}{\partial u_i} > 0 \), \( \frac{\partial K_i}{\partial u_j} < 0 \) for \( i \sim j \) and \( \frac{\partial K_i}{\partial u_j} = 0 \) for others.

Using this lemma, we can write discrete Laplace operator as a matrix, that is, \( \Delta = -\Sigma^{-1}L \) and \( \Delta f = -\Sigma^{-1}Lf \). Note that the term \( -\frac{\partial K_i}{\partial u_j} > 0 \) for \( i \sim j \) in the definition of the operator \( \Delta \), thus it could be taken as a weight on the edges. We will sometimes denote this weight as \( w_{ij} \) in the following if there is no confusion. This implies that \( \Delta \) is a standard Laplacian operator defined on the weighted triangulated surface \( (M, T, \Phi) \) with circle packing metric \( r \) and measure \( \mu \). And it is easy to check that the Laplacian operator \( \Delta \) is self-adjoint with respect to \( \langle \cdot, \cdot \rangle \).

When the circle packing metric evolves along the flow \( (2.11) \), so is the curvature \( R_i \). The evolution of \( R_i \) is very simple and it has almost the same form as the evolution of scalar curvature along the Ricci flow on surfaces derived by Hamilton in [29].

Lemma 2.5. Along the normalized combinatorial Ricci flow \( (2.11) \), the curvature \( R_i \) satisfies the evolution equation

\[
\frac{dR_i}{dt} = \Delta R_i + R_i(R_i - R_{av}). \tag{2.17}
\]

**Proof.** As \( u_i = \ln r_i^2 \), we have \( \frac{\partial}{\partial u_j} = \frac{1}{2} r_j \frac{\partial}{\partial r_j} \). Then we have

\[
\frac{dR_i}{dt} = \sum_j \frac{\partial R_i}{\partial u_j} \frac{du_j}{dt} = \sum_j \left( \frac{1}{r_i^2} \frac{\partial K_i}{\partial u_j} K_i - \frac{1}{r_i^2} r_i r_j \delta_{ij} \right) (R_{av} - R_j) = -\frac{1}{r_i^2} \sum_j \frac{\partial K_i}{\partial u_j} R_j + R_i(R_i - R_{av})
\]

In the last two steps, Lemma 2.4 is used.

Thus the evolution equation \( (2.17) \) of \( R_i \) is a reaction-diffusion equation. In fact, suppose \( \Omega \) is a subset of \( V \), set \( \nabla_{ij} f = f_j - f_i \) for \( f \in C(V) \) and \( i \sim j \), and denote

\[
\partial \Omega = \{ \{ ij \} | i \in \Omega, j \in \Omega^c \},
\]

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then it is easy to check that
\[
\frac{d}{dt} \int_\Omega R d\mu = \sum_{i \in \Omega, j \in \Omega, i \sim j} \left( -\frac{\partial K_i}{\partial u_j} \right) (R_j - R_i) = \int_{\partial \Omega} \nabla R dw.
\]

Then we have similar explanation as the smooth Ricci flow for equation (2.11).

### 2.2.3 Maximum principle

As the evolution equation (2.17) is a heat-type equation, we have the following discrete maximum principle for such equations. Such maximum principle is almost standard, but the authors have not found any reference that includes a general statement of the discrete version of the maximum principle, so we write it down here for completeness.

**Theorem 2.6.** (Maximum Principle) Let \( f : V \times [0,T) \to \mathbb{R} \) be a \( C^1 \) function such that
\[
\frac{\partial f_i}{\partial t} \geq \Delta f_i + \Phi(f_i), \quad \forall (i,t) \in V \times [0,T).
\]
Suppose there exists \( C_1 \in \mathbb{R} \) such that \( f_i(0) \geq C_1 \) for all \( i \in V \). Let \( \varphi \) be the solution to the associated ODE
\[
\begin{aligned}
\frac{d\varphi}{dt} &= \Phi(\varphi) \\
\varphi(0) &= C_1,
\end{aligned}
\]
then
\[
f_i(t) \geq \varphi(t)
\]
for all \( (i,t) \in V \times [0,T) \) such that \( \varphi(t) \) exists.

Similarly, suppose \( f : V \times [0,T) \to \mathbb{R} \) be a \( C^1 \) function such that
\[
\frac{\partial f_i}{\partial t} \leq \Delta f_i + \Phi(f_i), \quad \forall (i,t) \in V \times [0,T).
\]
Suppose there exists \( C_2 \in \mathbb{R} \) such that \( f_i(0) \leq C_2 \) for all \( i \in V \). Let \( \psi \) be the solution to the associated ODE
\[
\begin{aligned}
\frac{d\psi}{dt} &= \Phi(\psi) \\
\psi(0) &= C_2,
\end{aligned}
\]
then
\[
f_i(t) \leq \psi(t)
\]
for all \( (i,t) \in V \times [0,T) \) such that \( \psi(t) \) exists.

We omit the proof of the theorem and refer to [14] for the details, as the proof is almost the same.
Remark 1. In fact, Theorem 2.6 is valid for general Laplacian operators defined as

\[ \Delta f_i = \sum_{j \sim i} a_{ij}(t) (f_j - f_i), \]

where \( a_{ij} > 0 \), but not required to satisfy the symmetric condition \( a_{ij} = a_{ji} \).

Applying Theorem 2.6 with vanishing initial value, we can easily get the following conclusion.

Corollary 2.7. If \( R_i(0) \geq 0 \) (\( R_i(0) \leq 0 \)) for all \( i \in V \), then \( R_i(t) \geq 0 \) (\( R_i(t) \leq 0 \)) for \( i \in V \) as long as the flow exists, i.e. the positive and negative curvatures are preserved along the normalized flow (2.11).

Set \( R_{\text{max}}(t) = \max_{i \in V} R_i(t) \) and \( R_{\text{min}}(t) = \min_{i \in V} R_i(t) \). Applying Theorem 2.6 to general initial data, we get the following lower bounds for \( R_i(t) \).

Lemma 2.8. (Lower Bound) Let \( r_i(t) \) be a solution to the normalized combinatorial Ricci flow (2.17) on a closed triangulated surface \( (M,T,\Phi) \).

(1) If \( \chi(M) < 0 \), then

\[ R_i - R_{\text{av}} \geq \frac{R_{\text{av}}}{1 - (1 - \frac{R_{\text{av}}}{R_{\text{min}}(0)})e^{R_{\text{av}}t}} - R_{\text{av}} \geq (R_{\text{min}}(0) - R_{\text{av}})e^{R_{\text{av}}t}. \]

(2) If \( \chi(M) = 0 \), then

\[ R_i \geq \frac{R_{\text{min}}(0)}{1 - R_{\text{min}}(0) t} > -\frac{1}{t}. \]

(3) If \( \chi(M) > 0 \) and \( R_{\text{min}}(0) < 0 \), then

\[ R \geq \frac{R_{\text{av}}}{1 - (1 - \frac{R_{\text{av}}}{R_{\text{min}}(0)})e^{R_{\text{av}}t}} \geq R_{\text{min}}(0)e^{-R_{\text{av}}t}. \]

Notice that in each case, the right hand side of the lower bound estimate tends to 0 as \( t \to \infty \). However, for the upper bound, the situation is not so good. The solution of the ODE

\[ \begin{cases} \frac{ds}{dt} = s(s - R_{\text{av}}) \\ s(0) = s_0 \end{cases} \]

corresponding to (2.17) is

\[ s(t) = \begin{cases} 0, & s_0 = 0 \\ \frac{s_0}{1 - s_0 R_{\text{av}}}, & s_0 \neq 0, \chi(M) = 0 \\ \frac{1 - (1 - \frac{R_{\text{av}}}{s_0})e^{R_{\text{av}}t}}{1 - (1 - \frac{R_{\text{av}}}{s_0})e^{R_{\text{av}}t}}, & s_0 \neq 0, \chi(M) \neq 0 \end{cases}. \]
If \( s_0 > \max\{R_{\text{av}}, 0\} \), there is \( T < \infty \) given by
\[
T = \begin{cases} 
-\frac{1}{R_{\text{av}}} \ln \left(1 - \frac{R_{\text{av}}}{s_0}\right) > 0, & \chi(M) \neq 0 \\
\frac{1}{s_0}, & \chi(M) = 0
\end{cases}
\]
such that
\[
\lim_{t \to T^-} s(t) = +\infty.
\]
The implies that, in the case \( R_{\text{max}}(0) > \max\{R_{\text{av}}, 0\} \), Theorem 2.6 will not give us good upper bound for the curvature along the combinatorial Ricci flow (2.17). However, in the case \( R_{\text{max}}(0) < 0 \), we have good upper bounds.

**Lemma 2.9.** If \( R_i(0) < 0 \) for all \( i \in V \), then we have
\[
R_i(t) - R_{\text{av}} \leq R_{\text{av}} \left(1 - \frac{R_{\text{av}}}{R_{\text{max}}(0)}\right) e^{R_{\text{av}} t}.
\]
These results are almost parallel to that of the smooth Ricci flow on surfaces.

### 2.2.4 Convergence of the 2-dimensional combinatorial Ricci flow

Combining Lemma (2.8) and (2.9) gives

**Theorem 2.10.** Suppose \((M, \mathcal{T}, \Phi)\) is a weighted triangulated surface with initial circle packing metric \( r(0) \) satisfying \( R_i(0) < 0 \) for all \( i \in V \), then there exists a negative constant curvature metric \( r^* \) on \((M, \mathcal{T}, \Phi)\). Furthermore, the solution \( r(t) \) to the normalized combinatorial Ricci flow (2.11) exists for all time and converges exponentially fast to \( r^* \) as \( t \to +\infty \).

Theorem 2.10 derived by combinatorial maximum principle implies more than it seems. It claims that, if there is a metric with all curvatures negative, then there always exists a negative constant curvature metric, and vice versa. Using this fact, we will give an combinatorial and topological condition which is equivalent to the existence of negative constant curvature metric in section 2.5. For the convergence of combinatorial Ricci flow, the negative initial curvature condition in Theorem 2.10 is a little restrictive. It is natural to consider Hamilton’s approach to generalize this result. However, we found that there is some technical difficulty to go further with Hamilton’s approach in this case. By studying critical points of (2.11), which is an ODE system, we can get the following local convergence result.

**Theorem 2.11.** Suppose \( r^* \) is a constant \( R \)-curvature metric on weighted triangulated surface \((M, \mathcal{T}, \Phi)\). If the first positive eigenvalue \( \lambda_1 \) of \( \Lambda = \Sigma^{-\frac{1}{2}} L \Sigma^{-\frac{1}{2}} \) at \( r^* \) satisfies
\[
\lambda_1(\Lambda) > R_{\text{av}}^* = \frac{2\pi \chi(M)}{||r^*||^2}
\]
and \(\|r(0) - r^*\|^2\) is small enough, then the solution to the normalized combinatorial Ricci flow (2.11) exists for \(t \in [0, +\infty)\) and converges exponentially fast to \(r^*\).

**Proof.** We can rewrite the normalized combinatorial Ricci flow (2.11) as

\[
\frac{dr_i}{dt} = \frac{1}{2}(R_{av} - R_i)r_i.
\]

Set \(\Gamma_i(r) = \frac{1}{2}(R_{av} - R_i)r_i\), then the Jacobian matrix of \(\Gamma(r)\) is given by

\[
(D_r\Gamma(r))_{ij} = \frac{\partial}{\partial r_j} \left( \frac{1}{2}(R_{av} - R_i)r_i \right) = \frac{1}{2}(R_{av} - R_i)\delta_{ij} + R_{av}(\delta_{ij} - \frac{r_ir_j}{\|r\|^2}) - \frac{1}{r_i} \frac{\partial K_i}{\partial u_j}.
\]

At the constant curvature metric point \(r^*\), we have

\[
D_r\Gamma|_{r^*} = R_{av} \left( I - \frac{rr^T}{\|r\|^2} \right) - \Sigma^{-\frac{1}{2}}L\Sigma^{-\frac{1}{2}} = R_{av} \left( I - \frac{rr^T}{\|r\|^2} \right) - \Lambda.
\]

Select an orthonormal matrix \(P\) such that

\[
P^T\Lambda P = \text{diag}\{0, \lambda_1(\Lambda), \cdots, \lambda_{N-1}(\Lambda)\}.
\]

Suppose \(P = (e_0, e_1, \cdots, e_{N-1})\), where \(e_i\) is the \((i+1)\)-column of \(P\). Then \(\Lambda e_0 = 0\) and \(\Lambda e_i = \lambda_i e_i, 1 \leq i \leq N - 1\), which implies \(e_0 = r/\|r\|\) and \(r \perp e_i, 1 \leq i \leq N - 1\). Hence

\[
(I_N - \frac{rr^T}{\|r\|^2})e_0 = 0 \quad \text{and} \quad (I_N - \frac{rr^T}{\|r\|^2})e_i = e_i, 1 \leq i \leq N - 1,
\]

which implies \(P^T(I_N - \frac{rr^T}{\|r\|^2})P = \text{diag}\{0, 1, \cdots, 1\}\). Therefore,

\[
D_r\Gamma|_{r^*} = P \cdot \text{diag}\{0, R_{av} - \lambda_1(\Lambda), \cdots, R_{av} - \lambda_{N-1}(\Lambda)\} \cdot P^T.
\]

If \(\lambda_1(\Lambda) > R_{av}^* = \frac{2\pi\chi(M)}{\|r^*\|^2}\), then \(D_r\Gamma|_{r^*}\) is negative semi-definite with kernel \(\{cr|c \in \mathbb{R}\}\) and \(\text{rank}(D_r\Gamma|_{r^*}) = N - 1\). Note that, along the flow, \(\|r\|^2\) is invariant. Thus the kernel is transversal to the flow. This implies that \(D_r\Gamma|_{r^*}\) is negative definite on \(\mathbb{S}^{N-1}\) and \(r^*\) is a local attractor of the normalized combinatorial Ricci flow (2.11). Then the conclusion follows from the Lyapunov Stability Theorem(41, Chapter 5).

If \(\chi(M) \leq 0\), we always have \(\lambda_1(\Lambda) > R_{av}^* = \frac{2\pi\chi(M)}{\|r^*\|^2}\), thus we get

**Corollary 2.12.** Suppose there is a nonpositive constant curvature metric \(r^*\) on a weighted triangulated surface \((M, T, \Phi)\) with \(\chi(M) \leq 0\). If \(\|r(0) - r^*\|^2\) is small enough, then the solution to the normalized combinatorial Ricci flow (2.11) exists for \(t \in [0, +\infty)\) and converges exponentially fast to \(r^*\).

Before proving the global results, we first check that the convergence of flow (2.11) ensures the existence of the constant curvature metric.
Lemma 2.13. Suppose that the solution to flow (2.11) lies in a compact region in $\mathbb{R}^N_{>0}$, then there exists a constant curvature metric $r^*$. Moreover, there exists a sequence of metrics which converges to $r^*$ along this flow.

Proof. Consider the Ricci potential

$$F(u) = \int_{u_0}^{u} \sum_{i=1}^{N} (K_i - R_{av} r_i^2) du_i,$$

where $u_i = \ln r_i^2$ and $u_0 \in \mathbb{R}^N$ is an arbitrary point. If we set $\varphi_i = K_i - \frac{2 \pi \chi(M)}{||r||^2} r_i^2$, by direct calculation, we have

$$\frac{\partial \varphi_i}{\partial u_j} = \frac{\partial \varphi_j}{\partial u_i}.$$

As the domain of $u$ is simply connected, this implies that the integration is path independent and then the Ricci potential is well defined. Suppose $t \in [0, T)$ and $T$ is the maximal existing time of $u(t)$. Then $\{u(t)\} \subset \mathbb{R}^N$ implies $T = +\infty$, otherwise $u(t)$ will run out of the compact region. $\{u(t)\} \subset \mathbb{R}^N$ also implies that $F(u(t))$ is bounded. Moreover,

$$\frac{d}{dt} F(u(t)) = (\nabla_u F)^T \cdot \dot{u} = - \sum_i \left( \frac{K_i}{r_i^2} - \frac{2 \pi \chi(M)}{||r||^2} r_i^2 \right)^2 = - \sum_i (R_i - R_{av})^2 r_i^2 \leq 0.$$

Hence $\lim_{t \to +\infty} F(u(t))$ exists. Then by the mean value theorem, there exists a sequence $\xi_n \in (n, n+1)$ such that

$$F(u(n+1)) - F(u(n)) = (F(u(t)))'|_{t=\xi_n} = - \sum_i (R_i - R_{av})^2 r_i^2 (\xi_n) \to 0$$

as $n \to +\infty$. Since $\{r(t)\} \subset \mathbb{R}^N_{>0}$, we can further select a subsequence $t_k = \xi_{n_k}$ such that $r(t_k) \to r^*$ as $k \to +\infty$. Combining $\sum_i r_i^2 (R_i - R_{av})^2 (\xi_{n_k}) \to 0$, we know that $r^*$ determines a constant $R$-curvature. \hfill $\square$

Corollary 2.14. Suppose that the solution to flow (2.11) exists for all time and converges to $r(+\infty)$, then $r(+\infty)$ is a constant curvature metric.

For the Ricci potential $F$ introduced in the proof of Lemma 2.13, we further have the following property.

Lemma 2.15. Given a weighted triangulated surface $(M, T, \Phi)$. Assuming there is a constant curvature circle packing metric $r^*$. Define Ricci potential as

$$F(u) = \int_{u_0}^{u} \sum_{i=1}^{N} (K_i - R_{av} r_i^2) du_i.$$  \hspace{1cm} (2.19)

Denote $\mathcal{U}_a \triangleq \{u \in \mathbb{R}^N | \sum_i u_i = a\}$, $a \in \mathbb{R}$. If $\lambda_1(A) > R_{av}$ at all $r \in \mathbb{R}^N_{>0}$, then
(1) \(Hess_u F\) is positive semi-definite with rank \(N - 1\) and kernel \({t1\mid t \in \mathbb{R}}\).

(2) Restricted to \(\mathscr{U}_a\), \(F\mid\mathscr{U}_a\) is strictly convex and proper. \(F\mid\mathscr{U}_a\) has a unique zero point which is also the unique minimum point. Moreover, \(\lim_{u \in \mathscr{U}_a, u \to \infty} F(u) = +\infty\).

**Proof.** By direct calculation, we have

\[
Hess_u F = L - R_{av} \Sigma \frac{1}{2} \left( I - \frac{rr^T}{||r||^2} \right) \Sigma \frac{1}{2} \left( \Lambda - R_{av} \left( I - \frac{rr^T}{||r||^2} \right) \right) \Sigma \frac{1}{2}.
\]

(2.20)

By the same analysis as that of Theorem 2.11, we know that, if \(\lambda_1(\Lambda) > R_{av}\), \(Hess_u F\) is positive semi-definite with \(\text{rank}(Hess_u F) = N - 1\) and \(\text{Ker}(Hess_u F) = \{t1\mid t \in \mathbb{R}\}\).

For the second part of the proof, we follow Chow and Luo’s work [13]. It’s easy to see \(F(u) = F(u + t1)\) for any \(t \in \mathbb{R}\) and \(u \in \mathbb{R}^N\). \(F\) is invariant when moving along the direction \(1\), we just need to prove it on \(\mathscr{U}_0 = \{u \in \mathbb{R}^N\mid \sum u_i = 0\}\). A rigorous proof is formulated in Theorem B.2 of [13]. In addition, \(u^* + \frac{a-1}{N}u^* 1\) is the unique zero point and minimum point of \(F\mid\mathscr{U}_a\). \(\square\)

**Theorem 2.16.** Suppose \((M, \mathcal{T}, \Phi)\) is a weighted triangulated surface and \(\lambda_1(\Lambda) > R_{av}\) at all \(r \in \mathbb{R}^N\).

Then the solution to the flow (2.11) converges if and only if there exists a constant curvature metric \(r^*\). Furthermore, if the solution converges, it converges exponentially fast to the metric of constant curvature.

**Proof.** The necessary part can be seen from Corollary 2.11. For the sufficient part, without loss generality, we may assume \(r^* \in S^{N-1}\). Consider flow (2.11) with initial metric \(r(0) \in S^{N-1} \cap \mathbb{R}^N\). Then \(r(t) \in S^{N-1} \cap \mathbb{R}^N\) and \(u(t) \in \mathcal{V}_1^* \triangleq \{u \in \mathbb{R}^N\mid \sum e^{u_i} = 1\}\) for all \(t\). Let \(\Pi\) be the orthogonal projection from \(\mathcal{V}_1^*\) to the plane \(\mathscr{U}_0 = \{u \in \mathbb{R}^N\mid \sum u_i = 0\}\). Then \(F(u) = F(\Pi(u))\) and \(\Pi(u) \to \infty\) as \(u \to \infty\). In fact, for a sequence \(\{u^{(n)}\}\), \(\Pi(u^{(n)})\) is unbounded if and only if \(|\Pi(u_i^{(n)}) - \Pi(u_j^{(n)})| = |u_i - u_j|\) is unbounded. Then Lemma 2.15 implies

\[
\lim_{u \to \infty, u \in \mathcal{V}_1^*} F(u) = +\infty,
\]

(2.21)

which means that \(F(u)\) is still proper on \(\mathcal{V}_1^*\). Since \(F(u(t))' \leq 0\), \(u(t)\) must lies in a compact region in \(\mathcal{V}_1^*\). By Lemma 2.13, the solution exists for \(t \in [0, +\infty)\) and there exists a sequence of metrics \(r(t_k)\) which converges to \(r^*\) as \(t_k \uparrow +\infty\). Hence \(F(u(+\infty)) = \lim_{k \to +\infty} F(u(t_k)) = F(u^*) = 0\). While Lemma 2.15 says that \(u^*\) is the unique zero and minimum point of \(F\), thus we get \(u(t)\) converges to \(u^*\) as \(t \to +\infty\). The exponentially convergent rate comes from Theorem 2.11. \(\square\)

**Corollary 2.17.** Suppose \((M, \mathcal{T}, \Phi)\) is a weighted triangulated surface with \(\chi(M) \leq 0\). Then the solution to the flow (2.11) converges if and only if there exists a constant \(R\)-curvature metric \(r^*\). Furthermore, if the solution converges, it converges exponentially fast to the metric of constant curvature.
We expect Corollary 2.17 still true for surfaces with $\chi(M) > 0$. However, things are not so satisfactory for general triangulations. In fact, things may become very different and complicated. Assuming there exists a constant curvature metric $r^*$, we may compare the Ricci potential (2.19) with

$$G(u) = \int_{u^*}^{u} \sum_{i=1}^{N} (K_i - K_i^*) du_i,$$

where $K_i^* = \frac{2\pi \chi(M)}{\|r^*\|} r_i^2$, then

$$F(u) - G(u) = \int_{u^*}^{u} \sum_{i=1}^{N} (K_i^* - R_{av} r_i^2) du_i$$

$$= 2\pi \chi(M) \left( \sum_{i=1}^{N} r_i^2 (u_i - u_i^*) \right) - \int_{u^*}^{u} e^{u_1} du_1 + \cdots + e^{u_N} du_N$$

$$= 2\pi \chi(M) \sum_{i=1}^{N} r_i^2 (u_i - u_i^*) - 2\pi \chi(M) \ln | e^{u_1} + \cdots + e^{u_N} |_{u^*}.$$  

When restricted to the hypersurface $\mathcal{V}^* \equiv \{ u \in \mathbb{R}^N | \sum_i e^{u_i} = \sum_i e^{u_i^*} \}$, the last term become zero, hence

$$F(u) = G(u) + 2\pi \chi(M) \sum_{i=1}^{N} r_i^2 (u_i - u_i^*) \quad \forall u \in \mathcal{V}^*.$$  

By similar arguments in the proof of Theorem 2.16 we further have

$$\lim_{u \to \infty, u \in \mathcal{V}^*} G(u) = +\infty.$$  

One may expect that the growth behavior of $F$ is similar to $G$. Unfortunately, there is an obstruction which makes things very complicated. It’s easy to see

$$\lim_{u \to \infty, u \in \mathcal{V}^*} \sum_{i=1}^{N} r_i^2 (u_i - u_i^*) = -\infty.$$  

The growth behavior of $F$ is uncertain unless one can compare the growth rates of these two terms concretely. It seems that we can not expect that $G(u)$ succeeds the last term due to the following example.

**Example 2.** Given a topological sphere, triangulate it to four faces of a single tetrahedron. Fix weights $\Phi \equiv 0$. Denote $r^* = 1$, then $r^*$ is a constant curvature metric with $R_i \equiv \pi$. By direct calculation, we get $Hess_u F|_{u^*} = (\frac{\sqrt{3}}{6} - \frac{\pi}{4})(4I_4 - 11^T)$, which is negative semi-definite with three negative eigenvalues. Up to scaling, we may think $Hess_u F$ is negative.
definite at \( r^* \). Thus the fixed point \( r^* = 1 \) of the ODE system \( r'_i(t) = \frac{1}{2}(R_{av} - R_i)r_i \) (i.e. flow (2.11)) is a source. For this particular weighted triangulation, the flow (2.11) can never converge to the constant curvature metric \( r^* \) for any initial value \( r(0) \), when \( t \to +\infty \). However, when \( t \to -\infty \), the solution of flow (2.11) converges to \( r^* \) assuming that \( r(0) \in S^{N-1}(2) \) is close enough to \( r^* \).

From the example we know that it’s impossible to get an analog version of Corollary 2.17 when \( \chi(M) > 0 \). However, we can get the following long time existence of (2.11) by maximum principle.

**Theorem 2.18.** Given a weighted triangulated surface \((M, T, \Phi)\) with \( \chi(M) > 0 \) and

\[
- \sum_{(e,v) \in Lk(I)} (\pi - \Phi(e)) + 2\pi \chi(F_I) < 0, \quad \forall I : \emptyset \subsetneq I \subsetneq V. \tag{2.26}
\]

Suppose \( r_0 \) is a circle packing metric with \( R_i \geq 0 \). Then the solution of (2.11) with initial metric \( r_0 \) exists for all time and lies in a compact region in \( \mathbb{R}^N_{>0} \). Furthermore, there exists \( t_n \uparrow +\infty \), such that \( r(t_n) \) converges to a constant curvature metric \( r^* \).

**Proof.** By Lemma 2.13 we just need to prove that \( \{r(t)\} \subset \mathbb{R}^N_{>0} \). We prove it by contradiction. Suppose

\[
\{r(t)\} \cap \partial(\mathbb{R}^N_{>0} \cap S^{N-1}) \neq \emptyset,
\]

then there exists a sequence \( t_n \) such that

\[
r(t_n) \to r^* \in \partial(\mathbb{R}^N_{>0} \cap S^{N-1}),
\]

where \( r^* = (0, \ldots, 0, r^*_{|I|+1}, \ldots, r^*_N) \) for some nonempty proper subset \( I \) of \( V \). By Proposition 4.1 of [13], we have

\[
\lim_{n \to +\infty} \sum_{i \in I} K_i(r(t_n)) = - \sum_{(e,v) \in Lk(I)} (\pi - \Phi(e)) + 2\pi \chi(F_I) < 0. \tag{2.27}
\]

However, by Corollary 2.7 \( R_i \geq 0 \) is preserved along the flow (2.11). Thus \( K_i \geq 0 \) for all \( i \in V \), which implies \( \lim_{n \to +\infty} \sum_{i \in I} K_i(r(t_n)) \geq 0 \). This contradicts (2.27). \( \square \)

**Remark 2.** It’s easy to see that the triangulation of sphere in Example 2 doesn’t satisfy the condition (2.26).

### 2.2.5 The prescribing curvature problem

Using modified combinatorial Ricci flow, we can consider the prescribing curvature problem on weighted triangulated surface \((M, T, \Phi)\).
**Definition 2.19.** Suppose $(M, \mathcal{T}, \Phi)$ is a weighted triangulated surface with circle packing metric $r$, $\mathcal{R} \in C(V)$ is a function defined on $M$. The modified combinatorial Ricci flow with respect to $\mathcal{R}$ is defined to be

\[
\frac{d g_i}{dt} = (\mathcal{R}_i - R_i) g_i,
\]

(2.28)

where $g_i = r_i^2$ as before.

Note that the total measure $\mu(M) = ||r||^2$ may change along the modified combinatorial Ricci flow (2.28), which is different from that of (2.11).

$\mathcal{R}$ is called admissible if there is a circle packing metric $\mathcal{r}$ with curvature $\mathcal{R}$. For given function $\mathcal{R} \in C(V)$, we can introduce the following modified Ricci potential

\[
\mathcal{F}(u) = \int_{u_0}^{u} \sum_{i=1}^{N} (K_i - \mathcal{R}_i r_i^2) \, du.
\]

(2.29)

It is easy to check that the modified Ricci potential $\mathcal{F}$ is well-defined. Furthermore, by direct calculation, we have

\[
\text{Hess}_u \mathcal{F} = L - \Sigma \frac{1}{2} \left( \begin{array}{c} \mathcal{R}_1 \\ \vdots \\ \frac{1}{2} \sqrt{\Sigma} \\ \end{array} \right) \Sigma \frac{1}{2}.
\]

The following lemma is useful.

**Lemma 2.20.** ([13]) Suppose $\Omega \subset \mathbb{R}^N$ is convex, the function $h : \Omega \to \mathbb{R}$ is strictly convex, then the map $\nabla h : \Omega \to \mathbb{R}^N$ is injective.

**Theorem 2.21.** Suppose $(M, \mathcal{T}, \Phi)$ is a weighted triangulated surface and $\mathcal{R} \in C(V)$ is a function defined on $M$.

**(1)** If the solution to the modified flow (2.28) converges, then $\mathcal{R}$ is admissible.

**(2)** If $\mathcal{R}_i \leq 0$ for all $i$, but not identically zero, and $\mathcal{R}$ is admissible by a metric $\mathcal{r}$. Then $\mathcal{r}$ is the unique metric in $\mathbb{R}^N_{>0}$ such that it’s curvature is $\mathcal{R}$. Moreover, the solution to the modified flow (2.28) converges exponentially fast to $\mathcal{r}$.

**Proof.** The first part is obviously, and $\mathcal{R}$ is admissible by metric $r(+\infty)$. For the second part, notice that

\[
\text{Hess}_u \mathcal{F} = L - \Sigma \frac{1}{2} \text{diag} \{ \mathcal{R}_1, \cdots, \mathcal{R}_N \} \Sigma \frac{1}{2},
\]
It is easy to check that, if $\bar{R}_i \leq 0$ for $i = 1, \cdots, N$ and not identically zero, $\text{Hess}_u \bar{F}$ is positive definite. By Lemma 2.20, $\nabla_u \bar{F} = (K_1 - \bar{R}_1 r_1^2, \cdots, K_N - \bar{R}_N r_N^2)^T$ is an injective map from $u \in \mathbb{R}^N$ to $\mathbb{R}^N$. Hence $\bar{r}$ is the unique zero point of $\nabla_u \bar{F}$. This fact implies that $\bar{r}$ is the unique metric in $\mathbb{R}^N_{>0}$ such that it’s curvature is $\bar{R}$. By Lemma B.1 in [18], we know that $\bar{F}$ is proper and $\lim_{u \to \infty} \bar{F}(u) = +\infty$. Furthermore, $\frac{d}{dt} F(u(t)) = -\sum_i r_i^{-2} (K_i - \bar{R}_i r_i^2)^2 \leq 0$ implies that the solution to (2.28) lies in a compact region. The following of the proof is the same as that of Theorem 2.16, so we omit it here.

\[\square\]

**Remark 3.** Theorem 2.21 could be taken as a discrete version of the result obtained by Kazdan and Warner in [32].

**Remark 4.** The second part of Theorem 2.21 implies that $\chi(M) < 0$. If $\bar{R}_i = 0$ for all $i$, then the corresponding prescribing curvature problem is already solved in Corollary 2.17. In this case, the metric $\bar{r}$ is not unique. However, it’s unique up to scaling. This is slightly different from Theorem 2.21.

### 2.3 Combinatorial Calabi flow on surfaces

Given a compact complex manifold admitting at least one Kähler metric, to find the extreme metric which minimizes the $L^2$ norm of the curvature tensor in a given principal cohomology class, Calabi [6] introduced the Calabi flow, which could be written as

$$\frac{\partial g}{\partial t} = \Delta_g K \cdot g$$

(2.30)
on a Riemannian surface. Chruściel [15] proved that the Calabi flow (2.30) exists for all time and converges to a constant Gaussian curvature metric on closed surfaces using the Bondi mass estimate, assuming the existence of the constant Gaussian curvature in the background which is ensured by the the uniformization theorem. Chang [9] pointed out that Chruściel’s results still hold for arbitrary initial metric, which implies the uniformization theorem on closed surfaces with genus greater than one. Chen [10] gave a geometrical proof of the long-time existence and the convergence of the Calabi flow on closed surfaces.

The first author [18] first introduced the notion of combinatorial Calabi flow in Euclidean background geometry and proved that the convergence of the flow is equivalent to the existence of constant classical discrete Gauss curvature. Then the authors [19] studied the combinatorial Calabi flow in hyperbolic background geometry. We also use the combinatorial Calabi flow to study some constant combinatorial curvature problem on 3-dimensional triangulated manifolds [20]. In this subsection, we will study the constant $R$-curvature problem by combinatorial Calabi flow.
Definition 2.22. For a weighted triangulated surface \((M, T, \Phi)\) with circle packing metric \(r\), the combinatorial Calabi flow is defined as

\[
\frac{dg_i}{dt} = \Delta R_i \cdot g_i,
\] (2.31)

where \(g_i = r_i^2\) and \(\Delta\) is the Laplacian operator given by (2.16).

It is easy to check that the total measure \(\mu(M) = \|r\|^2\) of \(M\) is invariant along the combinatorial Calabi flow (2.31). Interestingly, the combinatorial curvature \(R_i\) evolves according to

\[
\frac{dR_i}{dt} = -R_i \Delta R_i - \Delta^2 R_i,
\]

which is almost the same as that of the scalar curvature along the smooth Calabi flow on surfaces. We can rewrite the combinatorial Calabi flow (2.31) as

\[
\frac{dr_i}{dt} = -\frac{1}{2r_i} \sum_j \frac{\partial K_i}{\partial u_j} R_j = -\frac{1}{2r_i} \sum_j \frac{\partial K_i}{\partial u_j} (R_j - R_{av}).
\]

Set

\[
\Gamma(r)_i = -\frac{1}{2r_i} \sum_j \frac{\partial K_i}{\partial u_j} (R_j - R_{av}).
\]

If there exists a constant curvature metric \(r^*\), then we have

\[
D_j \Gamma_i|_{r^*} = -\frac{1}{r_i r_j} \left( \sum_k \frac{1}{r_k^2} \frac{\partial K_i}{\partial u_k} \frac{\partial K_j}{\partial u_k} - R_{av} \frac{\partial K_i}{\partial u_j} \right).
\]

If we further assume that the condition (2.18), i.e. \(\lambda_1(\Lambda) > R_{av}^2\), is satisfied, then \(D\Gamma|_{r^*}\) is negative semi-definite with rank \(N - 1\) and kernel \(\{t1|t \in \mathbb{R}\}\). From this fact we can get local convergence results similar to Theorem 2.11, Corollary 2.12. We can also get results similar to Lemma 2.13, Corollary 2.14, Theorem 2.16, and Corollary 2.17. We just state the following main theorem for this subsection here.

Theorem 2.23. Suppose \((M, T, \Phi)\) is a weighted triangulated surface with \(\chi(M) \leq 0\), then the combinatorial Calabi flow (2.31) converges if and only if there exists a constant curvature metric \(r^*\).

Proof. If the solution of (2.31) converges, \(u(+\infty)\) must be a critical point of this ODE system, i.e. \(\Delta R(+\infty) = -\Sigma^{-1} LR(+\infty) = 0\), which implies that \(R(+\infty)\) belongs to the kernel of \(L\) and hence \(r(+\infty)\) is a constant curvature metric.
Assuming there exists a constant curvature metric $r^*$. Write the flow \( \dot{u} = \Delta R = -\Sigma^{-1} LR \) or \( \dot{r}^2 = -LR \), where \( r^2 = (r_1^2, \ldots, r_N^2)^T \). It’s easy to see that
\[
\frac{d}{dt} F(u(t)) = -(K - R_{uv} r^2)^T \Sigma^{-1} LR = -R^T LR \leq 0.
\]
Using the properties of the Ricci potential, we can get the convergence result by similar arguments used in the proof of Theorem 2.16.

Analogue to the combinatorial Ricci flow, we can also use the combinatorial Calabi flow to study the combinatorial prescribing curvature problem. In fact, we have the following result.

**Theorem 2.24.** Suppose \((M, \mathcal{T}, \Phi)\) is a weighted triangulated surface, \( \mathcal{R} \in C(V) \) is a function defined on \( M \) with \( \mathcal{R}_i \leq 0 \) for all \( i \). Then \( \mathcal{R} \) is admissible if and only if the solution of the modified combinatorial Calabi flow
\[
\frac{dg_i}{dt} = \Delta(R - \mathcal{R})_i g_i \tag{2.32}
\]
exists for all time and converges.

The proof is just a combination of that of Theorem 2.21, Remark 4, and Theorem 2.23, we omit it here.

Following the way in [18], we can also introduce a notion of energy to study the curvature \( R_i \).

**Definition 2.25.** For a weighted triangulated surface \((M, \mathcal{T}, \Phi)\) with circle packing metric \( r \), the combinatorial Calabi energy is defined as
\[
\tilde{C}(r) = \sum_{i=1}^{N} \phi_i^2, \tag{2.33}
\]
where \( \phi_i = K_i - \frac{2\pi \chi_M}{||r||^2} r_i^2 \).

Consider the combinatorial Calabi energy \( \tilde{C} \) as a function of \( u \), we have \( \nabla_u \tilde{C} = 2A^T \varphi \), where
\[
A = \frac{\partial(\varphi_1, \ldots, \varphi_N)}{\partial(u_1, \ldots, u_N)} = \left( \begin{array}{ccc}
\frac{\partial \varphi_1}{\partial u_1} & \cdots & \frac{\partial \varphi_1}{\partial u_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial \varphi_N}{\partial u_1} & \cdots & \frac{\partial \varphi_N}{\partial u_N}
\end{array} \right).
\]
Note that \( A \) is in fact given by \( (2.20) \). Thus, if \( \chi(M) \leq 0 \), \( A \) is symmetric, positive semidefinite with rank \( N - 1 \) and kernel \( \{t1| t \in \mathbb{R}\} \). Using this fact, we can define a new flow, which is the gradient flow of \( \tilde{C} \). We call it the modified combinatorial Calabi flow.
Definition 2.26. For a weighted triangulated surface $(M, T, \Phi)$ with circle packing metric $r$, the modified combinatorial Calabi flow is defined as
\[ \dot{u} = -\frac{1}{2} \nabla_u \tilde{C}, \quad (2.34) \]
or equivalently,
\[ \dot{u} = -A^T \varphi, \quad (2.35) \]
where $\varphi = (\varphi_1, \cdots, \varphi_N)^T$.

Note that $\sum_i \varphi_i = 0$, so in the case of $\chi(M) \leq 0$, if the flow (2.34) converges, it converges to the constant curvature metric. Along the flow (2.34), we have
\[ \dot{\varphi} = -AA^T \varphi, \quad \dot{\tilde{C}} = -2\|A^T \varphi\|^2 \leq 0 \]
and
\[ \frac{d}{dt} F(u(t)) = (\nabla_u F)^T \cdot \dot{u} = \varphi^T \cdot \dot{u} = -\varphi^T A^T \varphi \leq 0, \]
which implies that $F(u(t))$ is decreasing along the flow (2.34).

Following the arguments in the proof of Theorem 2.16 and 2.23, we can derive the following result.

Theorem 2.27. Suppose $(M, T, \Phi)$ is a weighted triangulated surface with $\chi(M) \leq 0$, then the existence of constant curvature metric $r^*$ is equivalent to the convergence of the modified combinatorial Calabi flow (2.34).

2.4 Combinatorial $\alpha$-curvature and combinatorial $\alpha$-flows

In the previous sections, we investigated the properties of the modified Gauss curvature $R_i = K_i/r_i^2$. Since the area of the disk packed at $i$ is just $\pi r_i^2$, the denominator of the curvature $r_i^2$ (omitting the efficient $\pi$) may also be considered as an “area element” attached to vertex $i$. We want to know whether there exists other types of “area element”. Suppose $A_i$ is a general “area element”, which is an analogy of the volume element in the smooth case, the combinatorial Gauss curvature could be defined as $R_i = K_i/A_i$. The average curvature should be $2\pi \chi(M)/\sum A_i$, which is an analogy of $\int Rd\mu/\int d\mu$ in the smooth case. If we expect the functional
\[ F(u) = \int_{u_0}^u \sum_{i=1}^N (K_i - \frac{2\pi \chi(M)}{\sum A_i} A_i) du_i \]
to be well defined, the following formula
\[ \frac{\partial}{\partial u_i} \left( \frac{A_j}{\sum_k A_k} \right) = \frac{\partial}{\partial u_j} \left( \frac{A_i}{\sum_k A_k} \right) \quad (2.36) \]
should be satisfied for all \(i\) and \(j\). It’s a trivial observation that \(A_i = r_i^\alpha\) always satisfies (2.36) for all \(\alpha \in \mathbb{R}\).

When \(\chi(M) \leq 0\), we had already seen in the previous sections that combinatorial Ricci flow and Calabi flow are good enough to evolve circle packing metric to constant \(R\)-curvature metric. However, in the case \(\chi(M) > 0\), we don’t know how to evolve it. Maybe this is because that the growth behavior of \(F\) is uncertain (in integral level), or equivalently, \(Hess_u F\) is not positive semi-definite (in differential level), even for the simplest triangulation of \(S^2\) (see Example 2).

The above two reasons motivate us to consider a new “area element” \(A_i = r_i^\alpha\). We find that, for \(\chi(M) > 0\), if \(\alpha < 0\), combinatorial flow methods are good enough to evolve the curvature to a constant. For \(\chi(M) \leq 0\) case, if \(\alpha > 0\), properties of combinatorial flows are almost the same with previous sections. It’s very interesting that \([13]\) and \([18]\) can be included into the case of \(\alpha = 0\).

**Definition 2.28.** For a weighted triangulated surface \((M, T, \Phi)\) with circle packing metric \(r\), the combinatorial \(\alpha\)-curvature is defined as

\[
R_{\alpha,i} = \frac{K_i}{r_i^\alpha},
\]

where \(\alpha\) is a real number.

For this type of curvature, we can also consider the corresponding constant curvature problem and prescribing curvature problem. As the methods are all the same as that we dealt with \(R\)-curvature, we will give only the outline and skip the details in the following.

The measure defined on \(M\) is now \(\mu_\alpha(i) = r_i^\alpha\) and the average curvature is now

\[
R_{\alpha,\text{av}} = \frac{\int_M R_\alpha d\mu_\alpha}{\int_M d\mu_\alpha} = \frac{2\pi \chi(M)}{||r||_\alpha},
\]

where \(||r||_\alpha = (\sum_{i=1}^N r_i^\alpha)^{\frac{1}{\alpha}}\). In this subsection, we set \(u = \ln r_i\) and define the Ricci potential as

\[
F_\alpha(u) = \int_{u_0}^u \sum_{i=1}^N (K_i - R_{\alpha,\text{av}} r_i^\alpha) du_i,
\]

where \(u_0 \in \mathbb{R}^N\) is an arbitrary point. By direct calculation, it is easy to check that the Ricci potential \(F_\alpha\) is well-defined. We further have \(\nabla_u F_\alpha = K - R_{\alpha,\text{av}} r_i^\alpha\) and

\[
Hess_u F_\alpha = \tilde{L} - \alpha R_{\alpha,\text{av}} \Sigma_\alpha \left( I - \frac{\varphi_r^\alpha \cdot (\varphi_r^\alpha)^T}{||r||_\alpha} \right) \Sigma_\alpha^{-1}
\]

\[
= \begin{pmatrix}
\varphi_r^\alpha & \cdots & \varphi_r^\alpha \\
\vdots & \ddots & \vdots \\
\varphi_r^\alpha & \cdots & \varphi_r^\alpha
\end{pmatrix} \left( \Lambda_\alpha - \alpha R_{\alpha,\text{av}} \left( I - \frac{\varphi_r^\alpha \cdot (\varphi_r^\alpha)^T}{||r||_\alpha} \right) \right) \left( \begin{pmatrix}
\varphi_r^\alpha \\
\vdots \\
\varphi_r^\alpha
\end{pmatrix}
\right).
\]

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where $r^\alpha = (r_1^\alpha, \ldots, r_N^\alpha)^T$, $\Lambda^\alpha = \Sigma^{-2} \tilde{\Sigma}^{-2}$  and $\tilde{L}_{ij} = \frac{\partial K_i}{\partial u_j}$ with $u_j = \ln r_j$. Note that the matrix $I - \frac{r^\alpha}{||r^\alpha||} (r^\alpha)^T$ has eigenvalues 1 ($N - 1$ times) and 0 (1 time) and kernel $\{tr^\alpha t \mid t \in \mathbb{R}\}$. Following the arguments in the proof of Lemma 2.15 we have, if the first positive eigenvalue of $\Lambda^\alpha$ satisfies
\[
\lambda_1(\Lambda^\alpha) > \alpha R_{\alpha,av},
\]
$Hess_u F^\alpha$ is positive semi-definite with rank $N - 1$ and kernel $\{t1 \mid t \in \mathbb{R}\}$. Especially, if $\alpha \chi(M) \leq 0$, $Hess_u F^\alpha$ is positive semi-definite with rank $N - 1$ and kernel $\{t1 \mid t \in \mathbb{R}\}$.

We can also define the following modification of the combinatorial Ricci flow, which is called $\alpha$-Ricci flow.

**Definition 2.29.** For a weighted triangulated surface $(M, T, \Phi)$ with circle packing metric $r$, the $\alpha$-Ricci flow is defined to be
\[
\frac{dr_i}{dt} = -R_{\alpha,i} r_i.
\]
(2.38)

The normalization of the $\alpha$-Ricci flow is
\[
\frac{dr_i}{dt} = (R_{\alpha,av} - R_{\alpha,i}) r_i.
\]
(2.39)

Notice that, when $\alpha = 0$, the flow (2.38) and the normalized flow (2.39) are just Chow and Luo’s combinatorial Ricci flows [13]. In this case, $\Pi_{i=1}^N r_i$ (or $\sum_{i=1}^N u_i$) is invariant along the normalized flow (2.39). When $\alpha \neq 0$, $\|r\|_\alpha$ (or $\sum_{i=1}^N e^{\alpha u_i}$) is invariant along the normalized flow (2.39). Along the normalized $\alpha$-Ricci flow, the $R_{\alpha}$-curvature evolves according to
\[
\frac{dR_{\alpha,i}}{dt} = -\frac{1}{r_i^\alpha} \sum_{j=1}^N \frac{\partial K_i}{\partial u_j} R_{\alpha,j} + \alpha R_{\alpha,i}(R_{\alpha,i} - R_{\alpha,av}).
\]
(2.40)

For $\alpha = 2$, this is just the evolution equation (2.17) derived in Lemma 2.5. As the evolution equation (2.40) is still a heat-type equation, the discrete maximum principle, i.e. Theorem 2.6, could be applied to this equation. The evolution of curvature (2.40) under the normalized $\alpha$-Ricci flow suggests us to define the $\alpha$-Laplacian as
\[
\Delta_\alpha f_i = -\frac{1}{r_i^\alpha} \sum_{j=1}^N \frac{\partial K_i}{\partial u_j} f_j,
\]
(2.41)

where $f \in C(V)$ and $u_i = \ln r_i$. Using this Laplacian, we can define the combinatorial $\alpha$-Calabi flow as
\[
\frac{dr_i}{dt} = (\Delta_\alpha R_\alpha)_i r_i.
\]
(2.42)
Similarly, denote $\varphi_{\alpha,i} = K_i - \frac{2\pi \chi(M)}{|r_i|^{\alpha}}$ and $\tilde{C}_\alpha(r) = \sum_{i=1}^{N} \varphi_{\alpha,i}^2$, we can define a modified $\alpha$-Calabi flow as
\[
\dot{u} = -\frac{1}{2} \nabla u \tilde{C}_\alpha. \tag{2.43}
\]

Using the normalized $\alpha$-Ricci flow (2.39), the combinatorial $\alpha$-Calabi flow (2.42) and the modified $\alpha$-Calabi flow (2.43), we can give a characterization of the existence of constant $\alpha$-curvature metric.

**Theorem 2.30.** Suppose $(M, \mathcal{T}, \Phi)$ is a weighted triangulated surface with $\alpha \chi(M) \leq 0$. Then the existence of constant $R_\alpha$-curvature circle packing metric, the convergence of $\alpha$-Ricci flow (2.39), the convergence of $\alpha$-Calabi flow (2.42) and the convergence of modified $\alpha$-Calabi flow (2.43) are all equivalent. Furthermore, if the solutions of the flows converge, then they converge exponentially fast to constant $R_\alpha$-curvature metric.

**Proof.** Assuming there exists a constant curvature metric $r^*$, $u^*$ is the corresponding $u$-coordinate. Consider the $\alpha$-Ricci potential
\[
F_\alpha(u) = \int_{u^*}^{u} \sum_{i=1}^{N} (K_i - R_\alpha r_i^*)du_i. \tag{2.44}
\]
It has similar properties as stated in Lemma 2.15. When restricted to the hypersurface $U = \{u \in \mathbb{R}^N | \sum_i e^{\alpha u_i} = \sum_i e^{\alpha u_i(0)}\}$,
\[
\lim_{u \in U, u \to \infty} F_\alpha(u) = +\infty
\]
and $F_\alpha|U$ is also proper. The rest proof is similar to Theorem 2.16 and Theorem 2.23, so we omit it here. \square

We can also use the combinatorial $\alpha$-Ricci flow, combinatorial $\alpha$-Calabi flow and modified combinatorial $\alpha$-Calabi flow to study the prescribing $\alpha$-curvature problem. Specifically, we have the following result.

**Theorem 2.31.** Suppose $(M, \mathcal{T}, \Phi)$ is a weighted triangulated surface, $\alpha \in \mathbb{R}$ is a given real number, $\overline{R}_\alpha \in C(V)$ is a function defined on $M$ with $\alpha \overline{R}_{\alpha,i} \leq 0$ for $i = 1, \cdots, N$. Then $\overline{R}_\alpha$ is an admissible $\alpha$-curvature if and only if the combinatorial $\alpha$-Ricci flow with target
\[
\frac{dr_i}{dt} = (\overline{R}_{\alpha,i} - R_{\alpha,i})r_i,
\]
exists for all time and converges, if and only if the combinatorial $\alpha$-Calabi flow with target
\[
\frac{dr_i}{dt} = \Delta_\alpha (R_\alpha - \overline{R}_\alpha)_i r_i
\]
exists for all time and converges, if and only if the modified combinatorial \( \alpha \)-Calabi flow with target

\[
\dot{u} = -\frac{1}{2} \nabla u C
\]

exists for all time and converges, where \( C = \sum_i (K_i - R_{\alpha,i} \bar{r}_i^2)^2 \).

**Remark 5.** For \( \alpha = 0 \), the condition \( \alpha \chi(M) \leq 0 \) is always satisfied, so there is no restriction on \( \chi(M) \) and the equivalence in Theorem 2.30 and Theorem 2.31 is always valid. For \( \alpha = 0 \), the equivalent condition given by combinatorial Ricci flow is obtained in [13], and the equivalent condition given by combinatorial Calabi flow is obtained in [15].

2.5 Existence and uniqueness of constant curvature metric

2.5.1 Uniqueness of constant curvature metric

**Theorem 2.32.** Suppose \((M, T, \Phi)\) is a weighted triangulated surface. \( \forall c^* \in \mathbb{R} \),

1. If \( c^* < 0 \), there exists at most one circle packing metric with curvature \( R = c^* \mathbf{1} \).
2. If \( c^* = 0 \), there exists at most one metric with curvature \( R = c^* \mathbf{1} \) up to scaling.

**Proof.** The first part is just a corollary of the second part of Theorem 2.21. The second part is already proved by Thurston [16]. \( \square \)

If \( \chi(M) \leq 0 \), Theorem 2.32 implies that constant curvature metric (if exists) is unique or unique up to scaling. If \( \chi(M) > 0 \), there may be several different constant curvature metrics on \( S^{N-1} \) and we have the following example.

**Example 3.** Consider the weighted triangulation of the sphere in Example 2 with tangential circle packing metric. Let \( r_1 = 1, r_i = x > 0, 2 \leq i \leq 4 \) and then \( x = \frac{\cos \theta}{1 - \cos \theta} \).

![Figure 2: tetrahedron triangulation of sphere](image)

By direct calculation, \( K_1 = 6\theta - \pi, K_i = \frac{5}{3}\pi - 2\theta, 2 \leq i \leq 4 \). Constant \( R \)-curvature implies that

\[
\frac{\frac{5}{3}\pi - 2\theta}{x^2} = \frac{6\theta - \pi}{1^2}.
\]
Using MATLAB, we can solve the equation and get two solutions \( x = 1 \) and \( x \approx 5.9487 \). Thus we find at least two constant curvature metrics \( r^*_1 = 1 \) and \( r^*_2 \approx (1, 5.9487, 5.9487, 5.9487) \).

However, if we consider \( R_\alpha \) curvature instead, we can get similar result.

**Theorem 2.33.** Suppose \((M, \mathcal{T}, \Phi)\) is a weighted triangulated surface. If \( \alpha \chi(M) = 0 \), then there exists at most one constant curvature metric up to scaling. If \( \alpha \chi(M) < 0 \), then for any \( c^* \), there exists at most one metric with curvature \( R_{av} = c^*1 \).

**Remark 6.** For \( \alpha = 0 \), this is the result obtained by Thurston [46], Chow and Luo [13].

### 2.5.2 Existence of constant curvature metric

For classical combinatorial Gauss curvature \( K \), by the work of Thurston [46], Marden and Rodin [36], Chow and Luo [13], the existence of constant \( K \)-curvature metric is equivalent to

\[
2\pi \chi(M) \frac{|I|}{|V|} > - \sum_{(e,v) \in \text{Lk}(I)} (\pi - \Phi(e)) + 2\pi \chi(F_I)
\]

for any nonempty proper subset \( I \) of \( V \). They also proved that, the space of all admissible classical curvature \( K \) is

\[
\mathcal{Y} \triangleq \mathcal{X}_{GB} \cap \left( \bigcap_{\phi \neq I \subseteq V} \mathcal{Y}_I \right).
\]

If \( r^* \) determines a constant \( \alpha \)-curvature, then \( K^* = (K_1^*, \ldots, K_N^*) \) is admissible, where

\[
K_i^* = \frac{2\pi \chi(M)}{||r^*||_\alpha} r_i^\alpha.
\]

Hence \( K^* \in \mathcal{Y}_I \), i.e.

\[
2\pi \chi(M) \frac{\sum_{i \in I} r_i^\alpha}{||r^*||_\alpha} > - \sum_{(e,v) \in \text{Lk}(I)} (\pi - \Phi(e)) + 2\pi \chi(F_I)
\]

for any nonempty proper subset \( I \) of \( V \). Notice that (2.47) is necessary for the existence of constant \( \alpha \)-curvature metric, which involves the metric, topology and combinatorial structure of the triangulated surface. However, it is not sufficient for \( r^* \) to be a constant curvature metric since it is “open”, which means that, if one disturbs \( r^* \) slightly to a nonconstant curvature metric, condition (2.47) is still preserved.

For surface with \( \chi(M) < 0 \), discrete maximum principle provides an efficient way to derive a combinatorial and topological condition for the existence of constant curvature metric.

**Theorem 2.34.** Suppose \((M, \mathcal{T}, \Phi)\) is a weighted triangulated surface.

(1) There exists a negative constant curvature metric if and only if \( \mathcal{Y} \cap \mathbb{R}^N_{<0} \neq \phi \).
(2) There exists a zero curvature metric if and only if 0 ∈ ℳ.

(3) If there exists a positive constant curvature metric, then ℳ ∩ ℝ⁺Ｎ ≠ φ.

**Proof.** Zero R-curvature is exactly zero K-curvature, hence the existence of zero curvature metric is equivalent to say zero curvature is admissible, i.e. 0 ∈ ℳ. (3) and the “only if” part of (1) is trivial. We just need to prove the “if” part of (1). Since ℳ ∩ ℝ⁺Ｎ ≠ φ, we can always choose a initial metric r(0) with R(0) < 0, then the combinatorial Ricci flow (2.11) converges to a negative constant curvature metric by Theorem 2.10.

For surface with χ(M) > 0, we don’t know whether condition ℳ ∩ ℝ⁺Ｎ ≠ φ is sufficient. As is explained in section 2.4, consider α-curvature (α < 0) seems better for this case. By generalizing discrete maximal principle to α-curvature, we can get similar existence results for constant α-curvature metric.

**Theorem 2.35.** Suppose (M, T, Φ) is a weighted triangulated surface. If the initial metric r(0) satisfying αR_{α,i}(0) < 0 for all i ∈ V, then the normalized α-Ricci flow (2.39) converges to constant α-curvature metric.

**Proof.** Note that the α-curvature R_{α,i} evolves according to (2.40) along the normalized α-Ricci flow (2.39). The maximum principle, i.e. Theorem 2.6, is valid for this equation. By the maximum principle, if α > 0 and R_{α,i} < 0 for all i ∈ V, we have

\[(R_{α,min}(0) - R_{α,av})e^{αR_{α,av}t} \leq R_{α,i} - R_{α,av} \leq R_{α,av}(1 - \frac{R_{α,av}}{R_{α,max}(0)})e^{αR_{α,av}t}.\]

If α < 0 and R_{α,i} > 0 for all i ∈ V, we have

\[\frac{R_{α,av}}{R_{α,min}(0)}(R_{α,min}(0) - R_{α,av})e^{αR_{α,av}t} \leq R_{α,i} - R_{α,av} \leq (R_{α,max}(0) - R_{α,av})e^{αR_{α,av}t}.\]

In summary, if αR_{α,i} < 0 for all i ∈ V, there exists constants c₁ and c₂ such that

\[c₁e^{αR_{α,av}t} \leq R_{α,i} - R_{α,av} \leq c₂e^{αR_{α,av}t},\]

which implies the exponential convergence of the normalized α-Ricci flow (2.39).

**Corollary 2.36.** Suppose (M, T, Φ) is a weighted triangulated surface. If αχ(M) < 0, then there exists a constant α-curvature metric if and only if ℳ ∩ ℝ⁺Ｎ ≠ φ when α > 0 and ℳ ∩ ℝ⁺Ｎ ≠ φ when α < 0.

If α > 0, the property of constant α-curvature metric seems complicated and confusing as exhibited in Example 2 and Example 3. Even through, we can get some sufficient conditions for the existence of constant α-curvature metric by the maximum principle.
Theorem 2.37. Suppose \((M, T, \Phi)\) is a weighted triangulated surface. If there is a circle packing metric \(r\) satisfying \(R_{\alpha,i} \geq 0\) for all \(i \in V\), and

\[- \sum_{(e,v) \in \text{Lk}(I)} (\pi - \Phi(e)) + 2\pi \chi(F_I) < 0,\]

for any nonempty subset \(I\) of \(V\), then there exists a non-negative constant curvature metric \(r^*\).

Proof. By the maximum principle, i.e. Theorem 2.6, \(R_{\alpha,i} \geq 0\) is preserved along the normalized \(\alpha\)-Ricci flow \(2.39\). Using similar arguments in the proof of Theorem 2.18, we get the proof. \qed

Remark 7. Theorem 2.10 and 2.18 are now special cases of Theorem 2.35 and Theorem 2.37.

3. 3-dimensional combinatorial Yamabe problem

3.1 The definition of combinatorial scalar curvature

Suppose \(M\) is a 3-dimensional compact manifold with a triangulation \(\mathcal{T} = \{V, E, F, T\}\), where the symbols \(V, E, F, T\) represent the sets of vertices, edges, faces and tetrahedrons respectively. A sphere packing metric is a map \(r : V \to (0, +\infty)\) such that the length between vertices \(i\) and \(j\) is \(l_{ij} = r_i + r_j\) for each edge \(\{i, j\} \in E\), and the lengths \(l_{ij}, l_{ik}, l_{il}, l_{jl}, l_{kl}\) determine a Euclidean tetrahedron for each tetrahedron \(\{i, j, k, l\} \in T\).

We can take sphere packing metrics as points in \(\mathbb{R}^N > 0\) \(N\) times Cartesian product of \((0, \infty)\), where \(N = V\#\) denotes the number of vertices. It is pointed out [22] that a tetrahedron \(\{i, j, k, l\} \in T\) generated by four positive radii \(r_i, r_j, r_k, r_l\) can be realized as a Euclidean tetrahedron if and only if

\[Q_{ijkl} = \left(\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_l}\right)^2 - 2 \left(\frac{1}{r_i^2} + \frac{1}{r_j^2} + \frac{1}{r_k^2} + \frac{1}{r_l^2}\right) > 0.\] \hspace{1cm} (3.1)

Thus the space of admissible Euclidean sphere packing metrics is

\[\mathfrak{M}_T = \left\{ r \in \mathbb{R}^N_+ \mid Q_{ijkl} > 0, \forall \{i, j, k, l\} \in T \right\}.\]

Cooper and Rivin [17] called the tetrahedrons generated in this way conformal and proved that a tetrahedron is conformal if and only if there exists a unique sphere tangent to all of the edges of the tetrahedron. Moreover, the point of tangency with the edge \(\{i, j\}\) is of distance \(r_i\) to \(v_i\). They further proved that \(\mathfrak{M}_T\) is a simply connected open subset of \(\mathbb{R}^N_{>0}\), but not convex.
For a triangulated 3-manifold \((M, T)\) with sphere packing metric \(r\), there is also the notion of combinatorial scalar curvature. Cooper and Rivin \([17]\) defined combinatorial scalar curvature \(K_i\) at a vertex \(i\) as angle deficit of solid angles

\[
K_i = 4\pi - \sum_{\{i,j,k,l\} \in T} \alpha_{ijkl},
\]  

(3.2)

where \(\alpha_{ijkl}\) is the solid angle at the vertex \(i\) of the Euclidean tetrahedron \(\{i, j, k, l\} \in T\) and the sum is taken over all tetrahedrons with \(i\) as one of its vertices. \(K_i\) locally measures the difference between the volume growth rate of a small ball centered at vertex \(v_i\) in \(M\) and a Euclidean ball of the same radius. Cooper and Rivin’s definition of combinatorial scalar curvature is motivated by the fact that, in smooth case, the scalar curvature at a point \(p\) locally measures the difference of the volume growth rate of the geodesic ball with center \(p\) to the Euclidean ball \([33, 2]\).

Similar to the two dimensional case, Cooper and Rivin’s definition of combinatorial scalar curvature \(K_i\) is scaling invariant, which is not so satisfactory. The authors \([20]\) once defined a new combinatorial scalar curvature as \(K_i / r_i^2\) on 3-dimensional triangulated manifold \((M, T)\) with sphere packing metric \(r\). Motivated by the analysis in Section 2.1, we find that it is more natural to define the combinatorial scalar curvature in the following way.

**Definition 3.1.** For a triangulated 3-manifold \((M, T)\) with sphere packing metric \(r\), the combinatorial scalar curvature at the vertex \(i\) is defined as

\[
R_i = \frac{K_i}{r_i^2},
\]

(3.3)

where \(K_i\) is given by (3.2).

As \(R_i\) differs from \(K_i\) only by a factor \(1/r_i^2\), \(R_i\) still locally measures the difference between the volume growth rate of a small ball centered at vertex \(v_i\) in \(M\) and a Euclidean ball of the same radius. Furthermore, according to the analysis in Section 2.1, \(r_i^2\) is the analogue of the smooth Riemannian metric. If \(\tilde{r}_i^2 = cr_i^2\) for some positive constant \(c\), we have \(\tilde{R}_i = c^{-1}R_i\). This is similar to the transformation of scalar curvature in smooth case under scaling.

Analogous to the two dimensional case, we can define a measure on the vertices. As \(r_i^2\) is the analogue of the Riemannian metric, we can take the measure as \(\mu_i = r_i^3\), which corresponds to the volume element. Then the total combinatorial scalar curvature is

\[
S = \int_M Rd\mu = \sum_{i=1}^N R_i r_i^3 = \sum_{i=1}^N K_i r_i.
\]

(3.4)
Note that $S$ is just the functional introduced by Cooper and Rivin in [17]. For the total combinatorial scalar curvature $S$, we have the following important property.

**Lemma 3.2.** ([17], [43], [24]) Suppose $(M, T)$ is a triangulated 3-manifold with sphere packing metric $r$, $S$ is the total combinatorial scalar curvature. Then we have

$$\nabla_r S = K.$$ \hspace{1cm} (3.5)

If we set

$$\Lambda = Hess_r S = \frac{\partial(K_1, \cdots, K_N)}{\partial(r_1, \cdots, r_N)} = \begin{pmatrix}
\frac{\partial K_1}{\partial r_1} & \cdots & \frac{\partial K_1}{\partial r_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial K_N}{\partial r_1} & \cdots & \frac{\partial K_N}{\partial r_N}
\end{pmatrix},$$

then $\Lambda$ is positive semi-definite with rank $N - 1$ and the kernel of $\Lambda$ is the linear space spanned by the vector $r$.

It should be emphasized that, as pointed out by Glickenstein [22], the element $\frac{\partial K_i}{\partial r_j}$ for $i \sim j$ maybe negative, which is different from that of the two dimensional case.

By the definition of measure $\mu$, the total measure of $M$ is $\mu(M) = \sum_{i=1}^{N} r_i^3$. We will denote the total measure of $M$ by $V$ for simplicity in the following, if there is no confusion. The average combinatorial scalar curvature is

$$R_{av} = \frac{\int_M R dv g}{\int_M dv g} = \frac{S}{V} = \frac{\sum_{i=1}^{N} K_i r_i}{\sum_{i=1}^{N} r_i^3}.$$ \hspace{1cm} (3.6)

### 3.2 Combinatorial Yamabe problem in 3 dimension

For the curvature $R_i$, it is natural to consider the corresponding constant curvature problem. Suppose $R_i = \lambda, \forall i \in V$, for some constant $\lambda$, then we have $K_i = \lambda r_i^2$, which implies that $\lambda = \frac{S}{V}$. We can take $\frac{S}{V}$ as a functional of the sphere packing metric $r$. However, this functional is not scaling invariant in $r$. So we can modified the functional as $\frac{S}{\sqrt[3]{V}}$. This recalls us of the smooth Yamabe problem.

The classical Yamabe problem aims at solving the problem of existence of constant scalar curvature metric on a closed manifold. In order to study the constant scalar curvature problem on a closed Riemannian manifold $(M, g)$, Yamabe [48] introduced the so-called Yamabe functional $Q(g)$ and the Yamabe invariant $Y_{M,g_0}$, which are defined as

$$Q(g) = \frac{\int_M R dv g}{(\int_M dv g)^{\frac{n-2}{n}}},$$
\[ Y_{M,[g_0]} = \inf_{g \in [g_0]} Q(g), \]

where \([g_0]\) is the conformal class of Riemannian metric \(g_0\). Trudinger \[47\] and Aubin \[1\] made lots of contributions to this problem, and Schoen \[44\] finally gave the solution of the Yamabe Problem. We refer the readers to \[33\] for this problem.

For piecewise flat manifolds, one can introduce similar functionals and invariants. Champion, Glickenstein and Young \[7\] studied Einstein-Hilbert-Regge functionals and related invariants on triangulated manifolds with piecewise linear metrics. The authors \[20\] also introduced a type of combinatorial Yamabe functional and studied its properties. To study the curvature defined by (3.3), we can introduce the following definition of combinatorial Yamabe functional and Yamabe invariant on triangulated 3-manifolds with sphere packing metrics.

**Definition 3.3.** Suppose \((M, T)\) is a triangulated 3-manifold with a fixed triangulation \(T\). The combinatorial Yamabe functional is defined as

\[ Q(r) = \frac{S}{V^{1/3}} = \frac{\sum_{i=1}^{N} K_i r_i}{(\sum_{i=1}^{N} r_i^3)^{1/3}}, \quad r \in \mathcal{M}_T. \tag{3.7} \]

The combinatorial Yamabe invariant with respect to \(T\) is defined as

\[ Y_{M,T} = \inf_{r \in \mathcal{M}_T} Q(r). \]

The admissible sphere packing metric space \(\mathcal{M}_T\) for a given triangulated manifold \((M, T)\) is an analogue of the conformal class \([g_0]\) of a Riemannian manifold \((M, g_0)\), as every admissible sphere packing metric could be taken to be conformal to the metric with all \(r_i = 1\). We call \(\mathcal{M}_T\) the combinatorial conformal class for \((M, T)\). It is uniquely determined by the triangulation \(T\) of \(M\). \(Y_{M,T}\) is referred to as the Yamabe constant for \((M, T)\).

Note that the combinatorial Yamabe invariant \(Y_{M,T}\) is an invariant of the conformal class \(\mathcal{M}_T\) and is well defined, as we have

\[ |Q(r)| = \left| \frac{S}{V^{1/3}} \right| = \left| \frac{\sum K_i r_i}{(\sum r_i^3)^{1/3}} \right| \leq \left( \frac{\sum K_i^{3/2}}{(\sum r_i^3)^{1/3}} \right)^{2} = \|K\|_{3/2}, \tag{3.8} \]

and \(K_i\) is bounded by \((4 - 2d)\pi \leq K_i < 4\pi\), where \(d\) is the maximal degree and \(d \leq E^\#\). If the equality in (3.8) is achieved, the corresponding sphere packing metric must be a constant curvature metric.

By direct computation, we have

\[ \nabla_{r_i} Q = \frac{1}{V^{1/3}} (K_i - R_{av} r_i^2), \tag{3.9} \]

37
which implies that \( r \) is a constant combinatorial scalar curvature metric if and only if it is a critical point of combinatorial Yamabe functional \( Q(r) \).

Analogue to the smooth Yamabe problem, we can raise the following combinatorial Yamabe problem on 3-dimensional triangulated manifold.

**The Combinatorial Yamabe Problem.** Given a 3-dimensional manifold \( M \) with triangulation \( T \), find a sphere packing metric with constant combinatorial scalar curvature in the combinatorial conformal class \( \mathcal{M}_T \).

We could further consider finding a suitable triangulation for \( M \) which admits a constant combinatorial scalar curvature metric.

It is easy to see that the Platonic solids with tetrahedral cells all admits constant combinatorial scalar curvature metric, including the 5-cell, the 16-cell, the 600-cell, etc. In these cases, the constant combinatorial scalar curvature metrics arise from symmetry and taking the radii equal.

### 3.3 Combinatorial Yamabe flow

In 1980s, Hamilton [29, 30] proposed the Yamabe flow to the Yamabe problem. For a closed \( n \)-dimensional Riemannian manifold \( (M^n, g) \) with \( n \geq 3 \), the Yamabe flow is defined to be

\[
\frac{\partial}{\partial t} g_{ij} = -R g_{ij}
\]

with normalization

\[
\frac{\partial}{\partial t} g_{ij} = (r - R) g_{ij},
\]

where \( R \) is the scalar curvature of \( g \) and \( r \) is the average of the scalar curvature. Along the normalized Yamabe flow, the volume is invariant, the total scalar curvature is decreased, and the scalar curvature evolves according to

\[
\frac{\partial R}{\partial t} = (n - 1) \Delta R + R(R - r).
\]

It is proved by Hamilton [30] that the solution to the Yamabe flow (3.11) exists for all time and the solution converges exponentially fast to a metric with constant scalar curvature if \( R < 0 \) initially. Ye [49] then proved that, if the initial metric is locally conformally flat, then the solution to the normalized Yamabe flow (3.11) converges to a metric of constant scalar curvature. Brendle [4] proved that the solution of (3.11) also converges to a metric of constant curvature if \( 3 \leq n \leq 5 \). He [5] further handled the case of \( n \geq 6 \) and got some convergence results.
In the combinatorial case, Luo [34] first introduced the combinatorial Yamabe flow on surfaces and Glickenstein [22, 23] introduced the combinatorial Yamabe flow on 3-dimensional manifolds and studied its related properties. Luo [35] also introduced a combinatorial curvature flow for piecewise constant curvature metrics on compact triangulated 3-manifolds with boundary consisting of surfaces of negative Euler characteristic. To study the constant curvature problem of \( R_i \), we introduce the following combinatorial Yamabe flow.

**Definition 3.4.** Given a 3-dimensional triangulated manifold \((M, T)\) with sphere packing metric \( r \), the combinatorial Yamabe flow is defined to be

\[
\frac{dg_i}{dt} = -R_i g_i, \tag{3.13}
\]

with normalization

\[
\frac{dg_i}{dt} = (R_{av} - R_i) g_i, \tag{3.14}
\]

where \( g_i = r_i^2 \) and \( R_{av} \) is the average of the combinatorial scalar curvature given by (3.6).

Following the 2-dimensional case, it is easy to check that the solutions of (3.13) and (3.14) can be transformed to each other by a scaling procedure. And it is easy to see that, if the solution of (3.14) converges to a sphere packing metric \( r(\pm \infty) \), then \( r(\pm \infty) \) is a metric with constant combinatorial scalar curvature. Analogous to the smooth Yamabe flow, we have the following properties.

**Proposition 3.5.** Suppose \( r(t) \) is a solution of (3.14) on a triangulated 3-manifold \((M, T)\). Along the flow (3.14), the total measure \( \mu(M) = V = \sum_{i=1}^{N} r_i^3 \) is invariant and the total combinatorial scalar curvature \( S \) is decreased.

**Proof.** The normalized combinatorial Yamabe flow could be written as

\[
\frac{dr_i}{dt} = \frac{1}{2} (R_{av} - R_i) r_i.
\]

Using this equation, we have

\[
\frac{dV}{dt} = 3 \sum_{i=1}^{N} r_i^2 \frac{dr_i}{dt} = \frac{3}{2} \sum_{i=1}^{N} r_i^3 (R_{av} - R_i) = 0.
\]
For the total combinatorial scalar curvature $S$, we have
\[
\frac{dS}{dt} = \sum_{i=1}^{N} \frac{dK_i}{dt} r_i + \sum_{i=1}^{N} K_i \frac{dr_i}{dt}
= \frac{1}{2} \sum_{i,j=1}^{N} r_i \frac{\partial K_i}{\partial r_j} (R_{\text{av}} - R_j) + \frac{1}{2} \sum_{i=1}^{N} K_i r_i (R_{\text{av}} - R_i)
= -\frac{1}{2} \sum_{i=1}^{N} \frac{K_i (K_i - R_{\text{av}} r_i^2)}{r_i}
= -\frac{1}{2} \sum_{i=1}^{N} \frac{(K_i - R_{\text{av}} r_i^2)^2}{r_i},
\]
which implies that $S$ is decreased along the flow \eqref{3.14}. Note that Lemma \ref{3.2} is used in the third step. 

\textbf{Remark 8.} As the total measure of $M$ is invariant along \eqref{3.14}, we will focus on the properties of \eqref{3.14} in the following. Furthermore, we will assume $r(0) \in S^{N-1} = \{ r \in \mathbb{R}^N; ||r||_3 = (\sum r_i^3)^{\frac{1}{3}} = 1 \}$ in the following.

By direct calculations, we find that the curvature $R_i$ evolves according to the following equation along the normalized combinatorial Yamabe flow \eqref{3.14}
\[
\frac{dR_i}{dt} = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial K_i}{\partial r_j} r_j R_j + R_i (R_i - R_{\text{av}}).
\]
(3.16)

If we define the Laplacian as
\[
\Delta f_i = -\frac{1}{r_i^2} \sum_{j=1}^{N} \frac{\partial K_i}{\partial r_j} r_j f_j = \frac{1}{r_i^2} \sum_{j \sim i} (-\frac{\partial K_i}{\partial r_j}) (f_j - f_i)
\]
for $f \in C(V)$, then the equation \eqref{3.16} could be written as
\[
\frac{dR_i}{dt} = \frac{1}{2} \Delta R_i + R_i (R_i - R_{\text{av}}),
\]
(3.18)
which has almost the same form as the evolution equation \eqref{3.12} of scalar curvature along the Yamabe flow \eqref{3.11} in three dimension. The Laplacian defined by \eqref{3.17} satisfies $\int_M \Delta f d\mu = 0$ and $\Delta c = 0$ for any $f \in C(V)$ and constant $c \in \mathbb{R}$. Furthermore, by the calculations in \cite{22}, we have
\[
\Delta f_i = \frac{1}{r_i^2} \sum_{j \sim i} l_{ij}^* (f_j - f_i),
\]
(3.19)
where \( l_{ij}^* \) is the area dual to the edge \( \{i, j\} \). Note that this form of Laplacian is very similar to Hirani’s definition \cite{31} of Laplace-Beltrami operator

\[
\Delta f_i = \frac{1}{V_i^*} \sum_{j \sim i} l_{ij}^* (f_j - f_i),
\]

except the first factor. It should be mentioned that \( r_i^3 \) is a type of volume.

Though the definition (3.17) of Laplacian has lots of good properties, it is not a Laplacian on graphs in the usual sense, as \( l_{ij}^* \) may be negative, which makes the maximum principle for (3.18) not so good. This was founded by Glickenstein and he studied the properties of such Laplacian closely in \cite{22, 23}.

To study the long time behavior of (3.14), we need to classify the solutions of the flow.

**Definition 3.6.** A solution \( r(t) \) of the combinatorial Yamabe flow (3.14) is nonsingular if the solution \( r(t) \) exists for \( t \in [0, +\infty) \) and \( \{r(t)\} \subset \mathcal{M}_T \cap S^{N-1} \).

In fact, the condition \( \{r(t)\} \subset \mathcal{M}_T \cap S^{N-1} \) ensures the long time existence of the flow (3.14). Furthermore, we have the following property for nonsingular solutions.

**Theorem 3.7.** If there exists a nonsingular solution for the flow (3.14), then there exists at least one sphere packing metric with constant combinatorial scalar curvature \( R_i \) on \( (M, \mathcal{T}) \).

**Proof.** By (3.15), we know that \( Q(r) \) is decreasing along the flow (3.14). As \( Q(r) \) is uniformly bounded by (3.8), the limit \( \lim_{t \to +\infty} Q(r(t)) \) exists. Then there exists a sequence \( t_n \uparrow +\infty \) such that

\[
(Q(r))'(t_n) = -\frac{1}{2} \sum_{i=1}^{N} \frac{(K_i - R_{av} r_i^2)^2}{r_i} \to 0.
\]  

(3.20)

As \( \{r(t)\} \subset \mathcal{M}_T \cap S^{N-1} \), there exists a subsequence, denoted as \( r_n \), of \( r(t_n) \) such that \( r_n \to r^* \in \mathcal{M}_T \cap S^{N-1} \). Then (3.20) implies that \( r^* \) satisfies \( K_i^* = R_{av}(r_i^*)^2 \) and \( r^* \) is a sphere packing metric with constant combinatorial scalar curvature \( R_i \). \( \square \)

In fact, under some conditions, we find that the sphere packing metric with constant combinatorial scalar curvature is discrete on \( S^{N-1} \).

**Theorem 3.8.** The sphere packing metrics with nonpositive constant combinatorial scalar curvature are isolated in \( \mathcal{M}_T \cap S^{N-1} \).
Proof. We define the map

\[ G : \mathcal{M}_T \to \mathbb{R}^N \]

\[ r \mapsto \left( \frac{1}{r_1}(K_1 - \lambda r_1^2), \ldots, \frac{1}{r_N}(K_N - \lambda r_N^2) \right), \]

where \( \lambda = \frac{S}{V} \). It is easy to see that the zero point of \( G \) corresponds to the metric with constant combinatorial scalar curvature. By direct calculations, the Jacobian of \( G \) at the constant curvature metric is

\[ DG = \left( \Lambda - 2\lambda \Sigma^\frac{1}{2} \left( I - \frac{r^{3/2} \cdot (r^{3/2})^T}{V} \right) \Sigma^{\frac{1}{2}} \right) \Sigma^{-\frac{1}{2}}, \tag{3.21} \]

where \( r^{3/2} = (r_1^{3/2}, \ldots, r_N^{3/2})^T \). If \( r^* \) is a sphere packing metric with nonpositive constant combinatorial scalar curvature, then \( \lambda \leq 0 \) and hence \( DG \) is a positive semi-definite matrix with rank \( N - 1 \) and kernel \( tr^2 \), which is the normal of \( S^{N-1} \). Restricted to \( S^{N-1} \), \( DG \) is positive definite and then nondegenerate, which implies that the zero point of \( G \) with nonpositive curvature is isolated in \( \mathcal{M}_T \cap S^{N-1} \). \( \square \)

Suppose the solution of the flow (3.14) exists for \( t \in [0, T) \) with \( 0 < T \leq +\infty \), then we have \( \{r(t)\} \subseteq \mathcal{M}_T \cap S^{N-1} \). However, if

\[ \{r(t)\} \cap \partial(\mathcal{M}_T \cap S^{N-1}) \neq \emptyset, \]

the flow will raise singularities. They could be separated into two types as follows.

**Essential Singularity** There is a vertex \( i \in V \) such that there exists a sequence of time \( t_n \uparrow T \) such that \( r_i(t_n) \to 0 \) as \( n \to +\infty \);

**Removable Singularity** There exists a sequence of time \( t_n \uparrow T \) such that \( r_i(t_n) \subset \subset \mathbb{R}_{>0} \) for all vertices \( i \in V \), but there exists a tetrahedron \( \{ijkl\} \in T \) such that \( Q_{ijkl}(t_n) \to 0 \) as \( n \to +\infty \).

It seems that the following conjectures are likely to hold for the normalized combinatorial Yamabe flow (3.14) on 3-dimensional manifolds.

**Conjecture 1.** the normalized combinatorial Yamabe flow (3.14) will not develop essential singularity in finite time.

**Conjecture 2.** If no singularity develops along the normalized flow (3.14), the solution converges to a sphere packing metric with constant combinatorial scalar curvature as time approaches infinity.
It is interesting to note that Glickenstein \cite{23} made a small amount of progress on Conjecture 1. He proved that, for some special class of complexes, the flow he introduced would not develop singularities in finite time.

We have mentioned above that the existence of sphere packing metric with constant combinatorial scalar curvature is necessary for the convergence of the normalized combinatorial Yamabe flow \cite{3.14}. In fact, it is almost sufficient. And we have the following result.

**Theorem 3.9.** Suppose $r^*$ is a sphere packing metric on $(M, T)$ with nonpositive constant combinatorial scalar curvature. If $| | R(0) - R^* ||^2$ is small enough, the solution of the normalized combinatorial Yamabe flow \cite{3.14} exists for all time and converges to $r^*$.

**Proof.** We can rewrite the normalized combinatorial Yamabe flow \cite{3.14} as
\[
\frac{dr_i}{dt} = \frac{1}{2} (R_{av} - R_i) r_i.
\]
Set $\Gamma_i(r) = \frac{1}{2} (R_{av} - R_i) r_i$, then by the calculations in the proof of Theorem \cite{3.8}, we have
\[
D\Gamma|_{r^*} = -\frac{1}{2} Z \begin{pmatrix}
\frac{1}{r_1} & & \\
& \ddots & \\
& & \frac{1}{r_N}
\end{pmatrix},
\]
where
\[
Z = \Lambda - 2 R_{av} \begin{pmatrix}
\frac{1}{r_1} & & \\
& \ddots & \\
& & \frac{1}{r_N}
\end{pmatrix} \begin{pmatrix}
I - \frac{r_3/2 \cdot (r_3/2) I}{V} & \\
& \ddots & \\
& & \frac{1}{r_N}
\end{pmatrix}.
\]
If $R_{av} \leq 0$, then $D\Gamma|_{r^*}$ is a matrix with rank $N - 1$ and kernel $\{t r^2 | t \in \mathbb{R} \}$. Furthermore, the nonzero eigenvalues of $D\Gamma|_{r^*}$ are all negative. Note that, along the normalized flow \cite{3.14}, $||r||_3$ is invariant and thus the kernel $\{t r^2 | t \in \mathbb{R} \}$ is transversal to the flow. This implies that $D\Gamma|_{r^*}$ is negative definite on $S^{N-1}$ and $r^*$ is a local attractor of the normalized combinatorial Yamabe flow \cite{3.14}. Then the conclusion follows from the Lyapunov Stability Theorem. \qed

This theorem has the following interesting corollary.

**Corollary 3.10.** Given a 3-dimensional triangulated manifold $(M^3, T)$. If the initial total combinatorial scalar curvature functional $S(0) \leq 0$, and no singularity develops along the normalized combinatorial Yamabe flow \cite{3.14}, then the solution converges to a sphere packing metric $r^*$ with nonpositive constant combinatorial scalar curvature.
Proof. By the proof of Theorem 3.7, if the solution $r(t)$ of (3.14) is nonsingular, there exists a sequence $r_n \to r^*$, where $r^*$ is a sphere packing metric with constant combinatorial scalar curvature. As $S$ is decreasing along the flow (3.14) and $S(0) \leq 0$, $r^*$ has nonpositive combinatorial scalar curvature. Then Theorem 3.9 implies the conclusion of the corollary.

Remark 9. Especially, if $R_i(0) \leq 0$, Corollary 3.10 is still valid.

4 Some questions

There are several questions which we find interesting relating to the results in the paper.

1. From the results in Section 2, we see that the surfaces with positive Euler characteristic are particular for the studying of constant curvature problem on surfaces. Example 2 shows that combinatorial Ricci flow (1.2) may not converge to constant curvature metric, while Example 3 shows that constant curvature metric may not be unique. If we want to approximate a smooth geometric object by corresponding discrete object, the quantities on these objects should obey similar laws or exhibit similar properties. However, it has been proved [29, 12] that the Ricci flow on surfaces converges to a constant curvature metric for any initial metric. The differences between discrete case and the smooth case are very interesting. It deserve deeper studying. Maybe these differences are caused by triangulation. The tetrahedron triangulation used in Example 2 and 3 maybe too rough to approximate the sphere. We expect these differences will disappear as the triangulation becomes finer and finer. We want to know whether we can find a triangulation for the sphere such that the constant curvature metric uniquely exists. We further want to know how to evolve discrete curvature $R_i = K_i/r_i^2$ along discrete curvature flows to constant curvature or, more generally, evolve discrete $\alpha$-curvature to constant curvature when $\alpha \chi(M) > 0$.

2. Whether there are similar topological and combinatorial obstructions, similar to Thurston’s criterion (2.45), for the existence of constant $\alpha$-curvature metric? For negative constant curvature metric, we derived an existence result (Theorem 1.1) by discrete maximum principle, which is different from the other results. We want to further develop these methods to derive more existence results. We want to know whether we can get similar convergence results as that of the smooth surface Ricci flow and then give a discrete uniformization theorem for $R_i$. We believe that the results in [3] will play an important role in the procedure if it is feasible. We also want to know the relationship between our conditions and Thurston’s criterion (2.45).
3. The eigenvalue estimation of $\Lambda = \Sigma^{-\frac{1}{2}}L\Sigma^{-\frac{1}{2}}$ plays an important role in the proof of the main results in two dimension. And $\Lambda$ is closely related to the discrete Laplace operator $\Delta = -\Sigma^{-1}L$. It is interesting to estimate the first positive eigenvalue of discrete Laplace operator, i.e. $\lambda_1(-\Delta) = \lambda_1(\Sigma^{-1}L) = \lambda_1(\Lambda)$ and use it to derive some existence results.

4. Investigate the prescribing curvature problem when the target curvature $\overline{R_i} > 0$ or, more generally, $a\overline{R_{\alpha,i}} > 0$ for at least one vertex $i$.

5. Study the singularities of three dimensional combinatorial Yamabe flow (1.5), and find topological and combinatorial obstructions for the existence of sphere packing metric with constant combinatorial scalar curvature.

6. Study the discrete Yamabe functional (3.3) and solve the combinatorial Yamabe problem in three dimension.

7. Study the combinatorial Yamabe problem in higher dimensions.

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