On equality of derival and inner automorphisms of some p-groups

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Abstract: For a group $G$, $D(G)$ denotes the group of all derivation automorphisms of $G$. For a finite nilpotent group of class 2, it is shown that $D(G) \cong \text{Hom}(G/Z_2(G), Z_2(G))$. We prove that if $G$ is a nilpotent group of class $\geq 3$ such that $Z(G) \subseteq Z_2(G)$ and $D(G/Z_2(G)) = \text{Inn}(G/Z_2(G))$, then $D(G) = \text{Inn}(G)$ if and only if $\text{Aut}(G) = Z(\text{Inn}(G))$. Finally, for an odd prime $p$, we classify all p-groups of order $p^n$, $1 \leq n \leq 5$, for which $D(G) = \text{Inn}(G)$.

Keywords: automorphism; inner automorphism; class preserving automorphism; derivation automorphism; central automorphism; p-groups; nilpotent groups; isoclinic groups

1. Introduction

Let $G$ be a group. Notations used are standard, however, for the sake of completeness, by $e$ we denote the identity element of $G$. For $x, y \in G$, $x^y$ denotes the conjugate element $y^{-1}xy$ and $[x, y] = x^{-1}y^{-1}xy$ is the commutator of $x$ and $y$. $x^G$ denotes the conjugacy class of $x$ in $G$. The subgroup generated by the set of all commutators of $G$ is called derived group of $G$ and it is denoted by $[G, G]$ or $G'$ or $\gamma_2(G)$. $[x, G]$ denotes the set of all commutators $[x, g]$ for $g \in G$. Note that $x^g = x[x, g]$ for every $g \in G$ and $x^G = x[x, G]$. An endomorphism $f: G \rightarrow G$ is called a class preserving if for each $x \in G$, $f(x) \in x^G$. Note that if $f$ is a class-preserving endomorphism of $G$, then $x^{-1}f(x) \in [x, G]$. By $\text{Hom}_c(G)$, we denote the set $\{f \in \text{End}(G) | f(x) \in [x, G] \text{ for each } x \in G\}$. An automorphism $f: G \rightarrow G$ is called a class-preserving automorphism if for each $x \in G$, $f(x) \in x^G$. Note that the inner automorphism $T_{\gamma_2}(G): G \rightarrow G$ given by $T_{\gamma_2}(x) = x^{-1}xa$, for all $x \in G$, is a particular example of a class-preserving automorphism.

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PUBLIC INTEREST STATEMENT

Groups are the most basic algebraic structure which is the building block of modern algebra. Groups handle many practical problems like symmetries of objects and various problems of combinatorics. Automorphism group is one of the most fascinating object associated with a group. In recent past, many researchers proposed different automorphism groups and their equality has been established with each other. This article is one in this sequence. The notion of derivation automorphism group is introduced and its equality with the groups of inner automorphisms and class-preserving automorphisms is discussed. Results obtained are very fundamental in nature and will be quite helpful for researchers working in group theory.
group of all class-preserving automorphisms is denoted by $\text{Aut}_c(G)$. The group of all inner automorphisms of $G$ is denoted by $\text{Inn}(G)$ and it is a normal subgroup of $\text{Aut}_c(G)$. $\text{Out}_c(G)$ denotes the quotient group $\text{Aut}_c(G)/\text{Inn}(G)$. Let $N$ be a characteristic subgroup of $G$. Then each $\alpha \in \text{Aut}(G)$, induces an automorphism $\overline{\alpha}: G/N \to G/N$ given by $\overline{\alpha}(xN) = \alpha(x)N$. Thus the map $\theta: \text{Aut}(G) \to \text{Aut}(G/N)$ given by $\theta(\alpha) = \overline{\alpha}$ is a homomorphism of groups. The kernel of this homomorphism is precisely those automorphisms of $G$ which are identity on $G/N$. If we take $N = G$, then $\text{Ker}\theta$ is the group of all those automorphisms of $G$ which are identity on $G$. This group is abbreviated as $D(G)$ and elements of this group are called derial automorphisms of $G$ (Chiş, 2002). An automorphism of $G$ is called central if it is identity on $G/Z(G)$. The set of all central automorphisms of $G$ is a normal subgroup of $\text{Aut}(G)$ and it is denoted by $\text{Aut}_c(G)$. It has been shown by Sah (1968) that $\text{Aut}_c(G) = C_{\text{Aut}(G)}(\text{Inn}(G))$. In the recent past, interest of many mathematicians turned on the equalities of various automorphism groups viz. equalities of $\text{Aut}_c(G)$ and $\text{Inn}(G)$, $\text{Aut}_c(G)$ and $Z(\text{Inn}(G))$ and $\text{Aut}_c(G)$ and $\text{Out}_c(G)$ etc. Curran and McCaughan (2001) showed that if $G$ is a finite $p$-group, then $\text{Aut}_c(G) = \text{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. Further extending this work, Curran (2004) observed that $\text{Aut}_c(G)$ is minimum possible when $\text{Aut}_c(G) = Z(\text{Inn}(G))$ and he found that if $\text{Aut}_c(G) = Z(\text{Inn}(G))$, then $Z(G) \subseteq G$ and $Z(\text{Inn}(G))$ must be cyclic. Gumber and Sharma (2011) proved that if $G$ is a nilpotent group of class 2, then $\text{Aut}_c(G) = Z(\text{Inn}(G))$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. For a finite $p$-group, Jafari (2011) find out the necessary and sufficient condition when every central automorphism fix $Z(G)$ element-wise. Jain (2012) studied those finite $p$-groups for which $\text{Aut}(G) = \text{Aut}_c(G)$. Kalra and Gumber (2013) characterize all finite $p$-groups of order $p^n$, $1 \leq n \leq 7$, such that $\text{Aut}_c(G) = \text{Aut}_c(G)$. Further Yadav (2013) proved that if $G$ is a finite $p$-group such that $\text{Aut}_c(G) = \text{Aut}_c(G)$, then $G$ has even number of elements in any minimal generating set for $G$. Ghoraishi (2015) find out the necessary and sufficient condition for equality of class preserving and central automorphism of a finite group.

Kumar and Verma (2000, 2001) show that if $G$ is a group of order $p^5$, $1 \leq n \leq 4$, then $\text{Aut}_c(G) = \text{Inn}(G)$. In another note Yadav (2008) studied class-preserving automorphisms of group of order $p^5$, $p$ an odd prime and proved that $\text{Aut}_c(G) = \text{Inn}(G)$ for all groups $G$ of order $p^5$ except two isoclinism families. On the classification for group of order $p^6$ given by James (1980), Narain and Karan (2014) studied those groups of order $p^6$ for which $\text{Aut}_c(G) = \text{Inn}(G)$.

If $\phi$ is either an inner automorphism or a class-preserving automorphism of $G$, then one should note that for all $x \in G$, $x^{-1}\phi(x) \in [x, G] \subseteq G$, whereas if $\phi$ is a derivational automorphism of $G$, then for all $x \in G$, $x^{-1}\phi(x) \in G$. This shows that $\text{Inn}(G) \subseteq \text{Aut}_c(G) \subseteq D(G)$.

This motivate us to study those $p$-groups for which $D(G)$ coincides with $\text{Aut}_c(G)$ or $\text{Inn}(G)$. This paper is an attempt to study some $p$-groups for which $D(G) = \text{Aut}_c(G)$. One quite natural situation when $D(G) = \text{Aut}_c(G)$ is that $D(G) = \text{Inn}(G)$. Since for a $p$-group of order $p^5$, $1 \leq n \leq 4$, $\text{Aut}_c(G) = \text{Inn}(G)$, to establish the equality of $D(G)$ with $\text{Aut}_c(G)$ it is sufficient to show that $D(G) = \text{Inn}(G)$. We characterize all finite $p$-groups of class 2 for which $D(G) = \text{Inn}(G)$. For an odd prime $p$, we also classify those groups of order $p^3$ for which $D(G) = \text{Inn}(G)$.

2. Preliminaries and definitions

This section deals with some of the basic definitions and results which are used further.

**Definition 1** For a group $G$, the sequence $\{Z_i\}_{i=0}^\infty$ of subgroups of $G$ defined as follows

$$Z_0 = \{1\} \text{ and for } i > 0, \ Z_i/Z_{i-1} = [G/Z_{i-1}],$$

is called the upper central series of $G$; its $i$-th term is called the $i$-th center of $G$. Here $Z_1 = Z(G)$, the center of $G$. A group $G$ is said to be nilpotent if $Z_m = G$, for some positive integer $m$. The smallest integer $c$ such that $Z_c(G) = G$, is called the nilpotency class of $G$. 


Definition 2. For a group $G$, the sequence $\gamma_i(G)_{i \geq 1}$ of subgroups of $G$, defined by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$, is called the lower central series of $G$. Here $\gamma_1(G) = G$, the derived group of $G$.

A group $G$ is nilpotent if $\gamma_m(G) = \{e\}$, for some positive integer $m$. The smallest integer $c$ such that $\gamma_{c+1}(G) = \{e\}$, is called the nilpotency class of $G$.

Definition 3. A group $G$ is called nilpotent group of class 2 if $G$ has a lower central series of the form $G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \gamma_3(G) = \{e\}$.

The quaternion group $Q_8$ is an example of a nilpotent group of class 2. In fact every non-abelian group of order, $p^3$ is a nilpotent group of class 2.

Definition 4. A group is called a Camina group if and only if $G' \subseteq \{x, G\}$ for each $x \in G - G'$.

Every abelian group is a Camina group trivially. $Q_8 = \langle a, b | a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$, the quaternion group is a non-abelian p-group of order 8 with $|Z(Q_8)| = p$. Note that derived group of $Q_8$ coincides with the center and it is a non-abelian Camina p-group (for proof see Lemma 2.1).

Definition 5. Let $G$ be a group in which each element is of finite order, then the exponent of $G$ is the least common multiple of orders of all elements and it is denoted by $\text{exp}(G)$.

Following results are important for further study.

**Lemma 2.1** Let $G$ be a p-group of class 2. Then for each $x \in G$, $\langle x, G \rangle$ is a subgroup of $G$. Moreover if $|G| = |Z(G)| = p$, then $G$ is a Camina p-group.

**Proof** Let $G$ be a nilpotent group of class 2. Then $G' \subseteq Z(G)$. For $x \in G$, consider $H = \langle x, G \rangle$.

Let $\langle x, g_1, x, g_2 \rangle \in H$. Then $\langle x, g_1, x, g_2 \rangle = \langle x, g_1 \rangle \langle x, g_2 \rangle$. Since $G' \subseteq Z(G)$, $\langle x, g_1, x, g_2 \rangle = \langle x, g_1 \rangle \langle x, g_2 \rangle$. This shows that $\langle x, g_1, x, g_2 \rangle \in H$. Also for $(x, g) \in H$, $e = \langle x, e \rangle = \langle x, g^{-1} \rangle = \langle x, g^{-1} \rangle$. This shows that $\langle x, g^{-1} \rangle \in H$. Hence $\langle x, G \rangle$ is a subgroup of $G$. Let $x \in G - G'$. Then $\langle x, G \rangle$ is a non-trivial subgroup of $G$. Since $G'$ is a prime, then $G = \langle x, G \rangle$ and hence $G$ is a Camina group.

**Theorem 2.2** (Yadav, 2008) Let $G$ be a finite nilpotent group of class 2.

Then $\text{Aut}(G) \cong \text{Hom}(G/Z(G), \gamma_2(G))$, where $\text{Hom}(G/Z(G), \gamma_2(G))$ is the group $\{f \in \text{Hom}(G/Z(G), \gamma_2(G)) | f(xZ(G)) = \langle x, G \rangle \text{ for all } x \in G\}$.

**Theorem 2.3** (Yadav, 2008) Let $G$ be a finite p-group of class 2 such that $\gamma_2(G)$ is cyclic. Then, $\text{Out}(G) = 1$, i.e., $\text{Aut}(G) = \text{Inn}(G)$.

**Lemma 2.4** (Curran & McCaughan, 2001) Let $G$ be a nilpotent group of class 2. Let $Z(G), G / Z(G)$ and $G'$ have ranks $z, r$ and $d$ respectively. Then

(i) $|\text{Hom}(G/Z(G), G')| \geq |G/Z(G)|^{p^{(d-1)}}$.

(ii) $|\text{Hom}(G/Z(G), Z(G))| \geq |G/Z(G)|^{p^{(r-1)}}$.

**3. Nilpotent groups of class 2**

If $G$ is an abelian group, then derived group $G'$ for such a group is trivial and hence $\text{Inn}(G), \text{Aut}(G)$ and $D(G)$ contain merely the identity automorphism. Since abelian groups are precisely nilpotent group of class 1, this motivate us to study $D(G)$ for nilpotent groups of class 2.

**Theorem 3.1** Let $G$ be a finite nilpotent group of class 2. Then $D(G) \cong \text{Hom}(G/\gamma_2(G), \gamma_2(G))$. 

Proof Let $G$ be a nilpotent group of class 2. Then for each $f \in \text{D}(G)$, the map $\theta : G \rightarrow \gamma_2(G)/\gamma_4(Z(G))$ defined by $\theta_i(x) = x^{-1}f(x)$, is a homomorphism of groups. Since $\theta_i$ sends elements of $\gamma_2(G)$ to 1, it induces homomorphism $\overline{\theta_i} : G/\gamma_2(G) \rightarrow \gamma_4(Z(G))$ given by $\overline{\theta_i}(x\gamma_2(G)) = x^{-1}f(x)$. Thus we have the map $\alpha : \text{D}(G) \rightarrow \text{Hom}(G/\gamma_2(G), \gamma_4(Z(G)))$ given by $\alpha(f) = \overline{\theta_i}$. Let $f, g \in \text{D}(G)$ and $x \in G - \gamma_2(G)$. Then there exists $a \in \gamma_2(G)$ such that $g(x) = xa$. Since $\theta_i$ sends elements of $\gamma_2(G)$ to 1,

$$
\overline{\theta_i}g(x\gamma_2(G)) = x^{-1}f(g(x)) \\
= x^{-1}f(g(x)) \\
= x^{-1}f(xa) \\
= x^{-1}f(x)f(a) \\
= x^{-1}f(x)a a^{-1}f(a) \\
= x^{-1}f(x)a \theta_i(a) \\
= x^{-1}f(x)a \\
= x^{-1}f(x)x^{-1}g(x) \\
= \overline{\theta_i}(x\gamma_2(G)) \overline{\theta_i}(x\gamma_2(G))
$$

This shows that $\alpha$ is a homomorphism of groups.

Let $f \in \text{Hom}(G/\gamma_2(G), \gamma_4(Z(G))$. Then $f(x\gamma_2(G)) \in \gamma_4(Z(G))$.

Define a map $\phi : G \rightarrow G$ by $\phi(x) = xf(x\gamma_2(G))$. It is easy to see that $\phi$ is an endomorphism of $G$. Now $x^{-1}\phi(x) = f(x\gamma_2(G)) \in \gamma_4(Z(G))$. Let $x \in \ker(\phi)$. Then there are only two possibilities that either $x \in G - \gamma_2(G)$ or $x \in \gamma_2(G)$. Note that if $x \in G - \gamma_2(G)$, then $f(x\gamma_2(G)) \neq x^{-1}$, otherwise $x \in \gamma_2(G)$. Thus $x \in \ker(\phi)$ if and only if $x \in \gamma_2(G)$. But then $1 = \phi(x) = xf(x\gamma_2(G)) = xf(\gamma_2(G)) = x$. This shows that $\ker(\phi) = \{1\}$ and hence $\phi$ is a monomorphism. Since $G$ is a finite group, $\phi$ is an automorphism of $G$. Thus $\phi \in \text{D}(G)$.

Now $\overline{\theta_i}(x\gamma_2(G)) = x^{-1}f(x) = f(x\gamma_2(G))$. Thus $\overline{\theta_i} = f$. Since $\alpha(\phi) = \overline{\theta_i} = f$, $\alpha$ is an epimorphism. It is fairly easy to check that $\ker(\alpha) = \{1\}$. Thus $\alpha$ is an isomorphism and hence $D(G) \cong \text{Hom}(G/\gamma_2(G), \gamma_4(Z(G)))$.

Lemma 3.2 Let $G$ be a $p$-group of class 2. Then, the order of each non-trivial element $xZ(G)$ in $G/Z(G)$ is equal to the exponent of the subgroup $[x, G]$.

Proof Let $x \in G - Z(G)$ and $\exp(xZ(G)) = p^n$. If $|xZ(G)| = p^n$. Then $x^{p^n} \in Z(G)$ and hence $|x^{p^n}a| = 1$ for all $a \in G$. But then $|x, a|p^n = 1$ and hence $p^n \leq p^n$ i.e. $n \leq m$. Since $\exp(xZ(G)) = p^n$, for each $a \in G$ $1 = |x, a|p^n = |x^{p^n}a|$. This shows that $x^{p^n} \in Z(G)$ and hence $|xZ(G)| \leq p^n$ i.e. $m \leq n$. Thus $|xZ(G)| = \exp(x, G)$.

It is well known that in a finite $p$-group of class 2, $\exp(\gamma_2(G)) = \exp(G/Z(G))$.

Let $G$ be a $p$-group of order $p^r$. Let $\{x_1, x_2, \ldots, x_r\}$ be a minimal generating set for $G$. If $|\gamma_2(G)| = p^n$, then by Burnside basis theorem $r \leq n - m$. The following remarkable theorem (Theorem 5.1, Yadav, 2007) is quite useful in our context.

Theorem 3.3 Let $G$ be a finite $p$-group. If $|\text{Aut}_c(G)| = p^{m-n-r}$, then $G$ is either abelian $p$-group or a non-abelian Camina special $p$-group.

Theorem 3.4 Let $G$ be a finite $p$-group of class 2 such that $\text{Aut}_c(G) = D(G)$. Then the following holds

1. $|\text{Hom}(G/G', G')| = |\text{Hom}_c(G/Z(G), G')| = |\text{Hom}(G/Z(G), G')|
2. If $\{x_1, x_2, \ldots, x_r\}$ be a minimal generating set for $G$, then
   a. $\text{Hom}(G/Z(G), G') = \text{Hom}(G/Z(G), [x_i, G])$ for each $G = [x_i, G]$, for each $i, 1 \leq i \leq r$. 

Proof

(1) Since for a finite $p$-group of class 2, $D(G) \cong \text{Hom}(G/G', G')$ and $\text{Aut}_c(G) \cong \text{Hom}_c(G/Z(G), G')$. Thus

$$|\text{Hom}(G/Z(G), G')| \geq |\text{Hom}_c(G/Z(G), G')| = |\text{Hom}(G/G', G')| \geq |\text{Hom}(G/Z(G), G')|$$

But then

$$|\text{Hom}(G/Z(G), G')| = |\text{Hom}_c(G/Z(G), G')| = |\text{Hom}(G/G', G')|.$$

(2) Let $(x_1, x_2, \ldots, x_r)$ be a minimal generating set for $G$. Then $G/Z(G) = \langle x_1^{-1}, x_2^{-1}, \ldots, x_r^{-1} \rangle$.

(a) Since $|\text{Hom}(G/Z(G), G')| = |\text{Hom}_c(G/Z(G), G')|$, it follows that $\prod_{i=1}^{r} |\text{Hom}(\langle x_i^{-1} \rangle, G')| = \prod_{i=1}^{r} |\text{Hom}(\langle x_i^{-1} \rangle, [x_i, G])|$. But then $|\text{Hom}(\langle x_i^{-1} \rangle, G')| = |\text{Hom}(\langle x_i^{-1} \rangle, [x_i, G])|$.

(b) Since for a finite $p$-group of class 2, $\exp(G/Z(G)) = \exp(G/Z(G))$, from part (a), it follows that for each $i$, $1 \leq i \leq r$, $G' = [x_i, G]$.

\[\square\]

Theorem 3.5 Let $G$ be a finite $p$-group of class 2. Then $\text{Aut}_c(G) = D(G)$ if and only if $G$ is a Camina $p$-group.

Proof If $G$ is a Camina $p$-group, then $G \subseteq [x, G]$ for all $x \in G - G'$. Let $f \in D(G)$. Then for each $x \in G$, $x^{-1}f(x) \in G'$. For $x \in G - G, G' \subseteq [x, G]$ therefore $x^{-1}f(x) \in [x, G]$. If $x \in G'$, then $f(x) = x$. Thus we observe that for each $x \in G$, $x^{-1}f(x) \in G' \subseteq [x, G]$ and hence $D(G) \subseteq \text{Aut}_c(G)$.

Conversely suppose that $D(G) = \text{Aut}_c(G)$. Let $|G| = p^n$ and $|G'| = p^m$. Suppose $(x_1, x_2, \ldots, x_r)$ is a minimal generating set for $G$. Then $G/Z(G) = \langle x_1^{-1}, x_2^{-1}, \ldots, x_r^{-1} \rangle$. Now (from theorem 3.4 (1)), it follows that $|D(G)| = |\text{Aut}_c(G)| = \prod_{i=1}^{r} |\text{Hom}(\langle x_i^{-1} \rangle, [x_i, G])|$. Since $|\langle x_i^{-1} \rangle| = \exp([x_i, G]), |\text{Aut}_c(G)| = \prod_{i=1}^{r} |[x_i, G]|$. But for each $i$, $1 \leq i \leq r$, $G' = [x_i, G]$ it follows that $|\text{Aut}_c(G)| = |G'| = (p^n)^r = (p^m)^{n-m}$. Thus by theorem 3.3, $G$ is a Camina special $p$-group.

\[\square\]

Remark Since Camina special $p$-group is a particular kind of Camina groups, the above result holds good for Camina special $p$-groups.

Theorem 3.6 Let $G$ be a $p$-group of class 2. Then $D(G) = \text{Inn}(G)$ if and only if $G$ is a Camina group and $\gamma_2(G)$ is cyclic.

Proof If $D(G) = \text{Inn}(G)$, then $D(G) \subseteq \text{Aut}_c(G)$ and hence $G$ is a Camina group. Let $G/Z(G)$ and $\gamma_2(G)$ have exponent $p^d$ and ranks $r$ and $d$ respectively. Since $G$ is non-abelian, $r \geq 2$. It is well known that for a $p$-group, $|\text{Hom}(G/Z(G), \gamma_2(G))| \geq |G/Z(G)|^{p^{rd-1}}$. Now $|G/Z(G)| = |\text{Inn}(G)| = |D(G)| = |\text{Hom}(G, \gamma_2(G))| \geq |\text{Hom}(G/Z(G), \gamma_2(G))| \geq |G/Z(G)|^{p^{rd-1}}$. Thus $p^{rd-1} \leq 1$. But this is possible only when $d = 1$. This shows that $\gamma_2(G)$ is cyclic.

Conversely suppose that $G$ is a Camina group of class 2. Then $D(G) = \text{Aut}_c(G)$. Further if $\gamma_2(G)$ is cyclic then $\text{Aut}_c(G) = \text{Inn}(G)$ and hence $D(G) = \text{Inn}(G)$.

\[\square\]

Theorem 3.7 Let $G$ be a group such that $D(G) = \text{Inn}(G)$. If $Z(G) \subseteq \gamma_2(G)$ then $\text{Aut}(G) = Z(\text{Inn}(G))$. 

Proof Let $f \in \text{Autcent}(G)$. Then for each $x \in G$, $x^{-1}f(x) \in Z(G)$. Thus $\text{Autcent}(G) \subseteq D(G)$. Since $D(G) = \text{Inn}(G)$, $\text{Autcent}(G) \subseteq \text{Inn}(G)$. But $\text{Autcent}(G) = C_{\text{Aut}(G)}(\text{Inn}(G))$, this shows that $\text{Autcent}(G) = Z(\text{Inn}(G))$. \hfill $\Box$

**Theorem 3.8** Let $G$ be a finite $p$-group such that $Z(G) \subseteq \gamma_2(G)$ and $D(G/Z(G)) = \text{Inn}(G/Z(G))$. Then $|D(G)| = |\text{Autcent}(G)||G/Z_2(G)|$. Moreover $D(G) = \text{Inn}(G)$ if and only if $\text{Autcent}(G) = Z(\text{Inn}(G))$.

Proof Let $G$ be a finite $p$-group. Since $Z(G) \subseteq \gamma_2(G)$, each $f \in D(G)$ induces a derived automorphism $\bar{f}$ on $G/Z(G)$. Hence, we have a homomorphism $\alpha: D(G) \rightarrow D(G/Z(G))$ given by $\alpha(f) = \bar{f}$. It is easy to see that $\ker(\alpha) = D(G) \cap \text{Autcent}(G)$, since $Z(G) \subseteq \gamma_2(G), \text{Autcent}(G) \subseteq D(G)$ and hence $\ker(\alpha) = \text{Autcent}(G)$. But then $D(G)/\text{Autcent}(G)$ is isomorphic to a subgroup of $D(G/Z(G))$. If $\bar{h} \in D(G/Z(G))$, then there exists an inner automorphism $T_{\bar{a}} \in \text{Inn}(G/Z(G)) \subseteq D(G)\text{ and hence } \ker(\alpha) = \text{Autcent}(G)$. Then $\bar{h} = T_{\bar{a}} \bar{h}$. Thus $\alpha$ is an epimorphism. Thus $D(G)/\text{Autcent}(G) \cong \text{Inn}(G/Z(G))$. Now

$|D(G)| = |\text{Autcent}(G)||G/Z_2(G)|$

$= |Z_2(G)/Z(G)||G/Z_2(G)|$

$= |G/Z_2(G)|$

$= |\text{Inn}(G)|$.

Hence $D(G) = \text{Inn}(G)$. Conversely, suppose that $D(G) = \text{Inn}(G)$. Since, $Z(G) \subseteq G$, by theorem 3.7, $\text{Autcent}(G) = Z(\text{Inn}(G))$. \hfill $\Box$

4. Classification of groups of order $p^n$ ($1 \leq n \leq 5$)

Abelian groups satisfy $D(G) = \text{Inn}(G)$ trivially as the derived group for these groups is trivial. For an odd prime $p$, we classify all those groups of order $p^n$, $3 \leq n \leq 5$ for which $D(G) = \text{Inn}(G)$.

4.1. Groups of order $p^3$

Let $G$ be a non-abelian group of order $p^3$. Then $G$ is a group of class 2 with $|Z(G)| = p$. But then $|[X, G]| = |G| = |Z(G)|$. Thus $G$ is a Camina group with cyclic derived group and hence $D(G) = \text{Inn}(G)$. From the above discussion every group of order $p^3$ satisfies $D(G) = \text{Inn}(G)$.

4.2. Groups of order $p^4$

In next two sections, we study groups of order $p^4$ and $p^5$ ($p$ is an odd prime), on the basis of the classification given by James (1980). This classification is given in terms of isoclinism families. We start with the following definition of isoclinism of groups, given by Hall (1940).

Let $X$ be a finite group and $\bar{X} = X/Z(X)$. Then commutation in $X$ gives a well-defined map $\alpha_x : \bar{X} \times \bar{X} \rightarrow \gamma_2(X)$ such that $\alpha_x(xZ(X), yZ(X)) = [x, y]$ for $(x, y) \in X \times X$. Two finite groups $G$ and $H$ are called isoclinic if there exist isomorphisms

$\theta : G/Z(G) \rightarrow H/Z(H)$,

$\phi : \gamma_2(G) \rightarrow \gamma_2(H)$,

such that $\phi(\alpha, \beta) = [\alpha', \beta']$ for all $\alpha, \beta \in G$, where $\alpha' Z(H) = \theta(\alpha Z(G))$ and $\beta' Z(H) = \theta(\beta Z(G))$. The resulting pair $(\theta, \phi)$ is called an isoclinism of $G$ onto $H$. Clearly isomorphic groups are isoclinic but isoclinic groups need not be isomorphic. For example, $Q_8$ and $D_8$ are isoclinic groups which are not isomorphic. If $G$ and $H$ are isoclinic groups, then $\gamma_2(G) \cong \gamma_2(H)$ and $\delta_3(G) \cong \delta_3(H)$, whereas it is not necessary that $Z(G) \cong Z(H)$ (Hall, 1940). But one may observe that if $G$ and $H$ are finite isoclinic groups of equal order, then $|Z(G)| = |Z(H)|$. Since our further classification depends on the size of $\gamma_2(G)$ and that of $Z(G)$, it is sufficient to calculate $|D(G)|$ for only one member from each isoclinism family.
According to James (1980), for an odd prime, there are three isoclinism families of groups of order \( p^4 \) viz. \( \phi_1, \phi_2 \) and \( \phi_3 \). The family \( \phi_1 \) corresponds to the family of abelian groups and hence \( D(G) = \text{Inn}(G) \) for each member of this family.

The family \( \phi_2 \) consists of groups of class 2 such that \( G \) is a cyclic group of prime order. Thus again \( D(G) = \text{Inn}(G) \) for each member of this family.

In family \( \phi_3 \), each group is a nilpotent group of class 3. Thus if \( G \in \phi_3 \), then \( G \) is a group of maximal class and hence \( |Z(G)| = p \). Since \( G/Z(G) \) is a non-abelian group of order \( p^3 \), by Section 4.1, \( D(G/Z(G)) = \text{Inn}(G/Z(G)) \). For a group of maximal class, \( \text{Autc} (G) \neq Z(\text{Inn}(G)) \). Thus in view of theorem 3.8, \( D(G) \neq \text{Inn}(G) \).

From the above discussion, we conclude that a group of order \( p^4 \) (\( p \) is odd prime) satisfies \( D(G) = \text{Inn}(G) \) if and only if either \( G \) is abelian or it is a nilpotent group of class 2 with \( G \), a cyclic group of prime order.

4.3. Groups of order \( p^5 \)

For an odd prime there are 10 isoclinism families (\( \phi_1 - \phi_{10} \)) of groups of order \( p^5 \). The family \( \phi_1 \) consists of abelian groups and hence \( D(G) = \text{Inn}(G) \) for every group \( G \) lying in this family.

**Theorem 4.1** If \( G \) is a nilpotent group of class 2, then \( D(G) = \text{Inn}(G) \) if and only if \( G \) lie in the isoclinism families \( \phi_1 \) and \( \phi_2 \). Moreover in these cases \( D(G) = \text{Autc} (G) \) as well.

**Proof** Let \( G \) be a group of order \( p^5 \). Then, according to the classification given by James (1980), \( G \) is nilpotent group of class 2 if \( G \in \phi_2 \phi_3 \) and \( \phi_2 \).

If \( G \in \phi_3 \), then \( G = \phi_2 (22) \times (1) \) where \( \phi_2 (22) = \langle a, a_1, a_2 | [a, a] \rangle = a^p = a_2^5 = a_1^2 = 1 \rangle \). The group \( H = \phi_2 (22) \) is a group of order \( p^3 \) having nilpotency class 2 and derived group of prime order. But then \( G \) is also a nilpotent group of class 2 which has a derived group of prime order. Hence, by theorem 3.6, \( D(G) = \text{Inn}(G) \).

Let \( G \in \phi_2 \). Suppose \( G = \phi_2 (1^3) \). Then \( G = \langle a, a_1, a_2, \beta_1, \beta_2 \rangle \) such that \( [a, a] = \beta_1 \) and \( a^p = a_1^2 = a_2^5 = 1 \). Thus \( G \) is a nilpotent group of class 2. Now \( |D(G)| = |\text{Hom}(G, G) \rangle | = |\text{Hom}(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}, \langle \beta_1 \times \beta_2 \rangle) \rangle | = p^5 \). Thus \( D(G) \neq \text{Inn}(G) \).

If \( G = \phi_2 (1^3) \), then \( G = \langle a_1, a_2, a_3 | [a_3, a_2, a_1] = a_2 = 1 \rangle \). Since \( G \) is a nilpotent group of class 2 and \( |G| = p \), \( D(G) = \text{Inn}(G) = \text{Autc} (G) \).

**Theorem 4.2** If \( G \) is a group of class 3, then \( D(G) = \text{Inn}(G) \) if and only if \( G \in \phi_2 \).

**Proof** There are four isoclinism families \( \phi_4, \phi_5, \phi_6, \phi_7 \) in this category.

Let \( G = \phi_4 (1^3) \). Then \( G = \phi_4 (1^3) \times (1) \). \( \phi_4 (1^3) \) is a group of order \( p^4 \) having center of order \( p \) and nilpotency class 3. Thus \( G \) is also a nilpotent group of class 3 with \( |Z(G)| = p \). But then \( |G/Z(G)| = p^3 \) and hence \( D(G/Z(G)) = \text{Inn}(G/Z(G)) \).

If \( G = \phi_4 (1^3) \), then \( G = \langle a_1, a_2, \beta | [a_1, a_2] = a_1^2 = a_2 = 1 \rangle \). Clearly \( G \) is a nilpotent group of class 3 and \( |Z(G)| = p \). Thus, \( |G/Z(G)| = p^3 \) and hence \( D(G/Z(G)) = \text{Inn}(G/Z(G)) \).

If \( G = \phi_4 (32) \). Then \( G = \langle a_1, a_2, \beta | \beta = a_1^2 \rangle \). \( G \) is a nilpotent group of class 3 with \( |Z(G)| = p \). Hence, \( G/Z(G) \) is a group of order \( p^4 \) with nilpotency class 2. But then \( D(G/Z(G)) = \text{Inn}(G/Z(G)) \).
If $G$ is a group of order $p^3$, then $\text{Aut}(G) = Z(\text{Inn}(G))$ if and only if $G$ is isomorphic to $\phi_9(32)$ (Theorem 4.1, Gumber, 2011). Hence using theorem 3.8 in all the above three cases discussed above, we have $D(G) = \text{Inn}(G)$ if and only if $G = \phi_9(32)$.

We now left only with isoclinism family $\phi_p$. Let $G$ be a group $\phi_p(1^3)$ from the isoclinism family $\phi_p$. Then $G$ is a nilpotent group of class 3 with center of prime order. In this case, $|\text{Aut}(G)| = p^5$ and $\text{Aut}(G) \neq \text{Inn}(G)$ (Yadav, 2008). Thus $D(G) \neq \text{Inn}(G)$, otherwise $\text{Aut}(G) = \text{Inn}(G)$.

**Theorem 4.3**  
If $G$ is a group from the isoclinism families $\phi_3$ or $\phi_{10}$ then $D(G) \neq \text{Inn}(G)$.

**Proof**  
Let $G$ be a group $\phi_p(1^3)$ from the isoclinism family $\phi_p$. Then $G$ is a nilpotent group of class maximal class with center of prime order. Again in this case $|\text{Aut}(G)| = p^5$ and $\text{Aut}(G) \neq \text{Inn}(G)$ (Lemma 5.2, Yadav, 2008). Thus $D(G) \neq \text{Inn}(G)$.

If $G$ is the group $\phi_p(1^3)$, then $G$ is a nilpotent group with class 4 and $|Z(G)| = p$. Thus $Z(G) \leq \gamma_2(G)$. Clearly $G/Z(G)$ is a group of order $p^3$ having nilpotency class 3. Thus $D(G/Z(G)) \neq \text{Inn}(G/Z(G))$. Since $Z(G) \leq \gamma_2(G)$, $D(G) \neq \text{Inn}(G)$, otherwise $D(G/Z(G)) = \text{Inn}(G/Z(G))$.

5. Conclusion

On the basis of classification of groups of order $p^n, 1 \leq n \leq 5$ (for an odd prime $p$), it is proved that if $|G| = p^r, 1 \leq r \leq 3$, then $D(G) = \text{Inn}(G)$. If $|G| = p^5$, then $D(G) = \text{Inn}(G)$ and only if $G$ is abelian or it is a nilpotent group of class 2 with a cyclic derived group of order $p$ and if $|G| = p^7$, then $D(G) = \text{Inn}(G)$, if and only if $G \in \phi_2, \phi_3, \phi_4$ and $\phi_5$. A necessary and sufficient condition for nilpotent groups of class 2 is also obtained when $D(G) = \text{Inn}(G)$.
