The connectedness of some varieties and the Deligne-Simpson problem

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To the memory of my mother

Abstract

The Deligne-Simpson problem (DSP) (resp. the weak DSP) is formulated like this: give necessary and sufficient conditions for the choice of the conjugacy classes \( C_j \subset GL(n, \mathbb{C}) \) or \( c_j \subset gl(n, \mathbb{C}) \) so that there exist irreducible (resp. with trivial centralizer) \((p + 1)\)-tuples of matrices \( M_j \in C_j \) or \( A_j \in c_j \) satisfying the equality \( M_1 \cdots M_{p+1} = I \) or \( A_1 + \cdots + A_{p+1} = 0 \).

The matrices \( M_j \) and \( A_j \) are interpreted as monodromy operators of regular linear systems and as matrices-residua of Fuchsian ones on Riemann’s sphere. For \((p+1)\)-tuples of conjugacy classes one of which is with distinct eigenvalues 1) we prove that the variety \( \{ (M_1, \ldots, M_{p+1}) | M_j \in C_j, M_1 \cdots M_{p+1} = I \} \) or \( \{ (A_1, \ldots, A_{p+1}) | A_j \in c_j, A_1 + \cdots + A_{p+1} = 0 \} \) is connected if the DSP is positively solved for the given conjugacy classes and 2) we give necessary and sufficient conditions for the positive solvability of the weak DSP.

Key words: generic eigenvalues, monodromy operator, (weak) Deligne-Simpson problem.

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1 Introduction

The Deligne-Simpson problem (DSP) is formulated like this: give necessary and sufficient conditions upon the choice of the conjugacy classes \( c_j \subset gl(n, \mathbb{C}) \) (resp. \( C_j \subset GL(n, \mathbb{C}) \), \( j = 1, \ldots, p + 1, \ p \geq 2 \), so that there exist irreducible \((p+1)\)-tuples of matrices \( A_j \in c_j \) (resp. \( M_j \in C_j \)) such that \( A_1 + \cdots + A_{p+1} = 0 \) (resp. \( M_1 \cdots M_{p+1} = I \)). “Irreducible” means “without common proper invariant subspace”. The weak DSP is obtained from the DSP by replacing the requirement of irreducibility by the weaker requirement the centralizer of the \((p+1)\)-tuple to be trivial, i.e. reduced to scalars. In what follows we write “tuple” instead of \("(p+1)\)-tuple". 

The DSP or the weak DSP is solvable for given conjugacy classes \( c_j \) or \( C_j \) if there exist tuples of matrices \( A_j \in c_j \) whose sum is 0, resp. of matrices \( M_j \in C_j \) whose product is \( I \), irreducible or with trivial centralizer.

The matrices \( A_j \) (resp. \( M_j \)) are interpreted as matrices-residua (resp. as monodromy matrices) of Fuchsian (resp. regular) linear systems of differential equations on Riemann’s sphere; the conjugacy classes \( C_j \) are interpreted as local monodromies around the poles of a regular linear system and for matrices \( M_j \) the problem admits the interpretation: for which tuples of local monodromies do there exist irreducible monodromy groups with such local monodromies; see the details in [Ko1] or [Ko2].

The multiplicative version of the problem (i.e. for matrices \( M_j \)) has been stated by P. Deligne (the additive, i.e. for matrices \( A_j \), by the author) and C. Simpson was the first to
obtain significant results towards the resolution of the problem, see [Si].

In what follows we assume that there hold the self-evident necessary conditions the sum of the traces of the classes \( c_j \) to be 0 (resp. the product of the determinants of the classes \( C_j \) to be 1). In terms of the eigenvalues \( \lambda_{k,j} \) (resp. \( \sigma_{k,j} \)) of the matrices from \( c_j \) (resp. \( C_j \)) repeated with their multiplicities, this condition reads

\[
\sum_{j=1}^{p+1} \sum_{k=1}^{n} \lambda_{k,j} = 0 \quad \text{resp.} \quad \prod_{j=1}^{p+1} \prod_{k=1}^{n} \sigma_{k,j} = 1.
\]

**Definition 1** An equality \( \sum_{j=1}^{p+1} \sum_{k \in \Phi_j} \lambda_{k,j} = 0 \) (resp. \( \prod_{j=1}^{p+1} \prod_{k \in \Phi_j} \sigma_{k,j} = 1 \)) is called a non-genericity relation; the non-empty sets \( \Phi_j \) contain one and the same number < \( n \) of indices for all \( j \). Eigenvalues satisfying none of these relations are called generic.

**Remark 2** Reducible tuples exist only for non-generic eigenvalues (the eigenvalues of each diagonal block of a block upper-triangular tuple satisfy some non-genericity relation).

For generic eigenvalues the problem is completely solved in [Ko1], [Ko2] and [Ko3], and the solution is a criterium upon the Jordan normal forms (JNFs) defined by the conjugacy classes, i.e. it does not depend on the concrete choice of the eigenvalues provided that the latter remain generic. The additive version of the problem is solved also for arbitrary eigenvalues, see [C-B].

Denote by \( J(Y) \) (resp. \( J(C) \)) the JNF of the matrix \( Y \) (resp. the JNF defined by the conjugacy class \( C \)). For a conjugacy class \( C \) in \( \mathfrak{gl}(n, \mathbb{C}) \) or \( GL(n, \mathbb{C}) \) denote by \( d(C) \) its dimension (as a variety in \( \mathfrak{gl}(n, \mathbb{C}) \) or \( GL(n, \mathbb{C}) \)) and for a matrix \( Y \in C \) set \( r(C) := \min_{\lambda \in \mathbb{C}} \text{rank}(Y - \lambda I) \).

The integer \( n - r(C) \) is the maximal number of Jordan blocks of \( J(Y) \) with one and the same eigenvalue. Set \( d_j := d(c_j) \) (resp. \( d(C_j) \)), \( r_j := r(c_j) \) (resp. \( r(C_j) \)). The quantities \( r(C) \) and \( d(C) \) depend only on the JNF \( J(C) \), not on the eigenvalues.

**Proposition 3** (C. Simpson, see [Si].) The following couple of inequalities is a necessary condition for solvability of the DSP in the case of matrices \( M_j \):

\[
d_1 + \ldots + d_{p+1} \geq 2n^2 - 2 \quad (\alpha_n) \quad \text{for all} \quad j, \quad r_1 + \ldots + r_j + \ldots + r_{p+1} \geq n \quad (\beta_n).
\]

The proposition holds also for matrices \( A_j \) whose sum is 0, see [Ko2] or [Ko4].

**Proposition 4** Conditions \((\alpha_n)\) and \((\beta_n)\) are necessary for the solvability of the weak DSP in the case when one of the conjugacy classes has \( n \) distinct eigenvalues.

The proposition is proved at the end of part B) of Subsection 2.1. C. Simpson proves in [Si] the following

**Theorem 5** For generic eigenvalues and when one of the conjugacy classes \( C_j \) is with distinct eigenvalues, conditions \((\alpha_n)\) and \((\beta_n)\) together are necessary and sufficient for the existence of irreducible tuples of matrices \( M_j \in C_j \) whose product is \( I \).

The same is true for matrices \( A_j \in c_j \) whose sum is 0 (see [Ko4] and compare with [Ko5], Theorems 19 and 32 in which one of the matrices is supposed to have eigenvalues of multiplicity \( \leq 2 \), not necessarily distinct ones).

In the present paper we consider sets of the form

\[
\mathcal{V}(c_1, \ldots, c_{p+1}) = \{(A_1, \ldots, A_{p+1}) | A_j \in c_j, A_1 + \ldots + A_{p+1} = 0\}
\]
and

\[ \mathcal{W}(C_1, \ldots, C_{p+1}) = \{(M_1, \ldots, M_{p+1})| M_j \in C_j, M_1 \ldots M_{p+1} = I\} \]

or just \( \mathcal{V} \) and \( \mathcal{W} \) for short. The aim of the present paper is to prove (in Section 3) the following

**Theorem 6** 1) For generic eigenvalues and when one of the conjugacy classes \( c_j \) (resp. \( C_j \)) is with distinct eigenvalues the set \( \mathcal{V} \) (resp. \( \mathcal{W} \)) is a smooth and connected variety.

2) For arbitrary eigenvalues, when one of the conjugacy classes \( c_j \) (resp. \( C_j \)) is with distinct eigenvalues, and if there exist irreducible tuples, then the closure of the set \( \mathcal{V} \) (resp. \( \mathcal{W} \)) is a connected variety. The algebraic closures of these sets coincide with their topological closures; these are the closures of the subvarieties consisting of irreducible tuples; these subvarieties are connected. The singular points of the closures are precisely the tuples of matrices with non-trivial centralizers.

3) If one of the matrices \( A_j \) or \( M_j \) is with distinct eigenvalues, then conditions \( (\alpha_n) \) and \( (\beta_n) \) together are necessary and sufficient for the solvability of the weak DSP.

**Convention.** In what follows we assume that the conjugacy class \( c_{p+1} \) (resp. \( C_{p+1} \)) is with distinct eigenvalues.

**Example 7** Consider the case of matrices \( A_j \) for \( p = 2, n = 2 \), with eigenvalues of the three diagonalizable non-scalar matrices respectively \( (a, b), (c, d), (g, h) \), where \( a + c + g = b + d + h = 0 \) and there are no non-genericity relations which are not corollaries of these ones. Then the set \( \mathcal{V} \) is a stratified variety with three strata – \( S_0, S_1 \) and \( S_2 \). The stratum \( S_i \) consists of triples which up to conjugacy equal

\[ A_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad A_2 = \begin{pmatrix} c & \varepsilon_i \\ \eta_i & d \end{pmatrix}, \quad A_3 = \begin{pmatrix} g & -\varepsilon_i \\ -\eta_i & h \end{pmatrix} \]

where \( \varepsilon_0 = \eta_0 = 0, \varepsilon_1 = 1, \eta_1 = 0, \varepsilon_2 = 0, \eta_2 = 1 \). Hence, \( S_0 \) lies in the closure of \( S_1 \) and \( S_2 \). A theorem by N. Katz (see [Kaj]) forbids coexistence of irreducible and reducible triples in the so-called rigid case (i.e. when there is equality in \( (\alpha_n) \) which is the case here – one has \( d_1 = d_2 = d_3 = 2 \)). Hence, there are no strata of \( \mathcal{V} \) other than \( S_0, S_1 \) and \( S_2 \), and \( \mathcal{V} \) is connected. However, it is not smooth at \( S_0 \). Indeed, one can deform analytically a triple from \( S_0 \) into one from \( S_1 \) and into one from \( S_2 \); the triples from \( S_1 \) and \( S_2 \) defining different semi-direct sums, the strata \( S_1 \) and \( S_2 \) cannot be parts of one and the same smooth variety containing \( S_0 \).

**Example 8** Again for \( p = n = 2 \), consider the case when \( c_1 \) is nilpotent non-scalar and \( c_2 = -c_3 \) is with eigenvalues 1, 2. Then the algebraic and topological closure of the variety \( \mathcal{V}(c_1, c_2, c_3) \) consists of three strata – \( T_0, T_1 \) and \( T_2 \). Up to conjugacy, the triples from \( T_0 \) are diagonal and equal \( \text{diag}(0, 0), \text{diag}(1, 2), \text{diag}(-1, -2) \). The ones from \( T_1 \) (resp. \( T_2 \)) up to conjugacy equal

\[ A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \quad \text{(resp.} A_2 = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}) \), \quad A_3 = -A_1 - A_2. \]

The stratum \( T_0 \) lies in the closures of \( T_1 \) and \( T_2 \) and like in Example 7, the closure of \( \mathcal{V} \) is singular along \( T_0 \). Notice that \( \mathcal{V} \) itself is not connected because \( T_0 \not\in \mathcal{V} \) and like in Example 7, the strata \( T_1, T_2 \) are not parts of one and the same smooth variety.

The reader will find other examples illustrating the stratified structure of the varieties \( \mathcal{V} \) or \( \mathcal{W} \) in [Ko7] and [Ko8], in particular, cases when the dimension of the variety is higher than the expected one when the centralizer is non-trivial.
Open questions and comments 9

1) It would be natural to ask the question whether parts 1) and 2) of Theorem 6 are true without the condition one of the classes $c_j$ or $C_j$ to be with distinct eigenvalues. The author is convinced that this is true (in the case when inequality $(\alpha_n)$ becomes equality the proof of this can be deduced from the results in [Ka] – rigid tuples are unique up to conjugacy and coexistence of irreducible and reducible tuples is impossible in the rigid case). In the present paper we use the condition one of the classes to be with distinct eigenvalues in the proofs (see Proposition 20 and its proof).

2) It would be interesting to prove the connectedness of the closures of the varieties $V$ or $W$ without the assumption that there are irreducible tuples, see part 2) of Theorem 6. All examples known to the author are of connected closures, see [Ko7] and [Ko8].

3) In part 3) of the theorem the condition one of the matrices to be with distinct eigenvalues is essential. Even for double eigenvalues there is a counterexample – for $n = 2$ a triple of nilpotent non-zero matrices $A_j$ whose sum is 0, is upper-triangular up to conjugacy, and, hence, the centralizer of the triple contains all three matrices. Hence, for $n = 2$ the weak DSP is not solvable for a triple of nilpotent non-zero conjugacy classes.

4) The theorem implies that the moduli space of tuples of matrices from given conjugacy classes (one of which is with distinct eigenvalues), with zero sum or whose product is $I$, is connected provided that there exist irreducible tuples.

2 Preparation for the proof of Theorem 6

2.1 The known facts we use

A) The dimension of a variety $V$ or $W$.

To prove the theorem we need the following propositions:

Proposition 10 The centralizer of the $p$-tuple of matrices $A_j$ ($j = 1, \ldots, p$) is trivial if and only if the mapping $(sl(n, \mathbb{C}))^p \rightarrow sl(n, \mathbb{C}), (X_1, \ldots, X_p) \mapsto \sum_{j=1}^p [A_j, X_j]$ is surjective.

The proposition is proved in [Ko2].

Proposition 11 At a point defining a tuple with trivial centralizer a variety $V$ or $W$ is smooth and locally of dimension $d_1 + \ldots + d_p + 1 - n^2 + 1$.

The propositions of this subsection (except Propositions 10 and 15) are proved in the next one.

B) The basic technical tool.

Definition 12 Call basic technical tool the procedure described below whose aim is to deform analytically a given tuple of matrices $A_j$ or $M_j$ with trivial centralizer by changing their conjugacy classes in a desired way.

Set $A_j = Q^{-1}_j G_j Q_j$, $G_j$ being Jordan matrices. Look for a tuple of matrices $\hat{A}_j$ (whose sum is 0) of the form $\hat{A}_j = (I + \varepsilon X_j(\varepsilon))^{-1} Q^{-1}_j (G_j + \varepsilon V_j(\varepsilon)) Q_j (I + \varepsilon X_j(\varepsilon))$ where $\varepsilon \in (\mathbb{C}, 0)$ and $V_j(\varepsilon)$ are given matrices analytic in $\varepsilon$; they must satisfy the condition $\text{tr}(\sum_{j=1}^{p+1} V_j(\varepsilon)) \equiv 0$; set $N_j = Q^{-1}_j V_j Q_j$. The existence of matrices $X_j(\varepsilon)$ is deduced from the triviality of the centralizer, using Proposition 10 (see its proof and the details in [Ko2]).
Notice that one has \( \tilde{A}_j = A_j + \varepsilon[A_j, X_j(0)] + \varepsilon N_j + o(\varepsilon) \). Proposition 10 assures the existence of \( X_j(0) \), i.e. the existence in first approximation w.r.t. \( \varepsilon \) of the matrices \( X_j \); the existence of true matrices \( X_j \) analytic in \( \varepsilon \) follows from the implicit function theorem.

If for \( \varepsilon \neq 0 \) small enough the eigenvalues of the matrices \( \tilde{A}_j \) are generic, then their tuple is irreducible. In a similar way one can deform analytically tuples depending on a multi-dimensional parameter.

Deforming analytically tuples of matrices \( A_j \) is used sometimes to make some of the JNFs \( J(A_j) \) “more generic”:

**Example 13** Suppose that \( A_{p+1} \) is diagonal and has \( n - 2 \) simple eigenvalues and a double eigenvalue (the tuple being with trivial centralizer). Set

\[
A_{p+1} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & D \end{pmatrix}, \quad N_{p+1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

where \( D \) is \( n \times n \), diagonal, with distinct and different from \( a \) eigenvalues. Then for \( \varepsilon \neq 0 \) the matrix \( A_{p+1} \) is with the same eigenvalues and with a single Jordan block of size 2 corresponding to the eigenvalue \( a \).

In other cases deforming analytically of tuples changes the eigenvalues of the matrices \( A_j \):

**Example 14** Suppose that

\[
A_{p+1} = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & D \end{pmatrix}, \quad N_{p+1} = \begin{pmatrix} 0 & 0 & 0 \\ f & g & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

where \( D \) is as in Example 13 and \( f, g \in \mathbb{C} \). For generic \( f \) and \( g \) one obtains for \( \varepsilon \neq 0 \) small enough matrices \( \tilde{A}_{p+1} \) with \( n \) distinct eigenvalues.

Given a tuple of matrices \( M_j \) with trivial centralizer and whose product is \( I \), look for matrices \( \tilde{M}_j \) (whose product is \( I \)) of the form

\[
\tilde{M}_j = (I + \varepsilon X_j(\varepsilon))^{-1}(M_j + \varepsilon N_j(\varepsilon))(I + \varepsilon X_j(\varepsilon))
\]

where the given matrices \( N_j \) depend analytically on \( \varepsilon \in (\mathbb{C}, 0) \) and the product of the determinants of the matrices \( \tilde{M}_j \) is 1; one looks for \( X_j \) analytic in \( \varepsilon \). The existence of such matrices \( X_j \) follows again from the triviality of the centralizer, see [Ko2].

When applying the basic technical tool one often preserves the conjugacy classes of all matrices \( A_j \) or \( M_j \) but one, as this is done in the

**Proof of Proposition 4:**

Given a tuple with trivial centralizer of matrices \( A_j \) whose sum is 0 or of matrices \( M_j \) whose product is \( I \) the matrix \( A_{p+1} \) or \( M_{p+1} \) having distinct eigenvalues, deform it into a nearby such tuple with generic eigenvalues, hence, irreducible; for \( j = 1, \ldots, p \) the matrix \( A_j \) or \( M_j \) remains within its conjugacy class; \( A_{p+1} \) (resp. \( M_{p+1} \)) remains with distinct eigenvalues, but the latter change. For the deformed tuple conditions \((\alpha_n)\) and \((\beta_n)\) hold, hence, they hold for the initial one as well (the quantities \( d_j, r_j \) do not change under such a deformation and when checking condition \((\beta_n)\) one needs to consider only the sum \( r_1 + \ldots + r_p \) because \( r_{p+1} = n - 1 \)). \( \square \)

**C) Corresponding Jordan normal forms.**
A JNF is a collection of numbers \( \{ b_{i,l} \} \) where \( b_{i,l} \) stands for the size of the \( i \)-th Jordan block with the \( l \)-th eigenvalue. For a given JNF \( J^n = \{ b_{i,l} \} \) of size \( n \) define its corresponding diagonal JNF \( J^m \) (more generally, two JNFs are said to be corresponding to one another if they correspond to one and the same diagonal JNF). A diagonal JNF is a partition of \( n \) defined by the multiplicities of the eigenvalues. For each \( l \) fixed, the collection of numbers \( \{ b_{i,l} \} \) is a partition of \( \sum_{i \in I_l} b_{i,l} \) and \( J^m \) is the disjoint sum of the dual partitions; see the details in [Ko1], [Ko2] or [Kr], as well as the proof of the following

**Proposition 15** The quantities \( d \) and \( r \) defined in the Introduction are the same for two corresponding JNFs.

D) Realizing of monodromy groups by Fuchsian systems with different sets of eigenvalues of their matrices-residua. Procedure \((l,k)\).

**Notation 16** \( E_{i,k} \) denotes the matrix with a single non-zero entry which equals 1 and is in position \((i,k)\). In all other cases double subscripts denote matrix entries. E.g., \( A_{j;1,2} \) stands for the entry of the matrix \( A_j \) in position \((1,2)\).

When one makes the linear change of variables \( X \mapsto V(t)X \) (where \( V(t) \) is an \( n \times n \)-matrix-function meromorphic on the Riemann sphere and whose determinant does not vanish identically there) in the linear system \( dX/dt = A(t)X \), then the matrix \( A(t) \) undergoes the gauge transformation: \( A \mapsto -V^{-1}dV/dt+V^{-1}AV \). If the system is Fuchsian, then after such a change, in general, it is no longer Fuchsian, but for special choices of \( V \) one can obtain new Fuchsian systems (with the same monodromy group) whose matrices-residua belong to new conjugacy classes. For each matrix-residuum the eigenvalues of the new conjugacy class are shifted by integers w.r.t. the ones of the old conjugacy class. Indeed, the eigenvalues \( \lambda_{k,j} \) of \( A_j \) and \( \sigma_{k,j} \) of \( M_j \) are related by the equality \( \exp(2\pi i \lambda_{k,j}) = \sigma_{k,j} \).

We are interested here only in changes that shift the eigenvalues of \( A_{p+1} \) while preserving the conjugacy classes of the other matrices \( A_j \). This is the case when \( V \) is holomorphic and holomorphically invertible for \( t \neq a_{p+1} \). An admissible shift of the eigenvalues of \( A_{p+1} \) is a shift by an \( n \)-vector with integer components whose sum is 0. The basic result from [Ko6] implies the following

**Corollary 17** If the monodromy group defined by the operators \( M_j \) is irreducible, then for all but finitely many admissible shifts there exist linear changes \( X \mapsto V(t)X \) transforming the given Fuchsian system into a new one with the eigenvalues of \( A_{p+1} \) shifted like required.

Explain how to perform the simplest admissible shifts. Consider the Fuchsian system

\[
\frac{dX}{dt} = \left( \sum_{j=1}^{p+1} A_j/(t-a_j) \right)X
\]

Its Laurent series expansion at \( a_{p+1} \) looks like this:

\[
\frac{dX}{dt} = (A_{p+1}/(t-a_{p+1}) + \sum_{j=1}^{p} A_j/(a_{p+1}-a_j) + o(1))X
\]

Assume that \( A_{p+1} = \text{diag}(\lambda_{1,p+1}, \ldots, \lambda_{n,p+1}) \). If the entry \( c = (\sum_{j=1}^{p} A_j/(a_{p+1}-a_j))_{i,k} \) is non-zero, then in the Fuchsian system (1) one can perform the change \( w : X \mapsto (I+W/(t-a_{p+1}))X \).
can be chosen arbitrarily close to the initial ones. In the changed system, the new tuple of poles $(a, a, \cdots, a)$ and/or the positions of the poles $V_{ij}$ differ from those of the original system.

Observe that the matrix $I + W/(t - a_{p+1})$ is holomorphic and holomorphically invertible for $t \neq a_{p+1}$ and that its determinant is identically equal to 1. Call this change of eigenvalues Procedure $(l, k)$.

**Proposition 18** Given a system (1) with irreducible tuple of matrices-residua $A_j \in c_j$, $A_{p+1} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and given $l \neq k$, $1 \leq l, k \leq n$, one can either perform Procedure $(l, k)$ or one can change the matrices $A_j$, $1 \leq j \leq p$, within their conjugacy classes (their sum remaining the same) and/or the positions of the poles $a_j$ so that Procedure $(l, k)$ will be possible to perform in the changed system. The new tuple of poles $a_j$ and the new $p$-tuple of matrices $A_j$, $1 \leq j \leq p$, can be chosen arbitrarily close to the initial ones.

**E) The mapping $\phi$**
For a matrix $A_j \in c_j$ set $M(A_j) = \exp(2\pi i A_j)$. Denote by $C_j$ the conjugacy class of the matrix $M(A_j)$. The monodromy operators $M_j$ of the Fuchsian system (1) defined after a standard set of generators of $\pi_1(CP^1 \setminus \{a_1, \ldots, a_{p+1}\})$ (see the details in [Ko2]) equal $M(A_j)$ provided that there is no non-zero integer difference between two eigenvalues of $A_j$. We assume that a fixed point $a_0$ is chosen as well as an initial value of the solution, i.e. a non-degenerate $n \times n$-matrix $B = X|_{t=a_0}$. Thus the mapping $\phi: (A_1, \ldots, A_{p+1}) \mapsto (M_1, \ldots, M_{p+1})$ is a mapping $\mathcal{V}(c_1, \ldots, c_{p+1}) \to \mathcal{W}(C_1, \ldots, C_{p+1})$.

**Proposition 19**
1) Suppose that no conjugacy class $c_j$ has two eigenvalues differing by a non-zero integer. Then the mapping $\phi$ is a local diffeomorphism at every point of $\mathcal{V}$ where the centralizer of the tuple of matrices $A_j$ is trivial.

2) If in addition the eigenvalues of the conjugacy classes $c_j$ are generic, then $\phi$ is a global diffeomorphism of $\mathcal{V}$ onto its image in $\mathcal{W}$.

**F) Deforming direct sums into semi-direct ones and into irreducible representations.**

**Proposition 20**
1) A tuple of matrices $A_j$ from the closure of $\mathcal{V}$ (or of $W$) and with non-trivial centralizer defines a representation which is a direct sum.

2) If $A_{p+1}$ (or $M_{p+1}$) is with distinct eigenvalues, then any matrix from the centralizer of a tuple of the variety $\mathcal{V}$ (or $W$) is diagonalizable.

Denote by $P_i$, $i = 1, 2$, representations with trivial centralizers defined by matrices $A_j^i$ whose sum is 0 or by matrices $M_j^i$ whose product is $I$; $j = 1, \ldots, p + 1$. The two matrices $A_j^{p+1}$ (resp. $M_j^{p+1}$), $i = 1, 2$, are each with distinct eigenvalues and have no eigenvalue in common. Denote by $d_j^i$ (resp. $d_j^i$) the quantities $d$ of the conjugacy classes of the matrices $A_j^i$ or $M_j^i$ (resp. of the direct sums $A_j^i \oplus A_j^2$ or $M_j^1 \oplus M_j^2$). Set $\xi = \dim \text{Ext}^1(P_1, P_2)$.

**Proposition 21**
1) One has $\xi = (\sum_{j=1}^{p+1}(d_j^i - d_j^2 - d_j^i)/2) - 2m_1m_2$.

2) If $\xi \geq 1$, then there exists a semidirect sum of $P_1$ and $P_2$ which is not reduced to a direct one.

3) If the representations $P_1$ are irreducible and $\xi \geq 2$, then the semidirect sum of $P_1$ and $P_2$ can be deformed into an irreducible representation without changing the conjugacy classes of the matrices $A_j$ or $M_j$ (their sum remaining 0 or their product remaining $I$).
Remark 22 One knows a posteriori (i.e. after Theorem 6 is proved) that in the conditions of the proposition, when $P_1$, $P_2$ are irreducible, if $\xi = 1$ (and unlike for $\xi \geq 2$), then there exist no irreducible tuples. Indeed, in this case $\text{dim} \ V$ equals the dimension of its subvariety $U$ of semi-direct sums (this can be proved like Proposition 21), so $U$ cannot belong to the closure of the set of irreducible representations. The existence of irreducible representations would contradict part 2) of Theorem 6.

2.2 Proofs of the propositions

Proof of Proposition 11:

1). Consider first the case of matrices $A_j$. Assume without restriction that $c_j \subset sl(n, C)$. Consider the cartesian product $(c_1 \times \ldots \times c_p) \subset (sl(n, C))^p$. Define the mapping $\tau : (c_1 \times \ldots \times c_p) \rightarrow sl(n, C)$ by the rule $\tau : (A_1, \ldots, A_p) \mapsto A_{p+1} = -A_1 - \ldots - A_p$ (recall that the sum of the matrices $A_j$ is 0).

2). The variety $\mathcal{V}$ is the intersection of the two varieties in $c_1 \times \ldots \times c_p \times sl(n, C)$: the graph of the mapping $\tau$ and $c_1 \times \ldots \times c_p \times c_{p+1}$. This intersection is transversal which implies the smoothness of $\mathcal{V}$. Transversality follows from Proposition 10 – the tangent space to the conjugacy class $c_j$ at $A_j$ equals $\{[A_j, X] | X \in gl(n, C)\}$.

3). Recall that $\text{dim} c_j$ is denoted by $d_j$. One has

$$\text{dim} \ \mathcal{V} = \left(\sum_{j=1}^{p} d_j\right) - \text{codim}_{sl(n, C)} c_{p+1} = \left(\sum_{j=1}^{p} d_j\right) - [(n^2 - 1) - d_{p+1}] = \sum_{j=1}^{p+1} d_j - n^2 + 1.$$

4). In the case of matrices $M_j$ the only difference in the proof is that the mapping $(A_1, \ldots, A_p)$ $\mapsto A_{p+1} = -A_1 - \ldots - A_p$ from 2) has to be replaced by the mapping

$$(M_1, \ldots, M_p) \mapsto M_{p+1} = (M_1 \ldots M_p)^{-1}.$$

The reader will be able to restitute the missing technical details after examining the more detailed description of the basic technical tool given in [Ko2].

The proposition is proved. □

Proof of Proposition 18:

1). Assume for convenience that $(l, k) = (1, n)$. Procedure $(1, n)$ is defined only if $c = (\sum_{j=1}^{p} A_j/(a_{p+1} - a_j))_{1,n} \neq 0$. Hence, if at least one of the entries $A_{j;1,n}$, $1 \leq j \leq p$, is non-zero, one can change the positions of the poles $a_j$ a little to obtain the condition $c \neq 0$. So assume that for $j = 1, \ldots, p$ one has $A_{j;1,n} = 0$. (For $n = 2$ this means that the tuple is reducible which is a contradiction; so in what follows we assume that $n > 2$.) We show that there exist infinitesimal conjugations $A_j \mapsto (I + \varepsilon X_j)^{-1}A_j(I + \varepsilon X_j)$, $1 \leq j \leq p$, $\varepsilon \in (C, 0)$, the result of which is that for at least one $j$ one will have $A_{j;1,n} \neq 0$ for $\varepsilon \neq 0$ and $A_{p+1}$ does not change (in what follows we set $j = 1$). By analogy with the basic technical tool (Subsection 2.1, part B)) one can show that then there exist true (not only infinitesimal) conjugations.

2). These conjugations must be such that

$$\sum_{j=1}^{p} [A_j, X_j] = 0 \tag{3}$$

(first approximation w.r.t. $\varepsilon$). If no such $p$-tuple of conjugations changes the matrices $A_j$ so that $A_{1;1,n} \neq 0$ in first approximation w.r.t. $\varepsilon$, then the following condition must be a corollary of (3): $\text{tr}(E_{n,1}[A_1, X_1]) = 0$. This means that the linear form $\text{tr}(E_{n,1}[A_1, X_1]) = \text{tr}([E_{n,1}, A_1]X_1)$ (the
variables are the entries of $X_1$) must be representable as $\text{tr}(D \sum_{j=1}^{p}[A_j, X_j]) = \text{tr}(\sum_{j=1}^{p}[D, A_j]X_j)$ for some matrix $D \in \text{sl}(n, \mathbb{C})$. So from now on we want to prove that such a matrix $D$ does not exist.

3\textsuperscript{0}. Suppose it does. Hence, one has

$$[D, A_j] = 0 \text{ for } j = 2, \ldots, p, \quad [D, A_1] = [E_{n,1}, A_1]$$

Summing up these equalities one deduces that

$$-[D, A_{p+1}] = [E_{n,1}, A_1]$$

The matrix $[E_{n,1}, A_1]$ has non-zero entries only in the first column and in the last row. Hence, the possible non-zero entries of $D$ are in the first column, in the last row and on the diagonal (recall that $A_{p+1}$ is diagonal and with distinct eigenvalues).

4\textsuperscript{0}. Two cases are possible:

Case A). $D$ has at least two distinct eigenvalues.

Case B). $D$ is nilpotent.

In case A) permute the second, ..., $(n - 1)$-st eigenvalues of $A_{p+1}$ (by conjugation with a permutation matrix which is block-diagonal, with diagonal blocks of sizes $1, n - 2, 1$) so that the first $m$ eigenvalues of $D$ be equal (say, equal to $b$) and different from the $(m + 1)$-st, ..., $(n - 1)$-st one. If $m = 1$, then equalities (4) imply that the entries of $A_1, \ldots, A_p$ in positions $(1, i)$, $2 \leq i \leq n - 1$, equal 0. As $A_{j,1:n} = 0$ for $j = 1, \ldots, p$, the tuple of matrices $A_j$ is reducible – a contradiction.

If $m > 1$ and if $b$ is different from the $n$-th eigenvalue of $D$ as well, then it follows from (4) that the entries of $A_1, \ldots, A_p$ in the right upper corner $m \times (n - m)$ are zeros. Hence, the tuple of matrices $A_j$ is reducible – a contradiction.

If $m > 1$ and the $n$-th eigenvalue of $D$ equals $b$, then block-decompose the $n \times n$-matrices as follows:

$$\begin{pmatrix} B & M & Q \\ S & R & T \\ U & V & b \end{pmatrix}$$

where $B$ is $m \times m$ and $b$ is $1 \times 1$. Conjugate the matrices $A_j$ and $D$ by a matrix $I + F$ (where only the $S$-, $V$- and $U$-blocks of $F$ are non-zero) to annihilate the $S$- and $V$-blocks of $D$; note that $(I + F)^{-1}E_{n,1}(I + F) = E_{n,1}$. After the conjugation equalities (4) imply that the $M$- and $T$-blocks of the matrices $A_1, \ldots, A_p$ are 0. But then the $S$-blocks of these matrices are also 0. Hence, the tuple is reducible – a contradiction again. So case A) is impossible.

5\textsuperscript{0}. Consider case B). The form of $D$ (see 3\textsuperscript{0}) implies that all entries of $D$ on and above the diagonal are 0 and either $D^2 = 0$ and $\text{rk}D \leq 2$ or $D^2 \neq 0$, $\text{rk}D = 2$ and $D^3 = 0$. In the second case one has $D^2 = \alpha E_{n,1}$, $\alpha \neq 0$ and it follows from (4) that $[A_j, E_{n,1}] = 0$ for $j = 2, \ldots, p$; hence, $A_{j,1:n} = 0$ for $j = 2, \ldots, p$, $i = 2, \ldots, n$ and as these equalities hold also for $j = p + 1$ ($A_{p+1}$ is diagonal), the tuple of matrices $A_j$ is reducible.

So suppose that $D^2 = 0$ and $\text{rk}D \leq 2$. Let first $\text{rk}D = 2$ (this is possible only for $n > 3$). One can conjugate the matrices $A_j$ and $D$ by a block-diagonal matrix $W$ with three diagonal blocks, of sizes $1, n - 2, 1$, so that the matrix $D$ has at most three non-zero entries, in positions $(n - 1, 1)$, $(n, 2)$ and $(n, 1)$. The first two of them must be non-zero because $\text{rk}D = 2$. It follows from (4) that the entries in positions $(i, k)$ of the matrices $A_j$, $j = 1, \ldots, p$, are 0 for $i = 1, 2$; $k = 3, \ldots, n$ (the details are left for the reader). Hence, the tuple is reducible.

If $\text{rk}D = 1$, then by a similar conjugation $D$ is simplified to have non-zero entries only in one or both of positions $(n - 1, 1)$, $(n, 1)$ or $(n, 1)$, $(n, 2)$. In the first case the matrices $A_j$,
$j = 1, \ldots, p$ have zero entries in positions $(1,2), \ldots, (1,n)$, in the second case in positions $(1,n), \ldots, (n-1,n)$ and the tuple is reducible again.

The proposition is proved. □

**Proof of Proposition 19:**

1. The mapping $\phi$ is analytic. Indeed, one can fix the contours defining the generators of $\pi_1(\mathbb{C}P^1 \setminus \{a_1, \ldots, a_{p+1}\})$ which define the monodromy operators $M_1, \ldots, M_{p+1}$. In some neighbourhoods of the contours there are no poles and the dependence of the monodromy operators on the matrices-residua is analytic, i.e. $\phi$ is analytic.

2. Suppose that the tuples with trivial centralizers of matrices-residua of two different systems (1) are mapped onto one and the same tuple of monodromy operators $M_j$. Hence, the systems are obtained from one another by a linear change $X \mapsto W(t)X$ of the variables $X$ where the matrix $W(t)$ is holomorphic and holomorphically invertible outside the poles $a_j$ and is at most meromorphic at $a_j$. Indeed, one has $W = X_1(X_2)^{-1}$ where $X_i$ are fundamental solutions of the two systems; these solutions and their determinants grow no faster than some power of $(t - a_j)$ when $t \to a_j$. Hence, $W$ (and in the same way $W^{-1}$) is at most meromorphic at $a_j$. Outside $a_j$ both $X_1$ and $X_2$ are holomorphic and holomorphically invertible, hence, $W$ as well.

3. Under the change of the system $dX/dt = A_1(t)X$ changes into $dX/dt = A_2(t)X$ where $A_2(t) = -W^{-1}(dW/dt)W^{-1}A_1(t)W$. Hence, one must have $dW/dt = A_1(t)W - WA_2(t)$. Set $W = W_k/(t - a_j)^k + W_{k-1}/(t - a_j)^{k-1} + \ldots$ and suppose that $k \geq 1$. Denote by $A_j^i$ the matrices-residua of the two systems, $i = 1, 2$. Then one has $-kW_k = A_j^1W_k - W_kA_j^2$, i.e. $(A_j^1 + kI)W_k - W_kA_j^2 = 0$. The two matrices $A_j^1 + kI$ and $A_j^2$ have no eigenvalue in common; indeed, one has $A_j^1, A_j^2 \in c_j$ and $c_j$ has no eigenvalues differing by a non-zero integer. Hence, $W_k = 0$ (see about the matrix equation $AX - XB = 0$ in [Ga]), i.e. $W$ has no pole at $a_j$ for any $j$ and $W$ is a constant non-degenerate matrix.

4. The triviality of the centralizer of the tuple of matrices $A_j$ implies that $\phi$ is bijective (globally, from $V$ onto its image in $W$). An analytic bijective mapping is analytically invertible. This proves part 1) of the proposition. If the eigenvalues are generic, then every tuple of matrices $A_j$ from $V$ is irreducible, hence, with trivial centralizer from where part 2) of the proposition follows. □

**Proof of Proposition 20:**

1. Prove part 1) of the proposition. Two cases are possible:

   **Case 1** The centralizer contains a matrix with at least two distinct eigenvalues. Then the latter can be conjugated to a block-diagonal form with two diagonal blocks which have no eigenvalue in common. The commutation relations with such a matrix imply that the matrices $A_j$ or $M_j$ are themselves block-diagonal, i.e. they define a direct sum.

   **Case 2** Each matrix from the centralizer is with a single eigenvalue. Hence, the centralizer contains a nilpotent matrix $N$ and taking powers of it one can assume that $N^2 = 0$. One can conjugate $N$ to the form $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$. Hence, the matrices $A_j$ (or $M_j$) are blocked as follows: $\begin{pmatrix} L & Q & R \\ 0 & S & T \\ 0 & 0 & L \end{pmatrix}$ or $\begin{pmatrix} L & R \\ 0 & L \end{pmatrix}$. The presence of two equal diagonal blocks implies that $A_{p+1}$ (or $M_{p+1}$) has a multiple eigenvalue – a contradiction.

   Hence, one is necessarily in Case 1) and the representation defined by the matrices $A_j$ (or $M_j$) is a direct sum.

2. Prove part 2). If a matrix $Z$ from the centralizer is with non-trivial Jordan structure (i.e.
has at least one Jordan block of size $> 1$), then a suitable polynomial of $Z$ is a nilpotent matrix $N$, $N^2 = 0$, and like in case 2) one shows that $A_{p+1}$ or $M_{p+1}$ must have a multiple eigenvalue which is a contradiction. □

Proof of Proposition 21:

1'. Prove part 1) in the case of matrices $A_j$ (the case of matrices $M_j$ is left for the reader).

Note that one has $\text{Ext}^1(P_1, P_2) = R/Q$ (which implies that $\dim \text{Ext}^1(P_1, P_2) = \dim R - \dim Q$)

where

$$R = \{ (A_1^1 X_1 - X_1 A_2^1, \ldots, A_{p+1}^1 X_{p+1} - X_{p+1} A_{p+1}^2) | X_j \in M_{m_1 m_2}(C), \sum_{j=1}^{p+1} (A_j^1 X_j - X_j A_j^2) = 0 \}$$

$$Q = \{ (A_1^1 X - X A_2^1, \ldots, A_{p+1}^1 X - X A_{p+1}^2) | X \in M_{m_1 m_2}(C) \}$$

Indeed, if $A_j = \left( \begin{array}{cc} A_j^1 & 0 \\ 0 & A_j^2 \end{array} \right)$, $Y_j = \left( \begin{array}{cc} I & X_j \\ 0 & I \end{array} \right)$, then for the matrix $A_j' = (Y_j)^{-1} A_j Y_j$ one has $A_j' = \left( \begin{array}{cc} A_j^1 X_j - X_j A_j^2 \\ A_j^1 X_j - X_j A_j^2 \end{array} \right)$. Thus the space $R$ (resp. $Q$) is the one of tuples of right upper blocks of matrices $(Y_j)^{-1} A_j Y_j$ whose sum is 0 (resp. of tuples of matrices $Y^{-1} A_j Y$, i.e. right upper blocks obtained as a result of simultaneous conjugation by $Y$ of the form of $Y_j$).

On the other hand, the tangent space to the conjugacy class of the matrix $A_j$ (resp. of $A_j^1$ or of $A_j^2$) is $\{ [A_j, V_j] | V_j \in \mathfrak{gl}(m_1 + m_2, C) \}$ (resp. the latter’s restriction to the left upper or to the right lower block). The dimensions of these tangent spaces equal respectively $d_j^2$, $d_j^1$, $d_j^1$. The restrictions to the non-diagonal blocks being of equal dimension (hence, this dimension equals $(d_j^1 - d_j^2 - d_j^2)/2$), one deduces that $\dim R = (\sum_{j=1}^{p+1} (d_j^1 - d_j^2 - d_j^2)/2) - m_1 m_2$.

Subtracting $m_1 m_2$ corresponds to the condition the sum of the matrices $A_j^1 X_j - X_j A_j^2$ to equal 0. This condition is equivalent to a system of $m_1 m_2$ linear and linearly independent equations. Indeed, their linear dependence would imply the existence of a non-zero $m_2 \times m_1$-matrix $S$ such that

$$\text{tr}(S \sum_{j=1}^{p+1} (A_j^1 X_j - X_j A_j^2)) = \text{tr}(\sum_{j=1}^{p+1} (S A_j^1 - A_j^2 S) X_j) = 0$$

identically in the entries of the matrices $X_j$. Hence, for $j = 1, \ldots, p+1$ one has $SA_j^1 - A_j^2 S = 0$. For $j = p+1$ this implies that $S = 0$ because $A_{p+1}^1$ and $A_{p+1}^2$ have no eigenvalue in common – a contradiction. Obviously, $\dim Q = m_1 m_2$ (the map $X \mapsto A_{p+1}^1 X - X A_{p+1}^2$ is bijective because $A_{p+1}^1$ and $A_{p+1}^2$ have no eigenvalue in common) which implies the formula for $\xi$.

2'. Part 2) is trivial. To deform continuously a direct sum into a semidirect one it suffices to replace $X_j$ by $\varepsilon X_j$, $\varepsilon \in (C, 0)$.

3'. Prove part 3). The dimension of the variety $\mathcal{V}(c_1^1, \ldots, c_{p+1}^1)$ (resp. $\mathcal{V}(c_1^2, \ldots, c_{p+1}^2)$, $i = 1, 2$) where $c_j^i$ (resp. $c_j^i$) is the conjugacy class of $A_j$ (resp. $A_j^i$) equals $\nu' = \sum_{j=1}^{p+1} d_j^i - (m_1 + m_2)^2 + 1$ (resp. $\nu = \sum_{j=1}^{p+1} d_j^i - m_i^2 + 1$), see Proposition 11.

Show that if $\xi \geq 2$, then $\eta < \nu'$ where $\eta$ is the dimension of the variety $\mathcal{G}$ of semidirect sums of $P_1$ and $P_2$. One has $\eta = \eta_0 + m_1 m_2$ where $\eta_0$ is the dimension of the variety $\mathcal{G}_0$ of block upper-triangular matrices $A_j \in c_j^i$ whose diagonal blocks define the representations $P_i$. Adding $m_1 m_2$ corresponds to the possibility to obtain every tuple of matrices from $\mathcal{G}$ by conjugating a tuple from $\mathcal{G}_0$ by a matrix $\left( \begin{array}{cc} I & 0 \\ X & I \end{array} \right)$ where $X \in M_{m_2 m_1}(C)$.

One has $\eta_0 = \nu^1 + \nu^2 + \dim R$. We let the reader check oneself that the inequality $\xi > 1$ is equivalent to $\eta < \nu'$. The last inequality implies the existence of irreducible representations. □

11
3 Proof of Theorem 6

Proof of part 1):

1°. Consider the case of matrices $A_j$. Multiplying by $b \in \mathbb{C}^*$ or adding to $A_j$ of scalar matrices with zero sum changes an (ir)reducible tuple of matrices $A_j$ with zero sum into such a tuple, therefore it suffices to prove the theorem only in the case when $c_j \subset sI(n, \mathbb{C})$ and there is no non-zero integer difference between two eigenvalues (for every conjugacy class $c_j$).

Consider the mapping $\tau : (c_1 \times \ldots \times c_p) \rightarrow sI(n, \mathbb{C})$, $\tau : (A_1, \ldots, A_p) \mapsto -A_1 - \ldots - A_p$. Its graph $G$ is naturally isomorphic to $(c_1 \times \ldots \times c_p)$, hence, it is a connected smooth variety. Its subset $G_0$ consisting of $p$-tuples for which the matrix $A_{p+1} = -A_1 - \ldots - A_p$ is with distinct eigenvalues and the eigenvalues of the tuple $(A_1, \ldots, A_{p+1})$ are generic, is a Zariski open dense subset of $G$; hence, $G_0$ is connected.

2°. Consider the projection $\pi : G \rightarrow \mathbb{C}^{n-1}$ where $\mathbb{C}^{n-1}$ is the space of symmetric functions of the eigenvalues of $A_{p+1}$. For generic eigenvalues the fibres are non-empty smooth varieties; a priori they can consist of several components which are of one and the same dimension, and this dimension is the same for all generic eigenvalues. This follows from Theorem 5 (and the lines that follow it) and Proposition 11. The number of these components (denoted by $\chi$) is one and the same outside a proper algebraic subset $H \subset \mathbb{C}^{n-1}$. Hence, $\mathbb{C}^{n-1} \setminus H$ is connected.

Denote by $\mathbb{C}^{n-1}$ the space of eigenvalues of the matrix $A_{p+1}$, by $\beta$ the map “eigenvalues” $\mapsto "\text{symmetric functions of them}"$ and by $\Lambda$ the image (by $\beta$) in $\mathbb{C}^{n-1}$ of a hyperplane in $\mathbb{C}^{n+1}$ defined by an equation $\lambda_{i,1,p+1} = \lambda_{i,2,p+1}$ (recall that by $\lambda_{i,k}$ we denote the eigenvalues of $A_k$) and by $K$ the image in $\mathbb{C}^{n-1}$ of a hyperplane in $\mathbb{C}^{n-1}$ defined by some non-genericity relation. Denote by $\tilde{\Lambda}$ (resp. $\check{K}$) the union of all sets $\Lambda$ (resp. $K$). Consider a closed oriented contour $\gamma \subset \mathbb{C}^{n-1}\setminus(\tilde{\Lambda} \cup \check{K} \cup H)$. The monodromy of $\gamma$ is by definition the permutation operator acting on the components of the fibre $\pi^{-1}(a)$, $a \in \gamma$, when $\gamma$ is run once from $a$ to $a$. Up to conjugacy, this operator does not depend on the choice of $a \in \gamma$.

3°. We show that the monodromy of every such contour $\gamma$ is trivial. This together with the connectedness of $G_0$ implies that each fibre over $\mathbb{C}^{n-1}\setminus(\tilde{\Lambda} \cup \check{K} \cup H)$ consists of a single component which proves the connectedness of the varieties $\mathcal{V}$ for such eigenvalues.

Lemma 23 The fibres over $H\setminus(\tilde{\Lambda} \cup \check{K})$ are also non-empty and connected.

All lemmas from this section are proved in Section 4.

Proving the triviality of the monodromy of each contour $\gamma$ is tantamount to showing that the monodromy of each small lace $\gamma' \subset \mathbb{C}^{n-1}$ around each set $\Lambda$ (or $\check{K}$) in $\mathbb{C}^{n-1}$ and around each component of $H$ of codimension 1 in $\mathbb{C}^{n-1}$ is trivial. The lace is presumed to be contractible to a point $\Omega$ from $\tilde{\Lambda}$ (or $\check{K}$, or $H$). It suffices to consider two cases:

A) $\Omega \in \tilde{\Lambda}\setminus\check{K}$; we do not subtract $H$ because we cannot claim that $H$ does not contain components of $\tilde{\Lambda}$ of maximal dimension; it is sufficient to consider only the case when $A_{p+1}$ has a single double eigenvalue, its other eigenvalues being simple;

B) $\Omega \in (\check{K} \cup H)\setminus\tilde{\Lambda}$; in this case $A_{p+1}$ has distinct eigenvalues.

4°. Consider Case B) first (in 4° – 5°). The eigenvalues of $A_{p+1}$ are defined by their symmetric functions only up to permutation. Fix these eigenvalues at $\Omega$, i.e. choose one of their permutations. By continuity this defines such a permutation for all points from $\mathbb{C}^{n-1}$ sufficiently close to $\Omega$ because the eigenvalues are distinct. In other terms, one can consider the covering $\beta : \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}$. There are $n!$ sheats that cover the neighbourhood of any point from $\mathbb{C}^{n-1}\setminus\tilde{\Lambda}$. Fixing the permutation of the eigenvalues means fixing one of these sheats.

For each $n$-vector $\vec{v}$ with integer components whose sum is 0 shift the eigenvalues of $A_{p+1}$ by $\vec{v}$. This shifts their symmetric functions by some vector $\vec{v}'$ (recall part D) of Subsection 2.1).
Denote by $\gamma'(\vec{v}')$ (resp. by $c_{p+1}(\vec{v})$) the lace $\gamma'$ shifted by $\vec{v}'$ (resp. the semi-simple conjugacy class in $sl(n, \mathbb{C})$ whose eigenvalues are shifted w.r.t. the ones of $c_{p+1}$ by $\vec{v}$). Hence, there exists $\vec{v} = \vec{v}_0$ such that $\gamma'(\vec{v}_0)$ is contractible in $\mathbb{C}^{n-1}\setminus(\tilde{\Lambda} \cup \tilde{K} \cup H)$. This follows from the algebraicity of $\Lambda \cup \tilde{K} \cup H$. Hence, the monodromy of $\gamma'(\vec{v}_0)$ is trivial.

5°. Show that the monodromies of $\gamma'$ and $\gamma'(\vec{v}_0)$ are the same. By deforming $\gamma'$ in $\mathbb{C}^{n-1}\setminus(\tilde{\Lambda} \cup \tilde{K} \cup H)$ one can achieve the condition that for every vector $\vec{v}_0$ and for every point of $\gamma'(\vec{v}_0)$ the matrix $A_{p+1}$ has no eigenvalues differing by a non-zero integer.

Denote by $\mathbf{C}'$ the space of symmetric functions of the eigenvalues of the matrix $M_{p+1} = (M_1 \ldots M_p)^{-1}$ where $M_j \in C_j$ for $j \leq p$. Hence, $\mathbf{C}' = \mathbb{C}^n \cap \{ \sigma = 1 \}$, where $\sigma = \sigma_{1,p+1} \ldots \sigma_{n,p+1}$, and $\mathbf{C}'$ is a non-singular variety. Define its subvarieties $\Lambda^*, \tilde{K}^*$ by analogy with $\Lambda$, $\tilde{K}$ (see 2°) – consider the map $\zeta$: “eigenvalues” $\rightarrow$ “symmetric functions of them”. The set $\Lambda^*$ (resp. $\tilde{K}^*$) is the union of the intersections with $\mathbf{C}'$ of the images by $\zeta$ in $\mathbb{C}^n$ of hyperplanes $\sigma_{j_1,p+1} = \sigma_{j_2,p+1}$ (resp. of the varieties defined by non-genericity relations).

There exists a proper algebraic subset $H^* \subset \mathbf{C}'$ such that for $(\sigma_{1,p+1}, \ldots, \sigma_{n,p+1}) \in \mathbf{C}' \setminus (\Lambda^* \cup \tilde{K}^* \cup H^*)$ the sets $\mathcal{W}$ are smooth varieties with one and the same number $\chi^*$ of components. Notice that their dimension is the same and it is one and the same for all such sets of eigenvalues of $M_{p+1}$. This dimension is the same as the dimension of the components of $\mathcal{V}$ for eigenvalues of $c_{p+1}$ from $\mathbb{C}^{n-1}\setminus(\Lambda \cup \tilde{K} \cup H)$, see Proposition 11.

**Lemma 24** One has $\chi^* = \chi$.

The mapping $\phi$ (see part E) of Subsection 2.1) defines in a natural way images in $\mathbf{C}'$ of the contours $\gamma'$ and $\gamma'(\vec{v}_0)$. These images coincide; denote this common image by $\gamma^*$. The monodromies of $\gamma'$ and $\gamma'(\vec{v}_0)$ are the same as the monodromy of $\gamma^*$. Hence, these monodromies coincide and the monodromy of $\gamma'$ is trivial.

6°. Consider Case A). Choose like in 4° a permutation of the eigenvalues of $A_{p+1}$ at $\Omega$ (the permutations are $n!/2$ because there is one double eigenvalue). Denote by $\tilde{\Omega} \in \mathbb{C}^{n-1}$ the point defined by such eigenvalues. Any neighbourhood of $\tilde{\Omega}$ defines a two-fold covering of some neighbourhood of $\Omega$ via $\beta$.

The set $\mathcal{V}$ at $\tilde{\Omega}$ is a smooth non-empty algebraic variety. A point from it is a tuple of matrices $A_j$ with zero sum. If in this tuple the matrix $A_{p+1}$ is diagonalizable, then it can be analytically deformed into another one from $\mathcal{V}$ with “more generic” JNF $J(A_{p+1})$, i.e. one in which $A_{p+1}$ has a single Jordan block of size 2 corresponding to its double eigenvalue (see the description of the basic technical tool in part B) of Subsection 2.1 and Example 13 there). So assume that $A_{p+1}$ has this “more generic” JNF. Then deform the tuple into one where $A_{p+1}$ has distinct eigenvalues (see Example 14). The parameters of the deformation can be chosen to belong to a neighbourhood $\omega$ of $\tilde{\Omega}$ in some one-dimensional variety transversal at $\tilde{\Omega}$ to the lifting in $\mathbb{C}^{n-1}$ of $\tilde{\Lambda}$.

Thus one can identify the components of the variety $\mathcal{V}$ at $\tilde{\Omega}$ with some or all components of the varieties $\mathcal{V}$ at the other points from $\omega$. A priori the varieties $\mathcal{V}$ at points from $\omega \setminus \tilde{\Omega}$ have no less (say, $k_0$) components than the one at $\tilde{\Omega}$.

The union of varieties $\mathcal{V}$ over $\omega \setminus \tilde{\Omega}$ defines a trivial fibration over $\omega \setminus \tilde{\Omega}$. This implies that the monodromies along laces from Case A) are also trivial.

To prove the claim use similar ideas to the ones from Case B), see 4°. Namely, there exists a shift of the eigenvalues of $A_{p+1}$ defined at a point from $\omega$ by a vector $\vec{v}$ (with integer components whose sum is 0) so that the shifted neighbourhood $\omega(\vec{v})$ does not intersect with the lifting in $\tilde{\mathbb{C}}^{n-1}$ of the set $\tilde{\Lambda} \cup \tilde{K} \cup H$. Hence, all varieties $\mathcal{V}$ at points from $\omega(\vec{v})$ have $k_0$ components. They form a trivial fibration over $\omega(\vec{v})$ (to see this use again the basic technical tool).
The mapping $\phi$ (see part E) of Subsection 2.1) defined for eigenvalues from $\omega(\vec{v})$ (we denote this mapping by $\phi_1$) allows one to identify the components of the varieties $V$ over $\omega(\vec{v})$ and the varieties $W$ over $\phi(\omega(\vec{v}))$. Hence, the components of the varieties $W$ over $\phi(\omega(\vec{v}))$ define a trivial fibration over $\phi(\omega(\vec{v}))$.

On the other hand, one can define the mapping $\phi$ (and at least almost everywhere in the varieties $W$, i.e. on the complements of proper analytic subsets, the mapping $\phi^{-1}$) for eigenvalues from $\omega\setminus\bar{\Omega}$; denote this mapping by $\phi_2$.

This means that the components of the varieties $V$ define a trivial fibration over $\omega\setminus\bar{\Omega}$. Indeed, the varieties $V$ over points from $\omega(\vec{v})$ and the respective points from $\omega$ (i.e. shifted by $-\vec{v}$) define the matrices-residua of Fuchsian systems with one and the same monodromy; hence, there exists a linear change $X \mapsto W(t)X$ (which is the map $\phi_2^{-1} \circ \phi_1$) that transforms the first systems into the second ones. This change defines a birational diffeomorphism between varieties $V$ over $\omega(\vec{v})\setminus\Omega(\vec{v})$ and $\omega\setminus\bar{\Omega}$ (where $\Omega(\vec{v})$ is $\Omega$ shifted by $\vec{v}$; one excludes $\Omega$ because it is not clear whether the number of components of the variety $V$ over $\Omega$ is the same as for points from $\omega\setminus\bar{\Omega}$). The triviality of the fibration over $\omega(\vec{v})\setminus\Omega(\vec{v})$ implies the one over $\omega\setminus\bar{\Omega}$.

Hence, the monodromies along laces from Case A) are trivial. This proves part 1) of the theorem for matrices $A_j$.

Part 1) of the theorem is proved.

Proof of part 2):

Every tuple of matrices $A_j$ from the algebraic closure of $V$ and with non-trivial centralizer can be continuously deformed into one from $V$ and with trivial centralizer. The same is true for the varieties $W$.

The statement implies that the algebraic closure of $V$ belongs to its topological closure (for $W$ the proof of the statement is analogous and we skip it off). On the other hand, the topological closure belongs to the algebraic one – the eigenvalues of the classes $c_j$ remain the same on the topological closure of $V$ (by continuity); the ranks of the matrices $(A_j - \lambda_{k,j}I)^s$ remain the same on $V$ and on its topological closure they can only drop; hence, on the topological closure of $V$ the matrices $A_j$ belong to the topological closures of the conjugacy classes $c_j$ which are also their algebraic closures; the condition the sum of the matrices to be 0 is preserved (by continuity) on the topological closure; all this means that a tuple of matrices from the topological closure of $V$ belongs to its algebraic closure.

To prove the statement we use the following lemmas:

Lemma 25 A tuple of matrices $A_j$ from the algebraic closure of $V$ can be continuously deformed into one from $V$. The same is true for varieties $W$.

Lemma 26 Suppose that the tuples of matrices $A_j^i$, $i = 1, 2$, define two representations $P_i$ (of ranks $m_i$) with trivial centralizers where $\sum_{j=1}^{p+1} A_j^i = 0$, $i = 1, 2$, and the matrices $A_{p+1}^1$, $A_{p+1}^2$ have each distinct eigenvalues and no eigenvalue in common. Then there exist representations (defined by $p + 1$ matrices whose sum is 0) which are semi-direct sums of $P_1$ and $P_2$ (in both possible orders) and with trivial centralizers at least in the following cases:

1) $m_1 \geq 3$, $m_2 \geq 2$;

2) $m_1 = m_2 = 2$ and for at least one index $j$, $1 \leq j \leq p$, the matrices $A_j^1$ and $A_j^2$ belong to different conjugacy classes;
3) \( m_1 = m_2 = 1 \) and for at least two indices \( j, 1 \leq j \leq p \), the matrices \( A_j \) and \( A_j^2 \) are different;

4) \( m_1 = 2, m_2 = 1 \) and for at least one index \( j, 1 \leq j \leq p \), the matrices \( A_j \) have no eigenvalue in common;

5) \( m_1 > 1, m_2 = 1 \) and \( r_1^1 + \ldots + r_p^1 > m_1 \) where \( r_j^1 = r(A_j^1) \);

6) \( m_1 = m_2 = 2 \) and for at least three indices \( j, 1 \leq j \leq p \), at least one of the matrices \( A_j \), \( i = 1, 2 \), is not scalar.

A similar statement holds for matrices \( M_1^1, M_1^1 \ldots M_{p+1}^1 = I \).

Given a tuple of matrices from \( \mathcal{V} \) and with non-trivial centralizer conjugate it to a block-diagonal form the diagonal blocks defining representations with trivial centralizers, see Proposition 20. If one can find a couple of such representations for which Lemma 26 is applicable, then one can replace this couple by its semi-direct sum (which is with trivial centralizer) and thus reduce the number of diagonal blocks. When this number is 1, then the representation is with trivial centralizer. So describe all cases when the lemma is not applicable:

**Case A)** All diagonal blocks are of sizes 1 and/or 2.

**Case B)** There is one diagonal block of size > 2 and the rest are of size 1.

Indeed, there can be not more than one block of size > 2, otherwise we are in case 1) of the lemma, and if there is such a block, then there are no blocks of size 2 for the same reason.

**Remark 27** In both cases A) and B) if there is more than one diagonal block of size 1, then for all indices \( j = 1, \ldots, p \) but one (say, but for \( j = 1 \)) the restrictions of the matrices \( A_j \) to all diagonal blocks of size 1 are the same, otherwise case 3) of Lemma 26 is applicable.

In case A) for \( j = 1, \ldots, p \) the conjugacy classes of the restrictions of the matrices \( A_j \) to all blocks of size 2 are the same (to avoid case 2) of the lemma) and only for two of these indices (say, 1 and 2) are the restrictions of the matrices \( A_j \) to the diagonal blocks of size 2 non-scalar (to avoid case 6)). The one-dimensional blocks must have eigenvalues which for \( j = 1, \ldots, p \) are eigenvalues of the two-dimensional blocks (to avoid case 4)) and Remark 27 holds. Hence, the matrices (if any) \( A_3, \ldots, A_p \) are scalar and one can assume that \( p = 2 \); each of the matrices \( A_1, A_2 \) is either diagonalizable, with two eigenvalues, or is with a single eigenvalue and with Jordan blocks of sizes \( \leq 2 \). Hence, for \( j = 1, 2 \) one has \( d_j \leq n^2/2 \) and as \( d_3 = n^2 - n \), condition \((\alpha_n)\) does not hold for \( n > 2 \). Thus case A) is possible only for \( n = 2 \) and in this case either there is a single diagonal block and the centralizer is trivial or one can construct a semi-direct sum of the two one-dimensional blocks (one is in case 3) of the lemma, otherwise condition \((\alpha_2)\) fails.

In case B) for \( j = 2, \ldots, p \) the restrictions of \( A_j \) to the one-dimensional blocks are equal (see Remark 27). Moreover, for \( j = 1, \ldots, p \) these restrictions are eigenvalues \( \lambda_j \) of the restrictions \( A_j^0 \) of \( A_j \) to the block of size \( m_1 > 1 \) for which one has \( \text{rk}(A_j^0 - \lambda_j I) = r(A_j^0) \). (Indeed, if not, then for the representation \( P^0 \) defined by the matrices \( A_j^0 \) and for some diagonal block \( P^1 \) one will have \( \text{dim Ext}^1(P^0, P^1) > r(A_j^0) + \ldots + r(A_j^0) + m_1 - 2m_1 \geq 0 \) and one can construct a semi-direct sum of \( P^0 \) and \( P^1 \); compare with case 5) of the lemma.) But then one has \( r_1 + \ldots + r_p < n \), i.e. condition \((\beta_n)\) does not hold for the matrices \( A_j \) (the reader should go into the details oneself). Hence, case B) is also impossible.

9\(^9\). If a variety \( \mathcal{V} \) contains irreducible tuples, then its algebraic and topological closure is the one of its subset of irreducible tuples. The latter is a connected smooth variety. The same statements hold for varieties \( \mathcal{W} \).

Fix distinct points \( a_j \in \mathbb{C}, j = 1, \ldots, p + 1 \) which will be poles of Fuchsian systems. By multiplying the matrices by \( \alpha \in \mathbb{C}^* \) one can achieve the condition no sum of some of the eigenvalues of the matrices \( A_j \) and no difference \( \lambda_{k_1,j} - \lambda_{k_2,j} \) to be a non-zero integer. This
implies (see Theorem 5.1.2 from [Bo1]) that if a tuple of matrices \( A_j \) is irreducible, then the monodromy group of a Fuchsian system with poles \( a_j \) is irreducible as well.

Define the variety \( W(C_1, \ldots, C_{p+1}) \) after \( V(c_1, \ldots, c_{p+1}) \) like in Subsection 2.1, part E), i.e. \( C_j \) is the conjugacy class of the matrix \( M(A_j) \).

Denote by \( A^1, A^2 \) two tuples of matrices from \( V \), the first irreducible and the second with trivial centralizer. In what follows we allow \( A^2 \) to be replaced in the course of the proof by a tuple arbitrarily close to it, with trivial centralizer and from the same component of \( V \) as \( A^2 \). We show that the component of \( V \) to which \( A^2 \) belongs, lies in the closure of the component of \( A^1 \). Hence, the same will be true without the assumption the centralizer of \( A^2 \) to be trivial, see the statement from 80. Denote by \( M^1, M^2 \) the images of \( A^1, A^2 \) in \( W \) by \( \phi \), see Subsection 2.1, part E).

The monodromy group defined by \( M^1 \) can be realized by matrices-residua from \( V(c_1, \ldots, c_p, c_{p+1}(\vec{v})) \) for all but finitely many vectors \( \vec{v} \) (see 40 and Corollary 17).

**Lemma 28** The monodromy group defined by \( M^2 \) or a monodromy group with trivial centralizer close to it and from the same component of \( W \) as \( M^2 \), can be realized by matrices-residua from \( V(c_1, \ldots, c_p, c_{p+1}(\vec{v})) \) for infinitely many vectors \( \vec{v} \).

Hence, there exists a vector \( \vec{v}_0 \) for which the eigenvalues of \( V(c_1, \ldots, c_p, c_{p+1}(\vec{v}_0)) \) are generic and the latter variety contains tuples \( A^1, A^2 \) for which the corresponding tuples of monodromy operators equal \( M^1, M^2 \). Denote by \( w \) the linear change of variables \( X \mapsto W(t)X \) (where \( W \) is meromorphic on \( \mathbf{C}P^1 \) and holomorphically invertible for \( t \neq a_{p+1} \)) which transforms a system with matrices-residua from \( V_1 = V(c_1, \ldots, c_p, c_{p+1}) \) into one with matrices-residua from \( V_2 = V(c_1, \ldots, c_p, c_{p+1}(\vec{v}_0)) \). Recall that \( V_2 \) is a smooth and connected variety (this follows from part 1) of the theorem which is already proved). There are proper algebraic subsets \( U_i \subset V_i \) where \( w \) or \( w^{-1} \) is not defined.

One can connect \( A^1, A^2 \) in \( V_2 \) by a contour \( \gamma \) such that all points of \( \gamma \) define Fuchsian systems with trivial centralizers of their monodromy groups; indeed, the tuples of matrices-residua for which the centralizer of the monodromy group is non-trivial form a proper analytic subset of \( V_2 \). Moreover, one can require \( \gamma \) to avoid the set \( U_2 \). Hence, \( w^{-1}(\gamma) \) is a contour which connects \( A^1 \) with \( A^2 \) in \( V_1 \) and which contains no tuples with non-trivial centralizers. The tuple \( A^1 \) is irreducible; this means that the component of \( V_1 \) to which \( A^2 \) belongs, lies in the closure of the one of \( A^1 \).

If \( A^2 \) is irreducible as well, then the above reasoning implies that the components of \( A^1 \) and \( A^2 \) are parts of one and the same smooth variety. Hence, these components coincide.

Hence, one can connect \( M^1, M^2 \) by the contour \( \phi(\gamma) \) which avoids monodromy groups with non-trivial centralizers; this implies (the details are left for the reader) that if \( M^2 \) is reducible, then it belongs to a proper subvariety of \( W(C_1, \ldots, C_{p+1}) \) from the closure of the set of reducible tuples of \( W(C_1, \ldots, C_{p+1}) \). If \( M^2 \) is irreducible, then (like above, for \( A^1, A^2 \) it belongs to one and the same components of the set of irreducible tuples of \( W(C_1, \ldots, C_{p+1}) \) as \( M^1 \). There remains to observe that the reasoning about \( M^1, M^2 \) (including Lemma 28) can be performed without defining \( M^i \) as \( \phi(A^i) \). Implicitly we use here the following result (see [ArII], p. 132-133): if the monodromy operator \( M_{p+1} \) is diagonalizable, then the monodromy group is realizable by a Fuchsian system with \( J(A_j) = J(M_j) \) for \( j = 1, \ldots, p \).

The statement is proved.

**Proof of part 3):**

100. Prove part 3) of the theorem for matrices \( M_j \) first. Suppose that for the conjugacy classes \( C_j \) conditions \((\alpha_n)\) and \((\beta_n)\) hold (see Proposition 3 and Theorem 5) and that \( C_{p+1} \) is
with distinct eigenvalues. Suppose that the classes \( c_j \) are defined after \( C_j \) so that for \( A_j \in c_j \) one has \( \exp(2\pi i A_j) \in C_j \), see Subsection 2.1, part E). We suppose that for every \( j \) there is no non-zero integer difference between two eigenvalues of \( c_j \). In addition, one can choose the eigenvalues of all classes \( c_j \) to be generic (because one can shift by arbitrary integers whose sum is 0 the eigenvalues of \( c_{p+1} \)). Hence, the variety \( \mathcal{V}(c_1, \ldots, c_{p+1}) \) is non-empty (because the classes \( c_j \) also satisfy conditions \((\alpha_n)\) and \((\beta_n)\), see Proposition 3, Theorem 5 and the lines that follow it) and so is the variety \( \mathcal{W}(C_1, \ldots, C_{p+1}) \).

Every point of \( \mathcal{W}(C_1, \ldots, C_{p+1}) \) which is the image under the mapping \( \phi \), see Subsection 2.1, part E), is a tuple with trivial centralizer. Indeed, if the centralizer is non-trivial, then the monodromy group is a direct sum (see Proposition 20). The sum \( s_1 \) of the eigenvalues of the matrices \( A_j \) which are exponents relative to an invariant subspace \( S_1 \) must be 0. Indeed, by [Bo2], Lemma 3.6, this sum is a non-positive integer; in the case of a direct sum, there is an invariant subspace \( S_2 \) such that \( S_1 \oplus S_2 = \mathbb{C}^n \); the sum \( s_2 \) of the exponents relative to \( S_2 \) is also a non-positive integer and \( s_1 + s_2 = 0 \) (the sum of all eigenvalues of the matrices \( A_j \) is 0; one can assume that there are no eigenvalues participating both in \( s_1 \) and \( s_2 \) because there is no non-zero integer difference between any two eigenvalues of any matrix \( A_j \), hence, to each eigenvalue \( \sigma \) of \( M_j \) there corresponds a single eigenvalue \( \lambda \) of \( A_j \) such that \( \sigma = \exp(2\pi i \lambda) \)). Hence, \( s_1 = s_2 = 0 \).

However, these equalities contradict the genericity of the eigenvalues of the classes \( c_j \).

Hence, the weak DSP is solvable for the classes \( C_j \).

110. Prove part 3) of the theorem for matrices \( A_j \). It is necessary to prove it only for non-generic eigenvalues. Given the conjugacy classes \( c_j \) (satisfying conditions \((\alpha_n)\) and \((\beta_n)\)) fix a vector \( \vec{v} \) with integer components whose sum is 0 such that the conjugacy classes \( c_1, \ldots, c_p, c_{p+1}(\vec{v}) \) are with generic eigenvalues (see 40). Hence, the variety \( \mathcal{V}(c_1, \ldots, c_p, c_{p+1}(\vec{v})) \) is non-empty and smooth.

Fix a point \( A \in \mathcal{V}(c_1, \ldots, c_p, c_{p+1}(\vec{v})) \) and distinct points \( a_j \in \mathbb{C}, 1 \leq j \leq p + 1 \). Fix a sequence of vectors \( \vec{v}_i, i = 1, \ldots, \mu, \vec{v}_1 = \vec{v}, \vec{v}_\mu = \vec{0} \) where \( \vec{v}_{i+1} = \vec{v}_i + \vec{w}_i, \vec{w}_i \) being a vector two of whose components equal 1 and \(-1\) the others being 0. We want the vectors \( \vec{v}_i \) to satisfy

**Condition (R).** If for \( i = i_0 \) the eigenvalues of the conjugacy classes \( c_1, \ldots, c_p, c_{p+1}(\vec{v}_i) \) satisfy some non-genericity relation, then this relation is satisfied by their eigenvalues also for \( i > i_0 \).

Denote by \( P_i \) the Procedure \((l_i, k_i)\) which if possible to perform would shift the eigenvalues of \( A_{p+1} \) by \( \vec{w}_i \). Hence, if all procedures \( P_i \) are possible to perform (in the prescribed order), then they would lead from \( A \) to a point from \( \mathcal{V}(c_1, \ldots, c_p, c_{p+1}) \). Call Procedure \( Q_i \) the superposition \( P_i \circ \ldots \circ P_1 \).

If \( Q_i(A) \) is an irreducible tuple but \( P_{i+1} \) cannot be performed because the condition \( c \neq 0 \) does not hold (see Subsection 2.1, part D)), then one changes a little the positions of the poles \( a_j \) and one chooses an irreducible tuple close to \( Q_i(A) \) to which one applies \( P_{i+1} \), see Proposition 18.

If \( Q_i(A) \) is reducible, then conjugate it to a block upper-triangular form with diagonal blocks defining irreducible representations. Denote by \( B_i \) the tuple which has the same diagonal blocks as \( Q_i(A) \) and zeros elsewhere. Continue in the same way to perform the next Procedures \( P_i \) only to the necessary diagonal blocks (the fact that it will not be necessary to perform such procedures involving different diagonal blocks follows from Condition (R)). In the end one obtains a tuple from the closure of \( \mathcal{V}(c_1, \ldots, c_p, c_{p+1}) \) which by the statement from 80 can be deformed into a tuple from \( \mathcal{V}(c_1, \ldots, c_p, c_{p+1}) \) with trivial centralizer.

Part 3) of the theorem is proved. ☐
4 Proofs of the lemmas

Proof of Lemma 23:

The nonemptiness of the fibres follows from Theorem 5 and the lines that follow it. Suppose that such a fibre consists of at least 2 components. Choose two points – $A^1$, $A^2$ – from two different components. Assume that no sum of eigenvalues of some of the matrices $A_j$ is a non-zero integer; this can be achieved by multiplying all matrices $A_j$ by $b \in \mathbb{C}^*$. (Such a multiplication changes the conjugacy classes, but it is a diffeomorphism of fibres defined for the different tuples of conjugacy classes.) Hence, for fixed poles $a_j$ of system (1) the monodromy groups of the two systems $(F_1)$ and $(F_2)$, with tuples of matrices-residua $A^1$, $A^2$, are irreducible (this follows from Theorem 5.1.2 from [Bo1]).

By Corollary 17, there exist tuples of matrices-residua $A^1_1$, $A^2_1$ from one and the same variety $V(c_1, \ldots, c_p, c'_{p+1})$ (where the eigenvalues of $c'_{p+1}$ are obtained from the ones of $c_{p+1}$ by an admissible shift, see Subsection 2.1, part D) such that the Fuchsian systems (1) (denoted by $(F_1s)$, $(F_2s)$) with tuples of matrices-residua $A^1_1$, $A^2_1$ have the same monodromy groups as systems $(F_1)$, $(F_2)$. We say that the above operation “a couple of Jordan blocks with the same eigenvalue and of sizes $l, s$” (which implies that it is true in the general case as well.) Hence, for fixed poles $a_j$ of system (1) the monodromy groups of the two systems $(F_1)$ and $(F_2)$, with tuples of matrices-residua $A^1,s$, $A^2,s$, are irreducible (this follows from Theorem 5.1.2 from [Bo1]).

Denote by $w$ the linear change of variables $X \mapsto V(t)X$ bringing systems $(F_i)$ to systems $(F_is), i = 1, 2$. One can connect $A^1_i$ with $A^2_i$ by a contour $\gamma \subset V(c_1, \ldots, c_p, c'_{p+1})$ by bypassing the points where $w$ is not defined (these points form a proper subvariety of positive codimension). Hence, $w^{-1}(\gamma)$ connects $A^1$ with $A^2$ in $V(c_1, \ldots, c_{p+1})$ which is a contradiction. □

Proof of Lemma 24:

Fix a point (i.e. a tuple of matrices $M_{j_1}$) from each component of the variety $W(C_1, \ldots, C_{p+1})$ where the eigenvalues of $C_{p+1}$ are from $C^0(\Lambda^* \cup \hat{K}^* \cup H^*)$. There exists a vector $\vec{v}_1$ such that these tuples are realized as tuples of monodromy operators of Fuchsian systems with matrices-residua $A_0 \in c_j, j \leq p, A_{p+1} \in c_{p+1}(\vec{v}_1)$; the existence of $\vec{v}_1$ follows from Corollary 17. These tuples of matrices-residua belong to different components of the variety $V(c_1, \ldots, c_{p+1}(\vec{v}_1))$; and conversely, points from different (from same) components of $V(c_1, \ldots, c_{p+1}(\vec{v}_1))$ are mapped by $\phi$ into points from different (from same) components of $W(C_1, \ldots, C_{p+1})$; this follows from Proposition 19. □

Proof of Lemma 25:

1º. We prove the lemma for matrices $A_j$, for matrices $M_j$ the proof is much the same. Conjugate the tuple to a block-diagonal form with diagonal blocks defining representations with trivial centralizers, see Proposition 20. In particular, there might be a single diagonal block.

If the given tuple is not from $V(c_1, \ldots, c_{p+1})$ but from its closure, then for some $j = j_1$ the conjugacy class of some of the matrices $A_{j_1}$ (we denote it by $c^{0}_{j_1}$) is not $c_{j_1}$ but belongs to its closure (one has $j_1 < p + 1$ because $A_{p+1}$ is with distinct eigenvalues). Given a conjugacy class from $gl(n, \mathbb{C})$ (which defines a JNF $J_1$), for any conjugacy class containing it in its closure and defining a JNF $J_2$ we say that $J_1$ is subordinate to $J_2$ (notation: $J_1 \prec J_2$).

2º. In what follows we use the following fact (see [He]): For $J_1$, $J_2$ as above (i.e. $J_1 \prec J_2$) one has that $J_2$ is obtained from $J_1$ as a superposition of one or several operations of the form “a couple of Jordan blocks with one and the same eigenvalue and of sizes $(l, s)$, $l \geq s$, is replaced by a couple of Jordan blocks with the same eigenvalue and of sizes $(l + 1, s - 1)$ while the other Jordan blocks (if any) remain the same”. (This fact is proved in [He] only for nilpotent conjugacy classes which implies that it is true in the general case as well.) We say that the above operation is of type $(l, s)$.

Example 29 Consider the family of matrices $A(\varepsilon) = \left( \begin{array}{cc} J' & \varepsilon D \\ 0 & J'' \end{array} \right)$ where $J', J''$ are two Jordan
blocks with the same eigenvalue, of sizes $s'$, $s''$, $s' + s'' = n$, and $\varepsilon \in (C, 0)$. If $s' \geq s''$ and $D$ has a single non-zero entry which is in position $(s', n)$ of $A$, then for $\varepsilon \neq 0$, $J(A)$ consists of two Jordan blocks, of sizes $s' + 1, s'' - 1$. If $s' < s''$, then for $D$ having a single non-zero entry, in position $(1, s' + 1)$, $J(A)$ consists of two Jordan blocks, of sizes $s'' + 1, s' - 1$.

3°. Fix a sequence of operations of type $(l, s)$ leading from $c_j^0$ to $c_j$ (and, hence, from $J(c_j^0)$ to $J(c_j)$). Suppose that an operation of this sequence can be performed between two Jordan blocks corresponding to the restriction of $A_j$ to one and the same diagonal block. The restrictions of the matrices $A_j$ to the given diagonal block defining a representation with trivial centralizer, one can apply the basic technical tool (see part B) of Subsection 2.1 – one deforms $A_j$ into $A_j + \varepsilon G$ where for $\varepsilon \neq 0$ one has $A_j + \varepsilon G \in c_j$ and one deforms analytically the restrictions $A_j^*$ of the other matrices $A_j$ to the given diagonal block by conjugating them so that the sum of the restrictions $A_j^*$ be 0.

If an operation of the sequence has to be performed between two Jordan blocks, from two different diagonal blocks, then without loss of generality one can assume that these are the first two diagonal blocks. To ease the notation we consider only the case when there are only two diagonal blocks (in the general case the proof is the same). The matrices $A_j$ can be presented in the form $A_j = \begin{pmatrix} B_j & 0 \\ 0 & F_j \end{pmatrix}$. One can deform $A_j$ into $A_j + \varepsilon G \in c_j$ by changing only the right upper block of $A_j$ (use the above example). After this one can conjugate the matrix $A_{p+1}$ by a matrix of the form $\begin{pmatrix} I & \varepsilon Y \\ 0 & I \end{pmatrix}$ where $Y$ is chosen such that the sum of the matrices remains 0. This conjugation changes the right upper block by $\varepsilon(B_{p+1}Y - YF_{p+1})$ and the possibility to choose $Y$ follows from the fact that $B_{p+1}$ and $F_{p+1}$ have no eigenvalue in common and, hence, the linear operator $Y \mapsto B_{p+1}Y - YF_{p+1}$ is bijective.

Hence, after finitely many operations of the form $(l, s)$ one obtains a tuple of matrices from $\mathcal{V}(c_1, \ldots, c_{p+1})$. □

**Proof of Lemma 26:**

1°. One has to check that in all 6 cases one has $\dim \text{Ext}^1(P_1, P_2) > 0$. We do this in detail only in case 1), the most difficult one. Recall that one has $\dim \text{Ext}^1(P_1, P_2) = R/Q$, hence, $\dim \text{Ext}^1(P_1, P_2) = \dim R - \dim Q$ where the spaces $R$ and $Q$ were defined in the proof of Proposition 21.

Define the linear operator $\xi_j : M_{m_1, m_2}(C) \rightarrow M_{m_1, m_2}(C)$ by $\xi_j : (.) \mapsto A_j^1( . ) - ( . ) A_j^2$. Set $R_j = \{ A_j^1X_j - X_jA_j^2 | X_j \in M_{m_1, m_2}(C) \}$. One has $\dim R_{p+1} = m_1m_2$ because $A_{i+1}^j$, $i = 1, 2$, have no eigenvalue in common.

Denote by $r_j^1, d_j^1$ the quantities $r_j, d_j$ computed for the matrices $A_j^1$, see the Introduction.

2°. For $j \leq p$ one has $\dim R_j \geq r_j^1m_2$. This is proved in 5°.

Hence,

$$\dim \text{Ext}^1(P_1, P_2) = \sum_{j=1}^{p+1} \dim R_j - m_1m_2 - \dim Q \geq m_1m_2 + (r_1^1 + \ldots + r_p^1)m_2 - m_1m_2 - m_1m_2 \geq 0$$

(6)

because $r_1^1 + \ldots + r_p^1 \geq m_1$, see Proposition 4. The first term $(-m_1m_2)$ corresponds to the condition $\sum_{j=1}^{p+1}(A_j^1X_j - X_jA_j^2) = 0$, the second equals $\dim Q$ (for no matrix $X \in M_{m_1, m_2}(C)$ does one have $A_{i+1}^jX - XA_{i+1}^j = 0$ because $A_{i+1}^j$ have no eigenvalue in common).

3°. Consider case 1) of the lemma. Try to understand when there might be equality in (6). Suppose first that $A_j^1$ are diagonalizable. Hence, for $j \leq p$ at least two matrices $A_j^2$ (say,
for \( j = 1, 2 \) are non-scalar, i.e. have at least two eigenvalues each, otherwise condition \((\alpha_{m_2})\) fails. The equality \( \dim R_j = r_j^1 m_2 \) is possible only if each eigenvalue of \( A_j^2 \) is eigenvalue of \( A_j^1 \) of multiplicity \( m_1 - r_j^1 \). Hence, for \( j = 1, 2 \), \( A_j^1 \) has at least two eigenvalues of maximal multiplicity which is \( \leq m_1/2 \).

In order to have \( r_1^1 + \ldots + r_p^1 = m_1 \), for \( 3 \leq j \leq p \) the matrices \( A_j^1 \) must be scalar and the matrices \( A_1^1, A_2^1 \) must have two eigenvalues, of multiplicity \( m_1/2 \) (i.e. \( m_1 \) must be even, \( m_1 \geq 4 \)). But in this case condition \((\alpha_{m_1})\) fails – one has \( d_{j+1}^1 = m_j^1 - m_1, d_1^1 = d_2^1 = m_1^2/2, d_j^1 = 0 \) for \( 3 \leq j \leq p \). Hence, for some \( 3 \leq j \leq p \) there is a non-scalar matrix \( A_j^1 \) which means that \( r_1^1 + \ldots + r_p^1 > m_1 \).

In the general case (when \( A_j^1 \) are not necessarily diagonal) use part C) of Subsection 2.1. The dimensions of the images of the operators \( \xi_j \) are the same when they are defined after a matrix \( A = \begin{pmatrix} A_1^1 & 0 \\ 0 & A_2^1 \end{pmatrix} \) or after a matrix \( B = \begin{pmatrix} B_1^1 & 0 \\ 0 & B_2^1 \end{pmatrix} \) where the JNF \( J(B) \) is the diagonal JNF corresponding to \( J(A) \) and \( J(B_j^1) \) corresponds to \( J(A_j^1) \). Indeed, this dimension equals \((d_j - d_j^1 - d_j^2)/2\) and the quantities \( d_j, d_j^1, d_j^2 \) are the same for both matrices \( A \) and \( B \).

This means that in case 1) there is always a strict inequality in (6).

4\( ^0 \). In cases 2), 3), 4), 5) and 6) one checks directly that \( \dim \text{Ext}^1(P_1, P_2) > 0 \); in case 2) one should notice that for at least three indices \( j \) the matrices \( A_j^1 \) must be non-scalar (for \( i = 1, 2 \)), otherwise condition \((\alpha_{m_i})\) fails. We leave the details for the reader.

5\( ^0 \). Without loss of generality assume that \( A_j^2 \) is in JNF. Then \( \xi_j \) splits into a direct sum of operators acting on \( M_{m_1, l_\nu}(C) \) where \( l_\nu \) is the multiplicity of an eigenvalue of \( A_j^2 \). So one can consider the case when there is a single eigenvalue \( \lambda \) of \( A_j^2 \). Hence, \( \xi_j : (\cdot) \mapsto (A_j^1 - \lambda I)(\cdot) - (\cdot)N \) where \( N \) is the nilpotent part of \( A_j^2 \). A change of variables \((\cdot) \mapsto (\cdot)L \) with a suitable diagonal matrix \( L \) brings the operator to the form \( \xi_j : (\cdot) \mapsto (A_j^1 - \lambda I)(\cdot) - (\cdot)\varepsilon N, \varepsilon \in \mathbb{C}^* \). Hence, when \( \varepsilon \) is small, \( \xi_j \) is a perturbation of the operator \( \xi_j^0 : (\cdot) \mapsto (A_j^1 - \lambda I)(\cdot) \) the dimension of whose image is \( \geq r_j^1 m_2 \). Hence, the same is true for \( \xi_j \) as well, i.e. \( \dim R_j \geq r_j^1 m_2 \). \( \square \)

**Proof of Lemma 28:**

1\( ^0 \). Consider the Fuchsian system \( dX/dt = (\sum_{j=1}^{p+1} A_j/(t-a_j))X \). Suppose that its matrices-residua are block upper-triangular: \( A_j = \begin{pmatrix} B_j & D_j \\ 0 & R_j \end{pmatrix} \). Suppose also that the tuples of matrices \( B_j \) and \( R_j \) are irreducible, that the only non-genericity relation modulo \( \mathbb{Z} \) which holds for the matrices \( A_j \) is the sum of the eigenvalues of the matrices \( B_j \) (resp. \( R_j \)) to be 0, and that \( A_{p+1} = \text{diag}(\lambda_{1,p+1}, \ldots, \lambda_{n,p+1}) \). Present the system by its Laurent series expansion at the pole \( a_{p+1} \): \( dX/dt = (A_{p+1}/(t-a_{p+1}) + F + o(1))X \) (\(*\)) where \( F = \sum_{j=1}^{p} A_j/(a_{p+1} - a_j) \).

Suppose that the block \( D \) of the matrix \( F \) has a non-zero entry \( \beta \) in position \((l,k)\). In what follows the reasoning is close to the one when Procedure \((l,k)\) is described in part D) of Subsection 2.1, but the monodromy group is not irreducible here. The change of variables \( X \mapsto (I + W/(t-a_{p+1}))X \) where the only non-zero entry of \( W \) is \( W_{k,l} = (\lambda_{k,p+1} - \lambda_{l,p+1} + 1)/\beta \) transforms the system into another Fuchsian system the eigenvalues of whose residuum at \( a_{p+1} \) have changed as follows: \( \lambda_{k,p+1} \mapsto \lambda_{k,p+1} + 1, \lambda_{l,p+1} \mapsto \lambda_{l,p+1} - 1 \), the other eigenvalues remain the same.

Hence, after the change of variables the new tuple of matrices-residua is irreducible – the change has destroyed the only non-genericity relation (but, of course, has not destroyed it modulo \( \mathbb{Z} \)).

2\( ^0 \). If no entry of the restriction of the matrix \( F \) to the block \( D \) is non-zero, then change a little the positions of the poles of the system so that one of the entries become non-zero. (This is possible to do because otherwise the entries of all blocks \( D_j \) must be 0, the monodromy group.
will be a direct sum and its centralizer will not be trivial.) The centralizer of the monodromy group remains trivial under small changes of the positions of the poles and the monodromy group remains within the same component of $W(C_1, \ldots, C_{p+1})$. One can change the poles $a_j$ so that at least one entry of the block $D$ becomes non-zero. After this one performs the change of variables from $t^0$.

3°. If the tuple of matrices $A_j$ has a more complicated block upper-triangular form, then one can destroy in the same way one by one all non-genericity relations satisfied by the eigenvalues of the matrices-residua. One combines changes of the positions of the poles with linear changes of the variables. Having destroyed all such relations, one can continue to construct similar linear changes which do not restore any of the relations. (The fact that the tuple of matrices-residua is no longer reducible does not affect the reasoning; we leave the details for the reader.) Thus one can construct infinitely many Fuchsian systems with $A_j \in c_j$ for $j \leq p$, with monodromy as required and with different sets of eigenvalues of $A_{p+1}$. The lemma is proved. □

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