The determination of a doubly resolving set with the minimum size for $C_n \Box P_k$ and some minimal resolving parameters for $(C_n \Box P_k) \Box P_2$

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Abstract

Applications of resolving sets in graph theory and other science have a long history, and if we consider a graph as a chemical compound then the determination of a doubly resolving set with the minimum size is very useful to analysis of chemical compound. In this work, we will consider the computational study of doubly resolving sets for the cartesian product $C_n \Box P_k$ and $(C_n \Box P_k) \Box P_2$. Indeed, we will show that if $n$ is an even or odd integer, then the minimum size of a doubly resolving set in $C_n \Box P_k$ is 3, and more we compute some minimal resolving parameters for $(C_n \Box P_k) \Box P_2$. In particular, we will show that if $n$ is an even or odd integer, then the minimum size of a doubly resolving set in $(C_n \Box P_k) \Box P_2$ is 4.

Keywords: cartesian product, resolving set, doubly resolving set.

1. Introduction

All graphs considered in this work are assumed to be finite and connected. A graphical representation of a vertex $v$ of a connected graph $G$ relative to an arranged subset $W = \{w_1, ..., w_k\}$ of vertices of $G$ is defined as the $k$-tuple $(d(v, w_1), ..., d(v, w_k))$, and this $k$-tuple is denoted by $r(v|W)$, where $d(v, w_i)$ is considered as the minimum distance of a shortest path from $v$ to $w_i$. If any vertices $u$ and $v$ that belong to $V(G) - W$ have various representations with respect to the set $W$, then $W$ is called a resolving set for $G$ [6]. Slater [20] considered the concept and notation of the metric dimension problem under the term locating set. Also, Harary and Melter [11] considered these problems under the term metric dimension as follows: A resolving set of the minimum size or cardinality is called the metric dimension of $G$ and this minimum size denoted by $\beta(G)$. Resolving parameters in graphs have been studied in [1, 4, 5, 14, 15, 16, 17, 21].

Cáceres [7] considered the concept and notation of a doubly resolving set of graph $G$, and we can see that a subset $W = \{w_1, w_2, ..., w_k\}$ of vertices of a graph $G$ is a doubly resolving set of $G$ if for any various vertices $x, y \in V(G)$ we have $r(x|W) \neq r(y|W)$, where $\lambda$ is an integer, and $I$ indicates the unit $\lambda$-vector $(1, ..., 1)$, see [2]. Doubly resolving sets have played a special role in the study of resolving sets. In particular, a doubly resolving set in graph $G$ with the minimum size, is denoted by $\psi(G)$. The applications of above concepts and related parameters are very useful to analysis of a chemical compound and note that these problems are NP hard, see [3, 8, 9, 10, 13].

The cartesian product of two graphs $G$ and $H$, denoted by $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ and with edge set $E(G \times H)$ so that $(g_1, h_1)(g_2, h_2) \in E(G \Box H)$, whenever $h_1 = h_2$ and $g_1g_2 \in E(G)$, or $g_1 = g_2$ and $h_1h_2 \in E(H)$.

Now, we use $C_n$ and $P_k$ to denote the cycle on $n \geq 3$ and the path on $k \geq 3$ vertices, respectively. In this article, we will consider the computational study of doubly resolving sets for the cartesian product $C_n \Box P_k$ and $(C_n \Box P_k) \Box P_2$. Indeed, in Section 3.1, we define a graph isomorphic to the cartesian product $C_n \Box P_k$, and we will consider the determination of a doubly resolving set with the minimum size of the cartesian product $C_n \Box P_k$, In particular, in Section

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2. Definitions and Preliminaries

Definition 2.1. Consider two graphs $G$ and $H$. If there is a bijection, $\theta : V(G) \rightarrow V(H)$ so that $u$ is adjacent to $v$ in $G$ if and only if $\theta(u)$ is adjacent to $\theta(v)$ in $H$, then we say that $G$ and $H$ are isomorphic.

Definition 2.2. [19] Let $G$ be a graph. A vertex $v$ of $G$ strongly resolves two vertices $u$ and $v$ of $G$ if $u$ belongs to a shortest $v - w$ path or $v$ belongs to a shortest $u - w$ path. A set $W = \{w_1, w_2, ..., w_m\}$ of vertices of $G$ is a strong resolving set of $G$ if every two distinct vertices of $G$ are strongly resolved by some vertices of $W$. A strong resolving set of the minimum size is called the strong metric dimension of $G$, and this minimum size is denoted by $sdim(G)$.

Remark 2.1. Suppose that $n$ is an even natural number greater than or equal to 6 and $G$ is the cycle graph $C_n$. Then $\beta(G) = 2$, $\psi(G) = 3$ and $sdim(G) = \lceil \frac{n}{2} \rceil$.

Remark 2.2. Suppose that $n$ is an odd natural number greater than or equal to 3 and $G$ is the cycle graph $C_n$. Then $\beta(G) = 2$, $\psi(G) = 2$ and $sdim(G) = \lceil \frac{n}{2} \rceil$.

Theorem 2.1. Suppose that $n$ is an odd integer greater than or equal to 3. Then the minimum size of a resolving set in the cartesian product $C_n \square P_k$ is 2.

Theorem 2.2. Suppose that $n$ is an even integer greater than or equal to 4. Then the minimum size of a resolving set in the cartesian product $C_n \square P_k$ is 3.

Theorem 2.3. If $n$ is an even or odd integer is greater than or equal to 3, then the minimum size of a resolving set in the cartesian product $C_n \square P_k$ is $n$.

3. Main Results

3.1. The determination of a doubly resolving set with the minimum size for $C_n \square P_k$

Although, some resolving parameters such as the minimum size of resolving sets and the minimum size of strong resolving sets calculated for the cartesian product $C_n \square P_k$, see [7, 18], but in this section we will determine the minimum size of a doubly resolving set in $C_n \square P_k$. Thus for this purpose, we first label the vertices of the $C_n \square P_k$ in a way that helps us and we introduce some notation which is used throughout this section. Suppose $n$ and $k$ are natural numbers greater than or equal to 3, and $[n] = \{1, ..., n\}$. Now, suppose that $G$ is a graph with vertex set $\{x_1, ..., x_n\}$ on layers $V_1, V_2, ..., V_k$, where $V_p = \{x_{i(p-1)+1}, x_{(p-1)+2}, ..., x_{(p-1)+n}\}$ for $1 \leq p \leq k$, and the edge set of graph $G$ is $E(G) = \{x_ix_j|x_i, x_j \in V_p, 1 \leq i < j \leq nk, j - i = 10rj - i = n - 1\} \cup \{x_ix_j|x_i \in V_q, x_j \in V_q+1, 1 \leq i < j \leq nk, 1 \leq q \leq k - 1, j - i = n\}$. We can see that this graph is isomorphic to the cartesian product $C_n \square P_k$. So, we can assume throughout this article $V(C_n \square P_k) = \{x_1, ..., x_{nk}\}$. Now, in this section, we give a more elaborate description of the cartesian product $C_n \square P_k$, that are required to prove of Theorems. We use $V_p$, $1 \leq p \leq k$, to indicate a layer of the cartesian product $C_n \square P_k$, where $V_p$, is defined already. Also, for every two vertices $x_i$ and $x_j$ in $C_n \square P_k$, we say that $x_i$ and $x_j$ are compatible in $C_n \square P_k$, if $n|j - i$. We can see that the degree of a vertex in the layers $V_1$ and $V_k$ is 3, also the degree of a vertex in the layer $V_p$, $1 < p < k$ is 4, and hence $C_n \square P_k$ is not regular. We say that two layers of $C_n \square P_k$ are congruous, if the degree of compatible vertices in two layers are identical. Note that, if $n$ is an even natural number, then $C_n \square P_k$ contains no cycles of odd length, and hence in this case $C_n \square P_k$ is bipartite. For more result of families of graphs with constant metric, see [3, 12]. The cartesian product $C_4 \square P_3$ is depicted in Figure 1.
Theorem 3.1. Consider the cartesian product $C_n \square P_k$. If $n$ is an odd integer greater than or equal to 3, then the minimum size of a doubly resolving set in the cartesian product $C_n \square P_k$ is 3.

Proof. In the following cases we show that the minimum size of a doubly resolving set in the cartesian product $C_n \square P_k$ is 3.

Case 1. First, we show that the minimum size of a doubly resolving set in the cartesian product $C_n \square P_k$ must be greater than 2. Consider the cartesian product $C_n \square P_k$ with the vertex set $\{x_1, \ldots, x_{n+1}\}$ on the layers $V_1, V_2, \ldots, V_k$, which is defined already. Based on Theorem 2.1, we know that $\beta(C_n \square P_k) = 2$. We can show that if $n$ is an odd integer then all the elements of every minimum resolving set of $C_n \square P_k$ must lie in exactly one of the congruous layers $V_i$ or $V_j$. Without lack of theory if we consider the layer $V_1$ of the cartesian product $C_n \square P_k$ then we can show that all the minimum resolving sets in the layer $V_1$ of $C_n \square P_k$ are the sets as to form $M_i = \{x_i, x_{i+1}\}$, $1 \leq i \leq \lceil \frac{n}{2} \rceil$ and $N_j = \{x_j, x_{j+1}\}$, $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$. On the other hand, we can see that the arranged subsets $M_i$ cannot be doubly resolving sets for $C_n \square P_k$ because for $1 \leq i \leq \lceil \frac{n}{2} \rceil$ and two compatible vertices $x_i$ and $x_{i+2}$ with respect to $x_i$, we have $r(x_i; M_i) - r(x_{i+2}; M_i) = -1$, where $I$ indicates the unit 2-vector $(1, 1)$. By applying the same argument we can show that the arranged subsets $N_j$ cannot be doubly resolving sets for $C_n \square P_k$. Hence, the minimum size of a doubly resolving set in $C_n \square P_k$ must be greater than 2.

Case 2. Now, we show that the minimum size of a doubly resolving set in the cartesian product $C_n \square P_k$ is 3. For $1 \leq i \leq \lceil \frac{n}{2} \rceil$, let $x_i$ be a vertex in the layer $V_1$ of $C_n \square P_k$ and $x_j$ be a compatible vertex with respect to $x_i$, where $x_i$ lie in the layer $V_2$ of $C_n \square P_k$, then we can show that the arranged subsets $A_i = M_i \cup x_i = \{x_i, x_{i+1}\}$ of vertices in the cartesian product $C_n \square P_k$ are the minimum doubly resolving sets for the cartesian product $C_n \square P_k$. It will be enough to show that for any compatible vertices $x_i$ and $x_j$ in $C_n \square P_k$, $r(x_i; A_i) - r(x_j; A_i) \neq \lambda I$. Suppose $x_i \in V_p$ and $x_d \in V_q$ are compatible vertices in the cartesian product $C_n \square P_k$, $1 \leq p < q \leq k$. Hence, $r(x_i; M_i) - r(x_q; M_i) = \lambda I$, where $I$ is a positive integer, and $I$ indicates the unit 2-vector $(1, 1)$. Also, for the compatible vertex $x_j$ with respect to $x_i$, $r(x_i; x_i) - r(x_d; x_i) = \lambda I$. So, $r(x_i; A_i) - r(x_d; A_i) \neq \lambda I$, where $I$ indicates the unit 3-vector $(1, 1, 1)$. Especially, for $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ if we consider the arranged subsets $B_j = N_j \cup x_c = \{x_j, x_{j+1}\}$ of vertices the cartesian product $C_n \square P_k$, where $x_c$ lie in the same layer $V_k$ of the cartesian product $C_n \square P_k$ and $x_c$ is a compatible vertex with respect to $x_j$, then by applying the same argument we can show that the arranged subsets $B_j = N_j \cup x_c = \{x_j, x_{j+1}\}$ of vertices in the cartesian product $C_n \square P_k$ are the minimum doubly resolving sets for the cartesian product $C_n \square P_k$.

Theorem 3.2. Suppose that $n$ is an even integer greater than or equal to 4. Then the minimum size of a doubly resolving set in the cartesian product $C_n \square P_k$ is 3.

Proof. Consider the cartesian product $C_n \square P_k$ with the vertex set $\{x_1, \ldots, x_{n+1}\}$ on the layers $V_1, V_2, \ldots, V_k$, which is defined already. Based on Theorem 2.2, if $n$ is even integer then $\beta(C_n \square P_k) = 3$ and it is well known that $\beta(C_n \square P_k) \leq \psi(C_n \square P_k)$. Especially, we show that $\psi(C_n \square P_k) = 3$. Suppose $S_1 = \{x_1, x_2\}$ is a set of vertices in the layer $V_1$ of the cartesian product $C_n \square P_k$ and $x_c$ is a compatible vertex with respect to $x_1$, where $x_c$ lie in the layer $V_k$ of the cartesian
product $C_n \square P_k$. We can show that the arranged subset $S_2 = S_1 \cup x_c = \{x_1, x_2, x_c\}$ of vertices in the cartesian product $C_n \square P_k$ is one of the minimum resolving sets for the cartesian product $C_n \square P_k$. In particular, we show that the arranged subset $S_2 = S_1 \cup x_c = \{x_1, x_2, x_c\}$ of vertices in the cartesian product $C_n \square P_k$ is one of the minimum doubly resolving sets for the cartesian product $C_n \square P_k$. It will be enough to show that for any compatible vertices $x_c$ and $x_d$ in $C_n \square P_k$, $r(x_c|S_2) - r(x_d|S_2) \neq \lambda I$. Suppose $x_c \in V_p$ and $x_d \in V_q$ are compatible vertices in the cartesian product $C_n \square P_k$, $1 \leq p < q \leq k$. Hence, $r(x_c|S_1) - r(x_d|S_1) = -\lambda I$, where $\lambda$ is a positive integer, and $I$ indicates the unit 2-vector $(1,1)$. Also, for $x_c \in S_2$, $r(x_c|S_2) - r(x_d|S_2) = \lambda I$, where $I$ indicates the unit 3-vector $(1,1,1)$.

\[ \square \]

**Remark 3.1.** It is noteworthy that, if $n$ is an odd integer greater than 3, then by the similar manner which is done in the previous Theorem we can show that the arranged subset $S_2 = S_1 \cup x_c = \{x_1, x_2, x_c\}$ of vertices in the cartesian product $C_n \square P_k$ is also one of the minimum doubly resolving sets for the cartesian product $C_n \square P_k$, where the set $S_2$ is defined in the previous Theorem.

**Lemma 3.1.** If $n$ is an even or odd integer is greater than or equal to 3, then the minimum size of a strong resolving set in the cartesian product $C_n \square P_k$ is $n$.

**Proof.** Although, the minimum size of strong resolving sets in the cartesian product $C_n \square P_k$ calculated, but by another way we show that the minimum size of a strong resolving set in the cartesian product $C_n \square P_k$ is $n$. Suppose $T_1 = V_2 \cup \ldots \cup V_{k-1}$ is an arranged subset of vertices in $C_n \square P_k$, where $V_p, 2 \leq p \leq k - 1$ which is defined already. If $k = 3$ then $T_1 = V_2$ cannot be a resolving set for $C_n \square P_k$. If $k \geq 4$ then we can prove that the set $T_1$ is a resolving set for $C_n \square P_k$. Now, by considering various vertices $x_1 \in V_1$ and $x_m \in V_k, n(k - 1) + 1 \leq m \leq nk$, there is not a $w \in T_1$ so that $x_1$ belongs to a shortest $x_m - w$ path or $x_m$ belongs to a shortest $x_1 - w$ path. Thus $T_1 = V_2 \cup \ldots \cup V_{k-1}$ cannot be a strong resolving set for $C_n \square P_k$. Now, suppose that $T_2$ is a subset of vertices in $V_1$ so that $T_2$ is a resolving set in $C_n \square P_k$ and the cardinality of $T_2$ is less than $n$. We can be concluded that $T_2$ cannot be a strong resolving set for $C_n \square P_k$. In particular, if the cardinality of $T_2$ is equal to $n - 1$, we prove that $T_2$ cannot be a strong resolving set for $C_n \square P_k$. In this case, without lack of theory assume that $T_2 = \{x_1, \ldots, x_{n-1}\}$. Now, by considering various vertices $x_m$ in $V_1$ and $x_1 \neq x_m$ in $V_2$, there is not a $w \in T_2$ so that $x_m$ belongs to a shortest $x_1 - w$ path or $x_1$ belongs to a shortest $x_m - w$ path. Thus the set $T_2 = \{x_1, \ldots, x_{n-1}\}$ of vertices in $C_n \square P_k$ cannot be a strong resolving set for $C_n \square P_k$. Hence, if $T$ is a strong resolving set in $C_n \square P_k$, then the minimum size of $T$ must be greater than or equal to $n$. So, suppose that $T = \{x_1, \ldots, x_n\}$ is an arranged subset of vertices in the layer $V_1$ of the cartesian product $C_n \square P_k$, we prove that this subset is a strong resolving set in $C_n \square P_k$. For $1 < p < q \leq k$, if both vertices $x_c \in V_p$ and $x_d \in V_q$ are compatible in $C_n \square P_k$ relative to $x_c$, $1 \leq r \leq n$, then $x_r$ belongs to a shortest $x_c - x_d$ path. For $1 < p < q \leq k$, if both vertices $x_c \in V_p$ and $x_d \in V_q$ are not compatible in $C_n \square P_k$ and lie in various layers in $C_n \square P_k$, then there is a exactly one compatible vertex in $V_1$ relative to $x_c$ say $x_c$ such that $x_c \in V_p$ and $x_d \in V_q$ are not compatible in $C_n \square P_k$ and lie in the same layer of $C_n \square P_k$ say $V_p$, then there is exactly one vertex in the layer $V_1$ say $x_c$ so that $x_c$ and $x_d$ are compatible in $C_n \square P_k$ and $x_c$ belongs to a shortest $x_c - x_d$ path. Thus the set $T = \{x_1, \ldots, x_n\}$ is one of the minimum strong resolving sets for $C_n \square P_k$, and hence the minimum size of a strong resolving set in the cartesian product $C_n \square P_k$ is $n$.

\[ \square \]

3.2. The determination of some minimal resolving parameters for $(C_n \square P_k) \square P_2$

Consider the cartesian product $C_n \square P_k$ with the vertex set $\{x_1, \ldots, x_n\}$ on the layers $V_1, V_2, \ldots, V_n$, where $V_p, 1 \leq p \leq k$, which is defined in Section 3.1. Also, we consider one copy of the cartesian product $C_n \square P_k$ with the vertex set $\{y_1, \ldots, y_n\}$ on the layers $U_1, \ldots, U_k$, where it can be defined $U_p$, as similar $V_p$ on the vertex set $\{y_1, \ldots, y_n\}$. Now, suppose that $H$ is a graph with vertex set $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ so that for $1 \leq t \leq nk$, the vertex $x_t$ is adjacent to $y_t$ in $H$, then we can see that the graph $H$ is isomorphic to $(C_n \square P_k) \square P_2$. So, we can assume that $(C_n \square P_k) \square P_2$ contains $k$ layers $Z_1, \ldots, Z_k$, where $Z_p = V_p \cup U_p, 1 \leq p \leq k$; also $V_p$ and $U_p$ denote internal and external layers of $(C_n \square P_k) \square P_2$, on the sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$, respectively. In particular, we can see that the degree of a vertex in the layers $Z_1$ and $Z_k$ is 4, also for $1 < p < k$, the degree of a vertex in the layer $Z_p$ is 5. The graph $(C_n \square P_k) \square P_2$ is depicted in Figure 2.
Theorem 3.3. If $n$ is an odd integer greater than or equal to 3, then the minimum size of a resolving set in $(C_n \Box P_k) \Box P_2$ is 3.

Proof. Suppose $V((C_n \Box P_k) \Box P_2) = \{x_1, ..., x_{nk}\} \cup \{y_1, ..., y_{lk}\}$. Based on Theorem 2.1, we know that if $n$ is an odd integer greater than or equal to 3, then the minimum size of a resolving set in $C_n \Box P_k$ is 2. Also, by definition of $(C_n \Box P_k) \Box P_2$ we can verify that for $1 \leq t \leq nk$, every vertex $y_t$ is adjacent to $x_t$, and hence none of minimal resolving sets of $C_n \Box P_k$ cannot be a resolving set for $(C_n \Box P_k) \Box P_2$. Therefore, the minimum size of a resolving set in $(C_n \Box P_k) \Box P_2$ must be greater than 2.

Now, we show that the minimum size of a resolving set in $(C_n \Box P_k) \Box P_2$ is 3. For $1 \leq i \leq \lceil \frac{n}{2} \rceil$, let $x_i$ be a vertex in internal layer $V_1$ of $(C_n \Box P_k) \Box P_2$ and $x_i$ be a compatible vertex with respect to $x_i$, where $x_i$ lie in the internal layer $V_2$ of $(C_n \Box P_k) \Box P_2$. Based on Theorem 3.1, we know that the arranged subsets $A_i = \{x_i, x_{i+1}, x_{i+2}\}$ of vertices in internal layers of $(C_n \Box P_k) \Box P_2$ are resolving sets for internal layers of $(C_n \Box P_k) \Box P_2$, and hence the arranged subsets $A_i = \{x_i, x_{i+1}, x_{i+2}\}$ are the minimum resolving sets for $(C_n \Box P_k) \Box P_2$ because for every vertex $y_i$ in external layer of $(C_n \Box P_k) \Box P_2$, we have $r(y_i | A_i) = (d(x_i, x_j) + 1, d(x_i, x_{i+j}), d(x_i, x_j) + 1, d(x_i, x_j) + 1)$, so all the vertices in the external layers $U_j$ have various representations with respect to the sets $A_i$. In the same way for $1 \leq j \leq \lceil \frac{n}{2} \rceil$, if we consider the arranged subsets $B_j = N_j \cup x_j = \{x_j, x_{i+j}, x_{i+j}, x_i\}$ of vertices in internal layers of $(C_n \Box P_k) \Box P_2$, where $x_i$ lie in the internal layer $V_2$ of $(C_n \Box P_k) \Box P_2$ and $x_j$ is a compatible vertex with respect to $x_j$, then by applying the same argument we can show that the arranged subsets $B_j = N_j \cup x_j = \{x_j, x_{i+j}, x_i\}$ of vertices in internal layers of $(C_n \Box P_k) \Box P_2$ are the minimum resolving sets for $(C_n \Box P_k) \Box P_2$. □

Lemma 3.2. If $n$ is an odd integer greater than or equal to 3, then the minimum size of a doubly resolving set in $(C_n \Box P_k) \Box P_2$ is greater than 3.
Proof. Suppose \( V((C_n \square P_k) \square P_2) = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\} \). For \( 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \), let \( x_i \) be a vertex in internal layer \( V_i \) of \( (C_n \square P_k) \square P_2 \), and \( x_i \) be a compatible vertex with respect to \( x_i \), where \( x_c \) lie in the internal layer \( V_i \) of \( (C_n \square P_k) \square P_2 \). Based on proof of Theorem 3.3, we know that the arranged subsets \( A_i = M_i \cup S = \{x_i, x_i + 1, x_i + 2, \ldots, x_i + k - 1\} \) of vertices in internal layers of \( (C_n \square P_k) \square P_2 \) cannot be doubly resolving sets for \( (C_n \square P_k) \square P_2 \) because \( r(y_j|A_i) = (d(x_i, x_i) + 1, d(x_i, x_i) + 1) \). In the same way for \( 1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil \), if we consider the arranged subsets \( B_j \) \( = N_j \cup S = \{x_j, x_j + 1, x_j + 2, \ldots, x_j + k - 1\} \) of vertices in internal layers of \( (C_n \square P_k) \square P_2 \), where \( x_i \) lie in the internal layer \( V_i \) of \( (C_n \square P_k) \square P_2 \) and \( x_c \) is a compatible vertex with respect to \( x_j \), then we can show that the arranged subsets \( B_j \) cannot be doubly resolving sets for \( (C_n \square P_k) \square P_2 \). Hence the minimum size of a doubly resolving set in \( (C_n \square P_k) \square P_2 \) is greater than 3.

Lemma 3.3. If \( n \) is an even integer greater than or equal to 4, then the minimum size of a resolving set in \( (C_n \square P_k) \square P_2 \) is greater than 3.

Proof. Suppose \( V((C_n \square P_k) \square P_2) = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\} \). Based on Theorem 2.2, we know that if \( n \) is an even integer greater than or equal to 4, then the minimum size of a resolving set in \( C_n \square P_k \) is 3. By the same manner which is done in Theorem 3.3, we can show that the minimum size of a resolving set in \( (C_n \square P_k) \square P_2 \) must be greater than 3.

Theorem 3.4. If \( n \) is an even or odd integer greater than or equal to 3, then the minimum size of a doubly resolving set in \( (C_n \square P_k) \square P_2 \) is 4.

Proof. Based on Lemma 3.3, we know that if \( n \) is an even integer greater than or equal to 4, then \( \beta((C_n \square P_k) \square P_2) > 3 \). Also based on Theorem 3.3, we know that if \( n \) is an odd integer greater than or equal to 3, then \( \beta((C_n \square P_k) \square P_2) = 3 \) and by Lemma 3.2, we know that, the minimum size of a doubly resolving set in \( (C_n \square P_k) \square P_2 \) is greater than 3. In particular, it is well known that \( \beta((C_n \square P_k) \square P_2) \leq \psi((C_n \square P_k) \square P_2) \). Now, we show that if \( n \) is an even or odd integer greater than or equal to 3, then the minimum size of a doubly resolving set in \( (C_n \square P_k) \square P_2 \) is 4. Let \( S_2 = \{x_1, x_2, x_k\} \) be an arranged subset of vertices in internal layers of \( (C_n \square P_k) \square P_2 \), where \( x_c \) is a compatible vertex with respect to \( x_1 \) and suppose that \( S_3 = S_2 \cup y_c = \{x_1, x_2, x_k, y_c\} \) is an arranged subset of vertices in \( (C_n \square P_k) \square P_2 \). Thus the minimum size of a doubly resolving set in \( (C_n \square P_k) \square P_2 \) is 4.

Theorem 3.5. If \( n \) is an even or odd integer greater than or equal to 3, then the minimum size of a strong resolving set in \( (C_n \square P_k) \square P_2 \) is 2n.

Proof. Suppose \( V((C_n \square P_k) \square P_2) = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\} \) and suppose that \( O_1 = Z_2 \cup \ldots \cup Z_{k-1} \) is an arranged subset of vertices in \( (C_n \square P_k) \square P_2 \), where \( Z_p, 2 \leq p \leq k - 1 \) which is defined already. It is easy to verify that, the subset \( O_1 = Z_2 \cup \ldots \cup Z_{k-1} \) cannot be a strong resolving set for \( (C_n \square P_k) \square P_2 \). By the same manner which is done in proof of Theorem 3.1, it is also easy to verify that, every subset of vertices in the layer \( Z_1 \) of \( (C_n \square P_k) \square P_2 \), of cardinality \( 2n - 1 \) cannot be a strong resolving set for \( (C_n \square P_k) \square P_2 \). Thus the minimum size of a strong resolving set in \( (C_n \square P_k) \square P_2 \) must be greater than or equal to \( 2n \). So, suppose that \( O_2 = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\} \) is an arranged subset of vertices in the layer \( Z_1 \) of \( (C_n \square P_k) \square P_2 \), by the same manner which is done in proof of Theorem 3.1, we can show that the subset \( O_2 \) is a strong resolving set in \( (C_n \square P_k) \square P_2 \), because for \( 1 \leq i \leq nk \), the vertex \( x_i \) is adjacent to \( y_i \), and hence the subset \( O_2 \) is one of the minimum strong resolving sets in \( (C_n \square P_k) \square P_2 \).

4. Conclusion

In this work, we considered the computational study of doubly resolving sets of the cartesian product \( C_n \square P_k \) and \( (C_n \square P_k) \square P_2 \). Indeed, we showed that if \( n \) is an even or odd integer, then the minimum size of a doubly resolving set in \( (C_n \square P_k) \square P_2 \) is 3, and we computed some minimal resolving parameters for \( (C_n \square P_k) \square P_2 \). In particular, we showed that if \( n \) is an even or odd integer, then the minimum size of a doubly resolving set in \( (C_n \square P_k) \square P_2 \) is 4.
Acknowledgements
This work was supported in part by Anhui Provincial Natural Science Foundation under Grant 2008085J01 and Natural Science Fund of Education Department of Anhui Province under Grant KJ2020A0478.

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