SHARP BOUNDS FOR THE SECOND SEIFFERT MEAN IN TERMS OF POWER MEANS

ZHEN-HANG YANG

Abstract. For $a, b > 0$ with $a \neq b$, let $T(a, b)$ denote the second Seiffert mean defined by

$$T(a, b) = \frac{a - b}{2 \arctan \frac{a - b}{a + b}}$$

and $A_r(a, b)$ denote the $r$-order power mean. We present the sharp bounds for the second Seiffert mean in terms of power means:

$$A_{p_1}(a, b) < T(a, b) \leq A_{p_2}(a, b),$$

where $p_1 = \log_{5/2} \pi$ and $p_2 = 5/3$ can not be improved.

1. Introduction

Throughout the paper, we assume that $a, b > 0$ with $a \neq b$. The power mean of order $r$ of the positive real numbers $a$ and $b$ is defined by

$$A_r(a, b) = \left( \frac{a^r + b^r}{2} \right)^{1/r}$$

if $r \neq 0$ and $A_0 = A_0(a, b) = \sqrt{ab}$.

It is well-known that the function $r \mapsto A_r(a, b)$ is continuous and strictly increasing on $\mathbb{R}$ (see [1]). As special cases, the arithmetic mean, geometric mean and quadratic mean are $A = A(a, b) = A_1(a, b)$, $G = G(a, b) = A_0(a, b)$ and $Q = Q(a, b) = A_2(a, b)$, respectively.

The Lehmer mean of order $r$ of the positive real numbers $a$ and $b$ is defined as

$$L_r = L_r(a, b) = \frac{a^{r+1} + b^{r+1}}{a^r + b^r}$$

(see [11]). It is seen that the function $r \mapsto L_r(a, b)$ is continuous and strictly increasing on $\mathbb{R}$. In particular, $L_0 = A$, $L_1 = C$ are the arithmetic mean, contra-harmonic mean, respectively. Clearly, Lehmer mean can be expressed by power means as $L_r = A_{r+1}^r A_r^{-r}$.

The first Seiffert mean [17] is defined by

$$P = P(a, b) = \frac{a - b}{2 \arcsin \frac{a - b}{a + b}}.$$
the first Seiffert mean $P$ in terms of power means proved by Jagers \[10\] and Hästö \[8\]:

$$A_{\log_2} (a, b) < P (a, b) < A_{2/3} (a, b).$$

In 1995, Seiffert \[18\] defined his second mean as

$$T = T (a, b) = \frac{a - b}{2 \arctan \frac{a - b}{a + b}},$$

and proved that

$$A < T < Q.$$

Sándor \[16\] pp. 265-267] showed that by a transformation of arguments, the mean $T$ can be reduced to the mean $P$:

$$T (a, b) = P (x, y),$$

where

$$x = \frac{\sqrt{2} \left( a^2 + b^2 \right) + a - b}{2}, \quad y = \frac{\sqrt{2} \left( a^2 + b^2 \right) - a + b}{2},$$

which implies

$$A (x, y) = Q (a, b), \quad G (x, y) = A (a, b).$$

Therefore, by using the transformations (1.3), the following transformations of means will be true:

$$G \rightarrow A, \quad A \rightarrow Q, \quad P \rightarrow T.$$

Thus, from the known inequalities involving $P$, $A$, $G$ he easily obtained corresponding ones involving $T$, $Q$, $A$, for example, (1.2) and the following inequalities:

$$Q^{2/3} A^{1/3} < Q \left( \frac{Q + A}{2} \right)^{2/3} < T < \frac{2Q + A}{3}.\tag{1.4}$$

Recently, Chu et al. in \[4\] proved the double inequality

$$p_1 Q + (1 - p_1) A < T < q_1 Q + (1 - q_1) A$$

holds if and only if $p_1 \leq (\sqrt{2} + 1) (4 - \pi) / \pi, q_1 \geq 2/3$, which shows that the constant 2/3 of the third inequality in (1.3) is the best.

Very recently, Witkowski \[21\] used some geometric ideas to prove a series of inequalities involving $T$, $Q$, $A$, such as

$$A < T < \frac{4}{\pi} A, \tag{1.6}$$

$$\frac{2\sqrt{2}}{\pi} Q < T < Q, \tag{1.7}$$

$$(1 - r_1)Q + r_1 A < T < \frac{2Q + A}{3} \tag{1.8},$$

where $r_1 = \frac{2(\pi - 2\sqrt{2})}{(2 - \sqrt{2})\pi} = 0.340341385...$. It is obvious that (1.8) is actually (1.5).

In 2010, Wang et al. \[20\] presented the optimal upper and lower Lehmer mean bounds for $T$ as follows:

$$L_0 < T < L_{1/3}.\tag{1.9}$$
In [5], Chu et al. demonstrated that the double inequality
\[ C(p_2 a + (1 - p_2) b, p_2 b + (1 - p_2) a) < T(a, b) < C(q_2 a + (1 - q_2) b, q_2 b + (1 - q_2) a) \]
if and only if \[ p_2 \leq \frac{1 + \sqrt{4/\pi - 1}}{2}, \quad q_2 \geq \frac{3 + \sqrt{3}}{6}. \]

It is interesting and useful to evaluate the second Seiffert mean \( T \) by power means \( A_p \). Until recently, the inequalities (1.2) has improved by Costin and Toader [6] as
\[ N < A_{3/2} < T < Q, \]
where \( N \) is the Neuman-Sándor mean defined in [13] by
\[ N = N(a, b) = \frac{a - b}{2 \arcsinh \frac{a - b}{\pi + b}}. \]

Up to now, this may be the best result for the bounds for the second Seiffert mean in terms of power means. For this reason, we are going to find the best \( p \in (3/2, 2) \) such that the inequality
\[ T(a, b) < A_p(a, b) \]
or its reverse inequality holds in this paper.

Our main results are the following

**Theorem 1.1.** The inequality (1.12) if and only if \( p \geq p_2 = 5/3 \). Moreover, we have
\[ \alpha_1 A_{5/3}(a, b) < T(a, b) < \alpha_2 A_{5/3}(a, b), \]
where \( \alpha_1 = 2^{8/5}\pi^{-1} = 0.964494... \) and \( \alpha_2 = 1 \) are the best possible constants.

**Theorem 1.2.** The inequality (1.12) is reversed if and only if \( p \leq p_1 = \log_{\pi/2} 2 = 1.5349... \). Moreover, we have
\[ \beta_1 A_{\log_{\pi/2} 2}(a, b) < T(a, b) < \beta_2 A_{\log_{\pi/2} 2}(a, b), \]
where \( \beta_1 = 1 \) and \( \beta_2 = 1.0136... \) are the best possible constants.

### 2. Lemmas

In order to prove our main results, we need the following lemmas.

**Lemma 2.1.** Let \( F_p \) be the function defined on \((0, 1)\) by
\[ F_p(x) = \ln \frac{T(1, x)}{A_p(1, x)} = \ln \frac{1 - x}{2 \arctan \frac{1 - x}{x + 1}} - \frac{1}{p} \ln \left( \frac{x^p + 1}{2} \right). \]
Then we have
\[ \lim_{x \to 1^-} \frac{F_p(x)}{(x - 1)^2} = -\frac{1}{24} (3p - 5), \]
\[ F_p(0^+) = \lim_{x \to 0^+} F_p(x) = \begin{cases} 
\frac{1}{p} \ln 2 - \ln \frac{p}{2} & \text{if } p > 0, \\
\infty & \text{if } p \leq 0,
\end{cases} \]
where \( F_0(x) := \lim_{p \to 0} F_p(x) \).
Proof. Using power series expansion we have
\[ F_p(x) = -\frac{1}{24} (3p - 5)(x - 1)^2 + O((x - 1)^3), \]
which yields (2.2).

Direct limit calculation leads to (2.3), which proves the lemma. \(\square\)

Lemma 2.2. Let \(F_p\) be the function defined on \((0, 1)\) by (2.1). Then \(F_p\) is strictly increasing on \((0, 1)\) if and only if \(p \geq 5/3\) and decreasing on \((0, 1)\) if and only if \(p \leq 1\).

Proof. Differentiation yields
\begin{equation}
F_p'(x) = \frac{x^{p-1} + 1}{x(1-x)(x^p+1)} \arctan \frac{1-x}{x+1},
\end{equation}
where
\begin{equation}
f_1(x) = \frac{(1-x)(x^p+1)}{(x^2+1)(x^{p-1}+1)} - \arctan \frac{1-x}{x+1}.
\end{equation}

Differentiation again leads to
\begin{equation}
f_1'(x) = -\frac{x(1-x)}{(x^2+1)^2(x^{p-1}+1)^2} f_2(x),
\end{equation}
where
\begin{equation}
f_2(x) = ((1-p)x^p + (p+1)x^{p-1} - 2x^{2p-3} - (p+1)x^{p-2} + (p-1)x^{p-3} + 2).
\end{equation}

(i) We now prove that \(F_p\) is strictly increasing on \((0, 1)\) if and only if \(p \geq 5/3\). From (2.4) it is seen that \(\text{sgn} F_p'(x) = \text{sgn} f_1(x)\) for \(x \in (0, 1)\), so it suffices to prove that \(f_1(x) > 0\) for \(x \in (0, 1)\) if and only if \(p \geq 5/3\).

Necessity. If \(f_1(x) > 0\) for \(x \in (0, 1)\) then there must be \(\lim_{x \to 1^-} (1-x)^{-3} f_1(x) \geq 0\). Application of L'Hopital rule leads to
\begin{equation}
\lim_{x \to 1^-} \frac{f_1(x)}{(1-x)^3} = \lim_{x \to 1^-} \frac{(1-x)(x^p+1)}{(x^2+1)(x^{p-1}+1)} - \arctan \frac{1-x}{x+1} = \frac{1}{24} (3p - 5),
\end{equation}
and so we have \(p \geq 5/3\).

Sufficiency. We now prove \(f_1(x) > 0\) for \(x \in (0, 1)\) if \(p \geq 5/3\). As mentioned previously, the function
\begin{equation}
p \mapsto L_{p-1}(1, x) = \frac{x^p + 1}{x^{p-1} + 1}
\end{equation}
is increasing on \(\mathbb{R}\), it is enough to show that \(f_1(x) > 0\) for \(x \in (0, 1)\) when \(p = 5/3\).
In this case, we have
\begin{equation*}
3x^{4/3} f_2(x) = -2x^3 + 8x^2 - 6x^{5/3} + 6x^{4/3} - 8x + 2.
\end{equation*}
Factoring yields
\begin{equation*}
3x^{4/3} f_2(x) = 2 \left(1 - \sqrt[3]{x}\right)^3 \left(x^{2/3} + 1\right) \left(x^{4/3} + 3x + 5x^{2/3} + 3x^{1/3} + 1\right) > 0.
\end{equation*}
It follows from (2.4) that \(f_1'(x) < 0\), that is, the function \(f_1\) is decreasing on \((0, 1)\).

(ii) We next prove that \(F_p\) is strictly decreasing on \((0, 1)\) if and only if \(p \leq 1\). Similarly, it suffices to show that \(f_1(x) < 0\) for \(x \in (0, 1)\) if and only if \(p \leq 1\).
Note that if \( f_1(x) < 0 \) for \( x \in (0, 1) \) then we have
\[
\lim_{x \to 0^+} f_1(x) = \begin{cases} 
1 - \frac{x}{2} > 0 & \text{if } p > 1, \\
\frac{x}{2} - \frac{x^2}{3} < 0 & \text{if } p = 1, \\
-\frac{x}{2} & \text{if } p < 1
\end{cases}
\]
which yields \( p \leq 1 \).

**Sufficiency.** We prove \( f_1(x) < 0 \) for \( x \in (0, 1) \) if \( p \leq 1 \). Due to the monotonicity of the function \( p \mapsto L_{p-1}(1, x) \), it suffices to demonstrate \( f_1(x) < 0 \) for \( x \in (0, 1) \) when \( p = 1 \). In this case, we have \( f_2(x) = 4 - 4x^{-1} < 0 \), then \( f'_1(x) > 0 \), and then for \( x \in (0, 1) \) we have \( f_1(x) < f_1(1) = 0 \), which proves the sufficiency and the proof of this lemma is finished.

**Lemma 2.3.** Let \( f_3 \) be the function defined on \( (0, 1) \) by
\[
f_3(x) = -p(p-1)x^3 + (p-1)(p+1)x^2 - 2(2p-3)x^p - (p+1)(p-2)x + (p-1)(p-3).
\]
Then \( f_3 \) is strictly increasing on \( (0, 1) \) if \( p \in (1, 5/3) \).

**Proof.** Differentiation yields
\[
f'_3(x) = -3p(p-1)x^2 + 2(p-1)(p+1)x - 2p(2p-3)x^{p-1} - (p+1)(p-2).
\]
Note that \( 1 < p < 5/3 \), using basic inequality for means \( x^{p-1} \leq (p-1)x + (2-p) \) \( (x>0) \)
to the last member of the third term in (2.9) we have
\[
f'_3(x) \geq -3p(p-1)x^2 + 2(p-1)(p+1)x - 2p(2p-3)((p-1)x + (2-p)) - (p+1)(p-2) = -3p(p-1)x^2 - 2(p-1)(2p^2 - 4p - 1)x + (p-2)(4p^2 - 7p - 1)
\]
\[
: = f_4(x).
\]
Thus, in order to prove \( f'_3(x) > 0 \), it needs to show that \( f_4(x) > 0 \) for \( x \in (0, 1) \).
Since \( f'_4(x) = -6p(p-1) < 0 \) and for \( p \in (1, 5/3) \)
\[
f_4(0^+) = (p-2)\left(p - \frac{\sqrt{5p+7}}{2}\right)\left(p + \frac{\sqrt{5p+7}}{2}\right) > 0,
\]
\[
f_4(1) = 6p\left(\frac{5}{3} - p\right) > 0,
\]
application of properties of concave functions yields for \( x \in (0, 1) \)
\[
f_4(x) > (1-x)f_4(0^+) + xf_4(1) > 0,
\]
which completes the proof.

**Lemma 2.4.** Let \( p \in (1, 5/3) \) and let the function \( x \mapsto F_p(x) \) be defined on \( (0, 1) \) by (2.7). Then the equation \( f_1(x) = 0 \) has a unique solution \( x_1 \) such that \( F_p \) is increasing on \( (0, x_1) \) and decreasing on \( (x_1, 1) \), where \( f_1(x) \) is defined by (2.2).

**Proof.** Differentiating \( f_2(x) \) defined by (2.7) gives
\[
x^{4-p}f'_2(x) = f_3(x),
\]
where \( f_3(x) \) is defined by (2.8).
Because that \( f_3 \) is strictly increasing on \((0,1)\) if \(p \in (1,5/3)\) by Lemma (2.3) and note that
\[
f_3(0^+) = (p-1)(p-3) < 0, \quad f_3(1) = 2(5-3p) > 0,
\]
there is a unique \(x_1 \in (0,1)\) such that \(f_3(x) < 0\) for \(x \in (0,x_1)\) and \(f_3(x) > 0\) for \(x \in (x_1,1)\). Then it is seen from (2.6) that \(f_2\) is decreasing on \((0,x_1)\) and increasing on \((x_1,1)\), which yields \(f_2(x) < f_2(1) = 0\) for \(x \in (x_1,1)\), which together with \(\text{sgn} f_2(0^+) = \text{sgn} (p-1) > 0\) reveals that there exits a unique \(x_2 \in (0,x_1)\) such that \(f_2(x) > 0\) for \(x \in (0,x_2)\) and \(f_2(x) < 0\) for \(x \in (x_2,1)\). It follows from (2.6) that \(f_1\) is decreasing on \((0,x_2)\) and increasing on \((x_2,1)\), and therefore \(f_1(x) < f_1(1) = 0\) for \(x \in (x_2,1)\), which in combination with \(f_1(0^+) = 1 - \frac{1}{3}x > 0\) indicates that there is a unique \(x_3 \in (0,x_2)\) such that \(f_1(x) > 0\) for \(x \in (0,x_3)\) and \(f_1(x) < 0\) for \(x \in (x_3,1)\). By (2.4) it is easy to see that the function \(x \mapsto F_p(x)\) is increasing on \((0,x_3)\) and decreasing on \((x_3,1)\), which proves the lemma. \(\square\)

3. Proofs of Main Results

Based on the lemmas in the above section, we can easily proved our main results.

**Proof of Theorem 1.1.** By symmetry, we assume that \(a > b > 0\). Then inequality (1.12) is equivalent to
\[
\ln T(1,x) - \ln A_p(1,x) = F_p(x) < 0,
\]
where \(x = b/a \in (0,1)\). Now we prove the inequality (3.1) holds for all \(x \in (0,1)\) if and only if \(p \geq 5/3\).

**Necessity.** If inequality (3.1) holds, then by Lemma 2.4 we have
\[
\begin{align*}
\lim_{x \to 1^-} F_p(x) &= -\frac{1}{p} (3p-5) \leq 0, \\
\lim_{x \to 0^+} F_p(x) &= \frac{1}{p} \ln 2 - \ln \frac{5}{2} \leq 0 \text{ if } p > 0,
\end{align*}
\]
which yields \(p \geq 5/3\).

**Sufficiency.** Suppose that \(p \geq 5/3\). It follows from Lemma 2.2 that \(F_p(x) < F_p(1) = 0\) for \(x \in (0,1)\), which proves the sufficiency. Using the monotonicity of the function \(x \mapsto F_{5/3}(x)\) on \((0,1)\), we have
\[
\ln \left(2^{8/5\pi^{-1}}\right) = F_{5/3}(0^+) < F_{5/3}(x) < F_{5/3}(1^-) = 0,
\]
which implies (1.13).

Thus the proof of Theorem 1.1 is finished. \(\square\)

**Proof of Theorem 1.2.** Clearly, the reverse inequality of (1.12) is equivalent to
\[
\ln T(1,x) - \ln A_p(1,x) = F_p(x) > 0,
\]
where \(x = b/a \in (0,1)\). Now we show that the inequality (3.2) holds for all \(x \in (0,1)\) if and only if \(p \leq \log_{x/2} 2\).

**Necessity.** The condition \(p \leq \log_{x/2} 2\) is necessary. Indeed, if inequality (3.2) holds, then we have
\[
\begin{align*}
\lim_{x \to 1^-} F_p(x) &= -\frac{1}{p} (3p-5) \geq 0, \\
\lim_{x \to 0^+} F_p(x) &= \frac{1}{p} \ln 2 - \ln \frac{5}{2} \geq 0 \text{ if } p > 0
\end{align*}
\]
or

\[
\begin{aligned}
\lim_{x \to 1-} \frac{F_p(x)}{x-1} &= - \frac{1}{24} (3p - 5) \geq 0, \\
\lim_{x \to 0+} F_p(x) &= \infty \quad \text{if } p \leq 0.
\end{aligned}
\]

Solving the above inequalities leads to \( p \leq \log_{\pi/2} 2 \).

**Sufficiency.** The condition \( p \leq \log_{\pi/2} 2 \) is also sufficient. Since the function \( r \mapsto A_r(1, x) \) is increasing, so the function \( p \mapsto F_p(x) \) is decreasing, thus it is suffices to show that \( F_p(x) > 0 \) for all \( x \in (0, 1) \) if \( p = p_1 = \log_{\pi/2} 2 \).

Lemma 2.4 reveals that for \( p \in (1, 5/3) \) there is a unique \( x_3 \) to satisfy

\[
f_1(x_3) = \frac{(1 - x_3) (x_3^p + 1)}{(x_3^p + 1) (x_3^{p-1} + 1)} - \arctan \frac{1 - x_3}{x_3 + 1} = 0
\]

such that the function \( x \mapsto F_p(x) \) is strictly increasing on \((0, x_3)\) and strictly decreasing on \((x_3, 1)\). It is acquired that for \( p_1 = \log_{\pi/2} 2 \in (1, 5/3) \)

\[
0 = F_{p_1}(0^+) < F_{p_1}(x) \leq F_{p_1}(x_3)
\]

\[
0 = F_{p_1}(1) < F_{p_1}(x_3) \leq F_{p_1}(x_3),
\]

which leads to

\[
A_{p_1}(1, x) < T(1, x) < (\exp F_{p_1}(x_3)) A_{p_1}(1, x).
\]

Solving the equation (3.3) for \( x_3 \) by mathematical computation software we find that \( x_3 \in (0.186930110570624, 0.186930110570625) \), and then

\[
\beta_2 = \exp (F_{p_1}(x_3)) \approx 1.0136,
\]

which proves the sufficiency and inequalities of (1.14).

\[\square\]

### 4. REMARKS

**Remark 4.1.** From the proof of Lemma 2.2, it is seen that \( f_1(x) > 0 \) if and only if \( p \geq 5/3 \), which implies that the inequality

\[
T(1, x) = \frac{x - 1}{2 \arctan \frac{x^2 + 1}{x^2 + 1}} > \frac{(x^2 + 1) (x^{p-1} + 1)}{2 (x^p + 1)}
\]

holds if and only \( p \geq 5/3 \). In a similar way, the inequality

\[
T(1, x) < \frac{(x^2 + 1) (x^{p-1} + 1)}{2 (x^p + 1)}
\]

is valid if and only if \( p \leq 1 \). The results can be restated as a corollary.

**Corollary 1.** The inequalities

\[
\frac{(a^2 + b^2) (a^{2/3} + b^{2/3})}{2 (a^{5/3} + b^{5/3})} < T(a, b) < \frac{a^2 + b^2}{a + b}
\]

with the best constants \( 5/3 \) and \( 1 \), and the function

\[
p \mapsto \frac{(a^2 + b^2) (a^{p-1} + b^{p-1})}{2 (a^p + b^p)}
\]

is decreasing.
In particular, putting \( p = 1, 1/2, \ldots \to -\infty \) and \( 5/3, 2, \ldots, \to \infty \) we get
\[
\frac{a^2 + b^2}{2 \max(a, b)} < \cdots < \frac{a + b}{2} < \frac{(a^2 + b^2) (a^{2/3} + b^{2/3})}{2 (a^{5/3} + b^{5/3})} < T(a, b) < \frac{a^2 + b^2}{2 \min(a, b)} < \cdots < \frac{a^2 + b^2}{2 \sqrt{ab}}.
\]

**Remark 4.2.** Using the monotonicity of the function defined on \((0, 1)\) by
\[
F_p(x) = \ln \frac{T(1, x)}{A_p(1, x)}
\]
given in Lemma 2.2, we can obtain a Fan Ky type inequality but omit the further details of the proof.

**Corollary 2.** Let \( a_1, a_2, b_1, b_2 > 0 \) with \( a_1/b_1 < a_2/b_2 < 1 \). Then the following Fan Ky type inequality
\[
T(a_1, b_1) < A_p(a_1, b_1) < A_p(a_2, b_2) < T(a_2, b_2)
\]
holds if \( p \geq 5/3 \). It is reversed if \( p \leq 1 \).

**Remark 4.3.** As sharp upper bounds for the second Seiffert mean, we have the following relations:

\[
(4.2) \quad T < \frac{2Q + A}{3} < A_{5/3} < L_{1/3}.
\]

In fact, it has been shown in [22, Conclusion 1] that the function \( r \mapsto A_r \) is strictly log-concave on \([0, \infty)\), and therefore
\[
A_{3/4}^{3/4} A_{1/3}^{1/4} < A_{2/3}^{4/3} = A_{4/3},
\]
which is equivalent with the third inequality in \((4.2)\). Now we prove the second one. Assume that \( a > b > 0 \) and set \((a/b)^{1/3} = x \in (0, 1)\). Then the inequality in question is equivalent to
\[
D(x) := \ln \left( \frac{2 \sqrt{x^6 + 1} + x^3 + 1}{3} - \ln \left( \frac{x^5 + 1}{2} \right)^{3/5} \right) < 0.
\]

Differentiating \( D(x) \) yields
\[
D'(x) = \frac{3x^2 (1-x)}{(x^5 + 1) \left( x^3 \sqrt{\frac{1}{2} x^6 + \frac{1}{2}} + \sqrt{\frac{1}{2} x^6 + \frac{1}{2} + 2 x^6 + 2} \right)} D_1(x),
\]
where
\[
D_1(x) = (1 + x) \sqrt{\frac{1}{2} x^6 + \frac{1}{2} - 2 x^2} = \frac{\left( (1 + x) \sqrt{\frac{1}{2} x^6 + \frac{1}{2}} - (2 x^2)^2 \right)}{(1 + x) \sqrt{\frac{1}{2} x^6 + \frac{1}{2} + 2 x^2}}
\]
\[
= \frac{(x - 1)^2 \left( x^6 + 4 x^5 + 8 x^4 + 12 x^3 + 8 x^2 + 4 x + 1 \right)}{2 (1 + x) \sqrt{\frac{1}{2} x^6 + \frac{1}{2} + 2 x^2}} > 0.
\]

Hence, \( D'(x) > 0 \) for \( x \in (0, 1) \), then \( D(x) < D(1) = 0 \).
Remark 4.4. By Theorem 1.1 and 1.2, the inequalities (1.2) and (1.11) can be improved as

\[(4.3) \; \; \; N < A_{3/2} < A_{\log_2 \pi/2} < T < A_{5/3} < A_2.\]

In our forthcoming paper, we shall establish the sharp bounds for the Neuman-Sándor mean in terms of power means as follows:

\[(4.4) \; \; \; A_{p_0} < N < A_{4/3},\]

where \(p_0 = \frac{\ln 2}{\ln \ln (3+2\sqrt{2})} = 1.2228...\)

Thus the chain of inequalities for bivariate means given in [6, (1)] can be refined as a more nice one:

\[(4.5) \; \; \; A_0 < L < A_{1/3} < P < A_{2/3} < I < A_{3/3} < N < A_{4/3} < T < A_{5/3},\]

where \(L, P, I, N, T\) are the logarithmic mean, the first Seiffert mean, the identric means, Neuman-Sándor mean, the second Seiffert mean, respectively.

**References**

[1] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and Their Inequalities*, Dordrecht, 1988.

[2] Y.-M. Chu, Y.-F. Qiu, and M.-K. Wang, Sharp power mean bounds for the combination of seiffert and geometric means, *Abstr. Appl. Anal.* 2010 (2010), Art. ID 108920, 12 pages.

[3] Y.-M. Chu, Y.-F. Qiu, M.-K. Wang and G.-D. Wang, The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert’s mean, *J. Inequal. Appl.* 2010 (2010), Art. ID 436457, 7 pages.

[4] Y.-M. Chu, M.-K. Wang, and W.-M. Gong, Two sharp double inequalities for Seiffert mean, *J. Inequal. Appl.* 2011 (2011): 44, 7 pages; available online at http://dx.doi.org/10.1186/1029-242X-2011-44.

[5] Y.-M. Chu and S.-W. Hou, Sharp bounds for Seiffert mean in terms of contraharmonic mean, *Abstr. Appl. Anal.* 2012 (2012), in press.

[6] I. Costin and G. Toader, A nice separation of some Seiffert type means by power means, Int. J. of Math. Math. Sci. 2012, in print.

[7] P. A. Hästö, A monotonicity property of ratios of symmetric homogeneous means, *J. Inequal. Pure Appl. Math.* 3 (5) (2002), Art. 71, 23 pages.

[8] P. A. Hästö, Optimal inequalities between Seiffert’s mean and power mean, *Math. Inequal. Appl.*, 7 (1) (2004) 47–53.

[9] D. He and Zh.-J. Shen, Advances in research on Seiffert mean, *Communications in inequalities research* 17 (4) (2010), Art. 26; Available online: http://old.irgoc.org/article/uploadFiles/201010/20101026014515652.pdf.

[10] A. A. Jagers, Solution of problem 887, *Nieuw Arch. Wisk.* 12 (1994), 2 30–231.

[11] D. H. Lehmer, On the compounding of certain means, *J. Math. Anal. Appl.* 36 (1971), 183–200.

[12] H. Liu and X.-J. Meng, The optimal convex combination bounds for Seiffert’s mean, *J. Inequal. Appl.* 2011 (2011), Art. ID 686834, 9 pages.

[13] E. Neuman and J. Sándor, On the Schwab-Borchardt mean, *Math. Pannon.* 17 (1) (2006) 49–59.

[14] J. Sándor, On certain inequalities for means III, *Arch. Math.* 76 (2001) 34–40.

[15] J. Sándor and E. Neuman, On certain means of two arguments and their extensions, *Int. J. Math. Math. Sci.* 2003 (16) (2003), 981–993, doi:10.1155/S0161171203208103.

[16] J. Sándor, *Selected Chapters of Geometry, Analysis and Number Theory: Classical Topics in New Perspectives*, LAP Lambert Academic Publishing, August 5, 2009.

[17] H.-J. Seiffert, Werte zwischen dem geometrischen und dem arithmetischen Mittel zweier Zahlen, * Elem. Math.* 42 (1987), 105–107.

[18] H.-J. Seiffert, *Aufgabe 16, Die Wurzel* 29 (1995) 221–222.

[19] S.-S. Wang and Y.-M. Chu, The best bounds of the combination of arithmetic and harmonic means for the Seiffert’s mean, *Int. J. Math. Anal. (Ruse)* 4 (21–24) (2010) 1079–1084.
[20] M.-K. Wang, Y.-F. Qiu, and Y.-M. Chu, Sharp bounds for Seiffert means in terms of Lehmer means, *J. Math. Inequal.* **4** (4) (2010) 581–586.

[21] A. Witkowski, Interpolations of Schwab-Borchardt mean, *Math. Inequal. Appl.* 2012, in print.

[22] Zh.-H. Yang, On the log-convexity of two-parameter homogeneous functions, *Math. Inequal. Appl.* **10**(3) (2007), 499-516.

System Division, Zhejiang Province Electric Power Test and Research Institute, Hangzhou, Zhejiang, China, 31001

E-mail address: yzhkm@163.com