ALMOST SIMILAR CONFIGURATIONS

IMRE BÁRÁNY AND ZOLTÁN FÜREDI

Abstract. Let \( h(n) \) denote the maximum number of triangles with angles between 59° and 61° in any \( n \)-element planar set. Our main result is an exact formula for \( h(n) \). We also prove \( h(n) = n^3/24 + O(n \log n) \) as \( n \to \infty \). However, there are triangles \( T \) and \( n \)-point sets \( P \) showing that the number of \( \varepsilon \)-similar copies of \( T \) in \( P \) can exceed \( n^3/15 \) for any \( \varepsilon > 0 \).

1. An exact result

Conway, Croft, Erdős, and Guy \cite{4} studied the distribution of angles determined by planar \( n \)-set. Motivated by their questions and results consider the following problem. We are going to set up this question more generally.

Let \( T \) be a fixed triangle with angles \( \alpha, \beta, \gamma \). Another triangle \( \Delta \) with angles \( \alpha', \beta', \gamma' \) is called \( \varepsilon \)-similar to \( T \) if \( |\alpha - \alpha'|, |\beta - \beta'|, \) and \( |\gamma - \gamma'| < \varepsilon \). Here \( \varepsilon > 0 \) is a small angle, smaller than any angle of \( T \). Let \( h(n, T, \varepsilon) \) denote the maximal number of triangles in a planar \( n \)-set that are \( \varepsilon \)-similar to \( T \).

The following construction gives a lower bound on \( h(n, T, \varepsilon) \) (see Figure 1). Place the points in three groups of as equal sizes as possible, with each group very close to the vertices of \( T \). This only gives the lower bound \( n^3/27 - O(n) \). Iterating this yields a better bound: splitting each of the three groups into three further groups gives the inequality (with notation \( f(n) = h(n, T, \varepsilon) \))

\[
f(a + b + c) \geq abc + f(a) + f(b) + f(c)
\]

where \( a, b, c \) are the sizes of the three groups. Define the sequence \( h(n) \) (for \( n = 0, 1, 2, \ldots \)) as the maximum lower bound what we can have using the above iterated threepartite construction. Let \( h(0) = h(1) = h(2) = 0, h(3) = 1 \) and for all \( n \geq 1 \) let

\[
h(n) := \max \{abc + h(a) + h(b) + h(c) : a + b + c = n, \ a, b, c \geq 0 \ \text{integers}\}.
\]
We show now by induction that $h(n) \leq \frac{1}{24}(n^3 - n)$.

$$h(n) = abc + h(a) + h(b) + h(c) \leq abc + \frac{a^3 - a}{24} + \frac{b^3 - b}{24} + \frac{c^3 - c}{24}$$

$$= \frac{n^3 - n}{24} + \frac{3}{4} \left( abc - \frac{a^2 b + b^2 a + b^2 c + c^2 b + c^2 a + a^2 c}{6} \right).$$

An application of the inequality between the arithmetic and geometric means yields that the second term is nonpositive. This proof show also that in $h(n) \leq \frac{1}{24}(n^3 - n)$ equality holds for $n \geq 3$ if and only if $n$ is a power for 3.

Standard induction shows that for some absolute constant $C > 0$ for all $n$ we have

$$\frac{n^3}{24} - Cn \log n < h(n) \leq \frac{1}{24}(n^3 - n).$$

It follows that for every triangle $T$ and for every $\varepsilon > 0$

$$h(n, T, \varepsilon) \geq h(n) \geq \frac{n^3}{24} - O(n \log n).$$

The constructions in Section 3 show that for some specific triangles better lower bounds hold. However, we prove in Section 6 the following theorem showing that the bound in (1.2) is very precise for almost regular triangles.

**Theorem 1.1.** Let $T$ be the regular triangle. There exists an $\varepsilon_0 \geq 1^\circ$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $n$ we have $h(n, T, \varepsilon) = h(n)$. In particular, when $n$ is a power of 3, $h(n, T, \varepsilon) = \frac{1}{24}(n^3 - n)$. 

![The iterated three-partite construction](image)
This implies that the following corollary.

**Corollary 1.2.** Let $T$ be a triangle whose angles between $60^\circ - \varepsilon_0/2$ and $60^\circ + \varepsilon_0/2$. Then $h(n,T,\varepsilon) = h(n)$ and $h(n,T,\varepsilon) = \frac{1}{24}(n^3 - n)$ if $n$ is a power of $3$.

2. **ONLY 0.3% ERROR FOR MOST OF THE TRIANGLES**

The space of triangles or rather triangle shapes can be identified with triples $(\alpha,\beta,\gamma)$ with $\alpha,\beta,\gamma > 0$ and $\alpha + \beta + \gamma = \pi$. Let $S$ be the subset of the plane $\alpha + \beta + \gamma = \pi$ defined by the inequalities $\alpha \geq \beta \geq \gamma > 0$, the domain $S$ represents every triangle by a single point. Thus we can talk about almost all triangles in measure theory sense. Theorem 1.1 or rather Corollary 1.2 gives an exact value for $h(n,t,\varepsilon_0)$ for at least $\Omega(\varepsilon_0^2)$ fraction of $S$. It shows that (as $n \to \infty$) at most about one quarter of the $\binom{n}{3}$ triangles could be almost regular and this bound is best possible.

We measure an angle $\alpha$ either in degrees or in radians, whatever is more convenient. We hope this is always clear from the context.

The next result uses extremal set theory, actually Turán theory of hypergraphs and flag algebra computations to give an upper bound for $h(n,T,\varepsilon)$ for almost every triangle $T$ that is only 0.3% larger than the lower bound in (1.2).

**Theorem 2.1.** For almost every triangle $T$ there is an $\varepsilon > 0$ such that

$$h(n,T,\varepsilon) \leq 0.25072 \left( \frac{n}{3} \right)^3 (1 + o(1)).$$

First, in Sections 7 and 8 we will prove a slightly weaker bound $h(n,T,\varepsilon) \leq 0.25108 \left( \frac{n}{3} \right)^3 (1 + o(1))$ and sketch in Section 9 how to get the better bound.

3. **CONSTRUCTING MANY ALMOST SIMILAR TRIANGLES**

In this section we give several examples of triangles $T$ where $h(n,T,\varepsilon)$ is larger than in the case of almost regular triangles. The construction is recursive just as in the previous section. Let $Q = \{q_1, \ldots, q_r\}$ be a finite set in the plane, and let $\mathcal{F}(Q,T)$ be the 3-uniform hypergraph with vertex set $\{1, \ldots, r\}$ and $ijk$ is an edge of $\mathcal{F}$ iff the triangle $q_iq_jq_k$ is similar to $T$. Further, let $x_1, \ldots, x_r$ be non-negative reals with $x_1 + \ldots + x_r = 1$. Given a large $n$ we place $n_i$ points in a very small disk $D_i$ centres at $q_i$ for all $i$; here $n_i$ is $x_in$ rounded up or down to the next integer so that $n = n_1 + \ldots + n_r$. If the disks are small enough then every triangle $p_ip_jp_k$ with $p_i \in D_i, p_j \in D_j, p_k \in D_k$ and $ijk \in \mathcal{F}$ is $\varepsilon$-similar to $T$. This gives $\sum_{ijk \in \mathcal{F}} n_in_jn_k \sim n^3 \sum_{ijk \in \mathcal{F}} x_ix_jx_k$ triangles that are $\varepsilon$-similar to $T$ on this level. We still have the freedom to place the $n_i$ points anywhere in $D_i$, and we recursively repeat the construction by placing a similar copy of $Q$ in each disk $D_i$, etc. Let $P_n$ denote the
Figure 2. Examples 1 and 2

\(n\)-element point set obtained at the end of the construction. A direct checking shows that, as \(n \to \infty\), \(P_n\) contains

\[n^3 \sum_{ijk\in F} x_ix_jx_k \frac{1}{1-(x_1^3 + \ldots x_r^3)} + O(n^2)\]

triangles \(\varepsilon\)-similar to \(T\). We will, of course, choose \(x_1, \ldots, x_r \geq 0\) to maximize the function

\[(3.1) \quad f(x) = f(x_1, \ldots, x_r) = \frac{\sum_{ijk\in F} x_ix_jx_k}{1-(x_1^3 + \ldots x_r^3)}\]

under the condition that \(x_1 + \ldots + x_r = 1\). Note that \(x_i = 0\) would mean that point \(q_i\) is not used in the construction in which case the underlying triple system \(F\) is different.

**Example 1.** \(T\) is right angled and \(Q = \{q_1, q_2, q_3, q_4\}\) is the set of four vertices of a rectangle such that any three vertices of \(P \subset Q\) form a triangle congruent to \(T\), see Figure 2. The function \(f(x)\) is symmetric in its 4 variables and its maximum is taken when \(x_1 = x_2 = x_3 = x_4 = 1/4\) where \(f(x) = 1/15\). Consequently

\[h(n, T, \varepsilon) \geq n^3 \left(\frac{1}{15} - O(n^2)\right),\]

a much larger lower bound than in (1.2).

**Example 2.** \(T\) is an isosceles right angled triangle and \(Q = \{q_1, q_2, q_3, q_4, q_5\}\) are the four vertices and the centre of a square, see Figure 2. The corresponding function is symmetric again in the variables \(x_1, x_2, x_3, x_4\) and takes its maximum when \(x_1 = x_2 = x_3 = x_4 = x\), say. Then \(x_5 = 1 - 4x\) and

\[f(x) = \frac{4x^3 + 4x^2(1 - 4x)}{1 - 4x^3 - (1 - 4x)^3} = \frac{x - 3x^2}{3(1 - 4x + 5x^2)}\]
where \( x \in [0, 1/4] \). The value of the maximum is \( 1/(6\sqrt{2} + 6) = 1/14.4852 \), and is reached at \( x = (3 - \sqrt{2})/7 \). This gives

\[
h(n, T, \varepsilon) = \frac{n^3}{14.4852} + O(n^2).
\]

Most likely this isosceles triangle gives the largest value for \( h(n, T, \varepsilon) \).

**Example 3.** The angles of \( T \) are 120°, 30°, 30° and \( Q = \{q_1, q_2, q_3, q_4\} \) are the three vertices and the centre of a regular triangle, see Figure 3. Here \( f(x) \) is again symmetric in its first three variables, so we choose \( x_1 = x_2 = x_3 = x \) and then \( x_4 = 1 - 3x \) and \( x \in [0, 1/3] \) and

\[
f(x) = \frac{3x^2(1 - 3x)}{1 - 3x^3 - (1 - 3x)^3} = \frac{x(1 - 3x)}{3 - 9x + 8x^2}.
\]

This function is maximized at \( x = (9 - \sqrt{24})/19 \) which gives

\[
h(n, T, \varepsilon) \geq \frac{n^3}{18.7979} + O(n^2).
\]

**Example 4.** The angles of \( T \) are \( \alpha = 40.2\ldots^\circ, 2\alpha = 80.4\ldots^\circ \) and \( \pi - 3\alpha = 59.3\ldots^\circ \) where \( \alpha \) is the root of the equation \( \sin(3\alpha)^3 = \sin(\alpha(\sin 2\alpha))^2 \). Let \( Q = \{q_1, q_2, q_3, q_4\} \) be a convex quadrilateral (see Figure 3) such that \( q_4q_1q_2 \) and \( q_4q_2q_3 \) are similar to \( T \). This means that the angles at \( q_4, \angle q_1q_4q_2 = \angle q_2q_4q_3 = \alpha \), and the angles at \( q_1 \) and \( q_2 \), i.e., \( \angle q_2q_4q_3 \) equal to 2\( \alpha \). Then the triangle \( q_3q_4q_1 \) is also similar to \( T \), so the structure of similar triangles in this \( Q \) is the same as in the previous Example 3. The same calculation leads to

\[
h(n, T, \varepsilon) \geq \frac{n^3}{18.7979} + O(n^2).
\]

**Example 5.** \( T \) is the triangle with angles 90°, 60°, 30°, \( Q \) is the set of vertices of the regular hexagon. Putting weights 1/6 on each vertex
the method gives

\[ h(n, T, \varepsilon) \geq \frac{n^3}{17.5} - O(n^2). \]

This is better than what we can get from the standard iterated three-partite construction, but slightly weaker than Example 1.

The next three examples give only \( h(n, T, \varepsilon) \geq \frac{n^3}{24} + O(n^2) \) for various \( T \) but we include them here for two reasons. First, although their order of magnitude is the same, their structure is completely different which shows that any proof to describe the extremal families could not be too simple. Second, in cases when \( n \) is a power of 5 (or 7, resp.) these examples yield \( h(n, T, \varepsilon) \geq \frac{1}{24}(n^3 - n) \) slightly exceeding \( h(n) \).

**Examples 6.** \( T \) has angles 108°, 36°, 36° and \( Q = \{q_1, q_2, q_3, q_4, q_5\} \) are the vertices of a regular pentagon, see Figure 4. The function \( f(x) \) is symmetric in its five variables and setting all \( x_i = 1/5 \) gives \( f(x) = 1/24 \). We have

\[ h(n, T, \varepsilon) \geq \frac{n^3}{24} + O(n^2). \]

**Examples 7.** \( T \) has angles 72°, 72°, 36° and \( Q = \{q_1, q_2, q_3, q_4, q_5\} \) are the vertices of a regular pentagon as in Example 6. The same argument yields

\[ h(n, T, \varepsilon) \geq \frac{n^3}{24} + O(n^2). \]
Example 8. The angles of $T$ are $\frac{4}{7}\pi, \frac{2}{7}\pi, \frac{1}{7}\pi$ and $Q$ is the set of vertices of a regular 7-gon. The corresponding $f(x)$ is symmetric and setting $x_i = 1/7$ gives $h(n, T, \varepsilon) \geq \frac{n^3}{24} - O(n^2)$.

4. Generalizations and extensions

The definition of $\varepsilon$-similar triangles can be carried over to planar $k$-set, $k \geq 4$. So let $A \subset \mathbb{R}^2$ be a fixed $k$-set, $A = \{a_1, \ldots, a_k\}$ and $\delta > 0$. Another $k$-set $B = \{b_1, \ldots, b_k\} \subset \mathbb{R}^2$ and $A$ are $\delta$-similar if there is a $\lambda > 0$ such that for all $i \neq j$

$$1 - \delta \leq \frac{|a_i a_j|}{|b_i b_j|} \leq 1 + \delta.$$ This is essentially the same as what we used for triangles. For any triangle $T$ and $\varepsilon > 0$ there exists a $\delta_1 = \delta_1(T, \varepsilon)$ such that a triangle $T'$ which is $\delta$-similar to $T$ with $\delta < \delta_1$ is also $\varepsilon$-similar to $T$. On the other hand, for every $\delta > 0$ there exists an $\varepsilon_1 = \varepsilon_1(T, \delta)$ such that a triangle $T'$ which is $\varepsilon$-similar to $T$, $\varepsilon < \varepsilon_1$, is also $\delta$-similar to $T$. Define $H(n, A, \delta)$ as the maximal number of $\delta$-similar copies of $A$ present in an $n$-element set in $\mathbb{R}^2$. Placing $k$ groups of points, each of size $n/k$, very close to the points of $A$ and iteration shows that for all $A$ and $\delta > 0$

$$H(n, A, \delta) \geq \frac{n^k}{k^k - k} + O(n^{k-1}).$$

The case of truly similar copies, that is when $\delta = 0$, is different. Then $H(n, A, 0) \leq 2n(n - 1)$. Elekes and Erdős [3] showed that $H(n, A, 0) \geq cn^{2-o(1)}$ for every $k$-set $A$, and $H(n, T, 0) \geq n^2/18$ for every triangle $T$. Laczkovich and Ruzsa [9] proved the remarkable result that $H(n, A, 0) = \Omega(n^2)$ if and only if the cross ratio of any four elements of $A$ is algebraic. Here $A$ is considered as a $k$-set in the complex plane and the cross ratio of four complex numbers $z_1, z_2, z_3, z_4$ is

$$\frac{(z_1 - z_3)/(z_3 - z_2)}{(z_1 - z_4)/(z_4 - z_2)}.$$ See more in [1].

The same question comes up in higher dimensions as well. Elekes and Erdős [3] and Pach [11] proved that for every $d$-dimensional simplex $\Delta^d$

$$n^{(d+1)/d-o(1)} \leq H(n, \Delta^d, 0) = O(n^{(d+1)/d}).$$

For results on equilateral triangles in $\mathbb{R}^d$, $d \leq 5$ see [2].

5. Optimal configurations

Let $T$ be any given triangle and assume $\varepsilon > 0$ is small. A point set $P \in \mathbb{R}^2$ with $|P| = n$ gives rise to a 3-uniform hypergraph $\mathcal{H}(P, T, \varepsilon)$:
its vertex set is \( P \) and \( xyz \in \mathcal{H} \) if the triangle with vertices \( x, y, z \in P \) is \( \varepsilon \)-similar to \( T \). So we have
\[
h(n, T, \varepsilon) \geq |\mathcal{H}(P, T, \varepsilon)|
\]
and \( P \) is called optimal (or optimal for \( T \)) if here equality holds. We write \( \deg(x) \) resp. \( \deg(x, y) \) for the degree of \( x \) and codegree of \( xy \), that is \( \deg(x) \) resp. \( \deg(x, y) \) is the number of triples in \( \mathcal{H} \) containing \( x \) and both \( x \) and \( y \).

We write \( B(x, r) \) for the Euclidean ball centered at \( x \) and having radius \( r \). There is a small \( \eta_0 = \eta_0(P) > 0 \) (depending only on \( P \)) such that for any \( \eta \in (0, \eta_0) \) the following holds. If \( xyz \in \mathcal{H} \), then the triangle with vertices \( x', y', z' \) is \( \varepsilon \)-similar to \( T \) for any \( x' \in B(x, \eta), y' \in B(y, \eta), z' \in B(z, \eta) \).

Assume next that \( x, y \in P \), \( \deg(x, y) = 0 \) and \( \deg(x) \geq \deg(y) \), and let \( x' \in B(x, \eta) \) an arbitrary point, distinct from \( x \). Define \( P' = P \cup \{x'\} \setminus \{y\} \).

**Lemma 5.1.** Under these conditions, \( |\mathcal{H}(P', T, \varepsilon)| \geq |\mathcal{H}(P, T, \varepsilon)| \). If \( \deg(x) > \deg(y) \) then \( |\mathcal{H}(P', T, \varepsilon)| > |\mathcal{H}(P, T, \varepsilon)| \).

The proof is simple: The triples in \( \mathcal{H} \) not containing \( y \) remain triples in \( \mathcal{H}' \). Write \( \deg'(.) \) for the degrees in \( \mathcal{H}' = \mathcal{H}(P', T, \varepsilon) \). Since \( xuv \in \mathcal{H} \) (here \( u, v \) are distinct from \( y \)) implies \( x'uv \in \mathcal{H}' \) and \( xuv \in \mathcal{H}' \), we have \( \deg'(x') \geq \deg(x) \) and \( \deg'(x) = \deg(x) \geq \deg(y) \). So indeed, \( |\mathcal{H}'| \geq |\mathcal{H}| \), and the inequality is strict if \( \deg(x) > \deg(y) \).

Assume next that \( \deg(x, y) = \deg(x, z) = 0 \). By the previous claim we can replace both \( y \) and \( z \) by \( x' \), \( x'' \in B(x, \eta) \) so that with \( P'' = P \cup \{x', x''\} \setminus \{y, z\} \) the new hypergraph \( \mathcal{H}'' \) satisfies \( |\mathcal{H}''| \geq |\mathcal{H}| \). Actually, \( x' \) and \( x'' \) can be chosen so that the triangle \( xx'x'' \) is \( \varepsilon \)-similar to \( T \) so \( |\mathcal{H}''| > |\mathcal{H}| \). We obtained the following

**Corollary 5.2.** If the planar \( n \)-set \( P \) is optimal, then \( \deg(x) = \deg(y) \) for every \( x, y \in P \) with \( \deg(x, y) = 0 \). Moreover, if \( \deg(x, y) = \deg(u, v) = 0 \), then \( \{x, y\} \) and \( \{u, v\} \) are disjoint or they coincide.

We are going to fix an optimal planar \( n \)-set \( P \) such that the diameter of \( P \) is of length one, and points \( x, y \in P \) with \( \deg(x, y) = 0 \) are very close to each other. This is accomplished with the next technical lemma.

**Lemma 5.3.** There is an optimal planar \( n \)-set \( P \) and an \( \eta \in (0, 10^{-3}) \) such that the diameter of \( P \) has length one, the points \( u, v \in P \) with \( \deg(u, v) = 0 \) satisfy \( |uv| < \eta \) and every disk of radius \( \eta \) contains at most two points from \( P \).

**Proof.** Start with an optimal planar \( n \)-set \( Q \) and let \( x_0, y_0 \in Q \) be the pair with maximal distance \( |x_0y_0| \) among all pairs \( u, v \) with \( \deg(u, v) \geq 1 \). As a homothety does not change \( \varepsilon \)-similarity we assume that \( |x_0y_0| = 1 \).
Next choose \( \eta > 0 \) smaller than \( \eta_0(P) \), and smaller than \( 10^{-3} \), and smaller than one tenth the minimal distance among pairs in \( Q \). Apply Lemma 5.1 to every pair \( u, v \in P \) with \( \deg(u, v) = 0 \). Such pairs are disjoint and \( \deg(u) = \deg(v) \) by Corollary 5.2. So we can replace \( v \) by \( u' \in B(u, \eta) \) or \( u \) by \( v' \in B(v, \eta) \). The choice between \( u' \) and \( v' \) is arbitrary except when \( x_0 \) or \( y_0 \) is present in the pair. Then we keep \( x_0 \) (resp. \( y_0 \)) and replace the other element of the pair by \( x_0' \in B(x_0, \eta) \) (and by \( y_0' \)). We get a new \( n \)-set \( Q' \) still maximizing \( h(n, T, \varepsilon) \). The diameter of \( Q' \) is between 1 and \( 1 + 2\eta \). Apply another homothety so that the diameter of the new \( n \)-set \( P \) obtained from \( Q' \) has diameter one. Then \( P \) satisfies the requirements. \( \square \)

6. Proof of Theorem 1.1

In this section all angles are measured in radians and we fix \( \varepsilon = 1/50 \). We need some definitions. We call a triangle \( \varepsilon \)-regular if it is \( \varepsilon \)-similar to the regular triangle. For distinct \( u, v \in \mathbb{R}^2 \) define \( q^+(u, v) \) (resp. \( q^-(u, v) \)) as the point in \( \mathbb{R}^2 \) obtained by rotating \( y \) about \( x \) by \( \pi/3 \) anti-clockwise (and clockwise), see Figure 6 for \( q^+(u, v) \). Then \( u, v, q^+(u, v) \) are the vertices of a regular triangle. Assume \( uwv \) is an \( \varepsilon \)-regular triangle. Then \( w \) is close to either \( q^+(u, v) \) or to \( q^-(u, v) \).

More formally, a simple computation using \( \varepsilon = 1/50 \) shows that

\[
(6.1) \quad w \in B(q^+(u, v), 1.2\varepsilon|uv|) \cup B(q^-(u, v), 1.2\varepsilon|uv|).
\]

We begin now the proof of Theorem 1.1. Fix a maximizer \( n \)-set \( P \subset \mathbb{R}^2 \) as in Lemma 5.3. The diameter of \( P \) is realized on points \( x, y \in P \) so \( |xy| = 1 \) and \( \deg(x, y) \geq 1 \) since pairs \( u, v \in P \) with \( \deg(u, v) = 0 \) are at distance less than \( \eta < 10^{-3} \). So there is \( z \in P \) with \( xyz \in \mathcal{H} \). Fix such a \( z \) and set \( s = 1.2\varepsilon = 0.024 \). According to (6.1), \( z \) is close to either \( q^+(x, y) \) or to \( q^-(x, y) \). We may assume that it is close to \( w = q^+(x, y) \) and so \( z \in B(w, s) \) implying that

\[
P \subset B(x, 1) \cap B(y, 1) \cap B(z, 1)
\subset B(x, 1) \cap B(y, 1) \cap B(w, 1 + s) := D
\]

because \( B(z, 1) \subset B(w, 1 + s) \).

Here \( D \) is a convex set, see Figure 6. Rotating \( D \) about \( x \) by angle \( \pi/3 \) anti-clockwise resp. clockwise we obtain the sets \( D^+(x) \) and \( D^-(x) \). The sets \( D^+(y), D^+(w) \) and \( D^-(y), D^-(w) \) are defined analogously. Set further \( x^* = q^+(w, y), y^* = q^+(x, w) \) and \( w^* = q^+(y, x) \) (see Figure 7) and define

\[
M(x) = B(x, 1 + 2s) \cap B(x^*, 1 + 2s),
M(y) = B(y, 1 + 2s) \cap B(y^*, 1 + 2s),
M(w) = B(w, 1 + 2s) \cap B(w^*, 1 + 2s).
\]

Set further \( N(w) = M(x) \cap M(y), N(y) = M(x) \cap M(w), \) and \( N(x) = M(w) \cap M(y) \).
Lemma 6.1. \( P \subset N(x) \cup N(y) \cup N(w) \).

**Proof.** Assume \( u \in P \). If \( \deg(x, u) = 0 \), then \( u \in B(x, \eta) \subset N(x) \). The same argument applies when \( \deg(y, u) = 0 \). If \( \deg(z, u) = 0 \), then \( u \in B(z, \eta) \) and \( B(z, \eta) \subset N(w) \).

Thus \( u \in P \) and we assume \( \deg(x, u), \deg(y, u), \deg(z, u) \geq 1 \). Here \( \deg(x, u) \) means there is \( v \in P \) with \( xyv \in H \) implying by (6.1) that \( v \in B(q^+(x, y), s) \cup B(q^-(x, y), s) \). In other words, rotating \( u \) about \( x \) by angle \( \pi/3 \) or \( -\pi/3 \) we arrive at a point at distance at most \( s \) from \( v \in P \subset D \). Going backwards, that is, rotating \( D \) about \( x \) by \( \pi/3 \) and \( -\pi/3 \) we obtain the sets \( D^+(x) \) and \( D^-(x) \) such that

\[
\begin{align*}
\text{Here } D^+(x) &= B(x, 1) \cap B(y^*, 1 + s) \cap B(w, 1), \text{ and then } \\
(D \cap D^+(x)) + sB &\subset B(y, 1 + s) \cap B(y^*, 1 + 2s) \subset M(y).
\end{align*}
\]

One proves similarly that

\[
(D \cap D^-(x)) + sB \subset B(w, 1 + 2s) \cap B(w^*, 1 + s) \subset M(w),
\]

implying that \( u \in M(y) \cup M(w) \). The same way \( u \in M(w) \cup M(x) \) follows from \( \deg(y, u) \geq 1 \), see Figure 7.

Finally, \( \deg(z, u) \geq 1 \) implies that there is \( t \in P \) such that \( zut \in H \) and then by (6.1)

\[
t \in B(q^+(z, u), s) \cup B(q^-(z, u), s) \subset B(q^+(w, u), 2s) \cup B(q^-(w, u), 2s)
\]
Figure 7. The sets $M(x)$ and $M(w)$ and their intersection

where the last containment follows from $q^\pm(z,u) \in B(q^\pm(w,u),s)$. Then

$$u \in [(D \cap D^+(w)) + 2sB] \cup [(D \cap D^-(w)) + 2sB]$$

and $D^+(w) = B(y,1) \cap B(w,1+s) \cap B(x^*,1)$. This shows that

$$(D \cap D^+(w)) + 2sB \subset B(x,1+2s) \cap B(x^*,1+2s) = M(x).$$

One proves the same way that $(D \cap D^-(w)) + sB \subset M(y)$, so $u \in M(x) \cup M(y)$.

We have shown so far that

$(6.2)$ \quad $u \in [M(x) \cup M(y)] \cap [M(y) \cup M(w)] \cap [M(w) \cup M(x)],$

not quite what we wanted but we are not far. It is easy to check that relation $(6.2)$ holds if and only if $u$ is contained in at least two of the sets $M(x), M(y), M(w)$. Observe now that $M(x) \cap M(y) \cap M(w) = \emptyset$. Indeed, if these three sets had a point in common, then their union would cover the triangle $xyw$ because the edges $xy, yw, wx$ resp. are contained in $M(w), M(x), \text{ and } M(y)$. But none of the sets contains the centre of the triangle $xyw$.

This implies that $(6.2)$ holds if and only if $u$ is contained either $M(x) \cap M(y)$ or in $M(y) \cap M(w)$ or in $M(w) \cap M(x)$.

Now we return to the proof of Theorem 1.1. Suppose $P$ has $a$ points in $N(x)$, $b$ points in $N(y)$ and $c$ in $N(w)$. By the lemma $n = a + b + c$. Write $f(n) = h(n,T,\varepsilon)$. We are going to show that $f(n) \leq h(n)$.

We prove next that no triangle in $\mathcal{H}$ has two point in $N(w)$ and one in $N(y)$. This is quite simple: assume $x_1y_1z_1 \in \mathcal{H}$ is a triple with $x_1, y_1 \in N(w)$ and in $z_1 \in N(y)$. A simple and generous computation shows that the diameter of $N(w)$ is smaller than $5s + \sqrt{2}s = 0.32$.

On the other hand, the distance between $N(w)$ and $N(y)$ is $1 - \sqrt{6}s = 0.346 \ldots$. So the ratio of the lengths of one edge ($x_1z_1$ or $y_1z_1$) to another...
edge (namely $x_1y_1$) is at least $0.346.../0.32 > 1.08$. On the other hand, the ratio of the length of any two edges in an $\varepsilon$-regular triangle is at most $\sin(\pi/3 + \varepsilon)/\sin(\pi/3 - \varepsilon) = 1.0233... < 1.03$ (where $\varepsilon = 1/50$). Consequently $x_1y_1z_1$ is not an $\varepsilon$-regular triangle, contrary to $x_1y_1z_1 \in \mathcal{H}$.

It follows that there are two kinds of triples in $\mathcal{H}$: either one vertex in each of $N(x)$, $N(y)$, and $N(w)$ or all three vertices are in one of the sets $N(x)$, $N(y)$, and $N(w)$.

The number of triangles with one vertex in each of $N(x)$, $N(y)$, and $N(w)$ is $abc$. The number of triangles with all vertices in $N(x)$, $N(y)$, resp. $N(w)$ is $f(a)$, $f(b)$, and $f(c)$. Thus

$$f(n) \leq abc + f(a) + f(b) + f(c)$$

and the argument (1.1) finishes the proof. \qed

7. Turán problems for hypergraph

Turán’s theory of extremal graphs and hypergraphs has several applications in geometry (see, e.g. [10]) and elsewhere [6]. Here we explain what we need for Theorem 2.1, the case of 3-uniform hypergraphs. Let $\mathcal{L}$ be a finite family of 3-uniform hypergraphs, the so-called forbidden hypergraphs. Turán’s problem is to determine the maximal number of edges that a 3-uniform hypergraph $H$ on $n$ vertices can have if it does not contain any member of $\mathcal{L}$ as a subhypergraph. This maximal number is usually denoted by $ex(n, \mathcal{L})$.

Define $K^-_4 = \{124, 134, 234\}$ which is the complete 3-uniform hypergraph on four vertices minus one edge, and $C_5 = \{123, 234, 345, 451, 512\}$ which is the 5-cycle, and let $\mathcal{L} = \{K^-_4, C_5\}$. We need the following result of Falgas-Ravry and Vaughan [7]:

$$(7.1) \quad (0.25 + o(1)) \left(\frac{n}{3}\right) \leq ex(n, \{K^-_4, C_5\}) \leq 0.251073 \left(\frac{n}{3}\right).$$

The upper bound part of this result will be used in the proof of Theorem 2.1 (weaker version). First some preparation is needed.

Given a triangle $T$ with angles $\alpha, \beta, \gamma$ an equation of the form

$$n_1\alpha + n_1\beta + n_2\gamma + n_4\pi = 0$$

is called a non-trivial linear equation of $T$ if the vector $(n_1, n_2, n_3, n_4)$ is linearly independent from $(1, 1, 1, -1)$, their coordinates are integers, and all are at most 5 in absolute value. Note that the equation $\alpha + \beta + \gamma - \pi = 0$ is satisfied by every triangle.

Here we extend the definition of the hypergraph $\mathcal{F}(Q,T)$ used in Section 3 given a finite multiset $Q = \{q_1, \ldots, q_r\} \subset \mathbb{R}^2$ and a triangle $T$, the vertex set of $\mathcal{F}(Q,T)$ is $\{1, \ldots, r\}$ and $ijk$ is an edge of $\mathcal{F}(Q,T)$ iff either $q_ijq_k$ is similar to $T$ or $q_i = q_j = q_k$. The multiset $Q$ is trivial if all of its points coincide. Otherwise we say that $Q$ realizes the hypergraph $\mathcal{F}$ and that $\mathcal{F}$ can be realized by $T$. 
Lemma 7.1. Assume $Q = \{q_1, q_2, q_3, q_4\}$ is a nontrivial multiset and $F(Q, T)$ contains a copy of $K_4^-$. Then the angles of $T$ satisfy a non-trivial linear equation.

Lemma 7.2. Assume $Q = \{q_1, \ldots, q_5\}$ is a nontrivial multiset and $F(Q, T)$ contains a copy of $C_5$. Then the angles of $T$ satisfy a non-trivial linear equation.

The proof of these lemmas are postponed into the next Section.

Proof of Theorem 2.1, weaker version. Assume $T$ is a triangle whose angles do not satisfy any non-trivial linear equation. Then there is an $\varepsilon(T) > 0$ such that no triangle which is $\varepsilon$-similar to $T$ satisfies any non-trivial linear equation. The reason is that, in the space of triangle shapes, $S$, $T$ is at positive distance from the closed set defined by the finitely many non-trivial linear equations.

This implies that, given a planar set $P$ of $n$ points, the hypergraph $H(P, T, \varepsilon)$ contains no copy of $K_4^-$ and no copy of $C_5$, provided $\varepsilon < \varepsilon(T)$.

Conjecture 7.3 (Falgas-Ravry and Vaughan, Conjecture 8 in [7]).

$$\lim_{n \to \infty} \text{ex}(n, \{K_4^-, C_5\}) \left(\frac{n}{3}\right)^{-1} = 1/4.$$  

This conjecture (if true) implies that $h(n, T, \varepsilon) = (1 + o(1))n^3/24$ for all triangles $T$ whose angles do not satisfy any non-trivial linear equation and a small enough $0 < \varepsilon < \varepsilon(T)$.

8. Proof of the two lemmas, realizations of $K_4^-$ and $C_5$

In both lemmas the triangles in $F(Q, T)$ cover all pairs of $Q$. So if $Q$ contains any point with multiplicity at least 2 then it contains a triangle of size 0, and one can easily see that all other triangles are of size 0, i.e., $Q$ is a trivial multiset. From now on, we may assume that $Q$ is a proper set and $T$ has angles $\alpha, \beta,$ and $\gamma$.

Proof of Lemma 7.1. Four distinct points $q_1, \ldots, q_4$ are given such that the three triangles of the form $q_i q_j q_4$, $1 \leq i < j \leq 3$, are similar to $T$. First, consider the case when $q_4$ lies on the boundary of the convex hull of $Q$. Then (with a possible relabeling of $q_1, q_2,$ and $q_3$) we obtain

$$\angle q_1 q_4 q_2 + \angle q_2 q_4 q_3 = \angle q_1 q_3 q_4.$$  

Here all the three angles belong to $\{\alpha, \beta, \gamma\}$ so we obtain either an equation like $\alpha + \beta = \gamma$ (implying $\gamma = \pi/2$) or an equation of the form $2\beta = \alpha$. In each case we got a non-trivial linear equation.

Actually, one can show that in this case $T$ is either right angled or the unique triangle $T$ defined in Example 4 whose angles are approximately $40.2^\circ, 80.4^\circ,$ and $59.3^\circ$. 
From now on, we may suppose that $q_4$ is in the interior of conv $Q$. Then conv $Q$ is a triangle with vertices $q_1$, $q_2$, and $q_3$ and we have

$$\angle q_1q_4q_2 + \angle q_2q_4q_3 + \angle q_3q_4q_1 = 2\pi.$$ 

Here all the three angles belong to $\{\alpha, \beta, \gamma\}$ so all three must be the same and equal to $\alpha$, say (otherwise we get a contradiction like $\alpha + \beta + \gamma = 2\pi$ or $2\pi > 2\alpha + \beta = 2\pi$). Then $3\alpha = 2\pi$ is a non-trivial linear equation. It is easy to see in this case that the angles of $T$ are $2\pi/3, \pi/6, \pi/6$, the case in Example 3.

Proof of Lemma 7.2. Let $\delta_i$ denote the angle $q_{i-1}q_iq_{i+1}$, subscripts taken mod 5. Here $\delta_i \in \{\alpha, \beta, \gamma\}$ because $q_{i-1}q_iq_{i+1}$ is an angle of a triangle similar to $T$. The polygonal path $q_1q_2q_3q_4q_5q_1$ is closed, see Figure 8, implying that

\[(8.1) \quad \pm \delta_1 \pm \delta_2 \pm \delta_3 \pm \delta_4 \pm \delta_5 \equiv 0 \mod 2\pi,\]

where we have to select the appropriate signs according to the polygonal path. We claim that each of the $2^5$ choices of signs lead to a non-trivial linear equation.

Denote by $n_1$ the coefficient of $\alpha$ in (8.1), and $n_2$ and $n_3$ are defined analogously. It follows that

\[(8.2) \quad n_1\alpha + n_2\beta + n_3\gamma = n_4\pi,\]

where each $n_i$ is an integer, $|n_1| + |n_2| + |n_3| \leq 5$, $|n_4| \leq 4$. Moreover $|n_1| + |n_2| + |n_3|$ is odd and $|n_4|$ is even, so the vector $(n_1, n_2, n_3, -n_4)$ is linearly independent from $(1, 1, 1, -1)$. \qed

9. The two ingredients of the proof of Theorem 2.1

To prove the stronger version of Theorem 2.1 further forbidden hypergraphs are needed. Let $\mathcal{L}$ consist of the following 9 hypergraphs:

1. $K^-_4 = \{123, 124, 134\}$
2. $C^-_5 = \{123, 124, 135, 245\}$, a cycle $C_5$ minus an edge,
3. $C^+_5 = \{126, 236, 346, 456, 516\}$, called 5-wheel,
4. $L_2 = \{123, 124, 125, 136, 456\}$
(5) \( L_3 = \{123, 124, 135, 256, 346\} \)
(6) \( L_4 = \{123, 124, 156, 256, 345\} \)
(7) \( L_5 = \{123, 124, 145, 346, 356\} \)
(8) \( L_6 = \{123, 124, 145, 346, 356\} \)
(9) \( L_7 = \{123, 145, 167, 246, 257, 347\} \), the set of lines on the Fano plane with one line removed.

For the proof of Theorem 2.1 we need the following fact.

Claim 9.1. \( \lim_{n \to \infty} \text{ex}(n, L) \left( \frac{n}{3} \right)^{-1} < 0.25072. \)

The proof of this claim is based on the flag-algebra method due to Razborov [12]. It requires computations by a computer: the “Flagmatic” package developed by Falgas-Ravry and Vaughan [7] (thanks to them). The actual computation was carried out independently by Manfred Scheucher and by John Talbot (thanks to both of them as well).

The other tool we need to complete the proof of Theorem 2.1 is to show that, for almost every triangle shape \( T \), there is an \( \varepsilon(T) > 0 \) such that for every finite set \( P \subset \mathbb{R}^2 \) the hypergraph \( \mathcal{H}(P, T, \varepsilon) \) contains no hypergraph from \( L \). For this purpose define, for every \( L \in \mathcal{L} \), the set \( S(L) \) of triangle shapes \( \triangle \) that can realize \( L \in \mathcal{L} \) as \( F(Q, \triangle) \) with a suitable (non-trivial multi)set \( Q \subset \mathbb{R}^2 \) of the same size as the vertex set of \( L \). Note that there are many hypergraphs, e.g., \( F_{3,2} := \{123, 124, 125, 345\} \), which can be realized by all triangles, esp. when we allow multiple vertices (see Figure 9). So \( S(F_{3,2}) = S \).

We remark that every triple system on at most 6 vertices which is not a subfamily of the standard iterated threepartite construction contains a member (1)–(8) from our list \( \mathcal{L} \).
Lemma 9.2. For every $L \in \mathcal{L}$ and for almost every triangle shape $T$ there is an $\varepsilon(T) > 0$ such that the distance between $T$ and $S(L)$ is larger than $\varepsilon(T)$.

This is in fact 9 lemmas, one for each $L \in \mathcal{L}$. Out of them the case $L = K_4^+$ is just Lemma 7.1. Also the case $L = C_5^+$ can be handled as Lemma 7.2. Indeed, if $F(Q, \triangle)$ is realizing $C_5^+$ with the central vertex $q_6$, and the triangles are $q_1q_2q_6$, $q_2q_3q_6$, $q_3q_4q_6$, $q_4q_5q_6$, and $q_5q_1q_6$, then with notation $\delta_i = \angle q_iq_6q_{i+1}$ ($i = 1, \ldots, 5$) we have $\pm\delta_1 \pm \delta_2 \pm \delta_3 \pm \delta_4 \pm \delta_5 \equiv 0 \mod 2\pi$. This is the same as (8.1), leading to a non-trivial linear equation.

10. Algebraic conditions for triangle realizations

This section is the continuation of the proof of Lemma 9.2. For the other $L \in \mathcal{L}$ linear equations do not suffice, we need non-trivial polynomial equations. We explain the proof method in details only for $C_5^+$. The other cases are similar and technical, and we leave them to the interested reader.

It is more convenient to work with a different representation of triangle shapes, namely with complex numbers. Given a triangle $T$ its shape is identified with a complex number $z \in \mathbb{C} \setminus \mathbb{R}$ such that the triangle with vertices $0, 1, z$ is similar to $T$. In fact there are twelve complex numbers $w$ such that the triangle $0, 1, w$ is similar to $T$ (unless $T$ is isosceles). The set of these twelve points is $T(z)$ and, as one can check easily,

$T(z) = \{z, 1-z, 1/z, 1-1/z, 1/(1-z), z/(z-1) \text{ and their conjugates}\}$.

Figure 9 shows some of these points. It follows that if $a, b \in \mathbb{C}$ are distinct, and $w \in T(z)$ then $a, b$ and $v = w(b-a) + a$ form a triangle similar to $T$. It is also true that if $a, b, u$ is a triangle similar to $T$, then $u$ must be obtained in this way, so it is a ratio of two linear functions of $z$, or its conjugate, $\overline{z}$. The coefficients of the linear functions depend on $a$ and $b$. Set $z = x + iy$. Hence the real and imaginary part of $u$ are a ratio of two quadratic polynomials in variables $x$ and $y$. 
Assume next that $F(Q, T)$ is $C^-_5$ and $Q = \{q_1, \ldots, q_5\}$. If $q_1 = q_2$, then the size of the triangle $q_1q_2q_3$ is 0. Eventually we obtain that $Q$ should be a trivial multiset. So we may suppose that $q_1 \neq q_2$. Then (after a proper affine transformation) we may suppose that $q_1 = 0$ and $q_2 = 1$.

As $q_1q_2q_3$ is similar to $T$, $q_3$ is one of those twelve points that can be expressed as the ratio of two linear functions in $z$ or in $\bar{z}$. Again, since $q_1q_3q_5$ is similar to $T$, $q_5$ can be expressed as a ratio of two quadratic functions in $z$ or in $\bar{z}$. Consequently, the real and imaginary part of $q_5$ can be written as the ratio of two degree four polynomials in variables $x, y$. Note that typically there are many, but of course finitely many, such points, an upper bound is $12^2$.

Analogously, via the chain of triangles $q_1q_2q_4$, $q_2q_4q_5$, the real and imaginary parts of $q_5$ are equal to the ratio of two (degree four) polynomials in $x, y$, again in at most $12^2$ ways. So the coordinates of $q_5$ are computed two different ways. Each one of the at most $12^2 \times 12^2$ possibilities gives two (degree 8) polynomial equations (with integer coefficients) for the pair $x, y$. Such a pair of equations is non-trivial if it is not the identity.

The target is then to show that for each of the $12^4 = 20,736$ such pairs of equations there is a $z$ not satisfying it. Then, by continuity, there is a small neighborhood of $z$ not satisfying the equations, so its solution set could not be full dimensional, it is an algebraic curve on $\mathbb{C}$. The union of these $12^4$ curves give us $S(C^-_5)$. Hence it is a small closed set and almost all $T$ avoids it. This part of the proof is geometric and is the content of the next lemma.

**Lemma 10.1.** Assume $Q = \{q_1, q_2, q_3, q_4, q_5\} \subset \mathbb{R}^2$ is a non-trivial multiset and $T$ is the regular triangle. Then $F(Q, T)$ does not contain $C^-_5$.

**Proof.** Recall that if $q_1$ and $q_2$ coincide then all the points in $Q$ coincide. So $q_1 \neq q_2$ and $q_1q_2q_3$ and $q_1q_2q_4$ are non-degenerate regular triangles. So either $q_3 = q_4$ or they are on opposite side of the line through $q_1, q_2$. On Figure 11 these two cases are shown, the points are from a triangular grid, and we use the notation there.

Observe first that in both cases $q_5$ is either $p_1$ or $p_2$ because $135 \in C^-_5$. When $q_3 = q_4$, $q_5$ coincides with either $q_1$ or $p_2$ because $245 \in C^-_5$. This is a contradiction since the sets $\{p_1, q_2\}$ and $\{q_1, p_2\}$ are disjoint.

When $q_3$ and $q_4$ are distinct, $q_5$ must coincide with either $q_1$ or $p_3$, a contradiction again because the sets $\{p_1, q_2\}$ and $\{q_1, p_3\}$ are disjoint. □

11. More problems

**Remark.** Theorem 1.1 was proved with $\varepsilon = 0.02$ that is for triangles whose angles are between $58.9^\circ$ and $61.1^\circ$. The computations were
generous and the statement of the theorem must be valid for a larger interval of angles, for instance between 56° and 64°.

The following Turán type conjecture (a weakening of Conjecture 7.3) would solve our problem asymptotically for all but a few triangles.

**Conjecture 11.1.** \( \lim_{n \to \infty} \frac{\text{ex}(n, \{K_4^-, C_5^-\})}{n^3} = \frac{1}{4}. \)

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MTA Rényi Institute, PO Box 127, H-1364 Budapest, Hungary, and Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, United Kingdom.

E-mail address: barany@renyi.hu

MTA Rényi Institute, PO Box 127, H-1364 Budapest, Hungary, and Department of Mathematics, University of Illinois at Urbana-Champaign, IL 62801, USA.

E-mail address: furedi@renyi.hu