Electromechanical buckling behavior of smart piezoelectrically actuated higher-order size-dependent graded nanoscale beams in thermal environment

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ABSTRACT
In the present work, thermo-electro-mechanical buckling behavior of functionally graded piezoelectric (FGP) nanobeams is investigated based on higher-order shear deformation beam theory. The FGP nanobeam is subjected to four types of thermal loading including uniform, linear, and sinusoidal temperature rise as well as heat conduction through the beam thickness. Thermo-electro-mechanical properties of FGP nanobeam are supposed to change continuously in the thickness direction based on power-law model. To consider the influences of small-scale sizes, Eringen’s nonlocal elasticity theory is adopted. Applying Hamilton’s principle, the nonlocal governing equations of an FGP nanobeam in thermal environments are obtained and are solved using Navier-type analytical solution. The significance of various parameters, such as thermal loadings, external electric voltage, power-law index, nonlocal parameter, and slenderness ratio on thermal buckling response of size-dependent FGP nanobeams is investigated.

1. Introduction
Increasing demands for high structural performance requirements, especially in severe temperature environments lead to generating a new kind of composite materials known as functionally graded materials (FGMs) which are designed to achieve a functional performance with gradually variable properties in one or more spatial directions. Containing various advantageous properties, FGMs are idoneous for various engineering applications, and has gained intense interest by several researchers [1-7]. Moreover, nanoscale beam structures have attracted the interest of some researchers in the field of nanomechanics. Therefore, it is significant to consider the small size influence in the mechanical analysis of nanostructures. Due to the lack of a material length scale, the classical continuum elasticity theory is not capable of describing the size influence. So, size-dependent continuum theories such as nonlocal elasticity theory proposed by Eringen [8-10] are developed to capture the size effects with supposing the stress at a reference point to be a functional of strain of all point of the body. Hence, the nonlocal...
elasticity theory has been extensively applied to analyze the mechanical responses of nanostructures [11-17].

Investigations on the static and dynamic characteristics of size-dependent FGM structures have been an area of intensive research over the last decade. Dynamic stability analysis of microbeams made of FGMs based on the modified couple stress theory (MCST), and Timoshenko beam theory (TBT) is presented by Ke and Wang [18]. As a consequence, they mentioned that non-classical Timoshenko beam model contains a material length scale parameter and can interpret the size effect. Also, Simsek and Reddy [19], suggested a study based on the MCST with a unified higher-order beam theory, which contains various beam theories as special cases for buckling of a functionally graded (FG) microbeam embedded in elastic Pasternak medium. Eltaher et al. [20] presented a finite element analysis for free vibration of FG nanobeams using nonlocal Euler–Bernoulli beam theory (EBT). Sharabiani and Yazdi [21] investigated nonlinear free vibration of functionally graded nanobeams within the framework of Euler–Bernoulli beam model including the von Kármán geometric nonlinearity. Rahmani and Pedram [22] analyzed the size effects on vibration of FG nanobeams based on nonlocal TBT. Most recently, Ebrahimi et al. [23] examined the applicability of differential transformation method in investigations on vibrational characteristics of FG size-dependent nanobeams. Recently, Rahmani and Jandaghian [24] presented Buckling analysis of functionally graded nanobeams based on a non-local third-order shear deformation theory. Also, nonlinear free vibration of axially FG Euler–Bernoulli microbeams with immovable ends is studied by Simsek [25], using the MCST. Zemri et al. [26] presented mechanical response of functionally graded nanoscale beam based on a refined nonlocal shear deformation beam model. Bounouara et al. [27] presented a nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation. Ebrahimi and Barati [28] presented dynamic modeling of a thermo-piezoelectrically actuated nanosize beam subjected to a magnetic field.

Thermal buckling can possess a destructive influence on the safety of structures, and hence it is regarded as an undesired phenomenon in several studies [29,30]. Referring to the thermo-mechanical analysis of size-dependent beam structures, the prediction of buckling behavior of size-dependent microbeams made of FGMs including thermal environment effects is studied by Sahmani and Ansari [31]. They incorporated strain gradient elasticity theory into the classical third-order shear deformation beam theory to develop a non-classical beam model which contains three additional internal material length scale parameters to consider the effects of size dependencies. Ansari et al. [32] investigated the thermal postbuckling characteristics of microbeams made of FGMs undergoing thermal loads based on the modified strain gradient theory. Nateghi and Salamat-talab [33] explored thermal effects on buckling and free vibration behavior of FG microbeams based on MCST. They adopted classical and first-order shear deformation beam theories to count for the effect of shear deformations. Also, Akgöz and Civalek [34] studied thermo-mechanical size-dependent buckling of embedded FG microbeams based on sinusoidal shear deformation beam and modified couple stress theories. Ebrahimi and Salari [35] investigated the thermal effects on buckling and free vibration characteristics of FG size-dependent Timoshenko nanobeams subjected to an in-plane
thermal loading by presenting a Navier-type solution. Ansari et al. [36] proposed an exact solution for the nonlinear forced vibration of FG nanobeams in thermal environment based on the surface elasticity theory.

Several higher-order theories are suggested for modeling of FGM structures [37]. It should be noted that the classical beam model fails to consider the influences of shear deformations. Hence, the buckling loads are overestimated. TBT can enumerate the influences of shear deformations for thick beams with presumption of a constant shear strain state in the direction of beam thickness. So, as a disadvantage of this theory, a shear correction factor is needed. Higher-order theories satisfy shear deformation effects without the need for shear correction factors. Ebrahimi and Barati [38] presented a nonlocal third-order beam theory for vibration analysis of size-dependent FG beams. Bourada et al. [39] presented a new simple shear and normal deformations theory for FG beams. Al-Basyouni et al. [40] investigated size-dependent bending and vibration analysis of higher-order FG microbeams based on MCST and neutral surface position.

Due to the extensive application of the smart materials such as piezoelectric materials in engineering structures, a remarkable attention is paid to the mechanical analysis of beam structures made of piezoelectric materials. Moreover, the piezoelectric materials are talented to produce rapid responses through electro-mechanical coupling. These materials can sustain deformation when an electric field is applied to the structure and also generate an electric field when they are exposed to a strain field. As one of the first investigations on the mechanical behavior of FGP beam, Shi and Chen [41], studied the problem of a FG piezoelectric cantilever beam exposed to different loadings. Also, Doroushi et al. [42] investigated the free and forced vibration characteristics of an FGPM beam subjected to thermo-electro-mechanical loads using the higher-order shear deformation beam theory. Kiani, Y., et al. [43] analyzed buckling behavior of FG beams with or without surface-bonded piezoelectric layers subjected to both thermal loading and constant voltage. Komijani et al. [44] studied free vibration of FGP beams with rectangular cross sections under in-plane thermal and electrical excitations in pre/postbuckling regimes. Lezgy-Nazargah et al. [45] suggested an efficient three-nodded beam element model for static, free vibration, and dynamic response of FG piezoelectric material beams. Large amplitude free flexural vibration of shear deformable FG beams with surface-bonded piezoelectric layers subjected to thermo-piezoelectric loadings with random material properties was presented by Shegokar and Lal [46]. The major disadvantage of this study is that they fail to consider small size influence. Therefore, no work has been reported yet on thermal buckling of FGP nanobeams based on higher-order shear deformation beam theory.

The present research examines the thermal buckling of nonlocal FGP beams under various types of thermal loadings, namely uniform, linear, and sinusoidal temperature rise and also heat conduction. The thermo-electro-mechanical material properties of the beam is supposed to be graded in the thickness direction according to the power-law distribution. Based on a higher-order beam theory as well as Hamilton’s principle, nonlocal governing equations for the thermal buckling of an FGP nanobeam are derived and are solved using Navier-type method. The small size effect is captured using Eringen’s nonlocal elasticity theory. The most beneficial feature of the
present beam model is to provide a parabolic variation of the transverse shear strains across the thickness direction and satisfies the zero traction boundary conditions on the top and bottom surfaces of the beam without using shear correction factors. Various numerical examples are presented investigating the influences of thermomechanical loadings, external electric voltage, power-law index, nonlocal parameter, and slenderness ratio on thermal buckling behavior of nanobeams made of FG piezoelectric materials.

2. Theoretical formulations

2.1. Nonlocal elasticity theory for the piezoelectric materials

Contrary to the constitutive equation of classical elasticity theory, Eringen’s nonlocal theory \[8,9,10\] notes that the stress state at a point inside a body is regarded to be a function of strains of all points in the neighbor regions. For a nonlocal homogeneous piezoelectric solid, the basic equations with zero body force may be defined as:

\[
\sigma_{ij} = \int_V \alpha(|x' - x|, \tau) \left[ C_{ijkl} e_{kl}(x') - e_{kji} E_k(x') - C_{ijkl} a_{kl} \Delta T \right] dV(x') \tag{1a}
\]

\[
D_i = \int_V \alpha(|x' - x|, \tau) \left[ e_{ikl} e_{kl}(x') + k_{ik} E_k(x') + p_i \Delta T \right] dV(x'), \tag{1b}
\]

where \(\sigma_{ij}, e_{ij}, D_i, E_i\) denote the stress, strain, electric displacement and electric field components, respectively; \(a_{kl}\) and \(\Delta T\) are the thermal expansion coefficient and temperature change, respectively; \(C_{ijkl}, e_{kij}, k_{ik}\) and \(p_i\) are elastic, piezoelectric, dielectric, and pyroelectric constant, respectively; \(\alpha(|x' - x|, \tau)\) is the nonlocal kernel function and \(|x' - x|\) is the Euclidean distance. \(\tau = e_0 a/l\) is defined as scale coefficient, where \(e_0\) is a material constant, which is determined experimentally or approximated by matching the dispersion curves of plane waves with those of atomic lattice dynamics; and \(a\) and \(l\) are the internal and external characteristic length of the nanostructures, respectively. Finally, it is possible to represent the integral constitutive relations given by Equation (4) in an equivalent differential form as:

\[
\sigma_{ij} - (e_0 a)^2 \nabla^2 \sigma_{ij} = C_{ijkl} e_{kl} - e_{kij} E_k - C_{ijkl} a_{kl} \Delta T \tag{2a}
\]

\[
D_i - (e_0 a)^2 \nabla^2 D_i = e_{ikl} e_{kl} + k_{ik} E_k + p_i \Delta T, \tag{2b}
\]

where \(\nabla^2\) is the Laplacian operator and \(e_0 a\) is the nonlocal parameter revealing the size influence on the response of nanostructures.

2.2. The material properties of FGP nanobeams

Assume an FG nanobeam composed of PZT – 4 and PZT – 5H piezoelectric materials exposed to an electric potential \(\Phi(x, z, t)\), with length \(L\) and uniform thickness \(h\), as shown in Figure 1. The effective material properties of the FGPM nanobeam are supposed to change continuously in the \(z\)-axis direction (thickness direction) based on the power-law model. So, the effective material properties, \(P\) including elastic,
piezoelectric, and dielectric constants and also thermal expansion coefficient can be stated in the following form [44]:

\[ P = P_2 V_2 + P_1 V_1 \]  

in which \( P_1 \) and \( P_2 \) denote the material properties of the bottom and higher surfaces, respectively. Also, \( V_1 \) and \( V_2 \) are the corresponding volume fractions related by:

\[ V_2 = \left( \frac{z}{h} + \frac{1}{2} \right)^p, \quad V_1 = 1 - V_2. \]  

Therefore, according to Equations (1) and (2), the effective electro-mechanical material properties of the FGP beam is defined as:

\[ P(z) = (P_2 - P_1) \left( \frac{z}{h} + \frac{1}{2} \right)^p + P_1, \]  

where \( p \) is power-law exponent which is non-negative and estimates the material distribution through the thickness of the nanobeam and \( z \) is the distance from the mid-plane of the graded piezoelectric beam. It must be noted that, the top surface at \( z = +h/2 \) of FGP nanobeam is assumed PZT – 4 rich, whereas the bottom surface (\( z = -h/2 \)) is PZT – 5H rich.

### 2.3. Nonlocal higher-order FG piezoelectric nanobeam model

Based on parabolic third-order beam theory, the displacement field at any point of the beam is supposed to be in the form:

\[ u_x(x, z) = u(x) + z \psi(x) - az^2 (\psi + \frac{\partial w}{\partial x}) \]  

\[ u_z(x, z) = w(x) \]  

in which \( u \) and \( w \) are displacement components in the mid-plane along the coordinates \( x \) and \( z \), respectively, while \( \psi \) denotes the total bending rotation of the cross-section.
To satisfy Maxwell’s equation in the quasi-static approximation, the distribution of electric potential along the thickness direction is supposed to change as a combination of a cosine and linear variation as follows [42]:

\[ \Phi(x,z,t) = -\cos(\xi z)\phi(x,t) + \frac{2z}{h}V \]  \hspace{1cm} (7)

where \( \xi = \pi/h \). Also, \( V \) is the initial external electric voltage applied to the FGP nano-beam; and \( \phi(x,t) \) is the spatial function of the electric potential in the \( x \)-direction. Considering strain–displacement relationships on the basis of parabolic beam theory, the non-zero strains can be stated as:

\[ \epsilon_{xx} = \epsilon_{xx}^{(0)} + z\epsilon_{xx}^{(1)} + z^2\epsilon_{xx}^{(3)} \]  \hspace{1cm} (8)

\[ \gamma_{xz} = \gamma_{xz}^{(0)} + z^2\gamma_{xz}^{(2)} \]  \hspace{1cm} (9)

where

\[ \epsilon_{xx}^{(0)} = \frac{\partial u}{\partial x}, \quad \epsilon_{xx}^{(1)} = \frac{\partial \psi}{\partial x}, \quad \epsilon_{xx}^{(3)} = -\alpha \left( \frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right). \]  \hspace{1cm} (10)

\[ \gamma_{xz}^{(0)} = \frac{\partial w}{\partial x} + \psi, \quad \gamma_{xz}^{(2)} = -\beta \left( \frac{\partial w}{\partial x} + \psi \right) \]  \hspace{1cm} (11)

And \( \beta = \frac{4}{h^2} \).

According to the defined electric potential in Equation (7), the non-zero components of electric field \( (E_x, E_z) \) can be obtained as:

\[ E_x = -\Phi_x = \cos(\xi z)\frac{\partial \phi}{\partial x}, \quad E_z = -\Phi_z = -\xi \sin(\xi z)\phi - \frac{2V}{h}. \]  \hspace{1cm} (12)

The Hamilton’s principle can be stated in the following form to obtain the governing equations of motion:

\[ \int_0^t \delta(\Pi_S + \Pi_W) \, dt = 0, \]  \hspace{1cm} (13)

where \( \Pi_S \) is strain energy and \( \Pi_W \) is work done by external applied forces. The first variation of strain energy \( \Pi_S \) can be calculated as:

\[ \delta \Pi_S = \int_{-h/2}^{h/2} \int_0^L \left( \sigma_{xx} \delta \epsilon_{xx} + \sigma_{xz} \delta \gamma_{xz} - D_x \delta E_x - D_z \delta E_z \right) \, dz \, dx \]  \hspace{1cm} (14)

Substituting Equations (8) and (9) into Equation (14) yields:

\[ \delta \Pi_S = \int_0^L \int_{-h/2}^{h/2} \left( (N\delta \epsilon_{xx}^{(0)} + M\delta \epsilon_{xx}^{(1)} + P\delta \epsilon_{xx}^{(3)}) + Q\delta \gamma_{xz}^{(0)} + R\delta \gamma_{xz}^{(2)} \right) \, dz \, dx \]

\[ + \int_0^L \int_{-h/2}^{h/2} \left( -D_x \cos(\xi z) \Delta \left( \frac{\partial \phi}{\partial x} \right) + D_z \xi \sin(\xi z) \delta \phi \right) \, dz \, dx \]  \hspace{1cm} (15)
in which \( N, M \) and \( Q \) are the axial force, bending moment, and shear force resultants, respectively. Relations between the stress resultants and stress component used in Equation (15) are defined as:

\[
\begin{align*}
N &= \int_A \sigma_{xx} \, dA, \\
M &= \int_A \sigma_{xz} \, dA, \\
P &= \int_A \sigma_{x}z \, dA, \\
Q &= \int_A \sigma_{xz} \, dA, \\
R &= \int_A \sigma_{xz}^2 \, dA 
\end{align*}
\] (16)

The work done due to external electric voltage, \( \Pi_W \), can be written in the form:

\[
\Pi_W = \int_0^l \left( (N_E + N_T) \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + q \delta w + f \delta u - N\delta \varepsilon_{xx}^{(0)} - \hat{M} \frac{\partial \delta \psi}{\partial x} \right) + aP \frac{\partial^2 \delta w}{\partial x^2} \right) dx
\] (17)

in which \( \hat{M} = M - aP, \hat{Q} = Q - \beta R \) and \( q(x) \) and \( f(x) \) are the transverse and axial distributed loads, and also \( N_T \) and \( N_E \) are the normal forces induced by various temperature change \( \Delta T \) and external electric voltage \( (V) \), respectively which are stated as:

\[
N_T = \int_{-h/2}^{h/2} c_{11} a_1 (T - T_0) \, dz, \quad N_E = -\int_{-h/2}^{h/2} e_{31} \frac{2V}{h} \, dz
\] (18)

For an FGPM nanobeam exposed to thermo-electro-mechanical loading in the one-dimensional case, the nonlocal constitutive relations (5a) and (5b) may be rewritten as:

\[
\sigma_{xx} - (e_0 a)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = c_{11} \varepsilon_{xx} - e_31 E_z - c_{11} a_1 \Delta T
\] (19)

\[
\sigma_{xz} - (e_0 a)^2 \frac{\partial^2 \sigma_{xz}}{\partial x^2} = c_{55} \gamma_{xz} - e_{15} E_x
\] (20)

\[
D_x - (e_0 a)^2 \frac{\partial^2 D_x}{\partial x^2} = e_{15} \gamma_{xz} + k_{11} E_x
\] (21)

\[
D_z - (e_0 a)^2 \frac{\partial^2 D_z}{\partial x^2} = e_{31} \varepsilon_{xx} + k_{33} E_z + p_3 \Delta T
\] (22)

Inserting Equations (15) and (17) in Equation (13) and integrating by parts, and gathering the coefficients of \( \delta u, \delta w \delta \psi \) and \( \delta \phi \), the following governing equations are obtained:

\[
\frac{\partial N}{\partial x} + f = 0
\] (23)

\[
\frac{\partial \hat{M}}{\partial x} - \hat{Q} = 0
\] (24)

\[
\frac{\partial \hat{Q}}{\partial x} + q - (N_E + N_T) \frac{\partial^2 w}{\partial x^2} + a \frac{\partial^2 p}{\partial x^2} = 0
\] (25)
where

\[ \mu_{76} \]

the second derivative of obtained as follows:

By integrating Equations (19)–(22) over the beam’s cross-section area, the force-strain and the moment-strain of the nonlocal third-order Reddy FGP beam theory can be obtained as follows:

\[
N - \mu \frac{\partial^2 N}{\partial x^2} = A_{xx} \frac{\partial u}{\partial x} + B_{xx} \frac{\partial \psi}{\partial x} - aE_{xx} \left( \frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + A_{31}^e \phi - N_E - N_T
\]

\[
M - \mu \frac{\partial^2 M}{\partial x^2} = B_{xx} \frac{\partial u}{\partial x} + D_{xx} \frac{\partial \psi}{\partial x} - aF_{xx} \left( \frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + E_{31} \phi
\]

\[
P - \mu \frac{\partial^2 P}{\partial x^2} = E_{xx} \frac{\partial u}{\partial x} + F_{xx} \frac{\partial \psi}{\partial x} - aH_{xx} \left( \frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + F_{31} \phi
\]

\[
Q - \mu \frac{\partial^2 Q}{\partial x^2} = (A_{xz} - \beta D_{xz}) \left( \frac{\partial w}{\partial x} + \psi \right) - E_{15} \frac{\partial \phi}{\partial x}
\]

\[
R - \mu \frac{\partial^2 R}{\partial x^2} = (D_{xz} - \beta F_{xz}) \left( \frac{\partial w}{\partial x} + \psi \right) - F_{15} \frac{\partial \phi}{\partial x}
\]

\[
\int_{-h/2}^{h/2} \left\{ D_{xx} - \mu \frac{\partial^2 D_{xx}}{\partial x^2} \right\} \cos(\xi z) dz = \left( E_{15} - \beta F_{15} \right) \left( \frac{\partial w}{\partial x} + \psi \right) + F_{11} \frac{\partial \phi}{\partial x}
\]

\[ (27) \]

\[
\int_{-h/2}^{h/2} \left\{ D_{xz} - \mu \frac{\partial^2 D_{xz}}{\partial x^2} \right\} \xi \sin(\xi z) dz = A_{31}^e \frac{\partial u}{\partial x} + (E_{31} - aF_{31}) \frac{\partial \psi}{\partial x} - aF_{31} \frac{\partial^2 w}{\partial x^2} - F_{33} \phi,
\]

where \( \mu = (e_0 a)^2 \) and quantities used in above equations are defined as:

\[
\{ A_{xx}, B_{xx}, D_{xx}, E_{xx}, F_{xx}, H_{xx} \} = \int_{-h/2}^{h/2} c_{11} \{1, z, z^2, z^3, z^4, z^6\} \, dz
\]

\[
\{ A_{xz}, D_{xz}, F_{xz} \} = \int_{-h/2}^{h/2} c_{55} \{1, z^2, z^4\} \, dz
\]

\[
\{ A_{31}^e, E_{31}, F_{31} \} = \int_{-h/2}^{h/2} e_{31} \{\xi \sin(\xi z), z \xi \sin(\xi z), z^3 \xi \sin(\xi z)\} \, dz
\]

\[
\{ E_{15}, F_{15} \} = \int_{-h/2}^{h/2} e_{15} \{\cos(\xi z), z^2 \cos(\xi z)\} \, dz
\]

\[ (31) \]

\[
\{ F_{11}, F_{33} \} = \int_{-h/2}^{h/2} k_{11} \cos^2(\xi z), k_{33} \xi^2 \sin^2(\xi z)\} \, dz.
\]

\[ (33) \]

The explicit relation of the nonlocal normal force can be derived by substituting for the second derivative of \( N \) from Equation (23) into Equation (27) as follows:
\[ N_x = A_{xx} \frac{\partial u}{\partial x} + K_{xx} \frac{\partial \psi}{\partial x} - a E_{xx} \frac{\partial^2 w}{\partial x^2} + A_{31}^e \phi + \mu \left( - \frac{\partial f}{\partial x} \right) - N_E - N_T \] (39)

Omitting \( \hat{Q} \) from Equations (24) and (25), we obtain the following equation:

\[ \frac{\partial^2 \hat{M}}{\partial x^2} = -a \frac{\partial^2 P}{\partial x^2} - q + (N_E + N_T) \frac{\partial^2 w}{\partial x^2} \] (40)

Also, the explicit relation of the nonlocal bending moment can be derived by substituting for the second derivative of \( \hat{M} \) from Equation (24) into Equation (28) and using Equations (28) and (29) as follows:

\[
\hat{M} = K_{xx} \frac{\partial u}{\partial x} + l_{xx} \frac{\partial \psi}{\partial x} - a J_{xx} \left( \frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + \left( E_{31} - a F_{31} \right) \phi + \mu \left( -a \frac{\partial^2 P}{\partial x^2} - q \right) + \frac{\partial}{\partial x} \left( (N_E + N_T) \frac{\partial w}{\partial x} \right),
\] (41)

where

\[
K_{xx} = B_{xx} - a E_{xx}, \quad l_{xx} = D_{xx} - a F_{xx}, \quad J_{xx} = F_{xx} - a H_{xx}.
\] (42)

By substituting for the second derivative of \( \hat{Q} \) from Equation (25) into Equation (30), and using Equations (30) and (31) the following expression for the nonlocal shear force is derived:

\[
\hat{Q} = \overline{A}_{xz} \left( \frac{\partial w}{\partial x} + \psi \right) - (E_{15} - \beta F_{15}) \frac{\partial \phi}{\partial x} + \mu \left( (N_E + N_T) \frac{\partial^2 w}{\partial x^2} - a \frac{\partial^2 P}{\partial x^2} - \frac{\partial q}{\partial x} \right)
\] (43)

Where

\[
\overline{A}_{xz} = A_{xz}^* - \beta l_{xz}^*, \quad A_{xz} = A_{xz} - \beta D_{xz}, \quad l_{xz} = D_{xz} - \beta F_{xz}
\] (44)

Now, we use \( \hat{M} \) and \( \hat{Q} \) from Equations (41) and (43) and the identity

\[
a \frac{\partial^2 P}{\partial x^2} = a \left( E_{xx} \frac{\partial^2 \psi}{\partial x^2} + F_{xx} \frac{\partial^2 \psi}{\partial x^2} - a H_{xx} \left( \frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^4 \phi}{\partial x^4} \right) + F_{31} \frac{\partial^2 \phi}{\partial x^2} \right).
\] (45)

It must be cited that inserting Equation (26) into Equations (32) and (33), does not provide an explicit expressions for \( D_x \) and \( D_z \). To overcome this problem, by using Equations (32) and (33), Equation (26) can be re-expressed in terms of \( u \), \( w \), \( \psi \) and \( \phi \). For a higher-order FGP nanobeam by substituting for \( N \hat{M} \) and \( \hat{Q} \) from Equations (39), (41), and (43) into Equations (23)–(25), the nonlocal governing equations can be obtained as below:

\[
A_{xx} \frac{\partial^2 u}{\partial x^2} + K_{xx} \frac{\partial^2 \psi}{\partial x^2} - a E_{xx} \frac{\partial^3 w}{\partial x^3} + A_{31}^e \frac{\partial \phi}{\partial x} + \mu \left( - \frac{\partial^2 f}{\partial x^2} \right) + f = 0
\] (46)

\[
K_{xx} \frac{\partial^2 u}{\partial x^2} + l_{xx} \frac{\partial^2 \psi}{\partial x^2} - a J_{xx} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^3 w}{\partial x^3} \right) - \overline{A}_{xz} \left( \phi + \frac{\partial w}{\partial x} \right) + \left( E_{31} - a F_{31} \right) \phi
\] (47)

\[\begin{align*}
+ (E_{15} - \beta F_{15}) \frac{\partial \phi}{\partial x} = 0
\end{align*}\]
\[
\mathbf{A}_{xz} \left( \frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + \mu \left( (N_E + N_T) \frac{\partial^4 w}{\partial x^4} - \frac{\partial^2 q}{\partial x^2} \right) + q - (N_E + N_T) \frac{\partial^2 w}{\partial x^2} - (E_{15} - \beta F_{15}) \frac{\partial \phi}{\partial x} \\
+ a \left( E_{xx} \frac{\partial^3 u}{\partial x^3} + J_{xx} \frac{\partial^3 \psi}{\partial x^3} - a H_{xx} \frac{\partial^4 w}{\partial x^4} + F_{31} \frac{\partial^2 \phi}{\partial x^2} \right) = 0
\]

(48)

\[
(E_{15} - \beta F_{15}) \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial \psi}{\partial x} \right) + F_{11} \frac{\partial^2 \phi}{\partial x^2} + A_{31} \frac{\partial u}{\partial x} + (E_{31} - a F_{31}) \frac{\partial \psi}{\partial x} - a F_{31} \frac{\partial^2 w}{\partial x^2} - F_{33} \phi = 0.
\]

(49)

3. Solution procedure

Here, on the basis the Navier method, an analytical solution of the governing equations for buckling of a simply supported FGP nanobeam is presented. To satisfy governing equations of motion and the simply supported boundary condition, the displacement variables are adopted to be of the form:

\[
u(x, t) = \sum_{n=1}^{\infty} U_n \cos \left( \frac{n\pi}{L} x \right) e^{i\omega_n t}
\]

(50)

\[
w(x, t) = \sum_{n=1}^{\infty} W_n \sin \left( \frac{n\pi}{L} x \right) e^{i\omega_n t}
\]

(51)

\[
\psi(x, t) = \sum_{n=1}^{\infty} \Psi_n \cos \left( \frac{n\pi}{L} x \right) e^{i\omega_n t}
\]

(52)

\[
\phi(x, t) = \sum_{n=1}^{\infty} \Phi_n \sin \left( \frac{n\pi}{L} x \right) e^{i\omega_n t},
\]

(53)

where, \( U_n, W_n, \Psi_n \) and \( \Phi_n \) are the unknown Fourier coefficients to be determined for each \( n \) value. The boundary conditions for simply supported FGP beam can be identified as:

\[
u(0) = 0, \quad \frac{\partial u}{\partial x}(L) = 0, \quad w(0) = w(L) = 0
\]

(54)

\[
\frac{\partial \psi}{\partial x}(0) = \frac{\partial \psi}{\partial x}(L) = 0, \quad \phi(0) = \phi(L) = 0
\]

Using Equations (50)–(53), the analytical solution can be obtained from the following equations to find critical buckling temperature:

\[
\begin{pmatrix}
  k_{1,1} & k_{1,2} & k_{1,3} & k_{1,4} \\
  k_{2,1} & k_{2,2} & k_{2,3} & k_{2,4} \\
  k_{3,1} & k_{3,2} & k_{3,3} & k_{3,4} \\
  k_{4,1} & k_{4,2} & k_{4,3} & k_{4,4}
\end{pmatrix}
\begin{pmatrix}
  U_n \\
  \Psi_n \\
  W_n \\
  \Phi_n
\end{pmatrix} = 0,
\]

(55)

where
\[ k_{1,1} = -A_{xx}\left(\frac{mL}{L}\right)^2, \quad k_{1,2} = -K_{xx}\left(\frac{mL}{L}\right)^2, \quad k_{1,3} = aE_{xx}\left(\frac{mL}{L}\right)^3, \quad k_{1,4} = -\frac{A_{e}}{C_0}\left(\frac{mL}{L}\right) \]

\[ k_{2,2} = -l_{xx}\left(\frac{mL}{L}\right)^2 + aJ_{xx}\left(\frac{mL}{L}\right)^2 - \bar{A}_{xz}, \quad k_{2,3} = -\bar{A}_{xz}\left(\frac{mL}{L}\right) + J_{xx}\left(\frac{mL}{L}\right)^3, \]

\[ k_{2,4} = -((E_{15} - \beta F_{15}) + (E_{31} - aF_{31}))\left(\frac{mL}{L}\right) \]

\[ k_{3,3} = (N_T + N_E)\left(\frac{mL}{L}\right)^2 \left(1 + \mu\left(\frac{mL}{L}\right)^2\right) - \bar{A}_{xz}\left(\frac{mL}{L}\right)^2 - \alpha^2 H_{xx}\left(\frac{mL}{L}\right)^4, \]

\[ k_{3,4} = -((E_{15} - \beta F_{15}) - aF_{31})\left(\frac{mL}{L}\right)^2, \quad k_{4,4} = -\left(F_{11}\left(\frac{mL}{L}\right)^2 + F_{33}\right) \]

4. Various thermal loadings

4.1. Uniform temperature rise (UTR)

For an FG nanobeam at reference temperature \(T_0\), the temperature is uniformly raised to a final value \(T\) which the temperature change is \(\Delta T = T - T_0\).

4.2. Linear temperature rise (LTR)

For an FG nanobeam, the temperature distribution is assumed to be varied linearly through the thickness as follows:

\[ T = T_1 + \Delta T \left(\frac{1}{2} + \frac{z}{h}\right), \quad (56) \]

where the buckling temperature difference in Equation (58) is \(\Delta T = T_2 - T_1\) and \(T_2\) and \(T_1\) are the temperature of the top surface and the bottom surface, respectively.

4.3. Nonlinear temperature rises

4.3.1. Heat conduction (HC)

The one-dimensional temperature distribution through the thickness can be obtained by solving the steady-state heat conduction equation with the boundary conditions on bottom and top surfaces of the beam across the thickness:

\[ -\frac{d}{dz}\left(\kappa(z, T) \frac{dT}{dz}\right) = 0 \]

\[ T \left(\frac{h}{2}\right) = T_2, \quad T \left(-\frac{h}{2}\right) = T_1 \quad (57) \]

The solution of the above equation is:

\[ T = T_1 + (T_2 - T_1) \left[ \frac{\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{dz}{\kappa(z, T)}}{\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{dz}{\kappa(z, T)}} \right], \quad (58) \]
where
\[ \Delta T = T_2 - T_1. \]

### 4.3.2. Sinusoidal temperature rise (STR)

The temperature field when FGP nanobeam is exposed to sinusoidal temperature rise across the thickness can be defined as [47]:

\[
T = T_1 + \Delta T \left( 1 - \cos \frac{\pi}{2} \left( \frac{z}{h} + \frac{1}{2} \right) \right),
\]

where \( \Delta T = T_2 - T_1 \) is temperature change.

### 5. Results and discussion

In this section, some numerical examples are presented to show the influence of electric field, material composition, nonlocal parameter, and slenderness ratio on the electromechanical buckling behavior of higher-order shear deformable FGP nanobeams. So, the nonlocal FGP beam composed of PZT \(-4\) and PZT \(-5H\), with electro-mechanical material properties listed in Table 1, is supposed. The beam geometry has the following dimensions: \( L \) (length) = 10 nm and \( h \) (thickness) = varied. Due to the fact that there is no available numerical result for the buckling responses of FG piezoelectric nanobeams based on the nonlocal elasticity theory, the only available work is thermal buckling of FG nanobeams under linear temperature rise presented by Ebrahimi and Salari [35]. Therefore, Table 2 compares critical buckling temperatures of the present theory with

| Properties       | PZT \(-4\) | PZT \(-5H\) |
|------------------|----------|----------|
| \( c_{11} \) (GPa) | 81.3     | 60.6     |
| \( c_{55} \) (GPa) | 25.6     | 23.0     |
| \( e_{31} \) (Cm\(^{-2}\)) | -10.0    | -16.604  |
| \( e_{15} \) (Cm\(^{-2}\)) | 40.3248  | 44.9046  |
| \( k_{11} \) (Cm\(^{-2}\)N\(^{-1}\)) | 0.6712e-8 | 1.5027e-8 |
| \( k_{33} \) (Cm\(^{-2}\)N\(^{-1}\)) | 1.0275e-8 | 2.554e-8 |
| \( a_1 \) (K\(^{-1}\))   | 2.0e-6   | 10.0e-6  |
| \( k \) (Wm\(^{-1}\)K\(^{-1}\)) | 2.1      | 1.5      |
| \( p_3 \) (Cm\(^{-2}\)K\(^{-1}\)) | 2.5e-5   | 0.548e-5 |

Table 2. Comparison of the critical buckling temperature \( \Delta T_{cr} \) of an \( S-S\) FG nanobeam subjected to linear temperature rise with various volume fraction index \( (L/h = 40) \).
those of nonlocal FGM Timoshenko beams. Also, it is supposed that the temperature rise in lower surface to reference temperature $T_0$ of the FGP nanobeam is $T_1 - T_0 = 5K$.

The influences of different parameters, such as various temperature rise (UTR, LTR, HC, and STR), external electric voltage ($V$), nonlocal parameter ($\mu$), gradient index, and slenderness ratio ($L/h$) on buckling responses of FGP nanobeams are tabulated in Tables 3–6. It is obvious that for all external voltages and power-law indexes, increasing the nonlocal parameter leads to reduction in buckling temperatures of FGP nanobeams, which indicates softening influence of nonlocality on the beam structure. Another remarkable observation from these tables is that for every temperature distribution, negative values of applied electric voltage generate larger values of the buckling temperatures ($\Delta T_{cr}$) compared to positive voltages. Also, similar to nonlocal parameter, power-law exponent shows a decreasing effect on the critical buckling temperatures. Moreover, comparing the obtained results of various temperature fields reveals that

### Table 3. The variation of the critical buckling temperature $\Delta T_{cr}$ [$K$] of an S–S FGP nanobeam under uniform temperature rise UTR for various nonlocal parameter and power-law indexes.

| $\mu$ | $p = 0.2$ | $p = 1$ | $p = 2$ | $L/h = 20$ | $L/h = 25$ | $L/h = 30$ |
|-------|-----------|-----------|-----------|-----------|-----------|-----------|
| $V = 0$ | 779.075 | 464.644 | 394.570 | 551.965 | 337.059 | 289.350 |
| $V = +0.5$ | 605.772 | 335.467 | 274.618 | 335.336 | 175.588 | 139.412 |
| $V = +1$ | 916.252 | 299.530 | 244.561 | 295.483 | 152.563 | 120.152 |
| $V = +2$ | 664.928 | 398.694 | 339.412 | 478.828 | 244.561 | 201.248 |

### Table 4. The variation of the critical buckling temperature $\Delta T_{cr}$ [$K$] of an S–S FGP nanobeam under linear temperature rise LTR for various nonlocal parameter and power-law indexes.

| $\mu$ | $p = 0.2$ | $p = 1$ | $p = 2$ | $L/h = 20$ | $L/h = 25$ | $L/h = 30$ |
|-------|-----------|-----------|-----------|-----------|-----------|-----------|
| $V = 0$ | 1841.03 | 1120.28 | 908.487 | 1300.88 | 809.321 | 663.111 |
| $V = +0.5$ | 1634.94 | 962.859 | 768.621 | 1043.27 | 612.546 | 488.279 |
| $V = +1$ | 1428.85 | 805.439 | 628.755 | 785.658 | 415.771 | 313.447 |
| $V = +2$ | 1693.09 | 1032.69 | 838.394 | 1206.09 | 753.201 | 618.201 |
| $V = +3$ | 1487.00 | 875.270 | 698.528 | 948.483 | 556.426 | 443.369 |
| $V = +4$ | 1280.92 | 717.850 | 558.663 | 690.873 | 359.651 | 268.537 |

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The in...
sinusoidal temperature rise (STR) provides larger values of $\Delta T_{cr}$ than UTR, LTR, and HC due to its non-linear essence, while UTR presents the lower values for critical temperature.

Figure 2 displays the buckling temperature difference of FGP nanobeam versus power-law index ($p$) for various thermal loadings and electric voltages at $\mu=0$ and $L/h = 25$. According to these figures, regardless of thermal loading type the critical temperature of FGP nanobeam decreases with the increase of power-law index, especially for lower power-law indexes. This is related to the reduction in stiffness of the beam with the rise of power-law index. It is apparent that the positive and negative applied electric voltage, respectively, reduces and increases the critical buckling temperature due to generated compressive and tensile in-plane forces in the nanobeams by imposing positive and negative voltages, respectively.

### Table 5. The variation of the critical buckling temperature $\Delta T_{cr}$ [$K$] of an S–S FGP nanobeam under heat conduction HC for various nonlocal parameter and power-law indexes.

| $\mu$ | $V = 0.5$ | $p = 0.2$ | $p = 1$ | $p = 2$ | $V = 0$ | $p = 0.2$ | $p = 1$ | $p = 2$ | $V = 0.5$ | $p = 0.2$ | $p = 1$ | $p = 2$ |
|-------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0     | 1790.47   | 1007.05   | 2024.15   | 1269.00   | 1950.04   | 1186.47   | 1424.98   | 1326.01   | 1125.58   | 936.288   | 540.876   | 381.076   |
| 1     | 64.6600   | 965.183   | 785.443   | 922.435   | 1245.74   | 670.924   | 523.379   | 845.449   | 599.570   | 422.050   | 260.570   | 192.857   |
| 2     | 1524.66   | 896.816   | 730.602   | 1095.99   | 1326.01   | 749.686   | 599.570   | 845.449   | 599.570   | 422.050   | 260.570   | 192.857   |

### Table 6. The variation of the critical buckling temperature $\Delta T_{cr}$ [$K$] of an S–S FGP nanobeam under sinusoidal temperature rise STR for various nonlocal parameter and power-law indexes.

| $\mu$ | $V = 0.5$ | $p = 0.2$ | $p = 1$ | $p = 2$ | $V = 0$ | $p = 0.2$ | $p = 1$ | $p = 2$ | $V = 0.5$ | $p = 0.2$ | $p = 1$ | $p = 2$ |
|-------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0     | 2712.32   | 1715.79   | 1372.84   | 1916.54   | 2408.70   | 1474.69   | 1161.49   | 1537.01   | 938.161   | 591.148   | 377.148   |
| 1     | 2494.38   | 1581.64   | 1266.92   | 1776.90   | 2105.08   | 1233.59   | 950.131   | 1157.48   | 636.785   | 473.658   | 303.658   |
| 2     | 2190.75   | 1340.54   | 1055.57   | 1397.37   | 2484.38   | 1581.64   | 1266.92   | 1776.90   | 636.785   | 473.658   | 303.658   | 190.301   |

**sinusoidal temperature rise (STR)** provides larger values of $\Delta T_{cr}$ than UTR, LTR, and HC due to its non-linear essence, while UTR presents the lower values for critical temperature.
To show the influence of small-scale parameter on the thermal buckling responses, Figure 3 demonstrates the variations of critical temperature difference of FGP nanobeam versus gradient index for various thermal loadings at a fixed slenderness ratio $L/h = 25$. It is clearly observable that for all kinds of thermal loads, the critical buckling temperature decreases with a more severe rate where the gradient index is in range from 0 to 2 than that where gradient index is in range between 2 and 10. Also, it can be seen that the nonlocal parameter diminishes the rigidity of nanostructures and reduces the critical buckling temperatures.

Moreover, Figure 4 represents the influence of slenderness ratio ($L/h$) on the critical buckling temperature of FGP nanobeam for every type of thermal loading for $\mu = 2$ (nm)$^2$ and $V = 0.5$. It is found that slenderness ratio has a considerable influence on the thermal buckling response of FGP nanobeams. Therefore, an increase in slenderness ratio (thinner nanobeam) results in reduction in critical buckling temperature. Also, it is seen that lower values of slenderness ratio have a greater effect on the buckling
temperature due to the reason that the reduction of critical temperature is more sensible.

Figure 5 depicts the variation of critical buckling temperature with respect to external electric voltage for different nonlocal parameters and thermal loads at $p = 0.5$ and $L/h = 25$. According to this figure, when the electric voltage rises from $V = -1$ to $V = +1$, the critical buckling temperature reduces linearly for all values of nonlocal parameter with a same manner. In fact, by increasing the external electric voltage, the differences between the results of various nonlocal parameters remain invariant. Therefore, it can be deduced that nonlocal scale parameter effect is not dependent on the values of applied electric voltage. Also, it must be cited that the value of critical buckling temperature for sinusoidal temperature rise (STR) reduces with a more slope than UTR, LTR, and HC.

The variation of the critical buckling temperature with respect to external electric voltage for different gradient indices at slenderness ratio $L/h = 25$ and $\mu = 2 (\text{nm})^2$ is plotted in Figure 6. It is observable that as the voltage changes from $V = -1$ to $V = +1$, the critical buckling temperature of FGP nanobeam decreases linearly for all
gradient indices, and also the buckling temperature results become closer together. The reason is that the FGP nanobeam is more influenced by the lower values of gradient index and hence the critical buckling temperature decreases more significantly. In addition, it can be seen that negative external voltages provide higher values for critical buckling temperature while positive voltages provide smaller values for buckling temperatures.

6. Conclusions

This paper investigates the thermo-electro-mechanical buckling behavior of nonlocal FGP beams by using a higher-order shear deformation beam theory and Eringen’s nonlocal elasticity theory. The governing differential equations are derived by implementing Hamilton’s principle, and also the Navier solution method is adopted to solve these stability equations. Thermo-electro-mechanical properties of FGP nanobeam are supposed to be variable through the thickness based on power-law model. Results are
presented in both tabular and graphical forms to examine the influences of various thermal loads, external electric voltage, small-scale parameter, power-law index, and slenderness ratio on the critical buckling temperature of higher-order FGP nanobeams. It is indicated that for all kinds of thermal loadings, increasing the nonlocal parameter leads to reduction in critical buckling temperatures due to its softening effect on the beam structure. Moreover, it is observed that the critical buckling temperature of FGP nanobeam reduces with the increase of the power-law index, and also it is found that this reduction is more considerable for positive electric voltages. So, as a consequence it must be cited that the critical buckling temperature of FGP nanobeams rely on the sign of electric voltage.

Disclosure statement

No potential conflict of interest was reported by the authors.
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