Dynamics of a Bose-Einstein condensate in optical trap

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Abstract

The dynamics of a 2D Bose-Einstein condensate in optical trap is studied taking into consideration fluctuations of the far-off-resonance laser field intensity. The problem is described in the frame of the mean field Gross-Pitaevskii equation with randomly varying trap potential. An analytic approach based on the moments method has been employed to describe the noise induced evolution of the condensate properties. Stochastic parametric resonance in oscillations of the condensate width is proved to exist. For the condensate with negative scattering length of atoms, it is shown that the noise can delay or even arrest the collapse. Analytical predictions are confirmed by numerical simulations of the underlying PDE and ODE models.

1 Introduction

Recently a Bose-Einstein condensate has been realized in all-optical far-off-resonance laser trap [1]. This important achievement opens up new perspectives in exploring the phenomenon. On the one hand, with the optical trap it is now possible to create more dense condensates, which is valuable for investigation of three body decay processes, to obtain different geometrical configurations including quasi-1D and 2D structures. On the other hand, in an optical trap atoms in arbitrary hyperfine states may be confined, therefore magnetic properties of atoms can be studied.

However, alongside with the above significant advantages, optical traps have also some drawbacks, the most pertinent of which is related to fluctuations of the laser field intensity. The last circumstance introduces stochasticity into dynamical behavior of the condensate, which has to be taken into account in real situations.
In the mean field theory of BEC based on the Gross-Pitaevskii (GP) equation, fluctuations of the laser field intensity can be regarded as modulations of the harmonic trap potential. The dynamics of BEC under regular variations of trap parameters has been considered by many authors (see e.g. [2] and references therein). From the experimental viewpoint the trap parameters are modulated in order to reveal the response spectrum of the condensate. Kagan et al., 3 have developed the scaling transformation of the order parameter, which in 2D leads to easily solvable ODE’s giving results very close to exact numerical solution of the GP equation. The existence of parametric resonances in the condensate response to time dependent perturbations, on the ground of variational methods and numerical simulations, was reported in Ref. [4].

In the present paper we study the dynamics of a BEC under randomly varying optical trap potential, caused by fluctuations of the laser field intensity. The basic idea consists in employing of differential relations between several integral quantities of the GP equation. This approach, known as moments method, will be applied to analysis of the evolution of the atomic cloud’s width driven by fluctuations of the laser field intensity. The issues of particular interest include the possibility of the stochastic parametric resonance in oscillations of the condensate width, and whether the noise induced expansion of the atomic cloud can delay or arrest the collapse of the BEC with attractive interaction between atoms.

2 Basic equations and the Method of Analysis

Starting point to further development will be the fact, that the dynamics of a nonlinear evolution equations (i.e. systems with infinite number of degrees of freedom) can be exactly reduced to the dynamics of a system with finite degrees of freedom 5. Namely, by introducing collective coordinates. This fact has been explored in Ref. 6 in order to illustrate the existence of extended parametric resonances in a two-dimensional GP equation. An advantage of the model proposed in 6 is that up to certain extent it allows exact solutions.

We will concentrate on radially symmetric solutions \( \psi \equiv \psi(r,t) \) to 2D GP equation, which describes the BEC having a cigar shape

\[
i \frac{\partial \psi}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \psi}{\partial r} + (1 + \epsilon(t))r^2 \psi + \chi |\psi|^2 \psi + i\gamma \psi.
\] (1)

The dimensionless variables of this equation are related to actual quantities through the following transformations: \( r' \rightarrow r/a_{ho} \), \( t' \rightarrow t \omega/2 \), \( \psi' \rightarrow (8\pi N|a|/a_{ho})^{1/2} \psi \), where \( a_{ho} = \sqrt{\hbar/m \omega} \) is the harmonic oscillator unit, \( \omega \) is the trap frequency, \( a \) is the s-wave scattering length of atoms which may be either positive or negative, \( \epsilon(t) \) is the random function on time simulating fluctuations of the laser trap frequency, \( \chi = \pm 1 \) carries the sign of interaction type: "+" corresponds to a repulsive interaction \( (a > 0) \) and "−" to attractive interaction \( (a < 0) \), \( \gamma \) is the coefficient characterizing the damping effects due to influence of non-condensed atoms (thermal cloud).
Modulation function $\epsilon(t)$ can be represented through the deviations of the laser field intensity $E(t)$ around its mean value $E_0$

$$\epsilon(t) = \frac{E(t) - E_0}{E_0}.$$  

(2)

In order to simplify the model, in what follows we concentrate on the case of delta correlated random process with zero mean value:

$$\langle \epsilon(t) \rangle = 0, \quad \langle \epsilon(t)\epsilon(t') \rangle = 2\sigma^2\delta(t-t'),$$  

(3)

where $\sigma > 0$ is the dispersion of fluctuations, $\delta(t)$ is the Dirac delta function, and angular brackets stand for statistical average over all realizations of the random process $\epsilon(t)$. Then the following quantities are adequate to describe the evolution of the Bose condensed atomic cloud

$$I_0 = 2\pi \int_0^\infty |\psi|^2 r dr,$$  

(4)

$$I_1 = 2\pi i \int_0^\infty (\psi \bar{\psi}_r - \bar{\psi} \psi_r) r^2 dr,$$  

(5)

$$I_2 = 2\pi \int_0^\infty \left(|\psi_r|^2 + \frac{\chi}{2}|\psi|^4\right) r dr,$$  

(6)

$$I_3 = 2\pi \int_0^\infty |\psi|^2 r^3 dr,$$  

(7)

$$I_4 = 2\pi\chi \int_0^\infty |\psi|^4 r dr.$$  

(8)

These quantities correspond to the total number of atoms in the condensate, radial field momentum, energy of the wave packet, mean square width of the atomic cloud and mean field interaction energy respectively. They are governed by the system of equations (hereafter a dot stands for the derivative with respect to time)

$$\dot{I}_1 = -2\gamma I_1 + 4I_2 - 4(1 + \epsilon(t))I_3,$$  

$$\dot{I}_2 = -2\gamma I_2 - 2(1 + \epsilon(t))I_1 - \gamma I_4,$$  

$$\dot{I}_3 = -2\gamma I_3 + 2I_1,$$  

(9)

and the equation for $I_0$ is singled out

$$\dot{I}_0 = -2\gamma I_0.$$  

The above system is not closed in the presence of the dissipation term, which will be considered below. We start with the nondissipative case, $\gamma = 0$ and derive the closed system for averaged quantities $J_k = \langle I_k \rangle$ using for this purpose the Furutsu-Novikov formula \footnote{Furutsu-Novikov formula} which gives

$$\langle \epsilon(t)I_k(t) \rangle = -4\sigma^2\delta_{k,3}\langle I_k(t) \rangle$$
$\delta_{k,l}$ is the Kronecker delta). Then the closed linear system of equations for the averaged values reads

$$\dot{J}_1 = 4J_2 - 4J_3, \quad \dot{J}_2 = -2J_1 + 8\sigma^2 J_3, \quad \dot{J}_3 = 2J_1.$$ (10)

This system is trivially solved. For the sake of definiteness we concentrate on the quantity $J_3$ which in the case at hand can be interpreted as a mean square width of the wave packet. Also, we consider the case of small noise intensity $\sigma^2$, consistent with the real experimental situations, then the solution has the form

$$J_3(t) = Ce^{2\xi t} + C_1e^{-\xi t}\cos(\eta t) + C_2e^{-\xi t}\sin(\eta t),$$ (11)

where $\xi$ and $\eta$ are introduced as a real and imaginary part of the roots of the equation for eigenvalues

$$\lambda^3 + 16\lambda - 64\sigma^2 = 0$$ (12)

($\lambda_1 = -2\xi$, $\lambda_2 = \xi + i\eta$, and $\lambda_3 = \xi - i\eta$) associated with the linear system (10). In particular for small dispersion $\sigma^2 \ll 1$ we have $\xi = 2\sigma^2 + O(\sigma^4)$, $\eta = 4 + O(\sigma^2)$.

Constants of integration $C_i$ are determined from initial conditions.

$$J_3(0) = x_0^2, \quad \dot{J}_3(0) = 0, \quad \ddot{J}_3(0) = 8J_2(0) - 8J_3(0).$$ (13)

If a Gaussian ansatz is assumed for the initial shape of the BEC

$$\psi(r,t) = A(t)e^{-\frac{r^2}{2x(t)}},$$ (14)

where, the amplitude $A$ and width $x$ are connected through normalization condition for the $\psi$ function, we have the following values for constants

$$C \approx x_0^2 - \frac{2}{\eta^2}(x_0^2 - \frac{1}{x_0^2}), \quad C_1 \approx \frac{2}{\eta^2}(x_0^2 - \frac{1}{x_0^2}), \quad C_2 \approx -2x_0^2 \frac{\xi}{\eta}.$$

Then it follows from (11), that the noise enhances the effects in the BEC. Noise induced expansion and contraction of the condensate are described by the same eq.(11), ”+” or ”−” being assigned to $\xi$, correspondingly. These conform with positive or negative value of the energy $I_2$, therefore the sign in front of the last (noise) term in eq.(12) will be different in these two cases. The noise results also in doubling of the frequency of oscillations of the condensate.

### 3 Numerical modelling

In order to verify predictions of the analytical approach we performed direct numerical solution of the GP equation (1), as well as stochastic ODE’s (9), resulting from the moments method. Due to increasing role of effective procedures for integration of nonlinear and stochastic PDE’s, we briefly outline the numerical approach which has been used.
To proceed with evolution problems at first we find the ground state solution to GP equation (1) with stationary trap potential \( \epsilon(t) = 0 \) in the absence of the damping term \( \gamma = 0 \). The corresponding ODE has the form

\[
\frac{d^2 \psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - r^2 \psi - |\psi|^2 \psi - \beta = 0,
\]

where \( \beta = \frac{2\mu}{\hbar\omega} \) is the normalized chemical potential of the condensate. In order to solve this equation the following boundary conditions are applied:

\( \psi(0) = \text{const}, \quad \psi'(0) = 0 \),

which state the regularity and continuity of the \( \psi \)-function at \( r = 0 \). Taking into account the fact, that asymptotic behavior of the \( \psi \)-function follows that of the 2D harmonic oscillator solution, one can find the value for \( \psi(0) \) as described in Ref. [8]. To this end we applied the nonlinear equation solving procedure ZEROIN from Ref. [9], which combines advantages of the bisection and secant techniques. After the both \( \psi(0) \) and \( \psi'(0) \) are known, it is straightforward to solve the equation using, for example Runge-Kutta procedure. It is worth to mention, that this approach reproduces results for the ground state wave functions presented in Refs. [8, 10].

The obtained ground state solution has been used as an initial condition for modelling of the time dependent GP equation (3) with randomly varying trap potential. In order to solve this PDE we employed the method of lines, in which the equation is discretized along the spatial variable. From 400 to 800 grid points are taken to discretize the spatial variable within the interval \( r \in [0..5] \). Stability of the ground state solution is checked by propagating eq. (4) with \( \epsilon(t) = 0, \ \gamma = 0 \) up to \( t = 50 \) in dimensionless time units. No detectable variation of the ground state solution is observed. The resulting set of ODE’s has been solved by the procedure DOPRIS [11], which is based on the Runge-Kutta method of 6(7)th order. This procedure with adaptive step-size control is proved to be efficient when providing the given accuracy in a less processor time is a crucial requirement, which is the case in solving of stochastic equations.

The system of equations (9) when the damping term is neglected, can be reduced to the singular Hill equation by introducing the variable \( X = \sqrt{I_3} \):

\[
\ddot{X} + 4(1 + \epsilon(t))X + \frac{Q}{X^3} = 0,
\]

where \( Q = 2I_2I_3 - I_1^2/2 \) is a constant, determined by integral quantities or the initial data. The solution of this equation can be represented in analytical form only for particular types of the modulation function \( \epsilon(t) \). For a stationary trap potential \( \epsilon(t) = 0 \) the solution is

\[
X(t) = \sqrt{\cos^2(2t) + \frac{Q}{4}\sin^2(2t)}.
\]

In the case of randomly varying trap potential the equation (10) has to be solved numerically. We solved it using the procedure DOPRIN [11], which is based on the Runge-Kutta method of 7(8)th order.

In Fig.1 the shapes of the trap potential

\[
U(x) = 2X^2 - \frac{Q}{2X^2},
\]

(18)
corresponding to equation (16) at $\epsilon(t) = 0$ is shown for cases of repulsive and attractive condensates. From this figure it is apparent the behavior of the above two types of condensates. In the "particle in a potential well" representation, departures of the particle from the stationary point (the minimum of the potential well), correspond to expansions and contractions of the atomic cloud with positive $a$, though it remains confined within the trap. Falling of the particle into the center of a singular potential corresponds to the rapid convergence of the condensate’s width to zero (collapse of the condensate with negative $a$).

Fig.2 illustrates expansion of the condensate with a repulsive interaction ($a > 0$) due to random variations of the trap potential. The gain parameter $\xi$ and oscillation frequency $\eta$ determined from this figure, which is the result of numerical solution of eq.(16), averaged over 400 realizations of random paths, are in close agreement with predictions of the moments approach eq.(11). The important conclusion coming out of this result is the existence of the stochastic parametric resonance in the system. Indeed, exponential growth of the amplitude and doubling of the oscillations frequency are clear signatures of the stochastic parametric resonance.

The evolution of the condensate wave function under randomly varying trap potential, obtained by direct numerical solution of the GP equation (1), corresponding to a particular random path, is shown in Fig.3. Exponential growth of the condensate’s width supplemented by doubling of the frequency of oscillations, is evident also from this figure.

The time dependence of the mean square width of the condensate, experiencing collapse under the action of randomly varying trap potential, is presented in Fig.4. Calculations are based on numerical solution of eq.(16), averaged over the ensemble of 400 realizations of random paths. In this case we have the interplay between two opposite effects: contraction of the condensate due to collapse and its stochastic expansion due to noise. As it is apparent from this figure, increasing of the noise intensity $\sigma^2$ leads to slowing down of the collapse. At some level of the noise intensity, contraction of the atomic cloud may be balanced, i.e the collapse can be arrested.
Figure 3: Evolution of the condensate's profile under the action of random variations of trap frequency. a) Numerical solution of eq. (1) with $\gamma = 0$, $\chi = +1$ for a particular set of random numbers. b) Variations of the condensate width, corresponding to this path. $\sigma = 0.5$.

Figure 4: Contraction of a 2D condensate with negative s-wave scattering length under different strengths $\sigma$ of the optical trap's noise. Initial conditions: $X(0) = 1, \dot{X}(0) = 0, Q = -1$.

4 Conclusion

In conclusion, we have studied the dynamics of a Bose-Einstein condensate confined in an optical trap with randomly varying trap potential. Applying the moments method the stochastic parametric resonance in oscillations of the condensate width has been revealed. It is also established, that the noise induced expansion of the condensate with attractive interaction between atoms, can delay or even arrest the collapse.

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\[ \langle X^2(t) \rangle \]
\[ \langle X^2(t) \rangle \] vs. time \( t \) for different values of \( \sigma \):

- \( \sigma = 0.6 \)
- \( \sigma = 0.4 \)
- \( \sigma = 0.2 \)
- \( \sigma = 0 \)