BÄCKLUND TRANSFORMATION
FOR INTEGRABLE SYSTEMS

A. N. Leznov\footnote{E-mail: leznov@mx.ihep.su}
Institute for High Energy Physics, 142284 Protvino, Moscow Region, Russia

Abstract
We establish an explicit form of the Backlund transformation for the most known integrable systems.
1. In this paper the Backlund transformation and its integration for the most known and applicable integrable systems is obtained. Here, by the Backlund transformation we mean any nonlinear mapping which transfers any given solution into another one. However, in this we do not investigate the properties of the transformation, its geometrical interpretation (if any), etc. To prove its validity one can make a direct check which use only one operation − differentiation.

As a hint for obtaining the Backlund transformation the author used a purely algebraic method for construction of the soliton type solutions, see [1, 2], modified for the case of the solvable algebras, [3].

2. The starting point of construction below use the following two facts. The integrable systems under consideration admit the transformation

\[ \theta \Rightarrow \tilde{\theta} \equiv S\theta = F(\theta, \theta^{(1)}_1...\theta^{(N)}_N), \quad S^N \neq 1.\]

Here \( \theta \) and \( \tilde{\theta} \) are unknown functions (variables) satisfying the corresponding PDEs, \( \theta^r_i \equiv \frac{\partial \theta}{\partial x^r_i} \).

There is the obvious solution of the nonlinear system in question which depends on a set of arbitrary functions. The soliton type solutions, reductions related with the discrete groups, solutions with definite boundary conditions are defined by the special choice of arbitrary functions mentioned above. Let us note that \( \theta_0 \) is a solution of a linear system of partial differential equations and it can be presented as a parametric integral on the plane of the complex variable \( \lambda \). This circumstance is just the main reason for applying to the integrable systems the methods of the theory of functions of complex variables, the technics of the Riemann problem, and, at last, the methods of the inverse scattering problem.

3. Here we give a list of integrable systems together with the Backlund transformations for them and the corresponding solutions.

1. Hirota equation

\[
\begin{align*}
v' + \alpha(\bar{v} - 6uv\bar{v}) - i\beta(\bar{v} - 2\bar{v}^2u) + \gamma \bar{v} + i\delta \bar{v} &= 0, \\
u' + \alpha(\bar{u} - 6uv\bar{u}) + i\beta(\bar{u} - 2\bar{u}^2v) + \gamma \bar{u} - i\delta \bar{u} &= 0;
\end{align*}
\]

\( \bar{v} \equiv sv = \frac{1}{u}, \quad \bar{u} \equiv su = u(uv - \ln u), \quad v_0 = 0, \quad u'_0 + \alpha \bar{u}_0 + i\beta \bar{u}_0 + \gamma \bar{u}_0 - i\delta u_0 = 0. \)

In this and in the other cases the main role will be played by the principal minors of the following matrix:

\[
\begin{pmatrix}
\phi^s & \phi^{s+1} & \phi^{s+2} & \ldots \\
\phi^{s+1} & \phi^{s+2} & \phi^{s+3} & \ldots \\
\phi^{s+2} & \phi^{s+3} & \phi^{s+4} & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

To denote the principal minors of these matrices we shall use the symbol \( D^n_r \). Here \( n \) is the rank of the matrix and \( r \) is the symbol of its element of left upper corner. For the solution
of the Backlund transformation we have
\[ v_n = (-1)^n \frac{D_0^{n-1}}{D_0^n}, \quad u_n = (-1)^{n+1} \frac{D_0^{n+1}}{D_0^{n+1}} \] (2)

The methods of theory function of complex variables gives the same expression where the role \( D_0^n \) play the nonlocal integral
\[ D_0^n = \int d\lambda_1 \ldots d\lambda_n c(\lambda_1) \ldots c(\lambda_n) W_n^2(\lambda_1, \ldots, \lambda_n), \] (3)

where \( W_n(\lambda) \) is the Vandermonde determinant; and \( c(\lambda) \) is the integrand in the representation for \( u_0 \).

2. Nonlinear Schrödinger equation
a)
\[ q' + \ddot{q} - 2rq^2 = 0, \quad \tilde{q} = 1 \frac{\ddot{r}}{r} = r [rq - \ln r]; \]
\[ -r' + \ddot{r} - 2qr^2 = 0, \quad q_0 = 0, \quad r_0' = \ddot{r}_0. \] (4)

The solution of the Backlund transformation is the same as in the previous section.

b)
\[ q' + \ddot{q} + 2(rq) \dot{q} = 0, \quad \tilde{q} = 1 \frac{\ddot{r}}{r} = r [(rq) + \ln \frac{r}{r'}]; \]
\[ -r' + \ddot{r} - 2(rq) \dot{r} = 0, \quad q_0 = 0, \quad r_0' = \ddot{r}_0. \] (5)

The solution of the Backlund transformation is as follows
\[ q_n = (-1)^n \frac{D_1^{n-1}}{D_0^n}, \quad r_n = (-1)^{n+1} \frac{D_0^{n+1}}{D_1^{n+1}} \] (6)

c)
\[ q' + \ddot{q} - 2(rq)^2 = 0, \quad \tilde{q} = r, \quad \tilde{r} = q - \frac{1}{r}; \]
\[ -r' + \ddot{r} + 2(r^2q) = 0, \quad q_0 = 0, \quad r_0' = \ddot{r}_0. \] (7)

The solution of the Backlund transformation is as follows
\[ q_{2n} = \frac{D_0^{n-1}D_0^n}{(D_0^n)^2}, \quad r_{2n} = \frac{D_0^{n-1}D_1^n}{(D_1^n)^2}, \]
\[ q_{2n+1} = \frac{D_0^{n-1}D_0^n}{(D_0^n)^2}, \quad r_{2n+1} = \frac{D_1^{n-1}D_1^n}{(D_1^n)^2}. \] (8)

3. One-dimensional Heisenberg ferromagnetic in classical region (XXX - model).
\[ S' = [S, \dot{S}], \quad S = (S_-, S_0, S_+), \quad S_0^2 + S_- S_+ = 1; \]
\[ \tilde{S}_- = S_- + 2 \left( \frac{1}{(s_1 + s_0)} \right) , \quad \tilde{S}_+ = S_+ + 2 \left( \frac{1}{(1 - s_0)} \right) , \]

\[ \tilde{S}_0 + 1 = -\tilde{S}_- \frac{S_+}{1 + S_0} , \quad S_0^0 = 0 , \quad S_0^0 = 1 , S_+^0 = 2 \tilde{S}_+ . \]

\[
S_-^n = \frac{D_2^{n-1} D_2^n}{(D_1^n)^2} , \quad S_0^n + 1 = 2 \frac{D_0^n D_2^n}{(D_1^n)^2} , \\
S_0^n - 1 = 2 \frac{D_2^{n-1} D_0^{n+1}}{(D_1^n)^2} , \quad S_+^n = -4 \frac{D_0^{n+1} D_0^n}{(D_1^n)^2} . \tag{9}
\]

4. XYZ-model in classical region. The Landau-Lifshits equation.

\[ \tilde{S}' = \tilde{S} \times \ddot{\tilde{S}} + \tilde{S} \times (JS) \]

\[ \tilde{S} = (S_1, S_2, S_3), (\tilde{S})^2 = 1 , \quad J = \text{diag}(J_1, J_2, J_3) \]

Under the steroiographic projection

\[ u = \frac{S_1 + i S_2}{1 + S_3} , \quad v = \frac{S_1 - i S_2}{1 + S_3} \]

and exchanging \(-it \to t\) it became a system of the equations:

\[ u' + \ddot{u} - 2v \frac{(\dot{u})^2 + R(u)}{1 + uv} + \frac{1}{2} \frac{\partial}{\partial u} R(u) = 0 \]

\[ -v' + \ddot{v} - 2u \frac{(\dot{v})^2 + R(v)}{1 + uv} + \frac{1}{2} \frac{\partial}{\partial v} R(v) = 0 \]

where \(R(x) = \alpha x^4 + \gamma x^2 + \alpha \frac{\partial R}{\partial x} = 4 \alpha x^3 + 2 \gamma x = 2^{2n+\alpha(x^4-1)} \quad \alpha = \frac{J_4 - J_3}{4} \quad \gamma = \frac{J_4 + J_3}{2} - J_3 \)

The system (10) is invariant under transformation \(u \to U, v \to V:\)

\[ U = \frac{1}{v} \frac{1}{1 + \frac{\nu v}{1 + uv}} \quad \frac{1}{1 + uv} \quad \frac{v \ddot{v} - (\dot{v})^2 + \alpha (v^4 - 1)}{\dot{v})^2 + R(v)} \tag{11} \]

which is the Backlund transformation for this system.

5. Lund-Pohlmeyer-Regge model

\[ y' - 4y + 2(xy)y' = 0 , \quad x = (\dot{y} + xy^2)^{-1} , \]

\[ x' - 4x - 2(xy)y' = 0 , \quad \ddot{y} = -((\dot{y} + xy^2) + y^{-1}(\dot{y} + xy^2)^2) , \tag{12} \]

\[ x_0 = 0 , \quad y_0 = 4y_0 . \]

\[ x_n = (-1)^{n+1} \frac{D_1^{n-1}}{D_1^n} , \quad y_n = (-1)^n \frac{D_0^{n+1}}{D_0^n} . \tag{13} \]
6. The main chiral field problem in a space of \( n \) dimensions (the case of an algebra \( A_1 \)).

The main chiral field problem in \( n \)-dimensional space is described by the following system of equations:

\[
(\theta_i - \theta_j) \frac{\partial^2 f}{\partial x_i \partial x_j} = [\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}],
\]

where the function \( f \) takes values in \( A_1 \) algebra, \( \theta_i \) are numerical parameters.

\[
F_- = \frac{1}{f_+},
\]

\[
\frac{\partial F_0}{\partial x_i} = (f_0 - F_0 + \theta_i) \frac{\partial}{\partial x_i} \ln f_+ - \frac{\partial f_0}{\partial x_i},
\]

\[
\frac{\partial F_+}{\partial x_i} = (f_0 - F_0 + \theta_i)^2 \frac{\partial f_+}{\partial x_i} - 2f_+(f_0 - F_0 + \theta_i) \frac{\partial f_0}{\partial x_i} - f_+^2 \frac{\partial f_-}{\partial x_i}.
\]

(15)

The last equations can be rewritten in the matrix form

\[
\frac{\partial F}{\partial x_i} = \exp \left[ -X\ln f_+ \right] \cdot r \frac{\partial f}{\partial x_i} r^{-1}
\]

\[
\exp \left[ -H\ln f_+ \right] \exp[X\ln f_+],
\]

(16)

where \( r \) is an automorphism of the algebra \( A_1 \) with the properties

\[
r X^\pm r^{-1} = -X^\pm, \quad r H r^{-1} = -H;
\]

\[
f_-^0 = 0, \quad f_0^0 = \tau, \quad f_+^0 = \alpha_0^0,
\]

where

\[
\frac{\partial^2 \tau}{\partial x_i \partial x_j} = 0, \quad (\theta_i - \theta_j) \frac{\partial^2 \alpha_0^0}{\partial x_i \partial x_j} = 2 \left[ \frac{\partial \tau}{\partial x_i} \frac{\partial \alpha_0^0}{\partial x_j} - \frac{\partial \tau}{\partial x_j} \frac{\partial \alpha_0^0}{\partial x_i} \right].
\]

(17)

To solve the Backlund transformation, let us consider the linear system of equations:

\[
\theta_i \frac{\partial \alpha^l}{\partial x_i} - 2 \frac{\partial \tau}{\partial x_i} \alpha^l = \frac{\partial \alpha^{l+1}}{\partial x_i}.
\]

(18)

From the last equations it follows that each function \( \alpha^l \) is a solution of the equation for \( \alpha^0 \).

We have an explicate expression for \( \alpha^s \):

\[
\tau = \sum \phi_i(x_i), \quad \alpha^s = \int d\lambda(\lambda)^s c(\lambda) \exp(\sum \frac{\phi_i(x_i)}{\lambda - \theta_i}).
\]

(19)

In terms of the \( \alpha^l \) the Backlund transformation have the solution

\[
f_0^n = \frac{D_0^{n-1}}{D_0^n}, \quad f_0^n = \tau - \frac{D_0^n}{D_0^{n-1}}, \quad f_+^n = \frac{D_0^{n+1}}{D_0^n}.
\]

(20)

In the determinant \( \dot{D}_0^n \) numbering of the indices of the last row is enlarged by unity.

7. The main chiral field problem for an arbitrary semisimple Lie algebra.
For the case of a semisimple Lie algebra and for an element $f$ being a solution of (12), the following statement takes place: There exists such an element $S$ taking values in a gauge group that

$$S^{-1} \frac{\partial S}{\partial x_i} = \frac{1}{f_-} \left[ \frac{\partial f}{\partial x_i}, X_M^+ \right] - \theta_i \frac{\partial S}{\partial x_i} f_-^{-1} X_M^+.$$  \hfill (21)

Here $X_M^+$ is the element of the algebra corresponding to its maximal root divided by its norm, i.e.,

$$[X_M^+, X^-] = H, \quad [H, X^\pm] = \pm 2X^\pm,$$

$-f_-$ is the coefficient function in the decomposition of $f$ of the element corresponding to the minimal root of the algebra. In this terms the Backlund transformation is as follows:

$$\frac{\partial F}{\partial x_i} = S \frac{\partial f}{\partial x_i} S^{-1} + \theta_i \frac{\partial S}{\partial x_i} S^{-1},$$  \hfill (22)

8. The system of self-dual equations in four-dimensional space (the case of the algebra $A_1$). The self-dual equations for an element $f$ with values in a semisimple Lie algebra have the following form:

$$\frac{\partial^2 f}{\partial y \partial \bar{y}} + \frac{\partial^2 f}{\partial z \partial \bar{z}} = \left[ \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right].$$  \hfill (23)

For this system the Backlund transformation has the form

$$F_- = \frac{1}{f_-}$$

$$\frac{\partial}{\partial y} F_0 = \frac{\partial}{\partial z} \ln f_- - \frac{\partial}{\partial y} f_0 + (f_0 - F_0) \frac{\partial}{\partial y} \ln f_-,$$

$$\frac{\partial}{\partial z} F_0 = - \frac{\partial}{\partial y} \ln f_- - \frac{\partial}{\partial z} f_0 + (f_0 - F_0) \frac{\partial}{\partial z} \ln f_-,$$

$$\frac{\partial}{\partial y} F_+ = - f_- \left\{ (f_0 - F_0) \frac{\partial}{\partial y} (f_0 - F_0) + \frac{\partial}{\partial z} (f_0 - F_0) \right\} - f_-^2 \frac{\partial}{\partial y} f_+,$$

$$\frac{\partial}{\partial z} F_+ = - f_- \left\{ (f_0 - F_0) \frac{\partial}{\partial z} (f_0 - F_0) - \frac{\partial}{\partial y} (f_0 - F_0) \right\} - f_-^2 \frac{\partial}{\partial z} f_+.$$  \hfill (24)

The substitution of (24) in the density of the topological charge gives the equality:

$$Q_F = q_f + \square \square \ln f_-$$

For the integration of the Backlund transformation we have the linear system of equation

$$\frac{\partial \alpha^l}{\partial \bar{y}} + 2 \frac{\partial \tau}{\partial z} \alpha^l = \frac{\partial \alpha^{l+1}}{\partial z}, \quad \frac{\partial \alpha^l}{\partial z} - 2 \frac{\partial \tau}{\partial y} \alpha = \frac{\partial \alpha^{l+1}}{\partial y}.$$  \hfill (25)

In this terms we have the solution of the self-dual system

$$f_-^n = \frac{D_0^{n-1}}{D_0}, \quad f_0^n = \frac{\dot{D}_0}{D_0} + \tau, \quad f_+^n = \frac{D_0^{n+1}}{D_0}.$$  \hfill (26)
9. The system of self-dual equations in the case of an arbitrary semisimple algebra. The following statement takes place:

There exists such an element $S$ taking the values in the gauge group, that

$$
S^{-1} \frac{\partial S}{\partial y} = \frac{1}{f_-} \left[ \frac{\partial f}{\partial y}, X'_M \right] - \frac{\partial \left( \frac{1}{f_-} \right) X'_M, \\
S^{-1} \frac{\partial S}{\partial z} = \frac{1}{f_-} \left[ \frac{\partial f}{\partial z}, X'_M \right] + \frac{\partial \left( \frac{1}{f_-} \right) X'_M.
$$

(27)

Here $X'_M$ is the element of the algebra corresponding to its maximal root, divided by its norm, i.e.,

$$
[X'_M, X^-] = H, \quad [H, X^\pm] = \pm 2X^\pm,
$$

$-f_-$ is the coefficient function in the decomposition of $f$ on the element corresponding to the minimal root of the algebra. The Backlund transformation have the form

$$
\frac{\partial F}{\partial y} = S \frac{\partial f}{\partial y} S^{-1}, \quad \frac{\partial F}{\partial z} = S \frac{\partial f}{\partial z} S^{-1} - \frac{\partial S}{\partial \bar{y}} S^{-1}.
$$

(28)

10. The main chiral field problem with the moving poles. Many integrable systems arise from (23) by imposing symmetry requirements on the solution. The cylindrically symmetric condition in four dimensional space restricts the form of the function $f$,

$$
f = \frac{1}{y} f(\xi, \bar{\xi}), \quad \xi = \frac{z - \bar{\xi}}{2} + \left[ \left( \frac{z + \bar{\xi}}{2} \right)^2 + y\bar{y} \right]^{1/2}, \quad \bar{\xi} = -\xi^*.
$$

(29)

This is the equation for the main chiral field with moving poles.

The result of integration of equation (27) is given in the form

$$
S = S(\xi, \bar{\xi}) \exp \frac{X'_M}{f_-} z;
$$

and the Backlund transformation has the following form:

$$
\frac{\partial F}{\partial \xi} = S \frac{\partial f}{\partial \xi} S^{-1} - \xi \frac{\partial S}{\partial \xi} S^{-1}, \quad \frac{\partial F}{\partial \bar{\xi}} = S \frac{\partial f}{\partial \bar{\xi}} S^{-1} - \xi \frac{\partial S}{\partial \bar{\xi}} S^{-1}.
$$

(31)

The relations (30), (31) realize the Backlund transformation for the main chiral field with moving poles.
11. The self dual equation under of cylindrical symmetric in three dimensional space. The condition of cylindrical symmetric in three dimensional space leads to the following form of the solution to equation:

\[ f = \frac{1}{y} f(ξ, \bar{ξ}), \quad ξ = \frac{z + \bar{z}}{2} + i(\bar{y}y)^{1/2}, \quad \bar{ξ} = -ξ^*; \]

\[ (ξ - \bar{ξ}) \frac{∂^2 f}{∂ξ∂ξ} = \frac{1}{2} \left( \frac{∂f}{∂ξ} - \frac{∂f}{∂ξ} \right) + \left[ \frac{∂f}{∂ξ} \frac{∂f}{∂ξ} \right]. \]  

(32)

The Backlund transformation for equation (32) arising from (27), (28) has the form

\[ S^{-1} \frac{∂}{∂ξ} S = \frac{1}{f_-} \left[ \frac{∂}{∂ξ} f, X^+_M \right] + \left( \frac{1}{f_-} - \frac{ξ - \bar{ξ}}{2} \frac{∂}{∂ξ} f_- \right) X^+_M, \]

\[ S^{-1} \frac{∂}{∂ξ} S = \frac{1}{f_-} \left[ \frac{∂}{∂ξ} f, X^+_M \right] + \left( \frac{1}{f_-} - \frac{ξ - \bar{ξ}}{2} \frac{∂}{∂ξ} f_- \right) X^+_M, \]

\[ \frac{∂}{∂ξ} F = S \left( \frac{∂}{∂ξ} f \right) S^{-1} + \frac{ξ - \bar{ξ}}{2} \frac{∂S}{∂ξ} S^{-1}, \quad \frac{∂}{∂ξ} F = S \left( \frac{∂}{∂ξ} f \right) S^{-1} - \frac{ξ - \bar{ξ}}{2} \frac{∂S}{∂ξ} S^{-1}. \]

In the case of algebra \( A_1 \) system (32) arises in the integration problem of the general relativity with two commuting Killing vectors.

12. The cylindrical symmetrically solution invariant under two orthogonal four-dimensional axis.

\[ x_1 \frac{∂^2 F}{∂x_1^2} + x_2 \frac{∂^2 F}{∂x_2^2} = \left[ \frac{∂F}{∂x_1}, \frac{∂F}{∂x_2} \right] \]  

(33)

The case of the algebra \( A_1 \) the explicate form of the Backlund transformation is the following \( F_- = \frac{1}{f_-} \)

\[ \frac{∂F_0}{∂x_1} = -1 + (f_0 - F_0) \frac{∂lnf_-}{∂x_1} + x_2 \frac{∂lnf_-}{∂x_2} - \frac{∂f_0}{∂x_1}, \]

\[ \frac{∂F_0}{∂x_2} = 1 + (f_0 - F_0) \frac{∂lnf_-}{∂x_2} - x_1 \frac{∂lnf_-}{∂x_1} - \frac{∂f_0}{∂x_2} \]

(34)

\[ \frac{∂F_+}{∂x_1} = -f_-[(f_0 - F_0) \frac{∂(f_0 - F_0)}{∂x_1} + x_2 \frac{∂(f_0 - F_0)}{∂x_2}] - f_-^2 \frac{∂f_-}{∂x_1}, \]

\[ \frac{∂F_+}{∂x_2} = -f_-[(f_0 - F_0) \frac{∂(f_0 - F_0)}{∂x_2} - x_1 \frac{∂(f_0 - F_0)}{∂x_1}] - f_-^2 \frac{∂f_-}{∂x_2}. \]

The linear system of the equations has the form

\[ x_1 \frac{∂α^l}{∂x_1} - α^l + 2α^l \frac{∂τ^l}{∂x_2} = \frac{∂α^{l+1}}{∂x_2}, \]

\[ x_2 \frac{∂α^l}{∂x_2} - α^l - 2α^l \frac{∂τ^l}{∂x_1} = -\frac{∂α^{l+1}}{∂x_1} \]

(35)

\[ τ^{l+1} = τ^l + \frac{x_1 - x_2}{2}. \]
The solution of the Backlund transformation coincide with the self-dual case (see (25) and (26)).

4. Let denote by $\theta_i$ the solution of the integrable system, which arise after application $i$-times the Backlund transformation to some given one. The Backlund transformation connect the solutions with different values of the index $i$ and so it arise the infinite chain of equations. In some cases this chain may be limited from ”left” or from the ”right” and as we have seen in the pervious section this possibility lead to some subclass of the solution of the integrable systems, which depend from the definite number of the arbitrary functions. The other possibility consists in the assuming of the periodical condition of the solution $\theta_{N+i} = \theta_i$, which choice some finite subclass of the solutions of the initial infinite chain. Let consider this possibility on the examples of the nonlinear Schrodinger equations (see (4)-(7)).

In that case we have only one variable for every value of the index $i$ $r_i \equiv \exp x_i$. In this notations the Backlund transformations are rewritten as follows

a) $q_{i+1} = \frac{1}{r_i} = \exp -x_i$, $r_{i+1} = r_i[q_i r_i - \ln r_i]$ 

or

$\ddot{x}_i + \exp(x_{i+1} - x_i) - \exp(x_i - x_{i-1}) = 0$

Under the condition $x_{i+N} = x_i$ this is the equations of the periodical Toda lattice for the algebra $A_N$

b) $(\ln \dot{x}_i)' + \exp(x_{i+1} - x_i) + \exp(x_i - x_{i-1}) = 0$

c) In that case $q_{i+1} = \exp x_i$ and for the variable $N_i \equiv r_i$ we receive the chain of the equations

$\dot{N}_i = N_i^2(N_{i-1} - N_{i+1})$

This is the Lotky-Volterra chain. In the work [4] the independent investigation of the periodical integrable chains was used for the construction of explicate form of the Backlund transformation for the definite class of the integrable systems. The solution of the Backlund transformation under the periodical condition may be connected with the double periodical solutions of such systems.

5. The main result of the present paper is in the formulas of the section 3 which realized the Backlund transformation for the integrable system under consideration. A more important consequences is in the assumption of a possibility to obtain the Backlund transformation on the base of the group theory. The Backlund transformation in question is the mapping whose invariants are the equations for integrable system. For this reason an independent construction of the Backlund transformation is equivalent to the problem of enumeration of all integrable systems or some class of them.

References

[1] S. M. Chumakov, A. N. Leznov, and V. I. Man’ko. Trudi PHIAN 167 (1986), 232-237.
[2] A.N.Leznov and M.V.Saveliev Acta. Appl. Math. v.16.(1-74) 1991.

[3] A. N. Leznov. Preprint IHEP 91-71 1991.

[4] A.B.Shabat and R.I.Yamilov Leningrad Math. J. Vol.2(1991), No.2.