Factoring Octonion Polynomials

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Abstract

We provide an analogue of Wedderburn’s factorization method for central polynomials with coefficients in an octonion division algebra, and present an algorithm for fully factoring polynomials of degree $n$ with $n$ conjugacy classes of roots, counting multiplicities.

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1. Introduction

For fields $F$ it is well-known that a polynomial $f(x) \in F[x]$ factors into linear factors $f(x) = c(x - \lambda_1) \cdots (x - \lambda_n)$ if and only if it is has $n$ roots in $F$ (counting multiplicities), in which case the roots are precisely $\lambda_1, \ldots, \lambda_n$.

For associative division rings $D$ the statement was generalized by Wedderburn in [7] (see also [5] and [4]). It is important to note that by “polynomials” over any algebra in this context, we mean central polynomials, i.e., polynomials in which the indeterminate is assumed to commute with any element in the ring. More precisely, the ring of (central) polynomials $D[x]$ over a division ring $D$ is defined to be the scalar extension $D \otimes_F F[x]$ of $D$ to the ring of polynomials $F[x]$ over the field $F = Z(D)$. For any element $\lambda \in D$, the substitution map $S_\lambda : D[x] \to D$ is defined by sending each polynomial $f(x) = c_n x^n + \cdots + c_1 x + c_0$ to $f(\lambda) = c_n \lambda^n + \cdots + c_1 \lambda + c_0$, thus placing the coefficients on the left-hand side of the indeterminate before plugging the value $\lambda$ in $x$, and therefore these polynomials are also known as “left” polynomials.

What Wedderburn showed was that if $\lambda_1 \in D$, $f(x) \in D[x]$ and $\lambda_1$ is a root of $f(x)$, then $f(x) = g(x)(x - \lambda_1)$ for some polynomial $g(x) \in D[x]$. Then, if $\lambda_2$ is
another root of \( f(x) \) in \( D \), \((\lambda_1 - \lambda_2)\lambda_2(\lambda_1 - \lambda_2)^{-1}\) is a root of \( g(x) \). This is called “Wedderburn’s method” in [5].

Wedderburn’s main motivation was the following observation: Let \( F \) be a field and \( D \) a central division \( F \)-algebra. Suppose \( f(x) \) is a monic irreducible polynomial in \( F[x] \). It is possible that \( f(x) \) becomes reducible over \( D[x] \). In particular, if \( f(x) \) has a root \( \lambda_1 \) in \( D \setminus F \), then \( f(x) \) decomposes in \( D[x] \) as \((x - \lambda_n)(x - \lambda_{n-1})\ldots(x - \lambda_1)\) where \( \lambda_1, \ldots, \lambda_n \) are inner conjugates in \( D \).

Our goal in this work is to study the analogous situation for octonion division algebras. Since by Kleinfeld’s theorem ([8, Chapter 7, Section 3]), every nonassociative alternative division algebra is an octonion algebra (see [6] and [3, Section 33] for reference), this will complete the picture for alternative division algebras in full generality. The main two results are thus the following:

1. An analogue of Wedderburn’s method for the ring of (central) polynomials \( A[x] \) over an octonion division algebra \( A \).
2. An algorithm for fully factoring polynomials \( f(x) \) of degree \( n \) in \( A[x] \) iteratively into linear factors \( f(x) = ((\ldots(c(x - \lambda_n))\ldots(x - \lambda_3))(x - \lambda_2))(x - \lambda_1) \), under the assumption there are \( n \) conjugacy classes for the roots of \( f(x) \), counting multiplicity.

2. Octonion Polynomials

Let \( A \) be an octonion division algebra with center \( F \). Recall that such an octonion algebra is a Cayley-Dickson doubling of a quaternion algebra \( Q \) over \( F \), i.e., \( A = Q \oplus Q\ell \) as an \( F \)-vector space, and the multiplication is given by

\[(q + r\ell) \cdot (s + t\ell) = qr + \gamma rt + (r\overline{s} + tq)\ell\]

for any \( q, r, s, t \in Q \), where \( \gamma \) is some fixed element in \( F^\times \), and \( z \mapsto \overline{z} \) is the canonical involution on \( Q \). The quaternion algebra \( Q \) in turn admits the structure

\[Q = F(i, j : i^2 = \alpha, j^2 = \beta, ij = -ji)\]

for some \( \alpha, \beta \in F^\times \) when \( \text{char}(F) \neq 2 \), in which case the canonical (symplectic) involution is given by

\[a + bi + cj + dk = a - bi - cj - dk,\]

and

\[Q = F(i, j : i^2 + i = \alpha, j^2 = \beta, jij^{-1} = i + 1)\]
for some $\alpha \in F$ and $\beta \in F^\times$ when $\text{char}(F) = 2$, in which case the canonical involution is given by

$$a + bi + cj + dk = a + b + bi + cj + dk.$$ 

The canonical involution extends from $Q$ to $A$ by

$$q + r\ell = \bar{q} - r\ell.$$ 

This involution gives rise to the trace and norm maps

$$\text{Tr} : A \rightarrow F$$

$$\text{Norm} : A \rightarrow F$$

defined by $\text{Tr}(z) = z + \bar{z}$ and $\text{Norm}(z) = z \cdot \bar{z}$. Every element $z \in A$ then satisfies $z^2 - \text{Tr}(z) \cdot z + \text{Norm}(z) = 0$, and thus the algebra $A$ is a quadratic algebra. Moreover, it is a composition algebra, for its norm form is multiplicative (see [3, Section 33]).

The algebra $A$ is also alternative, which means that every two elements $a, b$ in $A$ live in an associative subalgebra, and as a result, conjugation is well-defined.

Following the literature on (central) polynomials over associative division algebras, by the ring of polynomials $A[x]$ we mean the tensor product $A \otimes_F F[x]$ of $A$ and the ring of polynomials $F[x]$ over $F$. The center of $A[x]$ is thus $F[x]$. Therefore, every polynomial $f(x) \in A[x]$ can be written in the “standard form” $f(x) = c_n x^n + \cdots + c_1 x + c_0$ where $c_0, \ldots, c_n \in A$, in which case the coefficients are placed on the left-hand side of the indeterminate. For each $\lambda \in A$, the substitution map

$$S_\lambda : A[x] \rightarrow A$$

$$f(x) \mapsto f(\lambda)$$

is defined by sending $f(x) = c_n x^n + \cdots + c_1 x + c_0$ to $f(\lambda) = c_n \lambda^n + \cdots + c_1 \lambda + c_0$.

The element $\lambda \in A$ is thus a “root” of $f(x)$ when $f(\lambda) = 0$.

We start with factoring out one linear factor given a root:

**Proposition 2.1.** Given a polynomial $f(x) \in A[x]$ and an element $\lambda \in A$, $f(x)$ decomposes as $f(x) = g(x)(x - \lambda)$ if and only if $\lambda$ is a root of $f(x)$, in which case

$$g(x) = d_{n-1} x^{n-1} + \cdots + d_1 x + d_0,$$

where

$$d_k = c_n \lambda^{n-1-k} + c_{n-1} \lambda^{n-2-k} + \cdots + c_{k+1}, \ k \in \{0, \ldots, n - 1\}.$$
Proof. Consider the expression \( c_n x^n + \cdots + c_1 x - (c_n \lambda^n + \cdots + c_1 \lambda) \). On the one hand, since \( \lambda \) is a root of \( f(x) \), we have \( f(\lambda) = 0 \), i.e., \( c_n \lambda^n + \cdots + c_1 \lambda + c_0 = 0 \), and therefore \( c_0 = -(c_n \lambda^n + \cdots + c_1 \lambda) \), which means that \( f(x) = c_n x^n + \cdots + c_1 x - (c_n \lambda^n + \cdots + c_1 \lambda) \).

On the other hand, for each \( k \in \{1, \ldots, n\} \), the elements \( c_k \) and \( \lambda \) live in an associative subalgebra \( D \) of \( A \), and thus together with \( x \) they live in an associative subring \( D[x] \) of \( A[x] \). We therefore have

\[
c_k x^k - c_k \lambda^k = c_k (x^k - \lambda^k) = c_k (x^{k-1} + \lambda x^{k-2} + \cdots + \lambda^{k-1})(x - \lambda)
\]

\[
= (c_k x^{k-1} + c_k \lambda x^{k-2} + \cdots + c_k \lambda^{k-1})(x - \lambda).
\]

Hence, by rewriting \( c_n x^n + \cdots + c_1 x - (c_n \lambda^n + \cdots + c_1 \lambda) \) as \( (c_n x^n - c_n \lambda^n) + (c_n \lambda - c_n \lambda^n) + \cdots + (c_1 x - c_1 \lambda) \), we obtain that the latter is equal to \( g(x)(x - \lambda) \) where \( g(x) = d_{n-1} x^{n-1} + \cdots + d_1 x + d_0 \) and for each \( k \in \{0, \ldots, n-1\} \), \( d_k = c_n \lambda^{n-k} + c_{n-1} \lambda^{n-2-k} + \cdots + c_k \).

\[\square\]

**Remark 2.2.** The analogue of Wedderburn’s theorem [5, Theorem 0.4] holds true also in the octonionic case: let \( A \) be an octonion division algebra with center \( F \), a field, and let \( f(x) \) be a monic irreducible polynomial of degree \( n \) in \( F[x] \). Suppose it has a root \( \lambda_1 \) in \( A \). Then, since \( A \) is an octonion algebra, \( \lambda_1 \) lives inside a quaternion \( F \)-subalgebra \( D \) of \( A \). Now, by Wedderburn’s theorem, \( f(x) \) decomposes as \( (x - \lambda_n) \cdot \cdots \cdot (x - \lambda_1) \) in \( D[x] \), where \( \lambda_1, \ldots, \lambda_n \) are inner conjugates in \( D \). Since \( D[x] \subseteq A[x] \), the same decomposition applies also in \( A[x] \).

We present here the direct generalization of Wedderburn’s method in the alternative case:

**Proposition 2.3.** Given a polynomial \( f(x) \) in \( A[x] \), if \( \lambda_1 \) and \( \lambda_2 \) in \( A \) are two distinct roots of \( f(x) \), then \( \gamma = (\lambda_1 - \lambda_2) \lambda_2 (\lambda_1 - \lambda_2)^{-1} \) is a root of the polynomial

\[
h(x) = d_{n-1} x^{n-1} + \cdots + d_1 x + d_0, \quad \text{where}
\]

\[
d_k = \gamma^k \left( (\gamma^{-k} ((\lambda_1 - \lambda_2)^{-1} c_n)) \lambda_1^{n-1-k} + \cdots + (\gamma^{-k} ((\lambda_1 - \lambda_2)^{-1} c_{k+1})) \right), \quad k \in \{0, \ldots, n-1\}.
\]

**Proof.** Since both \( \lambda_1 \) and \( \lambda_2 \) are roots of \( f(x) \), we have

\[
c_n \lambda_1^n + \cdots + c_1 \lambda_1 + c_0 = 0, \quad \text{and}
\]

\[
c_n \lambda_2^n + \cdots + c_1 \lambda_2 + c_0 = 0.
\]

Subtracting the second from the first gives

\[
c_n (\lambda_1^n - \lambda_2^n) + \cdots + c_1 (\lambda_1 - \lambda_2) = 0.
\]
Now, for each $k \in \{1, \ldots, n\}$,
\[ \lambda_1^k - \lambda_2^k = (\lambda_1^{k-1} + \lambda_2^{k-2} \gamma + \cdots + \gamma^{k-1})(\lambda_1 - \lambda_2) \]
where $\gamma = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_2)^{-1}$. For each $k$, write $c'_k = (\lambda_1 - \lambda_2)^{-1}c_k$, and then
\[ c_k(\lambda_1^k - \lambda_2^k) = (\lambda_1 - \lambda_2) (c'_k(\lambda_1^{k-1} + \lambda_2^{k-2} \gamma + \cdots + \gamma^{k-1})(\lambda_1 - \lambda_2)^{-1} \]
by the Moufang identity $(rs)(tr) = r(st)r$. It remains to note that
\[ c'_k(\lambda_1^{k-1} + \lambda_2^{k-2} \gamma + \cdots + \gamma^{k-1}) = c'_k \lambda_1^{k-1} + \gamma((\gamma^{-1} c'_k) \lambda_1^{k-2}) \gamma + \cdots + \gamma^{k-2}((\gamma^{-k} c'_k) \lambda_1) \gamma^{k-2} + \gamma^{k-1}(\gamma^{-k} c'_k) \gamma^{k-1} \]
and then the sum $c_n(\lambda_1^n - \lambda_2^n) + \cdots + c_1(\lambda_1 - \lambda_2)$ turns into
\[ (\lambda_1 - \lambda_2) \left( d_{n-1} \gamma^{n-1} + \cdots + d_1 \gamma + d_0 \right) (\lambda_1 - \lambda_2)^{-1}, \]
which means $d_{n-1} \gamma^{n-1} + \cdots + d_1 \gamma + d_0 = 0$. \qed

Note that when all the coefficients $c_n, \ldots, c_0$ and the roots $\lambda_1$ and $\lambda_2$ belong to an associative subalgebra, the polynomial $h(x)$ from Proposition 2.4 coincides with the polynomial $g(x)$ for which $f(x) = g(x)(x - \lambda_1)$. However, this is not true in general.

**Example 2.4.** Consider the real octonion algebra $\mathbb{O}$ with standard generators $i, j, \ell$, and the polynomial $f(x) = ix^2 + jx + \ell$. It has two roots $\lambda_1 = \frac{1}{2}(1 + ij - il + j\ell)$ and $\lambda_2 = \frac{1}{2}(-1 + ij - il + j\ell)$. Then $\lambda_1 - \lambda_2 = 1 + il$, and $\gamma = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_2)^{-1} = \frac{1}{2}(1 + il) \cdot \frac{1}{2}(-1 + ij - il + j\ell) \cdot (1 - il) = \frac{1}{2}(1 + il)(-1 + j\ell) = \frac{1}{2}(-1 - ij - il + j\ell)$. The polynomial $f(x)$ decomposes as $g(x)(x - \lambda_1)$ where $g(x) = ix + (j + i\lambda_1) = ix + \frac{1}{2}(i + j - \ell - (ij)\ell)$, whose only root is $-\lambda_1$, which is not $\lambda_2$. On the other hand,
\[ h(x) = (\lambda_1 - \lambda_2)^{-1}ix + ((\lambda_1 - \lambda_2)^{-1}i)\lambda_1 + (\lambda_1 - \lambda_2)^{-1}j \]
\[ = \frac{1}{2}(i - \ell)x + \frac{1}{2}(-j - \ell) + \frac{1}{2}(j + (ij)\ell), \]
and then indeed $h(\gamma) = \frac{1}{2}(\ell - (ij)\ell) + \frac{1}{2}(-j - \ell) + \frac{1}{2}(j + (ij)\ell) = 0$, and therefore $\gamma$ is a root of $h(x)$.

So how do we keep factoring $g(x)$? The answer is in the following section and involves the companion polynomial.
3. The Companion Polynomial

In [1] the companion polynomial $C_f(x)$ was defined for any polynomial $f(x) = c_n x^n + \cdots + c_1 x + c_0$ in the following way:

$$C_f(x) = b_{2n} x^{2n} + b_{2n-1} x^{2n-1} + \cdots + b_1 x + b_0$$

where for each $k \in \{0, \ldots, 2n\}$, if $k$ is odd then $b_k$ is equal to the sum of all the elements $\text{Tr}(c^k c_j)$ with $0 \leq i < j \leq n$ and $i + j = k$, and if $k = 2m$ is even then $b_k$ is equal to the sum of all the elements $\text{Tr}(c^k c_j)$ with $0 \leq i < j \leq n$ and $i + j = k$ plus the element $\text{Norm}(c_m)$.

**Remark 3.1.** Since $\text{Tr}(c^k c_j) = c^k c_j + \overline{c}^k c_j$ and $\text{Norm}(c_m) = \overline{c}_m c_m$, by a straightforward computation (using the fact that $x$ is central) one sees that in fact

$$C_f(x) = \overline{f(x)} f(x),$$

where $\overline{f(x)} = c_n x^n + \cdots + c_1 x + c_0$.

In [1] it was proven that the roots of $C_f(x)$ coincide with the conjugacy classes of the roots of $f(x)$. This leads us to the following fact:

**Theorem 3.2.** A polynomial $f(x)$ of degree $n$ in $A[x]$ factors into linear factors

$$f(x) = ((\ldots (c(x - \lambda_n)) \ldots (x - \lambda_3))(x - \lambda_2))(x - \lambda_1)$$

if and only if $C_f(x)$ decomposes as $C_f(x) = \text{Norm}(c) \cdot \prod_{k=1}^{n} (x^2 - \text{Tr}(\gamma_k) + \text{Norm}(\gamma_k))$ for some $\gamma_1, \ldots, \gamma_n \in A$, in which case $\lambda_1, \ldots, \lambda_n$ are inner conjugates of $\gamma_1, \ldots, \gamma_n$, and represent the conjugacy classes of the roots of $f(x)$.

**Proof.** Recall that $\lambda_1$ is a root of $f(x)$ if and only if $f(x) = g(x)(x - \lambda_1)$. If that holds then $C_f = \overline{f(x)} f(x) = (x - \lambda_1) g(x) g(x)(x - \lambda_1) = C_g(x)(x^2 - \text{Tr}(\lambda_1) + \text{Norm}(\lambda_1))$. Since the roots of $C_g(x)$ are the conjugacy classes of the roots of $g(x)$, we conclude that the conjugacy classes of the $f(x)$ are the union of the conjugacy classes of the roots of $g(x)$ and the conjugacy class of $\lambda_1$.

Thus, by induction, if $f(x)$ decomposes completely as the product of linear factors $f(x) = ((\ldots (c(x - \lambda_n)) \ldots (x - \lambda_3))(x - \lambda_2))(x - \lambda_1)$, then $C_f = \text{Norm}(c) \cdot \prod_{k=1}^{n} (x^2 - \text{Tr}(\lambda_k) + \text{Norm}(\lambda_k))$, the roots of $C_f$ are both the conjugacy classes of $\lambda_1, \ldots, \lambda_n$, and the conjugacy classes of the roots of $f(x)$. Consequently, the roots of $f(x)$ are the conjugacy classes of $\lambda_1, \ldots, \lambda_n$.

In the opposite direction, suppose $C_f(x) = \text{Norm}(c) \cdot \prod_{k=1}^{n} (x^2 - \text{Tr}(\gamma_k) + \text{Norm}(\gamma_k))$ for some $\gamma_1, \ldots, \gamma_n \in A$. Take $\lambda_1$ to be a root of $f(x)$ in the conjugacy class of $\gamma_1$. Since for a root $\lambda_1$ of $f(x)$, $f(x) = g(x)(x - \lambda_1)$ and $C_f(x) = C_g(x)(x^2 -
Tr(\lambda_1) + \text{Norm}(\lambda_1))$, we conclude $C_g(x) = \text{Norm}(c) \cdot \prod_{k=2}^{n}(x^2 - \text{Tr}(\gamma_k) + \text{Norm}(\gamma_k))$, and thus for each $k \in \{2, \ldots, n\}$, there is a root of $g(x)$ in the conjugacy class of $\gamma_k$. Take in particular $\lambda_2$ to be the root of $g(x)$ in the conjugacy class of $\gamma_2$, and continue inductively.

**Remark 3.3.** The analogous statement clearly holds true for quaternion algebras as well, i.e., if $Q$ is a quaternion algebra over $F$ and $f(x)$ is a polynomial in $Q[x]$, then $f(x)$ decomposes into linear factors as $f(x) = c(x - \lambda_n) \ldots (x - \lambda_1)$ if and only if its companion polynomial $C_f(x) = \overline{f(x)} \cdot f(x)$ decomposes as $C_f(x) = \text{Norm}(c) \cdot \prod_{k=1}^{n}(x^2 - \text{Tr}(\gamma_k) + \text{Norm}(\gamma_k))$ for some $\gamma_1, \ldots, \gamma_n \in Q$. The argument is simpler here, because the algebra is associative: if $f(x)$ decomposes as $c(x - \lambda_n) \ldots (x - \lambda_1)$, then $C_f(x)$ is by a straightforward computation $\text{Norm}(c) \cdot \prod_{k=1}^{n}(x^2 - \text{Tr}(\lambda_k) + \text{Norm}(\lambda_k))$; and in the opposite direction, if $C_f(x) = \overline{f(x)} \cdot f(x)$ decomposes as $C_f(x) = \text{Norm}(c) \cdot \prod_{k=1}^{n}(x^2 - \text{Tr}(\gamma_k) + \text{Norm}(\gamma_k))$ for some $\gamma_1, \ldots, \gamma_n \in Q$, then by [2, Remark 3.5 and Theorem 3.6], $f(x)$ has roots $\lambda_1, \ldots, \lambda_n$ where each $\lambda_k$ is in the same conjugacy class as $\gamma_k$.

Note that in the decomposition $C_f(x) = \text{Norm}(c) \cdot \prod_{k=1}^{n}(x^2 - \text{Tr}(\gamma_k) + \text{Norm}(\gamma_k))$, each quadratic factor is central, and thus the product is well defined because it takes place in $F[x]$. The proof of Theorem 3.2 gives rise to an algorithm for factoring completely decomposable polynomials in $A[x]$:

**Algorithm 3.4.** Given a polynomial $f(x) \in A[x]$, in order to compute its factorization,

1. Compute the companion polynomial $C_f(x)$.

2. Decompose $C_f(x) = C_f = \text{Norm}(c) \cdot \prod_{k=1}^{n}(x^2 - \text{Tr}(\gamma_k) + \text{Norm}(\gamma_k))$ (on the way, make sure this decomposition exists, because it is necessary).

3. Take $\gamma_1$ and find a root $\lambda_1$ of the same conjugacy class for $f(x)$ by reducing the equation $f(\lambda_1) = 0$ into a linear equation using the equality $\lambda_1^2 - \text{Tr}(\gamma_1)\lambda_1 + \text{Norm}(\gamma_1)\lambda_1 = 0$ (in a similar way to [1, Theorem 3.4 and Algorithm 3.5]).

4. Decompose $f(x) = f_2(x)(x - \lambda_1)$.

5. For $k \in \{2, \ldots, n - 1\}$, take $\gamma_k$ and find a root $\lambda_k$ of the same conjugacy class for $f_k(x)$ by reducing $f_k(x) = 0$ into a linear equation, and decompose $f_k(x) = f_{k+1}(x)(x - \lambda_k)$.

6. We end up with a linear $f_n(x)$ that factors as $f_n(x) = c(x - \lambda_n)$. Then the obtained $\lambda_n, \ldots, \lambda_1$ satisfy $f(x) = ((\ldots (c(x - \lambda_n) \ldots (x - \lambda_1)(x - \lambda_2))(x - \lambda_2))(x - \lambda_1)$.

**Example 3.5.** Consider the real octonion algebra $\mathbb{O}$ and the polynomial $f(x) = \ell x^3 + i\ell x^2 + \ell x + i\ell$. Its companion polynomial is $C_f(x) = x^6 + 3x^4 + 3x^2 + 1 = (x^2 + 1)^3$. 

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Therefore the roots of \( f(x) \) are of trace 0 and norm 1. They include \( j \). Take \( \lambda_1 = j \). Then \( f(x) = (\ell x^2 + (\ell + i\ell) x + (\ell^2 + i\ell j + \ell))(x - j) \), \( g(x) = \ell x^2 + (i - j)\ell x - (ij)\ell \). Hence, \( C_g(x) = (x^2 + 1)^2 \), so its roots are still of trace 0 and norm 1. Suppose \( \lambda_2 \) is a root of \( g(x) \), then \( \lambda_2^2 = -1 \), and so \( 0 = g(\lambda_2) = -\ell + ((i - j)\ell)\lambda_2 - (ij)\ell \), which means \((1 + ij)\ell = ((i - j)\ell)\lambda_2 \), and so \( \lambda_2 = \frac{1}{2}((1 + ij)\ell)((i - j)\ell) = \frac{1}{2}((j - i)(1 + ij))\ell^2 = -j \).

Therefore, \( g(x) = (\ell x + (i - j)\ell + ell \cdot (j))(x + j) = (\ell x + i\ell)(x + j) = (\ell(x + i))(x + j) \). The obtained factorization of \( f(x) \) is thus

\[
f(x) = (((\ell(x + i))(x + j))(x - j)).
\]

Note that in this very specific polynomial, we have \( f(x) = \ell(x + i)(x^2 - 1) \), and thus there are plenty other possible ways of factoring it.

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