Sign of the Solutions of Linear Fractional Differential Equations and Some Applications

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Abstract
In this work we wish to highlight some consequences of a recent result proved in (J. Integral Equ. Appl. 29: 585–608, 2017). Particular emphasis will be given to its application on fractional variational problems of Herglotz type.

Keywords  Linear equation · Fractional derivative · Herglotz variational problem

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1 Preamble
Very recently, in 2017, Cong et al. [7] proved the following result (we refer the reader to Section 2 for the definitions appearing below):

Theorem 1.1 Consider the following fractional differential equation

\[ CD^\alpha_a [x](t) = f (t, x(t)), \quad 0 < \alpha \leq 1, \]  \hspace{1cm} (1.1)

where \( f : [a, \infty) \times \mathbb{R} \to \mathbb{R} \) is a continuous function satisfying the Lipschitz condition

\[ |f(t, x) - f(t, y)| \leq L(t)|x - y|, \quad t \in [a, \infty), \ x, y \in \mathbb{R}, \ L \in C([a, \infty), \mathbb{R}). \]  \hspace{1cm} (1.2)

Then, for any two different initial values \( x_{1a} \neq x_{2a} \) in \( \mathbb{R} \), the solutions \( x_1 \) and \( x_2 \) of (1.1) starting from \( x_{1a} = x_1(a) \) and \( x_{2a} = x_2(a) \) verify \( x_1(t) \neq x_2(t) \) for all \( t \in [a, \infty) \).

Remark 1.2 The existence and uniqueness of (continuous) solutions for (1.1) with the given initial conditions is guaranteed by, e.g., [3, Theorem 2].

Remark 1.3 Theorem 1.1 was conjectured in 2008 by Diethelm [9] and solved partially therein. However, it was only in 2017 that a complete and correct proof of it was given (see [7] for the historical developments regarding this result).

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In particular, for $0 < \alpha < 1$ consider the linear initial value problem (IVP)
\[ CD^\alpha a [x](t) = g(t)x(t), \quad x(a) = x_a > 0, \quad (1.3) \]
for which $f(t, x) = g(t)x$ (with $g \in C([a, \infty), \mathbb{R})$) obviously satisfies (1.2). Then we may conclude from Theorem 1.1 that the solution of (1.3) is positive on $[a, \infty)$. We find this result of much interest and, to the best of our knowledge, it was not sufficiently highlighted in the literature so far; therefore, we shall write it in the following:

**Theorem 1.4** Let $0 < \alpha < 1$ and $g \in C([a, \infty), \mathbb{R})$. Then the solution of the IVP (1.3) is positive on $[a, \infty)$.

We may extract several interesting consequences of Theorem 1.4 and we will refer some in Section 3 of this work. But before we go into it we want to observe that, following the same steps as those of the proof of Theorem 1.1, we may prove an analogous result where in the differential equation (1.1) we use the left Riemann–Liouville fractional derivative and the initial conditions are given by $I^{1-\alpha}_{a^+}[x](a) = x_a$ (again we refer the reader to Section 2 to understand the meaning of the symbols used in this section). Then, we may prove the following result:

**Theorem 1.5** Let $0 < \alpha < 1$ and $g \in C((a, \infty), \mathbb{R})$. Then the solution of the IVP
\[ D^\alpha_{a^+} [x](t) = g(t)x(t), \quad I^{1-\alpha}_{a^+} [x](a) = x_a > 0, \]
is positive on $(a, \infty)$.

However, in this work, we will be especially interested in the analogous theorem to Theorem 1.5 but using the right fractional derivative (cf. the proof of Theorem 3.11). In order to accomplish it we may appeal to the duality results (for left and right fractional operators) presented and proved in [6]. We, therefore, have:

**Theorem 1.6** Let $0 < \alpha < 1$ and $g \in C((\infty, b), \mathbb{R})$. Then the solution of the IVP
\[ D^\alpha_{b^-} [x](t) = g(t)x(t), \quad I^{1-\alpha}_{b^-} [x](b) = x_b > 0, \]
is positive on $(-\infty, b)$.

**Remark 1.7** It is clear that, if $x_b < 0$, then the solution in Theorem 1.6 is negative on $(-\infty, b)$.

The remaining of this manuscript is organized as follows: In Section 2 we provide the reader with the definitions and results of fractional calculus needed in this work. In Section 3 we present interesting applications of Theorems 1.4 and 1.6 enunciated above.

## 2 Preliminaries on Fractional Calculus

Let $I$ be an interval of $\mathbb{R}$ and $n \in \mathbb{N}$. Suppose that $E(I, \mathbb{R}^n)$ is a space of functions. We denote by $E_{loc}(I, \mathbb{R}^n)$ the space of functions $x : I \to \mathbb{R}^n$ such that $x \in E(I, \mathbb{R}^n)$ for every compact subinterval $J \subset I$.

We now introduce the left and right fractional integrals and derivatives used in this work.
Definition 2.1 Let $a < b$ be two real numbers and $[a, b] \subset I$. The left and right Riemann–Liouville fractional integrals of order $\alpha > 0$ of a function $f \in L^1_{loc}(I, \mathbb{R}^n)$ are defined, respectively by

$$I^\alpha_{a^+}[f](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \, ds, \quad t \geq a,$$

and

$$I^\alpha_{b^-}[f](t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) \, ds, \quad t \leq b,$$

provided that the right-hand side exists. For $\alpha = 0$ we set $I^0_{a^+}[f](t) = I^0_{b^-}[f](t) = f(t)$.

Definition 2.2 The left and right Riemann–Liouville (RL) fractional derivatives of order $0 < \alpha \leq 1$ of a function $f \in L^1_{loc}(I, \mathbb{R}^n)$ such that $I^{1-\alpha}_{a^+}[f] \in AC^\alpha_{a^+}(I, \mathbb{R}^n)$, respectively $I^{1-\alpha}_{b^-}[f] \in AC^\alpha_{b^-}(I, \mathbb{R}^n)$, are defined by

$$D^\alpha_{a^+}[f](t) = \frac{d}{dt} \left[I^{1-\alpha}_{a^+}[f]\right](t),$$

respectively,

$$D^\alpha_{b^-}[f](t) = -\frac{d}{dt} \left[I^{1-\alpha}_{b^-}[f]\right](t).$$

We denote by $AC^\alpha_{a^+}(I, \mathbb{R}^n)$, respectively $AC^\alpha_{b^-}(I, \mathbb{R}^n)$, the set of all functions $f \in L^1_{loc}(I, \mathbb{R}^n)$ possessing a left, respectively right, RL fractional derivative of order $0 < \alpha \leq 1$.

Definition 2.3 The left and right Caputo fractional derivatives of order $0 < \alpha \leq 1$ of a function $f \in C(I, \mathbb{R}^n)$ such that $f - f(a) \in AC^\alpha_{a^+}(I, \mathbb{R}^n)$, respectively $f - f(b) \in AC^\alpha_{b^-}(I, \mathbb{R}^n)$, are defined by

$$C D^\alpha_{a^+}[f](t) = D^\alpha_{a^+}[f - f(a)](t),$$

respectively,

$$C D^\alpha_{b^-}[f](t) = D^\alpha_{b^-}[f - f(b)](t).$$

We denote by $C AC^\alpha_{a^+}(I, \mathbb{R}^n)$, respectively $C AC^\alpha_{b^-}(I, \mathbb{R}^n)$, the set of all functions $f \in C(I, \mathbb{R}^n)$ possessing a left, respectively right, Caputo fractional derivative of order $0 < \alpha \leq 1$.

We introduce the two-parametric Mittag–Leffler function

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \ \beta \in \mathbb{C},$$

and we put $E_{\alpha}(z) = E_{\alpha,1}(z)$. It satisfies (cf. [18]),

$$E_{\alpha, \beta}(t) \geq 0 \quad \text{for } 0 < \alpha \leq 1, \ \beta \geq \alpha. \quad (2.1)$$

The following formulas may be found in [15].

Lemma 2.4 Suppose that $\alpha, \beta, \gamma$ are positive real numbers and $\lambda \in \mathbb{R}$. Then,

$$I^{\alpha}_{0^+}\left[\lambda^{\gamma-1} E_{\beta, \gamma}(\lambda t^\beta)\right](t) = t^{\alpha + \gamma - 1} E_{\beta, \alpha + \gamma}(\lambda t^\beta), \quad t \geq 0, \quad (2.2)$$

and

$$E_{\alpha, \beta}(t) = \frac{1}{\Gamma(\beta)} + t E_{\alpha + \beta}(t), \quad t \in \mathbb{R}. \quad (2.3)$$
The following result may be consulted in, e.g., [17, Theorem 5.15].

**Theorem 2.5** (Variation of constants formula) *The solution of the IVP*
\[
^CD_{a^+}^\alpha [x](t) = \lambda x(t) + f(t), \quad \lambda \in \mathbb{R}, \ 0 < \alpha \leq 1, \ t \geq a,
\]
\[
x(a) = x_a,
\]
can be given by
\[
x(t) = x_a E_\alpha (\lambda (t - a)^\alpha) + \int_a^t (t - s)^{\alpha - 1} E_{\alpha, \alpha} (\lambda (t - s)^\alpha) f(s) ds.
\]

We will also use the following result:

**Theorem 2.6** Let \( f : (-\infty, b] \to \mathbb{R} \) be a continuous function. Then, the solution \( x \in C((-\infty, b), \mathbb{R}) \) of the IVP
\[
D_b^{\alpha^*} [x](t) = f(t)x(t), \quad 0 < \alpha \leq 1, \ t < b,
\]
\[
l_{b^{-}}^1 [x](b) = x_b,
\]
can be given by
\[
x(t) = \frac{x_b}{\Gamma(\alpha)} \sum_{k=0}^{\infty} T^k_f [(b - s)^{\alpha - 1}](t), \quad t < b,
\]
where \( T^0_f [\phi] = \phi \) and \( T^{k+1}_f [\phi] = T_f [T^k_f \phi] \) (\( k \in \mathbb{N} \)), with \( T_f \) being the operator defined by \( T_f [\phi] = l_{b^{-}}^\alpha [f \phi] \).

**Proof** We only need to invoke the duality results of [6] and apply them to [8, Theorem 2.3].

\[ \square \]

### 3 Applications

#### 3.1 Two Direct Consequences of Theorem 1.4 and a Related Result

We start by stating a generalization of Theorem 1.4.

**Theorem 3.1** Let \( 0 < \alpha < 1 \) and \( f \in C([a, \infty) \times \mathbb{R}, \mathbb{R}) \) be such that \( f(t, 0) = 0 \) and it satisfies the Lipschitz condition (1.2). Then the solution of the IVP
\[
^CD_{a^+}^\alpha [x](t) = f(t, x(t)), \quad x(a) = x_a > 0,
\]
is positive on \([a, \infty)\).

**Proof** Just observe that the trivial solution \( x(t) = 0 \) solves the IVP \( ^CD_{a^+}^\alpha [x](t) = f(t, x(t)), \ x(a) = 0 \) on \([a, \infty)\) and apply Theorem 1.1.

\[ \square \]

**Example 3.2** Consider \( f(t, x) = g(t) \ln(x^2 + 1) \) with \( g \in C(\mathbb{R}^+_0, \mathbb{R}) \) and \( x \in \mathbb{R} \).

We have,
\[
|f(t, x) - f(t, y)| = |g(t)||\ln(x^2 + 1) - \ln(y^2 + 1)| \leq |g(t)||x - y|, \quad x, y \in \mathbb{R},
\]
where we have used the mean value theorem. Since \( f(t, 0) = 0 \), it follows from Theorem 3.1 that the solution of

\[
C D_0^\alpha [x](t) = g(t) \ln(x^2(t) + 1), \quad x(0) = 1, \quad t \geq 0,
\]
is positive.

The following result seems to be new in the literature.

**Theorem 3.3** Let \( x_a > 0 \). For \( g \in C([a, \infty], \mathbb{R}) \) and \( 0 < \alpha < 1 \) define

\[
k(t, s) = \frac{1}{\Gamma(\alpha)} (t - s)^{\alpha - 1} g(s),
\]
and the \( j \)th iterated kernel \( k_j \) for \( j = 1, 2, \ldots \) via the recurrence relation

\[
k_1(t, s) = k(t, s), \quad k_j(t, s) = \int_s^t k(t, \tau) k_{j-1}(\tau, s) d\tau, \quad j = 2, 3, \ldots
\]

Then the function

\[
x(t) = x_a \left( 1 + \int_a^t R(t, s) ds \right), \quad t \in [a, \infty),
\]

where \( R(t, s) = \sum_{j=1}^\infty k_j(t, s) \) is positive.

**Proof** This result follows from the representation for the solution of (1.3) given in [10, Theorem 7.10].

**Remark 3.4** Observe that the previous result is by no means obvious as the function \( g \) may be negative.

We end this section with a lateral but nevertheless interesting result, namely, we deduce a Bernoulli-type inequality using the theory of fractional calculus.\(^1\) The proof follows the same lines as the one in [11] (see also [12]), which was done using discrete fractional operators.

**Theorem 3.5** (Fractional Bernoulli’s inequality) Let \( \lambda \in \mathbb{R} \) and \( 0 < \alpha \leq 1 \). Then the following inequality holds:

\[
E_\alpha(\lambda t^\alpha) \geq \frac{\lambda t^\alpha}{\Gamma(\alpha + 1)} + 1, \quad t \geq 0.
\]  

**Proof** The formula trivially holds for \( t = 0 \). Let

\[
x(t) = \lambda \frac{t^\alpha}{\Gamma(\alpha + 1)}, \quad t \geq 0.
\]

We have \( x(0) = 0 \) and \( C D_0^\alpha x(t) = \lambda \) (cf. [17, Property 2.1]). Therefore,

\[
\lambda x(t) + \lambda = \lambda^2 \frac{t^\alpha}{\Gamma(\alpha + 1)} + \lambda \geq C D_0^\alpha x(t).
\]

\(^1\)We have never seen such type of result in the literature.
Define \( m(t) = \lambda x(t) + \lambda - C D_0^a x(t) \), which is nonnegative. Then, using Theorem 2.5, we get

\[
x(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)\alpha)[\lambda - m(s)] ds
\]

\[
= \lambda \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)\alpha) ds - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)\alpha) m(s) ds
\]

\[
\leq \lambda \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)\alpha) ds
\]

\[
= \lambda t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha),
\]

where we have used (2.1) and (2.2). Therefore,

\[
\lambda \frac{t^\alpha}{\Gamma(\alpha + 1)} \leq \lambda t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha),
\]

which for \( t > 0 \) is equivalent to

\[
\frac{\lambda}{\Gamma(\alpha + 1)} \leq \lambda E_{\alpha,\alpha+1}(\lambda t^\alpha),
\]

and upon using (2.3) and some rearrangements furnishes (3.1).

Remark 3.6 It is clear that, for \( \alpha = 1 \), inequality (3.1) reads as \( e^{\lambda t} \geq \lambda t + 1 \). This inequality is the continuous version of the Bernoulli inequality\(^2\) (cf. [1]), hence the name given in Theorem 3.5.

3.2 Herglotz’s Variational Problem

In this section, we will deduce necessary optimality conditions for a fractional variational problem of Herglotz type [14, 16] and, in particular, show the usefulness of Theorem 1.6. Let us first state what we mean here by the Herglotz variational problem in the classical case: Following [19, Problem \( PH \)], we consider:

\[
\begin{align*}
\text{z}(b) & \rightarrow \min \\
\text{subject to} & \quad \dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), \quad t \in [a, b], \\
x(a) = x_a, \quad z(a) = z_a, \quad x_a, z_a \in \mathbb{R}.
\end{align*}
\]

The aim is to find a couple \((x, z)\) in an appropriate space of functions that solves (3.2), i.e. that satisfy all the conditions in (3.2).

In our work we will consider the following fractional version of (3.2):

\[
\begin{align*}
\text{z}(b) & \rightarrow \min \\
\text{subject to} & \quad C D_{a+}^\alpha [z](t) = L(t, x(t), C D_{a+}^\alpha [x](t), z(t)), \quad t \in [a, b], \\
x(a) = x_a, \quad z(a) = z_a, \quad x_a, z_a \in \mathbb{R}.
\end{align*}
\]

\(^2\)That is, \((1 + x)^n \geq 1 + nx \) for \( x > -1 \) and \( n \in \mathbb{N} \).
Before proceeding let us just note that, if the function $L$ does not depend on its fourth variable, then $z$ is immediately determined and we may write (3.3) as

$$\frac{1}{\Gamma(\alpha)} \int_{a}^{b} (b-t)^{\alpha-1} \left[ L(t, x(t), C D_{a+}^{\alpha}[x](t)) + \frac{\Gamma(\alpha)z_{a}(b-t)^{1-\alpha}}{b-a} \right] dt \rightarrow \min$$

subject to $x(a) = x_{a}, \quad x_{a} \in \mathbb{R},$

in view of $I_{a+}^{\alpha}[C D_{a+}^{\alpha}[z]](t) = z(t) - z(a).$ So, if we define

$$\hat{L}(t, x, v) = L(t, x, v) + \frac{\Gamma(\alpha)z_{a}(b-t)^{1-\alpha}}{b-a},$$

we get the problem

$$\mathcal{L}(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{b} (b-t)^{\alpha-1} \hat{L}(t, x(t), C D_{a+}^{\alpha}[x](t)) dt \rightarrow \min$$

subject to $x(a) = x_{a}, \quad x_{a} \in \mathbb{R}.$

This is the basic problem of the fractional calculus of variations with fixed initial condition. The same problem with fixed initial and final conditions was recently studied in [13].

To the best of our knowledge the first work considering Herglotz-type problems involving fractional derivatives is [2]. However, the problem we consider here is different in nature from the one considered in [2] (when $0 < \alpha < 1$) as the authors considered the differential equation

$$\dot{z}(t) = L(t, x(t), C D_{a+}^{\alpha}[x](t), z(t)),$$

instead of $C D_{a+}^{\alpha}[z](t) = L(t, x(t), C D_{a+}^{\alpha}[x](t), z(t))$ considered above in (3.3). Moreover, the proofs of the necessary optimality conditions are quite different.

In this work, we will obtain first and second order necessary optimality conditions for (3.3) by using a recent result of [4], namely, the fractional version of the celebrated Pontryagin Maximum Principle (PMP). For the benefit of the reader we recall here the main result of [4].

Consider the Optimal Control Problem (OCP) of Bolza type given by

$$\varphi(x(a), x(b)) + I_{a+}^{\alpha}[F(.; x, u)](b) \rightarrow \min$$

subject to $x \in C AC_{a+}^{\alpha}([a, b], \mathbb{R}^{n}), \quad u \in L^{\infty}([a, b], \mathbb{R}^{m}),$

$C D_{a+}^{\alpha}[x](t) = f(t, x(t), u(t))$ a.e. $t \in [a, b],

g(x(a), x(b)) \in C,

u(t) \in U$ a.e. $t \in [a, b].$

A couple $(x^{*}, u^{*})$ is said to be an optimal solution to OCP if it satisfies all the above constraints and it minimizes the cost among all couples $(x, u)$ satisfying those constraints. Obviously, the functions involved in the OCP satisfy some regularity conditions, that we will skip here and refer the reader to [4, p. 8]. Under these hypothesis (regularity conditions), we have the following theorem.

**Theorem 3.7 (PMP)** **Assume that** $(x^{*}, u^{*}) \in C AC_{a+}^{\alpha}([a, b], \mathbb{R}^{n}) \times L^{\infty}([a, b], \mathbb{R}^{m})$ **is an optimal solution to the OCP. Then, there exists a nontrivial couple** $(p, p^{0})$, **where** $p \in AC_{b+}^{\beta}([a, b], \mathbb{R}^{n})$ **(called adjoint vector) and** $p^{0} \leq 0$, **such that the following conditions hold:**

---

3In the notation of [4] we are considering here $\beta = \alpha.$
(i) Fractional Hamiltonian system:
\[ CD^{\alpha}_a[x^*](t) = \partial_4 H(t, x^*(t), u^*(t), p(t), p^0), \]
\[ D^{\alpha}_b[p](t) = \partial_2 H(t, x^*(t), u^*(t), p(t), p^0), \]
for almost every \( t \in [a, b] \), where the Hamiltonian \( H : [a, b) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R} \) associated to Problem (OCP) is defined by
\[
H(t, x, u, p, p^0) = \langle p, f(t, x, u) \rangle_{\mathbb{R}^n} + p^0 (b - t)^{\alpha - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} F(t, x, u).
\]

(ii) Hamiltonian maximization condition:
\[
u^*(t) = \arg \max_{u \in U} H(t, x^*(t), u, p(t), p^0) \quad a.e. \ t \in [a, b].
\]

(iii) Transversality conditions on the adjoint vector: if in addition \( g \) is submersive\(^4\) at \((x^*(a), x^*(b))\), then the couple \((p, p^0)\) satisfy
\[
I^{1-\alpha}_b[p](b) = -p^0 \partial_2 \varphi(x^*(a), x^*(b)) + \partial_2 g(x^*(a), x^*(b))T \times \Psi,
\]
where \( \Psi \in N_C[g(x^*(a), x^*(b))] \), with \( N_C[x] = \{ z \in \mathbb{R}^j : \forall x' \in C, \langle z, x' - x \rangle_{\mathbb{R}^j} \leq 0 \} \).

A series of remarks is in order.

Remark 3.8 If \( U = \mathbb{R} \), i.e., there is no control constraint in the OCP, and the Hamiltonian is differentiable with respect to its third variable, then the maximization condition (ii) in Theorem 3.7 implies (cf. [4, Remark 3.18])
\[
\partial_3 H(t, x^*(t), u^*(t), p(t), p^0) = 0 \quad a.e. \ t \in [a, b].
\]
Moreover, if \( H \) is twice differentiable with respect to its third variable, we easily see that
\[
\partial_{33} H(t, x^*(t), u^*(t), p(t), p^0) \leq 0 \quad a.e. \ t \in [a, b]. \quad \text{(3.4)}
\]

Remark 3.9 If the initial point is fixed and if the final point is free in the OCP, then we may take (cf. [4, Remark 3.17])
\[
I^{1-\alpha}_b[p](b) = -\partial_2 \varphi(x^*(a), x^*(b)).
\]

Remark 3.10 It is mentioned in [4, p. 15] and shown in [5, Theorem 5.3] that \( p \in (C[a, b], \mathbb{R}^n) \).

It follows the main result of this section:

**Theorem 3.11** Consider the function \( L(t, x, u, z) \) in (3.3) to have continuous partial derivatives with respect to \( x, u \) and \( z \). Suppose that \((x^*, z^*) \in CAC^\alpha_{\alpha^+}([a, b], \mathbb{R}^n) \times \mathbb{R}^m \).
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\[ C^L_{\alpha^+}([a, b], \mathbb{R}), \text{ with } C^{G}_{\alpha^+}[x^*] \in C([a, b], \mathbb{R}), \text{ solves the Herglotz problem (3.3). Then} \]
\[ I^L_{b^-}\left[p\partial_2 L \left(\cdot, x^*, C^{G}_{\alpha^+}[x^*], z^*\right)\right](t) + p(t)\partial_3 L \left(t, x^*(t), C^{G}_{\alpha^+}[x^*](t), z^*(t)\right) = 0, \]
\[ \text{for all } t \in [a, b], \text{ where } p \text{ is the solution of} \]
\[ D^{G}_{b^-}[p](t) = p(t)\partial_4 L \left(t, x^*(t), C^{G}_{\alpha^+}[x^*](t), z^*(t)\right), \quad t \in [a, b], \quad I^L_{b^-}\left[p\right](b) = -1. \]

Moreover, if \( L \) is twice continuously differentiable with respect to \( u \), then the Legendre necessary optimality condition holds:
\[ \partial_{33} L \left(t, x^*(t), C^{G}_{\alpha^+}[x^*](t), z^*(t)\right) \geq 0, \quad t \in [a, b]. \]

**Proof** Suppose that \((x^*, z^*)\) is a solution of (3.3). Then, by letting \(x^1(t) = x^*(t), x^2(t) = z^*(t)\) and \(u^*(t) = C^{G}_{\alpha^+}[x^*](t)\), we conclude that \((x^1, x^2, u^*)\) solves the following OCP
\[ x^2(b) \to \min \]
subject to
\[ C^{G}_{\alpha^+}[x^1](t) = u(t), \quad t \in [a, b], \]
\[ C^{G}_{\alpha^+}[x^2](t) = L(t, x^1(t), u(t), x^2(t)), \quad t \in [a, b], \]
\[ x^1(a) = x_a, \quad x^2(a) = z_a, \quad x_a, z_a \in \mathbb{R}. \]

It follows from Theorem 3.7 and Remarks 3.8, 3.9 and 3.10 the existence of a vector \((p_1, p_2) \in C([a, b], \mathbb{R}^2)\) satisfying
\[ p_1(t) + p_2(t)\partial_3 L \left(t, x^1(t), u^*(t), x^2(t)\right) = 0 \quad \text{a.e. } t \in [a, b], \]
and
\[ D^L_{b^-}[p_1](t) = p_2(t)\partial_2 L \left(t, x^1(t), u^*(t), x^2(t)\right) \quad \text{a.e. } t \in [a, b], \quad I^L_{b^-}\left[p_1\right](b) = 0, \]
\[ D^L_{b^-}[p_2](t) = p_2(t)\partial_4 L \left(t, x^1(t), u^*(t), x^2(t)\right) \quad \text{a.e. } t \in [a, b], \quad I^L_{b^-}\left[p_2\right](b) = -1. \]

Observe that the continuity of \( p_1 \) and \( p_2 \) on \([a, b]\), together with the assumptions on \( L \) and \((x^*, u^*)\), imply that (3.7), (3.8) and (3.9) hold on \([a, b]\). Moreover, since \(p_1(t) = I^L_{b^-}\left[p_2\partial_2 L(\cdot, x^1, u^*, x^2)\right](t)\), then (3.5) follows from (3.7) immediately.

Suppose now that \( L \) is twice continuously differentiable with respect to \( u \). Then, by (3.4), we get
\[ p_2(t)\partial_{33} L \left(t, x^1(t), u^*(t), x^2(t)\right) \leq 0 \quad \text{a.e. } t \in [a, b], \]
and, upon using Theorem 1.6, Remark 1.7 and the continuity of \( p_2 \) on \([a, b]\),
\[ \partial_{33} L \left(t, x^1(t), u^*(t), x^2(t)\right) \geq 0 \quad \text{a.e. } t \in [a, b]. \]

The previous inequality holds on \([a, b]\) from the hypothesis on \( L \) and \((x^*, u^*)\). The proof is done. \( \square \)

We call (3.5) the **Euler–Lagrange equation in integral form** for the Herglotz variational problem (3.3).

**Remark 3.12** The function \( p \) of Theorem 3.11 has the representation (cf. Theorem 2.6):
\[ p(t) = -\frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} T^k[(b-s)^{\alpha-1}](t), \quad t < b, \]
where \( f(t) = \partial_4 L(t, x^*(t), C D_{a+}^\alpha[x^*](t), z^*(t)) \).

**Remark 3.13** We emphasize the importance of Theorem 1.6 in order to obtain the Legendre necessary condition (3.6). Because of it we were able to remove the dependence on the function \( p_2 \) in the inequality (3.10).

Also, suppose that \( L(t, x, u, z) \) does not depend on \( x \), i.e., \( L(t, x, u, z) = \hat{L}(t, u, z) \). Then, the Euler–Lagrange equation (3.5) becomes

\[
p(t) \partial_2^2 \hat{L}(t, C D_{a+}^\alpha[x^*](t), z^*(t)) = 0, \quad t \in [a, b).
\]

Again, we may use Theorem 1.6 and the continuity of the involved functions to conclude that

\[
\partial_2 \hat{L}(t, C D_{a+}^\alpha[x^*](t), z^*(t)) = 0, \quad t \in [a, b].
\]

If we let \( \alpha = 1 \) in Theorem 3.11, it follows the following:

**Corollary 3.14** The first and second order optimality conditions for the variational problem given by (3.2) are, respectively,

\[
\int_a^b e^{\int_t^s \partial_4 L[d\tau]} \partial_2 L[s]ds + e^{\int_t^s \partial_4 L[s]ds} \partial_3 L[t] = 0, \quad t \in [a, b],
\]

where \([s] = (s, x^*(s), \dot{x}^*(s), z^*(s))\), and

\[
\partial_{33} L(t, x^*(t), \dot{x}^*(t), z^*(t)) \geq 0, \quad t \in [a, b].
\]

**Proof** Just let \( \alpha = 1 \) in Theorem 3.11 and note that, in this case, \( p(t) = -e^{\int_t^s \partial_4 L[s]ds} \), for all \( t \in [a, b] \). 

We end this work by noting that we can obtain a differential form for (3.11). Indeed, since the integral on the left hand side of (3.11) and \( f(t) = e^{\int_t^s \partial_4 L[s]ds} > 0 \) are differentiable on \([a, b] \), then \( \partial_3 L \) is also differentiable and, hence, we easily obtain

\[
\partial_2 L(t, x^*(t), \dot{x}^*(t), z^*(t)) + \partial_4 L(t, x^*(t), \dot{x}^*(t), z^*(t)) \partial_3 L(t, x^*(t), \dot{x}^*(t), z^*(t))
- \frac{d}{dt} \partial_3 L(t, x^*(t), \dot{x}^*(t), z^*(t)) = 0, \quad t \in [a, b].
\]

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