Non-autonomous hybrid stochastic systems with delays *

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Abstract The aim of this paper is to study the dynamical behavior of non-autonomous stochastic hybrid systems with delays. By general Krylov-Bogolyubov’s method, we first obtain the sufficient conditions for the existence of an evolution system of measures of the non-autonomous stochastic system and also give some easily verifiable conditions. We then prove a sufficient condition for convergence of evolution systems of measures as the delay approaches zero. As an application of the abstract theory, we first prove the existence of evolution systems of measures for stochastic system with time-vary delays, which comes from feedback control problem based on discrete-time state observations. Furthermore, when observation interval goes to zero, we show every limit point of a sequence of evolution system of measures of the non-autonomous stochastic system must be an evolution system of measures of the limiting system.

Keywords. Non-autonomous; Markovian switching; Delay; Evolution system of measures; Limit measure.

1 Introduction

The existence of invariant measures of stochastic equations was obtained in [1, 19, 17, 16, 8]. Especially, the limiting behavior of invariant measures of stochastic delay systems was studied in [8] as delay approaches to zero. The existence of periodic measures was also obtained in [7, 2, 10] for the equations with periodic time-dependent forcings. Furthermore, in [8] the authors studied the limiting behavior of periodic measures of the stochastic equations with delays as delay goes to zero. To extended the notation of periodic measures to cover the equations with aperiodic external force, the concept of evolution system of measures was developed by [3, 14]. Recently, in [18], the

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limiting behavior of evolution system of measures of non-autonomous stochastic evolution systems was studied by Wang et. al.

We mention that hybrid stochastic differential equations (also known as stochastic differential equations with Markovian switching) have many applications in practice. For example, it has been used to model systems where they may experience abrupt changes in their structure. For hybrid stochastic differential equations without delays, the existence and asymptotic stability in distribution of invariant measures was studied in [24]. The asymptotic stability in distribution of invariant measures was also obtained in [25, 4, 20] for the hybrid stochastic differential equations with constant delays. In [6], the authors obtained the sufficient conditions of existence, stability and convergence of evolution system of measures of non-autonomous hybrid stochastic evolution systems.

This paper is concerned with the existence, stability and convergence of evolution system of measures of non-autonomous stochastic hybrid differential equations with delays. By general Krylov-Bogolyubov’s method, we first obtain the sufficient conditions for the existence of an evolution system of measures of the non-autonomous stochastic system and also give some easily verifiable conditions. For periodic Markov processes, we show the existence of periodic evolution systems of measures. We then prove a sufficient condition for convergence of evolution systems of measures as the delay approaches zero. As an application of our abstract results, we will investigate the existence, asymptotic stability in distribution, and the limiting behavior of evolution system of measures of stochastic system with time-vary delays, which comes from feedback control problem based on discrete-time state observations. When a given stochastic hybrid differential equation is not stable, Mao [11] discussed how to design a feedback control based on discrete-time state observations to stabilise the stochastic equation in the sense of the mean square exponential stability. Such a stabilisation problem has since then been studied by many authors, see, e.g., [9, 23, 12, 15, 5, 21, 22].

Throughout this paper, we let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and $X$ be a Polish space with a metric $d$. Let $\mathcal{B}(X)$ denote the family of all Borel measurable sets in $X$. If $x \in \mathbb{R}^n$, then $|x|$ is its Euclidean norm. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. Let $0 < \rho \leq 1$ and $C_\rho := C\left([-\rho, 0], \mathbb{R}^n\right)$ with the maximum norm $\|\phi\|_\rho = \sup_{-\rho \leq s \leq 0} |\phi(s)|$, $\phi \in C_\rho$. Let $W(t) = (W_1, \ldots, W_m)^T$ be an $m$-dimensional standard two-side Wiener process on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$ satisfying the usual condition and $r(t)$, $t \in \mathbb{R}$, be a right continuous irreducible Markov chain, independent of the Wiener process $W(t)$, on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$ taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator
\( \Gamma = (r_{ij})_{N \times N} \) given by

\[
P \{ r(t + \Delta) = j | r(t) = i \} = \begin{cases} 
  r_{ij} \Delta + o(\Delta), & i \neq j; \\
  1 + r_{ij} \Delta + o(\Delta), & i = j,
\end{cases}
\]

where \( \Delta > 0 \) and \( \lim_{\Delta \to 0} o(\Delta)/\Delta = 0 \). \( r_{ij} > 0 \) is the transition rate from \( i \) to \( j \) if \( i \neq j \) and \( r_{ii} = -\sum_{i \neq j} r_{ij} \). It is well known that almost every sample path of \( r(t) \) is a right-continuous step function and \( r(t) \) is ergodic.

In this paper, we study the long-term behavior of the following nonautonomous stochastic system with Markovian switching:

\[
du(t) = f(t, r(t), u(t), u(t - \rho_0(t))) \, dt + g(t, r(t), u(t), u(t - \rho_0(t))) \, dW(t), \quad t > s, \tag{1.1}
\]

with initial data

\[
u(s + \tau) = \xi(\tau), \quad -\rho \leq \tau \leq 0, \quad \text{and} \quad r(s) = j \in S, \tag{1.2}
\]

where \( s \in \mathbb{R} \), \( u(t) \in \mathbb{R}^n \) is an unknown state, \( \rho_0 : \mathbb{R} \to [0, \rho] \) is a Borel measurable function, \( \xi \in C_\rho \), and \( f : \mathbb{R} \times S \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) and \( g : \mathbb{R} \times S \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times m} \) are Borel measurable functions satisfying the following assumption.

\((A_0)\) There is a pair of positive constants \( L_f \) and \( L_g \) such that for all \( t \in \mathbb{R}, j \in S \) and \( x_i, y_i \in \mathbb{R}^n, i = 1, 2, \)

\[
|f(t, j, x_1, x_2) - f(t, j, y_1, y_2)| \leq L_f \left( |x_1 - x_2| + |y_1 - y_2| \right)
\]

and

\[
|g(t, j, x_1, x_2) - g(t, j, y_1, y_2)| \leq L_g \left( |x_1 - x_2| + |y_1 - y_2| \right),
\]

\((A_1)\) For any \( j \in S \)

\[
\sup_{t \in \mathbb{R}} |f(t, j, 0, 0)| < +\infty \quad \text{and} \quad \sup_{t \in \mathbb{R}} |g(t, j, 0, 0)| < +\infty.
\]

It is well know (see, e.g., [13]) that under Condition \((A_0)\) and \((A_1)\), we can show that for any \( \xi \in C_\rho \) and \( r(s) = j \in S \), system \((1.1)-(1.2)\) has a unique solution, which is written as \( u(t) \). To highlight the initial values, we let \( r_{s, j}(t) \) be the Markov chain starting from state \( i \in S \) at \( t = s \) and denote by \( u(t, s, \xi, j) \) the solution of Eq. \((1.1)-(1.2)\) with initial conditions \( u(s, s, \xi, j) = \xi \) and \( r(s) = j \). Moreover, for any bounded subset \( B \) of \( \mathbb{R}^n \),

\[
\sup_{(\xi, j) \in B \times S} \mathbb{E} \left[ \sup_{s \leq \tau \leq t} |u(\tau, s, \xi, j)|^2 \right] < \infty \quad \forall t > s.
\]
Recall that $u_t(s, \xi, j)$ is the segment of the solution $u(t, s, \xi, j)$ given by

$$u_t(s, \xi, j)(\tau) = u(t + \tau, s, \xi, j), \quad \text{for all } \tau \in [-\rho, 0].$$

Notice that $u_t(s, \xi, j) \in L^2(\Omega, C_{\rho})$ for all $t \geq s$.

The rest of this paper is organized as follows. Section 2 is devoted to the existence and periodicity of evolution system of measures of (1.1)-(1.2). In Section 3, we show the limiting behavior of evolution system of measures of time nonhomogeneous Markov processes as delay goes to zero. As an application, Section 4 is devoted to the existence, stability, periodicity and limiting behavior of evolution system of measures on $C_{\rho} \times S$ for a controlled problem.

2 Existence

Define $H = C_{\rho} \times S$ with distance $\|x_1 - x_2\|_H = \|\xi_1 - \xi_2\|_{\rho} + |j_1 - j_2|$, for $x_i = (\xi_i, j_i) \in H$, $i = 1, 2$, and $C_b(H)$ as the space of bounded continuous functions $\chi : H \to \mathbb{R}$ endowed with the norm

$$\|\chi\|_{\infty} = \sup_{x \in H} |\chi(x)|,$$

and denote by $L_b(H)$ the space of bounded Lipschitz functions on $H$. That is, of functions $\chi \in C_b(H)$ for which

$$\text{Lip}(\chi) := \sup_{x_1, x_2 \in H} \frac{|\chi(x_1) - \chi(x_2)|}{\|x_1 - x_2\|_H} < \infty.$$  

The space $L_b(H)$ is endowed with the norm

$$\|\chi\|_L = \|\chi\|_{\infty} + \text{Lip}(\chi).$$

Let us denote by $\mathcal{P}(H)$ the set of probability measures on $(H, \mathcal{B}(H))$. Define a metric on $\mathcal{P}(H)$ by

$$d^*_L(\mu_1, \mu_2) = \sup_{\|\chi\|_L \leq 1} |(\chi, \mu_1) - (\chi, \mu_2)|, \quad \mu_1, \mu_2 \in \mathcal{P}(H).$$

Let $y(t, s, \xi, j)$ denote the $H$-valued process $(u_t(s, \xi, j), r_{s,j}(t))$. $y(t, s, \xi, j)$ be a time nonhomogeneous Markov process. Let $p(t, s, \xi, j, (dy, k))$ denote the transition probability of the process $y(t, s, \xi, j)$. For $A \subset \mathcal{B}(C_{\rho})$ and $B \subset S$, let $P(t, s, \xi, j, A \times B)$ denote the probability of event $\{y(t, s, \xi, j) \in A \times B\}$ given initial condition $y(s, s, \xi, j) = (\xi, j)$ at time $t = s$, i.e.,

$$P(t, s, \xi, j, A \times B) = \int_{A \times B} p(t, s, \xi, j, (dy, k)).$$
where $\int_{A \times B} p(t, s, \xi, j, (dy, k)) = \sum_{k \in B} \int_A p(t, s, \xi, j, (dy, k))$. We define the transition evolution operator

$$P_{s,t} \varphi(\xi, j) = \mathbb{E}[\varphi(y(t, s, \xi, j))], \quad \varphi \in C_b(H).$$

It is easy to verify that $P_{s,t}$ is Feller, that is, $P_{s,t} : C_b(H) \to C_b(H)$, for $s < t$. Denote by $P^*_s : \mathcal{P}(H) \to \mathcal{P}(H)$ the duality operator of $P_{s,t}$. For $(\xi, j) \in H$, denote by $\delta_{\xi,j}$ the Dirac measure concentrating on $(\xi, j)$.

In this section, we show existence of an evolution system of measures $(\mu_t)_{t \in \mathbb{R}}$ indexed by $\mathbb{R}$. An evolution system of measures $(\mu_t)_{t \in \mathbb{R}}$ satisfies each $\mu_t, t \in \mathbb{R}$, is a probability measure on $H$ and

$$\int_H P_{s,t} \varphi(\xi, j) \mu_s(d\xi, j) = \int_H \varphi(\xi, j) \mu_t(d\xi, j), \quad \forall \varphi \in C_b(H), \quad s < t.$$

For $\omega > 0$, the evolution system of measures $\mu_t, t \in \mathbb{R}$, is $\omega$-periodic, if

$$\mu_t = \mu_{t+\omega}, \quad \forall t \in \mathbb{R}.$$

We will apply general Krylov-Bogolyubov’s method to prove the existence of evolution system of measures of (1.1)-(1.2). To that end, fixed $(\xi, j) \in H$, for each $n \in \mathbb{N}$ and $t \geq -n + \rho$, define a probability measure $\mu_{n,t} \in \mathcal{P}(H)$ by

$$\mu_{n,t} = \frac{1}{t - \rho + n} \int_{-n}^{t-\rho} P(t, \tau, \xi, j, \cdot \times \cdot )d\tau. \quad (2.1)$$

**Lemma 2.1.** Suppose for each $t \in \mathbb{R}$ the sequence $\{\mu_{n,t}\}_{n=1}^\infty$ is tight on $H$. Then (1.1)-(1.2) has an evolution system of measures.

**Proof.** The proof is similar to that of Theorem 3.1 in [14]. For the convenience of readers, we still prove it in detail. By the Prokhorov theorem there exists a probability measure $\mu_t$ on $H$ and a subsequence (which is still denoted by $\{\mu_{n,t}\}_{n=1}^\infty$) such that

$$\mu_{n,t} \to \mu_t, \quad \text{as} \quad n \to \infty. \quad (2.2)$$

Let $t \in \mathbb{R}$ and choose $s \leq t$. Define

$$\nu_t := P^*_{s,t} \mu_s.$$

Note that if this definition is indeed independent of $s$, $\nu_t = \mu_t$ and $\{\mu_t\}_{t \in \mathbb{R}}$ is an evolution system
of measures of (1.1)-(1.2). By the Feller property of $P_{s,t}$ we have for every $\varphi \in C_b(H)$

$$(\varphi, \nu_t) = (P_{s,t} \varphi, \mu_s) = \lim_{n \to \infty} (P_{s,t} \varphi, \mu_{n,s})$$

$$= \lim_{n \to \infty} \frac{1}{s - \rho + n} \int_{-n}^{s-\rho} P_{\tau, s} (P_{s,t} \varphi) (\xi, j) d\tau$$

$$= \lim_{n \to \infty} \frac{1}{t - \rho + n} \int_{-n}^{t-\rho} P_{\tau, t} \varphi (\xi, j) d\tau$$

$$= \lim_{n \to \infty} \frac{1}{t - \rho + n} \left( \int_{-n}^{t-\rho} P_{\tau, t} \varphi (\xi, j) d\tau - \int_{-n}^{s-\rho} P_{\tau, s} (P_{s,t} \varphi) (\xi, j) d\tau \right),$$

which is obviously independent of $s, s \leq t$. This completes the proof. 

**Corollary 2.2.** Suppose there exists a pair of $(\xi, j) \in H$ such that for each $t \in \mathbb{R}$ the laws of the process $\{u_t(s, \xi, j)\}_{s + \rho \leq t}$ is tight on $C_\rho$. Then (1.1)-(1.2) has an evolution system of measures.

**Proof.** $\{u_t(s, \xi, j)\}_{s + \rho \leq t}$ on $C_\rho$ is tight implies that the sequence $\{\mu_{n,t}\}_{n=1}^\infty$ defined in (2.1) is tight on $H$. Combining this fact and Lemma 2.1, the proof is completed. 

**Corollary 2.3.** Suppose there exists a pair of $(\xi, j) \in H$ such that for each $t \in \mathbb{R}, u_t(s, \xi, j)$ satisfies

$(A_1)$ for any $\delta > 0$, there exists a positive constant $R = R(\delta, \xi, j, t), \text{ independent of } s$, such that for $t \geq s$

$$P \{\|u_t(s, \xi, j)\|_\rho \leq R\} > 1 - \delta$$

and

$(A_2)$ for any $\delta_1, \delta_2 > 0$ and $s + \rho \leq t$, there exists $0 < \eta = (\delta_1, \delta_2, \xi, j) < \rho$, independent of $s$, such that

$$P \left\{ \sup_{t_2 - t_1 \leq n, \tau - \rho \leq t_1 \leq t_2 \leq t} |u(t_2, s, \xi, j) - u(t_1, s, \xi, j)| \leq \delta_1 \right\} > 1 - \delta_2.$$

Then (1.1)-(1.2) has an evolution system of measures.

**Proof.** By Corollary 2.2 it is sufficient to show $\{u_t(s, \xi, j)\}_{s + \rho \leq t}$ is tight on $C_\rho$. By $(A_1)$, we infer that for every $\delta > 0$, there exists $R = R(\delta, \xi, j, t) > 0$ such that for $s + \rho \leq t$

$$P \left\{ \|u_t(s, \xi, j)\|_\rho \leq R \right\} > 1 - \frac{1}{3} \delta.$$ (2.3)
By $(A_2)$, one can verify that given $\delta > 0$, for any $\delta^* > 0$, there exists $0 < \eta = \eta(\delta, \delta^*, \xi, j) < \rho$ such that for all $s \leq t - \rho$,

$$ P \left\{ \sup_{\tau_2 - \tau_1 < \eta - \rho \leq \tau_1 \leq \tau_2 \leq 0} |u(t + \tau_2, s, \xi, j) - u(t + \tau_1, s, \xi, j)| \leq \delta^* \right\} > 1 - \frac{1}{3}\delta. \tag{2.4} $$

Given $\delta > 0$, set

$$ Y_{1,\delta} = \left\{ v \in C_\rho : \|v\|_\rho \leq R \right\}, $$

$$ Y_{2,\delta} = \left\{ v \in C_\rho : \text{for any } \delta^* > 0, \text{ there exists a } \eta = \eta(\delta^*) > 0 \text{ such that} \right\} $$

$$ \sup_{\tau_2 - \tau_1 \leq \eta - \rho \leq \tau_1 \leq \tau_2 \leq 0} |v(\tau_2) - v(\tau_1)| \leq \delta^* \right\}, $$

and

$$ Y_\delta = Y_{1,\delta} \cap Y_{2,\delta}. $$

By (2.3) and (2.4) we get, for all $s + \rho \leq t$,

$$ P \left( \{u(t, s, \xi, j) \in Y_\delta\} \right) > 1 - \delta. $$

By the Arzela-Ascoli theorem $Y_\delta$ is a precompact subset of $C_\rho$. \qed

**Lemma 2.4.** Suppose for each $t \in \mathbb{R}$ the sequence $\{\mu_{n,t}\}_{n=1}^\infty$ is tight on $H$ and the $y(t, s, \xi, j)$ are \( \omega \)-periodic Markov processes. Then (1.1)-(1.2) has a \( \omega \)-periodic evolution system of measures.

**Proof.** It follows from periodic property of $y(t, s, \xi, j)$ that

$$ \mu_{n,t+\omega} = \frac{1}{t - \rho + n + \omega} \int_{t-n}^{t-\rho+\omega} P(t, s, \xi, j, \cdot \times \cdot)ds $$

$$ = \frac{1}{t - \rho + n + \omega} \int_{t-n-\omega}^{t-\rho} P(t, s+\omega, \xi, j, \cdot \times \cdot)ds $$

$$ = \frac{1}{t - \rho + n + \omega} \int_{t-n-\omega}^{t-\rho} P(t, s, \xi, j, \cdot \times \cdot)ds, $$

which means that for any subsequence $\{\mu_{n_k,t}\}_{k=1}^\infty$ of $\{\mu_{n,t}\}_{n=1}^\infty$, if $\lim_{k\to\infty} \mu_{n_k,t} = \mu_t$, then $\lim_{k\to\infty} \mu_{n_k,t+\omega} = \mu_{t+\omega} = \mu_t$. Following the proof process of Lemma 2.1, (1.1)-(1.2) has a \( \omega \)-periodic evolution system of measures. The proof is completed. \qed
3 Limits of evolution system of measures

In this section, we discuss the limiting behavior of evolution system of measures of problem (1.1) as \( \rho \rightarrow 0 \) where \( \rho \) is the length of delay in (1.1). We first prove an abstract theorem to guarantee any limiting point of evolution system of measures is still evolution system of measures.

Suppose for every \( \rho \in (0, 1], \psi \in C([-\rho, 0], X) \) and \( j \in S \), \( \{Z^\rho(t, s, \psi, j), t \geq s\} \) is a stochastic process in the state space \( C([-\rho, 0], X) \) with initial value \( Z^\rho(s, s, \psi, j) = \psi \) and \( r(s) = j \) at initial time \( s \). Similarly, assume for every \( z \in X \) and \( j \in S \), \( \{Z^0(t, s, z, j), t \geq s\} \) is a stochastic process in the state space \( X \) with initial value \( z \) at initial time \( Z^0(s, s, z, j) = z \) and \( j(s) = j \). Denote \( \mathcal{H}_\rho = (C([-\rho, 0], X) \times S) \) and \( \mathcal{H}_0 = X \times S \). Let \( Z^\rho(t, s, \xi, j), \rho \in [0, 1] \), denote the \( \mathcal{H}_\rho \)-valued process \( (Z^\rho(t, s, \xi, j), r_s(t)) \). \( Z^\rho(t, s, \xi, j), \rho \in [0, 1] \), are time nonhomogeneous Markov process and its probability transition operators are Feller.

Given \( \rho \in (0, 1] \), define an operator \( T_\rho : \mathcal{H}_\rho \rightarrow \mathcal{H}_0 \) by \( T_\rho(\psi, j) = (\psi(0), j) \) for \( (\psi, j) \in \mathcal{H}_\rho \), and \( T_\rho : \mathcal{H}_1 \rightarrow \mathcal{H}_\rho \) by \( T_\rho(\psi, j) = (\phi, j) \) for \( (\psi, j) \in \mathcal{H}_1 \) with \( \psi(s) = \phi(s) \) and \( s \in [-\rho, 0] \). In other words, \( T_\rho \) is a restriction operator from \( \mathcal{H}_1 \) to \( \mathcal{H}_\rho \).

Given \( D \subseteq C([-\rho, 0], X) \), we write \( T_\rho D = \{T_\rho(\psi, j) : \psi \in D, j \in S\} \). Since \( T_\rho \) is continuous, if \( D \) is compact, then so is \( T_\rho D \). Similarly, given \( D_1 \subseteq C([-1, 0], X) \), we write \( T_\rho D_1 = \{T_\rho(\psi, j) : \psi \in D_1, j \in S\} \). Note that if \( D_1 \) is compact, then so is \( T_\rho D_1 \).

Throughout this section, we assume that for every compact set \( K \subseteq C([-1, 0], X) \), \( t \geq s \) and \( \eta > 0 \),

\[
\lim_{\rho \rightarrow 0} \sup_{(\psi, j) \in T_\rho K} P \left( d \left( Z^\rho(t, s, \psi, j)(0), Z^0(t, s, \psi(0), j) \right) \geq \eta \right) = 0. \tag{3.1}
\]

**Theorem 3.1.** Assume (3.1) holds true and \( \rho_n \in (0, 1] \). Let \( \{\mu_t^{\rho_n}\}_{t \in \mathbb{R}} \) be an evolution system of measures of \( Z^{\rho_n} \) in \( \mathcal{H}_{\rho_n} \) for all \( n \in \mathbb{N} \) and \( \{\mu_t\}_{t \in \mathbb{R}} \) be a family of probability measures on \( \mathcal{H}_0 \). Suppose for each \( t \in \mathbb{R} \) \( \{\mu_t^{\rho_n}\}_{n=1}^\infty \) is tight in the sense that for every \( \epsilon > 0 \), there exists a compact set \( K_1 \subseteq C([-1, 0], X) \) such that

\[
\mu_t^{\rho_n}(T_{\rho_n} K_1) > 1 - \epsilon \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.2}
\]

Then we have:

(i) The sequence \( \{\mu_t^{\rho_n} \circ T_{\rho_n}^{-1}\}_{n=1}^\infty \) is tight on \( \mathcal{H}_0 \).

(ii) If \( \rho_n \rightarrow 0 \) and \( \mu_t \) is a probability measure in \( \mathcal{H}_0 \) such that \( \mu_t^{\rho_n} \circ T_{\rho_n}^{-1} \rightarrow \mu_t \) weakly, then \( \{\mu_t\}_{t \in \mathbb{R}} \) must be an evolution system of measures of \( Z^0 \).
Proof. (i). Given $t \in \mathbb{R}$ and $\epsilon > 0$, let $K_1 \subseteq C([-1, 0], X)$ be the compact set satisfying (3.2). Denote by $K_0 = \{\psi(0) : \psi \in K_1\}$. Then $K_0 \times S$ is a compact subset of $\mathcal{H}_0$ and for all $n \in \mathbb{N}$,

$$
\mu_t^{\rho_n} \circ T_{\rho_n}^{-1}(K_0 \times S) \geq \mu_t^{\rho_n}(T_{\rho_n} K_1) > 1 - \epsilon, \tag{3.3}
$$

which shows that $\{\mu_t^{\rho_n} \circ T_{\rho_n}^{-1}\}$ is tight.

(ii). We only need to verify that for all $\varphi \in L_b(\mathcal{H}_0)$ and $s \leq t$,

$$
\int_{\mathcal{H}_0} \mathbb{E}\varphi(Z^0(t, s, z, j)) \mu_s(dz, j) = \int_{\mathcal{H}_0} \varphi(z, j) \mu_t(dz, j). \tag{3.4}
$$

Notice that

$$
\int_{\mathcal{H}_0} \varphi(z, j) \mu_t^{\rho_n} \circ T_{\rho_n}^{-1}(dz, j) = \int_{\mathcal{H}_{\rho_n}} \varphi(T_{\rho_n}(\psi, j)) \mu_t^{\rho_n}(d\psi, j)
= \int_{\mathcal{H}_{\rho_n}} \mathbb{E}\varphi(T_{\rho_n}(Z^{\rho_n}(t, s, \psi, j), r_{s, j}(t)))) \mu_t^{\rho_n}(d\psi, j)
= \int_{\mathcal{H}_{\rho_n}} \mathbb{E}\varphi(Z^{\rho_n}(t, s, \psi, j)(0), r_{s, j}(t)) \mu_t^{\rho_n}(d\psi, j)
$$

which together with (3.3) yields that

$$
\begin{align*}
&\int_{\mathcal{H}_0} \mathbb{E}g(Z^0(t, s, z, j)) \mu_t^{\rho_n} \circ T_{\rho_n}^{-1}(dz, j) - \int_{\mathcal{H}_0} \varphi(z, j) \mu_t^{\rho_n} \circ T_{\rho_n}^{-1}(dz, j) \\
&\leq \int_{\mathcal{H}_{\rho_n}} \mathbb{E}|\varphi(Z^0(t, s, \psi(0), j), r_{s, j}(t)) - \varphi(Z^{\rho_n}(t, s, \psi, j)(0), r_{s, j}(t))| \mu_t^{\rho_n}(d\psi, j) \\
&\leq \int_{T_{\rho_n} K_1} \mathbb{E}|\varphi(Z^0(t, s, \psi(0), j), r_{s, j}(t)) - \varphi(Z^{\rho_n}(t, s, \psi, j)(0), r_{s, j}(t))| \mu_t^{\rho_n}(d\psi, j) \\
&+ \int_{\mathcal{H}_{\rho_n} \setminus T_{\rho_n} K_1} \mathbb{E}|\varphi(Z^0(t, s, \psi(0), j), r_{s, j}(t)) - \varphi(Z^{\rho_n}(t, s, \psi, j)(0), r_{s, j}(t))| \mu_t^{\rho_n}(d\psi, j) \\
&\leq \int_{T_{\rho_n} K_1} \mathbb{E}|\varphi(Z^0(t, s, \psi(0), j), r_{s, j}(t)) - \varphi(Z^{\rho_n}(t, s, \psi, j)(0), r_{s, j}(t))| \mu_t^{\rho_n}(d\psi, j) + 2\epsilon \sup_{x \in \mathcal{H}_0} |\varphi(x)|.
\end{align*}
\tag{3.5}
$$

Since $\varphi \in L_b(\mathcal{H}_0)$, given $\epsilon > 0$, there exists $\eta > 0$ such that $|\varphi(y, j) - \varphi(z, j)| < \epsilon$ if $d(y, z) < \eta$ and
where \( j \in S \). Then we get

\[
\int_{T_{\rho n}K_1} \mathbb{E} \left| \varphi(Z^0(t,s,\psi(0),j), r_{s,j}(t)) - \varphi(Z^{\rho n}(t,s,\psi,j)(0), r_{s,j}(t)) \right| \mu^n_s (d\psi, j)
\]

\[
= \int_{T_{\rho n}K_1} \left( \int_Y |\varphi(Z^0(t,s,\psi(0),j), r_{s,j}(t)) - \varphi(X^{\rho n}(t,s,\psi,j)(0), r_{s,j}(t))| P(d\omega) \right) \mu^n_s (d\psi, j)
\]

\[
+ \int_{T_{\rho n}K_1} \left( \int_{Y_C} |\varphi(Z^0(t,s,\psi(0),j), r_{s,j}(t)) - \varphi(Z^{\rho n}(t,s,\psi,j)(0), r_{s,j}(t))| P(d\omega) \right) \mu^n_s (d\psi, j)
\]

\[
\leq 2 \sup_{x \in \mathcal{H}_0} |\varphi(x)| \sup_{(\psi,j) \in T_{\rho n}K_1} P \left( \{|d(Z^{\rho n}(t,s,\psi,j)(0), Z^0(t,s,\psi(0),j))| \geq \eta\} \right) + \epsilon,
\]

(3.6)

where \( Y = \{ \omega \in \Omega : d(Z^{\rho n}(t,s,\psi,j)(0), Z^0(t,s,\psi(0),j)) \geq \eta\} \) and \( Y^C = \Omega - Y^C \). It follows from (3.1) and (3.5)–(3.6) that

\[
\lim_{n \to \infty} \left| \int_{\mathcal{H}_0} \mathbb{E} g(Z^0(t,s,z,j)) \mu^n_s \circ T_{\rho n}^{-1} (dz, j) - \int_{\mathcal{H}_0} \varphi(z,j) \mu^n_s \circ T_{\rho n}^{-1} (dz, j) \right|
\]

\[
\leq \epsilon + 2 \epsilon \sup_{x \in \mathcal{H}_0} |\varphi(x)|.
\]

(3.7)

Since \( \epsilon > 0 \) is arbitrary and \( \mu^n_s \circ T_{\rho n}^{-1} \to \mu_t \) weakly, we get (3.4) from (3.7). By (3.4) we know that \( \{\mu_t\}_{t \in \mathbb{R}} \) is an evolution system of measures of \( Z^0 \).

\[\square\]

4 Application

In this section, we apply the abstract results obtained in Section 2 and 3 to a hybrid stochastic differential equation with delays, which comes from a control problem.

In [9], Li et. al. investigated how to design a feedback control based on discrete-time state observations to stabilise a given unstable hybrid stochastic differential equation in the sense of asymptotic stability in distribution. The specific description of the problem is as follows. Consider an unstable hybrid stochastic differential equation

\[
du(t) = h(r(t), u(t)) dt + \sigma(r(t), u(t)) dW(t), \quad t > s,
\]

(4.1)

where \( s \in \mathbb{R} \) and \( u(t) \in \mathbb{R}^n \) is the state. The aim is to design a linear feedback control \( A(r(t)) u([t/\rho]) \) in the drift part so that the controlled system

\[
du(t) = (h(r(t), u(t)) + A(r(t)) u([t/\rho]) dt + \sigma(r(t), u(t)) dW(t), \quad t > s,
\]

(4.2)

has an evolution system of measures \( \{\mu_t\}_{t \in \mathbb{R}} \), which is asymptotically stable in distribution (defined later). Here \( A(j) \in \mathbb{R}^{n \times n} \), for \( j \in S \), \( 0 < \rho \leq 1 \) is a constant and \([t/\rho]\) is the integer of \( t/\rho \). Define

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\[ \rho_0 : \mathbb{R} \to [0, \rho] \] by for \( k \in \mathbb{Z} \)

\[ \rho_0 (t) = t - kp, \quad kp \leq t < (k + 1) \rho. \]

Then Eq. (4.2) can be written as

\[ du (t) = (h (r (t) , u (t)) + A (r (t)) u (t - \rho_0 (t))) \, dt + \sigma (r (t) , u (t)) \, dW (t), \quad t > s, \quad (4.3) \]

We assume \( h : S \times \mathbb{R}^n \to \mathbb{R}^n \) and \( \sigma : S \times \mathbb{R}^n \to \mathbb{R}^{n \times m} \) is globally Lipschitz in the second variable uniformly with respect to \( j \in S \). Denote \( f (r (t) , u (t)) = h (r (t) , u (t)) + A (r (t)) u (t - \rho_0 (t)) \) and \( g (r (t) , u (t)) = \sigma (r (t) , u (t)) \). Then the functions \( f, g \) satisfy assumptions \((A_0)\) and \((A_1)\). It is well known that under Condition \((A_0)\) and \((A_1)\), we can show that for any \( \xi \in C_\rho \) and \( r (s) = j \in S \), Eq. (4.3) has a unique solution \( u^\rho (t , s , \xi , j) \). Let \( y^\rho (t , s , \xi , j) \) denote the \( H \)-valued process \((u^\rho (s , \xi , j) , r_{s,j} (t)) \). Since \( \rho_0 (t) \) is \( \rho \)-periodic, \( y^\rho (t , s , \xi , j) \) is a time \( \rho \)-periodic Markov process.

We now recall the definition of asymptotic stability in distribution of the evolution system of measures.

**Definition 4.1.** The evolution system of measures \((\mu_t)_{t \in \mathbb{R}}\) of Eq. (4.3) is said to be asymptotic stability in distribution if for any \( \varphi \in C_b (H) \),

\[ \lim_{t \to +\infty} \left[ P_{s,t} \varphi (\xi , j) - \int_H \varphi (x , j) \mu_t (dx , j) \right] = 0, \quad \forall s \in \mathbb{R}, \quad (\xi , j) \in H. \]

In the sequence, let us assume

\( (A_2) \) There exists a positive number \( \beta \) and symmetric positive definite matrices \( Q_j (j \in S) \) such that

\[
2 (x - y) Q_j \left[ f (j , x , x) - f (j , y , y) \right] \\
+ \text{trace} \left[ (g (j , x) - g (j , y))^T Q_j (g (j , x) - g (j , y)) \right] \\
+ \sum_{i=1}^N \gamma_{ji} (x - y) Q_i (x - y) \leq -\beta \| x - y \|^2
\]

for all \((j , x , y) \in S \times \mathbb{R}^n \times \mathbb{R}^n \).

The main idea presented in \[9\] is as follows:

(i) Design a feedback control based on continuous-time state observations \( A (r (t)) u (t) \) in the drift term to stabilise the unstable hybrid stochastic differential equation (4.1). A controlled system is obtained:

\[ du^0 (t) = \left( h (r (t) , u^0 (t)) + A (r (t)) u^0 (t) \right) dt + \sigma (r (t) , u^0 (t)) \, dW (t), \quad t > s. \quad (4.4) \]
(ii) Show that when the observation interval \( \rho \) is sufficiently small, dynamical behaviors of the solutions of Eq. (4.2) and Eq. (4.4) have similar properties.

Repeating the scheme used in the proof Lemma 3.4, 3.5 and 3.6 in [9], we get the following lemmas in turn.

**Lemma 4.2.** Suppose \((A_0)-(A_1)\) hold. Then for every compact set \( K \subseteq C([-1,0],\mathbb{R}^n) \), \( t > s \) and \( \eta > 0 \),

\[
\lim_{\rho \to 0} \sup_{(\xi,j) \in T_\rho K} P \left( |u_\rho^0(t,s,\xi,j) - u_\rho^0(t,s,\xi(0),j)| \geq \eta \right) = 0.
\]

**Lemma 4.3.** Suppose \((A_0)-(A_2)\) hold. There exists a small enough \( \rho^* > 0 \), if \( 0 < \rho \leq \rho^* \), then for any \( \rho \in \mathbb{R}, (\xi,j) \in H \) any \( \eta > 0 \), there exists a positive constant \( R = R(\eta,\xi,j) \), independent of \( s \) and \( \rho \), such that for any \( t \geq s \) and \( 0 < \rho \leq \rho^* \),

\[
P \left\{ \|u_\rho^0(s,\xi,j)\| \leq R \right\} > 1 - \delta
\]

**Lemma 4.4.** Suppose \((A_0)-(A_2)\) hold. There exists a small enough \( \rho^* > 0 \), if \( 0 < \rho \leq \rho^* \), then for any \( \rho \in \mathbb{R}, \eta > 0 \) and bounded subset \( B \) of \( C_\rho \), there exists a \( T = T(\eta,B) \), independent of \( s \) and \( \rho \), such that for \( (\xi_1,\xi_2,j) \in B \times B \times S \) and \( 0 < \rho \leq \rho^* \),

\[
P \left\{ \| (u_\rho^0(s,\xi_1,j) - u_\rho^0(s,\xi_2,j)) \|_{\rho} < \eta \right\} \geq 1 - \eta, \quad \forall t \geq s + T.
\]

By the similar argument as that of Lemma 2.3 in [3], one can easily verify that

**Lemma 4.5.** Suppose \((A_0)-(A_2)\) hold. There exists a small enough \( \rho^* > 0 \), if \( 0 < \rho \leq \rho^* \), then for any \( (\xi,j) \in S, \delta_1,\delta_2 > 0 \) and \( s + \rho \leq t \), there exists \( 0 < \eta = (\delta_1,\delta_2,\xi,j) < \rho \), independent of \( s \), such that for \( 0 < \rho \leq \rho^* \),

\[
P \left\{ \sup_{t_2 - t_1 \leq \eta} \left| u_\rho^0(t_2,s,\xi,j) - u_\rho^0(t_1,s,\xi,j) \right| \leq \delta_1 \right\} > 1 - \delta_2.
\]

**Theorem 4.6.** Suppose \((A_0)-(A_1)\) hold. There exists a small enough \( \rho^* > 0 \), if \( 0 < \rho \leq \rho^* \), then \( (4.3) \) has a \( \rho \)-periodic evolution system of measures.

**Proof.** By Lemma 4.3 and 4.5, we get the result from Corollary 2.3 and Lemma 2.4 immediately. \( \blacksquare \)

Moreover, the following results gives information on the asymptotic stability in distribution of the \( \rho \)-periodic evolution system of measures.
Theorem 4.7. Suppose \((A_0)-(A_2)\) hold. There exists a small enough \(\rho^* > 0\), if \(0 < \rho \leq \rho^*\), then Eq. \((4.3)\) has a unique \(\rho\)-periodic evolution system of measures \(\{\mu_t\}_{t \in \mathbb{R}}\), which is asymptotic stability in distribution, i.e., for any \((\xi, j) \in H\)

\[
\lim_{t \to +\infty} d^*_L \left( P_{s,t}^* \delta_{\xi,j}, \mu_t \right) = 0, \quad \forall s \in \mathbb{R}.
\] (4.5)

Proof. By Lemma 4.3 and 4.4, we get (4.5) from Theorem 2.10 in [6] immediately.

Remark 4.8. In [9], the authors also investigated the asymptotic stability in distribution of the solutions of Eq. \((4.3)\). However, it is only proved that the discrete time points is asymptotic stability in distribution, i.e.,

\[
\lim_{n \to +\infty} d^*_L \left( P^*_{0,n\rho} \delta_{\xi,j}, \mu_{n\rho} \right) = 0.
\] (4.6)

Moreover, if the intervals are not equal, the controlled system \((4.3)\) is not periodic. By Theorem 2.10 in [6], we can get Eq. \((4.3)\) has a unique evolution system of measures \(\{\mu_t\}_{t \in \mathbb{R}}\), which is asymptotic stability in distribution.

It is well known (see, e.g., [24]) that under Conditions \((A_0)-(A_2)\), Eq. \((4.4)\) has a unique invariant measure \(\mu^0\), which is asymptotic stability in distribution.

Theorem 4.9. Suppose \((A_0)-(A_2)\) hold and \(\rho_n \to 0\). If \(\{\mu^\rho_n\}_{t \in \mathbb{R}}\) is the unique \(\rho_n\)-periodic evolution system of measures of problem \((4.3)\) with \(\rho\) replaced by \(\rho_n\) and \(\mu^0\) is the unique invariant measures of problem \((4.4)\), then for each \(t \in \mathbb{R}\), \(\mu^\rho_n \to \mu^0\) weakly.

Proof. (i). Since all uniform estimates given in Lemma 4.3 and Lemma 4.5 are uniform with respect to \(\rho_n \in (0, 1]\), by the arguments of Corollary 2.3 one can easily check that for each \(t \in \mathbb{R}\) the set

\[
\bigcup_{\rho \in (0, 1]} \mu^\rho_n
\]

is tight in the sense defined in Theorem 3.1.

(ii). By (i) we know that \(\{\mu^\rho_n\}\) is tight, and hence by Theorem 3.1 and Lemma 4.2 we infer that the sequence \(\{\mu^\rho_n \circ T_{\rho_n}^{-1}\}_{n=1}^\infty\) is also tight on \(H\). Consequently, there exists a subsequence \(\rho_{n_k}\) and a probability measure \(\mu^*_t\) such that \(\mu^\rho_{n_k} \circ T_{\rho_{n_k}}^{-1} \to \mu^*_t\) weakly. By Theorem 3.1 and Lemma 4.2 again, we find that \(\{\mu^*_t\}_{t \in \mathbb{R}}\) is the unique evolution system of measures of \((4.4)\), which coincides with the unique invariant measure.

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