Calculus of archimedean Rankin–Selberg integrals with recurrence relations

Taku Ishii,∗ and Tadashi Miyazaki†

June 9, 2020

Abstract
Let $n$ and $n'$ be positive integers such that $n - n' \in \{0, 1\}$. Let $F$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $K_n$ and $K_{n'}$ be maximal compact subgroups of $GL(n, F)$ and $GL(n', F)$, respectively. We give the explicit descriptions of archimedean Rankin–Selberg integrals at the minimal $K_n$- and $K_{n'}$-types for pairs of principal series representations of $GL(n, F)$ and $GL(n', F)$, using their recurrence relations. Our results for $F = \mathbb{C}$ can be applied to the arithmetic study of critical values of automorphic $L$-functions.

1 Introduction
The theory of automorphic $L$-functions via integral representations has its origin in the work of Hecke [7] for $GL(2)$, and the works of Rankin [22], Selberg [23] for $GL(2) \times GL(2)$. As a direct outgrowth of their works, the theory of Rankin–Selberg integrals for $GL(n) \times GL(n')$ were developed by Jacquet, Piatetski-Shapiro, and Shalika [12]. Our interest here is the archimedean local theory of their Rankin–Selberg integrals.

Let $F$ be either $\mathbb{R}$ or $\mathbb{C}$. We fix a maximal compact subgroup $K_n$ of $GL(n, F)$. Let $\Pi$ and $\Pi'$ be irreducible generic Casselman–Wallach representations of $GL(n, F)$ and $GL(n', F)$, respectively. We denote by $L(s, \Pi \times \Pi')$ the archimedean $L$-factor for $\Pi \times \Pi'$. The theory of archimedean Rankin–Selberg integrals for $\Pi \times \Pi'$ was developed by Jacquet and Shalika. In [15], they showed that any archimedean Rankin–Selberg integral for $\Pi \times \Pi'$ can be expressed as a linear combination of archimedean Rankin–Selberg integrals for

∗Faculty of Science and Technology, Seikei University, 3-3-1 Kichijoji-kitamachi, Musashino, Tokyo, 180-8633, Japan. E-mail: ishii@st.seikei.ac.jp
†Department of Mathematics, College of Liberal Arts and Sciences, Kitasato University, 1-15-1 Kitasato, Minamiku, Sagamihara, Kanagawa, 252-0373, Japan. E-mail: miyaza@kitasato-u.ac.jp
In the second main theorem (Theorem 2.9), we give a similar description for the
we leave it to Section 2.

describe \(\Psi\) explicitly in terms of Gelfand–Tsetlin type bases of
V defines an element of \(\text{Hom}_{K}\)
W ∈ W(\(\Pi,\psi\)) and \(W' \in W(\Pi',\psi^{-1})\), we define the archimedean Rankin–Selberg integral \(Z(s,W,W')\) by
\[
Z(s,W,W') = \int_{N_{n-1}\setminus GL(n-1,F)} W(\tau_{n}(g))W'(g)|\det g|_{F}^{s-1/2}\, dg \quad (\text{Re}(s) \gg 0),
\]
where \(W(\Pi,\psi)\), \(W(\Pi',\psi^{-1})\) are the Whittaker models of \(\Pi\), \(\Pi'\), respectively, \(N_{n-1}\) is the upper triangular unipotent subgroup of \(GL(n-1,F)\), and \(|\cdot|_{F}\) is the usual norm on \(F\). Let \((\tau_{\text{min}},V_{\text{min}})\) be the minimal \(K\)-type of \(\Pi\), and we fix a \(K\)-embedding \(W : V_{\text{min}} \to W(\Pi,\psi)\). Let \((\tau'_{\text{min}},V'_{\text{min}})\) be the minimal \(K\)-type of \(\Pi'\), and we fix a \(K\)-embedding \(W' : V'_{\text{min}} \to W(\Pi',\psi^{-1})\). Here we give \(W\) and \(W'\) concretely by the Jacquet integrals. We note that
\[
V_{\text{min}} \otimes_{\mathbb{C}} V'_{\text{min}} \ni v \otimes v' \mapsto Z(s,W(v),W'(v')) \in \mathbb{C}_{\text{triv}}
\]
defines an element of \(\text{Hom}_{K_{n-1}}(V_{\text{min}} \otimes_{\mathbb{C}} V'_{\text{min}},\mathbb{C}_{\text{triv}})\), where \(\mathbb{C}_{\text{triv}} = \mathbb{C}\) is the trivial \(K_{n-1}\)-module. In the first main theorem (Theorem 2.5), we give the explicit description of this \(K_{n-1}\)-homomorphism. More precisely, under the assumption \(\text{Hom}_{K_{n-1}}(V_{\text{min}} \otimes_{\mathbb{C}} V'_{\text{min}},\mathbb{C}_{\text{triv}}) \neq \{0\}\), we show the equality
\[
Z(s,W(v),W'(v')) = L(s,\Pi \times \Pi')\Psi(v \otimes v') \quad (v \in V_{\text{min}}, v' \in V'_{\text{min}})
\]  
(1.1)
with some nonzero \(\Psi \in \text{Hom}_{K_{n-1}}(V_{\text{min}} \otimes_{\mathbb{C}} V'_{\text{min}},\mathbb{C}_{\text{triv}})\) independent of \(s\), and describe \(\Psi\) explicitly in terms of Gelfand–Tsetlin type bases of \(V_{\text{min}}\) and \(V'_{\text{min}}\).
In the second main theorem (Theorem 2.9), we give a similar description for the \(GL(n) \times GL(n)\)-case. Since the statement of Theorem 2.9 is slightly complicated, we leave it to Section 2.
We introduce some applications of our results (Theorems 2.5 and 2.9) for $F = \mathbb{C}$. In the arithmetic study of critical values of automorphic $L$-functions for $\text{GL}(n) \times \text{GL}(n')$ with $n - n' \in \{0, 1\}$ by the cohomological method, the archimedean Rankin–Selberg integrals at the minimal $K_n$- and $K_{n'}$-types play important roles, and the hypothesis of the non-vanishing of them at critical points is called the non-vanishing hypothesis for $\text{GL}(n, F) \times \text{GL}(n', F)$. It is known that a local component at the complex place of irreducible regular algebraic cuspidal automorphic representation of $\text{GL}(n)$ is a cohomological principal series representation (cf. [21, Proposition 2.14]). Hence, Theorem 2.5 gives another proof of the non-vanishing hypothesis for $\text{GL}(n, \mathbb{C}) \times \text{GL}(n - 1, \mathbb{C})$ at all critical points, which were originally proved by Sun [27] and were used in Grobner–Harris [6] and Raghuram [21]. In [2], Dong and Xue proved the non-vanishing hypothesis for $\text{GL}(n, \mathbb{C}) \times \text{GL}(n, \mathbb{C})$ at the central critical point, and they indicate that it is hard to generalize their result to all critical points by the technique of the translation functor. Theorem 2.9 proves the non-vanishing hypothesis for $\text{GL}(n, \mathbb{C}) \times \text{GL}(n, \mathbb{C})$ at all critical points, and allows us to improve the archimedean part of Grenie’s theorem [5, Theorem 2] into more explicit form (cf. Remark 2.12). We expect that our explicit results will be applied to deeper study of special values of automorphic $L$-functions.

There are some related results to be mentioned here. In the cases of $\text{GL}(n) \times \text{GL}(n - 1)$ and $\text{GL}(n) \times \text{GL}(n)$, we expect that the archimedean Rankin–Selberg integrals for appropriate Whittaker functions are equal to the associated $L$-factors. This expectation was proved by Jacquet–Langlands [11] and Popa [20] for the $\text{GL}(2) \times \text{GL}(1)$-case; by Jacquet [13], Zhang [31] and the second author [19] for the $\text{GL}(2) \times \text{GL}(2)$-case; by Hirano and the authors [8] for the $\text{GL}(3) \times \text{GL}(2)$-case. The results of this paper, that is, the formula (1.1) and the analogous formula for $\text{GL}(n) \times \text{GL}(n)$ in Theorem 2.9 can be regarded as additional evidences of this expectation for the higher rank cases.

Let us briefly explain the idea of the proofs of our main theorems. The key ingredients are two kinds of the Godement sections for a principal series representation $\Pi$ of $\text{GL}(n, F)$. One is defined by Jacquet [14] as an integral transform of the standard section for some principal series representation of $\text{GL}(n - 1, F)$, and gives a recursive integral representation of a Whittaker function for $\Pi$. The other seems to be new, and is defined as an integral transform of the standard section for the same representation $\Pi$ of $\text{GL}(n, F)$. It gives an integral representation of a Whittaker function for $\Pi$, which is related to the local theta correspondence in Watanabe [30, §2]. Using two kinds of the Godement sections, we construct the recurrence relations of the archimedean Rankin–Selberg integrals for pairs of principal series representations of $\text{GL}(n, F)$ and $\text{GL}(n', F)$ with $n - n' \in \{0, 1\}$. Based on the representation theory of $K_n$, we write down these recurrence relations at the minimal $K_n$- and $K_{n'}$-types, explicitly, and prove the main theorems by induction. Here we remark that the explicit recurrence relations for the spherical case coincide with those in [10], which follow from explicit formulas of the radial parts of spherical Whittaker functions in [9].

This paper consists of five sections together with an appendix. In Section 2, we introduce basic notation and state the main theorems. In Section 3, we
define two kinds of the Godement sections and give the recurrence relations of the archimedean Rankin–Selberg integrals. Section 4 is devoted to some preliminary results on the theory of finite dimensional representations of $K_n$ and $GL(n, \mathbb{C})$. In Section 5, we prove the main theorems using the results in Sections 3 and 4. As an appendix, we generalize the explicit formulas of the radial parts of Whittaker functions in [9] using the Godement section.

This work was supported by JSPS KAKENHI Grant Numbers JP19K03452, JP18K03252.

2 Main results

In this section, we introduce basic notation and our main results. We describe each objects explicitly as possible, although not all of them are necessary to state our main theorems. The authors believe that they are of interest and useful for further studies.

2.1 Notation

We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. Let $\mathbb{R}^\times_+$ be the multiplicative group of positive real numbers. Let $\mathbb{N}_0$ be the set of non-negative integers. The real part, the imaginary part and the complex conjugate of a complex number $z$ are denoted by $\text{Re}(z)$, $\text{Im}(z)$ and $\overline{z}$, respectively. Throughout this paper, $F$ denotes the archimedean local field, that is, $F$ is either $\mathbb{R}$ or $\mathbb{C}$. It is convenient to define the constant $c_F$ by $c_\mathbb{R} = 1$ and $c_\mathbb{C} = 2$.

We define additive characters $\psi_t : F \to \mathbb{C}^\times$ ($t \in F$) and a norm $| \cdot |_F$ on $F$ by $\psi_t(z) = \exp(\pi c_F \sqrt{-1}(tz + \overline{tz})) = \begin{cases} \exp(2\pi \sqrt{-1}t^2) & \text{if } F = \mathbb{R}, \\ \exp(2\pi \sqrt{-1}(t^2 + \overline{t}^2)) & \text{if } F = \mathbb{C}, \end{cases}$ and $|z|_F = |z|^{c_F}$ for $z \in F$, where $| \cdot |$ is the ordinary absolute value. When $t = \varepsilon \in \{ \pm 1 \}$, we call $\psi_\varepsilon$ the standard character of $F$. We identify the additive group $F$ with its dual group via the isomorphism $t \mapsto \psi_t$, and denote by $d_F$ the self-dual additive Haar measure on $F$, that is, $d_\mathbb{R} = dx$ is the ordinary Lebesgue measure on $\mathbb{R}$ and $d_\mathbb{C} = 2dxdy$ ($z = x + \sqrt{-1}y$) is twice the ordinary Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$. For $m \in \mathbb{Z}$, we define a meromorphic function $\Gamma_F(s; m)$ of $s$ in $\mathbb{C}$ by $\Gamma_F(s; m) = c_F(\pi c_F)^{-sc_F+m/2} \Gamma \left( \frac{sc_F + m}{2} \right) = \begin{cases} \Gamma_\mathbb{R}(s + m) & \text{if } F = \mathbb{R}, \\ \Gamma_\mathbb{C}(s + m/2) & \text{if } F = \mathbb{C}, \end{cases}$ where $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$ and $\Gamma(s)$ is the usual Gamma function.

Throughout this paper, $n$ and $n'$ are positive integers. The space of $n \times n'$ matrices over $F$ is denoted by $M_{n,n'}(F)$. When $n' = n$, we denote $M_{n,n}(F)$
simply by $M_n(F)$. We denote by $d_F z$ the measure on $M_{n,n'}(F)$ defined by
\[ d_F z = \prod_{i=1}^{n} \prod_{j=1}^{n'} d_F z_{i,j} \quad (z = (z_{i,j}) \in M_{n,n'}(F)). \]

Let $O_{n,n'}$ be the zero matrix in $M_{n,n'}(F)$. Let $1_n$ be the unit matrix in $M_n(F)$. Let $e_n = (O_{1,n-1}, 1) \in M_{1,n}(F)$. When $n = 1$, we understand $e_1 = 1$.

### 2.2 Groups and the invariant measures

Let $G_n$ be the general linear group $GL(n, F)$ of degree $n$ over $F$. We fix a maximal compact subgroup $K_n$ of $G_n$ by
\[ K_n = \begin{cases} O(n) & \text{if } F = \mathbb{R}, \\ U(n) & \text{if } F = \mathbb{C}, \end{cases} \]

where $O(n)$ and $U(n)$ are the orthogonal group and the unitary group of degree $n$, respectively. Let $N_n$ and $U_n$ be the groups of upper and lower triangular unipotent matrices in $G_n$, respectively, that is,
\[ N_n = \{ x = (x_{i,j}) \in G_n \mid x_{i,j} = 0 \text{ (1} \leq j < i \leq n), \ x_{k,k} = 1 \text{ (1} \leq k \leq n) \}, \]
\[ U_n = \{ u = (u_{i,j}) \in G_n \mid u_{i,j} = 0 \text{ (1} \leq i < j \leq n), \ u_{k,k} = 1 \text{ (1} \leq k \leq n) \}. \]

We define subgroups $M_n$ and $A_n$ of $G_n$ by
\[ M_n = \{ m = \text{diag}(m_1, m_2, \ldots, m_n) \mid m_i \in G_1 = F^\times \quad (1 \leq i \leq n) \}, \]
\[ A_n = \{ a = \text{diag}(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R}_+^\times \quad (1 \leq i \leq n) \}. \]

Let $Z_n$ be the center of $G_n$. Then we have $Z_n = \{ t1_n \mid t \in G_1 = F^\times \}$. We denote by $C^\infty(G_n)$ the space of ($\mathbb{C}$-valued) smooth functions on $G_n$. We regard $C^\infty(G_n)$ as a $G_n$-module via the right translation
\[ (R(g)f)(h) = f(hg) \quad (g, h \in G_n, \ f \in C^\infty(G_n)). \]

Let $dk$, $dx$, $du$ and $da$ be the Haar measures on $K_n$, $N_n$, $U_n$ and $A_n$, respectively. In this paper, we normalize these Haar measures by
\[ \int_{K_n} dk = 1, \ \int dx = \prod_{1 \leq i < j \leq n} d_F x_{i,j}, \ \int du = \prod_{1 \leq j < i \leq n} d_F u_{i,j}, \ \int da = \prod_{i=1}^{n} \frac{2 c_F da_i}{a_i} \]

with $x = (x_{i,j}) \in N_n$, $u = (u_{i,j}) \in U_n$ and $a = \text{diag}(a_1, a_2, \ldots, a_n) \in A_n$. When $n = 1$, we understand $N_1 = U_1 = \{ 1 \}$ and
\[ \int_{N_1} f(x) dx = \int_{U_1} f(u) du = f(1) \]
for a function $f$ on $\{1\}$. We normalize the Haar measure $dg$ on $G_n$ so that
\[ \int_{G_n} f(g) dg = \int_{K_n} \int_{U_n} \int_{A_n} f(auk) da du dk = \int_{A_n} \int_{U_n} \int_{K_n} f(kua) dk du da \]
for any integrable function $f$ on $G_n$. We normalize the right $G_n$-invariant measure $dg$ on $N_n \backslash G_n$ so that
\[ \int_{G_n} f(g) dg = \int_{N_n \backslash G_n} \left( \int_{N_n} f(xg) dx \right) dg \] (2.1)
for any integrable function $f$ on $G_n$. We normalize the right $G_n$-invariant measure $dg$ on $Z_n N_n \backslash G_n$ so that
\[ \int_{N_n \backslash G_n} f(g) dg = \int_{Z_n N_n \backslash G_n} \left( \int_{G_1} f(hg) dh \right) dg \] (2.2)
for any integrable function $f$ on $N_n \backslash G_n$.

### 2.3 Principal series representations of $G_n$

Following Jacquet [14], we will define principal series representations of $G_n$ as representations induced from characters of the lower triangular Borel subgroup $U_n M_n$ of $G_n$ in this paper.

Let $d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}^n$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{C}^n$. For $l \in \mathbb{Z}$ and $t \in F^\times$, we set $\chi_l(t) = (t/|t|)^l$. We define characters $\chi_d$ and $\eta_\nu$ of $M_n$ by
\[ \chi_d(m) = \prod_{i=1}^{n} \chi_d, (m_i) = \prod_{i=1}^{n} \left( \frac{m_i}{|m_i|} \right)^{d_i}, \quad \eta_\nu(m) = \prod_{i=1}^{n} |m_i|^{\nu_i} = \prod_{i=1}^{n} |m_i|^{\nu_i/c_F} \]
for $m = \text{diag}(m_1, m_2, \ldots, m_n) \in M_n$. Let $\rho_n = (\rho_{n,1}, \rho_{n,2}, \ldots, \rho_{n,n}) \in \mathbb{Q}^n$ with
\[ \rho_{n,i} = \frac{n+1}{2} - i \quad (1 \leq i \leq n). \]

Let $I(d, \nu)$ be the subspace of $C^\infty(G_n)$ consisting of all functions $f$ such that
\[ f(umg) = \chi_d(m) \eta_{\nu-\rho_n}(m)f(g) \quad (u \in U_n, m \in M_n, g \in G_n), \]
(2.3)
on which $G_n$ acts by the right translation $\Pi_{d,\nu} = R$. We equip $I(d, \nu)$ with the usual Fréchet topology. We call $(\Pi_{d,\nu}, I(d, \nu))$ a (smooth) principal series representation of $G_n$. We denote by $I(d, \nu)_{K_n}$ the subspace of $I(d, \nu)$ consisting of all $K_n$-finite vectors. When $F = \mathbb{R}$, we note that
\[ \chi_{d+l} = \chi_d \quad I(d + l, \nu) = I(d, \nu) \quad (l \in 2\mathbb{Z}^n). \]
(2.4)

When $I(d, \nu)$ is irreducible, for any element $\sigma$ of the symmetric group $\mathfrak{S}_n$ of degree $n$, we have
\[ I(d, \nu) \simeq I((d_{\sigma(1)}, d_{\sigma(2)}, \ldots, d_{\sigma(n)}), (\nu_{\sigma(1)}, \nu_{\sigma(2)}, \ldots, \nu_{\sigma(n)})) \]
(2.5)
as representations of $G_n$ (cf. [24, Corollary 2.8]).

Let $I(d)$ be the space of smooth functions $f$ on $K_n$ satisfying
\[ f(mk) = \chi_d(m) f(k) \quad (m \in M_n \cap K_n, \ k \in K_n), \]
and we equip this space with the usual Fréchet topology. Because of the decomposition $G_n = U_n A_n K_n$ and (2.3), we can identify the space $I(d, \nu)$ with $I(d)$ via the restriction map $I(d, \nu) \ni f \mapsto f|_{K_n} \in I(d)$ to $K_n$. The inverse map $I(d) \ni f \mapsto f_{\nu} \in I(d, \nu)$ of the restriction map is given by
\[ f_{\nu}(uak) = \eta_{\nu - \rho_n}(a) f(k) \quad (u \in U_n, \ a \in A_n, \ k \in K_n). \quad (2.6) \]
We regard $I(d)$ as a $G_n$-module via this identification, and we denote the action of $G_n$ on $I(d)$ corresponding to $\Pi_{d, \nu}$ by $\Pi_{\nu}$, that is,
\[ (\Pi_{\nu}(g)f)(k) = f_{\nu}(kg) \quad (g \in G_n, \ k \in K_n, \ f \in I(d)). \]
Here we note that $\Pi_{\nu}|_{K_n}$ is the right translation and does not depend on $\nu$. We denote by $I(d)_{K_n}$ the subspace of $I(d)$ consisting of all $K_n$-finite vectors. For $f \in I(d)$, we call a map $\mathbb{C}^n \ni \nu \mapsto f_{\nu} \in C^\infty(G_n)$ defined by (2.6) the standard section corresponding to $f$.

**Remark 2.1.** For the study of automorphic forms such as the Eisenstein series, it is convenient to realize principal series representations of $G_n$ as representations $(\Pi_{B_n, d, \nu}, I_{B_n}(d, \nu))$ induced from characters of the upper triangular Borel subgroup $B_n = N_n M_n$, that is, $I_{B_n}(d, \nu)$ is the subspace of $C^\infty(G_n)$ consisting of all functions $f$ such that
\[ f(xmg) = \chi_d(m)\eta_{\nu + \rho_n}(m)f(g) \quad (x \in N_n, \ m \in M_n, \ g \in G_n), \]
and the action $\Pi_{B_n, d, \nu}$ of $G_n$ is the right translation $R$. The results in this paper can be translated into this realization via the $G_n$-isomorphism
\[ I_{B_n}(d, \nu) \ni f \mapsto f^{w_n} \in I((d_n, d_{n-1}, \cdots, d_1), (\nu_n, \nu_{n-1}, \cdots, \nu_1)) \]
with $f^{w_n}(g) = f(w_n g)$ $(g \in G_n)$. Here $w_n$ is the anti-diagonal matrix of size $n$ with 1 at all anti-diagonal entries.

### 2.4 Whittaker functions

Let $\varepsilon \in \{\pm 1\}$, and let $\psi_{\varepsilon}$ be the standard character of $F$ defined in §2.1. Let $\psi_{\varepsilon, n}$ be a character of $N_n$ defined by
\[ \psi_{\varepsilon, n}(x) = \psi_{\varepsilon}(x_{1,2} + x_{2,3} + \cdots + x_{n-1,n}) \quad (x = (x_{i,j}) \in N_n). \]
When $n = 1$, we understand that $\psi_{\varepsilon, 1}$ is the trivial character of $N_1 = \{1\}$.

Let $d \in \mathbb{Z}^n$ and $\nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{C}^n$. A $\psi_{\varepsilon}$-form on $I(d, \nu)$ is a continuous $\mathbb{C}$-linear form $T : I(d, \nu) \to \mathbb{C}$ satisfying
\[ T(\Pi_{d, \nu}(x)f) = \psi_{\varepsilon, n}(x)T(f) \quad (x \in N_n, \ f \in I(d, \nu)). \]
Kostant [18] shows that the space of \( \psi_z \)-forms on \( I(d, \nu) \) is one dimensional. Let us recall the construction of nonzero \( \psi_z \)-forms on principal series representations of \( G_n \), which are called the Jacquet integrals. If \( \nu \) satisfies
\[
\text{Re}(\nu_{i+1} - \nu_i) > 0 \quad (1 \leq i \leq n - 1),
\]
we define the Jacquet integral \( J_z : I(d, \nu) \to \mathbb{C} \) by the convergent integral
\[
J_z(f) = \int_{\Lambda_n} f(x) \psi_{z, n}(x) \, dx \quad (f \in I(d, \nu)).
\]
When \( n = 1 \), we understand \( J_z(f) = f(1) \) \((f \in I(d, \nu)) \). For \( \nu \in \mathbb{C} \) satisfying (2.7), we set \( J_z^{(d, \nu)}(f) = J_z(f_\nu) \) \((f \in I(d)) \), where \( f_\nu \) is the standard section corresponding to \( f \). By [29, Theorem 15.4.1], we know that \( J_z^{(d, \nu)}(f) \) has the holomorphic continuation to whole \( \nu \in \mathbb{C}^n \) for every \( f \in I(d) \), and \( \mathbb{C}^n \times I(d) \ni (\nu, f) \mapsto J_z^{(d, \nu)}(f) \in \mathbb{C} \) is continuous. Furthermore, this extends \( J_z^{(d, \nu)} \) to all \( \nu \in \mathbb{C}^n \) as a nonzero continuous \( \mathbb{C} \)-linear form on \( I(d) \) satisfying
\[
J_z^{(d, \nu)}(\Pi_\nu(x)f) = \psi_{z, n}(x) J_z^{(d, \nu)}(f) \quad (x \in \Lambda_n, f \in I(d)).
\]
We extends the Jacquet integral \( J_z : I(d, \nu) \to \mathbb{C} \) to whole \( \nu \in \mathbb{C}^n \) by
\[
J_z(f) = J_z^{(d, \nu)}(f|_{\Lambda_n}) \quad (f \in I(d, \nu))
\]
which is a nonzero \( \psi_z \)-form on \( I(d, \nu) \). We set
\[
W_z(f)(g) = J_z(\Pi_{d, \nu}(g)f) \quad (f \in I(d, \nu), g \in G_n).
\]
For \( f \in I(d, \nu) \), \( W_z(f) \) is called a Whittaker function for \( (\Pi_{d, \nu}, \psi_z) \), and satisfies
\[
W_z(f)(xg) = \psi_{z, n}(x) W_z(f)(g) \quad (x \in \Lambda_n, g \in G_n).
\]
We note that \( W_z(f_\nu)(g) = J_z^{(d, \nu)}(\Pi_\nu(g)f) \) is an entire function of \( \nu \) for \( g \in G_n \) and the standard section \( f_\nu \) corresponding to \( f \in I(d)_{\Lambda_n} \). Let
\[
\mathcal{W}(\Pi_{d, \nu}, \psi_z) = \{ W_z(f) \mid f \in I(d, \nu) \}.
\]
When \( \Pi_{d, \nu} \) is irreducible, this is a Whittaker model of \( \Pi_{d, \nu} \).

### 2.5 The Gelfand–Tsetlin type basis

In this subsection, we introduce a Gelfand–Tsetlin type basis of an irreducible holomorphic finite dimensional representation of \( GL(n, \mathbb{C}) \). Let \( gl(n, \mathbb{C}) = M_n(\mathbb{C}) \) be the associated Lie algebra of \( GL(n, \mathbb{C}) \). For \( 1 \leq i < j \leq n \), we denote by \( E_{i,j} \) the matrix unit in \( gl(n, \mathbb{C}) \) with 1 at the \((i, j)\)-th entry and 0 at other entries. We define the set \( \Lambda_n \) of dominant weights by
\[
\Lambda_n = \{ \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \}.
\]
Let \((\tau_\lambda, V_\lambda)\) be an irreducible holomorphic finite dimensional representation of \(GL(n, \mathbb{C})\) with highest weight \(\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n\), and we fix a \(U(n)\)-invariant hermitian inner product \(\langle \cdot, \cdot \rangle\) on \(V_\lambda\). Then we have
\[
\dim V_\lambda = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}
\]
by Weyl’s dimension formula [17, Theorem 4.48].

Let us recall the orthonormal basis on \(V_\lambda\), which is constructed by Gelfand and Tsetlin [3] (see Zhelobenko [32] for a detailed proof). We call
\[
M = (m_{i,j})_{1 \leq i \leq j \leq n} = \left( \begin{array}{cccc}
    m_{1,n} & m_{2,n} & \cdots & m_{n,n} \\
    m_{1,n-1} & m_{2,n-1} & \cdots & m_{n-1,n-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    m_{1,2} & m_{2,2} & \cdots & m_{n,1}
\end{array} \right) \quad (m_{i,j} \in \mathbb{Z})
\]
a integral triangular array of size \(n\), and call \(m_{i,j}\) the \((i, j)\)-th entry of \(M\). We denote by \(G(\lambda)\) the set of integral triangular arrays \(M = (m_{i,j})_{1 \leq i \leq j \leq n}\) of size \(n\) such that
\[
m_{i,n} = \lambda_i \quad (1 \leq i \leq n), \quad m_{j,k} \geq m_{j,k-1} \geq m_{j+1,k} \quad (1 \leq j < k \leq n).
\]
For \(M = (m_{i,j})_{1 \leq i \leq j \leq n} \in G(\lambda)\), we define \(\gamma^M = (\gamma_1^M, \gamma_2^M, \cdots, \gamma_n^M)\) by
\[
\gamma_j^M = \sum_{i=1}^{j} m_{i,j} - \sum_{i=1}^{j-1} m_{i,j-1} \quad (1 \leq j \leq n). \tag{2.10}
\]
We call \(\gamma^M\) the weight of \(M\). Gelfand and Tsetlin construct an orthonormal basis \(\{\zeta_M\}_{M \in G(\lambda)}\) of \(V_\lambda\) with the following formulas of \(\mathfrak{gl}(n, \mathbb{C})\)-actions:
\[
\tau_\lambda(E_{k,k})\zeta_M = \gamma_k^M \zeta_M, \tag{2.11}
\]
\[
\tau_\lambda(E_{j,j+1})\zeta_M = \sum_{1 \leq i \leq j} \tilde{a}_{i,j}^+(M)\zeta_{M+\Delta_{i,j}}, \tag{2.12}
\]
\[
\tau_\lambda(E_{j+1,j})\zeta_M = \sum_{1 \leq i \leq j} \tilde{a}_{i,j}^-(M)\zeta_{M-\Delta_{i,j}} \tag{2.13}
\]
for \(1 \leq k \leq n, \ 1 \leq j \leq n - 1\) and \(M = (m_{i,j})_{1 \leq i \leq j \leq n} \in G(\lambda)\), where \(\Delta_{i,j}\) is the integral triangular array of size \(n\) with 1 at the \((i, j)\)-th entry and 0 at other entries, and
\[
\tilde{a}_{i,j}^+ (M) = \left( \frac{\prod_{h=1}^{j+1} (m_{h,j+1} - m_{i,j} - h + i)}{\prod_{1 \leq h \leq j, \ h \neq i} (m_{h,j} - m_{i,j} - h + i)(m_{h,j} - m_{i,j} - h + i - 1)} \right)^{\frac{1}{2}},
\]
\[
\tilde{a}_{i,j}(M) = \left| \prod_{h=1}^{i-1} (m_{h,j+1} - m_{i,j} - h + i + 1) \prod_{h=1}^{i-1} (m_{h,j+1} - m_{i,j} - h + i) \right|^{1/2} \left| \prod_{h=1}^{j-1} (m_{h,j} - m_{i,j} - h + i + 1) \prod_{h=1}^{j-1} (m_{h,j} - m_{i,j} - h + i) \right|^{1/2}.
\]

We denote by \(H(\lambda)\) a unique element of \(G(\lambda)\) whose weight is \(\lambda\), that is,
\[
H(\lambda) = (h_{i,j})_{1 \leq i \leq j \leq n} \in G(\lambda) \quad \text{with} \quad h_{i,j} = \lambda_i.
\]

Then \(\zeta_{H(\lambda)}\) is a highest vector in \(V_\lambda\), that is,
\[
\tau_\lambda(E_{i,i}) \zeta_{H(\lambda)} = \lambda_i \zeta_{H(\lambda)} \quad (1 \leq i \leq n), \quad \tau_\lambda(E_{j,k}) \zeta_{H(\lambda)} = 0 \quad (1 \leq j < k \leq n).
\]

There is a \(\mathbb{Q}\)-rational structure of \(V_\lambda\) associated to the highest weight vector \(\zeta_{H(\lambda)}\). It comes from the natural \(\mathbb{Q}\)-rational structure of a tensor power of the standard representation of \(\text{GL}(n, \mathbb{C})\). We fix an embedding of \(V_i\) into a tensor power of the standard representation of \(\text{GL}(n, \mathbb{C})\) so that the image of \(\zeta_{H(\lambda)}\) is \(\mathbb{Q}\)-rational, and give a \(\mathbb{Q}\)-rational structure of \(V_\lambda\) via this embedding.

Let us construct a Gelfand–Tsetlin type \(\mathbb{Q}\)-rational basis of \(V_\lambda\). We set
\[
\xi_M = \sqrt{r(M)} \zeta_M \quad (M = (m_{i,j})_{1 \leq i < j \leq n} \in G(\lambda))
\]
with the rational constant
\[
r(M) = \prod_{1 \leq i < j \leq n} \frac{(m_{i,k} - m_{j,k-1} - i + j)!}{(m_{i,k} - m_{j,k-1} - i + j)!} \frac{(m_{i,k-1} - m_{j+1,k} - i + j)!}{(m_{i,k-1} - m_{j+1,k} - i + j)!}.
\]

Then \(\{\xi_M\}_{M \in G(\lambda)}\) is an orthogonal basis of \(V_\lambda\) such that \(\langle \xi_M, \xi_M \rangle = r(M)\) (\(M \in G(\lambda)\)). For an integral triangular array \(M = (m_{i,j})_{1 \leq i \leq j \leq n}\), we define the dual triangular array \(M^\vee = (m_{i,j}^\vee)_{1 \leq i \leq j \leq n}\) of \(M\) by \(m_{i,j}^\vee = -m_{j+1-i,j}\). The formulas corresponding to (2.11), (2.12) and (2.13) are given respectively by
\[
\tau_\lambda(E_{k,k}) \xi_M = \gamma_k^M \xi_M, \quad \tau_\lambda(E_{j,j+1}) \xi_M = \sum_{1 \leq i \leq j} a_{i,j}(M) \xi_{M+\Delta_{i,j}},
\]
\[
\tau_\lambda(E_{j+1,j}) \xi_M = \sum_{1 \leq i \leq j} a_{i,j}(M^\vee) \xi_{M+\Delta_{i,j}^\vee}
\]
for \(1 \leq k \leq n, 1 \leq j \leq n - 1\) and \(M = (m_{i,j})_{1 \leq i \leq j \leq n} \in G(\lambda)\), where \(a_{i,j}(M)\) is a rational number given by
\[
a_{i,j}(M) = \prod_{h=1}^{i-1} \frac{(m_{h,j+1} - m_{i,j} - h + i + 1)}{(m_{h,j+1} - m_{i,j} - h + i)} \prod_{h=1}^{j-1} \frac{(m_{h,j} - m_{i,j} - h + i + 1)}{(m_{h,j} - m_{i,j} - h + i)}.
\]

By these formulas and the equality \(\xi_{H(\lambda)} = \zeta_{H(\lambda)}\), we know that \(\{\xi_M\}_{M \in G(\lambda)}\) is a \(\mathbb{Q}\)-rational basis of \(V_\lambda\).
Until the end of this subsection, we assume $n > 1$. Let

$$\Xi^+(\lambda) = \{ \mu = (\mu_1, \mu_2, \cdots, \mu_{n-1}) \in \Lambda_{n-1} \mid \lambda_i \geq \mu_i \geq \lambda_{i+1} \ (1 \leq i \leq n-1) \}.$$ 

We regard $GL_{n-1}(\mathbb{C})$ as a subgroup of $GL_n(\mathbb{C})$ via the embedding

$$\iota_n: GL(n-1, \mathbb{C}) \ni g \mapsto \left( \begin{array}{cc} g & 0_{n-1,1} \\ 0_{1,n-1} & 1 \end{array} \right) \in GL(n, \mathbb{C}).$$

We set $\hat{\lambda}_M = (m_{i,j})$ for $M = (m_{i,j})_{1 \leq i \leq j \leq n-1} \in G(\lambda)$. By the construction of $\{ \xi_M \}_{M \in G(\lambda)}$, we know that $V_\lambda$ has the irreducible decomposition

$$V_\lambda = \bigoplus_{\mu \in \Xi^+(\lambda)} V_{\lambda,\mu}, \quad V_{\lambda,\mu} = \bigoplus_{M \in G(\lambda;\mu)} \mathbb{C} \xi_M \simeq V_\mu,$$

as a $GL(n-1, \mathbb{C})$-module, where

$$G(\lambda;\mu) = \{ M \in G(\lambda) \mid \hat{M} \in G(\mu) \}.$$

Let $\mu \in \Xi^+(\lambda)$. We define a $\mathbb{C}$-linear map $R^\lambda_\mu: V_\lambda \to V_\mu$ by

$$R^\lambda_\mu(\xi_M) = \begin{cases} \xi_{\hat{M}} & \text{if } M \in G(\lambda;\mu), \\ 0 & \text{otherwise} \end{cases} \quad (M \in G(\lambda)).$$

By the formulas (2.17), (2.18) and (2.19), we know that $R^\lambda_\mu$ is a surjective $GL(n-1, \mathbb{C})$-homomorphism. For $M \in G(\mu)$, we denote by $M[\lambda]$ the element of $G(\lambda;\mu)$ characterized by $\hat{M}[\lambda] = M$, that is,

$$M[\lambda] = \left( \begin{array}{c} \lambda \\ M \end{array} \right) \in G(\lambda;\mu).$$

Then we have $H(\mu)[\lambda] = \left( \begin{array}{c} \lambda \\ H(\mu) \end{array} \right)$, and $\xi_{H(\mu)[\lambda]}$ is the highest weight vector in the $GL(n-1, \mathbb{C})$-module $V_{\lambda,\mu}$. For later use, we prepare the following lemma.

**Lemma 2.2.** Retain the notation. A $\mathbb{C}$-linear map

$$V_\mu \ni \zeta_M \mapsto \zeta_{M[\lambda]} \in V_{\lambda,\mu} \quad (M \in G(\mu))$$

is a $GL(n-1, \mathbb{C})$-isomorphism which preserves the fixed inner products $\langle \cdot, \cdot \rangle$.

**Proof.** The assertion follows from (2.11), (2.12) and (2.13). \qed

### 2.6 Complex conjugate representations

For a finite dimensional representation $(\tau, V_\tau)$ of $GL(n, \mathbb{C})$, we define the complex conjugate representation $(\overline{\tau}, V_{\overline{\tau}})$ of $\tau$ as follows:
• Let \( \overline{V}_\tau \) be a set with a fixed bijective map \( V_\tau \ni v \mapsto \overline{v} \in \overline{V}_\tau \). We regard \( \overline{V}_\tau \) as a \( \mathbb{C} \)-vector space via the following addition and scalar multiplication:

\[
\overline{v_1} + \overline{v_2} = \overline{v_1 + v_2} \quad (v_1, v_2 \in V_\tau), \quad c\overline{v} = \overline{cv} \quad (c \in \mathbb{C}, \ v \in V_\tau),
\]

where \( \overline{v} \) is the complex conjugate of \( v \).

• The action \( \overline{\tau} \) is defined by \( \overline{\tau}(g)\overline{v} = \overline{\tau(g)v} \) \( \ (g \in \text{GL}(n, \mathbb{C}), \ v \in V_\tau) \).

By definition, the following assertions hold for finite dimensional representations \( (\tau, V_\tau) \) and \( (\tau', V_\tau') \) of \( \text{GL}(n, \mathbb{C}) \):

• The complex conjugate representation \( (\overline{\tau}, \overline{V}_\tau) \) of \( \tau \) is naturally identified with \( (\tau, V_\tau) \) via the correspondence \( \overline{\tau}(v) \mapsto v \) \( \ (v \in V_\tau) \).

• If \( \langle \cdot, \cdot \rangle \) is a \( U(n) \)-invariant hermitian inner product on \( V_\tau \), then

\[
V_\tau \otimes_\mathbb{C} \overline{V}_\tau \ni v_1 \otimes \overline{v}_2 \mapsto \langle v_1, v_2 \rangle \in \mathbb{C}
\]

is a non-degenerate \( \mathbb{C} \)-bilinear \( U(n) \)-invariant pairing.

• The complex conjugate representation \( (\tau \otimes \overline{\tau'}, \overline{V}_\tau \otimes_\mathbb{C} \overline{V}_\tau') \) of \( \tau \otimes \tau' \) is naturally identified with \( (\tau \otimes \overline{\tau'}, \overline{V}_\tau \otimes_\mathbb{C} \overline{V}_\tau') \) via the correspondence

\[
\overline{v}_1 \otimes \overline{v}_2 \leftrightarrow \overline{v_1} \otimes \overline{v_2} \quad (v_1 \in V_\tau, \ v_2 \in V_{\tau'}).
\]

• For any subgroup \( S \) of \( \text{GL}(n, \mathbb{C}) \), there is a bijective \( \mathbb{C} \)-antilinear map

\[
\text{Hom}_S(V_\tau, V_{\tau'}) \ni \Psi \mapsto \overline{\Psi} \in \text{Hom}_S(\overline{V}_\tau, \overline{V}_{\tau'})
\]

defined by \( \overline{\Psi}(v) = \overline{\Psi(v)} \in \overline{V}_{\tau'} \) \( \ (v \in V_\tau) \).

Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda_n \). We consider the complex conjugate representation \( (\overline{\tau}_\lambda, \overline{V}_\lambda) \) of \( \tau_\lambda \). We denote by \( u(n) \) the associated Lie algebra of \( U(n) \).

The complexification \( u(n)_\mathbb{C} = u(n) \otimes_\mathbb{R} \mathbb{C} \) of \( u(n) \) is isomorphic to \( \mathfrak{gl}(n, \mathbb{C}) \) via the correspondence \( E_{i,j} \mapsto E_{i,j} \) \( \ (1 \leq i, j \leq n) \) with

\[
E_{i,j}^{u(n)} = \frac{1}{2} \left\{ (E_{i,j} - E_{j,i}) \otimes 1 - \sqrt{-1}(E_{i,j} + E_{j,i}) \otimes \sqrt{-1} \right\} \in u(n)_\mathbb{C}.
\]

For \( 1 \leq i, j \leq n \) and \( v \in V_\lambda \), we have

\[
\tau_\lambda(E_{i,j}^{u(n)})v = \tau_\lambda(E_{i,j})v, \quad \overline{\tau}_\lambda(E_{i,j}^{u(n)})\overline{v} = -\overline{\tau}_\lambda(E_{i,j})\overline{v}. \quad (2.24)
\]

By the pairing \( V_\lambda \otimes_\mathbb{C} \overline{V}_\lambda \ni v_1 \otimes \overline{v}_2 \mapsto \langle v_1, v_2 \rangle \in \mathbb{C} \), we can identify \( (\overline{\tau}_\lambda, \overline{V}_\lambda) \) with the contragredient representation \( (\tau_\lambda, V_\lambda^\vee) \) of \( \tau_\lambda \) as a \( U(n) \)-module. Let \( \lambda^\vee = (-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1) \in \Lambda_n \). Since \( V_\lambda^\vee \simeq V_{\lambda^\vee} \) as \( \text{GL}(n, \mathbb{C}) \)-modules, we have \( \overline{V}_\lambda \simeq V_{\lambda^\vee} \) as \( U(n) \)-modules. In fact, by (2.17), (2.18), (2.19) and (2.24), we can confirm that the \( \mathbb{C} \)-linear map

\[
\overline{\xi}_M : \xi_M = (m_{i,j})_{1 \leq i, j \leq n} \in \mathbb{C}^n \mapsto (-1)^{\sum_{1 \leq i, j \leq n} m_{i,j}} \xi_{M^\vee} \in V_{\lambda^\vee} \quad (M = (m_{i,j})_{1 \leq i, j \leq n} \in \text{GL}(n))
\]

is a \( U(n) \)-isomorphism. Via this isomorphism, we derive the \( \mathbb{Q} \)-rational structure of \( \overline{V}_\lambda \) from that of \( V_{\lambda^\vee} \). Then \( \{ \xi_M \}_{M \in \text{GL}(n)} \) is a \( \mathbb{Q} \)-rational basis of \( \overline{V}_\lambda \).
Remark 2.3. We note that \( \{E_{i,j} - E_{j,i}\}_{1 \leq i<j \leq n} \) forms a basis of the associated Lie algebra \( \mathfrak{o}(n) \) of \( O(n) \). By (2.18), (2.19) and (2.24), we know that
\[
\overline{V_\lambda} \ni \xi_M \mapsto \xi_M \in V_\lambda \quad (M \in G(\lambda))
\]
defines a \( \mathbb{Q} \)-rational \( O(n) \)-isomorphism.

2.7 The minimal \( K_n \)- and \( K_n' \)-types

We define a subset \( \Lambda_{n,F} \) of \( \Lambda_n \) by \( \Lambda_{n,R} = \Lambda_n \cap \{0, 1\}^n \) and \( \Lambda_{n,C} = \Lambda_n \). In §4.2, we study the \( O(n) \)-module structure of \( V_\lambda \) for \( \lambda \in \Lambda_{n,F} \), and prove the following lemma.

Lemma 2.4. Let \( \lambda \in \Lambda_{n,F} \). Then \( V_\lambda \) is an irreducible \( K_n \)-module. Moreover, for any \( \lambda' \in \Lambda_{n,F} \) such that \( \lambda' \neq \lambda \), we have \( V_\lambda \not\cong V_{\lambda'} \) as \( K_n \)-modules.

Let \( (\Pi_{d,\nu}, I(d, \nu)) \) be a principal series representations of \( G_n \) with
\[
d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}^n, \quad \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{C}^n
\]
such that \( d \in \Lambda_{n,F} \). By the formula (2.17) and the Frobenius reciprocity law [17, Theorem 1.14], we know that \( \tau_d|_{K_n} \) is the minimal \( K_n \)-type of \( \Pi_{d,\nu} \), and \( \text{Hom}_{K_n}(V_d, I(d, \nu)) \) is 1 dimensional. Let \( f_{d,\nu}: V_d \to I(d, \nu) \) be the \( K_n \)-homomorphism normalized by \( f_{d,\nu}(\xi_{H(d)}(1_n)) = 1 \), that is,
\[
f_{d,\nu}(v)(uak) = \eta_{\nu-\rho_n}(a) \langle \tau_d(k)v, \xi_{H(d)} \rangle \quad (2.25)
\]
for \( u \in U_n, a \in A_n, k \in K_n \) and \( v \in V_d \). Here \( H(\lambda) \) (\( \lambda \in \Lambda_n \)) are defined by (2.14). For \( v \in V_d \), we note that \( f_{d,\nu}(v) \) is the standard section corresponding to \( f_d(v) \in I(d) \) defined by \( f_d(v)(k) = \langle \tau_d(k)v, \xi_{H(d)} \rangle \) (\( k \in K_n \)).

Let \( (\Pi_{d',\nu'}, I(d', \nu')) \) be a principal series representations of \( G_{n'} \) with
\[
d' = (d'_1, d'_2, \ldots, d'_n) \in \mathbb{Z}^{n'}, \quad \nu' = (\nu'_1, \nu'_2, \ldots, \nu'_n) \in \mathbb{C}^{n'}
\]
such that \(-d' \in \Lambda_{n',F} \). By the formula (2.17) and the Frobenius reciprocity law [17, Theorem 1.14], we know that \( \tau_{-d'}|_{K_{n'}} \) is the minimal \( K_{n'} \)-type of \( \Pi_{d',\nu'} \) and \( \text{Hom}_{K_{n'}}(V_{-d'}, I(d', \nu')) \) is 1 dimensional. Let \( \bar{f}_{d,\nu}: V_{-d'} \to I(d', \nu') \) be the \( K_{n'} \)-homomorphism normalized by \( \bar{f}_{d,\nu}(\xi_{H(-d')}(1_{n'})) = 1 \), that is,
\[
\bar{f}_{d,\nu}((uak)) = \eta_{\nu'-\rho_{n'}}(a) \langle \tau_{-d'}(k)v, \xi_{H(-d')} \rangle \quad (2.26)
\]
for \( u \in U_{n'}, a \in A_{n'}, k \in K_{n'} \) and \( v \in V_{-d'} \). For \( v \in V_{-d'} \), we note that \( \bar{f}_{d,\nu}(v) \) is the standard section corresponding to \( \bar{f}_{d}(v) \in I(d') \) defined by \( \bar{f}_{d}(v)(k) = \langle \tau_{-d'}(k)v, \xi_{H(-d')} \rangle \) (\( k \in K_{n'} \)).

We define the archimedean \( L \)-factor for \( \Pi_{d,\nu} \times \Pi_{d',\nu'} \) by
\[
L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'}) = \prod_{i=1}^{n} \prod_{j=1}^{n'} \Gamma_F(s + \nu_i + \nu_j; |d_i + d'_j|)
\]
In this subsection, we assume \( G \) principal series representations of \( G \). Theorem 2.5. Here \( 1/\Gamma_F(\nu; d) \) and \( 1/\Gamma_F(\nu'; d') \) are both nonzero if \( \Pi_{d,\nu} \) and \( \Pi_{d',\nu'} \) are irreducible. This fact follows from Corollary 5.4 in §5.1.

### 2.8 Archimedean Rankin–Selberg integrals for \( G_n \times G_{n-1} \)

In this subsection, we assume \( n > 1 \). Let \( (\Pi_{d,\nu}, I(d, \nu)) \) and \( (\Pi_{d',\nu'}, I(d', \nu')) \) be principal series representations of \( G_n \) and \( G_{n-1} \), respectively, with parameters

\[
d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}^n, \quad \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{C}^n, \\
d' = (d_1', d_2', \ldots, d_{n-1}') \in \mathbb{Z}^{n-1}, \quad \nu' = (\nu_1', \nu_2', \ldots, \nu_{n-1}') \in \mathbb{C}^{n-1}.
\]

We assume \( d \in \Lambda_{n,F} \) and \( -d' \in \Lambda_{n-1,F} \). If \( \Pi_{d,\nu} \) and \( \Pi_{d',\nu'} \) are irreducible, these are not serious assumptions because of (2.4) and (2.5). We take \( f_{d,\nu}, f_{d',\nu'} \), \( \Gamma_F(\nu; d), \Gamma_F(\nu'; d') \) and \( L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'}) \) as in §2.7 with \( n' = n - 1 \).

Let \( \varepsilon \in \{ \pm 1 \} \), \( W \in W(\Pi_{d,\nu}, \psi_\varepsilon) \) and \( W' \in W(\Pi_{d',\nu'}, \psi_{-\varepsilon}) \). Let \( s \in \mathbb{C} \) such that \( \text{Re}(s) \) is sufficiently large. We define the archimedean Rankin–Selberg integral \( Z(s, W, W') \) for \( \Pi_{d,\nu} \times \Pi_{d',\nu'} \) by

\[
Z(s, W, W') = \int_{N_{n-1}\backslash G_{n-1}} W(\iota_n(g)) W'(g) |\det g|_F^{s-1/2} dg, \quad (2.27)
\]

where \( \iota_n \) is defined by (2.20). Here we note

\[
Z(s, R(\iota_n(k)) W, R(k) W') = Z(s, W, W') \quad (k \in K_{n-1}). \quad (2.28)
\]

By (2.28), we know that

\[
v_1 \otimes v_2 \mapsto Z(s, W_\varepsilon(f_{d,\nu}(v_1)), W_{-\varepsilon}(f_{d',\nu'}(\overline{v_2}))) \quad (2.29)
\]

defines an element of \( \text{Hom}_{K_{n-1}}(V_d \otimes \mathbb{C}, V_{-d'} \otimes \mathbb{C}_{\text{triv}}) \). Here \( W_\varepsilon \) is defined by (2.8), and \( \mathbb{C}_{\text{triv}} = \mathbb{C} \) is the trivial \( K_{n-1} \)-module. The following theorem is the first main result of this paper, which gives the explicit expression of the \( K_{n-1} \)-homomorphism (2.29).

**Theorem 2.5.** Retain the notation. For \( v_1 \in V_d \) and \( v_2 \in V_{-d'} \), we have

\[
Z(s, W_\varepsilon(f_{d,\nu}(v_1)), W_{-\varepsilon}(f_{d',\nu'}(\overline{v_2}))) = \frac{(-\varepsilon \sqrt{-1})^{\sum_{i=1}^{n-1} (n-i)(d_i+d'_i)}}{(\dim V_{-d'}) \Gamma_F(\nu; d) \Gamma_F(\nu'; d')} L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'}) \langle R_d(v_1), v_2 \rangle
\]
if \(-d' \in \Xi^+(d)\), and \(Z(s, W_{d,\nu}(\xi_\Lambda(-d')|d|)), W_{-\epsilon}(\xi_{\Lambda}(-d')) = 0\) otherwise. Here \(R^d_{-\epsilon, d'}\) is given explicitly by (2.22). In particular, we have
\[
Z(s, W_{d,\nu}(\xi_\Lambda(-d')|d|)), W_{-\epsilon}(\xi_{\Lambda}(-d')) = \frac{(-\epsilon \sqrt{-1})^{-n(d,\nu)}(\xi_{\Lambda}(-d'))}{(\dim V_{-d'})!F_F(\nu; d)\Gamma(\nu; d')} L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'}) (2.30)
\]
if \(-d' \in \Xi^+(d)\). Here \(H(-d')\) and \(H(-d')|d|\) are defined by (2.14) and (2.23).

**Remark 2.6.** Retain the notation. The space \(\text{Hom}_{K_n}(V_d \otimes \mathbb{C} V_{-d'}, \mathbb{C}_{\text{triv}})\) is 1 dimensional if \(-d' \in \Xi^+(d)\), and is equal to \(\{0\}\) otherwise. This fact follows from Lemma 4.2 in \S4.2.

**Remark 2.7.** We set \(F = \mathbb{C}\). By [21, Proposition 2.14 and Theorem 2.21], we note that the compatible pairs of cohomological representations of \(G_n\) and \(G_{n-1}\) in Sun [27, \S6] can be regarded as pairs of some irreducible principal series representations \(\Pi_{d,\nu}\) and \(\Pi_{d',\nu'}\) with \(d \in \Lambda_{n,\mathbb{C}}\) and \(-d' \in \Xi^+(d)\). Hence, Theorem 2.5 gives another proof of the non-vanishing hypothesis for \(G_n \times G_{n-1}\), which were originally proved by Sun [27, Theorem C].

**Corollary 2.8.** Retain the notation, and assume \(-d' \in \Xi^+(d)\). Then
\[
\sum_{M \in G(-d')} r(M)^{-1} \xi_M^d \otimes \overline{\xi_M}
\]
is a unique \(\mathbb{Q}\)-rational \(K_{n-1}\)-invariant vector in \(V_d \otimes \overline{V_{-d'}}\) up to scalar multiple, and its image under the \(K_{n-1}\)-homomorphism (2.29) is
\[
\frac{(-\epsilon \sqrt{-1})^{-n(d,\nu)}(\xi_{\Lambda}(-d'))}{(\dim V_{-d'})!F_F(\nu; d)\Gamma(\nu; d')} L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'})
\]
Here \(r(M)\) and \(M[d]\) are defined by (2.16) and (2.23), respectively.

This corollary follows from Theorem 2.5 with Lemma 4.2, and is similar to Corollary 2.11 which gives the explicit description of the archimedean part of Grenié’s theorem [5, Theorem 2].

### 2.9 Schwartz functions

Let \(S(M_{n,n'}(F))\) be the space of Schwartz functions on \(M_{n,n'}(F)\). We define \(e_{(n,n')}(F) \in S(M_{n,n'}(F))\) by
\[
e_{(n,n')}(z) = \exp(-\pi c_F \text{Tr}(|zz|)) = \begin{cases} 
\exp(-\pi \text{Tr}(|zz|)) & \text{if } F = \mathbb{R}, \\
\exp(-2\pi \text{Tr}(|zz|)) & \text{if } F = \mathbb{C}
\end{cases} (2.31)
\]
for \(z \in M_{n,n'}(F)\). We denote \(e_{(n,n)}\) simply by \(e_{(n)}\). Let \(S_0(M_{n,n'}(F))\) be the subspace of \(S(M_{n,n'}(F))\) consisting of all functions \(\phi\) of the form
\[
\phi(z) = p(z, \overline{z}) e_{(n,n')}(z) \quad (z \in M_{n,n'}(F)),
\]
where \( p \) is a polynomial function. We call elements of \( \mathcal{S}_0(M_{n,n'}(F)) \) standard Schwartz functions on \( M_{n,n'}(F) \).

Let \( C(M_{n,n'}(F)) \) be the space of continuous functions on \( M_{n,n'}(F) \). We define actions of \( G_n \) and \( G_{n'} \) on \( C(M_{n,n'}(F)) \) by
\[
(L(g)f)(z) = f(g^{-1}z), \quad (R(h)f)(z) = f(zh)
\]
for \( g \in G_n, h \in G_{n'}, f \in C(M_{n,n'}(F)) \) and \( z \in M_{n,n'}(F) \). Since \( e_{(n,n')} \) is \( K_n \times K_{n'} \)-invariant, we note that \( \mathcal{S}_0(M_{n,n'}(F)) \) is closed under the action \( L \otimes R \) of \( K_n \times K_{n'} \), and all elements of \( \mathcal{S}_0(M_{n,n'}(F)) \) are \( K_n \times K_{n'} \)-finite.

Let \( l \in \mathbb{N}_0 \), and we consider the representation \( (\tau(l,0_{n-1}), V(l,0_{n-1})) \). Here we put \( 0_{n-1} = (0,0,\cdots,0) \in \Lambda_{n-1} \) if \( n > 1 \), and erase \( 0_{n-1} \) if \( n = 1 \). We set
\[
\ell(\gamma) = \gamma_1 + \gamma_2 + \cdots + \gamma_n \quad (\gamma = (\gamma_1,\gamma_2,\cdots,\gamma_n) \in \mathbb{Z}^n). \tag{2.32}
\]
For \( \gamma = (\gamma_1,\gamma_2,\cdots,\gamma_n) \in \mathbb{N}_0^n \) such that \( \ell(\gamma) = l \), we denote by \( Q(\gamma) \) a unique element of \( G((l,0_{n-1})) \) whose weight is \( \gamma \), that is,
\[
Q(\gamma) = (q_{i,j})_{1 \leq i \leq j \leq n} \quad \text{with} \quad q_{i,j} = \begin{cases} \sum_{k=1}^{\gamma_i} \gamma_k & \text{if } i = 1, \\ 0 & \text{otherwise}. \end{cases} \tag{2.33}
\]

Then we have \( G((l,0_{n-1})) = \{ Q(\gamma) \mid \gamma \in \mathbb{N}_0^n, \ell(\gamma) = l \} \). We define \( \mathbb{C} \)-linear maps \( \varphi_{1,n}^{(l)} : V(l,0_{n-1}) \to \mathcal{S}_0(M_{1,n}(F)) \) and \( \overline{\varphi}_{1,n}^{(l)} : V(l,0_{n-1}) \to \mathcal{S}_0(M_{1,n}(F)) \) by
\[
\varphi_{1,n}^{(l)}(\xi Q(\gamma))(z) = z_1^{\gamma_1} z_2^{\gamma_2} \cdots z_n^{\gamma_n} e_{(1,n)}(z), \tag{2.34}
\]
\[
\overline{\varphi}_{1,n}^{(l)}(\overline{\xi Q(\gamma)})(z) = z_1^{\gamma_1} z_2^{\gamma_2} \cdots z_n^{\gamma_n} e_{(1,n)}(z) \tag{2.35}
\]
for \( z = (z_1,z_2,\cdots,z_n) \in M_{1,n}(F) \) and \( \gamma = (\gamma_1,\gamma_2,\cdots,\gamma_n) \in \mathbb{N}_0^n \) such that \( \ell(\gamma) = l \). In §4.4, we show that \( \varphi_{1,n}^{(l)} \) and \( \overline{\varphi}_{1,n}^{(l)} \) are \( K_n \)-homomorphisms, where \( \mathcal{S}_0(M_{1,n}(F)) \) is regarded as a \( K_n \)-module via the action \( R \).

### 2.10 Injector

Let \( \lambda = (\lambda_1,\lambda_2,\cdots,\lambda_n) \in \Lambda_n \) and \( l \in \mathbb{N}_0 \). In this subsection, we specify each irreducible components of the tensor product \( V_\lambda \otimes \mathbb{C} V(l,0_{n-1}) \). Let
\[
\Xi^\circ(\lambda) = \{ \lambda' = (\lambda_1',\lambda_2',\cdots,\lambda_n') \in \Lambda_n \mid \lambda_1' \geq \lambda_1 \geq \lambda_2 \geq \lambda_2 \geq \cdots \geq \lambda_n' \geq \lambda_n \};
\]
and \( \Xi^\circ(\lambda;l) = \{ \lambda' \in \Xi^\circ(\lambda) \mid \ell(\lambda' - \lambda) = l \} \). Then Pieri’s rule [4, Corollary 9.2.4] asserts that \( V_\lambda \otimes \mathbb{C} V(l,0_{n-1}) \) has the irreducible decomposition
\[
V_\lambda \otimes \mathbb{C} V(l,0_{n-1}) \simeq \bigoplus_{\lambda' \in \Xi^\circ(\lambda;l)} V_{\lambda'} \tag{2.36}
\]
as \( \mathrm{GL}(n,\mathbb{C}) \)-modules.

For \( \lambda' = (\lambda_1',\lambda_2',\cdots,\lambda_n') \in \Xi^\circ(\lambda) \), we set
\[
S^\circ(\lambda',\lambda) = \frac{\prod_{1 \leq i \leq j \leq n} (\lambda_i' - \lambda_j - i + j)!}{\prod_{1 \leq i \leq j < n} (\lambda_i - \lambda_{i+1} - i + j)!}. \tag{2.37}
\]
When \( n > 1 \), for \( \mu = (\mu_1, \mu_2, \cdots, \mu_{n-1}) \in \Xi^+(\lambda) \), we set

\[
S^+(\lambda, \mu) = \prod_{1 \leq i \leq j < n} \frac{(\lambda_i - \mu_j - i + j)!}{(\mu_i - \lambda_j + i + j)!}.
\] (2.38)

Let \( \lambda' = (\lambda'_1, \lambda'_2, \cdots, \lambda'_n) \in \Xi^+(\lambda'; \lambda) \). Based on the result of Jucys [16], we will construct a \( \mathbb{Q} \)-rational \( \text{GL}(n, \mathbb{C}) \)-homomorphism \( \Gamma_{\lambda'}^{\lambda} : V_{\lambda'} \to V_{\lambda} \otimes \mathbb{C} V((l, 0_{n-1})) \) in §4.1. The explicit expression of \( \Gamma_{\lambda'}^{\lambda} \) is given by

\[
\Gamma_{\lambda'}^{\lambda}(\xi_{M'}) = \sum_{M \in G(\lambda)} \sum_{P \in G((l, 0_{n-1}))} c_{M, \lambda}^{M, P} \xi_M \otimes \xi_P \quad (M' \in G(\lambda')), \tag{2.39}
\]

where \( c_{M, \lambda}^{M, P} \) are rational numbers determined by the following conditions, recursively:

- When \( n = 1 \), we have \( c_{\lambda_1}^{\lambda_1, l} = 1 \) for \( \lambda_1, l \in \mathbb{Z} \).
- When \( n > 1 \), for \( \mu' \in \Xi^+(\lambda') \), \( M' \in G(\lambda'; \mu') \), \( \mu \in \Xi^+(\lambda) \), \( M \in G(\lambda; \mu) \), \( 0 \leq q \leq l \) and \( P \in G((l, 0_{n-1}); (q, 0_{n-2})) \), we have

\[
c_{M, \lambda}^{M', P} = \frac{\bar{\Lambda}^{M, \nu} \cdot S^\circ(\lambda', \lambda') S^\circ(\mu, \mu)}{q! \prod_{1 \leq i \leq j < n} (\lambda_i - \lambda_j - i + j)!} \times \left( \prod_{1 \leq i \leq j < n} \frac{(\mu'_i - \mu_j - i + j)!(\mu'_i - \lambda'_j - i + j)!}{(\mu'_i - \mu'_j - i + j)!}(\mu_i - \lambda_j + i + j)! \right) \times \sum_{\alpha \in \Xi^+(\lambda) \cap \Xi^+(\lambda')} (-1)^{(\alpha - \mu) \cdot S^\circ(\alpha, \alpha)} S^\circ(\lambda', \alpha) S^\circ(\mu', \alpha) S^\circ(\alpha, \mu)
\]

if \( \mu' \in \Xi^+(\mu; q) \), and \( c_{M, \lambda}^{M', P} = 0 \) otherwise.

We define a \( U(n) \)-invariant hermitian inner product on \( V_{\lambda} \otimes \mathbb{C} V((l, 0_{n-1})) \) by

\[
\langle v_1 \otimes v'_1, v_2 \otimes v'_2 \rangle = \langle v_1, v_2 \rangle \langle v'_1, v'_2 \rangle \quad (v_1, v_2 \in V_{\lambda}, v'_1, v'_2 \in V((l, 0_{n-1}))).
\]

Then we have

\[
\langle \Gamma_{\lambda'}^{\lambda}(\xi_{H(\lambda')}), \xi_{H(\lambda')} \otimes \xi_{Q(\lambda' - \lambda)} \rangle = C^\circ(\lambda'; \lambda),
\]

\[
\langle \Gamma_{\lambda'}^{\lambda}(v), \Gamma_{\lambda'}^{\lambda}(v') \rangle = b(\lambda' - \lambda) C^\circ(\lambda'; \lambda) \langle v, v' \rangle \quad (v, v' \in V_{\lambda'}),
\]

where

\[
C^\circ(\lambda'; \lambda) = \prod_{1 \leq i < j \leq n} \frac{(\lambda'_i - \lambda'_j - i + j)!(\lambda_i - \lambda_j - i + j)!(\lambda'_i - \lambda'_j - i + j - 1)!}{(\lambda'_i - \lambda'_j - i + j)!}(\lambda_i - \lambda_j - i + j - 1)! \tag{2.42}
\]

\[
b(\gamma) = \frac{\gamma_1 + \gamma_2 + \cdots + \gamma_n!}{\gamma_1! \gamma_2! \cdots \gamma_n!} \quad (\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{N}_0^\circ).
\]

(2.43)
2.11 Archimedean Rankin–Selberg integrals for $G_n \times G_n$

Let $(\Pi_{d,\nu}, I(d, \nu))$ and $(\Pi_{d',\nu'}, I(d', \nu'))$ be principal series representations of $G_n$ with parameters

\[ d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}^n, \quad \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{C}^n, \]

\[ d' = (d'_1, d'_2, \ldots, d'_n) \in \mathbb{Z}^n, \quad \nu' = (\nu'_1, \nu'_2, \ldots, \nu'_n) \in \mathbb{C}^n. \]

We assume $d \in \Lambda_{n,F}$ and $-d' \in \Lambda_{n,F}$. If $\Pi_{d,\nu}$ and $\Pi_{d',\nu'}$ are irreducible, these are not serious assumptions because of (2.4) and (2.5). We take $f_{d,\nu}$, $\tilde{f}_{d',\nu'}$, $\Gamma_F(\nu; d)$, $\Gamma_F(\nu'; d')$ and $L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'})$ as in §2.7 with $n' = n$.

Let $\varepsilon \in \{\pm 1\}$, $W \in \mathcal{W}(\Pi_{d,\nu}, \psi_\varepsilon)$, $W' \in \mathcal{W}(\Pi_{d',\nu'}, \psi_{-\varepsilon})$ and $\phi \in \mathcal{S}(M_{1,n}(F))$. Let $s \in \mathbb{C}$ such that $\text{Re}(s)$ is sufficiently large. We define the archimedean Rankin–Selberg integral $Z(s, W, W', \phi)$ for $\Pi_{d,\nu} \times \Pi_{d',\nu'}$ by

\[ Z(s, W, W', \phi) = \int_{N_n \backslash G_n} W(g)W'(g)\phi(\varepsilon_n g) \det g|_{F^d}^d \, dg, \quad (2.44) \]

where we put $e_n = (O_{1,n-1}, 1) \in M_{1,n}(F)$ as in §2.1. Here we note

\[ Z(s, W, W', \phi) = Z(s, W', W, \phi), \quad (2.45) \]

\[ Z(s, R(k)W, R(k)W', R(k)\phi) = Z(s, W, W', \phi) \quad (k \in K_n). \quad (2.46) \]

Let $l$ be an integer determined by

\[ l \in \{0, 1\} \text{ and } l \equiv -\ell(d + d') \mod 2 \quad \text{if } F = \mathbb{R} \text{ and } \ell(d + d') \leq 0, \]

\[ l \in \{0, 1\} \text{ and } l \equiv -\ell(d + d') \mod 2 \quad \text{if } F = \mathbb{R} \text{ and } \ell(d + d') \geq 0, \]

\[ l = -\ell(d + d') \quad \text{if } F = \mathbb{C}, \]

where $\ell(\gamma)$ ($\gamma \in \mathbb{Z}^n$) are defined by (2.32). By (2.46), we know that

\[ v_1 \otimes \overline{v_2} \otimes v_3 \mapsto Z(s, W_\varepsilon(f_{d,\nu}(v_1)), W_{-\varepsilon}(\tilde{f}_{d',\nu'}(\overline{v_2})), \varphi_{1,n}^{(l)}(v_3)) \quad (2.47) \]

defines an element of $\text{Hom}_{K_n}(V_d \otimes_{\mathcal{C}} \overline{V}_{-d} \otimes_{\mathcal{C}} V_{(l,0_{n-1})}, C_{\text{triv}})$ if $l \geq 0$, and

\[ v_1 \otimes \overline{v_2} \otimes \overline{v_3} \mapsto Z(s, W_\varepsilon(f_{d,\nu}(v_1)), W_{-\varepsilon}(\tilde{f}_{d',\nu'}(\overline{v_2})), \overline{\varphi}_{1,n}^{(-l)}(v_3)) \quad (2.48) \]

defines an element of $\text{Hom}_{K_n}(V_d \otimes_{\mathcal{C}} \overline{V}_{-d} \otimes_{\mathcal{C}} V_{(-l,0_{n-1})}, C_{\text{triv}})$ if $l \leq 0$. Here $W_\varepsilon$, $\varphi_{1,n}^{(l)}$, $\overline{\varphi}_{1,n}^{(-l)}$ are defined by (2.8), (2.34), (2.35), respectively, and $C_{\text{triv}} = \mathbb{C}$ is the trivial $K_n$-module. The following theorem is the second main result of this paper, which gives the explicit expressions of the $K_n$-homomorphisms (2.47) and (2.48).

**Theorem 2.9.** Retain the notation.

(1) Assume $l \geq 0$. For $v_1 \in V_d$, $v_2 \in V_{-d'}$ and $v_3 \in V_{(l,0_{n-1})}$, we have

\[ Z(s, W_\varepsilon(f_{d,\nu}(v_1)), W_{-\varepsilon}(\tilde{f}_{d',\nu'}(\overline{v_2})), \varphi_{1,n}^{(l)}(v_3)) \]

\[ = \frac{(-\varepsilon \sqrt{-1})^{\sum_{i=1}^{n}(n-i)(d_i+d'_i)}}{(\dim V_{-d'})\Gamma_F(\nu; d)\Gamma_F(\nu'; d')} L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'}) \langle v_1 \otimes v_3, l_{\tilde{d}}^d(v_2) \rangle. \]
if \(-d' \in \Xi^0(d)\), and \(Z(s, W_{\varepsilon}(I_{d',\nu'}(v_1)), W_{-\varepsilon}(\tilde{I}_{d',\nu'}(\overline{v_2})), \varphi_1^{(l)}(v_3)) = 0\) otherwise. Here \(\Gamma_d^{l,l'}\) is given explicitly by (2.39). In particular, if \(-d' \in \Xi^0(d)\), we have
\[
Z(s, W_{\varepsilon}(I_{d',\nu'}(\xi_H(d))), W_{-\varepsilon}(\tilde{I}_{d',\nu'}(\xi_H(-d'))), \varphi_1^{(l)}(\xi_Q(-d'-d)))
= \left(\frac{\varepsilon \sqrt{-1}}{n} \sum_{n=1}^{n-i} (n-i)(d_i+d_i') C^0(d_i'-d) \right) L(s, \Pi_{d',\nu} \times \Pi_{d',\nu'}). \tag{2.49}
\]

(2) Assume \(l \leq 0\). For \(v_1 \in V_d\), \(v_2 \in V_{-d'}\) and \(v_3 \in V_{(-l \cdot 0_{n-1})}\), we have
\[
Z(s, W_{\varepsilon}(I_{d',\nu'}(v_1)), W_{-\varepsilon}(\tilde{I}_{d',\nu'}(\overline{v_2})), \varphi_1^{(-l)}(\overline{v_3}))
= \left(\frac{\varepsilon \sqrt{-1}}{n} \sum_{n=1}^{n-i} (n-i)(d_i+d_i') C^0(d_i'-d) \varepsilon \right) L(s, \Pi_{d',\nu} \times \Pi_{d',\nu'}, (I_{d'}^{d'-l}(v_1), v_2 \otimes v_3)
if \(d \in \Xi^0(-d')\), and \(Z(s, W_{\varepsilon}(I_{d',\nu'}(v_1)), W_{-\varepsilon}(\tilde{I}_{d',\nu'}(\overline{v_2})), \varphi_1^{(-l)}(\overline{v_3})) = 0\) otherwise. Here \(\Gamma_d^{d'-l}\) is given explicitly by (2.39). In particular, if \(d \in \Xi^0(-d')\), we have
\[
Z(s, W_{\varepsilon}(I_{d',\nu'}(\xi_H(d))), W_{-\varepsilon}(\tilde{I}_{d',\nu'}(\xi_H(-d'))), \varphi_1^{(-l)}(\xi_Q(d'+d)))
= \left(\frac{\varepsilon \sqrt{-1}}{n} \sum_{n=1}^{n-i} (n-i)(d_i+d_i') C^0(d_i'+d) \varepsilon \right) L(s, \Pi_{d',\nu} \times \Pi_{d',\nu'}). \tag{2.50}
\]

Remark 2.10. Retain the notation. Under the assumption \(l \geq 0\), the space
\[
\text{Hom}_{K_n}(V_d \otimes \overline{\Lambda}^{-d'} \otimes \overline{V}_{(l \cdot 0_{n-1})}, C_{\text{triv}})
is 1 dimensional if \(-d' \in \Xi^0(d)\), and is equal to \(\{0\}\) otherwise. Under the assumption \(l \leq 0\), the space
\[
\text{Hom}_{K_n}(V_d \otimes \overline{\Lambda}^{-d'} \otimes \overline{V}_{(-l \cdot 0_{n-1})}, C_{\text{triv}})
is 1 dimensional if \(d \in \Xi^0(-d')\), and is equal to \(\{0\}\) otherwise. These facts follow from Lemma 4.3 in §4.2.

Let \(P_n\) be a maximal parabolic subgroup of \(G_n\) defined by
\[
P_n = \{p = (p_{i,j}) \in G_n \mid p_{i,j} = 0 \text{ for } i = 1, \ldots, n \text{ for } j = 1, \ldots, n-1\},
which contains the upper triangular Borel subgroup \(B_n = N_nM_n\). We put \(\chi(t) = (t/|t|)^{t} \in F^\times\) as in §2.3, and set \(\nu'' = -\sum_{i=1}^{n} (\nu_i + \nu_i')\). We define a subspace \(I_{P_n}(l, \nu'', s)\) of \(C^\infty(G_n)\) consisting of all functions \(f\) such that
\[
f(pg) = \chi(p_{n,n}) |p_{n,n}|_{F}^{\nu''-ns} \det p_{n,n}^{s} f(g) \quad (p = (p_{i,j}) \in P_n, g \in G_n),
on which \(G_n\) acts by the right translation \(\Pi_{P_n,l,\nu'',s} = R\). The representation \((\Pi_{P_n,l,\nu'',s}, I_{P_n}(l, \nu'', s))\) is called a degenerate principal series representation of \(G_n\). Similar to the proof of [5, Proposition 7], we can specify the minimal \(K_n\)-type of \(\Pi_{P_n,l,\nu'',s}\), which occur in \(\Pi_{P_n,l,\nu'',s}|_{K_n}\) with multiplicity 1. If \(l \geq 0,\)
we know that $\tau_{(l, \alpha_{n-1})}|K_n$ is the minimal $K_n$-type of $\Pi_{l, \nu''}$, and there is a $K_n$-homomorphism $f_{\Pi_{l, \nu''}}: V_{l, \alpha_{n-1}} \to I_{P_n}(l, \nu'', s)$ characterized by

$$f_{\Pi_{l, \nu''}}(\xi_{Q(\gamma)}) = \frac{|\det g|_F \prod_{i=1}^{n} \gamma_i}{\prod_{i=1}^{n} |g_{n,i}|^{n_{sF} - \nu'' s_{F} + 1}/2}$$

for $g = (g_{i,j}) \in G_n$ and $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{N}_0^n$ such that $\ell(\gamma) = l$. If $l \leq 0$, we know that $\tau_{(-l, \alpha_{n-1})}|K_n$ is the minimal $K_n$-type of $\Pi_{-l, \nu''}$, and there is a $K_n$-homomorphism $\bar{f}_{\Pi_{l, \nu''}}: V_{(-l, \alpha_{n-1})} \to I_{P_n}(l, \nu'', s)$ characterized by

$$\bar{f}_{\Pi_{l, \nu''}}(\xi_{Q(\gamma)}) = \frac{|\det g|_F \prod_{i=1}^{n} \gamma_i}{\prod_{i=1}^{n} |g_{n,i}|^{n_{sF} - \nu'' s_{F} - 1}/2}$$

for $g = (g_{i,j}) \in G_n$ and $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{N}_0^n$ such that $\ell(\gamma) = -l$.

For $f \in I_{P_n}(l, \nu'', s)$, we define an integral

$$Z_{P_n}(W, W', f) = \int_{Z_{nN_n} \setminus G_n} W(g) W'(g') f(g) \, dg.$$  \hspace{1cm} (2.51)

This integral is equivalent to (2.44) via the correspondence

$$Z(s, W, W', f) = Z_{P_n}(W, W', g_{\Pi_{l, \nu''}}(s))$$

with $g_{\Pi_{l, \nu''}}(s) \in I_{P_n}(l, \nu'', s)$ defined by

$$g_{\Pi_{l, \nu''}}(s)(g) = |\det g|_F \int_{G_{l}} \chi_{-l}(h) \phi(h_{n} g)|h|^{n_{sF} - \nu''} \, dh \quad (g \in G_n).$$

For $g \in G_n$ and $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{N}_0^n$ such that $\ell(\gamma) = |l|$, we have

$$g_{\Pi_{l, \nu''}}(\phi_{l,n}(\xi_{Q(\gamma)})) = \Gamma_F(n s - \nu; l) f_{\Pi_{l, \nu''}}(\xi_{Q(\gamma)}) \quad \text{if } l \geq 0,$n_{sF} - \nu'' s_{F} + 1)/2$$

$$g_{\Pi_{l, \nu''}}(\phi_{l,n}(\xi_{Q(\gamma)})) = \Gamma_F(n s - \nu; -l) \bar{f}_{\Pi_{l, \nu''}}(\xi_{Q(\gamma)}) \quad \text{if } l \leq 0$$

using

$$\int_{0}^{\infty} \exp(-\pi F r^2) t^{s_{F} + m - 2} \, dt = \frac{\Gamma_F(s; m)}{r^{s_{F} + m}/2} \quad (r \in \mathbb{R}_+^*, \, m \in \mathbb{Z}, \, \text{Re}(s_{F} + m) > 0).$$

Hence, Theorem 2.9 gives the explicit descriptions of the integral (2.51) at the minimal $K_n \times K_n \times K_n$-type of $\Pi_{l, \nu} \boxtimes \Pi_{l', \nu'} \boxtimes \Pi_{l'', \nu''}$. We note that

$$v_1 \otimes v_2 \otimes v_3 \mapsto Z_{P_n}(s, W_{\nu}(f_{\nu}(v_1)), W_{-\nu}(\bar{f}_{\nu'}(v_2)), f_{\Pi_{l, \nu''}}(v_3))$$

defines an element of $\text{Hom}_{K_n}(V_{d} \otimes_{\mathbb{C}} V_{d'} \otimes_{\mathbb{C}} V_{d''}, \mathbb{C}_{\text{triv}})$ if $l \geq 0$, and

$$v_1 \otimes v_2 \otimes v_3 \mapsto Z_{P_n}(s, W_{\nu}(f_{\nu}(v_1)), W_{-\nu}(\bar{f}_{\nu'}(v_2)), \bar{f}_{\Pi_{l, \nu''}}(v_3))$$

defines an element of $\text{Hom}_{K_n}(V_{d} \otimes_{\mathbb{C}} V_{d'} \otimes_{\mathbb{C}} V_{(-l, \alpha_{n-1})}, \mathbb{C}_{\text{triv}})$ if $l \leq 0$. By Theorem 2.9 with Lemma 4.3, we obtain the following corollary.
Corollary 2.11. Retain the notation.

(1) Assume \(-d' \in \Xi^0(d)\). Then

\[
\sum_{M \in G(d)} \sum_{M' \in G(-d')} \sum_{P \in G((l, 0_{n-1}))} \xi_{M'}^M r(M') \xi_M \otimes \xi_{M'} \otimes \xi_P,
\]

is a unique \(\mathbb{Q}\)-rational \(K_n\)-invariant vector in \(V_d \otimes_{\mathbb{C}} V_{-d'} \otimes_{\mathbb{C}} V_{(-l, 0_{n-1})}\) up to scalar multiple, and its image under the \(K_n\)-homomorphism (2.54) is

\[
\frac{(-\varepsilon \sqrt{-1})^N \sum_{i=1}^{N-1} (d_i + d'_i) b(-d - d') C^0 (-d' ; d) L(s, \Pi_{d, \nu} \times \Pi_{d', \nu'})}{\Gamma_F(\nu; d) \Gamma_F(\nu'; d') \Gamma_F(ns - \nu''; l)}.
\]

Here \(b(-d - d')\) and \(C^0(-d' ; d)\) are the nonzero rational constants, which are given by (2.43) and (2.42), respectively.

(2) Assume \(d \in \Xi^0(-d')\). Then

\[
\sum_{M \in G(d)} \sum_{M' \in G(-d')} \sum_{P \in G((-l, 0_{n-1}))} \xi_{M'}^M r(M) \xi_M \otimes \xi_{M'} \otimes \xi_P,
\]

is a unique \(\mathbb{Q}\)-rational \(K_n\)-invariant vector in \(V_d \otimes_{\mathbb{C}} V_{-d'} \otimes_{\mathbb{C}} V_{(-l, 0_{n-1})}\) up to scalar multiple, and its image under the \(K_n\)-homomorphism (2.55) is

\[
\frac{(-\varepsilon \sqrt{-1})^N \sum_{i=1}^{N-1} (d_i + d'_i) b(d + d') C^0 (d; -d') L(s, \Pi_{d, \nu} \times \Pi_{d', \nu'})}{\Gamma_F(\nu; d) \Gamma_F(\nu'; d') \Gamma_F(ns - \nu''; l)}.
\]

Here \(b(d + d')\) and \(C^0(d; -d')\) are the nonzero rational constants, which are given by (2.43) and (2.42), respectively.

Remark 2.12. We set \(F = \mathbb{C}\). By [2, Proposition 3.3], we note that the compatible pairs of cohomological representations of \(G_n\) in Grenié [5] can be regarded as pairs of some irreducible principal series representations \(\Pi_{d, \nu}\) and \(\Pi_{d', \nu'}\) with \(d, -d' \in \Lambda_{n, F}\) such that either \(-d' \in \Xi^0(d)\) or \(d \in \Xi^0(-d')\) holds.

Hence, Theorem 2.9 gives a proof of Grenié’s conjecture [5, Conjecture 1] at all critical points (Dong and Xue [2] proved this conjecture only at the central critical point by another method). Corollary 2.11 gives the explicit descriptions of the archimedean part of Grenié’s theorem [5, Theorem 2].

Remark 2.13. Although we use the orthonormal basis \(\{\xi_M\}_{M \in G(\lambda)}\) rather than \(\{\zeta_M\}_{M \in G(\lambda)}\) in the proofs, we state the main theorems in terms of the \(\mathbb{Q}\)-rational basis \(\{\xi_M\}_{M \in G(\lambda)}\) because of the applications in the above remark.

### 3 Recurrence relations

#### 3.1 The Godement section \((G_{n-1} \rightarrow G_n)\)

Let us recall the Godement section, which is defined by Jacquet in [14, §7.1]. Assume \(n > 1\). Let \(d = (d_1, d_2, \cdots, d_n) \in \mathbb{Z}^n\) and \(\nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{C}^n\).
We set \( \tilde{d} = (d_1, d_2, \ldots, d_{n-1}) \in \mathbb{Z}^{n-1} \) and \( \tilde{\nu} = (\nu_1, \nu_2, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1} \). Let \( f \in I(\tilde{d})_{K_{n-1}} \), and we denote by \( \tilde{f} \) the standard section corresponding to \( f \).

Let \( \phi \in S_0(M_{n-1, n}(F)) \). When \( \text{Re}(\nu_n - \nu_i) > -1 \) (\( 1 \leq i \leq n-1 \)), we define the Godement section \( g^{+}_{d_{n-1}, \nu_n}(f_{\tilde{d}}, \phi) \) by the convergent integral

\[
g^{+}_{d_{n-1}, \nu_n}(f_{\tilde{d}}, \phi)(g) = \chi_{d_n}(\det g) |\det g|_{F}^{\nu_n+(n-1)/2} \times \int_{G_{n-1}} \phi((h, O_{n-1, 1})g)f_{\tilde{d}}(h^{-1})\chi_{d_n}(\det h) |\det h|_{F}^{\nu_n+n/2} dh
\]

for \( g \in G_n \). Here we set \( \chi_i(t) = (t/|t|)^l \) \( (l \in \mathbb{Z}, \ t \in F^\times) \) as in §2.3. Jacquet shows that \( g^{+}_{d_{n-1}, \nu_n}(f_{\tilde{d}}, \phi)(g) \) extends to a meromorphic function of \( \nu_n \) in \( \mathbb{C} \), which is a holomorphic multiple of

\[
\prod_{1 \leq i \leq n-1} \Gamma_F(\nu_n - \nu_i + 1; |d_n - d_i|).
\]

Moreover, \( g^{+}_{d_{n-1}, \nu_n}(f_{\tilde{d}}, \phi) \) is an element of \( I(d, \nu)_{K_n} \) if it is defined. For later use, we prepare the following lemma.

**Lemma 3.1.** Retain the notation. Then we have

\[
\Pi_{d, \nu}(k)g^{+}_{d, \nu_n}(f_{\tilde{d}}, \phi) = (\det k)^{d_n}g^{+}_{d_{n-1}, \nu_n}(f_{\tilde{d}}, R(k)\phi) \quad (k \in K_n), \tag{3.1}
\]

\[
(\det k')^{d_n}g^{+}_{d_{n-1}, \nu_n}(\Pi_{d, \nu}(k')f_{\tilde{d}}, L(k')\phi) = g^{+}_{d_{n-1}, \nu_n}(f_{\tilde{d}}, \phi) \quad (k' \in K_{n-1}). \tag{3.2}
\]

**Proof.** When \( \text{Re}(\nu_n - \nu_i) > -1 \) (\( 1 \leq i \leq n-1 \)), the equalities (3.1) and (3.2) follow immediately from the definition. Hence, by the uniqueness of the analytic continuations, we obtain the assertion. \( \square \)

Let \( \varepsilon \in \{ \pm 1 \} \). In [14, §7.2], Jacquet gives convenient integral representations of Whittaker functions. If \( \nu \) satisfies (2.7), then for \( g \in G_n \), we have

\[
W_{\varepsilon}(g^{+}_{d, \nu_n}(f_{\tilde{d}}, \phi))(g) = \chi_{d_n}(\det g) |\det g|_{F}^{\nu_n+(n-1)/2} \times \int_{G_{n-1}} \left( \int_{M_{n-1, 1}(F)} \phi((h, hz)g)\psi_{-\varepsilon}(e_{n-1}z) dz \right) \tag{3.3}
\]

\[
\times W_{\varepsilon}(f_{\tilde{d}})(h^{-1})\chi_{d_n}(\det h) |\det h|_{F}^{\nu_n+n/2} dh,
\]

where \( e_{n-1} = (O_{1, n-2}, 1) \in M_{1, n-1}(F) \). The right hand side of (3.3) converges absolutely for all \( \nu \in \mathbb{C}^n \), and defines an entire function of \( \nu \). Thus the equality holds for all \( \nu \). In the appendix of this paper, we show that the integral representation (3.3) can be regarded as a generalization of the recursive formula [9, Theorem 14] of spherical Whittaker functions.

### 3.2 The Godement section \((G_n \to G_n)\)

In this subsection, we define a new kind of the Godement section. Let \( d \in \mathbb{Z}^n \) and \( \nu \in \mathbb{C}^n \). Let \( f \in I(d)_{K_n} \) and \( \phi \in S_0(M_n(F)) \). We denote by \( f_{\nu} \) the standard
Proposition 3.2. Let \( d \in \mathbb{Z}^n, l \in \mathbb{Z} \) and \( \epsilon \in \{\pm 1\} \). Let \( \Omega \) be an open, relatively compact subset of \( \mathbb{C}^n \). Then there is a constant \( c_0 \) such that, for any \( f \in I(d)K_n \) and \( \phi \in \mathcal{S}_0(M_n(F)) \), the following assertions (i) and (ii) hold:

(i) The integral (3.4) converges absolutely and uniformly on any compact subset of \( \{(s, \nu, g) \in \mathbb{C} \times \Omega \times G_n \mid \text{Re}(s) > c_0\} \).

(ii) Let \( \nu \in \Omega \) and \( s, \nu, g \in \mathbb{C} \) such that \( \text{Re}(s) > c_0 \). Then \( g_{l,s}^\nu(f_{\nu}, \phi) \) is an element of \( I(d)\nu K_n \), satisfying

\[
\Pi_{d,\nu}(k)g_{l,s}^\nu(f_{\nu}, \phi) = (\det k)^{-1}g_{l,s}^\nu(f_{\nu}, L(k)\phi) \quad (k \in K_n), \tag{3.5}
\]

\[
(\det k')^{-1}g_{l,s}^\nu(\Pi_{d,\nu}(k')f_{\nu}, R(k')\phi) = g_{l,s}^\nu(f_{\nu}, \phi) \quad (k' \in K_n). \tag{3.6}
\]

Moreover, for \( g \in G_n \), we have

\[
W_\epsilon(g_{l,s}^\nu(f_{\nu}, \phi))(g) = \int_{G_n} W_\epsilon(f_{\nu})(gh)\phi(h)\chi_l(\det h)|\det h|^{(n-1)/2} dh. \tag{3.7}
\]

Here \( f_{\nu} \) is the standard section corresponding to \( f \).

Proof. For \( g \in G_n \), we set \( ||g|| = \text{Tr}(g^t g) + \text{Tr}((g^{-1})^t (g^{-1})) \) and denote by

\[
g = u(g)a(g)k(g) \quad (u(g) \in U_n, \ a(g) \in A_n, \ k(g) \in K_n)
\]

the decomposition of \( g \) according to \( G_n = U_nA_nK_n \). It is easy to see that

\[
||a(g)|| \leq ||g|| = ||kgk'||, \quad ||gh|| \leq ||g|| ||h||, \quad a(gh) = a(g)a(k(g)h) \tag{3.8}
\]

for \( g, h \in G_n \) and \( k, k' \in K_n \). Since \( G_n \ni g \mapsto \eta_{\nu - \rho_n}(a(g)) \in \mathbb{C} \) is an element of \( I(0_n, \nu) \), we have

\[
\int_{N_n} |\eta_{\nu - \rho_n}(a(x))| dx < \infty \quad (\nu \in \mathbb{C}^n \text{ satisfying } (2.7)). \tag{3.9}
\]

by the absolute convergence of the Jacquet integral [29, Theorem 15.4.1].

We take \( d, l, \epsilon \) and \( \Omega \) as in the statement. Replacing \( \Omega \) with its superset if necessary, we may assume that \( \Omega \) contains an element \( \nu \) satisfying (2.7). By (3.8) and [14, Proposition 3.2], there are a constant \( c_1 \) and a continuous seminorm \( Q \) on \( I(d) \) such that, for any \( \nu \in \Omega, g \in G_n \) and \( f \in I(d) \), the following inequalities hold:

\[
|\eta_{\nu - \rho_n}(a(g))| \leq ||g||^{c_1}, \quad |W_\epsilon(f_{\nu})(g)| \leq ||g||^{c_1}Q(f). \tag{3.10}
\]

By [14, Lemma 3.3 (ii)], there is a positive constant \( c_0 \) such that, for any \( t > c_0 \) and \( \phi \in \mathcal{S}(M_n(F)) \), the integral

\[
\int_{G_n} ||h||^{c_1}\phi(h)|\det h|^{(n-1)/2} dh \tag{3.11}
\]
converges absolutely.

Let \( f \in I(d)_{K_n} \) and \( \phi \in S_0(M_n(F)) \). By (3.8), (3.10) and the definition of 
\( I(d, \nu) \), for \( \nu \in \Omega \), \( x \in \mathbb{N}_n \) and \( g, h \in G_n \), we have an estimate

\[
|f_\nu(xgh)| \leq |\eta_{\nu_\rho}(a(x))||g|^c_1||h|^c_1 \sup_{k \in K_n} |f(k)|. \tag{3.12}
\]

By the absolute convergence of (3.11) and the estimate (3.12) with \( x = 1_n \), we
obtain the assertion (i).

Let \( \nu \in \Omega \) and \( s \in \mathbb{C} \) such that \( \text{Re}(s) > c_0 \). By definition, we have the
equalities (3.5), (3.6) and

\[
g^{\nu}_{l,s}(f_\nu, \phi)(umg) = \chi_a(m)\eta_{\nu_\rho}(m)g^{\nu}_{l,s}(f_\nu, \phi)(g) \tag{3.13}
\]

for \( u \in U_n \), \( m \in M_n \) and \( g \in G_n \). Since \( \Pi_{d, \nu} \) is admissible and 
\( g^{\nu}_{l,s}(f_\nu, \phi) \) is a
continuous \( K_n \)-finite function on \( G_n \) satisfying (3.13), we know that \( g^{\nu}_{l,s}(f_\nu, \phi) \)
is smooth and an element of \( I(d, \nu)_{K_n} \) by [17, Propositions 8.4 and 8.5].

Let \( g \in G_n \). If \( \nu \in \Omega \) satisfies (2.7), we obtain the equality (3.7) as follows:

\[
W_\varepsilon(g^{\nu}_{l,s}(f_\nu, \phi))(g) = \int_{\mathbb{N}_n} g^{\nu}_{l,s}(f_\nu, \phi)(xg)\psi_{-\varepsilon,n}(x) \, dx
\]

\[
= \int_{G_n} \left( \int_{\mathbb{N}_n} f(xgh)\phi(h)\chi_l(\det h)|\det h|^{s+(n-1)/2} \, dh \right)\psi_{-\varepsilon,n}(x) \, dx
\]

\[
= \int_{G_n} \left( \int_{\mathbb{N}_n} f(xgh)\psi_{-\varepsilon,n}(x) \, dx \right)\phi(h)\chi_l(\det h)|\det h|^{s+(n-1)/2} \, dh
\]

\[
= \int_{G_n} W_\varepsilon(f_\nu)(gh)\phi(h)\chi_l(\det h)|\det h|^{s+(n-1)/2} \, dh.
\]

Here the third equality is justified by Fubini’s theorem, since the double integral
converges absolutely by (3.9), (3.12) and the absolute convergence of (3.11).

In order to complete the proof, it suffices to show that the both sides of (3.7)
are holomorphic functions of \((s, \nu)\) on a domain

\[\{(s, \nu) \in \mathbb{C} \times \Omega \mid \text{Re}(s) > c_0\}\.] \tag{3.14}\]

By (3.8), (3.10) and the absolute convergence of (3.11), the integral in the right
hand side of (3.7) converges absolutely and uniformly on any compact subset of
the domain (3.14), and defines a holomorphic function on the domain (3.14).

Let \( S_{\phi,l} \) be a subspace of \( S_0(M_n(F)) \) spanned by \( L(k)\phi \) \((k \in K_n)\), and we
regard \( S_{\phi,l} \) as a \( K_n \)-module via the action \( \det^{-l} \otimes L \). Let \( I_{\phi,l} \) be a subspace of
\( I(d)_{K_n} \) spanned by \( \{T(\phi') \mid \phi' \in S_{\phi,l}, T \in \text{Hom}_{K_n}(S_{\phi,l}, I(d)_{K_n})\} \). Then we
have \( g^{\nu}_{l,s}(f_\nu, \phi)|_{K_n} \in I_{\phi,l} \) by (3.5). Since \( \phi \) is \( K_n \)-finite and \( \Pi_{d, \nu} \) is admissible,
the space \( I_{\phi,l} \) is finite dimensional. Let \( \{f_{\phi,i}\}_{i=1}^m \) be an orthonormal basis of
\( I_{\phi,l} \) with respect to the \( L^2 \)-inner product

\[
\langle f_1, f_2 \rangle_{L^2} = \int_{K_n} f_1(k)\overline{f_2(k)} \, dk \quad \text{for} \quad (f_1, f_2 \in I(d)).
\]

24
Let $H$, hence we obtain the assertion by Proposition 3.2.

3.3 Recurrence relations with the Godement sections

On the domain (3.14), we know that the right hand side of (3.7) is holomorphic on the domain (3.14).

Remark 3.3. The equality (3.7) with $l = 0$ can be regarded as the local theta correspondence for a principal series representation $\Pi_{d,\nu}$ in [30, §2].

3.3 Recurrence relations with the Godement sections

Let $\varepsilon \in \{\pm 1\}$. For $\phi \in S(M_{n,1}(F))$, we define $\mathcal{F}_\varepsilon(\phi) \in S(M_{1,n}(F))$ by

$$\mathcal{F}_\varepsilon(\phi)(t) = \int_{M_{n,1}(F)} \phi(z) \psi_{\varepsilon}(tz) \, d_F z \quad (t \in M_{1,n}(F)).$$

Let

\[ d = (d_1, d_2, \cdots, d_n) \in \mathbb{Z}^n, \quad \nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{C}^n, \]

\[ d' = (d'_1, d'_2, \cdots, d'_{n'}) \in \mathbb{Z}^{n'}, \quad \nu' = (\nu'_1, \nu'_2, \cdots, \nu'_{n'}) \in \mathbb{C}^{n'}. \]

If $n > 1$, we set $\overrightarrow{d} = (d_1, d_2, \cdots, d_{n-1})$ and $\overrightarrow{\nu} = (\nu_1, \nu_2, \cdots, \nu_{n-1})$. If $n' > 1$, we set $\overrightarrow{d'} = (d'_1, d'_2, \cdots, d'_{n'-1})$ and $\overrightarrow{\nu'} = (\nu'_1, \nu'_2, \cdots, \nu'_{n'-1})$.

Proposition 3.4 ($G_n \times G_n \rightarrow G_n \times G_{n-1}$). Retain the notation, and assume $n' = n > 1$. Let $f \in I(d)_{K_n}$ and $f' \in I(\overrightarrow{d'})_{K_{n-1}}$. We denote by $f_\nu$ and $f'_\nu$ the standard sections corresponding to $f$ and $f'$, respectively. Let $\phi_1 \in S_0(M_{n-1,n}(F))$ and $\phi_2 \in S_0(M_{1,n}(F))$. For $s \in \mathbb{C}$ such that $\text{Re}(s)$ is sufficiently large, we have

\[
Z(s, \mathcal{F}_\varepsilon(f_\nu), W_{-\varepsilon}(g^\circ_{d'_{n',\nu'},n'}(f'_\nu, \phi_1)), \phi_2) = Z(s, \mathcal{F}_\varepsilon(g^\circ_{d',\nu',n}(f_\nu, \phi_0)), W_{-\varepsilon}(f'_\nu)),
\]

where $\phi_0 \in S_0(M_n(F))$ is defined by

$$\phi_0(z) = \phi_1((1_{n-1}, O_{n-1,1})z) \phi_2(e_n z) \quad (z \in M_n(F)).$$

Proof. Using (3.3), Jacquet shows the following equality [14, (8.1)]:

\[
Z(s, \mathcal{F}_\varepsilon(f_\nu), W_{-\varepsilon}(g^\circ_{d',\nu',n}(f'_\nu, \phi_1)), \phi_2) = \int_{N_{n-1} \backslash G_{n-1}} \left( \int_{G_n} W_{\varepsilon}(f_\nu(t_n(h)g)) \phi_0(g) \chi_{d_n}(\det g) \det g|_F^{s+\nu'+(n-1)/2} \, dg \right) \times W_{-\varepsilon}(f'_\nu(h)) \det h|_{F'}^{-1/2} \, dh.
\]

Hence, we obtain the assertion by Proposition 3.2.
Proposition 3.5 \((G_n \times G_{n-1} \to G_{n-1} \times G_{n-1})\). Retain the notation, and assume \(n' = n - 1\). Let \(f \in I(d)_{K_{n-1}}\) and \(f' \in I(d')_{K_{n-1}}\). We denote by \(f_\nu\) and \(f'_\nu\) the standard sections corresponding to \(f\) and \(f'\), respectively. Let \(\phi_1 \in S_0(M_{n-1}(F))\) and \(\phi_2 \in S_0(M_{n-1,1}(F))\). For \(s \in \mathbb{C}\) such that \(\text{Re}(s)\) is sufficiently large, we have

\[
Z(s, W_\varepsilon(g_{d_\nu,\nu}(f_\nu, \phi_0)), W_\varepsilon(f'_\nu)) = Z(s, W_\varepsilon(f_\nu), W_\varepsilon(g_{d_\nu,\nu}(f'_\nu, \phi_1)), W_\varepsilon(\phi_2)),
\]

where \(\phi_0 \in S_0(M_{n-1,n}(F))\) is defined by

\[
\phi_0(z) = \phi_1(z^l(1_{n-1}, O_{n-1,1}))\phi_2(z^l e_n) \quad (z \in M_{n-1,n}(F)).
\]

Proof. Using (3.3), Jacquet shows the following equality [14, (8.3)]:

\[
Z(s, W_\varepsilon(g_{d_\nu,\nu}(f_\nu, \phi_0)), W_\varepsilon(f'_\nu)) = \int_{G_{n-1} \backslash G_{n-1}} \left( \int_{G_{n-1}} W_\varepsilon(f'_\nu)(gh)\phi_1(h)\chi_{d_\nu}(\det h)\det h_1^{s+\nu+(n-2)/2} dh \right) \\
\times W_\varepsilon(f_\nu)(g)W_\varepsilon(\phi_2)(e_{n-1}g)\det g_1^\frac{n}{2} dg.
\]

Hence, we obtain the assertion by Proposition 3.2.

\[\square\]

4 Finite dimensional representations

4.1 The Clebsch–Gordan coefficients

Let \(\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t) \in \Lambda_t\), \(t \in \mathbb{N}_0\) and \(\lambda' = (\lambda'_1, \lambda'_2, \cdots, \lambda'_t) \in \Xi^0(\lambda; l)\). By Pieri’s rule (2.36), we can take a \(\text{GL}(n, \mathbb{C})\)-homomorphism \(\tilde{I}^\lambda_{\lambda'}: V_{\lambda'} \to V_{\lambda} \otimes_{\mathbb{C}} V_{(l,0_{n-1})}\) satisfying

\[
\langle \tilde{I}^\lambda_{\lambda'}(v), \tilde{I}^\lambda_{\lambda'}(v') \rangle = \langle v, v' \rangle \quad (v, v' \in V_{\lambda'}).
\]

Such \(\tilde{I}^\lambda_{\lambda'}\) is unique up to multiplication by scalars in \(U(1)\). We set

\[
\tilde{I}^\lambda_{\lambda'}(\zeta_{M'}) = \sum_{M \in G(\lambda)} \sum_{P \in G((l,0_{n-1}))} C^{M,P}_{M'} \zeta_M \otimes \zeta_P \quad (M' \in G(\lambda')).
\]

Then we call \(C^{M,P}_{M'}\) \((M \in G(\lambda), P \in G((l,0_{n-1})), M' \in G(\lambda'))\) the Clebsch–Gordan coefficients. When \(n = 1\), we may normalize \(C^{\lambda_1,l}_{\lambda_1+l,l} = 1\), since

\[
\Xi^0(\lambda_1;l) = \{\lambda_1 + l\}, \quad G(\lambda_1) = \{\lambda_1\}, \quad G(l) = \{l\}, \quad G(\lambda_1 + l) = \{\lambda_1 + l\}.
\]

We consider the case \(n > 1\). Let \(\mu \in \Xi^+(\lambda)\) and \(0 \leq q \leq l\). By Lemma 2.2, there are some constants

\[
\left( \begin{array}{c|c} \lambda & l \\ \mu & q \end{array} \right) \quad (\mu' \in \Xi^+(\lambda') \cap \Xi^0(\mu; q))
\]

(4.2)
such that, for any $M' \in G(\lambda'; \mu')$, $M \in G(\lambda; \mu)$, $P \in G((l, 0_{n-1}); (q, 0_{n-2}))$ and $\mu' \in \Xi^+(\lambda')$, the following equality hold:

$$C_{M,M'}^{M,P} = \begin{cases} \left( \begin{array}{ccc} \lambda, & l & \mu' \\ \mu, & q & \mu' \end{array} \right) C_{\tilde{M}, \tilde{M}'}^{\tilde{M}, \tilde{P}} & \text{if } \mu' \in \Xi^0(\mu; q), \\ 0 & \text{otherwise.} \end{cases}$$ (4.3)

The constants (4.2) are called the isoscalar factors. In [16] (see also [1] and [28, Chapter 18]), Jucys gives the following expressions of them under some normalization of $\tilde{I}^\lambda_{\lambda'}$:

$$\left( \begin{array}{ccc} \lambda, & l & \mu' \\ \mu, & q & \mu' \end{array} \right) = \sqrt{\frac{(l-q)!S^0(\lambda', \lambda)S^+(\lambda, \mu)S^0(\mu', \mu)S^+(\lambda', \mu')}{S^0(\lambda', \lambda)S^+(\lambda', \mu')}} \times \sum_{\alpha \in \Xi^+(\lambda') \cap \Xi^+(\lambda)} \frac{(-1)^{l(\alpha-\mu)}S^0(\alpha, \alpha)}{S^0(\mu', \alpha)S^0(\alpha, \mu)} S^+(\lambda, \alpha)$$ (4.4)

for $\mu' \in \Xi^+(\lambda') \cap \Xi^+(\mu; q)$, where the symbols $S^0(\lambda', \lambda)$ and $S^+(\lambda, \mu)$ are defined by (2.37) and (2.38), respectively. Hereafter, we assume that $\tilde{I}^\lambda_{\lambda'}$ is normalized so that (4.4) holds. Then we have

$$C_{M',Q(\mu)}^{M,Q((0_{n-1}, l))} = \sqrt{\frac{\Pi S^0(\lambda', \lambda)S^+(\lambda', \mu)}{S^0(\lambda', \lambda)S^+(\lambda, \mu)}} \quad \text{and} \quad C_{M',Q(\mu)}^{M,H(\lambda')} = \sqrt{S^0(\lambda; \lambda)}$$ (4.5)

for $M \in G(\lambda)$ and $M' \in G(\lambda')$ such that $\tilde{M} = \tilde{M}' \in G(\mu)$. Here $H(\lambda)$, $Q(\gamma)$ ($\gamma \in \mathbb{N}^n$) and $S^0(\lambda'; \lambda)$ are defined by (2.14), (2.33) and (2.42), respectively. All the Clebsch–Gordan coefficients $C_{M', P}$ are real numbers, and we have

$$\zeta_M \otimes \zeta_P = \sum_{\lambda' \in \Xi^+(\lambda; \lambda)} \sum_{M' \in G(\lambda')} \langle \zeta_M \otimes \zeta_P, \tilde{I}^\lambda_{\lambda'}(\lambda_M) \rangle \tilde{I}^\lambda_{\lambda'}(\lambda_{M'}) = \sum_{\lambda' \in \Xi^+(\lambda; \lambda)} \sum_{M' \in G(\lambda')} C_{M', P}^{M, P} \tilde{I}^\lambda_{\lambda'}(\lambda_{M'})$$ (4.6)

for $M \in G(\lambda)$ and $P \in G((l_0, 0_{n-1}))$. We set

$$\tilde{I}^\lambda_{\lambda'} = \sqrt{b(\lambda' - \lambda)C^0(\lambda'; \lambda)} \tilde{I}^\lambda_{\lambda'}.$$

where $b(\lambda' - \lambda)$ is defined by (2.43). Then the explicit expression (2.39) of $\tilde{I}^\lambda_{\lambda'}$ follows from (2.16) and (4.4), since

$$C_{M, P}^{M, P} = \sqrt{b(\lambda' - \lambda)C^0(\lambda'; \lambda) r(M) r(\lambda')} C_{M', P}^{M, P}$$

for $M \in G(\lambda)$, $P \in G((l, 0_{n-1}))$ and $M' \in G(\lambda')$. Here $r(M)$ is defined by (2.16). The equalities (2.41) and (2.40) follow from (4.1) and (4.5), respectively.
4.2 Some lemmas for representations of $K_n$

Let $C_{\text{triv}} = \mathbb{C}$ be the trivial $\text{GL}(n, \mathbb{C})$-module. The purpose of this subsection is to give proofs of Lemma 2.4 and the following three lemmas.

**Lemma 4.1.** Let $\lambda \in \Lambda_{n,F}$.

1. The space $\text{Hom}_{K_n}(V_\lambda \otimes_{\mathbb{C}} V_\lambda, C_{\text{triv}})$ is a 1 dimensional space spanned by the $\mathbb{C}$-linear map
   \[ V_\lambda \otimes_{\mathbb{C}} V_\lambda \ni v_1 \otimes v_2 \mapsto \langle v_1, v_2 \rangle \in \mathbb{C}. \]

2. Let $\lambda' \in \Lambda_{n,F} \cap \Xi^0(\lambda)$, and set $l = \ell(\lambda' - \lambda)$. For $\lambda'' \in \Xi^0(\lambda;l)$ such that $\lambda'' \neq \lambda'$, we have $\text{Hom}_{K_n}(V_{\lambda'} \otimes_{\mathbb{C}} V_{\lambda''}, C_{\text{triv}}) = \{0\}$.

**Lemma 4.2.** Assume $n > 1$, and we regard $K_{n-1}$ as a subgroup of $K_n$ via $(2.20)$. Let $\lambda \in \Lambda_{n,F}$ and $\mu \in \Lambda_{n-1,F}$. Then $\text{Hom}_{K_{n-1}}(V_\lambda \otimes_{\mathbb{C}} V_\mu, C_{\text{triv}})$ is a 1 dimensional space spanned by the $\mathbb{C}$-linear map
   \[ V_\lambda \otimes_{\mathbb{C}} V_\mu \ni v_1 \otimes v_2 \mapsto (R^\lambda_\mu(v_1), v_2) \in C_{\text{triv}} \]
   if $\mu \in \Xi^+(\lambda)$, and is equal to $\{0\}$ otherwise. Here $R^\lambda_\mu$ is defined by (2.22). Moreover, if $\mu \in \Xi^+(\lambda)$, then
   \[ \sum_{M \in G(\mu)} r(M)^{-1} \xi_{M[\lambda]} \otimes \xi_M \]
   is a unique $\mathbb{Q}$-rational $K_{n-1}$-invariant vector in $V_\lambda \otimes_{\mathbb{C}} V_\mu$ up to scalar multiple.

**Lemma 4.3.** Let $\lambda, \lambda' \in \Lambda_{n,F}$ such that $\ell(\lambda' - \lambda) \geq 0$. Let $l \in \mathbb{N}_0$. If $F = \mathbb{R}$, we assume $l \in \{0, 1\}$. Then $\text{Hom}_{K_n}(V_{\lambda'} \otimes_{\mathbb{C}} V_{\lambda} \otimes_{\mathbb{C}} V_{(1,0_{n-1})}, C_{\text{triv}})$ is a 1 dimensional space spanned by the $\mathbb{C}$-linear map
   \[ V_{\lambda'} \otimes_{\mathbb{C}} V_{\lambda} \otimes_{\mathbb{C}} V_{(1,0_{n-1})} \ni v_1 \otimes v_2 \otimes v_3 \mapsto (\Pi^\lambda_{\lambda'}(v_1), v_2 \otimes v_3) \in C_{\text{triv}} \]
   if $\lambda' \in \Xi^0(\lambda;l)$, and is equal to $\{0\}$ otherwise. Moreover, if $\lambda' \in \Xi^0(\lambda;l)$, then
   \[ \sum_{M' \in G(\lambda')} \sum_{M \in G(\lambda)} \sum_{P \in G(\{(1,0_{n-1})\})} \frac{\xi'_{M',P} \otimes \xi_M \otimes \xi_P}{r(M')} \]
   is a unique $\mathbb{Q}$-rational $K_{n}$-invariant vector in $V_{\lambda'} \otimes_{\mathbb{C}} V_{\lambda} \otimes_{\mathbb{C}} V_{(1,0_{n-1})}$ up to scalar multiple.

Since proofs of Lemmas 2.4, 4.1, 4.2 and 4.3 are easy for $F = \mathbb{C}$, the main concern is the case of $F = \mathbb{R}$. We have $\Lambda_{n,\mathbb{R}} = \{1_j, 0_{n-j} \mid 0 \leq j \leq n\}$ with $1_j = (1, 1, \cdots, 1) \in \mathbb{Z}^j$ and $0_{n-j} = (0, 0, \cdots, 0) \in \mathbb{Z}^{n-j}$. Here we erase the symbol $1_j$ if $j = 0$, and erase the symbol $0_{n-j}$ if $j = n$. Let $0 \leq l \leq n$, and we regard the $l$-th exterior power $\Lambda^l(M_{n,1}(\mathbb{C}))$ of $M_{n,1}(\mathbb{C})$ as a $\text{GL}(n, \mathbb{C})$-module via the action derived from the matrix multiplication. Then we have $V_{(1,0_{n-1})} \simeq \Lambda^l(M_{n,1}(\mathbb{C}))$ as $\text{GL}(n, \mathbb{C})$-modules via the correspondence
\[ \zeta_M \leftrightarrow \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_l} \quad (M \in G((1,0_{n-l}))) \]
with $1 \leq i_1 < i_2 < \cdots < i_t \leq n$ such that $\gamma_i^M = 1$ ($i \in \{i_1, i_2, \ldots, i_t\}$). Here $e_j$ is the matrix unit in $M_{n,1}(\mathbb{C})$ with 1 at the $(j, 1)$-th entry ($1 \leq j \leq n$) and 0 at other entries. We identify $V_{(1, o_{n-i})}$ with $\Lambda^t(M_{n,1}(\mathbb{C}))$ via this isomorphism.

We have $O(n) = SO(n) \sqcup SO(n)k_0$ with $k_0 = \text{diag}(1,1, \ldots, 1, -1) \in O(n)$ and $SO(n) = \{k \in O(n) \mid \det k = 1\}$. The complexification $so(n)_C$ of the associated Lie algebra $so(n)$ of $SO(n)$ is given by $so(n)_C = \bigoplus_{1 \leq i < j \leq n} CE_{i,j}$, with $E_{i,j} = E_{i,j} - E_{j,i}$. Here we understand $k_0 = -1$ and $so(1)_C = \{0\}$ if $n = 1$. Let us recall some facts in the highest weight theory [17, Theorem 4.28] for $SO(n)$. Let $m$ be the largest integer such that $2m \leq n$. When $n \geq 2$, for an irreducible representation $(\tau, V_{\tau})$ of $SO(n)$, there is a nonzero vector $v_0$ in $V_{\tau}$ such that, for $1 \leq i \leq m$ and $2i + 1 \leq j \leq n$,

\[
\tau(E_{2i-1,2i})v_0 = \sqrt{-1} \lambda_i v_0, \quad \tau(E_{2i-1,2i} + \sqrt{-1} E_{2i,2i})v_0 = 0
\]

with some $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{Z}^m$. Such vector $v_0$ is unique up to nonzero scalar multiple, and we call $v_0$ an $SO(n)$-highest weight vector of weight $\lambda$. The weight $\lambda = \lambda$ is called the highest weight of $\tau$, and $\tau \mapsto \lambda$ gives the bijection from the set of equivalence classes of irreducible representations of $SO(n)$ to the set of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m-n}, \lambda_m) \in \Lambda_m$ satisfying

\[
\begin{cases} 
(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, -\lambda_m) \in \Lambda_m & \text{if } n \text{ is even,} \\
\lambda_m \geq 0 & \text{if } n \text{ is odd.}
\end{cases}
\]

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m-n}, -\lambda_m) \in \Lambda_m$ such that $\lambda_m \geq 0$, we take a representation $(\tau_{so(n), \lambda}, V_{so(n), \lambda})$ of $SO(n)$ so that $\tau_{so(n), \lambda}$ is a direct sum of two irreducible representations with highest weights $\lambda$ and $(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, -\lambda_m)$ if $n = 2m$ and $\lambda_m > 0$, and $\tau_{so(n), \lambda}$ is an irreducible representation with highest weight $\lambda$ otherwise. By Weyl’s dimension formula [17, Theorem 4.48], we have

\[
\dim V_{so(n), (i+1, 1, \ldots, 0, \ldots, 1, o_{m-h})} = \frac{(2i + n)}{(i + l)(i + n - l)} \frac{1}{i!} \frac{1}{(n - l)!} \frac{1}{(l - 1)!} (4.8)
\]

for $1 \leq h \leq m$ and $i \in \mathbb{N}_0$.

**Lemma 4.4.** Retain the notation. Let $0 \leq l \leq n$. As an $O(n)$-module, $V_{(1, o_{n-l})}$ is irreducible and $V_{(1, o_{n-l})} \not\cong V_{(1, o_{n-l'})}$ for any $0 \leq l' \leq n$ such that $l' \neq l$. We set $h = \min\{l, n - l\}$. When $n \geq 2$, we have

\[
V_{(1, o_{n-l})} \cong V_{so(n), (1, o_{m-h})} \quad \text{as } SO(n)-\text{modules.} (4.9)
\]

**Proof.** For $1 \leq i_1 < i_2 < \cdots < i_t \leq n$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{\pm 1\}$, we have

\[
\tau_{(1, o_{n-l})}(\text{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)) \cdot e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_t} = \varepsilon_{i_1} \varepsilon_{i_2} \cdots \varepsilon_{i_t} \cdot e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_t}.
\]

By this equality, we know that $\text{Hom}_{O(n)}(V_{(1, o_{n-l})}, V_{(1, o_{n-l'})}) = \{0\}$ for any $0 \leq l' \leq n$ such that $l' \neq l$. Hence, our task is to show (4.9) and the irreducibility of $V_{(1, o_{n-l})}$ as an $O(n)$-module.
In the case of \( n \geq 2 \) and \( n \neq 2l \), the isomorphism (4.9) follows from [17, Examples in Chapter IV, §7], and we note that \( V_{(1,0_{n-i})} \) is an irreducible \( O(n) \)-module. In the case of \( n = 1 \), the irreducibility of an \( O(1) \)-module \( V_{(1,0_{n-i})} \) is trivial. Let us consider the case of \( n = 2l \). By direct computation, for \( \varepsilon \in \{ \pm 1 \} \), we can confirm that

\[
v_{\varepsilon} = (\varepsilon_1 + \sqrt{-1}\varepsilon_2) \wedge (\varepsilon_3 + \sqrt{-1}\varepsilon_4) \wedge \cdots \wedge (\varepsilon_{n-3} + \sqrt{-1}\varepsilon_{n-2}) \wedge (\varepsilon_{n-1} + \varepsilon\sqrt{-1}\varepsilon_n)
\]

is an \( SO(n) \)-highest weight vector of weight \((1_{l-1}, \varepsilon)\) in \( V_{(1,0_1)} \) and satisfies \( \tau_{(1,0_{n-i})}(k_0)v_{\varepsilon} = v_{-\varepsilon} \). Since \( \dim V_{(1,0_1)} = \dim V_{SO(n),1} \) by (4.8), we know that (4.9) holds and \( V_{(1,0_1)} \) is an irreducible \( O(n) \)-module.

**Proof of Lemma 2.4.** The assertion for \( F = \mathbb{R} \) follows immediately from Lemma 4.4. The assertion for \( F = \mathbb{C} \) follows immediately from the highest weight theory [17, Theorem 4.28] for \( U(n) \).

**Lemma 4.5.** Assume \( n \geq 2 \). Let \( 1 \leq l \leq n - 1 \). We set \( h = \min\{l, n - l\} \). Let

\[ I_l : V_{(1_{l-1},0_{n-l+1})} \to V_{(1,0_{n-1})} \otimes C V_{(1,0_{n-1})} \]

be a \( C \)-linear map defined by

\[
I_l(\varepsilon_1 \wedge \varepsilon_2 \wedge \cdots \wedge \varepsilon_{i-1}) = \sum_{j=1}^{n} (\varepsilon_1 \wedge \varepsilon_2 \wedge \cdots \wedge \varepsilon_{i-1} \wedge \varepsilon_j) \otimes \varepsilon_j
\]

for \( i_1, i_2, \ldots, i_{l-1} \in \{1, 2, \ldots, n\} \). Here we understand \( I_l(1) = \sum_{j=1}^{n} \varepsilon_j \otimes \varepsilon_j \) if \( l = 1 \). Then \( I_l \) is an \( O(n) \)-homomorphism. Moreover, there is an \( SO(n) \)-submodule \( V' \) of \( V_{(1,0_{n-1})} \otimes C V_{(1,0_{n-1})} \) such that \( V' \cong V_{SO(n),(2,1_{n-1},0_{m-h})} \) and

\[
I_l(V_{(1,0_{n-1})}) \otimes C V_{(1,0_{n-1})} = I_l(1_{l+1})V_{(1,0_{n-1})} \oplus I_l(V_{(1_{l-1},0_{n-l+1})}) \oplus V',
\]

(4.10)

where \( I_l(1_{l+1}) \) is the \( U(n) \)-homomorphism given by (2.39).

**Proof.** For \( v \in V_{(1_{l-1},0_{n-l+1})} \) and \( 1 \leq i < j \leq n \), we have

\[
I_l(\tau_{(1_{l-1},0_{n-l+1})}(F_{i,j}^{SO(n)}))v = (\tau_{(1_{l-1},0_{n-l+1})} \otimes \tau_{(1,0_{n-1})})(F_{i,j}^{SO(n)}))I_l(v),
\]

\[
I_l(\tau_{(1_{l-1},0_{n-l+1})}(k_0))v = (\tau_{(1_{l-1},0_{n-l+1})} \otimes \tau_{(1,0_{n-1})})(k_0)I_l(v)
\]

by direct computation. Hence, \( I_l \) is an \( O(n) \)-homomorphism.

For an \( SO(n) \)-highest weight vector \( v \) of weight \( \lambda \) in \( V_{(1,0_{n-1})} \), we note that \( v \otimes (\varepsilon_1 + \sqrt{-1}\varepsilon_2) \) is an \( SO(n) \)-highest weight vector of weight \( \lambda + (1,0_{n-1}) \) in \( V_{(1,0_{n-1})} \otimes C V_{(1,0_{n-1})} \). Hence, by Lemma 4.4, there is an \( SO(n) \)-submodule \( V' \) of \( V_{(1,0_{n-1})} \otimes C V_{(1,0_{n-1})} \) such that \( V' \cong V_{SO(n),(2,1_{n-1},0_{m-h})} \) and

\[
\{I_l(1_{l+1})V_{(1_{l+1},0_{n-l+1})} \oplus I_l(V_{(1_{l-1},0_{n-l+1})}) \} \cap V' = \{0\}
\]
By (4.8), we know that \( \dim V_{(1,0_{n-1})} \otimes_{\mathbb{C}} V_{(1,0_{n-1})} \) is equal to
\[
\dim V_{(1,0_{n-1})} + \dim V_{(1,0_{n-1}+1)} + \dim V_{\text{so}(n),(2,1_{h-1},0_{m-h})}.
\]
This implies that (4.10) holds. \( \square \)

**Lemma 4.6.** Let \((\tau, V_\tau)\) and \((\tau', V_{\tau'})\) be finite dimensional representations of \(\text{GL}(n, \mathbb{C})\). Let \(\langle \cdot, \cdot \rangle\) be a \(U(n)\)-invariant hermitian inner product on \(V_\tau\). Let \(\{v_i\}_{i=1}^d\) be an orthonormal basis of \(V_\tau\).

1. A \(\mathbb{C}\)-linear map \(\Psi_1 : \text{Hom}_\mathbb{C}(V_{\tau'}, V_\tau) \to \text{Hom}_\mathbb{C}(V_{\tau'}, \mathbb{C} V_{\tau}, \mathbb{C}_{\text{triv}})\) defined by
\[
\Psi_1(f)(v' \otimes \tau) = \langle f(v'), v \rangle \quad (f \in \text{Hom}_\mathbb{C}(V_{\tau'}, V_\tau), \ v' \in V_{\tau'}, \ v \in V_\tau)
\]
is bijective, and its inverse map is given by
\[
\Psi_1^{-1}(f)(v') = \sum_{i=1}^d f(v' \otimes \tau) v_i \quad (f \in \text{Hom}_\mathbb{C}(V_{\tau'}, \mathbb{C} V_{\tau}, \mathbb{C}_{\text{triv}}), \ v' \in V_{\tau'}).
\]
Moreover, we have \(\Psi_1(\text{Hom}_{K_n}(V_{\tau'}, V_\tau)) = \text{Hom}_{K_n}(V_{\tau'}, \mathbb{C} V_{\tau}, \mathbb{C}_{\text{triv}})\).

2. A \(\mathbb{C}\)-linear map \(\Psi_2 : V_{\tau'} \otimes_{\mathbb{C}} V_{\tau} \to \text{Hom}_\mathbb{C}(V_{\tau'}, V_\tau)\) defined by
\[
\Psi_2(v' \otimes \tau)(v_2) = \langle v_2, v_1 \rangle v' \quad (v_1, v_2 \in V_{\tau'}, \ v' \in V_{\tau'})
\]
is bijective, and its inverse map is given by
\[
\Psi_2^{-1}(f) = \sum_{i=1}^d f(v_i) \otimes \tau \quad (f \in \text{Hom}_\mathbb{C}(V_{\tau'}, V_\tau)).
\]
Moreover, we have \(\Phi_2((V_{\tau'} \otimes_{\mathbb{C}} V_{\tau})_{K_n}) = \text{Hom}_{K_n}(V_{\tau'}, V_\tau)\), where \((V_{\tau'} \otimes_{\mathbb{C}} V_{\tau})_{K_n}\) is the subspace of \(V_{\tau'} \otimes_{\mathbb{C}} V_{\tau}\) consisting of all \(K_n\)-invariant vectors.

**Proof.** The former part of the statement (1) follows from definition. The latter part of the statement (1) follows from
\[
\Psi_1(f)((\tau' \otimes \tau)(k)v' \otimes \tau) = \langle f((\tau' \otimes \tau)(k)v'), v \rangle = \langle \tau(k^{-1})f((\tau' \otimes \tau)(k)v'), v \rangle
\]
\[
= \Psi_1(\tau(k^{-1}) \circ f \circ \tau'(k))(v' \otimes \tau)
\]
for \(f \in \text{Hom}_\mathbb{C}(V_{\tau'}, V_\tau), \ v' \in V_{\tau'}, \ v \in V_\tau\) and \(k \in K_n\).

The former part of the statement (2) follows from definition. The latter part of the statement (2) follows from
\[
\Psi_2((\tau' \otimes \tau)(k)v' \otimes \tau)(v_2) = \langle v_2, \tau(k)v_1 \rangle \tau'(k)v' = \langle \tau(k^{-1})v_2, v_1 \rangle \tau'(k)v'
\]
\[
= \tau'(k)\Psi_2(v' \otimes \tau_2)(\tau(k^{-1})v_2)
\]
for \(v_1, v_2 \in V_{\tau'}, \ v' \in V_{\tau'}, \) and \(k \in K_n\). \( \square \)
Proof of Lemma 4.1. Let $\lambda \in \Lambda_{n,F}$. By Lemma 2.4, we note that the space $\text{Hom}_{K_n}(V_\lambda, V_\lambda)$ is a 1 dimensional space spanned by the identity map. Hence, we obtain the statement (1) by Lemma 4.6 (1).

Let $\lambda' \in \Lambda_{n,F} \cap \Xi^0(\lambda)$, and we set $l = \ell(\lambda' - \lambda)$. By the decompositions (2.36), (4.10) and Lemma 4.4, we have $\text{Hom}_{K_n}(V_\lambda', V_{\lambda''}) = \{0\}$ for $\lambda'' \in \Xi^0(\lambda;l)$ such that $\lambda'' \neq \lambda'$. Hence, we obtain the statement (2) by Lemma 4.6 (1).

Proof of Lemma 4.2. By (2.21) and Lemma 2.4, we know that $\text{Hom}_{K_n}(V_\lambda, V_\mu)$ is equal to $\mathbb{C} R^{\lambda}_{\mu}$ if $\mu \in \Xi^+(\lambda)$, and is equal to $\{0\}$ otherwise. By Lemma 4.6, we obtain the former part of the assertion, and know that, if $\mu \in \Xi^+(\lambda)$,

$$\sum_{\mu' \in G(\mu)} R^{\lambda}_{\mu'}(\zeta_{\mu'}) \otimes \overline{\zeta_{\mu'}}$$

is a unique $K_{n-1}$-invariant vector in $V_{\mu} \otimes_{\mathbb{C}} \overline{V_\lambda}$ up to scalar multiple. Hence, by (2.15) and the properties of complex conjugate representations in §2.6, we obtain the latter part of the assertion.

Proof of Lemma 4.3. By the decompositions (2.36), (4.10) and Lemma 4.4, we know that $\text{Hom}_{K_n}(V_{\lambda'}, V_\lambda \otimes_{\mathbb{C}} V_{(l; 0_{n-1})})$ is equal to $\mathbb{C} \Gamma^{\lambda}_{\lambda'}$ if $\lambda' \in \Xi^+(\lambda;l)$, and is equal to $\{0\}$ otherwise. By Lemma 4.6, we obtain the former part of the assertion, and know that, if $\lambda' \in \Xi^+(\lambda;l)$,

$$\sum_{\mu' \in G(\lambda')} \Gamma^{\lambda}_{\lambda'}(\zeta_{\mu'}) \otimes \overline{\zeta_{\mu'}}$$

is a unique $K_n$-invariant vector in $V_{\lambda} \otimes_{\mathbb{C}} V_{(l; 0_{n-1})} \otimes_{\mathbb{C}} \overline{V_{\lambda'}}$ up to scalar multiple. Hence, by (2.15) and the properties of complex conjugate representations in §2.6, we obtain the latter part of the assertion.

4.3 Polynomial functions

We set

$$\Lambda_{n}^{\text{poly}} = \{ \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n \mid \lambda_n \geq 0 \}.$$ 

We denote by $\mathcal{P}(M_{n,n'}(\mathbb{C}))$ the subspace of $C(M_{n,n'}(\mathbb{C}))$ consisting of all polynomial functions. Let $l \in \mathbb{N}_0$. We denote by $\mathcal{P}_l(M_{n,n'}(\mathbb{C}))$ the subspace of $\mathcal{P}(M_{n,n'}(\mathbb{C}))$ consisting of all degree $l$ homogeneous polynomial functions. We regard $\mathcal{P}_l(M_{n,n'}(\mathbb{C}))$ as $\text{GL}(n, \mathbb{C}) \times \text{GL}(n', \mathbb{C})$-module via the action $L \boxtimes R$ which is defined in §2.9. Let $q = \min\{n, n'\}$. Then the $\text{GL}(n)$-$\text{GL}(n')$ duality [4, Theorem 5.6.7] asserts that

$$\mathcal{P}_l(M_{n,n'}(\mathbb{C})) \simeq \bigoplus_{\lambda \in \Lambda_{n}^{\text{poly}}, \ell(\lambda) = l} V_{(\lambda, 0, n-q)} \boxtimes_{\mathbb{C}} V_{(\lambda, 0, n'-q)}$$

as $\text{GL}(n, \mathbb{C}) \times \text{GL}(n', \mathbb{C})$-modules. Since $V_{(\lambda, 0, n-q)} \simeq \overline{V_{(\lambda, 0, n-q)}}$ as $U(n)$-modules, we also have

$$\mathcal{P}_l(M_{n,n'}(\mathbb{C})) \simeq \bigoplus_{\lambda \in \Lambda_{n}^{\text{poly}}, \ell(\lambda) = l} \overline{V_{(\lambda, 0, n-q)}} \boxtimes_{\mathbb{C}} V_{(\lambda, 0, n'-q)} \quad (4.11)$$

32
as $U(n) \times \text{GL}(n', \mathbb{C})$-modules.

The purpose of this subsection is to construct polynomial functions, explicitly. We define $U(n) \times \text{GL}(n, \mathbb{C})$-homomorphisms $P_\lambda^n: \overline{V}_\lambda \otimes \mathbb{C} V_\lambda \rightarrow \mathcal{P}(M_n(\mathbb{C}))$ ($\lambda \in \Lambda_n^{\text{poly}}$) by the following lemma.

**Lemma 4.7.** Let $\lambda \in \Lambda_n^\text{poly}$. Then there is a $U(n) \times \text{GL}(n, \mathbb{C})$-homomorphism $P_\lambda^n: \overline{V}_\lambda \otimes \mathbb{C} V_\lambda \rightarrow \mathcal{P}(M_n(\mathbb{C}))$ characterized by

$$P_\lambda^n(\overline{\tau}(g)v_2)(g) = \langle \tau(g)v_2, v_1 \rangle \quad (v_1, v_2 \in V_\lambda, \ g \in \text{GL}(n, \mathbb{C})). \quad (4.12)$$

**Proof.** Because of (4.11), there is a nonzero $U(n) \times \text{GL}(n, \mathbb{C})$-homomorphism $P: \overline{V}_\lambda \otimes \mathbb{C} V_\lambda \rightarrow \mathcal{P}(M_n(\mathbb{C}))$. Since $\text{GL}(n, \mathbb{C})$ is dense in $M_n(\mathbb{C})$ and

$$P(\overline{\tau}(v_2)(g)) = P(\overline{\tau}(\tau(g)v_2)(1_n) \quad (v_1, v_2 \in V_\lambda, \ g \in \text{GL}(n, \mathbb{C})), \quad (4.13)$$

we note that $\overline{V}_\lambda \otimes \mathbb{C} V_\lambda \ni v_1 \otimes v_2 \mapsto P(v_1 \otimes v_2)(1_n) \in \mathbb{C}$ is a nonzero $\mathbb{C}$-bilinear pairing. Because of

$$P(\overline{\tau}(k)\overline{\tau}(\tau(g)v_2)(1_n) = P(\overline{\tau}(v_2))(1_n) \quad (k \in U(n))$$

and Lemma 4.1 (1) for $F = \mathbb{C}$, there is a nonzero constant $c$ such that

$$P(\overline{\tau}(v_2))(1_n) = c \langle v_2, v_1 \rangle \quad (v_1, v_2 \in V_\lambda). \quad (4.14)$$

By (4.13) and (4.14), we know that $P_\lambda^n = c^{-1}P$ satisfies (4.12). Since $\text{GL}(n, \mathbb{C})$ is dense in $M_n(\mathbb{C})$, we note that (4.12) characterizes $P_\lambda^n$. \qed

When $n > 1$, we define $U(n-1) \times \text{GL}(n, \mathbb{C})$-homomorphisms $P_\mu^n: \overline{V}_\mu \otimes \mathbb{C} V_{(\mu, 0)} \rightarrow \mathcal{P}(M_{n-1,n}(\mathbb{C}))$ ($\mu \in \Lambda_{n-1}^\text{poly}$) by the following lemma.

**Lemma 4.8.** Assume $n > 1$ and let $\mu \in \Lambda_{n-1}^\text{poly}$. There is a $U(n-1) \times \text{GL}(n, \mathbb{C})$-homomorphism $P_\mu^n: \overline{V}_\mu \otimes \mathbb{C} V_{(\mu, 0)} \rightarrow \mathcal{P}(M_{n-1,n}(\mathbb{C}))$ characterized by

$$P_\mu^n(\overline{\zeta}(M) \otimes v)((1_{n-1}, O_{n-1,1})z) = P_\mu^n((\zeta(M) \otimes v)(z))(1_{n-1}, O_{n-1,1}) \quad (4.15)$$

for $M \in G(\mu), \ v \in V_{(\mu, 0)}$ and $z \in M_n(\mathbb{C})$. Here $M([\mu, 0])$ is defined by (2.23). Furthermore, we have

$$P_\mu^n(\overline{\zeta}(M) \otimes v)(z) = P_\mu^n((\zeta(M) \otimes v)(z))(1_{n-1}, O_{n-1,1}) \quad (4.16)$$

for $v \in V_\mu, \ M \in G(\mu)$ and $z \in M_{n-1,n}(\mathbb{C})$.

**Proof.** We regard $\text{GL}(n-1, \mathbb{C})$ as a subgroup of $\text{GL}(n, \mathbb{C})$ via the embedding $\iota_n$ defined by (2.20). By the irreducible decomposition (4.11) and Lemma 2.2, the image of a $U(n-1) \times \text{GL}(n, \mathbb{C})$-homomorphism

$$\overline{V}_\mu \otimes \mathbb{C} V_{(\mu, 0)} \ni \overline{\zeta} \otimes v \mapsto P_\mu^n((\zeta(M) \otimes v)(z)) \in \mathcal{P}(M_n(\mathbb{C}))$$

33
is contained in the image of an injective $U(n-1) \times \GL(n, \mathbb{C})$-homomorphism

$$
P(M_{n-1,n}(\mathbb{C})) \ni p \mapsto (z \mapsto p((1_{n-1}, O_{n-1}, 1)z)) \in P(M_n(\mathbb{C})).$$

Hence, there is a $U(n-1) \times \GL(n, \mathbb{C})$-homomorphism $P_1^+: V_\nu \boxtimes \mathbb{C} V_{(\mu, 0)} \to P(M_{n-1,n}(\mathbb{C}))$ characterized by (4.15). By the irreducible decompositions (4.11) and Lemma 2.2, two injective $U(n-1) \times \GL(n-1, \mathbb{C})$-homomorphisms

$$\begin{align*}
V_\nu \boxtimes \mathbb{C} V_\nu \ni \tau \boxtimes \zeta M & \mapsto P_1^+(\tau \boxtimes \zeta M_{(\mu, 0)})) \in P(M_{n-1,n}(\mathbb{C})), \\
V_\nu \boxtimes \mathbb{C} V_\nu \ni \tau \boxtimes v_2 & \mapsto (z \mapsto P_1^+(\tau \boxtimes v_2)(z \cdot (1_{n-1}, O_{n-1}, 1))) \in P(M_{n-1,n}(\mathbb{C}))
\end{align*}$$

coincide up to scalar multiple. Hence, (4.16) follows from the equalities

$$
P_\mu^+(\zeta M \boxtimes \zeta M_{(\mu, 0)}))(1_{n-1}, O_{n-1}, 1) = (\zeta M_{(\mu, 0)}, \zeta M_{(\mu, 0)}) = 1$$

and $P_\mu^+(\zeta M \boxtimes \zeta M)(1_{n-1}) = (\zeta M, \zeta M) = 1$ for $M \in G(\mu)$. \qed

Let $l \in N_0$. We define $\mathbb{C}$-linear maps $p_{1,n}^{(l)}: V_{(l, 0_{n-1})} \to P(M_{1,n}(\mathbb{C}))$ and $p_{n,1}^{(l)}: V_{(l, 0_{n-1})} \to P(M_{n,1}(\mathbb{C}))$ by

$$
p_{1,n}^{(l)}(\zeta Q(\gamma))(z) = p_{n,1}^{(l)}(\zeta Q(\gamma))(z) = \sqrt{b(\gamma)} z_1^{\gamma_1} z_2^{\gamma_2} \cdots z_n^{\gamma_n},$$

for $z = (z_1, z_2, \ldots, z_n) \in M_{1,n}(\mathbb{C})$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in N_0^n$ such that $\ell(\gamma) = l$. Here $Q(\gamma)$ and $b(\gamma)$ are defined by (2.33) and (2.43), respectively.

**Lemma 4.9.** Let $l \in N_0$.

1. The group $\GL(n, \mathbb{C})$ acts on $P(M_{1,n}(\mathbb{C}))$ by the action $R$. Then $p_{1,n}^{(l)}$ is a $\GL(n, \mathbb{C})$-homomorphism such that, for $z \in M_n(\mathbb{C})$ and $v \in V_{(l, 0_{n-1})}$,

$$
p_{1,n}^{(l)}(v)(e_n z) = P_{(l, 0_{n-1})}^{(l)}(\zeta Q(\gamma))(z).
\quad (4.17)$$

2. The group $U(n)$ acts on $P(M_{n,1}(\mathbb{C}))$ by the action $L$. Then $p_{n,1}^{(l)}$ is a $U(n)$-homomorphism such that, for $z \in M_n(\mathbb{C})$ and $v \in V_{(l, 0_{n-1})}$,

$$
p_{n,1}^{(l)}(\tau)(z e_n) = P_{(l, 0_{n-1})}^{(l)}(\tau \boxtimes \zeta Q(\gamma))(z).
\quad (4.18)$$

**Proof.** Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in N_0^n$. By direct computation, we have

$$R(E_{i,j})p_{1,n}^{(l)}(\zeta Q(\gamma)) = \gamma_i p_{1,n}^{(l)}(\zeta Q(\gamma)),$$

$$R(E_{j,j+1})p_{1,n}^{(l)}(\zeta Q(\gamma)) = \sqrt{\gamma_{j+1}(\gamma_j + 1)} p_{1,n}^{(l)}(\zeta Q(\gamma_{j+1} - \delta_{j+1})), $$

$$R(E_{j+1,j})p_{1,n}^{(l)}(\zeta Q(\gamma)) = \sqrt{\gamma_j(\gamma_j + 1)} p_{1,n}^{(l)}(\zeta Q(\gamma_{j+1} + \delta_{j+1})),$$

$$L(E_{i,j})p_{n,1}^{(l)}(\zeta Q(\gamma)) = -\gamma_i p_{n,1}^{(l)}(\zeta Q(\gamma)),$$

$$L(E_{j,j+1})p_{n,1}^{(l)}(\zeta Q(\gamma)) = -\sqrt{\gamma_j(\gamma_{j+1} + 1)} p_{n,1}^{(l)}(\zeta Q(\gamma_{j+1} + \delta_{j+1})).$$
\[ L(E_{j+1,j})p_{n,1}^{(l)}(\zeta_{Q(\gamma)}) = -\sqrt{\gamma_{j+1}(\gamma_j + 1)} p_{n,1}^{(l)}(\zeta_{Q(\gamma) + \delta_j - \delta_{j+1}}) \]

for \( 1 \leq i \leq n \) and \( 1 \leq j \leq n - 1 \). Here we put \( p_{n,1}^{(l)}(\zeta_{Q(\gamma)}) = p_{n,1}^{(l)}(\zeta_{Q(\gamma)'}) = 0 \) if \( \gamma' \notin \mathbb{N}_0^n \), and denote by \( \delta_i \) the element of \( \mathbb{Z}^n \) with 1 at \( i \)-th entry (\( 1 \leq i \leq n \)) and 0 at other entries. Comparing these formulas with (2.11), (2.12) and (2.13), we know that \( p_{n,1}^{(l)} \) is a GL\((n, \mathbb{C})\)-homomorphism. Using (2.24) and

\[
L(E_{i,j}^{n(n)}) = L(E_{i,j}) \quad (1 \leq i, j \leq n) \quad \text{on} \quad \mathcal{P}(M_{n,1}(\mathbb{C})),
\]

we further know that \( p_{n}^{(l)} \) is a U\((n)\)-homomorphism.

Next, we will prove the equality (4.17). When \( n = 1 \), this equality follows from \( G(\ell) = \{l\} \) and

\[
p_{1,1}^{(l)}(\zeta_{Q(\gamma)})(g) = \langle \gamma(g)\zeta_l, \zeta_l \rangle = g^l \quad (g \in \text{GL}(1, \mathbb{C})).
\]

Assume \( n > 1 \). We regard GL\((n-1, \mathbb{C})\) as a subgroup of GL\((n, \mathbb{C})\) via the embedding \( \iota_n \) defined by (2.20). Because of the irreducible decompositions (4.11) and Lemma 2.2, two injective U\((n-1) \times \text{GL}(n, \mathbb{C})\)-homomorphisms

\[
\begin{align*}
&V_{\mathcal{O}_{n-1}} \otimes \mathbb{C} V_{(l,o_{n-1})} \ni v \mapsto (z \mapsto \nu_{n-1}^{(l)}(v)(z)) \in \mathcal{P}(M_{n}((\mathbb{C})),) \\
&V_{\mathcal{O}_{n-1}} \otimes \mathbb{C} V_{(l,o_{n-1})} \ni v \mapsto \mathcal{P}_{(l,o_{n-1})}^{(l)}(\zeta_{Q((0_{n-1},1,l))} \otimes \zeta_{Q((0_{n-1},1,l))}) \in \mathcal{P}(M_{n}((\mathbb{C})))
\end{align*}
\]

coincide up to scalar multiple. Hence, (4.17) follows from the equalities

\[
p_{1,1}^{(l)}(\zeta_{Q((0_{n-1},1,l))})(e_n) = 1, \quad \mathcal{P}_{(l,o_{n-1})}^{(l)}(\zeta_{Q((0_{n-1},1,l))} \otimes \zeta_{Q((0_{n-1},1,l))})(1_n) = 1.
\]

The proof of the equality (4.18) is similar. \( \square \)

For later use, we prepare the following lemmas.

**Lemma 4.10.** Assume \( n > 1 \). Let \( \mu \in A_{n-1}^{\text{poly}} \) and \( \gamma \in \mathbb{N}_0^n \). We set \( l = \ell(\gamma) \) and

\[
p_0(z) = P^+_{\mu}(\zeta_{H(\mu)} \otimes \zeta_{H((\mu,0))})(1_{n-1}, O_{n-1,1})z = P_{1,n}^{(l)}(\zeta_{Q(\gamma)})(e_n z)
\]

for \( z \in M_n(\mathbb{C}) \). Then we have

\[
p_0 = \sum_{\lambda' \in \Xi^+(\mu,0); l} \sum_{\lambda, \lambda' \in G(\lambda)} C_{H(\mu,0), Q((0_{n-1},1,l))}^{\lambda} C_{H((\mu,0)), Q(\gamma)}^{\lambda'} P_{\lambda'}^0(\zeta_{\lambda} \otimes \zeta_{\lambda'}),
\]

where \( C_{\lambda, \lambda'}^{\mu, \mu'} \) is the Clebsch–Gordan coefficient in §4.1.

**Proof.** We set \( Q_0 = Q((0_{n-1},1,l)) \) and \( Q_1 = Q(\gamma) \). Let \( g \in \text{GL}(n, \mathbb{C}) \). By Lemmas 4.7, 4.8 and 4.9, we have

\[
p_0(g) = P_{\mu}^0(\zeta_{H(\mu,0)} \otimes \zeta_{H((\mu,0))})(g) P_{(l,o_{n-1})}^0(\zeta_{Q_0} \otimes \zeta_{Q_1})(g)
\]

\[
= \langle \tau_{(\mu,0)}(g)\zeta_{H(\mu,0)}, \zeta_{H((\mu,0))} \rangle \langle \tau_{(l,o_{n-1})}(g)\zeta_{Q_1}, \zeta_{Q_0} \rangle
\]

35
By (4.1), (4.6) and Lemma 4.7, we have
\[ p_0(g) = \sum_{\lambda' \in \Xi((\mu, 0); l)} \sum_{N, N' \in G(\lambda')} C_N^{H((\mu, 0)), Q_0} C_{N'}^{H((\mu, 0)), Q_1} \langle \tau_{\lambda'}(g) \zeta_{N'}, \zeta_N \rangle \]
\[ = \sum_{\lambda' \in \Xi((\mu, 0); l)} \sum_{N, N' \in G(\lambda')} C_N^{H((\mu, 0)), Q_0} C_{N'}^{H((\mu, 0)), Q_1} P_\mu^0(\zeta_N \otimes \zeta_{N'})(g). \]

Since \( GL(n, \mathbb{C}) \) is dense in \( M_n(\mathbb{C}) \), we obtain the assertion.

\[ \square \]

Lemma 4.11. Assume \( n > 1 \). Let \( \mu \in \Lambda^{\text{poly}}_{n-1} \) and \( \gamma \in \mathbb{N}^{n-1}_0 \). We set \( l = \ell(\gamma) \) and
\[ p_0(z) = P_\mu^0(\zeta_{H((\mu, 0))} \otimes \zeta_{H((\mu, 0))})(z (1_{n-1}, O_{n-1, 1})) P_{n-1}^0(\zeta_{Q(\gamma)})(z \epsilon_n) \]
for \( z \in M_{n-1,n}(\mathbb{C}) \). Then we have
\[ p_0 = \sum_{\mu' \in \Xi((\mu, 0); l)} \sum_{N \in G((\mu', 0); \mu')} C_N^{H((\mu, 0)), Q((\gamma, 0))} C_{N'}^{H((\mu, 0)), Q((0_{n-1}, 1))} P_{\mu'}^0(\zeta_N \otimes \zeta_{N'}), \]
where \( C_{M, P}^{M, P} \) is the Clebsch–Gordan coefficient in §4.1.

Proof. We set \( Q_0 = Q((0_{n-1}, 1)) \) and \( Q_1 = Q((\gamma, 0)) \). Let \( z = (1_{n-1}, O_{n-1, 1})g \) with \( g \in G_n \). Then we have \( p_{n-1}^0(\zeta_{Q(\gamma)})(z \epsilon_n) = p_{n}^0(\zeta_{Q(\gamma)})(z \epsilon_n) \) by definition. Hence, by Lemmas 4.7, 4.8 and 4.9, we have
\[ p_0(z) = P_\mu^0(\zeta_{H((\mu, 0))} \otimes \zeta_{H((\mu, 0))})(g) P_{(l, 0_{n-1})}^0 (\zeta_{Q(\gamma)})(g) \]
\[ = \langle \tau_{(\mu, 0)}(g) \zeta_{H((\mu, 0))}, \zeta_{H((\mu, 0))} \rangle \langle \tau_{(l, 0_{n-1})}(g) \zeta_{Q(\gamma)}, \zeta_{Q(\gamma)} \rangle \]
\[ = \langle \tau_{(\mu, 0)}(g) \zeta_{H((\mu, 0))}, \zeta_{H((\mu, 0))} \rangle \langle \tau_{(l, 0_{n-1})}(g) \zeta_{H((\mu, 0))}, \zeta_{Q(\gamma)} \rangle \]
\[ = \langle \tau_{(\mu, 0)}(g) \zeta_{H((\mu, 0))} \otimes \zeta_{Q(\gamma)} \rangle \]
By (4.1) and (4.6), we have
\[ p_0(z) = \sum_{\lambda' \in \Xi(\mu, 0; l)} \sum_{N \in G(\lambda')} C_N^{H((\mu, 0)), Q_0} C_{N'}^{H((\mu, 0)), Q_1} \langle \tau_{\lambda'}(g) \zeta_{N'}, \zeta_N \rangle. \]
Because of \( H((\mu, 0)) \in G((\mu', 0); \mu') \), \( Q_1 \in G((l, 0_{n-1}); (l, 0_{n-2})) \) and (4.3), for \( \lambda' \in \Xi((\mu, 0); l) \) and \( N \in G(\lambda') \), we have \( C_N^{H((\mu, 0)), Q_1} = 0 \) unless \( \lambda' = (\mu', 0) \) and \( N \in G((\mu', 0); \mu') \) with some \( \mu' \in \Xi(\mu, l) \). Hence, we have
\[ p_0(z) = \sum_{\mu' \in \Xi(\mu, l)} \sum_{N \in G((\mu', 0); \mu')} C_N^{H((\mu, 0)), Q_0} C_{N'}^{H((\mu, 0)), Q_1} \langle \tau_{(\mu', 0)}(g) \zeta_{N'}, \zeta_N \rangle \]
\[ = \sum_{\mu' \in \Xi(\mu, l)} \sum_{N \in G((\mu', 0); \mu')} C_N^{H((\mu, 0)), Q_0} C_{N'}^{H((\mu, 0)), Q_1} P_{\mu'}^0(\zeta_N \otimes \zeta_{N'})(z) \]
by Lemmas 4.7 and 4.8. Since \( \{(1_{n-1}, O_{n-1}, 1)g \mid g \in GL(n, \mathbb{C})\} \) is dense in \( M_{n-1,n}(\mathbb{C}) \), we obtain the assertion.

\[ \square \]
4.4 Standard Schwartz functions

For $\lambda \in \Lambda_n^{\text{poly}}$, we define two $\mathbb{C}$-linear maps

$$
\Phi^\lambda_0 : \mathbb{C} \otimes V_\lambda \ni v \mapsto \Phi^\lambda_0 (v) = P^\lambda_0 (v) \in \mathcal{S}_0 (M_\lambda (F)),
$$

$$
\Phi^\lambda_1 : \mathbb{C} \otimes V_\lambda \ni v \mapsto \Phi^\lambda_1 (v) = P^\lambda_1 (v) \in \mathcal{S}_0 (M_\lambda (F))
$$

with $z \in M_\lambda (F)$. By the $K_n \times K_n$-invariance of $e_{(n)}$ and Lemma 4.7, we know that these are $K_n \times K_n$-homomorphisms.

When $n > 1$, for $\mu \in \Lambda_n^{\text{poly}}$, we define two $\mathbb{C}$-linear maps

$$
\Phi^\mu_0 : \mathbb{C} \otimes V_{(\mu,0)} \ni v \mapsto \Phi^\mu_0 (v) = P^\mu_0 (v) \in \mathcal{S}_0 (M_{n-1,n} (F)),
$$

$$
\Phi^\mu_1 : \mathbb{C} \otimes V_{(\mu,0)} \ni v \mapsto \Phi^\mu_1 (v) = P^\mu_1 (v) \in \mathcal{S}_0 (M_{n-1,n} (F))
$$

with $z \in M_{n-1,n} (F)$. By the $K_{n-1} \times K_n$-invariance of $e_{(n-1,n)}$ and Lemma 4.8, we know that these are $K_{n-1} \times K_n$-homomorphisms.

We regard $\mathcal{S}_0 (M_1, (F))$ and $\mathcal{S}_0 (M_{1,1} (F))$ as $K_n$-modules via the actions $R$ and $L$, respectively. Let $l \in \mathbb{N}_0$. Since $b(\gamma) = r(Q(\gamma))^{-1}$ for $\gamma \in \mathbb{N}_0^l$, the $\mathbb{C}$-linear maps $\varphi_{(1),l}$ and $\varphi_{(1),n}$ in §2.9 satisfy

$$
\varphi_{(1),l} (v) (z) = p_{(1),l} (v) (z) e_{(1),l} (z), \quad \varphi_{(1),n} (v) (z) = p_{(1),n} (v) (z) e_{(1),n} (z)
$$

for $v \in V_{(l,0_{n-1})}$ and $z \in M_1 (F)$. By the $K_n$-invariance of $e_{(1),n}$ and Lemma 4.9, we know that these are $K_n$-homomorphisms. We define two $\mathbb{C}$-linear maps $\varphi_{(1),l} : V_{(l,0_{n-1})} \to \mathcal{S}_0 (M_1, (F))$ and $\varphi_{(1),n} : V_{(l,0_{n-1})} \to \mathcal{S}_0 (M_{1,1} (F))$ by

$$
\varphi_{(1),l} (v) (z) = p_{(1),l} (v) (z) e_{(1),l} (z), \quad \varphi_{(1),n} (v) (z) = p_{(1),n} (v) (z) e_{(1),n} (z)
$$

with $z \in M_{1,1} (F)$. By the $K_n$-invariance of $e_{(1),l}$ and Lemma 4.9, we know that these are $K_n$-homomorphisms.

**Lemma 4.12.** Let $\epsilon \in \{\pm 1\}$ and $l \in \mathbb{N}_0$. Assume $l \in \{0,1\}$ if $F = \mathbb{R}$. Then, for $v \in V_{(l,0_{n-1})}$, we have

$$
\mathcal{F}_\epsilon (\varphi_{(1),l} (v)) (\tau) = (-\epsilon \sqrt{-1})^{l} \varphi_{(1),l} (v) (\tau), \quad \mathcal{F}_\epsilon (\varphi_{(1),n} (v)) (\tau) = (-\epsilon \sqrt{-1})^{l} \varphi_{(1),n} (v) (\tau),
$$

where the Fourier transform $\mathcal{F}_\epsilon$ is defined in (3.15).

**Proof.** It suffices to show the assertion for $v = \zeta Q(\gamma)$ with $\gamma \in \mathbb{N}_0^l$ such that $\ell (\gamma) = l$. For $t = (t_1, t_2, \ldots, t_n) \in M_{1,1} (F)$, we have

$$
\mathcal{F}_\epsilon (\varphi_{(1),l} (\zeta Q(\gamma))) (t) = \int_{M_{1,1} (F)} \varphi_{(1),l} (\zeta Q(\gamma)) (z) \psi_{-\epsilon} (tz) d_F z
$$

$$
= \sqrt{b(\gamma)} \prod_{i=1}^n \int_F z_i^{\gamma_i} \exp (-\pi c_F \zeta_i z_i - \pi \epsilon c_F \sqrt{-1} (t_i z_i + \overline{t_i z_i})) d_F z_i
$$
\[
= \sqrt{b(\gamma)}(\varepsilon \sqrt{-1} t_i^n) \exp(-\pi c_F t_i) = (-\varepsilon \sqrt{-1})^i \varphi_{1,n}^{(l)}(\zeta_{Q(\gamma)})(t).
\]

Here the third equality follows from the elementary formula
\[
\int_F z^m \exp(-\pi c_F z + \pi c_F \sqrt{-1}(zt + \overline{zt})) d_F z = (\sqrt{-1})^m \exp(-\pi c_F t) \quad (4.19)
\]
for \((t \in \mathbb{R}, m \in \{0, 1\})\) or \((t \in \mathbb{C}, m \in \mathbb{N}_0)\) according as \(F = \mathbb{R}\) or \(F = \mathbb{C}\).
Moreover, we have
\[
F_\varepsilon(\varphi_{1,n}^{(l)}(\zeta_{Q(\gamma)}))(t) = (-\varepsilon \sqrt{-1})^i \varphi_{1,n}^{(l)}(\zeta_{Q(\gamma)})(t).
\]
and completes the proof. \(\square\)

5 The proofs of the Main theorems

5.1 Explicit calculations for the Godement sections

In this subsection, we calculate the Godement sections in §3, explicitly, at the minimal \(K_n\)-types of principal series representations.

Lemma 5.1. Let \(a = \text{diag}(a_1, a_2, \cdots, a_n) \in A_n, u \in U_n, \lambda \in \Lambda_n\) and \(M \in G(\lambda)\). Then we have the following equalities

\[
\langle \tau_\lambda(ua) \zeta_M, \zeta_M \rangle = \langle \tau_\lambda(au) \zeta_M, \zeta_M \rangle = \prod_{i=1}^n a_i^{\gamma_i^M}, \quad (5.1)
\]

\[
\eta_{\rho_n}(a) \int_{U_n} e_{(n)}(ua) \, du = \eta_{-\rho_n}(a) \int_{U_n} e_{(n)}(au) \, du = \frac{e_{(n)}(a)}{\det a_1 (\lambda)}^{(n+1)/2}, \quad (5.2)
\]

where \(\gamma^M = (\gamma_1^M, \gamma_2^M, \cdots, \gamma_n^M)\) is the weight of \(M\) defined by (2.10).

Proof. By (2.11) and (2.13), we have

\[
\tau_\lambda(a) \zeta_M = \left( \prod_{i=1}^n a_i^{\gamma_i^M} \right) \zeta_M, \quad \tau_\lambda(u) \zeta_M = \zeta_M + \sum_{N \in G(\lambda), \gamma^M \geq \text{lex} \gamma^N} p_{M,N}(u) \zeta_N,
\]
where \(p_{M,N}\) is a polynomial function on \(U_n\) and \(>_{\text{lex}}\) is the lexicographical order. The equality (5.1) follows from these equalities and the orthonormality of \(\{\zeta_M\}_{M \in G(\lambda)}\). The equality (5.2) follows from direct computation

\[
\eta_{\rho_n}(a) \int_{U_n} e_{(n)}(ua) \, du = \eta_{-\rho_n}(a) \int_{U_n} e_{(n)}(au) \, du
\]

\[
= \prod_{i=1}^n a_i^{-(n+1-2i)c_F/2} \exp(-\pi c_F a_i^2) \prod_{j=1}^{i-1} \int_F \exp(-\pi c_F a_i^2 u_{i,j} u_{i,j}) \, d_F u_{i,j}
\]

38
For $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n$, $M = (m_{i,j})_{1 \leq i \leq j \leq n} \in G(\lambda)$ and $l \in \mathbb{Z}$, we define $\lambda + l \in \Lambda_n$ and $M + l \in G(\lambda + l)$ by

$$\lambda + l = (\lambda_1 + l, \lambda_2 + l, \cdots, \lambda_n + l), \quad M + l = (m_{i,j} + l)_{1 \leq i \leq j \leq n},$$

and denote $\lambda + (-l)$ and $M + (-l)$ simply by $\lambda - l$ and $M - l$, respectively. For $\lambda \in \Lambda_n$, $l \in \mathbb{Z}$, $g \in G(n, \mathbb{C})$ and $M, N \in G(\lambda)$, we have

$$(\det g)^l \langle \tau_N (g) \zeta_M, \zeta_N \rangle = \langle \tau_{\lambda + l} (g) \zeta_{M + l}, \zeta_N \rangle.$$

**Lemma 5.2.** Let $d = (d_1, d_2, \cdots, d_n) \in \mathbb{Z}^n$ and $\nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{C}^n$.

1. Assume $n > 1$. We take $\hat{\alpha}$ and $\hat{\nu}$ as in §3.1. If $d \in \Lambda_{n,F}$, we have

$$g_{d,n,\nu_n}^+ \left( \tilde{f}_{d,\nu}(\zeta_{H(\hat{\alpha})}), \Phi_{d-d_n}^+ (\overline{\zeta_{H(\hat{\alpha})}}) \otimes \overline{\zeta_{M-d_n}} \right) = \frac{1}{\dim V_d} \left( \prod_{i=1}^{n-1} \Gamma_F(\nu_i - \nu_i + 1; d_i - d_i) \right) f_{d,\nu}(\zeta_M)$$

for $M \in G(d)$. If $-d \in \Lambda_{n,F}$, we have

$$g_{d,n,\nu_n}^+ \left( \tilde{f}_{d,\nu}(\zeta_{H(\hat{\alpha})}), \Phi_{-d+d_n}^+ (\overline{\zeta_{H(\hat{\alpha})}}) \otimes \overline{\zeta_{M+d_n}} \right) = \frac{1}{\dim V_{-d}} \left( \prod_{i=1}^{n-1} \Gamma_F(\nu_i - \nu_i + 1; d_n - d_i) \right) \tilde{f}_{d,\nu}(\zeta_M)$$

for $M \in G(-d)$. Here $f_{d,\nu}$ and $\tilde{f}_{d,\nu}$ are defined by (2.25) and (2.26), respectively.

2. Let $l \in \mathbb{Z}$ and $s \in \mathbb{C}$ such that $\text{Re}(s)$ is sufficiently large. If $d \in \Lambda_{n,F}$ and $d + l \in \Lambda_{n,\mathbb{C}}$, we have

$$g_{d,s}^+ \left( f_{d,\nu}(\zeta_{H(\hat{\alpha})}), \Phi_{d+1} (\overline{\zeta_{M+1}}) \otimes \overline{\zeta_{H(\hat{\alpha})}} \right) = \frac{1}{\dim V_d} \left( \prod_{i=1}^n \Gamma_F(s + \nu_i; d_i + l) \right) f_{d,\nu}(\zeta_M) \quad (M \in G(d)).$$

If $-d \in \Lambda_{n,F}$ and $-d - l \in \Lambda_{n,\mathbb{C}}$, we have

$$g_{d,s}^+ \left( \tilde{f}_{d,\nu}(\zeta_{H(\hat{\alpha})}), \Phi_{d-l}^+ (\overline{\zeta_{M-l}}) \otimes \overline{\zeta_{H(\hat{\alpha})}} \right) = \frac{1}{\dim V_{-d}} \left( \prod_{i=1}^n \Gamma_F(s + \nu_i; -d_i - l) \right) \tilde{f}_{d,\nu}(\zeta_M) \quad (M \in G(-d)).$$
Proof. First, we consider the proof of the statement (1). Since the proofs of (5.3) and (5.4) are similar, here we will prove only (5.3). Assume \( n > 1 \) and \( d \in \Lambda_{n,F} \). We define a \( \mathbb{C} \)-linear map \( g_+: V_d \to I(d, \nu) \) by

\[
g_+(\zeta_M) = g^+_{d,n,\nu}(\tilde{f}_d, \zeta_H(d)), \quad \Phi^+_{d-d_n}(\zeta_H(d)-d_n) \quad (M \in G(d)).
\]

Then \( g_+ \) is a \( K_n \)-homomorphism because of (3.1). Since \( \text{Hom}_{K_n}(V_d, I(d, \nu)) \) is 1 dimensional, there is a constant \( c_+ \) such that \( g_+ = c_+ f_{d,\nu} \). Let us calculate \( c_+ \). Since (4.12) and (4.16) imply

\[
\Phi^+_{d-d_n}(\zeta_H(d)-d_n) \zeta_H(d) = (h, O_{n-1})
\]

we have

\[
c_+ = c_+ f_{d,\nu}(\zeta_H(d))(1_1) = g_+(\zeta_H(d))(1_1)
\]

\[
= \int_{G_{n-1}} \langle \tau_{d-d_n}^{-1}(h)\zeta_H(d)-d_n, \zeta_H(d)-d_n \rangle \epsilon(n-1)(h)
\]

\[
= \int_{G_{n-1}} \langle \tau_{d-d_n}^{-1}(h)\zeta_H(d)-d_n, \zeta_H(d)-d_n \rangle \epsilon(n-1)(h)
\]

\[
\times f_{d,\nu}(\zeta_H(d))(h^1) \chi_{d_n}(\det h) |\det h|^{n+n/2} dh.
\]

Decomposing \( h = kua \) \( (k \in K_{n-1}, u \in U_{n-1}, a \in A_{n-1}) \) and applying Schur’s orthogonality [17, Corollary 1.10] for the integration on \( K_{n-1} \) with the equalities

\[
\chi_{d_n}(\det h) f_{d,\nu}^2(\zeta_H(d))(h^1) = \eta_\nu |\det d_n|^{-n+n/2} da.
\]

and \( \dim V_{d-d_n} = \dim V_d \), we have

\[
c_+ = \frac{1}{\dim V_d} \int_{A_{n-1}} \left( \int_{U_{n-1}} \langle \tau_{d-d_n}^{-1}(u a)h, \zeta_H(d)-d_n, \zeta_H(d)-d_n \rangle \epsilon(n-1)(u a) du \right)
\]

\[
\times \eta_\nu \det d_n |\det h|^{n+n/2} da.
\]

By Lemma 5.1 and (2.33), we have

\[
c_+ = \frac{1}{\dim V_d} \prod_{i=1}^{n-1} \int_0^\infty \exp(-\pi c_F a_i^2) a_i^{(\nu_n-\nu_i+1)a_F+d_i-d_n} \frac{2c_F d_i}{a_i}
\]

\[
= \frac{1}{\dim V_d} \prod_{i=1}^{n-1} \Gamma_F(\nu_n-\nu_i+1; d_i-d_n).
\]

Hence, the equality (5.3) follows from \( g_+ = c_+ f_{d,\nu} \).

Next, we consider the proof of the statement (2). Since the proofs of (5.5) and (5.6) are similar, here we will prove only (5.5). Assume \( d \in \Lambda_{n,F} \) and \( d + t \in \Lambda_{n,F}^\text{pol} \). We define a \( \mathbb{C} \)-linear map \( g_+: V_d \to I(d, \nu) \) by

\[
g_+(\zeta_M) = g^+_{d,\nu}(\tilde{f}_d, \zeta_H(d)), \quad \Phi^+_{d+t}(\zeta_M+t) \zeta_H(d) \quad (M \in G(d)).
\]
Then $g_\circ$ is a $K_n$-homomorphism because of (3.5). Since $\text{Hom}_{K_n}(V_d, I(d, \nu))$ is 1 dimensional, there is a constant $c_0$ such that $g_\circ = c_0 f_{d, \nu}$. Let us calculate $c_0$. By (4.12), we have

$$c_0 = c_0 f_{d, \nu}(H(d))(1_n) = g_\circ(\zeta_{H(d)})(1_n)$$

$$= \int_{G_n} f_{d, \nu}(\zeta_{H(d)})(h) \langle \tau_{d+1}(h)(\zeta_{H(d)+1}, \zeta_{H(d)+1})e_{(n)}(h) \rangle \times \chi_l(\det h) |\det h|^{\nu_+ + (n-1)/2} dh.$$ 

Decomposing $h = au k$ ($a \in A_n, u \in U_n, k \in K_n$) and applying Schur’s orthogonality [17, Corollary 1.10] for the integration on $K_n$ with the equalities

$$\chi_l(\det h) f_{d, \nu}(\zeta_{H(d)})(h) = \eta_{\nu - \rho_n}(a) \langle \tau_{d+l}(k)(\zeta_{H(d)+1}, \zeta_{H(d)+1}) \rangle,$$

$$\tau_{d+1}(h)\zeta_{H(d)+1} = \sum_{M \in G(d)} \langle \tau_{d+l}(k)(\zeta_{H(d)+1}, \zeta_{M+l}) \rangle \tau_{d+l}(au)\zeta_{M+l}$$

and $\dim V_{d+1} = \dim V_d$, we have

$$c_0 = \frac{1}{\dim V_d} \int_{A_n} \left( \int_{U_n} \langle \tau_{d+l}(au)\zeta_{H(d)+1}, \zeta_{H(d)+1} \rangle e_{(n)}(au) du \right) \times \eta_{\nu - \rho_n}(a) |\det a|^{\nu_+ + (n-1)/2} da.$$ 

By Lemma 5.1 and (2.53), we have

$$c_0 = \frac{1}{\dim V_d} \prod_{i=1}^{n} \int_{0}^{\infty} \exp(-\pi c_F a_i^2) a_i^{s + \nu_i} |\det a|^{\nu_+ + (n-1)/2} a_i \times \prod_{i=1}^{n} \Gamma_F(s + \nu_i; d_i + l).$$

Hence, the equality (5.5) follows from $g_\circ = c_0 f_{d, \nu}$. 

**Corollary 5.3.** We use the notation in Lemma 5.2 (1). If $d \in \Lambda_{n, F}$, we have

$$W_{\zeta}(f_{d, \nu}(\zeta_M))(g) = \left( \frac{\text{dim } V_{\hat{d}}}{\prod_{i=1}^{n-1} \Gamma_F(\nu_n - \nu_i + 1; d_i - d_n)} \right) \times \int_{G_{n-1}} \int_{M_{n-1, 1}(F)} \Phi_{\hat{d} - d_n}^+ \left( \zeta_{H(d) - d_n} \otimes \zeta_{M - d_n} \right)(h, hz) g(\psi_{\zeta}(e_{n-1} z) dz) \times W_{\zeta}(f_{\hat{d}, \nu}(\zeta_{H(d)})(h^{-1})) \chi_{d_n}(\det h) |\det h|^{\nu_n + n/2} dh$$

for $M \in G(d)$ and $g \in G_n$. If $-d \in \Lambda_{n, F}$, we have

$$W_{\zeta}(f_{\hat{d}, \nu}(\zeta_M))(g) = \left( \frac{\text{dim } V_{-\hat{d}}}{\prod_{i=1}^{n-1} \Gamma_F(\nu_n - \nu_i + 1; d_n - d_i)} \right)$$
\[ \times \int_{G_{n-1}} \left( \int_{M_{n-1,1}(F)} \frac{\Phi_{-\hat{d}+d_n}(\zeta_{H(-\hat{d})}+d_n \boxplus \zeta_{M+d_n})}{\zeta} ((h, hz) \, g \, \psi \epsilon (e_{n-1}z)) \, dz \right) \]
\[ \times W_{\epsilon}((\Phi_{-\hat{d}+d_n}(\zeta_{H(-\hat{d})})) (h^{-1}) \chi_{d_n}((\det h) \, \det h)^{m+n/2} \, dh, \]

for \( M \in G(-d) \) and \( g \in G_n \).

**Proof.** The assertion follows immediately from (3.3) and Lemma 5.2 (1). □

**Corollary 5.4.** Let \( \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{C}^n \) and \( d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}^n \) such that either \( d \in \Lambda_{n,F} \) or \( -d \in \Lambda_{n,F} \) holds. We take \( \Gamma_F(\nu; d) \) as in §2.7. Then \( 1/\Gamma_F(\nu; d) \) is nonzero if \( \Pi_{d,\nu} \) is irreducible.

**Proof.** Let us give the proof for the case of \( d \in \Lambda_{n,F} \). Let \( g \in G_n \). By Corollary 5.3 and the entireness of the right hand side of (3.3), it is easy to show that \( W_{\epsilon}(f_{d,\nu}(\zeta_{H(d)})) (g) \) is the product of \( 1/\Gamma_F(\nu; d) \) and some entire function of \( \nu \) by induction. The irreducibility of \( \Pi_{d,\nu} \) implies that there is some \( g \in G_n \) such that \( W_{\epsilon}(f_{d,\nu}(\zeta_{H(d)})) (g) \neq 0 \). Hence, we have \( 1/\Gamma_F(\nu; d) \neq 0 \). The proof for the case of \( -d \in \Lambda_{n,F} \) is similar. □

### 5.2 Explicit recurrence relations

Let \( \epsilon \in \{ \pm 1 \} \). Let \( (\Pi_{d,\nu}, I(d, \nu)) \) and \( (\Pi_{d',\nu'}, I(d', \nu')) \) be principal series representations of \( G_n \) and \( G_{n'} \), respectively, with parameters

\[
d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}^n, \quad \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{C}^n,
\]
\[
d' = (d'_1, d'_2, \ldots, d'_n) \in \mathbb{Z}^{n'}, \quad \nu' = (\nu'_1, \nu'_2, \ldots, \nu'_n) \in \mathbb{C}^{n'}.
\]

**Proposition 5.5.** Retain the notation. Assume \( n' = n > 1 \), \( -d' \in \Lambda_{n,F} \) and \( d' \in \mathbb{Z}^{n'}(-d') \cap \Lambda_{n,F} \). Let \( l = \ell(d + d') \). \( s \in \mathbb{C} \) such that \( \text{Re}(s) \) is sufficiently large. Then we have

\[
Z(s, W_{\epsilon}(f_{d,\nu}(\zeta_{H(d)})), W_{-\epsilon}(f_{d',\nu'}(\zeta_{H(-d')})), \phi)_{1,n}(\zeta_{Q(d+d')})
\]
\[
= \zeta_{H(-d')}+d'_n, Q(0_{n-1,1}) \zeta_{H(-d')}+d'_n, Q(d+d')
\]
\[
\times \dim V_{d,\hat{d}} \prod_{i=1}^{n} \Gamma_F(s + \nu_i + \nu'_i; d_i + d'_i) \n \dim V_d \prod_{i=1}^{n} \Gamma_F(\nu'_i - \nu'_i + 1; d'_i + d'_i)
\]
\[
\times Z(s, W_{\epsilon}(f_{d,\nu}(\zeta_{H(-d')})), W_{-\epsilon}(f_{d',\nu'}(\zeta_{H(-d')})), \phi)_{1,n}(\zeta_{Q(d+d')}).
\]

**Proof.** Let \( \phi_1 = \Phi_{-\hat{d}+d_n}(\zeta_{H(-\hat{d})}+d'_n \boxplus \zeta_{H(-\hat{d})}+d'_n \boxplus \zeta_{M+d_n}) \) and \( \phi_2 = \varphi_{1,n}(\zeta_{Q(d+d')}) \). By Proposition 3.4, we have

\[
Z(s, W_{\epsilon}(f_{d,\nu}(\zeta_{H(d)})), W_{-\epsilon}(f_{d',\nu'}(\zeta_{H(-d')})), \phi_1), \phi_2)
\]
\[
= Z(s, W_{\epsilon}(g_{d_n,s+\nu_n}(f_{d,\nu}(\zeta_{H(d)}), \phi_0)), W_{-\epsilon}(f_{d',\nu'}(\zeta_{H(-d')})), \phi_2).
\]
where \( \phi_0(z) = \phi_1((1_{n-1}, O_{n-1,1})z) \phi_2(\epsilon_n z) \) (\( z \in M_n(F) \)). Since we have

\[
g^+_{d_n',\nu_n',\nu_n}(\bar{f}_{d',\nu'}, (\zeta_H(-d'), \phi_1)) = \prod_{i=1}^{n-1} \frac{\Gamma_F(\nu_n' - \nu_i' + 1; d_n' - d_i')}{\dim V_{-d_i'}} \bar{f}_{d',\nu'}(\zeta_H(-d'))
\]

by (5.4), it suffices to prove the equality

\[
Z(s, W_{\Xi}(g^0_{d_n',\nu_n',\nu_n}(f_{d',\nu'}(\zeta_H(d)), \phi_0)), W_{-\varepsilon}(\bar{f}_{d',\nu'}(\zeta_H(-d')))) = \sum_{\lambda' \in \mathcal{X}(-d' + d_n'; l)} \sum_{N,N' \in G(\lambda')} C^H(-d' + d_n', Q((0_{n-1,1})) C^H(-d' + d_n', Q(d + d'))
\times \prod_{i=1}^{n-1} \frac{\Gamma_F(s + \nu_i + \nu_n' - \nu_i' + 1; d_i' + d_n')}{\dim V_{d_i'}}
\times Z(s, W_{\Xi}(g^0_{d_n',\nu_n,\nu_n}(f_{d',\nu'}(\zeta_H(d)), \phi_0)), W_{-\varepsilon}(\bar{f}_{d',\nu'}(\zeta_H(-d')))).
\]

By Lemma 4.10, we have

\[
g^0_{d_n,\nu_n}(f_{d',\nu'}(\zeta_H(d)), \phi_0)
= \sum_{\lambda' \in \mathcal{X}(-d' + d_n'; l)} \sum_{N,N' \in G(\lambda')} C^N(-d' + d_n', Q((0_{n-1,1})) C^N(-d' + d_n', Q(d + d'))
\times g^0_{d_n,\nu_n}(f_{d',\nu'}(\zeta_H(d)), \Phi_{\lambda'}(\zeta_N \boxtimes \zeta_{N'})).
\]

By (3.6), we note that

\[
v \otimes \zeta_M \mapsto g^0_{d_n',\nu_n}(f_{d',\nu'}(\zeta_H(d)), \phi_0)
\]

defines an element of Hom\(_{K_n}(V_d \otimes C_{triv}, C_{triv})\) for \( \lambda' \in \mathcal{X}(-d' + d_n'; l), v_1 \in V_\lambda, \) and \( g \in G_n \). Hence, by Lemma 4.1, for \( \lambda' \in \mathcal{X}(-d' + d_n'; l) \) and \( N,N' \in G(\lambda') \), we have

\[
g^0_{d_n',\nu_n}(f_{d',\nu'}(\zeta_H(d)), \Phi_{\lambda'}(\zeta_N \boxtimes \zeta_{N'}))(g) = 0
\]

unless \( \lambda' = d + d_n' \) and \( N' = H(d) + d_n' \). By (5.8) and this equality, we have

\[
g^0_{d_n',\nu_n}(f_{d',\nu'}(\zeta_H(d)), \phi_0)
= \sum_{N \in G(d + d_n')} C^N(-d' + d_n', Q((0_{n-1,1})) C^N(-d' + d_n', Q(d + d'))
\times g^0_{d_n',\nu_n}(f_{d',\nu'}(\zeta_H(d)), \Phi_{d+d_n'}(\zeta_N \boxtimes \zeta_{H(d) + d_n'})).
\]

By (2.28) and (3.5), we note that

\[
\zeta_M \otimes \bar{\varpi} \mapsto Z(s, W_{\Xi}(g^0_{d_n',\nu_n}(f_{d',\nu'}(v_1), \Phi_{d+d_n'}(\zeta_M + d_n' \boxtimes \bar{\varpi}))), W_{-\varepsilon}(\bar{f}_{d',\nu'}(\bar{\varpi})))
\]

defines an element of Hom\(_{K_{n-1}}(V_d \otimes C_{triv}, C_{triv})\) for \( v_1 \in V_d \) and \( v_2 \in V_d + d_n' \). Hence, by Lemma 4.2, for \( N \in G(d + d_n') \), we have

\[
Z(s, W_{\Xi}(g^0_{d_n',\nu_n}(f_{d',\nu'}(\zeta_H(d)), \Phi_{d+d_n'}(\zeta_N \boxtimes \zeta_{H(d) + d_n'}))), W_{-\varepsilon}(\bar{f}_{d',\nu'}(\zeta_H(-d')))) = 0
\]

unless \( \hat{N} = H(-d') + d_n' \). By (5.5), (5.9) and this equality, we obtain (5.7). \( \square \)
Proposition 5.6. Retain the notation. Assume \( n' = n - 1, d \in \Lambda_{n,F} \) and \(-d' \in \Xi^+(d) \cap \Lambda_{n-1,F} \). Let \( l = \ell(d + d') \). Then we have

\[
Z(s, W_{\varepsilon}(f_{d,d'}(\zeta_{H(-d')}[d])), W_{-\varepsilon}(\tilde{f}_{d,d'}(\zeta_{H(-d')}))) \\
= (-\varepsilon \sqrt{-1})^{l} \left( C_{H(d-d_n)}^{H((-d'-d_n,0))}, Q((\tilde{d}+d',0)) C_{H(-d')}^{H((-d'-d_n,0))}, Q((0_{n-1,l})) \right)^{-1} \\
\times \dim V_d \frac{\prod_{i=1}^{n-1} \Gamma_F(s + \nu_n + \nu'_i; -d_n - d'_i)}{\dim V_{-d'} \frac{\prod_{i=1}^{n-1} \Gamma_F(\nu_n - \nu_i + 1; -d_n + d'_i)}{\dim V_{-d'}}} \\
\times Z(s, W_{\varepsilon}(f_{d,d'}(\zeta_{H(\tilde{d}))}), W_{-\varepsilon}(\tilde{f}_{d,d'}(\zeta_{H(-d')}))), \mathcal{F}_{\varepsilon}(\phi_2)).
\]

Proof. Let \( \phi_1 = \Phi_{d-d_n}^{+}(\zeta_{H(-d')}) \) and \( \phi_2 = \varphi_{n-1,1}^{(l)}(\zeta_{H(\tilde{d})}) \). By Proposition 3.5, we have

\[
Z(s, W_{\varepsilon}(g_{d_{-d_n},\nu_n}(f_{d,d'}(\zeta_{H(\tilde{d}))}, \phi_0)), W_{-\varepsilon}(\tilde{f}_{d,d'}(\zeta_{H(-d')}))).
\]

By (5.6). Because of these equalities and Lemma 4.12, it suffices to prove

\[
Z(s, W_{\varepsilon}(g_{d_{-d_n},\nu_n}(f_{d,d'}(\zeta_{H(\tilde{d}))}, \phi_0)), W_{-\varepsilon}(\tilde{f}_{d,d'}(\zeta_{H(-d')}))).
\]

By Lemma 4.11, we have

\[
g_{d_{-d_n},\nu_n}(f_{d,d'}(\zeta_{H(\tilde{d}))}, \phi_0) = \sum_{\lambda' \in \Xi^+(d-d_n,l)} \sum_{N' \in G(\lambda',0)} C_{\Lambda}^{H((-d'-d_n,0))}, Q((\tilde{d}+d',0)) \times C_{\Lambda}^{H(-d')} Q((0_{n-1,l})) g_{d_{-d_n},\nu_n}(f_{d,d'}(\zeta_{H(\tilde{d}))}, \phi_\lambda(\zeta_{\lambda'} \otimes \zeta_{N})).
\]

By (3.2), we note that

\[
v \otimes \zeta_M \rightarrow g_{d_{-d_n},\nu_n}(f_{d,d'}(\zeta_{H(\tilde{d}))}, \phi_\lambda(\zeta_{\lambda'} \otimes \zeta_{N}))(g)
\]
defines an element of \( \text{Hom}_{\mathcal{K}_{n-1}}(V_{\lambda'} \otimes_{C} V_{\lambda'}^{+} + d_n, \mathcal{C}_{\text{triv}}) \) for \( \lambda' \in \Xi^+(d-d_n,l) \), \( v_1 \in V_{\lambda'} \) and \( g \in G_n \). Hence, by Lemma 4.1, for \( N \in G((\lambda',0); \lambda') \), \( N' \in G((\lambda',0)) \) and \( \lambda' \in \Xi^+(d-d_n,l) \), we have

\[
g_{d_{-d_n},\nu_n}(f_{d,d'}(\zeta_{H(\tilde{d}))}, \phi_\lambda(\zeta_{\lambda'} \otimes \zeta_{N}))) = 0.
\]
By (2.28) and (3.1), we note that

\[ N_H = (\sum_{d} \phi_d(d,\nu)) \]

Here consider the case of \( n \) let us prove the statement (1) by induction with respect to \( n \).

Proof.

Theorem 5.7. Retain the notation. Assume \( d \in A_{n,F} \) and \( -d' \in A_{n',F} \). We take \( \Gamma_F(\nu';d) \) and \( \Gamma_F(\nu';d') \) as in §2.7.

(1) Assume \( n' = n \) and \( d \in \Xi^-(d') \). Let \( l = \ell(d + d') \). Then we have

\[
Z(s, W_\epsilon(g_{d,v}(\hat{f}_{d},\nu)(v_1), \Phi_{d-d_n}^+(\hat{\zeta}_{H(d)}) \otimes \Phi_{d}^+(\hat{\zeta}_{H(d)-d_n} \otimes \zeta_{N'})))
\]

\[
= \sum_{N' \in G(d-d_n)} C_{N'}^H((-d'+d_n,0),Q((-d'+d_n,0))) C_{N'}^H((-d'-d_n,0),Q((0_{n-1},1))
\]

\[
\times g_{d,v}(\hat{f}_{d},\nu(\zeta_{H(d)}), \Phi_{d-d_n}^+(\hat{\zeta}_{H(d)-d_n} \otimes \zeta_{N'})).
\]

By (2.28) and (3.1), we note that

\[
\zeta_M \otimes \tau \mapsto Z(s, W_\epsilon(g_{d,v}(\hat{f}_{d},\nu)(v_1), \Phi_{d-d_n}^+(\hat{\zeta}_{H(d)}) \otimes \Phi_{d}^+(\hat{\zeta}_{H(d)-d_n})))
\]

defines an element of \( \text{Hom}_{K_{n-1}}(V_d \otimes C \mathcal{V}_{d'}, C_{\text{triv}}) \) for \( v_1 \in V_d \) and \( v_2 \in V_{d-d_n} \). Hence, by Lemma 4.2, for \( N' \in G(d - d_n) \), we have

\[
Z(s, W_\epsilon(g_{d,v}(\hat{f}_{d},\nu(\zeta_{H(d)}), \Phi_{d-d_n}^+(\hat{\zeta}_{H(d)-d_n} \otimes \zeta_{N'}))) \]

\[
W_\epsilon(\hat{f}_{d'},\nu(\zeta_{H(-d')}))) = 0
\]

unless \( \tilde{N}' = H(-d') - d_n \). By (5.3), (5.12) and this equality, we obtain (5.10).

\[
\square
\]

(2) Assume \( n' = n - 1 \) and \( -d' \in \Xi^+(d) \). Then we have

\[
Z(s, W_\epsilon(f_{d,v}(\zeta_{H(-d')}(d)), \Phi_{d-d_n}^+(\hat{\zeta}_{H(-d')})))
\]

\[
= \frac{(-\xi^\text{F} \sum_{i=1}^{n-1} n-d') \sqrt{b(d + d')} \zeta \Gamma(\zeta_{H(d)} \otimes \zeta_{H(-d')})}{(\dim V_d)} \Gamma_F(\nu';d) \Gamma_F(\nu';d')
\]

Here \( r(H(-d')[d]) \) is defined by (2.16).

Proof. Let us prove the statement (1) by induction with respect to \( n \). First, we consider the case of \( n = 1 \). Since

\[
W_\epsilon(f_{d_1,v}(\zeta_{d_1}))(ak) = a^{n_{cF}} k^{d_1}, \quad W_\epsilon(\tilde{f}_{d_1,v_1}(\zeta_{d_1}))(ak) = a^{n_{cF}} k^{d_1},
\]

\[
\gamma_{d_1+\nu_1}(\zeta_{d_1+\nu_1})(ak) = (ak)^{\nu_1+\nu_1} \exp(-\pi c_F a^2)
\]

for \( a \in A_1 = \mathbb{R}_+^\times \) and \( k \in K_1 \), we have

\[
Z(s, W_\epsilon(f_{d_1,v}(\zeta_{d_1})), \Phi_{d-d_n}^+(\hat{\zeta}_{H(d)}) \otimes \Phi_{d}^+(\hat{\zeta}_{H(d)-d_n} \otimes \zeta_{N'})).
\]
\[
= \int_0^\infty \exp(-\pi c_F a^2) u^{s+\nu_1 + \nu'_1} c_F + d_1 + d'_1 \frac{2 c_F}{a} \left( \int_{K_1} dk \right)
= \Gamma_F(s + \nu_1 + \nu'_1; d_1 + d'_1) = L(s, \Pi_{d_1, \nu_1} \times \Pi_{d'_1, \nu'_1}).
\]

Here the second equality follows from (2.53). Next, we consider the case of \( n \geq 2 \). Let \( q = \ell(d + d') \). By Propositions 5.5 and 5.6, we have

\[
Z(s, W_\varepsilon(f_\delta, \nu_1 \zeta_{H(d)})), W_{-\varepsilon}(f^\delta, \nu_1 \zeta_{H(d)})), \bar{\varphi}^{(l)}_{1,n}(\zeta_{Q(d + d')}))
= (-\varepsilon \varepsilon - 1)^q \frac{C^H(-d') + d'_n Q(d + d')} {C^H(-d') + d'_n Q(d + d')} \frac{C^H(-d') + d'_n Q((0_{d-1}, l))} {C^H(-d') + d'_n Q((0_{d-1}, l))} \\
\times \dim V_d \prod_{n=1}^q \Gamma_F(s + \nu_1 + \nu'_1; d_1 + d'_n) \prod_{n=1}^q \Gamma_F(s + \nu_1 + \nu'_1; -d_n - d'_1) \prod_{n=1}^q \Gamma_F(s + \nu_1 + \nu'_1; -d_n - d'_1)
\times Z(s, W_\varepsilon(f_\delta, \nu_1 \zeta_{H(d)})), W_{-\varepsilon}(f^\delta, \nu_1 \zeta_{H(d)})), \bar{\varphi}^{(l)}_{1,n}(\zeta_{Q(d + d')}))
\]

Moreover, by (4.5), we have

\[
\frac{C^H(-d') + d'_n Q(d + d')} {C^H(-d') + d'_n Q(d + d')} \frac{C^H(-d') + d'_n Q((0_{d-1}, l))} {C^H(-d') + d'_n Q((0_{d-1}, l))} \\
= \frac{1}{\sqrt{\gamma(H(\mu)[\lambda])}} \prod_{h=1}^{n-1} \frac{(d_h - d_n - h + n)!(-d'_h + d'_n - h + n - 1)!} {d_h + d'_n - h + n)!(-d'_h - d_n - h + n - 1)!}
\]

By the above equalities and the induction hypothesis, we obtain the formula in the statement (1).

The statement (2) follows from (4.5), Proposition 5.6, the statement (1) and

\[
\frac{1}{\sqrt{\gamma(H(\mu)[\lambda])}} = \prod_{1 \leq i \leq j \leq n-1} (\mu_i - \mu_j - i + j)! (\lambda_i - \lambda_j + 1 - i + j)! (\lambda_i - \lambda_j + i + j)! (\mu_i - \lambda_j + 1 - i + j)!
\]

for \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n \) and \( \mu = (\mu_1, \mu_2, \cdots, \mu_{n-1}) \in \Xi^+(\lambda). \)

\[\square\]

**Proof of Theorem 2.5.** The equality (2.30) follows from Theorem 5.7 (2). Since (2.29) is an element of \( \text{Hom}_{K_{n-1}}(V_d \otimes \overline{\nu}, C_{\text{triv}}) \), we completes the proof by Lemma 4.2.

\[\square\]

**Proof of Theorem 2.9 (2).** The equality (2.50) follows from Theorem 5.7 (1). Since (2.48) is an element of \( \text{Hom}_{K_n}(V_d \otimes \overline{\nu} \otimes \overline{\nu} \otimes \overline{\nu} \otimes (0_{n-1}), C_{\text{triv}}) \), we completes the proof by Lemma 4.3.

\[\square\]

Similar to Propositions 5.5 and 5.6, we obtain the following propositions.
Proposition 5.8. Retain the notation. Assume \( n' = n > 1, d' \in \Lambda_{n,F} \) and 
\( -d \in \Xi^n(d') \cap \Lambda_{n,F} \). Let \( l = \ell(-d - d') \). Then we have

\[
Z(s, W_{d'}(\zeta_H(d'))), W_{-d'}(\zeta_H(d'))(\zeta_Q(-d-d'))
= C_{d}^{\mathcal{H}(d')-d''}, Q((0_{n-1},), \mathcal{H}(d')-d'', Q(d+d')
\times \frac{\dim V_{-d} \prod_{i=1}^{n-1} \Gamma_F(s + \nu_i + \nu_i', -d'' + d_i')}{\dim V_{d'} \prod_{i=1}^{n-1} \Gamma_F(\nu_i - \nu_i' + 1; -d'' + d_i')}
\times Z(s, W_{d'}(\zeta_H(d'))), W_{-d'}(\zeta_H(d'))(\zeta_Q(-d-d'))
\]

Proposition 5.9. Retain the notation. Assume \( n' = n - 1, -d \in \Lambda_{n,F} \) and 
\( d' \in \Xi^n(-d) \cap \Lambda_{n-1,F} \). Let \( l = \ell(-d - d') \). Then we have

\[
Z(s, W_{d'}(\zeta_H(d'))), W_{-d'}(\zeta_H(d'))(\zeta_Q(-d-d'))
= (-\varepsilon \sqrt[\ell - 1]{\prod_{i=1}^{n-1} \Gamma_F(s + \nu_i + \nu_i'; d_n + d_i')}
\times \frac{\dim V_{-d} \prod_{i=1}^{n-1} \Gamma_F(s + \nu_i - \nu_i + 1; d_n - d_i)}{\dim V_{d'} \prod_{i=1}^{n-1} \Gamma_F(\nu_i - \nu_i' + 1; -d'' + d_i')}
\times Z(s, W_{d'}(\zeta_H(d'))), W_{-d'}(\zeta_H(d'))(\zeta_Q(-d-d'))
\]

Similar to Theorem 5.7, we obtain the following theorem using Propositions 5.8 and 5.9.

Theorem 5.10. Retain the notation. Assume \( -d \in \Lambda_{n,F} \) and \( d' \in \Lambda_{n',F} \). We 
take \( \Gamma_F(\nu; d') \) and \( \Gamma_F(\nu'; d') \) as in \( \mathfrak{n}2.7 \).

1. Assume \( n' = n \) \( \text{and} \) \( d' \in \Xi^n(-d) \). Let \( l = \ell(-d - d') \). Then we have

\[
Z(s, W_{d'}(\zeta_H(d'))), W_{-d'}(\zeta_H(d'))(\zeta_Q(-d-d'))
= (-\varepsilon \sqrt[\ell - 1]{\prod_{i=1}^{n-1} \Gamma_F(s + \nu_i - \nu_i + 1; d_n - d_i)}
\times \frac{\dim V_{-d} \prod_{i=1}^{n-1} \Gamma_F(\nu_i - \nu_i' + 1; -d'' + d_i')}{\dim V_{d'} \prod_{i=1}^{n-1} \Gamma_F(\nu_i - \nu_i' + 1; -d'' + d_i')}
\times Z(s, W_{d'}(\zeta_H(d'))), W_{-d'}(\zeta_H(d'))(\zeta_Q(-d-d'))
\]

2. Assume \( n' = n - 1 \) \( \text{and} \) \( d' \in \Xi^n(-d) \). Then we have

\[
Z(s, W_{d'}(\zeta_H(d'))), W_{-d'}(\zeta_H(d'))(\zeta_Q(-d-d'))
= (-\varepsilon \sqrt[\ell - 1]{\prod_{i=1}^{n-1} \Gamma_F(s + \nu_i - \nu_i' + 1; -d'' + d_i')}
\times \frac{\dim V_{-d} \prod_{i=1}^{n-1} \Gamma_F(\nu_i - \nu_i' + 1; -d'' + d_i')}{\dim V_{d'} \prod_{i=1}^{n-1} \Gamma_F(\nu_i - \nu_i' + 1; -d'' + d_i')}
\times Z(s, W_{d'}(\zeta_H(d'))), W_{-d'}(\zeta_H(d'))(\zeta_Q(-d-d'))
\]

Proof of Theorem 2.9 (1). The equality (2.49) follows from Theorem 5.10 (1) and (2.45). Since (2.47) is an element of \( \text{Hom}_{K}\mathcal{O}(V_{d} \otimes_{\mathcal{C}} V_{d'} \otimes_{\mathcal{C}} V_{l,0_{n-1}} \otimes_{\mathcal{C}} \mathcal{C}_{\text{triv}}) \), we completes the proof by Lemma 4.3 and the properties of complex conjugate representations in \( \mathfrak{n}2.6 \).
A  Explicit formulas of Whittaker functions

In this appendix, we consider the explicit formulas of the radial parts of Whittaker functions on $G_n$. Let $\varepsilon \in \{\pm 1\}$, $d = (d_1, d_2, \cdots, d_n) \in \mathbb{Z}^n$, and $\nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{C}^n$. Assume that either $d \in \Lambda_{n,F}$ or $-d \in \Lambda_{n,F}$ holds. We set

$$\tilde{W}^{(\varepsilon)}_{d,\nu}(a) = \begin{cases} \eta_{-\rho_n}(a)W_\varepsilon(f_{d,\nu}(\xi_{H(d)}))(a) & \text{if } d \in \Lambda_{n,F}, \\
-\eta_{-\rho_n}(a)W_\varepsilon(f_{d,\nu}(\xi_{H(-d)}))(a) & \text{if } -d \in \Lambda_{n,F} \end{cases} \quad (a \in A_n).$$

Then we have the following theorem, which is the generalization of the explicit formulas [9, Theorem 14] of spherical Whittaker functions on GL$(n, \mathbb{R})$.

**Theorem A.1.** Retain the notation, and we assume $n > 1$. We take $\hat{a}$ and $\hat{\nu}$ as in §3.1. Let $a = \text{diag}(a_1, a_2, \cdots, a_n) \in A_n$. Then we have

$$\tilde{W}^{(\varepsilon)}_{d,\nu}(a) = \frac{\prod_{i=1}^n a_i^{\nu_i+|d_i|-d_n}}{\prod_{i=1}^{n-1} \Gamma_F(\nu_i - \nu_i + 1; |d_i| - d_n)} \times \int_{(\mathbb{R}_+^*)^{n-1}} \tilde{W}^{(\varepsilon)}_{d,\hat{\nu}}(t) \prod_{i=1}^{n-1} \exp \left(-\pi c_F \left(\frac{t_i^2}{a_i^{\nu_i+|d_i|-d_n}} + \frac{\beta_i^2}{t_i^2}\right) \right) t_i^{-\nu_i+|d_i|-d_n} \frac{2c_F dt_i}{t_i}$$

with $t = \text{diag}(t_1, t_2, \cdots, t_{n-1}) \in A_{n-1}$.

**Proof.** We will prove here only the case of $d \in \Lambda_{n,F}$, since the proof for the case of $-d \in \Lambda_{n,F}$ is similar. By Lemmas 4.7 and 4.8, we have

$$\begin{aligned}
\Phi^+_{d-d_n}((\xi_{H(d)}-d_n)|H(d)-d_n) &\mathbb{E} \xi_{H(d)-d_n}((h, hz)a) \\
&= \left(\prod_{i=1}^{n-1} a_i^{d_i-d_n}\right) \langle \tau^{\hat{a}-d_n}(h)\xi^{\hat{a}+d_n}|H^{\hat{a}}-d_n, \xi^{\hat{a}+d_n}|H^{\hat{a}}-d_n, e_{(n-1,n)}((h, hz)a) \rangle
\end{aligned}$$

for $h \in G_{n-1}$ and $z \in M_{n-1,1}(F)$. Hence, by Corollary 5.3, we have

$$\begin{aligned}
\tilde{W}^{(\varepsilon)}_{d,\nu}(a) &= \eta_{-\rho_n}(a)W_\varepsilon(f_{d,\nu}(\xi_{H(d)}))(a) \\
&= \frac{\dim V_{\hat{d}}}{\prod_{i=1}^n a_i^{\nu_i+|d_i|-d_n}} \prod_{i=1}^{n-1} \Gamma_F(\nu_i - \nu_i + 1; |d_i| - d_n) \\
&\times \int_{G_{n-1}} \left(\int_{M_{n-1,1}(F)} e_{(n-1,n)}((h, hz)a) \psi_{-\varepsilon(e_{n-1}z)} dz \right) W_\varepsilon(f_{d,\nu}(\xi_{H(d)}))(h^{-1}) \\
&\times \langle \tau^{\hat{a}-d_n}(h)\xi^{\hat{a}+d_n}|H^{\hat{a}}-d_n, \xi^{\hat{a}+d_n}|H^{\hat{a}}-d_n\rangle \chi_{d_n}(\det h) |\det h|^{\nu_n+n/2} dh.
\end{aligned}$$

Decomposing $h^{-1} = xt_k$ ($x \in N_{n-1}$, $t = \text{diag}(t_1, t_2, \cdots, t_{n-1}) \in A_{n-1}$, $k \in K_{n-1}$) and applying Schur’s orthogonality [17, Corollary 1.10] for the integration on $K_{n-1}$ together with the equalities

$$\langle \tau^{\hat{a}-d_n}(h)^{\varepsilon}|H^{\hat{a}}-d_n, \xi^{\hat{a}+d_n}|H^{\hat{a}}-d_n\rangle \chi_{d_n}(\det h)$$

48
By the elementary formula (4.19) and the equality

$$\langle \tau_{\hat{d}}(k)\zeta_{H(\hat{d})}, \zeta_{H(\hat{d})} \rangle$$

we have

and

$$W_\epsilon(f_{\hat{d},\hat{\nu}}(\zeta_{H(\hat{d})}))(h^{-1}) = \psi_{\epsilon, n-1}(x)W_\epsilon(f_{\hat{d},\hat{\nu}}(\tau_{\hat{d}}(k)\zeta_{H(\hat{d})}))(t)$$

$$= \sum_{M \in \hat{G}(\hat{d})} (\tau_{\hat{d}}(k)\zeta_{H(\hat{d})}, \zeta_{M}) \psi_{\epsilon, n-1}(x)W_\epsilon(f_{\hat{d},\hat{\nu}}(\zeta_{M}))(t),$$

we have

$$\tilde{W}_{d,\nu}^{(\epsilon)}(a) = \prod_{i=1}^{n-1} a_i^{(\nu_n+i-1)c_F+d_i-d_n} \times \frac{\int_{[\mathbb{R}_+^{n-1}]} \left( \int_{N_n-1} \int_{M_{n-1,1}(F)} e_{(n-1,n)}((t^{-1}x^{-1}, t^{-1}x^{-1}z)a)\psi_{\epsilon, n-1}(x) \times \psi_{-\epsilon}(e_{n-1}z) dz dx \right) \tilde{W}_{d,\nu}^{(\epsilon)}(t) \prod_{i=1}^{n-1} t_i^{-(\nu_n+n-i)c_F-d_i+d_n} \frac{2\epsilon F dt_i}{t_i}}{\Gamma_F(\nu_n - \nu_n + 1; d_i - d_n)}.$$

Let us consider the integral

$$\int_{N_n-1} \int_{M_{n-1,1}(F)} e_{(n-1,n)}((t^{-1}x^{-1}, t^{-1}x^{-1}z)a)\psi_{\epsilon, n-1}(x)\psi_{-\epsilon}(e_{n-1}z) \, dz \, dx.$$

Substituting \( \begin{pmatrix} x^{-1} \\ O_{1,n-1} \\ \frac{x^{-1}z}{1} \end{pmatrix} \rightarrow x \), this integral becomes

$$\int_{N_n} e_{(n-1,n)}((1_{n-1}, O_{n-1,1}) \nu_n(t^{-1})xa)\psi_{-\epsilon, n}(x) \, dx. \quad (A.1)$$

By the elementary formula (4.19) and the equality

$$e_{(n-1,n)}((1_{n-1}, O_{n-1,1}) \nu_n(t^{-1})xa)$$

$$= \prod_{i=1}^{n-1} \exp(-\pi c_F t_i^{-2}a_i^2) \prod_{j=i+1}^{n} \exp(-\pi c_F t_j^{-2}a_j^2 x_{i,j} \tau_{i,j}),$$

we know that (A.1) is equal to

$$\prod_{i=1}^{n-1} \exp\left(-\pi c_F \left( \frac{t_i^2}{a_{i+1}^2} + \frac{a_i^2}{t_i^2} \right) \right) t_i^{(n-i)c_F a_{i+1}^{-c_F}}.$$

Therefore, we obtain the assertion. \( \square \)
References

[1] S. J. Ališauskas, A.-A. A. Jucys, and A. P. Jucys. On the symmetric tensor operators of the unitary groups. *J. Mathematical Phys.*, 13:1329–1333, 1972.

[2] Chao-Ping Dong and Huajian Xue. On the nonvanishing hypothesis for Rankin-Selberg convolutions for $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$. *Represent. Theory*, 21:151–171, 2017.

[3] I. M. Gel’fand and M. L. Cetlin. Finite-dimensional representations of the group of unimodular matrices. *Doklady Akad. Nauk SSSR (N.S.*), 71:825–828, 1950.

[4] Roe Goodman and Nolan R. Wallach. *Symmetry, representations, and invariants*, volume 255 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2009.

[5] Loïc Grenié. Critical values of automorphic $L$-functions for $GL(r) \times GL(r)$. *Manuscripta Math.*, 110(3):283–311, 2003.

[6] Harald Grobner and Michael Harris. Whittaker periods, motivic periods, and special values of tensor product $L$-functions. *J. Inst. Math. Jussieu*, 15(4):711–769, 2016.

[7] Erich Hecke. *Mathematische Werke*. Herausgegeben im Auftrag der Akademie der Wissenschaften zu Göttingen. Vandenhoeck & Ruprecht, Göttingen, 1959.

[8] Miki Hirano, Taku Ishii, and Tadashi Miyazaki. Archimedean zeta integrals for $GL(3) \times GL(2)$. *To appear in Memoir of the AMS.*

[9] Taku Ishii and Eric Stade. New formulas for Whittaker functions on $GL(n, \mathbb{R})$. *J. Funct. Anal.*, 244(1):289–314, 2007.

[10] Taku Ishii and Eric Stade. Archimedean zeta integrals on $GL_n \times GL_m$ and $SO_{2n+1} \times GL_m$. *Manuscripta Math.*, 141(3-4):485–536, 2013.

[11] H. Jacquet and R. P. Langlands. *Automorphic forms on GL(2).* Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin-New York, 1970.

[12] H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika. Rankin-Selberg convolutions. *Amer. J. Math.*, 105(2):367–464, 1983.

[13] Hervé Jacquet. *Automorphic forms on GL(2). Part II.* Lecture Notes in Mathematics, Vol. 278. Springer-Verlag, Berlin-New York, 1972.

[14] Hervé Jacquet. Archimedean Rankin-Selberg integrals. In *Automorphic forms and $L$-functions II. Local aspects*, volume 489 of *Contemp. Math.*, pages 57–172. Amer. Math. Soc., Providence, RI, 2009.
[15] Hervé Jacquet and Joseph Shalika. Rankin-Selberg convolutions: Archimedean theory. In Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989), volume 2 of Israel Math. Conf. Proc., pages 125–207. Weizmann, Jerusalem, 1990.

[16] A. A. Jucis. The isoscalar factors of the Clebsch-Gordan coefficients of unitary groups. Litovsk. Fiz. Sb., 10:5–12, 1970.

[17] Anthony W. Knapp. Representation theory of semisimple groups. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001. An overview based on examples, Reprint of the 1986 original.

[18] Bertram Kostant. On Whittaker vectors and representation theory. Invent. Math., 48(2):101–184, 1978.

[19] Tadashi Miyazaki. The local zeta integrals for $GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$. Proc. Japan Acad. Ser. A Math. Sci., 94(1):1–6, 2018.

[20] Alexandru A. Popa. Whittaker newforms for Archimedean representations. J. Number Theory, 128(6):1637–1645, 2008.

[21] A. Raghuram. Critical values of Rankin-Selberg $L$-functions for $GL_n \times GL_{n-1}$ and the symmetric cube $L$-functions for $GL_2$. Forum Math., 28(3):457–489, 2016.

[22] R. A. Rankin. Contributions to the theory of Ramanujan’s function $\tau(n)$ and similar arithmetical functions. I. The zeros of the function $\sum_{n=1}^\infty \tau(n)/n^s$ on the line $\Re s = 13/2$. II. The order of the Fourier coefficients of integral modular forms. Proc. Cambridge Philos. Soc., 35:351–372, 1939.

[23] Atle Selberg. Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist. Arch. Math. Naturvid., 43:47–50, 1940.

[24] Birgit Speh and David A. Vogan, Jr. Reducibility of generalized principal series representations. Acta Math., 145(3-4):227–299, 1980.

[25] Eric Stade. Mellin transforms of $GL(n, \mathbb{R})$ Whittaker functions. Amer. J. Math., 123(1):121–161, 2001.

[26] Eric Stade. Archimedean $L$-factors on $GL(n) \times GL(n)$ and generalized Barnes integrals. Israel J. Math., 127:201–219, 2002.

[27] Binyong Sun. The nonvanishing hypothesis at infinity for Rankin-Selberg convolutions. J. Amer. Math. Soc., 30(1):1–25, 2017.

[28] N. Ja. Vilenkin and A. U. Klimyk. Representation of Lie groups and special functions. Vol. 3, volume 75 of Mathematics and its Applications (Soviet Series). Kluwer Academic Publishers Group, Dordrecht, 1992. Classical and quantum groups and special functions, Translated from the Russian by V. A. Groza and A. A. Groza.
[29] Nolan R. Wallach. *Real reductive groups. II*, volume 132 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1992.

[30] Takao Watanabe. Global theta liftings of general linear groups. *J. Math. Sci. Univ. Tokyo*, 3(3):699–711, 1996.

[31] Shou-Wu Zhang. Gross-Zagier formula for GL₂. *Asian J. Math.*, 5(2):183–290, 2001.

[32] D. P. Zhelobenko. On Gel’fand-Zetlin bases for classical Lie algebras. In *Representations of Lie groups and Lie algebras (Budapest, 1971)*, pages 79–106. Akad. Kiadó, Budapest, 1985.

52