Attitude Control of Rigid Body with Inertia Uncertainty and Saturation Input

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Attitude Control of Rigid Body with Inertia Uncertainty and Saturation Input

Xi Ma, Fuchun Sun*, Hongbo Li, and Bing He

Abstract: In this paper, the attitude control problem of rigid body is addressed with considering inertia uncertainty, bounded time-varying disturbances, angular velocity-free measurement, and unknown non-symmetric saturation input. Using a mathematical transformation, the effects of bounded time-varying disturbances, uncertain inertia, and saturation input are combined as total disturbances. A novel finite-time observer is designed to estimate the unknown angular velocity and the total disturbances. For attitude control, an observer-based sliding-mode control protocol is proposed to force the system state convergence to the desired sliding-mode surface; the finite-time stability is guaranteed via Lyapunov theory analysis. Finally, a numerical simulation is presented to illustrate the effective performance of the proposed sliding-mode control protocol.

Key words: attitude control; inertial uncertainty; angular velocity-free measurement; saturation input; finite-time observer; sliding-mode control

1 Introduction

The problem of rigid body attitude control has attracted much interest over past decades due to its wide theoretical and practical applications. The attitude model is represented by a set of two vector equations, namely, the kinematic equation and the dynamic equation\(^1\). From a practical point of view, the design of an effective and robust attitude control protocol with high-precision is an important issue for industrial application. Due to its inherent highly nonlinear properties, some nonlinear control methods, such as feedback control\(^2\), sliding-mode control\(^3\), adaptive control\(^4\), backstepping control\(^5\), and robust control\(^6\), have been studied to address the problem with complicated conditions, such as external disturbance and/or dynamic uncertainty and/or saturation input. For more details on these topics, we refer the readers to Refs. [7–10] and the references therein.

It is worth mentioning that research on the rigid-body attitude control problem under practical conditions is ongoing. As is well-known, system disturbances are inevitable due to environmental factors\(^11\), which may deteriorate the system’s stability and force the control protocol design to become more intractable. In Ref. [12], an adaptive sliding-mode control scheme for the attitude control problem with dynamic uncertainty and external disturbance was proposed. This scheme regulates the relative attitude and angular velocity of the rigid body within a desired small region. In Ref. [13], the inertial uncertainty and external disturbance were lumped together as the total disturbances, and an Extended State Observer (ESO) was introduced to estimate the total disturbances and compensate for them by designing a sliding-mode controller. For the dynamic uncertainty, Chebyshev neural networks were used as universal approximator to learn unknown
nonlinear functions in Ref. [14]. In addition, a robust control term, using the hyperbolic tangent function, was applied to counteract neural-network approximation errors and external disturbances, which made the proposed controller continuous and hence chattering-free.

In the above literature, the major assumption is the full states (i.e., attitude and angular-velocity) measurement. However, the above results may not be used directly to handle the attitude control problem, if the angular velocity-free measurement is considered. It should also be pointed out that the full states measurement assumption is rigorous because the angular velocity-free measurement is frequently encountered in various engineering systems, such as some low-cost satellites. A feasible way to handle the unmeasured angular velocity is to design an observer to approximate these unknown states. A nonlinear state observer was designed in Refs. [15, 16] and the smooth structure property was ensured, meaning that all the estimated states were in $C_\infty$ continuity. In Ref. [17], a passivity-based control scheme was proposed to ensure system attitude asymptotic convergence without the angular velocity measurement. In Ref. [18], the attitude control problem with angular velocity-free measurement was also considered. A certainty-equivalence passivity-based controller was developed to guarantee the attitude convergence, with an adaptive observer to estimate the angular velocity.

All the results proposed in the above literature focus on systems without considering the saturation input. It is well known that saturation control is the most important non-smooth nonlinear property, and should be explicitly considered in control design. Compared with unrestrained control input, the existence of saturation input may deteriorate the control performance and even result in system instability [19]. Lu et al. [20] considered actuator saturation for attitude control problems. In Ref. [21], hyperbolic tangent functions were employed to prevent actuator saturation, whereas saturation functions were used in Ref. [22]. In Refs. [23, 24], the actuator saturation was taken into account for the first time and a global saturated finite-time control scheme was proposed.

The main purpose of this paper is to investigate the rigid-body attitude control problem in the presence of the external disturbances, the inertial uncertainty, the angular velocity-free measurement, and unknown non-symmetrical saturation input. Here, the system uncertain inertia and saturation input are combined with external disturbance and a novel finite-time observer is used to estimate the angular velocity and the disturbances. An observer-based sliding-mode control protocol is proposed and its finite-time stability is guaranteed by Lyapunov analysis. The main contributions of this paper are as follows:

1. In contrast to the existing literature on rigid-body attitude systems, the inertial uncertainty, the angular velocity-free measurement, and the unknown non-symmetrical saturation input are considered; these are combined with external disturbances and the effect is estimated by the designed finite-time observer. However, the angular velocity-free measurement and input saturation are not considered in Ref. [14], and only external disturbance is taken into account in Ref. [3]. The inertial uncertainty, angular velocity-free measurement, external disturbances, and input saturation were all explicitly considered when designing the control protocol in this paper.

2. Combined with the designed finite-time observer, an observer-based sliding-mode control protocol is designed and the finite-time stability is guaranteed using Lyapunov theorem analysis. Moreover, the proposed controller is robust against not only structured uncertainty but also unstructured uncertainty.

This paper is organized as follows. In Section 2, a typical rigid-body attitude control problem is formulated. Preliminaries and the main assumptions are also given. Section 3 discusses the finite-time observer and the sliding-mode controller for the rigid-body attitude control problem under complicated conditions. The elaborate proof of the closed-loop system stability is also provided. A numerical simulation is performed to validate the proposed control strategy in Section 4, followed by conclusions in Section 5.

2 Problem Formulation and Preliminaries

The rigid-body attitude control system can be described by two sets of equations. These are the kinematic and dynamic equations that relate the time derivatives of the angular coordinate to the angular velocity vector, and the dynamic equation that describes the evolution of the velocity vector [25, 26]
is

\[
\begin{align*}
\dot{q}_0 &= -\frac{1}{2} q_v^T \omega, \\
\dot{q}_v &= \frac{1}{2} (q_0 J_3 + q_v^x) \omega, \\
J \dot{\omega} &= -\omega^x J \omega + u + d
\end{align*}
\]
holds situations. Assumption 2 is reasonable while most of the external the inertial uncertainty used in most of the literature. Assumption 1 is a common assumption regarding control systems, spacecraft attitude dynamics, and so on. Assumption 1 is a common assumption regarding

d.t/<d 

bounded, i.e.,

\[ J \]

\( \text{disturbance; } \)

\[ B \]

\( \text{expressed in the body frame } \)

\[ I \]

\( \text{velocity with respect to the inertia frame } \)

\[ C \]

\( \text{and inertia uncertainty } \)

\( \text{and lemmas are necessary.} \)

To simplify our analysis, we define

\[ q' := \left( \begin{array}{c} q_0 \\ q_v \end{array} \right), \quad E (q) := \left( \begin{array}{c} -\frac{1}{2} q_v^T \\ \frac{1}{2} (q_0 I_3 + q_v^\times) \end{array} \right). \]

Therefore, the attitude kinematic equation can be rewritten as

\[ \dot{q} = E (q) \omega. \]

Before our progression, the following assumptions and lemmas are necessary.

**Assumption 1** The symmetric positive definite inertia matrix satisfies \( J = J_0 + \Delta J \), where \( \Delta J \) denotes the inertial uncertainty.

**Assumption 2** The time-varying disturbance \( d(t) \) is bounded, i.e., \( d(t) < d_m \), where \( d_m \) is a known positive constant.

**Remark 1** Many practical attitude control systems can be described by Eq. (1), such as satellite attitude control systems, spacecraft attitude dynamics, and so on. Assumption 1 is a common assumption regarding the inertial uncertainty used in most of the literature. Assumption 2 is reasonable while most of the external disturbances can be considered as bounded in practical situations.

The following lemmas are introduced.

**Lemma 1** For \( x \in \mathbb{R}^4 \), the following equality holds

\[ \| E (x) \|_2 = \frac{1}{2} \| x \|_2. \]

**Lemma 2** Assume \( x_1, x_2, \ldots, x_n \geq 0 \) and \( 0 < p < 1 \), then the following inequality holds:

\[ \left( \sum_{i=1}^{n} x_i \right)^p \leq \sum_{i=1}^{n} x_i^p. \]

Now consider the system’s unknown disturbance \( d \) and inertia uncertainty \( \Delta J \). The attitude dynamic equations can be rewritten as

\[ (J_0 + \Delta J) \dot{\omega} = -\omega^\times (J_0 + \Delta J) \omega + u + d \quad (2) \]

Note that \( (J_0 + \Delta J)^{-1} \) can be expressed as

\[ (J_0 + \Delta J)^{-1} = J_0^{-1} + \Delta J. \]

So, with some translations, system (2) can be rewritten as

\[ \dot{\omega} = F + J_0^{-1}u + G \quad (3) \]

where

\[ F = -J_0^{-1} \omega^\times J_0 \omega, \]

\[ G = J_0^{-1}d - J_0^{-1} \omega^\times \Delta J_0 \omega - \Delta J \omega^\times J_\omega + \Delta J u + \Delta J d. \]

**Remark 2** In Eq. (3), \( F \) is the known dynamics and \( G \) is the unknown dynamics, which can be considered as the effect of the inertia uncertainty term \( \Delta J \) and the unknown time-varying disturbance \( d \). It should be pointed out that the presence of uncertain inertia and time-varying disturbances makes the attitude control problem more complicated.

**Remark 3** Compared with the previous work in Ref. [13], we consider a more complicated condition. However, control saturation and model uncertainty are inevitable in some practical applications, which makes the control problem more intractable. Additionally, from a practical viewpoint, focus on synchronization time is more important than achieving synchronization, which implies better robustness and disturbance rejection properties. Here, we consider the finite-time case instead of the asymptotic converging case.

### 3 Main Result

Our aim is to design a control protocol \( u \) for system (1) under some practical conditions to make the attitude system stable, i.e.,

\[ q_0 \to \pm 1, \quad q_v \to 0, \quad \omega \to 0, \text{ as } t \to \infty. \]

#### 3.1 Novel extended state observer

A feasible way of handling an unknown state is to design an observer to approximate its real value. The ESO mentioned in Ref. [27] shows high efficiency when accomplishing the nonlinear dynamic estimation. In addition, this ESO regards the system disturbance as the extended state for estimation. Inspired by Ref. [28], system (1) can be rewritten as

\[
\begin{aligned}
\dot{q}_v &= f (q) \omega, \\
\dot{\omega} &= F + J_0^{-1}u + G, \\
\dot{G} &= g (t)
\end{aligned}
\]
error dynamics can be obtained as extended state observer (5), the corresponding observer where
\[ \dot{\hat{\omega}} = f(q) \hat{\omega} + \rho_1 \left( |e_1|^{\alpha_1} + |e_1|^{\beta_1} \right) \text{sign}(e_1), \]
\[ \dot{\hat{\hat{\omega}}} = \hat{\hat{F}} + J_o^{-1} u + \hat{\hat{G}} + \rho_2 \left( |e_1|^{\alpha_2} + |e_1|^{\beta_2} \right) \text{sign}(e_1), \]
\[ \dot{\hat{\hat{\omega}}} = \rho_3 \left( |e_1|^{\alpha_3} + |e_1|^{\beta_3} \right) \text{sign}(e_1) \tag{5} \]
where \( e_v = q_v - \hat{q}_v \) is the estimation error, 0.75 < \( \alpha_1 < 1, \alpha_2 = 2\alpha_1 - 1, \alpha_3 = 3\alpha_1 - 2, \beta_1 = \alpha_1^{-1}, \beta_2 = \alpha_1^{-1} + \alpha_1 - 1, \) and \( \beta_3 = \beta_1^{-1} + 2\alpha_1 - 2. \)

From the system attitude (4) and the designed extended state observer (5), the corresponding observer error dynamics can be obtained as
\[ \dot{e}_1 = f(q) \rho_1 \left( |e_1|^{\alpha_1} + |e_1|^{\beta_1} \right) \text{sign}(e_1), \]
\[ \dot{e}_2 = \hat{F} + e_3 - \rho_2 \left( |e_1|^{\alpha_2} + |e_1|^{\beta_2} \right) \text{sign}(e_1), \]
\[ \dot{e}_3 = g(t) - \rho_3 \left( |e_1|^{\alpha_3} + |e_1|^{\beta_3} \right) \text{sign}(e_1) \tag{6} \]

where \( e_2 = \omega - \hat{\omega} \) and \( e_3 = G - \hat{\hat{G}}. \)

Based on the fact that the function \( F \) is continuous and differentiable for \( \omega \in \mathbb{R}^3, \) according to the differential mean value theorem, the following equation can be obtained:
\[ F - \hat{F} = F'(\zeta) e_2, \zeta \in (\omega, \hat{\omega}). \]

Let \( a \) denote \( F'(\zeta), \) then substituting the above equation into Eqs. (6) yields
\[ \dot{\hat{e}}_1 = f(q) e_2 - \rho_1 \left( |e_1|^{\alpha_1} + |e_1|^{\beta_1} \right) \text{sign}(e_1), \]
\[ \dot{\hat{e}}_2 = a e_2 + e_3 - \rho_2 \left( |e_1|^{\alpha_2} + |e_1|^{\beta_2} \right) \text{sign}(e_1), \]
\[ \dot{\hat{e}}_3 = g(t) - \rho_3 \left( |e_1|^{\alpha_3} + |e_1|^{\beta_3} \right) \text{sign}(e_1) \tag{7} \]

The finite-time convergence of the estimation errors in Eq. (7) can be presented in the following theorem.

**Theorem 1** Consider the attitude control system (4) and the observer defined in Eq. (5), then the state estimation errors \( e_1, e_2, \) and \( e_3 \) will converge to a residual set \( U \) governed by
\[ U = \{(e_1, e_2, e_3) | \|e_1\| \leq \Delta, \|e_2\| \leq \Delta, \|e_3\| \leq \Delta \} \]
where \( \Delta \) is defined by the observer parameter.

**Proof** See Theorem 1 in Ref. [28].

**Remark 4** It should be noted that the estimation term \( \hat{G} \) is the total disturbance’s estimation and is used to compensate for the uncertainty in the control protocol. Compared with the observer designed in Refs. [29, 30], one of the most advantages is that it can approximate the real-time action of system uncertainty, which is more practical in engineering applications.

### 3.2 Control protocol design and stability analysis

To facilitate the analysis, a translation form is considered that
\[ x = \omega + \sigma q_v, \]
where \( \sigma \) is a positive constant.

Then, the attitude control system (4) can be transformed as
\[ \dot{x} = F + J_o^{-1} u + G + \sigma f(q_v) \omega \tag{8} \]

It has been proven in Ref. [31] that for system (8), \( \lim_{t \to \infty} q_v(t) = 0 \) can be achieved if there exists a control law \( u(t) \) for Eq. (8) ensuring \( \lim_{t \to \infty} x(t) = 0 \) with any initial state \( \|x(0)\| = 1. \) This means while \( \lim_{t \to \infty} x(t) = 0, \lim_{t \to \infty} q_v(t) = 0 \) is also achieved as well as \( \lim_{t \to \infty} \omega(t) = 0 \) due to Eq. (1).

Similarly, if the system states \( \lim_{t \to \infty} q_v(t) = 0 \) and \( \lim_{t \to \infty} \omega(t) = 0 \) are satisfied by the control law \( u(t), \) the state \( \lim_{t \to \infty} x(t) = 0 \) can also be guaranteed.

**Remark 5** Note that, it is only proven that \( q_v \to 0, \omega \to 0. \) The state \( q_0 \) is not considered. This is due to the fact that \( q_0 \to \pm 1 \) once \( q_v \to 0 \) from the constraint condition \( q_0^2 + q_v^T q_v = 1. \) In the physical space, these two equilibrium points \( q = [1.0, 0, 0]^T \) and \( q = [-1.0, 0, 0]^T \) correspond to the same physical point[32]. Hence, it is only needed to guarantee that \( q_v \to 0. \)

#### 3.2.1 Sliding-mode control protocol design

In the following, we introduce the sliding-mode variables,
\[ S = Cx, \]
where \( C \in \mathbb{R}^{3 \times 3}. \)

Without losing generality, we assume matrix \( C \) is of full rank and matrix \( C J_o^{-1} \) is nonsingular.

The sliding-mode control protocol is now given by
\[ u = -J_o C^{-1} \left( \hat{F} \hat{G} + \sigma f(q_v) \hat{\omega} + c_1 S + c_2 S^v + \psi \right) \tag{9} \]
where \( \hat{F} \) and \( \hat{G} \) are the estimation of system terms \( F \) and \( G, \) respectively; \( c_1 > 0, c_2 > 0, \) and \( 0 < v < 1 \) are control parameters and \( \psi = [\psi_1, \psi_2, \psi_3]^T \) are hyperbolic tangent functions defined by Ref. [33].

\[ \psi_i = k_i \text{tanh} \left( \frac{k_{ui} \psi_i S_i}{e} \right), i = 1, \ldots, 3, \]
where \( k_{ui} = 0.2785, k_i \) is a positive constant satisfying \( k_i > \lambda_{\text{max}}(C) \left( \Delta^2 + \left( 1 + \frac{\sigma}{2} \right) \Delta \right) \) and \( \lambda_{\text{max}}(C) \) is the
maximum eigenvalue of matrix $C$. It is easy show that the robust term $\kappa_i$ has the following property:

$$
\lambda_{\max} (C) \|S\| (\Delta^2 + \left(1 + \frac{\sigma}{2}\right) \Delta) - \sum_{i=1}^{n} S_i \psi_i \leq 3\epsilon
$$

(10)

**Remark 6** The structure of the closed-loop system is depicted in Fig. 1. Note that the proposed control protocol (9), the compensating terms $\hat{F}$, $\hat{G}$, and $f(q) \hat{\omega}$ are used to eliminate the effect of system nonlinear term $F$ and unknown terms $G$ and $\sigma f(q) \omega$; the robust term $\psi$ is used to counteract the effects of the estimation errors. And, the sliding-mode feedback term $c_1\dot{S} + c_2\dot{S}^u$ is used to drive the system states to zero. Furthermore, compared with Ref. [14], the proposed controller is continuous with respect to time and chattering-free, by avoiding the use of the signum function.

### 3.2.2 Stability analysis of closed-loop system

In this section, the stability of the closed-loop system (8) can be established by the following theorem.

**Theorem 2** Considering Assumptions 1 and 2, and the proposed sliding-mode protocol (9). Then for any initial conditions, the finite-time stability of closed-loop system (8) can be achieved.

**Proof** Consider the following Lyapunov function candidate,

$$
V = \frac{1}{2} S^T S
$$

(11)

By integrating the designed control protocol (9), the time derivative of the Lyapunov function can be obtained:

$$
\dot{V} = S^T \dot{S} = S^T (CF + C \dot{J}_q^{-1} u + CG + \sigma f(q) \omega) = S^T (\dot{F} + \dot{G} + \sigma f(q) e_2 - c_1\dot{S} - c_2\dot{S}^u - \psi)
$$

(12)

where $\dot{F} = F - \hat{F}$, and $\dot{G} = G - \hat{G}$

Due to the property of the observer, we can obtain

$$
\|C F\| \leq \lambda_{\max} (C) \Delta^2,
\|C G\| \leq \lambda_{\max} (C) \Delta,
\|C f(q) e_2\| \leq 0.5\lambda_{\max} (C) \Delta.
$$

Then, we can obtain that

$$
\dot{V} \leq \lambda_{\max} (C) (\Delta^2 + \Delta + 0.5\sigma \Delta) \|S\| - S^T (c_1\dot{S} + c_2\dot{S}^u + \psi) \leq -c_1S^T S - c_2S^T S^u + 3\epsilon
$$

(13)

From Lemma 2, the following inequality is obtained.

$$
-c_2S^T S^u = -c_2 \sum_{i=1}^{3} S_i^{v+1} \leq -2^{v+1} \frac{c_2}{3} \left(0.5 \sum_{i=1}^{3} S_i^2\right) = -\gamma_2 V^\frac{v+1}{2}
$$

(14)

where $\gamma_2 = 2^{v+1} c_2 > 0$. Then, Formula (13) can be rewritten as

$$
\dot{V} \leq -\gamma_1 V - \gamma_2 V^\frac{v+1}{2} + \gamma_3
$$

(15)

where $\gamma_1 = 2c_1$ and $\gamma_3 = 3\epsilon$ are positive constants. Then, Formula (15) can be expressed as two forms:

$$
\dot{V} \leq -\left(\gamma_1 - \frac{\gamma_3}{V}\right) V - \gamma_2 V^\frac{v+1}{2}
$$

(16)

$$
\dot{V} \leq -\gamma_1 V - \left(\gamma_2 - \frac{\gamma_3}{V^\frac{v+1}{2}}\right) V^\frac{v+1}{2}
$$

(17)

From Formula (16), if the condition $\gamma_1 - \frac{\gamma_3}{V} > 0$ is satisfied, then the finite-time stability is still guaranteed, which implies that $V \leq \frac{\gamma_3}{\gamma_1}$ in finite time. Then, the system state $x$ converges to

$$
\|x\| \leq \frac{1}{\lambda_{\min} (C)} \sqrt{\frac{2\gamma_3}{\gamma_1}} = \frac{1}{\lambda_{\min} (C)} \sqrt{\frac{3\epsilon}{c_1}}
$$

From Formula (17), if the condition $\gamma_2 - \frac{\gamma_3}{V^\frac{v+1}{2}} > 0$ is satisfied, then the finite-time stability is still guaranteed, which implies that $V < \left(\frac{\gamma_3}{\gamma_2}\right)^\frac{v+1}{2}$ in finite time. Then, the system state $x$ converges to

$$
\|x\| \leq \frac{1}{\lambda_{\min} (C)} \sqrt{2 \left(\frac{\gamma_3}{\gamma_2}\right)^\frac{v+1}{2}} = \sqrt{2} \frac{\epsilon}{\lambda_{\min} (C)} \left(\frac{c_2}{3\epsilon}\right)^{v+1}
$$

From Formulas (16) and (17), it is concluded that the designed system state $x$ could converge to the bounded region in finite time, which implies the system state $x$ is finite-time stable. Furthermore, the control parameters $c_1$ and $c_2$ and hyperbolic tangent function parameter
\( e \) depend on the size of the bounded region, that is, the smaller bounded region, the smaller \( e \) and bigger control parameters \( c_1 \) and \( c_2 \) are required.

Remark 7 In system (4), the proposed sliding-mode protocol (9) can be guaranteed bounded motion in a bounded region. What is more, the size of the bounded region can be adjusted by tuning the observer and control parameters. Thus, observer parameter selection is also vital, because it determines not only the performance of the observer but also the motion of the attitude tracking system.

3.2.3 Attitude control with input saturation

In practical engineering, input saturation is the most important non-smooth nonlinear property, and should be explicitly considered in control design. Here, we propose a sliding-mode control protocol for the unknown parameters of the control input saturation.

**Remark 7** In system (4), the proposed sliding-mode protocol (9) can be guaranteed bounded motion in a bounded region. What is more, the size of the bounded region can be adjusted by tuning the observer and control parameters. Thus, observer parameter selection is also vital, because it determines not only the performance of the observer but also the motion of the attitude tracking system.

3.2.3 Attitude control with input saturation

In practical engineering, input saturation is the most important non-smooth nonlinear property, and should be explicitly considered in control design. Here, we propose a sliding-mode control protocol for the attitude system (4) in the presence of external disturbance, inertial uncertainty, angular velocity-free measurement, and unknown non-symmetric input saturation. Considering the unknown input saturation constraints, the control input \( u_k \) can be expressed as

\[
u_k = \begin{cases} u_{\text{max}}, & \text{if } v_{rk} > u_{\text{max}}; \\ v_{rk}, & \text{if } u_{\text{min}} < v_{rk} < u_{\text{max}}; \\ -u_{\text{min}}, & \text{if } v_{rk} < u_{\text{min}}. \end{cases}
\]  

(18)

where \( k = 1, 2, 3 \) and \( v_r = [v_{r1}, v_{r2}, v_{r3}]^T \in \mathbb{R}^3 \) is the designed control input command. \( u_{\text{min}} \) and \( u_{\text{max}} \) are the unknown parameters of the control input saturation. Here, \( u_{\text{min}} \neq u_{\text{max}} \) denotes the non-symmetric input saturation.

For further analysis, with considering the effect of the unknown non-symmetric input saturation, the attitude dynamics (4) can be transformed as

\[
\begin{align*}
\dot{q}_v &= f(q_v) \omega, \\
\dot{\omega} &= F + J_0^{-1} v_r + J_0^{-1} \delta u + G, \\
\dot{G} &= g(t)
\end{align*}
\]  

(19)

where \( \delta u = u - v_r \).

Since the upper and the lower limits of the non-symmetric input saturation are unknown, \( \delta u \) is unknown. To handle the unknown term \( J_0^{-1} \delta u \), we define the compound unknown disturbance as

\[
\dot{G} = G + J_0^{-1} \delta u
\]  

(20)

Considering Eqs. (19) and (20), we obtain

\[
\begin{align*}
\dot{q}_v &= f(q_v) \omega, \\
\dot{\omega} &= F + J_0^{-1} v_r + \dot{G}, \\
\dot{G} &= g(t)
\end{align*}
\]  

(21)

Due to the unknown compound disturbance \( \dot{G} \), the observer (5) is modified as

\[
\begin{align*}
\dot{q}_v &= f(q_v) \omega + \rho_1 \left( |e_1|^{\alpha_1} + |e_1|^{\beta_1} \right) \text{sign}(e_1), \\
\dot{\omega} &= \hat{F} + J_0^{-1} u + \hat{G} + \rho_2 \left( |e_1|^{\alpha_2} + |e_1|^{\beta_2} \right) \text{sign}(e_1), \\
\dot{G} &= \rho_3 \left( |e_1|^{\alpha_3} + |e_1|^{\beta_3} \right) \text{sign}(e_1)
\end{align*}
\]  

(22)

where \( \hat{G} \) is the observer estimation of \( G \).

Based on the output of the finite-time observer, the sliding-mode attitude control protocol is designed as

\[
v_r = -J_0 C^{-1} \left( \hat{C} \hat{F} + \hat{C} \hat{G} + \sigma C f(q_v) \omega + c_1 S + c_2 S^T + \psi \right)
\]  

(23)

The above design procedure and analysis can be summarized in the following theorem, which contains the results for the attitude control dynamics (4) with unknown non-symmetric input saturation.

**Theorem 3** Considering Assumptions 1 and 2 and the attitude dynamics (19) with saturation input (23). With the finite-time observer in Eq. (22) and the proposed sliding-mode control protocol (23), the state of the dynamic system (20) can converge to a bounded region in finite time.

**Proof** The proof is similar to Theorem 2 and is omitted here.

**Remark 8** For Eq. (20), the effect of unknown non-symmetric input saturation is treated as a part of the total disturbances, which is approximated using the designed finite-time observer (18). Although the attitude control dynamics have possibility of unknown non-symmetric input saturation, the finite-time stability is still guaranteed via Lyapunov theory analysis.

4 Illustrative Example

In this section, the performance of the proposed control scheme is investigated with a given numerical simulation. Consider the rigid-body system (1) with the nominal inertia matrix,

\[
J_0 = \begin{bmatrix} 20 & 1.2 & 0.9 \\ 1.2 & 1.7 & 1.4 \\ 0.9 & 1.4 & 15 \end{bmatrix} \text{kg \cdot m}^2
\]

and uncertain inertia \( \Delta J = \text{diag} \{ \sin(0.5t), 2 \sin(0.6t), 3 \sin(0.4t) \} \) kg\cdot m\(^2\).

The unknown disturbances are described as

\[
d(t) = [0.5 \sin(0.1t), 0.7 \sin(0.2t), 0.3 \sin(0.3t)] \text{N} \cdot \text{m}.
\]

The initial attitude orientation of the unit quaternion is \( q(0) = [0.3, -0.2, -0.3, 0.8832]^T \), the initial value of angular velocity is \( \omega(0) = [0.2, 0.3, 0.5]^T \text{rad/s} \), and the initial values of the state observers are \( \dot{q}(0) = [0, 0, 0, 1]^T \text{rad/s} \) and \( \dot{\omega}(0) = (0, 0, 0)^T \text{rad/s} \).
The simulation parameters are $c_1 = 2$, $c_2 = 1.5$, $v = 0.6$, $\sigma = 0.5$, $u_{\text{min}} = 5$, $u_{\text{max}} = 10$, and $C = I_3$, which can regulate the convergence rate of the state trajectory.

The angular velocity, unit quaternion, and total disturbance estimation errors are shown in Figs. 2–4. Figure 5 shows the unit quaternion finally converging to $q = [1, 0, 0, 0]^T$, which shows that the proposed control scheme is effective. Figures 6 and 7 show the history of the sliding-mode surface and the control torque. The sliding-mode surface versus time converges in accordance with the simulation result and also validates the stability analysis in Theorem 3. It should be noted that the design parameters $c_1$, $c_2$, and $\varepsilon$ determine the band of the bounded region, and we can choose approximate $c_1$, $c_2$, and $\varepsilon$ to be small enough to guarantee motion along the sliding surface.

5 Conclusion

In this paper, the rigid-body attitude control problem with time-varying disturbance, inertial uncertainty, and saturation input is considered, as the angular velocity cannot be directly used in the control law. A finite-time observer is designed to obtain the real system state and total disturbance estimation information. An observer-based sliding-mode control law is investigated to force the system converge to a bounded region of origin. A numerical simulation is presented to verify the effectiveness of the proposed method.

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