MONOIDAL TRANSFORMATIONS OF SINGULARITIES IN POSITIVE CHARACTERISTIC.

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Abstract.
A sequence of monoidal transformations is defined, in terms of invariants, for a singular hypersurface embedded in a smooth scheme of positive characteristic. Some examples are added to illustrate the improvement of singularities by this procedure.

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1. Introduction

1.1. Let $V^{(d)}$ be a smooth scheme of dimension $d$ over an algebraically closed field $k$; and let $X(\subset V^{(d)})$ be an embedded hypersurface.

The first invariant considered for resolution of singularities of a hypersurface is the multiplicity. The set of points $F_b$ where $X$ has the highest multiplicity, say $b$, is closed.

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Here $X$ is defined by a locally principal sheaf of ideals $I(X)$ and $x \in F_b$ if and only if the order of $I(X)$ at the local regular ring $\mathcal{O}_{V^{(d)},x}$ is $b$. When $x$ is a closed point, $gr_{\mathfrak{m}}(\mathcal{O}_{V^{(d)},x})$ is a polynomial ring in $d$-variables over the field $k$ and an homogeneous polynomial of degree $b$ is defined (the initial form).

The $\tau$-invariant attached to $X$ at $x$, say $\tau_x$, is the least number of variables needed to express the initial form. This naive infinitesimal invariant has a strong significance on the local behavior of the singularity.

The integer $\tau_x$ is a lower bound on the codimension of $F_b$ in $V^{(d)}$ at $x$. If equality holds, $F_b$ is smooth. In such case, the multiplicity drops after a monoidal transformation at such center.

In general, set $\tau_x = \tau$, and $\beta : V^{(d)} \longrightarrow V^{(d-\tau)}$ a sufficiently general smooth morphism defined in a neighborhood of $x$. We can assume that $\beta$ is a composition of smooth morphisms

$$V^{(d)} \longrightarrow V^{(d-1)} \longrightarrow \ldots \longrightarrow V^{(d-\tau)}$$

A local presentation of $X$ (at $x$) is defined by

1. Positive integers $0 \leq e_1 \leq e_2 \leq \cdots \leq e_{\tau}$.
2. Monic polynomials,
   
   $$f_1^{(p_{\tau}^e)}(x_1) = x_1^{p_{\tau}^e} + a_{11}^{(1)} x_1^{p_{\tau}^e-1} + \cdots + a_{1e_1}^{(1)} \in \mathcal{O}_{V^{(d-1)}}[x_1]$$
   $$f_2^{(p_{\tau}^e)}(x_2) = x_2^{p_{\tau}^e} + a_{21}^{(2)} x_2^{p_{\tau}^e-1} + \cdots + a_{2e_2}^{(2)} \in \mathcal{O}_{V^{(d-2)}}[x_2]$$
   
   $$\vdots$$
   $$f_\tau^{(p_{\tau}^e)}(x_\tau) = x_\tau^{p_{\tau}^e} + a_{11}^{(\tau)} x_\tau^{p_{\tau}^e-1} + \cdots + a_{\tau e_\tau}^{(\tau)} \in \mathcal{O}_{V^{(d-\tau)}}[x_\tau].$$

3. $I^{(s)}$: an ideal in $\mathcal{O}_{V^{(d-\tau)}}$ and a positive integer $s$.

Local presentations are defined with the property that they express the set $F_b$ of $b$-fold points, and that this expression is preserved, or stable, by permissible monoidal transformations.

In fact, they are defined so that

$$F_b = \bigcap_{i=1}^{\tau} \{ x \in V^{(d)} \mid \nu_x(f_i^{(p_{\tau}^e)}) \geq p_{\tau}^e \} \cap \{ x \in V^{(d)} \mid \nu_x(\beta^{*(I^{(s)})}) \geq s \}.$$ 

Let $Z(\subset V^{(d)})$ be the complete intersection scheme defined (locally at $x$) by the ideal $\langle f_1^{(p_{\tau}^e)}, \ldots, f_\tau^{(p_{\tau}^e)} \rangle$. The morphism $\beta : V^{(d)} \longrightarrow V^{(d-\tau)}$ induces a finite morphism
\( \beta : Z \longrightarrow V^{(d-\tau)} \), there is a natural inclusion \( F_b \subset Z \). Moreover, a theorem of Zariski ensures that the induced map \( F_b \xrightarrow{\beta} \beta(F_b) \) is a set theoretical bijection.

A property of local presentations is that \( I(s) \) is zero as ideal in \( O_{V^{(d-\tau)}} \) if and only if \( F_b \) has codimension \( \tau \) locally at \( x \). As indicated above, in such case \( F_b \) is locally smooth and \( b \)-fold points of the hypersurface \( X \) are resolved by blowing-up at such center.

If \( Y(\subset F_b \subset V^{(d)}) \) is a smooth permissible center, it is mapped isomorphically into \( \beta(Y) \subset V^{(d-\tau)} \) and \( I(s) \) has order \( \geq s \) along points of this smooth subscheme.

Let \( V^{(d-\tau)} \leftarrow V_1^{(d-\tau)} \) be the monoidal transformation with center \( \beta(Y) \), and exceptional locus, say \( H_1^{(d-\tau)}(\subset V_1^{(d-\tau)}) \). There is a natural factorization

\[
I(s)O_{V_1^{(d-\tau)}} = I(H_1^{(d-\tau)})^s \cdot I_1^{(s)}
\]

for a sheaf of ideals \( I_1^{(s)} \subset O_{V_1^{(d-\tau)}} \).

Let \( V^{(d)} \leftarrow V_1^{(d)} \) be the monoidal transformation with center \( Y \), and exceptional locus, say \( H_1^{(d)}(\subset V_1^{(d)}) \).

Let \( X_1 \) be the strict transform of the hypersurface \( X, F^{(1)}_b \) the set of \( b \)-fold points of \( X_1 \), and \( x_1 \in F^{(1)}_b \) a closed point mapping to \( x \in F_b \). Locally at \( x_1 \), there is a smooth morphism \( \beta_1 : V_1^{(d)} \longrightarrow V_1^{(d-\tau)} \) (compatible with \( \beta : V^{(d)} \longrightarrow V^{(d-\tau)} \)). Here \( \beta_1 \) is a composition

\[
V_1^{(d)} \longrightarrow V_1^{(d-1)} \longrightarrow \ldots \longrightarrow V_1^{(d-\tau)},
\]

compatible with

\[
V^{(d)} \longrightarrow V^{(d-1)} \longrightarrow \ldots \longrightarrow V^{(d-\tau)}.
\]

Set now:

1. Positive integers \( 0 \leq e_1 \leq e_2 \leq \cdots \leq e_\tau \) as before.
2. Monic polynomials, \( g_i^{(p^e_i)} \) for \( i = 1, \ldots, \tau \) (the strict transforms of the monic polynomials \( f_i^{(p^e_i)} \)).
3. The ideal \( I_1^{(s)} \) in \( O_{V_1^{(d-\tau)}} \) with the same positive integer \( s \).

The property of stability of local presentations is that, with this notion of transformations we obtain again a local presentation of \( X_1 \) locally at \( x_1 \).

Local presentations appeared in [13] and [23], and they also played a relevant role in the recent progress on resolution of singularities by Kawanoue and Matsuki (see [24], [25]).
Here we fix notation as in [6], where $I^{(s)}$ is defined in terms of elimination. There it is proved that a sequence of permissible transformations over $X$, say

$$ V^{(d)} \leftarrow V_1^{(d)} \leftarrow \cdots \leftarrow V_r^{(d)} $$

induces a sequence

$$ V^{(d-\tau)} \leftarrow V_1^{(d-\tau)} \leftarrow \cdots \leftarrow V_r^{(d-\tau)}, $$

and $I^{(s)}_i$ in $\mathcal{O}_{V_i^{(d-\tau)}}$ are defined by successive transformations.

When $k$ is a field of characteristic zero, then $e_1 = e_2 = \cdots = e_r = 0$ (so $f_i = x_i$ for $i = 1, \ldots, \tau$).

In this case, the natural identification of $F_b$ with $\beta(F_b)$ (the induced set theoretical bijection $F_b \rightarrow \beta(F_b)$) is that of $F_b(\subset V^{(d)})$ with

$$ \{x \in V^{(d-\tau)} | \nu_x(I^{(s)}) \geq s\}. $$

Elimination of $b$-fold points (and resolution of singularities) in characteristic zero is achieved by a sequence of monoidal transformations over $V^{(d)}$ defined in two stages:

Stage A): A sequence of monoidal transformations

(1.1.1) \begin{align*}
X & \quad X_1 & \quad \cdots & \quad X_n \\
V^{(d)} & \leftarrow V_1^{(d)} & \leftarrow \cdots & \leftarrow V_n^{(d)}
\end{align*}

is defined so that, setting

$$ V^{(d-\tau)} \leftarrow V_1^{(d-\tau)} \leftarrow \cdots \leftarrow V_n^{(d-\tau)} $$

and $I^{(s)}$ as above, then either (1.1.1) is a resolution (i.e. $X_n$ does not have points of multiplicity $b$) or

(1.1.2) \begin{align*}
I_n^{(s)} = I(H_1^{(d-\tau)})^{a_1} \cdot I(H_2^{(d-\tau)})^{a_2} \cdots I(H_n^{(d-\tau)})^{a_n},
\end{align*}

where the $H_i^{(d-\tau)}$ are the irreducible components of the exceptional locus. Here the sequence is defined so that these smooth hypersurfaces have only normal crossings.

For this reason, we say that $I_n^{(s)}$ is “in the monomial case”, in fact, it is locally defined by a monomial in a suitable regular system of parameters.

Stage B): A sequence of monoidal transformations

$$ V_n^{(d)} \leftarrow V_{n+1}^{(d)} \leftarrow \cdots \leftarrow V_N^{(d)} $$

is defined so that setting

$$ V_n^{(d-\tau)} \leftarrow V_{n+1}^{(d-\tau)} \leftarrow \cdots \leftarrow V_N^{(d-\tau)} $$
and $I_i^{(s)} \subset \mathcal{O}_{V^{(d-\tau)}}$ as before, then

$$\{ x \in V_N^{(d-\tau)} \mid \nu_x(I_i^{(s)}) \geq s \} = \emptyset.$$ 

The stability of local presentation show that the strict transform of $X$ at $V_N$ has no $b$-fold point.

If we start with $I_n^{(s)}(\subset \mathcal{O}_{V_n^{(d-\tau)}})$ as in (1.1.2), it is simple to define:

\[(1.1.3) \quad \begin{array}{cccc}
V_n^{(d-\tau)} & \leftarrow & V_{n+1}^{(d-\tau)} & \leftarrow \cdots \leftarrow V_N^{(d-\tau)} \\
I_n^{(s)} & \leftarrow & I_{n+1}^{(s)} & \leftarrow \cdots \leftarrow I_N^{(s)}
\end{array} \]

by blowing-up combinatorial centers (defined as intersection of smooth exceptional hypersurfaces), so that $\{ x \in V_N^{(d-\tau)} \mid \nu_x(I_N^{(s)}) \geq s \} = \emptyset$.

Note here that the local presentation in characteristic zero will allow us to lift this sequence to

$$\begin{array}{cccc}
V_n^{(d)} & \leftarrow & V_{n+1}^{(d)} & \leftarrow \cdots \leftarrow V_N^{(d)} \\
I_n^{(s)} & \leftarrow & I_{n+1}^{(s)} & \leftarrow \cdots \leftarrow I_N^{(s)}
\end{array}$$

with the desired property. In fact, we can take $f_i^{(p_{s_1})} = x_i$ and $\{ x \in V^{(d)} \mid \nu(x_i) \geq 1 \} = \{ x_1 = 0, \ldots, x_\tau = 0 \}$ is naturally identified with $V^{(d-\tau)}$.

In this paper, $k$ will be of positive characteristic. In this context, Stage A) has been proved in [6], so we consider here, as starting point, the case of local presentations in positive characteristic where the setting of (1.1.2) holds. Namely, where (1.1.1) is not a resolution and $I^{(s)}$ is locally monomial in a smooth $d - \tau$-dimensional scheme.

A resolution of $I^{(s)}$, in the sense of (1.1.3) can be defined in the same way. However, local presentations in positive characteristic are not as friendly as they are in characteristic zero, and (1.1.3), defined over a $d - \tau$ dimensional schemes, does not have a natural lifting as a sequence of permissible centers as before. There is no direct extension of Stage B) to positive characteristic, at least using the outcome of [6]. So new invariants should be required in the locally monomial case, and in this paper we introduce the notion of "strong monomial contact" in Section 8.

We show in Theorem 8.2 that there is a naturally defined monomial ideal $I_n^{(s)}$, say

$$I_n^{(s)} = I(H_1^{(d-\tau)})^{a_1} \cdot I(H_2^{(d-\tau)})^{a_2} \ldots I(H_n^{(d-\tau)})^{a_n}$$

with exponents $0 \leq \alpha_i' \leq \alpha_i$ (see (1.1.2)). It has the property that whenever

$$\begin{array}{cccc}
V_n^{(d-\tau)} & \leftarrow & V_{n+1}^{(d-\tau)} & \leftarrow \cdots \leftarrow V_N^{(d-\tau)} \\
I_n^{(s)} & \leftarrow & I_{n+1}^{(s)} & \leftarrow \cdots \leftarrow I_N^{(s)}
\end{array}$$
is a combinatorial resolution, then there is a lifting to a permissible sequence, say
\[ V_n^{(d)} \leftarrow V_{n+1}^{(d)} \leftarrow \cdots \leftarrow V_N^{(d)}. \]
This new monomial ideal \( I_n^{(s)} \) is defined in terms of the \( \tau \)-invariant. In this way, we obtain an extension of the sequence in Stage A) defined in [6], to a larger sequence of permissible transformation, say
\[ V^{(d)} \leftarrow V_1^{(d)} \leftarrow \cdots \leftarrow V_n^{(d)} \leftarrow V_{n+1}^{(d)} \leftarrow \cdots \leftarrow V_N^{(d)}. \]
We do not achieve resolution of \( b \)-fold points. Our extension is defined entirely in terms of the \( \tau \)-invariant, and it seems to improve the singularities of the embedded hypersurface from the point of view of \( \tau \). We include some examples to illustrate this fact.

In Example [10.2] we study our sequence of monoidal transformations when applied to a singularity studied in detail by H. Hauser in [17]. The example shows the relation of the invariants introduced in this paper with those in works of V. Cossart, H. Hauser, and T. Moh.

A smooth morphism \( \beta : V^{(d)} \rightarrow V^{(d-\tau)} \) factorizes as a composition
\[ V^{(d)} \rightarrow V^{(d-1)} \rightarrow \cdots \rightarrow V^{(d-\tau)}. \]
The arguments in [6] show that it suffices to study the case \( \beta : V^{(d)} \rightarrow V^{(d-1)} \) (the case \( \tau = 1 \)). So in this paper we discuss the construction of the sequence of monoidal transformations for a hypersurface with \( \tau \geq 1 \), and the paper is organized so as to give a careful motivation of the invariants involved.

We profited from discussions with Ana Bravo and Santiago Encinas.

2. Rees algebras

2.1. We define a Rees algebra over a smooth scheme \( V \) to be a graded noetherian subring of \( \mathcal{O}_V[W] \), say:
\[ \mathcal{G} = \bigoplus_{k \geq 0} I_k W^k, \]
where \( I_0 = \mathcal{O}_V \) and each \( I_k \) is a sheaf of ideals. We assume that at every affine open set \( U(\subset V) \), there is a finite set \( \mathcal{F} = \{ f_1 W^{n_1}, \ldots, f_s W^{n_s} \} \), \( n_i \in \mathbb{Z}_{\geq 1} \) and \( f_i \in \mathcal{O}_V(U) \), so that the restriction of \( \mathcal{G} \) to \( U \) is \( \mathcal{O}_V(U)[f_1 W^{n_1}, \ldots, f_s W^{n_s}] \subset \mathcal{O}_V(U)[W] \).

A closed set is attached to \( \mathcal{G} \) called the singular locus:
\[ \text{Sing}(\mathcal{G}) := \{ x \in V \mid \nu_x(I_k) \geq k, \text{ for each } k \geq 1 \}, \]
Remark 2.2. Rees algebras are related to Rees rings. A Rees algebra is a Rees ring if, given an affine open set \( U \subset V \), \( \mathcal{F} = \{ f_1 W^{n_1}, \ldots, f_s W^{n_s} \} \) can be chosen with all degrees \( n_i = 1 \). Rees algebras are integral closures of Rees rings in a suitable sense. In fact, if \( N \) is a positive integer divisible by all \( n_i \), it is easy to check that 

\[
\mathcal{O}_V(U)[f_1 W^{n_1}, \ldots, f_s W^{n_s}] = \bigoplus_{k \geq 0} I_k W^k \subset \mathcal{O}_V(U)[W],
\]

is integral over the Rees sub-ring \( \mathcal{O}_V(U)[I_N W^N](\subset \mathcal{O}_V(U)[W^N]) \).

Proposition 2.3. Given an affine open \( U \subset V \), and \( \mathcal{F} = \{ f_1 W^{n_1}, \ldots, f_s W^{n_s} \} \) as above,

\[
\text{Sing}(\mathcal{G}) \cap U = \bigcap_{1 \leq i \leq s} \{ x \in U \mid \text{ord}_x(f_i) \geq n_i \}.
\]

Proof. Since \( \nu_x(f_i) \geq n_i \) for \( x \in \text{Sing}(\mathcal{G}) \), \( 0 \leq i \leq s \);

\[
\text{Sing}(\mathcal{G}) \cap U \subset \bigcap_{1 \leq i \leq s} \{ x \in U \mid \text{ord}_x(f_i) \geq n_i \}.
\]

On the other hand, for any index \( N \geq 1 \), \( I_N(U)W^N \) is generated by elements of the form \( G_N(f_1 W^{n_1}, \ldots, f_s W^{n_s}) \), where \( G_N(Y_1, \ldots, Y_s) \in \mathcal{O}_V(U)[Y_1, \ldots, Y_s] \) is weighted homogeneous of degree \( N \), provided each \( Y_j \) has weight \( n_j \). The reverse inclusion is now clear.

2.4. A monoidal transformation of \( V \) on a smooth sub-scheme \( Y \), say \( V \overset{\pi}{\leftarrow} V_1 \) is said to be permissible for \( \mathcal{G} \) if \( Y \subset \text{Sing}(\mathcal{G}) \). In such case, for each index \( k \geq 1 \), there is a sheaf of ideals, say \( I_1^{(1)} \subset \mathcal{O}_{V_1} \), so that \( I_k \mathcal{O}_{V_1} = I(H)^k I_1^{(1)} \), where \( H \) denotes the exceptional locus of \( \pi \). One can easily check that

\[
\mathcal{G}_1 = \bigoplus_{k \geq 0} I_k^{(1)} W^k
\]

is a Rees algebra over \( V_1 \), which we call the transform of \( \mathcal{G} \), and denote by:

\[
(2.4.1) \quad V \overset{\pi}{\leftarrow} V_1 \quad \mathcal{G} \overset{\pi}{\leftarrow} \mathcal{G}_1
\]

A sequence of transformations will be denoted as

\[
(2.4.2) \quad V \overset{\pi_1}{\leftarrow} V_1 \overset{\pi_2}{\leftarrow} \ldots \overset{\pi_k}{\leftarrow} V_k.
\]
2.5. We now introduce an equivalence relation between Rees algebras. This notion is closely related with the equivalence relation Hironaka defined in the ambient of couples \((J,b)\).

Definition 2.6. Two Rees algebras over \(V\), say \(G = \bigoplus_{k \geq 0} I_k W^k\) and \(G' = \bigoplus_{k \geq 0} J_k W^k\), are integrally equivalent, if both have the same integral closure.

Proposition 2.7. Let \(G\) and \(G'\) be two integrally equivalent Rees algebras over \(V\). Then:

1. \(\text{Sing}(G) = \text{Sing}(G')\).

   Note, in particular, that every monoidal transform \(V \leftarrow V_1\) on a center \(Y \subset \text{Sing}(G) = \text{Sing}(G')\) defines transforms, say \((G)_1\) and \((G')_1\) on \(V_1\).

2. \((G)_1\) and \((G')_1\) are integrally equivalent on \(V_1\).

If \(G\) and \(G'\) are integrally equivalent on \(V\), the same holds for any open restriction, and also for pull-backs by smooth morphisms \(W \rightarrow V\).

On the other hand, as \((G)_1\) and \((G')_1\) are integrally equivalent, they define the same closed set on \(V_1\) (the same singular locus), and the same holds for further monoidal transformations, pull-backs by smooth schemes, and concatenations of both kinds of transformations.

2.8. Differential Rees algebras is a class of Rees algebras that play a particularly relevant role in singularity theory.

Let \(V\) be a smooth scheme over a field \(k\), so for each non-negative integer \(s\) there is a locally free sheaf of differential operators of order \(s\), say \(\text{Diff}_k^s\). For \(s = 0\) there is a natural identification, say \(\text{Diff}_k^0 = \mathcal{O}_V\), and for each \(s \geq 0\) \(\text{Diff}_k^s \subset \text{Diff}_k^{s+1}\).

We define an extension of a sheaf of ideals \(J \subset \mathcal{O}_V\), \(\text{Diff}_k^s(J)(U)\), so that over the affine open set \(U\), \(\text{Diff}_k^s(J)(U)\) is the extension of \(J(U)\) defined by adding \(D(f)\), for all \(D \in \text{Diff}_k^s(U)\) and \(f \in J(U)\), that is

\[
\text{Diff}_k^s(J)(U) = \{D(f) \mid D \in \text{Diff}_k^s(U) \text{ and } f \in J(U)\}.
\]

So \(\text{Diff}_k^0(J) = J\), and \(\text{Diff}_k^s(J) \subset \text{Diff}_k^{s+1}(J)\) as sheaves of ideals in \(\mathcal{O}_V\). Let \(V(I) \subset V\) denote the closed set defined by an ideal \(I \subset \mathcal{O}_V\). The order of the ideal \(J\) at the local regular ring \(\mathcal{O}_{V,x}\) is \(\geq s\) if and only if \(x \in V(\text{Diff}_k^{s-1}(J))\).

Definition 2.9. We say that a Rees algebra \(\bigoplus_{n \geq 0} I_n W^n\), on a smooth scheme \(V\), is a Diff-algebra relative to the field \(k\), if:

i) \(I_n \supset I_{n+1}\) for \(n \geq 0\).
ii) There is open covering of $V$ by affine open sets $\{U_i\}$, and for every $D_k \in Diff^r(U_i)$, and $h \in I_n(U_i)$, then $D_k(h) \in I_{n-r}(U_i)$ provided $n \geq r$.

Note that (ii) can be reformulated by:

\[ \text{ii') } \forall n, D_k \in Diff^r(U_i) \implies \forall n, \exists \tilde{D}_k \in Diff^r(U_i) \text{ such that } \tilde{D}_k(h) \in I_{n-r}(U_i) \text{ for each } n, \text{ and } 0 \leq r \leq n. \]

2.10. Fix a closed point $x \in V$, and a regular system of parameters $\{x_1, \ldots, x_n\}$ at $O_{V,x}$. The residue field, say $k'$ is a finite extension of $k$, and the completion $\hat{O}_{V,x} = k'[x_1, \ldots, x_n]$.

The Taylor development is the continuous $k'$-linear ring homomorphism given by:

\[ Tay : k'[x_1, \ldots, x_n] \to k'[x_1, \ldots, x_n, T_1, \ldots, T_n] \quad x_i \mapsto x_i + T_i \]

So for $f \in k'[x_1, \ldots, x_n]$, $Tay(f) = \sum_{\alpha \in \mathbb{N}^n} g_\alpha T^\alpha$, with $g_\alpha \in k'[x_1, \ldots, x_n]$. Define, for each $\alpha \in \mathbb{N}^n$, $\Delta^\alpha(f) = g_\alpha$. This is a differential operator on the ring of formal power series. There is a natural inclusion of $O_{V,x}$ in its completion, and $\Delta^\alpha(O_{V,x}) \subset O_{V,x}$. Moreover, the set $\{\Delta^\alpha | \alpha \in (\mathbb{N})^n, 0 \leq |\alpha| \leq c\}$ generates the $O_{V,x}$-module $Diff_{k'}(O_{V,x})$, i.e. generates $Diff_{k'}$ locally at $x$ (See [14] Theorem 16.11.2).

Theorem 2.11. For every Rees algebra $G$ over a smooth scheme $V$, there is a Diff-algebra, say $G(G)$ such that:

i) $G \subset G(G)$.

ii) If $G \subset G'$ and $G'$ is a Diff-algebra, then $G(G) \subset G'$.

Furthermore, if $x \in V$ is a closed point, $\{x_1, \ldots, x_n\}$ is a regular system of parameters at $O_{V,x}$, and if $G$ is locally generated by $\mathcal{F} = \{g_n W^{n_i} | n_i > 0, 1 \leq i \leq m\}$, then

\[ \mathcal{F}' = \{\Delta^\alpha(g_n) W^{n_i - |\alpha|} | g_n W^{n_i} \in \mathcal{F}, \alpha \in \mathbb{N}^n, 0 \leq |\alpha| < n_i \leq n_i\} \]

generates $G(G)$ locally at $x$.

(see [36], Theorem 3.4).

Remark 2.12. The local description in the Theorem shows that $Sing(G) = Sing(G(G))$.

In fact, as $G \subset G(G)$, it is clear that $Sing(G) \supset Sing(G(G))$. For the converse note that if $\nu_x(g_n) \geq n_i$, then $\Delta^\alpha(g_n)$ has order at least $n_i - |\alpha|$ at the local ring $O_{V,x}$.

The following Main Lemma, due to Jean Giraud, relates $G$-extensions with monoidal transformations.
Lemma 2.13. (J. Giraud) Let $G$ be a Rees algebra on a smooth scheme $V$, and let $V ←→ V_1$ be a permissible (monoidal) transformation for $G$. Let $G_1$ and $G(G)_1$ denote the transforms of $G$ and $G(G)$. Then:

1. $G_1 ⊂ G(G)_1$.
2. $G(G(G)_1) = G(G(G))$.

(see [11], Theorem 4.1.)

Remark 2.14. By applying Theorem 2.11 in Giraud Lemma, the following inclusions are obtained:

$G_1 ⊂ G(G)_1 ⊂ G(G(G)_1)$. In particular, by Remark 2.12, $\text{Sing}(G(G)_1) = \text{Sing}(G_1)$. Moreover, the same argument applies for a sequence of permissible monoidal transformation, say $V \overset{\pi_1}{←} V_1 \overset{\pi_2}{←} \cdots \overset{\pi_k}{←} V_k$ and in this case $\text{Sing}(G_k) = \text{Sing}(G(G)_k)$.

2.15. Fix now a smooth morphism of smooth schemes, say $V → V'$. Let $\text{Diff}^r_{V'}(V)$ denote the locally free sheaf of relative differential operators of order $r$.

We say that a Rees algebra $\bigoplus I_k W^k$ over $V$ has Diff-structure relative to $V'$, if conditions in Definition 2.9 hold, where we now require that $D ∈ \text{Diff}^r_{V'}(U_i)$ in (ii).

Since $\text{Diff}^r_{V'}(V) ⊂ \text{Diff}^r_k(V)$ it follows that any Diff-structure relative to $k$ is also relative to $V'$.

Any Rees algebra can be extended to a smallest Diff-structure relative to $V'$, for that, note that given an ideal $I ⊂ O_V$, and a smooth morphism $V → V'$. An extension of ideals $I ⊂ \text{Diff}^r_{V'}(I)$,

$\text{Diff}^r_{V'}(I)(U) = \{ D(f) \mid f ∈ I(U), D ∈ \text{Diff}^r_{V'}(U) \}$

is defined for each open set $U$ in $V$.

Note finally that a Rees algebra $\bigoplus I_k W^k$ over $V$ has Diff-structure relative to $V'$, if and only if, for any positive integers $r ≤ n$, $\text{Diff}^r_{V'}(I_n) ⊂ I_{n-r}$.

3. Elimination algebras

3.1. Let $S$ be a ring. Consider the polynomial ring $S[Z]$ and a monic polynomial $f(Z) ∈ S[Z]$ such that $f(Z) = Z^n + a_1Z^{n-1} + \cdots + a_n$. We are searching for polynomial functions in the coefficients which are stable after changes of the form $Z ↦ Z - s$
with $s \in S$. For this reason we discuss some aspects of invariant and elimination theory, and obtain some results in a universal way.

Let us consider $F_n(Z) = (Z - Y_1)(Z - Y_2) \ldots (Z - Y_n)$, the universal monic polynomial of degree $n$ in the polynomial ring of $n + 1$ variables $k[Y_1, \ldots, Y_n, Z]$. The group of permutations of $n$ elements, $S_n$, acts on $k[Y_1, \ldots, Y_n]$ by permuting the index of the variables $Y_1, \ldots, Y_n$; and this action extends to $k[Y_1, \ldots, Y_n, Z]$ by fixing $Z$.

The subring of invariants, say $k[Y_1, \ldots, Y_n]^{S_n}$, is generated, as a $k$-algebra, by the symmetric elemental functions of order $i$, $s_{n,i}$, for $1 \leq i \leq n$:

$$s_{n,1} = Y_1 + \cdots + Y_n$$

$$s_{n,2} = \sum_{1 \leq i < j \leq n} Y_i Y_j$$

$$\vdots$$

$$s_{n,n} = Y_1 Y_2 \ldots Y_n$$

That is, $k[Y_1, \ldots, Y_n]^{S_n} = k[s_{n,1}, \ldots, s_{n,n}]$ and

$$k[Y_1, \ldots, Y_n, Z]^{S_n} = k[s_{n,1}, \ldots, s_{n,n}][Z].$$

It is easy to check that the monic polynomial $F_n(Z) = (Z - Y_1) \ldots (Z - Y_n) \in k[s_{n,1}, \ldots, s_{n,n}][Z]$, since this polynomial is invariant under the action of $S_n$.

Let $S$ be a $k$-algebra and fix $f(Z) = Z^n + a_1 Z^{n-1} + \cdots + a_n$, a monic polynomial of degree $n$ in $S[Z]$. This polynomial arises from the universal polynomial $F_n(Z)$ by the base change morphism

$$k[s_{n,1}, \ldots, s_{n,n}] \longrightarrow S$$

$$s_{n,i} \longrightarrow (-1)^i \cdot a_i$$

The group $S_n$ acts linearly over the polynomial ring $k[Y_1, \ldots, Y_n, Z]$ and this action preserves the grading of the ring. So, the invariant subring $k[s_{n,1}, \ldots, s_{n,n}][Z]$ can be considered as a graded sub-ring (with the grading inherited from that of $k[Y_1, \ldots, Y_n, Z]$). The group $S_n$ also acts linearly in the graded sub-ring $k[Y_i - Y_j]_{1 \leq i,j \leq n} \subset k[Y_1, \ldots, Y_n]$, defining an inclusion of graded sub-rings

$$k[Y_i - Y_j]^{S_n}_{1 \leq i,j \leq n} \subset k[Y_1, \ldots, Y_n]^{S_n}.$$
The algebra \( k[Y_i - Y_j]^{S_n} \) will be called the *universal elimination algebra*, any polynomial in this ring provides, for each \( f(Z) = Z^n + a_1Z^{n-1} + \cdots + a_n \) in \( S[Z] \), and each base change as above, a function on the coefficients \( a_i \) which is invariant by changes of variable of the form \( Z \to Z - s, \ s \in S \).

**3.2.** A morphism \( Tay \) is defined as in [2, 10]. Let \( S \) be a \( k \)-algebra and consider the \( S \)-homomorphism

\[
Tay : \ S[Z] \to S[Z, T]
\]

\[
Z \mapsto Z + T
\]

For \( f(Z) \in S[Z] \), we have \( Tay(f(Z)) = \sum_{e \in \mathbb{N}} g_e(Z)T^e \), with \( g_e(Z) \in S[Z] \) and finally define, for each \( e \in \mathbb{N} \), \( \Delta^{(e)}(f(Z)) = g_e(Z) \).

**Remark 3.3.** Since \( F_n(Z) = (Z - Y_1) \cdot (Z - Y_2) \cdots (Z - Y_n) \in k[Y_1, \ldots, Y_n][Z] \), then

\[
F_n(T + Z) = (T + (Z - Y_1)) \cdot (T + (Z - Y_2)) \cdots (T + (Z - Y_n)).
\]

The coefficients of this polynomial in the variable \( T \) are the symmetric polynomials evaluated on the element \( (Z - Y_1, \ldots, Z - Y_n) \). So

\[
\Delta^{(e)}(F_n(Z)) = (-1)^{n-e}s_{n,n-e}(Z - Y_1, Z - Y_2, \ldots, Z - Y_n).
\]

Here \( S_n \) acts on the graded sub-ring \( k[Z - Y_1, \ldots, Z - Y_n] \subset k[Y_1, \ldots, Y_n, Z] \); setting, \( \sigma(Z) = Z \) for every \( \sigma \in S_n \) and preserving the graded structure. Note that

\[
k[Z - Y_1, \ldots, Z - Y_n]^{S_n} = k[F_n(Z), \{\Delta^{(e)}(F_n(Z))\}_{e=1,\ldots,n-1}],
\]

and that each \( \Delta^{(e)}(F_n((Z))) \) is homogeneous of degree \( n - e \).

As \( Y_i - Y_j = (Z - Y_j) - (Z - Y_i) \) we extract, from the inclusion \( k[Y_i - Y_j]_{1 \leq i, j \leq n} \subset k[Z - Y_1, \ldots, Z - Y_n] \), an inclusion of graded sub-rings

\[
k[H_{n_1}, \ldots, H_{n_r}] = k[Y_i - Y_j]^{S_n} \subset k[F_n(Z), \{\Delta^{(e)}(F_n(Z))\}_{e=1,\ldots,n-1}],
\]

so each \( H_{n_i} \) is also a weighted homogeneous in \( k[F_n(Z), \{\Delta^{(e)}(F_n(Z))\}_{e=1,\ldots,n-1}] \).

**3.4.** We will now assign, to \( f(Z) = Z^n + a_1Z^{n-1} + \cdots + a_n \) monic of degree \( n \) in \( S[Z] \), a Rees algebras of the ring \( S[Z][W] \) (i.e. finitely generated sub-algebras of \( S[Z][W] \) for a dummy variable \( W \)). To be precise, we attach to a graded subring in \( k[s_{n,1}, \ldots, s_{n,n}][Z] \) a subring in \( S[Z][W] \), so that whenever \( H \) is a weighted homogeneous polynomial of degree, say \( m \), in \( k[s_{n,1}, \ldots, s_{n,n}][Z] \), we assign to it an element of the form \( hW^m \), with \( h \in S[Z] \). 
Given \( f(Z) = Z^n + a_1 Z^{n-1} + \cdots + a_n \) in \( S[Z] \) we define a \( k \)-algebra homomorphism on \( S[Z][W] \) by setting

\[
k[s_{n,1}, \ldots, s_{n,n}][Z] \longrightarrow S[Z][W]
\]

\[
s_{n,i} \longmapsto (-1)^i \cdot a_i W^i
\]

\[
Z \longmapsto ZW.
\]

Any graded sub-ring in \( k[s_{n,1}, \ldots, s_{n,n}][Z] \) defines now a graded sub-algebra in \( S[Z][W] \), and from (3.3.2) we obtain

\[
(3.3.1) \quad S[h_n W^{n_1}] \subset S[f(Z) W^n, \Delta^{(e)}(f(Z)) W^{n-e}]_{1 \leq e \leq n-1}
\]

Note that \( k[Y_i - Y_j]^S_n \) does not involve the variable \( Z \), so that \( S[h_n W^{n_1}] \subset S[W] \). For this reason, the algebra \( S[h_n W^{n_1}] \) is called the elimination algebra.

**Remark 3.5.** The \( \Delta^{(e)}(f(Z)) W^{n-e} \) in (3.3.1) are exactly the relatives differential operators applied to \( f(Z) \) with weight \( (n-e) \). Consider the Taylor morphism

\[
Tay : k[s_{n,1}, \ldots, s_{n,n}][Z] \longrightarrow k[s_{n,1}, \ldots, s_{n,n}][Z, T]
\]

and the base change morphism \( k[s_{n,1}, \ldots, s_{n,n}] \longrightarrow S \) defined as above. Because of the good behaviour of differentials with base change and by (3.2), we obtain \( \Delta^{(e)}(f(Z)) W^{n-e} \) from \( \Delta^{(e)}(F_n(Z)) \)

**3.6.** Consider \( F_n(Z) = (Z - Y_1) \cdot (Z - Y_2) \cdots (Z - Y_n) \in k[Y_1, \ldots, Y_n][Z, V] \) where \( V \) is a new variable, and set formally

\[
F'_n \left( \frac{Z}{V} \right) = \left( \frac{1}{V} \right)^n F_n(Z) = \left( \frac{Z}{V} - \frac{Y_1}{V} \right) \cdot \left( \frac{Z}{V} - \frac{Y_2}{V} \right) \cdots \left( \frac{Z}{V} - \frac{Y_n}{V} \right)
\]

in \( k[Y_1, \ldots, Y_n][V, (V)^{-1}][Z] \). In fact we may view \( F'_n \left( \frac{Z}{V} \right) \) as a monic polynomial in \( k[Y_1, \ldots, Y_n][V, (V)^{-1}][\frac{Z}{V}] \), or as a monic polynomial in the sub-ring \( k[\frac{Y_1}{V}, \frac{Y_2}{V}, \ldots, \frac{Y_n}{V}] \).

Let \( \Delta^{(e)}_i \) denote the relative differential operators on \( k[\frac{Y_1}{V}, \frac{Y_2}{V}, \ldots, \frac{Y_n}{V}][\frac{Z}{V}] \) (relative to the subring \( k[\frac{Y_1}{V}, \frac{Y_2}{V}, \ldots, \frac{Y_n}{V}] \)). It follows from (3.3.1) that

\[
(3.6.1) \quad \Delta^{(e)}_i \left( F'_n \left( \frac{Z}{V} \right) \right) = \left( \frac{1}{V} \right)^{n-e} \cdot \Delta^{(e)}(F_n(Z)).
\]

Let \( S_n \) act on \( k[Y_1, \ldots, Y_n][V, (V)^{-1}][Z] \) by permutation on the variables \( Y_i \) and fixing both \( V \) and \( Z \). In this way it acts also by permutation on the variables of \( k[\frac{Y_1}{V}, \frac{Y_2}{V}, \ldots, \frac{Y_n}{V}] \), and trivially on \( \frac{Z}{V} \). Therefore

\[
k \left[ \frac{Y_1}{V} - \frac{Y_i}{V} \right]^{S_n} = k[H'_{n_1}, \ldots, H'_{n_i}].
\]
where
\[(3.6.2) \quad H'_{n_i} = \frac{1}{V_{n_i}} \cdot H_{n_i}\]
for \(H_{n_i}\) as in \((3.3.2)\).

**3.7.** A universal elimination algebra was defined in \((3.1)\) for one universal polynomial. These ideas have a natural extension to the case of several polynomials. Here we only consider the case of two polynomials, but the arguments are similar for the case of more than two.

Fix two positive integers, \(r, s \in \mathbb{N}\) such that \(r + s = n\) and consider \(F_r(Z) = (Z - Y_1) \ldots (Z - Y_r)\) and \(F_s(Z) = (Z - Y_{r+1}) \ldots (Z - Y_n)\). The permutation group \(S_r\) acts on \(k[Y_1, \ldots, Y_r]\) and \(S_s\) acts on \(k[Y_{r+1}, \ldots, Y_n]\). Define
\[k[H'_{m_1}, \ldots, H'_{m_{r,s}}] := k[Y_j - Y_j] \cap k[H'_{m_1}, \ldots, H'_{m_{r,s}}],\]
as the universal elimination algebra for two polynomials. Since \(S_r \times S_s \subset S_n\), there is a natural inclusion
\[k[H_{m_1}, \ldots, H_{m_n}] := k[Y_j - Y_j] \subset k[H'_{m_1}, \ldots, H'_{m_{r,s}}],\]
which is a finite extension of graded algebras. On the other hand, one can check that
\[k[Z-Y_1, \ldots, Z-Y_n]^{S_r \times S_s} = k[F_r(Z), \{\Delta^e(F_r(Z))\}_{e=1}^{r-1}, F_s(Z), \{\Delta^\ell(F_s(Z))\}_{\ell=1}^{s-1}].\]

The inclusion of finite groups \(S_r \times S_s \subset S_n\) also shows that there is an inclusion of invariant algebras:
\[k[F_r(Z), \{\Delta^j(F_r(Z))\}_{j=1}^{n-1}] \subset k[F_r(Z), \{\Delta^e(F_r(Z))\}_{e=1}^{r-1}, F_s(Z), \{\Delta^\ell(F_s(Z))\}_{\ell=1}^{s-1}],\]
which is a finite extension of graded rings.

Note that \(\Delta^\ell(F_s(Z))\) is homogeneous of degree \(r - e\) for \(e = 1, \ldots, n - 1\), and that \(k[Z-Y_1, \ldots, Z-Y_n]^{S_r \times S_s}\) is a graded subring in \(k[Y_1, \ldots, Y_n, Z]\). Moreover, there is a natural inclusion:
\[(3.7.1) \quad k[H'_{m_1}, \ldots, H'_{m_{r,s}}] \subset k[F_r(Z), \{\Delta^e(F_r(Z))\}_{e=1}^{r-1}, F_s(Z), \{\Delta^\ell(F_s(Z))\}_{\ell=1}^{s-1}],\]
that arises from \(k[Y_j - Y_j] \subset k[Z-Y_1, \ldots, Z-Y_n]\).

Here \(F_r(Z)F_s(Z) = F_n(Z)\). The rings \(k[Y_1, \ldots, Y_r]^{S_r} = k[v_1, \ldots, v_r]\), and \(k[Y_{r+1}, \ldots, Y_n]^{S_s} = k[w_1, \ldots, w_s]\) are graded subrings in \(k[Y_1, \ldots, Y_n]\), \(F_r(Z) \in k[v_1, \ldots, v_r][Z], F_s(Z) \in k[w_1, \ldots, w_s][Z]\), and there is an inclusion
\[k[H'_{m_1}, \ldots, H'_{m_{r,s}}] \subset k[v_1, \ldots, v_r, w_1, \ldots, w_s] \]
arising from \( k[Y_i - Y_j] \subset k[Y_1, \ldots, Y_n] \). In particular each \( H_i' \) is also a weighted homogeneous polynomial in the universal coefficients \( \{v_1, \ldots, v_r, w_1, \ldots, w_s\} \).

3.8. The previous discussion, for the case of two polynomials extends to the case of several polynomials. These algebras specialize into subalgebra of the form

\[
(3.8.1) \quad S[Z][f_i(Z)W^{n_i}, \{\Delta^e_i(f_i(Z))W^{n_i-e_i}\}_{1 \leq e_i \leq n_i-1}]_{1 \leq i \leq r},
\]

in the sense of (3.4), where \( f_i(Z) \) are monic polynomials of the form

\[
f_i(Z) = Z^{n_i} + a_{1,i}Z^{n_i-1} + \cdots + a_{n_i,i}
\]

for \( i = 1, \ldots, r \). This same specialization, applied to the universal elimination algebras (free of the variable \( Z \)), define algebras, say

\[
(3.8.2) \quad S[h^{(j)}_{n_1, \ldots, n_r} W^{N^{(j)}_{n_1, \ldots, n_r}}]_{1 \leq j \leq R_{n_1, \ldots, n_r}} \subset S[W],
\]

for suitable positive integers \( R_{n_1, \ldots, n_r} \).

An important property of specializations in (3.4) is their compatibility with finite extensions of graded algebras. So, for example, in the case two polynomial discussed in (3.7), we conclude that if \( f_n(Z) \in S[Z] \) is a monic polynomial of degree \( n \) which factorizes as a product of monic polynomials, say \( f_n(Z) = f_r(Z)f_s(Z) \), then there is a natural (and finite!) inclusion of graded rings:

\[
S[Z][f_n(Z)W^n, \{\Delta^j(f_n(Z))W^{n-j}, j \leq n\}] \subset S[Z][f_r(Z)W^r, \{\Delta^e(f_r(Z))W^{r-e}\}_{e=1, \ldots, r}, f_s(Z), \{\Delta^\ell(f_s(Z))W^{s-\ell}\}_{\ell=1, \ldots, s-1}]
\]

(as subalgebras of \( S[Z][W] \)). Similarly, a finite extension of graded subalgebras of \( S[W] \) is defined by the specialization of the corresponding elimination algebras. The same holds for more then two polynomials.

4. Projections

4.1. Let \( V^{(d)} \) denote a smooth scheme of dimension \( d \), and let \( X \subset V^{(d)} \) be a hypersurface such that \( X = V(< f >) \) locally at a \( n \)-fold point \( x \in V^{(d)} \). So \( n = \text{max} - \text{ord} f \), the maximum order of the hypersurface in a neighborhood of \( x \).

We claim that for a sufficiently \( \text{generic} \) projection

\[
V^{(d)} \xrightarrow{\beta} V^{(d-1)}
\]

the hypersurface \( X \) can be express, in \( \text{étale} \) topology, as \( X = V(f(Z)) \), where \( f(Z) \in \mathcal{O}_{V^{(d-1)},\beta(x)}[Z] \) is a monic polynomial of degree \( n \) in a variable \( Z \). This will hold under a suitable geometric condition, that will be expressed on \( T_{V^{(d)},x} \), the tangent
space at the point. In fact, we will show that such conditions on $f$ can be achieved whenever the tangent line, at $x$, of the smooth curve $\beta^{-1}(\beta(x))$, say $\ell \subset \mathbb{T}_{V(d),x}$ and the tangent cone of the hypersurface at the point, say $\mathcal{C}_f \subset \mathbb{T}_{V(d),x}$, are in general position (intersect only at the origin).

Let $\{x_1, \ldots, x_d\}$ be a regular system of parameters at $\mathcal{O}_{V(d),x}$. Recall that

$$
\mathbb{T}_{V(d),x} = \text{Spec}(gr_{\mathfrak{m}}(\mathcal{O}_{V(d),x})),
$$

where $X_i$ denotes the class of $x_i$ in $\mathfrak{m}/\mathfrak{m}^2$, that is, the initial form. Let us compute now the initial form of $f$, to do so consider the following exact sequences:

$$
\begin{align*}
0 \to & \left< f \right> \quad \to \quad \mathcal{O}_{V(d),x} \quad \to \quad \mathcal{O}_{x,x} \quad \to \quad 0 \\
0 \to & \left< \mathcal{O}_{V(d),x} \right> \quad \to \quad \mathcal{O}_{x,x} \quad \to \quad \left< \left< f \right> \right> \quad \to \quad 0 \\
0 \to & \left[ \left< \left< f \right> \right> \right]_r \quad \to \quad \mathcal{M}/\mathcal{M}^r+1 \quad \to \quad \mathcal{M}/\mathcal{M}^r+1 \quad \to \quad 0
\end{align*}
$$

where $\left[ \left< \left< f \right> \right> \right]_r$ denotes the ideal of the homogeneous forms of degree $r$ in the homogeneous ideal $\left< \left< f \right> \right>$.

Here $\mathcal{M}/\mathcal{M}^r+1 = \mathcal{M}/\mathcal{M}^r+1$ for every $r < b$, and the first time that equality fails to hold is at $r = b$; that is, in degree $b$, where the initial form of $f$, say $\text{In}(f)$, appears. So $gr_{\mathfrak{m}}(\mathcal{O}_{x,x}) = k[X_1, \ldots, X_d]/\left< \text{In}(f) \right>$, and the tangent cone of $X$ at $x$ is

$$
\mathcal{C}_f = \text{Spec}(gr_{\mathfrak{m}}(\mathcal{O}_{x,x})) = \text{Spec}(k[X_1, \ldots, X_d]/\left< \text{In}(f) \right>)(\subset \mathbb{T}_{V(d),x}).
$$

4.2. Fix now a smooth morphism $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$, defined at a neighborhood of $x$, and let $\ell$ denote the smooth curve $\beta^{-1}(\beta(x))$.

As $f$ has multiplicity $n$ at $\mathcal{O}_{V(d),x}$, the class of $f$, say $\mathcal{T}$, has order at least $n$ at the local regular ring $\mathcal{O}_{\ell,x}$. Moreover, the order is $n$ if and only if the tangent line of $\ell$ at $x$ and $\mathcal{C}_f$ intersect only at the origin of $\mathbb{T}_{V(d),x}$.

If we fix a regular system of parameters, say $\{x_1, \ldots, x_{d-1}\}$, at $\mathcal{O}_{V^{(d-1)},\beta(x)}$, the ideal defining $\ell$ is given by $\left< x_1, \ldots, x_{d-1} \right>$, and a new parameter $Z$ can be added so as to define a regular system of parameters at $\mathcal{O}_{V(d),x}$. Note finally that the geometric condition imposed at the point $x$ can also be expressed by $\Delta^{(n)}(f)(x) \neq 0$, where $\Delta^{(n)}$ is a suitable differential operator of order $n$, relative to $\beta : V^{(d)} \to V^{(d-1)}$. The advantage of this new reformulation is that it shows that if the geometric condition holds, for $X$ and $\beta$ at $x$, it also holds at any $n$-fold point of $X$ in a neighborhood of $x$. 
Consider now the completion of the local rings in the previous projection, that is, \( \hat{O}_{V^{(d)}, x} \) and \( \hat{O}_{V^{(d-1), \beta(x)}} \), and apply Weierstrass Preparation Theorem, so the polynomial \( f \) can be expressed as

\[
u \cdot f(x_1, \ldots, x_{d-1}, Z) = Z^n + a_1 Z^{n-1} + \cdots + a_n,
\]

where \( u \) is a unit, \( \{x_1, \ldots, x_{d-1}\} \) is a regular system of parameters at \( \hat{O}_{V^{(d-1), \beta(x)}} \), and after adding the variable \( Z, \{x_1, \ldots, x_{d-1}, Z\} \) is a regular system of parameters at \( \hat{O}_{V^{(d), x}} \), and \( a_i \in \hat{O}_{V^{(d-1), \beta(x)}} \).

An similar result holds replacing completion by henselization. In this case, the coefficients \( a_i \) are functions in a étale neighbourhood of the point \( \beta(x) \) in \( V^{(d-1)} \).

### 4.3. The linear space of vertices.

Let \( V^{(d)} \) be a smooth scheme, \( X \) a hypersurface locally described by \( f \), and \( \mathcal{C}_f \subset \mathbb{T}_{V^{(d)}, x} \) the tangent cone associated to \( X \) at \( x \). Given a vector space \( V \), a vector \( v \in V \) defines a translation, say \( tr_v(w) = w + v \) for \( w \in V \).

There is a largest linear subspace, say \( \mathcal{L}_f \), so that \( tr_v(C_f) = C_f \) for any \( v \in \mathcal{L}_f \), this subspace is called the **linear space of vertices**.

An important property of this subspace \( \mathcal{L}_f \) is that for any smooth center \( Y \) in \( X \), such that \( x \in Y \) and \( X \) has multiplicity \( n \) along \( Y \), the tangent space of \( Y \), say \( \mathbb{T}_{Y, x} \), is a subspace of \( \mathcal{L}_f \).

There is a characterization of this linear space in algebraic terms. An homogeneous ideal \( I \) is said to be **closed by differential operators** if for any homogeneous element \( g \in I \) of order \( n \) and any differential operator \( \Delta^e \) of order \( |e| \leq b - 1 \), then \( \Delta^e(g) \in I \). If we extend \( \langle In(f) \rangle \) to the smallest ideal closed by differential operators, say \( \widehat{I} \), then the zero-set of this homogeneous ideal defines the subspace \( \mathcal{L}_f \) we have just defined. In these arguments we are assuming that the underlying field \( k \) is perfect.

Similar notions can be defined for Rees algebras. Let \( \mathcal{G} \) be a Rees algebra on the smooth scheme \( V^{(d)} \). An homogeneous ideal is defined by \( \mathcal{G} \) at \( x \), say \( In_x(\mathcal{G}) \), included in \( gr_{\mathbb{N}}(O_{V^{(d)}, x}) \); namely that generated by the class of \( I_s \) at the quotient \( \mathfrak{M}_s^e/\mathfrak{M}_s^{e+1} \), for all \( s \). Denote this ideal by \( \mathbb{I}_{\mathcal{G}, x} \). The ideal \( \mathbb{I}_{\mathcal{G}, x} \) defines a cone, say \( \mathcal{C}_{\mathcal{G}}, \) at \( \mathbb{T}_{V^{(d)}, x} \).

Recall that there is a largest subspace, say \( \mathcal{L}_\mathcal{G} \), included and acting by translations on \( \mathcal{C}_\mathcal{G} \).

One can check that \( In_x(G(\mathcal{G})) \) is the smallest (homogeneous) extension of \( \mathbb{I}_{\mathcal{G}, x} = In_x(\mathcal{G}) \), closed by the action of the differential operators \( \Delta^e \); that is, with the property that if \( F \) is an homogeneous polynomial of degree \( N \) in the ideal, and if \( |e| \leq N - 1 \), then also \( \Delta^e(F) \) is in the ideal. This homogeneous ideal defines the subspaces \( \mathcal{L}_\mathcal{G} \), included in \( \mathcal{C}_\mathcal{G} \), with the properties stated before.
Recall that $\text{Sing}(G) = \text{Sing}(G')$ (see (2.12)). The previous discussion shows how the homogeneous ideal at $x$ attached to $G$, say $I_n(G)$, relates to the one attached to $G'$, say $I_n(G')$: If $\mathcal{C}_G$ is the cone associated with $G$, then the cone associated to $G'$ is the linear subspace $\mathcal{L}_G$.  

**Definition 4.4.** (Hironaka $\tau$-invariant). $\tau_{G,x}$ will denote the minimum number of variables required to express generators of the ideal $I$. This algebraic definition can be reformulated geometrically: $\tau_{G,x}$ is the codimension of the linear subspace $\mathcal{L}_G$.

**Proposition 4.5.** If $G$ and $G'$ are two Rees algebras with the same integral closure, then for each $x \in \text{Sing} G = \text{Sing} G'$, there is an equality between their $\tau$-invariants, that is, $\tau_{G,x} = \tau_{G',x}$.

Some auxiliary results will be needed:

**Lemma 4.6.** If the Rees algebra $G$ is defined locally at $x$ by $G = \oplus I_n W^n = \mathcal{O}_V[f_1 W^{n_1}, \ldots, f_s W^{n_s}]$, then $\mathcal{I}_{G,x} = \langle I_{n_1}(f_{n_1}), \ldots, I_{n_s}(f_{n_s}) \rangle$.

**Proof.** Take $h_n W^n \in I_n W^n$. There is a weighted homogeneous polynomial of degree $n$, say $G_n(Y_1, \ldots, Y_s) \in \mathcal{O}_V[Y_1, \ldots, Y_s]$, where each $Y_i$ has weight $n_i$, such that $G_n(f_{n_1} W^{n_1}, \ldots, f_{n_s} W^{n_s}) = h_n W^n$. Considering the initial form of this last expression it follows that $I_{n_i}(f_{n_i})$ generate every initial form of $I_{n_i}(I_n)$ for any $n$, therefore the equality holds.

**Lemma 4.7.** With the previous settings, consider an integer $N > 0$ such that $N$ is a common multiple of every $n_i$, with $i \in \{1, \ldots, s\}$. Define by $G_N = \mathcal{O}_V[I_N W^N]$ the Rees ring attached to $I_N$. Then,

1. $\mathcal{I}_{G_N,x} = \langle I_{m_N}(I_N) \rangle$.
2. $\sqrt{\mathcal{I}_{G_N,x}} = \sqrt{\langle I_{m_N}(I_N) \rangle} = \sqrt{\mathcal{I}_{G,x}}$.

**Proof.**  
1. $\mathcal{I}_{G_N,x} = \langle I_{m_N}(I_{m_N}) \rangle_{m \geq 0}$, and note that $\langle I_{m_N}(I_{m_N}) \rangle = \langle I_{m_N}(I_N) \rangle^m \subset \langle I_{m_N}(I_N) \rangle$.
2. Let us check $\sqrt{\langle I_{m_N}(I_N) \rangle} = \sqrt{\mathcal{I}_{G,x}}$. The first inclusion is immediate since $\langle I_{m_N}(I_N) \rangle \subset \mathcal{I}_{G,x}$. By Lemma 4.6, it is enough to check the other inclusion for the generators of $G$. As $f_{n_i}^{\alpha_i} \in I_N$ for $\alpha_i = \frac{N}{n_i}$, then $I_{n_i}(f_{n_i})^{\alpha_i} \in I_N(I_N)$ and the equality holds.

**Lemma 4.8.** Set $N$, $G_N$ and notation as before. Then, $\tau_{G,x} = \tau_{G_N,x}$.
Proof. By definition, $\tau_{G,x} = \text{codim}(L_{G,x})$. $L_{G,x}$ is the linear space of vertices of the zeroset of $I_{G,x}$, so it is enough to consider the zeroset of $\sqrt{I_{G,x}}$ to define $L_{G,x}$. By Lemma 4.4, $\sqrt{I_{G,x}} \cap \sqrt{I_{G,N,x}}$ so $V(\sqrt{I_{G,x}})$ and $V(\sqrt{I_{G,N,x}})$ have the same subspace of vertices, that is, $L_{G,x} = L_{G,N,x}$. So, finally, $\tau_{G,x} = \tau_{G,N,x}$.

Proof of Proposition 4.5 Assume that locally at $x$, $G = \oplus I_n W^n = O_{V(x)}[f_{n_1},f_{n_2},\ldots,f_{n_r}]$ and $G' = \oplus I'_n W^n = O_{V(x)}[g_{m_1},g_{m_2},\ldots,g_{m_s}]$. Choose a positive integer $N$, such that $N$ is a common multiple of every $n_i$ and every $m_j$, for $i = 1,\ldots,r$ and $j = 1,\ldots,s$. Consider the Rees rings attached to $G$ and $G'$ given by $G_N = O_{V(x)}[I_N W^N]$ and $G'_N = O_{V(x)}[I'_N W^N]$, respectively.

It suffices to consider the case where $G \subset G'$, so then $I_N \subset I'_N$ and this inclusion is an integral extension of ideals. It follows that $\langle \text{In}_N(I_N) \rangle \subset \langle \text{In}_N(I'_N) \rangle$ is an integral extension of ideals, as one can check that the conditions of integral dependence hold for the generators. Then, $\sqrt{\langle \text{In}_N(I_N) \rangle} = \sqrt{\langle \text{In}_N(I'_N) \rangle}$ and by Lemmas 4.7 and 4.8, it follows that $\tau_{G,x} = \tau_{G_N,x} = \tau_{G'_N,x} = \tau_{G',x}$, which proves the Proposition.

5. Zariski’s Multiplicity Formula

In this section we apply a result of Zariski concerning multiplicities of local rings to our particular setting. A generic projection of a hypersurface, embedded in a smooth scheme of dimension $d$, is locally defined by a monic polynomial with coefficients which are functions in a smooth scheme of dimension $d - 1$. That is, $\xrightarrow{\beta} V^{(d-1)}$, where here $\beta$ is the restriction of a smooth projection as described in the previous section.

Theorem 5.1 (Zariski Multiplicity Theorem). Let $A$ be a local ring, $\mathfrak{M}$ its maximal ideal, $\mathfrak{q}$ a $\mathfrak{M}$-primary ideal in $A$, and $B$ an overring of $B$ which is a finite $A$-module. Then $B$ is a semi-local ring, and $\mathfrak{q}B$ an open ideal on $B$. Let $\{\mathfrak{p}_i\}$ be the set of maximal ideals of $B$ and let $\mathfrak{q}_i$ be the primary component of $\mathfrak{q}B$ relative to $\mathfrak{p}_i$. If no element different from 0 in $A$ is a zero divisor in $B$ and all the local rings $B_{\mathfrak{p}_i}$ have the same dimension as $A$, then

$$[B : A]e(\mathfrak{q}) = \sum_i [B/\mathfrak{p}_i : A/\mathfrak{M}]e(\mathfrak{q}_i)$$

where we denote by $e(\mathfrak{q})$ the multiplicity of the ideal $\mathfrak{q}$ and by $[B : A]$ the maximum number of elements of $B$ which are linearly independent over $A$, that is, the dimension of the total quotient ring of $B$ consider as a vector space over the quotient field of $A$. 

\[\square\]
Proof. See [38], Ch. VII §10 Corollary 1 to Theorem 24 \( \Box \)

5.2. Consider the previous morphism \( X \xrightarrow{\beta} V^{(d-1)} \), that is, the restriction of the projection map. Locally, in an open affine neighbourhood, \( X \) is defined by a ring, say \( B \), and \( V^{(d-1)} \) by another ring, say \( A \), such that \( B = A[Z] < f(Z) > \), where \( f(Z) \) is monic. As \( f(Z) \) is monic, \( \beta \) is a finite morphism.

We start by computing the value of \([B : A]\), it is enough to consider the tensor product \( B \otimes_A K \), where \( K \) is the fraction field of \( A \), so \( B \otimes_A K = K[Z] < f(Z) > \); and, as \( f(Z) \) is monic of degree \( n \), then \([B : A] = n\).

Consider a smooth variety \( Y \subset X \) such that \( X \) is equimultiple along \( Y \), of multiplicity \( n \). Let \( P \) be the generic point of \( Y \) and \( q \) the contraction of \( P \) in \( A \). Localizing \( A \) at \( q \), we obtain a local ring \( A_q \), where \( A_q \) has multiplicity 1, that is, \( e(q) = 1 \).

Let \( P_1, \ldots, P_n \) be the prime ideals in \( B \) that dominate \( q \), that is, the points of the fibre \( B \otimes_A k(q) \), one of these prime ideals is the ideal \( P \) we have considered before. Take the local rings \( B_{P_i} \) and the ideal spanned by \( q \), \( qB_{P_i} \). In this context, Zariski’s formula states that

\[
b = [B : A]e(q) = \sum_i [k(P_i) : k(q)]e_{B_{P_i}}(qB_{P_i}),
\]

where \( e_{B_{P_i}}(qB_{P_i}) \) denotes the multiplicity of \( qB_{P_i} \) in \( B_{P_i} \). Note that \( qB_{P_i} \) is a proper ideal and its radical is \( P_iB_{P_i}. \)

We have assumed that \( P \) is the generic points of a variety of multiplicity \( n \). Suppose now, for simplicity, that this \( P \) corresponds to \( P_1 \) in the previous discussion. Then \( e_{B_{P_i}}(P_1) = n \) and \( e_{B_{P_i}}(P_1) \leq e_{B_{P_i}}(qB_{P_i}) \), so

\[
n = [B : A]e(q) \leq n \cdot [k(P_1) : k(q)] + \sum_{i \geq 2} [k(P_i) : k(q)]e_{B_{P_i}}(qB_{P_i});
\]

that is, \( P_1 \) is the unique prime that dominates \( q \) and \( [k(P_1) : k(q)] = 1 \).

Proposition 5.3. Under the previous hypothesis, denote by \( F \) the set of points of multiplicity \( n \) of the hypersurface \( X \). Then,

1. \( \beta \) induces a set bijection between \( F (\subset X) \) and its image, \( \beta(F) \), in \( V^{(d-1)} \).
2. If \( P \) is a point of \( F \), then the residual fields \( k(P) \) and \( k(\beta(P)) \) are isomorphic.

Proof. Follows from the previous discussion. \( \Box \)

Theorem 5.4. Let \( X \) be a hypersurface in \( V^{(d)} \). Let \( x \in X \) be a closed point of multiplicity \( n \) at \( X \). Consider \( Y \subset X \) an equimultiple regular centre, such that \( x \in Y \). If \( V^{(d)} \xrightarrow{\beta} V^{(d-1)} \) is the projection described in [4, 1], then \( \beta(Y) \) is smooth.
Proof. We may reduce our problem to étale topology, and assume that the setting is as before, that is, that $X$ is defined by a monic polynomial of degree $n$, so $X \xrightarrow{\beta} V^{(d-1)}$ is finite. In particular, $Y \rightarrow \beta(Y)$ is also finite and birational by $(5.3)$.

Let $D$ be the ring of $Y$ and $C$ the one of $\beta(Y)$. From $(5.3)$, this two domains have the same quotient field and $C \rightarrow D$ is a finite extension. We want to prove, in fact, that $C = D$. We apply the following well known Lemma

Lemma 5.5. Let $M$ be a finite $R$-module and $(R, M)$ a local domain. Suppose that $R/M \otimes_R M$ is a $R/M$-vectorial space of dimension $e$ and $K \otimes_R M$ is a $K$-vectorial space of dimension $e'$, where $K$ is the quotient field of $R$. In general, $e \leq e'$ and the equality holds if and only if $M \cong R \oplus \cdots \oplus R = R^e$, that is, if $M$ is a free $R$-module of rank $e$.

The previous Lemma will be used to prove that $D$ is a locally free $C$-module of rank 1 and then conclude that $C = D$. From $(5.3)$ $(2)$ it follows that $\text{Spec}(Y) \leftarrow \text{Spec}(\beta(Y))$ is a set bijection, and if we consider a prime $q$ in $C$, then there is a unique prime $q'$ that dominates $q$. Now, the semilocal ring $D \otimes_C C_q$ has a unique maximal, so $D_{q'} = D \otimes_C C_q$ and $C_q \rightarrow D_{q'}$ is finite.

Denote by $K$ the quotient field of $C$. As $D \otimes_C K$ is a finite extension of $K$ and a domain, then $D \otimes_C K$ is a field and, in particular, the quotient field of $D$. From $(5.3)$ the rings $C$ and $D$ have the same quotient field, so

$$\dim_K(D \otimes_C K) = 1.$$  

Take $q = M_{\beta(x)}$, where $x$ is the point of multiplicity $n$ in the hypothesis of the theorem. The condition of transversality of $\beta$ ensures that $qD_{q'} = q'D_{q'}$.

From this assertion we deduce that $D_{q'} \otimes_{C_q} C_q/q = D_{q'}/qD_{q'} = D_{q'}/q'D_{q'} = k(q')$ and from $(5.3)$ we know that $k(q) = k(q')$. Then

$$\dim_{C_q/q} D_{q'} \otimes_{C_q} C_q/q = 1.$$  

So, applying Lemma 5.5 we obtain that $D_{q'}$ is a free $C_q$-module of rank 1. From Nakayama’s Lemma we can lift a basis of $D_{q'} \otimes_{C_q} C_q/q$ as a $C_q/q$-vectorial space to a minimal basis of $D_{q'}$ as a $C_q$-module. So,

$$C_q = D_{q'},$$  

since the rank is 1. We can repeat that for every $x$ of multiplicity $n$ and then $C = D$ in étale topology, so $\beta(Y)$ is smooth. $\circ$
5.6. Let $X$ and $V^{(d)}$ be as before, $x \in X$ of multiplicity $n$ at $X$. Consider $Y \subset X$ an equimultiple regular centre of multiplicity $n$, such that $x \in Y$. Let $\{x_1, \ldots, x_{d-1}\}$ be a regular system of parameters of $\mathcal{O}_{V^{(d-1)},\beta(x)}$, then it can be lifted to a regular system of parameters $\{x_1, \ldots, x_{d-1}, z\}$ of $\mathcal{O}_{V^{(d)}},x$.

Suppose that the smooth center $\beta(Y)$ can be expressed locally as $I(\beta(Y)) = \langle x_1, \ldots, x_r \rangle$. We now show that after a change of variables of the form $z \mapsto z - \alpha$, $Y$ can be expressed locally as $I(Y) = \langle z, x_1, \ldots, x_r \rangle$.

Consider the class of the function $z$ in $\mathcal{O}_{X,x} = \mathcal{O}_{V^{(d-1)},\beta(x)}[z]/\langle f(z) \rangle$, say $\overline{z}$. As $\mathcal{O}_{Y,x} = \mathcal{O}_{\beta(Y),\beta(x)}$ (see the proof of 5.4), we can consider the class of $\overline{z}$ in $\mathcal{O}_{\beta(Y),\beta(x)}$, that is, $\overline{x}$. Then $\overline{z}$ is the class of some $\alpha \in \mathcal{O}_{V^{(d-1)},\beta(x)}$, so after a change of variable of the form $z \mapsto z - \alpha$, it follows that $z \in I(Y)$ locally in a neighborhood of $x$ and is equal to $0$ after restriction to $\mathcal{O}_{\beta(Y),\beta(x)}$.

6. Commutativity of projections and monoidal transformations.

6.1. As before, let $X \subset V^{(d)}$ be a hypersurface, $x \in X$ be a point of multiplicity $n$ and $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ a generic projection defined as in (12), so locally $X = V(f(z))$, where $z$ is equal to $0$ in the local ring $\mathcal{O}_{\beta(X),\beta(x)}$, $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ and $a_i \in \mathcal{O}_{V^{(d-1)},\beta(x)}$ of order $\geq i$.

A relative differential Rees algebra is attached locally to the hypersurface $X$, it is defined as $\mathcal{G} = \mathcal{O}_{V^{(d)},x}[f(z)W^k, \Delta^{(e)}(f(z))W^{n-e}]_{1 \leq e \leq n-1}$. The singular locus of $\mathcal{G}$ is exactly the set of points of multiplicity $n$ of $X$ in a neighborhood of $x$.

As in (6.1.1), we can define an elimination algebra associate with this projection (i.e. associate with the polynomial expression of $f(z)$). Denote this elimination Rees algebra by

$$\mathcal{R}_{\mathcal{G},\beta} = \mathcal{O}_{V^{(d-1)},\beta(x)}[h_{n_1}W^{n_1}, \ldots, h_{n_s}W^{n_s}].$$

Consider now a equimultiple regular centre $Y \subset X$ of multiplicity $n$ such that $x \in Y$ and $I(Y) = \langle z, x_1, \ldots, x_r \rangle$. A monoidal transformation of $V^{(d)}$ on $Y$ can be defined and also one of $V^{(d-1)}$ on $\beta(Y)$. In this section we prove that there exists a commutative diagram between projections and these blow-ups, in an open subset.

Let $V^{(d)} \xrightarrow{\pi} V_1^{(d)}$ be the monoidal transformation of $V^{(d)}$ on $Y$, and let $\mathcal{G}_1$ be the transform of $\mathcal{G}$ defined as in (24). Generators of $\mathcal{G}_1$ are defined as transforms (up to units) of the generators of the Rees algebra $\mathcal{G}$.

**Remark 6.2.** Every point of multiplicity $n$ is in the open set defined by $U = \bigcup_{i=1}^{r} U_{x_i}$, where $U_{x_i} = \text{Spec}(k[x_1, x_2, \ldots, x_r, x_{r+1}, \ldots, x_{d-1}, x_i])$. 
To check this claim it is enough to study the transforms of the equations at $U_z = \text{Spec}(k[x_1, \ldots, x_r, x_{r+1}, \ldots, x_{d-1}, z])$, here $f(z) = z^n (1 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n})$ and the strict transform $f_1 = 1 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n}$. So points of multiplicity $n$ can be seen in the other charts.

In the open subset $U_{x_i}$, the transform of the Rees algebra, $\mathcal{G}_1$, is given by

$$\mathcal{G}_1 = \mathcal{O}_{V_1^{(d-1)}}(U_{x_i})[f_1 W^b, (\Delta^{(e)}(f))_1]_{1 \leq e \leq n-1},$$

with $f_1 = f_1(\frac{x_i}{x_1}) = (\frac{x_i}{x_1})^n + \frac{a_1}{x_1} (\frac{x_i}{x_1})^{n-1} + \cdots + \frac{a_n}{x_1^n}$ and $(\Delta^{(e)}(f))_1 = (\Delta^{(e)}(f)(\frac{x_i}{x_1}))$, where $\Delta^{(e)}_1$ denotes the relative differential operator of order $e$ in this chart (See (3.6.1)).

This ensures that a generic projection from $\mathcal{U}$ to $V_1^{(d-1)}$ can be defined, say $\mathcal{U} \xrightarrow{\beta_1} V_1^{(d-1)}$. We claim that $\beta_1$ is compatible with the projection $\beta$ (see Proposition 6.4). A new elimination algebra is defined in terms of this projection $\beta_1$, say $\mathcal{R}_{\mathcal{G}_1, \beta_1} = \mathcal{O}_{V_1^{(d-1)}}(U_{x_i})[\tilde{h}_{m_1} W^{m_1}, \ldots, \tilde{h}_{m_l} W^{m_l}].$

It follows from the definition of this algebra, that each $\tilde{h}_{m_j}$ is a weighted polynomial of degree $m_j$ on the coefficients of $f_1(\frac{x_i}{x_1})$. That is, on $\frac{a_j}{x_1^n}$ (with weight $j$), where the $a_j$ are the coefficients of the polynomial $f(z)$.

Moreover, it follows from (3.6.2) that $n_j = m_j$ for $j = 1, \ldots, s$ and indeed $s = l$ (where the $n_j$ are the weights of the elimination algebra in (6.1.1)) and

$$\tilde{h}_{n_i} = \left(\frac{1}{x_1}\right)^{n_i} h_{n_i}.$$

Consider now the monoidal transformation of $V^{(d-1)}$ on $I(\beta(Y)) = < x_1, \ldots, x_r >$. The transform of the elimination algebra $\mathcal{R}_{\mathcal{G}, \beta}$, say $(\mathcal{R}_{\mathcal{G}, \beta})_1$, is defined. One can check in the chart $U_{x_i}$ that the transform is

$$(\mathcal{R}_{\mathcal{G}, \beta})_1 = \mathcal{O}_{V_1^{(d-1)}}(U_{x_i})[(h_{n_1})_1 W^{n_1}, \ldots, (h_{n_s})_1 W^{n_s}],$$

where $(h_{n_i})_1 = \left(\frac{1}{x_1}\right)^{n_i} h_{n_i}$, for $i = 1, \ldots, r$. This proves that

$$(\mathcal{R}_{\mathcal{G}, \beta})_1 = \mathcal{R}_{\mathcal{G}_1, \beta_1}.$$

**Theorem 6.3.** Let $X$ and $V^{(d)}$ be as before, $x \in X$ of multiplicity $n$ at $X$. Let $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ be a generic projection defined as in (4.2). Consider $Y \subset X$ an equimultiple regular centre of multiplicity $n$ and $\beta(Y)$ its image in $V^{(d-1)}$. Denote by $\pi$ the monoidal transformation of $V^{(d)}$ on $Y$ and by $\tilde{\pi}$ the one of $V^{(d-1)}$ on $\beta(Y)$. Then,
1. the following diagram commutes,

\[
\begin{array}{ccc}
V^{(d)} & \xrightarrow{\pi} & V^{(d)}_1 \supset \mathcal{U} \\
\downarrow{\beta} & \circ & \downarrow{\beta_1} \\
V^{(d-1)} & \xleftarrow{{\tilde{\pi}}} & V^{(d-1)}_1
\end{array}
\]

where \(\mathcal{U}\) is an open subset of \(V^{(d)}_1\) that contains \(\text{Sing} (\mathcal{G}_1)\) and \(\beta_1\) is uniquely determined.

2. \((\mathcal{R}_{\mathcal{G},\beta})_1 = \mathcal{R}_{\mathcal{G}_1,\beta_1}\) in \(\mathcal{U}\), where \((\mathcal{R}_{\mathcal{G},\beta})_1\) denotes the transform of the elimination algebra of \(\mathcal{R}_{\mathcal{G},\beta}\) and \(\mathcal{R}_{\mathcal{G}_1,\beta_1}\) the elimination algebra of \(\mathcal{G}_1\) (referred to the projection \(\beta_1\)).

**Proof.** It follows from the previous discussion.

6.4. Stability of the projections.

A transversal projection \(\beta : V^{(d)} \rightarrow V^{(d-1)}\) defined as in (4.2) is stable under monoidal transformations. That is, if \(\pi : V^{(d)}_1 \rightarrow V^{(d)}\) and \(\pi : V^{(d-1)}_1 \rightarrow V^{(d-1)}\) are the monoidal transformations defined before, then an induced projection \(\beta_1 : \mathcal{U} \rightarrow V^{(d-1)}_1\) is defined on an open subset \(\mathcal{U} \subset V^{(d)}_1\) such that \(\beta_1\) is a generic transversal projection.

**Proof.** With the previous setting, the projection is defined, in every chart \(U_{x_i}\), by the monic polynomial

\[
f_1(x) = \left(\frac{\hat{x}}{x_i}\right)^n + a_{1i}\left(\frac{\hat{x}}{x_i}\right)^{n-1} + \cdots + a_{ni}.
\]

Note that to prove that geometric condition it suffices to show that \(\Delta_1^{(n)}(f_1)(\bar{x})\) is a unit for \(\bar{x} \in U_{x_i}\) and \(\Delta_1^{(n)}\) a certain differential operator of order \(n\) relative to \(\beta_1\).

Take \(\Delta_1^{(b)}\) the differential relative operator defined by the Taylor morphism (see (2.10)). From (3.6.1), we deduce that

\[
\Delta_1^{(n)}(f_1)(\bar{x}) = \Delta_1^{(n)}(f)(x) = u,
\]

where \(u\) is a unit and \(\bar{x}\) maps into \(x\) via the blow-up. So we conclude that \(\beta_1\) is transversal.

6.5. Suppose the same setting as before, i.e., let \(X \subset V^{(d)}\) be a hypersurface, \(x \in X\) a point of multiplicity \(n\) and \(V^{(d)} \xrightarrow{\beta} V^{(d-1)}\) a generic projection, such that, locally, \(X\) is defined by a monic polynomial with coefficients in \(\mathcal{O}_{\beta(X),\beta(x)}\), say \(f(Z)\). Consider the monoidal transformation \(V^{(d)}_1 \leftarrow V^{(d)}_1\) as before. Let \(\mathcal{G} = \mathcal{O}_{V^{(d)}(f(Z))W^n, \Delta^{(e)}(f(Z))W^n_{-e}|_{1 \leq e \leq n-1}\) be the relative differential Rees algebra and \(\mathcal{G}_1\) the transform.
In the open subset \( \mathcal{U} = \bigcup U_{x_i} \), the transform is defined by
\[
\mathcal{G}_1 = \mathcal{O}_{V(\mathcal{G})}(U_{x_i}) \left[ f_1 \left( \frac{Z}{x_i} \right) W^n, \Delta_1^{(e)} \left( f_1 \left( \frac{Z}{x_i} \right) \right) W^{n-e} \right]_{1 \leq e \leq n-1},
\]
where \( f_1 \) is a monic polynomial in \( \frac{Z}{x_i} \) and \( \Delta_1^{(e)} \) are the differential operators relatives to the induced projection \( \beta_1 \). So, if \( \mathcal{G} \) is a relative differential Rees algebra, then \( \mathcal{G}_1 \) is also a relative differential Rees algebra.

In contrast with this last assertion, if \( \mathcal{G} \) is a differential Rees algebra (absolute), then, in general, it doesn’t hold that \( \mathcal{G}_1 \) is differential. Consider, for example, in \( k[X, Y, Z] \) the differential Rees algebra \( \mathcal{G} = k[(Z^2 + X^2 Y)W^2, (2Z)W, (X^2)W, (2XY)W] \). If we make the blow-up in the origin of \( \mathbb{A}_k^3 \), then the transform of \( \mathcal{G} \) in the chart \( U_X \) is given by \( \mathcal{G}_1 = k[(Z + XY)W^2, (2Z)W, (X)W, (2XY)W] \). Apply the differential operator \( \Delta^{(1,0,0)} \) to \( Z + XY \), then \( (Y)W \) belongs to the differential Rees algebra of the transform and \( (Y)W \not\in \mathcal{G}_1 \).

Another important fact is to understand the behavior of the singular locus of Rees algebras after projections. We have proved that there exists a bijection between the set of points of multiplicity \( b \) of the hypersurface (i.e, \( \text{Sing} (\mathcal{G}) \)) and its image by \( \beta \) (see Proposition 5.3). Moreover, in [35] it is shown that if \( \mathcal{G} \) is differential, then the following equality holds
\[
\beta (\text{Sing} (\mathcal{G})) = \text{Sing} (\mathcal{R}_{\mathcal{G}, \beta})
\]
But, in a more general case, when \( \mathcal{G} \) is relative differential (the property that is compatible with monoidal transformations), \( \beta (\text{Sing} (\mathcal{G})) \subset \text{Sing} (\mathcal{R}_{\mathcal{G}, \beta}) \) (see [35]), but the equality does not hold.

6.6. Example: Consider the hypersurface defined by \( T^2 + XY Z \) in \( \mathbb{A}_k^4 \), where \( k \) is a perfect field of characteristic 2. Here \( \mathcal{G} = G(\mathcal{G}) = \mathcal{O}_{V(\mathcal{G})}[(T^2+XY Z)W^2, (XY)W, (XZ)W, (YZ)W] \) is the differential Rees algebra attached to this hypersurface. The singular locus of \( \mathcal{G} \) is the union of three lines, say \( L_1, L_2 \) and \( L_3 \) (corresponding to the axes \( X, Y \) and \( Z \)).

Consider the projection \( \beta \) defined by the elimination of the variable \( T \). Then \( \beta (\text{Sing} (\mathcal{G})) = \tilde{L}_1 \cup \tilde{L}_2 \cup \tilde{L}_3 \), that is the union of three lines, corresponding to the three axes of coordinates.

The elimination algebra, say \( \mathcal{R}_{\mathcal{G}, \beta} \), is given by \( \mathcal{R}_{\mathcal{G}, \beta} = \mathcal{O}_{V(\beta)}[(XY)W, (XZ)W, (YZ)W] \) and the singular locus is \( \text{Sing} (\mathcal{R}_{\mathcal{G}, \beta}) = \tilde{L}_1 \cup \tilde{L}_2 \cup \tilde{L}_3 \), so in this case \( \beta (\text{Sing} (\mathcal{R}_{\mathcal{G}, \beta})) = \beta (\text{Sing} (\mathcal{G})) \).

Define the quadratic transformation of \( \mathbb{A}_k^4 \) at the origin, say \( 0 \); and the quadratic transformation of \( \mathbb{A}_k^3 \) at \( \beta (0) \) (the origin in \( \mathbb{A}_k^3 \)).
The transform of $G$ in the chart $U_X$ is $G_1 = \mathcal{O}_{V'_{\{4\}}}(U_X)[(T^2 + XYZ)W^2, (XY)W, (XZ)W]$ (note the symmetry among the three charts $U_Z$, $U_Y$ and $U_X$ which cover $U$, see (6.2)).

The singular locus of $G_1$ at $U_X$ is the union of three lines.

Note also that after this quadratic transformation, the transform of the differential Rees algebra is no longer differential, although it is a relative differential algebra.

After the monoidal transformation on $\beta(\overline{0})$, the transform of $R_{G, \beta}$ in the open chart $U_X$ is given by $(R_{G, \beta})_1 = \mathcal{O}_{V'_{\{3\}}}(U_X)[(XY)W, (XZ)W]$, so the singular locus of $(R_{G, \beta})_1 = R_{G_1, \beta_1}$ is the union of the three axis and the exceptional hypersurface. So,

$$\beta_1(Sing(G_1)) \subseteq Sing(R_{G_1, \beta_1}).$$

7. Local Presentation.

7.1. Given a Rees algebra $G$ and a closed point $x \in Sing(G)$, then a tangent cone $C_G$, at $T_{V(d)}x = Spec(gr_{\mathfrak{m}}(\mathcal{O}_{V(d)}x))$ is defined by an homogeneous ideal $I_{G,x}$ in $gr_{\mathfrak{m}}(\mathcal{O}_{V(d)}x)$ (see [1.3]). If $G = \mathcal{O}_V[f_{n_1}W^{n_1}, \ldots, f_{n_s}W^{n_s}]$, locally at $x$, then $I_{G,x} = \langle I_{n_1}(f_{n_1}), \ldots, I_{n_s}(f_{n_s}) \rangle$.

Recall that there is a largest subspace, say $L_G$, included and acting by translations on $C_G$, and $\tau_{G,x}$ is defined as the codimension of this linear subspace. Furthermore, $\tau_{G,x} \geq 1$ whenever $I_{G,x}$ is non-zero. A projection $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ is said to be transversal to $G$ at $x$ if the tangent line of the fiber $\beta^{-1}(\beta(x))$ at $x$ is not included in the subspace $L_G$. Fix a regular system of parameters $\{y_1, y_2, \ldots, y_{d-1}\}$ at $\mathcal{O}_{V^{(d-1)}, \beta(x)}$, and choose an element $Z$ so that $\{y_1, y_2, \ldots, y_{d-1}, Z\}$ is a regular system of parameters at $\mathcal{O}_{V^{(d)}x}$. Recall that Rees algebras are to be considered up to integral closure. So if the condition of transversality holds one can modify the local generators $\{f_{n_1}W^{n_1}, \ldots, f_{n_s}W^{n_s}\}$ of $G$ so that each

$$(7.1.1) \quad f_{n_i} = Z^n + a^{(i)}_1Z^{n_i-1} + \cdots + a^{(i)}_{n_i} \in \mathcal{O}_{V^{(d-1)}, \beta(x)}[Z]$$

is a monic polynomial in $Z$.

Assume now that the Rees algebra $G$ is a Diff-algebra relative to $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$. In such case $G$ can be identified, locally at $x$, by

$$\mathcal{O}_{V^{(d-1)}, x}[Z][f_{n_1}(Z)W^{n_1}, \{\Delta^{(e_i)}(f_{n_1}(Z))W^{n_i-e_i}\}_{1 \leq e_i \leq n_i-1}]_{1 \leq i \leq s},$$

via the natural inclusion $\mathcal{O}_{V^{(d-1)}, x}[Z] \subset \mathcal{O}_{V^{(d)}, x}$, which is a specialization of a the universal elimination algebra defined for $s$ monic polynomials as in (3.8.1). In particular an elimination algebra, say $R_{G, \beta} \subset \mathcal{O}_{V^{(d-1)}, \beta(x)}[W]$ is defined as in (3.8.2).
Proposition 7.2. (Local relative presentation) Let $x \in \text{Sing}(G)$ be a close point where $\tau_{G,x} \geq 1$. Consider a projection $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ which is transversal at $x$. Assume that $G$ is a Diff-algebra relative to $\beta$ and that there is an element $f_n W^n \in G$ such that $f_n$ has order $n$ at the local regular ring $O_{V^{(d)},x}$, and $f_n = f_n(Z)$ is a monic polynomial of degree $n$ in $O_{V^{(d-1)},\beta(x)}$. Then, at a suitable neighborhood of $x$:

$$G \sim O_{V^{(d)}}[f_n(Z)W^n, \Delta^{(e)}(f_n(Z))W^{n-e}]_{0 \leq e \leq n-1} \odot \mathcal{R}_{G,\beta},$$

where $\mathcal{R}_{G,\beta}$ in $O_{V^{(d)}}[W]$ is identified with $\beta^*(\mathcal{R}_{G,\beta})$.

Proof. We may assume that $f_n(Z) \in \{f_{n_1}W^{m_1}, \ldots, f_{n_s}W^{m_s}\}$ as in (7.1.1). Let us check these assertions in the universal case. In order to simplify notation we consider here the case of two generators (i.e., the case $s = 2$). So consider variables $Z, Y_i$ and $V_j$ over a field $k$, and

$$F_n(Z) = (Z - Y_1) \cdot (Z - Y_2) \ldots (Z - Y_n).$$

This polynomial is universal of degree $n$, since $f_n = f_n(Z)$ is a pull-back of $F_n(Z)$. Let

$$G_m(Z) = (Z - V_1) \cdot (Z - V_2) \ldots (Z - V_m)$$

be the universal polynomial of degree $m$. A natural inclusion $\mathcal{R}_G \subseteq G$ arises from (3.7.1).

The product of the permutation groups $S_n \times S_m$ acts on $k[Z, Y_1, \ldots, Y_n, V_1, \ldots, V_m]$. This group also acts on

$$S = k[Z - Y_1, Z - Y_2, \ldots, Z - Y_n, Z - V_1, Z - V_2, \ldots, Z - V_m].$$

The subring of invariants of $S$, $S_{S_n \times S_m}$, is

$$k[\Delta^{(e)}(F_n(Z)), \Delta^{(e')} (G_m(Z))]_{0 \leq e \leq n-1, 0 \leq e' \leq m-1},$$

where $\Delta^{(e)}(F_n(Z))$ is an homogeneous polynomial of degree $n - e$ and $\Delta^{(e')} (G_m(Z))$ is homogeneous of degree $m - e'$. We add a dummy variable $W$ that simply will express the degree. Then, the subring of invariants $S_{S_n \times S_m}$ is

$$k[\Delta^{(e)}(F_n(Z))W^{n-e}, \Delta^{(e')} (G_m(Z))W^{m-e'}]_{0 \leq e \leq n-1, 0 \leq e' \leq m-1}.$$

The universal elimination algebra is defined as the invariant ring of $S_n \times S_m$ acting on

$$S' = k[(Z - Y_2) - (Z - Y_1), \ldots, (Z - Y_n) - (Z - Y_1), (Z - V_1) - (Z - Y_1), \ldots, (Z - V_m) - (Z - Y_1)].$$

The key observation to prove the assertion is that $S$ is spanned by two subrings: $k[Z - Y_1, \ldots, Z - Y_n]$ and $S'$. The subring of invariants on the first is $k[\Delta^{(e)}(F_n(Z))W^{n-e}]_{0 \leq e \leq n-1}$.
and the one of the second is the universal elimination algebra. So both invariant algebras are included in $S^{\mathfrak{g} \times \mathfrak{s}_m}$; and in order to prove the claim it suffices to show that $S^{\mathfrak{g} \times \mathfrak{s}_m}$ is an integral extension of the subalgebra spanned by the two invariant subalgebras. To prove this last condition note that $S$ is an integral extension of the subalgebra spanned by the two invariant subalgebras. This proves the claim. ☐

**Remark 7.3.** With the setting as before, it is sometimes convenient to search for the smallest integer $n$ for which there is a monic polynomial of degree $n$ and a local relative presentation as in Proposition 7.2. Recall that we may assume that $\mathcal{G}$ is integrally closed. So we look for the smallest possible integer $n_1$ that can arise in a set of local generators $\{f_{n_1}W^{n_1}, \ldots, f_{n_s}W^{n_s}\}$ of $\mathcal{G}$ as in (7.1.1).

The discussion in 3.7 already shows that such monic polynomial $f_{n_1}(Z) \in \mathcal{O}_{V^{d-1}}[Z]$ must be irreducible. On the other hand as the algebra is closed by the action of relative differential operators it also follows that $n_1 = p^e$, is a power of the characteristic of the underlying field.

**Theorem 7.4.** Let $V^{(d)}$ be a smooth scheme of dimension $d$, $\mathcal{G}$ a differential Rees algebra, $x \in \text{Sing}(\mathcal{G})$ a closed point, and suppose that $\tau_{\mathcal{G},x} \geq 1$. Fix a generic projection $V^{(d)} \overset{\beta}{\longrightarrow} V^{(d-1)}$ (see 4.2). Then an elimination algebra $\mathcal{R}_{\mathcal{G},\beta}$ is defined at $V^{(d-1)}$ locally at $\beta(x)$, and the $\tau$-invariant drops by one, that is, $\tau_{\mathcal{R}_{\mathcal{G},\beta},\beta(x)} = \tau_{\mathcal{G},x} - 1$.

**Proof.** Fix a regular system of parameters $\{x_1, \ldots, x_d\}$ of $\mathcal{O}_{V^{(d)},x}$. So the graded ring is given by $k'[X_1, \ldots, X_d]$ where $X_i$ denotes the initial form of $x_i$. Here $k'$ is the residue field of the local ring at the closed point. If we assume that $k'$ is a perfect field, it is well known that for a suitable choice of $\{x_1, \ldots, x_d\}$, then $l_{\mathcal{G},x} = \langle X_1^{p^{e_1}}, \ldots, X_d^{p^{e_r}} \rangle$ for certain non-negative integers $e_i, i = 1, \ldots, r$.

So, there is an element $f_i W^{p^{e_i}} \in \mathcal{G}$ for $i = 1, \ldots, r$, such that $\text{In}_{p^{e_1}}(f_i) = X_i^{p^{e_i}}$. Since $\beta$ is generic, then $\mathcal{C}_{\mathcal{G}} \cap \ell = \{0\}$, where $\ell$ denotes the tangent line to the fiber of the projection. It follows that $\mathcal{C}_{f_i} \cap \ell = \{0\}$ for some $i \in \{1, \ldots, r\}$. Assume that this condition is achieved by $i = 1$.

By Weierstrass Preparation Theorem it follows that in an étale neighborhood of $x$, $f_1(x_1) = x_1^{p^{e_1}} + a_1 x_1^{p^{e_1} - 1} + \ldots + a_{p^{e_1}}$, where $a_i \in \mathcal{O}_{V^{(d-1)},\beta(x)}$ and the order of each $a_i$ is $> i$. Note that we obtain a strict inequality since $\text{In}_{p^{e_1}}(f_1) = X_1^{p^{e_1}}$.

By 7.2 we may assume that there is a local relative presentation of the form

$$\mathcal{G} \sim \mathcal{O}_{V^{(d)}}[f_1(x_1)W^{p^{e_1}}, \Delta^{(\alpha)}(f_1(x_1))W^{p^{e_1} - \alpha}]_{1 \leq \alpha \leq p^{e_1} - 1} \oplus \beta^*(\mathcal{R}_{\mathcal{G},\beta}).$$
Denote by $\tilde{G}$ the Rees algebra defined as $\mathcal{O}_{V(d)}[f_1(x_1)W^{p^{e_1}}, \Delta^{(\alpha)}(f_1(x_1))W^{p^{e_1-\alpha}}]_{1 \leq \alpha \leq p^{e_1} - 1}$. Then, $\tau_{\tilde{G},x} = 1$ and the only variable needed to define the generators of $\mathbb{P}_{\tilde{G},x}$ is $X_1$. On the other hand, we have eliminated the variable $x_1$ at $\beta^*(\mathcal{R}_{\tilde{G},\beta})$, so $X_1$ is not needed to define generators of $\mathbb{P}_{\beta^*(\mathcal{R}_{\tilde{G},\beta})}$.

Finally, since $\mathcal{G} \sim \tilde{G} \odot \beta^*(\mathcal{R}_{\tilde{G},\beta})$ and by Proposition 4.5, it follows that $\tau_{\mathcal{G}} = \tau_{\tilde{G}} \odot \mathcal{R}_{\tilde{G},\beta}$, and by the previous arguments, $\tau_{\mathcal{G}} = 1 + \tau_{\mathcal{R}_{\tilde{G},\beta}}$.

Remark 7.5. Let $\mathcal{G}$ be a differential Rees algebra, $x \in \text{Sing}(\mathcal{G})$ a closed point, and suppose that $\tau_{\mathcal{G},x} \geq 1$. We can find a power of the characteristic, say $p^{e_1}$, so that there is a local relative presentation

$$\mathcal{G} \sim \mathcal{O}_{V(d)}[f_1(x_1)W^{p^{e_1}}, \Delta^{(\alpha)}(f_1(x_1))W^{p^{e_1-\alpha}}]_{1 \leq \alpha \leq p^{e_1} - 1} \odot \beta^*(\mathcal{R}_{\tilde{G},\beta}).$$

with $f_1(x_1) = x_1^{p^{e_1}} + a_1^{(1)}x_1^{p^{e_1}-1} + \cdots + a_1^{(p^{e_1})} \in \mathcal{O}_{V(d-1),\beta(x)}[x_1]$ and where $p^{e_1}$ is the smallest integer $n$ for which there is an element $f_n \in I_n$ of order $n$ at the point $x$.

As $\mathcal{R}_{\tilde{G},\beta} \subset \mathcal{G}$ it follows that $\mathcal{R}_{\tilde{G},\beta}$ is a differential algebra in $V^{(d-1)}$. Suppose now that $\tau_{\mathcal{R}_{\tilde{G},\beta}(x)} \geq 1$. Then there is a transversal smooth morphism $\beta_1 : V^{(d-1)} \to V^{(d-2)}$, and a positive integer $e_2(\geq e_1)$ so that

$$\mathcal{R}_{\tilde{G},\beta} \sim \mathcal{O}_{V(d)}[f_2(x_2)W^{p^{e_2}}, \Delta^{(\alpha)}(f_2(x_2))W^{p^{e_2-\alpha}}]_{1 \leq \alpha \leq p^{e_2} - 1} \odot \beta_1^*(\mathcal{R}_{\mathcal{R}_{\tilde{G},\beta} \odot \beta_1}),$$

for a monic polynomial $f_2(x_2) = x_2^{p^{e_2}} + a_1^{(1)}x_2^{p^{e_2}-1} + \cdots + a_2^{(p^{e_2})} \in \mathcal{O}_{V(d-2),\beta(x)}[x_2]$.

So if $\mathcal{G}$ is a differential algebra and if $\tau = \tau_{\mathcal{G},x} \geq 1$, taking a smooth transversal morphism $\beta : V^{(d)} \to V^{(d-\tau)}$ we get for each index $1 \leq i \leq \tau$ a monic polynomial

$$f_i(x_i) = x_i^{p^{e_i}} + a_1^{(i)}x_i^{p^{e_i}-1} + \cdots + a_i^{(p^{e_i})} \in \mathcal{O}_{V^{(d-1)},\beta(x)}[x_i],$$

where $e_1 \geq e_2 \geq \cdots \geq e_{\tau}$, and

$$\mathcal{G} \sim \mathcal{O}_{V(d)}[f_i(x_i)W^{p^{e_i}}, \Delta^{(\alpha)}(f_i(x_i))W^{p^{e_i-\alpha}}]_{1 \leq i \leq \tau, 1 \leq \alpha \leq p^{e_i} - 1} \odot \beta^*(\mathcal{R}_{\mathcal{G},\beta}).$$

Note that $\mathcal{G}$ contains the algebra

$$P_\mathcal{G} = \mathcal{O}_{V(d)}[f_i(x_i)W^{p^{e_i}}]_{1 \leq i \leq \tau} \odot \beta^*(\mathcal{R}_{\mathcal{G},\beta}),$$

but $\text{Sing}(\mathcal{G}) = \text{Sing}(P_\mathcal{G})$. Moreover the equality of the singular loci is preserved after any sequence of permissible transformations. $P_\mathcal{G}$ is called the local presentation algebra.
8. THE MONOMIAL CASE.

8.1. Let $V$ be a smooth scheme and $E = \{H_1, \ldots, H_r\}$ be smooth hypersurfaces with normal crossings. A monomial ideal supported on $E$ will be a sheaf of ideals of the form $\mathcal{M} = I(H_1)^{\alpha_1} \cdot I(H_2)^{\alpha_2} \cdots I(H_r)^{\alpha_r}$. A monomial algebra is a Rees algebra of the form $O_V[\mathcal{M}W^s]$ for some positive integer $s$.

If $V'$ is smooth and $\pi : V' \rightarrow V$ is a smooth morphism, then the pull-back of the hypersurfaces of $E$ has normal crossings and a monomial ideal supported on $E$ has a natural lifting to $V'$.

In our setting, we will take a smooth morphism $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ transversal to a simple algebra $G \subset O_V^{(d)}[W]$ (i.e. $\tau_{G,x} \geq 1$) and a monomial ideal $\mathcal{M}$ in $V^{(d)}$ which is the pull-back of a monomial ideal in $V^{(d-1)}$.

We say that the Rees algebra $O_V^{(d-1)}[\mathcal{M}W^s]$ has strong monomial contact with $G$ if locally at any closed point $x \in \text{Sing}(G)$ there is a smooth section defined by an element $z \in O_{V^{(d)}}$ so that $G \subset \langle z \rangle W \odot \mathcal{M}W^s$.

We reformulate this condition in terms of local coordinates. Let $\{y_1, \ldots, y_{d-1}\}$ be a regular system of parameters at $O_{V^{(d-1)},\beta(x)}$, then $z$ is such that

(i) $\{z, y_1, \ldots, y_{d-1}\}$ is a regular system of parameters at $O_{V^{(d)},x}$.

(ii) $G \subset \langle z \rangle W \odot \langle y_1^{h_1} \ldots y_{d-1}^{h_{d-1}} \rangle W^s$ (locally at $x$), where $y_1^{h_1} \ldots y_{d-1}^{h_{d-1}}$ generates the monomial ideal at $O_{V^{(d-1)},\beta(x)}$.

In the setting of resolution of singularities we start with a simple differential algebra $G \subset O_V^{(d)}[W]$ which we may assume to be integrally closed, together with a transversal projection $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$. In such case we may assume that there is a presentation of the form

$$G \sim O_V^{(d)}[f_{p^e}W^{p^e}, \Delta^{(\alpha)}(f_{p^e})W^{p^e-\alpha}]_{1 \leq \alpha \leq p^e-1} \odot \mathcal{R}_{G,\beta},$$

where:

(i) $\mathcal{R}_{G,\beta}$ denotes the elimination algebra in $O_{V^{(d-1)}}[W]$, identified here with its pull-back in $O_{V^{(d)}}[W]$.

(ii) $f_{p^e}$ is analytically irreducible of order $p^e$ at each closed point $x \in \text{Sing}(G)$.

Recall that a sequence of permissible transformations of $G$, say

$$G \xrightarrow{\pi_1} G_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_r} G_r$$
induces a sequence

\[ \begin{array}{ccc}
G & \xleftarrow{\pi_1} & G_1 & \xleftarrow{\pi_r} & G_r \\
V^{(d)} & \xleftarrow{\beta} & V_1^{(d)} & \xleftarrow{\beta_1} & V_r^{(d)} \\
V^{(d-1)} & \xleftarrow{\pi_1'} & V_1^{(d-1)} & \xleftarrow{\pi_r'} & V_r^{(d-1)}
\end{array} \]

where:

(i) each vertical morphism \( \beta_i: V_i^{(d)} \rightarrow V_i^{(d-1)} \) is transversal to \( G_i \),

(ii) the lower sequence induces a sequence of transformations of \( R_{G_i, \beta} \), say

\[ \begin{array}{ccc}
R_{G_i, \beta} & \xleftarrow{\pi_1} & (R_{G_i, \beta})_1 & \xleftarrow{\pi_r} & (R_{G_i, \beta})_r \\
V^{(d-1)} & \xleftarrow{\beta_1} & V_1^{(d-1)} & \xleftarrow{\beta_r} & V_r^{(d-1)}
\end{array} \]

and furthermore, each \( (R_{G_i, \beta})_i \) is the elimination algebra of \( G_i \) relative to \( V_i^{(d)} \xrightarrow{\beta_i} V_i^{(d-1)} \), that is, \( (R_{G_i, \beta})_i = R_{G_i, \beta_i} \).

In particular, for each index \( i \),

\[ G_i \sim O_{V_i^{(d)}} [f_{p^e}^{(i)}W^{p^e}, \Delta^{(a)}(f_{p^e}^{(i)})W^{p^e-\alpha}|_{1 \leq \alpha \leq p^e-1} \odot (R_{G_i, \beta})_i, \]

where \( \sim \) means same integral closure and \( f_{p^e}^{(i)} \) is the strict transform of \( f_{p^e} \).

The main theorem in [6] states that there is a natural way to define such sequence, so that \( (R_{G_i, \beta})_r \) is a monomial ideal supported on the exceptional locus. Furthermore, that sequence and the monomial ideal \( (R_{G_i, \beta})_r \) are independent of \( \beta \).

In what follows, we take:

\[ \begin{array}{ccc}
G & \xleftarrow{\pi_1} & G_1 & \xleftarrow{\pi_r} & G_r \\
V^{(d)} & \xleftarrow{\beta} & V_1^{(d)} & \xleftarrow{\beta_1} & V_r^{(d)} \\
V^{(d-1)} & \xleftarrow{\pi_1'} & V_1^{(d-1)} & \xleftarrow{\pi_r'} & V_r^{(d-1)}
\end{array} \]

\[ \begin{array}{ccc}
R_{G_i, \beta} & \xleftarrow{\pi_1} & (R_{G_i, \beta})_1 & \xleftarrow{\pi_r} & (R_{G_i, \beta})_r \\
V^{(d-1)} & \xleftarrow{\beta_1} & V_1^{(d-1)} & \xleftarrow{\beta_r} & V_r^{(d-1)}
\end{array} \]

as in the formulation of the theorem in [6]. So here, \( (R_{G_i, \beta})_r \subset O_{V^{(d-1)}} [W] \) is monomial and supported on the exceptional locus, and so is its pull-back to \( V_r^{(d)} \).

We identify \( (R_{G_i, \beta})_r \) with its pull-back, say \( (R_{G_i, \beta})_r = I(H_1)^{\alpha_1} \ldots I(H_r)^{\alpha_r} W^s = NW^s \). We also identify:

\[ G_r \sim O_{V_r^{(d)}} [f_{p^e}^{(r)}W^{p^e}, \Delta^{(a)}(f_{p^e}^{(r)})W^{p^e-\alpha}|_{1 \leq \alpha \leq p^e-1} \odot NW^s \]
Theorem 8.2. With the setting as before, there is a monomial ideal $\mathcal{M}$ such that

(i) $\mathcal{M}$ divides $\mathcal{N}$, or say $\mathcal{M} = I(H_1)^{h_1} \cdots I(H_r)^{h_r}$ and $0 \leq h_i \leq \alpha_i$ for $i = 1, \ldots, r$.

(ii) $\mathcal{M} W^s$ has strong monomial contact relative to $\beta_r$ with $\mathcal{G}_r$ locally at any closed point in $\text{Sing}(\mathcal{G}_r)$.

Remark 8.3. Condition (ii) says that locally at $x \in \text{Sing}(\mathcal{G}_r)$ there is an element $z$ of order one at $\mathcal{O}_{V_r(d), x}$ defining a local section of $V_r(d) \rightarrow V_r(d-1)$, so that

$$\mathcal{G}_r \subset \langle z \rangle W \odot \mathcal{M} W^s.$$ 

This implies that

$$\text{Sing}(\langle z \rangle W \odot \mathcal{M} W^s) \subset \text{Sing}(\mathcal{G}_r),$$

and furthermore, that a resolution of $\langle z \rangle W \odot \mathcal{M} W^s$ induces a sequence of transformations of $\mathcal{G}_r$.

Note that there is a natural identification of $\langle z \rangle W \odot \mathcal{M} W^s \subset \mathcal{O}_{V_r(d-1)}[W]$ with $\mathcal{M} W^s \subset \mathcal{O}_{V_r(d-1)}[W]$, so that a resolution of one is equivalent to a resolution of the other.

A resolution of a monomial Rees algebra $\mathcal{M} W^s \subset \mathcal{O}_{V_r(d-1)}[W]$ is simple to achieve. This is done by blowing-up combinatorial centers defined in terms of the exponents $h_i$ of $\mathcal{M}$.

Any such center in $V_r(d-1)$ is also permissible for the elimination algebra $\mathcal{N} W^s$ (or its transforms) and can be lifted uniquely to a smooth permissible center of $\mathcal{G}_r$, independently of the choice of the transversal parameter $z$ in the previous expression. In fact, Zariski’s formula applied here for the restriction $X_r \xrightarrow{\beta_r} V_r(d-1)$ ($X_r$ is the hypersurface defined by $f_r^{(r)}$) states that such lifting is unique.

In particular, a resolution of $\mathcal{M} W^s$ in $\mathcal{O}_{V_r(d-1)}$ induces a sequences of transformations, say

$$\mathcal{G}_r \quad \mathcal{G}_{r+1} \quad \cdots \quad \mathcal{G}_R$$

$$V_r(d) \quad V_{r+1}(d) \quad \cdots \quad V_R(d)$$

defined entirely in terms of the exponents of $\mathcal{M}$.

9. Construction of the strong monomial algebra.

9.1. We now address the construction of the monomial algebra $\mathcal{M} W^s$ of Theorem 8.2 following its notation.
Let \( i \leq r \) be an index so that \( \alpha_i \geq 1 \). Note that \( H_{i}^{(d)}(\subset V_{r}^{(d)}) \) is the strict transform of the exceptional component of \( V_{i-1}^{(d)} \leftarrow V_{i}^{(d)} \), and recall that there is a square diagram

\[
\begin{array}{ccc}
V_{i-1}^{(d)} & \xleftarrow{\pi_i} & V_{i}^{(d)} \\
\downarrow{\beta_{i-1}} & & \downarrow{\beta_i} \\
V_{i-1}^{(d-1)} & \xleftarrow{\pi'_i} & V_{i}^{(d-1)}
\end{array}
\]

where \( V_{i-1}^{(d-1)} \leftarrow V_{i}^{(d-1)} \) induces a transformation of elimination algebras, and \( H_i = H_{i}^{(d)} \) is the pullback of the exceptional hypersurface of \( V_{i-1}^{(d-1)} \leftarrow V_{i}^{(d-1)} \), say \( H_{i}^{(d-1)} \).

Let \( H_{i}^{(d)} \xrightarrow{\pi_i} H_{i}^{(d-1)} \) be the restriction of the smooth morphism \( V_{i}^{(d)} \xrightarrow{\beta_i} V_{i}^{(d-1)} \) and let \( \bar{G}_i \) be the restriction of \( G_i \) to the smooth hypersurface \( H_{i}^{(d)} \). Here

\[
\bar{G}_i \sim \mathcal{O}_{H_{i}^{(d)}}[\overline{f_{\bar{p}}^{(i)}}W^p, \Delta^{\alpha}(\overline{f_{\bar{p}}^{(i)}}W^p-\alpha)]_{1 \leq \alpha \leq \bar{p}^s-1} \otimes (\mathcal{R}_{G_i}) W^s,
\]

(where the bars denote restrictions). By general properties of elimination algebras, the following are equivalents:

1. The exponent \( \alpha_i \geq 1 \).
2. \(((\mathcal{R}_{G_i})_i)\) is trivial (i.e., is the algebra defined by the zero ideal).

Moreover, under these equivalent conditions,

\[
\bar{G}_i \sim \mathcal{O}_{H_{i}^{(d)}}[\overline{f_{\bar{p}}^{(i)}}W^p]_{1 \leq \alpha \leq \bar{p}^s-1}
\]

and the finite morphism

\[
(H_{i}^{(d)} \supset V(\overline{f_{\bar{p}}^{(i)}}) \longrightarrow H_{i}^{(d-1)}
\]

is purely inseparable.

We will define an integer \( 0 \leq h_i \leq \alpha_i \) so that \( 0 < h_i \). We take into account the following condition among components of the exceptional locus:

\[
(C) \quad \text{Sing}(\bar{G}_i) \text{ is of pure codimension one in } H_{i}^{(d)}.
\]

This geometric condition \((C)\) is closely related to the \( \tau \)-invariant, and it ensures that the equivalent conditions \((1)\) and \((2)\) hold.

The proof of the next Proposition will show that if \( C_i \) denotes the center of \( V_{i}^{(d)} \leftarrow V_{i+1}^{(d)} \), then condition \((C)\) holds if and only if

1. the order of \(((\mathcal{R}_{G_i})_i)\) is \( > 1 \) along \( C_i \) (or equivalently, \(((\mathcal{R}_{G_i})_{i+1})\) is the zero algebra).
Proposition 9.2. Condition (C) holds if and only if the \( \tau \)-invariant of \( G_i \) is one along closed points of \( C_i \).

Proof. Locally at a closed point \( x \in C_i \) there is a transversal morphism

\[ V_i^{(d)} \xrightarrow{\beta_i} V_i^{(d-1)} \]

and a presentation of \( G_i \), say

\[ G_i \sim O_{V_i^{(d)}} \left[ f^{(i)}_p \Delta^\alpha(f^{(i)}_p) W^{p^e-\alpha} \right]_{1 \leq \alpha \leq p^e-1} \otimes (R_{G_i})_i, \]

so \( x \) is a \( p^e \)-fold point of \( f^{(i)}_p \) and \( \beta_i \) is transversal to the hypersurface defined by \( f^{(i)}_p \) locally at the point.

It is known that under these conditions, if \( \tau = 1 \), then the \( \tau \)-invariant of the elimination algebra is zero. Equivalently, the order of the algebra \( (R_{G_i})_i \) is \( > 1 \) at \( O_{V_i^{(d-1)}, \beta_i(x)} \).

So if \( \tau = 1 \) along points of \( C_i \), the order of \( (R_{G_i})_i \) is \( > 1 \) along points of \( \beta_i(C_i) \subset V_i^{(d-1)} \), and hence the order of \( (R_{G_i})_i \) is \( > 1 \) along points of \( C_i \).

We start by proving the equivalence for the case in which \( C_i \) is a closed point. Assume first that \( \tau = 1 \) and that \( C_i \) is a closed point. In this case, \( \text{In}_{I(C_i)}(f^{(i)}_p) \) is a \( p^e \)-th power of a linear form at \( \text{gr}_{I(C_i)}(O_{V_i^{(d)}}) = k[\overline{z}, \overline{y}_1, \ldots, \overline{y}_{d-1}] \) where bars denote the initial forms of the regular system of parameters \( \{ z, y_1, \ldots, y_{d-1} \} \) at \( O_{V_i^{(d)}, C_i} \), where \( \{ y_1, \ldots, y_{d-1} \} \) is a regular system of parameters at \( O_{V_i^{(d-1)}, C_i} \) and \( z \) is a transversal parameter.

As \( \text{In}_{I(C_i)}(f^{(i)}_p) \) is a \( p^e \)-th power of a linear form, all \( \Delta^{(s)}(f^{(i)}_p) \) have zero initial class at \( I(C_i)^{p^e-s}/I(C_i)^{p^e-s+1} \). So, \( \overline{G_i} \) is the algebra at \( \text{Proj}(\text{gr}_{I(C_i)}(O_{V_i^{(d)}})) \) induced by \( \text{In}(f^{(i)}_p)W^{p^e} \) (at the projective space).

In this case, \( \text{Sing}(\overline{G_i}) \) is a linear space of pure codimension one.

Assume now that \( C_i \) is a closed point and that \( \text{Sing}(\overline{G_i}) \) is of pure codimension one. Then, \( \text{In}_{I(C_i)}(f^{(i)}_p) \in \text{gr}_{I(C_i)}(O_{V_i^{(d)}}) \) (non zero-homogeneous polynomial of degree \( p^e \)) induces a hypersurface at \( \text{Proj}(\text{gr}_{I(C_i)}(O_{V_i^{(d)}})) \) which contains a hypersurface of \( p^e \)-fold points. This can only happen if \( \text{In}_{I(C_i)}(f^{(i)}_p) \) is a \( p^e \)-th power of a linear form.

This proves the Proposition under the assumption that \( C_i \) is a closed point.

For the general case, in which \( C_i \) is not a closed point, consider

\[ H_i^{(d)} = \text{Proj}(\text{gr}_{I(C_i)}(O_{V_i^{(d)}})) \xrightarrow{\text{C}_i} \]
The fiber at each closed point $x \in C_i$ is the projective space $\text{Proj} \left( \mathcal{O}_{V_i^{(d)}} / \mathfrak{m}_x \otimes \text{gr} I(C_i) \left( \mathcal{O}_{V_i^{(d)}} \right) \right)$. If $\{z_1, \ldots, z_d\}$ is a regular system of parameters at $\mathcal{O}_{V_i^{(d)}}$, so that $I(C_i) = \langle z_1, \ldots, z_s \rangle$, then

$$\text{gr} \mathfrak{m}_x \left( \mathcal{O}_{V_i^{(d)},x} \right) = k[\overline{z}_1, \ldots, \overline{z}_s],$$

$$\text{gr} I(C_i) \left( \mathcal{O}_{V_i^{(d)},x} \right) = \mathcal{O}_{V_i^{(d)},x} / \langle z_1, \ldots, z_s \rangle \overline{[\overline{z}_1, \ldots, \overline{z}_s]}.$$  

There is a natural inclusion

$$\mathcal{O}_{V_i^{(d)}} / \mathfrak{m}_x \otimes \text{gr} I(C_i) \left( \mathcal{O}_{V_i^{(d)}} \right) \hookrightarrow \text{gr} \mathfrak{m}_x \left( \mathcal{O}_{V_i^{(d)}} \right)$$

and $\text{In}_x (G_i)$ (in $\text{gr} \mathfrak{m}_x \left( \mathcal{O}_{V_i^{(d)}} \right)$) is generated by the image of the class of $\text{In}_I(C_i) (G_i)$ at $\mathcal{O}_{V_i^{(d)}} / \mathfrak{m}_x \otimes \text{gr} I(C_i) \left( \mathcal{O}_{V_i^{(d)}} \right)$.

In particular, the $\tau$-invariant of $\text{In}_x (G_i)$ is the $\tau$ invariant of $\left( \text{In}_I(C_i) (G_i) \right)_x$ in $\mathcal{O}_{V_i^{(d)}} / \mathfrak{m}_x \otimes \text{gr} I(C_i) \left( \mathcal{O}_{V_i^{(d)}} \right)$.

The equivalence follows now from the case in which $C_i$ is a closed point.

We now define the exponents $h_i$, following the notation in Theorem 8.2 by induction on $i$.

Note here that if $H_1^{(d)}, \ldots, H_i^{(d)} \subset V_i^{(d)}$ (and $H_1^{(d-1)}, \ldots, H_i^{(d-1)} \subset V_i^{(d-1)}$) denote the exceptional hypersurface, then

$$(R_{G, \beta})_i \subset I(H_1)^{\alpha_1} \ldots I(H_i)^{\alpha_i} W^s$$

**Remark 9.3.** Recall that Condition (C) holds at $H_i^{(d)}$ when the restriction of the Rees algebra $G_i$ to $H_i^{(d)}$, say $\overline{G}_i$, is such that $\text{Sing}(\overline{G}_i)$ is a hypersurface in $H_i^{(d)}$. If $x \in \text{Sing}(G_i)$ is a closed point and if

$$\mathcal{O}_{V_i^{(d)}} \left[ f_{\rho^e}^{(i)} W^{p^e}, \Delta_{\alpha}^{(f_{\rho^e}^{(i)} W^{p^e})} |_{1 \leq \alpha \leq \rho^e-1} \right] \otimes (R_{G, \beta})_i$$

is a local relative presentation of $G_i$ at $x$, then

$$\mathcal{O}_{H_i^{(d)}} \left[ f_{\rho^e}^{(i)} W^{p^e} \right]$$

is a local relative presentation of $\overline{G}_i$, and $f_{\rho^e}^{(i)}$ is a $p^e$-th power of a smooth hypersurface at $H_i^{(d)}$ (locally at $x$).

Let $G_i \subset \langle z \rangle W \otimes \mathcal{M} W^s$ and $\mathcal{M} = y_1^{h_1} . . . y_i^{h_i} W^s$ (with $h_j \leq \alpha_j$), so in local coordinates if

$$f_{\rho^e}^{(i)} = z^{p^e} + a_1 z^{p^e-1} + \cdots + a_{p^e},$$

then $a_j W^j \in \langle y_1^{h_1} . . . y_i^{h_i} \rangle W^s$. That is,

$$h_i \geq 1 \iff f_{\rho^e}^{(i)} |_{y_i=0} = z^{p^e} \iff H_i^{(d)} = \{ y_i = 0 \}$$

satisfies Condition (C).
Fix a sequence of permissible transformations

\[
\begin{array}{cccc}
G & G_1 & \ldots & G_r \\
V^{(d)} & \pi_1 & \ldots & \pi_r \\
V_1^{(d)} & V_r^{(d)}
\end{array}
\]

and assume that Condition (C) holds for \(G_i\) at \(H_i^{(d)} (\subset V_i^{(d)})\). One can check that for \(j > i\) the restriction of \(G_j\) to \(H_i^{(d)} (\subset V_j^{(d)})\), say \((G_j)_i\), is a transform of the restriction of \(G_i\) at \(H_i^{(d)} (\subset V_i^{(d)})\) and that Condition (C) also holds at \((G_j)_i\).

We may assume, by induction on \(i\):

(H0) \(h_1, \ldots, h_i\) are defined.

(H1) For each index \(j \leq i\) for which \(h_j \geq 1\), setting \(G_i|_{H_j}\) by restriction, then \(G_i|_{H_j} = \mathcal{O}_{H_j}[f_p^{(i)} W^{p_e}]\) and \(\langle f_p^{(i)} \rangle\) is a \(p_e\)-th power of an ideal of a smooth hypersurface in \(H_j\) locally at each point of \(\text{Sing}(G_i|_{H_j})\).

(H2) For any \(x \in \text{Sing}(G_i)\), if \(\{H_{j_1}, \ldots, H_{j_\ell}\}\) with \(1 \leq j_1 < \cdots < j_\ell \leq i\) are the exceptional components containing \(x\), then

\[G_i \subset \langle z \rangle W \odot \left( y_{j_1}^{h_{j_1}} \cdots y_{j_\ell}^{h_{j_\ell}} \right) W^s\]

locally at \(x\).

In (H2) we assume that \(\{y_1, \ldots, y_{d-1}\}\) is a suitable regular system of parameters at \(\mathcal{O}_{V_i^{(d-1)}}, x\), that \(\{z, y_1, \ldots, y_{d-1}\}\) is a regular system of parameters at \(\mathcal{O}_{V_i^{(d)}, x}\) and that \(y_{j_t}\) defines \(H_{j_t}\) for \(t = 1, \ldots, \ell\).

Set

\[
\begin{array}{cccc}
V_i^{(d)} & \pi_{i+1} & V_{i+1}^{(d)} \\
\beta_{i} & & \beta_{i+1} \\
V_i^{(d-1)} & \pi_{i+1}^{(d-1)} & V_{i+1}^{(d-1)}
\end{array}
\]

and \(H_{i+1}^{(d-1)}\) by blowing up \(C_i\). Note that

\(\langle R_{g,\beta} \rangle_{i+1} \subset I(H_1)^{\alpha_{i+1}} \cdots I(H_{i+1})^{\alpha_{i+1}} W^s\).

We now define \(h_{i+1}(\leq \alpha_{i+1})\) so that previous conditions can be lifted to \(i+1\).

Set \(G_{i+1}|_{H_{i+1}}\) by restriction. Recall that condition (C) holds for this restriction if and only if the \(\tau\)-invariant of \(G_i\) is one along points in \(C_i (\subset V_i)\) and that in such case \(\alpha_{i+1} \geq 1\).

We will define \(h_{i+1} = 0\) whenever Condition (C) does not hold. In particular \(h_{i+1} \geq 1\) implies \(\alpha_{i+1} \geq 1\).
9.4. Strategy:

We will first show that at a closed point \(x' \in \text{Sing}(G_{i+1}) \cap H_{i+1}\) there exists a regular system of parameters \(\{y_1, \ldots, y_{d-1}\}\) at \(\mathcal{O}_{V_i^{(d-1)}, \beta_{i+1}(x')}\) so that

\[
G_{i+1} \subset \langle z \rangle W \circ \left( y_{j_1}^{h_{j_1}} \cdots y_{j_{i+1}}^{r_{j_{i+1}}} \right) W^s,
\]

where \(\{y_j\}\) define the exceptional hypersurfaces different from \(H_{i+1}\) containing \(x'\), \(y_{i+1}\) defines \(H_{i+1}\) and \(r_{i+1} \geq 1\). We then define \(h_{i+1}\) as the biggest possible integer \(r_{i+1}\) so that an expression of the form (9.4.1) holds with \(r_{i+1} = h_{i+1}\) at any closed point in \(H_{i+1}\).

Condition (9.4.1) is a corollary of the following Lemma and Remark 9.6.

Recall that locally at any closed point \(x\) of \(C_i (\subset V_i^{(d)})\) there is an inductively defined expression

\[
G_i \subset \langle z \rangle W \circ \left( y_{j_1}^{h_{j_1}} \cdots y_{j_{i+1}}^{r_{j_{i+1}}} \right) W^s,
\]

**Lemma 9.5.** We can assume that

(i) \(\langle z \rangle \subset I(C_i)\) in (9.4.2)

(ii) \(\text{In}_x(\{f_p^{(i)}\})\) is a \(p^i\)-th power of \(\text{In}_x(z)\) at \(\text{gr}_x(\mathcal{O}_{V_i^{(d)}, x})\).

Replace \(\mathcal{O}_{V_i^{(d)}, x}\) and \(\mathcal{O}_{V_i^{(d-1)}, \beta_{i}(x)}\) by their completions \(k[[z, y_1, \ldots, y_{d-1}]]\) and \(k[[y_1, \ldots, y_{d-1}]]\). By assumptions and (9.4.2) we may take:

\[
f_p^{(i)} = z^p + a_1 z^{p-1} + \cdots + a_p \in k[[z, y_1, \ldots, y_{d-1}]]
\]

and

\[
a_k W^k \in \left( y_{j_1}^{h_{j_1}} \cdots y_{j_{i+1}}^{r_{j_{i+1}}} \right) W^s,
\]

In fact, a local relative presentation of \(G_i\) at \(x\) is given by

\[
\mathcal{O}_{V_i^{(d)}, x}[Y^{p^i}] W^{p^i}, \Delta^{(\alpha)}(f_p^{(i)} W^{p^i-\alpha})_{1 \leq \alpha \leq p^i-1} \circ (R \mathcal{G}, \beta_i)
\]

So (9.5.1) follows from the fact that

\[
\Delta^{(\alpha)}(f_p^{(i)} W^{p^i-\alpha}) \in \langle z \rangle W \circ \left( y_{j_1}^{h_{j_1}} \cdots y_{j_{i+1}}^{r_{j_{i+1}}} \right) W^s
\]

We claim that there exists \(z'\) so that

\[
\langle z \rangle W \circ \left( y_{j_1}^{h_{j_1}} \cdots y_{j_{i+1}}^{r_{j_{i+1}}} \right) W^s = \langle z' \rangle W \circ \left( y_{j_1}^{h_{j_1}} \cdots y_{j_{i+1}}^{r_{j_{i+1}}} \right) W^s
\]

and \(z' \in I(C_i)\).

By assumption of transversality \(\beta_i(C_i)\) is a smooth center at \(V_i^{(d-1)}\) and we may assume \(I(\beta(C_i)) = \langle y_1, \ldots, y_r \rangle\).
Proof of the Lemma \[9.5\] Fix notation as above. We first prove (i) by considering two cases:

Case (i1): There is an index \( j_i \leq r \) (i.e. the center \( C_i \) is included in \( H_{j_i}^{(d)} \)).

In this case, setting \( y_{j_i} = 0 \), it follows that \( f_{p^e}^{(i)} = f_{p^e}^{(i)}(z, y_1, \ldots, y_{d-1}) |_{y_{j_i}=0} \) is a \( p^e \)-th power. Since \( f_{p^e}^{(i)} \) is monic of degree \( p^e \) in \( z \), it follows that

\[
\overline{f_{p^e}^{(i)}} = (z - \alpha)^{p^e} \quad \text{for} \quad \alpha \in k[[y_1, \ldots, y_{d-1}]]
\]

So, \( \alpha^{p^e} \) is a sum of monomials in \( a_{p^e} \). Therefore,

\[
\alpha^{p^e} W^{p^e} \in \left( y_{j_1}^{h_{j_1}} \cdots y_{j_r}^{h_{j_r}} \right) W^s
\]

and up to integral closure \( \alpha W \in \left( y_{j_1}^{h_{j_1}} \cdots y_{j_r}^{h_{j_r}} \right) W^s \). This proves (i) within this case.

Case (i2): All \( j_i > r \) (i.e. \( C_i \) is not included in any \( H_{j_i}^{(d)} \)).

As \( C_i \) is permissible for \( G_i \), then \( \beta_i(C_i) \) is permissible for \( (R_{G_i, \beta})_i \) in \( V_{i}^{(d-1)} \). Here \( \langle y_1, \ldots, y_r \rangle \) is the ideal defining \( \beta_i(C_i) \) locally at \( \beta_i(x) \).

\( (R_{G_i, \beta})_i \) is generated over \( O_{V_i^{(d-1)}} \) by homogeneous elements, say \( g_1 W^{s_1}, \ldots, g_q W^{s_q} \) and each \( g_j \) has order \( s_j \geq 1 \). In particular, all \( g_j \in \langle y_1, \ldots, y_r \rangle \) and the same holds for the elimination algebra of \( f_{p^e}^{(i)} = z^{p^e} + a_1 z^{p^e-1} + \cdots + a_{p^e} \).

In particular,

\[
\overline{f_{p^e}^{(i)}} = f_{p^e}^{(i)} |_{y_1=0, \ldots, y_r=0} \in k[[y_{r+1}, \ldots, y_{d-1}]][z]
\]

has an elimination algebra which is zero. Under these conditions it is proved in \[35\] that

\[
\overline{f_{p^e}^{(i)}} = (z^{p^e'} + a_{p^e'})^{p^e''},
\]

where \( e' + e'' = e \), \( a_{p^e'} \in k[[y_{r+1}, \ldots, y_{d-1}]] \) and \( z^{p^e'} + a_{p^e'} \) is irreducible.

We claim now that \( e' = 0 \), namely that \( f_{p^e}^{(i)} = (z - a)^{p^e} \) for some \( a \in k[[y_{r+1}, \ldots, y_{d-1}]] \). If this claim holds note that \( a^{p^e} \) is a sum of some monomials of \( a_{p^e} \). Then, the same argument as in Case (i1) would show that

\[
a W \in \left( y_{j_1}^{h_{j_1}} \cdots y_{j_r}^{h_{j_r}} \right) W^s
\]

and that \( z - a \in I(C_i) \).

So, it remains to show that \( e' = 0 \).

\( C_i \) is included in points of multiplicity \( p^e \) of the hypersurface defined by \( f_{p^e}^{(i)} \). Zariski’s multiplicity formula ensures that the finite morphism

\[
C_i \longrightarrow \beta_i(C_i)
\]
is an isomorphism locally at \( x \) (and \( \beta_i(x) \)). Taking completions at these points, \( \beta_i(C_i) \) is defined by \( \text{Spec}(k[[y_1, \ldots, y_{d-1}]]/(y_1, \ldots, y_r)) \) and \( C_i \) is defined by an irreducible component of \( f_{p^e}^{(i)} \). Since the only irreducible component is \( z^{p^e} + a_{p^e'} \) it follows that \( e' = 0 \).

The proof of part (ii) is similar to that of Case (i1): Consider again the formal expression of \( f_{p^e}^{(i)} \) in \( k[[z, y_1, \ldots, y_{d-1}]] \), say

\[
f_{p^e}^{(i)} = \sum_{j \geq p^e} F_j^{(i)}(z, y_1, \ldots, y_{d-1}),
\]

where each \( F_j^{(i)} \) is homogeneous as polynomial in \( \{z, y_1, \ldots, y_{d-1}\} \).

By assumption \( F_{p^e}^{(i)} \) is a monic polynomial in the variable \( z \) and also a \( p^e \)-th power, so

\[
F_{p^e}^{(i)} = (z - \alpha)^{p^e}
\]

and the same argument as before show that \( \alpha W \in \left( y_{j_1}^{h_{j_1}} \ldots y_{j_\ell}^{h_{j_\ell}} \right) W^s \).

Note finally that given a closed point \( x \), we may assume that \( F_{p^e}^{(i)} = y_1 \) and the arguments in Case (i1) and (i2) are such that this condition is not affected by further changes of \( z \) (so as to guarantee that \( z \in I(C_i) \)).

Remark 9.6. We now proof formula (9.4.1) for \( r_{i+1} \geq 1 \). Recall that \( I(C_i) \subset \mathcal{O}_{V_{i+1},x} \) is the ideal spanned by \( \langle y_1, \ldots, y_r \rangle \) and \( G_i \subset \langle y_1 \rangle \cap H_{i+1} \) mapping to \( x \). Assume that \( y_1 \) defines \( H_{i+1} \) at \( \mathcal{O}_{V_{i+1},x} \). By Lemma 9.5 (ii), \( \frac{z}{y_1} \) is a non-invertible element of \( \mathcal{O}_{V_{i+1},x'} \).

Check that

\[
G_{i+1} \subset \left( \frac{z}{y_1} \right) W \cap \left( \left( \frac{y_{j_1}}{y_1} \right)^{h_{j_1}} \ldots \left( \frac{y_{j_\ell}'}{y_1} \right)^{h_{j_\ell}'} \right) W^s,
\]

where \( \left( \frac{y_{j_1}}{y_1}, \ldots, \frac{y_{j_\ell}'}{y_1} \right) \) correspond to the strict transforms of the exceptional components containing the point \( x' \).

In order to prove (9.4.1) it suffices to check that

\[
G_{i+1} \subset \langle z' \rangle W \cap \left( y_1 \left( \frac{y_{j_1}}{y_1} \right)^{h_{j_1}} \ldots \left( \frac{y_{j_\ell}'}{y_1} \right)^{h_{j_\ell}'} \right) W^s,
\]

for a suitable choice of \( z' \) so that \( z' \in \langle \frac{z}{y_1} \rangle W \cap \left( \left( \frac{y_{j_1}}{y_1} \right)^{h_{j_1}} \ldots \left( \frac{y_{j_\ell}'}{y_1} \right)^{h_{j_\ell}'} \right) W^s \).

This follows using the same arguments as in the proof of Lemma 9.5.
10. Examples

10.1. We discuss some examples to illustrate the construction of the strong monomial algebra $\mathcal{M}$ (see Theorem 8.2). The first example is treated by Hauser in [17].

Example 10.2. Consider the hypersurface defined by $f(Z, X, Y) = Z^2 + Y^7 + X^4Y$ in $\mathbb{A}^3_k$, where $k$ is a perfect field of characteristic 2. Here $G = G(\mathcal{G}) = \mathcal{O}_{V(3)}[(Z^2 + Y^7 + X^4Y)W^2, (Y^6 + X^4)W]$ is the differential Rees algebra attached to the hypersurface.

The singular locus of $G$ is the cusp $\{(0, \lambda^3, \lambda^2) \mid \lambda \in k\}$. The projection $\beta : \mathbb{A}^3_k \to \mathbb{A}^2_k$ induced by the inclusion $k[X, Y] \hookrightarrow k[X, YZ]$ is transversal to $G$, and gives rise to the elimination of the variable $Z$. The elimination algebra is $\mathcal{R}_{G, \beta} = \mathcal{O}_{V(3)}[(Y^6 + X^4)W]$.

Fix $C$ as the origin, so $I(C) = \langle Z, X, Y \rangle$. The element $(Z^2 + Y^7 + X^4Y)W^2$ is a generator of the algebra, which means that $f = Z^2 + Y^7 + X^4Y$ is considered here with weight 2. We view $f$ as a monic polynomial on $Z$, and the constant term has order 5 at the origen. We would like to assign to this constant term a fractional order $\lambda \in k$, and define the ”slope” of $f$ at the close point $C$ as $\frac{5}{2}$. In fact this is the biggest slope in the sense of Newton polygons for all choices of transversal parameter $Z$.

Take the quadratic transformation with center $C$ at the $U_Y$-chart (notation as in 6.2). To simplify notation we consider again coordinates $\{Z, X, Y\}$ ($Z = \frac{X}{Y}, X = X, Y = Y$ at $U_Y$, so the restriction to this chart defines:

$$G_1 = \mathcal{O}_{V(3)}[(Z^2 + Y^5 + X^4Y^3)W^2, (Y^5 + X^4Y^3)W^1]$$

$$\mathcal{R}_{G, \beta})_1 = \mathcal{O}_{V(3)}[(Y^5 + X^4Y^3)W^1].$$

The maximum power of the exceptional hypersurface which can be extracted is $\frac{3}{2}(-\frac{5}{2} = 1)$. In this case, the integers $s$ and $h_1$ in Theorem 8.2 are $s = 2$ and $h_1 = 3$.

Note that, $G_1 \subset \langle Z \rangle W \cap I(H_1)^3W^2$ and $3$ is the highest exponent for which this inclusion holds (see 9.4).

We take now the quadratic transformation at $C_1$, the origin of $U_X$, so $I(C_1) = \langle Z, X, Y \rangle$. The transforms at the $U_X$-chart of this transformation are given by,

$$G_2 = \mathcal{O}_{V(3)}[(Z^2 + X^3Y^5 + X^5Y^3)W^2, (X^4Y^5 + X^6Y^3)W^1]$$

$$\mathcal{R}_{G, \beta})_2 = \mathcal{O}_{V(2)}[(X^4Y^5 + X^6Y^3)W^1].$$

In this case, $G_2 \subset \langle Z \rangle W \cap I(H_1)^3I(H_2)^3W^2$. Check that $3$ is the maximum exponent for which the inclusion holds, so following 9.4 we take $h_2 = 3$. 


Take again a quadratic transformation, now at $C_2$ the origin of the chart. So $I(C_2) = \langle Z, X, Y \rangle$. In the $U_Y$-chart:

$$G_3 = \mathcal{O}_{V_3^{(3)}}[(Z^2 + X^3Y^6 + X^5Y^6)W^2, (X^4Y^8 + X^6Y^8)W^1]$$

and

$$(R_{G,\beta})_3 = \mathcal{O}_{V_3^{(3)}}[(X^4Y^6 + X^6Y^6)W^1].$$

Note that in this chart, $H_1 = \emptyset$ (the strict transform of $H_1$ does not appear in this chart), that

$$G_3 \subset \langle Z \rangle W \odot I(H_2)^3 I(H_3)^6 W^2,$$

and that $h_3 = 6$. The singular locus of $G_3$ at this chart is the union of three lines,

$$\{ Z = 0, X = 0 \} \cup \{ Z = 0, Y = 0 \} \cup \{ Z = 0, X + 1 = 0 \}.$$

In order to achieve monomialization of $R_{G,\beta}$, consider the center $I(C_3) = \langle Z, Y, 1 + X \rangle$. After the change of variables $X_1 = 1 + X$, $I(C_3) = \langle Z, X_1, Y \rangle$ and

$$f^{(3)} = Z^2 + Y^6X^3(1 + X)^2 = Z^2 + Y^6X_1^2 + Y^6X_1^3 + Y^6X_1^4 + Y^6X_1^5.$$

Recall here the cleaning process of $p^e$-th powers we use in Lemma [9.5] so after a change of variable of the form $Z \mapsto Z - Y^3(X + X^2)$, $f^{(3)} = Z^2 + Y^6X_1^3 + Y^6X_1^5$. Recall also that elimination algebra are invariants under changes of the form $Z - \alpha$.

Consider the quadratic transformation with center $C_3$ at the $U_{X_1}$-chart:

$$G_4 = \mathcal{O}_{V_4^{(3)}}[(Z^2 + Y^6(1 + X_1)^2X_1^7)W^2, (Y^8(1 + X_1)^4X_1^9)W^1]$$

and

$$(R_{G,\beta})_4 = \mathcal{O}_{V_4^{(3)}}[(Y^8(1 + X_1)^4X_1^9)W^1].$$

At this chart the elimination algebra is within the monomial case. Now $G_4 \subset \langle Z \rangle W \odot I(H_2)^3 I(H_3)^6 I(H_4)^7 W^2$, and $h_4 = 7$. So

$$\mathcal{M}W^2 = I(H_2)^3 I(H_3)^6 I(H_4)^7 W^2$$

is the strong monomial algebra of Theorem [5.2] at least at an open set containing the charts we have considered.

The monomial $\mathcal{M}W^2$ induces a new sequence of monoidal transformations. Rewrite first $f^{(4)}$ in terms of the regular system of parameters $\{ Z, X, Y \}$, where $X = 1 + X_1$; then $f^{(4)} = Z^2 + Y^6X_3(1 + X)^6$ and locally at the origin $\mathcal{M}W^2 = X^3Y^6(1 + X)^3W^2$, where $(1 + X)^3$ is a unit.

Recall that a resolution of $\mathcal{M}W^2$ (defined over a 2-dimensional scheme) will lift to a sequence of permissible transformations of $G_4$.

To resolve $\mathcal{M}W^2$ consider successively a monoidal transformation at $X = 0$ and three at $Y = 0$ (roughly speaking, they induce a monoidal transformation at $\langle Z, X \rangle$.)
and three at \( \langle Z, Y \rangle \), then the strict transform of \( f^{(4)} \), say \( f' \), is \( f = Z^2 + X(1 + X)^6 \) and \( M'W^2 = X(1 + X)^7W^2 \).

Now, after the change of variable \( X_1 = 1 + X \), \( f' = Z^2 + X_1^6 + X_1^7 \), so after cleaning the 2-th power, \( f' = Z^2 + X_1^6 \) and \( M'W^2 = (1 + X_1)X_1^7W^2 \), where \((1 + X_1)\) is a unit. After three monoidal transformations at \( \langle Z, X_1 \rangle \) (i.e., the three monoidal transformations at \( \langle X_1 \rangle \) enough to resolve \( M'W^2 \)), the strict transform of \( f' \), say \( f'' \), is \( f'' = (Z^2 + X_1)W^2 \) which is a non-singular hypersurface.

**Example 10.3.** Consider now another pathological example, that defined by the hypersurface \( f(T, X, Y, Z) = T^2 + XYZ \) in \( \mathbb{A}_k^4 \), where \( k \) is a perfect field of characteristic 2 (see 6.6). The inclusion \( k[X, Y, Z] \to k(X, Y, Z, T) \) induces a smooth transversal projection \( \beta \) (elimination of the variable \( T \)). The differential Rees algebra and the elimination algebra are

\[
\mathcal{G} = \mathcal{O}_{V(0)}[(T^2 + XYZ)W^2, XYW, XZW, YZW] \quad \text{and} \quad \mathcal{R}_{\mathcal{G}, \beta} = \mathcal{O}_{V(3)}[XYW, XZW, YZW]
\]

After the quadratic transformation at the origin and restriction to the chart \( U_X \):

\[
\mathcal{G}_1 = \mathcal{O}_{V_1(4)}[(T^2 + XYZ)W^2, XYW, XZW] \quad \text{and} \quad (\mathcal{R}_{\mathcal{G}, \beta})_1 = \mathcal{O}_{V_1(3)}[XYW, XZW].
\]

Now, \( \mathcal{G}_1 \subset \langle T \rangle W \odot I(H_1)W^2 \), and in this case \( s = 2 \) and \( h_1 = 1 \) in Theorem 8.2.

Take the quadratic transformation at the origin, and the restriction to the chart \( U_Y \):

\[
\mathcal{G}_2 = \mathcal{O}_{V_2(4)}[(T^2 + XYZ)W^2, XYW] \quad \text{and} \quad (\mathcal{R}_{\mathcal{G}, \beta})_2 = \mathcal{O}_{V_2(3)}[XYW].
\]

Note that \( \mathcal{G}_2 \subset \langle T \rangle W \odot I(H_1)I(H_2)W^2 \) and that \( h_2 = 1 \). Note that \( (\mathcal{R}_{\mathcal{G}, \beta})_2 \) is monomial and the singular locus of \( \mathcal{G}_2 \) is not combinatorial, that is, the singular locus is the union of three lines and only one of them is intersection of the exceptional hypersurfaces \( H_i \).

In order to resolve the monomial algebra \( MW^2 \) it is enough to take the monoidal transformation at \( \langle X, Y \rangle \). After such transformation, and restriction to the \( U_X \)-chart, \( \mathcal{G}_3 = \mathcal{O}_{V(0)}[(T^2 + YZ)W^2, XYW] \), and the \( \tau \) invariant has increased. The same holds at the \( U_Y \)-chart.
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