Fractional D1-Branes at Finite Temperature

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Abstract

The supergravity dual of $N$ regular and $M$ fractional D1-branes on the cone over the Einstein manifold $Q^{1,1,1}$ has a naked singularity in the infrared. The supergravity dual of $N$ regular and $M$ fractional D3-branes on the conifold also has such a singularity. Buchel suggested and Gubser et al. have shown that in the D3-brane case, the naked singularity is cloaked by a horizon at a sufficiently high temperature. In this paper we derive the system of second-order differential equations necessary to find such a solution for $Q^{1,1,1}$. We also find solutions to this system in perturbation theory that is valid when the Hawking temperature of the horizon is very high.

April 2001
1 Introduction

The AdS/CFT correspondence [1, 2, 3] has produced a wealth of new information about strongly coupled conformal gauge theories. Considerable effort has also been invested into extending it to non-conformal theories. One recent development is the fascinating story that has emerged surrounding a certain four dimensional $\mathcal{N} = 1$ supersymmetric $SU(N) \times SU(N + M)$ gauge theory [4, 5]. The goal of the present work is to attempt to retell at least part of the same story [5] for a $\mathcal{N} = 2$ supersymmetric $SU(N) \times SU(N) \times SU(N + M)$ gauge theory living in two dimensions.

We begin by summarizing the story surrounding the $\mathcal{N} = 1$ $SU(N) \times SU(N + M)$ gauge theory. The theory may be realized by adding $M$ fractional D3-branes (wrapped D5-branes) to $N$ regular D3-branes at the apex of the conifold, which is defined by the constraint $\sum_{i=1}^{4} z_{i}^{2} = 0$ in $\mathbb{C}^{4}$ [3]. For $M = 0$, this gauge theory reduces to the superconformal theory dual to the $AdS_{5} \times T^{1,1}$ background of type IIB string theory [7, 8].

In the supergravity dual, the $M$ fractional branes correspond to $M$ units of RR 3-form flux through the 3-cycle of the compact space $T^{1,1}$. This flux changes the background and introduces the logarithmic running of $\int S_{2} B_{2}$, which is related to the running of field theoretic couplings [3]. In turn, this running causes the RR 5-form flux, which corresponds to the number of ordinary D3-branes, to grow logarithmically with the radius [9], due to the equation $dF_{5} = H_{3} \wedge F_{3}$. In [4], this behavior was attributed to a cascade of Seiberg dualities in the dual gauge theory.

While the Klebanov-Tseytlin (KT) solution [9] is smooth in the UV (for large $\rho$), it has a naked singularity in the IR. Two complementary ways have been found of removing the singularity, and it is with the removal of the singularity that the story becomes very interesting. In [4], Klebanov and Strassler (KS) proposed to deform the conifold, i.e. to replace the constraint with $\sum_{i=1}^{4} z_{i}^{2} = \epsilon^{2}$. The resulting solution, a warped deformed conifold, is perfectly non-singular and without a horizon in the IR, while it asymptotically approaches the KT solution [9] in the UV. The mechanism that removes the naked singularity is related to the breaking of the chiral symmetry in the dual $SU(N) \times SU(N + M)$ gauge theory. The $Z_{2M}$ chiral symmetry, which may be approximated by $U(1)$ for large $M$, is realized geometrically as $z_{i} \rightarrow -z_{i}$ [4] although supersymmetry is preserved.

The second mechanism for removing the singularity from the KT solution was proposed by Buchel [10] and later worked out in detail [11, 5]. It was suggested that a non-extremal and hence supersymmetry breaking generalization of the KT solution, with unbroken $U(1)$ symmetry, may have a regular Schwarzschild horizon “cloaking” the naked singularity. The dual field theory interpretation is restoration of chiral symmetry above some critical temperature $T_{c}$. In [11, 5], the authors were able to show that at least at high temperatures, where the differential equations could be analyzed through perturba-
tion theory in the number of fractional D3-branes, a well behaved supergravity solution exists with restored $U(1)$ symmetry which in the IR involves a regular Schwarzschild horizon but which in the UV approaches the asymptotic KT geometry.

These two methods of removing the singularity from the KT solution form an attractive and consistent picture for the gauge theory dual. At high temperature, above $T_c$, the chiral $U(1)$ symmetry is present. As we lower the temperature, the horizon distance shrinks until we reach the critical temperature $T_c$ where the horizon can no longer “shield” the singularity. At this point, a phase transition occurs and a KS type solution becomes preferred. The chiral symmetry is broken.

In [12], generalizations of the KT solution were found involving fractional M2-branes and fractional D$p$-branes, $p = 0, 1, 2, 4$. Like in the KT solution, the transverse space is conical and moreover these generalizations are typically smooth in the UV but possess a naked singularity as the radius of the cone shrinks below some critical value. This singularity renders the dual gauge theory poorly defined in the IR. A detailed understanding of the gauge/gravity correspondence for these fractional branes could shed light on gauge theories in dimensions other than four, and so removing the IR singularities of these generalizations is an important challenge. In the case of fractional D2-branes, a KS type [4] solution has already been found [13] that resolves the singularity while preserving supersymmetry. The IR limit $\rho \rightarrow 0$ of this fractional D2-brane solution is thought to correspond to confinement in the dual three dimensional gauge theory.

It is natural to wonder if the mechanism for singularity resolution at high Hawking temperature in the fractional D3-brane system will work for fractional D$p$-branes with $p$ other than 3. We shall focus on the case $p = 1$, the fractional D-string solution. This solution consists of a warped product of $\mathbb{R}^{1,1}$ flat space-time directions and a Ricci flat, eight dimensional cone $Y_8$. The base of the cone is then a seven dimensional Einstein space, which in this paper will be $Q^{1,1,1}$ [14]. The manifold $Q^{1,1,1}$ can be described as a coset:

$$Q^{1,1,1} = \frac{SU(2)^3}{U(1)^2}.$$  

The ordinary D-strings fill the space-time dimensions. D3-branes are wrapped over one of the 2-cycles of $Q^{1,1,1}$, and the remaining two dimensions of these D3-branes fill $\mathbb{R}^{1,1}$, creating the fractional D-strings. In the dual supergravity description, the fractional D-strings correspond to turning on an RR five form flux, $F_5$, which pierces one of the five-cycles of $Q^{1,1,1}$. Meanwhile, the ordinary D-strings correspond to an electric form flux $F_3$ in the $\mathbb{R}^{1,1}$ and radial directions.

Our non-extremal generalization of the fractional D1-brane solution is well-behaved and singularity free at high Hawking temperature. Moreover, our ansatz preserves the

\[1\] A related finite temperature solution was discussed in [15], which used the wrapped brane solution of [10] to identify appropriate boundary conditions in the IR.
U(1) symmetry of $Q^{1,1,1}$. Hopefully, a $U(1)$ breaking deformation of the cone over $Q^{1,1,1}$, which eliminates the singularity of the KT type fractional D-string solution, will someday be found. It seems reasonable to conjecture that there is some chiral symmetry breaking phase transition, just as in the fractional D3-brane case.

Below the critical temperature $T_c$, this hypothetical deformed cone over $Q^{1,1,1}$ would be preferred. Above $T_c$, the ansatz we present below describes a fractional D-string solution where the naked singularity is cloaked and the $U(1)$ symmetry restored.

We begin by presenting our ansatz for this $U(1)$ symmetry preserving fractional D-string. Next, we develop a perturbation theory that is valid at high temperature, and thereby show that at sufficiently high temperature, the naked singularity of the corresponding KT type fractional D-string is “shielded” by an event horizon.

2 Non-Extremal Generalization of the Fractional D1-Brane Ansatz

We start with an ansatz for the non-extremal fractional D1-branes. Our strategy is similar to that used in finding the non-extremal generalization of the fractional D3-branes [11]. We will add additional warping functions to the metric that preserve the underlying $U(1) \times SU(2)^3$ symmetry of $Q^{1,1,1}$. At the same time, we will leave unchanged the geometric dependence of the RR and NSNS field strengths on the two- and five-cycles of $Q^{1,1,1}$.

The general ansatz for a 10-d Einstein-frame metric consistent with the underlying symmetries of $Q^{1,1,1}$ involves five functions $x, y, z, w_1, w_2$ of a radial coordinate $u$

$$ds^{2}_{10E} = e^{3z}(e^{-2x}dX_0^2 + e^{2x}dX_1^2) + e^{-z}ds^{2}_8$$

where

$$ds^{2}_8 = e^{14y}du^2 + e^{2y}(dM_7)^2,$$

$$(dM_7)^2 = e^{-12w_1}g_\psi^2 + e^{2w_1-2w_2}\sum_{i=1}^{2}(g_{\theta_i}^2 + g_{\phi_i}^2) + e^{2w_1+4w_2}(g_{\theta_3}^2 + g_{\phi_3}^2)$$

and

$$g_\psi = \frac{1}{4}(d\psi + \sum_{i=1}^{3}\cos \theta_i d\phi_i), \quad g_{\theta_i} = \frac{1}{2\sqrt{2}}d\theta_i, \quad g_{\phi_i} = \frac{1}{2\sqrt{2}}\sin \theta_i d\phi_i.$$
This metric can be brought into a more familiar D-string form
\[
\text{ds}_{10E}^2 = h(\rho)^{-3/4}[A(\rho)dX_6^2 + dX_7^2] + h(\rho)^{1/4} \left[ \frac{d\rho^2}{B(\rho)} + \rho^2(dM_7)^2 \right]
\]
with the redefinitions
\[
h = e^{-4z-8x/3}, \quad \rho = e^{y+x/3}, \quad A = e^{-4x}, \quad e^{14y+2x/3}du^2 = B^{-1}d\rho^2.
\]
When \(w_1 = w_2 = 0\) and \(e^{6y} = \rho^6 = \frac{1}{6w}\), the transverse 8-d space is the cone over \(Q^{1,1,1} = M_7\). Small \(u\) thus corresponds to large distances (where we shall assume that \(h, A, B \rightarrow 1\), and \(\rho \rightarrow \infty\)) and vice versa.

The function \(w_1\) squashes the \(U(1)\) fiber of \(Q^{1,1,1}\) relative to the three two spheres while the function \(w_2\) squashes one of the two spheres relative to the other two. For comparison, the non-extremal generalization of the KT solution, which involved the 5-d Einstein manifold \(T^{1,1}\), made use of only one squashing function \(w\), not two. This \(w\), which squashed the \(U(1)\) fiber of \(T^{1,1}\) relative to rest of the manifold, roughly corresponds to our \(w_1\). The most general volume preserving squashing of \(Q^{1,1,1}\) consistent with the symmetries would involve two more functions \(w_2\) and \(w_3\). Note, however, from (8), that the harmonic two-forms on \(Q^{1,1,1}\) involve linear combinations of the \(SU(2)'s\). In order to keep one of these harmonic two-forms \(u\) independent, we may only add one more squashing function \(w_2\). The additional Einstein equations in the \(Q^{1,1,1}\) directions give additional constraints when compared to the five dimensional \(T^{1,1}\) case; fortunately \(w_1\) and \(w_2\) give us the freedom necessary to satisfy the constraints. Ricci-flat 8-d spaces with nontrivial \(w_i\) correspond to the resolutions of \(Q^{1,1,1}\) considered by [18].

The extremal D-string solution and the more general fractional D-string solution on \(Y_8\) have \(x = w_i = 0\). Adding a non-constant \(x(u)\) drives the non-extremality. For example, the non-extremal version of a D-string on the cone over \(Q^{1,1,1}\) \((w = 0)\) has \(x = au, e^{-4x} = A = 1 - \frac{2a}{3\rho^3}, e^{-4z-8x/3} = h = 1 + \frac{4}{\rho^3}, \rho = e^{y+x/3}\). Our aim will be to understand how switching on the non-extremality \((x = au)\) changes the extremal fractional D-string solution.

Our ansatz for the \(p\)-form fields is dictated by the geometry and thus is exactly the same as in the extremal fractional D-string case [12]:
\[
F_3 = K(u)e^{6z-\Phi}d^2x \wedge du,
\]
\[
B_2 = f(u)\omega_2, \quad \omega_2 = (g_{\theta_1} \wedge g_{\phi_1} - g_{\theta_2} \wedge g_{\phi_2})/\sqrt{2},
\]
\[
F_5 = P(\omega_5 + *\omega_5), \quad \omega_5 = -g_{\theta_5} \wedge \omega_2 \wedge g_{\theta_3} \wedge g_{\phi_3}.
\]
Moreover, the dilaton \(\Phi\) is assumed to be a function of the radial variable \(u\) only. The \(M\) fractional D-strings (wrapped D3-branes) thus correspond \(M\) units of flux through the five-cycle of \(Q^{1,1,1}\), and \(P \sim g_sM\). The function \(K(u)\) corresponds roughly to the flux of
ordinary D-strings through the compact space $Q^{1,1,1}$ itself. The equation of motion for $F_3$, $d*e^\Phi F_3 = F_5 \wedge H_3$, implies

$$K(u) = Q + Pf(u). \quad (10)$$

In what follows, we shall use this ansatz to reduce the type IIB supergravity equations of motion to a system of nonlinear, coupled ordinary differential equations describing the radial evolution of $x, y, z, w_1, w_2, K$, and $\Phi$.

### 3 Derivation of the Equations of Motion

We have seven warping functions and hence will require a system of seven ordinary differential equations. From analogy with the non-extremal generalization of the KT solution \[11\], we also expect a zero energy constraint, giving eight equations total. Consideration of the $p$-form field strengths yields two nontrivial equations of motion, one for the dilaton and one for $H_3 = dB_2$. The $H_3$ equation of motion, $d*e^{-\Phi} H_3 = -F_5 \wedge F_3$, reduces to the ordinary differential equation

$$\left(e^{2z-4y-4w_1+4w_2-\Phi}K'\right)' = P^2 K e^{6z-\Phi}, \quad (11)$$

while the dilaton equation of motion, $2d*e^\Phi H_3 = -e^\Phi F_3 \wedge *H_3$, reduces to

$$2\Phi'' = -e^{-\Phi+2z-4y-4w_1+4w_2} \frac{K'^2}{P^2} - K^2 e^{6z-\Phi}. \quad (12)$$

Einstein’s equations are $R_{MN} = T_{MN}$ where $R_{MN}$ is the Ricci curvature and

$$T_{MN} = \frac{1}{2} \partial_M \Phi \partial_N \Phi + \frac{1}{96} \tilde{F}_{MPQRS} \tilde{F}_N^{PQRS}$$
$$+ \frac{1}{4} (e^{-\Phi} H_{MPQ} H_N^{PQ} + e^\Phi \tilde{F}_{MPQ} \tilde{F}_N^{PQ})$$
$$- \frac{1}{48} G_{MN} (e^{-\Phi} H_{PQR} H^{PQR} + e^\Phi \tilde{F}_{PQR} \tilde{F}^{PQR}). \quad (13)$$

In order to write down these equations in a convenient form, we will work in an orthonormal frame basis:

$$e^0 = e^{\frac{3}{2}z-x} dX_0, \quad e^1 = e^{\frac{3}{2}z+x} dX_1, \quad e^\nu = e^{-\frac{3}{2}z+7y} du,$$
$$e^\psi = e^{-\frac{3}{2}z+y-6w_1} g_\psi, \quad e^{\theta_3} = e^{-\frac{3}{2}z+y+w_1+2w_2} g_{\theta_3}, \quad e^{\phi_3} = e^{-\frac{3}{2}z+y+w_1+2w_2} g_{\phi_3},$$
$$e^{\theta_i} = e^{-\frac{3}{2}z+y+w_1-w_2} g_{\theta_i}, \quad e^{\phi_i} = e^{-\frac{3}{2}z+y+w_1-w_2} g_{\phi_i}, \quad i = 1, 2. \quad (15, 16)$$

In this basis, Einstein’s equations are diagonal. Moreover, the equations corresponding to $R_{\theta_1 \theta_1}, R_{\theta_2 \theta_2}, R_{\phi_1 \phi_1}$, and $R_{\phi_2 \phi_2}$ are identical and similarly for the equations corresponding
to $R_{\theta_3 \theta_3}$ and $R_{\phi_3 \phi_3}$. Thus, we are left with six relations. Our strategy will be to put aside $R_{uu}$ at first and use the remaining five relations along with the two field strength equations to derive a second order, nonlinear system in the seven warping functions. At the end, we will find that the $R_{uu}$ relation provides a zero energy constraint, analogous to the one found in \cite{11}.

From the field strengths, it is relatively easy to see that $T_{00} = T_{11}$. However, $R_{00} = -e^{z-14y}(\frac{3}{2}z'' + x'')$, while $R_{11} = -e^{z-14y}(\frac{3}{2}z'' - x'')$. Hence, the first two Einstein equations allow us to solve for $x(u)$ exactly:

$$x'' = 0, \quad x = au, \quad a > 0.$$ \hspace{1cm} (17)

In the case of non-extremal fractional D3-branes \cite{11}, the same behavior was found for this $x$ function, and the factor $a$ was identified with the degree of non-extremality.

Having solved for $x$, we can use either of the first two Einstein equations to find an equation for $z$:

$$12z'' = 2P^2 e^{4z+4y+4w_1-4w_2} + 3K^2 e^{6z-\Phi} + \frac{(K')^2}{P^2} e^{2z-4y-4w_1+4w_2-\Phi}.$$ \hspace{1cm} (18)

Comparing (12) and (18) with the corresponding equations (3.18) and (3.19) in \cite{11}, there is some interesting similarity. Both here and in \cite{11}, the equations involve sums of the same three terms proportional to $P^2$, $K^2$, and $(K'/P)^2$. In fact, there exists a linear transformation of (12) and (18),

$$z = z_n + \frac{1}{4} \Phi_n, \quad z_n = \frac{3}{4} z - \frac{1}{8} \Phi,$$

$$\Phi = \frac{3}{2} \Phi_n - 2z_n, \quad \Phi_n = \frac{1}{2} \Phi + z,$$ \hspace{1cm} (19)

such that the new differential equations (23) and (24) correspond almost precisely with those in \cite{11}. To take advantage of the calculations in \cite{11} and \cite{5}, we shall use the transformed variables $z_n$ and $\Phi_n$ in what follows. Note that if $\Phi_n'' = 0$, then $z'' = -\Phi''/2$. Moreover, in the extremal case $x = 0$, $z = -\Phi/2$ and $h^{-3/4} = e^{3z} = h^{-1/2}e^{-\Phi/2}$. Hence, we see that the Einstein frame metric (3) in this particular case can be obtained from the string frame metric through the usual procedure of multiplying by a factor of $e^{-\Phi/2}$.

To derive similar equations for $y$, $w_1$, and $w_2$, we need to take linear combinations of the Einstein equations involving

$$R_{\psi \psi} = e^{z-14y}(\frac{1}{2}z'' - y' + 6w'') + 2e^{z-2y}(e^{-16w_1-8w_2} + 2e^{-16w_1+4w_2}),$$

$$R_{\theta_1 \theta_1} = e^{z-14y}(\frac{1}{2}z'' - y' - w_1'' + w_2'') + 2e^{z-2y}(-e^{-16w_1+4w_2} + 4e^{-2w_1+2w_2}),$$

$$R_{\theta_2 \theta_2} = e^{z-14y}(\frac{1}{2}z'' - y' - w_1'' - 2w_2'') + 2e^{z-2y}(-e^{-16w_1-8w_2} + 4e^{-2w_1-4w_2}).$$
We leave it to the reader to derive the field strength contributions \([\mathbf{13}]\) to Einstein’s equations and give only the results

\[
7y'' = -\Phi_n'' + 2e^{12y}(-e^{-16w_1-8w_2} - 2e^{-16w_1+4w_2} + 8e^{-2w_1-4w_2} + 16e^{-2w_1+2w_2}), \tag{20}
\]

\[
7w_1'' = \Phi_n'' + \frac{8}{3}e^{12y}(-e^{-16w_1-8w_2} - 2e^{-16w_1+4w_2} + e^{-2w_1-4w_2} + 2e^{-2w_1+2w_2}), \tag{21}
\]

\[
2w_2'' = -\Phi_n'' - \frac{4}{3}e^{12y}(-e^{-16w_1-8w_2} - e^{-16w_1+4w_2} - 4e^{-2w_1-4w_2} + 4e^{-2w_1+2w_2}), \tag{22}
\]

\[
6\Phi_n'' = P^2 e^{4z_n+4y+\Phi_n+4w_1-4w_2} - \frac{(K')^2}{P^2} e^{4z_n-4y-\Phi_n-4w_1+4w_2}, \tag{23}
\]

\[
8z_n'' = 2K^2 e^{8z_n} + P^2 e^{4z_n+4y+\Phi_n+4w_1-4w_2} + \frac{(K')^2}{P^2} e^{4z_n-4y-\Phi_n-4w_1+4w_2}, \tag{24}
\]

\[
(e^{4z_n-4y-\Phi_n-4w_1+4w_2} K')' = P^2 Ke^{8z_n}. \tag{25}
\]

As yet, we have ignored the \(R_{uu}\) Einstein equation. For our metric

\[
R_{uu} = e^{z-14y} \left(\frac{1}{2} z'' - 6(z')^2 - 7y'' + 42(y')^2 - 2(x')^2 - 42(w_1')^2 - 12(w_2')^2\right) \tag{26}
\]

which, using \([20], [24], [23], [19] and [17]\), yields the zero energy constraint

\[
84(y'')^2 - 16(z')^2 - 3(\Phi_n')^2 - 84(w_1')^2 - 24(w_2')^2 + 6\Phi_n'' + K^2 e^{8z_n} - 4e^{12y}(-e^{-16w_1-8w_2} - 2e^{-16w_1+4w_2} + 8e^{-2w_1-4w_2} + 16e^{-2w_1+2w_2}) = 4a^2. \tag{27}
\]

In later sections, it will be important to keep track of the dimensions of the various parameters involved. From the form of the metric \([1]\) it is natural to require that \(e^y\) and \(u^{-1/6}\) should have dimension of length, while \(x, z, w\) be dimensionless. Since we have set the 10-d gravitational constant to be 1 (i.e. we measure the scales in terms of the 10-d “Planck scale” \(L_P \sim (g_s\alpha'^2)^{1/4}\)), then from \([27]\), we conclude that \(K\) and \(Q\) in \([10]\) have dimension (length)\(^6\) while \(P\) has dimension (length)\(^4\) and \(f\) has dimension (length)\(^2\). It is easy to restore the dependence on the Planck length by rescaling \(Q \rightarrow L_P^6 Q\), \(P \rightarrow L_P^4 P\), etc. To restore the dependence on the string coupling one should further rescale \(P^2 \rightarrow g_s P^2\). At the end, \(Q \sim g_s\alpha'^3 N\), \(P \sim g_s\alpha'^2 M\), where \(N\) and \(M\) are the number of ordinary and fractional D-strings, respectively.

### 4 Three Simple Solutions

In addition to the extremal D-string solution, there are three other relatively simple solutions to the system of equations \([21]–[23], [27]\). Some of these solutions were discussed briefly in the Introduction and in Section 2. There is the analog of the KT
solution for fractional D-strings, what we will call the extremal fractional D-string solution. Next, there is the non-extremal ordinary D-string solution. These two solutions will be very important when we try to find a non-extremal fractional D-string solution through perturbing in the number of $P$ of fractional D-strings. We will find a solution which interpolates between the extremal fractional D-string solution in the UV and the non-extremal ordinary D-string solution in the infrared.

The third solution is more of a mathematical curiosity. It is the analog of the singular, non-extremal D3-brane solution found in [10]. This third solution, although non-extremal, is singular because it has a naked singularity in the far infrared. Although it approaches the extremal fractional D-string solution as the non-extremality parameter $a \to 0$, it does not approach the non-extremal ordinary D-string solution as the number of fractional D-strings $P \to 0$.

4.1 Singular Non-Extremal Fractional D-Strings

As a first attempt at finding a non-extremal solution, we might be tempted to try to preserve the geometry of the $Q^{1,1,1}$ base space by turning off the squashing functions $w_i$; this approach will lead us to the D-string analog of the Buchel solution. Our motivation is two-fold. First, because this solution is singular, we will see the necessity of squashing the $Q^{1,1,1}$. Second, to obtain the extremal fractional D-strings in the next subsection, we need only take the limit in which the non-extremality parameter $a \to 0$.

So let us suppose that the $Q^{1,1,1}$ is not squashed, i.e. $w_1 = w_2 = 0$. Then the equations (20)–(25), (27) simplify dramatically. From (22), we have $\Phi'_n = p$, where $p$ is a constant, so (23) becomes

$$f' = -Pe^{4y+\Phi_n}.$$  \hfill (28)

From (25), it is straightforward to see that $z_n$ must then also satisfy a first order equation:

$$(e^{-4z_n})' = Q + Pf.$$ \hfill (29)

Then (20) reduces to $y'' = 6e^{12y}$. The zero-energy constraint (27) sets one of the integration constants of this differential equation:

$$y' = -\sqrt{b^2 + e^{12y}}, \quad 84b^2 = 4a^2 + 3p^2,$$ \hfill (30)

which integrates to give $e^{6y} = \frac{b}{\sinh 6b}$. Without loss of generality, we may choose $b > 0$. The differential equation for $f$ (28) becomes

$$f' = -Pe^{(p-4b)y} \left( \frac{2b}{1 - e^{-12by}} \right)^{2/3}.$$ \hfill (31)

Once $f$ is known, $z_n$ is easily found by integrating (29).
The precise analog with the Buchel solution is found by taking $p = -2a/3$ which in turn implies that $b = 2a/3\sqrt{7}$ (30), but we shall keep $p$ and $b$ general in the discussion that follows. To demonstrate the pathological nature of these solutions, let us take a look at the Ricci scalar:

$$R = \frac{1}{4} e^{z_n - \frac{14y + pu}{4}} \left( P^2 e^{4y + 4z_n + pu} - \frac{1}{2} \left( (Q + P f)^2 e^{8z_n} - 6p(Q + P f) e^{4z_n} - 9p^2 \right) \right)$$

(32)

In analogy with the behavior of the Buchel solution for D3-branes, we expect this solution will exhibit divergences at large $u$, so we develop asymptotics for $f$ and $z_n$ far in the IR:

$$f = f_0 - \frac{P(2b)^{2/3}}{p - 4b} e^{(p - 4b)u} + O(e^{(p - 16b)u})$$

(33)

and hence

$$e^{-4z_n} = C_1 + u(Q + P f_0) - \frac{P^2(2b)^{2/3}}{(p - 4b)^2} e^{(p - 4b)u} + O(e^{(p - 16b)u})$$

(34)

where $C_1$ is some integration constant. The asymptotics are clearly very sensitive to the relative magnitudes of $p$ and $b$. Hence we will consider the two cases, $p - 4b \geq 0$ and $p - 4b < 0$ separately.

In the case $p - 4b \geq 0$, the exponential term in (34) is dominant and causes $e^{-4z_n}$ to be large and negative at large $u$. The subsequent exponential terms contribute very little because of the constraint (30): in particular, in order for $a$ to be real, $|p| \leq \sqrt{28b}$ and hence $p - 16b < 0$. The function $e^{-4z_n}$ is continuous in the region $0 < u < \infty$, and note that near $u = 0$, $e^{-4z_n} \sim C_2 + u(Q + P f_0) + O(u^{4/3})$. Hence, close to $u = 0$, $e^{-4z_n}$ is nonnegative (in order to have well behaved asymptotics at large distances) and growing. From these facts, it is clear that $e^{-4z_n}$ becomes zero at some finite $u_{cr}$ and the Ricci scalar blows up. Defining the horizon as the locus where $G_{00} = \exp(3z_n - 2au + 3pu/4)$ vanishes, a horizon will only occur, if at all, in the limit $u \to \infty$. Hence, the singularity is naked and the solution is pathological.

For the case $p - 4b < 0$, the constant and linear terms in (33) and (34) dominate at large $u$. Hence, $f \sim f_0$ and $e^{-4z_n} \sim u$. Also, $e^{-y} \sim e^{bu}$. As was noted previously, in order for $a$ to be real, $\sqrt{28b} \geq |p|$. Hence, the $e^y$ terms will dominate the Ricci scalar and send it off to infinity at large $u$, even if we consider the limit $P \to 0$. In other words, this solution has a naked singularity in the IR, even in the absence of fractional D-strings. The best we can do is to set the relative magnitudes of the $a$ and $p$ so that there is also a horizon at $u = \infty$. These considerations show that leaving the $Q^{1,1,1}$ unsquashed leads to a naked singularity, so to find a well behaved non-extremal fractional D-string solution, we will have to look elsewhere.
4.2 The Extremal Fractional D-string Solution

As was mentioned above, the limit $a, p \to 0$ of the singular non-extremal D-string solution recovers the extremal fractional D-string solution. Let us take this limit:

\[ \Phi_n \to 0, \quad e^{6y} \to \frac{1}{6u}, \quad f \to f_0 - \frac{6^{1/3}P}{2}u^{1/3}, \quad e^{-4z_n} \to C_1 + (Q + Pf_0)u - \frac{3 \cdot 6^{1/3}P^2}{8}u^{4/3}. \]  

(35)

(36)

This solution is well behaved in the UV, in the limit $u \to 0$. If $C_1 = 1$, then the metric (8) becomes asymptotically flat at large distance $u \to 0$. However, in what follows, it will be more convenient to choose $C_1 = 0$, thus eliminating the asymptotically flat regime, “zooming in” on the low-energy dynamics of the dual gauge theory. As was noted in [12], the solution gives rise to a naked singularity in the infrared at $u_{cr}^{1/3} = (Q+Pf_0)4^{2/3}/9P^2$.

In analogy with [11], we shall define $L_P$ as the value of $u^{-1/6}$ at which this solution develops a singularity.

4.3 The Non-Extremal Ordinary D-String Solution

We are searching for a solution without fractional D-strings, so we set $f = P = 0$. Moreover, we know from the literature [19] that this solution does not require the extra degrees of freedom provided by $w_1$ and $w_2$ so we set $w_1 = w_2 = 0$. The system of equations (20)–(25), (27) simplifies substantially in this case and becomes easily integrable:

\[ y'' = 6e^{12y}, \quad z''_n = \frac{1}{4}Q^2 e^{8z_n}, \quad x'' = 0, \quad \Phi_n'' = 0. \]  

(37)

Integrating these equations once, we find

\[ x' = a, \quad \Phi'_n = p, \quad y'^2 = b^2 + e^{12y}, \quad z'^2_n = c^2 + \frac{Q^2}{16}e^{8z_n}. \]  

(38)

From the constraint (27), we find that the integration constants must obey $a^2 - 21b^2 + 4c^2 + \frac{3}{4}p^2 = 0$. Integrating the equations for $y$ and $z_n$, we find that

\[ e^{6y} = \frac{b}{\sinh 6bu}, \quad e^{4z_n} = \frac{4c}{Q \sinh 4c(u + k)}. \]  

(39)

Cast in a more familiar form,

\[ \rho^6 = e^{6y + 2x} = \frac{2be^{-2(3b-a)u}}{1 - e^{-12bu}}, \quad A(u) = e^{-4x} = e^{-4au}. \]  

(40)

(41)
As has become customary in gauge/gravity duality, we set the boundary condition that $h$ approach zero as $u \to 0$. In other words $k = 0$. This boundary condition removes the asymptotically flat region so that we “zoom in” on the low-energy dynamics of the dual gauge theory.

To match the standard non-extremal D-string solution, we require that $3b = 2c = a$, and $p = -2a/3$. This choice satisfies the zero-energy constraint and (40)–(42) become

$$h(u) = e^{-4zn - \Phi_n - 8x/3} = \frac{Q}{4c} e^{-pau - 8au/3} \sinh 4c(u + k) \ .$$

(42)

In other words, we recover the standard non-extremal D-string metric. To summarize, the non-extremal ordinary D-string solution consists of $w_i = 0$ and

$$\Phi_n = -2au/3, \quad e^{4x} = e^{4au}, \quad e^{-6y} = 3a^{-1} \sinh 2au, \quad e^{-4zn} = \frac{Q}{2a} \sinh 2au \ .$$

(44)

5 Asymptotics of the Regular Non-Extremal Fractional D-String

We have not succeeded in finding an analytic solution to the system of differential equations (20)–(25), (27). To proceed further, we can either integrate the equations numerically or seek an approximate solution in perturbation theory. In any case we must be certain that our solution satisfies the correct boundary conditions in the short-distance ($u \to \infty$) and long-distance ($u \to 0$) limits, where we understand the physics. The procedure we will follow is very similar to that in [5].

For $P \to 0$ we must obtain the black D-string solution, which has a regular Schwarzschild horizon. If the horizon is preserved as we add fractional D1-branes, we expect the following asymptotics to hold as $u \to \infty$ (44):

$$x = au, \quad y \to -au/3 + y_s, \quad z_n \to -au/2 + z_s, \quad w_i \to w_{i*}, \quad \Phi_n \to -2au/3 + \Phi_s, \quad K \to K_s \ .$$

(45)

The metric for $u \to \infty$ in the $u - X_0$ directions is given by

$$ds^2 = e^{-4au + 3z_s + \frac{3}{4}\Phi_s} dX_0^2 + e^{-4au - z_s - \frac{3}{4}\Phi_s + 14y_s} du^2$$

(46)

so that with the natural near-horizon variable $U = e^{-2au}$, the usual procedure of choosing periodicity of the Euclidean time $X_0$ to avoid a conical singularity fixes the Hawking temperature $T_H$: 11
\[ T_H = \frac{a}{\pi} e^{2z_n - 7y_n + \Phi_n/2}. \] (47)

At large distances \((u \to 0)\) the non-extremal solution should approach the extremal solution (35), (36), i.e. we require that

\[ u \to 0 : \ x, w, \Phi_n \to 0, \ y \to -\frac{1}{6} \log 6u. \] (48)

(Note that this behavior is also in agreement with the small \(u\) asymptotics (44) of the regular non-extremal D-string solution.) The behaviors of the effective D-string charge and of the warp factor at small \(u\) are

\[ K(u) \to \frac{6^{1/3} P^2}{2} \left( \frac{3}{4} L_p^{-2} - u^{1/3} \right), \quad e^{-4z_n} \to \frac{3 \cdot 6^{1/3} P^2}{8} u \left( L_p^{2} - u^{1/3} \right). \] (49)

We expect that the physics of this fractional D-string system should be very similar to that of the fractional D3-brane solution considered in [5]. On the supergravity side, when \(T < T_c\) the solution which preserves the \(U(1)\) symmetry of \(Q^{1,1,1}\) is singular and one needs an appropriate deformation, perhaps of the KS-type [4], that breaks this \(U(1)\) symmetry and removes the singularity. For \(T > T_c\) we should be able to construct a nonsingular solution which preserves the \(U(1)\) chiral symmetry, and we will do this in perturbation theory in the next section.

The flux corresponding to the number of D-strings is given by \( *F_3 = K(u)e^{-\Phi} \omega_2 \wedge \omega_5 \), so on the gauge theory side, we want to think of \(K(u)e^{-\Phi}\), the effective D-string charge, as an effective number of unconfined color degrees of freedom. As we run the scale of the theory into the infrared \((u \to \infty)\), the number of colors should decrease. Above the critical temperature, this number will be positive everywhere, but below the critical temperature, the effective number of colors will vanish at finite \(u\). This behavior is the same as for fractional D3-branes. One potentially bothersome difference from the D3-brane case is the dependence of the D-string flux on the dilaton. However, things are all right because

\[ e^{-\Phi} = e^{-\frac{3}{2}\Phi_n + 2z_n} \propto \frac{e^{au}}{\sqrt{\sinh 2a(u + k)}} \] (50)

is decreasing for all positive \(u\). Thus if \(K(u)\) decreases with increasing \(u\), the flux will still decrease as well, provided that the fractional D-strings are a small enough perturbation that the variation of the dilaton is dominated by the presence of the ordinary D-strings. Moreover, \(K(u)e^{-\Phi}\) is well-defined for all values of \(u\), in particular as \(u \to \infty\). Notice that in (50) we have temporarily restored the integration constant \(k\) from (39). In Section 4.3, we set \(k = 0\) to zoom in on the IR physics. This approximation introduces a singularity at \(u = 0\) \(e^{-\Phi}\) diverges but this is strictly an artifact of having removed the asymptotically flat region.
While the number of colors decreases as we move toward the horizon, from (50) one can see that the string coupling $e^\Phi$ increases. From [20], we expect that the string coupling can be expressed in terms of gauge theory parameters as $e^\Phi \sim g_{YM}^3 N^{1/2}/\Lambda^3$ where $\Lambda$ sets the energy scale of the gauge theory, and $N \sim K e^{-\Phi}$ gives the number of ordinary D-strings at a given scale. Hence, $e^\Phi \sim g_{YM}^3 K^{1/3}/\Lambda^2$. In the case where $K$ is constant, i.e. there are no fractional D-strings, we expect the string coupling to become larger in the IR. Indeed, (50) holds exactly in this case. However, once we add fractional D-strings, $K$ should decrease as $u \to \infty$. Thus, in perturbation theory, we expect to see corrections to (50) that tend to decrease $e^\Phi$ as $u \to \infty$.

6 Perturbation Theory in $P$

One useful approach to constructing the required non-extremal fractional brane solution is to start with the non-extremal ordinary D-string solution (44), which is valid for $P = 0$, and find its deformation order by order in $P^2$. A remarkable feature of perturbation theory in $P^2$ near the extremal ($a = 0$) D-string background is that already the first-order correction gives the exact form of the extremal fractional D-string solution (35), (36). Therefore, it is natural to expect that a similar expansion near the non-extremal D-string solution will capture the basic features of the non-extremal generalization of the extremal fractional D-string.

More precisely, our starting point will be the well-known non-extremal ordinary D-string solution (44) with $Q$ replaced by the effective D-string charge $K_*$, so that we automatically match onto the near horizon asymptotics (45). Perturbing in $P^2$ around the non-extremal D-string solution of charge $K_*$, we will see that the $O(P^2)$ modification is already enough to match onto the extremal fractional D-string long-distance asymptotics. The small parameter governing this expansion is actually the dimensionless ratio $\lambda \equiv P^2 K_*^{-1} a^{-1/3}$, i.e. for this method to work the horizon value of the effective D-string charge $K_*$ has to be sufficiently large. In view of the discussion in Section 5, this means that this perturbation theory is applicable for $T \gg T_c$. Unlike [3], $\lambda$ here depends also on the non-extremality parameter $a$.

It is useful to rescale the fields by appropriate powers of $P^2$, setting

$$K(u) = K_* + P^2 F(u) \ , \ \ \Phi_n(u) = -2au/3 + P^2 \phi(u) \ , \ \ w_i(u) = P^2 \omega_i(u) \ , \ (51)$$

and

$$y \to y + P^2 \xi \ , \ \ z_n \to z_n + P^2 \eta \ , \ (52)$$

where $y$ and $z$ represent the pure D-string solution (44): $e^{-6y} = 3a^{-1} \sinh 2au$, $e^{-4z_n} = \frac{K_*}{2a} \sinh 2au$, and $\xi$ and $\eta$ are corrections to it. To match onto the small $u$ extremal

---

3 Note that $P \sim g_s M$ and $K \sim g_s N$ where $M$ and $N$ are the numbers of fractional and regular D-strings respectively.
fractional D-string asymptotics (19), we require that
\[ \omega_i(0) = \xi(0) = \phi(0) = 0 , \quad F \rightarrow -\frac{(6u)^{1/3}}{2} , \quad \eta \rightarrow \frac{u^{1/3}L_0^2}{4} . \] (53)

Now the system (20)–(25) takes the following explicit form:
\begin{align*}
7\xi'' - \phi'' - 504e^{12y}\xi + O(P^2) &= 0 , \quad (54) \\
7\omega'' - \phi'' - 112e^{12y}\omega_1 + O(P^2) &= 0 , \quad (55) \\
2\omega'' + \phi'' + 16e^{12y}\omega_2 + O(P^2) &= 0 , \quad (56) \\
6\phi'' + e^{4z_n-4y+2au/3}(F'^2 - e^{8y-4au/3}) + O(P^2) &= 0 , \quad (57) \\
(e^{4z_n-4y+2au/3}F')' - K_0 e^{8z_n} + O(P^2) &= 0 \quad (58) \\
8\eta'' - 4K_0 e^{8z_n}F - 16K_0^2 e^{8z_n}\eta - e^{4z_n-4y+2au/3}(F'^2 + e^{8y-4au/3}) + O(P^2) &= 0 . \quad (59)
\end{align*}

The constraint (27) becomes
\begin{align*}
168y'\xi' - 32z'\eta' + 4a\phi' - e^{4z_n-4y+2au/3}(F'^2 - e^{8y-4au/3}) \\
+ 8K_0^2 e^{8z_n}\eta + 2K_0 e^{8z_n}F - 84 \cdot 12e^{12y}\xi + O(P^2) &= 0 . \quad (60)
\end{align*}

### 6.1 Leading-order solution for \( K \)

Using the fact that \( K_0 e^{8z_n} = 4K_0^{-1}z_n'' \) (17), we get from (58) that
\[ F' = e^{-4z_n-2au/3+4y}(C + 4K_0^{-1}z_n'). \] (61)

For large \( u \) (near the horizon), we must have \( F' \rightarrow 0 \) in order to satisfy (13). This along with the explicit form of \( z_n \) (14) fixes the integration constant to be \( C = 2a/K_0 \). Hence
\[ F' = -e^{-4au}\left(\frac{2a}{3(1-e^{-4au})}\right)^{2/3} , \] (62)

and thus
\[ F = \frac{3}{4a^{1/3}}\left(\frac{2}{3}\right)^{2/3}\left(1 - (1-e^{-4au})^{1/3}\right) . \] (63)

As required by \( K(u) = K_0 + P^2 F(u) \) (51), this expression satisfies \( F(u \rightarrow \infty) \equiv F_* = 0 \). In other words, the fractional D-string charge \( K(u) \rightarrow K_* \) in the large \( u \) limit, as desired. Notice that \( F' < 0 \) so that, as advertised in Section 5, \( K(u) \) decreases as \( u \) increases. Moreover, if \( P^2 K_0^{-1}a^{-1/3} \ll 1 \), the perturbation caused by \( F(u) \) is small for all values of \( u \). Even at small \( u \), \( F \) is well behaved
\[ F(u) = \frac{3}{4a^{1/3}}\left(\frac{2}{3}\right)^{2/3} - \frac{6^{1/3}}{2} u^{1/3} + O(u^{4/3}) . \] (64)
In addition, from these small $u$ asymptotics, we find that at small $u$, we have recovered the extremal fractional D-string solution \(^{[35]}\)!

Thus already at the leading order this perturbation theory produces a solution with the correct extremal fractional D-string asymptotics. This remarkable fact strengthens our confidence that an exact solution interpolating between the extremal fractional D-string at small $u$ and the regular non-extremal D-string horizon at large $u$ indeed exists.

Our perturbed solution should be a good approximation to it provided that $P^2 K_*^{-1} \ll a^{1/3}$. This limit corresponds to high Hawking temperatures, as we now show by matching \((64)\) with \((49)\) for small $u$. We find that

$$\frac{3 \cdot 6^{1/3}}{8} L_p^{-2} K_* = \frac{K_*}{P^2} + \left(\frac{2}{3}\right)^{2/3} \frac{3}{4 a^{1/3}}.$$  \hfill (65)

On the other hand, the Hawking temperature is determined in terms of the non-extremality $a$ and the charge near the horizon $K_*$ by the usual D-string formula \((47)\)

$$T = \frac{3}{\pi} \left(\frac{3}{2}\right)^{1/6} a^{1/3} K_*^{-1/2}.$$  \hfill (66)

Comparing with the extremal fractional D-string solution, we expect that the critical temperature roughly corresponds to the value of $a$ for which the naked singularity of the extremal fractional D-string solution and the horizon of the regular non-extremal fractional D-string coincide. In other words, $a^{1/3} \sim L_p^2$. Thus, the perturbation theory will correspond to high Hawking temperatures so long as $a^{1/3} \gg L_p^2$. Comparing with \((65)\), we recover the naive inequality needed for our perturbation theory to be valid, namely $P^2 / K_* \ll a^{1/3}$. To summarize, in our perturbative regime, $T \gg T_c$.

### 6.2 Solutions for other fields

We begin by solving for the correction to the dilaton $\phi$. Using \((62)\) and \((14)\), the equation for the dilaton correction \((57)\) becomes

$$\phi'' = \frac{1}{K_*} \left(\frac{2a}{3}\right)^{5/3} \frac{e^{-4au}}{(1 - e^{-4au})^{2/3}}.$$  \hfill (67)

Integrating once, and keeping in mind the boundary condition $\phi' \to 0$ as $u \to \infty$, we find

$$\phi' = -\frac{2a}{3K_*} F = \frac{3}{4 a K_*} \left(\frac{2a}{3}\right)^{5/3} (1 - e^{-4au})^{-1/3} - 1.$$  \hfill (68)

Hence, $\phi$ is a decreasing function of $u$. In other words, the string coupling decreases as we approach the horizon. To solve for $\phi$, we may integrate once more:

$$\phi = \frac{1}{8 K_*} \left[ -3 v^{1/3} + \frac{3}{2} \log \frac{1 - v}{1 - v^{1/3}} + \sqrt{3} \tan^{-1} \frac{2 v^{1/3} + 1}{\sqrt{3}} - \frac{\pi \sqrt{3}}{6} \right]$$  \hfill (69)
From (18), we have fixed the boundary condition such that \( \phi = 0 \) at \( u = 0 \). Moreover, to make the formula neater, we have introduced

\[
v = 1 - e^{-4au}
\]
(70)

and

\[
\frac{1}{k^*} \equiv \frac{1}{K^*a^{1/3}} \left( \frac{2}{3} \right)^{2/3}.
\]
(71)

To show that \( \phi \) is well behaved at both small \( u \) and large \( u \), we can write down some limiting expressions

\[
u \to \infty : \quad \phi = \phi_* + \frac{1}{24k^*}(1 - v) + O[(1 - v)^2],
\]
(72)

\[
u \to 0 : \quad \phi = -\frac{1}{8k^*}v + \frac{3}{32k^*}v^{4/3} + +O(v^2),
\]
(73)

where

\[
\phi_* = \frac{1}{8k^*} \left( -3 + \frac{\pi \sqrt{3}}{6} + \frac{3}{2} \log 3 \right).
\]
(74)

The relations describing the corrections to the other fields (54)–(56), (59) are all ordinary, second order, linear differential equations. There exists a powerful technique, called the Lagrange method of variation of parameters, which is useful for dealing with this type of ODE. In particular, given an ODE of the form

\[
\frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = g(x),
\]
(75)

and two linearly independent solutions \( y_1 \) and \( y_2 \) to the corresponding homogenous equation \( g(x) = 0 \), we can construct a general solution corresponding to the case \( g(x) \neq 0 \):

\[
Y = -Y_1 \int \frac{Y_2 g}{W} dx + Y_2 \int \frac{Y_1 g}{W} dx + c_1 Y_1 + c_2 Y_2.
\]
(76)

where \( W = Y_1 \frac{dy_2}{dx} - \frac{dy_1}{dx} Y_2 \) is the Wronskian. (Linear independence corresponds to the fact that \( W \neq 0 \).)

As a first step, we cast the differential equations for \( \omega_i, \xi, \) and \( \eta \), (54)–(56) and (59), into the form (75), i.e.

\[
\omega''_1 - \frac{64a^2}{9} \frac{e^{-4au}}{(1 - e^{-4au})^2} \omega_1 = \frac{2a^2}{21k^* (1 - e^{-4au})^{2/3}},
\]
(77)

\[
\omega''_2 + \frac{32a^2}{9} \frac{e^{-4au}}{(1 - e^{-4au})^2} \omega_2 = \frac{a^2}{3k^* (1 - e^{-4au})^{2/3}},
\]
(78)
\[
\xi'' - \frac{32a^2 e^{-4au}}{(1 - e^{-4au})^2} \xi = -\frac{2a^2}{21k_*} \frac{e^{-4au}}{(1 - e^{-4au})^{2/3}},
\]

\[
\eta'' - \frac{32a^2 e^{-4au}}{(1 - e^{-4au})^2} \eta = \frac{a^2}{2k_*} \left[ \frac{12e^{-4au}}{(1 - e^{-4au})^2} - \frac{11e^{-4au}}{(1 - e^{-4au})^{5/3}} + \frac{e^{-8au}}{(1 - e^{-4au})^{5/3}} \right].
\]

To analyze these differential equations, we again introduce the new radial variable \( v = 1 - e^{-4au} \). Then, (77)–(80) become

\[
\begin{align*}
\dot{\omega}_1 &- \frac{\omega_1}{1 - v} - \frac{4/9}{v^2(1 - v)} \omega_1 = \frac{1}{168k_* v^{2/3}(1 - v)}, \\
\dot{\omega}_2 &- \frac{\omega_2}{1 - v} + \frac{2/9}{v^2(1 - v)} \omega_2 = -\frac{1}{48k_* v^{2/3}(1 - v)}, \\
\dot{\xi} &- \frac{\xi}{1 - v} - \frac{2}{v^2(1 - v)} \xi = -\frac{1}{168k_* v^{2/3}(1 - v)}, \\
\ddot{\eta} &- \frac{\eta}{1 - v} - \frac{2}{v^2(1 - v)} \eta = \frac{1}{32k_*} \left[ \frac{12}{v^2(1 - v)} - \frac{11}{v^{5/3}(1 - v)} + \frac{1}{v^{8/3}} \right],
\end{align*}
\]

where the dots denote \( d/dv \). Now, the homogenous equation

\[
\ddot{f} - \frac{\dot{f}}{1 - v} - \frac{A}{v^2(1 - v)} f = 0
\]

is solved for generic \( A \) by \( f(v) = v^{\nu} 2F_1(\nu, \nu; 2\nu; v) \), where \( 2F_1 \) is the hypergeometric function and \( \nu(\nu - 1) = A \). As it happens, \( A = 2 \) is a degenerate case where the solutions to the homogenous equation are elementary functions of \( v \) (namely, \( \frac{1}{v} - \frac{1}{2} \) and \( -2 + \frac{v^2}{v} \log(1 - v) \)). Perhaps surprisingly, we have been able to find analytic expressions for \( \xi \) and \( \eta \) using these homogenous solutions. Let us start with \( \xi \). In particular

\[
\xi = \frac{1}{56k_* v} \left[ 3v^{1/3}(v - 4) + (v - 2) \log \frac{(1 - v^{1/3})^3}{1 - v} + 2\sqrt{3}(2 - v) \tan^{-1} \frac{2v^{1/3} + 1}{\sqrt{3}} \right] + \]

\[
+ b_1 \left( \frac{1}{v} - \frac{1}{2} \right) + b_2 \left( -2 + \frac{v^2}{v} \log(1 - v) \right).
\]

The boundary conditions for \( \xi \) are that \( \xi = 0 \) at \( v = 0 \) (83) and \( d\xi/du \to 0 \) as \( u \to \infty \) (83). This second boundary condition, in terms of the \( v \) variable, means that \( d\xi/dv \) must be finite or zero in the limit \( v \to 1 \) because \( dv/du = 4a(1 - v) \). As a result, we find that

\[
b_1 = -\frac{3\pi}{84k_*}; \quad b_2 = -\frac{1}{28k_*}.
\]

Next, we solve for \( \eta \):

\[
\eta = \frac{3}{64k_* v} \left[ 6v^{1/3}(v - 2) - 8 + (v - 2) \log \frac{(1 - v^{1/3})^3}{1 - v} + 2\sqrt{3}(2 - v) \tan^{-1} \frac{2v^{1/3} + 1}{\sqrt{3}} \right]
\]
\[ c_1 \left( \frac{1}{v} - \frac{1}{2} \right) + c_2 \left( -2 + \frac{v - 2}{v} \log(1 - v) \right) . \] (88)

Recall at the end of Section 5, we expected that the correction to the original dilaton \( \Phi \) should decrease as \( u \to \infty \), reflecting the corresponding decrease in \( K \). Indeed, from (69) and (88), it is clear that this correction \( 3\phi/2 - 2\eta \) (see (19)) is indeed a decreasing function of \( u \). The boundary conditions for \( \eta \) are again that \( d\eta/du \to 0 \) as \( u \to \infty \) (45). However the small \( u \) boundary condition is merely the fact that \( \eta \) does not diverge at small \( u \). The integration constants which satisfy these conditions are

\[ c_1 = \frac{1}{32k_*} (12 - \pi \sqrt{3}) ; \quad c_2 = -\frac{3}{32k_*} . \] (89)

To show that \( \xi \) and \( \eta \) are well behaved at both large \( u \) and small \( u \), we look at the asymptotics. First, we examine the long distance asymptotics \( (v \to 0 \text{ or equivalently } u \to 0) \) of \( \xi \) and \( \eta \):

\[ \xi = \frac{3}{784k_*} v^{4/3} - \frac{1}{168k_*} v^2 + O(v^{7/3}) , \] (90)

\[ \eta = -\frac{3}{16k_*} + \frac{9}{64k_*} v^{1/3} + \frac{9}{896k_*} v^{4/3} - \frac{1}{64k_*} v^2 + O(v^{7/3}) . \] (91)

The first and second terms in the \( \eta \) expansion agree with the extremal fractional D-string solution, as one can see by comparing with (65) and (49).

Next we consider the expansions of \( \xi \) and \( \eta \) near the horizon, \( u \to \infty \) or equivalently \( v \to 1 \):

\[ \xi = \xi_* + \left( 2\xi_* - \frac{1}{168k_*} \right) (1 - v) + O[(1 - v)^2] , \] (92)

\[ \eta = \eta_* + \left( 2\eta_* + \frac{1}{32k_*} \right) (1 - v) + O[(1 - v)^2] . \] (93)

The horizon values \( \eta_* \) and \( \xi_* \) are

\[ \xi_* = \frac{1}{168k_*} (-15 + \pi \sqrt{3} + 9 \log 3) , \] (94)

\[ \eta_* = \frac{1}{64k_*} (-18 + \pi \sqrt{3} + 9 \log 3) . \] (95)

We were careful to calculate \( \xi_* \), \( \eta_* \) and \( \phi_* \) because they show up as corrections to observables such as the Hawking temperature (17) and the horizon area. On the other hand, the volume of the compact space, this squashed \( Q^{1,1,1} \), does not depend on the squashing factors \( \omega_1 \) and \( \omega_2 \); the squashing factors cancel from the observables we are interested in calculating. As a result, and because the homogenous solutions to (81) and (82) are more complicated, we shall be more cavalier in our treatment of the asymptotics for \( \omega_1 \) and \( \omega_2 \). At small \( u \), (77) and (78) become

\[ \omega_1 = p_1 u^{4/3} + \frac{a^{4/3} u^{4/3} / 70k_*}{u^{4/3}} + O(u^{7/3}) . \] (96)
\[ \omega_2 = p_2 u^{1/3} + p_3 u^{2/3} - \frac{a^{4/3} z^{2/3}}{8k_+} u^{4/3} + O(u^{7/3}) . \]  

(97)

The \( p_i, i = 1, 2, 3 \), are undetermined constants of integration which can be determined numerically, keeping in mind the boundary conditions \( w_i(0) = 0 \) and \( w_i'(\infty) = 0 \). The large \( u \) asymptotics are

\[ \omega_1 = \omega_{1*} + \left( \frac{4}{9} \omega_{1*} + \frac{1}{168k_+} \right) e^{-4au} + O(e^{-8au}) , \]  

(98)

\[ \omega_2 = \omega_{2*} + \left( -\frac{2}{9} \omega_{2*} - \frac{1}{48k_+} \right) e^{-4au} + O(e^{-8au}) . \]  

(99)

We can now go a bit further than was done in the previous sections and determine the order \( P^2 \) corrections to the temperature and the entropy. The metric (1) can be recast into the form

\[
d s_{10E}^2 = \left( \frac{4a}{K_*} \right)^{3/4} e^{3P^2(\eta + \phi/4)} v^{-3/4} \left[ (1 - v) dX_0^2 + dX_1^2 \right] + \\
\left( \frac{4a}{K_*} \right)^{-1/4} \left( \frac{2a}{3} \right)^{1/3} v^{-1/12} \left( \frac{1}{36} e^{P^2(14\xi - \eta - \phi/4)} \frac{dv^2}{v^2(1 - v)} + e^{P^2(2\xi - \eta - \phi/4)} (dM_7)^2 \right) .
\]  

(100)

Using the large \( u \) asymptotics for \( \xi \) (92), \( \eta \) (93), and \( \phi \) (72), we obtain an explicit expression for the entropy per unit volume divided by the temperature squared:

\[
\frac{S}{VT^2} = \alpha K_*^{3/2} e^{3P^2(7\xi_* - 2\eta_* - \phi_/2)} = \alpha K_*^{3/2} \left( 1 + \frac{3P^2}{8k_*} + O(P^4/k_*^2) \right) ,
\]  

(101)

where \( \alpha \) is a constant of order unity. In the last equality, we have used the values for \( \xi_* \) (44), \( \eta_* \) (53), and \( \phi_* \) (74). Note that the transcendental numbers \( \log 3 \) and \( \pi \) drop out of the linear combination of \( \xi_* \), \( \phi_* \), and \( \eta_* \).

Using (66) and (65), one finds that

\[
K_* \sim \frac{P^2}{L_P^2} \left[ 1 - \frac{2^{5/2} L_P^3}{\pi^{3/2} PT} + \ldots \right] .
\]  

(102)

Hence, from (101),

\[
\frac{S}{VT^2} \sim \frac{P^3}{L_P^3} \left[ 1 - \frac{2^{5/2} L_P^3}{\pi^{3/2} PT} + \ldots \right] .
\]  

(103)

As expected, we find that the entropy ratio \( S/VT^2 \) increases toward a limiting value as \( T \) increases. The important point is that both the number of D-strings at the horizon \( K_* \) and the entropy ratio \( S/VT^2 \) depend in the same way on the temperature. This picture is consistent with gauge theory where the number of D-strings should correspond roughly to the number of degrees of freedom in the theory. As the number of D-strings grows, so should the entropy.
7 Remarks

We have presented, within the framework of perturbation theory in the number of fractional D-strings, a well behaved non-extremal fractional D-string solution. This finite temperature solution breaks supersymmetry but preserves the $U(1)$ symmetry of the transverse conical space. It would be good if an additional KS type solution could be found for these fractional D-strings in a $Q^{1,1,1}$ background. In analogy with the KS solution for fractional D3-branes, we would expect the corresponding solution to involve blowing up the tip of the cone over $Q^{1,1,1}$ in a way that keeps the five-cycle finite but allows the two cycle to become vanishingly small. In addition, we expect this solution would preserve supersymmetry but would break the $U(1)$ symmetry of the $Q^{1,1,1}$. If such a KS type solution is found, then we can use the finite temperature solution found here as evidence for a chiral symmetry breaking phase transition. We leave construction of such a deformation for future work.

Another interesting direction to pursue is construction of a finite temperature solution for the fractional D2-branes. In [13], two examples of a KS type deformation of the fractional D2-branes were found. Each example involves a different six dimensional Einstein manifold – an $S^2$ bundle over either $S^4$ or $\text{CP}^4$. It should be straightforward to introduce appropriate squashing functions to the cones over these two Einstein spaces, thereby producing an ansatz which admits well behaved fractional D2-branes at finite temperature, at least in perturbation theory.

Finally, it would be extremely interesting to find a way to use the systems of differential equations found here (20)–(25), (27) and in [11] to see what happens close to the expected phase transition. Unfortunately, to date we know of no such analytic solutions. Moreover, there are many integration constants involved, some of which are set in the IR, some in the UV, which makes any kind of numerical shooting algorithm a tedious prospect. Still, the authors of the current paper were mildly surprised that the set of differential equations (20)–(25), (27) proved tractable in the first order in perturbation theory, and it seems likely that there are some more surprises waiting.

Acknowledgments

We are grateful to I. R. Klebanov and M. Rangamani for many useful discussions. The work of C. P. H. was supported in part by the DoD. P. O. is supported in part by an NSF Graduate Research Fellowship.

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