BIFURCATION DIAGRAMS OF POSITIVE SOLUTIONS FOR ONE-DIMENSIONAL MINKOWSKI-CURVATURE PROBLEM AND ITS APPLICATIONS

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Abstract. In this paper, we study the classification and evolution of bifurcation curves of positive solutions for one-dimensional Minkowski-curvature problem

\[
- \left( \frac{u'}{\sqrt{1-u'^2}} \right)' = \lambda f(u), \quad \text{in } (-L, L),
\]

\[ u(-L) = u(L) = 0, \]

where \( \lambda > 0 \) is a bifurcation parameter, \( L > 0 \) is an evolution parameter, \( f \in C[0, \infty) \cap C^2(0, \infty) \) and there exists \( \beta > 0 \) such that \((\beta - z) f(z) > 0\) for \( z \neq \beta \). In particular, we find that the bifurcation curve \( S_L \) is monotone increasing for all \( L > 0 \) when \( f(u)/u \) is of Logistic type, and is either C-shaped or S-shaped for large \( L > 0 \) when \( f(u)/u \) is of weak Allee effect type. Finally, we can apply these results to obtain the global bifurcation diagrams in some important applications including ecosystem model.

1. Introduction. In this paper, we study the classification and evolution of bifurcation curves of positive solutions for one-dimensional Minkowski-curvature problem

\[
\begin{align*}
- \left( \frac{u'}{\sqrt{1-u'^2}} \right)' &= \lambda f(u), \quad \text{in } (-L, L), \\
 u(-L) = u(L) &= 0,
\end{align*}
\]

where \( \lambda > 0 \) is a bifurcation parameter, \( L > 0 \) is an evolution parameter, \( f \in C[0, \infty) \cap C^2(0, \infty) \) and there exists \( \beta > 0 \) such that \((\beta - z) f(z) > 0\) for \( z \neq \beta \). It is well-known that studying the exact shape of bifurcation curve \( S_L \) of (1) is equivalent to studying of the exact multiplicity of positive solutions of problem (1) where

\[
S_L = \left\{ (\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \in C^2(-L, L) \cap C[-L, L] \text{ is a positive solution of (1)} \right\}.
\]

Notice that the problem (1) plays an important role in certain fundamental issues in differential geometry and in the special theory of relativity, see for example [3, 5]. We refer the readers, for motivations and results, to [1] and the references cited therein.

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For comparison, we consider the semilinear problem
\begin{equation}
\begin{aligned}
- u'' &= \lambda f(u), \quad \text{in} \ (L, L), \\
u(-L) &= u(L) = 0.
\end{aligned}
\end{equation}
(3)
The solutions of (3) are the steady state solutions of a reaction–diffusion population model in one space dimension. A typical form of reaction–diffusion population model equation is
\[ \frac{\partial}{\partial t} u = d \Delta u + f(u), \]
where \(u(x, t)\) is the population density, \(d > 0\) is the diffusion constant and \(f(u)/u\) is the growth rate per capita. We refer to the work of McCabe, Leach and Needham [14], Shi and Shivaji [17], Wang and Kot [19], and Xin [20] and the references therein. In (3), the constant \(L\) can be scaled out so one defines the bifurcation curve \(\tilde{S}\) of (3) on the \((\lambda, \|u\|_{\infty})\)-plane by
\[ \tilde{S} \equiv \{ (\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (3)} \} . \]
Next, we give some terminologies related to the shapes of bifurcation curves \(S_L\) on the \((\lambda, \|u\|_\infty)\)-plane (while similar terminologies for \(\tilde{S}\) also holds).

**Monotone increasing:** We say that, on the \((\lambda, \|u\|_\infty)\)-plane, the bifurcation curve \(S_L\) is monotone increasing if \(S_L\) is a continuous curve and for each pair of points \((\lambda_1, \|u_{\lambda_1}\|_\infty)\) and \((\lambda_2, \|u_{\lambda_2}\|_\infty)\) of \(S_L\), \(\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty\) implies \(\lambda_1 \leq \lambda_2\), see Figure 1(i).

**⊂-shaped:** We say that, on the \((\lambda, \|u\|_\infty)\)-plane, the bifurcation curve \(S_L\) is \(⊂\)-shaped if \(S_L\) is a continuous curve, initially continues to the left and eventually continues to the right, see Figure 1(ii).

**S-shaped:** We say that, on the \((\lambda, \|u\|_\infty)\)-plane, the bifurcation curve \(S_L\) is \(S\)-shaped if \(S_L\) is a continuous curve, initially continues to the right (or starts from (0, 0)), eventually continues to the right and has a turning point which turns to the left, see Figure 1(ii).

There are many references in studying the bifurcation curve \(\tilde{S}\) of (3), cf. [2, 10, 11, 18, 12, 15, 16, 21, 22]. For instance, Ouyang and Shi [15] obtained the bifurcation diagrams for the problem (3) with nonlinearity
\[ f(u) = u^p - q^q, \quad 1 < p < q. \]
They proved that the corresponding bifurcation curve $\tilde{S}$ is either monotone increasing or $\subset$-shaped. Lee et al. [12] studied the problem (3) with nonlinearity

$$f(u) = u - \frac{u^2}{K} - \frac{cu^2}{1 + u^2}, \quad K, c > 0.$$  \hspace{1cm} (5)

This model describes the steady states of a logistic growth model with grazing in a spatially homogeneous ecosystem. It also describes the dynamics of the fish population with natural predation. They proved that the corresponding bifurcation curve $\tilde{S}$ is $S$-shaped for $c \in (\sqrt{\frac{8}{27}}, 2)$ and large $K > 0$. Huang and Wang [11, Theorem 2.1] studied the problem (3) with nonlinearity

$$f(u) = -au^3 + bu^2 + cu + d, \quad a, b, d > 0 \text{ and } c \geq 0.$$  \hspace{1cm} (6)

They proved that there exists $a_0 > 0$ such that corresponding bifurcation curve $\tilde{S}$ is $S$-shaped for $0 < a < a_0$ and monotone increasing for $a \geq a_0$.

There are a few references in studying the Minkowski-curvature problem, cf. [6, 7, 13]. Recently, Huang [6] studies the shape of bifurcation curve for Minkowski-curvature problem with positive nonlinearity. However, these results can not be applied in problem (1) because $f(u)$ in (1) is not positive. Thus for completeness in research, studying the problem (1) is interesting and essential. In addition, we have some investigations when $g(u) = f(u)/u$ may take one of the following forms:

(i) **Logistic type.** $0 < g(0^+) \leq \infty$, $g'(u) < 0$ on $(0, \beta)$, $g(\beta) = 0$ and $g(u) < 0$ on $[\beta, \infty)$ for some $\beta > 0$.

(ii) **Weak Allee effect type.** $0 \leq g(0^+) < \infty$, $g'(u) > 0$ on $(0, \beta)$, $g'(u) < 0$ on $(\beta, \beta)$, $g(\beta) = 0$ and $g(u) < 0$ on $(\beta, \infty)$ for some $\beta > \beta$.

We prove that the bifurcation curve $S_L$ is monotone increasing for all $L > 0$ when $g$ is of **Logistic type**, and is either $\subset$-shaped or $S$-shaped for large $L > 0$ when $g$ is of **weak Allee effect type**, see Theorems 2.1, 2.2 and Proposition 1. As applications, we use these results to obtain the bifurcation diagrams for the problem (1), (5), the problem (1), (6) and the problem (1) with

$$f(u) = au^q - bu^q + cu, \quad q \geq 1, q > p > 0, \quad a, b > 0 \text{ and } c \geq 0,$$  \hspace{1cm} (7)

see Theorems 3.1–3.3 stated below. Obviously, (4) is the special case of (7).

Throughout this paper, we let

$$\eta \equiv \lim_{u \to 0^+} \frac{F(u)}{u^2},$$

where $F(u) \equiv \int_0^u f(t)dt$. Then there are seven possibilities:

(C1) $\eta = 0$.
(C2) $\eta = \infty$.
(C3) $\eta \in (0, \infty)$ and $f''(0^+) \in (0, \infty]$.
(C4) $\eta \in (0, \infty)$ and $f''(0^+) \in [-\infty, 0]$.
(C5) $\eta \in (0, \infty)$, $f''(0^+) = 0$ and $f^{(3)}(0^+) = \infty$ (if $f^{(3)}(u)$ exists).
(C6) $\eta \in (0, \infty)$, $f''(0^+) = 0$ and $f^{(3)}(0^+) \in [-\infty, 0]$ (if $f^{(3)}(u)$ exists).
(C7) $\eta \in (0, \infty)$, $f''(0^+) = 0$ and $f^{(3)}(0^+) \in (0, \infty)$ (if $f^{(3)}(u)$ exists).

The paper is organized as follows. Section 2 contains statements of main results. Section 3 contains three important applications. Section 4 contains preparatory lemmas. Finally, section 4 contains the proofs of the main results.
2. Main results. In this section, we present our main results. First, we classify the bifurcation curves $S_L$ of (1) for $L > 0$.

**Theorem 2.1.** Consider (1). Then the bifurcation curve $S_L$ of (1) is continuous on the $(\lambda, \|u_\lambda\|_\infty)$-plane, starts from the point $(\kappa, 0)$ and goes to $(\infty, m_{L, \beta})$ for $L > 0$ where

$$
\kappa \equiv \begin{cases} 
\infty & \text{if } \eta = 0, \\
\frac{\pi^2}{8\eta^2} & \text{if } \eta \in (0, \infty), \\
0 & \text{if } \eta = \infty,
\end{cases}
and \quad m_{L, \beta} \equiv \min\{L, \beta\}.
$$

Furthermore,

(i) if one of (C1), (C3) and (C5) holds, then $S_L$ is $C$-shaped for all $L > 0$;
(ii) if one of (C2), (C4) and (C6) holds, then $S_L$ is either monotone increasing or $S$-shaped for $L > 0$. Moreover, $S_L$ is $S$-shaped for large $L > 0$ if the bifurcation curve $\bar{S}$ of (3) is not monotone increasing.
(iii) if (C7) holds, then $S_L$ is $C$-shaped for $L > \tilde{L}$, and is either monotone increasing or $S$-shaped for $L > \tilde{L}$ where

$$
\tilde{L} \equiv \pi \sqrt{\frac{3\eta}{2f''(0^+)}}.
$$

In Theorem 2.1, we find that the shape of bifurcation curve $S_L$ may change with varying $L > 0$. In order to determine the evolution of bifurcation curve $S_L$ with varying $L > 0$, we need the following assumptions (H1)–(H3):

(H1) $f(u)/u < 0$ on $(0, \beta)$ ($f(u)/u$ is of logistic type).
(H2) $f(u)/u < 0$ for small $u > 0$.
(H3) there exists $\tau \in (0, \beta)$ such that $3f(u) + uf'(u) > 0$ for $0 < u \leq \tau$ and $\theta(\alpha) > \theta(u)$ for $\tau \leq \alpha < \beta$ and $0 < u < \alpha$ where

$$
\theta(u) \equiv 2 \int_0^u f(u)du - uf(u).
$$

**Theorem 2.2.** Consider (1). Then the following statements (i) and (ii) hold:

(i) If (H1) holds, the bifurcation curve $S_L$ of (1) is monotone increasing for all $L > 0$.
(ii) If one of the following conditions holds:

(H2), (C4) or ((C6) and (H3)),

then the bifurcation curve $S_L$ of (1) is monotone increasing for small $L > 0$.

**Theorem 2.3.** Consider (1). Assume that (H3) and one of (C2), (C4) and (C6) hold. Then the following statements (i) and (ii) hold:

(i) If the bifurcation curve $\bar{S}$ of (3) is monotone increasing, then the bifurcation curve $S_L$ of (1) is monotone increasing for all $L > 0$.
(ii) If the bifurcation curve $\bar{S}$ of (3) is not monotone increasing, then there exists a number $\tilde{L}$ where

$$
\tilde{L} \begin{cases} 
\geq 0 & \text{if (C2) holds,} \\
> 0 & \text{if either (C4) or (C6) holds,}
\end{cases}
$$

such that the bifurcation curve $S_L$ of (1) is $S$-shaped for $L > \tilde{L}$ and monotone increasing for $0 < L \leq \tilde{L}$. Furthermore, if (C2) and (H2) hold, then $\tilde{L} > 0$, see Figure 2.
Remark 1. (i) Note that $\tilde{L}$ may be 0 when (C2) holds. For instance, we assume that
\[
f(u) = \begin{cases} 
\sqrt{u} \left[ \frac{u}{2} + \sin \left( \frac{u}{2} \right) \right] & \text{for small } u > 0, \\
0 & \text{for } u = 0. 
\end{cases}
\]
Clearly, (C2) holds. By numerical simulations, we find that the bifurcation curve $S_L$ of (1) is S-shaped for all $L > 0$. It implies that $\tilde{L} = 0$.

(ii) The shape of bifurcation curve $\bar{S}$ of (3) has been widely studied. Thus we can apply these results and techniques to determine the monotonicity of the bifurcation curve $\bar{S}$ of (3), see Proposition 1.

Proposition 1. Consider (3). Then the following statements (i) and (ii) hold:

(i) If (H1) holds, then the bifurcation curve $\bar{S}$ of (3) is monotone increasing.

(ii) If $[f(u)/u]' > 0$ for small $u > 0$, then the bifurcation curve $\bar{S}$ of (3) is not monotone increasing.

Remark 2. Assume that $f(u)/u$ is of weak Allee effect type. By Theorems 2.1, 2.3 and Proposition 1(ii), we find that the bifurcation curve $S_L$ is $\subset$-shaped for large $L > 0$ when (C1), (C3), (C5) or (C7) holds, and is S-shaped for large $L > 0$ when (C2), (C4) or (C6) holds.

Theorem 2.4 (See Figure 3). Consider (1). Assume that (C7) and (H3) hold. Then the following statements (i) and (ii) hold.

(i) If the bifurcation curve $S_L$ of (1) is monotone increasing, then the bifurcation curve $S_L$ of (1) is $\subset$-shaped for $L > \hat{L}$ and monotone increasing for $0 < L \leq \hat{L}$.

(ii) If the bifurcation curve $S_L$ of (1) is not monotone increasing, then there exists $\hat{L} \in (0, \bar{L})$ such that the bifurcation curve $S_L$ of (1) is $\subset$-shaped for $L > \hat{L}$, S-shaped for $\hat{L} < L < \bar{L}$ and monotone increasing for $0 < L \leq \hat{L}$.

3. Applications. In this section, we apply Theorems 2.1–2.4 to obtain the bifurcation diagrams for the problem (1), (5), problem (1), (6) and problem (1), (7), respectively.
Theorem 3.1 (See Figure 4). Consider (1), (5). Let
\[ \Phi(K, c) \equiv \frac{4 - c^2}{4} K^4 + c (c^2 - 5) K^3 + (3c^2 + 2) K^2 + 3cK + 1. \]
Assume that \((K, c) \in \Gamma\) where 
\[ \Gamma \equiv \left\{ (K, c) \in \mathbb{R}^2_+ : c \geq \frac{K^2 - 3}{3K} \right\} \cup \left\{ (K, c) \in \mathbb{R}^2_+ : c < \frac{K^2 - 3}{3K} \text{ and } \Phi(K, c) > 0 \right\}. \]
Then the bifurcation curve \(S_L\) is either monotone increasing or S-shaped, starts from the point \((\frac{\pi}{4}, 0)\) and goes to \((\infty, m_{L,\beta})\) for \(L > 0\). Furthermore,

(i) \(S_L\) is monotone increasing for small \(L > 0\).
(ii) If \(Kc < 8\), then \(S_L\) is monotone increasing for all \(L > 0\);
(iii) If \(c \in (\frac{8}{\sqrt{27}}, 2)\) and \(K > 0\) is large, then \(S_L\) is S-shaped for large \(L > 0\).

Remark 3. By elementary analysis, we can prove that \(f(u)\), defined by (5), has exactly one positive zero on \((0, \infty)\) if, and only if, \((K, c) \in \Gamma\). But this proof is easy and rather length, we put it in [9]. Furthermore, by numerical simulations, we find two functions \(\phi_1, \phi_2 \in C(\sqrt{27}, \infty)\) satisfying \(\phi_1(K) > \phi_2(K)\) and \(\Phi(K, \phi_1(K)) = \Phi(K, \phi_2(K)) = 0\) for \(K > \sqrt{27}\), see Figure 4. In [9], we obtain that 
\[ 2 < \phi_2(K) < \frac{2K^2 + 18}{9K} < \phi_1(K) < \frac{K^2 - 3}{3K}. \]

Theorem 3.2 (See Figure 5). Consider (1), (6). Then the bifurcation curve \(S_L\) starts from \((0, 0)\) and goes to \((\infty, m_{L,\beta})\) for \(L > 0\). Furthermore, there exists \(a_0 > 0\) such that

(i) If \(a \geq a_0\), then \(S_L\) is monotone increasing for all \(L > 0\).
(ii) If \(0 \leq a < a_0\), then there exists \(\tilde{L}_a > 0\) such that \(S_L\) is S-shaped for \(L > \tilde{L}_a\) and monotone increasing for \(0 < L \leq \tilde{L}_a\).

Theorem 3.3 (See Figure 6). Consider (1), (7). Then the bifurcation curve \(S_L\) starts from \((\kappa, 0)\) and goes to \((\infty, m_{L,\beta})\) for \(L > 0\) where

\[ \kappa \equiv \begin{cases} 0 & \text{if } 0 < p < 1 \text{ and } c \geq 0, \\ \frac{c^2}{4(\alpha + c)L^2} & \text{if } p = 1 \text{ and } c \geq 0, \\ \frac{c^2}{\pi^2L^2} & \text{if } p > 1 \text{ and } c > 0, \\ \infty & \text{if } p > 1 \text{ and } c = 0. \end{cases} \]

Furthermore,
Figure 4. Graphs of bifurcation curve $S_L$ of (1), (5). $\phi_1, \phi_2 \in C(\sqrt{27}, \infty)$ satisfy $\phi_1(K) > \phi_2(K)$ and $\Phi(K, \phi_1(K)) = \Phi(K, \phi_2(K)) = 0$.

Figure 5. Graphs of bifurcation curve $S_L$ of (1), (6).

(i) Assume that $c = 0$. Then, for $L > 0$, $S_L$ is monotone increasing if $0 < p \leq 1$ and $\subset$-shaped if $p > 1$.

(ii) Assume that $c > 0$. Let

$$\Psi(p, q, a, b, c) = \frac{a(q-p)}{p+1} \left( \frac{q+7}{q-1} - p \right) + 4c \left[ \frac{b(q-1)(p+1)}{a(q+1)(p-1)} \right] \frac{q-1}{q+p}$$

if $p > 1$. (10)
The following statements (a)–(d) hold:
(a) If $0 < p \leq 1$, then $S_L$ is monotone increasing for all $L > 0$.
(b) If $1 < p < 3$, then $S_L$ is $\subset$-shaped for all $L > 0$.
(c) If $p = 3$, then $S_L$ is $\subset$-shaped for $L > \frac{\pi}{2} \sqrt{\frac{2a}{c^2}}$ and either monotone increasing or S-shaped for $\frac{\pi}{2} \sqrt{\frac{2a}{c^2}} \geq L > 0$. Furthermore, if $\Psi(3, q, a, b, c) > 0$, then one of the following assertions holds:
(c1) $S_L$ is monotone increasing for $\frac{\pi}{2} \sqrt{\frac{2a}{c^2}} \geq L > 0$.
(c2) there exists $\tilde{L} \in (0, \frac{\pi}{2} \sqrt{\frac{2a}{c^2}})$ such that $S_L$ is S-shaped for $\frac{\pi}{2} \sqrt{\frac{2a}{c^2}} > L > \tilde{L}$ and monotone increasing for $\tilde{L} \geq L > 0$.
(d) If $p > 3$, then $S_L$ is either monotone increasing or S-shaped for $L > 0$.
Furthermore, if $\Psi(p, q, a, b, c) > 0$, then there exists $\tilde{L} > 0$ such that $S_L$ is S-shaped for $L > \tilde{L}$ and monotone increasing for $0 < L \leq \tilde{L}$.

Figure 6. Graphs of bifurcation curve $S_L$ of (1), (7). (i) $c = 0$.
(ii) $c > 0$.

4. Lemmas. To prove Theorems 2.1–2.4, we introduce the time-map method used in Corsato [4, p. 127]. We define the time-map formula for (1) by

$$T_\lambda(\alpha) \equiv \int_0^\alpha \frac{\lambda [F(\alpha) - F(u)] + 1}{\sqrt{\lambda^2 [F(\alpha) - F(u)]^2 + 2\lambda [F(\alpha) - F(u)]}} du \text{ for } \alpha \in (0, \beta) \text{ and } \lambda > 0.$$ (11)

Recall the definition of $S_L$ in (2), we may see that

$$S_L = \{(\lambda, \alpha) : T_\lambda(\alpha) = L \text{ for some } \alpha \in (0, \beta) \text{ and } \lambda > 0\}.$$ (12)

Thus, in order to draw the graph of bifurcation curves $S_L$, it is important to understand the fundamental properties of the time-map $T_\lambda(\alpha)$ on $(0, \beta)$. Similarly, we define the time-map formula for (3) by

$$\bar{T}(\alpha) \equiv \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{\sqrt{F(\alpha) - F(u)}} du \text{ for } \alpha \in (0, \beta).$$ (13)

Then we have that $\|u_\lambda\|_\infty = \alpha$ and $\bar{T}(\alpha) = \sqrt{\lambda}$. So by the definition of $\bar{S}$ in (3), we see that

$$\bar{S} = \left\{(\lambda, \alpha) : \sqrt{\lambda} = \bar{T}(\alpha) \text{ for some } \alpha \in (0, \beta) \right\}.$$ (14)
Since \( f \in C^2(0, \infty) \), it can be proved that \( T_\lambda(\alpha) \) is a twice continuously differentiable function of \( \alpha > 0 \) and \( \lambda > 0 \). The proofs are easy but tedious and hence we omit them. For the sake of convenience, we let
\[
A(\alpha, u) \equiv \alpha f(\alpha) - u f(u), \quad B(\alpha, u) \equiv \int_u^\alpha f(t) dt \quad \text{and} \quad C(\alpha, u) \equiv \alpha^2 f'(\alpha) - u^2 f'(u).
\]
Clearly, \( B(\alpha, u) > 0 \) for \( 0 < u < \alpha < \beta \). Then we have that
\[
T_\lambda(\alpha) = \int_0^\alpha \frac{\lambda B(\alpha, u) + 1}{\sqrt{\lambda B(\alpha, u) + 1}} du > \int_0^\alpha 1 du = \alpha \quad \text{for} \quad \alpha \in (0, \beta) \quad \text{and} \quad \lambda > 0.
\]

(15)

Lemma 4.1 ([7, Lemmas 3.1 and 3.2]). Consider (1). The following statements (i)-(iii) hold:

(i) \[
\lim_{\alpha \to 0^+} T_\lambda(\alpha) = \begin{cases} 
\frac{\pi^2}{2\sqrt{2}\lambda} & \text{if} \ \eta = \infty, \\
\infty & \text{if} \ \eta \in (0, \infty), \\
\frac{\pi^2}{2\sqrt{2}\lambda} & \text{if} \ \eta = 0
\end{cases} \quad \text{for} \ \lambda > 0.
\]

(ii) \[
\lim_{\alpha \to 0^+} T'_\lambda(\alpha) = \begin{cases} 
-\infty & \text{if} \ f''(0^+) = \infty, \\
-\frac{1}{6\sqrt{2}\lambda} f''(0^+) & \text{if} \ f''(0^+) \text{ exists}, \\
\infty & \text{if} \ f''(0^+) = -\infty
\end{cases} \quad \text{for} \ \lambda > 0.
\]

(iii) Assume that \( f''(0^+) = 0 \) and \( f^{(3)}(u) \) exists for \( u > 0 \). Then
\[
\lim_{\alpha \to 0^+} T''_\lambda(\alpha) = \begin{cases} 
-\infty & \text{if} \ \lim_{u \to 0^+} f^{(3)}(0^+) = \infty, \\
\frac{3\pi^2}{8\sqrt{2\lambda}} \left[ \lambda - \frac{1}{12\pi^2} f^{(3)}(0^+) \right] & \text{if} \ \lim_{u \to 0^+} f^{(3)}(0^+) \text{ exists}, \\
\infty & \text{if} \ \lim_{u \to 0^+} f^{(3)}(0^+) = -\infty
\end{cases} \quad \text{for} \ \lambda > 0.
\]

Lemma 4.2. Consider (1). Then \( \lim_{\alpha \to \beta^-} T_\lambda(\alpha) = \infty \) for \( \lambda > 0 \).

Proof. Since \( f(\beta) = 0 \), there exists a nonnegative function \( h \in C[0, \beta] \) such that \( f(u) = (\beta - u) h(u) \) on \( [0, \beta] \). By L'Hôpital rule, we see that
\[
\lim_{u \to \beta^-} h(u) = \lim_{u \to \beta^-} \frac{f(u)}{\beta - u} = -f'(\beta) \text{ exists}.
\]

Then there exists \( M > 0 \) such that \( 0 \leq h(u) < M \) for \( 0 \leq u < \beta \). So we observe that, for \( 0 \leq t < 1 \),
\[
B(\beta, \beta t) = \int_{\beta t}^\beta f(s) ds = \int_{\beta t}^\beta (\beta - s) h(s) ds < M \int_{\beta t}^\beta (\beta - s) ds = \frac{1}{2} M \beta^2 (1 - t)^2,
\]
from which it follows that there exists \( t_* \in (0, 1) \) such that \( B(\beta, \beta t) < 1 \) for \( t_* < t < 1 \). By (11) and (16), we see that
\[
\lim_{\alpha \to \beta^-} T_\lambda(\alpha) = \lim_{\alpha \to \beta^-} \alpha \int_0^1 \frac{\lambda B(\alpha t) + 1}{\sqrt{\lambda^2 B^2(\alpha t) + 2\lambda B(\alpha t)}} dt \\
\geq \lim_{\alpha \to \beta^-} \alpha \int_{t_*}^1 \frac{1}{\sqrt{\lambda^2 B^2(\alpha t) + 2\lambda B(\alpha t)}} dt
\]
\[ \geq \int_{t}^{1} \frac{\beta}{\sqrt{(\lambda^2 + 2\lambda) B(\beta, \beta t)}} dt \geq \sqrt{\frac{2}{(\lambda^2 + 2\lambda) M}} \int_{t}^{1} \frac{1}{1 - t} dt = \infty. \]

The proof is complete. \( \square \)

By similar arguments in [6, Lemma 4.2 and (4.19)], we immediately have the following Lemma 4.3.

**Lemma 4.3.** Consider (1). Fix \( \alpha \in (0, \beta) \). Then the following statements (i)–(iii) hold:

(i) \( \partial T(\alpha)/\partial \lambda < 0 \) for \( \lambda > 0 \).

(ii) \( \lim_{\lambda \to 0^+} T(\alpha) = \infty \) and \( \lim_{\lambda \to \infty} T(\alpha) = \alpha \).

(iii) \( \lim_{\lambda \to 0^+} \sqrt{\lambda} T(\alpha) = T(\alpha) \) and \( \lim_{\lambda \to \infty} \sqrt{\lambda} T(\alpha) = 1 \).

**Lemma 4.4.** Consider (1). Assume that (H3) holds. Then

\[ \frac{\partial}{\partial \lambda} \left[ \sqrt{\lambda} T(\alpha) \right] > 0 \quad \text{on} \quad (0, \tau) \quad \text{and} \quad T(\alpha) > 0 \quad \text{on} \quad [\tau, \beta) \quad \text{for} \quad \lambda > 0. \]

**Proof.** By (H3), we see that

\[ \frac{\partial}{\partial u} [A(\alpha, u) + 2B(\alpha, u)] = -[3f(u) + uf'(u)] < 0 \quad \text{for} \quad 0 < u < \alpha \leq \tau. \quad (17) \]

Since \( A(\alpha, \alpha) + 2B(\alpha, \alpha) = 0 \), and by (17), we observe that

\[ A(\alpha, u) + 2B(\alpha, u) > 0 \quad \text{for} \quad 0 < u < \alpha \leq \tau. \quad (18) \]

By (11), we compute that

\[ T(\alpha) = \frac{1}{\alpha} \int_{0}^{\alpha} \frac{\lambda B^3 + 3\lambda^2 B^2 + \lambda (2B - A)}{[\lambda B^2 + 2\lambda B]^{3/2}} du. \quad (19) \]

Since \( B(\alpha, u) > 0 \) for \( 0 < u < \alpha \leq \tau \), and by (18) and (19), we see that, for \( \alpha \in (0, \tau] \) and \( \lambda > 0 \),

\[ \frac{\partial}{\partial \lambda} \left[ \sqrt{\lambda} T(\alpha) \right] = \sqrt{\lambda} \frac{\partial}{\partial \lambda} T(\alpha) + \frac{1}{2\sqrt{\lambda}} T(\alpha) \]

\[ = \frac{1}{2\alpha\sqrt{\lambda}} \int_{0}^{\alpha} \frac{\lambda B^2 + 5B^2 \lambda + 3A + 6B}{(\lambda B^2 + 2\lambda B)^{3/2}} du > 0. \]

Since \( B(\alpha, u) > 0 \) for \( 0 < u < \alpha \leq \tau \), and by (H3) and (19), we see that, for \( \alpha \in [\tau, \beta) \) and \( \lambda > 0 \),

\[ T(\alpha) \geq \frac{1}{\alpha} \int_{0}^{\alpha} \frac{\lambda (2B - A)}{[\lambda B^2 + 2\lambda B]^{3/2}} du = \int_{0}^{\alpha} \frac{\lambda [\theta(\alpha) - \theta(u)]}{[\lambda B^2(\alpha, u) + 2\lambda B(\alpha, u)]^{3/2}} du > 0. \]

The proof is complete. \( \square \)

**Lemma 4.5.** Consider (1). Assume that one of the following conditions holds:

(H2), (C4) or ((C6) and (H3)).

Then there exists \( \varsigma \in (0, \tau) \) such that \( T(\alpha) > 0 \) for \( 0 < \alpha \leq \varsigma \) and all \( \lambda > 0 \).

**Proof.** We consider three cases:

**Case 1.** Assume that (H2) holds. Let \( g(u) = f(u)/u \). There exists \( \varsigma \in (0, \tau) \) such that \( g'(u) < 0 \) for \( 0 < u \leq \varsigma \). It follows that \( \theta'(u) = -u^2 g'(u) > 0 \) for \( 0 < u \leq \varsigma \). Then

\[ 2B(\alpha, u) - A(\alpha, u) = \theta(\alpha) - \theta(u) > 0 \quad \text{for} \quad 0 < u < \alpha \leq \varsigma. \quad (20) \]
By (19) and (20), we observe that \( T'_\lambda(\alpha) > 0 \) for \( 0 < \alpha \leq \varsigma \) and \( \lambda > 0 \).

**Case 2.** Assume that (C4) holds. Since \( f''(0^+) < 0 \), there exists \( \varsigma \in (0, \tau) \) such that \( f''(u) < 0 \) for \( 0 < u \leq \varsigma \). We compute and find that
\[
\theta''(u) = -uf''(u) > 0 \quad \text{for} \quad 0 < u \leq \varsigma.
\] (21)

Since \( \eta \in (0, \infty) \), and by L'Hôpital rule, we see that \( f(0) = 0 \) and \( f'(0^+) = 2\eta \in (0, \infty) \). By (21), we further see that
\[
\theta'(u) > \lim_{u\to 0^+} \theta'(u) = \lim_{u\to 0^+} [f(u) - uf'(u)] = 0 \quad \text{for} \quad 0 < u \leq \varsigma,
\]
which implies that (20) holds. So by (19), we observe that \( T'_\lambda(\alpha) > 0 \) for \( 0 < \alpha \leq \varsigma \) and \( \lambda > 0 \).

**Case 3.** Assume that (C6) and (H3) hold. Since (C6) holds, and by modifying the proof of [7, Lemma 3.2], we observe that
\[
\lambda > \varsigma, \quad \varsigma > 0.
\]

Using \( \alpha \) and \( \varsigma \) in parts (i) and (iii) of Theorem 2.1, we see that
\[
\lambda > \varsigma, \quad \varsigma > 0.
\]

By (22), Lemma 4.3(i) and implicit function theorem. Then by (12), we obtain that
\[
\lim_{\lambda \to 0^+} T_\lambda(\alpha) = \alpha < m_{L,\beta} \leq L < \infty = \lim_{\lambda \to 0^+} T_\lambda(\alpha).
\] (22)

By (22), Lemma 4.3(i) and continuity of \( T_\lambda(\alpha) \) with respect to \( \lambda \), there exists unique \( \lambda_L(\alpha) > 0 \) such that \( T_{\lambda_L(\alpha)}(\alpha) = L \). Furthermore, \( \lambda_L(\alpha) \in C^1(0, L) \) by Lemma 4.3(i) and implicit function theorem. Then by (12), we obtain that
\[
S_L = \{ (\lambda, \alpha) : T_\lambda(\alpha) = L \quad \text{for} \quad \lambda > 0 \} = \{ (\lambda_L(\alpha), \alpha) : \alpha \in (0, m_{L,\beta}) \}
\] (23)

is continuous for \( L > 0 \). So statement (i) holds.

**Lemma 4.6.** Consider (1) with fixed \( L > 0 \). Then the following statements (i)–(iii) hold:

(i) There exists a positive and continuously differentiable function \( \lambda_L(\alpha) \) on \( (0, m_{L,\beta}) \) such that \( T_{\lambda_L(\alpha)}(\alpha) = L \). Moreover, the bifurcation curve
\[
S_L = \{ (\lambda_L(\alpha), \alpha) : \alpha \in (0, m_{L,\beta}) \}
\]
is continuous on the \( (\lambda, \|u\|_\infty) \)-plane.

(ii) \( \text{sgn}(\lambda_L(\alpha)) = \text{sgn}(T'_{\lambda_L(\alpha)}(\alpha)) \) for \( \alpha \in (0, m_{L,\beta}) \) where \( \text{sgn}(u) \) is the signum function.

(iii) \( \lim_{\alpha \to 0^+} \lambda_L(\alpha) = \kappa \) and \( \lim_{\alpha \to m_{L,\beta}} \lambda_L(\alpha) = \infty \) where \( \kappa \) is defined in Theorem 2.1. Moreover, the bifurcation curve \( S_L \) starts from the point \( (\kappa, 0) \) and goes to \( (\infty, m_{L,\beta}) \) for \( L > 0 \).

**Proof.** (I) Assume that there exists \( \alpha \geq m_{L,\beta} \) such that \( L = T_\lambda(\alpha) \) for some \( \lambda > 0 \). Since \( 0 < \alpha < \beta \), and by (15), we see that \( L = T_\lambda(\alpha) > \alpha \geq m_{L,\beta} = L \), which is a contradiction. Thus,
\[
\{ \alpha \geq m_{L,\beta} : L = T_\lambda(\alpha) \text{ for some } \lambda > 0 \} = \emptyset.
\]

Let \( \alpha \in (0, m_{L,\beta}) \). By Lemma 4.3(ii), we see that
\[
\lim_{\lambda \to \infty} T_\lambda(\alpha) = \alpha < m_{L,\beta} \leq L < \infty = \lim_{\lambda \to 0^+} T_\lambda(\alpha).
\] (22)

By (22), Lemma 4.3(i) and continuity of \( T_\lambda(\alpha) \) with respect to \( \lambda \), there exists unique \( \lambda_L(\alpha) > 0 \) such that \( T_{\lambda_L(\alpha)}(\alpha) = L \). Furthermore, \( \lambda_L(\alpha) \in C^1(0, L) \) by Lemma 4.3(i) and implicit function theorem. Then by (12), we obtain that
\[
S_L = \{ (\lambda, \alpha) : T_\lambda(\alpha) = L \quad \text{for} \quad \lambda > 0 \} = \{ (\lambda_L(\alpha), \alpha) : \alpha \in (0, m_{L,\beta}) \}
\] (23)

is continuous for \( L > 0 \). So statement (i) holds.
(II) Since $T_{\lambda L}(\alpha) = L$ for $\alpha \in (0, L)$, we observe that

$$0 = \frac{\partial}{\partial \alpha} T_{\lambda L}(\alpha) = T'_{\lambda L}(\alpha) + \frac{\partial}{\partial \lambda} T_{\lambda}(\alpha) \bigg|_{\lambda = \lambda L(\alpha)} \lambda'_{L}(\alpha) \quad \text{for} \quad \alpha \in (0, m_{L, \beta}),$$

which implies that statement (ii) holds by Lemma 4.3(i).

(III) By similar arguments in [6, proof of Lemma 4.5(iii)], we obtain that

$$\lim_{\alpha \to 0^+} \lambda L(\alpha) = \kappa.$$ In addition, we assume that $\liminf_{\alpha \to m_{L, \beta}} \lambda L(\alpha) < \infty$. Then there exist $M > 0$ and $\{\alpha_n\}_{n \in \mathbb{N}} \subset (0, m_{L, \beta})$ such that

$$\lim_{n \to \infty} \alpha_n = m_{L, \beta} \quad \text{and} \quad \lambda L(\alpha_n) < M \quad \text{for} \quad n \in \mathbb{N}. \quad (24)$$

If $m_{L, \beta} = \beta \leq L$, by Lemmas 4.2, 4.3(i) and 4.6(i), we see that

$$L = \lim_{n \to \infty} T_{\lambda L}(\alpha_n) \geq \lim_{n \to \infty} T_M(\alpha_n) = \lim_{\alpha \to m_{L, \beta}} T_M(\alpha) = \infty,$$

which is a contradiction. On the other hand, if $m_{L, \beta} < \beta$, by Lemmas 4.3(i), 4.6(i), and (15), we see that

$$L = \lim_{n \to \infty} T_{\lambda L}(\alpha_n) \geq \lim_{n \to \infty} T_M(\alpha_n) = T_M(m_{L, \beta}) > m_{L, \beta} = L,$$

which is a contradiction. So $\lim_{\alpha \to \lambda L(\alpha)} \geq \liminf_{\alpha \to m_{L, \beta}} \lambda L(\alpha) = \infty$. By (23), the bifurcation curve $S_L$ starts from the point $(\kappa, 0)$ and goes to $(\infty, m_{L, \beta})$ for $L > 0$. Thus statement (iii) holds. The proof is complete. \hfill \Box

**Lemma 4.7.** Consider (1). Assume that (H3) and one of (C2), (C4) or (C6) hold, and that $\bar{T}'(\hat{\alpha}) < 0$ for some $\hat{\alpha} \in (0, \beta)$. Then there exists $\bar{L} \geq 0$ such that

$$\Theta = \{ L > 0 : \lambda L(\alpha) \text{ has a local maximum at } \hat{\alpha}_M \text{ and a local minimum at } \hat{\alpha}_m (> \hat{\alpha}_M) \text{ on } (0, m_{L, \beta}) \} = (\bar{L}, \infty).$$

**Proof.** By Lemmas 4.5 and 4.6(ii)(iii), we see that

$$\lim_{\alpha \to 0^+} \lambda L(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to m_{L, \beta}} \lambda L(\alpha) = \infty \quad \text{if (C2) holds},$$

$$\lambda'_{L}(\alpha) > 0 \quad \text{for small } \alpha > 0 \quad \text{and} \quad \lim_{\alpha \to m_{L, \beta}} \lambda L(\alpha) = \infty \quad \text{if either (C4) or (C6) holds}.$$

So we obtain that

$$\Theta = \{ L > 0 : \lambda'_{L}(\alpha) < 0 \quad \text{for some } \alpha \in (0, m_{L, \beta}) \}. \quad (25)$$

Next, we divide this proof into the following three steps.

**Step 1.** We prove that $\Theta$ is nonempty. By Lemma 4.3(iii), we observe that

$$\lim_{\alpha \to 0^+} \sqrt{\lambda L(\alpha)} = T'_{\lambda L}(\hat{\alpha}) < 0.$$ So $T'_{\lambda L}(\hat{\alpha}) < 0$ for some $\lambda L > 0$. Let $L_1 = T_{\lambda_1}(\hat{\alpha})$. By Lemma 4.6(i)(ii), we have that

$$\lambda_1 = \lambda L(\hat{\alpha}) \quad \text{and} \quad \lambda'_{L_1}(\hat{\alpha}) < 0,$$

which implies that $L_1 \in \Theta$. So $\Theta$ is nonempty.

**Step 2.** We prove that $\Theta$ is open. Let $L_2 \in \Theta$. It implies that $\lambda'_{L_2}(\alpha_1) < 0$ for some $\alpha_1 \in (0, m_{L_2, \beta})$. Since $T_{\lambda L_{\alpha_1}}(\alpha_1) = L_2$ by Lemma 4.6(i), and by Lemma 4.3(i) and implicit function theorem, there exists $\delta_1 > 0$ such that $\lambda L(\alpha_1)$ is a
continuously differentiable function of $L \in (L_2 - \delta_1, L_2 + \delta_1)$. Since $\lambda'_{L_2}(\alpha_1) < 0$, and by continuity of $\lambda'_{L}(\alpha)$ with respect to $L$, we observe that $\Theta$ is open.

**Step 3.** We prove Lemma 4.7. First, we assert that

$$[L, \infty) \subset \Theta \text{ if } L \in \Theta. \quad (26)$$

By Steps 1, 2 and (26), there exists $\tilde{L} \geq 0$ such that $\Theta = (\tilde{L}, \infty)$. Next, we prove assertion (26). Let $L_3 \in \Theta$. Assume that there exists $L_4 \in (L_3, \infty)$ such that $L_4 \notin \Theta$. Then we see that

$$\lambda'_{L_3}(\alpha_2) < 0 \text{ for some } \alpha_2 \in (0, m_{L_3, \beta}), \text{ and } \lambda'_{L_4}(\alpha) \geq 0 \text{ for } \alpha \in (0, m_{L_4, \beta}). \quad (27)$$

Since $\alpha_2 < m_{L_3, \beta} \leq m_{L_4, \beta}$, and by (27) and Lemma 4.6(ii), we observe that

$$T'_{\lambda_{L_3}(\alpha_2)}(\alpha_2) < 0 \leq T'_{\lambda_{L_4}(\alpha_2)}(\alpha_2).$$

So by Lemma 4.4, we obtain that $0 < \alpha_2 < \tau$ and $\lambda_{L_3}(\alpha_2) < \lambda_{L_4}(\alpha_2)$. By Lemma 4.3(i), we see that

$$L_4 = T_{\lambda_{L_4}(\alpha_2)}(\alpha_2) < T_{\lambda_{L_3}(\alpha_2)}(\alpha_2) = L_3,$$

which is a contradiction. So assertion (26) holds.

The proof is complete. \qed

5. Proofs of theorems.

**Proof of Theorem 2.1.** Let $L > 0$ be given. By Lemma 4.6(i)(iii), the bifurcation curve $S_L$ of (1) is continuous in $(\lambda, \|u_\lambda\|_\infty)$-plane, starts from the point $(\kappa, 0)$ and goes to $(\infty, m_{L, \beta})$ for $L > 0$. Next, we divide the remainder proofs of Theorem 2.1 into the following three steps.

**Step 1.** We prove statement (i). Assume that (C1) holds. By Lemma 4.6(iii), we have that $\lim_{\alpha \to 0^+} \lambda_L(\alpha) = \lim_{\alpha \to m_{L, \beta}} \lambda_L(\alpha) = \infty$. Then $S_L$ is $C$-shaped. Assume that either (C3) or (C5) holds. By Lemma 4.1, we have that, for $\lambda > 0$,

$$\begin{cases} 
\lim_{\alpha \to 0^+} T'_X(\alpha) < 0 & \text{if (C3) holds,} \\
\lim_{\alpha \to 0^+} T'_X(\alpha) = 0 & \text{and } \lim_{\alpha \to 0^+} T''_X(\alpha) = -\infty & \text{if (C5) holds.}
\end{cases}$$

Then there exists $\delta_1 \in (0, m_{L, \beta})$ such that $T'_X(\alpha) < 0$ for $0 < \alpha < \delta_1$ and $\lambda > 0$. By Lemma 4.6(ii)(iii), we see that

$$\lambda_L(\alpha) < 0 \text{ for } 0 < \alpha < \delta_1, \text{ and } \lim_{\alpha \to m_{L, \beta}} \lambda_L(\alpha) = \infty.$$

So $S_L$ is $C$-shaped.

**Step 2.** We prove statement (ii). Assume that (C2) holds. By Lemma 4.6(iii), we have that

$$\lim_{\alpha \to 0^+} \lambda_L(\alpha) = 0 \text{ and } \lim_{\alpha \to m_{L, \beta}} \lambda_L(\alpha) = \infty.$$

Then $S_L$ is either monotone increasing or $S$-shaped. Assume that either (C4) or (C6) holds. By Lemma 4.1, we have that, for $\lambda > 0$,

$$\begin{cases} 
\lim_{\alpha \to 0^+} T'_X(\alpha) > 0 & \text{if (C4) holds,} \\
\lim_{\alpha \to 0^+} T'_X(\alpha) = 0 & \text{and } \lim_{\alpha \to 0^+} T''_X(\alpha) > 0 & \text{if (C6) holds.}
\end{cases}$$
Then there exists $\delta_2 \in (0, m_{L,\beta})$ such that $T'_L(\alpha) > 0$ for $0 < \alpha < \delta_2$ and $\lambda > 0$. By Lemma 4.6(ii)(iii), we see that

$$
\lambda_L'(\alpha) > 0 \text{ for } 0 < \alpha < \delta_2, \text{ and } \lim_{\alpha \to m_{L,\beta}^-} \lambda_L(\alpha) = \infty.
$$

So $S_L$ is either monotone increasing or S-shaped. In addition, assume that the bifurcation curve $\tilde{S}$ of (3) is not monotone increasing. By (14), there exists $\hat{\alpha} \in (0, \beta)$ such that $\tilde{T}'(\hat{\alpha}) < 0$. So by Lemma 4.3(iii), we see that

$$
\lim_{\lambda \to 0^+} \sqrt{\lambda} T'_L(\hat{\alpha}) = \tilde{T}'(\hat{\alpha}) < 0,
$$

from which it follows that there exists $\tilde{\lambda} > 0$ such that

$$
T'_L(\tilde{\lambda}) < 0 \text{ for } 0 < \lambda \leq \tilde{\lambda}. \tag{28}
$$

Let $\tilde{L} = T_L(\tilde{\lambda})$. It implies that $\tilde{L} > \hat{\alpha}$ by (15). Assume that $L > \tilde{L}$. It follows that

$$
T_{L,\hat{\alpha}}(\tilde{\lambda}) = L > \tilde{L} = T_L(\tilde{\lambda}).
$$

So by Lemma 4.3(i), we obtain that $\lambda_{L,\hat{\alpha}}(\tilde{\lambda}) < \tilde{\lambda}$. By (28), we see that $T_{L,\hat{\alpha}}'(\tilde{\lambda}) < 0$. It follows that $\lambda_{L,\hat{\alpha}}'(\tilde{\lambda}) < 0$ by Lemma 4.6(ii). Then statement (ii) holds.

**Step 3.** We prove statement (iii). Assume that (C7) holds. By Lemma 4.1, we have that

$$
\lim_{\alpha \to 0^+} T'_L(\alpha) = 0 \text{ and } \lim_{\alpha \to 0^+} T''_L(\alpha) \begin{cases} < 0 & \text{if } 0 < \lambda < \frac{1}{12\eta^2} f^{(3)}(0^+), \\ = 0 & \text{if } \lambda = \frac{1}{12\eta^2} f^{(3)}(0^+), \\ > 0 & \text{if } \lambda > \frac{1}{12\eta^2} f^{(3)}(0^+). \end{cases} \tag{29}
$$

Assume that $L > \hat{L}$ where $\hat{L}$ is defined in (8). By Lemma 4.6(iii), we see that

$$
\lim_{\alpha \to 0^+} \lambda_L(\alpha) = \frac{\pi^2}{8\eta L^2} < \frac{\pi^2}{8\eta \hat{L}^2} = \frac{f^{(3)}(0^+)}{12\eta^2}.
$$

So by (29), we observe that $T_{L,\hat{\alpha}}'(\alpha) < 0$ for small $\alpha > 0$. Then by Lemma 4.6(ii), we obtain that

$$
\lambda_{L,\hat{\alpha}}'(\alpha) < 0 \text{ for small } \alpha > 0 \text{ when } L > \hat{L}. \tag{30}
$$

Assume that $0 < L < \hat{L}$. Similarly, we obtain that

$$
\lambda_{L,\hat{\alpha}}'(\alpha) > 0 \text{ for small } \alpha > 0 \text{ when } L < \hat{L}. \tag{31}
$$

Since $\lim_{\alpha \to m_{L,\beta}^-} \lambda_L(\alpha) = \infty$ by Lemma 4.6(iii), and by (30) and (31), we see that $S_L$ is C-shaped for $L > \hat{L}$ and either monotone increasing or S-shaped for $0 < L < \hat{L}$.

The proof is complete. \hfill $\square$

**Proof of Theorem 2.2.** (1) Assume that (H1) holds. Let $g(u) = f(u)/u$. Then $\theta'(u) = -u^2 g'(u) > 0$ for $0 < u < \beta$. It follows that

$$
2B(\alpha, u) - A(\alpha, u) = \theta(\alpha) - \theta(u) > 0 \text{ for } 0 < u < \alpha < \beta.
$$

By (19), we see that

$$
T'_L(\alpha) \geq \frac{1}{\alpha} \int_0^\alpha \frac{\lambda (2B - A)}{[\lambda^2 B^2 + 2\lambda B]^{3/2}} du > 0 \text{ for } 0 < \alpha < \beta \text{ and } \lambda > 0.
$$

Then by Lemma 4.6(ii), we obtain that $\lambda_{L,\beta}'(\alpha) > 0$ for $0 < \alpha < m_{L,\beta}$ and $L > 0$, which implies that $S_L$ is monotone increasing for $L > 0$. Thus statement (i) holds.
(II) Assume that one of the following conditions holds:

(H2), (C4) or ((C6) and (H3)).

By Lemma 4.5, there exists \( \zeta \in (0, \tau) \) such that \( T'_{\lambda}(\alpha) > 0 \) for \( 0 < \alpha \leq \zeta \) and \( \lambda > 0 \). Then we observe that

\[
T'_{\lambda}(\alpha) > 0 \quad \text{for} \quad 0 < \alpha < m_{L,\beta} \leq L < \zeta.
\]

So by Lemma 4.6(ii), we obtain that \( \lambda'_{L}(\alpha) > 0 \) for \( 0 < \alpha < m_{L,\beta} \) and \( 0 < L < \zeta \), which implies that \( S_{L} \) is monotone increasing for \( 0 < L < \zeta \). Thus statement (ii) holds. The proof is complete.

Proof of Theorem 2.3. (I) Assume that the bifurcation curve \( \bar{S} \) of (3) is monotone increasing. By (14), \( \bar{T}(\alpha) \) is also monotone increasing on \((0, \beta)\). Then by Lemmas 4.3(iii) and 4.4, we see that

\[
\left\{ \begin{array}{ll}
\sqrt{T'_{\lambda}}(\alpha) > \lim_{\lambda \to 0^+} \sqrt{T'_{\lambda}}(\alpha) = \bar{T}'(\alpha) \geq 0 & \text{on} \ (0, \tau) \\
T'_{\lambda}(\alpha) > 0 & \text{on} \ [\tau, \beta) \text{ for} \ \lambda > 0.
\end{array} \right.
\]

By (32) and Lemma 4.6(ii), we obtain that \( \lambda'_{L}(\alpha) > 0 \) for \( 0 < \alpha < m_{L,\beta} \). Thus statement (i) holds.

(II) Assume that the bifurcation curve \( \bar{S} \) of (3) is not monotone increasing. By (14), we see that \( \bar{T}(\alpha) \) is not monotone increasing on \((0, \beta)\). It implies that \( \bar{T}'(\hat{\alpha}) < 0 \) for some \( \hat{\alpha} \in (0, \beta) \). So by Lemma 4.7, there exists \( \bar{L} \geq 0 \) such that

\[
\Theta = \left\{ L > 0 : \lambda_{L}(\alpha) \text{ has a local maximum at} \ \hat{\alpha}_{m} \text{ and a local minimum at} \ \alpha_{m} (\text{where} \ \alpha_{m} (\text{and} \ \alpha_{M}) \text{on} \ (0, m_{L,\beta}) \text{is} \ (\bar{L}, \infty) \right\}.
\]

By Theorem 2.1, the bifurcation curve \( S_{L} \) is S-shaped for \( L > \bar{L} \) and monotone increasing for \( 0 < L \leq \bar{L} \). Assume that one of the following conditions holds:

- ((C2) and (H2)), (C4) or (C6).

Since (H3) holds, and by Theorem 2.2(ii), we see that \( S_{L} \) is monotone increasing for small \( L > 0 \). So \( \bar{L} > 0 \). The proof is complete.

Proof of Proposition 1. (I) Assume that (H1) holds. By similar proof of Theorem 2.2, we see that \( 2B(\alpha, u) - A(\alpha, u) > 0 \) for \( 0 < u < \alpha < \beta \). So by (13), we obtain that

\[
\bar{T}'(\alpha) = \frac{1}{2\sqrt{2\alpha}} \int_{0}^{\alpha} \frac{2B(\alpha, u) - A(\alpha, u)}{B^{3/2}(\alpha, u)} du > 0 \ \text{for} \ 0 < \alpha < \beta.
\]

Then by (14), the bifurcation curve \( \bar{S} \) of (3) is monotone increasing. So statement (i) holds.

(II) Assume that \( [f(u)/u]' > 0 \) for small \( u > 0 \). Let \( g(u) = f(u)/u \). There exists \( \delta > 0 \) such that \( g'(u) > 0 \) for \( 0 < u \leq \delta \). It follows that \( \theta'(u) = -u^2 g'(u) < 0 \) for \( 0 < u \leq \delta \). Then we see that

\[
2B(\alpha, u) - A(\alpha, u) = \theta(\alpha) - \theta(u) < 0 \ \text{for} \ 0 < u < \alpha \leq \delta.
\]

So by (13), we obtain that

\[
\bar{T}'(\alpha) = \frac{1}{2\sqrt{2\alpha}} \int_{0}^{\alpha} \frac{2B(\alpha, u) - A(\alpha, u)}{B^{3/2}(\alpha, u)} du < 0 \ \text{for} \ 0 < \alpha \leq \delta.
\]
By (14), the bifurcation curve $\tilde{S}$ of (3) is not monotone increasing. So statement (ii) holds. The proof is complete. □

**Proof of Theorem 2.4.** (I) Assume that $S_L$ is monotone increasing. By Lemma 4.6(i), we see that $\lambda'_L(\alpha) \geq 0$ for $0 < \alpha < m_{L,\beta}$. Then $T'_{\lambda_L}(\alpha) > 0$ for $0 < \alpha < m_{L,\beta}$ by Lemma 4.6(ii). Let $L \in (0, \tilde{L})$ be given. Since

$$T_{\lambda_L}(\alpha) = \tilde{L} > L = T_{\lambda_L}(\alpha) \quad \text{for } 0 < \alpha < m_{L,\beta},$$

and by Lemma 4.3(i), we see that $\lambda_L(\alpha) > \tilde{\lambda}_L(\alpha)$ for $0 < \alpha < m_{L,\beta}$. So by Lemma 4.4, we observe that if $0 < L \leq \tau$,

$$\sqrt{\lambda_L(\alpha)} T'_{\lambda_L}(\alpha) > 0 \quad \text{for } 0 < \alpha \leq L;$$

and if $L > \tau$,

$$\left\{ \begin{array}{ll}
\sqrt{\lambda_L(\alpha)} T'_{\lambda_L}(\alpha) > 0 \quad & \text{for } 0 < \alpha \leq \tau, \\
T'_{\lambda_L}(\alpha) > 0 \quad & \text{for } \tau \leq \alpha < m_{L,\beta}.
\end{array} \right.$$

Then we obtain that $T'_{\lambda_L}(\alpha) > 0$ for $0 < \alpha < m_{L,\beta}$. It follows that $\lambda'_L(\alpha) > 0$ for $0 < \alpha < m_{L,\beta}$ by Lemma 4.6(ii). Thus, by Theorem 2.1, the bifurcation curve $S_L$ is $\subset$-shaped for $L > \tilde{L}$ and monotone increasing for $0 < L \leq \tilde{L}$. Then the statement (i) holds.

(II) Assume that $S_L$ is not monotone increasing. By Lemma 4.6(i), we have that

$$\lambda'_L(\alpha_1) < 0 \quad \text{for some } \alpha_1 \in (0, m_{L,\beta}).$$

Let

$$\Theta' \equiv \{ L \in (0, \tilde{L}) : \lambda_L(\alpha) \text{ has a local maximum at } \tilde{\alpha}_M \text{ and a local minimum at } \tilde{\alpha}_m (\neq \tilde{\alpha}_M) \text{ on } (0, L) \}.$$ 

By similar proof of Lemma 4.7, we obtain that $\Theta' = (\tilde{L}, \tilde{L})$ for some $\tilde{L} \in [0, \tilde{L})$. Then by Theorem 2.1, the bifurcation curve $S_L$ is $\subset$-shaped for $L > \tilde{L}$, $S$-shaped for $\tilde{L} < L < \tilde{L}$ and monotone increasing for $0 < L \leq \tilde{L}$. Finally, we need to prove that $\tilde{L} > 0$. By (29), there exist $\lambda_1 > \frac{1}{127\pi} f^{(3)}(0^+) \text{ and } \delta \in (0, \tau)$ such that

$$T'_{\lambda_1}(\alpha) > 0 \quad \text{for } 0 < \alpha \leq \delta.$$  

(34)

So by Lemma 4.4, we see that

$$\sqrt{\lambda'_{\lambda_1}(\alpha)} > 0 \quad \text{for } 0 < \alpha \leq \delta \text{ and } \lambda > \lambda_1.$$  

Assume that

$$0 < L < \min \left\{ \lim_{\alpha \to 0^+} T_{\lambda_1}(\alpha), \delta \right\}.$$  

So by (34), we observe that

$$T_{\lambda_L}(\alpha) = L \leq \lim_{\alpha \to 0^+} T_{\lambda_1}(\alpha) < T_{\lambda_1}(\alpha) \quad \text{for } 0 < \alpha < m_{L,\beta} \leq \delta,$$

which implies that $\lambda_L(\alpha) > \lambda_1$ for $0 < \alpha < m_{L,\beta}$ by Lemma 4.3(ii). So by (35) and Lemma 4.6(ii), we obtain that $\lambda'_L(\alpha) > 0$ for $0 < \alpha < m_{L,\beta}$. Thus, $\tilde{L} > 0$. The proof is complete. □
Proof of Theorem 3.1. Assume that \((K,c) \in \Gamma\). By [9], there exists \(\beta > 0\) such that \(f(u) > 0\) on \((0,\beta)\), \(f(\beta) = 0\) and \(f(u) < 0\) on \((\beta,\infty)\). We compute that
\[
\eta = \frac{1}{2} \quad \text{and} \quad f''(0) = \frac{2(Kc + 1)}{K} < 0,
\]
which implies that (C4) holds. So by Theorem 2.1, the bifurcation curve \(S_L\) is either monotone increasing or S-shaped, starts from the point \((\frac{\pi^2}{4\tau^2},0)\) and goes to \((\infty,m_{L,\beta})\) for \(L > 0\). Let
\[
g(u) = \frac{f(u)}{u} = 1 - \frac{cu}{1 + u^2}.
\]
We compute that
\[
g'(u) = \frac{-u^4 + (Kc - 2)u^2 - (Kc + 1)^2}{K(u^2 + 1)^2} = \frac{G(u^2)}{K(u^2 + 1)^2},
\]
where \(G(v) \equiv -v^2 + (Kc - 2)v - (Kc + 1)\). Obviously, \(g'(u) < 0\) for small \(u > 0\). It implies that (H2) holds. So by Theorem 2.2(ii), \(S_L\) is monotone increasing for small \(L > 0\). Then statement (i) holds. Assume that \(Kc < 8\). Since \(G(v)\) is a quadratic polynomial of \(v\), its discriminant \(\Delta_1 = Kc(Kc - 8) > 0\). It follows that \(G(u^2) < 0\) for \(u > 0\). By (36), we see that \(g'(u) < 0\) for \(u > 0\). So by Theorem 2.2(i), \(S_L\) is monotone increasing for all \(L > 0\). Then statement (ii) holds.

Finally, assume that \(c \in (\frac{8}{\sqrt{27}},2)\) and \(K > 0\) is large. By [12], we see that \((K,c) \in \Gamma\) and the bifurcation curve \(S\) of (3), (5) is not monotone increasing. Since (C4) holds, and by Theorem 2.1(ii), we see that \(S_L\) is S-shaped for large \(L > 0\). Then statement (iii) holds. The proof is complete.

Proof of Theorem 3.2. Clearly, there exists \(\beta > 0\) such that \(f(u) > 0\) on \((0,\beta)\), \(f(\beta) = 0\) and \(f(u) < 0\) on \((\beta,\infty)\), cf. [11, Theorem 2.1]. We compute that \(\eta = \infty\). It implies that (C2) holds. By [11, Theorem 2.1], there exists \(a_0 > 0\) such that \(\bar{S}\) is S-shaped for \(0 < a < a_0\) and monotone increasing for \(a \geq a_0\). Then we assert that (H3) holds. So Theorem 3.2 holds by Theorem 2.3.

Next, we prove that (H3) holds. Let \(\tau = \frac{2\theta}{3a} > 0\). We observe that
\[
f(\tau) = \frac{27a^2d + 4b^3 + 18abc}{27a^2} > 0,
\]
which implies that \(\tau < \beta\). We compute that
\[
3f(u) + uf'(u) = -6au^3 + 5bu^2 + 4cu + 3d.
\]
Clearly, \(3f(u) + uf'(u)\) is a increasing and then decreasing function on \((0,\infty)\). Since
\[
3f(\tau) + \tau f'(\tau) = \frac{27a^2d + 4b^3 + 24abc}{9a^2} > 0,
\]
we see that
\[
3f(u) + uf'(u) > 0 \quad \text{for} \quad 0 < u \leq \tau.
\]
(37)

In addition, we compute that
\[
\theta(u) = \frac{1}{6u} (3au^3 - 2bu^2 + 6d).
\]
Then by [8, 10], we have the following results:

(G1) when \(a \geq \bar{a} \equiv \sqrt{\frac{b^3}{27a}}\), \(\theta(u)\) is strictly increasing on \((0,\beta)\).
(G2) when \(0 < a < \bar{a}\), there exist \(0 < p_1(a) < p_2(a) < \beta\) such that

\[
\theta'(u) \begin{cases} 
< 0 & \text{for } u \in (0, p_1(a)) \cup (p_2(a), \beta), \\
0 & \text{for } u = p_1(a) \text{ and } p_2(a), \\
> 0 & \text{for } u \in (p_1(a), p_2(a)).
\end{cases}
\]

Furthermore,

\[
\lim_{a \to \bar{a}^-} p_1(a) = \lim_{a \to \bar{a}^-} p_2(a) = \frac{b}{3\bar{a}}.
\]

Assume that \(a \geq \bar{a}\). Then (H3) follows by (37) and (G1). Assume that \(0 < a < \bar{a}\).

For the sake of convenience, we let \(\theta(u, a) = \theta(u)\). Then we observe that

\[
\frac{\partial}{\partial a} \theta(p_1(a), a) = \theta'(p_1(a), a) \frac{\partial p_1(a)}{\partial a} + \frac{\partial \theta(u, a)}{\partial a} \bigg|_{u=p_1(a)} = \frac{\partial \theta(u, a)}{\partial a} \bigg|_{u=p_1(a)} = \frac{1}{2} p_1''(a) > 0.
\]

By (G2) and (38), we see that, for \(0 < a < \bar{a}\),

\[
\theta(p_1(a), a) < \lim_{a \to \bar{a}^-} \theta(p_1(a), a) = \theta(\frac{b}{3\bar{a}}, \bar{a}) = \frac{bd}{3a} - \frac{b^4}{162a^3} < \frac{2bd}{3a} = \theta(\tau).
\]

It follows that \(\theta(\tau) > \theta(u)\) for \(0 < u < \tau\). Moreover, \(\theta(\alpha) > \theta(u)\) for \(\tau \leq \alpha < \beta\) and \(0 < u < \alpha\). Then (H3) holds by (37).

The proof is complete.

**Proof of Theorem 3.3.** By elementary analysis, there exists \(\beta > 0\) such that \(f(u) > 0\) on \((0, \beta)\), \(f(\beta) = 0\) and \(f(u) < 0\) on \((\beta, \infty)\). We compute that

\[
\eta = \begin{cases} 
\infty & \text{if } 0 < p < 1, \\
\frac{a+c}{2} & \text{if } p = 1, \\
\frac{c}{2} & \text{if } p > 1.
\end{cases}
\]

By Theorem 2.1, the bifurcation curve \(S_L\) of (7) starts from \((\kappa, 0)\) and goes to \((\infty, m_{L, \beta})\) for \(L > 0\) where \(\kappa\) is defined by (9). Let \(g(u) = f(u)/u = au^{p-1} - bu^{q-1} + c\). Then we have that

\[
g'(u) = \begin{cases} 
u^{p-2} [a(p-1) - b(q-1)u^{q-p}] & \text{if } p \neq 1, \\
b(q-1)u^{q-2} & \text{if } p = 1.
\end{cases}
\]

Next, we divide this proof into the following six steps.

**Step 1.** We prove statement (i). Assume that \(c = 0\). Then we consider two cases.

**Case 1.** Assume that \(0 < p \leq 1\). Since \(q > p\), and by (40), we see that \(g'(u) < 0\) for \(u > 0\). So by Theorem 2.2(i), the bifurcation curve \(S_L\) is monotone increasing for all \(L > 0\).

**Case 2.** Assume that \(p > 1\). Since (C1) holds by (39), and by Theorem 2.1(i), the bifurcation curve \(S_L\) is \(C\)-shaped for all \(L > 0\).

Thus statement (i) holds by Cases 1 and 2.

**Step 2.** We prove statement (ii)(a). Assume that \(0 < p \leq 1\). Since \(q > p\), and by (40), we see that \(g'(u) < 0\) for \(u > 0\). So by Theorem 2.2(i), the bifurcation curve \(S_L\) is monotone increasing for all \(L > 0\). Then statement (a) holds.

**Step 3.** We prove statement (ii)(b). Assume that \(1 < p < 3\). Since

\[
\begin{cases} 
\ f''(0^+) > 0 & \text{if } 1 < p \leq 2, \\
\ f''(0^+) = 0 & \text{if } 2 < p < 3,
\end{cases}
\]

then
we see that (C3) holds if $1 < p \leq 2$, and (C5) holds if $2 < p < 3$. So by Theorem 2.1, the bifurcation curve $S_L$ of (7) is $<$-shaped for all $L > 0$. Then statement (ii)(b) holds.

**Step 4.** We prove that (H3) holds if $\Psi(p, q, a, b, c) > 0$ and $p > 1$. Assume that $\tau < \beta$. Since $\Psi(p, q, a, b, c) > 0$, we see that, for $0 < u \leq \tau$, we further observe that $\theta(u)$ is strictly decreasing and then strictly increasing on $(0, \infty)$. Since

$$\theta(u) = \frac{b(q-1)u^{p+1}}{(q+1)} \left\{ \begin{array}{ll}
< 0 & \text{for } 0 < u < \tau, \\
= 0 & \text{for } u = \tau, \\
> 0 & \text{for } \tau < u < \beta,
\end{array} \right.$$

we compute and find that

$$f(\tau) = \tau^p \left( a - b\tau^{q-p} \right) + ct = \frac{2a(q-p)\tau^p}{(q-1)(p+1)} + ct > 0,$$

which implies that $\tau < \beta$. Since $\theta'(u) = -u^2g'(u)$, and by (40), we observe that $\theta(u)$ is strictly decreasing and then strictly increasing on $(0, \infty)$. Since

$$\theta(u) = \frac{b(q-1)u^{p+1}}{(q+1)} \left\{ \begin{array}{ll}
< 0 & \text{for } 0 < u < \tau, \\
= 0 & \text{for } u = \tau, \\
> 0 & \text{for } \tau < u < \beta,
\end{array} \right.$$

we further observe that $\theta(\alpha) > \theta(u)$ for $\tau < \alpha < \beta$ and $0 < u < \alpha$. In addition, since $\Psi(p, q, a, b, c) > 0$, we see that, for $0 < u \leq \tau$,

$$3f(u) + uf'(u) = u^p \left[ a(p+3) - b(q+3)u^{q-p} + \frac{4c}{u^{p-1}} \right]$$

$$\geq u^p \left[ a(p+3) - b(q+3)\tau^{q-p} + \frac{4c}{\tau^{p-1}} \right]$$

$$= u^p \left[ \frac{a(q-p)}{p+1} \left( \frac{q+7}{q-1} - p \right) + \frac{4c}{\tau^{p-1}} \right]$$

$$= u^p \Psi(p, q, a, b, c) > 0.$$}

Thus (H3) holds.

**Step 5.** We prove statement (ii)(c). Assume that $p = 3$. Since $f''(0^+) = 0$ and $f'''(0^+) = 6a > 0$, we see that (C7) holds. In addition, assume that $\Psi(3, q, a, b, c) > 0$. Then (H3) holds by Step 4. So by Theorems 2.1 and 2.4, statement (ii)(c) holds.

**Step 6.** We prove statement (ii)(d). Assume that $p > 3$. Since $f''(0^+) = 0$ and $f'''(0^+) = 0$, we see that (C6) holds. In addition, assume that $\Psi(p, q, a, b, c) > 0$. Then (H3) holds by Step 4. So by Theorem 2.3, statement (ii)(d) holds.

The proof is complete.

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