HYPERREFLECTION GROUPS

DAVID G RADCLIFFE

Abstract. We introduce the concept of hyperreflection groups, which are a generalization of Coxeter groups. We prove the Deletion and Exchange Conditions for hyperreflection groups, and we discuss special subgroups and fundamental sectors of hyperreflection groups. In the second half of the paper, we prove that Coxeter groups and graph products of groups are examples of hyperreflection groups.

1. Introduction

This article introduces the concept of hyperreflection groups, which are a fruitful generalization of Coxeter groups (also called reflection groups). A hyperreflection is a kind of multiple reflection. In the case of reflection symmetry on a connected space, the fixed points of the reflection separate the space into two components which are interchanged by the reflection. A hyperreflection is a group action on a connected space whose fixed points separate the space into many components, and for any two components there is a unique group element that maps one to the other. A hyperreflection group is a group that is generated by hyperreflections.

Since hyperreflections need not have order two, they are much more general than reflections. We will prove that graph products of groups are hyperreflection groups. The graph product is a very general construction that includes the weak direct product and the free product as special cases. We will also show that Coxeter groups are hyperreflection groups.

On the other hand, hyperreflection groups are not hopelessly general. They retain many of the properties of Coxeter groups, and many results in the theory of Coxeter groups can be translated to this more general setting.

2. Hypergraphs and Cayley Hypergraphs

In this section, we state the basic definitions concerning hypergraphs, and we define the Cayley hypergraph. The reader who wishes to learn more about hypergraphs is referred to [2]. It should be noted that the term “Cayley hypergraph” is not standard, but it was named by analogy to Cayley graphs, which will be discussed in section 8.

A hypergraph is a pair \((V, E)\) where \(E\) is a set of nonempty subsets of \(V\). \(V\) is called the vertex set and \(E\) is called the edge set. A hypergraph differs from a graph insofar as an edge of a hypergraph can contain an arbitrary number of vertices, but an edge of a graph always contains exactly two vertices. A subgraph of a hypergraph \((V, E)\) is a hypergraph \((V', E')\) such that \(V' \subseteq V\) and \(E' \subseteq E\). A hypergraph \((V, E)\) is disconnected if there exists a partition of \(V\) into two nonempty disjoint subsets such that no edge contains elements from both subsets. A hypergraph is connected if it is not disconnected. A component of a hypergraph is a maximal connected subgraph that contains at least one vertex. The vertex set of a component will also be called a component. A disconnected hypergraph is the disjoint union of its components.

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A walk in \((V,E)\) is an alternating sequence of vertices and edges
\[ p = (v_0, e_1, v_1, e_2, \ldots, v_n) \]
such that \(\{v_{i-1}, v_i\} \subseteq e_i\) for all \(1 \leq i \leq n\). A walk may have repeated vertices or edges. We say that the \(p\) is a walk of length \(n\) from \(v_0\) to \(v_n\). Two vertices \(u\) and \(v\) belong to the same component if and only if there exists a walk from \(u\) to \(v\).

If \(G\) is a group and \(\Sigma\) is a collection of nontrivial subgroups of \(G\), then the Cayley hypergraph of \((G, \Sigma)\), denoted \(\text{Cay}(G, \Sigma)\), is the hypergraph whose vertex set is \(G\) and whose edge set is
\[ \{gS : (g, S) \in G \times \Sigma\}. \]
The Cayley hypergraph is connected if and only if \(\bigcup \Sigma\) generates \(G\). \(G\) acts on \(\text{Cay}(G, \Sigma)\) by left multiplication, and the action is simply transitive on the vertex set.

3. Hyperreflections

In this section, we define hyperreflections and introduce their most elementary properties. We will use hyperreflections in the next section to define hyperreflection groups, which are groups that are generated by hyperreflections.

Let \(X\) be a connected space. The term “space” is meant to include graphs, hypergraphs, and topological spaces, but it may refer to any sort of geometric object for which a notion of connectivity can be meaningfully defined (see [12]). If \(G\) is a group of automorphisms of \(X\) then let \(\text{Fix}(G)\) denote the fixed set of \(G\), i.e.
\[ \text{Fix}(G) = \{x \in X : \forall g \in G, gx = x\}. \]

A nontrivial subgroup \(R\) of \(\text{Aut}(X)\) is a called a hyperreflection if \(R\) acts simply transitively on the components of \(X \setminus \text{Fix}(R)\). This means that \(\text{Fix}(R) \neq X\), and if \(C_1\) and \(C_2\) are two components of \(X \setminus \text{Fix}(R)\) then there is a unique \(r \in R\) such that \(rC_1 = C_2\). The fixed sets of hyperreflections are called walls. A hyperreflection of order two is called a reflection.

**Theorem 3.1.** Let \(R\) be a hyperreflection on \(X\) and let \(1 \neq r_1 \in R\). Then \(\text{Fix}(R) = \text{Fix}(r_1) := \{x \in X : r_1x = x\}\).

*Proof.* If \(x \in \text{Fix}(R)\) then \(rrx = x\) for all \(r \in R\), so in particular \(r_1x = x\). Now suppose that \(x \notin \text{Fix}(R)\), and let \(C\) be the component of \(X \setminus \text{Fix}(R)\) that contains \(x\). Then \(r_1C\) is also a component of \(X \setminus \text{Fix}(R)\). Since \(R\) acts simply transitively on the set of components, \(r_1C \neq C\). Therefore \(r_1C \cap C = \emptyset\), hence \(r_1x \neq x\). \(\Box\)

**Theorem 3.2.** If \(R\) and \(S\) are hyperreflections on \(X\), and there exists \(t \in R \cap S\) with \(t \neq 1\), then \(\text{Fix}(R) = \text{Fix}(S)\).

*Proof.* By Theorem 3.1, \(\text{Fix}(R) = \text{Fix}(t) = \text{Fix}(S)\). \(\Box\)

**Theorem 3.3.** If \(R\) is a hyperreflection on \(X\) and if \(\sigma \in \text{Aut}(X)\) then \(R^\sigma := \sigma R \sigma^{-1}\) is a hyperreflection on \(X\).

*Proof.* Note that \(\text{Fix}(R^\sigma) = \sigma \text{Fix}(R)\), since \(x \in \text{Fix}(R) \iff R x = x \iff \sigma R \sigma^{-1} \cdot \sigma x = \sigma x\). Also, \(X \setminus \text{Fix}(R^\sigma)\) is disconnected since \(X \setminus \text{Fix}(R^\sigma) = \sigma (X \setminus \text{Fix}(R))\). Let \(C_1\) and \(C_2\) be components of \(X \setminus \text{Fix}(R^\sigma)\). Then \(\sigma^{-1}C_1\) and \(\sigma^{-1}C_2\) are components of \(X \setminus \text{Fix}(R^\sigma)\). So, there is a unique \(r \in R\) such that \(r \cdot \sigma^{-1}C_1 = \sigma^{-1}C_2\). Let \(s = \sigma \sigma^{-1} \in R^\sigma\). Then \(sC_1 = C_2\), and \(s\) is unique, because \(r\) is unique. Therefore \(R^\sigma\) is a hyperreflection on \(X\). \(\Box\)

**Theorem 3.4.** If \(R\) and \(S\) are hyperreflections on \(X\) and if \(R \subseteq S\) then \(R = S\).
Proof. Fix(\(R\)) = Fix(\(S\)) by Theorem 3.2. Let 1 \(\neq\) \(s\) \(\in\) \(S\) and let \(C\) be a component of \(X \setminus\) Fix(\(S\)). Then \(C\) and \(sC\) are distinct components of \(X \setminus\) Fix(\(S\)), so they are also distinct components of \(X \setminus\) Fix(\(R\)). Therefore there is a unique element \(r \in R\) such that \(rC = sC\). Since \(S\) acts freely on the components of \(X \setminus\) Fix(\(S\)), it follows that \(r = s\). Therefore \(S \subseteq R\), hence \(R = S\). \(\square\)

It is not true that distinct hyperreflections are always disjoint. For example, let \(X\) be the union of the coordinate axes in the plane, viewed as a topological space. Let \(R\) be the subgroup of Aut(\(X\)) that is generated by the 90° rotation \(r(x, y) = (-y, x)\), and let \(S\) be the subgroup generated by \(s(x, y) = (-2y, x/2)\). Then \(R\) and \(S\) are both hyperreflections on \(X\), but \(r^2 = s^2 \neq 1\).

4. Hyperreflection Systems

Let \(G\) be a group, let \(\Sigma\) be a set of nontrivial subgroups of \(G\) whose union generates \(G\), and let \(X = \text{Cay}(G, \Sigma)\). We say that \((G, \Sigma)\) is a hyperreflection system if the action of each element of \(\Sigma\) by left multiplication on \(X\) is a hyperreflection. If \((G, \Sigma)\) is a hyperreflection system, then the elements of \(\Sigma\) are called fundamental hyperreflections.

Note that \((G, \{G\})\) is a hyperreflection system for any nontrivial group \(G\). Such a hyperreflection system is called trivial. We say that \(G\) is a hyperreflection group if there exists a set \(\Sigma\) of proper subgroups of \(G\) such that \((G, \Sigma)\) is a hyperreflection system.

We assume for the remainder of the section that \((G, \Sigma)\) is a hyperreflection system.

Theorem 4.1. A subgroup \(T\) of \(G\) fixes the edge \(gS\) if and only if \(T \subseteq S^g\).

Proof. \(T\) fixes \(gS\) \(\iff\) \(TgS = gS\) \(\iff\) \(TgSg^{-1} = gSg^{-1}\) \(\iff\) \(T \subseteq S^g\). \(\square\)

We will assume for the remainder of this section that \((G, \Sigma)\) is a hyperreflection system.

Theorem 4.2. Let \(A, B \in \Sigma\) and let \(h, k \in G\). Then either \(A^h = B^k\) or \(A^h \cap B^k = \{1\}\). In particular, if \(A \neq B\) then \(A \cap B \neq \{1\}\).

Proof. Suppose that 1 \(\neq\) \(g \in A^h \cap B^k\). Then \(g\) fixes the edges \(hA\) and \(kB\) by Theorem 4.1. By Theorem 3.3, \(A^h\) and \(B^k\) are hyperreflections. By Theorem 3.2, \(A^h\) fixes the edge \(kB\), and \(B^k\) fixes the edge \(hA\). Therefore, \(A^h \subseteq B^k\) and \(B^k \subseteq A^h\) by Theorem 4.1, hence \(A^h = B^k\). \(\square\)

Theorem 4.3. If \(T\) is a subgroup of \(G\), and \(T\) is a hyperreflection that fixes the edge \(gA\), then \(T = A^g\). In particular, \(T\) is a hyperreflection if and only if it is a conjugate of a fundamental hyperreflection.

Proof. \(T \subseteq A^g\) by Theorem 4.1, therefore \(T = A^g\) by Theorems 3.3 and 3.4. Since every hyperreflection fixes an edge, and conjugates of hyperreflections are hyperreflections, it follows that \(T\) is a hyperreflection if and only if it is conjugate to some \(A \in \Sigma\). \(\square\)

Theorem 4.4. If \(R, S \in \Sigma, \ g, h \in G, \) and Fix(\(R^g\)) = Fix(\(S^h\)), then \(R^g = S^h\).

Proof. If \(e \in \text{Fix}(R^g) \cap \text{Fix}(S^h)\) then \(e = kA\) for some \(A \in \Sigma\) and \(k \in G\). Therefore \(R^g = A^k\) and \(S^h = A^k\) by Theorem 4.3, hence \(R^g = S^h\). \(\square\)

5. Words and Reduced Words

Let \(G\) be a group and let \(\Sigma\) be a set of nontrivial subgroups of \(G\). A word in \((G, \Sigma)\) of length \(n\) is a pair of sequences

\[ s = ((s_1, \ldots, s_n), (S_1, \ldots, S_n)) \]
such that \( 1 \neq s_i \in S_i \in \Sigma \) for all \( i \). The elements \( s_i \) are called letters. If the subgroups in \( \Sigma \) are pairwise disjoint then the \( S_i \) are determined uniquely by the \( s_i \). In that case, we will call \((s_1, \ldots, s_n)\) a word, since there is no ambiguity. Recall that if \((G, \Sigma)\) is a hyperreflection system then the elements of \( \Sigma \) are disjoint by Theorem 4.2.

A word determines certain other important sequences, which we will describe. The word \( s \) determines a sequence of partial products \((g_0, \ldots, g_n)\) which may be defined recursively as follows:

\[
\begin{align*}
g_0 &= 1_G, \\
g_i &= g_{i-1}s_i & \text{for } 1 \leq i \leq n.
\end{align*}
\]

The word \( s \) is said to represent \( g_n \). Two words are equivalent if they represent the same group element. A word is reduced if there is no shorter word that represents the same element. The length of a group element \( g \) is denoted \( \ell(g) \), and is defined as the length of a reduced word that represents \( g \).

The word \( s \) also determines a dual word

\[
t = ((t_1, \ldots, t_n), (T_1, \ldots, T_n))
\]

defined as follows:

\[
\begin{align*}
t_i &= g_i g_{i-1}^{-1} = g_{i-1}s_i g_{i-1}^{-1}, \\
T_i &= g_i^{-1}S_ig_{i-1}.
\end{align*}
\]

The reader may verify that

\[
g_i = s_1 \cdots s_i = t_i \cdots t_1
\]

for all \( 1 \leq i \leq n \), and that the dual word of \( t \) is \( s \).

These definitions are best understood in the context of the Cayley hypergraph \( \text{Cay}(G, \Sigma) \). The word \( s \) corresponds to a walk \((g_0, e_1, g_1, e_2, \ldots, g_n)\) from 1 to \( g_n \), where \( e_i = g_{i-1}S_i = g_iS_i \). Since \( g_i = g_{i-1}s_i = t_ig_{i-1} \), we have two different ways to describe how we move from one vertex \( g_{i-1} \) to the next vertex \( g_i \). We can either multiply by \( s_i \) on the right, or multiply by \( t_i \) on the left. The subgroup \( T_i \) is the stabilizer of the \( i \)th edge along the walk.

We will maintain these notations for the remainder of the article, so if a word \( s \) is defined, then we consider the sequences \((S_i), (g_i), (t_i), \) and \((T_i)\) to be defined as well.

6. The Deletion and Exchange Conditions

In this section we describe two conditions, called the Deletion and Exchange Conditions, that are satisfied by any hyperreflection system. These conditions illustrate the important role of dual words in the reduction of words in hyperreflection systems. We will assume throughout this section that \((G, \Sigma)\) is a hyperreflection system.

The first theorem shows that any word in a hyperreflection system can be reduced by successive deletion and replacement of letters. We call this theorem the Deletion Condition, because it generalizes the Deletion Condition for Coxeter groups \([4, 8]\).

**Theorem 6.1.** Let \( s = (s_1, \ldots, s_n) \) be a word representing \( g \in G \). Then the following statements hold.

1. If \( t_i = t_j^{-1} \) and \( i < j \), then \( g = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_n \), where the hats indicate that the letters \( s_i \) and \( s_j \) are to be deleted.
2. If \( T_i = T_j, \) \( t_i \neq t_j^{-1} \), and \( i < j \), then there exists \( 1 \neq \tilde{s}_i \in S_i \) such that \( g = s_1 \cdots \tilde{s}_i \cdots \hat{s}_j \cdots s_n \). In other words, we replace \( s_i \) with another non-identity element of \( S_i \), and we delete \( s_j \).
Proof. We begin by observing that $s_1 \cdots s_n = t_j t_i s_1 \cdots s_i \cdots s_j \cdots s_n$. 

If $t_i = t_j^{-1}$ then the last expression is equal to $s_1 \cdots s_i \cdots s_j \cdots s_n$, which proves (1).

Suppose that $T_i = T_j$ and $t_i \neq t_j^{-1}$. Then $t_j t_i g_{i-1} S_i \in T_i g_{i-1} S_i = g_{i-1} S_i$, so there exists $1 \neq \tilde{s}_i \in S_i$ such that $t_j t_i g_{i-1} = g_{i-1} \tilde{s}_i$. Consequently, $s_1 \cdots s_n = s_1 \cdots \tilde{s}_i \cdots s_j \cdots s_n$, which proves (2).

One can prove part (3) by applying part (2) to the inverse word $s^{-1} = (s_n^{-1}, \ldots, s_1^{-1})$. If $s$ is reduced then $T_i \neq T_j$ for all $i \neq s$, since otherwise we could reduce the word length by applying (2) or (3). The converse is proved in the next theorem.

**Theorem 6.2.** Let $s = (s_1, \ldots, s_n)$ and $s' = (s'_1, \ldots, s'_n)$ be two words representing $g$.

1. If $T_i \neq T_j$ for all $i \neq j$ then $\{T_1, \ldots, T_n\} \subseteq \{T'_1, \ldots, T'_n\}$.
2. If $T_i \neq T_j$ for all $i \neq j$ then $s$ is reduced.
3. If $s$ and $s'$ are both reduced, then $m = n$ and $\{T_1, \ldots, T_m\} = \{T'_1, \ldots, T'_n\}$.

**Proof.** Suppose that $T_i \neq T_j$ for all $i < j$. The word $s$ corresponds to a walk $(g_0, e_1, g_1, \ldots, g_n)$ in Cay($G$, $\Sigma$), and this walk crosses each wall Fix($T_i$) exactly once. Since these walls separate 1 and $g$, they must be crossed by every walk from 1 to $g$. Therefore $T_i \in \{T'_1, \ldots, T'_n\}$ for all $i$, which implies that $\{T_1, \ldots, T_m\} \subseteq \{T'_1, \ldots, T'_n\}$. This also implies that $m \leq n$, which proves (2).

If $s$ and $s'$ are both reduced, then $T_i \neq T_j$ for all $i < j$, and $T'_i \neq T'_j$ for all $i < j$. Part (1) implies that $\{T_1, \ldots, T_m\} \subseteq \{T'_1, \ldots, T'_n\}$ and $\{T'_1, \ldots, T'_n\} \subseteq \{T_1, \ldots, T_m\}$, hence the two sets are equal.

**Theorem 6.3.** If $(s_1, \ldots, s_n)$ and $(s'_1, \ldots, s'_n)$ are two reduced words representing $g$ then $\{t_1, \ldots, t_n\} = \{t'_1, \ldots, t'_n\}$.

**Proof.** Let $i \in \{1, 2, \ldots, n\}$. The walk $(g_0, e_0, g_1, \ldots, g_n)$ crosses each wall Fix($T_i$) exactly once, since $T_i \neq T_j$ for $i \neq j$ by Theorem 6.1. The walk $(g'_0, g'_1, \ldots, g'_n)$ must also cross each wall Fix($T'_i$) exactly once, since $T'_i \neq T'_j$ for $i \neq j$. Both walks travel from $C_i$ to $t_i C_i$, so there exists $j$ such that $g'_j \in C_i$ and $g'_j \in t_i C_i$, which implies that $t'_j = t_i$.

Since $i$ was arbitrary, it follows that $\{t_1, \ldots, t_n\} \subseteq \{t'_1, \ldots, t'_n\}$, therefore $\{t_1, \ldots, t_n\} = \{t'_1, \ldots, t'_n\}$ by symmetry.

**Theorem 6.4.** If $g \in G$ and $S \in \Sigma$ then the right coset $Sg$ has a unique element of minimal length, and the left coset $gS$ also has a unique element of minimal length. (cf. Theorem 7.6.)

**Proof.** Let $w$ and $w'$ be two distinct elements of minimal length in $Sg$, and choose a reduced word $(s_1, \ldots, s_n)$ representing $w$. Then there exists $1 \neq s \in S$ such that $w' = sw = ss_1 \cdots s_n$. 

$\square$
The word \((s, s_1, \ldots, s_n)\) is not reduced, so we can reduce the length via deletion. The deletion must involve the first letter \(s\), since \((s_1, \ldots, s_n)\) is reduced. Therefore, there exists \(1 \leq i \leq n\) and \(s' \in S\) such that \(w' = s's_1 \ldots \widehat{s_i} \ldots s_n\). But \((s')^{-1}w' \in Sg\) and \(\ell((s')^{-1}w') < n\), which is a contradiction. This proves that \(Sg\) has a unique element of minimal length.

Note that if \(w\) is the unique element of minimal length in \(Sg\) then \(w^{-1}\) is the unique element of minimal length in \(g^{-1}S\), and so every left coset of a fundamental hyperreflection has a unique element of minimal length. \(\square\)

We conclude the section with a theorem that we call the Exchange Condition, which is a generalization of the Exchange Condition for Coxeter groups \([4, 8]\).

**Theorem 6.5.** Let \(s = (s_1, \ldots, s_n)\) be a reduced word for \(g\), and let \(1 \neq s_0 \in S_0 \in \Sigma\). Then \(|\ell(s_0^{-1}g) - \ell(g)| \leq 1\), and the following statements hold.

1. If \(\ell(s_0^{-1}g) = \ell(g) - 1\) then there exists \(1 \leq i \leq n\) such that \(g = s_0s_1 \ldots \widehat{s_i} \ldots s_n\).
2. If \(\ell(s_0^{-1}g) = \ell(g)\) then there exists \(1 \leq i \leq n\) and \(\widehat{s_i} \in S_i \setminus \{1, s_i\}\) such that \(g = s_0s_1 \ldots \widehat{s_i} \ldots s_n\).
3. If \(\ell(s_0^{-1}g) = \ell(g) + 1\) then no reduced word for \(g\) begins with an element of \(S_0\).

**Proof.** \(|\ell(s^{-1}g) - \ell(g)| \leq 1\) is a consequence of the triangle inequality for word length.

Suppose that \(\ell(s_0^{-1}g) = \ell(g) - 1\). Note that \(s_0^{-1}g\) is the unique element of minimal length in \(S_0g\). The word \((s_0^{-1}, s_1, \ldots, s_n)\) for \(s_0^{-1}g\) is not reduced, so it can be reduced using the Deletion Condition. Since \((s_1, \ldots, s_n)\) is reduced, the deletion must involve the first letter. Therefore, there exists \(1 \leq i \leq n\) and \(s \in S_0\) such that \(s_0^{-1}g = ss_1 \ldots \widehat{s_i} \ldots s_n\), hence \(g = (s_0s)s_1 \ldots \widehat{s_i} \ldots s_n\).

Let \(s' = (s_0s)^{-1}\). Then \(s'g \in S_0g\) and \(\ell(s'g) \leq n - 1\), so \(s'g\) has minimal length in \(S_0g\). Therefore \(s' = s_0^{-1}\), which implies \(s = 1\). Thus, \(g = s_1 \ldots \widehat{s_i} \ldots s_n\), which proves part (1).

We prove part (2) by applying the Deletion Condition to \(s_0^{-1}g = s_0^{-1}s_1 \ldots s_n\). As before, the deletion must involve the first letter, so \(s_0^{-1}g = s_1 \ldots \widehat{s_i} \ldots s_n\) for some \(\widehat{s_i} \in S_i\). Therefore, \(g = s_0s_1 \ldots \widehat{s_i} \ldots s_n\). But \(\widehat{s_i} \neq 1\) because \(\ell(g) = n\), and \(\widehat{s_i} \neq s_i\) because \(s_0 \neq 1\).

If \(g\) has a reduced word of length \(n\) that begins with \(s \in S_0\), then \(s_0^{-1}g\) has a word of length \(n\) that begins with \(s_0^{-1}s\), which proves part (3). \(\square\)

### 7. Special Subgroups and Sectors

Let \((G, \Sigma)\) be a hyperreflection system. If \(A \subseteq \Sigma\) then let \(G_A\) denote the subgroup of \(G\) that is generated by \(\bigcup A\). If \(A = \{A\}\) then we will sometimes write \(G_A\) instead of \(G_{\{A\}}\). We define \(G_\emptyset\) to be the identity subgroup. A subgroup of the form \(G_A\) is called a special subgroup.

**Theorem 7.1.** If \(A \subseteq \Sigma\), \(R \in \Sigma\), and \(G_A \cap R \neq \{1\}\) then \(R \in A\).

**Proof.** Let \(1 \neq g \in R \subseteq G_A\), and let \((s_1, \ldots, s_n)\) be a word of minimal length such that \(1 \neq s_i \in S_i \in A\) and \(g = s_1 \ldots s_n\).

We claim that \((s_1, \ldots, s_n)\) is reduced. Suppose not; then by Theorem 6.1 we can reduce the word to a shorter word by successive deletions. But each letter in the shorter word also belongs to \(\bigcup A\), and this contradicts minimality. The shortest possible word for \(g\) has length 1, since \(g \in R \subseteq \Sigma\). Therefore, \(g \in R \cap S_1\), so \(R = S_1 \in A\) by Theorem 4.2. \(\square\)

**Theorem 7.2.** If \((s_1, \ldots, s_n)\) and \((s'_1, \ldots, s'_n)\) are two reduced words for \(g\) then \(\{S_1, \ldots, S_n\} = \{S'_1, \ldots, S'_n\}\).
Proof. Let $A = \{S_1, \ldots, S_n\}$ and $B = \{S'_1, \ldots, S'_n\}$. Then $t_i \in G_A$ for all $i$, so $t'_i \in G_A$ for all $i$ by Theorem 6.3. Since $w_i' = t'_i \cdots t'_1$ for all $i$, it follows that $w_i' \in G_A$ for all $i$. But $w_i = w_i'-1s_i$, thus $s_i' \in G_A$ for all $i$. Therefore, $S'_i \cap G_A \neq \{\emptyset\}$ for all $i$, hence $S'_i \in A$ for all $i$ by Theorem 7.1. This shows that $B \subseteq A$. By symmetry, $A \subseteq B$, therefore $A = B$. \hfill \Box

**Theorem 7.3.** If $(s_1, \ldots, s_m)$ is a reduced word for $g$, and $(s'_1, \ldots, s'_n)$ is any other word representing $g$, not necessarily reduced, then $\{S_1, \ldots, S_m\} \subseteq \{S'_1, \ldots, S'_n\}$.

Proof. If $(s'_1, \ldots, s'_n)$ is not reduced then it may be transformed to an equivalent reduced word $(s''_1, \ldots, s''_p)$ by successive deletions (Theorem 6.1). It is clear that $\{S''_1, \ldots, S''_p\} \subseteq \{S_1, \ldots, S_m\}$. By Theorem 7.2, $\{S''_1, \ldots, S''_p\} = \{S_1, \ldots, S_m\}$. Therefore, $\{S_1, \ldots, S_m\} \subseteq \{S'_1, \ldots, S'_n\}$. \hfill \Box

**Theorem 7.4.** If $A \subseteq \Sigma$ then $(G_A, A)$ is a hyperreflection system.

Proof. Let $X_0 = \text{Cay}(G_A, A)$ and $X = \text{Cay}(G, \Sigma)$. Note that $X_0$ is a connected subgraph of $X$. Let $A \in A$ be given. Let $C$ be the identity component of $X \setminus \text{Fix}(A)$ and let $C_0 = C \cap X_0$.

Now $X = \text{Fix}(A) \cup \bigcup_{a \in A} aC$, so $X_0 = (\text{Fix}(A) \cap X_0) \cup \bigcup_{a \in A} (aC \cap X_0) = (\text{Fix}(A) \cap X_0) \cup \bigcup_{a \in A} aC_0$. Moreover, $\text{Fix}(A) \cap X_0$ is identical to the fixed set of the action of $A$ on $X_0$, since $X_0$ is invariant under $A$. Since the unions in the expression of $X$ are disjoint, the unions in the expression of $X_1$ are also disjoint. So it only remains to show that $C_0$ is connected.

Let $g$ be any vertex of $C_0$. This implies that $g \in G_A$, and there exists a word $(s_1, \ldots, s_m)$ representing $g$ whose path does not cross $\text{Fix}(A)$, i.e. $T_i \neq A$ for all $1 \leq i \leq n$. Let $(s'_1, \ldots, s'_n)$ be a reduced word representing $g$. Then $S'_i \in A$ for all $i$ by Theorem 7.3, and $T'_i \neq A$ for all $i$ by Theorem 6.2. Therefore, the walk $(g_0, e'_1, g'_1, \ldots, g'_n)$ lies in $C_0$, hence $C_0$ is connected. \hfill \Box

**Theorem 7.5.** If $A, B \subseteq \Sigma$ then $G_{A \cap B} = G_A \cap G_B$.

Proof. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$ it follows that $G_{A \cap B} \subseteq G_A$ and $G_{A \cap B} \subseteq G_B$. Therefore, $G_{A \cap B} \subseteq G_A \cap G_B$.

To prove the reverse inclusion, let $g \in G_A \cap G_B$, and let $(s_1, \ldots, s_n)$ be a reduced word for $g$. Then $S_i \in A$ for all $i$ by Theorem 7.3, and $S_i \in B$ for the same reason. Therefore, $S_i \in A \cap B$ for all $i$, hence $g \in G_{A \cap B}$. \hfill \Box

**Theorem 7.6.** (cf. [4, p. 47]). Suppose that $A, B \subseteq \Sigma$ and that $w$ is an element of minimum length in the double coset $G_A w G_B$. Then any element $w'$ in this double coset can be written in the form $w' = awb$ where $a \in G_A$, $b \in G_B$, and $\ell(w') = \ell(a) + \ell(w) + \ell(b)$. In particular, the double coset has a unique element of minimum length.

Proof. Choose $a \in G_A$ and $b \in G_B$ such that $w' = awb$ and $\ell(a) + \ell(b)$ is as small as possible. Choose reduced words $r$, $s$, and $t$ for $a$, $w$, and $b$, respectively. If the concatenation $rst$ is not reduced, then we can produce an equivalent word of shorter length by deleting two letters, or by deleting one letter and replacing another letter. The two letters cannot occur in the same subword ($r$, $s$, or $t$), because these subwords are reduced. There are three cases to consider:

1. Delete a letter from $s$, and delete or replace a letter from $r$.
2. Delete a letter from $s$, and delete or replace a letter from $t$.
3. Delete a letter from $r$, and delete or replace a letter from $t$.

Note that deleting or replacing a letter from $r$ or $t$ yields another element of the same special subgroup. Thus, the first two cases are impossible as they would yield an element of the double coset that is shorter than $w$, and the third case is impossible because it contradicts the minimality of $\ell(a) + \ell(b)$. Therefore, $rst$ is reduced, hence

\[
\ell(w') = \ell(awb) = \ell(a) + \ell(w) + \ell(b).
\]
If \( w \) and \( w' \) both have minimal length in the double coset, then \( \ell(a) = \ell(b) = 0 \), thus \( w' = w \).

Given a subset \( A \) of \( \Sigma \), let \( G^A \) be the set of all \( g \in G \) such that \( g \) is the unique element of minimal length in the coset \( gG_A \). \( G^A \) is called the fundamental \( A \)-sector.

**Theorem 7.7.** If \( g \in G \) and \( A \subseteq \Sigma \) then there exist unique elements \( h \in G^A \) and \( k \in G_A \) such that \( g = hk \).

*Proof.* Let \( h \) be the unique minimal element \( gG_A \). Then \( h \in G^A \) because \( gG_A = hG_A \). Since \( g \in hG_A \), there exists a unique \( k \in G_A \) such that \( h = hk \). \( \square \)

8. **Coxeter Groups as Hyperreflection Systems**

A Coxeter group is a group having a presentation of the form

\[ W = \langle S \mid (st)^{m(s,t)} = 1 \ (s, t \in S) \rangle \]

where \( S \) is a finite set of generators of \( W \), \( m(s, s) = 1 \) for all \( s \in S \), and \( m(s, t) = m(t, s) \in \{2, 3, 4, ..., \infty \} \) for all \( s, t \in S \) with \( s \neq t \). If \( m(s, t) = \infty \) then the corresponding relation is omitted. It can be shown that \( m(s, t) \) is the order of \( st \) in \( W \) [8, p. 110]. The pair \( (W, S) \) is called a Coxeter system, and \( S \) is a set of Coxeter generators for \( W \).

Coxeter groups appear in nature as the symmetry groups of regular polytopes, and they are important in the theory of Lie algebras, where they arise as subgroups of the isometry groups of root systems [7]. The reader who wishes to learn more about Coxeter groups is referred to [4] or [8].

There is an alternative characterization of Coxeter groups, due to Michael Davis. He proves in [4, Thm 3.3.4] that \( (W, S) \) is a Coxeter system if and only if the Cayley graph of \( (W, S) \) is a reflection system. The Cayley graph \( \text{Cay}(W, S) \) is defined as the graph whose vertex set is \( W \) and whose edge set is \( \{\{w, ws\} : w \in W, s \in S\} \). Observe that \( \{w, ws\} = w\langle s \rangle \), so we may identify \( \text{Cay}(W, S) \) with \( \text{Cay}(W, \Sigma) \) where \( \Sigma = \{\langle s \rangle : s \in S\} \). The definition of a reflection system is rather involved, but in the case of a Cayley graph it reduces to the assertion that each element of \( S \) acts by reflection on the Cayley graph. Therefore, if \( (W, S) \) is a Coxeter system then \( (W, \Sigma) \) is a hyperreflection system.

9. **Graph Products of Groups**

*The next two sections incorporate material from a preprint by the present author [13].*

Given a graph with nontrivial groups as vertices, a group is formed by taking the free product of the vertex groups, with added relations implying that elements of adjacent groups commute. This group is said to be the graph product of the vertex groups. If the graph is discrete then the graph product is the free product of the vertex groups; while if the graph is complete then the graph product is the weak direct product of the vertex groups. See [9] for the definitions of the free product and weak direct product of groups. Graph products were first defined in Elizabeth Green’s Ph.D. thesis [5], and have been studied by many other authors [3,6,11]. In this section, we will characterize graph products of groups by a universal mapping property, and we will present two constructions of the graph product.

Let \( \Gamma = (V, E) \) be a graph, and let \( \{G_v\}_{v \in V} \) be a collection of groups which is indexed by the vertex set of \( \Gamma \). We say that \( (\Gamma, \{G_v\}_{v \in V}) \) is a graph of groups. (This differs from the usual definition, which has vertex groups and edge groups, together with monomorphisms from the edge groups to the vertex groups. See [1].)
A graph product of a graph of groups consists of a group $G$ and a collection of homomorphisms $e_v: G_v \to G$ such that the following conditions hold.

1. If $\{u, v\} \in E$ then $[e_u(x), e_v(y)] = 1$ for all $x \in G_u, y \in G_v$.
2. If $h_u: G_u \to H$ is a collection of homomorphisms such that $[h_u(x), h_v(y)] = 1$ whenever $\{u, v\} \in E$, then there is a unique homomorphism $\phi: G \to H$ such that $\phi \circ e_v = h_v$ for all $v \in V$.

**Theorem 9.1.** The homomorphisms $e_v$ in the definition of graph product are injective, and $G$ is generated by the union of the images of the $e_v$.

**Proof.** Let $v \in V$, and define a homomorphism $h_u: G_u \to G_v$ for each $u \in V$ as follows. If $u = v$ then $h_u(g) = g$ for all $g \in G_u$, otherwise $h_u(g) = 1$ for all $g \in G_u$. Since $[h_u(x), h_v(y)] = 1$ for all $u \neq w$, there is a unique homomorphism $\phi: G \to G_v$ such that $\phi \circ e_u = h_u$ for all $u$. Therefore $\phi \circ e_v = h_v$ for all $v \in V$ by the universal mapping property of graph products, there is a unique homomorphism $\psi$ such that $\psi \circ e_v = e_v$. Therefore we have $\psi \circ \phi \circ e_v = \psi \circ e_v = e_v$. But $\psi \circ \phi$ is injective, hence $e_v$ is also injective. (Note that it also follows that $\phi$ is surjective.)

As for the other assertion, let $G_0$ denote the subgroup of $G$ which is generated by the union of the images of the $e_v$. It is required to prove that $G_0 = G$. Let $j: G_0 \to G$ be the inclusion homomorphism, and let $h_v$ denote the co-restriction of $e_v$ to $G_0$. That is, $j \circ h_v = e_v$ for all $v \in V$. By the universal mapping property of graph products, there is a unique homomorphism $\phi: G \to G_0$ such that $\phi \circ e_v = h_v$ for all $v \in V$. Therefore $(j \circ \phi) \circ e_v = j \circ h_v = e_v$ for all $v \in V$. But $id_G \circ e_v = e_v$, so it follows from the universal mapping property that $j \circ \phi = id_G$. Therefore $G_0 = G$ as claimed.

**Theorem 9.2.** If $(G, e_v)$ and $(H, f_v)$ are two graph products of $(\Gamma, \{G_v\}_{v \in V})$ then there exists an isomorphism $\phi: G \to H$ such that $\phi \circ e_v = f_v$. In other words, the graph product of a graph of groups is unique up to isomorphism.

**Proof.** By the definition of graph product there is a unique homomorphism $\phi: G \to H$ so that $f_v = \phi \circ e_v$, and there is a unique homomorphism $\psi: H \to G$ so that $e_v = \psi \circ f_v$. Therefore $(\psi \circ \phi) \circ e_v = e_v$ and $(\phi \circ \psi) \circ f_v = f_v$. On the other hand, $id_G \circ e_v = e_v$ and $id_H \circ f_v = f_v$, so by the uniqueness property we have $\psi \circ \phi = id_G$ and $\phi \circ \psi = id_H$. Therefore $\phi$ is an isomorphism from $G$ to $H$.

If each $G_v$ is a subgroup of $G$, and each $e_v$ is an inclusion homomorphism, then we say that $G$ is the internal graph product of the $G_v$. In this case we suppress mention of the $e_v$, and say that $G$ is the graph product of the subgroups $G_v$. In general, if $(G, e_v)$ is the graph product of $(\Gamma, \{G_v\}_{v \in V})$, then $G$ is the internal graph product of the subgroups $e_v(G_v)$.

It remains to prove that graph products exist. We will give two different constructions.

Let $F$ denote the free product of the $G_v$. By definition, there exist monomorphisms $\iota_v: G_v \to F$ such that the following condition is satisfied: for any family of homomorphisms $h_v: G_v \to H$ there is a unique homomorphism $\psi: F \to H$ such that $\psi \circ \iota_v = h_v$ for all $v \in V$.

Let $N$ denote the normal closure in $F$ of the set of all commutators $[\iota_u(x), \iota_v(y)]$ where $u$ and $v$ are adjacent vertices, $x \in G_u$ and $y \in G_v$. Let $\pi: F \to F/N$ be the quotient homomorphism and let $e_v = \pi \circ \iota_v$ for all $v \in V$.

**Theorem 9.3.** With the above definitions, $(F/N, e_v)$ is the graph product of $(\Gamma, \{G_v\}_{v \in V})$.

**Proof.** Let $H$ be a group, and let $h_v: G_v \to H$ be a collection of homomorphisms so that $[h_u(x), h_v(y)] = 1$ whenever $\{u, v\} \in E$, $u \in G_u$ and $v \in G_v$. By the definition of free product there is a unique homomorphism $\psi: F \to H$ such that $\psi \circ \iota_v = h_v$ for all $v \in V$.

If $\{u, v\} \in E$, $x \in G_u$ and $y \in G_v$ then $\psi([\iota_u(x), \iota_v(y)]) = [h_u(x), h_v(y)] = 1$, so $N \subseteq \ker(\psi)$. Therefore there is an induced homomorphism $\phi: F/N \to H$ such that $\phi = \psi \circ \pi$. It follows that $\phi \circ e_v = h_v$ for all $v \in V$, as $\phi \circ e_v = \phi \circ \pi \circ \iota_v = \psi \circ \iota_v = h_v$. 


It remains to show that \( \phi \) is unique. To that end, let \( \phi' : F/N \to H \) be a homomorphism such that \( \phi' \circ e_v = h_v \) for all \( v \in V \), and let \( \psi' = \phi' \circ \pi \). Then \( \psi' \circ i_v = \phi' \circ \pi \circ i_v = \phi' \circ e_v = h_v \) for all \( v \in V \). Therefore \( \psi' = \psi \), by uniqueness of \( \psi \). Since \( \pi \) is surjective and \( \phi \circ \pi = \phi' \circ \pi \), it follows that \( \phi = \phi' \).

Therefore \((F/N, e_v)\) is the graph product of \((\Gamma, \{G_v\}_{v \in V})\). \( \square \)

We describe another construction of the graph product. Let \( X \) denote the set of all finite sequences \((g_1, \ldots, g_n)\) where \( 1 \neq g_i \in G_{v_i} \) and \( v_i \in V \) for all \( i \) from 1 to \( n \). We assume that the \( G_v \) are pairwise disjoint except for a common identity element 1. A sequence of this type is called a word, and each entry is a syllable. The length of a word is the number of syllables. We admit the empty word \((\lambda)\), which has length 0.

Given two words \( w = (g_1, \ldots, g_n) \) and \( x = (h_1, \ldots, h_m) \), the product \( wx \) is defined by concatenation: \( wx = (g_1, \ldots, g_n, h_1, \ldots, h_m) \). This product is associative and it has an identity element \( \lambda \), so it gives \( X \) the structure of a monoid. The inverse of \( w \) is defined as \( w^{-1} = (g_1^{-1}, \ldots, g_n^{-1}) \), although it must be noted that \( w^{-1} \) is not the multiplicative inverse of \( w \) in \( X \). In fact \( \lambda \) is the only element of \( X \) which has a multiplicative inverse.

We define non-negative integer powers by the following recursive definition.

\[
w^n = \begin{cases} 
\lambda & \text{if } n = 0, \\
wwn^{n-1} & \text{if } n \geq 1.
\end{cases}
\]

This is extended to negative integer powers by defining \( w^{-n} = (w^{-1})^n \) for \( n \geq 2 \).

Let \( R = R_1 \cup R_2 \cup R_3 \), where

\[
\begin{align*}
R_1 &= \bigcup_{v \in V} \{ (g, g^{-1}), \lambda : g \in G_v \}, \\
R_2 &= \bigcup_{v \in V} \{ (g, h), (gh) : g, h \in G_v, gh \neq 1 \}, \quad \text{and} \\
R_3 &= \bigcup_{\{u, v\} \in E} \{ (g, h), (h, g) : g \in G_u, h \in G_v \}.
\end{align*}
\]

We say that two words \( r \) and \( s \) are elementarily equivalent, denoted \( r \approx s \), if there exist words \( w, x, y, z \) such that \( r = wxz \), \( s = wyz \), and either \( (x, y) \in R \) or \( (y, x) \in R \). Furthermore, \( r \) and \( s \) are said to be equivalent, denoted \( r \sim s \), if there exists a finite sequences of words \( w_0, w_1, \ldots, w_n \) such that \( w_0 = r \), \( w_n = s \), and \( w_{i-1} \approx w_i \) for all \( i \) from 1 to \( n \).

In more intuitive terms, two words are equivalent if the first word can be transformed to the second word by means of the following moves and their inverses.

1. If a syllable \( g \) is followed by its inverse \( g^{-1} \), then delete both syllables.
2. If two successive syllables \( g \) and \( h \) belong to the same vertex group, and if \( gh \neq 1 \), then replace the two syllables with the single syllable \( gh \).
3. If two successive syllables \( g \) and \( h \) belong to adjacent vertex groups, then swap \( g \) and \( h \).

Sometimes we will allow words to contain the identity element 1 as a syllable. In that case we add another rule stating that 1’s can be deleted.

It is clear that the relation defined above is an equivalence relation. Moreover, it preserves multiplication\(^1\) — if \( w \sim x \) and \( y \sim z \) then \( wy \sim xz \). Let \( \Omega \) be the set of equivalence classes of \( X \). Then \( \Omega \) inherits from \( X \) the structure of a monoid. In fact \( \Omega \) is a group, since \( w_{-1}w \sim \lambda \) and \( w^{-1}w \sim \lambda \) for all \( w \in X \). We will write 1 for the identity element \([\lambda]\) of \( \Omega \).

\(^1\)In other words, \( \sim \) is a congruence.
Define $\pi: X \to \Omega$ by $\pi(x) = [x]$. There are natural homomorphisms $e_v: G_v \to \Omega$ defined by $e_v(g) = [(g)]$ for $g \neq 1$ and $e_v(1) = 1$.

**Theorem 9.4.** With the above definitions, $(\Omega, e_v)$ is the graph product of $(\Gamma, \{G_v\}_{v \in V})$.

**Proof.** Let $\{u, v\} \in E$, $1 \neq x \in G_u$, and $1 \neq y \in G_v$. Then $e_v(xy) = e_v(yx)$, since $(x, y) \approx (y, x)$. Therefore $e_v([x, y]) = 1$, and the first condition in the definition of graph product is verified.

Let $h_u: G_v \to H$ be any collection of homomorphisms with the property that $[h_u(x), h_v(y)] = 1$ for all $x \in G_u$, $y \in G_v$ when $\{u, v\} \in E$.

Define $\psi: X \to H$ as follows. If $w = (g_1, \ldots, g_n)$ and $g_i \in G_{v_i}$ for all $i$, then let $\psi(w) = h_{v_1}(g_1) \cdots h_{v_n}(g_n)$. If $w = \lambda$ then let $\psi(w) = 1$.

It is clear that $\psi$ is a monoid homomorphism and that it respects equivalence. Therefore there is a group homomorphism $\phi: \Omega \to H$ such that $\psi = \phi \circ \pi$.

Now if $1 \neq g \in G_v$ then $\phi \circ e_v(g) = \phi( [(g)] ) = \psi( (g) ) = h_v(g)$. Therefore $\phi \circ e_v = h_v$ for all $v \in V$. It remains to show that $\phi$ is unique. To that end, let $\phi': \Omega \to H$ be a homomorphism such that $\phi \circ e_v = h_v$. If $1 \neq g \in G_v$ then $\phi'([(g)]) = \phi' \circ e_v(g) = h_v(g) = \phi \circ e_v(g) = \phi([(g)])$. But $\Omega$ is generated by elements of the form $[(g)]$. Therefore $\phi = \phi'$, and the proof is complete. \qed

10. Normal Forms for Elements of Graph Products

Let $(\Gamma, \{G_v\}_{v \in V})$ be a graph of groups, with graph product $G$. We will realize $G$ as the group of equivalence classes of words from Theorem 9.4. Let $X$ be the set of words used in this construction.

Choose an arbitrary linear ordering $\prec$ of $V$. Let $w = (g_1, \ldots, g_n)$ be a word, where $1 \neq g_i \in G_{v_i}$ and $v_i \in V$ for all $i$. We say that $w$ is reduced if it is not equivalent to any shorter word. We say that $w$ is normal if it satisfies the following conditions:

1. $v_i \neq v_{i+1}$ for all $i$ between 1 and $n - 1$, and
2. if $\{v_i, v_{i+1}\} \in E$ then $v_i \prec v_{i+1}$.

We also consider $\lambda$ to be a normal word.

**Theorem 10.1.** Every element of $G$ is represented by exactly one normal word. The unique normal word representing $g$ is called the normal form of $g$.

**Proof.** The following argument is modeled on the proof by van der Waerden of the normal form theorem for free products [10, 14]. The theorem was first proved by Green [5].

For each $v \in V$ we define $\mu_v: G_v \times X \to X$ by the following recursive algorithm. Let $g \in G_v$ and let $x \in R$.

1. If $g = 1$ then $\mu_v(g, x) = x$.
2. If $g \neq 1$ and $x = \lambda$ then $\mu_v(g, x) = (g)$.
3. Suppose that $g \neq 1$ and $x \neq \lambda$. Let $g_1$ be the first syllable of $x$. Select $v_1 \in V$ such that $g_1 \in G_{v_1}$, and select $y \in X$ such that $x = (g_1) y$.
   a. If $v = v_1$ and $gg_1 = 1$ then $\mu_v(g, x) = y$.
   b. If $v = v_1$ and $gg_1 \neq 1$ then $\mu_v(g, x) = (gg_1) y$.
   c. If $v_1 \prec v$ and $\{v, v_1\} \in E$ then $\mu_v(g, x) = (g_1) \mu_v(g, y)$.
   d. Otherwise $\mu_v(g, x) = (g) x$.

Let $R$ denote the set of normal words of $X$. We claim that if $x \in R$, $v \in V$ and $g \in G_v$ then $\mu_v(g, x) \in R$. The proof is by induction on word length. Suppose that $x \in R$, and that $\mu_v(g, y)$ is normal for every reduced word $y$ which is shorter than $x$. We need to show that
\[ \mu_v(g, x) \in R. \] This is done by checking each of the six cases in the recursive definition. The verification of these cases is left to the reader.

A similar case-by-case analysis shows that if \( g, h \in G_v \) and \( x \in R \) then \( \mu_v(g, \mu_v(h, x)) = \mu_v(gh, \mu_v(x)) \). Since \( \mu_v(g, \mu_v(g^{-1}, x)) = x \), it follows that \( \mu_v(g, \cdot) \) is a permutation of \( R \) for each \( g \in G \). Therefore there is a homomorphism \( h_v: G_v \to \text{Perm}(R) \) defined by \( h_v(g) = \mu_v(g, \cdot) \).

It is a routine matter to verify that \([h_u(g), h_v(k)] = 1\) whenever \( \{u, v\} \in E \). Therefore there exists a homomorphism \( \phi: G \to \text{Perm}(R) \) such that \( \phi(g) = \mu_v(g, \cdot) \) when \( g \in G_v \).

This homomorphism allows us to compute for any word \( x \) an equivalent normal word \( w \). Let \( x = (x_1, \ldots, x_n) \) and let \( w = \phi([x])(\lambda) \). Then \( w \) is a normal word, and \( w \) is equivalent to \( x \). On the other hand, \( w = \phi([w])(\lambda) \), so \( w \) is the only normal word which is equivalent to \( x \). For if \( w' \) is another normal word equivalent to \( x \), then \( w' = \phi([w'])(\lambda) = \phi([w])(\lambda) = w \). \( \square \)

**Theorem 10.2.** Every normal word is reduced.

**Proof.** Let \( x \) be a normal word, and let \( y \) be a reduced word which is equivalent to \( x \). If \( y \) contains two successive syllables \( g_i \) and \( g_{i+1} \) such that \( \{v_i, v_{i+1}\} \in E \) and \( v_{i+1} \prec v_i \), then swap these syllables. Repeat this until no more swaps are possible. This must terminate because no pair of syllables can be swapped more than once. Let \( z \) be the word which results. Now \( z \) cannot have two successive syllables belonging to the same vertex group, else \( y \) would not have minimal length. Therefore \( z \) is normal, hence \( z = x \) by the previous theorem. Since \( y \) and \( z \) have the same length, it follows that \( x \) is reduced. \( \square \)

**Corollary 10.3.** A reduced word can be transformed into an equivalent normal word by swapping syllables belonging to adjacent vertex groups. \( \square \)

**11. Graph Products as Hyperreflection Systems**

Let \((V, E)\) be a graph, and suppose that \( G \) is the internal graph product of a collection of subgroups \( \{G_v\}_{v \in V} \). Also suppose that \((G_v, \Sigma_v)\) is a hyperreflection system for each \( v \in V \). The main objective of this section is to prove that \((G, \Sigma)\) is a hyperreflection system, where \( \Sigma = \bigcup_{v \in V} \Sigma_v \).

Define a weight function \( \text{wt} \) on \( G \) as follows. For each \( v \in V \) let \( \ell_v \) be the length function associated to \((G_v, \Sigma_v)\), and let \( \ell = \bigcup_{v \in V} \ell_v \). If \( (g_1, \ldots, g_n) \) is the normal form for \( g \), and if \( g_i \in G_{v_i} \) for each \( i \), then define \( \text{wt}(g) = \sum_{i=1}^{n} \ell(g_i) \).

Let \( v \in V \) and \( S \in \Sigma_v \) be given. Choose a linear ordering \( \prec \) on \( V \) such that \( v \) is minimal with respect to \( \prec \). This linear ordering determines a normal form for the elements of \( G \).

**Theorem 11.1.** If \( g \in G \) then \( Sg \) has a unique element of minimum weight, and \( g \) is the minimum weight element of \( Sg \) if and only if \( g_1 \in (G_v)^S \) or \( g_1 \notin G_v \), where \( g_1 \) is the first syllable of the normal form for \( g \).

**Proof.** Let \((g_1, \ldots, g_n)\) be the normal form for \( g \). We consider three cases.

1. Suppose that \( g_1 \notin G_v \). If \( 1 \neq s \in S \) then \((s, g_1, \ldots, g_n)\) is the normal form for \( sg \). Therefore, \( \text{wt}(sg) = \text{wt}(g) + 1 \) for all \( 1 \neq s \in S \), so \( g \) is the unique element of minimum weight in \( Sg \).

2. Suppose that \( g_1 \in G_v \) and \( g_1 \notin S \). If \( s \in S \), then \((sg_1, \ldots, g_n)\) is the normal form for \( sg \). Therefore, \( \text{wt}(sg) \) is minimized when \( \ell(sg_1) \) is minimized. But the coset \( Sg_1 \) of \( G_v \) has a unique element of minimum length, hence \( Sg \) has a unique element of minimum weight. Furthermore, \( g \) is the minimum weight element of \( Sg \) if and only if \( g_1 \) is the minimum length element of \( Sg_1 \), which occurs precisely when \( g_1 \in (G_v)^S \).
(3) Suppose that $g_1 \in S$. If $s = g_1^{-1}$ then $(g_2, \ldots, g_n)$ is the normal form for $sg$, otherwise $(sg_1, g_2, \ldots, g_n)$ is the normal form for $sg$. The weight is uniquely minimized when $s = g_1^{-1}$, hence $Sg$ has a unique element of minimum weight. Note that $g$ cannot be the minimum weight element of $Sg$ in this case, since $\text{wt}(g_1^{-1}) = \text{wt}(g) - 1$.

\[ \square \]

**Theorem 11.2.** $(G, \Sigma)$ as defined above is a hyperreflection system.

**Proof.** Let $C$ be the set of all elements $g \in G$ such that $g$ is the unique element of minimum weight in $Sg$. Since every coset $Sg$ has a unique element of minimum weight, it follows that $G$ is the disjoint union of $sC$ for $s \in S$.

Let $g \in C$ and let $(g_1, \ldots, g_n)$ be the normal form for $g$. Choose a reduced word $s_i$ for each $g_i$ and let $s = (s_1, \ldots, s_N)$ be the concatenation of the $s_i$. Then $s$ determines a walk $\pi$ from 1 to $g$ in $\text{Cay}(G, \Sigma)$.

Suppose that $\pi$ crosses from $C$ to $sC$ for some $s \in S \setminus \{1\}$, i.e. there exists $k$ such that

$$s(s_1 \cdots s_{k-1}) = (s_1 \cdots s_k).$$

Then $s$ is equivalent to a new word

$$s' = (s, s_1, \ldots, \hat{s}_k, \ldots, s_N).$$

Let $i$ be the index such that $s_k$ lies in the subword $s_i$, which represents the syllable $g_i$. If $g_i = s_k$ then $g$ can be represented by $(s, g_1, \ldots, \hat{g}_i, \ldots, g_n)$. Since this word has $n$ syllables, it is reduced; so its normal form is obtained by swapping adjacent syllables. The first syllable $s$ cannot be swapped, because $v$ is the first vertex in the chosen linear ordering of $V$. Therefore, $s$ must be the first syllable in the normal form of $g$, which contradicts the assumption that $g \in C$.

If $g_i \neq s_k$ then $g$ can be represented by $(s, g_1, \ldots, g'_i, \ldots, g_n)$, where $g'_i$ is obtained by deleting $s_k$ from $s_i$. Since this word has $n + 1$ syllables, it is not normal. However, $(g_1, \ldots, g'_i, \ldots, g_n)$ is normal, and this implies (by the normal form algorithm) that $g_1 \in S$ and that

$$(sg_1, g_2, \ldots, g'_i, \ldots, g_n)$$

is the normal form for $g$. Therefore $sg_1 = g_1$ by the uniqueness of normal forms, which is a contradiction. Therefore, every $g \in C$ can be joined to 1 by a walk that does not cross $\text{Fix}(S)$.

It remains to prove that if $1 \neq s_0 \in S$ then every walk from 1 to $s_0$ must cross $\text{Fix}(S)$. Let $\pi$ be any walk from 1 to $s_0$, and let $(s_1, s_2, \ldots, s_n)$ be the corresponding word representing $s_0$. Since $1 \in C$ and $s_0 \notin C$, there exists an index $k$ and $1 \neq s \in S$ such that $u := s_1 \cdots s_{k-1} \in C$ and $us_k \in sC$.

Let $(g_1, \ldots, g_n)$ be the normal form for $u$. Then either $g_1 \in (G_v)^S$ or $g_1 \notin G_v$, whereas the normal form for $us_k$ begins with an element of $s(G_v)^S$ since $us_k \in sC$. In order to affect the first syllable of the normal form, $s_k$ must commute with $g_2, \ldots, g_n$, hence $t_k = g_1 s_k g_1^{-1}$. If $g_1 \notin G_v$, then $s_k$ must commute with $g_1$ as well. There are three cases to consider.

1. If $g_1 \notin G_v$, then $s_k$ commutes with $u$, so $t_k = us_k u^{-1} = s_k$. Therefore, $\pi$ crosses $\text{Fix}(S)$ while passing from $u$ to $us_k$.
2. If $g_1 \in G_v$ and $g_1 \neq s_k^{-1}$, then $(g_1 s_k, g_2, \ldots, g_n)$ is the normal form for $us_k$. Therefore $g_1 s_k \notin (G_v)^S$, which implies that $\pi$ crosses $\text{Fix}(S)$ while passing from $u$ to $us_k$.
3. If $g_1 = s_k^{-1}$, then $(g_2, \ldots, g_n)$ is the normal form for $us_k$. But the normal form for $us_k$ must start with an element of $G_v$, so this case cannot occur.
Therefore, each component of $\text{Cay}(W, \Sigma) \setminus \text{Fix}(R)$ contains exactly one element of $S$, which implies that $S$ is a hyperreflection. Since $S$ is an arbitrary element of $\Sigma$, it follows that $(G, \Sigma)$ is a hyperreflection system. □

**Corollary 11.3.** If $G$ is the internal graph product of $\{G_v\}_{v \in V}$ then $(G, \{G_v\}_{v \in V})$ is a hyperreflection system.

**Proof.** Each $G_v$ has a trivial hyperreflection system $(G_v, \{G_v\})$, so the previous theorem implies that $(G, \{G_v\}_{v \in V})$ is a hyperreflection system. □

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E-mail address: dradcliffe@gmail.com