Constructing algebraic Lie algebras

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Abstract

We give an algorithm for constructing the algebraic hull of a given matrix Lie algebra in characteristic zero. It is based on an algorithm for finding integral linear dependencies of the roots of a polynomial, that is probably of independent interest.

1 Introduction

One of the major tools in the theory of algebraic groups is their correspondence with Lie algebras. Many problems regarding algebraic groups can be reformulated in terms of the corresponding Lie algebras, for which they are generally easier to solve. There is considerable interest in studying algebraic groups computationally (cf., e.g., [6], [10]). Also for this it would be of great interest to exploit the connection with Lie algebras. In this paper we treat a question that arises in this context, namely the problem to decide whether a given Lie algebra corresponds to an algebraic group. In particular, a positive solution to this problems enables us to decide which subalgebras of a Lie algebra of an algebraic group correspond to algebraic subgroups. To tackle this problem we restrict to base fields of characteristic 0, because for that case there is a well developed theory of the connection between algebraic groups and Lie algebras (see [5]). In particular, a connected algebraic group is completely determined by its Lie algebra.

Let $V$ be a finite-dimensional vector space. A subgroup $G \subset \text{GL}(V)$ is said to be algebraic if there is a set of polynomial functions $P$ on $\text{End}(V)$ such that $G$ consists of all $g \in \text{GL}(V)$ with $f(g) = 0$ for all $f \in P$. To such a group corresponds a Lie algebra,
Lie(G) ⊂ gl(V) ([5], Chapter II, §8), where by gl(V) we denote the Lie algebra of all endomorphisms of V. Now a given Lie subalgebra g ⊂ gl(V) is called algebraic if there is an algebraic subgroup G ⊂ GL(V) such that g = Lie(G). In [5], Chevalley studied this concept in characteristic 0, and gave several sufficient criteria for a g ⊂ gl(V) to be algebraic.

Let g ⊂ gl(V) be any Lie algebra. Then by [5], Chapter II, Theorem 13, there is a unique smallest algebraic Lie algebra containing g. This algebraic Lie algebra is called the algebraic hull of g. In this paper we consider the problem of constructing the algebraic hull for a given g ⊂ gl(V).

Based on results of Chevalley we describe an algorithm for constructing the algebraic hull. The computationally hardest step is to construct the splitting field of a polynomial. Since this can be a rather formidable task, we subsequently give an algorithm which is similar in nature, but avoids the problem of having to construct splitting fields. This is based on algorithms for finding integral dependencies of algebraic integers. Combining complex and p-adic approximations to the roots, and the technique of lattice reduction (LLL), we obtain an algorithm for computing the Z-module of integral relations among a given set of algebraic integers. In the literature, several somewhat similar methods for solving this problem are known (cf., e.g., [7] §2.7.2, [12]). These methods focus on finding one linear dependency, while our algorithms find (a basis of) the whole module of linear dependencies.

This paper is arranged as follows. In Section 2 we introduce the notation that we use, and summarize a number of results of Chevalley. Then in Section 3 we describe the algorithm that makes use of splitting fields of polynomials. In Section 4 we show how Galois groups can in some instances be of help with constructing the algebraic hull. This is used in Section 5, where we give the algebraic hull of the Lie algebra spanned by a semisimple 4 × 4-matrix. Then in Section 6 algorithms are given for finding integral linear dependencies among the roots of a polynomial. These algorithms are then used in Section 7, where an algorithm is given for constructing the algebraic hull of a Lie algebra, avoiding the construction of splitting fields. Finally, in Section 8 we report on some practical experiences with an implementation of the algorithms in the computer algebra system Magma [4, 3].

## 2 Preliminaries

Here F will be a field of characteristic 0. We will use the language of matrices, rather than that of endomorphisms, as this is more convenient for calculations. In particular, gl(n, F) is the Lie algebra of all n × n-matrices over F. By [5], Chapter II, Theorem 14, a Lie algebra g ⊂ gl(n, F) is algebraic if it is generated by algebraic Lie algebras. It follows that g is algebraic if and only if the algebraic hull of the subalgebra spanned by each basis element of g is contained in g. Hence we can compute the algebraic hull of g if we can compute it in the case where g is spanned by one matrix X.

Let X ∈ gl(n, F). Then by gF(X) we denote the algebraic hull of the Lie algebra spanned by X. Let X = S + N be the Jordan decomposition of X. Then from [5], Chapter...
In Theorem II, Theorem 10 (see also \[2\], §7), it follows that \( \mathfrak{g}_F(X) = \mathfrak{g}_F(S) \oplus \mathfrak{g}_F(N) \). Moreover, \( \mathfrak{g}_F(N) \) is spanned by \( N \), by \[5\], Chapter II, §13, Proposition 1. So the problem is reduced to finding \( \mathfrak{g}_F(X) \) when \( X \) is semisimple.

The following theorem is proved in \[5\]:

**Theorem 1 (Chevalley)** Let \( X \in \mathfrak{gl}(n, F) \) be semisimple, and let \( K \supseteq F \) be an algebraic extension containing the eigenvalues \( \alpha_1, \ldots, \alpha_n \) of \( X \). Let \( U \in \text{GL}(n, K) \) be such that \( Y = UXU^{-1} \) is in diagonal form, with the \( \alpha_i \) on the diagonal. Set \( \Lambda = \{ (e_1, \ldots, e_n) \in \mathbb{Z}^n \mid \sum_i e_i \alpha_i = 0 \} \). Then

1. \( \mathfrak{g}_K(X) = U^{-1} \mathfrak{g}_K(Y)U \) and 
\[
\mathfrak{g}_K(Y) = \{ \text{diag}(a_1, \ldots, a_n) \mid a_i \in K \text{ and } \sum_i e_i a_i = 0 \text{ for all } (e_1, \ldots, e_n) \in \Lambda \}.
\]
2. \( \mathfrak{g}_F(X) \otimes K \cong \mathfrak{g}_K(X) \).
3. \( \mathfrak{g}_F(X) \subset A_F(X) \) where \( A_F(X) \) is the associative \( F \)-algebra with one generated by \( X \).

The first part of 1. is straightforward. Let \( G_K(X) \) denote the smallest algebraic subgroup of \( \text{GL}(n, K) \) such that its Lie algebra contains \( X \). Then \( \mathfrak{g}_K(X) = U^{-1} \mathfrak{g}_K(Y)U \) and \( \mathfrak{g}_K(X) = \text{Lie}(G_K(X)) = U^{-1} \text{Lie}(G_K(Y))U = U^{-1} \mathfrak{g}_K(Y)U \). The second part of 1. is \[5\], §13, Proposition 2. 2. follows from the proof of \[5\], §13, Theorem 10. Furthermore, 3. is \[5\], §14, Proposition 14. (There it is shown that \( \mathfrak{g}_F(X) \) is contained in the associative algebra (not necessarily with one) generated by \( X \). However, for us it will be more convenient to add the identity.)

### 3 An algorithm for the algebraic hull

In this section we use the same notation as in the previous section. In particular we let \( X \) be a semisimple \( n \times n \)-matrix with coefficients in the field \( F \) of characteristic 0. We let \( K \) be a finite extension of \( F \) containing the eigenvalues \( \alpha_1, \ldots, \alpha_n \) of \( X \). Furthermore, \( \Lambda = \{ (e_1, \ldots, e_n) \in \mathbb{Z}^n \mid \sum_i e_i \alpha_i = 0 \} \), and \( \Lambda_Q = \{ (e_1, \ldots, e_n) \in \mathbb{Q}^n \mid \sum_i e_i \alpha_i = 0 \} \). By \( A_F(X) \) we denote the associative algebra with one generated by \( X \). The algorithm is based on the following lemma.

**Lemma 2** For \( \varepsilon = (e_1, \ldots, e_n) \in \mathbb{Q}^n \) and \( i \geq 0 \) set \( \Delta_i(\varepsilon) = \sum_{k=1}^n e_k\alpha_k^i \). Let \( I = X^0, X, \ldots, X^t \) be a basis of \( A_F(X) \). Set 
\[
\Upsilon = \{ (\gamma_0, \ldots, \gamma_t) \in F^{t+1} \mid \sum_{i=0}^t \Delta_i(\varepsilon)\gamma_i = 0 \text{ for all } \varepsilon \in \Lambda_Q \}.
\]
Then \( \mathfrak{g}_F(X) = \{ \sum_{i=0}^t \gamma_i X^i \mid (\gamma_0, \ldots, \gamma_t) \in \Upsilon \} \).
Let \( Y = \text{diag}(\alpha_1, \ldots, \alpha_n) \). Then there is a \( U \in \text{GL}(n, K) \) with \( UXU^{-1} = Y \). Here \( t + 1 \) is the degree of the minimal polynomial of \( X \). Then since the minimal polynomial of a semisimple matrix is the square free part of its characteristic polynomial, the minimal polynomial of \( Y \) (over \( K \)) is the same as the minimal polynomial of \( X \) (over \( F \)). Hence \( A_K(Y) \) is spanned by \( I, Y, Y^2, \ldots, Y^t \).

Set \( y = \sum_{i=0}^t \gamma_i Y^i \). Write \( y(k, k) \) for the entry in \( y \) on position \((k,k)\). Then by Theorem 1 we get that \( e \) in order to compute \( \Upsilon \) it is enough to consider \( \Upsilon' \). Then since the minimal polynomial \( K \) is spanned by \( I, Y, Y^2, \ldots, Y^t \).

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Hence \( A_K(Y) \) is spanned by \( I, Y, Y^2, \ldots, Y^t \).
Example 4 Let \( X \in \mathfrak{gl}(4, \mathbb{Q}) \) have minimum polynomial \( T^4 + bT^2 + c \) with \( D = b^2 - 4c \) not a square in \( \mathbb{Q} \). Then the eigenvalues of \( X \) are \( \alpha_1 = \alpha, \alpha_2 = -\alpha, \alpha_3 = \beta, \alpha_4 = -\beta \), where \( \alpha^2 = \frac{1}{2}(-b + \sqrt{D}) \) and \( \beta^2 = \frac{1}{2}(-b - \sqrt{D}) \). Then \( \alpha \) and \( \beta \) cannot be proportional over \( \mathbb{Q} \) (otherwise \( \alpha^2 \) and \( \beta^2 \) would be as well). Hence the \( \alpha_i \) span a 2-dimensional subspace of \( K \). So \( \dim \Lambda = 2 \), and is spanned by \( e^1 = (1, 1, 0, 0), e^2 = (0, 0, 1, 1) \). Then \( \Delta_0(e^1) = 2, \Delta_1(e^1) = \Delta_3(e^1) = 0, \Delta_2(e^1) = 2\alpha^2 \). For \( e^2 \) we get the same except that \( \Delta_2(e^2) = 2\beta^2 \). So
\[
\mathcal{Y} = \{(\gamma_0, \ldots, \gamma_3) \in \mathbb{Q}^3 | 2\gamma_0 + 2\alpha^2\gamma_2 = 2\gamma_0 + 2\beta^2\gamma_2 = 0\}.
\]
Hence \( \mathcal{Y} \) consists of \((0, \gamma_1, 0, \gamma_3)\). We conclude that \( \mathfrak{g}(X) \) is spanned by \( X, X^3 \).

4 The permutation module

Here we use the same notation as in the previous section. In this section we make some observations that on some occasions directly give a basis of \( \mathfrak{g}_F(X) \).

Let \( f \) be the characteristic polynomial of \( X \). Let \( K \) be the splitting field of \( f \), and \( G = \text{Gal}(K/F) \). We represent \( G \) as a permutation group on the roots \( \alpha_1, \ldots, \alpha_n \) of \( f \). Let \( M \) be the permutation module of \( G \) over \( \mathbb{Q} \), i.e., \( M \) has basis \( w_1, \ldots, w_n \) and \( \sigma \cdot w_i = w_{\sigma(i)} \). On many occasions we will write the elements of \( M \) as row vectors. Then \( \sigma(a_1, \ldots, a_n) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)}) \). There is a direct sum decomposition of \( G \)-modules \( M = M_0 \oplus M_1 \), where \( M_0 = \{\sum a_iw_i \mid \sum a_i = 0\} \) and \( M_1 \) is spanned by \( w_1 + \cdots + w_n \).

Let \( (e_1, \ldots, e_n) \in \Lambda_Q \) and \( \sigma \in G \) then \( 0 = \sigma(\sum e_i\alpha_i) = \sum e_i\alpha_{\sigma(i)} = \sum e_i\alpha_{\sigma^{-1}(i)} \). It follows that \( \Lambda_Q \) is a \( G \)-submodule of \( M \). So by Maschke’s theorem \( \Lambda_Q = V_1 \oplus \cdots \oplus V_s \), where the \( V_r \) are irreducible \( G \)-submodules.

From Lemma 2 we recall that \( \Delta_i(\underline{e}) = \sum_{k=1}^n e_k \alpha_k^i \), where \( \underline{e} \in \mathbb{Q}^n \).

**Lemma 5** Write \( f = x^n + a_1x^{n-1} + \cdots + a_n \). Then the \( G \)-submodule \( M_1 \subset M \) occurs in \( \Lambda_Q \) if and only if \( a_1 = 0 \). Furthermore, \( \Delta_i(\underline{e}) = \text{Tr}(X^i) \), where \( \underline{e} = (1, 1, \ldots, 1) \) spans \( M_1 \).

**Proof.** We have \( a_1 = 0 \) if and only if \( \sum \alpha_i = 0 \), hence the first statement. Set \( e = (1, 1, \ldots, 1) \). Let \( Y \) be as in the proof of Lemma 2. Then \( \Delta_i(\underline{e}) = \sum_k \alpha_k^i = \text{Tr}(Y^i) = \text{Tr}(X^i) \).

**Lemma 6** Suppose that \( f \) is square-free and that \( M_0 \) is irreducible. Then \( a_1 = 0 \) implies \( \Lambda_Q = M_1 \) and \( a_1 \neq 0 \) implies \( \Lambda_Q = 0 \).

**Proof.** Note that \( \Lambda_Q \) cannot contain \( M_0 \) since in that case a vector like \((1, -1, 0, \ldots, 0)\) would be contained in \( \Lambda_Q \), implying \( \alpha_1 = \alpha_2 \) (which is impossible because \( f \) is square free). Hence the lemma follows by Lemma 5.

**Corollary 7** Suppose that \( f \) is irreducible. Let \( A_F(X) \) denote the associative algebra generated by \( X \). Suppose that \( G \) is 2-transitive, or that \( F = \mathbb{Q} \) and \( n \) is prime. If \( \text{Tr}(X) = 0 \) then \( \mathfrak{g}_F(X) \) consists of all \( X' \in A_F(X) \) with \( \text{Tr}(X') = 0 \), otherwise \( \mathfrak{g}_F(X) = A_F(X) \).
Proof. If $G$ is 2-transitive then $M_0$ is irreducible, by [11], Corollary 29.10. If $n = p$ is prime then $M_0$ is irreducible over $\mathbb{Q}$. This can be proved as follows. First of all, since $G$ is transitive it contains a $p$-cycle. Now we let $H$ be the subgroup generated by this $p$-cycle. Then $M$ is also an $H$-module. Moreover, as $H$-module it is isomorphic to the regular module, i.e., to the module afforded by the left action of $H$ on the group algebra $\mathbb{Q}H$. The $H$-submodules of $\mathbb{Q}H$ are exactly the ideals of $\mathbb{Q}H$. But $\mathbb{Q}H$ is isomorphic to $\mathbb{Q}[x]/(x^p - 1)$, which by the Chinese Remainder Theorem is isomorphic to $\mathbb{Q} \oplus Q[x]/(x^{p-1} + x^{p-2} + \cdots + 1)$. We conclude that $\mathbb{Q}H$ splits as the direct sum of two simple ideals. Hence the $H$-module $M$ is a direct sum of two simple submodules. So the same holds for the $M$ when viewed as $G$-module.

Now the result follows by Lemmas [5] [6].

In particular, if $G = S_n$ or $G = A_n$ ($n \geq 4$) then we can easily compute $\mathfrak{g}(X)$.

5 Degree 4

Here we use the observations of the previous section to give a complete description of $\mathfrak{g}_F(X)$, where $X$ is a semisimple $4 \times 4$-matrix, with irreducible characteristic polynomial.

Let $f = x^4 + ax^3 + bx^2 + cx + d$ be the characteristic polynomial of $X$, and suppose that it is irreducible. Let $G$ denote the Galois group $\text{Gal}(K/F)$, where $K$ is the splitting field of $f$. We remark that if $F = \mathbb{Q}$ then it is straightforward to determine $G$, e.g., by the procedure outlined in [18], Theorem 106. Note that the case where $G = S_4$, $A_4$ is settled by Corollary [7].

Proposition 8 Suppose that $G$ is not isomorphic to $S_4$ or $A_4$. Then

1. if $a = 0$ and $a^3 - 4ab + 8c \neq 0$ then $\mathfrak{g}_F(X) = \{X' \in A_F(X) \mid \text{Tr}(X') = 0\}$,
2. if $a = 0$ and $a^3 - 4ab + 8c = 0$ then $\mathfrak{g}_F(X)$ is spanned by $X, X^3$,
3. if $a \neq 0$ and $a^3 - 4ab + 8c \neq 0$, then $\mathfrak{g}_F(X)$ is spanned by $I, X, X^2, X^3$,
4. if $a \neq 0$ and $a^3 - 4ab + 8c = 0$, then $\mathfrak{g}_F(X)$ is spanned by $I, X, X^2 + \frac{1}{3a}X^3$.

Proof. Since $G$ is a transitive permutation group on 4 points, not isomorphic to $S_4$, $A_4$, there remain the possibilities $G \cong \mathbb{Z}/4\mathbb{Z}$, $G \cong D_8$, $G \cong V_4$. These groups have respective generating sets $\{(1,2,3,4), (1,2,3,4), (1,3), ((1,2)(3,4), (1,4)(2,3))\}$. In the first two cases the module $M_0$ decomposes as a direct sum of two submodules with bases $\{(1,-1,1,-1)\}$, $\{(1,0,-1,0), (0,1,0,-1)\}$ (this holds for both cases). Now $\Lambda_Q$ cannot contain the second module (as in that case some roots would be equal). If $G = V_4$ then $M_0$ decomposes as a direct sum of three submodules, respectively spanned by $(1,1,-1,-1)$, $(1,-1,1,-1)$, $(1,-1,-1,1)$. The $G$-module $\Lambda_Q$ cannot contain two of these vectors, as otherwise after adding it would follow that two roots are equal.

So in all cases, after maybe renumbering the roots, there are the following possibilities for $\Lambda_Q$: $\Lambda_Q = 0$, $\Lambda_Q$ is spanned by $(1,1,1,1)$, or by $(1,1,1,1)$, $(1,1,-1,-1)$, or by $(1,1,1,1)$.
Let \( \alpha_1, \ldots, \alpha_4 \) be the roots of \( f \). Set \( \alpha_1 = \alpha_2 - \alpha_3 - \alpha_4, \alpha_2 = \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4, \alpha_3 = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 \). Then the product \( \alpha_1 \alpha_2 \alpha_3 \) is a symmetric polynomial in the \( \alpha_i \), hence can be expressed in terms of the coefficients of \( f \). It turns out that \(-a_1 a_2 a_3 = a^3 - 4ab + 8c\). So this number is zero if and only if \( \Lambda \) contains \( (1, 1, -1, -1) \). This proves 1. and 3 (cf. Lemma 5).

Suppose that \( a^3 - 4ab + 8c = 0 \). Then we can assume that \( \Lambda \) contains \( \varepsilon = (1, 1, -1, -1) \). In order to obtain a basis of \( \mathcal{Y} \) (cf. Algorithm 3) we have to solve the equation \( \sum_{i=0}^{3} \Delta_i(\varepsilon) \gamma_i = 0 \). Note that \( \Delta_0(\varepsilon) = \Delta_1(\varepsilon) = 0 \). We know that \( \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0 \), and also that \( \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -a \). These two relations are equivalent to \( \alpha_1 + \alpha_2 + \frac{1}{2}a = 0 \) and \( \alpha_3 + \alpha_4 + \frac{1}{2}a = 0 \). Now \( \Delta_2(\varepsilon) = 2\alpha_2^2 + a\alpha_2 - 2\alpha_4^2 - a\alpha_4 \) as the difference is equal to

\[\frac{1}{2}(\alpha_1 - \alpha_2 - \frac{1}{2}a)(\alpha_1 + \alpha_2 + \frac{1}{2}a) + \frac{1}{2}(\alpha_3 + \alpha_4 + \frac{1}{2}a) \] 

Similarly, \( \Delta_3(\varepsilon) = -\frac{3}{4}a(2\alpha_2^2 + a\alpha_2 - 2\alpha_4^2 - a\alpha_4) \) as the difference is equal to

\[\frac{1}{2}\alpha_1\alpha_2 - \frac{1}{2}a\alpha_1 + \frac{1}{4}a^2(\alpha_1 + \alpha_2 + \frac{1}{2}a) + \frac{1}{2}(\alpha_3 + \alpha_4 + \frac{1}{2}a) \] 

From this it follows that \( 3a\Delta_2(\varepsilon) + 4\Delta_3(\varepsilon) = 0 \). Furthermore, \( \Delta_2(\varepsilon) = 2a_2^2 + a\alpha_2 - 2\alpha_4^2 - a\alpha_4 = 2(\alpha_2 - \alpha_4)(\alpha_2 + \alpha_4 + \frac{1}{2}a) \). From this we conclude that \( \Delta_2(\varepsilon) \neq 0 \). Indeed, \( \alpha_2 - \alpha_4 \neq 0 \) as \( f \) is irreducible. Secondly, \( \alpha_2 + \alpha_4 = -\frac{1}{2}a \) would entail \( \alpha_1 - \alpha_4 = 0 \) as \( \alpha_1 + \alpha_2 = -\frac{1}{2}a \).

Suppose \( a \neq 0 \). Then the equation \( \sum_{i=0}^{3} \Delta_i(\varepsilon) \gamma_i = 0 \) is equivalent to \( (-\frac{4}{3a}\gamma_2 + \gamma_3)\Delta_3(\varepsilon) = 0 \), and we have just seen that \( \Delta_3(\varepsilon) \neq 0 \). So \( \gamma_3 = \frac{4}{3a}\gamma_2 \) and 4. is proved.

If \( a = 0 \), then \( \Delta_3(\varepsilon) = 0 \) and the equation \( \sum_{i=0}^{3} \Delta_i(\varepsilon) \gamma_i = 0 \) reduces to \( \gamma_2 = 0 \). Also \( \varepsilon' = (1, 1, 1, 1) \in \Lambda \). Then by adding \( \varepsilon \) and \( \varepsilon' \) we see that \( \alpha_2 = -\alpha_1 \) and \( \alpha_4 = -\alpha_3 \). So \( \Delta_1(\varepsilon') = \Delta_3(\varepsilon') = 0 \), and \( \Delta_0(\varepsilon') = 4 \). So we get the equation \( 4\gamma_0 = 0 \). This proves 2. □

The calculations in the final part of the proof have been done with the help of Magma. Using similar calculations more results of the same flavour can be derived. Without proof we state the following result.

**Proposition 9** Let \( X \) be a 6 \times 6-matrix with irreducible characteristic polynomial \( p = x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f \). Suppose that the Galois group has number 4,6,7,8, or 11 in the classification of transitive groups in Magma. Set

\[ r_1 = c + \frac{5}{27}(a^3 - \frac{18}{5}ab) \] 

and

\[ r_2 = e - \frac{1}{81}a^5 + \frac{1}{27}a^3b - \frac{1}{3}ad. \]

Then

1. If \( r_1 = r_2 = 0 \) then \( g_F(X) \) is spanned by

\[ I, X, \frac{a}{2}X^2 + X^3, -\frac{5a^3}{54}X^2 + \frac{5a}{6}X^4 + X^5. \]

2. If one of \( r_1, r_2 \) is nonzero then \( g_F(X) \) is equal to \( A_F(X) \) if \( a \neq 0 \), and equal to \( \{ X' \in A(X_F) \mid \text{Tr}(X') = 0 \} \) if \( a = 0 \).
6 Finding integral dependencies of roots

Let \( f \in \mathbb{Q}[x] \) be a square free polynomial with roots \( \alpha_1, \ldots, \alpha_n \) in some splitting field \( \Gamma \supset \mathbb{Q} \). Elements of \( \Gamma \supset K := \mathbb{Q}[\alpha_1, \ldots, \alpha_n] \) can be represented as polynomials \( g \in \mathbb{Q}[X_1, \ldots, X_n] \) coming from a representation \( K \cong \mathbb{Q}[X_1, \ldots, X_n]/I \) for some zero-dimensional ideal \( I \subset \mathbb{Q}[X_1, \ldots, X_n] \). Although constructive methods for the construction of \( I \) or \( K \) are known, e.g. \([17]\), in general they are limited to small examples: the splitting field can have degree as large as \( n! \) over \( \mathbb{Q} \) and generically, it has. In what follows we assume \( f \) to be monic and integral, so that \( \alpha_i \) are algebraic integers. We will give algorithms for the following tasks:

1. Given some \( g \in \mathbb{Z}[X_1, \ldots, X_n] \) decide if \( g(\alpha_1, \ldots, \alpha_n) = 0 \)

2. Given \( g_j \in \mathbb{Z}[X_1, \ldots, X_n] \) (\( 1 \leq j \leq s \)) find a \( \mathbb{Z} \)-module basis for \( \Lambda := \{ \sum_{j=1}^{s} e_j g_j(\alpha_1, \ldots, \alpha_n) = 0 \} \).

Obviously, both tasks are trivial if exact representations for \( K \) or \( I \) are known, so we essentially assume that \((K : \mathbb{Q})\) is too large to allow direct algebraic constructions to succeed. Our method will be based on approximate representations of the \( \alpha_i \), i.e. we are going to use the field \( \mathbb{C} \) of complex numbers and certain unramified \( p \)-adic extensions of \( \mathbb{Q}_p \) for our work. For basic properties of \( p \)-adic numbers, we refer to \([20, 14]\).

Let \( p \in \mathbb{Z} \) be a prime number. For any \( r \in \mathbb{Z} \), we can write \( r = p^l r' \) for some \( r' \) not divisible by \( p \). The function

\[
v_p : \mathbb{Z} \setminus \{0\} \to \mathbb{Z} : r = p^l r' \mapsto l
\]

is called the \( p \)-adic valuation on \( \mathbb{Z} \). We extend \( v_p \) to all of \( \mathbb{Z} \) by defining \( v_p(0) := \infty \) and extend further to \( \mathbb{Q} \) by setting \( v_p(a/b) = v_p(a) - v_p(b) \). Via

\[
|.|_p : \mathbb{Q} \to \mathbb{Q} : x \mapsto p^{-v_p(x)}, 0 \mapsto 0
\]

this gives rise to the (normalised) \( p \)-adic absolute value and thus the \( p \)-adic topology on \( \mathbb{Q} \). The completion \( \mathbb{Q}_p \) of \( \mathbb{Q} \) wrt to \( |.|_p \) is called the field of \( p \)-adic numbers, it contains the \( p \)-adic integers, the completion \( \mathbb{Z}_p \) of \( \mathbb{Z} \).

Suppose now that over \( \mathbb{F}_p \), the field with \( p \)-elements, \( f \) factors as

\[
f = \prod_{i=1}^{l} f_i \pmod{p}
\]

with irreducible, pairwise coprime \( f_i \in \mathbb{F}_p[t] \). Then there is an (unramified) extension \( \Gamma/\mathbb{Q}_p \) of degree \( f_p := \deg \prod_{i=1}^{l} \deg f_i \) where \( f \) splits into linear factors. Furthermore, there is a unique extension of \( |.|_p \) to \( \Gamma \) which is again denoted by \( |.|_p \). Similarly to \( \mathbb{R} \) or \( \mathbb{C} \), elements in \( \Gamma \) cannot, in general, be represented exactly, instead approximations with a given fixed
precision have to be used. The advantage of using $\Gamma$ as a splitting field rather than $\mathbb{C}$ or $K$ directly, lies in the fact that arithmetic operations in $\Gamma$ incur less numerical loss of precision that operations with real numbers, while the algebraic degree of $\Gamma/\mathbb{Q}_p$ is still much smaller than the degree of $K/\mathbb{Q}$. The main disadvantage of using $\Gamma$ or $\mathbb{C}$ is that, since there is no exact representation of elements, in general we cannot decide if an element is zero without additional information.

Lastly, we note that there is exactly one prime ideal $P$ of $\mathbb{Z}_K$ (the ring of integers of $K$) such that $\Gamma = K_P$ the $p$-adic completion at $P$. For elements $x \in \mathbb{Z}_K$, we have $x \in P^k$ if and only if $|x|_p \leq p^{-k}$.

In addition to the $p$-adic information mainly encoded in $\Gamma$ we are also going to need complex information about elements in $K$. As a number field $K/\mathbb{Q}$, $K$ admits $(K : \mathbb{Q})$ many distinct embeddings $(\cdot)^{(j)}$ $(1 \leq j \leq (K : \mathbb{Q}))$ into the complex numbers. For any $x \in K$ we define a length:

$$T_2 : K \to \mathbb{R} : x \mapsto \sum_{j=1}^{(K:Q)} |x^{(j)}|^2.$$ 

Note that $\sqrt{T_2}$ is an Euclidean norm on the $\mathbb{Q}$-vectorspace $K$. Elementary Galois theory and the inequality between arithmetic and geometric means can be used to derive non-trivial lower bounds on $T_2(x)$:

$$\sqrt{N_{K/\mathbb{Q}}(x^2)} \leq \frac{1}{K : \mathbb{Q}} T_2(x)$$

which implies for algebraic integers $x \in \mathbb{Z}_K$ that

$$T_2(x) \geq (K : \mathbb{Q}).$$

**Remark.** Let $\beta_1, \ldots, \beta_n \in \mathbb{C}$ be the complex roots of $f$. In general it is extremely difficult to sort the complex roots in such a way that $\alpha_i$ corresponds to $\beta_i$ which means that for example from $\sum_{i=1}^{n} e_i \alpha_i = 0$ we cannot not, in general, conclude that $\sum_{i=1}^{n} e_i \beta_i = 0$.

After these preliminaries we can now state our algorithm for the first problem:

**Algorithm 10** Let $\alpha_1, \ldots, \alpha_n \in \Gamma/\mathbb{Q}_p$ be the roots of some monic polynomial $f \in \mathbb{Z}[t]$ and assume that $\Gamma$ is unramified over $\mathbb{Q}_p$. Set $K := \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ and let $g \in \mathbb{Z}[x_1, \ldots, x_n]$ be arbitrary. This algorithm decides if

$$g(\alpha_1, \ldots, \alpha_s) = 0.$$ 

1. Compute a bound $M > 0$ such that $|g(\alpha_1, \ldots, \alpha_n)^{(j)}| \leq M$ for all complex embeddings $(\cdot)^{(j)} : K \to \mathbb{C}$. Such a bound can be obtained by first computing a bound $M'$ on the complex roots $\beta_i \in \mathbb{C}$ of $f$ and then estimating $|g(\gamma_1, \ldots, \gamma_n)|$ for all choices of $|\gamma_i| \leq M'$. 

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2. Compute a bound \( r \geq (K : \mathbb{Q}) \).

3. Set \( k := \left\lceil \frac{r \log M}{(K : \mathbb{Q}) \log p} \right\rceil \).

4. Compute \( \tilde{\alpha}_j \) such that \( |\tilde{\alpha}_j - \alpha_j|_p \leq p^{-k} \).

5. Evaluate \( \tilde{G} := g(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) \).

6. If \( |\tilde{G}|_p > p^{-k} \) return not Zero, otherwise return IsZero.

**Proof.** Throughout this proof, we write \( \tilde{\alpha}_i \) for finite precision approximations to the exact root \( \alpha_i \in \Gamma \) that we cannot exactly represent. Similarly, \( G := g(\alpha_1, \ldots, \alpha_n) \) is the exact element that we cannot compute but need to decide if \( G = 0 \) and \( \tilde{G} := g(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) \) is a finite precision approximation.

We first note that since \( f \in \mathbb{Z}[t] \) is monic, we have \( \alpha_i \in \mathbb{Z}_p \Gamma \), the integral closure of \( \mathbb{Z}_p \) in \( \Gamma \). Now \( g \in \mathbb{Z}[x_1, \ldots, x_n] \) implies \( G \in \mathbb{Z}_p \Gamma \) as well. Writing \( \tilde{\alpha}_i = \alpha_i + p^k \beta_i \) with some \( \beta_i \in \mathbb{Z}_p \Gamma \) we obtain from the ultrametric property of \( |\cdot|_p \):

\[
|g(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)|_p \leq |g(\alpha_1, \ldots, \alpha_n)|_p,
\]

i.e., there is no precision loss in the evaluation.

Let \( K := \mathbb{Q}(\alpha_1, \ldots, \alpha_n) \), as above and \( P \) be the unique prime ideal \( \mathbb{Z}_K \supset P|p \) such that \( \Gamma = K_P \). Then for \( x \in \mathbb{Z}_K \) such that \( |x|_p \leq p^{-k} \) we obtain \( x \in P^k \), thus \( N_{K/\mathbb{Q}}(x) \in N_{K/\mathbb{Q}}(P)^k \) and, since \( N_{K/\mathbb{Q}}(P) \) is an ideal in \( \mathbb{Z} \) generated by \( p^{r(K:Q_p)} \):

\[
p^k r(K:Q_p) \leq N_{K/\mathbb{Q}}(x).
\]

Now, let’s assume we have \( k \) and \( M \) as in the algorithm and \( |\tilde{G}|_p \leq p^{-k} \). From

\[
\sqrt{(K : \mathbb{Q})} N_{K/\mathbb{Q}}(G^2) \leq \frac{T_2(G) K : \mathbb{Q}}{K : \mathbb{Q}} \leq \frac{M^2(K : \mathbb{Q})}{K : \mathbb{Q}} = M^2
\]

we get either \( G = \tilde{G} = 0 \) or \( N_{K/\mathbb{Q}}(G) \geq p^{k r(K:Q_p)} \). And thus

\[
k r(K:Q_p) \leq \frac{\log M}{\log p}
\]

which contradicts our choices. Thus we conclude, \( G = 0 \) as claimed. \( \square \)

While the above algorithm can verify a relation, it does not tell us how to find one. Also, the precision necessary to verify relations can be extremely large, it is essentially linear in \( (K : \mathbb{Q}) = \#\text{Gal}(f) \). In order to use similar ideas to find relations we first need a result allowing us to get a bound on a basis of the relation lattice:

**Theorem 11** Let \( \alpha_1, \ldots, \alpha_n \) be algebraic integers, \( K := \mathbb{Q}(\alpha_1, \ldots, \alpha_n) \), \( r := (K : \mathbb{Q}) \) and define

\[
\Lambda := \{ \xi \in \mathbb{Z}^n \mid \sum_{i=1}^n c_i \alpha_i = 0 \}.
\]
Suppose that $|\alpha_i^{(j)}| \leq M$ for all complex embeddings $(.)^{(j)} : K \rightarrow \mathbb{C}$, then $\Lambda$ has a $\mathbb{Z}$-basis $\mathbf{b_i} \in \mathbb{Z}^n$, $1 \leq i \leq l$ with

$$\|\mathbf{b_i}\|_\infty \leq n^{n-1}M^{n-1}.$$ 

Proof. The function

$$f : \mathbb{Z}^n \rightarrow \mathbb{R} : \mathbf{e} \mapsto \sqrt{T_2(\sum_{i=1}^{n} e_i \alpha_i)}$$

is a convex distance function in the sense of [15, p. 250]. Let $m$ be a standard basis element of $\mathbb{Z}^n$, i.e. $m = (m_i)_{1 \leq i \leq n}$ and $m_i = 0$ for all $i \neq i_0$ while $m_{i_0} = 1$. Then $f(m) = \sqrt{T_2(\alpha_{i_0})} \leq \sqrt{rM}$. From (2) we get for non-zero algebraic integers $x \in K$ that $T_2(x) \geq r$, thus $f(m) \geq \sqrt{r}$, for all $m \in \mathbb{Z}^n$ with $f(m) \neq 0$. The rest now follows directly from the Proposition of [15, p. 250].

The next essential ingredient is the LLL algorithm for lattice reduction. We need the following property of a reduced basis [7, Theorem 2.6.2.(5)]:

**Lemma 12** Let $\Lambda \subseteq \mathbb{Z}^n$ be a lattice. Suppose that $\Lambda$ contains linear independent elements $x_1, \ldots, x_l$ of norm $\|x_i\|_2 \leq M$. Then for a LLL-reduced basis $b_1, \ldots, b_n$ of $\Lambda$ we have $\|b_i\|_2^2 \leq 2^{n-1}M^2$ for $1 \leq i \leq l$.

Combining the previous results we can now give a first algorithm for linear dependencies:

**Algorithm 13** Let $f \in \mathbb{Z}[t]$ be monic and $\alpha_1, \ldots, \alpha_n \in \Gamma/\mathbb{Q}_p$ be the roots of $f$ in some unramified extension of $\mathbb{Q}_p$ of degree $f_p$. We assume that elements in $\Gamma$ are represented as vectors in $\mathbb{Q}_p^{f_p}$ wrt. some fixed basis $\omega_1, \ldots, \omega_{f_p}$. Furthermore, let $g_i \in \mathbb{Z}[x_1, \ldots, x_n]$ be arbitrary ($1 \leq i \leq s$) and define

$$\Lambda := \{ \mathbf{e} \in \mathbb{Z}^s \mid \sum_{i=1}^{s} e_i g_i(\alpha_1, \ldots, \alpha_n) = 0 \}.$$ 

This algorithm computes a $\mathbb{Z}$-basis for $\Lambda$.

1. Compute a bound $M' > 0$ such that for each $i$ we have $|g_i(\alpha_1, \ldots, \alpha_n)^{j}| \leq M'$, for all complex embeddings $(.)^{(j)} : K \rightarrow \mathbb{C}$.
2. Set $N := s^{s-1}M^{s-1}$
3. Set $k := \left\lceil \frac{\log NM}{\log p} \right\rceil$
4. Set $\lambda := N^22^{s-1}$
5. Compute $\tilde{\alpha}_j$ such that $|\tilde{\alpha}_j - \alpha_j|_p \leq p^{-k}$
6. Compute $\tilde{\beta}_i := g(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ for $1 \leq i \leq s$, and form a matrix $B$ where the $i$th row contains the lift of coefficients of $\tilde{\beta}_i$ as elements to $\mathbb{Z}$. 

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7. Form a big matrix \( \tilde{B} \in \mathbb{Z}^{(s+f_p) \times (s+f_p)} \) by first concatenating \( I_s \) and \( \lambda B \) to get \( (I_s | \lambda B) \) and then appending a matrix \( (0 I_s | \lambda p^k I_{f_p}) \) to the bottom.

8. Apply the LLL algorithm to the rows of \( \tilde{B} \) obtaining a new matrix \( L = (L_{i,j})_{1 \leq i, j \leq f_p + s} \).

9. The lattice \( \Lambda \) is generated by \( (L_{i,j})_{1 \leq i \leq l, 1 \leq j \leq s} \) where \( l \) is the index of the last row \( L_i \) of \( L \) with norm \( \| L_i \|_2 < \lambda \).

\[ \begin{align*}
\text{Proof.} \quad \text{Using Theorem 11 we see that } N \text{ is a bound for the maximum norm of a length of a basis-relation, so that } NM s \text{ is a bound for the complex embedding } |(,)^{(j)}| \text{ of a possible relation. The precision is now chosen in the same way as in Algorithm 10 so that a possible relation } e \in \mathbb{Z}^s \text{ with } \| e \|_2 < N \text{ and } | \sum_{i=1}^s e_i g_i(\alpha_1, \ldots, \alpha_n) |_p < p^{-k} \text{ has to be zero.}

\text{In the matrix } L, \text{ the } s \text{-leftmost columns encode the transformations applied to } B \text{ while the rightmost columns give the evaluated relation:}
\lambda \sum_{i=1}^s L_{j,i} g_i(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{f_p} L_{j,i+s} \tilde{\omega}_i + p^k x
\end{align*}\]

(for some \( x \in \mathbb{Z}^f \)). So we see that if there is a relation \( \sum_{i=1}^s e_i g_i(\alpha_1, \ldots, \alpha_n) = 0 \), then the \( \mathbb{Z} \)-span of the first \( s \) rows of \( \tilde{B} \) contains a vector \( (e_1, \ldots, e_n, u_1, \ldots, u_{f_p}) \), with \( u_i \in \lambda p^k \mathbb{Z} \).

So by adding suitable multiples of the last \( f_p \) rows of \( \tilde{B} \) we get that \( (e_1, \ldots, e_n, 0, \ldots, 0) \) lies in the \( \mathbb{Z} \)-span of the rows of \( \tilde{B} \).

Our choice of \( k \) and \( \lambda \) now ensures the following facts:

1. If and only if \( (L_{j,i})_{1 \leq j \leq s} \) is a true relation, \( (L_{j,i+s})_{1 \leq j \leq f} \) is zero.

2. If \( (L_{j,i+s})_{1 \leq j \leq f} \) is not zero, then \( \| (L_{i,j})_{1 \leq j \leq s+f} \|_2 \geq \lambda > N \)

3. If there are relations within the bounds of Theorem 11 then the LLL will find them since by Lemma 12 there must be rows in \( L \) with norm bounded by \( 2^{s-1} N < \lambda \) which implies that they are relations.

Find the proof of the correctness shows that this method must terminate with the correct answer.

\[ \square \]
If the Galois-action on the $p$-adic roots $\alpha_1, \ldots, \alpha_n$ is known, then we can improve the runtime substantially, by using the fact that if $\sum_{i=1}^s g_i(\alpha_1, \ldots, \alpha_n) = 0$ then we also have $\sum_{i=1}^s g_i(\sigma \alpha_1, \ldots, \sigma \alpha_n) = 0$ for all $\sigma \in G = \text{Gal}(f)$. This allows us to replace LLL by much faster echelon algorithms over $\mathbb{Z}/p^k\mathbb{Z}$ followed by rational reconstruction.

Algorithm 14 Let $f \in \mathbb{Z}[t]$ be monic and $\alpha_1, \ldots, \alpha_n \in \Gamma/\mathbb{Q}_p$ be the roots of $f$ in some unramified extension of $\mathbb{Q}_p$ of degree $p$. Furthermore, let $G = \text{Gal}(f) < S_n$ be given explicitly, ie $\sigma \alpha_i = \alpha_{\sigma i}$. Now, let $g_i \in \mathbb{Z}[x_1, \ldots, x_n]$ be arbitrary ($1 \leq i \leq s \leq \#G$) and define

$$\Lambda := \{ e \in \mathbb{Z}^s \mid \sum_{i=1}^s e_i g_i(\alpha_1, \ldots, \alpha_n) = 0 \}.$$

This algorithm computes a $\mathbb{Z}$-basis for $\Lambda$.

1. Compute a bound $M > 0$ such that for each $i$ we have $|g_i(\beta_1, \ldots, \beta_n)^{(j)}| < M'$
2. Set $N := s^{s-1} M^{s-1}$
3. Set $k := \lceil 2 \log NM/\log p \rceil$
4. Select a set $S \subseteq G$ of size $s$, containing the identity of $G$.
5. repeat
6. Compute $\tilde{\alpha}_j$ such that $|\tilde{\alpha}_j - \alpha_j|_p \leq p^{-k}$
7. Set $B := ()$ a matrix with $s$ rows and 0 columns.
8. for $\sigma \in S$ do
9. Compute $\tilde{\beta}_i := g_i(\tilde{\alpha}_{\sigma 1}, \ldots, \tilde{\alpha}_{\sigma n})$ for $1 \leq i \leq s$, and form a matrix $\tilde{B}$ where the $i$th row contains the lift of coefficients of $\tilde{\beta}_i$ as elements to $\mathbb{Z}$.
10. Set $B := (B|\lambda \tilde{B})$, ie. append $\lambda \tilde{B}$ to the right of $B$.
11. Apply HNF techniques to compute a the nullspace $N$ of $B \in (\mathbb{Z}/p^k\mathbb{Z})^{s \times s}$ in echelon form.
12. Use rational reconstruction to find (if possible) the unique $\tilde{N} \in \mathbb{Q}^{l \times s}$ such that $\tilde{N} \cong N$ mod $p^k$. If this fails, increase the set $S$ by randomly selecting at most $0.2 \#S$ elements in $G \setminus S$ and $k := \lceil 1.2k \rceil$ and go back to step 5.
13. Compute a matrix $S \in \mathbb{Z}^{l \times n}$ such that $S$ is a $\mathbb{Z}$-basis for the intersection of the $\mathbb{Q}$-vectorspace with basis $\tilde{N}$ and $\mathbb{Z}^s$ (using some saturation method).
14. Apply the LLL algorithm to $S$ to obtain a LLL reduced basis $L$.
15. Set $k := \lceil 1.2k \rceil$ and increase the set $S$ by randomly selecting at most $0.2 \#S$ elements in $G \setminus S$. 

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16. until all rows $L_i$ of $L$ are norm bounded: $\|L_i\|_2 < N$ and are true relations by Algorithm 10

**Proof.** Let $K := \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$, then, since $\Gamma$ is a splitting field for $f$, we have $K \otimes_{\mathbb{Q}} \Gamma \cong \Gamma^{(K:Q)}$ and the embedding is given via $\alpha_i \mapsto (\sigma(\alpha_i))_{\sigma \in G}$. Furthermore, as $\mathbb{Q}_p$-vectorspace we have $\Gamma \cong \mathbb{Q}_p^f$ and $K \otimes_{\mathbb{Q}} \Gamma \cong \mathbb{Q}_p^{f(K:Q)}$ and the embedding extends via composition with $\Gamma \ni \gamma = \sum_{i=1}^f \gamma_i \omega_i \mapsto (\gamma_i)_{1 \leq i \leq f} \in \mathbb{Q}_p^f$. If we apply this embedding to $V := [g_1(\alpha), \ldots, g_s(\alpha)]_\mathbb{Q}$ we get $V \otimes \Gamma = [(g_1(\sigma\alpha))_{\sigma \in G}, \ldots, (g_s(\sigma\alpha))_{\sigma \in G}]_\Gamma$. The fact that extensions of scalars preserves dimension,

$$\dim \mathbb{Q}(V) = \dim \Gamma(V \otimes \Gamma) = \dim \mathbb{Q}_p(V \otimes \Gamma)$$

implies that eventually $\text{rk}_{\mathbb{Q}_p} B = \dim \mathbb{Q}(V)$. Similarly, the increasing precision ensures that the rational reconstruction will be successful, eventually. To be more precise: Assume that $S$ is large enough so that $\dim \mathbb{Q}(V) = \text{rk}_{\mathbb{Q}_p} B$ and let $M \in \mathbb{Q}^{l \times s}$ be a $\mathbb{Q}$-basis matrix for the $\mathbb{Q}$-nullspace of $B$. Without loss of generality, we can furthermore assume that $M = (M_{i,j})$ is in echelon form, thus $M$ is uniquely defined by $V$. If the precision $k$ is chosen so that $p^k > \max h(M_{i,j})^2$ for the naive height $h : \mathbb{Q} \to \mathbb{Z} : p/q \mapsto \max(p, q)$, then $N$ in step 11 will allow to compute $M$ by reconstruction.

If however $S$ is too small, it may happen that $\text{rk}_{\mathbb{Q}_p} B < \dim \mathbb{Q}(V)$ which means that the matrix $N$ computed in step 11 does not represent an approximation to $M$. In this case, either the reconstruction fails or the reconstructed relations cannot be verified in step 16.

In steps 4, 12 and 15 we increase the size of the matrix by exploiting the Galois action. This is done hoping that generically the new columns are independent from the previous ones as to increase the rank of $B$. It is clear that if $S = G$ then the $\mathbb{Z}$-nullspace of $B$ would be precisely $\Lambda$, however if $S = \{I_G\}$ then there will always be spurious relations, some of which may be small in size.

Also, using well known techniques that generalize rational reconstruction to number fields and omitting steps 13 and 14 the algorithms can easily be extended to find $R$-relations instead of $\mathbb{Z}$-relations for $R$ being any order in some number field.

**Remark.** If the Galois action is known then on some occasions we can use Algorithm 10 to give a more efficient algorithm for finding a basis of $\Lambda = \{(e_1, \ldots, e_n) \mid \sum_i e_i \alpha_i = 0\}$. As in Section 4 we denote the permutation module of $G$ by $M$ (where $G$ is the Galois group).

We assume that $M$ has a unique decomposition as direct sum of irreducible $G$-modules, $M = V_1 \oplus \cdots \oplus V_r$. The uniqueness of this decomposition is equivalent to all of the $V_i$ being non-isomorphic. In that case we can compute the direct sum decomposition by computing a maximal set of orthogonal primitive idempotents in the centre of the algebra

$$\text{End}_G(M) = \{T : M \to M \mid T \text{ is linear and } T(\sigma(v)) = \sigma(T(v)) \text{ for } v \in M \text{ and } \sigma \in G\}.$$

It also follows that $\Lambda_{\mathbb{Q}} = V_{i_1} \oplus \cdots \oplus V_{i_k}$. Now for each $V_i$ we do the following. For each element $(e_1, \ldots, e_n)$ in a basis of $V_i$ we check whether $\sum_i e_i \alpha_i = 0$ with Algorithm 10. Then $\Lambda$ is equal to the direct sum of the $V_i$ that pass this test.
7 A second algorithm for the algebraic hull

In this section we assume that the base field $F$ is $\mathbb{Q}$. We describe a second algorithm for constructing $\mathfrak{g}_\mathbb{Q}(X)$, for a semisimple matrix $X$. It is similar in nature to the algorithm of Section 3. But instead of constructing the splitting field we make use of the algorithms in the previous section. For simplicity we assume that the characteristic polynomial is square free. The generalisation to the general case is straightforward. We use the same notation as in Section 3.

First we find $\Lambda := \{ e \in \mathbb{Z}^n \mid \sum_{i=1}^n e_i \alpha_i = 0 \}$ using $g_i := x_i$ and either Algorithm 13 or Algorithm 14. Let $\underline{e} = (e_1, \ldots, e_n)$ be a basis element of $\Lambda$. The second step consists in solving the equations that define $\Upsilon$ (cf., Lemma 2). For $i \geq 0$ set $g_i(\underline{e}) = \sum_{k=1}^n e_k x_k^i$, and $\Delta_i(\underline{e}) := g_i(\underline{e})(\alpha_1, \ldots, \alpha_n)$. Let $t + 1$ be the degree of the minimal polynomial of $X$. Then again with Algorithm 13 or 14 we find all integral (or equivalently, rational) linear dependencies of the $\Delta_i(\underline{e})$, i.e. all vectors $u = (u_1, \ldots, u_t) \in \mathbb{Q}^t$ with $\sum u_i \Delta_i(\underline{e}) = 0$. Let $M(\underline{e})$ denote the $\mathbb{Q}$-vector space spanned by all those vectors $u$. Then $\Upsilon$ is equal to the intersection of all $M(\underline{e})$, where $\underline{e}$ runs through a basis of $\Lambda$. So this way we find a basis of $\Upsilon$, and hence a basis of $\mathfrak{g}_\mathbb{Q}(X)$ (cf. Algorithm 3).

8 Examples

To generate a set of input examples we used the database of polynomials over the rationals with given Galois groups by Klüners and Malle [13]. In this database the $n$-th transitive permutation group on $d$ points is denoted $dT_n$. For each polynomial of degree $d$ ($6 \leq d \leq 12$) with Galois group isomorphic to $dT_n$ we computed the companion matrix $X$ of $f$ and used this as input to our algorithms. In Figure 1 we plot the running times for the computation of $\mathfrak{g}_\mathbb{Q}(X)$ using both the algorithm in Section 3 with an exact, algebraic representation of the splitting field of $f$ as well as the algorithms in Section 7 against the logarithm of the group size. From the data presented, it is clear that the runtime of all three algorithms depends mainly on the size of the Galois group of $f$, i.e. the degree of the splitting field. Also, clearly, the algebraic representation of the splitting field has the worst runtime behaviour. In the second figure, we use a variation of the algorithms in Section 7 where instead of using the bounds from Algorithm 10 we compute the relations with a much smaller bound and “verify” them using twice the $p$-adic precision. While this of course does not give guaranteed results, nevertheless, in all cases where the bounds were small enough to use them, the output obtained thus was correct. Since this approach does not directly depend on the size of the splitting field, we can use this for larger degrees.

From both the figures we notice that for the purpose of computing algebraic hulls, it does not matter if Algorithm 13 or 14 is used. For proven results, the time is always dominated by the proof step while the actual computation takes only negligible time - even in large degrees and large Galois groups.
Figure 1: Time vs. log #Gal(f) for f of degree 6, 8, 9 and 10 and all transitive groups with proven bounds. C is used for data coming from the algebraic, exact representation of the splitting field, B is time using [14] and A is using [13].

Figure 2: Time vs. log #Gal(f) for f of degree 6, 8, 9, 10, 12, 14 and 15 and all transitive groups, using heuristic bounds. C is used for data coming from the algebraic, exact representation of the splitting field, B is time using [14] and A is using [13].
References

[1] K Belabas. A relative van Hoeij algorithm over number fields. *J. Symbolic Computation*, 37(5):641–668, 2004.

[2] A. Borel. *Linear algebraic groups*. Springer-Verlag, Berlin, Heidelberg, New York, second edition, 1991.

[3] Wieb Bosma, John J. Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Computation*, 24(3–4):235–266, 1997.

[4] John J. Cannon. MAGMA. [http://magma.maths.usyd.edu.au](http://magma.maths.usyd.edu.au) 2006.

[5] Claude Chevalley. *Théorie des groupes de Lie. Tome II. Groupes algébriques*. Actualités Sci. Ind. no. 1152. Hermann & Cie., Paris, 1951.

[6] Arjeh M. Cohen, Scott H. Murray, and D. E. Taylor. Computing in groups of Lie type. *Math. Comp.*, 73(247):1477–1498 (electronic), 2004.

[7] Henri Cohen. *A Course in Computational Algebraic Number Theory*, volume 138 of *Graduate Texts in Mathematics*. Springer, Berlin, erste edition, 1993.

[8] C. Fieker and C. Friedrichs. On reconstruction of algebraic numbers. In W. Bosma, editor, *Proceedings of the 4th International Symposium (ANTS-IV), Leiden, Netherlands, July 2–7, 2000*, volume 1838 of *LNCS*, pages 285–296, Berlin, 2000. Springer.

[9] K Geißler. *Berechnung von Galoisgruppen über Zahl- und Funktionenkörpern*. PhD Thesis, TU-Berlin, 2003.

[10] Fritz Grunewald and Daniel Segal. Some general algorithms. I. Arithmetic groups. *Ann. of Math. (2)*, 112(3):531–583, 1980.

[11] Gordon James and Martin Liebeck. *Representations and characters of groups*. Cambridge University Press, New York, second edition, 2001.

[12] Bettina Just. Integer relations among algebraic numbers. *Math. Comp.*, 54(189):467–477, 1990.

[13] J. Klüners and G. Malle. A database for field extensions of the rationals. *LMS J. of Comput. and Math*, 4:182–196, 2001.

[14] Serge Lang. *Algebraic Number Theory*, volume 110 of *GTM*. Springer, Berlin, second edition, 1994.

[15] D.W. Masser. Linear relations in algebraic groups. In Alan Baker, editor, *New Advances in transcendence theory*, pages 248–262, New York, 1988. Cambridge University Press.
[16] P. Nguyen and D. Stehlé. Floating-point LLL revisited. In Proceedings of Eurocrypt 2005, volume 3494 of LNCS, pages 215–233, Berlin, 2005. Springer.

[17] Guéna el Renault and Kazuhiro Yokoyama. A modular method for computing the splitting field of a polynomial. In Florian Hess, Sebastian Pauli, and Michael Pohst, editors, Algorithmic Number Theory Symposium (ANTS VII), volume 4076 of LNCS, pages 124–140, Berlin, 2006. Springer.

[18] Joseph Rotman. Galois theory. Universitext. Springer-Verlag, New York, second edition, 1998.

[19] Eugene Schenkman. Group theory. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1965.

[20] Jean-Pierre Serre. Local Fields. Springer, Berlin, 1979.