Bokstein homomorphism as a universal object

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Introduction.

For any prime $p$, the homology $H_\ast(X,k)$ of a topological space $X$ with coefficients in the residue field $k = \mathbb{Z}/p\mathbb{Z}$ is naturally a module over the Steenrod algebra $A$ of stable cohomological operations at $p$. While the operations themselves are linear endomorphisms of the functor $H_\ast(\cdot,k)$, their construction is highly non-linear. In particular, if one represents $H_\ast(\cdot,k)$ as the homology of any of the standard functorial complexes, e.g. the singular chain complex $C_\ast(\cdot,k)$, then there is no meaningful way to lift the Steenrod algebra to a DG algebra $A_\ast$ that acts naturally on $C_\ast(\cdot,k)$. Indeed, were it possible to do it, one would obtain a triangulated functor from the stable homotopy category localized at $p$ to the derived category $\mathcal{D}(A_\ast)$ of DG modules over $A_\ast$, and Adams spectral sequence would then show that the functor is an equivalence. Thus we would have a DG enhancement for the stable homotopy category localized at $p$. This is known not to exist (see e.g. [S] for an excellent overview, and in particular for a proof of non-existence).

A seeming exception to the rule is the very first of the homological operations — the Bokstein homomorphism $\beta : H_\ast(X,k) \to H_{\ast-1}(X,k)$. To construct it, all one has to do is to consider the chain complex $C_\ast(X,\mathbb{Z}/p^2)$ with coefficients in the ring $\mathbb{Z}/p^2$. This is still a linear object, at least in the sense of algebraic triangulated categories as in [K], and the Bokstein homomorphism is just the connecting homomorphism in the long exact sequence associated to changing the coefficients from $\mathbb{Z}/p^2$ to $k = \mathbb{Z}/p$.

However, this exception is in fact an illusion: just as all the other homological operations, the Bokstein homomorphism does not lift to a functorial map $C_\ast(X,k) \to C_{\ast-1}(X,k)$.

If one tries to understand what goes wrong, one realizes that the problem is in the ring map $a : \mathbb{Z}/p^2 \to k$. This is a square-zero extension of rings. If for a moment we replace $\mathbb{Z}/p^2$ with the trivial square-zero extension $k[t]/t^2$, then for any complex $V$ of flat $k[t]/t^2$-modules, the tensor product complex
\( V_\cdot = V \otimes_{k[t]/t^2} k \) does have a functorial endomorphism \( \beta : V_\cdot \to V_{\cdot -1} \). Moreover, sending \( V \) to \( (V_\cdot, \beta) \) descends to an equivalence

\[
D(k[t]/t^2) \cong D(k[\beta]),
\]

where \( k[\beta] \) is the free associative DG algebra generated by one element \( \beta \) of degree 1 (this is one of the simplest instances of Koszul duality, see e.g. [BBD]). Then by the general homological yoga, the non-trivial extension \( a : \mathbb{Z}/p^2 \to k \) ought to correspond to a class in the second Hochschild cohomology group of the ring \( k \), and we might expect to have a version of Koszul duality twisted by such a class. But the class does not exist — in fact, the group in question is trivial. While \( a \) is a square-zero extension of rings, it is not a square-zero extension of \( k \)-algebras. So, while the derived category \( D(\mathbb{Z}/p^2) \) of \( \mathbb{Z}/p^2 \)-modules of course has a DG enhancement, it does not have a enhancement over \( k \) — something that would have been automatic by Koszul duality were we to have a DG lifting of the Bokstein homomorphism \( \beta \).

One can then pose the following somewhat philosophical question. Assume that we are working with the category \( C_\cdot(k) \) of complexes of \( k \)-vector spaces, with its derived category \( D(k) \) and the rest of standard homological machinery. Assume at the same time that our vision is limited to the world where \( p = 0 \): we do not know that \( \mathbb{Z} \) exists, nor that there are any primes. Can we nevertheless describe, or even discover, the category \( D(\mathbb{Z}/p^2) \), purely in terms of complexes of \( k \)-vector spaces and various operations on them?

The question of course might look spurious; our justification for posing it is that it seems to admit an interesting answer. A one-sentence formulation is that one should simply allow non-linearity, as long as we have linearity on the level of homology. Specifically, we mean the following. Any DG algebra \( A_\cdot \) over \( k \) defines an endofunctor \( F \) of the category \( C_\cdot(k) \) by setting

\[
F(V_\cdot) = A_\cdot \otimes_k V_\cdot,
\]

and since \( A_\cdot \) is a DG algebra, the functor \( F \) is a monad (that is, an algebra in the monoidal category of endofunctors, with the monoidal structure given by composition). The functor \( F \) is by definition linear, so that it sends quasiisomorphic complexes to quasiisomorphic complexes, and induces a \( k \)-linear endofunctor of the derived category \( D(k) \). However, the converse it not true: there are endofunctors of the category \( C_\cdot(k) \) with these two properties that do not come from any complex \( A_\cdot \). On the level of homology, such endofunctors are necessarily linear, but on the level of complexes, they...
are not. Nevertheless, they can be monads, too; and it is such a monad that should give a functorial lifting of the Bokstein homomorphism to chain complexes. After localization with respect to quasiisomorphisms, the category of algebras over such a monad would then give the derived category $\mathcal{D}(\mathbb{Z}/p^2)$.

Of course, the above is just a sketch of a possible argument. But note that our question can already be posed on the level of objects, without going to complexes: how do we describe modules over $\mathbb{Z}/p^2$ in terms of vector spaces over $k$? More concretely, can we make such a description functorial enough so that it allows us, for example, to treat square-zero extension of the form $\mathbb{Z}/p^2 \to k$ on a par with usual $k$-linear square-zero extensions of $k$-algebra? In this context, a lot is already known since the pioneering work of M. Jibladze and T. Pirashvili in the late 80-ies (see e.g. [JP] that also contains references to earlier works). In particular, the appropriate replacement of the second Hochschild homology group $HH^2(R, M)$ of a $k$-algebra $R$ with coefficients in an $R$-bimodule $M$ is the so-called MacLane cohomology group $H^2_M(R, M)$. By themselves, MacLane cohomology groups have been discovered by MacLane [M] back in 1956, but the definition used an explicit complex. Among other things, Jibladze and Pirashvili show that just as Hochschild cohomology groups $HH^*(R, M)$, the groups $H^*_M(R, M)$ can be interpreted as Ext-groups from $R$ to $M$. To do this, one needs to enlarge the category where the Ext-groups are computed. Specifically, $R$-bimodules correspond to additive endofunctors of the additive category $\mathcal{E} = R$-proj of finitely generated projective $R$-modules, and instead of them, one should use the category of all endofunctors.

Recently, an extension of this story to a more general exact or abelian category $\mathcal{E}$ has been studied in [KL], and several MacLane-type cohomology theories have been constructed. A natural next thing to do would be to show that second MacLane cohomology classes correspond to square-zero extensions. In the case of rings and projective modules over them, this has also been shown by Jibladze and Pirashvili in [JP]. However, their approach was focused on rings and the associated “algebraic theories” rather that just the category $\mathcal{E} = R$-proj, and it is not clear how to generalize it to the abelian category case. Thus a simple purely categorical construction of the correspondence would be useful.

The first goal of the present paper is to provide such a construction. We start with a ring $R$ and a bimodule $M$, interpret MacLane cohomology classes $\varepsilon \in H^2_M(R, M)$ as Ext-classes in the category of endofunctors of $R$-proj, and show how they correspond to square-zero extensions of $R$ by $M$. 
In addition to this, we also consider the category $C_{pf}^R(R)$ of perfect complexes of $R$-modules. We introduce a convenient class of endofunctors of $C_{pf}^R(R)$ that we call admissible; these are “additive on the level of homology”. We then show that extensions in this category of admissible endofunctors are also related to square-zero extensions of the ring $R$.

Our motivating example for working with complexes is a certain natural admissible endofunctor $C_*$ of the category $C_{pf}^R(k)$ that appeared recently in the work on cyclic homology done in [Ka3]. The precise definition of the functor $C_*$ is given in Subsection 6.2. It is based on an old observation: for any $k$-vector space $V$ and any integer $i$, we have a functorial isomorphism

$$\tilde{H}_i(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \cong V,$$

where $\tilde{H}_i$ stands for Tate homology, and the cyclic group $\mathbb{Z}/p\mathbb{Z}$ acts on the $p$-th tensor power $V^{\otimes p}$ by the longest permutation. In [Ka3], this isomorphism is extended to complexes $V_\bullet \in C_{pf}^R(k)$, and this gives rise to the admissible endofunctor $C_*$. For applications to cyclic homology, it is important to figure out how $C_*$ is related to the square-zero extension $\mathbb{Z}/p^2 \to k$. This is the second goal of our paper.

What we discover along the way is that $C_*$ can be generalized to an endofunctor of the category $C_{pf}^R(R)$ for an arbitrary commutative ring $R$, and the corresponding square-zero extension is then the second Witt vectors ring $W_2(R)$ (if $R = k = \mathbb{Z}/p$, then $W_2(R) = \mathbb{Z}/p^2$). We note that while the Witt vectors ring is a very classic object, it continues to attract attention (see for example a recent paper [CD]). Thus yet another interpretation of $W_2(R)$ might prove useful.

Finally, in the interests of full disclosure, we should mention that our results for complexes are not as strong as for objects, and were we to restrict our attention to objects, the paper could have been shortened quite significantly. Nevertheless, we do spend some time on the case of complexes. Aside from the pragmatic application to cyclic homology and Witt vectors, we feel that it is important to study this case in as much detail as possible, since potentially, it provides technology that can be used for higher homological operations. Of course, those can be also studied by a variety of standard methods of algebraic topology, but it never hurts to have yet another point of view. In particular, polynomial endofunctors of the category of $k$-vector spaces have been the subject of a lot of research in the last decades, and we now have a lot of structural information on them (see e.g. [Pf] for an overview). Constructing non-trivial admissible polynomial endofunctors of
$C, (k)$ similar to our cyclic powers functor $C$, and feeding this information back into algebraic topology might be a worthwhile exercise.

Here is an outline of the paper. Section 1 is essentially devoted to linear algebra: we assemble several standard facts on objects and complexes in abelian categories, in order to set up the definitions and notation. Mostly, these deal with representing classes in Ext by extensions and splitting these extensions. All the proofs are about as difficult as the snake lemma, so we omit them. In Section 2 we fix an associative unital ring $R$, and we discuss the categories of endofunctors $\mathcal{B}(R), \mathcal{B}_{q}(R)$ of the category $R$-proj of finitely generated projective $R$-modules and the category $C_{pf}^q(R)$ of bounded complexes in $R$-proj. For $R$-proj, simply considering pointed functors is good enough. For $C_{pf}^q(R)$, we introduce the notion of an admissible endofunctor (Definition 2.2). This notion is rather ad hoc and probably not the best possible, but it works for our purposes, so we leave it at that. In Section 3 we introduce square-zero extensions of the ring $R$, and show how to classify them by extensions in the category $\mathcal{B}(R)$. We also explain why if one wants an absolute theory — that is, square-zero extensions of rings and not of algebras — then it is necessary to consider non-additive functors. The main technical point in the correspondence is Proposition 3.4 that shows that a certain explicit functorial extension of abelian groups automatically splits. It might have an abstract interpretation as a vanishing of a certain MacLane-type cohomology group, but we have not pursued this: instead, we opt for a direct construction. After that, in Section 4 we assume known the correspondence between square-zero extension and extensions in the endofunctor category, and we show how this allows to describe modules and complexes over a square-zero extension $R'$ of a ring $R$ in terms of data purely in the categories $R$-proj and $C_{p}'(R)$. Then in Section 5 we study multiplication: we assume that the ring $R$ is commutative, and show that its commutative square-zero extensions $R'$ correspond to extensions in $\mathcal{B}(R)$ that are multiplicative in a certain natural way. In particular, this allows us to show that liftings of an $R$-algebra $A$ to a square-zero extension $R'$ are classified by Hochschild cohomology classes. We then attempt to generalize this to complexes and DG algebras, and explain why it does not work too well (this is Subsection 5.4).

Finally, in Section 6 we turn to our specific example of a non-linear extension that appears in the study of cyclic homology: the functor $C$, corresponding to the second Witt vectors ring $W_2(R)$. In this special case, we are able to go further than in the general situation, and in particular, we can describe liftings of a DG algebra $A$, over a perfect field $k$ of characteristic...
p in terms of splittings of a certain canonical square-zero extension of $A$, in the category of DG algebras over $k$.

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1 Elementary extensions.

1.1 Objects. We start by recalling some standard linear algebra. For any abelian category $E$, we will denote by $C_\ast(E)$ the category of chain complexes in $E$ (our indexes are homological, so that the differential acts as $d: M_{i+1} \to M_i$). For any complex $E_\ast$ and integer $m$, we denote by $E_\ast[m] \in C_\ast(E)$ the homological shift by $m$ of the complex $E_\ast$ (that is, the $n$-th term of the complex $E_\ast[m]$ is $E_{n-m}$). For any complex $E_\ast$, we denote by $\text{Cone}(E_\ast) \in C_\ast(E)$ the cone of the identity map $\text{id}: E_\ast \to E_\ast$, so that we have a short exact sequence

$$0 \to E_\ast \to \text{Cone}(E_\ast) \to E_\ast[1] \to 0$$

in $C_\ast(E)$. For any integer $m$, we will denote by $C_{\geq m}(E), C_{\leq m}(E) \subset C_\ast(E)$ the full subcategories spanned by complexes $E_\ast$ such that $E_i = 0$ for $i < m$ resp. $i > m$. For any two integers $m \leq n$, we will denote $C_{[m,n]}(E) = C_{\geq m}(E) \cap C_{\leq n}(E)$, and we will say that a complex $E_\ast$ has amplitude $[m,n]$ if it lies in $C_{[m,n]}(E) \subset C_\ast(E)$. Any object $A \in E$ can be treated as a complex by placing it in degree 0; this gives an equivalence

$$E \cong C_{[0,0]}(E) \subset C_\ast(E).$$

For any integer $m$, we denote by $\tau_{\geq m}: C_\ast(E) \to C_\ast(E)$ the composition of the embedding $C_{\geq m}(E) \subset C_\ast(E)$ with its right-adjoint functor, and we denote by $\tau_{\leq m}: C_\ast(E) \to C_\ast(E)$ the composition of the embedding $C_{\leq m}(E) \subset C_\ast(E)$ with its left-adjoint functor. For any two integers $m \leq n$, we denote $\tau_{[m,n]} = \tau_{\leq n} \tau_{\geq m} \cong \tau_{\geq m} \tau_{\leq n}$. These are the canonical truncation functors. For any complex $E_\ast$ and any integer $m$, $\tau_{[m,m]}E_\ast \in C_{[m,m]}(E) \cong E$ is the $m$-th homology object $H_m(E_\ast)$ of the complex $E_\ast$. We record right away the following obvious fact.
Lemma 1.1. Assume given an exact sequence
\[ 0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0 \]
of complexes in \( \mathcal{E} \), and assume that for some integer \( m \), the connecting differential \( \delta : H_m(A) \rightarrow H_{m-1}(B) \) of the corresponding long exact sequence of homology is equal to 0. Then the sequences
\[ 0 \longrightarrow \tau \geq m B \longrightarrow \tau \geq m C \longrightarrow \tau \geq m A \longrightarrow 0, \]
\[ 0 \longrightarrow \tau \leq m-1 B \longrightarrow \tau \leq m-1 C \longrightarrow \tau \leq m-1 A \longrightarrow 0 \]
are exact.

Proof. Clear. \( \square \)

Definition 1.2. A sequence
\[ (1.3) \quad 0 \longrightarrow B \underset{b}{\longrightarrow} E \underset{a}{\longrightarrow} A \longrightarrow 0 \]
of complexes in an abelian category \( \mathcal{E} \) is \textit{quasiexact} if \( b \) is injective, \( a \) is surjective, and the complex \( \ker a/\text{im } b \) is acyclic.

We note right away that a quasiexact sequence generates a long exact sequence in homology: if for some integer \( i \), we denote by \( Z_i \subset E_i \) the preimage \( Z_i = d^{-1}(b(B_{i-1})) \) of \( b(B_{i-1}) \subset E_{i-1} \), then the natural map \( Z_i \rightarrow H_i(A) \) induced by \( a \) is surjective, and the natural map \( Z_i \rightarrow H_{i-1}(B) \) induced by the differential \( d \) factors through \( H_i(A) \).

Definition 1.3. An \textit{elementary extension} of an object \( A \in \mathcal{E} \) by an object \( B \in \mathcal{E} \) is a quasiexact sequence
\[ (1.4) \quad 0 \longrightarrow B[1] \underset{b}{\longrightarrow} C \underset{a}{\longrightarrow} A \longrightarrow 0 \]
in \( C_{[0,1]}(\mathcal{E}) \), where \( A \) and \( B \) are treated as complexes via the embedding \( (1.2) \).

Equivalently, an elementary extension \( (1.4) \) is given by two objects \( C_0, C_1 \) in the category \( \mathcal{E} \) and a four-term exact sequence
\[ (1.5) \quad 0 \longrightarrow B \underset{b}{\longrightarrow} C_1 \underset{d}{\longrightarrow} C_0 \underset{a}{\longrightarrow} A \longrightarrow 0, \]
where \( d : C_1 \rightarrow C_0 \) is the differential in the complex \( C^* \). Thus by Yoneda, it defines an element \( \psi \in \text{Ext}^2(A, B) \). Elementary extensions of \( A \) by \( B \) form
a category $\mathcal{E}l(A, B)$ in an obvious way, and we denote by $\text{El}(A, B)$ the set of its connected components. We note that \textit{a priori} it is a class, not a set, but in all our examples it will actually be a set, and the same will be true for various other connected component sets that will appear. The set $\text{El}(A, B)$ is naturally identified with $\text{Ext}^2(A, B)$ — that is, elementary extensions have the same class $\psi$ if and only if they can be connected by a chain of maps. In this case, we say that elementary extensions are \textit{equivalent}. A canonical elementary extension with trivial class $\psi = 0$ is obtained by setting $C_1 = B$, $C_0 = A$, $d = 0$. An elementary extension is \textit{split} if it is equivalent to the trivial one.

**Definition 1.4.** A \textit{splitting} of an elementary extension (1.4) is an object $C_{01} \in \mathcal{E}$ equipped with two maps $c_1 : C_1 \to C_{01}$, $c_0 : C_{01} \to C_0$ that fit into a cartesian commutative diagram

$$
\begin{array}{ccc}
C_1 & \longrightarrow & C_{01} \\
\downarrow & & \downarrow \\
\text{Im} \, d & \longrightarrow & C_1,
\end{array}
$$

where $\text{Im} \, d$ is the image of the differential $d : C_1 \to C_0$.

Equivalently, a splitting is an object $C_{01}$ equipped with a three-step filtration $W_0C_{01} \subset W_1C_{01} \subset C_{01}$ and identifications

$$W_1C_{01} \cong C_1, \quad C_{01}/W_0C_{01} \cong C_0$$

such that under these identifications, $W_0C_{01} \subset W_1C_{01}$ is identified with $B \subset C_1$, the quotient $C_{01}/W_1C_{01} = (C_{01}/W_0C_{01})/(W_1C_{01}/W_0C_{01})$ is identified with $B = C_1/\text{Im} \, d$, and the natural map $W_1C_{01} \to C_{01} \to C_{01}/W_0C_{01}$ is identified with the differential $d : C_1 \to C_0$. It is well-known and easy to check that an elementary extension is split if and only if it admits a splitting in the sense of Definition 1.4.

Morphisms of splittings of a fixed elementary extension $\varphi = \langle C_*, a, b \rangle \in \mathcal{E}l(A, B)$ are defined in the obvious way, and they are obviously invertible. We denote by $\mathcal{S}pl(\varphi)$ the groupoid of such splittings, and we denote by $\text{Spl}(\varphi)$ the set of isomorphism classes of objects in $\mathcal{S}pl(\varphi)$.

The category $\mathcal{E}^o$ opposite to $\mathcal{E}$ is also abelian, and $C_*(\mathcal{E}^o) \cong C_*(\mathcal{E})$ is opposite to $C_*(\mathcal{E})$. Since the diagram (1.5) is obviously self-dual, an elementary extension $\varphi \in \mathcal{E}l(A, B)$ defines an elementary extension of $B$ by $A$ in the opposite category, so that we have a natural antiequivalence

$$\mathcal{E}l(A, B) \cong \mathcal{E}l(B^o, A^o), \quad \varphi \mapsto \varphi^o$$
where \( A^o \) and \( B^o \) are \( A \) and \( B \) considered as objects of \( E^o \), and \( \varphi^o \) is the extension given the sequence (1.3) corresponding to \( \varphi \) but considered as a sequence in \( E^o \). Alternatively, in terms of the diagram (1.4), one needs to pass to the opposite category and apply a homological shift by 1. The notion of a splitting is also self-dual, so that we have a natural equivalence

\[
Spl(\varphi) \cong Spl(\varphi^o), \quad \varphi \in \mathcal{E}(A, B)
\]

sending \( \langle C_{01}, c_0, c_1 \rangle \) to \( \langle C^o_{01}, c_1, c_0 \rangle \).

Finally, assume given an elementary extension \( \varphi = \langle C_*, a, b \rangle \in \mathcal{E}(A, B) \), and a map \( f : B \to B' \) to some other object \( B' \in \mathcal{E} \). Then we define the composition \( f \circ C_* \) by the short exact sequence

\[
\begin{array}{c}
0 \to B[1] \xrightarrow{f \circ b} B'[1] \oplus C_* \xrightarrow{f \circ C_*} 0
\end{array}
\]

in \( C_*(\mathcal{E}) \), and we note that \( f \circ C_* \) is naturally an elementary extension of \( A \) by \( B' \). The construction is functorial, so that we have a functor

\[
(1.7) \quad f \circ - : \mathcal{E}(A, B) \to \mathcal{E}(A, B').
\]

On the level of the connected component sets, this functor corresponds to the map \( f \circ - : \text{Ext}^2(A, B) \to \text{Ext}^2(A, B') \) obtained by composition with \( f \). Moreover, sending a splitting \( \langle C_{01}, c_0, c_1 \rangle \) of the elementary extension \( \varphi \) to the cokernel of the natural map \( (-f) \oplus (c_1 \circ b) : B \to B' \oplus C_0 \) defines a functor

\[
(1.8) \quad f \circ - : Spl(\varphi) \to Spl(f \circ \varphi).
\]

Dually, for any map \( f : A' \to A \), one obtains a functor

\[
(1.9) \quad - \circ f : \mathcal{E}(A, B) \to \mathcal{E}(A', B)
\]

that induces the composition map \( - \circ f : \text{Ext}^2(A, B) \to \text{Ext}^2(A', B) \) on the connected component sets, and one has a natural functor

\[
(1.10) \quad - \circ f : Spl(\varphi) \to Spl(\varphi \circ f)
\]

for any elementary extension \( \varphi \in \mathcal{E}(A, B) \).

1.2 Complexes. More generally, assume given two complexes \( A_*, B_* \in C_*(\mathcal{E}) \), and define an elementary extension of \( A_* \) by \( B_* \) as a quasiexact sequence

\[
(1.11) \quad 0 \to B_*[1] \xrightarrow{b} C_* \xrightarrow{a} A_* \to 0
\]
in \( \mathcal{C}_*^{\mathcal{E}} \). Elementary extensions again form a category in the obvious way; we denote this category by \( \mathcal{E}_l(A, B) \), and we denote by \( \mathcal{E}_l(A, B) \) the set of its connected components. Extensions are equivalent if they lie in the same connected component of \( \mathcal{E}_l(A, B) \). A trivial extension is obtained by taking \( C_* = A_* \oplus B_* \). An extension is split if it is equivalent to the trivial one.

Every elementary extension (1.11) defines an extension of \( A_* \) by the complex \( \ker a \subset C_* \), and \( \ker a \) is by definition quasiisomorphic to \( B_*[1] \), so that we again obtain a canonical element

\[
\psi \in \text{Hom}_{\mathcal{D}(\mathcal{E})}(A, B[2]),
\]

where \( \mathcal{D}(\mathcal{E}) \) denotes the derived category of the abelian category \( \mathcal{E} \). Sending an extension (1.11) to its class \( \psi \) identifies the set \( \mathcal{E}_l(A, B) \) with the group \( \text{Hom}_{\mathcal{D}(\mathcal{E})}(A, B[2]) \).

We note that if \( A_* \) and \( B_* \) are objects \( A, B \) placed in degree 0, then \( \mathcal{E}_l(A, B) \) is bigger than \( \mathcal{E}_l(A, B) \) since the complex \( C_* \) in (1.11) is not required to have amplitude \([0, 1]\). However, we have a fully faithful embedding \( \mathcal{E}_l(A, B) \subset \mathcal{E}_l(A, B) \), and replacing a complex \( C_* \) with its canonical truncation \( \tau_{[0, 1]}C_* \) defines a natural functor

\[
\tau_{[0, 1]} : \mathcal{E}_l(A, B) \to \mathcal{E}_l(A, B)
\]

inverse to the embedding. Both the embedding and the inverse functor induce an isomorphism on the sets of connected components, so that every elementary extension in \( \mathcal{E}_l(A, B) \) is canonically equivalent to an extension of the form (1.4). The notion of an elementary extension of complexes is also self-dual, so that we have a natural antiequivalence

\[
\mathcal{E}_l(A, B)^\circ \cong \mathcal{E}_l(A^\circ, B^\circ),
\]

an extension of (1.6).

**Definition 1.5.** A left DG splitting of an elementary extension \( \varphi \) represented by a diagram (1.11) is a complex \( C^\circ_* \) equipped with maps \( l : C^\circ_* \to C_* \), \( b^\circ : \text{Cone}(B_*) \to C^\circ_* \) so that, with the map \( \alpha \) being the map of (1.11), the diagram

\[
\begin{array}{ccc}
\text{Cone}(B_*) & \xrightarrow{b^\circ} & C^\circ_* \\
\downarrow{\alpha} & & \downarrow{l} \\
B_*[1] & \xrightarrow{b} & C_* \\
\end{array}
\]

(1.12)
is commutative, and the sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Cone}(B_q) & \overset{b^!}{\longrightarrow} & C^t_q & \overset{a^*}{\longrightarrow} & A_q & \longrightarrow & 0 \\
\end{array}
\]

is quasiexact. A right DG splitting of the elementary extension $\varphi$ is a complex $C^r$ equipped with maps $r : C_\cdot \rightarrow C^r$, $a^r : C^r \rightarrow \text{Cone}(A_\cdot)$ such that $\langle C^r, r, a^r \rangle$ give a left DG splitting of the extension $\varphi^0$ in the opposite category $C_\cdot(\mathcal{E})^o$.

As a corollary of the definition, we see that for left DG splitting $C^l_q$, the map $a \circ l : C^l_q \rightarrow A_q$ is a quasiisomorphism, and for a right DG splitting $C^r$, the map $r \circ b : B_q[1] \rightarrow C^r$ is a quasiisomorphism. An elementary extension admits a right DG splitting iff it admits a left DG splitting iff it is split. For any left DG splitting $\langle C^l_q, l, b^! \rangle$ of an elementary extension, the cone $C^l_r$ of map $l$ is a right DG splitting of the same extension, with the obvious maps $r$, $a^r$. Conversely, for a right DG splitting $\langle C^r_q, r \rangle$, the shift $C^{rl}_r$ of the cone of the map $r$ is a left DG splitting. If we denote by $\text{Spl}^l_q(\varphi)$, $\text{Spl}^r_q(\varphi)$ the categories of left and right DG splittings of an elementary extension $\varphi$, then this defines functors

\[
\begin{align*}
\text{Spl}^l_q(\varphi) & \rightarrow \text{Spl}^r_q(\varphi), \\
\text{Spl}^r_q(\varphi) & \rightarrow \text{Spl}^l_q(\varphi).
\end{align*}
\]

The functors are not mutually inverse, but we do have functorial quasiisomorphisms

\[
C^{rl}_r \cong C^r, \quad C^{ltr}_l \cong C^l.
\]

Thus if we denote by $\text{Spl}^l_q(\varphi)$ resp. $\text{Spl}^r_q(\varphi)$ the sets of connected components of the categories $\text{Spl}^l_q(\varphi)$ resp. $\text{Spl}^r_q(\varphi)$, then we have a natural identification $\text{Spl}^l_q(\varphi) \cong \text{Spl}^r_q(\varphi)$.

**Definition 1.6.** A left DG splitting $\langle C^l_q, l, b^! \rangle$ of an extension $\langle 1.11 \rangle$ is strict if the diagram (1.12) is a Cartesian square in $C_\cdot(\mathcal{E})$. A right DG splitting is strict if it is strict as a left DG splitting in $C_\cdot(\mathcal{E})^o$.

For a strict left DG splitting $\langle C^l_q, l, b^! \rangle$ of an extension (1.11), the middle homology of the quasiexact sequence (1.13) is canonically isomorphic to the middle homology of the sequence (1.11). For a general left DG splitting, we just have a map between two acyclic complexes, and not necessarily an isomorphism. The same is true for right DG splittings. Any map between strict DG splittings is automatically an isomorphism, so that, if we denote by

\[
\begin{align*}
\text{Spl}^l_q(\varphi) & \subset \text{Spl}^l_q(\varphi), \\
\text{Spl}^r_q(\varphi) & \subset \text{Spl}^r_q(\varphi)
\end{align*}
\]

we have

\[
\text{Spl}^l_q(\varphi) \subset \text{Spl}^l_q(\varphi), \quad \text{Spl}^r_q(\varphi) \subset \text{Spl}^r_q(\varphi).
\]
the full subcategories spanned by strict splittings, then both \( S^{pl \ell}(\varphi) \) and \( S^{pl \ell'}(\varphi) \) are groupoids.

For any two objects \( A, B \in \mathcal{E} \), a splitting \( \langle C_{01}, c_0, c_1 \rangle \) of an elementary extension \( \mathcal{E} \) in the sense of Definition 1.4 defines its left and right DG splittings \( C^l, C^r \) in the sense of Definition 1.5: these are the natural complexes

\[
\begin{align*}
C_1 & \xrightarrow{c_1} C_{01}, & C_{01} & \xrightarrow{c_0} C_0,
\end{align*}
\]

placed in homological degrees 0 and 1, with the maps \( l \) resp. \( r \) induced by the maps \( c_0, \text{id} \) resp. \( c_1, \text{id} \), and the maps \( b', a^l \) induced by the maps \( b, c_1 \circ b \) resp. \( a \circ c_1, a \). Both these DG splittings are strict and functorial, so that we obtain natural functors

\[
S^{pl}(\varphi) \rightarrow S^{pl \ell}(\varphi) \subset S^{pl \ell}(\varphi), \quad S^{pl}(\varphi) \rightarrow S^{pl \ell'}(\varphi) \subset S^{pl \ell'}(\varphi).
\]

Automatically, \( C^r \) is quasisomorphic to \( C^r \), and \( C^l \) is quasiisomorphic to \( C^r \), but both these quasiisomorphisms are not isomorphisms on the nose.

Conversely, given a left DG splitting \( \langle C^l, l, b' \rangle \) of an elementary extension \( \mathcal{E} \), we can canonically construct its splitting \( C_{01} \) in two steps. First, we observe that the canonical truncation \( \tilde{C}^l = \tau_{[0,1]} C^l \) is also a left DG splitting, with the maps \( \tilde{l} = \tau_{[0,1]}(l), \tilde{b} = \tau_{[0,1]}(b') \). Second, we define \( C_{01} \) by the short exact sequence

\[
\begin{align*}
0 & \longrightarrow \tilde{C}^l_1 \xrightarrow{d \oplus \tilde{l}} \tilde{C}^l_0 \oplus C_1 \longrightarrow C_{01} \longrightarrow 0,
\end{align*}
\]

where \( d : \tilde{C}^l_1 \rightarrow \tilde{C}^l_0 \) is the differential. The map \( c_1 \) is induced by the natural embedding \( C_1 \rightarrow \tilde{C}^l_0 \oplus C_1 \), and the map \( c_0 \) is induced by the map

\[
\tilde{C}^l_0 \oplus C_1 \xrightarrow{\tilde{l} \oplus d} C_0,
\]

where \( d \) is the differential in the complex \( C \), (since \( l \) commutes with differentials, this map factors through \( C_{01} \)). The original left DG splitting \( C^l \) is then automatically equivalent, but not necessarily isomorphic, to the left-hand side complex in \( (1.15) \). For right DG splittings, one has a parallel construction that we leave to the reader.

**1.3 Gerbs.** For any objects \( A, B \in \mathcal{E} \) and an elementary extension \( \varphi \in \mathcal{E}(A, B) \), the set \( \text{Spl}(\varphi) \) of isomorphism classes of splittings of \( \varphi \) is naturally a torsor over the extension group \( \text{Ext}^1(A, B) \). This can be lifted to the level of categories: if we denote by \( \mathcal{E}(A, B) \) the groupoid of short exact sequences

\[
\begin{align*}
0 & \longrightarrow B \xrightarrow{b} E \xrightarrow{a} A \longrightarrow 0
\end{align*}
\]

(1.17)
in the category $\mathcal{E}$, then $\mathcal{S}pl(\varphi)$ is a gerb over $\mathcal{E}_x(A, B)$. Specifically, for any two objects $(C'_{01}, c'_1, c'_0), (C''_{01}, c''_1, c''_0) \in \mathcal{S}pl(\varphi)$, the diagram
\[
\begin{array}{ccc}
C_1 & \xrightarrow{c'_1 \oplus c''_1} & C'_{01} \oplus C''_{01} & \xrightarrow{c'_0 \oplus c''_0} & C_0
\end{array}
\]
is a complex, and its middle homology is naturally an extension of $A$ by $B$, so that we obtain a difference functor
\[
(1.18) \quad - : \mathcal{S}pl(\varphi) \times \mathcal{S}pl(\varphi) \to \mathcal{E}_x(A, B).
\]
Analogously, for any $(C_{01}, c_1, c_0) \in \mathcal{S}pl(\varphi)$ and any short exact sequence (1.17), the diagram
\[
\begin{array}{ccc}
B & \xrightarrow{c_1 \oplus b} & C_{01} \oplus E & \xrightarrow{c_0 \oplus -a} & A
\end{array}
\]
is a complex, and its middle homology is naturally an object in $\mathcal{S}pl(\varphi)$, so that we obtain a difference functor
\[
(1.19) \quad - : \mathcal{S}pl(\varphi) \times \mathcal{E}_x(A, B) \to \mathcal{S}pl(\varphi).
\]
The standard algebraic relations between differences then extend to isomorphisms between the relevant compositions of the functors (1.18) and (1.19). In particular, we have natural isomorphisms
\[
(1.20) \quad C \cong C' - (C' - C), \quad E \cong C - (C - E),
\]
for any $C, C' \in \mathcal{S}pl(\varphi), E \in \mathcal{E}_x(A, B)$.

To extend the functors (1.18), (1.19) to DG splittings, for any two complexes $A_*, B_* \in C_*(\mathcal{E})$, we denote by $\mathcal{E}_x(A_*, B_*)$ the category of quasiexact sequences (1.13). Then for any left DG splitting $(C'_l, l, b')$ of a DG elementary extension $\varphi = \langle C_*, a, b \rangle \in \mathcal{E}_L(A_*, B_*)$, and any object $\langle E_*, a', b' \rangle \in \mathcal{E}_x(A_*, B_*)$, we have a natural sequence
\[
\begin{array}{ccc}
0 & \longrightarrow & B_* & \xrightarrow{(b' \circ \beta) \oplus b'} & C'_l \oplus E_* & \xrightarrow{(a \circ l) \oplus -a'} & A_* & \longrightarrow & 0
\end{array}
\]
that is exact on the left and on the right, and its middle homology is naturally a left DG splitting of the extension $\varphi$. We denote this splitting by $C'_l - E_*$. The construction is functorial, so that we obtain a difference functor
\[
(1.21) \quad - : \mathcal{S}pl^l(A_*, B_*) \times \mathcal{E}_x(A_*, B_*) \to \mathcal{S}pl^l(\varphi).
\]
We note that since any exact sequence of complexes is quasiexact, we have a natural fully faithful embedding $\mathcal{E}_x(A_*, B_*) \subset \mathcal{E}_x(A_*, B_*)$, and if a DG
splitting $C$ is strict, then for any $E \in \mathcal{E}_x(A_\cdot, B_\cdot) \subset \mathcal{E}_x(A_\cdot, B_\cdot)$, the difference $C - E$ is also strict, so that (1.21) induces a functor

$$- : S pl^l(\varphi) \times \mathcal{E}_x(A_\cdot, B_\cdot) \rightarrow S pl^l(\varphi).$$

Conversely, assume given another left DG splitting $\langle \tilde{C}_l^l, \tilde{l}, \tilde{b} \rangle \in S pl^l(\varphi)$, and denote by $\overline{C}_l$ the cone of the map

$$\tilde{C}_l^l \oplus \tilde{C}_l^l \xrightarrow{\text{id} \oplus -\tilde{l}} C_\cdot.$$

Then by definition, we have a quasiexact sequence

$$0 \rightarrow \overline{B}_\cdot \rightarrow \overline{C}_\cdot \rightarrow \overline{A}_\cdot \rightarrow 0,$$

where $\overline{B}_\cdot$ is the cone of the map

$$\text{Cone}(B_\cdot) \oplus \text{Cone}(B_\cdot) \xrightarrow{\alpha \oplus -\alpha} B_\cdot[1],$$

and $\overline{A}_\cdot$ is the cone of the map

$$A_\cdot \oplus A_\cdot \xrightarrow{\text{id} \oplus -\text{id}} A_\cdot.$$

Furthermore, $\text{id} \oplus \text{id} : \text{Cone}(B_\cdot) \rightarrow \text{Cone}(B_\cdot) \oplus \text{Cone}(B_\cdot)$ extends to a natural embedding $\text{Cone}(B_\cdot) \subset \overline{B}_\cdot[-1]$, and the projection $\text{id} \oplus -\text{id} : A_\cdot \oplus A_\cdot \rightarrow A_\cdot$ extends to a natural surjection $\overline{A}_\cdot \rightarrow \text{Cone}(A_\cdot)$. Thus altogether, the sequence (1.24) induces a sequence

$$0 \rightarrow \text{Cone}(B_\cdot) \rightarrow \overline{C}_\cdot[-1] \rightarrow \text{Cone}(A_\cdot)[-1] \rightarrow 0.$$ 

Its middle homology is then naturally an object in $\mathcal{E}_x(A_\cdot, B_\cdot)$ that we denote by $C_l^l - \tilde{C}_l^l$. Again, the construction is functorial, so that we obtain a natural functor

$$- : S pl^l(\varphi) \times S pl^l(\varphi) \rightarrow \mathcal{E}_x(A_\cdot, B_\cdot).$$

If DG splittings $C_l^l, \tilde{C}_l^l$ are strict, then one can refine the construction by using the kernel of the map (1.23) instead of its cone. What we end up with, then, is the middle cohomology of a natural sequence

$$0 \rightarrow \text{Cone}(B_\cdot) \xrightarrow{b_l \oplus \tilde{b}} C_l^l \oplus \tilde{C}_l^l \xrightarrow{\text{id} \oplus -\tilde{l}} C_\cdot \rightarrow 0.$$

We denote it by $C_l^l - \tilde{C}_l^l$. It fits naturally into a quasiexact sequence

$$0 \rightarrow B_\cdot \rightarrow (C_l^l - \tilde{C}_l^l) \rightarrow A_\cdot \rightarrow 0.$$

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whose middle homology is naturally identified with the middle homology of (1.11) homologically shifted by 1. In particular, $C_l^q \cong \check{C}_l^q$ is naturally an object in $\mathcal{E}_*(A_*, B_*)$. Moreover, we have a natural map

$$(1.26) \quad C_l^q \cong \check{C}_l^q \to C_l^q - \check{C}_l^q$$

induced by the embedding of the kernel of a surjective map of complexes into the homological shift of its cone.

The functors (1.21) and (1.25) are related by natural maps

$$(1.27) \quad C_l^q \to \check{C}_l^q - (\check{C}_l^q - C_l^q), \quad E_* \to \check{C}_l^q - (C_l^q - E_*),$$

a DG refinement of the isomorphisms (1.20). All of this has a completely parallel version for right DG splittings that we leave to the reader.

## 2 Functors.

### 2.1 Pointed functors. For any small category $\mathcal{C}$ and any abelian category $\mathcal{E}$, the category $\text{Fun}(\mathcal{C}, \mathcal{E})$ of all functors from $\mathcal{C}$ to $\mathcal{E}$ is abelian. If the category $\mathcal{C}$ is additive, then say that a functor $F \in \text{Fun}(\mathcal{C}, \mathcal{E})$ is pointed if $F(0) = 0$, and denote by

$$(2.1) \quad \text{Fun}_0(\mathcal{C}, \mathcal{E}) \subset \text{Fun}(\mathcal{C}, \mathcal{E})$$

the full subcategory spanned by pointed functors. Note that since $0 \in \mathcal{C}$ is both the initial and the terminal object, every $F \in \text{Fun}(\mathcal{C}, \mathcal{E})$ canonically splits as

$$(2.2) \quad F = F' \oplus F(0),$$

where $F'$ is pointed and $F(0)$ is the constant functor. Thus $\text{Fun}_0(\mathcal{C}, \mathcal{E}) \subset \text{Fun}(\mathcal{C}, \mathcal{E})$ is an abelian subcategory closed under extensions. Another remark is that for any two objects $V_1, V_2 \in \mathcal{C}$, both $V_1$ and $V_2$ are distinguished in $V_1 \oplus V_2$ as the images of orthogonal idempotents $e_1, e_2 \in \text{End}(V_1 \oplus V_2)$, $e_1^2 = e_1, e_2^2 = e_2, e_1 e_2 = e_2 e_1 = 0$, and for any pointed functor $F : \mathcal{C} \to \mathcal{E}$, the idempotents $F(e_1)$ and $F(e_2)$ are also orthogonal. Therefore we have a functorial decomposition

$$(2.3) \quad F(V_1 \oplus V_2) = F(V_1) \oplus F(V_2) \oplus F(V_1, V_2),$$

where $F(-, -) : \mathcal{C} \times \mathcal{C} \to \mathcal{E}$ is a certain functor known as cross-effect functor. A pointed functor $F$ is additive if and only if the cross-effects functor $F(-, -)$ is trivial; therefore the full subcategory

$$(2.4) \quad \text{Fun}_{add}(\mathcal{C}, \mathcal{E}) \subset \text{Fun}_0(\mathcal{C}, \mathcal{E})$$

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is again an abelian subcategory closed under extensions.

Assume given an associative unital ring $R$. Denote by $R\text{-mod}$ the abelian category of left $R$-modules, and let $R\text{-proj} \subset R\text{-mod}$ be the full subcategory spanned by finitely generated projective modules. Then the category $R\text{-proj}$ is small and additive, and the category $R\text{-mod}$ is abelian, so we can take $\mathcal{C} = R\text{-proj}$ and $\mathcal{E} = R\text{-mod}$. We denote

$$B(R) = \text{Fun}_{a}(R\text{-proj}, R\text{-mod}).$$

More generally, given another associative unital ring $R'$, we denote

$$B(R, R') = \text{Fun}_{a}(R\text{-proj}, R'\text{-mod}) \subset \text{Fun}(R\text{-proj}, R'\text{-mod}).$$

Any left module $M$ over the product $R' \otimes R^o$ of $R'$ and the algebra $R^o$ opposite to $R$ defines a functor $l(M) \in B(R, R')$ by setting

$$l(M)(V) = M \otimes_R V, \quad V \in R\text{-proj}.$$

The functor $l(M)$ is additive. Conversely, for any additive functor $F : R\text{-proj} \to R'\text{-mod}$, the value $F(R)$ at the free $R$-module $R$ is naturally a module over $R' \otimes R^o$, and we have $F \cong l(F(R))$. Thus $(R' \otimes R^o)\text{-mod} \cong \text{Fun}_{add}(R\text{-proj}, R'\text{-mod})$, and (2.4) induces an exact fully faithful embedding

$$l(-) : (R' \otimes R^o)\text{-mod} \subset B(R, R')$$

whose essential image is closed under extensions. If $R = R'$, what we get is a functor

$$l(-) : R\text{-bimod} \subset B(R),$$

where $R\text{-bimod}$ is the category of $R$-bimodules. This is a fully faithful additive exact functor between abelian categories, and its essential image consists of additive functors $F : R\text{-proj} \to R\text{-mod}$. In addition, this essential image is closed under extensions, so that the natural map

$$\text{Ext}^i(M, N) \to \text{Ext}^i(l(M), l(N)), \quad M, N \in R\text{-bimod}$$

is an isomorphism for $i = 0$ and $i = 1$. For $i \geq 2$, this not necessarily true.

To simplify notation, we denote $l = l(R) \in B(R)$, where $R$ is the diagonal bimodule. The object $l$ corresponds to the natural embedding functor $R\text{-proj} \to R\text{-mod}$.

**Definition 2.1.** For any associative unital ring $R$, and any $R$-bimodule $M$, an elementary extension of $R$ by $M$ is an elementary extension (1.4) of $l$ by $l(M)$ in the abelian category (2.5).
By definition, an elementary extension $\varphi$ of Definition 2.1 has a cohomology class lying in the group

\begin{equation}
H^2_M(R, M) = \text{Ext}^2_{B(R)}(1, 1(M)).
\end{equation}

This group coincides with the so-called second MacLane cohomology group of $R$ with coefficients in $M$. Indeed, by virtue of the canonical decomposition (2.2), the embedding (2.2) induces a fully faithful functor on the derived categories, so that we can compute the right-hand side of (2.9) in the abelian category $\text{Fun}(R\text{-proj}, R\text{-mod})$; the result coincides with MacLane cohomology by the famous result of Ji Bladze and Pirashvili [JP].

2.2 Admissible functors. In order to obtain a version of Definition 2.1 for complexes, we need to introduce an appropriate class of functors between them. For any associative unital ring $R$, we denote by $C_*(R) = C_*(R\text{-mod})$ the category of complexes of left $R$-modules, and let $C^p_*(R) \subset C_*(R)$ be the full subcategory spanned by finite-length complexes of finitely generated projective modules. Inverting quasiisomorphisms in $C_*(R)$ gives the derived category $\mathcal{D}(R)$ of the abelian category $R\text{-mod}$, and $C^p_*(R) \subset C_*(R)$ spans the full subcategory $\mathcal{D}^p(R) \subset \mathcal{D}(R)$ of compact objects in the triangulated category $\mathcal{D}(R)$.

For any rings $R, R'$, the category $C^p_*(R)$ is additive and small, and the category $C_*(R')$ is abelian, so that we can consider pointed functors from $C^p_*(R)$ to $C_*(R')$. Note that we have

$$\text{Fun}_o(C^p_*(R), C_*(R')) \cong C_*(\text{Fun}_o(C^p_*(R), R'\text{-mod})).$$

We are interested in functors that moreover descend to the derived categories, so we need to impose some further conditions. The following seems a reasonable choice.

**Definition 2.2.** A pointed functor $F_* : C^p_*(R) \to C_*(R')$ is admissible if it sends acyclic complexes in $C^p_*(R)$ to acyclic complexes in $C_*(R')$, and exact sequences of complexes in $C^p_*(R)$ to quasiexact sequences of complexes in $C_*(R')$. The category

\begin{equation}
\mathcal{B}_*(R, R') \subset \text{Fun}_o(C^p_*(R), C_*(R')) \cong C_*(\text{Fun}_o(C^p_*(R), R'\text{-mod}))
\end{equation}

is the full subcategory spanned by admissible functors.
Admissibility is not invariant under quasiisomorphisms. However, the subcategory \( \mathcal{B}_q(R, R') \subset \mathcal{C}_q(\text{Fun}_0(C^p(R), R' \text{-mod})) \) is obviously closed under extensions and homological shifts, thus also under taking cones of maps. Therefore the full subcategory

(2.11) \[ \mathcal{D}^{adm}(R, R') \subset \mathcal{D}(\text{Fun}_0(C^p(R), R' \text{-mod})) \]

spanned by admissible functors is triangulated.

**Lemma 2.3.** Any admissible functor \( F : C^p(R) \to \mathcal{C}_q(R') \) sends quasiisomorphisms to quasiisomorphisms, hence descends to a functor

(2.12) \[ F : \mathcal{D}^{p\mathfrak{f}}(R) \to \mathcal{D}(R'), \]

and the functor \( F \) is triangulated.

**Proof.** For any map \( f : E'_i \to E_i \) in \( C^p(R) \), the exact sequence

(2.13) \[ 0 \longrightarrow E'_i \longrightarrow \text{Cone}(f) \longrightarrow E_i[1] \longrightarrow 0 \]

in \( C^p(R) \) maps to the exact sequence \[ \text{(1.1)} \]

so that we have a commutative diagram

(2.14) \[
\begin{array}{cccccc}
0 & \longrightarrow & F_* (E'_i) & \longrightarrow & F_* (\text{Cone}(f)) & \longrightarrow & F_* (E_i[1]) & \longrightarrow & 0 \\
& & F_* (f) \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_* (E_i) & \stackrel{F_* (\beta)}{\longrightarrow} & F_* (\text{Cone}(E_i)) & \stackrel{F_* (\alpha)}{\longrightarrow} & F_* (E_i[1]) & \longrightarrow & 0
\end{array}
\]

in \( \mathcal{C}_q(R') \) with quasiexact rows. If \( f \) is a quasiisomorphism, then the middle terms are acyclic, so that \( F_* (f) \) is indeed a quasiisomorphism. Hence \( F_* \) descends to a functor \[ \text{(2.12)} \]. Moreover, even without \( E'_i \) and the map \( f \), we still have the second row of \[ \text{(2.14)} \], and it induces a diagram

\[ F_* (E_i)[1] \longleftarrow \text{Cone}(F_* (\beta)) \longrightarrow F_* (E_i[1]) \]

in \( \mathcal{C}_q(R') \). The diagram is functorial in \( E_i \), and both maps are quasiisomorphisms. Therefore the functor \( F \) of \[ \text{(2.12)} \] commutes with homological shifts. Furthermore, for any \( E_i, E'_i \in C^p(R) \), the image under \( F_* \) of the split exact sequence

(2.15) \[ 0 \longrightarrow E_i \longrightarrow E_i \oplus E'_i \longrightarrow E'_i \longrightarrow 0 \]

must be quasiexact, so that the cross-effects complex \( F_* (E_i, E'_i) \) is acyclic, and \( F \) is additive. Finally, since every distinguished triangle in \( \mathcal{D}^{p\mathfrak{f}}(R) \) can
be represented by an exact sequence (2.13), the diagram (2.14) shows that $F$ sends distinguished triangles to distinguished triangles.

By virtue of Lemma 2.3, the category $D^{adm}(R, R')$ of (2.11) acts naturally by triangulated functors from $D^{\mathcal{P}}(R)$ to $D(R')$. In this sense, it can serve as a reasonable axiomatization of the “triangulated category of enhanced triangulated functors from $D^{\mathcal{P}}(R)$ to $D(R')$”.

### 2.3 Expansion and restriction

One source of examples of admissible functors is the following. For any complex $F_\ast$ of pointed functors from $R$-proj to $R'$-mod, define its expansion

$$
\varepsilon_\ast(F_\ast) : C_*^{\mathcal{P}}(R) \rightarrow C_*^{\mathcal{P}}(R')
$$

by applying $F_\ast$ termwise and taking the sum-total complex of the resulting bicomplex — that is, we set

$$
(2.16) \quad \varepsilon_\ast(F_\ast)(E_\ast)_n = \bigoplus_{i+j=n} F_i(E_j), \quad E_\ast \in C_\ast^{\mathcal{P}}(R).
$$

The differential in the bicomplex squares to 0 precisely because $F_i$ are pointed functors.

**Definition 2.4.** A complex $F_\ast$ in $\mathcal{B}(R,R')$ is quasiadditive if for any $V,V' \in R$-proj, the complex $F_\ast(V,V')$ of cross-effects functors is acyclic.

**Lemma 2.5.** Assume given a quasiadditive complex $F_\ast$ in $\mathcal{B}(R,R')$. Then its expansion $F'_\ast = \varepsilon_\ast(F_\ast)$ is admissible in the sense of Definition 2.2.

**Proof.** Since by definition, complexes in $C_*^{\mathcal{P}}(R)$ are made out of projective $R$-modules, every exact sequence in $C_*^{\mathcal{P}}(R)$ is termwise-split. Then to show that $F'_\ast$ sends it to a quasiexact sequence, it suffices to apply the standard spectral sequence of the bicomplex defining $\varepsilon_\ast(F_\ast)$.

It remains to show that $F'_\ast$ sends acyclic complexes to acyclic complexes. But every acyclic complex in $C_*^{\mathcal{P}}(R)$ is a sum of complexes of the form $\mathrm{Cone}(E)[i]$ for some integer $i$ and projective $R$-module $E \in R$-proj. Since $F'_i$ obviously commutes with shifts, it suffices to prove that $F'_i(\mathrm{Cone}(E))$ is acyclic. This is also obvious, since by definition, we have $F'_i(\mathrm{Cone}(E)) \cong \mathrm{Cone}(F'_i(E))$.

Since (2.16) is functorial with respect to $F_\ast$, Lemma 2.5 provides a natural functor

$$
(2.17) \quad \varepsilon_\ast : C_\ast^{\mathcal{P}}(\mathcal{B}(R,R')) \rightarrow \mathcal{B}_\ast(R,R'),
$$
where \( C^{\text{add}}(\mathcal{B}(R,R')) \subset C_*(\mathcal{B}(R,R')) \) denotes the full subcategory spanned by quasiadditive complexes. Quasiadditivity is by definition invariant under quasiisomorphisms, so that inverting quasiisomorphisms in \( C^{\text{add}}(\mathcal{B}(R,R')) \), we obtain a full triangulated subcategory

\[
D^{\text{add}}(R,R') \subset D(\mathcal{B}(R,R'))
\]

in the derived category \( D(\mathcal{B}(R,R')) \). Expansion then descends to a functor

\[
\varepsilon : D^{\text{add}}(R,R') \to D^{\text{adm}}(R,R'),
\]

where \( D^{\text{adm}}(R,R') \) is the triangulated category \((2.11)\). It seems likely that the functor \( \varepsilon \) is actually an equivalence — in particular, up to a quasiisomorphism, every admissible functor from \( C^{\text{pf}}q(R) \) to \( Cq(R') \) comes from a quasiadditive complex of pointed functors from \( R\text{-proj} \) to \( R'\text{-mod} \). However, we have not pursued this (in any case, the construction of such a complex is unlikely to be either explicit or particularly useful).

On the chain level, the expansion functor \((2.17)\) is definitely not essentially surjective. In particular, for any pointed functor \( F_q : C^{\text{pf}}q(R) \to Cq(R') \) and any integer \( i \), define the homological twist \( \tau^i(F_q) \) by

\[
\tau^i(F_q)(V_q) = F_q(V_q[i])[-i].
\]

Then if \( F_q \) is admissible, \( \tau^i(F_q) \) is also admissible and induces the same functor from \( D^{\text{pf}}(R) \) to \( D(R') \). However, \( F_q \) and \( \tau^i(F_q) \) are in general different as functors from \( C^{\text{pf}}q(R) \) to \( Cq(R') \). But if \( F_q = \varepsilon_q(F'_q) \) for some quasiadditive complex \( F'_q \), they are the same — as mentioned in the proof of Lemma \((2.5)\), we have a natural isomorphism

\[
\tau^i(\varepsilon_q(F'_q)) \cong \varepsilon_q(F'_q)
\]

for any complex \( F'_q \in C^{\text{add}}_*(\mathcal{B}(R,R')) \).

We can also go in the other direction: composing a pointed functor \( F_* : C^{\text{pf}}_*(R) \to C_*(R') \) with the embedding \((1.2)\) gives an object

\[
\rho_*(F_*) \in \text{Fun}_o(R\text{-proj}, C_*(R')) \cong C_*(\mathcal{B}(R,R')),
\]

and it is easy to see from \((2.15)\) that if \( F_* \) is admissible, \( \rho_*(F_*) \) is quasiadditive. We thus obtain a restriction functor

\[
r_* : \mathcal{B}_*(R,R') \to C^{\text{adm}}_*(\mathcal{B}(R,R')).
\]

This is a one-sided inverse to the expansion functor \((2.17)\) — we have a natural identification \( \varepsilon_* \circ \rho_* \cong \text{id} \). We cannot say anything about the composition in the other direct. In particular, for an arbitrary \( F_* \in \mathcal{B}_*(R,R') \), we do not even have a natural map between \( F_* \) and \( \varepsilon_*(\rho_*(F_*)) \).
2.4 DG elementary extensions. We can now give our DG version of
Definition 2.1. For any associative unital ring \( R \) and \( R \)-module \( M \), the
functor \( \mathcal{I}(M) \in \mathcal{B}(R) \) is additive, hence quasiadditive in the sense of Definition 2.4. We simplify notation by letting \( \mathcal{B}_{*}(R) = \mathcal{B}_{*}(R, R) \), and we denote by
\[
\mathcal{I}_{*}(M) = \varepsilon_{*}(\mathcal{I}(M)) \in \mathcal{B}_{*}(R)
\]
the expansion (2.17) of the functor \( \mathcal{I}(M) \). If \( M \) is the diagonal bimodule \( R \),
we further simplify notation by writing \( \mathcal{I}_{*} = \mathcal{I}_{*}(R) = \varepsilon_{*}(\mathcal{I}) \).

Definition 2.6. For any associative unital ring \( R \) and any \( R \)-bimodule \( M \),
an elementary DG extension of \( R \) by \( M \) is a quasiexact sequence
\[
0 \longrightarrow \mathcal{I}_{*}(M)[1] \longrightarrow C_{*} \longrightarrow \mathcal{I}_{*} \longrightarrow 0
\]
in the category \( \mathcal{B}_{*}(R) \subset \mathcal{C}_{*}(\text{Fun}_{o}(\mathcal{C}_{pf}^{qf}(R), R\text{-mod})) \) such that for any integer \( m \), the functor \( \mathcal{C}_{*} : \mathcal{C}_{pf}^{qf}(R) \rightarrow \mathcal{C}_{*}(R) \) sends the subcategory \( \mathcal{C}_{\leq m}^{qf}(R) \subset \mathcal{C}_{pf}^{qf}(R) \) into the subcategory \( \mathcal{C}_{\leq m+1}(R) \subset \mathcal{C}_{*}(R) \).

In other words, an elementary DG extension is an elementary extension (1.11) of \( \mathcal{I}_{*} \) by \( \mathcal{I}(M) \) with admissible \( C \), satisfying the additional assumption:
we require that
\[
\mathcal{C}_{*}(\mathcal{C}_{\leq m}^{qf}(R)) \subset \mathcal{C}_{\leq m+1}(R)
\]
for any integer \( m \). Note that since (2.21) is required to be quasiexact, and
\( \mathcal{I}_{*}, \mathcal{I}(M) \) are admissible, one does not have to check all the conditions of Definition 2.2 to see that the functor \( \mathcal{C}_{*} : \mathcal{C}_{pf}^{qf}(R) \rightarrow \mathcal{C}_{*}(R) \) is admissible. In fact, it suffices to check that it sends termwise-split injections to termwise-split injections, and termwise-split surjections to termwise-split surjections; the rest is automatic.

In particular, if \( \mathcal{C}_{*} \) in (2.21) is an expansion of some complex in the
category \( \text{Fun}_{o}(\mathcal{C}_{pf}^{qf}(R), R\text{-mod}) \), it is automatically admissible. Therefore
any elementary extension of \( R \) by \( M \) in the sense of Definition 2.1 generates
an elementary DG extension given by
\[
0 \longrightarrow \mathcal{I}_{*}(M) = \varepsilon_{*}(\mathcal{I}(M)) \longrightarrow \varepsilon_{*}(\mathcal{C}_{*}) \longrightarrow \mathcal{I}_{*} = \varepsilon_{*}(I) \longrightarrow 0,
\]
where \( \varepsilon_{*} \) is the expansion functor (2.17). Conversely, every elementary DG
extension (2.21) gives an elementary extension
\[
0 \longrightarrow \mathcal{I}(M) \longrightarrow \tau_{\geq 0} \rho_{*}(\mathcal{C}_{*}) \longrightarrow 1 \longrightarrow 0,
\]
where $\rho$ is the restriction functor (2.20), and $\tau_{\geq 0}$ is the canonical truncation at 0 (the complex $\rho_{\ast}(C_{\ast})$ automatically lies in $C_{\leq 1}(B(R))$ by (2.22), and the truncation is needed to insure that it also lies in $C_{\geq 0}(B(R))$. Both elementary extensions and elementary DG extensions form categories in the obvious way, denoted $\mathcal{E}l(R, M)$ resp. $\mathcal{E}l_{\ast}(R, M)$, and the correspondence between them is sufficiently functorial to give expansion and restriction functors

(2.23) $\varepsilon : \mathcal{E}l(R, M) \rightarrow \mathcal{E}l_{\ast}(R, M), \quad \rho : \mathcal{E}l_{\ast}(R, M) \rightarrow \mathcal{E}l(R, M)$.

As for the functors (2.17) and (2.20), we have $\rho \circ \varepsilon \cong \text{id}$. Slightly more generally, for any integer $i$, the homological twist $\tau^{i}(C_{\ast})$ of an elementary DG extension $C_{\ast}$ is also an elementary DG extension, and we can define a twisted restriction functor $\rho_{i} : \mathcal{E}l_{\ast}(R, M) \rightarrow \mathcal{E}l(R, M)$ by

(2.24) $\rho_{i}(C_{\ast}) = \rho(\tau^{i}(C_{\ast}))$.

Then by (2.19), we still have $\rho_{i} \circ \varepsilon \cong \text{id}$. We denote the sets of connected components of the categories $\mathcal{E}l(R, M)$, resp. $\mathcal{E}l_{\ast}(R, M)$ by $\text{El}(R, M)$, resp. $\text{El}_{\ast}(R, M)$.

3 Square-zero extensions.

3.1 The algebra case. Assume given an associative unital ring $R$ and an $R$-bimodule $M$. As usual, by a square-zero extension $R'$ of $R$ by $M$ we will understand an associative unital ring $R'$ equipped with a short exact sequence

(3.1) $0 \rightarrow M \xrightarrow{i} R' \xrightarrow{q} R \xrightarrow{} 0$

of abelian groups such that $q$ is a unital ring map, and we have

$$i(m)r' = i(mp(r')), \quad r'i(m) = i(p(r')m), \quad i(m)i(m') = 0$$

for any $m, m' \in M$ and $r' \in R'$. Square-zero extensions of $R$ by $M$ form a category $\text{Sq}(R, M)$ in the obvious way. This category is obviously a groupoid, and we denote by $\text{Sq}(R, M)$ the set of isomorphism classes of its objects.

If $R$ and $R'$ are algebras over a commutative ring $k$, and $R$ is flat over $k$, then it is well-known that square-zero extensions are classified by Hochschild cohomology classes: we have a natural identification

(3.2) $\text{Sq}(R, M) \cong \text{HH}_{k}^{2}(R, M) = \text{Ext}_{R \otimes_{k} R'}^{2}(R, M)$,
where the Ext-group on the right is computed in the category of $k$-linear $R$-bimodules.

Let us recall what is probably the most direct way to construct the correspondence (3.2) (it seems to be a well-known folklore result). We begin with the following general fact. For any square-zero extension (3.1), restriction with respect to $p$ gives a natural exact functor $q_* : R\text{-mod} \to R'\text{-mod}$. It has a left-adjoint functor $q^* : R'\text{-mod} \to R\text{-mod}$. By adjunction, $q^*$ is right-exact, so it has derived functors $L^i q^* : R'\text{-mod} \to R\text{-mod}$.

**Lemma 3.1.** Assume given a square-zero extension (3.1) of an associative unital ring $R$ by an $R$-bimodule $M$. Then for any projective left $R$-module $V$, we have natural identifications

$$q^* q_* V \cong V, \quad L^1 q^* q_* V \cong M \otimes_R V,$$

and both are functorial in $V$.

**Proof.** The first isomorphism is clear: for any $V \in R\text{-mod}$, we have

$$q^* q_* V \cong R \otimes_{R'} V,$$

and the right-hand side is by definition identified with $V$. For the second isomorphism, consider the sequence (3.1) as a short exact sequence of $R'$-bimodules. Then for any $V' \in R'\text{-mod}$, we have a long exact sequence

$$
\begin{align*}
0 = \text{Tor}_1^{R'}(R', V') & \longrightarrow \text{Tor}_1^{R'}(R, V') \quad \delta \longrightarrow M \otimes_{R'} V' \quad \longrightarrow a \longrightarrow \\
\quad \longrightarrow a \longrightarrow R' \otimes_{R'} V' = V' \quad b \longrightarrow R \otimes_{R'} V' \longrightarrow 0
\end{align*}
$$

functorial in $V'$. If $V' = q_* V$ for some $V \in R\text{-mod}$, then the map $b$ is an isomorphism. Therefore the map $a$ vanishes, and the map $\delta$ is an isomorphism. Since $L^1 q^*(-) \cong \text{Tor}_1^{R'}(R, -)$ and $M \otimes_{R'} V \cong M \otimes_R V$, this proves the claim. \hfill \square

Assume now that the ring $R$ is a flat algebra over a commutative ring $k$, and that a square-zero extension (3.1) is $k$-linear. Then in particular, $R$ is naturally a left module over $R' \otimes_k R'$. Moreover, for any projective $R' \otimes_k R'$-module $P$ and any $R$-module $V$, the tensor product $P \otimes_R V$ is a projective $R'$-module. Thus if we take a projective resolution $P'_r$ of the $R' \otimes_k R'$-module $R$, then for any projective $R$-module $V$, the complex $P'_r(V) = P_r \otimes_R V$ is a projective resolution of the $R'$-module $q_* V$. We then obtain a functor

$$P'_r(-) : R\text{-proj} \to C_1(R'\text{-mod})$$

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that sends any \( V \in R\text{-proj} \) to a projective resolution of \( q_* V \). Composing it with \( q^* \), we obtain a functor

\[
P_\cdot(-): R\text{-proj} \to C_\cdot(R\text{-mod}), \quad P_\cdot(V) = q^* P'_\cdot(V) = R \otimes_{R'} P'_\cdot \otimes_R V
\]

represented by a complex

\[
P_\cdot = R \otimes_{R'} P'_\cdot
\]

of modules of \( R \otimes_k R^o \). Moreover, by definition, for any \( V \in R\text{-proj} \), the \( i \)-th homology of the complex \( P_\cdot(V) \) is identified with \( L^i q^* q_* V \). Thus if we let

\[
\overline{P}_\cdot = \tau_{[0,1]} P_\cdot
\]

be the canonical truncation of the complex \( P_\cdot \), then Lemma 3.1 shows that \( \overline{P}_\cdot \) fits into an exact sequence

\[
0 \longrightarrow M \longrightarrow \overline{P}_1 \longrightarrow \overline{P}_0 \longrightarrow R \longrightarrow 0
\]

of modules over \( R \otimes_k R^o \). This exact sequence represents by Yoneda a class in \( HH_k^2(R, M) \). This is what corresponds to the square-zero extension \( R' \) under the identification (3.2).

### 3.2 The absolute case

Unfortunately, the construction presented above does not work in the absolute case, that is, in the absence of a good base ring \( k \). What breaks down is (3.3). In general, there is no way to lift the functor \( q_* \) to an additive functor from \( R\text{-proj} \) to \( C_\cdot(R'\text{-mod}) \) that sends a finitely generated projective \( R\text{-module} \) \( V \) to a projective resolution of \( q_* V \).

**Example 3.2.** Assume that \( R = k \) is a finite field of some characteristic \( p \), and let \( R' = W_2(k) \) be its ring of second Witt vectors. Then \( R\text{-proj} \) is the category of finite-dimensional \( k\)-vector spaces, so that its objects are free modules \( k^n \), \( n \geq 1 \). For any such module \( V = k^n \), we of course have many projective resolutions over \( W_2(k) \) — for example, we can take the minimal resolution

\[
\cdots \longrightarrow p \longrightarrow W_2(k)^n \longrightarrow p \longrightarrow W_2(k)^n \longrightarrow p \longrightarrow W_2(k)^n,
\]

with all the differentials given by multiplication by \( p \). However, in order to make this functorial in \( V \), one would have, at the very least, to find a group map \( GL_n(k) \to GL_n(W_2(k)) \) splitting the reduction map \( GL_n(W_2(k)) \to GL_n(k) \). This is not possible already for \( n = 2 \) and \( k = \mathbb{Z}/p\mathbb{Z} \). More generally, it is easy to see that in fact *any* additive functor from \( k\text{-proj} \) to
$W_2(k)$-proj, hence also to $C_1(W_2(k))$ is trivial (such a functor must correspond to a module $M$ over $W_2(k) \otimes k$ that is projective over $W_2(k)$, and since $W_2(k) \otimes k \cong k$, this implies $M = 0$).

One way to correct the situation is to use non-additive functors and MacLane cohomology instead of Hochschild cohomology. Namely, while the minimal resolution of Example 3.2 is hopeless, it is nevertheless perfectly possible to lift the functor $\pi_*$ to a functor (3.3) as long as we allow non-additive functors. This is based on the following observation.

**Lemma 3.3.** For any object $c \in C$ of a small category $C$, and any projective object $P \in \text{Fun}(C, E)$ in the abelian category of functors from $C$ to an abelian category $E$ satisfying AB4*, the value $P(c) \in E$ of the functor $P$ at the object $c$ is a projective object in $E$.

**Proof.** Define a functor $i_{cs} : E \to \text{Fun}(C, E)$ by setting

$$i_{cs}(E)(c') = \prod_{f : c \to c'} E, \quad E \in E, c' \in C,$$

where the right-hand side stand for the product of copies of $E$ numbered by maps from $c$ to $c'$. Then $i_{cs}$ is obviously right-adjoint to the evaluation functor $E \mapsto E(c)$, and since $E$ satisfies AB4*, $i_{cs}$ is exact. \qed

By virtue of Lemma 3.3, we can now do the following. Assume that $R'$ is a square-zero extension (3.1) of a ring $R$ by a bimodule $M$, and consider the category $\mathcal{B}(R, R')$ of (2.6). This is an abelian category that satisfies AB4 and AB4* and has enough projectives. Composing a functor $F \in \mathcal{B}(R, R')$ with $q^*$ gives a pointed functor $q^* F : R$-proj $\to R$-mod, so that we have a natural functor

$$q^* : \mathcal{B}(R, R') \to \mathcal{B}(R),$$

and analogously, composition with $q_*$ gives a natural functor

$$q_* : \mathcal{B}(R) \to \mathcal{B}(R, R').$$

With this notation, the object $q_*(1) \in \mathcal{B}(R, R')$ corresponds to the functor $q_*$ itself. Choose a projective resolution $P'_* \in \mathcal{B},$ of $q_*(1)$, consider the complex

$$P_* = q^* P'_*$$

in the category $\mathcal{B}(R)$, and let $\overline{P}_* = \tau_{[0,1]} P_*$. Then by Lemma 3.3, for any $V \in R$-proj, $P'_*(V)$ is a projective resolution of $q_*(V)$ in the category $R'$-mod, so that as before, Lemma 3.1 shows that we have an elementary extension

$$0 \longrightarrow \text{id}(M)[1] \longrightarrow \overline{P}_* \longrightarrow 1 \longrightarrow 0$$
of $R$ by $M$ in the sense of Definition 2.1. This extension depends both on
$R'$ and on the choice of a resolution $P'$, so it is not quite functorial in $R'$
and does not define a functor $Sq(R, M) \to \mathcal{E}(R, M)$. However, since any
two projective resolutions of the same objects are quasiisomorphic, we do
have a well-defined map

$$
(3.4) \quad Sq(R, M) \to \mathcal{E}(R, M) \cong HH^2_M(R, M)
$$
on the sets of connected components. This associates a canonical MacLane
cohomology class of degree 2 to any square-zero extension of $R$ by $M$.

### 3.3 Canonical splitting.

We now want to prove that the map (3.4) is
in fact an isomorphism — that is, square-zero extensions (3.1) correspond
bijectively to their MacLane cohomology classes. To do this, we first need
to consider functors from $R$-proj to abelian groups.

Let $e : \mathbb{Z} \to R$ be the tautological map sending 1 to 1, so that the
restriction functor $e^*$ is just the forgetful functor from $R$-modules to abelian
groups. Then $e_*$ is pointed, thus gives an object in $\mathcal{B}(R, \mathbb{Z})$. To keep notation
consistent, we extend $e^*$ to a functor

$$
(3.5) \quad e_* : \mathcal{B}(R) \to \mathcal{B}(R, \mathbb{Z})
$$
by applying it pointwise; then the object in $\mathcal{B}(R, \mathbb{Z})$ corresponding to $e_*$ is
$e_*(I)$, where $I \in \mathcal{B}(R)$ corresponds to the taugological functor. Moreover,
$e_*$ is additive, so that $e_*(I)$ corresponds to an $R^o$-module — namely, to $R$
considered as a right module over itself. Since $R$ is projective as a module
over $R^o$, we have

$$
(3.6) \quad \Ext^1_{\mathcal{B}(R, \mathbb{Z})}(e_*(I), l(W)) \cong \Ext^1_{R^o}(R, W) = 0
$$
for any $R^o$-module $W$ and the corresponding additive functor $l(W)$ in the
category $\mathcal{B}(R, \mathbb{Z})$. Somewhat surprisingly, an analogous statement also holds
for $\Ext^2(e_*(I), -)$. In fact, we even have a stronger statement.

**Proposition 3.4.** Assume given a unital associative ring $R$, an $R^o$-module
$W$, and an elementary extension $\varphi \in \mathcal{E}(l_*(I), l(W))$ represented by a quasi-
exact sequence

$$
(3.7) \quad 0 \longrightarrow l(W)[1] \longrightarrow C_* \longrightarrow b \longrightarrow e_*(I) \longrightarrow 0
$$
in $C_*(\mathcal{B}(R, \mathbb{Z}))$. Then $\varphi$ admits a splitting $C_{01}$, and this splitting is unique
up to an isomorphism.
Proof. Uniqueness immediately follows from (3.6). Indeed, for any two splittings $C_{01}, C'_{01}$ of the extension $\varphi$, we have

$$C'_{01} \cong C_{01} - T$$

for some object $T \in \mathcal{E}(e_*(I), I(W))$, where in the right-hand side we take the difference functor (1.19). But by (3.6), we have $T \cong 0$ and $C'_{01} \cong C_{01}$.

To prove existence, we need to define a splitting $C_{01}(V)$ of the elementary extension $\varphi(V)$ of abelian groups for any $V \in R$-proj, and we need to do it in a way that is functorial in $V$. Evaluating (3.7) at the free module $R \in R$-proj, we obtain in particular a surjective map $b : C_0(R) \to e_*(R) = R$. Choose an element $s \in C_0(R)$ such that $b(s) = 1$. Since $R \in R$-proj represents the forgetful functor $e_*$, by Yoneda, the element $s \in C_0(R)$ defines a map

$$(3.8) \quad \tilde{s} : V \to C_0(V), \quad V \in R\text{-proj}$$

of sets functorial in $V$, and since $b(s) = 1$, we have $b \circ \tilde{s} \cong \text{id}$. Let

$$(3.9) \quad C_{01}(V) = C_1(V) \times V$$

as a set, and define $c_1 : C_1(V) \to C_{01}(V)$, $c_0 : C_{01}(V) \to C_0(V)$ by

$$(3.10) \quad c_1(c) = c \times 0, \quad c_0(c \times v) = \delta(c) + \tilde{s}(v), \quad c \in C_1(V), v \in V,$$

where $\delta : C_1(V) \to C_0(V)$ is the differential. To turn $C_{01}(V)$ into the required splitting, we need to endow it with a structure of an abelian group in such a way that $c_0$ and $c_1$ become group maps. It suffices to construct a functorial cocycle map

$$(3.11) \quad c(-, -) : V \times V \to C_1(V)$$

such that

$$(3.12) \quad c(v_1, 0) = c(0, v_1) = 0,$$

$$(3.13) \quad c(v_1, v_2) = c(v_2, v_1),$$

$$(3.14) \quad c(v_1, v_2) + v_3 + c(v_1 + v_2, v_3) = c(v_1, v_2 + v_3) + v_1 + c(v_2, v_3),$$

$$(3.15) \quad \delta(c(v_1, v_2)) = \tilde{s}(v_1) + \tilde{s}(v_2) - \tilde{s}(v_1 + v_2)$$

for any $v_1, v_2, v_2 \in V$. Since the functor sending $V$ to the set $V \times V$ is represented by $R^2$, it suffice to construct the element $c(v_1, v_2) \in C_1(V)$ in the universal case $V = R^2$, $v_1$ and $v_2$ the generators of the two copies of
Then (3.12) is equivalent to saying that $c(v_1, v_2)$ lies in the cross-effect component $C_1(R, R) \subset C_1(R^2)$ of the decomposition (2.3). But since both $e_*(I)$ and $I(W)$ are additive functors, their cross-effect functors are trivial, and since (3.7) is quasiexact, its cross-effect component

$$C_1(R, R) \xrightarrow{\delta} C_0(R, R)$$

is an acyclic complex. Therefore (3.15) uniquely determines $c(v_1, v_2)$. Then (3.13) immediately follows from uniqueness. To check (3.14), it suffices to consider the case $V = R^3$ spanned by $v_1, v_2, v_3$, and again, the claim immediately follows from (3.15) and the acyclicity of cross-effects.

3.4 Regular endomorphisms. We note that the splitting of Proposition 3.4 is only canonical in a weak sense: isomorphisms between different splittings are not canonical. This is in fact an inherent feature of the construction, and it is this feature that allows us to recover a non-trivial square-zero extension from an elementary extension.

Namely, assume given an associative unital ring $R$, an $R$-bimodule $M$, and an elementary extension $\varphi \in E(R, M)$ represented by a quasiexact sequence

$$0 \longrightarrow I(M)[1] \longrightarrow C_* \longrightarrow I \longrightarrow 0$$

in the category $C_{[0,1]}(B(R))$. Then applying the functor (3.5) to (3.16) gives an elementary extension

$$0 \longrightarrow e_*(I(M)) \longrightarrow e_*(C_*) \longrightarrow e_*(I) \longrightarrow 0$$

in the category $B(R, Z)$. Denote by $C_{01}$ the canonical splitting of this extension provided by Proposition 3.4 and consider the algebra $\text{End}(C_{01})$ of additive endomorphisms of the functor $C_{01} \in B(R, Z)$. Say that an endomorphism $r' \in \text{End}(C_{01})$ is regular if it fits into a commutative diagram

$$
\begin{array}{ccc}
e_*(C_1) & \xrightarrow{c_1} & C_{01} & \xrightarrow{c_0} & e_*(C_0) \\
\downarrow r & & \downarrow r' & & \downarrow r \\
e_*(C_1) & \xrightarrow{c_1} & C_{01} & \xrightarrow{c_0} & e_*(C_0)
\end{array}
$$

for some element $r \in R$. The set of all regular endomorphisms obviously forms a unital subalgebra in $\text{End}(C_{01})$ that we denote by $R'$. 

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Lemma 3.5. In the assumptions above, the algebra $R'$ is a square-zero extension of $R$ by $M$.

Proof. Note that since $c_0$ in (3.17) is surjective, $r$ is unique. Sending an element $r' \in R'$ to this unique $r$ defines an algebra map

$$(3.18) \quad q : R' \to R.$$ 

The kernel of the map $q$ is the space of map $r'$ that vanish on $\text{Im} \, a$, thus factor through the quotient $e_*(I) = C_{01}/c_1(e_*(C_1))$, and take values in $e_*(I(M)) = \text{Ker} \, c_0 \subset C_{01}$. Therefore

$$\text{Ker} \, p \cong \text{Hom}(e_*(I), e_*(I(M))),$$

and by Yoneda, this is identified with $e_*(I(M))(R) = M$. To finish the proof, it remains to check that the map (3.18) is surjective — that is, every element $r \in R$ lifts to some regular endomorphism $r' \in \text{End}(C_{01})$. Indeed, fix an element $r$. Since the elementary extension $\varphi = e_*(C_1)$ comes from an elementary extension in $\mathcal{B}(R)$, we have

$$r \circ \varphi \cong \varphi \circ r,$$

where the compositions are given by the composition functors (1.7), (1.9). Then a lifting $r'$ exists if and only if we have

$$(3.19) \quad r \circ C_{01} \cong C_{01} \circ r \in S\text{pl}(r \circ \varphi) \cong S\text{pl}(\varphi \circ r),$$

where the compositions on the left-hand side are given by the composition functors (1.8), (1.10). But the extension $r \circ \varphi$ is also of the form (3.7), so that the existence of an isomorphism (3.19) immediately follows from the uniqueness statement of Proposition 3.4. \qed

3.5 The inversion theorem. By virtue of Lemma 3.5 and Proposition 3.4 for any associative unital ring $R$ and $R$-bimodule $M$, we have a well-defined map

$$(3.20) \quad \text{El}(R, M) \to \text{Sq}(R, M)$$

that sends an elementary extension $\alpha$ to the corresponding square-zero extension $R'$.

Theorem 3.6. The maps (3.4) and (3.20) are inverse to each other.
For the proof of Theorem 3.6 it is convenient to introduce an additional piece of notation. For any square-zero extension $R' \in \text{Sq}(R, M)$, and any $R'$-module $V'$, the embedding $M \subset R'$ induces a natural multiplication map
\begin{equation}
M \otimes_{R'} V' \to R' \otimes_{R'} V' \cong V',
\end{equation}
and by (3.1), $q^*V' = R \otimes_{R'} V'$ is the cokernel of this map. Moreover, by adjunction, we have an identification $M \otimes_{R'} V' \cong M \otimes_R q^*V'$. We will denote by $m_{V'}$ the canonical map
\begin{equation}
m_{V'} : M \otimes_{R} q^*V' \to V'
\end{equation}
induced by (3.21) via this identification. The module $V'$ is of the form $q_*V$ for some $V \in \text{R}-\text{mod}$ if and only if $m_{V'} = 0$.

**Proof of Theorem 3.6.** Assume first given a square-zero extension (3.1), and consider the corresponding elementary extension $C_q = \tau_{[0,1]}q^*P_q'$ in $B(R)$, where $P_q'$ is a projective resolution of the object $q_*(l) \in \mathcal{B}(R, R')$. Denote $C_q' = \tau_{[0,1]}P_q'$. Then by adjunction, we have a natural map $C_q' \to q_*C_q$.

in the category $B(R, R')$, and its composition with the projection $C_q \to q_*l$ is a quasiisomorphism. Then taking $C^l_q = C_q'$ in (1.16) provides a splitting $C_01$ of the elementary extension $q_*C_q$. Denote by $e' : \mathbb{Z} \to R'$ the tautological map, and consider the induced splitting $e'_*(C_01)$ of the elementary extension $e'_*(q_*(C_q)) \cong e_*(C_q)$. By uniqueness, it must be isomorphic to the splitting provided by Proposition 3.4. But by construction, the ring $R'$ acts on $e'_*(C_01)$ by regular endomorphisms. Therefore it maps to the square-zero extension provided by Lemma 3.5. Since all maps between square-zero extensions are isomorphisms, this proves that (3.20) sends $C_q$ to $R'$.

Conversely, assume given an elementary extension $(C_q, a, b)$ with the corresponding splitting $\langle C_01, c_0, c_1 \rangle$ of Proposition 3.4 and let $R' \subset \text{End}(C_01)$ be the square-zero extension provided by Lemma 3.5. Moreover, consider the corresponding left DG splitting $C^l_q$ of (1.15). Then by definition, $R'$ acts on $C_01$ and $C^l_q$, so that $C_01$ actually defines a splitting of the extension $q_*(C_q)$ in the category $B(R, R')$, and $C^l_q$ is a complex in $B(R, R')$ quasiisomorphic to $q_*(l)$. By the usual property of projective resolutions, for any projective resolution $P_q'$ of $q_*(l)$, we then have a quasiisomorphism $P_q' \to C^l_q$, and it induces a map $P_q' = \tau_{[0,1]}q^*P_q' \to \tau_{[0,1]}q^*C^l_q = q^*C^l_q$. 

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To finish the proof, it remains to show that $q^*C^l_i \cong C_i$ — indeed, then $P_i$ and $C_i$ lie in the same connected component of the category $\mathcal{E}l(R, M)$, thus correspond to the same element in $\text{El}(R, M)$. In homological degree 1, we have $C^l_1 = q_* C_1$, so that $q^* C^l_1 = q^* q_* C_1 \cong C_1$. In degree 0, $q^* C^l_0 = q^* C_0$ is the cokernel of the map $m_{C_0}$ of (3.22). However, by construction, this maps factors as

\begin{equation}
M \otimes_R q^* C_{01} \xrightarrow{\text{id} \otimes (a \circ q^* c_0)} M \otimes_R I \cong I(M) \xrightarrow{c_1 \circ b} C_{01},
\end{equation}

where the map $\text{id} \otimes (a \circ q^* c_0)$ is surjective, and the map $c_1 \circ b$ is injective. Therefore indeed $q^* C^l_0 \cong \text{Coker } m_{C_0} \cong \text{Coker } (c_1 \circ b) \cong C_0$. □

4 Splittings and liftings.

4.1 Modules. Assume given a unital associative ring $R$, an $R$-bimodule $M$, and an elementary extension $C_*$ of $R$ by $M$, and let $R' \in \text{Sq}(R, M)$ be the corresponding square-zero extension. As we have mentioned in the proof of Theorem 3.6, the canonical splitting $C_{01}$ of the elementary extension $e_*(C_*)$ is naturally a left module over $R'$, so that it actually defines a splitting of the elementary extension $q_*(C_*)$ in the category $\mathcal{B}(R, R')$. We denote this splitting by the same letter $C_{01}$ since it is effectively the same object. We now want to explain how $C_{01}$ helps to describe modules over $R'$ in terms of modules over $R$.

We start with the following observation. By definition, for any two associative unital rings $R$, $R'$, a functor $F \in \mathcal{B}(R, R')$ is defined on the category $R$-$\text{proj}$ of finitely generated projective left $R$-modules. However, we can extend its domain of definition in the standard way. Namely, for any $R$-module $V$, the category $I(V)$ of finitely presented $R$-modules $V_i$ equipped with a map $i : V_i \to V$ is small and filtering, and we have

\begin{equation}
V = \lim_{\text{lim}} V_i.
\end{equation}

If $V$ is flat, then the full subcategory $J(V) \subset I(V)$ spanned by $\langle V_i, i \rangle$ with $V_i \in R$-$\text{proj}$ is cofinal (see e.g. [PG] Lemma 3.5] that assumes commutativity of $R$ but does not use it). Therefore $J(V)$ is also filtering, and we can replace the colimit in (4.1.1) with the colimit of the same functor over $J(V)$. Therefore for any $F \in \mathcal{B}(R, R')$ and flat $R$-module $V$, we can set

\begin{equation}
F(V) = \lim_{\text{lim}} F(V_i) \in R'$-$\text{mod},
\end{equation}

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and if we denote by $R$-flat the category of flat $R$-modules, then this is a well-defined functor from $R$-flat to $R'$-mod. Moreover, the extension operation is functorial in $F$, and since the categories of modules satisfy Grothendieck’s axiom AB5, the operation is exact.

In particular, in our setting, the elementary extension $C_\ast$ defines an elementary extension

\begin{equation}
0 \longrightarrow M \otimes_R V[1] \longrightarrow C_\ast(V) \longrightarrow V \longrightarrow 0
\end{equation}

of $R$-modules for any $V \in R$-flat, and the splitting $C_{01}$ provides a splitting $C_{01}(V)$ of the corresponding elementary extension $q_\ast(C_\ast(V))$ in $R'$-mod. Both are functorial in $V$.

**Definition 4.1.** A lifting of a flat $R$-module $V \in R$-flat to a square-zero extension $R'$ is a flat $R'$-module $V' \in R'$-flat equipped with an isomorphism $q_\ast V' = R \otimes_{R'} V' \cong V$.

All liftings of a given $V \in R$-flat form a groupoid in an obvious way. We denote this groupoid by $\mathcal{L}ift(V,R')$, and we denote by $\text{Lift}(V,R')$ the set of isomorphism classes of its objects.

**Proposition 4.2.** For any $V \in R$-flat, we have a natural equivalence of categories

$$\mathcal{L}ift(V,R') \cong \text{Spl}(C_\ast(V))$$

between liftings of $V$ to $R'$ and splittings of the elementary extension (4.2).

**Proof.** For any lifting $V' \in \mathcal{L}ift(R,V)$, the canonical map (3.22) is injective and fits into a short exact sequence

$$0 \longrightarrow q_\ast(M \otimes_R V) \longrightarrow V' \longrightarrow q_\ast V \longrightarrow 0.$$

Therefore every lifting is in particular an extension of $q_\ast V$ by $q_\ast(M \otimes_R V)$, so that we have a natural functor

\begin{equation}
\mathcal{L}ift(R,V) \rightarrow \mathcal{E}x(q_\ast V,q_\ast(M \otimes_R V)).
\end{equation}

Moreover, for any extension $V' \in \mathcal{E}x(q_\ast V,q_\ast(M \otimes_R V))$, the canonical map $m_{V'}$ factors through $q_\ast(M \otimes_R V) \subset V'$ by means of a map

$$\overline{m}_{V'} : q_\ast(M \otimes_R V) \rightarrow q_\ast(M \otimes_R V),$$

and $V'$ comes from a lifting if and only if $\overline{m}_{V'} = \text{id}$ is the identity map. Therefore in particular, (4.3) is a fully faithful embedding.
On the other hand, the functor \( q_* \) induces a fully faithful embedding

\[
q_* : \text{Spl}(C_\bullet(V)) \rightarrow \text{Spl}(q_* C_\bullet(V)).
\]

To describe its image, note that since the maps \( m_{q_* C_0(V)} \) and \( m_{q_* C_1(V)} \) both vanish, the map \( m_{C'} \) for any splitting \( C' \in \text{Spl}(q_* C_\bullet(V)) \) factors as

\[
M \otimes_R q^* C' \xrightarrow{\text{id} \otimes (a q^* c_0)} M \otimes_R V \xrightarrow{\overline{m}_{C'}} M \otimes_R V \xrightarrow{c_1 \circ b} C'(V),
\]

where as in (3.23), \( \text{id} \otimes (a \circ q^* c_0) \) is surjective, \( c_1 \circ b \) is injective, and \( \overline{m}_{C'} \) is some map. Then \( C' \) lies in \( q_*(\text{Spl}(C_\bullet(V))) \subset \text{Spl}(q_* C_\bullet(V)) \) if and only if \( m_{C'} = 0 \).

It remains to observe that the canonical splitting \( C_{01} \in \text{Spl}(q_* C_\bullet) \) provides a pair of functors

\[
\begin{align*}
(4.4) \quad \text{Spl}(q_* C_\bullet) & \rightarrow \mathcal{E}(q_* V, q_*(M \otimes_R V)), \quad C' \mapsto C_{01} - C', \\
(4.5) \quad \mathcal{E}(q_* V, q_*(M \otimes_R V)) & \rightarrow \text{Spl}(q_* C_\bullet), \quad E \mapsto C_{01} - E,
\end{align*}
\]

where in the right-hand side, we have the difference functors (1.18) and (1.19), and these functors are mutually inverse by (1.20). Moreover, whenever \( C' \cong C_{01} - E \), we have

\[
\overline{m}_{C_{01}} = \overline{m}_{C'} + \overline{m}_E.
\]

But since by (3.23), we have \( \overline{m}_{C_{01}} = \text{id} \), this means that the functors \((4.4)\) and \((4.5)\) induce an equivalence between the subcategories \( \text{Lift}(V, R') \subset \mathcal{E}(q_* V, q_*(M \otimes_R V)) \) and \( \text{Spl}(C_\bullet(V)) \subset \text{Spl}(q_* C_\bullet(V)) \). \( \square \)

### 4.2 DG sections

We next want to extend the correspondence between liftings and splitting obtained in Proposition 4.2 to complexes of \( R \)-modules and elementary DG extensions of Definition 2.6. In order to do this, we first need an appropriate version of Proposition 3.4.

Fix an associative unital ring \( R \), and consider the category \( \mathcal{B}_*(R, \mathbb{Z}) \) of (2.10). The forgetful functor \( e_* : \mathcal{C}^{p,f}_*(R) \rightarrow \mathcal{C}_*(\mathbb{Z}) \) is admissible in the sense of Definition 2.2 hence gives an object \( e_*(1) = 1, (R) \in \mathcal{B}_*(R, \mathbb{Z}) \). Denote

\[
K_* = \text{Cone}(R) \in \mathcal{C}^{p,f}_*(R),
\]

or explicitly, \( K_0 = K_1 = R, K_i = 0 \) otherwise, the differential \( d : K_1 \rightarrow K_0 \) given by the identity map. Then for any \( m \), we have the tautological map \( \delta : K_*[m] \rightarrow K_*[m + 1] \) given by the composition

\[
K_*[m] \xrightarrow{\alpha} R[m + 1] \xrightarrow{\beta} K_*[m + 1]
\]

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of the tautological maps $\{1\}$. We have $\delta^2 = 0$, so that $\langle K_\ast(i), \delta \rangle$ is a complex in the additive category $C_\ast(R)$.

**Definition 4.3.** A DG section $s_\ast$ of a surjective map $f : F_\ast \to e_\ast(I)$ in the category $B_\ast(R, \mathbb{Z})$ is a collection of elements $s_m \in F_m(K_\ast[m-1])$, $m \in \mathbb{Z}$ such that

\begin{equation}
F_\ast(\delta)(s_m) = ds_{m+1}, \quad f(s_m) = 1 \in R = e_\ast(I)(K_\ast[m-1])_m
\end{equation}

for any integer $m$.

For any integer $m$, the complex $K_\ast[m-1]$ represents the forgetful functor sending a complex $V_\ast$ to the set $V_m$, so that a DG section $s_\ast$ represents by Yoneda a collection of functorial maps

\begin{equation}
\tilde{s}_m : V_m \to F_m(V_\ast), \quad V_\ast \in C^{p,f}_\ast(R)
\end{equation}

such that $s_m \circ d = d \circ s_{m-1}$. A DG section does not always exists. For example, let $F_\ast = \text{Cone}(e_\ast(I)[-1])$ be the functor that sends a complex $V_\ast \in C_\ast(R)$ to the underlying complex of $\text{Cone}(V_\ast)[-1]$, and let $f = \kappa_0$ be the tautological map of $\{1\}$. Then it is easy to check that $f : F_\ast \to e_\ast(I)$ does not admit a DG section in the sense of Definition 4.3. However, as the following result shows, it is easy to exclude situations of this kind.

**Lemma 4.4.** Assume given surjective map $f : F_\ast \to e_\ast(I)$ in the category $B_\ast(R, \mathbb{Z})$, and assume that there exists an element $t \in F_0(R)$ such that $f(t) = 1 \in R$ and $dt = 0$. Then $f$ admits a DG section in the sense of Definition 4.3.

**Proof.** First of all, observe that once we have an element $t_m \in F_m(R[m])$ for some integer $m$ such that $dt_m = 0$ and $f(t_m) = 1$, it is trivial to construct an element $s_m \in F_{m+1}(K_\ast[m])$ such that $ds_m = F_\ast(\alpha)(t_m)$. Indeed, the admissible functor $F_\ast$ must send the acyclic complex $K_\ast[m]$ to an acyclic complex, and $dF_\ast(\alpha)(t_m) = F_\ast(\alpha)(dt_m) = 0$. Moreover, $ds_m = F_\ast(\alpha)(t_m)$ implies $df(s_m) = \alpha(f(t_m)) = 1$, and since the relevant differential in $K_\ast[m]$ is the identity map, this yields $f(s_m) = 1$. We can then let $t_{m+1} = F_\ast(\beta)(s_m)$, and repeat the procedure. By induction, we can therefore assume given elements $s_n \in F_n(K_\ast[n-1])$ satisfying (4.7) for all $n \geq m$, with $d(s_m) = F_\ast(\alpha)(t_m)$.

To finish the proof, we need to apply induction in the other direction. Thus it suffices to construct $s_{m-1} \in F_m(K_\ast[m-1])$ and $t_{m-1} \in F_{m-1}(R[m-1])$. Then it is easy to check that...
such that \( F_\ast(\beta)(s_{m-1}) = t_m, \)
\( ds_{m-1} = F_\ast(\alpha)(t_{m-1}), \)
\( dt_{m-1} = 0, \)
and \( f(t_{m-1}) = 1, \)
\( f(s_{m-1}) = 1. \)

To do this, note that since \( F_\ast \) is admissible, it must send the exact sequence (1.1) for the complex \( R[m-1] \) to a quasiexact sequence. Then as we have observed right after the definition of a quasiexact sequence, we can always choose \( s_{m-1} \in F_m(K_q[m-1]) \) such that \( F_q(\alpha)(s_{m-1}) = t_m, \)
\( ds_{m-1} = F_q(\beta)(t_{m-1}) \) for some \( t_{m-1} \in F_{m-1}(R[m-1]). \) Moreover, \( F_q(\beta) \) is injective. Therefore \( F_q(\beta)(dt_{m-1}) = d(ds_{m-1}) = 0 \) implies \( dt_{m-1} = 0. \)

It remains to notice that the maps \( \alpha \) and \( \beta \) are isomorphisms in relevant degrees, so that \( \alpha(f(s_{m-1})) = f(t_m) = 1 \) implies \( f(s_{m-1}) = 1, \) and this in turns implies \( f(t_{m-1}) = 1. \)

\[ \square \]

**Corollary 4.5.** Assume given a surjective map \( f : F_\ast \to e_\ast(l_\ast) \) in the category \( B(R, \mathbb{Z}) \) satisfying the assumptions of Lemma 4.4, and assume further that

\[ F_\ast(C_{\leq m}(R)) \subset C_{\leq m}(\mathbb{Z}) \]

for any integer \( m. \) Then \( f \) admits a one-side inverse \( g : e_\ast(l_\ast) \to F_\ast, \)
\( f \circ g = id. \)

**Proof.** By Lemma 4.4, \( f \) admits a DG section \( s_\ast, \) and it suffices to prove that the corresponding maps (1.8) are additive. As in the proof of Proposition 3.4, it suffices to consider the universal situation: we let \( V_\ast = K_q[m-1] \oplus K_q[m-1] \) for some integer \( m, \) with \( v_1, v_2 \in R \oplus R = V_m \) given by \( v_1 = 1 \oplus 0, \)
\( v_2 = 0 \oplus 1, \) and must prove that

\[ p_m = \bar{s}_m(v_1 + v_2) - \bar{s}_m(v_1) - \bar{s}_m(v_2) \]

is equal to 0. By the definition of a DG section, we have \( F_\ast(\delta \oplus \delta)(p_m) = d(p_{m+1}), \) so that \( q_m = F_\ast(\alpha \oplus \alpha)(p_m) \in F_m(R[m], R[m]) \) is closed. But by definition, \( q_m \) lies in the cross-effects component \( F_\ast(R[m], R[m]). \) Since \( F_\ast \) is admissible, this is an acyclic complex, and by assumption, it is trivial in homological degrees > \( m. \) Therefore \( q_m = 0 \) for any \( m. \) But then \( p_m \) is closed for any \( m, \) and by the same argument, it also vanishes. \( \square \)

**Proposition 4.6.** Assume given an \( R^0 \)-module \( W \) and a DG elementary extension

\[ (4.10) \quad 0 \longrightarrow l_\ast(W) \xrightarrow{b} C_\ast \xrightarrow{a} e_\ast(l_\ast) \longrightarrow 0 \]
in the category $\mathcal{B}_i(R, Z)$ such that $C_i$ satisfies (2.22) with $C_{m+1}(R)$ replaced by $C_{m+1}(Z)$. Then the extension (4.10) admits a strict right DG splitting, and such a strict right DG splitting is unique up to an isomorphism.

**Proof.** As in the proof of Proposition 3.4, uniqueness immediately follows from Corollary 4.3. Indeed, by virtue of (2.22), the difference $E_i = C^r_i - \tilde{C}^r_i$ between any two strict right DG splitting of (4.10) satisfies the assumptions of Corollary 4.3 so that we have a map $l_i(W) \oplus e_*(l_i) \rightarrow E_i$ from the trivial extension $l_i(W) \oplus e_*(l_i) \in \mathcal{E}(e_*(l_i), l_i(W)) \subset \mathcal{E}(e_*(l_i), l_i(W))$. Together with the natural maps (1.26) and (1.27), it induces a map $C^r_i \rightarrow \tilde{C}^r_i$, and since all maps between strict DG splittings are isomorphisms, $C^r_i$ is isomorphic to $\tilde{C}^r_i$.

To prove existence, note that by the definition of a DG elementary extension, the map $a : C_i(R) \rightarrow e_*(l_i)(R) = R$ is an isomorphism on homology in homological degree 0. Therefore $a$ satisfies assumptions of Lemma 4.4 and admits a DG section $s$. As in the proof of Proposition 3.4, fix such a section $s$, and consider the corresponding functorial maps (4.8). These are not necessarily additive, but they do commute with the differentials. Now let

\[(4.11) \quad C^r_i(V_i) = C_i(V_i) \times V_{i-1}, \quad i \in \mathbb{Z},\]

with the differential given by $d(c \times v) = dc + \bar{s}(v)$, and with the maps $r, a^r$ given by $r(c) = c \times 0$, $a^r(c \times v) = a(c) + v$. As in the proof of Proposition 3.4, in order to turn $C^r_i$ into a strict right DG splitting, we need to equip $C^r_i(V_i)$, $i \in \mathbb{Z}$ with functorial structures of abelian groups compatible with the map $r$ and the differential, and this amounts to constructing functorial maps

\[c_i(-, -) : V_i \times V_i \rightarrow C_{i+1}(V_i), \quad i \in \mathbb{Z}\]

satisfying (3.12), (3.13), (3.14), and

\[(4.12) \quad dc_i(v_1, v_2) = c_{i-1}(dv_1, dv_2) + \bar{s}_i(v_1 + v_2) - \bar{s}_i(v_1) - \bar{s}_i(v_2)\]

for any $v_1, v_2 \in V_i$, $i \in \mathbb{Z}$. As in the proof of Proposition 3.4, to construct $c_i(-, -)$, it suffices to consider the universal case $V_i = K, [i-1] \oplus K, [i-1]$, with $v_1, v_2 \in R \oplus R = V_i$ given by $v_1 = 1 \oplus 0$, $v_2 = 0 \oplus 1$. Then we have

\[(4.13) \quad c_{i-1}(dv_1, dv_2) = C_i(\delta \oplus \tilde{\delta})(c_{i-1}(v_1, v_2)),\]

where $\delta$ is the canonical map (4.6). Moreover, as in (2.18), the DG elementary extension (4.10) generates an elementary extension $\rho_i(C_i) = \rho(\tau^i(C_i))$
for any integer $i$, and the DG section $s_*$ gives an element $s = C_* (\alpha)(s_i) \in C_i (R[i])$ such that $a(s) = 1$. Then Proposition 3.4 provides functorial maps

$$\tilde{c}_i (-, -) : V \times V \to C_{i+1} (V[i])$$

satisfying (3.12)–(3.15) with $\tilde{s}$ being the map represented by $s$. Let us look for the elements $c_i (v_1, v_2) \in C_i (K_*[i-1] \oplus K_*[i-1])$ with the additional assumption

$$C_* (\alpha \oplus \alpha) (c_i (v_1, v_2)) = \tilde{c}_i (v_1, v_2).$$

Then by virtue of (4.13) and (4.6), (4.12) can be rewritten as

$$dc_i (v_1, v_2) = C_* (\beta \oplus \beta) (\tilde{c}_{i-1} (v_1, v_2)) + \tilde{s}_i (v_1 + v_2) - \tilde{s}_i (v_1) - \tilde{s}_i (v_2),$$

and since $\tilde{s}_i$ commutes with the differentials while the maps $\tilde{c}_i (-, -)$ satisfy (3.15), the right-hand side is annihilated by $d$. Again as in the proof of Proposition 3.4 the condition (3.12) then means that $c_i (-, -)$ takes values in the cross-effects component $C_{i+1} (K_* [i-1], K_* [i-1])$, and (4.15) with (4.14) prescribe its differential and image $C_* (\alpha \oplus \alpha)(c_i (-, -))$ in $C_{i+1} (R[i], R[i])$. However, the map

$$C_* (K_* [i-1], K_* [i-1]) \xrightarrow{C_* (\alpha \oplus \alpha)} C_* (R[i], R[i])$$

is a surjective maps of acyclic complexes, so that its kernel $\overline{C}_*$ is acyclic, and moreover, by (2.22), we have $C_{i+2} (K_* [i-1]) = 0$, so that the differential $d : \overline{C}_{i+1} \to \overline{C}_i$ in the acyclic complex $\overline{C}_*$ is injective. Therefore (3.12), (4.15) and (4.14) uniquely determine $c_i (-, -)$. As in the proof of Proposition 3.4 (3.13) and (3.14) now immediately follow from this uniqueness.

**4.3 Complexes.** Assume now given an $R$-bimodule $M$, and a DG elementary extension $\varphi = \langle C_*, a, b \rangle \in \mathcal{E}l_1 (R, M)$. Then by Theorem 3.6 its restriction $r(\varphi) \in \mathcal{E}l(R, M)$ in the sense of (2.23) corresponds to a square-zero extension $R'$ of $R$ by $M$. On the other hand, we have the forgetful functor $e_* : \mathcal{B}_* (R) \to \mathcal{B}_* (R, \mathbb{Z})$, and Proposition 4.6 provides a canonical strict right DG splitting $C'_r$ of the extension $e_* (\alpha)$.

**Lemma 4.7.** With the notation above, the algebra $R'$ acts naturally on the complex $C'_r$, thus turning it into a strict right DG splitting of the elementary extension $q_* (\alpha)$ in the category $\mathcal{B}_* (R, R')$.  

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Proof. As in the proof of Lemma \[3.5\] define regular endomorphisms of the splitting \(C_r^q\) as those that fit into a diagram \([3.17]\), and denote by \(R''\) the algebra of regular automorphisms. Then the uniqueness statement of Proposition \[4.6\] immediately implies that \(R''\) is a square-zero extension of \(R\) by \(M\). Since restriction is functorial, we have a natural map \(R'' \to R'\), and since all maps in \(S_q(R, M)\) are isomorphisms, we have \(R' \cong R''.\) \[\square\]

Now as in Subsection 4.1, we observe that for any complex \(V_q\) of flat \(R\)-modules, the category \(J(V_q)\) of complexes \(V' \in C_{qf}(R)\) equipped with a map \(i : V'_q \to V_q\) is small and filtering, and we have

\[
V_q = \lim_{V'_q \in J(V_q)} V'_q.
\]

Therefore setting

\[
C_\ast(V_q) = \lim_{V'_q \in J(V_q)} C_\ast(V'_q)
\]

extends the functor \(C_\ast\) to a functor

\[
(4.16) \quad C_\ast : C_{qf}(R) \to C_\ast(R),
\]

where \(C_{qf}(R) \subset C_\ast(R)\) is the full subcategory spanned by complexes of flat \(R\)-modules, and for any complex \(V_q \in C_{qf}(R)\), we have a natural elementary extension \(\varphi(V_q)\) of \(V_q\) by \(M \otimes_R V_q\) in the category \(C_\ast(R)\). Moreover, the right DG splitting \(C_r^q\) of the extension \(q \ast C_q\) provided by Lemma \[4.7\] extends to a splitting of the extension of \([1.16]\) in the category of admissible functors from \(C_{qf}(R)\) to \(C_\ast(R')\).

Definition 4.8. A DG lifting of a complex \(V_q \in C_{qf}(R)\) to the square-zero extension \(R'\) is a complex \(V_q' \in C_\ast(R')\) equipped with a map \(V_q' \to q \ast V_q\) such that the multiplication map \(m_{V_q'}\) factors as

\[
q \ast M \otimes_R V_q' \longrightarrow q \ast (M \otimes_R V_q) \longrightarrow \overline{m} \longrightarrow V_q',
\]

and the map \(\overline{m}\) fits into a quasiexact sequence

\[
0 \longrightarrow q \ast (M \otimes_R V_q) \longrightarrow \overline{m} \longrightarrow V_q' \longrightarrow q \ast V_q \longrightarrow 0.
\]

The category of DG liftings of \(V_q\) to \(R'\) is denoted \(\text{Lift}_\ast(V_q, R')\), and the set of its connected components is denoted \(\text{Lift}_\ast(V_q, R')\).

Proposition 4.9. For any complex \(V_q \in C_{qf}(R)\) of flat \(R\)-modules, there is a natural bijection between the sets \(\text{Lift}_\ast(V_q, R')\) and \(\text{Spl}_\ast(\varphi(V_q))\).
Proof. The proof is essentially the same as Proposition \[1.2\]. We consider the right DG splitting $C^r_\bullet$ of the extension $q_*(\varphi)$ provided by Lemma \[1.7\], we let $C^l_r$ be the left DG splitting associated to it by the functor \[1.14\], and we define functors

\[
\begin{align*}
S\text{pl}^l_\bullet(\varphi(V_\bullet)) &\to \text{Ex}_\bullet(q_*(V_\bullet), q_*(M \otimes_R V_\bullet)), \\
\text{Ex}_\bullet(q_*(V_\bullet), q_*(M \otimes_R V_\bullet)) &\to S\text{pl}^l_\bullet(\varphi(V_\bullet))
\end{align*}
\]

by

\[
C'_\bullet \mapsto C^l_r - C'_\bullet, \quad E_\bullet \mapsto C^l_r - E_\bullet,
\]

where we use the difference functors \[1.21\], \[1.25\]. Then we observe that by virtue of the canonical maps \[1.27\], these functors induced mutually inverse maps between $\text{Spl}^l_\bullet(\varphi(V_\bullet))$ and $\text{Lift}_\bullet(V_\bullet, R')$. \hfill \Box

Remark 4.10. The reader might notice that contrary to what we said in the Introduction, we do not recover complexes in $C_\bullet(R')$ as algebras over a monad in $C_\bullet(R)$ extending the endofunctor $C_\bullet$. The reason for this is that the monad in question is essentially freely generated by $C_\bullet$. Our splittings extend to algebras over this monad canonically, and technically, there is no reason to actually work this out: it adds a lot of complexity but gives the same end result. However, if one wants to go beyond square-zero extensions, the structure of a monad or a comonad becomes essential.

5 Multiplication.

5.1 Multiplicative extensions. Now assume that our ring $R$ is commutative, so that the category $R$-proj of projective finitely generated $R$-modules is a tensor category, with the unit object $R$.

Definition 5.1. For any commutative ring $R'$, a multiplicative structure on a pointed functor $F \in \mathcal{B}(R, R')$ is given by a map $e : R' \to F(R)$ and a collection of functorial maps

\[
m(V, V') : F(V) \otimes_{R'} F(V') \to F(V \otimes_R V'), \quad V, V' \in R\text{-proj},
\]

subject to obvious associativity and unitality conditions.

For example, the tautological functor $! \in \mathcal{B}(R)$ carries an obvious multiplicative structure. More generally, for any associative $R$-algebra $A$, we can treat $A$ as a diagonal $R$-bimodule — that is, take the same right $R$-action.
as the left $R$-action, — and then $l(A)$ has a natural multiplicative structure induced by the multiplication in $A$.

Conversely, for any associative $R$-algebra $A$ and any functor $F \in \mathcal{B}(R)$ equipped with a multiplicative structure, $F(A)$ is naturally an $R$-algebra. In particular, $F(R)$ is an $R$-algebra. Then one can show that if $F = l(M)$ is additive, then $F = l(F(R))$ (so that in particular, $M$ must be a diagonal $R$-bimodule). Since we will not need it, we do not give a proof.

More generally, given a complex $F_*$ in $\mathcal{B}(R,R')$, we say that a multiplicative structure on $F_*$ is given a map $e : R' \to F_0(R)$ and a collection of functorial maps

$$m : F_*(V) \otimes_{R'} F_*(V') \to F_*(V \otimes_R V'), \quad V,V' \in R\text{-proj},$$

again subject to associativity and unitality conditions.

Equivalently, one can consider the tensor product of $R$-modules as a functor

$$\mu : R\text{-proj} \times R\text{-proj} \to R\text{-proj},$$

and one defines a tensor structure on $\text{Fun}(R\text{-proj},R'\text{-mod})$ by setting

$$(5.1) \quad F \circ F' = \mu_!(F \boxtimes_{R'} F'),$$

where $\mu_!$ is the left Kan extension with respect to the functor $\mu$. By adjunction, the product of pointed functors is pointed, so that the subcategory $\mathcal{B}(R,R') \subset \text{Fun}(R\text{-proj},R'\text{-mod})$ also acquires a tensor structure. Then equipping $F \in \mathcal{B}(R,R')$ with a multiplicative structure is equivalent to turning it into an algebra object in $\langle \mathcal{B}(R,R'), \circ \rangle$, and giving a multiplicative structure on a complex $F_*$ is equivalent to turning it into a DG algebra in $\mathcal{B}(R,R')$.

**Definition 5.2.** An elementary extension $\varphi = \langle C_*,a,b \rangle$ of a commutative ring $R$ by an $R$-bimodule $M$ is multiplicative if $C_*$ is equipped with a multiplicative structure such that $a : C_* \to I$ is a multiplicative map.

**Lemma 5.3.** Assume given a multiplicative elementary extension $\varphi$ of a commutative ring $R$ by an $R$-bimodule $M$. Then the bimodule $M$ is diagonal.

**Proof.** By definition, since the extension $\varphi = \langle C_*,a,b \rangle$ is multiplicative, $C_0$ is an algebra object in $\langle \mathcal{B}(R), \circ \rangle$, and $C_1$ is a module over $C_0$. By restriction, $l(M) \subset C_1$ is also a module, so that we have functorial maps

$$(5.2) \ m(V,V') : C_0(V) \otimes_R (M \otimes_R V') \to M \otimes_R (V \otimes_R V'), \quad V,V' \in R\text{-proj}.$$
But for any $m \in M \otimes \mathbb{R} V$, $c \in C_1(V')$, we have

$$m(V, V')(dc \otimes m) = d(m(V, V')(c \otimes m)) + m(V, V')(c \otimes dm) = 0,$$

so that the $C_0$-action on $I(M)$ vanishes on $d(C_1) \subset C_0$ and factors through $I = C_0/d(C_1)$. Then if we take $V' = R$, the map (5.2) is induced by a map

$$(5.3) \quad V \otimes R M = V \otimes_R (M \otimes_R R) \to M \otimes_R (V \otimes_R R) = M \otimes_R V.$$

This map is functorial in $V$, thus $\text{End}_R(V)$-equivariant. If $V = R$, both its source and its target are identified with $M$ by unitality, and the map itself is the identity map. However, $R = \text{End}_R(R)$ acts via the left action on the bimodule $M$ on the left-hand side of (5.3), and via the right action in the right-hand side.

**Proposition 5.4.** Assume given a square-zero extension $R' \in S_q(R, M)$ of a commutative ring $R$ by an $R$-bimodule $M$, and assume that $R'$ is commutative (in particular, $M$ is diagonal). Then the elementary extension $\varphi \in E(R, M)$ that corresponds to $R'$ under the equivalence of Theorem 3.6 can be chosen to be multiplicative.

**Proof.** Note that the functor $q_* : R\text{-proj} \to R'\text{-mod}$ has a natural multiplicative structure, so that the object $q_*(I) \in B(R, R')$ is an algebra with respect to the tensor product (5.1). On the other hand, the adjoint functor $q^* : B(R, R') \to B(R)$ is a tensor functor. Thus by the construction of the correspondence (3.4), it suffices to check that one can choose a projective resolution $P$, of the algebra object $q_*(I) \in B(R, R')$ that is a DG algebra with respect to the tensor product (5.1). To do this, observe that by adjunction, the tensor product $P \otimes P'$ of two projective objects $P, P' \in B(R, R')$ is projective. Therefore for any projective $P \in B(R, R')$, the tensor algebra

$$T' P = \bigoplus_{i \geq 0} P^{\otimes i}$$

is a projective object in $B(R, R')$. Now we can construct a multiplicative resolution of the algebra $q_*(I)$ by the standard inductive procedure. □

**5.2 Multiplicative splittings.** We now want to invert Proposition 5.4 by showing that the correspondence (3.20) sends multiplicative elementary extensions to commutative square-zero extensions. It is convenient to generalize the context slightly.
Definition 5.5. An elementary extension \((1.4)\) in a tensor abelian category \(\mathcal{E}\) is multiplicative if \(A\) is an associative unital algebra in \(\mathcal{E}\), \(C_*\) is a DG algebra, and \(a : C_* \to A\) is an algebra map. A multiplicative splitting of a multiplicative elementary extension \((1.4)\) in \(\mathcal{E}\) is its splitting \(\langle C_0, c_0, c_1 \rangle\) equipped with a DG algebra structure on the complex \(C_1^l\) of \((1.15)\) such that the map \(l : C_1^l \to C_*\) is a DG algebra map.

We note that for any splitting \(C_{01}\), a DG algebra structure on the complex \(C_1^l\) is completely determined on its degree-0 part, that is, an algebra structure on \(C_{01}\). In order for it to define a multiplicative splitting, the map \(c_0 : C_{01} \to C_0\) must be an algebra map, the left and right \(C_{01}\)-actions on itself must factor through \(c_0\), so that \(C_{01}\) is a \(C_0\)-bimodule, and the map \(c_1 : C_1 \to C_{01}\) must be a \(C_0\)-bimodule map.

For any commutative ring \(R\), the forgetful functor \(e_* : \mathcal{B}(R) \to \mathcal{B}(R, \mathbb{Z})\) is multiplicative. Therefore for any multiplicative elementary extension \(\varphi = \langle C_*, a, b \rangle \in \mathcal{E}(R, M)\) of \(R\) by a diagonal \(R\)-bimodule \(M\), the elementary extension \(e_*(\varphi)\) is a multiplicative elementary extension in \(\mathcal{B}(R, \mathbb{Z})\).

Lemma 5.6. For any multiplicative elementary extension \(\varphi \in \mathcal{E}(R, M)\), the canonical splitting \(\langle C_{01}, c_0, c_1 \rangle\) of the elementary extension \(e_*(\varphi)\) provided by Proposition 3.4 is multiplicative in a natural way.

Proof. Since \(\varphi = \langle C_*, a, b \rangle\) is multiplicative, a natural choice of the element \(s \in C_0(R)\) with \(a(s) = 1\) is provided by the multiplicative structure on \(C_0\), namely, by the unity map \(R \to C_0\). The corresponding map \((3.8)\) is then a multiplicative map. By \((3.9)\), we have
\[
C_{01}(V) = C_1(V) \times V, \quad V \in R\text{-proj},
\]
with maps \(c_0, c_1\) given by \((5.10)\). To turn \(C_{01}\) into an algebra in \(\mathcal{B}(R, \mathbb{Z})\), we need to construct multiplication maps
\[
\cdot : C_{01}(V) \times C_{01}(V') \to C_{01}(V \otimes_R V'), \quad V, V' \in R\text{-proj}
\]
that are associative, unital, and bilinear in each argument. We note that by multiplicativity of the extension \(C_*\), we have
\[
(5.4) \quad c \cdot \delta(c') - \delta(c) \cdot c' = \delta(c \cdot c') = 0, \quad c, c' \in C_1(V),
\]
where \(\delta : C_1(V) \to C_0(V)\) is the differential, and we let
\[
(5.5) \quad (c \times v) \cdot (c' \times v') = (c \cdot \delta(c') + c \cdot \check{s}(v') + \check{s}(v) \cdot c') \times (v \cdot v')
\]
\[
= (\delta(c) \cdot c' + c \cdot \check{s}(v') + \check{s}(v) \cdot c') \times (v \cdot v')
\]
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for any \( v \in V, v' \in V', f \in C_1(V), f' \in C_1(V') \). This product is obviously unital, with \( 0 \times 1 \in C_0(R) \) being the unit. To check that it is bilinear in the first argument, we need to check that for any \( v_1, v_2 \in V, c_1, c_2 \in C_1(V) \), we have

\[
(c_1 + c_2 + c(v_1, v_2)) \cdot (\delta(c') + \tilde{s}(v')) + \tilde{s}(v_1 + v_2) \cdot c' = (c_1 + c_2) \cdot (\delta(c') + \tilde{s}(v')) + (\tilde{s}(v_1) + \tilde{s}(v_2)) \cdot c' + c(v_1 \cdot v', v_2 \cdot v'),
\]

where \( c(-, -) \) is the cocycle (3.11). By (3.15), this can be rewritten as

\[
c(v_1, v_2) \cdot \tilde{s}(v') + c(v_1, v_2) \cdot \delta(c') - \delta(c(v_1, v_2)) \cdot c' = c(v_1 \cdot v', v_2 \cdot v'),
\]

and taking into account (5.4), we further reduce it to

\[
c(v_1, v_2) \cdot \tilde{s}(v') = c(v_1 \cdot v', v_2 \cdot v').
\]

By the same cross-effects argument as in the proof of Proposition 3.4, it suffices to check this equality after applying the differential \( \delta \), and then it immediately follows from (3.15) and multiplicativity of the map \( \tilde{s} \).

The proof that the multiplication is bilinear in the second argument is exactly the same — all one has to do is to use the second of the equivalent expressions in the right-hand side of (5.5) instead of the first one.

Now, by bilinearity, to prove that the multiplication (5.5) is associative, it suffices to check associativity separately for elements of the form \( c \times 0 \) and \( 0 \times v \), and out of the eight possibilities that arise, the only non-trivial one is the equality

\[
((c \times 0) \cdot (c' \times 0)) \cdot (c'' \times 0) = (c \times 0) \cdot ((c' \times 0) \cdot (c'' \times 0))
\]

for any \( V, V', V'' \in R\text{-proj}, c \in C_1(V), c' \in C_1(V'), c'' \in C_1(V'') \). Then by (5.5) and (5.4), the left-hand side of (5.6) is given by

\[
((c \cdot \delta(c')) \cdot \delta(c'')) \times 0 = ((\delta(c) \cdot c') \cdot \delta(c'')) \times 0,
\]

and this clearly coincides with the right-hand side.

To finish the proof, it remains to prove that \( c_0 : C_{01} \rightarrow C_0 \) is an algebra map, and \( c_1 : C_1 \rightarrow C_{01} \) is a \( C_{01} \)-module map. This is a straightforward check that we leave to the reader; the only non-trivial observation is that we have

\[
\delta(c) \cdot \delta(c') = \delta(c \cdot \delta(c'))
\]

for any \( V, V' \in R\text{-proj}, c \in C_1(V), c' \in C_1(V') \). \( \square \)
5.3 The inverse construction. We can now prove a converse to Proposition 5.2. We keep the notation of last subsection.

Proposition 5.7. Assume given a multiplicative elementary extension \( \varphi = \langle C, a, b \rangle \in \mathcal{E}(R, M) \) of a commutative ring \( R \) by a an \( R \)-bimodule \( M \), and let \( R' \in \mathcal{S}(R, M) \) be the square-zero extension of \( R \) by \( M \) corresponding to \( \varphi \) by (3.20). Then \( R' \) is commutative, and the multiplicative splitting \( C_{01} \) of the induced extension \( e_{\ast}(\varphi) \) provided by Lemma 5.6 extends to a multiplicative splitting of the extension \( q_{\ast}(\varphi) \).

Proof. Since by assumption, \( C_0 \) is a multiplicative functor, \( C_0(R) \) is a unital associative ring, and the augmentation map \( C_0(R) \to R \) admits a natural splitting \( R \to C_0(R) \). Since \( C_{01} \) is multiplicative, \( C_{01}(R) \) is also a unital associative ring, and \( c_0 : C_{01}(R) \to C_0(R) \) is a ring map. Consider the sub-ring \( R'' = c_0^{-1}(R) \subset C_{01}(R) \), where \( R \subset C_0(R) \) is the image of the splitting \( R \to C_0(R) \). Then \( R'' \) is a square-zero extension of \( R \) by \( M \). Moreover, since \( M \) is a diagonal \( R \)-bimodule by Lemma 5.3, (5.5) immediately shows that \( R'' \) is commutative. Now, every \( c \in C_{01}(R) \) induces by (5.5) a functorial map

\[
-c : C_{01}(V) \to C_{01}(V), \quad V \in R\text{-proj},
\]

and this map is obviously regular in the sense of (3.17). Therefore we have a natural ring map \( C_{01}(R) \to R' \). The restriction of this map to the square-zero extension \( R'' \subset C_{01}(R) \) must be an isomorphism, so that \( R' \cong R'' \) is indeed commutative. Moreover, the \( R' \)-action on \( C_{01} \) can be described by (5.7), and this shows that the product in \( C_{01} \) is \( R' \)-linear, so that \( C_{01} \) gives a multiplicative splitting of \( q_{\ast}(\varphi) \). \( \square \)

We can also prove a multiplicative refinement of Proposition 4.2. Namely, assume given a flat associative unital \( R \)-algebra \( A \), and say that a lifting \( A' \) of \( A \) to \( R \) is a flat associative unital \( R' \)-algebra \( A' \) equipped with isomorphism \( A' \otimes_R R \cong A \). Denote the groupoid of all liftings of \( A \) to \( R' \) by \( \text{Lift}^0(A, R') \). On the other hand, note that since \( A \) is an algebra, \( C_\ast(A) \) is a DG algebra in \( R\text{-mod} \), and moreover, it is a multiplicative elementary extension of \( A \) by \( M \otimes_R A \) in the sense of Definition 5.5. Denote by \( \text{Spl}^0(C_\ast(A)) \) the groupoid of multiplicative splittings of this extension.

Proposition 5.8. The equivalence of Proposition 4.2 induces an equivalence of categories

\[
\text{Lift}^0(A, R') \cong \text{Spl}^0(C_\ast(A))\].

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Proof. By Proposition 5.7, the splitting $C_{01}(A)$ of the extension $q_* C_*(A)$ is multiplicative. It remains to observe that the difference functors (1.18) and (1.19) are obviously compatible with multiplicative structures. □

We note that the DG algebra $C_*(A)$ is a square-zero extension of $A$ by the complex $(M \otimes_R A)[1]$, and multiplicative splittings considered in Proposition 5.8 are by definition splittings of this extension in the category of DG algebras over $R$. In particular, it is easy to construct an obstruction to the existence of such a splitting: it is a certain class in the third Hochschild cohomology group $HH^3(A, M \otimes_R A)$. If the obstruction vanishes, then the set of isomorphism classes of splittings is a torsor over $HH^2(A, M \otimes_R A)$. As far as we know, this seems to be a new result.

5.4 DG extensions. A natural next thing to do would be to generalize the notion of a multiplicative structure to DG elementary extensions of Definition 2.6, with the eventual goal of extending Proposition 5.8 to complexes of flat $R$-modules (or equivalently, finding a multiplicative version of Proposition 4.9). By itself, a multiplicative version of Definition 5.1 presents no problems: all we need to do is to add homological indices.

Definition 5.9. For any two commutative rings $R$, $R'$, a multiplicative structure on an admissible functor $F_* \in B_*(R, R')$ is given by a map $e : R' \to F_*(R)$ and a collection of functorial maps

$$m(V, V') : F_*(V) \otimes_R F_*(V') \to F_*(V \otimes_R V'), \quad V, V' \in C_{*\otimes}(R),$$

subject to obvious associativity and unitality conditions.

Analogously, one can define multiplicative DG elementary extensions and their multiplicative splitting. Moreover, for any multiplicative DG extension $\varphi_* \in \mathcal{E}_l(R, M)$, the restriction $\rho(\varphi_*) \in \mathcal{E}_l(R, M)$ is also multiplicative, so that the associated square-zero extension $R' \in \text{Sq}(R, M)$ must be a commutative ring.

But unfortunately, this is where the story ends, for the following reason. The canonical strict DG splitting $C^r_*$ of the extension $q(r)(\varphi_*)$ provided by Lemma 4.7 is a right DG splitting, nor a left one; and the corresponding left DG splitting $C^l_*$ is not strict. On the other hand, if we take the multiplicative splitting $C_{01}$ of Proposition 5.7, then it is the corresponding strict left
DG splitting $C^l$ that is multiplicative. While $C^l$ is quasiisomorphic to $C^{rl}$, it is not isomorphic to it, and it seems that $C^{rl}$ does not have a meaningful multiplicative structure.

Because of this, in the case of a general DG elementary extension, we cannot proceed any further, and in particular, a DG version of Proposition 5.8 is beyond out reach.

6 Cyclic powers.

To finish the paper, we study in detail one particular DG elementary extension $C_q$ of an arbitrary commutative ring $R$ annihilated by a prime. We show how the general constructions work in this case, and prove some additional results not available in the general situation.

6.1 Objects. Fix a commutative associative unital ring $R$ annihilated by a prime $p > 1$. For any flat $R$-module $V \in R$-flat, the $p$-th tensor power $V^{\otimes RP}$ is naturally a module over the group algebra $R[\mathbb{Z}/p\mathbb{Z}]$ of the cyclic group $\mathbb{Z}/p\mathbb{Z}$, with the generator of the group acting by the order-$p$ permutation $\sigma : V^{\otimes RP} \to V^{\otimes RP}$. Moreover, the spaces of invariants and coinvariants of the permutation $\sigma$ are related by a natural trace map

$$(6.1) \quad (V^{\otimes RP})_\sigma \xrightarrow{\text{tr}} (V^{\otimes RP})^\sigma$$

given by $\text{tr} = \text{id} + \sigma + \cdots + \sigma^{p-1}$.

Denote by $\tilde{C}_q(V)$ the complex of $R$-modules obtained by placing (6.1) in homological degrees 0 and 1. Then the homology of the complex $\tilde{C}_q(V)$ has been computed in [Ka3]. To state the answer, denote by $R^{(1)}$ the group $R$ considered as a module over itself via the Frobenius map — that is, let

$$r \cdot r' = r^{p}r', \quad r, r' \in R^{(1)},$$

and for any flat $R$-module $V$, let $V^{(1)} = R^{(1)} \otimes_R V$. Note that the Frobenius map induces an $R$-module map $R \to R^{(1)}$, hence a functorial map

$$(6.2) \quad \text{Fr} : V \to V^{(1)}.$$

Lemma 6.1 ([Ka3, Lemma 6.9]). The complex $\tilde{C}_q(V)$ fits into an exact sequence

$$0 \longrightarrow V^{(1)} \xrightarrow{\psi} (V^{\otimes RP})_\sigma \xrightarrow{\text{tr}} (V^{\otimes RP})^\sigma \xrightarrow{\tilde{\psi}} V^{(1)} \longrightarrow 0,$$

and this sequence is functorial with respect to $V$. □
Explicitly, the map $\psi$ in Lemma 6.1 is given by

\[ \psi(r \cdot v) = rv \otimes R_p \]

(this is additive modulo $\text{Im}(\text{id} - \sigma)$). If $V \in R\text{-proj}$, then $\hat{\psi}$ is the dual map, and for a general flat $R$-module $V$, it is obtained by taking the filtered colimit. In any case, both $\psi$ and $\hat{\psi}$ are completely functorial.

**Definition 6.2.** For any $V \in R\text{-flat}$, the complex $C_*(V)$ is obtained by the pullback square

\[
\begin{array}{ccc}
C_*(V) & \xrightarrow{a} & V \\
\downarrow & & \downarrow \text{Fr} \\
\tilde{C}_*(V) & \xrightarrow{\tilde{a}} & V^{(1)},
\end{array}
\]

where $\tilde{a}$ is the natural map induced by the map $\hat{\psi}$ of Lemma 6.1, and $\text{Fr}$ is the Frobenius map (6.2).

By definition, $C_*$ is a complex of pointed functors from $R\text{-proj}$ to $R\text{-mod}$, thus a complex in the category $B(R)$. Moreover, $C_*(V)$ comes equipped with a functorial map $a : C_*(V) \to V$, and a functorial map $b : V^{(1)}[1] \to C_*(V)$ is induced by the map $\psi$ of Lemma 6.1. Altogether, $\langle C_*, a, b \rangle$ define an elementary extension of $R$ by the diagonal bimodule $R^{(1)}$ that we denote by

\[
\varphi \in \mathcal{E} \ell(R, R^{(1)}).
\]

Therefore by (3.20), there is a square-zero extension $R' \in \mathcal{E} \ell(R, R^{(1)})$ associated to the elementary extension $\varphi$.

Moreover, for any flat $R$-modules $V_1, V_2 \in R\text{-flat}$, we have natural maps

\[
V_1 \otimes_R \left( V_2 \otimes_R \right)^{\sigma} \to V_1 \otimes_R \left( V_2 \right)^{\sigma} \otimes_R \left( V_1 \otimes_R V_2 \right)^{\otimes R_p},
\]

\[
\left( V_1 \otimes_R \right)^{\sigma} \otimes_R \left( V_2 \otimes_R \right)^{\sigma} \to \left( \left( V_1 \otimes_R V_2 \right) \right)^{\sigma},
\]

and these maps are $\sigma$-equivariant, thus descend to functorial maps

\[
\left( V_2 \otimes_R \right)^{\sigma} \otimes_R \left( V_2 \otimes_R \right)^{\sigma} \to \left( \left( V_1 \otimes_R V_2 \right) \right)^{\sigma},
\]

\[
\left( V_2 \otimes_R \right)^{\sigma} \otimes_R \left( V_2 \otimes_R \right)^{\sigma} \to \left( \left( V_1 \otimes_R V_2 \right) \right)^{\sigma},
\]

\[
\left( V_2 \otimes_R \right)^{\sigma} \otimes_R \left( V_2 \otimes_R \right)^{\sigma} \to \left( \left( V_1 \otimes_R V_2 \right) \right)^{\sigma}.
\]
compatible with the trace maps \( \text{tr} \). Thus the complex of functors \( \tilde{C}_*(-) \) has a natural multiplicative structure. Then so does the complex \( C_*(-) \) of Definition 6.2 so that the elementary extension \( \varphi \) of (6.4) is multiplicative. By Proposition 5.7, the square-zero extension \( R' \) must then be commutative.

As it happens, one commutative square-zero extension of \( R \) by \( R^{(1)} \) is very well-known: this is the ring \( W_2(R) \) of second Witt vectors of \( R \). Let us prove that this is exactly what we get from the extension (6.4).

**Proposition 6.3.** For any commutative ring \( R \) annihilated by a prime \( p \), the square-zero extension \( R' \in \text{Sq}(R, R^{(1)}) \) corresponding to the elementary extension (6.4) is isomorphic to the second Witt vectors ring \( W_2(R) \).

**Proof.** Note that by definition, we have \( C_0(R) \cong R \), so that by the same argument as in the proof of Proposition 5.7, we have \( R' \cong C_01(R) \), where \( C_01 \) is the multiplicative splitting of the extension \( C_* \) provided by Lemma 5.6. We also have \( C_1(R) \cong R^{(1)} \), so that as a set, we have

\[
R' \cong R \times R^{(1)}.
\]

The additive structure is given by Proposition 3.4 — we have

\[
\sum (x_0 \times x_1) + (y_0 \times y_1) = (x_0 + y_0) \times (x_1 + y_1 + c(x_0, y_0))
\]

for any \( x_0, y_0 \in R \), \( x_1, y_1 \in R^{(1)} \), where \( c(-, -) \) is the cocycle (3.11). To compute \( c(-, -) \) explicitly, note that the multiplicative splitting map (3.8) is given by \( \tilde{s}(v) = v^{\otimes p} \), and consider the universal situation \( V = Rv_1 \oplus Rv_2 \). Then (3.15) reads as

\[
(id + \sigma + \cdots + \sigma^{p-1})c(v_1, v_2) = \sum_{s \in \mathcal{S}} \mu_s(v_1, v_2),
\]

where the sum is over the set \( \mathcal{S} \) of all degree-\( p \) non-commutative monomials in \( v_1 \) and \( v_2 \) except for \( v_1^{\otimes p} \) and \( v_2^{\otimes p} \). In other words, we have \( \mathcal{S} = S^p \setminus S \), where \( S = \{1, 2\} \) is the set of indices, and \( S \subset S^p \) is the diagonal. The permutation action of \( G = \mathbb{Z}/p\mathbb{Z} \) on \( \mathcal{S} \) is free, and to write down an explicit formula for \( c(v_1, v_2) \), it suffices to chose a splitting \( \kappa : \mathcal{S}/G \to \mathcal{S} \) of the quotient map \( \mathcal{S} \to \mathcal{S}/G \). We then have

\[
c(v_1, v_2) = \sum_{s \in \mathcal{S}/G} \mu_{\kappa(s)}(v_1, v_2) \in V_\sigma^{\otimes np},
\]

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and this does not depend on the choice of $\kappa$. Projecting back onto $V = R$, and noting that $R$ is commutative, we see that

$$c(x,y) = \sum_{1 \leq i \leq p-1} \frac{1}{p} \binom{p}{i} x^i y^{p-i}, \quad x, y \in R,$$

so that (6.5) becomes

$$(6.7) \quad (x_0 \times x_1) + (y_0 \times y_1) = (x_0 + y_0) \times \left( x_1 + y_1 + \sum_{1 \leq i \leq p-1} \frac{1}{p} \binom{p}{i} x_0^i y_0^{p-i} \right).$$

As for the product in $R' = C_{01}(R)$, then by Lemma 5.6, it is given by (5.5), and in our situation, it reads as

$$(6.8) \quad (x_0 \times x_1) \cdot (y_0 \times y_1) = x_0 y_0 \times (x_0^p y_1 + x_1 y_0^p).$$

It remains to recall that $R$ is assumed to be annihilated by $p$, and to notice that modulo $p$, (6.7) and (6.8) are exactly the standard formulas for the sum and product of Witt vectors. \qed

### 6.2 Complexes

The elementary extension $\varphi \in \mathcal{E}(R, R^{(1)})$ defined by the complex $C_\ast$ of Definition 6.2 also has a very natural lifting to a DG elementary extension. To describe it, we first need another interpretation of the complex $\tilde{C}_\ast(V)$. Recall that for any $R[\mathbb{Z}/p\mathbb{Z}]$-module $E$, the Tate cohomology complex $\tilde{C}_\ast(\mathbb{Z}/p\mathbb{Z}, E)$ is obtained by taking $\tilde{C}_i(\mathbb{Z}/p\mathbb{Z}, E) = E$ for any integer $i$, with the differential $d_i : \tilde{C}_i(\mathbb{Z}/p\mathbb{Z}, E) \to \tilde{C}_{i-1}(\mathbb{Z}/p\mathbb{Z}, E)$ given by

$$d_i = \begin{cases} 
\text{id} - \sigma, & i = 2j, \\
\text{tr}, & i = 2j + 1.
\end{cases}$$

In other words, $\tilde{C}_\ast(\mathbb{Z}/p\mathbb{Z}, E)$ is the 2-periodic complex

$$(6.9) \quad \begin{array}{cccc}
\text{id} - \sigma & E & \text{tr} & E & \text{id} - \sigma & E & \text{tr} & E & \text{id} - \sigma \\
\end{array}.$$  

Then one immediately observes that we have a functorial identification

$$(6.10) \quad \tilde{C}_\ast(V) \cong \tau_{\{0,1\}} \tilde{C}_\ast(\mathbb{Z}/p\mathbb{Z}, V^{\otimes R_p})$$

for any flat $R$-module $V$. To generalize (6.10) to complexes, one needs to find appropriate versions of the Tate cohomology complex $C_\ast(\mathbb{Z}/p\mathbb{Z}, -)$ and of the truncation functor $\tau_{\{01\}}$.
The former is easy: for any complex of \( R[\mathbb{Z}/p\mathbb{Z}] \)-modules \( E_* \), (6.9) is naturally a bicomplex, and we denote by \( \tilde{\mathcal{C}}_*(\mathbb{Z}/p\mathbb{Z}, E_*) \) its total complex.

For the latter, we use the filtered truncation functors introduced in [Ka3, Section 1.3]. Namely, for any abelian category \( \mathcal{E} \), denote by \( CF_*(\mathcal{E}) \) the category of complexes in \( \mathcal{E} \) equipped with a decreasing filtration \( F^* \) numbered by all integers.

**Definition 6.4.** For any integer \( n \) and any abelian category \( \mathcal{E} \), the filtered truncation functors \( \tau_{\geq n}^F, \tau_{\leq n}^F : CF_*(\mathcal{E}) \to CF_*(\mathcal{E}) \) are given by

\[
\begin{align*}
\tau_{\geq n}^F E_i &= d^{-1}(F^{n+1-i}E_{i-1}) \cap F^{n-i}E_i \subset E_i, \\
\tau_{\leq n}^F E_i &= E_i/(F^{n+1-i}E_i + d(F^{n-i}E_{i+1}))
\end{align*}
\]

for any \( E_* \in CF_*(\mathcal{E}) \).

This is essentially [Ka3, (1.8)], with a difference of notation: \( \tau_{\geq n}^F \) is \( \tau^n \) of [Ka3], and \( \tau_{\leq n}^F E_* \) is the quotient \( E_*/\beta^n E_* \). For any filtered complex \( E \), and any \( n \), we have natural maps \( \tau_{\geq n}^F E_* \to E_* \rightarrow E_* \to \tau_{\leq n}^F \). Given two integers \( n \leq m \), we denote \( \tau_{[n,m]}^F E_* = \tau_{\geq n}^F \tau_{\leq m}^F E_* \cong \tau_{\leq m}^F \tau_{\geq n}^F E_* \). If the filtration \( F^* \) on a complex \( E_* \) is termwise-split, then we have natural identifications

\[
\begin{align*}
\text{gr}^i_F(\tau_{\geq n}^F E_*) &\cong \tau_{\geq n+i} \text{gr}^i_F E_*, \\
\text{gr}^i_F(\tau_{\leq n}^F E_*) &\cong \tau_{\leq n+i} \text{gr}^i_F E_*, \\
\text{gr}^i_F(\tau_{[n,m]}^F E_*) &\cong \tau_{[n+i,m+i]} \text{gr}^i_F E_*
\end{align*}
\]

(6.11)

for any integer \( i \). In particular, \( \tau_{\geq n}^F \), \( \tau_{\leq n}^F \) and \( \tau_{[n,m]}^F \) send filtered quasiisomorphisms to filtered quasiisomorphisms — in effect, inverting filtered quasiisomorphisms of filtered complexes gives the filtered derived category \( DF_*(\mathcal{E}) \), and \( \tau_{\leq n}^F, \tau_{\geq n}^F \) are truncation functors with respect to a natural t-structure on \( DF_*(\mathcal{E}) \).

We now note that any filtration \( F^* \) on a complex \( E_* \) of \( R[\mathbb{Z}/p\mathbb{Z}] \)-modules induces a filtration on the Tate complex \( \tilde{\mathcal{C}}_*(\mathbb{Z}/p\mathbb{Z}, E_*) \). We denote by \( \mathcal{F}^* \) the stupid filtration on \( E_* \) — we recall that by definition, it is given by

\[
\mathcal{F}^i E_j = \begin{cases} 
E_j, & i + j \geq 0, \\
0, & i + j < 0,
\end{cases} \quad i, j \in \mathbb{Z}.
\]

Then we define a filtration \( F^* \) on \( E_* \) by rescaling \( \mathcal{F}^* \) by \( p \) — that is, we let \( F^i E_* = \mathcal{F}^{ip} E_* \). With these definitions in mind, the following generalization of Lemma 6.1 has been obtained in [Ka3].
Lemma 6.5 ([Ka3]). Assume given a complex \( V_\ast \) of flat \( R \)-modules, and consider the \( p \)-fold tensor power \( E_\ast = V_\ast \otimes_{R}^{p} \). Then for any integer \( i \) not divisible by \( p \), the Tate complex \( \tilde{C}_\ast(\mathbb{Z}/p\mathbb{Z}, E_\ast) \) is acyclic. Moreover, equip \( E_\ast \) with the \( p \)-th rescaling \( F^\ast \) of the stupid filtration \( F^\ast \), and consider the induced filtration on the Tate complex \( \tilde{C}_\ast(\mathbb{Z}/p\mathbb{Z}, E_\ast) \). Then for any integer \( i \), we have an functorial isomorphism

\[
(6.12) \quad \tau_{[i]} F^\ast \tilde{C}_\ast(\mathbb{Z}/p\mathbb{Z}, E_\ast) \cong V_\ast^{(1)}[i],
\]

this isomorphism is functorial in \( V_\ast \), and the induced filtration \( F^\ast \) on \( V_\ast^{(1)}[i] \) is the stupid filtration shifted by \( i \).

Proof. This is [Ka3, Proposition 6.10] (the first and the last claims are a part of the definition of a “tight” complex given in [Ka3, Definition 5.2]). □

As an application of Lemma 6.5, for any complex \( V_\ast \in C^{\text{fl}}_\ast(R) \) of flat \( R \)-modules, let us denote by

\[
(6.13) \quad \tilde{C}_\ast(V_\ast) = \tau_{[0,1]} F^\ast \tilde{C}_\ast(\mathbb{Z}/p\mathbb{Z}, V_\ast \otimes_{R}^{p} \mathbb{Z})
\]

the filtered truncation of the Tate cohomology complex \( \tilde{C}_\ast(\mathbb{Z}/p\mathbb{Z}, V_\ast \otimes_{R}^{p} \mathbb{Z}) \).

Then Lemma 6.5 and (6.11) show that the complex \( \tilde{C}_\ast(V_\ast) \) fits into a natural functorial quasiexact sequence

\[
0 \longrightarrow V_\ast^{(1)}[1] \overset{b}{\longrightarrow} \tilde{C}_\ast(V_\ast) \overset{\tilde{a}}{\longrightarrow} V_\ast^{(1)} \longrightarrow 0,
\]

where as before, we let \( V_\ast^{(1)} = R^{(1)} \otimes_{R} V_\ast \). If we now define \( C_\ast(V_\ast) \) by the pullback square (6.3), then we have a functorial quasiexact sequence

\[
(6.14) \quad 0 \longrightarrow V^{(1)}[1] \overset{b}{\longrightarrow} C_\ast(V_\ast) \overset{a}{\longrightarrow} V_\ast \longrightarrow 0.
\]

Lemma 6.6. The sequence (6.14) defines a DG elementary extension \( \varphi_\ast \) of \( R \) by \( R^{(1)} \).

Proof. It is immediately clear from (6.11) that the functor \( C_\ast = C_\ast(-) \) satisfies (2.22), so it remains to prove that it is admissible in the sense of Definition 2.2. As we have remarked after stating the condition (2.22), the quasiexact sequence (6.14) insures that it suffices to prove that \( C_\ast \) sends termwise-split injections to termwise-split injections, and termwise-split surjections to termwise-split surjections. In fact, since the square (6.3) is Cartesian, it suffices to prove the same for the functor \( \tilde{C}_\ast \) of (6.13). We will do the injections — the argument for surjections is exactly the same.
Assume given a map $f : V \to V'$ of complexes of flat $R$-modules, and denote

$$E_* = V_*^\otimes RP, \quad E'_* = V'_*^\otimes RP.$$  

Assume that $f$ is a termwise-split injection. Then the $p$-th tensor power $f^\otimes p : E_* \to E'_*$ is a termwise-split injection of complexes of $R[\mathbb{Z}/p\mathbb{Z}]$-modules, so that for any integers $i, j$, the map

$$(6.15) \quad f^\otimes p : \tau_{[i,i+1]} \check{C}_* (\mathbb{Z}/p\mathbb{Z}, E_j) \to \tau_{[i,i+1]} \check{C}_* (\mathbb{Z}/p\mathbb{Z}, E'_j)$$

is an injection. Fix an integer $i$, and consider the associated graded quotient $\text{gr}_F E_*$. By the definition of the filtration $F^*$, this quotient has amplitude $[(i - 1)p + 1, ip]$, and its associated graded quotients with respect to the stupid filtration $F'$ are the complexes $E_j[j], (i - 1)p < j \leq ip$. Then the Tate complex $\check{C}_* (\mathbb{Z}/p\mathbb{Z}, \text{gr}_F E_*)$ is an iterated extension of shifts of Tate complexes $\check{C}_* (\mathbb{Z}/p\mathbb{Z}, E_j), (i - 1)p < j \leq ip$. But by Lemma 6.5 all these complexes are acyclic, except possibly for the one corresponding to $j = ip$. Then applying inductively Lemma 1.1 we conclude that

$$\text{gr}_F \tau_{[i,i+1]} \check{C}_* (\mathbb{Z}/p\mathbb{Z}, \text{gr}_F E_*) \cong \tau_{[i,i+1]} \check{C}_* (\mathbb{Z}/p\mathbb{Z}, E_j)[j], \quad (i - 1)p < j \leq ip.$$  

Moreover, we have the same identification for $E'_*$, and since the maps (6.15) are injections, we conclude that the map

$$\tau_{[i,i+1]} \check{C}_* (\mathbb{Z}/p\mathbb{Z}, \text{gr}_F E_*) \to \tau_{[i,i+1]} \check{C}_* (\mathbb{Z}/p\mathbb{Z}, \text{gr}_F E'_*)$$

induced by $f$ is an injection. Collecting these maps for all $i$ and applying (6.11), we further conclude that the map

$$\tau_{[0,1]} F \check{C}_* (\mathbb{Z}/p\mathbb{Z}, E_*) \to \tau_{[0,1]} F \check{C}_* (\mathbb{Z}/p\mathbb{Z}, E'_*)$$

induced by $f$ is an injection, and this is exactly what we had to prove. \[\square\]

6.3 Multiplication. It is clear from (6.10) and (6.11) that the restriction $\rho(\varphi_\cdot) \in \mathcal{E}_l(R, R^{(1)})$ of the DG elementary extension $\varphi_\cdot \in \mathcal{E}_l(R, R^{(1)})$ provided by Lemma 6.6 coincides with the elementary extension $\varphi_\cdot$ of (6.4). As it happens, one can also extend the multiplicative structure on $\varphi_\cdot$ to a multiplicative structure on $\varphi_\cdot$. To do this, we need to recall a more invariant definition of Tate cohomology.

For any finite group $G$ and any bounded complex $E_\cdot$ of $R[G]$-modules, the Tate cohomology groups $\check{H}^\cdot(G, E_\cdot)$ are given by

$$\check{H}^\cdot(G, E_\cdot) = \text{Ext}_{D^b(R[G])/(D^p(R[G]))}^\cdot (R_* E_\cdot),$$

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where \( R \) is the trivial \( R[G] \)-module, and the Ext-groups are computed in the quotient of the bounded derived category \( \mathcal{D}^b(R[G]) \) of \( R[G] \)-modules by its full subcategory \( \mathcal{D}^{bf}(R[G]) \subset \mathcal{D}^b(R[G]) \) spanned by perfect complexes of \( R[G] \)-modules (equivalently, \( \mathcal{D}^{bf}(R[G]) \subset \mathcal{D}^b(R[G]) \) is spanned by compact objects in the triangulated category \( \mathcal{D}(R[G]) \)). In particular, \( \check{H}^*(G, R) \) is always an algebra.

In the case \( G = \mathbb{Z}/p\mathbb{Z} \), the Tate cohomology complex \( \check{C}_*(\mathbb{Z}/p\mathbb{Z}, E) \) of (6.9) computes exactly the groups \( \check{H}^q(\mathbb{Z}/p\mathbb{Z}, E) \). Unfortunately, the multiplication in \( \check{H}^*(\mathbb{Z}/p\mathbb{Z}, R) \) does not lift to a DG algebra structure on \( \check{C}_*(\mathbb{Z}/p\mathbb{Z}, R) \). However, this problem can be solved by changing the complex. Namely, consider a finite group \( G \), choose a projection resolution \( P_q \) of the trivial \( \mathbb{Z}[G] \)-module \( \mathbb{Z} \), and let \( \mathbf{P}_q \) be the cone of the augmentation map \( P_q \to \mathbb{Z} \). Then it is easy to show (see e.g. [Ka2, Subsection 7.2] but the claim is completely standard) that for bounded complex \( E_q \) of \( R[G] \)-modules, we have a natural identification

\[
\check{H}^*(G, E_q) \cong \lim_{\rightarrow} H^q(G, E_q \otimes F^{-i}\mathbf{P}_q),
\]

where \( H^*(G, -) = \text{Ext}^*_D(R[G]) (R, -) \) is the cohomology of the group \( G \), and \( F^{-i}\mathbf{P}_q \) is the \((-i)\)-th term of the stupid filtration on the complex \( \mathbf{P}_q \). Moreover, the right-hand side of (6.16) is canonically independent of the choice of a resolution \( P_q \) — indeed, for any two resolutions \( P_q, P'_q \), we have natural maps \( \mathbf{P}_q \to \mathbf{P}_q \otimes \mathbf{P}'_q, \mathbf{P}'_q \to \mathbf{P}_q \otimes \mathbf{P}'_q \), and both maps induce isomorphisms in the right-hand side of (6.16). If one wants to represent Tate cohomology by an explicit functorial complex, one also needs to represent the usual cohomology \( H^*(G, E) \) by such a complex; the standard way to do is is to choose an injective resolution \( I' \) of the trivial \( \mathbb{Z}[G] \)-module \( \mathbb{Z} \), and consider the cohomology complex

\[
C^*(G, E) = (E \otimes I')^G.
\]

Altogether, the following sums up the situation.

**Definition 6.7.** (i) *Resolution data* for a finite group \( G \) is a pair \( \nu = \langle P, I' \rangle \) of a projective resolution \( P \), and an injective resolution \( I' \) of the trivial \( \mathbb{Z}[G] \)-module \( \mathbb{Z} \).

(ii) For any associative unital ring \( R \), any bounded complex \( E_* \) of \( R[G] \)-modules, and any resolution data \( \nu \), the *Tate cohomology complex* of \( G \) with coefficients in \( E_* \) is given by

\[
\check{C}_*(G, \nu, E_*) = (E_* \otimes \mathbf{P}_* \otimes I_*')^G.
\]

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where \( \overline{P} \) is the cone of the augmentation map \( P \to \mathbb{Z} \).

Then for any \( G \) and \( E_* \), and for any choice of the resolution data \( \nu \), we obviously have

\[
\check{C}^*_\nu(G, E_*) \cong \lim_{\to} C^*\left(G, E_* \otimes F^{-i}P\right),
\]

where \( C^* \) in the right-hand side is the complex (6.17), and by (6.16), \( \check{C}^*_\nu \) computes the Tate cohomology \( \check{H}^* \) and does not depend on \( \nu \) up to a canonical quasiisomorphism.

Alternatively, given resolution data \( \nu = \langle P_*, I^* \rangle \), one can consider the composition \( P_* \to \mathbb{Z} \to I^* \) of the augmentation maps, and let \( \widetilde{P} \) be its cone. Then for any bounded complex \( E_* \) of \( R[G] \)-modules, one can consider the complex

\[
\check{C}'_\nu(G, E_*) = \left(E_* \otimes \widetilde{P}\right)^G.
\]

Then we have an obvious map \( \widetilde{P} \to P \otimes I^* \), and one easily shows that the induced map

\[
\check{C}'_\nu(G, E_*) \to \check{C}^*_\nu(G, E_*),
\]

is a quasiisomorphism. Thus the complex (6.19) also computes Tate cohomology groups \( \check{H}^* \) (in fact, historically, this was their original definition).

Now let us say that resolution data \( \nu = \langle P_*, I^* \rangle \) are multiplicative if both \( I^* \) and \( \widetilde{P} \) are equipped with a structure of a unital associative DG algebra. Note that multiplicative resolution data do exist for any finite group \( G \). Indeed, for \( I^* \), this is well-known, and for \( P_* \), it suffices to take a free \( \mathbb{Z}[G] \)-module \( P_0 \) equipped with a surjective map \( d : P_0 \to \mathbb{Z} \), and let \( P_i = P_0 \otimes \mathbb{Z}^i \), \( i \geq 1 \), with the differential induced by \( d \).

**Lemma 6.8.** Assume given a set of multiplicative resolution data \( \nu \) for the cyclic group \( \mathbb{Z}/p\mathbb{Z} \), and for any complex \( V_* \in C_{p}^*(R) \) with tensor power \( E_* = V_* \otimes^R \mathbb{Z}/p\mathbb{Z} \), let

\[
\check{C}_\nu(V_* \otimes \mathbb{Z}/p\mathbb{Z} \nu, E_*),
\]

where the filtration \( F^* \) on \( E_* \) is the same as in Lemma 6.5. Then the admissible functor \( \check{C}_\nu : C_p^*(R) \to C^*_\nu(R) \) has a natural multiplicative structure in the sense of Definition 5.9.
Proof. The tensor power $V^\otimes R^p$ has a trivial multiplicative structure, and since $\nu$ is multiplicative, the functor $\check{C}_*(\mathbb{Z}/p\mathbb{Z}, \nu, -)$ is also multiplicative. It remains to notice that this multiplicativity is compatible with the filtrations, and the filtered truncation functor $\tau_{(0,1]}^F: CF_*(R) \to CF_*(R)$ also has an obvious multiplicative structure. \hfill \Box

As an immediately corollary of Lemma 6.8, we see that the functor $C_*(\nu, V_q)$ obtained from $\check{C}_*(\nu, V_q)$ by the pullback square (6.3) is also multiplicative, so that we obtain a multiplicative DG elementary extension $\varphi^\nu_\nu$ of $R$ by $R^{(1)}$. Moreover, while the periodic Tate complex $\check{C}_*(\mathbb{Z}/p\mathbb{Z}, E_\cdot)$ is not of the form (6.18) for any choice of the resolution data, it is of the form (6.19) — as resolution data, one takes the pair of the standard periodic projective and injective resolutions of $\mathbb{Z}$. Therefore we have a natural functorial quasiisomorphism $C_*(V_\cdot) \to C_*(\nu, V_\cdot)$, where $C_*(V_\cdot)$ is the functorial complex (6.14). Taking restrictions, we also obtain a map from the elementary extension $\varphi$ of (6.4) to the restriction $\rho(\varphi^\nu_\nu)$ of the DG elementary extension $\varphi^\nu_\nu$. An interested reader can easily check that this map is multiplicative, so that $\varphi^\nu_\nu$ indeed extends $\varphi$ to a multiplicative functor.

6.4 Splittings. We will now show that under some assumptions, the situation for the multiplicative DG elementary extension $\varphi^\nu_\nu$ provided by Lemma 6.8 is better than for a general multiplicative DG elementary extension — namely, we do have a good multiplicative strict left DG splitting of the induced extension $q_\nu(\varphi^\nu_\nu)$.

First of all, assume that the commutative ring $R$ is perfect — that is, the Frobenius map $Fr: R \to R^{(1)}$ is bijective. In this case, we have $\check{C}_*(V) \cong C_*(V)$ and $\check{C}_*(V_\cdot) \cong C_*(V_\cdot)$, without the need to apply the pullback square (6.3). Moreover, the augmentation ideal in the second Witt vectors ring $W_2(R) = R'$ is generated by $p$, so that we have $R \cong R'/p$.

Next, consider the category $C_{\nu}^p(R')$, and for any complex $V_\cdot \in C_{\nu}^p(R')$, denote

$$C_*(V_\cdot) = \tau_{(0,1]}^F \check{C}_*(\mathbb{Z}/p\mathbb{Z}, V_\cdot^\otimes R^p),$$

where as in Lemma 6.5 the filtration $F_\cdot$ on $V_\cdot^\otimes R^p$ is the $p$-th rescaling of the stupid filtration.
Lemma 6.9. Assume given a complex \( V \in C^{pf}(R') \), let \( q : V \to V/p \) be the quotient map, and let \( p : V/p \to V \) be the embedding induced by the multiplication by \( p \).

(i) We have a natural exact sequence

\[
0 \longrightarrow C_\ast(V/p) \overset{p}{\longrightarrow} C_\ast(V) \overset{q}{\longrightarrow} C_\ast(V/p) \longrightarrow 0
\]

of complexes of \( R' \)-modules.

(ii) Let \( q_i : \tau_{[i,i]}^F C_\ast(V) \to \tau_{[i,i]}^F C_\ast(V/p) \) be the map induced by the quotient map \( q \) for \( i = 0, 1 \). Then \( q_0 \) is an isomorphism, and \( q_1 = 0 \).

Proof. For (i), note that it suffices to prove that the sequence becomes exact after taking the associated graded quotient \( \text{gr}^F \) for an arbitrary integer \( i \).

We always have an exact sequence

\[
0 \longrightarrow \check{C}_\ast(Z/pZ, (V/p) \otimes_{R'} p) \overset{p}{\longrightarrow} \check{C}_\ast(Z/pZ, V \otimes_{R'} p) \overset{q}{\longrightarrow} \check{C}_\ast(Z/pZ, (V/p) \otimes_{R'} p) \longrightarrow 0
\]

of Tate complexes, so that by (6.11) and Lemma 1.1, it suffices to check that the connecting differential

\[
\delta_{i,i} : \check{H}_j(Z/pZ, \text{gr}^F_\ast(V/p) \otimes_{R'} p) \to \check{H}_{j-1}(Z/pZ, \text{gr}^F_\ast(V/p) \otimes_{R'} p)
\]

in the corresponding long exact sequence of homology vanishes for any \( i \) and \( j = i, i + 2 \). Since the Tate complex is 2-periodic, it suffices to consider the case \( j = i \). By Lemma 6.5, we have a functorial identification

\[
(6.22) \quad \check{H}_i(Z/pZ, \text{gr}^F_\ast(V/p) \otimes_{R'} p) \cong \check{H}_{i-1}(Z/pZ, \text{gr}^F_\ast(V/p) \otimes_{R'} p) \cong V_i^{(1)}/p,
\]

so that \( \delta_{i,i} \) is an endomorphism of \( V_i^{(1)}/p \). Then by functoriality, we have replace the complex \( V \) with \( V_i[i] \), so it suffices to consider the case when \( V \) is concentrated in a single homological degree.

For (ii), note that since by Lemma 6.5 the filtration \( F' \) induces a shift of the stupid filtration on \( \tau_{[j,j]}^F C_\ast(V) \) for any \( j \), it again suffices to prove the claim after passing to \( \text{gr}^F_\ast \). Moreover, if we know that \( \delta_{i,i} = 0 \), then both claims reduce to checking that \( \delta_{i,i+1} \) is an isomorphism, and by virtue of (6.22), it suffices to check this for complexes concentrated in a single homological degree.

Moreover, since both homology groups in (6.22) are additive with respect to \( V \), we may further assume that \( V_i \cong R' \), so that \( V = R'[i] \) for some
integer $i$. Then $V^\otimes_{R^p} \cong R'[ip]$, and the permutation $\sigma$ acts by $(-1)^{(p-1)}id$. If $i(p-1)$ is even, then, since the Tate complex (6.9) is $2$-periodic, we have
\[
(6.23) \quad \check{C}_.(\mathbb{Z}/p\mathbb{Z}, (R'[i])^\otimes_{R^p}) \cong \check{C}_.(\mathbb{Z}/p\mathbb{Z}, R'[i]).
\]
If $i(p-1)$ is odd, then necessarily $p = 2$, and the differentials in (6.9) are given by $id - \sigma$, $id + \sigma$. Therefore replacing $\sigma$ by $-\sigma$ is equivalent to shifting the complex by $1$, and we still have the identification (6.23). It shows that we may further assume that $i = 0$. Then the complex (6.9) is the complex
\[
0 \rightarrow R' \xrightarrow{p} R' \rightarrow 0 \rightarrow R' \xrightarrow{p} R' \rightarrow 0,
\]
with $d : \check{C}_.(\mathbb{Z}/p\mathbb{Z}, R') \rightarrow \check{C}_.(\mathbb{Z}/p\mathbb{Z}, R')$ given by $0$ for even $i$ and $p$ for odd $i$, and both claims (i), (ii) are obvious. □

We now note that the proof of Lemma 6.9 only depends on the homology groups $\check{H}_.(\mathbb{Z}/p\mathbb{Z}, -)$, so it remains valid if we change (6.20) by considering one of the different versions of the Tate complex given in Subsection 6.3. So, fix once and for all a set $\nu$ of multiplicative resolution data for the cyclic group $\mathbb{Z}/p\mathbb{Z}$, and let $\check{C}_.(\mathbb{Z}/p\mathbb{Z}, \nu, -) = \check{C}_.(\mathbb{Z}/p\mathbb{Z}, -)$, with (6.20) reinterpreted accordingly (we will not need the periodic complex (6.9) anymore, so there is no danger of confusion).

Now, for any complex $W_. \in C^p_{\nu}(R)$, denote by $\check{C}^p_.(W_.) \subset C_.(W_.)$ the kernel of the map $a : C_.(W_.) \rightarrow W_.$, and denote by $\check{C}^p_. = C_.(W_.)/b(W_.[1])$ the cokernel of map $b : W_[1] \rightarrow C_.(W_.)$. Then by Lemma 6.9 (i), for any $V_. \in C^p_{\nu}(R')$, we can define complexes $C^p_.(V_.), C^p_.(V_.)$ by exact sequences
\[
(6.24) \quad 0 \rightarrow \check{C}^p_.(V_.) \xrightarrow{p} C^p_.(V_.) \rightarrow \check{C}^p_.(V_.) \rightarrow 0,
\]
\[
0 \rightarrow C^p_.(V_.) \rightarrow C_.(V_.) \rightarrow \check{C}^p_.(V_.) \rightarrow 0.
\]
By Lemma 6.9 (ii), $C^p_.(V_.)$ resp. $\check{C}^p_.(V_.)$ is a strict left resp. right DG splitting of the DG elementary extension $C_.(V_.)$. Both $C^p_.(V_.)$ and $\check{C}^p_.(V_.)$ are functorial in $V_.$. The functor $C^p_.(-)$ has an obvious multiplicative structure induced by the multiplicative structure on $C_.(-)$, and the action of $C_.(-)$ on $C^p_.(-) \subset C_.(-)$ factors through its quotient $C^p_.(-)$ — for any two complexes $V_., V'_p \in C^p_{\nu}(R')$, we have functorial action maps
\[
C^p_.(V_.) \otimes_{R'} C^p_.(V_.) \rightarrow C^p_.(V'_p \otimes_{R'} V_.),
\]
\[
C^p_.(V_.) \otimes_{R'} \check{C}^p_.(V_.) \rightarrow C^p_.(V'_p \otimes_{R'} V_.).
\]
We will need one somewhat technical result on the functor $C^p_.(-)$. Take the unity element $t_0 = 1 \in C^p_.(R')$, and as in the proof of Lemma 4.4 lift it to an
element \( s_0 \in C^0_0(\text{Cone}(R')[-1]) \) such that \( C^i_*(\alpha)(s_0) = t_0 \) and \( ds_0 = C^i_*(\beta)(t) \) for some \( t_1 \in C_1(R'[-1]) \). Then by adjunction, \( s_0 \) induces a map

\[
\tilde{s} : V_0 \to C^0_0(V_*)
\]

for any complex \( V_* \in C^{pf}_*(R') \), and this map is functorial in \( V_* \). Moreover, since \( C^i_*(\alpha)(s_0) = 1 \), we have

\[
\tilde{s}(v) = v^{\otimes_p}
\]

for any closed \( v \in V_0, \ dv = 0 \). If we project \( C^i_*(V_*) \) to \( C_0(V_*/p) \), then the composition map \( V_0 \to C_0(V_*/p) \) factors through a functorial map

\[
\overline{s} : V_0/p \to C_0(V_*/p)
\]

corresponding to the image of the element \( s_0 \) in \( C_0(R) \), and we still have

\[
\overline{s}(v) = v^{\otimes_p}
\]

for any closed \( v \in V_0/p, \ dv = 0 \). In particular, both \( \tilde{s} \) and \( \overline{s} \) are multiplicative on closed elements. On the whole \( V_0 \), the map \( \tilde{s} \) does not have to multiplicative, but the following is sufficient for our purposes.

**Lemma 6.10.** (i) Assume given two complexes \( V_*, V'_* \in C^{pf}_*(R') \) and elements \( v \in V_0, v' \in V'_0 \) such that \( dv = dv' = 0 \) mod \( p \). Then we have

\[
\tilde{s}(v \cdot v') = \tilde{s}(v) \cdot \tilde{s}(v') \in C^0_0(V_* \otimes R' V'_*).
\]

(ii) Assume in addition that \( V_* = V'_* \) and \( v = v' \) mod \( p \). Then we have

\[
\tilde{s}(v) = \tilde{s}(v').
\]

**Proof.** For (i), note that we can always replace the complexes \( V_*, V'_* \) with the 0-th terms of their stupid filtration; assume therefore that both lie in \( C^{pf}_0(R') \). Then so does the product \( V''_* = V_0 \otimes R' V'_0 \), and the natural projection \( \lambda'' : V''_* \to V''_0 \cong V_0 \otimes R' V'_0 \) is the tensor product of the projections \( \lambda : V_* \to V_0, \lambda' : V'_* \to V'_0 \). Since \( C^i_*(V''_*) \) is a strict left DG splitting of \( C_*(V''_*/p) \), the map

\[
C^i_*(V''_*) \xrightarrow{q \otimes C^i_*(\lambda'')} C_*(V''_*/p) \oplus C^i_*(V''_*)
\]

is injective in degree 0 for dimension reasons. Therefore is suffices to check \( \tilde{s}(v \cdot v') = \tilde{s}(v) \cdot \tilde{s}(v') \) after projecting to \( C_0(V''_*/p) \) and to \( C^0_0(V''_0) \). For the former, note
that by assumption, \(v\) and \(v'\) are closed modulo \(p\), so that (6.29) immediately follows from (6.28). For the latter, note that then we can replace \(V, V'\) with \(V_0, V'_0\), and then \(v\) and \(v'\) become closed, and (6.29) follows from (6.27).

For (ii), note that it suffices to prove that \(\tilde{s}(v) = \tilde{s}(v')\) in the universal situation \(V = \text{Cone}(R'v_1 \oplus R'v_2)[-1], \ v = v_1, \ v' = v_1 + pv_2\). In this situation, the map (6.30) for the complex \(V\) is still injective, and moreover, since \(v = v' \mod p\), we have \(q(\tilde{s}(v)) = \tilde{s}(v) = \tilde{s}(v') = q(\tilde{s}(v'))\), so that we may further project to \(V_0 = R'v_1 \oplus R'v_2\). Then (3.15) and (6.6) show that

\[
(6.31) \quad \tilde{s}(v') - \tilde{s}(v) = \sum_{1 \leq i \leq p-1} \sum_{s \in S_i/G} \delta(\mu_{s}(v_1,v_2)),
\]

where \(S_i \subset S\) is the subset parametrizing monomials of degree \(i\) in \(v_2\) (and \(p - i\) in \(v_1\)). Since for any \(i \geq 1\), \(s \in S_i/G\), \(p^i\) is divisible by \(p\), while \(\delta(\mu_{s}(v_1,v_2))\) lies in \(C_0(V_0/p) \subseteq C_0(V_0/p)\), the right-hand of (6.31) gives 0 in the quotient \(C_0^l(V_0) = C_0(V_0)/pC_0(V_0/p)\).

\[\square\]

6.5 Liftings. We can now prove the following surprising general property of the functors \(C^l, C^r\) of (6.24) (we note that the idea for this is essentially due to V. Vologodsky). We need to assume further that the perfect ring \(R\) is a field.

Proposition 6.11. The functorial complexes \(C^l(V), C^r(V)\) of (6.24) only depend on the quotient \(V/p\), so that the corresponding admissible functors

\[
C^l, C^r : C^p(R') \to C_*(R')
\]

factor through the projection \(q^* : C^p(R') \to C^p(R), \ q^*(V) = V/p\).

Proof. By definition, morphisms from \(V\) to \(V'\) in the category \(C^p(R')\) are degree-0 classes

\[
f \in \text{Hom}^0(V, V')
\]

in the complex \(\text{Hom}^*(V, V')\) such that \(df = 0\). Such a morphism acts on \(V\) via the action map

\[
(6.32) \quad a : \text{Hom}^*(V, V') \otimes_{R'} V \to V'.
\]

On the other hand, on \(p\)-th tensor powers, such a morphism \(f\) acts by \(f^\otimes p\). Then (6.27) shows that the morphisms \(C^l(f), C^r(f)\) can be expressed
in terms of the map \( \tilde{s} \) of (6.26) and the maps (6.25). Namely, for any \( f : V \to V' \) and \( c \in C^l(V) \), \( c' \in C^r(V) \), we have

\[
(6.33) \quad C^l(f)(c) = C^l(a)(\tilde{s}(f) \cdot c), \quad C^r(f)(c') = C^r(a)(\tilde{s}(f) \cdot c'),
\]

where \( \cdot \) denotes the product (6.25), and \( a \) is the action map (6.32).

Now, since \( R \) is by assumption a field, the projection \( q^* : C^{p_1}(R') \to C^{p_1}(R) \) is essentially surjective. Therefore one can describe the category \( C^{p_0}(R') \) in the following way: objects are complexes \( V \in C^{p_0}(R') \), morphisms from \( V \) to \( V' \) are closed degree-0 classes

\[
(6.34) \quad f \in \text{Hom}^0(V / p, V' / p) = \text{Hom}^0(V, V') / p
\]
in the complex \( \text{Hom}^*(V, V') / p \). But by Lemma 6.10 for any \( V, V' \in C^{p_0}(R') \), the map \( \tilde{s} \) actually factors as

\[
\text{Hom}^0(V, V') \overset{q}{\longrightarrow} \text{Hom}^0(V, V') / p \overset{\tilde{s}}{\longrightarrow} C^l_0(\text{Hom}^*(V, V')),
\]

and the map \( \tilde{s} \) is multiplicative on closed classes. Then to factor the functors \( C^l(-), C^r(-) \) through \( q^* \), just let morphisms \( f \) of (6.34) act by (6.33). \( \Box \)

**Remark 6.12.** The only place where we have used the assumption that \( R \) is field is in concluding that \( q^* : C^{p_1}(R') \to C^{p_1}(R) \) is essentially surjective. It might be that homological properties of perfect rings are good enough to insure this in a more general situation, but I have not pursued this.

Now, by virtue of Proposition 6.11 we can redefine the functors \( C^l(-), C^r(-) \) as admissible functors

\[
C^l, C^r \in \mathcal{B}(R, R')
\]

from \( C^{p_1}(R) \) to \( \mathcal{C}_*(R') \), so that what we denoted earlier by \( C^l(V_1), C^r(V_1) \) will now become \( C^l(V_1/p), C^r(V_1/p) \). Then \( C^l, C^r \) are strict left resp. right DG splittings of the extension \( q_* \mathcal{C}_* \), where \( \mathcal{C}_* \) is the cyclic powers extension (6.14) of Lemma 6.6. By the unicity clause of Proposition 4.6, the splitting \( C^r \) must coincide with the one constructed in that Proposition in the context of a general DG elementary extension. The splitting \( C^l \) is new: we do not know how to construct it in the general case. However, for the cyclic powers extension \( \mathcal{C}_* \), it is perfectly well-defined. Moreover, by construction, it has a natural multiplicative structure. As in Subsection 4.3 we can further extend to a functor

\[
C^l : C^{p_1}(R) \to \mathcal{C}_*(R'),
\]

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and the extended functor inherits a multiplicative structure. In particular, for any DG algebra $A$, over $R$, we have a functorial map

\begin{equation}
C_l(A) \to q_*C_l(A)
\end{equation}

of DG algebras over $R'$. This opens the way to various DG versions of Proposition 5.8. We will only record the simplest possible result in this direction.

**Proposition 6.13.** Assume given a DG algebra $A$ over $R$, and consider the DG algebra $C_l(A)$ with the augmentation map $a : C_l(A) \to A$. Assume further that there exists a DG algebra $A'$ flat over $R'$ such that $A \cong A' \otimes_{R'} R = A'/p$. Then there exists a DG algebra $C_l(A)$ over $R$ and a DG algebra map $l : C_l(A) \to C_l(A')$ such that the composition map

\begin{equation}
\tilde{C}_l(A) \xrightarrow{l} C_l(A) \xrightarrow{a} A
\end{equation}

is a quasiisomorphism.

**Proof.** We have the DG algebra $C_l(A)$ over $R'$ and the DG algebra map (6.35). Moreover, $C_l(A)$ is a multiplicative strict left DG splitting $\psi \in \mathcal{S}p_l(\varphi)$ of the multiplicative elementary extension $q_*\varphi = q_*C_l(A)$ in the category $C_l(R')$. On the other hand, the DG algebra $A'$ is an extension of $A$, by $A^{(1)}$, thus defines an object $\epsilon$ in the extension groupoid $E_x(A, A^{(1)})$. Applying the difference functor (1.22), we obtain an object

$$\psi - \epsilon = (\tilde{C}_l(A), l, b) \in \mathcal{S}p_l(q_*\varphi).$$

As in Proposition 4.2, $\tilde{C}_l(A)$ is actually a complex over $R = R'/p$, and as in Proposition 5.8 it has a natural DG algebra structure. \qed

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