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RATIONALITY OF BERSHADSKY-POLYAKOV VERTEX ALGEBRAS

TOMOYUKI ARAKAWA

Abstract. We prove the conjecture of Kac-Wakimoto on the rationality of exceptional \( W \)-algebras for the first non-trivial series, namely, for the Bershadsky-Polyakov vertex algebras \( W_{3}^{(2)} \) at level \( k = p/2 - 3 \) with \( p = 3, 5, 7, 9, \ldots \). This gives new examples of rational conformal field theories.

1. Introduction

Recently, a remarkable family of \( W \)-algebras associated with simple Lie algebras and their non-principal nilpotent elements, called exceptional \( W \)-algebras, has been discovered by Kac and Wakimoto [10]. In [10] it was conjectured that with an exceptional \( W \)-algebra one can associate a rational conformal field theory.

As a first step to resolve the Kac-Wakimoto conjecture we have proved in the previous article [3] that exceptional \( W \)-algebras are lisse, or equivalently, \( C_2 \)-cofinite. Therefore it remains [15, 6] to show that exceptional \( W \)-algebras are rational, i.e., that the representations are completely reducible, in order to prove the conjecture. In this article we prove the rationality of the first non-trivial series of exceptional \( W \)-algebras, that is, the Bershadsky-Polyakov (vertex) algebras \( W_{3}^{(2)} \) at level \( k = p/2 - 3 \) with \( p = 3, 5, 7, 9, \ldots \). The vertex algebra \( W_{3}^{(2)} \) is the \( W \)-algebra associated with \( g = \mathfrak{sl}_3 \) and its minimal nilpotent element.

Let us state our main result more precisely: Let \( W_k \) denote the unique simple quotient of \( W_{3}^{(2)} \) at level \( k \neq -3 \).

Main Theorem (Conjectured by Kac and Wakimoto [10]). Let \( p \) be an odd integer equal or greater than 3, \( k = p/2 - 3 \). Then the vertex algebra \( W_k \) is rational. The simple \( W_k \)-modules are parameterized by the set of integral dominant weights of \( \hat{\mathfrak{sl}}_3 \) of level \( p - 3 \). These simple modules can be obtained by the quantum BRST reduction from irreducible admissible representations of \( \hat{\mathfrak{sl}}_3 \) of level \( k \).

For \( p = 3 \), \( W_{3/2-3} \) is one-dimensional. In the remaining cases \( W_{p/2-3} \) are conformal with negative central charges.

We note that Zhu’s algebra of \( W_3^{(2)} \) is closely related with Smith’s algebra [14], which is a deformation of the universal enveloping algebra \( U(\mathfrak{sl}_3(\mathbb{C})) \) of \( \mathfrak{sl}_3(\mathbb{C}) \), and that the rational quotient \( W_{p/2-3} \) has features in common with the \( \mathfrak{sl}_2 \)-integrable affine vertex algebras in the sense that the following relations hold:

\[
G^+(z) p^{-2} := G^-(z) p^{-2} := 0,
\]

where \( G^+(z) \) and \( G^-(z) \) are the standard generating fields of \( W_{p/2-3} \), see below.

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2. BERSHADSKY-POLYAKOV ALGEBRAS AT EXCEPTIONAL LEVELS.

Let $W_k$ denote the Bershadsky-Polyakov (vertex) algebra $W_3^{(2)}$ at level $k \neq -3$, which is the vertex algebra freely generated by the fields $J(z), G^\pm(z), T(z)$ with the following OPE’s:

$$J(z)J(w) \sim \frac{2k+3}{3(z-w)^2}, \quad G^\pm(z)G^\pm(w) \sim 0,$$

$$J(z)G^\pm(w) \sim \pm \frac{1}{z-w} G^\pm(w),$$

$$T(z)T(w) \sim -\frac{(2k+3)(3k+1)}{2(k+3)(z-w)^2} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w} \partial T(w),$$

$$T(z)G^\pm(w) \sim \frac{3}{2(z-w)^2}G^\pm(w) + \frac{1}{z-w} \partial G^\pm(w),$$

$$T(z)J(w) \sim \frac{1}{(z-w)^2}J(w) + \frac{1}{z-w} \partial J(w),$$

$$G^+(z)G^- (w) \sim \frac{(k+1)(2k+3)}{(z-w)^3} + \frac{3(k+1)}{(z-w)^2}J(w) + \frac{1}{z-w} \left(3 : J(w)^2 : \frac{3(k+1)}{2} \partial J(w) - (k+3)T(w)\right).$$

As in introduction we denote by $W_k$ the unique simple quotient of $W_k$.

**Theorem 2.1** ([3]). Let $k$, $p$ be as in Main Theorem. Then $W_k$ is lisse, or equivalently, $C_2$-cofinite.

Set

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = T(z) + \frac{1}{2} \partial J(z).$$

This defines a conformal vector of $W_k$ with central charge

$$c(k) = -\frac{4(k+1)(2k+3)}{k+3} = -\frac{4(p-4)(p-3)}{p},$$

which gives $J$, $G^+$, $G^-$ conformal weights 1, 1, and 2, respectively. Hence $W_k$ is $\mathbb{Z}_{\geq 0}$-graded with respect to the Hamiltonian $L_0$. We expand the corresponding fields accordingly:

$$J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}, \quad G^+(z) = \sum_{n \in \mathbb{Z}} G^+_n z^{-n-1}, \quad G^-(z) = \sum_{n \in \mathbb{Z}} G^-_n z^{-n-2}.$$

We have

$$[J_m, J_n] = \frac{2k+3}{3} m \delta_{m+n,0}, \quad [J_m, G_n] = G_{m+n}, \quad [J_m, F_n] = -F_{m+n},$$

$$[L_m, J_n] = -n J_{m+n} - \frac{(2k+3)(m+1)m}{6} \delta_{m+n,0},$$

$$[L_m, G^+_n] = -n G^+_{m+n}, \quad [L_m, G^-_n] = (m-n)G^-_{m+n},$$

$$[G^+_m, G^-_n] = 3(J^2)_{m+n} + (3(k+1)m - (2k+3)(m+n+1)) J_{m+n} - (k+3) L_{m+n} + \frac{(k+1)(2k+3)m(m+1)}{2} \delta_{m+n,0},$$
Proposition 2.3. Suppose that $G$ obtained $W$ such that $\sum$ It is clear that $L$.

For $(\xi, \chi) \in \mathbb{C}^2$, let $L(\xi, \chi)$ be the irreducible representation of $\mathcal{W}^k$ generated by the vector $(\xi, \chi)$ such that

\[ J_0(\xi, \chi) = \xi(\xi, \chi), \quad J_n(\xi, \chi) = 0 \quad \text{for} \quad n > 0, \]
\[ L_0(\xi, \chi) = \chi(\xi, \chi), \quad L_n(\xi, \chi) = 0 \quad \text{for} \quad n > 0, \]
\[ G_n(\xi, \chi) = 0 \quad \text{for} \quad n \geq 0, \quad G_n^+(\xi, \chi) = 0 \quad \text{for} \quad n \geq 1. \]

By Theorem 2.1, any simple $\mathcal{W}_k$-module is of the form $L(\xi, \chi)$ with some $\xi$ and $\chi$. (It is important that the lisse condition is defined independent of the choice of a conformal vector.)

For a $\mathcal{W}^k$-module $M$ set

\[ M_{a,d} = \{ m \in M; J_0m = am, \quad L_0m = dm \}. \]

It is clear that $L(\xi, \chi) = \bigoplus_{(a,d) \in \mathbb{C}^2, \quad d \in \mathbb{Z} + 2g \geq 0} L(\xi, \chi)_{a,d}, \quad \text{dim} \quad L(\xi, \chi)_{\xi, \chi} = 1$. Let

\[ L(\xi, \chi)_{\text{top}} = \{ v \in L(\xi, \chi); \quad L_0v = \chi v \} = \bigoplus_{a} L(\xi, \chi)_{a, \chi}. \]

By definition $L(\xi, \chi)_{\text{top}}$ is spanned by the vectors $(G_0^i)^i|\xi, \chi\rangle$ with $i \geq 0$.

Following [11] set

\[ g(\xi, \chi) = -(3\xi^2 - (2k + 3)\xi - (k + 3)\chi), \]

so that $G_0^{-}G_0^{+}|\xi, \chi\rangle = g(\xi, \chi)|\xi, \chi\rangle$. We have

\[ G_0^{-} (G_0^+)^i|\xi, \chi\rangle = ih_i(\xi, \chi)(G_0^+)^{-i-1}|\xi, \chi\rangle, \]

where

\[ h_i(\xi, \chi) = \frac{1}{i} \left( g(\xi, \chi) + g(\xi + 1, \chi) + \cdots + g(\xi + i - 1, \chi) \right) \]

\[ = -i^2 + ki - 3\xi i + 3i - 3\xi^2 - k + 2k\xi + 6\xi + k\chi + 3\chi - 2. \]

Hence we have the following assertion.

**Proposition 2.2.** If the space $L(\xi, \chi)_{\text{top}}$ is $n$-dimensional, then $h_n(\xi, \chi) = 0$.

Define

\[ \Delta(-J, z) = z^{-J_0} \exp \left( \sum_{k=1}^{\infty} (-1)^{k+1} \frac{-J_k}{kz^k} \right), \]

and set

\[ \sum_{n \in \mathbb{Z}} \psi(a(n)) z^{-n - 1} = Y(\Delta(-J, z) a, z) \]

for $a \in \mathcal{W}^k$. For any $\mathcal{W}^k$-module $M$, we can define on $M$ a new $\mathcal{W}^k$-module structure by twisting the action of $\mathcal{W}^k$ as $a(n) \mapsto \psi(a(n))$ ([11]). We denote by $\psi(M)$ thus obtained $\mathcal{W}^k$-module from $M$.

**Proposition 2.3.** Suppose that $\dim L(\xi, \chi)_{\text{top}} = i$. Then

\[ \psi(L(\xi, \chi)) \cong L(\xi + i - 1 - \frac{2k + 3}{3}, \chi - (\xi - i + 1) + \frac{2k + 3}{3}). \]
Proof. The assertion follows from the fact that 
\[
\psi(J_n) = J_n - \frac{2k+3}{3} \delta_{n,0}, \quad \psi(L_n) = L_n - J_n + \frac{2k+3}{3}, \\
\psi(G_n^+) = G_{n-1}^+, \quad \psi(G_n^-) = G_{n+1}^-.
\]
\[\square\]

By solving the equation 
\[h_i(\xi, \chi) = h_j(\xi + i - 1 - \frac{2k+3}{3}, \chi - (\xi - i + 1) + \frac{2k+3}{3})\]
we obtain the following assertion.

**Proposition 2.4.** Suppose that \(\dim L(\xi, \chi)_{\text{top}} = i\) and \(\dim \psi(L(\xi, \chi))_{\text{top}} = j\).
Then 
\[
\xi = \xi_{i,j} \overset{\text{def}}{=} \frac{1}{3}(-2i - j + 2k + 6), \\
\chi = \chi_{i,j} \overset{\text{def}}{=} \frac{j^2 + ji - 2i - 6j - 2jk + 3k + 6}{3(k+3)}.
\]

**Proposition 2.5.** Let \(k, p\) be as in Main Theorem. Then \((G_{-1}^+)_{p-2}1\) belongs to the maximal ideal of \(W^k\).

**Proof.** Since \(\xi_{1,p-2} = \chi_{1,p-2} = 0\), the correspondence \(1 \mapsto (\xi_{1,p-2}, \chi_{1,p-2})\) gives an isomorphism \(W_k \cong L(\xi_{1,p-2}, \chi_{1,p-2})\).
Because 
\[h_{p-2}(\xi_{1,p-2} - (2k+3)/2, \chi_{1,p-2} + (2k+3)/3) = 0,\]
from Proposition 2.3 it follows that \(\psi(W_k)_{\text{top}}\) is at most \(p - 2\)-dimensional. Hence 
\((G_{-1}^+)_{p-2}1 = 0.\]

**Remark 2.6.** One can show that in fact \((G_{-1}^+)_{p-2}\) generates the maximal ideal of \(W^k\). However we do not need this fact.

**Proposition 2.7.** Let \(k, p\) be as in Main Theorem. Then any simple \(W_k\)-module is isomorphic to \(L(\xi_{i,j}, \chi_{i,j})\) for some \((i, j)\) such that \(1 \leq i \leq p - 2, 1 \leq j \leq p - i - 1\).

**Proof.** Let \(L(\xi, \chi)\) be a simple \(W_k\)-module. As : \(G^+(z)^{p-2} := 0\) on \(L(\xi, \chi)\) by Proposition 2.3 \(L(\xi, \chi)_{\text{top}}\) is at most \((p - 2)\)-dimensional. Since \(\psi(L(\xi, \chi))\) is also a \(W_k\)-module we have \((\xi, \chi) = (\xi_{i,j}, \chi_{i,j})\) for some \(1 \leq i, j \leq p - 2\). Because 
\[\psi(\psi(L(\xi_{i,j}, \chi_{i,j})))\] is also a \(W_k\)-module it follows that \(\xi_{i,j} + i - 1 - \frac{2k+3}{3} = \frac{j^2 + ji - 2i - 6j - 2jk + 3k + 6}{3(k+3)} \leq -2j - 1 + 2k + 6 = \frac{2j - 1}{3}.\) Hence \(j \leq p - i - 1.\]

The simple \(W_k\)-modules \(L(\xi_{i,j}, \chi_{i,j})\) with \(1 \leq i \leq p - 2, 1 \leq j \leq p - i - 1\), are mutually non-isomorphic since their highest weights are distinct.

### 3. Proof of Main Theorem

Let \(k, p\) be as in Main Theorem.

Let \(\mathfrak{g} = sl_3\) as in introduction, \(\mathfrak{h} \subset \mathfrak{g}\) be the Cartan subalgebra of \(\mathfrak{g}\) consisting of diagonal matrices. Set 
\[
h_i = E_{i,i} - E_{i+1,i+1}, \quad h_\theta = h_1 + h_2, \quad e_i = e_{\alpha_i} = E_{i,i+1}, \quad f_i = f_{\alpha_i} = E_{i+1,i}, \quad f_i \neq f_i, \quad f_i = f_i, \quad \text{for } i = 1, 2, \quad e_\theta = E_{1,3}, \quad f_\theta = E_{3,1}, \quad \text{where } E_{i,j} \text{ is the matrix element. We equip } \mathfrak{g} \text{ the invariant form } (x|y) = \text{tr}(xy). \quad \text{Set } \Lambda_1 = (2h_1 + h_2)/3, \quad \Lambda_1 = (h_1 + 2h_2)/3, \text{ so that } (\Lambda_1|h_\theta) = \delta_{i,j}.\]
Let \( \hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D \) be the (non-twisted) affine Kac-Moody algebra associated with \( \mathfrak{g} \), where \( K \) is the central element and \( D \) is the degree operator. Let \( \hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D \subset \hat{\mathfrak{g}} \) the standard Cartan subalgebra, \( \hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta \) the dual of \( \mathfrak{h} \), where \( \Lambda_0 \) and \( \delta \) are elements dual to \( K \) and \( D \), respectively.

The vector \( \psi \) is the corresponding Dynkin grading: \( \psi = \{ u \in \mathfrak{g} ; [h_\theta, u] = 2ju \} \). Denote by \( H^{\hat{\mathfrak{g}}}_{\hat{f}_0} (? ) \) the BRST cohomology of the generalized quantized Drinfeld-Sokolov reduction associated with \( ( \mathfrak{g}, \hat{f}_0 ) \) and the Dynkin grading. We have \( [7, 9] \) the vertex algebra isomorphism

\[
V_k \cong H^{\hat{\mathfrak{g}}}_{\hat{f}_0} (V^k (\mathfrak{g})),
\]

which is given by the following assignment:

\[
J^u (z) \mapsto J^{-\hat{\lambda} + \hat{\lambda}_2} (z) : \Phi_1 (z) \Phi_2 (z) :,
\]

\[
G^\alpha (z) \mapsto J^{h_\alpha (z)} = J^{k_1 (z)} \Phi_2 (z) : + : \Phi_1 (z) \Phi_2 (z)^2 : -(k + 1) \partial \Phi_2 (z),
\]

\[
G^\alpha (z) \mapsto - J^{h_2 (z)} \Phi_1 (z) : - : \Phi_1 (z)^2 \Phi_2 (z) : -(k + 1) \partial \Phi_1 (z),
\]

Here

\[
J^u (z) = u(z) - \sum_{\alpha, \gamma \in \{ \alpha_1, \alpha_2, \theta \}} c_{\alpha, \gamma}^f : \psi_\alpha^* (z) \psi_\gamma (z) :.
\]

For \( u \in \mathfrak{g} \), \( c_{\alpha_1, \alpha_2}^f \) is the structure constant, \( \psi_\alpha (z) \), \( \psi_\alpha^* (z) \) with \( \alpha \in \{ \alpha_1, \alpha_2, \theta \} \) are fermionic ghosts satisfying

\[
\psi_\alpha (z) \psi_\beta^* (w) \sim \frac{\delta_{\alpha, \beta}}{z - w}, \quad \psi_\alpha (z) \psi_\beta (w) \sim \psi_\alpha^* (z) \psi_\beta^* (w) \sim 0,
\]

\( \Phi_1 (z) \), \( \Phi_2 (z) \) are bosonic ghosts satisfying

\[
\Phi_1 (z) \Phi_2 (w) \sim \frac{1}{z - w}, \quad \Phi_1 (z) \Phi_2 (w) \sim 0,
\]

and the BRST differential is the zero mode of the field

\[
Q (z) = \sum_{\alpha \in \{ \alpha_1, \alpha_2, \theta \}} e_{\alpha} (z) \psi_\alpha^* (w) : \psi_{\alpha_1} (z) \psi_{\alpha_2} (z) \psi_\theta (z) : + \Phi_1 (z) \psi_{\alpha_1} (z) + \Phi_2 (z) \psi_{\alpha_2} (z) + \psi_\theta (z).
\]

Let \( O_k \) be the category of \( \hat{\mathfrak{g}} \) at level \( k \), \( L_\lambda \) the irreducible representation of \( \hat{\mathfrak{g}} \) with highest weight \( \lambda \). Denote by \( W_k \).Mod the category of \( W_k \)-modules.

**Theorem 3.1** \((1)\).

(i) The functor \( H^{\hat{\mathfrak{g}}}_{\hat{f}_0} (\hat{\mathfrak{g}}) \) \( \rightarrow \) \( W_k \).Mod, \( \hat{M} \rightarrow H^{\hat{\mathfrak{g}}}_{\hat{f}_0} (\hat{\mathfrak{g}}) \), is exact.

(ii) For \( \lambda \in \hat{\mathfrak{h}}^* \) we have \( H^{\hat{\mathfrak{g}}}_{\hat{f}_0} (L_\lambda) = 0 \) if and only if \( \lambda (\alpha_0^\vee) \in \{ 0, 1, 2, 3, \ldots \} \). Otherwise \( H^{\hat{\mathfrak{g}}}_{\hat{f}_0} (L_\lambda) \) is irreducible.

Let \( Adm_k \) be the set of admissible weights \((8)\) of \( \hat{\mathfrak{g}} \) of level \( k \), and put

\[
Adm^k_+ = \{ \lambda \in Adm_k ; \hat{\lambda} \text{ is an integral dominant weight of } \mathfrak{g} \},
\]

where \( \hat{\mathfrak{h}}^* \ni \lambda \mapsto \hat{\lambda} \in \mathfrak{h}^* \) is the restriction. Then

\[
Adm^k_+ = \{ \hat{\mu} + k \Lambda_0 ; \mu \in \hat{P}^{p-3} \}.
\]
where $\mathcal{P}_{p-3}$ is the set of integral dominant weights of $\mathfrak{g}$ of level $p - 3$. Explicitly, we have
\[
\text{Adm}_+^k = \{\lambda_{i,j}; 1 \leq i \leq p - 2, \ 1 \leq j \leq p - i - 1\},
\]
where
\[
\lambda_{i,j} = (i - 1)\bar{\Lambda}_1 + (p - i - j - 1)\bar{\Lambda}_2 + k\Lambda_0.
\]
Note that
\[
(\xi_{i,j} = (\lambda_{i,j} - \bar{\Lambda}_1 + \bar{\Lambda}_2), \ \chi_{i,j} = \frac{(\lambda_{i,j}|\lambda_{i,j} + 2\bar{\rho})}{2(k + 3)} - (\lambda_{i,j}|\bar{\Lambda}_2),
\]
where $\bar{\rho} = \bar{\Lambda}_1 + \bar{\Lambda}_2$.

Recall the following result of Malikov and Frenkel [12].

**Theorem 3.2** ([12] Corollary 5.2.2). For $\lambda \in \text{Adm}_+^k$, $L_\lambda$ is a module over $L_{k\Lambda_0}$.

**Proposition 3.3.** For $\lambda_{i,j} \in \text{Adm}_+^k$, $H_{f_0}^{\mathfrak{g}+0}(L_{\lambda_{i,j}})$ is a simple $\mathcal{W}_k$-module isomorphic to $L(\xi_{i,j}, \chi_{i,j})$.

**Proof.** By Theorem 3.1 we have $\mathcal{W}_k \cong H_{f_0}^{\mathfrak{g}+0}(L_{k\Lambda_0})$. Hence by the functoriality of $H_{f_0}^{\mathfrak{g}+0}(?)$, Theorem 3.2 immediately gives that $H_{f_0}^{\mathfrak{g}+0}(L_{\lambda_{i,j}})$ is a module over $\mathcal{W}_k$. By Theorem 3.1 $H_{f_0}^{\mathfrak{g}+0}(L_{\lambda_{i,j}})$ is (nonzero and) irreducible. Let $v$ be the image of the highest weight vector of $L_{\lambda_{i,j}}$ in $H_{f_0}^{\mathfrak{g}+0}(L_{\lambda_{i,j}})$. By (2) and the fact [9] that the image of $L(z)$ in $\mathcal{W}_k$ is cohomologous to
\[
L_\phi(z) + L_{\partial \chi}(z) + L_\Phi(z) + \partial J^{\lambda_2}(z),
\]
where $L_\phi(z)$ is the Sugawara operator of $\mathfrak{g}$, $L_{\partial \chi}(z) = -\sum_{\alpha=\alpha_1, \alpha_2, \phi} :\phi_\alpha(z)\partial\phi_\alpha^*(z)$, $L_\Phi(z) = \frac{1}{2} \left( :\Phi_2(z)\partial\Phi_1(z) : - \partial\Phi_1(z)\Phi_2(z) \right)$, it is straightforward to check that the assignment $\left(\xi_{i,j}, \chi_{i,j}\right) \mapsto v$ gives a $\mathcal{W}_k$-module homomorphism. By the irreducibility, this must be an isomorphism.

By Propositions 2.7 and 3.3 the set $\left\{H_{f_0}^{\mathfrak{g}+0}(L_\lambda); \lambda \in \text{Adm}_+^k \right\}$ gives the complete set of isomorphism classes of simple $\mathcal{W}_k$-modules. Therefore Main Theorem now follows immediately from the following important result of Gorelik and Kac [5].

**Theorem 3.4** ([5] Corollary 8.8.9). For any $\lambda, \mu \in \text{Adm}_+^k$, we have
\[
\text{Ext}^1_{\mathcal{W}_k-\text{Mod}}(H_{f_0}^{\mathfrak{g}+0}(L_\lambda), H_{f_0}^{\mathfrak{g}+0}(L_\mu)) = 0.
\]

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