ABSTRACT

The large $N_c$ limit of Yang-Mills gauge theory is the dynamics of a closed gluonic chain, but this fact does not obviate the inherently strong coupling nature of the dynamical problem. However, we suggest that a single link in such a chain might be reasonably described in the quasi-perturbative language of gluons and their interactions. To implement this idea, we use the MIT bag to model the physics of a nearest neighbor bond.
The sum of (planar) diagrams which survives ‘t Hooft’s large $N_c$ limit of $SU(N_c)$ gauge theory can be interpreted as the quantum dynamics of a closed chain of gluons. The limit establishes a cyclic ordering of the gluons in a color singlet state, and only nearest neighbors on each cycle interact. As long as one stays with the continuum theory, there are no further simplifications possible in the asymptotically free situation: dimensional transmutation eliminates the dimensionless coupling, trading it for a scale, which sets the mass gap if there is one. In that case there are no free parameters in the theory, and its single particle spectrum is knowable only in the context of a complete solution; it is an all or nothing affair. The case of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is an exception because that theory is superconformal and hence scale invariant. Then the ‘t Hooft coupling $N_c g^2$ is a free parameter, which can be varied in the hope of further simplification. This is the setting for the dramatic developments of the past year surrounding Maldacena’s proposal that the large coupling limit is equivalent to supergravity on a five dimensional Anti de Sitter space-time.

A quantitative understanding of the single particle mass spectrum at $N_c \to \infty$ in a non-scale-invariant gauge theory, such as QCD, remains an apparently formidable problem, although there are suggestions and conjectures that some form of the Maldacena conjecture might yield a solution. Alternatively, one can introduce a cutoff of some type, thereby temporarily adding a parameter and providing the theory with a flexibility for further simplification, albeit at the cost of straying from the continuum limit. The existing proposals for the QCD/5 dimensional string theory connection a la Maldacena effectively do this. The large $N_c$ limit of lattice gauge theory is a more conventional version of this approach. Yet another is our early work with a light-front (infinite momentum frame) description, in which the $P^+$ carried by each gluon is discretized. In this latter work the large coupling limit in the presence of this cutoff leads to a string interpretation, although the physical nature of the resulting string is highly sensitive to the details of the cutoff procedure.

In this paper we put aside attempts at a detailed quantitative solution of large $N_c$ gauge theory, and instead aim at an understanding of the qualitative features of the gluon chain dynamics which will play a crucial role in a more complete treatment. The overriding question, of course, is whether the gluon interactions are sufficiently attractive to form a bond between each nearest neighbor pair in the chain. After that there is the issue of the fate of the gluon spin. A potentially disastrous outcome would be a ferromagnetic tendency for the spins to align yielding low mass glueballs with enormous spin. The spin-spin interaction between neighboring gluons should have an anti-ferromagnetic character.

With regard to the first point, it is certain that the perturbative, renormalization group improved, Coulomb force $N_c \alpha_s(R) / 2R^2 \sim 3\pi / (11R^2 \ln R\Lambda)$ as $R \to 0$ is attractive between neighbors due to the color structure imposed by the large $N_c$ limit. However, to bond a pair of massless gluons, the coefficient of $1/R^2$ must be of order 1 (in order to balance the gradient of the kinetic energy of order $1/R$), which goes beyond the validity of perturbation theory. As $R$ increases from small values, this coefficient grows, and it is plausible that at some critical $R_c$ the attraction is strong enough to bond, although this speculation is out of perturbative control. However, if bonding of a neighboring pair is established, a chain of many gluons can then be built from similar bonds forming between each nearest neighbor pair. An appealing feature of this scheme is that it does not require extrapolating the basic perturbative gluon dynamics of any given pair too much beyond $R_c$ where $\alpha_s = O(1)$.

We are close here to the philosophy of the MIT bag model of low mass hadrons. In principle, the complete description of a light single glueball state should involve all scales, and there would be no a priori reason to expect the bare gluon interactions derived from the classical
Lagrangian to have much relevance. However the phenomenological success of the MIT Bag model indicates that this gloomy expectation is too pessimistic. In that model the large distance effects on a given single hadron state are lumped into a boundary condition on the fields which are confined to a cavity (for simplicity, usually but not necessarily taken spherical) with a state dependent size $R$. The fields within the cavity are then treated perturbatively, and the color magnetic part of the interactions yields a qualitatively correct pattern of hyperfine splittings within a multiplet with a fixed number of partons; for instance, this model works very well for the 3 quark baryonic states. The actual measured size of the splittings unfortunately requires the coupling to be uncomfortably large: $N_c \alpha_s / \pi \approx 2$. Nonetheless the idea that confinement effects are flavor and spin independent and that the spin dependent forces are well described by basic tree level interactions seems a reasonable working hypothesis for our qualitative analysis.

We therefore examine the magnetic interaction between two neighboring gluons on a glueball chain. Since it will be important to compare the size of the gluon exchange contribution to the quartic contribution, we confine the two gluons to a spherical cavity. This provides an infrared cutoff in space, and also takes rough account of the interaction of the two gluons with the environment of the rest of the chain. The first step in this analysis is to find the modes of noninteracting gluons in the cavity. This is a standard textbook problem (see for example chapter 16 of [18]). One simply solves Maxwell’s equations with time dependence $e^{-i \omega t}$.

\[ \nabla \times \mathbf{E} = i \omega \mathbf{B} \quad \nabla \times \mathbf{B} = -i \omega \mathbf{E} \quad \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0. \]  

Then

\[ (-\nabla^2 - \omega^2)\mathbf{E} = (-\nabla^2 - \omega^2)\mathbf{B} = 0, \]

the solutions of which are, of course, $j_l(\omega r) Y_{lm}(\theta, \phi)$. To solve the divergence conditions, it is convenient to divide the modes into (TE) and (TM) modes:

\[
\begin{align*}
\mathbf{E}_{lm}^{TE} &= K j_l(\omega r) L Y_{lm} \\
\mathbf{B}_{lm}^{TE} &= \frac{1}{i \omega} \nabla \times \mathbf{E}_{lm}^{TE} \\
\mathbf{B}_{lm}^{TM} &= K j_l(\omega r) L Y_{lm} \\
\mathbf{E}_{lm}^{TM} &= \frac{i}{\omega} \nabla \times \mathbf{B}_{lm}^{TM}
\end{align*}
\]

where we have taken only the regular solutions at $r = 0$.

The frequencies $\omega$ are determined by the color confining boundary conditions of a perfect magnetic conductor: $\mathbf{E}_{\text{norm}} = \mathbf{B}_{\text{tang}} = 0$. The lowest non-zero solution for $\omega$ occurs for the (TE) case at $l = 1$, with the boundary condition $\tilde{z}_0 j'_1(\tilde{z}_0) = -j_1(\tilde{z}_0)$ the lowest solution of which is $\omega_0 R \equiv \tilde{z}_0 \approx 2.7437$. If the cavity were an electric conductor, the boundary conditions would have the roles of electric and magnetic fields reversed, but we stick to the color confining conditions in this article. Finally, we note that we may take the vector potential of each mode to be simply $\mathbf{E}/i \omega$ in which case it automatically satisfies the Coulomb gauge condition $\nabla \cdot A = 0$. It is clear from the construction of the modes that the angular momentum carried by each mode is just $l$.

As already mentioned the lowest frequency mode is the (TE) case with $l = 1$:

\[
\mathbf{A}_m^{0} = \frac{K}{i \omega_0 j_1(\omega_0 r)} L Y_{1m} = \frac{K}{i \omega_0 j_1(\omega_0 r)} \frac{3}{4 \pi} \hat{r} \times \mathbf{e}_m
\]  

* The elements of the following calculations were drawn from unpublished bag model calculations I did in the mid seventies for the lightest glueball states with spin 0 and 2. Some of my results were cited in [16]. Similar calculations were independently done in [17]. We repeat some of that early work here because the new context requires adaptations which are most easily explained within a complete self-contained account.
where the polarization vectors are $\epsilon_1 = \epsilon^*_1 = i(1, i, 0)/\sqrt{2}$ and $\epsilon_0 = (0, 0, -i)$, and $K$ is determined by the normalization condition

$$1 = \int d^3 x A^*_n \cdot A_m = 2 \frac{K^2}{\omega_0^2} \int_0^R r^2 \, dr \, j^2_1(\omega_0 r) = \frac{K^2}{\omega_0^2} R^3 \left(j^2_0(z_0) + j^2_0(z_0) - 3j_0(z_0)j_1(z_0)/z_0\right). \quad (5)$$

From the above considerations we know that these modes carry angular momentum $j = 1$. We would like to put two gluons in this lowest mode and then study the spin-spin interaction induced by the QCD interactions. At zeroth order the total energy of the system is $2\omega_0 = 2z_0/R$. This energy level is 9-fold degenerate with $j = 2, 1, 0$. We are next interested in how the interactions resolve this degeneracy.

To find out, we set up cavity perturbation theory, in which the zeroth order fields are precisely the modes we have just discussed. We therefore expand the vector potential in a complete set of normal modes:

$$A(x, t) = \sum_n \frac{1}{\sqrt{2\omega_n}}[a_n x_n(x)e^{-i\omega_n t} + a^+_n x_n^*(x)e^{i\omega_n t}] \quad (6)$$

where $\omega_n$ is the frequency for the $n$th mode, and the $x_n$ are just the vector potentials of the normal modes, normalized according to

$$\int d^3 x x_n(x) \cdot x^*_m(x) = \delta_{nm}. \quad (7)$$

They also satisfy the completeness relation

$$\sum_n x^k_n(x) x^l_n(y) = (\delta_{kl} - \frac{\nabla_k \nabla_l}{\nabla^2})\delta(x - y), \quad (8)$$

where the delta function includes a transverse projector as required for Coulomb gauge. The gluon propagator has the representation

$$i\langle 0|T[A_k(x)\alpha^\beta_A(t)y^\beta_A]\rangle|0\rangle = \delta\alpha\delta\gamma \int \frac{d\omega}{2\pi} e^{-i\omega(x^0 - y^0)} \sum_n \left[\frac{\alpha^k_n(x)\alpha^{l*}_n(y)}{2\omega_n(\omega_n - \omega - i\epsilon)} + \frac{\alpha^{k*}_n(x)\alpha^l_n(y)}{2\omega_n(\omega_n + \omega - i\epsilon)}\right] \quad (9)$$

where we made use of the fact that $\sum_{\omega_n}$ fixed $\alpha^k_n(x)\alpha^{l*}_n(y)$ can be shown to be real.

We are interested in the spin-spin interaction which has its origin in the magnetic part of the Hamiltonian,

$$H_{mag} = \int d^3 x \frac{1}{4} Tr F^2_{kl} = \int d^3 x \left\{ \frac{1}{4} \left[ T(\partial_k A_l - \partial_l A_k) \right]^2 + 2ig Tr(\partial_k A_l - \partial_l A_k)[A_k, A_l] - g^2 Tr[A_k, A_l]^2 \right\} = \int d^3 x \left\{ \frac{1}{2} T(\partial_k A_l)^2 + ig Tr\partial_k A_l[A_k, A_l] - \frac{g^2}{2} Tr A_k A_l[A_k, A_l]\right\}, \quad (10)$$
where in the last line we have committed to Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. The lowest order splittings are $O(g^2)$, but of course the cubic terms will only enter at this order in second order perturbation theory. We first evaluate the contribution of the quartic terms for which first order perturbation theory suffices: we simply must evaluate the matrix elements of the quartic terms on the zeroth order states and diagonalize the resulting matrix.

We choose the color structure of the zeroth order states corresponding to a single link in a glueball chain:

$$|m_1, m_2\rangle^\beta_\alpha = \frac{1}{\sqrt{N_c}} (a_{m_1}^\dagger a_{m_2}^\dagger)^\beta_\alpha |0\rangle. \quad (11)$$

Note that this state is in the adjoint representation of $SU(N_c)$, as is appropriate for a single link in a gluonic chain. Contrast this with a typical bag calculation which is restricted to color singlets only. The operators in this state each create a gluon in the lowest cavity mode. We have suppressed all labels except spin $m_{1,2} = \pm 1, 0$, and we recall that they are matrices in color space, $\alpha, \beta \in \mathbb{I}$ being color matrix indices. The norm of this state is unity in the limit $N_c \to \infty$ by virtue of the factor $1/\sqrt{N_c}$. Next we evaluate the matrix element in the limit $N_c \to \infty$:

$$\langle m_1', m_2' | \Delta_{\text{quartic}} | m_1, m_2 \rangle = \frac{g^2 N_c}{2} \frac{K^4}{\omega_0^4} \int_0^R n^2 dr j_1^2(\omega_0 r) \left[ (\epsilon_{m_1} \times \epsilon_{m_1}'^*) \cdot (\epsilon_{m_2} \times \epsilon_{m_2}') + (\epsilon_{m_1} \times \epsilon_{m_2}) \cdot (\epsilon_{m_1}'^* \times \epsilon_{m_2}') \right]. \quad (12)$$

The two terms in square brackets are commuting $9 \times 9$ spin matrices. The first has eigenvalues $-1, 1, 2$, where a spin 2 quintet belongs to $-1$, a spin 1 triplet belongs to 1, and a spin 0 singlet belongs to 2. The second matrix has respective eigenvalues 0, 2, 0, so it only serves to raise the triplet two units yielding the pattern $-1, 3, 2$ for respectively spin 2, 1, and 0. This inverted level structure means that the quartic interaction yields ferromagnetic spin-spin couplings. We next turn to the contribution of the cubic terms for which we must use second order perturbation theory.

To do second order perturbation theory we expand the interaction picture evolution operator to second order

$$T e^{-i \int_0^T dt' H_I(t')} = I - i \int_0^T dt' H_I(t') - \frac{1}{2} \int_0^T dt' \int_0^T dt'' T[H_I(t')H_I(t'')], \quad (13)$$

and consider only the contribution of the cubic terms of $H_I$ in the last term. To extract the second order level shift we take matrix elements between the zeroth order states and isolate the coefficient of $-i T$ for $T \to \infty$. Since we are stopping at order $g^2$, terms with higher powers of $T$ will not appear, so this procedure gives the energy shift unambiguously. We will also exploit the simplifications of the limit $N_c \to \infty$. Four of the six factors of $\mathbf{A}$ in the last term contract against operators in the states, leaving the last contraction to produce a factor of the gluon propagator

$$D_{kl}(\mathbf{x}, y, x^0 - y^0) \delta_\alpha^\delta_\gamma^\delta_\gamma \equiv i(0) T[A_k(x)^\beta_\alpha A_l(y)^\delta_\gamma] |0\rangle. \quad (14)$$
which satisfies:

\[(\partial_0^2 - \nabla^2)D_{kl} = \delta_{kl}^\text{tr}(x - y).\] (16)

The cubic term in \(H_{\text{mag}}\) can be written

\[H_{\text{mag,cubic}} = ig \int d^3x \text{Tr} \partial_k A_l[A_k, A_l].\] (17)

Inserting this into the second order term in the evolution operator, and contracting out a pair of gluon fields in all possible ways leads to

\[-\frac{1}{2} \int_0^T dt' \int_0^T dt'' T[H_1(t')H_1(t'')] \rightarrow -ig^2 \frac{1}{2} \int_0^T d^4x \int_0^T d^4y D_{kl}(x,y) \text{Tr} T[J_k(x)J_l(y)],\] (18)

where the current operator \(J_k(x)\) is found to be

\[J_k(x) = A_l \partial_k A_l + [A_k, \nabla \cdot A] + 2 \nabla A_k \cdot A - 2 A \cdot \nabla A_k.\] (19)

A contraction which involves \(\partial_k A_l\) requires an integration by parts, but then the surface term is proportional to \([n_k A_k, A_l]\) which is proportional to \([A_{\text{norm}}, A_{\text{tang}}]\) and so vanishes for both electric \((A_{\text{tang}} = 0)\) and magnetic \((A_{\text{norm}} = 0)\) conductor boundary conditions.

When (18) is sandwiched between two states of the same energy, a factor of \(T\) arises as \(T \to \infty\) because then the integrand is only a function of \(x^0 - y^0\), but \(x^0, y^0\) are integrated independently over the large interval \(T\). Thus for \(E_f = E_i\) we have for the coefficient of \(-iT\) as \(T \to \infty\),

\[\langle f | \Delta E | i \rangle = g^2 \frac{1}{2} \int_{-\infty}^{\infty} dx^0 \int d^3x d^3y D_{kl}(x, y)|_{y^0=0} \langle f | \text{Tr} T[J_k(x)J_l(y)|_{y^0=0}] | i \rangle.\] (20)

Taking \(|i\rangle, |f\rangle\) to be of the form (11), we find that in the large \(N_c\) limit there are only two essentially different contraction patterns that survive. Note that the equivalence of \(k, x\) to \(l, y\) means that we can insist that the first creation operator in \(|i\rangle\), say, contracts with one of the two fields in \(J(y)\) provided we multiply by 2. After that contraction, which can be taken in two distinct ways, the remaining contractions are all uniquely determined when \(N_c \to \infty\). There is an annihilation graph in which both operators in \(|i\rangle\) contract against the fields in \(J(y)\). And there is an exchange graph in which each current contracts with one operator in \(|i\rangle\) and one in \(|f\rangle\).

We first evaluate the exchange graph. Since the two gluons in our chosen states have the same mode frequency, the currents coupling to the propagator are time independent so the \(x^0\) integral can be immediately done:

\[\int dx^0 D_{kl}(x,y)|_{y^0=0} = \sum_n \frac{1}{2\omega_n^2} \left[\alpha_n^k(x)\alpha_n^{l*}(y) + \alpha_n^k(x)\alpha_n^{l}(y)\right] \equiv G_{kl}(x, y).\] (21)

Clearly \(G\) satisfies the static Green function equation

\[-\nabla^2 G_{kl}(x, y) = \delta_{kl}^\text{tr}(x - y).\] (22)

The matrix element reduces to

\[\langle f | \Delta E | i \rangle_{\text{exchange}} = N_c g^2 \delta_n^{\alpha'} \delta_{\beta'} \int d^3x d^3y G_{kl}(x, y) \mathcal{J}_k^2(x) \mathcal{J}_l^1(y) = N_c g^2 \delta_n^{\alpha'} \delta_{\beta'} \int d^3x \mathcal{J}_k^2(x) A_k(x)\] (23)
Here $J$ is the matrix element, stripped of color factors, of $J$ between one gluon states:

$$
\mathcal{J}_1^1(y) = \frac{1}{2\omega_0} (\alpha_{m_1,r} \delta_{\alpha_{m_1,r}} + \alpha_{m_1,t} \nabla \cdot \alpha_{m_1,t}^* + 2\nabla \alpha_{m_1,t} \cdot \alpha_{m_1,t}^* - 2\alpha_{m_1} \cdot \nabla \alpha_{m_1,t}^*)
$$

$$
\mathcal{J}_k^2(x) = \frac{1}{2\omega_0} (\alpha_{m_2,r} \delta_{\alpha_{m_2,r}} + \alpha_{m_2}^* \nabla \cdot \alpha_{m_2}^* - \alpha_{m_2} \cdot \nabla \alpha_{m_2}^* + 2\nabla \alpha_{m_2}^* \cdot \alpha_{m_2}^* - 2\alpha_{m_2}^* \cdot \nabla \alpha_{m_2})
$$

(24)

Notice that $\mathcal{J}_k^2$ is the same as $-\mathcal{J}_1^1$ with the substitution $m_1, m_1' \rightarrow m_2, m_2'$.

We do not need a closed form expression for $G_{kl}$. Instead we can work with $A_k(x) \equiv \int d^2 y G_{kl} \mathcal{J}_1^1$, which satisfies

$$
-\nabla^2 A_k = \mathcal{J}_1^1 = -\frac{9K^2}{4\pi \omega_0} \frac{j_1(\omega_0 r)^2}{2\omega_0 r^2} \text{[r} \times (\epsilon_{m_1} \times \epsilon_{m_1}^*)]_k,
$$

(25)

together with the magnetic conductor boundary conditions $A_{norm} = 0$ and $(\nabla \times A)_{tang} = 0$. The r.h.s. doesn’t require the transverse projector because $\nabla \cdot \mathcal{J}^1 = 0$. This equation is easily solved by the ansatz $A = A(r) \times (\epsilon_{m_1} \times \epsilon_{m_1}^*)$, whence $A(r)$ satisfies:

$$
A'' + \frac{4}{r} A' = \frac{9K^2}{4\pi \omega_0} \frac{j_1(\omega_0 r)^2}{2\omega_0 r^2}.
$$

(26)

This last equation can be directly integrated to obtain

$$
A(r) = \frac{3K^2}{8\pi \omega_0^3} \left[ -\frac{1}{r^3} \int_0^r r^2 dr' j_1(\omega_0 r')^2 - \int_0^R \frac{dr'}{r} j_1(\omega_0 r')^2 - \frac{1}{2R^3} \int_0^R r^2 dr' j_1(\omega_0 r')^2 \right],
$$

(27)

where the integration constant is fixed by the boundary condition $RA'(R) + 2A(R) = 0$, which assures the vanishing of the tangential magnetic field. In terms of these quantities the level shift becomes

$$
\langle f|\Delta E|i\rangle_{\text{exchange}} = N_c g^2 \delta_{\alpha} \delta_{\beta'} \frac{3K^2}{\omega_0^3} \int_0^R r^2 dr A(r) j_1(\omega_0 r)^2 (\epsilon_{m_1} \times \epsilon_{m_1}^*) \cdot (\epsilon_{m_2} \times \epsilon_{m_2}^*) = N_c g^2 \delta_{\alpha} \delta_{\beta'} \frac{9K^4}{8\pi \omega_0^6} \int_0^r \frac{dr}{r} j_1(\omega_0 r)^2 (\epsilon_{m_1} \times \epsilon_{m_1}^*) \cdot (\epsilon_{m_2} \times \epsilon_{m_2}^*)
$$

$$
\left[ -2 \int_0^r r^2 dr' j_1(\omega_0 r')^2 - \frac{r^3}{2R^3} \int_0^R r^2 dr' j_1(\omega_0 r')^2 \right]
$$

(28)

To simplify this integral we note that

$$
\chi(z) \equiv \int_0^z z' z' j_1(z')^2 = \frac{z^2}{2} [z j_1(z)^2 + z j_0(z)^2 - 3 j_0(z) j_1(z)].
$$

Putting $z = z_0$ gives the integral required to determine the normalization constant $2(K^2/\omega_0^3)\chi(z_0) = \omega_0^3$. Numerically $\chi(z_0) \approx 1.1385$. Stripping off the color factors, we have the level shift matrix for the exchange graph:

$$
\langle m_1', m_2'|\Delta_{\text{exchange}}|m_1, m_2 \rangle = N_c g^2 \frac{9\omega_0}{16\pi} \chi(z_0)^2 \left[ -\int_0^{z_0} \frac{dz}{z} j_1(z)^2 \chi(z) - \frac{\chi(z_0)^2}{4z_0^3} \right] (\epsilon_{m_1} \times \epsilon_{m_1}^*) \cdot (\epsilon_{m_2} \times \epsilon_{m_2}^*)
$$

$$
\approx -0.4958 \frac{N_c g^2 \omega_0}{16\pi} \delta_{\alpha} \delta_{\beta'} (\epsilon_{m_1} \times \epsilon_{m_1}^*) \cdot (\epsilon_{m_2} \times \epsilon_{m_2}^*).
$$

(29)
For comparison, we put in the numbers for the quartic contribution:

\[ \langle m_1', m_2' | \Delta_{\text{quartic}} | m_1, m_2 \rangle \approx 0.1129 \frac{N_c g^2 \omega_0}{16\pi} [((\epsilon_{m_1} \times \epsilon_{m_1}') \cdot (\epsilon_{m_2} \times \epsilon_{m_2}')) + (\epsilon_{m_1} \times \epsilon_{m_2}) \cdot (\epsilon_{m_1}' \times \epsilon_{m_2}')] \quad (30) \]

Combining the quartic and exchange contributions gives the level pattern \((0.3829, -0.1571, -0.7658)\) in units of \(N_c g^2 \omega_0/16\pi\), restoring an anti-ferromagnetic coupling.

It remains to evaluate the annihilation graph. In this case the currents carry the time dependence \(e^{i\omega t}\), so the propagator is no longer the static Green function. For this graph the currents turn out to be

\[ J^1_k = -\frac{9K^2}{4\pi \omega_0^2} \frac{j_1(\omega_0 r)^2}{2\omega_0 r^2} [\mathbf{r} \times (\epsilon_{m_1} \times \epsilon_{m_2})]_k \quad J^2_k = e^{2i\omega t} \frac{9K^2}{4\pi \omega_0^2} \frac{j_1(\omega_0 r)^2}{2\omega_0 r^2} [\mathbf{r} \times (\epsilon_{m_1}' \times \epsilon_{m_2}')]_k, \quad (31) \]

and

\[ A_k(x) \equiv \int \frac{dy}{d}\int dt e^{2i\omega t} D_{kl}(x, t; y) J^1_k(y) \]

satisfies the inhomogeneous Helmholtz equation

\[ (-\nabla^2 - \omega^2) A_l = J^1_l = -\frac{9K^2}{4\pi \omega_0^2} \frac{j_1(\omega_0 r)^2}{2\omega_0 r^2} [\mathbf{r} \times (\epsilon_{m_1} \times \epsilon_{m_1}')]_l, \quad (32) \]

with \(\omega = 2\omega_0\), but it is clearer to write the solution for general \(\omega\). Again with the ansatz \(A_l = A(r)[\mathbf{r} \times (\epsilon_{m_1} \times \epsilon_{m_2})]_l\), this becomes

\[ A'' + \frac{4}{r} A' + \omega^2 A = \frac{9K^2}{4\pi \omega_0^2} \frac{j_1(\omega_0 r)^2}{2\omega_0 r^2}, \quad (33) \]

with the boundary condition remaining \(R A'(R) + 2A(R) = 0\). Putting these into the formula for the contribution to the energy shift gives

\[ \langle f | \Delta E | i \rangle_{\text{annih}} = N_c g^2 \delta^\alpha_1 \delta^\beta_2 \frac{3K^2}{\omega_0} \int^R_0 r' dr' j_1(\omega_0 r') (\epsilon_{m_1} \times \epsilon_{m_2}) \cdot (\epsilon_{m_1}' \times \epsilon_{m_2}'). \quad (34) \]

We see immediately that this graph only contributes to the shift of the triplet spin 1 state. This is, of course, not surprising because it describes a direct mixing between two gluon and one gluon states and the one gluon state is pure spin 1.

The solution for \(A\) with the correct boundary conditions is easily found to be, for general \(\omega\),

\[ A(r) = \frac{9K^2}{4\pi \omega_0^2} \frac{\omega}{2\omega_0} \int^R_0 r' dr' j_1(\omega_0 r') \left[ y_1(\omega r_>) - \frac{\omega R y_0(\omega R) - y_1(\omega R)}{\omega R j_0(\omega R) - j_1(\omega R)} j_1(\omega_0 r') \right] j_1(\omega_0 r')^2, \quad (35) \]

where we use the standard notation \(r_<(r_>)\) for the lesser(greater) of the pair \(r, \ r'\). We want \(\omega = 2\omega_0\), but with \(\omega\) general we should recover the previous case for \(\omega \to 0\), a useful check on numerical work. Stripping off the color factors, the annihilation graph contribution to the level shift becomes

\[ \langle f | \Delta_{\text{annih}} | i \rangle = \frac{N_c g^2 \omega_0}{16\pi} \frac{27\eta}{\chi(z_0)} \int^z_0 zdz j_1(z)^2 \left[ y_1(\eta z) - \frac{\eta y_0(\eta z) - y_1(\eta z)}{\eta y_0(\eta z) - j_1(\eta z)} j_1(\eta z) \right] \]

\[ + \int^z_0 z' dz' j_1(\eta z') j_1(z')^2 (\epsilon_{m_1} \times \epsilon_{m_2}) \cdot (\epsilon_{m_1}' \times \epsilon_{m_2}'). \quad (36) \]
where we have defined $\eta \equiv \omega / \omega_0 \rightarrow 2$ for this graph. For this value of $\eta$, the numerical evaluation of the above expression gives
\[
\langle f | \Delta_{\text{annih}} | i \rangle \approx 0.1430 \frac{N_c g^2 \omega_0}{16 \pi} (\epsilon_{m_1} \times \epsilon_{m_2}) \cdot (\epsilon^*_{m_1} \times \epsilon^*_{m_2}).
\] (37)

Finally putting all contributions together we arrive at the total level shift matrix
\[
\langle m'_1, m'_2 | \Delta_{\text{total}} | m_1, m_2 \rangle \approx \frac{N_c g^2 \omega_0}{16 \pi} \left[ -0.3829 (\epsilon_{m_1} \times \epsilon^*_{m'_1}) \cdot (\epsilon_{m_2} \times \epsilon^*_{m'_2}) + 0.2559 (\epsilon_{m_1} \times \epsilon_{m_2}) \cdot (\epsilon^*_{m'_1} \times \epsilon^*_{m'_2}) \right],
\] (38)

which translates to the level pattern $(0.3829, 0.1289, -0.7658)$ in units of $N_c g^2 \omega_0 / 16 \pi$ for spins $2, 1, 0$ respectively.

The upshot of our calculations is that the perturbative interactions between nearest neighbors on the gluonic chain described by ’t Hooft’s large $N_c$ limit have the correct sign and spin dependence to provide a satisfactory pattern of single glueball states. But of course perturbation theory is only valid for weak coupling (at short distances), and a near neighbor bond of massless gluons requires this coupling to be at least of order 1. As the coupling increases at larger distances, perturbation theory will break down, but it is perhaps not too much to hope that the qualitative pattern of interactions is not drastically changed. The anti-ferromagnetic character of the spin-spin interactions is very reassuring. Given that the bonds do form, even a relatively weak residual anti-ferromagnetic interaction should be enough to guarantee that the lowest excitations of a long chain have low spin.

It is noteworthy that this pattern would not hold if the only interaction were due to the $A^4$ term: gluon exchange is essential. Thus, for example, the fishnet diagrams, based on a quartic coupling and analyzed in the large coupling light-front studies of [11], cannot contain the essential physics of large $N_c$ QCD. Those diagrams were singled out by setting up a cutoff model in which the strong coupling limit “froze” the $P^+$ distribution to be uniform amongst all the gluons in the chain. In fact, this frozen $P^+$ distribution led to a string picture of the gluonic chain with the degrees of freedom of a critical string, whereas QCD must be subcritical ($4 < 26$). To bring into a strong coupling expansion the exchange effects necessary for anti-ferromagnetic interactions, a more sophisticated cutoff model which allows fluctuations in the $P^+$ distribution at large coupling must be devised. It is from these fluctuations that we can expect the Liouville world sheet field, necessary for subcritical string theory, to emerge. As described in [4] the Liouville field should also be the “5th” dimension in any alleged QCD/5d String connection.

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