Solving \textbf{UNIQUE-SAT} in a Modal Quantum Theory

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Abstract

In recent work, Benjamin Schumacher and Michael D. Westmoreland investigate a version of quantum mechanics which they call \textit{modal quantum theory}. This theory is obtained by instantiating the mathematical framework of Hilbert spaces with a finite field instead of the field of complex numbers. This instantiation collapses much of the structure of actual quantum mechanics but retains several of its distinguishing characteristics including the notions of superposition, interference, and entanglement. Furthermore, modal quantum theory excludes local hidden variable models, has a no-cloning theorem, and can express natural counterparts of quantum information protocols such as superdense coding and teleportation.

We show that the problem of \textbf{UNIQUE-SAT} — which decides whether a given Boolean formula is unsatisfiable or has exactly one satisfying assignment — is deterministically solvable in any modal quantum theory \textit{in constant time}. The solution exploits the lack of orthogonality in modal quantum theories and is not directly applicable to actual quantum theory.

1 Modal Quantum Theory

In their recent work, Schumacher and Westmoreland \cite{2} argue that much of the structure of traditional quantum mechanics is maintained in the presence of finite fields. In particular, they establish that the quantum theory based on the finite field of booleans retains the following characteristics of quantum mechanics: the notions of superposition, interference, entanglement, and mixed states of quantum systems; the time evolution of quantum systems using invertible linear operators; the complementarity of incompatible observables; the exclusion of local hidden variable theories and the impossibility of cloning quantum states; and the presence of natural counterparts of quantum information protocols such as superdense coding and teleportation.

\textbf{Fields.} A field is an algebraic structure with notions of addition and multiplication that satisfy the usual axioms. The rationals, reals, complex numbers, and quaternions form fields that are infinite. There are also finite fields that satisfy the same set of axioms. Finite fields are necessarily “cyclic.”

The simplest field is the field of booleans $\mathbb{F}_2$ consisting of two scalars \{false, true\}. The elements false and true are associated with the probabilities of quantum events, with false interpreted as \textit{definitely no} and true interpreted as \textit{possibly yes}.\footnote{Everything works if we switch the interpretation with false interpreted as \textit{possibly no} and true as \textit{definitely yes}.} The field $\mathbb{F}_2$ comes with an addition operation $\lor$ (which in this case must be exclusive-or) and a multiplication operation $\land$ (which in this case must be conjunction). In particular we have:

\[
\begin{array}{ccc}
\text{false} \lor \text{false} &=& \text{false} \\
\text{false} \lor \text{true} &=& \text{true} \\
\text{true} \lor \text{false} &=& \text{true} \\
\text{true} \lor \text{true} &=& \text{false}
\end{array}
\]

\[
\begin{array}{ccc}
\text{false} \land \text{false} &=& \text{false} \\
\text{false} \land \text{true} &=& \text{false} \\
\text{true} \land \text{false} &=& \text{false} \\
\text{true} \land \text{true} &=& \text{true}
\end{array}
\]
The definitions are intuitively consistent with the interpretation of scalars as probabilities for quantum events except that it appears strange to have \texttt{true} \lor \texttt{true} be defined as \texttt{false}, i.e., to have a twice-possible event become impossible. This results from the cyclic property intrinsic to finite fields requiring the existence of an inverse to \lor. This inverse must be \texttt{true} itself which means that \texttt{true} essentially plays both the roles of “possible with phase 1” and “possible with phase -1” and that the two occurrences cancel each other in a superposition.

**Vector Spaces.** Consider the simple case of a 1-qubit system with bases \(|0\rangle\) and \(|1\rangle\). The Hilbert space framework allows us to construct an infinite number of states for the qubit all of the form \(\alpha|0\rangle + \beta|1\rangle\) with \(\alpha\) and \(\beta\) elements of the underlying field of complex numbers and with the side condition that \(|\alpha|^2 + |\beta|^2 = 1\). Moving to a finite field immediately limits the set of possible states as the coefficients \(\alpha\) and \(\beta\) are now drawn from a finite set. In the field \(\mathbb{F}_2\), there are exactly three valid states for the qubit: \texttt{false}|0\rangle + \texttt{true}|1\rangle (which is equivalent to \(|1\rangle\)), \texttt{true}|0\rangle + \texttt{false}|1\rangle (which is equivalent to \(|0\rangle\)), and \texttt{true}|0\rangle + \texttt{true}|1\rangle (which we write as \(|+\rangle\)). The fourth possibility is the zero vector which is not an allowed quantum state. In a larger field with three scalars, there would be eight possible states for the qubit which intuitively suggests that one must “pay” for the amount of desired superpositions: the larger the finite field, the more states are present with the full Bloch sphere seemingly appearing at the “limit.”

Interestingly, we can easily check that the three possible vectors for a 1-qubit state are linearly dependent with any pair of vectors expressing the third as a superposition:

\[
\begin{align*}
|0\rangle + |1\rangle &= |+\rangle \\
|0\rangle + |+\rangle &= |1\rangle \\
|1\rangle + |+\rangle &= |0\rangle
\end{align*}
\]

In other words, other than the standard basis consisting of \(|0\rangle\) and \(|1\rangle\), there are just two other possible bases for this vector space, \{\(|1\rangle, |+\rangle\}\} and \{\(|+\rangle, |0\rangle\}\}.

**Inner Products.** A Hilbert space comes equipped with an inner product \((v_1 | v_2)\) which is an operation that associates each pair of vectors with a complex scalar value that quantifies the “closeness” of the two vectors. The inner product induces a norm \(\sqrt{(v | v)}\) that can be thought of as the length of vector \(v\). In a finite field, we can still define an operation \((v_1 | v_2)\) which, following our interpretation of the scalars, would need to return \texttt{false} if the vectors are definitely not the same and \texttt{true} if the vectors are possibly the same. This operation however does not yield an inner product, as the definition of inner products requires that the field has characteristic 0. This is not the case for the field \(\mathbb{F}_2\) (nor for any finite field for that matter) as the sum of positive elements must eventually “wrap around.” In other words, if we choose to instantiate the mathematical framework of Hilbert spaces with a finite field, we must therefore drop the requirement for inner products and content ourselves with a plain vector space.

**Invertible Linear Maps.** In actual quantum computing, the dynamic evolution of quantum states is described by unitary maps which preserve inner products. As modal quantum theory lacks inner products, the dynamic evolution of quantum states is described by any invertible linear map, i.e., by any linear map that is guaranteed never to produce the zero vector.

As an example, there are 16 linear (not necessarily invertible) functions in the space of 1-qubit functions. Out of these, six are permutations on the three 1-qubit vectors; the remaining ten maps all map one of the vectors to the zero vector which makes them non-invertible. On one hand, this space is quite impoverished compared to the full set of 1-qubit linear maps in the Hilbert space. In particular, even some of the elementary unitary maps such as the Hadamard matrix are not expressible in that space. On the other hand, the space includes non-unitary maps that are not allowed in actual quantum computing. Of particular interest are the following two maps:

\[
\begin{align*}
s |0\rangle &= |+\rangle \\
s |1\rangle &= |1\rangle \\
s |+\rangle &= |0\rangle
\end{align*}
\]

\[
\begin{align*}
s^\dagger |0\rangle &= |0\rangle \\
s^\dagger |1\rangle &= |+\rangle \\
s^\dagger |+\rangle &= |1\rangle
\end{align*}
\]
The space of 1-qubit maps also includes the identity map which we refer to as $X_0$ below, and the negation map which we refer to as $X_1$ below.

**Measurement.** Measurement in arbitrary bases is complicated but measurement in the standard basis is fairly simple. In a 1-qubit system, we have:

- $\text{measure } |0\rangle$ deterministically produces $|0\rangle$;
- $\text{measure } |1\rangle$ deterministically produces $|1\rangle$;
- $\text{measure } |+\rangle$ non-deterministically produces $|0\rangle$ or $|1\rangle$.

There is no probability distribution associated with the non-deterministic choice between $|0\rangle$ and $|1\rangle$ in the last case.

**Entanglement and Superdense Coding.** Despite the restriction to finite fields and the drastic reduction in the state space of qubits and their maps, the theory built on the field of booleans has a definite quantum character: it can, for example, express quantum protocols such as superdense coding [2].

## 2 UNIQUE-SAT

The problem of UNIQUE-SAT is the problem of deciding whether a given Boolean formula known to have either 0 or 1 satisfying assignment has 0 or 1 assignment. Surprisingly this problem is, in a precise sense [3], just as hard as the general satisfiability problem and hence all problems in the NP complexity class.

## 3 Solving UNIQUE-SAT in the Field of Booleans

We are given a classical function $f : \text{Bool}^n \rightarrow \text{Bool}$ that takes $n$-bits and returns at most one true result. In the following, we use $\vec{x}$ to denote a sequence $x_1, x_2, \ldots, x_n$ of $n$ bits. Given a function $f : \text{Bool}^n \rightarrow \text{Bool}$, we construct the Deutsch quantum black box $U_f$ as follows:

$$U_f |y \rangle |\vec{x}\rangle = |y \lor f(\vec{x}) \rangle |\vec{x}\rangle$$

It is straightforward to verify that $U_f$ is an invertible map, and hence an acceptable map for the evolution of states in a modal quantum theory.

**The Algorithm.** The algorithm is pictorially presented in Figure 1. It consists of the following steps:

1. Initialize an $n+1$ qubit state to $|0\rangle |\overline{0}\rangle$.
2. Apply the map $s$ to each qubit in the second component of the state.
3. Apply the quantum black box version $U_f$ to the entire state.
4. Apply the map $s$ to each qubit in the second component of the state.
5. Apply the map $s^\dagger$ to the first component of the state.
6. Let the first component of the state be $a$. Apply the map $X_a$ to each qubit in the second component of the state.
7. Apply the map $s^\dagger$ to the first component of the state.
8. Measure the resulting state in the standard basis for $n+1$ qubits. If the measurement is $|0\rangle |\overline{0}\rangle$ then the function $f$ is unsatisfiable. If the measurement is anything else then the function $f$ satisfiable.
Correctness (case I). Assume the function $f$ is unsatisfiable:

1. In the first step, the initial state is $|0\rangle |\overline{0}\rangle$.
2. In the second step, the state becomes $|0\rangle |\overline{1}\rangle$.
3. In the third step, the function $U_f$ is the identity and the state remains $|0\rangle |\overline{1}\rangle$.
4. Applying $s$ to each qubit in the second component of the state produces $|0\rangle |\overline{0}\rangle$.
5. Applying $s^\dagger$ to the first component leaves the state unchanged as $|0\rangle |\overline{0}\rangle$.
6. As the first component of the state is 0, applying the map $X_0$ (which is the identity) leaves the state unchanged as $|0\rangle |\overline{0}\rangle$.
7. Applying $s^\dagger$ to the first component leaves the state unchanged as $|0\rangle |\overline{0}\rangle$.
8. Measuring the state will deterministically produce $|0\rangle |\overline{0}\rangle$.

Correctness (case II). Assume the function $f$ is satisfiable at some input $a_1, a_2, \ldots, a_n$ denoted $\overline{\pi}$:

1. In the first step, the initial state is $|0\rangle |\overline{0}\rangle$.
2. In the second step, the state becomes $|0\rangle |\overline{1}\rangle$. We can write this state as $|0\rangle |\overline{1}\rangle + \sum_\pi |0\rangle |\overline{\pi}\rangle$ where the superscript $\pi$ is to remind us that the summation is over all the $2^n$ combinations except $a_1, a_2, \ldots, a_n$.
3. In the third step, applying $U_f$ produces $|1\rangle |\overline{\pi}\rangle + \sum_\pi |0\rangle |\overline{\pi}\rangle$. Because $|0\rangle |\overline{\pi}\rangle + |0\rangle |\overline{\pi}\rangle$ is the zero vector, we can rewrite this state as $|+\rangle |\overline{\pi}\rangle + \sum_\pi |0\rangle |\overline{\pi}\rangle$ where the summation is now over all the vectors, i.e., we can write the state as $|+\rangle |\overline{\pi}\rangle + |0\rangle |\overline{\pi}\rangle$.
4. Applying $s$ to each qubit in the second component produces $|+\rangle |\overline{s(a)}\rangle + |0\rangle |\overline{0}\rangle$.
5. Applying $s^\dagger$ to the first component produces: $|1\rangle |\overline{s(a)}\rangle + |0\rangle |\overline{0}\rangle$.
6. Applying $X_b$ where $b$ is the first component of the state to each qubit in the second component to the state produces $|1\rangle |\overline{\text{not}(s(a))}\rangle + |0\rangle |\overline{0}\rangle$.
7. Applying $s^\dagger$ to the first component produces $|+\rangle |\overline{\text{not}(s(a))}\rangle + |0\rangle |\overline{0}\rangle$.
8. For the measurement of $|+\rangle |\overline{\text{not}(s(a))}\rangle + |0\rangle |\overline{0}\rangle$ to be guaranteed to be never $|0\rangle |\overline{0}\rangle$, we need to verify that $|+\rangle |\overline{\text{not}(s(a))}\rangle$ has one occurrence $|0\rangle |\overline{0}\rangle$. This can be easily proved as follows. Since each $a_i$ is either 0 or 1, then each $s(a_i)$ is either + or 1, and hence each $\text{not}(s(a_i))$ is either + or 0. The result follows since any state with a combination of + and 0 — when expressed in the standard basis — would consist of a superposition containing the state $|0\ldots\rangle$. 

![Modal Quantum Circuit for UNIQUE-SAT in $F_2$](image-url)
4 Conclusion

We have presented an algorithm for solving \texttt{UNIQUE-SAT} in the modal quantum theory based over the finite field of booleans. The existence of this algorithm suggests that modal quantum theories, as defined, collapse too much of the structure of quantum mechanics and that they should be restricted to retain some notion of orthogonality.

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References

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