Difference of facial achromatic numbers between two triangular embeddings of a graph

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Abstract

A facial 3-complete $k$-coloring of a triangulation $G$ on a surface is a vertex $k$-coloring such that every triple of $k$-colors appears on the boundary of some face of $G$. The facial 3-achromatic number $\psi_3(G)$ of $G$ is the maximum integer $k$ such that $G$ has a facial 3-complete $k$-coloring. This notion is an expansion of the complete coloring, that is, a proper vertex coloring of a graph such that every pair of colors appears on the ends of some edge.

For two triangulations $G$ and $G'$ on a surface, $\psi_3(G)$ may not be equal to $\psi_3(G')$ even if $G$ is isomorphic to $G'$ as graphs. Hence, it would be interesting to see how large the difference between $\psi_3(G)$ and $\psi_3(G')$ can be. We shall show that the upper bound for such difference in terms of the genus of the surface.

1 Introduction

In this paper, we consider finite and undirected graph. A graph is called simple if it has no loops and multiple edges. We mainly focus on simple graphs unless we particularly mention it. An embedding of a graph $G$ on a surface $\mathbb{F}$ is a drawing of $G$ on $\mathbb{F}$ with no pair of crossing edges. Technically, we regard an embedding as injective continuous map $f : G \to \mathbb{F}$, where $G$ is regarded as a one-dimensional topological space. We sometime consider that $G$ is already mapped on a surface and denote its image by $G$ itself to simplify the notation, while if we deal with two or more embeddings of $G$ on a surface, we denote them by $f_1(G), f_2(G), \ldots$ to distinguish them.

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The faces of a graph $G$ embedded on a surface $F$ are the connected components of the open set $F - G$. We denote by $V(F)$ the set of vertices in the boundary of a face $F$ of $G$, and by $F(G)$ the set of faces of $G$. A triangulation on a surface $F$ is an embedding of a graph on $F$ so that each face is bounded by a 3-cycle. A graph $G$ is said to have a triangulation on a surface, if $G$ is embeddable on the surface as a triangulation.

A (vertex) $k$-coloring of a graph $G$ is a map $c : V(G) \rightarrow \{1, 2, \ldots, k\}$. A $k$-coloring $c$ of $G$ is proper if $c(u) \neq c(v)$ whenever two vertices $u$ and $v$ are adjacent. For a subset $S \subseteq V(G)$, we denote by $c(S)$ the set of colors of the vertices in $S$.

Colorings of graphs embedded on surfaces with facial constraints have attracted a lot of attention. In particular, facially-constrained colorings of plane graphs were overviewed by Czap and Jendrol’ [2]. Many facially-constrained colorings can be translated into colorings of some kind of hypergraphs, called “face-hypergraphs”. The face-hypergraph $H(G)$ of a graph $G$ embedded on a surface is the hypergraph with vertex-set $V(G)$ and edge-set $\{V(F) : F \in F(G)\}$, whose concept was introduced in [6].

A complete $k$-coloring of a graph $G$ is a proper $k$-coloring such that each pair of $k$-colors appears on at least one edge of $G$. The achromatic number of $G$ is the maximum integer $k$ such that $G$ has a complete $k$-coloring. This notion was introduced by Harary and Hedetniemi [4], and has been extensively studied (see [5] for its survey). Recently, Matsumoto and the second author [7] introduced a new facially-constrained coloring, called the “facial complete coloring”, which is an expansion of the complete coloring. A $k$-coloring, which is not necessarily proper, of a graph $G$ embedded on a surface is facially $t$-complete if for any $t$-element subset $X$ of the $k$ colors, there is a face $F$ of $G$ such that $X \subseteq c(V(F))$. The maximum integer $k$ such that $G$ has a facial $t$-complete $k$-coloring is the facial $t$-achromatic number of $G$, denoted by $\psi_t(G)$. It seems to be natural to consider facial $t$-complete colorings for graphs embedded on a surface so that each face is bounded by a cycle of length $t$.

We should notice that the facial $t$-achromatic number of an embedded graph depends on the embedding of the graph in general. That is, if a graph $G$ has two distinct embeddings $f_1(G)$ and $f_2(G)$ on $F$, then $\psi_t(f_1(G))$ may not be equal to $\psi_t(f_2(G))$. Hence, it would be interesting to see how large the difference between $\psi_t(f_1(G))$ and $\psi_t(f_2(G))$ can be. In this paper, we focus on facial 3-complete colorings of triangulations on a surface from this point of view, and show the upper bound for such difference as follows.

**Theorem 1.** Let $G$ be a graph which has two triangulations $f_1(G)$ and $f_2(G)$ on a surface $F$, and let $g$ be the Euler genus of $F$. If $F$ is orientable, then

$$|\psi_3(f_1(G)) - \psi_3(f_2(G))| \leq \begin{cases} 9g/2 & (g \leq 1) \\ 27g/2 - 27 & \text{(otherwise)} \end{cases}.$$
If \( \mathbb{F} \) is non-orientable, then

\[
|\psi_3(f_1(G)) - \psi_3(f_2(G))| \leq \begin{cases} 
3g & (g = 1) \\
21g - 27 & (\text{otherwise})
\end{cases}
\]

Note that we can easily construct a triangulation on each surface so that its facial 3-achromatic number is an arbitrarily large, while Theorem 1 implies that the difference of the facial 3-achromatic numbers between two triangulations \( f_1(G) \) and \( f_2(G) \) on a given surface, which is obtained from the same graph \( G \), can be bounded by a constant.

On the other hand, the upper bounds in Theorem 1 do not seem to be sharp. Unfortunately, we have no construction of a graph which has two triangulations on a surface whose facial 3-achromatic numbers differ. So one may suspect that \( \psi_1(f_1(G)) = \psi_1(f_2(G)) \) whenever a graph \( G \) has two triangulations \( f_1(G) \) and \( f_2(G) \) on a surface. However, we do not believe that. Actually, we shall show in Section 5, the non-simple graphs having two triangulations on a surface whose facial 3-achromatic numbers differ (the definition of the facial complete coloring can be extended to non-simple graphs naturally). Hence, we hope that there exist such graphs for simple graphs.

We introduce some useful lemmas in Section 3 to prove Theorem 1 in Section 4. Before these sections, we would like to introduce some related results dealing with other facially-constrained colorings. For not only the facial complete coloring but also other facially-constrained colorings, the possibility of such a coloring depends on the embedding in general. We survey some results from this point of view in the next section.

## 2 Related results

A rainbow coloring (or a cyclic coloring) is a coloring of a graph \( G \) embedded on a surface so that each face is rainbow, that is, any two distinct vertices on its boundary have disjoint colors. The minimum integer \( n \) such that \( G \) has a rainbow \( n \)-coloring is the rainbowness of \( G \), denoted by \( \text{rb}(G) \). An antirainbow coloring (or a valid coloring) is a coloring of a graph \( G \) embedded on a surface so that no face is rainbow. The maximum integer \( n \) such that \( G \) has a surjective antirainbow \( n \)-coloring is the antirainbowness of \( G \), denoted by \( \text{arb}(G) \).

Let \( G \) be the graph consisting of \( m \geq 3 \) cycles of length 3 with one common vertex, which has two embeddings \( f_1(G) \) and \( f_2(G) \) on the sphere as shown in Fig. 1. Then \( G \) has \( 2m + 1 \) vertices.

As there is a face incident with all vertices in \( f_1(G) \), we have \( \text{rb}(f_1(G)) = |V(G)| = 2m + 1 \). It is also easy to see that \( \text{rb}(f_2(G)) = 5 \). Hence, the difference between \( \text{rb}(f_1(G)) \) and \( \text{rb}(f_2(G)) \) is \( 2m - 4 \). It implies that the rainbowness of a graph embedded on a surface depends on the embedding. Moreover, such a difference can be arbitrarily large. On the other hand, it is easy to see that \( \text{rb}(G) = \chi(G) \)
for every triangulation on a surface, where $\chi(G)$ is the chromatic number of $G$. This implies that the rainbowness of a triangulation does not depend on the embedding.

Ramamurthi and West [11] observed that for the above two embeddings $f_1(G)$ and $f_2(G)$ of $G$ on the sphere, $\text{arb}(f_1(G)) = m + 1$ and $\text{arb}(f_2(G)) = \lceil 3m/2 \rceil$. Then the difference of these antirainbownesses is $\lceil m/2 \rceil - 1$, and hence the antirainbowness of a graph embedded on a surface also depends on the embedding. Ramamurthi and West [11] conjectured this difference is the maximum difference for two embeddings of a graph on the sphere, that is, for every planar graph $G$ of order $n$, there is no pair of embeddings of $G$ on the sphere whose antirainbownesses differ from at least $\lfloor (n - 2)/4 \rfloor$.

Arocha, Bracho and Neumann-Lara [1] studied the antirainbow 3-colorability of triangulations obtained from complete graphs, which they called the tightness. They proved that the complete graph of order 30 has both of a tight triangulation and an untight one on the same surface. This implies that the antirainbowness of triangulations depends on the embedding. As the generalization of their work, Negami [10] introduced the looseness of a triangulation $G$ on a surface, which corresponds to $\text{arb}(G) + 2$. He proved that for any graph having two triangulations $f_1(G)$ and $f_2(G)$ on a surface $F$ of Euler genus $g$, $|\text{arb}(f_1(G)) - \text{arb}(f_2(G))| \leq 2\lfloor g/2 \rfloor$.

A weak coloring of a graph $G$ embedded on a surface is a coloring of $G$ such that no face is monochromatic, that is, all vertices on its boundary have the same color. Note that a weak coloring of an embedded graph corresponds to a proper coloring of its face-hypergraph. The weak chromatic number of $G$, denoted by $\chi_w(G)$, is the minimum integer $k$ such that $G$ has a weak $k$-coloring. Kündgen and Ramamurthi [6] studied weak colorings of graphs embedded on surfaces from various viewpoints and conjectured that for each positive integer $k$, there is a graph that has two different embeddings on the same surface whose weak chromatic numbers differ by at least $k$. Recently, the first author and Noguchi [3] answered this conjecture affirmatively in two ways. They first constructed two distinct embeddings of a simple graph on a surface such that one of them has a weak 2-coloring but the other has arbitrarily large weak chromatic number. They second showed that there are non-simple graphs $G$ having two triangulations $f_1(G)$ and $f_2(G)$ on a surface with $\chi_w(f_1(G)) = |V(G)|/2$ and $\chi_w(f_2(G)) \leq |V(G)|/3$. 

Fig. 1: Two embeddings of $G$ on the sphere.
3 Cycles in a triangulation

To prove Theorem 1, we give some notations and introduce some lemmas.

Let $G$ be a graph and $H$ be a subgraph of $G$. An edge not in $H$ but with both ends in $H$ is called chord of $H$. A subgraph $H$ of a graph $G$ is induced if $H$ has no chord. An $H$-bridge is a subgraph of $G$ induced by a chord of $H$, or a component of $G - V(H)$ together with all edges joining it to $H$. In an $H$-bridge, a vertex belongs to $V(H)$ is called a vertex of attachment. Note that any two $H$-bridges are edge-disjoint and meet only the common vertices of attachment. (See [8] for more details of $H$-bridges.)

**Lemma 2.** Let $G$ be a triangulation on a surface, and $C_1, C_2, \ldots, C_k$ be vertex-disjoint facial cycles of $G$. If there is no chord in the union $H = C_1 \cup C_2 \cup \cdots \cup C_k$, then there is only one $H$-bridge in $G$.

**Proof.** Let $C = uvw$ be a facial cycle of $G$ bounded by three vertices $u, v$ and $w$. Suppose that $C$ is not contained in $H$. Since $H$ consists of vertex-disjoint cycles and has no chord, $C$ meets at most one cycle of $H$. Suppose that $C$ meets $C_i$ at a vertex, say $u$, and $v, w \notin V(C_i)$ for any $1 \leq i \leq k$. If $v$ and $w$ belong to different $H$-bridges in $G$, then the edge $vw$ joins these $H$-bridges, a contradiction. Hence, $v$ and $w$ belongs to the same $H$-bridge in $G$. It implies that all vertices and edges around $C_i$ belongs to one $H$-bridge in $G$. Suppose that $C$ meets none of $C_1, C_2, \ldots, C_k$. Then it is clear that $u, v$ and $w$ belong to the same $H$-bridge in $G$. Therefore, there is only one $H$-bridge in $G$. \hfill \square

Let $G$ be a graph embedded on a surface $F$. A cycle $C$ of $G$ is contractible if it bounds a disk in $F$, and separating if it separates $F$ into two parts. We say that $C$ is 2-sided if it divides its annular neighbourhood into two parts, and is 1-sided otherwise. Note that a non-separating cycle of $G$ must be non-contractible, and if a separating cycle $C$ of $G$ is not facial then there are at least two $C$-bridges in $G$.

**Lemma 3.** Let $G$ be a graph which has two triangulations $f_1(G)$ and $f_2(G)$ on a surface, and $C$ be a 3-cycle of $G$. If $f_1(C)$ is facial in $f_1(G)$ but $f_2(C)$ is not facial in $f_2(G)$, then $f_2(C)$ is non-contractible in $f_2(G)$.

**Proof.** Suppose to that $f_3(C)$ is contractible in $f_2(G)$. Since $f_2(C)$ is not facial in $f_2(G)$, it separates $f_2(G)$ into two components. On the other hand, since $f_1(C)$ is facial in $f_1(G)$, it follows from Lemma 2 that $G$ has only one $C$-bridge in $G$, a contradiction. \hfill \square

For two disjoint cycles $C_1$ and $C_2$ of a graph embedded on a surface $F$, cut the surface $F$ along them. When one of the component of the resulting surface is an annulus with boundary components $C_1$ and $C_2$, we say that $C_1$ and $C_2$ are homotopic.

We introduce two lemmas about sets of pairwise non-homotopic cycles. The second lemma closely follows from the proof of [8 Proposition 3.7], which corresponds to the first one. However, to keep the paper self-contained, we give its proof.
Lemma 4 (Malnič and Mohar [8]). Let $G$ be a graph embedded on a surface $\mathbb{F}$, and let $g$ be the Euler genus of $\mathbb{F}$. Let $\Gamma$ be a set of pairwise disjoint, non-contractible and pairwise non-homotopic cycles of $G$. If $\mathbb{F}$ is orientable, then

$$|\Gamma| \leq \begin{cases} g/2 & (g \leq 2) \\ 3g/2 - 3 & \text{(otherwise)} \end{cases}.$$ 

If $\mathbb{F}$ is non-orientable, then

$$|\Gamma| \leq \begin{cases} g & (g \leq 1) \\ 3g - 3 & \text{(otherwise)} \end{cases}.$$ 

Lemma 5. Let $G$ be a graph embedded on a non-orientable surface $\mathbb{F}$ of Euler genus $g$. Let $\Gamma_1$ (resp. $\Gamma_2$) be a set of pairwise disjoint, non-contractible and pairwise non-homotopic 1-sided (resp. 2-sided) cycles of $G$. Then $|\Gamma_1| \leq g$ and

$$|\Gamma_2| \leq \begin{cases} 0 & (g = 1) \\ 2g - 3 & \text{(otherwise)} \end{cases}.$$ 

Proof. It is easy to see that this lemma holds for $g \leq 2$. Hence, we may assume that $g \geq 3$. Moreover, we may assume that $\Gamma_1$ is maximal, that is there is no 1-sided cycle in $G$ disjoint from $\Gamma_1$. Cutting $\mathbb{F}$ along the cycles in $\Gamma_1$, we obtain a connected surface, denoted by $\mathbb{F}'$, which has $|\Gamma_1|$ boundary components. Thus, $\chi(\mathbb{F}') \leq 2 - |\Gamma_1|$. Since $\chi(\mathbb{F}') = \chi(\mathbb{F}) = 2 - g$, we have $|\Gamma_1| \leq g$. 

We may also assume that $\Gamma_2$ is maximal, that is, all 2-sided cycles in $G$ disjoint from $\Gamma_2$ is contractible or homotopic to some element of $\Gamma_2$. Cut $\mathbb{F}$ along the cycles in $\Gamma_2$. Then $\mathbb{F}$ is separated into some connected surfaces, denoted by $\mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_k$. Note that they are all compact and with non-empty boundary. We denote by $b(\partial \mathbb{F}_i)$ the number of boundary components of $\mathbb{F}_i$ for $1 \leq i \leq k$. Since each cycle in $\Gamma_2$ gives rise to two boundary components, we have $\sum_{i=1}^k b(\partial \mathbb{F}_i) = 2|\Gamma_2|$. 

Let $\mathbb{F}_1^*, \mathbb{F}_2^*, \ldots, \mathbb{F}_k^*$ be the surfaces obtained from $\mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_k$ by pasting a disk to each boundary component. By the maximality of $\Gamma_2$, $\mathbb{F}_i^*$ is the sphere or the projective plane for $1 \leq i \leq k$. We denote by $n_s$ and $n_p$ the numbers of the spheres and the projective planes among $\mathbb{F}_i^*$’s, respectively. Then we have $n_p \leq g$ and $\sum_{i=1}^k \chi(\mathbb{F}_i^*) = 2n_s + n_p$. 

Now we shall show that if $\mathbb{F}_i^*$ is the sphere, then $b(\partial \mathbb{F}_i) \geq 3$. If $b(\partial \mathbb{F}_i) = 1$, then $\mathbb{F}_i$ is a closed disk, that is, the cycle bounding $\mathbb{F}_i$ is contractible in $\mathbb{F}$, a contradiction. Suppose that $b(\partial \mathbb{F}_i) = 2$. Then $\mathbb{F}_i$ is an annulus. If two cycles of $\Gamma_2$ corresponding to the boundary components $\mathbb{F}_i$ are the same, then $\mathbb{F}$ must be the Klein bottle, a contradiction. Thus, these two cycles are different from each other. However, in this situation, they are homotopic in $\mathbb{F}$, a contradiction. Therefore, we may assume that $b(\partial \mathbb{F}_i) \geq 3$. It implies that $3n_s + n_p \leq 2|\Gamma_2|$. 

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Since \( \chi(\mathcal{F}) \) is equal to the sum of all \( \mathcal{F}_i \)'s, we have
\[
\chi(\mathcal{F}) = \sum_{i=1}^{k} \chi(\mathcal{F}_i) = \sum_{i=1}^{k} \chi(\mathcal{F}_i^*) - \sum_{i=1}^{k} b(\partial \mathcal{F}_i) = 2n_s + n_p - 2|\Gamma_2|.
\]
\[
= \frac{2}{3}(3n_s + n_p - 2|\Gamma_2|) + \frac{1}{3}n_p - \frac{2}{3}|\Gamma_2|
\]
\[
\leq \frac{1}{3}g - \frac{2}{3}|\Gamma_2|.
\]
Since \( \chi(\mathcal{F}) = 2 - g \), we have \( |\Gamma_2| \leq 2g - 3 \). \( \square \)

4. Proof of Theorem 1

Proof of Theorem 1. Suppose that \( \psi_3(f_1(G)) = k \) and \( \psi_3(f_2(G)) < k \). Let \( c : V(G) \to \{1, 2, \ldots, k\} \) be a facial 3-complete \( k \)-coloring of \( f_1(G) \). Then, every triple of \( k \)-colors appears in some face of \( f_1(G) \). On the other hand, some triples do not appear in the faces of \( f_2(G) \). Let \( \mathcal{T} \) be a set of triples in \( k \) colors such that any triple in \( \mathcal{T} \) does not appear in the faces of \( f_2(G) \), and for any pair of triples \( T \) and \( T' \) in \( \mathcal{T} \), \( T \cap T' = \emptyset \). Moreover, we choose \( \mathcal{T} \) so that \( |\mathcal{T}| \) is as large as possible. Let \( T_1, T_2, \ldots, T_m \) be the triples in \( \mathcal{T} \), and so \( |\mathcal{T}| = m \). By the maximality of \( \mathcal{T} \), we can choose \( k - 3m \) colors so that every triple in these colors appear in some face of \( f_2(G) \). It implies that \( f_2(G) \) has a facial 3-complete \( (\max\{3, k - 3m\}) \)-coloring. Then, \( |\psi_3(f_1(G)) - \psi_3(f_2(G))| \leq 3m \).

Let \( \mathcal{C} = \{C_1, C_2, \ldots, C_m\} \) be a set of facial cycles in \( f_1(G) \) such that \( c(V(C_i)) = T_i \) for \( 1 \leq i \leq m \). Since every \( C_i \) is not facial in \( f_2(G) \), it follows from Lemma 3 that every \( C_i \) is non-contractible in \( f_2(G) \).

Claim 6. There are at most three pairwise homotopic cycles of \( \mathcal{C} \) in \( f_2(G) \).

Proof. Suppose that \( C_1, C_2, C_3 \) and \( C_4 \) are pairwise homotopic in \( f_2(G) \), and appear on the annulus bounded by \( C_1 \) and \( C_4 \) in this order. Thus, the union \( C_2 \cup C_4 \) separates \( C_1 \) from \( C_3 \), and hence there are no chords of \( C_1 \cup C_3 \). Similarly, \( C_1 \cup C_3 \) also separates \( C_2 \) from \( C_4 \). It implies that there are at least two \( C_1 \cup C_3 \)-bridges in \( G \). On the other hand, since both of \( C_1 \) and \( C_3 \) are facial in \( f_1(G) \) and \( C_1 \cup C_3 \) has no chord, it follows from Lemma 2 that there is only one \( C_1 \cup C_3 \)-bridge in \( G \), a contradiction. Therefore, there are at most three pairwise homotopic cycles of \( \mathcal{C} \) in \( f_2(G) \). \( \square \)

Now we shall give the upper bound for \( |\mathcal{T}| = m \), which induces the upper bound for \( |\psi_3(f_1(G)) - \psi_3(f_2(G))| \). We first consider the case when the surface \( \overline{\mathcal{F}} \) is homeomorphic to one of the sphere, the projective plane, and the torus. Suppose that \( \overline{\mathcal{F}} \) is the sphere. All cycles in \( G \) is contractible, and hence \( \mathcal{C} = \emptyset \). Actually, it follows Lemma 3 that \( f_1(G) \) and \( f_2(G) \) are essentially equivalent embeddings. (In general, Whitney [13] showed that every 3-connected planar graph has essentially unique embedding in the sphere.) Suppose that \( \overline{\mathcal{F}} \) is the projective plane. There is
no pair of disjoint non-contractible cycles in $f_2(G)$, and hence $m \leq 1$. Suppose that $F$ is the torus. All non-contractible and pairwise disjoint cycles in $G$ are pairwise homotopic. Then, all cycles in $C$ are pairwise homotopic by Lemma 4, and hence it follows from Claim 6 that $m \leq 3$.

Second, suppose that $F$ is an orientable surface of genus at least two. If $m > 9g - 9$, then there are at least four pairwise homotopic cycles in $C$ by Lemma 4, which contradicts Claim 6. Hence, we have $m \leq 9g - 9$. Finally, suppose that $F$ is a non-orientable surface of genus at least two. If $m > 7g - 9$, then there are at least $6g - 8$ 2-sided cycles in $C$, and hence some four of them are pairwise homotopic by Lemma 5, which contradicts Claim 6. Therefore, in any case, the desired inequality holds.

5 Facial complete colorings of non-simple graphs

In this section, we consider graphs which may have multiple edges. We denote by $K_n$ the complete graph of order $n$, and denote by $K_m^n$ the non-simple graph obtained from $K_n$ by replacing each edge with $m$ multiple edges.

The first author [3] constructed two triangulations $f_1(G)$ and $f_2(G)$ obtained from the graph $G = K_{12m-1}^{6m-1}$ on a surface for any positive integer $m$. The weak chromatic numbers of these triangulations differ by at least $2m$, and hence his construction gives an affirmatively answer of K¨undgen and Ramamurthi’s conjecture [6, Conjecture 8.1] (see also Section 2 in this paper). We now show that the facial 3-achromatic numbers of these triangulations also differ.

For details of constructions of $f_1(G)$ and $f_2(G)$, see [3, Section 3]. The face-hypergraph $H(f_1(G))$ of $f_1(G)$ is isomorphic to a complete 3-uniform hypergraph. That is, the triangulation $f_1(G)$ has exactly $\left|V(G)\right|^3/3$ faces and there is a face bounded by each triple of vertices. (Such a triangulation is called complete, whose notion was defined in [6].) Then it is easy to see that $\psi_3(f_1(G)) = |V(G)| = 12m$.

Let $T$ be a triangulation on a surface obtained from $K_{12m}$ (by Ringel’s Map Color Theorem [12], $K_{12m}$ has a triangulation on a surface). The edge-set of $H(f_2(G))$ coincides with that of $H(T)$ by ignoring the multiplicity of the edge-sets. It implies that $\psi_3(f_2(G)) = \psi_3(T)$. Suppose that $T$ is facially 3-complete $k$-colorable. Then, $T$ must have at least $k^3$ faces, and hence we obtain the following inequality:

$$|\mathcal{F}(T)| = 4m(12m - 1) \geq k(k - 1)(k - 2)/6$$
$$288m^2 - 24m \geq (k - 2)^3$$
$$\sqrt[3]{288} \frac{m^2}{3} \geq k - 2$$
$$7m + 2 \geq k.$$  

Then, $\psi_3(f_2(G)) \leq 7m + 2$ (this bound might be loose), and hence we have

$$\psi_3(f_1(G)) - \psi_3(f_2(G)) \geq 5m - 2.$$  

Since $G$ is isomorphic to $K_{12m}^{6m-1}$, both of two triangulations $f_1(G)$ and $f_2(G)$ are embedded on a surface of Euler genus $(m - 1)(m - 2)(2m + 3)/3$. It implies that for
any non-negative integer \( g \), there is a graph having two triangulations on a surface of Euler genus at least \( g \), whose facial 3-achromatic numbers differ from \( \Omega(\sqrt[3]{g}) \).

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