UNI-DIRECTIONAL SYNCHRONIZATION AND FREQUENCY DIFFERENCE LOCKING INDUCED BY A HETEROCLINIC RATCHET

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Abstract
A system of four coupled oscillators that exhibits unusual synchronization phenomena has been analyzed. Existence of a one-way heteroclinic network, called heteroclinic ratchet, gives rise to uni-directional (de)synchronization between certain groups of cells. Moreover, we show that locking in frequency differences occur when a small white noise is added to the dynamics of oscillators.

Key words
Heteroclinic ratchets, synchronization, coupled oscillators, uni-directional desynchronization.

1 Introduction
Phase oscillators are used as approximations for the phase dynamics of coupled limit cycle oscillators in the case of weak coupling [Kuramoto, 1984; Pikovsky et al., 2001; Hoppensteadt and Izhikevich, 1997]. They exhibit synchronization and clustering phenomena [Kuramoto, 1984; Sakaguchi and Kuramoto, 1986], even if coupling function consist of the first harmonic only. If the second harmonic of the coupling function is considered, it is possible to observe switchings between different clusterings as a result of an asymptotically stable robust heteroclinic cycle [Hansel et al., 1993]. It is known that heteroclinic cycles are not structurally stable but they may exist robustly for coupled systems. This is due to the existence of robust invariant subspaces for certain coupling structures that may support robust heteroclinic connections that are saddle-to-sink on the invariant subspaces and form a heteroclinic cycles [Krupa, 1997; Ashwin and Field, 1999; Aguiar et al., 2011]. Existence of robust heteroclinic cycles or more generally heteroclinic networks in a system of three and four globally coupled phase oscillators have been analyzed in [Ashwin et al., 2008] and in [Ashwin et al., 2006], where an extreme sensitivity phenomenon to detuning of natural frequencies has been observed. Namely, oscillators loose synchrony even for very small detuning of natural frequencies. [Karabacak and Ashwin, 2010] have considered the third harmonic of the coupling function and observed one-way heteroclinic networks, which are called heteroclinic ratchets. A heteroclinic ratchet is a heteroclinic network that, for some axis, contains trajectories winding in one direction only. Heteroclinic ratchets give rise to extreme sensitivity to detuning of certain sign. Namely, synchronization of a pair of oscillators is possible only when the natural frequency of a certain oscillator is larger than the other. We call this phenomenon uni-directional (de)synchronization.

In the sequel, we identify a heteroclinic ratchet for a system of four coupled oscillators in Section 2. Although the system is less complicated than the original ratcheting system considered in [Karabacak and Ashwin, 2010], it exhibits more complicated dynamics: uni-directional synchronization between groups of oscillators, explained in Sections 3 and 4.

2 A Model of Four Coupled Oscillators That Supports Heteroclinic Networks
Consider the following well-known model of coupled phase oscillators:

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} c_{ij} g(\theta_i - \theta_j). \quad (1)$$

Here, $\theta_i \in \mathbb{T} = [0, 2\pi)$ denotes the phase of oscillator $i$ and $\omega_i$ is its natural frequency. The connection matrix
\{c_{ij}\} represents the coupling. \(c_{ij} = 1\) if oscillator \(i\) receives an input from oscillator \(j\) and \(c_{ij} = 0\) otherwise.

Since \(g(\cdot)\) is a \(2\pi\)-periodic function, it can be written as a sum of Fourier harmonics: \(g(x) = \sum_{k=1}^{\infty} r_k \sin(kx + \alpha_k)\). Without loss of generality, we may set \(K = N\) and \(r_1 = -1\) by a scaling of time. Let us choose the following coupling function with two harmonics only:

\[
g(x) = -\sin(x + \alpha_1) + r_2 \sin(2x) \quad (2)
\]

The model (1) is used as the approximate phase dynamics of weakly coupled limit cycle oscillators, and the weak coupling gives rise to a \(T^1\) phase-shift symmetry in the phase model (1). Hence, the dynamics of (1) is invariant under the phase shift

\[
(\theta_1, \theta_2, \ldots, \theta_N) \mapsto (\theta_1 + \epsilon, \theta_2 + \epsilon, \ldots, \theta_N + \epsilon)
\]

for any \(\epsilon \in \mathbb{T}\). Below, this symmetry is used to reduce the dynamics to an \((N - 1)\)-dimensional phase difference system.

Let us consider the coupled phase oscillator system (1) with the coupling structure given in Figure 1. This gives rise to the following system:

\[
\begin{align*}
\dot{\theta}_1 &= \omega_1 + g(\theta_1 - \theta_2) + g(\theta_1 - \theta_3) \\
\dot{\theta}_2 &= \omega_2 + g(\theta_2 - \theta_1) + g(\theta_2 - \theta_4) \\
\dot{\theta}_3 &= \omega_3 + g(\theta_3 - \theta_2) + g(0) \\
\dot{\theta}_4 &= \omega_4 + g(\theta_4 - \theta_1) + g(\theta_4 - \theta_2) .
\end{align*}
\quad (3)
\]

Defining phase difference variables as \(\phi_1 := \theta_1 - \theta_2\), \(\phi_2 := \theta_2 - \theta_3\), and \(\phi_3 := \theta_3 - \theta_4\), we obtain the following dynamical system for phase differences:

\[
\begin{align*}
\dot{\phi}_1 &= \omega_{13} + g(\phi_1 + \phi_3 - \phi_2) + g(\phi_1) \\
\dot{\phi}_2 &= \omega_{24} + g(-\phi_1 - \phi_3 + \phi_2) + g(\phi_2) \\
\dot{\phi}_3 &= \omega_{34} + g(\phi_3 - \phi_2) + g(0) \\
\dot{\phi}_4 &= \omega_{14} + g(-\phi_3 - \phi_1) - g(-\phi_2) .
\end{align*}
\quad (4)
\]

\(\omega_{ij}\) denotes the detuning between oscillator \(i\) and oscillator \(j\), namely \(\omega_{ij} = \omega_i - \omega_j\).

We first assume identical oscillators, that is

\[
\omega_1 = \cdots = \omega_4 = \omega \implies \omega_{ij} = 0 \quad \forall i, j .
\quad (5)
\]

Oscillators with different natural frequencies will be considered in Section 3.

\[\text{Figure 1. An asymmetric coupled cell system: this gives the coupled system of form } (1).\]

\[\text{Figure 2. Two balanced colorings of the coupled cell system in Figure 1. These give rise to 2-dimensional invariant subspaces } X_1 \text{ and } X_2 \text{ of the system (1).}\]

\subsection{2.1 Invariant Subspaces}

The assumption that the oscillators are identical makes it possible to use the balanced coloring method [Stewart et al., 2003] to obtain invariant subspaces of the system (1). A coloring of cells in a coupled cell system is called balanced if cells with identical color receives the same number of inputs from cells of any given color. A balanced coloring gives rise to an invariant subspace obtained by assuming that the cells of same color have identical states. The converse of this statement is also true. Namely, for a given coupling structure, a coloring is balanced if the corresponding subspace is invariant under any system having that coupling structure. Therefore, the invariant subspaces obtained by the balanced coloring method are robust under the perturbations that preserve the coupling structure. For an introduction to this theory, see [Golubitsky and Stewart, 2006].

Using the balanced coloring method the invariant subspaces of the coupled cell system given in Figure 1 can be found as in Table 1. Using the above-mentioned phase difference reduction, the corresponding invariant subspace in \(T^1\) for the system (1) are also listed in Table 1. Note that for the system (1), there are only two 2-dimensional invariant subspaces, namely \(X_1\) and \(X_2\). Balanced colorings for these invariant subspaces are given in Figure 2. The invariant subspaces \(X_1\), \(X_2\) and their intersection \(X_3\) can support a robust heteroclinic cycle (see [Ashwin et al., 2011] for robustness criteria of heteroclinic cycles).
2.2 Existence and Stability of a Heteroclinic Ratchet

A heteroclinic ratchet (first defined in [Karabacak and Ashwin, 2010]) is a heteroclinic network that contains a heteroclinic cycle winding in some direction and does not contain another heteroclinic cycle winding in the opposite direction. To be precise, a heteroclinic cycle \( C \subset \mathbb{T}^N \) parametrized by \( x(s) \) \( (x: [0, 1) \to \mathbb{T}^N) \) is winding in some direction if there is a projection map \( P: \mathbb{R}^N \to \mathbb{R} \) such that the parametrization \( \dot{x}(s) \) \( (\dot{x}: [0, 1) \to \mathbb{R}^N) \) of the lifted heteroclinic cycle \( C \subset \mathbb{R}^N \) satisfies

\[
\lim_{s \to 1} P(\dot{x}(s)) - P(\dot{x}(0)) = 2k\pi \quad \text{for some positive integer } k.
\]

A heteroclinic cycle winding in the opposite direction would satisfy the same condition for a negative integer \( k \) (see [Ashwin and Karabacak, 2011] for general properties of heteroclinic ratchets).

As discussed above, the system (4) may have a robust heteroclinic network on the invariant subspaces \( X_1 \) and \( X_2 \). Such a heteroclinic network should be connecting saddles on \( X_3 = X_1 \cap X_2 \). Reducing the equations in (4) to \( X_3 \) and considering identical natural frequencies, we get

\[
\dot{\phi}_3 = g(\phi_3) - g(-\phi_3).
\]

This system is \( \mathbb{Z}_2 \)-equivariant, and therefore it can admit a codimension-1 pitchfork bifurcation of the zero solution under some nondegeneracy conditions. The saddles emanating from this bifurcation are on \( X_3 \) and they are of the form \( p = (0, 0, p_3) \) and \( q = (0, 0, -p_3) \). The value \( p_3 \) can be obtained by solving

\[
g(p_3) - g(-p_3) = 0
\]

as

\[
p_3 = \cos^{-1}\left(\frac{\cos \alpha_1}{2r_2 \cos \alpha_2}\right). \tag{7}
\]

In order to show that there exist heteroclinic connections between \( p \) and \( q \) we use the simulation program XPPAUT [Ermentrout, 2002]. We identify a heteroclinic ratchet for the parameter values

\[
\alpha_1 = -5, \quad r_2 = 0.15 \tag{8}
\]

(see Figure 3). On \( X_1 \), the heteroclinic ratchet contains a non-winding trajectory and a trajectory winding along \( +\phi_1 \) and \( -\phi_3 \) directions (see Figure 3a). On \( X_2 \), it contains a non-winding trajectory and a trajectory winding along \( +\phi_2 \) direction (see Figure 3b). These four connections and the saddles \( p \) and \( q \) form a heteroclinic ratchet (see Figure 3). For the parameters given in (8), \( p \) can be found as \((0, 0, 0.3315)\). Considering the Jakobien of (4) at \( p \), we can find the eigenvalues of the saddle \( p \) as \( \lambda_1^{(p)} = -0.3112 \) and \( \lambda_2^{(p)} = 0.2967 \) and \( \lambda_3^{(p)} = -0.0636 \). Similarly, the eigenvalues of \( q = -p = (0, 0, 0.3315) \) can be found as \( \lambda_1^{(q)} = 0.3130 \) and \( \lambda_2^{(q)} = -0.3276 \) and \( \lambda_3^{(q)} = -0.0636 \). These eigenvalues of \( p \) and \( q \) correspond to the eigenvectors \( v_1 = (1, 0, 0) \in X_1 \), \( v_2 = (0, 1, 0) \in X_2 \) and \( v_3 = (0, 0, 1) \in X_3 \). A heteroclinic cycle is attracting if the saddle quantity, defined as the absolute value of the ratio between the product of the eigenvalues corresponding to the expanding connections and the product of the
chronized if the observed frequency differences can be broken by noise and/or detuning of natural frequencies.\[\Omega_{ij} = \lim_{t \to \infty} \frac{|\theta_i^t - \theta_j^t|}{t}, \tag{9}\]
is equal to zero. Here $\theta_i^t \in \mathbb{R}$ is the lifted phase variable for $\theta_i \in \mathbb{T}$. It is known that coupled oscillators can get frequency synchronized when the distance between their natural frequencies, namely $|\omega_{ij}| := |\omega_i - \omega_j|$ is small enough. If frequency synchronization of oscillators $i$ and $j$ occurs only when a specific one of the oscillators has greater natural frequency, namely for a certain sign of $\omega_{ij}$, we say that synchronization is uni-directional. Uni-directional synchronization phenomenon has been shown to occur for oscillator pairs when an asymptotically stable heteroclinic ratchet exists in the phase space [Karabacak and Ashwin, 2010; Ashwin and Karabacak, 2011].

\[\text{Figure 4. A solution of the system } (4) \text{ converging to the heteroclinic ratchet. Initial states are chosen as } (2, 1, 0.5). \text{ The solution shows the peculiar property for heteroclinic cycles that the residence time near equilibria increases as } t \to \infty, \text{ before } \phi_1 \text{ and } \phi_2 \text{ get locked at zero due to the precision errors.}\]

\[\text{Figure 5. Uni-directional synchronization with respect to } \omega_{13}. \text{ The main graph shows the frequency differences } \Omega_{13}, \Omega_{24} \text{ and } \Omega_{34} \text{ for } (4) \text{ with parameters given in } (8) \text{ as a function of detuning } \omega_{13} \text{ when } \omega_{24} = \omega_{34} = 0. \text{ Oscillators are frequency synchronized when } \omega_{13} \leq 0 \text{ and the synchronization fails for oscillator pairs } (13) \text{ and } (24) \text{ whenever } \omega_{13} > 0. \text{ The insets show time evolution of the phase differences } \phi_i \text{ for a positive value of } \omega_{13}.\]

\[\text{Figure 6. Uni-directional synchronization with respect to } \omega_{24}. \text{ The frequency differences } \Omega_{13}, \Omega_{24} \text{ and } \Omega_{34} \text{ for } (4) \text{ with parameters given in } (8) \text{ are shown as a function of detuning } \omega_{24} \text{ when } \omega_{13} = \omega_{34} = 0. \text{ Oscillators are frequency synchronized when } \omega_{24} \leq 0 \text{ and the synchronization fails for the oscillator pair } (24) \text{ whenever } \omega_{24} > 0.\]
4 Locking in Frequency Differences

Noise induced uni-directional desynchronization of oscillators has been observed in [Karabacak and Ashwin, 2010] for a coupled system admitting a heteroclinic ratchet. Here, we show that existence of a heteroclinic ratchet for the system (4) leads to a locking in frequency differences when a small noise is applied. Figure 8 shows a solution of the system (4) under white noise with amplitude $10^{-12}$. The noisy solution exhibits approximately non-winding trajectories. On the other hand, the number of windings in $\phi_1$ and $-\phi_3$ directions are exactly the same because of the structure of the heteroclinic ratchet in Figure 3.

As a result, the solution gives rise to the following frequency locking between frequency differences:

$$\Omega_{13} = \Omega_{24} = -\Omega_{34}. \quad (10)$$

This is in agreement with the simulation results given in Figure 8. Therefore, the observed oscillator frequencies $\Omega_i := \lim_t |\theta_i^L(t)|/t$ are in the following form:

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{pmatrix} = \begin{pmatrix} \Omega + \delta \\ \Omega \\ \Omega + \delta \\ \Omega + 2\delta \end{pmatrix}, \quad (11)$$

where $\delta$ is a positive number. This type of a result cannot be seen directly from the connection structure of the coupled system, and is a consequence of the particular heteroclinic ratchet that the system admits. Although the noise induces synchronization of oscillators 1 and 3, the balanced coloring method explained in Section 2.1 does not give $\{\theta \in T^4 \mid \theta_1 = \theta_3\}$ as an invariant subspace.

5 Conclusion

We have analyzed a system of four coupled phase oscillators. The existence of an asymptotically stable heteroclinic ratchet gives rise to uni-directional synchronization of certain groups of oscillators and induce a particular locking in the frequency differences of oscillators when small amplitude white noise is introduced to the system. These phenomena also lead to frequency synchronization of some oscillators, that can not be found by using the balanced coloring method, therefore does not correspond to any synchrony subspace.

For future works, the relation between the connection structure and possible synchronization groups can
be studied. Although the synchronization groups can not be inferred from the coupling structure directly, the coupling structure serves to create invariant subspaces on which heteroclinic ratchets can be supported. For this reason, the coupling structure plays an indirect role on the existence of possible synchronization groups. Another direction could be to study bifurcations of heteroclinic ratchets that result in winding periodic orbits on torus. This can explain the effect of small detunings of natural frequencies on the observed frequencies in a complete way.

References
Aguiar, M., Ashwin, P., Dias, A., and Field, M. (2011). Dynamics of coupled cell networks: synchrony, heteroclinic cycles and inflation. *J. Nonlinear Sci.*, 21:271–323.
Ashwin, P., Burylko, O., and Maistrenko, Y. (2008). Bifurcation to heteroclinic cycles and sensitivity in three and four coupled phase oscillators. *Physica D*, 237:454–466.
Ashwin, P., Burylko, O., Maistrenko, Y., and Popovych, O. (2006). Extreme sensitivity to detuning for globally coupled phase oscillators. *Phys. Rev. Lett.*, 96(5):054102.
Ashwin, P. and Field, M. (1999). Heteroclinic networks in coupled cell systems. *Arch. Ration. Mech. Anal.*, 148(2):107–143.
Ashwin, P. and Karabacak, O. (2011). Robust heteroclinic behaviour, synchronization, and ratcheting of coupled oscillators. In *Dynamics, Games and Science II*, pages 125–140. Springer Berlin Heidelberg.
Ashwin, P., Karabacak, O., and Nowotny, T. (2011). Criteria for robustness of heteroclinic cycles in neural microcircuits. *J. Math Neurosci.*, 1(1):1–18.
Ermentrout, G. (2002). *A Guide to XPPAUT for Researchers and Students*. SIAM, Pittsburgh.
Golubitsky, M. and Stewart, I. (2006). Nonlinear dynamics of networks: the groupoid formalism. *Bull. Amer. Math. Soc. (N.S.)*, 43(3):305–364 (electronic).
Hansel, D., Mato, G., and Meunier, C. (1993). Clustering and slow switching in globally coupled phase oscillators. *Phys. Rev. E*, 48(5):3470–3477.
Hoppensteadt, F. and Izhikevich, E. (1997). *Weakly Connected Neural Networks*. Springer-Verlag, New York.
Karabacak, O. and Ashwin, P. (2010). Heteroclinic ratchets in networks of coupled oscillators. *J. Nonlinear Sci.*, 20(1):105–129.
Krupa, M. (1997). Robust heteroclinic cycles. *J. Nonlinear Sci.*, 7(2):129–176.
Kuramoto, Y. (1984). *Chemical Oscillations, Waves and Turbulence*. Springer-Verlag, Berlin.
Melbourne, I. (1989). Intermittency as a codimension-three phenomenon. *J. Dyn. Differ. Equ.*, 1(4):347–367.
Pikovsky, A., Rosenblum, M., and Kurths, J. (2001). *Synchronization: A Universal Concept in Nonlinear Science*. Cambridge University Press, Cambridge.
Sakaguchi, H. and Kuramoto, Y. (1986). A soluble active rotator model showing phase transitions via mutual entrainment. *Prog. Theor. Phys.*, 76(3):576–581.
Stewart, I., Golubitsky, M., and Pivato, M. (2003). Symmetry groupoids and patterns of synchrony in coupled cell networks. *SIAM J. Appl. Dyn. Sys.*, 2(4):609–646.
