EXTENSIONS OF HOMOGENEOUS COORDINATE RINGS TO $A_\infty$-ALGEBRAS

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ABSTRACT. We study $A_\infty$-structures extending the natural algebra structure on the cohomology of $\oplus_{n \in \mathbb{Z}} L^n$, where $L$ is a very ample line bundle on a projective $d$-dimensional variety $X$ such that $H^i(X, L^n) = 0$ for $0 < i < d$ and all $n \in \mathbb{Z}$. We prove that there exists a unique such nontrivial $A_\infty$-structure up to a strict $A_\infty$-isomorphism (i.e., an $A_\infty$-isomorphism with the identity as the first structure map) and rescaling. In the case when $X$ is a curve we also compute the group of strict $A_\infty$-automorphisms of this $A_\infty$-structure.

1. Introduction

Let $X$ be a projective variety over a field $k$, $L$ be a very ample line bundle on $X$. Recall that the graded $k$-algebra $R_L = \oplus_{n \geq 0} H^0(X, L^n)$ is called the homogeneous coordinate ring corresponding to $L$. More generally, one can consider the bigraded $k$-algebra

$$A_L = \oplus_{p,q \in \mathbb{Z}} H^q(X, L^p).$$

We call $A_L$ the extended homogeneous coordinate ring corresponding to $L$.

Since $A_L$ can be represented naturally as the cohomology algebra of some dg-algebra (say, using injective resolutions or Čech cohomology with respect to an affine covering), it is equipped with a family of higher operations called Massey products. A better way of recording this additional structure uses the notion of $A_\infty$-algebra due to Stasheff. Namely, by the theorem of Kadeishvili the product on $A_L$ extends to a canonical (up to $A_\infty$-isomorphism) $A_\infty$-algebra structure with $m_1 = 0$ (see [4] 3.3 and references therein). More precisely, this structure is unique up to a strict $A_\infty$-isomorphism, i.e., an $A_\infty$-isomorphism with the identity map as the first structure map (see section 2.1 for details). Note that the axioms of $A_\infty$-algebra use the cohomological grading on $A_L$ (where $H^q(X, L^p)$ has cohomological degree $q$), and all the operations ($m_n$) have degree zero with respect to the internal grading (where $H^q(X, L^p)$ has internal degree $p$). The natural question is whether it is possible to characterize intrinsically this canonical class of $A_\infty$-structures on $A_L$. This question is partly motivated by the homological mirror symmetry. Namely, in the case when $X$ is a Calabi-Yau manifold, the $A_\infty$-structure on $A_L$ is supposed to be $A_\infty$-equivalent to an appropriate $A_\infty$-algebra arising on a mirror dual symplectic side. An intrinsic characterization of the $A_\infty$-isomorphism class of our $A_\infty$-structure could be helpful in reducing the problem of constructing such an $A_\infty$-equivalence to constructing

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an isomorphism of the usual associative algebras. More generally, it is conceivable that the algebra $A_L$ can appear as cohomology algebra of some other dg-algebras (for example, if there is an equivalence of the derived category of coherent sheaves on $X$ with some other such category), so one might be interested in comparing corresponding $A_\infty$-structures on $A_L$.

Thus, we want to study all $A_\infty$-structures $(m_n)$ on $A_L$ (with respect to the cohomological grading), such that $m_1 = 0$, $m_2$ is the standard double product and all $m_n$ have degree 0 with respect to the internal grading. Let us call such an $A_\infty$-structure on $A_L$ admissible. As we have already mentioned before, there is a canonical strict $A_\infty$-isomorphism class of such structures coming from the realization of $A_L$ as cohomology of the dg-algebra $\oplus_n \mathcal{C}^*(L^n)$ where $\mathcal{C}^*(?)$ denotes the Čech complex with respect to some open affine covering of $X$. By definition, an $A_\infty$-structure belongs to the canonical class if there exists an $A_\infty$-isomorphism from $A_L$ equipped with this $A_\infty$-structure to the above dg-algebra inducing identity on the cohomology. The simplest picture one could imagine would be that all admissible $A_\infty$-structures are strictly $A_\infty$-isomorphic, i.e., that $A_L$ is intrinsically formal. It turns out that this is not the case. However, our main theorem below shows that if the cohomology of the structure sheaf on $X$ is concentrated in degrees 0 and $\dim X$ then for sufficiently ample $L$ the situation is not too much worse.

We will recall the notion of a homotopy between $A_\infty$-morphisms in section 2.1 below. \footnote{All $A_\infty$-morphisms and homotopies between them are assumed to respect the internal grading on $A_L$.} Let us say that an $A_\infty$-structure is nontrivial if it is not $A_\infty$-isomorphic to an $A_\infty$-structure with $m_n = 0$ for $n \neq 2$. By rescaling of an $A_\infty$-structure we mean the change of the products $(m_n)$ to $(\lambda^{n-2}m_n)$ for some constant $\lambda \in k^\ast$. Our main result gives a classification of admissible $A_\infty$-structures on $A_L$ up to a strict $A_\infty$-isomorphism and rescaling (under certain assumptions).

**Theorem 1.1.** Let $L$ be a very ample line bundle on a $d$-dimensional projective variety $X$ such that $H^q(X,L^p) = 0$ for $q \neq 0$, $d$ and all $p \in \mathbb{Z}$. Then

(i) up to a strict $A_\infty$-isomorphism and rescaling there exists a unique non-trivial admissible $A_\infty$-structure on $A_L$; moreover, $A_\infty$-structures on $A_L$ from the canonical strict $A_\infty$-isomorphism class are nontrivial;

(ii) for every pair of strict $A_\infty$-isomorphisms $f, f' : (m_i) \to (n_i)$ between admissible $A_\infty$-structures on $A_L$ there exists a homotopy from $f$ to $f'$.

**Remarks.** 1. One can unify strict $A_\infty$-isomorphisms with rescalings by considering $A_\infty$-isomorphisms $(f_n)$ with the morphism $f_1$ of the form $f_1(a) = \lambda^{\deg(a)}$ for some $\lambda \in k^\ast$ (where $a$ is a homogeneous element of $A_L$). In particular, part (i) of the theorem implies that all non-trivial admissible $A_\infty$-structures on $A_L$ are $A_\infty$-isomorphic.

2. As we will show in section 2.1, part (ii) of the theorem is equivalent to its particular case when $f' = f$. In this case the statement is that every strict $A_\infty$-automorphism of $f$ is homotopic to the identity.

3. If one wants to see more explicitly how a canonical $A_\infty$-structure on $A_L$ looks like, one has to choose one of the natural dg-algebras with cohomology $A_L$ (an obvious algebraic choice is the Čech complex; in the case $k = \mathbb{C}$ one can also use the Dolbeault complex).
choose a projector \( \pi \) from the dg-algebra to some space of representatives for the cohomology such that \( \pi = 1 - dQ - Qd \) for some operator \( Q \), and then apply formulas of [5] for the operations \( m_n \) (they are given by certain sums over trees).

The above theorem is applicable to every line bundle of sufficiently large degree on a curve. In higher dimensions it can be used for every sufficiently ample line bundle on a \( d \)-dimensional projective variety \( X \) such that there exists a dualizing sheaf on \( X \) and \( H^i(X, \mathcal{O}_X) = 0 \) for \( 0 < i < d \). For example, this condition is satisfied for complete intersections in projective spaces. At present we do not know how to extend this theorem to the case when \( \mathcal{O}_X \) has some nontrivial middle cohomology. Note that for a smooth projective variety \( X \) over \( \mathbb{C} \) the natural (up to a strict \( A_\infty \)-isomorphism) \( A_\infty \)-structure on \( H^*(X, \mathcal{O}_X) \) is trivial as follows from the formality theorem of [1]. This suggests that for sufficiently ample line bundle \( L \) one could try to characterize the canonical \( A_\infty \)-structure on \( A_L \) (up to a strict \( A_\infty \)-isomorphism and rescaling) as an admissible \( A_\infty \)-structure whose restriction to \( H^*(X, \mathcal{O}_X) \) is trivial.

In the case when \( X \) is a curve we can also compute the group of strict \( A_\infty \)-automorphisms of an \( A_\infty \)-structure on \( A_L \). As we will explain below 2.1, strict \( A_\infty \)-isomorphisms on \( A_L \) form a group \( HG \), which is a subgroup of automorphisms of the free coalgebra \( \text{Bar}(A_L) \) (preserving both gradings). The dual to the degree zero component of \( \text{Bar}(A_L) \) (with respect to both gradings) can be identified with the completed tensor algebra \( \hat{T}(H^1(X, \mathcal{O}_X)^*) = \prod_{n \geq 0} T^n(H^1(X, \mathcal{O}_X)^*) \). Therefore, we obtain a natural homomorphism from \( HG \) to to the group \( G \) of continuous automorphisms of \( \hat{T}(H^1(X, \mathcal{O}_X)^*) \).

**Theorem 1.2.** Let \( L \) be a very ample line bundle on a projective curve \( X \) such that \( H^1(X, L) = 0 \). Let also \( HG(m) \subset HG \) be the group of strict \( A_\infty \)-automorphisms of an admissible \( A_\infty \)-structure \( m \) on \( A_L \). Then the above homomorphism \( HG \to G \) restricts to an isomorphism of \( HG(m) \) with the subgroup \( G_0 \subset G \) consisting of inner automorphisms of \( \hat{T}(H^1(X, \mathcal{O}_X)^*) \) by elements in \( 1 + \prod_{n > 0} T^n(H^1(X, \mathcal{O}_X)^*) \).

Assume that \( X \) is a smooth projective curve. Then there is a canonical noncommutative thickening \( \tilde{J} \) of the Jacobian \( J \) of \( X \) (see [3]). As was shown in [10], a choice of an \( A_\infty \)-structure in the canonical strict \( A_\infty \)-isomorphism class gives rise to a formal system of coordinates on \( \tilde{J} \) at zero. More precisely, by this we mean an isomorphism of the formal completion of the local ring of \( \tilde{J} \) at zero with \( \hat{T}(H^1(X, \mathcal{O}_X)^*) \) inducing the identity map on the tangent spaces. Formal coordinates associated with two strictly isomorphic \( A_\infty \)-structures are related by the coordinate change given by the image of the corresponding \( A_\infty \)-isomorphism under the homomorphism \( HG \to G \). Now Theorem 1.2 implies that two \( A_\infty \)-structures in the canonical class that induce the same formal coordinate on \( \tilde{J} \) can be connected by a unique strict \( A_\infty \)-isomorphism. Indeed, two such isomorphisms would differ by a strict \( A_\infty \)-automorphism inducing the trivial automorphism of \( \hat{T}(H^1(X, \mathcal{O}_X)^*) \), but such an \( A_\infty \)-automorphism is trivial by Theorem 1.2.

**Convention.** Throughout the paper we work over a fixed ground field \( k \). The symbol \( \otimes \) without additional subscripts always denotes the tensor product over \( k \).

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2. Preliminaries

2.1. Strict $A_\infty$-isomorphisms and homotopies. We refer to [4] for an introduction to $A_\infty$-structures. We restrict ourselves to several remarks about $A_\infty$-isomorphisms and homotopies between them.

A strict $A_\infty$-isomorphism between two $A_\infty$-structures $(m)$ and $(m')$ on the same graded space $A$ is an $A_\infty$-morphism $f = (f_n)$ from $(A, m)$ to $(A, m')$ such that $f_1 = \text{id}$. The equations connecting $f$, $m$ and $m'$ can be interpreted as follows. Recall that $m$ and $m'$ correspond to coderivations $d_m$ and $d_{m'}$ of the bar-construction $\text{Bar}(A) = \oplus_{n \geq 1} T^n(A[1])$ such that $d^2_m = d^2_{m'} = 0$. Now every collection $f = (f_n)_{n \geq 1}$, where $f_n : A^{\otimes n} \to A$ has degree $1 - n$, $f_1 = \text{id}$, defines a coalgebra automorphism $\alpha_f : \text{Bar}(A) \to \text{Bar}(A)$, with the component $\text{Bar}(A) \to A[1]$ given by $(f_n)$. The condition that $f$ is an $A_\infty$-morphism is equivalent to the equation $\alpha_f \circ d_m = d_{m'} \circ \alpha_f$. In other words, strict $A_\infty$-isomorphisms between $A_\infty$-structures precisely correspond to the action of the group of automorphisms of $\text{Bar}(A)$ as a coalgebra on the space of coderivations $d$ such that $d^2 = 0$. More precisely, we consider only automorphisms of $\text{Bar}(A)$ of degree 0 inducing the identity map $A \to A$. Let us denote by $HG = HG(A)$ the group of such automorphisms which we will also call the group of strict $A_\infty$-isomorphisms on $A$. We will denote by $m \to g \ast m$, where $g \in HG$, the natural action of this group on the set of all $A_\infty$-structures on $A$.

One can define a decreasing filtration $(HG_n)$ of $HG$ by normal subgroups by setting

\[ HG_n = \{ f = (f_i) \mid f_i = 0, 1 < i \leq n \}. \]

Note that $f \in HG_n$ if and only if the restriction of $\alpha_f$ to the sub-coalgebra $\text{Bar}(A)_{\leq n} = \oplus_{i \leq n} (A[1])^\otimes i$ is the identity homomorphism. Furthermore, it is also clear that $HG \simeq \text{proj lim}_n HG/HG_n$. In particular, an infinite product of strict $A_\infty$-isomorphisms 

\[ f(3) \ast f(2) \ast f(1) \]

is well-defined as long as $f(n) \in HG_{i(n)}$, where $i(n) \to \infty$ as $n \to \infty$. The notion of a homotopy between $A_\infty$-morphisms is best understood in a more general context of $A_\infty$-categories. Namely, for every pair of $A_\infty$-categories $C, D$ one can define the $A_\infty$-category $\text{Fun}(C, D)$ having $A_\infty$-functors $F : C \to D$ as objects (see [6], [8]). In particular, there is a natural notion of closed morphisms between two $A_\infty$-functors $F, F' : C \to D$. Specializing to the case when $C$ and $D$ are $A_\infty$-categories with one object corresponding to $A_\infty$-algebras $A$ and $B$ we obtain a notion of a closed morphism between a pair of $A_\infty$-morphisms $f, f' : A \to B$. Following [4] we call such a closed morphism a homotopy between $A_\infty$-morphisms $f$ and $f'$. More explicitly, a homotopy $h$ is given by a collection of maps $h_n : A^{\otimes n} \to B$ of degree $-n$, where $n \geq 1$, satisfying some equations. These equations are written as follows: there exists a unique linear map $H : \text{Bar}(A) \to \text{Bar}(B)$ of degree $-1$ with the component $\text{Bar}(A) \to B$ given by $(h_n)$, such that

\[ \Delta \circ H = (\alpha_f \otimes H + H \otimes \alpha_{f'}) \circ \Delta, \]

(2.1.1)

where $\alpha_f, \alpha_{f'} : \text{Bar}(A) \to \text{Bar}(B)$ are coalgebra homomorphisms corresponding to $f$ and $f'$, $\Delta$ denotes the comultiplication. Then the equation connecting $h$, $f$ and $f'$ is

\[ \alpha_f - \alpha_{f'} = d_A \circ H + H \circ d_B, \]

(2.1.2)

where $d_A$ (resp., $d_B$) is the coderivation of $\text{Bar}(A)$ (resp., $\text{Bar}(B)$) corresponding to the $A_\infty$-structure on $A$ (resp., $B$). It is not difficult to check that for a given $A_\infty$-morphism
f from A to B the equations (2.1.1) and (2.1.2) imply that $\alpha_f$ is a homomorphism of dg-coalgebras, so it defines an $A_{\infty}$-morphism $f'$ from A to B. Moreover, similarly to the case of strict $A_{\infty}$-isomorphisms we have the following result.

**Lemma 2.1.** Let A and B be $A_{\infty}$-algebras and $f = (f_n)$ be an $A_{\infty}$-morphism from A to B. For every collection $(h_n)_{n \geq 1}$, where $h_n : A^{\otimes n} \to B$ has degree $-n$, there exists a unique $A_{\infty}$-morphism $f'$ from A to B such that $h$ is a homotopy from $f$ to $f'$.

**Proof.** It is easy to see that equation (2.1.1) is equivalent to the following formula

$$H[a_1|\ldots|a_n] = \sum_{i_1<\ldots<i_k<m<j_1<\ldots<j_l=n} \pm [f_{i_1}(a_1, \ldots, a_{i_1})|\ldots|f_{i_k-i_{k-1}}(a_{i_{k-1}+1}, \ldots, a_i)]$$

$$h_{m-i}(a_{i+1}, \ldots, a_m)[f'_{j_1-m}(a_{m+1}, \ldots, a_{j_1})|\ldots|f'_{j_l-j_{l-1}}(a_{j_{l-1}+1}, \ldots, a_{j_l})],$$

(2.1.3)

where $a_1, \ldots, a_n \in A$, $n \geq 1$. We are going to construct the maps $H|_{Bar(A)_{\leq n}}$ and $\alpha_f|_{Bar(A)_{\leq n}}$ recursively, so that at each step the equations (2.1.2) and (2.1.3) are satisfied when restricted to $Bar(A)_{\leq n}$. Of course, we also want $H$ to have $(h_n)$ as components. Then such a construction will be unique. Note that $H|_{A[1]}$ is given by $h_1$ and $\alpha_f|_{A[1]}$ is given by $f'_1 = f_1 - m_1 \circ h_1 - h_1 \circ m_1$. Now assume that the restrictions of $H$ and $\alpha_f$ to $Bar(A)_{\leq n-1}$ are already constructed, so that the maps $f'_i : A^{\otimes i} \to B$ are defined for $i \leq n-1$. Then the formula (2.1.3) defines uniquely the extension of $H$ to $Bar(A)_{\leq n}$ (note that in the RHS of this formula only $f'_i$ with $i \leq n-1$ appear). It remains to apply formula (2.1.2) to define $\alpha_f|_{Bar(A)_{\leq n}}$. \hfill \Box

Let $HG$ be the group of strict $A_{\infty}$-isomorphisms on a given graded space $A$. In other words, $HG$ is the group of degree 0 coalgebra automorphisms of $Bar(A)$ with the component $A \to A$ equal to the identity map. This group acts on the set of all $A_{\infty}$-structures on $A$. The stabilizer subgroup of some $A_{\infty}$-structure $m$ is the group of strict $A_{\infty}$-automorphisms $HG(m)$. We can consider the set of all strict $A_{\infty}$-automorphisms $f_h \in HG(m)$ such that there exists a homotopy $h$ from the trivial $A_{\infty}$-automorphism $f^m$ to $f_h$ (where $f^m_{i+1} = 0$ for $i > 1$). It is easy to see that $A_{\infty}$-automorphisms of the form $f_h$ constitute a normal subgroup in $HG(m)$ that we will denote by $HG(m)^0$. Furthermore, for every $g \in HG$ we have $HG_{g^m} = gHG(m)^0g^{-1}$. Also, for a pair of elements $g_1, g_2 \in HG$ such that $m' = g_1 \ast m = g_2 \ast m$, there exists a homotopy between $g_1$ and $g_2$ (where $g_i$ are considered as $A_{\infty}$-morphisms from $(A, m)$ to $(A, m')$) if and only if $g_1^{-1}g_2 \in HG(m)^0$.

**2.2. Obstructions.** Below we use Hochschild cohomology $HH(A) := HH(A, A)$ for a graded associative algebra $A$ (see [7] for the corresponding sign convention). When considering $A = A_L$ as a graded algebra we equip it with the cohomological grading, so in the situation of Theorem 1.1 this grading has only 0-th and $d$-th non-trivial graded components.

The following lemma is well known and its proof is straightforward.

**Lemma 2.2.** Let $m$ and $m'$ be two admissible $A_{\infty}$-structures on $A$. Assume that $m_i = m'_i$ for $i < n$, where $n \geq 3$. 


(i) Set \( c(a_1, \ldots, a_n) = (m'_n - m_n)(a_1, \ldots, a_n) \). Then \( c \) is a Hochschild \( n \)-cocycle, i.e.,

\[
\delta c(a_1, \ldots, a_{n+1}) = \sum_{j=1}^{n} (-1)^{j} c(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) + (-1)^{n+1} c(a_1, \ldots, a_n) a_{n+1} = 0.
\]

(ii) If \( m' = f \star m \) for a strict \( A_\infty \)-isomorphism \( f \) such that \( f_i = 0 \) for \( 1 < i < n - 1 \), then setting \( b(a_1, \ldots, a_{n-1}) = (-1)^{n-1} f_{n-1}(a_1, \ldots, a_{n-1}) \) we get

\[
c(a_1, \ldots, a_n) = \delta b(a_1, \ldots, a_n),
\]

where \( c \) is the \( n \)-cocycle defined in (i). Hence, \( c \) is a Hochschild coboundary.

Thus, the study of admissible \( A_\infty \)-structures on \( A \) is closely related to the study of certain components of Hochschild cohomology of \( A \). More precisely, let us denote \( C_{p,q}^n(A) \) (resp. \( HH_{p,q}^n(A) \)) the space of reduced Hochschild \( n \)-cochains (resp. \( n \)-th Hochschild cohomology classes) of internal grading \(-p\) and of cohomological grading \(-q\). In other words, \( C_{p,q}^n(A) \) consists of cochains \( c : A^\otimes n \to A \) such that \( \text{intdeg} c(a_1, \ldots, a_n) = \text{intdeg} a_1 + \ldots + \text{intdeg} a_n - p \), \( \deg c(a_1, \ldots, a_n) = \deg a_1 + \ldots + \deg a_n - q \). Since, all the operations \( m_n \) respect the internal grading and have (cohomological) degree \( 2 - n \), we see that the cocycle \( c \) defined in Lemma 2.2 lives in \( C_{0,n-2}^n(A) \).

There is an analogue of Lemma 2.2 for strict \( A_\infty \)-isomorphisms.

**Lemma 2.3.** Let \( m \) and \( m' \) be admissible \( A_\infty \)-structures on \( A \), \( f, f' \) be a pair of strict \( A_\infty \)-isomorphisms from \( m \) to \( m' \). Assume that \( f_i = f'_i \) for \( i < n \), where \( n \geq 2 \).

(i) Set \( c(a_1, \ldots, a_n) = (f'_n - f_n)(a_1, \ldots, a_n) \). Then \( c \) is a Hochschild \( n \)-cocycle in \( C_{0,n-1}^n(A) \).

(ii) If \( \phi : f \to f' \) is a homotopy such that \( \phi_i = 0 \) for \( i < n - 1 \), then for \( b(a_1, \ldots, a_{n-1}) = \pm \phi_{n-1}(a_1, \ldots, a_{n-1}) \) one has \( c = \delta b \).

### 3. Calculations

#### 3.1. Hochschild cohomology

In this subsection we calculate the components of the Hochschild cohomology of \( A = A_L \) that are relevant for the proof of Theorem 1.1.

Let us set \( R = R_L \) and let \( R_+ = \oplus_{n \geq 1} R_n \) be the augmentation ideal in \( R \), so that \( R/R_+ = k \). Recall that the bar-construction provides a free resolution of \( k \) as \( R \)-module of the form

\[
\ldots \to R_+ \otimes R_+ \otimes R \to R_+ \otimes R \to R \to k.
\]  

(3.1.1)

For graded \( R \)-bimodules \( M_1, \ldots, M_n \) we consider the bar-complex

\[
B^\bullet(M_1, \ldots, M_n) = M_1 \otimes T(R_+) \otimes M_2 \otimes \ldots \otimes T(R_+) \otimes M_n,
\]

where \( T(R_+) \) is the tensor algebra of \( R_+ \). This is just the tensor product over \( T(R_+) \) of the bar-complexes of \( M_1, \ldots, M_n \) (where \( M_1 \) is considered as a right \( R \)-module, \( M_2, \ldots, M_{n-1} \) as \( R \)-bimodules, and \( M_n \) as a left \( R \)-module). The grading in this complex is induced by the cohomological grading of the tensor algebra \( T(R_+) \) defined by \( \deg T^i(R_+) = -i \), so that \( B^\bullet(M_1, \ldots, M_n) \) is concentrated in nonnegative degrees and the differential has degree 1. For example, \( B^\bullet(k, R) \) is the bar-resolution (3.1.1) of \( k \).
Proposition 3.1. Under the assumptions of Theorem 1.1 let us consider the graded \( R \)-
module \( M = \oplus_{i \in \mathbb{Z}} H^d(X, L^i) \). Let \( M_1, \ldots, M_n \) be graded \( R \)-bimodules such that each of
them is isomorphic to \( M \) as a (graded) right \( R \)-module and as a left \( R \)-module.

(i) The complex \( B^\bullet(k, M) = T(R_+) \otimes M \) has one-dimensional cohomology, which is
concentrated in degree \(-d - 1\) and internal degree 0.

(ii) \( H^i(B^\bullet(M_1, M_2)) = 0 \) for \( i \neq -d - 1 \) and \( H^{-d-1}(B^\bullet(M_1, M_2)) \) is isomorphic to \( M \) as
a (graded) right \( R \)-module and as a left \( R \)-module.

(iii) \( H^i(B^\bullet(M_1, \ldots, M_n)) = 0 \) for \( i > -(n - 1)(d + 1) \).

(iv) \( H^i(B^\bullet(k, M_1, \ldots, M_n, k)) = 0 \) for \( i > -(n(d + 1)) \). In addition, the space
\( H^{-d-1}(B^\bullet(k, M_1, k)) \) is one-dimensional and has internal degree 0.

Proof. (i) Localizing the exact sequence (3.1.1) on \( X \) and tensoring with \( L^m \), where \( m \in \mathbb{Z} \), we obtain the following exact sequence of vector bundles on \( X \):

\[
\ldots \oplus_{n_1, n_2 > 0} R_{n_1} \otimes R_{n_2} \otimes L^{m-n_1-n_2} \to \oplus_{n > 0} R_n \otimes L^{m-n} \to L^m \to 0.
\] (3.1.2)

Each term in this sequence is a direct sum of a number of copies of line bundles \( L^n \): for
a finite-dimensional vector space \( V \) we denote by \( V \otimes L^n \) the direct sum of \( \dim V \) copies
of \( L^n \). Now let us consider the spectral sequence with \( E_1 \)-term given by the cohomology
of all sheaves in this complex and abutting to zero (this sequence converges since we can
calculate cohomology using Čech resolutions with respect to a finite open affine covering
of \( X \)). The \( E_1 \)-term consists of two rows: one obtained by applying the functor \( H^0(X, \cdot) \)
to (3.1.2), another obtained by applying \( H^d(X, \cdot) \). The row of \( H^0 \)'s has form

\[
\ldots \oplus_{n_1, n_2 > 0} R_{n_1} \otimes R_{n_2} \otimes R_{m-n_1-n_2} \to \oplus_{n > 0} R_n \otimes R_{m-n} \to R_m \to 0
\]

which is just the \( m \)-th homogeneous component of the bar-resolution. Hence, this complex
is exact for \( m \neq 0 \). Since the sequence abuts to zero the row of \( H^d \)'s should also be exact
for \( m \neq 0 \). For \( m = 0 \) the row of \( H^0 \)'s reduces to the single term \( H^0(X, \mathcal{O}_X) = k \), hence,
the row of \( H^d \)'s in this case has one-dimensional cohomology at \(-(d + 1)\)-th term and is
exact elsewhere.

(ii) Consider the filtration on \( B^\bullet(M_1, M_2) \) associated with the \( \mathbb{Z} \)-grading on \( M_2 \). By part
(i) the corresponding spectral sequence has the term \( E_1 \simeq H^{-d-1}(M_1 \otimes T(R_+)) \otimes M_2 \simeq M_2 \).
Hence, it degenerates in this term and

\[
H^*(K^\bullet) \simeq H^{-d-1}(K^\bullet) \simeq M
\]
as a right \( R \)-module. Similarly, the spectral sequence associated with the filtration on \( K^\bullet \)
induced by the \( \mathbb{Z} \)-grading on \( M_2 \) gives an isomorphism of left \( R \)-modules \( H^{-d-1}(K^\bullet) \simeq M \).

(iii) For \( n = 2 \) this follows from (ii). Now let \( n > 2 \) and assume that the assertion holds
for \( n' < n \). We can consider \( B^\bullet(M_1, \ldots, M_n) \) as the total complex associated with a
bicomplex, where the bidegree \((\deg_0, \deg_1)\) on \( M_1 \otimes T(R_+) \otimes \cdots \otimes T(R_+) \otimes M_n \) is given by

\[
\deg_0(x_1 \otimes t_1 \otimes \ldots \otimes t_{n-1} \otimes x_n) = \sum_{i \equiv 0(2)} \deg(t_i),
\]

\[
\deg_1(x_1 \otimes t_1 \otimes \ldots \otimes t_{n-1} \otimes x_n) = \sum_{i \equiv 1(2)} \deg(t_i),
\]

7
where \( t_i \in T(R_+) \), \( x_i \in M_i \), deg denotes the cohomological degree on \( T(R_+) \). Therefore, there is a spectral sequence abutting to cohomology of \( B^\bullet(M_1, \ldots, M_n) \) with the \( E_1 \)-term
\[
H^\bullet(M_1 \otimes T(R_+) \otimes M_2) \otimes T(R_+) \otimes H^\bullet(M_3 \otimes T(R_+) \otimes M_4) \otimes \ldots,
\]
where the last factor of the tensor product is either \( M_n \) or \( H^\bullet(M_{n-1} \otimes T(R_+) \otimes M_n) \). Using part (ii) we see that
\[
\left( H^\bullet(M_1', \ldots, M'_{n'}) \left[ (n - n')(d + 1) \right] \right)
\]
with \( n' < n \). It remains to apply the induction assumption.

(iv) Consider first the case \( n = 1 \). The complex \( B^\bullet(k, M_1, k) = T(R_+) \otimes M_1 \otimes T(R_+) \) is the total complex of the bicomplex \((\partial_1 \otimes \text{id}, \text{id} \otimes \partial_2)\), where \( \partial_1 \) and \( \partial_2 \) are bar-differentials on \( T(R_+) \otimes M_1 \) and \( M_1 \otimes T(R_+) \). Our assertion follows immediately from (i) by considering the spectral sequence associated with this bicomplex.

Now assume that for some \( n > 1 \) the assertion holds for all \( n' < n \). As before we view \( B^\bullet(k, M_1, \ldots, M_n, k) \) as the total complex of a bicomplex by defining the bidegree on \( T(R_+) \otimes M_1 \otimes \ldots \otimes M_n \otimes T(R_+) \) as follows:
\[
\text{deg}_0(t_0 \otimes x_1 \otimes t_1 \ldots \otimes x_n \otimes t_n) = \sum_{i \equiv 0(2)} \text{deg}(t_i),
\]
\[
\text{deg}_1(t_0 \otimes x_1 \otimes t_1 \ldots \otimes x_n \otimes t_n) = \sum_{i \equiv 1(2)} \text{deg}(t_i).
\]

Assume first that \( n \) is even. Then there is a spectral sequence associated with the above bicomplex abutting to the cohomology of \( B^\bullet(k, M_1, \ldots, M_n, k) \) and with the \( E_1 \)-term
\[
T(R_+) \otimes H^\bullet(M_1 \otimes T(R_+) \otimes M_2) \otimes T(R_+) \otimes \ldots \otimes H^\bullet(M_{n-1} \otimes T(R_+) \otimes M_n) \otimes T(R_+).
\]
Using (ii) we see that \( E_1 \) is isomorphic to the complex of the form
\[
B^\bullet(k, M'_1, \ldots, M'_{n/2}, k)[n(d + 1)/2],
\]
so we can apply the induction assumption. If \( n \) is odd then we consider another spectral sequence associated with the above bicomplex, so that
\[
E_1 = H^\bullet(T(R_+) \otimes M_1) \otimes T(R_+) \otimes H^\bullet(M_2 \otimes T(R_+) \otimes M_3) \otimes T(R_+) \otimes \ldots \otimes H^\bullet(M_{n-1} \otimes T(R_+) \otimes M_n) \otimes T(R_+).
\]
By (i) and (ii) this complex is isomorphic to \( B^\bullet(k, M'_1, \ldots, M'_{(n-1)/2}, k)[(n + 1)(d + 1)/2] \). Again we can finish the proof by applying the induction assumption. \( \square \)

We will also need the following simple lemma.

**Lemma 3.2.** Let \( C^\bullet \) be a complex in an abelian category equipped with a decreasing filtration \( C^\bullet = F^0C^\bullet \supset F^1C^\bullet \supset F^2C^\bullet \supset \ldots \) such that \( C^n = \text{proj. lim.} \lim_{i} C^n/F^iC^n \) for all \( n \). Let \( \text{gr}_i C^\bullet = F^iC^\bullet/F^{i+1}C^\bullet \), \( i = 0, 1, \ldots \) be the associated graded factors. Assume that \( H^n \text{gr}_i C^\bullet = 0 \) for all \( i > 0 \) and for some fixed \( n \). Then the natural map \( H^n C^\bullet \to H^n \text{gr}_0 C^\bullet \) is injective.
Proof. Considering an exact sequence of complexes

$$0 \rightarrow F^1C^\bullet \rightarrow C^\bullet \rightarrow \text{gr}_0 C^\bullet \rightarrow 0$$

one can easily reduce the proof to the case $H^i \text{gr}_i C^\bullet = 0$ for all $i \geq 0$. In this case we have to show that $H^n C^\bullet = 0$. Let $c \in C^n$ be a cocycle and let $c_i$ be its image in $C^n/F^i C^n$. It suffices to construct a sequence of elements $x_i \in C^{n-1}/F^i C^{n-1}$, $i = 1, 2, \ldots$, such that $x_{i+1} \equiv x_i \mod F^i C^{n-1}$ and $c_i = d(i)x_i$, where $d(i)$ is the differential on $C^\bullet/F^i C^\bullet$. Since $n$-th cohomology of $C^\bullet/F^i C^\bullet = \text{gr}_0 C^\bullet$ is trivial we can find $x_1$ such that $c_1 = d(1)x_1$. Then we proceed by induction: once $x_1, \ldots, x_i$ are chosen an easy diagram chase using the exact triple of complexes

$$0 \rightarrow \text{gr}_i C^\bullet \rightarrow C^\bullet/F^{i+1} C^\bullet \rightarrow C^\bullet/F^i C^\bullet \rightarrow 0$$

and the vanishing of $H^n(\text{gr}_i C^\bullet)$ show that $x_{i+1}$ exists. \qed

**Theorem 3.3.** Under the assumptions of Theorem 1.1 one has $HH^i_{0,md}(A) = 0$ for $i < m(d + 2)$ and $\dim HH^{d+2}_{0,d}(A) \leq 1$, where $A = A_L$.

**Proof.** Set $C^i = C^i_{0,md}(A)$ (see 2.2). Note that Hochschild differential maps $C^i$ into $C^{i+1}$ (since $m_2$ preserves both gradings on $A$). Recall that the decomposition of $A$ into graded pieces with respect to the cohomological degree has form $A = R \oplus M$, where $R$ has degree 0 and $M = \bigoplus_{i \in \mathbb{Z}} H^d(X, L^i)$ has degree $d$. The natural augmentation of $A$ is given by the ideal $A_+ = R_+ \oplus M$. Each of the spaces $C^i$ decomposes into a direct sum $C^i = C^i(0) \oplus C^i(d)$, where $C^i(0) \subset \text{Hom}(A^{\otimes i}_+, R)$, $C^i(d) \subset \text{Hom}(A^{\otimes i}_+, M)$. More precisely, the space $C^i(0)$ consists of linear maps

$$[T(R_+) \otimes M \otimes T(R_+) \otimes \ldots \otimes M \otimes T(R_+)]_i \rightarrow R \quad (3.1.3)$$

preserving the internal grading, where there are $m$ factors of $M$ in the tensor product and the index $i$ refers to the total number of factors $H^*(L^*)$ (so that the LHS can be considered as a subspace of $A^{\otimes i}_+$). Similarly, the space $C^i(d)$ consists of linear maps

$$[T(R^+) \otimes M \otimes T(R^+) \otimes \ldots \otimes M \otimes T(R^+)]_i \rightarrow M$$

preserving the internal grading, where there is $m + 1$ factors of $M$ in the tensor product. Clearly, $C^\bullet(d)$ is a subcomplex in $C^\bullet$, so we have an exact sequence of complexes

$$0 \rightarrow C^\bullet(d) \rightarrow C^\bullet \rightarrow C^\bullet(0) \rightarrow 0.$$ 

Therefore, it suffices to prove that $H^i(C^\bullet(0)) = H^i(C^\bullet(d)) = 0$ for $i < m(d + 2)$, and that in the case $m = 1$ one has in addition $H^{d+2}(C^\bullet(d)) = 0$ and $\dim H^{d+2}(C^\bullet(0)) \leq 1$.

To compute the cohomology of these two complexes we can use spectral sequences associated with some natural filtrations to reduce the problem to simpler complexes. First, let us consider the decomposition

$$C^\bullet(0) = \prod_{j \geq 0} C^\bullet(0)_j,$$
where \( C^i(0)_j \subset C^i(0) \) is the space of maps (3.1.3) with the image contained in \( H^0(L^j) \subset R \). The differential on \( C^\bullet(0) \) has form
\[
\delta(x_j)_{j \geq 0} = (\sum_{j' \leq j} \delta_{j',j} x_{j'})_{j \geq 0}
\]
for some maps \( \delta_{j',j} : C^\bullet(0)_{j'} \to C^\bullet(0)_j \), where \( j' \leq j \). By Lemma 3.2 it suffices to prove that one has \( H^i(C^\bullet(0)_j, \delta_{j,j}) = 0 \) for \( i < m(d+2) \) and all \( j \), while for \( m = 1 \) one has in addition \( H^{d+2}(C^\bullet(0)_j, \delta_{j,j}) = 0 \) for \( j > 0 \) and \( \dim H^{d+2}(C^\bullet(0)_0, \delta_{0,0}) \leq 1 \). But
\[
(C^\bullet(0)_j, \delta_{j,j}) = \text{Hom}(K^\bullet_{m,j}, R_j)[-m],
\]
where \( K^\bullet_m = B^\bullet(k, M, \ldots, M, k) \) (\( m \) copies of \( M \)) and \( K^\bullet_{m,j} \) is its \( j \)-th graded component with respect to the internal grading. Here we use the following convention for the grading on the dual complex: \( \text{Hom}(K^\bullet, R)^i = \text{Hom}(K^{-i}, R) \). Therefore, Proposition 3.1(iv) implies that cohomology of \( C^\bullet(0)_j \) is concentrated in degrees \( \geq m(d+1) + m = m(d+2) \). Moreover, for \( m = 1 \) the \( (d+2) \)-th cohomology space is non-zero only for \( j = 0 \), in which case it is one-dimensional.

For the complex \( C^\bullet(d) \) we have to use a different filtration (since \( M \) is not bounded below with respect to the internal grading). Consider the decreasing filtration on \( C^\bullet(d) \) induced by the following grading on \( T(R^+) \otimes M \otimes T(R) \otimes \ldots \otimes M \otimes T(R) \):
\[
\text{deg}(t_1 \otimes x_1 \otimes t_2 \otimes \ldots \otimes x_{m+1} \otimes t_{m+2}) = \text{deg}(t_1) + \text{deg}(t_{m+2}),
\]
where \( t_i \in T(R) \), \( x_i \in M \), the degree of \( R \) is taken to be \(-1\). The associated graded complex is
\[
\text{Hom}_{gr}(T(R^+) \otimes B^\bullet(M, \ldots, M) \otimes T(R^+) \otimes M)[-m - 1],
\]
where there are \( m+1 \) factors of \( M \) in the bar-construction. It remains to apply Proposition 3.1(iii).

\[ \square \]

3.2. Some Massey products. In this subsection we will show the nontriviality of the canonical class of \( A_\infty \)-structures on \( A_L \) and combine it with our computations of the Hochschild cohomology to prove the main theorem.

Note that the canonical class of \( A_\infty \)-structures can be defined in a more general context. Namely, if \( \mathcal{C} \) is an abelian category with enough injectives then we can define the canonical class of \( A_\infty \)-structures on the derived category \( D^+(\mathcal{C}) \) of bounded below complexes over \( \mathcal{C} \). Indeed, one can use the equivalence of \( D^+(\mathcal{C}) \) with the homotopy category of complexes with injective terms and apply Kadeishvili’s construction to the dg-category of such complexes (see [10],1.2 for more details). In the case when \( \mathcal{C} \) is the category of coherent sheaves the resulting \( A_\infty \)-structure is strictly \( A_\infty \)-isomorphic to the structure obtained using Čech resolutions (since the relevant dg-categories are equivalent). In this context we have the following construction of nontrivial Massey products.

**Lemma 3.4.** Let \( \mathcal{C} \) be an abelian category with enough injectives,
\[
0 \to F_0 \xrightarrow{\alpha_0} F_1 \xrightarrow{\alpha_2} F_2 \to \ldots \xrightarrow{\alpha_n} F_n \to 0
\]
be an exact sequence in \( \mathcal{C} \), where \( n \geq 2 \), and let \( \beta : F_n \to F_0[n - 1] \) be a morphism in the derived category \( D^b(\mathcal{C}) \) corresponding to the Yoneda extension class in \( \text{Ext}^{n-1}(F_n, F_0) \)
represented by the above sequence. Assume that \( \text{Ext}^{j-i-1}(F_j, F_i) = 0 \) when \( 0 \leq i < j \leq n - 1 \). Then

\[
m_{n+1}(\alpha_1, \ldots, \alpha_n, \beta) = \pm \text{id}_{F_0}
\]

for any \( A_\infty \)-structure \( (m_i) \) on \( D^b(C) \) from the canonical class.

**Proof.** Assume first that \( n = 2 \). Then \( m_3(\alpha_1, \alpha_2, \beta) \) is the unique value of the well-defined triple Massey product in \( D^b(C) \) (see [9], 1.1). Using the standard recipe for its calculation (see [2], IV.2) we immediately get that \( m_3(\alpha_1, \alpha_2, \beta) = \text{id} \).

For general \( n \) we can proceed by induction. Assume that the statement is true for \( n - 1 \). Set \( F_{n-1}' = \ker(\alpha_n) \). Then we have exact sequences

\[
0 \to F_0 \xrightarrow{\alpha} F_1 \to \ldots \to F_{n-2} \xrightarrow{\alpha_{n-1}} F_{n-1}' \to 0,
\]

\[
0 \to F_{n-1}' \xrightarrow{i} F_{n-1} \xrightarrow{\alpha} F_n \to 0,
\]

where \( m_2(\alpha_{n-1}', i) = i \circ \alpha_{n-1}' = \alpha_{n-1} \). By the definition, we have \( \beta = m_2(\gamma, \beta') \), where \( \beta' \in \text{Ext}^{n-2}(F_{n-1}', F_0) \) and \( \gamma \in \text{Ext}^n(F_n, F_{n-1}') \) are the extension classes corresponding to these exact sequences. Applying the \( A_\infty \)-axiom to the elements \( (\alpha_1, \ldots, \alpha_n, \gamma, \beta') \) and using the vanishing of \( m_{n-1+i}(\alpha_{i+1}, \ldots, \alpha_n, \gamma, \beta') \in \text{Ext}^{i-1}(F_i, F_0) \) for \( 0 < i < n \), we get

\[
m_{n+1}(\alpha_1, \ldots, \alpha_n, \beta) = \pm m_n(\alpha_1, \ldots, \alpha_{n-1}, m_3(\alpha_{n-1}, \alpha_n, \gamma), \beta').
\]

Furthermore, applying the \( A_\infty \)-axiom to the elements \( (\alpha_{n-1}', i, \alpha_n, \gamma) \) we get

\[
m_3(\alpha_{n-1}, \alpha_n, \gamma) = \pm m_2(\alpha_{n-1}', m_3(i, \alpha_n, \gamma)).
\]

Next, we claim that sequences (3.2.1) and (3.2.2) satisfy the assumptions of the lemma. Indeed, for (3.2.1) this is clear, so we just have to check that \( \text{Hom}(F_{n-1}, F_{n-1}') = 0 \). The exact sequence (3.2.1) gives a resolution \( F_0 \to \ldots \to F_{n-2} \) of \( F_{n-1}' \) and we can compute \( \text{Hom}(F_{n-1}, F_{n-1}') \) using this resolution. Now the required vanishing follows from our assumption that \( \text{Ext}^{n-i-2}(F_{n-1}, F_i) = 0 \) for \( 0 \leq i \leq n - 2 \). Therefore, we have

\[
m_3(i, \alpha_n, \gamma) = \text{id}.
\]

Together with (3.2.4) this implies that

\[
m_3(\alpha_{n-1}, \alpha_n, \gamma) = \pm \alpha_{n-1}'.
\]

Substituting this into (3.2.3) we get

\[
m_{n+1}(\alpha_1, \ldots, \alpha_n, \beta) = \pm m_n(\alpha_1, \ldots, \alpha_{n-1}, \alpha_{n-1}', \beta').
\]

It remains to apply the induction assumption to the sequence (3.2.1). \( \Box \)

**Proof of Theorem 1.1.** (i) Since the algebra \( A = A_L \) is concentrated in degrees 0 and \( d \), the first potentially nontrivial higher product of an admissible \( A_\infty \)-structure \( (m_i) \) on \( A \) is \( m_{d+2} \). Therefore, by Lemma 2.2 for every such \( A_\infty \)-structure \( (m_i) \) on \( A \) the map \( m_{d+2} \) induces a cohomology class \( [m_{d+2}] \in HH_{d+2}^d(A) \). We claim that if \( (m_i') \) is another admissible \( A_\infty \)-structure on \( A \) then \( (m_i) \) is strictly \( A_\infty \)-isomorphic to \( (m_i') \) if and only if \( [m_{d+2}] = [m_{d+2}'] \). Indeed, this follows from Lemma 2.2 and from the vanishing of higher obstructions due to Theorem 3.3 (these obstructions lie in \( HH_{0,md+2}^d(A) \) where \( m > 1 \), and the vanishing follows since \( md + 2 < m(d + 2) \)). In particular, an admissible \( A_\infty \)-structure
(m_i) is nontrivial if and only if [m_{d+2}] \neq 0. Since by Theorem 3.3 the space \(HH_{0,d}^{d+2}(A)\) is at most one-dimensional, it remains to prove the nontriviality of an admissible \(A_\infty\)-structure from the canonical class. Replacing \(L\) by its sufficiently high power if necessary we can assume that there exists \(d+1\) sections \(s_0, \ldots, s_{d+1} \in H^0(L)\) without common zeroes. The corresponding Koszul complex gives an exact sequence

\[0 \to O \to O^{\oplus(d+1)} \otimes_O L \to O^{\oplus(d+1)} \otimes_O L^2 \to \ldots \to O^{\oplus(d+1)} \otimes_O L^d \to L^{d+1} \to 0.\]

By our assumptions this sequence satisfies the conditions required in Lemma 3.4, hence we get a nontrivial \((d+2)\)-ple Massey product for our \(A_\infty\)-structure.

(ii) Applying Lemma 2.3 we see that obstructions for connecting two strict \(A_\infty\)-isomorphisms by a homotopy lie in \(\oplus_{m \geq 1} HH_{0,md}^{md+1}(A)\). But this space is zero by Theorem 3.3. \(\square\)

**Corollary 3.5.** Under assumptions of Theorem 1.1 the space \(HH_{0,d}^{d+2}(A_L)\) is one-dimensional.

**Proof.** Indeed, from Theorem 3.3 we know that \(\dim HH_{0,d}^{d+2}(A_L) \leq 1\). If this space were zero then the above argument would show that all admissible \(A_\infty\)-structures on \(A_L\) are trivial. But we know that \(A_\infty\)-structures on \(A_L\) from the canonical class are nontrivial. \(\square\)

**Remark.** One can ask whether there exists an \(A_\infty\)-structure on \(A_L\) from the canonical class such that \(m_n = 0\) for \(n > d + 2\) or at least \(m_n = 0\) for all sufficiently large \(n\). However, even in the case of smooth curves of genus \(\geq 1\) the answer is “no”. The proof can be obtained using the construction of a universal deformation of a coherent sheaf (when it exists) using the canonical \(A_\infty\)-structure, outlined in [10]. For example, it is shown there that the products

\[m_{n+2} : H^1(O_X) \otimes H^0(L^{n_1}) \otimes H^0(L^{n_2}) \to H^0(L^{n_1+n_2})\]

appear as coefficients in the universal deformation of the structure sheaf. The base of this family is \(\text{Spec } R\), where \(R \simeq k[[t_1, \ldots, t_g]]\) is the completed symmetric algebra of \(H^1(O_X)\). If all sufficiently large products were zero, this family would be induced by the base change from some family over an open neighborhood \(U\) of zero in the affine space \(A^g\). But this would imply that the embedding of \(\text{Spec } R\) into the Jacobian (corresponding to the isomorphism of \(R\) with the completion of the local ring of the Jacobian at zero) factors through \(U\), which is false.

### 3.3. Proof of Theorem 1.2.

Theorem 1.1(i) easily implies that every admissible \(A_\infty\)-structure on \(A = A_L\) is (strictly) \(A_\infty\)-isomorphic to some strictly unital \(A_\infty\)-structure. Therefore, it is enough to prove our statement for strictly unital structures. Recall that the group of strict \(A_\infty\)-isomorphisms \(HG\) is the group of coalgebra automorphisms of \(\text{Bar}(A_L)\) inducing the identity map \(A_L \to A_L\) and preserving two grading on \(\text{Bar}(A_L)\) induced by the two gradings of \(A_L\). Thus, we can identify \(HG\) with a subgroup of algebra automorphisms of the completed cobar-construction \(\text{Cobar}(A_L) = \prod_{n \geq 0} T^n(A_L)[n]\) (our convention is that passing to dual vector space changes the grading to the opposite one).

By Theorem 1.1(ii) for every strict \(A_\infty\)-automorphism \(f\) of an \(A_\infty\)-structure \(m\) there exists a homotopy from \(f\) to the trivial \(A_\infty\)-automorphism \(f^{tr}\). Let \(\alpha = \alpha_f^m\) be the automorphism of \(\text{Cobar}(A_L)\) corresponding to \(f\) and \(h = H^* : \text{Cobar}(A_L) \to \text{Cobar}(A_L)[-1]\)
be the map giving the homotopy from $f$ to $f^{tr}$. The equations dual to (2.1.1) and (2.1.2) in our case have form

$$h(xy) = h(x)y \pm \alpha(x)h(y),$$

$$\alpha = \text{id} + d \circ h + h \circ d,$$

where $d$ is the differential on Cobar$(A_L)$ associated with $m$. Recall that $A_L = H^0 \oplus H^1$, where $H^0 = \oplus_{n \geq 0} H^0(X, L^n)$, $H^1 = \oplus_{n \leq 0} H^1(X, L^n)$. Since $h$ has degree $-1$ we have $h((H^1)^*[−1]) = 0$ and $h((H^0)^*[−1]) \subset \bar{T}((H^1)^*[−1])$. Furthermore, since $h$ preserves the internal degree, we have $h(H^0(X, L^n)^*[−1]) = 0$ for all $n > 0$. Let $\epsilon \in (H^0)^*[−1] \subset \text{Cobar}(A_L)$ be an element corresponding to the natural projection $H^0 \rightarrow H^0(X, O_X) \simeq k$. Then we have $A_L^*[−1] = k\epsilon \oplus V$, where $V = (H^1)^*[−1] \oplus (\oplus_{n > 0} H^0(X, L^n))^*[−1]$, and $h(V) = 0$. Let $\langle V \rangle \subset \text{Cobar}(A_L)$ be the subalgebra topologically generated by $V$. Then $h$ vanishes on $\langle V \rangle$. Also, for every $x \in V$ we have

$$dx = \epsilon x + x\epsilon \mod \langle V \rangle$$

since our $A_{\infty}$-structure is strictly unital. Hence, for $x \in V$ we have

$$\alpha(x) = x + h(dx) = x + h(\epsilon)x - \alpha(x)h(\epsilon),$$

which implies that

$$\alpha(x) = (1 + h(\epsilon))x(1 + h(\epsilon))^{-1}.$$

In particular, the restriction of $\alpha$ to the subalgebra $\bar{T}(H^1(X, O_X)^*[−1])$ is the inner automorphism associated with the invertible element $1 + h(\epsilon) \in \bar{T}(H^1(X, O_X)^*[−1])$. On the other hand, we have

$$d\epsilon = \epsilon^2 \mod \langle V \rangle.$$

Hence,

$$\alpha(\epsilon) = \epsilon + dh(\epsilon) + h(\epsilon)\epsilon - \alpha(\epsilon)h(\epsilon),$$

so that

$$\alpha(\epsilon) = (1 + h(\epsilon))\epsilon(1 + h(\epsilon))^{-1} + dh(\epsilon) \cdot (1 + h(\epsilon))^{-1}.$$

Thus, $\alpha$ is uniquely determined by $h(\epsilon)$. Also, by Lemma 2.1 $h(\epsilon)$ can be an arbitrary element of $\prod_{n \geq 1} T^n((H^1(X, O_X)^*[−1])$.}

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