THE HOPF ALGEBRAS OF SYMMETRIC FUNCTIONS
AND QUASISYMMETRIC FUNCTIONS IN NON-COMMUTATIVE
VARIABLES ARE FREE AND COFREE

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Abstract. We uncover the structure of the space of symmetric functions in non-commutative variables by showing that the underlined Hopf algebra is both free and co-free. We also introduce the Hopf algebra of quasi-symmetric functions in non-commutative variables and define the product and coproduct on the monomial basis of this space and show that this Hopf algebra is free and cofree. In the process of looking for bases which generate the space we define orders on the set partitions and set compositions which allow us to define bases which have simple and natural rules for the product of basis elements.

1. Introduction

Recent progress has been made on unraveling the structure of the algebra of symmetric functions in noncommutative variables. This algebra was originally looked at by Wolf [13] showing that the algebra is freely generated by elements of the monomial basis. In the paper by Rosas and Sagan [9], bases analogous to those found in the commutative algebra of symmetric functions were defined showing that the algebra has similar features to the algebra of symmetric functions (in commuting variables). In [4], along with Rosas and Reutenauer, we introduced a Hopf algebra structure and computed some of the structure of the coinvariants in the non-commutative polynomials. In [2], along with Rosas and Hohlweg, we found a connection between the representations of the partition algebra and the dual of the algebra symmetric functions in non-commutative variables. In [6] it is remarked that this graded dual can be realized as a subalgebra of a commutative Hopf algebra of endofunctions and the techniques there can be used to derive independent proofs that this Hopf algebra is both free and cofree. Aguiar and Mahajan [1] are currently writing a monograph on combinatorial Hopf algebras and most of these results are covered with a geometric perspective.

In this paper we continue to uncover the structure of the space of symmetric functions in non-commutative variables by showing that the Hopf algebra is both free and co-free by exhibiting bases which generate the algebra and co-algebra. Using similar methods, we also introduce the Hopf algebra of quasi-symmetric functions in non-commutative variables and define the product and coproduct on the monomial basis of this space and show that this Hopf algebra is free and cofree.
cofree. The quasi-symmetric functions in non-commutative variables appears in the thesis of F. Hivert [5] as an example of combinatorial Hopf algebras that arise in relation to symmetrization operations. A personal communication by J-C. Novelli and J-Y. Thibon informs us [7] that this Hopf algebra is bidendriform and can be shown by consequence to be self-dual, free and co-free.

In the process of looking for bases which generate the space we define orders on the set partitions and set compositions which allow us to define bases which have simple and natural rules for the product of elements. These orders have interesting properties in their own right and are closely related to the structure of the Hopf algebra.

This article is divided into sections and an appendix. Sections 2 and 5 introduce notation and the algebras of symmetric functions and quasi-symmetric functions in non-commuting variables and their graded duals (respectively \( \text{NCSym} \), \( \text{NCQSym} \), \( \text{NCSym}^* \) and \( \text{NCQSym}^* \)). In addition, we use these sections to quickly run through combinatorial enumerative results which we use to understand the indexing sets for the generators of the algebras and their graded duals.

In section 3 we show that \( \text{NCSym} \) is freely generated by the analogue of the power basis defined in \([9]\) in \( \text{NCSym} \). We also define a second order on set partitions which defines a multiplicative basis which does not seem to have a natural anlogue in the algebra of symmetric functions. This basis can be used to show that the matrix of \( [\delta_{A \geq B}]_{A,B \vdash [n]} \) (where \( \leq \) is the refinement order on set partitions) factors in a very natural way over the integers. In section 4 we show that the graded dual of the algebra \( \text{NCSym} \) is freely generated by a subset of the basis elements which are dual to the monomial basis. A combinatorial interpretation for the indexing sets follows from the enumerative results that were found in section 2.

In the last two sections we find similar results for \( \text{NCQSym} \) and its graded dual. We introduce two new orders on set partitions which are used to define natural bases on these spaces. These bases have very elegant and simple expressions for the product of two basis elements and are likely to play an important role in the structure of these algebras. The appendix holds images of one of the orders on set compositions defined in section 7. This order can be generalized to any Coxeter system, where one finds that the intervals of the poset are Cohen-Macauley and several other interesting properties. With C. Hohlweg, this is shown in a recent paper [3].

2. Set partitions and \( \text{NCSym} \)

2.1. Set Partitions. Throughout this article, the notation \([n]\) represent the set \( \{1, 2, 3, \ldots, n\} \). We say then that \( A \subseteq 2^{[n]} \setminus \{\emptyset\} \) with \( A = \{A_1, A_2, \ldots, A_k\} \) is a set partition of \( n \) (denoted \( A \vdash [n] \)) if \( A_i \cap A_j = \emptyset \) for \( i \neq j \) and \( A_1 \cup A_2 \cup \cdots \cup A_k = [n] \). For \( A \subseteq 2^{[n]} \setminus \{\emptyset\} \) with \( A_i \cap A_j = \emptyset \) for \( i \neq j \) let \( st(A) \) represent the set partition formed by lowering the entries of \( A \) keeping the values in relative order. The subsets \( A_i \) are referred to as the parts of the set partition. In this case we say the size of the set partition is \( |A| = n \) and the length of \( A \) is \( \ell(A) = k \) referring to the number of parts of the set partition. The number of set partitions of size \( n \) are given by the Bell numbers \( B_n \).
The set partitions of \( n \) of the second kind formulas: \( (12) \) sequence A000110, and the number of set partitions of length \( k \) are the Stirling numbers of the second kind \( S_{n,k} \) \( (12) \) sequence A008277). These numbers are given by the following recursive formulas:

\[
B_n = \sum_{i=0}^{n-1} \binom{n-1}{i} B_{n-1-i} \text{ with } B_0 = B_1 = 1
\]

\[
S_{n,k} = kS_{n-1,k} + S_{n-1,k-1} \text{ with } S_{n,1} = S_{n,n} = 1
\]

The set partitions of \( n \) are ordered by refinement order where \( A \leq B \) if and only if for each \( i, A_i \subseteq B_j \) for some \( j \). Under this natural order the set \( \{A : A \vdash [n]\} \) is a lattice where we denote the g.l.b. of \( A \) and \( B \) as \( A \land B = \{A_i \cap B_j : 1 \leq i \leq \ell(A), 1 \leq j \leq \ell(B)\} \) while \( A \lor B \) represent the l.u.b. of \( A \) and \( B \).

We use the symbol \( | \) to represent an associative operation on set partitions \( A \triangleright [n] \) and \( B \triangleright [k] \). Let \( A|B \triangleright [n+k] \) which represents \( \{A_1, A_2, \ldots, A_{\ell(A)}, B_1+n, B_2+n, \ldots, B_{\ell(B)}+n\} \) where \( B_i+n \) are the entries of \( B_i \) with \( n \) added to each.

A set partition \( A \triangleright [n] \) is called splittable if there is a number \( r < n \) such that \( A_i \cup A_{i+1} \cup \cdots \cup A_r = [r] \) for some proper subset of the parts of \( A \) and \( A \) is called atomic if it is non-empty and not splittable. Alternatively we have that \( A \) is splitable if there exists non-empty set partitions \( B \triangleright [k] \) and \( C \triangleright [n-k] \) such that \( C = B|C \). For a set partition \( A \), we define the split of \( A \) to be \( A' = (A^{(1)}, A^{(2)}, \ldots, A^{(d)}) \) where each \( A^{(i)} \) is an atomic set partition of size \( \alpha_i > 0 \) and \( A = A^{(1)} | A^{(2)} | \cdots | A^{(d)} \). For example, for \( A = \{\{1,3\}, \{2\}, \{4\}, \{5,8\}, \{6,7\}\} \) we have \( A' = (\{\{1,3\}, \{2\}\}, \{\{1\}\}, \{\{1,4\}, \{2,3\}\}) \). By definition we have that \( A \) is atomic if and only if \( A' = (A) \).

The bijection between \( A \) and \( A' = (A^{(1)}, A^{(2)}, \ldots, A^{(d)}) \) has the property that \( \ell(A) = \ell(A^{(1)}) + \ell(A^{(2)}) + \cdots + \ell(A^{(d)}) \) and that implies the following combinatorial result. It is a hint of the type of algebraic structure we find when we consider the symmetric functions in non-commutative variables.

**Proposition 1.** Let \( a_{k,i} \) represent the number of atomic set partitions of size \( k \) and of length \( i \) \( (12) \) sequence A087903), then we have

\[
\frac{1}{1 - \sum_{k \geq 1} \sum_{i=1}^{k-1} a_{k,i} t^i q^k} = \sum_{n \geq 0} \sum_{r \geq 1} S_{n,r} t^r q^n
\]

By setting \( t = 1 \) in the expression above we also have the following corollary.

**Corollary 2.** Let \( a_k \) represent the number of atomic set partitions of size \( k \) \( (12) \) sequence A074664), then

\[
\frac{1}{1 - \sum_{k \geq 1} a_k q^k} = \sum_{n \geq 0} B_n q^n
\]
2.2. Lyndon words. Let \( x_1, x_2, x_3, \ldots \) be a totally ordered alphabet. A word in this alphabet \( w = x_{i_1}x_{i_2}\cdots x_{i_{|w|}} \) is called Lyndon if it is lexicographically strictly smaller than all of the cyclic shifts of \( w \).

We have the following proposition about a decomposition of words into Lyndon words. This is exercise 7.89.d in [10]:

**Proposition 3.** ([8] (7.4.1)) For every word \( w \) in a totally ordered alphabet, there is a unique decomposition of \( w \) as a concatenation of Lyndon words \( w^{(i)} \) such that \( w = w^{(1)}w^{(2)}\cdots w^{(d)} \) with \( w^{(i)} \geq_{\text{lex}} w^{(i+1)} \).

Recall that the size of the set partition is denoted by \( |A| \) and the number of parts is \( \ell(A) \). We consider a total order on set partitions such that \( A <_T B \) if \( |A| < |B| \), or \( |A| = |B| \) and \( \ell(A) < \ell(B) \), or \( |A| = |B| \) and \( \ell(A) = \ell(B) \) and \( w(A) <_{\text{lex}} w(B) \) where \( w(A) \) is a word with \( w(A)_i = j \) if \( i \in A^j \) where \( A^j = (A^{(1)}, A^{(2)}, \ldots, A^{(d)}) \).

The following purely combinatorial result will be used in our discussion of the primitives of the Hopf algebra of symmetric functions in non-commutative variables. We say that \( A \), a set partition of \( [n] \), is Lyndon if \( A^1 \) is Lyndon in lexicographic order in the atomic set partitions which are ordered using the total order on all set partitions listed above. Remark that every atomic set partition is Lyndon since \( A^1 \) has length 1.

Now by viewing \( A^1 \) as a word in the alphabet of atomic set partitions we may use Proposition 3 and the bijection between \( A \) and \( A^1 \) to arrive at the following two corollaries.

**Proposition 4.** Let \( b_{k,i} \) be the number of Lyndon set partitions of size \( k \) and length \( i \) (12 sequence A112340). We have

\[
\prod_{k \geq 1} \prod_{i \geq 1} \frac{1}{(1 - tq^k)^{b_{k,i}}} = \sum_{n \geq 0} \sum_{r \geq 1} S_{n,r} t^r q^n.
\]

**Corollary 5.** Let \( b_k \) be the number of Lyndon set partitions of size \( k \) (12 sequence A085686).

\[
\prod_{k \geq 1} \frac{1}{(1 - q^k)^{b_k}} = \sum_{n \geq 0} B_n q^n.
\]

2.3. The Hopf algebra \( NCSym \) and the graded dual. We refer the reader to [9] for some of the structure of the algebra of the symmetric functions in non-commuting variables and to [4] for the motivation for the following definition of the Hopf algebra.

Define \( NCSym = \bigoplus_{n \geq 0} \mathcal{L}\{ m_A : A \vdash [n] \} \) as a graded vector space. \( NCSym \) is endowed with the following product and coproduct which makes it a Hopf algebra (see [4] Theorem 5). For \( A \vdash [n] \) and \( B \vdash [k] \), set

\[
m_A m_B = \sum_{C \cap ([n] \cup [k]) = A \cup B} m_C.
\]
The coproduct is defined by
\[ \Delta_{\text{NCSym}}(m_A) = \sum_{S \subseteq \ell(A)} m_{st(A_S)} \otimes m_{st(A_{Sc})} \]
where for each subset \( S = \{i_1, i_2, \ldots, i_k\} \subseteq \ell(A) \), we denote \( A_S = \{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\} \).

Using this product and coproduct, \( \text{NCSym} \) is a Hopf algebra with the unit, counit defined in the usual manner and the antipode is determined from the graded bialgebra structure. This algebra is named the symmetric functions in non-commutative variables because of the following result:

**Proposition 6.** (Corollary 2) Define \( \text{NCSym}^n \subseteq \mathbb{Q}(x_1, x_2, \ldots, x_n) \) as the linear span of the elements
\[ m_A[X_n] = \sum_{\alpha} x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_{|A|}} \text{ where } \alpha_r = \alpha_s \text{ iff } r, s \in A_i \text{ for some } i. \]

The map \( \phi_n : \text{NCSym} \to \text{NCSym}^n \) defined by \( \phi_n(m_A) = m_A[X_n] \) is an algebra morphism.

The elements \( m_A[X_n] \) are invariant under the action of the symmetric group \( S_n \) which permutes the variables and may be considered a non-commutative analogue of the monomial symmetric polynomials, hence we refer to the Hopf algebra \( \text{NCSym} \) as the symmetric functions in non-commuting variables (even though we do not reference the variables when we consider the algebra as the linear span of elements \( m_A \)).

The graded dual Hopf algebra is denoted \( \text{NCSym}^* = \bigoplus_{n \geq 0} \mathcal{L}\{w_A : A \vdash [n]\} \) where the basis \( w_A \) are the dual elements to the basis \( \{m_A\} \) in the sense that there exists a pairing between \( \text{NCSym} \) and \( \text{NCSym}^* \) with \( [m_A, w_B] = \delta_{AB} \). The product and coproduct are defined on this space so that

\[ [m_C, w_A w_B] = [\Delta_{\text{NCSym}}(m_C), w_A \otimes w_B] \]

and

\[ [m_B \otimes m_C, \Delta_{\text{NCSym}}(w_A)] = [m_B m_C, w_A]. \]

To this end we explicitly define for \( A \vdash [n] \) and \( B \vdash [m] \),

\[ w_A w_B = \sum_{S \subseteq \binom{n+m}{n}} w_{A \uparrow S \uparrow B \setminus [n+m]} \cdot S \]

where \( \binom{n+m}{n} \) is the set of \( n \) element subsets of \( [n+m] \) and \( A \uparrow S \) represents raising the entries in \( A \) so that they remain in the same relative order but so that the union of all of the parts is equal to \( S \).

**Remark 1.** Because there is a surjective graded map from \( \text{NCSym} \) to \( \text{Sym} \) defined as \( \chi(m_A) = c_A m_{\lambda(A)} \) (see [9] and [4]), this means that there is an injection of \( \text{Sym}^* \) to \( \text{NCSym}^* \) which is easy to calculate explicitly, namely, \( \chi^*(h_\lambda) = c_A \sum_{\lambda(A) = \lambda} w_A \). This does not immediately associate the elements \( w_A \) as a ‘homogeneous’ basis and this is a reason we choose to label them with a \( w \) as upside down ‘m’ rather than an ‘h.’
It is easy to give an explicit expression for the coproduct as well. For \( A \vdash [n] \),

\[
\Delta_{\text{NCSym}}^* (w_A) = \sum_{k=0}^{n} \sum_{(k||n-k) \wedge A = (B\upharpoonright_C)} w_B \otimes w_C
\]

(9)

\[
= \sum_{i=0}^{n} w_{st(A\downarrow_{[i, \ldots, n]})} \otimes w_{st(A\downarrow_{i+1, \ldots, n})}
\]

(10)

where in the sum \( B \vdash [k] \) and \( C \vdash [n-k] \) and the notation \( A\downarrow_S \) is the set partition \( A \) restricted to the entries which are in \( S \) (throwing away any empty parts).

Since \( \text{NCSym} \) is non-commutative and co-commutative, \( \text{NCSym}^* \) is commutative and non-co-commutative (as seen in the product \( \boxtimes \) and coproduct \( \otimes \) formulas). We leave as an exercise to the reader to show that these formulas verify the definitions of (9) and (10).

We give the following two examples of the product and co-product to help demonstrate the formulas above. The set partitions in the first example below have their parts ordered so that it is easy to read the set \( S \in \bigl[\begin{smallmatrix} 3 \cr 2 \end{smallmatrix}\bigr] \) from the first 3 numbers that appear in the set partition.

\[
w_{\{1,23\}}w_{\{1,2\}} = w_{\{1,23,4,5\}} + w_{\{1,24,3,5\}} + w_{\{1,34,2,5\}} + w_{\{2,34,1,5\}} + w_{\{1,25,3,4\}} + w_{\{1,35,2,4\}} + w_{\{2,35,1,4\}} + w_{\{1,45,2,3\}} + w_{\{2,45,1,3\}} + w_{\{3,45,1,2\}}
\]

\[
= w_{\{1,23,4,5\}} + w_{\{1,24,3,5\}} + w_{\{1,25,3,4\}} + 2w_{\{1,2,3,4,5\}} + 2w_{\{1,2,3,5,4\}} + 3w_{\{1,2,3,4,5\}}
\]

\[
\Delta_{\text{NCSym}}^* (w_{\{13,256,4\}}) = w_{\{13,256,4\}} \otimes 1 + w_{\{13,25,4\}} \otimes w_{\{1\}} + w_{\{13,2,4\}} \otimes w_{\{12\}} + w_{\{13,2\}} \otimes w_{\{1,23\}} + w_{\{1,2\}} \otimes w_{\{1,34,2\}} + w_{\{1\}} \otimes w_{\{2,145,3\}} + 1 \otimes w_{\{13,256,4\}}
\]

3. \( \text{NCSym} \) is free

One of the earliest references to the algebra of \( \text{NCSym} \) is in a paper by Wolfe [13] who considered it with the goal of finding an analogue of the fundamental theorem of symmetric functions. In that paper it was proved that \( \text{NCSym} \) is freely generated by a subset of basis elements \( m_A \).

In [4] we give a combinatorial description of the basis elements which generate this algebra and give a modern restatement of one of her set of generators (the other is similar). To state the result precisely, define for set partitions \( A \vdash [n] \) and \( B \vdash [m] \), \( A* B = \{ A_1 \cup (B_1+n), A_2 \cup (B_2+n), \ldots, A_r \cup (B_r+n) \} \) (with conventions that \( A_r \) and \( B_r \) are empty sets for \( r \) greater than the respective lengths). Then say \( A \) is non-splitable if it cannot be written as \( A = B* C \) for non-empty set partitions \( B \) and \( C \). Note that non-splitable is not the same as atomic and even the number of non-splitable elements of size \( n \) and length \( k \) is not equal to the number of atomic set partitions of size \( n \) and length \( k \).

**Proposition 7.** (see [13] and [4]) The algebra \( \text{NCSym} \) is freely generated by the basis elements \( m_A \) where \( A \) is non-splitable.
Here we give a very short combinatorial proof of the result that $NCSym$ is free.

Define a basis, $\{p_A\}_A$ of $NCSym$ as

$$p_A = \sum_{B \geq A} m_B.$$  

This basis was considered in \cite{9} as the natural analogue of the power basis of $Sym$ in the algebra $NCSym$. Indeed, in \cite{2} we show that this basis is fundamental to $NCSym$ for representation theoretic reasons and we prove that it has the following property.

**Proposition 8.** (Lemma 4.1.(i) of \cite{2})

$$p_A p_B = p_{A \mid B}$$

We can conclude from this result the following corollary.

**Corollary 9.** $NCSym$ is freely generated by the elements $p_A$ where $A$ is atomic.

*Proof*. Since the $p_A$ are multiplicative, we have that for $A^t = (A^{(1)}, A^{(2)}, \ldots, A^{(k)})$,

$$p_A = p_{A^{(1)}} p_{A^{(2)}} \cdots p_{A^{(k)}}.$$  

Because the $p_A$ are linearly independent, the $\{p_A : A \text{ atomic}\}$ must be algebraically independent. 

□

Before departing from the question of freeness of $NCSym$ we would like to introduce another basis which is multiplicative and arises naturally in the study of this space. To this end we define the following order on the collection of set partitions of size $n$. Say that $B$ covers $A$ in our order if and only if there are two parts of $A$, $A_i$ and $A_i'$, with all elements in $A_i$ less than all elements in $A_i'$ and $B = (A/\{A_i, A_i'\}) \cup \{A_i \cup A_i'\}$. Define $A \leq_s B$ as the closure of this covering relation.

For $n = 3$ and $n = 4$ we have the following poset diagrams.

![Hasse diagram of set partitions of size 3 and 4 with $\leq_s$ order.](image)

**Figure 1.** Hasse diagram of set partitions of size 3 and 4 with $\leq_s$ order.
Notice that the maximal elements in these two posets are all atomic, although not all atomic set partitions are maximal (e.g. \{14, 2, 3\} is atomic and not maximal).

**Proposition 10.** The order \( \leq_s \) is ranked by \( n - \ell(A) \) and the poset is Eulerian. Moreover, fixing a set partition \( A \), the sub-poset consisting of the elements \( \{B : B \leq_s A\} \) with the \( \leq_s \) order is a boolean lattice.

**Proof.** In the covering relation defining the \( \leq_s \) order we have that \( B \) is of length 1 less than the length of \( A \) and hence the poset is ranked. The minimal element of this poset is \( \{1, 2, \ldots, n\} \) and hence the rank of any set partition \( A \) in this poset is \( n - \ell(A) \).

Now let \( \mathcal{P}_A = \{B : B \leq_s A\} \) be the sub-poset with maximal element \( A \). For any element \( B \in \mathcal{P}_A \), let \( \phi(B) \) be a subset of \([n]\) which consists of the elements which are not maximal in their parts in \( B \). Now if \( B \) covers \( C \in \mathcal{P}_A \), then \( \phi(C) \cup \{\max(C_i)\} = \phi(B) \) since \( B \) is formed from \( C \) by joining two parts \( C_i \) and \( C_i' \) where all elements of \( C_i \) are less than all elements of \( C_i' \). This is the covering relation for the boolean lattice and hence \( \mathcal{P}_A \) is isomorphic to the boolean lattice on the set \( \phi(A) \).

This also implies that any interval of the poset is Eulerian since the Möbius function at \( A \) depends only on the interval \( \mathcal{P}_A \). Since this is a boolean lattice the Möbius function is equal to \((-1)^{rank(A)}\).

\[ \square \]

We observe in this poset that the number of elements covered by \( A \) is equal to \( n - \ell(A) \). This is because the number of places that a part of \( A \) can be split is equal to the number of elements of \( \phi(A) \).

Define now a new basis of \( \text{NCSym} \) by

\[ q_A = \sum_{A \leq_s B} m_B \]

The property that makes this basis intrinsic to this algebra is that the elements \( q_A \) where \( A \) is atomic are multiplicative, hence are algebraic generators of this basis. We leave the proof of the following theorem to the diligent reader.

**Theorem 11.** For \( A \vdash [n] \) where \( A^l = (A^{(1)}, A^{(2)}, \ldots, A^{(k)}) \) we have that

\[ q_A = q_{A^{(1)}} q_{A^{(2)}} \cdots q_{A^{(k)}}. \]

Moreover, the algebra \( \text{NCSym} \) is freely generated by the elements

\[ \{q_A | A \vdash [n], n \geq 0, A \text{ is atomic}\}. \]
We show in this section that the algebra $\text{NCSym}^*$ is generated algebraically by the elements $w_A$ such that $A$ is Lyndon (recall that $A$ is Lyndon if $A^1$ is a Lyndon word in the alphabet of set partitions). To begin, we consider the shuffle of two words $u \shuffle v$ to be the multi-set of all shuffles of the words $u$ and $v$. In this definition we set for $S \in \binom{\ell(u)+\ell(v)}{\ell(u)}$, $u \shuffle_S v$ to be the word with the $i^{th}$ letter in $u$ if $i \in S$ and a letter of $v$ if $i \notin S$ (the letters in relative order of the word $u$ and $v$). We are then considering $u \shuffle v$ to be the multi-set of words $\{u \shuffle_S v : S \in \binom{\ell(u)+\ell(v)}{\ell(u)}\}$.

Given a totally ordered alphabet $X = x_1, x_2, x_3, \ldots$, we denote by $X^\shuffle$ the shuffle algebra. This is the commutative algebra of all words in the alphabet $X$ with the shuffle product. We will use the following proposition about shuffle algebras.

**Proposition 12.** (\cite{8} Theorem 6.1) For any alphabet $X$, the algebra $X^\shuffle$ is freely generated by the elements of the set of words $\{u | u \text{ is Lyndon}\}$

As mentioned in the first section, a set partition $A$ corresponds to a list of atomic set partitions $A^1$ which we may consider as a word in the alphabet of atomic set partitions.

**Lemma 13.** Let $A \vdash [n]$ and $B \vdash [m]$. For $S \in \binom{[n+m]}{n}$ then

$$\ell((A \uparrow_S \cup B \uparrow_{S'})^1) \leq \ell(A^1) + \ell(B^1)$$

with equality if and only if $(A \uparrow_S \cup B \uparrow_{S'})^1 \in A^1 \shuffle B^1$.

**Proof.** Let $A^1 = (A^{(1)}, A^{(2)}, \ldots, A^{(d)})$ and $B^1 = (B^{(1)}, B^{(2)}, \ldots, B^{(p)})$. For any $S \in \binom{[n+m]}{n}$, the atomic parts $A^{(i)} \uparrow_S$ resp. $B^{(j)} \uparrow_{S'}$ of $(A^1)^1$ resp. $(B^1)^1$ can never be split in smaller parts in $(A^1 \uparrow_S \cup B^1 \uparrow_{S'})^1$. This implies that the atomic parts of $(A \uparrow_S \cup B \uparrow_{S'})^1$ combines parts of $(A^1)^1$ and $(B^1)^1$. These correspond to the parts of $A^1$ and $B^1$. The inequality follows from this fact. For the equality, we remark that this happen if and only if no atomic parts of $A^1$ and $B^1$ are combined, hence $(A \uparrow_S \cup B \uparrow_{S'})^1$ is a shuffle of $A^1$ and $B^1$. For the converse we remark that any word in the shuffle $A^1 \shuffle B^1$ is obtained by a unique choice of $S$. \qed

Now because $w_A w_B = \sum_{S \in \binom{[n+m]}{n}} w_{A \uparrow_S \cup B \uparrow_{S'}}$ this tells us that the sum may be broken up into terms $w_C$ where $C^1$ has maximal length and all other terms.

**Corollary 14.**

$$w_A w_B = \sum_{C : C^1 \in A^1 \shuffle B^1} w_C + \sum_{D : \ell(D') < \ell(A^1) + \ell(B^1)} w_D$$

In order to find algebraic generators, we find a total order such that $\text{NCSym}^*$ is isomorphic to a shuffle algebra. To this end we remark that given a total order on the atomic set partitions we get
a lexicographic order $\leq_{\text{lex}}$ on words in the atomic set partitions. Now for $A^i = (A^{(1)}, A^{(2)}, \ldots, A^{(k)})$ we will denote the number of atomic set partitions in $A^i$ by $\ell(A^i) = k$. We then define $A <_\tau B$ if $\ell(A^i) < \ell(B^i)$, or $\ell(A^i) = \ell(B^i)$ and $A^i <_{\text{lex}} B^i$. Using this order in Corollary 14 we have that all terms in the second sum are strictly smaller than all terms in the first sum. By triangularity, this shows that $\text{NCSym}^*$ is isomorphic to the shuffle algebra in the alphabet given by atomic set partitions. Combining this with Proposition 12, we have shown the following theorem.

**Theorem 15.** The algebra $\text{NCSym}^*$ is freely generated by the elements

$$\{w_B \mid B \vdash [n], n \geq 0, B \text{ is Lyndon}\}.$$

5. **Set compositions and $\text{NCQSym}$**

5.1. **Set compositions.** There is an algebra related to $\text{NCSym}$ which is indexed by set compositions (ordered set partitions). A set composition of $n$ is a sequence $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_{\ell(\Phi)})$ where each $\Phi_i$ is a non empty subset of $[n]$ and $\Phi_i \cap \Phi_j = \emptyset$ if $i \neq j$ (note that we are using $\ell(\Phi)$ to represent the number of parts or the length of the set composition $\Phi$). As with set partitions, the size of the set composition will be denoted $|\Phi| = |\Phi_1| + |\Phi_2| + \cdots + |\Phi_{\ell(\Phi)}|$. The number of set compositions of size $n$ and of length $k$ is equal to $k!S_{n,k}$ (sequence A019538) the total number of set compositions of $n$ is of course $\sum_{k=1}^{n} k!S_{n,k}$ (sequence A000670) are sometimes known as the ordered Bell numbers.

As we used symbols $A, B, C, D$ to represent set partitions, we will generally use symbols $\Phi, \Psi, \Pi$ and $\Gamma$ to represent set compositions and use the notation $\Phi \vdash [n]$ to indicate that $\Phi$ is a set composition of $n$. In addition, when writing explicit set compositions we will leave off any $\{\}$ around the parts and commas removed from between the elements of each of the sets for brevity. For example, the set composition $([3, 4], [2], [1, 5])$ will be represented by $(34, 2, 15)$.

The set compositions of $n$ are endowed with a natural order similar to the refinement order on set partitions. We will say that for $\Phi, \Psi \vdash [n]$, $\Phi \leq \Psi$ if for each $i$, $\Phi_i \subseteq \Psi_j$ for some $j$ and if $\Phi_i \subseteq \Psi_j$, then either $\Phi_{i+1} \subseteq \Psi_j$ or $\Phi_{i+1} \subseteq \Psi_{j+1}$.

This is a order with covering relations $\Phi \lessdot (\Phi_1, \ldots, \Phi_i \cup \Phi_{i+1}, \Phi_{i+2}, \ldots, \Phi_{\ell(\Phi)})$ for each $1 \leq i < \ell(\Phi)$. This order does not form a lattice on the set of set compositions of $n$, but it is a ranked poset with no single minimal element (the permutations correspond to all of the minimal elements of this poset) and a maximal element $([n])$. As an example, we show the diagram of the poset of set compositions of size 3 defined by this relation here.

Although there is not a lattice structure, there are two operations similar to the meet and join operations on set partitions for the poset of set compositions which we shall exploit for notational purposes. For two set partitions $\Phi, \Psi \vdash [n]$, there is a well defined element $\Phi \lor \Psi$ which is the smallest element which is larger than both $\Phi$ and $\Psi$. There is not necessarily a single element which is a greatest element less than both $\Phi$ and $\Psi$, so we will choose one canonical one as a
representative. Define
\( (11) \Phi \land \Psi = (\Phi_1 \cap \Psi_1, \Phi_1 \cap \Psi_2, \ldots, \Phi_2 \cap \Psi_1, \Phi_2 \cap \Psi_2, \ldots, \Phi_{\ell(\Phi)} \cap \Psi_{\ell(\Psi)}) \).

Note that in general \( \Phi \land \Psi \neq \Psi \land \Phi \) as one would normally have in a lattice structure. It will hold that \( \Phi \land \Psi \leq \Phi \), but in general we do not have \( \Phi \land \Psi \leq \Psi \).

The operations which exist on set partitions can be extended in a natural way to set compositions. For instance, define for \( \Phi \models [n] \) and \( \Psi \models [k] \),
\[ \Phi \upharpoonright \Psi = (\Phi_1, \ldots, \Phi_{\ell(\Phi)}, \Psi_1 + n, \ldots, \Psi_{\ell(\Psi)} + n). \]

It is easily checked that each of the operations \( \lor, \land \) and \( \upharpoonright \) are associative. Of the three, only the \( \lor \) operation is commutative. Also, for \( S \in \binom{[n+m]}{n} \) we denote by \( \Phi \uparrow S \) the set composition obtained by raising the entries in \( \Phi \) so that they remain in the same relative order but the entries are the elements of \( S \). As well if \( \Psi \) is a set composition of an arbitrary finite set of integers, \( st(\Psi) \) represent the set composition formed by lowering the entries of \( \Psi \) keeping the values in the same relative order.

With the definition of the \( \upharpoonright \) operation we may also define the concept of ‘splitable’ with respect to this operation. We will say that \( \Phi \models [n] \) is splitable if there exists non-empty set compositions \( \Psi \models [k] \) and \( \Gamma \models [n-k] \) such that \( \Phi = \Psi \upharpoonright \Gamma \). If \( \Phi \) is not splitable then it will be called atomic. Just as we did for set partitions, we will define a bijection between set compositions \( \Phi \) and sequences of set compositions \( \Phi^r = (\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(k)}) \) where \( \Phi^{(r)} \) are each non-empty atomic set compositions such that \( \Phi = \Phi^{(1)} \upharpoonright \Phi^{(2)} \upharpoonright \cdots \upharpoonright \Phi^{(k)} \).

Since we have that \( \ell(\Phi) = \ell(\Phi^{(1)}) + \ell(\Phi^{(2)}) + \cdots + \ell(\Phi^{(k)}) \) this bijection implies the following enumerative result regarding set compositions.

**Proposition 16.** Let \( c_{k,i} \) represent the number of atomic set compositions of size \( k \) and of length \( i \) (12 sequence A109062), then we have
\[ (12) \frac{1}{1 - \sum_{k \geq 1} \sum_{i=1}^{k-1} c_{k,i} i^k q^k} = \sum_{n \geq 0} \sum_{r \geq 1} r! S_{n,r} t^r q^n \]
There is a natural map from the set compositions to the set partitions where one 'forgets' the order on the parts of the set composition. We will use the notation $A(\Phi)$ to represent the set partition \(\{\Phi_1, \Phi_2, \ldots, \Phi_{\ell(\Phi)}\}\). It follows that this map is compatible with the lattice of set partitions in the sense that $A(\Phi | \Psi) = A(\Phi) | A(\Psi)$, $\ell(A(\Phi)) = \ell(\Phi)$, $A(\Phi \lor \Psi) = A(\Phi) \lor A(\Psi)$, $A(\Phi \land \Psi) = A(\Phi) \land A(\Psi)$ and if $\Phi \leq \Psi$ then $A(\Phi) \leq A(\Psi)$.

In the next section we will define a bialgebra related to $NCSym$ which is both non-commutative and non-cocommutative (that is, the graded dual of this algebra will also be non-commutative).

5.2. The Hopf algebra of $NCQSym$ and its graded dual. As we defined an analogue of the symmetric functions which are non-commutative, one may also define an analogue of the quasi-symmetric functions in non-commuting variables. Recall that a polynomial is quasi-symmetric if for every increasing sequence of indices $i_1 < i_2 < \cdots < i_k$ and every composition $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ of length $k$ (with $\alpha_i > 0$) we have that the coefficient of the monomial $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ is equal to the coefficient of the monomial $x_{\gamma_1}^{\alpha_1} x_{\gamma_2}^{\alpha_2} \cdots x_{\gamma_k}^{\alpha_k}$.

Now for a sequence $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{N}^n$, we let $\Psi = \Delta(\gamma)$ be a set composition $\Psi | = [n]$ such that $i \in \Psi \#(\gamma_r \gamma_r < \gamma_i) + 1$. For example, $\Delta(2, 1, 1, 7, 9, 1, 2, 7) = (236, 17, 48, 5)$.

We will define a polynomial in $n$ non-commuting variables $\{x_1, x_2, \ldots x_n\}$ to be quasi-symmetric if for every pair of sequences $\gamma, \tau \in [n]^k$ such that $\Delta(\gamma) = \Delta(\tau)$, the coefficient of $x_{\gamma_1} x_{\gamma_2} \cdots x_{\gamma_k}$ is equal to the coefficient of $x_{\tau_1} x_{\tau_2} \cdots x_{\tau_k}$.

There is a natural basis of the space of quasi-symmetric polynomials which are similar to monomial symmetric functions and it is this basis for which we define an analogue. For an ordered set of variables $X$, and for a set composition of $n$ which is of length $k$ define

$$M_\Phi[X] = \sum_{\gamma} x_{\gamma_1} x_{\gamma_2} \cdots x_{\gamma_n}$$

where the sum is over all sequences $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{N}^n$ such that $\Delta(\gamma) = \Phi$.

Just as we did for the symmetric functions in non-commutative variables we can define a product and coproduct on this space which endows it with a Hopf algebra structure. Without giving reference to the variables we define an abstract vectors space by

$$NCQSym_n = \mathcal{L}\{M_\Phi : \Phi | = [n]\}$$

and then set $NCQSym = \bigoplus_{n \geq 0} NCQSym_n$.

The product is defined so that it inherits the product $M_\Phi[X]M_\Psi[X]$ from the space of non-commutative polynomials. For $\Phi | = [n]$ and $\Psi | = [k]$

$$M_\Phi M_\Psi = \sum_{(([n]) ([k])) \land \Gamma = \Phi | \Psi} M_\Gamma.$$
In addition we can define a coproduct structure which is the natural analogue of the coproduct structure on the other spaces since it follows by essentially replacing one set of variables by two. Define the map \( \Delta : NCQSym_n \to \bigoplus_{k=0}^n NCQSym_k \otimes NCQSym_{n-k} \) by

\[
\Delta(M_\Phi) = \sum_{i=0}^{\ell(\Phi)} M_{st(\Phi_1,\ldots,\Phi_i)} \otimes M_{st(\Phi_{i+1},\ldots,\Phi_{\ell(\Phi)})}.
\]

There is a natural embedding of \( NCSym \) to \( NCQSym \) just as there is a relationship between the symmetric functions and the quasi-symmetric functions. The monomial basis of symmetric functions of \( NCSym \) are related to the monomial basis of \( NCQSym \) by the map \( \theta : NCSym \to NCQSym \) which is defined by

\[
\theta(m_A) = \sum_{A=A(\Phi)} M_\Phi.
\]

We will also be interested in the graded dual of the algebra \( NCQSym \). We will define this algebra as the linear span of the elements \( W_\Phi \) which are the basis which is dual to the monomial basis \( M_\Phi \). Set \( NCQSym^*_n = \mathcal{L}\{W_\Phi : \Phi \models [n]\} \) and then \( NCQSym^* = \bigoplus_{n \geq 0} NCQSym^*_n \). The pairing between \( NCQSym \) and \( NCQSym^* \) defined by \( [M_\Phi, W_\Psi] = \delta_{\Phi \Psi} \) defines the product and coproduct on \( NCQSym^* \) through the duality relations

\[
[M_\Phi M_\Psi, W_\Gamma] = [M_\Phi \otimes M_\Psi, \Delta^*(W_\Gamma)].
\]

From these two relations it is easy to determine that for set compositions \( \Phi \models [n] \) and \( \Psi \models [k] \) we have

\[
W_\Phi W_\Psi = \sum_{S \in \binom{[n+m]}{n}} W_{\Phi S \Psi S^c}.
\]

where \( \cdot \) represents concatenation. Finally, the coproduct \( \Delta^* \) is given explicitly on the dual basis as

\[
\Delta^*(W_\Phi) = \sum_{k=0}^n \sum_{\Phi \models [k], \Psi \models [n-k]} W_\Psi \otimes W_\Gamma.
\]

**Remark 2.** It is interesting to remark that one can define on \( NCQSym \) a second internal comultiplication corresponding to the substitution \( M_\Phi[X] \mapsto M_\Phi[XY] \). Here we use the lexicographic order to order the alphabet \( XY \). This gives

\[
\Delta^\odot(W_\Phi) = \sum_{\Psi \wedge \Gamma = \Psi} W_\Psi \otimes W_\Gamma.
\]

The dual to this operation gives an algebra on \( NCQSym^*_n \) for all \( n \). This is precisely the dual of the Solomon-Tits algebra initially defined by Tits in \( [11] \) (see also \([2]\)).
Note that because there is an embedding of \( \theta : NCSym \rightarrow NCQSym \) then the dual of this map is a projection from \( NCQSym^* \) to \( NCSym^* \) which may be given explicitly as the surjection \( \theta^* : NCQSym^* \rightarrow NCSym^* \) by \( \theta^*(W_\Phi) = w_{A(\Phi)} \). This of course follows from the defining relation 
\[
\theta(m_A), W_\Phi |_{NCQSym} = [m_A, \theta^*(W_\Phi)]_{NCSym}.
\]

6. **NCQSym is free**

There is another order on set compositions defined with the covering relations \( \Phi \lessdot \Phi_1, \ldots, \Phi_i \cup \Phi_{i+1}, \Phi_{i+2}, \ldots, \Phi_{\ell(\Phi)} \) for each \( 1 \leq i < \ell(\Phi) \) such that every integer in \( \Phi_i \) is less than every integer in \( \Phi_{i+1} \). This new order is analogous to the second order that we defined on set partitions and so we use the notation \( \leq^* \) to denote the closure of these covering relations.

Under this order the set composition \((n, n-1, \ldots, 1)\) is not comparable to any other set composition. In general, the connected component to any particular set composition will be a boolean lattice. This is easy to see since any set composition is greater than or equal to a minimal element \( \Pi \) which has all of the number in \([n]\) in separate parts (essentially, a permutation). It is easy to see that the elements which are above each of the permutations form a boolean lattice on this order which is isomorphic to the boolean lattice of the set of \( \{i \mid \Pi_i < \Pi_{i+1}\} \). In the diagram below we see how the set compositions of size 3 are simply the union of boolean lattices.

![Figure 3. Hasse diagram of \( \{\Phi \models [3]\}, \geq^* \) ](image)

Now we can define a basis \( \{Q_\Phi\}_\Phi \) using this order by
\[
Q_\Phi = \sum_{\Phi' \geq \Phi} M_{\Phi'}.
\]

This basis is not multiplicative as was the \( q_A \) basis of \( NCSym \), but it does have the following elegant rule for the product of two basis elements. For two set compositions \( \Phi \models [n] \) and \( \Psi \models [m] \), we define \( \Phi \shuffle \Psi \) to be the (non-commutative) shuffle \( \Phi \shuffle \Psi \uparrow_n \) where \( \Phi \) is viewed as a word in the subset of \([n]\) and \( \Psi \uparrow_n \) is viewed as the word in the subset of \( \{n+1, n+2, \ldots, n+m\} \) obtained from \( \Psi \) by adding \( n \) to every entries.
**Theorem 17.** For $\Phi \models [n]$ and $\Psi \models [m]$, 

$$Q_\Phi Q_\Psi = \sum_{\Gamma \in \Phi \sqcup \Psi} Q_{\Gamma}$$

**Proof.** From the definitions we have 

$$Q_\Phi Q_\Psi = \sum_{\Phi' \geq \Phi, \Psi' \geq \Psi} M_{\Phi'} M_{\Psi'} = \sum_{\Phi' \geq \Phi, \Psi' \geq \Psi} \sum_{(([n]) \land \Gamma') = \Phi' \land \Psi'} M_{\Gamma'}.$$ 

On the other hand we have 

$$\sum_{\Gamma \in \Phi \sqcup \Psi} Q_{\Gamma} = \sum_{\Gamma \in \Phi \sqcup \Psi} \sum_{\Gamma' \geq \Gamma} M_{\Gamma'}.$$ 

We first remark that both equations are multiplicity free. In the right hand side of the second equation, we have that $\Gamma'$ satisfies $\Gamma' \geq \Gamma$ for $\Gamma \in \Phi \sqcup \Psi$. This happen if and only if $(\Phi \land \Psi) \land \Gamma' = \Phi \land \Psi$ which is equivalent to $(([n]) \land ([m])) \land \Gamma' = \Phi' \land \Psi$. This gives us the terms in the right hand side of the first equation and conclude the desired equality. 

For example, 

$$Q_{(13, 2)} Q_{(1, 2)} = Q_{(13, 2, 4, 5)} + Q_{(13, 4, 2, 5)} + Q_{(13, 4, 5, 2)} + Q_{(4, 13, 2, 5)} + Q_{(4, 13, 5, 2)} + Q_{(4, 5, 13, 2)}.$$ 

Notice that this is also a non-commutative product even though we are using the shuffle operation.

This result then shows the consequence which is one of the main goals of this paper, namely,

**Theorem 18.** The algebra $NCQSym$ is freely generated by the elements 

$$\{ Q_\Phi \mid \Phi \models [n], n \geq 0, \Phi \text{ is atomic}\}.$$ 

**Proof.** For $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_\ell)$ we define $w(\Phi) = w_1 w_2 \cdots w_\ell$ to be the word such that $w_i = j$ if and only if $i \in \Phi_j$. For two set composition $\Phi$ and $\Psi$, we say that $\Phi \preceq_{\text{lex}} \Psi$ if $w(\Phi) \leq_{\text{lex}} w(\Psi)$.

We have that if $\Gamma \in \Phi^{(1)} \sqcup \cdots \sqcup \Phi^{(k)}$, then $\Phi^{(1)} \mid \Phi^{(2)} \mid \cdots \mid \Phi^{(k)} \preceq_{\text{lex}} \Gamma$. This implies that if $\Phi = \Phi^{(1)} \mid \Phi^{(2)} \mid \cdots \mid \Phi^{(k)}$ is the unique minimal atomic decomposition of $\Phi$, then 

$$Q_{\Phi^{(1)}} Q_{\Phi^{(2)}} \cdots Q_{\Phi^{(k)}} = Q_\Phi + \sum_{\Phi' \preceq_{\text{lex}} \Gamma} c_{\Gamma'} Q_{\Gamma'}.$$ 

The theorem follows by triangularity. 

It is also interesting that because the order that we have chosen is an Eulerian poset, we know that the dual elements in $NCQSym^*$ are easy to calculate. In fact, we have 

$$Q_\Phi^* = \sum_{\Psi \geq \Phi} (-1)^{\ell(\Phi) - \ell(\Psi)} W_\Psi.$$
7. NCQSym is cofree

So that we can again establish a basis which has elegant multiplicative properties in the algebra of NCQSym*, we need to introduce another order on set compositions. The orders which we define for these bases arise because they interact nicely with respect to the product operations on the algebras, what is remarkable is that at the same time the posets have properties which make them interesting to study in their own right. In a related paper with C. Hohlweg [3] we consider some of the properties of this poset in more detail.

We will define a order in which only elements which have have the same image under α are comparable. Consider the order which we define \( \Phi \leq_{\#} \Psi \) if \( \alpha(\Phi) = \alpha(\Psi) \) and if \( \Phi^i = (\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(k)}) \) then \( \Phi^{(i)} = st(\Psi_{\ell_{i-1}+1}, \Psi_{\ell_{i-1}+2}, \ldots, \Psi_{\ell_i}) \) where \( \ell_0 = 0 \) and \( \ell_i = \ell(\Phi^{(1)}) + \ell(\Phi^{(2)}) + \cdots + \ell(\Phi^{(i)}) \). In the appendix we include the Hasse diagrams for this poset for the set compositions of size 3 and 4.

For example, \((123,4,56,7)\) is smaller than or equal to all set compositions \(\Psi\) such that \(\alpha(\Psi) = (3,1,2,1)\) since \((123,4,56,7)^{\top} = ((123), (1), (12), (1))\) and \(st(\Psi_1) = (123)\), \(st(\Psi_2) = (1)\), \(st(\Psi_3) = (12)\) and \(st(\Psi_4) = (1)\) for all \(\Psi | [7]\) with \(\alpha(\Psi) = (3,1,2,1)\). \((134,2,56,7)\) is smaller than \((156,2,37,4)\) because \((134,2,56,7)^{\top} = ((134,2), (12), (1)) = (st(156,2), st(37), st(1))\), but it is not comparable to say \((124,5,36,7)\) since \((st(124,5), st(36), st(7)) = ((123,4), (12), (1))\).

In the appendix we give drawings of the poset for \(n = 4\) representing the elements of this order. In particular, we note that for the set compositions \(\Phi | [n]\) with \(\alpha(\Phi) = (1^n)\) this defines a order on permutations (since there is a natural correspondence between permutations and set compositions with \(\alpha(\Phi) = (1^n)\)) where the number of elements of rank \(k\) are given by the coefficient of \(x^k\) in the generating function \(g(x)^{\frac{1}{1+g(x)-tg(x)}}\) where \(g(x) = \sum_{k \geq 0} k! x^k\) (OEIS sequence A059438). See [3] for more results about this order and other related ones.

This order appears naturally in the algebra of NCQSym* because if we define a basis
\[
V_{\Phi} = \sum_{\Phi' \geq_{\#} \Phi} W_{\Phi'}
\]
then the basis has very elegant rule for computing the product.

**Theorem 19.** For \(\Phi | [n]\) and \(\Psi | [k]\), then
\[
V_{\Phi} V_{\Psi} = V_{\Phi | \Psi}
\]

**Proof.** We have
\[
V_{\Phi} V_{\Psi} = \sum_{\Phi' \geq_{\#} \Phi} \sum_{\Psi' \geq_{\#} \Psi} W_{\Phi'} W_{\Psi'} = \sum_{\Phi' \geq_{\#} \Phi} \sum_{\Psi' \geq_{\#} \Psi} \sum_{S \in {[n+m] \choose n}} W_{\Phi'^S \Psi' \Psi}. \]
On the other hand we have
\[
V_{\Phi | \Psi} = \sum_{\Gamma \geq_{\#} \Phi | \Psi} W_{\Gamma}.
\]
We remark that both equations are multiplicity free. For any $S \in \binom{[n+m]}{n}$, $\Phi' \geq_\# \Phi$ and $\Psi' \geq_\# \Psi$ we have that
\[
\alpha(\Phi' \uparrow_S \cdot \Psi' \uparrow_{S^c}) = \alpha(\Phi'|\Psi') = \alpha(\Phi') \cdot \alpha(\Psi') = \alpha(\Phi) \cdot \alpha(\Psi) = \alpha(\Phi|\Psi).
\]
Also we have that $(\Phi|\Psi)^! = \Phi^! \cdot \Psi^! = (\Phi^{(1)}, \ldots, \Phi^{(k)}, \Psi^{(1)}, \ldots, \Psi^{(r)})$. Since
\[
\Phi^{(i)} = st(\Phi'_{\ell_i -1+1}, \ldots, \Phi'_{\ell_i}) = st(\Phi'_{\ell_i -1+1} \uparrow_S, \ldots, \Phi'_{\ell_i} \uparrow_S)
\]
and
\[
\Psi^{(j)} = st(\Psi'_{t_j -1+1}, \ldots, \Phi'_{t_j}) = st(\Psi'_{t_j -1+1} \uparrow_S, \ldots, \Phi'_{t_j} \uparrow_S)
\]
for $\ell_i = \ell(\Phi^{(1)}) + \cdots + \ell(\Phi^{(i)})$ and $t_j = \ell(\Psi^{(1)}) + \cdots + \ell(\Psi^{(j)})$, we have that $\Phi' \uparrow_S \cdot \Psi' \uparrow_{S^c} \geq_\# \Phi|\Psi$. Conversely, if $\Gamma \geq_\# \Phi|\Psi$, there is a unique $S$ such that $\Gamma = \Phi' \uparrow_S \cdot \Psi' \uparrow_{S^c}$. This shows the desired equality.

For example if we compute $V_{(1,2)} = W_{(1,2)}$ and $V_{(1,2)} = W_{(1,2)} + W_{(2,1)}$ then
\[
V_{(1,2)}V_{(2,1)} = W_{(1,2)}W_{(2,1)} = W_{(12,4,3)} + W_{(13,4,2)} + W_{(14,3,2)} + W_{(23,4,1)} + W_{(24,3,1)} + W_{(34,2,1)} = V_{(12,4,3)}.
\]

In light of Theorem 19 it is clear that the set $\{V_{\Phi} \mid \Phi = [n], n \geq 0, \Phi \text{ is atomic}\}$ freely generate $NCQSym^*$. This implies our last theorem.

**Theorem 20.** The algebra $NCQSym$ is cofree.

## 8. Appendix A: Poset of $\{\{\Phi = [4]\}, \leq_\#\}$ and $\{\{\Phi = [3]\}, \leq_\#\}$

For a set composition $\Phi$, let $\alpha(\Phi)$ be the composition $([\Phi_1], [\Phi_2], \ldots, [\Phi_{\ell(\Phi)}])$. The poset of $\{\{\Phi = [n]\} \mid \alpha(\Phi) = (\alpha_1, \alpha_2, \ldots, \alpha_k)\}$ is isomorphic to the poset $\{\{\Phi = [n]\} \mid \alpha(\Phi) = (\alpha_1, \alpha_2, \ldots, \alpha_k)\}$ by reversing the entries in $\Phi$ and complementing the entries. This appendix includes the Hasse diagrams for the set compositions of size 3 and 4.

\[
\begin{align*}
\begin{array}{ccc}
(3,1,2) & (3,2,1) & (2,3,1) \\
(2,1,3) & (1,3,2) & (13,2) & (23,1) & (2,13) & (3,12) \\
(12,3) & (1,23) & (123) & (123) & (123) & (123)
\end{array}
\end{align*}
\]

**Figure 4.** Poset of $\{\Phi = [3]\}$
Figure 5. Poset of $\{ \Phi \models [4] : \alpha(\Phi) = (3, 1) \}$

Figure 6. Poset of $\{ \Phi \models [4] : \alpha(\Phi) = (2, 2) \}$

Figure 7. Poset of $\{ \Phi \models [4] : \alpha(\Phi) = (1, 2, 1) \}$

Figure 8. Poset of $\{ \Phi \models [4] : \alpha(\Phi) = (2, 1, 1) \}$

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