Open end correction for a flanged circular tube using the diffusion process

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Received 9 April 2013, in final form 16 May 2013
Published 27 June 2013
Online at stacks.iop.org/EJP/34/1159

Abstract

In physics lessons on waves and resonance phenomena in high school and college, we usually consider sound waves in a tube with open or closed ends (Halliday \textit{et al} 2002 \textit{Fundamentals of Physics} (Hoboken, NJ: Wiley)). However, it is well known that we need a tube with open end correction $\Delta L$. The correction for a flanged circular tube was first given by Rayleigh (1870 \textit{Phil. Trans. R. Soc. Lond.} 161 77–118; 1926 \textit{The Theory of Sound} (New York: Dover)) and experimentally checked by several authors (Bosanquet 1877 \textit{Phil. Mag.} 4 216; Blaikley 1879 \textit{Phil. Mag.} 7 339; Boehm 1910 \textit{Phys. Rev.} 31 341; Bate 1930 \textit{Phil. Mag.} 10 617; Bate 1937 \textit{Phil. Mag.} 24 453; Anderson and Ostensen 1928 \textit{Phys. Rev.} 31 267). In this paper, we show the different methods of obtaining the end correction for a circular tube by using a diffusion process.

1. Introduction

A standing sound wave in a tube with an open end has an antinode outside its end point. This small space is called the open end correction (hereafter abbreviated as the OEC). The end correction $\Delta L$ was calculated for an infinite flanged circular tube by Rayleigh [2], and after that, Levine and Schwinger calculated the unflanged end correction as a function of wavelength [4]. The OEC for an infinite flanged circular tube as a function of wavelength was determined by Nomura \textit{et al} by using the radiation impedance method [5]. A recent application of the OEC was given by Howe [6] (and related references therein).

The idea of obtaining the OEC as discussed by Rayleigh comes from the consideration of fluid mechanics [2, 6]. Let us suppose a half infinite space with a connected circular tube filled with nonrotational and noncompressive fluid, and assume a piston moving at a speed of $V$ in the tube, as shown in figure 1. Then, the total kinetic energy of the fluid is given by the
Figure 1. A moving piston in a tube connected to a half infinite space filled with noncompressive fluid.

Figure 2. A circular well with depth $L$. The particle server is at the bottom. Particles diffuse from the bottom of the well into the outer region.

definition of the OEC.

\[
\frac{1}{2} \rho \int (\nabla \phi)^2 \, d^3 x = \frac{1}{2} \rho V^2 S(L + \Delta L),
\]

where $\rho$ is the fluid mass density, $\phi$ is the velocity potential, $S$ is the cross-sectional area of the tube, and $\Delta L$ is the OEC. The integration region is taken towards the piston’s inner circular tube and half infinite 3D space. The nonzero velocity region is not only inside the tube but also spread around a branching bay with a speed lower than $V$. This region contributes to the OEC.

Another way to derive the OEC is given by using the radiation impedance (hereafter referred to as the radiation impedance method). [5, 7] Although the theory of the method is slightly complicated, a qualitative explanation is possible. When a pressure field inside a tube oscillates, the air inside the tube is affected by the reaction force from the air outside. This is different from the free edge boundary condition. As a result, the pressure field out of the open end oscillates with the one inside the tube. Therefore the antinode appears a little away from the end of the tube, and forms the OEC. This scenario appears mathematically in the radiation impedance method, as we see in section 4.

In this paper we consider the static diffusion process to derive the OEC and discuss its analogy to the one for a sound wave. As shown in figure 2, we place a circular well at $x = y = 0$ with a radius $a$ and a depth $L$, and consider the particle diffusion of density $n(x, y, z)$. We put a particle bath with a density $n_s$ at the bottom of the well; we thus have $n = 0$ at infinity. By using the above boundary conditions, we approximately solve the static diffusion equation
under the condition that one-dimensional diffusion is occurring in the well. It is important
that the particle density does not vanish at the open end of the well \( z = 0, r < a \), and that it
diffuses into many directions in the open region. The particle density at the open end can be
identified as being similar to the reaction force for the diffusion flow in the well. Because the
diffusion flow in the well is given by the difference of the density at the top and bottom. The
OEC is defined as the length from the top of the well to the point where the density equals
zero just above the open end. The expression of the OEC given above is quite analogous to the
one in sound wave theory. This is done only by changing the density into the pressure (which
is related to the reaction force) at the open end. We show that value of the OEC is \( 8a/3\pi \),
consistent with that given by Rayleigh and Nomura et al., when the tube radius is small enough
compared with the wavelength \( a/\lambda \ll 1 \) [5].

By considering this analogy, the OEC can be calculated simply and the understanding
of physical content is deepened. We believe that these two OEC theories teach us important
educational lessons on the importance of intuitive understanding and analogy. Due to the
difficulty of the mathematical treatment involved, this paper is suitable for undergraduate
students.

2. Estimation of diffusion in a well

In a well, we approximate the diffusion process by using a one-dimensional diffusion
equation (see figure 2). This is very simple, but is essential for our further discussion. At
the top of the well, the space is opens widely and it is therefore possible to suppose that
\( n_{\text{top}} \equiv n(z = 0, r < a) = 0 \). Then, the constant diffusion flow comes up from the bottom.

\[
n(z) = n_s - \frac{J}{D}(z + L), \quad J = \frac{n_s L}{L/D}.
\]

Our purpose is to correct this flow formula.

We take \( n_{\text{top}} \neq 0 \) and then obtain the diffusion field and constant diffusion flow.

\[
n(z) = n_s - \frac{J'}{D}(z + L), \quad J' = \frac{n_s - n_{\text{top}} L}{L/D}.
\]

\( \pi a^2 J' \) is the number of particles supplied per unit time from the well to the open region
while leaving \( n_{\text{top}} \) undetermined. Then, we solve the diffusion equation in open space with the
boundary conditions

\[
n(r, z = \infty) = n(r = \infty, z > 0) = 0,
\]

\[
\frac{\partial n}{\partial z} = \begin{cases} 0 & r \geq a, z = 0 \\
-J'/D & r < a, z = 0.
\end{cases}
\]

The additional condition

\[
n(z = 0, r < a) = n_{\text{top}}
\]

is given below using the solution of the diffusion equation in an open region. Then, two
diffusion fields in the inner and outer wells are connected, and the diffusion equation is
completely solved.

3. Diffusion in the open region

We consider the diffusion equation \( \Delta n = 0 \) in the open region \( z \geq 0 \). The boundary conditions
are (4) and (5).
By supposing an axial symmetry, we obtain the following static diffusion equation for the diffusion field \( n \):
\[
\frac{\partial^2 n}{\partial r^2} + \frac{1}{r} \frac{\partial n}{\partial r} + \frac{\partial^2 n}{\partial z^2} = 0.
\]  
(7)

Then, we have a solution in the form
\[
n(r, z) = \int_0^\infty d\lambda f_\lambda J_0(\lambda r) e^{-\lambda z}.
\]  
(8)

On the plane \( z = 0 \), the diffusion flow in the \(+z\) direction is given by
\[
J(z = 0) = -D \frac{\partial n}{\partial z} \bigg|_{z=0} = D \int_0^\infty \lambda f_\lambda J_0(\lambda r) d\lambda.
\]  
(9)

\( f_\lambda \) should be selected to obtain the boundary condition (5). The following formula solves the problem.
\[
\int_0^\infty J_1(a\lambda) J_0(\lambda r) d\lambda = \begin{cases} 
1/a & (a > r > 0) \\
0 & (r > a > 0).
\end{cases}
\]  
(10)

From (5), (9), and (10), we determine the function \( f_\lambda \) given as
\[
f_\lambda = \frac{a J'_1}{D\lambda} J_1(a\lambda).
\]  
(11)

Then, we obtain the following solution for \( n \):
\[
n(r, z) = \frac{a J'}{D} \int_0^\infty d\lambda \frac{x}{\lambda} J_1(x) J_0 \left( \frac{\lambda r}{a} \right) e^{-\lambda z}.
\]  
(12)

Next, we must determine \( n_{\text{top}} = n(r < a, z = 0) \). From (12), we obtain
\[
n(r, 0) = \frac{a J'}{D} \int_0^\infty \frac{dx}{x} J_1(x) J_0 \left( \frac{r}{a} x \right) \equiv \frac{a J'}{D} N(r/a).
\]  
(13)

The function \( N(r/a) \) is expressed by the following hypergeometric function:
\[
N(r/a) = \begin{cases} 
F(1/2, -1/2, 1; (r/a)^2) & r < a \\
2/\pi & r = a \\
F(1/2, 1/2, 2; (a/r)^2)/(2\pi a) & r > a.
\end{cases}
\]  
(14)

The form of this function is given in figure 3. The smoothed curve is the function \( N(r/a) \). To obtain a consistent solution with a diffusion field in the well, the matching condition (6) is required at \( (r < a, z = 0) \). However, as shown in figure 3, the distribution at \( r/a < 1 \) is not constant. This shows that our solution (12) is not consistent with the inner-well solution, since our approximated solution (3) counts the diffusion out in the \( r \)-direction. To further achieve what it is in this approximation, we approximate the form of \( N(r/a) \) by its mean in the region \( r/a < 1 \).
\[
\langle n \rangle = \frac{2}{a^2} \int_0^a dr F(1/2, -1/2, 1; (r/a)^2) \equiv K = \frac{8}{3\pi} \sim 0.8488.
\]  
(15)

This mean is shown by a solid square line in figure 3. Then, an additional boundary condition (6) leads to
\[
n_{\text{top}} = \frac{aK}{D} J'.
\]  
(16)

From (3) and (16), we obtain
\[
J' = \frac{n_s}{(L+aK)/D}.
\]  
(17)
This result should be compared with the second equation of (2); it shows that the effective depth of the well is slightly larger than $L$ and that the OEC is defined by

$$\Delta L = aK = \frac{8a}{3\pi}.$$ (18)

This is the same OEC for sound in a long-wavelength limit [5]. The point $z = \Delta L$ upwards of the well is the place where the density $n$ vanishes effectively.

Note that, from equation (16) the OEC is defined by

$$\Delta L = \frac{\langle n \rangle}{J/D},$$ (19)

where

$$\langle n \rangle = \frac{1}{\pi a^2} \int_{r<a} n(r, z = 0) dS.$$

4. Discussion

Why do we have the OEC for sound in a long-wavelength limit by using our method? We discuss this point and show that the definition of the OEC by the radiation impedance method is quite similar to that given in this paper.

First we review the radiation impedance method of obtaining the OEC. Let us use a disk as a sound source oscillating in the $z$ direction and having a radius $a$ on the surface $z = 0$. The sound emitted from the source satisfies the wave equation,

$$\left( \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) p = 0,$$ (20)

where $p$ is the pressure field and $c$ is the speed of sound.

The linearized Euler equation is necessary to consider the speed of sound source:

$$\rho \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} p,$$ (21)

where $\rho$ is the mass density.

From the solution of equations (20) and (21), we obtain the fluid velocity $\vec{u}$.
The boundary conditions for $p$ at the surface $z = 0$ are as follows:

$$\frac{\partial p}{\partial z} = \begin{cases} 0 & (u_z = 0) \quad \cdots \quad r \geq a, z = 0, \\ -i\omega \rho u_z \text{ (source)} & \cdots \quad r < a, z = 0, \end{cases}$$  \hspace{1cm} (22)$$

where the time dependence is supposed to be $\exp i\omega t$.

From the emitted sound $p(\vec{x}, t)$, we obtain the force of reaction $F$ to the sound source as

$$F = \int_{r < a} p(z = 0) \, dS. \hspace{1cm} (23)$$

Then, the radiation impedance is defined by

$$Z = \frac{F}{u_z \text{ (source)}} = -i\rho \omega \int_{r < a} p(z = 0) \, dS. \hspace{1cm} (24)$$

From the parallelism to the mechanical impedance, the additional inertial mass due to the radiation is defined using the imaginary part of the radiation impedance:

$$\delta m = \frac{1}{\omega} \Im(Z). \hspace{1cm} (25)$$

This additional mass is supplied by the OEC as

$$\delta m = \rho \Delta L (\pi a^2). \hspace{1cm} (26)$$

So, we obtain the OEC by using the formula

$$\Delta L = \frac{\Im(Z)}{\rho \pi a^2 \omega} = -3\pi \left( \frac{\langle p \rangle}{\partial_z p(r < a)} \right). \hspace{1cm} (27)$$

where

$$\langle p \rangle = \frac{1}{\pi a^2} \int_{r < a} p(z = 0) \, dS.$$

Now, let us check the similarity of this OEC in a diffusion process. The long-wavelength limit is given as

$$\lambda \to \infty, \quad \omega \to 0.$$

Then equation (20) leads us to the Laplace equation. The linearized Euler equation (21) should be compared with the equation of a diffusion flow.

$$\vec{J}/D = -\vec{\nabla} n.$$

Therefore, we have the correspondences; $n \sim p$ and $\vec{J}/D \sim i\omega \rho \vec{u}$. The boundary condition (22) corresponds to

$$\frac{\partial n}{\partial z} = \begin{cases} 0 & \quad \cdots \quad r \geq a, z = 0, \\ -\vec{J}/D & \quad \cdots \quad r < a, z = 0, \end{cases}$$  \hspace{1cm} (28)$$

which is the same as (5).

The definition of the OEC as equation (27) corresponds to

$$\Delta L = -\frac{\langle n \rangle}{\partial_z n(r < a)} = \frac{\langle n \rangle}{\vec{J}/D}. \hspace{1cm} (29)$$

This is the same as the definition of the OEC for the diffusion equation (19). Therefore, the equations, boundary conditions and definitions of the OEC are the same in the long-wavelength limit of these two methods. This is the reason why we can calculate the OEC of sound waves simply by using a diffusion equation. The difficulty in obtaining the OEC is considerably simplified in the long-wavelength limit by considering the diffusion equation.
5. Conclusion

We have calculated the OEC in the diffusion model and shown that its value is the same as the one obtained in the sound wave theory in a long wavelength limit. The OEC of a sound wave occurs by the reaction force from the air in the open region. In the case of diffusion theory, the particle density at the open end works just like reaction force. We showed the strong similarity between these two phenomena and obtained the OEC in more simple way. This shows that analogy can be helpful in understanding physical phenomena, and it is very important for physics education.

Acknowledgments

The authors would like to thank Professor Yokoyama at Gakushuin University for the helpful suggestion.

References

[1] Halliday D, Resnik R and Walker J 2001 Fundamentals of Physics 6th edn (Hoboken, NJ: Wiley)
[2] Rayleigh L 1870 Phil. Trans. R. Soc. Lond. 161 77–118
Rayleigh L 1926 The Theory of Sound vol 2 (New York: Dover)
[3] Bosanquet R H M 1877 Phil. Mag. 4 216
Blakley D J 1879 Phil. Mag. 7 339
Boehm W M 1910 Phys. Rev. 31 341
Bate A E 1930 Phil. Mag. 10 617
Bate A E 1937 Phil. Mag. 24 453
Anderson S H and Ostensen F C 1928 Phys. Rev. 31 267
[4] Levine H and Schwinger J 1948 Phys. Rev. 73 383–406
[5] Nomura Y, Yamamura I and Inawashiro S 1960 J. Phys. Soc. Japan 15 510–7
[6] Howe M S 1999 J. Fluid Mech. 385 63–78
[7] Morse P M and Ingard K U 1987 Theoretical Acoustics (Princeton, NJ: Princeton University Press)