October 6, 2014

On 81 symplectic resolutions of a 4-dimensional quotient by a group of order 32

Maria Donten-Bury and Jarosław A. Wiśniewski

Abstract. We establish a general layout for resolving quotient singularities via constructing the total coordinate, or Cox ring, of their resolutions. In particular we describe the total coordinate ring of symplectic resolutions of a 4-dimensional quotient singularity obtained by an action of a group of order 32. The existence of such resolutions is known by a result of Bellamy and Schedler. We infer the geometric structure of all such resolutions and their flops. Moreover, we represent the group in question as a group of automorphisms of an abelian 4-fold so that the resulting quotient has singularities with symplectic resolutions. As the result we construct a new Kummer-type symplectic 4-fold.

1. Introduction

1.A. Background. Long before the notion of the Mori Dream Space was brought to life by Hu and Keel in [HK00] and put in the context of the Mumford’s Geometric Invariant Theory (GIT), [MFK94], and related to homogeneous coordinate rings, which were introduced for toric varieties by David Cox in [Cox95], a similar concept emerged in the local study of contractions in the Minimal Model Program (MMP). In particular, in his 1992 inspiring talk [Rei92] Miles Reid explained how to view flips in terms of variation of GIT. Although the notion of the total coordinate ring, known also as the Cox ring, has been extensively studied for projective varieties in the last decade, see for example [CT06], [STV07], [AHL10], apparently it has not been used for understanding local contractions nor for resolution of higher dimensional singularities, as it was proposed in [Rei92].

In the present paper we embrace the concept of the total coordinate ring for understanding resolutions of quotient singularities, in the spirit of [Rei92]. The preceding papers about total coordinate rings of resolutions of quotient singularities, [FGAL11], [DB13], concerned the case of surface singularities for which the resolutions are classically known. The excellent book [ADHL14] which provides the exhaustive overview of the present state of knowledge about Cox rings and their applications tackles this problem for toric and complexity-1 case.

The present paper is about symplectic resolutions of higher dimensional quotient singularities. In particular, we focus on a 4-dimensional singularity which is known to admit such a resolution by Bellamy and Schedler, [BS13]. The result of Bellamy and Schedler is based on relation of symplectic resolutions to smoothings of the singularity by a Poisson deformation, see [Nam11], [GK04]. The methods which we provide in the present paper are of completely different nature. They reveal an explicit description of the resolutions and their birational modifications.

This research was conducted within the framework of the Polish National Science Center project 2013/08/A/ST1/00804 with support from grants 2012/07/N/ST1/03202 (first author) and 2012/07/B/ST1/0334 (second author).
which is a significant advantage over the previous approach which was non-effective in this respect.

1.B. Resolutions via Cox rings. The main result of the present paper is the following.

THEOREM 1.1. Let $V$ be a 4-dimensional vector space with a symplectic form $\omega$. Assume that $G$ is a group of order 32 defined in Section 2.C acting on $V$ and preserving $\omega$. By $T$ we denote a 5-dimensional algebraic torus with coordinates $t_i$, $i = 0, \ldots, 4$ associated to 5 classes of symplectic reflections generating $G$.

Then the total coordinate ring $R$ of (every) crepant resolution of the quotient singularity $V/G$ is generated in $\mathbb{C}[V] \otimes \mathbb{C}[T]$ by $t_i^{-2}$ for $i = 0, \ldots, 4$ and $\phi_{ij}t_it_j$ for $0 \leq i < j \leq 4$, where $\phi_{ij}$ are eigen-functions of the action of $Ab(G)$ on $\mathbb{C}[V]^{[G,G]}$, as defined in Lemma 3.12.

There are 81 GIT quotients of $\text{Spec } R$ which yield all crepant resolutions of $V/G$. A distinguished resolution of $V/G$ has a 2-dimensional fiber which is a union of a $\mathbb{P}^2$ blown-up in 4 points and of ten copies of $\mathbb{P}^2$.

The same number of different symplectic resolutions of $V/G$ has been calculated by Bellamy [Bel14], by using the results of Namikawa [Nam10].

Our construction of the total coordinate ring of a symplectic resolution is built up on the base of the coordinate ring of the quotient $V/G$ which is equal to $\mathbb{C}[V]^{[G,G]} \subset \mathbb{C}[V]$, [AG10]. The generators of the Cox ring $R$ of the resolution are the classes of exceptional divisors of the resolution and strict transforms of Weil divisors associated to homogeneous generators of $\mathbb{C}[V]^{[G,G]}$. In our theorem these are the generators of the type $t_i^{-2}$ and $\phi_{ij}t_it_j$, respectively. We note that by the McKay correspondence [Kal02], the exceptional divisors are in correspondence with classes of symplectic reflections in $G$.

1.C. Contents of the paper. In the preliminary section of the paper we introduce definitions and recall results which are needed in subsequent sections. In particular, in Section 2.C we recall the definition of the group $G$ for which the resolution of $V/G$ is known to exist by [BS13]. Subsections 2.D and 2.E provide notions needed for constructing total coordinate rings of quotient singularities. A vast part of subsection 2.D in which monomial valuations are introduced, is about blowing-up a cyclic quotient, an example which illustrates a general construction which we develop later in the paper.

Next, in subsections 3.A and 3.B we introduce the layout for constructing Cox ring of resolutions of quotient singularities. The idea is as follows. It is known that the Cox ring of a quotient singularity $V/G$ is the ring of invariants of the commutator $\mathbb{C}[V]^{[G,G]}$ which decomposes into eigenspaces of the action of the abelianization $Ab(G)$ which can be identified with the class group $\text{Cl}(V/G)$. That is $\mathbb{C}[V]^{[G,G]} = \bigoplus_{\mu \in G^\vee} \mathbb{C}[V]_{\mu}^G$ where $\mathbb{C}[V]_{\mu}^G$ is a rank 1 reflexive $\mathbb{C}[V]^G$ module associated to a character $\mu \in G^\vee$. Given a resolution $\varphi : X \to V/G$ the push-forward map allows to identify spaces of sections of line bundles over $X$ with submodules of these eigen-modules, $\Gamma(X, \mathcal{O}_X(D)) \to \Gamma(V/G, \mathcal{O}_{V/G}(\varphi_*D))$.

If $D = \sum a_iE_i$, where $E_i$'s are exceptional divisors of $\varphi$ and $a_i$'s are integers, then it follows that $\Gamma(X, \mathcal{O}_X(D)) = \{ f \in \Gamma(V/G, \mathcal{O}_{V/G}) : \forall i, \nu_{E_i} \geq -a_i \}$ where $\nu_{E_i}$'s are divisorial valuations on the field $\mathbb{C}(X) = \mathbb{C}(V)^G$. This observation does
not make much sense for a general $D$ but, fortunately, in case of symplectic resolutions each valuation $\nu_{E_i}$ can be related to a monomial valuation $\nu_{T_i}$ on the field of fractions of $\mathbb{C}[V]$ which comes from the action of the symplectic reflection $T_i$ of $V$ associated to $E_i$ via the McKay correspondence. We use this idea in subsection 3.E where we prove that a ring $\mathcal{R}$ defined in 3.C is in fact the total coordinate ring $\mathcal{R}(X)$ of a resolution of $X \to V/G$, where $G$ is the group introduced in 2.C.

Note that $\mathcal{R} = \mathcal{R}(X)$ is constructed before $X$ is known and now $X$ can be recovered as a GIT quotient of $\text{Spec} \mathcal{R}(X)$ with respect to the action of the algebraic torus $\mathbb{T}_{\text{Cl}(X)}$ which is associated to grading of $\mathcal{R}$ in the class group $\text{Cl}(X)$. We do it in Section 4. Although the construction of a GIT quotient and verification of its smoothness is, in general, conceptually clear it is still computationally involved. Firstly, we find out that a natural choice of a character $\kappa$ of $\mathbb{T}_{\text{Cl}(X)}$ yields a good GIT quotient, see subsection 4.A. Secondly, we verify that the resulting GIT quotient is indeed smooth, see subsection 4.C. To perform the calculations we use an embedding of $\text{Spec} \mathcal{R}$ in an affine space, in which the action of $\mathbb{T}_{\text{Cl}(X)}$ is diagonal, see subsection 3.D. Next, we consider a covering of $\text{Spec} \mathcal{R}$ by sets associated to orbits of the big torus of the affine space. After all reductions and using symmetries we are left with a manageable computational problem which is calculated and cross-calculated with standard algebraic software: $\text{GS}$, $\text{DGPS12}$ and $\text{S}^{+13}$.

A striking consequence of our calculation of the resolution $\varphi^\kappa : X^\kappa \to V/G$ is that it has the unique 2-dimensional fiber containing, as a component, the blow-up of $\mathbb{P}^2$ at four general points. In terms of McKay correspondence this component is related to the only nontrivial central element of the group $G$. The Picard group of $X^\kappa$ can be identified with the Picard group of this surface. In fact, different resolutions $\varphi : X \to V/G$ are in relation to Zariski decomposition of divisors on that surface; we describe the geometry of all resolutions and their flops, see Section 5.

We note that the geometric arguments in Section 5 work once $\mathcal{R}$ is identified as the Cox ring of some symplectic resolution of $V/G$, which is known to exist by the result of Bellamy and Schedler, $\text{BS13}$. However, we stress that the GIT calculations and verification of smoothness in Section 4 not only make our paper self-contained but also provide a general pattern for dealing with cases in which the approach based on $\text{Nam11}$ and $\text{GK04}$ does not work.

A complete framework for our Cox ring constructions and calculations is presented in a recent book by Arzhantsev, Derenthal, Hausen and Laface, $\text{ADHL14}$. The notion of monomial valuations which is essential for proving the main theorem, has been used in the context of McKay correspondence for resolutions of singularities by Reid, Ito and Kaledin, $\text{IR96}$, $\text{Rei97}$, $\text{Kal02}$.

In the final section of the paper we follow the program initiated in $\text{AW10}$ and $\text{Don11}$. We produce a representation of the group $G$ in the group of automorphisms of an abelian 4-fold which is the product of 4 elliptic curves with complex multiplication. The resulting quotient admits a symplectic resolution hence we obtain a new Kummer-type symplectic 4-fold, 6.3.

1.D. Notation. We use the standard notation in set and group theory. A commutator of $G$ is denoted by $[G,G]$ and by $\text{Ab}(G)$ we denote its abelianization $G/[G,G]$. A group of characters of $G$ is $G^\vee = \text{Hom}(G, \mathbb{C}^\times)$. The quotient group $\mathbb{Z}/\langle r \rangle$ will be denoted by $\mathbb{Z}_r$ with $[d]_r$ denoting the class of $d \in \mathbb{Z}$. By $\lfloor d/r \rfloor$ we denote the integral part (round-down) of the fraction $d/r$. If $G$ acts on a set $B$
then by $B^G$ we denote the set of the fixed points of the action. In particular if $G$ acts by homomorphisms on a ring $B$ then by $B^G$ we denote the ring of invariants of the action.

A torus $\mathbb{T}$ means an algebraic torus with finite lattice of characters (called also monomials) $M = M_\mathbb{T} = \text{Hom}_{\text{alg}}(\mathbb{T}, \mathbb{C}^*)$ and dual lattice of 1-parameter subgroups $N = M^*$. Given a finitely generated free abelian group $M$ we define associated torus $\mathbb{T}_M = \text{Hom}(M, \mathbb{C}^*)$ with lattice of characters equal $M$. We drop subscripts whenever it makes no confusion. The pairing $M \times N \to \mathbb{Z}$ is denoted by $(u, v) \mapsto \langle u, v \rangle$.

For $u \in M_\mathbb{T}$ by $\chi^u$ we denote the character $\chi^u: \mathbb{T} \to \mathbb{C}^*$. By $\mathbb{C}[M_\mathbb{T}]$ we denote the ring of Laurent polynomials graded in the lattice $M_\mathbb{T}$. More generally, for a cone $\sigma \subset M_\mathbb{Q}$ by $\mathbb{C}[M \cap \sigma]$ we understand the respective subalgebra of $\mathbb{C}[M]$. For the lattice $N$ or $M$ with a given basis by $\sigma_N^+$ or, respectively, $\sigma_M^+$ we will denote the convex cone generated by the basis, to which we refer as positive orthant. Thus, for a $\mathbb{C}$-linear space $V$ we have $\mathbb{C}[V] \subset \mathbb{C}[M_\mathbb{T}]$ where $\mathbb{C}[V]$ is the ring of polynomials in linear coordinates of $V$ and $\mathbb{T} = \mathbb{T}_V$ is the standard torus acting diagonally on coordinates of $V$.

All varieties are defined over the field of complex numbers. By $\mathbb{C}(X)$ we denote the field of rational functions on a variety $X$ while by $\mathbb{C}[X]$ we denote the algebra of global functions on $X$. By $\text{Spec } A$ we understand the maximal spectrum of a ring $A$; in particular if $X$ is an affine variety then $X = \text{Spec } \mathbb{C}[X]$.

For a $\mathbb{Q}$-factorial variety $X$ by $N^1(X)$ and $N_1(X)$ we will denote the $\mathbb{R}$-linear space of divisors and, respectively, of proper 1-cycles on $X$, modulo numerical equivalence. The class of a divisor $D$ or a curve $C$ will be denoted by $[D]$ and $[C]$, respectively. We have a natural intersection pairing of these two spaces.

A cone in a $\mathbb{R}$-vector space $N$ generated by elements $v_1, v_2, \ldots$ is, by definition, \( \text{cone}(v_1, v_2, \ldots) = \sum \mathbb{R}_{\geq 0} v_i \). By $\text{Nef}(X) \subset N^1(X)$ we denote the cone of divisors which are non-negative on effective 1-cycles. By $\text{Eff}(X)$ and $\text{Mov}(X)$ we denote the cones in $N^1(X)$ which are $\mathbb{R}_{\geq 0}$-spanned by the classes of effective divisors and, respectively, by divisors whose linear systems have no fixed component.

A blow-up of $\mathbb{P}^2$ in $r$ general points will be denoted by $\mathbb{P}^2_r$. The blow-up of $\mathbb{P}^2$ in three (different) collinear points will be denoted by $\mathbb{P}^2_3$.

2. Preliminaries

2.A. The total coordinate ring. Let $X$ be a normal $\mathbb{Q}$-factorial variety over the field of complex numbers. We assume that $\mathbb{C}[X]$ is finitely generated, its invertible elements are in $\mathbb{C}^*$, and $X$ is projective (hence proper) over $\text{Spec } \mathbb{C}[X]$. In what follows we also assume that the divisor class group $\text{Cl}(X)$ is finitely generated.

In order to define the total coordinate ring of $X$, called also the Cox ring of $X$, we set

\[
R(X) = \bigoplus_{[D] \in \text{Cl}(X)} \Gamma(X, \mathcal{O}_X(D))
\]

where $\Gamma(X, \mathcal{O}_X(D))$ denotes the space of global sections of the reflexive sheaf associated to the linear equivalence class of the divisor $D$. It is standard to use the identification $\Gamma(X, \mathcal{O}_X(D)) = \{ f \in \mathcal{O}(X)^* : \text{div}(f) + D \geq 0 \} \cup \{ 0 \}$ and to say that a non-zero $f \in \Gamma(X, \mathcal{O}_X(D))$ is associated with an effective divisor $D' = \text{div}(f) + D$. The divisor $D'$ is the zero locus of this section of $\mathcal{O}_X(D)$ which we will usually
denote by $f_D$. Note that by our assumptions the relation $D' \leftrightarrow f_D$ is unique up to multiplication by a constant from $\mathbb{C}^*$.

Now, to define the multiplication in $\mathcal{R}(X)$ properly we need to set the inclusion $\Gamma(X, \mathcal{O}_X(D)) \subset \mathbb{C}(X)$ so that it does not depend on the choice of the divisor $D$ in its linear equivalence class. If $\text{Cl}(X)$ is a free finitely generated group (no torsions) then we choose divisors $D_1, \ldots, D_m$ whose classes make a basis of $\text{Cl}(X)$ and for $D$ linearly equivalent to $\sum_i a_i D_i$, with $a_i \in \mathbb{Z}$, we define

$$\Gamma(X, \mathcal{O}_X(D)) = \{ f \in \mathbb{C}(X)^* : \text{div}(f) + \sum_i a_i D_i \geq 0 \} \cup \{ 0 \} = \{ f \in \mathcal{O}_X(X \setminus \bigcup_i D_i) \subset \mathbb{C}(X) : \forall_i \nu_{D_i}(f) \geq -a_i \}$$

where the second equality makes sense if $D_i$’s are prime divisors and for every $i$ we take $\nu_{D_i} : \mathbb{C}(X) \to \mathbb{Z} \cup \{ \infty \}$, a valuation centered at the divisor $D_i$. Therefore we represent $\mathcal{R}(X)$ as a sub-module of $\mathbb{C}(X)$ and consequently we define the multiplication in $\mathcal{R}(X)$ as inherited from $\mathbb{C}(X)$. It can be checked that this defines a ring structure on $\mathcal{R}(X)$ which not depend on the choice of $D_i$’s (different choices give isomorphic rings). This definition has to be adjusted if $\text{Cl}(X)$ has torsions. We advise the reader to consult [ADHL14, Ch. 1] for details.

We recall that, for a Weil divisor $D$ on a normal variety $X$, the sheaf $\mathcal{O}_X(D)$ is reflexive of rank one. The association $D \to \mathcal{O}_X(D)$ determines the bijection between the class group $\text{Cl}(X)$ and the isomorphism classes of reflexive rank-1 sheaves over $X$. Locally, in a similar manner, we can define a graded sheaf of divisorial algebras over $X$ which we will denote by $\mathcal{R}_X$.

Suppose that $\mathcal{R}(X)$ is a finitely generated $\mathbb{C}$-algebra. Then $\text{Spec} \mathcal{R}(X)$ is an affine variety and the grading of $\mathcal{R}(X)$ in $\text{Cl}(X)$ determines the action of the algebraic quasi-torus $\mathbb{T}_{\text{Cl}(X)} = \text{Hom}(\text{Pic} X, \mathbb{C}^*)$.

As we assumed that $X$ is projective over Spec $\mathbb{C}[X]$, we can use geometric invariant theory, or GIT, [MFK94], to recover $X$ as a quotient of Spec $\mathcal{R}_X$. Namely, a relatively ample divisor $H$ on $X$ determines a linearization of this action on the trivial line bundle over Spec $\mathcal{R}_X$ and the irrelevant ideal $\mathcal{I}_{\text{tr}H} = \sqrt{(\Gamma(X, \mathcal{O}(mH)) : m > 0)}$ which determines the set of unstable points with respect to this linearization. The quotient of its complement is $X$. The choice of a big but not necessarily ample divisor on $X$ yields a GIT quotient which is birational to $X$.

This line of argument works in much broader set up as it is explained in [ADHL14, Ch. 1]. In the present paper we will concentrate on a special situation of quotient singularity and its resolution which will be discussed in detail in subsequent sections.

2.B. Small blow-ups of $\mathbb{P}^2$. The surface obtained by blowing up $r$ general points on $\mathbb{P}^2$ will be denoted by $\mathbb{P}^2_r$. If $r \leq 3$ then $\mathbb{P}^2_r$ is a toric surface whose geometry and Cox ring are well known. Here, for the sake of completeness, we recall properties of $\mathbb{P}^2_r$.

If $\mathbb{P}^2_4$ is obtained by blowing $\mathbb{P}^2$ at points $p_1, \ldots, p_4$ then by $F_{0i}$ we denote the exceptional ($-1$)-curve over $p_i$ and by $F_{ij}$, with $1 \leq i < j \leq 4$, we denote strict transform of the line passing through $\{ p_1, \ldots, p_4 \} \setminus \{ p_i, p_j \}$. With this notation we have the following well known facts, see e.g. [Man86].

**Lemma 2.3.** The surface $\mathbb{P}^2_4$ has the following geometry:

1. For two pairs of numbers $0 \leq i < j \leq 4$ and $0 \leq p < q \leq 4$ we have $F_{ij}, F_{pq} = |\{ i, j \} \cup \{ p, q \}| - 3$. 


(2) The surface $\mathbb{P}_4^2$ admits five distinct birational morphisms $\beta_i : \mathbb{P}_4^2 \to \mathbb{P}^2$ such that $\beta_i$ contracts four $(-1)$-curves $F_{pq}$ for $i \in \{p, q\}$.

(3) The surface $\mathbb{P}_4^2$ admits five conic fibrations $\alpha_i : \mathbb{P}_4^2 \to \mathbb{P}^1$, each of them having three reducible fibers $F_{rs} \cup F_{pq}$, where $\{i, p, q, r, s\} = \{0, \ldots, 4\}$.

Let us consider a 5-dimensional $\mathbb{R}$-vector space $W$ with a basis $e_0, \ldots, e_4$. We define the intersection product on $W$ by setting $e_i^2 = -3$ and $e_i \cdot e_j = 1$, if $i \neq j$.

For $0 \leq i < j \leq 4$ we set $f_{ij} = (e_i + e_j)/2$. If moreover we set $\kappa = \sum_i e_i$ and $c_i = [\kappa - e_i]/2 = [(\sum_j e_j) - e_i]/2$, then $\kappa \cdot e_i = \kappa \cdot f_{ij} = 1$, $\kappa^2 = 5$, and $f_{ij}^2 = -1$.

Also $e_i \cdot c_i = 2$ and $e_i \cdot e_j = 0$ for $i \neq j$ hence the base $e_i/2$, for $i = 0, \ldots, 4$ is dual, in terms of the intersection product, to the base consisting of $c_i$'s. In particular, $\text{cone}(e_0, \ldots, e_4)$ is where the intersection with $c_i$'s is positive.

By $\Lambda \subset W$ we denote the lattice spanned by $f_{ij}$'s. We note that $\sum_i a_i(e_i/2) \in \Lambda$ if $a_i$'s are integral and $\sum_i a_i$ is even. The following can be easily verified.

**Lemma 2.4.** The space $N^1(\mathbb{P}_4^2) = N_1(\mathbb{P}_4^2)$ with intersection product and the lattice of integral divisors $\text{Pic}(\mathbb{P}_4^2)$ can be identified with $W \supset \Lambda$, so that $[F_{ij}] = f_{ij}$ and $[-K_{\mathbb{P}_4^2}] = \kappa$. Under this identification the cone $\text{Eff}(\mathbb{P}_4^2)$ is spanned by $f_{ij}$'s. It has 10 facets:

- five facets of $\text{Eff}(\mathbb{P}_4^2)$ are associated to morphisms $\alpha_i$ from 2.3 and they are contained in the facets of $\sigma^+$; given $i \in \{0, \ldots, 4\}$ a facet of this type is spanned by six $F_{pq}$'s such that $i \notin \{p, q\}$ and it is perpendicularly (in the sense of the intersection product) to $c_i = (\kappa - e_i)/2$;
- five simplicial facets of $\text{Eff}(\mathbb{P}_4^2)$ are associated to morphisms $\beta_i$ from 2.3 and they are obtained by cutting $\sigma^+$ with a hyperplane perpendicular to $(\kappa \pm e_i)/2$; this facet of $\text{Eff}(\mathbb{P}_4^2)$ is spanned by four $F_{pq}$'s such that $i \in \{p, q\}$.

As the consequence, the cone $\text{Nef}(\mathbb{P}_4^2)$ is spanned by the classes of $(\kappa \pm e_i)$'s.

The cone $\text{Eff}(\mathbb{P}_4^2)$ is divided into Zariski chambers depending on the Zariski decomposition of the divisors whose classes are inside the interior of each chamber, see [BFN10]. For example, the “central” chamber is the cone $\text{Nef}(\mathbb{P}_4^2)$ and if $[D] \in \text{Nef}(\mathbb{P}_4^2)$ then the linear system $|mD|, m \gg 0$ determines a morphism into the projective space whose image is $\mathbb{P}_4^2$. Equivalently, $\text{Proj}(\bigoplus_{m \geq 0} \mathcal{O}(\mathbb{P}_4^2, mD)) = \mathbb{P}^4$. For $D$ outside the nef cone the image of the rational map defined by $|mD|$ (or this projective spectrum) will depend on the intersection of $D$ with $(-1)$ curves $F_{ij}$. This determines the division of $\text{Eff}(\mathbb{P}_4^2)$ into the chambers in question. We summarize the information in Table 1.

**Table 1. Zariski chambers and birational images of $\mathbb{P}_4^2$**

| $\text{Proj}(\bigoplus_{m \geq 0} \mathcal{O}(\mathbb{P}_4^2, mD))$ | number of chambers with $D$ of this type |
|-------------------------------------------------------------|---------------------------------------------|
| $\mathbb{P}_4^2$                                           | one, nef cone                               |
| $\mathbb{P}_2 \times \mathbb{P}_2$                        | ten                                         |
| $\mathbb{P}_4^2$                                           | thirty                                      |
| $\mathbb{P}_4^2$                                           | twenty                                      |
| $\mathbb{P}_4^2$                                           | five, associated to simplicial facets of $\text{Eff}(\mathbb{P}_4^2)$ |
| $\mathbb{P}^1 \times \mathbb{P}^1$                        | ten                                         |

The following result is known, see e.g. [STV07].
**Proposition 2.5.** The total coordinate ring of $\mathbb{P}^2_4$ coincides with the projective coordinate ring of the Grassmann variety $\text{Gr}(2,W)$ of planes in a 5 dimensional vector space $W$ which is embedded in $\mathbb{P}(\Lambda^2 W^*)$ via the Plücker embedding. That is, $\mathcal{R}(\mathbb{P}^2_4)$ is the quotient of the polynomial ring in variables $w_{ij}$, for $0 \leq i < j \leq 4$, by the ideal generated by the following quadratic trinomials

\[
\begin{align*}
& w_{14}w_{23} + w_{13}w_{24} - w_{12}w_{34} \quad w_{04}w_{23} - w_{03}w_{24} - w_{02}w_{34} \\
& w_{04}w_{13} + w_{03}w_{14} - w_{01}w_{34} \quad w_{04}w_{12} - w_{02}w_{14} - w_{01}w_{24} \\
& w_{03}w_{12} + w_{02}w_{13} - w_{01}w_{23}
\end{align*}
\]

The action of the Picard torus can be identified with that of $\mathcal{T}_\Lambda = \mathcal{T}_W/(−I_W)$, where $\mathcal{T}_W$ is the standard torus of the space $W$ with which acts on $\bigwedge^2 W$ with isotropy $(-I_W)$. In other words, it is given by a grading in $\Lambda \subset \mathbb{Z}^5$ such that

$\text{deg } w_{01} = (1,1,0,0,0), \text{deg } w_{02} = (1,0,1,0,0), \ldots, \text{deg } w_{34} = (0,0,0,1,1)$.

The GIT quotients of the affine variety $\text{Spec } \mathcal{R}(\mathbb{P}^2_4)$ depend on the choice of the linearization of the torus action given by a character in the lattice $\Lambda$. In particular, the choice of a divisor class in the interior of each of the Zariski chambers of $\mathbb{P}^2_4$ determines the quotient as in Table 7.

A similar discussion can be done for the blow-up of $\mathbb{P}^2$ in three collinear points, which we denote by $\mathbb{P}^2_3$. If by $E_1$ we denote the unique $(-2)$-curve on $\mathbb{P}^2_3$ and by $E_2, E_3, E_4$ we denote $(-1)$-curves then the classes of $E_i$ form a standard basis of $\text{Pic } \mathbb{P}^2_3$ and generate the cone of effective divisors in $\text{N}^1(\mathbb{P}^2_3)$. The following result is pretty standard, see [Ott11].

**Proposition 2.6.** The total coordinate $\mathcal{R}(\mathbb{P}^2_3)$ is isomorphic to the quotient

$$\mathcal{O}[u_1, w_2, w_3, w_4, w_{34}, w_{24}, w_{23}] / (w_2w_{34} + w_3w_{24} + w_4w_{23})$$

Here the variables $u_1, w_2, w_3, w_4$ are associated to divisors $E_1, \ldots, E_4$ defined above. The grading of variables $u_1, w_2, w_3, w_4, w_{34}, w_{24}, w_{23}$ with respect to the standard basis of $\text{Cl } \mathbb{P}^2_3$ consisting of the classes of $E_i$’s is given by the columns of the following matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}
\]

(2.7)

**2.C. The group of order 32.** In what follows we will consider complex matrices acting on a linear space $V$. The case of our primary interest is when $V$ is of even dimension and admits a non-degenerate linear 2-form, which we will call a symplectic form. A group of linear transformations generated by matrices preserving such a form we will denote by $\text{Sp}(V)$.

By $I_V$ we denote the identity matrix and by $-I_V$ its opposite. We will usually drop the subscript. Also, for every matrix $A$ by $-A$ we denote its opposite, that is $-I \cdot A = A \cdot (-I)$. For two matrices $A$ and $B$ we set $[A,B] = A \cdot B \cdot A^{-1} \cdot B^{-1}$. If the linear space fixed by a matrix is of codimension 1 then we say that it is a quasi-reflection. The groups which we will consider contain no quasi-reflections. If $A \in \text{Sp}(V)$ and its fixed point set is of codimension 2 then we call it a symplectic reflection.
For $V$ of dimension 4 with coordinates $x_1, \ldots, x_4$ we consider the following matrices in $GL(V)$.

\[
T_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}
\]

(2.8)

\[
T_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\]

\[
T_4 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}
\]

We note that the symplectic form $\omega = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$ is preserved by each of the $T_i$'s. Moreover, they are symplectic reflections of order two.

We list the following facts which can be verified easily.

**Lemma 2.9.** If, for $i \neq j$ we set $R_{ij} = T_i \cdot T_j$, then $R_{ij}$'s are of order 4 and moreover the following holds:

1. $[T_i, T_j] = R^2_{ij} = -I$, $T_0 \cdot T_1 \cdot T_2 \cdot T_3 \cdot T_4 = I$, and $R_{ij} = -R_{ji} = R^{-1}_{ji}$
2. $[T_s, R_s] = -I$ if $s \in \{i, j\}$ and $[T_s, R_{ij}] = I$ if $s \notin \{i, j\}$
3. For two distinct pairs $i, j$ and $p, q$ we have $[R_{ij}, R_{pq}] = -I$ if $\{i, j\} \cap \{p, q\} = \emptyset$ and $[R_{ij}, R_{pq}] = I$ if $\{i, j\} \cap \{p, q\} \neq \emptyset$

Let $G$ be the group generated by the reflections $T_i$ in $Sp(V)$. It is worthwhile to note that this group is conjugate in $Sp(V)$ to the group generated by Dirac gamma matrices, c.f. [Tem30].

**Lemma 2.10.** The group $G$ is of order 32 and it has the following properties:

1. There are 17 classes of conjugacy of elements in $G$; two of the them consist of single elements $I$ and $-I$, five of them contain pairs of opposite reflections $\pm T_i$, for $i = 0, \ldots, 4$, and ten conjugacy classes consist of pairs of opposite elements $\pm R_{ij}$, with $0 \leq i < j \leq 4$.
2. The commutator $[G, G]$ of $G$ coincides with its center and it is generated by $-I$, the abelianization is $Ab(G) = G/[G, G] = \mathbb{Z}_2^4$.
3. If $N(T_i) < G$ is the normalizer (or centralizer) of the reflection $T_i$ then $N(T_i)/T_i \cong Q_8$, where $Q_8$ is the group of quaternions, or a binary-dihedral group, of order 8.

**2.D. Quotient singularities.** Let $G \subset GL(V)$ be a finite group with no quasi-reflections acting linearly and faithfully on $V \cong \mathbb{C}^n$. By $\mathbb{C}[V] \cong \mathbb{C}[x_1, \ldots, x_n]$ we understand the coordinate ring of the linear space with the ring of invariants denoted by $\mathbb{C}[V]^G$ which, by Hilbert-Noether theorem, is finitely generated $\mathbb{C}$-algebra. We set $Y = \text{Spec} \mathbb{C}[V]^G$.

**Proposition 2.11.** We have the isomorphisms: $\text{Pic}(Y) = 1$, $\text{Cl}(Y) = G/[G, G]$ and $\mathcal{R}(Y) = \mathbb{C}[V]^{[G, G]}$. 
their configuration is modeled on 2-dimensional Du Val singularities. Namely, if $E$ is an irreducible component of a general fiber of $\varphi|_{E_i}$, then $E_i$ is a

**Theorem 2.15.** There exist a natural basis of Borel-Moore homology $H^\bullet_\mathbb{R}(X, Q)$ whose elements are in bijection with the following objects:

- irreducible closed subvarieties $Z \subset X$ for which it holds $2 \text{codim}_X Z = \text{codim}_Y Z$;
- conjugacy classes of elements $g$ in $G$.

Under this correspondence every conjugacy class $[g]$ of an element $g \in G$ is related to $Z^{[g]} \subseteq X$ such that $\varphi(Z^{[g]}) = [V^g]$, where $[V^g]$ denotes the image in $V/G$ of the linear subspace $V^g$ of fixed points of $g$.

Since the exceptional divisors $E_i$ of $\varphi$ are contracted to codimension two set in $Y$, their configuration is modeled on 2-dimensional Du Val singularities. Namely, if $C_i \subset E_i$ is an irreducible component of a general fiber of $\varphi|_{E_i}$, then $C_i$ is a

**Proof.** The first part is classical nowadays, see e.g. [Ben93], the second part is in [AG10].

Recall that the abelianization $Ab(G) = G/[G, G]$ is isomorphic to the group of characters $G^\vee = \text{Hom}(G, \mathbb{C}^*)$. The ring $R(Y) = \mathbb{C}[V]^{G, G}$, as $\mathbb{C}[Y]$-module, is a direct sum of reflexive rank one modules associated to characters of the group $G$, see [Sta79] Thm. 1.3. That is, the grading of $R(Y)$ is into the eigenspaces associated to the characters of $G$:

$$(2.12) \quad R(Y) = \mathbb{C}[V]^{G, G} = \bigoplus_{\mu \in G^\vee} \mathbb{C}[V]_\mu$$

where $\mathbb{C}[V]_\mu$ is a rank-1 reflexive $\mathbb{C}[Y]$-module on which $G$ acts with character $\mu \in G^\vee$, for discussion see [DB13] Lemma 6.2.

The following fact is in [DB13] Sect. 2.

**Lemma 2.13.** Let $\varphi : X \to Y$ be a resolution of a quotient singularity. Then $\text{Pic} X = \text{Cl} X$ is a free finitely generated abelian group.

**Proof.** By Grauert-Riemenschneider we know that $H^1(X, \mathcal{O}) = 0$ hence it is enough to prove that $\text{Pic} X = H^2(X, \mathbb{Z})$ has no finite torsion. However by [Kol93] Thm. 7.8 we know that the fundamental group is trivial so, by universal coefficients theorem, $H^2(X, \mathbb{Z})$ has no torsion.

We note that the exceptional set of a resolution $\varphi : X \to Y = V/G$ is a divisor with $m$ irreducible components $E_i$ and thus $\text{Cl} X$ is a free abelian group of rank $m$. Moreover we have the following exact sequence

$$(2.14) \quad 0 \to \bigoplus_{i=1}^m \mathbb{Z}[E_i] \to \text{Cl} X \cong \mathbb{Z}^m \to \text{Cl} Y = Ab(G) \to 0$$

where the right arrow is the push-forward morphism $\varphi_*$. Now we assume that $\dim V = 2n$ and $G$ is a finite group in $Sp(V)$, and there exists a symplectic resolution $\varphi : X \to Y = V/G$. That is $X$ admits a closed everywhere non-degenerated 2-form which restricts to the invariant symplectic form defined on the smooth locus of $V/G$. Then the morphism $\varphi : X \to Y$ is semi-small which means, in particular, that every exceptional divisor of $\varphi$ in $X$ is mapped to a codimension 2 component of the singular set associated to the fixed point locus of a symplectic reflection. In fact $G$ has to be generated by symplectic reflections and we have the following version of McKay correspondence by Kaledin, see [Kal02].
rational curve and the intersection matrix \((E_i \cdot C_j)\) is a direct sum of known Cartan-type matrices, see [Die03 Thm. 1.3], [AW14 Thm. 4.1]. If \([L_i] \in N^1(X)\), for \(i = 1, \ldots, m\), is the basis dual (in terms of the intersection) to that consisting of \([C_i] \in N_1(X)\), then \(\text{Pic}(X)\) is a sublattice of the lattice \(\langle [L_1], \ldots, [L_m] \rangle\) and we note that the left-hand side arrow in the sequence \(2.14\) can be written as follows \(E_i \rightarrow \sum_j (E_i \cdot C_j)[L_j]\). Thus, the sequence \(2.14\) can be extended to the following diagram in which the left-hand side arrow in the lower sequence is a map of lattices of the same rank given by the intersection matrix \((E_i \cdot C_j)\)

\[
\begin{array}{cccc}
0 \rightarrow & \bigoplus_i \mathbb{Z}[E_i] & \rightarrow & \text{Cl}(X) & \rightarrow & \text{Cl}(V/G) & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \rightarrow & \bigoplus_i \mathbb{Z}[E_i] & \rightarrow & \bigoplus_i \mathbb{Z}[L_i] & \rightarrow & Q & \rightarrow & 0 \\
\end{array}
\]

Here \(Q\) is the quotient of lattices \(\bigoplus_i \mathbb{Z}[E_i] \hookrightarrow \bigoplus_i \mathbb{Z}[L_i]\). The homomorphism \(\text{Cl}(V/G) \hookrightarrow Q\) associates to a class of a Weil divisor \(D\) on \(V/G\) the class of \(\sum_i (\varphi_r^{-1} D \cdot C_i)[L_i]\) in the quotient group \(Q\).

### 2.E. Cyclic group quotients and monomial valuations

In this subsection we discuss a fundamental example of group action and a blow-up of the resulting quotient. Let \(\epsilon_r = \exp(2\pi i/r) \in \mathbb{C}^*\) be the primitive \(r\)-th root of unity, \(r > 1\). We consider the cyclic group \(\langle \epsilon_r \rangle \subset \mathbb{C}^*\). We assume that the group \(\langle \epsilon_r \rangle \cong \mathbb{Z}_r\) acts diagonally on the vector space \(V\) of dimension \(n\) with non-negative weights \((a_1, \ldots, a_n)\), that is \(\epsilon_r(x_1, \ldots, x_n) = (\epsilon_r^{a_1} x_1, \ldots, \epsilon_r^{a_n} x_n)\), with \(0 \leq a_i < r\), and at least two \(a_i\)’s positive. Moreover, we assume that the action of \(\mathbb{Z}_r\) is faithful which means that \((a_1, \ldots, a_n, r) = 1\). This action extends to the action of \(\mathbb{C}^* \supset \langle \epsilon_r \rangle\) with the same weights: for \(t \in \mathbb{C}\) we take \(t(x_1, \ldots, x_n) = (t^{a_1} x_1, \ldots, t^{a_n} x_n)\).

Let us describe this situation in toric terms. By \(M\) let us denote the lattice with the basis \((u_1, \ldots, u_n)\) consisting of characters of the standard torus \(T_V\) of \(V\) so that \(\chi^{a_i} = x_i\). If \(N = \mathbb{M}^*\) is the lattice of 1-parameter subgroups of the standard torus then the \(\mathbb{C}^*\)-action in question is defined by \(v \in N\) such that for every \(i\) we have \(\langle v, u_i \rangle = a_i\). Equivalently, the action of \(\mathbb{C}^*\) is associated to a grading \(\deg_v\) on \(\mathbb{C}[V] \subset \mathbb{C}[M]\) such that \(\deg_v(\chi^a) = \langle v, u \rangle\). The composition of \(\deg_v\) with the residue homomorphism \(\mathbb{Z} \rightarrow \mathbb{Z}_r\) determines \(\mathbb{Z}_r\)-grading \([\deg_v]_r\) on \(\mathbb{C}[V]\) associated to the action of \(\langle \epsilon_r \rangle\). By \([\deg_v]_r\), we will denote both, the grading on \(\mathbb{C}[V] = \mathbb{C}[M \cap \sigma^+]\) and the associated group homomorphism \(M \rightarrow \mathbb{Z}_r\).

Following [Re97, IR96, Kal02 Sect. 2] we state this definition.

**Definition 2.17.** The monomial valuation associated to the cyclic group action described above is \(\nu_v : \mathbb{C}(V) \rightarrow \mathbb{Z} \cup \{\infty\}\) such that for \(f = \sum_j c_j \chi^{m_j} \neq 0\) it holds \(\nu_v(f) = \min\{\langle v, m_j \rangle : c_j \neq 0\}\) and moreover \(\nu_v(f_1 f_2) = \nu_v(f_1) - \nu_v(f_2)\).

**Example 2.18.** In the situation introduced above we define a lattice \(N_{v/r} = N + (v/r) \cdot \mathbb{Z}\) where the sum is taken in \(N_{Q}\). Its dual \(M_{v/r} \subset M\) consists of characters on which \(v/r\) assumes integral values. In fact, \(M_{v/r}\) is the kernel of \([\deg_v]_r\). Thus \(\mathbb{C}[M_{v/r} \cap \sigma^+_M] \subset \mathbb{C}[V]\) is the ring of invariants of the action of \(\langle \epsilon_r \rangle\) on \(V\), or \(Y_{v/r} := \text{Spec} \mathbb{C}[M_{v/r} \cap \sigma^+_M]\) is the quotient \(V/\langle \epsilon_r \rangle\). Equivalently, \(Y_{v/r}\) is an affine toric variety associated to the cone \(\sigma^+_N\) and the lattice \(N_{v/r}\). We define a toric variety \(X_{v/r}\) whose fan is defined by taking a ray in \(N_{v/r}\) which is generated by \(v/r\) and sub-dividing the cone \(\sigma^+_N\) into a simplicial fan. The induced morphism
follows. Now we can use standard arguments in toric geometry, see e.g. (2.21) the homomorphism dual to the inclusion \( M \times Z \) spanned by these generators and by \( p_2 : \hat{M}_{v/r} \to Z \) we denote the projection onto the last coordinate. Let us note that the kernel of \( p_2 \) coincides with \( M_{v/r} \). If fact, if \( p_1 : M \times Z \to M \) is the projection to the first factor then on \( \hat{M}_{v/r} \) the composition \([d_{v/r}] \circ p_1 \) coincides with \( p_2 \) composed with the residue homomorphism \( Z \to Z_r \).

By \( \psi : \mathbb{C}[\hat{M}_{v/r} \cap \hat{\sigma}^+] \to \mathbb{C}[M \cap \sigma^+_M] \) let us denote the homomorphism of polynomial rings induced by the projection \( p_1 \), so that \( \psi(\chi^{(u_i,a_i)}) = \chi^{u} \) and \( \psi(\chi^{(0,-r)}) = 1 \).

**Proposition 2.19.** In the above situation the following holds:

1. \( \text{Cl}(Y_{v/r}) = Z_r \) and \( \mathcal{R}(Y_{v/r}) = \mathbb{C}[M \cap \sigma^+_M] \) with grading \([d_{v/r}] \).
2. \( \text{Cl}(X_{v/r}) = Z \) and \( \mathcal{R}(X_{v/r}) = \mathbb{C}[\hat{M}_{v/r} \cap \hat{\sigma}^+] \) with grading \( \mathcal{R}(X_{v/r}) = \bigoplus_{d \in \mathbb{Z}} \mathcal{R}(X_{v/r})_d \) associated to the projection \( p_2 \).
3. \( \mathcal{R}(Y_{v/r})_0 = \mathcal{R}(X_{v/r})_0 = \mathbb{C}[V]^{(r)} \) and if \( \mathcal{R}(X_{v/r})^+ = \bigoplus_{d \geq 0} \mathcal{R}(X_{v/r})_d \)
4. Then \( X_{v/r} = \text{Proj} \mathcal{R}(X_{v/r})^+ \) and \( \mathcal{O}_{X_{v/r}}(-E_{v/r}) = \mathcal{O}_{\text{Proj} \mathcal{R}(X_{v/r})^+}(r) \).

(4) The valuation \( \nu_v \) restricted to \( \mathbb{C}(X_{v/r}) = \mathbb{C}(Y_{v/r}) \subset \mathbb{C}(V) \) coincides with \( r \cdot \nu_E \), where \( \nu_E \) is the divisorial valuation centered at \( E_{v/r} \).

**Proof.** The proof uses toric geometry. Let \( \hat{N}_{v/r} \) be a lattice dual to \( \hat{M}_{v/r} \subset M \times Z \) with the basis \( v_0, v_1, \ldots, v_n \) such that for \( \langle v_i, (u_j,a_j) \rangle = 1 \) for \( 1 \leq i = j \leq n \) and \( \langle v_0, (0,-r) \rangle = 1 \) and the other products are zero. We check that the homomorphism dual to the inclusion \( M_{v/r} \hookrightarrow \hat{M}_{v/r} \) is \( \hat{N}_{v/r} \to N_{v/r} \) where \( v_i \)'s are send to the elements of the basis of \( N \) and \( v_0 \to v/r \). Indeed, if \( u = b_0(0,-r) + \sum b_i(a_i,r) \) is in \( M_{v/r} \subset \hat{M}_{v/r} \) then \( b_0 = \sum b_i(a_i,r) \) hence the claim follows. Now we can use standard arguments in toric geometry, see e.g. [CLS11].

**Corollary 2.20.** Suppose that the situation is as introduced above. Then the projection induced homomorphism

\[
\psi : \mathbb{C}[\hat{M}_{v/r} \cap \hat{\sigma}^+] = \mathcal{R}(X_{v/r}) \to \mathbb{C}[M \cap \sigma^+_M] = \mathcal{R}(Y_{v/r})
\]

is a homomorphism of graded \( \mathbb{C}[V]^{(r)} \) algebras. More precisely, for \( d \in \mathbb{Z} \) we have the induced injective homomorphism of \( d \)-th graded pieces \( \psi : \mathcal{R}(X_{v/r})_d \hookrightarrow \mathcal{R}(Y_{v/r})_d \), as modules over \( \mathbb{C}[V]^{(r)} = \mathcal{R}(X_{v/r})_0 = \mathcal{R}(Y_{v/r})_0 \), and the following holds

\[
\psi(\mathcal{R}(X_{v/r})_d) = \{ f \in \mathcal{R}(Y_{v/r})_d : \nu_v(f) \geq d \} = \{ f \in \mathcal{R}(Y_{v/r})_d : \nu_E(f) \geq \lfloor d/r \rfloor \}
\]

3. The total coordinate ring of symplectic resolutions of a quotient

**3.A. The push-forward map.** We begin this section by discussing a somewhat more general situation, than what is needed to tackle the problem of our interest. Let \( \varphi : X \to Y \) be a projective birational morphism of normal \( \mathbb{Q} \)-factorial varieties which satisfy assumptions formulated at the beginning of Section 2.7. By \( \mathbb{Q} \)-factoriality of \( Y \) we know that the exceptional set of the morphism \( \varphi \) is a \( \mathbb{Q} \)-divisor with components denoted by \( E_i \). The push-forward map of codimension one
cycles \( \varphi_* : \text{Cl}(X) \to \text{Cl}(Y) \) is surjective and its kernel is generated by the classes of \( E_i \)'s, c.f. \textbf{2.14}

Moreover, \( \varphi \) determines the morphism of the respective total coordinate rings which we will denote by \( \varphi_* \) as well. Namely, for a reflexive sheaf \( \mathcal{L} = \mathcal{O}_X(D) \) its reflexive push-forward \( \varphi_* \mathcal{L}^{\vee} \) is isomorphic to \( \mathcal{O}_Y(\varphi_*(D)) \). Thus, pushing down the sections determines the injective homomorphism of spaces

\[
\Gamma(X, \mathcal{O}_X(D)) \to \Gamma(Y, \mathcal{O}_Y(\varphi_*(D)))
\]

This follows from the construction in \textbf{ADHL14 Sect. 1.4}, see \textbf{HKL14}. For example, in case when \( \text{Cl}(X) \) is torsion-free, which is the case of our primary interest, c.f. Lemma \textbf{2.13} we choose divisors \( D_i \) on \( X \) whose classes generate \( \text{Cl}(X) \), and use the construction explained in Section \textbf{2.A} see also \textbf{ADHL14 1.4.1.1} to define \( \mathcal{R}(X) \). Subsequently we use divisors \( \varphi_*(D_i) \) and construction \textbf{ADHL14 1.4.2.1} to define \( \mathcal{R}(Y) \). We summarize this short general introduction by stating a result from a paper by Hausen, Keicher and Laface \textbf{HKL14 Prop. 2.2} to which we refer the reader for details.

\textbf{Proposition 3.3.} Let \( \varphi : X \to Y \) be a proper birational morphisms of varieties which satisfy assumptions stated at the beginning of Section \textbf{2.A} the exceptional set of \( \varphi \) equal to \( \bigcup_{i=1}^r E_i \), where \( E_i \)'s are prime divisor. Then there exists a canonical surjective homomorphism \( \varphi_* : \mathcal{R}(X) \to \mathcal{R}(Y) \) which agrees with the homomorphism of gradings \( \varphi_* : \text{Cl}(X) \to \text{Cl}(Y) \). The kernel of \( \varphi_* \) contains elements \( 1 - f_{E_i} \), where \( f_{E_i} \in \Gamma(X, \mathcal{O}(E_i)) \) is a section defining \( E_i \). Moreover, if \( \mathcal{R}(X) \) is finitely generated then in fact \( \ker \varphi_* = (1 - f_{E_i} : i = 1, \ldots, r) \).

The case of a blow-up of a cyclic singularity discussed in Section \textbf{2.1} is a particular example of this situation. In particular, the homomorphism \( \psi \) introduced there, is the push-forward \( \varphi_* \) of the respective Cox rings, c.f. Corollary \textbf{2.2.21}

\textbf{Example 3.4.} Let us consider the case of a resolution of a surface \( A_1 \) singularity. That is \( Y = V/\mathbb{Z}_2 \) where \( V = \mathbb{C}^2 \) and the non-trivial element \( \mathbb{Z}_2 \) acts as \(-I\) matrix. The \( \varphi : X \to Y \) is the blow-down of a \((-2)\)-curve. Clearly the situation is toric and the elements of both \( \mathcal{R}(Y) = \mathbb{C}[V] \) and of \( \mathcal{R}(X) \) are linear combinations of monomials which can be visualized as points in a lattice of rank 2 or 3, respectively.

In Figure 1 we present \( \varphi_*(\mathcal{R}(X))_d \) as a submodule of \( \mathcal{R}(Y)_{[d]_2} \subset \mathbb{C}[V] \). The monomials in \( \mathbb{C}[V] \) are integral lattice points in the positive quadrant on the plane which is indicated by the solid line segments; the dot in the left-lower corner is \((0,0)\). These monomials which are in \( \varphi_*(\mathcal{R}(X))_d \) are denoted by \( \bullet \) while those which are not in \( \varphi_*(\mathcal{R}(X))_d \) are denoted by \( \circ \). The skewed dotted line indicates where the monomial valuation \( \nu_{(1,1)} \) assumes value \( d \), c.f. \textbf{2.22}. For \( d \leq 1 \) we have \( \varphi_*(\mathcal{R}(X))_d = \mathcal{R}(Y)_{[d]_2} \).

\textbf{3.B. Cox ring of a resolution of a quotient singularity.} From now on we consider the case which is of our primary interest. Let \( G \subset GL(V) \) be a finite group without quasi-reflections with \( Y = V/G = \text{Spec} \mathbb{C}[V]^G \) the quotient. Suppose that \( \varphi : X \to Y \) is a resolution of singularities. Now, because of \textbf{2.13} \( \text{Cl}(X) = \text{Pic}(X) \) is
Figure 1. Resolution of $A_1$ singularity; the image $\varphi_*(\mathcal{R}(X)_d)$ in $\mathcal{R}(Y)_{[d]_2}$

\[
d \leq 1 \quad d = 2, 3 \quad d = 4, 5 \quad d = 6, 7
\]

$d$ even

$d$ odd

free abelian of rank, say, $m$. Then the group $\text{Hom} (\text{Cl}(X), \mathbb{C}^*)$ is an algebraic torus $T = T_{\text{Cl}(X)} \cong (\mathbb{C}^*)^m$ with the coordinate ring $\mathbb{C}[\text{Cl}(X)]$. The grading of $\mathcal{R}(X)$ in $\text{Cl}(X)$ is associated to the action of $T$ on $\mathcal{R}(X)$ with the ring of invariants equal to $\mathcal{R}(X)_0 = \mathbb{C}[V]^G$.

If $\mathcal{R}(X)$ is finitely generated $\mathbb{C}$-algebra then we can present the emerging objects in a single diagram

\[
\text{Spec} \mathcal{R}(X) \twoheadrightarrow X \quad V \\
\tau \quad G \\
V/G \quad V/[G, G]
\]

where $X \rightarrow V/G$ is the resolution of singularities. Moreover, $V \rightarrow V/[G, G] \rightarrow V/G$ and $\text{Spec} \mathcal{R}(X) \rightarrow V/G$ are the (categorical) quotients with respect to appropriate group actions (spectra of rings of invariants), and the rational map $\text{Spec} \mathcal{R}(X) \dashrightarrow X$ is a GIT quotient. The morphism of affine schemes $V/[G, G] = \text{Spec} \mathcal{R}(V/G) \rightarrow \text{Spec} \mathcal{R}(X)$ is the map of varieties associated to $\varphi_* : \mathcal{R}(X) \rightarrow \mathcal{R}(V/G)$ introduced above in 3.3.

The action of the torus $T = T_{\text{Cl}(X)}$ on $\text{Spec} \mathcal{R}(X)$ is associated to the multiplication homomorphism

\[
\mathcal{R}(X) \rightarrow \mathcal{R}(X) \otimes \mathbb{C}[\text{Cl}(X)]
\]

sending $f \in \Gamma(X, \mathcal{O}_X(D))$ to $f \cdot \chi^{|D|}$, with class $[D] \in \text{Cl}(X)$ defining the character $\chi^{|D|}$ of the torus in question. Here we make an identification $\mathcal{R}(X) \otimes \mathbb{C}[\text{Cl}(X)] = \mathcal{R}(X)[\text{Cl}(X)]$ of the tensor product with Laurent polynomials with coefficients in $\mathcal{R}(X)$.

We define a map

\[
\Theta : \mathcal{R}(X) \rightarrow \mathbb{C}[V]^{[G, G]} \otimes \mathbb{C}[\text{Cl}(X)]
\]
which to \( f \in \Gamma(X, \mathcal{O}_X(D)) \) associates \( f \cdot \chi^{[D]} \in \Gamma(Y, \mathcal{O}_Y(\varphi_*(D))) \otimes \chi^{[D]} \), where \( \chi^{[D]} \) denotes the character of \( \mathbb{T} \). That is, \( \Theta \) is a composition of the pushing down \( 3.1 \) and multiplication \( 3.6 \).

**Proposition 3.8.** The map \( \Theta \) defined above is injective.

**Proof.** If \( \Theta(f_1) = \Theta(f_2) \) then they are both in the same space \( \Gamma(X, \mathcal{O}_X(D)) \). However the map \( \Gamma(X, \mathcal{O}_X(D)) \to \Gamma(Y, \mathcal{O}_Y(\varphi_*(D))) \) is injective hence the claim. ■

Now we know that the total coordinate ring of \( X \) can be realized as a subring of the known ring \( \mathbb{C}[V]^{|G, G|} \otimes \mathbb{C}[Cl(X)] = R(V/G)[Cl(X)] \); the problem now is to construct generators of this subring.

If \( G \subset Sp(V) \) is a symplectic group and \( \varphi : X \to V/G \) a symplectic resolution then we are in situation of \( 2.16 \) and the description of elements of \( R(X) \) can be made even more transparent. Recall that \([L_i]\), with \( i = 1, \ldots, m \) is a \( \mathbb{Q} \)-basis of \( N^1(X) \), dual to the classes of \( C_i \)'s, components of fibers of \( \varphi|_{E_i} \). \( 2.10 \) then we have embedding of lattices \( Cl(X) \hookrightarrow \bigoplus_i \mathbb{Z}[L_i] \). This yields an embedding \( \mathbb{C}[Cl(X)] \hookrightarrow \mathbb{C}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \) where \( t_i \)'s are variables associated to \([L_i]\)'s, that is \( t_i = \chi^{[L_i]} \). By

\[
(3.9) \quad \overline{\Theta} : R(X) \to R(V/G)[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]
\]

we denote the composition of the homomorphism \( \Theta \) with the extension of coefficients \( \mathbb{C}[Cl(X)] \hookrightarrow \mathbb{C}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \). Let us note the following consequence of the construction of \( \overline{\Theta} \).

**Lemma 3.10.** The composition of \( \overline{\Theta} : R(X) \to R(V/G)[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \) with the evaluation homomorphism \( ev_1 : R(V/G)[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \to R(V/G) \) such that \( ev_1(t_i) = 1 \) for every \( i = 1, \ldots, m \), is equal to the push-forward homomorphism \( \varphi_* : R(X) \to R(V/G) \).

**Corollary 3.11.** Assume that we are in the situation discussed above. Let \( f_D \in R(V/G) = \mathbb{C}[V]^{|G, G|} \) be a non-zero element associated to an effective Weyl divisor \( D \) on \( V/G \). Let \( D = \varphi^{-1} D \) be its strict transform in \( X \). If \( f_D \in R(X) \) is associated to \( D \) in the total coordinate ring of \( X \) then

\[
\overline{\Theta}(f_D) = f_D \cdot \prod_i t_i^{\varphi_*(C_i)}
\]

**Proof.** The \( R(V/G) \)-coefficient of \( f_D \) is \( f_D \) and it remains to verify the degree of \( f_D \) with respect to \( Cl(X) \) which is provided by the homomorphism \( Cl(X) \hookrightarrow \bigoplus_i \mathbb{Z}[L_i] \) in \( 2.10 \) ■

**3.C. From the group \( G \) to a torus \( \mathbb{T} \).** From this point on by \( G \) we denote the group introduced in Section \( 2.C \). The ring of invariants of \( |G, G| = \langle -I \rangle \) is generated by quadratic forms in \( \mathbb{C}[x_1, x_2, x_3, x_4] \). We note that the linear space of forms is \( S^2 V^* \). The following observation can be verified easily.

**Lemma 3.12.** The action of \( Ab(G) = \mathbb{Z}_2^4 \) on \( S^2 V^* \) yields a decomposition of \( S^2 V^* \) into the sum of 1-dimensional eigenspaces generated by the functions \( \phi_{ij} \) given in the following table. The action of the class of \( T_i \) in \( Ab(G) \) on the function
The latter group can be interpreted as the group of characters of $\langle -T \rangle \subset \Lambda$. Let $t_0, \ldots, t_4$ be trivially so the action of $\Lambda$ is associated to the inclusion of lattices of characters. Since $-I_W$ acts trivially on $\bigwedge^2 W^*$ the action of $T_W$ on $\bigwedge^2 W^*$ descends to the action of $T_\Lambda$.

Let $\tilde{T}_i : W^* \to W^*$ be a homomorphism defined as follows $\tilde{T}_i(t_i) = -t_i$, $\tilde{T}_i(t_j) = t_j$, for $j \neq i$. We have injection $\bigoplus_{i=0}^4 \mathbb{Z} \tilde{T}_i \hookrightarrow T_W$ and thus a morphism $\bigoplus_{i=0}^4 \mathbb{Z} \tilde{T}_i \to T_\Lambda$ with kernel $(-I_W)$. We summarize this discussion in the following.

**Lemma 3.14.** The homomorphism of groups $\text{Ab}(G) = G/[G,G] \to T_\Lambda$ which maps the class of $\pm T_i$ in $\text{Ab}(G)$ to the class of $\pm \tilde{T}_i$ in $T_\Lambda$ makes the isomorphism $S^2 V^* \cong \bigwedge^2 W^*$ equivariant with respect to the action of $\text{Ab}(G)$.

We note that the above homomorphism $\text{Ab}(G) \to T_\Lambda$ can be described in terms of characters of these groups. Let $\bigoplus_{i=0}^4 \mathbb{Z}e_i \to \bigoplus_{i=0}^4 \mathbb{Z} \tilde{T}_i$ be the reduction modulo 2. The latter group can be interpreted as the group of characters of $\bigoplus_{i=0}^4 \mathbb{Z} \tilde{T}_i$. The morphism $\bigoplus_{i=0}^4 \mathbb{Z} \tilde{T}_i \to \text{Ab}(G)$ which maps $\tilde{T}_i$ to $[\pm T_i]$ implies inclusion $\text{Ab}(G)^\vee \hookrightarrow (\bigoplus_{i=0}^4 \mathbb{Z} \tilde{T}_i)^\vee \cong \bigoplus_{i=0}^4 \mathbb{Z} e_i$ and because of the inclusion $\Lambda \hookrightarrow \bigoplus_{i=0}^4 \mathbb{Z} e_i$ we get a surjective homomorphism of groups of characters $\Lambda \to \text{Ab}(G)^\vee$.

**Definition 3.15.** Let $\mathcal{R} \subset \mathbb{C}[V] \otimes \mathbb{C}[T_W] = \mathbb{C}[x_1, \ldots, x_4, t_0^{\pm 1}, \ldots, t_4^{\pm 1}]$ be the subring generated by the following functions:

- $\phi_{ij} \cdot t_i t_j$, where $0 \leq i < j \leq 4$,
- $t_i^{-2}$, where $i = 0, \ldots, 4$

The torus $T_W$ acts naturally on $\mathbb{C}[V] \otimes \mathbb{C}[T_W]$ by multiplication of the right factor and the inclusion $\mathcal{R} \subset \mathbb{C}[V] \otimes \mathbb{C}[T_W]$ is $T_W$ equivariant. We see that $-I_W$ acts on $\mathcal{R}$ trivially so the action of $T_W$ on $\mathcal{R}$ descends to the action of $T_\Lambda$. Note that we have a surjective homomorphism $\mathcal{R} \to \mathbb{C}[V]^{(-I)} \subset \mathbb{C}[V]$ obtained by setting $t_i \mapsto 1$. 

$$\phi_{rs}$$ is by multiplication by $\pm 1$, as indicated in the following table:

| function     | $T_0$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ |
|--------------|-------|-------|-------|-------|-------|
| $\phi_{01} = -2(x_1 x_4 + x_2 x_3)$ | $- - + + +$ |       |       |       |       |
| $\phi_{02} = 2\sqrt{-1}(-x_1 x_4 + x_2 x_3)$ | $- + - + +$ |       |       |       |       |
| $\phi_{03} = 2\sqrt{-1}(x_1 x_2 + x_3 x_4)$ | $- + + - +$ |       |       |       |       |
| $\phi_{04} = 2(-x_1 x_2 + x_3 x_4)$ | $- + + + -$ |       |       |       |       |
| $\phi_{12} = 2(x_1 x_3 - x_2 x_4)$ | $+ + - - +$ |       |       |       |       |
| $\phi_{13} = -x_1^2 - x_2^2 + x_3^2 + x_4^2$ | $+ + - - +$ |       |       |       |       |
| $\phi_{14} = \sqrt{-1}(x_1^2 + x_2^2 + x_3^2 + x_4^2)$ | $+ - + - -$ |       |       |       |       |
| $\phi_{23} = \sqrt{-1}(-x_1^2 + x_2^2 - x_3^2 + x_4^2)$ | $+ + - - +$ |       |       |       |       |
| $\phi_{24} = x_1^2 - x_2^2 - x_3^2 + x_4^2$ | $+ + - - +$ |       |       |       |       |
| $\phi_{34} = 2(x_1 x_3 + x_2 x_4)$ | $+ + + - -$ |       |       |       |       |
Proposition 3.16. The induced homomorphism $\mathcal{R}^T \rightarrow \mathbb{C}[V][G,G] \subset \mathbb{C}[V]$ is an injection onto the ring of invariants $\mathbb{C}[V]^G$. Therefore $\mathcal{R}^T \cong \mathbb{C}[V]^G$.

Proof. As noted in [3,4] we have an injection $Ab(G) \hookrightarrow T_\Lambda$ and we claim that the morphism $\mathcal{R} \rightarrow \mathbb{C}[V][G,G]$ is $Ab(G)$-equivariant. Indeed, the action of $Ab(G) \hookrightarrow T_W$ on $\phi_{ij}t_i$ agrees with that of $G$ on $\phi_{ij}$, while on $t_i^2$ the group $Ab(G) \hookrightarrow T_W$ acts trivially. Therefore, in particular, we have $\mathcal{R}^T \rightarrow \mathbb{C}[V]^G$. The injectivity of this homomorphism is clear since $(t_i - 1, i = 0, \ldots, 4) \cap \mathbb{C}[T_W]^{T_W} = (0)$. It remains to prove surjectivity. To this end, we note that a monomial $w_i^2 u^3 w^4$ is $G$-invariant if, for every $k = 0, \ldots, 4$, the sum $s_k = \sum_{k \in (i, j)} a_{ij}$ is divisible by 2. But then the monomial $w_i$ in question is the image of the $T_W$ invariant monomial $\prod_{i,j}(\phi_{ij}t_i)^{a_{ij}} \prod_k(t_k^2)^{s_k/2} \in \mathcal{R}$.

3.D. Generators of ideals. We present the ring $\mathcal{R}$ as the quotient ring of the graded polynomial ring $\mathbb{C}[w_{ij}, u_k : k = 0, \ldots, 4, 0 \leq i < j \leq 4]$ with the grading in $\text{Hom}(T_W, \mathbb{C}^*) \cong \bigoplus_{m=0}^4 \mathbb{Z} \cdot e_m$ given by the formula $\deg w_{ij} = e_i + e_j$, $\deg u_k = -2e_k$.

Proposition 3.17. The homomorphism $\mathbb{C}[w_{ij}, u_k : k = 0, \ldots, 4, 0 \leq i < j \leq 4] \rightarrow \mathcal{R}$ which sends $w_{ij}$ to $\phi_{ij}t_i t_j$ and $u_k$ to $t_k^2$ is surjective and preserves grading. Its kernel, denoted by $\mathcal{I}$, is generated by the following homogeneous polynomials

$w_{14} w_{23} + w_{13} w_{24} - w_{12} w_{34}$
$w_{04} w_{13} + w_{03} w_{14} - w_{01} w_{12}$
$w_{03} w_{12} + w_{02} w_{13} - w_{01} w_{23}$
$w_{02} w_{12} u_2 - w_{03} w_{13} u_3 + w_{04} w_{14} u_4$
$w_{01} w_{13} u_1 + w_{02} w_{23} u_2 + w_{04} w_{34} u_4$
$w_{03} w_{04} u_0 - w_{01} w_{14} u_1 + w_{23} w_{24} u_2$
$w_{01} w_{04} u_0 + w_{02} w_{24} u_2 + w_{13} w_{34} u_3$
$w_{01} w_{03} u_0 + w_{12} w_{23} u_2 + w_{14} w_{34} u_4$
$w_{02} u_2 + w_{03} u_3 + w_{04} u_4$
$w_{03} w_{04} u_0 + w_{02} u_2 + w_{13} u_3 + w_{14} u_4$
$w_{01} u_0 + w_{02} u_2 + w_{13} u_3 + w_{14} u_4$
$w_{01} u_0 + w_{02} u_2 + w_{13} u_3 + w_{14} u_4$

Proof. The first part is clear. The second part, that is the generators of the kernel $\mathcal{I}$, are obtained by computer calculation.

The next observation follows from [2,5].

Corollary 3.18. The ideal $\mathcal{I}_0 = \mathcal{I} + (u_0, \ldots, u_4)$ descends to an ideal in $\mathbb{C}[w_{ij}] = \mathbb{C}[w_{ij}, u_k]/(u_0, \ldots, u_4)$ which is an ideal of the affine cone over the Grassmann variety $Gr(2, W)$ embedded via Plücker embedding in $\mathbb{P}(W^*)$ with the associated grading coming from the action of $T_W$. In particular, we can identify $\text{Cl}(\mathbb{P}^2)$ with $\Lambda$ and we have a $T_\Lambda$ equivariant embedding $\text{Spec} \mathcal{R}(\mathbb{P}^2) \hookrightarrow \text{Spec} \mathcal{R}$.

The next lemma also follows by direct computer calculations.

Lemma 3.19. The primary decomposition of the ideal $\mathcal{I} + (u_2, u_3, u_4)$ contains two minimal prime ideals which are of dimension 7. One of these prime ideals is $\mathcal{I}_0$ while the other which we denote by $\mathcal{I}_{01}$ is generated by linear forms $u_2, u_3, u_4$ and $w_{01}$, and two trinomials

$w_{14} w_{23} + w_{13} w_{24} - w_{12} w_{34}$, $w_{04} w_{23} - w_{03} w_{24} - w_{02} w_{34}$
and nine binomials

\[
\begin{align*}
    w_{04}w_{13} + w_{03}w_{14}, \\
    w_{03}w_{04} - w_{13}w_{14}, \\
    w_{02}w_{04} - w_{12}w_{14}, \\
    w_{01}w_{03} + w_{11}w_{13}, \\
    w_{01}w_{02} - w_{11}w_{12}.
\end{align*}
\]

We note that the ideal \( \mathcal{I}_{01} \) determines an affine subvariety \( Z_{01} \) of \( \text{Spec } \mathcal{R} \subset \mathbb{C}^{15} \) associated to one of ten orbits of type 3C from Table 2.

Let us consider a 4-dimensional \( T_U \) torus with a lattice of characters \( U \cong \mathbb{Z}^4 \). We take linear homomorphisms \( \alpha : U \cong \mathbb{Z}^4 \rightarrow \Lambda \subset \mathbb{C}^5 \) and \( \beta : \Lambda \rightarrow U \cong \mathbb{Z}^4 \) defined by the respective matrices

\[
A = \begin{pmatrix}
    0 & 0 & 0 & 0 \\
    -2 & 1 & 1 & 1 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}, \quad
B = \begin{pmatrix}
    -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 1
\end{pmatrix}
\]

We note that \( \beta \circ \alpha = id_U \) and \( \alpha(U) \) is the kernel of the projection \( p_0 : \Lambda \rightarrow \mathbb{Z} \) on the zeroth coordinate while the kernel of \( \beta \) is generated by the vector \((1, -1, 0, 0, 0)\). Thus we have a decomposition \( T_\Lambda \cong T_U \times \lambda_0 \) where \( \lambda_0 \) is the 1-parameter subgroup of \( T_\Lambda \) associated to the projection projection \( p_0 \).

Let \( \tilde{Z}_{01} \subset Z_{01} \subset \text{Spec } \mathcal{R} \subset \mathbb{C}^{15} \) be the closed subset defined by the ideal \( \mathcal{I}_{01} := \{ u_0 + u_1, u_{02} - w_{12}, w_{03} + w_{13}, w_{04} - w_{14} \} \). We note that the variety \( \tilde{Z}_{01} \) is invariant with respect to the action of \( T_U \).

We can compare the ring \( \mathcal{R}(\mathbb{P}_3^d) \) from Lemma 2.4 with \( \mathcal{R}/\mathcal{I}_{01} \) (note that in 2.4 we intentionally use the notation overlapping with the notation used in the present set-up). Namely, by setting \( u_0 \mapsto u_0 = -u_1, u_2 \mapsto u_{02} - w_{12}, w_3 \mapsto w_{03} = -w_{13}, w_4 \mapsto w_{04} - w_{14} \) and \( w_{ij} \mapsto w_{ij} \) for \( 2 \leq i < j \leq 4 \) we define an isomorphism \( \mathcal{R}(\mathbb{P}_3^d) \rightarrow \mathcal{R}/\mathcal{I}_{01} \). Moreover, identifying \( U \) with the lattice \( Cl(\mathbb{P}_3^d) \) and using the fact that \( B \cdot A = I_U \) we conclude that the resulting isomorphism \( \text{Spec}(\mathcal{R}(\mathbb{P}_3^d)) \equiv \tilde{Z}_{01} = \text{Spec}(\mathcal{R}/\mathcal{I}_{01}) \) is \( T_{\text{Cl}(\mathbb{P}_3^d)} \) equivariant, see 2.7.

3.E. The total coordinate ring. In Section 4 we will produce a GIT quotient \( X \cong \text{Spec } \mathcal{R} \) and prove that it is smooth so that the resulting morphism \( \phi : X \rightarrow V/G \) is a resolution of singularities, see Theorem 4.11. The aim of the present subsection is to prove that the ring \( \mathcal{R} \) is the total coordinate ring of the resolution \( X \). That is, we will prove \( \mathcal{R}(X) \cong \mathcal{R} \) as graded rings.

To simplify the notation we set \( \mathcal{P} := \mathbb{C}[V](-t) = \mathcal{R}(V/G) \). The ring \( \mathcal{R} \) is constructed as a subring of the Laurent polynomial ring \( \mathcal{P}[t_0^{\pm 1}, \ldots, t_4^{\pm 1}] \) with grading in a lattice \( \Lambda \subset \mathbb{Z}^5 \) inherited from the ambient polynomial ring, see 3.15. In this construction each variable \( t_i \) is associated to the action of the symplectic reflection \( T_i, 3.14 \).

On the other hand the morphism \( \Theta : \mathcal{R}(X) \rightarrow \mathcal{P}[t_0^{\pm 1}, \ldots, t_4^{\pm 1}] \), defined in 3.9, is an embedding, see 3.8. The grading of \( \mathcal{R}(X) \) is in \( Cl(X) \) which embeds via 2.10 into \( \mathbb{Z}^5 \) so that \( Cl(X) = \Lambda \). Now, however, \( t_i \)'s are variables associated to exceptional divisors \( E_i \) of the symplectic resolution \( \varphi \), which by McKay correspondence 2.15 are in relation with the conjugacy classes of \( T_i \)'s. The composition of \( \Theta \) with evaluation \( ev_1 : t_0 \mapsto 1 \) is \( \varphi_* : \mathcal{R}(X) \rightarrow \mathcal{P} \), see 3.10. The restriction of the evaluation \( ev_1 \) to \( \mathcal{R} \) will be called \( \Phi : \mathcal{R} \rightarrow \mathcal{P} \).
We will prove that \( R = \overline{\Theta}(R(X)) \). As noted above, the definition of each side of this equality depends on the meaning of \( t_i \)'s and the link is McKay correspondence. Thus in order to relate these objects, following [Rei97] and [Kal02], we will use monomial valuations. Namely, by \( \nu_i : (\mathcal{P}) = \mathbb{C}(V)^{(+)} \to \mathbb{Z} \cup \{\infty\} \) we denote the monomial valuation on the field of fractions of \( \mathcal{P} \) associated to the action of \( T_i \). By [Kal02] we know that \( (\nu_i)|_{\mathbb{C}(V)^{(+)}} = 2\nu_{E_i} \), c.f. [2.17].

Because of [2.12] we have the following decomposition into a sum of \( \mathbb{C}[V]^G \)-modules of eigenfunctions of the action of \( Ab(G) \):

\[
(3.20) \quad \mathcal{P} = \bigoplus_{\mu \in G^\nu} \mathbb{C}[V]^\mu = \bigoplus_{\mu \in G^\nu} \mathcal{P}(\mu(\tau_0), \ldots, \mu(\tau_4))
\]

where \( \bar{\mu}(\tau_i) = 0 \) if \( \mu(\tau_i) = 1 \) and \( \bar{\mu}(\tau_i) = 1 \) if \( \mu(\tau_i) = -1 \). Note that this makes grading of \( \mathcal{P} \) in \( \mathbb{Z}^5 \). The grading on \( \mathcal{P} \) agrees with the \( \mathbb{Z}^5 \) grading on \( R \) and \( R(X) \) as well as with the valuations \( \nu_i \).

**Lemma 3.21.** For every \( f \in \mathcal{P} \) if a monomial \( f \cdot t_0^{d_0} \cdots t_4^{d_4} \) is in either \( R \) or \( R(X) \) then \( f \in \mathcal{P}_{(d_0, \ldots, d_4)} \). If \( f \in \mathcal{P}_{(d_0, \ldots, d_4)} \) then for every \( i = 0, \ldots, 4 \) the valuation \( \nu_i(f) \) has the same parity as \( d_i \). In particular, \( \nu_i(f_{ij}) = 1 \) if \( r \in \{i, j\} \) and \( \nu_i(f_{ij}) = 0 \) if \( r \notin \{i, j\} \).

**Proof.** For \( f \cdot t_0^{d_0} \cdots t_4^{d_4} \in R \) it is enough to check the statement for generators of \( R \), see [3.11]. If \( f \cdot t_0^{d_0} \cdots t_4^{d_4} \in R(X) \) then the statement follows from the definition of \( \Theta \), see [3.7]. The last part follows directly from the definition [2.17].

**Example 3.22.** Let \( \overline{D}_{ij} \) be a divisor on \( X \) which is strict transform via \( \varphi^{-1} \) of the Weil divisor \( D_{ij} \) on \( V/G \) associated to \( \phi_{ij} \in R(V/G) = \mathcal{P} \). Then the principal divisor on \( X \) of the function \( \phi_{ij}^2 \in \mathbb{C}[V]^G \subset \mathbb{C}(X) \) satisfies the following equality:

\[
\text{div}_{X} (\phi_{ij}^2) = 2\overline{D}_{ij} + E_i + E_j.
\]

Thus, if \( C_m \) is a general fiber of \( \varphi_{E_m} \) then \( \overline{D}_{ij} \cdot C_m = 1 \) if \( m \in \{i, j\} \) and \( \overline{D}_{ij} \cdot C_m = 0 \) if \( m \notin \{i, j\} \).

More generally we have the following.

**Lemma 3.23.** Let \( \overline{D} \) be a divisor in \( X \) which is strict transform via \( \varphi^{-1} \) of an effective Weil divisor \( D \) on \( V/G \). If \( f_D \in R(V/G) = \mathcal{P} \) is the element associated with \( D \) then \( \nu_i(f_D) = \overline{D} \cdot C_i \). Moreover, if \( f_{\overline{D}} \in R(X) \) is the element associated with \( \overline{D} \) then

\[
\overline{\Theta}(f_{\overline{D}}) = f_D \cdot t_0^{\nu_0(f_D)} \cdots t_4^{\nu_4(f_D)}
\]

**Proof.** Note that \( f_D^2 \in \mathbb{C}[V]^G \subset \mathbb{C}(X) \) and we have

\[
\text{div}_{X} (f_D^2) = 2\overline{D} + \nu_{E_0}(f_D^2)E_0 + \cdots + \nu_{E_4}(f_D^2)E_4
\]

Since \( \text{div}_{X} (f_D^2) \cdot C_i = 0 \) and \( \nu_i(C_i) = 2\nu_{E_i} \), we get

\[
2\overline{D} \cdot C_i = - (\nu_0(f_D)E_0 + \cdots + \nu_4(f_D)E_4) \cdot C_i
\]

and the first claim follows because \( D_i \cdot C_i = -2 \) and \( D_j \cdot C_i = 0 \) if \( j \neq i \). The second statement follows from [3.11].

**Corollary 3.24.** In the notation introduced above the following holds: \( \overline{\Theta}(f_{E_i}) = t_i^{-2} \) and \( \overline{\Theta}(f_{\overline{D}_ij}) = \phi_{ij} t_it_j \). Therefore \( R \subseteq \overline{\Theta}(R(X)) \).

The following result has been anticipated in the cyclic quotient case, see [2.22] and [3.4].
Proposition 3.25. The image via $\varphi_*$ of the graded pieces of $R(X)$ is determined by valuations $\nu_i$ in the following way:

$$\varphi_*(R(X)_{[d_0,\ldots,d_4]}) = \{ f \in P_{[d_0,\ldots,d_4]} : \forall_i \nu_i(f) \geq d_i \}$$

Proof. We use the notation from Lemma 3.23. If $f = f_D$ and $d_i \leq \nu_i(f)$ then the numbers $a_i = (\nu_i(f) - d_i)/2$ are non-negative integers because of 3.21 and

$$f_M := f_D \cdot f_{E_0}^{a_0} \cdots f_{E_4}^{a_4} \in R(X)_{[d_0,\ldots,d_4]}$$

is the element which is mapped to $f_D$ via $\varphi_*$. On the other hand, given an effective divisor $M$ on $X$ we can write it as $M = D + \sum u_i E_i$ where $D$ is the strict transform of $D := \varphi_*(M)$ and $a_i$ are non-negative integers. If $f_M = f_M \cdot f_{E_0}^{a_0} \cdots f_{E_4}^{a_4}$ is in $R(X)_{[d_0,\ldots,d_4]}$ then by the same arguments $\nu_i(\varphi_*(f_M)) = \nu_i(f_D) = d_i + 2a_i$. 

Now, in order to establish the inclusion $R \supseteq \overline{\Theta}(R(X))$ we need the following.

Lemma 3.27. Let $f \in P$ be homogeneous with respect to the decomposition 3.26. For $i = 0,\ldots,4$ we denote $d_i = \nu_i(f)$ and set $\overline{f} := f \cdot t_0^{d_0} \cdots t_4^{d_4} \in P[t_0,\ldots,t_4]$. Then $\overline{f} \in R([d_0,\ldots,d_4])$.

Proof. Since the functions $\phi_{ij}$ generate $P$ we can write $f = F(\phi_{ij})$ where $F \in \mathbb{C}[w_{ij} : 0 \leq i < j \leq 4]$ is a polynomial

$$F = \sum \alpha_{[k_{ij} : 0 \leq i < j \leq 4]} \prod w_{ij}^{k_{ij}}$$

where coefficients $\alpha_{[k_{ij} : 0 \leq i < j \leq 4]}$ depending on multi-indices $(k_{01}, k_{02}, k_{12}, k_{13}, k_{23})$ are in $\mathbb{C}$. Now we define $\hat{f} = F(\phi_{ij} t_j) \in P[t_0,\ldots,t_4]$, and if we write

$$\hat{f} = \sum \beta_{[l_{ij} : 0 \leq i < j \leq 4]} t_0^{l_0} \cdots t_4^{l_4}$$

then $\beta_{l_{ij} : 0 \leq i < j \leq 4} = \sum \alpha_{[k_{ij} : 0 \leq i < j \leq 4]} \prod w_{ij}^{k_{ij}}$ where the sum is taken over such set of indices $\alpha_{[k_{ij} : 0 \leq i < j \leq 4]}$ that for every $r = 0,\ldots,4$ we have $\sum_{r \in \{i,j\}} k_{ij} = i_r$. It follows that every non-zero $\beta_{l_{ij} : 0 \leq i < j \leq 4}$ is in $R([l_{01},\ldots,l_{13}])$.

We claim that for every $r = 0,\ldots,4$ the valuation $\nu_r(f) = d_r$ is the minimal $i_r$ such that the coefficient $\beta_{l_{ij} : 0 \leq i < j \leq 4}$ is non-zero. Indeed, we verify it directly like in the case of the cyclic group action, see Section 2.1. Note that $\nu_r(\phi_{ij}) = 1$ if $r \in \{i,j\}$ and $\nu_r(\phi_{ij}) = 0$ if $r \notin \{i,j\}$. On the other hand, the parity of $i_r$'s for which $\beta_{l_{ij} : 0 \leq i < j \leq 4} \neq 0$ is the same as that of $d_r$. Hence we can write

$$\overline{f} = \sum (\beta_{l_{ij} : 0 \leq i < j \leq 4} t_0^{l_0} \cdots t_4^{l_4}) \cdot (t_0^{d_0-d_r} \cdots t_4^{d_4-d_r})$$

which is a presentation of $\overline{f}$ in terms of the generators of $R$.

Theorem 3.28. The ring $R$ is the total coordinate ring of every symplectic resolution $\varphi : X \rightarrow V/G$. In fact $R = \overline{\Theta}(R(X))$.

Proof. By 3.24 it is enough to prove $R \supseteq \overline{\Theta}(R(X))$. Take $f \cdot t_0^{d_0} \cdots t_4^{d_4} \in \overline{\Theta}(R(X))$ with $f \in P$. Then, by 3.25 $\nu_i(f) \geq d_i$ and $\nu_i(f)$ has the same parity as $d_i$ by 3.24. Therefore, by 3.27

$$f \cdot t_0^{d_0} \cdots t_4^{d_4} = f \cdot (t_0^{\nu_0(f)} \cdots t_4^{\nu_4(f)}) \cdot t_0^{d_0-\nu_0(f)} \cdots t_4^{d_4-\nu_4(f)} \in R$$

$$\square$$

81 Resolutions 19
4. GIT quotients of Spec \( R \)

We study linearizations and corresponding GIT quotients of Spec \( R \). For a chosen one we prove its smoothness and this way we obtain an explicit description of a resolution of \( V/G \). In section \[5\] we will use these results to show how to modify this resolution to obtain all other ones.

4.A. Linearization, stability and isotropy. To construct a GIT quotient of Spec \( R \) we need to choose a suitable linearization of the trivial line bundle. It will be represented by a character \( \chi^u: T_A \to \mathbb{C}^* \) of the 5-dimensional torus. We investigate the sets of stable and semi-stable points of Spec \( R \) with respect to \( \chi \). In this section we explain how to check whether \( \chi^u \) and these sets have properties needed to have a good description of the quotient, i.e. satisfy condition \( 4.9 \). Note that we do not compute the irrelevant ideal, i.e. the ideal of the closed set of unstable points, explicitly – we prefer to deal with the set of semi-stable points using a description based on toric geometry, as explained below. The idea, similar as for the 2-dimensional quotients in [DB13] Sect. 4], is to look at the embedding

\[
\text{Spec } R \hookrightarrow \text{Spec } \mathbb{C}[w_{ij}, u_k : k = 0, \ldots, 4, 0 \leq i < j \leq 4] \simeq \mathbb{C}^{15}
\]
such that \( T_A \) is a subtorus of the big torus \((\mathbb{C}^*)^{15}\) of the affine space, as described in \[3.C \]. Then the set of semi-stable points can be presented as the intersection of Spec \( R \) with certain orbits of the big torus, see \[4.2 \] and Section \[1.B \].

We start from a few observations in a slightly more general setting. Let \( Z \) be an affine subvariety of \( A \simeq \mathbb{C}^r \), invariant under a (diagonal) action of a subtorus \( T \) of \( T_A \simeq (\mathbb{C}^*)^r \). By \( M_T \) and \( \hat{M}_T \) we denote monomial lattices of \( T \) and \( T_A \) respectively, and by \( \sigma^T \) and \( \hat{\sigma}^T \) their positive orthants. Then both \( \mathbb{C}[Z] \) and \( \mathbb{C}[A] \) have a grading by \( M_T \) associated with the action of \( T \). To analyze semi-stability we use the notion of orbit cones, see [BH06], Def. 2.1.

**Definition 4.1.** The orbit cone \( \omega_T(z) \subset M_T \otimes \mathbb{R} \) of \( z \in Z \) is a convex (polyhedral) cone generated by

\[
\{ u \in M_T : \exists f \in \mathbb{C}[Z]_u \text{ such that } f(z) \neq 0 \}
\]

where \( \mathbb{C}[Z]_u \) denotes the graded piece in degree \( u \) of \( \mathbb{C}[Z] \).

That is, to prove that \( z \) is semi-stable with respect to \( \chi^u \) it is sufficient to check that \( u \in \omega_T(z) \), hence we want to describe the orbit cones for considered action. We rely on a basic observation which follows directly from the definition of stability, see e.g. [Dol03], 8.1.

**Lemma 4.2.** Fix a character \( \chi^u, u \in M_T \), which gives linearizations of actions of \( T \) both on \( Z \) and on \( A \). Then the sets of stable and semi-stable points with respect to \( \chi^u \) satisfy \( Z^{ss} = Z \cap A^{ss} \) and \( Z \cap A^s \subseteq Z^s \).

The first part can be rephrased in terms of orbit cones, cf. [BH06], 2.5.

**Corollary 4.3.** The orbit cone for \( z \in Z \) and the action of \( T \) on \( Z \) is equal to the orbit cone of \( z \) and the action on \( A \).

Orbit cones for the affine space \( A \) are easy to describe. Let \( \pi: \hat{M}_T \to M_T \) be the homomorphism of lattices corresponding to \( T \subseteq T_A \); we will assume that it is given by the matrix \( U \) of weights of the action of \( T \) on \( A \). By \( \gamma_z \) we denote the face of \( \hat{\sigma}^T \) generated by monomials non-vanishing on \( T_A \cdot z \subset A \). The proof of the following statement is straightforward.

\[
\text{affine subvariety of } A \simeq \mathbb{C}^r \]

Lemma 4.4. Orbit cones for the action of $T$ on $A$, hence also on $Z$, are images of faces of $\tilde{\sigma}$ under $\pi$. More precisely, $\omega_T(z) = \pi(\gamma_z)$.

Corollary 4.5. A point $z$ is semi-stable with respect to the $T$-action on $Z$ linearized by $\chi^u$ if and only if $u \in \pi(\gamma_z)$ (see also [BH06] Lem. 2.7).

The next lemma follows from the fact that the actions of $T$ and $T_A$ on $A$ commute (or for semi-stability from [4.4], cf. [BH06] 2.5).

Lemma 4.6. Stability, semi-stability and isotropy groups of the action of $T$ on $A$ (hence also semi-stability and isotropy groups of the action on $Z$) are invariants of $T_A$, i.e. are properties of whole $T_A$-orbits.

These observations are very useful in algorithms dealing with sets of semi-stable points, see [Kei12]. Computations concerning stability are more subtle because of the orbit closedness condition: it may happen that a point is stable under the action on $Z$, but not on $A$. However, it turns out that for our purposes it is sufficient to check whether a point of $Z$ is in $A^\bullet - \{u\}$, such points are stable in $Z$.

Lemma 4.7. Consider a $T$-action on $A$ linearized by $\chi^u$, and assume that the isotropy group of a point $z \in A$ is finite. If $u$ is in the relative interior of $\pi(\gamma_z)$, then $z$ is stable.

Proof. A point in the closure of $T$-orbit is a limit of some one-parameter subgroup of $T$, hence it belongs to an orbit corresponding to a proper face of $\gamma_z$. Thus, to prove the stability of $z$ we want to find a $T$-invariant section $f$ of the trivial bundle on $A$ such that $z \in A_f = \{a \in A : f(a) \neq 0\}$ and $A_f$ does not contain any orbits corresponding to proper faces of $\gamma_z$. We will choose $f$ which is a character of $T$ (regular on $A$). If $u$ is in the relative interior of $\pi(\gamma_z)$ then there is some $\pi$ in the relative interior of $\gamma_z$ such that $\pi(\gamma_z) = u$, and we take $f = \chi^u$. Because $\pi$ is not contained in any face of $\gamma_z$, then $f$ vanishes on all orbits corresponding to faces of $\gamma_z$.

Next, we need to determine the orders of isotropy groups of points under the action of $T$. They can be computed using the Smith normal form of a matrix, see [New72 II.15], which is implemented e.g. in Singular, [DGPS12]. A matrix $U_z$ is obtained from the matrix $U$ of weights of the $T$-action by choosing columns corresponding to non-zero coordinates in the orbit $T_A \cdot z$.

Lemma 4.8. Let $a_1, \ldots, a_p$ be the non-zero entries on the diagonal of the Smith normal form (over $\mathbb{Z}$) of $U_z$ for some $z \in A$. If they fill the whole diagonal then the order of the isotropy group of $z$ under the $T$-action is $a_1 \cdots a_p$. If there are also zeroes on the diagonal then the isotropy group of $z$ is infinite.

Proof. Note that columns of $U_z$ are exactly the rays of the orbit cone $\omega_T(z)$, i.e. $U_z$ determines the homomorphism $M_z \to M_T$ of monomial lattices of $T_A \cdot z$ and $T$. Then $\text{Hom}(M_T/M_z, \mathbb{C}^\ast)$ is isomorphic to the kernel of the corresponding morphism of the tori, which is exactly the isotropy group of $z$. The Smith normal form of $U_z$ is obtained by multiplying it on both sides by some invertible integer matrices such that the result is a diagonal matrix with non-zero entries $a_1, \ldots, a_p$, where $a_i | a_{i+1}$ for $1 \leq i \leq p - 1$. Thus it gives the description of the quotient group $M_T/M_z$ as a product of finite cyclic groups of orders $a_1, \ldots, a_p$ and $\mathbb{Z}^q$, where $q$ is the number of zeroes on the diagonal.
Now we can describe the algorithm which we use to determine a good linearization for constructing a GIT quotient of $Z$ by $T$ explicitly. We are looking for a linearization $\chi^u$ satisfying the following condition:

$\text{(4.9)}$  
semi-stability of a point of $Z$ with respect to $\chi^u$ implies its stability with respect to $\chi^u$.

This is because in such a situation we obtain a geometric quotient together with a nice description of the set of stable points. By 4.8 this set (and also the set of zeroes of the irrelevant ideal) is a sum of intersections of certain $T_A$-orbits in $A$ with $Z$.

**Definition 4.10.** A $T_A$-orbit in $A$ which has non-empty intersection with $Z$ and whose points are semi-stable with respect to a fixed linearization $\chi^u$ will be called a $C[Z]$-relevant orbit with respect to $\chi^u$. (We will skip the information about the linearization whenever the choice is clear.)

Note that if a linearization $\chi^u$ satisfies condition 4.9, the intersections of $Z$ with all $C[Z]$-relevant orbits cover the set of stable points $Z^s$.

Algorithm 4.11 is implemented in the form of a small Singular package available at [www.mimuw.edu.pl/~marysia/gitcomp.lib](http://www.mimuw.edu.pl/~marysia/gitcomp.lib). The input data for the algorithm consists of

1. the ideal $I$ of $Z$,
2. the matrix $U$ defining the $T$-action on $A$ and $Z$,
3. a linearization of this action, given by a character $\chi^u$ of $T$, where $u \in M_T$.

The output is the information whether $\chi^u$ satisfies condition 4.9.

We start the computations from determining the set of $T_A$-orbits which have non-empty intersection with $Z$. They are represented by convex polyhedral cones: faces of the positive orthant $\hat{\sigma}^+$ of $\hat{M}$. Such cones are called $I$-faces in $[Kei12]$. Note that by 4.6 and 4.7 we need to check only properties of the whole $T$-orbits, hence the program operates on lists of $I$-faces or corresponding orbit cones.

**Algorithm 4.11.** The following actions are performed:

1. Determine the list $F$ of $I$-faces of $\hat{\sigma}^+$. 
   Here we use a Singular package GITfan.lib (see [Kei12]), which for given $I$ returns the desired list of cones.
2. Determine semi-stable points.
   For all $I$-faces from $F$ we compute rays of corresponding orbit cones using the matrix $U$. Then, by 4.5 we check whether $u$ is inside these cones. The result is the list $F^{ss}$ of orbit cones corresponding to $T_A$-orbits semi-stable with respect to $\chi^u$.
3. Check finiteness of the isotropy group.
   The order of the isotropy group is computed for each cone from $F^{ss}$, as described in 4.8. If for all cones from $F^{ss}$ points of corresponding orbits have finite isotropy groups then the next step is performed. Otherwise the negative answer is given immediately.
   Note this point of the computations is independent of the linearization.
4. Check stability of elements of $F^{ss}$.
   By 4.7 we check whether $u$ is in the relative interior of cones from $F^{ss}$. If it is true for all cones from this list then the linearization given by $\chi^u$ satisfies condition 4.9. In this case the program can output the list $F^{ss}$, which gives a
useful description of the set of $\chi^u$-stable points of $Z$. Also, the program returns
the information on orders of isotropy groups for all stable orbits.

Finally, we reveal the main application of Algorithm 4.11. We are looking for
a linearization of the $T := T_\Lambda$ action on $Z := \text{Spec } R$, embedded in $A \simeq \mathbb{C}^{15}$, which
allows to describe explicitly the geometry of the quotient. A very good candidate,
because of its symmetries, is $\chi^\kappa$ given by the weight vector $\kappa = (2, 2, 2, 2, 2)$. The
respective quotient makes a good starting point for performing flops leading to
other resolutions, see Section 5. We prove that this is indeed a right choice.

**Proposition 4.12.** The linearization of the $T_\Lambda$-action on $\text{Spec } R$ given by $\chi^\kappa$
for $\kappa = (2, 2, 2, 2, 2)$ satisfies condition 4.9. Therefore the corresponding quotient is geometric.
Moreover, all points of $\text{Spec } R$ which are semi-stable with respect to $\chi^\kappa$
have trivial isotropy group.

**Proof.** Computations performed using the implementation of Algorithm 4.11
give the result stated above. We use the weights of the $T_W$-action instead of $T_\Lambda$,
which changes just the order of the isorropy group multiplying it by 2.

**4.B. The set of stable points.** Using Algorithm 4.11 for a chosen linearization
satisfying condition 4.9 one obtains an explicit description of the set $(\text{Spec } R)^s$
of stable points. In general, it comes in the form of the list $F^s$ of $I$-faces corresponding to $R$-relevant orbits, see 4.10. We will describe these orbits in the
case of the linearization $\chi^\kappa$: by 4.12 their intersections with $\text{Spec } R$ cover the
whole $(\text{Spec } R)^s$.

It turns out that for $\chi^\kappa$ there are only 167 $R$-relevant orbits in $A$. Because of
the symmetries of generators of $I$ the result may be presented as a much shorter list.
Groups of these orbits are given by vanishing of sets of variables which differ by a
certain permutation of indices, hence we list just combinatorial types of possible
sets of vanishing variables, see Table 2.

Note that in each description of an orbit type in Table 2 letters $a, b, c, d, e$
stand for different elements of $\{0, 1, 2, 3, 4\}$. The division into orbit types is based on the
number of $u_i$’s equal to 0 and then on the set of $w_{ij}$’s equal to 0.

**4.C. Smoothness of the quotient.** The last element needed in the proof
that the geometric quotient $X = (\text{Spec } R)^s/T_\Lambda$ associated with the distinguished
linearization $\chi^\kappa$ is a resolution of singularities of $V/G$ is the smoothness of $X$. We
follow the idea explained in [DB13, Prop. 4.5]: we check that $X$ is a geometric
quotient of a smooth variety by a free torus action. Since by 4.12 the action of $T_\Lambda$
on $(\text{Spec } R)^s$ is free, it is sufficient to show that $X$ is nonsingular.

A natural approach is to compute the ideal of the set of singular points of $\text{Spec } R$
directly from the Jacobian criterion and show that it has empty intersection with $(\text{Spec } R)^s$.
However, the input data is too big for performing a direct computation in reasonable time. Hence we divide the process into a few separate cases and make use of the $T_\Lambda$ action and the description of $(\text{Spec } R)^s$ in terms of the toric structure of the ambient affine space $\mathbb{C}^{15}$ in Table 2.

We rely on two basic observations. First, it is sufficient to prove the smoothness
of one point in every $T_\Lambda$-orbit in $(\text{Spec } R)^s$. Thus we may consider only points with all $u_i$ equal to 0 or 1, that is, using the $T_\Lambda$-action we move non-zero $u_i$’s to 1. This
already simplifies the Jacobian matrix of $\text{Spec } R$ a lot. Then, since we do not want to
treat each orbit separately, we use the symmetries of equations of $\text{Spec } R$ so that
Table 2. $R$-relevant orbits in the ambient affine space of $\text{Spec } R$

| type id | equations | # orbits | dim | dim orbit $\cap \text{Spec } R$ |
|---------|-----------|----------|-----|-------------------------------|
| 5A      | $u_0 = u_1 = u_2 = u_3 = u_4 = 0$  $w_{ab} = w_{cd} = 0$ | 15  | 8  | 5 |
| 5B      | $u_0 = u_1 = u_2 = u_3 = u_4 = 0$  $w_{ab} = 0$ | 10  | 9  | 6 |
| 5C      | $u_0 = u_1 = u_2 = u_3 = u_4 = 0$  $w_{ab} = w_{cd} = 0$ | 1   | 10 | 7 |
| 3A      | $u_a = u_b = u_c = 0$  $w_{ab} = w_{ac} = w_{bc} = 0$  $w_{de} = 0$ | 10  | 8  | 5 |
| 3B      | $u_a = u_b = u_c = 0$  $w_{ab} = w_{de} = 0$ | 30  | 10 | 6 |
| 3C      | $u_a = u_b = u_c = 0$  $w_{de} = 0$ | 10  | 11 | 7 |
| 1A      | $u_a = 0$  $w_{ab} = w_{ac} = w_{bc} = 0$  $w_{de} = 0$ | 30  | 10 | 6 |
| 1B      | $u_a = 0$  $w_{bc} = w_{de} = 0$ | 15  | 12 | 7 |
| 1C      | $u_a = 0$  $w_{ab} = 0$ | 20  | 13 | 7 |
| 1D      | $u_a = 0$  $w_{ab} = 0$ | 5   | 14 | 8 |
| 0A      | $w_{ab} = w_{ac} = w_{bc} = w_{de} = 0$ | 10  | 11 | 7 |
| 0B      | $w_{ab} = 0$  $w_{ab} = 0$ | 10  | 14 | 8 |
| 0C      | 1  | 15 | 9 |

we can consider certain representatives of combinatorial types of $R$-relevant orbits. The following observation can be checked straightforwardly.

**Lemma 4.13.** The equations of $\text{Spec } R$, listed in 3.17, are invariant under a cyclic change of indices $i \mapsto i + 1 \mod 5$.

**Proposition 4.14.** The set $(\text{Spec } R)^s$ of stable points with respect to the linearization $\chi^c$ is nonsingular.

**Proof.** The argument is computational. We explain how to deal with the computations using basic functions of, for example, Macaulay2, [GS]. We assume that the equations of $\text{Spec } R$ are ordered as in 3.17. To compute the Jacobian matrix we differentiate with respect to variables ordered as follows:

$w_{01}, w_{02}, w_{03}, w_{04}, w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}, u_0, u_1, u_2, u_3, u_4,$

Take $z \in \text{Spec } R$ and assume that all $6 \times 6$ minors of the Jacobian matrix vanish at $z$. Then we have to show that $z \notin (\text{Spec } R)^s$, that is $z$ does not belong to any orbit from Table 2. Let us outline the computations. For each orbit type in Table 2 we choose representatives with respect to the cyclic group action, using 4.13 and consider only $z$ from these chosen orbits. We simplify the Jacobian matrix substituting 0 or 1 for some variables, looking at the $T_\lambda$-action on $z$ and its orbit type. Then we compute some (suitably chosen) $6 \times 6$ minors of this matrix and...
look for monomials. Finding a monomial minor means that the product of some coordinates vanishes at z. Usually this gives a few subcases to consider (vanishing of each coordinate from the product has to be considered separately). However, we obtain more precise information of the orbit type of z, which simplifies the Jacobian matrix even more. In each case, after a small number of such steps, we arrive at the conclusion that z is not contained in any R-relevant orbit, which finishes the proof.

We are left with providing the details of the computations. To shorten the description, by det(i_1,\ldots,i_k|j_1,\ldots,j_k) we will denote the minor of the rows i_1,\ldots,i_k and the columns j_1,\ldots,j_k of the Jacobian matrix of equations of Spec R. By Mon(x_{k_1},\ldots,x_{k_n}) we understand the set of all monomials in variables x_{k_1},\ldots,x_{k_n}. There are four cases depending on the type of the orbit from Table 2 in which z lies.

**Type 5.** We have u_0 = u_1 = u_2 = u_3 = u_4 = 0. After substituting into the Jacobian matrix check that det(7,\ldots,12|2,3,4,5,8,16) ∈ Mon(u_{01},u_{02},u_{12}), hence one of these variables is 0. By permuting indices we get two cases.

(a) w_{01} = 0.

Then we have det(0,1,4,11,13,14|0,1,2,9,11,13) ∈ Mon(w_{13},w_{14},w_{34}), and det(5,6,9,10,12,13|1,3,4,7,8,15) ∈ Mon(w_{02},w_{03},w_{23}). Hence at least 3 of w_{ij}'s are 0, which is impossible in orbits of type 5 in Table 2.

(b) w_{02} = 0.

Then we have det(0,1,4,10,13,14|0,1,2,6,7,9) ∈ Mon(w_{03},w_{04},w_{34}), and also det(2,3,9,11,12,14|0,3,4,10,11,14) ∈ Mon(w_{12},w_{14},w_{24}). Hence again at least three w_{ij}'s vanish.

**Type 3.** Applying the Λ action we may move to the point where these u_i's which are nonzero are equal to 1. By Remark 4.13 there are two cases.

(a) u_0 = u_1 = 1, u_2 = u_3 = u_4 = 0, w_{01} = 0.

Then det(0,\ldots,4,14|0,1,2,9,17,18) ∈ Mon(w_{04},w_{34}), but from the equations for cases 3A and 3B we see that only w_{34} = 0 could happen.

Next, det(0,\ldots,3,6,14|0,1,3,10,18,19) is a monomial, so at least three w_{ij}'s are 0. This means that we are in the case 3A and w_{23} = w_{24} = 0. However, in this case det(0,\ldots,3,7,12|0,9,10,11,17,18) is a monomial and too many variables vanish.

(b) u_0 = u_2 = 1, u_1 = u_3 = u_4 = 0, w_{02} = 0.

Then det(0,\ldots,4,11|0,1,2,9,15,18) ∈ Mon(w_{04},w_{12},w_{34}). Again, the only possibility consistent with equations of 3A and 3B is w_{34} = 0.

Now det(0,\ldots,3,5,11|0,9,10,11,17,18) ∈ Mon(w_{01},w_{04},w_{24}), but none of these variables can be 0 in the cases of type 3.

**Type 1.** Permuting indices and applying the Λ action we may assume that u_0 = u_1 = u_2 = u_3 = 1 and u_4 = 0. Compute the ideal generated by 6 × 6 minors of two sets of rows {0,\ldots,4,9} and {3,\ldots,8} and check that it contains w_{12}^6, that is w_{12} = 0. Hence we are in the case 1A or 1B and they both require w_{03} = 0. Then the ideal generated by 6 × 6 minors of the set of rows {0,\ldots,6} contains w_{02}^6, but this is impossible in any of these cases.

**Type 0.** Applying the Λ action we may assume that u_0 = u_1 = u_2 = u_3 = u_4 = 1. Compute the ideal generated by 6 × 6 minors of two sets of rows {0,\ldots,6} and {3,\ldots,8} and check that it contains w_{01}^5w_{02}, w_{01}^5w_{03} and w_{01}^5w_{04}. If w_{01} ≠ 0 then w_{02} = w_{03} = w_{04} = 0, which is impossible in orbits of type 0. Hence w_{01} = 0.
Now the ideal of $6 \times 6$ minors of two sets of rows $\{0, \ldots, 6\}$ and $\{3, \ldots, 8\}$ contains $w_{05}^2 w_{14}$. Thus we are in type $0A$ and there are two possibilities:

(a) $w_{01} = w_{02} = w_{12} = w_{34} = 0$.
Then $\det(0, 1, 2, 3, 5, 7[5, 10, 11, 14, 18, 19]) \in \text{Mon}(w_{04}, w_{13})$, so five variables vanish, which is impossible in orbits of type $0A$.

(b) $w_{01} = w_{14} = w_{04} = w_{23} = 0$.
Then $\det(0, \ldots, 4, 9[0, 1, 2, 9, 16, 18]) \in \text{Mon}(w_{03}, w_{34})$, a contradiction again.

\[ \mathbf{Theorem 4.15.} \] The GIT quotient $X$ of $\text{Spec } R$ by $T_A$ associated with the linearization $\chi^k$ is a resolution of singularities of $V/G$.

\[ \mathbf{Proof.} \] The isomorphism $R^{\mathbb{Z}^A} \simeq \mathbb{C}[V]^G$ proved in \cite{316} gives a proper birational morphism from $X$ to $V/G$. The properness follows by \cite{CLS11} 14.1.12 applied to the embedding $\text{Spec } R$ and its quotients in the toric ambient spaces. And by \cite{414} we know that $X$ is smooth.

5. The geometry of resolutions

5.A. The central resolution. Let us summarize the information which we get from the previous sections. The exceptional set of a resolution $\varphi : X \to V/G$ is covered by divisors $E_0, \ldots, E_4$ associated to the classes of symplectic reflections in $G$. There is a unique 2-dimensional fiber of $\varphi$ over $[0] \in V/G$ which has 11 components. \cite{215}, \cite{210} Each $E_i$ is contracted by $\varphi$ to a surface of $A_1$ singularities outside of $[0] \in V/G$. In terms of the ring $R$ the divisors $E_i$ are associated to functions $t_i^{2}, \ldots, t_{11}^{2}$.

By $C_i$ we denote a general fiber of $\varphi|_{E_i}$. Clearly $E_i \cdot C_j$ is $-2$ if $i = j$ and it is zero otherwise. Now we define $\kappa = \sum e_i$. In terms of the basis in $N^1(X) = \text{Cl}(X) \otimes \mathbb{R}$ dual to classes of $C_i$'s the class $\kappa$ is the vector $(2, 2, 2, 2, 2)$, see \cite{210}. For $i = 0, \ldots, 4$ in $N^1(X)$ we consider classes $e_i = [-E_i]$. By \cite{AW14} Thm. 3.5 we get the following.

\[ \mathbf{Lemma 5.1.} \] For every resolution $X \to V/G$ the cone of movable divisors $\text{Mov}(X)$ is spanned by the classes $e_i$.

By Theorem 4.15 the GIT quotient of $R$ by $T_A$ associated to the character $\kappa$ is a resolution $\varphi^\kappa : X^\kappa \to V/G$.

Recall that Table 2 presents a list of big torus orbits in the affine space containing $\text{Spec } R$ which are relevant with respect to the finite isotropy and the semistability condition associated to $\kappa$. Note that divisors $E_i$ are associated to relevant orbits of type 1D. The intersection $\bigcap_i E_i$ is associated to the unique relevant orbit of type 5C. In fact, from \cite{518} we see that this special orbit comes from an equivariant embedding $\text{Spec } R(\mathbb{P}^4_2) \hookrightarrow \text{Spec } R$.

This gives rise to an embedding of GIT quotients $\iota : \mathbb{P}^4_2 \hookrightarrow X^\kappa$ such that $\iota^* : \text{Pic } X^\kappa \to \text{Pic } \mathbb{P}^4_2$ is an isomorphism. By $F_0$ we will denote $\iota(\mathbb{P}^4_2)$. It follows that we can identify $N^1(X^\kappa) = N^1(\mathbb{P}^4_2)$ and we have $\text{Nef}(X^\kappa) \subseteq \text{cone}(\alpha_i, \beta_i : 0 \leq i \leq 4)$, where $\alpha_i := (e_i + \kappa)/2$ and $\beta_i := (-e_i + \kappa)/2$ c.f. \cite{214}.

Dually, we have isomorphism $\iota_* : N_1(X) \cong N_1(\mathbb{P}^4_2) = N_1(\mathbb{P}^4_2)$ and via this identification the classes of $(-1)$ curves on $F_0 \cong \mathbb{P}^4_2$ are $f_{ij} = (e_i + e_j)/2$. Let $C_{ij} \subset F_0$ be one of these $(-1)$-curves. Then the family of deformations of $C_{ij}$ is of dimension 2 at least, see \cite{WW03} 2.3, and it must cover a component of a
2-dimensional fiber of \( \varphi^\kappa \). Let us call such a component \( F_{ij} \). Since intersection of \( C_{ij} \) with the ample class \( \kappa \) is 1, the family of deformations of \( C_{ij} \) in \( F_{ij} \) is unsplit and of dimension 2, hence every curve in \( F_{ij} \) is numerically proportional to \( C_{ij} \), see e.g. [Ko96, IV.3.13.3].

Because the 2-dimensional fiber of \( \varphi^\kappa \) has 11 components we have a bijection between \( C_{ij} \)'s and components of this fiber different from \( F_0 \). Also, it follows that all curves in the 2-dimensional fiber of \( \varphi^\kappa \) have classes in \( \text{Eff}(F_0) \) and therefore, dually, \( \text{Nef}(X) = \text{Nef}(F_0) \). Therefore a contraction of the \((-1)\) curve \( C_{ij} \) in \( F_0 \) extends to a small contraction of \( X^\kappa \) and by [WW03, Thm. 1.1] \( F_{ij} \cong \mathbb{P}^2 \). Again, because \( C_{ij} \) has intersection 1 with the ample class it follows that it is a line on \( F_{ij} \). Thus we have proved the following.

**Proposition 5.2.** There exists a resolution \( X^\kappa \to V/G \) such that \( \kappa \) is a class of an ample divisor on \( X^\kappa \). The unique 2-dimensional fiber of the resolution \( X^\kappa \to V/G \) consists of 11 components:

- the unique component \( F_0 = \bigcap_i E_i \cong \mathbb{P}^2 \)
- 10 components \( F_{ij} \), for \( 0 \leq i < j \leq 4 \), which are contained in intersections of \( E_k \) 's such that \( k \notin \{i, j\} \); \( F_{ij} \cong \mathbb{P}^2 \)

The intersection \( F_0 \cap F_{ij} \) is a line on \( F_{ij} \) and \((-1)\)-curve on \( F_0 \). The line bundle associated to \( \kappa \) is \(-K_{F_0} \to F_0 \) and \( O(1) \) on every \( F_{ij} \).

We note that curves \( C_{ij} \) can be related to orbits of type 5B while the components \( F_{ij} \) can be related to orbits of type 3C in Table 2. In fact, the arguments above regarding \( F_{ij} \)'s can be replaced by direct calculations of quotients of respective closed subsets of \( \text{Spec} \mathcal{R}(X) \), see Lemma 3.19 and the subsequent discussion.

### 5.B. Flops.

We can use identification \( \varphi^\kappa \) introduced in the previous section to describe the other resolutions of \( V/G \) which will be obtained from \( X^\kappa \to V/G \) by Mukai flops, see [WW03].

Our description is similar to that of [AW14, Sect. 6.7]. Figure 2 illustrates the components of the 2-dimensional fiber of these resolutions and their incidence. The distinguished central component \( F_0 \) is always denoted by \( \bigstar \). Each of the diagrams in the table is described by the isomorphism class of this component. By \( (\mathbb{P}^2)^\lor \) we denote the central component after the flop. Also the other components of the 2-dimensional fiber are denoted by the same name \( F_{ij} \) for every resolution. Their isomorphisms types are described as follows:

- \( \bigstar = \mathbb{P}^2 \), \( \blacklozenge = \mathbb{P}^1 \), \( \blacklozenge = \mathbb{P}^2 \), \( \blacksquare = \mathbb{P}^1 \times \mathbb{P}^1 \) and \( \blacklozenge = \mathbb{P}^2 \) is the blow-up of \( \mathbb{P}^2 \) in three collinear points.

The incidence of components is denoted by line segments joining the respective symbols. The solid line denotes intersection along a rational curve while a dotted line denotes intersection at a point. For the sake of clarity we ignore the intersection (at a point) of components which will not be flopped.

The first diagram in Figure 2 illustrates the special fiber of the unique central resolution in which the central component is \( \mathbb{P}^2 \) and the remaining components are \( \mathbb{P}^2 \). The other resolutions are obtained by flopping some components which are isomorphic to \( \mathbb{P}^2 \), the direction of the flops is indicated by arrows. That is, via the identification of \( N^1(X) \) with \( N^1(F_0) = N^1(\mathbb{P}^2) \), the ample cone of \( \mathbb{P}^2 \) is placed in the center of the movable cone of \( X \) and the direction of our flops points out outside the central chamber.

The first two flops are along \( F_{04} \) and \( F_{03} \) and they lead to the central component isomorphic to, respectively, \( \mathbb{P}^2 \) (there are 10 different resolutions of this type) and \( \mathbb{P}^2 \).
Figure 2. Flops of symplectic resolutions of $V/G$

- $F_0 = \mathbb{P}^2 _{1}$
- $F_0 = \mathbb{P}^2 _{2}$
- $F_0 = \mathbb{P}^2 _{4}$
- $F_0 = \mathbb{P}^2 _{3}$
- $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$
(there are 30 different resolutions of this type). The surface $P^2_2$ has three $(-1)$-curves which make a chain, contracting the central one we get $P^1 \times P^1$ while contracting any of the other two we get $P^2$. Respectively, we can consider either a flop along $F_{24}$ which makes $F_0$ isomorphic to $P^1 \times P^1$ (there are 10 resolutions of this type) or flop along $F_{02}$ and get $F_0 = P^2_2$ (there are 20 resolutions of this type). This latter one type can be further flopped along $F_{01}$ to get $F_0 = P^2$ (5 resolutions of this type). Finally, $F_0$ can be flopped, we denote it then by $(P^2)^\dagger$; there are 5 resolutions of this type we will denote them $X^i \to V/G$. The cone $Nef(X^i)$ is simplicial and it is generated by $e_i$ and by four classes $(e_i + e_j)/2$, with $j \neq i$; we will call it an outer chamber of $Mov(X)$.

Note that, after the first flop, the surfaces $F_{23}$, $F_{24}$ and $F_{34}$ have a common point, which comes from contracting the $(-1)$ curve on $F_0$. We ignore it in our diagram since none of these three components will be flopped. Similarly, we will not put in the diagrams the intersection points which will be negligible from the point of view of possible flops.

Counting the number of resolutions we obtain a result already announced by Bellamy, [Bel14].

**Proposition 5.3.** There are 81 symplectic resolutions of the quotient singularity $V/G$.

In fact, from our construction it follows that symplectic resolutions of $V/G$ are in bijection with chambers in the cone $cone(e_0,\ldots,e_4)$ obtained by cutting this cone with hyperplanes perpendicular to classes $f_{ij}$ and $\beta_i$ which were defined in subsection 2.13.

### 6. A Kummer 4-fold

**6.A. A Kummer surface.** Let $E$ be an elliptic curve with the complex multiplication by $i = \sqrt{-1}$. That is we have a linear automorphism $i \in Aut(E)$ such that $i^2 = -id$. For simplicity, we can assume $E = \mathbb{C}/\mathbb{Z}[i]$ and $i$ acts by the standard complex multiplication. The automorphism $i$ of $\mathbb{C}$ has two fixed points $p_0 = [0]$ and $p_1 = [(1+i)/2]$ while it interchanges the other two order 2 points $i[1/2] = [i/2]$, $i[i/2] = [1/2]$; here the square brackets denote the classes in $\mathbb{C}/\mathbb{Z}[i]$. We see that, in fact, this multiplication yields an isomorphism of the group of order 2 points on $E$ with the ring $\mathbb{Z}_2[i]$ with $p_1$ identified with the unique zero divisor $1 + i$.

Let us recall that the standard representation of the binary dihedral group of order 8, or the quaternion group, $Q_8$ in $SL(2,\mathbb{C})$ is given by the following $2 \times 2$ matrices over $\mathbb{C}$:

$$A_i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad A_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad A_K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Since, in fact, this representation is in $SL(2,\mathbb{Z}[i])$, we have the action of $Q_8$ on $\mathbb{E}^2$. It is not hard to check (see the argument below) that the group $Q_8$ acts on $\mathbb{E}^2$ with 2 fixed points, namely $(p_0,p_0)$ and $(p_1,p_1)$, 6 point with isotropy $\mathbb{Z}_4$ and 8 points with isotropy $\mathbb{Z}_2$. This makes 16 points with non-trivial isotropy. In fact, they are all order 2 points on $\mathbb{E}^2$, because $-id$ is contained in every nontrivial subgroup of $Q_8$.

If $n_8$, $n_4$ and $n_2$ are the number of orbits of points with the isotropy $Q_8$, $\mathbb{Z}_4$ and $\mathbb{Z}_2$, respectively, then taking into consideration the ranks of the normalizers of these subgroups we get $n_8 = 2$, $n_4 = 3$, $n_8 = 2$ and $n_8 + 2n_4 + 4n_2 = 16$. Thus on
the quotient $\mathbb{E}^2/Q_8$ we have $n_8 = 2$ singularities of type $D_4$ and $n_4 = 3$ singularities of type $A_3$ and $n_2 = 4$ singular points of type $A_1$. Resolving these singularities we obtain a K3 surface with $4 \cdot n_8 + 3 \cdot n_4 + 1 \cdot n_2 = 19$ exceptional $(-2)$-curves. We note that $\dim_{\mathbb{C}} H^{1,1}(\mathbb{E}^2)^{Q_8} = 1$ completes this number to the dimension of $H^{1,1}$ of a Kummer surface.

More generally, by the above argument, for any representation of $Q_8$ in $\text{Aut}(\mathbb{E}^2)$ the numbers $n_8$, $n_4$ and $n_2$ defined above satisfy the following two equations

\begin{align*}
n_8 + 2n_4 + 4n_2 &= 16 \\
4n_8 + 3n_4 + n_2 &= 19
\end{align*}

Clearly the numbers $n_i$ are non-negative integers and $n_8 > 0$. We find out that there are two solutions of this system:

\begin{equation}
(n_8, n_4, n_2) = (2, 3, 2) \text{ or } (n_8, n_4, n_2) = (4, 0, 3).
\end{equation}

The latter solution is satisfied for the following representation of $Q_8$ in $\text{Aut}(\mathbb{E}^2)$:

\begin{align*}
B_I &= \begin{pmatrix} i & 0 \\ i+1 & -i \end{pmatrix} & B_J &= \begin{pmatrix} -i & i-1 \\ 0 & i \end{pmatrix} & B_K &= \begin{pmatrix} 1 & -1-i \\ 1-i & -1 \end{pmatrix}
\end{align*}

The fixed points for this representation are $(p_0, p_0)$, $(p_0, p_1)$, $(p_1, p_0)$, $(p_1, p_1)$. Therefore the remaining order two points have isotropy equal to $\mathbb{Z}_2$.

It is convenient to describe the action of the group $Q_8$ in terms of $\mathbb{Z}_2[i]$ modules. By $M'$ we will denote the free module $(\mathbb{Z}_2[i])^{\oplus r}$ and by $M'_r$ its sub-module $(1+i)\cdot M'$ which consists of elements annihilated by $1+i$. As noted above, the points with non-trivial isotropy with respect to the action of $Q_8$ on $\mathbb{E}^2$ are of order 2 as $-id$ is contained in every non-trivial subgroup of $Q_8$. Thus, in fact, in order to understand isotropy of the action of $Q_8$ on $\mathbb{E}^2$ we can look into the action of $Q_8/\langle -id \rangle$ on the set of order two points on $\mathbb{E}^2$ which the free $\mathbb{Z}_2[i]$-module $M^2$. That is, we focus on representations of $Q_8/\langle -id \rangle = \mathbb{Z}_2^{\oplus 2}$ in $\text{SL}(2, \mathbb{Z}_2[i])$.

From this point of view the first representation of $Q_8$ is reduced to

\begin{align*}
A_I &= \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & A_J &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & A_K &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
\end{align*}

while the second is

\begin{align*}
B_I &= \begin{pmatrix} i & 0 \\ i+1 & i \end{pmatrix} & B_J &= \begin{pmatrix} i & i+1 \\ 0 & i \end{pmatrix} & B_K &= \begin{pmatrix} 1 & i+1 \\ 1+i & 1 \end{pmatrix}
\end{align*}

Now the claims about the fixed points sets are easy to verify on $M^2$. Note that the second representation fixes the points whose coordinates are annihilated by $1+i$ which is the module $M'_0$.

**6.B. A symplectic Kummer 4-fold.** After discussing the 2-dimensional case we pass to dimension 4. We use the notation consistent with the preceding subsection.

**Proposition 6.2.** Suppose that $G' < \text{SL}(4, \mathbb{Z}[i])$ is a finite subgroup such that

- $G'$ is generated by five order 2 symplectic reflections $T^i_1$ and it is conjugate in $\text{SL}(4, \mathbb{C})$ to the group $G$;
- the reduction $G'/\langle -id \rangle \to \text{SL}(4, \mathbb{Z}_2[i])$ acts on $M^4$ so that its action on $M^4_0$ is trivial and every element in $M^4 \setminus M^4_0$ has isotropy generated by $\pm T^i_1$ for some $i$. 

Then $G'$ acts on $\mathbb{E}^4$ and the quotient $\mathbb{E}^4/G'$ admits a symplectic resolution $X$ such that its Betti numbers are as follows: $b_2(X) = b_0(X) = 23$ and $b_4(X) = 276$.

**Proof.** Clearly $G' < SL(4, \mathbb{Z}[i])$ yields an action of $G'$ on $\mathbb{E}^4$. We claim that any nontrivial isotropy group of this action which is different from a symplectic reflection $T'_i$ is actually isomorphic and conjugate in $SL(4, \mathbb{C})$ to $G$ or to a group $\langle T'_i, -T'_i \rangle$. This follows from the fact that every subgroup of $G$ which is different from $\langle T_i \rangle$ contains $-id$ hence it is isotropy of a point on $\mathbb{E}^4$ which is of order 2. Thus the isotropy of the action of $G'$ on $\mathbb{E}^4$ can be understood by looking at the action of $G'/\langle -id \rangle$ on the module $M^4$ and the claim follows.

Locally both $\mathbb{C}^4/G$ and $\mathbb{C}^4/(\pm T_i)$ admit symplectic resolutions. These resolutions can be glued to the global symplectic resolution $X$ of $\mathbb{E}^4/G'$ because outside isolated points which are images of the order 2 points in $\mathbb{E}^4$ the singularities are 2-dimensional families of $A_1$ surface singularities whose resolution is unique.

Thus it remains to calculate the Betti numbers of $X$. For this it is enough to find the number of the irreducible 2-dimensional components of the singular locus of $\mathbb{E}^4/G'$. The normalization of such components is a quotient of the fixed point set of some $T'_i$ by the group $\mathbb{N}(T'_i)/\langle T'_i \rangle = Q_8$ which, as we noted in the preceding subsection 6.1 has either 2 or 4 points of isotropy equal to $Q_8$. On the other hand, $\mathbb{E}^4/G'$ has 16 $= |M'_4|$ singular points of type $\mathbb{C}^4/G$ and each of them belongs to 5 different 2-dimensional components of the singular locus of $A^4/G'$. Since each such component contains 4 such points at most it follows that the number of these components is $16 \cdot 5/4 = 20$ at least. Now the result follows by [Cua01].

From the above calculations one can conclude that any resolution $X \rightarrow \mathbb{E}^4/G'$ contracts 20 exceptional divisors and it has 16 fibers of type discussed in Section 4 and 30 fibers which are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Although the number of such resolutions is 81$^16$ (some of these resolutions lead to non-projective varieties) there exists a unique special resolution which over each of the 16 points in $\mathbb{E}^4/G'$ with singularity of type $\mathbb{C}^4/G$ is of type $\varphi^*$ described in 5.4.

Let us consider the following 5 matrices defined over $\mathbb{Z}[i]$:

$$T'_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & -1 + i & 1 & 0 \\
1 - i & 0 & 0 & -1
\end{pmatrix} \quad T'_1 = \begin{pmatrix}
i & -1 - i & 0 & 1 - i \\
0 & i & -1 + i & 0 \\
0 & -1 - i & i & 0 \\
1 + i & 0 & -1 - i & -i
\end{pmatrix}
$$

$$T'_2 = \begin{pmatrix}
1 & 0 & 0 & -1 - i \\
0 & -1 & 1 + i & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \quad T'_3 = \begin{pmatrix}
i & 0 & 0 & 1 - i \\
1 - i & -i & -1 + i & 0 \\
0 & -1 - i & i & 1 - i \\
1 + i & 0 & 0 & -i
\end{pmatrix}
$$

$$T'_4 = \begin{pmatrix}
1 & -1 + i & 0 & -1 - i \\
-1 - i & -1 & 1 + i & 0 \\
0 & -1 + i & 1 & -1 - i \\
1 - i & 0 & -1 + i & -1
\end{pmatrix}
$$

**Lemma 6.3.** The group $G'$ generated by matrices $T'_i$ satisfies the assumption of Proposition 6.2.
Proof. Let us define
\[
W = \begin{pmatrix}
1 & 1/2 - i/2 & -1 & 0 \\
1 & -1/2 + i/2 & 0 & -1 - i \\
0 & -1/2 - i/2 & i & 0 \\
0 & 1/2 + i/2 & 0 & 0
\end{pmatrix}
\]
Then for \(i = 0, \ldots, 4\) it holds \(T_i^d = W^{-1} \cdot T_i \cdot W\) where \(T_i\)'s are the matrices from the list \([2,3]\). Clearly, the reduction of each of \(T_i^d\) to \(M^4\) (which by abuse we denote by \(T_i^d\) as well) is identity on \(M_0^4\) and it remains to check that the isotropy of every element from \(M^4 \setminus M_0^4\) is generated by some \(T_i^d\). This can be done as follows: one can write \(\ker(T_i^d - \text{id})\) as \(M_0^4 + K_i\) where \(K_i\) is a rank 2 free \(\mathbb{Z}_2[i]\)-module. Next one checks that for \(i \neq j\) it holds \(K_i + K_j = M^4\) and thus \(K_i \cap K_j = \{0\}\). Finally, because \([K_i \setminus M_0^4] = 60\) it follows that \(|M^4| = |M_0^4| + \sum_i |K_i \setminus M_0^4|\) and therefore every element of \(M^4 \setminus M_0^4\) belongs to exactly one \(K_i\). We omit calculations.

Corollary 6.4. The quotient \(E^4/G'\) has a resolution which is a Kummer symplectic 4-fold \(X\) with \(b_2(X) = b_0(X) = 23\) and \(b_4(X) = 276\).

References

[ADHL14] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface. Cox Rings. Cambridge University Press, New York, 2014.

[AG10] I. V. Arzhantsev and S. A. Gaiffullin. Cox rings, semi groups, and automorphisms of affine varieties. Mat. Sb., 201(1):3–24, 2010.

[AHL10] Michela Artebani, Jürgen Hausen, and Antonio Laface. On Cox rings of K3 surfaces. Compos. Math., 146(4):964–998, 2010.

[AW14] Marco Andreatta and Jarosław A. Wiśniewski. 4-dimensional symplectic contractions. Geom. Dedicata, 168(1):311–337, 2014.

[Bel14] Gwyn Bellamy. Counting resolutions of symplectic quotient singularities, http://arxiv.org/abs/1405.9625 2014.

[Ben93] D. J. Benson. Polynomial invariants of finite groups. Cambridge University Press, Cambridge, 1993.

[BFN10] Thomas Bauer, Michael Funke, and Sebastian Neumann. Counting Zariski chambers on Del Pezzo surfaces. J. Algebra, 324(1):92–101, 2010.

[BH06] Florian Berchtold and Jürgen Hausen. Git-equivalence beyond the ample cone. The Michigan Mathematical Journal, 54(3):483–516, 11 2006.

[BS13] Gwyn Bellamy and Travis Schedler. A new linear quotient of C^4 admitting a symplectic resolution. Math. Z., 273(3-4):753–769, 2013.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. Toric varieties, volume 124 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011.

[Cox95] David A. Cox. The homogeneous coordinate ring of a toric variety. J. Algebraic Geom., 4(1):17–50, 1995.

[CT06] Ana-Maria Castravet and Jenia Tevelev. Hilbert’s 14th problem and Cox rings. Compos. Math., 142(6):1479–1498, 2006.

[DB13] Maria Donten-Bury. Cox rings of minimal resolutions of surface quotient singularities, http://arxiv.org/abs/1301.2633 2013.

[DGPS12] Wolfram Decker, Gert-Martin Greuel, Gerhard Pfister, and Hans Schönemann. Singular 3-1-6 — A computer algebra system for polynomial computations, 2012.

[Dol03] Igor Dolgachev. Lectures on invariant theory, volume 296 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2003.

[Don11] Maria Donten. On Kummer 3-folds. Rev. Mat. Complut., 24(2):465–492, 2011.

[FGAL11] L. Facchini, V. González-Alonso, and M. Lasoń. Cox rings of du Val singularities. Matematiche (Catania), 66(2):115–136, 2011.
[GK04] Victor Ginzburg and Dmitry Kaledin. Poisson deformations of symplectic quotient singularities. *Adv. Math.*, 186(1):1–57, 2004.

[GS] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. [http://www.math.uiuc.edu/Macaulay2](http://www.math.uiuc.edu/Macaulay2).

[Gua01] Daniel Guan. On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four. *Math. Res. Lett.*, 8(5-6):663–669, 2001.

[HK00] Yi Hu and Sean Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[HKL14] Juergen Hausen, Keicher, and Antonio Laface. Computing Cox rings, [http://arxiv.org/abs/1305.4343](http://arxiv.org/abs/1305.4343), 2014.

[IR96] Yukari Ito and Miles Reid. The McKay correspondence for finite subgroups of SL(3, C). In *Higher-dimensional complex varieties (Trento, 1994)*, pages 221–240. de Gruyter, Berlin, 1996.

[Kal02] D. Kaledin. McKay correspondence for symplectic quotient singularities. *Invent. Math.*, 148(1):151–175, 2002.

[Kei12] Simon Keicher. Computing the GIT-fan. *IJAC*, 22(7), 2012.

[Kol93] János Kollár. Shafarevich maps and plurigenera of algebraic varieties. *Invent. Math.*, 113(1):177–215, 1993.

[Kol96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer-Verlag, Berlin, 1996.

[Man86] Yu. I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[Nam10] Yoshinori Namikawa. Poisson deformations of affine symplectic varieties, II. *Kyoto J. Math.*, 50(4):727–752, 2010.

[Nam11] Yoshinori Namikawa. Poisson deformations of affine symplectic varieties. *Duke Math. J.*, 156(1):51–85, 2011.

[New72] Morris Newman. *Integral matrices*. Academic Press, New York, 1972. Pure and Applied Mathematics, Vol. 45.

[Ott11] John Christian Ottem. On the Cox ring of $\mathbb{P}^2$ blown up in points on a line. *Math. Scand.*, 109(1):22–30, 2011.

[Rei92] Miles Reid. What is a flip? [http://homepages.warwick.ac.uk/staff/Miles.Reid/3folds](http://homepages.warwick.ac.uk/staff/Miles.Reid/3folds), 1992.

[Rei97] Miles Reid. McKay correspondence, [http://arxiv.org/abs/alg-geom/9702016](http://arxiv.org/abs/alg-geom/9702016), 1997.

[S¹³] W. A. Stein et al. *Sage Mathematics Software (Version 5.12)*. The Sage Development Team, 2013. [http://www.sagemath.org](http://www.sagemath.org).

[Sta79] Richard P. Stanley. Invariants of finite groups and their applications to combinatorics. *Bull. Amer. Math. Soc. (N.S.)*, 1(3):475–511, 1979.

[STV07] Mike Stillman, Damiano Testa, and Mauricio Velasco. Gröbner bases, monomial group actions, and the Cox rings of del Pezzo surfaces. *J. Algebra*, 316(2):777–801, 2007.

[Tem30] G. Temple. The group properties of Dirac’s operators. *Proc. R. Soc. Lond. A.*, 127:339–348, 1930.

[Wie03] Jan Wierzba. Contractions of symplectic varieties. *J. Algebraic Geom.*, 12(3):507–534, 2003.

[WW03] Jan Wierzba and Jarosław A. Wiśniewski. Small contractions of symplectic 4-folds. *Duke Math. J.*, 120(1):65–95, 2003.