Factorizations of one dimensional classical systems

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Abstract

A class of one dimensional classical systems is characterized from an algebraic point of view. The Hamiltonians of these systems are factorized in terms of two functions that together with the Hamiltonian itself close a Poisson algebra. These two functions lead directly to two time-dependent integrals of motion from which the phase motions are derived algebraically. The systems so obtained constitute the classical analogues of the well known factorizable one dimensional quantum mechanical systems.

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1 Introduction

The one dimensional systems, with a time independent Hamiltonian, are treated in a rather different way in classical and quantum mechanics. In quantum mechanics the Schrödinger equation for such systems determines the spectrum that can include a discrete part for bound states and a continuous sector for unbounded (scattering) states. For periodic potentials there are forbidden and allowed bands of energy whose borders constitute the spectrum in a wider sense. Only in some cases the eigenvalue problem for the stationary Schrödinger equation can be solved “exactly” so that we are able to find a “closed expression” for the spectrum (energy eigenvalues) as well as for the corresponding eigenfunctions. This situation arises (but it is not the only option) when there are lowering and raising operators that close a spectrum generating algebra with the Hamiltonian and factorize the Casimir of the algebra up to a constant [1, 2, 3, 4, 5].

From the point of view of classical mechanics, the Hamiltonians of one dimensional conservative systems are integrals of motion, hence all these systems are maximally integrable [6]. This property, in classical mechanics, implies that every closed and bounded trajectory must be periodic. For this reason, often they are considered almost trivial and usually have not received much attention [7]. The values of the energy are continuous for both, bound and unbounded motion states. While the motion is necessarily periodic with a certain frequency (in general depending on the energy) for bound states, it is non-periodic for the unbounded ones. In some special cases, we can find the explicit expression for the motion in the phase space by means of the algebraic structure underlying these classical systems. The main purpose of this paper is to show how this structure is presented for a class of one dimensional classical systems. The Hamiltonians that allow for this treatment are classical analogues of some quantum systems. We will see that the algebraic structure of these quantum and classical systems are similar, but at the same time there are important differences. Therefore, the general motivation of this work is to close the symmetry considerations of both, classical and quantum mechanics in this framework.

The organization of the paper is as follows. In section 2 we introduce our algebraic approach to the classical systems in terms of Poisson algebras [8, 9, 10]. For the factorizable classical systems we find two time-dependent integrals of motion similar to the Bohlin invariants [11, 12, 13]. These invariants allow us to obtain algebraically the solutions of motion that give rise to trajectories in the phase space. Usually, the time-independent integrals of motion have been used to simplify the differential equation of motion in order to get explicit solutions, but the time-dependent integrals of motion have been hardly used in studying stationary systems. This work is one step forward in the applications of such a type of invariants. This development is systematically applied to some examples in sections 3, 4 and 5. We will end the paper with some comments and conclusions on the results here obtained.

2 Factorizations and deformed algebras

Let us consider the Hamiltonian

\[ H(x, p) = \frac{p^2}{2m} + V(x) \] (1)

where \( x, p \) are canonical coordinates, i.e., \( \{ p, x \} = 1 \), being \( \{ \cdot, \cdot \} \) the notation for the Poisson brackets, and \( V(x) \) is the potential. Hereafter we will set \( 2m = 1 \) for simplicity. We will investigate
a kind of factorization of the Hamiltonian $H$ in terms of two complex-conjugate functions $A^\pm$ as

$$H = A^+ A^- + \gamma(H).$$

(2)

Here the term $\gamma(H)$, contrary to the usual factorizations in quantum mechanics, may depend on $H$. In the case of bound states with positive energy ($H > 0$), we will consider $A^\pm$ linear in $p$ and having the form, as suggested from (2),

$$A^\pm = \mp i f(x) p + \sqrt{H} g(x) + \varphi(x) + \phi(H)$$

(3)

where the functions $f(x)$, $g(x)$, $\varphi(x)$ and $\phi(H)$ will be determined in each case. It is clear that a global constant factor in the functions $A^\pm$ will produce an equivalent factorization so that in the following we will choose this factor to get simpler expressions.

We will ask the functions, $A^\pm$ and $H$, to define a deformed algebra with the Poisson brackets as follows

$$\{H, A^\pm\} = \pm i \alpha(H) A^\pm$$

(4)

$$\{A^+, A^-\} = -i \beta(H).$$

(5)

The auxiliary functions $\alpha(H)$, $\beta(H)$, and $\phi(H)$ will be expressed in terms of the powers of $\sqrt{H}$. In the case of bound motions with negative energy we must replace the square roots $\sqrt{H}$ by $\sqrt{-H}$ in the above construction. There are systems that can allow for bound and unbounded motions, in such cases the complex character of the factors $A^\pm$ can change in each energy sector, as well as the deformed algebra. This change also happens in the quantum frame for the algebras describing bound and scattering states of the same system [14].

Relation (4) is the most important since it implies both (2) and (5). In order to show this, we compute

$$\{H, A^+ A^-\} = A^+ \{H, A^-\} + \{H, A^+\} A^- = 0$$

(6)

where we have made use of (4). From the vanishing of this bracket we conclude that $A^+ A^-$ will depend only on $H$ and therefore, we can express this dependence as in (2). To show (5), we start from the Jacobi identity

$$\{H, \{A^+, A^-\}\} = -\{A^+, \{A^-, H\}\} - \{A^-, \{H, A^+\}\}$$

(7)

substituting on the r.h.s of this identity relation (4), we find

$$\{H, \{A^+, A^-\}\} = -i \{A^+ A^-, \alpha(H)\} = 0$$

so that the bracket $\{A^+, A^-\}$ can be expressed in the form (5).

Now, for a system allowing for this kind of factorization, we can construct two time-dependent integrals of motion,

$$Q^\pm = A^\pm e^{\mp i \alpha(H) t}.$$  

(8)

It is easy to check that the total time derivative of $Q^\pm$ is equal to zero:

$$\frac{dQ^\pm}{dt} = \{H, Q^\pm\} + \frac{\partial Q^\pm}{\partial t} = 0$$

(9)
This choice gives rise to

\[ q^\pm = ce^{\pm i \theta_0} \]  

(10)

where \( c = |q^\pm| \). Notice that this expression can also change for unbounded motions. Having in mind the factorization (2), the modulus \( c \) of \( q^\pm \) will depend on the (eigen)value \( E \) of the Hamiltonian \( H \): \( c = c(E) \).

The two independent integrals \( Q^\pm(x, p, t) = c(E) e^{\pm i \theta_0} \) allow us to find algebraically the trajectories \((x(t), p(t))\) in the phase space. In particular, we can appreciate from (8) that the frequencies of the motion for bound states are given by \( \alpha(E) \).

If we substitute \( p^2 = H - V(x) \) and \( A^\pm \) given by (3) in (4) we get two relations for the unknown functions \( g(x), f(x), \alpha(H), \phi(H), \varphi(x) \) and \( V(x) \):

\[ f(x) = \frac{2}{\alpha(H)} \left( \sqrt{H} g'(x) + \varphi'(x) \right) \]
\[ -2f'(x)(H - V(x)) + f(x) V'(x) = \alpha(H) \left( \sqrt{H} g(x) + \phi(H) + \varphi(x) \right) \]

(11)

where the prime denotes the derivative with respect to the corresponding argument of each function. If we use \( A^\pm \) in (2), we have the following equation for \( \gamma(H) \)

\[ H = f^2(x)(H - V(x)) + \left( \sqrt{H} g(x) + \phi(H) + \varphi(x) \right)^2 + \gamma(H) \]

(12)

which will determine completely the potential. Finally, substituting \( A^\pm \) into (5) we will get an equation for the remaining function \( \beta(H) \),

\[ \beta(H) = \sqrt{H} \left[ 2f(x) g'(x) - 2f'(x) g(x) \right] + \frac{1}{\sqrt{H}} \left[ f(x) g(x) V'(x) + 2f'(x) g(x) V(x) \right] \]
\[ -4f'(x) \phi'(H)(H - V(x)) + 2f(x) \left[ V'(x) \phi'(H) + \varphi'(x) \right] \]

(13)

In the next sections we will study systematically some examples. To get the potentials and the functions \( f(x), g(x), \varphi(x), \alpha(H), \gamma(H) \) and \( \beta(H) \), first we have chosen a suitable form of \( \phi(H), g(x) \) and \( \varphi(x) \), and we have also assumed that \( H \) may have a positive or a negative character.

3 Simple potentials: oscillator, Scarf and Pöschl-Teller

In this section, we will consider the simplest solutions of the problem which are obtained when the last term, \( \phi(H) \), in the expression (3) of \( A^\pm \) vanishes. Now, from relation (12) we see that there are two kind of solutions: those with \( g(x) = 0, \varphi(x) \neq 0 \) and the complementary set \( g(x) \neq 0, \varphi(x) = 0 \). The first choice leads to the harmonic oscillator, and the second one to the Scarf and Pöschl-Teller potentials as shown below.

3.1 The harmonic oscillator \((g = 0, \varphi \neq 0)\)

This choice gives rise to

\[ A^\pm = \mp if(x)p + \varphi(x) \]

(14)
and using it in (11), we have

\[ f(x) = \frac{2}{\alpha(H)} \varphi'(x) \]  

(15)

\[- 2 f'(x) (H - V(x)) + f(x) V'(x) = \alpha(H) \varphi'(x). \]  

(16)

Here, since we assume that \( H \) is a variable independent of \( x \), then Eq. (15) gives us \( \alpha(H) = \alpha_0 \) and (15)-(16) take the form

\[ f(x) = \frac{2}{\alpha_0} \varphi'(x) \]  

(17)

\[- 2 f'(x) H + 2 f'(x) V(x) + f(x) V'(x) = \alpha_0 \varphi'(x). \]  

(18)

Taking into account that the coefficient of \( H \) in (18) has to be equal to zero, since it is the only term depending on \( H \), from (17) we have \( \varphi(x) = a_0 x + b_0 \). The remaining part of (18) gives us

\[ V'(x) = \frac{\alpha_0^2 \varphi(x)}{2 \varphi'(x)}. \]  

(19)

Then, substituting \( \varphi(x) \) in this equation and integrating, we get the harmonic oscillator potential

\[ V(x) = \frac{\alpha_0^2}{2 a_0} \left( \frac{a_0 x^2}{2} + b_0 \right) + c \]  

(20)

where \( c \) is integration constant. From Eq. (12), we see that \( a_0 = \alpha_0/2 \), and \( \gamma_0 = c - b_0^2 \), so the potential (20) becomes

\[ V(x) = \left( \frac{\alpha_0}{2} x + b_0 \right)^2 + \gamma_0. \]  

(21)

Using (13), we get \( \beta(H) = \alpha_0 \) and therefore, in this case we can rewrite all the functions as

\[ f(x) = 1, \quad g(x) = 0, \quad \varphi(x) = \frac{\alpha_0}{2} x + b_0 \]  

\[ \alpha(H) = \alpha_0, \quad \beta(H) = \alpha_0, \quad \gamma(H) = \gamma_0. \]  

(22)

The explicit expressions of the factor functions are

\[ A^\pm = \mp i p + \left( \frac{\alpha_0}{2} x + b_0 \right) \]  

(23)

and the Hamiltonian is factorized in terms of these functions as

\[ H = A^+ A^- + \gamma_0. \]  

(24)

Finally, the Poisson brackets read

\[ \{H, A^\pm\} = \pm i \alpha_0 A^\pm, \quad \{A^+, A^-\} = -i \alpha_0. \]  

(25)

Of course, they constitute the oscillator algebra realized in terms of Poisson brackets. Now, from (8) we can write the time-dependent integrals of motion \( Q^\pm \) in the form

\[ Q^\pm = A^\pm e^{\mp i \alpha_0 t}. \]  

(26)
Having in mind that the Hamiltonian $H$ is an integral of motion whose value is the energy $E$ of the system, and the factorization (24), the value of $Q^\pm$ is given by

$$q^\pm = \sqrt{E - \gamma_0} e^{\pm i \theta_0}$$

(27)

where the energy is greater than the minimum of the potential, $E > \gamma_0$. Then, taking into account (23), (26) and (27), we get the following equations

$$\mp i p + \left(\frac{\alpha_0}{2} x + b_0\right) = \sqrt{E - \gamma_0} e^{\pm i (\theta_0 + \alpha_0 t)}.$$  

(28)

The phase trajectories $(x(t), p(t))$ can be found from these equations and have the well known expressions

$$x(t) = \frac{2}{\alpha_0} \left(\sqrt{E - \gamma_0} \cos (\theta_0 + \alpha_0 t) - b_0\right)$$

$$p(t) = -\sqrt{E - \gamma_0} \sin (\theta_0 + \alpha_0 t).$$

(29)

The constant $b_0$ can be eliminated by a translation in the $x$-axis, so hereafter in the next examples it will be set equal to zero. The values of $q^\pm$ fix completely the initial conditions $(x(0), p(0))$ of the motion.

### 3.2 The Scarf potential

Here we will consider $(g \neq 0, \varphi = 0)$, and the positive character $H > 0$ of the Hamiltonian for the bound states. Following the same procedure as in the previous case, after some calculations we get the Scarf potential

$$V(x) = \frac{\gamma_0}{\cos^2\left(\frac{\alpha_0}{2} x\right)}, \quad \gamma_0 > 0$$

(30)

where $\gamma_0$ and $\alpha_0$ are constants. We also obtain

$$f(x) = \cos\left(\frac{\alpha_0}{2} x\right), \quad g(x) = \sin\left(\frac{\alpha_0}{2} x\right), \quad \varphi(x) = 0$$

$$\alpha(H) = \alpha_0 \sqrt{H}, \quad \beta(H) = \alpha_0 \sqrt{H}, \quad \gamma(H) = \gamma_0.$$  

(31)

Therefore, we can write the factor functions

$$A^\pm = \mp i \cos\left(\frac{\alpha_0}{2} x\right) p + \sqrt{H} \sin\left(\frac{\alpha_0}{2} x\right)$$

(32)

the factorization,

$$H = A^+ A^- + \gamma_0$$

(33)

and the deformed algebra

$$\{H, A^\pm\} = \pm i \alpha_0 \sqrt{H} A^\pm, \quad \{A^+, A^-\} = -i \alpha_0 \sqrt{H}.$$  

(34)

This algebra can be easily rewritten, defining $A_0 \equiv \sqrt{H}$ as follows

$$\{A_0, A^\pm\} = \pm i \frac{\alpha_0}{2} A^\pm, \quad \{A^+, A^-\} = -i \alpha_0 A_0$$

(35)

which corresponds to the $su(1, 1)$ Poisson algebra. We remark that for the quantum system with Scarf potential the spectrum generating algebra is also $su(1, 1)$ Lie algebra [5].

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Now, the time-dependent integrals of motion satisfying relation (9) in terms of $A \pm$ are
\[ Q \pm = A \pm e^{\mp i \theta_0 \sqrt{H} t} \] (36)
with the values $q \pm = \sqrt{E - \gamma_0} e^{\pm i \theta_0}$. As the energy must be greater than the minimum of the potential, $E > \gamma_0$. Substituting (32) in (36) and taking into account the values of the time-dependent integrals of motion, we get
\[ \mp i p \cos \frac{\alpha_0}{2} x + \sqrt{E} \sin \frac{\alpha_0}{2} p = \sqrt{E - \gamma_0} e^{\pm i (\theta_0 + \alpha_0 \sqrt{E} t)} \] (37)
From these equations we have the following expressions
\[ \sin \frac{\alpha_0}{2} x = \sqrt{\frac{E - \gamma_0}{E}} \cos (\theta_0 + \alpha_0 \sqrt{E} t) \] (38)
\[ p \cos \frac{\alpha_0}{2} p = -\sqrt{E - \gamma_0} \sin (\theta_0 + \alpha_0 \sqrt{E} t) \]
and the phase trajectories $(x(t), p(t))$ can be easily written in the form:
\[ x(t) = \frac{2}{\alpha_0} \arcsin(\sqrt{\frac{E - \gamma_0}{E}} \cos (\theta_0 + \alpha_0 \sqrt{E} t)) \] (39)
\[ p(t) = -\frac{\sqrt{E(E - \gamma_0)}}{\sqrt{E - (E - \gamma_0) \cos^2(\theta_0 + \alpha_0 \sqrt{E} t)}}. \]

### 3.3 The Pöschl-Teller potential
In this section, we assume $(g \neq 0, \varphi = 0)$, and we choose the negative sign $H < 0$ of the Hamiltonian for the bound states. Therefore, we must change the form of $A \pm$ in (3),
\[ A \pm = \mp i f(x) p + \sqrt{-H} g(x) + \varphi(x) + \phi(H). \] (40)
The expressions (11), (12) and (13) must also adapt to this situation: replacing the root $\sqrt{H}$ by $\sqrt{-H}$ in all these equations, and in particular expression (13) takes the form
\[ \beta(H) = \sqrt{-H} \left[ 2 f(x) g'(x) - 2 f'(x) g(x) \right] - \frac{1}{\sqrt{-H}} \left[ f(x) g(x) V'(x) + 2 f'(x) g(x) V(x) \right] \]
\[ -4 f'(x) \phi'(H) \left( H - V(x) \right) + 2 f(x) \left[ V'(x) \phi'(H) + \varphi'(x) \right]. \] (41)
However, later we will also study the unbounded states with positive energy. Following the same procedure as in the above cases and having in mind that $\phi(H) = 0$, we arrive at the Pöschl-Teller potential
\[ V(x) = -\frac{\gamma_0}{\cosh^2(\frac{\alpha_0}{2} x)}, \quad \gamma_0 > 0 \] (42)
where $\gamma_0$ and $\alpha_0$ are constants. For this case, we get
\[ f(x) = \cosh \left( \frac{\alpha_0}{2} x \right), \quad g(x) = \sinh \left( \frac{\alpha_0}{2} x \right), \quad \varphi(x) = 0 \] \[ \alpha(H) = \alpha_0 \sqrt{-H}, \quad \beta(H) = \alpha_0 \sqrt{-H}, \quad \gamma(H) = -\gamma_0. \] (43)
Then, we have
\[ A^\pm = \mp i \cosh (\frac{\alpha_0}{2} x) p + \sqrt{-H} \sinh (\frac{\alpha_0}{2} x) \] (44)
the factorization
\[ H = A^+ A^- - \gamma_0 \] (45)
and the Poisson brackets
\[ \{H, A^\pm\} = \pm i \alpha_0 \sqrt{-H} A^\pm, \quad \{A^+, A^-\} = -i \alpha_0 \sqrt{-H}. \] (46)
This algebra can also be rewritten, defining \( A_0 \equiv -\sqrt{-H} \) as follows
\[ \{A_0, A^\pm\} = \pm i \frac{\alpha_0}{2} A^\pm, \quad \{A^+, A^-\} = i \alpha_0 A_0 \] (47)
which corresponds to the \( su(2) \) Poisson algebra. Here, we remark that for the quantum Pöschl-Teller potential the spectrum generating algebra can also be identified with the \( su(2) \) Lie algebra [5].

Hence, the time-dependent integrals of motion are
\[ Q^\pm = A^\pm e^{\mp i \alpha_0 \sqrt{-H} t}, \] with the values \( q^\pm = \sqrt{E + \gamma_0} e^{\pm i \theta_0}. \) The energy must be negative, but greater than the potential minimum, \(-\gamma_0 < E < 0\). Replacing (44) in \( Q^\pm \) and taking into account the values \( q^\pm \), we get
\[ \mp i p \cosh (\frac{\alpha_0}{2} x) + \sqrt{-E} \sinh (\frac{\alpha_0}{2} x) = \sqrt{E + \gamma_0} e^{\pm i (\theta_0 + \alpha_0 \sqrt{-E} t)}. \] (48)

Then, we have the following equations
\[ \sinh (\frac{\alpha_0}{2} x) = \sqrt{-\frac{E - \gamma_0}{E}} \cos (\theta_0 + \alpha_0 \sqrt{-E} t) \]
\[ p \cosh (\frac{\alpha_0}{2} x) = -\sqrt{E - \gamma_0} \sin (\theta_0 + \alpha_0 \sqrt{-E} t) \] (49)
and from these equations \( (x(t), p(t)) \) can be found as
\[ x(t) = \frac{2}{\alpha_0} \arcsinh \left( \sqrt{-\frac{E + \gamma_0}{E}} \cos (\theta_0 + \alpha_0 \sqrt{-E} t) \right) \]
\[ p(t) = -\frac{-\sqrt{-E(E + \gamma_0)} \sin (\theta_0 + \alpha_0 \sqrt{-E} t)}{\sqrt{-E + (E + \gamma_0) \cos^2 (\theta_0 + \alpha_0 \sqrt{-E} t)}} \] (50)

The potential (42) also allows for unbounded motions when \( E > 0 \). In this case we can write \( \sqrt{-E} = i \sqrt{|E|} \) (or \( \sqrt{-H} = i \sqrt{H} \)). Then, the factors \( A^\pm \) are no longer complex conjugate, but pure imaginary:
\[ A^\pm = \mp i \cosh (\frac{\alpha_0}{2} x) p + i \sqrt{H} \sinh (\frac{\alpha_0}{2} x). \] (51)

The algebra of the Poisson brackets (46) also changes for this energy sector
\[ \{H, A^\pm\} = \mp \alpha_0 \sqrt{H} A^\pm, \quad \{A^+, A^-\} = \alpha_0 \sqrt{H}. \] (52)
But, defining \( A_0 \equiv -i \sqrt{H} \), this algebra has also the form of an \( su(2) \) Poisson algebra
\[ \{A_0, A^\pm\} = \pm i \frac{\alpha_0}{2} A^\pm, \quad \{A^+, A^-\} = i \alpha_0 A_0. \] (53)
Dealing with this potential in quantum mechanics, the bound states are described by a finite-dimensional unitary representations of $su(2)$ but the scattering states by infinite-dimensional representations of the continuous series of $su(1, 1)$ [14]. However, if we use the same kind of lowering and raising operators, the scattering states belong to infinite-dimensional non-unitary representations of $su(2)$. Here, we have seen that in classical mechanics the bound motions are described by the unitary $su(2)$ poisson algebra (in the sense that the factor functions $A^\pm$ are complex-conjugate) and the unbounded motions by non-unitary $su(2)$ algebra (the factor functions $A^\pm$ are pure imaginary) by means of the Poisson brackets.

The integrals of motion $Q^\pm$ are imaginary and their eigenvalues can be written in the form $q^\pm = \mp i \sqrt{E + \gamma_0} e^{\mp \theta_0}$. After similar calculations, these integrals of motion have the explicit expressions

$\mp p \cosh \frac{\alpha_0}{2} x + \sqrt{E} \sinh \frac{\alpha_0}{2} x = \mp \sqrt{E + \gamma_0} e^{\mp (\theta_0 + \alpha_0 \sqrt{E} t)}.$

Therefore, we get

$\sinh \frac{\alpha_0}{2} x = \sqrt{\frac{E + \gamma_0}{E}} \sinh (\theta_0 + \alpha_0 \sqrt{E} t)$

$p \cosh \frac{\alpha_0}{2} x = \sqrt{E + \gamma_0} \cosh (\theta_0 + \alpha_0 \sqrt{E} t).$

Finally, the unbounded non-periodic solutions are

$x(t) = \frac{2}{\alpha_0} \arcsinh(\sqrt{\frac{E + \gamma_0}{E}} \sinh (\theta_0 + \alpha_0 \sqrt{E} t))$

$p(t) = \frac{\sqrt{E(E + \gamma_0)} \cosh (\theta_0 + \alpha_0 \sqrt{E} t)}{\sqrt{E + (E + \gamma_0) \sinh^2 (\theta_0 + \alpha_0 \sqrt{E} t)}}.$

Remark the different dependence of the frequencies on energy for the bound motion states of the above three systems. In the oscillator case, it is a constant independent of $E$, for the Scarf potential the frequency is an increasing function of $E$, while in the Pöschl-Teller potential the frequency is a decreasing function of $E$. This corresponds to the values of the bracket (4) of the Poisson algebra for each of these systems. The potentials and some trajectories are plotted in Figures 1, 2 and 3. We also notice that the values of $\gamma_0$ in the potentials (30) and (42) have been taken with suitable signs giving rise to bound states.

![Figure 1: The oscillator (continuous line), Scarf (dashed) and Pöschl-Teller (dotted) potentials.](image-url)
Figure 2: Some phase trajectories for the same three energies of motions in the oscillator (a) and Scarf potentials (b). The exterior trajectories of the latter have higher frequencies than the inner ones.

Figure 3: Now, it is shown the phase trajectories for the same three energies of periodic motion in the Pöschl-Teller potential. The exterior trajectories have slower frequencies than the inner ones. Also it is shown some unbounded trajectories.

4 Singular potentials

In this section we consider $\phi(H) \neq 0$, to obtain other classical systems. Here, we also propose some specific form of $\phi(H)$ and consider $\gamma(H)$ different from constant so that we will find the potentials of the previous section including a singular term.

4.1 The singular oscillator

If we choose $\phi(H) = \beta_0 H$, and $(g(x) = 0, \varphi(x) \neq 0)$, after similar calculations to the preceding cases, we get the singular oscillator potential

$$V(x) = \frac{(\beta_0 \alpha_0^2 x - 4 b_0)^2}{16 \beta_0^2 \alpha_0^2} + \frac{4 \alpha_0^2 (\gamma_0 + d_0^2)}{(\beta_0 \alpha_0^2 x - 4 b_0)^2} - \frac{d_0}{\beta_0}$$  \hspace{1cm} (57)
where $\beta_0, \alpha_0, b_0$ and $d_0$ are constants. Henceforth, we will choose the global constant $\beta_0 = 1$, the potential origin $d_0 = 1/2$, and set the space origin $b_0 = 0$, so that we will handle the simplified expression

$$V(x) = \frac{\alpha_0^2 x^2}{16} + \frac{4\gamma_0 + 1}{\alpha_0^2 x^2} - \frac{1}{2}. \quad (58)$$

For this choice, the unknown functions take the form

$$f(x) = -\frac{1}{2} \alpha_0^2 x, \quad \varphi(x) = -\frac{\alpha_0^2 x}{8} + \frac{1}{2}$$

$$\beta(H) = 2\alpha_0 H - \alpha_0, \quad \gamma(H) = -H^2 + \gamma_0 \quad (59)$$

with $\alpha(H) = \alpha_0$. Hence, the explicit expressions for the factors are

$$A^\pm = \pm i \frac{\alpha_0}{2} p - \frac{\alpha_0^2 x^2}{8} + H + \frac{1}{2} \quad (60)$$

and the relation between the Hamiltonian and these functions is given by

$$H = A^+ A^- - H^2 + \gamma_0. \quad (61)$$

Finally, we have the $su(1,1)$ Poisson algebra

$$\{H, A^\pm\} = \pm i \alpha_0 A^\pm, \quad \{A^+, A^-\} = -i \alpha_0 (2H - 1). \quad (62)$$

In this case, the time-dependent integrals of motion $Q^\pm$ has the form (26), with $c(E) = \sqrt{E^2 + E - \gamma_0}$. Then, the phase trajectories $(x(t), p(t))$ for $E > \frac{1 + \sqrt{1 + 4\gamma_0}}{2}$ are

$$x(t) = \frac{2}{\alpha_0} \sqrt{2E + 1 - 2 c(E) \cos(\theta_0 + \alpha_0 t)}$$

$$p(t) = \frac{c(E) \sin(\theta_0 + \alpha_0 t)}{\sqrt{2E + 1 - 2 c(E) \cos(\theta_0 + \alpha_0 t)}}. \quad (63)$$

These expressions show that the frequency of the motion for the singular oscillator is a constant $\alpha_0$ independent of the energy $E$.

### 4.2 The generalized Scarf potential ($H > 0$)

For the specific form

$$\phi(H) = \frac{\delta_0}{\sqrt{H}} + \beta_0 \sqrt{H} \quad (64)$$

and $\varphi(x) = 0$, we have the generalized Scarf potential. Here we will assume $\beta_0 \neq 0$, and by means of a global constant of the factor functions we can set it equal to one: $\beta_0 = 1$. In this way we get the potential

$$V(x) = \frac{1}{4} \left( \frac{2\delta_0 + \gamma_0}{\sin^2(\frac{\alpha_0}{2} x)} - \frac{2\delta_0 - \gamma_0}{\cos^2(\frac{\alpha_0}{2} x)} \right) \quad (65)$$
where $\gamma_0$, $\alpha_0$, and $\delta_0$ are constants. We have also

$$f(x) = -2 \cos \left(\frac{\alpha_0}{4} x\right) \sin \left(\frac{\alpha_0}{4} x\right), \quad g(x) = -2 \sin^2 \left(\frac{\alpha_0}{4} x\right), \quad \varphi(x) = 0$$

$$\beta(H) = \alpha_0 \sqrt{H} - \frac{\alpha_0 \delta_0^2}{H \sqrt{H}} \quad \gamma(H) = \gamma_0 - \frac{\delta_0^2}{\sqrt{H}} \quad \alpha(H) = \alpha_0 \sqrt{H}.$$ \hspace{1cm} (66)

Then, the two complex conjugate functions are

$$A^\pm = \mp 2 i \cos \left(\frac{\alpha_0}{4} x\right) \sin \left(\frac{\alpha_0}{4} x\right) + 2 \sqrt{H} \sin^2 \left(\frac{\alpha_0}{4} x\right) - \frac{\delta_0}{\sqrt{H}} - \sqrt{H}$$ \hspace{1cm} (67)

with the factorization relation

$$H = A^+ A^- + \gamma_0 - \frac{\delta_0^2}{H}$$ \hspace{1cm} (68)

and brackets

$$\{H, A^\pm\} = \pm i \alpha_0 \sqrt{H} A^\pm \quad \{A^+, A^-\} = -i \alpha_0 \left(\sqrt{H} - \frac{\delta_0^2}{H \sqrt{H}}\right).$$ \hspace{1cm} (69)

In this case, as well as in the remaining examples, the Poisson algebra is not a Lie algebra, but a deformed algebra due to the nonlinear function in the last bracket of (69). We can write the time-dependent integrals of motion in the form

$$Q^\pm = A^\pm e^\mp i \alpha_0 \sqrt{H} t,$$

with the value

$$q^\pm = c(E) e^{\pm i \theta_0}$$

where

$$c(E) = \sqrt{-\gamma_0 + E + \frac{\delta_0^2}{E}}.$$ \hspace{1cm} (70)

In order to have a positive potential with bound states, if we assume $\gamma_0 > 0$, then the coefficient $\delta_0$ must satisfy

$$-\gamma_0 < 2\delta_0 < \gamma_0.$$ \hspace{1cm} (71)

The energy of the bound states are given by the constraint $c(E) > 0$, that is, when $E$ is bigger than the minimum of $V(x)$,

$$E > \frac{\gamma_0 + \gamma_0^2 - 4\delta_0^2}{2}.$$ \hspace{1cm} (72)

Then, the trajectories $(x(t), p(t))$ are periodic for any allowed value of $E$:

$$x(t) = \frac{2}{\alpha_0} \arccos \left(\frac{-\delta_0 + c(E) \sqrt{E} \cos (\theta_0 + \alpha_0 \sqrt{E} t)}{E}\right)$$

$$p(t) = -\frac{c(E) E \sin (\theta_0 + \alpha_0 \sqrt{E} t)}{\sqrt{E^2 - (\delta_0 + c(E) \sqrt{E} \cos (\theta_0 + \alpha_0 \sqrt{E} t))^2}}.$$ \hspace{1cm} (72)

### 4.3 The generalized Pöschl-Teller potential ($H < 0$)

In this case, we consider $H < 0$ and accordingly, we change the sign of $H$ in the square roots of the function $\phi(H) = \frac{\delta_0}{\sqrt{H}} + \beta_0 \sqrt{-H}$, keeping $\varphi = 0$. We also make use of expression (40) of $A^\pm$, and equation (41) of Section 3.3. In this way we get the generalized Pöschl-Teller potential

$$V(x) = \frac{1}{4} \left(\frac{2\delta_0 + \gamma_0}{\sinh^2 \left(\frac{\alpha_0}{4} x\right)} + \frac{2\delta_0 - \gamma_0}{\cosh^2 \left(\frac{\alpha_0}{4} x\right)}\right).$$ \hspace{1cm} (73)
and the functions
\[ f(x) = 2 \cosh \left( \frac{\alpha_0}{4} x \right) \sinh \left( \frac{\alpha_0}{4} x \right), \quad g(x) = 2 \sinh^2 \left( \frac{\alpha_0}{4} x \right), \quad \varphi(x) = 0 \]
\[ \beta(H) = -\alpha_0 \sqrt{-H} - \frac{\alpha_0 \delta_0^2}{H \sqrt{-H}}, \quad \gamma(H) = \gamma_0 + 2H + \frac{\delta_0^2}{\sqrt{-H}}, \quad \alpha(H) = \alpha_0 \sqrt{-H} \]
where \( \alpha_0, \delta_0, \gamma_0 \) are constants, and we have set \( \beta_0 = 1 \). Now, the factors \( A^\pm \) are given by
\[ A^\pm = \mp 2i \cosh \left( \frac{\alpha_0}{4} x \right) \sinh \left( \frac{\alpha_0}{4} x \right) p + 2 \sqrt{-H} \sinh^2 \left( \frac{\alpha_0}{4} x \right) + \frac{\delta_0}{\sqrt{-H}} + \sqrt{-H} \]
with the factorization relation
\[ -H = A^+ A^- + \gamma_0 + \frac{\delta_0^2}{H} \]
and the deformed Poisson algebra takes the form
\[ \{H, A^\pm\} = \pm i \alpha_0 \sqrt{-H} A^\pm, \quad \{A^+, A^-\} = i \alpha_0 \left( \frac{\delta_0^2}{H \sqrt{-H}} + \sqrt{-H} \right). \]
The corresponding integrals of motion in terms of \( A^\pm \) are \( Q^\pm = A^\pm e^{\mp i \alpha_0 \sqrt{-H} t} \), with \( q^\pm = c(E) e^{\pm i \theta_0} \), where
\[ c(E) = \sqrt{\gamma_0 - E - \frac{\delta_0^2}{E}}. \]
If the potential has a positive singularity at \( x = 0 \), in order to allow for bound states, the coefficients in the potential, besides (71), they must satisfy \( |2 \delta_0 + \gamma_0| < |2 \delta_0 - \gamma_0| \). Then, the negative energy values of such states are given by the positive character in the square root (78) of \( c(E) \),
\[ \frac{-\gamma_0 + \sqrt{\gamma_0^2 - 4\delta_0^2}}{2} < E < 0. \]
Then, for these energies, the periodic trajectories \((x(t), p(t))\) have the following form:
\[ x(t) = \frac{2}{\alpha_0} \arccosh \left( \frac{\alpha_0}{c(E)} \sqrt{-E} \cos (\theta_0 + \alpha_0 \sqrt{-E} t) \right) \]
\[ p(t) = -\frac{c(E) E \sin (\theta_0 + \alpha_0 \sqrt{-E} t)}{\sqrt{(\delta_0 - c(E) \sqrt{-E} \cos (\theta_0 + \alpha_0 \sqrt{-E} t))^2 - E^2}}. \]
In order to get solutions for the unbounded motion, we choose \( E > 0 \). Then, the integrals of motion \( Q^\pm \) are imaginary and the eigenvalues can be written as \( q^\pm = i c(E) e^{\mp i \theta_0} \). In this case the algebra (77) changes its character. After similar calculations, we have the unbounded phase trajectories \((x(t), p(t))\):
\[ x(t) = \frac{2}{\alpha_0} \arccosh \left( \frac{\alpha_0}{c(E)} \sqrt{E} \cosh (\theta_0 + \alpha_0 \sqrt{E} t) \right) \]
\[ p(t) = \frac{c(E) E \sinh (\theta_0 + \alpha_0 \sqrt{E} t)}{\sqrt{(\delta_0 + c(E) \sqrt{E} \cosh (\theta_0 + \alpha_0 \sqrt{E} t))^2 - E^2}}. \]
where

\[ c(E) = \sqrt{\gamma_0 + E + \frac{\delta^2}{E}}. \]  

(82)

The figures of these singular potentials as well as some trajectories in the phase space are depicted in Figures 4 and 5. Notice that the frequency of the periodic motions in the above potentials is again governed by \( \alpha(E) \), and it has the same expression as in the non-singular potentials of section 3.

![Figure 4](image)

Figure 4: Plot of the oscillator (a), Scarf (b) and Pöschl-Teller (c) potentials (continuous lines) together with their singular counterparts: the singular oscillator, generalized Scarf and generalized Pöschl-Teller potentials.

![Figure 5](image)

Figure 5: Plot of three closed and three open trajectories in the phase space for the generalized Pöschl-Teller potential.

## 5 One dimensional Morse potentials

In order to get the one dimensional Morse potential and other related potentials, we will consider in this section the special form \( \phi(H) = \frac{\delta_0}{\sqrt{H}} \) with \( \varphi = 0 \). This choice corresponds to the case excluded in the previous section, when \( \beta_0 \to 0 \).
5.1 Hyperbolic case \((H < 0)\)

For \(\phi(H) = \frac{\delta_0}{\sqrt{-H}}\), taking into account (40) and (41), we get the potential

\[
V(x) = \frac{2 \delta_0 (C e^{\frac{\alpha_0 x}{2}} + D e^{-\frac{\alpha_0 x}{2}}) - \gamma_0}{(C e^{\frac{\alpha_0 x}{2}} - D e^{-\frac{\alpha_0 x}{2}})^2}
\]

where \(\alpha_0, \delta_0, \gamma_0\) and \(C, D\) are constants. The functions \(f(x), g(x), \beta(H)\) and \(\gamma(H)\) have the following form

\[
f(x) = C e^{\frac{\alpha_0 x}{2}} - D e^{-\frac{\alpha_0 x}{2}}, \quad g(x) = C e^{\frac{\alpha_0 x}{2}} + D e^{-\frac{\alpha_0 x}{2}}, \quad \varphi(x) = 0
\]

\[
\beta(H) = -\frac{\alpha_0 \delta_0^2}{H \sqrt{-H}} - 4 C D \alpha_0 \sqrt{-H}, \quad \gamma(H) = -\gamma_0 + (4 C D + 1) H + \frac{\delta_0^2}{H}
\]

with \(\alpha(H) = \alpha_0 \sqrt{-H}\). Then, the poisson brackets are

\[
\{H, A^\pm\} = \pm i \alpha_0 \sqrt{-H} A^\pm, \quad \{A^+, A^-\} = i \left( \frac{\alpha_0 \delta_0^2}{H \sqrt{-H}} + 4 C D \alpha_0 \sqrt{-H} \right).
\]

The functions \(A^\pm\) can be written as

\[
A^\pm = \mp i \left( C e^{\frac{\alpha_0 x}{2}} - D e^{-\frac{\alpha_0 x}{2}} \right) p + \sqrt{-H} (C e^{\frac{\alpha_0 x}{2}} + D e^{-\frac{\alpha_0 x}{2}}) + \frac{\delta_0}{\sqrt{-H}}
\]

and the time-dependent integrals of motion

\[
Q^\pm = A^\pm e^{\mp i \alpha_0 \sqrt{-H} t}, \quad q^\pm = c(E) \left( e^{\pm i \delta_0} \right).
\]

We shall consider different cases according to the values of \(C\) and \(D\), leading to regular potentials with bound states.

5.1.1 \(C = -D = 1/2\).

The option \(C = -D\) is equivalent to any other with opposite signs of \(C\) and \(D\); it is enough to consider an \(x\)-translation. By means of a constant factor we can choose \(C = -D = 1/2\) then, the corresponding potential is known as Scarf II (hyperbolic) \(^2\)

\[
V(x) = 2 \delta_0 \tanh(\frac{\alpha_0}{2} x) \text{sech}(\frac{\alpha_0}{2} x) - \gamma_0 \text{sech}^2(\frac{\alpha_0}{2} x).
\]

We have the following relations obtained for these particular values of \(C, D\):

\[
H = A^+ A^- - \gamma_0 + \frac{\delta_0^2}{H}
\]

\[
\{H, A^\pm\} = \pm i \alpha_0 \sqrt{-H} A^\pm, \quad \{A^+, A^-\} = i \alpha_0 \left( \frac{\delta_0^2}{H \sqrt{-H}} - \sqrt{-H} \right).
\]

For this case the functions \(A^\pm\) read

\[
A^\pm = \mp i \cosh(\frac{\alpha_0}{2} x) p + \sqrt{-H} \sinh(\frac{\alpha_0}{2} x) + \frac{\delta_0}{\sqrt{-H}}.
\]
Then, \((x(t), p(t))\) have the following form:

\[
x(t) = \frac{2}{\alpha_0} \text{arcsinh}\left[ \frac{\delta_0 - c(E) \sqrt{-E \cos (\theta_0 + \alpha_0 \sqrt{-E} t)}}{E} \right]
\]  
(92)

and

\[
p(t) = -\frac{\sqrt{E^2 + [\delta_0 - c(E) \sqrt{-E \cos (\theta_0 + \alpha_0 \sqrt{-E} t)}]^2}}{c(E) E \sin (\theta_0 + \alpha_0 \sqrt{-E} t)} \frac{c(E) E \sin (\theta_0 + \alpha_0 \sqrt{-E} t)}{\sqrt{E^2 + [\delta_0 - c(E) \sqrt{-E \cos (\theta_0 + \alpha_0 \sqrt{-E} t)}]^2}} \]

(93)

where \(c(E) = \sqrt{\gamma_0 + E - \frac{\delta_0^2}{E}}\). From \(c(E)\), we see that the range of the energy for bound states is given by \((\gamma_0 - \sqrt{\gamma_0^2 + 4\delta_0^2})/2 < E < 0\). When \(E > 0\), we have unbounded motion states with pure imaginary functions \(A^\pm\) and a different algebra from (90), similar to that of Pöschl-Teller potential.

For the choice \(C = \overline{D}\), we get a trigonometric singular potential without bound states.

### 5.1.2 \(D = 0\).

In this case, by taking \(D = 0\), the corresponding potential and the functions \(f(x), g(x), \beta(H), \) and \(\gamma(H)\) can be obtained from (83) and (84). We can also set \(C = 1\), since \(C\) is a global non-vanishing constant factor. The result is the Morse potential

\[
V(x) = 2 \delta_0 e^{-\frac{\alpha_0}{2}x} - \gamma_0 e^{-\alpha_0x}.
\]

(94)

Then, the complex conjugate functions are

\[
A^\pm = \mp i e^{\frac{\alpha_0}{2}x} p + \sqrt{H} e^{\frac{\alpha_0}{2}x} + \frac{\delta_0}{\sqrt{H}}.
\]

(95)

Now, we can build the usual time-dependent integrals of motion \(Q^\pm\) with eigenvalues \(q^\pm = c(E) e^{\pm i \theta_0}\) where \(c(E) = \sqrt{-\frac{\delta_0^2}{E} + \gamma_0}\). The periodic trajectories \((x(t), p(t))\) have the following form:

\[
x(t) = \frac{2}{\alpha_0} \text{log}\left[ \frac{\delta_0 - c(E) \sqrt{-E \cos (\theta_0 + \alpha_0 \sqrt{-E} t)}}{E} \right]
\]  
(96)

and

\[
p(t) = -\frac{\sqrt{E^2 + [\delta_0 - c(E) \sqrt{-E \cos (\theta_0 + \alpha_0 \sqrt{-E} t)}]^2}}{\delta_0 - c(E) \sqrt{-E \cos (\theta_0 + \alpha_0 \sqrt{-E} t)}} \frac{c(E) E \sin (\theta_0 + \alpha_0 \sqrt{-E} t)}{\sqrt{E^2 + [\delta_0 - c(E) \sqrt{-E \cos (\theta_0 + \alpha_0 \sqrt{-E} t)}]^2}} \]

(97)

In order to have periodic bounded motions we must take the coefficients in the potential with the signs \(\delta_0 < 0\) and \(\gamma_0 < 0\). In this way the range of the energies is \(\delta_0^2/\gamma_0 < E < 0\).

The case \(C = 0\) gives similar results, where the exponentials have opposite signs than in the case just considered above.

### 5.2 The trigonometric case \((H > 0)\)

When we take \(\phi(H) = \frac{\delta_0}{\sqrt{H}}\), we obtain the potential

\[
V(x) = -\frac{2 \delta_0 (C e^{i \frac{\alpha_0}{2}x} + D e^{-i \frac{\alpha_0}{2}x}) + \gamma_0}{(C e^{i \frac{\alpha_0}{2}x} - D e^{-i \frac{\alpha_0}{2}x})^2}
\]

(98)
where $\alpha_0$, $\delta_0$, $\gamma_0$ and $C$, $D$ are constants. The involved functions have the following expressions

$$f(x) = i(C e^{i \frac{\alpha_0}{2} x} - D e^{-i \frac{\alpha_0}{2} x}), \quad g(x) = C e^{i \frac{\alpha_0}{2} x} + D e^{-i \frac{\alpha_0}{2} x}, \quad \varphi(x) = 0$$

$$\beta(H) = -\frac{\alpha_0 \delta_0^2}{H \sqrt{-H}} + 4 CD \alpha_0 \sqrt{H}, \quad \gamma(H) = \gamma_0 + (1 - 4 CD) H - \frac{\delta_0^2}{H}$$

(99)

with $\alpha(H) = \alpha_0 \sqrt{H}$. Then, the associated Poisson brackets are

$$\{H, A^\pm\} = \pm i \alpha_0 \sqrt{H} A^\pm, \quad \{A^+, A^-\} = i \left( \frac{\alpha_0 \delta_0^2}{H \sqrt{H}} - 4 CD \alpha_0 \sqrt{H} \right).$$

(100)

For this case $A^\pm$ take the form

$$A^\pm = \pm (C e^{i \frac{\alpha_0}{2} x} - D e^{-i \frac{\alpha_0}{2} x}) p + \sqrt{H}(C e^{i \frac{\alpha_0}{2} x} + D e^{-i \frac{\alpha_0}{2} x}) + \frac{\delta_0}{\sqrt{H}}$$

(101)

and the time-dependent integrals of motion are given by $Q^\pm = A^\pm e^{\pm i \alpha_0 \sqrt{H} t}$, with $q^\pm = c(E) e^{\pm i \theta_0}$.

When we take $D = -C = i/2$, we obtain the potential called Scarf I (trigonometric) [2]

$$V(x) = 2\delta_0 \tan \left( \frac{\alpha_0}{2} x \right) \sec \left( \frac{\alpha_0}{2} x \right) + \gamma_0 \sec^2 \left( \frac{\alpha_0}{2} x \right).$$

(102)

Together with

$$f(x) = \cos \left( \frac{\alpha_0}{2} x \right), \quad g(x) = \sin \left( \frac{\alpha_0}{2} x \right), \quad \varphi(x) = 0$$

$$\beta(H) = -\frac{\alpha_0 \delta_0^2}{H \sqrt{-H}} + \alpha_0 \sqrt{H}, \quad \gamma(H) = \gamma_0 - \frac{\delta_0^2}{H}, \quad \alpha(H) = \alpha_0 \sqrt{H}.$$ 

(103)

Then, the Poisson brackets are

$$\{H, A^\pm\} = \pm i \alpha_0 \sqrt{H} A^\pm, \quad \{A^+, A^-\} = i \alpha_0 \left( \frac{\delta_0^2}{H \sqrt{H}} - \sqrt{H} \right).$$

(104)

Now the functions $A^\pm$ take the form

$$A^\pm = \mp i \cos \left( \frac{\alpha_0}{2} x \right) p + \sqrt{H} \sin \left( \frac{\alpha_0}{2} x \right) + \frac{\delta_0}{\sqrt{H}}$$

(105)
and

\[ c(E) = \sqrt{E + \frac{\delta_0^2}{E} - \gamma_0}. \]  

(106)

From this expression we find that bound motion states exist when \( \gamma_0 > 2\delta_0 \). Under this condition, the allowed energies are \( E > \left( \gamma_0 + \sqrt{\gamma_0^2 - 4\delta_0^2} \right)/2 \). The corresponding motions can be found without difficulty as in the Scarf potential.

![Figure 7: Plot of the trigonometric Scarf I potential for the values: \( \delta_0 = 1 \), \( \gamma_0 = 4 \).](image)

6 Conclusions

In this work we have studied a whole class of one-dimensional classical systems characterized by an underlying Poisson algebra which in general is a deformed Lie algebra. Here, the Poisson algebra is not made up of (time-independent) integrals of motion, as it is the usual case [8, 9, 10], but it includes functions, \( A^\pm \), directly related with time-dependent integrals of motion.

We have obtained some systems like the Scarf, Pöschl-Teller, Morse etc. which are clearly the classical analogues of the one-dimensional quantum systems that can be solved by means of the factorization method. However, the factorizations here employed are not exactly the factorizations of the corresponding quantum cases [2]. Instead, they are more related with the so called spectrum generating algebras in quantum mechanics [14, 3, 5], as can be seen replacing Poisson brackets by commutators: \( \{ \cdot, \cdot \} \rightarrow -i[\cdot, \cdot] \). Therefore, we have shown that these algebras can also be useful in classical mechanics to compute time-dependent integrals of motion of Bohlin type that give us the solutions of the motion in the phase space [11, 12, 13].

The algebraic structures here obtained for some classical systems correspond to well known ones for the analog quantum mechanical systems. Hence, this correspondence is very important to describe the coherent states of such quantum systems: for instance, the expected values of \( x \) and \( p \) are adequate for the harmonic oscillator, but in the case of the Scarf potential we should consider the expected values of \( \sin x \) and \( p \cos x \), which are the components of the functions \( A^\pm \), as we see from (29) and (38), respectively [5, 15].

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