An Efficient Algorithm of Logarithmic Transformation to Hirota Bilinear Form of KdV-type Bilinear Equation

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Abstract

In this paper, an efficient algorithm of logarithmic transformation to Hirota bilinear form of the KdV-type bilinear equation is established. In the algorithm, some properties of Hirota operator and logarithmic transformation are successfully applied, which helps to prove that the linear terms of the nonlinear partial differential equation play a crucial role in finding the Hirota bilinear form. Experimented with various integro-differential equations, our algorithm is proven to be more efficient than the algorithm referred by Zhou, Fu, and Li in getting the Hirota bilinear form, especially in achieving the coefficient of the logarithmic transformation.

Keywords: Hirota bilinear form, KdV-type bilinear equation, D-operator, Logarithmic transformation, Symbolic computation.

1. Introduction

In 1971, Hirota developed a direct method, Hirota bilinear method, to construct the exact multi-soliton solution of integrable nonlinear partial differential equation (NPDE) [1-3]. Once the Hirota bilinear form (HBF) of a NPDE is given, there are several ways to solve it. The solutions of NPDE can be constructed from HBF by the typical perturbation expansion method [1] and others [4,5]. Therefore, the key step of the Hirota direct method is to transform the NPDE into the Hirota bilinear form (HBF). Hietarinta designed a program on searching for integrable bilinear equations such as the KdV-type [6], mKdV-type [7], SG-type [8], NLS-type [9] equations. In 1992, Hereman and Zhuang [10] gave a summary on types of bilinear equations. In recent years, some algorithms for generating bilinear form of NPDE are described and Maple packages Bilinearization and HBFTTrans are established by Zhou, Fu, Li [11, 12] and Yang, Ruan [13], respectively. The package Bilinearization can construct the HBF of many NPDEs by solving a system of over-determined algebraic equations with respect to the combinatorial coefficients. However, the general ansatz of the bilinear form of NPDE in Ref. [11, 12], which relies on the WTC method [14] and the HBM method [15], is so complicated that the efficiency of performance is relatively low. Without depending on these two methods, our method can obtain the HBF of the KdV-type equations within shorter time. In this paper, some properties of Hirota-operator are taken advantage of, which brings forward to a more efficient algorithm for finding...
the HBF of KdV-type equations in NPDE. We take a series of classic KdV-type equations for instance, to demonstrate the validity of our algorithm. Furthermore, the implementation of the algorithm in Maple is applied to automate the tedious computation for the construction of the HBF of KdV-type equations in NPDE.

2. Hirota Bilinear Method

Now, let us briefly review the Hirota derivatives. In 1971, Hirota developed the Hirota derivative, which is also called D-operator \[1\]. For 1+1 dimensions, the D-operator is defined by

\[
D^n_x D^m_t f \cdot g = (\partial_x - \partial_x')^n (\partial_t - \partial_t')^m f(x,t) g(x', t')
\]

(2.1)

where \( f(x,t) \) and \( g(x,t) \) are differentiable functions of \( x \) and \( t \), respectively.

From the definition, there are some properties \[1\] of the D-operator,

\[
D^n_x D^m_t f \cdot g = D^m_t D^n_x f \cdot g
\]

(2.2)

\[
D^n_x D^m_t f \cdot 1 = \partial^n_x f, \quad \text{where } \partial^n_x = \partial^n / \partial x^n.
\]

(2.3)

\[
D^n_x D^m_t f \cdot g = (-1)^{m+n} D^n_x D^m_t g \cdot f
\]

(2.4)

\[
D^n_x D^m_t f \cdot f = 0, \quad \text{if } m + n \text{ is odd};
\]

(2.5)

furthermore, for arbitrary independent variable \( x \), a recursive definition of D-operator is,

\[
D^n_x f \cdot g = \begin{cases} f g, & \text{for } n = 0 \\ D_{n-1}^x f \cdot g - D_{n-1}^x f \cdot g, & \text{for } n > 0. \end{cases}
\]

(2.6)

As an example, the Korteweg de Viries (KdV) equation

\[
u_{xxx} + 6u_ux + u_t = 0
\]

(2.7)

where \( u = u(x,t) \), can be transformed through the dependent variable transformation,

\[
u = 2(\ln f)_{xx}
\]

(2.8)

into

\[
\frac{\partial}{\partial x} \left( \frac{f_{xx} f - f_{xx} f + f_{xxx} f - 4f_{xxx} f + 3f_{xx}^2}{f^2} \right) = 0
\]

(2.9)

from which the bilinear equation below is obtained

\[
f_{xx} f - f_{xx} f + f_{xxx} f - 4f_{xxx} f + 3f_{xx}^2 = cf^2,
\]

(2.10)

where \( c \) is a constant of integration. Equation \ref{eq:2.10}, with \( c = 0 \), may also be written concisely in terms of D-operators as

\[
(D_x^4 + D_x D_t)f \cdot f = 0.
\]

(2.11)

Thus, the Hirota bilinear form of the KdV equation is obtained.
3. Principles Of Our Algorithm

3.1. The Properties of Hirota Derivatives

Introducing vector notation
\[ \overrightarrow{D} = (D_x, D_y, D_z, \ldots) \] (3.1)
consider the KdV-type bilinear equation \[^1\]
\[ P(D_x, D_y, D_z, \ldots) f \cdot f = 0, \] (3.2)
where \( P \) is a general polynomial in \( D_x, D_y, D_z, \ldots \).

According to (2.4) and (2.5), it follows that
\[ P(-\overrightarrow{D}) = P(\overrightarrow{D}). \] (3.3)

Now assume that the degree of every term in \( P \) is even.

When \( n \) is even, it’s true that
\[ D^n f \cdot f = \left. \left( \sum_{i=0}^{n} \frac{\partial^n}{\partial x^n} \left( \overrightarrow{D}^{\cdot i} f(x') \right) \right) \right|_{x=x} \]
\[ = \sum_{i=0}^{n} \left( -1 \right)^i C_i^n \left( \overrightarrow{D}^{\cdot i} f(x) \right) \left( \partial^{\cdot i} f(x) \right) \]
\[ = 2 \sum_{i=0}^{n/2-1} \left( -1 \right)^i C_i^n \left( \overrightarrow{D}^{\cdot i} f(x) \right) \left( \partial^{\cdot i} f(x) \right) + (-1)^{n/2} C_n^{n/2} \left( \overrightarrow{D}^{\cdot n/2} f(x) \right)^2 \]
\[ = 2 (\overrightarrow{D}^n f) f + \overrightarrow{H}, \] (3.4)
where \( \overrightarrow{H} = H(f_x, f_y, \ldots) \) is a polynomial in \( f_x, f_y, \ldots \), WITHOUT \( f \), and \( H = D^n f \cdot f = 2 (\overrightarrow{D}^n f) f \). This simple observation implies the following crucial formula
\[ P(D_x, D_y, D_z, \ldots) f \cdot f = 2 \left( P(\partial_x, \partial_y, \partial_z, \ldots) \right) f + \overrightarrow{H} = 2 \left( P(\overrightarrow{\partial}) f \right) f + \overrightarrow{H}, \] (3.5)
where \( \overrightarrow{H} \) is a polynomial in \( f_x, f_y, \ldots \), WITHOUT \( f \), and \( P(\overrightarrow{\partial}) = P(\partial_x, \partial_y, \partial_z, \ldots) \). Some simple examples are listed as follows to illustrate equation (3.5)
\[ D_x^2 f \cdot f = 2(f_x, f_x - f_x^2), \quad \overrightarrow{H} = -2f_x, \] (3.6)
\[ D_y^2 f \cdot f = 2(f_y, f_y - f_y^2), \quad \overrightarrow{H} = -8f_yf_x + 6f_x^2, \] (3.7)
\[ D_z^2 f \cdot f = 2(f_z, f_z), \quad \overrightarrow{H} = -2f_z, \] (3.8)
where \( f_x \equiv \partial^k f \equiv \frac{\partial^k f}{\partial x^k}, k \in \mathbb{N} \). Furthermore, using equation (3.5), it is easy to find \( P(\overrightarrow{D}) f \cdot f \) is a quadratic polynomial in \( f, f_x, f_y, \ldots \), and then to obtain
\[ \frac{P(\overrightarrow{D}) f \cdot f}{f^2} = \frac{P_1}{f} + \frac{P_2}{f^2}. \] (3.9)
Here \( P_1 \) and \( P_2 \) are polynomials in \( f_x, f_y, \ldots \), WITHOUT \( f \). \( P_1 = 2P(\overrightarrow{\partial}) f \), is a linear polynomial; \( P_2 = \overrightarrow{H} \) is a quadratic polynomial.
3.2. Logarithmic Transformation

Consider the derivation formula,

\[
\left( \frac{f}{g} \right)_x = \frac{f_x g - f g_x}{g^2} = \frac{f_x}{g} - \frac{f g_x}{g^2},
\]  \hspace{1cm} (3.10)

Furthermore, for arbitrary independent variable \( t \), it’s easy to show that

\[
\left( \frac{f}{g} \right)_t = \frac{P_1}{f} + \frac{P_2}{f^2} + \ldots + \frac{P_{k+1}}{f^{k+1}},
\]  \hspace{1cm} (3.11)

where \( k \in \mathbb{N} \) and \( P_i (i = 1, 2, \ldots, k + 1) \) is a homogeneous polynomial of degree \( i \) in \( f_x, f_t, \ldots \), \( \text{WITHOUT} f \), and

\[
P_1 = \partial^1_t f_x, \quad P_{k+1} = (-1)^k k! f_x f_t.
\]  \hspace{1cm} (3.12)

are obtained by using equation (3.10) repeatedly.

Furthermore, when \( m + n \geq 1 \), it shows that

\[
\partial^m f^n (\ln f) = \frac{\partial^m f^n f}{f} + \cdots + (-1)^{m+n-1} (m+n-1)! \frac{f_x f_t}{f} \]  \hspace{1cm} (3.13)

Still, by using \( P \) as the denotation of a general polynomial and letting the lowest degree of the terms in \( P \) be equal to or greater than 1, we have

\[
\overrightarrow{P}(\overrightarrow{\partial})(\ln f) = \frac{P_1}{f} + \frac{P_2}{f^2} + \ldots + \frac{P_r}{f^r},
\]  \hspace{1cm} (3.14)

where \( r = \deg(P) \) and \( \overrightarrow{P_1} = \overrightarrow{P}(\overrightarrow{\partial}) f \). Here, \( \overrightarrow{P}_i (i = 1, 2, \ldots, r) \) also denotes a homogeneous polynomial of degree \( i \) in \( f_x, f_t, \ldots \), \( \text{WITHOUT} f \).

Using the logarithmic transformation \( u = 2 \ln f \), the following expressions are obtained:

\[
u_{xx} = 2 \left( \frac{f_{xx}}{f} - \frac{f_x^2}{f^2} \right) = \frac{D_x^2 f \cdot f}{f^2}
\]  \hspace{1cm} (3.15)

\[
u_{xt} = 2 \left( \frac{f_{xt}}{f} - \frac{f_x f_t}{f^2} \right) = \frac{D_x D_t f \cdot f}{f^2}
\]  \hspace{1cm} (3.16)

\[
u_{tt} = 2 \left( \frac{f_{tt}}{f} - \frac{f_t^2}{f^2} \right) = \frac{D_t^2 f \cdot f}{f^2}
\]  \hspace{1cm} (3.17)

\[
u_{tx} = 2 \left( \frac{f_{txx}}{f} - \frac{4 f_{xx} f_x + 3 f_{x}^2}{f^2} + \frac{12 f_{xx} f_t}{f^3} - \frac{6 f_x^2}{f^3} \right) = \frac{D_x^3 f \cdot f}{f^2} - 3 \left( \frac{D_x^2 f \cdot f}{f^2} \right)^2
\]  \hspace{1cm} (3.18)

\[
u_{tt} = 2 \left( \frac{f_{xxxx}}{f} + \cdots - \frac{120 f_{x}^6}{f^6} \right) = \frac{D_x^6 f \cdot f}{f^2} - 15 \frac{D_x^4 f \cdot f \cdot D_x^2 f \cdot f}{f^2} + 30 \left( \frac{D_x^2 f \cdot f}{f^2} \right)^3
\]  \hspace{1cm} (3.19)

Clearly, the expressions above can lead us to finding the KdV-type bilinear form associated with logarithmic transformation in NPDE.
3.3. The Relationship Between Logarithmic Transformation and KdV-type Bilinear Equation

In this paper, we consider the logarithmic transformation

\[ u = 2\alpha (\ln f)_x, \quad n = 0, 1, 2 \]  
(3.20)

where \( f = f(x, t, \cdots) \) and \( \alpha \) is a nonzero constant.

Substituting (3.20) to a NPDE

\[ W(u, u_x, u_t, \cdots) = 0, \]  
(3.21)

we get

\[ \overline{W}(f, f_x, f_t, \cdots; \alpha) \equiv W(u, u_x, u_t, \cdots)|_{u=2\alpha(\ln f)_x}. \]  
(3.22)

If NPDE \( \overline{W}(f, f_x, f_t, \cdots; \alpha) \) has the KdV-type bilinear form

\[ \left( \frac{P(\overrightarrow{D})f \cdot f}{f^2} \right)_{mx} = 0, \]  
(3.23)

where \( m \) is a nonnegative integer, using the properties of Hirota operator and logarithmic transformation, our algorithm will find the undetermined parameters \( n, m, P(\overrightarrow{D}), \) and \( \alpha \).

Because of (3.14), the terms in NPDE with degree \( k \) generate the homogeneous expression with logarithmic transformation (3.20)

\[ \overline{P}_k \overline{f}^k + \overline{P}_{k+1} \overline{f}^{k+1} + \cdots. \]  
(3.24)

That is, with logarithmic transformation (3.20), the terms in NPDE with degree \( k \) do NOT generate the terms \( \overline{P}_1 \overline{f}^i \) \((i = 1, 2, 3, \cdots, k - 1)\).

Consider KdV-type bilinear form

\[ \left( \frac{P(\overrightarrow{D})f \cdot f}{f^2} \right)_{mx} = \left( \frac{P_1f + P_2}{f^2} \right)_{mx} = \left( \frac{P_1}{f} + \frac{P_2}{f} \right)_{mx} = \left( \frac{P_1}{f} \right)_{mx} + \left( \frac{P_2}{f} \right)_{mx} \]

\[ = \frac{P_{1mx}}{f} + \cdots + \frac{(-1)^m1!P_1f^m_{mx}}{f^{m+1}} + \frac{P_{2mx}}{f^2} + \cdots + \frac{(-1)^m(m + 1)!P_2f^m_{mx}}{f^{m+2}} \]

\[ = \frac{P_{1mx}}{f} + \cdots + \frac{(-1)^m(m + 1)!P_2f^m_{mx}}{f^{m+2}} = 2\alpha P(\overrightarrow{D})f \]

\[ + \cdots + \frac{(-1)^m(m + 1)!P_2f^m_{mx}}{f^{m+2}}. \]  
(3.25)

Therefore, if an equation has the KdV-type bilinear form, with logarithmic transformation (3.20), the term \( \overline{P}_1 \overline{f}^i \) must be generated from the linear terms in NPDE (3.21).

Denoting the linear part in NPDE (3.21) by \( P(\overrightarrow{D})u \), with logarithmic transformation (3.20), according to (3.14), the linear part is transformed into

\[ \overline{P}(\overrightarrow{D})u \overline{u}^{2\alpha(\ln f)_x} \overline{f} \overline{f} + \cdots. \]

Thus, it follows that

\[ \overline{P}(\overrightarrow{D}) = \alpha \frac{f^m}{f} \overline{P}(\overrightarrow{D}). \]  
(3.26)
From the above equation, we find it important that the linear terms of the NPDE play a crucial role in finding the HBF. Hence, from the linear part in NPDE, \( P \), the function of HBF, can be obtained.

Let
\[
\left( \frac{P(D)f \cdot f}{f^2} \right)_{mx} = \hat{W}(f, f_x, f_t, \cdots; \alpha),
\]  
then the difference should be zero, namely,

\[
\text{Res}_{\hat{W}} \triangleq \hat{W}(f, f_x, f_t, \cdots; \alpha) - \left( \frac{P(\partial)f \cdot f}{f^2} \right)_{mx} = 0.
\]  

Clearly, \( n \) is a finite enumerated integer and \( m \) is also a finite enumerated integer (See Step 5 of algorithm in Section 5) and \( P(\partial) = \alpha n^m P(\partial) \).

With \( P \) satisfying (3.26), the coefficient of \( \frac{1}{f} \) in \( \text{Res}_{\hat{W}} \) is equal to zero. Then the homogeneous expression in \( f \) and its derivatives follows as:

\[
\text{Res}_{\hat{W}} = \hat{P}_2 f + \cdots + \hat{P}_r f^r.
\]  

Here, \( r \) is a finite positive integer.

Solving the numerators of (3.29), if there exists the nonzero numerical solution \( \alpha \), the undetermined parameters \( n, m, P(\partial) \) are also achieved. It can be proven that \( \alpha \) has no more than one nonzero numerical solution which satisfies (3.28) with certain \( n \) and \( m \). The proof will be given in our next paper.

In summary, if a NPDE has the KdV-type bilinear form associated with the logarithmic transformation (3.20), our algorithm will surely find it.

4. An Example Of our Algorithm

Now, let us take the KdV equation as an example.

\[
u_{xxx} + 6uu_x + u_t = 0.
\]  

Here, \( P(\partial) = \partial_x + \partial_t^3 \).

First substitute \( u = 2\alpha \ln f \) to (4.1) and simplify the equation, the homogeneous equation in \( f \) and its derivatives follows as:

\[
\frac{2\alpha(f_{xxx} + 12\alpha(\ln f)f_x + f_t)}{f} - \frac{6\alpha f_{xxx}f_x}{f^2} + \frac{4\alpha f_x^3}{f^3} = 0.
\]

Since the above equation has \( \ln(f) \) term, we goto the next logarithmic transformation \( u = 2\alpha(\ln f)_x \).

Likewise, substitute \( u = 2\alpha(\ln f)_x \) to (4.1) and simplify the equation, the homogeneous equation in \( f \) and its derivatives follows as:

\[
\frac{2\alpha f_{xxx} + f_{xx}}{f} - \frac{2\alpha(-12\alpha f_{xx}f_x + 4f_{xxx}f_x + 3f_{xx}^2 + f_x^3 + f_{xx}f_t)}{f^2} + \frac{24\alpha f_x^2(-\alpha f_x + f_{xx})}{f^3} - \frac{12\alpha f_x^4}{f^4} = 0.
\]
From the above equation, the coefficient of \( \frac{1}{f} \) is acquired: 
\[
P_1 = 2\alpha (f_{xxxx} + f_{xx}) = 2\alpha \left( \partial_x^4 + \partial_x^2 \partial_t \right) f = 2\alpha \partial_x P(\partial) f.
\]
Because the order of derivative is even, the Hirota bilinear form can be obtained: 
\[
P(\partial^2) f \cdot f = \alpha (D_x^4 + D_x D_t) f \cdot f,
\]
with which divided by \( f^2 \) and subtracts the above equation the difference is as follows:
\[
\frac{12f_{xx}(2\alpha f_x + f_{xx})}{f^2} - \frac{24\alpha f_x^2(-\alpha f_x + f_{xx})}{f^3} + \frac{12\alpha f_x^4}{f^4} = 0.
\]
Solving the numerators of the difference equation, it only has \( \alpha = 0 \). So we goto the next logarithmic transformation \( u = 2\alpha (\ln f)_{xx} \).

Similarly, substitute \( u = 2\alpha (\ln f)_{xx} \) to \( \alpha = 1 \) and simplify the equation, the homogeneous equation in \( f \) and its derivatives follows as:
\[
\frac{2\alpha (f_{xx} + f_{xxxx})}{f} - \frac{2\alpha(-12)(f_{xxxx}f_{xx} + f_{xx}f_t + 5f_{xxx}f_x + 10f_{xxt}f_{xx} + 2f_{txt})}{f^2} - \frac{4f_x(18f_x^3 + 6f_{xxxx}f_x - 15f_x^2 - 10f_{xxx}f_x - f_{xx}f_t)}{f^3} + \frac{120f_x f_{xx}f_t(\alpha - 1)}{f^4} - \frac{48\alpha f_x^2(\alpha - 1)}{f^5} = 0.
\]

From the above equation, the coefficient of \( \frac{1}{f} \) is acquired: 
\[
P_1 = 2\alpha (f_{xxxx} + f_{xx}) = 2\alpha \left( \partial_x^4 + \partial_x^2 \partial_t \right) f.
\]
Because the order of derivative is odd, we integrate \( P_1 \) and get the Hirota bilinear form \( P(\partial^2) f \cdot f = \alpha (D_x^4 + D_x D_t) f \cdot f \), with which divided by \( f^2 \) and derived in terms of \( x \) one time then subtracts the above equation, the difference is obtained:
\[
\frac{-24f_{xx}f_{xx}(\alpha - 1)}{f^2} + \frac{24f_x(\alpha - 1)(3f_x^3 + f_{xx}f_t)}{f^3} - \frac{120f_x f_{xx}f_t(\alpha - 1)}{f^4} + \frac{48\alpha f_x^2(\alpha - 1)}{f^5} = 0.
\]
Solving the numerators of the difference equation, a nonzero solution, \( \alpha = 1 \), is achieved.

In conclusion, by using the logarithmic transformation \( u = 2(\ln f)_{xx} \), it shows that the equation \( f_{xx}f_{xx} = \alpha (\partial_x^4 + \partial_x^2 \partial_t) f \) is equivalent to
\[
\left( \frac{f_{xx}f_{xx} - \alpha (\partial_x^4 + \partial_x^2 \partial_t) f}{f^2} \right)_x = 0.
\]

This example illustrates our method intuitively.

5. An Algorithm of Bilinear

Consider the general NPDE
\[
W(u, u_x, u_{xx}, \cdots) = 0,
\]
where \( u = u(x,t, \cdots) \), and \( W \) is a polynomial in \( u \) and its derivatives.

The algorithm for obtaining HBF goes in the following steps:

**Step 1** Let \( n = 0, f = f(x,t, \cdots) \) and \( \alpha \) is an undetermined nonzero constant.

**Step 2** Using logarithmic transformation \( u = 2\alpha (\ln f)_{xx} \), it follows that
\[
\bar{W} = \bar{W}(f, f_x, f_{xx}) = W(u, u_x, u_{xx}, \cdots)|_{u=2\alpha \ln f_{xx}}.
\]
Step 3 If \( \tilde{W} \) has \( \ln (f) \) term, then goto Step 6.

Step 4 Simplify the equation \( \tilde{W} \) to get the homogeneous expressions in \( f \) and its derivatives

\[
\tilde{W} = \frac{P_1}{f} + \frac{P_2}{f^2} + \cdots + \frac{P_m}{f^m},
\]

where \( P_i = P_i(f, f_1, \cdots) \) (\( i = 1, 2, \cdots, r \)) is a homogeneous polynomial of degree \( i \) in \( f, f_1, \cdots \).

Step 5 Get the coefficient of \( \frac{1}{f} \) in \( \tilde{W} \): \( P_1 = P_1(f, f_1, \cdots) \). Let the lowest order of the derivative of \( f \) with respect to \( x \) in \( P_1 \) be \( k \), and the corresponding Hirota bilinear form is \( P(D_x, D_t, \cdots) f \cdot f \). Then the following equation is obtained:

\[
P_1(f, f_1, \cdots) = 2 \alpha f P(f, f_1, \cdots),
\]

where \( 0 \leq m \leq k \) and the degrees of the terms in \( P(f, f_1, \cdots) \) are even. That is,

- If the lowest order derivative of \( f \) is even, then \( m = 0, 2, 4, \cdots, \) and \( m \leq k \).
- If the lowest order derivative of \( f \) is odd, then \( m = 1, 3, 5, \cdots, \) and \( m \leq k \).

Step 5.1 Let \( m = 0 \), if the lowest order of the derivative of \( f \) is even, or else let \( m = 1 \).

Step 5.2 Integrating \( P_1 \) by \( m \) times in terms of \( x \), we get \( P(f, f_1, \cdots) = \frac{1}{2} \alpha x^m P(f, f_1, \cdots) \). Hence the HBF is \( P(D_x, D_t, \cdots) f \cdot f \).

Step 5.3 Calculate the difference, \( Res_{\tilde{W}} = \tilde{W} - \left( \frac{P(D_x, D_t, \cdots) f \cdot f}{f^2} \right)_{m \times} \), and simplify the difference \( Res_{\tilde{W}} \) to get the homogeneous expressions in \( f \) and its derivatives

\[
Res_{\tilde{W}} = \frac{P_2}{f^2} + \cdots + \frac{P_m}{f^m}.
\]

Step 5.4 Get the equations \( \tilde{P}_i = 0 \) (\( i = 2, 3, \cdots, r \)), and solve the equations. If a nonzero \( \alpha \) exists, then using \( u = 2 \alpha (\ln f)_{m \times} \), the NPDE is equal to \( \frac{P(D_x, D_t, \cdots) f \cdot f}{f^2} = 0 \). Therefore, the HBF is \( P(D_x, D_t, \cdots) f \cdot f \), and exit of program.

Step 5.5 Let \( m = m + 2 \). If \( m \leq k \), goto Step 5.2, or else goto Step 6.

Step 6 Let \( n = n + 1 \). If \( n \leq 2 \) then goto Step 2, or else the NPDE has no HBF with logarithmic transformation, end program and exit.

6. Applications

Example 1. Boussinesq equation [16, 17]

\[
u_{tt} - u_{xx} - 3u_{tx}^2 - u_{tx} = 0 \quad (6.1)
\]

Solution. From the equation, Bilinear within 0.s outputs \( \alpha = 1, u = 2 \ln (f) \) and

\[
\frac{(D_x^2 - D_t^2 - D_t^4) f \cdot f}{f^2} = 0.
\]
Example 2. Sawada-Kotera equation \([18, 19]\)

\[ u_t + 45u_xu^2 - 15u_xuu_t + 15u_{xxx}u + u_{5x} = 0 \]  
\[ (6.2) \]

Solution. Within 0.032s our program outputs \( \alpha = -1 \), \( u = -2(\ln f)_{xx} \) and

\[ \left( \frac{(-D^2_x - D_xD_t)f \cdot f}{f^2} \right)_{xx} = 0. \]

Example 3. Kadomtsev-Petviashvili equation \([20]\)

\[ (u_{3x} + 6uu_x + u_t)_x + 3\delta^2u_{xy} = 0 \]  
\[ (6.3) \]

Solution. Bilinear within 0.016s outputs \( \alpha = 1 \), \( u = 2(\ln f)_{xx} \) and

\[ \left( \frac{(D^2_x + D_xD_t + 3\delta^2D^2_t)f \cdot f}{f^2} \right)_{xx} = 0. \]

Example 4. The shallow water waves equation \([21]\)

\[ u_t - u_{xxx} - 3uu_t + 3u_x \int_\infty^x u_t dx + u_t = 0 \]  
\[ (6.4) \]

Solution. By substituting \( u = wx \) and using the boundary condition \( u|_{x \to \infty} = 0 \) \((6.4)\) can be converted into the differential form

\[ w_{3xt} - w_{3xt} - 3w_xw_{xt} - 3w_xw_t + w_{xx} = 0, \]  
\[ (6.5) \]

Bilinear within 0.016s outputs \( \alpha = 1 \), \( w = 2(\ln f)_x \) and

\[ \left( \frac{(-D^2_x D_t + D^2_t + D_xD_t)f \cdot f}{f^2} \right)_{xx} = 0. \]

Example 5. Ito equation \([22]\)

\[ u_t + u_{3xx} + 6uu_x + 3u_{tx} + 3u_x \int_\infty^x u_t dx = 0 \]  
\[ (6.6) \]

Solution. By substituting \( u = w_x \) and using the boundary condition \( u|_{x \to -\infty} = 0 \) \((6.6)\) is transformed into the differential form

\[ w_{3xt} + w_{4xt} + 6w_xw_{xt} + 3w_xw_{tt} + 3w_{3xx}w_t + 3w_{3x}w_t = 0, \]  
\[ (6.7) \]

Bilinear within 0.031s outputs \( \alpha = 1 \), \( w = 2(\ln f)_x \) and

\[ \left( \frac{(D^2_x D_t + D^2_t)f \cdot f}{f^2} \right)_{xx} = 0. \]

Example 6. (2+1)-dimensional breaking soliton equation \([23]\)

\[ u_t + \beta u_{3xt} + 4\beta uu_t + 4\beta u_x \int_\infty^x u_t dx = 0 \]  
\[ (6.8) \]
Solution. The equation can be written as (6.9) by substituting $u = w_x$ and using the boundary condition $u_t|_{x \to -\infty} = 0$

\[ w_{xt} + \beta w_{3,xy} + 4\beta w_{x,wy} + 4\beta w_{tx,wy} = 0, \quad (6.9) \]

**Bilinear** within 0.016s outputs $\alpha = \frac{1}{2}$, $w = \frac{1}{\sqrt{2}}(\ln f)_x$ and

\[ \left( \frac{3}{4} \frac{(D_x D_t + \beta D_y^2 D_z) f \cdot f}{f^2} \right)_x = 0. \]

Example 7. The Bidirectional SK equation [24]

\[ 5 \int_{-\infty}^{\infty} u_0 dx + 5u_{xx} - 15u_t - 15u_x \int_{-\infty}^{\infty} u_\alpha dx - 45u_{xx}u^2 + 15u_{x}u_x + 15u_{3,xx}u - u_{5,x} = 0 \quad (6.10) \]

Solution. Similarly, we convert (6.10) into the differential form by substituting $u = w_x$ and using the boundary condition $u_t|_{x \to -\infty} = 0, u_t|_{x \to +\infty} = 0$

\[ 5w_{xt} + 5w_{3,xx} - 15w_xw_{xx} - 15w_{xx}w_t - 45w_{xx}w^2 + 15w_{3,xx}w_x + 15w_{4,xx}w_x - w_{6,x} = 0, \quad (6.11) \]

**Bilinear** within 0.015s outputs $\alpha = -1$, $w = -2(\ln f)_x$ and

\[ \left( \frac{(-5D_x^2 + D_y^2 - 5D_x^2 D_z) f \cdot f}{f^2} \right)_x = 0. \]

Example 8. (2+1)-dimensional SK equation [23]

\[ 9u_t + u_{5,x} + 15u_{4,xx}u_x + 15u_{3,xx}u + 45u_xu_x^2 - 5 \int_{-\infty}^{\infty} u_{xy} dx - 15uu_x - 15u_x \int_{-\infty}^{\infty} u_\alpha dx - 5u_{xy} = 0 \quad (6.12) \]

Solution. With the substitution $u = w_x$ and the boundary condition $u_t|_{x \to -\infty} = 0, u_{xy}|_{x \to +\infty} = 0$

\[ 9w_{xt} + w_{6,x} + 15w_{3,xx}w_{xx} + 15w_{4,xx}w_x + 45w_{xx}w_x^2 - 5w_{xy} - 15w_xw_{3,xy} - 15w_{xx}w_y - 5w_{3,xy} = 0, \quad (6.13) \]

**Bilinear** within 0.032s outputs $\alpha = 1$, $w = 2(\ln f)_x$ and

\[ \left( \frac{(D_y^6 - 5D_x^2 D_y - 5D_x^2 + 9D_z D_x) f \cdot f}{f^2} \right)_x = 0. \]

Example 9. (3+1)-dimensional KdV equation [13]

\[ u_t + 6u_xu_x + u_{2,xx} + u_{4,xx} + 60u_x^2u_x + 10u_{3,xx} + 20u_xu_{4,xx} = 0 \quad (6.14) \]

Solution. **Bilinear** within 0.016s outputs $\alpha = \frac{1}{2}$, $u = (\ln f)_x$ and

\[ \frac{1}{2} \left( \frac{(D_x D_t + D_y^2 D_z + D_y^2 D_z) f \cdot f}{f^2} \right) = 0. \]
7. The Program Code of Bilinear

with(PDEtools): with(DEtools):

## Hirota Bilinear Method
## Bilinear Derivative / Hirota Operator

BD:=proc(FF,DD) local f,g,x,m,opt;
if nargs=1 then return '*'(FF[]); fi; f,g:=FF[]; x,m:=DD[];
opt:=args[3..-1]; if m=0 then return proccame(FF,opt); fi;
proccame([diff(f,x),g],[x,m-1],opt)-proccame([f,diff(g,x)],[x,m-1],opt);
end:

'print/BD':=proc(FF,DD) local f,g,x,m,i; f,g:=FF[];
f:=cat(f,' ','g'); g:=product(D[args[i][1]]^args[i][2],i=2..nargs);
if g<>1 then f:=''(g)*''(f); fi; f; end:

## collect(expr,f); first!

getFnumer:=proc(df,f,pow::posint:=1) local i,g,fdenom;
if type(df,'+') then
  g:=[op(df)];
  fdenom:=map(denom,g);
  for i to nops(fdenom) while fdenom[i]<>f^pow do od;
  if i>nops(fdenom) then lprint(fdenom);
  error "no term(s) or numer=0 when denom=%1",op(0,f)^pow fi;
  g:=numer(g[i]);
  if not type(expand(g),'+') then lprint(g);
  error "Expected more than 1 term about Hirota D-operator" fi;
  return g; fi; lprint(df);
  error "expected 1st argument be type '+'."; end:

getvarpow:=proc(df::function) local i,f,var,dif,pow; if
op(0,dif)<>diff then lprint(df); error "expected diff function" fi;
f:=convert(df,D); var:=[op(f)]; dif:=[op(op([0,0],f))]);
pow:=[0$0<nops(var)]; f:=op(op(0,f))(var[]) for i to nops(var) do
dif:=selectremove(member,dif,{i});
pow[i]:=nops(dif[1]);
dif:=dif[2];
o; pow:=zip((x,y)->[x,y],var,pow); pow:=remove(has,pow,{0});
[[f,f],pow]]; end:

#convert to Hirota Bilinear Form

HBF:=proc(df) local i,c,f; if type(df,'+') then
  f:=[op(df)]; return map(proccame,f); fi;
if type(df,('*')) then f:=[op(df)];
f:=selectremove(hasfun,f,diff); c:=f[2]; f:=f[1];
if nops(f)<1 then lprint(df); error "need only one diff function factor." fi;
f := f[ ]; c := '*(c[ ]); f := getvarpow(f); f := [c, f]; return f; fi;
if op(0, df) = diff then f := getvarpow(df); f := [1, f]; return f; fi;
lprint(df); error "unexpected type."
end:

printHBF := proc(PL :: list) local j, DD, f, C, tmp, gcdC;
C := map2(op, 1, PL);
gcdC := 1;
if nops(C) > 1 then tmp := [seq(cat(_Z, i), i = 1..nops(C))];
gcdC := tmp + C; gcdC := ' + (gcdC[]); gcdC := factor(gcdC);
tmp := selectremove(has, gcdC, tmp);
gcdC := tmp[2];
if gcdC = 0 then gcdC := 1 fi;
gcdC := gcdC*content(tmp[1]); fi;
if gcdC <> 1 then C := C /~ gcdC; fi;
DD := map2(op, 2, PL);
f := op(0, DD[1][1][1]);
DD := zip('*, C, DD);
DD := ' + (DD[]); gcdC * '*(DD[]) * cat(f, ' ', f);
end:

# print Hirota Bilinear Transform
printHBT := proc(uf, u, f, i, j, PL, alpha := 1) local DD, g, C, tmp, pl;
pl := printHBF(PL);
if j > 0 then print(u = 2*alpha*'diff'(ln(f), x^j));
else print(u = 2*alpha*ln(f)); fi;
if i > 0 then print('diff'(pl/f^2, x^i) = 0); else
pl/f^2 = 0; fi;
NULL; end:

guessdifforder := proc(PL :: list, x :: name)
local L, minorder, maxorder, tmp;
L := map2(op, 2, PL);
L := map(z -> z[2..-1], L);
tmp := map2(op, 2, z), L);
tmp := map(z -> ' + ('z[]), tmp);
tmp := selectremove(type, tmp, even);
minorder := 0;
if nops(tmp[1]) < nops(tmp[2]) then minorder := 1 fi;
if has(tmp, []) then maxorder := 0; else tmp := map(op, tmp);
maxorder := min(tmp[]); fi;
if type(maxorder-minorder, odd) then maxorder := maxorder-1 fi;
[minorder, maxorder];
end:
guessalpha := proc(Res, uf, u, f, i, j, PL, alpha) local tmp, res, pl, flag, k;
flag := 1;
tmp := [op(Res)];
tmp := map(numer, tmp);
tmp := gcd(tmp[1], tmp[-1]);
if type(tmp, ' + ') then
tmp := remove(has, tmp, f); fi;
if tmp <> NULL and has(tmp, {alpha}) then
tmp := solve(tmp/alpha - difforder(uf), {alpha});
if tmp <> NULL and has(tmp, {alpha}) then lprint(tmp);
for k to nops([tmp]) while flag = 1 do
res := collect(expand(subs(tmp[k], Res)), f, factor);
if res = 0 then pl := subs(tmp[k], PL);
printHBT(uf, u, f, i, j, pl, rhs(tmp[k]));
flag := 0; fi;
od;
fi;
pl; end:

Bilinear := proc(uf, u, f, x, alpha) local su, h, i, j, g1, CB, PL, gdo, DD, Res;
if hasfun(uf, int) then error "Do not support integral function yet."
end:

f := f[ ]; c := '*(c[ ]); f := getvarpow(f); f := [c, f]; return f; fi;
if op(0, df) = diff then f := getvarpow(df); f := [1, f]; return f; fi;
lprint(df); error "unexpected type."
end:
Please substitute int function." fi; for j from 0 to 2 do
Res:=1; su:=u=2*alpha*diff(ln(f),[x$j]);
h:=collect(expand(dsubs(su,uf)),f,factor);
if hasfun(h,ln) then next; fi;
g1:=getFnumer(h,f)/2; g1:=expand(g1); CB:=HBF(g1);
gdo:=guessdifforder(CB,x);
for i from gdo[1] by 2 to gdo[2] do
if i=0 then PL:=CB; else PL:=HBF(int(g1,x$i)); fi;
DD:=add(PL[i][1]*BD(PL[i][2][1]),i=1..nops(PL));
Res:=collect(expand(diff(DD/f^2,diff(DD,f,x$i)))-h),f,factor);
if Res=0 then printHBT(uf,u,f,i,j,PL,alpha); break;
elif type(alpha,name) and has(DD,alpha) then
Res:=guessalpha(Res,uf,u,f,i,j,PL,alpha);
fi; od; if Res=0 then break; fi; od; PL; end:

8. Conclusions

To sum up, an algorithm for generating the Hirota bilinear form of NPDE with logarithm transformation has been proposed in this paper, and the bilinear forms of a class of NPDEs are obtained by using package Bilinear. Then we illuminate the availability of the algorithm by illustrating some examples.

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