GRADIENT FORMULA FOR LINEARLY SELF-INTERACTING BRANES

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Abstract

The computation of long range linear self-interaction forces in string and higher dimensional brane models requires the evaluation of the gradients of regularised values of divergent self-interaction potentials. It is shown that the appropriately regularised gradient in directions orthogonal to the brane surface will always be obtainable simply by multiplying the regularised potential components by just half the trace of the second fundamental tensor, except in the hypermembrane case for which the method fails. Whatever the dimension of the background this result is valid provided the codimension is two (the hyperstring case) or more, so it can be used for investigating brane-world scenarios with more than one extra space dimension.

1 Introduction

In recent studies of the linear self-interaction of superconducting and other cosmic string models in a standard (3+1) dimensional background spacetime, it has been found that the relevant divergences can be conveniently dealt with by the application of a simple universal gradient formula. This formula can be used to obtain the effective self-force from the appropriately regularized self-field and the precise form of this gradient formula does not depend on the physical nature of the fields to which the string may be coupled. These could, for example, be of the ordinary (observationally familiar) electromagnetic, and linearised gravitational kinds, but might also be of the theoretically predicted dilatonic and axionic kinds.

The purpose of this work is to generalise this gradient formula to higher dimensional branes in higher dimensional backgrounds. A particular motivation for doing this is to generalise the study of the original kind of brane-world scenario not just by dropping the postulate of reflection symmetry, but by allowing for more than one extra dimension, so as
to obtain models consisting of a 3-brane embedded in a spacetime of dimension greater than 5. More generally, our results are applicable to any $p$-brane in a $(q + 1)$ dimensional background spacetime, so long as the codimension $(q - p)$ is two (the “hyperstring” case, which includes ordinary strings in a (3+1) dimensional spacetime) or more. The method given here does however fail (due to long range “infrared” divergences) when the co-dimension is only one (the “hypermembrane” case, which includes ordinary membranes in a (3+1) dimensional spacetime), but in that case it is not really needed, since the short range “ultraviolet” divergences will have a relatively innocuous form that can be dealt with in terms of simple jump discontinuities.

The reason why one needs a gradient formula is that the relevant forces are typically obtained as gradients of potentials. Outside the brane worldsheet these potentials will be well behaved fields, but in the thin brane limit they will be singular on the worldsheet itself. One therefore needs to regularise these fields by some appropriate ultra-violet cut-off procedure, typically involving a length scale, $\epsilon$ say, that can be interpreted as characterising the underlying microstructure. When one has obtained the appropriately regularised potential on the worldsheet (which represents the cross-sectional average over an underlying microstructure) the next problem is to obtain the corresponding force by evaluating its gradient. The gradient components in directions tangential to the worldsheet can be obtained directly just by differentiating the regularised (macroscopic averaged) potential. The trouble is that, since the support of the regularised potential is confined to the worldsheet, differentiation in orthogonal directions is not directly meaningful.

To obtain the necessary orthogonal gradient components it is therefore necessary, in principle, to go back to the underlying microstructure and perform the ultra-violet limit process again, a rather arduous task that has frequently been performed independently by different authors in a wide range different physical contexts.

The present work has been prompted by the observation that, in the case of a string in an ordinary 3-dimensional background, the effect of the orthogonal gradient operation will always turns out to be equivalent just to multiplication of the regularised value of the relevant potential field by exactly half the extrinsic curvature vector $K_{\rho}$, which is defined to be the trace

$$K_{\rho} = g^{\mu\nu}K_{\mu\nu\rho},$$

of the second fundamental tensor $K_{\mu\nu\rho}$ of the worldsheet. The recognition of this simple universal rule makes it unnecessary to go back and perform the calculation over again every time such a problem arises in a different physical context. On the basis of dimensional considerations, and of the need to respect local Lorentz invariance, one would expect that multiplication by a factor proportional to $K_{\rho}$ would inevitably be what is required. But what is not so obvious in advance is whether the relevant proportionality factor should still always be exactly a half even for branes and backgrounds of higher dimensions.

The present work addresses this question and provides an unreservedly affirmative reply whenever $0 < p < q - 1$, on the basis of the consideration that whatever the relevant factor may be for a generic evolution of a brane of given space dimension $p$ in a back ground of given space (as opposed to spacetime) dimension $q$, this factor must evidently remain the same for any static configuration of the brane.

By restricting our attention to static configurations, for which a rigourous analysis is technically much simpler than in the dynamic case, what we have succeeded in showing here is that that – provided the codimension $(q - p)$ is two (the “hyperstring” case) or more, so that the quantities involved are at worst logarithmically divergent in the “infra red” (which excludes the
strongly divergent case of a membrane, but admits the case of a string in 3 dimensions) – the appropriate factor is indeed always exactly a half, no matter how high the dimension of the brane or the background. However our analysis also shows that this easily memorable result will no longer hold in the strongly “infra red” divergent extreme case of a “hypermembrane” (meaning a brane with codimension one, so that its supporting worldsheet is a hypersurface). What the present approach can not do is to provide any evidence at all, one way or the other, about value of the factor in question in the opposite extreme case of a zero-brane, meaning a simple point particle.

The article is organised as follows. In Section 2, we recall the definitions of the basic geometrical quantities, in particular the second fundamental tensor, that will appear in our discussion. We introduce the Poisson equation, both in the thin brane limit and for regularised configurations in Section 3 and discuss its solution in Section 4. The cross-sectional average of the field and its gradient have a zero-order contribution, in the case of a flat brane, that is evaluated in Section 5 and we illustrate our formalism on the choice of a canonical profile function in Section 6. The cross-sectional average of the gradient of the field vanishes in the flat configuration and our goal is to evaluate its value at first order in the curvature. After defining the procedure of averaging in Section 7, we study the effect of the bending of the brane on this procedure in Section 8. Section 9 finalises our demonstration by computing the cross-sectional average of the gradient of the field. A summary of the notation and some useful integrals are presented in the appendix.

2 Brane embedding geometry

Since it will play a central role in our analysis, it is worthwhile to recall the definition and some basic properties of the second fundamental tensor $K_{\mu\nu\rho}$.

We consider a $p$-dimensional surface (a $p$-brane) with internal coordinates $\sigma^i$ (with $i = 1 \ldots p$) smoothly embedded in a higher $q$-dimensional background endowed with a metric $g_{\mu\nu}$ with respect to coordinates $x^\mu$ (with $\mu = 1 \ldots q$). If the location of the surface is given in terms of a set of embedding functions $\bar{x}^\mu(\sigma^i)$, the internal surface metric, also referred to as first fundamental tensor, can be expressed as

$$\gamma_{ij} = g_{\mu\nu} \frac{\partial \bar{x}^\mu}{\partial \sigma^i} \frac{\partial \bar{x}^\nu}{\partial \sigma^j}. \quad (2)$$

Provided that this metric is non-singular (in the sense that the determinant of the component matrix does not vanish), as will always be the case for a strictly timelike surface in a Lorentz signature background spacetime, or for any surface if the background metric is positive definite, then it will have a well defined inverse with components $\gamma^{ij}$. It can be used to raise internal coordinate indices in the same way as the inverse background metric $g^{\mu\nu}$ is used for raising background coordinate indices. For any such non-singular embedding, $\gamma^{ij}$ can be mapped into the background tensor

$$\gamma^{\mu\nu} = \gamma^{ij} \frac{\partial \bar{x}^\mu}{\partial \sigma^i} \frac{\partial \bar{x}^\nu}{\partial \sigma^j}. \quad (3)$$

The corresponding mixed tensor $\gamma^\mu_\nu$ acts on vectors as the natural (rank $p$) surface tangential projector operator, while its (rank $q-p$) complement

$$\perp^\mu_\nu = g^\mu_\nu - \gamma^\mu_\nu, \quad (4)$$
acts similarly as the corresponding orthogonal projection operator. Fields with support confined to the \( p \)-surface \( \mathcal{S}^{(p)} \), such as \( \gamma_{\nu}^\mu \) and \( \perp_\mu^\nu \), can not be directly subjected to the unrestricted operation of partial differentiation with respect to the background coordinates, but only to differentiation in tangential directions, as performed by the operator

\[
\nabla_\mu = \gamma_\mu^\rho \nabla_\rho,
\]

where \( \nabla_\rho \) is the usual covariant derivative associated with \( g_{\mu\nu} \). The action of this differential operator on the first fundamental tensor defines the second fundamental tensor of the \( p \)-surface, according to the specification

\[
K_{\mu\nu\rho}^\sigma = \eta_{\nu\sigma} \nabla_\mu \eta_{\rho\sigma}.
\]

As a non-trivial integrability condition, it satisfies the generalised Weingarten symmetry condition

\[
K_{\mu\nu\rho}^\sigma = K_{\nu\mu\rho}^\sigma.
\]

It is also \( p \)-surface orthogonal on its last index and tangential on the other two, that is,

\[
K_{\mu\nu\sigma}^\eta = 0 = \perp_\sigma^\mu K_{\sigma\nu\rho},
\]

so that it has only a single non-identically vanishing self-contraction, namely the extrinsic curvature vector introduced in equation (1).

3 Regularisation of brane supported source distribution

Since we are essentially concerned with the treatment of “ultraviolet” regularisation, the effects of long range background curvature will be unimportant and can thus be neglected. As the framework for our analysis, it will be sufficient to consider a \( q \)-dimensional background space with a flat, static, positive definite metric \( g_{\mu\nu} \). It is therefore possible to choose a linear (but not necessarily orthogonal) system of space coordinates \( x^\mu \) so that the background metric will be constant, that is, \( \partial g_{\mu\nu} / \partial x^\rho = 0 \).

This choice has different implications. First, the tangentially projected covariant derivative will take the simple form

\[
\nabla_\mu = \gamma_\mu^\rho \frac{\partial}{\partial x^\rho} = g_{\mu\nu} \gamma_{ij} \frac{\partial \bar{x}^\nu}{\partial \sigma^i} \frac{\partial}{\partial \sigma^j},
\]

Second, in the linear system (for example Maxwellian or linearised gravitational), with which we are concerned here, each tensorial component decouples and evolves independently like a simple scalar field. Thus, it will be sufficient for our purpose to concentrate on the prototype problem of finding a scalar field, let us say \( \phi \{x\} \) (introducing the systematic use of curly brackets to indicate functional dependence), that satisfies the corresponding generalised Poisson equation to which the linear wave equation will be reduced in the static case to be dealt with here. Third, the background spacetime is assumed to be static so that the generalised Laplacian \( \nabla_\mu \nabla_\mu \) is to be regarded as as the static projection of what would be a Dalmertian type wave operator in a more complete \((p + 1)\)-dimensional spacetime description.
3.1 Generalised Poisson equation and its solutions in the thin brane limit

In unrationalised units for, let us say, a repulsive scalar source distribution $\rho\{x\}$ the generalised Poisson equation takes the form

$$\nabla^\mu \nabla_\mu \phi = -\Omega^{[q-1]} \rho\{x\},$$

(10)

where the Laplacian operator is simply given by

$$\nabla^\mu \nabla_\mu = g^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu},$$

(11)

where $\Omega^{[q-1]}$ is the surface area of a unit $(q-1)$-sphere in the $q$-dimensional background. The latter is given by the well known formula

$$\Omega^{[q-1]} = \frac{2\pi^{q/2}}{\Gamma\{q/2\}},$$

(12)

in which $\Gamma\{z\}$ is the usual Eulerian Gamma function (as specified in the appendix) which satisfies the recursion relation $\Gamma\{n+1\} = n\Gamma\{n\}$ and has the particular values $\Gamma\{1\} = 1$ and $\Gamma\{1/2\} = \sqrt{\pi}$. Thus in particular we shall have $\Omega^{[0]} = 2$, $\Omega^{[1]} = 2\pi$, $\Omega^{[2]} = 4\pi$, $\Omega^{[3]} = 2\pi^2$ and $\Omega^{[4]} = 8\pi^2/3$.

The motivation for working with a $p$-brane model of a physical system is of course to be able to use an economical description in which as many as possible of the fields involved are specified just on the supporting surface. Evidently, this requires less information than working with fields specified over the higher dimensional background. However, when long range interactions are involved then, as a price for such an economy, divergences are to be expected if the source fields are considered to be strictly confined to the supporting $p$-surface. To be more specific, the brane supported source distribution $\bar{\rho}\{\sigma\}$ (using a bar as a reminder to indicate quantities that are undefined off the relevant supporting surface) will correspond to the background source distribution

$$\rho\{x\} = \int \delta^{[q]}\{x, \bar{x}\{\sigma\}\} \bar{\rho}\{\sigma\} \, dS^{[q]},$$

(13)

where $\delta^{[q]}\{x, \bar{x}\{\sigma\}\}$ is the $q$-dimensional bi-scalar Dirac distribution. This is characterised by the condition that it vanishes wherever the evaluation points $x^\mu$ and $\bar{x}^\mu\{\sigma\}$ are different while nevertheless satisfying the unit normalisation condition

$$\int \delta^{[q]}\{x, \bar{x}\} \, dS^{[q]} = 1.$$

(14)

We use the notations $dS^{[p]}$ and $dS^{[q]}$ respectively for the $p$-dimensional brane surface measure and $q$-dimensional background space measure. The latter is given by an expression of the form

$$dS^{[q]} = |g|^{1/2} d^{[q]} x,$$

(15)

where the $q$-dimensional volume measure factor $|g|^{1/2}$ is just the square root of the determinant of the (positive definite) matrix of metric components $g_{\mu\nu}$. Similarly, we have

$$dS^{[p]} = |\gamma|^{1/2} d^{[p]} \sigma,$$

(16)

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where the $p$-dimensional volume measure factor $|\gamma|^{1/2}$ is the square root of the determinant of the matrix of surface metric components, as given by (2). The normalisation condition (14) is such that the measure factor means that the Dirac distributions specified in this way will transform as an ordinary bi-scalar, a property that distinguishes it from the corresponding Dirac type that would be given by the commonly used alternative convention in which the measure factor is omitted. This specifies distributions that behave not as ordinary bi-scalars but as weight one bi-scalar densities. Of course it does not matter which of these conventions is used if, as will be done below, the coordinates are restricted to be of the standard orthonormal type with respect to which the metric just has the Cartesian unit diagonal form.

For the source distribution given in the thin brane limit by (13), the corresponding solution of the linear field equation (10) is given in terms of a Green function by the expression

$$
\phi\{x\} = \int G^{[q]}\{x, \bar{x}\{\sigma}\} \bar{\rho}\{\sigma\} \, dS^{[p]},
$$

while the corresponding scalar field gradient – which is what determines the resulting force density – will be given by

$$
\frac{\partial \phi}{\partial x^\mu}\{x\} = \int \frac{\partial G^{[q]}\{x, \bar{x}\{\sigma}\}}{\partial x^\mu} \bar{\rho}\{\sigma\} \, dS^{[p]}.
$$

The Green function $G^{[q]}\{x, \bar{x}\{\sigma}\}$ satisfies the Laplace equation

$$
\nabla_\mu \nabla^\mu G^{[q]}\{x, \bar{x}\{\sigma}\} = -\Omega^{[q-1]} \delta^{[q]}\{x, \bar{x}\{\sigma}\}
$$

subject to the appropriate boundary conditions, which in the standard case for an isolated system will simply consist of the requirement that the field should tend to zero at large distances from the source. Like the Dirac “function”, it is a bi-scalar (not a bi-scalar density) that is well behaved as a distribution, but singular as a function wherever the two points on which it depends are coincident. Note that the Green function is determined up to an harmonic function, that is a solution of the homogeneous Poisson equation $\nabla_\mu \nabla^\mu G^{[q]}\{x, \bar{x}\{\sigma}\} = 0$, with the appropriate boundary conditions. In particular $G^{[q]}$ is determined up to an additive constant that induces a global shift in the potential and is not relevant for the computation of its gradient.

### 3.2 From the thin brane to a regular thick brane

Although convenient for many formal calculational purposes, the drawback to the use of the strict thin brane limit as represented in terms of such singular Dirac and Green distributions is that the ensuing value of the corresponding field $\phi$ and its gradient $\partial \phi / \partial x^\mu$ will also be singular on the brane $p$-surface, which unfortunately is just where one needs to evaluate them for purposes such as computing the corresponding energy and force density.

The obvious way to get round this difficulty – without paying the cost of introducing extra degrees of freedom – is to replace (13) by an expression of the form

$$
\bar{\rho}_{\{\epsilon\}}\{x\} = \int D^{[q]}_{\{\epsilon\}}\{x, \bar{x}\{\sigma}\} \bar{\rho}\{\sigma\} \, dS^{[p]},
$$

in which the strict Dirac distribution $\delta^{[q]}\{x, \bar{x}\}$ is replaced by a regular bi-scalar profile function, $D^{[q]}_{\{\epsilon\}}\{x, \bar{x}\}$ say, characterised by some sufficiently small but finite smoothing length scale, $\epsilon$ say,
which is subject to the normalisation condition
\[
\int D_{(c)}^{[q]} \{ x, \tilde{x} \} \, dS_{(c)}^{[q]} = 1. \tag{21}
\]

In order to respect the geometric isotropy and homogeneity of the background space, this regular “profile function” \( D_{(c)}^{[q]} \{ x, \tilde{x} \} \) can depend only on the relative distance, \( s \) say, that is defined – in the flat background under consideration – by
\[
s^2 = g_{\mu\nu} (x^\mu - \tilde{x}^\mu)(x^\nu - \tilde{x}^\nu), \tag{22}
\]
according to some ansatz of the form
\[
D_{(c)}^{[q]} \{ x, \tilde{x} \} = D_{(c)}^{[q]} \{ s^2 \}. \tag{23}
\]

The functionally singular Dirac distribution can be considered as the limit of such a function as the smoothing lengthscale tends to zero. Such a description corresponds to what may be described as “fuzzy” brane model characterised by an effective thickness whose order of magnitude will be a function of the value of the regularisation parameter \( \epsilon \). For some kinds of physical application one might wish to adjust the regularisation ansatz so to make it agree as well as possible with the actual internal structure (if known) of the extended physical system under consideration. The spirit of the present work is, however, rather to treat the regularisation parameter just as a provisional freely variable quantity for use at an intermediate stage of an analysis process whereby it would ultimately be allowed to tend to zero so as to provide a strict “thin” brane limit description. In order for our results to be meaningful for this latter purpose, it is important that they should be insensitive to the particular choice of the smoothing ansatz that is postulated. We shall take care to verify this

4 Field solution for regularised source distribution

For any source distribution of the regularised form (20), the function, \( G_{(c)}^{[q]} \{ s^2 \} \) that is the solution (with the appropriate boundary conditions) of the corresponding generalisation
\[
\nabla^\mu \nabla_\mu G_{(c)}^{[q]} \{ s^2 \} = -\Omega^{[q-1]} \tilde{D}_{(c)}^{[q]} \{ s^2 \}, \tag{24}
\]
of (19) will act as a generalised Green function. The solution of the Poisson equation (10) for the generic smoothed source distribution (20) will thus be given, using the natural abbreviation
\[
s\{ x, \sigma \} = s\{ x, \tilde{x}\{ \sigma \} \}, \tag{25}
\]
by
\[
\tilde{\phi}_{(c)} \{ x \} = \int G_{(c)}^{[q]} \{ s\{ x, \sigma \} \} \tilde{\rho}(\sigma) \, dS_{[p]}^{[\varphi]} . \tag{26}
\]

The gradient of this scalar field is evidently given by
\[
\frac{\partial \tilde{\phi}_{(c)} \{ x \}}{\partial x^\mu} \{ x \} = \int \frac{\partial G_{(c)}^{[q]} \{ s^2 \{ x, \sigma \} \}}{\partial x^\mu} \{ x, \tilde{x}\{ \sigma \} \} \tilde{\rho} \{ \sigma \} \, dS_{[p]}^{[\varphi]}, \tag{27}
\]
in which the integrand can be evaluated using the formula
\[
\frac{\partial G^{[q]}_{(\epsilon)}}{\partial x^\mu} = 2 s^\mu \frac{dG^{[q]}_{(\epsilon)}}{ds^2},
\] (28)
where \(s^\mu = x^\mu - \bar{x}^\mu\). One can evidently go on in the same way to obtain the second derivative
\[
\frac{\partial^2 \tilde{\phi}^{[q]}_{(\epsilon)}}{\partial x^\mu \partial x^\nu} \{x\} = \int \frac{\partial^2 G^{[q]}_{(\epsilon)}}{\partial x^\nu \partial x^\rho} \{s^2 \{x, \sigma\}\} \bar{\rho} \{\sigma\} d\mathcal{S}^{[q]}.
\] (29)
in which the integrand can be evaluated using the formula
\[
\frac{\partial^2 G^{[q]}_{(\epsilon)}}{\partial x^\mu \partial x^\nu} = 2 g_{\mu\nu} \frac{dG^{[q]}_{(\epsilon)}}{ds^2} + 4 s_\mu s_\nu \frac{d^2 G^{[q]}_{(\epsilon)}}{(ds^2)^2}.
\] (30)

It is to be noted that, by contraction of (30), the Laplace equation (24) for \(G^{[q]}_{(\epsilon)}\) takes the form
\[
\frac{d}{ds^2} \left(s^q \frac{dG^{[q]}_{(\epsilon)}}{ds^2}\right) = - \frac{\Omega^{[q-1]}}{4} s^{q-2} D^{[q]}_{(\epsilon)} \{s^2\}.
\] (31)
If one introduces the monotonically increasing function \(I^{[q]}_{(\epsilon)} \{s^2\}\) by
\[
I^{[q]}_{(\epsilon)} \{s^2\} \equiv \frac{\Omega^{[q-1]}}{4} \int_0^s u^{q-1} D^{[q]}_{(\epsilon)} \{u^2\} \, du,
\] (32)
then it will satisfy
\[
\frac{dI^{[q]}_{(\epsilon)} \{s^2\}}{ds^2} = \frac{\Omega^{[q-1]}}{2} s^{q-2} D^{[q]}_{(\epsilon)} \{s^2\}.
\] (33)
By substituting into (31), it follows that we shall have
\[
\frac{dG^{[q]}_{(\epsilon)}}{ds^2} = - \frac{I^{[q]}_{(\epsilon)} \{s^2\}}{2s^q},
\] (34)
and hence
\[
\frac{d^2 G^{[q]}_{(\epsilon)}}{(ds^2)^2} = \frac{q I^{[q]}_{(\epsilon)} \{s^2\}}{4s^{q+2}} - \frac{\Omega^{[q-1]} D^{[q]}_{(\epsilon)} \{s^2\}}{4s^2}.
\] (35)
The normalisation condition (21) implies by (32) that the function \(I^{[q]}_{(\epsilon)} \{s^2\}\) should satisfy the condition
\[
I^{[q]}_{(\epsilon)} \{s^2\} \to 1
\] (36)
in the limit \(s^2 \to \infty\), for all \(\epsilon\). It follows that any admissible ansatz for the smoothing profile will entail an asymptotic behaviour characterised by
\[
\frac{dG^{[q]}_{(\epsilon)}}{ds^2} \sim - \frac{1}{2s^q}
\] (37)
as $s^2 \to \infty$. The corresponding asymptotic formula for the regularised Green function $G_{(s)}^{[q]} \{ s^2 \}$ will therefore be given, as long as $q > 2$, by

$$G_{(s)}^{[q]} \{ s^2 \} \sim \frac{s^{2-q}}{q-2}$$

(38)
as $s \to \infty$. In the particular case $q = 2$, the integral over $s^2$ is marginally convergent, with asymptotic behaviour characterised by

$$G_{(s)}^{[2]} \{ s^2 \} \sim -\frac{1}{2} \ln\{ s^2 \}$$

(39)
as $s^2 \to \infty$.

5 Evaluation for flat brane configurations

Before considering the effect of the curvature of the brane, it will be instructive to apply the preceding formulae to the case of a flat $p$-brane configuration supporting a uniform source distribution.

The uniformity condition means that the value of $\bar{\rho}$ is independent of the internal coordinates $\sigma^i$, and the flatness condition means that the embedding mapping $\sigma^i \mapsto \bar{x}^{\mu}\{\sigma^i\}$ of the brane $p$-surface will be expressible in a system of suitably aligned Cartesian background space coordinates $\{\bar{x}^{\mu}\} = \{\bar{z}^i, \bar{r}^a\}$ (with $\mu = 1 \ldots q, i = 1 \ldots p, a = p+1 \ldots q$) in terms of a corresponding naturally induced system of internal coordinates $\sigma^i$ simply by

$$\bar{z}^i = \sigma^i, \quad \bar{r}^a = 0.$$ 

(40)

With such a coordinate system, the background metric components will be given by

$$g_{ij} = \gamma_{ij}, \quad g_{ia} = 0, \quad g_{ab} = \perp_{ab},$$ 

(41)

where $\gamma_{ij}$ and $\perp_{ab}$ are unit matrices of dimension $p$ and $(q-p)$ respectively.

The intrinsic uniformity of the configuration ensures that physical quantities depend only on the external coordinates $\bar{r}^a$, and the isotropy of the system ensures that scalar quantities can depend only on the radial distance $r$ from the brane $p$-surface given by

$$r^2 = \perp_{ab} r^a r^b.$$ 

(42)

In particular, the distance $s$ from a point with coordinates $x^{\mu}$ on the transverse $(q-p)$-plane through the origin (that is, with $\sigma^i = 0$) to a generic point with coordinates $\sigma$ on the flat $p$-brane locus will be given simply by

$$s^2\{x, \sigma\} = r^2 + \sigma^2,$$ 

(43)

where

$$\sigma^2 = \gamma_{ij} \sigma^i \sigma^j.$$ 

(44)

It can be seen that for a given fixed value of the source density $\bar{\rho}$ on the brane, the prescription (20) will provide a radial dependence of the form

$$\bar{\rho}\{ r^2 \} = \perp^{(p,q)}_{(s)} \{ r^2 \} \bar{\rho},$$ 

(45)
where the dimensionally reduced radial profile function \( \perp D_{(c)}^{[q,p]} \{ r^2 \} \) is given in terms of the original \( q \)-dimensional profile function \( D_{(c)}^{[q]} \{ s^2 \} \) by the expression

\[
\perp D_{(c)}^{[q,p]} \{ r^2 \} = \Omega_{(p-1)}^{[-p]} \int_{0}^{\infty} \sigma^{p-1} \perp D_{(c)}^{[q]} \{ \sigma^2 + r^2 \} \, d\sigma .
\]  

(46)

This integral will always be convergent as a consequence of the rapid asymptotic fall-off condition that must be satisfied by the function \( D_{(c)}^{[q]} \{ s^2 \} \) in order for it to obey the normalisation condition (21), which implies that

\[
\Omega_{(q-p-1)}^{[-p-1]} \int_{0}^{\infty} r^{q-p-1} \perp D_{(c)}^{[q,p]} \{ r^2 \} \, dr = 1 .
\]  

(47)

Since (37) implies that the associated regularised Green function \( G_{(c)}^{[q]} \{ s^2 \} \) will satisfy the comparatively weak asymptotic fall off condition (38), the field \( \tilde{\phi}_{(c)} \) will be given by

\[
\tilde{\phi}_{(c)} \{ r^2 \} = \rho \perp G_{(c)}^{[q,p]} \{ r^2 \} ,
\]  

(48)

using (26), where the dimensionally reduced radial Green function \( \perp G_{(c)}^{[q,p]} \{ r^2 \} \) is defined by

\[
\perp G_{(c)}^{[q,p]} \{ r^2 \} \equiv \Omega_{(p-1)}^{[-p]} \int_{0}^{\Delta} \sigma^{p-1} G_{(c)}^{[q]} \{ \sigma^2 + r^2 \} \, d\sigma ,
\]  

(49)

in the limit where \( \Delta \rightarrow \infty \), a limit that will be convergent only if \( q > p + 2 \).

In the prototype case \([1,2,3]\) of an ordinary string with \( p = 1 \) in a background of space dimension \( q = 3 \), the latter is only marginally too low compared with the dimension of the brane surface to achieve convergence. In that case, as in the more general hyperstring case, that is to say whenever \( q = p + 2 \), it is still possible to obtain a useful asymptotic formula by taking the upper limit \( \Delta \) to be a finite “infra-red” cut off length. It can be seen from (38) that in this marginally divergent codimension-2 case the resulting asymptotic behaviour will be given by

\[
\perp G_{(c)}^{[q,q-2]} \{ r^2 \} \sim \frac{\Omega_{(q-3)}^{[-q+3]}}{q-2} \ln \left( \frac{\Delta}{\sqrt{r^2 + \epsilon^2}} \right) ,
\]  

(50)

as \( \Delta \rightarrow \infty \).

One might go on to try obtain an analogous formula for what, as far as “infra-red” divergence behaviour is concerned, is the worst possible case of all, namely that of a hypermembrane (or “wall”) meaning the hyper-surface supported case characterised by \( q = p + 1 \). However in this codimension-1 case, the divergence will have the linear form

\[
\perp G_{(c)}^{[q,q-1]} \{ r^2 \} \sim \frac{\Omega_{(q-2)}^{[-q+2]}}{q-2} \Delta ,
\]  

(51)

which is particularly sensitive to the choice of cut-off \( \Delta \) and hence is not very useful.

For physical purposes the potential \( \tilde{\phi}_{(c)} \) will typically be less important than its (not so highly gauge dependent) gradient, which is what will determine the relevant force density. In the flat and uniform configuration considered in this section, the gradient \( \partial \tilde{\phi}_{(c)} / \partial x^\mu \) is not liable to the infrared divergence problems that beset the undifferentiated field when the background dimension \( p \) is not sufficiently high compared with the brane dimension \( q \). It is evident from the
uniformity of the configuration that the gradient components in directions parallel to the brane
$p$-surface will vanish, that is, we shall have $\partial \delta (\phi) / \partial z^i = 0$, while starting from the expression
\[ (27), \] it can be concluded that the orthogonal gradient components will be given by
\begin{equation}
\frac{\partial \phi^{(i)}}{\partial r^a} \{ r^2 \} = \bar{\rho} \frac{\partial \tilde{G}^{[q,p]}_{(i)}}{\partial r^a} \{ r^2 \}
\end{equation}
(52)
where, by Eq. \[ (44), \] even in the cases for which the undifferentiated integral \[ (49) \] would be
divergent, the required derivative is given by a well defined formula of the form
\begin{equation}
\frac{\partial \tilde{G}^{[q,p]}_{(i)}}{\partial r^a} \{ r^2 \} = -r_a J^{[q,p]}_{(i)} \{ r^2 \},
\end{equation}
in which the function $J^{[q,p]}_{(i)} \{ r^2 \}$ is defined as the integral
\begin{equation}
J^{[q,p]}_{(i)} \{ r^2 \} = \Omega^{[p-1]} \int_0^\infty \sigma^{p-1} \frac{I^{[q]}_{(i)}}{(r^2 + \sigma^2)q/2} \ d\sigma,
\end{equation}
(54)
which will always converge by virtue of \[ (36). \]
In a similar manner, starting from \[ (29), \] it can be seen that the corresponding second deriva-
tive components will be given in terms of the same convergent integral $J^{[q,p]}_{(i)} \{ r^2 \}$ by
\begin{equation}
\frac{\partial^2 \tilde{\phi}^{(i)}}{\partial r^a \partial r^b} \{ r^2 \} = \bar{\rho} \frac{\partial^2 \tilde{G}^{[q,p]}_{(i)}}{\partial r^a \partial r^b} \{ r^2 \}
\end{equation}
(55)
in which, by the definition \[ (43), \] it can be seen that we shall have
\begin{equation}
\frac{\partial^2 \tilde{G}^{[q,p]}_{(i)}}{\partial r^a \partial r^b} \{ r^2 \} = 2\Omega^{[p-1]} \int_0^\Delta \sigma^{p-1} \left( \frac{1}{\partial^{ab}_{(i)}} \frac{dG^{[q]}_{(i)}}{ds^2} + 2r_a r_b \frac{d^2 G^{[q]}_{(i)}}{(ds^2)^2} \right) \{ \sigma^2 + r^2 \} \ d\sigma,
\end{equation}
(56)
in which it is to be recalled that the use of curly brackets (as in the expression $\{ \sigma^2 + r^2 \}$
at the end) indicates functional dependence (not simple multiplication). Using \[ (35) \] to express
the second derivative in the integrand as
\begin{equation}
\frac{d^2 G^{[q]}_{(i)}}{(ds^2)^2} = \frac{(p-q) dG^{[q]}_{(i)}}{2\sigma^2} - \frac{\Omega^{[q-1]}_{(i)} p^{[q]}_{(i)}}{4\sigma^2} \left( \Omega^{[q-1]}_{(i)} \{ s^2 \} - \frac{\sigma^{2-p}}{r^2} \frac{d}{d\sigma} \left( \sigma^p \frac{dG^{[q]}_{(i)}}{ds^2} \right) \right),
\end{equation}
(57)
and observing that the asymptotic behaviour \[ (37) \] ensures that the last term in \[ (57) \] will con-
tribute nothing to the integral on the right of \[ (56), \] we finally obtain an expression of the
convenient form
\begin{equation}
\frac{\partial^2 \tilde{G}^{[q,p]}_{(i)}}{\partial r^a \partial r^b} \{ r^2 \} = -r_a r_b \Omega^{[q-1]}_{(i)} \frac{dG^{[q]}_{(i)}}{ds^2} + \left[ (q-p) \frac{r_a r_b}{r^2} - \partial^{ab}_{(i)} \right] J^{[q,p]}_{(i)} \{ r^2 \},
\end{equation}
(58)
in which $\hat{D}^{[q,p]}_{(i)} \{ r^2 \}$ is the dimensionally reduced profile function given by the integral \[ (46). \]
6 Illustration using a canonical profile function

Our main results will be derived for any generic smoothing ansatz, without restriction to any particular form for the profile function \( D_{\{q\}}^{\{s^2\}} \). However as a concrete illustration, it is instructive to consider the case of what we shall refer to as the “canonical” regularisation ansatz whereby the \( q \)-dimensional Dirac delta “function” \( \delta_{\{q\}}^{\{s^2\}} \) and the associated Green function \( G_{\{q\}}^{\{s^2\}}(x, \bar{x}) \) are considered as the respective limits – as the regularisation parameter \( \epsilon \to 0 \) – of one-parameter families of smooth bi-scalar functions

\[
D_{\{q\}}^{\{s^2\}} \{x, \bar{x}\} = \delta_{\{q\}}^{\{s^2\}}(x, \bar{x}), \quad \text{and} \quad G_{\{q\}}^{\{s^2\}} \{x, \bar{x}\} = G_{\{q\}}^{\{s^2\}}(x, \bar{x})
\]

that are characterised in terms of the squared distance \( s^2 \{x, \bar{x}\} \) and of the smoothing lengthscale \( \epsilon \) by the prescription

\[
\delta_{\{q\}}^{\{s^2\}} = \frac{\epsilon^2 q}{\Omega_{q-1}} (s^2 + \epsilon^2)^{-(q+2)/2}.
\]

This canonical profile function evidently satisfies \( \delta_{\{q\}}^{\{s^2\}} \to 0 \) when \( \epsilon^2 \to \infty \) for \( s^2 \neq 0 \), as well as the unit normalisation condition (21) using the fact that \( dS^{[q]} = \Omega_{q-1} s^{q-1} ds \). By expressing the canonical profile function as

\[
\delta_{\{q\}}^{\{s^2\}} = \frac{2}{\Omega_{q-1}} s^{2-q} \frac{d}{ds^2} \left[ \left( \frac{s^2}{s^2 + \epsilon^2} \right)^{q/2} \right],
\]

it can be seen that the resulting integral (32) will simply be given by

\[
I_{\{q\}}^{\{s^2\}} = \left( \frac{s^2}{s^2 + \epsilon^2} \right)^{q/2}.
\]

It is then straightforward, by integrating (34), to show that the corresponding regularised Green function, satisfying by the Poisson equation (24), will be given by the canonical Green function \( G_{\{q\}}^{\{s^2\}} \) defined by

\[
G_{\{q\}}^{\{s^2\}} = \frac{1}{(q - 2)}(s^2 + \epsilon^2)^{-(q-2)/2},
\]

as long as \( q > 2 \). As explained above, it is determined up to a constant that is fixed by the fall off requirement at infinity which is mandatory for the convergence of the integrals considered below but that will not be relevant for the gradients to be computed later on. In the marginally convergent case where \( q = 2 \), one finds that

\[
G_{\{q\}}^{[2]} = -\frac{1}{2} \ln \left( 1 + \frac{s^2}{\epsilon^2} \right).
\]

For a specific physical system this particular choice is not necessarily the ansatz that will be most accurate for the purpose of providing a realistic representation of its actual internal microstructure. However this “canonical” prescription – as characterised by the formulae (62) and (63) – is distinguished from other conceivable alternatives by the very convenient property
of having an analytic form that is preserved by the dimensional reduction procedure. The associated radial profile function is simply given by

$$\mathcal{D}_{\{q,p\}} \{r^2\} = \frac{\Omega^{\{p-1\}}}{\Omega^{\{q-1\}}} \frac{e^2 q}{(e^2 + r^2)^{(q-p+2)/2}} \int_0^\infty \frac{u^{p-1}}{(1 + u^2)^{(q+2)/2}} du.$$  (65)

Since we shall always have \(q > p - 2\), the latter integral will always be convergent. Using the well known properties (see (118) in the appendix) of the Euler Beta function

$$B\left\{\frac{p}{2}, \frac{(q - p)}{2} + 1\right\}$$

and (again using the formulae in the appendix) that the result will be expressible in terms of the Beta function as

$$\mathcal{S}_{\{q,p\}} \{r^2\} = \frac{\Omega^{\{q-p\}}}{2(q + e^2)^{(q-p)/2-1} B\left\{\frac{p}{2}, \frac{q - p}{2} - 1\right\}}.$$  (67)

According to (49) the corresponding dimensionally reduced Green functions will have a canonical form

$$\mathcal{G}_{\{q,p\}} \{r^2\} = \mathcal{G}_{\{q,p\}} \{r^2\}$$  (68)

that will similarly be given whenever they are well defined – that is whenever \(q > p + 2\) – by expressions of the form

$$\mathcal{G}_{\{q,p\}} \{r^2\} = \frac{\Omega^{\{q - 1\}}}{\Omega^{\{q-p-1\}}} \mathcal{G}_{\{q,p\}} \{r^2\}.$$  (70)

The form of the Green function, unlike that of the canonical profile function, is thus not exactly preserved by the dimensional reduction process but the only difference is an overall volume factor. The preceding expression is only valid when \(q > p + 2\), but in the marginally convergent hyperstring case where \(q = p + 2\) one can use the expression (64) for the canonical Green function to obtain the asymptotic relation

$$\mathcal{G}_{\{q,p\}} \{r^2\} \sim \frac{\Omega^{\{q - 3\}}}{(q - 2)} \ln \left\{\frac{\Delta}{\sqrt{r^2 + e^2}}\right\}$$  (71)

as \(\Delta \to \infty\). It is to be noted that the logarithmic term in this formula and in (50) will be expressible by (64) in the form

$$\ln \left\{\frac{\Delta}{\sqrt{r^2 + e^2}}\right\} = \mathcal{G}_{\{q,p\}} \{r^2\} + \ln \left\{\frac{\Delta}{\epsilon}\right\}$$  (72)

in which the first term is independent of the cutoff \(\Delta\). This means that in the asymptotic limit the dependence on \(r\) will drop out, so that we shall be left simply with

$$\mathcal{G}_{\{q,p\}} \{r^2\} \sim \frac{\Omega^{\{q - 3\}}}{(q - 2)} \ln \left\{\frac{\Delta}{\epsilon}\right\}$$  (73)

as \(\Delta \to \infty\).
7 Source weighted averages in flat canonical case

For the purposes of a macroscopic description, what really matters is not the detailed field distribution in a smoothed microscopic treatment as developed in the previous sections, but only its effective averaged values. For instance, in the example of the scalar field \( \tilde{\phi}_{(\epsilon)} \), we shall be interested in its cross-sectional average defined as

\[
\left\langle \tilde{\phi}_{(\epsilon)} \right\rangle \equiv \Omega^{[q-p-1]} \int_0^\infty r^{q-p-1} \tilde{\phi}_{(\epsilon)} \{r\} w\{r\} \, \text{d}r ,
\]

(74)

where \( w\{r\} \) is a weighting factor dependent only on the radial coordinate \( r \) subject to the normalization condition

\[
\Omega^{[q-p-1]} \int_0^\infty r^{q-p-1} w\{r\} \, \text{d}r = 1 .
\]

(75)

Let us first consider the flat uniform configurations studied in Section 5. Whatever the explicit form of the function \( w\{r\} \), it is obvious that the isotropy of the configuration will ensure that the average of the field gradient \((52)\) will cancel out, meaning that we shall have

\[
\left\langle \frac{\partial \tilde{\phi}_{(\epsilon)}}{\partial r^a} \right\rangle = 0 .
\]

Using the general relation

\[
\left\langle r^a r_b f\{r\} \right\rangle = \frac{1}{q-p} \, \perp_{ab} \left\langle \frac{r^2}{f\{r\}} \right\rangle ,
\]

(76)

which applies to any purely radial function \( f\{r\} \), it can thus be seen from \((55)\) and \((58)\), using the expression \((45)\) for the radial source density distribution, that

\[
\left\langle \frac{\partial^2 \tilde{\phi}_{(\epsilon)}}{\partial r^a \partial r^b} \right\rangle = -\frac{\Omega^{[q-1]}}{q-p} \left\langle \tilde{\rho}_{(\epsilon)} \right\rangle \, \perp_{ab} ,
\]

(77)

in which the value of \( \left\langle \tilde{\rho}_{(\epsilon)} \right\rangle \) also depends on the choice of the weighting function. It is to be remarked that (as a useful check on the preceding algebra) the trace this equation gives back the original Poisson equation \((54)\).

The kinds of energy and force contributions that are relevant in physical applications will typically be proportional to the product of the linear field \( \phi \) with the corresponding source term. This implies that the weighting of the field \( \tilde{\phi}_{(\epsilon)} \) considered here should be proportional to the source term \( \tilde{\rho}_{(\epsilon)} \). For a flat configuration, it can thus be deduced from \((13)\) that the appropriate weighting factor should be explicitly given by the formula

\[
w\{r\} = \frac{1}{D_{(\epsilon)}^{[q,p]}} \{r^2\} ,
\]

(78)

and hence that the average of the source term will be given by an expression of the form

\[
\left\langle \tilde{\rho}_{(\epsilon)} \right\rangle = \tilde{\rho} \Omega^{[q-p-1]} \int_0^\infty \left[ \frac{1}{D_{(\epsilon)}^{[q,p]}} \{r^2\} \right]^2 \, r^{q-p-1} \, \text{d}r ,
\]

(79)

which will be valid, whatever the profile function, as long as the configuration under consideration is flat. The corresponding expression for the average of the field \( \phi \) will have the form

\[
\left\langle \tilde{\phi}_{(\epsilon)} \right\rangle = \tilde{\phi} \Omega^{[q-p-1]} \int_0^\infty \frac{1}{D_{(\epsilon)}^{[q,p]}} \{r^2\} \frac{1}{G_{(\epsilon)}^{[q,p]}} \{r^2\} r^{q-p-1} \, \text{d}r .
\]

(80)
In the special canonical case introduced in Section 6, the profile factor will be given by (60), so that the (always convergent) integral (79) will be obtainable explicitly – as shown in the appendix – in terms of the Beta function. This leads to an expression of the form

$$\langle \rho \{\epsilon\} \rangle = \left( q - p \right) \left( q - p - 2 \right) \frac{\bar{\rho}}{\epsilon^{q - p}}$$

for the average source density. With further use of the formulae in the appendix, this can be written even more explicitly as

$$\langle \rho \{\epsilon\} \rangle = \frac{1}{16\pi \frac{q - p}{2}} \frac{(q - p)^2(q - p + 2)}{(q - p + 1)} \frac{\left[ \Gamma\{(q - p)/2\} \right]^3 \bar{\rho}}{\Gamma\{q - p\} \epsilon^{q - p}}.$$  

The simplest application of this result is to the case of a hypermembrane or “wall”, meaning a hypersurface forming brane as characterised by $q - p = 1$, for which the formula (82) just gives

$$\langle \rho \{\epsilon\} \rangle = \frac{3}{32\pi} \frac{\bar{\rho}}{\epsilon^{q - p}}.$$  

In terms of the canonically reduced Green functions (68), it it can be seen that the average of the regularised field $\phi \{\epsilon\}$ will be given in the flat case by a prescription of the form

$$\langle \phi \{\epsilon\} \rangle = \bar{\rho} \Omega [q^{-p - 2}] \int_{0}^{\infty} \frac{1}{r^{q - 2}} \frac{1}{\delta \{r \}^{q - p - 2}} \frac{\Gamma\{(q - p)/2\}^3}{\Gamma\{q - p\}} \frac{\bar{\rho}}{\epsilon^{q - p}}.$$  

Whenever the expression (70) is valid, that is whenever $q > p + 2$, the formulae in the appendix will provide a result expressible as

$$\langle \phi \{\epsilon\} \rangle = \frac{\bar{\rho} \Omega [q^{-p - 2}] \Gamma\{(q - p)/2\}^3}{\Gamma\{q - p\}} \frac{\bar{\rho}}{\epsilon^{q - p}}.$$  

The preceding formula will no longer be valid, but an analogous though weakly cutoff dependent expression will still be obtainable in in the marginal hyperstring case, when $q = p + 2$. In this case, instead of (70), the effective dimensionally reduced Green function will be given by the simple asymptotic formula (73) in the large value limit for the relevant infrared cutoff $\Delta$. The ensuing asymptotic formula for the averaged value of $\phi \{\epsilon\}(r)$ in a hyperstring is thus immediately seen to be given by

$$\langle \phi \{\epsilon\} \rangle \sim \frac{\bar{\rho} \Omega [q^{-3}] \Gamma\{q/2\}}{2(q - 2) \ln \left( \frac{\Delta}{\epsilon} \right)}$$

as $\Delta \to \infty$. 

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8 Allowance for curvature in the averaging process

We are now ready to tackle the main problem that motivates this study, which is the evaluation of the effect of curvature of the brane $p$-surface.

In order to do this, instead of the exactly flat configuration characterised by the coordinate system (40) and the particular form of the metric components (41), we need to consider a generically curved $p$-surface adjusted so as to be tangent to the reference flat $p$-surface at the coordinate origin.

The validity of the “thin brane” limit description with which we are concerned depends on the requirement that the relevant macroscopic lengthscale $\Delta$ (that was used to specify an appropriate “infra-red” cut off, in cases for which the convergence condition $q > p + 2$ does not hold) should be large compared with the microscopic brane thickness scale $\epsilon$. Subject to the requirement that

$$\alpha \equiv \frac{\epsilon}{\Delta} \ll 1$$

is satisfied (either because $\Delta$ is very large or because $\epsilon$ is very small), it will be sufficient for our present purpose to work at linear order in $\alpha$. This means that we need only consider up to quadratic order in $\sigma^i$ for an expansion where the brane locus will be specified in the relevant neighbourhood – of dimension large compared with $\epsilon$ but small compared with $\Delta$ – by

$$\bar{z}^i = \sigma^i, \quad \bar{r}^a = \frac{1}{2} K_{ij}^a \sigma^i \sigma^j [1 + o(\alpha)]$$

for an arbitrary set of constant coefficients $K_{ij}^a$. In this expansion, we have used the standard meaning of $o(\delta)$ denoting a quantity negligible with respect to $\delta$, that is satisfying $o(\delta)/\delta \to 0$ in the limit $\delta \to 0$. It can be easily verified that, as our notation (88) suggests, the constants of the development are exactly given by the non vanishing components of the second fundamental tensor $K_{\mu \nu \rho}^i$, defined in (6) at the origin, with respect to the orthonormal background coordinates (40).

In the simplest cases, the condition (87) might be satisfied simply by taking $\Delta$ to be arbitrarily large. In specific applications an upper bound on the admissible magnitude for $\Delta$ will commonly be imposed by the physical nature of the environment in which upper limits might be provided by lengthscales such as those characterising the gradients of relevant background fields or those characterising the separation between neighbouring branes. However, an upper limit that must always be respected will be provided, except in the strictly flat case, by the curvature lengthscale defined by the inverse of the magnitude of the mean curvature vector whose components are given by the formula

$$K^a = K_i^{ia}.$$  

As our notation suggests, these components $K^a$ are identifiable with respect to the orthonormal background coordinates (40) as the non-vanishing components of the complete curvature vector $K^\mu$ given by (1). In order to ensure that none of the coefficients $K_{ij}^a$ exceeds the order of magnitude of the inverse lengthscale $\Delta^{-1}$, the latter must thus be subject to the limitation

$$\Delta^{-2} \gtrsim K^a K_a.$$  

In particular, our formalism will break down near a kink or a cusp of the brane worldsheet.

The curvature of the worldsheet makes the evaluation of the physically appropriate weighting factor for the definition of effective sectional averages across the brane rather more delicate than
in the flat case dealt with in the preceding section. The kind of average that is ultimately relevant is a source density weighted mean taken not just over a \((q - p)\)-dimensional orthogonal section through the brane but rather over the \(q\)-dimensional volume between neighbouring \((q - p)\)-dimensional orthogonal sections through the brane. This distinction is irrelevant for a flat brane for which neighbouring sections are exactly parallel. However for a slightly curved brane, there will be a relative tilt between nearby orthogonal cross sections, which means that the effective weighting factor \(w\{r\}\) appropriate for the purpose of averaging over a \((q - p)\)-dimensional cross-section will no longer be exactly proportional to the density \(\rho\{r\}\) as in the flat case. This needs to be adjusted by a correction factor allowing for the relative compression or expansion of the relevant \(q\)-volume due to the relative tilting. By summing over the different directions in which the tilting can occur, it can be seen that the weighting factor will be given, at first order, by

\[
w\{r\} = \frac{\bar{\rho}_{\{1\}}(r)}{\bar{\rho}} \left[ 1 - K_a r^a + o(\alpha) \right]. \tag{91}\]

To obtain the corresponding generalisation of the formula (78) for the flat case, we must allow for the fact that the original formula (15) for the source density distribution also needs to be corrected to take into account the effect of the curvature. Using (88) to expand the distance function (22) as

\[
s^2\{x, \sigma\} = r^2 + \sigma^2 - \perp_{ab} K_{ij} r^a \sigma^i \sigma^j \left[ 1 + o(\alpha) \right], \tag{92}\]

we obtain the Taylor expansion

\[
D_{\{i\}}^{\{q\}} \{s^2\} = D_{\{i\}}^{\{q\}} \{r^2 + \sigma^2\} - \frac{d D_{\{i\}}^{\{q\}} \{r^2 + \sigma^2\}}{d \sigma^2} \perp_{ab} K_{ij} r^a \sigma^i \sigma^j \left[ 1 + o(\alpha) \right], \tag{93}\]

for the profile function (23). Since we are neglecting corrections of quadratic and higher order in the curvature, it can be seen that when the integration is carried out in two stages of which the first is just to take the integration over the \(p\)-sphere of constant radius \(\sigma\) centered on the origin, it is possible to replace the surface element \(dS^{[\sigma]}\) in (20) by \(\Omega^{[p-1]} \sigma^{p-1} d\sigma\) as in (16). Furthermore, in the anisotropic contribution proportional to \(\sigma^i \sigma^j\) it will be possible to replace \(\sigma^i \sigma^j dS^{[\sigma]}\) by the spherically averaged equivalent \(\gamma^{ij} \Omega^{[p-1]} \sigma^{p+1} d\sigma/p\). Therefore, to allow for the effect of the curvature at first order, the density distribution (15) needs to be replaced by

\[
\bar{\rho}_{\{\alpha\}} \{r\} = \bar{\rho} \left[ \perp_{i} D_{\{i\}}^{\{q,p\}} \{r^2\} - \frac{\Omega^{[p-1]}}{p} [K_a r^a + o(\alpha)] \int_{0}^{\infty} \sigma^{p+1} d\sigma \frac{d D_{\{i\}}^{\{q\}} \{r^2 + \sigma^2\}}{d \sigma^2} \right], \tag{94}\]

in which the second term can be evaluated using an integration by parts, using the definition (16) for \(\perp_{i} D_{\{i\}}^{\{q,p\}} \{r^2\}\) and the final result takes the simple form

\[
\bar{\rho}_{\{\alpha\}} \{r\} = \bar{\rho} \perp_{i} D_{\{i\}}^{\{q,p\}} \{r^2\} \left[ 1 + \frac{1}{2} K_a r^a + o(\alpha) \right]. \tag{95}\]

Thus, it can be seen that the geometric curvature adjustment factor in (11) is partially canceled by the density adjustment factor in (15) to give the corresponding curvature adjusted weighting factor

\[
w\{r\} = \perp_{i} D_{\{i\}}^{\{q,p\}} \{r^2\} \left[ 1 - \frac{1}{2} K_a r^a + o(\alpha) \right]. \tag{96}\]

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We note that as a corollary of this partial cancellation, the first order effect of the curvature is entirely canceled in the correspondingly weighted density function, which will be given simply by

\[ w\{r\} \tilde{\rho}_{(c)} \{r\} = \tilde{\rho} \left[ \frac{\partial D_{(c)}^{[\alpha]} \{r^2\}}{\partial r^2} \right]^2 [1 + o(\alpha)] . \]  

(97)

Thus to first order in the curvature, the average density will still be given by the same form as in the flat case.

9 Allowance for curvature in the potential gradient

In the flat configuration, the average of the field gradient \( \langle \partial \tilde{\phi}_{(c)} / \partial r^a \rangle \), vanishes. Thus, unlike \( \langle \tilde{\phi}_{(c)} \rangle \) and \( \langle \partial^2 \tilde{\phi}_{(c)} / \partial r^a \partial r^b \rangle \) for which the dominant contributions, respectively given by Eqs. (74) and (77), are of zeroth order in the curvature, it will have a dominant contribution that is of linear order in the curvature.

Starting from the general expression (26) for the field \( \tilde{\phi}_{(c)} \) in terms of the generalised Green function, the strict analogue of the integral (23) for the source density \( \tilde{\rho}_{(c)} \) is given, after Taylor expanding the Green function, by

\[ \tilde{\phi}_{(c)} \{r\} = \tilde{\rho} \Omega^{[q-1]} 0^p \int_0^\Delta \sigma^{p-1} d\sigma \left[ G_{(c)}^{[q]} \{r^2 + \sigma^2\} - \frac{\sigma^2}{p} \frac{d G_{(c)}^{[q]} \{r^2 + \sigma^2\}}{d\sigma^2} K_a r^a \right] [1 + o(\alpha)] . \]  

(98)

The truncation of the integration at the finite infra-red cut-off \( \Delta \) is only needed for the treatment of the marginally divergent case for which \( q = p + 2 \), but makes no difference, at first order in \( \epsilon / \Delta \). In the convergent cases \( q > p + 2 \), the result will be effectively the same as would be obtained simply by taking the limit \( \Delta \to \infty \). The corresponding expression for the gradient is found, after an integration by parts, to be

\[ \frac{\partial \tilde{\phi}_{(c)}}{\partial r^a} = \tilde{\rho} \Omega^{[q-1]} 0^p \int_0^\Delta \sigma^{p-1} d\sigma \left[ 2 + K_b r^b \right] \frac{d G_{(c)}^{[q]} \{r^2 + \sigma^2\}}{d\sigma^2} K_a r^a [1 + o(\alpha)] . \]  

(99)

The appropriately weighted potential function required for the evaluation of the relevant average can now be seen from (36) and (38) to be given, after another integration by parts, by

\[ w\{r\} \tilde{\phi}_{(c)} \{r\} = \tilde{\rho} \Omega^{[q-1]} 0^p \int_0^\Delta \sigma^{p-1} d\sigma \left[ G_{(c)}^{[q]} \{r^2 + \sigma^2\} \right] [1 + o(\alpha)] - \tilde{\rho} \Omega^{[q-1]} 0^p \int_0^\Delta \sigma^{p-1} d\sigma \left[ G_{(c)}^{[q]} \{r^2 + \sigma^2\} \right] [1 + o(\alpha)] \]  

(100)

When \( q > p + 2 \), the asymptotic behaviour (38) implies that the boundary contribution tends to zero, while the integral converges toward the radial Green function, as defined in (49), so that we obtain an expression of the simple cut-off independent form

\[ w\{r\} \tilde{\phi}_{(c)} \{r\} = \tilde{\rho} \Omega^{[q-1]} 0^p \int_0^\Delta \sigma^{p-1} d\sigma \left[ G_{(c)}^{[q]} \{r^2 + \sigma^2\} \right] [1 + o(\alpha)] . \]  

In the marginal case \( q = p + 2 \), we obtain an expression of the asymptotic form

\[ w\{r\} \tilde{\phi}_{(c)} \{r\} = \tilde{\rho} \Omega^{[q-3]} 0^p \int_0^\Delta \sigma^{p-1} d\sigma \left[ G_{(c)}^{[q]} \{r^2 + \sigma^2\} \right] [1 + o(\alpha)] . \]  

(101)

In the marginal case \( q = p + 2 \), we obtain an expression of the asymptotic form

\[ w\{r\} \tilde{\phi}_{(c)} \{r\} = \tilde{\rho} \Omega^{[q-3]} 0^p \int_0^\Delta \sigma^{p-1} d\sigma \left[ G_{(c)}^{[q]} \{r^2 + \sigma^2\} \right] [1 + o(\alpha)] . \]  

(102)
in which the boundary contribution at the end will provide a finite curvature adjustment term. However, since this adjustment term is an odd function of the radius vector \( r^a \) it will still provide no net contribution to the integrated average, which will thus be given simply by

\[
\left\langle \hat{\phi}(r) \right\rangle = \hat{\rho} \Omega^{[q-p-1]} \int_0^\infty r^{q_p-1} dr \hat{D}^{[q,p]} \{ r^2 \} \hat{G}^{[q,p]} \{ r^2 \} (1 + o(\alpha)),
\]

which to first order in \( \alpha \) is just the same as in the corresponding formula (80) for the flat case.

When we go on to work out the analogously weighted potential gradient distribution, it can be seen that the term with quadratic radial dependence in (99) will cancel out, leaving no net contribution to the integrated average, which will therefore be given simply

\[
w \{ r \} \frac{\partial \hat{\phi}(r)}{\partial r^a} = \hat{\rho} \Omega^{[p-1]} \hat{D}^{[q,p]} \{ r^2 \} \int_0^\Delta \sigma^{p-1} r^2 d\sigma \left[ 2r_0 - \frac{\sigma^2}{p} K_a \right] \frac{d\hat{G}^{[q]} \{ r^2 + \sigma^2 \}}{d\sigma^2} (1 + o(\alpha)).
\]

Since the first term in the previous integral is an odd function of the radius vector \( r^a \), it provides no net contribution to the corresponding integrated average, which will therefore be given simply by

\[
\left\langle \hat{\phi}(r) \right\rangle = \hat{\rho} \Omega^{[q-p-1]} K_a \int_0^\infty r^{q_p-1} dr \hat{D}^{[q,p]} \{ r^2 \} \hat{H}^{[q,p]} \{ r^2 \} (1 + o(\alpha)),
\]

where

\[
\hat{H}^{[q,p]} \{ r^2 \} \equiv -\frac{\Omega^{[p-1]}}{p} \int_0^\Delta \sigma^{p+1} d\sigma \frac{d\hat{G}^{[q]} \{ r^2 + \sigma^2 \}}{d\sigma^2}.
\]

Using an integration by parts, this integral can be rewritten in terms of the dimensionally reduced Green function \( \hat{G}^{[q,p]} \{ r^2 \} \) defined by (105) in the form

\[
\hat{H}^{[q,p]} \{ r^2 \} = \frac{1}{2} \hat{G}^{[q,p]} \{ r^2 \} - \frac{\Omega^{[p-1]}}{2p} \frac{\Delta^p}{\hat{D}^{[q,p]} \{ r^2 + \Delta^2 \}}.
\]

As in (106), the boundary contribution at the end will vanish in the large \( \Delta \) limit in consequence of the asymptotic behaviour (38) provided the convergence condition \( q > p + 2 \) is satisfied. In the marginally divergent hyperstring case \( q = p + 2 \), the boundary term will tend to a limit having a finite value (namely \( -1/(2p^2) \)) that will still be negligible compared with the first term in (107). Thus, whenever \( q - p \geq 2 \) we shall always have a relation of the form

\[
\hat{H}^{[q,p]} \{ r^2 \} = \frac{1}{2} \hat{G}^{[q,p]} \{ r^2 \} (1 + o(\alpha)).
\]

The only case to which this formula does not apply is that of a hypermembrane, meaning the hypersurface supported case characterised by \( q = p - 1 \), for which the boundary term will be linearly divergent, like the corresponding the integral for \( \hat{D}^{[q,p]} \{ r^2 \} \) as characterised by (51). We can still derive an asymptotic relation

\[
\hat{H}^{[q,p-1]} \{ r^2 \} \sim \frac{q-2}{2(q+1)} \hat{G}^{[q,p-1]} \{ r^2 \}
\]

as \( \Delta \to \infty \) even in this extreme case, but its utility is limited by the strongly cut off dependence of the quantities involved. It can be seen that it approaches agreement with the generic formula (108) when the space dimension \( q \) is very large.
When (108) is substituted back into (105) one obtains a result that can immediately be seen by comparison with the formula (103) for the averaged potential \( \langle \tilde{\phi}_{(\varepsilon)} \rangle \) to be expressible in terms of the latter by

\[
\left\langle \frac{\partial \tilde{\phi}_{(\varepsilon)}}{\partial r^\alpha} \right\rangle = \frac{1}{2} K_\alpha \left\langle \tilde{\phi}_{(\varepsilon)} \right\rangle \left[ 1 + o(\alpha) \right],
\]

(110)
a simple and easily memorable result whose derivation is the main purpose of this work.

10 Discussion

The preceding work shows not only that the relation (110) holds as an ordinary numerical equality for the strictly convergent cases for which the co-dimension, \((q - p)\), is greater than 2, but also that it holds as an asymptotic relation for the marginally divergent case of a hyperstring with \(q - p = 2\), including the previously studied examples [1, 2, 3] of application to a string with \(p = 1\) in the ordinary case of a background with space dimension \(q = 3\).

It is also clear from the preceding work that the formula (110) is not applicable to the case of hypermembrane, for which the co-dimension is given by \(q - p = 1\). In this case, it can be seen from (109) that the factor \(1/2\) in (110) should in principle be replaced by a factor \((p - 1)/2p\). However it is debatable whether any useful information can be extracted in this case since the result is strongly dependent on the cut-off due to the “infra-red” divergence, and in practice this issue is of little importance because the particularly good ultraviolet behaviour of the hypermembrane case makes it easily amenable to other, more traditional, methods such as the use [14] of Israel Darmois type jump conditions.

Dropping the explicit reminder that higher order adjustments would be needed if one wanted accuracy beyond first order in the ratio of brane thickness to curvature radius, the relation (110) translates into fully covariant notation as

\[
\left\langle \perp^{\mu\nu} \nabla_\nu \tilde{\phi}_{(\varepsilon)} \right\rangle = \frac{1}{2} K_\mu \langle \tilde{\phi}_{(\varepsilon)} \rangle.
\]

(111)
It is to be emphasised that the generic relation (111) is a robust result, whose validity, whenever the co-dimension is 2 or more, does not depend on the particular choice of the canonical regularisation ansatz on which the explicit formula (85) for \(\langle \tilde{\phi}_{(\varepsilon)} \rangle\) was based. It also shows that (111) is valid for any alternative non-canonical regularisation ansatz, whose mathematical properties would be less convenient for the purpose of explicit evaluation, but that might provide a more exact representation of the internal structure for particular physical applications in cases where information about this actual internal structure is available.

The question that remains to be discussed is how this result generalises from static Poisson configurations, as considered here, to dynamic configurations in a Lorentz covariant treatment for which the Laplacian operator would be replaced by a Dalembertian operator.

A priori, considerations just of Lorentz invariance imply that the only admissible alternative to the formula (111) would be a formula differing just by replacement of the factor \(1/2\) by some other numerical pre-factor. However, it must be taken into account that this numerical coefficient must match the coefficient that applies to any static configuration. It can thus be deduced from the postulate of consistency with what has been derived here that the formula (111) should be applicable to any \(p\)-brane with \(p + 1\)-dimensional timelike worldsheet in Lorentzian \(q + 1\)-dimensional background spacetime, for all values of \(p\) and \(q\) for which our present derivation applies, that is to say for \(q - 2 \geq p \geq 1\).
There are two interesting opposite extreme cases that are beyond this range. At one extreme, in the most strongly “infra red” divergent hyper-membrane case \( p = q - 1 \), the formula (111) needs a modified factor \( (p - 1)/2p \). However, as discussed above, the quantities involved are, in this case, too highly cut off dependent for the result in question to have much significance, and anyway, as remarked above, their ultraviolet misbehaviour is so mild that hypermembranes can be satisfactorily treated without recourse to regularisation, so no analogue of the gradient formula is needed.

On the other hand, at the opposite extreme, in the most strongly “ultra violet” divergent case, namely that of a simple point particle with \( p = 0 \), an appropriate regularised gradient formula is indeed something that is obviously needed. However, for a static configuration, the possibility of curvature does not arise at all in this “zero-brane” case, so that – while there is no reason to doubt the conjecture that it should still apply – there is no shortcut whereby the formula (111) can be derived without further work, which will require recourse to the technical complications (due to the dimension sensitive nature of the time-dependent Green functions [12]) involved in a fully dynamical treatment.

Appendix: Integral formulae and notation

The standard definition of the Euler Gamma function is provided by the integral formula

\[
\Gamma\{z\} \equiv \int_0^\infty e^{-t}t^{z-1}dt. \tag{112}
\]

When the argument has an integer value \( n \) it can be shown, using the obvious recursion relation \( \Gamma\{n + 1\} = n\Gamma\{n\} \), that it satisfies the well known relations

\[
\Gamma\{n + 1\} = n!, \quad \Gamma\left\{n + \frac{1}{2}\right\} = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}. \tag{113}
\]

In this article, we frequently need to evaluate integrals of the form

\[
I_{p,q} \equiv \int_0^\infty \frac{y^{p-1}}{(1+y^2)^{q/2}}dy. \tag{114}
\]

Whenever \( p > 1 \) and \( q > p \), this integral will be convergent and will be expressible – via a change of variables \( t = 1/(1+y^2) \) – in the form

\[
I_{p,q} = \frac{1}{2} B\left\{\frac{p}{2}, \frac{q-p}{2}\right\}. \tag{115}
\]

where \( B\{a,b\} \) is an Euler integral of the first kind – also known as the Beta function – that is specified by the formulae

\[
B\{a,b\} \equiv \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma\{a\}\Gamma\{b\}}{\Gamma\{a+b\}}. \tag{116}
\]

The values, as given by (12), of the surface area \( \Omega^{(q-1)} \) of a unit \( q \)-sphere are related for different values of \( q \) by the formulae

\[
\frac{2\Omega^{(q-1)}}{\Omega^{(p-1)}\Omega^{(q-p-1)}} = \frac{\Gamma\left\{\frac{p}{2}\right\}\Gamma\left\{\frac{q-p}{2}\right\}}{\Gamma\left\{\frac{q}{2}\right\}} = B\left\{\frac{p}{2}, \frac{q-p}{2}\right\}. \tag{117}
\]

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and thus satisfy the useful relation

\[
\frac{q}{2} B \left\{ \frac{p}{2}, \frac{q-p}{2} + 1 \right\} \Omega^{[n-1]} = \frac{q-p}{\Omega^{[n-p-1]}}. \tag{118}
\]

In order to facilitate the reading of the article, we conclude by providing the following table summarising the the notation used for the profile, Green, and other related functions in the three cases considered in this article.

| Notation                      | General notation | Infinitely thin limit | Canonical case |
|-------------------------------|------------------|-----------------------|----------------|
| Potential                     | \( \phi \)       | \( \phi \)            | \( \phi \)      |
| Source term                   | \( \rho \)       | \( \rho \)            | \( \rho \)      |
| Profile distribution          | \( \delta \)     | \( \delta \)          | \( \delta \)    |
| Green function                | \( G \)          | \( G \)               | \( G \)         |
| Radial profile function       | \( \delta \)     | \( \delta \)          | \( \delta \)    |
| Radial Green function         | \( G \)          | \( G \)               | \( G \)         |

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