A PARAMETRIZATION OF MATRIX CONJUGACY ORBIT SETS
AS UNIONS OF AFFINE PLANES

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Abstract. The problem of finding a canonical form of complex matrices up to conjugacy with
the set of canonical matrices being a union of affine planes in the matrix space is considered. A
solution of the problem is given producing a new canonical form for matrices over algebraically
closed fields.

Key words. Jordan form, rational form, eigenvalue, Weyr characteristic, affine plane.
MSC 15A21.

1. Introduction. It is known that sets of representatives of matrix conjugacy
classes in Jordan or rational canonical (elementary divisors version) forms in general
do not constitute algebraic varieties in the ambient matrix spaces.

Example 1.1. Consider 2 × 2 matrices over C. A set of the Jordan forms can
be chosen as J1 ∪ J2 where

\[ J_1 = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \right\}, \forall a, b \in \mathbb{C}, J_2 = \left\{ \begin{bmatrix} c & \cdot \\ 1 & c \end{bmatrix} \right\}, \forall c \in \mathbb{C}. \]

If a ≠ b then matrices \[ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \] and \[ \begin{bmatrix} b & a \\ a & b \end{bmatrix} \] are conjugate therefore J1 contains
two representatives of each matrix conjugacy class with two distinct eigenvalues.

M.L.Kontsevich [1], p.127, has proposed the problem of finding a canonical form
of matrices over C for which the set of representatives of matrix conjugacy classes
would be a union of disjoint affine planes - cosets in the ambient C-linear matrix space
modulo a subspace. In other words, for each n ≥ 2 find a union of disjoint affine
planes \( A \) in the C-linear space \( \text{Mat}(n, n, \mathbb{C}) \simeq \mathbb{C}^{n^2} \) so that each matrix conjugacy
orbit intersects \( A \) in exactly one point. The problem may be motivated by a desire
to provide more links between the representation theory of algebras and algebraic
geometry.

Note that neither the Jordan form nor the rational canonical forms satisfy this

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condition. The rational canonical form with invariant factors may be investigated if one wants to use affine varieties of higher degrees but this is not within the scope of this paper.

In this paper we present a solution of the above problem by developing a new canonical form of matrices which we call canonical plane form. Matrices considered are defined over \( \mathbb{C} \). Similar results can be proved for any algebraically closed field.

2. Description of canonical plane matrices and affine planes. The goal of this section is to describe complex matrices called canonical plane matrices which will be interpreted as points of affine planes in the \( \mathbb{C} \)-linear matrix spaces. In the first step we define matrices over multivariate polynomial rings. Then we obtain complex matrices using substitutions.

Matrices over polynomial rings are denoted using bold letters. \( E_m \) denotes the identity matrix.

2.1. Matrices over multivariate polynomial rings.

2.1.1. Partitions. Given \( n \in \mathbb{N}, n \geq 2 \), we consider all (nonincreasing) partitions of \( n \), i.e. sequences of natural numbers - addends \((n_1, ..., n_d)\) such that \( n_i \geq n_{i+1}, \forall i : 1 \leq i \leq d - 1 \) and \( \sum_{i=1}^{d} n_i = n \). Maximal constant contiguous subsequences, i.e. maximal subsequences \((n_r, ..., n_{r+s})\) with \( n_r = ... = n_{r+s} \) will be called stacks. A partition of \( n \) having \( t \) stacks of lengths \( l_1, ..., l_t \) with distinct addends \( m_1 > m_2 > ... > m_t \) may be denoted, symbolically, as \((m_1^{l_1}, ..., m_t^{l_t})\).

2.1.2. Diagonal matrices. Given a partition \( \pi = (n_1, n_2, ..., n_d) \) of \( n \) define a diagonal matrix

\[
D_{\pi} = \bigoplus_{i=1}^{d} X_i \cdot E_{n_i} \in Mat(n, n, \mathbb{C}[X_1, ..., X_d]).
\]
We use the convention that indices of the variables in diagonal blocks are arranged in the increasing order starting from the upper left corner as shown in the example.

**Example 2.1.** Let $n = 8$, $\pi = (3, 3, 2)$, then

\[
D_\pi = X_1 E_3 \oplus X_2 E_3 \oplus X_3 E_2 = \begin{bmatrix}
X_1 & X_1 & X_2 & X_2 & X_3 & X_3 \\
X_1 & X_2 & X_2 & X_3 & X_3
\end{bmatrix}.
\]

**2.1.3. Generalized companion matrices.** For any $l \leq d$ define a $l \times l$ matrix $R_l(X_{i_1}, \ldots, X_{i_l})$ as follows:

\[
R_l(X_{i_1}, \ldots, X_{i_l}) = \begin{bmatrix}
0 & 0 & \ldots & 0 & (-1)^{l+1} e_l(X_{i_1}, \ldots, X_{i_l}) \\
1 & 0 & \ldots & 0 & (-1)^l e_{l-1}(X_{i_1}, \ldots, X_{i_l}) \\
0 & 1 & \ldots & 0 & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & e_1(X_{i_1}, \ldots, X_{i_l})
\end{bmatrix}
\]

where $e_i$ is the elementary symmetric polynomial of degree $i$ in $l$ variables. It is the $l \times l$ companion matrix having $l$ distinct "eigenvalues" $X_{i_1}, \ldots, X_{i_l}$.

**Example 2.2.** $R_3(X_1, X_2, X_3) = \begin{bmatrix}
0 & 0 & X_1 X_2 X_3 \\
1 & 0 & -X_1 X_2 - X_1 X_3 - X_2 X_3 \\
0 & 1 & X_1 + X_2 + X_3
\end{bmatrix}$. 
2.1.4. Block lower triangular matrices. Given a partition of $\pi = (n_1, ..., n_d)$ of $n$ define a lower block triangular matrix $P_\pi$ by modifying $D_\pi$ in two steps as follows:

1. For each pair $(n_j, n_{j+1})$ with $n_j > n_{j+1}$ insert the matrix

$$F_{n_{j+1}, n_j} = \begin{bmatrix} O_{n_{j+1}, n_{j+1}} & E_{n_{j+1}} \\ E_{n_{j+1}} & X_{n_{j+1}} \end{bmatrix}$$

as shown:

$$\begin{bmatrix} X_j \cdot E_n \\ X_{j+1} \cdot E_{n_{j+1}} \end{bmatrix} \rightarrow \begin{bmatrix} X_j \cdot E_{n_{j+1}} \\ F_{n_{j+1}, n_j} \cdot X_{n_{j+1}} \cdot E_{n_{j+1}} \end{bmatrix}.$$

2. For each stack $(n_r, n_{r+1}, ..., n_{r+u})$ of $\pi$ with $n_r = ... = n_r = m$ substitute the submatrix

$$\begin{bmatrix} X_r \cdot E_m \\ X_{r+1} \cdot E_m \\ \vdots \\ X_{r+u} \cdot E_m \end{bmatrix}$$

by

$$R_{u+1}(X_r, ..., X_{r+u}) \otimes E_m = \begin{bmatrix} E_m \\ (-1)^{u+2} e_{u+1}(X_r, ..., X_{r+u}) \cdot E_m \\ (-1)^u e_u(X_r, ..., X_{r+u}) \cdot E_m \\ \vdots \\ (-1)^u e_u(X_r, ..., X_{r+u}) \cdot E_m \end{bmatrix}.$$

Thus stacks of $\pi$ bijectively correspond to diagonal blocks of $P_\pi$ and off-diagonal blocks of $P_\pi$ are identity matrices coming from $F_{n_{j+1}, n_j}$.

Example 2.3. Let $n = 8$, $\pi = (3, 3, 2)$, then

$$P_\pi = \begin{bmatrix} -X_1 X_2 & -X_1 X_2 & -X_1 X_2 \\ 1 & X_1 + X_2 & X_1 + X_2 \\ 1 & X_1 + X_2 & X_1 + X_2 \\ 1 & 1 & X_3 \end{bmatrix}. $$
2.2. Complex canonical plane matrices and affine planes.

2.2.1. Substitutions maps. Define the substitution map
\[ \varphi : \text{Mat}(n, n, \mathbb{C}[X_1, ..., X_d]) \times \mathbb{C}^d \rightarrow \text{Mat}(n, n, \mathbb{C}), \]
\[ ([f_{ij}], (a_1, ..., a_d)) \mapsto [f_{ij}(a_1, ..., a_d)]. \]
Denote the image of \( P_\pi \times \mathbb{C}^d \) under \( \varphi \) by \( P_\pi \).

2.2.2. Description of canonical plane matrices. Let \( f = X^l - \sum_{i=0}^{l-1} a_i X^i \in \mathbb{C}[X] \). Define a companion matrix \( C_f \) of \( f \) in a standard form:
\[
C_f = C(a_0, ..., a_{l-1}) = \begin{bmatrix}
0 & 0 & ... & 0 & a_0 \\
1 & 0 & ... & 0 & a_1 \\
0 & 1 & ... & ... & ... \\
... & ... & ... & ... & ... \\
0 & 0 & ... & 1 & a_{l-1}
\end{bmatrix}
\]
Each \( A \in P_\pi \) is a block lower triangular matrix with the diagonal blocks in form \( C(a_0, ..., a_{l-1}) \otimes E_m \) for some \( l \) and \( (a_0, ..., a_{l-1}) \in \mathbb{C}^l \).

Example 2.4. Let \( n = 8, \pi = (3,3,2) \), then
\[
A = \begin{bmatrix}
-2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\
1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} = \varphi(P_\pi, (1, 2, 1)).
\]

2.2.3. Inverse image of a canonical plane matrix. Given \( A \in P_\pi \) we can find an element of its inverse image under \( \varphi \) as follows. For each diagonal block \( C(a_0, ..., a_{l-1}) \otimes E_m \) of \( A \) solve the equation \( X^l - \sum_{i=0}^{l-1} a_i X^i = 0 \), get the multiset of roots \([\lambda_1, ..., \lambda_l] \), its elements are the complex numbers which must be substituted (in any order) into the polynomial arguments in the corresponding diagonal block of \( P_\pi \). Note that the vector \((a_0, ..., a_{l-1}) \) uniquely determines the multiset of roots \([\lambda_1, ..., \lambda_l] \) and vice versa.
2.2.4. Canonical affine planes. Affine planes in a linear space $L$ are identified with additive cosets modulo a subspace $V$ of $L$. Thus to define an affine plane $\mathcal{P} \subseteq L$ we need to fix one element $l \in \mathcal{P}$ (a constant shift) and describe $V$ (linear part). Dimension of $\mathcal{P}$ is equal to $\dim V$.

Theorem 2.5. $n \in \mathbb{N}$, $n \geq 2$, $\pi = (n_1, ..., n_d)$ - a partition of $n$. Then $\mathcal{P}_\pi$ is an affine plane in $\text{Mat}(n, n, \mathbb{C})$ of dimension $d$.

Proof. We will express $A \in \mathcal{P}_\pi$ as a sum

$$A = S + L(A)$$

of a constant matrix $S$ (the shift) and a variable matrix $L(A)$ (the linear part) as follows. The nonzero elements of $S$ are the off-diagonal blocks together with the identity submatrices of generalized companion matrices. Thus $S$ is uniquely defined for all matrices in $\mathcal{P}_\pi$. We define $L(A) = A - S$ and observe that the possibly nonzero elements of $L(A)$ are the blocks of form $[a_0|...|a_{l-1}]^T \otimes E_m$ corresponding to last columns of generalized companion matrices.

Suppose $A \in \mathcal{P}_\pi$ has the $t$ diagonal blocks $C(a_{j0}, ..., a_{j,t-1}) \otimes E_{m_j}$, $\forall j \in \{1, ..., t\}$. For each $j \in \{1, ..., t\}$ solve the $j$-th stack equation $X^j - \sum_{i=0}^{l_j-1} a_{ji}X^i = 0$, get the multiset of roots $[\lambda_{j1}, ..., \lambda_{j,l_j}]$, $\forall j \in \{1, ..., t\}$. We have that

$$A = \varphi_M(\mathcal{P}_\pi, (\lambda_{11}, ..., \lambda_{1,t_1}, ..., \lambda_{t1}, ..., \lambda_{tt}))$$

Note that the ordering of the roots within each stack does not change the value of $\varphi$.

We see that $A \in \mathcal{P}_\pi$ can be constructed for any sequence of vectors $(a_{10}, ..., a_{1,t_1-1}), ..., (a_{t0}, ..., a_{t,t_1-1})$ thus $A - S$ runs through a linear subspace $V_\pi$ of $\text{Mat}(n, n, \mathbb{C})$ as $A$ runs through $\mathcal{P}_\pi$. We see that $\dim V_\pi$ is the sum of the dimensions of vectors $(a_{10}, ..., a_{1,t_1-1}), ..., (a_{t0}, ..., a_{t,t_1-1})$ which is equal to $d$. □

Example 2.6. Consider the matrix $A$ of example 2.4. In this case

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad L(A) = \begin{bmatrix} -2 & -2 & -2 \\ 3 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}.$$
dim \( V_{\pi} = 3 \), \( V_{\pi} = \langle V_1, V_2, V_3 \rangle \) where

\[
V_1 = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}, \quad V_2 = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}, \\
V_3 = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}.
\]

**Theorem 2.7.** \( n \in \mathbb{N}, n \geq 2, \pi \) and \( \pi' \) - distinct partitions of \( n \). Then

\[ P_{\pi} \cap P_{\pi'} = \emptyset. \]

**Proof.** We show that supports of constant shift matrices are different. Given \( \pi \neq \pi' \) with \( \pi = (n_1, n_2, ...) \) and \( \pi' = (n'_1, n'_2, ...) \) consider the distinct pair \( n_j \neq n'_j \) with the minimal \( j \). There are two possibilities: (1) new stacks start from \( n_j \) and \( n'_j \) or (2) a new stack starts from one of \( n_j \) or \( n'_j \). In the first case the off-diagonal blocks of \( P_{\pi} \) and \( P_{\pi'} \) inserted to the left of diagonal blocks for the new stacks have 1’s in different positions. In the second case in \( P_{\pi} \) and \( P_{\pi'} \) we have diagonal blocks of distinct sizes having the same position of upper left corners, by considering identity matrix blocks of generalized companion matrices it follows that at least one 1’ in the off-diagonal block below the shorter diagonal block in, say, \( P_{\pi} \) is absent in \( P_{\pi'} \). \( \square \)
3. Main results.

3.1. Jordan forms of canonical plane matrices.

3.1.1. Preparations. Let a partition $\pi$ have $t$ stacks of lengths $l_1, \ldots, l_t$ with distinct addends $m_1 > m_2 > \ldots > m_t$. Define $m_{t+1} = 0$. Define $s_j = \sum_{i=1}^{j} l_i$. Define $P_j = \{m_1 - m_j + 1, \ldots, m_1 - m_{j+1}\}$.

Let $A \in P_{\pi}$ have diagonal blocks $C(a_{j0}, \ldots a_{jlj} - 1) \otimes E_{m_j}$, $j \in \{1, \ldots, t\}$. For each $j \in \{1, \ldots, t\}$ solve the $j$th stack equation

$$X^{l_j} - \sum_{i=0}^{l_j-1} a_{ji}X^i = 0,$$

get the multiset of roots $[\lambda_{j1}, \ldots, \lambda_{jlj}]$, $j \in \{1, \ldots, t\}$. Denote the multiplicity of $\lambda$ as a root for the $i$th stack equation by $\mu(\lambda, i)$. Denote $\sum_{i=1}^{j} \mu(\lambda, i)$ by $\alpha(\lambda, j)$.

We think of $A$ acting in $\mathbb{C}^n$ - the $n \times 1$ column space with the standard basis $B = \{e_1, \ldots, e_n\}$, $e_i = E_{1i}$ where $E_{ij}$ is a matrix unit.

The goal of this subsection is to find the Jordan form of $A \in P_{\pi}$. This is done in several steps:

1. by inspecting the digraph of $A$ we find a decomposition of $\mathbb{C}^n$ into a direct sum of $m_1$ $A$-invariant subspaces which are generated by $e_1, \ldots, e_{m_1}$ as $\mathbb{C}[A]$-modules,
2. decompose $A$ into a direct sum by permutations,
3. decompose the obtained direct summands of $A$ into the Jordan form using their block structure.

3.1.2. Invariant subspaces. For each $i \in \{1, \ldots, m_1\}$ define a $A$-invariant subspace $V_i = \mathbb{C}[A] \cdot e_i$. For each $j \in \{1, \ldots, t\}$ define a $s_j \times s_j$ matrix

$$G_j = \bigoplus_{i=1}^{j} C(c_{i0}, \ldots, c_{il_i-1}) + \sum_{i=1}^{j-1} E_{s_i+1,s_i}.$$ 

Here we also use the convention about the block ordering as in 2.1.2, see example 3.2 below.

**Theorem 3.1.** In the notations of 3.1.1 we have
1. $C^n = \bigoplus_{i=1}^{m_1} V_i$,
2. $\dim V_i = s_j$ for $i \in P_j$,
3. If $i \in P_j$ then the restriction of $A$ to $V_i$ with respect to $B$ is $G_j$.

**Proof.** The statements are proved by considering the images of $e_1, \ldots, e_{m_1}$ under powers of $A$, using the digraph of $A$ and induction. The digraph of $A$ with respect to $B$ decomposes into $m_1$ weakly connected components corresponding to $V_1, \ldots, V_{m_1}$. Vertex sets of these components and restrictions of $A$ can be explicitly described to prove the restriction statement. The number of elements of $B$ in the component corresponding to $V_i$ is equal to $s_j$ if $i \in P_j$ which implies the dimension statement. Further details are omitted. \(\square\)

**Example 3.2.** Consider the matrix $A$ of example 2.4. Its digraph is shown in Fig.1.

In this case $C^8 = V_1 \oplus V_2 \oplus V_3$ where $V_1 = \langle e_1, e_4 \rangle$, $V_2 = \langle e_2, e_5, e_7 \rangle$, $V_3 = \langle e_3, e_6, e_8 \rangle$.

$$G_1 = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

**3.1.3. Decomposition by permutation.** Denote the matrix conjugacy relation by $\simeq$.

**Theorem 3.3.** In the notations of 3.1.1 we have

$$A \simeq \bigoplus_{j=1}^{t} (m_j - m_{j+1}) G_j.$$
Proof. The described direct sum is obtained by permuting the rows and columns of $A$ following theorem 3.1.

Example 3.4. The matrix $A$ of example 2.4 is permutation conjugate to

$$G_1 \oplus G_2 \oplus G_2 = \begin{bmatrix}
-2 & 1 & -2 & 1 & -2 \\
1 & 3 & 1 & 3 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}.$$ 

3.1.4. Jordan forms. Denote the Jordan block with eigenvalue $\lambda$ and size $i$ by $J_i(\lambda)$.

Theorem 3.5. In the notations of 3.1.1 we have

$$G_j \simeq \bigoplus_{\lambda} J_{\alpha(\lambda,j)}(\lambda).$$

Proof. The result follows from the lower triangular block structure of $G_j$. $G_j$ is in block lower triangular form therefore its characteristic polynomial $\chi(G_j, x)$ is equal to the product of the characteristic polynomials of diagonal blocks. It follows that $\chi(G_j, x) = \prod_{\lambda} (\lambda - x)^{\alpha(\lambda,j)}$. The diagonal blocks of $G_j$ have subdiagonals with all elements equal to 1 and only zeros under subdiagonals. The only nonzero elements outside diagonal blocks of $G_j$ are 1’s in the corner positions between diagonal blocks. It follows that $G_j$ has the global nonzero subdiagonal. It follows that $G_j$ has a cyclic vector $[1\mid 0\mid 0\mid ...\mid 0]^T$. Hence the minimal polynomial of $G_j$ is $\pm \chi(G_j, x)$. It follows $G_j$ has one Jordan block for each eigenvalue.

Theorem 3.6. In the notations of 3.1.1 we have

$$A \simeq \bigoplus_{j=1}^t \left( m_j - m_{j+1} \right) \left( \bigoplus_{\lambda} J_{\alpha(\lambda,j)}(\lambda) \right).$$

(3.1)

Proof. It follows from theorems 3.3 and 3.5.
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Example 3.7. The Jordan form of \( A \) from example 2.4 is shown in Fig.2.

\[
\begin{align*}
&\begin{array}{c}
\circ
\end{array} \quad 2 \\
&\begin{array}{c}
\circ
\end{array} \quad e_1' \\
&\begin{array}{c}
\circ
\end{array} \quad i \\
&\begin{array}{c}
\circ
\end{array} \quad e_4' \\
&\begin{array}{c}
\circ
\end{array} \quad 2 \\
&\begin{array}{c}
\circ
\end{array} \quad e_2' \\
&\begin{array}{c}
\circ
\end{array} \quad i \\
&\begin{array}{c}
\circ
\end{array} \quad e_5' \\
&\begin{array}{c}
\circ
\end{array} \quad 2 \\
&\begin{array}{c}
\circ
\end{array} \quad e_3' \\
&\begin{array}{c}
\circ
\end{array} \quad i \\
&\begin{array}{c}
\circ
\end{array} \quad e_6' \\
&\begin{array}{c}
\circ
\end{array} \quad 2 \\
&\begin{array}{c}
\circ
\end{array} \quad e_7' \\
&\begin{array}{c}
\circ
\end{array} \quad i \\
&\begin{array}{c}
\circ
\end{array} \quad e_8'
\end{align*}
\]

Fig.2. - the Jordan form digraph of \( A \) from example 2.4

3.2. Canonical plane forms of Jordan matrices. The goal of this subsection is to show that a direct sum of Jordan matrices is conjugate to a canonical plane matrix.

3.2.1. Weyr characteristic of canonical plane matrices. The Jordan form of a matrix \( B \) with the single eigenvalue \( \lambda \) is determined by its Weyr characteristic sequence \( \Omega(B, \lambda) = (\omega_1, \omega_2, ...) \) where \( \omega_i \) is the number of Jordan blocks of size at least \( i \) (see [2] for a discussion and recent applications).

Example 3.8. If \( B \cong 2J_1(\lambda) \oplus J_3(\lambda) \oplus J_4(\lambda) \) then \( \Omega(B, \lambda) = (4, 2, 2, 1, 0, ...), \omega_u = 0 \) for all \( u \geq 5 \). The Jordan form of any matrix \( B \) with eigenvalues \( \lambda_1, \lambda_2, ... \) is determined by the Weyr array \( \Omega(B) = (\Omega(B, \lambda_1), \Omega(B, \lambda_2), ...) \).

Theorem 3.9. In the notations of 3.1.1 let an eigenvalue \( \lambda \) of \( A \) have nonzero multiplicity \( \mu(\lambda, j) \) if and only if \( j \in \{j_1, j_2, ..., j_v\}, j_1 < j_2 < ... < j_v \). Then

\[
\Omega(A, \lambda) = \left( m_{j_1}, ..., m_{j_1}, m_{j_2}, ..., m_{j_2}, ..., m_{j_v}, ..., m_{j_v}, 0, ... \right) \underbrace{\mu(\lambda, j_1) \text{ copies}}_{\mu(\lambda, j_2) \text{ copies}} \underbrace{\mu(\lambda, j_v) \text{ copies}}_{\mu(\lambda, j_v) \text{ copies}}.
\]

Proof. The statement follows from theorem 3.6 by reordering the direct sum (3.1), counting the number of Jordan blocks of \( A \) of given \( \lambda \) and size and using induction. We have that

\[
A \cong \bigoplus_{j=1}^{t} (m_{j} - m_{j+1}) \left( \bigoplus_{\lambda} J_{(\lambda, j)}(\lambda) \right) \cong \bigoplus_{\lambda} \left( \bigoplus_{j=1}^{t} (m_{j} - m_{j+1}) J_{(\lambda, j)}(\lambda) \right)
\]
Thus for a given $\lambda$

1. the minimal size of a Jordan block is $\alpha(\lambda, j_1) = \mu(\lambda, j_1)$, there are $m_{j_1}$ Jordan blocks of sizes at least $1, ..., \mu(\lambda, j_1)$, start $\Omega(A, \lambda)$ as $(m_{j_1}, ..., m_{j_1})$, $\mu(\lambda, j_1)$ copies.

2. the next possible size of a Jordan block is $\alpha(\lambda, j_2)$, there are $m_{j_1} - (m_{j_1} - m_{j_2}) = m_{j_2}$ Jordan blocks of sizes at least $\mu(\lambda, j_1) + 1, ..., \alpha(\lambda, j_2)$, continue $\Omega(A, \lambda)$ as $(m_{j_1}, ..., m_{j_1}, m_{j_2}, ..., m_{j_2})$, $\mu(\lambda, j_1)$ copies $\mu(\lambda, j_2)$ copies.

3. ...

3.2.2. From Jordan forms to canonical plane matrices. We describe an algorithm for finding a canonical plane matrix which is conjugate to a given matrix $B$. Rearrange $\Omega(B)$ array as follows:

1. sort the nonzero entries of $\Omega(B)$ in nonincreasing order,
2. collect the equal nonzero entries into a sequence of multisets $M(B) = (M_1, M_2, ..., M_t)$, all elements of $M_i$ are equal to some $m_i$, $m_1 > m_2 > ... > m_t$.

Denote $|M_i|$ by $l_i$, denote the multisets of eigenvalues corresponding to the elements of $M_i$ by $\Lambda_i = [\lambda_{i1}, ..., \lambda_{i, l_i}]$. Define $\pi = (m_1^{l_1}, ..., m_t^{l_t})$.

Theorem 3.10. In the above notations

$$B \simeq \tilde{B} = \varphi(P_\pi, (\lambda_{11}, ..., \lambda_{1,l_1}, ..., \lambda_{t1}, ..., \lambda_{t,l_t})).$$

Proof. Using the theorem 3.9 we construct $\Omega(\tilde{B})$ by considering diagonal blocks of $\tilde{B}$:

1. from the 1st block we get the initial elements equal to $m_1$ of $\Omega(\tilde{B}, \lambda)$ for $\lambda \in \Lambda_1$, thus get initial subsequences of $\Omega(\tilde{B}, \lambda)$ of form $(m_1, ..., m_1)$ for each $\lambda$ with $\mu(\lambda, 1) \neq 0$.
2. from the 2nd block we get elements equal to $m_2$ of $\Omega(\tilde{B}, \lambda)$ for $\lambda \in \Lambda_2$, thus
form subsequences of $\Omega(\tilde{B}, \lambda)$ of length $\alpha(\lambda, 2)$ by adding $(m_2, \ldots, m_2)$ for each 
$\lambda$ with $\mu(\lambda, 1) \neq 0, \mu(\lambda, 2) \neq 0$ or initiate $\Omega(\tilde{B}, \lambda)$ as $(m_2, \ldots, m_2)$ for each $\lambda$
with $\mu(\lambda, 1) = 0, \mu(\lambda, 2) \neq 0$.

3. ...

By induction on $|M|$ we see that $\Omega(\tilde{B}) = \Omega(B)$ therefore $\tilde{B} \simeq B$. □

**Definition 3.11.** For any matrix $B$ denote its canonical plane matrix $\tilde{B}$ by $\text{Aff}(B)$.

**Remark 3.12.** Note that the canonical plane form of a matrix in general is different from the Jordan form and the rational form (either in invariant factors or elementary divisors version) as example 2.4 shows.

### 3.3. Bijectivity of the correspondence between canonical plane matrices and Jordan forms.

**Theorem 3.13.**

1. $A, A'$ - canonical plane $n \times n$ matrices. Then

   $$A \neq A' \implies A \neq A'.$$

2. $B, B'$ - any $n \times n$ matrices. Then

   $$B \neq B' \implies \text{Aff}(B) \neq \text{Aff}(B').$$

**Proof.**

1. Let $A \in \mathcal{P}_\pi$, $A' \in \mathcal{P}_{\pi'}$, $A \neq A'$. If $\pi \neq \pi'$ then the sets of addends of $\pi$ and $\pi'$ are different which by theorem 3.9 implies $\Omega(A) \neq \Omega(A')$. If $\pi = \pi'$ then $A = S + L(A)$, $A' = S + L(A')$ as in the proof of theorem 2.5. It follows that $L(A) \neq L(A')$ hence for at least one $\lambda \in \mathbb{C}$ the multiplicity functions $\mu(\lambda, i)$ for $A$ and $A'$ are different. By theorem 3.9 it follows that $\Omega(A, \lambda) \neq \Omega(A', \lambda)$.

2. Let $B \neq B'$. Then $\Omega(B) \neq \Omega(B')$. Hence there exist $m$ and $\lambda \in \mathbb{C}$ such that the multiplicities of $m$ in $\Omega(B, \lambda)$ and $\Omega(B', \lambda)$ are different. Considering $M(B)$ and $M(B')$ as explained at the beginning of 3.2.2 and theorem 3.10 we see that $L(B) \neq L(B')$. It follows that $\text{Aff}(B) \neq \text{Aff}(B')$. □
3.4. Conclusion. For any \( n \in \mathbb{N}, n \geq 2 \), define \( P = \bigcup_{\pi} P_{\pi} \), where the union is taken over all partitions of \( n \).

**Theorem 3.14.** \( P \) contains each \( n \times n \) matrix conjugacy class exactly once.

**Proof.** The statement follows from theorems 3.10 and 3.13. \( \square \)

3.5. Appendix - canonical plane matrices for two values of \( n \).

3.5.1. \( n = 2 \). There are two partitions of 2: \( \pi_1 = (2) \) and \( \pi_2 = (1, 1) \). The affine planes are

1. \( P_{\pi_1} = \left\{ \left[ \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right] : \forall a \in \mathbb{C} \right\} \), each matrix in \( P_{\pi_1} \) is in its Jordan form \( 2J_1(a) \),
2. \( P_{\pi_2} = \left\{ \left[ \begin{array}{ccc} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 1 & b \end{array} \right] : \forall a, b \in \mathbb{C} \right\} \), the Jordan form depends on the roots of the characteristic polynomial \( x^2 - bx - a \):
   (a) if there are 2 simple roots \( \lambda_1, \lambda_2 \) then \( J_1(\lambda_1) \oplus J_1(\lambda_2) \),
   (b) if there is 1 double root \( \lambda_0 \) then \( J_2(\lambda_0) \).

3.5.2. \( n = 5 \). We describe all canonical plane matrices for \( n = 5 \) and give the corresponding Jordan forms. One can check that there are 27 subcases which correspond to the 27 distinct Jordan forms for \( 5 \times 5 \) complex matrices. Below for each subcase distinct arguments for polynomial roots and Jordan blocks mean distinct complex numbers.

There are seven partitions of 5: \( \pi_1 = (5) \), \( \pi_2 = (4, 1) \), \( \pi_3 = (3, 2) \), \( \pi_4 = (3, 1, 1) \), \( \pi_5 = (2, 2, 1) \), \( \pi_6 = (2, 1, 1, 1) \), \( \pi_7 = (1, 1, 1, 1, 1) \). For each partition \( \pi \) we give the description of \( P_{\pi} \):

1. \( P_{\pi_1} = \left\{ \left[ \begin{array}{ccc} a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & a \end{array} \right] : \forall a \in \mathbb{C} \right\} \), each matrix in \( P_{\pi_1} \) is in its Jordan form \( 5J_1(a) \);
2. \( P_{\pi_2} = \left\{ \left[ \begin{array}{ccccc} a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & b \end{array} \right] : \forall a, b \in \mathbb{C} \right\} \), the Jordan forms:
   (a) \( 4J_1(a) \oplus J_1(b) \) if \( a \neq b \),
   (b) \( 3J_1(a) \oplus J_2(a) \) if \( a = b \).
3. $\mathcal{P}_{x_3} = \{ \begin{bmatrix} a & a \\ 1 & b \end{bmatrix} \}, \forall a, b \in \mathbb{C} \}$, the Jordan forms:
- (a) $3J_1(a) \oplus 2J_1(b)$ if $a \neq b$,
- (b) $J_1(a) \oplus 2J_2(a)$ if $a = b$.

4. $\mathcal{P}_{x_4} = \{ \begin{bmatrix} a & a \\ 1 & b \end{bmatrix} \}, \forall a, b, c \in \mathbb{C} \}$, the Jordan form depends on the roots of the polynomial $x^2 - cx - b$ - the characteristic polynomial of the block $\begin{bmatrix} 1 & b \\ c & a \end{bmatrix}$:
- (a) 2 simple roots $\lambda_1, \lambda_2, \lambda_1 \neq a - 3J_1(a) \oplus J_1(\lambda_1) \oplus J_1(\lambda_2)$,
- (b) 2 simple roots $\lambda_1, \lambda_2, \lambda_2 = a, \lambda_2 \neq a - 2J_1(a) \oplus J_2(a) \oplus J_1(\lambda_2)$,
- (c) 1 double root $\lambda_1 \neq a - 3J_1(a) \oplus J_2(\lambda_1)$,
- (d) 1 double root $\lambda_1 = a - 2J_1(a) \oplus J_3(a)$,

5. $\mathcal{P}_{x_5} = \{ \begin{bmatrix} a & a \\ 1 & b \end{bmatrix} \}, \forall a, b, c \in \mathbb{C} \}$, the Jordan form depends on the roots of the polynomial $x^3 - bx - a$ - the characteristic polynomial of the block $\begin{bmatrix} 1 & a \\ b & c \end{bmatrix}$:
- (a) 2 simple roots $\lambda_1, \lambda_2, \lambda_1 \neq c - J_1(c) \oplus 2J_1(\lambda_1) \oplus 2J_1(\lambda_2)$,
- (b) 2 simple roots $\lambda_1, \lambda_2, \lambda_1 = c, \lambda_2 \neq c - J_1(c) \oplus J_2(c) \oplus 2J_1(\lambda_2)$,
- (c) 1 double root $\lambda_1 \neq c - 2J_2(\lambda_1) \oplus J_1(c)$,
- (d) 1 double root $\lambda_1 = c - J_2(c) \oplus J_3(c)$,

6. $\mathcal{P}_{x_6} = \{ \begin{bmatrix} a & a \\ 1 & b \end{bmatrix} \}, \forall a, b, c, d \in \mathbb{C} \}$, the Jordan form depends on the roots of the polynomial $x^3 - dx^2 - cx - b$ - the characteristic polynomial of the block $\begin{bmatrix} 1 & b \\ c & d \end{bmatrix}$:
- (a) 3 simple roots $\lambda_1, \lambda_2, \lambda_3, \lambda_1 \neq a - 2J_1(a) \oplus J_1(\lambda_1) \oplus J_1(\lambda_2) \oplus J_1(\lambda_3)$,
- (b) 3 simple roots $\lambda_1, \lambda_2, \lambda_3, \lambda_1 = a, \lambda_2, \lambda_3 \neq a - J_1(a) \oplus J_2(a) \oplus J_1(\lambda_2) \oplus J_1(\lambda_3)$,
- (c) 1 double root $\lambda_1 \neq a$ and 1 simple root $\lambda_3 \neq a - 2J_1(a) \oplus J_2(\lambda_1) \oplus J_1(\lambda_3)$,
- (d) 1 double root $\lambda_1 = a$ and 1 simple root $\lambda_3 \neq a - J_1(a) \oplus J_3(a) \oplus J_1(\lambda_3)$,
- (e) 1 double root $\lambda_1 \neq a$ and 1 simple root $\lambda_3 = a - J_1(a) \oplus J_2(a) \oplus J_2(\lambda_1)$,
(f) 1 triple root $\lambda_1 \neq a - 2J_1(a) \oplus J_3(\lambda_1)$,
(g) 1 triple root $\lambda_1 = a - J_1(a) \oplus J_4(a)$;

7. $\mathcal{P}_{\pi_7} = \left\{ \begin{bmatrix} 1 & a \\ b & 0 \\ c & 1 \\ d & 1 \\ e & 1 \end{bmatrix} : \forall a, b, c, d, e \in \mathbb{C} \right\}$, the Jordan form depends on the roots of the characteristic polynomial of the matrix $x^5 - ex^4 - dx^3 - cx^2 - bx - a$:

(a) 5 simple roots $\lambda_i$, $i \in \{1, \ldots, 5\} - \bigoplus_{i=1}^{5} J_1(\lambda_i)$,
(b) 1 double root $\lambda_1$ and 3 simple roots $\lambda_i$, $i \in \{3, \ldots, 5\} - J_2(\lambda_1) \oplus J_1(\lambda_3) \oplus J_1(\lambda_4) \oplus J_1(\lambda_5)$,
(c) 2 double roots $\lambda_1, \lambda_3$ and 1 simple root $\lambda_5 - J_2(\lambda_1) \oplus J_2(\lambda_3) \oplus J_1(\lambda_5)$,
(d) 1 triple root $\lambda_1$ and 2 simple roots $\lambda_4, \lambda_5 - J_3(\lambda_1) \oplus J_1(\lambda_4) \oplus J_1(\lambda_5)$,
(e) 1 triple root $\lambda_1$ and 1 double root $\lambda_4 - J_3(\lambda_1) \oplus J_2(\lambda_4)$,
(f) 1 root of order 4 $\lambda_1$ and 1 single root $\lambda_5 - J_4(\lambda_1) \oplus J_1(\lambda_5)$,
(g) 1 root of order 5 $\lambda_1 - J_5(\lambda_1)$.

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