Singular Vectors by Fusions in $A_1^{(1)}$

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**ABSTRACT**

Explicit expressions for the singular vectors in the highest weight representations of $A_1^{(1)}$ are obtained using the fusion formalism of conformal field theory.
We study in this letter some aspects of the representation theory of $A^{(1)}_1$. The subject was studied thoroughly by mathematicians [1] and physicists [2] in the last years. It is known that a Verma module (i.e., highest weight representation) is reducible for certain values of the highest weight. This indicates the existence of a singular vector in the module, that generates a submodule. An irreducible representation is obtained by quotienting the submodules out, or equivalently by setting the singular vectors to zero. The later point of view is the most appropriate to physical applications.

Conformal field theory Ward identities permit us to express any correlation function that contains descendents as a differential operator that acts on the correlation function with the descendent replaced by its primary [3]. A singular vector is a descendent that we equate to zero. We get, therefore, a differential equation for any correlation function that includes the corresponding primary field. The explicit form of this equation is fixed by the form of the singular vector. Explicit forms for singular vectors in Verma modules for the Virasoro algebra were obtained recently by Bauer, Di Francesco, Itzykson, and Zuber [4]. Forms for $A^{(1)}_N$ were given by Malikov, Feigin and Fuks [5]. The form that M.F.F. found reflects the symmetries of the $A^{(1)}_N$ algebra, but it is not very suitable to physical application. Our approach is based on basic facts and methods in conformal field theory, with which the form of the singular vectors for $A^{(1)}_1$ is derived efficiently. This approach is in the same spirit of B.D.I.Z. and we have analogously a mysterious relation between the singular vectors and the classical w-algebras [4].

We present the infinite-dimensional Lie algebra $A^{(1)}_1$ as a current algebra. The generators are $J^n_a \ n \in \mathbb{Z} \ a = +, 0, -$ and $K$. The non-vanishing commutators are

\[ [J^+_n, J^-_m] = 2J^0_{n+m} + nK\delta_{n+m} \]
\[ [J^0_n, J^\pm_m] = \pm J^\pm_{n+m} \]
\[ [J^0_n, J^0_m] = \frac{1}{2} nK\delta_{n+m} \]

The generators $J^n_a$ are doubly graded, with respect to $a$ (generated by $J^0_0$) and to $n$ (generated by $L_0$). It is convenient to introduce a combined gradation $d(n, a) = 2n + a$. We are interested in the representations generated by a highest weight vector. The highest weight vector is defined by

\[ J^n_a |J, t> = 0 \ m(n, a) \geq 1 \]
\[ J^0_0 |J, t> = J|J, t> \]
\[ K|J, t> = (t - 2)|J, t> \quad J, t \in \mathbb{C} \]
Notice that $J$ is not a half-integer as usual but takes value in the complex plane, and we work with complex central charge as well. All the states are reached by applying repeatedly $J_n^a$ with $d(n,a) \leq -1$ on the highest weight, and taking linear combinations. A combination $|R> = \sum A(\{a_i\}, \{n_i\})J_{n_1}^{a_1} \cdots J_{n_s}^{a_s} |J,t>$ is called a descendent. A descendent $|R>$ that satisfies $J_n^a |R> = 0$ for $d(n,a) \geq 1$ is called a singular vector. It is easy to see that singular vectors are linear combinations of homogeneous singular vectors w.r.t the graduation $L_0$ and $J_0^0$. We denote the homogeneous part as $|R> \equiv |J,t,n,m>$

\[
|R> \equiv |J,t,n,m> = \sum_{\sum a_i = m, \sum n_i = n} A(\{a_i\}, \{n_i\})J_{n_1}^{a_1} \cdots J_{n_s}^{a_s} |J,t>
\]

**Theorem:** (Kac-Kazhdan [6])

The highest weight $|J,t \neq 0>$ generates a reducible module iff $J$ takes the value $J_{r,s,\pm}$ defined by

\[
2J_{r,s,\pm} + 1 = \pm(r - (s-1)t) \quad r, s \in \mathbb{N}
\]

(1)

The singular vector is in $|J_{r,s,\pm}, t \neq 0, r(s-1), \pm r>$.  

**Theorem** (Malikov, Feigin and Fuks [5])

The form of the singular vector in $|J_{r,s,+}, t \neq 0>$ is

\[
(J_0^+)^{r+(s-1)t}(J_{-1}^+)^{r+(s-2)t} \cdots (J_{-1}^+)^{r-(s-2)t}(J_0^-)^{r-(s-1)t} |J_{r,s,+}, t \neq 0>
\]

(2)

and $J_0^- \leftrightarrow J_{-1}^+$ for the sign $(-)$ in (1). This is not to be interpreted as multiplication of operators raised to complex powers, but in the following way: for $t \in \mathbb{N}$ (2) is well defined, and by ordering the generators and applying the Poincaré Birkhoff Witt theorem, this vector can be expressed as

\[
\sum_{p,q=0}^{\infty} P_{p,q}(\{J_n^a\}, t)(J_{-1}^+)^{r(s-1)-p}(J_0^-)^{r(s-1)-q} |J_{r,s,+}, t>
\]

(3)

where $p, q$ are non negative integers, $P_{p,q}$ are polynomials in the $J_n^a$ with $d(n,a) \leq -2$. The heart of the proof is to show that $P_{p,q}$ are polynomials in $t$, and this allows to analytically continue to the whole complex plane. Actually calculating the r.h.s of (3), by applying successively the analytical continued commutation relations turns out to be tedious. It is the aim of this letter to give a short-cut for it.

Let us consider for the moment only the $+$ sign in (1) then $J = j - j' t$ where $2j + 1, 2j' + 1 \in \mathbb{N}$. We denote the corresponding highest weight representation as $(j, j')$. The
singular vector in the class of representation of type \((0, j)\) admits a matrix form, the \(A_1^{(1)}\) analog of the formulae of Benoit and Saint-aubin \(\overline{[7]}\) and of B.D.I.Z in the case of Virasoro.

Let \(n = 2j + 1\). We define the \(2n - 1\) dimensional vectors:

\[
\vec{f} = (f_{J+n-1}, g_{J+n-1}, f_{J+n-2}, \cdots, g_{J+1}, f_J)^t
\]
\[
\vec{F} = (-g_{J+n}, 0, \cdots, 0)^t
\]

where \(f_J \equiv |J, t>\) and the other components are defined by solving the triangular system

\[
\vec{F} = \left( J^- + \sum_{s=0}^{\infty} R(s+1) t^s K^s \right) \vec{f}
\]

(4)

where

\[
(J^-)_{i,j} = \delta_{j,i+1} \quad i, j = 1, \ldots, 2n - 1
\]
\[
R(s+1)_{i,j} = (J_0^0 + \frac{s}{2} \delta_{s, i}^{(2)} + (J^+_{0+\frac{s}{2}} \delta_{i,0}^{(2)} + J^-_{-\frac{s}{2}} \delta_{i,1}^{(2)}) \delta_{s,0}^{(2)})(-1[\frac{s}{2}]) \delta_{i,j}
\]
\[
(K)_{i,j} = ((n - \frac{i+1}{2}) \delta_{i,1}^{(2)} + \frac{i}{2} \delta_{i,0}^{(2)}) \delta_{j,i+1}
\]

where \([x]\) is the integer part of \(x\), and

\[
\delta_{i,j}^{(2)} = \begin{cases} 
1 & i = j \mod 2 \\
0 & \text{otherwise}
\end{cases}
\]

In fact \(K^{2n-1} \equiv 0\), so the sum in (4) is finite. One can show that the components of \(\vec{f}, \vec{F}\) satisfy

\[
J_0^+ f_{J+r} = 0 \quad \quad \quad \quad J_1^- f_{J+r} = -rtg_{J+r}
\]
\[
J_0^+ g_{J+r} = (n-r)t f_{J+r-1} \quad \quad \quad \quad J_1^- g_{J+r} = 0
\]

from which is clear that \(g_{J+n}\) is a singular vector. Take for example the representation \((0, \frac{1}{2})\) (that is \(J = -\frac{1}{2}t\)), then (4) reads

\[
\begin{pmatrix}
-g_{J+2} \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
J_0^- & tJ_{-1}^0 & -t^2J_{-1}^- \\
1 & J_{+1}^+ & tJ_{-1}^0 \\
0 & 1 & J_0^-
\end{pmatrix} \begin{pmatrix}
f_{J+1} \\
g_{J+1} \\
f_J
\end{pmatrix}
\]

which gives the singular vector

\[
g_{J+2} = (J_0^- J_{-1}^+ J_0^- - tJ_0^- J_{-1}^0 - tJ_{-1}^0 J_0^- - t^2 J_{-1}^-) f_J
\]
Notice that in the “classical limit”, and after “gauging” in the spirit of Hamiltonian reduction
\[
J_0^- \to d \\
J_0^0 \to 0 \\
J_{-n}^+ \to 0 \quad n \neq -1 \\
J_{-1}^+ \to 1 \\
(-1)^{\frac{n+1}{2}} t^{2n} J_n^- \to w_{n+1}
\]
this matrix, that maps \( f_J \) to \( g_{J+n} \), becomes the covariant differential operator of Drinfeld-Sokolov \([8]\) that maps \((1-n)/2\) forms to \((1+n)/2\) forms.

We show now how the matrix form appears naturally from fusions of primary fields in conformal field theory. Our main result will be a recursive formula for the components of \( \vec{f} \) from which the matrix can be recovered. The advantage of this approach is that it gives us also the form of the singular vector in a general \((j,j')\) reducible representation. It gives us important information about fusion rules as well.

We introduce a chiral primary field \( \phi^J_m(z) \) w.r.t the Virasoro algebra as well as w.r.t \( A^{(1)}_1 \). It transforms as a vector under the horizontal algebra (the zero modes algebra):

\[
[J^a_0, \phi^J_m(z)] = R^a_{mn} \phi^J_n(z)
\]

Following Zamolodchikov and Fateev \([9]\) we introduce an auxiliary parameter in order to have

\[
[J^a_0, \phi^J(z, x)] = R^a(x) \phi^J(z, x)
\]

where \( R^a(x) \) is a differential operator.

The correspondence fields-states is given by \( \lim_{z \to 0} \lim_{x \to 0} \phi^J(z, x)|\Omega, t > = |J, t >. \)
Here \( |\Omega, t > \) is the vacuum which is characterized as a highest weight state that is annihilated by the whole horizontal algebra.

The Virasoro algebra and \( A^{(1)}_1 \) are related by the Sugawara construction

\[
L_n = \sum (\ : J^0_{n-m} J^0_m + \frac{1}{2} J^+_{n-m} J^-_m + \frac{1}{2} J^-_{n-m} J^+_m :)
\]

In this formulation \( L_{-1} \) and \( J_0^- \) generate translations in \( z \) and in \( x \) respectively. Thus, we can write

\[
\phi^J(z, x) = e^{x J_0^- + z L_{-1}} \phi^J(0, 0) e^{-x J_0^- - z L_{-1}}
\]
It follows that

\[
[J_n^-, \phi^J(z, x)] = z^n \frac{d}{dx} \phi^J(z, x)
\]

\[
[J_n^0, \phi^J(z, x)] = z^n (-x \frac{d}{dx} + J) \phi^J(z, x)
\]

\[
[J_n^+, \phi^J(z, x)] = z^n (-x^2 \frac{d}{dx} + 2x J) \phi^J(z, x)
\]

Let us look now at the short distance operator product expansion for these chiral primary fields. For this aim it is more convenient to write

\[
\phi^J(z, x) = z^{-h+L_0 x^J - J^0_0 \phi^J(1, 1)} x^J_0 \hat{z}^{-L_0}
\]

which is a consequence of (5). Imagine that we are interested only in the \( J \) sector in the fusion of \( J_0 \) and \( J_1 \) then

\[
\phi^{J_0}(z, x)|J_1, t> = \phi^{J_0}(z, x)\phi^{J_1}(0, 0)|\Omega, t>
\]

\[
= z^{-h_0 - h_1 + L_0 x^J_0 + J_1 - J^0_0 \phi^{J_0}(1, 1)} \phi^{J_1}(0, 0)|\Omega, t>
\]

\[
\text{in sector } \frac{J}{2} \sum_{n=0}^{\infty} \sum_{m=-n}^{\infty} z^{h_0 - h_1 + n x^J_0 + J_1 + J_0 + m} \psi^J_{n,m}(0, 0)|\Omega, t>
\]

\( \psi^J_{n,m} \) are fixed by the requirement that the two sides of (6) transform in the same manner under the Vir and \( A_1^{(1)} \) algebras. Thus, for example

\[
\phi^{J_0}(z, x)L_0 \phi^{J_1}(0, 0)|\Omega, t> = h_1 \phi^{J_0}(z, x)\phi^{J_1}(0, 0)|\Omega, t>
\]

\[
= (L_0 - \frac{d}{dz}) \phi^{J_0}(z, x)\phi^{J_1}(0, 0)|\Omega, t>
\]

Plugging (6), we obtain

\[
L_0 \psi^J_{n,m} = (h + n) \psi^J_{n,m}
\]

and in the same way

\[
J^0_0 \psi^J_{n,m} = (J - m) \psi^J_{n,m}
\]

This fixes \( \psi^J_{n,m} \) to be in the homogeneous subspace of conformal degree \( n \) and charge \(-m\), as expected from (6). For the generators \( J^a_n \ d(n, a) \geq 1 \) there are only two independent equations

\[
J^+_0 \psi^J_{n,m} = (J - J_0 + J_1 - m + 1) \psi^J_{n,m-1}
\]

\[
J^-_1 \psi^J_{n,m} = (-J + J_0 + J_1 + m + 1) \psi^J_{n-1,m+1}
\]
These descent equations determine $\psi^J_{n,m}$ as long as the kernel of $J^+_0, J^-_1$ is trivial. The existence of a singular vector in $|J, t, n_0, m_0>$ means that the kernel of $J^+_0, J^-_1$ is non-trivial, and as a consequence $J_0$, $J_1$ and $J$ must verify a relation in order for (7) to have a solution. This relation is the fusion rule. Take the vacuum sector for example, $\psi^0_{o,o} \equiv |\Omega, t>$ and $\psi^0_{o,1} = A_{o,1}J_0^-\psi^0_{o,o}$ is a singular vector. The non-trivial descent equation gives $0 = J^+_0\psi^0_{o,1} = (J_1 - J_0)\psi^0_{o,o}$ and we see that only the fusion of a primary field with itself contains the vacuum sector.

A solution to the descent equations is obtained using the Knizhnik- Zamolodchikov equation [10] combined with the fusion procedure. We write

$$(tL_{-1} - (J^+_1 J^-_0 + 2J_1 J^0_{-1}))\phi J_1(0, 0)|\Omega, t > = 0$$

(8)

multiply on the left by $\phi^J_0(z, x)$ to get

$$0 = \phi^J_0(z, x)(tL_{-1} - (J^+_1 J^-_0 + 2J_1 J^0_{-1}))\phi J_1(0, 0)|\Omega, t >$$

(9)

We commute the $A^{(1)}_1$ and Vir operators to the left using (5), plugging (6), and after some manipulations we obtain

$$(nt + m(2J + 1 - m))\psi^J_{n,m} = (-J + J_0 + J_1 + m + 1) \sum_{k+l=n, k \geq 1} J^+_{-k}\psi^J_{l,m+1}$$

$$+ 2(J - J_0 - m) \sum_{k+l=n, k \geq 1} J^0_{-k}\psi^J_{l,m}$$

$$+ (J - J_0 + J_1 + m + 1) \sum_{k+l=n, k \geq 0} J^-_{-k}\psi^J_{l,m-1}$$

(10)

If $nt + m(2J + 1 - m)$ does not vanish for any pair of integers $n \geq 0$ $m \geq -n$ then the $\psi^J_{n,m}$ are uniquely determined starting from $\psi^J_{0,0}$, and can be shown recursively to satisfy the descent equations. If $J$ is such that $n_0 t + m_0(2J + 1 - m_0) = 0$ $n_0 \geq 0$ $m_0 \geq -n_0$ the same is true provided $n < n_0$ or $m + n < m_0 + n_0$, and for $(n, m) = (n_0, m_0)$, the right hand side of (10) is annihilated by $J^+_0$ and $J^-_1$. In this case there are two possibilities. If $m_0$ does not divide $n_0$ then both sides of (10) vanish. If $m_0$ divides $n_0$ then the r.h.s is not trivial in general, and it is in the kernel of $J^+_0$ and $J^-_1$, that is, it is the wanted singular vector. The descendents $\psi^J_{n,m}$ $0 < n < n_0$, $-n < m < 2n_0 - n$ and $-n_0 < m < m_0$ for $n = n_0$ can be arranged as a column vector. Equation (10) then can be written in a matrix form, analogous to (4). Details and proofs will be given elsewhere. Quantum Hamiltonian reduction [11] between the physical spaces of Vir and $A^{(1)}_1$, and the relation between the $A^{(1)}_1$ singular vectors and classical w-algebra are under investigation.
References

[1] See for example:
V. Kac, Infinite dimensional Lie algebras. Cambridge: Cambridge University Press 1985, and references therein.

[2] Fractional levels appear naturally in 2d quantum gravity:
A.M. Polyakov, Mod. Phys. Lett. A 2 (1987) 893.;
V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys Lett A 3 (1988) 819.;
D. Bernard and G. Felder, Comm. Math. Phys. 127 (1990) 145.

[3] A. Belavin, A.M Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1983) 333.

[4] M. Bauer, Ph.Di Francesco, C. Itzykson and J.-B. Zuber, “Covariant Differential Equations and Singular Vectors in Virasoro Representations”, Saclay Preprint SPht/91-030. to appear in Nucl. Phys. B.;
See also
M. Bauer, “Singular Vectors in Virasoro Verma Modules”, Saclay Preprint SPht/91-096, to appear in the Proceeding of the Trieste Workshop on String Theory, April 24-26, 1991, Trieste, Italy.

[5] F.G. Malikov, B.L. Feigin and D.B Fuks, Funkt.Anal. Prilozhen, 20 No. 2 (1987) 25

[6] V.G. Kac and D.A. Kazhdan, Adv. Math 34 (1979) 97.

[7] L. Benoit and Y. Saint-Aubin, Phys. Lett. 215B (1988) 517.

[8] V.G. Drinfeld and V.V. Sokolov, Journ. Sov. Math. 30 (1985) 1975.

[9] A.B. Zamolodchikov and V.A. Fateev, Sov. J. Nucl. Phys. 43 (1986) 657.

[10] V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B247 (1984) 83.

[11] A.A Belavin, Advanced Studies in Pure Mathematics 19 (1989) 117.;
M. Bershadsky and H.Ooguri, Comm. Math. Phys. 126 (1989) 49.;
For approach akin to ours, see
P. Furlan, A.Ch. Ganchev, R. Paunov and V.B. Petkova, “Reduction of the Rational Spin $sl(2,C)$ WZNW conformal Theory”, Preprint SISSA-67/91/FM, KA-THEP-1991-3.