RIGIDITY OF STRICTLY CONVEX DOMAINS IN EUCLIDEAN SPACES

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Abstract. In this paper, we prove a rigidity theorem for smooth strictly convex domains in Euclidean spaces.

1. Introduction

In [3], the authors proved a rigidity theorem for geodesic balls in Euclidean spaces [3, Theorem 1.7]. The main purpose of this paper is to generalize such a rigidity theorem to all strictly convex domains with smooth boundary in \( \mathbb{R}^n \). More precisely, we have the following main theorem of the paper.

Theorem 1.1. Let \((M, g)\) be a strictly convex domain with smooth boundary in \(\mathbb{R}^n\) \((n \geq 2)\). Let \((N, \bar{g})\) be a spin Riemannian manifold with boundary and \(f : N \to M\) be a spin map. If

1. \(\text{Sc}(\bar{g})_x \geq \text{Sc}(g)_{f(x)} = 0\) for all \(x \in N\),
2. \(H_{\bar{g}}(\partial N)_y \geq H_g(\partial M)_{f(y)}\) for all \(y \in \partial N\),
3. \(f\) is distance-non-increasing on \(N\),
4. the degree of \(f\) is nonzero,

then \(f\) is an isometry.

As a special case of the above theorem, we see that, given a strictly convex domain with smooth boundary in \(\mathbb{R}^n\), one cannot increase its metric, its scalar curvature and the mean curvature of its boundary simultaneously.

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2. Proof of Theorem 1.1

The odd dimensional case reduces to the even dimensional case by considering \(f \times \text{id} : N \times [0, 1] \to M \times [0, 1]\) and applying the index theory for manifolds with corners developed in [4]. Hence without loss of generality, we may assume that \(N\) and \(M\) are even-dimensional.

Let \(S_N\) and \(S_M\) be the spinor bundles over \(N\) and \(M\). Consider the vector bundle \(S_N \otimes f^*S_M\) over \(N\), which carries a natural Hermitian metric and a unitary connection \(\nabla\) compatible with the Clifford multiplication by elements of \(\Cl(TN) \otimes f^*\Cl(TM)\). We will denote the Clifford multiplication of a vector \(\overline{v} \in TN\) by \(c(\overline{v})\) and the Clifford multiplication of a vector \(v \in f^*TM\) by \(c(v)\).

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Consider the associated Dirac operator $D$ on $S_N \otimes f^* S_M$:

$$D = \sum_{i=1}^{n} \bar{e}(e_i) \nabla e_i,$$

where $\{e_i\}$ is a local orthonormal basis of $TN$ and

$$\nabla = \nabla^{SN} \otimes 1 + 1 \otimes f^*(\nabla^{SM})$$

is the connection induced by the spinor connections on $S_N$ and $f^* S_M$.

Let $B$ be the local boundary condition on $\partial N$ given by

$$(\bar{\epsilon} \otimes \epsilon)(\bar{e}(e_n(x)) \otimes c(\epsilon_n(x)))\varphi(x) = -\varphi(x), \quad \forall x \in \partial N,$$

for all smooth sections $\varphi$ of $S_N \otimes f^* S_M$, where $\bar{e}$ (resp. $\epsilon$) is the $\mathbb{Z}_2$-grading operator on $S_N$ (resp. $f^* S_M$), and $\bar{e}_n(x)$ (resp. $\epsilon_n(x)$) is the unit inner normal vector at $x$ (resp. at $f(x)$).

**Proof of Theorem 1.1.** In even dimensions, it follows from [2, Theorem 1.1] that

1. $\text{Sc}(g)_x = \text{Sc}(f)_{f(x)} = 0$ for all $x \in N$,
2. $H^g(\partial N)_y = H^f(\partial M)_{f(y)}$ for all $y \in \partial N$.

In fact, the proof of [2, Theorem 1.1] shows that there exists a non-zero parallel section $\varphi$ of $S_N \otimes f^* S_M$ satisfying the boundary condition $B$, i.e., $\nabla \varphi = 0$. As mentioned above, the odd dimensional case can be reduced to the even dimensional case by applying [4, Theorem 1.8]. Moreover, since $\partial M$ is strictly convex, it follows from [3, Proposition 3.1] that $f$: $\partial N \rightarrow \partial M$ is a local isometry.

By [3, Theorem 2.3], $(N, \mathcal{F})$ is also flat. Since we shall need part of the argument of [3, Theorem 2.3] for the proof of the current theorem, let us repeat some key steps from the proof of [3, Theorem 2.3] for the convenience of the reader. Since $\nabla$ is a Hermitian connection and $\varphi$ is a nonzero parallel section, we may assume that $|\varphi| = 1$ everywhere on $N$. Since $(M, g)$ is a strictly convex domain in $\mathbb{R}^n$, let $\{v_1, v_2, \ldots, v_n\}$ be the standard basis of $\mathbb{R}^n$, which we shall also view as parallel sections of $TM$. As $M$ is strictly convex, there exist $y_1, y_2, \ldots, y_n$ in $\partial M$ such that the inner normal vector of $\partial M$ at $y_i$ is equal to $v_i$. Let $x_i \in \partial N$ be a point in the preimage $f^{-1}(y_i)$.

Let $\Lambda$ be the collection of all subsets of $\{1, 2, \ldots, n\}$. For $\lambda \in \Lambda$, we define

$$w_\lambda = \wedge_{i \in \lambda} v_i \in \Lambda^* TM$$

Note that $\{w_\lambda\}_{\lambda \in \Lambda}$ are parallel sections of $\Lambda^* TM$ that form an orthonormal basis of $\Lambda^* T_x M$ at every point $x \in M$.

With the section $\varphi$ of $S_N \otimes f^* S_M$ above, we define

$$\varphi_\lambda = (1 \otimes c(w_\lambda))\varphi.$$  

Since $w_\lambda$ is parallel, we see that $\varphi_\lambda$ is also parallel with respect to the connection $\nabla$. Note that $\nabla$ is a Hermitian connection that preserves the inner product on $S_N \otimes f^* S_M$. Therefore, for any pair of elements $\lambda, \mu \in \Lambda$, the function $\langle \varphi_\lambda(x), \varphi_\mu(x) \rangle$ (as $x$ varies over $N$) is a constant function.

**Claim.** The parallel sections $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ are mutually orthogonal.
Note that dim$(S_N \otimes f^*S_M) = 2^n = |\Lambda|$, where $|\Lambda|$ is the cardinality of the set $\Lambda$. Thus if the claim holds, then the curvature form of $S_N \otimes f^*S_M$ vanishes. Since $M$ is flat, the curvature form of $S_N \otimes f^*S_M$ is equal to the curvature form $R_{\gamma}^{S_N}$ of $S_N$. By [1, Theorem 2.7], we have

$$R_{\gamma}^{S_N} = \frac{1}{2} R^{\gamma}_{X,Y} \cdot \sigma, \quad \text{for all } \sigma \in \Gamma(S_N) \text{ and } X, Y \in \Gamma(TN),$$

where $R^{\gamma}$ is the curvature form of the Levi–Civita connection on $TN$ with respect to $\gamma$. It follows that $R^{\gamma} = 0$, that is, $\gamma$ is flat.

Now we prove the claim. For each $\lambda \in \Lambda$ and $x \in M$, we denote by $V_\lambda$ the subspace in $T_xM \cong \mathbb{R}^n$ spanned by $\{v_i\}_{i \in \lambda}$.

Let $\lambda$ and $\mu$ be two distinct members of $\Lambda$. Without loss of generality, we assume there exists $1 \leq k \leq n$ such that $k \in \mu$ and $k \notin \lambda$. Equivalently, we have $v_k \in V_\lambda^\perp \cap V_\mu$. Let $\bar{v}_k$ be the unit inner normal vector at $x_k$ of $\partial N$. Recall that the section $\varphi$ satisfies the following boundary condition at $x_k$:

$$(\bar{\varphi} \otimes \epsilon)(\bar{\varphi}(\bar{v}_k) \otimes c(v_k))\varphi(x_k) = -\varphi(x_k).$$

Note that for a vector $v \in \mathbb{R}^n$, we have

$$c(w_\lambda)c(v) = \begin{cases} (-1)^{|\lambda|}c(v)c(w_\lambda), & \text{if } v \in V_\lambda^\perp, \\ (-1)^{|\lambda|-1}c(v)c(w_\lambda), & \text{if } v \in V_\lambda, \end{cases}$$

where $|\lambda|$ is the cardinality of the set $\lambda$. Therefore $\varphi_\lambda$ and $\varphi_\mu$ satisfy the following equations at $x_k$:

$$(\bar{\varphi} \otimes \epsilon)(\bar{\varphi}(\bar{v}_k) \otimes c(v_k))\varphi_\lambda(x_k) = -\varphi_\lambda(x_k),$$

$$(\bar{\varphi} \otimes \epsilon)(\bar{\varphi}(\bar{v}_k) \otimes c(v_k))\varphi_\mu(x_k) = \varphi_\mu(x_k).$$

It follows that $\langle \varphi_\lambda, \varphi_\mu \rangle$ vanishes at $x_k$, hence everywhere on $N$. This proves the claim, hence shows that $(N, \gamma)$ is flat.

We will prove that $f$ preserves the second fundamental forms of $\partial N$ and $\partial M$. Before that, we shall need the following key observation. For any two points $x, y \in \partial N$, we denote by $\vec{v}_x$ and $\vec{v}_y$ the inner normal vectors at $x$ and $y$ in $\partial N$, and $v_x$ and $v_y$ the unit inner normal vectors at $f(x)$ and $f(y)$ in $\partial M$, respectively. Pick a path $\gamma$ connecting $x$ and $y$, then the parallel transport of $v_y$ along $\gamma$ defines a vector field. In particular, $v_y$ at $x$ is independent of the choice of the path, since $N$ is flat. We claim that

$$\langle \vec{v}_x, \vec{v}_y \rangle = \langle v_x, v_y \rangle.$$

Recall that the section $\varphi$ satisfies the boundary condition at $y$:

$$(\bar{\varphi}(\bar{v}_y) \otimes 1)\varphi(y) = -(1 \otimes \epsilon c(v_y))\varphi(y).$$

Since $\varphi$ is parallel and $v_y$ is parallel along $\gamma$, we have that $(\bar{\varphi}(\bar{v}_y) \otimes 1)\varphi$ and $(1 \otimes \epsilon c(v_y))\varphi$ are both parallel along $\gamma$. By the uniqueness of the parallel transport, we see that at $x$

$$(\bar{\varphi}(\bar{v}_y) \otimes 1)\varphi(x) = -(1 \otimes \epsilon c(v_y))\varphi(x).$$
The boundary condition at $x$ yields that

$$(\partial_x (\bar{\nu}_x) \otimes 1) \varphi(x) = -(1 \otimes \epsilon c(v_x)) \varphi(x).$$

Therefore for any two real numbers $a_1$ and $a_2$, we have

$$(\partial_x (a_1 \bar{v}_x + a_2 \bar{v}_y) \otimes 1) \varphi(x) = -(1 \otimes \epsilon c(a_1 v_x + a_2 v_y)) \varphi(x).$$

Since $\varphi(x) \neq 0$, by taking the norm of both sides of the above equation, we see that

$$|a_1 \bar{v}_x + a_2 \bar{v}_y| = |a_1 \bar{v}_x + a_2 \bar{v}_y|$$

for all $a_1, a_2 \in \mathbb{R}$. It follows that $\langle \bar{v}_x, \bar{v}_y \rangle = \langle v_x, v_y \rangle$.

We now show that $f$ preserves the second fundamental forms of $\partial N$ and $\partial M$. For any $x \in \partial N$, we choose $n$ points $\{x_1, \ldots, x_n\}$ near $x$ in $\partial M$ so that the set $\{v_1, \ldots, v_n\}$ is linearly independent in $\mathbb{R}^n$, where $v_i$ is the unit inner normal vector of $\partial M$ at $f(x_i)$. Such a set of points always exits since $\partial M$ is strictly convex. Let $\bar{v}_i$ be the inner normal vector of $\partial N$ at $x_i$. The parallel transport of each $\bar{v}_i$ gives a parallel vector field near $x$, which we still denote by $\bar{v}_i$. The same argument above shows that

$$\langle \bar{v}_i, \bar{v}_j \rangle = \langle v_i, v_j \rangle, \forall i, j.$$  

In particular, the set of vectors $\{\bar{v}_1, \ldots, \bar{v}_n\}$ is also linearly independent.

Let $\bar{v}$ (resp. $v$) be the unit inner normal vector field of $\partial N$ (resp. $\partial M$) near $x$ (resp. $f(x)$). The same argument above again shows that $\langle \bar{v}, \bar{v}_i \rangle = \langle v, v_i \rangle$ for all $1 \leq i \leq n$. Therefore, $\bar{v}$ and $v$ can be written as linear combinations of $\bar{v}_i$’s and $v_i$’s with the same coefficients. In other words, there are smooth functions $k_1, \ldots, k_n$ defined on a neighborhood of $x$ in $\partial M$ such that

$$v = \sum_{i=1}^n k_i v_i \text{ and } \bar{v} = \sum_{i=1}^n (f^* k_i) \bar{v}_i.$$  

Let $\bar{w}$ be an arbitrary vector field tangent to $\partial N$. Since $v_i$ and $\bar{v}_i$ are parallel, we have

$$\nabla_{f^* \bar{w}}^M v = \sum_{i=1}^n f_*(\bar{w}(k_i)) \cdot v_i \text{ and } \nabla_{\bar{w}}^N \bar{v} = \sum_{i=1}^n \bar{w}(f^* k_i) \cdot v_i.$$  

Note that $\bar{w}(f^* k_i) = f^*(f_* \bar{w}(k_i))$ by the chain rule. Therefore for any vector fields $\bar{w}, \bar{v}$ tangent to $\partial N$, we have

$$\langle \nabla_{\bar{w}}^N \bar{v}, \nabla_{\bar{w}}^N \bar{v} \rangle = \langle \nabla_{f^* \bar{w}}^M v, \nabla_{f^* \bar{w}}^M v \rangle.$$  

Let $\{\bar{w}_1, \ldots, \bar{w}_{n-1}\}$ be a local orthonormal basis of $T\partial N$ near $x$. As $f$ is a local isometry from $\partial N$ to $\partial M$, $\{f_* \bar{w}_1, \ldots, f_* \bar{w}_{n-1}\}$ is also a local orthonormal basis of $T\partial M$ near $f(x)$. Let $A = (A_{jk})$ and $\bar{A} = (\bar{A}_{jk})$ be the second fundamental forms of $\partial M$ and $\partial N$, that is,

$$\nabla_{\bar{w}_j}^N \bar{v} = -\sum_{k=1}^{n-1} \bar{A}_{jk} \bar{w}_k, \text{ and } \nabla_{f_* \bar{w}_j}^M v = -\sum_{k=1}^{n-1} A_{jk} f_* \bar{w}_k.$$  

Since $M$ is strictly convex, we assume without loss of generality that $A$ is a diagonal matrix with positive diagonal entries.
By rewriting \( \langle \nabla_{\pi}^N w, \nabla_{\pi}^N v \rangle = \langle \nabla_{f,\pi}^M u, \nabla_{f,\pi}^M v \rangle \) in terms of the above matrix entries, we obtain that
\[
A^2 = A^2.
\]
Since \( \bar{A} \) is symmetric, we have that \( O = \bar{A}A^{-1} \) is an orthogonal matrix. Note that
\[
\text{tr}(A) = \text{tr}(O) = \sum_{j=1}^{n-1} O_{jj} A_{jj} \leq \sum_{j=1}^{n-1} A_{jj} = \text{tr}(A),
\]
where the second equality is because \( A \) is diagonal, and the third inequality is because \( |O_{jj}| \leq 1 \), since \( O \) is orthogonal. Recall that the mean curvature of \( N \) and \( M \) are equal, that is, \( \text{tr}(A) = \text{tr}(A) \). This implies that \( O_{jj} = 1 \) for each \( 1 \leq j \leq n - 1 \). Therefore, \( O \) is the identity matrix. It follows that \( f \) preserves the second fundamental forms.

Let \( \tilde{N} \) be the universal cover of \( N \) with the lift metric. Fix a point \( \tilde{x} \) in \( \tilde{N} \) and an orthonormal frame at \( \tilde{x} \). The parallel transport of this orthonormal frame at \( \tilde{x} \) defines a set of global orthonormal basis \( \{ \tilde{e}_1, \ldots, \tilde{e}_n \} \) of \( T\tilde{N} \), where each \( \tilde{e}_i \) is parallel. For any \( \tilde{y} \in \tilde{N} \), choose a smooth path \( \gamma \) connecting \( \tilde{x} \) and \( \tilde{y} \), and define
\[
\tilde{y}_i := \int_{\gamma} \langle \dot{\gamma}, \tilde{e}_i \rangle
\]
where \( \dot{\gamma} \) is the tangent vector of \( \gamma \). Since \( \tilde{N} \) is simply connected and flat, the above integral is independent of the choice of \( \gamma \) among all smooth curves connecting \( \tilde{x} \) and \( \tilde{y} \). These functions \( \tilde{y}_i: \tilde{N} \to \mathbb{R} \) together give rise to a map \( p: \tilde{N} \to \mathbb{R}^n \) that is locally isometric.

Our next step is to show that \( p: \tilde{N} \to p(\tilde{N}) \) is a Riemannian covering map. First we show that \( \partial N \) has only one connected component. Otherwise, fix a connected component \( C \) of \( \partial N \). The distance from \( C \) to \( \partial N - C \) is positive, and is attained by some geodesic \( \gamma \) connecting \( x \in C \) and \( y \in C' \), where \( C' \) is a connected component of \( \partial N - C \). Since the length of \( \gamma \) is the minimum among all curves connecting \( C \) to \( \partial N - C \), it follows that \( \gamma \) is orthogonal to both \( C \) and \( C' \), and lies in the interior of \( N \) except the two end points. Let \( U \) be a small neighborhood of \( \gamma \). Since \( N \) is flat, \( U \) embeds isometrically into \( \mathbb{R}^n \). Such an embedding maps \( \gamma \) to a line segment. Now since both \( C \) and \( C' \) are strictly convex, any line segment from \( C \) to \( C' \) inside \( U \) parallel to \( \gamma \) shorten the distance. This contradicts the minimality of the chosen geodesic \( \gamma \) and proves the claim.

The exact same argument above also shows that \( \partial \tilde{N} \) has only one connected component. Therefore \( p(\partial \tilde{N}) \) is connected in \( \mathbb{R}^n \), and has the same metric and second fundamental form as \( \partial M \). Thus \( p(\partial \tilde{N}) \) is a subset in \( \mathbb{R}^n \) that only differs from \( \partial M \) by an affine isometry. Indeed, this follows from the uniqueness of solutions to the partial differential equations describing \( p(\partial \tilde{N}) \) and \( \partial M \) in \( \mathbb{R}^n \). Moreover, \( p \) restricted to \( \partial \tilde{N} \) is a covering map, which in fact is an isometry if \( n = \dim N \geq 3 \).
We claim that if \( p(x) \in p(\partial \tilde{N}) \), then \( x \in \partial \tilde{N} \). In other words, the map \( p \) will never map an interior point of \( \tilde{N} \) to \( p(\partial \tilde{N}) \). Assume to the contrary that there exists \( x \) in the interior of \( \tilde{N} \) such that \( p(x) \in p(\partial \tilde{N}) \). The distance from \( x \) to \( \partial \tilde{N} \) is attained by a unique geodesic segment \( \gamma \) from \( x \) to a point \( y \in \partial \tilde{N} \). Note that \( \gamma \) is orthogonal to \( \partial \tilde{N} \). As \( p \) is a local isometry, \( p(\gamma) \) is a non-trivial line segment in \( \mathbb{R}^n \) from \( p(x) \) to \( p(y) \), which is orthogonal to \( p(\partial \tilde{N}) \) at \( p(y) \). Since \( \partial \tilde{N} \) is convex, the vector in \( \mathbb{R}^n \) from \( p(y) \) to \( p(x) \) is pointing inward (with respect to \( p(\partial \tilde{N}) \)). Therefore \( p(\gamma) \) lies entirely in the inside\(^1\) of \( p(\partial \tilde{N}) \). Let \( \alpha : [0, 1] \to p(\partial \tilde{N}) \) be a smooth path in \( p(\partial \tilde{N}) \) with \( \alpha(0) = p(y) \) and \( \alpha(1) = p(x) \). Since \( p \) is a covering map on \( \partial \tilde{N} \), \( \alpha \) lifts uniquely to a path \( \tilde{\alpha} \) such that \( \tilde{\alpha}(0) = y \). As \( p \) is a local isometry near \( y \), there is a unique geodesic \( \gamma_t \) connecting \( y \) and \( \tilde{\alpha}(t) \) for all sufficiently small \( t \in [0, 1] \), which is mapped isometrically under the map \( p \) to the line segment connecting \( p(y) \) and \( \alpha(t) \). Since \( p \) is a local isometry everywhere, we can continue the construction of such geodesics \( \gamma_t \) for all \( t \in [0, 1] \). In particular, \( \gamma_1(1) \) coincides with \( p(\gamma) \). By construction, \( \gamma_1 \) and \( \gamma \) have the same length and point towards the same direction starting from \( y \). It follows that \( x \) coincides with the other end point \( \tilde{\alpha}(1) \) of \( \gamma_1 \), which lies in \( \partial \tilde{N} \) by construction. This contradicts the assumption that \( x \) lies in the interior of \( \tilde{N} \). This finishes the proof of the claim. Note that the same argument also proves that every point in the inside of \( p(\partial \tilde{N}) \) admits at least one preimage in \( \tilde{N} \).

The interior \( \tilde{N} - \partial \tilde{N} \) of \( \tilde{N} \) is connected and \( p(\tilde{N} - \partial \tilde{N}) \) is disjoint from \( p(\partial \tilde{N}) \), so \( p(\tilde{N} - \partial \tilde{N}) \) lies entirely in the inside of \( p(\partial \tilde{N}) \). To summarize, we see that \( p(\tilde{N}) \) is precisely the region enclosed by the hypersurface \( p(\partial \tilde{N}) \) in \( \mathbb{R}^n \). As \( p(\partial \tilde{N}) \) coincides with \( \partial M \) up to an affine isometry, \( p(\tilde{N}) \) coincides with \( M \) up to an affine isometry. Without loss of generality, we may assume that \( p(\tilde{N}) = M \).

Now we show that \( p : \tilde{N} \to p(\tilde{N}) = M \) is a covering map. Indeed, if \( z \) is a point in the interior of \( M \), then its preimage \( p^{-1}(z) \) consists of only interior points of \( \tilde{N} \). Let \( \varepsilon \) be the distance from \( z \) to \( \partial M \). Then the \( \varepsilon \)-neighborhood of each point in \( p^{-1}(z) \) is mapped isometrically under the map \( p \) to the \( \varepsilon \)-neighborhood of \( z \). In particular, the (\( \varepsilon/2 \))-neighborhoods of points in \( p^{-1}(z) \) are disjoint in \( \tilde{N} \). The same holds when \( z \) lies in \( \partial M \), as each point in its preimage lies in \( \partial \tilde{N} \).

As \( M \) is simply connected, \( p \) has to be the trivial covering map, hence an isometry. In particular, \( p : \partial \tilde{N} \to \partial M \) is a homeomorphism. Let \( \pi : \tilde{N} \to N \) be the corresponding covering map for the universal cover \( \tilde{N} \) of \( N \). Note that \( f \circ \pi = p \) on \( \partial \tilde{N} \). Therefore the restriction \( \pi \) on \( \partial \tilde{N} \) is injective. It follows that \( \tilde{N} = N \) and \( \pi \) is the identity map.

\(^1\)We have already shown that \( p(\partial \tilde{N}) \) is strictly convex smooth compact hypersurface in \( \mathbb{R}^n \). It follows that \( p(\partial \tilde{N}) \) separates \( \mathbb{R}^n \) into two parts. That is, \( \mathbb{R}^n - p(\partial \tilde{N}) \) consists of two connected components, exactly one of which is compact. We call the compact connected component of \( \mathbb{R}^n - p(\partial \tilde{N}) \) the inside of \( p(\partial \tilde{N}) \), and the noncompact connected component of \( \mathbb{R}^n - p(\partial \tilde{N}) \) the outside of \( p(\partial \tilde{N}) \).
Now the map $h := f \circ p^{-1}: M \to \tilde{N} = N \to M$ is distance non-increasing, and equal to the identity map when restricted to $\partial M$. To prove the theorem, it suffices to show that any such map has to be the identity map on $M$. Let $x_1$ and $x_2$ be two arbitrary points on $\partial M$. Since $M$ is strictly convex, there is a unique line segment $\ell$ connecting $x_1$ and $x_2$ that lies entirely in $M$. Then $h(\ell)$ is a curve in $M$ connecting $x_1$ and $x_2$, hence its length is at least the length of $\ell$. Since $h$ is distance non-increasing, it follows that $h$ maps $\ell$ to itself isometrically. Note that all such line segments cover the whole $M$. This completes the proof.

□

REFERENCES

[1] Jean-Pierre Bourguignon, Oussama Hijazi, Jean-Louis Milhorat, Andrei Moroianu, and Sergiu Moroianu. A spinorial approach to Riemannian and conformal geometry. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2015.

[2] John Lott. Index theory for scalar curvature on manifolds with boundary. Proc. Amer. Math. Soc., 149(10):4451–4459, 2021.

[3] Jinmin Wang and Zhizhang Xie. On Gromov's dihedral rigidity conjecture and Stoker's conjecture. arXiv:2203.09511, 2022.

[4] Jinmin Wang, Zhizhang Xie, and Guoliang Yu. On Gromov's dihedral extremality and rigidity conjectures. 2021. arXiv:2112.01510.

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