A PERSONAL SURVEY ON RECENT AND LESS RECENT RESULTS ON TILTING THEORY

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Dedicated to the memory of Sheila Brenner and Michael C. R. Butler

Introduction

In the introduction to Ringel’s book “Tame Algebras and Integral Quadratic Forms”, the Author makes the following comment about references: “however we should point out that some general ideas, which have influenced the results and the methods presented here, are not available in official publications, or not even written up.” The above remark also explains very well my personal experiences with respect to many situations concerning both Places and People, to use the terminology of FDLIST [F]. For me one of the first examples of the complexity of “general ideas” (and their zigzag journeys) is given by the beginning of Tilting Theory in Italy. The best written reference for the knowledge of the big role played by Adalberto Orsatti - and his Algebra Team in Padova - is Menini’s paper [M]. This paper contains a very interesting account of both important public events and official publications, as well as information, never previously written up, on private conversations, classical letters and unexpected connections between distant places and people. On the other hand, the first paper by Italian authors on Tilting Theory, that is [MO], is due to Menini and Orsatti. The title of the paper (“Representable equivalences between categories of modules and applications”), suggests that classes of modules are the most important ingredients. In the introduction of [MO] the authors thank Masahisa Sato, Enrico Gregorio and me. I regret that I have never expressed officially - in a paper - my thanks (and surprise) for this unexpected reference, as I should have done long ago. The aim of this paper is to fill this gap, at least partially. In a sense, this note is the written version of conversations with young colleagues on unofficial history, “general ideas”, unexpected facts and open problems. I may sum up as follows the lessons learned by making pictures of tilting–type objects:

(a) “Simple” and combinatorial objects may have unexpected concealed topological properties.

(b) “Non simple” objects may have unexpected concealed discrete properties.

In the following $K$ denotes an algebraically closed field, and we assume that all vector spaces and algebras are defined over $K$. This note is organized as follows. In Section 1, I will describe my first homework on Tilting Theory in a non technical way. This homework, given by Orsatti, was to study and explain the example of tilting module constructed by Happel and Ringel at the end of [HR], hence “coming from Bielefeld”, as Orsatti told me. Next, in Sections 2 and 3, I describe the naïve strategies, used, from the very beginning, to answer some questions on more or less abstract tilting objects, and I make some personal remarks on definitions. Finally, in Sections 4 and 5, I collect some motivations and examples for the observations and/or conjectures made in (a) and (b). In addition to the names of C. Menini and A. Orsatti (the authors of [MO] already mentioned), any partial list of people

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I would like to thank should contain at least the following names: L. Angeleri - Hügel, S. Bazzoni, R. Colpi, E. Gregorio, F. Mantese, A. Tonolo and J. Trlifaj. Thanks to all of you for your problems – explained in the clearest possible way – and your hope that there ought to be a solution somewhere!

I described some facts and results contained in this note in some seminars at the universities of Milano (June 2012), Milano Bicocca (October 2012) and Ferrara (June 2013). I wish to thank the organizers of these talks (Gilberto Bini, Cecilia Cavaterra, Elisabetta Rocca, Maria Gabriella Kuhn and Claudia Menini) for the opportunity of presenting my work to an interested audience, and all the participants for their interest and questions. Next, I would like to thank Ibrahim Assem, together with all the organizers of the "XVIIth Meeting on representation theory of algebras" (Bishop’s University, Sherbrooke, October 2013) dedicated to the memory of Michael C.R. Butler, for the idea of putting a preliminary version of this note in the home page of the meeting, with the title "Preprint dedicated to Brenner Butler". I will always remember with gratitude many conversations with the creators of tilting theory and their attention to the work of other people of any age and countries.

1. A letter from Japan and my first homework on Tilting Theory

Menini’s paper \([M\) page 11] explains the contribution given by Sato to her joint paper \([MO\) with Orsatti in the following comments: “I would like to recall here that it was Masahisa Sato that pointed out to us that tilting modules might provide examples of . . . . In fact Orsatti explained this problem to Sato during a NATO meeting held in Antwerpen (Belgium) in the period July 20 - 29 , 1987 . After some time Sato wrote to Orsatti showing an example of a tilting module”. I remember very well what happened next. Orsatti told me to make a copy of the letter received from Japan. My homework was to look at Happel - Ringel’s example of a tilting module, say \(T\), considered in the last page of \([HR\), and to give a talk about it. Hence my unique and small contribution to \([MO\] was just a talk. This explains my surprise for those unexpected thanks. Moreover, now I realize that I should have thanked Menini and Orsatti. Indeed the preparation of the seminar was very useful for me for several reasons. For instance, I had the pleasant surprise of a direct experience that Auslander–Reiten quivers can really help to guess and see possible equivalences, before making a proof of their existence. Indeed tilting equivalences and cotilting dualities also have a combinatorial nature, inherited and suggested by that of quivers and modules. In addition to this, I was able to discover the magic power of Auslander’s formula (see \([AuReS\) Proposition 4.6 and Corollary 4.7 or \([A4\) conditions (5) and (6) , pages 75 - 76]) for verifying the vanishing of certain Ext\(^1\) groups, by simply looking at the Auslander–Reiten quiver. Without this formula, I am (and was) not able to check that the module \(T\) is selforthogonal. The next picture (of a very combinatorial object with several symmetries) illustrates the shape of Happel–Ringel’s bimodule \(_AT_B\).
According to [HR], the underlying vector space of $T$ has dimension 23, while $A$ is the algebra given by the Dynkin quiver $E_6$ with “subspace orientation”, that is of the form

$$\begin{array}{cccccc}
1 & 2 & 6 & 4 & 3 \\
\end{array}$$

On the other hand, the algebra $B = \text{End}_T A$ is isomorphic to the algebra given by the fully commutative quiver

$$\begin{array}{cccccc}
e & c \\
e & b & d & f \\
a & d \end{array}$$

Finally, in Picture 1 and in the next pictures of bimodules, we adopt the following conventions. First of all, every square of the picture indicates an element $v$ of a fixed basis of the underlying vector space of $T$. Next, the index $x$ on the left (resp. $y$ on the right) of a small square corresponding to the vector $v$ indicates that $e_x v = v = ve_y$, where $e_i$ is the path of length zero around the vertex $i$. Following Ringel’s suggestion during my staying in Bielefeld, the small squares have a special position, so that they describe in an obvious way also the composition factors of the same module. (See, for instance, [R1, R2] and [R3, page 126] for descriptions and/or pictures of complicated modules.) On the other hand, the bimodules associated to “valued” arrows in [DIR] and [D] gave me the idea of adding two indices (on the left and on the right of the small squares). In this way we can see the action of left or right multiplication by the primitive idempotents, corresponding to vertices of some quiver. Finally, I want to say that the above picture is not as old as my homework. I used similar pictures to visualize rather small bimodules for the first time, and more or less by chance. Of course, in case of big and complicated bimodules, it would be better to replace a 2–dimensional “global” visualization (of the main properties of the left and right underlying modules) by a 3–dimensional one, without or with few self–intersections. However, even flat pictures, like Picture 1, are powerful enough to give for free a lot of indirect information in a compact way. As an example, the above picture tells us how the tilting equivalence represented by $T$ (between the modules generated by the left $A$–module $T$ and the modules cogenerated by the left $B$–module $D(T_B) = \text{Hom}_K(T_B, K)$) acts on some indecomposable modules. For instance, the indecomposable summands of $A T$ are sent to the following indecomposable summands of $B B$:

$$\begin{array}{cccc}
5 & 6 & a, \\
6 & 6 & b, \\
5 & 6 & c, \\
6 & 6 & a \\
1 & 2 w & d, \\
2 & 5 & w, \\
1 & 6 & a, \\
3 & 6 & d, \\
4 & 6 & c. \\
\end{array}$$

On the other hand Picture 1 also describes how the cotilting duality induced by $T$ (between the modules cogenerated by left $A$–module $T$ and the modules cogenerated by the right $B$–module $T$) acts on some indecomposable modules. For
instance, the indecomposable summands of $A_T$ are sent to following indecomposable summands of $B_B$:

\[
\begin{array}{cccc}
5 & \mapsto & a & \ b & c \\
6 & \mapsto & e & \ b & f \\
2 & 5 & 6 & \mapsto & b & e & f \\
5 & 4 & 6 & \mapsto & c & f \\
1 & 3 & 2 & 5 & 6 & \mapsto & d & e & f \\
3 & 5 & 4 & 6 & \mapsto & e \\
\end{array}
\]

2. What happened next: talk with big matrices, more or less abstract cancellations...

The analysis of Happel–Ringel’s bimodule was both my first homework on Tilting Theory, and the subject of my first talk on this subject. The talk was not my first talk in Padova containing some quivers. However, all the quivers used earlier were much smaller. Hence, it was easier to describe without pictures at least the corresponding path algebras. When I tried to do the same with Happel–Ringel’s example, I realized how even basic techniques of representation theory of finite dimensional algebras can make otherwise invisible things become visible. For instance, in order to give a definition of $A$ and $B$ without quivers, I distributed some pages with the largest matrices I ever used, namely 23 by 23 matrices, that I did not wanted to draw at the blackboard. After some time I noticed that I could use more reasonable matrices to describe the $K$–linear maps from $T$ to $T$, corresponding to multiplications by elements of $A$ or $B$. Indeed, $T$ has the strong property that the groups of all morphisms between two indecomposable summands of $A_T$ and $T_B$ respectively are $K$–vector spaces of dimension at most one. Hence, after cancellation of many “inessential” rows and columns, it is easy to see that $A$ and $B$ are isomorphic to subalgebras of the algebra of all 6 by 6 matrices of the following form:

\[
\begin{pmatrix}
K & 0 & 0 & 0 & 0 & 0 \\
K & K & 0 & 0 & 0 & 0 \\
0 & 0 & K & 0 & 0 & 0 \\
0 & 0 & K & K & 0 & 0 \\
0 & 0 & 0 & K & 0 & 0 \\
K & K & K & K & K & K
\end{pmatrix}
= \begin{pmatrix}
K & K & K & K & K & K \\
0 & K & 0 & 0 & K & 0 \\
0 & 0 & K & 0 & 0 & K \\
0 & 0 & 0 & K & K & 0 \\
0 & 0 & 0 & 0 & K & 0 \\
0 & 0 & 0 & 0 & 0 & K
\end{pmatrix}.
\]

It turned out that endomorphism rings of abelian groups are the subject of my master degree thesis on “Abelian groups whose endomorphism ring is locally compact in the finite topology”, written under the direction of Adalberto Orsatti. Moreover, some of my first papers deal with endomorphism rings. However, without using quivers, this previous abstract experience on (usually large) endomorphism rings wouldn’t have been useful for verifying that $B = \text{End}_A T$ actually had the indicated form. The cancellation of rows and columns was only the first of many other (more abstract) cancellations made in the sequel, concerning modules and complexes, as in 3.5, 3.6 and 3.7. For the theoretical importance of cancellations in Tilting Theory, we refer to Ringel’s lecture on the occasion of the 20th anniversary of the Department of Mathematics of the University of Padova [see the section “Fully documented Lectures” in Ringel’s home page]. We note that the suggestive title of this lecture is “Tilting Theory: the Art of Losing Modules”.

3. Short or long definitions with or without classes of modules

It is well-known that some definitions of tilting and/or partial tilting modules (the most compact and elegant ones) consist of precisely one rather short property. For instance, any “classical” tilting (resp. cotilting) $R$–module $M$ (hence also Happel–Ringel’s module $T$), has the property that the class of all modules generated (resp. cogenerated) by $M$ coincides with the kernel of $\text{Ext}^1_R(M, -)$ (resp. $\text{Ext}^1_R(-, M)$). This previous global property is equivalent to three discrete properties of two modules, namely of $M$ and of the regular module $R$ (resp. an injective cogenerator $Q$), as in the definition given by Brenner and Butler in [BB].

3.1. A long definition (for the classical case). We say that a module $M$ is a classical tilting or cotilting module (more precisely, a 1–tilting or 1–cotilting module) respectively, if the following conditions hold:

- The projective (resp. injective) dimension of $M$ is at most 1.
- $\text{Ext}^1_R(M, \bigoplus M) = 0$ (resp. $\text{Ext}^1_R(\Pi M, M) = 0$), where $\bigoplus M$ (resp. $\Pi M$) is any direct sum (resp. product) of copies of $M$.
- There is a short exact sequence of the form $0 \rightarrow R \rightarrow M' \rightarrow M'' \rightarrow 0$ (resp. $0 \rightarrow M' \rightarrow M'' \rightarrow Q \rightarrow 0$), where $M'$ and $M''$ are direct summands of direct sums (resp. products) of copies of $M$.

For me it was always easier to deal with more than one (but finitely many) elementary properties instead of dealing with just one property on classes of modules. We refer to [Ba] Proposition 3.6 and Lemma 3.12 for the beautiful technical condition (on the relationship between two classes of modules) which characterizes “non classical” tilting or cotilting modules and their generalizations (for instance, their direct summands and more precisely partial tilting (resp. cotilting) modules of projective (resp. injective) dimension $> 1$). We only recall that - as in the classical case - the usual definition of all these modules $M$ consists of two conditions on just two modules, namely on $M$ and a very special projective or injective module. One of the reasons why it may be difficult to check equalities or inclusions of classes of modules which play a key role in the characterization of “non classical” tilting and cotilting - type modules $M$ is the following. Even in dealing with algebras of finite representation type, in the “non classical” case, these modules $M$ have the property that one of these two classes of modules (namely an orthogonal class) is closed under direct summands, while the other does not necessarily have this closure property. More precisely, according to Bazzoni’s paper [Ba], quoted at the beginning of this section, $n$ - tilting modules (or just tilting modules, for short), admit the following definition.

3.2. A short definition (for the general case). Given an $R$–module $M$ and a natural number $n > 0$, we denote by $\text{Gen}_n(M)$ the class of all modules $X$ such that there is an exact sequence of the form

$M(1) \rightarrow \ldots \rightarrow M(n) \rightarrow X \rightarrow 0$,

where the $M(i)$’s are direct summands of direct sums of copies of $M$. Following [Ba], we say that $M$ is a tilting (resp. partial tilting) module of projective dimension at most $n$ if $\text{Gen}_n(M)$ is equal to (resp. is contained in) the orthogonal class $M^\perp = \bigcap_{i > 0} \ker \text{Ext}^i_R(M, -)$. 

6 GABRIELLA D’ESTE
3.3. Example: A class of modules not always closed under direct summands. Let $R$ be the algebra given by the (fully commutative) quiver

\[
\begin{array}{ccc}
1 & \xrightarrow{\quad 3 \quad} & 3 \\
\quad 2 \quad & \xleftarrow{\quad 4 \quad} & \quad 5 \quad
\end{array}
\]

such that the composition of any two arrows is zero. Then we deduce from [D4 Example B] that the injective module $T = \{ 4 \oplus 5 \oplus 2 \oplus 3 \oplus 1 \oplus 2 \}$ is a partial tilting module (of projective dimension 3) such that $\text{Gen}_3(T)$ contains the module $1 \oplus 1$, but not its summand 1. Moreover both $\text{Gen}_3(T)$ and the class $\text{Add}(T)$, formed by all injective $R$-modules without simple summands, have the same indecomposable modules, namely the 4 indecomposable summands of $T$. This means that we cannot determine the class $\text{Gen}_3(T)$ by just looking at the Auslander–Reiten quiver. In other words, this means that the operation of making direct sums (that is the “only really well-understood construction” in the words of [V, page 476]) is not enough to investigate an important class of modules generated by a maximal direct summand of a tilting module of projective dimension 3.

3.4. Cancellation of an injective non-projective summand (to obtain a “large” partial tilting module). We may roughly speaking say that we obtain the faithful module $T$ constructed in 3.3 from the minimal injective cogenerator $D(R_R) = \text{Hom}_K(R_R, K)$ by means of cancellation of its injective summand 1 (of projective dimension 3). On the other hand, $\text{Ext}^2_R(T, 5) \neq 0$ and 5 is the unique indecomposable module $X$ such that $\text{Hom}_R(T, X) = 0$. Consequently, we have

\[\text{(⋆) } \text{Ker} \text{Hom}_R(T, -) \cap T^i = 0.\]

In other words, $T$ is a large partial tilting module [D3]. We recall some facts concerning these modules. First of all, any tilting module is a large partial tilting module [Ba, page 371]. Second, any finitely generated large partial tilting module of projective dimension at most 1 is a tilting module [C1, Theorem 1]. Finally, property (⋆) implies that any large partial tilting module, say again $T$, is sincere [KT], that is with the property that $\text{Hom}(P, T) \neq 0$ for every projective module $P \neq 0$. Consequently, a module $T$ of finite length is sincere (AuReS and [I4]) if every simple module is a composition factor of $T$. After we point out the theoretical and practical importance of cancellations in Tilting Theory, we show that sometimes cancellation of a projective-injective summand (that is, of an obvious summand) of a tilting module gives rise to a large partial tilting module.

3.5. Cancellation of a projective-injective summand of a tilting module with minimal orthogonal class (to obtain a “large” partial tilting module). Let $R$ be a finite dimensional algebra or, more generally, a noetherian and semiperfect ring such that every indecomposable injective module has a simple socle. Let $M$ be an injective tilting $R$-module (of projective dimension $> 1$) such that the orthogonal class $M^i = \bigcap_{n \geq 0} \text{Ker} \text{Ext}^n_R(M, -)$ is the class of all injective modules. Let $T$ be a sincere summand of $M$, obtained from $M$ after cancellation of a projective summand $P$. Then we deduce from [D3 Theorem 4] that $T$ is a large partial tilting module.

We will give in 3.7 a “minimal” example of the above result, where $T$ is
(1) the unique indecomposable injective module which is not projective;
(2) a uniserial module such that every simple module has multiplicity one as a
composition factor of \( T \), that is a sincere module of minimal dimension.

3.6. Remark on complexes. It turns out that various types of cancellations seem
to be useful also by dealing with more abstract partial tilting objects. For instance,
this often happens with partial tilting complexes, say \( T^\circ \), in the sense of Rickard
with the following property:

(a) \( T^\circ \) is the projective resolution of a large partial modules, say \( T \), which is
not a tilting module.

By Rickard, this hypothesis guarantees the existence of a non-zero right bounded com-
plex (of projective modules) \( X^\circ \) with the following property:

(b) \( X^\circ \) is not the projective resolution of a module and every morphism from
\( T^\circ \) to any shift of \( X^\circ \) is homotopic to zero.

The examples constructed in [D5], [D6] and [D7] suggest that there is no canonical
way to obtain \( X^\circ \) from \( T^\circ \). Moreover, the same holds by confining ourselves to
complexes \( T^\circ \) and \( X^\circ \) satisfying (a) and (b) respectively and with the following
additional “very combinatorial” property (in the words of [Sc-ZI]) :

(c) Any non–zero component of the indecomposable summands of \( T^\circ \) and \( X^\circ \)
is an indecomposable module.

Indeed, an indirect proof of the intricacy of complexes with respect to modules
is that – up to shift – the choices of the indecomposable complexes \( X^\circ \) satisfying
both (b) and (c) may be quite different. For instance, they may be either zero
Example C (iii) and (iv)], or one [D7] Remark after Example 1], or infinitely
but countably many [D7] Remark after Example 3], or uncountably many [D7
Example 4]. The following example shows that by deleting some components of
an indecomposable complex \( T^\circ \), with properties (a) and (c), we may obtain all the
shortest complexes \( X^\circ \) with properties (b) and (c), that is the “elementary” com-
plexes in the sense of [Sc-ZI] of the form

(d) \( 0 \rightarrow P \rightarrow Q \rightarrow 0 \) with \( P \) and \( Q \) indecomposable projective modules.

3.7. Example of cancellations concerning partial tilting modules and com-
plexes. For every even integer \( m > 2 \), there is a uniserial non faithful injective
module \( T \) (of projective dimension \( m \)) such that we obtain all the indecomposable complexes \( X^\circ \) with the above properties (b) and (d) by means of various types of
cancellations of some components of \( T^\circ \), that is left cancellations, right cancella-
tions and, sometimes, also central cancellations. Indeed, let \( A \) be the Nakayama
algebra, considered in [M12], given by the quiver

```
1 ← a_1 ← 2 ← ... ← a_{n-1} ← n ← a_n
```

with relation \( a_n \cdots a_1 = 0 \), where \( 2n = m + 2 \). Next, let \( T \) denote the injective
module of the form
Then we obtain $T$ from the minimal injective cogenerator $D(A_A) = \text{Hom}_K(A_A, K)$ after cancellation of its $n-1$ indecomposable projective summands. Thus we deduce from 3.5 (or from [D3 Example 6]) that $T$ is a large partial tilting module of projective dimension $m$. Moreover its projective resolution $T^\circ$ satisfies (a) and (c), and the complexes $X^\circ$ satisfying (b) and (d) are of the form $0 \to I(i) \to I(j) \to 0$, where $I(*)$ denotes the indecomposable injective module corresponding to the vertex $*$ and $i > j > 1$ [D6 Proposition 1]. Hence they are exactly the complexes with two non-isomorphic injective components different from zero, obtained from $T^\circ$ after suitable cancellations.

4. Do finite dimensional bimodules have a concealed topology?

Thanks to the method of visualizing bimodules by means of pictures, as in Section 1, I could first “see” and then prove that even rather small bimodules have bad behaviour with respect to quite natural possible constructions, namely embeddings into bimodules with an underlying left (or right) injective module. Indeed, by the results proved in [Ma1] on the socle of $E(C)/C$ [Ma1 Lemma 2.2] and on the modules cogenerated by $E(C)/C$ [Ma1 Propositions 1.7 and 2.1 and Theorem 1.17], it is natural to measure the gap between a classical cotilting module $C$ and its injective envelope $E(C)$, at least in case of modules which are finite dimensional vector spaces. However, in this special situation, where the discrete topology should be the canonical topology, two radically different situations show up.

4.1. Bad case. It is not always possible to embed a finite dimensional cotilting bimodule $C$ in another bimodule $D$ with the property that $D$, as a left (resp. right) module, is the injective envelope of $C$ [D2 Example B (c), (d)]. Moreover, no left–right symmetry exists, because only one of the constructions may be possible [D2 Example A (c), (d)].

4.2. Good case. When such an embedding exists, the structure of $D$, namely its structure as a right (resp. left) module seems to be the most obvious one. Indeed also multiplications on the opposite side, that is right (resp. left) multiplications, are described by nice matrices with many entries equal to zero. In other words, they seem to be “continuous” extensions of their restriction to $C$. However, as the following toy example shows, the property of being an indecomposable bimodule is neither hereditary nor left–right symmetric.

4.3. Toy example of a finite dimensional cotilting bimodule [D2 Example D]. Let $S$ (resp. $R$) be the algebra given by the quiver \[\bullet \to 4 \to 5 \to 6 \to \bullet \] (resp. \[\bullet \to 1 \to 3 \to \bullet \]), and let $SC_R$ be an indecomposable cotilting bimodule with

\[SC = \begin{array}{c}
\begin{array}{cccc}
4 & 5 & \oplus & 5 \\
6 & & & \\
\end{array}
\end{array}
\]

(resp. $CR = \begin{array}{c}
\begin{array}{cccc}
1 & 2 & \oplus & 1 \\
3 & & & \\
\end{array}
\end{array}\oplus 2\right)

of the form
Then $C$ has codimension 2 in its left (resp. right) injective envelope

$$E(C) = \frac{4}{5} \oplus \frac{4}{5} \oplus \frac{4}{6} \quad \text{(resp. } E(C) = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{1}{3}).$$

Moreover $E(C)$ is the support of an indecomposable (resp. a decomposable) bimodule $D$, containing $C$ as a bimodule, of the form

4.4. Remarks on the action of primitive idempotents and nilpotent elements in the good case. In all the examples constructed in [D2], where $sC_R$ is a cotilting bimodule such that $E(sC)$ admits a structure of $S - R$ bimodule, containing the cotilting bimodule $sC_R$, the following facts hold:

1. The ring $R$ is hereditary.
2. If $X$ is an indecomposable summand of $sC$ and $e$ is a primitive idempotent of $R$ such that $xe = x$ for every element $x \in X$, then we also have $x'e = x'$ for every element $x' \in E(sX)$. 
(3) The nilpotent elements of $S$ and $R$ corresponding to arrows act on the elements of $E(C) \setminus C$ in the easiest possible way, by a kind of shift. Indeed, if $s \in S$, $r \in R$ and $u, v, w$ (resp. $u, w, w, x$) are linearly independent elements of $C$ (resp. of $E(C)$) such that $x \in E(C) \setminus C$, $sx = v$, $sw = u$, $vr = u$, then we have $xr = w$, as illustrated by the following picture.

![Picture 5](image1)

4.5. Remarks on the action of new concealed rings in the bad case. In all the examples constructed in [D2], where $SC_R$ is a cotilting bimodule such that $E(SC)$ does not admit a structure of $S$–$R$ bimodule, containing the cotilting bimodule $SC_R$, the following facts hold:

(1) The ring $R$ is not hereditary.
(2) There are a ring $R^*$, a ring epimorphism $F: R^* \to R$ and a bimodule $SU_{R^*}$, containing $SC_{R^*}$, such that $E(SC) = SU$.

We give two examples, where Ker $F$ is a $K$–vector space of dimension 1 or 2.

4.6. An example (of the bad case) with $R^*$ hereditary. As in [D2] Example A, let $R$ (resp. $S$) be the algebra given by the quiver

\[ \begin{array}{cccc}
1 & \rightarrow & 2 & \rightarrow \quad \text{with relation } ba = 0
\end{array} \]

Let $SC_R$ be the cotilting bimodule such that $SC = 6 \oplus 4 \oplus 4$ and $C_R = 3 \oplus 2 \oplus 2$. Next, let $R^*$ be the hereditary algebra given by the Dynkin diagram

\[ \begin{array}{cccc}
1 & \quad 2 & \quad 3
\end{array} \]

Finally, let $F: R^* \to R$ be the obvious ring epimorphism.

Then dim Ker $F = 1$ and $C$, regarded as a $S$–$R^*$ bimodule, is contained in the $S$–$R^*$ bimodule $U$, satisfying (1) and (2), described by the following picture.

![Picture 6](image2)
4.7. **An example (of the bad case) with \( R^* \) non hereditary.** As in \[D2\], let \( R \) (resp. \( S \)) be the algebra given by the quiver \( \bullet \xrightarrow{a} \bullet \) with relation \( ab = 0 \) (resp. \( \bullet \xrightarrow{c} \bullet \xleftarrow{d} \bullet \) with relation \( cd = 0 \)). Next, let \( sC_R \) be the cotilting bimodule such that \( sC = \frac{3}{4} \bigoplus \frac{3}{3} \) and \( C_R = \frac{1}{2} \bigoplus \frac{1}{1} \). Next, let \( R^* \) denote the algebra given by the quiver \( \bullet \xrightarrow{a} \bullet \) with relation \( aba = 0 \). Finally, let \( F : R^* \to R \) be the obvious ring epimorphism. Then \( \dim \ker F = 2 \) and \( C \), regarded as an \( S–R^* \) bimodule, is contained in the \( S–R^* \) bimodule \( U \), satisfying (1) and (2), described by the following picture

![Picture 7](image)

4.8. **Two open problems on bimodules.**

**Problem 1.** Are conditions (1), (2) and (3) of Remark 4.4 satisfied by any bimodule as in the good case described in 4.2?

**Problem 2.** Are conditions (1) and (2) of Remark 4.5 satisfied by any bimodule as in the bad case described in 4.1?

5. **Do infinite dimensional modules need no topological tools?**

Another reason why the presence of a kind of topology, pointed out in Section 4, is a strange fact is the absence of topology in the proof of a result concerning dualities induced by cotilting bimodules of infinite dimension. Before we discuss this, we recall that a left \( S \)-module (resp. a right \( R \)-module) \( M \) is reflexive with respect to the bimodule \( sU_R \) (or just \( U \)-reflexive or reflexive, for short) if \( M \) is canonically isomorphic to its double dual with respect to \( U \), that is to the group \( \Delta(\Delta(M)) \), where \( \Delta \) denotes both the contravariant functors \( \text{Hom}(-, sU_R) \) for \( ? = R, S \) and the group \( \Delta(X) \) is equipped with its bimodule structure (see \[AF\] Proposition 4.4 or \[J\] Propositions 3.4 and 3.5) for any left \( S \)-module and any right \( R \)-module \( X \).
5.1. **Obvious and non obvious reflexive modules.** Even in special cases, that is given a faithfully balanced bimodule $U$, there is a big gap between

(*) the well–known indecomposable reflexive modules, which are either projective or summands of $U$ [AP Propositions 20.13 and 20.14 and Corollary 20.16] ;

(**) the rest of the world, that is the non obvious indecomposable reflexive modules.

Concerning (*), we first note that the cotilting bimodule described in 4.3 admits 4 indecomposable reflexive left (resp. right) modules. Moreover, comparing Auslander–Reiten quivers (and looking at Picture 2), we see that all of them are obvious reflexive modules and the duality $\Delta$ acts as follows:

$$
\begin{align*}
4 &\mapsto 5, & 5 &\mapsto 6, & 6 &\mapsto 1 \\
3 &\mapsto 1, & 1 &\mapsto 3, & 2 &\mapsto 4.
\end{align*}
$$

A similar situation holds for the cotilting module described in 4.6 (resp. 4.7), where the cotilting duality $\Delta$ acts as follows:

$$
\begin{align*}
6 &\mapsto 3, & 4 &\mapsto 6, & 5 &\mapsto 1 \\
2 &\mapsto 4, & 1 &\mapsto 5, & 6 &\mapsto 2.
\end{align*}
$$

However, to give an example of a cotilting bimodule admitting non obvious reflexive modules, it is enough to take Happel–Ringel’s cotilting (and tilting) bimodule $T$ described in Section 1. Indeed, by comparing Auslander–Reiten quivers (and by looking at Picture 1), it is easy to see that $T$ admits 14 indecomposable reflexive left and right modules respectively. On the other hand, $\frac{5}{6}$ is the unique indecomposable projective summand of the left $\Lambda$–module $T$. Therefore $T$ admits 11 obvious (resp. 3 non obvious) indecomposable reflexive left and right modules respectively. We already described (at the end of Section 1) how the cotilting duality $\Delta$ acts on the indecomposable summands of $\Lambda T$. On the other hand, $\Delta$ acts as follows on the remaining indecomposable obvious reflexive left $\Lambda$–modules, that is on the five indecomposable projective modules which are not summands of $\Lambda T$:

$$
\begin{align*}
1 &\mapsto 2, & 2 &\mapsto 6, & 3 &\mapsto 4 \\
4 &\mapsto 6, & 6 &\mapsto 1 \\
3 &\mapsto 2, & 4 &\mapsto 3, & 3 &\mapsto 2.
\end{align*}
$$

Finally, $\Delta$ acts as follows on the three indecomposable non obvious reflexive left $\Lambda$–modules:

$$
\begin{align*}
1 &\mapsto 2, & 2 &\mapsto 6, & 3 &\mapsto 4 \\
4 &\mapsto 6, & 6 &\mapsto 1 \\
3 &\mapsto 2, & 4 &\mapsto 3.
\end{align*}
$$

Concerning (***) at the beginning of this section, we refer to [C3 Sections 2 and 3 ] (or to [C2 CbCF CF ] and to the other papers quoted in [C3]) for important results obtained by means of rather technical topological tools. That’s why it was a pleasant surprise to see that a discrete bimodule was enough to give the following answer to a question posed by Colpi with the hint: “You cannot use finite dimensional algebras and modules!” (See [CbCF Theorem 1] for the nice behaviour of submodules of reflexive modules over artin algebras.)

5.2. **Proposition** [D1 Lemma 2.4 and Theorem 2.5 (ii)]. Reflexive modules with respect to a cotilting bimodule are not necessarily closed under submodules. Moreover, even the well–known reflexive modules with respect to a faithfully balanced bimodule $sU_R$, that is the indecomposable summands of both the left (resp. right) regular module $R$ (resp. $S$) and of the module $U$ are not necessarily closed under submodules.
The next example - more precisely, the next picture - shows that the above result has a purely combinatorial motivation, coming from basic linear algebra. Indeed, the result follows from the same reason why any infinite dimensional vector over $K$ space cannot be isomorphic to its double dual, and so it cannot be reflexive with respect to the regular bimodule $KK$ (See [C3] Proposition 1.8 for a general result on direct sum of infinitely many non-zero reflexive modules with respect to a cotilting bimodule.)

5.3. Toy example of a finite dimensional cotilting bimodule as in Proposition 5.2 [D1] Lemma 2.4 and Theorem 2.5 (ii)]. With terminology suggested by [HU], assume $R = S$ is the “generalized” Kronecker algebra given by the quiver

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\]

Then the indecomposable projective non simple left (resp. right) module is a reflexive module with respect to the cotilting bimodule $\mathcal{A}A\mathcal{A}$. However, as indicated in the following picture, its maximal submodule, i.e. the Jacobson radical of $A$, generated by the infinitely many arrows from 1 to 2, is not reflexive with respect to $A$.

\[
\begin{array}{c}
\begin{array}{c}
\Box \\
\downarrow \\
\Box
\end{array} \\
\Box \\
\Box
\end{array}
\]

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