A SHAPE CALCULUS APPROACH FOR TIME HARMONIC SOLID–FLUID INTERACTION PROBLEM IN STOCHASTIC DOMAINS

DEBOPRIYA MUKHERJEE AND THANH TRAN

Abstract. The present paper deals with the interior solid-fluid interaction problem in harmonic regime with randomly perturbed boundaries. Analysis of the shape derivative and shape Hessian of vector- and tensor-valued functions is provided. Moments of the random solutions are approximated by those of the shape derivative and shape Hessian, and the approximations are of third order accuracy in terms of the size of the boundary perturbation. Our theoretical results are supported by an analytical example on a square domain.

Keywords and phrases: solid–fluid interaction; stochastic domain; shape derivative; shape Hessian.

1. Introduction

In this paper, we consider the time harmonic forced vibrations of an elastic solid encircling in its interior an inviscid compressible fluid with randomly located boundaries. Since the domain and its perturbed boundaries are stochastic, the solution depends on the ‘random event’ $\omega$ and the parameter $\varepsilon \geq 0$ controlling the amplitude of the perturbation. The usual approach to generate a large number $N$ of ‘sample’ domains and to solve the deterministic boundary value problem on each sample is overpriced. To overcome this costly computation, we approximate the moments of the solution by those of its shape derivative and shape Hessian, as has been done for other simpler models; see [12, 25, 26]. The problem considered in this article is much more complex involving vector-valued functions and tensors which requires careful analysis.

Over the past few years, the solid–fluid interaction problems gain much attention due to its applications in different engineering fields [11, 16, 35], magneto-hydrodynamic flows [24], electro-hydrodynamics [29], etc. The model problem is represented by a vector–valued equation describing time harmonic elastodynamic equations in the solid domain and the Helmholtz equation in the fluid region. On the common boundary the two systems are coupled via adequate transmission conditions.

It is well known that mathematical models are approximations of physical phenomena. Most often, the base model is too complicated or the scales are too disparate to include all the parameters successfully. The neglected parameters are often replaced by some randomness in the deterministic model. In this way, loadings, coefficients and the underlying domains are considered as stochastic input parameters. In the present article, we focus on randomness in the domains. The authors of [25, 26] exploit shape calculus tools to compute statistical moments of the random solutions of elliptic boundary value problems on uncertain domains. The authors of [12] use the same tool to solve elliptic transmission problems in unbounded stochastic domains. They also provide rigorous derivation and properties of shape derivatives.

Shape calculus tools involving the computation of shape derivative and shape Hessian (known as the so-called material derivative approach of continuum mechanics) is studied in the books [27, 36] in the deterministic framework. There is a growing literature rationalized to shape optimization problems in estimating the first and second order shape derivatives in the deterministic setup. For a quick survey we refer to, for example, [3, 15, 37].

Authors in [14] have derived asymptotic expansions of the first moments of the distribution of the output functional considered on a random domain through a boundary value problem.
Computation and use of second order derivative for vector valued states in the context of linear elasticity goes back to the works of Murat and Simon. Application of shape optimization methods in fluid mechanics in deterministic set-up is well-known in the literature; see, for example [1] for the Stokes equations with both Dirichlet and Neumann boundary conditions, [10] for stationary Navier-Stokes equations with non-homogeneous Dirichlet boundary conditions, [28] for transmission boundary value problem.

The solid–fluid interaction problems have been studied by Estecahandy and her co-authors in a nice series of works in deterministic set-up. To be more specific, we refer to [5] (Discontinuous Galerkin based approach for higher-order polynomial-shape functions with the high frequency propagation regime) and [6] (finite element method approach to the Lipschitz continuous polygonal domains) and references therein.

In this article, we develop a precise mathematical theory for computing the statistics of the solution of the solid-fluid interaction problem with randomly perturbed boundaries. Our main contribution in this article is the derivation of the second order shape Taylor expansion of vector-valued and tensor-valued functions. The results are presented in Theorem 4.3 (material derivative), Theorem 4.4 (shape derivative) and Theorem 4.5 (shape Hessian). As a consequence, we obtain the stochastic shape Taylor expansion for the moments of the solution (Theorem 4.6).

To the best of our knowledge, this current work appears to be the first systematic treatment of second order shape calculus for vector-valued and tensor-valued functions which form the solution of the solid–fluid interaction problem under consideration.

In order to apply shape calculus to our particular model problem (the solid-fluid problem) a technical issue requires us to study the spectrum of the solution operator of the problem (Proposition 3.2). This result, another contribution of the paper, has its own interest.

Let us briefly describe the content of this paper. Section 2 deals with the description of the model problem with perturbed random boundaries and details of the function spaces involved during the course of analysis. Section 3 consists of the spectral properties of the solution operator. Section 4 contains details of first and second order shape calculus. It also shows the approximation of the solution moments with those of its shape derivatives. Section 5 provides an analytical example on a square domain perturbed by a uniform distribution. This example illustrates the accuracy of the approximation. Finally in the Appendix we recollect some basic definitions of tensors and their properties, present some technical lemmas, and recall elementary concepts of material and shape derivatives.

In the paper, $C$ stands for (with or without subscripts) a generic constant independent of the discretization parameter and the wave number. These constants may take different values at different places.

2. Time harmonic solid–fluid interaction problem on perturbed domain

In this section we describe the problem and provide preliminaries for the forthcoming analysis.

2.1. Statistical moments. Throughout this paper, we denote by $(\Omega, \mathcal{U}, P)$ a generic complete probability space. Let $D$ be a bounded domain in $\mathbb{R}^3$ with boundary $\partial D$ of class $C^k$, $k \geq 2$.

**Definition 2.1.** For a random field $v \in L^k(\Omega, D)$, its $k$-order moment $\mathcal{M}^k[v]$ is an element of $D^{(k)}$ defined by

$$\mathcal{M}^k[v] := \int_{\Omega} \left( v(\omega) \otimes \cdots \otimes v(\omega) \right) dP(\omega).$$

In the case $k = 1$, the statistical moment $\mathcal{M}^1[v]$ is same as the *mean value* of $v$ and is denoted by $E[v]$. If $k \geq 2$, the statistical moment $\mathcal{M}^k[v]$ is known as the *$k$-point autocorrelation function* of $v$. The quantity $\mathcal{M}^k[v - E[v]]$ is termed the $k$-th central moment of $v$. In particular, the second order moments: the *correlation* and *covariance* are defined by

$$\text{Cor}[v] := \mathcal{M}^2[v] \quad \text{and} \quad \text{Cov}[v] := \mathcal{M}^2[v - E[v]].$$ (2.1)
2.2. Representation of random interfaces. Let us consider a solid body represented by a $C^2$ domain $O_S \subset \mathbb{R}^d$, $d = 2, 3$, with $\partial O_S = \Gamma_D \cup \Gamma_N \cup \Sigma_C$, where $\Gamma_D$, $\Gamma_N$, and $\Sigma_C$ are disjoint parts of $\partial O_S$. We assume that the solid structure is fixed at $\Gamma_D \neq \emptyset$ and free of stress on $\Gamma_N$. The solid interacts through the interface $\Sigma_C$ with a homogeneous, inviscid and compressible fluid occupying a bounded domain $O_F$. The boundary $\partial O_F$ of the fluid domain is $\Sigma_C$. We denote by $n_S$ ($n_F$ respectively) the outward-pointing unit normal vector to the boundary $\partial O_S$ ($\partial O_F$ respectively) of the fluid-solid domain $O := O_S \cup O_F$; see Figure 1. It can be observed that on $\Sigma_C$, one has $n_S = -n_F$. For more details about the model problem, we refer to [19, 32] and the references cited therein.

Following [12] and the references therein, we present the random domain. Suppose $\kappa \in L^2(\mathcal{U}, C^{2,1}(\partial O_S))$ is a random field. For some sufficiently small value $\varepsilon \geq 0$, we consider a family of random interfaces of the form

$$\partial O_S^\varepsilon(\omega) = \{x + \varepsilon \kappa(x, \omega)n(x) : x \in \partial O_S\}, \quad \omega \in \mathcal{U},$$

(2.2)

where $n$ is given by

$$n = \begin{cases} n_F & \text{on } \Sigma_C, \\ n_S & \text{on } \Gamma_D \cup \Gamma_N. \end{cases}$$

Here, the randomness of the surfaces $\partial O_S^\varepsilon(\omega)$ is represented by the randomness in $\kappa(\cdot, \omega)$. We observe that the interface $\partial O_S^\varepsilon(\omega)|_{\varepsilon=0}$ coincides with $\partial O_S$ and therefore is a deterministic closed manifold. If we identify $\partial O_S^\varepsilon$ and $\partial O_S$ with the functions defining their graphs, then

$$\|\partial O_S^\varepsilon - \partial O_S\|_{L^2(\mathcal{U}, C^{2,1})} \leq \varepsilon \|\kappa\|_{L^2(\mathcal{U}, C^{2,1}(\partial O_S))}\|n\|_{C^{2,1}(\partial O_S)}.$$

We will specify the required smoothness assumptions on $\kappa$ in shape calculus in Section 4. From (2.2) we observe that the mean random interface is represented by

$$\mathbb{E}[\partial O_S^\varepsilon] = \{x + \varepsilon \mathbb{E}[\kappa(x, \cdot)]n(x), \ x \in \partial O_S\}.$$

Without loss of generality, we may assume that the random perturbation amplitude $\kappa(x, \omega)$ is centred, i.e.,

$$\mathbb{E}[\kappa(x, \cdot)] = 0 \quad \forall x \in \partial O_S.$$

(2.3)

In this case

$$\mathbb{E}[\partial O_S^\varepsilon] = \partial O_S$$

and

$$\text{Cov}[\kappa](x, y) = \text{Cor}[\kappa](x, y).$$

Figure 1. Solid domain $O_S$ and fluid domain $O_F$

2.3. Model problem. For some sufficiently small and nonnegative $\varepsilon$ we aim to compute the linear oscillations of an elastic structure encircling in its interior an inviscid fluid appearing in the fluid-solid perturbed domain $O^\varepsilon(\omega) := O_S^\varepsilon(\omega) \cup \Sigma_C^\varepsilon(\omega) \cup \partial O_F^\varepsilon(\omega)$, under the action of a given time sinusoidal body force prescribed in the solid domain whose amplitude is $f : B_R \to \mathbb{R}^d$, which is assumed to be independent of $\omega$. Having introduced these perturbed domains and boundaries, the model problem is to find the solid displacement field $u^\varepsilon$ and the fluid pressure $p^\varepsilon$ satisfying

$$\text{div } \sigma^\varepsilon(x, \omega) + \mu^2 \rho_S u^\varepsilon(x, \omega) = f(x) \quad \text{in } O_S^\varepsilon(\omega),$$

(2.4a)

$$\sigma^\varepsilon(x, \omega) = \mathcal{E}(u^\varepsilon(x, \omega)) \quad \text{in } O_S^\varepsilon(\omega),$$

(2.4b)

$$u^\varepsilon(x, \omega) = 0 \quad \text{on } \Gamma_D^\varepsilon(\omega),$$

(2.4c)

$$\sigma^\varepsilon(x, \omega)n^\varepsilon = 0 \quad \text{on } \Gamma_N^\varepsilon(\omega),$$

(2.4d)

$$\Delta p^\varepsilon(x, \omega) + \frac{\mu^2}{c^2} p^\varepsilon(x, \omega) = 0 \quad \text{in } O_F^\varepsilon(\omega),$$

(2.4e)

$$\sigma^\varepsilon(x, \omega)n^\varepsilon + p^\varepsilon(x, \omega)n^\varepsilon = 0 \quad \text{on } \Sigma_C^\varepsilon(\omega),$$

(2.4f)
\[
\frac{\partial p^e}{\partial n^e}(x,\omega) - \mu^2 \rho_F u^e(x,\omega) \cdot n^e = 0 \quad \text{on } \Sigma_C^e(\omega), \quad (2.4g)
\]

where \( \mathbf{0} \) stands for a generic null vector or tensor. Here the stress tensor \( \sigma^e \) is defined by the linearised strain tensor \( \mathcal{E}(\mathbf{u}^e) \) and the Hooke operator \( \mathcal{C} : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d} \) defined by

\[
\mathcal{E}(\mathbf{u}^e) := \frac{1}{2} \left( \nabla \mathbf{u}^e + (\nabla \mathbf{u}^e)^\top \right) \quad \text{and} \quad \mathcal{C} \tau := \lambda(\text{Tr} \, \tau) I + 2 \nu \tau \quad \forall \tau \in \mathbb{R}^{d \times d}.
\]

Here \( \lambda, \nu > 0 \) are the Lamé constants and \( \text{Tr} \, \tau \) denotes the trace of \( \tau \). The remaining physical coefficients are the solid and fluid densities \( \rho_0, \rho \) and \( \omega > 0 \), respectively.

In the next subsection, we introduce the function spaces in the deterministic set-up needed for the analysis.

### 2.4. Function spaces and weak formulation.

We denote by \( C_c^\infty(\mathbb{R}^d; \mathbb{R}^d) \) as the space of all \( \mathbb{R}^d \)-valued compactly supported \( C^\infty \) functions in \( \mathbb{R}^d \). In what follows we will denote the vectorial and tensorial counterparts of order \( d \) \( (d = 2, 3) \) of a given Hilbert space \( \mathbf{H} \) by \( \mathbf{H}_d \) and \( \mathbf{H}^{d \times d} \), respectively. We use standard notation for the Hilbertian Sobolev space \( \mathbf{H}^s(D), s \geq 0 \), defined on a \( C^1 \) bounded domain \( D \subset \mathbb{R}^d \) and denote by \( \| \cdot \|_{s,D} \) the norms in \( \mathbf{H}^s(\Omega), \mathbf{H}^s(D)^d \) and \( \mathbf{H}^s(D)^{d \times d} \).

For \( \sigma : D \to \mathbb{R}^{d \times d} \) and \( \mathbf{u} : D \to \mathbb{R}^d \), we define the row-wise divergence \( \text{div} \, \sigma : D \to \mathbb{R}^d \) and the gradient \( \nabla \mathbf{u} : D \to \mathbb{R}^{d \times d} \) by,

\[
(\text{div} \, \sigma)_i := \sum_j \partial_j \sigma_{ij} \quad \text{and} \quad (\nabla \mathbf{u})_{ij} := (\nabla \mathbf{u})_{ij} = \partial_j u_i.
\]

We introduce for \( s \geq 0 \) the Hilbert space

\[
\mathbf{H}^s(\text{div}; D) := \left\{ \tau \in \mathbf{H}^s(D)^{d \times d} : \text{div} \, \tau \in \mathbf{H}^s(D)^d \right\}
\]

endowed with the norm \( \| \tau \|^2_{\mathbf{H}^s(\text{div}; D)} := \| \tau \|^2_{s,D} + \| \text{div} \, \tau \|^2_{s,D} \), and we use the convention \( \mathbf{H}(\text{div}; D) := \mathbf{H}^0(\text{div}; D) \).

The stress tensor \( \sigma^e \), which is imposed here as a primary unknown in the solid, will be sought in the Sobolev space

\[
\mathcal{W}^e := \left\{ \tau \in \mathbf{H}(\text{div}, \partial \Omega^e_F) : \tau \mathbf{n}^e = 0 \quad \text{on } \Gamma_N^e \right\}.
\]

The fluid main variable is the pressure \( p^e \in \mathbf{H}^1(\partial \Omega^e_F) \). For convenience we introduce the product space

\[
\tilde{X}^e := \mathcal{W}^e \times \mathbf{H}^1(\partial \Omega^e_F)
\]

endowed with the Hilbertian norm

\[
\|(\tau, q)\|^2_{\tilde{X}^e} := \|\tau\|^2_{\mathbf{H}(\text{div}, \partial \Omega^e_F)} + \|q\|^2_{1, \partial \Omega^e_F} \quad \forall (\tau, q) \in \tilde{X}^e.
\]

In articles \cite{17} \cite{17}, displacement formulation in the solid combined with a formulation using the acoustic pressure (or the fluid displacement) as main variables in the fluid domain is studied. In recent years, there are extensive studies of the stress-pressure formulation weakly imposing the symmetry of the stress tensor; see for instance \cite{11} \cite{21} \cite{23}. The dual-mixed formulation which approximates the elastic Cauchy stress tensor is emphasized in the literature; see e.g. \cite{11} \cite{31} and the references therein.

As we are dealing with a dual formulation in \( \partial \Omega^e_F \), the transmission condition \( 2.3 b \) becomes essential (cf. \cite{23} \cite{32} ), it should then be strongly imposed in the continuous energy space

\[
\tilde{X}^e := \left\{ (\tau, q) \in \tilde{X}^e : \tau \mathbf{n}^e + q \mathbf{n}^e = 0 \quad \text{on } \Sigma_C^e \right\}.
\]

It is natural \cite{11} \cite{8} \cite{13} \cite{22} to take into consideration the symmetry of the stress tensor weakly through the introduction of a Lagrange multiplier, which is given by the rotation \( r^e := \frac{1}{2} \left( \nabla \mathbf{u}^e - \left( \nabla \mathbf{u}^e - \mathcal{C} \mathbf{u}^e \right) \right) \).
Let us introduce the orthogonal complement to $\ker(\nabla_\text{u}^\varepsilon)$ and belongs to the space $\mathcal{Q}^\varepsilon$ of skew symmetric tensors

$$\mathcal{Q}^\varepsilon := \{ s \in L^2(\Omega)^{d \times d} : \ s^\top = -s \}.$$ 

For brevity of notations we denote the Hilbertian product norm in $\tilde{X}^\varepsilon \times \mathcal{Q}^\varepsilon$ by

$$\| ( ( \tau, q ), s ) \|_{\varepsilon}^2 := \| ( \tau, q ) \|_{\varepsilon}^2 + \| s \|_{\mathcal{Q}^\varepsilon}^2 \ \forall (( \tau, q ), s) \in \tilde{X}^\varepsilon \times \mathcal{Q}^\varepsilon.$$ 

We define the following bounded bilinear forms

$$a_1^\varepsilon : H(\text{div}, \Omega) \times H(\text{div}, \Omega) \to \mathbb{R} \ \text{by} \ \ a_1^\varepsilon(\sigma, \tau) = \int_{\Omega} \frac{1}{\rho_s} \text{div} \sigma(x) \cdot \text{div} \tau(x) \, dx,$$

(2.5a)

$$a_2^\varepsilon : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R} \ \text{by} \ a_2^\varepsilon(p, q) = \int_{\Omega} \frac{1}{\mu_s} \nabla p(x) \cdot \nabla q(x) \, dx,$$ 

(2.5b)

$$d_1^\varepsilon : H(\text{div}, \Omega) \times H(\text{div}, \Omega) \to \mathbb{R} \ \text{by} \ d_1^\varepsilon(\sigma, \tau) = \int_{\Omega} C^{-1} \sigma(x) : \tau(x) \, dx,$$ 

(2.5c)

$$d_2^\varepsilon : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R} \ \text{by} \ d_2^\varepsilon(p, q) = \int_{\Omega} \frac{1}{\mu_s \mu_c^2} p(x) q(x) \, dx,$$ 

(2.5d)

$$b^\varepsilon : H(\text{div}, \Omega) \times \mathcal{Q}^\varepsilon \to \mathbb{R} \ \text{by} \ b^\varepsilon(\tau, s) = \int_{\Omega} \tau(x) : s(x) \, dx,$$ 

(2.5e)

$$\ell^\varepsilon : H(\text{div}, \Omega) \to \mathbb{R} \ \text{by} \ \ell^\varepsilon(\tau^\varepsilon) = \int_{\Omega} \frac{1}{\rho_s} f \cdot \text{div} \tau^\varepsilon,$$ 

(2.5f)

and denote

$$a^\varepsilon((\sigma, p), (\tau, q)) := a_1^\varepsilon(\sigma, \tau) + a_2^\varepsilon(p, q),$$ 

(2.6a)

$$A^\varepsilon((\sigma, p), (\tau, q)) := a^\varepsilon((\sigma, p), (\tau, q)) + d_1^\varepsilon(\sigma, \tau) + d_2^\varepsilon(p, q),$$ 

(2.6b)

$$\mathcal{A}^\varepsilon\left( ((\sigma^\varepsilon, p^\varepsilon), (\tau^\varepsilon, q^\varepsilon)), ((\tau^\varepsilon, q^\varepsilon), (s^\varepsilon)) \right) := A^\varepsilon((\sigma^\varepsilon, p^\varepsilon), (\tau^\varepsilon, q^\varepsilon)) + b^\varepsilon(\tau^\varepsilon, r^\varepsilon) + b^\varepsilon(\sigma^\varepsilon, s^\varepsilon),$$

(2.6c)

$$\mathcal{B}^\varepsilon\left( ((\sigma^\varepsilon, p^\varepsilon), (\tau^\varepsilon, q^\varepsilon)), ((\tau^\varepsilon, q^\varepsilon), (s^\varepsilon)) \right) := d_1^\varepsilon(\sigma^\varepsilon, \tau^\varepsilon) + d_2^\varepsilon(p^\varepsilon, q^\varepsilon) + b^\varepsilon(\tau^\varepsilon, r^\varepsilon) + b^\varepsilon(\sigma^\varepsilon, s^\varepsilon),$$

(2.6d)

$$\mathcal{D}^\varepsilon\left( ((\sigma^\varepsilon, p^\varepsilon), (\tau^\varepsilon, q^\varepsilon)), ((\tau^\varepsilon, q^\varepsilon), (s^\varepsilon)) \right) = \mathcal{A}^\varepsilon\left( ((\sigma^\varepsilon, p^\varepsilon), (\tau^\varepsilon, q^\varepsilon)), ((\tau^\varepsilon, q^\varepsilon), (s^\varepsilon)) \right) - (1 + \mu_s^2) \mathcal{B}^\varepsilon\left( ((\sigma^\varepsilon, p^\varepsilon), (\tau^\varepsilon, q^\varepsilon)), ((\tau^\varepsilon, q^\varepsilon), (s^\varepsilon)) \right).$$

(2.6e)

We point out that the kernel $\ker(a^\varepsilon) := \{(\tau, q) \in X^\varepsilon : a^\varepsilon((\tau, q), (\tau, q)) = 0\}$ of the bilinear form $a^\varepsilon(\cdot, \cdot)$ in $X^\varepsilon$ is given by

$$\ker(a^\varepsilon) = \{(\tau, q) \in X^\varepsilon : \ \text{div} \ \tau = 0 \ \text{in} \ \Omega \ \text{and} \ q \ \text{constant in} \ \Omega \}. $$

Let us introduce the orthogonal complement to $\ker(a^\varepsilon) \times \mathcal{Q}^\varepsilon$ in $X^\varepsilon \times \mathcal{Q}^\varepsilon$ with respect to the bilinear form $\mathcal{B}^\varepsilon$ by

$$[\ker(a^\varepsilon) \times \mathcal{Q}^\varepsilon]^\perp := ((\sigma^\varepsilon, p^\varepsilon), (\tau^\varepsilon)) \in X^\varepsilon \times \mathcal{Q}^\varepsilon : \mathcal{B}^\varepsilon\left( ((\sigma^\varepsilon, p^\varepsilon), (\tau^\varepsilon)), ((\tau^\varepsilon, q^\varepsilon), (s^\varepsilon)) \right) = 0 \ \forall (\tau^\varepsilon, q^\varepsilon, s^\varepsilon) \in \ker(a^\varepsilon) \times \mathcal{Q}^\varepsilon.$$

(2.7)

With all the bilinear forms defined in (2.6), we are now able to write the weak formulation of problem (2.4). Considering the body force $f \in L^2(B_R)^d$, it is direct to write in the equivalent tensorial form (see [23, 31] for more details): Find $((\sigma^\varepsilon, p^\varepsilon), (\tau^\varepsilon)) \in X^\varepsilon \times \mathcal{Q}^\varepsilon$ such that

$$\mathcal{D}^\varepsilon\left( ((\sigma^\varepsilon, p^\varepsilon), (\tau^\varepsilon)), ((\tau^\varepsilon, q^\varepsilon), (s^\varepsilon)) \right) = \ell^\varepsilon(\tau^\varepsilon) \ \forall ((\tau^\varepsilon, q^\varepsilon), (s^\varepsilon)) \in X^\varepsilon \times \mathcal{Q}^\varepsilon.$$ 

(2.8)

**Remark 1.**

(i) We note that variational formulation (2.8) has been designed in terms of the stress tensor $\sigma$ of the solid (not displacement vector field $u$) and pressure $p$ of the fluid. However, using the equilibrium equation (2.4a) one can recover the displacement vector field.
(ii) According to [32, and references cited therein, for $O_S \in C^2$, the displacement field $u$ that solves this problem belongs to $H^{1+\alpha}(O_S)^d$ for all $\alpha \in (1/2, 1)$.

3. The solution operators $S^\varepsilon$

The shape calculus technique to be used in the next section is originated from shape optimization in the deterministic framework; see [27, 36]. For this reason and for simplicity, we temporarily escape randomness and consider only deterministic perturbed interfaces.

3.1. Representation of perturbed deterministic model. We now present some properties of perturbed interfaces which are required for the subsequent analysis. Let $\tilde{\kappa}$ and $\tilde{n}$ be any smoothness preserving extension of $\kappa$ and $n$ on $\mathbb{R}^3$ such that $\tilde{\kappa} \in \mathcal{W}^{1,\infty}(\mathbb{R}^3) \cap C^{2,1}(\mathbb{R}^3)$. For $\varepsilon \geq 0$, we define $T^\varepsilon : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$T^\varepsilon(x) = x + \varepsilon \tilde{\kappa}(x)\tilde{n}(x) \quad \forall x \in \mathbb{R}^3.$$  

Without loss of generality we assume that the extension $\tilde{\kappa}$ vanishes outside a sufficiently large ball $B_R$ (with origin as the centre and radius $R$) containing $O_S^\varepsilon \cup O_F^\varepsilon$ for $0 \leq \varepsilon \leq \varepsilon_0$, for some $\varepsilon_0 > 0$. This implies that the perturbation mapping $T^\varepsilon(x)$ is an identity in the complement $B_R^\varepsilon := \mathbb{R}^3 \setminus \overline{B_R}$, i.e.

$$T^\varepsilon(x) = x \quad \forall x \in B_R^\varepsilon.$$  

For ease of notation, throughout the paper, we denote by $O^\varepsilon := T^\varepsilon(O)$ either $O_S^\varepsilon$ or $O_F^\varepsilon$ when there is no ambiguity. For convenience, we abbreviate

$$V(x) := \tilde{\kappa}(x)\tilde{n}(x), \quad x \in \mathbb{R}^3.$$  

In [36], the field $V$ is called the velocity field of the mapping $T^\varepsilon$ and in [25, 26], $V$ is known as the boundary perturbation field in the normal direction. From (3.3), one can observe that $\kappa = \langle V, n \rangle$.

Remark 2. When $\varepsilon = 0$, we omit the superscript $\varepsilon$ in the notations of the spaces, norms and bilinear forms.

3.2. The solution operators and their spectra. In this section we define the solution operators $S^\varepsilon$ and study their properties which will be used to prove the existence of the material derivative in the next section. These properties have been studied in [31, 34]. However, since we need to apply these results with different values of $\varepsilon > 0$ and pass to the limit when $\varepsilon \to 0$, it is important to check the estimates to ensure that they are independent of $\varepsilon$.

For each $\varepsilon \geq 0$, let us introduce the operator

$$S^\varepsilon : \mathcal{E}^\varepsilon \times \mathcal{Q}^\varepsilon \rightarrow \mathcal{E}^\varepsilon \times \mathcal{Q}^\varepsilon,$$

$$\left((F^\varepsilon, f^\varepsilon), G^\varepsilon\right) \mapsto \left((\sigma^\varepsilon_+, p^\varepsilon_+), r^\varepsilon_+\right) = S^\varepsilon((F^\varepsilon, f^\varepsilon), G^\varepsilon)$$

where $((\sigma^\varepsilon_+, p^\varepsilon_+), r^\varepsilon_+) \in \mathcal{E}^\varepsilon \times \mathcal{Q}^\varepsilon$ satisfies, for all $((\tau^\varepsilon, q^\varepsilon), s^\varepsilon) \in \mathcal{E}^\varepsilon \times \mathcal{Q}^\varepsilon$,

$$A^\varepsilon \left( ((\sigma^\varepsilon_+, p^\varepsilon_+), r^\varepsilon_+), ((\tau^\varepsilon, q^\varepsilon), s^\varepsilon) \right) = \mathbb{E}^\varepsilon \left( ((F^\varepsilon, f^\varepsilon), G^\varepsilon), ((\tau^\varepsilon, q^\varepsilon), s^\varepsilon) \right).$$  

The well definiteness and symmetry with respect to the bilinear form $A^\varepsilon(\cdot, \cdot)$ of this operator $S^\varepsilon$ is proved in [32, Lemma 3.2]. To focus on the solution of the problem, we first characterize the spectral properties of the operator $S^\varepsilon$ for each $\varepsilon \geq 0$.

Lemma 3.1. For $\varepsilon \geq 0$, the spectrum $\text{sp}(S^\varepsilon)$ of $S^\varepsilon$ decomposes as follows

$$\text{sp}(S^\varepsilon) = \{0, 1\} \cup \{\eta_k(\varepsilon)\}_{k \in \mathbb{N}}$$

where $\{\eta_k(\varepsilon)\}_{k \in \mathbb{N}}$ satisfying

$$1 > \eta_1(\varepsilon) \geq \cdots \geq \eta_k(\varepsilon) \geq \cdots > 0$$  

is a decreasing sequence of finite-multiplicity eigenvalues of $S^\varepsilon$ which converges to 0. Moreover, 1 is an infinite-multiplicity eigenvalue of $S^\varepsilon$ while 0 is not an eigenvalue. The associated eigenspace of the eigenvalue 1 is $\text{ker}(a^\varepsilon) \times \mathcal{Q}^\varepsilon$. 


Proof. See [34, Section 4].

It is proved in [32, Theorem 3.1] that if the input frequency \( \mu \), see (2.6a), is chosen such that \( 1/(1 + \mu^2) \notin \text{sp}(S^c) \), the problem (2.8) is well posed. To ensure that such a choice of \( \mu \) is possible for all \( \varepsilon \geq 0 \) sufficiently small, it is necessary to prove that there exists \( \varepsilon_0 > 0 \) such that

\[
\bigcup_{0 \leq \varepsilon \leq \varepsilon_0} \text{sp}(S^c) \neq [0, 1].
\]

In fact, we will prove a stronger result that there exists \( \delta > 0 \) such that, for all nonnegative \( \varepsilon \) sufficiently small, all the eigenvalues \( \eta_k(\varepsilon) \) are crowded to the left of \( \eta_1(0) + \delta \). This result, which has its own interest, is stated in the following proposition.

**Proposition 3.2.** For each \( \delta \in (0, 1 - \eta_1(0)) \), there exists \( \varepsilon_0 > 0 \) such that

\[
\eta_1(0) + \delta, 1) \subset [0, 1] \setminus B
\]

where \( B := \bigcup_{0 \leq \varepsilon \leq \varepsilon_0} \text{sp}(S^c) \).

Proof. Noting the decrease property (3.5), it suffices to prove that

\[
\liminf_{\varepsilon \to 0} \frac{1}{\eta_1(\varepsilon)} \geq \frac{1}{\eta_1(0)}.
\]

Indeed, assume that (3.7) holds. Let us show that (3.6) holds. For each \( \delta \in (0, 1 - \eta_1(0)) \), let \( \delta_1 = \delta/|\eta_1(0) + \eta_1(0)\delta| > 0 \). By the definition of \( \liminf \), there exists \( \varepsilon > 0 \) such that

\[
\frac{1}{\eta_1(0)} - \delta_1 < \frac{1}{\eta_1(\varepsilon)} \quad \forall 0 < \varepsilon \leq \varepsilon_0,
\]

proving that

\[
\eta_1(\varepsilon) < \eta_1(0) + \delta \quad \forall 0 < \varepsilon \leq \varepsilon_0,
\]

which concludes (3.6) due to (3.5).

We now prove (3.7). Since \( \ker(a^\varepsilon) \times Q^\varepsilon \) is the eigenspace associated with the eigenvalue \( 1 \), see Lemma 3.1, the eigenspace associated with \( \eta_1(\varepsilon) \) is a subspace of \( \ker(a^\varepsilon) \times Q^\varepsilon \) which is defined in (2.7). As a consequence, we derive that for all \( ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) \in \ker(a^\varepsilon) \times Q^\varepsilon \)

\[
\mathbb{A}^\varepsilon\left(((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon), ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon)\right) = \frac{1}{\eta_1(\varepsilon)} \mathbb{B}^\varepsilon\left(((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon), ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon)\right)
\]

where \( ((\sigma^1, p^1), r^1) \) is an eigenvector associated with the eigenvalue \( \eta_1(\varepsilon) \). The Rayleigh quotient gives

\[
\frac{1}{\eta_1(\varepsilon)} = \min_{0 \neq ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) \in \ker(a^\varepsilon) \times Q^\varepsilon} \mathbb{A}^\varepsilon\left(((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon), ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon)\right) / \mathbb{B}^\varepsilon\left(((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon), ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon)\right).
\]

Denoting

\[
\mathbb{R}^\varepsilon((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) := \frac{\mathbb{A}^\varepsilon\left(((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon), ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon)\right)} {\mathbb{B}^\varepsilon\left(((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon), ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon)\right)}
\]

and using (2.5) and (2.6) we deduce

\[
\mathbb{R}^\varepsilon((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) = \frac{a^\varepsilon_1(\sigma^\varepsilon, \sigma^\varepsilon) + a^\varepsilon_2(\sigma^\varepsilon, \sigma^\varepsilon) + d^\varepsilon_1(\sigma^\varepsilon, \sigma^\varepsilon) + d^\varepsilon_2(\sigma^\varepsilon, \sigma^\varepsilon) + 2b^\varepsilon(\sigma^\varepsilon, r^\varepsilon)} {d^\varepsilon_1(\sigma^\varepsilon, \sigma^\varepsilon) + d^\varepsilon_2(\sigma^\varepsilon, \sigma^\varepsilon) + 2b^\varepsilon(\sigma^\varepsilon, r^\varepsilon)}.
\]

We will show that (noting the notation convention in Remark 2)

\[
\mathbb{R}^\varepsilon((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) = \mathbb{R}((\sigma^\varepsilon \circ T^c, p^\varepsilon \circ T^c), r^\varepsilon \circ T^c) + \varepsilon G((\sigma^\varepsilon \circ T^c, p^\varepsilon \circ T^c), r^\varepsilon \circ T^c)
\]

where \( G \) is a mapping from \( \ker(a) \times \mathbb{Q}^{1/2} \) to \( \mathbb{R} \). Letting \( \mathbb{Q}^{1/2}_a := \ker(a^\varepsilon) \times \mathbb{Q}^\varepsilon \), then \( \mathbb{Q}^{1/2}_a \). Since

\[
T^c|\mathbb{Q}_a : \mathbb{Q}_a \to \mathbb{Q}^\varepsilon_a \quad \text{and} \quad T^c|\mathbb{Q}^{1/2}_a : \mathbb{Q}^{1/2}_a \to \mathbb{Q}^{1/2}_a.
\]
are bijective, it follows from (3.8) that
\[
\inf_{0 \neq ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) \in \mathbb{Q}^d} \mathbb{R}^d((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) \geq \inf_{0 \neq ((\sigma, p), r) \in \mathbb{Q}^d} \mathbb{R}((\sigma, p), r) + \varepsilon \inf_{0 \neq ((\sigma, p), r) \in \mathbb{Q}^d} G((\sigma, p), r)
\]
which proves (3.7) by letting \( \varepsilon \to 0 \).

We now move to prove (3.9). Following the notations of Kronecker product mentioned in Appendix A.2, we now employ change of variables where Proposition 2.4 in [31] states that the constant is independent of \( \varepsilon \) for some bounded functions \( \rho, \gamma, J_{T^\varepsilon}, J_{T^\varepsilon}^{-1}, A \) and \( \tilde{A} \).

We now move to prove (3.9). Following the notations of Kronecker product mentioned in Appendix A.2, we now employ change of variables where Proposition 2.4 in [31] states that the constant is independent of \( \varepsilon \) for some bounded functions \( \rho, \gamma, J_{T^\varepsilon}, J_{T^\varepsilon}^{-1}, A \) and \( \tilde{A} \). Using \( |\text{div}(\sigma(x))|^2 = |\mathcal{L}_1(\sigma(x))|^2 \) in \( a_1^2(\sigma^\varepsilon, \sigma^\varepsilon) \) and equation (A.4) and Lemma A.4, we see that

\[
a_1^2(\sigma^\varepsilon, \sigma^\varepsilon) = \int_{0^d} \frac{1}{\rho S} \gamma(x, y) |\mathcal{L}_1(\sigma^\varepsilon \circ T^\varepsilon(y))| \, dy
\]

where

\[
g_1(\sigma^\varepsilon \circ T^\varepsilon) = \int_{0^d} \frac{1}{\rho S} \gamma(x, y) |\mathcal{L}_1(\sigma^\varepsilon \circ T^\varepsilon(y))| \, dy + \int_{0^d} \frac{1}{\rho S} (1 + \hat{\gamma}(\varepsilon, y)) \left[ 2|\mathcal{L}_1(\sigma^\varepsilon \circ T^\varepsilon(y))| : \mathcal{L}_V(\sigma^\varepsilon \circ T^\varepsilon(y)) \right] \, dy
\]

Repeating the similar arguments for each bounded bilinear maps on the right hand side of (3.8), one achieve

\[
a_2^2(p^\varepsilon, p^\varepsilon) = a_2(p^\varepsilon \circ T^\varepsilon, p^\varepsilon \circ T^\varepsilon) + \varepsilon g_2(p^\varepsilon \circ T^\varepsilon),
\]

\[
d_1^2(\sigma^\varepsilon, \sigma^\varepsilon) = d_1(\sigma^\varepsilon \circ T^\varepsilon, \sigma^\varepsilon \circ T^\varepsilon) + \varepsilon g_3(\sigma^\varepsilon \circ T^\varepsilon),
\]

\[
d_2^2(p^\varepsilon, p^\varepsilon) = d_2(p^\varepsilon \circ T^\varepsilon, p^\varepsilon \circ T^\varepsilon) + \varepsilon g_4(p^\varepsilon \circ T^\varepsilon),
\]

\[
b_1^2(\sigma^\varepsilon, r^\varepsilon) = b(\sigma^\varepsilon \circ T^\varepsilon, r \circ T^\varepsilon) + \varepsilon g_5(\sigma^\varepsilon \circ T^\varepsilon, r \circ T^\varepsilon).
\]

for some bounded functions \( g_i; i = 2, \ldots, 5 \). This proves (3.9), finishing the proof of the proposition.

The following result is similar to [31] Proposition 2.4. However, here it is necessary to check that the constant is independent of \( \varepsilon \).

**Proposition 3.3.** If \( 1/(1 + \mu^2) > \eta_1(0) \) then there exist \( \varepsilon_0 > 0 \) and a constant \( C \) depending only on \( \varepsilon_0 \) such that for all \( \varepsilon \in [0, \varepsilon_0] \) the following inequality holds

\[
\left\| \frac{1}{1 + \mu^2} - \mathcal{S}^\varepsilon \right\|_{\varepsilon} \left\| ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) \right\|_{\varepsilon} \geq C \left\| ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) \right\|_{\varepsilon} \forall ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) \in \mathbb{X}^\varepsilon \times \mathbb{Q}^\varepsilon. \tag{3.10}
\]

**Proof.** Proposition 2.4 in [31] states that

\[
\left\| \frac{1}{1 + \mu^2} - \mathcal{S}^\varepsilon \right\|_{\varepsilon} \left\| ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) \right\|_{\varepsilon} \geq C^\varepsilon(\varepsilon) \delta_\mu(\mathcal{S}^\varepsilon) \left\| ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) \right\|_{\varepsilon} \forall ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) \in \mathbb{X}^\varepsilon \times \mathbb{Q}^\varepsilon,
\]
where \( C^*(\varepsilon) \) is a positive constant independent of \( ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) \) and
\[
0 < \delta_\mu(S^\varepsilon) := \text{dist}\left(\frac{1}{1+\mu^2}, \text{sp}(S^\varepsilon)\right) < 1
\]
represents the distance between \( 1/(1+\mu^2) \) and the spectrum of \( S^\varepsilon \). First we show that \( \delta_\mu(S^\varepsilon) \) is bounded below by a constant independent of \( \varepsilon \). Due to the assumption \( 1/(1+\mu^2) > \eta_1(0) \), we can invoke Proposition 3.2 to obtain \( \varepsilon_0 > 0 \) satisfying
\[
\eta_1(0) < \eta_1(\varepsilon) < \eta_1(0) + \delta < \frac{1}{1+\mu^2} < 1 \quad \forall \varepsilon \in [0, \varepsilon_0]
\]
where \( \delta \) is some positive number. By virtue of (3.5), we have
\[
\eta_k(\varepsilon) < \eta_1(0) + \delta < \frac{1}{1+\mu^2} < 1 \quad \text{for all} \quad k \geq 1 \quad \text{and} \quad \varepsilon \in [0, \varepsilon_0].
\]
Hence,
\[
\delta_\mu(S^\varepsilon) = \min_{k \geq 1} \left(1 - \frac{1}{1+\mu^2}, \frac{1}{1+\mu^2} - \eta_k(\varepsilon)\right) \geq \min \left(\frac{\mu^2}{1+\mu^2}, \frac{1}{1+\mu^2} - (\eta_1(0) + \delta)\right) := c.
\]

Next we trace the constant \( C^*(\varepsilon) \) in (3.11) to show that it is bounded below by a constant independent of \( \varepsilon \). Following the proof of Proposition 2.4 in [11] this constant \( C^* \) depends on the constant \( c_1 \) in Proposition 2.1 of the same paper. This constant in turn depends on \( \alpha \) in [33, Lemma 2.1]. This constant \( \alpha \) depends on the constant \( c \) in [9, Proposition IV.3.1] and \( c_2 \) in [20, Lemma 2.2]. Proposition IV.3.1 of [9] is in fact Lemma III.3.2 of [18]. Tracing all these constants one can check that they depend continuously on the measure of the domain, namely \( |\mathcal{O}^\varepsilon_\delta| \). Since \( |\mathcal{O}^\varepsilon_\delta| \) depends continuously on \( \varepsilon \), see [18], so does the constant \( C^*(\varepsilon) \). Because \( \varepsilon \in [0, \varepsilon_0] \), this continuity implies that \( C^*(\varepsilon) \) has a minimum value which is positive. This proves the proposition.

**Proposition 3.4.** Let \( \varepsilon_0 \) and \( B \) be given in Proposition 3.3. If \( \left\{\frac{1}{1+\mu^2}\right\} \subset [0, 1] \setminus B \), then there exists a positive constant \( C \) depending only on \( \varepsilon_0 \) such that, for any \( f \in (L^2(B_R))^d \) and any \( \varepsilon \in [0, \varepsilon_0] \), the solution \( ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) \in X^\varepsilon \times Q^\varepsilon \) of (2.8) satisfies
\[
\left\|\left((\sigma^\varepsilon \circ T^\varepsilon, p^\varepsilon \circ T^\varepsilon), r^\varepsilon \circ T^\varepsilon\right)\right\| \leq \frac{C}{1+\mu^2} \|f\|_{0, \Omega_0}.
\]
(3.12)

**Proof.** By following the proof of [32, Theorem 3.1] we can prove that
\[
\left\|\left((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon\right)\right\| \leq \frac{C}{1+\mu^2} \|f\|_{0, \Omega_0},
\]
where the constant \( C \) comes from Proposition 3.3 which is independent of \( \varepsilon \). Then by change of variable formula, we have (3.12). This completes the proof.

4. **Shape calculus**

In the present section we derive the shape derivative and shape Hessian for the solution \( ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) \) of (2.8).

4.1. **Material derivative.** This section is devoted to a rigorous proof and characterization of the material derivative of (2.8).

**Proposition 4.1.** Let \( ((\sigma^\varepsilon, p^\varepsilon), r^\varepsilon) \in X^\varepsilon \times Q^\varepsilon \) be solution of (2.8) and \( ((\sigma, p), r) \in X \times Q \) be solution of the unperturbed problem (i.e., (2.8) for \( \varepsilon = 0 \)). Assume \( f \in (L^2(B_R))^d \cap (X \times Q)^* \) and \( \kappa \in C^1(\partial \Omega_0) \). Then
\[
\lim_{\varepsilon \to 0} \left\|\left((\sigma^\varepsilon \circ T^\varepsilon, p^\varepsilon \circ T^\varepsilon), r^\varepsilon \circ T^\varepsilon\right) - ((\sigma, p), r)\right\| = 0.
\]
(4.1)
Proof. To prove (4.1), we first aim to prove that
\[
A\left( (1 - (1 + \mu^2)S)(\langle \sigma^c \circ T^c - \sigma, p^c \circ T^c - p \rangle, r^c \circ T^e - r \rangle, ((\tau, q), s) \right) \to 0 \quad \text{as} \quad \varepsilon \to 0. 
\] (4.2)
As a next step, we deduce from the well-posedness of the unperturbed problem (i.e., (2.8) for \( \varepsilon = 0 \)) the existence of \( C > 0 \), independent of \( \varepsilon \) such that
\[
C \| (1 - (1 + \mu^2)S)(\langle \sigma^c \circ T^c - \sigma, p^c \circ T^c - p \rangle, r^c \circ T^e - r \rangle \| \leq \sup_{0 \neq ((\tau, q), s) \in X \times Q} \frac{A(1 - (1 + \mu^2)S)((\sigma^c \circ T^c - \sigma, p^c \circ T^c - p), r^c \circ T^e - r), ((\tau, q), s)}}{\|((\tau, q), s)\|}. 
\] (4.3)
Hence, there exists \( ((\tau, q), s) \in X \times Q \) such that
\[
\| (1 - (1 + \mu^2)S)((\sigma^c \circ T^c - \sigma, p^c \circ T^c - p), r^c \circ T^e - r) \| - \gamma \leq \frac{A(1 - (1 + \mu^2)S)((\sigma^c \circ T^c - \sigma, p^c \circ T^c - p), r^c \circ T^e - r), ((\tau, q), s)}}{C \|((\tau, q), s)\|}, \] (4.3)
where \( \gamma > 0 \) is arbitrary. On letting \( \varepsilon \to 0 \), using (4.2), we have
\[
\limsup_{\varepsilon \to 0} \| (1 - (1 + \mu^2)S)((\sigma^c \circ T^c - \sigma, p^c \circ T^c - p), r^c \circ T^e - r) \| - \gamma \leq 0. \] (4.4)
Since \( \gamma > 0 \) is arbitrary,
\[
\limsup_{\varepsilon \to 0} \| (1 - (1 + \mu^2)S)((\sigma^c \circ T^c - \sigma, p^c \circ T^c - p), r^c \circ T^e - r) \| = 0. \] (4.5)
We then use equation (3.10) in Proposition 3.3 to obtain (4.1).

We are now left to prove (4.2). To begin with, we first estimate
\[
D^\varepsilon((\sigma^c, p^c), (\tau^c, q^c), s^c) - D((\sigma, p), (\tau, q), s) = \ell^\varepsilon(\tau^e) - \ell(\tau), \] (4.6)
Using the fact that \( f \cdot \text{div} \tau(y) = (I \otimes f(y)) : (\nabla \otimes \tau(y)) \), and exploiting Lemmas A.A, A.B, we have for all \( \tau \in W \),
\[
\ell^\varepsilon(\tau^e) - \ell(\tau) = \int_{\Omega_k} \rho S \left[ \left( \gamma(\varepsilon, y)(J_{T^c}^{-1} \otimes f(T^c y)) : (\nabla \otimes \tau(y)) \right) - f \cdot \text{div} \tau(y) \right] dy \\
= \int_{\Omega_k} \rho S \left[ \left( \gamma(\varepsilon, y)(J_{T^c}^{-1} \otimes f(T^c y)) : (\nabla \otimes \tau(y)) \right) - \left( (I \otimes f(y)) : (\nabla \otimes \tau(y)) \right) \right] dy \\
\to 0 \quad \text{as} \quad \varepsilon \to 0. 
\]
Passing to the limit as \( \varepsilon \to 0 \) in (4.6), we arrive at (4.3). This completes the proof. \( \square \)

**Lemma 4.2.** There exists a unique solution \( ((\hat{\sigma}, \hat{p}), \tilde{r}) \in X \times Q \) to the following equation for all \( ((\tau, q), s) \in X \times Q \)
\[
D((\hat{\sigma}, \hat{p}), (\tau, q), s) = - \int_{\Omega_k} \rho S \left[ \tilde{A}'(0, y) \sigma(y) : (I \otimes (I \cdot \nabla)^T \tau(y))^T \right] dy \\
+ \int_{\Omega_k} \rho S \left[ \text{div} V(y)(I \otimes f(y)) + \tilde{V}_1(y) \otimes f(y) + I \otimes (\nabla f(y) \cdot V) : (\nabla \otimes \tau(y)) \right] dy \\
- \int_{\Omega_k} \rho F \left[ \tilde{A}'(0, y) \nabla p(y) \cdot \nabla q(y) \right] dy + \mu^2 \int_{\Omega_k} C^{-1}(y) \sigma(y) : \tau(y) dy \\
+ \int_{\Omega_k} \rho F \gamma(y)p(y)q(y)dy + \int_{\Omega_k} \gamma(y)\tau(y)dy + \int_{\Omega_k} \gamma(y)\sigma(y) : s(y)dy. 
\]
(4.7)
where \( \tilde{A}', \tilde{A}, \gamma_1, \Gamma \) and \( \tilde{V}_1 \) are given in Appendix A.2.

**Proof.** Existence and uniqueness of the solution of (4.7) in the space \( X \times Q \) relies on the well-known Babuška-Brezzi theory (see [32, Theorem 3.1]). \( \square \)
Theorem 4.3. Assume that \( f \in H^1(\Omega) \cap (X \times \mathcal{Q})^\varepsilon \) and \( \kappa \in C^1(\partial \Omega_S) \). Let \( (\sigma^\varepsilon, p^\varepsilon, r^\varepsilon) \in X^\varepsilon \times \mathcal{Q}^\varepsilon \) and \((\sigma, p, r) \in X \times \mathcal{Q}\) be solutions of (2.3) and the unperturbed problem (i.e., (2.3) for \( \varepsilon = 0 \)), respectively. Then \( (\sigma^\varepsilon, p^\varepsilon, r^\varepsilon) \) has a material derivative \((\dot{\sigma}, \dot{p}, \dot{r})\) in \( X \times \mathcal{Q} \) which satisfies (4.7).

Proof. It suffices to show that
\[
\lim_{\varepsilon \to 0^+} \left\| \left( \frac{\sigma^\varepsilon - \sigma}{\varepsilon}, \frac{p^\varepsilon - p}{\varepsilon}, \frac{r^\varepsilon - r}{\varepsilon} \right) - \left( \dot{\sigma}, \dot{p}, \dot{r} \right) \right\| = 0, \tag{4.8}
\]
where \((\dot{\sigma}, \dot{p}, \dot{r})\) is the solution of (4.7). Using (2.6c) and (3.3) we have
\[
\mathbb{D}\left( \left( \frac{\sigma^\varepsilon - \sigma}{\varepsilon}, \frac{p^\varepsilon - p}{\varepsilon}, \frac{r^\varepsilon - r}{\varepsilon} \right), ((\tau, q), s) \right)
= \frac{1}{\varepsilon} \int_{\Omega_S} \left[ \text{div} \left( \frac{\sigma^\varepsilon - \sigma}{\varepsilon} \right) - \dot{\sigma} \right] \cdot \text{div} \tau dy + \frac{1}{\varepsilon} \int_{\Omega_F} \nabla \left( \frac{p^\varepsilon - p}{\varepsilon} \right) \cdot \nabla q dy
- \mu^2 \int_{\Omega_S} \mathcal{C}^{-1} \left( \frac{\sigma^\varepsilon - \sigma}{\varepsilon} \right) \cdot \tau dy - \mu^2 \int_{\Omega_F} \mathcal{C}^{-1} \left( \frac{p^\varepsilon - p}{\varepsilon} \right) q dy
- \mu^2 \int_{\Omega_S} \left( \frac{r^\varepsilon - r}{\varepsilon} \right) dy
= \int_{\Omega_S} \frac{1}{\varepsilon} \left[ \gamma^\varepsilon(y) \right] J_{11} \left( \frac{f(T^\varepsilon y) - f(y)}{\varepsilon} \right) - I \otimes f(y) dy
+ \frac{1}{\varepsilon} \left[ \nabla f(y) \cdot V \right] dy - \int_{\Omega_S} \frac{1}{\varepsilon} \left[ \big( \mathcal{A}(\varepsilon, y), \dot{\sigma} \big) \right] - \mathcal{A}(0, y) \sigma \right] dy
+ \mu^2 \int_{\Omega_F} \mathcal{C}^{-1} \left( \frac{\gamma^\varepsilon(y) - 1}{\varepsilon} \right) \nabla \left( \frac{p^\varepsilon - p}{\varepsilon} \right) \cdot \tau dy
+ \mu^2 \int_{\Omega_F} \mathcal{C}^{-1} \left( \frac{\gamma^\varepsilon(y) - 1}{\varepsilon} \right) \nabla \left( \frac{r^\varepsilon - r}{\varepsilon} \right) \cdot \tau dy
+ \int_{\Omega_S} \left( \frac{\gamma^\varepsilon(y) - 1}{\varepsilon} \right) \nabla \left( \frac{r^\varepsilon - r}{\varepsilon} \right) \cdot \tau dy \tag{4.9}
\]
Using equation (A.4), Proposition 4.1, Lemma A.4, Lemma A.5 and letting \( \varepsilon \to 0 \), we see that the right hand side of (4.9) \( \to 0 \). Repeating the arguments as in (4.3) - (4.5) for \( \big( \frac{\sigma^\varepsilon - \sigma}{\varepsilon}, (\frac{p^\varepsilon - p}{\varepsilon}, \frac{r^\varepsilon - r}{\varepsilon}) \big) \) and exploiting equation (3.10), we prove (4.8). This completes the proof. \( \square \)

4.2. Shape derivative. This section is devoted to the existence and characterization of shape derivative and shape Hessian of the deterministic solution of the considered model problem. We denote
\[
\mathcal{J}_1(v^\varepsilon, \Omega^\varepsilon, w) := \int_{\Omega^\varepsilon} v^\varepsilon \star w \quad \text{and} \quad \mathcal{J}_2(v^\varepsilon, \partial \Omega^\varepsilon, w) := \int_{\partial \Omega^\varepsilon} v^\varepsilon \star w
\]
where \( \star \) stands for the usual product of two scalar functions, or the dot product of two vector functions, or the component-wise inner product of two tensor functions. Let us define the following spaces:
\[
\mathcal{D}_1 := \left\{ \mathbf{r} \in C^\infty_c(\mathbb{R}^d; \mathbb{R}^{d \times d}) : \langle \mathbf{D} \mathbf{r}^{(k)}, \mathbf{n} \rangle = 0 \text{ on } \partial \Omega_S, \forall k = 1, \ldots, d \right\},
\]
\[
\mathcal{D}_2 := \left\{ q \in C^\infty_c(\mathbb{R}^d; \mathbb{R}) : \frac{\partial q}{\partial \mathbf{n}} = 0 \text{ on } \partial \Omega_F \right\}.
\]
Theorem 4.4. Under the assumptions of Theorem 4.3, \((\sigma^\varepsilon, p^\varepsilon, r^\varepsilon)\) has a shape derivative \(((\sigma', p'), r')\) belonging in \((H(\div; \Omega_S) \times H^1(\Omega_F)) \times \mathcal{Q})\) that satisfies

\[
\begin{align*}
\text{div} \sigma' + \mu p_S \sigma' &= 0 \quad \text{in } \Omega_S, & (4.10a) \\
\sigma' &= C\mathcal{E}(u') \quad \text{in } \Omega_S, & (4.10b) \\
u' &= -\kappa(\nabla u)n + \frac{1}{\rho_S \mu^2} \div_{\partial \Omega_S}(n)G \quad \text{on } \Gamma_D, & (4.10c) \\
\sigma'n &= (\nabla^\top \otimes \sigma)(V \otimes n) \quad \text{on } \Gamma_N, & (4.10d) \\
\Delta p' + \frac{\mu^2}{c^2} p' &= 0 \quad \text{in } \Omega_F, & (4.10e) \\
\sigma'n + p'n &= -(\nabla^\top \otimes \sigma)(V \otimes n) - (V^\top \nabla p)n \quad \text{on } \Sigma_C, & (4.10f) \\
\frac{1}{\rho_F} \frac{\partial p'}{\partial n} - \mu^2 u' \cdot n &= \frac{1}{\rho_S} (\div_{\partial \Omega_F}(n)G \cdot n - \mu^2 \kappa(\nabla un) \cdot n) \\
+ \frac{1}{\rho_F} \left[ \div(\kappa \nabla p) - \frac{\partial}{\partial n}(\kappa \nabla p) \cdot n - \div_{\partial \Omega_F}(n)\kappa \frac{\partial p}{\partial n} + \frac{\mu^2}{c^2} \kappa \right] \quad \text{on } \Sigma_C, & (4.10g)
\end{align*}
\]

where

\[G := \div(\kappa \sigma) - \frac{\partial \kappa}{\partial n} \sigma n + \kappa \sigma \left( \frac{\partial}{\partial n}(n) \right) - \sigma \left( \nabla_{\partial \Omega_S} \kappa \right).\]  

Proof. We split the proof of (4.10) in three steps. In the first step, we will derive equations satisfied by the shape derivative \(((\sigma', p'), r')\) on the domains \(\Omega_S\) and \(\Omega_F\). In the next two steps, we will derive the boundary conditions satisfied by \(((\sigma', p'), r')\). We use Lemma A.3 in Appendix A.3 to prove (4.10).

Step I: In this step, our goal is to prove (4.10a), (4.10b) and (4.10c). Existence of the shape derivative \(((\sigma', p'), r')\) follows from Theorem 4.3 and Definition A.4. We choose \(((\tau, q), s) \in \left( (D_1 \times D_2) \cap X \right) \times \mathcal{Q}, \text{ in } \mathbb{R}^8 \). In order to apply Lemma A.3, we consider each of the bilinear forms present in the definition of \(D^\varepsilon\). Using integration by parts (see Lemma A.2) in Appendix A.1, we have

\[
a_1(\sigma^\varepsilon, \tau) = -\int_{\Omega_S} \frac{1}{\rho_S} \sigma^\varepsilon : \nabla(\div \tau) + \int_{\partial \Omega_S} \frac{1}{\rho_S} \sigma^\varepsilon n^\varepsilon \cdot \div \tau dS.
\]

Using part (iv) of Lemma A.8, we have

\[
dJ_1(\sigma^\varepsilon, \Omega_S, \frac{\nabla(\div \tau)}{\rho_S})_{e=0} = -\int_{\Omega_S} \frac{1}{\rho_S} \sigma' : \nabla(\div \tau) + \int_{\partial \Omega_S} \frac{1}{\rho_S} \sigma : \nabla(\div \tau) \kappa dS, \tag{4.12}
\]

\[
dJ_2(\sigma^\varepsilon n^\varepsilon, \partial \Omega_S, \frac{\div \tau}{\rho_S})_{e=0} = \int_{\partial \Omega_S} \frac{1}{\rho_S} \left[ (\sigma' \cdot \div \tau + \sigma n^\varepsilon \cdot \div \tau) + \kappa \frac{\partial}{\partial n} (\sigma n \cdot \div \tau) \right. \\
+ \kappa \left( \div(n) \sigma n + \div \tau \right) \right] dS. \tag{4.13}
\]

From (4.12) and (4.13) and using tangential Green’s formula (see Lemma A.2), we have

\[
dJ_1(\sigma^\varepsilon, \Omega_S, \frac{\nabla(\div \tau)}{\rho_S})_{e=0} + dJ_2(\sigma^\varepsilon n^\varepsilon, \partial \Omega_S, \frac{\div \tau}{\rho_S})_{e=0} \]

\[
= \int_{\Omega_S} \frac{1}{\rho_S} \div \sigma' \cdot \div \tau + \int_{\partial \Omega_S} \frac{1}{\rho_S} \div (\kappa \sigma) \cdot \div \tau - \int_{\partial \Omega_S} \frac{1}{\rho_S} \sigma n \cdot \div \tau dS \\
- \int_{\partial \Omega_S} \frac{1}{\rho_S} \sigma(\nabla_{\partial \Omega_S} \kappa) \cdot \div \tau dS + \int_{\partial \Omega_S} \frac{1}{\rho_S} \sigma \left( \frac{\partial}{\partial n}(n) \right) \cdot \div \tau \kappa dS. \tag{4.14}
\]
As in \(a_1^\varepsilon(\sigma^\varepsilon, \tau)\), using integration by parts (see Lemma A.2 in Appendix A.1), we have
\[
a_2^\varepsilon(p^\varepsilon, q) = -\int_{\partial F} \frac{1}{\rho_F} p^\varepsilon \Delta q + \int_{\partial F} \frac{1}{\rho_F} p^\varepsilon \frac{\partial q}{\partial n} dS.
\]

Exploiting parts (iv) and (v) of Lemma A.8, we have
\[
d_j(p^\varepsilon, \Omega^\varepsilon, \frac{\Delta q}{\rho_F}) = -\int_{\partial F} \frac{1}{\rho_F} p^\varepsilon \Delta q dS, \quad \text{for} \quad j = 1, 2.
\]

Next, we choose \((\tau, q)\) such that \(-\mu^2 \int_{\partial F} \Delta q \kappa dS - \mu^2 \int_{\partial F} \nabla q \cdot \nabla q \kappa dS = 0\), and integrating by parts, we have
\[
a_2^\varepsilon(p^\varepsilon, q) = -\int_{\partial F} \frac{1}{\rho_F} p^\varepsilon \Delta q dS.
\]

Exploiting parts (iv) and (v) of Lemma A.8, we have
\[
d_j(p^\varepsilon, \Omega^\varepsilon, \frac{\Delta q}{\rho_F}) = -\int_{\partial F} \frac{1}{\rho_F} p^\varepsilon \Delta q dS, \quad \text{for} \quad j = 1, 2.
\]

In the second term of \(d_j(p^\varepsilon, \Omega^\varepsilon, \frac{\Delta q}{\rho_F})\) we use the identity for \(\Delta q\), which is
\[
\Delta q = \Delta_{\partial F}(q) + \nabla_{\partial F}(n) \cdot \frac{\partial q}{\partial n} \frac{1}{(\partial n)^2}.
\]

Hence further using Green’s and tangential Green’s formula, (4.15) and (4.16) reduce to
\[
d_j(p^\varepsilon, \Omega^\varepsilon, \frac{\Delta q}{\rho_F}) = -\int_{\partial F} \frac{1}{\rho_F} p^\varepsilon \Delta q dS, \quad \text{for} \quad j = 1, 2.
\]

Results similar to (4.14) and (4.17) can be obtained for the other bilinear forms in the definition of \(D^\varepsilon\). Hence we obtain
\[
\int_{\Omega^\varepsilon} \frac{1}{\rho_S} \nabla \cdot \sigma' \cdot \nabla \tau + \int_{\partial F} \frac{1}{\rho_F} G \cdot \nabla \tau dS + \int_{\partial F} \frac{1}{\rho_F} \nabla p^\varepsilon \cdot \nabla q - \mu^2 \int_{\partial F} C^{-1} \sigma : \tau
\]
\[
- \mu^2 \int_{\Omega^\varepsilon} \frac{1}{\rho_S} \nabla^2 q - \mu^2 \int_{\partial F} \nabla p^\varepsilon \cdot \nabla q \kappa dS - \mu^2 \int_{\partial F} C^{-1} \sigma : \tau \kappa dS
\]
\[
- \mu^2 \int_{\partial F} \frac{1}{\rho_F} \nabla q \kappa dS - \mu^2 \int_{\partial F} \tau \kappa dS = 0.
\]

with
\[
\int_{\Omega^\varepsilon} \sigma' : s + \int_{\partial F} \sigma : s \kappa dS = 0.
\]

We now choose \((\tau, q, s)\) such that \((\nabla \cdot \sigma' : \nabla \tau) + (\nabla q \cdot \nabla q) = 0\), and integrating by parts, we have
\[
\int_{\Omega^\varepsilon} \frac{1}{\rho_S} \nabla \cdot \sigma' \cdot \nabla \tau + \int_{\partial F} \frac{1}{\rho_F} \nabla p^\varepsilon \cdot \nabla q - \mu^2 \int_{\Omega^\varepsilon} C^{-1} \sigma : \tau
\]
\[
+ \int_{\Omega^\varepsilon} \tau' = 0.
\]

Using density argument we obtain (4.10a) and (4.10c). Next, using (2.4f) and integration by parts we achieve
\[
\int_{\Omega^\varepsilon} (C^{-1} \sigma^\varepsilon - \tau^\varepsilon) : \tau = -\int_{\Omega^\varepsilon} u^\varepsilon \cdot \nabla \tau + \int_{\partial F} u^\varepsilon \cdot \tau n^\varepsilon dS.
\]

Again exploiting Lemma A.8 to (4.21) and using Tangential Green’s formula, we have (4.10b). **Step II:** In this step, we aim to prove (4.10g). In order to find the required boundary
condition, we choose \( ((\tau, q), s) \in \left( (\bar{D}_1 \times \bar{D}_2) \cap \mathbb{X} \right) \times \mathcal{Q} \), and use \((4.20)\) to obtain from \((4.18)\) and \((4.19)\),

\[
\int_{\partial \Omega_s} \frac{1}{\rho_S} \text{div } \sigma' \cdot \text{div } \tau dS + \int_{\partial \Omega_s} \frac{1}{\rho_S} G \cdot \text{div } \tau dS - \int_{\partial \Omega_F} \frac{1}{\rho_F} \Delta p' \cdot q - \mu^2 \int_{\partial \Omega_F} \frac{1}{\rho_F} \partial q \cdot \partial q' q
\]

\[
+ \int_{\partial \Omega_F} \frac{1}{\rho_F} \nabla p' \cdot \nabla q' q dS + \mu^2 \int_{\partial \Omega_F} u' \cdot \text{div } \tau - \mu^2 \int_{\partial \Omega_F} \tau \cdot u' dS
\]

\[
- \mu^2 \int_{\partial \Omega_s} \nabla u : \tau \kappa dS - \mu^2 \int_{\partial \Omega_F} \frac{1}{\rho_F} \partial q \cdot \partial q' q dS = 0.
\]

Using \((4.10a)\) and \((4.10c)\) we have

\[
\int_{\partial \Omega_s} \frac{1}{\rho_S} G \cdot \text{div } \tau dS - \mu^2 \int_{\partial \Omega_s} \tau \cdot u' dS - \mu^2 \int_{\partial \Omega_s} \nabla u : \tau \kappa dS + \int_{\partial \Omega_F} \frac{1}{\rho_S} \partial q \cdot \partial q' q dS
\]

\[
+ \int_{\partial \Omega_F} \frac{1}{\rho_F} \nabla p' \cdot \nabla q' q dS - \mu^2 \int_{\partial \Omega_F} \frac{1}{\rho_F} \partial q \cdot \partial q' q dS = 0.
\]  

Using Lemma \(A.2\) and using \( \frac{\partial \tau}{\partial n} n = 0 \) and \( \frac{\partial q}{\partial n} = 0 \) to the first and fifth terms of \((4.22)\) respectively, we achieve

\[
\int_{\partial \Omega_s} \frac{1}{\rho_S} G \cdot \text{div } \tau dS = - \int_{\partial \Omega_s} \frac{1}{\rho_S} \left[ \nabla G : \tau + \tau n \cdot \frac{\partial G}{\partial n} + \text{div}_{\partial \Omega_s}(n) \tau n \cdot G \right] dS,
\]  

\[
\int_{\partial \Omega_F} \frac{1}{\rho_F} \nabla p \cdot \nabla q k dS = - \int_{\partial \Omega_F} \frac{1}{\rho_F} \left[ q \text{div}(\nabla p \kappa) + \left( \frac{\partial}{\partial n}(\nabla p \kappa) \cdot n \right) q + \text{div}_{\partial \Omega_F}(n) \frac{\partial p}{\partial n} \kappa q \right] dS.
\]  

We now choose \( \tau \in \mathcal{W} \) such that \( \tau = 0 \) on \( \Gamma_D \cup \Gamma_N \). This yields \( q = -(\tau n) \cdot n \) on \( \Sigma_C \). Substituting this value of \( q \), equations \((4.23)\) and \((4.24)\) in \((4.22)\) on \( \Sigma_C = \partial \Omega_F \), we have \((4.10g)\).

**Step III:** In this step, we prove \((4.10c)\), \((4.10d)\) and \((4.10f)\). Using \((4.10g)\) and \( \tau n = 0 \) on \( \Gamma_N \), we have

\[
-\mu^2 \int_{\Gamma_D} \tau n \cdot u' dS = \int_{\Gamma_D \cup \Gamma_N} \frac{1}{\rho_S} \nabla G : \tau dS - \int_{\Gamma_D} \frac{1}{\rho_S} \tau n \cdot \nabla q dS
\]

\[
- \int_{\Gamma_D \cup \Gamma_N} \frac{1}{\rho_S} \text{div}_{\partial \Omega_s}(n) \tau n : G dS + \mu^2 \int_{\Gamma_D \cup \Gamma_N} \kappa \nabla u : \tau dS.
\]

Since \( \Gamma_D \cap \Gamma_N = \emptyset \), choosing \( \tau = 0 \) on \( \Gamma_N \), we have

\[
-\mu^2 n^T \otimes u' = \frac{1}{\rho_S} \left[ \nabla G - n^T \otimes \nabla q - \text{div}_{\partial \Omega_s}(n) n^T \otimes G \right] + \mu^2 \kappa \nabla u, \quad \text{on } \Gamma_D
\]

which proves \((4.10c)\). By Definition \(A.4\) we have

\[
\sigma'_{ij} = \sigma_{ij} - \nabla \sigma_{ij} \cdot V.
\]

Using \( \sigma n = 0 \) on \( \Gamma_N \), we have

\[
(\sigma' n)_i = - \sum_{j=1}^d \nabla \sigma_{ij} \cdot V n^j,
\]

and this directly implies \((4.10d)\). Again since \( (\hat{\sigma}, \hat{p}) \in \mathcal{X} \), \( \hat{\sigma} n + \hat{p} n = 0 \), we have

\[
(\sigma' n + p' n)_i = - \sum_{j=1}^d \left( \nabla \sigma_{ij} \cdot V n^j + \nabla p \cdot V n^j \right).
\]
which directly implies (4.10). Hence combining all the three steps, we conclude that the shape derivative \( (\sigma', p'), r' \) satisfies (4.10) for all \( ((\tau, q), s) \in \left( (D_1 \times D_2) \cap X \right) \times \mathcal{Q} \). This completes the proof.

Before proceeding to the next theorem, let us consider the perturbation of the domain \( \Omega := \Omega_S \cup \Omega_F \) with respect to \( T^\delta \) (where \( T^\delta \) is given by (3.11) and (3.21)). We consider another boundary variation \( V_1 \) which is of the form

\[
V_1(x) := \kappa_1(x)n(x), \quad i = 1, 2
\]

where \( \kappa_1 \) has the same regularity as in (A.11). We refer to Appendix A.3 for further details about second order variations. We will show that shape Hessian \( ((\sigma'', p''), r'') \) exists. For this, we need to consider the shape derivative \( ((\sigma'_s, p'_s), r'_s) \) exists in \( \mathbf{H}(\text{div}; \Omega_S^2) \times \mathbf{H}^1(\Omega_F^2) \times \mathcal{Q}^2 \) and satisfies (4.10), (4.11).

Theorem 4.5. Under the above mathematical settings as in Theorem 4.4, shape Hessian \( ((\sigma'', p''), r'') \) belonging in \( (\mathbf{H}(\text{div}; \Omega_S) \times \mathbf{H}^1(\Omega_F) \times \mathcal{Q}) \) that satisfies

\[
\begin{align*}
\text{div} \sigma'' + \mu^2 \rho_S u'' &= 0 \quad \text{in } \Omega_S, \\
\sigma'' &= \mathcal{E}(u'') \quad \text{in } \Omega_S, \\
u'' &= \frac{1}{\mu^2} (\text{div}_{\partial \Omega_S} n)\mathbb{H}_1 + \frac{1}{\mu^2} \mathbb{H}_3 n \quad \text{on } \Gamma_D, \\
\sigma'' n &= \bar{\sigma} n - (\nabla^T \otimes \sigma)(V \otimes n) - (\nabla^T \otimes \sigma')(V_1 \otimes n) - M(\nabla^T \otimes \sigma)(V \otimes n) \quad \text{on } \Gamma_N, \\
\Delta p'' + \frac{\mu^2}{c} p'' &= 0 \quad \text{in } \Omega_F, \\
\sigma'' n + p'' n &= \bar{\sigma} n - (\nabla^T \otimes \sigma)(V \otimes n) - (\nabla^T \otimes \sigma')(V_1 \otimes n) - M(\nabla^T \otimes \sigma)(V \otimes n) \\
&\quad + \bar{p} n - (V^T(\nabla V) + V_1^T(\nabla V') + V_1^T p V) n \quad \text{on } \Sigma_C, \\
\frac{1}{\rho_F} \frac{\partial p''}{\partial n} - \mu^2 u'' \cdot n &= \frac{1}{\rho_S} (\text{div}_{\partial \Omega_F} n)\mathbb{H}_1 \cdot n - \mathbb{H}_3 n \cdot n - \text{div} \mathbb{H}_2 + \text{div}_{\partial \Omega_F}(n)\mathbb{H}_2 \cdot n \quad \text{on } \Sigma_C,
\end{align*}
\]

where

\[
\begin{align*}
\mathbb{H}_1 &= -\frac{1}{\rho_S} \left[ \text{div}(\kappa \sigma') - \frac{\partial(\kappa)}{\partial n} \sigma' n - \sigma'(\nabla_{\partial \Omega_S} \kappa) + \sigma \frac{\partial(n)}{\partial n} + G' + \frac{\partial G}{\partial n} + \kappa_1 \text{div}_{\partial \Omega_S}(n)G \right], \\
\mathbb{H}_2 &= -\left[ \frac{\kappa}{\rho_F} (\nabla p') + \frac{\kappa_1 \kappa}{\rho_F} \frac{\partial(p)}{\partial n} + \frac{\kappa_1}{\rho_F} \nabla p + \kappa_1 \kappa \text{div}_{\partial \Omega_F}(n)\nabla p \right], \\
\mathbb{H}_3 &= -\mu^2 \left[ (\kappa_1 \nabla u' + \kappa \nabla u' + \kappa_1 \left( \frac{\partial}{\partial n}(\kappa \nabla u) + \text{div}_{\partial \Omega_S}(n) \kappa \nabla u \right) \right], \\
\mathbb{H}_4 &= \mu^2 \left[ \frac{1}{\rho_F^2} (\kappa_1 p' + \kappa p') + \frac{\kappa_1}{\rho_F^2} \frac{\partial(p)}{\partial n}(\kappa p) + \text{div}_{\partial \Omega_S}(n)\kappa p \right],
\end{align*}
\]

and \( M(f) \) denotes the material derivative of a given function \( f \).

Proof. Existence of shape Hessian \( ((\sigma'', p''), r'') \) follows from Theorem 4.4 and (A.13). We choose \( ((\tau, q), s) \in \left( (D_1 \times D_2) \cap X \right) \times \mathcal{Q} \), and proceed in the similar lines as in (4.18) and (4.19) to have

\[
\int_{\Omega_S} \frac{1}{\rho_S} \text{div} \sigma'_S \cdot \text{div} \tau + \int_{\partial \Omega_S} \frac{1}{\rho_S} G^\delta \cdot \text{div} \tau dS + \int_{\Omega_F} \frac{1}{\rho_F} \nabla p'_S \cdot \nabla q - \mu^2 \int_{\Omega_S} C^{-1} \sigma'_S : \tau
\]
\[-\mu^2 \int_{\partial \Omega_\nu} \frac{1}{p F^2} p'_\delta q - \mu^2 \int_{\partial S} \tau : r'_\delta + \int_{\partial \Omega_\nu} \frac{1}{p F} \nabla p^\delta \cdot \nabla q \kappa dS - \mu^2 \int_{\partial \Omega_\nu} C^{-1} \sigma^\delta : \tau \kappa dS \]

\[-\mu^2 \int_{\partial \Omega_\nu} \frac{1}{p F^2} p^\delta q \kappa dS - \mu^2 \int_{\partial S} \tau : r^\delta \kappa dS + \int_{\partial S} \sigma^\delta : s + \int_{\partial \Omega_\nu} \sigma^\delta : s \kappa dS = 0. \]

Proceeding in the same lines as in Step I following (4.18) and (4.19) in the proof of Theorem 4.4 for \((\sigma'_\delta, p'_\delta, r'_\delta)\) and passing to the limit as \(\delta \to 0\), we achieve (4.25a), (4.25b) and (4.25c).

Now we move to prove the boundary conditions satisfied by \((\sigma''^\nu, p''^\nu, r''^\nu)\). Recurrence use of the same arguments used in Step II, we have

\[
\int_{\partial \Omega_\nu} \frac{1}{p F} \tau n \cdot \text{div} \sigma''^\nu dS + \int_{\partial \Omega_\nu} \frac{1}{p F} \frac{\partial \sigma''^\nu}{\partial n} dS = \int_{\partial S} \frac{1}{p F} \text{div} \tau dS + \int_{\partial \Omega_\nu} \frac{1}{p F} \frac{\partial \sigma''^\nu}{\partial n} dS
\]

where \(H_i\) are given by (4.20) for \(i = 1, 2, 3, 4\). Using Lemma A.2 tangential Green’s formula, \(\frac{\partial \tau}{\partial n} n = 0\) and \(\frac{\partial q}{\partial n} = 0\), we obtain from (4.27)

\[
\int_{\partial \Omega_\nu} \frac{1}{p F} \tau n \cdot \text{div} \sigma''^\nu dS + \int_{\partial \Omega_\nu} \frac{1}{p F} \frac{\partial \sigma''^\nu}{\partial n} dS = -\int_{\partial \Omega_\nu} \left[ \nabla H_1 : \tau + \tau n \cdot \frac{\partial H_1}{\partial n} + \text{div} \frac{\partial \sigma''^\nu}{\partial n} (n) \cdot H_1 + H_3 : \tau \right] dS
\]

\[
-\int_{\partial \Omega_\nu} \left[ \frac{1}{p F} \text{div} H_2 + \frac{1}{p F} \text{div} \frac{\partial \sigma''^\nu}{\partial n} (n) H_2 \cdot n + H_4 q \right] dS. \tag{4.28}
\]

Recalling that \(\partial \Omega_\nu = \Gamma_\nu \cap \Gamma_F\), we obtain the same equation as (4.18) with all integrals replaced by integrals over \(\Sigma_C\). Now by choosing \(q = -(\tau n) \cdot n\) in this resulting equation, we obtain (4.25). Using (4.25) and the same argument as in the proof of (4.10), we obtain (4.25c). With the help of (4.13), we have (4.25d) and (4.25f). Hence, the shape Hessian \((\sigma''^\nu, p''^\nu, r''^\nu)\) satisfies (4.25), completing the proof.

**Remark 3.** Both problems (4.11) and (4.25) for the shape derivative and shape Hessian, respectively can be solved by using the methods in [19] [20] [21] [22] etc.

4.3. Computation of stochastic moments. In Sections 4.1 and 4.2, we have defined material derivative, shape derivative and shape Hessian in which the quantities \(\kappa\) and \(\kappa_1\) are deterministic. Since (2.4) is posed on a domain with uncertainty located boundaries (see (2.2)), these derivatives also depend on \(\omega\). Thus, we compute the mean and the variance of the random solutions. The main result of this paper is stated below.

**Theorem 4.6.** Let \(((\sigma^\varepsilon(\omega), p^\varepsilon(\omega)), r^\varepsilon(\omega))\) be solution of (2.3) with the random interface \(\partial \Omega^\varepsilon(\omega)\) given by (2.2), and let \(((\sigma, p), r)\) be solution of the unperturbed problem (i.e., (2.3) for \(\varepsilon = 0\)) with reference interface \(\partial \Omega_\nu\). Assume that the perturbation function \(\kappa = \kappa_1\) belongs to \(L^2(\Omega, C^{2,1}(\partial \Omega_\nu))\) for an integer \(k\) and \(f \in (L^2(B_R))^d \cap (\Omega \times \mathbb{Q})^a\). Then for sufficiently small \(\varepsilon \geq 0\), there exists compact set \(K \subset (\Omega_S \cup \Omega_F) \cap (\Omega^\varepsilon_S \cup \Omega^\varepsilon_F)\) such that

1. \(((\sigma^\varepsilon(x, \omega), p^\varepsilon(x, \omega)), r^\varepsilon(x, \omega))\) admits the asymptotic expansion \(\mathbb{P} - \text{a.e.} \omega \in \Omega, \text{ i.e. for } x \in (\Omega_S \cup \Omega_F) \cap (\Omega^\varepsilon_S \cup \Omega^\varepsilon_F)\)

\[
((\sigma^\varepsilon(x, \omega), p^\varepsilon(x, \omega)), r^\varepsilon(x, \omega)) = ((\sigma(x, p(x)), r(x)) + \varepsilon((\sigma'(x, \omega), p'(x, \omega)), r'(x, \omega)) + \varepsilon \frac{\varepsilon}{2} ((\sigma''(x, \omega), p''(x, \omega)), r''(x, \omega)) + O(\varepsilon^3). \tag{4.29}
\]

2. The mean and variance of the solution \(((\sigma^\varepsilon(\omega), p^\varepsilon(\omega)), r^\varepsilon(\omega))\) can be approximated, respectively, by

\[
\mathbb{E}[(\sigma^\varepsilon, p^\varepsilon, r^\varepsilon)] = ((\sigma, p), r) + O(\varepsilon^2), \tag{4.30}
\]

\[
\mathbb{E}[(\sigma^\varepsilon, p^\varepsilon, r^\varepsilon)^2] = ((\sigma, p), r) + O(\varepsilon^3). \tag{4.31}
\]
Using stochastic Taylor expansion (4.29) we note that

\[
\text{Var}(\sigma^\varepsilon) = \varepsilon^2 \left( \mathbb{E}[\sigma(x)]^2 \right) + O(\varepsilon^3),
\]

where for any tensor \( \tau \), we denote by \( \tau^2 \) the tensor product \( \tau : \tau \).

**Proof.** We first start with proving (4.29). With the shape derivative and shape Hessian of \((\sigma, p, r)\) given in Theorems 4.3 and 4.5 and equation (A.11), using the Taylor expansion (A.15) for an arbitrary, fixed realization \( \kappa(\cdot, \omega), \omega \in \Omega \), we have the stochastic counterpart (4.30).

We now move to prove (4.30). On taking expectation, we have

\[
\mathbb{E}[\sigma^\varepsilon(x, \cdot)] = \mathbb{E}[\sigma(x) + \varepsilon \sigma'(x, \cdot) + \frac{\varepsilon^2}{2} \sigma''(x, \cdot) + O(\varepsilon^3)] \quad \text{for} \quad x \in (O_S \cup O_F) \cap (O_S \cup O_F).
\]

Since \((\sigma', p', r')\) depends linearly on \( \kappa \), exploiting \( \mathbb{E}[\kappa] = 0 \), see (2.3), it can be seen that \( \mathbb{E}[(\sigma', p', r')] \) satisfies (4.10) with zero boundary data, and hence \( \mathbb{E}[(\sigma', p', r')] = 0 \). This proves (4.30).

We note that the quantity \( \text{Var}(\sigma^\varepsilon) \) for a tensor \( \sigma^\varepsilon \) is given by the following

\[
\text{Var}(\sigma^\varepsilon) := \mathbb{E}\left[ (\sigma^\varepsilon - \mathbb{E}[\sigma^\varepsilon]) : (\sigma^\varepsilon - \mathbb{E}[\sigma^\varepsilon]) \right] = \mathbb{E}[\sigma^\varepsilon : \sigma^\varepsilon] - \left( \mathbb{E}[\sigma^\varepsilon] : \mathbb{E}[\sigma^\varepsilon] \right).
\]

In similar manner, \( \text{Var}(r^\varepsilon) \) is given by

\[
\text{Var}(r^\varepsilon) = \mathbb{E}[r^\varepsilon : r^\varepsilon] - \left( \mathbb{E}[r^\varepsilon] : \mathbb{E}[r^\varepsilon] \right).
\]

Using stochastic Taylor expansion (4.29) we note that \( \mathbb{P} - \text{a.e.} \), \( \omega \in \Omega \)

\[
\sigma^\varepsilon(x, \omega) = \sigma(x) + \varepsilon \sigma'(x, \omega) + \frac{\varepsilon^2}{2} \sigma''(x, \omega) + O(\varepsilon^3).
\]

Keeping in mind \( \sigma^2(x) = \sigma(x) : \sigma(x) \) and \( (\sigma'(x, \omega))^2 = \sigma'(x, \omega) : \sigma'(x, \omega) \), we see that \( \mathbb{P} - \text{a.e.} \), \( \omega \in \Omega \)

\[
(\sigma^\varepsilon(x, \omega))^2 = \left( \sigma(x) + \varepsilon \sigma'(x, \omega) + \frac{\varepsilon^2}{2} \sigma''(x, \omega) + O(\varepsilon^3) \right)^2
\]

\[
= \sigma(x) + \varepsilon^2 \sigma'(x, \omega)^2 + 2 \varepsilon \sigma(x) : \sigma(x) + \sigma(x) \cdot \sigma(x) + O(\varepsilon^3).
\]

Hence, on taking expectation on both sides of (4.32), we have

\[
\mathbb{E}[\sigma^\varepsilon(x)^2] = \sigma^2(x) + \varepsilon^2 \mathbb{E}[\sigma'(x)^2] + O(\varepsilon^3).
\]

On taking expectation and then squaring on both sides of (4.33) we have

\[
\left( \mathbb{E}[\sigma^\varepsilon(x)] \right)^2 = \left( \sigma(x) + \frac{\varepsilon^2}{2} \mathbb{E}[\sigma''(x)] + O(\varepsilon^3) \right)^2 = \sigma^2(x) + \varepsilon^2 \sigma(x) : \sigma''(x) + O(\varepsilon^3).
\]

This essentially concludes from (4.32)

\[
\text{Var}(\sigma^\varepsilon(x)) = \mathbb{E}[\sigma^\varepsilon(x)^2] - \left( \mathbb{E}[\sigma^\varepsilon(x)] \right)^2 = \varepsilon^2 \mathbb{E}[\sigma'(x)^2] + O(\varepsilon^3).
\]

Similarly for \( p^\varepsilon \) and \( r^\varepsilon \) we have

\[
\text{Var}(p^\varepsilon(x)) = \varepsilon^2 \mathbb{E}\left[ (p^\varepsilon)^2 \right] + O(\varepsilon^3), \quad \text{Var}(r^\varepsilon(x)) = \varepsilon^2 \mathbb{E}\left[ (r^\varepsilon)^2 \right] + O(\varepsilon^3).
\]

Combining (4.31) - (4.36) we arrive at (4.31), finishing the proof of this theorem. \( \Box \)

**Remark 4.** Observing (2.1) and \( \mathbb{E}[\sigma'(x)] = 0 \), it can be observed that

\[
\text{Var}(\sigma(x)) = \text{Cor}(\sigma(x), \sigma(x))|_{x=y}.
\]

Similarly, \( \text{Var}(p(x)) = \text{Cor}(p(x), p(y))|_{x=y} \), \( \text{Var}(r(x)) = \text{Cor}(r(x), r(y))|_{x=y} \).
In this section, we present a particular example of the solid–fluid problem in a square domain. In this example we will solve a slightly different problem \( 2.4 \) with \( 2.41 \) replaced by
\[
\sigma^\varepsilon(\omega)n^\varepsilon + p^\varepsilon(\omega)n^\varepsilon = g \quad \text{on } \Sigma_{C}^\varepsilon(\omega)
\] (5.1)
and \( \Gamma_N = \emptyset \).

**Figure 2.**

All the theoretical results in Section 4.2 still hold, except that \( 2.41 \) will have a correction term due to the non-homogeneous condition \( 5.1 \). We consider \( 2.4 \) for \( \varepsilon = 0 \) on the domains \( \Omega_F := [-1,1]^2 \subset \mathbb{R}^2 \) and \( \Omega_S := [-2,2]^2 \setminus \Omega_F \) and take the parameters \( \mu^2 = 6\pi^2, \rho_S = 3, \rho_F = \lambda = \nu = 1 \). Then we choose the data \( f \) so that the exact solution for the displacement, pressure and the stress tensor of the considered unperturbed problem are given, respectively, by
\[
u(x, y) = \begin{pmatrix} \sin \pi x & \sin \pi y \\ \sin \pi x & \sin \pi y \end{pmatrix} \forall (x, y) \in \Omega_S, \quad p(x, y) = \cos \pi x \cos \pi y, \quad \forall (x, y) \in \Omega_F
\]
\[
\sigma(x, y) = \begin{pmatrix} \sin \pi (x + y) + 2 \cos \pi x \sin \pi y & \sin \pi (x + y) \\ \sin \pi (x + y) + 2 \cos \pi x \sin \pi y \end{pmatrix} \forall (x, y) \in \Omega_S.
\]
Let the random interface \( \Gamma^\varepsilon(\omega) \) be given by
\[
\Gamma^\varepsilon(\omega) = \{ x + \varepsilon \kappa(x, \omega)n(x) : x \in \Gamma \}
\]
where \( \Gamma := (\cup_{i=1}^{4} \Gamma_i) \cup (\cup_{i=1}^{4} \Sigma_{C}^i) \). Next, let us consider the perturbed domain
\[
\Omega^\varepsilon_F(\omega) := [-1 + \varepsilon a(\omega), 1 + \varepsilon a(\omega)]^2 \quad \text{and} \quad \Omega^\varepsilon_S(\omega) := [-2 + \varepsilon a(\omega), 2 + 2\varepsilon a(\omega)]^2 \setminus \Omega^\varepsilon_F(\omega)
\]
where the perturbation parameter \( \kappa(x, \omega) = a(\omega) \) has a constant (but random) value over the whole \( \partial \Omega_F \). The random variable \( a(\omega) \) takes values in \([-1,1]\) and is centred so that \( \mathbb{E}[a] = 0 \). Further, we consider \( a(\omega) \) is a uniformly distributed random variable with values in \([-1,1]\) and probability density function (PDF) \( \rho_1(t) = 1/2 \), so that
\[
\text{Cov}[\kappa](x, y) = \mathbb{E}[a^2] = \int_{-1}^{1} t^2 \rho_1(t) \, dt = \frac{1}{3}, \quad (5.2)
\]
Solution for the displacement and pressure, of \( 2.4 \) are given, respectively, by
\[
u^\varepsilon(x, y, \omega) = \begin{pmatrix} \sin \pi (x - \varepsilon a(\omega)) & \sin \pi (y - \varepsilon a(\omega)) \\ \sin \pi (x - \varepsilon a(\omega)) & \sin \pi (y - \varepsilon a(\omega)) \end{pmatrix} \forall (x, y) \in \Omega^\varepsilon_S
\]
\[
p^\varepsilon(x, y, \omega) = \cos \pi (x - \varepsilon a(\omega)) \cos \pi (y - \varepsilon a(\omega)), \quad \forall (x, y) \in \Omega_F.
\]
We now split the verification in three steps. In the first two steps, we will verify equations satisfied by the shape derivative (Theorem 4.4) and shape Hessian (Theorem 4.5). In the last step, we will verify Theorem 4.6.

**Step I:** Exploiting (4.12) and Lemma A.4, we see that the shape derivative of \( u \) and \( p \) denoted, respectively, by \( u' \) and \( p' \) are given by
\[
u'(x, y, \omega) = -a(\omega) \pi \begin{pmatrix} \sin \pi (x + y) \\ \sin \pi (x + y) \end{pmatrix} \forall (x, y) \in \Omega_S,
\]
\[
l(x, y, \omega) = -a(\omega) \pi^2 \begin{pmatrix} 4 \cos \pi (x + y) & 2 \cos \pi (x + y) \\ 2 \cos \pi (x + y) & 4 \cos \pi (x + y) \end{pmatrix} \forall (x, y) \in \Omega_S,
\]
\[
p'(x, y, \omega) = a(\omega) \pi \sin \pi (x + y) \forall (x, y) \in \Omega_F.
\]
Hence elementary calculations reveal that \( ((u', l'), p') \) satisfies
\[
\text{div } l' + 6\pi^2 u' = 0 \quad \text{in } \Omega_S,
\]
\[
l' = \mathcal{CE}(u') \quad \text{in } \Omega_S,
\]
\[ \frac{1}{\rho S \mu^2} \text{div}_{\partial \Omega_S}(n)G - \kappa(\nabla u)n = \begin{cases} a(\omega)\pi \left( \frac{\sin \pi x}{\sin \pi x} \right) & \text{on } \Gamma_1 \cup \Gamma_3, \\
\left( \frac{\sin \pi y}{\sin \pi y} \right) & \text{on } \Gamma_2 \cup \Gamma_4, \end{cases} \]

\[ \Delta p' + 2\pi^2 p' = 0 \]

in \( \Omega_F \),

\[ \frac{1}{\rho S}(\text{div}_{\partial \Omega_F} n) \cdot n - \mu^2 \kappa((\nabla u)n) \cdot n + \frac{1}{\rho E} \left[ \text{div}(\kappa \nabla p) - \frac{\partial}{\partial n}(\kappa \nabla p) \cdot n - \text{div}_{\partial \Omega_F}(n)\frac{\partial p}{\partial n} \right] + \frac{\mu^2}{C^2} = \begin{cases} a(\omega)\pi^2 \cos \pi x - 6\pi^3 a(\omega) \sin \pi x & \text{on } \Sigma^1_C, \\
\left( \frac{\sin \pi y}{\sin \pi y} \right) & \text{on } \Sigma^2_C, \\
\left( \frac{\cos \pi x}{\cos \pi x} \right) & \text{on } \Sigma^3_C, \\
\left( \frac{\cos \pi y}{\cos \pi y} \right) & \text{on } \Sigma^4_C. \end{cases} \]

Therefore, computation of the shape derivative agrees with our result (4.10) in Theorem 4.4.

We also note that \( G \) defined in (4.11) is given by

\[ G = 2a(\omega)\pi^2 \begin{pmatrix} \cos \pi(x + y) & -\sin \pi x \sin \pi y \\ \cos \pi(x + y) & -\sin \pi x \sin \pi y \end{pmatrix}. \]

**Step II:** Let us consider another perturbation parameter \( \kappa_1(x, \omega) = b(\omega) \) which has a constant (but random) value over the whole \( \partial \Omega_F \). The random variable \( b(\omega) \) possesses similar properties as \( a(\omega) \). Hence, \( E[\kappa_1] = 0 \) and \( \text{Cov}[\kappa_1](x, y) = E[b^2] = \frac{1}{3} \). Again using (A.13) and Lemma A.7, we now calculate the shape Hessian of \( u, \sigma \) and \( p \). We denote, respectively, by \( u'', \sigma'' \) and \( p'' \).

One can clearly see that \( \text{div}_{\partial \Omega_F} n = 0 \) and \( \text{div}_{\partial \Omega_F} n = 0 \). Hence on computing the right hand side of (4.25) and (4.25g), there is no contribution coming from the term \( H_1 \). We also observe that on \( \Sigma^1_C \), using (4.26) we obtain

\[ H_2 = a(\omega)b(\omega)\pi^2 \begin{pmatrix} \cos \pi x \\ 0 \end{pmatrix}, \quad H_3 = -12a(\omega)b(\omega)\pi^4 \cos \pi x, \quad H_4 = -4a(\omega)b(\omega)\pi^3 \sin \pi x. \]

In a similar manner, one can compute \( H_i \) for \( i = 2, 3, 4 \) on \( \Sigma^2_C \cup \Sigma^3_C \cup \Sigma^4_C \). Therefore, by elementary calculations one can see that \((u'', \sigma'', p'')\) is given by

\[ u''(x, y, \omega) = 2a(\omega)b(\omega)\pi^2 \begin{pmatrix} \cos \pi(x + y) \\ \cos \pi(x + y) \end{pmatrix}, \forall (x, y) \in \Omega_S, \]

\[ \sigma''(x, y, \omega) = -\begin{pmatrix} 8a(\omega)b(\omega)\pi^3 \sin \pi(x + y) & -4a(\omega)b(\omega)\pi^3 \sin \pi(x + y) \\ -4a(\omega)b(\omega)\pi^3 \sin \pi(x + y) & -8a(\omega)b(\omega)\pi^3 \sin \pi(x + y) \end{pmatrix}, \forall (x, y) \in \Omega_S, \]

\[ p''(x, y, \omega) = -2a(\omega)b(\omega)\pi^2 \cos \pi(x + y), \forall (x, y) \in \Omega_F, \]

and satisfies

\[ \text{div} \sigma'' + 6\pi^2 u'' = 0 \]

\[ \sigma'' = C \mathcal{E} (u'') \]

\[ \frac{1}{\mu^2} (\text{div}_{\partial \Omega_S} n)H_1 + \frac{1}{\mu^2}H_3 n = \begin{cases} 2ab\pi^2 \begin{pmatrix} \cos \pi x \\ \cos \pi x \end{pmatrix} & \text{on } \Gamma_1 \cup \Gamma_3, \\
2ab\pi^2 \begin{pmatrix} \cos \pi y \\ \cos \pi y \end{pmatrix} & \text{on } \Gamma_2 \cup \Gamma_4, \end{cases} \]

\[ \Delta p'' + 2\pi^2 p'' = 0 \]

\[ -\frac{1}{\rho S}(\text{div}_{\partial \Omega_F} n)H_1 \cdot n - H_3 n \cdot n - \text{div} H_2 + \text{div}_{\partial \Omega_F}(n)H_2 \cdot n + H_4 \]
Therefore, computation of the shape Hessian agrees with our result (4.25) in Theorem 4.5.

**Step III:** In this step, we choose \( \kappa = \kappa_1 \). Using \( \mathbb{E}[a] = 0 \), equation (5.2), and \( \mathbb{E} \left[ \cos \pi (x - \varepsilon a(\omega)) \cos \pi (y - \varepsilon a(\omega)) \right] = \cos \pi x \cos \pi y + O(\varepsilon^2) \), one can derive

\[
p^\varepsilon(x, y, \omega) = p(x, y) + \pi \varepsilon a(\omega) \sin \pi (x + y) - \pi^2 a^2(\omega) \varepsilon^2 \cos \pi (x + y) + a^3 O(\varepsilon^3),
\]

which verifies (4.29) for \( \varepsilon a \). Proceeding in similar lines, it is easy to observe that \( u^\varepsilon(\omega) \) and \( \sigma^\varepsilon(\omega) \) admit the asymptotic shape Taylor expansion given by (4.29).

Again taking the expectation on (5.3) and using \( \mathbb{E}[a] = 0 \), \( \mathbb{E}[a^2] = 1/3 \), rudimentary calculations reveal that \( \mathbb{E}[p^\varepsilon(x, y, \cdot)] = p(x, y) + O(\varepsilon^2) \) for \( (x, y) \in \mathcal{O}_F \). In a similar way, one can verify

\[
\mathbb{E}[u^\varepsilon(x, y, \cdot)] = u(x, y) + O(\varepsilon^2), \quad \mathbb{E}[\sigma^\varepsilon(x, y, \cdot)] = \sigma(x, y) + O(\varepsilon^2), \quad (x, y) \in \mathcal{O}_S.
\]

This verifies (4.30) in Theorem 4.6.

Also exploiting \( \mathbb{E}[a] = 0 \), equation (5.2), and

\[
\mathbb{E} \left[ \cos \pi (x - \varepsilon a(\omega)) \cos \pi (y - \varepsilon a(\omega)) \right] = \cos \pi x \cos \pi y - \frac{\pi^2 \varepsilon^2}{3} \cos \pi (x + y) + a^3 O(\varepsilon^3),
\]

one can observe that for \( (x, y) \in \mathcal{O}_F \)

\[
\mathbb{E}[p^\varepsilon(x, y)] = p(x, y) - \frac{\pi^2 \varepsilon^2}{3} \cos \pi (x + y) + a^3 O(\varepsilon^3)
\]

\[
\mathbb{E}[p^\varepsilon(x, y)]^2 = p^2(x, y) + \frac{\pi^2 \varepsilon^2}{3} \sin^2 \pi (x + y) - \frac{\pi^2 \varepsilon^2}{3} p(x, y) \cos \pi (x + y) + O(\varepsilon^3).
\]

This implies, \( \text{Var}[p^\varepsilon] = \mathbb{E}[p^\varepsilon]^2 - [\mathbb{E}[p^\varepsilon]]^2 = \frac{\pi^2 \varepsilon^2}{3} \sin^2 \pi (x + y) + O(\varepsilon^3) \).

Since \( p'(x, y) = \pi a(\omega) \sin \pi (x + y) \), one has \( \mathbb{E}[p'^2] = \frac{\pi^2}{3} \sin^2 \pi (x + y) \). Hence, we have \( \text{Var}[p^\varepsilon] = \varepsilon^2 \mathbb{E}[p'^2] + O(\varepsilon^5) \). In a similar manner, one can check that \( \text{Var}[u^\varepsilon] = \varepsilon^2 \mathbb{E}[u'^2] + O(\varepsilon^5) \). Therefore, approximation of the variance agrees with (4.31) in Theorem 4.6. Thus we corroborate the theoretical results achieved in this paper on a square domain.

**APPENDIX A. APPENDIX**

This section has been split into 3 parts. The very first subsection consists of basic tensor algebra notations and integration by parts formula for tensor-valued functions. The second subsection presents a series of lemmas which are involved in the analysis. The last subsection is devoted to the introduction of necessary concepts regarding shape derivative and shape Hessian for \( H^\alpha \) functions when \( \alpha > 0 \).

### A.1. Tensor algebra.

This section is based on Kronecker product and some of its properties; see [30].

**Definition A.1.** Let \( A = (a_{ij})_{i,j=1,2} \) be a \( 2 \times 2 \) matrix and \( \sigma = (\sigma_{ij})_{i,j=1,2} \) be a \( 2 \times 2 \)-tensor-valued function. We define

\[
\mathcal{L}_A \sigma := \left( A^T \begin{bmatrix} \partial_1 & \partial_2 \end{bmatrix} \right)^T \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} := \begin{bmatrix} a_{11} \partial_1 \sigma_{11} + a_{21} \partial_2 \sigma_{11} + a_{12} \partial_1 \sigma_{12} + a_{22} \partial_2 \sigma_{12} \\ a_{11} \partial_1 \sigma_{21} + a_{21} \partial_2 \sigma_{21} + a_{12} \partial_1 \sigma_{22} + a_{22} \partial_2 \sigma_{22} \end{bmatrix}.
\]

We note that when \( A \) is the identity matrix, the operator \( \mathcal{L}_I \) is not the gradient of the tensor \( \sigma \).
Again using integration by parts formula, we have

\[ K = \int_D \sigma \cdot \text{div} \tau \, dx = -\int_D \sigma : \nabla(\text{div} \tau) \, dx + \int_{\partial D} \sigma n \cdot \text{div} \tau \, dS(x) \]

where \( \sigma \) and \( \tau \) are matrix-valued functions defined on \( D \) and \( \partial D \), respectively.

**Proof.** Let \( D \) be a \( C^2 \) domain in \( \mathbb{R}^d \). Then for \( \tau, \sigma \in (\mathbb{H}^2(D))^d \),

\[ \int_{\partial D} \text{div} \sigma \cdot \text{div} \tau \, dS = \int_{\partial D} \nabla(\text{div} \tau) \cdot \sigma n \, dS + \int_{\partial D} \text{div} \sigma \cdot \text{div} \tau \, dS \]

where \( \sigma n \) is the unit outward normal vector to the boundary \( \partial D \) and

\[ \mathbf{D} \sigma := \left[ D\sigma^{(1)} \ldots D\sigma^{(N)} \right]^T \]

\[ \mathbf{F}(\sigma) := \left[ \langle D\sigma^{(1)} n, n \rangle_{\mathbb{R}^N} \ldots \langle D\sigma^{(N)} n, n \rangle_{\mathbb{R}^N} \right]^T \]

To evaluate \( K_1 \), using \( \left( \nabla(\text{div} \tau) \right)_{ik} = \sum_j \frac{\partial^2 \tau_{ik}}{\partial x_j \partial x_j} \), we obtain

\[ \sigma \cdot \nabla(\text{div} \tau) = \sum_{i,j} \sigma_{ij} \frac{\partial}{\partial x_j} \text{div} (\sum_k \sigma_{ik} \frac{\partial \tau_{kj}}{\partial x_k}) = \sigma_{ij} \frac{\partial}{\partial x_j} (\sum_k \sigma_{ik} \frac{\partial \tau_{kj}}{\partial x_k}) \]

Again using integration by parts formula, we have

\[ K_2 = \sum_{i,j,k} \int_{\partial D} \frac{\partial \sigma_{ij}}{\partial n} \frac{\partial \tau_{kj}}{\partial x_k} \, n_i \, dS + \int_{\partial D} \sum_{i,j,k} \frac{\partial \sigma_{ij}}{\partial n} \frac{\partial^2 \tau_{kj}}{\partial x_j \partial x_k} \, n_i \, dS, \]

which on further simplification yields

\[ K_{21} = \sum_{i=1}^k \int_{\partial D} \frac{\partial \sigma_{ki}}{\partial n} n_i \, \text{div}(\tau^{(k)}) \, dS = \sum_{i,l} \int_{\partial D} \frac{\partial \sigma_{ki}}{\partial x_l} n_i \, n_l \, \text{div}(\tau^{(k)}) \, dS = \int_{\partial D} \langle \mathbf{D}\sigma^{(k)} n, n \rangle_{\mathbb{R}^N} \, dS, \]

Note that \( A \otimes B \neq B \otimes A \).

The component-wise outer product of two matrices \( A, B \in \mathbb{R}^{m \times n} \) is denoted by

\[ A : B = \mathbf{T}r(A^T B) \]

where \( \mathbf{T}r \) denotes the trace of the matrix.
where $D\sigma$ and $F(\sigma)$ are given by (A.1) and (A.2) respectively. Combining all these we have the required result.

A.2. Technical Lemmas. If $V$ is given by (3.3), until the end of this subsection let us assume that $T^ε$ is defined by (3.1) and (3.2) with $κ ∈ C^1(\mathbb{R}^3)$, and denote its Jacobian matrix and Jacobian determinant by $J_{T^ε}$ and $γ(ε, ⋅)$, respectively. The following result is straightforward.

Lemma A.3. Assuming $κ ∈ W^{1,∞}(\mathbb{R}^3)$ and $κ(x) = 0$ for $x ∈ B^c_R$, there hold $V ∈ (H^1(\mathbb{R}^3))^3$ and 

$$\frac{∂^m V(x)}{∂ x^m} = 0 \quad ∀x ∈ B^c_R, \quad l = 1, 2, 3, \quad m = 0, 1.$$ 

We denote $V(x) := (V_1(x), V_2(x), V_3(x))^T$. For the explicit forms of $J_{T^ε}(⋅)$ and $γ(ε, ⋅)$, we refer to [12]. Exploiting Lemma A.3, we derive that for sufficiently small $ε > 0$, there holds

$$γ(ε, x) = 1 + ε γ_1(x) + ε^2 γ_2(x) + ε^3 γ_3(x) ≥ c > 0 \quad ∀x ∈ \mathbb{R}^3.$$  

(A.4)

Lemma A.4. For any $σ ∈ \mathbb{X}$,

1. $∥J_{T^ε}^{-T} - I∥_{L^∞(\mathbb{R}^3)} ≤ Cε,$

2. $∥L^{-1}_{T^ε} σ - L^{-1} σ∥_{L^2(\mathbb{R}^3)} ≤ Cε.$

(A.5) (A.6)

Proof. Recalling the form of $J_{T^ε}$ in [12], we see that

$$\text{Adj} J_{T^ε}(x) = \left( I + ε \tilde{V}_1(x) + ε^2 \tilde{V}_2(x) \right)^T$$

where

$$\tilde{V}_1 = \begin{bmatrix} V_{2,2} + V_{3,3} & -V_{1,2} & -V_{1,3} \\ -V_{2,1} & V_{1,1} + V_{3,3} & -V_{1,3} \\ -V_{3,1} & -V_{3,2} & V_{1,1} + V_{2,2} \end{bmatrix}$$

and

$$\tilde{V}_2 = \begin{bmatrix} V_{2,2} V_{3,3} - V_{2,3} V_{3,2} & V_{1,3} V_{3,2} - V_{1,2} V_{3,3} & V_{1,2} V_{2,3} - V_{1,3} V_{1,2} \\ V_{2,1} V_{3,1} - V_{2,1} V_{3,3} & V_{1,1} V_{3,3} - V_{1,3} V_{3,1} & V_{1,3} V_{2,1} - V_{1,1} V_{2,3} \\ V_{2,1} V_{3,2} - V_{2,2} V_{3,1} & V_{1,2} V_{3,1} - V_{1,1} V_{3,2} & V_{1,1} V_{2,2} - V_{1,2} V_{2,1} \end{bmatrix}$$

where $V_{i,j} = \frac{∂V_i}{∂ x_j}$. This implies

$$J_{T^ε}^{-1}(x) = \frac{1}{γ(ε, x)} \text{Adj} J_{T^ε}(x) = \frac{I + ε \tilde{V}_1(x) + ε^2 \tilde{V}_2(x)}{1 + ε γ_1(x) + ε^2 γ_2(x) + ε^3 γ_3(x)} = I + ε \tilde{V}_1(x, ε)$$

where $\tilde{V}_1(x, ε) = \tilde{V}_1(x) - γ_1(x) + O(ε)$. This concludes (A.5). Furthermore, we have

$$L^{-1}_{T^ε} σ = L^{-1} σ + ε L\tilde{V}_1 σ$$

and this concludes (A.6). This completes the proof.

(A.7) (A.8)

In view of Lemma 3.2 of [12], let us consider $A(ε, ⋅) := γ(ε, ⋅)J_{T^ε}^{-1}J_{T^ε}^{-T}$. Furthermore, we denote $A'(0, ⋅)$ is the Gâteaux derivative of $A(ε, ⋅)$ at $ε = 0$, namely

$$A'(0, x) = \lim_{ε → 0} \frac{A(ε, x) - I(x)}{ε}, \quad x ∈ \mathbb{R}^3.$$
Lemma A.5. Define operators $\tilde{A} : \mathbb{X} \to L^2(\mathcal{O}_S)$ and $\tilde{J} : \mathbb{X} \to L^2(\mathcal{O}_S)$ by

$$\tilde{A}(\varepsilon, y)\sigma(y) := \gamma(\varepsilon, y)J^{-1}_{T^e} \otimes L_{T^e} \sigma(y), \quad \varepsilon > 0,$$

(A.7)

$$\tilde{J}(y)\sigma(y) := \mathbf{I} \otimes L_I \sigma(y), \quad \forall \sigma(y) \in \mathbb{X}.$$  

(A.8)

Then

$$\lim_{\varepsilon \to 0} \|\tilde{A}(\varepsilon, \cdot)\sigma - \tilde{J}(\cdot)\sigma\|_{L^2(\mathbb{R}^3)} = 0,$$

(A.9)

$$\lim_{\varepsilon \to 0} \left\| \frac{\tilde{A}(\varepsilon, \cdot)\sigma - \tilde{J}(\cdot)\sigma - \tilde{A}'(0, \cdot)\sigma}{\varepsilon} \right\|_{L^2(\mathcal{O}_S)} = 0,$$

(A.10)

where we denote $\tilde{A}'(0, \cdot)$ as the Gâteaux derivative of $\tilde{A}(\varepsilon, \cdot)$ at $\varepsilon = 0$, namely

$$\tilde{A}'(0, y)\sigma(y) = \lim_{\varepsilon \to 0} \frac{\tilde{A}(\varepsilon, y)\sigma(y) - \tilde{J}(y)\sigma(y)}{\varepsilon}, \quad y \in \mathcal{O}_S.$$  

Proof. Using (A.7) and (A.8) we have

$$\tilde{A}(\varepsilon, y)\sigma(y) - \tilde{J}(y)\sigma(y)$$

$$= (\gamma(\varepsilon, y) - 1)(J^{-1}_{T^e} \otimes L_{T^e} \sigma(y)) + (J^{-1}_{T^e} - \mathbf{I}) \otimes L_{T^e} \sigma(y) + \mathbf{I} \otimes \left( L_{T^e} \sigma(y) - L_I \sigma(y) \right).$$

Now using Lemma A.4 we achieve

$$\|\tilde{A}(\varepsilon, \cdot)\sigma - \tilde{J}(\cdot)\sigma\|_{L^2(\mathbb{R}^3)} \leq O(\varepsilon),$$

which proves (A.9). Again, using (A.7) and Dominated Convergence Theorem we have (A.10). 



A.3. Material and shape derivatives. In this section we give a general overview on first and second order shape calculus (see [36]). These definitions and Lemmas can be introduced for the stress tensor $\sigma$, pressure $p$ and the skew-symmetric tensor $s$ in the spaces $\mathbb{X}$ and $\mathbb{Q}$ respectively.

Let $D$ be a deterministic bounded domain in $\mathbb{R}^3$ with boundary $\partial D$ of class $C^k$, $k \geq 2$. For $\varepsilon > 0$, let $D^\varepsilon$ be the perturbed domain with respect to $T^e$ (where $T^e$ is defined by (3.1) and (3.2)). Let us define the boundary variations $V_1$ and $V_2$ by

$$V_i(x) := \kappa_i(x)\mathbf{n}(x), \quad i = 1, 2, \quad \text{where} \quad \|\kappa_i\|_{W^{2, \infty}(\partial D) \cap C^{2,1}(\partial D)} \leq 1.$$  

(A.11)

We employ second order variations of the type

$$D_{\varepsilon, \delta} [\kappa_1, \kappa_2] := \{D_{\varepsilon} [\kappa_1] \delta [\kappa_2] : x + \varepsilon \kappa_1 (x) \mathbf{n}(x) + \delta \kappa_2 (x) \mathbf{n}(x) : x \in D\},$$

$$\partial D_{\varepsilon, \delta} [\kappa_1, \kappa_2] := \{\partial D_{\varepsilon} [\kappa_1] \delta [\kappa_2] : x + \varepsilon \kappa_1 (x) \mathbf{n}(x) + \delta \kappa_2 (x) \mathbf{n}(x) : x \in \partial D\}.$$  

Definition A.3. Let $\alpha > 0$. For any sufficiently small $\varepsilon$, let $v^\varepsilon(D^\varepsilon)$ be an element in $H^\alpha(D^\varepsilon)$. The material derivative weak (strong) of $v^\varepsilon(D^\varepsilon)$ in the direction of a vector field $V_1$ (given by (A.11)) denoted by $\dot{v}(D) := \dot{v}[\kappa_1, D]$ and is defined by

$$\dot{v}(D) := \lim_{\varepsilon \to 0} \frac{v^\varepsilon(D^\varepsilon) \circ T^e - v^0(D)}{\varepsilon},$$

provided the limit exists in weak (strong) sense in the corresponding space $H^\alpha(D)$.

Remark 5. The function $\dot{v}(D)$ is the weak (strong) material derivative of $v^\varepsilon(D^\varepsilon)$ in $H^\alpha(D^\varepsilon)$ if

$$\frac{v^\varepsilon(D^\varepsilon) \circ T^e - v^0(D)}{\varepsilon}$$

is weakly (strongly) convergent to $\dot{v}(D)$ in $H^\alpha(D)$ as $\varepsilon \to 0$.
Proposition A.6. Let \( \dot{v}(D) \) be the weak (strong) material derivative of an element \( v^\varepsilon(D^\varepsilon) \in H^\alpha(D^\varepsilon) \), in the direction of a vector field \( V_1 \) (given by (A.11)). Then for \( \alpha > 1/2 \), there exists the weak (strong) material derivative \( \dot{v}(\partial D) \) of the element \( v^\varepsilon(\partial D^\varepsilon) = v^\varepsilon(D^\varepsilon)\big|_{\partial D^\varepsilon} \) which is given by

\[
\dot{v}(\partial D) = \dot{v}(D)|_{\partial D} \quad \text{in} \quad H^{\alpha-1/2}(\partial D).
\]

For proof, we refer to Proposition 2.75 in [36].

Definition A.4. Let \( \alpha > 1/2 \). Let the weak material derivative \( \dot{v}(D) \) exists in \( H^\alpha(D) \) (or \( \dot{v}(\partial D) \in H^{\alpha-1/2}(\partial D) \)) and \( \nabla v^0 \cdot V_1 \in H^\alpha \) for vector field \( V_1 \) (given by (A.11)). The shape derivative of \( v^\varepsilon(D^\varepsilon) \in H^\alpha(D^\varepsilon) \) (or \( v^\varepsilon(\partial D^\varepsilon) \in H^{\alpha-1/2}(\partial D^\varepsilon) \)) is given by

\[
v^\varepsilon = \begin{cases} 
\dot{v}(D) - \nabla v^0(D) \cdot V_1, & \text{if } v^\varepsilon(D^\varepsilon) \in H^\alpha(D^\varepsilon), \\
\dot{v}(\partial D) - \nabla_{\partial D_0} v^0(\partial D) \cdot V_1, & \text{if } v^\varepsilon(\partial D^\varepsilon) \in H^{\alpha-1/2}(\partial D^\varepsilon).
\end{cases}
\]  

(A.12)

If \((V_1, V_2)\) are pairs of boundary perturbation fields given by (A.11), let us consider \((\dot{v})^\varepsilon \delta(D) \in H^\alpha(D^\delta) \) defined in the direction of the vector field \( V_2 \). Then the second order material derivative of \((v^\varepsilon)^\delta\) which is a bilinear form on the pair of vector fields \((V_1, V_2)\) denoted by \(\ddot{v}(D) = \ddot{v}[\kappa_1, \kappa_2, D]\) and is given by

\[
\ddot{v}(D) := \lim_{\delta \to 0} \frac{(\dot{v})^\varepsilon \delta(D) \circ T^\delta - \dot{v}(D)}{\delta}.
\]

The shape Hessian is the second order shape derivative denoted by \( v'' = v''[\kappa_1, \kappa_2] \) and is defined by

\[
v'' = \begin{cases} 
\ddot{v}(D) - (\nabla v)(D) \cdot V_1 - \nabla v(D) \cdot \dot{V}_1 - \nabla v'(D) \cdot V_2, & \text{if } \dot{v} \in H^\alpha, \\
\ddot{v}(\partial D) - M(\nabla_{\partial D_0} v)(D) \cdot V_1 - \nabla_{\partial D_0} v(D) \cdot \dot{V}_1 - \nabla_{\partial D_0} v'(D) \cdot V_2, & \text{if } \dot{v} \in H^{\alpha-1/2},
\end{cases}
\]

where \( M(f) \) denotes the material derivative of a function \( f \).

Lemma A.7. Let \( \alpha > 0 \). Let \( v' \) and \( v'' \) be shape derivative and shape Hessian of \( v(D) \in H^\alpha(D) \), then for any compact set \( K \subset \subset D \) we have

\[
v' = \lim_{\varepsilon \to 0} \frac{v - v^0}{\varepsilon} \quad \text{and} \quad v'' = \lim_{\delta \to 0} \frac{(v')^\delta - v'}{\delta} \quad \text{in} \quad H^\alpha(K).
\]

(A.14)

For proof see Lemma 3.6 of [12].

With (A.14) at hand, we obtain for all \( 0 \leq \varepsilon < \varepsilon_0 \), the ‘shape Taylor expansion’

\[
v^\varepsilon(x) = v(x) + \varepsilon v'[\kappa_1](x) + \frac{\varepsilon^2}{2} v''[\kappa_1, \kappa_1](x) + O(\varepsilon^3) \quad \text{for} \quad x \in K \subset \subset D \cap D_\varepsilon.
\]

(A.15)

Lemma A.8. Let \( \alpha > 0 \). Let \( \dot{v}, \dot{w} \) be material derivatives, and \( v', w' \) be shape derivatives of \( v^\varepsilon, w^\varepsilon \) in \( H^\alpha(D^\varepsilon), \varepsilon \geq 0 \), respectively. Then the following statements are true.

(i) The material and shape derivatives of the product \( v^\varepsilon w^\varepsilon \) are \( \dot{v}w^0 + v^0\dot{w} \) and \( v'w^0 + v^0w' \), respectively.

(ii) The material and shape derivatives of the quotient \( \frac{v^\varepsilon}{w^\varepsilon} \) are

\[
\frac{(\dot{v}w^0 - v^0\dot{w})}{(w^0)^2} \quad \text{and} \quad \frac{(v'w^0 - v^0w')}{(w^0)^2},
\]

respectively, provided that all the fractions are well-defined.

(iii) If \( v^\varepsilon = v \) for all \( \varepsilon \geq 0 \), then \( \dot{v} = \nabla v^0 \cdot V_1 = \nabla v \cdot V_1 \) and \( v' = 0 \).

(iv) If

\[
J_1(D^\varepsilon) := \int_{D^\varepsilon} v^\varepsilon \, dx, \quad J_2(D^\varepsilon) := \int_{\partial D^\varepsilon} v^\varepsilon \, d\sigma,
\]

respectively.
and

\[ dJ_i(D^\varepsilon)|_{\varepsilon=0} := \lim_{\varepsilon \to 0} \frac{J_i(D^\varepsilon) - J_i(D^0)}{\varepsilon}, \quad i = 1, 2, \]

then

\[ dJ_1(D^\varepsilon)|_{\varepsilon=0} = \int_{D^0} v^\varepsilon' \, dx + \int_{\partial D^0} \left( \frac{\partial v^\varepsilon}{\partial n} + \text{div}_{\partial D^0}(n) \nu^0 \right) \langle V_1, n \rangle \, d\sigma \]

and

\[ dJ_2(D^\varepsilon)|_{\varepsilon=0} = \int_{\partial D^0} v^\varepsilon \, d\sigma + \int_{\partial D^0} \left( \frac{\partial v^\varepsilon}{\partial n} \right) \langle V_1, n \rangle \, d\sigma. \]

(v) The shape derivatives of \( \frac{\partial v}{\partial n^\varepsilon} \big|_{\partial D^\varepsilon} \) and \( w^\varepsilon \frac{\partial v}{\partial n^\varepsilon} \big|_{\partial D^\varepsilon} \) are, respectively,

\[ \nabla_{\partial D^\varepsilon} v \cdot \nabla_{\partial D^0}(V_1, n) \quad \text{and} \quad w^\varepsilon \frac{\partial v}{\partial n} \big|_{\partial D^0} - w^0 \left( \nabla_{\partial D^\varepsilon} v \cdot \nabla_{\partial D^0}(V_1, n) \right). \]

Proof. Statements (i)–(iii) and (v) can be obtained by using elementary calculations. Statement (iv) are proved in [36] pages 113–116]. □

Lemma A.9. The material and shape derivatives of the normal field \( n^\varepsilon \) are given by

\[ \dot{n} = n^\varepsilon = -\nabla_{\partial D^0} \kappa. \]

Proof. See [12] Lemma 3.9]. □

References

[1] D. N. Arnold, R. S. Falk, and R. Winther, Mixed finite element methods for linear elasticity with weakly imposed symmetry, Math. Comp. 76 (2007), no. 260, 1699–1723.
[2] D. N. Arnold and R. Winther, Mixed finite elements for elasticity, Numer. Math. 92 (2002), no. 3, 401–419.
[3] J. B. Bacani and J. F. T. Rabago, On the second-order shape derivative of the Kohn-Vogelius objective functional using the velocity method, Int. J. Differ. Equ. (2015) Art. ID 954836, 10.
[4] M. Badra, F. Caubet, and M. Dambrine, Detecting an obstacle immersed in a fluid by shape optimization methods, Math. Models Methods Appl. Sci. 21 (2011), no. 10, 2069–2101.
[5] H. Barucq, R. Djellouli, and E. Estecahandy, Efficient DG-like formulation equipped with curved boundary edges for solving elasto-acoustic scattering problems, Internat. J. Numer. Methods Engrg. 8 (2000), no. 4, 1312–1336.
[6] H. Barucq, R. Djellouli, E. Estecahandy, and M. Moussaoui, Mathematical determination of the Fréchet derivative with respect to the domain for a fluid-structure scattering problem: case of polygonal-shaped domains, SIAM J. Control Optim. 51 (2013), no. 4, 2949–2975.
[7] S. K. Chakrabarti, Numerical models in fluid-structure interaction, WIT (2005).
[8] A. Chernov, D. Pham, and T. Tran, A shape calculus based method for a transmission problem with a random interface, Comput. Math. Appl. 70 (2015), no. 7, 1401–1424.
[9] B. Cockburn, J. Gopalakrishnan, and J. Guzmán, A new elasticity element made for enforcing weak stress symmetry, Math. Comp. 79 (2010), no. 271, 1331–1349.
[10] M. Dambrine, H. Harbrecht, and B. Puig, Computing quantities of interest for random domains with second order shape sensitivity analysis, ESAIM Math. Model. Numer. Anal. 49 (2015), no. 5, 1285–1302.
[11] M. C. Delfour and J.-P. Zolésio, Shape Hessian by the velocity method: a Lagrangian approach, in Stabilization of flexible structures (Montpellier, 1989), Vol. 147 of Lect. Notes Control Inf. Sci., 255–279, Springer, Berlin (1990).
[12] E. H. Dowell and K. C. Hall, Modeling of fluid-structure interaction, Annual review of fluid mechanics 33 (2001), no. 1, 445–490.
[13] X. Feng, Analysis of finite element methods and domain decomposition algorithms for a fluid-solid interaction problem, SIAM J. Numer. Anal. 38 (2000), no. 4, 1312–1336.
[18] G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations, Springer Monographs in Mathematics, Springer, New York, second edition (2011), ISBN 978-0-387-09619-3. Steady-state problems.

[19] C. García, G. N. Gatica, and S. Meddahi, Finite element semidiscretization of a pressure-stress formulation for the time-domain fluid-structure interaction problem, IMA J. Numer. Anal. 37 (2017), no. 4, 1772–1799.

[20] G. N. Gatica, Analysis of a new augmented mixed finite element method for linear elasticity allowing $RT_0-P_1-P_0$ approximations, M2AN Math. Model. Numer. Anal. 40 (2006), no. 1, 1–28.

[21] G. N. Gatica, A. Márquez, and S. Meddahi, Analysis of the coupling of primal and dual-mixed finite element methods for a two-dimensional fluid-solid interaction problem, SIAM J. Numer. Anal. 45 (2007), no. 5, 2072–2097.

[22] ———, A new dual-mixed finite element method for the plane linear elasticity problem with pure traction boundary conditions, Comput. Methods Appl. Mech. Engrg. 197 (2008), no. 9-12, 1115–1130.

[23] ———, Analysis of the coupling of Lagrange and Arnold-Falk-Winther finite elements for a fluid-solid interaction problem in three dimensions, SIAM J. Numer. Anal. 50 (2012), no. 3, 1648–1674.

[24] D. G. E. Grigoriadis, S. C. Kassinos, and E. V. Votyakov, Immersed boundary method for the MHD flows of liquid metals, J. Comput. Phys. 228 (2009), no. 3, 903–920.

[25] H. Harbrecht, On output functionals of boundary value problems on stochastic domains, Math. Methods Appl. Sci. 33 (2010), no. 1, 91–102.

[26] H. Harbrecht, R. Schneider, and C. Schwab, Sparse second moment analysis for elliptic problems in stochastic domains, Numer. Math. 109 (2008), no. 3, 385–414.

[27] J. Haslinger and R. A. E. Mäkinen, Introduction to shape optimization, Vol. 7 of Advances in Design and Control, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (2003), ISBN 0-89871-536-9. Theory, approximation, and computation.

[28] F. Hettlich and W. Rundell, The determination of a discontinuity in a conductivity from a single boundary measurement, Inverse Problems 14 (1998), no. 1, 67–82.

[29] J. F. Hoburg and J. R. Melcher, Internal electrohydrodynamic instability and mixing of fluids with orthogonal field and conductivity gradients, Journal of Fluid Mechanics 73 (1976), no. 2, 333–351.

[30] R. Horn and C. Johnson, Matrix analysis, second ed (2013).

[31] A. Márquez, S. Meddahi, and T. Tran, Analyses of mixed continuous and discontinuous Galerkin methods for the time harmonic elasticity problem with reduced symmetry, SIAM J. Sci. Comput. 37 (2015), no. 4, A1909–A1933.

[32] ———, Frequency-explicit asymptotic error estimates for a stress-pressure formulation of a time harmonic fluid-solid interaction problem, Comput. Math. Appl. 76 (2018), no. 9, 2090–2109.

[33] S. Meddahi, D. Mora, and R. Rodríguez, Finite element spectral analysis for the mixed formulation of the elasticity equations, SIAM J. Numer. Anal. 51 (2013), no. 2, 1041–1063.

[34] ———, Finite element analysis for a pressure-stress formulation of a fluid-structure interaction spectral problem, Comput. Math. Appl. 68 (2014), no. 12, part A, 1733–1750.

[35] H. J.-P. Morand and R. Ohayon, Fluid structure interaction, John Wiley (1995).

[36] J. Sokolowski and J.-P. Zolésio, Introduction to shape optimization, Vol. 16 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin (1992), ISBN 3-540-54177-2. Shape sensitivity analysis.

[37] T. Tiihonen, Shape optimization and trial methods for free boundary problems, RAIRO Modél. Math. Anal. Numér. 31 (1997), no. 7, 805–825.

School of Mathematics and Statistics, The University of New South Wales, Sydney 2052, Australia.

E-mail address: debopriya249@gmail.com

School of Mathematics and Statistics, The University of New South Wales, Sydney 2052, Australia.

E-mail address: thanh.tran@unsw.edu.au