Permute, Graph, Map, Derange

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Abstract. We study decomposable combinatorial labeled structures in the exp-log class, specifically, two examples of type \( a = 1 \) and two examples of type \( a = 1/2 \). Our approach is to establish how well existing theory matches experimental data. For instance, the median length of the longest cycle in a random \( n \)-permutation is \((0.6065...)n\), whereas the median length of the largest component in a random \( n \)-mapping is \((0.7864...)n\). Unsolved problems are highlighted, in the hope that someone else might address these someday.

Permutations and derangements decompose into cycles; undirected 2-regular labeled graphs and mappings decompose into connected components. Among the most striking features of a combinatorial object are

- the number of cycles or components,
- the size of the longest cycle or largest component,
- the size of the shortest cycle or smallest component.

We shall focus on the latter two topics. Throughout this paper, a random object is chosen uniformly from a set, e.g., all permutations/derangements on \( n \) symbols, all 2-regular graphs with \( n \) vertices, and all mappings \( \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \), weighted with equal probability. Define \( b_n \) to be the number of \( n \)-objects:

\[
b_n = \begin{cases} 
  n! & \text{for permutations}, \\
  (n-1)b_{n-1} + \binom{n-1}{2}b_{n-3} & \text{if } n \geq 3, \\
  b_0 = 1, b_1 = b_2 = 0 & \text{for graphs}, \\
  n^n & \text{for mappings}, \\
  n b_{n-1} + (-1)^n & \text{if } n \geq 1, \\
  b_0 = 1 & \text{for derangements}
\end{cases}
\]
and $c_n$ to be the number of $n$-objects that are connected, i.e., who possess exactly one component (having $n$ nodes):

$$c_n = \begin{cases} 
(n-1)! & \text{for permutations,} \\
(n-1)!/2 & \text{if } n \geq 3, \\
0 & \text{otherwise for graphs,} \\
n! \sum_{j=1}^{n} \frac{n^{n-j-1}}{(n-j)!} & \text{for mappings,} \\
(n-1)! & \text{if } n \geq 2, \\
0 & \text{otherwise for derangements.} 
\end{cases}$$

Our starting point is a recursive formula for $L_{k,n}$, the number of $n$-objects whose largest component has exactly $k$ nodes, $1 \leq k \leq n$. The initial conditions are

$$L_{0,n} = \delta_{0,n}, \quad L_{1,n} = 1 - \delta_{0,n}$$

for permutations and mappings, and

$$L_{0,n} = \delta_{0,n}, \quad L_{1,n} = 0$$

for graphs and derangements. Letting

$$m_{j,k,n} = \min\{k-1, n-k-j\}$$

(although we suppress the dependence on $j$, $k$, $n$), we have

$$L_{k,n} = \sum_{j=1}^{[n/k]} \frac{n!c_k^j}{j!(k!)^j(n-k-j)!} \begin{cases} 
\sum_{i=1}^{m} L_{i,n-k-j} & \text{if } m \geq 1, \\
\sum_{i=0}^{1} L_{i,n-k-j} & \text{if } m = 0. 
\end{cases}$$

The separation of cases $m \geq 1$ and $m = 0$ clarifies what was surely intended in [1, 2].

Next is a recursive formula for $S_{k,n}$, the number of $n$-objects whose smallest component has exactly $k$ nodes, $1 \leq k \leq n$. The initial conditions are

$$S_{0,n} = \delta_{0,n}$$

for permutations and mappings, and

$$S_{0,n} = 0$$
for graphs and derangements. Letting
\[ \theta_{k,n} = \begin{cases} 1 & \text{if } k \text{ is a divisor of } n, \\ 0 & \text{otherwise} \end{cases} \]
we have [2]
\[ S_{k,n} = \sum_{j=1}^{[n/k]} \frac{n! c_{k}^{j}}{j!(k!)^{j}(n - k j)!} \cdot \sum_{i=k+1}^{n-k j} S_{i,n-k j} \theta_{k,n} \frac{n! c_{n/k}^{i}}{(n/k)! (k!)^{n/k}}. \]
Clearly
\[ \sum_{k=1}^{n} L_{k,n} = \sum_{k=1}^{n} S_{k,n} = b_{n} \]
and \( L_{n,n} = S_{n,n} = c_{n} \) (the latter following from the empty sum convention). A computer algebra software package (e.g., Mathematica) makes exact integer calculations for ample \( n \) of \( L_{k,n} \) and \( S_{k,n} \) feasible.

Permutations and derangements belong to the exp-log class of type \( a = 1 \), whereas graphs and mappings belong to the exp-log class of type \( a = 1/2 \). Explaining the significance of the parameter \( a > 0 \) would take us too far afield [3]. Let
\[ E(x) = \int_{x}^{\infty} \frac{e^{-t}}{t} \, dt = -\text{Ei}(-x), \quad x > 0 \]
be the exponential integral. Define [4, 5, 6, 7, 8, 9]
\[ L_{G_{a}}(r, h) = \frac{\Gamma(a+1) a^{r-1}}{\Gamma(a+h)(r-1)!} \int_{0}^{\infty} x^{h-1} E(x) x^{-r} \exp \left[ -a E(x) - x \right] \, dx, \]
\[ S_{G_{a}}(r, h) = \begin{cases} e^{-h} \gamma a^{r-1}/r! & \text{if } h = a, \\ \frac{\Gamma(a+1)}{(h-1)!(r-1)!} \int_{0}^{\infty} x^{h-1} \exp \left[ a E(x) - x \right] \, dx & \text{if } h > a \end{cases} \]
which are related to the \( h \text{th} \) moment of the \( r \text{th} \) largest/smallest component size (in this paper, rank \( r = 1 \); height \( h = 1 \) or 2). Our notation \( S_{G_{a}} \) is deceiving. While permutation and derangement moments coincide for \( L \) (both being \( L_{G_{a}} \) with \( a = 1 \)), they are not equal for \( S \) (they differ by a factor \( e \)). In the same way, graph and mapping moments coincide for \( L \) but differ for \( S \) (by a factor \( e^{3/4}/\sqrt{2} \)).

Finally, we ask the first (of several) questions. Does a general recursion (similar to that for \( L_{k,n} \) and \( S_{k,n} \)) exist for the number \( N_{k,n} \) of \( n \) objects who possess exactly \( k \) components? Specific formulas are known for each of our four examples, but they are quite dissimilar. It seems unlikely that generalization to arbitrary \( N_{k,n} \) is possible, however it is still well-worth contemplating.
1. **Permute**

For fixed $n$, the sequences $\{L_{k,n} : 1 \leq k \leq n\}$ and $\{S_{k,n} : 1 \leq k \leq n\}$ constitute probability mass functions (upon normalization by $b_n = n!$). These have corresponding means $L\mu_n$, $S\mu_n$ and variances $L\sigma_n^2$, $S\sigma_n^2$ given in Table 1. We also provide the median $L\nu_n$; note that $S\nu_n = 1$ for $n \geq 3$ is trivial. For convenience (in table headings only), the following notation is used:

$$L\tilde{\mu}_n = \frac{L\mu_n}{n}, \quad L\tilde{\sigma}_n^2 = \frac{L\sigma_n^2}{n^2}, \quad L\tilde{\nu}_n = \frac{L\nu_n}{n},$$

$$S\tilde{\mu}_n = \begin{cases} 
\frac{s\mu_n}{\ln(n)} & \text{if } a = 1, \\
\frac{s\mu_n}{n^{1/2}} & \text{if } a = 1/2;
\end{cases} \quad S\tilde{\sigma}_n^2 = \begin{cases} 
\frac{s\sigma_n^2}{n} & \text{if } a = 1, \\
\frac{s\sigma_n^2}{n^{3/2}} & \text{if } a = 1/2.
\end{cases}$$

| $n$   | $L\tilde{\mu}_n$ | $L\tilde{\sigma}_n^2$ | $L\tilde{\nu}_n$ | $S\tilde{\mu}_n$ | $S\tilde{\sigma}_n^2$ |
|------|------------------|---------------------|-----------------|------------------|---------------------|
| 1000 | 0.624642         | 0.036945            | 0.6060          | 0.717352         | 1.307043            |
| 2000 | 0.624486         | 0.036926            | 0.6060          | 0.703135         | 1.307125            |
| 3000 | 0.624434         | 0.036920            | 0.6063          | 0.695960         | 1.307153            |
| 4000 | 0.624408         | 0.036917            | 0.6062          | 0.691295         | 1.307167            |

Table 1: Statistics for Permute ($a = 1$)

We have

$$\lim_{n \to \infty} \frac{L\mu_n}{n} = LG_1(1, 1) = 0.62432998854355087099..., $$

$$\lim_{n \to \infty} \frac{L\sigma_n^2}{n^2} = LG_1(1, 2) - LG_1(1, 1)^2 = 0.03690783006485220217..., $$

$$\lim_{n \to \infty} \frac{L\nu_n}{n} = \frac{1}{\sqrt{e}} = 0.60653065971263342360..., $$

$$\lim_{n \to \infty} \frac{s\mu_n}{\ln(n)} = e^{-\gamma} = 0.56145948356688516982..., $$

$$\lim_{n \to \infty} \frac{s\sigma_n^2}{n^{3/2}} = SG_1(1, 2) = 1.30720779891056809974.... $$

It is not surprising that $S\sigma_n^2$ enjoys linear growth: $S_{1,n} \sim (1 - 1/e)n!$ and $S_{n,n} = (n-1)!$ jointly place considerable weight on the distributional extremes. The unusual logarithmic growth of $S\mu_n$ is due to $S_{1,n}$ nevertheless overwhelming all other $S_{k,n}$. 


A one-line proof of the $L_{\nu_n}$ result is [10][11]

$$
\lim_{n \to \infty} \sum_{k=\left[\frac{n}{\sqrt{e}}\right]}^{n} \frac{1}{k} = \lim_{n \to \infty} \ln(n) - \ln \left( \frac{n}{\sqrt{e}} \right) = \ln \left( \frac{1}{\sqrt{e}} \right) = \frac{1}{2}.
$$

Alternatively, the asymptotic probability that the longest cycle has size $> nx$ is [11]

$$
\int_{x}^{1} \frac{1}{y} dy = \frac{1}{2}
$$

hence $\ln(1/x) = 1/2$ and $x = 1/\sqrt{e}$. We will see a variation of this approach later.

No formula for the covariance between sizes of the longest cycle and shortest cycle is known. Interplay between the number of cycles and either of the extremes likewise remains inscrutable. We earlier examined not permutations, but instead integer compositions, finding a complicated recursion for a certain bivariate probability distribution [12][13]. Thus a cross-correlation can be estimated for compositions, but not yet for permutations [14].

2. Graph

Let us explain why $L_{3,6} = 10, L_{6,6} = 60; L_{4,7} = 105, L_{7,7} = 360; \text{ and } L_{4,8} = 315, L_{5,8} = 672, L_{8,8} = 2520$. When $n = 6$, a 2-regular graph is either a hexagon, with $5!/2$ distinct labelings, or the disjoint union of two triangles, with $\binom{6}{3}/2$ labelings. When $n = 7$, a 2-regular graph is either a heptagon, with $6!/2$ distinct labelings, or the disjoint union of a triangle and a square, with $3\binom{7}{3}$ labelings (since $3!/2 = 3$). When $n = 8$, a 2-regular graph is either an octagon, with $7!/2$ distinct labelings; the disjoint union of a triangle and a pentagon, with $12\binom{8}{3}$ labelings (since $4!/2 = 12$); or the disjoint union of two squares, with $9\binom{8}{4}/2$ labelings (since $(3!/2)^2 = 9$). Circumstances become more complicated when $n = 9$: a 2-regular graph is either an enneagon, or the disjoint union of a square and a pentagon, or the disjoint union of three triangles, or the disjoint union of a triangle and a hexagon [15].

Upon normalization by $b_n$, we obtain

| $n$   | $L_{\tilde{\mu}_n}$ | $L_{\tilde{\sigma}_n}^2$ | $L_{\tilde{\nu}_n}$ | $s_{\tilde{\mu}_n}$ | $s_{\tilde{\sigma}_n}^2$ |
|-------|---------------------|----------------------|---------------------|-------------------|----------------------|
| 1000  | 0.7558771          | 0.037099             | 0.7860             | 3.007677         | 2.097084            |
| 2000  | 0.758297           | 0.037053             | 0.7865             | 3.029960         | 2.096470            |
| 3000  | 0.758139           | 0.037038             | 0.7860             | 3.039930         | 2.096262            |
| 4000  | 0.758060           | 0.037030             | 0.7865             | 3.045902         | 2.096157            |

Table 2: Statistics for Graph ($a = 1/2$)
and
\[
\lim_{n \to \infty} \frac{L\mu_n}{n} = L G_{1/2}(1, 1) = 0.75782301126849283774..., \\
\lim_{n \to \infty} \frac{L\sigma_n^2}{n^2} = L G_{1/2}(1, 2) - L G_{1/2}(1, 1)^2 = 0.0370072165829030320..., \\
\lim_{n \to \infty} \frac{L\nu_n}{n} = \frac{4e}{(1 + e)^2} = 0.78644773296592741014..., \\
\lim_{n \to \infty} \frac{s\mu_n}{n^{1/2}} = e^{3/4} s G_{1/2}(1, 1) = 3.0850424756314922958..., \\
\lim_{n \to \infty} \frac{s\sigma_n^2}{n^{3/2}} = e^{3/4} s G_{1/2}(1, 2) = 2.09583743942571712967....
\]

Some non-explicit formulas for the latter two results arise in Section 7. A proof for the median result is deferred to Section 3.

3. Map

Let us explain why \( S_{1,2} = 1, S_{2,2} = 3 \) and \( S_{1,3} = 10, S_{3,3} = 17 \). The unique 2-map with totally disconnected nodes is the identity map; we associate this map with its image sequence 12, i.e., two isolated loops (two components of size 1). The maps 11 and 22 are each pictured as one loop attached to a 1-tail (a component of size 2); the map 21 is pictured as a 2-cycle (again, a component of size 2). For 3-maps, we have

123 i.e., three isolated loops;

113, 121, 122, 133, 223, 323 i.e., one isolated loop and one loop attached to a 1-tail;

213, 321, 132 i.e., one isolated loop and one 2-cycle

which give ten cases, and

112, 131, 221, 322, 233, 313 i.e., one loop attached to a 2-tail;

111, 222, 333 i.e., one loop attached to two 1-tails;

211, 212, 232, 311, 331, 332 i.e., one 2-cycle attached to a 1-tail;

231, 312 i.e., one 3-cycle

which give seventeen cases \([16, 17]\). The complexity grows when \( n = 4 \): it can be shown that \( S_{1,4} = 87, S_{2,4} = 27, S_{4,4} = 142 \).

Upon normalization by \( b_n = n^n \), we obtain
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| n  | $L\bar{\mu}_n$ | $L\bar{\sigma}^2_n$ | $L\bar{\nu}_n$ | $S\bar{\mu}_n$ | $S\bar{\sigma}^2_n$ |
|----|----------------|--------------------|----------------|----------------|-------------------|
| 1000 | 0.762505 | 0.036968 | 0.7920 | 1.969526 | 1.384968 |
| 2000 | 0.761122 | 0.036980 | 0.7905 | 1.991932 | 1.389355 |
| 3000 | 0.760512 | 0.036985 | 0.7900 | 2.002505 | 1.391309 |
| 4000 | 0.760149 | 0.036988 | 0.7895 | 2.009048 | 1.392477 |

Table 3: Statistics for Map ($a = 1/2$)

and

$$\lim_{n \to \infty} \frac{L\bar{\mu}_n}{n} = L\bar{G}_{1/2}(1, 1) = 0.75782301126849283774...,$$

$$\lim_{n \to \infty} \frac{L\bar{\sigma}^2_n}{n^2} = L\bar{G}_{1/2}(1, 2) - L\bar{G}_{1/2}(1, 1)^2 = 0.03700721658229030320...,$$

$$\lim_{n \to \infty} \frac{L\bar{\nu}_n}{n} = \frac{4e}{(1 + e)^2} = 0.78644773296592741014...,$$

$$\lim_{n \to \infty} \frac{S\bar{\mu}_n}{n^{1/2}} = \sqrt{2}S\bar{G}_{1/2}(1, 1) = 2.06089224152016653900...,$$

$$\lim_{n \to \infty} \frac{S\bar{\sigma}^2_n}{n^{3/2}} = \sqrt{2}S\bar{G}_{1/2}(1, 2) = 1.40007638550124502818....$$

As before, some non-explicit formulas for the latter two results arise in Section 7.

The asymptotic probability that the largest component has size $> n x$ is $[18, 19]$

$$\int_0^x \frac{1}{2y\sqrt{1-y}} dy = \frac{1}{2}$$

hence

$$\frac{1}{2} \ln \left( \frac{1 + \sqrt{1 - x}}{1 - \sqrt{1 - x}} \right) = \frac{1}{2}$$

and $x = 4e/(1 + e)^2$.

4. Derange

Derangements are permutations with no fixed points $[20]$. It is easy to show that $L_{3,5} = 20$ (a longest cycle in a 5-derangement cannot have size 2 or 4) and $S_{2,5} = 20$. Upon normalization by $b_n$, we obtain

| n  | $L\bar{\mu}_n$ | $L\bar{\sigma}^2_n$ | $L\bar{\nu}_n$ | $S\bar{\mu}_n$ | $S\bar{\sigma}^2_n$ |
|----|----------------|--------------------|----------------|----------------|-------------------|
| 1000 | 0.625266 | 0.037018 | 0.6060 | 1.701217 | 3.551193 |
| 2000 | 0.624798 | 0.036963 | 0.6065 | 1.685257 | 3.552276 |
| 3000 | 0.624642 | 0.036945 | 0.6067 | 1.677202 | 3.552637 |
| 4000 | 0.624564 | 0.036935 | 0.6065 | 1.671965 | 3.552818 |

Table 4: Statistics for Derange ($a = 1$)
and
\[
\lim_{n \to \infty} \frac{L \mu_n}{n} = \ell G_1(1, 1) = 0.62432998854355087099, \\
\lim_{n \to \infty} \frac{L \sigma^2_n}{n^2} = \ell G_1(1, 2) - \ell G_1(1, 1)^2 = 0.03690783006485220217, \\
\lim_{n \to \infty} \frac{L \nu_n}{n} = \frac{1}{\sqrt{e}} = 0.6065306597126334236, \\
\lim_{n \to \infty} \frac{S \mu_n}{\ln(n)} = e^{1-\gamma} = 1.52620511159586388047, \\
\lim_{n \to \infty} \frac{S \sigma^2_n}{n} = e \cdot sG_1(1, 2) = 3.5533592057985429740.
\]

The asymptotic expression for the average shortest cycle length follows from
\[
\frac{s \mu_n \cdot b_n + 1 \cdot (n! - b_n)}{n!} \sim e^{-\gamma} \ln(n)
\]
and the fact that \(b_n/n! \to 1/e\) as \(n \to \infty\); similarly for higher moments.

5. Generalized Dickman Rho (I)

Define \(b_{n,m}\) to be the number of \(n\)-objects whose largest component has size \(\leq m\); thus \(b_n = b_{n,n}\). Given \(a > 0\), let
\[
\rho_a(x) = \begin{cases} 
1 & \text{if } 0 \leq x < 1, \\
1 - a \int_{1}^{x} \rho_a(t-1) \cdot (t-1)^{a-1} \frac{dt}{t^a} & \text{if } x \geq 1
\end{cases}
\]
and observe that the standard Dickman function \(\rho(x) = \rho_1(x)\). A theorem proved in [21] asserts that
\[
\lim_{m \to \infty} \frac{b_{\lfloor x m \rfloor, m}}{b_{\lfloor x m \rfloor}} = \rho_a(x)
\]
for any \(x > 1\). Of course,
\[
\sum_{j=1}^{m} L_{j,n} = b_{n,m}
\]
hence we can easily verify this result experimentally.
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| $m \setminus x$ | 2       | 3       | 4       | 5       | 2       | 3       | 4       |
|-----------------|---------|---------|---------|---------|---------|---------|---------|
| 100             | 0.309347| 0.049634| 0.0050952| 0.0003748| 0.117715| 0.0082644| 0.0003680|
| 200             | 0.308101| 0.049121| 0.0050026| 0.0003646| 0.118178| 0.0082399| 0.0003616|
| 300             | 0.307685| 0.048950| 0.0049719| 0.0003597| 0.118404| 0.0082262| 0.0003616|
| 400             | 0.307477| 0.048864| 0.0049566| 0.0003597| 0.118404| 0.0082262| 0.0003616|
| 500             | 0.307353| 0.048813| 0.0049475| 0.0003587| 0.118449| 0.0082233| 0.0003616|
| 600             | 0.307269| 0.048779| 0.0049414| 0.0003580| 0.118478| 0.0082214| 0.0003616|
| 700             | 0.307210| 0.048755| 0.0049370| 0.0003575| 0.118500| 0.0082200| 0.0003616|
| 800             | 0.307165| 0.048736| 0.0049337| 0.0003572| 0.118516| 0.0082190| 0.0003616|
| $\infty$        | 0.306853| 0.048608| 0.0049109| 0.0003547| 0.118626| 0.0082115| 0.0003594|

Table 5A: Ratio $b_{\lfloor x \rfloor, m} / b_{\lfloor x \rfloor}$ for Permute ($a = 1$) and Graph ($a = 1/2$)

| $m \setminus x$ | 2       | 3       | 4       | 2       | 3       | 4       | 5       |
|-----------------|---------|---------|---------|---------|---------|---------|---------|
| 100             | 0.111305| 0.0074576| 0.0003185| 0.304359| 0.048597| 0.0049699| 0.0003645|
| 200             | 0.112756| 0.0076060| 0.0003258| 0.305604| 0.048605| 0.0049409| 0.0003596|
| 300             | 0.113579| 0.0076901| 0.0003302| 0.306020| 0.048607| 0.0049310| 0.0003580|
| 400             | 0.114124| 0.0077458| 0.0003332| 0.306228| 0.048608| 0.0049260| 0.0003572|
| 500             | 0.114518| 0.0077862| 0.0003354| 0.306353| 0.048608| 0.0049230| 0.0003567|
| 600             | 0.114822| 0.0078173| 0.0003371| 0.306436| 0.048608| 0.0049210| 0.0003564|
| 700             | 0.115065| 0.0078423| 0.0003384| 0.306496| 0.048608| 0.0049196| 0.0003561|
| 800             | 0.115265| 0.0078629| 0.0003396| 0.306540| 0.048608| 0.0049185| 0.0003559|
| $\infty$        | 0.118626| 0.0082115| 0.0003594| 0.306853| 0.048608| 0.0049109| 0.0003547|

Table 5B: Ratio $b_{\lfloor x \rfloor, m} / b_{\lfloor x \rfloor}$ for Map ($a = 1/2$) and Derange ($a = 1$)

Why have we devoted effort to evaluating Dickman’s rho? Answer: the function $\rho_a(x)$ is fundamentally connected to $L_{k,n}$ asymptotics in Sections 1–4. The $h^{th}$ moments of the largest component size are

$$\int_0^\infty \frac{\rho_1(x)}{(x + 1)^{h+1}} dx \quad \text{and} \quad \frac{1}{2} \int_0^\infty \frac{\rho_{1/2}(x)}{\sqrt{x}(x + 1)^{h+1/2}} dx, \quad h = 1, 2, \ldots$$

for $a = 1$ and $a = 1/2$, respectively. Extension to arbitrary $a > 0$ is possible. Of course, we also have integrals $L_{G_a(r,h)}$ available.
6. Generalized Buchstab Omega (I)

Define \( b_{n,m} \) to be the number of \( n \)-objects whose smallest component has size \( \geq m \); note that \( c_n = b_{n,n} \). Given \( a > 0 \), let

\[
\Omega_a(x) = \begin{cases} 
1 & \text{if } 1 \leq x < 2, \\
1 + a \int_2^x \frac{\Omega_a(t-1)}{t-1} \, dt & \text{if } x \geq 2
\end{cases}
\]

and observe that the standard Buchstab function \( \omega(x) = \Omega_1(x)/x \). A theorem proved in [22] asserts that

\[
\lim_{m \to \infty} \frac{c_{\lfloor x m \rfloor}}{b_{\lfloor x m \rfloor,m}} = \frac{1}{\Omega_a(x)}
\]

for any \( x > 1 \). Of course,

\[
\sum_{j=m}^n S_{j,n} = b_{n,m}
\]

hence we can easily verify this result experimentally.

As an aside, \( c_n/b_{n,m} \) is called the probability of connectedness in [22], i.e., the odds that an \( n \)-object, whose smallest component has size at least \( m \), is connected. No analogous name has been proposed for \( b_{n,m}/b_n \) from Section 5, i.e., the odds that all components of an \( n \)-object have size at most \( m \). Maybe probability of smoothness would be appropriate ("smooth" coming from prime number theory). For Section 7, the same ratio might be called the probability of roughness, wherein all components of an \( n \)-object have size at least \( m \).

| \( m \setminus x \) | 2   | 3   | 4   | 5   | 2   | 3   | 4   | 5   |
|---------------------|-----|-----|-----|-----|-----|-----|-----|-----|
| 100                 | 0.990 | 0.587992 | 0.443034 | 0.354438 | 0.995 | 0.740555 | 0.628689 | 0.555092 |
| 200                 | 0.995 | 0.589306 | 0.444151 | 0.355327 | 0.997 | 0.741591 | 0.629581 | 0.555860 |
| 300                 | 0.997 | 0.589743 | 0.444523 | 0.355624 | 0.998 | 0.741936 | 0.629878 | 0.556116 |
| 400                 | 0.997 | 0.589962 | 0.444710 | 0.355772 | 0.999 | 0.742108 | 0.630027 | 0.556244 |
| 500                 | 0.998 | 0.590093 | 0.444822 | 0.355862 | 0.999 | 0.742212 | 0.630116 | 0.556321 |
| 600                 | 0.998 | 0.590180 | 0.444896 | 0.355921 | 0.999 | 0.742281 | 0.630175 | 0.556372 |
| 700                 | 0.999 | 0.590242 | 0.444949 | 0.355963 | 0.999 | 0.742330 | 0.630218 | 0.556409 |
| 800                 | 0.999 | 0.590289 | 0.444989 | 0.355995 | 0.999 | 0.742367 | 0.630250 | 0.556436 |
| ...                 | ... | ... | ... | ... | ... | ... | ... | ... |
| \( \infty \)       | 1   | 0.590616 | 0.445269 | 0.356218 | 1   | 0.742626 | 0.630473 | 0.556628 |

Table 6A: Ratio \( c_{\lfloor x m \rfloor}/b_{\lfloor x m \rfloor,m} \) for Permute \( (a = 1) \) and Graph \( (a = 1/2) \).
7. Generalized Buchstab Omega (II)

Define \( b_{n,m} \) to be the number of \( n \)-objects whose smallest component has size \( \geq m \) (as in Section 6). When restricting attention to permutations, Panario & Richmond \([3]\) obtained that

\[
\lim_{m \to \infty} \frac{m b_{[x m],m}}{b_{[x m]}} = \omega(x)
\]

for any \( x > 1 \), where \( \omega(x) \) is the standard Buchstab function. They seemed to presume that the same limit would occur for derangements (since both permutations and derangements have \( a = 1 \)), which is not true. Replace now the initial factor \( m \) in the numerator by \( m^{1/2} \). Panario & Richmond realized that 2-regular graphs and mappings would possess a limit different from \( \omega(x) \). They seemed, however, to presume that equivalent limits would occur (since both graphs and maps have \( a = 1/2 \)), which is again untrue. In Section 5, we studied two functions \( \Omega_a, a \in \{1, 1/2\} \); here we have four omega (lowercase “o”) functions \( \omega_A, A \in \{P, G, M, D\} \), one for each structure under consideration. Upon multiplication of limits, we discover

\[
\frac{\omega_A(x)}{\Omega_a(x)} = \lim_{m \to \infty} \frac{m^a c_{[x m]}}{b_{[x m]}} = \begin{cases} 
1 & \text{if } A = P \text{ and } a = 1, \\
e^{3/4} \sqrt{\pi/2} & \text{if } A = G \text{ and } a = 1/2, \\
\sqrt{\pi/2} & \text{if } A = M \text{ and } a = 1/2, \\
e & \text{if } A = D \text{ and } a = 1
\end{cases}
\]

using known \( c_n/b_n \) asymptotics as \( n \to \infty \) for graphs and mappings \([23, 24, 25]\). Perhaps, for fixed \( a > 0 \), \( \omega_A \) varies only up to multiplicative constant. These formulas allow us to provide numerical values in the final rows of Tables 6A and 6B.
Return now to Panario & Richmond. Especially puzzling is a claim (for permutations) that

$$\lim_{n \to \infty} \frac{s^2}{n} = \int_2^\infty \frac{\omega(x)}{x^2} \, dx.$$  

From Section 1, the left-hand side is $sG_1(1, 2) = 1.307207...$ whereas the right-hand side is $\frac{1}{2}(0.556816...)$ $[20]$. Thus predictions in $[3]$ for $A \in \{P, D\}$ are evidently mistaken.

| $m \setminus x$ | 2       | 3       | 4       | 5       | 2       | 3       | 4       | 5       |
|-----------------|---------|---------|---------|---------|---------|---------|---------|---------|
| 100             | 0.50500 | 0.56690 | 0.56429 | 0.56427 | 1.33744 | 1.46573 | 1.49445 | 1.51342 |
| 200             | 0.50250 | 0.56664 | 0.56287 | 0.56286 | 1.33203 | 1.46216 | 1.49116 | 1.51038 |
| 300             | 0.50166 | 0.56522 | 0.56240 | 0.56239 | 1.33023 | 1.46097 | 1.49007 | 1.50937 |
| 400             | 0.50125 | 0.56501 | 0.56216 | 0.56216 | 1.32933 | 1.46037 | 1.48952 | 1.50887 |
| 500             | 0.50100 | 0.56488 | 0.56202 | 0.56202 | 1.32879 | 1.46002 | 1.48920 | 1.50856 |
| 600             | 0.50083 | 0.56480 | 0.56193 | 0.56192 | 1.32843 | 1.45978 | 1.48898 | 1.50836 |
| 700             | 0.50072 | 0.56474 | 0.56186 | 0.56186 | 1.32817 | 1.45961 | 1.48882 | 1.50822 |
| 800             | 0.50063 | 0.56470 | 0.56181 | 0.56181 | 1.32798 | 1.45948 | 1.48871 | 1.50811 |
| ∞               | 0.5     | 0.56438 | 0.56146 | 0.56145 | 1.32663 | 1.45860 | 1.48789 | 1.50735 |

Table 7A: Ratio $m^a b_{\lfloor x \rfloor, m} / b_{\lfloor x \rfloor}$ for Permute ($a = 1$) and Graph ($a = 1/2$)

| $m \setminus x$ | 2       | 3       | 4       | 5       | 2       | 3       | 4       | 5       |
|-----------------|---------|---------|---------|---------|---------|---------|---------|---------|
| 100             | 0.87413 | 0.95520 | 0.97361 | 0.98570 | 1.37273 | 1.54100 | 1.53390 | 1.53385 |
| 200             | 0.87676 | 0.96022 | 0.97895 | 0.99132 | 1.36594 | 1.53756 | 1.53005 | 1.53001 |
| 300             | 0.87816 | 0.96259 | 0.98148 | 0.99397 | 1.36367 | 1.53642 | 1.52876 | 1.52874 |
| 400             | 0.87906 | 0.96406 | 0.98303 | 0.99559 | 1.36254 | 1.53585 | 1.52812 | 1.52810 |
| 500             | 0.87971 | 0.96508 | 0.98411 | 0.99672 | 1.36186 | 1.53551 | 1.52774 | 1.52772 |
| 600             | 0.88021 | 0.96584 | 0.98492 | 0.99756 | 1.36141 | 1.53528 | 1.52748 | 1.52746 |
| 700             | 0.88060 | 0.96644 | 0.98555 | 0.99823 | 1.36109 | 1.53512 | 1.52730 | 1.52728 |
| 800             | 0.88092 | 0.96693 | 0.98607 | 0.99876 | 1.36084 | 1.53500 | 1.52716 | 1.52715 |
| ∞               | 0.88623 | 0.97438 | 0.99395 | 1.00695 | 1.35914 | 1.53415 | 1.52620 | 1.52619 |

Table 7B: Ratio $m^a b_{\lfloor x \rfloor, m} / b_{\lfloor x \rfloor}$ for Map ($a = 1/2$) and Derange ($a = 1$)
Why have we devoted effort to evaluating Buchstab’s omega? Answer: an array of formulas, parallel to those involving $\rho_a(x)$, corresponding to $h$th moments of the smallest component size, were proposed in [3]:

$$\int_{2}^{\infty} \frac{\omega_A(x)}{x^{h+1/2}} \, dx, \quad h = 1, 2, \ldots, \quad \text{for } A \in \{G, M\}$$

and would be exceedingly attractive. Unfortunately the potential for fulfillment is not good. No high-precision numerical estimates of these integrals are currently known; thus we are not certain that any of the various $\omega_A(x)$ are necessarily allied with $S_{k,n}$ asymptotics in Sections 1–4. For now, the formulas remain frustratingly non-explicit and unverified.

8. Generalized Dickman Rho (II)

Define $b_{n,m}$ to be the number of $n$-objects whose largest component has size $\leq m$ (as in Section 5). When restricting attention to permutations, we observe that

$$\lim_{m \to \infty} \frac{\lfloor x \cdot m \rfloor \cdot c_{\lfloor x \cdot m \rfloor}}{b_{\lfloor x \cdot m \rfloor, m}} = \frac{1}{\rho(x)}$$

for any $x > 1$, where $\rho(x)$ is the standard Dickman function. When restricting attention to derangements, a factor of $e$ needs to be included (just as in Sections 4 and 7). Replace now the initial factor $\lfloor x \cdot m \rfloor$ in the numerator by $\lfloor x \cdot m \rfloor^{1/2}$. Again 2-regular graphs and mappings possess non-equivalent limits different from $1/\rho(x)$. Just as we found the defining limit for Buchstab’s omega in terms of $\Omega_a(x)$ earlier, here we discover the limit in terms of $\rho_a(x)$:

$$\lim_{m \to \infty} \frac{\lfloor x \cdot m \rfloor^a \cdot c_{\lfloor x \cdot m \rfloor}}{b_{\lfloor x \cdot m \rfloor, m}} = \frac{1}{\rho_a(x)}, \quad \begin{cases} 1 \\ e^{3/4} \sqrt{\pi}/2 \\ \sqrt{\pi}/2 \\ e \end{cases} \text{ if } A = P \text{ and } a = 1, \\
\text{if } A = G \text{ and } a = 1/2, \\
\text{if } A = M \text{ and } a = 1/2, \\
\text{if } A = D \text{ and } a = 1. \quad \text{if } A = G \text{ and } a = 1/2, \\
\text{if } A = M \text{ and } a = 1/2, \\
\text{if } A = D \text{ and } a = 1.$$

We know that $\rho_a$ is important (Section 5) and believe that $\omega_A$ deserves further study (Section 7). It is hoped that someone else might succeed in carrying on research where we have stopped.
At the conclusion of Section 5, we gave expressions for the \( h \)th moments of largest component size, given \( a = 1 \) or \( a = 1/2 \), without justification. Here is a plausibility argument. Assuming (absent any proof) that the first moment for arbitrary \( a > 0 \) is 

\[
\lambda_a = 1 - \int_1^\infty \frac{\rho_a(x)}{x^2} dx,
\]

we reverse integration-by-parts:

\[
\begin{align*}
\frac{d}{dx} x^{1-a} & = x^{-2-a} dx, & \quad v = x^a \rho_a(x), \\
\int_{x=1}^\infty x^{-1-a} dx & = \frac{1}{1 + a} x^{1-a} & \quad dv = a [x^{-1} + (x-1)^{-1} + a (x-1)] dx
\end{align*}
\]
and obtain
\[-\frac{a}{1 + a} \left( \int_{1}^{\infty} \frac{\rho_a(x)}{x^2} dx - \int_{1}^{\infty} \frac{\rho_a(x - 1)}{(x - 1)^{1-a}x^{1+a}} dx \right) = -\frac{1}{1 + a} \left( \frac{\rho_a(x)}{x} \right)_{1}^{\infty} - \int_{1}^{\infty} \frac{\rho_a(x)}{x^2} dx \]
i.e.,
\[-\frac{a}{1 + a} \int_{0}^{\infty} \frac{\rho_a(x)}{x^{1-a}(x + 1)^{1+a}} dx = \frac{1}{1 + a} - (1 - \lambda_a) + \frac{a}{1 + a}(1 - \lambda_a) = \frac{\lambda_a}{1 + a}.\]

From here, we infer (again, absent any proof) that the $h^{th}$ moment is
\[a \int_{0}^{\infty} \frac{\rho_a(x)}{x^{1-a}(x + 1)^{h+a}} dx, \quad h = 1, 2, \ldots.

An explanation for $s\mu_n$ and $s\sigma_n^2$ asymptotics in Sections 2 and 3 was not given. Reason: it is more complicated than the proof in Section 4. At some later point, we hope to study combinatorial objects called cyclations [28] for which moments are known to be precisely $sG_{1/2}$. Since $\sqrt{n}c_n/b_n \to \sqrt{\pi}/2$ as $n \to \infty$ for these, and because corresponding limits for graphs and mappings appear in Section 7, the factors $e^{3/4}$ and $\sqrt{2}$ emerge.

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