Large volume susy breaking with a chiral solution to the decompactification problem

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Abstract

We study heterotic ground states in which supersymmetry is broken by coupling the momentum and winding charges of two large extra dimensions to the R-charges of the supersymmetry generators. The large dimensions give rise to towers of heavy string thresholds that contribute to the running of the gauge couplings. In the general case, these contributions are proportional to the volume of the two large dimensions and invalidate the perturbative string expansion. The problem is evaded if the susy breaking sectors arise as a spontaneously broken phase of \(\mathcal{N} = \frac{4}{2} \rightarrow \mathcal{N} = 0\) supersymmetry, provided that \(\mathcal{N} = 4\) supersymmetry is restored on the boundary of the moduli space. We discuss the mechanism in the case of \(\mathbb{Z}_2 \times \mathbb{Z}_2\) orbifolds, which requires that the twisted sector that contains the large extra dimensions has no fixed points. We analyse the full string partition function and show that the twisted sectors distribute themselves in non aligned \(\mathcal{N} = 2\) orbits, hence preserving the solution to the string decompactification problem. Remarkably, we find that the contribution to the vacuum energy from the \(\mathcal{N} = 2 \rightarrow \mathcal{N} = 0\) sectors is suppressed, and the only substantial contribution arises from the breaking of the \(\mathcal{N} = 4\) sector to \(\mathcal{N} = 0\). Implementation of the mechanism in three net chiral generations quasi-realistic models requires that the two other twisted sectors produce 1 and 2 chiral generations, respectively.

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1 Introduction

String theory is the leading contender for a unified theory of all known interactions \[1\], and numerous string models exhibiting rich phenomenological properties have been constructed. They utilize various compactification technics, like for instance the Calabi-Yau compactifications \[1\], the orbifold compactifications \[2\], the 2d-fermionic constructions \[3\], the self-dual lattice constructions \[4\], the asymmetric orbifold compactifications \[5\], the \(\mathcal{N} = (2,2)\) superconformal constructions \[6\], as well as the \(\mathcal{N} = (2,0)\) constructions \[3,5\].

However, all of the quasi-realistic string models that have been constructed to date, namely with the correct standard model spectrum, possess an \(\mathcal{N} = 1\) spacetime supersymmetry (susy), and the question of how this symmetry is broken is still an open problem. The mechanisms that have been proposed to address this point are either perturbative \[7–10\] or non-perturbative \[11–14\]. One can consider:

- A non-perturbative breaking \textit{via} gaugino condensation \[11\], which up till now has to be discussed at the level of the effective supergravity. Due to the non-perturbative nature of the mechanism, one looses the predictability associated to the underlying string model. One then has to resort to an effective parametrization of the susy breaking parameters.

- Perturbative and/or non-perturbative flux compactifications, where internal fluxes are introduced and break susy suitably. These models can be explored using the non-perturbative \(S, T, U\)-dualities between the heterotic, Type IIA, Type IIB and orientifold superstring vacua \[13–15\].

- An interesting example of geometrical fluxes is the one associated with a Stringy Scherk-Schwarz (SSS) susy breaking compactification, which has the advantage to be implemented at the perturbative string level \[9\]. Here, the symmetry breaking parameters are obtained directly from the perturbative string theory.

In this last approach, the Scherk-Schwarz mechanism \[16\] defined in supergravity theories is promoted at the superstring level \[8–10\]. Denoting the string scale as \(M_s = 1/\sqrt{\alpha'}\), the mechanism entails that some of the compactified dimensions of characteristic size \(R/M_s\) (measured in string frame) of the internal manifold are large, \textit{i.e.} of the order of the inverse
of the supersymmetry breaking scale. In Einstein frame, we have $m_\frac{3}{2}^{(E)} = \mathcal{O}(M_{\text{Planck}}/R) = \mathcal{O}(1-10) \text{ TeV}$. This follows from the fact that supersymmetry is broken by coupling a $\mathbb{Z}_2$ freely acting shift in these compactified directions, with the R-charges of the supersymmetry generators. These large dimensions give rise to tower of states, charged under low-energy gauge groups, that populate the energy range between the susy breaking scale and the Planck scale. They induce thresholds, whose analysis was recently pioneered in [17], that contribute to the running of the gauge couplings, Yukawa couplings and soft susy breaking parameters.

However, a problem arises when the threshold corrections are proportional to the volume of the large dimensions. When the $\beta$-function coefficient is negative, they drive the theory to strong coupling at energies lower than the unification (or string) scale [18]. This problem is known as the \textit{decompactification problem} and some proposals exist on how to avoid it [9, 10, 15, 18]. A first idea supposes the existence of models without $\mathcal{N} = 2$ sectors, so that the threshold corrections are independent of the volume moduli of the internal theory [10]. Alternatively, one can suppose the thresholds of different spin states cancel among each other at one-loop in the perturbative expansion [10]. However, the stability of this mechanism against higher loop corrections has not been demonstrated. Moreover, no quasi-realistic model realizing one of the above two proposals has been constructed so far.

In this paper, we examine a different possibility, which was introduced in Ref. [18] in the context of $\mathcal{N} = 2$ supersymmetric models. Due to the properties of the $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ spontaneous breaking \textit{via} freely acting orbifolds, the behavior of thresholds as functions of the moduli of the internal manifold is radically different from that of the generic orbifold models, where the breaking from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ is not spontaneous [18]. The reason for this distinction is that $\mathcal{N} = 4$ supersymmetry is restored on the boundary of the moduli space. In this case, for large values of the relevant moduli, the thresholds vanish (up to logarithmic corrections).

In order to extend the above idea to non-supersymmetric models, we first present in Sect. 2 the class of string theories we consider, namely the heterotic $\mathbb{Z}_2 \times \mathbb{Z}_2$ asymmetric orbifold models realized in fermionic constructions, where $\mathcal{N} = 1$ supersymmetry is further spontaneously broken to $\mathcal{N} = 0$ by a SSS mechanism. Then, we provide in Sect. 3 some kind of introduction on how the gauge coupling threshold corrections in simple $\mathcal{N} = 4$ models spontaneously broken to $\mathcal{N} = 0$ do not develop dangerous linear dependences on volume
moduli. We turn back to the general case of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ models from Sect. 4 to the end of the article.

In Sect. 4, we evaluate the threshold corrections and effective potential generated at one-loop in the sectors arising from the action of a single $\mathbb{Z}_2$, namely the $\mathcal{N} = 4$ sector and the so-called $\mathcal{N} = 2$ 1st complex plane. For the associated $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ susy breaking to be spontaneous, the $\mathbb{Z}_2$ twist acts simultaneously as a shift along the two untwisted internal directions. The SSS mechanism responsible of the final spontaneous susy breaking to $\mathcal{N} = 0$ is implemented by an additional $\mathbb{Z}_2^{\text{shift}}$. The action of the latter on the above two untwisted internal directions introduces sub-sectors we analyse carefully. We find that only the $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$ sub-sector (denoted as $B$), together with two sub-sectors (denoted as $C$ and $D$) preserving distinct $\mathcal{N} = 2$ supersymmetries contribute substantially.

Sect. 5 discusses physically the formal results obtained in the sub-sectors $B, C, D$. Three moduli-dependent mass scales $M^{(E)}_{B,C,D}$ are introduced, the lowest of which being in the TeV region in realistic models. These scales, which are different from the gravitini masses present in each sector, control the contributions of the whole towers of Kaluza-Klein states that contribute to the running effective gauge couplings. Some examples are also presented.

Sect. 6 completes the sector by sector analysis of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ models by considering the additional sectors arising from the action of the second $\mathbb{Z}_2$, namely the 2nd and 3rd complex planes, together with the $\mathcal{N} = 1$ sector. Actually, imposing the constraint that the models are chiral, with 3 families, we find that the SSS susy breaking to $\mathcal{N} = 0$ must only involve the 1st plane moduli. This has two consequences. First, the gravitino mass $m_3/2$ of the $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$ model is of order $1/\text{Im} T_1$, the inverse of the volume of the internal 1st plane. Moreover, the 2nd plane, 3rd plane and $\mathcal{N} = 1$ sectors preserve exact supersymmetries and the threshold scales $M^{(E)}_I$ associated to the complex planes $I = 2, 3$ must be of the order of the Planck scale.

Sect. 7 collects the above results in order to write the expression of the running coupling constants in the $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$ models. Moreover, it is remarkable that the effective potential arises only from the $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$ sector $B$, the other sectors being either supersymmetric or exponentially suppressed when $m_3^{(E)}/2$ is lower than the Planck scale. Finally, our conclusions are presented in Sect. 8.
2 The $\mathbb{Z}_2 \times \mathbb{Z}_2$ models with spontaneously broken susy

The context in which we will propose a solution to the decompactification problem consists in $\mathbb{Z}_2 \times \mathbb{Z}_2$ asymmetric orbifolds obtained via fermionic constructions and describing a spontaneous $\mathcal{N} = 1 \to \mathcal{N} = 0$ susy breaking. As we will see in Sect. [1] the relevant models rely on an underlying $\mathcal{N} = 4$ structure. Specifically, one of the two $\mathbb{Z}_2$’s must act freely, so that an $\mathcal{N} = 2$ sector will have the desired properties of spontaneously broken $\mathcal{N} = 4 \to \mathcal{N} = 2$ [18]. One has to be careful however. If all $\mathcal{N} = 2$ sectors appear as a spontaneous breaking of $\mathcal{N} = 4$, then the resulting $\mathcal{N} = 1$ model will be non-chiral, since in $\mathbb{Z}_2 \times \mathbb{Z}_2$ asymmetric orbifolds, the chiral families arise from the $\mathcal{N} = 2$ twisted sectors with non-trivial fixed points. Therefore, to make contact with realistic models, we will consider $\mathcal{N} = 1$ models, where only one of the two $\mathbb{Z}_2$ actions is free, which implies only one $\mathcal{N} = 2$ sector comes from an $\mathcal{N} = 4 \to \mathcal{N} = 2$ spontaneous breaking. The final implementation of the $\mathcal{N} = 1 \to \mathcal{N} = 0$ spontaneous breaking is done by coupling another $\mathbb{Z}_2$ freely acting shift in the large internal directions, with the supersymmetric R-symmetry charges (e.g. the four $SO(1,9)$ helicity charges of the ten dimensional mother theory).

For a gauge group factor $G^i$ at Kac-Moody level $k^i$, the running effective field theory coupling constant of a string model is [15,18–21]

$$\frac{16 \pi^2}{g_s^2(\mu)} = k^i \frac{16 \pi^2}{g_s^2} + b^i \log \frac{M_s^2}{\mu^2} + \Delta^i,$$  \hspace{1cm} (2.1)

where $b^i$ is the $\beta$-function coefficient, $g_s$ is the string coupling and $\mu$ plays the role of renormalization scale in the effective field theory. In string calculations, a mass gap $\mu$ is introduced to regularize the infrared [20]. The analytic expression of the threshold corrections takes the form

$$\Delta^i = \int_X \frac{d^2 \tau}{\tau_2} \left( \frac{1}{2} \sum_{a,b} \mathcal{Q} \left( \mathcal{P}_i^2 - \frac{k^i}{4\pi \tau_2} \right) C[^a_b](2\nu) - b^i \right) \bigg|_{v=0} + b^i \log \frac{2 e^{1-\gamma}}{\pi \sqrt{27}},$$  \hspace{1cm} (2.2)

where $C[^a_b](2\nu)$ is related to the partition function for given spin structures $(a,b)$ of the worldsheet fermionic supercoordinates,

$$C[^a_b](2\nu) = \tau_2 Z[^a_b](2\nu).$$  \hspace{1cm} (2.3)

$(a,b)$ are integer modulo 2: Spacetime bosons have $a = 0$, while spacetime fermions have $a = 1$. As indicated by the presence of the variable $v$, $Z[^a_b](2\nu)$ is actually a refined partition
function, on which the helicity operator $Q$ acts on the left-moving part,

$$Q = \frac{i}{\pi} \partial_\tau \left( \log \frac{\theta[^a_0](2\tau)}{\eta} \right) = \frac{1}{16\pi^2} \frac{\partial^2 \theta[^a_0](2\tau)}{\theta[^a_0](2\tau)} - \frac{i}{\pi} \partial_\tau \log \eta. \quad (2.4)$$

Our conventions for the $\theta[^a_0](v|\tau)$-functions can be found in Appendix C of Ref. 22, and it is understood that $\theta[^a_0](v)$ denotes $\theta[^a_0](v|\tau)$, while $\theta[^a_0]$ stands for $\theta[^a_0](0|\tau)$. On the contrary, $P_i$ is the charge operator of the gauge group factor $G^i$, thus acting on the right-moving sector of the heterotic string. Finally, no infrared divergence occurs in the expression of $\Delta^i$, due to the relation

$$b^i = \lim_{\tau_2 \to \infty} \frac{1}{2} \sum_{a,b} Q P_i^2 C[^a_0][v](2\tau)|_{v=0}. \quad (2.5)$$

In $\mathbb{Z}_2 \times \mathbb{Z}_2$ asymmetric orbifolds which preserve $\mathcal{N} = 1$ supersymmetry, the $\mathcal{N} = 4$ sector gives vanishing contribution. In this case, only the $\mathcal{N} = 2$ sectors contribute, so that

$$\Delta^i = \sum_{I=1}^3 \Delta^i_I(T_I, U_I), \quad (2.6)$$

where the threshold corrections $\Delta^i_I(T_I, U_I)$ are coming from the three different $\mathcal{N} = 2$ planes. $T_I, U_I, I = 1, 2, 3$, are the six moduli associated to the three $\Gamma_{2,2}$-lattices of the six internal dimensions. Notice that all $\mathcal{N} = 1$ sectors are absent. Thus, the full $\beta$-function coefficient of the $\mathbb{Z}_2 \times \mathbb{Z}_2 \mathcal{N} = 1$ theory is

$$b^i = \sum_{I=1}^3 b^i_I, \quad b^i_I = \frac{1}{2} \sum_{a,b} Q P_i^2 C[^a_0][I], \quad (2.7)$$

where $C[^a_0][I]$ is the contribution from the plane $I$, and the modular covariant helicity operator $Q$ can be replaced by $\frac{i}{\pi} \partial_\tau \log \theta[^a_0]$, since the $-\frac{i}{\pi} \partial_\tau \log \eta$ contribution is proportional to zero, due to the preservation of supersymmetry.

Our goal is to derive the analogous structure of the threshold corrections and the effective potential in general $\mathbb{Z}_2 \times \mathbb{Z}_2$ asymmetric orbifolds, where $\mathcal{N} = 1$ supersymmetry is spontaneously broken “à la Stringy Scherk-Schwarz”, in the context of the fermionic constructions, once we include the moduli deformations $T_I, U_I$ of the three $\Gamma_{2,2}$-lattices. The partition function is

$$Z(2\tau) = \frac{1}{\tau_2 |\eta|^4} \sum_{a,b} \frac{1}{4} \sum_{H_1, G_I} \frac{(-)^{a+b+ab}}{\theta[^a_0](2\tau)} \frac{\theta[^a_0+H_2]}{\theta[^b_0+G_2]} \frac{\theta[^a_0+H_1]}{\theta[^b_0+G_1]} \frac{\theta[^a_0-H_1-H_2]}{\theta[^b_0-G_1-G_2]} \times$$

$$\frac{1}{2N} \sum_{h_{I,J}^i, g_{I,J}^i} S[^a,h_{I,J}^i,h_I^i,h_J^i,g_I^i,g_J^i][^a,h_{I,J}^i,h_I^i,h_J^i,g_I^i,g_J^i] Z_{2,2}[^a,h_{I,J}^i,h_I^i,h_J^i,g_I^i,g_J^i] Z_{2,2}[^a,h_{I,J}^i,h_I^i,h_J^i,g_I^i,g_J^i] Z_{0,16}[^a,h_{I,J}^i,h_I^i,h_J^i,g_I^i,g_J^i], \quad (2.8)$$
in terms of which the effective potential can be expressed as

\[ V_{\text{eff}} = -\frac{1}{(2\pi)^4} \int \frac{d^2 \tau}{2\tau^2} Z|_{v=0}. \]  

(2.9)

Our notations in the expression (2.8) are as follows:

- \((H_1, G_1)\) and \((H_2, G_2)\) are integer modulo 2 associated to the \(\mathbb{Z}_2 \times \mathbb{Z}_2\) twists. We have:
  - The \(\mathcal{N} = 4\) sector, which corresponds to \((H_1, G_1) = (H_2, G_2) = (0, 0)\).
  - Three \(\mathcal{N} = 2\) twisted sectors, i.e. the so-called complex planes, namely:
    - Complex plane \(I = 1\): \((H_1, G_1) \neq (0, 0)\) with \((H_2, G_2) = (0, 0)\).
    - Complex plane \(I = 2\): \((H_2, G_2) \neq (0, 0)\) with \((H_1, G_1) = (0, 0)\).
    - Complex plane \(I = 3\): \((H_1, G_1) = (H_2, G_2) \neq (0, 0)\).
  - The \(\mathcal{N} = 1\) twisted sector, which corresponds to \((H_1, G_1) \neq (0, 0), (H_2, G_2) \neq (0, 0)\), \((H_1, G_1) \neq (H_2, G_2)\).

In the above list, \(\mathcal{N} = 4, 2, 1\) denotes the number of fermionic zero modes present in each sector, when no spontaneous breaking to \(\mathcal{N} = 0\) is implemented. Indeed, the (extended) supersymmetry of each sector may or may not be in a spontaneously broken phase, \(\mathcal{N} = 4, 2, 1 \rightarrow \mathcal{N} = 0\), depending on the choice of \(S\) introduced below.

- \((h^i_I, g^i_I), i = 1, 2, I = 1, 2, 3\), are integer modulo 2. They define the shifts of the three untwisted \(\Gamma_{2,2}\)-lattices, which are sums over two momenta \(m^i_I\) and two winding numbers \(n^i_I\) associated to each complex plane \(I\).

- The contribution of the six internal coordinates (shifted by \((h^i_I, g^i_I)\) and twisted by \((H_I, G_I)\)) is given in the second line of Eq. (2.8) in terms of blocks \(Z_{2,2}^{[h^i_I | H_I ]}_{[g^i_I | G_I ]}, I = 1, 2, 3\), where we denote \((H_3, G_3) \equiv (-H_1 - H_2, -G_1 - G_2)\).

- The fact that the shifts \((h^i_I, g^i_I)\) and the twists \((H_I, G_I)\) are not in general independent give an effective normalization factor \(1/2^N\) for the partition function, with \(N\) the number of independent shift pairs \((h^i_I, g^i_I)\).

- \(S\) is a phase that can implement the breaking of the \(\mathcal{N} = 1\) spacetime supersymmetry to \(\mathcal{N} = 0\). When \(S^{[a, h^i_I, H_I ]}_{[b, g^i_I, G_I ]} \equiv 1\), the theory is \(\mathcal{N} = 1\) supersymmetric. The supersymmetry can be broken spontaneously “à la Stringy Scherk-Schwarz” once the
10-dimensional helicity characters (R-parity charges)

\[(a\ b), \ (a+H_1\ b), \ (a+H_2\ b+G_1), \ (a+H_1\ b+G_2)\] 

are correlated with the lattice charges, i.e. with some shift pairs \((h^i_j, g^i_j)\).

- Finally, the contribution of the 32 extra right-moving worldsheet fermions is denoted \(Z_{0,16}[^{h^i_j}_j; \ [g^i_j, G^i_j]\]\). In the absence of twists and shifts, \(Z_{0,16}\) is the partition function associated to the \(E_8 \times E_8\) or \(SO(32)\) root lattices. When shifts and twists are non-trivial, the initial gauge group is broken to a product of lower dimensional subgroups (modulo some stringy extended symmetry points). Therefore, the role of the non-trivial twists and shifts is to generate non-zero discrete and continuous Wilson lines. In realistic models, the choice of shifts and twists must be such that the gauge group contains an \(SO(10)\) chiral factor, which is further broken to a subgroup that contains the desired standard model gauge group coupled to acceptable particle content in three generations.

If no particular attention is devoted to the choice of shifts \((h^i_j, g^i_j)\), the models where \(\mathcal{N} = 1\) supersymmetry is spontaneously broken to \(\mathcal{N} = 0\) “à la Stringy Scherk-Schwarz” may be \(i\) not chiral and/or \(ii\) suffer from the so-called decompactification problem. The reason of point \(ii\) is that the scale of susy breaking is fixed by the inverse of the characteristic size \(R\) of the internal compactified dimensions involved in the breaking, \(m_2 = O(M_s/R)\). In order to have small supersymmetry breaking scale compared to the string scale, \(m_2 = 10^{-14} M_s\), \(R\) must be enormous. Consequently, when the threshold corrections due to the tower of Kaluza-Klein states are proportional to the volume of the large extra dimensions and dressed with a negative \(\beta\)-function coefficient, the perturbative expansion is invalidated [15, 18]. However, this is not always the case. The next section is devoted to the presentation of the simplest case, where such a volume term is not generated.

3 \(\mathcal{N} = 4 \rightarrow \mathcal{N} = 0\) sector

The partition function (2.8) can be separated in sectors according to the \(\mathbb{Z}_2 \times \mathbb{Z}_2\) action. In this section, we focus on the \(\mathcal{N} = 4\) sector \((H_1, G_1) = (H_2, G_2) = (0, 0)\), which can be spontaneously broken to \(\mathcal{N} = 0\), when the SSS phase \(S\) is non-trivial. In this case, the induced contribution to the thresholds yields a logarithmic dependance on the volume of the
internal directions involved in the susy breaking. Actually, the threshold corrections of the $\mathcal{N} = 4 \to \mathcal{N} = 0$ sector appearing in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models are smaller by a factor 4, compared to those of the full “mother” $\mathcal{N} = 4 \to \mathcal{N} = 0$ theory. Thus, we will compute in this section the threshold corrections in an $\mathcal{N} = 4 \to \mathcal{N} = 0$ theory and will remind in the final result that a factor of $\frac{1}{4}$ arising from a $\mathbb{Z}_2 \times \mathbb{Z}_2$ projection may be included. We will present in detail the simple case, where a single factorized circle is involved in the process of supersymmetry breaking. This can be considered as an introductory section, since Sects 4–7 will present the analysis valid in generic $\mathbb{Z}_2 \times \mathbb{Z}_2$ models, with three chiral families.

In an $\mathcal{N} = 4$ model, two possibilities may arise once a phase $S$ is introduced. If $S$ is independent of $(a,b)$, then the $\mathcal{N} = 4$ supersymmetry is unbroken. In this case, the contribution of the worldsheet fermions to the partition function yields

$$\frac{1}{2} \sum_{a,b} (-)^{a+b+ab} \theta_{[a]}^{[a]}(2v) \theta_{[b]}^{[a]} = \theta_{[1]}^{[1]}(v) = \mathcal{O}(v^{4}) ,$$

(3.1)

where we use the Jacobi $\theta$-function identity and the relation $\theta_{[1]}^{[1]}(v|\tau) = 2\pi \eta^{3}(\tau) v + \mathcal{O}(v^{3})$. Therefore, the partition function (and effective potential) vanish. Similarly, the helicity insertion, which defines the corrections to the coupling constants, gives

$$\frac{1}{2} \sum_{a,b} (-)^{a+b+ab} Q \theta_{[a]}^{[a]}(2v) \theta_{[b]}^{[a]} = \frac{1}{16\pi^{2}} \partial_{v}^{2}(\theta_{[1]}^{[1]}(v)) = \mathcal{O}(v^{2}) ,$$

(3.2)

which shows that the gauge coupling thresholds vanish as well.

The second possibility is when the phase $S$ couples non-trivially the helicity charges $(a, b)$, with the shiftings of the internal lattice. This will break spontaneously the $\mathcal{N} = 4$ supersymmetry to $\mathcal{N} = 0$. In order to simplify our discussion in this section, we restrict ourselves to the case where only one $S^{1}$ cycle is involved in the susy breaking, and is very large. Denoting the associated shifts $(h_{1}^{1}, g_{1}^{1})$ as $(h, g)$, we have

$$S = e^{i\pi (ah + bg + hg)} .$$

(3.3)

Moreover, we specialize to the case where the $S^{1}$ shifted lattice is factorized,

$$\Gamma_{6,6+16}^{[h]} = \Gamma_{1,1}^{[h]}(R_{1}) \Gamma_{5,21}^{[h]} ,$$

(3.4)

where $\Gamma_{5,21}^{[h]}$ is a shifted lattice associated to the remaining 5 internal coordinates and the 32 right-moving worldsheet fermions of the heterotic string.\footnote{In $\mathbb{Z}_2 \times \mathbb{Z}_2$ models, $\Gamma_{5,21}$ is further factorized as in Eq. (2.8).} For instance, the dependance
of the $\Gamma_{5,21}$-lattice on $(h,g)$ may induce a Higgs mechanism by acting on the right-moving worldsheet degrees of freedom. In any case, due to our assumptions, this dependance must not imply a participation of the $\Gamma_{5,21}$ moduli in the super-Higgs mechanism, which would otherwise induce a very large gravitino mass. The $S^1$ lattice, $\Gamma_{1,1}^{[h]}$, admits two representations, Hamiltonian or Lagrangian, which are related to one another by Poisson resummation on the momentum quantum number $m$ [15][18]:

$$\Gamma_{1,1}^{[h]}(R_1) = \sum_{m,n} (-)^m q^{\frac{3}{2}p_L^2} \bar{q}^{\frac{3}{2}p_R^2}, \quad \text{where} \quad p_L = \frac{1}{\sqrt{2}} \left[ m \pm \left( n + \frac{h}{2} \right) R_1 \right]$$

$$= \frac{R_1}{\sqrt{\tau_2}} \sum_{n,m} e^{-\frac{nR_1^2}{2\tau_2} \left( (\bar{m}+\frac{h}{2})+(n+\frac{g}{2}) \right)}.$$  (3.5)

In fact, restricting the internal lattice to the above factorized form will not affect the asymptotic behavior of the threshold corrections for large $R_1$.

Because of the non-trivial correlation of the helicity and lattice charges through the SSS susy breaking phase, both the partition function and the coupling constant corrections are not zero. Indeed, in the partition function, the worldsheet fermions and SSS phase give

$$\frac{1}{2} \sum_{a,b} (-)^{a+b+ab} e^{i\pi (ag+bh+bg)} \theta[a] \theta[b] = \frac{1}{2} \sum_{A,B} e^{i\pi (A+B+AB+h+g)} \theta[A+\frac{1}{2}h] \theta[B+\frac{1}{2}] = e^{i\pi (h+g+1)} \theta[1-g] \theta[1-h],$$  (3.6)

which contribute to the effective potential when $(h,g) \neq (0,0)$ [23]. Moreover, using the above equation, the integrand involved in the gauge threshold corrections becomes

$$\frac{1}{2} \sum_{a,b} \mathcal{Q} \left( P_i^2 - \frac{k_i}{4\pi \tau_2} \right) C^{[g]}(2v) \bigg|_{v=0} = \frac{1}{2} \sum_{h,g} e^{i\pi (h+g+1)} \frac{i}{\pi} \left( \frac{1}{4} \partial_\tau \theta [1-g] - \left( \partial_\tau \log \eta \right) \theta [1-h] \right) \times$$

$$\frac{1}{\eta^{12} \eta^{24}} \Gamma_{1,1}^{[h]}(R_1) \left( \mathcal{P}_i^2 - \frac{k_i}{4\pi \tau_2} \right) \Gamma_{5,21}^{[h]}.$$  (3.7)

The second part of the helicity operator $\mathcal{Q}$ proportional to $\partial_\tau \log \eta$ gives non-trivial contribution, when supersymmetry is broken to $\mathcal{N} = 0 \ i.e. \ (h,g) \neq (0,0)$.

To perform the integral over the fundamental domain, one can use the unfolding method introduced in Ref. [24] and used in [15][18][19][25]. Defining $N = 2n + h$ and $\bar{M} = 2\bar{m} + g$, when $R_1$ is sufficiently large to guaranty the absolute convergences of the series, one can map the integral over the fundamental domain $\mathcal{F}$ into an integral over $\mathcal{F}$ restricted to the pair $(\bar{N}, \bar{M}) = (0,0)$, plus an integral over the “upper half strip” ($\frac{1}{2} < \tau_2 < \frac{3}{2}, \tau_2 > 0$) restricted to $N = 0$, $\bar{M} \neq 0$. In the strip representation, the winding contributions to the
fundamental domain integral are mapped to the momentum contributions in the ultraviolet region of the strip, \( \tau_2 < 1 \). In our case, all integrands with \( N = 0 \) (i.e. \( n = h = 0 \)) and \( \tilde{M} \) even (i.e. \( g = 0 \)) preserve \( \mathcal{N} = 4 \) and therefore vanish, as shown in Eq. (3.2). This is fundamental, since the key point to not have a contribution to the thresholds proportional to a large volume (\( R_1 \) in the present case) is that the integrand with \((N, \tilde{M}) = (0, 0)\) vanishes. Thus, we are left with an integral over the strip, with \((h, g) = (0, 1)\),

\[
\Delta^i = \lim_{\mu \to 0} \left[ \int \frac{d^2 \tau}{\tau_2} \frac{i}{2 \pi} \left( \frac{1}{4} \frac{\partial_r \theta[1][0]}{\eta^{12}} - (\partial_r \log \eta) \frac{\theta[1][0]}{\eta^{12}} \right) \frac{R_1}{\sqrt{\tau_2}} \sum_m e^{-\frac{\pi R_1^2}{\tau_2} (2\tilde{m}+1)^2 - \pi \mu^2 \tau_2} \times \left( \mathcal{P}_i^2 - \frac{k_i}{4\pi \tau_2} \right) \Gamma_{5,21}[10] - b^i \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} e^{-\pi \mu^2 \tau_2} \right] + b^i \ln \frac{2 e^{1-\gamma}}{\pi \sqrt{27}}.
\tag{3.8}
\]

In Eq. (3.8), we introduced a small mass \( \mu \) in order to regulate the infrared divergences in the large \( \tau_2 \) limit [20]. Other ways to regularize the infrared regime have been proposed recently [25] and have the advantage of preserving in a very elegant way both worldsheet and target space dualities. Our results, however, do not depend of the regularization scheme.

The would be tachyonic level appearing in the right-moving sector is projected out by the level matching condition induced via \( \tau_1 \)-integration over the strip. In the large \( R_1 \) limit, the massive string states give exponentially suppressed contributions to the integral over \( \tau_2 \) and can be consistently neglected. The dominant contribution comes from the massless level and even if supersymmetry is broken, there are no-tachyons arising from the left-moving sector. More specifically, we have

\[
\left( \frac{i}{\pi} \partial_r \log \theta[1][0] - \frac{i}{\pi} \partial_r \log \eta \right) \frac{\theta[1][0]}{\eta^{12}} = \left( -\frac{1}{4} + \frac{1}{12} \right) 16 + \mathcal{O}(q) = -\frac{8}{3} + \mathcal{O}(q), \tag{3.9}
\]

which is an expected result, since the constant term in the above \( q \)-expansion must be proportional to the \( \beta \)-function contribution of the bosons of the \( \mathcal{N} = 4 \) vector multiplets.

On the contrary, the gauge group contribution comes from the \( \mathcal{P}_i^2 \) charge operator, which acts on the right-moving sector. Actually, in our conventions, the \( \beta \)-function contributions of massless degrees of freedom are:

\[
b(\text{gauge boson}) = -\frac{11}{3} C(\mathcal{R}), \quad b(\text{real scalar}) = \frac{1}{6} C(\mathcal{R}), \quad b(\text{Majorana fermion}) = \frac{2}{3} C(\mathcal{R}), \tag{3.10}
\]

where \( C(\mathcal{R}) \delta^{ab} = \text{Tr}(T^a T^b) \) is the group factor coefficient associated to the generators \( T^a \) in the representation \( \mathcal{R} \) of \( G^i \). In an \( \mathcal{N} = 4 \) vector multiplet, \( \mathcal{R} \) is the adjoint representation,
and there are 6 real scalars and 4 Majorana gauginos per gauge boson, leading to $b(\text{bosons}) = -\frac{8}{3} C(R)$ and $b(\text{fermions}) = \frac{8}{3} C(R)$. When supersymmetry is unbroken, the $\mathcal{N} = 4$ $\beta$-functions vanish. However, in our case, supersymmetry is spontaneously broken via the SSS mechanism. The gravitinos and gauginos are getting masses that can be read in the Hamiltonian form of the $\Gamma_{1,1}^{[0]}$-lattice in Eq. (3.5) and are proportional to the inverse of the internal radius,

$$m^2_{\frac{1}{2}} = m^2_{\frac{1}{2}} = \frac{M_s^2}{R_1^2},$$  \hspace{1cm} (3.11)

while the gauge bosons and scalars remain massless,

$$m^2_1 = m^2_0 = 0.$$  \hspace{1cm} (3.12)

Thus, the logarithmic behavior of the $\beta$-function is fully controlled by the massless bosons, while the main corrections in the thresholds come from the tower of states organized by the shifted $\Gamma_{1,1}^{[0]}(R_1)$-lattice.

Neglecting in Eq. (3.8) the exponentially suppressed contributions for large radius, $\Delta^i$ gets simplified enormously,

$$\Delta^i = b^i \Delta - k^i Y,$$  \hspace{1cm} (3.13)

where $b^i \Delta$ comes from the $\mathcal{P}^2_i$ action and $k^i Y$ is the universal contribution arising from its modular covariant term $\frac{k^i}{4\pi \tau_2}$. The former is

$$\Delta = \lim_{\mu \to 0} \left[ R_1 \sum_{\tilde{m}} \int_0^{+\infty} \frac{d\tau_2}{\tau_2^{3/2}} e^{-\frac{\pi k^i}{4\tau_2} (2\tilde{m} + 1)^2} e^{-\pi \tau_2 \mu^2} - \int_1^{+\infty} \frac{d\tau_2}{\tau_2} e^{-\pi \tau_2 \mu^2} \right] - \ln \pi - \gamma + \cdots$$

$$= \lim_{\mu \to 0} \left[ 2 \sum_{\tilde{m}} \frac{1}{2\tilde{m} + 1} e^{-\pi R_1 |2\tilde{m} + 1| \mu} - \Gamma(0, \pi \mu^2) \right] - \ln \pi - \gamma + \cdots,$$  \hspace{1cm} (3.14)

where the dots stand for $\mathcal{O}(e^{-cR_1})$ corrections, with $c$ positive and of the order of the lowest mass $M_0$ of the massive spectrum divided by $M_s$\footnote{$M_0$ depends on the moduli appearing in the $\Gamma_{5,21}^{[h]}$-lattice and is at most equal to $M_s$.}. In the above expression, $\Gamma(s, x)$ is the upper incomplete $\Gamma$-function. Using the fact that $\Gamma(0, x) = -\ln(x) - \gamma + \mathcal{O}(x)$, one finally finds

$$\Delta = \lim_{\mu \to 0} \left[ 2 \ln \left( \frac{1 + e^{-\pi R_1 \mu}}{1 - e^{-\pi R_1 \mu}} \right) + \ln \mu^2 \right] + \cdots = -\log \left( \frac{\pi^2}{4} R_1^2 \right) + \cdots.$$  \hspace{1cm} (3.15)

For the determination of $Y$, the infrared regulator $\mu$ is not needed since the integral is infrared convergent,

$$Y = \frac{C_0}{4\pi} \sum_{\tilde{m}} \int_0^{+\infty} \frac{d\tau_2}{\tau_2^{5/2}} R_1 e^{-\frac{\pi k^i}{4\tau_2} (2\tilde{m} + 1)^2} + \cdots = \frac{7\zeta(3)}{4\pi^2} \frac{C_0}{R_1^2} + \cdots.$$  \hspace{1cm} (3.16)
In (3.16), \( C_0 \) is the product of the contribution of the helicity operator \( Q \) acting on the left-moving sector, \( Q = -\frac{8}{3} \), with a coefficient \( 2 + d_G - n_F \) associated to the right-moving sector,

\[
C_0 = \frac{1}{2} \sum_{a,b=0,1} QC^{[a]}_{[b]} \bigg|_{q^n q^n} = -\frac{8}{3} (2 + d_G - n_F).
\]  

(3.17)

\( d_G \) is the number of vector bosons in the \( \mathcal{N} = 4 \) vector multiplets of the parent \( \mathcal{N} = 4 \) theory that remain massless after spontaneous breaking to \( \mathcal{N} = 0 \). In other words, \( d_G \) is the dimension of the gauge group. Similarly, \( 4n_F \) is the number of Majorana fermions in the \( \mathcal{N} = 4 \) vector multiplets of the parent \( \mathcal{N} = 4 \) theory that remain massless after spontaneous breaking to \( \mathcal{N} = 0 \). Therefore, the corrections to the coupling constants in this \( \mathcal{N} = 4 \to \mathcal{N} = 0 \) model is

\[
\Delta^i = b^i \Delta - k^i Y = -b^i \log \left( \frac{\pi^2}{4} R_1^2 \right) + k^i \frac{14 \zeta(3)}{3 \pi^2} \frac{2 + d_G - n_F}{R_1^2} + \mathcal{O} \left( e^{-c R_1} \right).
\]  

(3.18)

The dangerous volume dependence (linear term in \( R_1 \)) is absent, and the reason for this is the restoration of the \( \mathcal{N} = 4 \) supersymmetry in the \( R_1 \to \infty \) limit. Since the universal contribution \( Y \) scales like \( m_2^2 / M_s^2 \), it is a tiny correction to the logarithmic term and may be neglected.

As said at the beginning of this section, the contribution of the \( \mathcal{N} = 4 \to \mathcal{N} = 0 \) sector in a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) model is obtained from Eq. (3.18) by changing \( b^i \to b^i / 4 \) and \( C_0 \to C_0 / 4 \), where the \( \beta \)-function \( b^i \) and \( C_0 \) refer to the \( \mathcal{N} = 4 \to \mathcal{N} = 0 \) theory. However, the presence of \( \mathcal{N} = 2 \) sectors requires more attention in the choice of susy breaking shifts. An \( \mathcal{N} = 2 \to \mathcal{N} = 0 \) model containing a sector of the form

\[
\frac{S^1}{\mathbb{Z}_2^{\text{shift}}} \times \frac{T^4}{\mathbb{Z}_2},
\]  

(3.19)

where the circle of radius \( R_1 \) is shifted as before to break susy spontaneously to \( \mathcal{N} = 0 \), will contain a contribution to the thresholds arising from the integration over \( \mathcal{F} \) of the lattice term with \( (N, \tilde{M}) = (0, 0) \), which is proportional to the large radius \( R_1 \). This contribution arises from an \( \mathcal{N} = 2 \) preserving sector, which therefore does not vanish as is the case when \( \mathcal{N} = 4 \) is preserved. On the contrary, an \( \mathcal{N} = 2 \to \mathcal{N} = 0 \) model based on an internal space containing a factor

\[
\frac{S^1/\mathbb{Z}_2^{\text{shift}}}{} \times \frac{T^3}{\mathbb{Z}_2},
\]  

(3.20)
is safe. The reason for this is that the only $R_1$-dependent contribution to the partition function arises from the untwisted sector, which realizes an $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$ spontaneous breaking. Unfortunately, there are no model based on a single large $S^1$ shifted direction that realizes a SSS spontaneous breaking of $\mathcal{N} = 1$ supersymmetry to $\mathcal{N} = 0$ that solves the decompactification problem. Therefore, we proceed in the next section with the more sophisticated case where two internal shifted directions involved in the breaking are large.

4 $\mathcal{N} = 4$ and 1st plane contributions: $(H_2, G_2) = (0, 0)$

From now on, we come back to the generic $Z_2 \times Z_2$ model defined in Eq. (2.8). In this section and the following, we develop a sector by sector analysis of the contributions to the gauge threshold corrections and effective potential. The susy breaking is defined by the SSS phase $S^{[a,h_i][b,g_i]}$ that correlates non-trivially the shifts and the twists charges with the helicity and R-symmetry charges. However, $S$ being sector-dependent, it can be trivial ($S = 1$) in some sectors, thus preserving supersymmetry, and non-trivial ($S \neq 1$) in some others, thus inducing a spontaneous breaking of supersymmetry.

In the present section, we focus on the $\mathcal{N} = 4$ sector $(H_2, G_2) = (H_1, G_1) = (0, 0)$, together with the 1st $\mathcal{N} = 2$ plane $(H_2, G_2) = (0, 0), (H_1, G_1) \neq (0, 0)$. We derive here the formal results, and will comment on them physically in Sect. 5. Both sectors contain subsectors, which preserve or break supersymmetry. The contribution of the untwisted internal coordinates ($(H_2, G_2) = (0, 0)$) in the partition function (2.8) is

$$Z_{2,2}^{[h^1_1,0]} = \frac{\Gamma_{2,2}^{[h^1_1,g^2_1]}}{(\eta\bar{\eta})^2},$$

where the shifted lattice dependance on the $T_1, U_1$ moduli (denoted $T, U$ in this section and Sect. 5) is

$$\Gamma_{2,2}^{[h^1_1,g^2_1]} = \sum_{m_i,n_i} (-)^{m_1g^1_1 + m_2g^2_1} e^{2i\pi \tau \left[m_1 \left(n_1 + \frac{h^1_1}{2}\right) + m_2 \left(n_2 + \frac{h^2_1}{2}\right)\right]} \times$$

$$e^{-\frac{\pi \tau_2}{\text{Im} T \text{Im} U} \left[T \left(n_1 + \frac{h^1_1}{2}\right) + TU \left(n_2 + \frac{h^2_1}{2}\right) - Um_1 + m_2\right]^2}$$

$$= \sqrt{\text{det} G} \sum_{\tilde{m}_i, \tilde{n}_i} e^{-\frac{\tau_2}{2} \left[\tilde{m}_i + \frac{\tilde{g}^1_1}{2} + (n_i + \frac{h^1_1}{2})\right] \tau} \left(G_{ij} + B_{ij}\right) \left[\tilde{m}_j + \frac{\tilde{g}^2_1}{2} + (n_j + \frac{h^2_1}{2})\right] \tau},$$

(4.2)
where the dictionary between $T, U$ and the internal metric and antisymmetric tensor in the two associated compact directions is

$$G_{ij} = \frac{\text{Im}T}{\text{Im}U} \begin{pmatrix} 1 & \text{Re}U \\ \text{Re}U & |U|^2 \end{pmatrix}, \quad B_{ij} = \text{Re}T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.3)$$

As explained before, our solution of the decompactification problem requires the breaking of $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ to be spontaneous. This is implemented by imposing the twist action labeled by $(H_1, G_1) \equiv (H, G)$ to act simultaneously as a shift in the above $\Gamma_{2,2}$-lattice. As in the previous section, independent $\Gamma_{2,2}$-lattice charges $(h, g)$ must be used to define the SSS $\mathcal{N} = 2 \rightarrow \mathcal{N} = 0$ susy breaking phase. In the sectors we consider here, two options parameterized by $\zeta' = 0$ or 1 can be chosen:

In the sectors $(H_2, G_2) = (0,0), \quad S = e^{i\pi [ag+bh+h\zeta(\alpha G+bH+HG)]} \quad (4.4)$

Therefore, both shifts $(h_i^1, g_i^1), i = 1, 2,$ are involved and three classes of two models (labeled by $\zeta = 0$ or 1) can be analyzed:

$$a) \quad \Gamma_{2,2}' \frac{h+\zeta H, h^0}{g+\zeta G, G} \quad \text{i.e.} \quad (h_1^1, g_1^1) \equiv (h, g) + \zeta(H, G), \quad (h_2^1, g_2^1) \equiv (H, G)$$

$$b) \quad \Gamma_{2,2} \frac{H, h+\zeta H}{g+\zeta G, G} \quad \text{i.e.} \quad (h_1^1, g_1^1) \equiv (H, G), \quad (h_2^1, g_2^1) \equiv (h, g) + \zeta(H, G)$$

$$c) \quad \Gamma_{2,2} \left( \frac{h+\zeta H, h+1(1-\zeta)H}{g+\zeta G, g+(1-\zeta)G} \right) \quad \text{i.e.} \quad (h_1^1, g_1^1) \equiv (h, g) + \zeta(H, G), \quad (h_2^1, g_2^1) \equiv (h, g) + (1-\zeta)(H, G). \quad (4.5)$$

In the absence of SSS phase and $\mathbb{Z}_2^{\text{shift}}$ action parameterized by $(h, g)$, the models would describe the partial spontaneous breaking of supersymmetry from $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$, which was considered in [18]. In this reference, it was shown that the pathological volume behaviors of the gauge couplings are absent, thanks to the restoration of $\mathcal{N} = 4$ supersymmetry in the large volume limit. In the presence of non-trivial SSS phase, the $\mathbb{Z}_2^{\text{shift}}$ action parameterized by $(h, g)$ breaks further the supersymmetry to $\mathcal{N} = 0$. In this case, the decompactification problem becomes more involved, due to extra contributions coming from the sectors with non-trivial charges $(h, g)$.

The separation of the $(H_2, G_2) = (0,0)$ sector of the partition function (2.8) in sub-sectors is more transparent once we perform the summation over the helicity charges $(a,b)$, keeping

$^3$The \textit{a priori} remaining cases $\Gamma_{2,2}' \frac{h+H, 0}{g+G, 0}, \Gamma_{2,2} \frac{0, h+H}{0, g+G}$ and $\Gamma_{2,2} \frac{h+H, h+H}{g+G, g+G}$ lead to a volume dependence in the gauge thresholds, arising from the sub-sector $(h, g) = (H, G) \neq (0,0), \quad \text{which preserves } \mathcal{N} = 2 \text{ supersymmetry.}$

14
the non-trivial characters \((h, g)\) and \((H, G)\) fixed\(^4\)

\[
\frac{1}{2} Z^{[h,H], [g,G]}_{[2]} (2v) = \frac{1}{4 \eta^8} \sum_{a,b} e^{\pi (a+b+ab)} e^{i \pi (ag+bh+gh+\zeta' (aG+bH+HG))} \times \theta_{[g]}^{[2]}(2v) \theta_{[b]}^{[2]} \theta_{[h+b+G]}^{[2]} \theta_{[a-H]}^{[2]} \Gamma_{2,2} \left[ \frac{h_1^i, h_2^i}{g_1^i, g_2^i} \right] \frac{1}{\eta^i} Z_{4,20}^{[h,H], [g,G]} \\
= \frac{1}{2 \eta^8} e^{i \pi (hg+G(1+h+H))} \theta_{[1-g]}^{[1]}(v) \theta_{[1-g+H]}^{[1]}(v) \Gamma_{2,2} \left[ \frac{h_1^i, h_2^i}{g_1^i, g_2^i} \right] \frac{1}{\eta^i} Z_{4,20}^{[h,H], [g,G]} \quad (4.6)
\]

The above result is obtained by redefining 
\(a = A - h - \zeta' H, \ b = B - g - \zeta' G\) and summing over \(A, B\) equal to 0 or 1. Note that \(\zeta'\) has disappeared, which shows that the two SSS phases \(S\) in Eq. (4.4) are actually equivalent, the different sectors of the theory being simply reshuffled. In Eq. (4.6), the conformal block \(Z_{4,20}^{[h,H], [g,G]}\) for \((H,G) = (0,0)\) involves an untwisted lattice \(\Gamma_{4,20}^{[h,g]}\), which depends on moduli. Anticipating the results of Sect. 6, the order of magnitude of these moduli must not be too far from 1, which we will suppose here and in Sect. 5. Therefore, \(\Gamma_{4,20}^{[h]}\) must be such that its moduli do not participate in the super-Higgs mechanism that breaks susy to \(\mathcal{N} = 0\). Otherwise, a gravitino mass close to \(M_{\text{Planck}}\) would be generated in the sub-sector \((h,g) \neq (0,0), (H,G) = (0,0)\) (denoted as \(B\) in the following), \(i.e.\) far above the acceptable 1–10 TeV region. However, the dependance of the \(\Gamma_{4,20}\)-lattice on \((h,g)\) may induce a Higgs mechanism arising from an left/right-asymmetric action on the right-moving worldsheet degrees of freedom. Several examples will be given in Sect. 6.

In Eq. (4.6), the number of odd \(\theta\)-functions \(\theta_{[1+X]}^{[1+Y]}(v)\), with \((X,Y) = (0,0)\), counts the preserved supersymmetries, according to the number of fermionic zero modes in each sub-sector. In the following, we use this number of preserved supersymmetries to classify the sub-sectors and derive the effective potential and gauge couplings corrections in each case.

4.1 A: The exact \(\mathcal{N} = 4\) sector \((h,g) = (0,0), (H,G) = (0,0)\)

In this sector we denote \(A, \mathcal{N} = 4\) supersymmetry is unbroken. Therefore, the contributions \(V_{\text{eff}, A}\) and \(\Delta^i_A\) to the partition function (or effective potential) and to the gauge couplings vanish. This is due to the fact that the partition function (4.6) is in this case proportional

\(^4\)The factor \(\frac{1}{2}\) in the l.h.s. refers to the \(\mathbb{Z}_2^{\text{shift}}\) projection obtained once the sum over \(h\) and \(g\) is performed. The analogous \(\frac{1}{4}\) factor associated to the \(\mathbb{Z}_2 \times \mathbb{Z}_2\) twist (or \(\frac{1}{2}\) for a single \(\mathbb{Z}_2\) twist) will be included later.
to $\theta_1^{[4]}(v) = \mathcal{O}(v^4)$ and the $\beta$-functions are of order $\mathcal{O}(v^2)$,

$$\Delta^i_A = 0, \quad V_{\text{eff},A} = 0.$$  \hfill (4.7)

The four gravitini in this sector are massless,

$$m^i_3 = 0, \quad i = 1, 2, 3, 4.$$  \hfill (4.8)

**4.2 $B$: The $\mathcal{N} = 4 \to \mathcal{N} = 0$ sector $(h, g) \neq (0, 0), (H, G) = (0, 0)$**

In this sector we denote $B$, all arguments of the $\theta$-functions in Eq. (4.6) are identical but not equal to $[1]$. The partition function being proportional to $\theta^{[1+h]}_1(v)$, both corrections $V_{\text{eff},B}$ and $\Delta^i_B$ to the effective potential and to the $\beta$-functions are non-vanishing. The four gravitini have equal non zero masses, which can be read from the Hamiltonian form of the lattice (4.2) (the first equality) \footnote{We display the masses for $\Re(U)$ in the range $(-1, 1)$.}

$$m^i_3 \equiv m_B = \frac{|\alpha_B U - \text{sign}(\Re U)\beta_B|}{\sqrt{\Im T \Im U}} M_s = \frac{(\alpha_B \Im U)^2 + (\alpha_B |\Re U| - \beta_B)^2}{\sqrt{\Im T \Im U}} M_s, \quad i = 1, 2, 3, 4,$$ \hfill (4.9)

where we define

$$(\alpha_B, \beta_B) = \begin{cases} (1, 0) & \text{in case } a) \\ (0, 1) & \text{in case } b) \\ (1, 1) & \text{in case } c). \end{cases}$$ \hfill (4.10)

In Sect. 3 we evaluated the coupling constant correction in case $a)$, when only one radius denoted by $R_1$ was very large. In this regime, the contribution of the remaining $\Gamma_{5,21}$-lattice was trivial. However, there are extra contributions when both compact directions in the 1st plane are large. In the following, utilizing the techniques of Ref. \footnote{We define $\text{sign}(0) = +1$.} we compute the thresholds in cases $a), b)$ and $c)$ in the regime where the complex moduli $T$ and $U$ satisfy $\Im T \gg 1, U$ finite, that guaranties $m_B \ll 1$.

Thanks to the Lagrangian expression of the lattice (4.2) (the second equality), the sector $h = 1$ is exponentially suppressed. Keeping explicitly the sector $(h, g) = (0, 1)$, the threshold corrections in sector $B$ are

$$\Delta^i_B = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \left\{ \frac{1}{\eta^4 \eta_1^4} \frac{1}{4\pi} \partial_i \left( \theta_1^{[4]}(\eta) \right) \frac{1}{2} \Gamma_{2,2}^{[0,0]} \left[ \frac{k}{4\pi \tau_2} \right] Z_{4,20}^{[0,0]} - \tilde{b}^i_B \right\} + \tilde{b}^i_B \log \frac{2e^{1-\gamma}}{\pi \sqrt{27}} + \cdots, \hfill (4.11)$$
where the coefficient $b^i_B$ is introduced to cancel the infrared divergence and the dots stand for exponentially small contributions for large $\text{Im} \, T$ and finite $U$. Similarly, the effective potential based on the partition function (4.6) with $(H, G) = (0, 0)$ is

$$V_{\text{eff}B} = -\frac{1}{(2\pi)^4} \int_F \frac{d^2 \tau}{2\tau_2^3} \frac{4^{\frac{1}{8}}}{\pi} \frac{1}{\pi} \Gamma_{2, 2} \left[ \alpha_B, \beta_B \right] Z_{4, 20} \left[ 0, 0 \right] + \cdots .$$

(4.12)

In the above two expressions, the dressing with the Lagrangian form of the $\Gamma_{2, 2}$-lattice implies the non-level matched modes as well as the massive (level-matched) physical states to yield exponentially suppressed contributions. As a result, the universal form of the thresholds in sector $B$,

$$\Delta^i_B = b^i_B \Delta_B - k^i Y_B,$$

(4.13)

as well as the effective potential take the simple forms obtained from the massless states:

$$\Delta_B = \int_F \frac{d^2 \tau}{\tau_2} \left( \Gamma_{2, 2} \left[ 0, 0 \right] - 1 \right) + \log \frac{2e^{1-\gamma}}{\pi \sqrt{27}} + \cdots ,$$

$$Y_B = \frac{C_B}{8\pi} \int_F \frac{d^2 \tau}{\tau_2^2} \Gamma_{2, 2} \left[ 0, 0 \right] + \cdots ,$$

$$V_{\text{eff}B} = -\frac{C_V}{2(2\pi)^4} \int_F \frac{d^2 \tau}{\tau_2^3} \Gamma_{2, 2} \left[ 0, 0 \right] + \cdots ,$$

(4.14)

where $C_B = -\frac{2}{3} (2 + d_G - n_F)$ and $C_V = 8(2 + d_G - n_F)$. In these coefficients, $d_G$ is the number of vector bosons in the $\mathcal{N} = 4$ vector multiplets of the parent $\mathcal{N} = 4$ theory that remain massless after spontaneous breaking to $\mathcal{N} = 0$, i.e. the dimension of the gauge group realized in the sector $B$. Similarly, $4n_F$ is the number of Majorana fermions in the $\mathcal{N} = 4$ vector multiplets of the parent $\mathcal{N} = 4$ theory that remain massless after spontaneous breaking to $\mathcal{N} = 0$. In other words, $C_V$ is the index that counts the number of bosonic degrees of freedom minus the number of fermionic degrees of freedom in the $\mathcal{N} = 0$ sector $B$,

$$C_V = 8(2 + d_G - n_F) \equiv \text{Bosons} - \text{Fermions} \text{ in the sector } B .$$

(4.15)

A simple way to evaluate $\Delta_B$ is based on the relation between the shifted lattices.
\( \Gamma_{2,2}^{[0,0]}(T,U) \) and the unshifted one, \( \Gamma_{2,2}(T,U) \). For the cases \( a, b \) and \( c \), we use respectively

\[
\Gamma_{2,2}^{[0,0]}(T,U) = \sum_{h,g} \Gamma_{2,2}^{[h,0]}(T,U) + \cdots = 2 \Gamma_{2,2}(T,U) + \cdots,
\]

\[
\Gamma_{2,2}^{[0,0]}(T,U) = \sum_{h,g} \Gamma_{2,2}^{[0,h]}(T,U) + \cdots = 2 \Gamma_{2,2}(T,U) + \cdots,
\]

\[
\Gamma_{2,2}^{[0,0]}(T,U) = \sum_{h,g} \Gamma_{2,2}^{[h,h]}(T,U) + \cdots = 2 \Gamma_{2,2}(T,U) + \cdots, \tag{4.16}
\]

where the primes indicate the sums are over \( (h, g) \neq (0,0) \). Using the well-known integral [19]

\[
\int_{\mathcal{F}} \frac{d^2 \tau}{(2\pi)^2} (\Gamma_{2,2}(T,U) - 1) + \log \frac{2e^{1-\gamma}}{\pi \sqrt{24}} = - \log \left( 4\pi^2 |\eta(T)|^4 |\eta(U)|^4 \text{Im} T \text{Im} U \right), \tag{4.17}
\]

one obtains

\[
\Delta_B = - \log \left( \frac{\pi^2}{4} |\theta_{11}^0(T)|^4 |\theta_{1-\alpha_B}^1(U)|^4 \text{Im} T \text{Im} U \right) + \mathcal{O} \left( e^{-c\pi \text{Im} T} \right), \tag{4.18}
\]

where \( c \) is positive and of the order of the lowest mass of the massive spectrum divided by \( M_a \). This lowest non-vanishing mass depends on the modulus \( U \), together with the moduli of the \( \Gamma_{4,20}^{[h]} \)-lattice present in the sector \( B \) and introduced below Eq. (4.6). Supposing that the order of magnitude of \( U \) is not too far from 1, a fact that will be justified in Sect. 3 and given the fact that the \( \Gamma_{4,20}^{[h]} \)-lattice moduli are also not too far from 1, we have \( c = \mathcal{O}(1) \). Moreover, since

\[
\log |\theta_{11}^0(T)|^4 = \mathcal{O} \left( e^{-\pi \text{Im} T} \right), \tag{4.19}
\]

this contribution can be omitted in Eq. (4.18). Thus, the \( \text{Im} T \) volume dependence of \( \Delta_B \) is only logarithmic. The key point for this is the following. In the integral (4.17), the contribution \( \tilde{m}_i = n_i = 0 \) in the unshifted lattice (4.2) is proportional to \( \sqrt{\text{det} G} = \text{Im} T \), which is responsible for a \( \frac{\pi}{3} \text{Im} T \) dominant contribution in the result. On the contrary, the shifted lattice in \( \Delta_B \) is expressed in Eq. (4.16) as a difference of two unshifted lattices, where the contribution \( \tilde{m}_i = n_i = 0 \) cancels out.

For the second part of the thresholds, \( Y_B \), and the effective potential, we use the fact that the contributions with non-trivial winding numbers \( n_i \) in the lattice (4.2) are exponentially suppressed,

\[
\Gamma_{2,2}^{[0,0]}(T,U) = \frac{\text{Im} T}{\mathcal{T}_2} \sum_{\tilde{m}_1, \tilde{m}_2} e^{-\frac{\pi \text{Im} T}{\mathcal{T}_2} |\tilde{m}_1 + \alpha_B + (\tilde{m}_2 + \beta_B)U|^2} + \cdots. \tag{4.20}
\]
This expression also justifies that, at our level of approximation, we are free to extend the integration domain from $F$ to the full upper half strip. This leads

$$Y_B = -\frac{2 + d_{GB} - n_{FB}}{3\pi^3} \frac{1}{\text{Im} T} E_{\alpha_B, \beta_B}(U|2) + \mathcal{O}\left(e^{-c\sqrt{\text{Im} T}}\right),$$

$$V_{\text{eff} B} = -\frac{2 + d_{GB} - n_{FB}}{2\pi^3} \frac{1}{(\text{Im} T)^2} E_{\alpha_B, \beta_B}(U|3) + \mathcal{O}\left(e^{-c\sqrt{\text{Im} T}}\right),$$

(4.21)

where we have defined “shifted real analytic Eisenstein series” as

$$E_{(g_1, g_2)}(U|s) = \sum_{\tilde{m}_1, \tilde{m}_2} \frac{(\text{Im} U)^s}{\tilde{m}_1 + \frac{g_1}{2} + (\tilde{m}_2 + \frac{g_2}{2})U^{2s}}.$$  

(4.22)

In these functions, $g_1$ and $g_2$ are integer modulo 2 and the prime means $\tilde{m}_1 = \tilde{m}_2 = 0$ is excluded from the sum when $g_1 = g_2 = 0$. They satisfy modular properties as follows:

$$E_{(g_1, g_2)}(M(U)|s) = E_{(g_1, g_2)}(M^T(U)|s),$$

where $M(U) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $M \in SL(2, \mathbb{Z})$.

(4.23)

Note that the sign of the index $C_V = \text{Bosons} - \text{Fermions}$ in the sector $B$ is essential to discuss questions moduli stabilization [23].

4.3 **C: The exact $\mathcal{N} = 2$ sector with $(h, g) = (0, 0), (H, G) \neq (0, 0)$**

The partition function (4.6) associated to this sector, which we will denote by $C$, is proportional to $\theta[1]^2(0)\theta[1+H]^2(0) = \mathcal{O}(v^2)$. Thus, the contribution $V_{\text{eff} C}$ to the effective potential is zero, while the threshold correction $\Delta^{i C}$ is not vanishing and proportional to an $\mathcal{N} = 2 \beta$-function coefficient $b^i_C$. Two of the four gravitini are massless, while the masses of the other two are given in terms of the $T$ and $U$ moduli,

$$m^{1,2}_{\frac{3}{2}} = 0, \quad m^{3,4}_{\frac{3}{2}} \equiv m_C = \frac{(\alpha_C \text{Im} U)^2 + (\alpha_C |\text{Re} U| - \beta_C)^2}{\sqrt{\text{Im} T \text{Im} U}} M_s,$$

(4.24)

where we have

$$(\alpha_C, \beta_C) = \begin{cases} (\zeta, 1) & \text{in case } a \\ (1, \zeta) & \text{in case } b \\ (\zeta, 1 - \zeta) & \text{in case } c \end{cases}.$$  

(4.25)

The threshold corrections in this sector are those of $\mathcal{N} = 2$ theories that are obtained by an $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ spontaneous susy breaking via a free $\mathbb{Z}_2$ orbifold action. They have been computed in Ref. [18] but we briefly rederive the results we need here.
The Lagrangian form of the lattice (4.2) implies the sector $H = 1$ to be exponentially suppressed, when $\text{Im} T \gg 1$ and $U$ is finite. Keeping explicitly the sector $(H, G) = (0, 1)$, one obtains using again $\theta^{[1]}(v|\tau) = 2\pi \eta^3(\tau) v + O(v^3),$

$$\Delta^i_C = \int d^2 \tau \frac{1}{\tau_2} \left\{ \Gamma_{2,2}^{[0,0]} \left( \mathcal{P}_i^2 - \frac{k_i}{4\pi \tau_2} \right) \Omega - b^i_C \right\} + b^i_C \log \frac{2e^{1-\gamma}}{\pi \sqrt{27}} + \cdots,$$

where \( \Omega = \frac{\theta^{[1]}_0^2}{4\eta^2 \tilde{\eta}^4} Z_{4.20}^{[0,0]} \). (4.26)

In fact, since the 4 directions associated to the second and third plane are twisted, $Z_{4.20}^{[0,0]}$ contains an overall factor $\eta^2 / \theta^{[1]}_0^2$ making $\Omega$ an antiholomorphic function. The contribution $b^i_C$ to the full $\beta$-function coefficient subtracts the infrared divergence. Proceeding as in the sector $B$, only the massless contributions dressed by the $\Gamma_{2,2}^{[0,0]}$-lattice are non-negligible, leading to formally identical results:

$$\Delta^i_C = b^i_C \Delta_C - k^i Y_C,$$

where

$$\Delta_C = -\log \left( \frac{\pi^2}{4} |\theta^{[1]}_1(T)|^4 |\theta^{[1-\beta_C]}_1(U)|^4 \text{Im} T \text{Im} U \right) + O \left( e^{-c \sqrt{\text{Im} T}} \right),$$

$$Y_C = -\frac{2}{3\pi^3} + \frac{n_{V_C} - n_{H_C}}{\text{Im} T} \text{E}_{ac,\beta_C}(U|2) + O \left( e^{-c \sqrt{\text{Im} T}} \right).$$

(4.28)

In the above expression, $n_{V_C}$ and $n_{H_C}$ are the numbers of massless vector multiplets and hypermultiplets in the sector $C$. Thus $n_{V_C}$ is the dimension of the gauge group $G_C$ realized in this sector, while

$$I_C = n_{V_C} - n_{H_C}$$

is an index arising naturally from the extended supersymmetry we will denote $N_C = 2$. As in sector $B$, the $|\theta^{[1]}_1(T)|^4$-term can be omitted and the thresholds are only logarithmic in $\text{Im} T$. As said before, it is interesting enough that in this sector the cosmological term vanishes, $V_{\text{eff}C} = 0$, thanks to the exact $N_C = 2$ supersymmetry.

### 4.4 D: The exact $\mathcal{N} = 2$ sector with $(h, g) = (H, G), (G, H) \neq (0, 0)$

We denote this sector as $D$. As in sector $C$, the partition function (4.6) vanishes, since it is proportional to $\theta^{[1+[H]2]}(v)\theta^{[1]}(v) = O(v^3)$. There is an exact $\mathcal{N} = 2$ supersymmetric
spectrum, which is not that of the sector $C$, the two $\mathcal{N} = 2$ supersymmetries being not aligned. The two massless and two massive gravitini are not the same,

$$m^2_{1,2} \equiv m_D = \frac{(\alpha_D \text{Im} U)^2 + (\alpha_D |\text{Re} U| - \beta_D)^2}{\sqrt{\text{Im} T \text{Im} U}} M_s, \quad m^2_{3,4} = 0 \quad (4.30)$$

and the non-vanishing masses are even different to those in sector $C$. This is due to the fact that the pairs $(\alpha_D, \beta_D)$ and $(\alpha_C, \beta_C)$ are not equal,

$$(\alpha_D, \beta_D) = \begin{cases} 
(1 - \zeta, 1) & \text{in case } a) \\
(1, 1 - \zeta) & \text{in case } b) \\
(1 - \zeta, \zeta) & \text{in case } c). 
\end{cases} \quad (4.31)$$

Actually, we see that the sectors $C$ and $D$ are replaced by one another under the change

$$\zeta \rightarrow 1 - \zeta,$$

sector $C \leftrightarrow \text{sector } D \iff \zeta \rightarrow 1 - \zeta. \quad (4.32)$$

As a result, the threshold corrections to the gauge couplings are

$$\Delta^i_D = b^i_D \Delta_D - k^i Y_D, \quad (4.33)$$

where

$$\Delta_D = -\log \left( \frac{\pi^2}{4} |\theta^{[0]}_{[1]}(T)|^4 |\theta^{[1-\beta_D]}_{[1-\alpha_D]}(U)|^4 \text{Im} T \text{Im} U \right) + \mathcal{O} \left( e^{-c \sqrt{\text{Im} T}} \right),$$

$$Y_D = -\frac{2 + n_{V_D} - n_{H_D}}{3\pi^3} \frac{1}{\text{Im} T} E_{\alpha_D, \beta_D}(U) 2 + \mathcal{O} \left( e^{-c \sqrt{\text{Im} T}} \right). \quad (4.34)$$

$n_{V_D}$ and $n_{H_D}$ count the massless vector multiplets and hypermultiplets in the sector $D$, while

$$\mathcal{I}_C = n_{V_D} - n_{H_D} \quad (4.35)$$

is the index arising from the second non-aligned extended supersymmetry we will denote $\mathcal{N}_D = 2$. Of course, $n_{V_D}$ is nothing but the dimension of the gauge group $G_D$ realized in this sector. As before, the $|\theta^{[0]}_{[1]}(T)|^4$-term in $\Delta_D$ can be omitted and the contribution to the cosmological term vanishes: $V_{\text{eff}, D} = 0$.

### 4.5 $E \& F$: The $\mathcal{N}_{C,D} = 2 \to \mathcal{N}_{C,D} = 0$ sectors $hG - gH \neq 0$

The previous sectors $A, B, C, D$ have $(H, G)$ or $(h, g)$ equal to $(0, 0)$, or $(H, G) = (h, g)$. All these conditions are equivalent to saying that the determinant $|h^H_g G|$ vanishes. In the
remaining sectors, namely $E$ and $F$, one has $|\frac{h}{g} H| \neq 0$, which implies not only that $(H,G) \neq (0,0)$, but also that $(h,g) \neq (0,0)$ and $(h,g) \neq (H,G)$. In other words, the supersymmetries $\mathcal{N}_C = 2$ of sector $C$ and $\mathcal{N}_D = 2$ of sector $D$ are both broken to $\mathcal{N}_C = 0$ and $\mathcal{N}_D = 0$. Indeed, one finds that in the partition function (4.6), the left-moving part (including the four twisted left-moving internal coordinates) is not vanishing and universal, modulo the $\Gamma_{2,2}$ shifted lattice. We display below the partition function in case $a$), for the $\mathcal{N}_C = 0$ sector $E$,

$$
\frac{1}{2} \left( Z_{[1,0]} + Z_{[1,1]} + Z_{[0,1]} \right)_{\Gamma_{2,2}} = -8 \frac{\theta_3^2(v) \theta_3^2(v)}{\eta^6 \theta_2^2} \Gamma_{2,2} \left[ \zeta,1 \right] \bar{Z}_{[0]} \\
-8 \frac{\theta_3^2(v) \theta_3^2(v)}{\eta^6 \theta_4^2} \Gamma_{2,2} \left[ 1-\zeta,0 \right] \bar{Z}_{[1]} \\
+8 \frac{\theta_3^2(v) \theta_3^2(v)}{\eta^6 \theta_3^2} \Gamma_{2,2} \left[ \zeta,1 \right] \bar{Z}_{[0]} \right), \quad (4.36)
$$

and for the $\mathcal{N}_D = 0$ sector $F$,

$$
\frac{1}{2} \left( Z_{[1,0]} + Z_{[1,1]} + Z_{[0,1]} \right)_{\Gamma_{2,2}} = +8 \frac{\theta_3^2(v) \theta_3^2(v)}{\eta^6 \theta_2^2} \Gamma_{2,2} \left[ 1-\zeta,0 \right] \bar{Z}_{[0]} \\
+8 \frac{\theta_3^2(v) \theta_3^2(v)}{\eta^6 \theta_4^2} \Gamma_{2,2} \left[ \zeta,1 \right] \bar{Z}_{[1]} \\
-8 \frac{\theta_3^2(v) \theta_3^2(v)}{\eta^6 \theta_3^2} \Gamma_{2,2} \left[ 1-\zeta,1 \right] \bar{Z}_{[0]} \right). \quad (4.37)
$$

In these expressions, the $Z$-factors are purely antiholomorphic. The partition functions in case $b$) are obtained from the above ones by exchanging the columns of the $\Gamma_{2,2}$-lattices. In case $c$), the first columns of the $\Gamma_{2,2}$-lattices are as above, while the second columns are obtained by changing $\zeta \rightarrow 1 - \zeta$ in the first ones.

The key point here is that due to the fact that once $|\frac{h}{g} H| \neq 0$, it is forbidden to have $h = H = 0$ in the sectors $E$ and $F$. Therefore, all individual terms in the associated partition functions are coupled with exponentially suppressed shifted lattices (see Eq. (4.5)), when $\text{Im} T$ is large and $U$ finite. This shows explicitly that in sectors $E$ and $F$, the contributions to the cosmological term and coupling constants can be neglected,

$$
\Delta_{E,F} = \mathcal{O} \left( e^{-c \sqrt{\text{Im} T}} \right), \quad Y_{E,F} = \mathcal{O} \left( e^{-c \sqrt{\text{Im} T}} \right), \quad V_{\text{eff},E,F} = \mathcal{O} \left( e^{-c \sqrt{\text{Im} T}} \right). \quad (4.38)
$$
5 Analysis of the $\mathcal{N} = 4$ and 1$^{\text{st}}$ plane contributions

Before investigating the second and third plane contributions in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ models, we would like to comment further on the structure of the corrections coming from the $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2 \rightarrow \mathcal{N} = 0$ susy breaking associated to the sectors $A$ to $F$. Some explicit examples will also be given. Let us start by collecting the results found in the previous section:

- In sector $A$, the contributions to the effective potential, $V_{\text{eff}}^A$, and to the gauge thresholds, $\Delta^i_A$, are always zero due to the “mother” $\mathcal{N} = 4$ theory.

- There are two non-aligned $\mathcal{N}_C = 2$ and $\mathcal{N}_D = 2$ “daughter” supersymmetries in the sectors $C$ and $D$. In the former, the first two gravitini are massless, while in the latter the third and fourth gravitini are massless. Gauge coupling corrections $\Delta^i_{C,D}$ occur, while there are no contributions to the effective potential, $V_{\text{eff}}^{C,D} = 0$.

- The sectors $E$ and $F$ are not supersymmetric and correspond to the breaking $\mathcal{N}_{C,D} = 2 \rightarrow \mathcal{N}_{C,D} = 0$. However, their contributions $V_{\text{eff}}^{E,F}$ and $\Delta^i_{E,F}$ are exponentially suppressed, when Im $T$ is large and $U$ finite.

- The contributions $V_{\text{eff}}^B$ and $\Delta^i_B$ of the sector $B$ are the only ones arising from a non-supersymmetric sector. The latter realizes a spontaneous breaking of $\mathcal{N} = 4$ to $\mathcal{N} = 0$. Moreover, the sector $B$ is the only one that gives a non-vanishing (or non-negligible) cosmological term, which is proportional to $m_B^4 \equiv m_B^4 \propto 1/(\text{Im } T)^2$.

- The non-trivial contributions to the gauge thresholds arise from the sectors $B$, $C$ and $D$. For any model $a)$, $b)$ or $c)$, with $\zeta = 0$ or 1, $(\alpha_B, \beta_B)$, $(\alpha_C, \beta_C)$ and $(\alpha_D, \beta_D)$ take distinct values among the set $\{(1, 0), (0, 1), (1, 1)\}$. In fact, the 6 models realize the 3! allowed permutations of these parameters.

The contributions of the sectors $A$ to $F$ are what is required to write the corrections to the gauge coupling constants and to the cosmological term in a generic $\mathbb{Z}_2$ orbifold model, realizing an $\mathcal{N} = 2 \rightarrow \mathcal{N} = 0$ spontaneous susy breaking à la SSS. The running gauge couplings can be expressed in terms of the redefined infrared regulator $Q^2 = \mu^2 \pi^2 / 4$, and are
valid for \( Q < m_B, m_C, m_D < cM_s \), where \( c \) is defined below Eq. (4.18). They take the form

\[
\frac{16 \pi^2}{g_s^2(Q)} = k_i \frac{16 \pi^2}{g_s^2} - \frac{1}{2} \left( b_i^B + b_i^C + b_i^D \right) \log \frac{Q^2}{M_s^2} - \frac{1}{2} \frac{b_i^B}{M_B^2} \log \left( \left| \theta^{[1-\beta_B]}_{[1-\alpha_B]}(U) \right|^4 \text{Im} T \text{Im} U \right) - \frac{1}{2} \frac{b_i^C}{M_C^2} \log \left( \left| \theta^{[1-\beta_C]}_{[1-\alpha_C]}(U) \right|^4 \text{Im} T \text{Im} U \right) - \frac{1}{2} \frac{b_i^D}{M_D^2} \log \left( \left| \theta^{[1-\beta_D]}_{[1-\alpha_D]}(U) \right|^4 \text{Im} T \text{Im} U \right) + \mathcal{O} \left( \frac{1}{\text{Im} T} \right),
\]

while the effective potential is

\[
V_{\text{eff}} = \frac{1}{2} V_{\text{eff}B} + \mathcal{O} \left( e^{-c \sqrt{\text{Im} T}} \right) = -\frac{1}{2} + \frac{d_G - n_{EB}}{2 \pi^2} \frac{1}{(\text{Im} T)^2} E_{\alpha_B, \beta_B}(U, 3) + \mathcal{O} \left( e^{-c \sqrt{\text{Im} T}} \right). \tag{5.2}
\]

The \( \frac{1}{2} \) factors in front of the \( \beta \)-function coefficients and in the expression of the potential come from the normalization arising from the \( \mathbb{Z}_2 \) orbifold projection. The gravitino mass \( m_3 \) of the \( \mathcal{N} = 2 \to \mathcal{N} = 0 \) model being equal to that of sector \( B \),

\[
m_3 \equiv m_B = \frac{(\alpha_B \text{Im} U)^2 + (\alpha_B |\text{Re} U| - \beta_B)^2}{\sqrt{\text{Im} T \text{Im} U}} M_s, \tag{5.3}
\]

the cosmological term is proportional to \( m_3^4 \). Note that no correction of order \( M_0^2 m_3^2 \) occurs.

In order to make the physical interpretation of the gauge coupling threshold corrections more transparent, it is convenient to introduce moduli-dependent mass scales,

\[
\frac{1}{M_B^2} = \frac{1}{M_s^2} \left| \theta^{[1-\beta_B]}_{[1-\alpha_B]}(U) \right|^4 \text{Im} T \text{Im} U, \tag{5.4}
\]

\[
\frac{1}{M_C^2} = \frac{1}{M_s^2} \left| \theta^{[1-\beta_C]}_{[1-\alpha_C]}(U) \right|^4 \text{Im} T \text{Im} U, \tag{5.4}
\]

\[
\frac{1}{M_D^2} = \frac{1}{M_s^2} \left| \theta^{[1-\beta_D]}_{[1-\alpha_D]}(U) \right|^4 \text{Im} T \text{Im} U,
\]

in terms of which the coupling constant corrections for \( Q < M_B, M_C, M_D \) take the form

\[
\frac{16 \pi^2}{g_s^2(Q)} = k_i \frac{16 \pi^2}{g_s^2} - \frac{1}{2} b_i^B \log \frac{Q^2}{M_B^2} - \frac{1}{2} b_i^C \log \frac{Q^2}{M_C^2} - \frac{1}{2} b_i^D \log \frac{Q^2}{M_D^2} + \mathcal{O} \left( \frac{1}{\text{Im} T} \right). \tag{5.5}
\]

As we are going to see, the behavior of these thresholds depends crucially on the complex structure \( U \). In particular, the hierarchy between the moduli-dependent scales \( M_B, M_C, M_D \) depends only on \( U \). To further investigate the qualitative features of the \( U \)-dependance, we can focus on the particular susy breaking pattern of model \( a \), with \( \zeta = 0 \), keeping in mind
that the gauge coupling thresholds in all six cases $a), b), c)$, with $\zeta = 0, 1$, are obtained by permutation of the defining expressions of the threshold scales $M_{B,C,D}$. In this case, the shifted lattice involved in the threshold corrections is $\Gamma_{2,2}^{0,0}_{[g,G]}$ and the susy breaking scales in sectors $B, C, D$ are

$$m_B = \frac{|U|}{\sqrt{\text{Im} T \text{Im} U}} M_s, \quad m_C = \frac{1}{\sqrt{\text{Im} T \text{Im} U}} M_s, \quad m_D = \frac{(\text{Im} U)^2 + (1 - |\text{Re} U|)^2}{\sqrt{\text{Im} T \text{Im} U}} M_s. \quad (5.6)$$

This shows that the scale at which $\mathcal{N} = 4$ is spontaneously broken to $\mathcal{N} = 2$ is $m_C$, since $(h, G) = (0, 1)$ is the value taken by $(\alpha_C, \beta_C)$. Similarly, the scale at which supersymmetry is spontaneously broken to $\mathcal{N} = 0$ is $m_B$, since $(h, G) = (1, 0)$ is the value taken by $(\alpha_B, \beta_B)$. These two scales are relatively small compared to $M_s$, as is also the third one, $m_D$, which emerges for $(g, G) = (1, 1) = (\alpha_D, \beta_D)$.

To proceed, we specialize further to the situation where $\text{Re} U = 0$ and define

$$t = \text{Im} T = R_1 R_2, \quad u = \text{Im} U = \frac{R_2}{R_1}, \quad (5.7)$$

where $R_1$ and $R_2$ are the radii of the shifted squared untwisted internal 2-torus. The susy breaking scales become

$$m_B^2 = \frac{u}{t} M_s^2, \quad m_C^2 = \frac{1}{tu} M_s^2, \quad m_D^2 = m_B^2 + m_C^2, \quad (5.8)$$

which implies $m_D$ is the largest one. The moduli-dependent scales $M_B$ and $M_C$ become

$$\frac{1}{M_B^2} = \frac{1}{M_s^2} |\theta_2(iu)|^4 tu = \frac{1}{m_B^2} |\theta_4(iu)|^4, \quad \frac{1}{M_C^2} = \frac{1}{M_s^2} |\theta_4(iu)|^4 tu = \frac{1}{m_C^2} |\theta_4(iu)|^4. \quad (5.9)$$

Utilizing the identity $|\theta_3(iu)|^4 = |\theta_2(iu)|^4 + |\theta_4(iu)|^4$, which is valid for pure imaginary arguments, we obtain the moduli-dependent threshold scale related to the $\mathcal{N}_D = 2$ supersymmetric sector $D$ as a function of $M_B$ and $M_C$,

$$\frac{1}{M_D^2} = \frac{1}{M_s^2} |\theta_3(iu)|^4 tu = \frac{1}{M_B^2} + \frac{1}{M_C^2}. \quad (5.10)$$

This shows that in the present case, $M_D$ is the lowest threshold scale. This example is illuminating. It shows that the scales at which supersymmetry is restored in the sectors $B, C, D$ are not the associated gravitini masses $m_{B,C,D}$. Instead, the relevant scales for
supersymmetry restoration are the full threshold scales $M_{B,C,D}$, whose hierarchy differs from
that of the scales $m_{B,C,D}$. For instance, since

\[ M_B^2 \sim m_B^2, \quad M_C^2 \sim \frac{m_B^2}{16} e^{\pi/u}, \quad M_D^2 \sim m_B^2, \quad \text{when } u \ll 1, \quad (5.11) \]

the full hierarchy of the threshold scales for small enough $u$ is $Q < M_D \leq M_B \leq M_C$, while we have $m_B < m_C \leq m_D$. Moreover, in the limit where $u$ is very small, the scale $M_C$ grows exponentially, which gives large corrections to the gauge couplings in Eq. (5.5), proportional to $1/u = R_1/R_2$. On the contrary, since

\[ M_B^2 \sim \frac{m_C^2}{16} e^{\pi u}, \quad M_C^2 \sim m_C^2, \quad M_D^2 \sim m_C^2, \quad \text{when } u \gg 1, \quad (5.12) \]

the hierarchy of the threshold scales for large enough $u$ is $Q < M_D \leq M_C \leq M_B$, while $m_C < m_B \leq m_D$. Furthermore, when $u$ is very large, the scale $M_B$, which grows exponentially with $u$, gives rise to large corrections to the couplings in Eq. (5.5), proportional to $u = R_2/R_1$. In the end, in both extreme limits summarized by the condition $u + 1/u \gg 1$, large linear corrections can destroy the string perturbative expansion, when dressing $\beta$-function coefficients are negative. In such cases, one must assume that $u$ is not too small or large.

In our low energy description, the range of permitted ratios $u = R_2/R_1$ can be derived by the requirement that the higher threshold scale must be smaller than the scale of the massive states we neglected i.e. $cM_s$. In general, the lowest threshold scale among $M_B$, $M_C$ and $M_D$ in Eq. (5.4) is the one that contains $\theta_3(U)$ in its definition. As we have just shown, this scale has a simple relation with the highest threshold scale in the extreme limits $u \gg 1$ or $u \ll 1$. The validity constraint in these two limits becomes

\[ \frac{1}{16} e^{\pi(u+1/u)} = \frac{M_{\text{high}}^2}{M_{\text{low}}^2} < \frac{e^2}{2} \frac{M_{\text{Planck}}^2}{M_{\text{low}}^2}, \quad (5.13) \]

where $M_{\text{low}}^{(E)}$ is the lowest scale measured in the Einstein frame. Notice that the ratio $M_{\text{high}}/M_{\text{low}}$ is independent of the frame. This gives the condition:

\[ u + \frac{1}{u} < \frac{2}{\pi} \log \left( 4 e \frac{M_{\text{Planck}}}{M_{\text{low}}^{(E)}} \right). \quad (5.14) \]

Assuming the lowest supersymmetry breaking scale measured in Einstein frame to be in the 1–10 TeV region, we take $M_{\text{low}}^{(E)} = \mathcal{O}(10^4)$ GeV, and given the gravity scale $M_{\text{Planck}} = 2.4 \cdot 10^{18}$ GeV, one finds for $c = \mathcal{O}(1)$ the permitted values of $u$:

\[ u + \frac{1}{u} < 22. \quad (5.15) \]
Once $u$ is in this region, we can write the following interpolating expression for the running gauge couplings, in terms of the physical energy scale measured in string frame, $Q < cM_s$ (or $Q^{(E)} = Q/g_s < cM_{\text{Planck}}$ in the Einstein frame). It is valid for all supersymmetry breaking patterns i.e. models $a)$, $b)$ or $c)$, with $\zeta = 0$ or $1$, and independently of the $U$-dependent hierarchy among the threshold scales $M_B, M_C$ and $M_D$:

$$
\frac{16 \pi^2}{g^2_i(Q)} = k^i \frac{16 \pi^2}{g^2_s} - \frac{1}{2} b^i_B \log \left( \frac{Q^2}{Q^2 + M_B^2} \right) - \frac{1}{2} b^i_C \log \left( \frac{Q^2}{Q^2 + M_C^2} \right) - \frac{1}{2} b^i_D \log \left( \frac{Q^2}{Q^2 + M_D^2} \right).
$$

The above expression implements the successive decouplings of the effective threshold mass scales $M_{B,C,D}$, which occur when the infrared cut-off scale $Q$ crosses them. $Q$ plays the role of a scattering energy scale. For $Q$ smaller than the three threshold scales, it can be neglected compared to them and one recovers the threshold formula for small $Q$, Eq. (5.5). Once $Q$ becomes larger than one of the threshold scales, the latter can be neglected compared to $Q$, which is consistent with the fact that the whole tower of associated thresholds give negligible contribution. In particular:

- In the cases where the susy breaking pattern and the complex structure $U$ imply $M_B$ to be the lowest threshold scale, when the physical scale satisfies $M_B < Q < M_C, M_D$, the two non-aligned $\mathcal{N}_C = 2$ and $\mathcal{N}_D = 2$ supersymmetries are restored. The full $\mathcal{N} = 4$ supersymmetry is recovered when $Q$ is above $M_C$ and $M_D$.

- In the cases where the susy breaking pattern and the complex structure $U$ imply $M_B$ to be the highest threshold scale, then the model describes a total $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$ spontaneous susy breaking, when the physical scale satisfies $M_C, M_D < Q < M_B$. When $Q > M_B$, the full $\mathcal{N} = 4$ supersymmetry is restored.

### 5.1 Example 1: Gauge group factor $E_8$

Before analyzing the contributions of the 2nd and 3rd planes in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ models, we would like to present typical examples in the $\mathbb{Z}_2$ case i.e. where $\mathcal{N} = 4$ supersymmetry is spontaneously broken to $\mathcal{N} = 2$ and further broken to $\mathcal{N} = 0$. In fact, the $\beta$-function coefficients we are going to focus on can either be deduced by computing those associated to the sectors $B, C$ and $D$, or directly by considering the massless spectrum of the $\mathcal{N} = 0$ theory.
In our first example, we consider the models whose gauge groups contain a factor $G^i = E_8$. The associated affine character in the adjoint representation, $\bar{E}_8(\bar{\tau})$, is realized by 16 right-moving Majorana-Weyl worldsheet fermions,

$$\bar{E}_8(\bar{\tau}) = \frac{1}{2} \sum_{\gamma, \delta} \bar{\theta}_{\gamma}^{\bar{i}} \bar{\theta}_{\delta}^{\bar{i}}.$$

(5.17)

The latter is factorized in the right-moving part of the partition function, whose relevant conformal block takes the form

$$Z_{4,20}[^{[\ell_H]}_{[\ell_G]}] = Z_{4,12}[^{[\ell_H]}_{[\ell_G]}] \bar{E}_8.$$

(5.18)

The adjoint character $\bar{E}_8$ can be written as the sum of $\bar{O}^{16}$ and $\bar{S}^{16}$, the characters associated to the adjoint and spinorial representations of $SO(16)$:

$$\bar{E}_8 = \bar{O}^{16} + \bar{S}^{16}, \quad \bar{O}^{16} = \frac{\bar{\theta}_2^8 + \bar{\theta}_4^8}{2\bar{\eta}^8}, \quad \bar{S}^{16} = \frac{\bar{\theta}_2^8 + \bar{\theta}_4^8}{2\bar{\eta}^8}.$$

(5.19)

Since the character $\bar{E}_8$ is factorized, the gauge groups realized in the sectors $B, C, D$ contain a common factor, $G^i_B = G^i_C = G^i_D = E_8$. In sector $B$, the $\beta$-function coefficient arises from the bosonic part of an $N = 4$ vector multiplet (1 gauge boson + 6 real scalars) in the adjoint of $G^i_B$. In sectors $C$ and $D$, the $\beta$-function coefficients correspond to $N_C = 2$ and $N_D = 2$ vector multiplets in the adjoint of $G^i_C$ and $G^i_D$. Thus, we have

$$b^i_B = -\frac{8}{3} C(E_8), \quad b^i_C = -2 C(E_8), \quad b^i_D = -2 C(E_8),$$

(5.20)

where $C(E_8) = 14 + 16 = 30$. The contribution 14 in $C(E_8)$ comes from the adjoint of $SO(16)$, $C(O^{16}) = 14$, while the contribution 16 comes from the spinorial of $SO(16)$, $C(S^{16}) = 16$. Thus, the sector by sector analysis leads to a $\beta$-function coefficient in the $N = 0$ theory given by

$$b^i = \frac{1}{2} (b^i_B + b^i_C + b^i_D) = -\frac{10}{3} C(E_8) = -100,$$

(5.21)

which shows that the gauge theory is asymptotically free.

To cross check this value, we can directly compute $b^i$ from the point of view of an $N = 2 \to N = 0$ spontaneously broken theory. The massless spectrum contains the bosonic part of an $N = 2$ vector multiplet in the adjoint representation of $G^i$, namely 1 gauge boson for 2 real scalars, while the gauginos have become massive:

$$b^i = \left(-\frac{11}{3} + \frac{2}{6}\right) C(E_8) = -\frac{10}{3} C(E_8).$$

(5.22)
5.2 Example 2: Gauge group factor $SO(16)$

The second models we would like to present have a gauge group factor $G^i = SO(16)$. The latter is obtained by coupling non-trivially the lattice shift $(h, g)$, with the $SO(16)$ spinorial representation initially present in the character $\bar{E}_8$. The coupling is implemented by a phase as follows:

$$Z_{4.20}[h|G] = Z_{4.12}[h|G]Z_{0.8}[h] \quad \text{where} \quad Z_{0.8}[h] = \frac{1}{2} \sum_{\gamma, \delta} \bar{\theta}^{[\gamma]}{\delta}^{[\bar{h}]}e^{i\pi(g\gamma + h\delta + hg)} ,$$

which breaks simultaneously $E_8 \rightarrow SO(16)$ and supersymmetry to $\mathcal{N} = 0$.

The SSS phase changes effectively to

$$S = e^{i\pi[g(a+\gamma)+h(b+\delta)]} .$$

This shows clearly that in the sector $B$, the fermions of the initially massless $\mathcal{N} = 4$ vector multiplets in the $\bar{O}^{16}$ representation (i.e. for $\gamma = 0$) become massive, while the bosons remain massless. However, compared to Example 1, the new thing is that the situation is reversed for the states in the $\bar{S}^{16}$ representation (i.e. for $\gamma = 1$): The bosons of the originally massless $\mathcal{N} = 4$ vector multiplets become massive, while the fermions remain massless. In total, the gauge group factor in the non-supersymmetric sector $B$ is $G^i_B = SO(16)$ and the $\beta$-function coefficient is

$$b^i_B = -\frac{8}{3} \{ C(O^{16}) - C(S^{16}) \} .$$

Notice that since $(h, g) = (0, 0)$ in sector $C$, the gauge group factor $G^i_C = E_8$ is unbroken and the associated $\mathcal{N}_C = 2$ supersymmetric $\beta$-function coefficient is identical to that of Example 1,

$$b^i_C = -2C(E_8) \equiv -2 \{ C(O^{16}) + C(S^{16}) \} .$$

However, in sector $D$, where $(h, g) \neq (0, 0)$, the $E_8$ gauge group is broken to $G^i_D = SO(16)$, with massless hypermultiplets in the spinorial representation $\bar{S}^{16}$. The $\mathcal{N}_D = 2$ supersymmetric $\beta$-function coefficient is thus

$$b^i_D = -2 \{ C(O^{16}) - C(S^{16}) \} .$$

Taking into account the above sector by sector contributions, the $\beta$-function coefficient of the $G^i = SO(16)$ non-supersymmetric gauge theory is

$$b^i = \frac{1}{2}(b_B + b_C + b_D) = -\frac{10}{3} C(O^{16}) + \frac{4}{3} C(S^{16}) = \frac{76}{3} .$$
Even if in this example the gauge theory is non-asymptotically free, it remains a good exercise that illustrates the sector by sector analysis of the gauge threshold corrections.

Here also, the agreement with the direct evaluation of the $\beta$-function coefficient of the $\mathcal{N} = 2 \to \mathcal{N} = 0$ theory can be checked. This can be done in two steps. At the $\mathcal{N} = 2$ level obtained by applying the $\mathbb{Z}_2$ action that breaks spontaneously $\mathcal{N} = 4 \to \mathcal{N} = 2$, the massless spectrum contains an $\mathcal{N} = 2$ vector multiplet in the adjoint representation of $G^i = SO(16)$, coupled to a hypermultiplet in the spinorial representation. Applying the final $\mathbb{Z}_2^{shift}$ responsible for the $\mathcal{N} = 2 \to \mathcal{N} = 0$ spontaneous breaking, the massless spectrum charged under the $G^i = SO(16)$ gauge group factor are the bosons of the $\mathcal{N} = 2$ vector multiplet in the adjoint representation of $SO(16)$, together with the fermions of the hypermultiplet in the spinorial representation. Consistently, one finds

$$b^i = \left(-\frac{11}{3} + \frac{2}{6}\right) C(O^{16}) + \frac{4}{3} C(S^{16}) = -\frac{10}{3} C(O^{16}) + \frac{4}{3} C(S^{16}).$$

(5.29)

### 5.3 Example 3: Gauge group factor $SO(8) \times SO(8)$

The third example we would like to present has a $G^i = SO(8) \times SO(8)$ gauge subgroup. It is obtained by coupling non-trivially both $(g, h)$ and $(G, H)$, with the vectorial and spinorial representations of $SO(8) \times SO(8)$ initially present in the $E_8$ character:

$$Z_{4,20}^{|g|H,G} \cdot Z_{4,12}^{|g|H,G} \cdot Z_{0,8}^{|h|H,G} \cdot Z_0^{|g|H,G} = \frac{1}{2} \sum_{\gamma, \delta} \tilde{g}^{4[\gamma]} \tilde{g}^{4[\gamma + H]} \tilde{g}^{4[\gamma + H]} \tilde{g}^{4[\gamma + H]} \tilde{g}^{4[\gamma + H]} e^{i\pi(g\gamma + h\delta + gh + GH)}.$$

(5.30)

As in Example 2, the coupling to $(h, g)$ breaks $E_8 \to SO(16)$ and supersymmetry to $\mathcal{N} = 0$, while the coupling to $(H, G)$ breaks further $SO(16) \to SO(8) \times SO(8)$. Here also, the SSS phase is effectively

$$S = e^{i\pi g(\gamma + \delta + h\delta + h\gamma + b \delta + b \gamma)}.$$

(5.31)

Since $(H, G) = (0, 0)$ in sector $B$, the latter is identical to that of Example 2. Therefore, we have $G^i_B = SO(16)$, with $\beta$-function coefficient

$$b^i_B = -\frac{8}{3} \left\{C(O^{16}) - C(S^{16})\right\}.$$

(5.32)

However, since the overall gauge group factor of the model is $G^i = SO(8) \times SO(8)$, it is
instructive to express the characters of $G_B^i = SO(16)$ in terms of those of $SO(8) \times SO(8)$:

$$\bar{O}^{16} = \bar{O}^8 \bar{O}^8 + \bar{V}^8 \bar{V}^8, \quad \bar{S}^{16} = \bar{S}^8 \bar{S}^8 + \bar{C}^8 \bar{C}^8.$$  

(5.33)

Thus, the bosons of the initially massless $\mathcal{N} = 4$ vector multiplets in the $\bar{O}^{16}$ representation (i.e. for $\gamma = 0$) are in the adjoint representation $(28, 1) \oplus (1, 28)$ as well as in the bi-vectorial $(8_v, 8_v)$ of $SO(8) \times SO(8)$. Moreover, the fermions of the initially massless $\mathcal{N} = 4$ vector multiplets in the $\bar{S}^{16}$ representation (i.e. for $\gamma = 1$) are in the $(8_s, 8_s)$ and $(8_c, 8_c)$ bi-spinorial representations of $SO(8) \times SO(8)$.

As said before, the model can be constructed by successive breakings,

$$E_8 \rightarrow SO(16) \rightarrow SO(8) \times SO(8),$$  

(5.34)

by first coupling the $SO(8) \times SO(8)$ characters initially present in $\bar{E}_8$,

$$\bar{E}_8 = \bar{O}^{16} + \bar{S}^{16} = \bar{O}^8 \bar{O}^8 + \bar{V}^8 \bar{V}^8 + \bar{S}^8 \bar{S}^8 + \bar{C}^8 \bar{C}^8,$$

(5.35)

with $(h, g)$, and then with $(H, G)$. In the intermediate step, which is nothing but the sector $B$, the $G_B^i = SO(16)$ gauge theory is non-supersymmetric. However, the analysis of the sectors $C$ and $D$ is more conveniently done by considering the model from two other viewpoints:

- The breaking (5.34) can be realized by first coupling the $SO(8) \times SO(8)$ characters with $(H, G)$, and then with $(h, g)$. In the intermediate step, which is nothing but the sector $C$, we have an $\mathcal{N}_C = 2$ supersymmetric $G_C^i = SO(16)$ gauge theory.

- The breaking (5.34) can also be realized by first coupling the $SO(8) \times SO(8)$ characters with $(h, g) = (H, G)$, and then with $(h - H, g - G)$. In the intermediate step, which is nothing but the sector $D$, we have an $\mathcal{N}_D = 2$ supersymmetric $G_D^i = SO(16)$ gauge theory.

Actually, the three intermediate gauge group factors $G_{B,C,D}^i = SO(16)$ are not aligned, so that the resulting unbroken gauge group of the combined final theory is $G^i = SO(8) \times SO(8)$. Correspondingly, thanks to the triality symmetry of the three $SO(8)$ representations $8_v, 8_s, 8_c$, there are three alternative decompositions of the $SO(16)$ characters in terms of $SO(8) \times SO(8)$ ones. If desired, these decompositions can be used to describe the spectra.
in sectors $B, C, D$ in terms of $SO(8) \times SO(8)$ representations. They are

\[
\begin{align*}
\text{in sector } B: \quad & \bar{O}^{16} = \bar{O}^8 \bar{O}^8 + \bar{V}^8 \bar{V}^8, \quad \bar{S}^{16} = \bar{S}^8 \bar{S}^8 + \bar{C}^8 \bar{C}^8, \\
\text{in sector } C: \quad & \bar{O}^{16} = \bar{O}^8 \bar{O}^8 + \bar{S}^8 \bar{S}^8, \quad \bar{S}^{16} = \bar{C}^8 \bar{C}^8 + \bar{V}^8 \bar{V}^8, \\
\text{in sector } D: \quad & \bar{O}^{16} = \bar{O}^8 \bar{O}^8 + \bar{C}^8 \bar{C}^8, \quad \bar{S}^{16} = \bar{V}^8 \bar{V}^8 + \bar{S}^8 \bar{S}^8.
\end{align*}
\] (5.36)

In any case, what we are interested in is the massless spectrum in sector $C$, charged under the gauge group factor $G_C = SO(16)$. To find it, we start from the parent $\mathcal{N} = 4$ theory, where the spectrum amounts to an $\mathcal{N} = 4$ vector multiplet in the representation $\bar{E}_8$. Implementing the $(H, G)$-projection and using the fact that $\bar{E}_8 = \bar{O}^{16} + \bar{S}^{16}$, we obtain the sector $C$, whose massless spectrum lies schematically in the representation

\[
(\mathcal{N}_C = 2 \text{ vector multiplet}) \cdot \bar{O}^{16} \oplus (\mathcal{N}_C = 2 \text{ hypermultiplet}) \cdot \bar{S}^{16}.
\] (5.37)

We have an $\mathcal{N}_C = 2$ vector multiplet in the adjoint representation and a hypermultiplet in the spinorial representation, so that

\[
b_C^i = -2 \left\{ C(O^{16}) - C(S^{16}) \right\}. \quad (5.38)
\]

By symmetry between the sectors $C$ and $D$, we also have in sector $D$ for the gauge group factor $G_D = SO(16)$,

\[
b_D^i = -2 \left\{ C(O^{16}) - C(S^{16}) \right\}. \quad (5.39)
\]

Combining the above results, the $\beta$-function coefficient of the $G^i = SO(8) \times SO(8)$ non-supersymmetric gauge theory is

\[
b^i = \frac{1}{2} (b_B + b_C + b_D) = -\frac{10}{3} \left\{ C(O^{16}) - C(S^{16}) \right\} = \frac{20}{3}. \quad (5.40)
\]

Here also, the gauge theory is non-asymptotically free.

To check the above value of $b^i$, we can derive the massless spectrum of the theory that is charged under $G^i = SO(8) \times SO(8)$. We have just seen that the implementation of the $(H, G)$-projection on the parent $\mathcal{N} = 4$ theory leads to the massless spectrum of sector $C$, given in Eq. (5.37). Using the decomposition of the $SO(16)$ characters in terms of $SO(8) \times SO(8)$ ones valid in sector $C$, this spectrum can be written as

\[
(\mathcal{N}_C = 2 \text{ vector multiplet}) \cdot (\bar{O}^8 \bar{O}^8 + \bar{S}^8 \bar{S}^8) \oplus (\mathcal{N}_C = 2 \text{ hypermultiplet}) \cdot (\bar{C}^8 \bar{C}^8 + \bar{V}^8 \bar{V}^8).
\] (5.41)
We can now implement the final \((h,g)\)-projection, which let us with massless states schematically as follows:

\[
(\text{bosons of the vector multiplet}) \cdot \bar{O}^8 O^8 \oplus (\text{fermions of the vector multiplet}) \cdot S^8 S^8 \oplus \\
(\text{bosons of the hypermultiplet}) \cdot \bar{V}^8 V^8 \oplus (\text{fermions of the hypermultiplet}) \cdot \bar{C}^8 C^8 .
\]

(5.42)

We have 1 gauge boson and 2 real scalars in the adjoint representation of \(SO(8) \times SO(8)\), \((28,1) \oplus (1,28)\), together with 4 real scalars in the \((8_v, 8_v)\), and 4 Majorana fermions in the \((8_s, 8_s) \oplus (8_c, 8_c)\). Since the gauge coupling of \(G\) is equal to that of each of its \(SO(8)\) subgroups, it is sufficient to calculate the \(\beta\)-function coefficient associated to one of them:

\[
\beta = \left( - \frac{11}{3} + \frac{2}{6} \right) C(O^8) + \frac{4n(V^8)}{6} C(V^8) + \frac{4n(S^8)}{3} C(S^8) + \frac{4n(C^8)}{3} C(C^8) ,
\]

where \(C(O^8) = 6\), \(C(V^8) = C(S^8) = C(C^8) \equiv C^8 = 1\) and the multiplicities arising from the second \(SO(8)\) factor are all equal, \(n(V^8) = n(S^8) = n(C^8) = 8\). In total, one has

\[
\beta = - \frac{10}{3} C(O^8) + \frac{80}{3} C^8 = \frac{20}{3} ,
\]

(5.44)

which is in agreement with the sector by sector contributions.

### 5.4 The generic case

The above examples illustrate the universal structure of the running effective gauge couplings valid in the \(\mathbb{Z}_2\) asymmetric orbifold models that realize a spontaneous \(\mathcal{N} = 4 \rightarrow \mathcal{N} = 2 \rightarrow \mathcal{N} = 0\) supersymmetry breaking à la SSS. In these models, no dangerous linear dependance on the internal volume appears in the threshold corrections. The result is given in Eq. (5.5) for \(Q < M_{B,C,D}\) (or Eq. (5.16) for \(Q < cM_s\)), with the sector by sector \(\beta\)-function coefficients given by:

\[
b^i_B = - \frac{8}{3} \{ C(O_B) - C(R_B) \} , \quad b^i_C = -2 \{ C(O_C) - C(R_C) \} , \quad b^i_D = -2 \{ C(O_D) - C(R_D) \} .
\]

(5.45)

The structures of the sectors \(C\) and \(D\) are simple to understand, since both of them describe \(\mathcal{N} = 2\) supersymmetric gauge theories. The associated gauge groups contain factors \(G^i_C\) and \(G^i_D\), which may be different. The individual \(\beta\)-function coefficients are given in terms of vector multiplets contributions in the adjoint representations of \(G^i_{C,D}\), denoted by
−2C(O_{C,D}), together with hypermultiplets contributions in the representations \( \mathcal{R}_{C,D} \), denoted by \( 2C(\mathcal{R}_{C,D}) \).

On the contrary, the structure of sector \( B \), which describes a non-supersymmetric gauge theory with a gauge group factor \( G_B \), is something new. The \( -\frac{8}{3}C(O_B) \) contribution to \( b_B \) comes from the bosons of initially massless \( \mathcal{N} = 4 \) vector multiplets in the parent \( \mathcal{N} = 4 \) model, that remain massless. These bosons (1 vector and 2 real scalars) are in the adjoint representation of \( G_B \). The second contribution, \( \frac{8}{3}C(\mathcal{R}_B) \), arises from the fermions of initially massless \( \mathcal{N} = 4 \) vector multiplets in the parent theory, that remain massless. They are 4 Majorana fermions in a spinorial representation \( \mathcal{R}_B \). If as in Examples 2 and 3, \( \mathcal{R}_B \) is a spinorial representation of \( SO(16) \), it is in general the spinorial representation of a subgroup of \( E_8 \), such as \( SO(16), SO(8) \times SO(8), E_7 \times SU(2), SO(12) \times SO(4) \) or even \( SO(4)^4 \). All these cases can be easily realized by fermionic constructions.

6 2\textsuperscript{nd} plane, 3\textsuperscript{rd} plane and \( \mathcal{N} = 1 \) sector contributions: \((H_2, G_2) \neq (0, 0)\)

In Sects \( 4 \) and \( 5 \) we have extensively analyzed the threshold corrections in \( \mathbb{Z}_2 \) asymmetric orbifold models, where an \( \mathcal{N} = 4 \rightarrow \mathcal{N} = 2 \rightarrow \mathcal{N} = 0 \) spontaneous breaking of supersymmetry is implemented so that the running gauge couplings develop only logarithmic dependancies on the volume of the untwisted internal 2-torus. Up to an additional overall factor of \( \frac{1}{2} \), these results are the contributions of the \( \mathcal{N} = 4 \) and 1\textsuperscript{st} complex plane in \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold models. In the present section, we proceed with the evaluations of the contributions arising from the remaining sectors, namely the 2\textsuperscript{nd} and 3\textsuperscript{rd} complex planes, and the \( \mathcal{N} = 1 \) sector. All of them are twisted, with \( (H_2, G_2) \neq (0, 0) \). Moreover, the 2\textsuperscript{nd} plane has \( (H_1, G_1) = (0, 0) \), the 3\textsuperscript{rd} plane has \( (H_1, G_1) = (H_2, G_2) \) and the \( \mathcal{N} = 1 \) sector has \( (H_1, G_1) \neq (0, 0), (H_1, G_1) \neq (H_2, G_2) \).

If in all complex planes the \( \mathcal{N} = 2 \) supersymmetries appear as a spontaneously broken version of \( \mathcal{N} = 4 \), then the resulting \( \mathcal{N} = 1 \) model (further spontaneously broken to \( \mathcal{N} = 0 \) à la SSS) will not contain a net number of chiral families. Indeed, in the \( \mathcal{N} = 1 \) supersymmetric models constructed \textit{via} \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) asymmetric orbifolds, the chiral families always come from the \( \mathcal{N} = 2 \) twisted sectors that have non-trivial fixed points. Therefore, in order to make
contact with realistic models, we must have at least one plane in which there are non-trivial fixed points.

However, an even stronger restriction on the models comes from the fact that the net number of families per complex plane can either be 0 or \(2^n\), where \(n = 0, 1, 2, 3\) or 4. Thus, if we request the total number of families to be 3, given the fact that the 1st plane is non-chiral, the two other planes must provide 1 and 2 chiral families, respectively. In particular, this means that we must restrict ourselves to \(\mathcal{N} = 1\) models, where only one of three complex planes (say the 1st) realize an \(\mathcal{N} = 4 \rightarrow \mathcal{N} = 2\) spontaneous breaking of supersymmetry.

Another important point in our considerations is that in the 2nd and 3rd complex planes, as well as in the \(\mathcal{N} = 1\) sector, the internal shifted 2-torus associated to the 1st plane is twisted, with \((H_2, G_2) \neq (0, 0)\). Actually, its bosonic contribution to the partition function (2.8) is

\[
Z_{2,2} \left[ \begin{array}{c} h_1^1, h_2^1 \\ g_1^1, g_2^1 \end{array} \right] = \frac{\Gamma_{2,2} \left[ \begin{array}{c} h_1^1, h_2^1 \\ g_1^1, g_2^1 \end{array} \right] (T_1, U_1)}{(\eta \bar{\eta})^2}, \quad \text{when } (H_2, G_2) = (0, 0),
\]

\[
\frac{\eta \bar{\eta}}{\theta(1-H_2^1) \theta(1-H_2^2)} \delta_{0, h_1^1 G_2 - g_1^1} \delta_{0, h_2^2 G_2 - g_2^2}, \quad \text{when } (H_2, G_2) \neq (0, 0),
\]

(6.1)

where the shifts \((h_1^i, g_1^i)\) are defined in Eq. (4.5). Thus, the 2nd plane, 3rd plane and \(\mathcal{N} = 1\) sectors are independent of the moduli \(T_I, U_I\) i.e. independent of the gravitino masse \(m_3^{(E)} \equiv m_3/g_s\), which is in the desired 1–10 TeV region. We now proceed by arguing that the SSS phase in the sectors \((H_2, G_2) \neq (0, 0)\) must not break supersymmetry to \(\mathcal{N} = 0\). The reason for this comes in three steps. First, in the 2nd and 3rd planes, the sub-sectors with \((h, g) = (0, 0)\) always preserve \(\mathcal{N} = 2\) supersymmetry, and since they arise from non-free \(\mathbb{Z}_2\) actions, the order of magnitude of the moduli \(T_I, U_I, I = 2, 3\), must be close enough to 1 for the decompactification problem not to arise. Second, if the \((h, g) \neq (0, 0)\) sub-sectors of the 2nd and 3rd planes were non-supersymmetric, the respective gravitini mass scales would be determined by \(T_I, U_I, I = 2, 3\), and thus of order \(M_{\text{Planck}}\), when measured in Einstein frame, which is something we want to exclude. Third, the \((h, g) \neq (0, 0)\) sub-sector of the \(\mathcal{N} = 1\) sector must preserve supersymmetry as well, in order to not lead to an extremely large gravitino mass. To summarize, in our solution to the decompactification problem, the SSS phase \(S\) in the sectors \((H_2, G_2) \neq (0, 0)\) must not contain the factor \(e^{i\pi(a g + b h + h g)}\) introduced
in Eq. (4.4), which would otherwise break $\mathcal{N} = 0$. Note that since the sectors $(H_2, G_2) = (0, 0)$ and $(H_2, G_2) \neq (0, 0)$ are independent orbits of the worldsheet modular group, the associated choices of SSS phases do not need to be correlated to guarantee the consistency of the whole $\mathbb{Z}_2 \times \mathbb{Z}_2$ model. In the sectors $(H_2, G_2) \neq (0, 0)$, a certainly valid susy preserving choice is $S \equiv 1$. However, playing with the quantum numbers $(H_1, G_1)$ and $(H_2, G_2)$, we can have

$$S = e^{i\pi [\zeta_1(aG_1+bH_1)+\zeta_2(aG_2+bH_2+H_2G_2)]},$$

where $\zeta_1$ and $\zeta_2$ can be fixed to 0 or 1. As we just noticed, $\zeta_1$ may not be equal to $\zeta'$ we introduced in Eq. (4.4). To see that $(\zeta_1, \zeta_2) = (0, 0)$ is not the only allowed choice, we consider the conformal block associated to the left-moving fermionic degrees of freedom,

$$\frac{1}{2} \sum_{a,b} e^{i\pi(a+b+ab)} e^{i\pi[\zeta_1(aG_1+bH_1+H_1G_1)+\zeta_2(aG_2+bH_2+H_2G_2)]} \theta_{[a]}^2(2\nu) \theta_{[b+G_2]}^2 \theta_{[b+G_1]} \theta_{[a-H_1-H_2]}^2$$

$$= e^{i\pi(\zeta_1+\zeta_2)(H_1G_2-G_1H_2)} e^{i\pi(G_1+G_2)(1+H_1+H_2)} \theta_{[1]}^2(v) \theta_{[1-G_2]}^2(v) \theta_{[1-G_1]}^2(v) \theta_{[1+H_1+H_2]}^2(v).$$

To show this equality, one can redefine $a = A - \zeta_1H_1 - \zeta_2H_2$, $b = B - \zeta_1G_1 - \zeta_2G_2$ and sum over $A, B$ equal to 0 or 1. Given that $(H_2, G_2) \neq (0, 0)$, we see that $\mathcal{N} = 2$ supersymmetry is preserved in the 2nd plane, $(H_1, G_1) = (0, 0)$, and in the 3rd plane, $(H_1, G_1) = (H_2, G_2)$. Supersymmetry is also preserved in the $\mathcal{N} = 1$ sector, $|H_1H_2| \neq 0$. Two distinct cases arise however, $\zeta_1 = \zeta_2$ or $\zeta_1 = 1 - \zeta_2$, corresponding to different choices of discrete torsions that yield opposite contributions of the $\mathcal{N} = 1$ sector to the partition function.

The $\mathcal{N}_I = 2$, $I = 2, 3$, unbroken supersymmetries of the 2nd and 3rd planes are not aligned to one another, as well as non aligned with the $\mathcal{N}_C = 2$ and $\mathcal{N}_D = 2$ supersymmetries appearing in the sectors $C$ and $D$ of the 1st complex plane. Being supersymmetric, the 2nd plane, 3rd plane and $\mathcal{N} = 1$ sector do not contribute to the effective potential. Moreover, their contributions to the gauge coupling thresholds are identical to those present in the $\mathcal{N} = 1$ supersymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ asymmetric orbifold models. In this class of theories, the $\mathcal{N} = 1$ sectors do not contribute. The reason for this is that the helicity operator $Q$ acting on an $\mathcal{N} = 1$ sector involves

$$\partial^2_v \left( \theta_{[1]}(v) \theta_{[1-G_2]}(v) \theta_{[1-G_1]}(v) \theta_{[1+H_1+H_2]}(v) \right)_{v=0} \propto \partial^2_v \left( \theta_1(v) \theta_2(v) \theta_3(v) \theta_4(v) \right)_{v=0} = 0,$$

(6.4)
thanks to the fact that $\theta_1(v)$ is odd and $\theta_{2,3,4}(v)$ are even. Therefore, corrections to the gauge couplings occur only from the $\mathcal{N} = 2$ planes. The case of $\mathcal{N} = 2$ planes in symmetric
orbifolds, which are characterized by \((2, 2)\) superconformal symmetry, have been analyzed extensively in the literature \[18\]. However, even if the analysis for asymmetric orbifolds, which possess \((2, 0)\) superconformal symmetry, has not yet been fully completed, our conclusions will remain valid in this case, as discussed at the end of this section.

Let us start by considering our 2\(^{nd}\) and 3\(^{rd}\) planes in the \((2, 2)\) case. As was shown in Refs \[18,19\], the gauge coupling corrections are given in terms of two threshold functions,

\[
\Delta_i = b_i^I \Delta(T_I, U_I) - k_i Y(T_I, U_I), \quad I = 2, 3, \tag{6.5}
\]

where \(b_i^I\) are the \(\mathcal{N} = 2\) \(\beta\)-function coefficients in each plane\[7\],

\[
\Delta(T_I, U_I) = -\log \left(4\pi^2|\eta(T_I)|^4|\eta(U_I)|^4 \text{Im} T_I \text{Im} U_I\right),
\]

\[
Y(T_I, U_I) = -\frac{\xi}{12} \int \frac{d^2 \tau}{\tau_2} \Gamma_{2,2}(T_I, U_I) \left[ \left( E_2 - \frac{3}{\pi \tau_2} \right) \frac{E_4 E_6}{\eta^{24}} - j + 1008 \right]. \tag{6.6}
\]

In these expressions, \(E_{2,4,6}\) are holomorphic Eisenstein series, with modular weights 2,4,6,

\[
E_2 = \frac{12}{i\pi} \partial_\tau \log \eta = 1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n},
\]

\[
E_4 = \frac{1}{2} (\theta_2^8 + \theta_3^8 + \theta_4^8) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n},
\]

\[
E_6 = \frac{1}{2} (\theta_2^4 \theta_3^4 + \theta_3^4 \theta_4^4 + \theta_4^4 \theta_2^4 - \theta_2^4 \theta_3^4 - \theta_3^4 \theta_2^4 - \theta_4^4 \theta_3^4) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}, \tag{6.7}
\]

while \(j = \frac{1}{q} + 744 + \mathcal{O}(q)\) is holomorphic and modular invariant. \(\xi\) is a constant that can be expressed in terms of the numbers of massless vector multiplets and hypermultiplets per plane. Using the relation between gauge and \(R^2\)-term renormalizations \[15\], it is fixed to \(\xi = -1\), thanks to the anomaly cancellation conditions \[26\] valid in the six dimensional decompactification limits \[15,18\]. This property being general in all \(\mathcal{N} = 2\) theories with underlying \((2, 2)\) superconformal symmetries, the threshold corrections are universal in this case \[21\], modulo the \(\beta\)-function coefficients and Kac-Moody levels.

As anticipated below Eq. (6.1), what is relevant to note is that these threshold corrections scale linearly with the volume of the untwisted 2-tori. For \(\text{Im} T_I \gg 1\) and \(U_I\) finite, one has

\[
\Delta(T_I, U_I) = \frac{\pi}{3} \text{Im} T_I - \log(\text{Im} T_I) + \mathcal{O}(1), \quad Y(T_I, U_I) = 4\pi \text{Im} T_I + \mathcal{O}\left(\frac{1}{\text{Im} T_I}\right), \tag{6.8}
\]

\[\text{In our conventions, } b_i^I, I = 2, 3, \text{ are } \beta\text{-function coefficients in the parent theories obtained by acting with a single } \mathbb{Z}_2. \text{ In the } \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ models we are interested in, overall factors } \frac{1}{2} \text{ must be included in the r.h.s. of Eq. (6.5), for the thresholds to the correctly normalized.}\]
which invalidates the string perturbative expansion (when the dressing \( \beta \)-function coefficient is negative). As follows from target space duality, similar dangerous behaviors occur in all limits, where the Kähler and/or complex structures of the untwisted 2-tori are large or small: 

\[ T_I \rightarrow \infty \text{ or } 0, \text{ and/or } U_I \rightarrow \infty \text{ or } 0. \]

This is not a surprise, since we have seen in the previous sections (and also in Ref. [18]) that for the linear terms not to arise, \( \mathcal{N} = 4 \) supersymmetry must be restored on the moduli space boundary. However, this cannot be the case in our 2\textsuperscript{nd} and 3\textsuperscript{rd} complex planes, since the breaking from \( \mathcal{N} = 4 \) to \( \mathcal{N} = 2 \) in these sectors is not spontaneous. These considerations force us to assume that the order of magnitude of the moduli of the 2\textsuperscript{nd} and 3\textsuperscript{rd} planes, \( T_I \) and \( U_I \), \( I = 2, 3 \), are not to far from 1. This justifies that we took the order of magnitude of the coefficient \( c \) introduced in Eq. (4.18) to not be far from 1. Moreover, the moduli-dependent scales \( M_I \)’s that control the threshold corrections are

\[
\frac{1}{M_I^2} = \frac{16}{M_s^2} \left| \eta(T_I) \right|^4 \left| \eta(U_I) \right|^4 \text{Im} T_I \text{Im} U_I, \quad I = 2, 3
\]

and are close to the string scale \( M_s \). In the above expression, we introduce the string coupling constant, which is related to the dilaton field, \( g_s^2 = 1/\text{Im} S_{\text{dil}} \), in order to display the threshold masses in units of gravitational scale.

The contributions \( b_i \Delta(T_I, U_I) \) controlled by the \( M_I \)’s have to be completed by the universal contribution \(-k_i Y(T_I, U_I)\), whose order of magnitude is close to 1. Being infrared finite, these corrections are continuous functions that remain finite even at special values of \((T_I, U_I)\), where additional massless states arise. Thus, we are free to absorb them in a redefinition of the string coupling [27]:

\[
\frac{16 \pi^2}{g_{\text{renor}}^2} = \frac{16 \pi^2}{g_s^2} - \frac{1}{2} Y(T_2, U_2) - \frac{1}{2} Y(T_3, U_3),
\]

where the factors \( \frac{1}{2} \) arise from the action of the second \( \mathbb{Z}_2 \) (see Footnote [7]) and the “renormalized” string coupling is

\[
g_{\text{renor}}^2 = \frac{g_s^2}{1 - \frac{1}{32 \pi^2} (Y(T_2, U_2) + Y(T_3, U_3)) g_s^2}. \tag{6.11}
\]

When the 2\textsuperscript{nd} and 3\textsuperscript{rd} complex planes are realized as \((2, 0)\) asymmetric compactifications \textit{via} fermionic constructions, the natural values for \( \text{Im} T_I \) and \( \text{Im} U_I \) are of order 1.
Moreover, the target space dualities $SL(2,\mathbb{Z})_{T_I} \times SL(2,\mathbb{Z})_{U_I}$ of the $(2,2)$ case are broken to some sub-groups. Consequently, $|\eta(T_I)|^4$ and $|\eta(U_I)|^4$ are replaced by products of other modular functions, with however identical weights. In all cases, $(2,2)$ and $(2,0)$, the orders of magnitude of the dressed threshold scales $M_I$, $I = 2, 3$, remain close to the string scale.

7 Chiral heterotic string vacua with $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$

In this section, we would like to comment on our perturbative string solution to the long standing decompactification problem, with relatively small supersymmetry breaking scale in the TeV regime, $m^{(E)}_3 = O(1–10)$ TeV. Our framework is based on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ asymmetric orbifold constructions, obtained via fermionic formulation. Our main preoccupation is to obtain realistic vacua containing the standard model gauge group and three chiral families. Thirty years of effort led to several string constructions that satisfy these requirements. Among these models, are those in which an initial $SO(10)$ gauge symmetry is broken at the string level by discrete Wilson lines to the Pati-Salam gauge group, $G_{PS} = SU(4) \times SU(2)_L \times SU(2)_R$, which is further broken to the standard model gauge group, $G_{SM} = SU(3) \times SU(2)_L \times U(1)_Y$, by a conventional Higgs mechanism.

While the early examples of realistic free fermionic models consisted in isolated examples [28], in more recent years, systematic classification methods have been developed that enable scanning large classes of three generations models, with viable phenomenological properties [29]. However, in all these vacua [28, 29], as well as in other quasi-realistic heterotic string models [30], $\mathcal{N} = 1$ supersymmetry is unbroken and its spontaneous breaking to $\mathcal{N} = 0$ needs to be implemented. Moreover, in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models, the net chirality arises from the three underlying $\mathcal{N} = 2$ complex planes in two different possible ways. In one class of vacua, the chiral families appear symmetrically, namely $(1,1,1)$ in the $\mathcal{N} = 2$ planes. In the second class of models, the families appear asymmetrically, namely $(0,1,2)$ in the $\mathcal{N} = 2$ planes. As said in the previous section, $(0,0,3)$ models are not allowed, since the number of families per plane is either 0 or $2^n$, with $n = 0, 1, 2, 3$ or 4.

In the present paper, we explore the possibility to break $\mathcal{N} = 1$ supersymmetry by a SSS mechanism. In the $(0,1,2)$ models, the untwisted 2-torus present in the 0-chirality plane can be used to implement the $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$ breaking, in a way that solves the
decompactification problem in an elegant way. The key point is that the $\mathbb{Z}_2$ twist associated to the 0-chirality plane acts simultaneously as a shift in the untwisted 2-torus, so that the $\mathcal{N} = 2$ supersymmetry of the plane arises as a spontaneously broken phase of $\mathcal{N} = 4$. As a result, when the 2-torus is large for $m_2$ to be small, the underlying $\mathcal{N} = 4$ structure of the plane is restored, thus protecting the gauge coupling thresholds from growing linearly with the 2-torus volume.

In these models, we find remarkable that the $\mathcal{N} = 4$ sector spontaneously broken to $\mathcal{N} = 0$, which is referred as sector $B$, is the only one leading to a substantial contribution to the effective potential (the cosmological term), when $m_B \equiv m_2$ is small compared to $cM_s$, 

$$V_{\text{eff}} = \frac{1}{4} V_{\text{eff}}^{B} + \mathcal{O}\left(e^{-c \sqrt{\text{Im}T_1}}\right) = -\frac{1}{4} \left[ \frac{2 + d_{G_B} - n_{F_B}}{2\pi^2} \frac{1}{(\text{Im}T_1)^2} E_{\alpha_B,\beta_B}(U_1|3) + \mathcal{O}\left(e^{-c \sqrt{\text{Im}T_1}}\right) \right],$$

which is proportional to $m_2^4$. Moreover, the relevant threshold corrections to the gauge couplings arise from the sector $B$, as well as from four sectors exhibiting exact $\mathcal{N} = 2$ supersymmetries: The sectors $C$ and $D$, which are actually sub-sectors of the 0-chirality 1st complex plane, and the 2nd and 3rd complex planes, which are chiral. The associated $\mathcal{N}_C$, $\mathcal{N}_D$, $\mathcal{N}_2$, $\mathcal{N}_3 = 2$ supersymmetries are all non-aligned.

The above five contributions to the gauge coupling thresholds are characterized by effective mass scales: $M_{B,C,D}$ depend on the 0-chirality 1st plane moduli $T_1, U_1$, while $M_I$, $I = 2, 3$, depends on the $I$th chiral plane moduli $T_I, U_I$. These dressed mass scales are built with the functions $\theta_2, \theta_3, \theta_4$ and $\eta$, and are modular invariant, with respect to some target space duality sub-group of $SL(2,\mathbb{Z})_{T_1} \times SL(2,\mathbb{Z})_{U_1}$. The running effective coupling constants in the $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$ models take a very simple form, once expressed in terms of the dressed mass scales and coupling $g_{\text{renor}}$,

$$\frac{16\pi^2}{g_i^2(Q)} = k_i \frac{16\pi^2}{g_{\text{renor}}^2} - \frac{1}{4} b^i_B \log \left(\frac{Q^2}{Q^2 + M_B^2}\right) - \frac{1}{4} b^i_C \log \left(\frac{Q^2}{Q^2 + M_C^2}\right) - \frac{1}{4} b^i_D \log \left(\frac{Q^2}{Q^2 + M_D^2}\right) - \frac{1}{2} b^i_2 \log \left(\frac{Q^2}{M_2^2}\right) - \frac{1}{2} b^i_3 \log \left(\frac{Q^2}{M_3^2}\right),$$

(7.2)

where $Q < cM_s$ is the energy scale measured in string frame ($Q^{(E)} < cM_{\text{Planck}}$ in the Einstein
frame) and the sector by sector $\beta$-coefficients are

$$b_i^B = -\frac{8}{3} \left\{ C(O_B) - C(R_B) \right\}, \quad b_i^C = -2 \left\{ C(O_C) - C(R_C) \right\}, \quad b_i^D = -2 \left\{ C(O_D) - C(R_D) \right\},$$

$$b_i^2 = -2 \left\{ C(O_2) - C(R_2) \right\}, \quad b_i^3 = -2 \left\{ C(O_3) - C(R_3) \right\}.$$

(7.3)

The $O_{B,C,D,2,3}$ and $R_{B,C,D,2,3}$ symbols refer to adjoint and matter representations of gauge group factors $G^{i}_{B,C,D,2,3}$ that are realized in the sectors $B, C, D$ and $I = 2, 3$, respectively. In total, the $\beta$-function coefficient of the $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$ model, for $Q$ smaller than all threshold scales, is given by

$$b_i = \frac{1}{4} \left( b_i^B + b_i^C + b_i^D \right) + \frac{1}{2} \left( b_i^2 + b_i^3 \right).$$

(7.4)

When $\text{Im} U_1 = \mathcal{O}(1)$, the dressed masses measured in Einstein frame, $M_{B,C,D}^{(E)} = M_{B,C,D}/g_s$, are all in the TeV region. Thus, they decouple in Eq. (7.2), when $Q^{(E)} = Q/g_s$ reaches larger energy scales, thanks to the restoration of $\mathcal{N} = 4$ supersymmetry in the sector $B$ and the non-chiral 1st plane. When $\text{Im} U_1$ or $1/\text{Im} U_1$ is larger, say up to 20 or so, only two scales among $M_{B,C,D}^{(E)}$ are in the TeV region, while the remaining one can be up to $cM_{\text{Planck}}$. In this case, the full restoration of $\mathcal{N} = 4$ supersymmetry in the sector $B$ and 1st plane occurs only at energies above this highest threshold scale. In Eq. (7.2), the reason why we do not add $Q^2$-terms in front of the $M_I^2$s, $I = 2, 3$, is that the order of magnitude of these two threshold masses is close to the string scale $M_s$, and that in our effective description, the physical energy $Q$ must not exceed $cM_s$.

From the effective field theory viewpoint, the SSS breaking gives rise to a specific $\mathcal{N} = 1$ supergravity no-scale model, with so-called “$S_{\text{dil}}T_1U_1$”-breaking mechanism [31]. We remind that $S_{\text{dil}}$ is the four dimensional dilaton, while $T_1, U_1$ are the moduli of the zero-chirality complex plane. The moduli of the 2nd and 3rd planes do not participate in the supersymmetry breaking. As explained in Ref. [31], the determination via radiative corrections of the vacuum expectation value of the “no-scale modulus” and thus of the $\mathcal{N} = 1$ gravitino mass $m_3^{(E)}$ [32], at relatively low scale of order 1–10 TeV, requires that the genus-1 effective potential is free from terms that scale like $(m_3^{(E)})^2 \Lambda^2$. In such terms, $\Lambda$ is the cut-off of the effective field theory, which in principle can be as large as $M_{\text{Planck}}$ or $M_{\text{s}}$. Thus, it is remarkable that in the setup we consider to break spontaneously $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$ in this work, such terms are absent, thanks to the underlying $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$ supersymmetry breaking structure of the sector $B$, which imposes the genus-1 effective potential to scale like $(m_3^{(E)})^4$.
8 Conclusions

In this paper, our concern is to implement a low scale spontaneous breaking of supersymmetry in phenomenologically interesting $\mathcal{N} = 1$ models, while maintaining the validity of gauge coupling perturbation theory. We address this question within the context of $\mathcal{N} = 1 \mathbb{Z}_2 \times \mathbb{Z}_2$ asymmetric orbifolds, realized by fermionic constructions. At the $\mathcal{N} = 1$ supersymmetric level, it is known that an $\mathcal{N} = 2$ complex plane realized as an $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ spontaneous breaking of supersymmetry yields threshold corrections to the gauge couplings with a mild logarithmic dependence on the complex plane volume \cite{18}. This contrasts with the case where the $\mathbb{Z}_2$ action responsible of the $\mathcal{N} = 4$ breaking to $\mathcal{N} = 2$ is not freely acting. Indeed, a linear dependence of the thresholds on the complex plane volume arises in this case, invalidating perturbation theory once the volume is large. What we shown in the present work is that the above solution to the “decompactification problem” can be extended to the case where $\mathcal{N} = 1$ supersymmetry is further spontaneously broken to $\mathcal{N} = 0$ at a low scale, by implementing an additional $\mathbb{Z}_2^{\text{shift}}$ orbifold shift acting along the large internal dimensions and coupled with the helicity charges $(a, b)$.

To arrive at this conclusion, we develop a sector by sector analysis of the models and analyze systematically the associated induced threshold corrections. We find that one of the $\mathbb{Z}_2$ twists, which for instance preserves the 1st complex plane, must act freely by shifting simultaneously the complex plane. In this case, the volume of the plane is allowed to be large and we can further implement the $\mathbb{Z}_2^{\text{shift}}$ shift responsible for the susy breaking to $\mathcal{N} = 0$ along this plane. As desired, the gravitino mass $m^{(E)}_3$ generated in this way is low. We find that taking into account the first $\mathbb{Z}_2$ and the additional $\mathbb{Z}_2^{\text{shift}}$ only, three subsectors denoted as $B$, $C$ and $D$ contribute substantially to the thresholds. What is meant by “substantially” is that other sub-sectors, which are non-supersymmetric, contribute in the 1st complex plane, whose effects are however exponentially suppressed when the gravitino mass is small, $m^{(E)}_3 \ll M_{\text{Planck}}$. Moreover, this hierarchy allows another great simplification, since it implies the contributions of the massive excitations of the string are also exponentially suppressed, compared to those arising from the Kaluza-Klein towers of states above the charged massless states.

On the contrary, the constraint that the full $\mathbb{Z}_2 \times \mathbb{Z}_2$ model be realistic imposes the second $\mathbb{Z}_2$ twist (and thus the diagonal of the two $\mathbb{Z}_2$'s) to have fixed points, in order to
generate 1 chiral family in the 2nd plane, and 2 chiral families in the 3rd one. Indeed, since models with 3 families in the same plane do not exist, the 3 generations must be treated asymmetrically, giving the possibility to understand why two of them, (u_d) and (c_s), are relatively light compared to the last one, (t_b). Moreover, since the 2nd and 3rd planes do not arise from a spontaneous breaking of $\mathcal{N} = 4$ supersymmetry, their volume moduli must be close to 1, in Planck units, in order not to introduce the decompactification problem back. In addition, supersymmetry has to be preserved in these sectors, since otherwise an extremely large gravitino mass would be generated. These two planes are the remaining sectors that contribute to the thresholds.

In total, the five relevant sectors in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ chiral models, where $\mathcal{N} = 1$ supersymmetry is spontaneously broken to $\mathcal{N} = 0$ at low scale à la SSS are as follows:

- The sector $B$, describes the $\mathcal{N} = 0$ spontaneously broken phase of the $\mathcal{N} = 4$ spectrum of the initial parent theory. Surprisingly, this sector is the only non-supersymmetric one that is relevant for the gauge coupling thresholds and effective potential. In fact, the other sectors relevant for the gauge couplings being supersymmetric, the sector $B$ is solely responsible for the generation of the cosmological term, which is proportional to $(m_{\frac{E}{2}})^4$. No $M_{\text{Planck}}^2 (m_{\frac{E}{2}})^2$ term is induced, as would be the case in field theory, when a finite number of modes are taken into account.

- The sectors $C$ and $D$, which are both sub-sectors of the non-chiral 1st complex plane, preserve $\mathcal{N}_C = 2$ and $\mathcal{N}_D = 2$ supersymmetries, respectively.

- The 2nd and 3rd chiral complex planes preserve $\mathcal{N}_2 = 2$ and $\mathcal{N}_3 = 2$ supersymmetries, respectively.

The gauge coupling thresholds arising from the above sectors are controlled by associated mass scales, which are functions of the Kähler and complex structures $T_I, U_I$ of the corresponding planes, $I = 1, 2, 3$. In the 1st plane, the smallest of the masses $M_{B}^{(E)}, M_{C}^{(E)}$ and $M_{D}^{(E)}$ is about 1–10 TeV (as is the case for all of them is $U_1 \simeq i$). However, any hierarchy among these scales can be achieved by permuting the formal expressions taken by $M_{B,C,D}^{(E)}$, which can be done by changing the pattern of shifts along the 1st complex plane. On the contrary, in the chiral planes, $M_2^{(E)}$ and $M_3^{(E)}$ are close to the Planck scale. Finally, addi-
tional universal contributions of order 1 arising from these 2nd and 3rd planes correct slightly the large inverse bare coupling, $k^i/g_s^2$.

What we have found is the complete dependance of the running effective gauge couplings on the physical scale $Q^{(E)}$, up to $cM_{\text{Planck}}$, including when $Q^{(E)}$ crosses the thresholds scales $M^{(E)}_{B,C,D}$ and that the associated Kaluza-Klein towers of states decouple from the thresholds. The upper bound $cM_{\text{Planck}}$, where $c$ is not far from 1, is the order of magnitude of the massive string modes in Planck units, whose exponentially suppressed contributions have been neglected. The result, displayed in Eq. (7.2), takes a universal form that depends only on the $\beta$-function coefficients associated to the above listed five relevant sectors. Moreover, the form itself of the $\beta$-function coefficients is universal, Eqs (7.3). The factors $\mp 2$ in the coefficients $b^i_C, b^i_D, b^2, b^3$ arise from the massless vector multiplets and hypermultiplets charged under the gauge group factors $G^{i}_{C,D,2,3}$, which are realized in each sectors. The factors $\mp \frac{8}{3}$ in $b^i_B$ follow from specific truncations to $N=0$ of the massless $N=4$ vector multiplets in the parent models: 1 vector boson plus 6 real scalars contribute $-\frac{8}{3}$, while 4 Majorana fermions contribute $\frac{8}{3}$. All these states are charged under a gauge group factor $G^i_B$, realized in the sector $B$.

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