Integral mixed Cayley graphs over abelian groups

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Abstract

A mixed graph is said to be integral if all the eigenvalues of its Hermitian adjacency matrix are integer. Let \( \Gamma \) be an abelian group. The mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is a mixed graph on the vertex set \( \Gamma \) and edge set \( \{(a, b) : b - a \in S\} \), where \( 0 \notin S \). We characterize integral mixed Cayley graph \( \text{Cay}(\Gamma, S) \) over an abelian group \( \Gamma \) in terms of its connection set \( S \).

Mathematics Subject Classifications: 05C50, 05C25

1 Introduction

We only consider graphs without loops and multi-edges. A graph \( G \) is denoted by \( G = (V(G), E(G)) \), where \( V(G) \) and \( E(G) \) are the vertex set and edge set of \( G \), respectively. Here \( E(G) \subset V(G) \times V(G) \setminus \{(u, u) | u \in V(G)\} \) such that \( (u, v) \in E(G) \) if and only if \( (v, u) \in E(G) \). A graph \( G \) is said to be oriented if \( (u, v) \in E(G) \) implies that \( (v, u) \notin E(G) \). A graph \( G \) is said to be mixed if \( (u, v) \in E(G) \) does not always imply that \( (v, u) \in E(G) \), see [15] for details. In a mixed graph \( G \), we call an edge with end vertices \( u \) and \( v \) to be undirected (resp. directed) if both \( (u, v) \) and \( (v, u) \) belong to \( E(G) \) (resp. only one of \( (u, v) \) and \( (v, u) \) belongs to \( E(G) \)). An undirected edge \( (u, v) \) is denoted by \( u \leftrightarrow v \), and a directed edge \( (u, v) \) is denoted by \( u \to v \). A mixed graph can have both directed and undirected edges. Note that, if all edges of a mixed graph \( G \) are directed (resp. undirected) then \( G \) is an oriented graph (resp. a simple graph). For a mixed graph \( G \), the underlying graph \( G_U \) of \( G \) is the simple undirected graph in which all edges of \( G \) are considered undirected. By the terms of order, size, number of components, degree of a vertex, distance between two vertices etc., we mean that they are the same as in their underlying graphs.
The *Hermitian adjacency matrix* of a mixed graph $G$ is denoted by $H(G) = (h_{uv})_{n \times n}$, where $h_{uv}$ is given by

$$h_{uv} = \begin{cases} 
1 & \text{if } (u, v) \in E \text{ and } (v, u) \in E, \\
i & \text{if } (u, v) \in E \text{ and } (v, u) \notin E, \\
-i & \text{if } (u, v) \notin E \text{ and } (v, u) \in E, \\
0 & \text{otherwise.}
\end{cases}$$

Here $i = \sqrt{-1}$ is the imaginary number unit. This matrix was introduced by Liu and Li [15] in the study of hermitian energies of mixed graphs, and also independently by Guo and Mohar [9]. Hermitian adjacency matrix of a mixed graph incorporates both adjacency matrix of simple graph and skew adjacency matrix of an oriented graph. The Hermitian spectrum of $G$, denoted by $Sp_H(G)$, is the multi set of the eigenvalues of $H(G)$. It is easy to see that $H(G)$ is a Hermitian matrix and so $Sp_H(G) \subseteq \mathbb{R}$.

A mixed graph is said to be integral if all the eigenvalues of its Hermitian adjacency matrix are integers. Integral graphs were first defined by Harary and Schwenk in 1974 [10] and proposed a classification of integral graphs. See [5] for a survey on integral graphs.

Let $\Gamma$ be a group, $S \subseteq \Gamma$ and $S$ does not contain the identity element of $\Gamma$. The set $S$ is said to be symmetric (resp. skew-symmetric) if $S$ is closed under inverse (resp. $a^{-1} \notin S$ for all $a \in S$). Define $\overline{S} = \{u \in S : u^{-1} \notin S\}$. Clearly $S \setminus \overline{S}$ is symmetric and $\overline{S}$ is skew-symmetric. The *mixed Cayley graph* $G = Cay(\Gamma, S)$ is a mixed graph, where $V(G) = \Gamma$ and $E(G) = \{(a, b) : a, b \in \Gamma, a^{-1}b \in S\}$. Since we have not assumed that $S$ is symmetric, so a mixed Cayley graph can have directed edges. If $S$ is symmetric, then $G$ is a (simple) *Cayley graph*. If $S$ is skew-symmetric then $G$ is an *oriented Cayley graph*.

In 1982, Bridge and Mena [6] introduced a characterization of integral Cayley graphs over abelian groups. Later on, the exact characterization was rediscovered by Wasin So [17] for cyclic groups in 2005. In 2009, Abdollahi and Vatandoost [1] proved that there are exactly seven connected cubic integral Cayley graphs. In the same year, Klotz and Sander [13] proved that if the Cayley graph $Cay(\Gamma, S)$ over abelian group $\Gamma$ is integral, then $S$ belongs to the Boolean algebra $\mathbb{B}(\Gamma)$ generated by the subgroups of $\Gamma$, and its converse proved by Alperin and Peterson [3]. In 2013, DeVos et al. [8] gave a sufficient condition for the integrality of Cayley multigraphs and proved the necessary part for abelian groups, which in turn, is an alternative, character-theoretic proof of the characterization of Bridges and Mena [6]. In 2014, Cheng et al. [14] proved that normal Cayley graphs (its generating set $S$ is closed under conjugation) of symmetric groups are integral. Alperin [2] gave a characterization of integral Cayley graphs over finite groups. In 2017, Lu et al. [16] gave necessary and sufficient conditions for the integrality of Cayley graphs over dihedral groups $D_n$. In particular, they completely determined all integral Cayley graphs of the dihedral group $D_p$ for a prime $p$. In 2019, Cheng et al. [7] obtained several simple sufficient conditions for the integrality of Cayley graphs over the dicyclic group $T_{3n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$. In particular, they also completely determined all integral Cayley graphs over the dicyclic group $T_{3p}$ for a prime $p$. In [12], the authors have characterized integral mixed circulant graphs in terms of their connection set. In this paper, we give a characterization of integral mixed Cayley graphs over abelian...
groups in terms if its connection set. In what follows, $\Gamma$ is always taken to be a finite abelian group.

This paper is organized as follows. In second section, we express the eigenvalues of a mixed Cayley graph as a sum of eigenvalues of a simple Cayley graph and an oriented Cayley graph. In third section, we obtain a sufficient condition on the connection set $S$ for integrality of the mixed Cayley graph $Cay(\Gamma, S)$ over an abelian group $\Gamma$. In fourth section, we prove the necessity of the sufficient condition obtained in Section 3.

2 Mixed Cayley graph and group characters

A representation of a finite group $\Gamma$ is a homomorphism $\rho : \Gamma \to GL(V)$, where $GL(V)$ is the group of automorphisms of a finite dimensional vector space $V$ over the complex field $\mathbb{C}$. The dimension of $V$ is called the degree of $\rho$. Two representations $\rho_1$ and $\rho_2$ of $\Gamma$ on $V_1$ and $V_2$, respectively, are equivalent if there is an isomorphism $T : V_1 \to V_2$ such that $T\rho_1(g) = \rho_2(g)T$ for all $g \in \Gamma$.

Let $\rho : \Gamma \to GL(V)$ be a representation. The character $\chi_\rho : \Gamma \to \mathbb{C}$ of $\rho$ is defined by setting $\chi_\rho(g) = Tr(\rho(g))$ for $g \in \Gamma$, where $Tr(\rho(g))$ is the trace of the representation matrix of $\rho(g)$. By degree of $\chi_\rho$ we mean the degree of $\rho$ which is simply $\chi_\rho(1)$. If $W$ is a $\rho(g)$-invariant subspace of $V$ for each $g \in \Gamma$, then we say $W$ a $\rho(\Gamma)$-invariant subspace of $V$. If the only $\rho(\Gamma)$-invariant subspaces of $V$ are $\{0\}$ and $V$, we say $\rho$ an irreducible representation of $\Gamma$, and the corresponding character $\chi_\rho$ an irreducible character of $\Gamma$. For a group $\Gamma$, we denote by $\text{IRR}(\Gamma)$ and $\text{Irr}(\Gamma)$ the complete set of non-equivalent irreducible representations of $\Gamma$ and the complete set of non-equivalent irreducible characters of $\Gamma$, respectively.

Let $\Gamma$ be a finite abelian group under addition with $n$ elements, and $S$ be a subset of $\Gamma$ with $0 \notin S$, where $0$ is the additive identity of $\Gamma$. Then $\Gamma$ is isomorphic to the direct product of cyclic groups of prime power order, i.e.

$$\Gamma \cong \mathbb{Z}_{n_1} \otimes \cdots \otimes \mathbb{Z}_{n_k},$$

where $n = n_1 \cdots n_k$, and $n_j$ is a power of a prime number for each $j = 1, \ldots, k$. We consider an abelian group $\Gamma$ as $\mathbb{Z}_{n_1} \otimes \cdots \otimes \mathbb{Z}_{n_k}$ of order $n = n_1 \cdots n_k$. The exponent of $\Gamma$, denoted by $\exp(\Gamma)$, is defined to be the least common multiple of $n_1, n_2, \ldots, n_k$. We consider the elements $x \in \Gamma$ as elements of the cartesian product $\mathbb{Z}_{n_1} \otimes \cdots \otimes \mathbb{Z}_{n_k}$, i.e.

$$x = (x_1, x_2, \ldots, x_k),$$

where $x_j \in \mathbb{Z}_{n_j}$ for all $1 \leq j \leq k$.

Addition in $\Gamma$ is done coordinate-wise modulo $n_j$. For a positive integer $k$ and $a \in \Gamma$, we denote by $ka$ or $a^k$ the $k$-fold sum of $a$ to itself, $(−k)a = k(−a)$, $0a = 0$, and inverse of $a$ by $−a$.

Lemma 1. [18] Let $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$ be a cyclic group of order $n$. Then $\text{IRR}(\mathbb{Z}_n) = \{\phi_k : 0 \leq k \leq n - 1\}$, where $\phi_k(j) = \omega^j_n$ for all $0 \leq j, k \leq n - 1$, and $\omega_n = \exp(\frac{2\pi i}{n})$. 

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Lemma 2. [18] Let $\Gamma_1$ and $\Gamma_2$ be two abelian groups of order $m, n$, respectively. Let

$$\text{IRR}(\Gamma_1) = \{\phi_1, \ldots, \phi_m\}, \text{ and } \text{IRR}(\Gamma_2) = \{\rho_1, \ldots, \rho_n\}.$$ 

Then

$$\text{IRR}(\Gamma_1 \times \Gamma_2) = \{\psi_{kl} : 1 \leq k \leq m, 1 \leq l \leq n\},$$

where $\psi_{kl} : \Gamma_1 \times \Gamma_2 \to \mathbb{C}^*$ and $\psi_{kl}(g_1, g_2) = \phi_k(g_1)\rho_l(g_2)$ for all $g_1 \in \Gamma_1, g_2 \in \Gamma_2$.

Consider $\Gamma = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$. By Lemma 1 and Lemma 2, $\text{IRR}(\Gamma) = \{\psi_\alpha : \alpha \in \Gamma\}$,

where

$$\psi_\alpha(x) = \prod_{j=1}^{k} \omega_{n_j}^{\alpha_j x_j} \text{ for all } \alpha = (\alpha_1, \ldots, \alpha_k), x = (x_1, \ldots, x_k) \in \Gamma,$$

and $\omega_{n_j} = \exp\left(\frac{2\pi i}{n_j}\right)$. Since $\Gamma$ is an abelian group, every irreducible representation of $\Gamma$ is 1-dimensional and thus can be identified with its characters. Hence $\text{IRR}(\Gamma) = \text{Irr}(\Gamma)$. For $x \in \Gamma$, let $\text{ord}(x)$ denote the order of $x$. The following lemma can be easily proved.

Lemma 3. Let $\Gamma$ be an abelian group of order $n$, and $\text{Irr}(\Gamma) = \{\psi_\alpha : \alpha \in \Gamma\}$ be the set of all $n$ characters of $\Gamma$. Then the following statements are true.

(i) $\psi_\alpha(x) = \psi_x(\alpha) \text{ for all } x, \alpha \in \Gamma$.

(ii) $(\psi_\alpha(x))^{\text{ord}(x)} = (\psi_\alpha(x))^{\text{ord}(\alpha)} = 1 \text{ for all } x, \alpha \in \Gamma$.

(iii) $\psi_\alpha(x)^l = 1 \text{ for all } x, \alpha \in \Gamma, \text{ where } l = \text{exp}(\Gamma)$.

Let $f : \Gamma \to \mathbb{C}$ be a function. The Cayley color digraph of $\Gamma$ with color function $f$, denoted by $\text{Cay}(\Gamma, f)$, is defined to be the directed graph with vertex set $\Gamma$ and arc set \{(x, y) : x, y \in \Gamma\} such that each arc $(x, y)$ is colored by $f(x^{-1}y)$. The adjacency matrix of $\text{Cay}(\Gamma, f)$ is defined to be the matrix whose rows and columns are indexed by the elements of $\Gamma$, and the $(x, y)$-entry is equal to $f(x^{-1}y)$. The eigenvalues of $\text{Cay}(\Gamma, f)$ are simply the eigenvalues of its adjacency matrix.

Theorem 4. [4] Let $\Gamma$ be a finite abelian group. Then the spectrum of the Cayley color digraph $\text{Cay}(\Gamma, f)$ is $\{\gamma_\alpha : \alpha \in \Gamma\}$, where

$$\gamma_\alpha = \sum_{y \in \Gamma} f(y)\psi_\alpha(y) \text{ for all } \alpha \in \Gamma.$$ 

Lemma 5. [4] Let $\Gamma$ be an abelian group. Then the spectrum of the mixed Cayley graph $\text{Cay}(\Gamma, S)$ is $\{\gamma_\alpha : \alpha \in \Gamma\}$, where $\gamma_\alpha = \lambda_\alpha + \mu_\alpha$ and

$$\lambda_\alpha = \sum_{s \in S \setminus \mathbb{S}} \psi_\alpha(s), \quad \mu_\alpha = i \sum_{s \in \mathbb{S}} \left(\psi_\alpha(s) - \psi_\alpha(-s)\right), \text{ for all } \alpha \in \Gamma.$$
Proof. Define \( f_S : \Gamma \to \{0, 1, i, -i\} \) such that

\[
f_S(s) = \begin{cases} 
1 & \text{if } s \in S \setminus \overline{S}, \\
i & \text{if } s \in \overline{S}, \\
-i & \text{if } s \in \overline{S}^{-1}, \\
0 & \text{otherwise}.
\end{cases}
\]

The adjacency matrix of Cayley color digraph \( \text{Cay}(\Gamma, f_S) \) agrees with the Hermitian adjacency matrix of mixed Cayley graph \( \text{Cay}(\Gamma, S) \). Thus the result follows from Theorem 4.

Next two corollaries are special cases of Lemma 5.

**Corollary 6.** \cite{13} Let \( \Gamma \) be an abelian group. Then the spectrum of the Cayley graph \( \text{Cay}(\Gamma, S) \) is \( \{\lambda_\alpha : \alpha \in \Gamma\} \), where \( \lambda_\alpha = \lambda_{-\alpha} \) and

\[
\lambda_\alpha = \sum_{s \in S} \psi_\alpha(s) \text{ for all } \alpha \in \Gamma.
\]

**Proof.** Note that \( \overline{S} = \emptyset \), and so \( s \in S \) if and only if \( -s \in S \). Using Lemma 5, we have

\[
\lambda_\alpha = \sum_{s \in S} \psi_\alpha(s) = \sum_{s \in S} \psi_{-\alpha}(-s) = \sum_{s \in S} \psi_{-\alpha}(s) = \lambda_{-\alpha}.
\]

**Corollary 7.** Let \( \Gamma \) be an abelian group. Then the spectrum of the oriented Cayley graph \( \text{Cay}(\Gamma, S) \) is \( \{\mu_\alpha : \alpha \in \Gamma\} \), where \( \mu_\alpha = -\mu_{-\alpha} \) and

\[
\mu_\alpha = i \sum_{s \in S} \left( \psi_\alpha(s) - \psi_\alpha(-s) \right) \text{ for all } \alpha \in \Gamma.
\]

**Proof.** Note that \( S \setminus \overline{S} = \emptyset \). Using Lemma 5, we have

\[
\mu_\alpha = i \sum_{s \in S} \left( \psi_\alpha(s) - \psi_\alpha(-s) \right) = i \sum_{s \in S} \left( \psi_{-\alpha}(-s) - \psi_{-\alpha}(s) \right) = -\mu_{-\alpha}.
\]

**Theorem 8.** Let \( \Gamma \) be an abelian group. The mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is integral if and only if both Cayley graph \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) and oriented Cayley graph \( \text{Cay}(\Gamma, S) \) are integral.

**Proof.** Assume that the mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is integral. Let \( \gamma_\alpha \) be an eigenvalue of mixed Cayley graph \( \text{Cay}(\Gamma, S) \). By Lemma 5, Corollary 6 and Corollary 7, we have \( \gamma_\alpha = \lambda_\alpha + \mu_\alpha \) and \( \gamma_{-\alpha} = \lambda_\alpha - \mu_\alpha \) for all \( \alpha \in \Gamma \), where \( \lambda_\alpha \) is an eigenvalue of the Cayley graph \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) and \( \mu_\alpha \) is an eigenvalue of the oriented Cayley graph \( \text{Cay}(\Gamma, S) \). Thus \( \lambda_\alpha = \frac{\gamma_\alpha + \gamma_{-\alpha}}{2} \in \mathbb{Q} \) and \( \mu_\alpha = \frac{\gamma_\alpha - \gamma_{-\alpha}}{2} \in \mathbb{Q} \). As \( \lambda_\alpha \) and \( \mu_\alpha \) are rational algebraic integers, so \( \lambda_\alpha, \mu_\alpha \in \mathbb{Q} \) implies that \( \lambda_\alpha \) and \( \mu_\alpha \) are integers. Thus the Cayley graph \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) and the oriented Cayley graph \( \text{Cay}(\Gamma, S) \) are integral.

Conversely, assume that both Cayley graph \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) and oriented Cayley graph \( \text{Cay}(\Gamma, S) \) are integral. Then Lemma 5 implies that \( \text{Cay}(\Gamma, S) \) is integral.

Let $n \geq 2$ be a fixed integer. Define $G_n(d) = \{k : 1 \leq k \leq n - 1, \gcd(k, n) = d\}$. It is clear that $G_n(d) = dG_{\frac{n}{d}}(1)$.

Alperin and Peterson [3] considered a Boolean algebra generated by a class of subgroups of a group in order to determine the integrality of Cayley graphs over abelian groups. Suppose $\Gamma$ is a finite group, and $\mathcal{F}_\Gamma$ is the family of all subgroups of $\Gamma$. The Boolean algebra $\mathcal{B}(\Gamma)$ generated by $\mathcal{F}_\Gamma$ is the set whose elements are obtained by arbitrary finite intersections, unions, and complements of the elements in the family $\mathcal{F}_\Gamma$. The minimal non-empty elements of this algebra are called atoms. Thus each element of $\mathcal{B}(\Gamma)$ is the union of some atoms. Consider the equivalence relation $\sim$ on $\Gamma$ such that $x \sim y$ if and only if $y = x^k$ for some $k \in \mathcal{G}_{m}(1)$, where $m = \text{ord}(x)$.

**Lemma 9.** [3] The equivalence classes of $\sim$ are the atoms of $\mathcal{B}(\Gamma)$.

For $x \in \Gamma$, let $[x]$ denote the equivalence class of $x$ with respect to the relation $\sim$. Also, let $\langle x \rangle$ denote the cyclic group generated by $x$.

**Lemma 10.** [3] The atoms of the Boolean algebra $\mathcal{B}(\Gamma)$ are the sets $[x] = \{y : \langle y \rangle = \langle x \rangle\}$.

By Lemma 10, each element of $\mathcal{B}(\Gamma)$ is a union of some sets of the form $[x]$. Thus, for all $S \in \mathcal{B}(\Gamma)$, we have $S = [x_1] \cup \cdots \cup [x_k]$ for some $x_1, \ldots, x_k \in \Gamma$.

The next result provides a complete characterization of integral Cayley graphs over an abelian group $\Gamma$ in terms of the atoms of $\mathcal{B}(\Gamma)$.

**Theorem 11.** ([3], [6]) Let $\Gamma$ be an abelian group. The Cayley graph $\text{Cay}(\Gamma, S)$ is integral if and only if $S \in \mathcal{B}(\Gamma)$.

### 3 A sufficient condition for integrality of mixed Cayley graphs over abelian groups

Unless otherwise stated, we consider $\Gamma$ to be an abelian group of order $n$. Due to Theorem 8, to find characterization of the integral mixed Cayley graph $\text{Cay}(\Gamma, S)$, it is enough to find characterization of the integral Cayley graph $\text{Cay}(\Gamma, S \setminus \overline{S})$ and the integral oriented Cayley graph $\text{Cay}(\Gamma, S \setminus \overline{S})$. The integral Cayley graph $\text{Cay}(\Gamma, S \setminus \overline{S})$ is characterized by Theorem 11. So our attempt is to characterize the integral oriented Cayley graph $\text{Cay}(\Gamma, S)$.

Define $\Gamma(4)$ to be the set of all $x \in \Gamma$ which satisfies $\text{ord}(x) \equiv 0 \pmod{4}$. It is clear that $\text{exp}(\Gamma) \equiv 0 \pmod{4}$ if and only if $\Gamma(4) \neq \emptyset$. For all $x \in \Gamma(4)$ and $r \in \{0, 1, 2, 3\}$, define

$$M_r(x) := \{x^k : 1 \leq k \leq \text{ord}(x), k \equiv r \pmod{4}\}.$$  

For all $a \in \Gamma$ and $S \subseteq \Gamma$, define $a + S := \{a + s : s \in S\}$ and $-S := \{-s : s \in S\}$. Note that $-s$ denotes the inverse of $s$, that is $-s = s^{m-1}$, where $m = \text{ord}(s)$.

**Lemma 12.** Let $\Gamma$ be an abelian group and $x \in \Gamma(4)$. Then the following statements are true.
Proof. (i) It follows from the definitions of Lemma 13.

(ii) Both \( M_1(x) \) and \( M_3(x) \) are skew-symmetric subsets of \( \Gamma \).

(iii) \(-M_1(x) = M_3(x) \) and \(-M_3(x) = M_1(x)\).

(iv) Let \( a + M_1(x) = M_3(x) \) and \( a + M_3(x) = M_1(x) \) for all \( a \in M_2(x) \).

(v) \( a + M_1(x) = M_1(x) \) and \( a + M_3(x) = M_3(x) \) for all \( a \in M_0(x) \).

Proof. (i) It follows from the definitions of \( M_r(x) \) and \( \langle x \rangle \).

(ii) If \( x^k \in M_1(x) \) then \(-x^k = x^{n-k} \not\in M_1(x)\), as \( k \equiv 1 \) (mod 4) implies \( n - k \equiv 3 \) (mod 4). Thus \( M_1(x) \) is a skew-symmetric subset of \( \Gamma \). Similarly, \( M_3(x) \) is also a skew-symmetric subset of \( \Gamma \).

(iii) As \( k \equiv 1 \) (mod 4) if and only if \( n - k \equiv 3 \) (mod 4), we get \(-x^k = x^{n-k}\). Therefore \(-M_1(x) = M_3(x) \) and \(-M_3(x) = M_1(x)\).

(iv) Let \( a \in M_2(x) \) and \( y \in a + M_1(x) \). Then \( a = x^{k_1} \) and \( y = x^{k_1} + x^{k_2} = x^{k_1+k_2} \), where \( k_1 \equiv 2 \) (mod 4) and \( k_2 \equiv 1 \) (mod 4). Since \( k_1 + k_2 \equiv 3 \) (mod 4), we have \( y \in M_3(x) \) implying that \( a + M_1(x) \subseteq M_3(x) \). Since size of both sets \( M_1(x) \) and \( M_3(x) \) are same, hence \( a + M_1(x) = M_3(x) \). Similarly, \( a + M_3(x) = M_1(x) \) for all \( a \in M_2(x) \).

(v) The proof is similar to Part (iv). \( \square \)

Lemma 13. Let \( x \in \Gamma(4) \). Then \( i \left( \sum_{s \in M_1(x)} \psi_\alpha(s) - \sum_{s \in M_3(x)} \psi_\alpha(s) \right) \in \mathbb{Z} \) for all \( \alpha \in \Gamma \).

Proof. Let \( x \in \Gamma(4) \), \( \alpha \in \Gamma \) and

\[
\mu_\alpha = i \left( \sum_{s \in M_1(x)} \psi_\alpha(s) - \sum_{s \in M_3(x)} \psi_\alpha(s) \right).
\]

Case 1: There exists \( a \in M_2(x) \) such that \( \psi_\alpha(a) \neq -1 \). Then

\[
\mu_\alpha = -i \left( \sum_{s \in M_3(x)} \psi_\alpha(s) - \sum_{s \in M_1(x)} \psi_\alpha(s) \right)
= -i \left( \sum_{s \in a + M_1(x)} \psi_\alpha(s) - \sum_{s \in a + M_3(x)} \psi_\alpha(s) \right)
= -i \left( \sum_{s \in M_1(x)} \psi_\alpha(a + s) - \sum_{s \in M_3(x)} \psi_\alpha(a + s) \right)
= -i \psi_\alpha(a) \left( \sum_{s \in M_1(x)} \psi_\alpha(s) - \sum_{s \in M_3(x)} \psi_\alpha(s) \right)
= -\psi_\alpha(a) \mu_\alpha.
\]
We have \((1 + \psi_\alpha(a))\mu_\alpha = 0\). Since \(\psi_\alpha(a) \neq -1\), so \(\mu_\alpha = 0 \in \mathbb{Z}\).

Case 2: There exists \(a \in M_0(x)\) such that \(\psi_\alpha(a) \neq 1\). Applying the same process as in Case 1, we get \(\mu_\alpha = 0 \in \mathbb{Z}\).

Case 3: Assume that \(\psi_\alpha(a) = -1\) for all \(a \in M_2(x)\) and \(\psi_\alpha(a) = 1\) for all \(a \in M_0(x)\). Then \(\psi_\alpha(a) = -\psi_\alpha(x)\) for all \(a \in M_3(x)\) and \(\psi_\alpha(a) = \psi_\alpha(x)\) for all \(a \in M_1(x)\). Therefore

\[
\mu_\alpha = i \left( \sum_{s \in M_1(x)} \psi_\alpha(s) - \sum_{s \in M_3(x)} \psi_\alpha(s) \right) = 2i\psi_\alpha(x)|M_1(x)|.
\]

Since \(\psi_\alpha(x)^4 = 1\) and \(\mu_\alpha\) is real, we have \(\psi_\alpha(x) = \pm i\). Thus \(\mu_\alpha = \pm 2|M_1(x)| \in \mathbb{Z}\). \(\square\)

For \(m \equiv 0 \pmod{4}\) and \(r \in \{1, 3\}\), define

\[
G_m^r(1) = \{k : k \equiv r \pmod{4}, \gcd(k, m) = 1\}.
\]

Define an equivalence relation \(\approx\) on \(\Gamma(4)\) such that \(x \approx y\) if and only if \(y = x^k\) for some \(k \in G_m^1(1)\), where \(m = \operatorname{ord}(x)\). Observe that if \(x, y \in \Gamma(4)\) and \(x \approx y\) then \(x \sim y\), but the converse need not be true. For example, consider \(x = 5 \pmod{12}\), \(y = 11 \pmod{12}\) in \(\mathbb{Z}_{12}\). Here \(x, y \in \mathbb{Z}_{12}(4)\) and \(x \sim y\) but \(x \not\approx y\). For \(x \in \Gamma(4)\), let \([x]\) denote the equivalence class of \(x\) with respect to the relation \(\approx\).

**Lemma 14.** Let \(\Gamma\) be an abelian group, \(x \in \Gamma(4)\) and \(m = \operatorname{ord}(x)\). Then the following are true.

(i) \([x] = \{x^k : k \in G_m^1(1)\}\).

(ii) \([-x] = \{x^k : k \in G_m^3(1)\}\).

(iii) \([x] \cap [-x] = \emptyset\).

(iv) \([x] = [x] \cup [-x]\).

**Proof.**

(i) Let \(y \in [x]\). Then \(x \approx y\), and so \(\operatorname{ord}(x) = \operatorname{ord}(y) = m\) and there exists \(k \in G_m^1(x)\) such that \(y = x^k\). Thus \([x] \subseteq \{x^k : k \in G_m^1(1)\}\). On the other hand, let \(z = x^k\) for some \(k \in G_m^1(1)\). Then \(\operatorname{ord}(x) = \operatorname{ord}(z)\) and so \(x \sim z\). Thus \(\{x^k : k \in G_m^1(1)\} \subseteq [x]\).

(ii) Note that \(-x = x^{m-1}\) and \(m - 1 \equiv 3 \pmod{4}\). By Part (i),

\[
[-x] = \{(-x)^k : k \in G_m^1(1)\} = \{x^{(m-1)k} : k \in G_m^1(1)\}
= \{x^{-k} : k \in G_m^1(1)\}
= \{x^k : k \in G_m^3(1)\}.
\]

(iii) Since \(G_m^1(1) \cap G_m^3(1) = \emptyset\), so by Part (i) and Part (ii), \([x] \cap [-x] = \emptyset\) holds.
(iv) Since \( [x] = \{ x^k : k \in G_m(1) \} \) and \( G_m(1) \) is a disjoint union of \( G_m^1(1) \) and \( G_m^3(1) \), by Part (i) and Part (ii), \([x] = [x] \cup [-x]\) holds.

Let \( D_g \) be the set of all odd divisors of \( g \), and \( D_g^1 \) (resp. \( D_g^3 \)) be the set of all odd divisors of \( g \) which are congruent to 1 (resp. 3) modulo 4. It is clear that \( D_g = D_g^1 \cup D_g^3 \).

**Lemma 15.** Let \( \Gamma \) be an abelian group, \( x \in \Gamma(4) \), \( m = \text{ord}(x) \) and \( g = \frac{m}{4} \). Then the following are true.

(i) \( M_1(x) \cup M_3(x) = \bigcup_{h \in D_g} [x^h] \).

(ii) \( M_1(x) = \bigcup_{h \in D_g^1} [x^h] \cup \bigcup_{h \in D_g^3} [-x^h] \).

(iii) \( M_3(x) = \bigcup_{h \in D_g^3} [-x^h] \cup \bigcup_{h \in D_g^1} [x^h] \).

**Proof.**

(i) Let \( x^k \in M_1(x) \cup M_3(x) \), where \( k \equiv 1 \) or \( 3 \) (mod 4). To show \( x^k \in \bigcup_{h \in D_g} [x^h] \), it is enough to show \( x^k \sim x^h \) for some \( h \in D_g \). Let \( h = \gcd(k, g) \in D_g \). Note that

\[
\text{ord}(x^k) = \frac{m}{\gcd(m, k)} = \frac{m}{\gcd(g, k)} = \frac{m}{h} = \text{ord}(x^h).
\]

Also, as \( h = \gcd(k, m) \), we have \( \langle x^k \rangle = \langle x^h \rangle \), and so \( x^k = x^{jh} \) for some \( j \in G_q(1) \), where \( q = \text{ord}(x^h) = \frac{m}{h} \). Thus \( x^k \sim x^h \) where \( h = \gcd(k, g) \in D_g \). Conversely, let \( z \in \bigcup_{h \in D_g} [x^h] \). Then there exists \( h \in D_g \) such that \( z = x^{jh} \) where \( j \in G_q(1) \) and \( q = \frac{m}{\gcd(m, h)} \). Now \( h \in D_g \) and \( q \equiv 0 \) (mod 4) imply that both \( h \) and \( j \) are odd integers. Thus \( hj \equiv 1 \) or \( 3 \) (mod 4) and so \( \bigcup_{h \in D_g} [x^h] \subseteq M_1(x) \cup M_3(x) \). Hence \( M_1(x) \cup M_3(x) = \bigcup_{h \in D_g} [x^h] \).

(ii) Let \( x^k \in M_1(x) \), where \( k \equiv 1 \) (mod 4). By Part (i), there exists \( h \in D_g \) and \( j \in G_q(1) \) such that \( x^k = x^{jh} \), where \( q = \frac{m}{\gcd(m, h)} \). Note that \( k = jh \). If \( h \equiv 1 \) (mod 4) then \( j \in G_q^1(1) \), otherwise \( j \in G_q^3(1) \). Thus using parts (i) and (ii) of Lemma 14, if \( h \equiv 1 \) (mod 4) then \( x^k \approx x^h \), otherwise \( x^k \approx -x^h \). Hence \( M_1(x) \subseteq \bigcup_{h \in D_g^1} [x^h] \cup \bigcup_{h \in D_g^3} [-x^h] \). Conversely, assume that \( z \in \bigcup_{h \in D_g^1} [x^h] \cup \bigcup_{h \in D_g^3} [-x^h] \). This gives \( z \in [x^h] \) for an \( h \in D_g^1 \) or \( z \in [-x^h] \) for an \( h \in D_g^3 \). In the first case, by part (i) of Lemma 14, there exists \( j \in G_q^1(1) \) with \( q = \frac{m}{\gcd(m, h)} \) such that \( z = x^{jh} \). Similarly, for the second case, by part (ii) of Lemma 14, there exists \( j \in G_q^3(1) \) with \( q = \frac{m}{\gcd(m, h)} \) such that \( z = x^{jh} \). In both the cases, \( hj \equiv 1 \) (mod 4). Thus \( z \in M_1(x) \).

(iii) The proof is similar to Part (ii).

\[ \square \]
Lemma 16. Let \( x \in \Gamma(4) \). Then \( i \left( \sum_{s \in [x]} \psi_{\alpha}(s) - \sum_{s \in [-x]} \psi_{\alpha}(s) \right) \in \mathbb{Z} \) for all \( \alpha \in \Gamma \).

Proof. Note that there exists \( x \in \Gamma(4) \) with \( \text{ord}(x) = 4 \). Apply induction on \( \text{ord}(x) \). If \( \text{ord}(x) = 4 \), then \( M_1(x) = [x] \) and \( M_3(x) = [-x] \). Hence by Lemma 13,

\[
i \left( \sum_{s \in [x]} \psi_{\alpha}(s) - \sum_{s \in [-x]} \psi_{\alpha}(s) \right) = i \left( \sum_{s \in [x]} \psi_{\alpha}(s) - \sum_{s \in [-x]} \psi_{\alpha}(s) \right)
+ \sum_{h \in D_3^1, h > 1} i \left( \sum_{s \in [x]} \psi_{\alpha}(s) - \sum_{s \in [-x]} \psi_{\alpha}(s) \right)
+ \sum_{h \in D_3^3, h > 1} i \left( \sum_{s \in [x]} \psi_{\alpha}(s) - \sum_{s \in [-x]} \psi_{\alpha}(s) \right).
\]

Hence

\[
i \left( \sum_{s \in [x]} \psi_{\alpha}(s) - \sum_{s \in [-x]} \psi_{\alpha}(s) \right) = i \left( \sum_{s \in M_1(x)} \psi_{\alpha}(s) - \sum_{s \in M_3(x)} \psi_{\alpha}(s) \right)
- \sum_{h \in D_3^1, h > 1} i \left( \sum_{s \in [x]} \psi_{\alpha}(s) - \sum_{s \in [-x]} \psi_{\alpha}(s) \right)
+ \sum_{h \in D_3^3, h > 1} i \left( \sum_{s \in [x]} \psi_{\alpha}(s) - \sum_{s \in [-x]} \psi_{\alpha}(s) \right)
\]

is also an integer for all \( \alpha \in \Gamma \) because of Lemma 13 and induction hypothesis. \( \Box \)

For \( \exp(\Gamma) \equiv 0 \pmod{4} \), define \( \mathbb{D}(\Gamma) \) to be the set of all skew-symmetric subsets \( S \) of \( \Gamma \) such that \( S = [x_1] \cup \cdots \cup [x_k] \) for some \( x_1, \ldots, x_k \in \Gamma(4) \). For \( \exp(\Gamma) \not\equiv 0 \pmod{4} \), define \( \mathbb{D}(\Gamma) = \{ \emptyset \} \).
Theorem 17. Let $\Gamma$ be an abelian group. If $S \in \mathbb{D}(\Gamma)$ then the oriented Cayley graph $\text{Cay}(\Gamma, S)$ is integral.

Proof. Assume that $S \in \mathbb{D}(\Gamma)$. Then $S = \{x_1, \ldots, x_k\}$ for some $x_1, \ldots, x_k \in \Gamma(4)$. Let $\mathbb{S}_H(\text{Cay}(\Gamma, S)) = \{\mu_\alpha : \alpha \in \Gamma\}$. We have

$$
\mu_\alpha = i \sum_{s \in S} \left( \psi_\alpha(s) - \psi_\alpha(-s) \right) = \sum_{j=1}^k \sum_{s \in \{x_j\}} i \left( \psi_\alpha(s) - \psi_\alpha(-s) \right).
$$

Now by Lemma 16, $\mu_\alpha \in \mathbb{Z}$ for all $\alpha \in \Gamma$. Hence the oriented Cayley graph $\text{Cay}(\Gamma, S)$ is integral.

Theorem 18. Let $\Gamma$ be an abelian group. If $S \setminus \mathbb{S} \in \mathbb{B}(\Gamma)$ and $\mathbb{S} \in \mathbb{D}(\Gamma)$ then the mixed Cayley graph $\text{Cay}(\Gamma, S)$ is integral.

Proof. By Theorem 8, $\text{Cay}(\Gamma, S)$ is integral if and only if both $\text{Cay}(\Gamma, S \setminus \mathbb{S})$ and $\text{Cay}(\Gamma, \mathbb{S})$ are integral. Thus the result follows from Theorem 11 and Theorem 17.

4 Characterization of integral mixed Cayley graphs over abelian groups

The cyclotomic polynomial $\Phi_n(x)$ is the monic polynomial whose zeros are the primitive $n^{th}$ root of unity. That is

$$
\Phi_n(x) = \prod_{a \in \mathbb{G}_n(1)} (x - \omega_n^a),
$$

where $\omega_n = \exp\left(\frac{2\pi i}{n}\right)$. Clearly the degree of $\Phi_n(x)$ is $\varphi(n)$. See [11] for more details about cyclotomic polynomials.

Theorem 19. [11] The cyclotomic polynomial $\Phi_n(x)$ is irreducible in $\mathbb{Z}[x]$.

The polynomial $\Phi_n(x)$ is irreducible over $\mathbb{Q}(i)$ if and only if $[\mathbb{Q}(i, \omega_n) : \mathbb{Q}(i)] = \varphi(n)$. Also $\mathbb{Q}(\omega_n)$ does not contain the number $i = \sqrt{-1}$ if and only if $n \not\equiv 0 \pmod{4}$. Thus, if $n \not\equiv 0 \pmod{4}$ then $[\mathbb{Q}(i, \omega_n) : \mathbb{Q}(\omega_n)] = 2 = [\mathbb{Q}(i), \mathbb{Q}]$, and therefore

$$
[\mathbb{Q}(i, \omega_n) : \mathbb{Q}(i)] = \frac{[\mathbb{Q}(i, \omega_n) : \mathbb{Q}(\omega_n)] [\mathbb{Q}(\omega_n) : \mathbb{Q}]}{[\mathbb{Q}(i) : \mathbb{Q}]} = [\mathbb{Q}(\omega_n) : \mathbb{Q}] = \varphi(n).
$$

Hence for $n \not\equiv 0 \pmod{4}$, the polynomial $\Phi_n(x)$ is irreducible over $\mathbb{Q}(i)$.

Let $n \equiv 0 \pmod{4}$. Then $\mathbb{Q}(i, \omega_n) = \mathbb{Q}(\omega_n)$, and so

$$
[\mathbb{Q}(i, \omega_n) : \mathbb{Q}(i)] = \frac{[\mathbb{Q}(i, \omega_n) : \mathbb{Q}]}{[\mathbb{Q}(i) : \mathbb{Q}]} = \frac{\varphi(n)}{2}.
$$

Hence the polynomial $\Phi_n(x)$ is reducible over $\mathbb{Q}(i)$.
We know that $G_n(1)$ is a disjoint union of $G_n^1(1)$ and $G_n^3(1)$. Define

$$
\Phi_n^1(x) = \prod_{a \in G_n^1(1)} (x - \omega_n^a) \text{ and } \Phi_n^3(x) = \prod_{a \in G_n^3(1)} (x - \omega_n^a).
$$

It is clear from the definition that $\Phi_n(x) = \Phi_n^1(x)\Phi_n^3(x)$.

**Theorem 21.** Let $l$ and $Cay$.

**Proof.**

Note that, so that the mixed Cayley graph $S$ for $\exp$ irreducible monic polynomials in $\mathbb{Q}(i)[x]$ of degree $\frac{x^{\text{deg}}}{4}$.

In this section, first we prove that there is no integral oriented Cayley graph $Cay(\Gamma, S)$ for $\exp(\Gamma) \not\equiv 0 \pmod{4}$ and $S \neq \emptyset$. After that we find a necessary condition on the set $S$ so that the mixed Cayley graph $Cay(\Gamma, S)$ is integral.

**Theorem 20.** [12] Let $n \equiv 0 \pmod{4}$. The factors $\Phi_n^1(x)$ and $\Phi_n^3(x)$ of $\Phi_n(x)$ are irreducible monic polynomials in $\mathbb{Q}(i)[x]$ of degree $\frac{x^{\text{deg}}}{2}$.

In this section, first we prove that there is no integral oriented Cayley graph $Cay(\Gamma, S)$ for $\exp(\Gamma) \not\equiv 0 \pmod{4}$ and $S \neq \emptyset$. After that we find a necessary condition on the set $S$ so that the mixed Cayley graph $Cay(\Gamma, S)$ is integral.

**Proof.** Let $l = \exp(\Gamma)$ and $Sp_H(Cay(\Gamma, S)) = \{\mu_\alpha : \alpha \in \Gamma\}$. Assume that $l \not\equiv 0 \pmod{4}$ and $Cay(\Gamma, S)$ is integral. By Corollary 7, $\mu_\alpha = -\mu_{-\alpha} \in \mathbb{Q}$ and

$$
\mu_\alpha = i \sum_{s \in S} \left(\psi_\alpha(s) - \psi_\alpha(-s)\right) \text{ for all } \alpha \in \Gamma.
$$

Note that, $\psi_\alpha(s)$ and $\psi_\alpha(-s)$ are $l^{th}$ roots of unity for all $\alpha \in \Gamma, s \in S$. Fix a primitive $l^{th}$ root $\omega$ of unity and express $\psi_\alpha(s)$ in the form $\omega^j$ for some $j \in \{0, 1, \ldots, l-1\}$. Thus

$$
\mu_\alpha = i \sum_{s \in S} \left(\psi_\alpha(s) - \psi_\alpha(-s)\right) = \sum_{j=0}^{l-1} a_j \omega^j,
$$

where $a_j \in \mathbb{Q}(i)$. Since $\mu_\alpha \in \mathbb{Q}$, so $p(x) = \sum_{j=0}^{l-1} a_j x^j - \mu_\alpha \in \mathbb{Q}(i)[x]$ and $\omega$ is a root of $p(x)$. Since $l \not\equiv 0 \pmod{4}$, so $\Phi_l(x)$ is irreducible in $\mathbb{Q}(i)[x]$. Thus $p(\omega) = 0$ and $\Phi_l(x)$ is the monic irreducible polynomial over $\mathbb{Q}(i)$ having $\omega$ as a root. Therefore $\Phi_l(x)$ divides $p(x)$, and so $\omega^{-1} = \omega^{l-1}$ is also a root of $p(x)$. Note that, if $\psi_\alpha(s) = \omega^j$ for some $j \in \{0, 1, \ldots, l-1\}$ then $\psi_{-\alpha}(s) = \omega^{-j}$. We have

$$
0 = p(\omega^{-1}) = \sum_{j=0}^{l-1} a_j \omega^{-j} - \mu_\alpha = \mu_{-\alpha} - \mu_\alpha \Rightarrow \mu_\alpha = \mu_{-\alpha}.
$$

Since $\mu_{-\alpha} = -\mu_\alpha$, we get $\mu_\alpha = 0$, for all $\alpha \in \Gamma$. Hence $S = \emptyset$.

Conversely, if $S = \emptyset$ then all the eigenvalues of $Cay(\Gamma, S)$ are zero. Thus $Cay(\Gamma, S)$ is integral. \qed
Lemma 14 says that corresponding to each equivalence class of the relation $\sim$ we get two equivalence classes of the relation $\approx$. Define $E$ to be the matrix of size $n \times n$, whose rows and columns are indexed by elements of $\Gamma$ such that $E_{x,y} = i\psi_x(y)$. Note that each row of $E$ corresponds to a character of $\Gamma$ and $EE^* = nI_n$, where $E^*$ is the conjugate transpose of $E$. Let $v_{[x]}$ be the vector in $\mathbb{Q}^n$ whose coordinates are indexed by the elements of $\Gamma$, and the $a^{th}$ coordinate of $v_{[x]}$ is given by

$$v_{[x]}(a) = \begin{cases} 1 & \text{if } a \in [x], \\ -1 & \text{if } a \in [x^{-1}], \\ 0 & \text{otherwise}. \end{cases}$$

By Lemma 16, we have $Ev_{[x]} \in \mathbb{Q}^n$.

**Lemma 22.** Let $\Gamma$ be an abelian group, $v \in \mathbb{Q}^n$ and $Ev \in \mathbb{Q}^n$. Let the coordinates of $v$ be indexed by elements of $\Gamma$. Then

(i) $v_x = -v_{-x}$ for all $x \in \Gamma$.

(ii) $v_x = v_y$ for all $x, y \in \Gamma(4)$ satisfying $x \approx y$.

(iii) $v_x = 0$ for all $x \in \Gamma \setminus \Gamma(4)$.

**Proof.** Let $E_x$ and $E_y$ denote the column vectors of $E$ indexed by $x$ and $y$, respectively, and assume that $u = Ev \in \mathbb{Q}^n$. For $z \in \mathbb{C}$, let $\overline{z}$ denote the complex conjugate of $z$.

(i) We use the fact that $\overline{\psi_x(y)} = \psi_{-x}(y) = \psi_x(-y)$ for all $x, y \in \Gamma$. Again

$$u = Ev \Rightarrow E^*u = E^*Ev = (nI_n)v \Rightarrow \frac{1}{n}E^*u = v \in \mathbb{Q}^n.$$ 

Thus

$$v_x = \frac{1}{n}(E^*u)_x = \frac{1}{n} \sum_{a \in \Gamma} E_{x,a}^*u_a = \frac{1}{n} \sum_{a \in \Gamma} i\psi_a(x)u_a = -\frac{1}{n} \sum_{a \in \Gamma} i\psi_a(-x)u_a = -\frac{1}{n} \sum_{a \in \Gamma} i\psi_a(-x)u_a = -\frac{1}{n} \sum_{a \in \Gamma} E_{x,a}^*u_a = -\frac{1}{n}(E_{-x}^*u)_x = -v_{-x}.$$ 

(ii) If $\Gamma(4) = \emptyset$ then there is nothing to prove. Now assume that $\Gamma(4) \neq \emptyset$, so that $\text{exp}(\Gamma) \equiv 0 \pmod{4}$. Let $x, y \in \Gamma(4)$ and $x \approx y$. Then there exists $k \in G^1_m(1)$ such that $y = x^k$, where $m = \text{ord}(x) = \text{ord}(y)$. Assume that $x \neq y$, so that $k \geq 2$. Using
Lemma 3, entries of $E_x$ and $E_y$ are $i$ times an $m$th root of unity. Fix a primitive $m$th root of unity $\omega$, and express each entry of $E_x$ and $E_y$ in the form $i\omega^j$ for some $j \in \{0, 1, \ldots, m - 1\}$. Thus

$$nv_x = (E^*u)_x = \sum_{j=0}^{m-1} a_j \omega^j,$$

where $a_j \in \mathbb{Q}(i)$ for all $j$. Thus $\omega$ is a root of $p(x) = \sum_{j=0}^{m-1} a_j x^j - nv_x \in \mathbb{Q}(i)[x]$. Therefore, $p(x)$ is a multiple of the irreducible polynomial $\Phi_m^1(x)$, and so $\omega^k$ is also a root of $p(x)$, because of $k \in G^1_m(1)$. As $y = x^k$ implies that $\psi_y(a) = \psi_x(a)^k$ for all $a \in \Gamma$, we have $(E^*u)_y = \sum_{j=0}^{m-1} a_j \omega^{kj}$. Hence

$$0 = p(\omega^k) = \sum_{j=0}^{m-1} a_j \omega^{kj} - nv_x = (E^*u)_y - nv_x = nv_y - nv_x \Rightarrow v_x = v_y.$$

(iii) Let $x \in \Gamma \setminus \Gamma(4)$ and $r = \text{ord}(x) \not\equiv 0 \pmod{4}$. Fix a primitive $r$th root $\omega$ of unity, and express each entry of $E_x$ in the form $i\omega^j$ for some $j \in \{0, 1, \ldots, r - 1\}$. Thus

$$nv_x = (E^*u)_x = \sum_{j=0}^{r-1} a_j \omega^j,$$

where $a_j \in \mathbb{Q}(i)$ for all $j$. Thus $\omega$ is a root of $p(x) = \sum_{j=0}^{r-1} a_j x^j - nv_x \in \mathbb{Q}(i)[x]$. Therefore, $p(x)$ is a multiple of the irreducible polynomial $\Phi_r(x)$, and so $\omega^{-1}$ is also a root of $p(x)$. Since $\psi_{-x}(a) = \psi_x(a)^{-1}$ for all $a \in \Gamma$, therefore $(E^*u)_{-x} = \sum_{j=0}^{r-1} a_j \omega^{-j}$. Hence

$$0 = p(\omega^{-1}) = \sum_{j=0}^{r-1} a_j \omega^{-j} - nv_x = (E^*u)_{-x} - nv_x = nv_{-x} - nv_x,$$

implies that $v_x = v_{-x}$. This together with Part (i) imply that $v_x = 0$ for all $x \in \Gamma \setminus \Gamma(4)$. \hfill \Box

**Theorem 23.** Let $\Gamma$ be an abelian group. The oriented Cayley graph $\text{Cay}(\Gamma, S)$ is integral if and only if $S \in \mathbb{D}(\Gamma)$.

**Proof.** Assume that the oriented Cayley graph $\text{Cay}(\Gamma, S)$ is integral. If $\Gamma(4) = \emptyset$ then by Theorem 21, we have $S = \emptyset$, and so $S \in \mathbb{D}(\Gamma)$. Now assume that $\exp(\Gamma) \equiv 0 \pmod{4}$ so...
that $\Gamma(4) \neq \emptyset$. Let $v$ be the vector in $\mathbb{Q}^n$ whose coordinates are indexed by the elements of $\Gamma$, and the $x^{th}$ coordinate of $v$ is given by

$$\begin{align*}
v_x &= \begin{cases} 
1 & \text{if } x \in S, \\
-1 & \text{if } x \in S^{-1}, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}$$

We have

$$(Ev)_a = \sum_{x \in \Gamma} E_{a,x} v_x = \sum_{x \in S} E_{a,x} - \sum_{x \in S^{-1}} E_{a,x} = i \sum_{x \in S} (\psi_a(x) - \psi_a(-x)).$$

Thus $(Ev)_a$ is an eigenvalue of the integral oriented Cayley graph $\text{Cay}(\Gamma, S)$. Therefore $Ev \in \mathbb{Q}^n$, and hence all the three conditions of Lemma 22 hold.

By the third condition of Lemma 22, $v_x = 0$ for all $x \in \Gamma \setminus \Gamma(4)$, and so we must have $S \cup S^{-1} \subseteq \Gamma(4)$. Again, let $x \in S$, $y \in \Gamma(4)$ and $x \approx y$. The second condition of Lemma 22 gives $v_x = v_y$, which implies that $y \in S$. Thus $x \in S$ implies $[x] \subseteq S$. Hence $S \in \mathcal{D}(\Gamma)$. The converse part follows from Theorem 17.

The following example illustrates Theorem 23.

**Example 24.** Consider $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$ and $S = \{(0, 1), (1, 3)\}$. The graph $\text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_4, S)$ is shown in Figure 1a. We see that $\mathit{J}(0, 1) = \{(0, 1)\}$ and $\mathit{J}(1, 3) = \{(1, 3)\}$. Therefore $S \in \mathcal{D}(\Gamma)$. Further, using Corollary 7 and Equation 1, the eigenvalues of $\text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_4, S)$ are obtained as

$$\mu_{\alpha} = i(\psi_{\alpha}(0, 1) - \psi_{\alpha}(0, 3)) + i(\psi_{\alpha}(1, 3) - \psi_{\alpha}(1, 1))$$

for each $\alpha \in \mathbb{Z}_2 \times \mathbb{Z}_4$, where

$$\psi_{\alpha}(x) = (-1)^{\alpha_1 x_1} x_2^{\alpha_2}$$

for all $\alpha = (\alpha_1, \alpha_2), x = (x_1, x_2) \in \mathbb{Z}_2 \times \mathbb{Z}_4$.

It can be seen that $\mu_{(0,0)} = \mu_{(0,1)} = \mu_{(0,2)} = \mu_{(0,3)} = \mu_{(1,0)} = \mu_{(1,2)} = 0$, $\mu_{(1,1)} = -4$ and $\mu_{(1,3)} = 4$. Thus $\text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_4, S)$ is integral.

![Figure 1: The graph $\text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_4, S)$](image)

(a) $S = \{(0, 1), (1, 3)\}$

(b) $S = \{(0, 1), (0, 3), (1, 3)\}$
Theorem 25. Let $\Gamma$ be an abelian group. The mixed Cayley graph $\text{Cay}(\Gamma, S)$ is integral if and only if $S \setminus \overline{S} \in \mathcal{B}(\Gamma)$ and $\overline{S} \in \mathcal{D}(\Gamma)$.

Proof. By Theorem 8, the mixed Cayley graph $\text{Cay}(\Gamma, S)$ is integral if and only if both $\text{Cay}(\Gamma, S \setminus \overline{S})$ and $\text{Cay}(\Gamma, S)$ are integral. Note that $S \setminus \overline{S}$ is a symmetric set and $\overline{S}$ is a skew-symmetric set. Thus by Theorem 11, $\text{Cay}(\Gamma, S \setminus \overline{S})$ is integral if and only if $S \setminus \overline{S} \in \mathcal{B}(\Gamma)$. By Theorem 23, $\text{Cay}(\Gamma, \overline{S})$ is integral if and only if $S \in \mathcal{D}(\Gamma)$. Hence the result follows.

The following example illustrates Theorem 25.

Example 26. Consider $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$ and $S = \{(0,1), (0,3), (1,3)\}$. The graph $\text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_4, S)$ is shown in Figure 1b. Observe that $S = \{(1,3)\} = [(1,3)] \in \mathcal{D}(\Gamma)$ and $S \setminus \overline{S} = \{(0,1), (0,3)\} = [(0,1)] \in \mathcal{B}(\Gamma)$. Further, using Lemma 5 and Equation 1, the eigenvalues of $\text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_4, S)$ are obtained as

$$
\mu_{\alpha} = [\psi_{\alpha}(0,1) + \psi_{\alpha}(0,3)] + i[\psi_{\alpha}(1,3) - \psi_{\alpha}(1,1)] \quad \text{for each} \ \alpha \in \mathbb{Z}_2 \times \mathbb{Z}_4,
$$

where

$$
\psi_{\alpha}(x) = (-1)^{\alpha_1 x_1}i^{\alpha_2 x_2} \quad \text{for all} \ \alpha = (\alpha_1, \alpha_2), x = (x_1, x_2) \in \mathbb{Z}_2 \times \mathbb{Z}_4.
$$

One can see $\mu_{(0,0)} = \mu_{(0,1)} = \mu_{(1,0)} = \mu_{(1,3)} = 2$ and $\mu_{(0,2)} = \mu_{(0,3)} = \mu_{(1,1)} = \mu_{(1,2)} = -2$. Hence $\text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_4, S)$ is integral.

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