A Complete Real-Variable Theory of Hardy Spaces on Spaces of Homogeneous Type

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Abstract  Let \((X, d, \mu)\) be a space of homogeneous type, with the upper dimension \(\omega\), in the sense of R. R. Coifman and G. Weiss. Assume that \(\eta\) is the smoothness index of the wavelets on \(X\) constructed by P. Auscher and T. Hytönen. In this article, when \(p \in (\omega/(\omega + \eta), 1]\), for the atomic Hardy spaces \(H^p_{cw}(X)\) introduced by Coifman and Weiss, the authors establish their various real-variable characterizations, respectively, in terms of the grand maximal function, the radial maximal function, the non-tangential maximal functions, the various Littlewood-Paley functions and wavelet functions. This completely answers the question of R. R. Coifman and G. Weiss by showing that no any additional (geometrical) condition is necessary to guarantee the radial maximal function characterization of \(H^1_{cw}(X)\) and even of \(H^p_{cw}(X)\) with \(p\) as above. As applications, the authors obtain the finite atomic characterizations of \(H^p_{cw}(X)\), which further induce some criteria for the boundedness of sublinear operators on \(H^p_{cw}(X)\). Compared with the known results, the novelty of this article is that \(\mu\) is not assumed to satisfy the reverse doubling condition and \(d\) is only a quasi-metric, moreover, the range \(p \in (\omega/(\omega + \eta), 1]\) is natural and optimal.

1 Introduction

The real-variable theory of Hardy spaces plays a fundamental role in harmonic analysis. The classical Hardy space on the \(n\)-dimensional Euclidean space \(\mathbb{R}^n\) was initially developed by Stein and Weiss [48] and later by Fefferman and Stein [11]. Hardy spaces \(H^p(\mathbb{R}^n)\) have been proved to be a suitable substitute of Lebesgue spaces \(L^p(\mathbb{R}^n)\) with \(p \in (0, 1]\) in the study of the boundedness of operators. Indeed, any element in the Hardy space can be decomposed into a sum of some basic elements (which are called atoms); see Coifman [5] for \(n = 1\) and Latter [36] for general \(n \in \mathbb{N}\). Characterizations of Hardy spaces via Littlewood-Paley functions were due to Uchiyama [49]. For more study on classical Hardy spaces on \(\mathbb{R}^n\), we refer the reader to the well-known monographs [47, 41, 16, 17, 19]. Modern developments regarding the real-variable theory of Hardy spaces are so deep and vast that we can only list a few literatures here, for example, the theory of Hardy spaces associated with operators (see [2, 3, 30, 10]), Hardy spaces with variable exponents (see [20, 21, 22]).
The real-variable theory of Musielak-Orlicz Hardy spaces (see [35, 51]), and also Hardy spaces for ball quasi-Banach spaces (see [46]).

In this article, we focus on the real-variable theory of Hardy spaces on spaces of homogeneous type. It is known that the space of homogeneous type introduced by Coifman and Weiss [6, 7] provides a natural setting for the study of both functions spaces and the boundedness of operators. A quasi-metric space \((X, d)\) is a non-empty set \(X\) equipped with a quasi-metric \(d\), that is, a non-negative function defined on \(X \times X\), satisfying that, for any \(x, y, z \in X\),

(i) \(d(x, y) = 0\) if and only if \(x = y\);

(ii) \(d(x, y) = d(y, x)\);

(iii) there exists a constant \(A_0 \in [1, \infty)\) such that \(d(x, z) \leq A_0[d(x, y) + d(y, z)]\).

The ball \(B\) on \(X\) centered at \(x_0 \in X\) with radius \(r \in (0, \infty)\) is defined by setting

\[
B := B(x_0, r) := \{x \in X : d(x, x_0) < r\}.
\]

For any ball \(B\) and \(r \in (0, \infty)\), denote by \(\tau B\) the ball with the same center as that of \(B\) but of radius \(\tau\) times that of \(B\). Given a quasi-metric space \((X, d)\) and a non-negative measure \(\mu\), we call \((X, d, \mu)\) a space of homogeneous type if \(\mu\) satisfies the doubling condition: there exists a positive constant \(C_{(\mu)} \in [1, \infty)\) such that, for any ball \(B \subset X\),

\[
\mu(2B) \leq C_{(\mu)} \mu(B).
\]

The above doubling condition is equivalent to that, for any ball \(B\) and \(\lambda \in [1, \infty)\),

\[
\mu(\lambda B) \leq C_{(\mu)} \lambda^{\omega} \mu(B),
\]

where \(\omega := \log_2 C_{(\mu)}\) is called the upper dimension of \(X\). If \(A_0 = 1\), we call \((X, d, \mu)\) a doubling metric measure space.

According to [7] pp. 587–588, we always make the following assumptions throughout this article. For any point \(x \in X\), assume that the balls \(B(x, r)\) for \(r \in (0, \infty)\) form a basis of open neighborhoods of \(x\); assume that \(\mu\) is Borel regular, which means that open sets are measurable and every set \(A \subset X\) is contained in a Borel set \(E\) satisfying that \(\mu(A) = \mu(E)\); we also assume that \(\mu(B(x, r)) \in (0, \infty)\) for any \(x \in X\) and \(r \in (0, \infty)\). For the presentation concision, we always assume that \((X, d, \mu)\) is non-atomic [namely, \(\mu(\{x\}) = 0\) for any \(x \in X\)] and diam\((X) := \sup\{d(x, y) : x, y \in X\} = \infty\). It is known that diam\((X) = \infty\) implies that \(\mu(X) = \infty\) (see, for example, [1] Lemma 8.1]).

Let us recall the notion of the atomic Hardy space on spaces of homogeneous type introduced by Coifman and Weiss [7]. For any \(\alpha \in (0, \infty)\), the Lipschitz space \(L_{\alpha}(X)\) is defined to be the collection of all measurable functions \(f\) such that

\[
\|f\|_{L_{\alpha}(X)} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\mu(B(x, d(x, y)))} < \infty.
\]

Denote by \((L_{\alpha}(X))'\) the dual space of \(L_{\alpha}(X)\) equipped with the weak-* topology.

**Definition 1.1.** Let \(p \in (0, 1]\) and \(q \in (p, \infty] \cap [1, \infty]\). A function \(a\) is called a \((p, q)\)-atom if
(i) \( \text{supp } a := \{ x \in X : a(x) \neq 0 \} \subset B(x_0, r) \) for some \( x_0 \in X \) and \( r \in (0, \infty) \);

(ii) \( \left[ \int_X |a(x)|^q \, d\mu(x) \right]^\frac{1}{q} \leq \left[ \mu(B(x_0, r)) \right]^{\frac{1}{q} - \frac{1}{r}} \);

(iii) \( \int_X a(x) \, d\mu(x) = 0. \)

The atomic Hardy space \( H_{cw}^{p,q}(X) \) is defined as the subspace of \( (L_{1/p-1}(X))' \) when \( p \in (0, 1) \) or of \( L^1(X) \) when \( p = 1 \), which consists of all the elements \( f \) admitting an atomic decomposition

\[
f = \sum_{j=0}^{\infty} \lambda_j a_j,
\]

where \( \{a_j\}_{j=0}^{\infty} \) are \( (p,q) \)-atoms, \( \{\lambda_j\}_{j=0}^{\infty} \subset \mathbb{C} \) satisfies \( \sum_{j=0}^{\infty} |\lambda_j|^p < \infty \) and the series in (1.2) converges in \( (L_{1/p-1}(X))' \) when \( p \in (0, 1) \) or in \( L^1(X) \) when \( p = 1 \). Define

\[
\|f\|_{H_{cw}^{p,q}(X)} := \inf \left\{ \left( \sum_{j=0}^{\infty} |\lambda_j|^p \right)^{\frac{1}{p}} \right\},
\]

where the infimum is taken over all the representations of \( f \) as in (1.2).

It was proved in [7] that the atomic Hardy space \( H_{cw}^{p,q}(X) \) is independent of the choice of \( q \) and hence we sometimes write \( H_{cw}^p(X) \) for short. It was also proved in [7] that the dual space of \( H_{cw}^p(X) \) is the Lipschitz space \( L_{1/p-1}(X) \) when \( p \in (0, 1) \), and the space \( \text{BMO}(X) \) of bounded mean oscillation when \( p = 1 \).

It is well known that the most basic result in the real-variable theory of Hardy spaces is their characterizations in terms of maximal functions. Coifman and Weiss [7] pp. 641–642] observed that a proof of the duality result between \( H^1(\mathbb{R}^n) \) and \( \text{BMO}(\mathbb{R}^n) \) from Carleson [4] can be extended to the general setting of spaces of homogeneous type provided a certain additional geometrical assumption is added, from which one can then obtain a radial maximal function characterization of \( H_{cw}^1(X) \). Coifman and Weiss [7] p. 642] then asked that to what extent their geometrical condition is necessary for the validity of the radial maximal characterization of \( H_{cw}^1(X) \). Since then, lots of efforts are made to build various real-variable characterizations of the atomic Hardy spaces on spaces of homogeneous type with few geometrical assumptions. In this article, we completely answer the aforementioned question of Coifman and Weiss by showing that no additional (geometrical) condition is necessary to guarantee the radial maximal function characterization of \( H_{cw}^1(X) \) and even of \( H_{cw}^p(X) \) with \( p \leq 1 \) but near to 1.

Recall that a triple \((X,d,\mu)\) is said to be Ahlfors-\( n \) regular if \( \mu(B(x,r)) \sim r^n \) for any \( x \in X \) and \( r \in (0, \text{diam } X) \) with equivalent positive constants independent of \( x \) and \( r \). When \((X,d,\mu)\) is Ahlfors-\( n \) regular, upon assuming the quasi-metric \( d \) satisfying that there exists \( \theta \in (0, 1) \) such that, for any \( x, x', y \in X \),

\[
|d(x,y) - d(x',y)| \leq (d(x,x'))^\theta |d(x,y) + d(x',y)|^{1-\theta},
\]

Macías and Segovia [43] characterized Hardy spaces via the grand maximal functions, and Li [37] obtained another grand maximal function characterization via test functions introduced in [28].
Also, Duong and Yan [9] characterized Hardy spaces via the Lusin area function associated with certain semigroup.

Recall that an RD-space \((X, d, \mu)\) is a doubling metric measure space with the measure \(\mu\) further satisfying the reverse doubling condition, that is, there exist a positive constant \(C \in (0, 1)\) and \(\kappa \in (0, \omega]\) such that, for any ball \(B(x, r)\) with \(x \in X, r \in (0, \text{diam } X/2)\) and \(\lambda \in [1, \text{diam } X/(2r)]\),

\[
\tilde{C} \lambda^\kappa \mu(B(x, r)) \leq \mu(B(x, \lambda r)).
\]

Indeed, any path connected doubling metric measure space is an RD-space (see [27, 55]). Characterizations of Hardy spaces on RD-spaces via various Littlewood-Paley functions were established in [26, 27]. Also, characterizations of Hardy spaces on RD-spaces via various maximal functions can be found in [20, 21, 54]. It should be mentioned that local Hardy spaces can be used to characterize more general scale of function spaces like Besov and Triebel-Lizorkin spaces on RD-spaces (see [55]). For a systematic study of Besov and Triebel-Lizorkin spaces on RD-spaces, we refer the reader to [27]. More on analysis over Ahlfors-n regular metric measure spaces or RD-spaces can be found in [18, 22, 33, 34, 52, 32, 55, 8, 56].

The main motivation of studying the real-variable theory of function spaces and the boundedness of operators on spaces of homogeneous type comes from the celebrated work of Auscher and Hytönen [11], in which they constructed an orthonormal wavelet basis \(\{\psi_k^{\alpha} : k \in \mathbb{Z}, \alpha \in \mathcal{G}_k\}\) of \(L^2(X)\) with Hölder continuity exponent \(\eta \in (0, 1)\) and exponential decay by using the system of random dyadic cubes. The first creative attempt of using the idea of [11] to investigate the real-variable theory of Hardy spaces on spaces of homogeneous type was due to Han et al. [23] (see also Han et al. [24]). Indeed, in [23], Hardy spaces via wavelets on spaces of homogeneous type were introduced and then these spaces were proved to have atomic decompositions. The method used in [23] is based on a new Calderón reproducing formula on spaces of homogeneous type (see [23, Proposition 2.5]). But there exists an error in the proof of [23 Proposition 2.5], namely, since the regularity exponent of the approximations of the identity in [23, p. 3438] is \(\theta\) [indeed, \(\theta\) is from the regularity of the quasi-metric \(d\) in [13]], it follows that the regularity exponent in [23 (2.6)] should be \(\min\{\theta, \eta\}\) and hence the correct range of \(p\) in [23 Proposition 2.5] (indeed, all results of [23]) seems to be \((\omega)/(\omega + \min\{\theta, \eta\}), 1]\) which is not optimal. Moreover, the criteria of the boundedness of Calderón-Zygmund operators on the dual of Hardy spaces were established in [25]. Also, Fu and Yang [14] obtained an unconditional basis of \(H_{\text{cw}}^1(X)\) and several equivalent characterizations of \(H_{\text{cw}}^p(X)\) in terms of wavelets.

Another motivation of this article comes from the Calderón reproducing formulae established in [29]. Indeed, the work of [29] was partly motivated by the wavelet theory of Auscher and Hytönen in [11] and a corresponding wavelet reproducing formula (which can converge in the distribution space) in [29]. The already existing works (see [26, 27, 20, 54, 55]) regarding Hardy spaces on RD-spaces show the feasibility of establishing various real-variable characterizations of the atomic Hardy spaces on spaces of homogeneous type via the Calderón reproducing formulae. It should be mentioned that a characterization of the atomic Hardy spaces via the Littlewood-Paley functions was established in [25] via the aforementioned wavelet reproducing formula; see also [25] for some corresponding conclusions of product Hardy spaces on spaces of homogeneous type.

In this article, motivated by [23, 29], for the atomic Hardy spaces \(H_{\text{cw}}^p(X)\) with any \(p \in (\omega)/(\omega + \eta), 1]\), we establish their various real-variable characterizations, respectively, in terms of the grand maximal function, the radial maximal function, the non-tangential maximal function,
the various Littlewood-Paley functions and wavelets. Observe that these characterizations are true for $H^p_{cw}(X)$ with $p \in (\omega/(\omega + \eta), 1]$ and $X$ being any space of homogeneous type without any additional (geometrical) conditions, which completely answers the aforementioned question asked by Coifman and Weiss [7, p. 642]. As an application, we obtain the finite atomic characterizations of Hardy spaces, which further induce some criteria for the boundedness of sublinear operators on Hardy spaces. Compared with the known results, the novelty of this article is that $\mu$ is not assumed to satisfy the reverse doubling condition and $d$ is only a quasi-metric. Moreover, the range of $p \in (\omega/(\omega + \eta), 1]$ for the various maximal function characterizations and the Littlewood-Paley function characterizations of the atomic Hardy spaces $H^p_{cw}(X)$ is natural and optimal. The key tool used through this article is those Calderón reproducing formulae from [29].

In addition, we point out that, when $X$ is a doubling metric measure space, the finite atomic characterizations of Hardy spaces are also useful in establishing the bilinear decomposition of the product space $H^1_{cw}(X) \times \text{BMO}(X)$ and $H^p_{cw}(X) \times L^{1/p-1}(X)$, with $p \in (\omega/(\omega + \eta), 1]$ in [15, 40, 13, 14], and also in the study of the endpoint boundedness of commutators generated by Calderón-Zygmund operators and BMO($X$) functions in [38, 39].

The organization of this article is as follows.

In Section 2 we recall the notions of the space of test functions and the space of distributions introduced in [26], as well as the random dyadic cubes in [11] and the approximation of the identity with exponential decay introduced in [29]. Then we restate the Calderón reproducing formulae established in [29].

Section 3 concerns Hardy spaces defined via the grand maximal function, the radial maximal function and the non-tangential maximal function. We show that these Hardy spaces are all equivalent to the Lebesgue space $L^p(X)$ when $p \in (1, \infty)$ (see Section 3.1), and they are all mutually equivalent when $p \in (\omega/(\omega + \eta), 1]$ (see Section 3.2), all in the sense of equivalent (quasi-)norms. The proof for the latter borrows some ideas from [54] and uses the Calderón reproducing formulae built in [29]. Moreover, we prove that the Hardy space $H^{r,p}(X)$ defined via the grand maximal function is independent of the choices of the distribution space $(\mathcal{G}^0_r(\beta, \gamma))'$ whenever $\beta, \gamma \in (\omega[1/p - 1], \eta)$; see Proposition 3.3 below.

Section 4 is devoted to the atomic characterization of $H^{r,p}(X)$. Notice that, if a distribution has an atomic decomposition, then it belongs to $H^{r,p}(X)$ obviously by the definition of atoms; see Section 4.1. All we remain to do is to establish the converse relationship. In Section 4.2 by modifying the definition of the grand maximal function $f^*$ to $\tilde{f}^*$ so that the level set $\{x \in X: f^*(x) > \lambda\}$ with $\lambda \in (0, \infty)$ is open, we then apply the partition of unity to the open set $\Omega_\lambda$ and obtain a Calderón-Zygmund decomposition of $f \in H^{r,p}(X)$. This is further used in Section 4.3 to construct an atomic decomposition of $f$. In Section 4.4 we compare the atomic Hardy spaces $H^{r,p}_{at}(X)$ with $H^{r,p}_{cw}(X)$ and prove that they are exactly the same space in the sense of equivalent (quasi-)norms.

Section 5 deals with the Littlewood-Paley theory of Hardy spaces. In Section 5.1 we show that the Hardy space $H^p(X)$, defined via the Lusin area function, is independent of the choices of exp-ATIs. In Section 5.2 we use the homogeneous continuous Calderón reproducing formula and the molecular characterizations of the atomic Hardy spaces (see [39]) to establish the atomic decompositions of elements in $H^p(X)$, and then we connect $H^p(X)$ with $H^{r,p}(X)$. In Section 5.3 we characterize Hardy spaces $H^p(X)$ via the Lusin area function with aperture, the Littlewood-Paley $g$-function and the Littlewood-Paley $g^*_\lambda$-function.
In Section 6, we consider the Hardy space $H^p_0(X)$ defined via wavelets, which was introduced in [23]. We improve the result of [25, Theorem 4.3] and prove that $H^p_0(X)$ coincides with $H^p(X)$ in the sense of equivalent (quasi-)norms.

In Section 7, as an application, we obtain criteria of the boundedness of the sublinear operators from Hardy spaces to quasi-Banach spaces. To this end, we first establish the finite atomic characterizations, namely, we show that, if $q \in (p, \infty) \cap [1, \infty)$, then $\| \cdot \|_{H^{p,q}_\text{fin}(X)}$ and $\| \cdot \|_{H^{p,q}_\text{at}(X)}$ are equivalent (quasi-)norms on a dense subspace $H^{p,q}_\text{fin}(X)$ of $H^{p,q}_\text{at}(X)$; the above equivalence also holds true on a dense subspace $H^{p,\infty}_\text{fin}(X) \cap UC(X)$ of $H^{p,\infty}_\text{at}(X)$, where $UC(X)$ denotes the space of all uniformly continuous functions on $X$.

At the end of this section, we make some conventions on notation. We always assume that $\omega$ is as in (1.1) and $\eta$ is the smoothness index of wavelets (see Theorem 7.1 or Definition 2.3 below). We assume that $\delta$ is a very small positive number, for example, $\delta \leq (2A_0)^{-\alpha}$ in order to construct the dyadic cube system and the wavelet system on $X$ (see [31, Theorem 2.2] or Lemma 2.3 below). For any $x, y \in X$ and $r \in (0, \infty)$, let

$$V_r(x) := \mu(B(x, r)) \quad \text{and} \quad V(x, y) := \mu(B(x, d(x, y))),$$

where $B(x, r) := \{ y \in X : d(x, y) < r \}$. We always let $\mathbb{N} := \{1, 2, \ldots \}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For any $p \in [1, \infty]$, we use $p'$ to denote its conjugate index, namely, $1/p + 1/p' = 1$. The symbol $C$ denotes a positive constant which is independent of the main parameters, but it may vary from line to line. We also use $C_{(\alpha, \beta, \ldots)}$ to denote a positive constant depending on the indicated parameters $\alpha, \beta, \ldots$. The symbol $A \lesssim B$ means that there exists a positive constant $C$ such that $A \leq CB$. The symbol $A \sim B$ is used as an abbreviation of $A \lesssim B \lesssim A$. We also use $A \lesssim_{\alpha, \beta, \ldots} B$ to indicate that here the implicit positive constant depends on $\alpha, \beta, \ldots$ and, similarly, $A \sim_{\alpha, \beta, \ldots} B$. For any $s, t \in \mathbb{R}$, denote the minimum of $s$ and $t$ by $s \wedge t$. For any finite set $\mathcal{J}$, we use $\#\mathcal{J}$ to denote its cardinality. Also, for any set $E$ of $X$, we use $\chi_E$ to denote its characteristic function and $E^C$ the set $X \setminus E$.

2 Calderón reproducing formulae

This section is devoted to recalling Calderón reproducing formulae obtained in [29]. To this end, we first recall the notions of both the space of test functions and the distribution space.

**Definition 2.1.** Let $x_1 \in X$, $r \in (0, \infty)$, $\beta \in (0, 1]$ and $\gamma \in (0, \infty)$. A function $f$ defined on $X$ is called a test function of type $(x_1, r, \beta, \gamma)$, denoted by $f \in \mathcal{G}(x_1, r, \beta, \gamma)$, if there exists a positive constant $C$ such that

(i) (the size condition) for any $x \in X$,

$$|f(x)| \leq C \frac{1}{V_r(x_1) + V(x_1, x)} \left(\frac{r}{r + d(x_1, x)}\right)^\gamma;$$

(ii) (the regularity condition) for any $x, y \in X$ satisfying $d(x, y) \leq (2A_0)^{-1}[r + d(x_1, x)]$,

$$|f(x) - f(y)| \leq C \left[\frac{d(x, y)}{r + d(x_1, x)}\right]^{\beta} \frac{1}{V_r(x_1) + V(x_1, x)} \left(\frac{r}{r + d(x_1, x)}\right)^\gamma.$$
For any \( f \in \mathcal{G}(x_1, r, \beta, \gamma) \), define the norm
\[
\|f\|_{\mathcal{G}(x_1, r, \beta, \gamma)} := \inf \{ C \in (0, \infty) : C \text{ satisfies (i) and (ii)} \}.
\]

Define
\[
\hat{\mathcal{G}}(x_1, r, \beta, \gamma) := \left\{ f \in \mathcal{G}(x_1, r, \beta, \gamma) : \int_X f(x) \, d\mu(x) = 0 \right\}
\]
equipped with the norm \( \| \cdot \|_{\hat{\mathcal{G}}(x_1, r, \beta, \gamma)} := \| \cdot \|_{\hat{\mathcal{G}}(x_1, r, \beta, \gamma)} \).

Observe that the above version of \( \hat{\mathcal{G}}(x_1, r, \beta, \gamma) \) was originally introduced by Han et al. \[27\] (see also \[26\]).

Fix \( x_0 \in X \). For any \( x \in X \) and \( r \in (0, \infty) \), we know that \( \mathcal{G}(x, r, \beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma) \) with equivalent norms, but the equivalent positive constants depend on \( x \) and \( r \). Obviously, \( \mathcal{G}(x_0, 1, \beta, \gamma) \) is a Banach space. In what follows, we simply write \( \mathcal{G}(\beta, \gamma) := \mathcal{G}(x_0, 1, \beta, \gamma) \) and \( \hat{\mathcal{G}}(\beta, \gamma) := \hat{\mathcal{G}}(x_0, 1, \beta, \gamma) \).

Fix \( \varepsilon \in (0, 1] \) and \( \beta, \gamma \in (0, \varepsilon) \). Let \( \mathcal{G}_0^\varepsilon(\beta, \gamma) \) [resp., \( \hat{\mathcal{G}}_0^\varepsilon(\beta, \gamma) \)] be the completion of the set \( \mathcal{G}(\varepsilon, \beta, \gamma) \) [resp., \( \hat{\mathcal{G}}(\varepsilon, \beta, \gamma) \)] in \( \mathcal{G}(\beta, \gamma) \), that is, if \( f \in \mathcal{G}_0^\varepsilon(\beta, \gamma) \) [resp., \( f \in \hat{\mathcal{G}}_0^\varepsilon(\beta, \gamma) \)], then there exists \( \{ \phi_j \}_{j=1}^{\infty} \subset \mathcal{G}(\varepsilon, \beta, \gamma) \) [resp., \( \{ \phi_j \}_{j=1}^{\infty} \subset \hat{\mathcal{G}}(\varepsilon, \beta, \gamma) \)] such that \( \| \phi_j - f \|_{\mathcal{G}(\beta, \gamma)} \to 0 \) as \( j \to \infty \). If \( f \in \mathcal{G}_0^\varepsilon(\beta, \gamma) \) [resp., \( f \in \hat{\mathcal{G}}_0^\varepsilon(\beta, \gamma) \)], we then let
\[
\|f\|_{\mathcal{G}_0^\varepsilon(\beta, \gamma)} := \|f\|_{\mathcal{G}(\beta, \gamma)} \quad \text{[resp., } \|f\|_{\hat{\mathcal{G}}_0^\varepsilon(\beta, \gamma)} := \|f\|_{\hat{\mathcal{G}}(\beta, \gamma)} \].
\]

The dual space \( (\mathcal{G}_0^\varepsilon(\beta, \gamma))' \) [resp., \( (\hat{\mathcal{G}}_0^\varepsilon(\beta, \gamma))' \)] is defined to be the set of all continuous linear functionals on \( \mathcal{G}_0^\varepsilon(\beta, \gamma) \) [resp., \( \hat{\mathcal{G}}_0^\varepsilon(\beta, \gamma) \)] and equipped with the weak-* topology. The spaces \( \mathcal{G}_0^\varepsilon(\beta, \gamma) \) and \( \hat{\mathcal{G}}_0^\varepsilon(\beta, \gamma) \) are called the spaces of distributions.

Let \( L^1_{\text{loc}}(X) \) be the space of all locally integrable functions on \( X \). Denote by \( \mathcal{M} \) the Hardy-Littlewood maximal operator, that is, for any \( f \in L^1_{\text{loc}}(X) \) and \( x \in X \),
\[
\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y),
\]
where the supremum is taken over all balls \( B \) of \( X \) that contain \( x \). For any \( p \in (0, \infty) \), the Lebesgue space \( L^p(X) \) is defined to be the set of all \( \mu \)-measurable functions \( f \) such that
\[
\|f\|_{L^p(X)} := \left[ \int_X |f(x)|^p \, d\mu(x) \right]^{\frac{1}{p}} < \infty
\]
with the usual modification made when \( p = \infty \); the weak Lebesgue space \( L^{p, \infty}(X) \) is defined to be the set of all \( \mu \)-measurable functions \( f \) such that
\[
\|f\|_{L^{p, \infty}(X)} := \sup_{\lambda \in (0, \infty)} \lambda \mu(\{ x \in X : |f(x)| > \lambda \})^{\frac{1}{p}} < \infty.
\]
It is known (see \[7\]) that \( \mathcal{M} \) is bounded on \( L^p(X) \) when \( p \in (1, \infty) \) and bounded from \( L^1(X) \) to \( L^{1, \infty}(X) \). Then we state some estimates from \[27\] Lemma 2.1, which are proved by using \[1.1\].
Lemma 2.2. Let $\beta, \gamma \in (0, \infty)$.

(i) For any $x, y \in X$ and $r \in (0, \infty)$, $V(x, y) \sim V(y, x)$ and

$$V_r(x) + V_r(y) + V(x, y) \sim V_r(x) + V(x, y) \sim V_r(y) + V(x, y) \sim \mu(B(x, r + d(x, y))),$$

where the equivalent positive constants are independent of $x$, $y$ and $r$.

(ii) There exists a positive constant $C$ such that, for any $x_1 \in X$ and $r \in (0, \infty),$

$$\int_X \frac{1}{V_r(x_1) + V(x_1, x)} \left[ \frac{r}{r + d(x_1, x)} \right]^\gamma d\mu(x) \leq C.$$

(iii) There exists a positive constant $C$ such that, for any $x \in X$ and $R \in (0, \infty),$

$$\int_{d(x, y) \leq R} \frac{1}{V(x, y)} \left[ \frac{d(x, y)}{R} \right]^\beta d\mu(y) \leq C \quad \text{and} \quad \int_{d(x, y) \geq R} \frac{1}{V(x, y)} \left[ \frac{R}{d(x, y)} \right]^\beta d\mu(y) \leq C.$$

(iv) There exists a positive constant $C$ such that, for any $x_1 \in X$ and $r \in (0, \infty),$

$$\int_{d(x_1, x) \geq R} \frac{1}{V_r(x_1) + V(x_1, x)} \left[ \frac{r}{r + d(x_1, x)} \right]^\gamma d\mu(x) \leq C \left( \frac{r}{r + R} \right)^\gamma.$$

(v) There exists a positive constant $C$ such that, for any $r \in (0, \infty)$, $f \in L^1_{loc}(X)$ and $x \in X,$

$$\int_X \frac{1}{V_r(x) + V(x, y)} \left[ \frac{r}{r + d(x, y)} \right]^\gamma |f(y)| d\mu(y) \leq CM(f)(x).$$

Next we recall the system of dyadic cubes established in [31, Theorem 2.2] (see also [1]), which is restated in the following version.

Lemma 2.3. Fix constants $0 < c_0 \leq C_0 < \infty$ and $\delta \in (0, 1)$ such that $12A_0^2C_0\delta \leq c_0$. Assume that a set of points, $\{z^k_\alpha : k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\} \subset X$ with $\mathcal{A}_k$ for any $k \in \mathbb{Z}$ being a countable set of indices, has the following properties: for any $k \in \mathbb{Z},$

(i) $d(z^k_\alpha, z^k_\beta) \geq c_0 \delta^k$ if $\alpha \neq \beta;$

(ii) $\min_{\alpha \in \mathcal{A}_k} d(x, z^k_\alpha) \leq C_0 \delta^k$ for any $x \in X.$

Then there exists a family of sets, $\{Q^k_\alpha : k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\},$ satisfying

(iii) for any $k \in \mathbb{Z}$, $\bigcup_{\alpha \in \mathcal{A}_k} Q^k_\alpha = X$ and $\{Q^k_\alpha : \alpha \in \mathcal{A}_k\}$ is disjoint;

(iv) if $k, l \in \mathbb{Z}$ and $l \geq k,$ then either $Q^k_\beta \subset Q^l_\alpha$ or $Q^k_\beta \cap Q^l_\alpha = \emptyset;$

(v) for any $k \in \mathbb{Z}$ and $\alpha \in \mathcal{A}_k,$ $B(z^k_\alpha, c_0 \delta^k) \subset Q^k_\alpha \subset B(z^k_\alpha, C_0 \delta^k),$ where $c_0 := (3A_0^2)^{-1}c_0,$ $C_0 := 2A_0C_0$ and $z^k_\alpha$ is called “the center” of $Q^k_\alpha.$
Throughout this article, we keep the notation used in Lemma 2.3. Moreover, for any $k \in \mathbb{Z}$, let
\[ \mathcal{X}^k := \{ z^k_α \}_{α \in A_k}, \quad \mathcal{G}_k := \mathcal{A}_{k+1} \setminus \mathcal{A}_k \quad \text{and} \quad \mathcal{Y}^k := \{ z^{k+1}_δ \}_{δ \in G_k} =: \{ y^k_α \}_{α \in G_k}. \]

Next we recall the notion of approximations of the identity with exponential decay introduced in [29].

**Definition 2.4.** A sequence $\{ Q_k \}_{k \in \mathbb{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is called an approximation of the identity with exponential decay (for short, exp-ATI) if there exist constants $C, \nu \in (0, \infty), a \in (0, 1)$ and $η \in (0, 1)$ such that, for any $k \in \mathbb{Z}$, the kernel of operator $Q_k$, which is still denoted by $Q_k$, satisfying

(i) (the identity condition) $\sum_{k=-\infty}^{\infty} Q_k = I$ in $L^2(\mathcal{X})$, where $I$ is the identity operator on $L^2(\mathcal{X})$;

(ii) (the size condition) for any $x, y \in \mathcal{X}$,
\[
|Q_k(x, y)| \leq C \frac{1}{\sqrt{V_\phi(x) V_\phi(y)}} \exp \left\{ -\nu \left[ \frac{d(x, y)}{\delta^k} \right]^a \right\} \times \exp \left\{ -\nu \left[ \max \{d(x, y^k), d(y, y^k)\} \right]^a \right\};
\]

(iii) (the regularity condition) for any $x, x', y \in \mathcal{X}$ with $d(x, x') \leq \delta^k$,
\[
|Q_k(x, y) - Q_k(x', y)| + |Q_k(y, x) - Q_k(y, x')| \leq C \left[ \frac{d(x, x')}{\delta^k} \right]^q \frac{1}{\sqrt{V_\phi(x) V_\phi(y)}} \exp \left\{ -\nu \left[ \frac{d(x, y)}{\delta^k} \right]^a \right\} \times \exp \left\{ -\nu \left[ \max \{d(x, y^k), d(y, y^k)\} \right]^a \right\};
\]

(iv) (the second difference regularity condition) for any $x, x', y, y' \in \mathcal{X}$ with $d(x, x') \leq \delta^k$ and $d(y, y') \leq \delta^k$, then
\[
\|Q_k(x, y) - Q_k(x', y) - [Q_k(x, y') - Q_k(x', y')]\| 
\leq C \left[ \frac{d(x, x')}{\delta^k} \right]^q \left[ \frac{d(y, y')}{\delta^k} \right]^q \frac{1}{\sqrt{V_\phi(x) V_\phi(y)}} \exp \left\{ -\nu \left[ \frac{d(x, y)}{\delta^k} \right]^a \right\} \times \exp \left\{ -\nu \left[ \max \{d(x, y^k), d(y, y^k)\} \right]^a \right\};
\]

(v) (the cancelation condition) for any $x, y \in \mathcal{X}$,
\[
\int_\mathcal{X} Q_k(x, y') \, d\mu(y') = 0 = \int_\mathcal{X} Q_k(x', y) \, d\mu(x').
\]
Remark 2.5. By [29] Remark 2.8, we know that the factor \( \frac{1}{\sqrt{\nu(x)} V_\nu(y)} \) in (2.1), (2.2) and (2.3) can be replaced by \( \frac{1}{\nu(x)} \) or \( \frac{1}{\nu(y)} \), and max\(d(x, y^j), d(y, y^k)\) by \(d(x, y^k)\) or by \(d(y, y^k)\), with \(\exp(-v|\frac{d(x, y^k)}{\beta}|^p)\) replaced by \(\exp(-v'|\frac{d(x, y^k)}{\beta}|^p)\), where \(v' \in (0, v)\) only depends on \(a\) and \(A_0\). Moreover, the condition in Definition 2.4 (iii) [resp., (iv)] can be replaced by \(d(x, x') \leq (2A_0)^{-1}[\delta^k + d(x, y)]\) (resp., \(d(x, x') \leq (2A_0)^{-2}[\delta^k + d(x, y)]\)) and \(d(y, y') \leq (2A_0)^{-2}[\delta^k + d(x, y)]\). For their proofs, see [29] Proposition 2.9.

With the above exp-ATI, we have the following homogeneous continuous Calderón reproducing formula established in [29].

Theorem 2.6. Let \(\{Q_k\}_{k \in \mathbb{Z}}\) be an exp-ATI and \(\beta, \gamma \in (0, \eta)\). Then there exists a sequence \(\{\tilde{Q}_k\}_{k \in \mathbb{Z}}\) of bounded linear operators on \(L^2(X)\) such that, for any \(f \in (\mathcal{G}_0^\beta(\beta, \gamma))^\prime\),

\[
f = \sum_{k = -\infty}^{\infty} \tilde{Q}_k f,
\]

where the series converges in \((\mathcal{G}_0^\beta(\beta, \gamma))^\prime\). Moreover, there exists a positive constant \(C\) such that, for any \(k \in \mathbb{Z}\), the kernel of \(\tilde{Q}_k\) satisfies the following conditions:

(i) for any \(x, y \in X\),

\[
\left| \tilde{Q}_k(x, y) \right| \leq C \frac{1}{\nu(x)} \left[ \frac{\delta^k}{\delta^k + d(x, y)} \right]^{\gamma};
\]

(ii) for any \(x, x', y \in X\) with \(d(x, x') \leq (2A_0)^{-1}[\delta^k + d(x, y)]\),

\[
\left| \tilde{Q}_k(x, y) - \tilde{Q}_k(x', y) \right| \leq C \left[ \frac{d(x, x')}{\delta^k + d(x, y)} \right]^{\beta} \frac{1}{\nu(x)} \left[ \frac{\delta^k}{\delta^k + d(x, y)} \right]^{\gamma};
\]

(iii) for any \(x \in X\),

\[
\int_X \tilde{Q}_k(x, y) \, d\mu(y) = 0 = \int_X \tilde{Q}_k(y, x) \, d\mu(y).
\]

Next, we recall the homogeneous discrete Calderón reproducing formulae established in [29].

To this end, let \(j_0 \in \mathbb{N}\) be a sufficiently large integer such that \(\delta^{j_0} \leq (2A_0)^{-4}C^2\), where \(C^2\) is as in Lemma 2.3. Based on Lemma 2.3, for any \(k \in \mathbb{Z}\) and \(\alpha \in \mathcal{A}_k\), we let

\[
N(k, \alpha) := \{\tau \in \mathcal{A}_{k+j_0} : Q_{\tau}^{k+j_0} \subset Q_{\alpha}^k\}
\]

and \(N(k, \alpha)\) be the cardinality of the set \(N(k, \alpha)\). For any \(k \in \mathbb{Z}\) and \(\alpha \in \mathcal{A}_k\), we rearrange the set \(\{Q_{\tau}^{k+j_0} : \tau \in N(k, \alpha)\}\) as \(\{Q_{\alpha}^m\}_{m=1}^{N(k, \alpha)}\), whose centers are denoted, respectively, by \(\{z_{k,m}^{N(k, \alpha)}\}_{m=1}^{N(k, \alpha)}\).

Theorem 2.7. Let \(\{Q_k\}_{k \in \mathbb{Z}}\) be an exp-ATI and \(\beta, \gamma \in (0, \eta)\). For any \(k \in \mathbb{Z}\), \(\alpha \in \mathcal{A}_k\) and \(m \in \{1, \ldots, N(k, \alpha)\}\), suppose that \(y_{\alpha}^{k,m}\) is an arbitrary point in \(Q_{\alpha}^m\). Then, for any \(i \in \{1, 2\}\), there exists a sequence \(\{\tilde{Q}_k^{(i)}\}_{k=\infty}^{\infty}\) of bounded linear operators on \(L^2(X)\) such that, for any \(f \in (\mathcal{G}_0^\beta(\beta, \gamma))^\prime\),

\[
f(\cdot) = \sum_{k=\infty}^{\infty} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(k, \alpha)} \tilde{Q}_k^{(i)}(\cdot, y_{\alpha}^{k,m}) \int_{Q_{\alpha}^m} Q_k f(y) \, d\mu(y)
\]
\[ \sum_{k=-\infty}^{\infty} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(\alpha)} \mu \left( \mathcal{Q}_k^{(m)} \right) \mathcal{Q}_k^{(2)} \left( \cdot, y_{k,m}^{\alpha} \right) Q_k f \left( y_{k,m}^{\alpha} \right), \]

where the equalities converge in \( (G^1_0(\beta, \gamma))' \). Moreover, for any \( k \in \mathbb{Z} \), the kernels of \( \mathcal{Q}_k^{(1)} \) and \( \mathcal{Q}_k^{(2)} \) satisfy (i), (ii) and (iii) of Theorem 2.6.

To recall the inhomogeneous discrete Calderón reproducing formulae established in [29], we introduce the following 1-exp-A TI and exp-IA TI.

**Definition 2.8.** A sequence \( \{P_k\}_{k=-\infty}^{\infty} \) of bounded linear operators on \( L^2(X) \) is called an approximation of the identity with exponential decay and integration 1 (for short, 1-exp-ATI) if \( \{P_k\}_{k=-\infty}^{\infty} \) has the following properties:

(i) for any \( k \in \mathbb{Z} \), \( P_k \) satisfies (ii), (iii) and (iv) of Definition 2.4 but without the exponential decay factor

\[ \exp \left\{ -y \left[ \max \left\{ d(x, y^k), d(y, y^\beta) \right\} \right]^q \right\} \]

(ii) for any \( k \in \mathbb{Z} \) and \( x \in X \), \( \int_X P_k(x, y) \, d\mu(y) = 1 = \int_X P_k(y, x) \, d\mu(y) \);

(iii) for any \( k \in \mathbb{Z} \), letting \( Q_k := P_k - P_{k-1} \), then \( \{Q_k\}_{k \in \mathbb{Z}} \) is an exp-ATI.

**Remark 2.9.** The existence of the 1-exp-ATI is guaranteed by [11 Lemma 10.1]. Moreover, by the proofs of [29] Proposition 2.9 and [27] Proposition 2.7(iv), we know that, for any \( f \in L^2(X) \), \( \lim_{k \to \infty} P_k f = f \) in \( L^2(X) \).

**Definition 2.10.** A sequence \( \{Q_k\}_{k=0}^{\infty} \) of bounded linear operators on \( L^2(X) \) is called an inhomogeneous approximation of the identity with exponential decay (for short, exp-ATI) if there exists a 1-exp-ATI \( \{P_k\}_{k=-\infty}^{\infty} \) such that \( Q_0 = P_0 \) and \( Q_k = P_k - P_{k-1} \) for any \( k \in \mathbb{N} \).

Next we recall the following inhomogeneous discrete reproducing formula established in [29].

**Theorem 2.11.** Let \( \{Q_k\}_{k=0}^{\infty} \) be an exp-ATI and \( \beta, \gamma \in (0, \eta) \). Then there exists a sequence \( \{\mathcal{Q}_k\}_{k=0}^{\infty} \) of bounded linear operators on \( L^2(X) \) such that, for any \( f \in (G^1_0(\beta, \gamma))' \),

\[ f(.) = \sum_{k=0}^{N} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(\alpha)} \int_{\mathcal{Q}_m^k} \mathcal{Q}_k(.) y \, d\mu(y) Q_{\alpha,1}^{k,m}(f) \]

\[ + \sum_{k=1}^{\infty} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(\alpha)} \mu \left( \mathcal{Q}_m^k \right) \mathcal{Q}_k \left( ., y_{k,m}^\alpha \right) Q_k f \left( y_{k,m}^\alpha \right), \]

where the equality converges in \( (G^1_0(\beta, \gamma))' \), every \( y_{k,m}^\alpha \) is an arbitrary point in \( \mathcal{Q}_m^k \), and, for any \( k \in \{0, \ldots, N\} \),

\[ Q_{\alpha,1}^{k,m}(f) := \frac{1}{\mu(\mathcal{Q}_m^k)} \int_{\mathcal{Q}_m^k} Q_k f(u) \, d\mu(u). \]

Moreover, for any \( k \in \mathbb{Z}_+ \), \( \mathcal{Q}_k \) satisfies (i) and (ii) of Theorem 2.6 and, for any \( x \in X \),

\[ \int_X \mathcal{Q}_k(x, y) \, d\mu(y) = \int_X \mathcal{Q}_k(y, x) \, d\mu(y) = \begin{cases} 1 & \text{if } k \in \{0, \ldots, N\}, \\ 0 & \text{if } k \in \{N + 1, N + 2, \ldots\}, \end{cases} \]

where \( N \in \mathbb{N} \) is some fixed constant independent of \( f \) and \( y_{k,m}^\alpha \).
3 Hardy spaces via various maximal functions

Let $\beta, \gamma \in (0, \eta)$ and $f \in (G^\theta_0(\beta, \gamma))'$. Let $\{P_k\}_{k \in \mathbb{Z}}$ be a 1-exp-ATI as in Definition 2.8 Define the radial maximal function $M^*(f)$ of $f$ by setting

$$M^*(f)(x) := \sup_{k \in \mathbb{Z}} |P_k f(x)|, \quad \forall \ x \in X.$$ 

Define the non-tangential maximal function $M_0(f)$ of $f$ with aperture $\theta \in (0, \infty)$ by setting

$$M_0(f)(x) := \sup_{k \in \mathbb{Z}} \sup_{y \in B(x, \theta)} |P_k f(y)|, \quad \forall \ x \in X.$$ 

Also, define the grand maximal function $f^*$ of $f$ by setting

$$f^*(x) := \sup \{ |(f, \varphi)| : \varphi \in G^\theta_0(\beta, \gamma) \text{ and } \|\varphi\|_{G^\theta_0(x, r_0, \beta, \gamma)} \leq 1 \text{ for some } r_0 \in (0, \infty) \}, \quad \forall \ x \in X.$$ 

Correspondingly, for any $p \in (0, \infty)$, the Hardy spaces $H^{+p}(X)$, $H^p_0(X)$ with $\theta \in (0, \infty)$ and $H^{-p}(X)$ are defined, respectively, by setting

$$H^{+p}(X) := \left\{ f \in (G^\theta_0(\beta, \gamma))' : \|f\|_{H^{+p}(X)} := \|M^+(f)\|_{L^p(X)} < \infty \right\},$$

$$H^p_0(X) := \left\{ f \in (G^\theta_0(\beta, \gamma))' : \|f\|_{H^p_0(X)} := \|M_0(f)\|_{L^p(X)} < \infty \right\},$$

and

$$H^{-p}(X) := \left\{ f \in (G^\theta_0(\beta, \gamma))' : \|f\|_{H^{-p}(X)} := \|f^*\|_{L^p(X)} < \infty \right\}.$$ 

Based on [20] Remark 2.9(ii), we easily observe that, for any $f \in (G^\theta_0(\beta, \gamma))'$ and $x \in X,$

$$(3.1) \quad M^+ f(x) \leq M_0(f)(x) \leq C f^*(x),$$

where $C$ is a positive constant only depending on $\theta$.

The aim of this section is to prove that the Hardy spaces $H^{+p}(X)$, $H^p_0(X)$ and $H^{-p}(X)$ are mutually equivalent when $p \in \omega/((\omega + \eta), \infty]$ in the sense of equivalent (quasi-)norms (see Section 3.2); in particular, they all are equivalent to the Lebesgue space $L^p(X)$ when $p \in (1, \infty]$ in the sense of equivalent norms (see Section 3.1). Moreover, we prove that $H^{-p}(X)$ is independent of the choices of the distribution space $(G^\theta_0(\beta, \gamma))'$ whenever $\beta, \gamma \in (\omega(1/p - 1), \eta)$; see Proposition 3.8 below.

3.1 Equivalence to the Lebesgue space $L^p(X)$ when $p \in (1, \infty]$ 

In this section, we show that the Hardy spaces $H^{+p}(X)$, $H^p_0(X)$ and $H^{-p}(X)$ are all equivalent to the Lebesgue space $L^p(X)$, when $p \in (1, \infty]$, in the sense of both representing the same distributions and equivalent norms. First we give some basic properties of $H^{+p}(X)$.

**Proposition 3.1.** Let $p \in (0, \infty]$. Then $H^{+p}(X)$ is a (quasi-)Banach space, which is continuously embedded into $(G^\theta_0(\beta, \gamma))'$, where $\beta, \gamma \in (0, \eta)$. 

Proof. Let \( f \in H^{s,p}(X) \) and \( \varphi \in G_{0}^{\beta}(\beta, \gamma) \) with \( \|\varphi\|_{\mathcal{G}(\beta, \gamma)} \leq 1 \). For any \( x \in B(x_{0}, 1) \), by Definition 2.1, we easily know that \( \|\varphi\|_{\mathcal{G}(\beta, \gamma)} \leq 1 \) with the implicit positive constant independent of \( x \) and hence \( |(f, \varphi)| \leq f^{*}(x) \). Therefore, for any \( \varphi \in G_{0}^{\beta}(\beta, \gamma) \) with \( \beta, \gamma \in (0, \eta) \), we have

\[
|(f, \varphi)|^{p} \leq \frac{1}{V_{1}(x_{0})} \int_{B_{x_{0},1}} |f^{*}(x)|^{p} \, d\mu(x) \leq \|f^{*}\|_{L^{p}(X)}^{p} \sim \|f\|_{H^{s,p}(X)}^{p}.
\]

This implies that \( H^{s,p}(X) \) is continuously embedded into \( (G_{0}^{\beta}(\beta, \gamma))' \).

To see that \( H^{s,p}(X) \) is a (quasi-)Banach space, we only prove its completeness. Indeed, suppose that \( \{f_{k}\}_{k=0}^{\infty} \) in \( H^{s,p}(X) \) is a Cauchy sequence, which is also a Cauchy sequence in \( (G_{0}^{\beta}(\beta, \gamma))' \) with \( \beta, \gamma \in (0, \eta) \). By the completeness of \( (G_{0}^{\beta}(\beta, \gamma))' \), the sequence \( \{f_{k}\}_{k=0}^{\infty} \) converges to some element \( f \in (G_{0}^{\beta}(\beta, \gamma))' \) as \( k \to \infty \). If \( \varphi \in G_{0}^{\beta}(\beta, \gamma) \) satisfies \( \|\varphi\|_{\mathcal{G}(x_{0}(x_{0}, \beta, \gamma))} \leq 1 \) for some \( x \in X \) and \( r_{0} \in (0, \infty) \), then \( |(f_{k+l} - f_{k}, \varphi)| \leq (f_{k+l} - f_{k})^{*}(x) \) for any \( k, l \in \mathbb{N} \). Letting \( l \to \infty \), we obtain

\[
|(f - f_{k}, \varphi)| \leq \liminf_{l \to \infty} (f_{k+l} - f_{k})^{*}(x),
\]

which further implies that, for any \( x \in X \),

\[
(f - f_{k})^{*}(x) \leq \liminf_{l \to \infty} (f_{k+l} - f_{k})^{*}(x).
\]

By the Fatou lemma, we conclude that

\[
\|(f - f_{k})^{*}\|_{L^{p}(X)} \leq \liminf_{l \to \infty} \|(f_{k+l} - f_{k})^{*}\|_{L^{p}(X)} \to 0
\]

as \( k \to \infty \), which, together with the sublinearity of \( \| \cdot \|_{H^{s,p}(X)} \), further implies that \( f \in H^{s,p}(X) \) and \( \lim_{k \to \infty} \|f_{k} - f\|_{H^{s,p}(X)} = 0 \). Therefore, \( H^{s,p}(X) \) is complete. This finishes the proof of Proposition 3.1.

To show the equivalence of \( H^{s,p}(X) \), \( H^{0}_{0}(X) \) and \( H^{s,p}(X) \) to the Lebesgue space \( L^{p}(X) \) when \( p \in (1, \infty) \) in the sense of both representing the same distributions and equivalent norms, we need the following technical lemma.

**Lemma 3.2.** Let \( \{P_{k}\}_{k \in \mathbb{Z}} \) be a 1-exp-ATI as in Definition 2.8 Assume that \( \beta, \gamma \in (0, \eta) \). Then the following statements hold:

(i) there exists a positive constant \( C \) such that, for any \( k \in \mathbb{Z} \) and \( \varphi \in \mathcal{G}(\beta, \gamma) \), \( \|P_{k}\varphi\|_{\mathcal{G}(\beta, \gamma)} \leq C\|\varphi\|_{\mathcal{G}(\beta, \gamma)} \);

(ii) for any \( f \in \mathcal{G}(\beta, \gamma) \) and \( \beta' \in (0, \beta) \), \( \lim_{k \to \infty} P_{k}f = f \) in \( \mathcal{G}(\beta', \gamma) \);

(iii) if \( f \in G_{0}^{\beta}(\beta, \gamma) \) [resp., \( f \in G_{0}^{\beta}(\beta, \gamma)' \)], then \( \lim_{k \to \infty} P_{k}f = f \) in \( G_{0}^{\beta}(\beta, \gamma) \) [resp., \( G_{0}^{\beta}(\beta, \gamma)' \)].

**Proof.** The proof of (i) can be obtained by the method used in the proof of [29] Lemma 4.14. The proof of (ii) is given in [20] Lemma 3.6, whose proof does not rely on the reverse doubling condition of \( \mu \) and the metric \( d \). We obtain (iii) directly by (i), (ii) and a standard duality argument. This finishes the proof of Lemma 3.2.

Then we have the following proposition.
Proposition 3.3. Let \( p \in [1, \infty) \), \( \beta, \gamma \in (0, \eta) \) and \( \{P_k\}_{k \in \mathbb{Z}} \) be a 1-exp-ATI. If \( f \in (G_0^p(\beta, \gamma))' \) belongs to \( H^{+p}(X) \), then there exists \( \tilde{f} \in L^p(X) \) such that, for any \( \varphi \in G_0^p(\beta, \gamma) \),

\[
\langle f, \varphi \rangle = \int_X \tilde{f}(x) \varphi(x) \, d\mu(x)
\]

and \( \|\tilde{f}\|_{L^p(X)} \leq \|M^+(f)\|_{L^p(X)} \); moreover, if \( p \in [1, \infty) \), then, for almost every \( x \in X \), \( |\tilde{f}(x)| \leq M^+(f)(x) \).

Proof. Let \( f \in (G_0^p(\beta, \gamma))' \) and \( M^+(f) = \sup_{k \in \mathbb{Z}} |P_k f| \in L^p(X) \), where \( \{P_k\}_{k \in \mathbb{Z}} \) is a 1-exp-ATI as in Definition 2.8. Then \( \{P_k f\}_{k \in \mathbb{Z}} \) is uniformly bounded in \( L^p(X) \). If \( p \in (1, \infty] \), then \( p' \in [1, \infty) \) and \( L^{p'}(X) \) is separable. Thus, by the Banach-Alaoglu theorem (see, for example, [50, Theorem 3.17]), we find a function \( \tilde{f} \in L^p(X) \) and a sequence \( \{k_j\}_{j=1}^\infty \subset \mathbb{Z} \) such that \( k_j \to \infty \) and \( P_{k_j} f \to \tilde{f} \) as \( j \to \infty \) in the weak-* topology of \( L^p(X) \). By this and the Hölder inequality, for any \( g \in L^{p'}(X) \), we have

\[
\left| \int_X \tilde{f}(x) g(x) \, d\mu(x) \right| = \lim_{j \to \infty} \left| \int_X P_{k_j} f(x) g(x) \, d\mu(x) \right| \leq \|M^+(f)\|_{L^p(X)} \|g\|_{L^{p'}(X)},
\]

which further implies that \( \|\tilde{f}\|_{L^p(X)} \leq \|M^+(f)\|_{L^p(X)} \).

If \( p = 1 \), notice that \( \|\sup_{k \in \mathbb{Z}} |P_k f|\|_{L^1(X)} = \|M^+(f)\|_{L^1(X)} \) is finite. Then, by the proof of [50, Theorem III.C.12], \( \{P_k f\}_{k \in \mathbb{Z}} \) is relatively compact in \( L^1(X) \). Therefore, by the Eberlein-Šmulian theorem (see [50, II.C]), we know that \( \{P_k f\}_{k \in \mathbb{Z}} \) is weakly sequentially compact, that is, there exist a function \( \tilde{f} \in L^1(X) \) and a subsequence \( \{P_{k_j} f\}_{j=1}^\infty \) such that \( P_{k_j} f \to \tilde{f} \) weakly in \( L^1(X) \). As the arguments for the case \( p \in (1, \infty] \), we still have \( \|\tilde{f}\|_{L^1(X)} \leq \|M^+(f)\|_{L^1(X)} \).

Moreover, for any \( \varphi \in G_0^p(\beta, \gamma) \), by the fact \( G_0^p(\beta, \gamma) \subseteq L^p(X) \) for any \( p \in [1, \infty) \) and Lemma 3.2(iii), we conclude that

\[
\langle f, \varphi \rangle = \lim_{k \to \infty} \langle P_k f, \varphi \rangle = \lim_{j \to \infty} \int_X P_{k_j} f(x) \varphi(x) \, d\mu(x) = \int_X \tilde{f}(x) \varphi(x) \, d\mu(x).
\]

Let \( p \in [1, \infty) \). For any \( j \in \mathbb{N} \) and \( x \in X \), we have \( P_{k_j}(\cdot, \cdot) \in \mathcal{G}(\eta, \eta) \) (see the proof of [29, Proposition 2.10]), which, together with (3.3), implies that

\[
P_{k_j} f(x) = \langle f, P_{k_j}(\cdot, \cdot) \rangle = \int_X P_{k_j}(x, y) \tilde{f}(y) \, d\mu(y) = P_{k_j} \tilde{f}(x).
\]

From this and [27, Proposition 2.7(iv)], we deduce that \( \{P_{k_j} f\}_{j \in \mathbb{N}} \) converges to \( \tilde{f} \) in the sense of \( \|\cdot\|_{L^p(X)} \). Then, by the Riesz theorem, we find a subsequence of \( \{P_{k_j} f\}_{j \in \mathbb{N}} \), still denoted by \( \{P_{k_j} f\}_{j \in \mathbb{N}} \), such that \( P_{k_j} f(x) \to \tilde{f}(x) \) as \( k_j \to \infty \) for almost every \( x \in X \). Therefore, \( |\tilde{f}(x)| \leq M^+(f)(x) \) for almost every \( x \in X \). This finishes the proof of Proposition 3.3. \( \square \)

Finally, we show the following main result of this section.

Theorem 3.4. Let \( p \in (1, \infty) \) and \( \beta, \gamma \in (0, \eta) \). Then the following hold true:

(i) if \( f \in (G_0^p(\beta, \gamma))' \) belongs to \( H^{+p}(X) \), then there exists \( \tilde{f} \in L^p(X) \) such that (3.2) holds true and \( \|\tilde{f}\|_{L^p(X)} \leq \|f\|_{H^{+p}(X)} \);
Lemma 3.6. any $f \in L^p(X)$ induces a distribution on $G_0^p(\beta, \gamma)$ as in (3.2), still denoted by $f$, such that

$$f \in H^{+p}(X) \text{ and } \|f\|_{H^{+p}(X)} \leq C\|f\|_{L^p(X)},$$

where $C$ is a positive constant independent of $f$.

Consequently, for any fixed $\theta \in (0, \infty)$, $H^{+p}(X) = H_0^p(X) = H^{+p}(X) = L^p(X)$ in the sense of both representing the same distributions and equivalent norms.

Proof. We obtain (i) directly by Proposition 3.3. Now we prove (ii). Suppose that $p \in (1, \infty]$ and $f \in L^p(X)$. Clearly, $f$ induces a distribution on $G_0^p(\beta, \gamma)$ as in (3.2). By [20 Proposition 3.9], we find that, for almost every $x \in X$, $f^\ast(x) \leq M(f)(x)$, with the implicit positive constant independent of $f$ and $x$. Therefore, from the boundedness of $M$ on $L^p(X)$, we deduce that $\|f^\ast\|_{L^p(X)} \leq \|M(f)\|_{L^p(X)} \leq \|f\|_{L^p(X)}$. This finishes the proof of (ii).

By (i), (ii) and (3.1), we obtain $H^{+p}(X) = H_0^p(X) = H^{+p}(X) = L^p(X)$, which completes the proof of Theorem 3.4.

3.2 Equivalence of Hardy spaces defined via various maximal functions

The main aim of this section concerns the equivalence of Hardy spaces defined via various maximal functions for the case $p \in (\omega/(\omega + \eta), 1]$. Indeed, our goal is to show the following equivalence theorem.

Theorem 3.5. Assume that $p \in (\omega/(\omega + \eta), 1]$ and $\theta \in (0, \infty)$. Then, for any $f \in (G_0^p(\beta, \gamma))'$ with $\beta, \gamma \in (\omega(1/p - 1), \eta)$,

$$\|f\|_{H^{+p}(X)} \sim \|f\|_{H_0^p(X)} \sim \|f\|_{H^{+p}(X)},$$

with equivalent positive constants independent of $f$. In other words, $H^{+p}(X) = H_0^p(X) = H^{+p}(X)$ with equivalent (quasi-)norms.

To prove Theorem 3.5, we borrow some ideas from [54]. To this end, we need the following two technical lemmas.

Lemma 3.6. Assume that $\phi \in G_0^p(\beta, \gamma)$ with $\beta, \gamma \in (0, \eta)$. Let $\sigma := \int_X \phi(x) \, d\mu(x)$. If $\psi \in G(\eta, \eta)$ with $\int_X \psi(x) \, d\mu(x) = 1$, then $\phi - \sigma \psi \in G_0^p(\beta, \gamma)$.

Proof. Since $\phi \in G_0^p(\beta, \gamma)$ with $\beta, \gamma \in (0, \eta)$, it follows that there exists $\{\phi_n\}_{n=1}^{\infty} \subset G(\eta, \eta)$ such that $\lim_{n \to \infty} \|\phi - \phi_n\|_{G(\beta, \gamma)} = 0$. Letting $\sigma_n := \int_X \phi_n(x) \, d\mu(x)$ for any $n \in \mathbb{N}$, by Definition 2.1 and Lemma 2.2(ii), we conclude that $\lim_{n \to \infty} |\sigma - \sigma_n| = 0$, where $\sigma := \int_X \phi(x) \, d\mu(x)$. Let $\varphi_n := \phi_n - \sigma \psi$ for any $n \in \mathbb{N}$. Then $\varphi_n \in G(\eta, \eta)$ and

$$\|\phi - \sigma \psi - \varphi_n\|_{G(\beta, \gamma)} \leq \|\phi - \phi_n\|_{G(\beta, \gamma)} + |\sigma - \sigma_n|\|\psi\|_{G(\beta, \gamma)} \to 0 \quad \text{as } n \to \infty.$$

Thus, $\phi - \sigma \psi \in G_0^p(\beta, \gamma)$. This finishes the proof of Lemma 3.6.

The next lemma comes from [27], Lemma 5.3, whose proof remains true for a quasi-metric $d$ and also does not rely on the reverse doubling condition of $\mu$. 
Lemma 3.7. Let all the notation be as in Theorem 2.7. Let \( k, k' \in \mathbb{Z} \), \( \{d_{\alpha}^{k,m}\}_{k \in \mathbb{Z}, \alpha \in A_k, m \in \{1, \ldots, N(k, \alpha)\}} \subset \mathbb{C}, \gamma \in (0, \eta) \) and \( r \in (\omega/(\omega + \gamma), 1) \). Then there exists a positive constant \( C \), independent of \( k, k' \), \( y_{\alpha}^{k,m} \in Q_{\alpha}^{k,m} \) and \( d_{\alpha}^{k,m} \) with \( k \in \mathbb{Z}, \alpha \in A_k \) and \( m \in \{1, \ldots, N(k, \alpha)\} \), such that, for any \( x \in X \),

\[
\sum_{\alpha \in A_k} \sum_{m=1}^{N(k,\alpha)} \mu \left( Q_{\alpha}^{k,m} \right) \frac{1}{V(x, y_{\alpha}^{k,m})} \left[ \frac{d_{\alpha}^{k,m}}{|\delta^{k,k'}| + d(x, y_{\alpha}^{k,m})} \right]^\gamma |d_{\alpha}^{k,m}|
\leq C \delta^{(k-k')|\omega(1-\frac{1}{r})} \left[ M \left( \sum_{\alpha \in A_k} \sum_{m=1}^{N(k,\alpha)} |d_{\alpha}^{k,m}|^r X_{Q_{\alpha}^{k,m}} \right) (x) \right]^\frac{1}{r}.
\]

Now we show Theorem 3.5 by using the above two technical lemmas. In what follows, the symbol \( \epsilon \to 0^+ \) means that \( \epsilon \in (0, \infty) \) and \( \epsilon \to 0 \).

Proof of Theorem 3.5. Let \( f \in (G_0^\beta(\beta, \gamma))' \) with \( \beta, \gamma \in (\omega(1/p - 1), \eta) \). Fix \( \theta \in (0, \infty) \). By (3.1), we have

\[
\|M^\theta(f)\|_{L^p(X)} \leq \|M_\theta(f)\|_{L^p(X)} \leq \|f^*\|_{L^p(X)}.
\]

Thus, the proof of Theorem 3.5 is reduced to showing

(3.4) \[ f^* \leq M^\theta(f) + \left( M \left( \left[M^\theta(f)\right]' \right)(x) \right)^\frac{1}{r}. \]

To obtain (3.4), it suffices to prove that, for some \( r \in (0, p) \) and any \( x \in X \),

(3.5) \[ f^*(x) \leq M^\theta(f)(x) + \left( M \left( \left[M^\theta(f)\right]' \right)(x) \right)^\frac{1}{r}. \]

If (3.5) holds true, then, by the boundedness of \( M \) on \( L^{p/r}(X) \), we conclude that

\[
\|f^*\|_{L^p(X)} \leq \|M^\theta(f)\|_{L^p(X)} + \left( M \left( \left[M^\theta(f)\right]' \right)(x) \right)^\frac{1}{r} \leq \|M^\theta(f)\|_{L^p(X)} \sim \|M^\theta(f)\|_{L^p(X)},
\]

which proves (3.4).

We now fix \( x \in X \) and show (3.5). Let \( \{P_k\}_{k \in \mathbb{Z}} \) be a 1-exp-ATI. For any \( k \in \mathbb{Z} \), define \( Q_k := P_k - P_{k-1} \). Then \( \{Q_k\}_{k \in \mathbb{Z}} \) is an exp-ATI. Assume for the moment that, for any \( \varphi \in G_0^\beta(\beta, \gamma) \) with \( \|\varphi\|_{\hat{G}(x,\delta,\beta,\gamma)} \leq 1 \) for some \( l \in \mathbb{Z} \),

(3.6) \[ |\langle f, \varphi \rangle| \leq \left[ M \left( \left[M^\theta(f)\right]' \right)(x) \right]^\frac{1}{r}. \]

We now use (3.6) to show (3.5). For any \( \varphi \in G_0^\beta(\beta, \gamma) \) with \( \|\varphi\|_{\hat{G}(x,0,\beta,\gamma)} \leq 1 \) for some \( r_0 \in (0, \infty) \), choose \( l \in \mathbb{Z} \) such that \( \delta^{l+1} \leq r_0 < \delta^l \). Clearly, \( \|\varphi\|_{\hat{G}(x,\delta^l,\beta,\gamma)} \leq 1 \). Let \( \sigma := \int \varphi(y) \, d\mu(y) \) and \( \varphi := \varphi - \varphi P_l(x, \cdot) \). Notice that \( \int P_l(x, y) \, d\mu(y) = 1 \) and \( P_l(x, \cdot) \in G(\eta, \eta) \) (see the proof of [29 Proposition 2.10]). From Lemma 3.6, it follows that \( \varphi \in G_0^\beta(\beta, \gamma) \). Moreover, \( \|\varphi\|_{\hat{G}(x,\delta^l,\beta,\gamma)} \leq \|\varphi\|_{\hat{G}(x,\delta^l,\beta,\gamma)} + |\sigma|P_l(x, \cdot) \| \leq 1 \). By (3.6), we know that

\[
|\langle f, \varphi \rangle| \leq |\langle f, \varphi \rangle| + |\sigma| |\langle f, P_l(x, \cdot) \rangle|
\leq \left[ M \left( \left[M^\theta(f)\right]' \right)(x) \right]^\frac{1}{r} + |P_l f(x)| \leq \left[ M \left( \left[M^\theta(f)\right]' \right)(x) \right]^\frac{1}{r} + M^\theta(f)(x),
\]
which is exactly (3.5).

It remains to prove (3.6). For any \( \epsilon \in (0, \infty) \), choose \( \gamma^{k_m} \in Q^{k_m}_\alpha \) such that
\[
\left| Q_k f \left( k_m \right) \right| \leq \inf_{z \in Q^{k_m}_\alpha} |Q_k f(z)| + \epsilon \leq 2 \inf_{z \in Q^{k_m}_\alpha} M^+(f)(z) + \epsilon.
\]

Let \( g := f|_{\tilde{G}^0_\alpha(\beta, \gamma)} \) be the restriction of \( f \) on \( \tilde{G}^0_\alpha(\beta, \gamma) \). Obviously, \( g \in (\tilde{G}^0_\alpha(\beta, \gamma))' \) and \( \|g\|_{(\tilde{G}^0_\alpha(\beta, \gamma))'} \leq \|f\|_{(\tilde{G}^0_\alpha(\beta, \gamma))'} \). By Theorem 2.7 we conclude that
\[
\langle f, \varphi \rangle = \langle g, \varphi \rangle = \sum_{k=-\infty}^{\infty} \sum_{a \in A_k} \sum_{m=1}^{N(k,\alpha)} \mu \left( Q^{k_m}_\alpha \right) \bar{Q}^*_k \varphi \left( k_m \right) Q_k g \left( k_m \right)
\]
\[
= \sum_{k=-\infty}^{\infty} \sum_{a \in A_k} \sum_{m=1}^{N(k,\alpha)} \mu \left( Q^{k_m}_\alpha \right) \bar{Q}^*_k \varphi \left( k_m \right) Q_k f \left( k_m \right),
\]
where \( \bar{Q}^*_k \) denotes the dual operator of \( \bar{Q}_k \). By the proof of (3.2), which remains true for a quasi-metric \( d \) and does not rely on the reverse doubling condition of \( \mu \), we find that, for any fixed \( \beta' \in (0, \beta \wedge \gamma) \) and any \( k \in \mathbb{Z} \),
\[
\left| \bar{Q}^*_k \varphi \left( k_m \right) \right| \leq \delta^{k-lp'} \frac{1}{V_{\beta,\alpha}(x) + V(x, y^{k_m}_\alpha)} \left[ \frac{\delta^{k+1}}{\delta^{k+1} + d(x, y^{k_m}_\alpha)} \right]^\gamma.
\]

Choose \( \beta' \in (0, \beta \wedge \gamma) \) such that \( \omega/\omega + \beta' < p \). From this and Lemma 3.7 we deduce that, for any fixed \( r = (\omega/\omega + \beta', p) \),
\[
|\langle f, \varphi \rangle| \leq \sum_{k=-\infty}^{\infty} \delta^{k-lp'} \sum_{a \in A_k} \sum_{m=1}^{N(k,\alpha)} \mu \left( Q^{k_m}_\alpha \right) \left[ \inf_{z \in Q^{k_m}_\alpha} M^+(f)(z) + \epsilon \right] \left[ \frac{\delta^{k+1}}{\delta^{k+1} + d(x, y^{k_m}_\alpha)} \right]^\gamma
\]
\[
\leq \sum_{k=-\infty}^{\infty} \delta^{k-lp'} \delta^{k-(k+1)l} \left[ M \left( \inf_{a \in A_k} \sum_{m=1}^{N(k,\alpha)} \left[ \frac{\delta^{k+1}}{\delta^{k+1} + d(x, y^{k_m}_\alpha)} \right]^\gamma \right) \right]^{\frac{1}{r}}
\]
\[
\leq \sum_{k=-\infty}^{\infty} \delta^{k-lp'} \delta^{k-(k+1)l} \left[ M \left( \left[ M^+(f) + \epsilon \right]^r \right) \right]^{\frac{1}{r}} \rightarrow \left[ M \left( \left[ M^+(f) + \epsilon \right]^r \right) \right]^{\frac{1}{r}} \quad \text{as} \quad \epsilon \rightarrow 0^+.
\]

This proves (3.6) and hence finishes the proof of Theorem 3.5.

To conclude this section, we show that the Hardy space \( H^{r,p}(X) \) is independent of the choices of \( (G^0_\alpha(\beta, \gamma))' \) whenever \( \beta, \gamma \in (\omega(1/p - 1), \eta) \).

**Proposition 3.8.** Let \( p \in (\omega(\omega + \eta), 1) \) and \( \beta_1, \beta_2, \gamma_1, \gamma_2 \in (\omega(1/p - 1), \eta) \). If \( f \in (G^0_\alpha(\beta_1, \gamma_1))' \) and \( f \in H^{r,p}(X) \), then \( f \in (G^0_\alpha(\beta_2, \gamma_2))' \).
Proof. Let \( f \in \mathcal{G}_0^1(\beta_1, \gamma_1)' \) with \( \|f\|_{L^{p}(X)} < \infty \). We first prove that there exists \( \theta \in (0, \infty) \) such that, for any \( \varphi \in \mathcal{G}(\eta, \eta) \) with \( \|\varphi\|_{\mathcal{G}(\beta_2, \gamma_2)} \leq 1 \),

\[
(3.9) \quad |\langle f, \varphi \rangle| \leq \|\mathcal{M}_{\theta}(f)\|_{L^{p}(X)}.
\]

Notice that \( \varphi \in \mathcal{G}(\eta, \eta) \subset \mathcal{G}_0^1(\beta_1, \gamma_1) \) and \( f \in (\mathcal{G}_0^1(\beta_1, \gamma_1))' \). With all the notation involved as in Theorem \ref{thm:main}, we have

\[
\langle f, \varphi \rangle = \sum_{k=0}^{N} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{\mathcal{N}(k, \alpha)} \int_{Q_{\alpha}^m} \tilde{Q}_{\alpha}^k \varphi(y) \, d\mu(y) Q_{\alpha, 1}^{k,m}(f) \\
+ \sum_{k=N+1}^{\infty} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{\mathcal{N}(k, \alpha)} \mu(Q_{\alpha}^m) \tilde{Q}_{\alpha}^k \varphi(y_{\alpha}^{k,m}) Q_{\alpha}(y_{\alpha}^{k,m}) =: Z_1 + Z_2.
\]

Choose \( \theta := 2A_0 C^3 \) with \( C^3 \) as in Lemma \ref{lem:main}. By the definition of \( Q_{\alpha}^{k,m} \) and Lemma \ref{lem:main}, we have \( Q_{\alpha}^{k,m} \subset B(z_{\alpha}^{k,m}, C^3 \delta^{k+1j_0}) \subset B(z, 2A_0 C^3 \delta^k) \) for any \( z \in Q_{\alpha}^{k,m} \).

Fix \( x \in B(x_0, 1) \). Then \( \|\varphi\|_{\mathcal{G}(\beta_1, \beta_2, \gamma_2)} \sim \|\varphi\|_{\mathcal{G}(\beta_0, 1, \beta_2, \gamma_2)} \leq 1 \). If \( k \in \{0, \ldots, N\} \), then we have \( \|\varphi\|_{\mathcal{G}(\beta_1, \beta_2, \gamma_2)} \sim \|\varphi\|_{\mathcal{G}(\beta_0, 1, \beta_2, \gamma_2)} \leq 1 \), where the implicit constants are independent of \( x \) but can depend on \( N \). Let \( \beta_- := \min(\beta_1, \gamma_1, \beta_2, \gamma_2) \). By \[27 \] (3.2), we conclude that, for any \( y \in Q_{\alpha}^{k,m} \),

\[
|\tilde{Q}_{\alpha}^k \varphi(y)| \leq \frac{1}{V_1(x) + V(x, y)} \left[ \frac{1}{1 + d(x, y)} \right]^{\beta_-} \sim \frac{1}{V_1(x) + V(x, y^{k,m})} \left[ \frac{1}{1 + d(x, y^{k,m})} \right]^{\beta_-}.
\]

Moreover, for any \( k \in \{0, \ldots, N\} \) and \( z \in Q_{\alpha}^{k,m} \), we have

\[
|Q_{\alpha, 1}^{k,m}(f)| \leq \frac{1}{\mu(Q_{\alpha}^m)} \int_{Q_{\alpha}^m} \left[ |P_k f(y)| + |P_{k-1} f(y)| \right] d\mu(y) \leq 2M_{\theta}(f)(z).
\]

Thus, we obtain

\[
(3.10) \quad |Z_1| \leq \sum_{k=0}^{N} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{\mathcal{N}(k, \alpha)} \frac{1}{V_1(x) + V(x, y^{k,m}_{\alpha})} \left[ \frac{1}{1 + d(x, y^{k,m}_{\alpha})} \right]^{\beta_-} \inf_{z \in Q_{\alpha}^{k,m}} M_{\theta}(f)(z).
\]

If \( k \in \{N + 1, N + 2, \ldots\} \), then \( |Q_k f(y_{\alpha}^{k,m})| \leq 2 \inf_{z \in Q_{\alpha}^{k,m}} M_{\theta}(f)(z) \). Again, by \|\varphi\|_{\mathcal{G}(\beta_1, \beta_2, \gamma_2)} \leq 1 and \[27 \] (3.2), we find that, for any fixed \( \beta' \in (0, \beta_-) \),

\[
|\tilde{Q}_{\alpha}^k \varphi(y_{\alpha}^{k,m})| \leq \delta^{k \beta'} \frac{1}{V_1(x) + V(x, y^{k,m}_{\alpha})} \left[ \frac{1}{1 + d(x, y^{k,m}_{\alpha})} \right]^{\beta_-},
\]

because now \( k \in \mathbb{Z}_+ \) and we do not need the cancelation of \( \varphi \). Therefore, we have

\[
(3.11) \quad |Z_2| \leq \sum_{k=N+1}^{\infty} \delta^{k \beta'} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{\mathcal{N}(k, \alpha)} \frac{1}{V_1(x) + V(x, y^{k,m}_{\alpha})} \left[ \frac{1}{1 + d(x, y^{k,m}_{\alpha})} \right]^{\beta_-} \inf_{z \in Q_{\alpha}^{k,m}} M_{\theta}(f)(z).
\]
Following the estimation of (3.8), from (3.10) and (3.11), we deduce that, for some \( r \in (\omega/(\omega + \eta), p) \),

\[
|\langle f, \varphi \rangle|^p \leq \left|\mathcal{M}(\mathcal{M}_0(f))\right|^p(x) \leq \left|\mathcal{M}(\mathcal{M}_0(f))\right|^p(\omega/|\omega| + \eta).
\]

Notice that the above inequality holds true for any \( x \in B(x_0, 1) \). Then, by the boundedness of \( \mathcal{M} \) on \( L_p^{1/p}(X) \), we further conclude that

\[
|\langle f, \varphi \rangle|^p \leq \frac{1}{V_1(x_0)} \int_X \left|\mathcal{M}(\mathcal{M}_0(f))\right|^p \, d\mu(x) \leq \|\mathcal{M}_0(f)\|^p_{L_p^{1/p}(X)},
\]

which is exactly (3.9).

Combining (3.9) and (3.11), we find that, for any \( \varphi \in G(\eta, \eta) \),

(3.12)

\[
|\langle f, \varphi \rangle| \leq \|\mathcal{M}_0(f)\|_{L_p(X)} \|\varphi\|_{G(\beta_2, \gamma_2)} \leq \|f\|_{H^{s,p}(X)} \|\varphi\|_{G(\beta_2, \gamma_2)}.
\]

Now let \( g \in G(\beta_2, \gamma_2) \). By the definition of \( G(\beta_2, \gamma_2) \), we know that there exist \( \{\varphi_j\}_{j=1}^\infty \subset G(\eta, \eta) \) such that \( \|g - \varphi_j\|_{G(\beta_2, \gamma_2)} \to 0 \) as \( j \to \infty \), which implies that \( \{\varphi_j\}_{j=1}^\infty \) is a Cauchy sequence in \( G(\beta_2, \gamma_2) \). By (3.12), we find that, for any \( j, k \in \mathbb{N} \),

\[
|\langle f, \varphi_j - \varphi_k \rangle| \leq \|f\|_{H^{s,p}(X)} \|\varphi_j - \varphi_k\|_{G(\beta_2, \gamma_2)}.
\]

Therefore, \( \lim_{j \to \infty} \langle f, \varphi_j \rangle \) exists and the limit is independent of the choice of \( \{\varphi_j\}_{j=1}^\infty \). Thus, it is reasonable to define \( \langle f, g \rangle := \lim_{j \to \infty} \langle f, \varphi_j \rangle \). Moreover, by (3.12), we conclude that

\[
|\langle f, g \rangle| = \lim_{j \to \infty} |\langle f, \varphi_j \rangle| \leq \|f\|_{H^{s,p}(X)} \liminf_{j \to \infty} \|\varphi_j\|_{G(\beta_2, \gamma_2)} \sim \|f\|_{H^{s,p}(X)} \|g\|_{G(\beta_2, \gamma_2)}.
\]

This implies \( f \in (G(\beta_2, \gamma_2))' \) and \( \|f\|_{(G(\beta_2, \gamma_2))'} \leq \|f\|_{H^{s,p}(X)} \), which completes the proof of Proposition (3.8). \( \square \)

### 4 Grand maximal function characterizations of atomic Hardy spaces

In this section, we establish the atomic characterizations of \( H^{s,p}(X) \) with \( p \in (\omega/(\omega + \eta), 1] \).

**Definition 4.1.** Let \( p \in (\omega/(\omega + \eta), 1] \), \( q \in (p, \infty] \cap [1, \infty) \) and \( \beta, \gamma \in (\omega(1/p - 1), \eta) \). The **atomic Hardy space** \( H_{at}^{p,q}(X) \) is defined to be the set of all \( f \in (G(\beta, \gamma))' \) such that \( f = \sum_{j=1}^\infty \lambda_j a_j \), where \( \{a_j\}_{j=1}^\infty \) is a sequence of \( (p, q) \)-atoms and \( \{\lambda_j\}_{j=1}^\infty \subset \mathbb{C} \) satisfies \( \sum_{j=1}^\infty |\lambda_j|^p < \infty \). Moreover, let

\[
\|f\|_{H_{at}^{p,q}(X)} := \inf \left( \sum_{j=1}^\infty |\lambda_j|^p \right)^{1/p},
\]

where the infimum is taken over all the decompositions of \( f \) as above.

Observe that the difference between \( H_{cw}^{p,q}(X) \) and \( H_{at}^{p,q}(X) \) mainly lies on the choices of distribution spaces. When \( (X, d, \mu) \) is a doubling metric measure space, it was proved in [40] Theorem 4.4] that \( H_{cw}^{p,q}(X) \) and \( H_{at}^{p,q}(X) \) coincide with equivalent (quasi-)norms. Since now \( d \) is a quasi-metric, for the completeness of this article, we include a proof of their equivalence in Section 4.4 below.

The main aim in this section is to prove the following conclusion.
\textbf{Theorem 4.2.} Let \( p \in (\omega/(\omega + \eta), 1) \), \( q \in (p, \infty]\cap [1, \infty] \) and \( \beta, \gamma \in (\omega(1/p - 1), \eta) \). As subspaces of \((\mathcal{G}_0^q(\beta, \gamma))'\), \( H^{*}\rho(X) = H^{\rho}_{at}(X) \) with equivalent (quasi-)norms.

We divide the proof of Theorem 4.2 into three sections. In Section 4.1, we prove that \( H^{\rho}_{at}(X) \subset H^{*}\rho(X) \) directly by the definition of \( H^{\rho}_{at}(X) \). The next two sections mainly deal with the proof of \( H^{*}\rho(X) \subset H^{\rho}_{at}(X) \). In Section 4.2, we obtain a Calderón-Zygmund decomposition for any \( f \in H^{*}\rho(X) \). Then, in Section 4.3, we show that any \( f \in H^{*}\rho(X) \) has a \((p, \infty)\)-atomic decomposition. In Section 4.4, we reveal the equivalent relationship between \( H^{\rho}_{at}(X) \) and \( H^{\rho}_{CW}(X) \).

\section{Proof of \( H^{\rho}_{at}(X) \subset H^{*}\rho(X) \)}

In this section, we prove \( H^{\rho}_{at}(X) \subset H^{*}\rho(X) \), as subspaces of \((\mathcal{G}_0^q(\beta, \gamma))'\) with \( \beta, \gamma \in (\omega(1/p - 1), \eta) \). To do this, we need the following technical lemma.

\textbf{Lemma 4.3.} Let \( p \in (\omega/(\omega + \eta), 1) \) and \( q \in (p, \infty]\cap [1, \infty] \). Then there exists a positive constant \( C \) such that, for any \((p, q)\)-atom \( a \) supported on \( B := B(x_B, r_B) \), with \( x_B \in X \) and \( r_B \in (0, \infty) \), and any \( x \in X \),

\begin{equation}
\tag{4.1}
a^+(x) \leq C M(a(x)\chi_{B(x_B, 2A_0r_B)}(x) + C \left[ \frac{r_B}{d(x_B, x)} \right]^\beta \frac{\mu(B)}{V(x_B, x)} \chi_{B(x_B, 2A_0r_B)}(x)
\end{equation}

and

\begin{equation}
\tag{4.2}
||a^+||_{L^p(X)} \leq C,
\end{equation}

where the atom \( a \) is viewed as a distribution on \( \mathcal{G}_0^q(\beta, \gamma) \) with \( \beta, \gamma \in (\omega(1/p - 1), \eta) \).

\textbf{Proof.} First, we show (4.1). Let \( \varphi \in \mathcal{G}_0^q(\beta, \gamma) \) be such that \( ||\varphi||_{\mathcal{G}_0^q(\beta, \gamma)} \leq 1 \) for some \( r \in (0, \infty) \), where \( \beta, \gamma \in (\omega(1/p - 1), \eta) \). When \( x \in B(x_B, 2A_0r_B) \), by Lemma 2.2(v), we find that

\begin{equation}
||a^+||_{L^p(X)} \leq M(a(x)),
\end{equation}

which consequently implies that \( a^+(x) \leq M(a(x)) \).

Let \( x \notin B(x_B, 2A_0r_B) \). Then, for any \( y \in B \), we have \( d(x, x_B) \geq 2A_0r_B > 2A_0d(x_B, y) \). Therefore, by the definition of \((p, q)\)-atoms and Definition 2.1(ii), we conclude that

\begin{equation}
||a^+||_{L^p(X)} \leq \int_B |a^+(y)| \varphi(y) d\mu(y) \leq \int_B |a(y)||\varphi(y) - \varphi(x_B)| d\mu(y)
\end{equation}

Taking the supremum over all such \( \varphi \in \mathcal{G}_0^q(\beta, \gamma) \) satisfying \( ||\varphi||_{\mathcal{G}_0^q(\beta, \gamma)} \leq 1 \) for some \( r \in (0, \infty) \), we obtain (4.1).
Now, we use \((4.1)\) to show \((4.2)\). When \(q \in (1, \infty)\), from the Hölder inequality and the boundedness of \(M\) on \(L^p(X)\), we deduce that
\[
\int_{B(x_B, 2A_0 r_B)} [M(a)(x)]^p \, d\mu(x) \leq [\mu(B(x_B, 2A_0 r_B))]^{1-p/q} [M(a)]_p^p \leq [\mu(B)]^{1-p/q} [a]_{L^p(X)}^p \leq 1.
\]
If \(q = 1\), then, by \(p \in (\omega/(\omega + \eta), 1)\) and the boundedness of \(M\) from \(L^1(X)\) to \(L^{1,\infty}(X)\), we conclude that
\[
\int_{B(x_B, 2A_0 r_B)} [M(a)(x)]^p \, d\mu(x) = \int^\infty_0 \mu(\{x \in B(x_B, 2A_0 r_B) : M(a)(x) > \lambda\}) \, d\lambda^p
\]
\[
\leq \int^\infty_0 \min\left\{\mu(B), \frac{\|a\|_{L^1(X)}}{\lambda}\right\} \, d\lambda^p
\]
\[
\leq \int^\infty_{\|a\|_{L^1(X)}/\mu(B)} \mu(B) \, d\lambda^p + \int^\infty_{\|a\|_{L^1(X)/\mu(B)}} \|a\|_{L^1(X)} \lambda^{-1} \, d\lambda^p
\]
\[
\leq \|a\|_{L^p(X)}^p [\mu(B)]^{1-p} \leq 1.
\]
By the fact \(\beta > \omega(1/p - 1)\) and the doubling condition \((1.1)\), we have
\[
\int_{d(x, x_B) \geq 2A_0 r_B} \left[\frac{r_B}{d(x_B, x)}\right]^{\beta p} \left[\frac{1}{\mu(B)}\right]^{1-p} \left[\frac{1}{V(x_B, x)}\right]^p \, d\mu(x)
\]
\[
\leq \sum_{k=1}^{\infty} 2^{-k\beta p} 2^{k\omega(1-p)} \int_{2^{k+1} A_0 r_B \leq d(x, x_B) < 2^{k+1} A_0 r_B} \frac{1}{V(x_B, x)} \, d\mu(x) \leq 1.
\]
Combining the last three formulae with \((4.1)\), we obtain \((4.2)\), which then completes the proof of Lemma 4.3 \(\square\)

**Proof of \(H_{at}^{p,q}(X) \subset H^{s,p}(X)\).** Assume that \(f \in (G_{0}^{p}(\beta, \gamma))^\prime\) is non-zero and it belongs to \(H_{at}^{p,q}(X)\) with \(\beta, \gamma \in (\omega(1/p - 1), \eta)\). Then \(f = \sum_1^\infty \lambda_j a_j\), where \(|a_j|_{1}\) are \((p, q)\)-atoms and \(|\lambda_j|_{\infty} \subset \mathbb{C}\) satisfy \(\sum_1^\infty |\lambda_j|_{p} \sim \|f\|_{H_{at}^{p,q}(X)}^p\). By the definition of the grand maximal function, we conclude that, for any \(x \in X\),
\[
f^\ast(x) \leq \sum_1^\infty |\lambda_j| a_j^\ast(x).
\]
From this and \((4.2)\), we deduce that
\[
\|f^\ast\|_{L^p(X)}^p \leq \sum_1^\infty |\lambda_j|_{p}^p \|a_j^\ast\|_{L^p(X)} \leq \sum_1^\infty |\lambda_j|_{p}^p \|f\|_{H_{at}^{p,q}(X)}^p.
\]
This finishes the proof of \(H_{at}^{p,q}(X) \subset H^{s,p}(X)\). \(\square\)

**4.2 Calderón-Zygmund decomposition of a distribution from \(H^{s,p}(X)\)**

In this section, we obtain a Calderón-Zygmund decomposition of any \(f \in H^{s,p}(X)\). First we establish a partition of unity for an open set \(\Omega\) with \(\mu(\Omega) < \infty\).
Proposition 4.4. Suppose \( \Omega \subset X \) is a proper open set with \( \mu(\Omega) \in (0, \infty) \) and \( A \in [1, \infty) \). For any \( x \in \Omega \), let
\[
 r(x) := \frac{d(x, \Omega^c)}{2AA_0} \in (0, \infty).
\]
Then there exist \( L_0 \in \mathbb{N} \) and a sequence \( \{x_k\}_{k \in I} \subset \Omega \), where \( I \) is a countable index set, such that

(i) \( \{B(x_k, r_k/(5AA_0^3))\}_{k \in I} \) is disjoint. Here and hereafter, \( r_k := r(x_k) \) for any \( k \in I \);

(ii) \( \bigcup_{k \in I} B(x_k, r_k) = \Omega \) and \( B(x_k, Ar_k) \subset \Omega \);

(iii) for any \( x \in \Omega \), \( Ar_k \leq d(x, \Omega^c) \leq 3AA_0^2r_k \) whenever \( x \in B(x_k, r_k) \) and \( k \in I \);

(iv) for any \( k \in I \), there exists \( y_k \notin \Omega \) such that \( d(x_k, y_k) < 3AA_0r_k \);

(v) for any given \( k \in I \), the number of balls \( B(x_j, Ar_j) \) that intersect \( B(x_k, Ar_k) \) is at most \( L_0 \);

(vi) if, in addition, \( \Omega \) is bounded, then, for any \( \sigma \in (0, \infty) \), the set \( \{k \in I : r_k > \sigma \} \) is finite.

Proof. We show this proposition by borrowing some ideas from [47, pp. 15–16]. Let \( \epsilon := (5AA_0^3)^{-1} \) and \( \{B(x, \epsilon r(x))\}_{x \in \Omega} \) be a covering of \( \Omega \). Now we pick the maximal disjoint subcollection of \( \{B(x, \epsilon r(x))\}_{x \in \Omega} \), denoted by \( \{B_k\}_{k \in I} \), which is at most countable, because of \( (1,1) \) and \( \mu(\Omega) \in (0, \infty) \). For any \( k \in I \), denote the center of \( B_k \) by \( x_k \) and \( r(x_k) \) by \( r_k \). Then we obtain (i).

Properties (iii) and (iv) can be shown by the definition of \( r_k \), the details being omitted. Now we show (ii). Obviously, \( B(x_k, Ar_k) \subset \Omega \) for any \( k \in I \). It suffices to prove that \( \Omega \subset \bigcup_{k \in I} B(x_k, r_k) \). For any \( x \in \Omega \), since \( \{B_k\}_{k \in I} \) is maximal, it then follows that there exists \( k \in I \) such that \( B(x, \epsilon r(x)) \cap B(x_k, Ar_k) \neq \emptyset \). We claim that \( r_k \geq r(x)/(4AA_0^2) \). If not, then \( r_k < r(x)/(4AA_0^2) \). Suppose that \( x_0 \in B(x_k, \epsilon r_k) \cap B(x, \epsilon r(x)) \). Then, for any \( y \in B(x_k, 3AA_0r_k) \), we have
\[
 d(y, x) \leq A_0[d(y, x_0) + d(x_0, x)] \leq A_0^2[d(y, x_k) + d(x_k, x_0)] + A_0d(x_0, x) \leq 6AA_0^3r_k + A_0\epsilon r(x)
\]
\[
 \leq \frac{3}{2}AA_0r(x) + \frac{1}{5}AA_0r(x) = \frac{17}{10}AA_0r(x)
\]
and hence \( B(x_k, 3AA_0r_k) \subset B(x, \frac{17}{10}AA_0r(x)) \subset \Omega \), which is a contradiction to (iv). This proves the claim.

Further, by the fact that \( r(x) \leq 4AA_0^2r_k \), we have
\[
 d(x, x_k) \leq A_0[d(x, x_0) + d(x_0, x_k)] < A_0\epsilon r(x) + A_0\epsilon r_k \leq 5AA_0^3\epsilon r_k = r_k,
\]
that is, \( x \in B(x_k, r_k) \). This finishes the proof of (ii).

Now we prove (v). Fix \( k \in I \). Suppose that \( B(x_j, Ar_j) \cap B(x_k, Ar_k) \neq \emptyset \). We claim that \( r_j \geq 8AA_0^2r_k \). If not, then \( r_j > 8AA_0^2r_k \). Choose \( y_0 \in B(x_j, Ar_j) \cap B(x_k, Ar_k) \). For any \( y \in B(x_k, 3AA_0r_k) \), we have
\[
 d(y, x_j) \leq A_0[d(y, y_0) + d(y_0, x_j)] \leq A_0^2[d(y, x_k) + d(x_k, y_0)] + A_0d(y_0, x_j)
\]
\[
 \leq 3AA_0^3r_k + AA_0^2r_k + AA_0r_j \leq \frac{3}{2}AA_0r_j,
\]
which further implies that \( y \in B(x_j, \frac{3}{2} A_0 r_j) \). Therefore, \( B(x_k, 3 A_0 r_k) \subseteq B(x_j, \frac{3}{2} A_0 r_j) \subseteq \Omega \), which contradicts to (iv). Thus, we have \( r_j \leq 8 A_0^3 r_k \). By symmetry, we also have \( r_k \leq 8 A_0^3 r_j \). Let

\[
\mathcal{F} := \{ j \in I : B(x_j, A r_j) \cap B(x_k, A r_k) \neq \emptyset \}.
\]

Then, for any \( j \in \mathcal{F} \), \( d(x_j, x_k) < A A_0 (r_j + r_k) \leq 9 A A_0^3 r_k \), which further implies that

\[
B \left( x_j, (5 A_0^3)^{-1} r_j \right) \subset B \left( x_k, A_0 \left[ d(x_j, x_k) + (5 A_0^3)^{-1} r_j \right] \right) \subset B(x_k, 11 A A_0^4 r_k).
\]

Then, from the fact \( d(x_j, x_k) \leq \min(r_j, r_k) \) and (i), we deduce that

\[
\mu \left( B \left( x_j, (5 A_0^3)^{-1} r_j \right) \right) \sim \mu(B(x_j, r_j)) \sim \mu(B(x_k, r_k)) \sim \mu(B(x_k, 11 A A_0^4 r_k))
\]

with the equivalent positive constants depending on \( A \). Thus, we obtain (v) by (i).

Finally we prove (vi). Since \( \Omega \) is bounded, it follows that there exist \( x_0 \in X \) and \( R \in (0, \infty) \) such that \( \Omega \subset B(x_0, R) \). If (vi) fails, then there exists \( \sigma_0 \in (0, \infty) \) such that \( \mathcal{K} := \{ k \in I : r_k > \sigma_0 R \} \) is infinite. Then, for any \( k \in \mathcal{K} \),

\[
\mu(B(x_k, r_k/(5 A_0^3))) \sim \mu(B(x_k, e_0 R)) \geq \mu(B(x_0, R)) \geq \mu(\Omega) > 0.
\]

By this and (i), we have \( \mu(\Omega) \geq \sum_{k \in \mathcal{K}} \mu(B(x_k, r_k/(5 A_0^3))) = \infty \). That is a contradiction. This proves (vi) and hence finishes the proof of Proposition 4.4. \( \square \)

**Proposition 4.5.** Let \( \Omega \subset X \) be an open set and \( \mu(\Omega) < \infty \). Suppose that sequences \( \{x_k\}_{k \in I} \) and \( \{r_k\}_{k \in I} \) are as in Proposition 4.4 with \( A := 16 A_0^4 \). Then there exist non-negative functions \( \{\phi_k\}_{k \in I} \) such that

(i) for any \( k \in I \), \( 0 \leq \phi_k \leq 1 \) and \( \text{supp} \ \phi_k \subset B(x_k, 2 A_0 r_k) \);

(ii) \( \sum_{k \in I} \phi_k = \chi_{\Omega} \);

(iii) for any \( k \in I \), \( \phi_k \geq L_0^{-1} \) in \( B(x_k, r_k) \), where \( L_0 \) is as in Proposition 4.4;

(iv) there exists a positive constant \( C \) such that, for any \( k \in I \), \( \|\phi_k\|_{L^\infty(\Omega)} \leq CV_{\psi_k}(x_k) \).

**Proof.** By Corollary 4.2, for any \( k \in I \), we find a function \( \psi_k \) such that \( \chi_{B(x_k, r_k)} \leq \psi_k \leq \chi_{B(x_k, 2 A_0 r_k)} \) and \( \|\psi_k\|_{L^\infty(\Omega)} \leq r_k^{-\eta} \). Here and hereafter, for any \( s \in (0, \eta] \) and a measurable function \( f \), define

\[
\|f\|_{C^s(\Omega)} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|d(x, y)|^s}.
\]

Since \( A \geq 2 A_0 \), from (ii) and (v) of Proposition 4.4 it follows that, for any \( x \in \Omega \), \( 1 \leq \sum_{k \in I} \psi_k(x) \leq L_0 \). For any \( k \in I \) and \( x \in X \), let

\[
\phi_k(x) := \begin{cases} 
\psi_k(x) \left[ \sum_{j \in I} \psi_j(x) \right]^{-1} & \text{when } x \in \Omega, \\
0 & \text{when } x \notin \Omega.
\end{cases}
\]
Then, for any $k \in I$, we have $0 \leq \phi_k \leq 1$, supp $\phi_k \subset B(x_k, 2A_0r_k)$ and $\sum_{k \in I} \phi_k(x) = 1$ when $x \in \Omega$. Moreover, for any $k \in I$, we have $\phi_k \geq L_{0}^{-1}$ in $B(x_k, r_k)$. Thus, we prove (i), (ii) and (iii).

It remains to prove (iv). Fix $k \in I$. For any $y \in X$, we have

$$|\phi_k(y)| \leq \chi_{B(x_k, 2A_0r_k)}(y) \leq V_{r_k}(x_k) \frac{1}{V_{r_k}(x_k) + V(x_k, y)} \left[ \frac{r_k}{r_k + d(x_k, y)} \right]^\eta.$$ 

Now we prove that $\phi_k$ satisfies the regularity condition. Suppose that $d(y, y') \leq (2A_0)^{-1}[r_k + d(x_k, y)]$. If $|\phi_k(y) - \phi_k(y')| \neq 0$, then $d(x_k, y) \leq (3A_0)^2 r_k$. If not, then $d(x_k, y) \geq (3A_0)^2 r_k$, so that $\phi_k(y) = 0$ and

$$d(y', x_k) \geq A_0^{-1} d(x_k, y) - d(y, y') \geq (2A_0)^{-1} d(x_k, y) - (2A_0)^{-1} r_k > 2A_0 r_k$$

and hence $\phi_k(y') = 0$, which contradicts to $|\phi_k(y) - \phi_k(y')| \neq 0$. Notice that $\psi_k(y')|\psi_j(y) - \psi_j(y')| \neq 0$ implies that $y' \in B(x_k, 2A_0r_k)$ and also $y$ or $y'$ belongs to $B(x_j, 2A_0r_j)$, which further implies that $B(x_k, Ar_j) \cap B(x_j, Ar_j) \neq \emptyset$. Then, by the proof of Proposition 4.4(v), the number of $j$ satisfying $\psi_k(y')|\psi_j(y) - \psi_j(y')| \neq 0$ is at most $L_0$ and $r_j \sim r_k$. Therefore,

$$|\phi_k(y) - \phi_k(y')| \leq \left| \frac{\psi_k(y)}{\sum_{j \in I} \psi_j(y)} - \frac{\psi_k(y')}{\sum_{j \in I} \psi_j(y')} \right| \leq \left( \frac{\psi_k(y) - \psi_k(y')}{\sum_{j \in I} \psi_j(y)} \right) + \left( \frac{\sum_{j \in I} |\psi_j(y) - \psi_j(y')|}{\sum_{j \in I} \psi_j(y')} \right) \leq \frac{d(y, y')}{r_k} \sum_{j \in I; B(x_k, Ar_k) \cap B(x_j, Ar_j) \neq \emptyset} \left( \frac{d(y, y')}{r_j} \right) \leq \frac{d(y, y')}{r_k} V_{r_k}(x_k) \left[ \frac{d(y, y')}{r_k + d(x_k, y)} \right]^{\eta} \frac{1}{V_{r_k}(x_k) + V(x_k, y)} \left[ \frac{r_k}{r_k + d(x_k, y)} \right]^{\eta}.$$ 

Then we obtain the desired regularity condition of $\phi_k$. This finishes the proof of (iv) and hence of Proposition 4.5.

Assume that $f \in (G_{0}^{\eta}(\beta, \gamma))'$ belongs to $f \in H^{s, p}(X)$, where $p \in (\omega/(\omega + \eta), 1]$ and $\beta, \gamma \in (\omega(1/p - 1), \eta)$. To obtain the Calderón-Zygmund decomposition of $f$, we apply Propositions 4.4 and 4.5 to the level set $\{x \in X : f^*(x) > \lambda\}$ with $\lambda \in (0, \infty)$. The encountering problem is that such a level set may not be open even in the case that $d$ is a metric. To solve this problem in the case that $d$ is a metric, a variant of the notion of the space of test functions is adopted in [20 Definition 2.5] so that to ensure that the level set is open (see [20 Remark 2.9]). Here, we borrow some idea from [20].

By the proof of [42 Theorem 2], we know that there exist $\theta \in (0, 1)$ and a metric $d'$ such that $d' \sim d^\theta$. For any $x \in X$ and $r \in (0, \infty)$, define the $d'$-ball $B'(x, r) := \{y \in X : d'(x, y) < r\}$. Then $(X, d', \mu)$ is a doubling metric measure space. Moreover, for any $x, y \in X$ and $r \in (0, \infty)$, we have

$$\mu(B(y, \theta r + d(x, y))) \sim \mu\left( B\left( y, \left[ r + d(x, y) \right]^\theta \right) \right) \sim \mu\left( B\left( y, r^\theta + d'(x, y) \right) \right),$$

where the equivalent positive constants are independent of $x$ and $r$. Using the metric $d'$, we introduce a variant of the space of test functions as follows.
**Definition 4.6.** For any \( x \in X, \rho \in (0, \infty) \) and \( \beta', \gamma' \in (0, \infty) \), define \( G(x, \rho, \beta', \gamma') \) to be the set of all functions \( f \) satisfying that there exists a positive constant \( C \) such that

(i) (the size condition) for any \( y \in X \),

$$
|f(y)| \leq C \left( \frac{1}{\mu(B'(y, \rho + d'(x, y)))} \left[ \frac{\rho}{\rho + d'(x, y)} \right]^\gamma' \right);
$$

(ii) (the regularity condition) for any \( y, y' \in X \) satisfying \( d(y, y') \leq \frac{1}{2} \),

$$
|f(y) - f(y')| \leq C \left[ \frac{d'(y, y')}{\rho + d'(x, y)} \right]^\beta' \left[ \frac{1}{\mu(B'(y, \rho + d'(x, y)))} \left[ \frac{\rho}{\rho + d'(x, y)} \right]^\gamma' \right].
$$

Also, define

$$
\|f\|_{G(x, \rho, \beta', \gamma')} := \inf \{ C \in (0, \infty) : \text{(i) and (ii) hold true} \}.
$$

By the previous argument, we find that \( G(x, r, \beta, \gamma) = G(x, r^\beta, \beta/(\theta, \gamma/\theta)) \) with equivalent norms, where the equivalent positive constants are independent of \( x \) and \( r \). For any \( \beta, \gamma \in (0, \eta) \) and \( f \in (G_{0}^{\beta}(\beta, \gamma))' \), define the modified grand maximal function of \( f \) by setting, for any \( x \in X \),

$$
f^* (x) := \sup \{ (f, \varphi) : \varphi \in G_{0}^{\beta}(\beta, \gamma) \text{ with } \|\varphi\|_{G(x, r^\beta, \beta/(\theta, \gamma/\theta)} \leq 1 \text{ for some } r \in (0, \infty) \}.
$$

Then \( f^* \sim f^* \) pointwisely on \( X \). For any \( \lambda \in (0, \infty) \) and \( j \in \mathbb{Z} \), define

$$
\Omega_{j} := \{ x \in X : f^*(x) > \lambda \} \quad \text{and} \quad \Omega_{j} := \Omega_{2j}.
$$

By the argument used in [20] Remark 2.9(ii), we find that \( \Omega_{j} \) is open under the topology induced by \( d' \), so it is under the topology induced by \( d \).

Now suppose that \( p \in (\omega/(\omega + 1), 1), \beta, \gamma \in (\omega(1/p - 1), \eta) \) and \( f \in H^{p}(X) \) and every \( \Omega_{j} \) with \( j \in \mathbb{Z} \) has finite measure. Consequently, there exist \( \{ I_{k} \}_{k \in I_{j}} \subset X \) with \( I_{j} \) being a countable index set, \( \{ r_{k} \}_{k \in I_{j}} \subset (0, \infty) \), \( L_{0} \in \mathbb{N} \) and a sequence \( \{ \varphi_{k} \}_{k \in I_{j}} \) of non-negative functions satisfying all the conclusions of Propositions 4.4 and 4.5. For any \( j \in \mathbb{Z} \) and \( k \in I_{j} \), define \( \Phi_{j}^{k} \) by setting, for any \( \varphi \in G_{0}^{\beta}(\beta, \gamma) \) and \( x \in X \),

$$
\Phi_{j}^{k}(\varphi)(x) := \varphi_{k}(x) \left[ \int_{X} \varphi_{k}(z) d\mu(z) \right]^{-1} \int_{X} [\varphi(x) - \varphi(z)] \varphi_{k}(z) d\mu(z).
$$

It can be seen that \( \Phi_{j}^{k} \) is bounded on \( G_{0}^{\beta}(\beta, \gamma) \) with operator norm depending on \( j \) and \( k \); see [20] Lemma 4.9]. Thus, it makes sense to define a distribution \( b_{j}^{k} \) on \( G_{0}^{\beta}(\beta, \gamma) \) by setting, for any \( \varphi \in G_{0}^{\beta}(\beta, \gamma) \),

$$
\langle b_{j}^{k}, \varphi \rangle := \langle f, \Phi_{j}^{k}(\varphi) \rangle.
$$

To estimate \( (b_{j}^{k})^* \), we have the following result. For its proof, see, for example, [37] Lemma 3.7].
Proposition 4.7. For any $j \in \mathbb{Z}$ and $k \in I_j$, $b_k^j$ is defined as in (4.3). Then there exists a positive constant $C$ such that, for any $j \in \mathbb{Z}$, $k \in I_j$ and $x \in X$,

\[
(b_k^j)^+(x) \leq C2^{j} \frac{\mu(B(x_k^j, r_k^j))}{\mu(B(x_k^j, r_k^j)) + V(x_k^j, x)} \left[ \frac{r_k^j}{r_k^j + d(x_k^j, x)} \right]^\beta \chi_{B(x_k^j, 16A_k r_k^j)}(x) + Cf^\ast(x)\chi_{B(x_k^j, 16A_k r_k^j)}(x).
\]

The next lemma is exactly [20] Lemma 4.10. The proof remains true if $d$ is a quasi-metric and $\mu$ does not satisfy the reverse doubling condition.

Lemma 4.8. Let $\beta \in (0, \infty)$, $p \in (\omega/(\omega + \beta), \infty)$, $L_0 \in \mathbb{N}$ and $I$ be a countable index set. Then there exists a positive constant $C$ such that, for any sequences $\{x_k\}_{k \in I} \subset X$ and $\{r_k\}_{k \in I} \subset (0, \infty)$ satisfying $\sum_{k \in I} \chi_{B(x_k, r_k)} \leq L_0$,

\[
\int_{X} \left\{ \sum_{k \in I} \frac{V_{r_k}(x_k)}{V_{r_k}(x_k) + V(x_k, x)} \left[ \frac{r_k}{r_k + d(x_k, x)} \right]^\beta \right\}^p \mu(x) \leq C \mu \left( \bigcup_{k \in I} B(x_k, r_k) \right).
\]

Then, by Proposition 4.7 and Lemma 4.8, we have the following result.

Proposition 4.9. Let $p \in (\omega/(\omega + \eta), 1)$. For any $j \in \mathbb{Z}$ and $k \in I_j$, let $b_k^j$ be as in (4.3). Then there exists a positive constant $C$ such that, for any $j \in \mathbb{Z}$,

\[
\int_{X} \sum_{k \in I_j} \left[ (b_k^j)^+(x) \right]^p \mu(x) \leq C \left\| f^\ast \chi_{\Omega_j} \right\|_{L^p(X)}^p;
\]

moreover, there exists $b^j \in H^{+p}(X)$ such that $b^j = \sum_{k \in I_j} b_k^j$ in $H^{+p}(X)$ and, for any $x \in X$,

\[
(b^j)^+(x) \leq C2^{j} \sum_{k \in I_j} \frac{\mu(B(x_k^j, r_k^j))}{\mu(B(x_k^j, r_k^j)) + V(x_k^j, x)} \left[ \frac{r_k^j}{r_k^j + d(x_k^j, x)} \right]^\beta + Cf^\ast(x)\chi_{\Omega_j}(x);
\]

if $g^j := f - b^j$ for any $j \in \mathbb{Z}$, then, for any $x \in X$,

\[
g^j)^+(x) \leq C2^{j} \sum_{k \in I_j} \frac{\mu(B(x_k^j, r_k^j))}{\mu(B(x_k^j, r_k^j)) + V(x_k^j, x)} \left[ \frac{r_k^j}{r_k^j + d(x_k^j, x)} \right]^\beta + Cf^\ast(x)\chi_{\Omega_j}(x).
\]

Proof. Fix $j \in \mathbb{Z}$. We first prove (4.4). Indeed, by Proposition 4.7, we find that

\[
\int_{X} \sum_{k \in I_j} \left[ (b_k^j)^+(x) \right]^p \mu(x) \leq 2^{jp} \int_{X} \sum_{k \in I_j} \left\{ \frac{\mu(B(x_k^j, r_k^j))}{\mu(B(x_k^j, r_k^j)) + V(x_k^j, x)} \left[ \frac{r_k^j}{r_k^j + d(x_k^j, x)} \right]^\beta \right\}^p \mu(x)
\]

\[
+ \int_{\bigcup_{k \in I_j} B(x_k^j, 16A_k r_k^j)} [f^\ast(x)]^p \mu(x).
\]

By Proposition 4.4(ii), we have $\Omega_j = \bigcup_{k \in I_j} B(x_k^j, 16A_k r_k^j)$. Applying this and Lemma 4.8, the first term in the right-hand side of the above formula is bounded by a harmlessly positive constant multiple of $2^{jp} \mu(\Omega_j)$. Combining this with $f^\ast \sim f^\ast$ implies that

\[
\int_{X} \sum_{k \in I_j} \left[ (b_k^j)^+(x) \right]^p \mu(x) \leq 2^{jp} \mu(\Omega_j) + \int_{\Omega_j} [f^\ast(x)]^p \mu(x) \leq \left\| f^\ast \chi_{\Omega_j} \right\|_{L^p(X)}^p,
\]
which proves \(4.4\).

Next we prove \(4.5\). By \(4.4\), the dominated convergence theorem and the completeness of \(H^{s,q}(X)\) (see Proposition 4.1), we know that there exists \(b^j \in H^{s,q}(X)\) such that \(b^j = \sum_{k \in I_j} b^j_k\) in \(H^{s,q}(X)\). Moreover, from Proposition 4.7 and \(\Omega^j = \bigcup_{k \in I_j} B(x^j_k, 16 \delta^j_k r^j_k)\), we deduce that, for any \(x \in X\),

\[
(b^j)^*(x) \leq \sum_{k \in I_j} (b^j_k)^*(x) \leq 2^j \sum_{k \in I_j} \frac{\mu(B(x^j_k, r^j_k))}{\mu(B(x^j_k, r^j_k)) + V(x^j_k, x)} \left[ \frac{r^j_k}{r^j_k + d(x^j_k, x)} \right]^\beta + f^*(x) \chi_{\Omega^j}(x).
\]

This finishes the proof of \(4.5\).

It remains to prove \(4.6\). If \(x \in (\Omega^j)^C\), then, by \(4.5\), we conclude that

\[
(g^j)^*(x) \leq f^*(x) + (b^j)^*(x) \leq 2^j \sum_{k \in I_j} \frac{\mu(B(x^j_k, r^j_k))}{\mu(B(x^j_k, r^j_k)) + V(x^j_k, x)} \left[ \frac{r^j_k}{r^j_k + d(x^j_k, x)} \right]^\beta + f^*(x),
\]

as desired.

Now we consider the case \(x \in \Omega^j\). According to Proposition 4.1(v), for any \(n \in I_j\), we choose a point \(y^j_n \in \Omega^j\) satisfying 32 \(A^j_0 r^j_n \leq d(x^j_n, y^j_n) < 48 \delta^j_k r^j_k\). Since \(x \in \Omega^j\), it follows that there exists \(k_0 \in I_j\) such that \(x \in B(x^j_{k_0}, r^j_{k_0})\). Let \(\mathcal{F}\) be the set of all \(n \in I_j\) such that \(B(x^j_n, 16 \delta^j_k r^j_k) \cap B(x^j_{k_0}, 16 \delta^j_k r^j_k) \neq 0\). Then, by the proof of Proposition 4.1(v), \(\#\mathcal{F} \leq L_0\) and \(r^j_n \sim r^j_{k_0}\) whenever \(n \in \mathcal{F}\).

Suppose that \(\varphi \in G^j(\beta, \gamma)\) with \(\|\varphi\|_{G^j(\beta, \gamma)} \leq 1\) for some \(r \in (0, \infty)\). We then estimate \(\langle g^j, \varphi \rangle\) by considering the cases \(r \leq r^j_{k_0}\) and \(r > r^j_{k_0}\), respectively.

**Case 1** \(r \leq r^j_{k_0}\). In this case, we write

\[
\langle g^j, \varphi \rangle = \langle f, \varphi \rangle - \sum_{n \in \mathcal{F}} \langle b^j_n, \varphi \rangle = \langle f, \varphi \rangle - \sum_{n \in \mathcal{F}} \langle b^j_n, \varphi \rangle - \sum_{n \notin \mathcal{F}} \langle b^j_n, \varphi \rangle = \langle f, \varphi \rangle - \sum_{n \notin \mathcal{F}} \langle f, \varphi \rangle - \sum_{n \notin \mathcal{F}} \langle b^j_n, \varphi \rangle,
\]

where \(\varphi := (1 - \sum_{n \notin \mathcal{F}} \phi^j_n) \varphi\) and, for any \(n \in \mathcal{F}\),

\[
\varphi_n := \phi^j_n \left[ \int_X \phi^j_n(z) \, d\mu(z) \right]^{-1} \int_X \varphi(z) \phi^j_n(z) \, d\mu(z).
\]

We first consider the term \(\sum_{n \notin \mathcal{F}} \langle b^j_n, \varphi \rangle\). Indeed, from \(x \in B(x^j_{k_0}, r^j_{k_0})\), it follows that \(x \notin B(x^j_n, 16 \delta^j_k r^j_k)\) when \(n \notin \mathcal{F}\). Applying Proposition 4.7 implies that

\[
\left| \langle b^j_n, \varphi \rangle \right| \leq \left| (b^j_n)^*(x) \right| \leq 2^j \frac{\mu(B(x^j_n, r^j_n))}{\mu(B(x^j_n, r^j_n)) + V(x^j_n, x)} \left[ \frac{r^j_n}{r^j_n + d(x^j_n, x)} \right]^\beta,
\]

and hence

\[
\sum_{n \notin \mathcal{F}} \left| \langle b^j_n, \varphi \rangle \right| \leq 2^j \sum_{n \notin \mathcal{F}} \frac{\mu(B(x^j_n, r^j_n))}{\mu(B(x^j_n, r^j_n)) + V(x^j_n, x)} \left[ \frac{r^j_n}{r^j_n + d(x^j_n, x)} \right]^\beta,
\]
as desired.

Next we consider the term $\sum_{n \in \mathcal{J}} \langle f, \overline{\varphi}_n \rangle$. Notice that $\|\overline{\varphi}_n\|_{\ell^2(B^0_{y'_n}, \beta, \gamma)} \leq 1$. By $d(x'_n, y'_n) \sim r'_n$, we then have $\|\overline{\varphi}_n\|_{\ell^2(B^0_{y'_n}, \beta, \gamma)} \leq 1$. Therefore,

$$
\| \langle f, \overline{\varphi}_n \rangle \| \lesssim f^\ast \left( \frac{y'_n}{r'_n} \right) \sim f^\ast \left( \frac{y'_n}{r'_n} \right) \lesssim 2^j \sim 2^j \left( \frac{r'_n}{\mu(B(x'_n, r'_n))} + V(x'_n, x) \right) \left( \frac{r'_n}{r'_n + d(x'_n, x)} \right)^\beta,
$$

where, in the last step, we used the facts that $x \in B(x'_n, r'_n)$ and $d(x'_n, x_n) \leq r'_n + r'_{k_0} \sim r'_n$ whenever $n \in \mathcal{J}$. Then, summing all $n \in \mathcal{J}$, we obtain the desired estimate.

Finally, we consider the term $\langle f, \overline{\varphi} \rangle$. Since $\varphi \in \ell^2(B^0_{\beta, \gamma})$, it is easy to see that $\overline{\varphi} \in \ell^2(B^0_{\beta, \gamma})$. Once we have proved that

$$
(4.7) \quad \| \overline{\varphi} \|_{\ell^2(B^0_{y'_n}, \beta, \gamma)} \leq 1,
$$

then

$$
\| \langle f, \overline{\varphi} \rangle \| \lesssim f^\ast \left( \frac{y'_n}{r'_n} \right) \sim f^\ast \left( \frac{y'_n}{r'_n} \right) \lesssim 2^j \sim 2^j \left( \frac{r'_n}{\mu(B(x'_n, r'_n))} + V(x'_n, x) \right) \left( \frac{r'_n}{r'_n + d(x'_n, x)} \right)^\beta,
$$

as desired.

To prove (4.7), we first consider the size condition. For any $z \in B(x'_n, 16A^0_{y'_n}r'_n)$, by Proposition 4.5, we have $\sum_{n \in \mathcal{J}} \phi'_n(z) = \sum_{n \in \mathcal{J}} \phi'_n(z) = 1$ and hence $\overline{\varphi}(z) = 0$. When $d(z, x'_n) \geq 16A^0_{y'_n}r'_n$, by the fact $d(x'_n, z) \geq 2A_0d(x, x'_n)$, we have

$$(4.8) \quad r'_n + d\left( z, y'_n \right) \leq r'_n + d\left( z, x'_n \right) + d\left( x'_n, y'_n \right) \leq (2A_0)^6 \left[ r'_n + d\left( z, x'_n \right) \right]$$

$$
\quad \leq (2A_0)^7 d\left( z, x'_n \right) \leq (2A_0)^8 d(z, x) \lesssim (2A_0)^8 \left[ r + d(z, x) \right]
$$

and hence $\mu(B(y'_n, r'_n)) + V(y'_n, z) \lesssim V_r(x) + V(x, z)$, which, together with the size condition of $\varphi$ and the fact that $r \leq r'_n$, further implies that

$$
\| \overline{\varphi}(z) \| \leq 1 \leq \frac{1}{V_r(x) + V(x, z)} \left( \frac{r}{r + d(x, z)} \right)^\gamma \leq \frac{1}{\mu(B(y'_n, r'_n)) + V(y'_n, z)} \left( \frac{r'_n}{r'_n + d(y'_n, z)} \right)^\gamma.
$$

This finishes the proof of the size condition.

Now we consider the regularity of $\overline{\varphi}$. Suppose that $z, z' \in X$ with $d(z, z') \leq (2A_0)^{-1} \left[ r'_n + d(z, y'_n) \right]$. Due to the size condition, we only need to consider the case $d(z, z') \leq (2A_0)^{-1} \left[ r'_n + d(z, y'_n) \right]$. If $\overline{\varphi}(z) - \overline{\varphi}(z') \neq 0$, then either $d(z, x'_n) \geq 16A^0_{y'_n}r'_n$ or $d(z', x'_n) \geq 16A^0_{y'_n}r'_n$, which always implies that $d(z, x'_n) \geq 8A^0_{y'_n}r'_n$.

Indeed, if $d(z, x'_n) < 8A^0_{y'_n}r'_n$, then $d(z, y'_n) \leq A_0 \left[ d(z, x'_n) + d(x'_n, y'_n) \right] < (2A_0)^6 r'_n$ and hence $d(z, z') \leq (2A_0)^3 r'_n$, which further implies that $d(z', x'_n) \leq A_0 \left[ d(z', z) + d(z, x'_n) \right] < 16A^0_{y'_n}r'_n$ and it is a contraction.
Notice that \( d(z, x_j^0) \geq 8A_0^3 r_j \), which, together with an argument as in the estimation of (4.8), implies \( r_k + d(z, y_j^0) \leq (2A_0)^3 [r + d(z, x)] \), so that \( d(z, z') \leq (2A_0)^{-1} [r + d(z, x)] \). By the definition of \( \vphi \), we find that

\[
|\vphi(z) - \vphi(z')| \leq \left( 1 - \sum_{n \in \cJ} \phi_n(z) \right) |\vphi(z) - \vphi(z')| + |\vphi(z')| \sum_{n \in \cJ} \left| \phi_n(z) - \phi_n(z') \right|.
\]

Using the regularity condition of \( \vphi \) and the fact \( d(z, z') \leq (2A_0)^{-1} [r + d(z, x)] \), we obtain

\[
\left( 1 - \sum_{n \in \cJ} \phi_n(z) \right) |\vphi(z) - \vphi(z')| \leq \left[ \frac{d(z, z')}{r + d(z, x)} \right]^{\beta} \frac{1}{V_r(x) + V(x, z)} \left[ \frac{r}{r + d(z, x)} \right]^{\gamma}
\]

\[
\leq \left[ \frac{d(z, z')}{r_k + d(z, y_j^0)} \right]^{\beta} \frac{1}{\mu(B(y_j^0, r_k)) + V(y_j^0, z)} \left[ \frac{r_k}{r_k + d(y_j^0, z)} \right]^{\gamma},
\]

where, in the last step, we used \( r_k + d(z, y_j^0) \leq r + d(z, x) \), \( r \leq r_k \), \( x \in B(x_k^j, r_k) \), and \( d(y_j^0, z) \sim r_k \).

We now estimate \( |\vphi(z')| \sum_{n \in \cJ} |\phi_n(z) - \phi_n(z')| \). If \( \vphi(z')|\phi_n(z) - \phi_n(z')| \neq 0 \), then \( z' \notin B(x_k^j, 16A_0^3 r_k^j) \) and \( z \) or \( z' \) belongs to \( B(x_k^j, 2A_0 r_k^j) \). When \( n \in \cJ \), we have \( r_n^j \sim r_k^j \sim r_k^j + d(y_k^j, z) \). Also, \( r_k^j + d(z, y_k^j) \leq r + d(z, x) \sim r + d(z', x) \). By these, \( \#\cJ \leq L_0 \) and \( r \leq r_k^j \), we conclude that

\[
|\vphi(z')| \sum_{n \in \cJ} |\phi_n(z) - \phi_n(z')| \leq \left[ \frac{d(z, z')}{r_k^j + d(y_k^j, z)} \right]^{\beta} \frac{1}{\mu(B(y_k^j, r_k^j)) + V(y_k^j, z)} \left[ \frac{r_k^j}{r_k^j + d(y_k^j, z)} \right]^{\gamma}.
\]

This finishes the proof of the regularity condition and hence of (4.7). Thus, we complete the proof of Case 1).

Case 2) \( r > r_k^j \). In this case, we write

\[
|\langle g^j, \varphi \rangle| \leq |\langle f, \varphi \rangle| + \sum_{n \notin \cJ} |\langle b_n^j, \varphi \rangle| + \sum_{n \notin \cJ} |\langle b_n^j, \varphi \rangle|.
\]

The estimation of \( \sum_{n \notin \cJ} |\langle b_n^j, \varphi \rangle| \) has already been given in Case 1).

From \( x \in B(x_k^j, r_k^j) \) and \( d(y_k^j, x_k^j) \sim r_k^j \leq r \), it follows that \( |\phi| \|g(y_k^j, r, \beta, y)\| \leq 1 \) and hence

\[
|\langle f, \varphi \rangle| \leq \left[ \frac{\mu(B(x_k^j, r_k^j))}{\mu(B(x_k^j, r_k^j)) + V(x_k^j, x)} \left[ \frac{r_k^j}{r_k^j + d(x_k^j, x)} \right]^{\beta} \right].
\]

If \( n \in \cJ \), then \( r_n^j \sim r_k^j \) and hence \( d(y_n^j, x_k^j) \leq r_k^j \). This, together with the fact \( r_k^j < r \) and \( x \in B(x_k^j, r_k^j) \), implies that \( |\phi| \|g(y_n^j, r, \beta, y)\| \leq 1 \). Thus, by Proposition 4.7, we have

\[
\sum_{n \notin \cJ} |\langle b_n^j, \varphi \rangle| \leq \sum_{n \notin \cJ} |\langle b_n^j, \varphi \rangle| \leq 2^j \sum_{n \notin \cJ} \frac{\mu(B(x_n^j, r_n^j))}{\mu(B(x_n^j, r_n^j)) + V(x_n^j, x)} \left[ \frac{r_n^j}{r_n^j + d(x_n^j, x)} \right]^{\beta}.
\]
Then we obtain the desired estimate for $\langle g^j, \varphi \rangle$ in the case $r > r^j_{k_0}$.

Combining the two cases above, we find that, for any $x \in \Omega^j$,

$$
(g^j)^*(x) \leq 2^j \sum_{k \in I_j} \frac{\mu(B(x^j_k, r^j_k))}{\mu(B(x^j_k, r^j_k)) + V(x^j_k, x)} \left[ \frac{r^j_k}{r^j_k + d(x^j_k, x)} \right]^\beta.
$$

Thus, (4.6) holds true. This finishes the proof of Proposition 4.9.

\[ \square \]

4.3 Atomic characterization of $H^{s,p}(X)$

In this section, we prove $H^{s,p}(X) \subset H_{at}^{s,q}(X)$ and complete the proof of Theorem 4.2. First, we obtain dense subspaces of $H^{s,p}(X)$.

**Lemma 4.10 ([20] Proposition 4.12).** Let $p \in (\omega/(\omega + \eta), 1]$, $\beta, \gamma \in (\omega(1/p - 1), \eta)$ and $q \in [1, \infty)$. If regard $H^{s,p}(X)$ as a subspace of $(G^p_d)^ \cdot (\beta, \gamma)'$, then $L^q(X) \cap H^{s,p}(X)$ is dense in $H^{s,p}(X)$.

In the next two lemmas, we suppose that $f \in L^2(X) \cap H^{s,p}(X)$. Based on Proposition 3.3 and (3.1), we may follow [20] Remark 4.14 and assume that there exists a positive constant $C$ such that, for any $x \in X$, $|f(x)| \leq C f^\gamma(x)$. With all the notation as in the previous section, for any $j \in \mathbb{Z}$ and $k \in I_j$, define

$$
(4.9) \quad m^j_k := \frac{1}{\| \phi^j_k \|_{L^1(X)}} \int_X f(\xi) \phi^j_k(\xi) \, d\mu(\xi) \quad \text{and} \quad b^j_k := (f - m^j_k) \phi^j_k.
$$

Then we have the following technical lemma.

**Lemma 4.11 ([20] Proposition 4.13).** For any $j \in \mathbb{Z}$ and $k \in I_j$, let $m^j_k$ and $b^j_k$ be as in (4.9). Then

(i) there exists a positive constant $C$, independent of $j$ and $k \in I_j$, such that $|m^j_k| \leq C 2^j$;

(ii) $b^j_k$ induces the same distribution as defined in (4.3);

(iii) $\sum_{k \in I_j} b^j_k$ converges to some function $b^j$ in $L^2(X)$, which induces a distribution that coincides with $b^j$ as in Proposition 4.9;

(iv) let $g^j := f - b^j$. Then $g^j = f_X(\Omega_j) + \sum_{k \in I_j} m^j_k \phi^j_k$. Moreover, there exists a positive constant $C$, independent of $j$, such that, for any $x \in X$, $|g^j(x)| \leq C 2^j$.

For any $j \in \mathbb{Z}$, $k \in I_j$ and $l \in I_{j+1}$, define

$$
(4.10) \quad L^{j+1}_{k,l} := \frac{1}{\| \phi^j_l \|_{L^1(X)}} \int_X \left[ f(\xi) - m^{j+1}_l \right] \phi^j_l(\xi) \phi^{j+1}_l(\xi) \, d\mu(\xi)
$$

Then $L^{j+1}_{k,l}$ has the following properties.

**Lemma 4.12.** For any $j \in \mathbb{Z}$, $k \in I_j$ and $l \in I_{j+1}$, let $L^{j+1}_{k,l}$ be as in (4.10). Then
(i) there exists a positive constant $C$, independent of $j, k$ and $l$, such that
\[
\sup_{x \in X} |L_{k,l}^j \phi_{j+1}^l(x)| \leq C 2^j;
\]

(ii) $\sum_{k \in I} \sum_{l \in I_{j+1}} L_{k,l}^j \phi_{j+1}^l = 0$ both in $(G_0^j(\beta, \gamma))'$ and everywhere.

**Proof.** We first show (i). Indeed, for any $j \in \mathbb{Z}, k \in I, l \in I_{j+1}$ and $x \in X$,
\[
\left| L_{k,l}^j \phi_{j+1}^l(x) \right| \leq \left| m_l^j \right| \phi_{j+1}^l(x) + \phi_{j+1}^l(x) \left\| f(\xi) \phi_{j+1}^l(\xi) \right\|_{L_1(X)} d\mu(\xi) =: Y_1 + Y_2.
\]

By Lemma 4.4(i) and the definition of $\phi_{j+1}^l$, it is easy to obtain $Y_1 \leq 2^j$.

Now we consider $Y_2$. If $\phi_{j+1}^l$ is a non-zero function, then $B(x_k^j, 2A_0 r_k^j) \cap B(x_{j+1}^j, 2A_0 r_{j+1}^j) \neq \emptyset$, which further implies that $r_{j+1}^j \leq 3A_0 r_k^j$. Otherwise, if $r_{j+1}^j > 3A_0 r_k^j$, then, for any $y \in B(x_k^j, 48A_0^5 r_k^j)$,
\[
d(\eta, x_{j+1}^j) \leq A_0 \left[ d(y, x_k^j) + d(x_k^j, x_{j+1}^j) \right] < 48A_0^5 r_k^j + 2A_0^2(2A_0 r_k^j + 2A_0 r_{j+1}^j)
\]
\[
< 16A_0^5 r_{j+1}^j + \frac{2}{3}A_0^2 r_{j+1}^j + 2A_0^2 r_{j+1}^j < 20A_0^5 r_{j+1}^j,
\]
which implies that $B(x_j^{j+1}, 48A_0^5 r_j^{j+1}) \subset B(x_{j+1}^{j+1}, 2A_0 r_{j+1}^j) \subset \Omega^{j+1} \subset \Omega^j$ and hence contradicts to Proposition 4.4(iv).

Define $\varphi := \phi_{j+1}^l/\|\phi_{j+1}^l\|_{L_1(X)}$. According to Proposition 4.4(iv) with $A := 16A_0^4$, we can choose $y_{j+1}^j \in (\Omega^{j+1})^c$ such that $d(y_{j+1}^j, x_{j+1}^j) \leq 48A_0^5 r_{j+1}^j$. We now show $\varphi \in G(y_{j+1}^j, r_{j+1}^j, \eta, \eta)$ and $\|\varphi\|_{g_{y_{j+1}^j, r_{j+1}^j, \eta, \eta}} \leq 1$. Notice that $\sup \varphi \subset B(x_{j+1}^j, 2A_0 r_{j+1}^j)$. Moreover, by this and the choice of $y_{j+1}^j$, we conclude that, for any $x \in B(x_{j+1}^j, 2A_0 r_{j+1}^j)$,
\[
|\varphi(x)| \leq |\phi_{j+1}^l(x)| \leq \frac{1}{\mu(B(x_{j+1}^j, r_{j+1}^j)) + V(x_{j+1}^j, x)} \left[ \frac{r_{j+1}^j}{r_{j+1}^j + d(x_{j+1}^j, x)} \right]^\eta
\]
\[
\sim \frac{1}{\mu(B(x_{j+1}^j, r_{j+1}^j)) + V(y_{j+1}^j, x)} \left[ \frac{r_{j+1}^j}{r_{j+1}^j + d(y_{j+1}^j, x)} \right]^\eta.
\]

This shows the size condition of $\varphi$.

To consider the regularity condition of $\varphi$, we suppose that $x, x' \in X$ satisfying $d(x, x') \leq (2A_0)^{-1}[r_{j+1}^j + d(y_{j+1}^j, x)]$. Due to the size condition, we may assume $d(x, x') \leq (2A_0)^{-3}[r_{j+1}^j + d(y_{j+1}^j, x)]$. We claim that $\varphi(x) - \varphi(x') \neq 0$ implies that $d(x, x_{j+1}^j) \leq 96A_0 r_{j+1}^j$.

Indeed, if $d(x, x_{j+1}^j) > 96A_0 r_{j+1}^j$, then $\varphi(x) = 0$. By $d(x_{j+1}^j, y_{j+1}^j) \leq 48A_0^5 r_{j+1}^j$, we find that $d(x, y_{j+1}^j) > 48A_0^5 r_{j+1}^j$ and hence $d(x, x_{j+1}^j) \leq (2A_0)^{-2}(2A_0)^{-2}(2A_0)^{-3}d(x, x_{j+1}^j) \leq (2A_0)^{-1}d(x, x_{j+1}^j)$. Consequently, $d(x, x_{j+1}^j) \geq A_0^3 d(x, x_{j+1}^j) - d(x, x') > 48A_0^5 r_{j+1}^j$ and $\varphi(x') = 0$. This contradicts to $\varphi(x) - \varphi(x') \neq 0$.

By the above claim, $r_{j+1}^j \leq 3A_0 r_k^j$ and $d(y_{j+1}^j, x_{j+1}^j) \sim r_{j+1}^j$, we know that
\[
|\varphi(x) - \varphi(x')| \leq \frac{1}{\mu(B(x_{j+1}^j, r_{j+1}^j))} \left| \phi_{j+1}^l(x) \phi_{j+1}^l(x) - \phi_{j+1}^l(x') \phi_{j+1}^l(x') \right| + \left| \phi_{j+1}^l(x) - \phi_{j+1}^l(x') \phi_{j+1}^l(x') \right|.
\]
and hence

\[
\sum \frac{1}{\mu(B(x', r_j^{i+1}))} \left( \left| \frac{d(x,x')}{r_j^{i+1}} \right|^\eta + \left| \frac{d(x,x')}{r_k'} \right|^\eta \right) \leq \sum \frac{d(x,x')}{{r_j^{i+1} + d(y^{i+1}, x)}} \mu(B(y^{i+1}, r_j^{i+1}))) + V(y^{i+1}, x) \left| \frac{r_j^{i+1}}{r_j^{i+1} + d(y^{i+1}, x)} \right|^\eta.
\]

Thus, we obtain \( v \in \mathcal{G}(y^{i+1}, r_j^{i+1}, \eta, \eta) \) and \( \|v\|_{\mathcal{G}(y^{i+1}, r_j^{i+1}, \eta, \eta)} \leq 1 \), which further implies that \( \|v\|_{\mathcal{G}(y^{i+1}, r_j^{i+1}, \beta, \gamma)} \leq 1 \) and hence

\[
Y_2 = T(f, v) \leq 2^j.
\]

This finishes the proof of (i).

Next we prove (ii). If \( L_{j,k}^{j+1} \neq 0 \), then the proof in (i) implies \( B(x_k', 2A_0r_j^{j+1}) \cap B(x_j^{i+1}, 2A_0r_j^{i+1}) \neq \emptyset \) and \( r_j^{j+1} \leq 3A_0r_j^j \). Further, for any \( y \in B(x_j^{i+1}, 2A_0r_j^{j+1}) \), we have

\[
d(y, x_k') \leq A_0 \left| d(y, x_j^{i+1}) \right| + d(x_j', x_j^{i+1}) < 2A_0^2r_j^{j+1} + A_0^2 \left( 2A_0r_j^j + 2A_0r_j^{j+1} \right) \\
< 6A_0^3r_j^j + 2A_0^4r_j^{j+1} + 6A_0^4r_j^j \leq 14A_0^4r_j^j < 16A_0^4r_j^{j+1},
\]

which implies that \( B(x_k', 2A_0r_j^{i+1}) \cap B(x_j', 16A_0^4r_j^{j+1}) \subseteq \Omega_i^j \) by Proposition \( 4.4 \). Thus, for any \( k \in I_j \) and \( x \in X \), we find that

\[
\sum_{k \in I_j} \sum_{l \in I_{j+1}} \left| L_{j,k}^{j+1} \phi_l^{i+1} \right| \leq 2^j \chi_{B(x_k', 16A_0^4r_j^{j+1})}(x)
\]

and hence

\[
\sum_{k \in I_j} \sum_{l \in I_{j+1}} \left| L_{j,k}^{j+1} \phi_l^{i+1} (x) \right| \leq 2^j \sum_{k \in I_j} \chi_{B(x_k', 16A_0^4r_j^{j+1})}(x) \leq 2^j \chi_{\Omega_i^j}(x).
\]

Consequently,

\[
\sum_{k \in I_j} \sum_{l \in I_{j+1}} L_{j,k}^{j+1} \phi_l^{i+1} = \sum_{l \in I_{j+1}} \left( \sum_{k \in I_j} L_{j,k}^{j+1} \right) \phi_l^{i+1} = \sum_{l \in I_{j+1}} \frac{\phi_l^{i+1}}{||\phi_l^{i+1}||_{L^1(X)}} \int_X \left[ f(\xi) - m_{j}^{i+1} \right] \phi_l^{i+1}(\xi) \mu(\xi)
\]

\[
= \sum_{l \in I_{j+1}} \frac{\phi_l^{i+1}}{||\phi_l^{i+1}||_{L^1(X)}} \int_X \left[ f(\xi) - m_{j}^{i+1} \right] \phi_l^{i+1}(\xi) \mu(\xi)
\]

\[
= \sum_{l \in I_{j+1}} \frac{\phi_l^{i+1}}{||\phi_l^{i+1}||_{L^1(X)}} \int_X b_l^{i+1}(\xi) \mu(\xi) = 0.
\]

By the fact that \( \sum_{k \in I_j} \sum_{l \in I_{j+1}} \int_X L_{j,k}^{j+1} \phi_l^{i+1}(\xi) \mu(\xi) \leq 2^j \mu(\Omega_i^j) < \infty \) and the dominated convergence theorem, we find that \( \sum_{k \in I_j} \sum_{l \in I_{j+1}} L_{j,k}^{j+1} \phi_l^{i+1} = 0 \) in \( L^1(X) \) and hence in \( (\mathcal{G}_0^j(\beta, \gamma))^j \). This finishes the proof of Lemma 4.12. \( \square \)
Now we show another side of Theorem 4.2.

**Proof of** $H^{s,p}(X) \subset H_{at}^{p,q}(X)$. By Lemma 4.10 we first suppose $f \in L^2(X) \cap H^{s,p}(X)$. We may also assume $|f(x)| \leq f^*(x)$ for any $x \in X$. We use the same notation as in Lemmas 4.11 and 4.12. For any $j \in \mathbb{N}$, let $h^j := g^{j+1} - g^j = b^j - b^{j+1}$. Then $f - \sum_{m=-\infty}^j h^j = b^{m+1} - g^m$. For any $m \in \mathbb{Z}$, by Lemma 4.11, we conclude that $\|g^m\|_{L^\infty(X)} \leq 2^{-m}$. Moreover, by (4.5), we find that $\|(b^{m+1})^*\|_{L^p(X)} \leq \|f^*\|_{L^p(X)} \to 0$ as $m \to \infty$. Thus, $f = \sum_{j=-\infty}^\infty h^j$ in $(G_0^p(\beta, \gamma))^\prime$. Besides, by the definition of $b^m_k$, we know that supp $b^{m+1} \subseteq \Omega^{m+1}$, which then implies that $\sum_{j=-\infty}^\infty h^j$ converges almost everywhere. Notice that, by Lemma 4.12(ii), for any $j \in \mathbb{Z}$, we have

$$h^j = b^j - b^{j+1} = \sum_{k \in I_j} b^j_k - \sum_{l \in I_{j+1}} b^j_l + \sum_{k \in I_j} \sum_{l \in I_{j+1}} L^j_{k,l} \phi^j_{k,l},$$

which converges in $(G_0^p(\beta, \gamma))^\prime$ and almost everywhere. Moreover, for any $j \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$h^j_k = b^j_k - \sum_{l \in I_{j+1}} (b^j_l \phi^j_{k,l} - L^j_{k,l} \phi^j_{k,l+1}) = (f - m^j_k) \phi^j - \sum_{l \in I_{j+1}} [(f - m^j_{l+1}) \phi^j - L^j_{k,l} \phi^j_{k,l+1}].$$

The fourth term is supported on $B^j_k := B(x^j_k, 16\mathcal{A}^j_{0,k})$, which is deduced from (4.11). Thus, supp $h^j_k \subseteq B^j_k$. Moreover, by Lemmas 4.11(i) and 4.12(i), we conclude that there exists a positive constant $C$, independent of $j$ and $k$, such that $\|h^j_k\|_{L^\infty(X)} \leq C 2^j$. Now, let

$$\lambda^j_k := C 2^j \left|\mu(B^j_k)^{1/2}\right| \quad \text{and} \quad a^j_k := \left(\lambda^j_k\right)^{-1} h^j_k.$$ Then $a^j_k$ is a $(p, \infty)$-atom supported on $B^j_k$ and $f = \sum_{j=-\infty}^\infty \sum_{k \in I_j} \lambda^j_k a^j_k$ in $(G_0^p(\beta, \gamma))^\prime$. Moreover, we have

$$\sum_{j=-\infty}^\infty \sum_{k \in I_j} |\lambda^j_k|^p \lesssim \sum_{j=-\infty}^\infty \sum_{k \in I_j} \mu(B^j_k) \lesssim \sum_{j=-\infty}^\infty \sum_{j=-\infty}^\infty 2^{-jp} \mu(\Omega^j) \sim \|f^*\|_{L^p(X)} \sim \|f\|_{H^{s,p}(X)},$$

which further implies that $\|f\|_{H^{s,p}(X)} \lesssim \|f\|_{H^{s,p}(X)}$.

When $f \in H^{s,p}(X)$, using Lemma 4.10 and a standard density argument and following the proof in [43] pp. 301–302, we obtain the atomic decomposition of $f$, the details being omitted. This finishes the proof of $H^{s,p}(X) \subset H_{at}^{p,q}(X)$ and hence of Theorem 4.2. \qed

**Remark 4.13.** By the argument used in the proof of $H^{s,p}(X) \subset H_{at}^{p,q}(X)$, we find that, if $f \in L^q(X) \cap H^{s,p}(X)$ with $q \in [1, \infty]$, then $f = \sum_{j=1}^\infty \sum_{k \in I_j} h^j_k$ in $(G_0^p(\beta, \gamma))^\prime$ and almost everywhere, where, for any $j \in \mathbb{Z}$ and $k \in I_j$, $h^j_k$ is as in (4.12).
4.4 Relationship between $H_{at}^{p,q}(X)$ and $H_{cw}^{p,q}(X)$

In this section, we consider the relationship between $H_{at}^{p,q}(X)$ and $H_{cw}^{p,q}(X)$. To see this, we need the following two technical lemmas.

**Lemma 4.14** ([1], p. 592). Let $p \in (0,1)$, $q \in (p, \infty] \cap [1, \infty]$ and $a$ be a $(p,q)$-atom. Then, for any $\varphi \in L_{1/p-1}(X)$, $|\langle a, \varphi \rangle| \leq \|\varphi\|_{L_{1/p-1}(X)}$.

**Lemma 4.15.** Let $\beta \in (0, \eta]$ and $\gamma \in (0, \infty)$. If $\varphi \in G(\beta, \gamma)$, then $\varphi \in L_{\beta/(\omega)}(X)$ and there exists a positive constant $C$, independent of $\varphi$, such that $\|\varphi\|_{L_{\beta/(\omega)}(X)} \leq C\|\varphi\|_{G(\beta, \gamma)}$.

**Proof.** Suppose that $\|\varphi\|_{G(\beta, \gamma)} \leq 1$. If $d(x,y) \leq (2A_0)^{-1}[1 + d(x_0, x)]$, then, by the regularity condition of $\varphi$ and (11), we have

$$|\varphi(x) - \varphi(y)| \leq \left[ \frac{d(x,y)}{1 + d(x_0, x)} \right]^{\beta} \left[ \frac{1}{V_1(x_0) + V(x_0, x)} \right] \left[ \frac{1}{1 + d(x_0, x)} \right]^\gamma \leq \left[ \frac{\mu(B(x,d(x,y)))}{\mu(B(x,1 + d(x_0, x)))} \right]^{\beta/\omega} \leq \left[ V(x,y) \right]^{\beta/\omega}.$$

If $d(x,y) > (2A_0)^{-1}[1 + d(x_0, x)]$, then, from the size condition of $\varphi$, we deduce that

$$|\varphi(x) - \varphi(y)| \leq 1 \sim [\mu(B(x_0, 1))]^{\beta/\omega} \leq [\mu(B(x_0, 1 + d(x_0, x)))]^{\beta/\omega} \sim [\mu(B(x, 1 + d(x_0, x)))]^{\beta/\omega} \leq [V(x,y)]^{\beta/\omega}.$$

Thus, for any $x, y \in X$, we always have $|\varphi(x) - \varphi(y)| \leq \|\varphi\|_{G(\beta, \gamma)} [V(x,y)]^{\beta/\omega}$. This implies $\varphi \in L_{\beta/(\omega)}(X)$ and $\|\varphi\|_{L_{\beta/(\omega)}(X)} \leq \|\varphi\|_{G(\beta, \gamma)}$, which completes the proof of Lemma 4.15.

Now we establish the relationship between two kinds of atomic Hardy spaces.

**Theorem 4.16.** Let $p \in (\omega/(\omega + \eta), 1)$, $q \in (p, \infty] \cap [1, \infty]$ and $\beta, \gamma \in (\omega(1/p - 1), \eta)$. If regard $H_{at}^{p,q}(X)$ as a subspace of $(G_0^{q}(\beta, \gamma))'$, then $H_{cw}^{p,q}(X) = H_{at}^{p,q}(X)$ with equal (quasi-)norms.

**Proof.** We only consider the case $p \in (\omega/(\omega + \eta), 1)$. The proof of $p = 1$ is similar and the details are omitted.

We first prove $H_{cw}^{p,q}(X) \subset H_{at}^{p,q}(X)$. By Lemma 4.15, we have $G_0^{q}(\beta, \gamma) \subset G(\omega(1/p - 1), \gamma) \subset L_{1/p-1}(X)$ and hence $(L_{1/p-1}(X))' \subset (G_0^{q}(\beta, \gamma))'$. For any $f \in H_{cw}^{p,q}(X)$, by Definition 1.1, we know that there exist $(p,q)$-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ with $\sum_{j=1}^\infty |\lambda_j|^p < \infty$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $(L_{1/p-1}(X))'$ and hence in $(G_0^{q}(\beta, \gamma))'$. Let $g := f|_{G_0^{q}(\beta, \gamma)}$. Then, for any $\varphi \in G_0^{q}(\beta, \gamma) \subset L_{1/p-1}(X)$, we have

$$\langle g, \varphi \rangle = \langle f, \varphi \rangle = \sum_{j=1}^\infty \lambda_j \langle a_j, \varphi \rangle.$$

Thus, $g = \sum_{j=1}^\infty \lambda_j a_j$ in $(G_0^{q}(\beta, \gamma))'$ and $\|g\|_{H_{at}^{p,q}(X)} \leq (\sum_{j=1}^\infty |\lambda_j|^p)^{\frac{1}{q}}$. If we take the infimum over all the atomic decompositions of $f$ as above, we obtain $\|g\|_{H_{at}^{p,q}(X)} \leq \|f\|_{H_{cw}^{p,q}(X)}$. Thus, $H_{cw}^{p,q}(X) \subset H_{at}^{p,q}(X)$.
To show $H^{p,q}_{cw}(X) \supset H^{p,q}_{at}(X)$, following the proof of [7] p. 593, Theorem B, we conclude that the dual space of $H^{p,q}_{at}(X)$ is $L_{1/p-1}(X)$ in the following sense: each bounded linear functional on $H^{p,q}_{at}(X)$ is a mapping of the form

$$f \mapsto \sum_{j=1}^{\infty} \lambda_j \int_X a_j(x)g(x) \, d\mu(x),$$

where $g \in L_{1/p-1}(X)$ and $f$ has an atomic decomposition

$$f = \sum_{j=1}^{\infty} \lambda_j a_j$$

in $(\mathcal{G}_0^q(\beta, \gamma))'$ with $(p,q)$-atoms $(a_j)^{\infty}_{j=1}$ and $(\lambda_j)^{\infty}_{j=1} \subset \mathbb{C}$ satisfying $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Therefore, it is reasonable to define the pair $(f, g)$ as follows:

$$(f, g) := \sum_{j=1}^{\infty} \lambda_j \int_X a_j(x)g(x) \, d\mu(x).$$

In this way, we find that (4.14) also converges in $(L_{1/p-1}(X))'$, and hence $f \in H^{p,q}_{cw}(X)$ and $\|f\|_{H^{p,q}_{cw}(X)} \leq (\sum_{j=1}^{\infty} |\lambda_j|^p)^{1/p}$. Taking the infimum over all the atomic decompositions of $f$ as above, we obtain $\|f\|_{H^{p,q}_{cw}(X)} \leq \|f\|_{H^{p,q}_{at}(X)}$. Thus, $H^{p,q}_{at}(X) \subset H^{p,q}_{cw}(X)$, which completes the proof of Theorem 4.16.

5 Littlewood-Paley function characterizations of atomic Hardy spaces

In this section, we consider the Littlewood-Paley function characterizations of Hardy spaces. Differently from Sections 3 and 4, we use $(\mathcal{G}_0^q(\beta, \gamma))'$ as underlying spaces to introduce Hardy spaces. Let $p \in (\omega/(\omega + \eta), 1], \beta, \gamma \in (\omega(1/p - 1), \omega), f \in (\mathcal{G}_0^q(\beta, \gamma))'$ and $\{Q_k\}_{k \in \mathbb{Z}}$ be an exp-ATI. For any $\theta \in (0, \infty)$, define the Lusin area function of $f$, with aperture $\theta$, $S_\theta(f)$, by setting, for any $x \in X$,

$$S_\theta(f)(x) := \left[ \sum_{k=-\infty}^{\infty} \int_{B(x, \theta^k)} |Q_kf(y)|^2 \frac{d\mu(y)}{V_{\theta^k}(x)} \right]^{1/2}.$$  

In particular, when $\theta = 1$, we write $S_\theta$ simply as $S$. Define the Hardy space $H^p(X)$ via the Lusin area function by setting

$$H^p(X) := \left\{ f \in (\mathcal{G}_0^q(\beta, \gamma))' : \|f\|_{H^p(X)} := \|S(f)\|_{L^p(X)} < \infty \right\}.$$  

In Section 5.1, we show that $H^p(X)$ is independent of the choices of exp-ATIs. In Section 5.2, we connect $H^p(X)$ with $H^{p,p}(X)$ by considering the molecular and the atomic characterizations of elements in $H^p(X)$. Section 5.3 deals with equivalent characterizations of $H^p(X)$ via the Littlewood-Paley g-function

$$g(f)(x) := \left[ \sum_{k=-\infty}^{\infty} |Q_kf(x)|^2 \right]^{1/2}.$$
and the Littlewood-Paley $g^*_A$-function

$$g^*_A(f)(x) := \left\{ \sum_{k=-\infty}^{\infty} \left( \int_X |Q_k f(y)|^2 \left[ \frac{\delta^k}{\delta^k + d(x, y)} \right] dy \right)^{\frac{1}{2}} \frac{d\mu(y)}{V_\delta(x) + V_\delta(y)} \right\}^{\frac{1}{2}}.$$  

where $f \in (\mathcal{G}_\theta^\prime(\beta, \gamma))^\prime$ with $\beta, \gamma \in \omega(1/p - 1, \eta)$, $x \in X$ and $\lambda \in (0, \infty)$.

### 5.1 Independence of exp-ATIs

In this section, we show that $H^p(X)$ is independent of the choices of exp-ATIs. If $\mathcal{E} := \{E_k\}_{k \in \mathbb{Z}}$ and $Q := \{Q_k\}_{k \in \mathbb{Z}}$ are two exp-ATIs, then we denote by $S_\mathcal{E}$ and $S_Q$ the Lusin area functions via $\mathcal{E}$ and $Q$, respectively.

**Theorem 5.1.** Let $\mathcal{E} := \{E_k\}_{k \in \mathbb{Z}}$ and $Q := \{Q_k\}_{k \in \mathbb{Z}}$ be two exp-ATIs. Suppose that $p \in (\omega/(\omega + \eta), 1]$ and $\beta, \gamma \in \omega(1/p - 1, \eta)$. Then there exists a positive constant $C$ such that, for any $f \in (\mathcal{G}_\theta^\prime(\beta, \gamma))^\prime$,

$$C^{-1} \|S_\mathcal{E}(f)\|_{L^p(X)} \leq \|S_Q(f)\|_{L^p(X)} \leq C \|S_Q(f)\|_{L^p(X)}.$$  

To show Theorem 5.1, the Fefferman-Stein vector-valued maximal inequality is necessary.

**Lemma 5.2** ([22] Theorem 1.2]). Suppose that $p \in (1, \infty)$ and $u \in (1, \infty]$. Then there exists a positive constant $C$ such that, for any sequence $\{f_j\}_{j=1}^\infty$ of measurable functions,

$$\left\| \left\{ \mathcal{M}(f_j)^u \right\} \right\|_{L^p(X)} \leq C \left\| \left\{ f_j^u \right\} \right\|_{L^p(X)}$$  

with the usual modification made when $u = \infty$.

**Proof of Theorem 5.1** By symmetry, we only need to prove $\|S_\mathcal{E}(f)\|_{L^p(X)} \leq \|S_Q(f)\|_{L^p(X)}$. For any $k \in \mathbb{Z}$, $f \in (\mathcal{G}_\theta^\prime(\beta, \gamma))^\prime$ with $\beta, \gamma$ as in Theorem 5.1 and $z \in X$, define

$$m_k(f)(z) := \left[ \frac{1}{V_\delta(z)} \int_{B(z, \delta^k)} |Q_k f(u)|^2 d\mu(u) \right]^{\frac{1}{2}}.$$  

Now suppose that $l \in \mathbb{Z}$, $x \in X$ and $y \in B(x, \delta^l)$. By Theorem 2.7, we conclude that

$$E_l f(y) = \sum_{k=-\infty}^{\infty} \sum_{a \in A_k} \sum_{m=1}^{N(k,a)} E_l \tilde{Q}_k \{ f, \gamma \} \int_{Q_k^m} Q_k f(u) d\mu(u),$$  

where all the notation is as in Theorem 2.7 and $\{\tilde{Q}_k\}_{k=-\infty}^\infty$ satisfy the conditions of Theorem 2.7. Notice that, if $z \in Q_k^m$, then $Q_k^m \subset B(z, \delta^k)$ and $\mu(Q_k^m) \sim V_\delta(z)$. Therefore, we have

$$\left\| \frac{1}{\mu(Q_k^m)} \int_{Q_k^m} Q_k f(u) d\mu(u) \right\| \leq \left[ \frac{1}{V_\delta(z)} \int_{B(z, \delta^k)} |Q_k f(u)|^2 d\mu(u) \right]^{\frac{1}{2}} \sim m_k(f)(z),$$  

in the sense of $\mathcal{G}_\theta^\prime(\beta, \gamma)$.
which further implies that
\[
\left| \frac{1}{\mu(Q_{\alpha}^{m})} \int_{Q_{\alpha}^{m}} Q_{k} f(u) \, d\mu(u) \right| \leq \inf_{z \in Q_{\alpha}^{m}} m_{k}(f)(z).
\]
Moreover, by the proof of (3.7), we find that, for any fixed \( \beta' \in (0, \beta) \),
\[
\left| E_{f} Q_{k} (y, y_{\alpha}^{m}) \right| \leq \delta^{k-\beta'} \frac{1}{V_{\delta^{k} \chi}(y) + V(y, y_{\alpha}^{m})} \left[ \frac{\delta^{k \cdot \Lambda} + d(y, y_{\alpha}^{m})}{\delta^{k \cdot \Lambda} + d(x, y_{\alpha}^{m})} \right]^{\gamma} \]
\[
\sim \delta^{k-\beta'} \frac{1}{V_{\delta^{k} \chi}(x) + V(x, y_{\alpha}^{m})} \left[ \frac{\delta^{k \cdot \Lambda} + d(y, y_{\alpha}^{m})}{\delta^{k \cdot \Lambda} + d(x, y_{\alpha}^{m})} \right]^{\gamma},
\]
where only the regularity condition of \( \tilde{Q}_{k} \) on the first variable is used. Therefore, by Lemma 3.7 for any fixed \( r \in (\omega/\omega + \gamma, 1) \), we have
\[
|E_{f} Q(y)\rangle \leq \sum_{k=-\infty}^{\infty} \delta^{k-\beta'} \sum_{\alpha \in A_{k}} \sum_{m=1}^{N(k, \alpha)} \mu(Q_{\alpha}^{m}) \frac{1}{V_{\delta^{k} \chi}(x) + V(x, y_{\alpha}^{m})} \left[ \frac{\delta^{k \cdot \Lambda} + d(y, y_{\alpha}^{m})}{\delta^{k \cdot \Lambda} + d(x, y_{\alpha}^{m})} \right]^{\gamma} \inf_{z \in Q_{\alpha}^{m}} m_{k}(f)(z)
\]
\[
\leq \sum_{k=-\infty}^{\infty} \delta^{k-\beta'} \delta^{k-(k \cdot \Lambda) u(1-\frac{1}{p})} \left\{ M \left( \sum_{\alpha \in A_{k}} \sum_{m=1}^{N(k, \alpha)} \inf_{z \in Q_{\alpha}^{m}} \left[ m_{k}(f)(z) \right]^{\gamma} \chi Q_{\alpha}^{m} \right) (x) \right\}^{\frac{1}{p}}.
\]
Choose \( \beta' \) and \( r \) such that \( r \in (\omega/\omega + \beta', p) \). Then, by the Hölder inequality, we conclude that
\[
|S_{E}(f)(x)|^{2} = \sum_{k=-\infty}^{\infty} \int_{B(x, \delta^{k \cdot \Lambda})} \frac{|E_{f} Q(y)\rangle^{2}}{V_{\delta^{k} \chi}(x)} \, dy
\]
\[
\leq \sum_{k=-\infty}^{\infty} \sum_{\alpha \in A_{k}} \sum_{m=1}^{N(k, \alpha)} \inf_{z \in Q_{\alpha}^{m}} \left[ m_{k}(f)(z) \right]^{\gamma} \chi Q_{\alpha}^{m} \right) (x) \right\}^{\frac{1}{p}}^{2}
\]
\[
\leq \sum_{k=-\infty}^{\infty} \sum_{\alpha \in A_{k}} \sum_{m=1}^{N(k, \alpha)} \inf_{z \in Q_{\alpha}^{m}} \left[ m_{k}(f)(z) \right]^{\gamma} \chi Q_{\alpha}^{m} \right) (x) \right\}^{\frac{1}{p}}^{2}
\]
\[
\leq \sum_{k=-\infty}^{\infty} \left\{ M \left( \sum_{\alpha \in A_{k}} \sum_{m=1}^{N(k, \alpha)} \inf_{z \in Q_{\alpha}^{m}} \left[ m_{k}(f)(z) \right]^{\gamma} \chi Q_{\alpha}^{m} \right) (x) \right\}^{\frac{1}{p}} \leq \sum_{k=-\infty}^{\infty} \left\{ M \left( \left[ m_{k}(f) \right]^{\gamma} \right) (x) \right\}^{\frac{1}{p}}.
\]
Therefore, from Lemma 5.2 we deduce that
\[
\| S_{E}(f) \|_{L^{p}(X)} \leq \left\{ \sum_{k=-\infty}^{\infty} \left[ M \left( \left[ m_{k}(f) \right]^{\gamma} \right) \right]^{\frac{1}{p}} \right\}^{\frac{1}{\gamma}} \leq \left\{ \sum_{k=-\infty}^{\infty} \left[ m_{k}(f) \right]^{\frac{1}{p}} \right\}^{\frac{1}{p}} \sim \| S_{Q}(f) \|_{L^{p}(X)}.
\]
This finishes the proof of Theorem 5.1.
5.2 Atomic characterizations of \( H^p(X) \)

The main aim of this section is to obtain the atomic characterizations of \( H^p(X) \) when \( p \in (\omega/(\omega + \eta), 1] \).

For any \( p \in (\omega/(\omega + \eta), 1] \), \( q \in (p, \infty] \cap [1, \infty] \) and \( \beta, \gamma \in (\omega(1/p - 1), \eta) \), we define the homogeneous atomic Hardy space \( \dot{H}^{p,q}_{\text{at}}(X) \) in the same way of \( H^{p,q}_{\text{at}}(X) \), but with the distribution space \((\mathcal{G}_0^{\beta,\gamma}(\beta, \gamma))'\) replaced by \((\mathcal{G}_0^{\beta,\gamma}(\beta, \gamma))'\). Then the following relationship between \( H^{p,q}_{\text{at}}(X) \) and \( \dot{H}^{p,q}_{\text{at}}(X) \) can be found in [20] Theorem 5.4.

**Proposition 5.3.** Suppose \( p \in (\omega/(\omega + \eta), 1] \), \( \beta, \gamma \in (\omega(1/p - 1), \eta) \) and \( q \in (p, \infty] \cap [1, \infty] \). Then \( \dot{H}^{p,q}_{\text{at}}(X) = H^{p,q}_{\text{at}}(X) \) with equivalent (quasi)-norms. More precisely, if \( f \in H^{p,q}_{\text{at}}(X) \), then the restriction of \( f \) on \( \mathcal{G}_0^{\beta,\gamma}(\beta, \gamma) \) belongs to \( \dot{H}^{p,q}_{\text{at}}(X) \); Conversely, if \( f \in \dot{H}^{p,q}_{\text{at}}(X) \), then there exists a unique \( \tilde{f} \in \dot{H}^{p,q}_{\text{at}}(X) \) such that \( \tilde{f} = f \) in \((\mathcal{G}_0^{\beta,\gamma}(\beta, \gamma))'\).

Due to the fact that the kernels \( \tilde{Q}_b \) in the homogeneous continuous Calderón formula in Theorem 2.6 has no compact support, we can only use Theorem 2.6 to decompose an element of \( H^p(X) \) into a linear combination of the following molecules.

**Definition 5.4.** Suppose that \( p \in (0, 1] \), \( q \in (p, \infty] \cap [1, \infty] \) and \( \varepsilon := \{\varepsilon_\ell\}_{\ell=1}^{\infty} \subset [0, \infty) \) satisfying

\[
\sum_{\ell=1}^{\infty} m(\varepsilon_\ell) p < \infty.
\]

A function \( M \in L^q(X) \) is called a \((p, q, \varepsilon)-molecule\) centered at a ball \( B := B(x_0, r_0) \) for some \( x_0 \in X \) and \( r \in (0, \infty) \) if \( m \) has the following properties:

\[\begin{align*}
(\text{i}) & \quad \|M_X B\|_{L^q(X)} \leq [\mu(B)]^\frac{1}{q - \frac{1}{p}}; \\
(\text{ii}) & \quad \text{for any } m \in \mathbb{N}, \quad \|M_{X B(x_0, \delta^{-m} r_0)} \chi_{B(x_0, \delta^{-m+1} r_0)}\|_{L^q(X)} \leq \varepsilon_m [\mu(B(x_0, \delta^{-m} r_0))]^\frac{1}{q - \frac{1}{p}}; \\
(\text{iii}) & \quad \int_X M(x) \, d\mu(x) = 0.
\end{align*}\]

By (i) and (ii) of Definition 5.4, the Hölder inequality, (5.4) and the fact \( p \in (0, 1] \), we find that, if \( M \) satisfies (i) and (ii) of Definition 5.4, then \( M \in L^1(X) \) and hence Definition 5.4(iii) makes sense.

After carefully checking the proof of [39] Theorem 3.4, we obtain the following molecular characterization of the atomic Hardy space \( H^{p,q}_{\text{at}}(X) \) of Coifman and Weiss [7], the details being omitted.

**Proposition 5.5.** Suppose that \( p \in (0, 1] \), \( q \in (p, \infty] \cap [1, \infty] \) and \( \varepsilon := \{\varepsilon_\ell\}_{\ell=1}^{\infty} \) satisfying (5.4). Then \( f \in H^{p,q}_{\text{at}}(X) \) if and only if there exist \((p, q, \varepsilon)-molecules\) \( \{M_j\}_{j=1}^{\infty} \) and \( \{\ell_j\}_{j=1}^{\infty} \subset \mathbb{C} \), with \( \sum_{j=1}^{\infty} |\ell_j|^p < \infty \), such that

\[
f = \sum_{j=1}^{\infty} \lambda_j M_j.
\]
converges in $(L_1/p−1(X))'$ when $p ∈ (0, 1)$ or in $L^1(X)$ when $p = 1$. Moreover, there exists a positive constant $C$, independent of $f$, such that, for any $f ∈ H_{cw}^{p,q}(X)$,

$$C^{-1}∥f∥_{H_{cw}^{p,q}(X)} ≤ \inf \left\{ \sum_{j=1}^{∞} |A_j|^p \right\}^{\frac{1}{p}} ≤ C∥f∥_{H_{cw}^{p,q}(X)},$$

where the infimum is taken over all the molecular decompositions of $f$ as in $(5.5)$.

Let $p ∈ (ω/(ω + η), 1]$ and $q ∈ (p, ∞] ∩ [1, ∞]$. By Proposition 5.3, $H_{cw}^{p,q}(X) = H_{cw}^{p,q}(X)$ and the already known fact that $H_{cw}^{p,q}(X)$ is independent of the choice of $q ∈ (p, ∞] ∩ [1, ∞]$, we know that $H_{at}^{p,q}(X) = H_{at}^{p,q}(X)$. With this observation, we show $H_{at}^{p,q}(X) ∈ H^p(X)$ as follows.

**Proposition 5.6.** Let $p ∈ (ω/(ω + η), 1]$, $β, γ ∈ (ω(1/p − 1), η)$, $q ∈ (p, ∞] ∩ [1, ∞]$ and $\{Q_k\}_{k∈Z}$ be an exp-ATI. Let $θ ∈ (0, ∞)$ and $S_θ$ be as in $(5.1)$. Then there exists a positive constant $C$, independent of $θ$, such that, for any distribution $f ∈ (G_0^{p,q}(β, γ))'$ belonging to $H_{at}^{p,q}(X)$,

$$(5.6) \quad \|S_θ(f)\|_{L^p(X)} ≤ C \max \{θ^{−ω/2}, θ^{ω/2}\} \|f\|_{H_{at}^{p,q}(X)}.$$

In particular, $H_{at}^{p,q}(X) = H_{at}^{p,q}(X) ∈ H^p(X)$.

**Proof.** Let $β, γ ∈ (ω(1/p − 1), η)$. It suffices to show $(5.6)$ for the case $θ ∈ [1, ∞)$, because both $(5.6)$ with $θ = 1$ and $S_θ(f) ≤ θ^{−ω/2}S(f)$ for any $f ∈ (G_0^{p,q}(β, γ))'$ whenever $θ ∈ (0, 1)$ imply that $(5.6)$ also holds true for any $θ ∈ (0, 1)$.

We start with the proof of the fact that the Littlewood-Paley $g$-function as in $(5.2)$ is bounded on $L^2(X)$. Indeed, for any $h ∈ L^2(X)$, we write

$$\|g(h)\|_{L^2(X)}^2 = \sum_{k=-∞}^{∞} \int_X |Q_k h(z)|^2 \, dμ(z) = \sum_{k=-∞}^{∞} \left\langle Q_k^* Q_k h, h \right\rangle.$$

By Theorem 2.6 and the proof of [27 (3.2)], we find that, for any fixed $β' ∈ (0, β ∧ γ)$, any $k_1, k_2 ∈ Z$ and $x, y ∈ X$, we have

$$(5.7) \quad |Q_{k_1} Q_{k_2}^*(x, y)| ≤ \delta^{k_1−k_2} β' \frac{1}{V_{β_1, β_2}(x) + V(x, y)} \left[ \frac{\delta^{k_1 ∧ k_2}}{V_{β_1, β_2}(x) + V(x, y)} \right]^γ.$$

Notice that, in $(5.7)$, only the regularity of $Q_k$ with respect to the second variable is used. Thus, by Lemma 2.2 v) and the boundedness of $M$ on $L^2(X)$, we conclude that, for any $k_1, k_2 ∈ Z$,

$$\left\|\left(Q_{k_1} Q_{k_1}^* \right) \left(Q_{k_2} Q_{k_2}^* \right)\right\|_{L^2(X)→L^2(X)} ≤ \left\|Q_{k_1} Q_{k_2}^*\right\|_{L^2(X)→L^2(X)} ≤ \delta^{k_1−k_2} β'.$$

Therefore, by the fact that $Q_{k_1} Q_{k_2}$ is self-adjoint and the Cotlar-Stein lemma (see [47 pp. 279–280] and [29 Lemma 4.5]), we obtain the boundedness of $\sum_{k=-∞}^{∞} Q_k^* Q_k$ on $L^2(X)$ and hence the boundedness of $g$ on $L^2(X)$.

Suppose that $a$ is a $(p, 2)$-atom supported on a ball $B := B(x_0, r_0)$ with $x_0 ∈ X$ and $r_0 ∈ (0, ∞)$. By the Fubini theorem and the boundedness of $g$ on $L^2(X)$, we find that

$$\|S_θ(a)\|_{L^2(X)} ≤ \left\| \left\{ \sum_{k_0} |Q_k a|^2 \right\}^{1/2} \right\|_{L^2(X)} \sim \|g(a)\|_{L^2(X)} ≤ \|a\|_{L^2(X)} ≤ \|μ(B)\|^{1−\frac{1}{p}},$$
which further implies that

\[
\int_{B(x_0, 4A_0^2\theta r_0)} |S_\theta(a)(x)|^p \, d\mu(x) \leq \|S_\theta(a)\|_{L^p(\mathbb{X})}^p \left[ \mu \left( B \left( x_0, 4A_0^2\theta r_0 \right) \right) \right]^{1 - \frac{p}{2}} \lesssim \theta^{\omega(1 - \frac{p}{2})}.
\]

Let \( x \notin B(x_0, 4A_0^2\theta r_0) \) and \( y \in B(x, \theta d_1) \). Since now \( \theta \in [1, \infty) \), for any \( u \in B = B(x_0, r_0) \), we have \( d(u, x) < (4A_0^2\theta)^{-1}d(x_0, x) < (2A_0)^{-1}[\delta^k + d(x_0, y)] \) and hence

\[
|Q_k(a)(y)| = \left| \int_X Q_k(y, u)a(u) \, d\mu(u) \right| \leq \int_B |Q_k(y, u) - Q_k(y, x_0)| \, d\mu(u)
\]

\[
\lesssim \int_B \left[ \frac{d(x_0, u)}{\delta^k + d(x_0, y)} \right] \frac{1}{V_{\delta^k}(x_0) + V(x_0, y)} \left[ \frac{\delta^k}{\delta^k + d(x_0, y)} \right]^y |a(u)| \, d\mu(u)
\]

\[
\lesssim [\mu(B)]^{-1 - \frac{\theta}{2}} \left[ \frac{r_0}{d(x_0, x)} \right] \frac{1}{V(x_0, x)} \left[ \frac{\delta^k}{d(x_0, x)} \right]^y.
\]

On the one hand, if \( \delta^k < (4A_0^2\theta)^{-1}d(x_0, x) \), then \( d(x_0, y) \geq (4A_0)^{-1}d(x_0, x) \) and hence

\[
|Q_k(a)(y)| \leq [\mu(B)]^{-1 - \frac{\theta}{2}} \left[ \frac{r_0}{d(x_0, x)} \right] \frac{1}{V(x_0, x)} \left[ \frac{\delta^k}{d(x_0, x)} \right]^y,
\]

which further implies that

\[
\sum_{\delta^k < (4A_0^2\theta)^{-1}d(x_0, x)} \int_{d(x, y) < \theta \delta^k} |Q_k(a)(y)|^2 \, d\mu(y) \leq [\mu(B)]^{-1 - \frac{\theta}{2}} \left[ \frac{r_0}{d(x_0, x)} \right] \frac{1}{V(x_0, x)} \left[ \frac{\delta^k}{d(x_0, x)} \right]^2
\]

\[
\lesssim [\mu(B)]^{-1 - \frac{\theta}{2}} \left[ \frac{r_0}{d(x_0, x)} \right] \frac{1}{V(x_0, x)} \left[ \frac{\delta^k}{d(x_0, x)} \right]^2.
\]

On the other hand, if \( \delta^k \geq (4A_0^2\theta)^{-1}d(x_0, x) \), then \( V(x_0, x) \leq \mu(B(x_0, \theta d_1)) \leq \theta^{\omega}V_{\delta^k}(x_0) \) and

\[
|Q_k(a)(y)| \leq \theta^{\omega[\mu(B)]^{-1 - \frac{\theta}{2}} \left( \frac{r_0}{\delta^k} \right) \frac{1}{V(x_0, x)},
\]

which further implies that

\[
\sum_{\delta^k \geq (4A_0^2\theta)^{-1}d(x_0, x)} \int_{d(x, y) < \theta \delta^k} |Q_k(a)(y)|^2 \, d\mu(y) \leq \theta^{2\omega[\mu(B)]^{-1 - \frac{\theta}{2}} \left[ \frac{1}{V(x_0, x)} \right]^2} \sum_{\delta^k \geq (4A_0^2\theta)^{-1}d(x_0, x)} \left( \frac{r_0}{\delta^k} \right)^{2\eta}
\]

\[
\sim \theta^{2\omega + 2\eta[\mu(B)]^{-1 - \frac{\theta}{2}} \left[ \frac{1}{V(x_0, x)} \right]^2}.
\]

Therefore, when \( x \notin B(x_0, 4A_0^2\theta r_0) \), we have

\[
S_\theta(a)(x) \leq \theta^{\omega + \eta[\mu(B)]^{-1 - \frac{\theta}{2}} \left[ \frac{r_0}{d(x_0, x)} \right] \frac{1}{V(x_0, x)}.
\]
Consequently, using \( p \in (\eta/(\omega + \eta), 1) \), \( B = B(x_0, r_0) \) and \((1.1)\), we obtain

\[
\int_{[B(x_0, 2A_0 \eta r_0)]^c} [S\theta(a)(x)]^p \, d\mu(x) \\
\leq \theta^{(\omega+\eta)p} [\mu(B)]^{p-1} \int_{[B(x_0, 4A_0 \eta r_0)]^c} \left[ \frac{r_0}{d(x_0, x)} \right]^{p\eta} \left[ \frac{1}{V(x_0, x)} \right]^{p} \, d\mu(x) \\
\leq \theta^{p\omega} \mu(B)]^{p-1} \sum_{j=2}^{\infty} 2^{-j\eta} \int_{(2A_0 \eta r_0 \leq d(x_0, x) < (2A_0 \eta r_0)]^c} \left[ \frac{1}{\mu(B(x_0, (2A_0 \eta r_0)]^c) \right]^{p} \, d\mu(x) \\
\leq \theta^p \sum_{j=2}^{\infty} 2^{-j\eta} \leq \theta^p.
\]

Combining \((5.8)\) and \((5.9)\) implies that, when \( \theta \in [1, \infty) \),

\[
(5.10) \quad ||S\theta(a)||_{L^p(X)} \leq \theta^{\omega/p}.
\]

Let \( f \in \dot{H}^{p,2}_{at}(X) \). By the definition of \( \dot{H}^{p,2}_{at}(X) \), we know that, for any \( \epsilon \in (0, \infty) \), there exist \((p, 2)\)-atoms \( \{a_j\}_{j=1}^{\infty} \) and \( \{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C} \) such that \( f = \sum_{j=1}^{\infty} \lambda_j a_j \) in \( (\dot{G}^p_0(\beta, \gamma))' \) and \( \sum_{j=1}^{\infty} |\lambda_j|^p \leq \|f\|_{\dot{H}^{p,2}_{at}(X)}^p + \epsilon \). By \((5.10)\) and the fact \( S\theta(f) \leq \sum_{j=1}^{\infty} |\lambda_j| S\theta(a_j) \), we conclude that

\[
\|S\theta(f)\|_{L^p(X)} \leq \sum_{j=1}^{\infty} |\lambda_j|^p \|S\theta(a_j)\|_{L^p(X)} \leq \theta^p \sum_{j=1}^{\infty} |\lambda_j|^p \leq \theta^p \|\|f\|_{\dot{H}^{p,2}_{at}(X)}^p + \epsilon \rightarrow \theta^p \|f\|_{\dot{H}^{p,2}_{at}(X)}^p \quad (\epsilon \rightarrow 0^+) \]

as \( \epsilon \rightarrow 0^+ \). This finishes the proof of \((5.6)\) and hence of Proposition \((5.6)\). \( \square \)

Next, we use Proposition \((5.3)\) to show the following converse of Proposition \((5.6)\).

**Proposition 5.7.** Let \( p \in (\omega/(\omega + \eta), 1) \), \( \beta, \gamma \in (\omega(1/p - 1), \eta) \) and \( f \in (\dot{G}^p_0(\beta, \gamma))' \) belong to \( H^p(X) \). Then there exist a sequence \( \{a_j\}_{j=1}^{\infty} \) \((p, 2)\)-atoms and \( \{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C} \) such that \( f = \sum_{j=1}^{\infty} \lambda_j a_j \) in \( (G^p_0(\beta, \gamma))' \) and \( \sum_{j=1}^{\infty} |\lambda_j|^p \leq C \|f\|_{H^p(X)}^p \), where \( C \) is a positive constant independent of \( f \). Consequently, \( H^p(X) \subset \dot{H}^{p,2}_{at}(X) \).

**Proof.** Assume that \( f \in (\dot{G}^p_0(\beta, \gamma))' \) belongs to \( H^p(X) \). To avoid the confusion of notation, we use \( \{E_k\}_{k \in \mathbb{Z}} \) to denote an exp-ATI and then define \( S(f) \) as in \((5.1)\) but with \( Q_k \) therein replaced by \( E_k \). Denote by \( D \) the set of all dyadic cubes. For any \( k \in \mathbb{Z} \), we define \( \Omega_k := \{x \in X : S(f)(x) > 2^k\} \) and

\[
\mathcal{D}_k := \left\{ Q \in \mathcal{D} : \mu(Q \cap \Omega_k) > \frac{1}{2} \mu(Q) \right\}.
\]

It is easy to see that, for any \( Q \in \mathcal{D} \), there exists a unique \( k \in \mathbb{Z} \) such that \( Q \in \mathcal{D}_k \). A dyadic cube \( Q \in \mathcal{D}_k \) is called a maximal cube in \( \mathcal{D}_k \) if \( Q' \in \mathcal{D} \) and \( Q' \supset Q \), then \( Q' \notin \mathcal{D}_k \). Denote the set of all maximal cubes in \( \mathcal{D}_k \) at level \( j \in \mathbb{Z} \) by \( \{Q_{r_{jk}}\}_{r_{jk} \in I_{jk}} \), where \( I_{jk} \subset \mathcal{A}_j \) may be empty. The center of \( Q_{r_{jk}} \) is denoted by \( z_{r_{jk}} \). Then \( \mathcal{D} = \bigcup_{r_{jk} \in I_{jk}} \bigcup_{r_{jk} \in I_{jk}} \{Q \in \mathcal{D}_k : Q \subset Q_{r_{jk}}\} \).
From now on, we adopt the notation $E_Q := E_l$ and $\overline{E_Q} := \overline{E_l}$ whenever $Q = Q_{l}^{\alpha}$ for some $l \in \mathbb{Z}$ and $\alpha \in \mathcal{A}_{l+1}$. Then, by Theorem 2.6, we find that
\[
(5.11) \quad f() = \sum_{l=-\infty}^{\infty} \overline{E_l} E_l f() = \sum_{l=-\infty}^{\infty} \sum_{\alpha \in \mathcal{A}_{l+1}} \int_{Q_{l}^{\alpha}} \overline{E_l}(\cdot, y) E_l f(y) \, d\mu(y)
\]
\[
= \sum_{Q \in D} \int \overline{E_Q}(\cdot, y) E_Q f(y) \, d\mu(y)
\]
\[
= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{\tau \in \mathcal{A}_k} \sum_{Q \in D_k, Q \subset Q_{l}^{j}} \int \overline{E_Q}(\cdot, y) E_Q f(y) \, d\mu(y)
\]
\[
= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{\tau \in \mathcal{A}_k} \lambda_{l,k} \beta_{l,k}(\cdot),
\]
where all the equalities converge in $(\mathcal{G}_0^\beta(\beta, \gamma))'$,
\[
\lambda_{l,k} := [\mu(Q_{l,k})]^{\frac{1}{2}} \left[ \sum_{Q \in D_k, Q \subset Q_{l}^{j}} \int |E_Q f(y)|^2 \, d\mu(y) \right]^{\frac{1}{2}}
\]
and
\[
(5.12) \quad \beta_{l,k}(\cdot) := \frac{1}{\lambda_{l,k}} \sum_{Q \in D_k, Q \subset Q_{l}^{j}} \int \overline{E_Q}(\cdot, y) E_Q f(y) \, d\mu(y).
\]

For any $Q \in D_k$ and $Q \subset Q_{l,k}^{j}$, assume that $Q = Q_{l}^{\alpha}$ for some $l \in \mathbb{Z}$ and $\alpha \in \mathcal{A}_{l+1}$. Since $\delta$ is assumed to satisfy $\delta < (2A_0)^{-10}$, it then follows that $2A_0 C^2 \delta < 1$ so that $Q = Q_{l}^{\alpha} \subset B(y, \delta')$ for any $y \in Q$. By this and the fact that $\mu(Q \cap \Omega_{k+1}) \leq \frac{1}{2} \mu(Q)$, we obtain
\[
\mu(B(y, \delta') \cap (Q_{l,k}^{j} \setminus \Omega_{k+1})) \geq \mu(B(y, \delta') \cap (Q \setminus \Omega_{k+1})) = \mu(Q \setminus \Omega_{k+1}) \geq \frac{1}{2} \mu(Q) - V_{\delta'}(y).
\]
Thus, we have
\[
\sum_{Q \in D_k, Q \subset Q_{l,k}^{j}} \int_{Q} |E_Q f(y)|^2 \, d\mu(y)
\]
\[
\leq \sum_{l=j-1}^{\infty} \sum_{\alpha \in \mathcal{A}_{l+1}, D_k \supset Q_{l}^{\alpha} \subset Q_{l,k}^{j}} \int_{Q_{l}^{\alpha}} \frac{\mu(B(y, \delta') \cap (Q_{l,k}^{j} \setminus \Omega_{k+1}))}{V_{\delta'}(y)} |E_l f(y)|^2 \, d\mu(y)
\]
\[
\leq \sum_{l=j-1}^{\infty} \int_{Q_{l,k}^{j}} \frac{\mu(B(y, \delta') \cap (Q_{l,k}^{j} \setminus \Omega_{k+1}))}{V_{\delta'}(y)} |E_l f(y)|^2 \, d\mu(y)
\]
\[
\sim \int \sum_{l=j-1}^{\infty} \int_{B(y, \delta') \cap (Q_{l,k}^{j} \setminus \Omega_{k+1})} |E_l f(y)|^2 \frac{d\mu(x)}{V_{\delta'}(y)} \, d\mu(y)
\]
\[ \int_{Q_{r,k}} |S(f)(x)|^2 \, d\mu(x) \leq 2^{2k} \mu(Q_{r,k}). \]

From this and the fact \( \mu(Q_{r,k}) < 2\mu(Q_{r,k} \cap \Omega_k) \), it follows that

\[
(5.13) \quad \sum_{k=\infty}^{\infty} \sum_{j=\infty}^{\infty} \sum_{\tau \in I_{jk}} (A_{r,k}^j) \leq \sum_{k=\infty}^{\infty} 2^{kp} \sum_{j=\infty}^{\infty} \mu(Q_{r,k} \cap \Omega_k) \leq 2^{kp} \mu(\Omega_k) \sim \|S(f)\|_{L^p(X)},
\]

Choose \( \gamma' \in (\omega(1/p-1), \gamma) \) and let \( \bar{\varepsilon} := \{q^m|\gamma'-\omega(1/p-1)|\}_{m \in \mathbb{H}} \). Assume for the moment that every \( b^j_{r,k} \) as in (5.12) is a \((p,2,\bar{\varepsilon})\)-molecule centered at a ball \( B_{r,k}^j := B(z_{r,k}, 4A_0^j \delta^{-1}) \), whose proof is given in Lemma 5.8 below. Further, applying Proposition 5.5, we conclude that \( \|b^j_{r,k}\|_{H^p_{\alpha}^2(X)} \leq 1 \).

Thus, \( b_{r,k}^j \) can be written as a linear combination of \((p,2)\)-atoms which converges in \((L_1/p-1)(X)')\) when \( p \in (\omega/(\omega + \eta), 1) \) or in \( L^1(X) \) when \( p = 1 \), and hence converges in \((\tilde{g}^p_0(\beta, \gamma)')\) because \( \tilde{g}^p_0(\beta, \gamma) \subset L_{1/p-1}(X) \) (see Lemma 4.15). Invoking this, (5.11) and (5.13), we find that \( f \in \tilde{H}^p_{\alpha}^2(X) \) and \( \|f\|_{\tilde{H}^p_{\alpha}^2(X)} \leq \|S(f)\|_{L^p(X)} \). This finishes the proof of Proposition 5.7.

**Lemma 5.8.** Let all the notation be as in the proof of Proposition 5.7. Then every \( b^j_{r,k} \) as in (5.12) is a harmlessly positive constant multiple of a \((p,2,\bar{\varepsilon})\)-molecule centered at the ball \( B_{r,k}^j := B(\tau_{r,k}, 4A_0^j \delta^{-1}) \), where \( \bar{\varepsilon} := \{q^m|\gamma'-\omega(1/p-1)|\}_{m \in \mathbb{H}} \) and \( \gamma' \in (\omega(1/p-1), \gamma) \).

**Proof.** Let \( b_{r,k}^j \) be as in (5.12). For any \( h \in L^2(X) \) with \( \|h\|_{L^2(X)} \leq 1 \), by the Fubini theorem and the Hölder inequality, we conclude that

\[
\left| \int_X b_{r,k}^j(x)h(x) \, d\mu(x) \right| \leq \frac{1}{\lambda_{r,k}^j} \sum_{Q \in D_{r,k}, Q \subset Q_{r,k}^j} \left| E_Q f(y) \right| \left( \int_X \left| E_Q h(x) \right| \, d\mu(x) \right) \, d\mu(y)
\]

\[
\leq \frac{1}{\lambda_{r,k}^j} \left[ \sum_{Q \in D_{r,k}, Q \subset Q_{r,k}^j} \left| E_Q f(y) \right|^2 \, d\mu(y) \right]^\frac{1}{2} \left[ \sum_{Q \in D_{r,k}, Q \subset Q_{r,k}^j} \int_X \left| E_Q h(x) \right|^2 \, d\mu(x) \right]^\frac{1}{2}
\]

\[
\leq \left\{ \mu(Q_{r,k}^j) \right\}^{1/2} \left( \|\tilde{g}(h)\|^2_{L^2(X)} \right)^{1/2},
\]

where \( \tilde{g}(h) := \sum_{k=\infty}^{\infty} \left| E_k^* h \right|^2 \). Noticing that the kernel of \( E_k^* \) has the regularity with respect to the second variable, we follow the argument used in the beginning of the proof of Proposition 5.6 to deduce that \( \tilde{g} \) is bounded on \( L^2(X) \). Thus, we have

\[
\left| \int_X b_{r,k}^j(x)h(x) \, d\mu(x) \right| \leq \left\{ \mu(Q_{r,k}^j) \right\}^{1/2} \left( \|\tilde{g}(h)\|_{L^2(X)} \right) \leq \left\{ \mu(B_{r,k}) \right\}^{1/2} \left( \|\tilde{g}(h)\|^2_{L^2(X)} \right)^{1/2}.
\]
Taking supremum over all \( h \in L^2(X) \) with \( \|h\|_{L^2(X)} \leq 1 \), we further find that
\[
\|b^j_{r,k}\|_{L^2(X)} \leq \left[ \mu(B^j_{r,k}) \right]^{\frac{1}{p} - \frac{1}{p'}}.
\]

Let \( \gamma' \in (\omega(1/p - 1), \gamma) \). Fix \( m \in \mathbb{N} \) and let \( R_m := (\delta^{-m} B^j_{r,k}) \setminus (\delta^{-m+1} B^j_{r,k}) \). Then, for any \( x \in R_m \), by the Hölder inequality and the size condition of \( \{E_l\}_{l \in \mathbb{Z}} \), we conclude that
\[
|b^j_{r,k}(x)| \leq \frac{1}{\lambda^j_{r,k}} \sum_{Q \in D_{r,k}} \sum_{Q' \subset Q_{r,k}^j} \int_Q \left| \overline{E}_Q(x,y)E^j f(y) \right| \, d\mu(y)
\]
\[
\leq \frac{1}{\lambda^j_{r,k}} \sum_{l=j-1}^{\infty} \left( \sum_{\alpha \in A_{r,k}} \sum_{d^l_{a} \subset Q^j_{r,k}} \int_{Q_{a}^{j+1}} \frac{1}{V_{d^l_{a}}(x) + V(x,y)} \left[ \frac{\delta^l_{d^l_{a}}}{\delta^l_{d^l_{a}} + d(x,y)} \right]^{\gamma'} |E_l f(y)| \, d\mu(y) \right)^2
\]
\[
\times \left( \sum_{l=j-1}^{\infty} \sum_{\alpha \in A_{r,k}} \sum_{d^l_{a} \subset Q^j_{r,k}} \int_{Q_{a}^{j+1}} \frac{1}{V_{d^l_{a}}(x) + V(x,y)} \left[ \frac{\delta^l_{d^l_{a}}}{\delta^l_{d^l_{a}} + d(x,y)} \right]^{2(y'-\gamma')} |E_{l+1} f|^2 \, d\mu(y) \right)^{\frac{1}{2}}
\]
\[
= \frac{1}{\lambda^j_{r,k}} Y(x)Z(x).
\]
Notice that, for any \( x \in R_m \), we have \( 4A_0^2 \delta^{j-m-1} \leq d(x,z^j_{r,k}) < 4A_0^2 \delta^{j-m-2} \) and, for any \( y \in Q^j_{r,k} \), we have \( \delta^l_{d^l_{a}} + d(x,y) \sim d(x,y) \sim \delta^{-m+j} \) and hence
\[
Y(x) \leq \left[ \sum_{l=j-1}^{\infty} \sum_{\alpha \in A_{r,k}} \sum_{d^l_{a} \subset Q^j_{r,k}} \int_{Q_{a}^{j+1}} \frac{1}{\mu(B(y,\delta^{-m+j}))} \left[ \frac{\delta^l_{d^l_{a}}}{\delta^{-m+j}} \right]^{2\gamma'} \, d\mu(y) \right]^{\frac{1}{2}}
\]
\[
\leq \left[ \sum_{l=j-1}^{\infty} \left( \frac{\delta^l_{d^l_{a}}}{\delta^{-m+j}} \right)^{2\gamma'} \int_{Q^j_{r,k}} \frac{1}{\mu(B(z^j_{r,k},\delta^{-m+j}))} \, d\mu(y) \right]^{\frac{1}{2}} \leq \delta^m \left[ \frac{\mu(B^j_{r,k})}{\mu(\delta^{-m} B^j_{r,k})} \right]^{\frac{1}{2}}.
\]
Thus, for any \( x \in R_m \), we have
\[
|b^j_{r,k}(x)| \leq \frac{1}{\lambda^j_{r,k}} \delta^{m \gamma'} \left[ \frac{\mu(B^j_{r,k})}{\mu(\delta^{-m} B^j_{r,k})} \right]^{\frac{1}{2}} Z(x),
\]
which, together with the Fubini theorem and Lemma 2.2 ii), implies that
\[
\|b^j_{r,k} \chi_{R_m}\|_{L^2(X)} \leq \frac{1}{\lambda^j_{r,k}} \delta^{m \gamma'} \left[ \frac{\mu(B^j_{r,k})}{\mu(\delta^{-m} B^j_{r,k})} \right]^{\frac{1}{2}} \left( \int_{R_m} [Z(x)]^2 \, d\mu(x) \right)^{\frac{1}{2}}.
\]
Let \( p \) be independent of \( f \).

Conversely, if \( \lambda \parallel S \) provided that either one in (5.14) to obtain \( \| S \|_p \) as in (5.4) and \( b_{k}^{j} \) is a harmlessly positive constant multiple of a \((p, 2, \varepsilon)\)-molecule. This finishes the proof of Lemma 5.8.

Combining Propositions 5.6 and 5.7 we immediately obtain the following main result of this section, the details being omitted.

**Theorem 5.9.** Suppose that \( p \in \left( \frac{\omega}{(\omega + \eta)}, 1 \right) \) and \( \beta, \gamma \in (\omega(1/p - 1), \eta) \) and \( q \in (p, \infty) \cap [1, \infty] \). As subspaces of \((\mathcal{G}^{q}_{0}(\beta, \gamma))^{*} \), it holds true that \( \hat{H}^{p,q}_{\text{at}}(X) = H^{p}(X) \) with equivalent (quasi-)norms.

### 5.3 Hardy spaces via various Littlewood-Paley functions

In this section, we characterize Hardy spaces \( H^{p}(X) \) via the Lusin area functions with apertures, the Littlewood-Paley \( g \)-functions and the Littlewood-Paley \( g_{\text{at}} \)-functions, respectively.

**Theorem 5.10.** Let \( p \in (\omega/(\omega + \eta), 1] \) and \( \beta, \gamma \in (\omega(1/p - 1), \eta) \). Assume that \( \theta \in (0, \infty) \) and \( \lambda \in (\omega[1 + 2/p], \infty) \). Then, for any \( f \in (\mathcal{G}^{q}_{0}(\beta, \gamma))^{*} \), it holds true that

\[
\| f \|_{H^{p}(X)} \sim \| S_{\theta}(f) \|_{L^{p}(X)} \sim \| g_{\text{at}}^{*}(f) \|_{L^{p}(X)} \sim \| g(f) \|_{L^{p}(X)},
\]

provided that either one in (5.14) is finite. Here, the positive equivalent constants in (5.14) are independent of \( f \).

**Proof.** Let \( f \in (\mathcal{G}^{q}_{0}(\beta, \gamma))^{*} \) with \( \beta, \gamma \in (\omega(1/p - 1), \eta) \). With \( \{Q_{k}\}_{k \in \mathbb{Z}} \) being an exp-ATI, we define \( S_{\theta}(f) \), \( g_{\text{at}}^{*}(f) \) and \( g(f) \), respectively, as in (5.1), (5.2) and (5.3), where \( \theta \in (0, \infty) \) and \( \lambda \in (\omega[1 + 2/p], \infty) \).

By Proposition 5.6 and Theorem 5.9 if \( f \in H^{p}(X) \), then \( \| S_{\theta}(f) \|_{L^{p}(X)} \leq \| f \|_{H^{p}_{\text{at}}(X)} \sim \| f \|_{L^{p}(X)} \). Conversely, if \( \| S_{\theta}(f) \|_{L^{p}(X)} < \infty \), then we proceed as the proof of Proposition 5.7 to deduce that \( f = \sum_{j=1}^{\infty} L_{j} a^{j} \in (\mathcal{G}^{q}_{0}(\beta, \gamma))^{*} \), where \( \{a_{j}\}_{j=1}^{\infty} \) are \((p, 2)\)-atoms and \( \{L_{j}\}_{j=1}^{\infty} \subset \mathbb{C} \) satisfying \( \sum_{j=1}^{\infty} |L_{j}|^{p} \leq \| S_{\theta}(f) \|_{L^{p}(X)} \).

Combining this with Theorem 5.9 implies that

\[
\| f \|_{H^{p}(X)} = \| S_{\theta}(f) \|_{L^{p}(X)} \sim \| f \|_{H^{p}_{\text{at}}(X)} \leq \| S_{\theta}(f) \|_{L^{p}(X)}.
\]

Therefore, we have \( \| f \|_{H^{p}(X)} \sim \| S_{\theta}(f) \|_{L^{p}(X)} \) whenever \( \| f \|_{H^{p}(X)} \) or \( \| S_{\theta}(f) \|_{L^{p}(X)} \) is finite.

Noticing that \( S(f) \leq g_{\text{at}}^{*}(f) \leq \sum_{j=1}^{\infty} 2^{j(\omega - 1)/2} S_{2}(f) \), we then apply (5.6) and \( \lambda \in (\omega[1 + 2/p], \infty) \) to obtain

\[
\| S(f) \|_{L^{p}(X)} \leq \| g_{\text{at}}^{*}(f) \|_{L^{p}(X)} \leq \sum_{j=1}^{\infty} 2^{j(\omega - 1)/2} \| S_{2}(f) \|_{L^{p}(X)}
\]
\[ \delta \in \alpha \] and hence

\[
\| \| H^p(X) \sim \| g_A(\delta) \| L^p(\mathbb{R}) \quad \text{whenever } \| f \| H^p(X) \text{ or } \| g_A(\delta) \| L^p(\mathbb{R}) \text{ is finite.}
\]

If \( f \in H^p(X) = \hat{H}^m_{\alpha}(X) \), then, by following the proof of (5.6), we also obtain

\[
\| g(\delta) \| L^p(\mathbb{R}) \leq \| f \| \tilde{H}^m_{\alpha}(X) \sim \| f \| H^p(X).
\]

To finish the proof of (5.14), it remains to prove \( \| f \| H^p(X) \leq \| g(\delta) \| L^p(\mathbb{R}) \). Indeed, for any \( x \in X \), we have

\[
S(f)(x) = \left[ \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{A} \setminus \mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \int_{d(x,y) < \delta^k} |Q_kf(y)|^2 \chi_{Q^m_d}(x) \frac{d\mu(y)}{V^\alpha(x)} \right]^{\frac{1}{2}} \leq \left\{ \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \left[ \sup_{z \in B_{Q^m_d}(\alpha,\delta^{-1})} |Q_kf(z)|^2 \chi_{Q^m_d}(x) \right] \right\}^{\frac{1}{2}}.
\]

where \( Q^m_{\alpha} \) is as in Section 2 and \( z_{m,\alpha}^m \) the center of \( Q^m_{\alpha} \). With all the notation as in Theorem 2.7 we know that, for any \( z \in B_{Q^m_{\alpha}}(\alpha,\delta^{-1}), \)

\[
Q_kf(z) = \sum_{k' \in \mathbb{Z}} \sum_{\alpha' \in \mathcal{A}_{K}} \sum_{m'=1}^{N(k',\alpha')} \mu \left( Q^m_{\alpha'} \right) Q_{k'} \left( \chi_{Q^m_{\alpha'}}(x) \right) Q_{k'}f(y_{\alpha'}),
\]

where \( y_{\alpha'}^m \) is an arbitrary point in \( Q^m_{\alpha'} \). Fix \( \beta' \in (0, \beta, \gamma) \). Then, similarly to the proof of (3.7) (see also (27) (2.21)), we conclude that, for any \( z \in B_{Q^m_{\alpha}}(\alpha,\delta^{-1}), \)

\[
|Q_kf(z)| \leq \frac{1}{V^\alpha(z)} \frac{1}{V(z,y_{\alpha'})} \left[ \delta^{k+k'} \right]^{\gamma}.
\]

The variable \( z \) in (5.16) can be replaced by any \( x \in Q^m_{\alpha'} \), because \( |d(z,x) - d(z,x_{m,\alpha})| \leq \delta \leq \delta^{k+k'} \). Further, from Lemma 3.7 we deduce that, for any fixed \( r \in (\omega/\omega + \eta, 1] \), any \( k' \in \mathbb{Z} \) and \( \zeta \in B_{Q^m_{\alpha}}(\alpha,\delta^{-1}), \)

\[
\left| \sum_{\alpha' \in \mathcal{A}_{K}} \sum_{m'=1}^{N(k',\alpha')} \mu \left( Q^m_{\alpha'} \right) Q_{k'} \left( \chi_{Q^m_{\alpha'}}(x) \right) Q_{k'}f(y_{\alpha'}^m) \right| \leq \delta^{(k+k') \omega^{-1} - 1} \left[ M \left( \sum_{\alpha' \in \mathcal{A}_{K}} \left| Q_{k'}f(y_{\alpha'}^m) \right| \chi_{Q^m_{\alpha'}} \right) \right] \frac{1}{2}
\]

and hence

\[
|Q_kf(z)| \leq \delta^{(k+k') \omega^{-1} - 1} \left[ M \left( \sum_{\alpha' \in \mathcal{A}_{K}} \sum_{m'=1}^{N(k',\alpha')} \left| Q_{k'}f(y_{\alpha'}^m) \right| \chi_{Q^m_{\alpha'}} \right) \right] \frac{1}{2}.
\]
Combining (5.15) and (5.17), choosing \( r \) and \( \beta' \) such that \( r \in (\omega/(\omega + \beta'), p) \) and applying the Hölder inequality, we further conclude that, for any \( x \in X \),

\[
[S(f)(x)]^2 \leq \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{K}_x} \sum_{m=1}^{N(k, \alpha)} \left( \sum_{k' \in \mathbb{Z}} \delta^{k-k'} \delta^{(k \land k') \omega \frac{1}{\omega'-1}} \right) \times \left[ \mathcal{M} \left( \sum_{\alpha' \in \mathcal{A}_x} \sum_{m'=1}^{N(k', \alpha')} \left| Q_{k'} f \left( y_{\alpha'}^{k', m'} \right) \right| \chi_{Q_{k'}^{a, \alpha}} \right)(x) \right]^{\frac{2}{1}} \chi_{Q_{k}^{a, \alpha}}(x)
\]

\[
\leq \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{K}_x} \sum_{m=1}^{N(k, \alpha)} \left( \sum_{k' \in \mathbb{Z}} \delta^{k-k'} \delta^{(k \land k') \omega \frac{1}{\omega'-1}} \right) \times \left[ \mathcal{M} \left( \sum_{\alpha' \in \mathcal{A}_x} \sum_{m'=1}^{N(k', \alpha')} \left| Q_{k'} f \left( y_{\alpha'}^{k', m'} \right) \right| \chi_{Q_{k'}^{a, \alpha}} \right)(x) \right]^{\frac{2}{1}} \chi_{Q_{k}^{a, \alpha}}(x)
\]

\[
\leq \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} \delta^{k-k'} \left[ \left| Q_{k'} f \left( y_{\alpha}^{k, m} \right) \right| \chi_{Q_{k'}^{a, \alpha}} \right](x) \times \left[ \mathcal{M} \left( \sum_{\alpha' \in \mathcal{A}_x} \sum_{m'=1}^{N(k', \alpha')} \left| Q_{k'} f \left( y_{\alpha'}^{k', m'} \right) \right| \chi_{Q_{k'}^{a, \alpha}} \right)(x) \right]^{\frac{1}{1}} \chi_{Q_{k}^{a, \alpha}}(x)
\]

From this and Lemma 5.2, we deduce that

\[
\|f\|_{H^p(X)} = \|S(f)\|^\frac{1}{p}_{L^{p/(2)}(X)} \leq \left\| \mathcal{M} \left( \sum_{\alpha' \in \mathcal{A}_x} \sum_{m'=1}^{N(k', \alpha')} \left| Q_{k'} f \left( y_{\alpha'}^{k', m'} \right) \right| \chi_{Q_{k'}^{a, \alpha}} \right) \right\|_{L^{p/(2)}(X)}^{\frac{1}{2}} \leq \left\| \sum_{k' \in \mathbb{Z}} \sum_{\alpha' \in \mathcal{A}_x} \sum_{m'=1}^{N(k', \alpha')} \left| Q_{k'} f \left( y_{\alpha'}^{k', m'} \right) \right| \chi_{Q_{k'}^{a, \alpha}} \right\|_{L^{p/(2)}(X)}^{\frac{1}{2}}
\]

\[
\approx \left\| \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{K}_x} \sum_{m=1}^{N(k, \alpha)} \left| Q_{k} f \left( y_{\alpha}^{k, m} \right) \right| \chi_{Q_{k}^{a, \alpha}} \right\|_{L^{p/(2)}(X)}^{\frac{1}{2}}.
\]

By this and the arbitrariness of \( y_{\alpha}^{k, m} \), we finally conclude that

\[
\|f\|_{H^p(X)} \approx \left\| \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{K}_x} \sum_{m=1}^{N(k, \alpha)} \inf_{z \in Q_{k}^{a, \alpha}} \left| Q_{k} f(z) \right| \chi_{Q_{k}^{a, \alpha}} \right\|_{L^{p/(2)}(X)}^{\frac{1}{2}} \leq \|g(f)\|_{L^p(X)}.
\]

This finishes the proof of \( \|f\|_{H^p(X)} \leq \|g(f)\|_{L^p(X)} \) and hence of Theorem 5.10. \( \Box \)
Remark 5.11. If $X$ is a homogeneous group, Folland and Stein [12] showed that, for any given $p \in (0, 2]$ and any $f \in \mathscr{S}'(X)$, $\|g^\lambda_\alpha(f)\|_{L^p(X)} \leq \|S(f)\|_{L^p(X)}$ whenever $\lambda \in (2\omega/p, \infty)$, where $\mathscr{S}'(X)$ denotes the space of tempered distributions on $X$ (see [12] Corollary 7.4) by observing that $\lambda$ in (5.3) be equal to $2\lambda$ with $\lambda$ as in the Littlewood-Paley $g^\lambda_\alpha$-function in [12]. Comparing with this, the range of $\lambda$ in Theorem 5.10 is narrower, this is because it was proved in [12] Theorem 7.1 that, for any given $p \in (0, 2]$, any $\theta \in [1, \infty)$ and $f \in \mathscr{S}'(X)$, 

\begin{equation}
\|S(f)\|_{L^p(X)} \lesssim \|\theta^{\omega(1/p-1/2)}S(f)\|_{L^p(X)}
\end{equation}

while, in the proof of Theorem 5.10 we only show that (5.18) for an arbitrary space of homogeneous type $X$ holds true, with $\omega(1/p - 1/2)$ replaced by $\omega/p$, when $p \in (\omega/(\omega + \eta), 1]$ and $f \in (\mathcal{G}^\eta_0(\beta, \gamma)')$ with $\beta, \gamma \in (\omega(1/p - 1), \eta)$. However, it is still unclear whether or not (5.18) for an arbitrary space of homogeneous type $X$ (and hence Theorem 5.10 with $\lambda \in (2\omega/p, \omega(1 + 2/p))$) holds true.

6 Wavelet characterizations of Hardy spaces

In this section, we characterize the Hardy space via the wavelet orthogonal system $\{\psi^k_\alpha : k \in \mathbb{Z}, \alpha \in \mathcal{G}_k\}$ introduced in [1] Theorem 7.1. The sequence $\{D_k\}_{k \in \mathbb{Z}}$ of operators on $L^2(X)$ associated with integral kernels

\begin{equation}
D_k(x, y) := \sum_{\alpha \in \mathcal{G}_k} \psi_\alpha(x)\psi_\alpha(y), \quad \forall \ x, \ y \in X
\end{equation}

turns out to be an exp-ATI; see [25, 29]. Thus, all the conclusions in Section 5 hold true for $\{D_k\}_{k \in \mathbb{Z}}$.

For any $f \in (\mathcal{G}^\eta_0(\beta, \gamma)')$ with $\beta, \gamma \in (0, \eta)$, define the \textit{wavelet Littlewood-Paley function} $S(f)$ by setting, for any $x \in X$,

\begin{equation}
S(f)(x) := \left\{\sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{G}_k} \mu(Q_{\alpha}^{k+1})^{-1}\left|\left(\psi^k_\alpha, f\right)\right|^2 \chi_{Q_{\alpha}^{k+1}}(x)\right\}^{1/2}.
\end{equation}

For any $p \in (0, \infty)$, define the corresponding \textit{wavelet Hardy space} $H^p_\alpha(X)$ by

\begin{equation}
H^p_\alpha(X) := \left\{f \in (\mathcal{G}^\eta_0(\beta, \gamma))' : \|f\|_{H^p_\alpha(X)} := \|S(f)\|_{L^p(X)} < \infty\right\}.
\end{equation}

For any $p \in (\omega/(\omega + \eta), \infty)$, the $L^p(X)$-norm equivalence between the wavelet Littlewood-Paley function $S(f)$ and the Littlewood-Paley $g$-function $g(f)$ was proved in [25] Theorem 4.3 whenever $f$ is a distribution. The proof of [25] Theorem 4.3 seems problematic because the authors therein used an unknown fact that, when $f \in (\mathcal{G}^\eta_0(\beta, \gamma))'$ and $n \in \mathbb{N}$,

\begin{equation}
\sum_{|k| \leq n} \sum_{a \in \mathcal{G}_k} \left\langle f, \psi^k_\alpha \right\rangle \psi^k_\alpha \in L^2(X).
\end{equation}

Although (6.2) may not be true for distributions, it is obviously true when $f \in L^2(X)$. Indeed, the argument used in the proof of [25] Theorem 4.3 proves the following result.
Theorem 6.1. Suppose \( p \in (\omega/(\omega + \eta), \infty) \) and \( \beta, \gamma \in (0, \eta) \). Then there exists a positive constant \( C \) such that, for any \( f \in (G^0_0(\beta, \gamma))' \),

\[
\|G(f)\|_{L^p(X)} \leq C \|S(f)\|_{L^p(X)}
\]

and, if \( f \in L^2(X) \), then

\[
C^{-1} \|S(f)\|_{L^p(X)} \leq \|G(f)\|_{L^p(X)} \leq C \|S(f)\|_{L^p(X)}.
\]

Here and hereafter, \( G(f) \) is defined as in (5.2), but with \( Q_k \) therein replaced by \( D_k \) in (6.1).

To show that (6.4) holds true for all distributions, we need the following basic property of \( H^p_\mu(X) \).

Proposition 6.2. Let \( p \in (\omega/(\omega + \eta), 1] \) and \( \beta, \gamma \in (\omega(1/p - 1), \eta) \). Then \( H^p_\mu(X) \) is a (quasi-)Banach space that can be continuously embedded into \( (G^0_0(\beta, \gamma))' \).

Proof. Assume that \( f \in (G^0_0(\beta, \gamma))' \) belongs to \( H^p_\mu(X) \). By (6.3), Theorems 5.10 and 5.9 we have \( \|f\|_{H^p_\mu(X)} \leq \|f\|_{H^p_\mu(X)} \). Consequently, for any \( \varepsilon \in (0, \infty) \), there exist \( (\alpha_j)_{j=1}^\infty \) and \( \{|\alpha_j|\}_{j=1}^\infty \subset \mathbb{C} \) satisfying \( (\sum_{j=1}^\infty |\alpha_j|^p)^{1/p} \leq \|f\|_{H^p_\mu(X)} + \varepsilon \) such that \( f = \sum_{j=1}^\infty \alpha_j \ell_j \) in \( (G^0_0(\beta, \gamma))' \).

Combining this with Lemmas 4.14 and 4.15, we find that, for any \( \varphi \in G^0_0(\beta, \gamma) \),

\[
\langle f, \varphi \rangle \leq \sum_{j=1}^\infty |\alpha_j| \langle \ell_j, \varphi \rangle \leq \sum_{j=1}^\infty |\alpha_j| \|\varphi\|_{L^1/p^p-1(X)} \leq \|\varphi\|_{G^0_0(\beta, \gamma)} \left( \sum_{j=1}^\infty |\alpha_j|^p \right)^{1/p}
\]

Letting \( \varepsilon \to 0^+ \), we obtain \( \|f\|_{G^0_0(\beta, \gamma)'} \leq \|f\|_{H^p_\mu(X)} \). Thus, \( H^p_\mu(X) \) can be continuously embedded into \( (G^0_0(\beta, \gamma))' \).

To prove that \( H^p_\mu(X) \) is a (quasi-)Banach space, we only prove its completeness. Let \( \{f_n\}_{n=1}^\infty \) be a Cauchy sequence in \( H^p_\mu(X) \). Then \( \{f_n\}_{n=1}^\infty \) is also a Cauchy sequence in \( (G^0_0(\beta, \gamma))' \), so it converges to some element \( f \) in \( (G^0_0(\beta, \gamma))' \). For any \( n \in \mathbb{N} \) and \( x \in X \), applying the Fatou lemma twice, we conclude that

\[
S(f - f_n)(x) = S \left( \lim_{m \to \infty} [f_m - f_n] \right)(x) = \left[ \sum_{k \in \mathbb{Z}} \sum_{a \in G_k} \left| \psi_a^k, \lim_{m \to \infty} [f_m - f_n] \right| \chi_{G_a^{k+1}}(x) \right]^2
\]

and hence

\[
\|f - f_n\|_{H^p_\mu(X)} = \int_X |S(f - f_n)(x)|^p \, d\mu(x)
\]
7.1 Finite atomic characterizations of Hardy spaces

sequence establishing finite atomic characterizations of

\[ \lim_{m \to \infty} \int_X |S(f_m - f_n)(x)|^p \, d\mu(x) \]
\[ \leq \lim_{m \to \infty} \int_X |S(f_m - f_n)(x)|^p \, d\mu(x) = \lim_{m \to \infty} \|f_m - f_n\|^p_{H^p_{\text{loc}}(X)}. \]

Letting \( n \to \infty \), we find that \( f \in H^p_{\text{loc}}(X) \) and \( \lim_{n \to \infty} \|f - f_n\|_{H^p_{\text{loc}}(X)} = 0 \). Therefore, \( H^p_{\text{loc}}(X) \) is complete. This finishes the proof of Proposition 6.2.

Applying Theorem 6.1 and Proposition 6.2, we show the following wavelet characterizations of Hardy spaces.

\textbf{Theorem 6.3.} Suppose \( p \in (\omega/\omega + \eta), 1 \) and \( \beta, \gamma \in (\omega/1/p - 1, \eta) \). As subspaces of \((\tilde{G}^p_0(\beta, \gamma))^\prime\), \( H^p(X) = H^p_{\text{loc}}(X) \) with equivalent (quasi-)norms.

\textbf{Proof.} Due to (6.3), Theorems 5.10 and 5.9, we obtain \( H^p_{\text{loc}}(X) \subset H^p(X) \) and \( \| \cdot \|_{H^p(X)} \leq \| \cdot \|_{H^p_{\text{loc}}(X)} \).

It remains to show \( H^p(X) \subset H^p_{\text{loc}}(X) \). To this end, by Theorem 5.9, we conclude that \( L^2(X) \cap H^p(X) \) is dense in \( H^p(X) \). Thus, for any \( f \in H^p(X) \), there exist \( \{f_n\}_{n=1}^\infty \subset L^2(X) \cap H^p(X) \) such that \( \lim_{n \to \infty} \|f - f_n\|_{H^p(X)} = 0 \). Obviously, \( \{f_n\}_{n=1}^\infty \) is a Cauchy sequence of \( H^p(X) \). Noticing that \( \{f_n\}_{n=1}^\infty \subset L^2(X) \), we use (6.4) and Theorem 5.10 to conclude that

\[ \|f_m - f_n\|_{H^p_{\text{loc}}(X)} = \|S(f_m - f_n)\|_{L^p(X)} \sim \|\mathcal{G}(f_m - f_n)\|_{L^p(X)} \sim \|f_m - f_n\|_{H^p(X)} \to 0 \]

as \( m, n \to \infty \), so that \( \{f_n\}_{n=1}^\infty \) is also a Cauchy sequence of \( H^p_{\text{loc}}(X) \). By Proposition 6.2, there exists \( \tilde{f} \in H^p_{\text{loc}}(X) \) such that \( f_n \to \tilde{f} \) as \( n \to \infty \) in \( H^p_{\text{loc}}(X) \), also in \((\tilde{G}^p_0(\beta, \gamma))^\prime\). Meanwhile, \( f_n \to f \) as \( n \to \infty \) in \( H^p(X) \), also in \((\tilde{G}^p_0(\beta, \gamma))^\prime\). Therefore, \( \tilde{f} = f \in (\tilde{G}^p_0(\beta, \gamma))^\prime \) and \( f \in H^p_{\text{loc}}(X) \). Moreover,

\[ \|f\|_{H^p_{\text{loc}}(X)} \leq \|f - f_n\|_{H^p_{\text{loc}}(X)} + \|f_n\|_{H^p_{\text{loc}}(X)} \sim \|f - f_n\|_{H^p(X)} + \|f_n\|_{H^p(X)} \leq \|f\|_{H^p(X)} \]

when \( n \) is sufficiently large. Thus, we obtain \( H^p(X) \subset H^p_{\text{loc}}(X) \) and \( \| \cdot \|_{H^p_{\text{loc}}(X)} \leq \| \cdot \|_{H^p(X)} \). This finishes the proof of Theorem 6.3. \qed

7 Criteria of the boundedness of sublinear operators

Let \( p \in (\omega/\omega + \eta), 1 \). By the argument used in Sections 3 through 6, we conclude that the Hardy spaces \( H^{+\theta}(X), H^p_{\text{al}}(X) \) with \( \theta \in (0, \infty) \), \( H^{+\beta}(X), H^p_{\text{al}}(X), H^p_{\text{al}}(X), \hat{H}^p_{\text{al}}(X) \) with \( q \in (p, \infty) \cap [1, \infty] \) and \( H^p_{\text{loc}}(X) \) are essentially the same space in the sense of equivalent (quasi-)norms. From now on, we simply use \( H^p(X) \) to denote either one of them if there is no confusion. In this section, we give criteria of the boundedness of sublinear operators on Hardy spaces via first establishing finite atomic characterizations of \( H^p(X) \).

7.1 Finite atomic characterizations of Hardy spaces

For any \( p \in (\omega/\omega + \eta), 1 \) and \( q \in (p, \infty) \cap [1, \infty] \), we say \( f \in H^p_{\text{al}}(X) \) if there exist \( N \in \mathbb{N} \), a sequence \( \{a_j\}_{j=1}^N \) of \((p, q)\)-atoms and \( \{\lambda_j\}_{j=1}^N \subset \mathbb{C} \) such that

\[ f = \sum_{j=1}^N \lambda_j a_j. \]
Also, define
\[ \|f\|_{H^{p,q}_{\text{fin}}(X)} := \inf_{\sum_{j=1}^{N} |\lambda_j|^p} \left( \sum_{j=1}^{N} |\lambda_j|^p \right)^{\frac{1}{p}}, \]
where the infimum is taken over all the decompositions of \( f \) above. It is easy to see that \( H^{p,q}_{\text{fin}}(X) \) is a dense subset of \( H^{p,q}_{\text{at}}(X) \) and \( \|\cdot\|_{H^{p,q}_{\text{fin}}(X)} \leq \|\cdot\|_{H^{p,q}_{\text{at}}(X)} \). Denote by the symbol \( UC(X) \) the space of all uniformly continuous functions on \( X \), that is, a function \( f \in UC(X) \) if and only if, for any fixed \( \varepsilon \in (0, \infty) \), there exists \( \sigma \in (0, \infty) \) such that \( |f(x) - f(y)| < \varepsilon \) whenever \( d(x, y) < \sigma \). The next theorem characterizes \( H^{p,q}_{\text{at}}(X) \) via \( H^{p,q}_{\text{fin}}(X) \).

**Theorem 7.1.** Suppose \( p \in (\omega/\omega + \eta), 1 \). Then the following statements hold true:

(i) if \( q \in (p, \infty) \cap [1, \infty) \), then \( \|\cdot\|_{H^{p,q}_{\text{fin}}(X)} \) and \( \|\cdot\|_{H^{p,q}_{\text{at}}(X)} \) are equivalent (quasi)-norms on \( H^{p,q}_{\text{fin}}(X) \);

(ii) \( \|\cdot\|_{H^{p,q}_{\text{fin}}(X)} \) and \( \|\cdot\|_{H^{p,q}_{\text{at}}(X)} \) are equivalent (quasi)-norms on \( H^{p,q}_{\text{fin}}(X) \cap UC(X) \);

(iii) \( H^{p,q}_{\text{fin}}(X) \cap UC(X) \) is a dense subspace of \( H^{p,q}_{\text{at}}(X) \).

**Proof.** First, we prove (i). It suffices to show that \( \|f\|_{H^{p,q}_{\text{fin}}(X)} \lesssim \|f\|_{H^{p,q}_{\text{at}}(X)} \|f\|_{H^{p,q}_{\text{fin}}(X)} \) for any \( f \in H^{p,q}_{\text{fin}}(X) \) with \( q \in (p, \infty) \cap [1, \infty) \). We may as well assume that \( \|f\|_{H^{p,q}_{\text{at}}(X)} = 1 \). Let all the notation be as in the proof that \( \|f\|_{H^{p,q}_{\text{at}}(X)} \leq 1 \). Then
\[
f = \sum_{j \in \mathbb{Z}} \sum_{k \in I_j} \lambda_k a_k^j = \sum_{j \in \mathbb{Z}} \sum_{k \in I_j} h_k^j = \sum_{j \in \mathbb{Z}} h_j
\]
both in \( (G^d_0(\beta, \gamma))' \) and almost everywhere. Here and hereafter, for any \( j \in \mathbb{Z} \) and \( k \in I_j \), the quantities \( h_j, h_k^j, \lambda_k, a_k^j \) as in \( (4.12) \) and \( (4.13) \). Since \( f \in H^{p,q}_{\text{fin}}(X) \), it follows that there exist \( x_1 \in X \) and \( R \in (0, \infty) \) such that \( \text{supp } f \subset B(x_1, R) \). We claim that there exists a positive constant \( \tilde{c} \) such that, for any \( x \not\in B(x_1, 16A^d_0 R) \),

\[ f^*(x) \leq \tilde{c} [\mu(B(x_1, R))]^{-\frac{1}{p}}. \]

We admit \( (7.1) \) temporarily and use it to prove (i) and (ii). Let \( j' \) be the maximal integer such that \( 2^{j'} \leq \tilde{c} [\mu(B(x_1, R))]^{-\frac{1}{p}} \) and define
\[ h := \sum_{j \leq j'} \sum_{k \in I_j} \lambda_k a_k^j \quad \text{and} \quad \ell := \sum_{j > j'} \sum_{k \in I_j} \lambda_k a_k^j \]
In what follows, for the sake of convenience, we elide the fact whether \( I_j \) or not is finite and simply write the summation \( \sum_{k \in I_j} \) in \( (7.2) \) as \( \sum_{k=1}^{\infty} \). If \( j > j' \), then \( \Omega_j = \{ x \in X : f^*(x) > 2^j \} \subset B(x_1, 16A^d_0 R) \), which implies that \( \text{supp } \ell \subset B(x_1, 16A^d_0 R) \) because \( \text{supp } a_k^j \subset \Omega_j \). From \( f = h + \ell \), it then follows that \( \text{supp } h \subset B(x_1, 16A^d_0 R) \) because \( f \) is a harmless constant multiple of a \( (p, \infty) \)-atom.

\[ \|h\|_{L^\infty(X)} \leq \sum_{j \leq j'} \|h_j\|_{L^\infty(X)} \leq \sum_{j \leq j'} 2^j \sim [\mu(B(x_1, R))]^{-\frac{1}{p}} \]

and \( \int_X h(x) d\mu(x) = 0 \), we conclude that \( h \) is a harmless constant multiple of a \( (p, \infty) \)-atom.

}\]
Next we deal with \( \ell \). For any \( N := (N_1, N_2) \in \mathbb{N}^2 \), define
\[
\ell_N := \sum_{j=j'+1}^{N_1} \sum_{k=1}^{N_2} A^j_k a^j_k = \sum_{j=j'+1}^{N_1} \sum_{k=1}^{N_2} h^j_k.
\]
Then \( \ell_N \) is a finite linear combination of \((p, \infty)\)-atoms and \( \sum_{j=j'+1}^{N_1} \sum_{k=1}^{N_2} |A^j_k|^p \leq 1 \). Notice that \( \text{supp}(\ell - \ell_N) \subset B(x_1, 16A_0^j R) \) and \( \int_X |\ell(x) - \ell_N(x)| \, d\mu(x) = 0 \). It suffices to show that \( ||\ell - \ell_N||_{L^p(X)} \rightarrow 0 \) can be sufficiently small when \( N_1 \) and \( N_2 \) are big enough. Noticing that \( \ell = \sum_{j=N_1+1}^\infty h^j + \sum_{j=j'+1}^{N_1} \sum_{k=1}^{N_2} h^j_k \), we have
\[
||\ell - \ell_N||_{L^p(X)} \leq \left\| \sum_{j=N_1+1}^\infty h^j \right\|_{L^p(X)} + \sum_{j=j'+1}^{N_1} \left\| \sum_{k=1}^{N_2} h^j_k \right\|_{L^p(X)}.
\]
For any \( j \in \mathbb{Z} \) and \( k \in \mathbb{N} \), we recall that \( \text{supp} h^j_k \subset B^j_k \subset \Omega^j \) and \( ||h^j||_{L^p(X)} \leq 2^j \). By \( f = \sum_{j=-\infty}^\infty h^j \) and \( \text{supp}(\sum_{j=N_1+1}^\infty h^j) \subset \Omega^{N_1} \), we conclude that, for any \( z \in \Omega^{N_1} \),
\[
\left| \sum_{j=N_1+1}^\infty h^j(z) \right| \leq \left| f(z) - \sum_{j \leq N_1} h^j(z) \right| \leq |f(z)| + \sum_{j \leq N_1} |h^j(z)| \leq |f(z)| + 2^{N_1}.
\]
Notice that, by [20] Proposition 3.9, there exists a constant \( \widetilde{C} > 1 \) such that \( f^* \leq \widetilde{C} \mathcal{M}(f) \). With \( f_1 := f \chi_{\{|x|: |f(x)| > 2^{N_1-1}/\tilde{C} \}} \) and \( f_2 := f - f_1 \), we have
\[
2^{N_1q} \mu(\Omega^{N_1}) \leq 2^{N_1q} \mu \left( \left\{ x \in X : \widetilde{C} \mathcal{M}(f)(x) > 2^{N_1-1} \right\} \right) \leq 2^{N_1q} \mu \left( \left\{ x \in X : \widetilde{C} \mathcal{M}(f_1)(x) > 2^{N_1-1} \right\} \right) \leq ||f_1||_{L^q(X)}^q \rightarrow 0
\]
as \( N_1 \rightarrow \infty \), because \( \mathcal{M} \) is bounded from \( L^q(X) \) to \( L^{q,\infty}(X) \) and \( f \in H_{\text{fin}}^{p,\infty}(X) \subset L^q(X) \). Therefore,
\[
\left\| \sum_{j=N_1+1}^\infty h^j \right\|_{L^q(X)}^q \leq \int_{\Omega^{N_1}} |f(z)|^q + 2^{N_1q} \, d\mu(z) \leq \left\| f \chi_{\Omega^{N_1}} \right\|_{L^q(X)}^q + 2^{N_1q} \mu(\Omega^{N_1}) \rightarrow 0
\]
as \( N_1 \rightarrow \infty \). Then, for any \( \epsilon \in (0, \infty) \), we choose \( N_1 \in \mathbb{N} \) such that \( \sum_{j=N_1+1}^\infty |h^j||_{L^p(X)} < \epsilon/2 \).

If we fix \( N_1 \in \mathbb{N} \) and \( N_1 \geq j > j' \), then the fact \( \sum_{k=1}^\infty |h^j_k| \leq 2^j X_{\Omega^j} \in L^q(X) \) implies that
\[
\lim_{N_1 \rightarrow \infty} \sum_{k=1}^\infty |h^j_k|_{L^p(X)} = 0.
\]
So, we further choose \( N_2 \in \mathbb{N} \) such that \( \sum_{j=j'+1}^{N_1} \sum_{k=1}^{N_2} h^j_k ||_{L^p(X)} < \epsilon/2 \). In this way, we have \( ||\ell - \ell_N||_{L^p(X)} < \epsilon \) for large \( N \). Then there exist a positive constant \( C_0 \), independent of \( N \) and \( \epsilon \), and a \((p,q)\)-atom \( a_{N_1} \) such that \( \ell - \ell_N = C_0 \epsilon a_{N_1} \). Therefore, we obtain \( ||f||_{H^{p,q}_{\text{fin}}(X)} \leq 1 \sim ||f||_{H^{p,q}_{\text{fin}}(X)} \) and complete the proof of (i) under the assumption (7.1).
To obtain (ii), we only need to prove that \( \|f\|_{H^{p,\infty}_\text{lin}} \lesssim \|f\|_{H^{p,\infty}_a} \) whenever \( f \in H^{p,\infty}_\text{lin}(X) \cap \text{UC}(X) \). We may also assume that \( \|f\|_{H^{p,\infty}_a(X)} = 1 \). Notice that \( f \in L^{\infty}(X) \) and \( \|f^*\|_{L^{\infty}(X)} \lesssim \|\mathcal{M}(f)\|_{L^{\infty}(X)} \leq c_0 \|f\|_{L^{\infty}(X)} \), where \( c_0 \) is a positive constant independent of \( f \). Let \( j' > j \) be the largest integer such that \( 2^j \leq c_0 \|f\|_{L^{\infty}(X)} \). We write \( f = h + \ell \) with \( h \) as in (7.2), but now \( \ell = \sum_{j' < j} \sum_{k=1}^{\infty} h_k^j \). As in the proof of (i), we know that \( h \) is a harmlessly positive constant multiple of some \((p, \infty)\)-atom.

Now we consider \( \ell \). Notice that \( f \in \text{UC}(X) \). Then, for any \( \epsilon \in (0, \infty) \), there exists \( \sigma \in (0, \infty) \) such that \( |f(x) - f(y)| \leq \epsilon \) whenever \( d(x, y) \leq \sigma \). Split \( \ell = \ell_1^\sigma + \ell_2^\sigma \) with

\[
\ell_1^\sigma := \sum_{(j,k) \in G_1} h_k^j = \sum_{(j,k) \in G_1} \lambda_k^j a_k^j \quad \text{and} \quad \ell_2^\sigma := \sum_{(j,k) \in G_2} h_k^j,
\]

where

\[
G_1 := \{(j, k) : 12A_0^3 r_k^j \geq \sigma, \ j' < j \leq j'' \} \quad \text{and} \quad G_2 := \{(j, k) : 12A_0^3 r_k^j < \sigma, \ j' < j \leq j'' \}.
\]

Notice that, for any \( j' < j \leq j'' \), \( \Omega^j \) is bounded. Thus, by Proposition 4.4(vi), we find that \( G_1 \) is a finite set, which further implies that \( \ell_1^\sigma \) is a finite linear combination of \((p, \infty)\)-atoms and

\[
\sum_{(j,k) \in G_1} \left| a_k^j \right|^p \leq 1.
\]

To consider \( \ell_2^\sigma \), it is obvious that \( \text{supp} \ell_2^\sigma \subset B(x_1^j, 16A_0^3 R) \) and \( \int_X \ell_2^\sigma(x) \, d\mu(x) = 0 \), so it remains to estimate \( \|\ell_2^\sigma\|_{L^{\infty}(X)} \). For any \( (j, k) \in G_2 \), applying the definition of \( h_k^j \) in (4.12) implies that

\[
| h_k^j | \leq | b_k^j | + \sum_{l \in I_{j+1}} | b_{l+1}^j \phi_k^j | + \sum_{l \in I_{j+1}} | L_{j+1} \phi_k^j |.
\]

By the definition of \( b_k^j \), we have \( \text{supp} b_k^j \subset B(x_k^j, 2A_0 r_k^j) \). Moreover, for any \( x \in B(x_k^j, 2A_0 r_k^j) \),

\[
| b_k^j(x) | \leq \left| f(x) - f\left( x_k^j \right) \right| + \frac{1}{\|\phi_k^j\|_{L^1(X)}^p} \int_{B(x_k^j, 2A_0 r_k^j)} f(\xi) | \phi_k^j(\xi) | \, d\mu(\xi)
\leq | f(x) - f\left( x_k^j \right) | + \frac{1}{\|\phi_k^j\|_{L^1(X)}^p} \int_{B(x_k^j, 2A_0 r_k^j)} f(\xi) - f\left( x_k^j \right) | \phi_k^j(\xi) | \, d\mu(\xi) \leq \epsilon.
\]

If \( b_{l+1}^j \phi_k^j \neq 0 \), then \( B(x_{l+1}^j, 2A_0 r_{l+1}^j) \cap B(x_k^j, 2A_0 r_k^j) \neq \emptyset \), which further implies that \( r_{l+1}^j \leq 6A_0 r_k^j \). Thus, for any \( x \in B(x_{l+1}^j, 2A_0 r_{l+1}^j) \), we have \( d(x, x_k^j) < 12A_0 r_k^j \) and hence an argument similar to the estimation of (7.3) gives

\[
| b_{l+1}^j(x) | = \left| f(x) - f\left( x_k^j \right) \right| + \frac{1}{\|\phi_{l+1}^j\|_{L^1(X)}^p} \int_{B(x_{l+1}^j, 2A_0 r_{l+1}^j)} f(\xi) | \phi_{l+1}^j(\xi) | \, d\mu(\xi) \phi_{l+1}^j(x) \leq \epsilon \phi_{l+1}^j(x),
\]

so that

\[
\sum_{l \in I_{j+1}} | b_{l+1}^j(x) \phi_k^j(x) | \leq \epsilon \phi_k^j(x) \sum_{l \in I_{j+1}} \phi_{l+1}^j(x) \sim \epsilon \phi_k^j(x) \leq \epsilon.
\]
Using the definition of $L_{k,l}^{j+1}$ and arguing similarly as (7.3), we conclude that, for any $x \in X$,

\[
\sum_{l \in I_{k,l}} \left| L_{k,l}^{j+1} \phi_l^j(x) \right| \leq \varepsilon,
\]

where $L_{k,l}^{j+1}$ is as in (4.10). Summarizing all gives $\|h_k^l\|_{L^\infty(X)} \leq \varepsilon$. Recalling that $\supp h_k^l \subset B_k^l$ and $\sum_{k=1}^{\infty} \chi_{B_k^l} \leq L_0$, we obtain $\|h_k^l\|_{L^\infty(X)} \leq \varepsilon$. Therefore, there exist a positive constant $C_y$, independent of $\sigma$ and $\epsilon$, and a $(p, \infty)$-atom $a(\tau)$ such that $\ell_2 = C_y a(\tau)$. This proves that $\|f\|_{H_{lin}^{\infty}}(x) \leq 1$ and hence finishes the proof (ii) under the assumption (7.1).

Now we fix $x \in B(x_1, 16A_0^3R)$. Suppose that $\varphi \in G_0^0(B, \gamma)$ with $\|\varphi\|_{G(y, r, \beta, \gamma)} \leq 1$ for some $r \in (0, \infty)$. First we consider the case $r \geq 4A_0^2d(x_1)/3$. For any $y \in B(x, d(x, x_1))$, we have $\|\varphi\|_{G(y, r, \beta, \gamma)} \leq 1$, which implies that $|\langle f, \varphi \rangle| \leq f^*(y)$ and hence

\[
|\langle f, \varphi \rangle| \leq \left\{ \frac{1}{\mu(B(x, d(x, x_1)))} \int_{B(x, d(x, x_1))} [f^*(y)]^p \, d\mu(y) \right\}^{1/p} \leq \left[ \mu(B(x_1, R)) \right]^{-1/p}. \tag{7.4}
\]

Next we consider the case $r < 4A_0^2d(x_1)/3$. Choose a function $\xi$ satisfying $\chi_{B(x_1, 2A_0^{-4}d(x_1))} \leq \xi \leq \chi_{B(x_1, 2A_0^{-2}d(x_1))}$ and $\|\xi\|_{C^1(X)} \leq (d(x_1), x)^{-\gamma}$. Since $supp f \subset B(x_1, R)$, it follows that $f\xi = f$. Let $\tilde{\varphi} := \varphi\xi$. For any $y \in B(x, d(x, x_1))$, assuming for the moment that

\[
\|\tilde{\varphi}\|_{G(\gamma, r, \beta, \gamma)} \leq 1, \tag{7.5}
\]

we obtain

\[
|\langle f, \varphi \rangle| = \left| \int_X f(z)\varphi(z) \, d\mu(z) \right| = \left| \int_X f(z)\xi(z)\varphi(z) \, d\mu(z) \right| = |\langle f, \tilde{\varphi} \rangle| \leq f^*(y),
\]

which implies that (7.4) remains true in this case. Therefore, by the arbitrariness of $\varphi$ and the fact that $f^* \sim f^*$, we obtain (7.1).

Now we fix $y \in B(x_1, d(x_1, x))$ and prove (7.5). First we consider the size condition. Indeed, if $\tilde{\varphi}(z) \neq 0$, then $d(z, x_1) < (2A_0)^{-3}d(x_1, x)$ and hence $d(z, y) < (16A_0^3/7)d(x, z)$, which implies that

\[
|\tilde{\varphi}(z)| \leq |\varphi(z)| \leq \frac{1}{V_r(x) + V(x, z)} \left[ \frac{r}{r + d(x, z)} \right]^{\gamma} \sim \frac{1}{V_r(y) + V(y, z)} \left[ \frac{r}{r + d(y, z)} \right]^{\gamma}.
\]

To consider the regularity condition of $\tilde{\varphi}$, we may assume that $d(z, \tilde{z}) \leq (2A_0)^{-10}[r + d(y, z)]$ due to the size condition. For the case $d(z, x_1) > (2A_0)^{-1}d(x_1, x)$, we have $\tilde{\varphi}(z) = 0$ and, by $y \in B(x_1, d(x_1, x))$ and $r < 4A_0^2d(x_1, x)/3$, we further obtain

\[
d(z, \tilde{z}) \leq (2A_0)^{-10}[r + d(y, z)] \leq (2A_0)^{-10}[r + A_0d(y, x_1) + A_0d(x_1, z)] \leq (2A_0)^{-10}[4A_0^2d(x_1, x) + A_0d(x_1, z)] \leq (2A_0)^{-2}d(x_1, z),
\]

which further implies that $d(\tilde{z}, x_1) \geq \frac{1}{A_0^2}d(x_1, x) - d(z, \tilde{z}) \geq (2A_0)^{-2}d(x_1, x)$ and hence $\tilde{\varphi}(\tilde{z}) = 0$. So we only need to consider the case $d(z, x_1) \leq (2A_0)^{-1}d(x_1, x)$. Then we have $(2A_0)^{-1}d(x_1, x) \leq d(z, x) \leq 2A_0d(x_1, x)$ and

\[
d(y, z) \leq A_0^2[d(y, x_1) + d(x_1, x) + d(x, z)] \leq 2A_0^2d(x_1, x) + A_0^2d(x, z) \leq (2A_0)^3d(x, z),
\]
which implies that \(d(z, z') \leq (2A_0)^{-1}[r+d(x, z)]\) and \(r+d(y, z) \leq \min\{r+d(x, z), r+d(x, z'), d(x_1, x)\}\). Therefore, by the regularity of \(\varphi\) and the definition of \(\xi\), we conclude that

\[
\left|\varphi(z) - \varphi(z')\right| \leq \xi(z)|\varphi(z) - \varphi(z')| + |\varphi(z')||\xi(z) - \xi(z')|
\leq \left[\frac{d(z, z')}{r+d(x, z)}\right]^\beta \frac{1}{V_r(x) + V(x, z)} \left[\frac{r}{r+d(x, z)}\right]^\gamma
+ \frac{1}{V_r(x) + V(x, z')} \left[\frac{r}{r+d(x, z')}\right]^\gamma \frac{d(z, z')}{d(x_1, x)}
\leq \left[\frac{d(z, z')}{r+d(y, z)}\right]^\beta \frac{1}{V_r(y) + V(y, z)} \left[\frac{r}{r+d(y, z)}\right]^\gamma .
\]

This proves (7.5) and hence finishes the proofs of (i) and (ii).

Now we prove (iii). According to [23, pp. 3347–3348] (see also [27, Theorem 2.6]), there exists a sequence \(\{S_k\}_{k \in \mathbb{N}}\) of bounded operators on \(L^2(X)\) with their kernels satisfying the following conditions:

(i) \(S_k(x, y) = 0\) if \(d(x, y) \geq C_\delta \delta^k\) and, for any \(x, y \in X\),

\[|S_k(x, y)| \leq \frac{1}{V_{\delta^k}(x) + V_{\delta^k}(y)} .\]

where \(C_\delta\) is a fixed positive constant greater than 1;

(ii) for any \(x, x', y \in X\) with \(d(x, x') \leq C_\delta \delta^k\),

\[|S_k(x, y) - S_k(x', y)| + |S_k(y, x) - S_k(y, x')| \leq \left[\frac{d(x, x')}{\delta^k}\right]^\theta \frac{1}{V_{\delta^k}(x) + V_{\delta^k}(y)} .\]

where \(\theta\) is as in [23] Theorem 2.4; (iii) for any \(x \in X\), \(\int_X S_k(x, y) d\mu(y) = 1 = \int_X S_k(y, x) d\mu(y)\).

For any \(g \in \bigcup_{p \in [1, \infty]} L^p(X)\) and \(x \in X\), define

\[S_k g(x) := \int_X S_k(x, y) g(y) d\mu(y) .\]

Then, for any \((p, \infty)\)-atom \(a\) supported on \(B(z, r)\) with \(z \in X\) and \(r \in (0, \infty)\), we observe that \(S_k a\) satisfies the following properties:

(a) \(|S_k a|_{L^p(X)} \leq |a|_{L^p(X)}\) and \(\lim_{k \to \infty} |S_k a - a|_{L^2(X)} = 0\);

(b) when \(k\) is sufficiently large, \(\text{supp} \ S_k a \subset B(z, 2A_0 r)\);

(c) \(\int_X S_k a(x) d\mu(x) = 0\);

(d) \(S_k a \in UC(X)\).
Consequently, $S_k a$ is a harmlessly constant multiple of a $(p, \infty)$-atom and hence of a $(p, 2)$-atom. Thus, $$\|S_k a - a\|_{H^p_{\text{at}}(X)} \sim \|S_k a - a\|_{H^p_{\text{fin}}(X)} \to 0$$ as $k \to \infty$. For any $f \in H^p_{\text{at}}(X)$, there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset H^p_{\text{fin}}(X)$ such that $\lim_{n \to \infty} \|f_n - f\|_{H^p_{\text{at}}(X)} = 0$. Then, for any $n \in \mathbb{N}$, by the above (a) through (d), we find that $S_k(f_n) \in H^p_{\text{fin}}(X) \cap \text{UC}(X)$ and $\lim_{k \to \infty} \|S_k f_n - f_n\|_{H^p_{\text{at}}(X)} = 0$. This proves that $\|S_k f_n - f\|_{H^p_{\text{at}}(X)} \to 0$ as $n, k \to \infty$, which completes the proof of (iii) and hence of Theorem 7.1. \hfill $\square$

### 7.2 Criteria of the boundedness of sublinear operators on Hardy spaces

In this section, applying the finite atomic characterizations of Hardy spaces, we obtain two criteria on the boundedness of sublinear operators on Hardy spaces.

Recall that a complete vector space $\mathcal{B}$ is called a quasi-Banach space if its quasi-norm $\| \cdot \|_{\mathcal{B}}$ satisfies the following condition:

(i) for any $f \in \mathcal{B}$, $\|f\|_{\mathcal{B}} = 0$ if and only if $f$ is the zero element in $\mathcal{B}$;

(ii) for any $\lambda \in \mathbb{C}$ and $f \in \mathcal{B}$, $\|\lambda f\|_{\mathcal{B}} = |\lambda| \|f\|_{\mathcal{B}}$;

(iii) there exists $C \in [1, \infty)$ such that, for any $f, g \in \mathcal{B}$, $\|f + g\|_{\mathcal{B}} \leq C(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}})$.

Next we recall the definition of $r$-quasi-Banach spaces (see, for example, [35, 51, 53, 52, 20]).

**Definition 7.2.** Suppose that $r \in (0, 1)$ and $\mathcal{B}_r$ is a quasi-Banach space with its quasi-norm $\| \cdot \|_{\mathcal{B}_r}$. The space $\mathcal{B}_r$ is called an $r$-quasi-Banach space if there exists $\kappa \in [1, \infty)$ such that, for any $m \in \mathbb{N}$ and $(f_j)_{j=1}^m \subset \mathcal{B}_r$,

$$\left\| \sum_{j=1}^m f_j \right\|_{\mathcal{B}_r} \leq \kappa \left( \sum_{j=1}^m \|f_j\|_{\mathcal{B}_r} \right)^r.$$

Obviously, when $p \in (0, 1]$, $L^p(X)$ and $H^{\ast,p}(X)$ are $p$-quasi-Banach-spaces. Let $\mathcal{Y}$ be a linear space and $\mathcal{B}_r$ is an $r$-quasi-Banach space with $r \in (0, 1]$. An operator $T : \mathcal{Y} \to \mathcal{B}_r$ is said to be $\mathcal{B}_r$-sublinear if there exists a positive constant $\kappa \in [1, \infty)$ such that

(i) for any $f, g \in \mathcal{Y}$, $\|T(f) - T(g)\|_{\mathcal{B}_r} \leq \kappa \|T(f - g)\|_{\mathcal{B}_r}$;

(ii) for any $m \in \mathbb{N}$, $(f_j)_{j=1}^m \subset \mathcal{Y}$ and $(\lambda_j)_{j=1}^m \subset \mathbb{C}$,

$$\left\| T \left( \sum_{j=1}^m \lambda_j f_j \right) \right\|_{\mathcal{B}_r} \leq \kappa \sum_{j=1}^m |\lambda_j| \|T(f_j)\|_{\mathcal{B}_r}.$$

(see, for example, [35, Definition 2.5], [51, Definition 1.6.7], [53, Remark 1.1(3)], [52, Definition 1.6] and [20, Definition 5.8]).

The next theorem gives us a criteria for $\mathcal{B}_r$-sublinear operators that can be extended to bounded $\mathcal{B}_r$-sublinear operators from Hardy spaces to $\mathcal{B}_r$. It can be proved by following the proof of [20, Theorem 5.9] with slight modifications, the details being omitted.
Theorem 7.3. Let $p \in (\omega/(\omega + \eta), 1]$ and $r \in [p, 1]$. Suppose that $\mathcal{B}_r$ is an $r$-quasi-Banach space and either of the following holds true:

(i) $q \in (p, \infty) \cap [1, \infty)$ and $T : H^{p,q}_{\mathrm{fin}}(X) \to \mathcal{B}_r$ is a $\mathcal{B}_r$-sublinear operator with
$$\sup \{ \| T(a) \|_{\mathcal{B}_r} : a \text{ is any } (p,q)\text{-atom} \} < \infty;$$

(ii) $T : H^{p,\infty}_{\mathrm{fin}}(X) \cap UC(X) \to \mathcal{B}_r$ is a $\mathcal{B}_r$-sublinear operator with
$$\sup \{ \| T(a) \|_{\mathcal{B}_r} : a \text{ is any } (p,\infty)\text{-atom} \} < \infty.$$

Then $T$ can be uniquely extended to a bounded $\mathcal{B}_r$-sublinear operator from $H^{p,q}_{\mathrm{at}}(X)$ to $\mathcal{B}_r$.

References

[1] P. Auscher and T. Hytönen, Orthonormal bases of regular wavelets in spaces of homogeneous type, Appl. Comput. Harmon. Anal. 34 (2013), 266–296.
[2] T. A. Bui and X. T. Duong, Hardy spaces associated to the discrete Laplacians on graphs and boundedness of singular integrals, Trans. Amer. Math. Soc. 366 (2014), 3451–3485.
[3] T. A. Bui, X. T. Duong and F. K. Ly, Maximal function characterizations for new local Hardy type spaces on spaces of homogeneous type, Trans. Amer. Math. Soc., DOI: 10.1090/tran/7289.
[4] L. Carleson, Two remarks on $H^1$ and BMO, Adv. Math. 22 (1976), 269–277.
[5] R. R. Coifman, A real variable characterization of $H^p$, Studia Math. 51 (1974), 269–274.
[6] R. R. Coifman and G. Weiss, Analyse Harmonique Non-commutative sur Certains Espaces Homogènes, (French) Étude de certaines intégrales singulières, Lecture Notes in Mathematics, Vol. 242, Springer-Verlag, Berlin-New York, 1971.
[7] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis. Bull. Amer. Math. Soc. 83 (1977), 569–645.
[8] D. Deng and Y. Han, Harmonic Analysis On Spaces Of Homogeneous Type. With A Preface By Yves Meyer, Lecture Notes in Mathematics, 1966, Springer-Verlag, Berlin, 2009.
[9] X. Y. Duong and L. Yan, Hardy spaces of spaces of homogeneous type, Proc. Amer. Math. Soc. 131 (2003), 3181–3189.
[10] X. T. Duong and L. Yan, Duality of Hardy and BMO spaces associated with operators with heat kernel bounds, J. Amer. Math. Soc. 18 (2005), 943–973.
[11] C. Fefferman and E. M. Stein, $H^p$ spaces of several variables, Acta Math. 129 (1972), 137–193.
[12] G. B. Folland and E. M. Stein, Hardy Spaces on Homogeneous Groups, Mathematical Notes, 28. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982.
[13] X. Fu, D.-C. Chang and D. Yang, Recent progress in bilinear decompositions, Applied Analysis and Optimization 1 (2017), 153–210.
[14] X. Fu and D. Yang, Wavelet characterizations of the atomic Hardy space $H^1$ on spaces of homogeneous type, Appl. Comput. Harmon. Anal. 44 (2018), 1–37.
[15] X. Fu, D. Yang and Y. Liang, Products of functions in BMO($\mathcal{X}$) and $H^1_{\mathrm{at}}(\mathcal{X})$ via wavelets over spaces of homogeneous type, J. Fourier Anal. Appl. 23 (2017), 919–990.
[16] J. García-Cuerva and J. L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland Mathematics Studies, 116. Notas de Matemática [Mathematical Notes], 104. North-Holland Publishing Co., Amsterdam, 1985.

[17] L. Grafakos, Classical Fourier Analysis. Third edition, Graduate Texts in Mathematics, 249. Springer, New York, 2014.

[18] L. Grafakos, L. Liu, D. Maldonado and D. Yang, Multilinear analysis on metric spaces, Dissertationes Math. (Rozprawy Mat.) 497 (2014), 1–121.

[19] L. Grafakos, Modern Fourier Analysis, Third edition, Graduate Texts in Mathematics, 250. Springer, New York, 2014.

[20] L. Grafakos, L. Liu, and D. Yang, Maximal function characterizations of Hardy spaces on RD-spaces and their applications, Sci. China Ser. A 51 (2008), 2253–2284.

[21] L. Grafakos, L. Liu, and D. Yang, Radial maximal function characterizations for Hardy spaces on RD-spaces, Bull. Soc. Math. France 137 (2009), 225–251.

[22] L. Grafakos, L. Liu and D. Yang, Vector-valued singular integrals and maximal functions on spaces of homogeneous type, Math. Scand. 104 (2009), 296–310.

[23] Ya. Han, Yo. Han and J. Li, Criterion of the boundedness of singular integrals on spaces of homogeneous type, J. Funct. Anal. 271 (2016), 3423–3464.

[24] Ya. Han, Yo. Han and J. Li, Geometry and Hardy spaces on spaces of homogeneous type in the sense of Coifman and Weiss, Sci. China Math. 60 (2017), 2199–2218.

[25] Y. Han, J. Li and L. D. Ward, Hardy space theory on spaces of homogeneous type via orthonormal wavelet bases, Appl. Comput. Harmon. Anal. (2016), http://dx.doi.org/10.1016/j.acha.2016.09.002.

[26] Y. Han, D. Müller and D. Yang, Littlewood-Paley characterizations for Hardy spaces on spaces of homogeneous type, Math. Nachr. 279 (2006), 1505–1537.

[27] Y. Han, D. Müller and D. Yang, A theory of Besov and Triebel-Lizorkin spaces on metric measure spaces modeled on Carnot-Carathéodory spaces, Abstr. Appl. Anal. 2008, Art. ID 893409, 250 pp.

[28] Y. S. Han and E. T. Sawyer, Littlewood-Paley Theory on Spaces of Homogeneous Type and the Classical Function Spaces, Mem. Amer. Math. Soc. 110 (1994), vi+126 pp.

[29] Z. He, L. Liu, D. Yang and W. Yuan, New Calderón reproducing formulae with exponential decay on spaces of homogeneous type, arXiv: 1803.09924.

[30] S. Hofmann and S. Mayboroda, Hardy and BMO spaces associated to divergence form elliptic operators, Math. Ann. 344 (2009), 37–116.

[31] T. Hytönen and A. Kairema, Systems of dyadic cubes in a doubling metric space, Colloq. Math. 126 (2012), 1–33.

[32] G. Hu, D. Yang and Y. Zhou, Boundedness of singular integrals in Hardy spaces on spaces of homogeneous type, Taiwanese J. Math. 13 (2009), 91–135.

[33] P. Koskela, D. Yang and Y. Zhou, A characterization of Hajlasz-Sobolev and Triebel-Lizorkin spaces via grand Littlewood-Paley functions, J. Funct. Anal. 258 (2010), 2637–2661.

[34] P. Koskela, D. Yang and Y. Zhou, Pointwise characterizations of Besov and Triebel-Lizorkin spaces and quasiconformal mappings, Adv. Math. 226 (2011), 3579–3621.

[35] L. D. Ky, New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators. Integral Equations Operator Theory 78 (2014), 115–150.
[36] R. H. Latter, A characterization of $H^p(\mathbb{R}^n)$ in terms of atoms, Studia Math. 62 (1978), 93–101.

[37] W. Li, A maximal function characterization of Hardy spaces on spaces of homogeneous type, Approx. Theory Appl. (N.S.) 14 (1998), 12–27.

[38] L. Liu, D.-C. Chang, X. Fu and D. Yang, Endpoint boundedness of commutators on spaces of homogeneous type, Applicable Analysis 96 (2017), 2408–2433.

[39] L. Liu, D.-C. Chang, X. Fu and D. Yang, Endpoint estimates of linear commutators on Hardy spaces over spaces of homogeneous type, Submitted.

[40] L. Liu, D. Yang and W. Yuan, Bilinear decompositions for products of Hardy and Lipschitz spaces on spaces of homogeneous type, Dissertationes Math. (Rozprawy Mat.) (to appear).

[41] S. Z. Lu, Four Lectures on Real $H^p$ Spaces, World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[42] R. A. Macià and C. Segovia, Lipschitz functions on spaces of homogeneous type, Adv. in Math. 33 (1979), 257–270.

[43] R. A. Macià and C. Segovia, A decomposition into atoms of distributions on spaces of homogeneous type, Adv. in Math. 33 (1979), 271–309.

[44] E. Nakai and Y. Sawano, Hardy spaces with variable exponents and generalized Campanato spaces, J. Funct. Anal. 262 (2012), 3665–3748.

[45] W. Rudin, Functional Analysis, Second edition, International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.

[46] Y. Sawano, P.-K. Ho, D. Yang, and S. Yang, Hardy spaces for ball quasi-Banach function spaces, Dissertationes Math. (Rozprawy Mat.) 525 (2017), 1–102.

[47] E. M. Stein, Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Mathematical Series, 43, Monographs in Harmonic Analysis, III, Princeton University Press, Princeton, NJ, 1993.

[48] E. M. Stein and G. Weiss, On the theory of harmonic functions of several variables. I. The theory of $H^p$-spaces, Acta Math. 103 (1960), 25–62.

[49] A. Uchiyama, Characterization of $H^p(\mathbb{R}^n)$ in terms of generalized Littlewood-Paley $g$-functions, Studia Math. 81 (1985), 135–158.

[50] P. Wojtaszczyk, Banach Spaces for Analysts, Cambridge Studies in Advanced Mathematics, 25, Cambridge University Press, Cambridge, 1991.

[51] D. Yang, Y. Liang and L. D. Ky, Real-Variable Theory of Musielak-Orlicz Hardy Spaces, Lecture Notes in Mathematics, 2182. Springer, Cham, 2017.

[52] D. Yang and Y. Zhou, Boundedness of sublinear operators in Hardy spaces on RD-spaces via atoms, J. Math. Anal. Appl. 339 (2008), 622–635.

[53] D. Yang and Y. Zhou, A boundedness criterion via atoms for linear operators in Hardy spaces, Constr. Approx. 29 (2009), 207-218.

[54] D. Yang and Y. Zhou, Radial maximal function characterizations of Hardy spaces on RD-spaces and their applications, Math. Ann. 346 (2010), 307–333.

[55] D. Yang and Y. Zhou, New properties of Besov and Triebel-Lizorkin spaces on RD-spaces. Manuscripta Math. 134 (2011), 59–90.

[56] C. Zhuo, Y. Sawano and D. Yang, Hardy spaces with variable exponents on RD-spaces and applications. Dissertationes Math. (Rozprawy Mat.) 520 (2016), 1–74.
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