I. ABSTRACT/INTRODUCTION

Empirical distributions have their in-sample maxima as natural censoring. We look at the "hidden tail", that is, the part of the distribution in excess of the maximum for a sample size of $n$. Using extreme value theory, we examine the properties of the hidden tail and calculate its moments of order $p$.

The method is useful in showing how large a bias one can expect, for a given $n$, between the visible in-sample mean and the true statistical mean (or higher moments), which is considerable for $\alpha$ close to 1.

Among other properties, we note that the "hidden" moment of order 0, that is, the exceedance probability for power law distributions, follows an exponential distribution and has for expectation $\frac{1}{n}$ regardless of the parametrization of the scale and tail index.

II. THE INVISIBLE TAIL FOR A POWER LAW

Consider $K_n$ the maximum of a sample of $n$ independent identically distributed variables in the power law class; $K_n = \max(X_1, X_2, \ldots, X_n)$. Let $\phi(.)$ be the density of the underlying distribution. We can decompose the moments in two parts, with the "hidden" moment above $K$, as shown in Fig 1.

\[ \mu_{L,p} = \int_{K}^{\infty} x^p \phi(x) \, dx \]
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where $\mu_{L,p}$ is the observed part of the distribution and $\mu_{K,p}$ the hidden one. We note that $\phi(.)$ is not rescaled.

Fig. 1. The $p^{th}$ moment above $K$, the maximum in-sample observation.

Fig. 2. Proportion of the hidden mean in relation to the total mean, for different parametrizations of the tail exponent $\alpha$.

Fig. 3. Proportion of the hidden mean in relation to the total mean, for different sample sizes $n$. 

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**Proposition 1**

Let $K^*$ be point where the survival function of the random variable $X$ can be satisfactorily approximated by a factorized constant, that is $\mathbb{P}(X > x) \approx L^{-\alpha} x^{-\alpha}$.

Under the assumptions that $K > K^*$, the distribution for the hidden $\mu_n^{th}$ moment, $\mu_{K,n}$, for $n$ observation has for density $g_{n,\mu}(z)$:

$$g_{n,\mu}(z) = nL^{\frac{\alpha n}{2}} \left( z - \frac{p z}{\alpha} \right)^{\frac{n}{\alpha - p}} \exp \left( n \left( -\frac{p z}{\alpha} \right) \right) \left( z - \frac{p z}{\alpha} \right)^{-\frac{n}{\alpha - p}}$$

for $z \geq 0$, $\alpha > p$, and $L > 0$.

The mean becomes

$$E(\mu_{K,n}) = L n^{\frac{\alpha n}{2}} \Gamma \left( 1 - \frac{\beta}{\alpha} \right)$$

The proof is as follows. The expectation of the $\mu_n^{th}$ moment above $K$, with $K > L > 0$ can be derived as

$$\mu_{K,n} = \frac{\alpha L^\alpha K^{\beta - \alpha}}{\alpha - p}, \alpha > p$$

and we need to calculate its distribution.

For the full distribution $g_{n,\mu}(z)$, let us decompose the mean of a Pareto with scale $L$, so $K_{min} = L$.

By standard transformation, a change of variable, $K \sim F(\alpha, Ln^\frac{2}{\alpha})$ a Fréchet distribution with PDF: $f_K(K) = \alpha n K^{-\alpha - 1} L^{\alpha} e^{\alpha n(-z)}$, from which we get the required result.

**Remark 1**

We note that the distribution of the sample survival function (that is, $p = 0$) is an exponential distribution with pdf:

$$g_{n,0,\mu}(z) = ne^{-n z}$$

which we can see depends only on $n$. Exceedance probability does not depend on the thickness of the tails.

**III. Comparison with other distributions**

To get the expectation where closed forms are not available (say with the Lognormal), we just need to get the integral with a stochastic lower bound $K > K_{min}$

$$\int_{K_{min}}^{\infty} \left( \int_{\mu_{K,n}}^{\infty} x^p \phi(x) dx \right) f_K(K) dK.$$  

For a Gaussian with PDF $\phi(\cdot)$ indexed by $(g)$, $\mu_{K} = \int_{K}^{\infty} \phi(\cdot) dx = 2 \Phi^{-1} \left( \frac{1}{2} \sqrt{\frac{4 K^2}{\pi}} \right)$. As we saw earlier, without going through the Gumbel (rather EVT or "mirror-Gumbel"), it is preferable to the exact distribution of the maximum from the CDF of the Standard Gaussian $F(\cdot)$:

$$\frac{\partial F(\cdot)}{\partial K} = \frac{e^{-\frac{K^2}{2}} 2^{-n} - n \text{erf} \left( \frac{K}{\sqrt{2}} \right)^{n-1}}{\sqrt{\pi}},$$

where erfc is the complementary error function.

For $p = 0$, the expectation of the "invisible tail" $\approx \frac{1}{n}$, since:

$$\int_{0}^{\infty} e^{-\frac{K^2}{2}} 2^{-n} \nu n \text{erf} \left( \frac{K}{\sqrt{2}} \right)^{n-1} dK = 1 - \frac{2}{n+1}$$

For higher moments, it is not apparently possible to obtain results analytically, but $p = 1$ shows a rapid decline in line with the speed of convergence of the mean of the Gaussian under the law of large numbers.

**IV. Applications and conclusion**

The property of the hidden moments is useful to understand the "properties of the unseen", in environments where past data provides insufficient evidence –and we are aware of it; hence we can quantify such unknown (or at least get an idea of its magnitude). It thus has an epistemological value as we can tell beforehand the magnitude of the underestimation, and how confident we can be from past data.

This is particularly useful for one-tailed distributions where the sample mean will necessarily be underestimating the true mean if the tail is to the right, and overestimating if the tail is to the left. It applies, for instance, to the misestimation of the true P/L of strategy with long or short volatility profile, to the pricing of options with remote strike prices, to the true expected long term damage from hurricanes and other natural calamities, to the expected level of flooding, to the true properties of war and mean casualties from violence, and many more. Indeed using methods from extreme value theory critically changes the thinking and the conclusions as compared to "evidence based" methods that have statistical flaws under thicker tailed domains.
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