We present a simple model illustrating how a highly relativistic, compact object which is stable in isolation can be driven dynamically unstable by the tidal field of a binary companion. Our compact object consists of a test-particle in a relativistic orbit about a black hole; the binary companion is a distant point mass. Our example is presented in light of mounting theoretical opposition to the possibility that sufficiently massive, binary neutron stars inspiraling from large distance can collapse to form black holes prior to merger. Our strong-field model suggests that first order post-Newtonian treatments of binaries, and stability analyses of binary equilibria based on orbit-averaged, mean gravitational fields, may not be adequate to rule out this possibility.

I. INTRODUCTION

Binary neutron stars are among the most promising sources for gravitational wave detectors now under construction, like LIGO, VIRGO and GEO. This fact has motivated an intense theoretical effort to understand their dynamical evolution and predict the gravitational waveforms emitted during their inspiral and coalescence.

Fully general relativistic treatments of the binary problem are nontrivial because of the nonlinearity of Einstein’s field equations and the need for very large computational resources to solve them, together with the equations of relativistic hydrodynamics, in three spatial dimensions plus time. Recently, Wilson, Mathews and Maronetti (hereafter WWM) performed approximate relativistic simulations which suggest that neutron stars just below the maximum allowed rest mass and dynamically stable in isolation become unstable to collapse to black holes when placed in a close binary orbit. Their results are in disagreement with earlier calculations of stability performed in Newtonian gravitation, which show that tidal fields stabilize binary stars against radial collapse. The WWM results are also surprising in light of several recent, albeit simplified, post-Newtonian (PN) perturbation analyses, none of which indicate the presence of any relativistic radial instability in binaries, at least to first post-Newtonian (1PN) order.

To help clarify the issue, Baumgarte et. al. performed fully relativistic numerical calculations of binary neutron stars in quasi-equilibrium circular orbits. (The solutions correspond to “quasi”-equilibrium because of the inevitable slow, secular decay of the orbit due to the emission of gravitational waves). They found that the maximum allowed rest mass (i.e. baryon mass) of a neutron star in a synchronous binary is in fact slightly larger than in isolation. PN treatments of the same problem confirm this conclusion and extend it to nonsynchronous binaries as well (e.g. Ref [4]). This finding rules out at least one explanation for the results of WWM – namely, the possibility that binary equilibrium solutions might not exist for neutron stars with the highest values of rest mass allowed for isolated stars. Using their relativistic models, Baumgarte et. al. further showed that all stars which are stable to radial collapse in isolation are also stable when placed in a binary, at least to secular modes, all the way down to the innermost stable circular orbit. This result, together with the expectation that dynamical instabilities in binaries always arise at smaller separation than secular instabilities, as in Newtonian theory, suggests that a dynamical instability against radial collapse is just not present for binary neutron stars that inspiral from large distance. This conclusion has been considerably strengthened by an analytic, relativistic analysis by Thorne, who argues that a rigidly rotating, stable neutron star in an inspiraling binary system will be protected against secular and dynamical collapse by tidal interaction with its companion.

It would appear that the WWM result may not be correct and that their finding might be the result of inadequate computational resolution or one or more of the simplifications they adopt to solve this difficult numerical problem. Nevertheless, their work has raised the very interesting and nonobvious question about whether or not tidal fields in a relativistic binary can under any circumstances trigger the collapse of a compact object that is known to be stable in isolation.

In this paper, we offer a simple example to illustrate the possibility of binary-induced dynamical collapse of a compact object. The compact object consists of a test particle in a tight orbit around a black hole. This “com-
compact object” is itself placed in orbit about a distant mass and the “binary” — i.e., the “compact object” in circular orbit about the distant mass — is allowed to inspiral by the emission of gravitational radiation. We investigate whether the tidal field of the distant mass can drive such a compact object dynamically unstable even though the object, when in isolation, is stable. Here, dynamical instability manifests itself as a rapid plunge of the test particle toward the black hole.

Our analysis of this model is highly simplified and arguably heuristic in parts. Nevertheless, we are confident that our overall picture is reliable and that our essential conclusions will stand up to more rigorous scrutiny.

II. BASIC MODEL AND EQUATIONS

Consider at first a self-gravitating, N-body system in the post-Newtonian limit of general relativity. The (geodesic) equations of motion for body $i$ may be cast in the form

$$\ddot{r}_i = - \sum_{j \neq i} \frac{m_j}{r_{ij}^3} r_{ij} + F_i^{(PN)}(1, 2, \ldots, i, \ldots, N),$$

where $m_j$ is the total mass-energy of body $j$, $r_j$ is its coordinate position and $r_{ij} = r_i - r_j$. The first term on the right hand side of Eq. (1) is the usual Newtonian (Coulomb) expression, while the second term is the PN correction. Explicit expressions for this PN correction have been developed by many authors to various orders and in several gauges. Einstein, Infeld and Hoffman [1] provided the “EIH” equations of motion, which are valid to 1PN order. Damour and Shafer [2] derived an N-particle Lagrangian valid to 2PN order (see Ref. [1] for a review and references). In these treatments, the PN correction to the equation of motion for a particle typically depends on the positions, velocities and the accelerations of all of the $N$ particles.

Let us now specialize to $N = 3$. Assume that bodies 1 and 2 are appreciably closer to each other than they are to body 3, and treat the influence of 3 on the relative orbit of the close pair 1–2 as a small perturbation. Define the quantities $m = m_1 + m_2$ and

$$R_{cm} = \frac{m_1 r_1 + m_2 r_2}{m}, \quad r = r_1 - r_2, \quad r_{cm} = R_{cm} - r_3,$$

and use Eq. (1) to write the equation of motion for the relative orbit of the close pair in the form

$$\ddot{r} = - \frac{m}{r^3} r + f_{(tidal)} + F_1^{(PN)}(1, 2) - F_2^{(PN)}(1, 2),$$

where

$$f_{(tidal)} = \frac{m_3}{r_{cm}^3} \left[ \frac{3r_{cm} \cdot r}{r_{cm}^2} r_{cm} - r \right]$$

is the leading Newtonian tidal perturbation due to body 3 and where the leading post-Newtonian corrections $F_1$ and $F_2$ depend only on bodies 1 and 2. Here we assume that $r \ll r_{cm}$ and we drop tidal corrections of $O(m_3 r_{cm}^2)$. To the same order, the orbit of particle 3 about the close pair is determined by

$$\ddot{r}_{cm} = - \frac{M}{r_{cm}^3} r_{cm},$$

where $M = m + m_3$ and where we drop tidal terms of $O(M m_3 m_{cm} r_{cm}^2)$, as well as the very small PN corrections. Here $m_{<} \equiv max[m_1, m_2]$, and similarly for $m_{>}$. To this order, the orbit of body 3 about the close pair decouples from the relative orbit of that pair.

In this perturbative treatment, let us henceforth examine two extreme, opposite limits: In one limit, assume that the orbital plane of body 3 about the close pair moves in the plane describing the relative orbit of that pair (i.e., the inclination angle of the two orbital planes satisfies $i = 0^\circ$). In the other limit, assume that 3 moves in a plane perpendicular to the orbital plane of the pair ($i = 90^\circ$). Decompose Eqs. (3) and (4) into radial and angular components, yielding for the relative orbit of the pair

$$\ddot{r} = - \left( \frac{m}{r^3} \right) [A + B\dot{r}] + r\dot{\phi}^2 + f_{(tidal)}^r,$$

$$\ddot{\phi} = - \dot{\phi} \left( \frac{m}{r^3} \right) B + 2 \frac{\dot{r}}{r} + \frac{1}{r} f_{(tidal)}^\phi,$$

where

$$f_{(tidal)}^r = \begin{cases} \left( \frac{m_{<}}{r_{cm}} \right) \left[ 3 \cos^2(\phi - \theta) - 1 \right], & (i = 0^\circ) \\ \left( \frac{m_{<}}{r_{cm}} \right) \left[ 3 \cos^2(\theta) \cos^2(\phi) - 1 \right], & (i = 90^\circ) \end{cases}$$

and for the orbit of 3 about the pair

$$\ddot{r}_{cm} = - \frac{M}{r_{cm}^3} + r_{cm}\dot{\theta}^2,$$

$$\ddot{\theta} = - \frac{2\dot{\theta} \dot{r}_{cm}}{r_{cm}}.$$

In the Newtonian limit, $A = 1$ and $B = 0$ in Eqs. (3) and (4). In the case of isolated binaries, Lincoln and Will [12] derive post-Newtonian expressions for $A$ and $B$ for arbitrary masses, correct through 2.5PN order.
Kidder, Will and Wiseman (hereafter KWW: \[1\]) provide a “hybrid” set of equations in which the sum of the terms in \(A\) and \(B\) that are independent of the ratio \(\eta = \mu/m, \mu = m_1 m_2/m\), is replaced by the exact expression for geodesic motion in the Schwarzschild metric of a body of mass \(m\), while the terms dependent on \(\eta\) are left unaffected. Their resulting equation of motion is therefore exact in the test-body limit (\(\eta \to 0\)) and is valid to 2.5PN order when appropriately expanded for arbitrary masses. We shall utilize these same hybrid expressions for \(A\) and \(B\) and, for simplicity, work in the test-body limit by taking one member of our close pair to have a mass much smaller than the other. Adopting harmonic (or de Donder) coordinates, the resulting (Schwarzschild) expressions for \(A\) and \(B\) are given by

\[
A = \left[\frac{1 - m/r}{(1 + m/r)^2}\right] - \left[\frac{2 - m/r}{1 - (m/r)^2}\right] \frac{m}{r} \dot{r}^2 + v^2,
\]

\[
B = - \left[\frac{4 - 2m/r}{1 - (m/r)^2}\right] \dot{\theta},
\]

where

\[
v^2 = \dot{r}^2 + \dot{\theta}^2.
\]

We are interested in the dynamical behavior of the close pair, regarded as a single “compact object”, as it inspirals toward the distant mass \(m_3\) due to gravitational radiation emission. To treat the inspiral of this “binary” \((m_3 \text{ in orbit about the “compact object”})\), we must include radiation reaction terms in the lowest order (Newtonian) orbit Eqs. \(\[1\]\) and \(\[3\]\). Formally, such a treatment requires a consistent expansion up to 2.5PN order. In lieu of this, we shall analyze the inspiral by assuming that the binary is in a nearly circular, Keplerian orbit, which undergoes a slow inspiral due to gravitational radiation loss in the quadrupole limit. This assumption is equivalent to inserting a quadrupole radiation reaction potential in the binary orbit Eq. \(\[3\]\) and neglecting the lower-order, (non-dissipative) PN corrections in that particular equation. While such an expression is not formally consistent to 2.5PN order, we believe that it faithfully tracts the secular inspiral of the binary in the limit treated here in which the binary system is at wide (non-relativistic) separation. The details of the inspiral are not important here, only that the inspiral serves to bring a tidal perturber slowly in from infinity toward our compact object \(\[4\]\). The resulting equation for the binary inspiral is then

\[
r_{\text{cm}}(t)/r_{\text{cm}}(0) = (1 - t/T)^{1/4}
\]

and

\[
\theta(t) - \theta(0) = \frac{4}{3} \left(\frac{M}{r_{\text{cm}}(0)}\right)^{1/2} \frac{5}{16} (r_{\text{cm}}(0)/m)^{1/2} T[1 - (1 - t/T)^{3/4}],
\]

where the binary inspiral timescale \(T\) is given by

\[
T/m = \frac{5}{256} (M/m)(m_3/m)^{1/2}.
\]

We will use Eqs. \(\[1\]\)–\(\[3\]\) in analyzing the relative orbit Eqs. \(\[1\]\)–\(\[6\]\).

All orbital timescales in our problem are considerably shorter than \(T\). For some purposes it is instructive to perform an orbit average of the tidal terms appearing in Eq. \(\[6\]\) and \(\[7\]\), which yields \(\langle f_\text{tidal} \rangle = \alpha m_3 r/r_{\text{cm}}\), where \(\alpha = 1/2\) for \(i = 0^\circ, -1/4\) for \(i = 90^\circ\), and \(\langle f_\text{tidal} \rangle = 0\).

III. QUASI-EQUILIBRIUM AND STABILITY: MEAN-FIELD APPROACH

Use the orbit-averaged tidal force to consider the close pair as a compact object in the mean tidal field of its distant, binary companion. In this approximation the mean field, quasi-equilibrium state for the compact object object consists of a test particle in circular orbit at constant radius about a black hole. The degree of compaction of this quasi-equilibrium compact object is parametrized by the radius of the circular orbit, \(r_0/m\); for small values of this parameter \(\lesssim 10\), the object is relativistic. The quasi-equilibrium orbit in the mean tidal field is determined self-consistently by requiring \(\dot{r} = 0 = \dot{\theta} = \dot{\phi}\). Eq. \(\[1\]\) then implies the circular orbit condition

\[
\Omega_0^2 \equiv \dot{\phi}_0^2 = \frac{m A_0}{r_0^3} - \frac{a m_3}{r_{\text{cm}}^3},
\]

which can be solved along with Eqs. \(\[12\]\) and \(\[14\]\) to get \(\Omega_0\) as a function of \(r_0\). Because the orbiting particle is a test-particle of negligible mass, its orbit does not decay due to gravitational radiation, so the compact object is secularly stable. Dynamical stability of the circular orbit is determined by considering linear perturbations about the equilibrium values of \(r_0, \phi_0\) and \(\dot{\phi}_0 = 0\). The resulting stability analysis of our compact object is a simple extension of the analysis by KWW for PN binaries in the absence of a tidal field. We find by linearizing Eq. \(\[1\]\) in a mean tidal field that the condition for stable circular orbits may again be written as \(a + bc < 0\), where now

\[
a = 3\Omega_0^2 - \left(\frac{m}{r_0^3}\right) \left(\frac{\partial A}{\partial r}\right)_0 + 3\alpha \frac{m_3}{r_{\text{cm}}},
\]

\[
b = 2r_0\Omega_0 - \left(\frac{m}{r_0^3}\right) \left(\frac{\partial A}{\partial r}\right)_0
\]

\[
c = -\Omega_0 \left[\frac{2}{r_0} + \left(\frac{m}{r_0^3}\right) \left(\frac{\partial A}{\partial r}\right)_0\right]
\]

We can analyze some limiting cases analytically. In the absence of a tidal field \((r_{\text{cm}}/m \to \infty)\), the condition for dynamical stability reduces to

\[
\frac{r_0}{m} > 5, \quad \Omega_0^2 m^2 < \frac{1}{64} \left(\frac{r_{\text{cm}}}{m} \to \infty\right).
\]
FIG. 1. The innermost stable circular orbit (ISCO) as a function of the separation between the compact object and its binary companion in the mean-field approximation. The solid line is for coplanar test-particle and binary companion orbits \((i = 0^\circ)\); the dashed line is for orthogonal orbits \((i = 90^\circ)\). The dotted line shows the ISCO for an isolated compact object.

Eq. (20) reproduces the well-known value (in harmonic coordinates) of the innermost stable circular orbit (ISCO) in Schwarzschild geometry, \(r_0/m = 5\). Stability for a strictly Newtonian orbit \((r_0/m \gg 1)\) in the presence of a tidal field from a perturber in the orbital plane requires \(r_0/r_{\text{cm}} < \left(\frac{1}{2} \frac{m}{m_3}\right)^{1/3}, \quad \Omega_0^2 > \frac{2m_3}{r_{\text{cm}}^3}\) (Newtonian, \(i = 0^\circ\)),

which is the familiar Newtonian condition for avoiding tidal disruption. For a tidal perturber perpendicular to the orbital plane, all Newtonian orbits are stable.

The more interesting cases involve the effect of the mean tidal field of a companion on the stability of a relativistic compact object in binary equilibrium. To analyze these, we numerically evaluate Eq. (19) to determine the ISCO as a function of the strength of the tidal field; results are summarized in Fig 1. We see from Fig 2 that for a perturber in the orbital plane, the angular frequency of the ISCO moves to lower frequencies with increasing tidal field strength. Thus, a given compact object, specified by a fixed angular frequency, can be stable in isolation but driven dynamically unstable when in a binary system.

From Fig 1 we see, by contrast, that for a perturber perpendicular to the orbital plane the ISCO actually moves in. Likewise, we see from Fig 2 that the orbital frequency at the ISCO increases. Hence, in this case the mean tidal field due to a companion would appear to stabilize the compact object.

IV. STABILITY DURING INSPIRAL

To assess the true fate of our compact object inspiraling toward its binary companion, we abandon the mean-field approximation and integrate the relative orbit Eqs. (6)–(9) together with the inspiral Eqs. (15)–(17). At \(t = 0\) we place a test-particle in a circular orbit about a black hole, setting \(r/m = 5.9, \dot{r} = 0 = \phi\) and \(\dot{\phi}\) given by Eq. (18). We set \(m_3 = m\) and start the companion \(m_3\) at large separation \(r_{\text{cm}}/m = 40\). Hence, initially, the tidal field of the companion is negligible at the compact object and with so large a test particle orbit, the compact object begins in stable equilibrium, almost exactly as it would be were it in isolation. We treat two cases for the relative orientation of the two orbital planes, \(i = 0^\circ\) and \(i = 90^\circ\), setting \(\theta = 0^\circ\) at \(t = 0\) in the former by a distant observer.
FIG. 3. The orbital radius of the test particle about the black hole as a function of time. The solid curve shows the motion for coplanar test-particle and binary companion orbits ($i = 0^\circ$); the dashed curve shows the motion for orthogonal orbits ($i = 90^\circ$). The dotted curve shows the separation between the compact object and its binary companion as they inspiral together.

The results of our integrations are summarized in Figs 3-5, where we see that early on the compact object remains in stable equilibrium for many binary periods as the companion slowly approaches. The test-particle orbit during this phase is characterized by stable, small oscillations about the initial circular orbit; the oscillations occur at the beat frequency between the distant binary companion and test-particle orbits (cf. Eq (8)). When the companion gets sufficiently close ($r_{cm}/m \approx 20$ for $i = 0^\circ$, $r_{cm}/m \approx 30$ for $i = 90^\circ$), the amplitude of the oscillations becomes sufficiently large to cause the test-particle to plunge toward the hole and be captured. Hence in both cases the tidal field of the companion ultimately drives the dynamical collapse of the compact object.

When $i = 0^\circ$, this binary-induced collapse is anticipated from the mean-field stability analysis, since in this case, the tidal field destabilizes close circular test-particle orbits which are otherwise be stable in isolation (see Sec 3). The unexpected destabilization in the case $i = 90^\circ$ appears to result from the growing eccentricity case (companion aligned with the close pair) and $\theta = 90^\circ$ in the later case, (companion situated above the orbital plane of the close pair, along its normal) At such large binary separation, $\Omega_0 T \gg 1$, hence the binary orbits many times before the inspiral proceeds very far. We thus expect (and have verified numerically) that the fate of the compact object does not depend on our choices of initial values of $\phi$ and $\theta$.

FIG. 4. The trajectory of the test particle about the black hole during binary inspiral for a companion moving in a coplanar orbit ($i = 0^\circ$).

FIG. 5. The trajectory of the test particle about the black hole during binary inspiral for a companion moving in an orthogonal plane ($i = 90^\circ$).
of the test-particle orbit as the companion approaches, eventually driving the particle too close to the hole at pericenter and to ultimate capture. In this case, the mean-field analysis, based on orbit averaging and strict circular motion, does not account for the more complicated motion of the test particle and fails to predict its ultimate fate.

V. DISCUSSION

We have presented a simple illustration of a compact object which is stable in isolation but dynamically unstable to collapse when inserted in a binary. Admittedly, our compact object is highly idealized and our treatment is simplified. Our whole system consist of only 3 bodies moving on geodesics. We could trivially generalize our compact object to be a fully three-dimensional swarm of test particles, all in randomly oriented circular orbits at the same fixed radius about a central hole. Whenever this radius is close to, but slightly larger than 5M, then according to our previous analysis, the presence of a sufficiently close binary companion would drive most, if not all, of the test-particle orbits dynamically unstable during inspiral from large separation. Once again, such a compact object would be stable in isolation but could be driven dynamically unstable when placed in a binary system.

It is not clear what bearing our simple example has to the finding of WWM, especially in light of the counter evidence suggesting that binary-induced collapse does not occur in fluid stars. We have not incorporated any hydrodynamic forces in our model, and these forces are necessary to prevent catastrophic collapse in fluid systems. It is interesting nevertheless that a simple collisionless particle model exists which is stable in isolation but undergoes dynamical collapse during binary inspiral. Typically, collisionless equilibria in isolation are subject to the many of the same relativistic instabilities as fluid systems, including radial instability to collapse, (see, eg, Ref. 13 for a review and references), so it is certainly relevant for WWM and others to have considered the possibility of binary-induced collapse in the case of binary neutron stars.

Our simple model calculation offers a few cautions regarding the treatment of the relativistic binary problem. Crucial to our finding was our use of a high order relativistic equation for the dynamical behavior of of our compact object. Indeed, had we used 1PN expressions for the 3-body gravitational attraction, we would have found spurious behavior, because at 1PN (and even 3PN), the solution for the test-particle ISCO in a PN expansion of the Schwarzshild equations of motion yields a spurious root that does not converge to 5m for isolated holes 13. By using the hybrid expressions in Eqs. (8) and (9), our compact object is treated exactly to all orders, at least when it is isolation.

It is not surprising that a 1PN treatment is inadequate for this problem. Such an approximation is not even sufficient to determine reliably the onset of radial instability to collapse in an isolated fluid star. There are the highly nonlinear terms of Einstein’s equations play a crucial role and only for special equations of state (e.g. n = 3 polytropes) is a 1PN analysis quantitatively reliable 4.

We have also seen that stability analyses of equilibria based on orbit averaging and mean tidal fields may prove misleading, as they did for the i = 90° case treated above. By the same token, it is by no means clear that conclusions found for synchronous stellar binaries will apply to nonsynchronous binaries. (Here we recall that the viscosity in neutron star binaries is not sufficiently large to drive the system to corotation 16). As our calculations demonstrate, once we relax dynamical constraints and allow motion with greater degrees of freedom, the final outcome of binary inspiral can be quite different, even qualitatively.

There is mounting evidence which argues against the possibility that sufficiently massive, highly compact neutron stars in coalescing binaries can collapse to black holes prior to merger. However, we will be greatly reassured when this conclusion is finally corroborated by detailed hydrodynamical calculations in full general relativity.

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