Primordial black holes and third order scalar induced gravitational waves

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The process of Primordial black holess (PBHs) formation would be inevitably accompanied by scalar induced gravitational waves (SIGWs). This strong correlation between PBHs and SIGWs signals could be a promising approach to detecting PBHs in the upcoming gravitational wave (GW) experiments, such as Laser Interferometer Space Antenna (LISA). We investigate the third order SIGWs during a radiation-dominated (RD) era in the case of a monochromatic primordial power spectrum $\mathcal{P}_\zeta = A_\zeta k^3\delta(k - k_\*)$. For LISA observations, the relations between signal-to-noise ratio (SNR) and monochromatic primordial power spectrum are studied systematically. It shows that the effects of third order SIGWs extend the cutoff frequency from $2f_*$ to $3f_*$ and lead to about 200% increase of the SNR for frequency band from $10^{-5}$Hz to $1.6 \times 10^{-3}$Hz corresponding to PBHs with mass range $4 \times 10^{-12}M_\odot \sim 10^{-7}M_\odot$. We find that there exists a critical value $A_\zeta = 1.76 \times 10^{-2}$ for the amplitude of the monochromatic primordial power spectra, such that when $A_\zeta > A_*$, the energy density of third order SIGWs will be larger than the energy density of second order SIGWs.

Introduction.—The inflation theory predicts that the cosmological perturbations are originated from the quantum fluctuations during inflation, which means that valuable information about the early Universe is encoded in these perturbations. On large scales ($\gtrsim 1$ Mpc), the primordial curvature perturbations have been determined through the observations of the cosmic microwave background (CMB) and large scale structure (LSS) [1, 2]. It indicates a nearly scale-invariant power spectrum of primordial curvature perturbations with amplitude $\sim 2 \times 10^{-9}$. On small scales ($\lesssim 1$ Mpc), the constraints of primordial curvature perturbations are significantly weaker than those on large scales [3].

Over the past few years, the primordial scalar perturbations with large amplitudes on small scales have been attracting a lot of interests on account of their rich and profound phenomenology. If the amplitude of the small-scale primordial curvature perturbations are large enough, large-amplitude perturbation modes that enter the Hubble radius during the radiation-dominated (RD) era can lead to the production of Primordial black holess (PBHs) [4–25], which is a reasonable candidate for the whole or an appreciable portion of dark matter (DM). Moreover, the process during which the PBHs are formed would be inevitably accompanied by scalar induced gravitational waves (SIGWs) [26–29].

The SIGWs have been studied for many years [30]. The energy density spectra of SIGWs include valuable information about PBHs [31–55] and primordial non-Gaussianity [56–63]. Furthermore, recent studies on SIGWs were also extended to gauge issue [64–75], dumping effect [76–78], epochs of the Universe [79–90], and modified gravity [91–93]. The relation between the SIGWs and PBHs formation was first studied in Ref. [31] in the case of a monochromatic primordial power spectrum. In those previous studies, the second order SIGW were considered only. In Ref. [94], the authors partly considered the third order effects of SIGWs which are induced by first order scalar perturbation directly. They found that effects of the third-order correction lead to almost 20% increase of the signal-to-noise ratio (SNR) for Laser Interferometer Space Antenna (LISA) observations. Their results indicate that the higher order effects are not dispensable, and it is necessary to consider the higher order SIGWs when the small-scale primordial curvature perturbations are large enough. In Ref. [95], we studied the third order SIGWs during the RD era systematically. In addition to the source terms of the first order scalar perturbation, the source terms of three kinds of second order perturbations induced by the first order scalar perturbation were considered completely. We found that the third order gravitational waves sourced by the second order scalar perturbations dominate the energy density spectra of third order SIGWs. The direct contributions of the source term of the first order scalar perturbation which were studied in Ref. [94] are negligible compared to the total energy density spectrum of the third order SIGWs.

In this paper, we investigate the primordial curvature perturbations and PBHs in terms of the SIGWs. We find that the effects of the third order SIGWs have significant observational implications for LISA.

Third order SIGWs—The perturbed metric in the Friedmann-Lemaître-Robertson-Walker (FLRW)
spacetime with Newtonian gauge takes the form
\[
\begin{align*}
\text{ds}^2 = -a^2 \left[ \left( 1 + 2\phi^{(1)} + \phi^{(2)} \right) dt^2 + V_i^{(2)} dp_i dx^i \right] \\
+ \left( 1 - 2\phi^{(1)} - \psi^{(2)} \right) dx^i dx^j + \frac{1}{2} h_i^{(2)} + \frac{1}{6} h_{ij}^{(3)} \right] dx^i dx^j
\end{align*}
\]
where \(\phi^{(n)}\) and \(\psi^{(n)}\) \((n = 1, 2)\) are the \(n\)-order scalar perturbations, \(V_i^{(2)}\) is the second order vector perturbation, and \(h_i^{(n)}\) \((n = 2, 3)\) are the \(n\)-order tensor perturbations. In the RD era, the first order scalar perturbation is given by \([28, 29]\)
\[
\psi(\eta, k) = \frac{2}{3} \zeta k T_\phi(k\eta) ,
\]
where \(\zeta k\) is the primordial curvature perturbations. The transfer functions \(T_\phi(k\eta)\) in the RD era is
\[
T_\phi(x) = \frac{9}{x^2} \left( \sqrt{\frac{2}{3}} \sin \left( \frac{x}{\sqrt{3}} \right) - \cos \left( \frac{x}{\sqrt{3}} \right) \right) ,
\]
where we have defined \(x = k\eta\). We simplify the higher order cosmological perturbations in terms of the xPand package \([96]\) and obtain the equation of motion of third order SIGWs
\[
h_{ij}^{(3)} + 2h_i h_j^{(3)} - \Delta h_{ij}^{(3)} = -12A_{ij}^{lm} \phi_{lm}^{(3)} ,
\]
where \(A_{ij}^{lm} = T_i^l T_j^m - \frac{1}{2} T_{ij} T_{lm}\) is the transverse and traceless operator \([69, 95]\), and \(T_{ij}^l\) is defined as \(T_{ij}^l = \delta_i^l - \delta_i^l \Delta^{-1} \partial_i\). For the RD era, the conformal Hubble parameter \(H = a'/a = 1/\eta\). The source term \(S_{lm}^{(3)}\) takes the form
\[
S_{lm}^{(3)} = S_{lm,1}^{(3)} + S_{lm,2}^{(3)} + S_{lm,3}^{(3)} + S_{lm,4}^{(3)} ,
\]
where \(S_{lm,1}^{(3)}\), \(S_{lm,2}^{(3)}\), \(S_{lm,3}^{(3)}\), and \(S_{lm,4}^{(3)}\) are source terms of first order scalar perturbation, second order tensor perturbation, second order vector perturbation, and second order scalar perturbations, respectively. The explicit expressions of these source terms are given by
\[
S_{lm,1}^{(3)} = \frac{1}{H} \left( 12H\phi^{(1)} - \phi^{(1)r} \right) \partial_t \phi^{(1)} \partial_m \phi^{(1)} - \frac{1}{H^3} \left( 4\phi^{(1)} \partial_t \phi^{(1)} \partial_m \phi^{(1)r} + \partial_t \left( H\phi^{(1)} + \phi^{(1)r} \right) \partial_m H\phi^{(1)} + \phi^{(1)r} \right) \times \frac{1}{3H^2} \left( 2\Delta\phi^{(1)} - 9H\phi^{(1)r} \right) ,
\]
\[
S_{lm,2}^{(3)} = -\frac{1}{2} \phi^{(1)} \left( h_{lm}^{(2)} + 2h_{lm}^{(2)} \right) - \Delta h_{lm}^{(2)} - \phi^{(1)} \Delta h_{lm}^{(2)} - \frac{1}{3} \phi^{(1)} h_{lm}^{(2)} - \frac{1}{3} \phi^{(1)} h_{lm}^{(2)} ,
\]
\[
S_{lm,3}^{(3)} = \phi^{(1)} \partial_t \left( V_{lm}^{(2)} \right) + 2H V_{lm}^{(2)} + \phi^{(1)} \partial_m \left( V_{lm}^{(2)} \right) - \frac{1}{4H} \partial_t \partial_m \Delta V_{lm}^{(2)} ,
\]
\[
S_{lm,4}^{(3)} = \frac{1}{H} \left( \phi^{(1)} \partial_t \partial_m \psi^{(2)} + \phi^{(1)} \partial_m \partial_i \phi^{(2)} \right) + \frac{1}{H^3} \left( \phi^{(1)} \partial_t \partial_m \psi^{(2)} \right) + 3\phi^{(1)} \partial_t \partial_m \phi^{(2)} .
\]
In Eq. (8), we used the symbol of symmetric tensor \(T_{lm} = \frac{1}{2} (T_{lm} + T_{ml})\). Eq. (4) in momentum space can be solved by the Green’s function method, namely
\[
h_{k\lambda}^{(3)}(\eta) = \frac{12 k^3}{(k\eta)} \int \! d\eta' \sin(k\eta - k\eta') \eta^2 S_{k\lambda}(\eta') ,
\]
where we have defined \(h_{k\lambda}^{(3)}(\eta) = \bar{e}_{ij}^{(3)}(k) h_{ij}^{(3)}(k, \eta)\) and \(S_{k\lambda}(\eta) = -e_{\lambda}^{lm}(k) S_{lm}^{(3)}(k, \eta)\). And \(e_{ij}^{(3)}(k)\) is the polarization tensor, which satisfies \(e_{ij}^{(3)}(k) \bar{e}^{\lambda\mu}(k) \delta^{\lambda\mu} = 2\) and \(\delta^{\lambda\mu} e_{\lambda\mu}^{lm}(k) \bar{e}_{ij}^{lm}(k) = \Lambda_{ij}^{lm}(k) + A_{ij}^{lm}(k)\). Here, \(\Lambda_{ij}^{lm}(k)\) is defined as \([69]\)
\[
A_{ij}^{lm}(k) = \frac{1}{\sqrt{2}} \left( \bar{e}_{ij} e_{ij} - \bar{e}_{ij} e_{ij} \right) \frac{1}{\sqrt{2}} \left( \bar{e}_{ij} e_{ij} - \bar{e}_{ij} e_{ij} \right) ,
\]
where \((k/|k|, \bar{e}_{ij}, \bar{e}_{ij})\) is the normalized bases in three dimensional momentum space. Note that \(A_{ij}^{lm}(k)\) is an antisymmetric tensor, therefore, \(A_{ij}^{lm} T_{lm} = 0\) for arbitrary symmetric tensor \(T_{ij}\). In the calculations of the second and the third order SIGWs, \(A_{ij}^{lm}\) only acts on symmetric tensor. In this case, we obtain \(A_{ij}^{lm} = 0\) and \(\delta^{\lambda\mu} e_{\lambda\mu}^{lm}(k) \bar{e}_{ij}^{(3)}(k) = \Lambda_{ij}^{lm}(k)\). There are four kinds of source terms in Eq. (4), it is convenient to divide \(h_{k\lambda}^{(3)}(\eta)\) into four parts \(h_{k\lambda}^{(3)}(\eta) = \sum_{l=1}^{4} h_{k\lambda}(\eta)\). The formal expressions of \(h_{k\lambda}(\eta)\) are given by
\[
h_{k,1}^{(3)}(\eta) = \int \frac{d^3p}{(2\pi)^3/2} \int \frac{d^3q}{(2\pi)^3/2} e^{\lambda\mu}(k) (p_l - q_l) q_m \times I_{1}^{(3)}((k - p_1, |p_1|, q, |q|, \zeta) \zeta - p_\phi - p_\phi' - q_\phi' ,
\]
\[
h_{k,2}^{(3)}(\eta) = \int \frac{d^3p}{(2\pi)^3/2} \int \frac{d^3q}{(2\pi)^3/2} e^{\lambda\mu}(k) \Lambda_{lm}^{\tau}(p) q_m q_s \times I_{2}^{(3)}((k - p_1, |p_1|, q, |q|, \zeta) \zeta - p_\phi - p_\phi' - q_\phi' ,
\]
\[
h_{k,3}^{(3)}(\eta) = \int \frac{d^3p}{(2\pi)^3/2} \int \frac{d^3q}{(2\pi)^3/2} e^{\lambda\mu}(k) T_{lm}^{\tau}(p) q_m q_s \times I_{3}^{(3)}((k - p_1, |p_1|, q, |q|, \zeta) \zeta - p_\phi - p_\phi' - q_\phi' ,
\]
\[
h_{k,4}^{(3)}(\eta) = \int \frac{d^3p}{(2\pi)^3/2} \int \frac{d^3q}{(2\pi)^3/2} e^{\lambda\mu}(k) p_m p_{nm} \times I_{4}^{(3)}((k - p_1, |p_1|, q, |q|, \zeta) \zeta - p_\phi - p_\phi' - q_\phi' .
\]
where $I_i^{(3)} (i = 1, 2, 3, 4)$ are kernel functions of third order SIGW, which take the form of

$$I_i^{(3)} (u, \bar{u}, \bar{v}, x) = \frac{12}{k^2 \bar{z}} \int_0^x \bar{d} \bar{x} \frac{\bar{f}(x - \bar{x})}{x} f_i^{(3)} (u, \bar{u}, \bar{v}, \bar{x}),$$

\[ (i = 1, 2, 3, 4). \]

Here, we have defined $|\mathbf{k} - \mathbf{p}| = uk$, $|\mathbf{k} - \mathbf{q}| = wk$, $|\mathbf{p} - \mathbf{q}| = \bar{u}p = \bar{u}k$, and $q = \bar{v}p = \bar{v}k$. $f_i^{(3)} (u, \bar{u}, \bar{v}, \bar{x}) (i = 1, 2, 3, 4)$ are transfer functions of four kinds of source terms in Eq. (6)–Eq. (9), which are given in Appendix A. We consider a monochromatic primordial power spectrum

$$\mathcal{P}_\zeta = \zeta f (k - k_*) \ , \quad (17)$$

where $k_*$ is the wavenumber at which the power spectrum has a $\delta$-function peak. In the case of a monochromatic primordial power spectrum, the explicit expression of the energy density spectra of third order SIGWs in the RD era is given by [95]

$$\Omega^{(3)}_{GW} (\eta, k) = \frac{\delta^{(3)} (\eta, k)}{\rho_{tot} (\eta)} = \frac{x^2}{216} \sum_{i,j=1}^4 \mathcal{P}^{ij}_k \ , \quad (18)$$

where $\mathcal{P}^{ij}_k$ is the power spectra of third order SIGWs

$$\mathcal{P}^{ij}_k = \frac{A^3 k^3}{2\pi} \Theta (3 - \tilde{k}) \int_{1 - \frac{1}{4}}^{\min \{ \frac{3}{2}, 1 + \frac{1}{k} \}} dv \int_{w_-}^{w_+} dw \left( \frac{vw}{\sqrt{Y (1 - X^2)}} \right) f_i^{(3)} (u, v, \bar{u}, \bar{v}, x) \times \sum_{a=1}^3 \mathbb{P}^{a}_{ij} I_j^{(3), a} (u', v', \bar{u}', \bar{v}', \eta) \bigg|_{u = \frac{1}{t}, \bar{u} = \bar{v} = \frac{1}{\bar{t}}} \ , \quad (19)$$

where $\tilde{k} = k/k_*$ is a dimensionless parameter. The explicit expressions of $X, Y,$ and $w_{\pm}$ are given in Appendix B. The details of polynomials $\mathbb{P}^{a}_{ij}$ can be found in Ref. [95]. Note that the energy density of gravitational waves (GWs) decays as radiation, then the current total energy density spectra of SIGWs can be approximated by [97]

$$\Omega_{GW} (\eta_0, k) \approx \Omega_r \times \left( \Omega_{GW}^{(2)} (\eta, k) + \Omega_{GW}^{(3)} (\eta, k) \right) , \quad (20)$$

where $\Omega_{GW}^{(2)} (\eta, k)$ is the energy density spectrum of second order SIGWs which has been studied systematically in previous work [29, 98].

**SIGWs as a probe of PBHs**—The monochromatic primordial power spectrum corresponds to a monochromatic PBH formation. The possibility of forming a PBH can be calculated in terms of the amplitude of primordial power spectrum [7, 36]

$$\beta = \int_{\xi_c}^{+\infty} \frac{d\xi}{\sqrt{2\pi}\sigma} e^{-\xi^2/2\sigma^2} = \frac{1}{2} \text{erfc} \left( \frac{\xi_c}{\sqrt{2\sigma}} \right) , \quad (21)$$

where $\sigma^2 \equiv \langle \xi^2 \rangle = \int \mathcal{P}_\zeta (k) \ln k = A_\zeta$ is the variance of the primordial curvature perturbation and $\xi_c \approx 1$ is the threshold value to form a PBH [94, 99–101]. Moreover, the mass of the PBH can be approximated by the frequency of the $\delta$-function peak

$$\frac{m_{pbh}}{M_\odot} \approx 2.3 \times 10^{18} \left( \frac{3.91}{g_{form}^{\star}} \right)^{1/6} \left( \frac{H_0}{f_*} \right)^2 , \quad (22)$$

where $f_* = k_*/(2\pi)$ and $g_{form}^{\star}$ is the effective degrees of freedom when PBHs are formed. Then, the fraction of PBHs can be evaluated in terms of $\beta$ and $m_{pbh}$

$$f_{pbh} \simeq 2.5 \times 10^8 \beta \left( \frac{g_{form}^{\star}}{10^{-7.5}} \right)^{-\frac{1}{4}} \left( \frac{m_{pbh}}{M_\odot} \right)^{-\frac{1}{2}} . \quad (23)$$

In the case of the monochromatic PBH formation, Eq. (21)–Eq. (23) show that $\beta$ and $m_{pbh}$ are determined by the amplitude of primordial power spectrum $A_\zeta$ and the frequency of the $\delta$-function peak $f_*$, respectively. Besides, the fraction of PBHs $f_{pbh}$ increase with $k_*$ and $A_\zeta$. This strong correlation between PBHs and the concomitant SIGWs signals could be a promising approach to detecting PBHs in the upcoming GW experiments, such as LISA. Fig. 1 shows the energy density spectra of SIGWs compared with the sensitivity curves of LISA. We find that the effects of the third order SIGWs extend the cutoff frequency from $2f_*$ to $3f_*$ and have significant observational implications for LISA.

To quantify the effects of the third order SIGWs, we calculate the SNR $\rho$ for LISA, which is given by [102–106]

$$\rho = \sqrt{T} \left[ \int df \left( \frac{\Omega_{GW} (f)}{\Omega_n (f)} \right)^2 \right]^{1/2} , \quad (24)$$

where $\Omega_n (f) = 2\pi^2 f^3 S_n / 3H_0^2$ and $S_n$ is the strain noise power spectral density, $T$ is the observation time. Here, we set $T = 4$ years. In Fig. 2 and Fig. 3, we plot the SNR curves obtained for LISA experiment. More precisely, Fig. 2 shows the correlation between SNR and $m_{pbh}$ for a given $A_\zeta$. It shows that the effects of third order SIGWs lead to about 200% increase of the SNR for frequency band from $1.6 \times 10^{-3} \text{Hz}$ to $10^{-5} \text{Hz}$ corresponding to PBHs with mass range $4 \times 10^{-12} M_\odot \sim 10^{-7} M_\odot$. Fig. 3 shows the relation between SNR and $A_\zeta$. 
FIG. 1. The energy density spectra of SIGWs to the second order \( \Omega_{GW}^{(2)} \) (orange curve) and to the third order \( \Omega_{GW}^{(3)} = \Omega_{GW}^{(2)} + \Omega_{GW}^{(3)} \) (blue curve) as function of frequency \( f \). We have set \( f_s = 1.3 \times 10^{-3} \text{Hz} \) and \( A_\zeta = 0.02 \).

FIG. 2. The SNR of LISA as a function of \( m_{pbh} \) for \( \Omega_{GW}^{(2)} = \Omega_{GW}^{(2)} + \Omega_{GW}^{(3)} (\rho^\text{tot} = \rho^{(2)} + \rho^{(3)} \), blue solid curve) and \( \Omega_{GW}^{(2)} (\rho^{(2)} \), orange dashed curve). We have set \( A_\zeta = 0.02 \). We also give \( \Delta \rho/\rho^{\text{tot}} = \rho^\zeta/\rho^{(2)} - 1 \) in the bottom panel.

FIG. 3. The SNR of LISA as a function of \( A_\zeta \) for \( \Omega_{GW}^{\text{tot}} = \Omega_{GW}^{(2)} + \Omega_{GW}^{(3)} (\rho^\text{tot} = \rho^{(2)} + \rho^{(3)} \), blue solid curve) and \( \Omega_{GW}^{(2)} (\rho^{(2)} \), orange dashed curve). We have set \( f_s = 1.3 \times 10^{-3} \). We also give \( \Delta \rho/\rho^{\text{tot}} = \rho^\zeta/\rho^{(2)} - 1 \) in the bottom panel.

for a given \( m_{pbh} \). Obviously, SNR and corresponding third order correction increase with the amplitude of the small-scale primordial power spectra \( A_\zeta \).

Since the effects of the third order SIGWs significantly affect the total energy density spectrum of SIGWs, it is necessary to compare the energy density of the third order SIGWs \( \rho_{GW}^{(3)} \) with the energy density of the second order SIGWs \( \rho_{GW}^{(2)} \). If we assume \( \rho_{GW}^{(3)} < \rho_{GW}^{(2)} \), we will obtain a critical value \( A_\ast \) of the amplitude of the small-scale primordial power spectra. The critical value \( A_\ast \) can be calculated by

\[
\int_0^{2k_s} \Omega_{GW}^{(2)}(\eta, k) \, d\ln k = \int_0^{3k_s} \Omega_{GW}^{(3)}(\eta, k) \, d\ln k .
\]

Solving Eq. (25), we obtain \( A_\ast = 1.76 \times 10^{-2} \). If \( A_\zeta > A_\ast \), then the energy density of third order SIGWs will be larger than the energy density of second order SIGWs. Note that this critical value \( A_\ast \) is just an upper limit, because \( \Omega_{GW}^{(3)}(\eta, k) \) not only comes from \( \langle h_k^{(3)} h_k^{(3)} \rangle \), but also comes from \( \langle h_k^{(4)} h_k^{(2)} \rangle \). The study of fourth order SIGWs is extremely difficult since one has to calculate all three kinds of third order perturbations firstly. In this paper, we do not consider the effect of \( \langle h_k^{(4)} h_k^{(2)} \rangle \). Predictably, it will lead to a larger SNR and a smaller critical value \( A_\ast \).

Conclusion and discussion—In this paper, we studied the third order SIGWs and calculated corresponding energy density spectrum. We found that the third order correction extend the cutoff frequency from 2\( f_s \) to 3\( f_s \). The relations between SNR of LISA and the monochromatic primordial power spectrum were studied systematically. For PBHs with mass range \( 4 \times 10^{-12} M_\odot \sim 10^{-7} M_\odot \), the effects of third order SIGWs lead to around 200% increase of the SNR. For a given \( f_s \), SNR and third order correction increase with \( A_\zeta \). We pointed that there exists a critical value \( A_\ast \), such that when \( A_\zeta > A_\ast \), the energy density of third order SIGWs will be larger than the energy density of second order SIGWs. We conclude that the third order SIGWs are dominated by the source term of second order scalar perturbation \( S^{(3)} = \phi^{(1)}(\psi^{(2)} \rangle \). The third order SIGWs from a general primordial power spectrum were not considered here because of the difficulties of numerical calculation. The complete study of higher order SIGWs might be presented in the future work.

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Appendix A: Expressions of the third order transfer functions

In this appendix, we briefly summarize the explicit expressions of $f^{(3)}_i (u, \bar{u}, \bar{v}, \bar{w}, \bar{x})$ \((i = 1, 2, 3, 4)\)

\[
f^{(3)}_1 (u, \bar{u}, \bar{v}, \bar{x}, y) = \frac{8}{27} \left( 12 T_{\phi}(ux)T_{\phi}(uy)T_{\phi}(vy) - 4ux \frac{d}{d(ux)} T_{\phi}(ux)T_{\phi}(uy)T_{\phi}(vy) \right.
\]
\[
- \frac{2u^2x^2}{3} T_{\phi}(ux)T_{\phi}(uy)T_{\phi}(vy) - 6\bar{u}uxy \frac{d}{d(ux)} T_{\phi}(ux) \frac{d}{d(uy)} T_{\phi}(uy)T_{\phi}(vy) 
\]
\[
- \frac{4u^2\bar{u}x^2y}{3} T_{\phi}(ux) \frac{d}{d(uy)} T_{\phi}(uy)T_{\phi}(vy) - 4\bar{u}vy^2 T_{\phi}(ux) \frac{d}{d(uy)} T_{\phi}(uy) \frac{d}{d(vy)} T_{\phi}(vy) 
\]
\[
- \frac{2\bar{u}\bar{v}vy^2 x}{3} \frac{d}{d(ux)} T_{\phi}(ux) \frac{d}{d(uy)} T_{\phi}(uy) \frac{d}{d(vy)} T_{\phi}(vy) 
\]
\[
- \frac{2u^2\bar{u}\bar{v}x^2y^2}{3} T_{\phi}(ux) \frac{d}{d(uy)} T_{\phi}(uy) \frac{d}{d(vy)} T_{\phi}(vy) \right),
\]
\[\tag{26}\]

\[
f^{(3)}_2 (u, \bar{u}, \bar{v}, \bar{x}, y) = \frac{8}{27} \left( -6 T_{\phi}(ux)T_{\phi}(uy)T_{\phi}(vy) - 4\bar{u}vy^2 T_{\phi}(ux) \frac{d}{d(uy)} T_{\phi}(uy)T_{\phi}(vy) 
\right.
\]
\[
- T_{\phi}(ux)p^2 I^{(2)}_0 (\bar{u}, \bar{v}, \bar{v}) \frac{d}{d(uy)} T_{\phi}(uy) \frac{d}{d(vy)} T_{\phi}(vy) \right)
\]
\[
+ \frac{u}{v^2} \frac{d}{d(ux)} T_{\phi}(ux)p^2 I^{(2)}_0 (\bar{u}, \bar{v}, \bar{v}) \frac{d}{d(uy)} T_{\phi}(uy)T_{\phi}(vy) + \frac{u}{v^2} \frac{d}{d(ux)} T_{\phi}(ux)p^2 I^{(2)}_0 (\bar{u}, \bar{v}, \bar{v}) \right),
\]
\[\tag{27}\]

\[
f^{(3)}_3 (u, \bar{u}, \bar{v}, \bar{x}, y) = \frac{8}{27} \left( \frac{u}{v} \frac{d}{d(ux)} T_{\phi}(ux)p^2 I^{(2)}_0 (\bar{u}, \bar{v}, \bar{v}) + \frac{y}{v} T_{\phi}(ux)p^2 I^{(2)}_0 (\bar{u}, \bar{v}, \bar{v}) \right.
\]
\[
+ \frac{xuy}{8} \frac{d}{d(ux)} T_{\phi}(ux)p^2 I^{(2)}_0 (\bar{u}, \bar{v}, \bar{v}) - 4T_{\phi}(ux)T_{\phi}(uy)T_{\phi}(vy) 
\right.
\]
\[
+ 8\bar{u}yT_{\phi}(ux) \frac{d}{d(uy)} T_{\phi}(uy)T_{\phi}(vy) + 16T_{\phi}(ux)T_{\phi}(uy)T_{\phi}(vy) 
\left. + 4\bar{u}\bar{v}y^2 T_{\phi}(ux) \frac{d}{d(uy)} T_{\phi}(uy) \frac{d}{d(vy)} T_{\phi}(vy) \right),
\]
\[\tag{28}\]

\[
f^{(3)}_4 (u, \bar{u}, \bar{v}, \eta, \eta) = \frac{8}{27} \left( y \left( T_{\phi}(ux) \frac{\partial}{\partial y} I^{(2)}_0 (\bar{u}, \bar{v}, \bar{v}) \right) + uxy \left( \frac{d}{d(ux)} T_{\phi}(ux) \frac{\partial}{\partial y} I^{(2)}_0 (\bar{u}, \bar{v}, \bar{v}) \right) 
\right.
\]
\[
+ ux \left( \frac{d}{d(ux)} T_{\phi}(ux)(I^{(2)}_0 (\bar{u}, \bar{v}, \bar{v}) + f^{(2)}_0 (\bar{u}, \bar{v}, \bar{v})) \right) 
\left. + 3 \left( T_{\phi}(ux)(I^{(2)}_0 (\bar{u}, \bar{v}, \bar{v}) + f^{(2)}_0 (\bar{u}, \bar{v}, \bar{v})) \right) \right).
\]
\[\tag{29}\]

Appendix B: Details of the power spectra of third order SIGWs

\[
X = \left( -1 + u^2 + v^2 - 2u^2v^2 + v^2 \bar{v}^2 + u^2v^2 \bar{v}^2 - v^4 \bar{v}^2 + w^2 - u^2 \bar{w}^2 + v^2 \bar{w}^2 \right)
\times (1 - 2u^2 + u^4 - 2u^2v^2 + v^4)(1 - 2v^2\bar{v}^2 + v^4 - 2u^2 - 2v^2 \bar{v}^2 \bar{w}^2 + w^4)^{-\frac{1}{2}},
\]
\[\tag{30}\]

\[
Y = (1 - 2u^2 + u^4 - 2u^2v^2 + v^4)(1 - 2u^2 \bar{v}^2 + v^4 - 2u^2 - 2v^2 \bar{v}^2 \bar{w}^2 + w^4),
\]
\[\tag{31}\]

\[
w_{\pm} = \frac{1}{2} + \frac{3}{2k^2} - \frac{1}{2}v^2 \pm \frac{1}{2v} \sqrt{(v^2 - \frac{4}{k^2}) \left( v^2 - \left( 1 - \frac{1}{k} \right)^2 \right) \left( v^2 - \left( 1 + \frac{1}{k} \right)^2 \right)^{\frac{1}{2}}},
\]
\[\tag{32}\]