The “tetrad only” theory space: Nonperturbative renormalization flow and Asymptotic Safety

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Abstract

We set up a nonperturbative gravitational coarse graining flow and the corresponding functional renormalization group equation on the as to yet unexplored “tetrad only” theory space. It comprises action functionals which depend on the tetrad field (along with the related background and ghost fields) and are invariant under the semi-direct product of spacetime diffeomorphisms and local Lorentz transformations. This theory space differs from that of Quantum Einstein Gravity (QEG) in that the tetrad rather than the metric constitutes the fundamental variable and because of the additional symmetry requirement of local Lorentz invariance. It also differs from “Quantum Einstein Cartan Gravity” (QECG) investigated recently since the spin connection is not an independent field variable now. We explicitly compute the renormalization group flow on this theory space within the tetrad version of the Einstein-Hilbert truncation. A detailed comparison with analog results in QEG and QECG is performed in order to assess the impact the choice of a fundamental field variable has on the renormalization behavior of the gravitational average action, and the possibility of an asymptotically safe infinite cutoff limit is investigated. Implications for nonperturbative studies of fermionic matter coupled to quantum gravity are also discussed. It turns out that, in the context of functional flow equations, the “hybrid calculations” proposed in the literature (using the tetrad for fermionic diagrams only, and the metric in all others) are unlikely to be quantitatively reliable. Moreover we find that, unlike in perturbation theory, the non-propagating Faddeev-Popov ghosts related to the local Lorentz transformations may not be discarded but rather contribute quite significantly to the beta functions of Newton’s constant and the cosmological constant.
1 Introduction

In classical General Relativity there exists a remarkably rich variety of different variational principles which give rise to Einstein’s equation, or equations equivalent to it but expressed in terms of different field variables. The best known examples are the Einstein-Hilbert action expressed in terms of the metric, \( S_{EH}[g_{\mu \nu}] \), or the tetrad, respectively, \( S_{EH}[e^a_\mu] \). The latter action functional is obtained by inserting the representation of the metric in terms of vielbeins into the former:

\[
g_{\mu \nu} = \eta_{ab} e^a_\mu e^b_\nu.
\]

Another classically equivalent formulation, at least in absence of spinning matter, is provided by the first order Hilbert-Palatini action \( S_{HP}[e^a_\mu, \omega^{ab}_\mu] \) which, besides the tetrad, depends on the spin connection \( \omega^{ab}_\mu \) assuming values in the Lie algebra of \( O(1,3) \). Variation of \( S_{HP} \) with respect to \( \omega^{ab}_\mu \) leads, in vacuo, to an equation of motion which expresses that this connection has vanishing torsion. It can be solved algebraically as \( \omega = \omega(e) \) which, when inserted into \( S_{HP} \), brings us back to \( S_{EH}[e] \equiv S_{HP}[e, \omega(e)] \).

Another equivalent formulation is based upon the self-dual Hilbert-Palatini action \( S_{sdHP}[e^a_\mu, \omega^{(+)}_{\mu}ab] \) which depends only on the (complex, in the Lorentzian case) self-dual projection of the spin connection, \( \omega^{(+)}_{\mu}ab \) [1–4]. This action in turn is closely related to the Plebanski action [5], containing additional 2-form fields, and to the Capovilla-Dell-Jacobson action [6] which involves essentially only a self-dual connection. Similarly, Krasnov’s diffeomorphism invariant Yang-Mills theories [7] allow for a “pure connection” reformulation of General Relativity as well as deformations thereof.

The above variational principles are Lagrangian in nature; the fields employed provide a parametrization of configuration space. The corresponding Legendre transformation yields a Hamiltonian description in which the “carrier fields” of the gravitational interaction parametrize a phase-space now. In this way the ADM-Hamiltonian [8] and Ashtekar’s Hamiltonian [9], for instance, make their appearance.

Regarding the ongoing search for a quantum (field) theory of gravity this multitude of classical formalisms offers many equally plausible possibilities to explore. A priori it is
not clear which one of the above hamiltonian systems, if any, is linked to the as to yet unknown fundamental quantum theory in the simplest or most easy to guess way.

In the traditional approach of “quantizing” a known classical system more input than the field equations, such as the Lagrangian is needed, so that classically equivalent theories might possibly give rise to inequivalent quantum theories. Among those, at most one can be “correct”, in the sense of being realized in Nature. Of course, given the limitations of the available observational and experimental data it is not clear whether the most natural and/or simplest presentation of the classical limit emerging from this “correct” theory is in the above list, or even close to it. Up to now, because of the many well known conceptual and technical problems \[4\] we are not in a position to discriminate the various classical gravity theories on the basis of the quantum properties they imply.

Rather than trying to quantize a given classical dynamical system, there is another strategy one can adopt in order to search for a quantum theory consistent with the observed classical limit, the Asymptotic Safety program \[10\]–\[14\]. One of its advantages as compared to a “quantization” is that it depends on the classical input data to a lesser extent. The idea is to fix a certain theory space of action functionals, a coarse graining flow of it, and then search for a renormalization group (RG) fixed point (FP) on it at which the infinite ultraviolet (UV) cutoff limit can be taken in a “safe” way.

While originally motivated by the possibility of sidestepping the problem of perturbative nonrenormalizability, this search strategy in principle can predict the theory’s fundamental action. The only input needed is theory space. Once it is chosen one can “turn the crank” and, in case a suitable fixed point is found, construct a UV-regularized functional integral representation of the resulting theory \[15\]. Only at this very last stage we can identify the hamiltonian system which, implicitly, was quantized by taking the continuum limit at the respective fixed point.

To characterize a theory space \( \mathcal{T} \) we must pick a certain set of fields, collectively denoted \( \Phi \), a space of action functionals \( A[\Phi] \), and a group \( \mathbf{G} \) of symmetry transformations under which they are required to be invariant. In this sense, the above classical gravity
theories motivate us to explore, for instance, the case where $\Phi \equiv g_{\mu\nu}$ is the metric and $G$ the diffeomorphism group, or, as in Einstein-Cartan theory, $\Phi \equiv (e^a_\mu, \omega^{ab}_\mu)$ where $G$ is the semidirect product of local Lorentz transformations and spacetime diffeomorphisms.

We emphasize that these spaces and symmetries are the only “inspiration” drawn from the classical examples. Their dynamics, i.e. the specific classical action they postulate, plays no special rôle in the Asymptotic Safety program. It is just one special point in the pertinent theory space, and usually not the sought for fixed point of the RG flow.

Most of the work on Asymptotic Safety has been done in “Einstein” gravity which, by definition, is based upon a theory space $\mathcal{T}_E$ of functionals $A[g_{\mu\nu}, \cdots]$ invariant under $G = \text{Diff}(\mathcal{M})$, the diffeomorphisms of the spacetime manifold $\mathcal{M}$. Recently also first investigations of the “Einstein-Cartan” choice

$$\mathcal{T}_{EC} = \{A[e^a_\mu, \omega^{ab}_\mu, \cdots]; G = \text{Diff}(\mathcal{M}) \ltimes O(4)_{\text{loc}}\}$$

(1.1)

were published [16]. Here $O(4)$ plays the rôle of the Euclidean Lorentz group and $O(4)_{\text{loc}}$ denotes the group of the corresponding local gauge transformations.

The present paper instead is devoted to the “tetrad only” theory space pertaining to a $d$ dimensional spacetime $\mathcal{M}$:

$$\mathcal{T}_{tet} = \{A[e^a_\mu, \cdots]; G = \text{Diff}(\mathcal{M}) \ltimes O(d)_{\text{loc}}\}.$$  

(1.2)

With actions depending on the vielbein only, this space is intermediate between $\mathcal{T}_E$ and $\mathcal{T}_{EC}$: Coming from the “Einstein” side it generalizes $\mathcal{T}_E$ by declaring $e^a_\mu$ the fundamental field and the metric a composite thereof, $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$. Conversely, coming from the “Einstein-Cartan” side, every $A[e, \omega, \cdots] \in \mathcal{T}_{EC}$ implies a certain $A[e, \cdots] \in \mathcal{T}_{tet}$ upon inserting $\omega = \omega_{\text{LC}}(e)$, where $\omega_{\text{LC}}(e)$ is the torsion-free (Levi-Civita) connection the vielbein gives rise to.

There are various independent motivations for this investigation.

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1The dots stand for the background fields and Faddeev-Popov ghosts to be introduced later.
2We shall consider the case of Euclidean signature throughout.
(A) The first functional RG based results obtained on the Einstein-Cartan theory space $\mathcal{T}_{EC}$, in a truncation with a scale dependent Hilbert-Palatini action (including a running Immirzi term), show certain characteristic differences in comparison with the familiar case of $\mathcal{T}_E$ truncated with a running Einstein-Hilbert action; in particular, the $\mathcal{T}_{EC}$ results show a stronger RG scheme and gauge fixing dependence than the older ones on the “Einstein” case [16]. It would be interesting to know whether these differences are mainly due to the use of the different truncations, different field variables, or both. In the present paper we shall change only the field variable (and the group $G$ correspondingly), but not the truncation, and so it should be possible to disentangle the two sources of deviations to some extent.

We note here that, like most settings of quantum field theory, the flow equation of the average action is not invariant under diffeomorphisms in field space, $\Phi \mapsto \Phi'(\Phi)$. Thus, at intermediate steps, as long as one does not compute observables, there is no reason to expect any field parametrization independence. Moreover, and perhaps this is even more important, the gauge fixing and ghost sectors are quite different for $\text{Diff}(\mathcal{M})$ and $\text{Diff}(\mathcal{M}) \ltimes \text{O}(d)_{\text{loc}}$, respectively. Therefore the $\beta$-functions for the running Newton constant $G_k$ or cosmological constant $\bar{\lambda}_k$, for instance, may well depend on whether the functional renormalization group equation (FRGE) is formulated in terms of the metric or tetrad. Similar remarks apply also to a recent study of the perturbative RG running of $G_k$ and the Immirzi parameter [17].

(B) On theory spaces involving fermions coupled to gravity introducing vielbeins is compulsory. Besides the pure gravity couplings, such as $G_k$, $\bar{\lambda}_k$, etc. the average action will then depend on additional couplings related to the matter field monomials. If we collectively denote these couplings by $u_{\text{grav}}$ and $u_{\text{mat}}$, respectively, their $\beta$-functions are of the form

$$\beta_{\text{grav}} = \beta_{\text{grav}}^{\text{grav}}(u_{\text{grav}}) + \beta_{\text{grav}}^{\text{grav}}(u_{\text{grav}}, u_{\text{mat}}) \quad (1.3)$$

$$\beta_{\text{mat}} = \beta_{\text{mat}}^{\text{mat}}(u_{\text{mat}}) + \beta_{\text{mat}}^{\text{grav}}(u_{\text{grav}}, u_{\text{mat}}) \quad (1.4)$$
Diagrammatically speaking, the two parts $\beta_{\text{grav}}$ and $\beta_{\text{mat}}$ of the pure gravity $\beta$-functions stem from the graviton and matter loops, respectively. Conversely, the running of the matter couplings has a part due to pure matter loops, $\beta_{\text{mat}}$, plus mixed matter-gravity contributions, $\beta_{\text{grav}}^{\text{mat}}$.

In order to get a first impression of the impact the fermions have on the gravitational RG flow one might neglect the running of the matter couplings, and try to compute $\beta_{\text{grav}}$ only. While the evaluation of $\beta_{\text{mat}}$ from the fermion loops clearly requires a vielbein and a spin connection, the pure gravity part $\beta_{\text{grav}}$ does not obviously do so. From a pragmatic point of view it is therefore tempting to take the $\beta_{\text{grav}}$ part from a (much simpler, and already available) computation in the metric formalism. The invariants $I[g_{\mu\nu}]$ occurring in the latter one would interpret as $I[e] \equiv I[g_{\mu\nu} = \eta_{ab} e^{a}_\mu e^{b}_\nu]$. For some (but not all) field monomials in $\mathcal{T}_{\text{tet}}$ this establishes a correspondence to monomials in $\mathcal{T}_E$, and one can try to identify their running prefactors; for instance, $\bar{\lambda}_k \int d^d x \sqrt{g} \in \mathcal{T}_E \leftrightarrow \bar{\lambda}_k \int d^d x \ e \in \mathcal{T}_{\text{tet}}$, when $g$ and $e$ denote the determinants of $g_{\mu\nu}$ and $e^{a}_\mu$, respectively.

Thus it seems that only the fermion loops, $\beta_{\text{grav}}^{\text{mat}}$, need to be calculated. This requires fixing a Lorentz gauge in order to associate a unique $e^{a}_\mu$ to a given $g_{\mu\nu}$, and for $\omega^{ab}_\mu$ one might take the unique Levi-Civita connection associated to this vielbein, $\omega_{\text{LC}}^{ab}_\mu(e)$.

We shall refer to this procedure as a hybrid calculation. Clearly it can be meaningful at most within a truncation of $\mathcal{T}_E$ and $\mathcal{T}_{\text{tet}}$ that allows an identification of monomials; an example is the Einstein-Hilbert action regarded as a functional of $g_{\mu\nu}$ and $e^{a}_\mu$, respectively, with the same two couplings $G_k$ and $\bar{\lambda}_k$ occurring in both cases. At the exact level there exists certainly no such one-to-one correspondence between action monomials in $\mathcal{T}_E$ and $\mathcal{T}_{\text{tet}}$. Nevertheless, if it was possible to establish the “hybrid” scheme as a reliable approximation, this would be of considerable importance for the feasibility of practical calculations.

As to yet, all investigations for the gravity + fermions theory space, in particular in the Asymptotic Safety context, are, in fact, hybrid computations of this form \cite{18,20}. They combine the metric-formalism $\beta$-functions for $G_k$ and $\bar{\lambda}_k$ in the Einstein-Hilbert
truncation with certain matter contributions $\beta^\text{mat}_{\text{grav}}$, solve for $u_{\text{grav}}(k)$, and insert the result into (1.4) to obtain the running of the matter couplings. In ref. [20] the gravity corrections to certain 4-fermion couplings $u_{\text{mat}}$ were studied in this way.

A necessary condition for the consistency of the hybrid approach is that the pure gravity part $\beta^\text{grav}_{\text{grav}}$ does not change much when we switch from $g_{\mu\nu}$ to $e^a_{\mu}$ as the fundamental field variable in the Einstein-Hilbert truncation. In the present paper we shall be able to explicitly test whether or not this is actually the case. It will be one of our main results that the hybrid scheme is very hard, if not impossible to justify, at least at the quantitative level. We shall demonstrate in detail that if one aims at some degree of numerical precision, one should consistently work with the vielbein and its corresponding ghost system already at the pure gravity level.

(C) Picking the vielbein as the fundamental field variable requires fixing a $O(d)_{\text{loc}}$ gauge. In perturbation theory, a popular choice is the Deser-van Nieuwenhuizen algebraic gauge fixing condition where the antisymmetric part of the $d \times d$ matrix $e^a_{\mu}$ is required to vanish [21, 22]. As $O(d)$ has $\frac{1}{2}d(d-1)$ parameters, this reduces the $d^2$ independent components of $e^a_{\mu}$ to $\frac{1}{2}d(d+1)$, which is precisely the number of independent fields $g_{\mu\nu}$ has in $d$ dimensions.

It was shown that, for this gauge, and in perturbation theory, no Faddeev-Popov ghosts need to be introduced for the $O(d)_{\text{loc}}$ factor of $G$, and that it allows to explicitly express vielbein fluctuations purely in terms of metric fluctuations [23]. Therefore the point of view was advocated that even in presence of fermions the vielbein can be eliminated in favor of the metric.

While this method was proven to be correct in a well defined perturbative context, recently it has been proposed to use this same procedure, in particular the omission of the $O(d)_{\text{loc}}$ ghosts, also in the context of a nonperturbative flow equation for the gravity–fermion system [19, 20]. If applicable, it would provide a very economic framework for hybrid computations of the type sketched above.
However, as we are going to discuss in detail there are reasons to doubt that the perturbative arguments justifying the omission of the $O(d)_{\text{loc}}$ ghosts carry over to the nonperturbative setting of the FRGE. In fact, in perturbation theory the ghosts are omitted since their inverse propagator contains no derivatives, they are non-propagating, leading to a trivial Faddeev-Popov determinant. In the FRGE, instead, a straightforward evaluation of the functional traces cuts off all field modes in a uniform fashion, no matter if their kinetic term contains 2, or more, or no derivatives at all.

In the present paper we shall explicitly evaluate the contributions to $\beta_{\text{grav}}$ from the non-propagating $O(d)_{\text{loc}}$ ghosts pertaining to the symmetric vielbein gauge, and we shall analyze whether they really can be discarded in setting up the flow equation for the average action.

Fortunately, the particular fermionic $\beta$-functions computed in [20] happen to be independent on whether the $O(d)_{\text{loc}}$ ghosts are retained or not. However, in future extensions of such studies it will be important to know how to treat them correctly.

The remaining sections of this article are organized as follows. In Section 2 we summarize various preliminaries on the gravitational average action and its FRGE which will be needed later on. In Section 3 we focus on the “tetrad only” theory space $T_{\text{tet}}$ in the Einstein-Hilbert truncation, and calculate the corresponding $\beta$-functions. The resulting RG flow is analyzed with numerical methods in Section 4 then. Our results, in particular on the issues (A)-(C) raised above, are summarized in Section 5.

2 The average action approach to quantum gravity

Introducing the scale-dependent effective average action $\Gamma_k$ it has been possible to construct a functional RG flow for quantum gravity [11]. This “running action” can be considered the generating functional of the 1PI correlation functions that take into account quantum fluctuation of all scales between the UV and an infrared cutoff scale $k$. For $k \to \infty$ it is closely related to the bare action $S$ and for vanishing cutoff it coincides
with the usual effective action $\Gamma = \Gamma_{k=0}$. Its scale-dependence is governed by an exact renormalization group equation:

$$\partial_t \Gamma_k = \frac{1}{2} \text{STr} \left[ \frac{\partial \hat{R}_k}{\Gamma^{(2)}_k + \hat{R}_k} \right].$$

Here $t \equiv \ln k$, and $\Gamma^{(2)}_k$ denotes the matrix of the second functional derivative of $\Gamma_k$ with respect to the dynamical fields. Furthermore, $\hat{R}_k$ is an operator that implements the infrared cutoff in the path integral by replacing the bare action $S$ with $S + \Delta_k S$ where $\Delta_k S$ is quadratic in the fluctuations, $\Delta_k S \propto \int \varphi \hat{R}_k \varphi$. Finally, the supertrace in (2.1) comprises a trace over all internal indices as well as an integral/sum over all modes of $\varphi$; for fermionic fields it contains an additional minus sign [24].

As $\Gamma_k$ is a generic point in “theory space”, i.e. a functional of a given set of fields restricted only by the required symmetries, solving this exact equation is usually a formidable task. For this reason one has to resort to truncations of theory space in order to find approximate solutions to eq. (2.1). This is done by expanding $\Gamma_k$ in a basis of integrated field monomials $I_\alpha$, i.e. $\Gamma_k[\cdot] = \sum \alpha c_\alpha(k) I_\alpha[\cdot]$ and restricting the sum to a finite number of terms. The scale-dependence of $\Gamma_k$ is then described by a finite number of running couplings $c_\alpha(k)$. If we project the RHS of (2.1) onto this subspace of theory space the functional equation reduces to a coupled system of ordinary differential equations in these couplings.

If we describe pure gravity with the metric as field variable, the simplest truncation is the Einstein-Hilbert truncation with only two running couplings: Newton's constant $G_k$ and the cosmological constant $\bar{\lambda}_k$. As gravity is a gauge theory we also have to add a gauge fixing and a ghost term to the truncation ansatz; its running shall be ignored in our approximation. Our ansatz for $\Gamma_k$ can therefore be decomposed into a “bosonic part” $\bar{\Gamma}_k$ and the classical ghost contribution $S_{gh}$:

$$\Gamma_k = \Gamma_k^{EH} + \Gamma_{gf} + S_{gh} \equiv \bar{\Gamma}_k + S_{gh}$$

(2.2)
Using this decomposition the FRGE \((2.1)\) can be written in the following form:

\[
\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \left[ \frac{\partial_t \hat{R}_k}{\Gamma_k^{(2)} + \hat{R}_k} \right] - \text{Tr} \left[ \frac{\partial_t \hat{R}_{k\text{gh}}}{S_{k\text{gh}}^{(2)} + \hat{R}_{k\text{gh}}} \right]. \tag{2.3}
\]

The gravitational average action heavily relies on the background field method \([25]\). The field chosen to represent gravity is split arbitrarily into a background part and a fluctuation: \(\phi = \bar{\phi} + \phi\). \(\Gamma_k\) is constructed as a background gauge invariant functional of both fields, \(\Gamma_k[\phi, \bar{\phi}]\), i.e. it is invariant under a simultaneous action of \(G\) on both \(\phi\) and \(\bar{\phi}\). As we only deal with a so-called single metric truncation \([26]\) in this paper, we will set the fluctuations to zero after the second derivative \(\Gamma^{(2)}\) with respect to the fluctuations has been taken. At the end we therefore arrive at a system of differential equations for the running couplings parametrizing \(\Gamma_k[\bar{\phi}] = \Gamma_k[\bar{\phi}, \bar{\phi}]\).

For example, in the “tetrad only” case the average action is a curve \(k \mapsto \Gamma_k\) in the theory space \(\mathcal{T}_{\text{tet}}\) which, to be precise now, consists of \(\text{Diff}(\mathcal{M}) \times O(d)_{\text{loc}}\) invariant functionals of the type \(A[e^a_\mu, \bar{e}^a_\mu, C^\mu, \bar{C}_\mu, \Sigma^{ab}, \bar{\Sigma}_{ab}]\); besides the vielbein and its background, they depend on the diffeomorphism ghosts \((C^\mu, \bar{C}_\mu)\) and \(O(d)_{\text{loc}}\) ghosts \((\Sigma^{ab}, \bar{\Sigma}_{ab})\). Instead of \(e^a_\mu\) we shall often consider the vielbein fluctuation \(\varepsilon^a_\mu \equiv e^a_\mu - \bar{e}^a_\mu\), the independent argument of the action.

## 3 Tetrad theory space in Einstein-Hilbert truncation

At this point we take two decisions. One of them refers to the deeper level of the exact theory, the other to the practical (computational) level of concrete approximations.

First, we fix the theory space to be the “tetrad only” one, \(\mathcal{T}_{\text{tet}}\), so that all actions to be considered depend only on \(e^a_\mu\), along with the corresponding background and ghost fields.

Second, to be able to perform practical calculations we decide to truncate \(\mathcal{T}_{\text{tet}}\) by an ansatz for \(\Gamma_k\) which is essentially a \(k\)-dependent version of the Einstein-Hilbert action reexpressed in terms of the tetrad, \(S_{\text{EH}}[g(e)]\).
3.1 The FRGE on $\mathcal{T}_{\text{tet}}$

In this subsection we derive the RG flow of tetrad gravity in the Einstein-Hilbert truncation

$$\tilde{\Gamma}_k[e, \bar{e}] = -\frac{1}{16\pi G_k} \int d^d x \sqrt{g(e)} \left( R(g(e)) - 2\bar{\lambda}_k \right) + \Gamma_{gf}[e, \bar{e}]. \tag{3.1}$$

This action involves two running couplings, the cosmological constant $\bar{\lambda}_k$ and Newton’s constant $G_k$; the latter is frequently expressed in terms of the dimensionless function $Z_{Nk}$ according to $G_k \equiv Z_{Nk}^{-1} \bar{G}$ with a constant $\bar{G}$.

To be as general as possible we re-express the metric in terms of the new field variable $e^a_{\mu}$ in the following way:

$$g_{\mu\nu} = \xi^{-1} e^a_{\mu} e^b_{\nu} \eta_{ab}. \tag{3.2}$$

This representation resembles the usual vielbein decomposition of the metric, except for the additional free parameter $\xi > 0$. For this reason we will refer to the field $e^a_{\mu}$ as a generalized vielbein for a given $g_{\mu\nu}$. Treating $e^a_{\mu}$ as the independent variable we assume that the basis 1-forms $e^a = e^a_{\mu} dx^\mu$ indeed form a non-degenerate co-frame. The parameter $\xi$ is merely a mathematical tool that enables us to study a continuous class of field redefinitions at a time.

As for the usual vielbein this generalized decomposition of the metric is not unique, but there exists an $O(d)$ manifold of vielbein fields corresponding to the same metric. We will treat this arbitrariness as an additional gauge freedom, such that the total group of gauge transformations is given by $G = \text{Diff}(\mathcal{M}) \ltimes O(d)_{\text{loc}}$. Compared to the metric formulation we therefore have to add a second gauge fixing term; the corresponding background gauge invariant ghost-action can be constructed using the formalism introduced in [27].

If we decompose both the metric $g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \bar{h}_{\mu\nu}$ and the vielbein $e^a_{\mu} \equiv \bar{e}^a_{\mu} + \bar{\varepsilon}^a_{\mu}$ into background fields and fluctuations, we find

$$\bar{g}_{\mu\nu} + \bar{h}_{\mu\nu} = g_{\mu\nu} = \xi^{-1} \bar{e}^a_{\mu} \bar{e}^b_{\nu} \eta_{ab} = \xi^{-1} e^a_{\mu} e^b_{\nu} \eta_{ab} + 2\xi^{-1} \bar{\varepsilon}_{(\mu\nu)} + \mathcal{O}(\bar{\varepsilon}^2). \tag{3.3}$$
Here and in the following we use the background vielbein $\tilde{e}^a_\mu$ to change the type of the first (i.e., frame) index of the vielbein fluctuation: $\bar{\epsilon}_{\mu
u} = \eta_{ab}e^a_\mu \tilde{\epsilon}^b_\nu$. We see that the symmetric part of the vielbein fluctuations, $\bar{\epsilon}_{(\mu\nu)}$, is proportional to the metric fluctuations $\bar{h}_{\mu\nu}$ in lowest order, while we can relate the additional $d(d-1)/2$ gauge degrees of freedom carried by $e^a_\mu$ to the antisymmetric part of the fluctuations, $\bar{\epsilon}_{[\mu\nu]}$.

This observation motivates the following choice of gauge conditions. For the diffeomorphisms we choose the usual harmonic gauge fixing function for metric fluctuations, replacing $\bar{h}_{\mu\nu} \rightarrow 2\xi^{-1}\bar{\epsilon}_{(\mu\nu)}$, with $\kappa \equiv (32\pi \tilde{G})^{-1/2}$:

$$F_\mu = 2\sqrt{2}\kappa \xi^{-1}\left(\bar{D}^\nu \bar{\epsilon}_{(\mu\nu)} - \frac{1}{2} \bar{D}_\mu \bar{\epsilon}_\nu\right).$$  \hspace{1cm} (3.4)

The $O(d)$ transformations are gauge fixed using

$$G^{ab} = 2\xi^{-\frac{1}{2}} \bar{g}^{\mu\nu} \bar{\epsilon}_{(\mu}^{[a} \bar{e}^{b]}_\nu} = 2\xi^{-\frac{1}{2}} \bar{\epsilon}^{[ab]},$$ \hspace{1cm} (3.5)

corresponding to a suppression of the antisymmetric vielbein fluctuations.

With these gauge conditions the gauge fixing term in the effective average action assumes the usual form, involving parameters $\alpha_D$ and $\alpha_L$:

$$\Gamma_{gf,k}[\bar{e}, \bar{\epsilon}] = \frac{1}{2\alpha_D} \int d^d x \sqrt{\bar{g}} \bar{g}^{\mu\nu} F_\mu F_\nu + \frac{1}{2\alpha_L} \int d^d x \sqrt{\bar{g}} G^{ab} G_{ab}. \hspace{1cm} (3.6)$$

In the following we fix the diffeomorphism gauge parameter $\alpha_D$ to $\alpha_D = 1/Z_{Nk}$ which leads to the same cancelation in the kinetic operator as in metric gravity [11].

In order to obtain a background $G$-invariant ghost action with respect to both $O(d)_{loc}$ transformations and diffeomorphisms, we can make use of the Faddeev-Popov construction only if we first reparametrize the gauge transformations in such a way, that the new generators of diffeomorphisms and $O(d)$ transformations commute. This corresponds to an $O(d)$ covariantization of the Lie derivative. Following this procedure, described in detail in [16,27], while treating the ghost sector classically (i.e. we can set $e = \bar{e}$ already at the level of the ghost action) we arrive at

$$S_{gh}[C, \bar{C}, C', \Sigma'; \bar{\epsilon}] = -\int d^d x \bar{e} \begin{pmatrix} \bar{C}_\mu \\ \bar{\Sigma}^{\mu\nu} \end{pmatrix}^T \begin{pmatrix} \sqrt{2\xi^{-1}(\delta_\mu \bar{D}^2 + \bar{R}_\mu)} & 0 \\ 2\xi^{-\frac{1}{2}} \bar{\mu} \delta_\mu \bar{D}_\nu & 2\xi^{-\frac{1}{2}} \bar{\mu} \delta_\mu \delta^{[\mu} \delta^{\nu]} \end{pmatrix} \begin{pmatrix} C^\mu \\ \Sigma^{\mu\nu} \end{pmatrix}.$$ \hspace{1cm} (3.7)
Here $\tilde{C}^\mu$, $C^\mu$ represent the diffeomorphism ghosts and $\tilde{\Sigma}^{\mu\nu}$, $\Sigma^{\mu\nu}$ the $O(d)$ ghost fields.

As the infinitesimal transformation under diffeomorphisms contains a derivative, while the corresponding $O(d)$ transformation does not, the diffeomorphism ghosts have a canonical mass dimension of one unit less compared to the $O(d)$ ghosts. In order to obtain a Hessian operator of a well-defined mass dimension we have rescaled the fields $\tilde{\Sigma}^{\mu\nu} = \tilde{\mu}\tilde{\Sigma}'^{\mu\nu}$, $\Sigma^{\mu\nu} = \tilde{\mu}\Sigma'^{\mu\nu}$ with an arbitrary mass parameter $\tilde{\mu}$; consequently the Hessian operator obtains a mass dimension of 2.

3.2 Structure of the vielbein sector

After having presented the details of our truncation we can now pass on to the evaluation of the FRGE (2.3) in this truncation. On the LHS of the equation, after setting $\tilde{\epsilon} = \epsilon$, we obtain the same result as in the metric version of the Einstein-Hilbert truncation [11]:

$$\partial_t \Gamma_k[e,e] = 2\kappa^2 \int d^d x \sqrt{g(e)} \left[ - R(g(e)) \partial_t Z_{Nk} + 2\partial_t \left( Z_{Nk} \tilde{\lambda}_k \right) \right]$$  \hspace{1cm} (3.8)

On the RHS of the FRGE, however, we find two types of additional contributions to the supertrace as compared to those already present in the metric description. While the second type of contributions is due to the extended gauge group of the theory, the first type is closely linked to the off-shell character of the FRGE. This can be seen as follows.

In order to obtain $\check{\Gamma}^{(2)}$ we expand $\check{\Gamma}_k$ to second order in the vielbein fluctuations and read off the operator from the quadratic term $\check{\Gamma}_k^{\text{quad}}$. As $\Gamma_{gf}$ is already quadratic in the fluctuations we only have to expand $\Gamma_{EH,k}$. For

$$\Gamma_{EH}^{\text{quad}} = \frac{1}{2} \delta_\epsilon^2 \Gamma_{EH}|_{\epsilon=\tilde{\epsilon}}$$  \hspace{1cm} (3.9)

we find

$$\Gamma_{EH}^{\text{quad}} = \frac{1}{2} \cdot \frac{4}{\xi^2} \int d^d x_1 d^d x_2 \left. \frac{\delta^2 \Gamma_{EH}}{\delta g_{\rho\sigma}(x_2) \delta g_{\mu\nu}(x_1)} \right|_{g=\tilde{g}} \tilde{\epsilon}_{(\mu\nu)}(x_1) \tilde{\epsilon}_{(\rho\sigma)}(x_2)$$

$$+ \frac{1}{2} \cdot \frac{2}{\xi} \int d^d x \left. \frac{\delta \Gamma_{EH}}{\delta g_{\mu\nu}(x)} \right|_{g=\tilde{g}} \tilde{\epsilon}_{a(\nu}(x) \tilde{\epsilon}_{a)\mu}(x)$$  \hspace{1cm} (3.10)
Here we have used the chain rule for functional derivatives. Obviously, the first term on the RHS of (3.10) corresponds exactly to the one known from the metric calculation, while the second term is due to the field redefinition. We note that those two terms come with different powers of $\xi$, which enables us to keep track of their respective origin during the entire calculation and in the final result. This was in fact our main motivation for introducing this book-keeping device.

Note also that in (3.10) the term due to the field redefinition is proportional to the first variation $\delta \Gamma_{\text{EH}}/\delta g_{\mu\nu}$. So it would vanish if we were to go “on shell”, i.e. to insert a special metric or vielbein which happens to be a stationary point of $\Gamma_{\text{EH}}$. We emphasize that in the process of computing $\beta$-functions this would be a severe mistake. To see this, consider an (exact) average action expanded as

$$\Gamma_k[\phi, \bar{\phi}] = \sum_{\alpha} c_\alpha(k) I_\alpha[\phi, \bar{\phi}], \quad (3.11)$$

where $c_\alpha(k)$ denote the running couplings and the $I_\alpha$’s are $G$-invariant basis functionals (integrated field monomials, say) independent of $k$. When represented in this fashion one may think of $\Gamma_k$ as a “generating function” for the set of running couplings, $\{c_\alpha(k)\}$, which are “projected out” by expanding $\Gamma_k$ in the basis $\{I_\alpha[\cdot, \cdot]\}$. In this picture the fields $\phi, \bar{\phi}$ have a subordinate status only. They serve as arguments of the $I_\alpha$’s, and their only rôle is that of a dummy variable needed in order to define the basis functionals $I_\alpha$. Therefore, in order for the set $\{I_\alpha\}$ to remain complete it is in general not possible to narrow down the function space $\phi, \bar{\phi}$ are drawn from in any way, for instance by stationary point conditions or the like. In this sense, the average action and its associated FRGE are intrinsically “off shell” in nature.

At most at the level of truncations where the set $\{I_\alpha\}$ is incomplete anyhow we may opt for special choices of the fields (e.g. satisfying convenient symmetry conditions) as long as the invariants in the truncation ansatz when calculated for these fields can still be distinguished from all other invariants and from each other. This is an often used
computational trick that simplifies practical calculations without affecting the result in any way.

For the total quadratic part of the action \( \bar{\Gamma}_k \) we obtain, with \( \sqrt{\bar{g}} \equiv \bar{e} \),

\[
\bar{\Gamma}_k^{\text{quad}} \left[ \bar{e}; \bar{e} \right] = \frac{4 Z_{N_k} K^2}{\xi^2} \int d^d x \sqrt{\bar{g}} \bar{\epsilon}_{(\mu \nu)} \left[ - K^\rho_{\rho \sigma} \bar{D}^2 + U^\mu_{\rho \sigma} \right] \bar{\epsilon}^{(\rho \sigma)} \tag{3.12}
\]

\[
+ \frac{2 Z_{N_k} K^2}{\xi} \int d^d x \sqrt{\bar{g}} \left( \bar{R}^\mu_{\nu} + \Lambda \bar{g}^\mu_{\nu} - \frac{\bar{R}}{2} \bar{g}^\mu_{\nu} \right) \bar{\epsilon}_{a(\mu} \bar{\epsilon}^a_{\nu)} + \frac{1}{2 \alpha L} \frac{4}{\xi} \int d^d x \sqrt{\bar{g}} \bar{\epsilon}^{[ab]} \bar{\epsilon}_{[ab]}
\]

where

\[
K^\mu_{\rho \sigma} \equiv \frac{1}{4} \left( \delta^\mu_{\rho} \delta^\nu_{\sigma} + \delta^\mu_{\sigma} \delta^\nu_{\rho} - \bar{g}^\mu_{\nu} \bar{g}^\rho_{\sigma} \right) \tag{3.13}
\]

and

\[
U^\mu_{\rho \sigma} \equiv \frac{1}{4} \left[ \delta^\rho_{\mu} \delta^\nu_{\sigma} + \delta^\rho_{\sigma} \delta^\nu_{\mu} - \bar{g}^\mu_{\nu} \bar{g}^\rho_{\sigma} \right] \left( \bar{R} - 2 \bar{\lambda}_k \right) + \frac{1}{2} \left[ \bar{g}^\mu_{\nu} \bar{R}_{\rho \sigma} + \bar{g}_{\rho \sigma} \bar{R}^\mu_{\nu} \right] - \delta^\mu_{(\rho} \bar{R}^\nu_{\sigma)} - \bar{R}^\nu_{(\rho} \bar{R}^\nu_{\sigma)} \tag{3.14}
\]

We observe that the first term on the RHS of (3.12) is exactly the contribution known from the metric computation \[11\]; in particular thanks to \( \alpha_D = 1/Z_{N_k} \) all non-minimal terms in the differential operator canceled. The second and third terms in (3.12) correspond to the already mentioned first and second type of new contributions, respectively.

In a next step we decompose the vielbein fluctuations \( \bar{\epsilon}_{\mu \nu} \) into their symmetric traceless part \( \bar{\epsilon}_{\mu \nu} \), antisymmetric part \( \bar{\epsilon}_{\mu \nu} \), and trace part \( \phi \), according to

\[
\bar{\epsilon}_{\mu \nu} = \hat{\bar{\epsilon}}_{\mu \nu} + \tilde{\bar{\epsilon}}_{\mu \nu} + \frac{1}{d} \bar{g}_{\mu \nu} \phi
\]

with \( \bar{\epsilon}_{\mu \nu} = \hat{\bar{\epsilon}}_{\mu \nu} \), \( \hat{\bar{\epsilon}}_{\mu \nu} = 0 \) and \( \bar{\epsilon}_{\mu \nu} = \bar{\epsilon}_{[\mu \nu]} \). In addition we specify the background spacetime to be a maximally symmetric Einstein space with

\[
\bar{R}_{\mu \nu \rho \sigma} = \frac{1}{d(d-1)} \left[ g_{\mu \rho} g_{\nu \sigma} - g_{\mu \sigma} g_{\nu \rho} \right] \bar{R} \quad \text{and} \quad \bar{R} = \frac{1}{d} \bar{g}_{\mu \nu} \bar{R}.
\]

This spacetime is still sufficiently general to identify the contributions to the relevant invariants \( \int \sqrt{\bar{g}} \) and \( \int \sqrt{g} \bar{R} \) unambiguously. Within the present truncation it is thus a permissable restriction of the function space of the metric; it does not affect the generality of the calculation and so is an example of the computational trick mentioned above.
Using the relations (3.15) and (3.16) the quadratic part of the action reads

\[ \tilde{\Gamma}^{\text{quad}}[\bar{\varepsilon}; \bar{e}] = \frac{Z_{Nk}\kappa^2}{2} \int d^4x \sqrt{g} \left\{ \bar{\varepsilon}_{\mu\nu} \left[ -\bar{D}^2 + (\xi - 2)\bar{\lambda}_k + C_T(\xi)\bar{R} \right] \bar{\varepsilon}^{\mu\nu} 
\right. \\
+ \bar{\varepsilon}_{\mu\nu} \xi \left[ \frac{1}{Z_{Nk}\kappa^2\alpha_L} + \bar{\lambda}_k - \frac{d - 2}{2d} \bar{R} \right] \bar{\varepsilon}^{\mu\nu} \\
\left. - \frac{d - 2}{2d} \phi_e \left[ -\bar{D}^2 - \left( 2 + \frac{2\xi}{d} \right) \bar{\lambda}_k + C_S(\xi)\bar{R} \right] \phi_e \right\} \] (3.17)

with the constants

\[ C_T(\xi) \equiv \frac{d(d - 3) + 4}{d(d - 1)} - \frac{d - 2}{2d} \xi, \quad C_S(\xi) \equiv \frac{d - 4 + \xi}{d}. \] (3.18)

Note that whereas the symmetric tensor \( \hat{\varepsilon}_{\mu\nu} \) has a standard positive definite kinetic term, its antisymmetric counterpart is non-propagating; the \( \tilde{\varepsilon}_{\mu\nu} \)-bilinear contains no derivatives at all, but only a (gauge dependent) mass term. Note also that in \( d > 2 \) the trace part \( \phi_e \) has a “wrong sign” kinetic term, reflecting the well known conformal factor instability \[11\].

Let us now fix the precise form of the cutoff operator \( \hat{R}_k \) in the various sectors of field space. Generically it has the structure

\[ \hat{R}_k = Z_k k^2 R^{(0)}(-\bar{D}^2/k^2), \] (3.19)

where \( Z_k \) is a matrix in field space, and \( R^{(0)}(u) \) is a dimensionless “shape function” that interpolates smoothly between \( R^{(0)}(0) = 1 \) and \( \lim_{u \to \infty} R^{(0)}(u) = 0 \). At least in simple matter field theories on a rigid background spacetime, there is a simple rule for finding a suitable \( Z_k \), and this rule has also been used in the metric calculation in \[11\]: If a certain field mode has a kinetic operator of the form \( [-\bar{D}^2 + \cdots] \), the \( Z_k \) is fixed in such a way that in the sum \( \Gamma_k + \Delta_k S \) this operator gets replaced by \( [-\bar{D}^2 + k^2 R^{(0)}(-\bar{D}^2/k^2) + \cdots] \).

In the case at hand it is straightforward to implement this rule for \( \tilde{\varepsilon}_{\mu\nu} \) and \( \phi_e \). In the different sectors we choose

\[ (Z_k)_{\varepsilon\varepsilon} = 2Z_{Nk}\xi\kappa^2, \quad (Z_k)_{\varepsilon\xi} = 2\xi^{-1}Z_{Nk}\kappa^2, \quad (Z_k)_{\phi_e\phi_e} = -2\xi^{-2}Z_{Nk}\kappa^2\frac{d - 2}{2d}. \] (3.20)
As for the antisymmetric tensor $\tilde{\varepsilon}_{\mu
u}$, we fixed the corresponding $Z_k$ in such a way that, taking the overall prefactor into account, the addition of $\tilde{R}_k$ to the inverse propagator replaces the square brackets in the $\tilde{\varepsilon}_{\mu\nu}$-bilinear of (3.17) by

\[
\left[ k^2 R^{(0)}(-\tilde{D}^2/k^2) + \frac{1}{Z_{Nk}k^2\alpha_L} + \tilde{\lambda}_k - \frac{d-2}{2d} \tilde{R} \right].
\]  

(3.21)

Now we have specified all ingredients entering the supertrace on the RHS of (2.3) in the different sectors.

First of all we note that the contributions of the antisymmetric sector vanish in the limit of $\alpha_L \to 0$, as this part of the trace is given by

\[
\frac{1}{2} \text{Tr} \tilde{\varepsilon} \partial_t \left( Z_{Nk} k^2 R^{(0)}(-\tilde{D}^2/k^2) \right) \left[ Z_{Nk} \left( k^2 R^{(0)}(-\tilde{D}^2/k^2) + \tilde{\lambda}_k + 1/(Z_{Nk}k^2\alpha_L) - \tilde{R}(d-2)/(2d) \right) \right] \right] 
= \frac{\alpha_L}{2} \text{Tr} \tilde{\varepsilon} \left[ \partial_t \left( Z_{Nk} k^2 R^{(0)}(-\tilde{D}^2/k^2) \right) \left[ Z_{Nk} \left( \alpha_L k^2 R^{(0)}(-\tilde{D}^2/k^2) + \alpha_L \tilde{\lambda}_k + 1/(Z_{Nk}k^2) - \alpha_L \tilde{R}(d-2)/(2d) \right) \right] \right] 
\rightarrow \frac{\alpha_L}{2} \to 0.
\]  

(3.22)

This behavior is easy to understand as the limit $\alpha_L \to 0$ corresponds to a sharp implementation of the $O(d)$ gauge condition that introduces a delta functional $\delta[\tilde{\varepsilon}_{\mu\nu}]$ into the path integral. Since the domain of tensors with $\tilde{\varepsilon}_{\mu\nu} = 0$ is invariant under the coarse graining operation it is obvious that the antisymmetric fluctuations should not contribute to any RG running in this limit. From now on we will choose the gauge $\alpha_L = 0$ in order to simplify the discussion.

In this particularly simple gauge the quadratic form (3.17) is structurally similar to the corresponding equation in the metric formalism, see eq. (4.12) in [11]. However, the prefactors of $\tilde{\lambda}_k$ in the various terms of $\Gamma_k^{\text{quad}}$ and the now $\xi$-dependent coefficients $C_5(\xi)$, $C_T(\xi)$ of the curvature scalar $\tilde{R}$ are different and this will have a rather significant impact on the resulting RG flow. Replacing these constants appropriately in the original metric calculation we can obtain the “bosonic” contributions to the $\beta$-functions without a new calculation from those of [11].
### 3.3 Propagating and non-propagating ghosts

Let us move on and discuss the ghost sector. Here we choose the cutoff operator to be

\[
\widehat{R}_k^{gh} = \begin{pmatrix}
\sqrt{2}\xi^{-1}\delta\mu k^2 R(0)(-\bar{D}^2/k^2) & 0 \\
0 & Z_L^{gh}\delta[^\mu^\nu^\rho^\sigma]k^2 R(0)(-\bar{D}^2/k^2)
\end{pmatrix}.
\]

(3.23)

In the diffeomorphism-ghost sector we have adjusted \(Z_k^{gh}\) to the kinetic term according to the above rule.

In the \(O(d)\) ghost sector, however, there is no kinetic term; the ghosts do not propagate. Nevertheless, a consistent application of the FRGE requires us not to ignore, but to systematically integrate out these non-propagating modes in the same way as all the others, i.e. ordered, and eventually cut off according to their \(\bar{D}^2\)-eigenvalue. Therefore we introduce a cutoff-operator (with a prefactor unrelated to the couplings in \(\Gamma_k\), denoted by \(Z_L^{gh}\)) in this sector as well.\(^3\)

In the gauge chosen, the inverse ghost propagator \(S^{(2)}_{gh} + \widehat{R}_k^{gh}\) is a triangular matrix, such that the contributions of the different sectors to the trace decouple.

For any constant choice of \(Z_L^{gh} = Z_L^{gh}\) we obtain contributions of the \(O(d)\) ghost sector of the form

\[
\text{Tr} \left[ \frac{\partial_k(Z_L^{gh}k^2 R(0))}{-M^2 + Z_L^{gh}k^2 R(0)} \delta[^\mu^\nu^\rho^\sigma] \right] = \text{Tr} \left[ \frac{k^{-2}\partial_k(k^2 R(0))}{-M^2/Z_L^{gh}k^2 + R(0)} \delta[^\mu^\nu^\rho^\sigma] \right]
\]

(3.24)

with the abbreviation \(M^2 \equiv 2\mu^2\xi^{-1/2}\). Introducing the dimensionless mass parameter \(\mu \equiv \bar{\mu}/k\), and then neglecting any further running of \(\mu\), we observe that the trace

\(3.24\)

depends only on the \(k\)-independent dimensionless quantity

\[
-\frac{M^2}{Z_L^{gh}k^2} \equiv -\frac{2\mu^2}{Z_L^{gh}\xi^{1/2}}.
\]

(3.25)

In order to avoid divergences due to a vanishing denominator in \(3.24\) we have to choose a negative value for \(Z_L^{gh}\), as known from the conformal sector. Since both parameters, \(\mu\)

\(\text{Recall that ideally, at the exact level, the cutoff action } \Delta_k S \text{ would be independent of the running couplings present in } \Gamma_k \text{ [24].}\)
and $Z_{gh}^L$, occur only in the combination (3.25) we can mimic any choice of $Z_{gh}^L < 0$ by choosing a suitable $\mu$. (In particular $Z_{gh}^L \to -1$, upon replacing $\mu^2 \to -\mu^2/Z_{gh}^L$.)

In the following we will discuss three distinguished choices of $Z_{gh}^L$:

(i) $Z_{gh}^L = -1$: the cutoff term is unrelated to $\Gamma_k$, the $O(d)$ ghost contribution will therefore depend on $\mu$ and $\xi$.

(ii) $Z_{gh}^L = -M^2/k^2 = -2\mu^2\xi^{-1/2}$: the cutoff is optimally adapted to the form of $\Gamma_k$ leading to a cancelation of the parameters $\mu$ and $\xi$. This procedure is closest to the above rule for usual kinetic term adaptation and we therefore expect the most reliable results for this choice.

(iii) $Z_{gh}^L \to 0$: no cutoff term introduced. This choice corresponds to neglecting the $O(d)$ ghost modes completely, the trace (3.24) vanishes.

As explained above, these three choices are equivalent to using $Z_{gh}^L = -1$ and setting $\mu^2$ equal to $\mu^2$, $\xi^{1/2}/2$, and $\mu \to \infty$, respectively. We shall refer to them as the ghost adaptation schemes (i)–(iii) from now on.

3.4 The interpolating beta functions

The remaining part of the calculation consists of projecting out the invariants $\int \sqrt{g}$ and $\int \sqrt{g}R$ from the supertrace in order to find the $\beta$-functions for $G_k$ and $\bar{\lambda}_k$; it follows exactly the metric calculation in [11].

If we turn over to dimensionless couplings

$$g_k = \frac{k^{d-2}}{32\pi Z_{Nk}\kappa^2} = k^{d-2}G_k, \quad \lambda_k = k^{-2}\bar{\lambda}_k$$  (3.26)

the resulting system of coupled RG equations is autonomous and has the structure

$$\partial_t g_k = \beta_g(g_k, \lambda_k) \equiv \left[ d - 2 + \eta_N(g_k, \lambda_k) \right] g_k,$$  (3.27)

$$\partial_t \lambda_k = \beta_\lambda(g_k, \lambda_k)$$  (3.28)
with the anomalous dimension \( \eta_N = -\partial_t \ln Z_{Nk} \). We shall employ the standard threshold functions \( \Phi, \bar{\Phi} \) of [11] along with a new type of threshold function, \( \tilde{\Phi} \), defined according to

\[
\Phi_n^p(w) = \frac{1}{\Gamma(n)} \int_0^\infty dz \ z^{n-1} \frac{R^{(0)}(z) - z R^{(0)'}(z)}{[z + R^{(0)}(z) + w]^p}
\]

(3.29)

\[
\tilde{\Phi}_n^p(w) = \frac{1}{\Gamma(n)} \int_0^\infty dz \ z^{n-1} \frac{R^{(0)}(z)}{[z + R^{(0)}(z) + w]^p}
\] for \( n > 0 \) \hspace{1cm} (3.30)

\[
\tilde{\Phi}_n^0(w) = \Phi_n^0(w) = \Phi_n^p(w) = (1 + w)^{-p}. \]

We can write down an explicit expression for \( \eta_N \) in terms of the couplings \( g, \lambda \) then:

\[
\eta_N(g, \lambda) = \frac{g \tilde{B}_1(\lambda)}{1 - g \tilde{B}_2(\lambda)}
\] \hspace{1cm} (3.32)

The functions \( \tilde{B}_1 \) and \( \tilde{B}_2 \) are \( \xi \)-dependent generalizations of similar ones occurring in [11]:

\[
\tilde{B}_1(\lambda) = \frac{1}{3} (4\pi)^{1-d/2} \left[ (d - 1)(d + 2) \Phi_{d/2-1}^1((\xi - 2)\lambda) 
\right.

+ 2\Phi_{d/2-1}^1 \left( -2\lambda \frac{d - 2 + \xi}{d - 2} \right) - 4d \Phi_{d/2-1}^1(0) 
- 6(d - 1)(d + 2) C_T(\xi) \Phi_{d/2}^2((\xi - 2)\lambda)

- 12 C_S(\xi) \Phi_{d/2}^2 \left( -2\lambda \frac{d - 2 + \xi}{d - 2} \right) 
- 24 \Phi_{d/2}^2(0) \right]
\] \hspace{1cm} (3.33)

and

\[
\tilde{B}_2(\lambda) = -\frac{1}{6} (4\pi)^{1-d/2} \left[ (d - 1)(d + 2) \Phi_{d/2-1}^1((\xi - 2)\lambda) 
\right.

+ 2\Phi_{d/2-1}^1 \left( -2\lambda \frac{d - 2 + \xi}{d - 2} \right) - 6(d - 1)(d + 2) C_T(\xi) \Phi_{d/2}^2((\xi - 2)\lambda)

- 12 C_S(\xi) \Phi_{d/2}^2 \left( -2\lambda \frac{d - 2 + \xi}{d - 2} \right) \right].
\] \hspace{1cm} (3.34)
For the $\beta$-function of the cosmological constant we obtain

$$\beta_{\lambda} = -(2 - \eta_N)\lambda$$

$$+ \frac{1}{2} g_{k}(4\pi)^{1-d/2} \left[ 2(d - 1)(d + 2) \Phi_{d/2}^{1}((\xi - 2)\lambda) + 4\Phi_{d/2}^{1}\left(-2\lambda \frac{d - 2 + \xi}{d - 2}\right) - 8d \Phi_{d/2}^{1}(0) - 4d(d - 1) \tilde{\Phi}_{d/2}^{1}\left(\frac{2\mu^{2}}{\sqrt{\xi}}\right) - (d - 1)(d + 2) \eta_N \tilde{\Phi}_{d/2}^{1}((\xi - 2)\lambda) - 2 \eta_N \tilde{\Phi}_{d/2}^{1}\left(-2\lambda \frac{d - 2 + \xi}{d - 2}\right) \right].$$

(3.35)

These general $\xi$-dependent expressions are in exact correspondence to the eqns. (4.40) and (4.43) of ref. [11] for metric gravity. Analyzing the $\xi$-dependence of the RG flow they give rise to is a convenient way of exploring the field parametrization dependence of the flow.

An important observation is that for a constant, $\xi$-independent choice of $\mu$ (i.e. in the ghost adaptation schemes (i) and (iii)) the above $\beta$-functions reduce precisely to those of the metric result in the limit $\xi \to 0$. All prefactors and arguments of the threshold functions coincide and the function $\tilde{\Phi}$ vanishes in this limit. Although this result is far from obvious when considering the definition of $\xi$ in eq. (3.2), we can now regard this one parameter family of field redefinitions as an interpolation between the metric description (for $\xi \to 0$) and the usual vielbein decomposition (for $\xi = 1$).

In scheme (ii) however, the argument of $\tilde{\Phi}$ is constant, so that in the limit $\xi \to 0$ the $\beta$-functions match the metric result except for the additional $\tilde{\Phi}$ contributions, which are precisely the terms due to the $O(d)$ ghosts.

4 Numerical analysis of the RG flow

In this section we will analyze the RG flow in $d = 4$ dimensions. We will compare results of different cutoff schemes, namely with the optimized shape function, $R^{(0)}(z) = (1 - z)\theta(1 - z)$, and the exponential one, $R^{(0)}(z) = sz/(e^{sz} - 1)$, for shape parameters $s$ ranging from 2 to 20.
4.1 The standard vielbein case $\xi = 1$

To start with, let us consider the usual vielbein representation of the metric in (3.2) and set $\xi = 1$ for the time being. With $\xi$ fixed the flow continues to depend on the mass parameter $\mu \equiv \bar{\mu}/k$. We shall analyze this dependence in the following, highlighting especially the implications of those choices of $\mu$ that correspond to the three ghost adaptation schemes (i)–(iii).

A first encouraging result is that there exists a non-Gaussian fixed point (NGFP), for any value of the dimensionless constant $\mu \neq 0$, and in all cutoff schemes we studied.

![Figure 1](image_url)

**Figure 1.** The critical exponents $\theta_i = \theta'_i + i\theta''_i$ split into real and imaginary part (solid and dashed line, respectively) of the NGFP and the product $g^*\lambda^*$ (dotted line) as a function of the mass parameter $\mu$ for the optimized cutoff. The straight horizontal lines represent the values of the corresponding quantities in the metric calculation.

(A) Fixed point properties. Figure 1 shows, for the case of the optimized cutoff, the $\mu$-dependence of three quantities one might expect to be universal, namely the critical exponents $\theta_i$ at the fixed point and the product $g^*\lambda^*$. We notice that, while the very existence of the fixed point is indeed universal, its properties heavily depend on the value
of \( \mu \): For \( \mu \lesssim 0.8 \) we find a UV attractive FP with two real critical exponents, which then turn into a complex conjugated pair. At \( \mu \approx 1.35 \) the FP changes its character and becomes UV repulsive in both directions. For large \( \mu \)-values the dependence on \( \mu \) weakens for all three “universal” quantities.

Employing the exponential cutoff (not shown here) essentially leads to the same picture: real critical exponents turn into a complex pair before the otherwise UV attractive FP gets UV repulsive for large \( \mu \). In all cases the product \( g^*\lambda^* \) changes its sign from negative to positive within the interval of \( \mu \), in which the FP is attractive and has complex critical exponents.

It is important to stress that even in a much better truncation with many more invariants we would not expect these quantities to become independent of \( \mu \): The parameter \( \mu \) should not be considered a free parameter corresponding e.g. to different cutoff schemes, but it rather corresponds to an additional coupling. In principle its running is prescribed by an additional \( \beta \)-function which however is not determined by the present calculation. Therefore one should not worry too much about the \( \mu \)-dependence of the “universal” quantities.

In the ghost adaptation scheme (i) the best we can do, as we did not calculate the running of \( \mu \) in our truncation, is to sensibly choose a fixed value for the constant \( \mu \). Most naturally we would choose a value of the order of 1 as any other choice would correspond to the introduction of an additional unmotivated physical scale other than \( k \).

Strikingly, in all cutoff schemes studied there exists indeed a \( \mu \)-interval including, or at least close to \( \mu = 1 \) in which the situation is similar to the metric theory: We find the NGFP, it is UV attractive, has \( g^*\lambda^* > 0 \), and a pair of complex conjugate critical exponents.

As an alternative to choosing \( \mu = 1 \) it is therefore tempting to find the “best fit” to the metric calculation by selecting a \( \mu \)-value such that there is also a quantitative agreement of the universal quantities.
In Fig. 1 the values corresponding to the metric calculation are given by the horizontal lines. We observe that the crossings of the lines of the same type are quite close to each other and are all located at a $\mu$ of the order of 1. For the optimized cutoff we find the crossing for the real part of the critical exponent $\theta'_i$ very much at $\mu \approx 1$ and for the product $g^*\lambda^*$ at about $\mu \approx 1.1$; the imaginary part $\theta''_i$ takes on its metric value at $\mu \approx 1.45$ in a region where the FP turned UV repulsive already. Taking the average of these values we arrive at $\mu \approx 1.2$, for which we expect the best agreement between the vielbein and the metric theory.

The fact that we find this agreement of metric and vielbein values in a relatively small $\mu$-interval close to the most natural value of $\mu = 1$ can be interpreted as an indication that also the full quantum theories are similar to each other or perhaps equivalent.

The adaptation scheme (ii) is expected to be the most reliable one. It yields the value $\mu = 1/\sqrt{2} \approx 0.7$ for $\xi = 1$. However, in this scheme the results of scheme (i) are not confirmed: For the smaller value of the parameter $\mu$ we find a UV attractive NGFP at $g^*\lambda^* < 0$, with two real critical exponents.

The adaptation scheme (iii) corresponds to large $\mu \rightarrow \infty$, so that we find a UV repulsive FP in this scheme.

(B) The phase portrait. Let us now discuss the entire RG flow. In Fig. 2 we have plotted its phase portrait for different values of $\mu$. Figs. 2(b) and 2(c) correspond to the first adaptation scheme (i). We observe that the best fit case $\mu = 1.2$ is indeed most similar to the metric flow known from Quantum Einstein Gravity (QEG) in the Einstein-Hilbert truncation [11–13]: We find a NGFP in the positive $(g, \lambda)$-quadrant with two attractive directions; the trajectories spiral into it due to the nonzero imaginary part of the critical exponents. It is in an interplay with the Gaussian fixed point (GFP) and there exists a “separatrix” that separates trajectories with positive and negative IR values for the cosmological constant $\lambda$. Also a major difference to the metric case is to be noted: The UV repulsive direction of the GFP has changed and points now into the negative $\lambda$-
(a) $\mu = 1/\sqrt{2}$, ghost adaptation scheme (ii): The phase portrait resembles the one in QECG.

(b) $\mu = 1$: This value is the most natural choice when using ghost adaptation scheme (i).

(c) $\mu = 1.2$: For this value we obtain a situation most similar to the metric theory in scheme (i).

(d) $\mu = 2$: The limit cycle is a qualitatively new feature of the phase portrait in scheme (iii).

Figure 2. RG phase portraits for different values of the mass parameter $\mu$ at $\xi = 1$. The figures show the impact of the $O(d)$ ghost contribution: While for large $\mu$, when it is suppressed, we obtain a limit cycle, for smaller $\mu$ we find flow diagrams similar to the ones known from QEG and QECG, respectively.
halfplane. Therefore the separatrix starts off with negative \( \lambda \) before heading to the NGFP at \( \lambda^* > 0 \). This effect can be traced back to be due to the \( O(d) \) ghost contributions.

For smaller \( \mu \) (as e.g. \( \mu = 1/\sqrt{2} \) in adaptation scheme (ii)) these contributions are enhanced in such a way, that the NGFP itself lies at \( \lambda^* < 0 \) (cf. Fig. 2(a)). Now the fixed point has real critical exponents, but is still UV attractive. Qualitatively this picture resembles much the RG flow of Quantum Einstein Cartan Gravity (QECG) in the planes of vanishing and infinite Immirzi parameter as found in [16].

For large \( \mu \) (scheme (iii)), as exemplarily shown for the case of \( \mu = 2 \) in Fig. 2(d), we find a rather different behavior. Although the flow looks similar to the metric case in large parts of the \( (g, \lambda) \)-plane, the NGFP is repulsive now; the critical exponents form a complex conjugate pair with a negative real part. These two circumstances lead to the formation of a limit cycle around the NGFP. This limit cycle is UV attractive for trajectories approaching it both from outside and from the interior.

Clearly such a limit cycle is an interesting and intriguing new possibility for the nonperturbative UV completion of a quantum field theory. It is “asymptotically safe” in a novel sense. However, in this concrete case the picture of a limit cycle is hardly credible against the background of all RG flow studies of gravity to date. Nevertheless it is inspiring to see its formation for the first time in quantum gravity.

(C) Non-propagating ghosts. The fact that we should not choose the parameter \( \mu \) too large teaches us another important lesson: Consider the \( \beta \)-functions as given in the previous section. They involve the new threshold function \( \tilde{\Phi}_{d/2}^1(w) \) that vanishes for \( w \to \infty \) and diverges for \( w \to 0 \). In both \( \beta \)-functions, the terms with \( \tilde{\Phi}_{d/2}^1(w) \) are exactly the ghost contributions of the \( O(d) \) gauge group. Since the \( \tilde{\Phi} \) argument is always \( w = 2\mu^2/\sqrt{\xi} \) we can control the magnitude of these contributions by changing \( \mu \): We obtain a suppression for large \( \mu \) and an infinite enhancement in the limit \( \mu \to 0 \). If we had not added a cutoff for the \( O(d) \) ghosts, the situation would correspond to the limit \( \mu \to \infty \), i.e. adaptation scheme (iii). In this case we find a UV repulsive fixed point,
quite different from all results known from metric calculations. We therefore conclude that contrary to the situation of perturbation theory \cite{23} it is crucial to include all modes of the non-background fields into the renormalization procedure, whether they are propagating or not, by introducing a cutoff-operator for all of them and retaining their contribution to the supertrace in the FRGE. This implies that we should choose adaptation scheme (i) or (ii) but not (iii).

Similar remarks might also apply to perturbative calculations with regularization schemes which retain power divergences.

4.2 Field parametrizations with $\xi \neq 1$

When altering the value of $\xi$, we do not change theory space as both field content and symmetries remain the same. Therefore we expect to find the same fixed point properties in the RG flow for all values of $\xi$, resulting in universal quantities that, in case of a good approximation to the exact flow, are largely independent of $\xi$. We will use this criterion in order to test the reliability of the different ghost adaptation schemes in this section.

(A) Adaptation scheme (i). In Fig. 3 we have plotted the universal quantities (critical exponents and the product of the fixed point coordinates) for various values of the mass parameter $\mu$ as functions of $\xi$. As $\mu$ does not depend on $\xi$ in these examples, all of them correspond the adaptation scheme (i), although Fig. 3(d) already shows typical characteristics of the large $\mu$ limit and can therefore also be seen as an example of scheme (iii).

In all four cases we start with the values of the metric theory at $\xi = 0$ and find for each of them a quite pronounced dependence on $\xi$. While for small $\mu$ as shown in Fig. 3(a) the critical exponents turn from complex to real and $g^*\lambda^*$ turns negative as we move towards $\xi = 1$, for large $\mu$ (Fig. 3(d)) $g^*\lambda^*$ stays positive but the fixed point gets repulsive. Only in the region of $\mu \approx 1$ the situation improves a little, as no quantity changes its sign

\footnote{It might be interesting to reconsider the calculation \cite{17} in this light since there a propertime regulator has been used.}
Figure 3. Critical exponents and $g^* \lambda^*$ calculated using the optimized cutoff, for different values of the mass parameter $\mu$, as functions of $\xi$: The real part of the critical exponents $\theta_i'$ (solid), its imaginary part $\theta_i''$ (dashed) and the product of the fixed point coordinates $g^* \lambda^*$ (dotted).

in the interval of $\xi \in [0, 1]$. However the quantities plotted are far from being constant with respect to $\xi$; furthermore if we compare the analogous results obtained with the family of $s$-dependent exponential cutoffs (as is done in the Appendix) we find that these results still show a substantial cutoff scheme dependence. Can we do better than this?

(B) Adaptation scheme (ii). If we employ the optimally adapted cutoff (ii) instead (Fig. 4(b)), the $O(d)$ ghost contribution is now independent of $\xi$. Therefore $\xi = 0$ does not correspond to the metric theory any more. In this case we find the universal quantities almost independent of $\xi$. 

27
Figure 4. Critical exponents and $g^*\lambda^*$ for different values of an adapted mass parameter $\mu$ as a function of $\xi$ ($\theta'$ solid, $\theta''$ dashed, $g^*\lambda^*$ dotted), calculated with the optimized cutoff.

Variants of this cutoff adaptation differing by a factor of $\sqrt{2}$ (Figs. 4(a), 4(c)) show that the universality can even be improved when choosing a smaller $\mu$. This effect, however, does not really improve the reliability of the flow.: In the limit $\mu \to 0$ the constant $O(d)$ ghost contribution diverges and governs the RG flow, so that the effect of the physical field modes becomes negligible. Therefore it is evident that the $\xi$ dependence weakens when going to smaller values of $\mu$, but only at the cost of losing the physics content of the flow.

(C) Discussion. The properties of the universal quantities calculated in this chapter show, that the influence of the $O(d)$ ghosts on the fixed point properties is quite significant.
While neglecting these contributions (adaptation scheme (iii)) leads to the implausible result that the fixed point changes its character and gets UV repulsive for some $\xi \in [0, 1]$, the simple unadapted cutoff (scheme (i)) leads to universal quantities strongly dependent on $\xi$.

Only the optimally adapted ghost cutoff (scheme (ii)) predicts relatively stable values for the universal quantities. These values indicate a fixed point at $\lambda^* < 0$ with real critical exponents, that therefore may not be the one known from the metric theory.

If this picture is correct, part of the $\xi$-dependence found in scheme (i) is clearly due to the fact, that in this scheme the quantities are forced to take on their metric values at $\xi = 0$. This way we would have constructed an interpolation between theories of different universality class which obviously leads to a $\xi$-dependence of the “universal quantities”.

Nevertheless, all results show a cutoff scheme, i.e. $R^{(0)}(\cdot)$-dependence that is more severe than in the metric case. It is analyzed further in the Appendix to which the reader might turn at this point.

Apparently the truncation chosen is less reliable than the Einstein-Hilbert truncation of metric gravity, although it can be considered as its exact “translation” to the tetrad theory space. Together with the different FP properties this indicates that the quantum theories of metric and tetrad gravity (if both should turn out nonperturbatively renormalizable) are perhaps not similar to each other. For this reason it is crucial to use tetrads as fundamental field variables whenever an RG study of fermions coupled to gravity is performed even if only the pure gravity $\beta$-functions are investigated. Our results can be considered a warning that in a nonperturbative RG analysis the $O(d)$ ghost sector cannot be ignored (as opposed to perturbation theory [23]). Seen in this light, the status of hybrid calculations which add fermionic contributions to metric QEG seems questionable.
5 Summary and Conclusion

In this paper we performed a first survey of the renormalization group flow on the “tetrad only” theory space $\mathcal{T}_{\text{tet}} = \{ A[e^a_\mu, \cdots ] \}$. Its points are action functionals which, besides the indispensable background and ghost fields, depend on the vielbein $e^a_\mu$ only, and which are invariant under the semidirect product of spacetime diffeomorphisms and local Lorentz transformations. Contrary to the Einstein-Cartan theory the spin connection is not an independent field, but rather is identified with the Levi-Civita connection implied by $e^a_\mu$. This excludes the possibility of field configurations with torsion. We truncated $\mathcal{T}_{\text{tet}}$ so as to consist of a running Einstein-Hilbert term, along with the classical gauge fixing and ghost terms. As a result, the only difference in comparison to Quantum Einstein Gravity (QEG) in the Einstein-Hilbert truncation \cite{11,13} is the use of $e^a_\mu$ rather than the metric $g_{\mu\nu}$ as the fundamental field variable and the larger group of gauge transformations $\text{Diff}(\mathcal{M}) \ltimes O(d)_{\text{loc}}$ replacing $\text{Diff}(\mathcal{M})$. In the present treatment the latter has the status of a composite field: $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$. Our main tool was the gravitational average action on $\mathcal{T}_{\text{tet}}$, and in particular the FRGE which governs its scale dependence. Since this framework is not covariant under field reparametrizations, and since the respective groups $G$ are different, the RG flow on $\mathcal{T}_{\text{tet}}$ is likely to differ from the one of QEG, even at the exact level.

This expectation was confirmed by our explicit calculation. The details of the Einstein-Hilbert flows with $e^a_\mu$ and $g_{\mu\nu}$, respectively, as fundamental fields are indeed different in a significant way. However, their gross topological features are still similar, nevertheless. In particular we found on $\mathcal{T}_{\text{tet}}$ one, and only one, non-Gaussian fixed point, exactly as in QEG. Provided it is not a truncation artifact it seems suitable for taking a nonperturbative continuum limit there, thus defining an asymptotically safe field theory.

To assess the reliability of the approximations made we investigated the dependence of “universal” quantities such as critical exponents and the product $g^*\lambda^*$ on the cutoff shape function $R^{(0)}(\cdot)$ and the mass parameter $\mu$. The latter had to be introduced in order
to give equal canonical dimensions to diffeomorphism and $O(d)_{\text{loc}}$ ghosts. While in a more complete truncation it would be treated as a running coupling with its own $\beta$-function, we neglected its running in the present investigation. The upshot of the analysis is that the very existence of the NGFP indeed seems to be a universal feature, in the sense that it exists for all admissible cutoffs and values of $\mu$.

However, further details, even the critical exponents show a variability with $R^{(0)}(\cdot)$ and $\mu$ which is significantly larger than in QEG with the same truncation. (In particular, in QEG there is no analog of the parameter $\mu$.) Thus we must conclude that, using the same type of flow equation and the same (Einstein-Hilbert) truncation, the use of the vielbein instead of the metric leads to a less robust RG flow.

Can we understand on general grounds why the flow of the metric theory might have better robustness properties than the one based upon the tetrad? A possible explanation is as follows.

The running couplings parametrizing a general functional $\Gamma_k$ are, per se, not measurable quantities, that is, typical observables are complicated combinations of these couplings, and in forming these combinations the scheme dependence which the individual couplings have (even in the exact theory!) cancels among them. Consider now a theory space whose actions are constrained to be invariant under a group $G$ of gauge transformations which we make larger and larger. As a result, more and more excitations carried by the (fixed) set of fields considered are declared “unphysical” gauge modes. Nevertheless all those modes continue to contribute to the supertrace in the FRGE, but are counteracted by an increasing number of ghosts needed to gauge-fix $G$. Loosely speaking, increasing the size of $G$ reduces the amount of “physical” (in the sense of “non-gauge”) or “observable” contents encoded in the running couplings. In diagrammatic terms, the ratio of physical excitations relative to gauge excitations gets smaller when $G$ grows. However, since those features of the RG flow which are due to the gauge modes have no reason to be scheme independent, one can expect that the larger is $G$ the more scheme dependent is even the exact RG flow.
While, in $d = 4$, metric gravity has 4 gauge parameters per spacetime point related to the diffeomorphisms, this number increases to 4+6 in tetrad gravity since local Lorentz invariance is demanded in addition. If we assume that both theories have the same number of physical degrees of freedom, it is clear that tetrad gravity has a smaller ratio of physical to unphysical field modes, and this might explain to some extent why its RG flow has the more delicate scheme dependence we observed.

To close with, let us come back to the issues raised in the Introduction which motivated the present work.

(A) In ref. [16] the RG flow was computed for a 3D truncation of $\mathcal{T}_{EC} = \{A[e, \omega, \cdot \cdot \cdot]\}$ which has the same gauge transformations $\text{Diff}(\mathcal{M}) \ltimes \text{O}(d)_{\text{loc}}$ as $\mathcal{T}_{\text{tet}} = \{A[e, \cdot \cdot \cdot]\}$, but treats the spin connection as an independent field. There, too, the very existence of a NGFP is a robust feature which obtains for all cutoff and gauge choices, but the quantitative details are more scheme dependent then we are used to from QEG. In this respect the results of [16] are very reminiscent of what we saw in the present paper. In [16] both the truncated action and the fundamental variables are different from QEG ("Holst" instead of "Einstein-Hilbert", and $(e, \omega)$ instead of the metric). In the light of our present results we can say that the hitherto unexplained relatively strong scheme dependence seen in [16] could be entirely due to the different variables used and the related larger group of gauge transformations; even though the running actions used were quite different in the two cases (first vs. second order in derivatives, etc.) this is not necessarily the cause for the observed differences.

(B) In the literature [18–20] “hybrid” calculations were proposed in order to avoid recalculating parts of the $\beta$-functions for the gravitational couplings in presence of fermionic matter. The idea is to use the tetrad formalism only when it comes to evaluating fermion loops, but to keep the metric as the fundamental variable for the gravity loops. While this can be legitimate in perturbation theory [23], the present investigation revealed that the quantitative details of the flow of Newton’s constant and the cosmological constant are significantly different in the metric and the vielbein formalism. Hence, adding the
fermionic loops to the “old” metric $\beta$-functions does not seem a consistent procedure, even within the limited scope of a truncation. Thus we must conclude that one should refrain from such hybrid calculations when one aims at quantitative results. (C) In the symmetric vielbein gauge the $O(d)_{\text{loc}}$ ghosts are non-propagating. It was therefore argued, in perturbation theory, that they simply may be ignored in practical calculations [23]. As we saw quite explicitly, the same is not true in the FRGE framework. The Lorentz ghosts do have a considerable impact on the RG flow we found, and moreover the arguments put forward in [23] are easily seen not to carry over to $\Gamma_k$ at $k > 0$.

Several semi-quantitative calculations [28, 29] have shown that the Standard Model coupled to asymptotically safe gravity may lead to a theory with enhanced predictivity, that is some of the perturbatively undetermined parameters of the Standard Model (like the mass of the Higgs boson [28] or the fine-structure constant [29]) can be calculated in the coupled gravity + matter theory. The present paper has identified possible pitfalls in RG calculations of such coupled systems of gravity and fermions and indicated how to avoid them. This paves the way for a fully quantitative treatment of the considerations in refs. [28] and [29]. Even though this might require more work than thought before, the chance to compute the Higgs mass or the fine-structure constant clearly will be worth the effort.

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Appendix

A The family of exponential cutoffs

In this appendix we study the cutoff scheme dependence of \( g^* \lambda^* \) and the critical exponents within the family of exponential cutoff functions \( R^{(0)}(z) = sz/(e^{sz} - 1) \) for shape parameters \( s \) ranging from 2 to 20 in more detail.

The first set of figures, Figs. 5 to 7, corresponds to the application of ghost adaptation scheme (i), with each figure representing a different choice of \( \mu \) (\( \mu = 0.5, 1, 5 \) in Figs. 5, 6, 7 respectively). Fig. 7 where \( \mu \) is already rather large can also be seen as representing ghost adaptation scheme (iii). Each of the figures contains a series of plots ordered from small to large shape parameters employed in the exponential cutoff function.

In a good approximation of the exact flow we would expect the plots to show only little \( \xi \)-dependence of the universal quantities, resulting in horizontal lines, as well as only small variations of the same picture for different shape parameters, i.e. almost equal plots within each of the figures. The dependence on \( \mu \), on the other hand, could be more pronounced, as it should be seen as an additional coupling set to a fixed value.

We see, however, that there is a severe dependence on the parameter \( \xi \) in all three figures; it leads to a change of sign of \( g^* \lambda^* \), changing critical exponents from complex to real, and even a change of character of the fixed point from UV attractive to repulsive. Although we already made similar observations for the optimized cutoff function (cf. Fig. 3), only the exponential cutoff functions reveal the full degree of scheme dependence in these results: While for the optimized cutoff we were able to choose a value of \( \mu \approx 1 \) such that none of the above problematic changes occurred in the interval \( \xi \in [0, 1] \) (cf. Fig. 3 b, c), we now find that changing \( s \) has an effect similar to choosing a different \( \mu \). For that reason we find qualitatively the same plots (Fig. 3 a, b, c, d) obtained for the optimized cutoff function and distinguished choices of \( \mu \) all within the family of exponential cutoff functions for the same value of \( \mu = 1 \) (Fig. 6 a, c, d, f).
Only for large $\mu$ the variation of the plots within Fig. 7 is relatively weak. But here we find a large $\xi$-dependence of the critical exponents leading to a change of character of the fixed point in all plots, as we already found for the optimized cutoff in the same limit.

Taken together these observations show, that both adaptation schemes (i) and (iii) lead to severely scheme dependent results, that make it almost impossible draw any universally valid conclusion besides the existence of a NGFP.

Let us therefore go on and discuss the second set of figures (Figs. 8 to 10). Again each figure represents a certain choice of the parameter $\mu$ and contains a series of plots showing values of the same quantities obtained for different shape parameters $s$ of the exponential cutoff function. In this case however, we employed the three variants of the ghost adaptation scheme (ii) differing by a factor of $\sqrt{2}$, that we already introduced when we discussed this scheme for the optimized cutoff function in the main part of this paper (cf. Fig. 4).

The first and most prominent observation is that the $\xi$-dependence in the plots is considerably weaker for all three variants of scheme (ii) compared to schemes (i) and (iii). While we still find some dependence on the shape parameter $s$ (in all the three figures there are real critical exponents for small $s$ turning complex for larger $s$), except for the plots 10(e) and 10(f) all plots show almost horizontal lines, i.e. virtually no $\xi$-dependence of the universal quantities.

Secondly, as for the optimized cutoff, we find the weakest scheme dependence for the variant of adaptation scheme (ii) with the smallest value of $\mu$ (cf. Fig. 8). This, however, is probably due to the effective suppression of the physical degrees of freedom in the limit of small $\mu$, as explained in the main part of the paper.

For these reasons we conclude that, within the limits of the present truncation, the most reliable results are from the plots employing adaptation scheme (ii) as shown in Fig. 9. They suggest, in accordance with the optimized cutoff result in Fig. 4(b), a UV attractive FP in the $\lambda^* < 0$ region, presumably with real critical exponents.
Figure 5. Critical exponents and $g^*\lambda^*$ for different shape parameters $s$ depending on $\xi$ with mass parameter $\mu = 0.5$ ($\theta'$ solid, $\theta''$ dashed, $g^*\lambda^*$ dotted).
Figure 6. Critical exponents and $g^* \lambda^*$ for different shape parameters $s$ depending on $\xi$ with mass parameter $\mu = 1$ ($\theta'$ solid, $\theta''$ dashed, $g^* \lambda^*$ dotted).
Figure 7. Critical exponents and $g^* \lambda^*$ for different shape parameters $s$ depending on $\xi$ with mass parameter $\mu = 5$ ($\theta'$ solid, $\theta''$ dashed, $g^* \lambda^*$ dotted).
Figure 8. Critical exponents and $\lambda^*$ for different shape parameters $s$ depending on $\xi$ with an adapted mass parameter $\mu = (\xi/4)^{1/4}/\sqrt{2}$ (solid, $\theta'$ solid, $\theta''$ dashed, $g^*\lambda^*$ dotted).
Figure 9. Critical exponents and $g^*\lambda^*$ for different shape parameters $s$ depending on $\xi$ with an adapted mass parameter $\mu = (\xi/4)^{1/4}$ ($\theta'$ solid, $\theta''$ dashed, $g^*\lambda^*$ dotted).
Figure 10. Critical exponents and $g^*\lambda^*$ for different shape parameters $s$ depending on $\xi$ with an adapted mass parameter $\mu = \sqrt{2} \cdot (\xi/4)^{1/4}$ ($\theta'$ solid, $\theta''$ dashed, $g^*\lambda^*$ dotted).
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