Groups with decidable word problem that do not embed in groups with decidable conjugacy problem

Arman Darbinyan

Abstract We show the existence of finitely presented torsion-free groups with decidable word problem that cannot be embedded in any finitely generated group with decidable conjugacy problem. This answers a well-known question of Collins from the early 1970’s.

1 Introduction

Introduced by Max Dehn in 1911, [7], the word problem and the conjugacy problem, along with the group isomorphism problem, are considered as the central algorithmic problems in combinatorial and geometric group theory.

Let \( G = \langle X \rangle \) be a group with a given finite generating set \( X \). If we consider the elements of \( X \cup X^{-1} \) as formal letters and denote by \( (X \cup X^{-1})^* \) the set of finite words from the alphabet \( X \cup X^{-1} \), then each finite word from \( (X \cup X^{-1})^* \) in a natural way represents an element in \( G \), and vice versa: for each element \( g \in G \) there exists a word (in fact, countably many words) \( w \in (X \cup X^{-1})^* \) that represent \( g \), in which case we write \( w =_G 1 \). It is said that \( G \) has decidable word problem if there exists an algorithm that for each \( w \in (X \cup X^{-1})^* \) decides whether or not \( w =_G 1 \). It is said that \( G \) has decidable conjugacy problem
if there exists an algorithm that for each pair of words $u, v \in (X \cup X^{-1})^*$ decides whether or not $u$ and $v$ represent conjugate elements in $G$.

The connections between the word and conjugacy problems have been studied extensively by various authors. It is easy to see that decidability of the conjugacy problem implies decidability of the word problem, nevertheless, in general, the converse is not true. The first examples of finitely generated groups with decidable word problem but undecidable conjugacy problem were constructed by Miller III [14] and by Collins [5]. For other results about connections between the word and conjugacy problems in groups see, for example, [13–15].

If $G$ is a finitely generated group with decidable word problem, then it is an easy observation that all its finitely generated subgroups have decidable word problem as well. In contrast, there are known examples of finitely generated groups with decidable conjugacy problem that have subgroups of index two with undecidable conjugacy problem. See [6,10].

Given a finitely generated group $G$ with decidable word problem, it is natural to ask whether or not $G$ can be embedded into a finitely generated group with decidable conjugacy problem. As observed by Collins, [11], Macintyre constructed groups with torsions that have decidable word problem but cannot be embedded in groups with decidable conjugacy problem. However, that construction in an essential way makes use of the fact that conjugate elements in groups must have the same order. This naturally leads to the following question, asked by Collins in early 1970’s.

**Question 1.** Can every torsion-free group with decidable word problem be embedded in a group with decidable conjugacy problem?

Probably the first source where this question was posed in a written form is [3]. It also appears in the 1976 edition of The Kourovka Notebook as Problem 5.21, [11]. Other sources with this question include [2,16,17]. The strongest affirmative result towards this question is obtained by Olshanskii and Sapir [17], who showed that if in addition to the word problem the power problem is also solvable in a group, then the group embeds into a group with decidable conjugacy problem.

We resolve Question 1 with Theorem 1 below.

**Theorem 1** There exists a finitely generated torsion-free group $G$ with decidable word problem such that $G$ cannot be embedded into a group with decidable conjugacy problem. Moreover, $G$ can be chosen to be a solvable group of derived length 4 or a finitely presented group.

One of the key ingredients in the proof of Theorem 1 is the utilization of the recursively enumerable recursively inseparable sets. Two disjoint sets of natural numbers $S_1, S_2 \subset \mathbb{N}$ are called **recursively inseparable** if there is no recursive set $T \subset \mathbb{N}$ such that $S_1 \subseteq T$ and $S_2 \subseteq \mathbb{N} \setminus T$. The set $T$ is
called *separating set*. Clearly, if two disjoint sets are recursively inseparable, then both of them are non-recursive. Indeed, if, say, \( S_1 \) is recursive, then as a recursive separating set one could simply take \( S_1 \). Nevertheless, it is well-known that there exist recursively enumerable and recursively inseparable pairs of sets of natural numbers, see [19]. Up to our knowledge this is the second time after a result of Miller III [15, Corollary 3.9] when this concept was utilized to produce a new result in a group theoretical setting. The method used for obtaining the main result of the current paper was later applied by the author to answer other open questions as well, [9].

2 Countable groups with decidable word problem

Suppose that \( G = \langle X \rangle \), where \( X = \{x_1, x_2, \ldots \} \). Then let us denote

\[
WP_X(G) = \{ w \in (X \cup X^{-1})^* \mid w =_G 1 \}.
\]

We call the set \( WP_X(G) \) the word problem of \( G \) with respect to the enumerated generating set \( X = \{x_1, x_2, \ldots \} \). We say that \( WP_X(G) \) is decidable if there exists an algorithm that for any input \((\pm n_1, \pm n_2, \ldots \pm n_k) \), \( k \in \mathbb{N} \), \( n_i \in \mathbb{N} \), \( i = 1, \ldots k \), decides whether or not the word \( x_{i(n_1)}^{\pm 1} x_{i(n_2)}^{\pm 1} \ldots x_{i(n_k)}^{\pm 1} \) represents the trivial element of \( G \). If \( WP_X(G) \) is decidable, then we say that the word problem of \( G \) is decidable with respect to the enumerated generating set \( X \).

Let \( i : \mathbb{N} \to \mathcal{I} \) be a bijection, where \( \mathcal{I} \) is a recursive set. Then we say that the word problem in \( G \) is decidable with respect to the enumeration \( X = \{x_{i(1)}, x_{i(2)}, \ldots \} \) (we denote \( WP_{(X,i)}(G) \)), if there exists an algorithm that for any input

\[
(\pm i(n_1), \pm i(n_2), \ldots \pm i(n_k)),
\]

\( k \in \mathbb{N} \), \( n_i \in \mathbb{N} \), \( i = 1, \ldots k \), decides whether or not the word \( x_{i(n_1)}^{\pm 1} x_{i(n_2)}^{\pm 1} \ldots x_{i(n_k)}^{\pm 1} \) represents the trivial element of \( G \). In case \( i : \mathbb{N} \to \mathcal{I} \) is computable, the decidability of \( WP_{(X,i)}(G) \) is equivalent to the decidability of \( WP_X(G) \).

Traditionally, the word problem in groups is defined with respect to finite generating sets. The main advantage of the finite generating sets is that in this case the decidability of the word problem does not depend on a particular choice of generators and their enumeration, hence the decidability becomes a group intrinsic property rather than being dependent on the enumerated generating set. The groups with decidable word problems with respect to countable infinite enumerated sets were independently introduced in slightly different terms by Rabin [18] and Mal’cev [12] and are called in literature by the more common name of *computable groups*. 
If a finitely generated group has decidable word problem with respect to some (equivalently, any) finite generating set, then it is said that the group has **decidable word problem**.

The following embedding theorem is from [8].

**Theorem 2.1** [8] Let $G = \langle X \rangle$, $X = \{x_1, x_2, \ldots\}$, such that the word problem $WP_X(G)$ is decidable. Then there exists an embedding $\Phi_X : G \hookrightarrow H$ of $G$ into a group $H$ generated by two elements $c$ and $s$, such that the following holds.

1. The word problem is decidable in $H$;
2. The map $n \mapsto \Phi_X(x_n)$ is computable (where $\Phi_X(x_n)$ is represented as a word from $\{c^{\pm 1}, s^{\pm 1}\}$);
3. If $G$ is torsion-free, then so is $H$;
4. If $G$ is a solvable group of derived length $l$, then $H$ is a solvable group of derived length $l + 2$.

**Remark 1** Parts (1) and (4) of Theorem 2.1 are explicitly stated in Theorem 1 and Corollary 2 in [8]. However, even though (2) and (3) are not stated explicitly in [8], they follow from the fact that the embedding in [8] embeds countable group $G = \langle X \rangle$ (notations differ there) into a subgroup $H = \langle c, s \rangle$ of the unrestricted wreath product $(G \wr \mathbb{Z}) \wr \mathbb{Z}$ by $\Phi_X : x_n \mapsto [c, c^{s^{2^n}-1}]$ (see Page 5 in [8]). The computability properties of the word problem in $H$ strongly depend on the generating set $X$ and its enumeration.

### 3 The construction

In order to show the existence of $G$ from Theorem 1, first, we will construct a countable but not finitely generated group $\hat{G}$ with decidable word problem. Then $G$ will be defined as a group in which $\hat{G}$ embeds in a certain way.

Let us fix two disjoint recursively enumerated and recursively inseparable sets $\mathcal{N} = \{n_1, n_2, \ldots\} \subset \mathbb{N}$ and $\mathcal{M} = \{m_1, m_2, \ldots\} \subset \mathbb{N}$. For the existence of a pair of sets with these properties see, for example, [19].

For all $n \in \mathbb{N}$, define the group $A_n$ as a free abelian additive group of rank two with basis $\{a_{n,0}, a_{n,1}\}$, that is

$$A_n = \langle a_{n,0} \rangle \bigoplus \langle a_{n,1} \rangle,$$

and such that the groups $A_1, A_2, \ldots$ are disjoint.

For all $n \in \mathbb{N}$, define the groups $\hat{A}_n$ as follows:

$$\hat{A}_n = \begin{cases} A_n & \text{if } n \notin \mathcal{N} \cup \mathcal{M}, \\ A_n/\langle a_{n,1} - 2^i a_{n,0} \rangle & \text{if } n = n_i \in \mathcal{N}, \\ A_n/\langle a_{n,1} - 3^i a_{n,0} \rangle & \text{if } n = m_i \in \mathcal{M}. \end{cases}$$
For all $n \in \mathbb{N}$, define the group $B_n$ as a free abelian additive group of rank 2, that is
\[ B_n = \langle b_{n,0} \rangle \bigoplus \langle b_{n,1} \rangle, \]
such that $B_1$, $B_2$, ... are disjoint.
Now, for all $n \in \mathbb{N}$, define the groups $\hat{B}_n$ as follows:
\[
\hat{B}_n = \begin{cases} 
B_n & \text{if } n \notin \mathcal{N} \cup \mathcal{M}, \\
B_n/(b_{n,1} - 2^i b_{n,0}) & \text{if } n = n_i \in \mathcal{N} \text{ or } n = m_i \in \mathcal{M}.
\end{cases}
\]

For all $n \in \mathbb{N}$ and $\epsilon \in \{0, 1\}$, let us denote the images of $a_{n,\epsilon}$ under the natural homomorphisms $\hat{A}_n \rightarrow \hat{A}_n$ by $\hat{a}_{n,\epsilon}$. Analogously, the image of $b_{n,\epsilon}$ under the natural homomorphism $\hat{B}_n \rightarrow \hat{B}_n$, we denote by $\hat{b}_{n,\epsilon}$. It follows from the definitions of $\hat{A}_n$ and $\hat{B}_n$ that, for all $n \in \mathbb{N}$, these groups are infinite and torsion free.

**Convention** Below, whenever a group $G$ is given with a generating set $S$, depending on the context, we either consider $S \cup S^{-1}$ as a subset of $G$ or we consider it as a set of formal letters (i.e. alphabet) that can form finite words. For example, if $G$ is an additive abelian group and $a, b \in S$, then $ab^{-1}$ and $b^{-1}a$ are finite words over the alphabet $S \cup S^{-1}$ that represent the element $a - b \in G$.

**Lemma 1** There exists an algorithm such that for each input $n \in \mathbb{N}$ and $w \in \{\hat{a}_{n,0}, \hat{a}_{n,1}\}^*$, it decides whether or not $w$ represents the trivial element in the group $\hat{A}_n$. Analogous statement also holds for the groups $\hat{B}_n$ with respect to the alphabet $\{\hat{b}_{n,0}, \hat{b}_{n,1}\}$.

**Proof** Indeed, since $\hat{A}_n$ is abelian with generating set $\{\hat{a}_{n,0}, \hat{a}_{n,1}\}$, each word $w$ from $\{\hat{a}_{n,0}, \hat{a}_{n,1}\}^*$ can be algorithmically transformed to a word of the form
\[ w' = (\hat{a}_{n,0})^{\lambda_0} (\hat{a}_{n,1})^{\lambda_1}, \lambda_0, \lambda_1 \in \mathbb{Z} \]
that represents the same element of $\hat{A}_n$ as the initial word $w$.
Now, assuming that $\lambda_0 \neq 0, \lambda_1 \neq 0$, in order for $w'$ to represent the trivial element in $\hat{A}_n$ it must be that $n \in \mathcal{N} \cup \mathcal{M}$, because if otherwise, then, by definition, the group $\hat{A}_n$ is free abelian of rank 2 with basis $\{\hat{a}_{n,0}, \hat{a}_{n,1}\}$.
In case $n \in \mathcal{N}$, by definition we have that $\hat{a}_{n,1} = 2^x \hat{a}_{n,0}$, where $x$ is the index of $n$ in $\mathcal{N}$, i.e. $n = n_x$. Similarly, in case $n \in \mathcal{M}$, by definition we have that $\hat{a}_{n,1} = 3^x \hat{a}_{n,0}$, where $x$ is the index of $n$ in $\mathcal{M}$, i.e. $n = m_x$.
Now, if $\lambda_0 = 0$ and $\lambda_1 = 0$, then clearly $w'$ (hence also $w$) represents the trivial element in $\hat{A}_n$. Therefore, without loss of generality let us assume that
at least one of $\lambda_0$ and $\lambda_1$ is not 0. Then, if we treat $x$ as an unknown variable, depending on whether $n = n_x$ or $n = m_x$, the equality $w' = 0$ would imply one of the following equations:

$$\lambda_0 + \lambda_1 2^x = 0$$

(1)

or

$$\lambda_0 + \lambda_1 3^x = 0,$$

(2)

respectively.

This observation suggests that in case $\lambda_0 \neq 0$ or $\lambda_1 \neq 0$, in order to verify whether or not $w' = 0$ in $\hat{A}_n$, we can first try to find $x$ satisfying (1) or (2), and in case such an $x$ does not exist, conclude that $w'$ (hence, also $w$) does not represent the trivial element in $\hat{A}_n$. Otherwise, if $x$ is the root of the equation (1), we can check whether or not $n = n_x$ (since $\mathcal{N}$ is recursively enumerable, this checking can be done algorithmically). Similarly, if $x$ is the root of the equation (2), we can check whether or not $n = m_x$. If as a result of this checking we get $n = n_x$ (respectively, $n = m_x$), then the conclusion will be that $w'$ (hence, also $w$) represents the trivial element in $\hat{A}_n$, otherwise, if $n \neq n_x$ (respectively, $n \neq m_x$), then the conclusion will be that $w'$ (hence, also $w$) does not represent the trivial element in $\hat{A}_n$.

The proof of the same statement for the groups $\hat{B}_n$ is identical. □

Define

$$\hat{A} = \bigoplus_{n=1}^{\infty} \hat{A}_n, \quad \hat{B} = \bigoplus_{n=1}^{\infty} \hat{B}_n, \quad \hat{G}_0 = \hat{A} \bigoplus \hat{B} \text{ and } C = \bigoplus_{n=1}^{\infty} \langle c_n \rangle,$$

where $\langle c_1 \rangle, \langle c_2 \rangle, \ldots$ are pairwise disjoint infinite cyclic groups. Denote

$$S_0 = \{\hat{a}_{n,0}, \hat{a}_{n,1}, \hat{b}_{n,0}, \hat{b}_{n,1} \mid n \in \mathbb{N}\}$$

and

$$S_1 = \{c_1, c_2, \ldots\}.$$

**Lemma 2** The word problems $WP_{S_0}(\hat{G}_0)$ and $WP_{S_1}(C)$ are decidable.

**Proof** Let us denote $S_{0,a} = \{\hat{a}_{n,0}, \hat{a}_{n,1} \mid n \in \mathbb{N}\}$ and $S_{0,b} = \{\hat{b}_{n,0}, \hat{b}_{n,1} \mid n \in \mathbb{N}\}$. Since by definition $\hat{G}_0$ is the direct sum of $\hat{A}$ and $\hat{B}$, the decidability of $WP_{S_0}(\hat{G}_0)$ is equivalent to the decidability of both $WP_{S_{0,a}}(\hat{A})$ and $WP_{S_{0,b}}(\hat{B})$. On its own turn, the decidability of $WP_{S_{0,a}}(\hat{A})$ and $WP_{S_{0,b}}(\hat{B})$ is an immediate consequence of Lemma 1.
The decidability of $WP_{S_1}(C)$ is obvious.

Let

$$\psi : C \to Aut(\hat{G}_0)$$

such that for $i \in \mathbb{N}$, $\psi(c_i)$ is the automorphism of $\hat{G}_0$ that fixes all elements from $\hat{G}_0$ except the ones from $\hat{A}_{n_i} \oplus \hat{B}_{n_i} \leq \hat{G}_0$, and for $\epsilon \in \{0, 1\}$, it flips $\hat{a}_{n_i,\epsilon}$ and $\hat{b}_{n_i,\epsilon}$, that is $\psi(c_i) : \hat{a}_{n_i,\epsilon} \mapsto \hat{b}_{n_i,\epsilon}$ and $\psi(c_i) : \hat{b}_{n_i,\epsilon} \mapsto \hat{a}_{n_i,\epsilon}$. The elements $\psi(c_i) \in Aut(\hat{G}_0), i \in \mathbb{N}$, pairwise commute. Thus $\psi$ is the induced homomorphism from $C$ to $Aut(\hat{G}_0)$. Now define $\hat{G}$ as the semidirect product of $\hat{G}_0$ and $C$ with respect to $\psi$, that is

$$\hat{G} = \hat{G}_0 \rtimes_\psi C.$$ 

Denote

$$S = \{\hat{a}_{n,0}, \hat{a}_{n,1}, \hat{b}_{n,0}, \hat{b}_{n,1}, c_n \mid n \in \mathbb{N}\}.$$ 

**Lemma 3** The group $\hat{G}$ is torsion-free and metabelian. Also, for every $i, n \in \mathbb{N}$ and $s \in S_0$, we have

$$c_i^{-1}s c_i = c_i s c_i^{-1} = \begin{cases} \hat{b}_{n,\epsilon} & \text{if } \epsilon \in \{0, 1\}, n = n_i \in \mathcal{N} \text{ and } s = \hat{a}_{n_i,\epsilon} \\ \hat{a}_{n,\epsilon} & \text{if } \epsilon \in \{0, 1\}, n = n_i \in \mathcal{N} \text{ and } s = \hat{b}_{n_i,\epsilon} \\ s & \text{otherwise.} \end{cases}$$

**Proof** The group $\hat{G}$ is a semidirect product of two torsion-free abelian groups, hence it is torsion-free and metabelian. The second statement follows from the definitions of the semidirect product, the automorphism $\psi(c_i) : \hat{G}_0 \to \hat{G}_0$, and from the identities $\hat{a}_{n,1} = 2^i \hat{a}_{n,0}, \hat{b}_{n,1} = 2^i \hat{b}_{n,0}$ and $c_i s c_i^{-1} = \psi(c_i)(s), c_i^{-1}s c_i = \psi(c_i)^{-1}(s) = \psi(c_i)(s)$.

**Lemma 4** The word problem $WP_{S}(\hat{G})$ is decidable.

**Proof** Note that every word from $(S \cup S^{-1})^*$ can be algorithmically transformed into a word of the form $e_1^{u_1} \ldots e_k^{u_k} u_{k+1}$ that represents the same element of $\hat{G}$, where $e_1, \ldots, e_k \in S_0 \cup S_0^{-1}$ and $u_1, \ldots, u_{k+1} \in (S_1 \cup S_1^{-1})^*$. On the other hand, by Lemma 3, for all $i \in \mathbb{N}$ and $s \in S_0 \cup S_0^{-1}, c_i \pm 1 s c_i^{\pm 1}$ is equal to some $\bar{s} \in S_0 \cup S_0^{-1}$. Moreover, since the set $\mathcal{N} = \{n_1, n_2, \ldots\}$ is recursively enumerated, by Lemma 3, the index of $\bar{s} \in S_0 \cup S_0^{-1}$ can be algorithmically computed. Therefore, every word of the form $e_1^{u_1} \ldots e_k^{u_k} u_{k+1}$, on its own turn, can be algorithmically transformed into a word of the form $e_1' \ldots e_k' u_{k+1}$ that represents the same element of $\hat{G}$ as the initial word, where
Let us introduce the enumeration $i : S \rightarrow \mathbb{N}$ of the set $S$:

$$i(s) = \begin{cases} 
5(n-1) + 1 & \text{if } s = \dot{a}_{n,0}, \\
5(n-1) + 2 & \text{if } s = \dot{a}_{n,1}, \\
5(n-1) + 3 & \text{if } s = \dot{b}_{n,0}, \\
5(n-1) + 4 & \text{if } s = \dot{b}_{n,1}, \\
5n & \text{if } s = c_n.
\end{cases}$$

Denote $X = \{x_1, x_2, \ldots\}$, where $x_n = i^{-1}(n), n \in \mathbb{N}$. ($X$ and $S$ coincide as sets but have differing enumerations.) The essential property of $i$ is that it is a computable bijection, which implies that $WP_X(\hat{G})$ is decidable.

Now suppose that $\Phi = \Phi_X : \hat{G} \hookrightarrow H$ is an embedding of the group $\hat{G}$ into a two-generated torsion-free group $H$ such that it satisfies the properties from Theorem 2.1. In particular, the maps $\phi_1$ and $\phi_2$ defined as

$$\phi_1 : (n, \epsilon) \mapsto \Phi(\dot{a}_{n,\epsilon}), \quad \phi_2 : (n, \epsilon) \mapsto \Phi(\dot{b}_{n,\epsilon}), \quad \text{and} \quad \phi_3 : n \mapsto \Phi(c_n),$$

are computable, and $H$ has decidable word problem.

As the next lemma shows, the group $H$ has the desirable properties we were looking for.

**Lemma 5** The group $H$ cannot be embedded in a finitely generated group with decidable conjugacy problem.

**Proof** By contradiction, let us assume that $H$ embeds in a finitely generated group $\tilde{H}$ that has decidable conjugacy problem. Then, without loss of generality we can assume that $H$ coincides with its isomorphic copy in $\tilde{H}$, that is $H \leq \tilde{H}$.

Below we show that the decidability of the conjugacy problem in $\tilde{H}$ contradicts the assumption that $\mathcal{N}$ and $\mathcal{M}$ are recursively inseparable.

Let us define $\mathcal{C} \subseteq \mathbb{N}$ as

$$\mathcal{C} = \{ n \in \mathbb{N} \mid \Phi(\dot{a}_{n,0}) \text{ is conjugate to } \Phi(\dot{b}_{n,0}) \text{ in } \tilde{H} \}.$$ 

Since the above described maps $\phi_1$, $\phi_2$ and $\phi_3$ are computable and $\tilde{H}$ has decidable conjugacy problem, there exists an algorithm that for any $n \in \mathbb{N}$
verifies whether or not $\Phi(\dot{a}_{n,0})$ is conjugate to $\Phi(\dot{b}_{n,0})$ in $\tilde{H}$. Therefore, the set $C$ is recursive.

Note that for all $i \in \mathbb{N}$, since by Lemma 3 the identity $c_i^{-1}\dot{a}_{n_i,0}c_i = \dot{b}_{n_i,0}$ holds, we get that $\Phi(\dot{a}_{n_i,0})$ is conjugate to $\Phi(\dot{b}_{n_i,0})$ in $\tilde{H}$. Therefore, $N \subseteq C$.

Since for groups with decidable word problem one can algorithmically find conjugator element for each pair of conjugate elements of the group, recursiveness of $C$ implies that there exists a computable map

$$f : C \to \tilde{H}$$

such that for all $n \in C$ we have

$$f(n)^{-1}\Phi(\dot{a}_{n,0})f(n) = \Phi(\dot{b}_{n,0}).$$

(For example, one can define $f(n)$ as the element of $\tilde{H}$ that corresponds to the lexicographically smallest conjugator word composed by finite generator letters of $H$.) For $n \in C$, let us denote

$$f(n) = h_n \in \tilde{H}.$$ 

Now let us define

$$A = \{ n \in C \mid h_n^{-1}\Phi(\dot{a}_{n,1})h_n = \Phi(\dot{b}_{n,1}) \} \subseteq \mathbb{N}.$$ 

$A$ is a recursive subset of $\mathbb{N}$, because the word problem in $\tilde{H}$ is decidable, the set $C$ is recursive, and the maps $\Phi$ and $f$ are computable. Also, since

$$\dot{a}_{n_i,1} = 2^i\dot{a}_{n_i,0}, \dot{b}_{n_i,1} = 2^i\dot{b}_{n_i,0} \quad \text{and} \quad c_i^{-1}\dot{a}_{n_i,0}c_i = \dot{b}_{n_i,0}, \quad \text{for} \ i \in \mathbb{N},$$

in $\dot{G}$, we get that each conjugator of the pair $\Phi(\dot{a}_{n_i,0}), \Phi(\dot{b}_{n_i,0})$ is also a conjugator for the pair $\Phi(\dot{a}_{n_i,1}), \Phi(\dot{b}_{n_i,1})$. Therefore, since $N \subseteq C$, we get

$$N \subseteq A.$$ 

On the other hand, since for each $m_i \in M$ we have

$$\dot{a}_{m_i,1} = 3^i\dot{a}_{m_i,0}, \dot{b}_{m_i,1} = 2^i\dot{b}_{m_i,0},$$

we get that the pairs of elements

$$\left( \Phi(\dot{a}_{m_i,0}), \Phi(\dot{b}_{m_i,0}) \right) \text{ and } \left( \Phi(\dot{a}_{m_i,1}), \Phi(\dot{b}_{m_i,1}) \right)$$
cannot be conjugated in $\tilde{H}$ by the same conjugator. Therefore, we get that

$$\mathcal{A} \cap \mathcal{M} = \emptyset.$$ 

Thus we got that $\mathcal{A} \subset \mathbb{N}$ is a recursive set such that $\mathcal{N} \subseteq \mathcal{A}$ and $\mathcal{A} \cap \mathcal{M} = \emptyset$. However, this contradicts the assumption that $\mathcal{N}$ and $\mathcal{M}$ are recursively inseparable. Lemma 5 is proved.

Finally, note that since $\hat{G}$ is metabelian, by property (4) of Theorem 2.1, $H$ is a solvable group of derived length 4. Therefore, Lemma 5 asserts that there exists a solvable group of derived length 4 that satisfies the statement of Theorem 1. Also by a version of Higman’s embedding theorem described by Aanderaa and Cohen in [1], the group $H$ can be embedded into a finitely presented group $\mathcal{G}$ with decidable word problem. As Chiodo and Vyas showed in [4], the group $\mathcal{G}$ defined this way will also inherit the property of torsion-freeness from the group $H$.

Since $H$ cannot be embedded into a group with decidable conjugacy problem, this property will be inherited by $\mathcal{G}$. Thus Theorem 1 is proved.

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