TIME-DEPENDENT EVOLUTION OF QUASI-SPHERICAL, SELF-GRAVITATING ACCRETION FLOW
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ABSTRACT
A self-similar solution for the time evolution of quasi-spherical, self-gravitating accretion flows is obtained under the assumption that the heat generated by viscosity is retained in the flow. The solutions are parameterized by the ratio of the mass of the accreting gas to the central object mass and the viscosity coefficient. While the density and the pressure are obtained simply by solving a set of ordinary differential equations, the radial and the rotational velocities are presented analytically. Profiles of the density and the rotational velocities show two distinct features. A low-density outer accreting flow with a relatively flat rotation velocity surrounds an inner high-density region. In the inner part, the rotational velocity increases from the center to a transition radius that separates the inner and outer portions. We show that the behavior of the solutions in the inner region depends on the ratio of the heat capacities, $\gamma$, and on the viscosity coefficient, $\alpha$.

Subject headings: accretion, accretion disks — hydrodynamics

1. INTRODUCTION
Accretion processes are now believed to play a major role in many astrophysical objects, from protostars to disks around compact stars and active galactic nuclei. Such systems have been studied at different levels depending on their physical properties. The geometry of the disk (thin or thick), transport of the thermal energy inside the disk, self-gravity of the accreting gas, and the magnetic fields are among the most important factors that shape any theory for such systems. For simplicity, traditional models of accretion disks assume a geometrically thin configuration and neglect the self-gravity of the accreting material (Shakura & Sunyaev 1973; Pringle 1981). These models have been extended by considering large-scale magnetic fields and a polytropic equation of state and using a better understanding of the mechanism of angular momentum transport (e.g., Livio & Pringle 1992; Ogilvie 1997; Duschl et al. 2000). However, the key ingredient in all such models is that the heat generated by turbulent viscosity does not remain in the flow. In other words, all of the viscously dissipated energy is assumed to be radiated away immediately.

Another type of accretion flow known as advection-dominated accretion flow (ADAF) has been proposed, in which the heat generated by viscosity cannot escape from the system and is retained in the flow (Ichimaru 1977; Narayan & Yi 1994). During recent years, ADAFs have been paid attention as plausible states of accretion flows around black holes, active galactic nuclei, or dim galactic nuclei (for a review, see Kato et al. 1998). Using a similarity method, Ogilvie (1999, hereafter OG99) extended the original steady state self-similar solutions to the time-dependent case. His solutions describe quasi-spherical, time-dependent, ADAFs.

On the other hand, many authors tried to study the effects related to the disk self-gravity. Recent observations show significant deviations from Keplerian rotation in objects that are believed to be accretion disks, and in some cases, there is strong evidence that the amount of mass in the disk is large (e.g., Drimmel 1996; Greenhill et al. 1996). The role of self-gravity in accretion disks was discussed by Paczynski (1978), who studied the vertical structure of disks under the influence of self-gravity. Some authors investigated the role of self-gravity on waves in disks (Lin & Pringle 1987, 1990). Within the framework of a geometrically thin configuration, Mineshige & Umemura (1996) extended the classical self-similar ADAF solution (Narayan & Yi 1994) and found global one-dimensional disk solutions influenced by self-gravity in both the radial and perpendicular directions of the disk. In another study, Mineshige & Umemura (1997) extended the previous steady state solutions to the time-dependent case while the effect of the self-gravity of the disk was taken into account. They used an isothermal equation of state, and so their solutions describe a viscous accretion disk in the slow accretion limit. Then, Mineshige et al. (1997) extended this study by obtaining solutions for polytropic viscous accretion disks. In addition, Tsuribe (1999) studied the self-similar collapse of an isothermal viscous accretion disk.

Although the solution of Mineshige & Umemura (1996) describes self-gravitating ADAFs, the applicability of the solution is restricted to geometrically thin configurations. The most appropriate geometrical configuration for advective flows, rather than thin, is quasi-spherical, as has been confirmed by many authors (e.g., Narayan & Yi 1995). OG99 presented the first semianalytical, quasi-spherical, time-dependent ADAF solution. Here we note that although OG99 considered ADAF in a point-mass potential, it is straightforward to relax this assumption so as to describe the effect of self-gravitation on quasi-spherical ADAFs using time-dependent similarity solutions.

This paper is organized as follows: In § 2 the general problem of constructing a model for a quasi-spherical, self-gravitating accretion flow is defined. The self-similar solutions are presented in § 3, and the effects of the input parameters are examined. We summarize the results in § 4.

2. FORMULATION OF THE PROBLEM
We start with the approach adopted by OG99, who studied quasi-spherical accretion flows without self-gravity. In his approach, the equations are written in spherical coordinates $(r, \theta, \varphi)$ by considering the equatorial plane $\theta = \pi/2$ and neglecting terms with any $\theta$- and $\varphi$-dependence. This implies that each physical variable represents an approximately spherically averaged quantity. Therefore, all the physical quantities depend
only on the spherical radius \( r \) and time \( t \). The self-gravity of the accreting gas can be described simply by the Poisson equation and the corresponding term in the momentum equation. The governing equations are the continuity equation,

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho v_r \right) = 0,
\]

the equations of motion,

\[
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho \right) + \frac{GM_*}{r^2} = \frac{v_r^2}{r},
\]

\[
\rho \left[ \frac{\partial}{\partial t} \left( r v_r \right) + v_r \frac{\partial}{\partial r} \left( r v_r \right) \right] = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ \nu \rho r^2 \frac{\partial}{\partial r} \left( \frac{v_r}{r} \right) \right],
\]

the Poisson equation,

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi}{\partial r} \right) = 4\pi G \rho,
\]

and the energy equation,

\[
\frac{1}{\gamma - 1} \left( \frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} \right) + \frac{\gamma}{\gamma - 1} \frac{p}{r^2} \frac{\partial}{\partial r} \left( r^2 v_r \right) = \nu r^2 \frac{\partial}{\partial r} \left( \frac{v_r}{r} \right)^2.
\]

In order to solve the equations, we need to assign the kinematic coefficient of viscosity \( \nu \). Although there is much uncertainty about the exact form of the viscosity, authors introduce some prescriptions for \( \nu \) through dimensional analysis or just based on phenomenological considerations (e.g., Duschl et al. 2000). We employ the usual \( \alpha \)-prescription for the viscosity, which we write as (Shakura & Sunyaev 1973)

\[
\nu = \alpha \frac{p}{\rho \Omega_K},
\]

where \( \Omega_K = (GM_*/r^3)^{1/2} \) is the Keplerian angular velocity at radius \( r \). This prescription was originally introduced for viscosity in a thin accretion disk; however, it has been widely used for studying the dynamics of thick accretion disks, such as ADAFs.

To simplify the equations, we make the following substitutions:

\[
\rho \rightarrow \tilde{\rho}, \quad p \rightarrow \tilde{p}, \quad v_r \rightarrow \tilde{v}, \quad v_{r,\varphi} \rightarrow \tilde{v}_{r,\varphi}, \quad \Psi \rightarrow \tilde{\Psi}, \quad r \rightarrow \tilde{r}, \quad t \rightarrow \tilde{t},
\]

where

\[
\tilde{v} = \sqrt{\frac{GM_*}{r}}, \quad \tilde{t} = \frac{t}{1/\sqrt{4\pi G \rho}}, \quad \tilde{p} = \tilde{\rho} \tilde{v}^2, \quad \tilde{\Psi} = \frac{GM_*}{r}.
\]

Under these transformations, the continuity equation does not change, and the rest of equations are cast into

\[
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{1}{\rho} \frac{\partial}{\partial r} \left( \rho \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2} = \frac{v_r^2}{r},
\]

\[
\rho \left[ \frac{\partial}{\partial t} \left( r v_r \right) + v_r \frac{\partial}{\partial r} \left( r v_r \right) \right] = \alpha \frac{1}{r^2} \frac{\partial}{\partial r} \left[ \nu r^{1/2} \frac{\partial}{\partial r} \left( \frac{v_r}{r} \right) \right],
\]

\[
\frac{1}{\gamma - 1} \left( \frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} \right) + \frac{\gamma}{\gamma - 1} \frac{p}{r^2} \frac{\partial}{\partial r} \left( r^2 v_r \right) = \alpha p r^{7/2} \left( \frac{\partial}{\partial r} \left( \frac{v_r}{r} \right) \right)^2.
\]

3. SELF-SIMILAR SOLUTIONS

We look for self-similar solutions of equations (1) and (9)–(12) and reduce this system into a set of ordinary differential equations. We define a self-similar variable

\[
\xi = \frac{r}{(t_0 - t)^{2/3}},
\]

where \( t < t_0 \), and demand that

\[
\rho(r, t) = (t_0 - t)^{-2} R(\xi),
\]

\[
p(r, t) = (t_0 - t)^{-8/3} P(\xi),
\]

\[
v_r(r, t) = (t_0 - t)^{-1/3} V(\xi),
\]

\[
v_{r,\varphi}(r, t) = (t_0 - t)^{-1/3} \Phi(\xi),
\]

\[
\Psi(r, t) = (t_0 - t)^{-2/3} S(\xi).
\]

Substituting these expressions into the above equations, we obtain the following set of self-similar equations:

\[
-\frac{2}{9} \xi + \frac{1}{R} \frac{dR}{d\xi} + \frac{1}{S} \frac{dS}{d\xi} + \frac{1}{\xi^2} = \frac{\Phi^2}{\xi},
\]

\[
-\xi^3 R \left( \Phi + 2\xi^2 \frac{d\Phi}{d\xi} \right) = 3\alpha \frac{d}{d\xi} \left[ P \xi^{11/2} \frac{d}{d\xi} \left( \frac{\Phi}{\xi} \right) \right],
\]

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{dS}{d\xi} \right) = R,
\]

\[
\frac{2}{3} \left( \frac{4 - 3\gamma}{\gamma - 1} \right) = \alpha \xi^{7/2} \left[ \frac{d}{d\xi} \left( \frac{\Phi}{\xi} \right) \right]^2.
\]

Interestingly, the continuity equation is integrable and gives

\[
\xi^2 R(\xi + 3V/2) = C,
\]

where \( C \) is a constant and should be zero. Thus,

\[
V = -\frac{2}{3} \xi.
\]

In addition, we can obtain the rotational similarity function \( \Phi(\xi) \) simply by integrating equation (22),

\[
\Phi(\xi) = \frac{\xi}{\xi_S} \Phi_S + \frac{4\xi}{3} \left( \xi^{1/4} - \xi_S^{-3/4} \xi \right),
\]

where

\[
\xi_S = \sqrt{\frac{2}{3\alpha} \left( \frac{4 - 3\gamma}{1 - \gamma} \right)}
\]

and \( \Phi_S \) is the rotational velocity at some \( \xi_S \). For simplicity, we assume that \( \xi_S \) defines the outer boundary of the accreting gas. This equation correctly gives \( \Phi(\xi = 0) = 0 \) and reaches a
maximum $\Phi_m$ by increasing $\xi$ from zero to a point at $\xi_m$. One can simply show that

$$\xi_m = \frac{1}{4} \sqrt[4]{\frac{\xi_s}{\sqrt{\xi_s} - \sqrt{\xi_0}}} \left(\frac{\xi_s}{\sqrt{\xi_s} - \sqrt{\xi_0}} \right)^{4/3},$$

(26)

where $\xi_0 = 3\Phi_s/4\lambda$, and

$$\Phi_m = \frac{\lambda}{\sqrt{4}} \left(\frac{\xi_s}{\sqrt{\xi_s} - \sqrt{\xi_0}} \right)^{1/3}.$$

(27)

Clearly, the behavior of the rotational velocity and the other physical variables, except for the radial velocity, depend on $\xi_s$ and $\Phi_s$. We can consider both $\xi_s$ and $\Phi_s$ as free parameters; however, it is possible to determine them uniquely, if we parameterize the self-similar solutions using conserved quantities, e.g., the mass of the system. As can be seen from the definition of $\lambda$, the above self-similar solutions are applicable within the range of $1 < \gamma < 4/3$ if we consider a positive value for $\alpha$.

We can derive asymptotic solutions when approaching the origin $\xi = 0$ as follows:

$$\Phi(\xi) \rightarrow \frac{4}{3} \lambda^{1/4},$$

(28)

$$R(\xi) \rightarrow 4 \left(\frac{2}{3} \lambda^2 + \frac{1}{11\alpha} \right) \xi^{-3/2},$$

(29)

$$P(\xi) \rightarrow \frac{32}{33\alpha} \left(\frac{2}{3} \lambda^2 + \frac{1}{11\alpha} \right) \xi^{-1}.$$  

(30)

These asymptotic solutions are valuable when performing numerical integrations to obtain similarity solutions starting from $x \rightarrow 0^+$. Note that the behavior of the solutions near the origin depends only on $\lambda$ and $\alpha$.

We now proceed to solve the rest of the above similarity equations. By substituting equation (24) into equation (20), we obtain

$$\frac{15}{4} P + \xi \frac{dP}{d\xi} = f(\xi) R,$$

(31)

where

$$f(\xi) = \frac{\xi^{1/4}}{3\alpha \lambda} \left[ 3 \frac{\Phi_s}{\xi_s} \xi + 4 \lambda \left(\frac{1}{2} \xi^{1/4} - \xi^{-3/4}\right) \right].$$

(32)

After some algebra, from this equation and equations (19) and (21), the following equation is obtained:

$$\frac{15}{4} f P \frac{d^2 P}{d\xi^2} = S_1 \left(\xi, P, \frac{dP}{d\xi}\right) + S_2 \left(\xi, P, \frac{dP}{d\xi}\right) + S_3 \left(\xi, P, \frac{dP}{d\xi}\right),$$

(33)

where

$$S_1 = \frac{19}{4} f \left(\frac{dP}{d\xi}\right)^2 - \left(\frac{df}{d\xi}\right) \left(\frac{dP}{d\xi}\right) \left(\frac{15}{4} P + \xi \frac{dP}{d\xi}\right),$$

(34)

$$S_2 = \frac{9}{\xi_s} \left(\frac{15}{4} P + \xi \frac{dP}{d\xi}\right)^2 - \frac{1}{f} \left(\frac{15}{4} P + \xi \frac{dP}{d\xi}\right)^3,$$

(35)

and

$$S_3 = -2 f \frac{dP}{d\xi} \left(\frac{15}{4} P + \xi \frac{dP}{d\xi}\right).$$

(36)

This equation can easily be integrated by considering appropriate boundary conditions. We can parameterize the solutions by $\xi_s$ as a function of the mass of the accreting gas and the viscosity parameter $\alpha$. For simplicity, we assume $R(\xi_s) = R_s$ and $P(\xi_s) = P_s$. The total mass of the accreting gas is

$$M = 4\pi \int_0^\infty \rho r^2 dr.$$  

(38)

Using similarity solutions, this equation reads

$$M = M_s \int_0^{\xi_s} R^2 \frac{dS}{d\xi}.$$

(39)

Substituting from Poisson equation (21), we have

$$M = M_s \xi_s^2 \frac{dS}{d\xi}.\frac{d\xi}{dS}.$$  

(40)

By inserting $dS/d\xi$ from equation (19) into the above equation, we finally obtain

$$M = M_s \left[ \xi_s \Phi_s^2 + \frac{2}{9} \xi_s^3 - \frac{2}{3\alpha \lambda} \xi_s \left(\frac{dP}{d\xi}\right) \right]^{-1}.1.$$  

(41)

Another relation is obtained from equation (31), as follows:

$$\frac{15}{4} P_s + \xi_s \left(\frac{dP}{d\xi}\right) = f(\xi_s) R_s,$$

(42)

where

$$f(\xi_s) = \frac{\sqrt{\xi_s}}{3\alpha \lambda} \left(3 \Phi_s - 2 \lambda \sqrt{\xi_s}\right).$$

(43)

Now we can use the standard fourth-order Runge-Kutta scheme to integrate the nonlinear ordinary differential equation (33) from sufficiently small $\xi$ to $\xi_s$. Given $\xi_s$ and $\Phi_s$ and using asymptotic solutions near the origin (i.e., eqs. [29] and [30]), one can start numerical integration, and equation (41) gives the corresponding ratio of masses, i.e., $M/M_s$. Before presenting the results of integration, it would be interesting to explore the typical behavior of the rotational velocity $\Phi(\xi)$.

Just as an illustrative example, we follow another approach: Let us assume that we know $R_s$, $P_s$, and $M/M_s$ from numerical integration and so can determine $\xi_s$ and $\Phi_s$ from equations (41) and (42) uniquely. For example, if we set $P_s = 0$ and $(dP/d\xi)_s = 0$ and consider a nonzero but negligible $R_s$, the outer boundary of the system is determined analytically,

$$\xi_s = \sqrt[4]{2\lambda^4 + 4.5 \left(\frac{M}{M_s} + 1\right)} - \sqrt[4]{4\lambda^4 + 4.5 \left(\frac{M}{M_s} + 1\right)},$$

(44)

$$\Phi_s = \frac{2}{3} \lambda \sqrt[4]{\xi_s}.$$  

(45)
In this case, Figure 1 shows \( \xi_8 \) and \( \xi_{24} \) as functions of \( \xi_{11} \) for \( M/M_* = 0.6, 1, 1.5, \) and \( 2.5 \). As this figure shows, for a fixed ratio of masses, the outer radius of the system increases by increasing the viscosity coefficient \( \xi_{11} \). In addition, if the ratio of masses increases, the outer radius increases irrespective of the exact value of \( \xi_{11} \). However, the rotational velocity at \( \xi_{8} \) (i.e., \( \Phi_{8} \)) is not very sensitive to values of \( \xi_{11} \) or \( M/M_* \) for large values of \( \xi_{11} \). If the viscosity coefficient \( \xi_{11} \) tends to small values, on the other hand, the rotational velocity \( \Phi_{8} \) increases significantly.

In Figure 2 we plot the rotational velocity of representative cases with \( \xi_{11} = 0.01 \) and \( 0.1 \) and \( M/M_* = 0.6, 1.5, \) and \( 2.5 \). As this figure shows, the rotation of the flow increases from the center to the outer radius. There are two regimes as far as the rotation of the flow is concerned. While in the outer part the profile of the rotational velocity in similarity space is nearly flat, in the inner portion this velocity strongly increases. However, for low values of \( \xi_{11} \) (e.g., 0.01), the rotational velocity reaches a maximum and then decreases with a nearly constant and small slope. In addition, the rotational velocity increases as the parameter \( \xi_{11} \) decreases, and the outer region with a flat rotation profile becomes larger, as the ratio of the mass of the accreting gas to the mass of the central object increases. In other words, while the rotational behavior of the inner part is nearly independent of the ratio of masses, the outer portion’s rotation and its extension are sensitive to that ratio.

In Figure 3 the typical behavior of the density and the pressure in similarity space is shown. We can see in the inner region that the solutions can clearly be described by asymptotic equations (29) and (30); however, the outer region has different density and pressure profiles. Interestingly, the behavior of the solutions in the inner region is independent of the extension of the system (\( \xi_{8} \)) and the rotational velocity \( \Phi_{8} \). As the ratio of \( M/M_* \) increases, the extension of the outer region becomes larger. Flows with large values of \( \xi_{11} \) have a lower central mass concentration compared to accretions with low values of \( \xi_{11} \). Generally, one can say that the quasi-spherical, self-gravitating accretion flows of our model consist of two parts: one inner portion with high density and an outer part with a lower density and a nearly flat rotation profile.

**Fig. 1.**—Typical behavior of \( \xi_8 \) and \( \Phi_8 \) as a function of \( \alpha \) corresponding to \( \gamma = 5/4 \) and \( R_s = 0 \). Left: Curves marked by the ratio of the mass of the accreting gas to the mass of the central object, \( M/M_* \). Right: Lowest curve corresponding to \( M/M_* = 0.6 \), shifting upward as this ratio increases.

**Fig. 2.**—Self-similar rotational velocity corresponding to \( \gamma = 5/4 \) and \( R_s = P_s = 0.01 \). Each curve is marked by the ratio of the mass of the accreting gas to the mass of the central object, \( M/M_* \), and \( \xi_8 \) and \( \Phi_8 \) as \( (M/M_*, \xi_8, \Phi_8) \).
Fig. 3.—Self-similar density and the pressure corresponding to $\gamma = 5/4$ for different values of the viscosity coefficient, i.e., $\alpha = 0.01$ and 0.04. Curves are labeled the same as in Fig. 2.

Fig. 4.—Ratio of the radial velocity to the rotational velocity for $\gamma = 5/4$, $R_s = P_s = 0.01$, and $\alpha = 0.1$ and 0.01. Curves are labeled by the ratio of the masses, i.e., $M/M_*$. 
However, in both regions the radial velocity is the same, irrespective of the input parameters.

From equations (16) and (17), we can simply show that the ratio of the radial velocity to the rotational velocity is \( v_r(r, t)/v_\Phi(r, t) = V/\Phi \). Figure 4 shows this ratio for \( \alpha = 0.1 \) and 0.01 and three different values of \( M/M_* \). Clearly, this ratio of velocities is much smaller than unity, implying that at each radius \( r \) and at any time \( t \), the rotational velocity is greater than the radial velocity. However, as the viscosity parameter \( \alpha \) increases, the ratio increases as well. This behavior is easy to understand, because we showed that the radial similarity velocity \( V \) is proportional to the similarity variable \( \xi \), independent of the input parameters. However, the rotational similarity velocity \( \Phi \) decreases with increasing parameter \( \alpha \), as can be seen from Figure 2. Therefore, the ratio \( V/\Phi \) increases in the case that \( \alpha \) increases.

As we showed, the total mass of the accreting gas is conserved. We can simply write the integral representing the angular momentum as 

\[
\frac{M}{M(M_* - M)} \int v_r(r, t) \, dr = \frac{1}{2} \int \frac{v_r(r, t)}{v_\Phi(r, t)} \, dr = \frac{1}{2} \int \frac{V}{\Phi} \, dr.
\]

Using the similarity solutions, we can show that the total angular momentum \( J \) is proportional to \((t_0 - t)^{1/3}\). In non–self-gravitating, quasi-spherical accretion flow, \( J \) is conserved, and the total mass \( M \) is proportional to \((t_0 - t)^{-1/3}\), as has been shown by OG99. But our solutions imply that the total mass of accreting material is conserved, and the angular momentum is decreasing. In addition, the central density \( \rho_c \) increases as \( \rho_c \propto (t_0 - t)^{-2}\). For a non–self-gravitating flow, we see another behavior: \( \rho_c \propto (t_0 - t)^{-7/3}\). However, the radius of the flow in both cases decreases in proportion to \((t_0 - t)^{2/3}\).

4. SUMMARY

In this paper, we have studied quasi-spherical accretion flow, in which the heat generated by viscosity is retained in the flow. In opposition to the usual studies performed up until now, we have considered the self-gravity of the flow. We derived similarity solutions for flows that are applicable within the range \( 0 < \gamma < 4/3 \), if we consider a positive value of \( \alpha \). Radial and rotational velocities have been obtained analytically. We obtained solutions parameterized by the ratio of the disk mass to the central object mass, \( M/M_* \), and the viscosity parameter, \( \alpha \). We showed that the extension of the accreting gas depends on this ratio and the viscosity parameter. More importantly, these input parameters have a direct effect on the rotational velocity.

Our solutions are different from the solutions by OG99 in various respects. We found that the radial similarity velocity is in proportion to \( \xi \), implying no critical point. This fortunate circumstance let us integrate the rest of the equations simply, although physically one should bear in mind that this kind of velocity (i.e., independent of input parameters) is a result of the mathematical limitations of the similarity method. At the outer edge of the accreting gas, the density and the pressure have low values compared to the central region.

Other viscosity laws have been proposed, for instance, the \( \beta \)-prescription, which is based on analogy with the turbulence observed in laboratory sheared flows and gives \( \nu \propto v_r r \) (Duschl et al. 2000). Although we have not explored self-gravitating accretion with this prescription, there are self-similar solutions with similarity indices the same as those found with the \( \alpha \)-prescription in this paper. However, ordinary differential equations governing the similarity physical variables are different and should be solved numerically. It would be interesting to compare the similarity solutions with a \( \beta \)-prescription with those we have obtained here.

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