Nuclear Equation of State and Neutron Star Cooling

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Abstract
We investigate the effects of the nuclear equation of state (EoS) to the neutron star cooling. New era for nuclear EoS has begun after the discovery of $\sim 2M_\odot$ neutron stars PSR J1614–2230 and PSR J0348+0432 \cite{1,2}. Also recent works on the mass and radius of neutron stars from low-mass X-ray binaries \cite{3} strongly constrain the EoS of nuclear matter. On the other hand, observations of the neutron star in Cassiopeia A (Cas A) more than 10 years confirmed the existence of nuclear superfluidity \cite{4,5}. Nuclear superfluidity reduces the heat capacities as well as neutrino emissivities. With nuclear superfluidity the neutrino emission processes are highly suppressed, and the existence of superfluidity makes the cooling path quite different from that of the standard cooling process. Superfluidity also allows new neutrino emission process, which is called ‘Pair Breaking and Formation’ (PBF). PBF is a fast cooling process and can explain the fast cooling rate of neutron star in Cas A. Therefore, it is essential to add the superfluidity effect in the neutron star cooling process. In this work, we simulate neutron star cooling curves using both non-relativistic and relativistic nuclear models. The existence of too early direct Urca process shows that some of nuclear models do not fit for the cooling simulation. After this first selection process, the nuclear pairing gaps are searched using the observational neutron star’s age and temperature data.

Keywords: Neutron star, Nuclear equation of state, Neutron star cooling, Nuclear superfluidity

1. Introduction
A neutron star (NS) is born as a result of the core collapse supernova explosion if the initial mass of the main sequence star is around 8 to 20 solar mass ($M_\odot$). Various EoS, relations between the density and the pressure, are used to describe the interior of neutron stars. The resulting central density of a neutron star is expected to reach up to several times of the nuclear saturation density ($n_0 \approx 0.16 \text{ fm}^{-3}$). Hence, a neutron star is one of the best astrophysical laboratories to study the physics of the extremely dense nuclear matter.

Figure 1 shows the structure of neutron stars in the theoretical point of view. Outside the envelope, there exists a very thin atmosphere which is composed of hydrogen and heavy elements \cite{6}. Depending on the temperature, its thickness ranges from a few centimeters to a few millimeters and the thermal radiation is expected to occur in this region. The envelope plays a role as a thermal insulator \cite{7} between the outer crust and the atmosphere. In the outer crust, heavy nuclei exist

\hspace{1cm} \text{Keywords: Neutron star, Nuclear equation of state, Neutron star cooling, Nuclear superfluidity}
with free gas of electrons, and the BCC (Body Centered Cubic) structure of nuclei is expected. As the density increases, neutrons drip out of heavy nuclei and can exist as free gas in the inner crust. The neutron $^1S_0$ superfluidity can also exist in the inner crust. In the boundary of the outer core and the inner crust, nuclear pasta phase may appear due to the nature of Coulomb interaction [8, 9], which favours a special geometrical shape to reduce the free energy. Above half of the nuclear saturation density uniform nuclear matter exists. Outer and inner cores occupy roughly 70% of neutron star’s volume. In the outer core the uniform nuclear matter can exist and neutron $^3P_2$ and proton $^1S_0$ superfluidities are expected. In the inner core exotic states with new degrees of freedom, such as kaons, hyperons, and deconfined quarks might appear. However, the inner structure of neutron star is still unknown due to the theoretical uncertainties, especially at the densities far beyond the nucleon saturation density.

The mass distributions of observed neutron stars are summarized in Table 1 of Lattimer [10]. Note that the mass distribution of NS-NS binaries has very sharp peaks since the error bars of mass measurements are relatively small, and the well measured neutron stars in NS-NS binaries have masses $< 1.5M_\odot$. On the other hand, $2M_\odot$ neutron stars have been observed in NS-WD (white dwarf) binaries. This implies that the mass distribution may depend on the binary evolution in addition to the neutron star equation of states [11]. Recent observations of $\sim 2M_\odot$ neutron stars ruled out many soft equations of states with which the maximum mass of neutron star becomes less than $2M_\odot$ [1, 2]. Another constraint on the nuclear equation of state comes from the X-ray binary observations and their analysis [3]. Mass distribution in X-ray binaries is very broad without sharp peak indicating that the uncertainties in the mass estimations are very large.

In Table 1 and Fig. 2 we plot the surface temperatures ($T_s^\infty$) and photon luminosities ($L^\infty$) of 18 observed neutron stars. Note that three different models are used to estimate the effective surface temperatures of neutron stars at infinity. For black body (BB) model, the photon luminosity...
\( L^\infty \) and the effective surface temperature \( T^\infty_s \) of a neutron star at infinity are related by

\[
L^\infty = 4\pi (R^\infty)^2 \sigma_{SB} (T^\infty_s)^4
\]

(1)

where \( R^\infty \) is the radius seen by an observer at infinity and \( \sigma_{SB} \) is the Stefan-Boltzmann constant. We can determine \( R^\infty \) by inserting \( T^\infty_s \) and \( L^\infty \) in the Table 1 into Eq. (1).

In recent works [12, 13], we have investigated the behavior of nuclear EoS at high densities by calculating the mass and radius of neutron stars with several Skyrme force models. We could confirm that the models that satisfy the large mass observation [1, 2] are consistent with the mass-radius zone in [3]. Moreover, the conclusion doesn’t change even if we include exotic degrees of freedom such as kaon condensation [12] or hyperons [13]. On the other hand, nuclear EoS is one of the key ingredients that determine the thermal evolution of neutron stars. Available nuclear models predict very diverse mass-radius relations, so it is well expected that the cooling behavior will be sensitive to nuclear models. Therefore, cooling of neutron stars provides a multi-test ground for the nuclear models; whether a model good for mass-radius relation is consistent with the observed temperature data or not.

In this work, we simulate the cooling curve of neutron stars with various EoS obtained from both relativistic and non-relativistic models. Without the superfluidity effect, the direct Urca process is a good indicator whether a specific nuclear model is suitable for neutron star cooling simulation. If the temperature drop is too fast even for the low mass neutron stars (\( \sim 1.2 \, M_\odot \)), such EoS is not consistent with most of the neutron star temperature observations. Even though the existence of superfluidity highly suppresses neutrino emissivity, once the direct Urca process is turned on before the critical temperature, the superfluidity cannot slow down the temperature drop so that the temperature of neutron stars is much below any observational data. Thus, the study of cooling curve of neutron star can provide hints about the inner structure of neutron stars and the EoS of dense nuclear matter. In addition to the nuclear EoS, physical conditions such as the composition of elements in the envelope, and existence of superfluidity in the core play crucial roles in determining the cooling curve of neutron stars. In general, chemical composition of the envelope seldom affects the cooling mechanism, but the surface temperature depends sensitively on whether the elements in the envelope are light or heavy. On the other hand, superfluidity directly determines the cooling rate. If nucleons form a cooper pair and transit to a superfluidic state, the rate of neutrino emissivity is suppressed exponentially. This may lead to a very slow cooling rate. However, below a critical temperature, creation and destruction of cooper pairs ignite a fast cooling mechanism, and this can give an abrupt decrease of temperature. Recent literature succeeds to reproduce the cooling curve of Cas A in terms of PBF [4, 5]. In this work, we incorporate PBF to various nuclear models and explore the extent to which the models can reconcile with PBF.

The structure of the paper is as follow. In Section 2 we summarize the equation of state of nuclear matter which we used in the study of thermal evolution of neutron stars. In this work, we do not consider exotic matter or quark matter in the core of neutron stars but assume that the core is composed of nucleons (protons and neutrons) and leptons (electrons and muons) in the form of uniform matter. In Section 3 we present the basic equations for neutron star cooling. Simple critical temperature formula is proposed and we analyze the pairing effects to neutron star cooling. In Section 4 we present the cooling curves for the standard cooling in which superfluidity is neglected with various nuclear models. In Section 5 we discuss the effect of the superfluidity to the cooling process. In Section 6 we give conclusions from neutron star cooling curves combined with various EoS. In Appendix, we summarize numerical methods to solve the diffusion equations for the thermal evolution of neutron stars.
Table 1: Thermal emission from isolated neutron stars. Temperatures were obtained using three different models; hydrogen atmosphere (HA), magnetized hydrogen atmosphere (mHA), and black-body (BB) model. For sources with no. 2 ∼ 7, two different models were used to link the effective temperature and the photon luminosity. Sources with no. 16 ∼ 17 have the limited observational data, thus have only one limit.

| No | Source         | Log($t_{sd}$/yr) | Log($t_{kin}$/yr) | Log($T_\infty$/K) | Log($L_\infty$/erg s$^{-1}$) | Model | Ref. |
|----|----------------|-----------------|-----------------|------------------|-----------------|-------|------|
| 1  | PSR J1119-6127 | 3.20            | -               | 6.08$^{+0.09}_{-0.07}$ | 32.82 - 33.66   | mHA   | [14] |
| 2  | RX J0822-4247† | 3.90            | 3.57$^{+0.04}_{-0.04}$ | 6.24$^{+0.04}_{-0.04}$ | 33.85 - 34.00   | HA    | [15, 16] |
| 3  | 1E 1207.4-5209 | 5.53$^{+0.44}_{-0.19}$ | 3.85$^{+0.48}_{-0.48}$ | 6.21$^{+0.07}_{-0.07}$ | 33.27 - 33.74   | HA    | [17] |
| 4  | PSR J1357-6429 | 3.86            | -               | 5.88$^{+0.04}_{-0.04}$ | 32.46 - 32.80   | mHA   | [18] |
| 5  | RX J0002+6246  | -               | 3.96$^{+0.08}_{-0.08}$ | 6.03$^{+0.03}_{-0.03}$ | 33.08 - 33.33   | HA    | [19] |
| 6  | PSR B0833-45†  | 4.05            | 4.26$^{+0.17}_{-0.31}$ | 5.83$^{+0.02}_{-0.02}$ | 32.41 - 32.70   | mHA   | [20] |
| 7  | PSR B1706-44   | 4.24            | -               | 5.80$^{+0.13}_{-0.13}$ | 31.81 - 32.93   | mHA   | [21] |
| 8  | PSR J0538+2817 | 4.47$^{+0.05}_{-0.06}$ | -               | 5.94$^{+0.08}_{-0.08}$ | 32.32 - 33.33   | mHA   | [22] |
| 9  | PSR B2334+61   | 4.61            | -               | 5.84$^{+0.08}_{-0.08}$ | 31.93 - 32.96   | mHA   | [23] |
| 10 | PSR B0656+14   | 5.04            | -               | 5.71$^{+0.03}_{-0.04}$ | 32.18 - 32.97   | BB    | [24] |
| 11 | PSR B0633+1748†| 5.53            | -               | 5.75$^{+0.04}_{-0.05}$ | 30.85 - 31.51   | BB    | [25] |
| 12 | RX J1856.4-3754| -               | 5.70$^{+0.05}_{-0.25}$ | 5.63$^{+0.08}_{-0.08}$ | 31.32 - 32.35   | mHA   | [26] |
| 13 | PSR B1055-52   | 5.73            | -               | 5.88$^{+0.08}_{-0.08}$ | 32.05 - 33.08   | BB    | [27] |
| 14 | PSR J0243+2740 | 6.08            | -               | 5.64$^{+0.08}_{-0.08}$ | 29.10 - 30.13   | mHA   | [28] |
| 15 | RX J0720.4-3125| 6.11            | -               | 5.70$^{+0.08}_{-0.08}$ | 31.37 - 32.40   | HA    | [29] |
| 16 | PSR J0205+6449††| -              | 2.91            | < 6.01           | < 33.29         | BB    | [30] |
| 17 | PSR B0531+21†† | -               | 3.0             | < 6.30           | < 34.45         | BB    | [31] |
| 18 | RX J0007.0+7303††| -              | 4.0-4.4         | < 5.82           | < 32.54         | BB    | [32] |

† Alternative names: Puppis A (PSR B0822-4247), Vela (PSR B0833-45), Geminga (PSR B0633+1748).
†† PSR J0205+6449 is a pulsar in supernova remnant 3C 58, PSR B0531+21 is in SN 1054 in Crab Nebula, and RX J0007.0+7303 is in the CTA1.
2. Neutron Star Equation of State

Nuclear matter properties beyond nuclear saturation density are still unknown and different EoSs give quite different masses and radii of neutron stars. In order to investigate the properties of neutron star matter, we first consider both relativistic and non-relativistic models for the neutron star core, which are consistent with $2.0 M_\odot$ neutron stars \(1, 2\). We consider the crust of neutron star separately because heavy nuclei can exist in the crust. The properties of neutron star crust is very important in understanding the neutron star properties in low-mass X-ray binaries.

2.1. Non-relativistic nuclear force model: energy density functional

For the non-relativistic nuclear force model, we use Skyrme force model to obtain the equation of state for nuclear matter \(3, 2\). In Skyrme force model, the interaction between two nucleons has the form of

$$
\hat{\sigma}_F(r_i, r_j) = t_0(1 + x_0 \hat{P}_\sigma)\delta(r_i - r_j) + \frac{t_1}{2}(1 + x_1\hat{P}_\sigma)\left[\delta(r_i - r_j)\hat{k}^2 + \hat{k}^2 \delta(r_i - r_j)\right] + t_2(1 + x_2\hat{P}_\sigma)\hat{k}^2 \cdot \delta(r_i - r_j)\hat{k} + \frac{1}{6}t_3(1 + x_3\hat{P}_\sigma)n^\alpha\delta(r_i - r_j) + iW_0\hat{k}^2 \delta(r_i - r_j) \times \hat{k} \cdot (\hat{\sigma}_i + \hat{\sigma}_j),
$$

where $\hat{P}_\sigma = \frac{1}{2}(1 + \hat{\sigma}_i \cdot \hat{\sigma}_j)$ is the spin-exchange operator, $t_i$, $x_i$ and $\alpha$ are the parameters of the interactions, and $\hat{k}$ is defined as

$$
\hat{k} = \frac{1}{2i}(\nabla_i - \nabla_j).
$$

Note that the interaction contains terms up to quadratic in derivatives, $t_3$ term is added to include many body effect beyond quadratic in density $n$, and $W_0$ term gives the spin-orbit interaction which is important to explain the nuclear structure. Parameters used for Skyrme force models are summarized in Table 2.
In Hartree-Fock level, the total energy can be expressed as

\[ E = \sum_{ij} \langle i | \hat{t} | j \rangle \rho_{ji} + \frac{1}{2} \sum_{ijkl} \tilde{v}_{ijkl} \rho_{ki} \rho_{lj} \]  

(4)

where \( \hat{t} \) is the kinetic energy operator and

\[ \tilde{v}_{ijkl} = \langle ij | \hat{v} (1 - \hat{P}_\alpha \hat{P}_\tau \hat{P}_\tau) | kl \rangle. \]  

(5)

Here \( \hat{P}_r \) is the parity operator and \( \hat{P}_\tau = \frac{1}{2} (1 + \hat{t}_i \cdot \hat{t}_j) \) is the iso-spin exchange operator. Due to the zero range property of Skyrme force model, the total energy can be easily obtained as

\[ E = \int d^3 r \mathcal{E} = \int d^3 r (\mathcal{E}_B + \mathcal{E}_C + \mathcal{E}_g + \mathcal{E}_J) \]  

(6)

where \( \mathcal{E}_B \) is the bulk part contribution, \( \mathcal{E}_C \) is the Coulomb contribution, \( \mathcal{E}_g \) is the contribution from the density gradient term, and \( \mathcal{E}_J \) is the contribution from the spin-orbit term. For a uniform matter in the neutron star core, \( \mathcal{E}_B \) is dominant. Hence the energy density can be approximated as

\[ \mathcal{E} \simeq \mathcal{E}_B = \frac{\hbar^2}{2 m_n} \tau_n + \frac{\hbar^2}{2 m_p} \tau_p + n(\tau_n + \tau_p) \left[ \frac{t_1}{4} \left( 1 + \frac{x_1}{2} \right) + \frac{t_2}{4} \left( 1 + \frac{x_2}{2} \right) \right] 
+ \left( \tau_n n_n + \tau_p n_p \right) \left[ \frac{t_2}{4} \left( \frac{1}{2} + x_2 \right) - \frac{t_1}{4} \left( \frac{1}{2} + x_1 \right) \right] 
+ \frac{t_0}{2} \left( 1 + \frac{x_0}{2} \right) n^2 - \left( \frac{1}{2} + x_0 \right) \left( n_n^2 + n_p^2 \right) 
+ \frac{t_3}{12} \left( 1 + \frac{x_3}{2} \right) n^2 - \left( \frac{1}{2} + x_3 \right) \left( n_n^2 + n_p^2 \right) n^\alpha, \]  

(7)
Table 3: Nuclear matter properties at the saturation density ($n_0$). Upper 6 models correspond to non-relativistic Skyrme force models and lower 6 models correspond to relativistic mean field models. $B$ is the binding energy of the symmetric nuclear matter, $S_v$ is the symmetry energy, $L$ is the slope of the symmetry energy, $K$ is the compression modulus, and $m^*_N$ is Landau effective nucleon mass (effective chemical potential).

| Model  | $n_0$ (fm$^{-3}$) | $B$ (MeV) | $S_v$ (MeV) | $L$ (MeV) | $K$ (MeV) | $m^*_N/m_N$ | Ref |
|--------|------------------|-----------|-------------|-----------|-----------|-------------|-----|
| SLy4   | 0.160            | 16.0      | 32.0        | 45.9      | 230       | 0.694       | 35  |
| SkI4   | 0.160            | 16.0      | 29.5        | 60.4      | 248       | 0.649       | 36  |
| SGI    | 0.155            | 15.9      | 28.3        | 63.9      | 262       | 0.608       | 37  |
| SV     | 0.155            | 16.1      | 32.8        | 96.1      | 306       | 0.383       | 38  |
| TOV-min| 0.161            | 15.9      | 32.3        | 76.2      | 222       | 0.934       | 39  |
| LS220  | 0.155            | 16.0      | 28.6        | 73.1      | 220       | 1.000       | 40  |
| IU FSU | 0.155            | 16.4      | 31.3        | 47.2      | 231       | 0.669       | 41  |
| DD-MEδ | 0.152            | 16.1      | 32.4        | 52.9      | 219       | 0.668       | 42  |
| SFHo   | 0.158            | 16.2      | 31.6        | 47.1      | 245       | 0.810       | 43  |
| NLρ    | 0.160            | 16.1      | 30.4        | 84.6      | 241       | 0.800       | 44  |
| TMA    | 0.147            | 16.0      | 30.7        | 90.1      | 318       | 0.691       | 45  |
| NL3    | 0.148            | 16.2      | 37.3        | 118       | 272       | 0.655       | 46  |

where $m_n$ and $m_p$ are neutron and proton masses, $n_n$ and $n_p$ are the densities of neutrons and protons, the total baryon density $n = n_n + n_p$, and $\tau_n$ and $\tau_p$ are kinetic energy densities of neutrons and protons, respectively. The pressure can be obtained by taking a density derivative of energy per baryon,

$$P = n^2 \frac{\partial (E/n)}{\partial n}.$$  

In the upper part of Table 3 we summarize the basic properties of nuclear matter for Skyrme force models which are used in this work. In the upper part of Figure 3 we plot the pressure for Skyrme force models for both symmetric and pure neutron matter. In the left panel of Figure 4 masses and radii of neutron stars are summarized for Skyrme force models.

2.2. Relativistic mean field model

Relativistic mean field models have been very successful in explaining the nuclear properties, such as binding energy, density profile, root mean square radius, etc. Based on these successes, many efforts have been given to build RMF models which are suitable for both finite nuclei and bulk nuclear matter properties. The typical RMF models are described by the Lagrangian [34],

$$\mathcal{L} = \bar{\psi} \left[ i \partial_\mu \bar{\psi} - \frac{1}{2} g_\rho \bar{\tau}^\rho \cdot \bar{\psi} + g_3 \bar{\delta} \cdot \bar{\tau} - m_N + g_\omega \sigma - \frac{1}{2} \epsilon(1 + \tau_3) \right] \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$+ \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} m_\sigma^2 \sigma^2 - \frac{1}{4} \Omega_{\mu\nu} \Omega^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega^\mu \omega_\mu - \frac{1}{4} \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} + \frac{1}{2} m_\tilde{\rho}^2 \tilde{\rho}^\mu \cdot \tilde{\rho}_\mu$$

$$+ \frac{1}{2} \partial_\mu \tilde{\delta} \cdot \partial^\mu \tilde{\delta} - \frac{1}{2} m_\tilde{\delta}^2 \tilde{\delta}^2 - V_{\text{eff}}(\sigma, \omega^\mu \omega_\mu, \tilde{\rho}^\mu \cdot \tilde{\rho}_\mu)$$

(9)
Figure 3: (Color online) Pressure of symmetric nuclear matter and pure neutron matter from both non-relativistic Skyrme force models (upper panels) and relativistic mean field models (lower panels). The shaded area is the result from Ref. [47]. For the pure neutron matter (right panels), the upper (lower) shaded area in each plot represents the stiff (soft) equation of state.
Figure 4: (Color online) Neutron star’s mass and radius relation from non-relativistic Skyrme force models (left) and relativistic mean field models (right). Thick horizontal lines indicate the masses of PSR J1614−2230 and PSR J0348+0432 [1, 2]. The shaded area is the most probable mass and radius, 1σ and 2σ region, from the analysis of Steiner et al. [3].

where \( \sigma \) is the scalar field, \( \omega^\mu \) is the vector-isoscalar field, \( \vec{b}^\mu \) is the vector-isovector field, \( \vec{A} \) is the photon field, \( \vec{\delta} \) is the scalar-isovector field, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), \( \Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \), \( \vec{R}_{\mu\nu} = \partial_\mu \vec{b}_\nu - \partial_\nu \vec{b}_\mu \), and \( V_{\text{eff}} \) is the general effective potential for meson fields. The equations of motion for meson fields can be obtained using the Euler-Lagrange equation

\[
\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0, \tag{10}
\]

where \( \phi = \sigma, \omega^\mu, b^\mu_i, \delta_i \). By taking expectation value of each field, one can define the scaled meson mean fields \( \Phi \equiv g_\sigma \langle \sigma \rangle, W \equiv g_\omega \langle \omega^0 \rangle, \) and \( R \equiv g_\rho \langle b^0_3 \rangle \). Then the equations of motion for the uniform nuclear matter become

\[
n_s = \frac{1}{c_\sigma^2} \Phi + \frac{\partial V_{\text{eff}}(\Phi, W, R)}{\partial \Phi}, \tag{11}
\]
\[
n = \frac{1}{c_\omega^2} W - \frac{\partial V_{\text{eff}}(\Phi, W, R)}{\partial W}, \tag{12}
\]
\[
\frac{1}{2} n_3 = \frac{1}{c_\rho^2} R - \frac{\partial V_{\text{eff}}(\Phi, W, R)}{\partial R}, \tag{13}
\]

where the scaled coupling \( c_i = g_i/m_i \), the baryon scalar density \( n_s = \langle \bar{\psi} \psi \rangle \), baryon density \( n = \langle \bar{\psi} \gamma^0 \psi \rangle = (k^3_{F_p} + k^3_{F_n})/(3\pi^2) \), and baryon isovector density \( n_3 = \langle \bar{\psi} \gamma_3 \gamma^0 \psi \rangle = (k^3_{F_p} - k^3_{F_n})/(3\pi^2) \). The pressure and energy density for nuclear matter can be obtained as

\[
P = \frac{1}{3\pi^2} \sum_{n,p} \int_0^{k_F} \frac{k^4}{\sqrt{k^2 + m_N^2}} dk - \frac{1}{2c_\sigma^2} \Phi^2 + \frac{1}{2c_\omega^2} W^2 + \frac{1}{2c_\rho^2} R^2 - V_{\text{eff}}(\Phi, W, R),
\]

1In some literature, meson mass terms are also included in the effective potential. However, in this work, mass terms are explicitly specified and the effective potentials have only higher order interaction terms beyond mass term.
The total energy density (without electron contribution) is given by

\[
\mathcal{E} = \frac{1}{\pi^2} \sum_{n,p} \int_0^{k_F} k^2 \sqrt{k^2 + m_N^2} \, dk + \frac{1}{2r^2} \Phi^2 - \frac{1}{2r^2} W^2 - \frac{1}{2r^2} R^2 + V_{\text{eff}}(\Phi, W, R)
\]

+ \Phi n + \frac{1}{2} R \rho_3,

(14)

where \( m_N = m_N - g_{\sigma} \sigma \pm g_{\delta} \delta \) \((+ : \text{proton, } - : \text{neutron})\). Various forms of meson effective potentials \( V_{\text{eff}}(\Phi, W, R) \) are used in the literature,

\[
V_{\text{eff}}(\sigma, \omega, \omega^\mu \mu, \tilde{\nu}^\mu \tilde{\nu}_\mu) = \frac{\kappa}{3!} (g_{\sigma} \sigma)^3 + \frac{\lambda}{24} (g_{\sigma} \sigma)^4 - \frac{\zeta}{4!} \sigma^2 (\omega^\mu \omega_\mu)^2 - \frac{\xi}{4!} \sigma^2 (\tilde{\nu}^\mu \tilde{\nu}_\mu)^2 - f(\sigma, \omega^\mu \omega_\mu) g_{\rho}^2 \tilde{\nu}^\mu \tilde{\nu}_\mu,
\]

(15)

with

\[
f(\sigma, \omega^\mu \omega_\mu) = \sum_{i=1}^{6} a_i \sigma^i + \sum_{j=1}^{3} b_j (\omega^\mu \omega_\mu)^j.
\]

(16)

The values of the parameters in various models are summarized in Table 4. In the density dependent coupling constant model (DD-ME6δ), the coupling constant has the form of

\[
g_\lambda(n) = g_\lambda(n_0) s_\lambda(x),
\]

(17)

where \( x = n/n_0 \), \( \lambda = \sigma, \omega, \rho, \delta \) and

\[
s_\lambda(x) = a_\lambda \frac{1 + b_\lambda (x + d_\lambda)}{1 + c_\lambda (x + e_\lambda)}.
\]

(18)

Numerical values of parameters are summarized in Table 5. Note that \( g_\lambda \)’s in Table 5 are equal to \( g_\lambda(n_0) \) in Eq. (17). In the lower part of Table 5, we summarize the basic nuclear matter properties of RMF models selected in this work. In the lower part of Figure 3, we plot the pressure for various RMF models for both symmetric and pure neutron matter. In the right panel of Figure 4, masses and radii of neutron stars are summarized for RMF models.

2.3. Neutron star crust

In the crust of neutron stars, heavy nuclei are expected to exist with free gas of neutrons and electrons. A simple but appropriate description of this situation is completed using liquid droplet formalism [12] [40]. The total energy density (without electron contribution) is given by

\[
F = u n_i f_i + \frac{3 s(u)}{2} \left[ (\sigma(x) + \mu_s) \nu_n \right] + \frac{4 \pi}{5} (r_N n_i x_i e)^2 c(u) + (1 - u) n_{no} f_o,
\]

(19)

where \( u \) is the volume fraction of heavy nuclei to Wigner-Seitz cell, \( n_i \) is the density inside of heavy nuclei, \( f_i \) is the energy per baryon of the heavy nuclei, \( s(u) \) is the surface shape factor, \( r_N \) is the radius of heavy nuclei, \( \sigma(x) \) is a surface tension as a function of proton fraction \( x \), \( \mu_s \) is the neutron chemical potential on the surface, \( \nu_n \) is the areal neutron density on the surface, \( x_i \) is the proton fraction of heavy nuclei, \( c(u) \) is the Coulomb shape function, \( n_{no} \) is the neutron density outside of heavy nuclei, and \( f_o \) is the energy per baryon outside of the heavy nuclei. Minimizing energy density, we have four equations to solve

\[
P_i - P_o - \beta \left( \mathcal{D}' - \frac{2D}{3u} \right) = 0,
\]

\[
w n_i x_i - n Y_p = 0,
\]

\[
w n_i + \frac{2}{3} \beta \mathcal{D} \nu_n + (1 - u) n_{no} - n = 0,
\]

\[
\mu_{ni} - \mu_{no} = 0,
\]

(20)
Table 4: Parameters for RMF models. We multiply $g_\rho$ by the factor of 2 in DD-ME$\delta$, NL$\rho$, and TMA models since the definition of Lagrangian is different from the standard Lagrangian here we used. For DD-ME$\delta$, all the quoted coupling constants ($g_\sigma$, $g_\omega$, $g_\rho$, $g_\delta$) are the ones calculated at the saturation density.

| Param. | Unit | IU-FSU | DD-ME$\delta$ | SHFo | NL$\rho$ | TMA | NL3 |
|--------|------|--------|--------------|------|--------|-----|----|
| $m_\sigma$ | MeV | 491.500 | 556.158 | 457.286 | 508.194 | 519.151 | 508.194 |
| $m_\omega$ | MeV | 782.500 | 783.000 | 762.500 | 782.501 | 781.950 | 782.501 |
| $m_\rho$ | MeV | 763.000 | 763.000 | 770.000 | 763.000 | 768.100 | 763.000 |
| $m_\delta$ | MeV | - | 983.000 | - | - | - | - |
| $g_\sigma$ | | 9.9713 | 10.3325 | 7.53631 | 8.27739 | 10.055 | 10.217 |
| $g_\omega$ | | 13.0321 | 12.2904 | 8.78166 | 9.23205 | 12.842 | 12.868 |
| $g_\rho$ | | 13.5900 | 2×6.3128 | 9.38351 | 2×3.76877 | 2×3.800 | 8.92188 |
| $g_\delta$ | | - | 7.1520 | - | - | - | - |
| $\kappa$ | fm$^{-1}$ | 0.017133 | - | 0.0710487 | 0.066 | 6.4529×10$^{-4}$ | 3.8599 |
| $\lambda$ | 0.000296 | - | 0.0245322 | -0.0288 | 0.0228112 | -0.015905 |
| $\zeta$ | | 0.03 | - | -0.0017013 | - | 0.0334419 | - |
| $\xi$ | | - | - | 0.0034525 | - | - | - |
| $a_1$ | fm$^{-1}$ | - | - | -0.23016 | - | - | - |
| $a_2$ | | - | - | 0.57972 | - | - | - |
| $a_3$ | fm | - | - | 0.34446 | - | - | - |
| $a_4$ | fm$^2$ | - | - | 3.4593 | - | - | - |
| $a_5$ | fm$^3$ | - | - | 1.3473 | - | - | - |
| $a_6$ | fm$^4$ | - | - | 0.66061 | - | - | - |
| $b_1$ | | 7.81241 | - | 5.8729 | - | - | - |
| $b_2$ | fm$^2$ | - | - | -1.6442 | - | - | - |
| $b_3$ | fm$^4$ | - | - | 314.64 | - | - | - |

Table 5: DD-ME$\delta$ density dependent parameters \[42\]. $\lambda$ indicates type of meson.

| $\lambda$ | $a_\lambda$ | $b_\lambda$ | $c_\lambda$ | $d_\lambda$ | $e_\lambda$ |
|-----------|--------------|--------------|--------------|--------------|--------------|
| $\sigma$ | 1.3927 | 0.1901 | 0.3679 | 0.9519 | 0.9519 |
| $\omega$ | 1.4089 | 0.1698 | 0.3429 | 0.9860 | 0.9860 |
| $\rho$ | 1.8877 | 0.0651 | 0.3469 | 0.9417 | 0.9737 |
| $\delta$ | 1.5178 | 0.3262 | 0.6041 | 0.4257 | 0.5885 |
with four unknowns, \( u \), \( n_i \), \( n_{no} \), and \( x_i \). Here, \( \beta = 9(\pi e^2 x_i^2 n_i^2 \sigma^2/15)^{1/3} \), \( D' = \partial D/\partial u \), and \( D = [c(u)s^2(u)]^{1/3} \) is a geometric shape function which corresponds to nuclear pasta phase in liquid droplet model \([8, 10]\). \( P_i \) \((P_o)\) is pressure inside (outside) of the heavy nuclei and the total pressure is given by

\[
P = P_o - \beta(D - uD') .
\]

The boundary between the crust and the core can be found by comparing the energy density or energy per baryon of uniform nuclear matter and the heavy nuclei with free neutron and proton gas. In general case, the energy difference between two phases near the boundary is so small that the pressure difference is negligible \([9]\).

3. Neutron Star Cooling Mechanism

Thermal evolution of neutron star can be obtained by solving the diffusion equations

\[
\frac{L_r}{4\pi \kappa r^2} = -\sqrt{1 - \frac{2Gm}{rc^2} e^{-\Phi_g} \frac{\partial}{\partial r}(T e^{\Phi_g})} ,
\]

\[
\frac{1}{4\pi r^2 e^{2\Phi_g}} \sqrt{1 - \frac{2Gm}{rc^2} \frac{\partial}{\partial r}(e^{2\Phi_g} L_r)} = -Q_\nu - \frac{C_v}{e^{\Phi_g}} \frac{\partial T}{\partial t},
\]

where \( L_r \) is the local photon luminosity, \( T \) is the local temperature, \( m = m(r) \) is the enclosed mass, and \( e^{\Phi_g} \) is the general relativistic metric function. \( \kappa \) is the total thermal conductivity, \( Q_\nu \) is the total neutrino emissivity, \( C_v \) is the total specific heat. The first equation is the general relativistic definition of photon luminosity and the second equation tells how the photon luminosity varies with neutrino emission.

3.1. Neutrino emission

In this subsection, we discuss the neutrino emission processes such as the direct Urca, modified Urca, nucleon bremsstrahlung, and other processes as summarized in Table 6. Note that the neutrino comes out from the entire region of a neutron star before and after its birth. Neutrino emission due to PFB will be discussed in Sect. 5.

3.1.1. Direct Urca process

The most efficient neutrino emission process is the direct Urca process :

\[
n \rightarrow p + e^- + \bar{\nu}_e ,
\]

\[
p + e^- \rightarrow n + \nu_e .
\]

With charge neutrality, the critical condition (momentum conservation) for the direct Urca process can be obtained by neglecting the neutrino momentum,\(^2\)

\[
\begin{align*}
p_{Fn} &= p_{Fp} + p_{Fe} , \\
p_{Fp}^2 &= p_{Fe}^3 + p_{F\mu}^3 , \\
p_{F\mu} &= \sqrt{p_{Fe}^2 - m_{\mu}^2 c^2} .
\end{align*}
\]

\(^2\) In the RMF model, we adopt the natural unit \( c = \hbar = 1 \), but for the cooling formalism, we follow the convention to use the cgs units.
Table 6: Neutrino emission processes in the core and crust of neutron stars. Emission rates were taken from Ref. [48]. The reduction factor $R$ was inserted to account for the pairing effects. $\tilde{n}$ and $\tilde{p}$ are quasi-particles ($np$ mixed states). Orders of $Q_{pl}$ and $Q_{pair}$ can be found in Ref. [48].

| Name             | Process                                             | $Q$ (erg cm$^{-3}$ s$^{-1}$) |
|------------------|-----------------------------------------------------|-------------------------------|
| **Core**         |                                                     |                               |
| Direct Urca      | $n \rightarrow p + e + \bar{\nu}_e$               | $Q_D \sim 4.0 \times 10^{27} R^6 T^6$ |
|                  | $p + e \rightarrow n + \nu_e$                      |                               |
| Modified Urca    | $n + n \rightarrow p + n + e + \bar{\nu}_e$       | $Q_{nn}^M \sim 8.1 \times 10^{21} R^8 T^8$ |
|                  | $p + n + e \rightarrow n + n + \nu_e$              |                               |
|                  | $n + p \rightarrow p + p + e + \bar{\nu}_e$       | $Q_{np}^M \sim 4.1 \times 10^{21} R^8 T^8$ |
| Bremsstrahlung   | $n + n \rightarrow n + n + \nu + \bar{\nu}$       | $Q_{nn} \sim 7.5 \times 10^{18} R^7 T^7$ |
|                  | $n + p \rightarrow n + p + \nu + \bar{\nu}$       | $Q_{np} \sim 1.5 \times 10^{20} R^8 T^8$ |
|                  | $p + p \rightarrow p + p + \nu + \bar{\nu}$       | $Q_{pp} \sim 7.5 \times 10^{19} R^9 T^9$ |
| Cooper pair decay| $n + n \rightarrow (nn) + \nu + \bar{\nu}$         | $Q_{PBF} \sim 10^{21} R^7 T^7$ |
|                  | $p + p \rightarrow (pp) + \nu + \bar{\nu}$        |                               |
| **Crust**        |                                                     |                               |
| $nn$ Bremsstrahlung| $n + n \rightarrow n + n + \nu + \bar{\nu}$ | $Q_{nn} \sim 7.5 \times 10^{18} R^7 T^7$ |
| $eZ$ Bremsstrahlung| $e + (Z,A) \rightarrow e + (Z,A) + \nu + \bar{\nu}$ | $Q_{eZ} \sim 10^{14} R^8 T^8$ |
| Plasmon decay    | $\gamma \rightarrow \nu + \bar{\nu}$             | $Q_{pl}$                      |
| $e^-$ $e^+$ Annihilation | $e^- + e^+ \rightarrow \nu + \bar{\nu}$ | $Q_{pair}$ |

Figure 5: (Color online) Left: schematic figure for the Urca process. The direct Urca process happens near the fermi surface of neutron, proton, and electron. The sum of magnitude of proton and electron (or muon) should be equal to or greater than the one of neutron. Right: critical fraction of proton to total number of baryons for the direct Urca process. If there exists muon, the critical fraction of proton becomes a function of density.
Figure 6: (Color online) The left figure shows the proton fraction for the EoS of Skyrme force models. In SV EoS, the direct Urca process for electron and muon turns on at lower baryon density. On the other hand, SLy4 does not allow both the electron and muon direct Urca process. On the right figure, we show the results with RMF models. All curves stop when the density reaches the central density of the maximum mass of neutron stars.

If there is no muon in the core of neutron star, the critical fraction of proton for the direct Urca process is simply $Y_{p,\text{crit}} = \frac{1}{9}$. For high dense matter where muon exists and $\mu_e \gg m_\mu$, 

$$Y_{p,\text{crit}} = \frac{1}{1 + \left(1 + \frac{1}{21/3}\right)^3} \approx 0.1477.$$  

(26)

In general case where both muon and electron exist, the critical fraction depends on baryon number density. In Figure 5 we summarize the critical proton fraction for the direct Urca process. The dashed line indicates the proton fraction when muon does not exist. The solid line represents the critical fraction for the electron direct Urca process. On the other hand, the dotted line shows the critical fraction for muon direct Urca process. In Figure 6, we summarize the proton fractions from various EoS in comparison with the critical proton fraction for the direct Urca process. In Table 7, we summarize the critical densities for the electron and muon direct Urca process.

The emission rate was firstly calculated by Lattimer et al. [49] and modified by Yakovlev et al. [48] adding superfluidity effects;

$$Q_D = 4.0 \times 10^{27} \frac{m_n^* m_p^*}{m_n m_p} \left(\frac{n_e}{n_0}\right)^{1/3} T_9^6 \Theta_{npe} R_D \text{ erg cm}^{-3} \text{ s}^{-1},$$  

(27)

where $m_n^*$ ($m_p^*$) is the effective mass of a neutron (proton), $T_9 \equiv T/10^9$ K, $R_D$ is the reduction factor from superfluidity, and $\Theta_{npe}$ is defined as,

$$\Theta_{npe} = \begin{cases} 
1 & \text{if } p_{Fn}, p_{Fp}, p_{Fe} \text{ satisfy the momentum conservation,} \\
0 & \text{otherwise.}
\end{cases}$$  

(28)
Table 7: Critical densities \( (n_c \text{ in fm}^{-3}) \) for the electron and muon direct Urca process and the maximum mass of neutron stars \( (M_{\text{max}}) \) for each model. Numbers in the parentheses \( (M_{\text{crit}} \text{ in unit of } M_\odot) \) correspond to the neutron star masses with which the direct Urca processes start to occur.

| Model   | \( n_c \text{ (M_{crit})} \) | \( M_{\text{max}} \text{ (M_\odot)} \) |
|---------|-------------------|------------------|
| SLy4    | -                 | 2.07             |
| SkI4    | 0.502 (1.63)      | 2.19             |
| SGI     | 0.492 (1.72)      | 2.25             |
| SV      | 0.253 (0.97)      | 2.44             |
| TOV min | 0.385 (1.12)      | 2.05             |
| LS220   | 0.433 (1.31)      | 2.04             |
| IU FSU  | 0.611 (1.77)      | 1.94             |
| DD-ME\( \delta \) | 0.764 (1.79) | 1.96             |
| SFHo    | 0.340 (1.11)      | 2.06             |
| NL\( \rho \) | 0.286 (1.14) | 1.99             |
| TMA     | 0.611 (1.77)      | 1.94             |
| DD-ME\( \delta \) | 0.764 (1.79) | 1.96             |
| SFHo    | 0.340 (1.11)      | 2.06             |
| NL3     | 0.205 (0.85)      | 2.78             |

3.1.2. Modified Urca

The second important neutrino emission process is, so called, modified Urca process,

\[
\begin{align*}
    n + n &\rightarrow p + n + e^- + \bar{\nu}_e, \quad p + n + e^- \rightarrow n + n + \nu_e, \\
    n + p &\rightarrow p + p + e^- + \bar{\nu}_e, \quad p + p + e^- \rightarrow n + p + \nu_e.
\end{align*}
\]

Compared with the direct Urca process, there is no threshold fraction for protons thus it always happens in the core of a neutron star. The emission rate for the modified Urca process is calculated by Friman and Maxwell [50] and recalculated by Yakovlev and Levenfish [51] to add the superfluidity effect

\[
Q_{nn}^M = 8.1 \times 10^{21} \left( \frac{m_n^*}{m_n} \right)^3 \left( \frac{m_p^*}{m_p} \right) \left( \frac{n_e}{n_0} \right)^{1/3} T_9^8 \alpha_n \beta_n R_{nn}^M \text{ erg cm}^{-3} \text{s}^{-1},
\]

\[
Q_{np}^M = 4.1 \times 10^{21} \left( \frac{m_p^*}{m_p} \right)^3 \left( \frac{m_n^*}{m_n} \right) \left( \frac{n_e}{n_0} \right)^{1/3} T_9^8 \alpha_p \beta_p R_{np}^M \text{ erg cm}^{-3} \text{s}^{-1},
\]

where \( \alpha_n \) and \( \alpha_p \) are correction terms to describe the momentum transfer dependence of the matrix elements in the Born approximation, and \( \beta_n \) and \( \beta_p \) are corrections to the non-Born and to NN interaction, which cannot be explained by one pion exchange and Landau theories [51]. We use the values of Yakovlev and Levenfish [51] for \( \alpha_n, \alpha_p, \beta_n \) and \( \beta_p \),

\[
\begin{align*}
    \alpha_n &= \alpha_p = 1.76 - 0.63 \left( \frac{n_0}{n_n} \right)^{2/3}, \\
    \beta_n &= \beta_p = 0.68.
\end{align*}
\]

The reduction factors \( R_{nn}^M \) and \( R_{np}^M \) comes from the superfluid nucleons in the core. We used the fitting functions provided by Yakovlev and Levenfish [51].
3.1.3. Nucleon bremsstrahlung

Emission process important next to the modified Urca process is nucleon bremsstrahlung. The difference between the modified Urca process and the nucleon bremsstrahlung is that during the process of nucleon bremsstrahlung the flavor of the reaction agent does not change so electron is not involved in the process compared with the modified Urca process,

\[
\begin{align*}
n + n &\rightarrow n + n + \nu + \bar{\nu}, \\
n + p &\rightarrow n + p + \nu + \bar{\nu}, \\
p + p &\rightarrow p + p + \nu + \bar{\nu}.
\end{align*}
\]

(32)

The emission rate for the nucleon bremsstrahlung was calculated by Friman and Maxwell [50], and Yakovlev and Levenfish [51] taking into account the superfluidity effect,

\[
\begin{align*}
Q_{nn}^B &= 7.5 \times 10^{19} \left( \frac{m_n^*}{m_n} \right)^4 \left( \frac{n_n}{n_0} \right)^{1/3} T_9^8 \alpha_{nn} \beta_{nn} N_\nu \mathcal{R}_{nn}^B \text{ erg cm}^{-3} \text{s}^{-1}, \\
Q_{np}^B &= 1.5 \times 10^{20} \left( \frac{m_n^* m_p}{m_n m_p} \right)^2 \left( \frac{n_p}{n_0} \right)^{1/3} T_9^8 \alpha_{np} \beta_{np} N_\nu \mathcal{R}_{np}^B \text{ erg cm}^{-3} \text{s}^{-1}, \\
Q_{pp}^B &= 7.5 \times 10^{19} \left( \frac{m_p^*}{m_p} \right)^4 \left( \frac{n_p}{n_0} \right)^{1/3} T_9^8 \alpha_{pp} \beta_{pp} N_\nu \mathcal{R}_{pp}^B \text{ erg cm}^{-3} \text{s}^{-1},
\end{align*}
\]

(33)

where \(\alpha_{nn}, \alpha_{pp}, \alpha_{np}\) are the correction factors from the matrix element in the Born approximation, and \(\beta_{nn}, \beta_{pp}, \beta_{np}\) are corrections which come from various reasons [51]. We use the same values in Yakovlev and Levenfish [51],

\[
\begin{align*}
\alpha_{nn} &= 0.59, \quad \alpha_{np} = 1.06, \quad \alpha_{pp} = 0.11, \\
\beta_{nn} &= 0.56, \quad \beta_{np} = 0.66, \quad \beta_{pp} = 0.7.
\end{align*}
\]

(34)

For the number of neutrino flavors, we take \(N_\nu = 3\) including \(\nu_\tau\). \(\mathcal{R}_{nn}^B, \mathcal{R}_{pp}^B, \mathcal{R}_{np}^B\) are reduction factors and given in Yakovlev and Levenfish [51].

3.1.4. Neutrino emission processes in the crust

In the crust of neutron stars, free protons do not exist and the neutrino emission occurs differently. In this work, we consider \(nn\) bremsstrahlung, \(eZ\) bremsstrahlung, plasmon decay, and \(e^-e^+\) pair annihilation in the crust.

In the inner crust, free neutrons are available and they are involved in the \(nn\) bremsstrahlung. The emissivity is the same as in the core except for the appearance of filling factor since the heavy nuclei in the crust has finite size,

\[
Q_{nn}^B = 7.4 \times 10^{19} \left( \frac{m_n^*}{m_n} \right)^4 \left( \frac{n_n}{n_0} \right)^{1/3} T_9^8 \alpha_{nn} \beta_{nn} N_\nu (1 - f) \mathcal{R}_{nn}^B \text{ erg cm}^{-3} \text{s}^{-1},
\]

(35)

where \(f\) is the volume fraction of the heavy nuclei in the Wigner-Seitz cell. In the crust \(\alpha_{nn}\) is given as [48, 50],

\[
\alpha_{nn} = 1 - \frac{3}{2} u \text{ arctan} \left( \frac{1}{u} \right) + \frac{u^2}{2(1 + u^2)}, \quad u = \frac{m_n c}{2p_{F_n}}.
\]

(36)
Since we assume that there exists a heavy nucleus at the center of the Wigner-Seitz cell, a neutrino emission process might happen through the collision between electron and the heavy nucleus,

\[ e + (Z, A) \rightarrow e + (Z, A) + \nu + \bar{\nu}. \] (37)

The above neutrino emissivity was calculated by Kaminker et al. [52] and the fitting function is given as

\[ Q_B^e Z = 10^6 h(\tau, r, \rho) \text{ erg cm}^{-3} \text{ s}^{-1}, \] (38)

with

\[ h(\tau, r, \rho) = 11.204 + 7.304 \tau + 0.2976 r - 0.370 r^2 + 0.188 \tau r - 0.103 \tau^2 \] + 0.0547 \tau^2 r - 6.77 \log_{10}(1 + 0.228 \rho/\rho_0), \] (39)

where \( \tau = \log_{10} T_8, \) \( r = \log_{10} \rho_{12}, \) \( (\rho_{12} = \rho/10^{12} \text{ g cm}^{-3}), \) and \( \rho_0 = 2.8 \times 10^{14} \text{ g cm}^{-3}. \)

Though a free electron cannot emit neutrino-antineutrino pair due to the lack of energy-momentum conservation, electrons with interacting medium can emit neutrinos via \( e \rightarrow e + \nu + \bar{\nu}. \) The process can be written as

\[ \gamma \rightarrow \nu + \bar{\nu}, \] (40)

where \( \gamma \) stands for a plamon. This process is highly efficient at high temperature and medium densities in the crust of neutron stars [48]. Simplified fit is given in Yakovlev et al. [48],

\[ Q_{pl} = \frac{Q_c}{96\pi^4 \alpha} \left( \frac{T}{T_r} \right)^9 (16.23 f_p^6 + 4.604 f_p^{7.5}) e^{-\frac{h}{k_B} \sum_\nu C^2_\nu}, \] (41)

where

\[ Q_c \equiv \frac{G_F^2}{\hbar} \left( \frac{m_e c}{\hbar} \right)^9 \approx 1.023 \times 10^{23} \text{ erg cm}^{-3} \text{ s}^{-1}, \quad \sum_\nu C^2_\nu = 0.9248, \quad T_r = \frac{m_e c^2}{k_B} \approx 5.93 \times 10^9 \text{ K}, \]

and \( f_p \) is the electron plasma parameter defined as

\[ f_p \equiv \frac{\hbar \omega_{pe}}{k_B T} = \frac{\hbar \sqrt{4\pi e^2 n_e/m_e^*}}{k_B T}. \]

\( e^- e^+ \) annihilation might happen in the crust of neutron stars,

\[ e^- + e^+ \rightarrow \nu + \bar{\nu}. \] (42)

The formula for \( e^- e^+ \) annihilation is given in Eq. (22) in Ref. [48].

### 3.2. Heat capacity

The specific heat in neutron star is given by the sum of its constituents,

\[ C_v = \sum_i C_i, \] (43)

where \( i \) denotes the type of particles. In general, \( C_i \) is given by

\[ C_i = \frac{m^*_i p_F i k_B^2 T}{3 \hbar^3}, \] (44)

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where \( m^*_i \) is the effective mass at the Fermi momentum \( p_{Fi} \). For neutron star core, we have the contribution from neutrons, protons, electrons, and muons,

\[
C^\text{core}_v = C_n + C_p + C_e + C_\mu.
\] (45)

Instead of using the simple formula for the specific heat of electrons [53], we use analytic formula for leptons in Lattimer-Swesty [40] as discussed below. The pressure, energy density, and entropy density contribution from leptons are simply functions of chemical potential. For example, the energy density is given by

\[
\mathcal{E}_l = \frac{g_l \mu_l}{8\pi^2} \left( \frac{\mu_l}{\hbar c} \right)^3 \left[ 1 + \mu_l^{-2} (2\pi^2 T^2 - m_l^2 c^4) + \frac{\pi^2 T^2}{\mu_l^4} \left( \frac{7}{15} \pi^2 T^2 - \frac{1}{2} m_l^2 c^4 \right) \right]
\] (47)

where \( g_l \) is the degeneracy factor for lepton and \( \mu_l \) is a chemical potential of lepton,

\[
\mu_l = r - \frac{q}{r}, \quad r = \left[ \sqrt{q^3 + t^2} + t \right]^{1/2},
\] (48)

where \( t = 3\pi^2 (\hbar c)^3 n_l / g_l \) and \( q = (\pi T)^2 / 3 - m_l^2 c^4 / 2 \). The electron specific heat can be obtained by evaluating \( \partial \mathcal{E}_l / \partial T \).

For neutron star crust, there is no contribution from protons but there are contributions from lattice ion,

\[
C^\text{crust}_v = C_e + C_n + C_{\text{ion}}.
\] (49)

Depending on the temperature, the ions are in the gas, liquid, or solid phase. The \( \Gamma \) factor which is the ratio between Coulomb energy and thermal energy, tells us in which phase the ionic lattice is as discussed in Ref. [54],

\[
\Gamma = \frac{Z^2 e^2}{k_B T r_c},
\] (50)

where \( r_c \) is the radius of Wigner-Seitz cell \( (\frac{4}{3} \pi r_c^3 n_i = 1) \). The contribution from the lattice ion becomes

\[
C_{\text{ion}} = \begin{cases} 
\frac{3 n_i k_B}{2} & \text{if } \Gamma \leq 1 \quad \text{(Gas Phase)} \\
\frac{3}{2} n_i k_B - n_i k_B \Gamma^2 \frac{\partial}{\partial \Gamma} \left( \frac{1}{n_i k_B T} \right) & \text{if } 1 < \Gamma \leq 178 \quad \text{(Liquid Phase)} \\
3 n_i k_B f_D (T/\theta_D) & \text{if } \Gamma > 178 \quad \text{(Solid Phase)}
\end{cases}
\] (51)

where the internal energy of the Coulomb liquid (\( \Gamma \leq \Gamma_m \), where \( \Gamma_m \) is the melting constant) is given by [55, 6]

\[
\frac{U}{n_i k_B T} = \Gamma^{3/2} \left( \frac{A_1}{\sqrt{A_2 + \Gamma}} + \frac{A_3}{1 + \Gamma} \right) + \frac{B_1 \Gamma^2}{B_2 + \Gamma} + \frac{B_3 \Gamma^4}{B_4 + \Gamma^4}.
\] (52)

We use values of \( A_{1,2,3} \) and \( B_{1,2,3,4} \) as in the Ref. [6]. \( \theta_D \) is a Debye temperature \( \theta_D = 5.6 \times \)

\footnote{A simplest formula for the specific heat of electrons [53] is given as}

\[
C_e = \frac{m^*_e p_F}{3 \hbar^2} \approx 5.67 \times 10^{19} \left( \frac{n_e}{n_0} \right)^{2/3} T_{9} \frac{\text{erg}}{\text{cm}^3 K}.
\] (46)
Table 8: The best fit parameters for $A$’s and $B$’s in Ref. [6].

|     | $A_1$ | $A_2$ | $A_3$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|-----|-------|-------|-------|-------|-------|-------|-------|
| $10^{22} Z (n_i \cdot \text{fm}^3/A)^{1/2} K$ |   |   |   |   |   |   |   |

The kinetic equation for thermal flux density is given by [59, 60]

$$
\mathbf{q}_i = \frac{1}{4\pi^3} \int d\mathbf{k}_i (\epsilon_i - \mu_i) \mathbf{v}_i F_i \equiv -\kappa_i \nabla T, \quad (55)
$$

where $\mathbf{k}_i$, $\mathbf{v}_i$, $\epsilon_i$, $\mu_i$ are the wave-vector, velocity, energy, and chemical potential of particle $i$, respectively. $\kappa_i$ is the thermal conductivity and $T$ is the temperature. $F_i$ is the distribution function [60]

$$
F_i = f_{FD,i} - \Phi_i \frac{\partial f_{FD,i}}{\partial \epsilon_i}, \quad (56)
$$

where $f_{FD,i}$ is the Fermi-Dirac distribution function and $\Phi_i$ is the deviation from the equilibrium distribution. This can be obtained from Boltzmann equation,

$$
\Phi_i = -\tau_i (\epsilon_i - \mu_i) \frac{\mathbf{v}_i \cdot \nabla T}{T}, \quad (57)
$$

where $\tau_i$ is the relaxation time (inverse of the collision frequency, $1/\tau_i = \nu_i$) and should be calculated numerically using collision integral. Then the thermal conductivity has the compact form [60],

$$
\kappa_i = \frac{\pi^2 k_B^2 T n_i \tau_i}{3 m_i^*}, \quad (58)
$$

For the smooth transition from liquid to solid phase, we use linear combination when $\Gamma$ is between $\Gamma = 178$ and $\Gamma = 220$ [57], $C_{ion} = C_L(T) \frac{220-\Gamma}{42} + C_S(T) \frac{\Gamma-178}{42}$ where $C_L(T) (C_S(T))$ is the specific heat of liquid (solid) phase.

The main contribution to the heat capacity comes from the core since it has large volume and mass fraction of a neutron star. If there exists nuclear superfluidity, $C_n$ and $C_p$ become highly suppressed as temperature decreases to much below the critical temperature [53]. In this case, the electrons (or muons) are the main contributor to the heat capacity of a neutron star. Thus the heat capacity is 20 times lower than the one without superfluidity since the electrons are not suppressed [58].

3.3. Thermal conductivity

The thermal conductivity arise from the collision phenomenon between particles for given density and temperature. The kinetic equation for thermal flux density is given by [59, 60]

$$
\mathbf{q}_i = \frac{1}{4\pi^3} \int d\mathbf{k}_i (\epsilon_i - \mu_i) \mathbf{v}_i F_i \equiv -\kappa_i \nabla T, \quad (55)
$$

where $\mathbf{k}_i$, $\mathbf{v}_i$, $\epsilon_i$, $\mu_i$ are the wave-vector, velocity, energy, and chemical potential of particle $i$, respectively, $\kappa_i$ is the thermal conductivity and $T$ is the temperature. $F_i$ is the distribution function [60]

$$
F_i = f_{FD,i} - \Phi_i \frac{\partial f_{FD,i}}{\partial \epsilon_i}, \quad (56)
$$

where $f_{FD,i}$ is the Fermi-Dirac distribution function and $\Phi_i$ is the deviation from the equilibrium distribution. This can be obtained from Boltzmann equation,

$$
\Phi_i = -\tau_i (\epsilon_i - \mu_i) \frac{\mathbf{v}_i \cdot \nabla T}{T}, \quad (57)
$$

where $\tau_i$ is the relaxation time (inverse of the collision frequency, $1/\tau_i = \nu_i$) and should be calculated numerically using collision integral. Then the thermal conductivity has the compact form [60],

$$
\kappa_i = \frac{\pi^2 k_B^2 T n_i \tau_i}{3 m_i^*}, \quad (58)
$$

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where \( n_i \) is the number density and \( m_i^* \) is the effective mass.

In the core, the thermal conductivity consists of neutron, electron and muon contribution,
\[
\kappa_{\text{core}} = \kappa_n + \kappa_e + \kappa_\mu \, .
\]  

(59)

Since a neutron can collide with both neutron and proton, the relaxation time is given by
\[
\tau_n = \frac{1}{\nu_{nn} + \nu_{np}} \, .
\]  

(60)

The electrons also collide with proton, electron, and muon. Since there are two different lepton species (electron and muon) in the core, the collision relaxation time and frequency relation is different from the single lepton (electron) matter \( [61] \):
\[
1 = \nu_e \tau_e + \nu'_e \mu \tau_\mu,
\]
\[
1 = \nu_\mu \tau_\mu + \nu'_\mu \tau_e \, ,
\]  

(61)

where the prime indicates the particle after collision. In \( \nu_e \), we have to consider the collision frequencies from \( ee \), \( e\mu \), and \( ep \),
\[
\nu_e = \nu_{ee} + \nu_{e\mu} + \nu_{ep} \, .
\]  

(62)

In the core, we use the results from Ref. \([60]\) for neutron thermal conductivity and Ref. \([61]\) for electron and muon thermal conductivity. Both papers consider the superfluid nucleon state so it gives reduction factor for a given temperature. In the crust, electrons are the main thermal conductivity factor and an electron collides with an electron or an ion. Thus, the thermal conductivity is given by
\[
\kappa_e = \frac{\pi^2 T_k^2 \nu_e}{3 m_e^* \nu_e}, \quad \nu_e = \nu_{ee} + \nu_{ei},
\]  

(63)

where \( m_e^* = \sqrt{m_e^2 c^2 + p_{Fe}^2} \). We use the result from Ref. \([62]\) for electron thermal conductivity in the crust of neutron star. Since the critical temperature for electron superconductivity is much lower than the internal temperature of neutron stars, it is not necessary to consider the superfluidity reduction.

As in the case of heat capacity, once the temperature drops down the critical temperature of nuclear superfluidity, the thermal conductivity caused by collisions between superfluid baryons or between electron, muon and superfluid baryon experience reductions. Thus the electron thermal conductivity dominates both in the core and crust.

4. Results from the Standard Cooling

4.1. Standard cooling and direct Urca

Simulation shows that the mass of most of the compact remnants after the supernova explosion is below \( 1.5 M_\odot \) \([63]\), and similar distribution of mass range is obtained from the precisely measured masses in NS-NS binary systems. Therefore, to a good accuracy, we can assume that most of the sources given in Table \([1]\) have masses less than \( 1.6 M_\odot \) or \( 1.7 M_\odot \).

Figures \([7]\) and \([8]\) show the cooling curves for Skyrme models. SLy4 model shows similar behavior regardless of the mass of neutron stars, which is mainly due to the absence of the direct Urca process. Were it not for the direct Urca, modified Urca is driving cooling mechanism in the standard cooling
scenario. We can see that the modified Urca is good at reproducing the data for young (below $10^4$ yrs) and old (above $10^5$ yrs) stars, but completely misses the middle age ($10^4 \sim 10^5$ yrs) data. This may imply that actual cooling will go through slow-quick-slow stages of neutrino emission process. For SkI4 and SGI models, temperature drops abruptly for masses $1.7M_\odot$ and $1.8M_\odot$, respectively. This abrupt decrease of temperature is the signal for the ignition of the direct Urca. In fact, the direct Urca is the fastest neutrino emission process ever known, so once it is turned on, regardless of the existence of PBF or exotic states, shape of the cooling curve is predominantly controlled by the direct Urca. The result shows that the direct Urca is too fast that it fails to pass through any observation data. On the other hand, if cooling is driven by the modified Urca, SLy4, SkI4 and SGI show very similar thermal evolution trajectory. In the SV model, the direct Urca occurs in all the neutron stars, so the model cannot explain the temperature data at all. For TOV min and LS220 models, the modified Urca is the main mechanism for low mass stars, but the direct Urca starts to occur also in the low mass stars. Assuming that most of the mass of the measured star in Figure 7 is in the range of $1.0M_\odot \sim 1.6M_\odot$, TOV min can hardly explain the observed temperature profile. It is striking that though TOV min model shows similar quality of mass-radius relation to SLy4, SkI4 and SGI (Figure 4), they predict quite different thermal evolution scenario. Combining the empirical data from both mass-radius relation and temperature, we can reduce the space for nuclear models which are applicable to the investigation of superdense nuclear matter. For this reason, we remove SV and TOV min models from the consideration hereafter.

Figures 9 and 10 present the cooling curves with RMF models. Now we can easily see that the direct Urca is not working in the SFHo model, and it is activated only in large mass stars in IU-FSU and DD-ME\(\delta\) models. We note that mass-radius relations for SFHo, IU-FSU and DD-ME\(\delta\) models are similar to those of SLy4, SkI4 and SGI models. The similarity is kept for the cooling curves, which may evidence a strong correlation between bulk properties of neutron stars and their thermal evolution. Three hard EoS models, NL\(\rho\), TMA and NL3 cannot reproduce the observation data at all, so we exclude them in the coming analyses.

A feature worthy of notice is that the cooling curve due to the modified Urca is sensitive to the fraction of protons in the core of neutron stars. This can be easily seen by comparing the cooling curves of SLy4, SkI4 and SGI, which are similar to each other, to those of TOV min and LS220. The latter models show fast cooling at young (less than $10^4$ yrs) and middle ($10^4 \sim 10^5$ yrs) ages, but the temperature drop slows down significantly at old ages (above $10^5$ yrs). One can observe a similar pattern from the results of RMF. This distinctive behavior in the modified Urca cooling can be partially understood from the proton fraction in Figure 6. With more protons (TOV min and LS220 in Skyrme force models and NL\(\rho\) and TMA in the RMF models), we have cooler stars in the young and middle ages.

4.2. Radius and symmetry energy properties

The radii of neutron stars have a close relation with the pressure around nuclear saturation density [65], \(R \propto P^{1/4}\). The energy per baryon and the pressure around saturation density can be expanded as

\[
E(n, \delta) = -B + \left(S_v + \frac{L}{3} \frac{n - n_0}{n_0} + \cdots \right) \delta^2 + \cdots, \tag{64}
\]

\[
P = n^2 \frac{\partial E}{\partial n} \approx \frac{L}{3} \frac{n^2}{n_0} \delta^2 \tag{65}
\]
Figure 7: (Color online) Surface temperature (left column) and photon luminosity (right column) vs age without superfluidity effects in non relativistic SLy4, SkI4, and SGI models. The symbol ‘⋆’ indicates the effective temperature of Cas A neutron star. Each curve in the plot corresponds to different neutron star mass in the range of $1.0 M_\odot$ to $2.0 M_\odot$. In case of SLy4, the direct Urca is not turned for any mass of neutron stars. We use the (GPE) $T_s - T_b$ relation in [64].
Figure 8: (Color online) Same plot as in Figure 7 for SV, TOV min, and LS 220 models. SV has high proton fraction even in the low mass neutron stars so it shows the fast cooling processes. In case of TOV min, the direct Urca process happens even in 1.2 M⊙. GPE $T_s - T_b$ relation was used.
Figure 9: (Color online) Same plot as in Figure 7 for relativistic mean field models; IU-FSU, DD-ME$\delta$, and SFHo. For RMF models which have maximum mass less than 2.0$M_{\odot}$, the cooling curve starts from 1.0$M_{\odot}$ and end up with 1.9$M_{\odot}$. For others, we draw up to 2.0$M_{\odot}$. Critical neutron star mass for the direct Urca process in IU-FSU model is 1.77$M_{\odot}$ but the significant effects are only for 1.9$M_{\odot}$ neutron star. GPE $T_s - T_b$ relation was used.
Figure 10: (Color online) Same plot as in Figure 7 for NLρ, TMA, and NL3. The direct Urca process for NL3 happens even less than 1.0$M_\odot$ so that all curves show fast cooling. GPE $T_s - T_b$ relation was used.
where $\delta = \frac{n_{n} - n_{p}}{n}$. This relation also gives a rough relation with the density derivative of symmetry energy $(L)$,

$$R \propto L^{1/4}. \quad (66)$$

The proton fraction is determined from the ground state energy of nuclear matter. The symmetry energy, which is a qualitative estimate of attraction between neutron and proton, has the relation with proton fraction. The algebraic relation $S(n) = S_v + \frac{L}{3} \frac{n_{n} - n_{p}}{n_0}$ indicates that the greater $L$, the greater proton fraction. This implies that the direct Urca process is connected with the radius of neutron stars even though the proton fraction at high density depends on the higher order terms of symmetry energy, and thus $L$ is still a good indicator of the direct Urca process in the core of neutron stars. Figure 11 shows that $R/L^{1/4}$ is indeed model independent. Roughly speaking, neutron stars’ radii would tell us the amount of proton fraction so, we can guess that the direct Urca process would be turned on or off. Thus, too large radius ($R_{1.4M_\odot} > 14$ km, or $L > 90$ MeV) is not favored since the direct Urca process happens even for the small mass of neutron stars. This is also consistent with Steiner et al. [3] in which they estimated the area of neutron stars’ masses and radii using X-ray burst data.

Lattimer and Lim [66] summarized symmetry energy properties $(S_v, L)$ both with experimental results and theoretical calculations. The analysis from the nuclear mass fits, neutron skins, heavy-ion collision, giant dipole resonances and dipole polarizabilities gives an overlapped region. Considering the theoretical calculation of pure neutron matter and astrophysical observations of neutron stars, the allowed ranges of symmetry energy $(S_v)$ and its density gradient $(L)$ are $29.0$ MeV $< S_v < 32.7$ MeV and $40.5$ MeV $< L < 61.9$ MeV. Our result for neutron star cooling indicates that $L < 85$ MeV so that the direct Urca process should not be turned on in the low mass neutron stars $(M < 1.2M_\odot)$. This is consistent with Lattimer and Lim’s conclusion.

![Figure 11](image-url)

Figure 11: (Color online) $R_{1.4M_\odot}/L^{1/4}$ for various nuclear models. $R_{1.4M_\odot}/L^{1/4}$ is independent of models.
Figure 12: (Color online) Band plot of $T_\infty$ from light elements envelope and heavy elements envelope. Each band has the mass range between 1.2 $M_\odot$ and 2.0 $M_\odot$. SLy4 cannot explain some of data even with the union calculation of light and heavy elements since the direct Urca process is not activated even in the maximum mass of a neutron star. On the other hand, SkI4 EoS can cover all the observed data.

4.3. Effect of envelope elements

The surface temperature highly depends on the chemical abundance of light elements [16, 67]. Figure 12 shows the band plot of $T_\infty$ both with the light elements and heavy elements. The bands in each plot indicate neutron star masses in the range of $1.2 M_\odot \sim 2.0 M_\odot$. At early age, the top curve represents the most massive star and later time, the curve from massive stars is the one with the lowest temperature.

For the real cooling process, one has to take into account the cooling from both light and heavy elements. The chemical evolution from the light elements to the heavy elements or pulsar injection of light elements into the magneto sphere [16] indicate that the real cooling curves may start from the band with light elements and moves towards the band with heavy elements as the neutron star evolves. The mass of light elements is defined as

$$\Delta M(t) = \Delta M(t = t_i)e^{-(t-t_i)/\tau_d}.$$  

(67)
Figure 13: (Color online) Light element decay and neutron star cooling curve. Left panel shows 1.4$M_\odot$ neutron star cooling path with SLy4 EoS. The initial mass of light elements is assumed to be $10^{-7}M_\odot$ and the decay starts at $t = 10^3$ years. The right figure shows the same evolution path but with two different masses of neutron stars (1.4$M_\odot$, 1.7$M_\odot$) using SkI4.

where $\tau_d$ is the decay time. In the accreted envelope, the surface temperature can be fitted as a function of the mass fraction of light elements to the total mass of neutron stars \[67\]. If the surface is made of pure irons,

$$T_{\text{eff,Fe}}^4 = g_{14}[(7\zeta)^{2.25} + (\zeta/3)^{1.25}],$$

(68)

where $\zeta = T_{b9} - (7T_{b9}\sqrt{g_{14}})^{1/2}/10^3$ and $g_{14} = \frac{1}{10^{14}}\frac{GM}{R} (1 - \frac{2GM}{Rc^2})^{-1/2}$. We define $T_b$ ($T_{b9} = T_b/10^9 K$) is the temperature where the energy density is $10^{10}$ g cm$^{-3}$. On the other hand, if the surface is fully accreted, i.e. there are only hydrogen,

$$T_{\text{eff,a}}^4 = g_{14}(18.1 T_{b9})^{2.42}.$$  

(69)

For partially accreted envelope, with the definition of $\eta \equiv g_{14}^2 \Delta M/M$, we have $T_s - T_b$ relation,

$$T_s = \left[ \frac{a T_{\text{eff,Fe}}^4 + T_{\text{eff,a}}^4}{a + 1} \right]^{1/4},$$

(70)

where $a = [1.2 + (5.3 \times 10^{-6}/\eta)^{0.38}] T_{b9}^{5/3}$. Using $T_s - T_b$ relation in Ref. \[67\], we can find the surface temperature for given mass of light elements on the surface.

In Figure 13, we show the cooling paths obtained by taking into account the light element decay. Depending on the amount of light elements and the decay time scale ($\tau_d$), the actual neutron star cooling curve will be located between the light and heavy elements bands. In this figure, the initial mass of light elements is assumed to be $10^{-7}M_\odot$ and the light elements start to decay when $t = 10^3$ years after the birth of neutron stars. In case of SLy4, the cooling curve is almost identical for all mass of neutron stars, thus a typical mass 1.4$M_\odot$ is chosen to see the evolution path. For SkI4, 1.4$M_\odot$ and 1.7$M_\odot$ cooling paths are shown to compare the effects of elements decay. This implies that elements composition, decay history, and the direct Urca process can be used selectively to explain all the observed data. Note that the EoS which doesn’t allow the direct Urca process (i.e.
Figure 14: (Color online) The direct Urca Process effects on LS200 and IU-FSU. For both models, most of the observations are in the very narrow mass range ($\Delta M = 0.01 M_\odot$). GPE $T_s - T_b$ relation was used.

SLy4 for example) cannot explain mid-old low temperature neutron stars without other fast cooling mechanism such as cooper pair emission or Bose condensation. The paths with light element decays also show that the rapid temperature drop can occur during the decay process of light elements.

5. Results with Superfluidity

It is believed that $^1S_0$ neutron superfluidity exits in the inner crust of neutron stars, and $^1S_0$ proton and $^3P_2$ neutron superfluidity state appear in the core of neutron star. Since there is no free proton in the crust of neutron star, the superfluidity involving proton exists only in the core of neutron star. Beta equilibrium condition allows the fraction of proton only up to 0.2 so the effect of the proton superfluidity itself is not as strong as that of the neutron superfluidity. However, the proton superfluidity delays the surface temperate drop mainly due to the reduction factors in the neutrino emissivity, heat capacity, and thermal conductivity. Once the local temperature drops below the critical temperature for superfluidity, the properties of nuclear matter change drastically. The neutrino emissivity, heat capacity, and thermal conductivity are reduced significantly and behave like $\sim \exp(-\Delta/T)$ where $\Delta$ is a paring gap energy. The numerical reduction factors for each physical quantities are given by Yakovlev et al. [53, 60]. As discussed in Page and Applegate [68] and Yakovlev et al. [53], the direct Urca process turns on in the really narrow mass range. As shown in Figures 7 - 10, the direct Urca process imposes huge effects on the cooling curve. If the mass of neutron star is slightly greater than the critical mass for the direct Urca process, ($M > M_D + 0.01M_\odot$), then the direct Urca effect is clear.

In Figure 14, the effects of the direct Urca process on neutron star cooling are summarized with two EoS LS220 and IU-FSU. The results show that most of the observed data can be explained by two curves 1.31$M_\odot$ and 1.32$M_\odot$ for LS220 and 1.78$M_\odot$ and 1.79$M_\odot$ for IU-FSU. Thus, if there's

\[ ^4 \text{Numerically, there needs more grid points in the core of neutron star to treat the direct Urca process properly. In this case, we used 16 times more grid points to see the split of the curves between 1.30M_\odot and 1.40M_\odot.} \]
no superfluidity effects, the mid age cold neutron stars (e.g. Vela, Geminga) should have the mass in the really narrow range. In the statistical point of view, it is unacceptable that almost all the observations are in the $0.01M_\odot/M_\odot \approx 1\%$ since the mass distribution of the observed neutron stars has the broad range \[10\]. This problem can be managed if we employ the pairing effects. The neutrino emissivity formula for PFB is given by \[69, 70\]

$$Q_{PFB} = 3.51 \times 10^{21} \left( \frac{m_i}{m_\star} \right) \left( \frac{p_{F_i}}{m_\star C} \right) T_9^7 a_{i,j} F_j \left[ \Delta_i(T) \right] \left( \frac{1}{T} \right) \frac{\text{erg}}{\text{cm} \cdot \text{s}},$$

where $i$ represents type of nucleons ($i = n, p$) and $j$ stands for singlet ($j = s$) or triplet ($j = t, m_J = 0$) pairing. $F_s$ and $F_t$ are given in Ref. \[69\]

$$F_s = y^2 \int_0^\infty \frac{z^4 \, dx}{(1 + e^z)^2}, \quad F_t = \frac{1}{4\pi} \int d\Omega \, y^2 \int_0^\infty \frac{z^4 \, dx}{(1 + e^z)^2},$$

where $x$ is an integration variable, $y = \Delta_i(T)/T$, $z = \sqrt{x^2 + y^2}$, and $\int d\Omega$ is the solid angle integration. The fitting functions for $F_s$ and $F_t$ are also given in Ref. \[69\], $a_{i,j}$'s are given by Ref. \[70\]

$$a_{n,s} = C_{V,n}^2 \left( \frac{4}{81} \right) \left( \frac{p_{F_n}}{m_n^2 C} \right)^4 + C_{A,n}^2 \left( \frac{p_{F_n}}{m_n^2 C} \right)^2 \left( 1 + \frac{11}{42} m_n^2 \right),$$

$$a_{p,s} = C_{V,p}^2 \left( \frac{4}{81} \right) \left( \frac{p_{F_p}}{m_p^2 C} \right)^4 + C_{A,p}^2 \left( \frac{p_{F_p}}{m_p^2 C} \right)^2 \left( 1 + \frac{11}{42} m_p^2 \right),$$

$$a_{n,t} = 2C_{A,n}^2, \quad a_{p,t} = 2C_{A,p}^2,$$

where $C_{V,n} = 1$, $C_{A,n} = g_A$, $C_{V,p} = 4\sin^2\theta_W - 1$, $C_{A,p} = -g_A$ with $g_A \approx 1.2$ and $\sin^2\theta_W \approx 0.23$. Since the core of neutron stars is mostly composed of neutrons ($p_{F_n} \gg p_{F_p}$) and the magnitude of $a_{n,t}$ and $a_{p,s}$ are comparable, the triplet PBF is the main neutrino emission agent once the superfluidity occurs. (Note that $^3P_2$ neutron and $^1S_0$ proton pairing are expected in the core of neutron stars.) When the temperature drops below the critical temperature, the modified Urca and bremsstrahlung neutrino emission processes are highly suppressed and PBF process overwhelmes the other neutrino emission \[70\].

Aside from the previous calculations of nuclear superfluidity, we introduce the phenomenological pairing gap formula to see the effect of gap size and the density range.

$$T_c(k_f) = \begin{cases} T_c^\text{max} \cdot \mathcal{N} \cdot (k_f - k_0)^{\alpha_c} (k_2 - k_f)^{\beta_c} & \text{if } k_0 < k_f < k_2; \\ 0 & \text{if otherwise}, \end{cases}$$

where $T_c^\text{max}$ is the maximum critical temperature for superfluidity for given $k_f$ (the fermi wave number for total bayron number density). $k_0$ ($k_2$) is the starting (ending) wave number for given type of pairing. $\mathcal{N}$ is the normalization factor for the critical temperature,

$$\mathcal{N} = \left( 1 + \frac{\beta_c}{\alpha_c} \right)^{\alpha_c} \left( 1 + \frac{\alpha_c}{\beta_c} \right)^{\beta_c} \frac{1}{(k_2 - k_0)^{\alpha_c + \beta_c}}.$$

Most of the neutron $^1S_0$ gap calculations can be fitted with $\alpha_c = 2$ and $\beta_c = 2$. Since the exact proton fraction in dense nuclear matter is not certain, the above formula is a good approximation to check the effects of the pairing on the specific heat and neutrino emission.
Figure 15: (Color online) Critical temperature for different type of pairing. $k_f$ is the wave number of Fermi-momentum (in unit of fm$^{-1}$) for the total baryon number density. The critical temperature is then evaluated for beta-stable nuclear matter. The core-crust boundary is obtained assuming the phase transition happens at $\rho \simeq 1/2\rho_0 = 0.08$ fm$^{-3}$. Each type of pairing can be obtained using different methods. $^1S_0$ neutron: CBF - Correlated Basis Function [72], PP - Polarization Potential [73], BHF - Brueckner Hartree Fock [74], RG - Renormalization Group [75]. $^1S_0$ proton: DBHF - Dirac Brueckner Hartree Fock [76], PCT - Parameterized Critical Temperature [77], BHF [74]. $^3P_2$ neutron: BHF [78], PCT [77], OPEG (BCS) - One Pion Exchange Gaussian with generalized BCS [79]. Note that $^1S_0$ proton superfluidity is not operative in the crust of neutron stars since free gas of proton is not available in the crust. The right bottom figure shows the critical temperatures for each type of pairing. All curves were obtained from the simple Eq. (74) with $\alpha_c = 2$ and $\beta_c = 2$ for $^1S_0$ and $^3P_2$ neutron pairing and with $\alpha_c = 6$ and $\beta_c = 1.2$ for $^1S_0$ proton pairing.
In Figure 16, the critical temperatures are summarized for given $k_f$ in beta-stable nuclear matter. Thus for $^3P_2$ neutron pairing (left bottom), $k_f$ may be different from the original paper since we use $k_f$ for the total baryon number density not for the pure neutron matter. If the gap calculation is done in the pure neutron matter ($k_{Fn}$), the proton fraction is given by APR EoS \cite{71} to recover $k_f$ of the total baryon number density for this figure.

In each gap calculation, the critical temperature strongly depend on the methodology. Considering this fact, we use the phenomenological critical temperature formula and see the cooling curve how it depends on it. For $^1S_0$ and $^3P_2$ neutron pairing, $\alpha_c = 2$ and $\beta_c = 2$ are suitable to represent the critical temperatures. For $^1S_0$ proton pairing, we adopt $\alpha_c = 6$ and $\beta_c = 1.2$ to mimic the behavior of the critical temperature in this example.

To see the effect of nuclear pairing, we choose SLy4 and SkI4 to simulate neutron star cooling, because both SLy4 and SkI4 satisfy the mass-radius constraint zone \cite{8} and SLy4 does not turn on the direct Urca process, while SkI4 does if the mass of neutron star is greater than 1.63 $M_\odot$. The cooling curves for the case of SLy4 in Figure 16 show that the early start of $^3P_2$ pairing gives the narrow band of cooling curves. The early start means $k_0$ is close to the crust and core boundary. Thus, the core of neutron stars has the superfluidity regardless of its mass. Therefore, small $k_0$ gives sharp drop of temperature in young age of neutron stars. In this case, the observational data for old age neutron star cannot be explained with cooling simulation. On the other hand, if the $^3P_2$ neutron pairing appears in some range of density, the low mass neutron stars do not show sharp temperature drops since $^3P_2$ pairing is not available in the core of neutron stars. It seems that if the EoS does not allow the direct Urca process, the combined area of both from the light and heavy element envelope cannot explain the mid age low temperature neutron stars or old age high temperature neutron stars at the same time. This might indicate that pion or kaon condensation might exist in the core of neutron stars if the massive neutron stars does not have enough proton fractions. As summarized in Figure 12 since SkI4 allows the direct Urca process, the bands from cooling curves can explain every observational data without pairing effects. If we assume pairing effects, however, reduction factors for heat capacity and neutrino emissivity play a role to restrict $k_0$ and $k_2$ for $^3P_2$ neutron pairing. The critical temperatures for SkI4 are summarized in Table 9. For each case, we use $\alpha_c = 2$ and $\beta_c = 2$ for $^1S_0$ and $^3P_2$ for neutron, and $\alpha_c = 8$ and $\beta_c = 2$ for

![Figure 16: (Color online) The effect of superfluidity for SLy4.](image-url)
Figure 17: (Color online) Superfluidity effects with SkI4 EoS. Each case has the different critical temperature so different \( \frac{d \ln T}{d \ln t} \). The error bars in the right bottom plot denote the analysis of ACIS-S (Graded Mode) in Ref. [80]. The solid lines in each plot show the cooling curve of \( 1.5M_\odot \) neutron star with \( \Delta M = 5 \times 10^{-13}M_\odot \).
Table 9: The critical temperature parameters for each case.

|            | Case I          | Case II         | Case III         |
|------------|-----------------|-----------------|------------------|
| $3P_2 n$   | $T_{c}^{\text{max}}$ (K) | $6.80 \times 10^8$ | $6.47 \times 10^8$ | $5.95 \times 10^8$ |
|            | $k_0$ (fm$^{-1}$) | 0.99            | 1.5              | 1.8              |
|            | $k_2$ (fm$^{-1}$) | 2.8             | 3.0              | 2.4              |
| $1S_0 p$   | $T_{c}^{\text{max}}$ (K) | $6.44 \times 10^9$ | $7.45 \times 10^9$ | $1.0 \times 10^9$ |
|            | $k_0$ (fm$^{-1}$) | 0.1             | 0.1              | 1.8              |
|            | $k_2$ (fm$^{-1}$) | 2.5             | 2.5              | 2.4              |
| $1S_0 n$   | $T_{c}^{\text{max}}$ (K) | $3.2 \times 10^9$ | $5.0 \times 10^9$ | $1.0 \times 10^9$ |
|            | $k_0$ (fm$^{-1}$) | 0.0             | 0.0              | 0.0              |
|            | $k_2$ (fm$^{-1}$) | 1.3             | 1.3              | 1.3              |

$1S_0$ for proton critical temperatures. The exact critical temperature might be obtained if we have the exact temperature drops of the Cas A neutron star for the next decade.

6. Conclusion

In succession to our prior works on the mass-radius relation of neutron stars [12, 13], in this work, we investigated the consistency of nuclear models with observation of neutron star temperatures. First, model selection was performed by constraining the model prediction of maximum mass of neutron stars larger than $2M_\odot$. As a result, we picked up 6 models among the non-relativistic Skyrme force models, and 6 models among the RMF models. In the second step, with the 12 models selected, we calculated the cooling curves with only standard cooling mechanisms and the direct Urca processes. The result shows clear dependence on the EoS of a model. If the direct Urca is not operating, the modified Urca is the most efficient way to emit neutrinos. We found that the modified Urca reproduces the observation data well for the ages less than $10^4$ years or more than $10^5$ years. However, no model could explain the data in the age of $10^4 \sim 10^5$ years. The modified Urca always gives temperature much higher than the observed ones. Once the direct Urca is working inside of neutron stars, the star cools down so fast that the cooling curves never touch the observation data. Thus, nuclear models which have large symmetry energy gradient ($L > 85$ MeV) cannot demonstrate any of astronomical data for neutron stars since the direct Urca process is turned on even for low mass neutron stars. As a result, we could sort out the nuclear models that satisfy both mass-radius relations and temperature data.

Surface temperature, which is a directly measured quantity, heavily depends on the composition of elements in the envelope. Models with moderate symmetry energy gradient can explain observational data with standard cooling processes. In the middle age, high temperature neutron star indicates that the surface of neutron star has some composition with light elements. With the decay of light elements to heavier ones, neutron star cooling curve can be any of one in the both light and heavy elements bands. Thus if EoS bands from light and heavy elements contain all of the observed data, it can be a good model for the neutron star cooling simulations.

We have explored the effect of nuclear pairing by combining the PBF to models that are qualified with the standard cooling mechanisms. We could reproduce the middle age low temperature data.
which are missed in the standard cooling mechanism, and at the same time could satisfy the Cas A temperature profile. However, looking into the detail, we find that the cooling curves with PBF are sensitive to the physical inputs such as pairing gaps, critical temperature for PBF, neutron star mass, and etc. Moreover, the temperatures predicted from the curves that are consistent with the middle age and Cas A data are too low for the stars with ages more than $10^5$ years, so the curves with PBF scarcely touch the old age data. Measurement of temperature change of Cas A in the next decade will shed some light on resolving these problems and we expect to reduce the uncertainties in the underlying physics.

In conclusion, we could confirm that the existing mass-radius relation and thermal evolution history of neutron stars provide critical test grounds with which we can narrow down the space of nuclear models which are good candidates for the application to dense nuclear matter. In this work, we concentrated on nuclear models and uncertainties due to envelope elements and nuclear pairing. In a recent work [13], we demonstrated that hyperons can exist in the core of neutron stars which satisfy the empirical mass-radius relations. Urca process accompanied with hyperons, though not so efficient as the direct Urca, is another source of fast cooling mechanism. Neutron star cooling with hyperons will be coming up in the near future.

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Appendix A. Numerical Solution for Neutron Star Cooling

The analysis for numerical solution of the diffusion equations can be found in the appendix of D. Page’s thesis [57]. Here we adopt the same notation with Page’s thesis in which tri-diagonal scheme is used to solve the coupled diffusion equations. To solve the diffusion equations, $L_r (T)$ is defined on the even (odd) grid. We present the numerical solutions for the neutron star cooling using penta-diagonal and tri-diagonal schemes and compare the convergency between them.\(^5\)

Appendix A.1. Penta-diagonal scheme

The penta-diagonal scheme appeared in ref. [56]. The first diffusion equation is

$$\mathcal{L} = -PS\frac{dT}{da} \quad (A.1)$$

\(^5\)In penta-diagonal scheme $L_r (T)$ is defined in even (odd) grid and intermediate time step, and in tri-diagonal scheme $L_r$ and $T$ are defined in all grid points.
where
\[ \mathcal{L} = e^{2\Phi} L_r, \quad \mathcal{T} = e^{\Phi} T, \quad P = \kappa, \quad S = (4\pi r^2) e^{\Phi} n. \] (A.2)

Here \( a(r) \) is the enclosed baryon number in radius \( r \) and it is related by
\[ \frac{da}{dr} = 4\pi r^2 n \sqrt{1 - \frac{2Gm}{rc^2}}, \] (A.3)
where \( n \) is a baryon number density. The thermal evolution of time index from \( n \) defined only on the odd grid in the numerical scheme, thus we use the average number of unknown are 4. In general the thermal conductivity \( \kappa \) where
\[ \mathcal{T} = \mathcal{T}^{n+1/2}_2, \mathcal{T}^{n+1/2}_1, \mathcal{T}^{n+1/2}_2, \mathcal{T}^{n+1/2}_3, \mathcal{T}^{n+1/2}_4 \]
where the subscript means spatial dimension and superscript represents time step. The total number of unknown are \( 4I + 6 \). Using these mid time interval variables and Henyey method, we write the thermal evolution of a neutron star time index from \( n \) to \( n + 1 \) as
\[ \mathcal{T}^{n+1}_2 = \mathcal{T}^{n+1}_2 - \Delta t \left[ Q^{n+1/2}_2 \frac{d\mathcal{L}}{da}^{n+1/2}_2 + R^{n+1/2}_2 \right], \] (A.6)
where
\[ \mathcal{L}^{n+1/2}_2 = -P^{n+1/2}_2 S^{n+1/2}_2 \frac{dT^{n+1/2}}{da}_2. \] (A.7)

The thermal evolution of time index from \( n + 1/2 \) to \( n + 1 \) is given by
\[ \mathcal{T}^{n+1}_2 = \mathcal{T}^{n+1}_2 - \frac{1}{2} \Delta t \left[ Q^{n+1}_2 \frac{d\mathcal{L}}{da}^{n+1}_2 + R^{n+1}_2 \right], \] (A.8)
where
\[ \mathcal{L}^{n+1}_2 = -P^{n+1}_2 S^{n+1}_2 \frac{dT^{n+1}}{da}_2. \] (A.9)

In general the thermal conductivity \( \kappa \) (\( = P \)) is a function of temperature \( T \), and the temperature is defined only on the odd grid in the numerical scheme, thus we use the average \( P_{2i} = \frac{1}{2}(P_{2i-1} + P_{2i+1}) \). From the above equations and \( P_{2i} \), we have equations to solve
\[ F_{4i+1} = F_{4i+1}(\mathcal{L}^{n+1/2}_2, \mathcal{T}^{n+1/2}_2, \mathcal{T}^{n+1/2}_1, \mathcal{L}^{n+1/2}_2, \mathcal{L}^{n+1/2}_1) \]
\[ = R^{n+1/2}_2 + Q^{n+1/2}_2 \frac{\mathcal{L}^{n+1/2}_2 - \mathcal{L}^{n+1/2}_1}{da_2 + da_2_1} + \frac{\mathcal{T}^{n+1}_2 - \mathcal{T}^{n}_2}{\Delta t} = 0, \] (A.10)
These four types of equations can be solved by the multi-dimensional Newton-Raphson method. The variations of $\delta \mathcal{L}$'s and $\delta \mathcal{T}$'s can be obtained by solving linearized Newton-Raphson method,

\begin{align*}
F_{4i+2} &= F_{4i+2}(\mathcal{L}_{2i}^{n+1}, T_{2i+1}^{n+1/2}, T_{2i+1}^{n+1}, \mathcal{L}_{2i+1}^{n+1}) \\
&= R_{2i+1}^{n+1} + Q_{2i+2}^{n+1} \frac{\mathcal{L}_{2i+1}^{n+1} - \mathcal{L}_{2i+1}^{n+1/2}}{\Delta t} + \frac{T_{2i+1}^{n+1} - T_{2i+1}^{n+1/2}}{\Delta t} = 0, \quad (A.11) \\
F_{4i-1} &= F_{4i-1}(T_{2i-1}^{n+1/2}, \mathcal{L}_{2i}^{n+1}, T_{2i+1}^{n+1/2}) \\
&= \mathcal{L}_{2i}^{n+1/2} + \frac{P_{2i-1}^{n+1} + P_{2i+1}^{n+1/2}}{2} \cdot S_{2i} \cdot \frac{T_{2i+1}^{n+1/2} - T_{2i-1}^{n+1/2}}{\Delta t} = 0, \quad (A.12) \\
F_{4i} &= F_{4i}(T_{2i-1}^{n+1}, \mathcal{L}_{2i}^{n+1}, T_{2i+1}^{n+1}) \\
&= \mathcal{L}_{2i}^{n+1} + \frac{P_{2i-1}^{n+1} + P_{2i+1}^{n+1/2}}{2} \cdot S_{2i} \cdot \frac{T_{2i+1}^{n+1} - T_{2i-1}^{n+1}}{\Delta t} = 0. \quad (A.13)
\end{align*}

where

\begin{align*}
A_{4i+1} &= \frac{\partial F_{4i+1}}{\partial \mathcal{L}_{2i}^{n+1/2}} = -\frac{Q_{2i+1}^{n+1/2}}{\Delta t}, \quad (A.18) \\
C_{4i+1} &= \frac{\partial F_{4i+1}}{\partial T_{2i+1}^{n+1/2}} = \frac{\partial R}{\partial T} \bigg|_{2i+1}^{n+1/2} + \frac{\partial Q}{\partial T} \bigg|_{2i+1}^{n+1/2} \cdot \frac{\mathcal{L}_{2i+1}^{n+1/2} - \mathcal{L}_{2i}^{n+1/2}}{\Delta t}, \quad (A.19) \\
D_{4i+1} &= \frac{\partial F_{4i+1}}{\partial T_{2i+1}^{n+1}} = \frac{1}{\Delta t}, \quad (A.20) \\
E_{4i+1} &= \frac{\partial F_{4i+1}}{\partial \mathcal{L}_{2i+2}^{n+1/2}} = \frac{Q_{2i+2}^{n+1/2}}{\Delta t}, \quad (A.21) \\
A_{4i+2} &= \frac{\partial F_{4i+2}}{\partial \mathcal{L}_{2i}^{n+1}} = -\frac{Q_{2i+1}^{n+1}}{\Delta t}, \quad (A.22) \\
B_{4i+2} &= \frac{\partial F_{4i+2}}{\partial T_{2i+1}^{n+1/2}} = -\frac{2}{\Delta t}, \quad (A.23) \\
C_{4i+2} &= \frac{\partial F_{4i+2}}{\partial T_{2i+1}^{n+1}} = \frac{\partial R}{\partial T} \bigg|_{2i+1}^{n+1} + \frac{\partial Q}{\partial T} \bigg|_{2i+1}^{n+1} \cdot \frac{\mathcal{L}_{2i+1}^{n+1} - \mathcal{L}_{2i}^{n+1}}{\Delta t} + \frac{2}{\Delta t}, \quad (A.24) \\
E_{4i+2} &= \frac{\partial F_{4i+2}}{\partial \mathcal{L}_{2i+2}^{n+1}} = \frac{Q_{2i+2}^{n+1}}{\Delta t}, \quad (A.25) \\
A_{4i-1} &= \frac{\partial F_{4i-1}}{\partial T_{2i-1}^{n+1/2}}
\end{align*}
In the above equations to solve $F_1, \ldots, F_{4I+2}$, we have unknowns $\mathcal{L}_0^{n+1/2}$, $\mathcal{L}_0^{n+1}$, $\mathcal{T}_1^{n+1/2}$, $\mathcal{T}_1^{n+1}$, $\ldots$, $\mathcal{L}_1^{n+1/2}$, and $\mathcal{L}_1^{n+1}$. There are $4I + 6$ unknowns quantities with only $4I + 2$ equations, but one can obtain the solutions with four additional boundary conditions. With the $T_s - T_b$ relation [64, 67, 81], we can have

$$\mathcal{L}_{2I+2} = e^{2\Phi_{2I+2}}(4\pi R^2 \sigma_B) T_s^{4I} = e^{2\Phi_{2I+2}}(4\pi R^2 \sigma_B) f(T_{2I+1}).$$

$^6$L_r(r = 0) = 0 reduces two unknowns ($\mathcal{L}_0^{n+1/2} = \mathcal{L}_0^{n+1} = 0$), and the uniform luminosity approximation ($L_{2I+2} = L_{2I+1}$) reduces $\mathcal{L}_{2I+2}$ and $\mathcal{L}_{2I+2}^{n+1}$ as a function of $T_{2I+1}$ and $T_{2I+1}$ respectively.
We have matrix equations to solve

\[
\begin{pmatrix}
C_1 & D_1 & E_1 & 0 & \cdots & 0 \\
B_2 & C_2 & E_2 & 0 & \cdots & 0 \\
A_3 & 0 & C_3 & E_3 & 0 & \cdots \\
0 & A_4 & 0 & C_4 & E_4 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & A_{4I-1} & 0 & C_{4I-1} & 0 & E_{4I-1} & 0 \\
0 & \cdots & 0 & A_{4I} & 0 & C_{4I} & 0 & E_{4I} & 0 \\
0 & \cdots & 0 & A_{4I+1} & 0 & C_{4I+1} & 0 & E_{4I+1} & 0 \\
0 & \cdots & 0 & A_{4I+2} & B_{4I+2} & 0 & C_{4I+2} & & \\
\end{pmatrix}
\begin{pmatrix}
\delta T_1^{n+1/2} \\
\delta T_2^{n+1} \\
\delta L_2^{n+1} \\
\vdots \\
\delta L_2^{n+1} \\
\delta L_2^{n+1} \\
\delta L_2^{n+1} \\
\delta T_2^{n+1} \\
\delta T_2^{n+1} \\
\end{pmatrix}
= -
\begin{pmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
F_{4I-1} \\
F_{4I} \\
F_{4I+1} \\
F_{4I+2} \\
\end{pmatrix}
\]

This penta-diagonal linear equation can be solved by \( L - U \) decomposition or Gaussian elimination method.

**Appendix A.2. Tri-diagonal scheme with every grid point**

Another method to solve the diffusion equation is to use every grid point \((L_0, L_1, \cdots, L_N, T_1, T_2, \cdots, T_N)\). In this case, we mix forward and backward numerical differentiation to make tri-diagonal matrix. For the luminosity equation,

\[
\mathcal{L} = -PS \frac{dT}{da} \rightarrow \mathcal{L}_i + P_i S_i \frac{T_{i+1} - T_i}{da_{i+1}} = 0.
\]  

and the temperature evolution equation becomes,

\[
\frac{dT}{dt} = -Q \frac{dL}{da} - R \rightarrow R_i + Q_i \frac{L_i - L_{i-1}}{da_i} + \frac{T_i - T_{i-1}}{\Delta t} = 0.
\]

Thus, the numerical equations to solve are

\[
F_{2i-1}(L_{i-1}^{n+1}, T_{i-1}^{n+1}, L_i^{n+1}) = R_i^{n+1} + Q_i^{n+1} \frac{L_i^{n+1} - L_{i-1}^{n+1}}{da_i} + \frac{T_i^{n+1} - T_i^n}{\Delta t},
\]

\[
F_{2i}(T_i^{n+1}, L_i^{n+1}, T_{i+1}^{n+1}) = L_i^{n+1} + P_i^{n+1} \frac{T_{i+1}^{n+1} - T_i^{n+1}}{da_{i+1}}.
\]

In this scheme, the unknowns are \((T_1, L_1, T_2, \cdots, L_{N-1}, T_N)\) and the final equations to solve are \(F_{2N-1}\) instead of \(F_{2N}\) since we don’t have \(T_{N+1}\) as unknown. We also use the same boundary condition as in Penta-diagonal scheme, \(L_0 = 0\) and \(L_N = e^{2\Phi 4\pi R^2 \sigma_B T_s^4} = e^{2\Phi 4\pi R^2 \sigma_B f(T_N)}\).

Therefore,

\[
F_1 = R_1^{n+1} + Q_1^{n+1} \frac{L_1}{a_1} + \frac{T_1^{n+1} - T_1^n}{\Delta t},
\]

\[
F_{2N-1} = R_N^{n+1} + Q_N^{n+1} \frac{L_N(T_N) - L_{N-1}}{da_N} + \frac{T_N^{n+1} - T_N^n}{\Delta t}.
\]
Newton-Raphson iteration method gives the equations,

\[ F_{2i-1} + A_{2i-1} \delta L_{i-1}^{n+1} + B_{2i-1} \delta T_i^{n+1} + C_{2i-1} \delta L_i^{n+1} = 0, \]  

\[ F_{2i} + A_{2i} \delta T_i^{n+1} + B_{2i} \delta L_i^{n+1} + C_{2i} \delta T_{i+1}^{n+1} = 0, \]

where

\[ A_{2i-1} = \frac{\partial F_{2i-1}}{\partial L_{i-1}^{n+1}} = -\frac{Q_{i-1}^{n+1}}{\Delta a_i}, \]  

\[ B_{2i-1} = \frac{\partial F_{2i-1}}{\partial T_i^{n+1}} = \frac{\partial R_i^n}{\partial T_i^n} + \frac{\partial Q_i^n}{\partial T_i^n} \cdot \frac{L_i^{n+1} - L_i^{n+1}}{\Delta a_i} + \frac{1}{\Delta t}, \]  

\[ C_{2i-1} = \frac{\partial F_{2i-1}}{\partial L_i^{n+1}} = \frac{Q_i^{n+1}}{\Delta a_i}, \]  

\[ A_{2i} = \frac{\partial F_{2i}}{\partial T_i^{n+1}} = \frac{\partial P_i^{n+1}}{\partial T_i^{n+1}} \cdot S_i \cdot \frac{T_{i+1}^{n+1} - T_i^{n+1}}{\Delta a_i} - \frac{P_i^{n+1} \cdot S_i^{n+1}}{\Delta a_i}, \]  

\[ B_{2i} = \frac{\partial F_{2i}}{\partial L_i^{n+1}} = 1, \]  

\[ C_{2i} = \frac{\partial F_{2i}}{\partial T_{i+1}^{n+1}} = \frac{P_i^{n+1} \cdot S_i^{n+1}}{\Delta a_i}. \]

Special case is needed for the boundary grid points.

\[ A_1 = 0, \]  

\[ C_{2N-1} = 0, \]  

\[ B_{2N-1} = \frac{\partial R_i^{n+1}}{\partial T_i^n} + \frac{\partial Q_i^{n+1}}{\partial T_i^n} \cdot \frac{L_i^{n+1} - L_{i-1}^{n+1}}{\Delta a_i} + \frac{1}{\Delta t} + \frac{Q_i^{n+1}}{\Delta a_N} \cdot \frac{\partial L_i^{n+1}}{\partial T_i^{n+1}}. \]

The tri-diagonal matrix becomes

\[
\begin{pmatrix}
B_1 & C_1 & 0 & \cdots & 0 \\
A_2 & B_2 & C_2 & 0 & \cdots \\
0 & A_3 & B_3 & C_3 & 0 & \cdots \\
& & & & & \\
0 & \cdots & 0 & A_{2N-3} & B_{2N-3} & C_{2N-3} & 0 \\
0 & \cdots & 0 & A_{2N-2} & B_{2N-2} & C_{2N-2} & 0 \\
0 & \cdots & 0 & A_{2N-1} & B_{2N-1} & C_{2N-1} & 0
\end{pmatrix}
\begin{pmatrix}
\delta T_1^{n+1} \\
\delta L_1^{n+1} \\
\delta T_2^{n+1} \\
\vdots \\
\delta T_{2N-1}^{n+1} \\
\delta L_{2N-1}^{n+1} \\
\delta T_{2N}^{n+1}
\end{pmatrix}
= \begin{pmatrix}
F_1 \\
F_2 \\
F_3 \\
\vdots \\
F_{2N-3} \\
F_{2N-2} \\
F_{2N-1}
\end{pmatrix}
\]

Appendix A.3. Comparison

Each numerical solution (tri-diagonal, penta-diagonal, and tri-diagonal all grids) gives the similar solution if the initial condition is identical for each simulation. In the point of view of numerical
Figure A.18: (Color online) Curves for each numerical method. SkI4 is used to simulate neutron star cooling. All three methods show the identical results. The left figure shows the normal non-pairing phase and the right figure shows the superconducting phase.

cost, tri-diagonal even ($L_r$)-odd ($T$) method is superior to penta-diagonal even ($L_r$)-odd ($T$) and tri-diagonal all grids method. Figure A.18 shows neutron star cooling curves with SkI4 model. Three different numerical methods show almost identical results. The difference in early stage is caused by the difference in the time step $\Delta t$ in each simulation. That is, penta-diagonal scheme, for example, for some case, $t_{n+1/2}$ is normal state and $t_{n+1}$ can be superfluidic phase because of temperature difference in each step. Thus the time step should be adjusted to solve the diffusion equations. For normal phase, all three methods give no difficulty in the simulation. However, in superconducting phase, the most stable numerical method is tri-diagonal with even ($L_r$) and odd ($T$) scheme since it is free from the intermediate time step for sudden decrease of temperature.

Appendix B. Spatial zone and time step

In neutron star cooling simulation, we make grids from the core to outer boundary of crust ($\rho = 10^{10} \text{g/cm}^3$) and connect the temperature $T_b$ with $T_s$ using uniform luminosity approximation and $T_s - T_b$ relation [64, 67, 81]. The density of the core is around $\rho \simeq 10^{14} \sim 10^{15} \text{g/cm}^3$ and the crust has the density in the range of $10^{10}$ to $10^{14} \text{g/cm}^3$. Even though, the size of crust is only $\sim 1\text{km}$, the nuclear phase changes from heavy nuclei with neutron and electron gas to heavy nuclei with electron gas. Since the different equation of state gives different central, core-crust boundary, and neutron drip density, it is reasonable to make mesh point,

$$N_1 = W_1 \log_{10} \left( \frac{\rho_c}{\rho_{core}} \right),$$

$$N_2 = W_2 \log_{10} \left( \frac{\rho_{core}}{\rho_{drip}} \right),$$

$$N_3 = W_3 \log_{10} \left( \frac{\rho_{drip}}{\rho_{env}} \right),$$

where $\rho_c$ is the central density, $\rho_{core}$ is the density for core-crust boundary, $\rho_{drip}$ is the neutron drip density, and $\rho_{env} = 10^{10} \text{g/cm}^3$ for density of boundary of crust and envelope.
Several constraints for time step $\Delta t$ are used. In general, as time goes, the numerical solution is more stabilized so that we can use large time step for the numerical solution. Here we use the $T_{\text{eff}}$ to determine the next time step. We choose different $t_{\text{scale}}$, $\Delta t_{n+1} = t_{\text{scale}} \Delta t^n$ for different conditions of $T_{\text{eff}}^n$ and $T_{\text{eff}}^{n-1}$.

\[
\begin{cases}
|\frac{T_{\text{eff}}^n - T_{\text{eff}}^{n-1}}{T_{\text{eff}}^n}| > 0.1, & t_{\text{scale}} = 1.02, \\
0.05 < \frac{|T_{\text{eff}}^n - T_{\text{eff}}^{n-1}|}{T_{\text{eff}}^n} \leq 0.1, & t_{\text{scale}} = 1.1, \\
0.01 < \frac{|T_{\text{eff}}^n - T_{\text{eff}}^{n-1}|}{T_{\text{eff}}^n} \leq 0.05, & t_{\text{scale}} = 1.2, \\
|\frac{T_{\text{eff}}^n - T_{\text{eff}}^{n-1}}{T_{\text{eff}}^n}| \leq 0.01, & t_{\text{scale}} = 1.5.
\end{cases}
\] (B.4)

Another constraint for the time step comes from total time. In our simulation the next time step is always less than one tenth of total time,

\[
\Delta t^{n+1} = \min(t_{\text{scale}} \Delta t^n, \frac{1}{10} t).
\] (B.5)

If the superfluidity happens, a neutron star experiences drastic changes in specific heat, thermal conductivity, and neutrino emission rate. Thus, when the internal temperature drops below the critical temperature for superfluidity, we use adaptive time step method. The $T_{\text{eff}}^{n+1}$ should change within maximum 5% of $T_{\text{eff}}^n$. For instance, if the numerical solution gives $T_{\text{eff}}^{n+1} < 0.95 T_{\text{eff}}^n$, we solve the diffusion equations again with the new time step $\Delta t^{n+1,i+1} = t_{\text{reduce}} \Delta t^{n+1,i}$ (where index $i$ indicates the $i$th trial time step) until $T_{\text{eff}}^{n+1} > 0.95 T_{\text{eff}}^n$. In our simulation $t_{\text{reduce}} = 0.75$ to reduce the time step. Once we find the solution, according to the temperature differences between $t^n$ and
In Fig. B.20, we compare results from three different numerical methods. If the superfluidity happens, penta-diagonal method needs a smaller size of time step to make the result similar with the ones from both the tri-diagonal methods (even \( L_r \) and odd \( T \)) and the tri-diagonal methods in which \( L_r \) and \( T \) are defined in all grid points.

Figure B.20: (Color online) Left figure shows the large scale cooling curve. Right figure shows the cooling curve near the critical temperature for superfluidity. Each curve shows different behavior near the critical temperature.

\[ t^{n+1}, \text{ we use the adaptive } t_{scale} \text{ for the next time } t^{n+2}. \text{ For superfluidity case, we use} \]

\[
\begin{align*}
0.01 < \frac{T_n^{eff} - T_{n-1}^{eff}}{T_{n-1}^{eff}} & \leq 0.05, & t_{s1} = 1.2, \\
\left| \frac{T_n^{eff} - T_{n-1}^{eff}}{T_{n-1}^{eff}} \right| & \leq 0.01, & t_{s2} = 1.5.
\end{align*}
\]

(B.6)
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