A Commuting Projector Model with a Non-zero Quantized Hall conductance

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By ungauging a recently discovered lattice rotor model for Chern-Simons theory, we create an exactly soluble path integral on the spacetime lattice for $U^\kappa(1)$ Symmetry Protected Topological (SPT) phases in 2 + 1 dimensions with a non-zero Hall conductance. We then convert the path integral on a 2 + 1d spacetime lattice into a 2d Hamiltonian lattice model, and show that the Hamiltonian consists of mutually commuting local projectors. We confirm the non-zero Hall conductance by calculating the Chern number of the exact ground state. It has recently been suggested that no commuting projector model can host a nonzero Hall conductance. We evade this no-go theorem by considering a rotor model, with a countably infinite number of states per site.

The discovery of the Quantum Hall Effect ignited a revolution in condensed matter physics [1–3]. Both the integral and fractional [4] forms showed the limits of the symmetry-breaking approach to phases of matter [5–7] and new research revealed states of matter hosting fractional particles [8, 9], protected edge modes [10], and topological ground state degeneracy [5]. Together with research directed at understanding high-$T_c$ superconductors [11–14], the new era of condensed matter has unveiled physics which is stranger, richer, and more entangled than before. Certainly, more is indeed different.

Commuting projector models have been a central tool for understanding the new zoo of theories. Employed most famously by Kitaev [15] to provide an exactly soluble model for the previously proposed 2+1d $Z_2$ topological order [12, 13] with emergent fermions and anyons, they now describe models for a wide class of string-net topological order [16], recently unleashed a flurry of research on fractons [17, 18], and continue to underlie our microscopic understanding of exotic phases.

It is quite surprising then that no commuting projector model has been discovered for gapped phases with non-zero Hall conductance. It was commonly believed that none could exist, and recently a no-go theorem has been proposed [19], ruling out a large class of potential theories with a finite Hilbert space on each site.

In this paper, we describe a commuting projector model for $U^\kappa(1)$ SPT phases with non-zero quantized Hall conductance [20–23], providing an exactly solved model for $U^\kappa(1)$ SPT states. Related to the evolution of the no-go theorem, the physical degrees of freedom in our model are $U^\kappa(1)$ rotors, with a countably infinite Hilbert space on each site. There is no clear way to reduce the on-site Hilbert space to finite dimension, while retaining $U^\kappa(1)$ symmetry and commuting projector property, because the Lagrangian or the Hamiltonian, while local, is not a smooth function of the rotor variables.

The phases we describe are short-range-entangled SPT phases, with a unique ground state on any manifold. Employing the recent discovery of a rotor path integral [24] for $K$-matrix Chern-Simons theory, we ungauged [25] that model to derive another rotor path integral for general $U^\kappa(1)$ SPT phases. In the Hamiltonian approach, the latter rotor path integral yields an exactly soluble commuting projector lattice model.

First, we review the Chern-Simons model and its properties. We then ungaug [25] it in the presence of a background field and show that integrating out the matter fields leads to a Chern-Simons response term. A conversion from the Lagrangian to Hamiltonian approaches then yields a commuting projector Hamiltonian for the most general $U^\kappa(1)$ SPT state. The Hamiltonian model leads to a unique wavefunction, which we show has the expected Chern number over the “holonomy torus.” Finally, we discuss the relation of our model to the no-go theorem and the theory of discontinuous group cocycles.

Chern-Simons Lattice Model. In Ref. 24, a local bosonic rotor model was constructed which realizes a 2+1d topological order described by $U^\kappa(1)$ Chern-Simons theory with an even $K$-matrix. The topological order has a chiral central charge given by the signature of the $K$-matrix, and hosts an exact $Z_k \times Z_k \times \cdots$ 1-symmetry, where $k_\ell$ are the diagonal entries of the Smith normal form of $K$.

The model is formulated in terms of cochains on a spacetime simplicial complex. A spacetime complex (lattice) is a triangulation of the three-dimensional spacetime with a branching structure. [20, 26], denoted as $M^3$. We denote vertices of the complex as $i,j,\ldots$, links as $\langle ij \rangle$, and so forth. For the Chern-Simons Lattice model with a $\kappa \times \kappa$ $K$-matrix, the physical degrees are rotor one-cochains, i.e. $a^{R/Z}_{i,j}$ on each link $\langle ij \rangle$ (which corresponds to lattice gauge field), $I = 1, \ldots, \kappa$. The ungauged model they will be rotor zero-cochains, i.e. a $\phi^{R/Z}_{i,j}$ on each site (which corresponds to lattice scalar field). We work in units where flux and cycles are quantized to unity; for instance, $U^\kappa(1)$ variables may be obtained in the form $U^\kappa(1) = e^{2\pi i a^{R/Z}_{i,j}}$, $\varphi^\kappa(1) = e^{2\pi i \phi^{R/Z}_{i,j}}$. Accordingly, we will require that all quantities be invariant under a gauge redundancy:

$$a^{R/Z}_{i,j} \rightarrow a^{R/Z}_{i,j} + n_{I,ij}, \quad \phi^{R/Z}_{i,j} \rightarrow \phi^{R/Z}_{i,j} + n_{I,i}$$

with $n_{I,ij}, n_{I,i} \in Z$, so that these variables are genuinely $R/Z$-valued. In this case, the lattice gauge fields $a^{R/Z}_{i,j}$

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will describe a compact $U^c(1)$ gauge theory, and the lattice scalar fields $\phi^R/Z_i$ will describe a bosonic model with $U^c(1)$ symmetry.

We will make use of the lattice differential $d$ [27] which sends $m$-cochains $f_m$ to $m+1$ cochains $df_m$ and satisfies $d^2 = 0$, and the lattice cup $\cup$-products, which take an $m$-cochain $f_m$ and a $n$-cochain $g_n$ and return a $n+m-k$ cochain $f_m \cup_k g_n$. We will abbreviate by omitting zero-cup products $f_m \cup_0 g_n = f_m g_n$. For a review of this notation, see [27] and the supplemental material of [24].

We can now write down the Chern-Simons lattice model. Given a bosonic $K$-matrix $K_{IJ}$ with even diagonal entries, define $k_{IJ} = K_{IJ}$ for $I \neq J$ and $k_{II} = \frac{1}{2} K_{II}$ for $I = J$. In terms of this reduced matrix $k_{IJ}$, the path integral is:

$$Z = \prod \left[ d\phi^R/Z_I \right] e^{i2\pi \sum_{i \leq j} k_{IJ} f_{M^3} \left[ a^R/Z_i \right]}$$

(1)

This will be the ground state of the commuting projector model.

The model has a $U^c(1)$ symmetry:

$$\phi^R/Z_{1,i} \rightarrow \phi^R/Z_{1,i} + \theta_I$$

(2)

for constant $\theta_I$. On a manifold with a boundary, the model is invariant under eq. (1) if and only if the reduced $k$-matrix $k_{IJ}$ is integral, i.e. if the original $K$-matrix $K_{IJ}$ is integral with even diagonals (which turns out to describe a quantized Hall conductance and a bosonic SPT order). In this case, the field variables are indeed $R/Z$ valued. A reduction of this model to a known one for $Z_n$ SPT states [20] is given in the Supplemental material.

In fact, the model realizes a $U^c(1)$ SPT state. To see the SPT order, we repeat the ungauging in the presence of a weak background gauge field $\bar{A}^R/Z$ and evaluate the effective action for $\bar{A}^R/Z$. In the presence of background $U^c(1)$ background gauge field $\bar{A}^R/Z$, the ungauging is done via

$$\bar{A}^R/Z = \bar{A}^R/Z + d\phi^R/Z.$$  

(3)

The new model is given by

$$Z = e^{i2\pi \sum_{i \leq j} k_{IJ} f_{M^3} \left[ a^R/Z_i \right] (\bar{A}^R/Z_i - \bar{A}^R/Z_j) - \left[ \bar{A}^R/Z_i + \bar{A}^R/Z_j \right]}$$

(4)

where $\phi^R/Z_i$ are zero-cochains defined on the vertices. The partition function is now given by:

$$Z = \prod \left[ d\phi^R/Z_i \right] e^{i2\pi \sum_{i \leq j} k_{IJ} f_{M^3} \left[ a^R/Z_i \right]}$$

(5)

where $M^3$ may have boundaries, and the measure is taken to be integration over all sites, $\prod \left[ d\phi^R/Z_i \right] = \prod \left[ d\phi^R/Z_i \right]$. Eq. (4) is the path integral description of the commuting projector model. As the action is a total derivative, the partition function is unity on any closed manifold, implying that the model describes a trivial topological order with zero central charge. On any spatial boundary, the path integral defines a wavefunction $|\psi\rangle$, where

$$\langle \phi^R/Z_{1,i} | \psi \rangle = \exp \left\{ -2\pi i \sum_{i \leq j} k_{IJ} \int_{M^2} d\phi^R/Z_i [d\phi^R/Z_j] \right\}$$

(6)

Noting that this implies that $d|\bar{A}^R/Z_i| = 0$, the path integral becomes:

$$Z = e^{i2\pi \sum_{i \leq j} k_{IJ} f_{M^3} \left[ a^R/Z_i \right] (\bar{A}^R/Z_i - \bar{A}^R/Z_j) - \left[ \bar{A}^R/Z_i + \bar{A}^R/Z_j \right]}$$

(7)

This is the Chern-Simons response on lattice. When $M^3$ is closed, the action is invariant under gauge transformations of the background gauge field $\bar{A}^R/Z_i \rightarrow \bar{A}^R/Z_i + d\phi^R/Z_i$.  


If $\mathcal{M}^3$ is a disk, the redundancy (1) can be used to set $|d\bar{a}_{1/2}| = 0$, and the response becomes:

$$Z = e^{i\pi \sum_{i,j} K_{ij} f_{\mathcal{M}^3} \bar{a}_{1/2} \bar{a}_{1/2}}$$

(11)

This Chern-Simons response, in terms of the unreduced $K$-matrix for $K_{ij}$, describes the Hall conductance and is the SPT invariant for our model.

Given that the bulk behavior of the path integral is trivial, we expect that we should be able to create an exactly soluble Hamiltonian model to describe the time-evolution. Furthermore, because the path integral defines a wavefunction on any spatial boundary independently of the bulk dynamics, we expect that this Hamiltonian model should be a commuting projector onto a ground state. As we shall now see, both are true.

**Commuting Projector Model.** We can use the spacetime formalism to construct a commuting projector Hamiltonian on a triangular lattice of the sort shown in Fig. 1a. To do so, we consider the time evolution of a single site, $\phi_{4/2}^{R/Z} \rightarrow \phi_{5/2}^{R/Z}$, while preserving the orientation of lattice links as shown in fig 1b. Evaluating the path integral on the complex shown yields the matrix elements for the transition $\phi_{4/2}^{R/Z} \rightarrow \phi_{5/2}^{R/Z}$ as a function of the surrounding $\phi_j^{R/Z}$:

$$M_{\phi_{4/2}^{R/Z} \rightarrow \phi_{5/2}^{R/Z}}(\phi_{1/1}^{R/Z} , ..., \phi_{1/8}^{R/Z}) = \exp \left\{ 2\pi i \sum_{i,j} k_{ij} \left[ \phi_{1/0}^{R/Z} \left( \phi_{j/5}^{R/Z} - \phi_{j/2}^{R/Z} \right) + \phi_{1/2}^{R/Z} \left( \phi_{j/6}^{R/Z} - \phi_{j/5}^{R/Z} + \phi_{j/5}^{R/Z} - \phi_{j/2}^{R/Z} \right) \right] \right\}$$

(12)

We will interpret this transition amplitude as the matrix element for an operator $\hat{M}_i$ acting on site-4. However, eq. (12) is somewhat daunting. Let us set consider $\hat{M}_4$ as an operator acting only on the Hilbert space on site-4. If $k_{ij} = 0$, then $\langle \phi_{1/4}^{R/Z} | \phi_{1/4}^{R/Z} \rangle = 1$, and $\hat{M}_4$ is simply the projector onto the state with zero angular momentum in each $U(1)$. For nonzero $k_{ij}$, we may rewrite the transition amplitude as $M_{\phi_{4/2}^{R/Z} \rightarrow \phi_{5/2}^{R/Z}} = \exp(2\pi i f(\phi_{1/5}^{R/Z} - f(\phi_{1/4}^{R/Z}))$ (note that this implies hermiticity) where $f(\phi)$ is a function defined in the appendix that depends on $\phi$ and takes as parameters $\phi_{1/1}^{R/Z} ... \phi_{1/8}^{R/Z}$, but not $\phi_{1/4}^{R/Z}$ or $\phi_{1/5}^{R/Z}$ . This implies that, up to an overall phase, the $\hat{M}_i$ act as

$$\hat{M} |\phi_i\rangle \propto \int d\phi \bar{e}^{2\pi i f(\phi_i)} |\phi_i\rangle$$

(13)

We see then that these projectors may be thought of as ‘twisting’ the zero angular momentum state by a phase function $f(\phi_i)$ which depends on the surrounding values of $\phi_i$. The phase itself is determined by the cocycle of the action in (4).

We may construct $\hat{M}_i$ for an entire lattice. As shown in the supplemental material, the $\hat{M}_i$ are hermitian and $U(1)$ symmetric under $\phi_{1/1}^{R/Z} \rightarrow \phi_{1/1}^{R/Z} + \theta_1 + n_1 i$.

The $\hat{M}_i$ inherit a number of remarkable properties from the fact that the $2 + 1d$ path integral action contains only a surface term. First, they mutually commute: consider the three spacetime complexes in figure 1c, which ad-

![FIG. 1. (Color Online). We construct a commuting projector model for the lattice in (a) by evaluating the spacetime path integral for the complex in (b) and turning the amplitude into operators. Because the path integral contains only a surface term, we can show that (c) the matrix is hermitian, (d) the operators commute, and (e) they are projectors.](image)
transforms as:

\[ \ell \rightarrow e^{-2\pi i \sum_{l<\ell} k_{l,m} \theta_y |\theta_x, \theta_y|} \quad (16) \]

In this gauge, one may replace the phase in eq. (15) by:

\[ e^{-2\pi i \sum_{l<\ell} k_{l,m} n_l m_j} \theta_x \ell_y \quad (17) \]

We should recognize this as the boundary conditions for a particle on a torus with flux:

\[ \sum_{l<\ell} k_{l,m} n_l m_j + \sum_{l<\ell} k_{l,m} m_j n_l = \sum_{l,J} K_{l,J} n_l m_J \quad (18) \]

To make sense of this, apply the gauge transformation:

\[ (\theta_x, \theta_y) \rightarrow e^{-2\pi i \sum_{l<\ell} k_{l,m} \theta_y |\theta_x, \theta_y|} \]

We see then that our Hamiltonian system has the (mixed) Hall conductance \( n \cdot K \cdot m \), in agreement with the Chern-Simons response function derived from the spacetime path integral. In the case of \( \kappa = 1 \), with \( n = m = 1 \), this becomes a system with integer hall conductance \( K \) which is an even integer.

Discussion. We have derived a model for the 2 + 1d \( U(1)^3 \) SPT states in terms of both a spacetime lattice path integral and a commuting projector model. We have confirmed the Hall response in two ways: by coupling the spacetime model to a background gauge field, and by examining the ground state wavefunction of the Hamiltonian model on a torus with twisted boundary conditions. Now we must understand how this model evades the no-go theorem.

The infinite dimensional on-site Hilbert space of the rotors, combined with an action that is not a continuous function of the field variables, is what allows this model to exist. The discontinuous action is critical for commuting-projectors; if it were continuous, then one could truncate the Hilbert space to low-angular momentum modes and render the on-site Hilbert space finite while retaining the full \( U^\kappa(1) \) symmetry and commuting-projector property, hence running afoul of the no-go theorem [19]. Conversely, the no-go theorem assumes that the ground state wavefunction is a finite Laurent polynomial in \( e^{i\theta_x}, e^{i\theta_y} \), an assumption which is violated in our model (See the details of the Chern number calculation in the Appendix for an example). This commuting projector model represents a fixed-point theory for \( U^\kappa(1) \) SPT phases with nonzero Hall conductance; it may be that any such fixed-point theory requires an infinite on-site Hilbert space.

There is a connection here to the theory of discontinuous group cocycles. SPT phases in \( d + 1 \) dimensions are classified by \( H^{d+1}(G, U(1)) \) [20], and their topological field theory actions are given by representative cocycles. Eq. (4) is an example of this sort of cocycle. For reasons similar to the underlying argument of the no-go theorem, if we restrict to continuous cochains, then corresponding \( H^2(\text{continuous}(G, U(1))) \) is not large enough to classify the group extensions of \( G \) by \( U(1) \) [28]. It is only by allowing piecewise-continuous cochains, like the action in eq. (4), does corresponding \( H^2(G, U(1)) \) classify the group extensions, as well as the projective representations of \( G \) and the 1+1d SPT orders [20].
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Appendix A: Chern Number for Hall Conductance

Working with the wavefunction on the lattice shown in Fig. 2, recall that we label lattice points by \((x,y)\in[0...L_x-1] \times [0...L_y-1]\). We twist the boundary conditions by \(\theta_x n_x, \theta_y m_y\), so that \(\phi_{I,x=0,y} = \phi_{I,x=L_x,y} - \theta_x n_1\) and \(\phi_{I,x,y=0} = \phi_{I,x=L_x,y} - \theta_y m_1\), with \(n_1, m_1 \in \mathbb{Z}^\kappa\).

Consider first the plaquettes marked in red. The contribution of these plaquettes to the path integral is of the form:

\[
\exp \left\{ -2\pi i \sum_{I \leq J} k_{I,J} \left[ \left( \phi_{I,x=L_x,y} + n_1 \theta_{I,x=0,y} - \phi_{I,x=L_x,y=1} - \phi_{I,x=L_x,y} \right) \left( \phi_{J,x=L_x,y+1} - \phi_{J,x=L_x,y} \right) \right. \right.
\]

\[
\left. \left. - \left( \left( \phi_{I,x=1,y} - \phi_{I,x=L_y-1,y} - \phi_{I,x=L_x-1,y} \right) \times \left[ \phi_{J,x=L_x,y+1} + n_1 \theta_{I,x=0,y} - \phi_{J,x=L_x,y} \right] \right) \right] \right\}
\]

Incrementing \(\theta_x\) by \(\ell_x\) changes the first term by an integer, while the second term changes by \(\left( \phi_{I,x=1,y} - \phi_{I,x=L_y-1,y} \right) \ell_x\), and so the multiplicative change on the path integral is:

\[
\epsilon^{2\pi i \sum_{I \leq J} k_{I,J} \left( \phi_{I,x=L_x,y} + n_1 \theta_{I,x=0,y} - \phi_{I,x=L_x,y=1} - \phi_{I,x=L_x,y} \right) n_1 \ell_x}
\]

Similarly, the plaquettes marked in blue change by:

\[
\epsilon^{-2\pi i \sum_{I \leq J} k_{I,J} \left( \phi_{I,x=1,y} - \phi_{I,x=L_y-1,y} - \phi_{I,x=L_x-1,y} \right) m_1 \ell_y}
\]

On the purple plaquettes at the corner, the contribution to the path integral is:

\[
\exp \left\{ -2\pi i \sum_{I \leq J} k_{I,J} \left[ \left( \phi_{I,x=L_x,y} = \phi_{I,x=L_x,y} - n_1 \theta_{I,x=0,y} - \phi_{I,x=L_x,y=1} - \phi_{I,x=L_x,y} \right) \times \phi_{J,x=L_x,y+1} + m_1 \theta_{I,x=L_y-1,y} - \phi_{J,x=L_x,y} \right) \right. \right.
\]

\[
\left. \left. \left. - \left( \phi_{I,x=1,y} - \phi_{I,x=L_y-1,y} - \phi_{I,x=L_x-1,y} \right) \times \phi_{J,x=L_x,y+1} + m_1 \theta_{I,x=L_y-1,y} - \phi_{J,x=L_x,y} \right] \right] \right\}
\]

which will change by:

\[
\exp \left\{ -2\pi i \sum_{I \leq J} k_{I,J} \left[ \left( \phi_{I,x=1,y} - \phi_{I,x=L_y-1,y} - n_1 \theta_{I,x=0,y} - \phi_{I,x=L_x-1,y} \right) m_1 \ell_y \right. \right.
\]

\[
\left. \left. \left. + \left( \phi_{I,x=L_x,y} = \phi_{I,x=L_x,y} - n_1 \theta_{I,x=0,y} - \phi_{I,x=L_x,y=1} - \phi_{I,x=L_x,y} \right) m_1 \ell_y \right] \right] \right\}
\]
We see that the model for the \( Z \) \( \phi \) curves vanish, and we are left with the result in the main text.

\[ (A3) \]

Combining the change on the red, blue, and purple plaquettes, the overall change to the wavefunction is:

\[ e^{-2\pi i} \sum_{\kappa} k_{IJ} [n_I\theta_v \ell \ell_v - n_I\theta_v \ell \ell_v - n_J\ell \ell_v \ell_v' \ell_J + n_J\ell \ell_v \ell_v' \ell_J ] \]  

(A4)

where \( \gamma_1, \gamma_2 \) are the red and blue loops in Fig. 2, respectively. As that \( \gamma_1, \gamma_2 \) are closed, the sums along those curves vanish, and we are left with the result in the main text.

Appendix B: Reducing to \( Z_n \) gauge theory

When \( \kappa = 1 \), the above becomes

\[ Z = \prod \int d\phi^R/Z e^{i2\pi k f_{A3} a^{z_n} d\phi^Z} \]  

(B1)

which describe a \( U(1) \) SPT state with Hall conductance \( \sigma_{xy} = \frac{2\pi k}{n} \).

Let us compare the model (B1) for \( U(1) \) SPT state with a model for \( Z_n \) SPT state [29]:

\[ Z = \sum_{\phi^Z} e^{i2\pi k f_{A3} a^{z_n} d\phi^Z} \]

\[ a^{z_n} = d\phi^Z - n\left[ \frac{1}{n} d\phi^Z \right] \]

where \( \phi^Z \) is a \( Z_n \)-valued 0-cochain. The above can be rewritten as

\[ Z = \sum_{\phi^Z} e^{-i2\pi k f_{A3} d\phi^Z} \]  

(B2)

We see that the model for the \( Z_n \) SPT state and the model for the \( U(1) \) SPT state have very similar forms. Here, \( d\phi^Z \) with \( \phi^Z = \frac{1}{n} \phi^Z \) is a cocycle in \( H^2(\mathbb{Z}_n, \mathbb{R}/Z) \), while \( d\phi^Z \) with \( \phi^Z = \frac{1}{n} \phi^Z \) is a cocycle in \( H^2(U(1), \mathbb{R}/Z) \).

Appendix C: Properties of the Commuting Projectors

Recall the definition of the \( M_i \):

\[ M_{\phi^R/Z \phi^R/Z}^{\phi^R/Z \phi^R/Z} (\phi_{I,1}^{R/Z}, ..., \phi_{I,8}^{R/Z}) = \]

\[ \exp \left\{ 2\pi i \sum_{I \leq J} k_{IJ} \left( \phi_{J,1}^{R/Z} \phi_{J,2}^{R/Z} + [\phi_{J,3}^{R/Z} - \phi_{J,5}^{R/Z}] \right) \right\} \]  

(C1)

Here we walk through the calculations to show analytically that the \( M_i \) are hermitian, \( U^c(1) \) symmetric, commuting projectors.

For hermiticity, we wish to show that

\[ M_{\phi^R/Z \phi^R/Z}^{\phi^R/Z \phi^R/Z} (\phi_{I,1}^{R/Z}, ..., \phi_{I,8}^{R/Z})^* = M_{\phi^R/Z \phi^R/Z}^{\phi^R/Z \phi^R/Z} (\phi_{I,1}^{R/Z}, ..., \phi_{I,8}^{R/Z}) \]  

(C2)

To see this, first note that the terms in eq. (C1) with coefficients of \( \phi_{I,1}^{R/Z}, \phi_{I,2}^{R/Z}, \phi_{I,3}^{R/Z} \) are antisymmetric under \( \phi_{I,1}^{R/Z} \leftrightarrow \phi_{I,5}^{R/Z} \). Next, consider the \( \phi_{I,5}^{R/Z} \) term:

\[ \phi_{I,1}^{R/Z} \phi_{I,5}^{R/Z} \phi_{I,3}^{R/Z} + \phi_{I,3}^{R/Z} \phi_{I,5}^{R/Z} \phi_{I,1}^{R/Z} \]  

under \( \phi_{I,1}^{R/Z} \leftrightarrow \phi_{I,5}^{R/Z} \), this becomes:

\[ -\phi_{I,1}^{R/Z} \phi_{I,5}^{R/Z} \phi_{I,3}^{R/Z} + \phi_{I,3}^{R/Z} \phi_{I,5}^{R/Z} \phi_{I,1}^{R/Z} \]  

(C3)

Which is precisely \(-1\) times the \( \phi_{I,5}^{R/Z} \) term. Similarly, the \( \phi_{I,4}^{R/Z} \) term becomes minus the \( \phi_{I,5}^{R/Z} \) term. Taking all of these together with the minus sign from the factor of \( i \), we see that the \( M_i \) are hermitian.

The symmetry \( \phi_{I,1}^{R/Z} \rightarrow \phi_{I,1}^{R/Z} + \theta_I \) arises essentially because only \( d\phi_{I}^{R/Z} \) appears in action

\[ 2\pi \sum_{I \leq J} k_{IJ} d\phi_{I}^{R/Z} \phi_{I,1}^{R/Z} \]  

(C4)

To see this explicitly in eq. C1, first note that the \( \theta_I \) cancel in the rounded terms.

What remains is:
\[ \exp \left\{ 2\pi i \sum_{I \leq J} k_{ij} \left( (\phi_{i1}^R + \phi_{i1}^Z)(\phi_{j5}^R - \phi_{j2}^R) + [\phi_{i3}^R - \phi_{i3}^Z] + [\phi_{i4}^R - \phi_{i4}^Z] + [\phi_{j2}^R - \phi_{j4}^R] \right) \right\} \]

\[ + \phi_{i1}(\phi_{j4}^R - \phi_{j3}^R) + [\phi_{i4}^R - \phi_{i4}^Z] + [\phi_{j1}^R - \phi_{j1}^Z] + [\phi_{j2}^R - \phi_{j2}^Z] + [\phi_{j4}^R - \phi_{j4}^Z] + [\phi_{j6}^R - \phi_{j6}^Z] \]

\[ = M_{i_1,i} \phi_{i,1}^R \phi_{i,5}^Z \exp \left\{ 2\pi i \sum_{I \leq J} k_{ij} \left( (\phi_{i1}^R + \phi_{i1}^Z)(\phi_{j5}^R - \phi_{j2}^R) + [\phi_{i3}^R - \phi_{i3}^Z] + [\phi_{i4}^R - \phi_{i4}^Z] + [\phi_{j2}^R - \phi_{j4}^R] \right) \right\} \] (C5)

\[ \text{FIG. 3.} \]

One may check that remaining rounded terms cancel one-by-one, essentially because this is \( d [\phi_{i1}^R] \) evaluated over the closed surface of the complex. To see that the \( M_i \) are symmetric under \( \phi_{i1}^R \to \phi_{i1}^R + n_{I,i} + \theta_1 \), first note that, because \( k_{I,j} \) is integral, we need only to worry about the rounded terms. However, one may check that each the effect \( n_{I,i} \) in each sum of rounded terms cancels, essentially because each sum is \( d [\phi_{i1}^R] \), and under \( \phi_{i1}^R \to \phi_{i1}^R + n_1 \) this becomes \( d [d \phi_{i1}^R] + d[d_1] = d [d \phi_{i1}^R] + d^2 n_1 \) and \( d^2 = 0 \). All told, we now see that the \( M_i \) are symmetric under \( \phi_{i1}^R \to \phi_{i1}^R + n_{I,i} + \theta_1 \), i.e. the \( M_i \) are \( U^R(1) \) symmetric.

Next we check commutation. The only nontrivial case occurs when the \( M_i \) are on adjacent sites. For this calculation, we will use a slightly different convention, indicating the time-evolved points with a prime as opposed to a new number, so that amplitudes take the form \( M_{i_1,i} \phi_{i,1}^R \phi_{i,5}^Z(\phi_{i_1}^R, \phi_{i_1}^Z) \). We also drop the \( R/Z \) superscripts.

Consider then the 2d spatial complex in Fig. 3. We wish to compare:

\[ \hat{M}_4 \hat{M}_5 = M_{\phi_{i1} \to \phi_{i1}'}(\phi_{1,1}, \ldots, \phi_{1,8}) \]

\[ \times M_{\phi_{i5} \to \phi_{i5}'}(\phi_{5,1}, \ldots, \phi_{5,8}) \]

to

\[ \hat{M}_5 \hat{M}_4 = M_{\phi_{i1} \to \phi_{i1}'}(\phi_{1,1}, \ldots, \phi_{1,8}) \]

\[ \times M_{\phi_{i4} \to \phi_{i4}'}(\phi_{4,1}, \ldots, \phi_{4,8}) \] (C7)

Expanding eq. (C7), it becomes:

\[ \hat{M}_5 \hat{M}_4 = \exp \left\{ 2\pi i \sum_{I \leq J} k_{ij} \left( \phi_{i1}(\phi_{j4} - \phi_{j3}) + [\phi_{i3} - \phi_{i3}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \right) \right\} \]

\[ + \phi_{i1}(\phi_{j4} - \phi_{j3}) + [\phi_{i3} - \phi_{i3}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j7} - \phi_{j6}) + [\phi_{j6} - \phi_{j5}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j5} - \phi_{j4}) + [\phi_{j4} - \phi_{j3}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j6} - \phi_{j5}) + [\phi_{j5} - \phi_{j4}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j9} - \phi_{j8}) + [\phi_{j8} - \phi_{j7}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j6} - \phi_{j5}) + [\phi_{j5} - \phi_{j4}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j9} - \phi_{j8}) + [\phi_{j8} - \phi_{j7}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j6} - \phi_{j5}) + [\phi_{j5} - \phi_{j4}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j9} - \phi_{j8}) + [\phi_{j8} - \phi_{j7}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j6} - \phi_{j5}) + [\phi_{j5} - \phi_{j4}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j9} - \phi_{j8}) + [\phi_{j8} - \phi_{j7}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j6} - \phi_{j5}) + [\phi_{j5} - \phi_{j4}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j9} - \phi_{j8}) + [\phi_{j8} - \phi_{j7}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j6} - \phi_{j5}) + [\phi_{j5} - \phi_{j4}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j9} - \phi_{j8}) + [\phi_{j8} - \phi_{j7}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j6} - \phi_{j5}) + [\phi_{j5} - \phi_{j4}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j9} - \phi_{j8}) + [\phi_{j8} - \phi_{j7}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j6} - \phi_{j5}) + [\phi_{j5} - \phi_{j4}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j9} - \phi_{j8}) + [\phi_{j8} - \phi_{j7}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

\[ + \phi_{i1}(\phi_{j6} - \phi_{j5}) + [\phi_{j5} - \phi_{j4}] + [\phi_{i4} - \phi_{i4}] + [\phi_{j2} - \phi_{j4}] \]

On the other hand,
\[ \hat{M}_4 \hat{M}_5 = \exp \left\{ 2\pi i \sum_{I \leq J} k_{IJ} \left( \phi_{I,0}(\phi'_{I,4} - \phi_{J,1}) + (\phi_{I,3} - \phi'_{I,4}) + (\phi_{I,4} - \phi_{J,3}) + (\phi_{I,1} - \phi_{J,4}) \right) \right\} \]

\[ + \phi_{I,1}(\phi'_{I,4} - \phi_{J,5}) + (\phi_{I,1} - \phi'_{I,4}) + (\phi_{I,4} - \phi_{J,1}) + (\phi_{I,5} - \phi_{J,4}) \]

\[ + \phi_{I,4}(\phi'_{I,4} - \phi_{J,5}) + (\phi_{I,4} - \phi_{J,1}) + (\phi_{I,5} - \phi_{J,4}) \]

Proceeding term-by-term, one may verify that these are equal. (To simplify the calculation, note that the terms with coefficients \( \phi_{I,0}, \phi_{I,2}, \phi_{I,3}, \phi_{I,5}, \) and \( \phi'_{I,5} \) are identical.)

We also verify that the \( \hat{M}_i \) are projectors. Retaining the same notation, we calculate:

\[ M_{b_{I,4} \rightarrow \phi'_{I,4}}(\phi_{I,1}, \ldots, \phi'_{I,4}, \ldots, \phi_{I,8}) M_{\phi_{I,4} \rightarrow \phi'_{I,4}}(\phi_{I,1}, \ldots, \phi_{I,8}) \]

\[ = \exp \left\{ 2\pi i \sum_{I \leq J} k_{IJ} \left( \phi_{I,0}(\phi'_{I,4} - \phi_{J,1}) + (\phi_{I,3} - \phi'_{I,4}) + (\phi_{I,4} - \phi_{J,3}) + (\phi_{I,1} - \phi_{J,4}) \right) \right\} \]

\[ + \phi_{I,1}(\phi'_{I,4} - \phi_{J,5}) + (\phi_{I,1} - \phi'_{I,4}) + (\phi_{I,4} - \phi_{J,1}) + (\phi_{I,5} - \phi_{J,4}) \]

\[ + \phi_{I,4}(\phi'_{I,4} - \phi_{J,5}) + (\phi_{I,4} - \phi_{J,1}) + (\phi_{I,5} - \phi_{J,4}) \]

\[ + \phi_{I,0}(\phi''_{I,4} - \phi_{J,1}) + (\phi'_{I,4} - \phi_{J,3}) + (\phi_{J,1} - \phi_{J,4}) + \phi_{I,4}(\phi''_{I,4} - \phi_{J,5}) + (\phi_{J,1} - \phi_{J,4}) + (\phi_{J,5} - \phi_{J,4}) \]

\[ + \phi_{I,3}(\phi''_{I,4} - \phi_{J,3}) + (\phi_{J,7} - \phi_{J,4}) + (\phi_{J,4} - \phi_{J,7}) + (\phi_{J,5} - \phi_{J,4}) + \phi_{I,4}(\phi_{J,7} - \phi_{J,4}) + (\phi_{J,8} - \phi_{J,7}) + (\phi_{J,5} - \phi_{J,8}) + (\phi'_{I,4} - \phi_{J,7}) \]

Whence \( \hat{M}_{I=4}^2 = \hat{M}_{I=4} \), and by translation \( \hat{M}_{I}^2 = \hat{M}_I \) for the entire lattice.

Finally, we also note that the \( \hat{M}_i \) are mutually independent, so there is no condition which could allow extra ground state degeneracy as in the toric code.

**Appendix D: Rewriting of the \( \hat{M}_i \)**

Defining

\[ f(\phi_x) = \sum_{I \leq J} k_{IJ} \left( \phi_{I,x}(\phi'_{I,5}^R - \phi_{J,6}^R) + (\phi_{I,6} - \phi_{J,5}) \right) \]

\[ + (\phi_{I,7} - \phi_{J,8}) + (\phi_{I,5} - \phi_{J,7}) \]

\[ + \phi_{I,0}(\phi_{J,7} - \phi_{J,2}) + (\phi_{J,3} - \phi_{J,5}) \]

\[ + (\phi_{J,7} - \phi_{J,2}) + (\phi_{J,3} - \phi_{J,5}) \]

we can see that eq. (C1) may be rewritten as:

\[ M_{\phi_{I,4} \rightarrow \phi_{I,4}}(\phi_{I,5}) = \exp(2\pi i f(\phi_{I,5})) \exp(-2\pi i f(\phi_{I,4})) \]