ON CERTAIN IRREDUCIBLE FINITE GROUP ACTIONS ON SURFACES

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Abstract. Ishizaka classified up to conjugation orientation preserving periodic diffeomorphisms of a surface which commute with a hyperelliptic involution. Here, an involution $\iota$ on a surface $\Sigma_g$ is hyperelliptic if and only if $\Sigma_g/\langle \iota \rangle \cong S^2$. In this article, we give a classification up to conjugacy for irreducible periodic diffeomorphisms of a surface $\Sigma_g$ which commute with involutions $\iota$ such that $\Sigma_g/\langle \iota \rangle \cong T^2$. More generally, we classify finite subgroups of centralizers of irreducible periodic diffeomorphism.

1. Introduction

An involution on an oriented closed surface $\Sigma_g$ of genus $g > 1$ is called hyperelliptic if it fixes $2g + 2$ points. It is well known that such involution is unique up to conjugacy, and has the maximum number of fixed points among involutions on $\Sigma_g$. A diffeomorphism of $\Sigma_g$ which commutes with a hyperelliptic involution is called hyperelliptic. Ishizaka [Is04b, Is07] classified hyperelliptic periodic diffeomorphisms up to conjugacy and gave the right-handed Dehn twist presentations of their mapping classes based on resolutions of singularities of families of Riemann surfaces.

An involution $\iota$ of $\Sigma_g$ is hyperelliptic if and only if the quotient space $\Sigma_g/\langle \iota \rangle$ is homeomorphic to a sphere. In this paper, we will classify periodic diffeomorphisms of $\Sigma_g$ which commute with an involution $\iota_g$ whose quotient space $\Sigma_g/\langle \iota_g \rangle$ is homeomorphic to a torus. In this case, there is a significant difference in the classification from the hyperelliptic case considered in [Is07]. Recall that a diffeomorphism of $\Sigma_g$ is called reducible if there exists a family of isotopy classes of essential simple closed curves on $\Sigma_g$ preserved by the diffeomorphism and irreducible if otherwise. By the classification in [Is07], every hyperelliptic periodic diffeomorphism of $\Sigma_g$ is a power of an irreducible periodic diffeomorphism. In turn, as we will see below, this is not the case for periodic diffeomorphisms which commute with an involution $\iota_g$ such that $\Sigma_g/\langle \iota_g \rangle \cong T^2$. (see Example 1.1). In this article, we focus on the classification in the irreducible case. Irreducible periodic diffeomorphisms of $\Sigma_g$ have been studied from various viewpoints (see e.g. [Hi10b, HK16, GR00, DNR20]).

To state the result, let us present an example of an irreducible periodic diffeomorphism $h_{n,p}$ of an oriented closed surface parametrized by a positive integer $n \geq 3$ and $p \in \{1, \ldots, n-1\}$. Consider a regular $2n$-gon. Let $\alpha_i$ and $\beta_j$ ($i = 0, \ldots, n-1$) be the edges of the $2n$-gon shown in Figure 1. By identifying $\alpha_i$ with $\beta_j$ for every pair of $i$ and $j \in \{0, \ldots, n-1\}$ with $i - j \equiv p \mod n$, we obtain a surface $\Sigma$ of genus $g = \frac{n - \gcd(n,p) - \gcd(n,p+1)}{2} + 1$. We denote $h_{n,p}$ the diffeomorphism of $\Sigma$ induced from the clockwise $\frac{2\pi}{n}$-rotation of the $2n$-gon. To simplify the figures, both $\alpha_0$ and $\beta_p$ will be denoted by $\alpha_p$ in the rest of the paper.

Example 1.1. We can construct an example of a reducible periodic diffeomorphism which commutes with an involution $\iota_{2g+1}$ from $h_{4g,2g}$ and $h_{4g+1,2g}$ by a connected sum.
Figure 1. A diffeomorphism $h_{n,p}$

Figure 2. Connected sum of some irreducible periodic diffeomorphisms (Figure 2): Remove four small disks centered at the fixed points of $h_{4g,2g}$ and $h_{4g-1,2g}$, respectively, and glue the complements of the disks along the circle boundaries so that $h_{4g,2g}$ and $h_{4g-1,2g}$ coincide with each other on the boundary. Then we get a periodic diffeomorphism $f$ on $(\Sigma_g - disks) \cup_\partial (\Sigma_g - disks)$ of period $4g$. Note that $f^{2g}$ is an involution whose quotient space is a torus which commutes with $f$.

Let us state the main result of this article. In this result, we classify irreducible periodic diffeomorphisms which commutes with $\iota_g$.

Theorem 1.2. Let $\iota_g$ be an involution of $\Sigma_g$ such that $\Sigma_g/\langle \iota_g \rangle$ is homeomorphic to a torus as shown in Figure 3. For any periodic diffeomorphism $f$ of $\Sigma_g$ which commutes with an involution $\iota_g$, if the subgroup $G = \langle f, \iota_g \rangle$ of $\text{Diff}^+(\Sigma_g)$ is irreducible, then $G$ is conjugate to a subgroup of one of following:

(i) $\langle h_{6,1}, I \rangle$ (Figure 4),
(ii) $\langle h_{8,1} \rangle$ (Figure 2),
(iii) $\langle h_{8,5} \rangle$ (Figure 5),
(iv) $\langle h_{12,2} \rangle$ (Figure 6),
(v) $\langle h_{12,3} \rangle$ (Figure 7).

It is remarkable that there are only finite cases in the classification of Theorem 1.2. It is in contrast to the case of the classification of hyperelliptic periodic classes, where we have three infinite families of conjugacy classes.

Figure 3. The involution $\iota_g$

For hyperelliptic involution $I$ and hyperelliptic periodic diffeomorphism $f$, the group $G = \langle f, I \rangle$ is cyclic except the case where $f$ is a power of $h_{2g+2,1}$ (see [1N20, Is07]). Motivated by this fact, we give a classification of the centralizer of irreducible diffeomorphisms as follows.
Theorem 1.3. Let $f$ be an irreducible periodic diffeomorphism of $\Sigma_g$ with $g > 1$, and $H$ a finite subgroup of the centralizer of $f$ in $\text{Diff}_+(\Sigma_g)$. Then one of the following occurs:

(i) $H$ is a subgroup of $\langle f \rangle$.

(ii) $H$ is conjugate to a subgroup of $\langle h_{4g+2, 2g+1} \rangle$ so that $\langle f \rangle$ is conjugate to $\langle h_{2g+1, 1} \rangle$.

(iii) $H$ is conjugate to a subgroup of $\langle h_{2g+2, 1} \rangle$ so that $\langle f \rangle$ is conjugate to $\langle h_{2g+2, 1} \rangle$.

As a corollary, we obtain the following characterization of $h_{n,1}$.

Corollary 1.4. If an irreducible periodic diffeomorphism $f$ on $\Sigma_g$ commutes with two involutions $I_1$ and $I_2$ which are not mutually conjugate, then $f$ is conjugate to a power of either $h_{2g+1, 1}$ or $h_{2g+2, 1}$.

2. Periodic Diffeomorphisms and Their Total Valencies

We first recall the classification of the conjugacy classes of periodic diffeomorphisms on $\Sigma_g$ in terms of the total valency introduced in [AI02]. Let $f \in \text{Diff}_+(\Sigma_g)$ be a periodic diffeomorphism of order $n$ and $C$ the cyclic subgroup of $\text{Diff}_+(\Sigma_g)$ generated by $f$. An $C$-orbit $Cx$ is called multiple if $|Cx| < |C|$, where $|\cdot|$ denote the cardinality. For $x \in \Sigma_g$, the minimum positive integer $\ell$ such that $f^\ell(x) = x$ is called the period of $Cx$. Let $\lambda = |C_x|$, where $C_x$ is the isotropy group of $x$. Note that $\frac{\lambda}{\mu}$ is equal to the period of $Cx$. Then, there uniquely exists $\mu \in \{1, 2, \ldots, \lambda - 1\}$ such that the restriction of $f$ to a small neighborhood of $x$ is the clockwise $\frac{2\pi \mu}{\lambda}$-rotation. Since $\mu$ and $\lambda$ are coprime, there exists an integer $\theta \in \{1, 2, \ldots, \lambda - 1\}$ such that $\mu \theta \equiv 1 \pmod{\lambda}$. The valency of $Cx$ is defined by $\frac{\theta}{\lambda}$. Let $\frac{\theta}{\lambda}, \frac{2\theta}{\lambda}, \ldots, \frac{\lambda\theta}{\lambda}$.
be the valencies of all multiple orbits of $f$. The data $[g, n; \theta_1 \lambda_1 + \theta_2 \lambda_2 + \cdots + \theta_s \lambda_s]$ is called the total valency of $f$.

Figure 10. A loop around a vertex $y$

By the following theorem, total valencies exactly determine periodic diffeomorphisms on $\Sigma_g$ up to conjugation.

**Theorem 2.1** (Nielsen [Ni37 Section 11], see also [AI02 Section 1.3], [Hi10a Theorem 2.1]). Let $f, f'$ be periodic diffeomorphisms of $\Sigma_g$ with total valencies $[g, n; \theta_1 \lambda_1 + \theta_2 \lambda_2 + \cdots + \theta_s \lambda_s]$ and $[g, n'; \theta'_1 \lambda'_1 + \theta'_2 \lambda'_2 + \cdots + \theta'_s \lambda'_s]$, respectively. $f$ is conjugate to $f'$ if and only if the following are satisfied:

(i) $s = s'$,
(ii) $n = n'$,
(iii) after changing indices, we have $\theta_i \lambda_i = \theta'_i \lambda'_i$ for $i = 1, 2, \ldots, s$.

By Theorem 2.1, we can identify the total valency of a periodic diffeomorphism with its conjugacy class. We will use the following well-known observations of Nielsen.

**Proposition 2.2** (Nielsen [Ni37 Equation(4.6)]). The sum of valencies of all multiple orbits of a periodic diffeomorphism is an integer.

**Remark 2.3.** Any multiple orbit of an involution is a fixed point whose valency is $\frac{1}{2}$. Thus, by Proposition 2.2, the number of fixed points of an involution is even.

In general, the composite of two periodic diffeomorphisms may not be periodic, and hence, the product of total valency does not make sense. But one can define powers of total valency of a periodic diffeomorphism $f$ by the total valency of powers $f$.

Let us recall the well-known Riemann-Hurwitz formula for the convenience of the reader.

**Proposition 2.4** (Riemann-Hurwitz formula). Consider the $n$-fold branched covering from $\Sigma_g$ to $\Sigma_{g'}$ with branched indices $\lambda_1, \ldots, \lambda_s$. Then we have the formula

$$2g - 2 = n \left( 2g' - 2 + \sum_{1 \leq i \leq s} \left( 1 - \frac{1}{\lambda_i} \right) \right).$$
We will use the following result due to Harvey, which gives a necessary condition for branched indices.

**Proposition 2.5 (Harvey [Hat66, Theorem 4]).** Assume $g > 1$. Set $M = \text{lcm}(\lambda_1, \lambda_2, \ldots, \lambda_s)$. If there is cyclic covering from $\Sigma_g$ to $\Sigma_{g'}$ with branched indices $\lambda_1, \ldots, \lambda_s$, then the following conditions are satisfied:

(i) $\text{lcm}(\lambda_1, \lambda_2, \ldots, \hat{\lambda_i}, \ldots, \lambda_s) = M$ for all $i \in \{1, 2, \ldots, s\}$, where $\hat{\lambda_i}$ denotes the omission of $\lambda_i$.

(ii) $M$ divides $n$, and if $g' = 0$, then $M = n$.

(iii) $s \neq 1$, and, if $g' = 0$, then $s \geq 3$.

We will use the following characterization of irreducibility of finite groups actions due to Gilman.

**Theorem 2.6 (Gilman [Gil83, Lemma 3.9]).** Let $G$ be a finite subgroup of $\text{Diff}_+(\Sigma_g)$. Then $G$ is irreducible if and only if $(\Sigma_g \setminus M_G)/G$ is homeomorphic to $S^2 \setminus \{3\text{-points}\}$.

### 3. Preliminaries

In this section, we will classify irreducible periodic diffeomorphisms which commutes with an involution $\iota_q$ such that $\Sigma_g / \iota_q$ is a torus to prove Theorem 1.2. Firstly, let us compute the total valency of $h_{n,p}$ defined in the introduction.

**Proposition 3.1.** The total valency of $h_{n,p}$ is $[g, n; \frac{1}{n} + \frac{p}{n} + \frac{n-p-1}{n}]$, where $g = \frac{n - \text{gcd}(n, p) - \text{gcd}(n, p + 1) + 1}{2}$.

**Proof.** The genus $g$ of the surface can be computed by counting the edges of the $2n$-gon $P$ (see Figure 1) as

$$g = \frac{n - \text{gcd}(n, p) - \text{gcd}(n, p + 1) + 1}{2}.$$ 

Clearly $h_{n,p}$ has 3 multiple orbits: The barycenter of $P$, the vertices of $P$ and the middle point of the edges of $P$. Thus, the total valency of $h_{n,p}$ is of the form $[g, n; \ell_1, \ell_2, \ell_3]$, where $\ell_1, \ell_2, \ell_3$ are coprime integers such that $\ell_i$ divides $n$ for $i = 1, 2, 3$.

It is clear that the valency of the fixed point at the barycenter of the $2n$-gon is $\frac{1}{n}$. Hence, we can set $\frac{\ell_1}{\lambda_1} = \frac{1}{n}$.

Let us compute $\frac{\ell_2}{\lambda_2}$ the valency of the multiple orbit which is consisted by the body-centered points of the barycenter of Figure 1.

Let $k = \text{gcd}(n, p)$. Let $n'$ and $p'$ be the integer such that $n = kn' + p = kp'$, respectively. The period of the multiple orbit is $k$, and hence, $\lambda_2 = n'$. Take a point $y$ in the orbit, and consider a loop which goes around the boundary of a small disk centered at $y$ as shown in Figure 1. The fundamental domains appear along this loop in the order $0, n, 2p, \ldots, (n' - 2)p, (n' - 1)p$ modulo $n$. Then, there uniquely exists $\nu \in \{1, 2, \ldots, n'\}$ such that $\nu p \equiv k \mod n$. Hence, we have $\nu p' \equiv 1 \mod n'$. By the definition of a valency, $\theta_2 = p'$, and hence, $\frac{\ell_2}{\lambda_2} = \frac{\ell_2}{\lambda_2} = \frac{n'}{n}$.

By Proposition 2.2, the valency $\frac{\ell_3}{\lambda_3}$ of another multiple orbit holds the condition

$$\frac{1}{n} + \frac{p}{n} + \frac{\theta_3}{\lambda_3} \in \mathbb{Z}.$$ 

Thus, we have $\frac{\ell_3}{\lambda_3} = \frac{n-p-1}{n}$.

Applying the Riemann-Hurwitz formula (Proposition 2.4), we can obtain the following well-known classification of periodic diffeomorphisms on tori.
Proposition 3.2. Let $f$ be a nontrivial orientation preserving periodic diffeomorphism on a torus $T$. Then $T/\langle f \rangle$ is homeomorphic to either $T^2$ or $S^2$.

If $T/\langle f \rangle \cong T^2$, then $f$ is multiple orbit free. If $T/\langle f \rangle \cong S^2$, then the total valency of $f$ is one of the following:

- $h_{2,1}^2 = h_{0,3}^3 = \left[1, 2; \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right]$,
- $h_{4,1} = \left[1, 4; \frac{1}{4} + \frac{1}{4} + \frac{1}{2}\right]$,
- $h_{6,3} = \left[1, 6; \frac{1}{6} + \frac{1}{3} + \frac{1}{2}\right]$,
- $h_{8,3} = \left[1, 8; \frac{1}{8} + \frac{1}{4} + \frac{1}{2}\right]$,
- $h_{3,1}^3 = h_{2,6}^3 = \left[1, 3; \frac{1}{3} + \frac{1}{3} + \frac{1}{2}\right]$,
- $h_{5,3}^3 = h_{2,6}^3 = \left[1, 5; \frac{2}{5} + \frac{1}{3} + \frac{1}{2}\right]$.

It is easy to see that we can identify $h_{3,1}$ with $h_{2,6}^3$ by deforming the 12-gon to define $h_{6,3}$ to a hexagon (see Figures 12, see also Lemma 5.3(1)).
Since any multiple point free periodic diffeomorphism on a torus is reducible, we get the consequence.

**Proposition 3.3.** Every irreducible periodic diffeomorphism on a torus is conjugate to one of the following:

(i) $h_{4,1}$ or its inverse,
(ii) $h_{6,3}$ or its inverse,
(iii) $h_{3,1}$ or its inverse.

Here $h_{4,1}$, $h_{6,3}$ and $h_{3,1}$ are shown in Figures 14, 15 and 13, respectively.

*Figure 14. $h_{4,1}$  Figure 15. $h_{6,3}$*

4. **Cyclic branched coverings of tori**

Take an involution $\iota_g$ of $\Sigma_g$ whose quotient space is homeomorphic to a torus. Let $f$ be a periodic diffeomorphism of $\Sigma_g$ of order $n$ which commutes with $\iota_g$. Let $\bar{f}$ be a periodic diffeomorphism of $\Sigma_g/\langle \iota_g \rangle$ induced by $f$. Let $n = \text{ord}(f)$ and $n = \text{ord}(\bar{f})$. We have the following simple observation.

**Lemma 4.1.**

(i) We have $n = 2\bar{n}$ if and only if $f^n = \iota_g$.

(ii) We have $n = \bar{n}$ if and only if $f^n \neq \iota_g$.

*Proof.* We have $f^n = \text{id}_{\Sigma_g/\langle \iota_g \rangle}$. Since the lifts of $\text{id}_{\Sigma_g/\langle \iota_g \rangle}$ to $\Sigma_g$ are $\iota_g$ and $\text{id}_{\Sigma_g}$, we have either $f^n = \iota_g$ or $f^n = \text{id}_{\Sigma_g}$. The latter is equivalent to $n = \bar{n}$, and the former implies $n = 2\bar{n}$. The proof is concluded. □

We will use the following well known result.

**Lemma 4.2.** There exists no cyclic branched covering $p : S^2 \to S^2$ which has more than two branched points $x_1, x_2, \ldots, x_s \in S^2$ such that $p^{-1}(x_i)$ is a one point set for $i = 1, 2, 3$. In particular, there exists no nontrivial periodic diffeomorphism of an orbifold homeomorphic to $S^2$ which fixes more than two points.

*Proof.* Assume that there exists such an $n$-fold branched covering $p : S^2 \to S^2$. By Riemann-Hurwitz formula (Proposition 2.4), we have

$$-2 = n \left( -2 + 3 \left( 1 - \frac{1}{n} \right) + \sum_{i=4}^{s} \left( 1 - \frac{1}{\lambda_i} \right) \right),$$

where $\lambda_i$ is the ramification index of $x_i$ for $i = 4, \ldots, s$. Then we have

$$1 - n = \sum_{i=4}^{s} \left( 1 - \frac{1}{\lambda_i} \right) \geq 0,$$
which implies \( n \leq 1 \), and hence a contradiction. \qed

Let \( G \) be the group generated by \( f \) and \( t_g \). In this section, we will classify the \( G \)-action up to conjugacy to prove Theorem 1.2.

The following simple observation is fundamental.

**Lemma 4.3.** (i) We have \( f(\text{Fix}(t_g)) = \text{Fix}(t_g) \), where \( \text{Fix}(t_g) \) is the fixed point set of \( t_g \).

(ii) The involution \( t_g \) maps every \( f \)-orbit to an \( f \)-orbit of the same valency.

**Proof.** It is easy to see that the valency of an \( f \)-orbit of \( x \) is determined by the isotropy action of \( f^k \), where \( k \) is the period of \( x \). The lemma follows from the conjugation invariance of the isotropy action. \qed

It is easy to determine the conjugacy class of the \( G \)-action in the case where \( G \) is cyclic. Let \( \bar{f} \) be the diffeomorphism on \( \Sigma_g/\langle t_g \rangle \) induced by \( f \).

Let \( p \) denote the double covering \( \Sigma_g \to \Sigma_g/\langle t_g \rangle \). In the sequel, we will use the following criterion.

**Lemma 4.4.** If \( p \) is ramified at a multiple point \( x \) of \( f \) of period \( k \) such that \( \frac{n}{k} \) is even, then \( G \) is cyclic.

**Proof.** By assumption, the isotropy group \( G_x \) of \( x \) is generated by \( f^k \) and \( t_g \). Since \( \frac{n}{k} \) is even, the subgroup of \( G_x \) generated by \( f^k \) is of even order. Thus, it contains \( f^{\frac{n}{k}} \), which is a \( \pi \)-rotation around \( x \). Therefore, it follows that \( f^{\frac{n}{k}} = t_g \), which implies that \( G \) is cyclic. \qed

It is easy to see that if \( \bar{f} \) is reducible then \( G \) is reducible: Indeed, if \( \bar{f} \) preserves a system of isotopy classes of essential simple closed curves, then \( G \) preserves its inverse image of \( p \). Hence, to classify irreducible group \( G \), it is sufficient to consider the case where \( \bar{f} \) is irreducible.

![Figure 16. A simple closed curve \( C \)](image-url)

**Lemma 4.5.** Assume that \( \bar{f} \) is conjugate to a power of \( h_{n,p} \). If the canonical projection \( p : \Sigma_g \to \Sigma_g/\langle t_g \rangle \) is branched at a free orbit of \( f \), then \( G \) is reducible.

**Proof.** Assume that \( p \) is branched at a simple point \( y \) of \( \bar{f} \). Take an \( \bar{f} \)-invariant simple closed curve \( C \) of \( \Sigma_g/\langle t_g \rangle \) so that it bounds a disk which contains the \( \bar{f} \)-orbit of \( y \) as shown in Figure 16. Then the preimage \( p^{-1}(C) \) is essential and preserved by \( f \) and \( t_g \), hence \( G \) is reducible. \qed
Now, we will determine irreducible \( f \) that induces \( h_{4,1} \) on the torus \( \Sigma_g/\langle \iota_g \rangle \).

Since \( \iota_g \) has 2\(g - 2 \) fixed points, \( p \) is branched at 2\(g - 2 \) points. By Lemma 4.3, \( f \) maps the branched locus of \( p \) to itself. By classifying nonempty \( f \)-invariant sets with an even number of points in the union of the multiple orbits of \( f \), we obtain the following lemma.

**Lemma 4.6.** If \( f = h_{4,1} \), then the branched locus \( B_p \) of \( p \) is one of the following:

(i) \( \{x, y\} \),
(ii) \( \{z_1, z_2\} \),
(iii) \( \{x, y, z_1, z_2\} \),

where \( x, y, z_1, z_2 \in \Sigma_g/\langle \iota_g \rangle \) are as shown in Figure 14.

**Proposition 4.7.** If \( f = h_{4,1} \), then \( G \) is conjugate to either of \( \langle h_{8,1} \rangle \) or \( \langle h_{8,5} \rangle \).

**Proof.** We can see that \( G \) is cyclic in all three cases in Lemma 4.6. Indeed, since \( \text{ord}(f) = 4 \), and the period of branch points is either 1 or 2, we can see that the cardinality of the isotropy group of every branch point is even. Then, by Lemma 4.4, \( G \) is cyclic. By Lemma 4.1, we have that \( n = 8 \).

Let us consider the case (i) in Lemma 4.6. We can see that \( p \) has three branched points of index 8, 8 and 2, respectively. Then the total valency of \( f \) is of the form \( [2, 8; \frac{a_1}{4} + \frac{a_3}{8} + \frac{a_1}{2}] \), where \( \theta_1, \theta_2, \theta_3 \in \{1, 3, 5, 7\} \). Here \( \frac{a_1}{4} + \frac{a_3}{8} + \frac{a_1}{2} \) is an integer by Nielsen’s condition (Proposition 2.2), and hence, \( (\theta_1, \theta_2) = (1, 3) \) or \( (5, 7) \). In both cases, since \( f^4 \) is a hyperelliptic involution, it contradicts Lemma 4.1. Thus, this case does not occur.

Finally, let us consider the case (iii) of Lemma 4.6. We can see that \( p \) has three branched points of index 8, 8 and 4, respectively. Let \( [2, 8; \frac{a_1}{4} + \frac{a_3}{8} + \frac{a_1}{2}] \) be the total valency of \( g \), where \( \theta_1, \theta_2 \in \{1, 3, 5, 7\} \) and \( \theta_3 \in \{1, 3\} \). Now \( \frac{a_1}{8} + \frac{a_3}{8} + \frac{a_1}{4} \) is an integer by Nielsen’s condition (Proposition 2.2), and hence, \( (\theta_1, \theta_2, \theta_3) = (k, k, 3k) \) or \( (k, 5k, k) \) mod 8 for \( k \in \{1, 3, 5, 7\} \). Thus, \( f \) is a power of either of \( h_{8,1} = [3, 8; \frac{3}{4} + \frac{3}{8} + \frac{3}{4}] \) or \( h_{8,5} = [3, 8; \frac{3}{5} + \frac{3}{8} + \frac{3}{4}] \).

Let us consider the case where \( f = h_{6,3} \). We obtain the following lemma by an argument similar to the case where \( f = h_{4,1} \).

**Lemma 4.8.** If \( f = h_{6,3} \), then branched locus \( B_p \) of \( p \) is one of the following:

(i) \( \{y_1, y_2\} \),
(ii) \( \{x, z_1, z_2, z_3\} \),
(iii) \( \{x, y_1, y_2, z_1, z_2, z_3\} \),

where \( x, y_1, y_2, z_1, z_2, z_3 \in \Sigma/(\iota_g) \) are as shown in Figure 15.

**Proposition 4.9.** If \( f = h_{6,3} \), then \( G \) is conjugate to a subgroup of either of \( \langle h_{6,1}, I \rangle \), \( \langle h_{1,2,2} \rangle \), or \( \langle h_{1,2,3} \rangle \).

**Proof.** Let us consider the case (iii) of Lemma 4.8. Since \( p \) is branched at a fixed point \( x \) of \( f \) and the order of \( f \) is even, by Lemma 4.4, \( G \) is cyclic. Then, by Lemma 4.1, we have \( n = 12 \) and the branch indices of \( f \) are 12, 4 and 3. Let \( [3, 12; \frac{a_1}{12} + \frac{a_3}{12} + \frac{a_1}{3}] \) be the total valency of \( f \), where \( \theta_1 \in \{1, 5, 7, 11\} \), \( \theta_2 \in \{1, 3\} \) and \( \theta_3 \in \{1, 2\} \). By Nielsen’s condition (Proposition 2.2), \( \frac{a_1}{12} + \frac{a_3}{12} + \frac{a_1}{3} \) is an integer, and hence, \( (\theta_1, \theta_2, \theta_3) = (k, k, 2k) \) mod 12 for \( k \in \{1, 5, 7, 11\} \). Thus, \( f \) is a power of \( h_{12,3} = [3, 12; \frac{3}{12} + \frac{3}{12} + \frac{3}{4}] \) shown in Figure 8. Note that \( h_{12,3} \) is equal to \( \iota_3 \)
Let us consider the case (iii) of Lemma 4.8. Since $p$ is branched at a fixed point $x$ of $\bar{f}$ and the order of $\bar{f}$ is even, by Lemma 4.4 $G$ is cyclic. Then, by Lemma 4.1, we have $n = 12$, and the branch indices of $\bar{f}$ are 12, 6 and 4. Let $[3, 12; \frac{1}{12} + \frac{1}{6} + \frac{1}{4}]$ be the total valency of $\bar{f}$, where $\theta_1 \in \{1, 5, 7, 11\}$, $\theta_2 \in \{1, 5\}$ and $\theta_3 \in \{1, 3\}$. By Nielsen’s condition (Proposition 2.2), $\theta_1 + \theta_2 + \theta_3$ is an integer, and hence, $(\theta_1, \theta_2, \theta_3) = (k, k, 3k)$ mod 12 for $k \in \{1, 5, 7, 11\}$. Thus, $f$ is a power of $h_{12,2} = [3, 12; \frac{1}{12} + \frac{1}{6} + \frac{1}{4}]$. Note that $h_{12,2}^5$ is equal to $\mu_4$.

Finally, let us consider the case (i) in Lemma 4.8. Since $p$ is branched at 2 points, we have $g = 2$. Since $\bar{f}$ fixes $x$, it follows that $p^{-1}(x)$ is either a multiple orbit of period two, or $f$ fixes $p^{-1}(x)$ pointwisely. In the former case, $\bar{f}$ or $f$ is a lift of $g$ which fixes $p^{-1}(x)$ pointwisely. Thus, to classify the conjugacy class of $G$, we can assume that $f$ fixes $p^{-1}(x)$ pointwisely without loss of generality. In this case, we can see that $n = 6$. Since the valency of $\{y_1, y_2\}$ as an $\bar{f}$-orbit is $\frac{1}{2}$, by a local argument, we can see that the valency of $p^{-1}(\{y_1, y_2\})$ is either $\frac{2}{3}$ or $\frac{1}{6}$. But $n = 6$ yields that only the former case occurs. Similarly, $p^{-1}(\{z_1, z_2, z_3\})$ is either a free orbit or the union of two multiple orbits of valency $\frac{1}{2}$. The Riemann-Hurwitz formula (Proposition 2.4) yields that only the former case occurs here. Indeed, if we substitute $g = 2$ and $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (6, 6, 3, 2, 2)$ to the formula, then we get $g' = -\frac{1}{2}$, which contradicts to the fact that $g'$ is the genus of $\Sigma_2/(f)$. Thus, it follows that the total valency of $f$ is $[2, 6; \frac{2}{3} + \frac{1}{6} + \frac{1}{6}]$. Thus $f$ is conjugate to $h_{6,1}$. Now $f^3 \circ h_2$ is a hyperelliptic involution which fixes all points in $p^{-1}(\{z_1, z_2, z_3\})$. By [1](N) Lemma 13, $G$ is conjugate to $(h_{6,1}, f)$, where $I$ is a hyperelliptic involution on $\Sigma_2$.

Finally, let us consider the case where $\bar{f} = h_{3,1}$. The multiple orbits of $h_{3,1}$ are three fixed points. A nonempty $\bar{f}$-invariant subset with an even number of points in the set of these three fixed points is a two point set. All of them are mutually conjugate, and hence we can assume that $p$ is branched at $y$ and $z$ in Figure 13. Thus, the branch locus of $p$ is the same as the case the case (i) in Lemma 4.8. By a parallel argument, we can see that $G$ is conjugate to a subgroup of $(h_{6,1}, f)$.

Theorem 1.2 follows from Propositions 4.7, 4.9 and the argument in the last paragraph.

5. Finite subgroups of the centralizer of irreducible periodic diffeomorphisms on surfaces

In this chapter, we will classify up to conjugacy finite subgroups of the centralizer of irreducible periodic diffeomorphisms to prove Theorem 1.3.

Let $f$ be an irreducible periodic diffeomorphism of $\Sigma_p$ of period $n$. By Theorem 2.6, the total valency of $f$ is of the form $\lfloor g, n; \frac{\theta_1}{n} + \frac{\theta_2}{n} + \frac{\theta_3}{n} \rfloor$ for some $\theta_1, \theta_2, \theta_3$ such that $1 \leq \theta_i \leq \frac{n}{2}$ for $i = 1, 2, 3$. Let $F$ be the subgroup of $\text{Diff}_+(\Sigma_p)$ generated by $f$, and let $H$ be a finite subgroup in the centralizer of $f$ in $\text{Diff}_+(\Sigma_p)$.

**Lemma 5.1.** If $\theta_1, \theta_2$ and $\theta_3$ are mutually different, then $H$ is a subgroup of $F$.

**Proof.** Take an arbitrary $d \in H$. The diffeomorphism $d$ maps a multiple orbit of $f$ to another multiple orbit of the same valency by Lemma 4.3. Therefore, by assumption, $d$ maps each multiple orbit of $f$ to itself. Then $d$ induces $d$ on $\Sigma/F \cong S^2$, which fixes 3 cone points. By Lemma 4.2, it follows that $d = 1d_{S^2}/F$. Since any lift of $1d_{S^2}/F$ is of the form $f^k$ for some integer $k$, we have $d = f^k$. Therefore $H$ is a subgroup of $F$.

Now let us consider the case where $\theta_1 = \theta_2$. We observe the following.

**Lemma 5.2.** If $\theta_1 = \theta_2 = \theta_3$, then $f$ is conjugate to $h_{3,1}$ or $h_{3,1}^2$. In particular, we have $g = 1$. 
Lemma 5.4. \( \frac{\theta n}{\theta} \) is an integer by Nielsen’s condition (Proposition 2.2). Since \( 0 < \frac{\theta n}{\theta} < 1 \), we have \( \frac{\theta n}{\theta} = 1 \) or \( 2 \). In the case where \( \frac{\theta n}{\theta} = 1 \), we have \((n, \theta) = (3, 1)\). Otherwise we have \( \frac{\theta n}{\theta} = 2 \), and hence \((n, \theta) = (3, 2)\). In both cases, by the Riemann-Hurwitz formula (Proposition 2.4), we have \( g = 1 \). Thus the total valency of \( f \) is equal to either \( \lfloor 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \rfloor \) or \( \lfloor 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \rfloor \). The former is the total valency of \( h_{3,1} \), and the latter is the total valency of \( h_{3,1}^2 \). Thus, \( f \) is conjugate to \( h_{3,1} \) or \( h_{3,1}^2 \) by Theorem 2.1. □

Thus, in order to prove Theorem 1.3, it is sufficient to consider the case where \( \theta_1 = \theta_2 \neq \theta_3 \).

By the first condition of Proposition 2.5, \( \gcd(\theta_1, n) = 1 \), and hence, \( f \) is conjugate to the \( \theta_1 \)-th power of \( h_{n,1} \). Clearly \( H \) is conjugate to a finite subgroup of the centralizer of \( h_{n,1} \). Thus, we can assume that \( f = h_{n,1} \) and \( H \) is a finite subgroup of the centralizer of \( h_{n,1} \) in the following. Since \( g = \frac{1}{2} - \gcd(n,p) - \gcd(n,n-p-1) + 1 \), we can compute the period \( n \) in terms of the genus \( g \) as follows:

\[
  n = \begin{cases} 
    2g + 1 & n : \text{odd}, \\
    2g + 2 & n : \text{even}.
  \end{cases}
\]

Since \([h_{n,1}] = [g, n; \frac{1}{n}, \frac{1}{n} + \frac{n-2}{n}],\) the diffeomorphism \( h_{n,1} \) have two fixed points of valency \( \frac{1}{n} \) and a multiple orbit of valency \( \frac{n-2}{n} \), which is a fixed point if \( n \) is odd and has period \( 2 \) if \( n \) is even. We have, for every \( n \), a (hyperelliptic) involution which commutes with \( h_{n,1} \) and exchanges two fixed points of valency \( \frac{1}{n} \) as follows:

Lemma 5.3.

(i) \( h_{4g+2,2g+1}^2 \) is conjugate to \( h_{2g+1,1} \), and \( h_{4g+2,2g+1}^{2g+1} \) is a hyperelliptic involution which commutes with \( h_{4g+2,2g+1}^2 \) and exchanges two fixed points of \( h_{4g+2,2g+1}^2 \) of valency \( \frac{1}{2g+1} \).

(ii) \( I \) shown in Figure 7 is a hyperelliptic involution which commutes with \( h_{2g+2,1} \) and exchanges two fixed points of \( h_{2g+2,1} \) of valency \( \frac{1}{2g+1} \).

Proof. By Proposition 3.1, the total valency of \( h_{4g+2,2g+1} \) is equal to \( \lfloor g, 4g + 2; \frac{1}{2g+1} + \frac{g}{2g+1} + \frac{1}{2} \rfloor \). Since \( \gcd(4g + 2, 2g) = 2 \), the period of \( h_{4g+2,2g+1} \) is \( 2g + 1 \). Figure 17 is a decomposition of \( \Sigma_g \) into the fundamental domain of \( h_{4g+2,2g+1} \).

The multiple orbit of \( h_{4g+2,2g+1} \) with valency \( \frac{1}{2} \) consists of \( 2g + 1 \) points, and it is a free orbit of \( h_{4g+2,2g+1}^2 \).

By construction, \( h_{4g+2,2g+1}^2 \) is induced from the clockwise \( \frac{2\pi}{2g+1} \)-rotation the \( 8g + 4 \)-gon. Thus, \( h_{4g+2,2g+1}^2 \) is induced from \( \frac{2\pi}{2g+1} \)-rotation (See Figure 18). Let \( x \) be the barycenter of Figure 17. Since we have

\[
g(2g + 1) \equiv -2g \equiv 1 \mod 2g + 1,
\]

by the definition of valency, the valency of \( x \) is equal to \( \frac{2g+1}{2g+1} \).

The period of the other multiple orbit of \( h_{4g+2,2g+1} \) is \( 2 \). Thus \( h_{4g+2,2g+1}^{2g} \) fixes this orbit pointwisely. Let \( \frac{\theta}{2g+1} \) be its valency. Then, by Nielsen’s condition (Proposition 2.2), \( \frac{\theta}{2g+1} + \frac{\theta}{2g+1} + \frac{2g+1}{2g+1} \) is an integer, and hence, we have \( \theta = 1 \). Therefore, we obtain

\[
[h_{4g+2,2g+1}^{2g}] = \left[ g, 2g + 1; \frac{1}{2g+1} + \frac{1}{2g+1} + \frac{2g+1}{2g+1} \right] = [h_{2g+1,1}],
\]

which proves the proof by Theorem 2.1. □

Lemma 5.4.

(i) \( H \) is conjugate to a subgroup of \( \langle h_{4g+2,2g+1} \rangle \), where \( h_{2g+1,1} \) is identified with \( h_{4g+2,2g+1}^{2g} \).
Figure 17. Fundamental domains of $h_{4g+2}$

Figure 18. Fundamental domains of $h_{4g+2,2g+1}$

(ii) $H$ is conjugate to a subgroup of $\langle h_{2g+2,1}, I \rangle$.

Proof. By Lemma 5.3, for every $n$, we have an involution $I_n$ which commutes with $h_{n,1}$ and two fixed points of valency $\frac{1}{n}$. By Lemma 4.3, one of the following two cases occurs:

(A) $d$ maps each of three multiple orbits of $h_{n,1}$ to itself.

(B) $d$ exchanges two fixed points of $h_{n,1}$ with valency $\frac{1}{n}$, and maps the multiple orbit of $f$ with valency $\frac{n-2}{n}$ to itself.

It is easy to see that, by a similar argument to Lemma 5.1, we have $d \in F$ in the case (A). Thus, it is sufficient to consider the case where there exists $d \in H$ which satisfies the condition (B). Note that, for any $d' \in H$ which satisfies the condition (B), $d^{-1} \circ d'$ maps every three multiple orbits to itself. Then, Lemma 4.2 yields that $d^{-1} \circ d' = h_{n,1}^{k}$ for some $k$. Therefore $d' \in \langle h_{n,1}, d \rangle$.

Consider the map $\bar{d}$ induced on $\Sigma_g/F$ by $d$. The map $I_n$ also induces a diffeomorphism $\bar{I}_n$ on $\Sigma_g/F$. Since both $\bar{d}$ and $\bar{I}_n$ are involutions which fix the same cone point, we have an orbifold diffeomorphism $\kappa: (\Sigma_g/F)/\bar{d} \to (\Sigma_g/F)/\bar{I}_n$.

Then, by lifting $\kappa$ to a map $\Sigma_g/F \to \Sigma_g/F$, we have an orbifold diffeomorphism $\kappa_1$ which makes the following diagram commutative:

\[ \Sigma_g/F \xrightarrow{\kappa_1} \Sigma_g/F \]

\[ (\Sigma_g/F)/\bar{d} \xrightarrow{n} (\Sigma_g/F)/\bar{I}_n, \]

where the vertical maps are the canonical projections. Then, since $\kappa$ is an isomorphism of double branched coverings, we have $d \circ \kappa = \kappa \circ I_n$. Therefore $d$ is conjugate to $I_n$. Then we may assume that $d = I_n$. Since $d \circ I_n^{-1} = id_{\Sigma_g/F}$, we can see that $d \circ I_n \in F$. It follows that $d \in \langle h_{n,1}, I_n \rangle$. 

Theorem 1.3 follows from the last lemma and the preceding argument.

We can see an exceptional phenomenon in the case of $g = 1$. In this case, $f$ may be conjugate to $h_{3,1}$ or $h_{3,1}^2$ by Lemma 5.2. Since the centralizer of $f$ coincides up to conjugacy in these two cases, we can assume that $f = h_{3,1}$. As explained in the paragraph after Proposition 3.2 (see Figures 12, 13), we can identify $h_{3,1}$ with $h_{6,3}$. Therefore $h_{6,3}$ commutes with $h_{3,1}$. In addition to $h_{6,3}$, we have one more periodic
diffeomorphism $J$ which commutes with $h_{3,1}$ as shown in Figure 19. This $J$ and $h_{6,3}$ generates a finite group in the centralizer of $h_{3,1}$.

Figure 19. A periodic diffeomorphism $J$ which commutes with $h_{3,1}$

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