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A Note on Iterations-based Derivations of High-order Homogenization Correctors for Multiscale Semi-linear Elliptic Equations

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Abstract

This Note aims at presenting a simple and efficient procedure to derive the structure of high-order corrector estimates for the homogenization limit applied to a semi-linear elliptic equation posed in perforated domains. Our working technique relies on monotone iterations combined with formal two-scale homogenization asymptotics. It can be adapted to handle more complex scenarios including for instance nonlinearities posed at the boundary of perforations and the vectorial case, when the model equations are coupled only through the nonlinear production terms.

Keywords: Corrector estimates, Homogenization, Elliptic systems, Perforated domains

2010 MSC: 35B27, 35C20, 76M30, 35B09

1. Background

Modern approaches to modeling focus on multiple scales. Given a multiscale physical problem, one of the leading questions is to derive upscaled model equations and the corresponding structure of effective model coefficients (e.g. \cite{1, 2}). This Note aims at exploring the quality of the upscaling/homogenization procedure by deriving whenever possible corrector (error) estimates for the involved unknown functions, fields, etc. and their gradients (i.e. of the transport fluxes). Ultimately, these estimates contribute essentially to the control of the approximation error of numerical methods to multiscale PDE problems.

Our starting point is a microscopic PDE model describing the motion of populations of colloidal particles in soils and porous tissues with direct applications in drug-delivery design and control of the spread of radioactive pollutants (\cite{3, 4, 5}). We have previously analyzed a reduced version of this system in \cite{6}. Here, we point out a short alternative proof based on monotone iterations (\cite{7}) of the corrector estimates derived in \cite{6} and extend them to higher asymptotic orders.

2. Problem setting

We are concerned with the study of the semi-linear elliptic boundary value problem of the form

\begin{align}
A^\varepsilon u^\varepsilon &= R(u^\varepsilon), & x &\in \Omega^\varepsilon, \\
u^\varepsilon &= 0, & x &\in \Gamma^\text{ext}, \\
\nabla u^\varepsilon \cdot n &= 0, & x &\in \Gamma^\varepsilon,
\end{align}

\text{(2.1)}
where the operator \( \mathcal{A}^\varepsilon u^\varepsilon := \nabla \cdot (-d^\varepsilon \nabla u^\varepsilon) \) involves \( d^\varepsilon \) termed as the molecular diffusion while \( R \) represents the volume reaction rate. We take into account the following assumptions:

\((A_1)\) the diffusion coefficient \( d^\varepsilon \in L^\infty(\mathbb{R}^d) \) for \( d \in \mathbb{N} \) is \( Y \)-periodic and symmetric, and it guarantees the ellipticity of \( \mathcal{A}^\varepsilon \) as follows:

\[ d^\varepsilon \xi^i \xi^j \geq \alpha \vert \xi \vert^2 \quad \text{for any } \xi \in \mathbb{R}^d; \]

\((A_2)\) the reaction coefficient \( R \in L^\infty(\Omega^\varepsilon \times \mathbb{R}) \) is globally \( L \)-Lipschitzian, i.e. there exists \( L > 0 \) independent of \( \varepsilon \) such that

\[ \Vert R(u) - R(v) \Vert \leq L \Vert u - v \Vert \quad \text{for } u, v \in \mathbb{R}. \]

It is worth noting that the domain \( \Omega^\varepsilon \in \mathbb{R}^d \) considered here approximates a porous medium. The precise description of \( \Omega^\varepsilon \) is showed in [6, Section 2] and [8]. In Figure 2.1 (left), we sketch an admissible geometry of our medium, pointing out the sample microstructure in Figure 2.1 (right). We follow the notation from [6].

\[ \Omega = \partial \Omega \]

**Figure 2.1:** An admissible perforated domain (left) and basic geometry of the microstructure (right).

**Remark 1.** In this paper, we denote the space \( V^\varepsilon \) by

\[ V^\varepsilon := \{ v \in H^1(\Omega^\varepsilon) \vert v = 0 \text{ on } \Gamma^{ext} \} \]

endowed with the norm

\[ \| v \|_{V^\varepsilon} = \left( \int_{\Omega^\varepsilon} \vert \nabla v \vert^2 \, dx \right)^{1/2}. \]

This norm is equivalent (uniformly in the homogenization parameter \( \varepsilon \)) to the usual \( H^1 \) norm by the Poincaré inequality.

### 3. Derivation of corrector estimates

We introduce the following \( M \)th-order expansion (\( M \geq 2 \)):

\[ u^\varepsilon(x) = \sum_{m=0}^{M} \varepsilon^m u_m(x, y) + \mathcal{O}(\varepsilon^{M+1}), \quad x \in \Omega^\varepsilon, \quad (3.1) \]
where \( u_m(x, \cdot) \) is \( Y \)-periodic for \( 0 \leq m \leq M \).

Following standard homogenization procedures, we deduce the so called auxiliary problems (see e.g. [9]). To do so, we consider the functional \( \Phi(x, y) \) depending on two variables: the macroscopic \( x \) and \( y = x/\varepsilon \) the microscopic presentation, and denote by \( \Phi^\varepsilon(x) = \Phi(x, y) \). The simple chain rule allows us to derive the fact that

\[
\nabla \Phi^\varepsilon(x) = \nabla_x \Phi \left(x, \frac{x}{\varepsilon}\right) + \varepsilon^{-1} \nabla_y \Phi \left(x, \frac{x}{\varepsilon}\right). \tag{3.2}
\]

The quantities \( \nabla u^\varepsilon \) and \( \mathcal{A}^\varepsilon u^\varepsilon \) must be expended correspondingly. In fact, it follows from (3.2) and (3.1) that

\[
\nabla u^\varepsilon = (\nabla_x + \varepsilon^{-1} \nabla_y) \left( \sum_{m=0}^M \varepsilon^m u_m + \mathcal{O}(\varepsilon^{M+1}) \right) \tag{3.3}
\]

Using the structure of the operator \( \mathcal{A}^\varepsilon \), we obtain the following:

\[
\mathcal{A}^\varepsilon u^\varepsilon = \varepsilon^{-2} \nabla_y \cdot (-d(y) \nabla_y u_0) + \varepsilon^{-1} \left[ \nabla_x \cdot (-d(y) \nabla_y u_0) + \nabla_y \cdot (-d(y) \nabla_x u_0 + \nabla_y u_1) \right] \tag{3.4}
\]

Concerning the boundary condition at \( \Gamma^\varepsilon \), we note:

\[
d^\varepsilon \nabla u^\varepsilon \cdot n = d_i(y) \left( \varepsilon^{-1} \nabla_y u_0 + \sum_{m=0}^{M-1} \varepsilon^m (\nabla_x u_m + \nabla_y u_{m+1}) \right) \cdot n. \tag{3.5}
\]

To investigate the convergence analysis, we consider the following structural property:

\[
R \left( \sum_{m=0}^M \varepsilon^m u_m \right) = \sum_{m=0}^M \varepsilon^{m-r} R(u_m) + \mathcal{O}(\varepsilon^{M-r+1}) \quad \text{for } r \in \mathbb{Z}, r \leq 2. \tag{3.6}
\]

At this point we see, if \( r \in \{1, 2\} \) it then generate spontaneously nonlinear auxiliary problems. To see the impediment, let us focus on \( r = 2 \). By collecting the coefficients of the same powers of \( \varepsilon \) in (3.4) and (3.5), we are led to the following systems, which we refer to the auxiliary problems:

\[
\begin{cases}
\mathcal{A}_0 u_0 = R(u_0), & \text{in } Y_1, \\
-d(y) \nabla_y u_0 \cdot n = 0, & \text{on } \partial Y_0,
\end{cases}
\tag{3.7}
\]

\[
\begin{cases}
\mathcal{A}_0 u_1 = R(u_1) - \mathcal{A}_1 u_0, & \text{in } Y_1, \\
-d(y) (\nabla_x u_0 + \nabla_y u_1) \cdot n = 0, & \text{on } \partial Y_0,
\end{cases}
\tag{3.8}
\]

\[
\begin{cases}
\mathcal{A}_0 u_{m+2} = R(u_{m+2}) - \mathcal{A}_1 u_{m+1} - \mathcal{A}_2 u_m, & \text{in } Y_1, \\
-d(y) (\nabla_x u_{m+1} + \nabla_y u_{m+2}) \cdot n = 0, & \text{on } \partial Y_0,
\end{cases}
\tag{3.9}
\]

\[\text{for } u_{m+2} \text{ is } Y \text{-periodic in } y,\]

\[\text{for } 0 \leq m \leq M - 1.\]

\[\text{for } 0 \leq m \leq M - 1.\]
for $0 \leq m \leq M - 2$.

Here, we have denoted by
\[
\mathcal{A}_0 := \nabla_y \cdot (-d(y) \nabla_y), \\
\mathcal{A}_1 := \nabla_x \cdot (-d(y) \nabla_y) + \nabla_y \cdot (-d(y) \nabla_x), \\
\mathcal{A}_2 := \nabla_x \cdot (-d(y) \nabla_x).
\]

**Remark 2.** In the case $r \leq 0$, it is trivial to not only prove the well-posedness of these auxiliary problems \((3.7)-(3.9)\), but also to compute the solutions by many approaches due to its linearity. For details, the reader is referred here to [10].

The idea is now to linearize the auxiliary problems. Inspired by the fact that a fixed-point homogenization argument seems to be applicable in this framework, and also by the way a priori error estimates are proven for difference schemes, we suggest an iteration technique to "linearize" the involved PDE systems. We start the procedure by choosing the initial point $u^{(0)}_m = 0$ for $m \in \{0, ..., M\}$. As next step, we consider the following systems corresponding to the nonlinear auxiliary problems:

\[
\begin{cases}
\mathcal{A}_0 u^{(n_0)}_0 = R(u^{(n_0-1)}_0), & \text{in } Y_1, \\
-d(y) \nabla_y u^{(n_0)}_0 \cdot n = 0, & \text{on } \partial Y_0, \\
\end{cases}
\]

\[
\begin{cases}
\mathcal{A}_0 u^{(n_1)}_1 = R(u^{(n_1-1)}_1) - \mathcal{A}_1 u^{(n_0)}_0, & \text{in } Y_1, \\
-d(y) \left( \nabla_x u^{(n_0)}_0 + \nabla_y u^{(n_1)}_1 \right) \cdot n = 0, & \text{on } \partial Y_0, \\
\end{cases}
\]

\[
\begin{cases}
\mathcal{A}_0 u^{(n_{m+2})}_{m+2} = R(u^{(n_{m+1}+1)}_{m+2}) - \mathcal{A}_1 u^{(n_{m+1})}_{m+1} - \mathcal{A}_2 u^{(n_m)}_m, & \text{in } Y_1, \\
-d(y) \left( \nabla_x u^{(n_{m+1})}_{m+1} + \nabla_y u^{(n_{m+2})}_{m+2} \right) \cdot n = 0, & \text{on } \partial Y_0, \\
\end{cases}
\]

for $0 \leq m \leq M - 2$. Note that the quantity $n_m$ is independent of $\varepsilon$.

Since the approximate auxiliary problems became linear, standard procedures are able to find the solutions $u^{(n_m)}_m$ for $0 \leq m \leq M$. Note that these problems admit a unique solution (see, e.g. [10, Lemma 2.2]) on $Y_1$, i.e. the quotient space of $V_{Y_1}$ defined by

\[
V_{Y_1} := \{ \phi \mid \phi \in H^1(Y_1), \phi \text{ is } Y-\text{periodic} \}.
\]

On the other side, if $\kappa_p := C_p L \alpha^{-1} < 1$ holds (here $C_p$ is the Poincaré constant depending only on the dimension of $Y_1$), we easily obtain that for every $m$, $\{u^{(n_m)}_m\}$ is a Cauchy sequence in $H^1(Y_1)$. Thereby, it naturally claims the existence and uniqueness of the nonlinear auxiliary problems \((3.7)-(3.9)\). Moreover, the convergence rate of the iteration procedure is given by

\[
\left\| u^{(n_m)}_m - u_m \right\|_{H^1(Y_1)} \leq \frac{\kappa_p^{n_m}}{1 - \kappa_p^{n_m}} \left\| u^{(1)}_m \right\|_{H^1(Y_1)}.
\]

**Remark 3.** For more details in this sense, see [1] Theorem 9 and [11] Theorem 2.2.

To prove the corrector estimate, we suppose that the solutions of the auxiliary problems \((3.7)-(3.9)\) belong to the space $L^\infty(\Omega^\varepsilon; \mathbb{V})$. Let us introduce the following function:

\[
\varphi^\varepsilon := u^\varepsilon - \sum_{m=0}^M \varepsilon^m u_m.
\]
Relying on the auxiliary problems (3.7)-(3.9), note that the function \( \varphi^\varepsilon \) satisfies the following system:

\[
\begin{aligned}
A^\varepsilon \varphi^\varepsilon &= R(u^\varepsilon) - \sum_{m=0}^{M-2} \varepsilon^{m-2} R(u_m), \\
-\varepsilon^{M-1} (A_1 u_M + A_2 u_{M-1}) - \varepsilon^M A_2 u_M, &\quad \text{in } \Omega^\varepsilon, \\
-\varepsilon^M \nabla_x \varphi^\varepsilon \cdot n &= \varepsilon^M \varepsilon^M \nabla_x u_M \cdot n, &\quad \text{on } \Gamma^\varepsilon.
\end{aligned}
\]  

(3.13)

Now, multiplying the PDE in (3.13) by \( \varphi \in V^\varepsilon \) and integrating by parts, we arrive at

\[
\langle d^\varepsilon \varphi^\varepsilon, \varphi \rangle_{V^\varepsilon} = \left\langle R(u^\varepsilon) - \sum_{m=0}^{M-2} \varepsilon^{m-2} R(u_m), \varphi \right\rangle_{L^2(\Omega^\varepsilon)} - \varepsilon^{M-1} \langle A_1 u_M + A_2 u_{M-1} + \varepsilon A_2 u_M, \varphi \rangle_{L^2(\Omega^\varepsilon)} - \varepsilon^M \int_{\Omega^\varepsilon_{int}} d^\varepsilon \nabla_x u_M \cdot n \varphi \, dS_x.
\]  

(3.14)

From here on, we have to estimate the integrals on the right-hand side of (3.14), which is a standard procedure; see [10] for similar calculations. Thus, we claim that

\[
\left| \left\langle R(u^\varepsilon) - \sum_{m=0}^{M-2} \varepsilon^{m-2} R(u_m), \varphi \right\rangle_{L^2(\Omega^\varepsilon)} \right| \leq C L \left\| u^\varepsilon - \sum_{m=0}^{M} \varepsilon^m u_m + \mathcal{O}(\varepsilon^{M-1}) \right\|_{V^\varepsilon} \left\| \varphi \right\|_{L^2(\Omega^\varepsilon)},
\]  

(3.15)

where we have essentially used the global Lipschitz condition on the reaction term, the assumption (3.6), and the Poincaré inequality. Next, we get

\[
\varepsilon^{M-1} \left| \langle A_1 u_M + A_2 u_{M-1} + \varepsilon A_2 u_M, \varphi \rangle_{L^2(\Omega^\varepsilon)} \right| \leq C \varepsilon^{M-1} \left\| \varphi \right\|_{L^2(\Omega^\varepsilon)},
\]  

(3.16)

while using the trace inequality (cf. [10] Lemma 2.31]) to deal with the the last integral, it gives

\[
\varepsilon^M \left| \int_{\Omega^\varepsilon_{int}} d^\varepsilon \nabla_x u_M \cdot n \varphi \, dS_x \right| \leq C \varepsilon^{M-1} \left\| \varphi \right\|_{L^2(\Omega^\varepsilon)}.
\]  

(3.17)

Combining (3.15)-(3.17), we provide that

\[
\alpha \left| \langle \varphi^\varepsilon, \varphi \rangle_{V^\varepsilon} \right| \leq C \varepsilon^{M-1} \left\| \varphi \right\|_{L^2(\Omega^\varepsilon)},
\]

which finally leads to \( \left\| \varphi^\varepsilon \right\|_{V^\varepsilon} \leq C \varepsilon^{M-1} \) by choosing \( \varphi = \varphi^\varepsilon \), very much in the spirit of energy estimates.

Summarizing, we state our results in the frame of the following theorems.

**Theorem 4.** Suppose (3.6) holds for \( r \in \{1, 2\} \) and assume \( \kappa_p := C_p \Lambda^\varepsilon < 1 \) for the given Poincaré constant. Let \( \{u_m^{(n_m)}\}_{n_m \in \mathbb{N}} \) be the schemes that approximate the nonlinear auxiliary problems (3.10)-(3.12). Then (3.10)-(3.12) admit a unique solution \( u_m \) for all \( m \in \{0, ..., M\} \) with the speed of convergence:

\[
\left\| u_m^{(n_m)} - u_m \right\|_{H^1(Y)} \leq \frac{C \kappa_p^{n_m}}{1 - \kappa_p} \quad \text{for all } n_m \in \mathbb{N} \text{ and } m \in \{0, ..., M\},
\]

where \( C > 0 \) is a generic \( \varepsilon \)-independent constant and \( n := \max \{n_0, ..., n_M\} \).

**Theorem 5.** Let \( u^\varepsilon \) be the solution of the elliptic system (2.1) with the assumptions (A1)–(A2) stated above and suppose that (3.6) holds for \( r \in \{1, 2\} \). For \( m \in \{0, ..., M\} \) with \( M \geq 2 \), we consider \( u_m \) the solutions of the auxiliary problems (3.10)-(3.12). Then we obtain the following corrector estimate:

\[
\left\| u^\varepsilon - \sum_{m=0}^{M} \varepsilon^m u_m \right\|_{V^\varepsilon} \leq C \varepsilon^{M-1},
\]

where \( C > 0 \) is a generic \( \varepsilon \)-independent constant.
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References

[1] E. Weinan, Principles of Multiscale Modeling, Cambridge University Press, 2011.

[2] M. Schmuck, G. Pavliotis, S. Kalliadasis, Effective macroscopic interfacial transport equations in strongly heterogeneous environments for general homogeneous free energies, Applied Mathematics Letters 35 (2014) 12–17.

[3] N. Ray, Colloidal transport in porous media modeling and analysis, Ph.D. thesis, University of Erlangen-Nuremberg (2013).

[4] O. Krehel, A. Muntean, P. Knabner, Multiscale modeling of colloidal dynamics in porous media including aggregation and deposition, Advances in Water Resources 86 (2015) 209–216.

[5] N. Ray, A. Muntean, P. Knabner, Rigorous homogenization of a Stokes-Nernst-Planck-Poisson system, Journal of Mathematical Analysis and Applications 390 (1) (2012) 374–393.

[6] V. Khoa, A. Muntean, Asymptotic analysis of a semi-linear elliptic system in perforated domains: well-posedness and corrector for the homogenization limit arXiv:1512.03735.

[7] C. Pao, Nonlinear Parabolic and Elliptic Equations, Springer, 1993.

[8] U. Hornung, W. Jäger, Diffusion, convection, adsorption, and reaction of chemicals in porous media, Journal of Differential Equations 92 (1991) 199–225.

[9] A. Bensoussan, J.-L. Lions, G. Papanicolaou, Asymptotic Analysis for Periodic Structures, AMS Chelsea Publishing, 1978.

[10] D. Cioranescu, J. Saint Jean Paulin, Homogenization of Reticulated Structures, Springer, 1999.

[11] V. Khoa, N. Tuan, L. Lan, N. Ngoc, A finite difference scheme for nonlinear ultra-parabolic equations, Applied Mathematics Letters 46 (2015) 70–76.