GLOBAL WELL-POSEDNESS OF REGULAR SOLUTIONS TO THE
THREE-DIMENSIONAL ISENTROPIC COMPRESSIBLE
NAVIER-STOKES EQUATIONS WITH DEGENERATE VISCOSITIES
AND VACUUM

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Abstract. In this paper, the Cauchy problem for the three-dimensional (3-D) isentropic compressible Navier-Stokes equations with degenerate viscosities is considered. By introducing some new variables and making use of the “quasi-symmetric hyperbolic”–“degenerate elliptic” coupled structure to control the behavior of the fluid velocity, we prove the global-in-time well-posedness of regular solutions with vacuum for a class of smooth initial data that are of small density but possibly large velocities. Here the initial mass density is required to decay to zero in the far field, and the spectrum of the Jacobi matrix of the initial velocity are all positive. The result here applies to a class of degenerate density-dependent viscosity coefficients, is independent of the BD-entropy, and seems to be the first on the global existence of smooth solutions which have large velocities and contain vacuum state for such degenerate system in three space dimensions.

1. Introduction

The time evolution of the mass density $\rho \geq 0$ and the velocity $u = (u^{(1)}, u^{(2)}, u^{(3)})^\top$ of a general viscous isentropic compressible fluid occupying a spatial domain $\Omega \subset \mathbb{R}^3$ is governed by the following isentropic compressible Navier-Stokes equations (ICNS):

$$\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P = \text{div}T.
\end{cases} \quad (1.1)$$

Here, $x = (x_1, x_2, x_3) \in \Omega$, $t \geq 0$ are the space and time variables, respectively. In considering the polytropic gases, the constitutive relation, which is also called the equations of state, is given by

$$P = A\rho^\gamma, \quad \gamma > 1, \quad (1.2)$$

where $A > 0$ is an entropy constant and $\gamma$ is the adiabatic exponent. $T$ denotes the viscous stress tensor with the form

$$T = \mu(\rho)\left(\nabla u + (\nabla u)^\top\right) + \lambda(\rho)\text{div}u I_3, \quad (1.3)$$

where $I_3$ is the $3 \times 3$ identity matrix.
\[ \mu(p) = \alpha \rho^\delta, \quad \lambda(p) = \beta \rho^\delta, \quad (1.4) \]

for some constant \( \delta \geq 0 \), \( \mu(p) \) is the shear viscosity coefficient, \( \lambda(p) + \frac{2}{3} \mu(p) \) is the bulk viscosity coefficient, \( \alpha \) and \( \beta \) are both constants satisfying

\[ \alpha > 0, \quad \text{and} \quad 2\alpha + 3\beta \geq 0. \quad (1.5) \]

Let \( \Omega = \mathbb{R}^3 \). We look for smooth solutions, \((\rho(t, x), u(t, x))\) to the Cauchy problem for \((1.1)-(1.5)\) with the initial data and far field behavior:

\[ \rho(t, x) \to 0, \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad t \geq 0. \quad (1.7) \]

In the theory of gas dynamics, the compressible Navier-Stokes equations can be derived from the Boltzmann equations through the Chapman-Enskog expansion, cf. Chapman-Cowling [3] and Li-Qin [18]. Under some proper physical assumptions, the viscosity coefficients and the heat conductivity coefficient \( \kappa \) are not constants but functions of the absolute temperature \( \theta \) such as:

\[ \mu(\theta) = a_1 \theta^\frac{\gamma}{\gamma - 1} F(\theta), \quad \lambda(\theta) = a_2 \theta^\frac{2}{\gamma - 1} F(\theta), \quad \kappa(\theta) = a_3 \theta^\frac{2}{\gamma - 1} F(\theta) \quad (1.8) \]

for some constants \( a_i \) \((i = 1, 2, 3)\). Actually in [3] for the cut-off inverse power force model, if the intermolecular potential varies as \( r^{-a} \), where \( r \) is intermolecular distance, then in \((1.8)\): \( F(\theta) = \theta^b \) with \( b = \frac{2}{a} \in [0, +\infty) \). In particular, for Maxwellian molecules, \( a = 4 \) and \( b = \frac{1}{2} \); while for elastic spheres, \( a = \infty \) and \( b = 0 \). As a typical model whose \( F \) is not a power function of \( \theta \), the Sutherland’s model is well known where

\[ F(\theta) = \frac{\theta}{\theta + s_0}, \quad (s_0 > 0 : \text{Sutherland’s constant}). \quad (1.9) \]

According to Liu-Xin-Yang [22], if we restrict the gas flow to be isentropic, such dependence is inherited through the laws of Boyle and Gay-Lussac:

\[ P = R \rho \theta = A \rho^\gamma, \quad \text{for constant} \quad R > 0, \]

i.e., \( \theta = AR^{-1} \rho^{\gamma - 1} \), and one finds that the viscosity coefficients are functions of the density. In this paper, we will focus on the cut-off inverse power force model whose viscosities have the forms shown in \((1.8)\). The corresponding conclusion for the isentropic flow of the Sutherland’s model will be shown in our forthcoming paper Xin-Zhu [31].

Throughout this paper, we adopt the following simplified notations, most of them are for the standard homogeneous and inhomogeneous Sobolev spaces:

\[ \|f\|_s = \|f\|_{H^s(\mathbb{R}^3)}, \quad \|f\|_p = \|f\|_{L^p(\mathbb{R}^3)}, \quad \|f\|_{m,p} = \|f\|_{W^{m,p}(\mathbb{R}^3)}, \]

\[ \|f\|_{C^k} = \|f\|_{C^k(\mathbb{R}^3)}, \quad \|f\|_{L^p L^q} = \|f\|_{L^p([0, T]; L^q(\mathbb{R}^3))}, \quad \|f\|_{X_1 \cap X_2} = \|f\|_{X_1} + \|f\|_{X_2}, \]

\[ D^{k, r} = \{f \in L^1_{loc}(\mathbb{R}^3) : \|f\|_{D^{k, r}} = \|\nabla^k f\|_r < +\infty\}, \quad D^k = D^{k,2}, \quad \int_{\mathbb{R}^3} f \, dx = \int f, \]

\[ \Xi = \{f \in L^1_{loc}(\mathbb{R}^3) : \|\nabla f\|_{\infty} + \|\nabla^2 f\|_2 < +\infty\}, \quad \|f\|_{\Xi} = \|\nabla f\|_{\infty} + \|\nabla^2 f\|_2, \]

\[ \|f(t, x)\|_{T; \Xi} = \|\nabla f(t, x)\|_{L^\infty([0, T] \times \mathbb{R}^3)} + \|\nabla^2 f(t, x)\|_{L^\infty([0, T]; H^2(\mathbb{R}^3))}. \]

A detailed study of homogeneous Sobolev spaces can be found in [8].
There is a lot of literature on the well-posedness of solutions to the problem (1.1)–(1.6) in multi-dimensional space. For 3-D constant viscous flow (δ = 0 in (1.4)) with inf \( x \rho_0(x) > 0 \), it is well-known that the local existence of classical solutions has been obtained by a standard Banach fixed point argument by Nash [26], which has been extended to be a global one by Matsumura-Nishida [25] for initial data close to a nonvacuum equilibrium in some Sobolev space \( H^s (s > \frac{5}{2}) \). Hoff [11] studied the global weak solutions with strictly positive initial density and temperature for discontinuous initial data. However, these approaches do not work when inf \( x \rho_0(x) = 0 \), which occurs when some physical requirements are imposed, such as finite total initial mass and energy in the whole space.

The main breakthrough for the well-posedness of solutions with generic data and vacuum is due to Lions [27], where he established the global existence of weak solutions with finite energy to the isentropic compressible flow provided that \( \gamma > \frac{9}{5} \) (see also Feireisl-Novotný-Petzeltová [6] for the case \( \gamma > \frac{3}{2} \)). However, the uniqueness problem of these weak solutions is widely open due to their fairly low regularities. Recently, Xin-Yan [30] proved that any classical solutions of viscous non-isentropic compressible fluids without heat conduction will blow up in finite time, if the initial data has an isolated mass group.

For density-dependent viscosities (δ > 0 in (1.4)) which degenerate at vacuum, system (1.1) has received extensive attentions in recent years. In this case, the strong degeneracy of the momentum equations in (1.2) near vacuum creates serious difficulties for well-posedness of both strong and weak solutions. A mathematical entropy function was proposed by Bresch-Desjardins [1] for \( \lambda(\rho) \) and \( \mu(\rho) \) satisfying the relation

\[
\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)),
\]

which offers an estimate \( \mu'(\rho)\sqrt{\rho} \in L^\infty([0,T];L^2(\mathbb{R}^d)) \) provided that \( \mu'(\rho_0)\sqrt{\rho_0} \in L^2(\mathbb{R}^d) \) for any \( d \geq 1 \). This observation plays an important role in the development of the global existence of weak solutions with vacuum for system (1.1) and some related models, see Bresh-Dejardins [2], Li-Xin [17], Mellet-Vassuer [24] and some other interesting results, c.f. [14, 32]. However, the regularities and uniquenes of such weak solutions remain open especially in multi-dimensional cases.

In this paper, we study the global well-posedness of regular solutions to the system (1.1) with density-dependent viscosities given in (1.4) and initial data such that inf \( x \rho_0(x) = 0 \). Then the analysis of the degeneracies in momentum equations (1.2) requires some special attentions. The major concerns are:

1. The degeneracy of time evolution in momentum equations (1.2). Note that the leading coefficient of \( u_t \) in momentum equations vanishes at vacuum, and this leads to infinitely many ways to define velocity (if it exits) when vacuum appears. Mathematically, this degeneracy leads to a difficulty that it is hard to find a reasonable way to extend the definition of velocity into vacuum region. For constant viscosities, a remedy was suggested by Cho-Choe-Kim (for example [4]), where they imposed initially a compatibility condition

\[-\text{div}T_0 + \nabla P(\rho_0) = \sqrt{\rho_0}g, \quad \text{for some} \quad g \in L^2(\mathbb{R}^3),\]

which is roughly equivalent to the \( L^2 \)-integrability of \( \sqrt{\rho_0}u_t(t = 0) \), and plays a key role in deducing that \( (\sqrt{\rho}u_t, \nabla u_t) \in L^\infty([0,T_*];L^2(\mathbb{R}^3)) \) for a short time \( T_* > 0 \). Then they established successfully the local well-posedness of smooth solutions with non-negative density in \( \mathbb{R}^3 \), which, recently, has been shown to be a global
one with small energy but large oscillations by Huang-Li-Xi [12] in $\mathbb{R}^3$ and Li-Xin [16] in $\mathbb{R}^2$.

(2) The strong degeneracy of the elliptic operator $\text{div} T$ caused by vacuum for $\delta > 0$. In [4, 12] for $\delta = 0$, the uniform ellipticity of the Lamé operator $L$ defined by

$$Lu = -\alpha \Delta u - (\alpha + \beta)\nabla \text{div} u$$

plays an essential role in the high order regularity estimates on $u$. One can use standard elliptic theory to estimate $|u|_{D^{k+2}}$ by the $D^k$-norm of all other terms in momentum equations. However, for $\delta > 0$, viscosity coefficients vanish as density function connects to vacuum continuously. This degeneracy makes it difficult to adapt the approach in the constant viscosity case in [4, 12] to the current case.

(3) The strong nonlinearity for the variable coefficients of the viscous term due to $\delta > 0$. It should be pointed out here that unlike the case of constant viscosities, despite the weak regularizing effect on solutions, the elliptic part $\text{div} T$ will also cause some troubles in the high order regularity estimates. For example, to establish some uniform a priori estimates independent of the lower bound of density in $H^3$ space, the key is to handle the extra nonlinear terms such as

$$\text{div}(\nabla^k \rho \delta S(u)) \quad \text{for} \quad S(u) = \alpha(\nabla u + (\nabla u)^\top) + \beta \text{div} u I_3,$$

where $k = 0, 1, 2, 3$. Therefore, much attentions need to be paid in order to control these strong nonlinearities, especially for establishing the global well-posedness of classical solutions.

Recently, there have some interesting works to overcome these difficulties mentioned above. In Li-Pan-Zhu [20] for the case $\delta = 1$, the degeneracies of the time evolution and the viscosity can be transferred to the possible singularity of the special source term $\left(\frac{\nabla \rho}{\rho}\right) \cdot S(u)$. Based on this observations, via establishing a uniform a priori estimates in $L^6 \cap D^1 \cap D^2$ space for the quantity $\nabla \frac{\rho}{\rho}$, the existence of the unique local classical solution in 2-D space (see also Zhu [33] for 3-D case) to (1.1) has been obtained under the assumptions

$$\rho_0(x) \rightarrow 0, \quad \text{as} \quad |x| \rightarrow \infty,$$

which also applies to the 2-D shallow water equations.

However, this result only allows vacuum at the far field, and the corresponding problem with vacuum appearing in some open sets, or even at a single point is still unsolved. Later, via introducing a proper class of solutions and establishing the uniform weighted a priori estimates for the higher order term $\rho^{\frac{\gamma - 1}{2}} \nabla^4 u$, the same authors [21, 34] gave the existence of 3-D local classical solutions to system (1.1) for the cases

$$1 < \delta \leq \min\{3, (\gamma + 1)/2\}.$$

Some other interesting results can also be seen in Ding-Zhu [5] and Li-Pan-Zhu [19].

1.1. Symmetric formulation. In order to deal with the issues mentioned above, we first need to analyze the structure of the momentum equations (1.1)_2 carefully, which can be decomposed into hyperbolic, elliptic and source parts as follows:

$$\underbrace{\rho(u_t + u \cdot \nabla u) + \nabla P}_{\text{Hyperbolic}} = \underbrace{-\rho^\delta Lu + \nabla \rho^\delta \cdot S(u)}_{\text{Elliptic Source}}. \quad (1.11)$$
For smooth solutions \((\rho, u)\) away from vacuum, these equations could be written into

\[
\begin{align*}
  u_t + u \cdot \nabla u + \frac{A\gamma}{\gamma - 1} \nabla \rho^{\gamma - 1} - \frac{\delta}{\delta - 1} \nabla \rho^{\delta - 1} \cdot S(u) &= -\rho^{\delta - 1} \mathcal{L}(u), \quad \text{Principal order} \\
  \quad \text{Higher order}
\end{align*}
\]  

(1.12)

Note that if \(\rho\) is smooth enough, one could pass to the limit as \(\rho \to 0\) on both sides of (1.12) and formally have

\[
u_t + u \cdot \nabla u = 0 \quad \text{as} \quad \rho(t, x) = 0.
\]  

(1.13)

So (1.12)-(1.13) indicate that the velocity \(u\) could be governed by a nonlinear degenerate parabolic system when vacuum appears in some open sets or at the far field.

Recently, in Geng-Li-Zhu [9], via introducing two new quantities:

\[
\varphi = \rho^{\delta - 1}, \quad \text{and} \quad \phi = \frac{4A\gamma}{(\gamma - 1)^2} \rho^{\gamma - 1},
\]

system (1.1) can be rewritten into a system that consists of a transport equation for \(\varphi\), and a “quasi-symmetric hyperbolic”–“degenerate elliptic” coupling system for \(U = (\phi, u)^T\):

\[
\begin{align*}
  &\varphi_t + u \cdot \nabla \varphi + \frac{\delta - 1}{2} \varphi \text{div} u = 0, \\
  &\sum_{j=1}^{3} A_j(U) \partial_j U = -\varphi^2 \mathcal{L}(u) + \mathcal{H}(\varphi) \cdot \mathcal{Q}(u),
\end{align*}
\]  

(1.14)

where \(\partial_j U = \partial x_j U\),

\[
A_j(U) = \begin{pmatrix}
  u(j) & \frac{\gamma - 1}{2} \phi e_j^T \\
  \frac{\gamma - 1}{2} \phi e_j & u(j) e_3^T
\end{pmatrix}, \quad j = 1, 2, 3, \quad \mathcal{L}(u) = \begin{pmatrix} 0 \\ L(u) \end{pmatrix},
\]

(1.15)

\[
\mathcal{H}(\varphi) = \begin{pmatrix} 0 \\ \nabla \varphi^2 \end{pmatrix}, \quad \mathcal{Q}(u) = \begin{pmatrix} 0 & 0 \\ 0 & Q(u) \end{pmatrix}, \quad Q(u) = \frac{\delta}{\delta - 1} S(u),
\]

and \(e_j = (\delta_{1j}, \delta_{2j}, \delta_{3j}) (j = 1, 2, 3)\) is the Kronecker symbol satisfying \(\delta_{ij} = 1\), when \(i = j\) and \(\delta_{ij} = 0\), otherwise. Based on this reformulation, a local well-posedness for arbitrary \(\delta > 1\) in some non-homogenous Sobolev spaces has been obtained for the compressible degenerate viscous flow in [9], which could be shown as follows:

**Theorem 1.1.** [9] Let \(\gamma > 1\) and \(\delta > 1\). If initial data \((\rho_0, u_0)\) satisfy

\[
\rho_0 \geq 0, \quad (\frac{\gamma - 1}{2} \rho_0^{\delta - 1}, \rho_0^{\delta - 1}, u_0) \in H^3,
\]

(1.16)

then there exists a time \(T_* > 0\) independent of the viscosity coefficients and a unique regular solution \((\rho, u)\) in \([0, T_*] \times \mathbb{R}^3\) to the Cauchy problem (1.1)-(1.7), where the regular
solution \((\rho, u)\) satisfies this problem in the sense of distribution and:

\[(A)\quad \rho \geq 0, \quad \rho^{\frac{\gamma-1}{2}} \in C([0, T_*]; H^3), \quad \rho^{\frac{\gamma+1}{2}} \in C([0, T_*]; H^3);
\]

\[(B)\quad u \in C([0, T_*]; H^{s'}) \cap L^\infty([0, T_*]; H^3), \quad \rho^{\frac{\gamma+1}{2}} \nabla^4 u \in L^2([0, T_*]; L^2),
\]

\[(C)\quad u_t + u \cdot \nabla u = 0 \quad \text{as} \quad \rho(t, x) = 0,
\]

where \(s' \in [2, 3)\) is an arbitrary constant.

1.2. Singularity formation. We consider first whether the local regular solution in Theorem 1.1 can be extended globally in time. In contrast to the classical theory for the constant viscosity case, we show the following somewhat surprising phenomenon that such an extension is impossible if the velocity field decays to zero as \(t \to +\infty\) and the initial total momentum is non-zero. More precisely, let

\[P(t) = \int \rho u \quad (\text{total momentum}).\]

**Theorem 1.2.** Let \(1 < \min\{\gamma, \delta\} \leq 2\). Assume \(|P(0)| > 0\). Then there is no global regular solution \((\rho, u)\) obtained in Theorem 1.1 satisfying the following decay

\[\limsup_{t \to +\infty} |u(t, x)|_\infty = 0. \quad (1.17)\]

Second, in the presence of vacuum region, there is another possibility of finite singularity defined as follows.

**Definition 1.1.** The non-empty open set \(V \subset \mathbb{R}^3\) is called a hyperbolic singularity set of \((\rho_0, u_0)(x)\), if \(V\) satisfies

\[
\begin{cases}
\rho_0(x) = 0, \quad \forall \ x \in V; \\
Sp(\nabla u_0) \cap \mathbb{R}^- \neq \emptyset, \quad \forall \ x \in V,
\end{cases}
\]

where \(Sp(\nabla u_0(x))\) denotes the spectrum of the matrix \(\nabla u_0(x)\).

By exploring the hyperbolic structure in (1.1) in vacuum region, one can confirm exactly that hyperbolic singularity set does generate singularities from local regular solutions in finite time.

**Theorem 1.3.** Let \(\gamma > 1\) and \(\delta > 1\). If the initial data \((\rho_0, u_0)(x)\) have a non-empty hyperbolic singularity set \(V\), then the regular solution \((\rho, u)(t, x)\) on \(\mathbb{R}^3 \times [0, T_m]\) obtained in Theorem 1.1 with maximal existence time \(T_m\) blows up in finite time, i.e., \(T_m < +\infty\).

1.3. Global-in-time well-posedness of smooth solutions. We now come to the major task to construct global smooth solutions for the system (1.1). Due to Theorems 1.2, 1.3 one needs to identify a class of initial data and a proper energy space to avoid the two singularity mechanisms shown above.

Let \(\tilde{u} = (\tilde{u}^{(1)}, \tilde{u}^{(2)}, \tilde{u}^{(3)})^\top\) be the solution in \([0, T] \times \mathbb{R}^3\) of the following Cauchy problem:

\[
\tilde{u}_t + \tilde{u} \cdot \nabla \tilde{u} = 0, \quad \tilde{u}(t = 0, x) = u_0(x)
\]

for \(x \in \mathbb{R}^3\). We give the definition of regular solutions considered in this paper:
Definition 1.2. Let $T > 0$ be a finite constant. A solution $(\rho, u)$ to the Cauchy problem (1.1)-(1.7) is called a regular solution in $[0, T] \times \mathbb{R}^3$ if $(\rho, u)$ satisfies this problem in the sense of distribution and:

- $(A)$ $\rho \geq 0$, $\left(\rho^{\frac{4}{3+\alpha}}, \rho^{\frac{2}{3+\alpha}}\right) \in C([0, T]; H^1_{loc} \cap L^{\infty}([0, T]; H^3))$;
- $(B)$ $u - \tilde{u} \in C([0, T]; H^1_{loc}) \cap L^{\infty}([0, T]; H^3)$, $\rho^{\frac{4}{3+\alpha}} \nabla u \in L^2([0, T]; L^2)$;
- $(C)$ $u_t + u \cdot \nabla u = 0$ as $\rho(t, x) = 0$,

where $s' \in [2, 3)$ is an arbitrary constant.

The main result on the global well-posedness of regular solutions to the three-dimensional isentropic compressible Navier-Stokes equations with degenerate viscosities and vacuum could be stated as follows.

Theorem 1.4. Let parameters $(\gamma, \delta, \alpha, \beta)$ satisfy

$$\gamma > 1, \quad \delta > 1, \quad \alpha > 0, \quad 2\alpha + 3\beta \geq 0, \quad (1.20)$$

and any one of the following conditions $(P_1)$-$(P_3)$:

- $(P_1)$ $2\alpha + 3\beta = 0$;
- $(P_2)$ $\delta \geq 2\gamma - 1$;
- $(P_3)$ $\delta = \gamma$.

If the initial data $(\rho_0, u_0)$ satisfy

- $(A_1)$ $\rho_0 \geq 0$ and $\left\|\rho_0^{\frac{4}{3+\alpha}}\right\|_3 + \left\|\rho_0^{\frac{2}{3+\alpha}}\right\|_3 \leq D_0(\gamma, \delta, \alpha, \beta, A, \kappa, \|u_0\|_3)$,
- $(A_2)$ $u_0 \in \Xi$ and there exists a constant $\kappa > 0$ such that

$\text{Dist}(\text{Sp}(\nabla u_0(x)), \mathbb{R}_+^3) \geq \kappa$ for all $x \in \mathbb{R}^3$,

where $D_0 > 0$ is some constant depending on $(\alpha, \beta, \delta, \gamma, \kappa, \|u_0\|_3)$, then for any $T > 0$, there exists a unique regular solution $(\rho, u)$ in $[0, T] \times \mathbb{R}^3$ to the Cauchy problem (1.1)-(1.7). Particularly, when condition $(P_2)$ holds, the smallness assumption on $\rho_0^{\frac{4}{3+\alpha}}$ could be removed.

Moreover, if $1 < \min(\gamma, \delta) \leq 3$, $(\rho, u)$ is a classical solution to the Cauchy problem (1.1)-(1.7) in $[0, T] \times \mathbb{R}^3$.

Remark 1.1. The conditions $(A_1)$-$(A_2)$ identify a class of admissible initial data that provides unique solvability to the Cauchy problem (1.1)-(1.7). Such initial data contain the following examples:

$$\rho_0(x) = \frac{\epsilon_1}{(1 + |x|)^2\sigma_1}, \quad \epsilon_1 g^{2\sigma_2}(x), \quad \epsilon_1 \exp\{-x^2\}, \quad \frac{\epsilon_1 |x|}{(1 + |x|)^2\sigma_3},$$

where $\epsilon_1 > 0$ is a sufficiently small constant, $0 \leq g(x) \in C^2(\mathbb{R}^3)$, and

$$\sigma_1 > \frac{3}{2} \max\left\{\frac{1}{\delta - 1}, \frac{1}{\gamma - 1}\right\}, \quad \sigma_2 > 3 \max\left\{\frac{1}{\delta - 1}, \frac{1}{\gamma - 1}\right\}, \quad \sigma_3 > \frac{3}{2} \max\left\{\frac{1}{\delta - 1}, \frac{1}{\gamma - 1}\right\} + \frac{1}{2};$$

$$u_0 = Ax + b + \epsilon_2 f(x),$$

where $A$ is a $3 \times 3$ constant matrix whose eigenvalues are all greater than $2\kappa$, $\epsilon_2 > 0$ is a sufficiently small constant, $f \in \Xi$ and $b \in \mathbb{R}^3$ is a constant vector.

Remark 1.2. It is worth pointing out that, for any $\gamma > 1$, even $2\alpha + 3\beta \neq 0$, that is to say $(P_1)$ fails, we can still deal with the corresponding well-posedness problem for a relatively wide class of $(\alpha, \beta, \delta)$ under the following condition:
(P₀) \(0 < M₁ = \frac{2α+3β}{2α+β} < \frac{3}{2} - \frac{1}{δ} \) and
\[M₂ = -3δ + 1 + \frac{1}{4} \left( \frac{(δ-1)^2}{3(2α+β)} + \frac{4δ(2α+β)^2}{(δ-1)^2} M₁^2 + 2M₁δ \right) < -1,\]
which can be seen in §5.

Next we indicate that the set of parameters \((α, β, δ)\) satisfying \((P₀)\) must be non-empty. First, for fixed \(δ > 1\), let \(α = a₁ \eta, β = a₂ \eta, \) where \(η > 0, a₁ > 0\) and \(2a₁ + 3a₂ ≥ 0\) are all constants. Thus \(M₁ = \frac{2a₁+3a₂}{2a₁+a₂} \). Then one can adjust the values of \(a₁\) and \(a₂\) to ensure that \(M₁ < \frac{3}{2} - \frac{1}{δ} \) holds.

Second, let \(x = (δ-1)^2/(4(2a₁+a₂)η)\), and consider the following function:
\[F(x) = x + \frac{1}{x} δ^2M₁^2 + 2M₁δ - 6δ + 4,\]
whose minimum value is
\[F(δM₁) = 4M₁δ - 6δ + 4 < 0\]
due to \(M₁ < \frac{3}{2} - \frac{1}{δ}\). Thus, one needs only to choose \(η\) to ensure that \(x\) belongs to a small neighbor of \(δM₁\) and \(F < 0\), which is equivalent to \(M₂ < -1\).

Remark 1.3. Without generality, we can assume that \(u₀(0) = 0\), which can be achieved by the following Galilean transformation:
\[t' = t, \ x' = x + u₀(0)t, \ \rho'(t', x') = ρ(t, x), \ \tilde{u}'(t', x') = \tilde{u}(t, x) - u₀(0).\]

The above well-posedness theory is still available for some other models such as the ones shown in the following two theorems.

Theorem 1.5. Let the viscous stress tensor \(T\) in (1.1) be given by
\[T = ρδ(2α∇u + β\text{div}uI₃).\]
Then the same well-posedness theory as in Theorem 1.4 holds in this case.

Theorem 1.6. Let (1.20) hold with \(β = 0\). If the viscous term \(\text{div}T\) is given by \(αρδΔu\),
then under the initial conditions shown in Theorem 1.4, there exists a unique regular solution \((ρ, u)\) in \([0, T] × \mathbb{R}^3\) to the Cauchy problem (1.1)-(1.2) with \((1.6)-(1.7)\).

Moreover, if \(1 < \min(γ, δ) ≤ 3\), the solution \((ρ, u)\) solves the Cauchy problem (1.1)-(1.2) with \((1.6)-(1.7)\) in \([0, T] × \mathbb{R}^3\) classically.

The rest of the paper is organized as follows: In §2, we show some decay estimates for the classical solutions of the multi-dimensional Burgers equations and the global well-posedness of an ordinary differential equation, which will be used later. In §3, we prove the finite time singularity formation in Theorems 1.2-1.3. §4-§8 are devoted to establishing the global well-posedness of regular solutions stated in Theorem 1.4. We start with the reformulation of the original problem (1.1)-(1.7) as (4.1) in terms of the new variables, and establish the local-in-time well-posedness of smooth solutions to (4.1) in the case that the initial density is compactly supported in §4. Note that one cannot apply the result in [9] directly here, the initial velocity here does not have a uniform upper bound in the whole space and stays in a homogenous sobolev space. It is also worth pointing...
out that recently, Li-Wang-Xin [15] prove that classical solutions with finite energy to the Cauchy problem of the compressible Navier-Stokes systems with constant viscosities do not exist in general inhomogeneous Sobolev space for any short time, which indicates in particular that the homogeneous Sobolev space is crucial as studying the well-posedness (even locally in time) for the Cauchy problem of the compressible Navier-Stokes systems in the presence of vacuum. This local smooth solution to (4.1) is shown to be global in time in §5-§6 by deriving a uniform (in time) a priori estimates independent of the size of the initial density’s support through energy methods based on suitable choice of time weights. Here, we have employed some arguments due to Grassin [10] and Serre [28] to deal with the nonlinear convection term \( u \cdot \nabla u \). Next, the assumption that the initial density has compact support is removed in §7. Finally, in §8, the proof of Theorem 1.4 is completed by making use of the results obtained in §7. Furthermore, we give an appendix to list some lemmas that are used in our proof, and outline some proofs of properties shown in §2 and some inequalities used frequently in this paper.

2. Preliminary

This section will be devoted to show some decay estimates for the classical solutions of the multi-dimensional Burgers equations and the global well-posedness of an ordinary differential equation, which will be used frequently in our proof.

First, let \( \hat{u} \) be the solution to the problem (1.19) in d-dimensional space. Then, along the particle path \( X(t; x_0) \) defined as

\[
\frac{d}{dt} X(t; x_0) = \hat{u}(t, X(t; x_0)), \quad x(0; x_0) = x_0,
\]

\( \hat{u} \) is a constant in t: \( \hat{u}(t, X(t; x_0)) = u_0(x_0) \) and

\[
\nabla \hat{u}(t, X(t; x_0)) = (I_d + t \nabla u_0(x_0))^{-1} \nabla u_0(x_0).
\]

Based on this observation, one can have the following decay estimates of \( \hat{u} \), which play important roles in establishing the global existence of the smooth solution to the problem considered in this paper, and their proof could be found in [10] or the appendix.

**Proposition 2.1.** [10] Let \( m > 1 + \frac{d}{2} \). Assume that

\[
\nabla u_0 \in L^\infty(\mathbb{R}^d), \quad \nabla^2 u_0 \in H^{m-1}(\mathbb{R}^d),
\]

and there exists a constant \( \kappa > 0 \) such that for all \( x \in \mathbb{R}^d \),

\[
\text{Dist}(\text{Sp}(\nabla u_0(x)), \mathbb{R}_-) \geq \kappa,
\]

then there exists a unique global classical solution \( \hat{u} \) to the problem (1.19), which satisfies

1. \( \nabla \hat{u}(t, x) = \frac{1}{1+t} I_d + \frac{1}{(1+t)^2} K(t, x), \quad \text{for all } x \in \mathbb{R}^d, \quad t \geq 0; \)

2. \( \|\nabla^l \hat{u}(\cdot, t)\|_{L^2(\mathbb{R}^d)} \leq C_0 (1 + t)^{\frac{d}{2} - (l+1)}, \quad \text{for } 2 \leq l \leq m + 1; \)

3. \( \|\nabla^2 \hat{u}(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leq C_0 (1 + t)^{-3} \|\nabla^2 u_0\|_{L^\infty(\mathbb{R}^d)}, \)

where the matrix \( K(t, x) = K_{ij} : \mathbb{R}^+ \times \mathbb{R}^d \to M_d(\mathbb{R}^d) \) satisfies

\[
\|K\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)} \leq C_0 (1 + \kappa^{-d} \|\nabla u_0\|_{L^\infty(\mathbb{R}^d)}^{d-1}).
\]
Here, $C_0$ is a constant depending only on $m$, $d$, $\kappa$ and $u_0$, and $C_{0,t}$ are all constants depending on $C_0$ and $l$. Moreover, if $u_0(0) = 0$, then it holds that
\[
|\hat{a}(t, x)| \leq |\nabla \hat{a}|_\infty |x|, \quad \text{for any} \quad t \geq 0. \tag{2.2}
\]

Secondly, we give a global well-posedness to the Cauchy problem of an ordinary differential equation:

**Proposition 2.2.** For the constants $b, C_i$ and $D_i$ ($i = 1, 2$) satisfying
\[
a > 1, \quad D_1 - (a - 1)b < -1, \quad D_2 < -1, \quad C_i \geq 0, \quad \text{for} \quad i = 1, 2, \tag{2.3}
\]
there exists a constant $\Lambda$ such that there exists a global smooth solution to the following Cauchy problem
\[
\begin{cases}
\frac{dZ}{dt}(t) + \frac{b}{1 + t}Z(t) = C_1(1 + t)^{D_1}Z^n(t) + C_2(1 + t)^{D_2}Z, \\
Z(x, 0) = Z_0 < \Lambda.
\end{cases} \tag{2.4}
\]

Its proof can be found in the appendix.

For simplicity, we use the following notations. For matrices $A_1, A_2, A_3$, $B = (b_{ij}) = (b_1, b_2, b_3)$, a vector $N = (n_1, n_2, n_3)^\top$, set $A = (A_1, A_2, A_3)$,
\[
\begin{align*}
\text{div} A &= \sum_{j=1}^3 \partial_j A_j, \quad W \cdot BW = \sum_{i,j=1}^3 b_{ij}w_iw_j, \\
|B|^2 &= B : B = \sum_{i,j=1}^3 b_{ij}^2, \quad N \cdot B = n_1b_1 + n_2b_2 + n_3b_3.
\end{align*} \tag{2.5}
\]

### 3. Singularity formation

In order to prove Theorem 1.2, we define:
\[
m(t) = \int \rho \quad (\text{total mass}), \quad E_k(t) = \frac{1}{2} \int \rho |u|^2 \quad (\text{total kinetic energy}).
\]

Then the regular solution $(\rho, u)(t, x)$ in $[0, T] \times \mathbb{R}^3$ defined in Theorem 1.1 has finite mass $m(t)$, momentum $\mathbb{P}(t)$ and kinetic energy $E_k(t)$. Indeed, due to $1 < \gamma \leq 2$,
\[
m(t) = \int \rho \leq C \int \phi^{\frac{2}{\gamma - 1}} \leq C|\phi|_2^2 < +\infty,
\]
which, together with the regularity shown in Theorem 1.1 implies that
\[
\mathbb{P}(t) = \int \rho u \leq |\rho|_2 |u|_2 < +\infty, \quad E_k(t) = \int \frac{1}{2} |\rho |u|^2 \leq C|\rho|_\infty |u|_2^2 < +\infty. \tag{3.1}
\]

The case for $1 < \delta \leq 2$ can be verified similarly.

Next it is shown that the total mass and momentum are conserved.

**Lemma 3.1.** Let $1 < \min\{\gamma, \delta\} \leq 2$, and $(\rho, u)$ be the regular solution obtained in Theorem 1.1 with $|\mathbb{P}(0)| > 0$, then
\[
\mathbb{P}(t) = \mathbb{P}(0), \quad m(t) = m(0), \quad \text{for} \quad t \in [0, T].
\]
Proof. The momentum equations imply that

$$P_t = -\int \text{div}(\rho u \otimes u) - \int \nabla P + \int \text{div}\mathbb{T} = 0,$$

where one has used the fact that $$\rho u^{(i)}u^{(j)}$$, $$\rho^\gamma$$ and $$\rho^\delta \nabla u \in W^{1,1}(\mathbb{R}^3)$$, for $$i, j = 1, 2, 3$$, due to the regularities of the solutions.

Similarly, one can also get the conservation of the total mass.

Now we are ready to prove Theorem 1.2. It follows from Lemma 3.1 that

$$|P(0)| \leq \int \rho(t,x)|u|(t,x) \leq \sqrt{2m^2}E^\frac{1}{2}(t) = \sqrt{2m^2}(0)E^\frac{1}{2}_k(t),$$

which yields that there exists a unique positive lower bound for $$E_k(t)$$,

$$E_k(t) \geq \frac{|P(0)|^2}{2m(0)} > 0 \quad \text{for} \quad t \in [0,T].$$

Thus one gets

$$C_0 \leq E_k(t) \leq \frac{1}{2}m(0)|u(t)|^2_\infty \quad \text{for} \quad t \in [0,T].$$

Obviously, that there exists a positive constant $$C_u$$ such that

$$|u(t)|_\infty \geq C_u \quad \text{for} \quad t \in [0,T].$$

Then Theorem 1.2 follows.

Finally, we prove Theorem 1.3. It follows from the definition of regular solutions given in Theorem 1.2 that in the vacuum domain, the velocity satisfies $$u_t + u \cdot \nabla u = 0$$, which, along with the formula

$$\nabla u(t, X(t;x_0)) = (I_d + t\nabla u_0(x_0))^{-1}\nabla u_0(x_0)$$

and (1.18), yields the desired conclusion.

4. Local-in-time well-posedness with compactly supported density

The rest of this paper is devoted to proving Theorem 1.4. In this section, we first reformulate the original Cauchy problem (1.1)-(1.7) as (4.1) below in terms of some variables, and then establish the local well-posedness of the smooth solutions to (4.1) in the case that the initial density has compact support.

Let $$\tilde{u}$$ be the unique classical solution to (1.19) obtained in Proposition 2.1. In terms of the new variables

$$(\varphi, W = (\phi, w = u - \tilde{u})) = \left(\frac{\varphi}{\rho^\frac{\gamma}{2}}, \sqrt{\frac{4A\gamma}{(\gamma - 1)^2\rho^\frac{\gamma - 1}{2}}}, u - \tilde{u}\right)$$

we have...
with \( w = (w^{(1)}, w^{(2)}, w^{(3)})^\top \), the Cauchy problem (1.1)-(1.7) can be reformulated into

\[
\begin{cases}
\varphi_t + (w + \hat{\varphi}) \cdot \nabla \varphi + \frac{\delta - 1}{2} \varphi \text{div}(w + \hat{\varphi}) = 0, \\
W_t + \sum_{j=1}^{3} A_j^\ast(W, \hat{\varphi}) \partial_j W + \varphi^2 \mathbb{L}(w) = \mathbb{H}(\varphi) \cdot \mathbb{Q}(w + \hat{\varphi}) + G(W, \varphi, \hat{\varphi}), \quad j = 1, 2, 3,
\end{cases}
\]

(4.1)

where

\[
A_j^\ast(W, \hat{\varphi}) = \begin{pmatrix} w^{(j)} + \hat{\varphi}^{(j)} & \frac{\gamma - 1}{2} \phi e_j \\ \frac{\gamma - 1}{2} \phi e_j^\top & (w^{(j)} + \hat{\varphi}^{(j)})I_3 \end{pmatrix},
\]

\( G(W, \varphi, \hat{\varphi}) = -B(\nabla \hat{\varphi}, W) - D(\varphi^2, \nabla^2 \hat{\varphi}), \)

(4.2)

and \( \mathbb{L}, \mathbb{H} \) and \( \mathbb{Q} \) are given in (1.15).

The main result in this section can be stated as follows:

**Theorem 4.1.** Let (1.20) hold. If initial data \((\varphi_0, \phi_0, u_0)\) satisfy

(A1) \( \varphi_0 \geq 0, \phi_0 \geq 0 \) and \((\varphi_0, \phi_0) \in H^3; \)

(A2) \( u_0 \in \Xi \) and there exists a constant \( \kappa > 0 \) such that for all \( x \in \mathbb{R}^3; \)

\[
\text{Dist}(\text{Sp}(\nabla u_0(x)), \mathbb{R}_-) \geq \kappa;
\]

(A3) \( \varphi_0 \) and \( \phi_0 \) are both compactly supported: \( \text{supp}_x \varphi_0 = \text{supp}_x \phi_0 \subset B_R; \)

where \( B_R \) is the ball centered at the origin with radius \( R > 0 \), then there exist a time \( T_* = T_*(\alpha, \beta, A, \gamma, \delta, \varphi_0, W_0) > 0 \) independent of \( R \) and a unique classical solution \((\varphi, \phi, w)\) in \([0, T_*] \times \mathbb{R}^3\) to (4.1) satisfying

\( (\varphi, \phi) \in C([0, T_*]; H^3), \ w \in C([0, T_*]; H^{s'}) \cap L^\infty([0, T_*]; H^3), \ \varphi \nabla^4 w \in L^2([0, T_*]; L^2), \)

for any constant \( s' \in [2, 3) \). Moreover, for \( t \in [0, T_*], \)

\( \varphi(t, z(t; \xi_0)) = \phi(t, z(t; \xi_0)) = 0, \text{ and } w(t, z(t; \xi_0)) = 0 \) for \( \xi_0 \in \mathbb{R}^3/\text{supp} \varphi_0, \)

(4.3)

where the curve \( z(t; \xi_0) \) is given via

\[
\frac{d}{dt} z(t; \xi_0) = (w + \hat{\varphi})(t, z(t; \xi_0)), \quad z(0; \xi_0) = \xi_0.
\]

(4.4)

The next three subsections will be devoted to prove Theorem 4.1.
4.1. **Uniform a priori estimates for the linear problem.** In order to show the local well-posedness for (4.1), we will consider the following linearized approximate problem:

\[
\begin{cases}
\varphi_t + (v + \hat{u}^N) \cdot \nabla \varphi + \frac{\delta - 1}{2} h\text{div}(v + \hat{u}) = 0, \\
W_t + \sum_{j=1}^{3} A_j^*(V, \hat{u}^N) \partial_j W + (\varphi^2 + \eta^2) \mathbb{L}(w) = \mathbb{H}(\varphi) \cdot \mathbb{Q}(v + \hat{u}) + G(W, \varphi, \hat{u}), \\
(\varphi, W)_{t=0} = (\varphi_0, W_0) = (\varphi_0, \phi_0, 0), \quad x \in \mathbb{R}^3,
\end{cases}
\]

where

\[
A_j^*(V, \hat{u}^N) = \begin{pmatrix}
v^{(j)} + (\hat{u}^N)^{(j)} & \frac{\gamma - 1}{2} \phi e_j \\
\frac{\gamma - 1}{2} \phi e_j^\top & (v^{(j)} + (\hat{u}^N)^{(j)}) I_3
\end{pmatrix}, \quad j = 1, 2, 3,
\]

with the vector \( \hat{u}^N = \hat{u}F(|x|/N) \). \( F(x) \in C^\infty_c(\mathbb{R}^3) \) is a truncation function satisfying

\[
0 \leq F(x) \leq 1, \quad \text{and} \quad F(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\
0 & \text{if } |x| \geq 2,
\end{cases}
\]

and \( N \geq 1 \) is a sufficiently large constant. \( \eta > 0 \) is a constant, and \( V = (\psi, v)^\top \). \( (h, \psi) \) are both known functions and \( v = (v^{(1)}, v^{(2)}, v^{(3)})^\top \in \mathbb{R}^3 \) is a known vector satisfying:

\[
(h, \psi, v)(0, x) = (\varphi_0, \phi_0, 0), \quad h \in C([0, T]; H^3), \quad \psi \in C([0, T]; H^3),
\]

\[
v \in C([0, T]; H^{s'}) \cap L^\infty([0, T]; H^3), \quad h\nabla^4 v \in L^2([0, T]; L^2),
\]

for any constant \( s' \in [2, 3) \). We also assume that for \( t \in [0, T] \),

\[
\psi(t, X(t; \xi_0)) = h(t, X(t; \xi_0)) = 0 \quad \text{and} \quad v(t, X(t; \xi_0)) = 0 \quad \text{for} \quad \xi_0 \in \mathbb{R}^3 / \text{supp} \varphi_0.
\]

Now the following global well-posedness in \([0, T] \times \mathbb{R}^3 \) of a classical solution \( (\varphi^{N\eta}, W^{N\eta}) = (\varphi^{N\eta}, \phi^{N\eta}, w^{N\eta}) \) to (4.1) can be obtained by the standard theory \([4, 7]\) at least when \( 0 < \eta < +\infty \) and \( 1 \leq N < +\infty. \)

**Lemma 4.1.** Let \( \eta > 0 \) and (A1)-(A3) in Theorem 4.1 hold. Then there exists a unique classical solution \( (\varphi^{N\eta}, W^{N\eta}) \) in \([0, T] \times \mathbb{R}^3 \) to (4.1) satisfying

\[
(\varphi^{N\eta}, \phi^{N\eta}) \in C([0, T]; H^3), \quad w^{N\eta} \in C([0, T]; H^3) \cap L^2([0, T]; H^4).
\]

Next we give some a priori estimates for solutions \( (\varphi^{N\eta}, \phi^{N\eta}, w^{N\eta}) \) in \( H^3 \) in the following Lemmas 4.2-4.3 which are independent of \( (R, N, \eta) \). For simplicity, we denote \( (\varphi^{N\eta}, \phi^{N\eta}, w^{N\eta}) \) as \( (\varphi, \phi, w) \), and \( W^{N\eta} \) as \( W \) in the rest of Subsection 4.1. For this purpose, we fix a positive constant \( c_0 \) large enough such that

\[
2 + \|\varphi_0\|_3 + \|\phi_0\|_3 + \|u_0\|_\infty \leq c_0,
\]
and
\[
\sup_{0 \leq t \leq T^*} (\|h(t)\|_3^2 + \|\psi(t)\|_3^2 + \|v(t)\|_2^2) + \text{ess sup}_{0 \leq t \leq T^*} |v(t)|_{D^3}^2 + \int_0^{T^*} |h\nabla^4 v|_2^2 dt \leq c_1^2,
\]  
(4.12)
for some constant
\[c_1 \geq c_0 > 1\]
and time \(T^* \in (0, T)\), which will be determined later (see (4.23)) and depend only on \(c_0\) and the fixed constants \((\alpha, \beta, \gamma, A, \delta, T)\).

In the rest of this section, \(c \geq 1\) will denote a generic positive constant depending only on fixed constants \((\alpha, \beta, \gamma, A, \delta, T)\), but independent of \((R, N, \eta)\), which may be different from line to line. It follows from the proof of Proposition 2.1 that
\[
\|\hat{u}\|_{T; \Xi} \leq Cc_0^3.
\]  
(4.13)
Based on this fact, one can establish the following estimates for \(\varphi\).

**Lemma 4.2.** Let \((\varphi, W)\) be the unique classical solution to (4.5) in \([0, T] \times \mathbb{R}^3\). Then
\[
1 + \|\varphi(t)\|_3^2 \leq Cc_0^3 \quad \text{for} \quad 0 \leq t \leq T_1 = \min(T^*, c_1^{-5}).
\]

**Proof.** Applying \(\nabla^k\) \((0 \leq k \leq 3)\) to (4.5), multiplying both sides by \(\nabla^k \varphi\), and integrating over \(\mathbb{R}^3\), one gets
\[
\frac{1}{2} \frac{d}{dt} \|\nabla^k\varphi\|_2^2 \leq C \|\text{div}(v + \hat{u}^N)\|_\infty \|\nabla^k\varphi\|_2 + C\Lambda^k_1 \|\nabla^k\varphi\|_2 + C\Lambda^k_2 \|\nabla^k\varphi\|_2,
\]  
(4.14)
where
\[
\Lambda^k_1 = |\nabla^k\{(v + \hat{u}^N) \cdot \nabla\varphi\} - (v + \hat{u}^N) \cdot \nabla^{k+1}\varphi|_2, \quad \Lambda^k_2 = |\nabla^k(\text{hdiv}(v + \hat{u}))|_2.
\]
It follows from Lemma 10.1 and Hölder’s inequality that
\[
|\Lambda^k_1|_2 \leq C(\|\hat{u}\|_{T; \Xi} + \|v\|_3) \|\varphi(t)\|_3, \quad |\Lambda^k_2|_2 \leq C(\|\hat{u}\|_{T; \Xi} + \|v\|_3) \|h(t)\|_3 + C \|h\nabla^4 v\|_2,
\]  
(4.15)
where one has used Proposition 2.1 and the fact that for \(0 \leq t \leq T^*\),
\[
|\hat{u}(t, x)| \leq 2N|\nabla\hat{u}|_\infty \quad \text{and} \quad |\nabla F(|x|/N)| \leq CN^{-1} \quad \text{for} \quad N \leq |x| \leq 2N.
\]  
(4.16)
Then, it follows from (4.14)-(4.15), Gronwall’s inequality and (4.12) that
\[
\|\varphi(t)\|_3 \leq \left(\|\varphi_0\|_3 + c_1^5 t^c + c_1 t^2\right) \exp(Cc_1^4 t) \leq Cc_0 \quad \text{for} \quad 0 \leq t \leq T_1 = \min\{T^*, c_1^{-5}\}. \]

**Lemma 4.3.** Let \((\varphi, W)\) be the unique classical solution to (4.5) in \([0, T] \times \mathbb{R}^3\). Then
\[
\|W(t)\|_2^2 + \int_0^t |\varphi\nabla^4 u|_2^2 ds \leq Cc_0^2 \quad \text{for} \quad 0 \leq t \leq T_2 = \min\{T_1, c_1^{-10}\}.
\]

**Proof.** Applying \(\nabla^k\) to (4.5), multiplying both sides by \(\nabla^k W\), and integrating over \(\mathbb{R}^3\) by parts, we have
\[
\frac{1}{2} \frac{d}{dt} \int |\nabla^k w|^2 + \alpha |\varphi^2 + \eta^2 \nabla^{k+1} w|^2 \leq (\alpha + \beta) |\varphi^2 + \eta^2 \text{div} \nabla^k w|^2
\]

\[
= \int (\nabla^k W) \cdot \text{div} A_j^*(V, \tilde{u}^N) \nabla^k W - \int \nabla \varphi^2 \cdot \nabla^k S(w) \cdot \nabla^k w
\]

\[
- \sum_{j=1}^3 \int \left( \nabla^k (A_j^*(V, \tilde{u}^N) \partial_j W) - A_j^*(V, \tilde{u}^N) \partial_j \nabla^k W \right) \cdot \nabla^k W
\]

\[
+ \int \left( - \nabla^k ((\varphi^2 + \eta^2) L) \right) \cdot \nabla^k w + \nabla \varphi^2 \cdot Q(\nabla^k v) \cdot \nabla^k w
\]

\[
+ \int \left( \nabla^k (\nabla \varphi^2 \cdot Q(v + \tilde{u})) - \nabla^k w \right) \cdot \nabla^k w
\]

\[
+ \int \nabla \varphi^2 \cdot Q(\nabla k u) \cdot \nabla k w + \int \nabla k G(W, \varphi, \tilde{u}) \cdot \nabla k W \equiv \sum_{i=1}^8 I_i.
\]

Now consider the terms on the right-hand side of (4.17). It follows from Lemmas \[10.1\] and \[4.2\] Proposition \[2.1\] Hölder’s and Young’s inequalities, \[4.14\] and \[4.16\] that

\[I_1 = \int (\nabla^k W)^\top \text{div} A_j^*(V, \tilde{u}^N) \nabla^k W \leq C(\|\nabla V\|_\infty + \|\nabla \tilde{u}\|_\infty) \|\nabla^k W\|_2^2 \leq Cc_1^2 \|\nabla^k W\|_2^2,
\]

\[I_2 = - \int \left( \nabla \varphi^2 \cdot S(\nabla^k w) \right) \cdot \nabla^k w
\]

\[\leq C |\nabla \varphi|_\infty \|\nabla^k w\|_2 \leq \frac{\alpha}{20} \|\nabla^k w\|_2^2 + Cc_0^2 \|\nabla^k w\|_2^2,
\]

\[I_3 = - \sum_{j=1}^3 \int \left( \nabla^k (A_j^*(V, \tilde{u}^N) \partial_j W) - A_j^*(V, \tilde{u}^N) \partial_j \nabla^k W \right) \nabla^k W
\]

\[\leq C(\|\nabla V\|_\infty + \|\nabla \tilde{u}\|_\infty) \|\nabla^k W\|_2 \leq Cc_1^2 \|\nabla^k W\|_2^2 \quad \text{when } k = 1,
\]

\[I_3 \leq C((\|\nabla V\|_\infty + \|\nabla \tilde{u}\|_\infty) \|\nabla W\|_1 + (\|\nabla^2 V\|_3 + \|\nabla^2 \tilde{u}\|_3) \|\nabla W\|_0) \|\nabla^k W\|_2
\]

\[\leq Cc_1^2 \|\nabla W\|_2^2 \quad \text{when } k = 2,
\]

\[I_3 \leq C((\|\nabla V\|_\infty + \|\nabla \tilde{u}\|_\infty) \|\nabla^2 W\|_1 + (\|\nabla^3 V\|_2 + \|\nabla^3 \tilde{u}\|_2) \|\nabla W\|_\infty) \|\nabla^3 W\|_2
\]

\[+ C((\|\nabla^2 V\|_3 + \|\nabla^2 \tilde{u}\|_3) \|\nabla^2 W\|_6 \|\nabla^3 W\|_2 \leq Cc_1^2 \|\nabla W\|_2^2 \quad \text{when } k = 3
\]

\[I_4 = - \int \left( \nabla^k ((\varphi^2 + \eta^2) L) \right) \cdot \nabla^k w
\]

\[\leq C |\varphi|_\infty |\nabla^2 w|_2 \|\nabla^k w\|_2 \leq Cc_0^2 \|\nabla^k w\|_2^2 \quad \text{when } k = 1,
\]

\[I_4 \leq C(|\varphi|_\infty |\nabla^3 w|_2 + (|\nabla^2 \nabla \varphi|_3 + |\varphi^2 \nabla \varphi|_3) |\nabla^2 w|_6) \|\nabla^2 w\|_2
\]

\[\leq Cc_0^2 \|\nabla^2 w\|_1^2 \quad \text{when } k = 2,
\]

\[I_4 \leq C(|\nabla^2 \varphi|_2 |\varphi^3 w|_6 |\nabla^2 w|_3 + |\nabla \varphi|_\infty |\nabla^2 \varphi|_3 |\nabla^2 w|_6 |\nabla^3 w|_2)
\]

\[+ C(|\nabla^3 \varphi|_2 |\varphi^3 w|_6 + |\nabla \varphi|_\infty |\varphi^4 w|_2) |\nabla^3 w|_2
\]

\[\leq \frac{\alpha}{20} \|\nabla^4 w\|_2^2 + Cc_0^2 \|\nabla^2 w\|_1^2 + Cc_0^2 \quad \text{when } k = 3
\]
and

\[ I_5 = \int \left( \nabla \varphi \cdot Q(\nabla^k v) \right) \cdot \nabla^k w \]

\[ \leq C |\varphi|_\infty |\nabla \varphi|_\infty |\nabla^{k+1} v|_2 |\nabla^k w|_2 \leq Cc_0^2 c_1 |\nabla^k w|_2, \quad \text{when } k \leq 2, \]

\[ I_5 = \int \left( \nabla \varphi \cdot Q(\nabla^k v) \right) \cdot \nabla^k w \]

\[ \leq C \left( |\nabla \varphi|_\infty |\nabla^3 w|_2 + |\nabla^2 \varphi|_3 |\varphi \nabla^3 w|_6 + |\nabla \varphi|_\infty |\varphi \nabla^4 w|_2 \right) |\nabla^3 v|_2 \]

\[ \leq Cc_0^2 c_1 |\nabla^3 w|_2 + Cc_0^2 c_1^2 + \frac{\alpha}{20} |\varphi \nabla^4 w|_2^2 \quad \text{when } k = 3, \]

\[ I_6 = \int \left( \nabla^k (\nabla \varphi \cdot Q(v + \tilde{u})) - \nabla \varphi \cdot Q(\nabla^k (v + \tilde{u})) \right) \cdot \nabla^k w \]

\[ \leq C \left( \|\tilde{u}\|_{T,\Xi} + \|v\|_3 \right) \|\varphi\|_3^2 \|w\|_2 \leq Cc_0^2 \|w\|_2 \quad \text{when } k \leq 2, \]

\[ I_6 \leq C \left( \|\tilde{u}\|_{T,\Xi} + \|v\|_3 \right) \|\varphi\|_3 \left( \|\varphi\|_3 |\nabla^3 w|_2 + |\varphi \nabla^3 w|_6 \right) + I_6^* \]

\[ \leq \frac{\alpha}{20} |\varphi \nabla^4 w|_2^2 + C \left( c_0^2 \|w\|_3 + c_0^{10} \right) + I_6^* \quad \text{when } k = 3, \]

\[ I_7 = \int \left( \nabla \varphi \cdot Q(\nabla^k \tilde{u}) \right) \cdot \nabla^k w \]

\[ \leq C |\tilde{u}|_{T,\Xi} \left( \|W\|_3^2 + \|W\|_3 \|\varphi\|_3^2 \right) + I_8^* \delta_{3,k}, \]

\[ I_8 = -\frac{\gamma - 1}{2} \int \nabla^k (\varphi \text{div} \tilde{u}) \nabla^k \phi - \int \nabla^k (\varphi \cdot \nabla \tilde{u}) \nabla^k w - \int \nabla^k (\varphi^2 L \tilde{u}) \nabla^k w \]

\[ \leq C \|\tilde{u}\|_{T,\Xi} \left( \|W\|_3^2 + \|W\|_3 \|\varphi\|_3^2 \right) + I_8^* \delta_{3,k}, \]

where integrations by parts have been used for the terms $I_5$-$I_6$ and $I_8$ when $k = 3$. And the terms $I_6^*$ and $I_8^* \delta_{3,k}$ can be estimated similarly by integration by parts as

\[ I_6^* = \int \varphi \nabla^4 \varphi \cdot Q(v + \tilde{u}) \cdot \nabla^3 w \leq \frac{\alpha}{20} |\varphi \nabla^4 w|_2^2 + Cc_0^2 \|w\|_3 + Cc_1^{10}, \]

\[ I_8^* \delta_{3,k} = \int \varphi^2 L \nabla^3 \tilde{u} \cdot \nabla^3 w \leq \frac{\alpha}{20} |\varphi \nabla^4 w|_2^2 + Cc_0^2 \|w\|_3 + Cc_0^{10}. \]

Due to (4.17)-(4.19), one gets that

\[ \frac{1}{2} \frac{d}{dt} \int |\nabla^k W|^2 + \frac{1}{2} \frac{\alpha}{\sqrt{\varphi^2 + \eta^2}} |\nabla^4 w|_2^2 \leq Cc_0^2 \|W\|_3^2 + Cc_1^{10}, \quad (4.20) \]

which, along with Gronwall’s inequality, implies that for $0 \leq t \leq T_2 = \min\{T_1, c_1^{-10}\}$,

\[ \|W(t)\|_3^2 + \int_0^t (\varphi^2 + \eta^2) |\nabla^4 W|_2^2 ds \leq \left( \|W_0\|_3^2 + Cc_1^{10} t \right) \exp(Cc_0^2 t) \leq Cc_0^2. \quad (4.21) \]

Combining the estimates obtained in Lemmas 4.2,4.3 shows that

\[ 1 + \|\varphi(t)\|_3^2 + \|W(t)\|_3^2 + \sum_{k=0}^3 \int_0^t (\varphi^2 + \eta^2) |\nabla^{k+1} w|_2^2 ds \leq Cc_0^2, \quad (4.22) \]

for $0 \leq t \leq \min\{T^*, c_1^{-10}\}$. Therefore, defining the constant $c_1$ and time $T^*$ by
\[ c_1 = C^2 c_0, \quad T^* = \min\{T, c_1^{-10}\}, \quad \text{for } 0 \leq t \leq T^*, \]

we then deduce that

\[ \|\varphi(t)\|_3^2 + \|\phi(t)\|_3^2 + \|w(t)\|_3^2 + \sum_{k=0}^{3} \int_0^t (\varphi^2 + \eta^2)|\nabla^{k+1}w|^2ds \leq c_1^2. \]  

In other words, given fixed \( c_0 \) and \( T \), there exist positive constant \( c_1 \) and time \( T^* \), depending solely on \( c_0, T \) and the generic constant \( C \) such that \( (4.24) \) holds for \( h \) and \( V \), then \( (4.23) \) holds for the solution to \( (4.25) \) in \([0, T^*] \times \mathbb{R}^3\). Here, it should be noted that the definitions of \((c_1, T^*)\) are all independent of the parameters \((R, N, \eta)\).

4.2. **Passing to the limits as \( \eta \to 0 \) and \( N \to +\infty \).** Due to the a priori estimate \( (4.24) \), one can solve the following problem:

\[
\begin{align*}
\varphi_t + (v + \hat{u}) \cdot \nabla \varphi + \frac{\delta - 1}{2} h \text{div}(v + \hat{u}) &= 0, \\
W_t + \sum_{j=1}^{3} A_j^s(V, \hat{u}) \partial_j W + \varphi^2 \mathbb{L}(w) &= \mathbb{H}(\varphi) \cdot \mathbb{Q}(v + \hat{u}) + G(W, \varphi, \hat{u}), \\
(\varphi, W)|_{t=0} &= (\varphi_0, W_0) = (\varphi_0, \phi_0, 0), \quad x \in \mathbb{R}^3, \\
(\varphi, W) &= (\varphi, \phi, w) \to (0, 0, 0) \quad \text{as } \quad |x| \to \infty \quad \text{for } \quad t \geq 0.
\end{align*}
\]

**Lemma 4.4.** Assume that the initial data \((\varphi_0, \phi_0, u_0)\) satisfy conditions \((A1)-(A3)\) shown in Theorem \(4.1\). Then there exist a time \( T^* = T^*(\alpha, \beta, A, \gamma, \delta, \varphi_0, W_0) > 0 \) independent of \( R \) and a unique classical solution \((\varphi, W)\) in \([0, T^*] \times \mathbb{R}^3\) to \( (4.23) \) such that

\((\varphi, \phi) \in C([0, T^*]; H^3), \quad w \in C([0, T^*]; H^s) \cap L^\infty([0, T^*]; L^2), \quad \varphi \nabla^4 w \in L^2([0, T^*]; L^2),\)

for any constant \( s' \in [2, 3) \). Moreover, \((\varphi, W)\) satisfies \( (4.24) \), and for \( t \in [0, T^*] \),

\[
\varphi(t, y(t; \xi_0)) = \phi(t, y(t; \xi_0)) = 0 \quad \text{and} \quad w(t, y(t; \xi_0)) = 0 \quad \text{for} \quad \xi_0 \in \mathbb{R}^3/\text{supp} \varphi_0. \quad (4.26)
\]

**Proof.** **Step 1:** Passing to the limit as \( \eta \to 0 \). Let \( N \geq 1 \) be a fixed constant. Due to Lemma \(4.1\) for every \( \eta \geq 0 \), there exists a unique classical solution \((\varphi^{N\eta}, W^{N\eta})\) to the linear Cauchy problem \( (1.5) \) satisfying \( (4.24) \) in \([0, T^*] \times \mathbb{R}^3\), where these estimates and the life span \( T^* > 0 \) are all independent of \((N, R, \eta)\).

Due to the estimate \( (4.24) \) and the equations in \( (1.5) \), it holds that

\[
\|\varphi_t^{N\eta}\|_2 + \|\phi_t^{N\eta}\|_2 + \|w_t^{N\eta}\|_1 + \int_0^t \|w_t^{N\eta}\|_2^2ds \leq C_0(N), \quad \text{for } \quad 0 \leq t \leq T^*, \quad (4.27)
\]

where the constant \( C_0(N) \) depends only on the generic constant \( C \), \((\varphi_0, \phi_0, u_0)\) and \( N \), and is independent of \((R, \eta)\).

By virtue of the uniform estimate \( (4.24) \) and \( (4.27) \) independent of \((R, \eta)\), there exists a subsequence of solutions (still denoted by) \((\varphi^{N\eta}, W^{N\eta})\), which converges to a limit
$$(\varphi^N, W^N) = (\varphi^N, \phi^N, w^N)$$ in weak or weak* sense as $\eta \to 0$:

\begin{align*}
(\varphi^N, W^N) &\to (\varphi^N, W^N) \quad \text{weakly* in } L^\infty([0, T^*]; H^1(\mathbb{R}^3)), \\
(\varphi^N, \phi^N) &\to (\varphi^N, \phi^N) \quad \text{weakly* in } L^\infty([0, T^*]; H^2(\mathbb{R}^3)), \\
w^N &\to w^N \quad \text{weakly* in } L^\infty([0, T^*]; H^1(\mathbb{R}^3)), \\
w^N &\to w^N \quad \text{weakly in } L^2([0, T^*]; H^2(\mathbb{R}^3)).
\end{align*}

(4.28)

The lower semi-continuity of weak convergence implies that $(\varphi^N, W^N)$ also satisfies the corresponding estimates (4.24) and (4.27) except that of $\varphi^N \nabla^4 w^N$.

Again by the uniform estimate (4.24) and (4.27) independent of $\eta$, and the compactness in Lemma 10.8 (see [29]), it holds that for any $R_0 > 0$, there exists a subsequence of solutions (still denoted by) $(\varphi^N, W^N)$, which converges to the same limit $(\varphi^N, W^N)$ as above in the following strong sense:

$$(\varphi^N, W^N) \to (\varphi^N, W^N) \quad \text{in } C([0, T^*]; H^2(B_{R_0})),$$

as $\eta \to 0$. (4.29)

Let $\beta_i = 1, 2, 3$ for $i = 1, 2, 3, 4$. Because $\|\varphi^N \nabla^4 w^N\|_{L^2 L^\infty}$ is uniformly bounded with respect to $N$ and $\eta$, there exists a vector $g = (g_1, g_2, g_3) \in L^2([0, T^*]; L^2(\mathbb{R}^3))$ such that,

$$
\int_{\mathbb{R}^3} \int_0^{T^*} \varphi^N \partial_{\beta_1 \beta_2 \beta_3 \beta_4} w^N f \, dx \, dt \to \int_{\mathbb{R}^3} \int_0^{T^*} g f \, dx \, dt \quad \text{as } \eta \to 0
$$

for any $f \in L^2([0, T^*]; L^2(\mathbb{R}^3))$.

For any $f^* \in C_c^\infty([0, T^*] \times \mathbb{R}^3)$, it follows from integration by parts, (4.24) and (4.29) that

$$
\lim_{\eta \to 0} \int_{\mathbb{R}^3} \int_0^{T^*} \varphi^N \partial_{\beta_1 \beta_2 \beta_3 \beta_4} w^N f^* \, dx \, dt
$$

\begin{align*}
= &\lim_{\eta \to 0} \left( - \int_{\mathbb{R}^3} \int_0^{T^*} \left( \partial_{\beta_1} \varphi^N \partial_{\beta_2 \beta_3 \beta_4} w^N f^* + \varphi^N \partial_{\beta_2 \beta_3 \beta_4} w^N \partial_{\beta_1} f^* \right) \, dx \, dt \right) \\
= &- \int_{\mathbb{R}^3} \int_0^{T^*} \left( \partial_{\beta_1} \varphi^N \partial_{\beta_2 \beta_3 \beta_4} w^N f^* + \varphi^N \partial_{\beta_2 \beta_3 \beta_4} w^N \partial_{\beta_1} f^* \right) \, dx \, dt \\
= &- \int_{\mathbb{R}^3} \int_0^{T^*} \partial_{\beta_2 \beta_3 \beta_4} w^N \partial_{\beta_1} (\varphi^N f^*) \, dx \, dt = \int_{\mathbb{R}^3} \int_0^{T^*} \varphi^N \partial_{\beta_1 \beta_2 \beta_3 \beta_4} w^N f^* \, dx \, dt.
\end{align*}

(4.31)

Because $C_c^\infty([0, T^*] \times \mathbb{R}^3)$ is dense in $L^2([0, T^*]; L^2(\mathbb{R}^3))$, one can get

$$
\varphi^N \nabla^4 w^N \to \varphi^N \nabla^4 w^N \quad \text{weakly in } L^2([0, T^*] \times \mathbb{R}^3), \quad \text{as } \eta \to 0.
$$

(4.32)

Thus, $(\varphi^N, W^N)$ satisfies (4.24).

Now, it is easy to show that $(\varphi^N, W^N)$ is a weak solution in the sense of distribution to the following problem:
Thus, \( (φ, W) \) Then, (4.36)-(4.37) and a similar proof as for (4.32) show that
\[
\left\{ \begin{aligned}
φ_t^N + (v + ̄u)^N) \cdot \nabla φ^N + \frac{δ - 1}{2} \text{div}(v + ̄u) &= 0, \\
W_t^N + \sum_{j=1}^{3} A_j^N(v, ̄u^N) \partial_j W^N + (φ^N)^2 \mathbb{L}(w^N) &= \mathbb{H}(φ^N) \cdot Q(v + ̄u) + G(W^N, φ^N, ̄u),
\end{aligned} \right.
\]
and has the following regularities
\[
(φ^N, φ_t^N, w^N) \in L^∞((0, T^*]; H^3), \quad (φ_t^N, φ_t^N) \in L^∞([0, T^*]; H^2),
φ^N \nabla w^N \in L^2([0, T^*]; L^2), \quad w_t^N \in L^∞([0, T^*]; H^1) \cap L^2([0, T^*]; H^2).
\]

**Step 2:** Passing to the limit as \( N \to +\infty \). One can prove the existence, uniqueness and time continuity of the classical solutions to (4.25) as follows.

**Step 2.1:** Existence. According to Step 1, for every \( N \geq 1 \), there exists a solution \((φ^N, W^N)\) to (4.33) with the a priori estimate (4.24) independent of \((R, N)\).

Due to (4.21) and (4.33), for any \( R_0 > 0 \), it holds that
\[
\|φ^N_t\|_{H^2(B_{R_0})} + \|φ^N_t\|_{H^2(B_{R_0})} + \|w_t^N\|_{H^1(B_{R_0})} + \int_0^t \|\nabla w_t^N\|_{L^2(B_{R_0})}^2 \, ds \leq C_0(R_0),
\]
where the constant \( C_0(R_0) \) is independent of \((R, N)\).

It follows from (4.21) and (4.33) that there exists a subsequence of solutions (still denoted by) \((φ^N, W^N)\), which converges to a limit \((φ, W)\) in weak or weak* sense as \( η \to 0 \):
\[
(φ^N, W^N) \to (φ, W) \text{ weakly* in } L^∞([0, T^*]; H^3(\mathbb{R}^3)),
(φ^N_t, φ^N_t) \to (φ_t, φ_t) \text{ weakly* in } L^∞([0, T^*]; H^2(B_{R_0})),
\]
\[
\text{weakly in } L^∞([0, T^*]; H^1(B_{R_0})),
\]
\[
\text{weakly in } L^2([0, T^*]; H^2(B_{R_0})).
\]

for any \( R_0 > 0 \). The lower semi-continuity of weak convergence implies that \((φ, W)\) also satisfies the estimate (4.24) except that for \( φ \nabla^4 w \).

Again by virtue of the uniform estimates (4.24) and (4.35) independent of \( N \) and the compactness in Lemma (10.8) (see [29]), for any \( R_0 > 0 \), there exists a subsequence of solutions (still denoted by) \((φ^N, W^N)\), which converges to the same limit \((φ, W) = (φ, φ, w)\) in the following strong sense:
\[
(φ^N, W^N) \to (φ, W) \text{ in } C([0, T^*]; H^2(B_{R_0})), \text{ as } N \to +\infty.
\]

Then, (4.36)-(4.37) and a similar proof as for (4.32) show that
\[
φ^N \nabla^4 w^N \to φ \nabla^4 w \text{ weakly in } L^2([0, T^*] \times \mathbb{R}^3), \text{ as } N \to +\infty.
\]
Thus, \((φ, W)\) satisfies (4.24).
It is easy to show that \((\varphi, W)\) is a weak solution in the sense of distribution to (4.25) with the properties that
\[
(\varphi, \phi, w) \in L^\infty([0, T^*]; H^3), \quad (\varphi_t, \phi_t) \in L^\infty([0, T^*]; H^2_{loc}),
\]
\[
\varphi \nabla^4 w \in L^2([0, T^*]; L^2), \quad w_t \in L^\infty([0, T^*]; H^1_{loc}) \cap L^2([0, T^*]; H^2_{loc}).
\] (4.39)

**Step 2.2:** Time-continuity. (4.25) and (4.9) imply that
\[
\frac{d}{dt}\varphi(t, X(t; \xi_0)) = 0, \quad \frac{d}{dt}\phi(t, X(t; \xi_0)) = 0, \quad \frac{d}{dt}w(t, X(t; \xi_0)) = -w \cdot \nabla \hat{u},
\]
for \(t \in [0, T^*]\) and \(\xi_0 \in \mathbb{R}^3/\text{supp} \varphi_0\), which yields that
\[
\varphi(t, X(t; \xi_0)) = \phi(t, X(t; \xi_0)) = 0, \quad w(t, X(t; \xi_0)) = 0,
\]
for \(t \in [0, T^*]\) and \(\xi_0 \in \mathbb{R}^3/\text{supp} \varphi_0\). Thus for any positive time \(t \in [0, T^*]\), the solution \((\varphi, \phi, w)\) is compactly supported with respect to the space variable \(x\).

Let \(\xi_0 \in \partial\text{supp} \varphi_0\) and \(|\xi_0| \leq R\). It follows form Proposition 2.4 that
\[
|X(t; \xi_0)| \leq R + \|\nabla \hat{u}\|_{L^\infty L^\infty_{\infty}} \int_0^t |X(\tau; \xi_0)| d\tau.
\] (4.40)

Then the Gronwall’s inequality implies that for \(0 \leq t \leq T^*\)
\[
|X(t; \xi_0)| \leq R \exp\left(\|\nabla \hat{u}\|_{L^\infty L^\infty_{\infty}} T^*\right) \leq C_0(R, T^*),
\] (4.41)
where \(C_0(R, T^*)\) depends on the generic constant \(C, T^*, (\varphi_0, \phi_0, u_0)\) and \(R\). Then for any \(0 < R < +\infty\), it follows from (4.25) that
\[
(\varphi_t, \phi_t) \in L^\infty([0, T^*]; H^2), \quad w_t \in L^\infty([0, T^*]; H^1) \cap L^2([0, T^*]; H^2).
\] (4.42)

(4.39) and (4.42) imply that for any \(s' \in [0, 3)\),
\[
(\varphi, \phi) \in C([0, T^*]; H^{s'} \cap \text{weak-}H^3), \quad w \in C([0, T^*]; H^{s'} \cap \text{weak-}H^3).
\] (4.43)

Using the same arguments as in the proof of Lemma 10.2 and 10.3, one has
\[
\lim_{t \to 0^+} \sup \|\varphi(t)\|_3 \leq \|\varphi_0\|_3, \quad \lim_{t \to 0^+} \sup \|\phi(t)\|_3 \leq \|\phi_0\|_3,
\] (4.44)
which, together with Lemma 10.9 and (4.43), implies that \((\varphi, \phi)\) is right continuous at \(t = 0\) in \(H^3\) space. The time reversibility of equations in (4.25) for \((\phi, \varphi)\) yields that
\[
(\varphi, \phi) \in C([0, T^*]; H^3).
\] (4.45)

**Step 2.3:** Uniqueness. Let \((\varphi_i, W_i) = (\varphi_i, \phi_i, w_i), \ i = 1, 2, 2\) be two solutions obtained in Step 2.1, and
\[
\begin{align*}
\overline{\varphi} &= \varphi_1 - \varphi_2, \quad \overline{\phi} = \phi_1 - \phi_2, \\
\overline{w} &= w_1 - w_2, \quad \overline{W} = W_1 - W_2.
\end{align*}
\]
Then \(\varphi_1 = \varphi_2\) follows from \(\overline{\varphi}_t + v \cdot \nabla \overline{\varphi} = 0\), and \(W_1 = W_2\) follows from
\[
\begin{align*}
\overline{W}_t + \sum_{j=1}^3 A_j^\pm(V, \hat{u}) \partial_j \overline{W} + \varphi_1^2 \overline{L} \overline{W} = G(\overline{W}, \varphi_1, \hat{u}).
\end{align*}
\] (4.46)
4.3. **Proof of Theorem 4.1.** The proof is based on the classical iteration scheme and the existence results for the linearized problem obtained in §4.2. As in §4.2, we define constants $c_0$ and $c_1$, and assume that

$$1 + ||\varphi_0||_3 + ||\phi_0||_3 + ||u_0||_2 \leq c_0.$$ 

Let $(\varphi^0, W^0) = (\phi^0, w^0)$ with the regularities

$$(\varphi^0, \phi^0) \in C([0, T^*]; H^3), \quad w^0 \in C([0, T^*]; H^{s'}) \cap L^\infty([0, T^*]; H^3), \quad \varphi^0 \nabla^4 w^0 \in L^2([0, T^*]; L^2)$$

for any $s' \in [2, 3)$ be the solution to the following problem:

\[
\begin{aligned}
&X_t + \hat{u} \cdot \nabla X = 0 \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^3, \\
&Y_t + \hat{u} \cdot \nabla Y = 0 \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^3, \\
&\Phi_t - X^2 \Delta \Phi = 0 \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^3, \\
&(X, Y, \Phi)|_{t=0} = (\varphi_0, \phi_0, 0) \quad \text{in} \quad \mathbb{R}^3, \\
&(X, Y, \Phi) \to (0, 0, 0) \quad \text{as} \quad |x| \to +\infty, \quad t > 0.
\end{aligned}
\]

(4.47)

Choose a time $T^{**} \in (0, T^*)$ small enough such that for $0 \leq t \leq T^{**}$,

$$||\varphi^0(t)||_3^3 + ||\phi^0(t)||_3^3 + ||w^0(t)||_3^3 + \int_0^t |\varphi^0 \nabla^4 w^0| ds \leq c_1^2. \quad (4.48)$$

Now the existence, uniqueness and time continuity can be proved as follows.

**Step 1:** Existence. Let $(h, \psi, v) = (\varphi^0, \phi^0, w^0)$ and $(\varphi, W)$ be a classical solution to (4.25). Then we construct approximate solutions $(\varphi^{k+1}, W^{k+1}) = (\varphi^{k+1}, \phi^{k+1}, w^{k+1})$ inductively as follows: Assume that $(\varphi^k, W^k)$ has been defined for $k \geq 1$, and let $(\varphi^{k+1}, W^{k+1})$ be the unique solution to (4.25) with $(h, V)$ replaced by $(\varphi^k, W^k)$ as:

\[
\begin{aligned}
\varphi^{k+1}_t + (w^k + \hat{u}) \cdot \nabla \varphi^{k+1} + \frac{\delta - 1}{2} \varphi^k \nabla (w^k + \hat{u}) &= 0, \\
W^{k+1}_t + \sum_{j=1}^3 A_j^* (W^k, \hat{u}) \partial_j W^{k+1} + (\varphi^{k+1})^2 \mathbb{I}(w^{k+1}) &= \mathbb{H}(\varphi^{k+1}) \cdot \mathcal{Q}(w^k + \hat{u}) + G(W^{k+1}, \varphi^{k+1}, \hat{u}), \\
(\varphi^{k+1}, W^{k+1})|_{t=0} &= (\varphi_0, W_0), \quad x \in \mathbb{R}^3, \\
(\varphi^{k+1}, W^{k+1}) \to (0, 0) \quad \text{as} \quad |x| \to +\infty, \quad t > 0.
\end{aligned}
\]

Here the sequence $(\varphi^k, W^k)$ satisfies the uniform a priori estimate (4.24) for $0 \leq t \leq T^{**}$, and is also compactly supported as long as it exists.

Now we prove the convergence of the whole sequence $(\varphi^k, W^k)$ to a limit $(\varphi, W)$ in some strong sense. Set

$$(\varphi^{k+1} - \varphi^k, W^{k+1} = (\varphi^{k+1} - \varphi^k, W^{k+1}, W^{k+1} - W^k, W^{k+1} - W^k).$$
then it follows from (4.39) that

\[
\begin{aligned}
\frac{\varphi_{t}^{k+1} + (w^{k} + \hat{u}) \cdot \nabla \varphi^{k+1} + \overline{w}^{k} \cdot \nabla \varphi^{k} - \delta - 1}{2} (\overline{\varphi}^{k} \operatorname{div}(w^{k-1} + \hat{u}) + \varphi^{k} \operatorname{div}w^{k})
+ \sum_{j=1}^{3} A_{j}^{*}(W^{k}, \hat{u}) \partial_{j} W^{k+1} + (\varphi^{k+1})^{2} \mathcal{L}(w^{k+1})
&= - \frac{3}{2} A_{j}(\overline{W}^{k}) \partial_{j} W^{k} - \overline{\varphi}^{k+1} (\varphi^{k+1} + \varphi^{k}) \mathcal{L}(w^{k})
+ (\mathbb{H}(\varphi^{k+1}) - \mathbb{H}(\varphi^{k})) \cdot Q(w^{k} + \hat{u}) + \mathbb{H}(\varphi^{k}) \cdot Q(w^{k})
-B(\nabla \hat{u}, \overline{W}^{k+1}) - D(\overline{\varphi}^{k+1}(\varphi^{k+1} + \varphi^{k}), \nabla^{2} \hat{u}).
\end{aligned}
\]  

(4.50)

First, multiplying (4.50) by $2\varphi^{k+1}$ and integrating over $\mathbb{R}^{3}$ yield that

\[
\begin{aligned}
\frac{d}{dt} |\varphi^{k+1}|_{2}^{2} &\leq C(|\nabla w^{k}|_{\infty} + |\nabla \hat{u}|_{\infty}) |\varphi^{k+1}|_{2}^{2} + C |\varphi^{k+1}|_{2} |\nabla \varphi^{k}|_{\infty}
\quad + C |\varphi^{k+1}|_{2} (|\nabla w^{k-1}|_{\infty} + |\nabla \hat{u}|_{\infty}) |\varphi^{k}|_{2} + |\varphi^{k} \operatorname{div}w^{k}|_{2} |\varphi^{k}|_{2}^{2}
\quad \leq C \nu^{-1} |\varphi^{k+1}(t)|_{2}^{2} + C |w^{k}|_{2}^{2} + |\varphi^{k}|_{2}^{2} + |\varphi^{k} \operatorname{div}w^{k}|_{2}^{2},
\end{aligned}
\]

(4.51)

where $0 < \nu \leq \frac{1}{2}$ is a constant to be determined.

Second, multiplying (4.50) by $2\overline{W}^{k+1}$ and integrating over $\mathbb{R}^{3}$, one has

\[
\begin{aligned}
\frac{d}{dt} \int |\overline{W}^{k+1}|^{2} + 2 \alpha |\varphi^{k+1} \nabla w^{k+1}|_{2}^{2} + 2 (\alpha + \beta) |\varphi^{k+1} \operatorname{div}w^{k+1}|_{2}^{2}
&= \int (\overline{W}^{k+1})^{\top} \operatorname{div}A^{*}(W^{k}, \hat{u}) \overline{W}^{k+1} - 2 \sum_{j=1}^{3} (\overline{W}^{k+1})^{\top} A_{j}(\overline{W}^{k}) \partial_{j} W^{k}
- 2 \frac{\delta - 1}{\delta} \int \nabla (\varphi^{k+1})^{2} \cdot Q(w^{k+1}) \cdot \overline{w}^{k+1} - 2 \int (\overline{\varphi}^{k+1}(\varphi^{k+1} + \varphi^{k}) L w^{k} \cdot \overline{w}^{k+1}
\quad + 2 \int \nabla (\overline{\varphi}^{k+1}(\varphi^{k+1} + \varphi^{k})) \cdot Q(w^{k} + \hat{u}) \cdot \overline{w}^{k+1} + 2 \int \nabla (\varphi^{k})^{2} \cdot Q(w^{k}) \cdot \overline{w}^{k+1}
\quad - 2 \int B(\nabla \hat{u}, \overline{W}^{k+1}) \cdot \overline{W}^{k+1} - 2 \int (\overline{\varphi}^{k+1}(\varphi^{k+1} + \varphi^{k}) L(\hat{u}) \cdot \overline{w}^{k+1} = \sum_{i=1}^{8} J_{i},
\end{aligned}
\]

The terms $J_{1}$-$J_{8}$ above can be estimated as follows

\[
\begin{aligned}
J_{1} &= \int (\overline{W}^{k+1})^{\top} \operatorname{div}A^{*}(W^{k}, \hat{u}) \overline{W}^{k+1} \leq C |\nabla (w^{k} + \hat{u})|_{\infty} |\overline{W}^{k+1}|_{2} \leq C |\overline{W}^{k+1}|_{2},
J_{2} &= -2 \sum_{j=1}^{3} A_{j}(\overline{W}^{k}) \partial_{j} W^{k} \cdot \overline{W}^{k+1}
\leq C |\nabla W^{k}|_{\infty} |\overline{W}^{k}|_{2} |\overline{W}^{k+1}|_{2} \leq C \nu^{-1} |\overline{W}^{k+1}|_{2}^{2} + \nu |\overline{W}^{k}|_{2}^{2},
\end{aligned}
\]  

(4.53)
It follows from (4.51) and (4.56) that
\[ J_3 = -2 \delta - \frac{1}{\delta} \int \nabla (\varphi^{k+1})^2 \cdot Q(\bar{w}^{k+1}) \cdot \bar{w}^{k+1} \]
\[ \leq C|\nabla \varphi^{k+1}|_\infty |\varphi^{k+1} \nabla \bar{w}^{k+1}|_2 |\bar{w}^{k+1}|_2 \leq C|\nabla \bar{w}^{k+1}|_2^2 + \frac{\alpha}{20} |\varphi^{k+1} \nabla \bar{w}^{k+1}|_2^2, \]
\[ J_4 = -2 \int \nabla (\varphi^{k+1} + \varphi^k) L \bar{w} \cdot \bar{w}^{k+1} \]
\[ \leq C|\nabla \bar{w}^{k+1}|_2 |\varphi^{k+1} \nabla \bar{w}^{k+1}|_2 + C|\nabla \bar{w}^{k+1}| \infty |\bar{w}^{k+1}|_2 \]
\[ \leq C|\nabla \bar{w}^{k+1}|_2^2 + \frac{\alpha}{20} |\varphi^{k+1} \nabla \bar{w}^{k+1}|_2^2 + C(1 + |\varphi^k \nabla \bar{w}^{k+1}|_2^2) |\nabla \bar{w}^{k+1}|_2^2, \]
\[ J_5 = 2 \int \nabla (\varphi^{k+1} + \varphi^k) \cdot Q(u^k + \hat{u}) \cdot \bar{w}^{k+1} \]
\[ \leq C(|\nabla^2 w^k|_6 + |\nabla^2 \bar{w}|_6) |\varphi^{k+1} \nabla \bar{w}^{k+1}|_2 |\varphi^{k+1}|_2 \]
\[ + C(|\varphi^k \nabla^2 w^k|_6 + |\varphi^k \nabla^2 \bar{w}|_6) |\nabla \bar{w}^{k+1}|_2 |\varphi^{k+1}|_2 \]
\[ + C(|\nabla w^k|_\infty + |\nabla \bar{w}|_\infty) |\varphi^{k+1}|_2 |\varphi^{k+1} \nabla \bar{w}^{k+1}|_2 + J_5^*, \]
\[ J_6 = 4 \int \varphi^k \nabla \varphi^k \cdot Q(\bar{w}^k) \cdot \bar{w}^{k+1} \]
\[ \leq C|\nabla \varphi^k|_6 |\varphi^k \nabla \bar{w}^{k+1}|_2 |\nabla \bar{w}^{k+1}|_2 \leq \nu |\varphi^k \nabla \bar{w}^{k+1}|_2^2 + C \nu^{-1} |\nabla \bar{w}^{k+1}|_2^2, \]
\[ J_7 = -2 \int B(\nabla \bar{u}, \nabla \bar{w}^{k+1}) \cdot \bar{w}^{k+1} \leq C|\nabla \bar{u}|_\infty |\nabla \bar{w}^{k+1}|_2, \]
\[ J_8 = -2 \int \varphi^{k+1} (\varphi^{k+1} + \varphi^k) \nabla \bar{u} \cdot \bar{w}^{k+1} \leq C|\varphi^{k+1}|_2 |\varphi^{k+1} + \varphi^k|_\infty |\nabla \bar{u}|_\infty |\nabla \bar{w}^{k+1}|_2, \]
where one has used the inequality (10.17) for the term |\varphi^k L \bar{w}^k|_\infty, and
\[ J_5^* = -2 \sum_{i,j} \varphi^{k+1} (\varphi^{k+1} - \varphi^k + \varphi^{k+1}) Q_{ij}(u^k + \hat{u}) \partial_i (\bar{w}^{k+1})(j) \]
\[ \leq C(|\nabla w^k|_\infty + |\nabla \bar{w}|_\infty) \left( |\nabla \bar{w}^{k+1}|_\infty |\varphi^{k+1}|_2^2 + |\varphi^{k+1} \nabla \bar{w}^{k+1}|_2^2 + |\varphi^{k+1}|_2^2 \right). \]
Then it follows from (4.52)-(4.55) and Young’s inequality that
\[ \frac{d}{dt} \int |\nabla \bar{w}^{k+1}|^2 + \alpha |\varphi^{k+1} \nabla \bar{w}^{k+1}|_2^2 \]
\[ \leq C(\nu^{-1} + |\varphi^k \nabla \bar{w}^{k+1}|_2^2) |\nabla \bar{w}^{k+1}|_2^2 + C |\varphi^{k+1}|_2^2 + \nu (|\varphi^k \nabla \bar{w}^{k+1}|_2^2 + |\varphi^k|_2^2 + |\bar{w}^k|_2^2). \]
Finally, define
\[ \Gamma^{k+1}(t) = \sup_{s \in [0,t]} |\nabla \bar{w}^{k+1}(s)|_2 + \sup_{s \in [0,t]} |\varphi^{k+1}(s)|_2. \]
It follows from (4.51) and (4.56) that
\[ \frac{d}{dt} \int \left( |\nabla \bar{w}^{k+1}|^2 + |\varphi^{k+1}(t)|_2^2 \right) + \alpha |\varphi^{k+1} \nabla \bar{w}^{k+1}|_2^2 \]
\[ \leq \nu \left( |\nabla \bar{w}^{k+1}|_2^2 + |\varphi^{k+1}|_2^2 + \nu (|\varphi^k \nabla \bar{w}^{k+1}|_2^2 + |\varphi^k|_2^2 + |\bar{w}^k|_2^2) \right). \]
Thus, by the above estimate for $\Gamma$ which implies that
\[
\int_0^t E^k_\nu(s) \, ds \leq C + C_\nu t \quad \text{for} \quad 0 \leq t \leq T^{**}.
\]

Then Gronwall’s inequality implies that
\[
\Gamma^{k+1} + \int_0^t \alpha |\varphi^{k+1} \nabla w^{k+1}|^2 \, ds \\
\leq \left( \nu \int_0^t |\varphi^k \nabla w^k|^2 \, ds + t \nu \sup_{s \in [0, t]} \left( |\nabla \varphi^k|^2 + |\varphi^k|^2 \right) \right) \exp (C + C_\nu t).
\]

Choose $\nu_0 > 0$ and $T_* \in (0, \min(1, T^{**}))$ small enough such that
\[
\nu_0 \exp C = \frac{1}{8} \min \{1, \alpha\}, \quad \exp(C_{\nu_0} T_*) \leq 2,
\]

which implies that
\[
\sum_{k=1}^{\infty} \left( \Gamma^{k+1}(T_*) + \int_0^{T_*} \alpha |\varphi^{k+1} \nabla w^{k+1}|^2 \, dt \right) \leq C < +\infty.
\]

Thus, by the above estimate for $\Gamma^{k+1}(T_*)$ and (4.24), the whole sequence $(\varphi^k, W^k)$ converges to a limit $(\varphi, W) = (\varphi, \phi, w)$ in the following strong sense:
\[
(\varphi^k, W^k) \to (\varphi, W) \text{ in } L^\infty([0, T_*]; H^2(\mathbb{R}^3)). \tag{4.57}
\]

Due to (4.24) and the lower-continuity of norm for weak convergence, $(\varphi, W)$ still satisfies (4.24). Now (4.57) implies that $(\varphi, W)$ satisfies problem (4.1) in the sense of distribution. So the existence of a classical solution is proved.

Step 2: Uniqueness. Similarly to the proof of Lemma 4.1, for any $t \in [0, T_*]$, the solution $(\varphi, \phi, w)$ is still compactly supported with respect to the space variable $x$, and the size of their supports is uniformly bounded by the constant $C_0(R, T_*)$.

Let $(\varphi_1, W_1)$ and $(\varphi_2, W_2)$ be two solutions to (4.1) satisfying the uniform a priori estimate (4.24). Set
\[
\varphi = \varphi_1 - \varphi_2, \quad W = (\varphi, W) = (\phi_1 - \phi_2, w_1 - w_2).
\]

Then according to (4.50), $(\varphi, W)$ solves the following system
\[
\begin{align*}
\varphi_t + (w_1 + \hat{u}) \cdot \nabla \varphi + \bar{w} \cdot \nabla \varphi_2 + \frac{\delta - 1}{2} (\nabla \text{div}(w_1 + \hat{u}) + \varphi_2 \text{div} \bar{w}) & = 0, \\
W_t + \sum_{j=1}^{3} A_j(W_1, \hat{u}) \partial_j \varphi + \varphi_2 L(\bar{w}) & = - \sum_{j=1}^{3} A_j(W) \partial_j W_2 \\
-\varphi(\varphi_1 + \varphi_2) L(w_2) + \left( \mathbb{H}(\varphi_1) - \mathbb{H}(\varphi_2) \right) \cdot Q(W_2) & = \frac{2}{2} \mathbb{H}(\varphi_1) \cdot Q(W) - B(\nabla \hat{u}, \bar{w}) - D(\nabla \varphi_1 + \varphi_2, \nabla^2 \hat{u}).
\end{align*}
\tag{4.58}
\]

Using the same arguments as in the derivation of (4.31)-(4.36), and letting
\[
\Lambda(t) = |W(t)|^2 + |\varphi(t)|^2,
\]
one can get that
\[
\frac{d}{dt} \Lambda(t) + C|\varphi_1 \nabla w(t)|^2 \leq I(t) \Lambda(t),
\]  
(4.59)
for some \( I(t) \) such that
\[
\int_0^t I(s) ds \leq C \quad \text{for } 0 \leq t \leq T_*. 
\]
So the Gronwall’s inequality yields \( \varphi = \phi = w = 0 \). Then the uniqueness is obtained.

**Step 3.** The time-continuity can be obtained via the same arguments as in Lemma 4.4.

5. **Global-in-time well-posedness with compactly supported initial density under \((P_0)\) or \((P_1)\)**

In this section, we will consider the global-in-time well-posedness of classical solutions to the Cauchy problem (4.1) with compactly supported \((\varphi_0,\phi_0)\). To this end, we first rewrite (4.1) as
\[
\begin{align*}
\varphi_t + w \cdot \nabla \varphi + \frac{\delta - 1}{2} \varphi \text{div} w &= -\hat{\varphi} \cdot \nabla \varphi - \frac{\delta - 1}{2} \varphi \text{div} \hat{u}, \\
W_t + \sum_{j=1}^3 A_j(W) \partial_j W + \varphi^2 \mathbb{I}(w) &= \mathbb{H}(\varphi) \cdot \mathbb{Q}(w + \hat{u}) + G^*(W, \varphi, \hat{u}), \\
(\varphi, \phi, w)(t=0, x) &= (\varphi_0, W_0) = (\varphi_0, \phi_0, 0), \quad x \in \mathbb{R}^3, \\
(\varphi, \phi, w) &\to (0, 0, 0) \quad \text{as } |x| \to \infty \quad \text{for } t \geq 0,
\end{align*}
\]  
(5.1)
where
\[
G^*(W, \varphi, \hat{u}) = -B(\nabla \hat{u}, W) - \sum_{j=1}^3 \hat{u}^{(j)} \partial_j W - D(\varphi^2, \nabla^2 \hat{u}).
\]  
(5.2)

Then the main result in this section can be stated as:

**Theorem 5.1.** Let (1.20) and any one of conditions \((P_0)-(P_3)\) hold. If the initial data \((\varphi_0, \phi_0, u_0)\) satisfies \((A_1)-(A_3)\) in Theorem 4.1 and
\[
\|\phi_0\|_3 + \|\varphi_0\|_3 \leq D_0(\alpha, \beta, \delta, A, \gamma, \kappa, \|u_0\|_\Xi),
\]
where \( D_0 > 0 \) is some constant depending on \((\alpha, \beta, \delta, A, \gamma, \kappa, \|u_0\|_\Xi)\), then for any \( T > 0 \), there exists a unique global classical solution \((\varphi, \phi, u)\) in \([0, T] \times \mathbb{R}^3\) to (4.1) satisfying
\[
(\varphi, \phi) \in C([0, T]; H^3), \quad w \in C([0, T]; H^{s'} \cap L^\infty([0, T]; H^3)), \quad \varphi \nabla^4 w \in L^2([0, T]; L^2),
\]  
(5.3)
for any constant \( s' \in [2, 3) \). Moreover, when \((P_2)\) holds, the smallness assumption on \( \varphi_0 \) could be removed.

We will prove Theorem 5.1 in the following five Subsections 5.1-5.5 via establishing a uniform-in-time weighted estimates under the condition \((P_0)\) or \((P_1)\) (see Theorem 4.4) and
Remark 5.2. The proof for the cases (P2)-(P3) will be given in Section 6. Introduce a time weighed norm $Z(t)$ for classical solutions as

$$Y_k(t) = |\nabla^k W(t)|_2, \quad Y^2(t) = \sum_{k=0}^{3}(1 + t)^{2\gamma_k} Y_k^2(t),$$

$$U_k(t) = |\nabla^k \varphi(t)|_2, \quad U^2(t) = \sum_{k=0}^{3}(1 + t)^{2\delta_k} U_k^2(t),$$

$$Z^2(t) = Y^2(t) + U^2(t), \quad \gamma_k = k - n, \quad \delta_k = k - m,$$

with $(n, m)$ to be determined in Subsections 5.3-5.4. Denote also

$$Z(0) = Z_0, \quad Y(0) = Y_0, \quad U(0) = U_0.$$

Hereinafter, $C \geq 1$ will denote a generic constant depending only on fixed constants $(\alpha, \beta, \gamma, A, \delta, \kappa)$, but independent of $(\varphi_0, W_0)$, which may be different from line to line, and $C_0 > 0$ denotes a generic constant depending on $(C, \varphi_0, W_0)$. Specially, $C(l)$ (or $C_0(l)$) denotes a generic positive constant depending on $(C, l)$ (or $(C, \varphi_0, W_0, l)$).

It then follows from Lemma 5.2 that:

**Lemma 5.1.**

$$|W(t)|_\infty \leq C(1 + t)^{2n-3} Y(t), \quad \text{and} \quad |\nabla W(t)|_\infty \leq C(1 + t)^{2n-5} Y(t),$$

$$|\varphi(t)|_\infty \leq C(1 + t)^{2m-3} U(t), \quad \text{and} \quad |\nabla \varphi(t)|_\infty \leq C(1 + t)^{2m-5} U(t).$$

### 5.1. Energy estimates on $W$. First, applying $\nabla^k$ to (5.1)2, multiplying by $\nabla^k W$ and integrating over $\mathbb{R}^3$, one can get

$$\frac{1}{2} \frac{d}{dt} |\nabla^k W|_2^2 + \int \left( \alpha \varphi^2 |\nabla^{k+1} w|^2 + (\alpha + \beta) \varphi^2 |\text{div} \nabla^k w|^2 \right) dt$$

$$= R_k(W) + S_k(W, \tilde{u}) + L_k(W, \varphi, \tilde{u}) + Q_k(W, \varphi, \tilde{u}),$$

where

$$R_k(W) = - \int \nabla^k W \cdot \left( \nabla^k \left( \sum_{j=1}^{3} A^j(W) \partial_j W \right) - \sum_{j=1}^{3} A^j(W) \partial_j \nabla^k W \right)$$

$$+ \frac{1}{2} \int \sum_{j=1}^{3} \nabla^k W \cdot \partial_j A^j(W) \nabla^k W,$$

$$S_k(W, \tilde{u}) = - \int \nabla^k W \cdot \nabla^k B(\nabla \tilde{u}, W) + \frac{1}{2} \int \sum_{j=1}^{3} \partial_j \tilde{u}^{(j)} \nabla^k W \cdot \nabla^k W$$

$$- \int \nabla^k W \cdot \left( \nabla^k \left( \sum_{j=1}^{3} \tilde{u}^{(j)} \partial_j W \right) - \sum_{j=1}^{3} \tilde{u}^{(j)} \partial_j \nabla^k W \right),$$

$$L_k(W, \varphi, \tilde{u}) = - \int \left( \nabla \varphi^2 \cdot S(\nabla^k w) - (\nabla^k (\varphi^2 L w) - \varphi^2 L \nabla^k w) \right) \cdot \nabla^k w$$

$$+ \int \nabla^k (\varphi^2 L \tilde{u}) \cdot \nabla^k w \equiv L^1_k + L^2_k + L^3_k,$$
and
\[
Q_k(W, \varphi, \hat{u}) = \int \left( \nabla \varphi^2 \cdot Q(\nabla k w) + (\nabla k (\nabla \varphi^2 \cdot Q(w)) - \nabla \varphi^2 \cdot Q(\nabla k w)) \right) \cdot \nabla k w \\
+ \int \nabla k (\nabla \varphi^2 \cdot Q(\hat{u})) \cdot \nabla k w \equiv: Q_k^1 + Q_k^2 + Q_k^3.
\] (5.7)

The right hand side of (5.5) will be estimated in the next lemma.

**Lemma 5.2 (Estimates on \( R_k \) and \( S_k \)).**
\[
\left| R_k(W)(t, \cdot) \right| \leq C|\nabla W|_\infty Y^2_k, 
\] (5.8)
\[
\frac{k + r}{1 + t} Y^2_k + S_k(W, \hat{u})(t, \cdot) \leq C_0 Y_k Z(1 + t)^{-\gamma_k - 2}, 
\] (5.9)
for \( k = 0, 1, 2, 3 \), where the constant \( r \) is given by:
\[
r = -\frac{1}{2} \quad \text{if} \quad \gamma \geq \frac{5}{3}, \quad \text{or} \quad \frac{3}{2} \gamma - 3 \quad \text{if} \quad 1 < \gamma < \frac{5}{3}. 
\] (5.10)

**Proof.** Let us prove the estimates on \( R_k \). Noticing that \( R_k \) is a sum of terms as
\[
\nabla k W \cdot \nabla l W \cdot \nabla k + 1 - l W 
\] for \( 1 \leq l \leq k \),
then (5.8) is obvious when \( k = 0 \) or 1.
For \( k \neq 0, 1 \), one can apply Lemma (10.5) to \( \nabla W \) to get that
\[
|\nabla^j W|_{p_j} \leq C |\nabla W|^{1-2/p_j}_\infty |\nabla^k W|^{2/p_j}_2, \quad \text{for} \quad p_j = \frac{2k - 1}{j - 1}. 
\] (5.11)
If \( l \neq k \) and \( l \neq 1 \), since \( 1/p_l + 1/p_{k-l+1} = \frac{1}{2} \), then Hölder’s inequality implies
\[
\int |\nabla^k W \nabla^l W \nabla^{k+1-l} W| \leq |\nabla^k W|_2 |\nabla^l W|_{p_l} |\nabla^{k+1-l} W|_{p_{k-l+1}} \leq C |\nabla W|_\infty |\nabla^k W|_2.
\]
The other cases could be handled similarly. Thus (5.8) is proved.

**Step 2:** Estimates on \( S_k \). The integrand of \( S_k(W, \hat{u}) \) in (5.6) can be rewritten as
\[
s_k(W, \hat{u}) = -\nabla^k W \cdot B(\nabla \hat{u}, \nabla^k W) + \frac{1}{2} \sum_{j=1}^3 \partial_j \hat{u}^{(j)} \nabla^k W \cdot \nabla^k W \\
- \nabla^k W \cdot \left( \nabla^k (B(\nabla \hat{u}, W)) - B(\nabla \hat{u}, \nabla^k W) \right) \\
- \nabla^k W \cdot \left( \nabla^k \left( \sum_{j=1}^3 \hat{u}^{(j)} \partial_j W \right) - \sum_{j=1}^3 \hat{u}^{(j)} \partial_j \nabla^k W \right) \equiv: s_k^1 + s_k^2,
\] (5.12)
where \( s_k^1 \) is a sum of terms with a derivative of order one of \( \hat{u} \), and \( s_k^2 \) is a sum of terms with a derivative of order at least two for \( \hat{u} \).
Lemma 5.3 (Estimates on $L_k$). For any suitably small constant $\eta > 0$, there are two constants $C(\eta)$ and $C_0(\eta)$ such that

$$L_k(W) \leq \eta|\nabla^{k+1}W|_2^2 + C(\eta)(1 + t)^{2m-5-\gamma_k}Z^3Y_k + C_0(1 + t)^{2m-n-4.5-\gamma_k}Z^2Y_k + C_0(1 + t)^{2m-10}Z^2.$$  

(5.18)
Proof. Step 1: Estimates on $L_k^1$. It is easy to check that,

$$L_k^1 \leq C|\varphi|_\infty^2 |\nabla \varphi|_\infty^2 |\nabla^k \varphi|_2^2 |\nabla^k w|_2 \leq C(1 + t)^{2m-5-\gamma_k} Z^3 Y_k,$$  \hspace{1cm} (5.19)

for $k \leq 2$,\n
$$L_3^1 \leq C|\varphi|_\infty^2 |\nabla \varphi|_\infty^2 |\nabla^3 \varphi|_2 \leq \eta |\varphi|_\infty^2 |\nabla^4 w|_2 + C(\eta)(1 + t)^{2m-5-\gamma_3} Z^3 Y_3.$$  \hspace{1cm} (5.20)

Step 2: Estimates on $L_k^3$. If $k = 0$ or 1, one has

$$L_0^3 \leq C|\varphi|_\infty |\nabla^2 \tilde{u}|_\infty |\nabla^2 \varphi|_2 \leq C_0(1 + t)^{2m-n-4.5-\gamma_0} Z^2 Y_0,$$

$$L_1^3 \leq C(|\nabla \varphi|_2 |\nabla^2 \tilde{u}|_\infty + |\varphi|_\infty |\nabla^3 \tilde{u}|_2) |\varphi|_\infty |\nabla w|_2 \leq C_0(1 + t)^{2m-n-4.5-\gamma_1} Z^2 Y_1.$$  \hspace{1cm} (5.21)

For the case $k = 2$, decompose $L_2^3 \triangleq L_2^3(0, 2) + L_2^3(1, 1) + L_2^3(2, 0)$. One can get

$$L_2^3(0, 2) \triangleq \int \varphi^2 |\nabla^2 \tilde{u}| \cdot |\nabla^2 w| \leq C|\varphi|_\infty^2 |\nabla^2 \tilde{u}|_2 |\nabla^2 w|_2 \leq C_0(1 + t)^{2m-n-4.5-\gamma_2} Z^2 Y_2,$$

$$L_2^3(1, 1) \triangleq \int \nabla \varphi^2 \cdot \nabla \tilde{u} \cdot |\nabla^2 w| \leq C|\varphi|_\infty |\nabla \varphi|_\infty |\nabla \tilde{u}|_2 |\nabla^2 w|_2 \leq C_0(1 + t)^{2m-n-4.5-\gamma_2} Z^2 Y_2,$$

$$L_2^3(2, 0) \triangleq \int \nabla^2 \varphi^2 \cdot \nabla \tilde{u} \cdot |\nabla^2 w| \leq C(|\nabla \varphi|_\infty^2 |\nabla \tilde{u}|_2 + |\varphi|_\infty |\nabla \varphi|_\infty |\nabla \tilde{u}|_2) |\nabla^2 w|_2 \leq C_0(1 + t)^{2m-n-4.5-\gamma_2} Z^2 Y_2.$$  \hspace{1cm} (5.22)

For the case $k = 3$, decompose $L_3^3 \triangleq L_3^3(0, 3) + L_3^3(1, 2) + L_3^3(2, 1) + L_3^3(3, 0)$. Then, by integration by parts, one can obtain

$$L_3^3(0, 3) \triangleq \int \varphi^2 \cdot \nabla^3 \tilde{u} \cdot |\nabla^3 w| \leq C|\varphi|_\infty |\nabla^3 \tilde{u}|_2 |\nabla \varphi|_\infty |\nabla^4 w|_2 + C_0(1 + t)^{2m-n-4.5-\gamma_3} Z^2 Y_3,$$

$$L_3^3(1, 2) \triangleq \int \nabla \varphi^2 \cdot \nabla^2 \tilde{u} \cdot |\nabla^3 w| \leq C|\varphi|_\infty |\nabla \varphi|_\infty |\nabla^2 \tilde{u}|_2 |\nabla^3 w|_2 \leq C_0(1 + t)^{2m-n-4.5-\gamma_3} Z^2 Y_3,$$

$$L_3^3(2, 1) \triangleq \int \nabla \varphi^2 \cdot \nabla \tilde{u} \cdot |\nabla^3 w| \leq C(|\nabla \varphi|_\infty^2 |\nabla^3 \tilde{u}|_2 + |\varphi|_\infty |\nabla \varphi|_\infty |\nabla^2 \tilde{u}|_2) |\nabla^3 w|_2 \leq C_0(1 + t)^{2m-n-4.5-\gamma_3} Z^2 Y_3,$$

$$L_3^3(3, 0) \triangleq \int \nabla \varphi^2 \cdot \tilde{u} \cdot |\nabla^3 w| \leq C(|\nabla \varphi|_\infty |\nabla^3 \varphi|_2 + |\nabla \varphi|_\infty |\nabla^2 \varphi|_2) |\nabla \tilde{u}|_\infty |\nabla^3 w|_2 \leq C_0(1 + t)^{2m-n-4.5-\gamma_3} Z^2 Y_3,$$

with $\eta > 0$ being any sufficiently small constant. It follows from (5.20)-(5.22) that

$$L_k^3 \leq \eta |\varphi|_\infty |\nabla^{k+1} w|_2 \delta_{3k} + C_0(1 + t)^{2m-n-4.5-\gamma_3} Z^2 Y_k + C_0(\eta)(1 + t)^{2m-10} Z^2.$$  \hspace{1cm} (5.23)
Step 3: Estimates on $L_k^2$. If $k = 1$, one gets

$$L_1^2 = \int \nabla \varphi^2 \cdot Lw \cdot \nabla w \leq C|\varphi|_{\infty} |\nabla \varphi|_{\infty} |\nabla^2 w|_2 |\nabla w|_2 \leq C(1 + t)^{2m-\gamma_3} Z^3 Y_1.$$  \hspace{1cm} (5.24)

Next for $k = 2$, decompose $L_2^2 \triangleq L_2^2(1, 1) + L_2^2(2, 0)$. In a similar way for $L_2^1$, one has

$$L_2^2(1, 1) \triangleq \int \nabla \varphi^2 \cdot \nabla Lw \cdot \nabla^2 w \leq C(1 + t)^{2m-5-\gamma_2} Z^3 Y_2,$$

$$L_2^2(2, 0) \triangleq \int \nabla^2 \varphi^2 \cdot Lw \cdot \nabla^2 w \leq C(|\nabla \varphi|_{\infty} |\nabla^2 w|_2 + |\varphi|_6 |\nabla^2 w|_6) |\nabla^2 w|_2 \leq C(1 + t)^{2m-5-\gamma_2} Z^3 Y_2.$$ \hspace{1cm} (5.25)

At last for $k = 3$, decompose $L_3^2 \triangleq L_3^2(1, 2) + L_3^2(2, 1) + L_3^2(3, 0)$. In a similar way for $L_3^1$, one can get

$$L_3^2(1, 2) \triangleq \int \nabla \varphi^2 \cdot \nabla^2 Lw \cdot \nabla^3 w \leq \eta|\varphi \nabla^4 w|_2 + C(\eta)(1 + t)^{2m-5-\gamma_3} Z^3 Y_3,$$

$$L_3^2(2, 1) \triangleq \int \nabla^2 \varphi^2 \cdot \nabla Lw \cdot \nabla^3 w \leq C(|\nabla \varphi|_{\infty} |\nabla^3 w|_2 + |\nabla^2 \varphi|_3 |\varphi \nabla Lw|_6) |\nabla^3 w|_2 \leq \eta|\varphi \nabla^4 w|_2 + C(\eta)(1 + t)^{2m-5-\gamma_3} Z^3 Y_3.$$ \hspace{1cm} (5.26)

Then combining the estimates \textit{[5.24]-[5.26]} yields

$$L_k^2 \leq \eta|\varphi \nabla^{k+1} w|_2^2 \delta_{3k} + C(\eta)(1 + t)^{2m-5-\gamma_k} Z^3 Y_k.$$ \hspace{1cm} (5.27)

Thus \textit{[5.18]} follows from above three steps.

\[\square\]

Lemma 5.4 (Estimates on $Q_k$). For any suitably small constant $\eta > 0$, there are two constants $C(\eta)$ and $C_0(\eta)$ such that

$$Q_k(W, \tilde{u}) \leq \eta|\varphi \nabla^{k+1} w|_2^2 \delta_{3k} + C(\eta)(1 + t)^{2m-5-\gamma_k} Z^3 Y_k + C_0(\eta)(1 + t)^{2m-10} Z^2 \leq \frac{(4\alpha + 6\beta)\delta}{(\delta - 1)(1 + t)} |\nabla^3 \varphi|_2 |\varphi \nabla^3 \text{div} w|_2.$$ \hspace{1cm} (5.28)

Proof. Step 1: Estimates on $Q_k^1$. In a similar way for $L_k^1$, it is easy to get

$$Q_k^1 \leq \eta|\varphi \nabla^{k+1} w|_2^2 \delta_{3k} + C(1 + t)^{2m-5-\gamma_k} Z^3 Y_k.$$ \hspace{1cm}

Step 2: Estimates on $Q_k^3$. If $k = 0$, one has

$$Q_0^3 \leq C|\varphi|_{\infty} |\nabla \tilde{u}|_6 |\nabla \varphi|_3 |w|_2 \leq C_0(1 + t)^{2m-n-4.5-\gamma_0} Z^2 Y_0.$$ \hspace{1cm} (5.29)
For $k = 1$, in a similar way for $L^3_1$, one obtains
\[
Q^3_1 \leq C \left( |\varphi|_\infty |\nabla^2 \varphi|_3 |\nabla \tilde{u}|_6 + |\nabla \varphi|_6^2 |\nabla \tilde{u}|_6 + |\varphi|_\infty |\nabla \varphi|_2 |\nabla^2 \tilde{u}|_\infty \right) |\nabla w|_2 \\
\leq C_0 (1 + t)^{2m-n-4.5-\gamma} Z^2 Y_1.
\]

Next for $k = 2$, decompose $Q^3_2 \triangleq Q^3_2(0,2) + Q^3_2(1,1) + Q^3_2(2,0)$. Then as for $L^3_2(1,1)$ and $L^3_2(2,0)$, one has
\[
\begin{align*}
Q^3_2(0,2) &\triangleq C \int \nabla \varphi^2 \cdot \nabla^3 \tilde{u} \cdot \nabla^2 w \leq C_0 (1 + t)^{2m-n-4.5-\gamma} Z^2 Y_2, \\
Q^3_2(1,1) &\triangleq C \int \nabla^2 \varphi^2 \cdot \nabla^2 \tilde{u} \cdot \nabla^2 w \leq C_0 (1 + t)^{2m-n-4.5-\gamma} Z^2 Y_2, \\
Q^3_2(2,0) &\triangleq C \int \nabla^3 \varphi^2 \cdot \nabla \tilde{u} \cdot \nabla^2 w \leq C \left( |\nabla^3 \varphi|_2 |\varphi|_\infty + |\nabla^2 \varphi|_6 |\nabla \varphi|_3 \right) |\nabla \tilde{u}|_\infty |\nabla^2 w|_2 \\
&\leq C_0 (1 + t)^{2m-n-3.5-\gamma} Z^2 Y_2.
\end{align*}
\]

For $k = 3$, let $Q^3_3 \triangleq Q^3_3(0,3) + Q^3_3(1,2) + Q^3_3(2,1) + Q^3_3(3,0)$. Then as for $L^3_3(1,2)$, $L^3_3(2,1)$ and $L^3_3(3,0)$, one can get
\[
\sum_{i=0}^{2} Q^3_3(i, 3-i) \triangleq C \int \nabla \varphi^2 \cdot \nabla^4 \tilde{u} + C \int \nabla^2 \varphi \cdot \nabla^3 \tilde{u} \cdot \nabla^3 w \\
+ C \int \nabla^3 \varphi^2 \cdot \nabla^2 \tilde{u} \cdot \nabla^3 w \leq C_0 (1 + t)^{2m-n-4.5-\gamma} Z^2 Y_3.
\]

Finally, it remains to handle the term $Q^3_3(3,0)$ defined below, for which some additional information is needed. Using integration by parts and Proposition 2.1, one can get
\[
\begin{align*}
Q^3_3(3,0) &\triangleq \frac{\delta}{\delta - 1} \sum_{i,j=1}^{3} \int \left( \alpha \nabla^3 \varphi \partial_i \tilde{u}^{(i)} + \partial_j \tilde{u}^{(i)} \right) \cdot \nabla^3 w^{(i)} \\
= Q^3_3(A) - \frac{\delta}{\delta - 1} \sum_{i,j=1}^{3} \int \alpha \nabla^3 \varphi \partial_i \tilde{u}^{(i)} + \partial_j \tilde{u}^{(i)} \cdot \nabla^3 \partial_j w^{(i)} \\
= Q^3_3(A) + Q^3_3(B) + Q^3_3(D) - \frac{\delta}{(\delta - 1)(1 + t)} \int (4\alpha + 6\beta) \varphi \nabla^3 \varphi \cdot \nabla^3 \text{div} w \\
\leq Q^3_3(A) + Q^3_3(B) + Q^3_3(D) + \frac{(4\alpha + 6\beta)\delta}{(\delta - 1)(1 + t)} |\nabla^3 \varphi|_2 |\varphi \nabla^3 \text{div} w|_2,
\end{align*}
\]

where
\[
\begin{align*}
Q^3_3(A) &= C \int \nabla^3 \varphi^2 \cdot \nabla^2 \tilde{u} \cdot \nabla^3 w = C \int \left( \varphi \nabla^3 \varphi + \nabla \varphi \cdot \nabla^2 \varphi \right) \cdot \nabla^2 \tilde{u} \cdot \nabla^3 w \\
&\leq C \left( |\varphi|_\infty |\nabla^3 \varphi|_2 + |\nabla \varphi|_\infty |\nabla^2 \varphi|_2 \right) |\nabla^2 \tilde{u}|_\infty |\nabla^3 w|_2 \\
&\leq C_0 (1 + t)^{2m-n-4.5-\gamma} Z^2 Y_3.
\end{align*}
\]
and

\[ Q_3^3(B) = C \int \nabla \varphi \cdot \nabla^2 \varphi \cdot \nabla \hat{u} \cdot \nabla^4 w \]

\[ = C \int \left( \nabla \varphi \cdot \nabla^3 \varphi + \nabla^2 \varphi \cdot \nabla^2 \varphi \right) \cdot \nabla \hat{u} + \nabla \varphi \cdot \nabla^2 \varphi \cdot \nabla^2 \hat{u} \cdot \nabla^3 w \]

\[ \leq C \left( |\nabla \varphi|_\infty |\nabla^3 \varphi|_2 |\nabla \hat{u}|_\infty + |\nabla^2 \varphi|_6 \left( |\nabla^2 \varphi|_6 |\nabla \hat{u}|_6 + |\nabla \varphi|_3 |\nabla^2 \hat{u}|_\infty \right) \right) |\nabla^3 w|_2 \]

\[ \leq C_0 (1 + t)^{2m-n-3.5-\gamma_3} Z^2 Y_3, \]

\[ Q_3^3(D) = - \frac{\delta}{(\delta - 1)(1 + t)^2} \sum_{i,j=1}^3 \int 2\alpha \varphi \nabla^3 \varphi (K_{ij} + K_{ji}) \cdot \nabla^3 \partial_j w^{(i)} \]

\[ - \frac{\delta}{(\delta - 1)(1 + t)^2} \sum_{i=1}^3 \int 2\beta \varphi \nabla^3 \varphi K_{ii} \cdot \nabla^3 \text{div} w \]

\[ \leq \eta |\varphi \nabla^4 w|_2^2 + C_0 (\eta)(1 + t)^{2m-10} Z^2. \]

Then combining the estimates (5.20) and (5.33) yields

\[ Q_3^3 \leq \eta |\varphi \nabla^{k+1} w|_{2d3k}^2 + C_0 (1 + t)^{2m-n-3.5-\gamma_k} Z^2 Y_k \]

\[ + C_0 (\eta)(1 + t)^{2m-10} Z^2 + \left( \frac{4\alpha + 6\beta}{(\delta - 1)(1 + t)} \right) |\nabla^3 \varphi|_2 |\varphi \nabla^3 \text{div} w|_2. \]  

**Step 3:** Estimates on \( Q_2^3 \). For \( k = 1 \), direct estimates give

\[ Q_1^3 = C \int (\varphi \nabla^2 \varphi + \nabla \varphi \cdot \nabla \varphi) \cdot \nabla w \cdot \nabla w \]

\[ \leq C \left( |\varphi|_3 |\nabla^2 \varphi|_6 |\nabla \varphi|_\infty + |\nabla^2 \varphi|_\infty |\nabla \varphi|_2 \right) |\nabla \varphi|_2 \leq C (1 + t)^{2m-5-\gamma_1} Z^3 Y_1. \]  

For \( k = 2 \), in a similar way for \( L_2^3(2,0) \), one gets easily

\[ Q_2^3 = C \int (\nabla^3 \varphi \cdot \nabla \varphi \cdot \nabla^2 w \cdot \nabla w) \cdot \nabla w \]

\[ \leq C \left( |\varphi|_\infty |\nabla^3 \varphi|_2 |\nabla \varphi|_\infty + |\nabla w|_\infty |\nabla \varphi|_6 |\nabla^2 \varphi|_3 \right) |\nabla^2 w|_2 \]

\[ + C (1 + t)^{2m-5-\gamma_2} Z^3 Y_2 \leq C (1 + t)^{2m-5-\gamma_2} Z^3 Y_2. \]  

For \( k = 3 \), as for \( L_3^3(3,0) \) and \( L_3^3(2,1) \), it follows from integration by parts that

\[ Q_3^3 = C \int (\nabla^4 \varphi \cdot \nabla w + \nabla^3 \varphi \cdot \nabla^2 w + \nabla^2 \varphi \cdot \nabla^3 w) \cdot \nabla^3 w \]

\[ \leq \eta |\varphi \nabla^4 w|_2^2 + C(\eta)(1 + t)^{2m-5-\gamma_3} Z^3 Y_3 + Q_3^3(A), \]  

with

\[ Q_3^3(A) \triangleq C \int \nabla^3 \varphi \cdot \nabla w \cdot \nabla^4 w = C \int (\varphi \nabla^3 \varphi + \nabla \varphi \cdot \nabla^2 \varphi) \cdot \nabla w \cdot \nabla^4 w \]

\[ \leq C |\nabla^3 \varphi|_2 |\nabla w|_\infty |\varphi \nabla^2 w|_2 + Q_3^3(B) \]

\[ \leq \eta |\varphi \nabla^4 w|_2^2 + C(\eta)(1 + t)^{2m-5-\gamma_3} Z^3 Y_3 + Q_3^3(B), \]
Lemma 5.5.

Then collecting the estimates (5.36)-(5.40) shows that

Lemma 5.6.

Estimates on Step 1.

Proof.

where the term $Q_3^2(B)$ can be estimated by using integration by parts again,

\[ Q_3^2(B) \triangleq C \int \nabla \varphi \cdot \nabla^2 \varphi \cdot \nabla w \cdot \nabla^4 w \]

\[ = C \int \left( \nabla^2 \varphi \cdot \nabla^2 \varphi \cdot \nabla w + \nabla \varphi \cdot \nabla^3 \varphi \cdot \nabla w + \nabla \varphi \cdot \nabla^2 \varphi \cdot \nabla^2 w \right) \cdot \nabla^3 w \]

\[ \leq C \left( |\nabla^2 \varphi|_6^2 |\nabla w|_6 + |\nabla^3 \varphi|_2 |\nabla \varphi|_\infty |\nabla w|_\infty + |\nabla \varphi|_6 |\nabla^2 \varphi|_6 |\nabla^2 w|_6 \right) |\nabla^3 w|_2 \]

\[ \leq C(1 + t)^{2m-5-\gamma k} Z^3 Y_k. \]

Then collecting the estimates \(5.36)-(5.40)\) shows that

\[ Q_3^2 \leq \eta |\varphi \nabla^{k+1} w|_2^2 \delta_{3k} + C(\eta)(1 + t)^{2m-5-\gamma k} Z^3 Y_k. \]  

Then \(5.28\) follows directly from above estimates.

\(\square\)

It follows from \(5.34\) and Lemmas \(5.2, 5.4\) that

Lemma 5.5.

\[ \frac{1}{2} \frac{d}{dt} Y_k^2 + \frac{k + r}{1 + t} Y_k^2 + \alpha \int |\varphi \nabla^{k+1} w|^2 + (\alpha + \beta) \int |\varphi \nabla^k \text{div} w|^2 \]

\[ \leq C(\eta)(1 + t)^{2m-5-\gamma k} Z^2 Y_k + C_0 \left( (1 + t)^{2m-n-3.5-\gamma k} + (1 + t)^{n-2.5-\gamma k} \right) Z^2 Y_k \]

\[ + C_0(\eta) \left( (1 + t)^{2m-10} Z^2 + (1 + t)^{-2\gamma k} ZY_k \right) \]

\[ + \eta |\varphi \nabla^{k+1} w|_2^2 \delta_{3k} + \frac{(4\alpha + 6\beta)\delta}{(\delta - 1)(1 + t)} |\nabla^3 \varphi|_2 |\varphi \nabla^k \text{div} w|_2 \delta_{3k}. \]

5.2. Estimates on $\varphi$. $\varphi$ is estimated in the following lemma.

Lemma 5.6.

\[ \frac{1}{2} \frac{d}{dt} |\nabla^k \varphi|^2 + \frac{3 \delta - 3 + k}{1 + t} |\nabla^k \varphi|^2 \]

\[ \leq C(1 + t)^{n-2.5-\delta k} Z^2 U_k + C_0(1 + t)^{-2-\delta k} ZU_k + \frac{\delta - 1}{2} |\varphi \nabla^3 \text{div} w|_2 |\nabla^3 \varphi|_2 \delta_{3k}. \]  

Proof. Step 1. Applying $\nabla^k$ to \(5.1\), multiplying by $\nabla^k \varphi$ and integrating over $\mathbb{R}^3$, one gets

\[ \frac{1}{2} \frac{d}{dt} |\nabla^k \varphi|^2 = \int \left( w \cdot \nabla^{k+1} \varphi + \frac{\delta - 1}{2} \nabla^k \varphi \text{div} w \right) \cdot \nabla^k \varphi + S_k^*(\varphi, \tilde{u}) - \Lambda_1^k - \Lambda_2^k, \]  

where $S_k^*(\varphi, \tilde{u})$ is given in Step 2 below, and

\[ \Lambda_1^k = \left( \nabla^k (w \cdot \nabla \varphi) - w \cdot \nabla^{k+1} \varphi \right) \cdot \nabla^k \varphi, \quad \Lambda_2^k = \frac{\delta - 1}{2} \left( \nabla^k (\varphi \text{div} w) - \nabla^k \varphi \text{div} w \right) \cdot \nabla^k \varphi. \]

Integration by parts yields immediately that

\[ \int \left( w \cdot \nabla^{k+1} \varphi + \frac{\delta - 1}{2} \nabla^k \varphi \text{div} w \right) \cdot \nabla^k \varphi \leq C |\nabla w|_\infty |\nabla^k \varphi(t)|_2. \]  

(5.45)
Step 2: Estimates on \( S_k^* = \int s_k^* \) with the integrand defined as

\[
s_k^*(\varphi, \tilde{u}) = -\frac{\delta - 1}{2} \nabla^k \varphi \cdot \nabla^k \varphi \text{div} \tilde{u} + \frac{1}{2} \text{div} \tilde{u} \nabla^k \varphi \cdot \nabla^k \varphi
- \frac{\delta - 1}{2} \nabla^k \varphi \cdot \left( \nabla^k (\varphi \text{div} \tilde{u}) - \text{div} \tilde{u} \nabla^k \varphi \right)
- \nabla^k \varphi \cdot \left( \nabla^k (\tilde{u} \cdot \nabla \varphi) - \tilde{u} \cdot \nabla^{k+1} \varphi \right) \equiv s_k^{1^*} + s_k^{2^*},
\]

where \( s_k^{1^*} \) is a sum of terms with a derivative of order one of \( \tilde{u} \), and \( s_k^{2^*} \) is a sum of terms with a derivative of order at least two of \( \tilde{u} \).

Step 2.1: Estimates on \( S_k^{1*} = \int s_k^{1*} \). Let \( \nabla^k = \partial_{\beta_1...\beta_k} \) with \( 0 \leq \beta_i = 1, 2, 3 \).

\[
S_k^{1*} = \int \left( -\frac{\delta - 1}{2} \nabla^k \varphi \cdot \nabla^k \varphi \text{div} \tilde{u} + \frac{1}{2} \text{div} \tilde{u} \nabla^k \varphi \cdot \nabla^k \varphi \right)
- \int \partial_{\beta_1...\beta_k} \varphi \cdot \sum_{i=1}^{k} \sum_{j=1}^{k} \partial_{\beta_i} \tilde{u} (\beta) \partial_j \partial_{\beta_1...\beta_{i-1}\beta_{i+1}...\beta_k} \varphi = J_1 + J_2 + J_3,
\]

where, by Proposition 2.1, \( J_1 - J_3 \) are estimated by

\[
J_1 = -\frac{3(\delta - 1)}{2(1 + t)} U_k^2 + N_1, \quad J_2 = \frac{3}{21 + t} U_k^2 + N_2, \quad J_3 = \frac{k}{1 + t} U_k^2 + N_3, \quad |N_j| \leq \frac{C_0}{(1 + t)^2} U_k^2, \quad j = 1, 2, 3.
\]

Therefore, it holds that

\[
S_k^{1*}(W, \tilde{u})(t, \cdot) \text{d}x \leq \frac{C_0}{(1 + t)^2} U_k^2 - \frac{3\delta - 3 + k}{1 + t} U_k^2.
\]

Step 2.2: Estimates on \( S_k^{2*} = \int s_k^{2*} \). \( s_k^{2*} \) is a sum of the terms defined as

\[
E_1(\varphi) = \nabla^k \varphi \cdot \nabla^l \tilde{u} \cdot \nabla^{k+1-l} \varphi \quad \text{for} \quad 2 \leq k \leq 3 \quad \text{and} \quad 2 \leq l \leq k;
E_2(\varphi) = \nabla^k \varphi \cdot \nabla^{l+1} \tilde{u} \cdot \nabla^{k-l} \varphi \quad \text{for} \quad 1 \leq k \leq 3 \quad \text{and} \quad 1 \leq l \leq k.
\]

Then it holds that

\[
S_k^{2*}(\varphi) \leq C |\varphi|_6 |\nabla^2 \tilde{u}|_2 |\nabla \varphi|_3 \leq C_0 (1 + t)^{-2-\delta_1} ZU_1,
S_k^{2*}(\varphi) \leq C (|\nabla^2 \tilde{u}|_\infty |\nabla \varphi|_2 + |\nabla^3 \tilde{u}|_2 |\nabla \varphi|_3) |\nabla^2 \varphi|_2 \leq C_0 (1 + t)^{-2-\delta_2} ZU_2,
S_k^{2*}(\varphi) \leq C (|\nabla^2 \tilde{u}|_\infty |\nabla^2 \varphi|_2 + |\nabla^3 \tilde{u}|_6 |\nabla \varphi|_3 + |\nabla^4 \tilde{u}|_2 |\nabla \varphi|_3) |\nabla^3 \varphi|_2 \leq C_0 (1 + t)^{-2-\delta_3} ZU_3,
\]

which implies immediately that

\[
S_k^{2*}(\varphi) \leq C_0 (1 + t)^{-2-\delta_k} ZU_k, \quad \text{for} \quad k = 1, 2, 3.
\]
**Global Well-Posedness of the Compressible Navier-Stokes Equations**

**Step 3:** Estimates on $\Lambda_1^k + \Lambda_2^k$. It follows from Lemma [5.2](#) and Hölder’s inequality that

\[
\begin{align*}
\Lambda_1^1 + \Lambda_2^1 & \leq C|\nabla w_\infty| |\nabla \varphi(t)|^2 \leq C(1 + t)^{n-2.5-\delta_2} Z^2 U_1, \\
\Lambda_1^2 + \Lambda_2^2 & \leq C(|\nabla \varphi|_\infty|\nabla^2 w(t)|_2 + |\nabla w_\infty| |\nabla^2 \varphi(t)|_2)|\nabla^2 \varphi(t)|_2 \\
& \leq C(1 + t)^{n-2.5-\delta_2} Z^2 U_2, \\
\Lambda_1^3 & \leq C(|\nabla \varphi|_\infty|\nabla^3 w(t)|_2 + |\nabla^2 \varphi|_6|\nabla^2 w(t)|_3 + |\nabla w_\infty| |\nabla^3 \varphi(t)|_2)|\nabla^3 \varphi(t)|_2 \\
& \leq C(1 + t)^{n-2.5-\delta_1} Z^2 U_3, \\
\Lambda_2^3 & \leq C \sum_{k=0}^\infty (1 + t)^{2\gamma k} |\nabla^k \varphi \nabla^4 w(t)|_2 + (\alpha + \beta) \sum_{k=0}^\infty (1 + t)^{2\gamma k} |\varphi \nabla^k \text{div} \varphi w(t)|_2 \\
& \leq C(1 + t)^{n-2.5} Z^3 + C_0(1 + t)^{2m-n-3.5} Z^3 \\
& \quad + C_0(1 + t)^{n-2.5} Z^3 + C_0(1 + t)^{2m-n-4} + (1 + t)^{-2} Z^2,
\end{align*}
\]

(5.51)

Then (5.44), (5.51) yield the desired (5.53).

Finally, set

\[
H(A^*, A, B) = \alpha(A^*)^2 + (\alpha + \beta) A^2 + \frac{3\delta - 3 + m}{1 + t} B^2 \\
- \left( \frac{\delta - 1}{2} (1 + t)^{n-m} + \frac{2\delta}{\delta - 1} (2\alpha + 3\beta)(1 + t)^{m-n-1} \right) AB,
\]

(5.52)

\[
A^* = (1 + t)^\gamma |\varphi \nabla^4 w|_2, \quad A = (1 + t)^\gamma |\varphi \nabla^3 \text{div} \varphi w|_2, \quad B = (1 + t)^\delta |\nabla^3 \varphi|_2,
\]

then the following lemma holds:

**Lemma 5.7.** There exist some positive constants $b_m$, such that

\[
\frac{1}{2} \frac{d}{dt} Z^2 + \frac{b_m}{1 + t} (Z^2 - B^2) + H(A^*, A, B) \\
+ \alpha \sum_{k=0}^\infty (1 + t)^{2\gamma k} |\varphi \nabla^{k+1} w(t)|^2_2 + (\alpha + \beta) \sum_{k=0}^\infty (1 + t)^{2\gamma k} |\varphi \nabla^k \text{div} \varphi w(t)|^2_2 \\
\leq \eta (1 + t)^{2\gamma} |\varphi \nabla^4 w|_2^2 + C(\eta)(1 + t)^{2m-5} Z^4 + C_0(1 + t)^{2m-n-3.5} Z^3 \\
+ C_0(1 + t)^{n-2.5} Z^3 + C_0(1 + t)^{2m-n-4} + (1 + t)^{-2} Z^2,
\]

(5.53)

with the constant $b_m$ given by

\[
b_m = \begin{cases} 
\min \left\{ n - 0.5, \frac{3\delta}{2} - 3 + m \right\} & \text{if } \gamma \geq \frac{\delta}{3}, \\
\min \left\{ \frac{3\gamma}{2} - 3 + n, \frac{3\delta}{2} - 3 + m \right\} & \text{if } 1 < \gamma < \frac{\delta}{3},
\end{cases}
\]

(5.54)

where $\eta > 0$ is any suitably small constant.

**Proof.** (5.53) can be obtained by multiplying (5.42) and (5.43) by $(1 + t)^{2\gamma k}$ and $(1 + t)^{2\delta k}$ respectively and summing the resulting inequalities together.
5.3. Proof of Theorem 5.1 under the condition \((P_0)\). Set
\[
M_1 = \frac{2\alpha + 3\beta}{2\alpha + \beta}, \quad M_2 = -3\delta + 1 + \frac{1}{2}M_3,
\]
\[
M_3 = \frac{(\delta - 1)^2}{4(2\alpha + \beta)} + \frac{4\delta^2(2\alpha + \beta)}{(\delta - 1)^2}M_1^2 + 2M_1\delta,
\]
\[
M_4 = \frac{1}{2} \min \left\{ \frac{3\gamma - 3}{2}, \frac{-M_2 - 1}{2}, 1 \right\} + M_2, \tag{5.55}
\]
\[
\Phi(A, B) = (2\alpha + \beta)A^2 + \frac{3\delta - 3 + m}{1 + t}B^2 \nonumber
\]
\[
- \left( \frac{\delta - 1}{2} (1 + t)^{n-m} + \frac{2\delta}{\delta - 1} (2\alpha + 3\beta)(1 + t)^{m-n-1} \right) AB.
\]

**Lemma 5.8.** Let condition \((P_0)\) and \(M_1 > 0\) hold. Then for \(m = n + 0.5 = 3\), there exist some positive constants \(\nu_*, b_*\) and \(\epsilon^*\), such that
\[
\frac{1}{2} \frac{d}{dt} Z^2 + \frac{(1 - \nu_*) b_*}{1 + t} Z^2 \leq C(1 + t)^{1+\epsilon^*} Z^4 + C_0(1 + t)^{1-\epsilon^*} Z^2, \tag{5.56}
\]
where
\[
\epsilon^* = \frac{1}{2} \min \left\{ \frac{3\gamma - 3}{2}, \frac{-M_2 - 1}{2}, 1 \right\} > 0,
\]
\[
\nu_* = \min \left\{ \frac{3\gamma - 3}{4(3\gamma - 1)}, \frac{-M_4 - 1}{6\delta - M_3}, \frac{1}{10} \right\} > 0, \tag{5.57}
\]
\[
b_* = \begin{cases} 
\min \left\{ \frac{3\gamma - 3}{2}, \frac{-M_2 - 1}{2}, 1 \right\} > 1 & \text{if } \gamma \geq \frac{5}{3}, \\
\min \left\{ \frac{3\gamma - 3}{2}, \frac{-M_2 - 1}{2}, 1 \right\} > 1 & \text{if } 1 < \gamma < \frac{5}{3}.
\end{cases}
\]

Moreover, there exists a constant \(\Lambda(C_0)\) such that \(Z(t)\) is globally well-defined in \([0, +\infty)\) if \(Z_0 \leq \Lambda(C_0)\).

**Proof.** \textbf{Step 1.} First, note that
\[
\Phi(A, B) = aA^2 + bB^2 - cAB = a\left( A - \frac{c}{2a} B \right)^2 + dB^2, \tag{5.58}
\]
with
\[
a = (2\alpha + \beta), \quad b = \frac{3\delta - 3 + m}{1 + t},
\]
\[
c = \left( \frac{\delta - 1}{2} (1 + t)^{n-m} + \frac{2\delta}{\delta - 1} (2\alpha + 3\beta)(1 + t)^{m-n-1} \right),
\]
\[
d = (1 + t)^{-1} \left( \frac{3}{2} \delta - 3 + m \right) - \frac{1}{4} \left( \frac{(\delta - 1)^2}{4(2\alpha + \beta)} (1 + t)^{2n-2m} + \frac{4\delta^2(2\alpha + \beta)}{(\delta - 1)^2} M_1^2 (1 + t)^{2m-2n-2} + 2M_1 \delta (1 + t)^{-1} \right).
\]
Then according to Lemma \(10.10\) and \((5.53)\), for any \(\nu \in (0, 1)\), one has

\[
\frac{1}{2} \frac{d}{dt} Z^2 + \frac{b_m}{1 + t} (Z^2 - B^2) + \nu \alpha (A^*)^2 + (1 - \nu) dB^2 + \nu \left( b - \frac{c^2}{4(\alpha + \beta)} \right) B^2 \\
+ \alpha \sum_{k=0}^{2} (1 + t)^{2\gamma_k} |\varphi \nabla^{k+1} w(t)|^2 + (\alpha + \beta) \sum_{k=0}^{2} (1 + t)^{2\gamma_k} |\varphi \nabla^k \text{div} w(t)|^2
\]

\[
\leq \eta (A^*)^2 + C(\eta)((1 + t)^{2m - 5} + (1 + t)^{2m - 5 - 2\gamma_3}) Z^4 \\
+ C_0 ((1 + t)^{2m - n - 3.5} + (1 + t)^{n - 2.5}) Z^3 \\
+ C_0(\eta)(1 + t)^{-2} Z^2.
\]

\[(5.59)\]

By Proposition \(2.2\) in order to obtain the uniform estimates on \(Z\), one needs

\[2m - 2n - 2 = 2n - 2m = -1, \quad (5.60)\]

which implies that

\[d = (1 + t)^{-1} d^*(m) = (1 + t)^{-1} \left( \frac{3}{2} \delta - 3 + m - \frac{1}{4} M_3 \right). \quad (5.61)\]

Choose \(\epsilon^* = \frac{1}{2} \min \left\{ \frac{3\gamma - 3}{2}, \frac{1}{2} M_3 - 1, \frac{1}{10} \right\} > 0\). Then it follows from \((5.59)\) that

\[
\frac{1}{2} \frac{d}{dt} Z^2 + \frac{b_m}{1 + t} (Z^2 - B^2) + \nu \alpha (A^*)^2 + (1 - \nu) d^*(m) B^2 \\
+ \nu \left( \frac{d^* - \frac{1}{4} M_3}{\alpha + \beta} \right) B^2
\]

\[
\leq \eta (A^*)^2 + C(\eta)((1 + t)^{2m - 5 + \epsilon^*} + (1 + t)^{2m - 5 - 2\gamma_3}) Z^4 + C_0(\eta) (1 + t)^{-\epsilon^*} Z^2.
\]

\[(5.62)\]

**Step 2.** On one hand, for fixed \(\delta > 1\), a necessary condition to guarantee that the set

\[
\Pi = \{(\alpha, \beta) | \alpha > 0, 2\alpha + 3\beta > 0, \text{ and } M_2 < -1\}
\]

is not empty is that \(M_1 < \frac{3}{2} - \frac{1}{\delta}\).

On the other hand, for fixed \(\alpha, \beta, \gamma, \delta\), there always exists a sufficiently large number \(m\) such that \(d^*(m) > 0\). Indeed, one needs that

\[d^*(m) = \frac{3}{2} \delta - 3 + m - \frac{1}{4} M_3 = -\frac{1}{2} M_2 + m - \frac{5}{2} > 0. \]

Here for the choice \(m = n + 0.5 = 3\), one has therefore \(d^* = \frac{3}{2} \delta - \frac{1}{4} M_3\).

**Step 3.** Due to

\[d^* = \frac{3}{2} \delta - \frac{1}{4} M_3 > 1 \quad \text{and} \quad M_4 = \epsilon^* + M_2 < -1, \]

so for \(\nu_* = \min \left\{ \frac{3\gamma - 3}{4(3\gamma - 1)}, \frac{M_4 - 1}{6M_3 - M_3} \frac{1}{20} \right\} \) and \(\nu = \min \left\{ \frac{1}{200}, \frac{4d^* \nu_* (\alpha + \beta)}{M_3 \alpha} \right\}\), it hold that

\[1 + \epsilon^* - 2(1 - \nu_*) d^* = M_4 + 2\nu_* d^* < -1, \]

\[(5.63)\]

\[ (1 - \nu) d^* + \nu \left( d^* - \frac{1}{4} M_3 (\alpha + \beta) \right) \geq (1 - \nu_*) d^*. \]
Based on this observation, we choose \( \eta = \frac{\nu_*}{100} \), which, together with \( \text{(5.53)} \), \( m = n + 0.5 = 3 \) and \( \text{(5.58)} \), implies that

\[
\frac{1}{2} \frac{d}{dt} Z^2 + \frac{(1 - \nu_*) b_*}{1 + t} Z^2 + \frac{\nu_*}{2} (1 + t)^{2\gamma_3} |\varphi \nabla^4 w(t)|^2 \\
+ \alpha \sum_{k=0}^2 (1 + t)^{2\gamma_k} |\varphi \nabla^{k+1} w(t)|^2 + (\alpha + \beta) \sum_{k=0}^2 (1 + t)^{2\gamma_k} |\varphi \nabla^k \text{div} w(t)|^2 = 0
\]

implies that

\[
Z \text{ where } M
\]

(5.64)

Based on this observation, we choose \( \eta = \frac{\nu_*}{100} \), which, together with \( \text{(5.53)} \), \( m = n + 0.5 = 3 \) and \( \text{(5.58)} \), implies that

\[
\frac{1}{2} \frac{d}{dt} Z^2 + \frac{(1 - \nu_*) b_*}{1 + t} Z^2 + \frac{\nu_*}{2} (1 + t)^{2\gamma_3} |\varphi \nabla^4 w(t)|^2 \\
+ \alpha \sum_{k=0}^2 (1 + t)^{2\gamma_k} |\varphi \nabla^{k+1} w(t)|^2 + (\alpha + \beta) \sum_{k=0}^2 (1 + t)^{2\gamma_k} |\varphi \nabla^k \text{div} w(t)|^2 = 0
\]

implies that

\[
Z \text{ where } M
\]

(5.64)

Moreover,

\[
\sum_{k=0}^2 (1 + t)^{2\gamma_k} |\varphi \nabla^{k+1} w(t)|^2 = 0
\]

implies that

\[
Z \text{ where } M
\]

(5.64)

Moreover,

\[
Z \text{ where } M
\]

(5.64)

Moreover,

\[
Z(t) \leq C_0 (1 + t)^{1 + \epsilon^*} Z^3 + C_0 (1 + t)^{-1 - \epsilon^*} Z, \quad (5.55)
\]

According to Proposition 2.2 and

\[
\epsilon^* > 0, \quad 1 + \epsilon^* - 2(1 - \nu_*) b_* < -1,
\]

then \( Z(t) \) satisfies

\[
Z(t) \leq \left( \frac{1 - (1 - \nu_*) b_*}{Z_0^2 - 2C \int_0^t (1 + s)^M \exp \left( \frac{2C^*\nu_*}{\epsilon} (1 + t)^{-\epsilon^*} - 1 \right) ds \right)^\frac{1}{2}, \quad (5.66)
\]

where \( M = 1 + \epsilon^* - 2(1 - \nu_*) b_* < -1 \).

Moreover, \( Z(t) \) is globally well-defined for \( t \geq 0 \) if and only if

\[
0 < Z_0 < \left( \frac{1}{2C \int_0^t (1 + s)^M \exp \left( \frac{2C^*\nu_*}{\epsilon} (1 + t)^{-\epsilon^*} - 1 \right) ds} \right)^\frac{1}{2}. \quad (5.67)
\]

Moreover,

\[
Z(t) \leq C_0 (1 + t)^{-(1 - \nu_*) b_*} \quad \text{for all } t \geq 0, \quad (5.68)
\]

which implies that

\[
Y_k(t) \leq C_0 (1 + t)^{-\gamma_k-(1 - \nu_*) b_*}, \quad \text{and} \quad U_k(t) \leq C_0 (1 + t)^{-\delta_k-(1 - \nu_*) b_*} \quad \text{for } \forall t \geq 0. \quad (5.69)
\]

Finally, it follows from \( \text{(5.64)} \) and \( \text{(5.68)} \) that for \( k = 0, 1, 2, 3 \):

\[
\sum_{k=1}^3 \int_0^t (1 + t)^{2\gamma_k} \int |\varphi \nabla^{k+1} w|^2 ds \leq C_0. \quad (5.70)
\]

5.4. Proof of Theorem 5.1 under the condition \( (P_1) \). For this case, we first choose \( 2n - 2m = -(1 + \epsilon) < -1 \) for sufficiently small constant \( 0 < \epsilon < \frac{1}{2} \min \{ 1, 3\gamma - 3 \} \). Then it follows from Lemma 5.7 that

Lemma 5.9. There exist some positive constants \( b_m, C \), and \( C_0 \) such that

\[
\frac{1}{2} \frac{d}{dt} Z^2 + \frac{b_m}{1 + t} Z^2 + \frac{\alpha}{2} \sum_{k=0}^3 (1 + t)^{2\gamma_k} |\varphi \nabla^{k+1} w(t)|^2 \leq C(1 + t)^{2n-4+\epsilon} Z^4 + C_0 (1 + t)^{-1 - \epsilon} Z^2, \quad (5.71)
\]
with the constant $b_m$ given by \cite{[5,54]}.

The proof is routine and thus omitted. The rest of the proof for Theorem 5.1 under the condition $(P_1)$ is similar to the one in Subsection 5.3 for the condition $(P_0)$.

6. GLOBAL-IN-TIME WELL-POSEDNESS WITH COMPACTLY SUPPORTED DENSITY UNDER $(P_2)$ OR $(P_3)$

Denote $\nu = (\delta - 1)/(\gamma - 1)$. It is always assumed that $\nu \geq 2$ or $\nu = 1$. Based on the analysis of the previous section, it remains to estimate the following two terms:

$$L_k(W, \tilde{u}) = -\int \left( \nabla \phi^{2\nu} \cdot S(\nabla^k w) - \left( \nabla^k (\phi^{2\nu} L w) - \phi^{2\nu} L \nabla^k w \right) \right) \cdot \nabla^k w$$

$$+ \int \nabla^k (\phi^{2\nu} L \tilde{u}) \cdot \nabla^k w \equiv: L_k^1 + L_k^2 + L_k^3,$$

$$Q_k(W, \tilde{u}) = \int \left( \nabla \phi^{2\nu} \cdot Q(\nabla^k w) + \left( \nabla^k (\nabla \phi^{2\nu} \cdot Q(w)) - \phi^{2\nu} \cdot Q(\nabla^k w) \right) \right) \cdot \nabla^k w$$

$$+ \int \nabla^k (\phi^{2\nu} \cdot Q(\tilde{u})) \cdot \nabla^k w \equiv: Q_k^1 + Q_k^2 + Q_k^3.$$

6.1. Estimates on $L_k$ and $Q_k$.

**Lemma 6.1.** For any suitably small constant $\eta > 0$, there are two constants $C(\eta)$ and $C_0(\eta)$ such that

$$L_k(W) \leq \eta |\phi|^{2\nu} \nabla^{k+1} w_2^2 \delta_{3,k} + C(\eta)(1 + t)^{2n - 3\nu - 2 - 2\gamma} Y^{2\nu + 1} Y_k$$

$$+ C_0(1 + t)^{2n - 3\nu - 2 - 2\gamma} Y^{2\nu + 1} Y_k + C(\eta)(1 + t)^{2n - 3\nu Y^{2\nu}} + C_0(1 + t)^{2n - 3\nu Y^{2\nu}} (6.1)$$

$$Q_k(W, \tilde{u}) \leq \eta |\phi|^{2\nu} \nabla^{k+1} w_2^2 \delta_{3,k} + C(\eta)(1 + t)^{2n - 3\nu - 2 - 2\gamma} Y^{2\nu + 1} Y_k$$

$$+ C_0(1 + t)^{2n - 3\nu - 2 - 2\gamma} Y^{2\nu + 1} Y_k + C(\eta)(1 + t)^{2n - 3\nu Y^{2\nu}} + C_0(1 + t)^{2n - 3\nu Y^{2\nu}} (6.1)$$

**Proof.** Now we give only the corresponding estimates on $Q_k$. The estimates on $L_k$ can be obtained similarly.

**Step 1:** Estimates on $Q_1^k$. Direct estimates yield that

$$Q_1^k \leq C|\phi|^{2\nu} |\nabla \phi| \nabla^{k+1} w_{22}^2 \nabla^{k+1} w_{22} \leq C(1 + t)^{2n - 3\nu - 2 - 2\gamma} Y^{2\nu + 1} Y_k, \text{ for } k \leq 2,$$

$$Q_3^1 \leq C|\phi|^{2\nu} \nabla^4 w_{22} |\phi| \nabla \phi| \nabla^{k+1} w_{22} \nabla^{k+1} w_{22} \leq C(1 + t)^{2n - 3\nu - 2 - 2\gamma} Y^{2\nu + 1} Y_3.$$

**Step 2:** Estimates on $Q_3^0$. If $k = 0$ or 1, one has

$$Q_0^3 \leq C|\phi|^{2\nu} \nabla \phi| \nabla^{k+1} w_{22}^2 \nabla^{k+1} w_{22} \leq C(1 + t)^{2n - 3\nu - 2 - 2\gamma} Y^{2\nu + 1} Y_0,$$

$$Q_1^3 \leq C|\phi|^{2\nu} \nabla \phi| \nabla^{k+1} w_{22}^2 \nabla^{k+1} w_{22} \leq C(1 + t)^{2n - 3\nu - 2 - 2\gamma} Y^{2\nu + 1} Y_1.$$

For $k = 2$, decompose $Q_2^3 \triangleq Q_3^3(L) + Q_3^3(2, 0)$, which can be estimated as

$$Q_2^3(L) \triangleq C \int (\nabla \phi^{2\nu} \cdot \nabla^3 \tilde{u} + \nabla \phi^{2\nu} \cdot \nabla^2 \tilde{u}) \cdot \nabla^2 w \leq C(1 + t)^{2n - 3\nu - 2 - 2\gamma} Y^{2\nu + 1} Y_2,$$

$$Q_3^3(2, 0) \leq C_0(1 + t)^{2n - 3\nu - 2 - 2\gamma} Y^{2\nu + 1} Y_2.$$
and
\[ Q_3^2(2, 0) = C \int \nabla^3 \phi \cdot \nabla \hat{u} \cdot \nabla^2 w \leq C(1) \leq C_0(1 + t)^{2n-3\epsilon - n - 0.5 - \gamma_2} Y^2 Y_3. \] (6.5)

For \( k = 3 \), set \( Q_3^3 \triangleq Q_3^3(L) + Q_3^3(3, 0) \), where
\[ Q_3^3(L) = C \int (\nabla \phi^2 \cdot \nabla^4 \hat{u} + \nabla^2 \phi^2 \cdot \nabla^3 \hat{u} + \nabla^3 \phi^2 \cdot \nabla^2 \hat{u}) \cdot \nabla^3 w \leq C \left( |\phi|_{\infty}^2 |\nabla^4 \hat{u}|_2 + |\phi|_{\infty} |\nabla^3 \hat{u}|_2 + |\phi|_{\infty}^2 |\nabla^2 \phi|_6 |\nabla^3 \hat{u}|_3 \right) \] (6.6)

with
\[ Q_3^3(A) \triangleq C \int \nabla^3 \phi^2 \cdot \nabla \hat{u} \cdot \nabla^2 w \leq C \left( |\phi|_{\infty}^3 |\nabla^3 \hat{u}|_2 + |\nabla \phi|_{3} |\nabla^2 \phi|_{6} |\phi|_{\infty}^{-2} |\nabla \hat{u}|_{\infty} |\phi|^{2} w|_{2} + Q_3^3(B) \right) \] (6.7)

where the term \( Q_3^3(B) \) can be estimated by using integration by parts again,
\[ Q_3^3(B) = C \int \phi^{2-3} \nabla \phi \cdot \nabla \phi \cdot \nabla \hat{u} \cdot \nabla^2 w \leq C \left( |\nabla \phi|_{\infty} |\nabla^2 \phi|_{2} + |\nabla^2 \phi|_{6} |\phi|_{\infty}^{2-3} |\nabla \hat{u}|_{\infty} |\nabla^3 w|_{2} \right) \] (6.8)

Then (6.3)-(6.8) give that
\[ Q_k \leq \eta |\phi|^{k+1} w^2 + C_0(1 + t)^{2n-3\epsilon - n - 0.5 - \gamma_2} Y^2 Y_3 \leq C_0(1 + t)^{2n-3\epsilon - n - 0.5 - \gamma_2} Y^2 Y_3. \] (6.9)

**Step 3:** Estimates on \( Q_k^2 \). For \( k = 1 \), one has
\[ Q_1^2 = C \int (\phi^{2i-1} \nabla^2 \phi + \phi^{2i} \nabla \phi) \cdot \nabla w \cdot \nabla w \leq C |\phi|_{\infty}^{2i-2} \left( |\phi|_{\infty} |\nabla^2 \phi|_{6} |\nabla w|_{3} + |\nabla \phi|_{\infty}^2 |\nabla w|_{2} \right) |\nabla w|_{2} \leq C(1 + t)^{2n-3\epsilon - 2 - \gamma_1} Y^{2i+1} Y_1. \] (6.10)
For $k = 2$, one has

\[ Q_2^2 = C \int (\nabla^3 \phi^2 \cdot \nabla w + \nabla^2 \phi^2 \cdot \nabla^2 w) \cdot \nabla^2 w \]

\[ \leq C \left( |\phi|_\infty |\nabla \phi|_\infty |\nabla w|_\infty |\nabla^2 \phi|_2 + |\phi|_\infty^2 |\nabla w|_\infty |\nabla^3 \phi|_2 \right) \]

\[ + |\nabla \phi|_\infty^2 |\nabla \phi|_3 |\nabla w|_6 + |\phi|_\infty |\nabla \phi|_\infty^2 |\nabla^2 w|_2 \]

\[ + |\nabla^2 w|_6 |\nabla^2 \phi|_3 |\phi|_\infty |\nabla^2 \phi|_2 \leq C(1 + t)^{2n-3 \mu - 2 \gamma_2 Y^{2+1} Y_2}. \]  \hspace{1cm} (6.11)

For $k = 3$, using integration by parts, one can get

\[ Q_3^2 = C \int (\nabla^4 \phi \cdot \nabla w + \nabla^3 \phi \cdot \nabla^2 w + \nabla^2 \phi \cdot \nabla^3 w) \cdot \nabla^3 w \]

\[ \leq C |\phi|_\infty^2 |\nabla \phi|_6 |\nabla \phi|_6 + |\phi|_\infty |\nabla^2 \phi|_6 |\nabla^2 w|_6 |\nabla^3 w|_2 \]

\[ + C |\phi|_\infty |\nabla \phi|_3 |\nabla w|_6 |\nabla^3 \phi|_2 \]

\[ + C |\phi|_\infty |\nabla \phi|_3 |\nabla^2 \phi|_6 |\nabla^3 w|_2 + |\nabla^2 \phi|_3 |\phi|_\infty |\nabla^3 w|_2 \]

\[ \leq \eta |\phi| |\nabla^4 w|^2 + C(\eta)(1 + t)^{2n-3 \mu - 2 \gamma_2 Y^{2+1} Y_3} + Q_3^2(A), \]  \hspace{1cm} (6.12)

with

\[ Q_3^2(A) = C \int (\nabla^3 \phi^2 \cdot \nabla w \cdot \nabla^4 w \]

\[ \leq C \left( |\nabla^3 \phi|_2 |\phi|_6 + |\nabla^2 \phi|_6 |\nabla \phi|_6 |\phi|_\infty |\nabla^4 w|_2 + Q_3^2(B) \right) \]

\[ \leq C(\eta)(1 + t)^{2n-3 \mu - 2 \gamma_2 Y^{2+1} Y_3} + \eta |\phi| |\nabla^4 w|^2 + Q_3^2(B), \]  \hspace{1cm} (6.13)

where the term $Q_3^2(B)$ can be estimated by using integration by parts again,

\[ Q_3^2(B) = C \int \phi^{2-3} \nabla \phi \cdot \nabla \phi \cdot \nabla \phi \cdot \nabla w \cdot \nabla^4 w \]

\[ \leq C \left( |\nabla^2 \phi|_3 |\nabla \phi|_2 + |\nabla \phi|_6 |\nabla \phi|_3 |\phi|_\infty |\phi|_\infty^2 |\nabla \phi|_6 |\nabla w|_\infty |\nabla^3 w|_2 \right) \]

\[ + C |\phi|_\infty^2 |\nabla \phi|_3 |\nabla^2 w|_6 |\nabla^3 w|_2 \]

\[ \leq C(1 + t)^{2n-3 \mu - 2 \gamma_2 Y^{2+1} Y_3}. \]  \hspace{1cm} (6.14)

Then the estimates (6.10) - (6.13) lead to

\[ Q_k^2 \leq \eta |\phi| |\nabla^{k+1} w|^2 + C(\eta)(1 + t)^{2n-3 \mu - 2 \gamma_2 Y^{2+1} Y_k}. \]  \hspace{1cm} (6.15)

6.2. Derivation of a global-in-time well-posedness for an ordinary differential inequality. Then, based on the above energy estimates, one can derive the following lemma.

Lemma 6.2. There exist some positive constants $a = r + n$, $C_0$ and $C$ such that

\[ \frac{dY(t)}{dt} + \frac{a}{1 + t} Y(t) \leq C_0(1 + t)^{1 - \eta} Y(t) + C(1 + t)^{2n-3 \mu - 1 + 2 \eta \gamma - \eta Y^{2+1}}, \]  \hspace{1cm} (6.16)

holds for any sufficiently small constant $0 < \eta < \frac{3(2n-1)}{2\mu+1}$. Moreover, there exists a constant $\Lambda(C_0)$ such that $Y(t)$ is globally well-defined in $[0, +\infty)$ if $Y_0 \leq \Lambda(C_0)$. 
Proof. First, it follows from Lemmas \([5.2, 6.1]\) and \((5.5)\) that

\[
\frac{1}{2} \frac{d}{dt} Y_2^2 + \frac{k + r}{1 + t} Y_2^2 + \frac{1}{2} \alpha \int |\phi^t \nabla^{k+1} w|^2 \leq CY_Y(1 + t)^{-\gamma - 2} + C(1 + t)^{n-2.5} Y_2^2 + C(1 + t)^{2n-3k-2-\gamma - 2} Y_2^{2+1} Y_k + C_0(1 + t)^{2n-3k-5} Y^{2^2}.
\]  

(6.17)

Second, multiplying \((6.17)\) by \((1 + t)^{2k}\) and simplifying by \(Y\) give

\[
\frac{dY(t)}{dt} + \frac{a}{1 + t} \leq C_0(1 + t)^{-2} Y + C(1 + t)^{n-2.5} Y^2 + C(1 + t)^{2n-3k-2} Y^{2+1} + C_0(1 + t)^{2n-3k-5} Y^{2+1} + C_0(1 + t)^{-1-\eta} Y(t) + C(1 + t)^{2n-3k-1 + 2\eta - 2} Y^{2+1}
\]

(6.18)

for any sufficiently small constant \(0 < \eta < 1\). A sufficient condition for the global existence of solutions to \((6.18)\) is

\[
K_{a,\gamma,\eta} = 2n - 3k - 2 + 1 + 2\eta - \eta - 2\eta a < -1.
\]

Indeed, it follows from the definition of \(a\) and \(\gamma \geq 2\) or \(\gamma = 1\) that

\[
K_{a,\gamma,\eta} = \begin{cases} 
-2\gamma - 1 - \eta + 2\eta < -1 & \text{if } \gamma \geq \frac{5}{3}, \\
3(1 - \gamma) + \eta(2\gamma - 1) < -1 & \text{if } 1 < \gamma < \frac{5}{3},
\end{cases}
\]

(6.19)

for the sufficiently small constant \(0 < \eta < \frac{3(\gamma - 1)}{2\gamma - 1}\).

Then, \((6.18)-(6.19)\), Proposition \(2.2\) and Lemma \(5.2\) give

\[
Y(t) \leq \frac{(1 + t)^{-a} \exp \left( \frac{C_0}{\eta} (1 - (1 + t)^{-\eta}) \right) \left( Z_0^{-2e - 2\xi C} \int_0^t (1 + s)^K_{a,\gamma,\eta} \exp \left( \frac{2C_0}{\eta} (1 - (1 + s)^{-\eta}) \right) ds \right)^{\frac{1}{\eta}}}{\left( 2\xi C \int_0^t (1 + s)^K_{a,\gamma,\eta} \exp \left( 2C_0 \eta (1 - (1 + s)^{-\eta}) \right) ds \right)^{\frac{1}{\eta}}}.
\]

(6.20)

Thus, \(Y(t)\) is defined for \(t \geq 0\) if and only if

\[
0 < Y_0 < \frac{1}{\left( 2\xi C \int_0^t (1 + s)^K_{a,\gamma,\eta} \exp \left( 2C_0 \eta (1 - (1 + s)^{-\eta}) \right) ds \right)^{\frac{1}{\eta}}}
\]

(6.21)

Moreover, it holds that

\[
Y(t) \leq C_0(1 + t)^{-\gamma - \eta}, \quad \text{and} \quad Y_k(t) \leq K(1 + t)^{-k - \eta} \quad \text{for all} \quad t \geq 0.
\]

(6.22)

Now we are ready to prove Theorem \(5.1\). Step 1: Estimates on \(\varphi \nabla^4 w\). Note that \((6.17)\) and \((6.22)\) for \(k = 3\) imply that

\[
\frac{1}{2} \frac{d}{dt} Y_3^2 + \frac{3 + r}{1 + t} Y_3^2 + \frac{1}{2} \alpha \int |\phi^t \nabla^{k+1} w|^2 \leq C(1 + t)^{-8.5 - 3e}.
\]

(6.23)

Multiplying \((6.23)\) by \((1 + t)^{2e}\) on both sides gives

\[
\frac{1}{2} \frac{d}{dt} ((1 + t)^{2e} Y_3^2) + \frac{3 + r - e}{1 + t} ((1 + t)^{2e} Y_3^2) + \frac{1}{2} \alpha (1 + t)^{2e} \int |\phi^t \nabla^{4} w|^2 \leq C(1 + t)^{-8.5 - 3e + 2e}.
\]

(6.24)
Under the assumptions that
\[ 3 + r - \varepsilon \geq 0, \quad -8.5 - 3r + 2 \varepsilon < -1, \]
i.e., \( e \leq 3 + r \), one can obtain
\[
(1 + t)^{2e} Y_3^2 + \frac{1}{2} \int_0^t (1 + t)^{2e} \int |\phi'| \nabla^4 w|^2 ds \leq C_0.
\] (6.25)

**Step 2:** Estimates on \( \varphi = \phi' \). Here we always assume that \( t \geq 2 \) in the rest of this proof. Due to (6.22), one can get
\[
(1 + t)^{-2.5} |\nabla^3 \varphi|^2 \leq C_0 (1 + t)^{-\frac{5}{2}(1 + t)} \varphi_i \varphi_j (1 + t)^{-1.5 - rt + 1.5 - k}. \] (6.26)

Applying \( \nabla^3 \) to (5.1), multiplying by \( \nabla^3 \varphi \) and integrating over \( \mathbb{R}^3 \), then similar to Lemma 5.6, one can get
\[
\frac{1}{2} \frac{d}{dt} |\nabla^3 \varphi|^2 + \frac{3\delta}{2} \int (1 + t)^{-2.5} |\nabla^3 \varphi|^2 + C(1 + t)^{-0.75t + 0.5rt - r - 3.25} |\nabla^3 \varphi(t)|^{3/2}_2 + C(1 + t)^{-1.5 - rt + 3.5} |\nabla^3 \varphi|^2 + C |\varphi \nabla^4 w|^2 \nabla^3 \varphi(t). \] (6.27)

Then, multiplying (6.27) by \( (1 + t)^{2p} \) and simplifying by \( C^2 = (1 + t)^{2p} |\nabla^3 \varphi|^2 \) lead to
\[
\frac{1}{2} \frac{d}{dt} G^2 + \frac{3\delta}{2} p G^2 \leq C_0 ((1 + t)^{-r - 2.5} + (1 + t)^{2p - 2e}) G^2 + C(1 + t)^{-0.75t + 0.5rt - r - 3.25 + 0.5 p G^{2/3}} \] (6.28)
\[ + C(1 + t)^{-1.5 - rt - 3.5 + p G} + C(1 + t)^{2e} |\varphi \nabla^4 w|^2. \]

Choose \( p \) satisfying the following conditions:
\[
\frac{3}{2} \delta - p > 0, \quad -0.75t - 0.5rt - r - 3.25 + 0.5 p < -1, \\
2p - 2e < -1, \quad -1.5 - rt - 3.5 + p < -1.
\]

Then (6.28) yields
\[
(1 + t)^{2p} |\nabla^3 \varphi|^2 \leq C_0.
\]

7. **Global-in-time well-posedness without compactly supported assumption**

In this section, we will extend the global-in-time well-posedness in Theorem 5.1 to the more general case whose initial mass density \( p_0 \) is still small, but is not necessary to be compactly supported, which can be stated as:

**Theorem 7.1.** Let (1.20) and any one of the conditions (P0)-(P3) hold. If initial data \((\varphi_0, \phi_0, u_0)\) satisfies (A1)-(A2), then for any positive time \( T > 0 \), there exists a unique global classical solution \((\varphi, \phi, u)\) in \([0, T] \times \mathbb{R}^3\) to the Cauchy problem (4.1) satisfying
\[
(\varphi, \phi, w) \in C([0, T]; H^s_{loc}) \cap L^\infty([0, T]; H^3), \quad \varphi \nabla^4 w \in L^2([0, T]; L^2), \quad (7.1)
\]
for any constant \( s' \in [2, 3) \). Moreover, when (P2) holds, the smallness assumption on \( \varphi_0 \) could be removed.

In the following subsections, we will prove Theorem 7.1 under the condition (P0). The proof for other cases is similar, and so is omitted.
7.1. **Existence.** According to Theorem 5.11, for the initial data

\[(\varphi_R^0, \phi_R^0, w_0) = (\varphi_0 F(|x|/R), \phi_0 F(|x|/R), 0),\]

there exists the unique global regular solution \((\varphi^R, \phi^R, w^R)\) satisfying:

\[
|\nabla^k \varphi^R(t)|_2 + |\nabla^k w^R(t)|_2 \leq C_0(1 + t)^{-1/(1 - \nu_s) b_+ + 2.5 - k};
\]

\[
\|(1 + t)^{2.5} \varphi^R \nabla^{k+1} w^R\|_{L^2_t L^2_x} \leq C_0, |\nabla^k \varphi^R(t)|_2 \leq C_0(1 + t)^{-1/(1 - \nu_s) b_+ + 3 - k},
\tag{7.2}
\]

for any positive time \(t > 0\), with \(C_0\) independent of \(R\).

Due to (7.2) and the following relations:

\[
\begin{cases}
\varphi_t^R = -(w^R + \hat{u}) \cdot \nabla \varphi^R - \frac{\delta - 1}{2} \varphi \text{div}(w^R + \hat{u}), \\
\phi_t^R = -(w^R + \hat{u}) \cdot \nabla \phi^R - \frac{\gamma - 1}{2} \phi \text{div}(w^R + \hat{u}), \\
w_t^R = -w^R \cdot \nabla w^R - \frac{\gamma - 1}{2} \phi^R \nabla \phi^R - (\varphi^R)^2 L w^R + \nabla ((\varphi^R)^2) \cdot Q(w^R + \hat{u}) - \hat{u} \cdot \nabla w^R - w^R \cdot \nabla \hat{u} - (\varphi^R)^2 \hat{u},
\end{cases}
\tag{7.3}
\]

it holds that for any finite constant \(R_0 > 0\) and finite time \(T > 0\),

\[
\|\varphi_t^R\|_{H^2(B_{R_0})} + \|\phi_t^R\|_{H^2(B_{R_0})} + \|w_t^R\|_{H^2(B_{R_0})} + \int_0^T \|\nabla^2 w_t^R\|_{L^2(B_{R_0})}^2 \, ds \leq C_0(R_0, T),
\tag{7.4}
\]

for \(0 \leq t \leq T\), where the constant \(C_0(R_0, T) > 0\) depending on \(C_0, R_0\) and \(T\).

Since (7.2) and (7.4) are independent of \(R\), there exists a subsequence of solutions (still denoted by) \((\varphi^R, \phi^R, w^R)\) converging to a limit \((\varphi, \phi, w)\) in the sense:

\[
(\varphi^R, \phi^R, w^R) \to (\varphi, \phi, w) \text{ strongly in } C([0, T]; H^2(B_{R_0})).
\tag{7.5}
\]

For \(k = 0, 1, 2, 3\), denote

\[
a_k = -(1 - \nu_s) b_+ + 2.5 - k, \quad b_k = -(1 - \nu_s) b_+ + 3 - k, \quad c_k = k - 2.5.
\]

Again due to (7.2), there exists a subsequence of solutions (still denoted by) \((\varphi^R, \phi^R, w^R)\) converging to the same limit \((\varphi, \phi, w)\) as above in the following weak* sense (for \(k = 0, 1, 2, 3\)):

\[
(1 + t)^{b_k} \varphi^R \to (1 + t)^{b_k} \varphi \text{ weakly* in } L^\infty([0, T]; H^3(\mathbb{R}^3)),
\]

\[
(1 + t)^{a_k} (\phi^R, w^R) \to (1 + t)^{a_k} (\phi, w) \text{ weakly* in } L^\infty([0, T]; H^3(\mathbb{R}^3)).
\tag{7.6}
\]

Combining the strong convergence in (7.3) and the weak convergence in (7.6) shows that \((\varphi, \phi, w)\) also satisfies the corresponding estimates (7.2) and (7.4) and (for \(k = 0, 1, 2, 3\)):

\[
(1 + t)^{c_k} \varphi^R \nabla^{k+1} w^R \to (1 + t)^{c_k} \varphi \nabla^{k+1} w \text{ weakly in } L^2([0, T] \times \mathbb{R}^3).
\tag{7.7}
\]

It is then obvious that \((\varphi, W)\) is a weak solution to problem (4.1) in the sense of distribution.
7.2. **Uniqueness.** Let \((\phi_1, W_1)\) and \((\phi_2, W_2)\) be two solutions to \((7.1)\) satisfying the uniform a priori estimates \((7.2)\). Set \(F_N = F(|x|/N)\), and

\[
\phi = \phi_1 - \phi_2, \quad W = (\phi, \overline{w}) = (\phi_1 - \phi_2, w_1 - w_2),
\]

\[
\phi^N = \phi F_N, \quad W^N = W F_N = (\overline{\phi}^N, \overline{w}^N),
\]

then \((\phi^N, W^N)\) solves the following problem

\[
\begin{cases}
\phi^N_t + (w_1 + \hat{u}) \cdot \nabla \phi^N + \overline{w}^N \cdot \nabla \phi_2 + \frac{\delta - 1}{2} (\overline{\phi}^N \div (w_2 + \hat{u}) + \phi_1 \div \overline{w}) \\
= \phi (w_1 + \hat{u}) \cdot \nabla F_N + \frac{\delta - 1}{2} \phi_1 \overline{w} \cdot \nabla F_N,
\end{cases}
\]

\[
\begin{aligned}
&- \sum_{j=1}^{3} A_j^*(W_1, \hat{u}) \partial_j \phi^N + \phi_1^2 \mathbb{L}(\overline{w})^N \\
&+ \mathbb{H}(\phi_1) \cdot \mathbb{Q}(\overline{w}^N) - B(\nabla \hat{u}, \overline{w}^N) - D(\overline{\phi}^N (\phi_1 + \phi_2), \nabla \hat{u}) \\
&+ \sum_{j=1}^{3} A_j^*(W_1, \hat{u}) \partial_j F_N + \phi_1^2 (\mathbb{L}(\overline{w}^N) - F_N \mathbb{L}(\overline{w})) - \mathbb{H}(\phi_1) \cdot \mathbb{Q}(\overline{w}) \cdot \nabla F_N,
\end{aligned}
\]

\[
(\phi^N, W^N)|_{t=0} = (0, 0), \quad x \in \mathbb{R}^3,
\]

\[
(\phi^N, W^N) \rightarrow (0, 0) \quad \text{as} \quad |x| \rightarrow +\infty, \quad t > 0.
\]

It is not hard to obtain that

\[
\begin{aligned}
dt \frac{d}{dt} \|\phi^N\|^2_2 &\leq C (|\nabla w_1|_\infty + |\nabla w_2|_\infty + |\nabla \hat{u}|_\infty) \|\phi^N\|^2_2 + C |\nabla \phi_2|_\infty \|\overline{w}^N\|_2 \|\phi^N\|_2 \\
&+ C |\nabla \phi_1 \div \overline{w}^N|_2 \|\phi^N\|_2 + I_N^1,
\end{aligned}
\]

\[
\begin{aligned}
dt \frac{d}{dt} \|W^N\|^2_2 + \frac{1}{2} \int \phi_1^2 |\nabla \overline{w}|^2 \leq C \left(1 + \|W_1\|^2_3 + \|W_2\|^2_3 + \|\nabla \hat{u}\|^2_3\right) \|\overline{w}^N\|^2_2 \\
&+ C (|\nabla \phi_2^2 w_2|_{\infty} |\overline{w}^N|_2 + |\nabla w_2|_3 (|\nabla \phi_1^2 \overline{w}^N|_2 + |\nabla \phi_1|_\infty |\overline{w}^N|_2)) \|\phi^N\|_2 + I_N^2,
\end{aligned}
\]

where the error terms \(I_N^1-I_N^2\) are given and estimated by

\[
I_N^1 = \int_{N \leq |x| \leq 2N} \frac{1}{N} \left(|w_1| + |\hat{u}| \right) |\overline{w}|^2 + |\overline{w}| |\phi_1| dx
\]

\[
\leq C \left(|w_1|_\infty + |\nabla \hat{u}|_\infty \|\phi\|_{L^2(\mathbb{R}^3/B_N)} + |\phi_1|_\infty \|\overline{w}\|_{L^2(\mathbb{R}^3/B_N)}\right) \|\overline{w}\|_{L^2(\mathbb{R}^3/B_N)}
\]

\[
I_N^2 = \int_{N \leq |x| \leq 2N} \frac{1}{N} \left(|W_1| + |\hat{u}| \right) |\overline{w}|^2 + |\phi_1|_\infty |\overline{w}|^2 + |\phi_1|^2 |\overline{w}|^2 \|\nabla \overline{w}\| dx
\]

\[
+ \int_{N \leq |x| \leq 2N} \frac{1}{N} \left(|\phi_1 + \phi_2 | \phi_1 \div \overline{w}^N + |\nabla \phi_1| \phi_1 \div \overline{w}\right) |\overline{w}| dx
\]

\[
\leq C (|w_1|_\infty + |\nabla \hat{u}|_\infty) \|\overline{w}\|^2_{L^2(\mathbb{R}^3/B_N)} + C |\phi_1|^2 \|\overline{w}\|^2_{L^2(\mathbb{R}^3/B_N)}
\]

\[
+ C |\phi_1 + \phi_2|_\infty |w_2|_\infty \|\overline{w}\|_{L^2(\mathbb{R}^3/B_N)} \|\phi\|_{L^2(\mathbb{R}^3/B_N)},
\]

for \(0 \leq t \leq T\).
Using the same arguments as in the derivation of (4.51)-(4.56), and letting
\[ \Lambda^N(t) = \|W^N(t)\|_{L^2}^2 + \frac{\alpha}{2C}\|\phi^N(t)\|_{L^2}^2, \]
one can have
\[
\begin{aligned}
\frac{d}{dt}\Lambda^N(t) + \frac{\alpha}{2}\|\nabla w^N(t)\|_{L^2}^2 \leq J(t)\Lambda^N(t) + I_N^1(t) + I_N^2(t), \\
\int_0^t (J(s) + I_N(s))ds \leq C_0 \quad &\text{for } 0 \leq t \leq T,
\end{aligned}
\]
where the constant \( C_0 > 0 \) is independent of \( N \).

It follows from
\[
\int_0^t \int (I_N^1 + I_N^2)dt \to 0, \quad \text{as } N \to +\infty,
\]
and Gronwall’s inequality that \( \overline{\phi} = \overline{\phi} = \overline{w} = 0 \). Then the uniqueness is obtained.

7.3. Time continuity. This follows from the uniform estimates (7.2) and equations in (4.1).

8. Proof of Theorem 1.4

Based on Theorem 7.1, now we are ready to give the global well-posedness of the regular solution to the original problem (1.1)-(1.7), i.e., the proof of Theorem 1.4. Moreover, we will show that this regular solution satisfies (1.1) classically in positive time \((0, T^*)\) when \( 1 < \min(\gamma, \delta) \leq 3 \).

\textbf{Proof.} First, Theorem 7.1 shows that there exists a unique global classical solution \((\phi, \phi, w)\) satisfying (7.2) and
\[
(\rho^{\frac{\gamma - 1}{2}}, \rho^{\frac{\gamma - 1}{2}}) = (\varphi, \phi) \in C^1((0, T) \times \mathbb{R}^3), \quad \text{and } (u, \nabla u) \in C((0, T) \times \mathbb{R}^3). \tag{8.1}
\]
Second, in terms of \((\varphi, \phi, u)\), one has
\[
\begin{aligned}
\phi_t + u \cdot \nabla \phi + \frac{\gamma - 1}{2} \phi \Delta u = 0, \\
u_t + u \cdot \nabla u + \frac{\gamma - 1}{2} \phi \nabla \phi + \varphi^2Lu = \nabla \varphi^2 \cdot Q(u). \tag{8.2}
\end{aligned}
\]
\textbf{Case 1:} \( 1 < \min(\gamma, \delta) \leq 3 \). We assume that \( 1 < \gamma \leq 3 \), and the other cases can be dealt with similarly. Since \( \rho = \phi^{\frac{2}{\gamma - 1}} \) and \( \rho^{\frac{2}{\gamma - 1}} \geq 1 \), so
\[ \rho \in C^1((0, T) \times \mathbb{R}^3). \]

Multiplying (8.2) by \( \rho \phi(t, x) = \frac{2}{\gamma - 1} \phi^{\frac{2}{\gamma - 1}}(t, x) \in C((0, T) \times \mathbb{R}^3) \) on both sides yields the continuity equation, (1.1).1
While multiplying (8.2) by \( \phi^{\frac{2}{\gamma - 1}} = \rho(t, x) \in C^1((0, T) \times \mathbb{R}^3) \) on both sides gives the momentum equations, (1.1)2.

Thus, \((\rho, u)\) solves the Cauchy problem (1.1)- (1.7) in the classical sense.

\textbf{Case 2:} \( \min(\gamma, \delta) > 3 \). For definiteness, we assume that \( \gamma > 3 \), and the other cases could be dealt with similarly. Since \( \phi = \rho^{\frac{2}{\gamma - 1}} \) and \( \rho^{\frac{2}{\gamma - 1}} > 0 \), so \( \rho \in C((0, T) \times \mathbb{R}^3). \)
It follows from (8.2) and $\frac{\gamma - 1}{2} > 1$ that
$$\rho_t + \text{div}(\rho u) = 0, \quad \text{when} \quad \rho(t, x) > 0.$$  
(8.3)

Now the whole space could be divided into two domains: the vacuum domain $V_0$ and its complement $V_p(t)$. Then it holds that for any smooth $f$,
$$\int_0^t \int_{V_p(s)} (\rho f_t + \rho u \cdot \nabla f) \, dx \, ds = \int_{V_p(0)} \rho_0 f(0, x) \, dx,$$
(8.4)

which means that
$$\int_0^t \int_{\mathbb{R}^3} (\rho f_t + \rho u \cdot \nabla f) \, dx \, dt = \int_{\mathbb{R}^3} \rho_0 f(0, x) \, dx.$$  
(8.5)

Multiplying (8.2) by $\phi^{\frac{\gamma - 1}{2}} = \rho(t, x)$ gives the momentum equations, (1.1). Thus, $(\rho, u)$ satisfies the Cauchy problem (1.1)-(1.7) in the sense of distributions.

Finally, it is easy to show that
$$\rho(t, x) \geq 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^3.$$  

That is to say, $(\rho, u)$ satisfies the Cauchy problem (1.1)-(1.7) in the sense of distributions and has the regularities shown in Definition 1.2, which means that the Cauchy problem (1.1)-(1.7) has a unique regular solution $(\rho, u)$.

9. Proof of Theorem 1.6

In this section, we prove Theorem 1.6 by modifying the proof of Theorem 1.4. For the Cauchy problem (1.1)-(1.2) with (1.6)-(1.7), if $\text{div} T = \alpha \rho^\delta \Delta u$, it can be rewritten into
$$\begin{cases}
\varphi_t + w \cdot \nabla \varphi + \frac{\delta - 1}{2} \varphi \text{div} w = -\widehat{u} \cdot \nabla \varphi - \frac{\delta - 1}{2} \varphi \text{div} \widehat{u}, \\
W_t + \sum_{j=1}^3 A_j(W) \partial_j W - \varphi^2 \Delta w = G_s(W, \varphi, \widehat{u}), \\
(\varphi, \phi, w)(t = 0, x) = (\varphi_0, \phi_0, 0), \quad x \in \mathbb{R}^3, \\
(\varphi, \phi, w) \to (0, 0, 0) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad t \geq 0,
\end{cases}$$  
(9.1)

where
$$G_s(W, \varphi, \widehat{u}) = - B(\nabla \widehat{u}, W) - \sum_{j=1}^3 \widehat{u}^{(j)} \partial_j W - \begin{pmatrix} 0 \\ \varphi^2 \Delta \widehat{u} \end{pmatrix}.\quad \quad (9.2)$$

Let $\gamma_k = k - \frac{2}{\gamma}$ and $\delta_k = k - m$, where $m$ will be determined in the end of this section, and $Z(t)$ be defined in (5.4). Then one has

**Lemma 9.1.** There exist some positive constants $b_m$, such that
$$\frac{dZ}{dt}(t) + \frac{b_m}{1 + t} Z(t) \leq \begin{cases}
C(1 + t)^{2m-5} Z^3(t) + C_0(1 + t)^{2m-9} Z & \text{if} \quad m \in [3.5, 4), \\
C(1 + t)^{2m-5} Z^3(t) + C_0(1 + t)^{5-2m} Z & \text{if} \quad m \in (3, 3.5),
\end{cases}$$  
(9.3)

where $b_m$ is given by (5.54).
The proof is similar to that of Lemma 5.7 with $Q_k = 0$. So the details are omitted here.

9.1. We treat the case $m \in [3.5, 4)$ first.

**Lemma 9.2.** Let $\delta > 1$, $\gamma > 4/3$ and $m \in [3.5, 4)$. For the following problem

$$
\begin{cases}
\frac{dZ(t)}{dt} + \frac{b_m}{1 + t} Z(t) = C_1 (1 + t)^{2m-5} Z^3(t) + C_2 (1 + t)^{2m-9} Z, \\
Z(x, 0) = Z_0,
\end{cases}
$$

there exists a constant $\Lambda$ such that $Z(t)$ is globally well-defined in $[0, +\infty)$ if $Z_0 \leq \Lambda$.

**Proof.** Set $M = 2m - 5 - 2b_m$. Due to Proposition 2.2 and its proof, a sufficient condition for the global existence of solution is

$$
M < -1, \quad \text{and} \quad 2m - 8 < 0,
$$

which requires that

$$
M < -1 \quad \text{if} \quad \gamma \geq \frac{5}{3}, \quad \text{or} \quad \frac{4}{3} \leq \frac{2}{3}m - 1 < \frac{5}{3}.
$$

Therefore, by choosing $Z_0$ small enough, one can obtain the global existence of the solution to (9.4). Moreover,

$$
Z(t) \leq C_0 (1 + t)^{-b_m} \quad \text{for all} \quad t \geq 0.
$$

□

Next we prove Theorem 1.6 for the cases $\gamma > 4/3$. First, according to Lemma 9.1, when $m \in [3.5, 4)$, the following inequality holds

$$
\frac{dZ(t)}{dt} + \frac{b_m}{1 + t} Z(t) \leq C(1 + t)^{2m-5} Z^3(t) + C_0 (1 + t)^{2m-9} Z.
$$

Then it follows from Lemma 9.2 that there exists a constant $\Lambda$ such that $Z(t)$ is globally well-defined in $[0, +\infty)$ if $Z_0 \leq \Lambda$, and (9.6) holds.

According to Lemma 5.5 and its proof with $Q_k = 0$, it is easy to see that

$$
\alpha \sum_{k=1}^{3} \int_{0}^{t} (1 + s)^{2\gamma_k} \int |\varphi \nabla^{k+1} w|^2 ds \leq C_0, \quad \text{for} \quad t \geq 0.
$$

The rest of the proof is similar to that for Theorem 1.4 and so is omitted.

9.2. We now treat the case $m \in (3, 3.5)$.

**Lemma 9.3.** Let $\delta > 1$, $\gamma > 1$ and $m \in (3, 3.5)$. For the following problem

$$
\begin{cases}
\frac{dZ(t)}{dt} + \frac{b_m}{1 + t} Z(t) = C(1 + t)^{2m-5} Z^3(t) + C_0 (1 + t)^{5-2m} Z, \\
Z(x, 0) = Z_0,
\end{cases}
$$

there exists a constant $\Lambda$ such that $Z(t)$ is globally well-defined in $[0, +\infty)$ if $Z_0 \leq \Lambda$. 


Proof. Set $M = 2m - 5 - 2b_m$. Due to Proposition 2.2 and its proof, a sufficient condition for a global solution with small data is

$$M < -1, \quad \text{and} \quad 6 - 2m < 0,$$

which means that $m$ must belong to $(3, 3.5)$, and requires that

$$M < -1 \quad \text{if} \quad \gamma \geq \frac{5}{3}, \quad \text{or} \quad 1 < \frac{2m}{3} - 1 < \gamma < \frac{5}{3}. \quad (9.10)$$

Therefore, by choosing $Z_0$ small enough, one can obtain the global existence of the solution to (9.9). Moreover, (9.6) still holds. □

Next we prove Theorem 1.6 for the cases $\gamma > 1$. First, according to Lemma 9.1, when $m \in (3, 3.5)$, it holds that

$$\frac{dZ}{dt}(t) + \frac{b_m}{1 + t}Z(t) \leq C(1 + t)^{2m-5}Z^3(t) + C_0(1 + t)^{5-2m}Z. \quad (9.11)$$

Then from Lemma 9.3, there exists a constant $\Lambda$ such that $Z(t)$ is globally well-defined in $[0, +\infty)$ if $Z_0 \leq \Lambda$, and (9.6) holds.

According to Lemma 5.5 and its proof with $Q_k = 0$, it is easy to see that (9.8) holds.

The rest of the proof is similar to that for Theorem 1.4, and so is omitted.

10. Appendix

In the first part of this appendix, we list some lemmas which were used frequently in the previous sections. The rest of the appendix will be devoted to show the proofs of Propositions 2.1-2.2 and some special Sobolev inequalities.

10.1. Some lemmas. The first one is the well-known Gagliardo-Nirenberg inequality.

Lemma 10.1. [13] For $p \in [2, 6]$, $q \in (1, \infty)$, and $r \in (3, \infty)$, there exists some generic constant $C > 0$ that may depend on $q$ and $r$ such that for

$$f \in H^1(\mathbb{R}^3), \quad \text{and} \quad g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3),$$

it holds that

$$|f|^p_p \leq C |f|^{(6-p)/2}_2 |\nabla f|^{(3p-6)/2}_2, \quad |g|_\infty \leq C |g|^{q(r-3)/(3r+q(r-3))}_q |\nabla g|^{3r/(3r+q(r-3))}_r. \quad (10.1)$$

Some common versions of this inequality can be written as

$$|u|_6 \leq C |u|_{D^1}, \quad |u|_\infty \leq C \|\nabla u\|_1, \quad |u|_\infty \leq C \|u\|_{W^{1,r}}, \quad \text{for} \quad r > 3. \quad (10.2)$$

The Sobolev imbedding theorem yields

Lemma 10.2. [13] Let $p > 3/2$ and $f \in H^p(\mathbb{R}^3)$. Then

$$|f|_\infty \leq C |f|^{1-\frac{3}{2p}}_2 |\nabla^p f|^{\frac{3}{2p}}_2, \quad (10.3)$$

where $C$ is a positive constant that may depend on $p$.

The next one can be found in Majda [23].
Lemma 10.3. \[23\] Let \( r, a \) and \( b \) be constants such that
\[
\frac{1}{r} = \frac{1}{a} + \frac{1}{b}, \quad \text{and} \quad 1 \leq a, b, r \leq \infty.
\]
\( \forall s \geq 1, \text{if} \ f, g \in W^{s,a} \cap W^{s,b}(\mathbb{R}^3), \text{then it holds that} \)
\[
|\nabla^s(fg) − f\nabla^s g|_r \leq C_s(|\nabla f|_a|\nabla^{s-1}g|_b + |\nabla^s f|_b|g|_a), \quad (10.4)
\]
\[
|\nabla^s(fg) − f\nabla^s g|_r \leq C_s(|\nabla f|_a|\nabla^{s-1}g|_b + |\nabla^s f|_a|g|_b), \quad (10.5)
\]
where \( C_s > 0 \) is a constant depending only on \( s \), and \( \nabla^s f \) \((s > 1)\) is the set of all \( \nabla_x^s f \) with \( |\xi| = s \). Here \( \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3 \) is a multi-index.

Next, the interpolation estimate, product estimate, composite function estimate and so on are given in the following four lemmas.

Lemma 10.4. \[23\] Let \( u \in H^s \), then for any \( s' \in [0, s] \), there exists a constant \( C_s \) depending only on \( s \) such that
\[
\|u\|_{s'} \leq C_s\|u\|_{0}^{1-\frac{s'}{s}}\|u\|_s^{\frac{s'}{s}}.
\]

Lemma 10.5. \[13\] Let \( r \geq 0, i \in [0, r], \) and \( f \in L^\infty \cap H^r \). Then \( \nabla^i f \in L^{2r/i} \), and there some generic constants \( C_{i,r} > 0 \) such that
\[
|\nabla^i f|_{2r/i} \leq C_{i,r}|f|^{1-\frac{i}{r}}|\nabla^r f|^{\frac{i}{r}}. \quad (10.6)
\]

Lemma 10.6. \[23\] Let functions \( u, v \in H^s \) and \( s > \frac{3}{2} \), then \( u \cdot v \in H^s \), and there exists a constant \( C_s \) depending only on \( s \) such that
\[
\|uv\|_s \leq C_s\|u\|_s\|v\|_s.
\]

Lemma 10.7. \[23\]

(1) For functions \( f, g \in H^s \cap L^\infty \) and \( |v| \leq s \), there exists a constant \( C_s \) depending only on \( s \) such that
\[
\|\nabla^v(fg)\|_s \leq C_s(|f|_\infty|\nabla^s g|_2 + |g|_\infty|\nabla^s f|_2). \quad (10.7)
\]

(2) Assume that \( g(u) \) is a smooth vector-valued function on \( \Omega \), \( u(x) \) is a continuous function with \( u \in H^s \cap L^\infty \), \( u(x) \in \Omega_1 \) and \( \overline{\Omega}_1 \subseteq \Omega \). Then for \( s \geq 1 \), there exists a constant \( C_s \) depending only on \( s \) such that
\[
|\nabla^s g(u)|_2 \leq C_s\left|\frac{\partial g}{\partial u}\right|_{s-1,\overline{\Omega}_1} |u|_{\infty}^{s-1}|\nabla^s u|_2. \quad (10.8)
\]

As a consequence of the Aubin-Lions Lemma, one has (c.f. \[23\]).

Lemma 10.8. \[23\] Let \( X_0, X \) and \( X_1 \) be three Banach spaces satisfying \( X_0 \subset X \subset X_1 \). Suppose that \( X_0 \) is compactly embedded in \( X \) and \( X \) is continuously embedded in \( X_1 \).

(1) Let \( G \) be bounded in \( L^p(0,T;X_0) \) with \( 1 \leq p < +\infty \), and \( \frac{\partial G}{\partial x} \) be bounded in \( L^1(0,T;X_1) \). Then \( G \) is relatively compact in \( L^p(0,T;X) \).

(2) Let \( F \) be bounded in \( L^\infty(0,T;X_0) \) and \( \frac{\partial F}{\partial x} \) be bounded in \( L^q(0,T;X_1) \) with \( q > 1 \). Then \( F \) is relatively compact in \( C(0,T;X) \).
The following lemma is useful to improve weak convergence to strong convergence.

**Lemma 10.9.** [23] If the function sequence \( \{w_n\}_{n=1}^{\infty} \) converges weakly to \( w \) in a Hilbert space \( X \), then it converges strongly to \( w \) in \( X \) if and only if

\[
\|w\|_X \geq \limsup_{n \to \infty} \|w_n\|_X.
\]

The last lemma will be used in the proof shown in Section 5.

**Lemma 10.10.** Let \((\varphi, \phi, w)\) be the regular solution to the Cauchy problem \((1.1)\), and \( Z \) be the time weighted energy defined in \((5.4)\). One has

\[
|\varphi \nabla^3 \text{div} w|_2^2 \leq |\varphi \nabla^4 w|_2^2 + J^*,
\]

where the term \( J^* \) can be controlled by

\[
J^* \leq \eta |\varphi \nabla^4 w|_2^2 + C(\eta)(1 + t)^{2m-5-2\gamma_3} Z^4,
\]

for any constant \( \eta > 0 \) small enough, and the constant \( C(\eta) > 0 \).

**Proof.** According to the definition of \( \text{div} \), one directly has

\[
|\varphi \nabla^3 \text{div} w|_2^2 = 3 \sum_{i=1}^{3} |\varphi \nabla^3 \partial_i w^{(i)}|_2^2 + 2 \sum_{i,j=1, i<j}^{3} \int \varphi^2 \nabla^3 \partial_i w^{(i)} \cdot \nabla^3 \partial_j w^{(j)}.
\]

Via the integration by parts, one can obtain that

\[
J_{ij} = \int \varphi^2 \nabla^3 \partial_i w^{(i)} \cdot \nabla^3 \partial_j w^{(j)} = \int \varphi^2 \nabla^3 \partial_i w^{(j)} \cdot \nabla^3 \partial_j w^{(i)}
\]

\[
+ \int (\partial_i \varphi^2 \nabla^3 \partial_j w^{(i)} \nabla^3 w^{(j)} - \partial_j \varphi^2 \nabla^3 \partial_i w^{(i)} \nabla^3 w^{(j)}).
\]

It is easy to see that

\[
J_{ij}^* = \int (\partial_i \varphi^2 \nabla^3 \partial_j w^{(i)} \nabla^3 w^{(j)} - \partial_j \varphi^2 \nabla^3 \partial_i w^{(i)} \nabla^3 w^{(j)})
\]

\[
\leq C|\nabla \varphi|_\infty |\varphi \nabla^4 w|_2 |\nabla^3 w|_2
\]

\[
\leq \eta |\varphi \nabla^4 w|_2^2 + C(\eta)(1 + t)^{2m-5-2\gamma_3} Z^4,
\]

for any constant \( \eta > 0 \) small enough, and the constant \( C(\eta) > 0 \), which, along with \((10.10)\)-(10.11), quickly implies \((10.9)\).

\[
\square
\]

10.2. **Proof of Proposition 2.1.** Here in this subsection, set

\[
\|f\|_\Xi = |\nabla f|_\infty + \|\nabla^2 f\|_{m-2} \text{ for any } m \geq 2.
\]

**Proof.** (I) Let \( G(t, x) = \nabla u(t, x) \) and \( G_0(x_0) = \nabla u_0(x_0) \). Then

\[
G(t, X(t; x_0)) = (I_d + tG_0(x_0))^{-1} G_0(x_0).
\]

Note that \( (I_d + tG_0(x_0))^{-1} = (\det(I_d + tG_0))^{-1} (\adj(I_d + tG_0))^T \), where \( \adj(I_d + tG_0) \) stands for the adjugate of \( (I_d + tG_0) \). Then it holds that

\[
|\nabla_x \tilde{u}|_\infty \leq \frac{(1 + t|\nabla u_0|_\infty)^{d-1}}{(1 + t\kappa)^d} |\nabla_x u_0|_\infty.
\]

(10.13)
Rewrite

\[ G(t, X(t; x_0)) = \frac{1}{1+t}I_d + \frac{1}{(1+t)^2}K(t, x_0), \]

where \( K(t, x_0) = (1+t)^2(I_d + tG_0)^{-1}G_0 - (1+t)I_d. \)

Since \( G_0^{-1} = (\text{det}(G_0))^{-1}(\text{adj}G_0)^\top, \) so

\[ |G_0^{-1}|_\infty \leq C\kappa^{-d}|G_0|^{d-1}. \]

Then for \( t \) large enough, one has \(|t^{-1}G_0^{-1}(x_0)| < 1\) for all \( x_0, \) and

\[ K(t, x_0) = \frac{(t+1)^2}{t}(I_d + t^{-1}G_0^{-1})^{-1} - (1+t)I_d 
= \frac{(t+1)^2}{t}(I_d - \frac{G_0^{-1}}{t} + O\left(\frac{1}{t^2}\right)) - (1+t)I_d 
= \frac{(t+1)^2}{t}I_d - \frac{(1+t)^2}{t^2}G_0^{-1} + O\left(\frac{1}{t}\right). \]

Remark 10.1. Note that

\[ G = \frac{1}{t}\left(I_d - (I_d + tG_0(x_0))^{-1}\right). \]

Then any eigenvalue \( \lambda_G \) of \( G \) has the following form

\[ \lambda_G = \frac{1}{t}\left(1 - (1 + t\lambda)^{-1}\right) = \frac{\lambda}{1 + t\lambda} > 0, \]

where \( \lambda \) is an eigenvalue of \( G_0. \)

(II) Let \( H(t, x_0) = G(t, X(t; x_0)). \) By induction, it is easy to show that, for \( k \geq 1: \)

\[ \nabla^k_{x_0}H = (I_d + tG_0)^{-1}\Lambda_k(I_d + tG_0)^{-1}, \]

where \( \Lambda_k \) is a sum of products of \( t(I_d + tG_0)^{-1} \) and \( \nabla^jG_0, \ j \in \{0, 1, ..., k\}, \) appearing \( \beta_j \) times with \( \sum_j j\beta_j = k. \)

On the one hand, by induction, it is also easy to show that, for \( k \geq 1: \)

\[ \nabla^k_{x_0}H(t, x_0) = \sum_{j=1}^{k} \nabla^jG(t, X(t; x_0))\left( \sum_{1 \leq k_i \leq k} \nabla^{k_i}_{x_0}X \otimes ... \otimes \nabla^{k_j}_{x_0}X \right) \]

with \( \sum_j k_i = k, \) and

\[ \nabla^j_{x_0}X = I_d + tG_0(x_0), \quad \nabla^l_{x_0}X = t\nabla^{l-1}G_0(x_0), \quad \text{for} \ l \geq 2. \]

On the other hand, one also show that for all \( j \geq 1 \) by induction: \( \nabla^j_G. \) \( \nabla^j_xG \) is a sum of terms which are products in a certain order of : \((I_d + tG_0)^{-1}, tI_d, I_d + tG_0 \) or \( \nabla^lG_0 \) appearing \( \beta_l \) times, with \( \sum_l l\beta_l = j. \) Moreover, the \( L^\infty \) norm of the terms with \( t \) is bounded by a constant times \((1+t)^{-j+2}, \) and

\[ |\nabla^j_{x_0}G(X(t; \cdot), t)|_2 \leq C_j(1+t)^{-(j+2)}, \quad \text{with} \quad C_j = C(\kappa, j, \|u_0\|\varepsilon). \]

Suppose \( \text{IH}(k-1), \) then

\[ \nabla^k_{x_0}G(t, X(t; x_0)) = \left( \nabla^k_{x_0}H(t, x_0) - \sum_{j=1}^{k-1} \nabla^j_{x_0}G(t, X(t; x_0))\left( \sum_{1 \leq k_i \leq k-1} \nabla^{k_i}_{x_0}X \otimes ... \otimes \nabla^{k_j}_{x_0}X \right) \right) \]

\[ \otimes (I_d + tG_0)^{-1)^\otimes k}. \]
In the right-hand side term, one has the norm of the following terms to estimate:

- a) \( J_1 = (I_d + tG_0)^{-1} \Lambda_k (I_d + tG_0)^{-k-1} \);
- b) \( J_2 = \nabla_x^2 G(I_d + tG_0)^{-s} \Pi_{k_i \neq 1} t^{|k_i|-1} G_0 (I_d + tG_0)^{-k} \), \( \sum_{k_i \neq 1} (k_i - 1) = k - j \),

where \( s \) is the number of \( k_j \) \( \neq 1 \).

For each of these terms, it needs to apply first the induction hypothesis and consider the \( L^{\infty} \)-norm in space for the terms with \( t \). It needs to use \( \text{IH}(j) \) for \( j \leq k - 1 \) to show that b) is a product of \( (\nabla^j G_0)^{\beta_j} \) with \( \sum_j j \beta_j = k \) and of \( tI_d, I_d + tG_0, (I_d + tG_0)^{-1} \), such that the \( L^{\infty} \)-norm of the terms with \( t \) is bounded by a constant times \( (1 + t)^{-(k+2)} \). Then one can find an upper bound in \( L^2 \)-norm for \( \Pi_{1 \leq j \leq k} (\nabla^j G_0)^{\beta_j} \) with \( \sum_j j \beta_j = k \), by using Gagliardo-Nirenberg inequality. Then one gets for \( \sum_j j \beta_j = k \):

\[
|J_1|_2 \leq C(1 + t)^{-k-2} |\Lambda_k|_2 \leq C(1 + t \kappa)^{-k-2} \Pi (\nabla^j G_0)^{\beta_j}|_2 \leq C(1 + t)^{-k-2} c_0^{k+1},
\]

\[
|J_2|_2 \leq C(1 + t)^{-k+j} |\nabla^j x G \Pi_{k_i \neq 1} t^{|k_i|-1} G_0|_2 \leq C(1 + t)^{-k-2} c_0^{k+1}.
\]

To conclude, one obtains the upper bound in \( \text{IH}(k) \) which depends on \( \kappa, |G_0|_\infty \), and \( |\nabla^k G_0|_2 \) for \( 1 \leq k \leq m \), that is to say on \( \kappa \) and \( \|u_0\|_\Xi \). Finally, it needs to make a change of variables to obtain:

\[
|\nabla^j x G(t, \cdot)|_2 \leq (1 + t)^{\frac{d}{2}} |\nabla^j x G(t, X(t; \cdot))|_2.
\]

Therefore (ii) is true for \( l \in \mathbb{N}, l \geq 2 \) since \( \nabla_x G = \nabla^2 \tilde{u} \). Then by interpolation one obtains the result for all \( l \in \mathbb{R}, l \geq 2 \).

(III) Since \( m - 1 > \frac{d}{2} \), one gets that \( \nabla^2 u_0 \in L^\infty \). Thus \( \nabla^2 \tilde{u} \in L^\infty \), and

\[
\nabla_x G(t, X(t; x_0)) = -(I_d + tG_0)^{-1} \nabla G_0(x_0)(I_d + tG_0)^{-2}.
\]

Using the estimates obtained in (I), we obtain: \( |\nabla_x G(t, X(t; x_0))|_\infty = O((1 + t)^{-3}) \).

(IV) If \( u_0(0) = 0 \), then it is obvious that \( \tilde{u}(t, 0) = 0 \) for any \( t \geq 0 \), and

\[
\tilde{u}(t, x) = \tilde{u}(t, 0) + \nabla \tilde{u}(t, ax) \cdot x = \nabla \tilde{u}(t, ax) \cdot x, \quad \text{for any } (t, x) \in [0, T] \times \mathbb{R}^3,
\]

where \( a \in [0, 1] \) is some constant. Thus (2.2) is proved.

\[
\Box
\]

10.3. Proof of Proposition 2.2

**Proof.** The solution \( Z(t) \) of the Cauchy problem (2.4) can be solved as:

\[
Z(t) = \frac{(1 + t)^{-b} \exp \left( \frac{C_2}{D_2+1} \left( (1 + t)^{D_2+1} - 1 \right) \right)}{Z_0^{-(a-1)} - (a - 1) C_1 \int_0^t (1 + s)^M \exp \left( \frac{(a-1)C_2}{D_2+1} \left( (1 + t)^{D_2+1} - 1 \right) \right) ds} \frac{1}{s^{a-1}},
\]

where \( M = D_1 - (a - 1)b < -1 \).

Thus, according to (2.3), \( Z(t) \) is globally well-defined for \( t \geq 0 \) if and only if

\[
0 < Z_0 < \frac{1}{(a - 1) C_1 \int_0^t (1 + s)^M \exp \left( \frac{(a-1)C_2}{D_2+1} \left( (1 + t)^{D_2+1} - 1 \right) \right) ds} \frac{1}{s^{a-1}}.
\]

Therefore, by choosing \( Z_0 \) small enough, one can obtain the global existence of our Cauchy problem (2.4). \( \Box \)
10.4. **Some Sobolev inequalities.** It follows from Lemma 10.1 that

\[ |\varphi \nabla w|_6 \leq C |\varphi \nabla^2 w|_{D^1} \leq C (|\varphi \nabla^2 w|_2 + |\nabla \varphi|_\infty |\nabla w|_2), \]

\[ |\varphi \nabla^2 w|_{\infty} \leq C |\varphi \nabla^2 w|_{D^1}^{\frac{1}{6}} |\nabla (\varphi \nabla^2 w)|_{D^1}^{\frac{1}{3}} \leq C |\varphi \nabla^2 w|_{D^1}^{\frac{1}{6}} |\nabla (\varphi \nabla^2 w)|_{D^1}^{\frac{1}{3}} \]

\[ \leq C \|\nabla (\varphi \nabla^2 w)\|_1 \leq C (\|\nabla \varphi\|_2 \|\nabla^2 w\|_1 + |\varphi \nabla^4 w|_2). \]  

(10.17)

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