Fundamental solutions for isotropic size-dependent couple stress elasticity

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1. Introduction

It has long been suggested that the strength of materials has size-dependency in smaller scales. Recently, in Hadjesfandiari and Dargush (2011), we have developed the consistent size-dependent couple stress theory for solids. Unlike previous work, this theory involves only true continuum kinematical quantities without recourse to any additional artificial degrees of freedom. By using the definition of admissible boundary conditions, along with kinematic and energy considerations, we have shown that the couple-stress tensor has a vectorial character and that the body couple is not distinguishable from the body force. The work also demonstrates that the stresses are fully determinate and the measure of deformation energetically-conjugate to couple-stress is the skew-symmetrical mean curvature tensor. This development can be extended quite naturally into many branches of continuum mechanics, including, for example, elastoplasticity and piezoelectricity. However, the first step is the development of infinitesimal linear isotropic elasticity, which involves only a single size-dependent constant.

Since this theory is much more complicated than Cauchy elasticity, analytical solutions are rare and, consequently, numerical formulations are needed to solve more general size-dependent couple stress elastic boundary value problems. Interestingly, it seems the boundary element method is a suitable numerical tool to solve a wide range of couple stress elastic boundary value problems. However, this requires the free space Green’s functions or fundamental solutions as the required kernels to transform the governing equations to a set of boundary integral equations. These fundamental solutions are the elastic solutions for an infinitely extended domain under the influence of unit concentrated forces and couples. The importance of these fundamental solutions is enhanced further, when we notice that the free space Green’s functions play a direct role in the solution of many practical problems for infinite domains.

In previous work, Mindlin and Tiersten (1962) have given the necessary potential functions for obtaining the three-dimensional displacement fundamental solutions in their indeterminate isotropic couple stress elasticity theory. Chowdhury and Glockner (1974) provide analogous functions by a matrix inversion technique for steady state vibration. Interestingly, we find that our linear equilibrium equations in terms of displacements in the determinate theory are identical to those in the indeterminate theory of Mindlin and Tiersten (1962) and Koiter (1964) for an isotropic material. This means the above displacement solutions are valid in our consistent, fully determinate theory. However, due to the indeterminacy of stresses, appearance of body couples independent of body force and existence of two size-dependent elastic constants, the corresponding stresses were not obtainable within the previous couple stress theory. It should be emphasized that the present consistent couple stress theory is not a special case of the Mindlin–Tiersten–Koiter theory. A full explanation is provided in Appendix A.

In the present work, we obtain all two- and three-dimensional displacement and stress fundamental solutions for our consistent isotropic couple stress elasticity. Within this theory, everything is fully determinate and depends on only a single size-dependent material constant. For the three-dimensional case, we derive the
displacement kernels directly by a decomposition method and then determine the corresponding stresses. Two-dimensional stress fundamental solutions have been presented in Huilgol (1967) only for concentrated force, based on the indeterminate couple-stress theory developed by Mindlin (1963a,b). Hadjesfandiari and Dargush (2011) have shown that some of these developments for plane problems remain useful in our determinate couple stress elasticity. However, we should remember in Mindlin's theory, there is an extra material constant, along with indeterminacy in the spherical part of the couple-stress tensor. Here we derive the complete two-dimensional fundamental solutions for point force and point couple, including displacements, force- and couple-stresses with a method similar to that used in the three-dimensional case.

Before continuing with the development for couple stress elasticity, we should mention the work to develop fundamental solutions within the framework of micropolar elasticity by Hirashima and Tomisawa (1977), Sandru (1966), Khan et al. (1971), Dragos (1984), Scarpetta (1990) and Sladek and Sladek (2003). Although their results have some similarities to our fundamental solutions, we should emphasize that microrotations are not a fully consistent continuum mechanics concept (Hadjesfandiari and Dargush, 2011). As another alternative, strain gradient theories have been developed. Here we mention the work on fundamental and singular solutions by Tsepoura et al. (2002), Polyzos et al. (2003), Karlis et al. (2007), Fannjiang et al. (2002) and Chan et al. (2006, 2008).

2. Basic equations of consistent couple stress theory

Let us assume the three-dimensional coordinate system $x_1 x_2 x_3$ as the reference frame with unit vectors $e_1, e_2,$ and $e_3$. Consider a material continuum occupying a volume $V$ enclosed by boundary surface $S$. In a continuum mechanical theory for size-dependent couple stress materials, the equations of equilibrium become

$$\sigma_{ij} + F_i = 0 \quad (1)$$

$$\mu_{ij} + \epsilon_{ikl} \sigma_{kl} = 0 \quad (2)$$

where $\sigma_{ij}$ and $\mu_{ij}$ are force- and couple-stress tensors, and $F_i$ is the body force per unit volume of the body. Within this theory, the true (polar) force-stress tensor is generally non-symmetric and can be decomposed as

$$\sigma_{ij} = \sigma_{(ij)} + \sigma_{ji} \quad (3)$$

where $\sigma_{(ij)}$ and $\sigma_{ji}$ are the symmetric and skew-symmetric parts, respectively. In Hadjesfandiari and Dargush (2011), we have shown that in a continuum mechanics theory, the pseudo (axial) couple-stress tensor is skew-symmetrical. Thus,

$$\mu_{ij} = -\mu_{ji} \quad (4)$$

Therefore, the true couple-stress vector $\mu_i$, dual to the tensor $\mu_{ij}$ can be defined by

$$\mu_i = \frac{1}{2} \epsilon_{ijk} \mu_{ij} \quad (5)$$

Then, the angular equilibrium equation gives the skew-symmetric part of the force-stress tensor as

$$\sigma_{(ij)} = -\mu_{ij} \quad (6)$$

The polar force-traction vector at a point on a surface with unit normal vector $n_i$ can be expressed as

$$t_i^{(n)} = \sigma_{ij} n_j \quad (7)$$

Similarly, the pseudo moment-traction vector can be written

$$m_i^{(n)} = \mu_{ij} n_j = \epsilon_{ijk} n_j \mu_{kj} \quad (8)$$

Hadjesfandiari and Dargush (2011) have shown that in couple stress materials, the body couple is not distinguishable from the body force. The body couple $C_j$ transforms into an equivalent body force $\frac{1}{2} \epsilon_{ijk} C_{kj}$ in the volume and a force-traction vector $\frac{1}{2} \epsilon_{ijk} C_{nj}$ on the bounding surface. In vectorial form, this means

$$F + \frac{1}{2} \nabla \times C = F \quad (9)$$

and

$$t^{(n)} + \frac{1}{2} C \times n = t^{(n)} \quad (10)$$

This is the first important result required for the development of the fully determinate couple stress theory.

In addition, there is a need to introduce the appropriate kinematical and constitutive relations. As an initial step, the displacement gradients are decomposed into symmetrical and skew-symmetrical components, such that

$$u_{ij} = e_{ij} + \omega_{ij} \quad (11)$$

where

$$e_{ij} = u_{ij} = \frac{1}{2} (u_{ij} + u_{ji}) \quad (12)$$

$$\omega_{ij} = u_{ij} - \frac{1}{2} (u_{ij} - u_{ji}) \quad (13)$$

Since the rotation tensor $\omega_{ij}$ is skew-symmetrical, one can introduce a dual axial rotation vector, such that

$$\omega_i = \frac{1}{2} \epsilon_{ijk} \omega_{kj} \quad (14)$$

In the usual infinitesimal Cauchy elasticity, only the symmetric strain tensor $e_{ij}$ contributes to the elastic energy. However, in the size-dependent couple stress elastic theory, mean curvatures $\kappa_i$ also play a role, where

$$\kappa_i = \epsilon_{ijk} \kappa_{kj} \quad (15)$$

Since the pseudo mean curvature tensor $\kappa_i$ is also skew-symmetrical, we can rewrite it in terms of its dual true mean curvature $\kappa_i$, where

$$\kappa_i = \frac{1}{2} \epsilon_{ijk} \kappa_{kj} \quad (16)$$

In Hadjesfandiari and Dargush (2011), we derive the general constitutive relations for an elastic material, where elastic energy is expressed in terms of the symmetrical strain tensor $e_{ij}$ and the mean curvature vector $\kappa_i$. Interestingly, for linear isotropic elastic media, the following constitutive relations can be written for the symmetric part of the force-stress tensor and couple-stress vector, respectively

$$\sigma_{(ij)} = \lambda e_{ijk} \delta_{ij} + 2 \mu e_{ij} \quad (17)$$

$$\mu_i = -8 \eta \kappa_i \quad (18)$$

and as a result for the skew-symmetric part of the force-stress tensor, we have

$$\sigma_{ji} = -\mu_{ij} = 8 \eta \kappa_{ij} = 2 \eta \epsilon_{ijk} \nabla^2 \omega_k \quad (19)$$

Therefore, the total force-stress tensor becomes

$$\sigma_{ij} = \lambda e_{ijk} \delta_{ij} + 2 \mu e_{ij} + 2 \eta \epsilon_{ijk} \nabla^2 \omega_k \quad (20)$$

Here $\lambda$ and $\mu$ are the usual Lamé elastic moduli, while $\eta$ is the sole additional parameter that accounts for couple stress effects in an isotropic material.
The elastic energy density \( W \) is defined by
\[
W = W_e + W_k = \frac{1}{2} \sigma_{ij} e_{ij} + \frac{1}{2} H_k K_{ki} = \frac{1}{2} \sigma_{ij} e_{ij} - \mu K_i
\]  
(21)

By using constitutive relations (17) and (18) in (21), this can be written as
\[
W = \frac{1}{2} \lambda (\varepsilon_{kk})^2 + \mu e_{ij} e_{ij} + 8\eta K_i K_i
\]  
(22)

The following restrictions on elastic constants are obtained by requiring positive definite elastic energy density:
\[
3\lambda + 2\mu > 0, \quad \mu > 0, \quad \eta > 0
\]  
(23)

The ratio
\[
\frac{\eta}{\mu} = \frac{\lambda}{2\mu}
\]  
(24)

specifies the characteristic material length \( l \) in the small deformation size-dependent elasticity theory under consideration here. Consequently, the constitutive relations (18) and (19) become
\[
\mu_i = -8\mu l^2 K_i
\]  
(25)
\[
\sigma_{ij} = -\mu \delta_{ij} = 2\mu l^2 e_{ij} \nabla^2 \omega_{ij}
\]  
(26)

It also should be noticed that
\[
\lambda = 2\mu \frac{v}{1 - 2v}
\]  
(27)

where \( v \) represents the usual Poisson ratio. Therefore, (17) can be written as
\[
\sigma_{ij} = 2\mu \left( \frac{v}{1 - 2v} e_{ij} \delta_{ij} + e_{ij} \right)
\]  
(28)

and the total stress tensor becomes
\[
\sigma_{ij} = 2\mu \left( \frac{v}{1 - 2v} e_{ij} \delta_{ij} + e_{ij} \right) + 2\mu l^2 e_{ij} \nabla^2 \omega_{ij}
\]  
(29)

After substituting (29) into (1), we can rewrite the governing differential equations in terms of the displacement and body force density fields as
\[
(\lambda + \mu + \mu l^2 \nabla^2) u_{ij} + \mu (1 - l^2 \nabla^2) \nabla^2 u_i + \nabla^2 F_i = 0
\]  
(30)

or in vectorial form as
\[
(\lambda + \mu + \mu l^2 \nabla^2)(\nabla \mathbf{u}) + (1 - l^2 \nabla^2) \nabla^2 \mathbf{u} + \mathbf{F} = 0
\]  
(31)

We use the direct decomposition method in the present work to derive the displacement fundamental solutions. Afterwards, we continue by developing the corresponding rotations, curvatures and fully-determinate force- and couple-stresses and tractions, which has previously not been possible. This, in turn, enables the formulation of new boundary integral representations and the development of boundary element methods to solve a broad range of couple stress elastostatic boundary value problems in both two- and three-dimensions.

3. Fundamental solutions for three-dimensional case

In this section, we derive the fundamental solutions for the three-dimensional case. As was mentioned, these are the elastic solutions of an infinitely extended domain under the influence of unit concentrated forces and couples. In Appendix B, we explore the singular behavior of these fundamental solutions.

3.1. Concentrated force

Assume that in the infinitely extended material, there is a unit concentrated force at the origin in an arbitrary direction specified by the unit vector \( \mathbf{a} \). This concentrated force can be represented as a body force
\[
\mathbf{F} = \delta^{(3)}(\mathbf{x}) \mathbf{a}
\]  
(32)

where \( \delta^{(3)}(\mathbf{x}) \) is the Dirac delta function in three-dimensional space. We know that
\[
\delta^{(3)}(\mathbf{x}) = -\nabla^2 \left( \frac{1}{4\pi r} \right)
\]  
(33)

where \( r \) is the distance from the origin to \( \mathbf{x} \). By applying the vectorial relation
\[
\nabla \times (\nabla \times \mathbf{G}) = \nabla(\nabla \cdot \mathbf{G}) - \nabla^2 \mathbf{G}
\]  
(34)

for the vector \( -\frac{\mathbf{a}}{r^2} \mathbf{a} \), we decompose the body force as
\[
\mathbf{F} = -\nabla^2 \left( \frac{1}{4\pi r} \right) \mathbf{a} = \nabla \times \left( \nabla \times \left( \frac{\mathbf{a}}{4\pi r} \right) \right) - \nabla \left( \nabla \cdot \frac{\mathbf{a}}{4\pi r} \right)
\]  
(35)

Consider the decomposition of the resulting displacement \( \mathbf{u}^{f} \) as
\[
\mathbf{u}^{f} = \mathbf{u}^{(1)} + \mathbf{u}^{(2)}
\]  
(36)

where \( \mathbf{u}^{(1)} \) and \( \mathbf{u}^{(2)} \) are the dilatational and solenoidal part of displacement \( \mathbf{u}^{f} \) satisfying
\[
\nabla \cdot \mathbf{u}^{(1)} = 0
\]  
(37)
\[
\nabla \times \mathbf{u}^{(2)} = 0
\]  
(38)

Then, it is seen that
\[
(\lambda + 2\mu) \nabla^2 \mathbf{u}^{(1)} = \nabla \left( \nabla \cdot \frac{\mathbf{a}}{4\pi r} \right)
\]  
(39)
\[
\mu l^2 \nabla^2 \nabla^2 \mathbf{u}^{(2)} - \mu \nabla^2 \mathbf{u}^{(2)} = \nabla \times \left( \nabla \times \frac{\mathbf{a}}{4\pi r} \right)
\]  
(40)

If we introduce two vectors \( \mathbf{A}^{(1)} \) and \( \mathbf{A}^{(2)} \), such that
\[
\mathbf{u}^{(1)} = \nabla \left( \nabla \cdot \mathbf{A}^{(1)} \right)
\]  
(41)
\[
\mathbf{u}^{(2)} = \nabla \times \left( \nabla \times \mathbf{A}^{(2)} \right) = \nabla \left( \nabla \cdot \mathbf{A}^{(2)} \right) - \nabla^2 \mathbf{A}^{(2)}
\]  
(42)

then we see that they satisfy the conditions in (37) and (38), respectively, and therefore
\[
(\lambda + 2\mu) \nabla^2 \mathbf{A}^{(1)} = \frac{\mathbf{a}}{4\pi r}
\]  
(43)
\[
\mu l^2 \nabla^2 \nabla^2 \mathbf{A}^{(2)} - \mu \nabla^2 \mathbf{A}^{(2)} = \frac{\mathbf{a}}{4\pi r}
\]  
(44)

These solutions should be in the form
\[
\mathbf{A}^{(1)} = \varphi \mathbf{a}
\]  
(45)
\[
\mathbf{A}^{(2)} = \psi \mathbf{a}
\]  
(46)

where \( \varphi \) and \( \psi \) are scalar functions of \( r \) having radial symmetry. Therefore,
\[
\nabla^2 \varphi = \frac{1}{4\pi(\lambda + 2\mu)r}
\]  
(47)
\[
\mu l^2 \nabla^2 \varphi - \nabla^2 \psi = \frac{1}{4\pi \mu r^3}
\]  
(48)

where \( \nabla^2 \) is the laplacian operator in three dimensions. Because of radial symmetry, this operator reduces to
\[
\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} = \frac{1}{r^2} \left( \frac{d}{dr} - \frac{2}{r} \frac{d}{dr} \right)
\]  
(49)
The regular solutions to (47) and (48) are
\[
\psi = \frac{1}{8\pi \mu (\lambda + 2\mu)} \rho
\]
\[
\psi = \frac{\rho}{4\pi \mu} \left( e^{-\sqrt{\lambda + \lambda/\mu}} - 1 \right) - \frac{1}{8\pi \mu} \rho
\]
Therefore, we have
\[
\mathbf{A}^{(1)} = \frac{1}{8\pi \mu (\lambda + 2\mu)} \mathbf{r}
\]
\[
\mathbf{A}^{(2)} = -\frac{\rho^2}{4\pi \mu} \frac{1 - e^{-\sqrt{\lambda + \lambda/\mu}} \mathbf{a} - \frac{1}{8\pi \mu r} \mathbf{r}
\]
Now from (41),
\[
u = \frac{1}{8\pi \mu (\lambda + 2\mu)} \mu \left[ \psi_{\lambda\xi} - \frac{\lambda X_{\xi}}{r^2} \right] \rho
\]
By using the relation in (27), we have
\[
\lambda + 2\mu = 2\mu \frac{1 - \nu}{1 - 2\nu}
\]
and (54) can be written as
\[
u = \frac{1 - 2\nu}{16\pi \mu (1 - \nu)} \left[ \frac{\lambda X_{\xi}}{r^2} \right] \rho
\]
It is also seen that from (42)
\[
u = \left\{ \psi_{\lambda\xi} + \frac{1}{4\pi \mu} \frac{1 - e^{-\sqrt{\lambda + \lambda/\mu}} \rho}{\rho} \right\} \rho
\]
and by using (51), we obtain
\[
u = \frac{1}{4\pi \mu r^2} \left\{ \left( 3 + 3 \frac{r^2}{1 + \frac{r^2}{3}} \right) e^{-\sqrt{\lambda + \lambda/\mu}} \frac{X_{\xi}}{r^2} \right. \]
\[
+ \left[ 1 - \left( 1 + r \frac{r^2}{1 + \frac{r^2}{3}} \right) \rho \right] \psi_{\lambda\xi} + \frac{1}{8\pi \mu r} \left( \frac{X_{\xi}}{r^2} + \frac{\lambda X_{\xi}}{r^2} \right) \rho
\]
and for the total displacement, we have
\[
u = \frac{1}{16\pi \mu (1 - \nu)} \frac{1}{r^2} \left[ (3 - 4\nu) \frac{X_{\xi}}{r^2} + \frac{\lambda X_{\xi}}{r^2} \rho \right]
\]
\[
+ \frac{1}{4\pi \mu r^2} \left\{ \left( 3 + 3 \frac{r^2}{1 + \frac{r^2}{3}} \right) e^{-\sqrt{\lambda + \lambda/\mu}} \frac{X_{\xi}}{r^2} \right. \]
\[
+ \left[ 1 - \left( 1 + r \frac{r^2}{1 + \frac{r^2}{3}} \right) \rho \right] \psi_{\lambda\xi} + \frac{1}{8\pi \mu r} \left( \frac{X_{\xi}}{r^2} + \frac{\lambda X_{\xi}}{r^2} \right) \rho
\]
Notice that the first part of this relation is the displacement from Cauchy elasticity.

In our determinate theory, we can now derive determinate stresses. These force- and couple-stresses are developed in the following. As a first step in that direction, we find that the gradient of displacement from (59) is
\[
u = \frac{1}{16\pi \mu (1 - \nu)} \frac{1}{r^2} \left[ (3 - 4\nu) \frac{X_{\xi}}{r^2} + \frac{\lambda X_{\xi}}{r^2} \rho \right]
\]
\[
+ \frac{1}{4\pi \mu r^2} \left\{ \left( 3 + 3 \frac{r^2}{1 + \frac{r^2}{3}} \right) e^{-\sqrt{\lambda + \lambda/\mu}} \frac{X_{\xi}}{r^2} \right. \]
\[
+ \left[ 1 - \left( 1 + r \frac{r^2}{1 + \frac{r^2}{3}} \right) \rho \right] \psi_{\lambda\xi} + \frac{1}{8\pi \mu r} \left( \frac{X_{\xi}}{r^2} + \frac{\lambda X_{\xi}}{r^2} \right) \rho
\]
Therefore, the strain tensor becomes
\[
\varepsilon = \frac{1}{16\pi \mu (1 - \nu)} \frac{1}{r^2} \left[ (3 - 4\nu) \frac{X_{\xi}}{r^2} + \frac{\lambda X_{\xi}}{r^2} \rho \right]
\]
\[
+ \frac{1}{4\pi \mu r^2} \left\{ \left( 3 + 3 \frac{r^2}{1 + \frac{r^2}{3}} \right) e^{-\sqrt{\lambda + \lambda/\mu}} \frac{X_{\xi}}{r^2} \right. \]
\[
+ \left[ 1 - \left( 1 + r \frac{r^2}{1 + \frac{r^2}{3}} \right) \rho \right] \psi_{\lambda\xi} + \frac{1}{8\pi \mu r} \left( \frac{X_{\xi}}{r^2} + \frac{\lambda X_{\xi}}{r^2} \right) \rho
\]
\[
= \frac{1}{16\pi \mu (1 - \nu)} \frac{1}{r^2} \left[ (3 - 4\nu) \frac{X_{\xi}}{r^2} + \frac{\lambda X_{\xi}}{r^2} \rho \right]
\]
\[
+ \frac{1}{4\pi \mu r^2} \left\{ \left( 3 + 3 \frac{r^2}{1 + \frac{r^2}{3}} \right) e^{-\sqrt{\lambda + \lambda/\mu}} \frac{X_{\xi}}{r^2} \right. \]
\[
+ \left[ 1 - \left( 1 + r \frac{r^2}{1 + \frac{r^2}{3}} \right) \rho \right] \psi_{\lambda\xi} + \frac{1}{8\pi \mu r} \left( \frac{X_{\xi}}{r^2} + \frac{\lambda X_{\xi}}{r^2} \right) \rho
\]
\[
\therefore \varepsilon = \frac{1}{16\pi \mu (1 - \nu)} \frac{1}{r^2} \left[ (3 - 4\nu) \frac{X_{\xi}}{r^2} + \frac{\lambda X_{\xi}}{r^2} \rho \right]
\]
\[
+ \frac{1}{4\pi \mu r^2} \left\{ \left( 3 + 3 \frac{r^2}{1 + \frac{r^2}{3}} \right) e^{-\sqrt{\lambda + \lambda/\mu}} \frac{X_{\xi}}{r^2} \right. \]
\[
+ \left[ 1 - \left( 1 + r \frac{r^2}{1 + \frac{r^2}{3}} \right) \rho \right] \psi_{\lambda\xi} + \frac{1}{8\pi \mu r} \left( \frac{X_{\xi}}{r^2} + \frac{\lambda X_{\xi}}{r^2} \right) \rho
\]
and the skew-symmetric part of the force-stress tensor becomes
\[
\sigma_{ij}^{\text{skew}} = - \frac{1}{4\pi r^2} \left[ (1 + \frac{r}{R}) e^{-r/R} \right] x_i \delta_{jk} - x_j \delta_{ik} \frac{d a_q}{r}
\]
(70)

As a result the total force-stress tensor is
\[
\sigma_{ij} = \sigma_{ij}^{\text{sym}} + \sigma_{ij}^{\text{skew}} = - \frac{1}{8\pi(1 - \nu) r^2}
\times \left[ (1 - 2\nu) x_i \delta_{jk} + x_j \delta_{ik} - x_k \delta_{ij} \frac{3x_k x_j}{r^3} \right] a_q
\]
\[
+ \frac{1}{2\pi r^2} \left[ \left( \frac{15}{2} - \left( \frac{15 + 15 \nu + 6 \nu^2}{r^2} + \frac{r}{r^2} \right) e^{-r/\nu} \right) \frac{x_k x_j}{r^3} \right] \delta_{jk} a_q
\]
\[
+ \left[ \left( \frac{3 + 3 \nu + \frac{r}{r^2} \right) e^{-r/\nu} - 3 \right] \delta_{ij} x_k + \delta_{ik} x_j + \delta_{jk} x_i \right] a_q
\]
\[
+ \frac{1}{2\pi r^2} \left[ \left( 1 + \frac{r}{R} \right) e^{-r/\nu} \right] x_i \delta_{jk} \frac{d a_q}{r}
\]
(71)

Therefore, the force-traction vector becomes
\[
\tau_{ij}^f = \sigma_{ij} n = - \frac{1}{8\pi(1 - \nu) r^2}
\times \left[ (1 - 2\nu) x_i n_j \delta_{jk} + x_j n_k \delta_{ik} - x_k n_i \frac{3x_k x_j}{r^3} \right] n_a
\]
\[
+ \frac{1}{2\pi r^2} \left[ \left( \frac{15}{2} - \left( \frac{15 + 15 \nu + 6 \nu^2}{r^2} + \frac{r}{r^2} \right) e^{-r/\nu} \right) \frac{x_k x_j}{r^3} \right] \delta_{jk} a_q
\]
\[
+ \left[ \left( \frac{3 + 3 \nu + \frac{r}{r^2} \right) e^{-r/\nu} - 3 \right] \delta_{ij} x_k + \delta_{ik} x_j + \delta_{jk} x_i \right] a_q
\]
\[
+ \frac{1}{2\pi r^2} \left[ \left( 1 + \frac{r}{R} \right) e^{-r/\nu} \right] x_i \delta_{jk} \frac{d a_q}{r}
\]
(72)

and the moment-traction vector is
\[
m_{ij}^{mf} = e_{ij} n \mu = \frac{1}{2\pi r^2} \left[ 3 - \left( \frac{3 + 3 \nu + \frac{r}{r^2} \right) e^{-r/\nu} \right]
\times \left( \frac{x_k n_j - x_j n_k}{r} \frac{x_k x_j}{r^3} \right) \delta_{jk} a_q
\]
\[
+ \frac{1}{2\pi r^2} \left[ \left( \frac{3 + 3 \nu + \frac{r}{r^2} \right) e^{-r/\nu} - 3 \right] \delta_{ij} x_k + \delta_{ik} x_j + \delta_{jk} x_i \right] a_q
\]
\[
+ \frac{1}{2\pi r^2} \left[ \left( 1 + \frac{r}{R} \right) e^{-r/\nu} \right] x_i \delta_{jk} \frac{d a_q}{r}
\]
(73)

Finally, we can consider the following relations
\[
U^f = U^f a_q
\]
(74)
\[
\omega^f = \Omega^f a_q
\]
(75)
\[
\sigma^f = \Sigma^f a_q
\]
(76)
\[
\kappa^f = \kappa^f a_q
\]
(77)
\[
\mu^f = M^f a_q
\]
(78)
\[
t_{ij}^{mf} = T_{ij}^f a_q
\]
(79)
\[
m_{ij}^{mf} = M_{ij}^f a_q
\]
(80)

where \(U^f, \omega^f, \sigma^f, \kappa^f, \mu^f, t_{ij}^{mf}, \) and \(m_{ij}^{mf}\) represent the corresponding displacement, rotation, force-stress, mean curvature, couple-stress, force-traction and moment-traction, respectively, at \(x\) due to a unit concentrated force in the \(g\)-direction at the origin. It is seen that these Green's functions are
\[
U^f = \frac{1}{16\pi \mu} \left( \frac{1}{1 - \nu} \right) \left[ (3 - 4\nu) \delta_{ij} + \frac{x_k x_j}{r^3} \right] + \frac{1}{4\pi \mu}
\times \frac{1}{r^2} \left[ \left( \frac{3 + 3 \nu + \frac{r}{r^2} \right) e^{-r/\nu} - 3 \right] \frac{x_k x_j}{r^3} \right] \delta_{ij}
\]
\[
+ \frac{1}{2\pi r^2} \left[ \left( \frac{15}{2} - \left( \frac{15 + 15 \nu + 6 \nu^2}{r^2} + \frac{r}{r^2} \right) e^{-r/\nu} \right) \frac{x_k x_j}{r^3} \right] \delta_{ij}
\]
\[
+ \frac{1}{2\pi r^2} \left[ \left( 1 + \frac{r}{R} \right) e^{-r/\nu} \right] \frac{x_k x_j}{r^3} \frac{d a_q}{r}
\]
\[
\Omega^f = \frac{1}{8\pi \mu} \left[ \left( 1 + \frac{r}{R} \right) e^{-r/\nu} - 1 \right] \frac{\epsilon_{ij} \epsilon_{kj}}{r}
\]
(81)

Next, \(\Sigma^f, \kappa^f, \mu^f, t_{ij}^{mf}, \) and \(m_{ij}^{mf}\) represent the corresponding rotation, displacement, force-stress, mean curvature, couple-stress, force-traction and moment-traction, respectively, at \(x\) due to a unit concentrated force in the \(g\)-direction at the origin. It is seen that these Green's functions are
\[
\Sigma^f = - \frac{1}{8\pi \mu} \left( \frac{1}{1 - \nu} \right) \left[ (3 - 4\nu) \delta_{ij} + \frac{x_k x_j}{r^3} \right] + \frac{1}{4\pi \mu}
\times \frac{1}{r^2} \left[ \left( \frac{3 + 3 \nu + \frac{r}{r^2} \right) e^{-r/\nu} - 3 \right] \frac{x_k x_j}{r^3} \right] \delta_{ij}
\]
\[
+ \frac{1}{2\pi r^2} \left[ \left( \frac{15}{2} - \left( \frac{15 + 15 \nu + 6 \nu^2}{r^2} + \frac{r}{r^2} \right) e^{-r/\nu} \right) \frac{x_k x_j}{r^3} \right] \delta_{ij}
\]
\[
+ \frac{1}{2\pi r^2} \left[ \left( 1 + \frac{r}{R} \right) e^{-r/\nu} \right] \frac{x_k x_j}{r^3} \frac{d a_q}{r}
\]
(82)

\[\forall \] Concentrated couple

Assume that in an infinitely extended couple stress elastic material there is a unit concentrated couple at the origin in the arbitrary direction specified by the unit vector \(a\). Therefore, we represent this concentrated couple by the body couple
\[
C = \delta^{(3)}(x) a
\]
(88)

As was mentioned previously, the effect of a body couple is equivalent to the effect of a body force represented by
\[
F^c = \frac{1}{2} \nabla \times C = \frac{1}{2} \nabla \delta^{(3)}(x) \times a
\]
(89)

or
\[
F^c = \frac{1}{2} \epsilon_{ij} \delta^{(3)} a_k
\]
(90)

with a vanishing surface effect at infinity. This shows that the displacement field of the concentrated couple \(C = \delta^{(3)}(x) a\) is equivalent to the rotation field of the concentrated force \(F = \delta^{(3)}(x) a\). Therefore, the solutions of the two problems are related, such that
\[
U^c = \frac{1}{2} \epsilon_{ij} \delta^{(3)} a_k
\]
(91)

which gives
\[
U^c = \frac{1}{8\pi \mu} \left( \frac{1}{1 - \nu} \right) \left[ (3 - 4\nu) \delta_{ij} + \frac{x_k x_j}{r^3} \right] + \frac{1}{4\pi \mu}
\times \frac{1}{r^2} \left[ \left( \frac{3 + 3 \nu + \frac{r}{r^2} \right) e^{-r/\nu} - 3 \right] \frac{x_k x_j}{r^3} \right] \delta_{ij}
\]
\[
+ \frac{1}{2\pi r^2} \left[ \left( \frac{15}{2} - \left( \frac{15 + 15 \nu + 6 \nu^2}{r^2} + \frac{r}{r^2} \right) e^{-r/\nu} \right) \frac{x_k x_j}{r^3} \right] \delta_{ij}
\]
\[
+ \frac{1}{2\pi r^2} \left[ \left( 1 + \frac{r}{R} \right) e^{-r/\nu} \right] \frac{x_k x_j}{r^3} \frac{d a_q}{r}
\]
(92)

This can be written as
Alternatively, we may write
\[ e^i_{0k} = \frac{1}{16\pi\mu} \frac{1}{r^3} \left[ \left( 3 - \frac{3 + 3\frac{r^2}{r^2} + \frac{r^2}{r^2} }{r^2} \right) e^{-r/l} \right] \frac{X_X}{r^2} \frac{e_{ipq}}{r^2} + \frac{X_X}{r^2} \frac{e_{ipq}}{r^2} \right) a_k \]
where the first part of this relation is the rotation from Cauchy elasticity.

Furthermore, the mean curvature vector is
\[ \kappa^i = \frac{1}{2} e_{0k} \alpha_{kj} = -\frac{1}{32\pi\mu} \frac{1}{r^2} \left[ \left( 1 + \frac{r}{l} \right) e^{-r/l} \right] \frac{e_{ipq}}{r^2} \frac{a_k}{r^2} \]
(101)
and therefore the couple-stress vector becomes
\[ \mu^c_{ij} = -8\mu^c \kappa^{kj} = \frac{1}{4\pi\mu} \frac{1}{r^2} \left[ \left( 1 + \frac{r}{l} \right) e^{-r/l} \right] \frac{e_{ipq}}{r^2} \frac{a_k}{r^2} \]
(102)
It should be noticed that there is no mean curvature in Cauchy elasticity for this loading.

The skew-symmetric part of \( \mu^c_{ij} \) is
\[ \mu^c_{ij} = -\frac{1}{8\pi\mu} \frac{1}{r^2} \left[ \left( 3 + 3\frac{r^2}{r^2} + \frac{r^2}{r^2} \right) e^{-r/l} \right] \frac{e_{ipq}}{r^2} \frac{X_X}{r^2} \frac{e_{ipq}}{r^2} \frac{a_k}{r^2} + \frac{1}{4\pi\mu} \frac{1}{r^2} \left[ \left( 1 + \frac{r}{l} \right) e^{-r/l} \right] \frac{e_{ipq}}{r^2} \frac{a_k}{r^2} \]
\[
T_{ij} = \frac{1}{8\pi} \left[ \frac{3}{4} \frac{X_{ij} r^2}{r^4} \epsilon_{ij} n_j + \frac{1}{8\pi} \right] \times \frac{1}{r^2} \left[ 3 - \left( 3 + \frac{r}{r^2} \right) e^{-r/r} \right] X_{ij} \frac{r^2}{r^2} \epsilon_{ij} n_j - \frac{1}{4\pi} \times \left( 1 + \frac{r}{r} \right) e^{-r/r} \epsilon_{ij} n_j
\]

(120)

\[
M_{ij} = \frac{1}{4\pi r^2} \left[ \left( 1 + \frac{r}{r} \right) e^{-r/r} \right] x n_i n_j - x n_i n_j \delta_{ij}
\]

(121)

Appendix B provides an investigation of the singular behavior of these three-dimensional fundamental solutions.

4. Fundamental solutions for two-dimensional case

In this section, we first present the governing equations of size-dependent couple stress elasticity in two-dimensions under plane strain conditions. Then, we derive the complete two-dimensional fundamental solutions with a similar method to that employed in the three-dimensional case. In Hadjesfandiari and Dargush (2012), we use these fundamental solutions within a two-dimensional boundary element method.

4.1. Governing equations for two dimensions

We suppose that the media occupies a cylindrical region, such that the axis of the cylinder is parallel to the \( x_3 \)-axis. Furthermore, we assume the body is in a state of planar deformation parallel to this plane, such that

\[
u_{x_3} = 0, \quad u_3 = 0 \quad \text{in} \quad V
\]

(122a, b)

where all Greek indices, here and throughout the remainder of the paper, will vary only over the range 1 and 2. Also, let \( V^{(2)} \) and \( S^{(2)} \) represent, respectively, the cross section of the body in the \( x_1x_2 \)-plane and its bounding edge in that plane.

As a result of these assumptions, all quantities are independent of \( x_3 \). Then, throughout the domain

\[
\omega_3 = 0, \quad \epsilon_{3i} = \epsilon_3 = 0, \quad \kappa_3 = 0
\]

(123a-c)

and

\[
\sigma_{3i} = \sigma_{3i} = 0, \quad \mu_3 = \mu_{21} = 0
\]

(124a, b)

Introducing the abridged notation

\[
\omega = \omega_3 = \frac{1}{2} \epsilon_{3j} u_{3j}
\]

(125)

where \( \epsilon_{3j} \) is the two-dimensional alternating symbol with

\[
\epsilon_{12} = -\epsilon_{21} = 1, \quad \epsilon_{11} = \epsilon_{22} = 0
\]

(126)

we see that the non-zero components of the curvature vector are

\[
K_3 = \frac{1}{2} \epsilon_{3j} \omega
\]

(127)

Therefore, the non-zero components of stresses are written

\[
\mu_3 = -4\mu r^2 \epsilon_{3j} \omega
\]

(128)

\[
\sigma_{23} = \lambda \epsilon_{21} \omega + 2\mu \epsilon_{3j} \omega
\]

(129)

\[
\sigma_{3i} = -\mu_{23} - 2\mu r^2 \omega
\]

(130)

and for total stress

\[
\sigma_{ij} = \lambda \epsilon_{ij} \omega + 2\mu \epsilon_{3j} \omega
\]

(131)

All the other stresses are zero, apart from \( \sigma_{33} \) and \( \mu_{33} \), which are given as

\[
\sigma_{33} = \nu \sigma_{22}
\]

(132)

\[
\mu_{33} = -4\mu r^2 \omega
\]

(133)

Notice that the stresses in (132) and (133), acting on planes parallel to the \( x_1x_2 \)-plane, do not enter directly into the solution of the boundary value problem.

For the planar problem, the stresses must satisfy the three equilibrium equations

\[
\sigma_{23} + F_2 = 0
\]

(134)

\[
\mu_{23} + \epsilon_{23} \sigma_{23} = 0
\]

(135)

with the obvious requirement \( F_3 = 0 \). The moment equation can be written as

\[
\sigma_{13} = -\mu_{13} = 0
\]

(136)

which actually gives the non-zero components

\[
\sigma_{21} = -\sigma_{12} = -\mu_{12}
\]

(137)

We also notice that the force-traction reduces to

\[
\tau_2^{(1)} = \sigma_{23} n_p
\]

(138)

and the moment-traction has only one component \( m_3 \). This can be conveniently denoted by the abridged symbol \( m \), where

\[
m^{(n)} = m_3^{(n)} = \epsilon_{3j} \mu_3 n_{3j} = 4\mu r^2 \frac{\partial \omega}{\partial n}
\]

(139)

4.2. Concentrated force

Assume that there is a line load on the \( x_3 \)-axis with an intensity of unity per unit length in the arbitrary direction specified by the unit vector \( \mathbf{a} \) in the \( x_1x_2 \)-plane. This distributed load can be represented by the body force

\[
\mathbf{F} = \delta^{(2)}(\mathbf{x}) \mathbf{a}
\]

(140)

where \( \delta^{(2)}(\mathbf{x}) \) is the Dirac delta function in two-dimensional space. It is known that

\[
\nabla^2 \left( \frac{1}{2\pi} \ln r \right) = \delta^{(2)}(\mathbf{x})
\]

(141)

where \( r \) is the distance from the origin to \( \mathbf{x} \) in the \( x_1x_2 \)-plane. By applying the vectorial relation in (34) for the vector \( \frac{1}{r\pi} \ln r \mathbf{a} \), we decompose the body force as

\[
\mathbf{F} = \nabla^2 \left( \frac{1}{2\pi} \ln r \right) \mathbf{a} = \nabla \left( \nabla \cdot \frac{\ln r \mathbf{a}}{2\pi} \right) - \nabla \times \left( \nabla \times \frac{\ln r \mathbf{a}}{2\pi} \right)
\]

(142)

If we consider the decomposition of displacement \( \mathbf{u}^f \) as

\[
\mathbf{u}^f = \mathbf{u}^{(1)} + \mathbf{u}^{(2)}
\]

(143)

where \( \mathbf{u}^{(1)} \) and \( \mathbf{u}^{(2)} \) are the dilatational and solenoidal part of the displacement vector \( \mathbf{u}^f \) satisfying

\[
\nabla \times \mathbf{u}^{(1)} = 0
\]

(144)

\[
\nabla \cdot \mathbf{u}^{(2)} = 0
\]

(145)

then it is seen that

\[
(\lambda + 2\mu) \nabla^2 \mathbf{u}^{(1)} = -\nabla \left( \nabla \cdot \frac{\ln r \mathbf{a}}{2\pi} \right)
\]

(146)

\[
\mu \nabla \nabla \times \mathbf{u}^{(2)} - \mu \nabla^2 \mathbf{u}^{(2)} = -\nabla \times \left( \nabla \times \frac{\ln r \mathbf{a}}{2\pi} \right)
\]

(147)

If we introduce two vectors \( \mathbf{A}^{(1)} \) and \( \mathbf{A}^{(2)} \), such that
\[ \mathbf{u}^{(1)} = \nabla ( \nabla \cdot \mathbf{A}^{(1)} ) \]
\[ \mathbf{u}^{(2)} = \nabla \times ( \nabla \times \mathbf{A}^{(2)} ) = \nabla ( \nabla \cdot \mathbf{A}^{(2)} ) - \nabla^{2} \mathbf{A}^{(2)} \]
which satisfy the conditions of (144) and (145), respectively, we obtain
\[ (\lambda + 2\mu)\nabla^{2} \mathbf{u}^{(1)} = -\frac{\ln r}{2\pi} \mathbf{a} \]
\[ \mu\nabla^{2} \nabla^{2} \mathbf{u}^{(2)} = -\frac{\ln r}{2\pi} \mathbf{a} \]
Then, the solutions are in the form
\[ \mathbf{A}^{(1)} = \varphi \mathbf{a} \]
\[ \mathbf{A}^{(2)} = \psi \mathbf{a} \]
where \( \varphi \) and \( \psi \) are scalar functions of \( r \) having two-dimensional radial symmetry. Therefore
\[ \nabla^{2} \varphi = -\frac{1}{2\pi(\lambda + 2\mu)} \ln r \]
\[ \nabla^{2} \psi - \nabla^{2} \varphi = -\frac{1}{2\pi} \ln r \]
In two-dimensions, with radial symmetry, the laplacian \( \nabla^{2} \) reduces to
\[ \nabla^{2} = \frac{d^{2}}{dr^{2}} + \frac{1}{r} \frac{d}{dr} \]
and the regular solutions to (154) and (155) are
\[ \varphi = -\frac{1}{8\pi(\lambda + 2\mu)} (r^{2} \ln r - r^{2}) \]
\[ \psi = \frac{\lambda}{2\pi \mu} \left[ K_{0}(\frac{r}{\lambda}) + \ln r \right] + \frac{1}{8\pi \mu} (r^{2} \ln r - r^{2}) \]
where \( K_{0} \) is the modified Bessel function of the second kind of zeroth order. Therefore,
\[ \mathbf{A}^{(1)} = -\frac{1}{8\pi(\lambda + 2\mu)} (r^{2} \ln r - r^{2}) \mathbf{a} \]
\[ \mathbf{A}^{(2)} = \frac{\lambda}{2\pi \mu} \left[ K_{0}(\frac{r}{\lambda}) + \ln r \right] \mathbf{a} + \frac{1}{8\pi \mu} (r^{2} \ln r - r^{2}) \mathbf{a} \]
Then, from (148)
\[ u_{a}^{(1)} = -\frac{1}{2\pi \mu(1-\nu)} \left[ \frac{2}{\nu} \mathbf{X}_{\mu} - (2 \ln r - 1) \delta_{\mu r} \right] a_{r} \]
and from (149)
\[ u_{a}^{(2)} = \left\{ \psi_{,\mu} - \frac{1}{2\pi \mu} \left[ K_{0}(\frac{r}{\lambda}) + \ln r \right] \delta_{\mu r} \right\} a_{r} \]
By using the relations
\[ \frac{\partial}{\partial r} K_{0}(\frac{r}{\lambda}) = -\frac{1}{\lambda} K_{1}(\frac{r}{\lambda}) \]
\[ \frac{\partial}{\partial r} K_{1}(\frac{r}{\lambda}) = -\frac{1}{\lambda} K_{0}(\frac{r}{\lambda}) - \frac{1}{\lambda} K_{1}(\frac{r}{\lambda}) \]
where \( K_{1} \) is the modified Bessel function of the second kind of first order, we obtain
\[ u_{c}^{(2)} = \frac{1}{8\pi \mu} \left\{ \frac{2}{\nu} \mathbf{X}_{\mu} - (2 \ln r + 1) \delta_{\mu r} \right\} a_{r} + \frac{1}{2\pi \mu} \left[ K_{0}(\frac{r}{\lambda}) + \ln r \right] \delta_{\mu r} a_{r} \]
\[ -\frac{1}{2\pi \mu} \left[ K_{0}(\frac{r}{\lambda}) + \ln r \right] \delta_{\mu r} a_{r} \]
It should be noticed that there are rigid body translation terms in \( u_{a}^{(1)} \) and \( u_{c}^{(2)} \) that cannot affect stress distributions. These terms can be neglected in this Green’s function for stress analysis. Therefore, by ignoring these rigid body terms and using
\[ u_{c}^{(2)} = u_{c}^{(1)} + u_{c}^{(2)} \]
we obtain
\[ u_{c}^{(2)} = -\frac{1}{8\pi \mu(1-\nu)} \left[ (3 - 4\nu) \ln r \delta_{\mu r} - \mathbf{X}_{\mu} \delta_{\mu r} + \mathbf{X}_{\mu} \delta_{\mu r} + \mathbf{X}_{\mu} \delta_{\mu r} + 2 \mathbf{X}_{\mu} \mathbf{X}_{\mu} \right] a_{r} + \frac{1}{2\pi \mu} \left[ K_{0}(\frac{r}{\lambda}) + \ln r \right] \delta_{\mu r} a_{r} \]
\[ -\frac{1}{2\pi \mu} \left[ K_{0}(\frac{r}{\lambda}) + \ln r \right] \delta_{\mu r} a_{r} \]
For the gradient of displacements, we have
\[ u_{\sigma,\mu}^{(2)} = -\frac{1}{8\pi \mu(1-\nu)} \left[ (3 - 4\nu) \mathbf{X}_{\sigma} \delta_{\mu r} - \mathbf{X}_{\sigma} \delta_{\mu r} - \mathbf{X}_{\sigma} \delta_{\mu r} - \mathbf{X}_{\sigma} \delta_{\mu r} + 2 \mathbf{X}_{\sigma} \mathbf{X}_{\mu} \right] a_{r} + \frac{1}{2\pi \mu} \left[ K_{0}(\frac{r}{\lambda}) + \ln r \right] \delta_{\mu r} a_{r} \]
\[ +\frac{1}{2\pi \mu} \left[ K_{0}(\frac{r}{\lambda}) + \ln r \right] \delta_{\mu r} a_{r} \]
Therefore, the strain tensor becomes
\[ u_{\sigma,\mu}^{(2)} = \varepsilon_{\sigma,\mu}^{(2)} \]
\[ = -\frac{1}{8\pi \mu(1-\nu)} \left[ (1 - 2\nu) \mathbf{X}_{\sigma} \delta_{\mu r} + \mathbf{X}_{\sigma} \delta_{\mu r} - \mathbf{X}_{\sigma} \delta_{\mu r} + \mathbf{X}_{\sigma} \delta_{\mu r} + 2 \mathbf{X}_{\sigma} \mathbf{X}_{\mu} \right] a_{r} + \frac{1}{2\pi \mu} \left[ K_{0}(\frac{r}{\lambda}) + \ln r \right] \delta_{\mu r} a_{r} \]
\[ +\frac{1}{2\pi \mu} \left[ K_{0}(\frac{r}{\lambda}) + \ln r \right] \delta_{\mu r} a_{r} \]
This relation shows that
\[ \varepsilon_{\sigma,\mu}^{(2)} = \varepsilon_{\sigma,\mu}^{(1)} = -\frac{1 - 2\nu}{4\pi \mu(1-\nu)} \mathbf{X}_{\mu} a_{r} \]
Therefore, the symmetric part of the force-stress tensor is
\[ \sigma_{\sigma,\mu}^{(2)} = 2\mu \left[ \varepsilon_{\sigma,\mu}^{(2)} - \varepsilon_{\mu,\sigma}^{(2)} \right] \]
\[ = -\frac{1}{4\pi \mu(1-\nu)} \left[ (1 - 2\nu) \delta_{\mu r} X_{\sigma} \delta_{\mu r} + \delta_{\mu r} X_{\sigma} \delta_{\mu r} + 2 X_{\sigma} X_{\mu} \right] a_{r} + \frac{1}{2\pi \mu} \left[ K_{0}(\frac{r}{\lambda}) + \ln r \right] \delta_{\mu r} a_{r} \]
\[ +\frac{1}{2\pi \mu} \left[ K_{0}(\frac{r}{\lambda}) + \ln r \right] \delta_{\mu r} a_{r} \]
Next, we consider rotations and note that the only non-zero in-plane component is
\[ \omega_{F}^{\varphi} = \frac{1}{2} \varepsilon_{\mu \varphi} \kappa_{\mu \varphi} - \frac{1}{4\pi \mu} \left[ K_{0}(\frac{r}{\lambda}) + \ln r \right] \varepsilon_{\mu \varphi} X_{\mu} a_{r} \]
Finally, we can consider the following relations

$$
\sigma_{\rho} = \Sigma_{\rho \sigma} a_{\rho}
$$

(182)

$$
\kappa'_{x} = K'_{y\sigma} a_{\rho}
$$

(183)

$$
\mu'_{x} = M'_{y\sigma} a_{\rho}
$$

(184)

$$
\xi_{\rho} = T'_{\rho \sigma} a_{\rho}
$$

(185)

and the skew-symmetric part of the force-stress tensor can be written

$$
\sigma'_{[\rho \sigma]} = -\mu'_{[\rho \sigma]} = \frac{1}{2\pi} K'_{[\rho \sigma]} \frac{X_{\rho} \delta_{\rho \sigma} - X_{\sigma} \delta_{\sigma \rho}}{r} a_{\rho}
$$

(175)

with non-zero components

$$
\sigma'_{[21]} = -\sigma'_{[12]} = \frac{1}{2\pi} K'_{[21]} \frac{X_{12} - X_{21}}{r} a_{\rho}
$$

(176)

Therefore, the total force-stress tensor is

$$
\sigma'_{\rho \sigma} = \sigma'_{[\rho \sigma]} + \sigma'_{[\rho \sigma]}
$$

$$
= \frac{1}{4\pi(1-v)\sqrt{r}} \left[ (1-2v) \frac{\delta_{\rho \sigma} X_{\rho} + \delta_{\rho \sigma} X_{\sigma} - \delta_{\rho \sigma} X_{\sigma} + 2X_{\rho}X_{\sigma}X_{\rho}}{r^3} \right] a_{\rho} + \frac{1}{2\pi} K'_{[\rho \sigma]} \frac{X_{\rho} \delta_{\rho \sigma} - X_{\sigma} \delta_{\sigma \rho}}{r} a_{\rho} + \frac{1}{4\pi} K'_{[\rho \sigma]} \frac{\delta_{\rho \sigma} X_{\rho} - X_{\sigma} \delta_{\sigma \rho}}{r^3} a_{\rho}
$$

(177)

and for the force-traction vector, we have

$$
T'_{z} = \sigma'_{z} n_{\rho} = -\frac{1}{4\pi(1-v)\sqrt{r}} \left[ (1-2v) \frac{n_{\rho} X_{\rho} + \delta_{\rho \rho} X_{\rho} n_{\rho} - n_{\rho} X_{\rho} + 2X_{\rho}X_{\rho} n_{\rho}}{r^3} \right] a_{\rho} + \frac{1}{2\pi} K'_{[\rho \rho]} \frac{X_{\rho} \delta_{\rho \rho} - X_{\rho} \delta_{\rho \rho}}{r} a_{\rho} + \frac{1}{4\pi} K'_{[\rho \rho]} \frac{\delta_{\rho \rho} X_{\rho} - X_{\rho} \delta_{\rho \rho}}{r^3} a_{\rho}
$$

(178)

Meanwhile, for moment-traction, we obtain

$$
\mu'_{z} = \varepsilon_{\rho \sigma} \mu'_{\rho \sigma} n_{\rho}
$$

$$
= \frac{1}{4\pi(1-v)\sqrt{r}} \left[ (1-2v) \frac{n_{\rho} X_{\rho} + \delta_{\rho \rho} X_{\rho} n_{\rho} - n_{\rho} X_{\rho} + 2X_{\rho}X_{\rho} n_{\rho}}{r^3} \right] a_{\rho} + \frac{1}{2\pi} K'_{[\rho \rho]} \frac{X_{\rho} \delta_{\rho \rho} - X_{\rho} \delta_{\rho \rho}}{r} a_{\rho} + \frac{1}{4\pi} K'_{[\rho \rho]} \frac{\delta_{\rho \rho} X_{\rho} - X_{\rho} \delta_{\rho \rho}}{r^3} a_{\rho}
$$

(179)

Finally, we can consider the following relations

$$
U'_{z} = U'_{x} a_{\rho}
$$

(180)

$$
\alpha' = \Omega'_{z} a_{\rho}
$$

(181)
4.3. Concentrated couple

Now, assume that there is a line distribution of couple load along the \(x_3\)-axis with unit intensity per unit length. This distributed load can be represented by a body couple

\[ C = \delta^{(2)}(\mathbf{x})e_3 \quad (194) \]

As we mentioned earlier, similarly to the three-dimensional case, the effect of a body couple in an infinite domain is equivalent to the result of a body force represented by

\[ \mathbf{F}^c = \frac{1}{2} \nabla \times C = \frac{1}{2} \nabla \delta^{(2)}(\mathbf{x}) \times e_1 = \frac{1}{2} \varepsilon_{xy} \delta^{(2)} \rho e_x \]

(195)

Therefore, the solutions of the two problems of concentrated force \( \mathbf{F} = \delta^{(2)}(\mathbf{x})a \) and concentrated couple \( C = \delta^{(2)}(\mathbf{x})e_3 \) are related, such that

\[ \mathbf{u}^c = \frac{1}{2} \varepsilon_{xy} U^c_{\gamma \beta} e_1 \]

(196)

or

\[ u^c_z = -\frac{1}{2} \varepsilon_{xy} U^c_{\gamma \beta} \]

(197)

Thus, we have

\[ u^c_z = \frac{1}{2\pi} \left[ K_0(\frac{r}{L}) - \frac{1}{r} \right] \varepsilon_{xy} X_y \]

(198)

and we can write the gradient of displacement as

\[ u^c_{x2} = -\frac{1}{4\pi} \frac{1}{l} \left[ K_0(\frac{r}{L}) + \frac{2}{r} K_1(\frac{r}{L}) - \frac{2}{r^2} \right] \varepsilon_{xy} X_y \]

\[ \times \left[ \frac{1}{r^2} \left[ K_1(\frac{r}{L}) - \frac{1}{r} \right] \varepsilon_{xy} \right] \]

(199)

Therefore, the strain tensor is

\[ \varepsilon_{c2} = -\frac{1}{8\pi} \frac{1}{l} \left[ K_0(\frac{r}{L}) + \frac{2}{r} K_1(\frac{r}{L}) - \frac{2}{r^2} \right] \varepsilon_{xy} X_y \]

(200)

It is interesting to note that

\[ \varepsilon_{c2} = \varepsilon_{c2} = 0 \]

(201)

which means the deformation field of a concentrated couple is equi-voluminal, a property shared by the three-dimensional case. Then, the symmetric part of the force-stress tensor is

\[ \sigma_{c2}^c = 2\mu \varepsilon_{c2} = -\frac{1}{4\pi} \frac{1}{l} \left[ K_0(\frac{r}{L}) + \frac{2}{r} K_1(\frac{r}{L}) - \frac{2}{r^2} \right] \varepsilon_{xy} X_y \]

(202)

We also have for in-plane rotation

\[ \omega^c = \frac{1}{2} \varepsilon_{xy} k^c \]

(203)

and the mean curvature vector is

\[ k^c = \frac{2}{\pi} \varepsilon_{xy} \omega^c \]

(204)

Therefore, the couple-stress vector becomes

\[ \mu^c = -8\mu^c \]

(205)

For the skew-symmetric part of force-stress tensor, we have

\[ \sigma_{c2}^c = \mu \varepsilon_{c2} \]

(206)

with the non-zero components

\[ \sigma_{c2}^c = -\sigma_{c2}^c = \frac{1}{4\pi} K_0(\frac{r}{L}) \]

(207)

Therefore, for the total force-stress tensor, we have

\[ \sigma^c = \sigma_{c2}^c + \sigma_{c2}^c \]

\[ = -\frac{1}{4\pi} \left[ K_0(\frac{r}{L}) + \frac{2}{r} K_1(\frac{r}{L}) - \frac{2}{r^2} \right] \varepsilon_{xy} X_y \]

\[ + \frac{1}{4\pi} K_0(\frac{r}{L}) \varepsilon_{xy} \]

(208)

The corresponding force-traction vector then becomes

\[ t^c = \varepsilon_{c2} \]

(209)

and the moment-traction is given by

\[ m^c = e_{ab} \varepsilon_{c2} n_b = -\frac{1}{2\pi} K_0(\frac{r}{L}) \frac{X_y n_b}{r} \]

(210)

Finally, we have all of the necessary Green’s functions or influence functions for the two-dimensional case, which can be written

\[ \mathbf{U}^c = \mathbf{u}^c = \frac{1}{4\pi} \frac{1}{l} \left[ K_0(\frac{r}{L}) - \frac{1}{r} \right] \varepsilon_{xy} X_y \]

(211)

\[ \Omega^c = \omega^c = \frac{1}{8\pi} \frac{1}{l} K_0(\frac{r}{L}) \]

(212)

\[ \Sigma^c = \sigma_{c2}^c \]

\[ = -\frac{1}{4\pi} \frac{1}{l} \left[ K_0(\frac{r}{L}) + \frac{2}{r} K_1(\frac{r}{L}) - \frac{2}{r^2} \right] \varepsilon_{xy} X_y \]

\[ + \frac{1}{4\pi} K_0(\frac{r}{L}) \varepsilon_{xy} \]

(213)

\[ M^c = \mu^c \]

\[ = \frac{1}{2\pi} K_0(\frac{r}{L}) \frac{X_y n_b}{r} \]

(214)

\[ T^c = t^c \]

\[ = \frac{1}{4\pi} \left[ K_0(\frac{r}{L}) + \frac{2}{r} K_1(\frac{r}{L}) - \frac{2}{r^2} \right] \varepsilon_{xy} X_y \]

\[ + \frac{1}{4\pi} K_0(\frac{r}{L}) \varepsilon_{xy} \]

(215)

\[ M^c = m^c \]

\[ = -\frac{1}{2\pi} K_0(\frac{r}{L}) \frac{X_y n_b}{r} \]

(216)

where \( U^c, \Sigma^c, K^c, M^c \) and \( T^c \) represent, respectively, the corresponding displacement, force-stress, mean curvature, couple-stress and force-traction at \( \mathbf{x} \) caused by a unit in-plane concentrated couple at the origin. Meanwhile, \( \Omega^c \) and \( M^c \) represent, respectively, the corresponding rotation and moment-traction at \( \mathbf{x} \) due to this unit in-plane concentrated couple at the origin.
5. Conclusions

We have derived the three- and two-dimensional fundamental solutions for isotropic size-dependent elasticity, based upon the fully determinate couple stress theory. Recall that this new theory resolves all of the difficulties present in previous attempts to construct a viable size-dependent elasticity. Furthermore, since in this theory, body couples do not appear in the constitutive relations and everything depends on only a single size-dependent material constant, all expressions for the fundamental solutions are elegantly consistent and quite useful in practice. In particular, these solutions can be used directly as influence functions to analyze infinite domain problems or as kernels in integral equations for numerical analysis. For example, a boundary element method for plane problems of couple stress elasticity is developed by Hadjesfandiari and Dargush (2012), based upon these fundamental solutions. Future work will include the formulation of boundary element methods for linear elastic fracture mechanics and for three-dimensional couple stress problems.

Acknowledgment

This research was supported in part by the U.S. National Science Foundation, grant number 0836768. The authors gratefully acknowledge this support.

Appendix A. Mindlin–Tiersten–Koiter indeterminate couple stress theory

In the indeterminate theory of Mindlin and Tiersten (1962) and Koiter (1964), the curvature tensor is defined by

\[ \mathbf{K}_q = \omega_{ij} \]  \hspace{1cm} (A.1)

This inconsistent curvature tensor, which is the source of all troubles in the indeterminate theory, does not require any restriction on the form of the couple-stress tensor \( \mu_q \). This is seen by decomposing the general tensor \( \mu_q \) into spherical \( \mu^{(s)}_q \) and deviatoric \( \mu^{(d)}_q \) parts in the following manner:

\[ \mu_q = \mu^{(s)}_q + \mu^{(d)}_q \]  \hspace{1cm} (A.2)

where

\[ \mu^{(s)}_q = \frac{1}{2} \mu_{kk} \delta_{ij} \]  \hspace{1cm} (A.3)

is indeterminate. By denoting

\[ Q = \frac{1}{3} \mu_{kk} \]  \hspace{1cm} (A.4)

the couple-stress tensor can be written as

\[ \mu_q = Q \delta_{ij} + \mu^{(d)}_q \]  \hspace{1cm} (A.5)

The indeterminacy \( Q \) then carries into the skew-symmetrical part of the force-stress tensor, such that

\[ \sigma_{ijkl} = \frac{1}{2} e_{ijk} Q_{kl} + \frac{1}{2} e_{i[kl} \mu^{(d)}_{jk]} \]  \hspace{1cm} (A.6)

However, this indeterminacy does not affect the force equilibrium equation (1), since

\[ \sigma_{ijkl} = \frac{1}{2} e_{ijk} \delta_{lj} + \frac{1}{2} \mu e_{ij} \]  \hspace{1cm} (A.8)

where the material constants \( \eta \) and \( \eta' \) account for couple-stress effects. Then, by using (A.5) and (A.6), we obtain

\[ \mu_q = Q \delta_{ij} + 4 \eta \omega_{ij} + 4 \eta' \omega_{ij} \]  \hspace{1cm} (A.9)

\[ \sigma_{ijkl} = \frac{1}{2} e_{ijk} Q_{kl} + 2 \eta e_{ijk} \nabla^2 \omega_{ik} \]  \hspace{1cm} (A.10)

\[ \sigma_{ijkl} = \frac{1}{2} e_{ijk} Q_{kl} + 2 \eta' e_{ijk} \nabla^2 \omega_{ik} \]  \hspace{1cm} (A.11)

\[ \sigma_{ijkl} = \frac{1}{2} e_{ijk} + 2 \mu e_{ij} + 2 \eta e_{ijk} \nabla^2 \omega_{ik} \]  \hspace{1cm} (A.12)

\[ \sigma_{ijkl} = \frac{1}{2} e_{ijk} + 2 \eta' e_{ijk} \nabla^2 \omega_{ik} \]  \hspace{1cm} (A.13)

\[ \sigma_{ijkl} = \frac{1}{2} \mu_{kk} + 2 \mu e_{ij} + 2 \eta e_{ijk} \nabla^2 \omega_{ik} \]  \hspace{1cm} (A.14)

\[ \sigma_{ijkl} = \frac{1}{2} \mu_{kk} + 2 \eta' e_{ijk} \nabla^2 \omega_{ik} \]  \hspace{1cm} (A.15)

Since \( \eta' \) only appears in the constitutive relation for couple stress (A.10), it might seem that the present fully consistent theory is just a special case of the isotropic linear indeterminate theory, obtained by letting

\[ \eta' = -\eta = -\mu \]  \hspace{1cm} (A.16)

and ignoring the indeterminacy term (i.e., \( Q = 0 \)). However, this is not mathematically valid for two reasons. First, the indeterminacy \( Q \) cannot simply be ignored in such an arbitrary manner and, second, the special case (A.16) is not even included in condition (A.15) for the indeterminate couple stress theory.

This peculiarity is the result of the definition of the different measures of curvature used in these theories. By inspecting the curvature tensor \( \mathbf{K}_q = \omega_{ij} \) in the indeterminate theory, we realize that this theory attributes couple-stresses to some deformations, which are not associated with couple-stresses in the present determinate theory. In particular, the indeterminate theory requires the existence of couple-stresses even when \( \omega_{ij} \) is symmetric, such as in the torsion of circular shafts and anti-plane deformation. For these cases, the couple-stress tensor \( \mu_q \) and its corresponding elastic energy \( W_c \) are

\[ \mu_q = Q \delta_{ij} + 4(\eta + \eta') \omega_{ij} \]  \hspace{1cm} (A.17)

\[ W_c = 2(\eta + \eta') \omega_{ij} \omega_{ij} \]  \hspace{1cm} (A.18)

Their existence requires one to exclude the special case (A.16).

However, in the present determinate theory, the mean curvature tensor is the consistent measure of curvature, where \( \kappa_q = \frac{1}{2} (\omega_{ij} - \omega_{ij}) \). For symmetric \( \omega_{ij} \), there is no mean curvature

\[ \kappa_q = \frac{1}{2} (\omega_{ij} - \omega_{ij}) = 0 \]  \hspace{1cm} (A.19)

which does not require existence of couple-stresses. This has been demonstrated for the torsion of circular shafts and anti-plane deformation in Hadjesfandiari and Dargush (2011).

Therefore, the present determinate theory is not mathematically a special case of indeterminate theory obtained by letting
Appendix B. Singular behavior of three-dimensional fundamental solutions

By using the Maclaurin series

\[
(1 + \frac{r}{r}) e^{-r/\ell} = 1 - \frac{r^2}{2\ell^2} + \frac{r^3}{3!\ell^3} - \cdots \\
(1 + \frac{r}{r} + \frac{r^2}{\ell^2}) e^{-r/\ell} = 1 + \frac{r^2}{2\ell^2} - \frac{2r^3}{3!\ell^3} + \cdots \\
(3 + 3\frac{r}{\ell} + \frac{r^2}{\ell^2}) e^{-r/\ell} = 3 - \frac{3r^2}{2\ell^2} + \frac{r^4}{8\ell^4} + \cdots \\
(15 + 15\frac{r}{\ell} + 6\frac{r^2}{\ell^2}) e^{-r/\ell} = 15 - 3\frac{3r^2}{2\ell^2} + \frac{r^4}{8\ell^4} + \cdots \\
\]

we obtain the Laurent series around the origin \(r = 0\) for the three-dimensional concentrated force fundamental solutions as follows:

\[
U_{ij}^f = \frac{1}{16\pi \ell} \left[ \frac{1}{1 - r/\ell} \left( \frac{3}{8} - \frac{3}{8} \ell + \cdots \right) \delta_{ij} - \frac{X_i X_j}{r^2} \right] \\
\Omega_{ij}^f = \frac{1}{8\pi \ell} \left[ \frac{1}{1 - r/\ell} \left( \frac{3}{2} \ell + 3 \frac{r^2}{2\ell^2} + \cdots \right) \frac{\varepsilon_{ijq} X_q}{r} \right] \\
K_{ij}^f = \frac{1}{32\pi \ell^2} \left[ \frac{1}{1 - r/\ell} \left( \frac{3}{2} \ell + 3 \frac{r^2}{2\ell^2} + \cdots \right) \frac{\varepsilon_{ijq} X_q}{r} \right] \\
M_{ij}^f = \frac{1}{3\pi \ell} \left[ \frac{1}{1 - r/\ell} \left( \frac{3}{8} - \frac{3}{8} \ell + \cdots \right) \frac{X_i X_j}{r^2} \right] \\
\Sigma_{ijq}^f = \frac{1}{8\pi (1 - r/\ell)} \left[ \frac{1}{r^2} \left( \frac{3}{2} \ell + 3 \frac{r^2}{2\ell^2} + \cdots \right) \frac{X_i X_j + \delta_{ijq}}{r^2} \right] + \frac{1}{2\pi} \\
\times \left( \frac{1}{8} + \cdots \right) \frac{X_i X_q X_j}{r^3} + \frac{1}{8\pi \ell^2} \left[ \frac{1}{1 - r/\ell} \left( 3 \frac{r^2}{2\ell^2} + \cdots \right) \frac{\delta_{ijq} + \delta_{pq} X_q}{r^2} \right] \\
- \frac{1}{8\pi \ell^2} \left[ \frac{1}{1 - r^2/4\ell^2} \cdots \right] \frac{X_i \delta_{jq} + X_j \delta_{iq}}{r^2} \\
\]  

Table 1

| Classical Cauchy elasticity | Size-dependent elasticity |
|----------------------------|--------------------------|
| \( U_{ij}^f \)            | Weak singular            |
| \( \varepsilon_{ij}^f \)  | Strong singular          |
| \( K_{ij}^f \)            | Hyper singular           |
| \( M_{ij}^f \)            | Not defined              |
| \( \Sigma_{ijq}^f \)      | Strong singular          |
| \( \Gamma_{ij}^f \)       | Strong singular          |

Meanwhile, for the three-dimensional concentrated couple fundamental solutions, we have:

\[
U_{ij}^c = \frac{1}{16\pi \ell^2} \left[ \frac{1}{1 - r/\ell} \left( \frac{3}{2} \ell + 3 \frac{r^2}{2\ell^2} + \cdots \right) \frac{\varepsilon_{ijq} X_q}{r} \right] \\
\Omega_{ij}^c = \frac{1}{4\pi \ell} \left[ \frac{1}{1 - r/\ell} \left( \frac{3}{2} \ell + 3 \frac{r^2}{2\ell^2} + \cdots \right) \frac{\varepsilon_{ijq} X_q}{r} \right] \\
K_{ij}^c = \frac{1}{16\pi \ell} \left[ \frac{1}{1 - r/\ell} \left( \frac{3}{2} \ell + 3 \frac{r^2}{2\ell^2} + \cdots \right) \frac{\varepsilon_{ijq} X_q}{r} \right] \\
M_{ij}^c = \frac{1}{8\pi \ell} \left[ \frac{1}{1 - r/\ell} \left( \frac{3}{2} \ell + 3 \frac{r^2}{2\ell^2} + \cdots \right) \frac{\varepsilon_{ijq} X_q}{r} \right] \\
\Sigma_{ijq}^c = \frac{1}{12\pi \ell^2} \left[ \frac{1}{1 - r/\ell} \left( \frac{3}{2} \ell + 3 \frac{r^2}{2\ell^2} + \cdots \right) \frac{\varepsilon_{ijq} X_q}{r} \right] \\
\]
In these series, as we can observe, there are terms with weak singularities $r_1$, strong singularities $r_2$, and hyper singularities $r_3$. Tables 1 and 2 summarize all singularities for the three-dimensional fundamental solutions, along with their corresponding ones in Cauchy elasticity.

### Tables 1 and 2

| Classical Cauchy elasticity | Size-dependent elasticity |
|-----------------------------|---------------------------|
| $U^c_{th}$                  | Strong singular           |
| $\Omega^c_{th}$             | Hyper singular            |
| $K^c_{th}$                  | 0                         |
| $M^c_{th}$                  | Not defined               |
| $\Sigma^c_{th}$             | Hyper singular            |
| $T^c_{th}$                  | Hyper singular            |
| $M^c_{th}$                  | Not defined               |

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