\textbf{PT} restoration via increased loss-gain in \textbf{PT}-symmetric Aubry-Andre model

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In systems with “balanced loss and gain”, the \textbf{PT}-symmetry is broken by increasing the non-hermiticity or the loss-gain strength. We show that finite lattices with oscillatory, \textbf{PT}-symmetric potentials exhibit a new class of \textbf{PT}-symmetry breaking and restoration. We obtain the \textbf{PT} phase diagram as a function of potential periodicity, which also controls the location complex eigenvalues in the lattice spectrum. We show that the sum of \textbf{PT}-potentials with nearby periodicities leads to \textbf{PT}-symmetry restoration, where the system goes from a \textbf{PT}-broken state to a \textbf{PT}-symmetric state as the average loss-gain strength is increased. We discuss the implications of this novel transition for the propagation of a light in an array of coupled waveguides.

\textit{Introduction.} Open systems with balanced loss and gain have gained tremendous interest in the past three years since their experimental realizations in optical \cite{1} \cite{2}, electrical \cite{3}, and mechanical \cite{4} systems. Such systems are described by non-hermitian Hamiltonians that are invariant under combined parity and time-reversal (\textbf{PT}) operations \cite{5}. Apart from their mathematical appeal, such non-hermitian Hamiltonians show non-intuitive properties such as unidirectional invisibility \cite{6} \cite{7} \cite{8} and are thus of potential technological importance.

Historically, \textbf{PT} Hamiltonians on an infinite line were the first to be investigated \cite{9} \cite{10}. The range of parameters where the spectrum of the Hamiltonian is purely real, $e_{\lambda} = c_{\lambda}^*$, and the eigenfunctions are simultaneous eigenfunctions of the \textbf{PT} operator, $\psi_{\lambda}(x) = \psi_{\lambda}^*(-x)$, is called the \textbf{PT}-symmetric phase. The emergence of complex conjugate eigenvalues when the parameters are not in this region is called \textbf{PT}-symmetry breaking. A positive threshold for \textbf{PT}-symmetry breaking implies that the system transitions \cite{11} \cite{12} from a quasi-equilibrium state at a small but nonzero non-hermiticity, to loss of reciprocity as the strength of the balanced loss-gain term crosses the threshold.

Although \textbf{PT}-symmetric Hamiltonian studies started with continuum Hamiltonians, all of their realizations are in finite lattices where the continuum, effective-mass approximation may not apply. This observation has led to tremendous interest in \textbf{PT}-symmetric lattice models \cite{13} \cite{14} \cite{15} that can be realized in coupled waveguide arrays \cite{16} \cite{17}.

A universal feature of all such systems is that \textbf{PT}-symmetry is broken by increasing the balanced loss-gain strength and restored by reducing it. Here, we present a tight-binding model that can exhibit exactly opposite behavior, via a family of \textbf{PT}-symmetric, periodic potentials.

A remarkable property of lattice models, absent in the continuum limit, is the effects of a periodic potential. The spectrum of a charged particle in constant magnetic field in two dimensions consists of Landau levels \cite{18} \cite{19}; a similar particle on a two-dimensional lattice displays a fractal, Hofstadter butterfly spectrum \cite{20} \cite{21}. In one dimensional lattices, a fractal spectrum emerges in the presence of a hermitian, periodic potential, and this model, known as the Aubry-Andre model \cite{22}, shows localization transition in a clean system when the strength of the incommensurate potential exceeds the nearest-neighbor hopping \cite{23}. Here, we consider a \textbf{PT}-symmetric Aubry-Andre model on an $N$-site lattice with hopping $J$ and complex potential $V_{\beta}(n) = V_0 \cos [2\pi \beta(n - n_c)] + i \gamma \sin [2\pi \beta(n - n_c)]$ where $n_c = (N + 1)/2$ is the lattice center and $\gamma > 0$.

Since $V_{\beta} = (-1)^{2n_c}V_{\beta}^* = (-1)^{2n_c}V_{1+\beta}$, it is sufficient to consider the family of potentials with $0 < \beta < 1$ (when $\beta = 0$ the problem reduces to the Aubrey-Andre model \cite{24} \cite{25}). We then consider the effect of two such potentials $V_{\beta_1} + V_{\beta_2}$ with $|\beta_1 - \beta_2| \sim 1/N \ll 1$.

Our salient results are follows: i) For a single potential $V_{\beta}$, the threshold loss-gain strength $\gamma_{\text{PT}}(N, V_0, \beta)$ shows $N$ local maxima along the $\beta$ axis; it is suppressed by a nonzero real modulation $V_0$. ii) The discrete index of pair of eigenvalues that become complex can be tuned stepwise by varying $0 < \beta < 1/2$. iii) For $V = V_{\beta_1} + V_{\beta_2}$, generically, the phase diagram in the $(\gamma_{\beta_1}, \gamma_{\beta_2})$ plane shows a re-entrant \textbf{PT}-symmetric phase: a broken \textbf{PT} symmetry is restored by increasing the non-hermiticity and broken again when $\gamma_{\beta_1}$ become sufficiently large. This behavior is absent in the extensively studied continuum Hamiltonians with complex potentials \cite{26} \cite{27}, and is a result of competition between the two lattice potentials $V_{\beta_1}$ and $V_{\beta_2}$.

We emphasize that $\gamma_i > 0$ means the gain-regions of the two potentials mostly align as do their respective loss-regions. Thus, the competition between $V_{\beta_1}$ and $V_{\beta_2}$ is not in their loss-gain profiles, but, as we will show below, due to the relative locations of \textbf{PT}-symmetry breaking energy-levels in the spectrum. \textbf{PT} phase diagram for a single potential. The tight-binding hopping Hamiltonian for an $N$-site lattice with open boundary conditions is $H_0 = -J \sum_{n=1}^{N-1} (|n+1\rangle \langle n | + |n+1\rangle \langle n |)$. Its particle-hole symmetric energy spectrum is given by $\epsilon_{0,p} = -2J \cos(k_p) = -\epsilon_0,\beta$ and the corresponding normalized eigenfunctions are $\psi_p(j) = \sin(k_p j) = (-1)^j \sin(\bar{k}_p j)$ where $0 < k_p = p \pi/(N+1) < \pi$ and $\bar{p} = N + 1 - p$. The properties that relate eigenval-
ues and eigenfunctions at indices $p, \tilde{p}$ remain valid in the presence of pure loss-gain potential $V_0 = -V_0^*$ [33]. The eigenvalue equation for an eigenfunction $f(n)$ of the non-Hermitian, $\mathcal{PT}$-symmetric Hamiltonian $H_\beta = H_0 + V_\beta$ is given by ($1 \leq n \leq N$)

$$-J [f(n+1) + f(n-1)] + V_\beta f(n) = E f(n), \quad (1)$$

with $f(0) = 0 = f(N+1)$. Since this difference equation is not analytically soluble for an arbitrary $\beta$, we numerically obtain the spectrum $E(\gamma)$ and the $\mathcal{PT}$-symmetry breaking threshold $\gamma_{\mathcal{PT}}(N, V_0, \beta)$ using different discretizations $\beta_k = k\delta\beta$ along the $\beta$-axis. Due to the $\beta \leftrightarrow 1 - \beta$ symmetry of the potential, it follows that the exact threshold loss-gain strength satisfies $\gamma_{\mathcal{PT}}(N, V_0, \beta) = \gamma_{\mathcal{PT}}(N, V_0, 1 - \beta) = \gamma_{\mathcal{PT}}(N, -V_0, \beta)$.

We consider a purely loss-gain potential, present results for an even lattice, and point out the salient differences that arise when lattice size $N$ is odd or when $V_0 \neq 0$. The left-hand panel in Fig. 1 shows the $\mathcal{PT}$-symmetric threshold $\gamma_{\mathcal{PT}}(N, \beta)/J$ for an $N = 50$ lattice obtained by using discretization $\delta\beta = 1/2N$ (blue squares), an $N = 100$ lattice with $\delta\beta = 1/4N$ (red markers), and an $N = 400$ lattice with $\delta\beta = 1/N$ (black stars). There is a monotonic suppression of the threshold strength with increasing $N$, and, crucially, the general shape of the phase diagram depends upon the size of $\delta\beta$ relative to $1/N$. A scaling of this threshold suppression for lattice sizes $N = 50 - 500$, shown in the inset, implies that $\gamma_{\mathcal{PT}}(N, \beta) = C_\beta / N$ where $C_\beta$ is a constant. Thus, the threshold strength is suppressed linearly and vanishes in the thermodynamic limit [33,35]. However, this algebraically fragile nature of the $\mathcal{PT}$-phase is not an impediment since $\mathcal{PT}$-systems to-date are only realized in small lattices with $N \ll 100$. The right-hand panel in Fig. 1 shows the phase diagram for $N = 50$ case with discretization $\delta\beta = 1/N^2$. The results for irrational values of $\beta$ and other discretizations lie on the same curve. The $\mathcal{PT}$ phase diagram shows $(N-2)$ local maxima located at $\beta_k = (2k+1)/2N$ and the two maxima at the end points, is symmetric about the center and has a local minimum in the threshold at $\beta = 1/2$. In addition, the function $\gamma_{\mathcal{PT}}(N, \beta)$ has $(N-1)$ minima at $\beta_k = k/N$ and smoothly oscillates over a period $\sim 1/N = 0.02$ as shown in the inset (solid red circles). These results are generic for any lattice size $N$ with discretization $\delta\beta \sim 1/N^2$.

When $N$ is odd, the non-Hermitian potential vanishes at $\beta = 1/2$ and the spectrum of the Hamiltonian $H_\beta$ is purely real. For an odd lattice, a similarly obtained phase diagram shows $(N-1)$ local maxima that are distributed equally on the two sides of $\beta = 1/2$, along with a substantial enhancement in the threshold strength as $\beta \to 1/2^\pm$. Adding a real potential modulation $V_0 \neq 0$, to the loss-gain potential, in general, suppresses the threshold strength.

The phase diagram can be understood as follows: for a small $\sim 1/N^2 \ll 1/N$, $V_\beta(n) = i\gamma(2\pi\beta)(n - n_c)$ and the enhanced $\mathcal{PT}$-breaking threshold, $\gamma_{\mathcal{PT}}/J \sim 0.3$, is consistent with a linear-potential threshold [36]. For an even lattice, the average of the gain-potential is given by $A_\beta(N) = \sum_{n}^N V_\beta(n)$, $\gamma = \sin^2(\pi\beta/2)/\sin(\pi\beta)$, and the $\mathcal{PT}$ threshold is greatest when the change in the average strength is maximum as $\beta$ is varied, $\partial_\beta A_\beta(\beta) = 0$. In the limit $N \gg 1$ and $\beta \gg 1/N^2$, it implies that the $N$ maxima of $\gamma_{\mathcal{PT}}(N, \beta)$ occur at $\beta_{n,\text{max}} = (2k+1)/2N$. On the other hand, $\gamma_{\mathcal{PT}}(N, \beta)$ is smallest when the change in the average strength is minimum, $\partial_\beta A_\beta(\beta) = 0$, and gives
the locations of \((N-1)\) minima as \(\beta_{k,\text{min}} = k/N\). A similar analysis applies to odd lattices, where the average potential is given by \(A_O(\beta) = \sin[\pi\beta(N-1)/2] \sin[\pi\beta(N+1)/2]/\sin(\pi\beta)\).

![Diagram](image)

**FIG. 2.** (color online) Index of eigenvalues that become complex as a function of \(\beta\) for an \(N = 20\) lattice with discretization \(\delta\beta = 1/2N = 0.025\) shows a \(\beta \leftrightarrow 1 - \beta\) symmetry, denoted by heavy and light red markers. When \(\beta \leq 0.08\), levels \((E_1, E_2)\) become degenerate and complex, and so do their particle-hole counterparts, \((E_{1g}, E_{2g})\) (red circles); in general, we can tune the location of \(\mathcal{PT}\)-breaking (blue squares, black stars, red markers) by appropriately choosing \(\beta\).

Next, we focus on the location of the \(\mathcal{PT}\) symmetry breaking. Due to the particle-hole symmetric spectrum of \(H_\beta\), two pairs of levels \((E_n, E_{n+1})\) and \((-E_n, -E_{n+1})\) become complex simultaneously. Figure 2 plots the indices of eigenvalues that become complex as a function of \(\beta\) for an \(N = 20\) lattice with discretization \(\delta\beta = 1/2N\). It shows that at small \(\beta\), the eigenvalues at the band edges become complex, whereas, as \(\beta \to 1/2\), the pairs of eigenvalues that become complex move to the center of the band. Thus the average range of \(\beta\)s with the same location for \(\mathcal{PT}\)-symmetry breaking is \(\sim 1/N\). It follows that by choosing an appropriate \(\beta\), one is able to control the location of \(\mathcal{PT}\)-symmetry breaking in the energy spectrum. As we will see next, this control allows us to introduce competition between potentials \(V_\beta\) with two different, but close, values of \(\beta\).

**\(\mathcal{PT}\) phase diagram with two potentials.** We now consider the \(\mathcal{PT}\)-symmetric phase of the Hamiltonian with two potentials, \(H = H_0 + V_{\beta_1} + V_{\beta_2}\), in the \((\gamma_1, \gamma_2)\) plane, where both axes are scaled by their respective threshold values \(\gamma_{\alpha,\mathcal{PT}} = \gamma_{\mathcal{PT}}(\beta_\alpha)\). Panel (a) in Fig. 3 shows the numerically obtained phase diagram for an \(N = 20\) lattice with \((\beta_1, \beta_2) = (0.20, 0.25)\) (blue stars and squares). It shows that, from a \(\mathcal{PT}\)-broken phase (point 1), it is possible to enter the \(\mathcal{PT}\)-symmetric phase by increasing the non-hermiticity \(\gamma_1\) (point 2). We emphasize that increasing \(\gamma_1\) increases the average gain- and loss- strength \(\gamma_1 A_E(\beta_1) + \gamma_2 A_E(\beta_2)\), and yet drives the system into a \(\mathcal{PT}\)-symmetric phase from a \(\mathcal{PT}\)-broken phase. Increasing \(\gamma_1\) further, eventually, drives the system into a \(\mathcal{PT}\) broken phase again (point 3). This re-entrant phase is due to competition between \(V_{\beta_1}\) and \(V_{\beta_2}\). For \((\beta_1, \beta_2) = (0.04, 0.08)\), the two co-operate and the phase boundary is an expected straight line. Panel (b): intensity \(I(k, t)\) of an initially normalized state shows that, starting from a \(\mathcal{PT}\)-broken phase (top panel), increasing \(\gamma_1\) initially restores bounded oscillations (center panel), followed by \(\mathcal{PT}\) breaking and amplification (bottom panel). Note the two-orders-of-magnitude difference in the total intensity.

![Diagram](image)

**FIG. 3.** (color online) Panel (a): \(\mathcal{PT}\)-phase diagram for potential \(V_{\beta_1} + V_{\beta_2}\). When \((\beta_1, \beta_2) = (0.20, 0.24)\) (blue stars and squares), a \(\mathcal{PT}\)-broken phase (point 1) is restored by increasing \(\gamma_1\) (point 2) and subsequently broken again (point 3). This re-entrant phase is due to competition between \(V_{\beta_1}\) and \(V_{\beta_2}\). For \((\beta_1, \beta_2) = (0.04, 0.08)\), the two co-operate and the phase boundary is an expected straight line. Panel (b): intensity \(I(k, t)\) of an initially normalized state shows that, starting from a \(\mathcal{PT}\)-broken phase (top panel), increasing \(\gamma_1\) initially restores bounded oscillations (center panel), followed by \(\mathcal{PT}\) breaking and amplification (bottom panel). Note the two-orders-of-magnitude difference in the total intensity.
restoration on the site- and time-dependent intensity $I(k,t) = |\langle \psi | \exp(-iHt/\hbar)|\psi_0 \rangle|^2$ of a state $|\psi_0 \rangle$ that is initially localized on site $N/2 = 10$; the time is measured in units of $\hbar/J$. The top-subpanel shows that intensity has a monotonic amplification in regions with gain sites, leading to a striated pattern (point 1). Center subpanel shows that by increasing $\gamma_1$, oscillatory behavior in the intensity is restored (point 2). Bottom subpanel shows that increasing $\gamma_1$ further breaks the $\mathcal{PT}$ symmetry again (point 3). Thus, we are able to restore $\mathcal{PT}$-symmetry by increasing the non-hermiticity and, achieve amplification by both reducing or increasing the average gain-strength. This novel behavior is absent in all lattice models with a single $\mathcal{PT}$ potential.

Since each potential $V_{\beta_j}$ breaks the $\mathcal{PT}$ symmetry for $\gamma_j/\gamma_j^{\mathcal{PT}} > 1$, naively, one may expect that the $\mathcal{PT}$ phase boundary for $V_{\beta_1} + V_{\beta_2}$ is given by $\gamma_1/\gamma_1^{\mathcal{PT}} + \gamma_2/\gamma_2^{\mathcal{PT}} = 1$. This is indeed the case for $(\beta_1, \beta_2) = (0.04, 0.08)$, shown by red dashed line in panel (a), even though the potential periodicities differ by a factor of two.

What is the key difference between the two sets of parameters, one of which shows a re-entrant $\mathcal{PT}$-symmetric phase? It is the indices of eigenvalues that become complex due to $V_{\beta_1}$ and $V_{\beta_2}$. The red rectangle in Fig. 2 shows that for $\beta \leq 0.8$, eigenvalues ($E_1, E_2$) become complex. In such a case, the two potentials $V_{\beta_1}$ and $V_{\beta_2}$ act in a cooperative manner effectively adding their strengths. Therefore, the $\mathcal{PT}$-phase boundary is a straight line. In contrast, the blue oval in Fig. 2 shows that for $\beta_1 = 0.20$, energy levels $E_3, E_5$ approach each other, become degenerate, and then complex as $\gamma \to \gamma_{1\mathcal{PT}}$; for $\beta_2 = 0.25$, the energy levels that become complex as $\gamma \to \gamma_{2\mathcal{PT}}$ is $E_5, E_6$. Thus, the level $E_5$ is lowered by potential $V_{\beta_1}$ and raised by the potential $V_{\beta_2}$ from its hermitian-limit value. This introduces competition between the two potentials $V_{\beta_1}$ and $V_{\beta_2}$ even though their gain-regions largely overlap and so do their respective loss regions.

This correspondence between competing potentials and $\mathcal{PT}$-restoration is further elucidated in Fig. 4. In conjunction with Fig. 2 it shows that re-entrant $\mathcal{PT}$-symmetric phase occurs when the two potentials compete (panels b-e, g, h). This restoration of $\mathcal{PT}$-symmetry can be due to increased loss-gain strength in $\gamma_1$ (panels d, e), $\gamma_2$ (panels b, c, g), or both (panel h). On the other hand, when the two potentials break the same set of eigenvalues, the $\mathcal{PT}$ phase boundary is a line (panels a, f, k).

Discussion. Competing potentials, a common theme in physics, often stabilize phases that would be unstable in the presence of only one of them [37]. A trivial definition of competing $\mathcal{PT}$-potentials is that the gain-region of one strongly overlaps with the loss-region of another, thus reducing the average gain (and loss) strength.

Here, we have unmasked the subtle competition between $\mathcal{PT}$ potentials whose gain regions largely overlap, based on the location of $\mathcal{PT}$-symmetry breaking induced by each. This competition results in $\mathcal{PT}$-restoration and subsequent $\mathcal{PT}$-breaking, leading to selective intensity suppression and oscillations at large loss-gain strength. Its hints were seen in a continuum model with complex $\delta$-function and constant potentials, but that continuum model is neither easily experimentally realizable nor can it tune between cooperative and competitive behavior [39]. The $\mathcal{PT}$-symmetric Aubry-Andre model provides a family of potentials with tunable competition or cooperation among them, and is thus ideal for investigating the consequences of such competition; even lattices as small as $N = 10$ that can be realized via coupled optical waveguides [19, 20] or cold atoms [38] may provide a comprehensive understanding of interplay between loss-gain strengths and $\mathcal{PT}$-symmetry breaking.

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