Information in probability:  
Another information-theoretic proof of a finite de Finetti theorem

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Abstract We recall some of the history of the information-theoretic approach to deriving core results in probability theory and indicate parts of the recent resurgence of interest in this area with current progress along several interesting directions. Then we give a new information-theoretic proof of a finite version of de Finetti’s classical representation theorem for finite-valued random variables. We derive an upper bound on the relative entropy between the distribution of the first $k$ in a sequence of $n$ exchangeable random variables, and an appropriate mixture over product distributions. The mixing measure is characterised as the law of the empirical measure of the original sequence, and de Finetti’s result is recovered as a corollary. The proof is nicely motivated by the Gibbs conditioning principle in connection with statistical mechanics, and it follows along an appealing sequence of steps. The technical estimates required for these steps are obtained via the use of a collection of combinatorial tools known within information theory as ‘the method of types.’

1 Entropy and information in probability

Shannon’s landmark 1948 paper \[76\] founded the field of information theory and ignited the fuse that led to much of the subsequent explosive development of communications theory and engineering in the 20th century. At the same time, it also led to a wave of applications of information theory to numerous other branches of science. Of those, some, e.g. those in bioinformatics and neuroscience, were successful, while some others, despite Shannon’s “bandwagon” warning \[77\], much less so.

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Within mathematics, information-theoretic ideas have had a major impact along several directions, perhaps most notably (although certainly not exclusively) in connection with probability theory. For our present purposes, the most relevant line of work is based on the idea of utilizing information-theoretic tools and ideas in order to prove core probabilistic results. Over the past 55 years, a great number of such proofs have appeared. These are often accompanied by new interpretations and rich intuition, thus providing new ways of understanding why fundamental probabilistic theorems are true, and sometimes also giving stronger versions of the original results.

In the rest of this introduction we describe some of the main landmarks along this path, and we indicate directions of current and likely near-future activity. This brief survey is necessarily incomplete and biased, due to our own subjective taste and bounded knowledge. Then in Section 2 we state and prove a new finite version of de Finetti’s classical representation theorem for finite-valued exchangeable random variables.

The first appearance of information-theoretic ideas in the proof of a genuinely probabilistic result was in 1958, when Hájek [41, 42] proved that the laws $\mu$ and $\nu$ of any two Gaussian processes are either absolutely continuous with respect to each other, or singular. Hájek exploited the implications of $D(\mu||\nu) + D(\nu||\mu)$ being finite or infinite, where $D(\mu||\nu)$ denotes the relative entropy or Kullback-Leibler divergence between $\mu$ and $\nu$,

$$D(\mu||\nu) := \begin{cases} \int \log \frac{d\mu}{d\nu} \, d\mu, & \text{if } \frac{d\mu}{d\nu} \text{ exists} \\ +\infty, & \text{otherwise.} \end{cases}$$ (1)

[Throughout, log denotes the natural logarithm.] In the same year, Kolmogorov [56] introduced entropy in ergodic theory. He provided a way to calculate the entropy of a transformation to conclude that Bernoulli shifts of different entropies are not metrically isomorphic. The importance of entropy in ergodic theory was also highlighted more than a decade later, when Ornstein [70, 72, 71] proved that Bernoulli shifts with the same entropy are necessarily isomorphic.

The following year, 1959, Linnik [63] gave an information-theoretic proof of the central limit theorem (CLT), showing that the law of the standardised sum $S_n = (1/\sqrt{n}) \sum_{i=1}^n X_i$ of $n$ independent and identically distributed (i.i.d.) random variables $X_1, \ldots, X_n$ with variance $\sigma^2$ (or, more generally, of independent random variables satisfying the Lindeberg condition) converges in distribution to a Gaussian. Linnik’s connection between the CLT and information-theoretic ideas was the first in a long series of works, along a path that remains active until today. Indeed, in a sequence of papers, including works by Shimizu [78], Brown [13], Barron [5], Johnson [48], Artstein et al. [2], Tulino and Verdú [82], and Madiman and Barron [65], it was shown that the differential entropy $h(S_n)$ of the standardised sums in fact increases with $n$, and its limiting value is the entropy $h(Z)$ of the standard Gaussian $Z \sim N(0, 1)$. This monotonic convergence in combination with the fact that the Gaussian has maximum entropy among all random variables with variance $\sigma^2$, presents an appealing analogy between the CLT and the second law of thermodynamics.
In the above discussion, the differential entropy of a continuous random variable $X$ with density $f$ is given by $h(X) = h(f) = -\int f \log f$, and the relative entropy between two probability measures $\mu, \nu$ on $\mathbb{R}$ with densities $f, g$ is $D(\mu\|\nu) = D(f\|g) = \int f \log(f/g)$. A simple computation shows that the “entropic CLT” just described can equivalently be stated as, $D(f_n\|\phi) \downarrow 0$ as $n \to \infty$, where $f_n$ is the density of $S_n$ and $\phi$ the standard normal density. This convergence in the sense of relative entropy implies convergence in $L^1$: Pinsker’s inequality, established by Csiszár [21], Kullback [59] and Kemperman [54], states that:

$$D(\mu\|\nu) \geq \frac{1}{2\log 2} \|\mu - \nu\|^2. \quad (2)$$

Subsequent work along these lines includes Carlen and Soffer’s dynamical systems approach [17], Johnson’s convergence to Haar measure on compact groups [53], Johnson and Barron’s rates of convergence in the entropic CLT [51], Bubeck and Ganguly’s entropic CLT for Wishart random matrices [15], and, most recently, an information-theoretic CLT for discrete random variables [36].

A year after Linnik’s paper made the first entropy-CLT connection, in 1960, Rényi [74] examined the convergence of Markov chains to equilibrium from an information-theoretic point of view, thus initiating another path of information-theoretic investigation in probability. Rényi showed that the relative entropy $D(P_n\|\pi)$ between the time-$n$ distribution $P_n$ of a finite-state chain with an all-positive transition matrix and its unique invariant distribution $\pi$, decreases to zero as $n \to \infty$. Similar and slightly more general results were established independently by Csiszár [20] in 1963, who also employed Rényi’s notion of $f$-divergence, an important generalisation of relative entropy. In the same year, Kendall [55] extended Rényi’s techniques and results, to include certain countable state space chains. A significant advance came with Fritz’s 1973 work [35], where he studied the asymptotic behavior of reversible Markov kernels and established their weak convergence to equilibrium. Barron in 2000 [7] extended Fritz’s result to convergence in relative entropy, and in 2009 Harremoës and Holst [44] used ideas related to information projections to further extend and generalise those earlier results.

The problem of Poisson approximation and convergence was first examined through the lens of information theory around 20 years ago, leading to a development analogous to that of the entropic CLT. Harremoës in 2001 identified the Poisson as the maximum entropy distribution among all laws that arise from sums of independent Bernoulli random variables with a fixed mean [43]. This characterisation was extended in 2007 by Johnson [50] to the class of ultra log-concave laws on the nonnegative integers. Meanwhile, in 2005 Kontoyiannis et al. [57] derived convergence results and nonasymptotic Poisson approximation bounds using entropy-theoretic methods. Interestingly, some of those results were based, in part, on a discrete modified logarithmic Sobolev inequality for the entropy established by Bobkov and Ledoux [11]. In a related direction, Harremoës et al. [45, 46] obtained Poisson approximation results under the thinning operation.
A similar program was carried out in the case of compound Poisson approximation.

Compound Poisson laws on the integers were again given a natural maximum-entropy interpretation by Johnson et al. \cite{52} and Yu \cite{89}, and compound Poisson approximation bounds and convergence results were established via information-theoretic techniques by Madiman et al. \cite{66} and Barbour et al. \cite{4}. Interestingly, in some cases the resulting nonasymptotic bounds give the best results to date.

**The method of types and large deviations.** Suppose \{X_n\} are i.i.d. random variables with common probability mass function (PMF) \( Q \) on a finite alphabet \( A \) of size \( m = |A| \). The type \( \hat{P}_n \) of a string \( x^n = (x_1, \ldots, x_n) \in A^n \) is simply the empirical PMF induced by \( x^n \) on \( A \). Let \( \mathcal{P}_n \) denote the collection of all \( n \)-types on \( A \), namely, all PMFs that arise as types of strings of length \( n \). Then, e.g., we have the obvious bound,

\[
|\mathcal{P}_n| \leq (n + 1)^m, \tag{3}
\]

and direct computation also shows that, for any \( x^n \in A^n \),

\[
Q^n(x^n_1) = e^{-n[H(\hat{P}_n) + D(\hat{P}_n \parallel Q)]}, \tag{4}
\]

Here, \( H(P) := -\sum_{x \in A} P(x) \log P(x) \) is the (discrete Shannon) entropy of a PMF \( P \) on \( A \), and the definition \[1\] of the relative entropy \( D(P \parallel Q) \) between two PMFs \( P, Q \) on the same discrete alphabet becomes \( D(P \parallel Q) = \sum_{x \in A} P(x) \log [P(x)/Q(x)] \).

Slightly more involved calculations lead to interesting and useful bounds. For example, for an \( n \)-type \( P \), let \( T(P) \) denote the type class of \( P \), consisting of all \( x^n \in A^n \) with type \( P \). Then the cardinality and probability of \( T(P) \) satisfy,

\[
(n + 1)^{-m} e^{nH(P)} \leq |T(P)| \leq e^{nH(P)} \tag{5}
\]

\[
(n + 1)^{-m} e^{-nD(P \parallel Q)} \leq Q^n(T(P)) \leq e^{-nD(P \parallel Q)}. \tag{6}
\]

The method of types is a collection of combinatorial estimates for probabilities associated with discrete i.i.d. random variables and memoryless channels, of which the examples in \[3\]–\[6\] above are the starting point. Based in part on preliminary ideas of Wolfowitz \cite{88}, the method of types was fully developed in 1981 by Csiszár and Körner \cite{28}. As described in Csiszár’s review \cite{25}, the method of types has been employed very widely and with great success in numerous information-theoretic problems arising from different communication-theoretic scenarios.

Based in part on the method of types, and also building on ideas from related work by Groeneboom et al. \cite{38}, Csiszár was able to establish a series of important results in large deviations. In 1975 \cite{22} he identified the exponent in Sanov’s theorem \cite{75} as an extremum of relative entropies, and in 1984 \cite{23} he proved a general, strong version of Sanov’s theorem, by a combination of the method of types, discretisation arguments, and a general Pythagorean inequality for the relative entropy established by Topsøe \cite{51}. He also gave a simpler proof along the same lines in his 2006 paper \cite{26}.
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Moreover, in the same paper \cite{23} Csiszár established a version of the Gibb's conditioning principle (also known as the conditional limit theorem) using the same tools. This was further extended by Csiszár et al. in 1987 \cite{27} to the case of Markov conditioning, and by Algoet et al. in 1992 \cite{11} to Markov types.

The method of types and the Gibbs conditioning principle will both play an important role in our proof of the finite de Finetti theorem in Section 2.

Exchangeability. Suppose \( \{X_n\} \) are i.i.d. random variables, and let \( E \) denote the exchangeable \( \sigma \)-algebra, that is, the sub-\( \sigma \)-algebra of \( \sigma(\{X_n\}) \) consisting of those events that are invariant under finite permutations of the indices in the sequence \( \{X_n\} \). In 2000, O’Connell \cite{69} gave a beautiful, elementary information-theoretic proof the Hewitt-Savage 0-1 law \cite{47}: \( E \) is trivial, in that all events in \( E \) have probability either zero or one.

Another aspect of exchangeability comes up in connection with de Finetti’s theorem. Let \( \{X_n\} \) be an exchangeable sequence of random variables with values in the same finite alphabet \( A \). Here, exchangeability means that, for any \( n \) and any permutation \( \pi \) on \( \{1, 2, \ldots, n\} \), the distribution of the random variables \( (X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)}) \) is the same as that of \( (X_1, X_2, \ldots, X_n) \). De Finetti’s theorem \cite{30, 31} states that \( \{X_n\} \) is exchangeable if and only if it is a mixture of i.i.d. sequences, that is, if and only if there is a measure \( \bar{\mu} \) on the simplex \( P \) of probability distributions on \( A \), such that, for any \( 1 \leq k \leq n \) and any \( x^k_1 = (x_1, \ldots, x_k) \in A^k \),

\[
P(X^k_1 = x^k_1) = M_{\bar{\mu}, k}(x^k_1) := \int_P Q^k(x^k_1) d\bar{\mu}(Q).
\] (7)

De Finetti’s theorem plays an important role in the foundations of subjective probability and Bayesian statistics, see, e.g., the discussions in \cite{33, 8}. But arguments about its practical relevance are limited by the fact that, as is well known \cite{34}, the representation (7) fails in general if it is only assumed that a finite collection of random variables \( (X_1, \ldots, X_n) \) is exchangeable for some fixed \( n \). Nevertheless, approximate versions of (7) remain valid in this case \cite{33, 34}. Such a ‘finite’ version of de Finetti’s theorem for binary random variables was recently established in \cite{37}, using information-theoretic ideas: It was shown that there is a mixing measure \( \bar{\mu} \) on \( P \) such that, for any \( 1 \leq k \leq n \), the distribution \( P_k \) of \( (X_1, \ldots, X_k) \) is close to \( M_{\bar{\mu}, k} \) in the precise sense that:

\[
D(P_k \| M_{\mu, k}) \leq \frac{5k^2 \log n}{n - k}.
\] (8)

A different information-theoretic proof of a different finite version of de Finetti’s theorem is given in Section 2.

Further connections. There are numerous other directions along which information-theoretic methods have been employed to establish either known or new probabilistic results. We briefly mention only a few more from the long list of relevant works, some of which go beyond probability theory. The interested reader may also consult Barron’s reviews of information-theoretic proofs and connections with statistics and
A natural and powerful connection has been drawn between information theory and concentration of measure inequalities, through what has come to be known as the entropy method. Often attributed to Herbst [29], the entropy method was primarily developed by Ledoux [60, 61, 62]. Marton’s 1996 work [68] had an early and significant influence in this direction as well. Entropy also appears naturally in connection with related work on transportation theory [10, 83]. Book-length accounts of measure concentration and related inequalities, including the entropy method, are given in [12] and [73].

A fascinating and multifaceted series of connections between information-theoretic ideas and functional inequalities started with Shannon’s entropy power inequality (EPI), stated in his original 1948 paper [76] and later proved by Stam [79] and Blachman [29]. Much of the relevant literature up to 1991 is summarised in Dembo et al.’s review [32], including the connection with Gross’ celebrated Gaussian logarithmic Sobolev inequality [39]. This paper also contains an early discussion of the strong ties between entropy inequalities and high-dimensional convex geometry, starting with Costa and Cover’s 1984 observation [18] that the Brunn-Minkowski inequality can be viewed as a special case of a generalised EPI.

Building on the technical ideas of Stam and Blachman, Bakry and Émery in a very influential 1985 paper [3] derived an important representation of the derivative of the relative entropy $D(P_t || Q_t)$ of the time-$t$ distributions of a diffusion with different initial conditions. Under appropriate assumptions, strong connections were established with logarithmic Sobolev inequalities, generalising the earlier connection between the EPI and Gross’ Gaussian logarithmic Sobolev inequality, and facilitating the study of the long-term behaviour of the underlying diffusion. An important observation, independently re-discovered by Barron [6], is that this derivative can be expressed as a “relative” Fisher information, which also admits an interpretation as a minimum mean squared error. This interpretation had been promoted earlier in work by Brown, see e.g. [14], and it was re-framed in more information theoretic terms by Guo et al. in 2005 [40], leading to a variety of subsequent developments.

More recently, a remarkable equivalence between the subadditivity property of entropy and the classical Brascamp-Lieb inequality was pointed out by Carlen and Cordero-Erausquin [16], and a unified information-theoretic treatment was given by Liu et al. [64]. In yet another direction, Tao in 2010 [80] developed a series of discrete entropy inequalities motivated by sumset and inverse sumset bounds in additive combinatorics, also leading to a discrete version of the EPI. More recent work in this direction includes [58, 67].

Finally, we mention that a natural analog of the entropy in free probability was introduced in a series of papers by Voiculescu [84, 85, 86, 87], where several properties of the free entropy are established, including a free version of the EPI. In related work, convergence results to maximum free entropy distributions is considered by Johnson in [49] Chapter 8].
2 Information-theoretic proof of a finite de Finetti theorem

Suppose \( X_1^n = (X_1, \ldots, X_n) \), for some fixed \( n \), are exchangeable, discrete random variables, with values in a finite alphabet \( A = |A| \) elements. Let \( \hat{P}_{X_1^n} \) denote the (random) type of \( X_1^n \), and let the measure \( \mu = \mu_n \) denote the law of \( \hat{P}_{X_1^n} \) on the probability simplex \( \mathcal{P} \). In this section we provide an information-theoretic proof of the following:

Theorem 1 (Finite de Finetti theorem). For any \( 1 \leq k \leq n \), let \( P_k \) denote the distribution of \( X_1^k = (X_1, \ldots, X_k) \) and \( M_{\mu, k} \) denote the mixture-of-i.i.d.s:

\[
M_{\mu, k}(x_1^k) = \int_{\mathcal{P}} Q^k(x_1^k) d\mu(Q), \quad x_1^k \in A^k.
\]

For any \( 1 \leq k \leq (n/100)^{1/3} \), we have,

\[
D(P_k \| M_{\mu, k}) \leq \epsilon(n, k) := 2\delta + ke^{-\frac{4}{2m} \left( \frac{n}{k} + 1 \right) 2m^k \log n}, \tag{9}
\]

with \( \alpha = \alpha_{n, k} = \left[ \frac{2k}{\sqrt{n}} \left( \frac{1 + 2k}{\sqrt{n}} + 1 \right) \right]^{1/2} \) and \( \delta = \delta_{n, k} = \alpha \log(m^k / \alpha) \).

Before giving the proof of the theorem, some remarks are in order:

1. It can be seen from (9) that, if \( k \) stays bounded as \( n \to \infty \), then:

\[
\epsilon(n, k) = O(\delta_{n, k}) = O\left( \left( \frac{k}{\sqrt{n}} \right)^{1/2} \log \frac{n}{k} \right) \to 0.
\]

Moreover, in order for \( \epsilon(n, k) \) to vanish, \( k \) can grow at most logarithmically with \( n \). This is, at least asymptotically, weaker than the bound \( \delta \) given in [37] for the binary case \( m = 2 \). What’s more, the proof of (9) given below is longer and more involved that the corresponding proof of (8) in [37]. So why bother? The reason is that the proof given here follows a completely different information-theoretic path than that in [37], and that path consists of an appealing sequence of steps making interesting connections. So we first present a heuristic outline, and then give the actual proof. In fact, as will be seen from the proof (especially Lemma 1), it is easy to improve the bound \( \epsilon(n, k) \), but our purpose here is to illustrate the ideas rather than to obtain optimal results.

2. We have cheated slightly in the statement of the theorem, in that the proof below is only given for the case when \( n \) is a multiple of \( k \). However, this is only a minor technical inconvenience; for example, we can replace \( n \) with an integer multiple of \( k \) which is no less than \( n - k \), leading to the same bound with \( \epsilon(n - k, k) \) in place of \( \epsilon(n, k) \).

3. Fe Finetti’s original theorem [7] easily follows from (9) by an application of Pinsker’s inequality (2) and a standard weak convergence argument.

Heuristic proof of de Finetti’s theorem [7].
Step 1: Since the sequence \( \{X_n\} \) is exchangeable it is also stationary, therefore, by the ergodic theorem \( \hat{P}_{X_n} \) converges as \( n \to \infty \) a.s. to a (random) \( P \) on \( A \). Let \( \bar{\mu} \) denote the law of \( P \), and let \( \{Y_n\} \) be i.i.d. random variables uniformly distributed on \( A \). Then, by exchangeability, we clearly have for any \( n \), any \( k \leq n \), any \( n \)-type \( Q_n \), and any \( a_1^k \in A^k \),

\[
\mathbb{P}(X_1^k = a_1^k | \hat{P}_{X_n} = Q_n) = \mathbb{P}(Y_1^k = a_1^k | \hat{P}_{Y_n} = Q_n). \tag{10}
\]

Step 2: Choose and fix any one of the almost all realisations \( \{Q_n\} \) along which \( \hat{P}_{X_n} \) converges to some \( Q \) as \( n \to \infty \). By (10) and symmetry we have,

\[
\mathbb{P}(X_1 = a | \hat{P}_{X_n} = Q_n) = \mathbb{E}\left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(Y_i = a) \right) \hat{P}_{Y_n} = Q_n, = Q_n(a),
\]

so that,

\[
\mathbb{P}(X_1 = a | \hat{P}_{X_n} = Q_n) = \mathbb{E}\left( \hat{P}_{Y_n} \right) = Q_n(a),
\]

for any \( a \in A \), and letting \( n \to \infty \) yields,

\[
\lim_{n \to \infty} \mathbb{P}(X_1 = a | \hat{P}_{Y_n} = Q_n) = Q(a). \tag{11}
\]

Step 3: Next we generalise (11) to blocks of random variables. As before, choose and fix any one of the almost all realisations \( \{Q_n\} \) of the random \( \hat{P}_{X_n} \) such that \( Q_n \to Q \) as \( n \to \infty \). Define a new sequence of i.i.d. random variables \( Z_n = (Y_{2n-1}, Y_{2n}) \), \( n \geq 1 \), so that each \( Z_n \) is uniformly distributed on \( A \times A \). From (10), taking \( k = 2 \) and an arbitrary even \( n = 2\ell \),

\[
\mathbb{P}(X_1, X_2 = (a_1, a_2) | \hat{P}_{X_{2\ell}} = Q_{2\ell}) = \mathbb{P}(Z_1 = (a_1, a_2) | \hat{P}_{Z_{2\ell}} \in E(Q_{2\ell})), \tag{12}
\]

where \( E(Q) \) denotes the set of probability distributions \( W \) on \( A \times A \) with the property that the average of the two marginals \( W_1 \) and \( W_2 \) of \( W \) equals \( Q \),

\[
E(Q) = \left\{ W \text{ on } A \times A : \frac{W_1 + W_2}{2} = Q \right\}.
\]

If we write \( U \) for the uniform distribution on \( A \times A \), it is easy to check that the distribution \( W_\ell \) that uniquely achieves the \( \min_{W \in E(Q)} D(W||U) \) is simply \( Q \times Q \). At this point, we would wish to apply the conditional limit theorem 10 to the i.i.d. process \( \{Z_n\} \), to obtain that,

\[
\lim_{\ell \to \infty} \mathbb{P}(Z_1 = (a_1, a_2) | \hat{P}_{Z_{2\ell}} \in E(Q_{2\ell})) = \lim_{\ell \to \infty} W_\ell(a_1, a_2) = \lim_{\ell \to \infty} Q_{2\ell}(a_1)Q_{2\ell}(a_2) = Q(a_1)Q(a_2),
\]

and combining this with (12) would yield:
\[ \mathbb{P}(X_1, X_2 = (a_1, a_2) | \hat{P}_{X_1^2} = Q_{2\ell}) \to Q(a_1)Q(a_2), \quad \ell \to \infty. \]

The same argument can be used without difficulty to show that for any \( k \geq 1 \) and any \( a_1^k \in A^k \),
\[ \lim_{\ell \to \infty} \mathbb{P}(X_1^k = a_1^k | \hat{P}_{X_1^k} = Q_{k\ell}) = Q^k(a_1^k). \quad (13) \]

**Step 4:** Since (13) holds for almost every sequence \( \{Q_n\} \), letting \( \ell \to \infty \), by the bounded converge theorem we have,
\[ \mathbb{P}(X_1^k = a_1^k) = \mathbb{E}(\mathbb{P}(X_1^k = a_1^k | \hat{P}_{X_1^k})) \to \int Q^k(a_1^k) d\bar{\mu}(Q), \]
as required. \( \square \)

The only problem with the above argument is that the set \( E(Q) \) has an empty interior so that the conditional limit theorem is not directly applicable. Nevertheless, in the next section where we take a finite-\( n \) approach, we are able to ‘imitate’ the proof of the conditional limit theorem and replace the step where the non-empty interior assumption is used with a different argument.

### 2.1 Proof

Recall the notation and terminology for types described in the Introduction. Let \( \mu = \mu_n \) denote the law of \( \hat{P}_{X_n} \) on \( \mathcal{P} \), and let \( \{Y_n\} \) be i.i.d. random variables uniformly distributed on \( A \). For any \( k \leq n \), any \( n \)-type \( Q_n \), and any \( a_1^k \in A^k \),
\[ \mathbb{P}(X_1^k = a_1^k | \hat{P}_{X_1^k} = Q_n) = \mathbb{P}(Y_1^k = a_1^k | \hat{P}_{Y_1^k} = Q_n). \]

For \( k = 1 \) and any \( a \in A \), by symmetry we have,
\[ \mathbb{P}(X_1 = a | \hat{P}_{X_1}) = \hat{P}_{X_1}(a), \]
and taking the expectation of both sides with respect to \( \mu \) shows that in fact \( P_1 = M_{\mu, 1} \).

For general \( 1 \leq k \leq n \) with \( n = k\ell \), for any \( n \)-type \( Q \) we have,
\[ \mathbb{P}(X_1^k = a_1^k | \hat{P}_{X_1^k} = Q) = \mathbb{P}(Z_1 = a_1^k | \hat{P}_{Z_1} \in E_k(Q)) \]
\[ = \mathbb{E}(\hat{P}_{Z_1}(a_1^k) | \hat{P}_{Z_1} \in E_k(Q)), \]
where now \( \{Z_n\} \) is a sequence of i.i.d. random variables uniformly distributed on \( A^k \), and \( E_k(Q) \) consists of all probability distributions \( W \) on \( A^k \) with the property that the average of the \( k \) one-dimensional marginals of \( W \) equals \( Q \). Taking expectations with respect to \( \hat{P}_{X_1^k} = \hat{P}_{X_1} \sim \mu = \mu_n \),
\[
\mathbb{P}(X^k_1 = a^k_1) = \int \mathbb{E}(\hat{P}_{Z^k_1}(a^k_1) | \hat{P}_{Z^k_1} \in E_k(Q)) d\mu(Q),
\]
and by the joint convexity of relative entropy,
\[
D(P_k \| M_{\mu,k}) = D\left( \int \mathbb{E}(\hat{P}_{Z^k_1} | \hat{P}_{Z^k_1} \in E_k(Q)) d\mu(Q) \bigg\| \int Q^k d\mu(Q) \right)
\leq \int D\left( \mathbb{E}(\hat{P}_{Z^k_1} | \hat{P}_{Z^k_1} \in E_k(Q)) \bigg\| Q^k \right) d\mu(Q)
\leq \int \mathbb{E}\left( D(\hat{P}_{Z^k_1} \| Q^k) | \hat{P}_{Z^k_1} \in E_k(Q) \right) d\mu(Q).
\tag{14}
\]

We will obtain an explicit bound for the relative entropy in (14). First, we construct a joint \( \ell \)-type \( W \) with desirable properties. Let \( \mathcal{P}_\ell \) denote the set of \( \ell \)-types on \( A^k \).

**Lemma 1.** For any \( \ell > k \geq 1 \) and any \( n \)-type \( Q \), there is a \( W \in E_k(Q) \cap \mathcal{P}_\ell \) with:
\[
\max_{a^k_1} |W(a^k_1) - Q^k(a^k_1)| \leq M := \left\lfloor \frac{2}{\ell} + \frac{4k}{\ell} + 2\sqrt{\frac{k}{\ell}} \right\rfloor. 
\]
Moreover, for \( 2 \leq k \leq \sqrt{\ell}/10 \),
\[
|H(W) - H(Q^k)| \leq -M \log \frac{M}{m^k}.
\]

**Proof.** Let \( x^{k\ell} \in A^{k\ell} \) have type \( Q \), let \( V^{k\ell} \) be a random permutation of \( x^{k\ell} \), and let \( \hat{W} \) denote its (random) \( \ell \)-type. Obviously we have that \( \hat{W} \in E_k(Q) \) by construction, and we will also show that \( \hat{W} \) satisfies the statement of the lemma with positive probability. Taking any \( k \leq \ell \) and \( \gamma > 0 \) arbitrary,
\[
\mathbb{P}\left( \max_{a^k_1} |\hat{W}(a^k_1) - Q^k(a^k_1)| > \gamma \right)
\leq \sum_{a^k_1} \mathbb{P}\left( |\hat{W}(a^k_1) - Q^k(a^k_1)| > \gamma \right)
\leq \sum_{a^k_1} \gamma^{-2} \mathbb{E}\left[ (\hat{W}(a^k_1) - Q^k(a^k_1))^2 \right]
= \gamma^{-2} \sum_{a^k_1} \left[ \rho_2(a^k_1) - 2Q^k(a^k_1)\rho_1(a^k_1) + Q^k(a^k_1)^2 \right],
\tag{15}
\]
where \( \rho_2(a^k_1) = \mathbb{E}[\hat{W}(a^k_1)^2] \) and \( \rho_1(a^k_1) = \mathbb{E}[\hat{W}(a^k_1)] \).

Now we find appropriate bounds so that the above probability is \( < 1 \). To get an upper bound on \( \rho_1(a^k_1) \) for some fixed \( a^k_1 \) note that,
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$$\rho_1(a_1^k) = \mathbb{P}(V_1^k = a_1^k) \leq \prod_{i=1}^{k} \frac{n(a_i)}{\ell k - i + 1},$$

where \(n(a_i)\) is the number of appearances of \(a_i\) in \(x^{k\ell}\), and hence,

$$\rho_1(a_1^k) \leq \prod_{i=1}^{k} \frac{n(a_i)}{\ell k - i + 1} \leq Q^k(a_1^k) \left(\frac{\ell k}{\ell k - k}\right)^k \leq Q^k(a_1^k) \left(1 - \frac{1}{\ell}\right)^{-k} \leq Q^k(a_1^k)(1 + k/\ell),$$

since \((1-x)^{-k} \leq 1+kx\) for \(x \in [0, 1)\). Similarly, writing \(a_1^k \ast a_1^k\) for the concatenation of \(a_1^k\) with itself, we can estimate,

$$\mathbb{P}(V_1^{2k} = a_1^k \ast a_1^k) \leq Q^k(a_1^k)^2(1 + 4k/\ell),$$

so that,

$$\rho_2(a_1^k) = \ell \frac{1}{\ell^2} \mathbb{P}(V_1^k = a_1^k) + \ell(\ell - 1) \frac{1}{\ell^2} \mathbb{P}(V_1^k = a_1^k \ast a_1^k) \leq (1/\ell)(1 + k/\ell)Q^k(a_1^k) + (1 + 4k/\ell)Q^k(a_1^k)^2 \leq (2/\ell)Q^k(a_1^k) + (1 + 4k/\ell)Q^k(a_1^k)^2. \quad (16)$$

Substituting the bound \((16)\) in \((15)\) we have,

$$\mathbb{P}\left( \max_{a_1^k} |W(a_1^k) - Q^k(a_1^k)| > \gamma \right) \leq \gamma^{-2} \sum_{a_1^k} Q^k(a_1^k) \left[ \frac{2}{\ell} + \left(2 + \frac{4k}{\ell}\right)Q^k(a_1^k) - 2\rho_1(a_1^k) \right] \leq \gamma^{-2} \max_{a_1^k} \left[ \frac{2}{\ell} + \left(2 + \frac{4k}{\ell}\right)Q^k(a_1^k) - 2\rho_1(a_1^k) \right]. \quad (17)$$

Finally, we get a lower bound on \(\rho_1(a_1^k)\). In the case where for some \(\beta > 0\) (to be chosen later), \(Q(a_i) > \beta\) for all \(a_i\), we have,

$$\rho_1(a_1^k) \geq \prod_{i=1}^{k} \frac{n(a_i) - i + 1}{\ell k - i + 1} \geq \prod_{i=1}^{k} \frac{n(a_i) - k + 1}{\ell k} \geq Q^k(a_1^k) \prod_{i=1}^{k} \left(1 - \frac{1}{\ell Q(a_i)}\right),$$

so that,
assuming $\beta \geq 1/2\ell$, since $(1-x)^k \geq 1-kx$ for all $k \geq 1$ and all $x \in [0, 2]$. For all such $a_1^k$, using (18) we can bound the expression in the maximum in (17) by
\[
\left| \frac{2}{\ell} + \left(2 + \frac{4k}{\ell}\right)Q^k(a_1^k) - 2\rho_1(a_1^k) \right| < \frac{2}{\ell} + \frac{4k}{\ell} + \frac{2k}{\ell\beta}.
\]
And in the case when at least one $a_i$ has $Q(a_i) \leq \beta$, simply omitting the negative term and noting that $Q^k(a_1^k) \leq \beta$ we bound the same term above by,
\[
\left[ \frac{2}{\ell} + \left(2 + \frac{4k}{\ell}\right)Q^k(a_1^k) \right] \leq \frac{2}{\ell} + \left(2 + \frac{4k}{\ell}\right)\beta.
\]
Combining the last three bounds,
\[
\mathbb{P}\left( \max_{a_i^k} |W(a_i^k) - Q^k(a_i^k)| > \gamma \right) \leq \gamma^{-2} \left[ \frac{2}{\ell} + \max \left\{ \frac{2k}{\ell} (2 + \frac{1}{\beta}), (2 + \frac{4k}{\ell})\beta \right\} \right],
\]
where the inequality is strict when the first term dominates the maximum. To obtain a good bound we take for $\beta$ a value approximately equal to the minimiser of the above expression: We set $\beta^* = \sqrt{\gamma/\ell}$. Note that for this $\beta^*$ it can be easily verified that the first term strictly dominates the maximum, giving,
\[
\mathbb{P}\left( \max_{a_i^k} |W(a_i^k) - Q^k(a_i^k)| > \gamma \right) < \gamma^{-2} \left[ \frac{2}{\ell} + \frac{4k}{\ell} + 2\sqrt{\frac{\ell}{\gamma}} \right],
\]
and taking $\gamma = M$ as in the lemma, completes the proof of the first statement.

For the second part, noting that for $2 \leq k \leq \sqrt{\ell}/10$ we have $M < 1/2$, the result follows from [28 Lemma 2.7].

Next we obtain an upper bound on the conditional expectation in (14).

**Lemma 2.** Suppose $n = \ell k$, with $2 \leq k \leq \sqrt{\ell}/10$. For any $n$-type $Q$ we have:

\[
\mathbb{E}\left( D(\hat{P}_{Z_1} \| Q^k) \middle| \hat{P}_{Z_1} \in E_k(Q) \right) \leq \epsilon(n, k).
\]

**Proof.** We follow the same steps as in the proof of the conditional limit theorem in [19]. Recall that if we write $U_k$ for the uniform distribution on $A^k$, then the $W_k^*$ that uniquely achieves $D^* = \min_{W \in E_k(Q)} D(W \| U_k)$ is $W_k^* = Q^k$. We partition $E_k(Q)$ into $B_{2\delta}$ and $C = E_k(Q) - B_{2\delta}$, where $B_{2\delta} = \{ W \in E_k(Q) : D(W \| U_k) \leq D^* + 2\delta \}$, with $\delta = \delta_{n,k}$. Then, writing $\nu_\ell$ for the distribution of $\hat{P}_{Z_1}$,

\[
\nu_\ell(C \cap E_k(Q)) = \frac{\nu_\ell(C \cap E_k(Q))}{\nu_\ell(E_k(Q))} \leq \frac{\nu_\ell(C)}{\nu_\ell(B_{2\delta})}.
\]
Thus, the claimed bound now follows by Lemma 3 on taking $\ell$ for relative entropy \[19\] to conclude that:

$$
\sum_{U_k} e^{-\ell D(W\|U_k)} 
$$

Since the set $P_\ell$ is closed and convex, we may apply the Pythagorean identity for relative entropy \[19\] to conclude that:

$$
\mathbb{P}(\hat{P}_{Z_L} \in C | \hat{P}_{Z_L} \in E_k(Q)) \leq (\ell + 1)^{2m^k} e^{-\ell \delta},
$$

or,

$$
\mathbb{P}(D(\hat{P}_{Z_L} \| Q^k) > D^* + 2\delta | \hat{P}_{Z_L} \in E_k(Q)) \leq (\ell + 1)^{2m^k} e^{-\ell \delta}.
$$

Since the set $E_k(Q)$ is closed and convex, we may apply the Pythagorean identity for relative entropy \[19\] to conclude that:

$$
\mathbb{P}(D(\hat{P}_{Z_L} \| Q^k) > 2\delta | \hat{P}_{Z_L} \in E_k(Q)) \leq (\ell + 1)^{2m^k} e^{-\ell \delta}.
$$

Thus,

$$
e D(\hat{P}_{Z_L} \| Q^k) | \hat{P}_{Z_L} \in E_k(Q) \leq (\ell + 1)^{2m^k} e^{-\ell \delta} \max_{P \in E_k(Q)} D(P \| Q^k) + 2\delta.
$$

The claimed bound now follows by Lemma 3 on taking $\ell = k/n$. 

Next we bound the above numerator and denominator. For the numerator, writing again $P_\ell$ for the set of $\ell$-types on $A^k$,

$$
\nu_\ell(C) = \sum_{W \in C \cap P_\ell} U_k(T(W)) \leq \sum_{W \in C \cap P_\ell} e^{-\ell D(W\|U_k)} \leq |E_k(Q) \cap P_\ell| e^{-\ell (D^*+2\delta)} \leq (\ell + 1)^{m^k} e^{-\ell (D^*+2\delta)}.
$$

where $T(W)$ in (a) denotes the type class of all strings of length $\ell$ in $A^k$ with type $W$, (b) is a standard property \[19\], (c) follows from the definition of $C$ and the fact that $E_k(Q) \cap P_\ell < E_k(Q)$, and (d) follows from the standard observation that $|E_k(Q) \cap P_\ell| \leq |P_\ell| \leq (\ell + 1)^{m^k}$. Similarly, letting $W_0$ denote the type from Lemma 1, we have

$$
\nu_\ell(B_{2\delta}) \geq \nu_\ell(B_{\delta}) = \sum_{W \in B_{\delta} \cap P_\ell} U_k(T(W)) \geq U_k(T(W_0)) \geq (\ell + 1)^{m^k} e^{-\ell D(W_0\|U)} \geq (\ell + 1)^{m^k} e^{-\ell (D^*+\delta)}.
$$

Combining these bounds, we obtain,

$$
\mathbb{P}(\hat{P}_{Z_L} \in C | \hat{P}_{Z_L} \in E_k(Q)) \leq (\ell + 1)^{2m^k} e^{-\ell \delta},
$$

or,

$$
\mathbb{P}(D(\hat{P}_{Z_L} \| Q^k) > D^* + 2\delta | \hat{P}_{Z_L} \in E_k(Q)) \leq (\ell + 1)^{2m^k} e^{-\ell \delta}.
$$

Since the set $E_k(Q)$ is closed and convex, we may apply the Pythagorean identity for relative entropy \[19\] to conclude that:

$$
\mathbb{P}(D(\hat{P}_{Z_L} \| Q^k) > 2\delta | \hat{P}_{Z_L} \in E_k(Q)) \leq (\ell + 1)^{2m^k} e^{-\ell \delta}.
$$

Thus,

$$
e D(\hat{P}_{Z_L} \| Q^k) | \hat{P}_{Z_L} \in E_k(Q) \leq (\ell + 1)^{2m^k} e^{-\ell \delta} \max_{P \in E_k(Q)} D(P \| Q^k) + 2\delta.
$$

The claimed bound now follows by Lemma 3 on taking $\ell = k/n$. 


Lemma 3. For any \( n \)-type \( Q \), \( \max_{W \in E_k (Q)} D(W\|Q^k) \leq k \log n \).

Proof. If \( a^k_i \in A^k \) is such that \( Q^k (a^k_i) = \prod_{i=1}^k Q(a_i) = 0 \), then \( Q(a_{i_0}) = 0 \) for some \( i_0 \). Since \( Q(a_{i_0}) = \frac{1}{n} \sum_{j=1}^k W_j (a_{i_0}) \), we must have \( W_1 (a_{i_0}) = \cdots = W_k (a_{i_0}) = 0 \), which implies that \( W(a^k_i) = 0 \).

On the other hand, if \( Q^k (a^k_i) > 0 \) then \( Q^k (a^k_i) \geq \frac{1}{n^k} \). Thus, for any \( W \in E_k (Q) \),

\[
D(W\|Q^k) = \sum_{a_i^k \in A^k} W(a_i^k) \log \frac{W(a_i^k)}{Q^k(a_i^k)} \\
\leq \sum_{a_i^k \in A^k} W(a_i^k) \log \frac{W(a_i^k)}{\left(\frac{1}{n}\right)^k} \\
= k \log n - H(W) \\
\leq k \log n,
\]

as required.

Theorem 1 follows from (14) combined with Lemma 2.

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