Semiclassical Scattering in Yang-Mills Theory

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Abstract

A classical solution to the Yang-Mills theory is given a new semiclassical interpretation. The boundary value problem on a complex time contour which arises from the semiclassical approximation to multiparticle scattering amplitudes is reviewed and applied to the case of Yang-Mills theory. The solution describes a classically forbidden transition between states with a large average number of particles in the limit $g \to 0$. It dominates a transition probability with a semiclassical suppression factor equal to twice the action of the well-known BPST instanton. Hence, it is relevant to the problem of high energy tunnelling. It describes transitions of unit topological charge for an appropriate time contour. Therefore, it may have a direct interpretation in terms of fermion number violating processes in electroweak theory. The solution describes a transition between an initial state with parametrically fewer particles than the final state. Thus, it may be relevant to the study of semiclassical initial state corrections in the limit of a small number of initial particles. The implications of these results for multiparticle production in electroweak theory are also discussed.

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1 Introduction

An intriguing feature of the Yang-Mills gauge theory is the periodic structure of its vacuum \[1, 2\]. In the semiclassical approximation, the topology of finite energy solutions leads to a classification of all gauge-inequivalent vacua in the theory. The discovery of this rich structure has had a profound impact on our understanding of non-perturbative aspects of the theory, notably low energy phenomena like the solution of the famous U(1) problem in QCD \[3\]. However, the role of the vacuum in the dynamics of particle scattering, and in particular high energy multiparticle scattering, is not yet as deeply understood. This deficit in our understanding has been confronted in recent years with the study of so-called “instanton-induced” cross-sections \[4, 5, 6, 7\].

The simplest semiclassical estimate of the contribution of the BPST instanton \[8\] to a total inclusive two particle cross-section in electroweak theory implies a result which grows exponentially with center-of-mass energy \[4, 5, 7, 9\]. The same behavior has also appeared in a large number of model field theories with instanton solutions throughout an extensive series of investigations \[6, 7\]. It has further been shown that the instanton is the basis of a systematic perturbative expansion of the final state radiative corrections to the cross-section \[9\]. This determines the leading semiclassical behavior, neglecting initial state radiative corrections,

\[
\sigma_{\text{tot}}(x) \sim \exp \left[ \frac{16\pi^2}{g^2} F(x) + o(\alpha^0) \right]
\]

as an expansion in powers of a small parameter, \(x \equiv E/E_0\), the ratio of the center-of-mass energy \(E\) and a mass scale of order the electroweak sphaleron mass, \(E_0 \simeq M_{w}/\alpha_{w} \simeq 10\ \text{TeV}\). The so-called “Holy Grail function”, \(F(x)\), is approximately \(-1\) for small \(x\), reflecting the severe ’tHooft suppression factor, \(\exp \left(-\frac{16\pi^2}{g^2} \right) \simeq 10^{-127}\), due to the large instanton action. The fact that the Holy Grail function is an increasing function of \(x\) for small \(x\) has led many to speculate about the possibility of overcoming the severe exponential suppression factor at energies of order \(E_0\). The possibility of strong multiparticle scattering in electroweak theory
at energies in the multi-TeV range has led to an enormous effort to understand the behavior of multiparticle cross-sections in the sphaleron energy regime [7].

One approach has been to consider a mechanism by which the exponentially growing cross-section unitarizes at high energies. In this regard, multi-instanton contributions to the cross-section (1.1) have been considered, as a means to unitarize the cross-section by s-channel iteration of the one-instanton contribution [10, 11, 12]. It has been argued that if one assumes the validity of (1.1) in the one-instanton sector, the strong multi-instanton contributions become important before the ’tHooft suppression is overcome.

However, all of these conclusions are based on semiclassical expansions around configurations which are not influenced by external sources. Indeed, the instanton and multi-instantons obey vacuum boundary conditions, and as such are relevant to this problem only in the approximation in which external sources are neglected. While the final state corrections can be taken into account in the perturbative expansion in $x$, the initial state corrections are more subtle. These involve radiative corrections to hard particles which are not a priori expressible semiclassically. However, there have been some indications [13, 15, 16] that the contributions to $F(x)$ of corrections involving hard initial legs may also be calculable in a semiclassical manner. It may then be possible to calculate the entire leading order semiclassical exponent in a saddle point approximation. What is needed is a new technique which accounts for external sources to make the semiclassical behavior of the total cross-section manifest.

A strategy for out-flanking the problem of initial state corrections was recently proposed by Rubakov, Son and Tinyakov [16, 17, 18, 19]. The basic idea is to consider transitions from states of a fixed large number of particles, say $N_{in} = \nu/g^2$. The instanton-induced transition probability from a multiparticle initial state is then calculable semiclassically, in the limit $g \to 0$ with $\nu$ fixed. Its leading semiclassical behavior is determined by the

\footnote{In addition, the distinction between corrections involving initial and final state particles is ambiguous at high orders of the low energy expansion [13].}
solution to a boundary value problem. The boundary conditions imposed at initial and final times correctly account for the energy transfer from the initial multiparticle state to the final multiparticle state. The leading semiclassical behavior of the \( N_{\text{in}} \)-particle transition probability has a form similar to (1.1)

\[
\sigma_{N_{\text{in}}}(x) \sim \exp \left[ -\frac{16\pi^2}{g^2} F(x, \nu) + o(g^0) \right].
\]

(1.2)

The function \( F(x, \nu) \) is a rigorous upper bound on the two-particle “Holy Grail” function (1.1),

\[
F(x, \nu) \geq F(x) \quad x \equiv E/E_0,
\]

(1.3)

and is related to a lower bound under less rigorous assumptions [16]. Since this function contains all initial state corrections for the \( N_{\text{in}} = \nu/g^2 \) particle transition, it is hoped that it reproduces the leading semiclassical behavior of two-particle transition when \( \nu \) is small, including initial and final state corrections. Initial indications from explicit calculations of initial and final state corrections are that the limit \( \nu \to 0 \) is smooth [14, 16], so that the contribution to a semiclassical transition probability from the solution of the boundary value problem contains the initial and final state corrections. The boundary value problem posed in this way also holds the promise of being amenable in principle to numerical computation of multiparticle transitions. It would now be useful to have some analytical examples to guide future efforts in this direction [19, 20].

For the calculation of an instanton-induced (i.e. tunneling) transition at fixed energy, the choice of a Minkowski or Euclidean time contour is too restrictive. The boundary value problem is instead conveniently formulated on a complex time contour, to be explained below. A few such solutions on a complex time contour have already been investigated.

A classical solution with two turning points on a complex time contour is the so-called periodic instanton [21]. The periodic instanton is a solution to the complex-time boundary

\footnote{This should not be confused with the periodic solution to the Euclidean formulation of finite-temperature Yang-Mills theory [22], nor with the periodic multi-instanton configurations [10, 11, 12].}
value problem which arises in the semiclassical approximation to the inclusive transition probability from all initial states at fixed energy, or a microcanonical distribution. It has been shown to determine the maximal probability for transition in the one-instanton sector from states of fixed energy \[21, 18\]. The periodic instanton in electroweak theory has so far been constructed only in a low energy approximation, and the resulting transition probability is determined in a perturbative expansion similar to that in (1.1). It has been found to describe transitions between states of equal number of particles which is large in the semiclassical limit, \(N_{\text{in}} = N_{\text{fin}} \sim 1/g^2\). So, this solution is irrelevant for describing \(2 \to n\) scattering processes at high energies, though it does play a role in determining the rate of tunnelling, and anomalous baryon number violation, at finite temperature \[23\].

A solution which describes transitions from a state of smaller number of particles to a state with a larger number of particles has also been constructed in a low energy expansion \[18\]. Similarly, it determines the maximum transition probability from states of fixed energy and particle number. However, it remains to construct solutions which describe such processes in general. This is a formidable task, requiring a solution of the Yang-Mills equations with arbitrary boundary conditions on a complex time contour. In this paper, we pursue more modest goals. We investigate the properties of a well-known, highly symmetric Minkowski time solution on a complex time contour. The solution in Minkowski time describes an energy density which evolves from early times as a thin collapsing spherical shell, bounces at an intermediate time, and expands outward again at late times. As yet, the role of this solution in scattering problems has not been fully developed \[24\].

We show that a subclass of the SO(4)-conformally invariant solutions found by Lüscher and Schechter exhibits a number of remarkable properties on a suitably chosen complex time contour:

1. The semiclassical suppression is equal to the action of the BPST instanton, \(\hbox{Im } S = \frac{8\pi^2}{g^2}\).

   This quantity controls the semiclassical exponential dependence of a transition proba-
bility between coherent states.

2. The topological charge of the solution is equal to the BPST instanton charge, \( Q = 1 \).
   Thus, the solution may have a direct interpretation for fermion number violating processes [24].

3. It solves the boundary value problem for the transition probability in the one-instanton sector from a coherent state with a smaller number of particles, to a state with a larger number of particles. This property makes the solution interesting for the investigation of \( 2 \to n \) processes with fermion number violation at high energies [16, 17, 18].

Thus, the Lüscher-Schechter solution considered in this paper provides an analytical benchmark for future numerical computations of many particle transition amplitudes in Yang-Mills theory. The work presented here makes explicit use of the conformal invariance of the Yang-Mills theory, both in the construction of the solution and in the calculation of its properties. To the extent that this theory represents the high energy limit of a spontaneously-broken gauge theory, the results also have implications for multiparticle cross-sections and high energy baryon number violation in electroweak theory. These implications will be discussed in the final section of this paper.

This paper is organized as follows. In Section 2, we show that the imaginary part of the action and the topological charge of the solution are determined solely by the number \( N \) of singularities of the solution in the complex time plane, enclosed between the complex time contour and the real time axis. These solutions have therefore the remarkable property that the imaginary part of the action and the topological charge obey

\[
\frac{g^2}{8\pi^2} \text{Im} S = Q = N.
\]

In Section 3, we show that the initial and final coherent states contain a different number of particles, the ratio being controlled by a parameter of the solution. We also explicitly demonstrate that the initial and final gauge field configurations belong to different topological
sectors, in agreement with the results of Section 2. In Section 4, we discuss the relationship of this classical solution to the saddle points of transition probabilities at fixed energy and particle number. Finally, Section 5 contains a discussion of the results and an outlook on unsolved problems.

2 The Action and Topological Charge of the Solution

In this section, we describe the classical solution and an appropriate choice of the complex time contour. Then, we compute its action and topological charge. The use of a Minkowski or Euclidean time contour for the semiclassical calculation of transition amplitudes in the one-instanton sector is too restrictive. Recall that computing tunneling contributions to fixed-energy (i.e. time-independent) Green functions in quantum mechanics can be performed in the WKB approximation only on a complex time contour, chosen to lie in Minkowski directions at early and late times, with a period of Euclidean evolution inserted at an intermediate time. They give the dominant WKB-contribution to classically forbidden processes. In the present case of quantum field theory, we will similarly be interested only in time-independent transition probabilities.

In the case of nonabelian gauge theories, the transitions between vacua with different topological number, signalled by fermion number or chirality violation, are analogous to the classically forbidden processes in quantum mechanics. The vacuum-to-vacuum transition amplitude is known to be maximized by instantons, in the semiclassical approximation. They, and in fact any finite action Euclidean solution, satisfy vacuum boundary conditions at infinity. However, for transitions involving many-particle initial and final states, vacuum boundary conditions are clearly not the correct ones. Considering solutions on a complex time contour, $C_T$, (fig.1) provides a natural description of the initial and final states in Minkowski space in terms of the free wave asymptotics of the solution at $|\text{Re} t| \to \infty$. 
The semiclassical calculation of the transition probability between arbitrary multiparticle states above neighboring topological gauge vacua is quite a formidable task. It requires solving the Yang-Mills equations on a suitably chosen complex time contour, with arbitrary boundary conditions imposed at the initial and final times. In order to simplify the problem, we will make use of the conformal symmetry of the classical Yang-Mills action and reduce the number of degrees of freedom to one.

Although such a drastic simplification will lead us astray from the problem of fermion number violation in high-energy collisions, it has the advantage of being tractable analytically and provides new insight into the role of complex time singularities. We find that they entirely determine the topological charge and imaginary part of the action of the solution. The imaginary part of the action enters the WKB-exponent for the transition probability between the initial and final states. The topological charge, through the anomaly equation, is the quantity determining the amount of fermion number or chirality violation in the process.

We consider the SO(4) conformally invariant Minkowski time \((t \in \mathbb{R})\) solutions of Lüscher and Schechter \([25, 26]\), analytically continued to a complex time contour (fig. 1). Our aim is the computation of many-particle transition amplitudes in the one-instanton sector. Therefore, we will consider only a subclass of these solutions which have integer topological
charge on the complex time contour $C_T$. Only the solutions with a turning point at say, $t = 0$ for all $\vec{x}$, have this property, as will be made clear at the end of this section. The Lüscher-Schechter solutions are real in Minkowski time. The turning point condition assures that their analytic continuation to the Euclidean time axis is real as well. Note that in general the fields will be complex on the $\text{Re} t < 0$ part of the contour, since $t = iT$ is not a turning point of the solution\(^3\).

In this section, we will work in Euclidean time and find a real solution with a turning point at zero Euclidean time. Its analytic continuation to Minkowski time will then be real as well. We define the Euclidean action of SU(2) pure Yang-Mills theory to be imaginary for a real Euclidean solution:

$$S = \frac{i}{4g^2} \int d^4x F_{\mu\nu}^a F_{\mu\nu}^a.$$  \hfill (2.1)

In order to make use of the conformal symmetry of the action (2.1), it is convenient to introduce new variables, which simplify the action of the conformal group. (For details, see [25, 26, 24].) The spatial radius $r \equiv |\vec{x}|$ and Euclidean time $\tau = it$ are mapped into two parameters of the Lobachevski plane $(w, \phi)$:

$$\{ 0 \leq r < \infty, -\infty < \tau < \infty \} \rightarrow \{ -\pi/2 \leq w \leq +\pi/2, -\infty < \phi < \infty \}$$

according to the relations:

$$\tan w = \frac{r^2 + \tau^2 - 1}{2r}, \quad \cosh \phi = \frac{1 + r^2 + \tau^2}{2r} \cos w, \quad \sinh \phi = \frac{\tau}{r} \cos w.$$  \hfill (2.2)

The following Jacobian relation holds:

$$\frac{dr \, d\tau}{r^2} = \frac{dw \, d\phi}{\cos^2 w}.$$  

Lüscher and Schechter have shown that the most general solution for which a $SO(4)$-conformal transformation can be compensated by a global $SU(2)$-gauge transformation is

\(^3\)It is easy to show that the $SO(4)$-conformally invariant solutions can have at most one turning point.
parameterized by a single function $q(\phi)$. Its action (2.1) is [25, 26]:

$$S = \frac{i}{2} \frac{12\pi}{g^2} \int_{-\infty}^{\infty} d\phi \int_{-\pi/2}^{+\pi/2} dw \cos^2 w \left[ \frac{1}{2} q^2 + \frac{1}{2} (q^2 - 1)^2 \right]$$

$$= \frac{i}{2} \frac{12\pi}{g^2} \int_0^{\infty} dr \int_{-\infty}^{+\pi/2} dw \cos^2 w \left[ \frac{1}{2} q^2 + \frac{1}{2} (q^2 - 1)^2 \right], \quad (2.3)$$

where $\dot{q} \equiv \frac{d}{d\phi} q(\phi)$.

The topological charge in terms of the Lüscher-Schechter Ansatz becomes:

$$Q \equiv \frac{1}{32\pi^2} \int d^4x F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\phi \int_{-\pi/2}^{+\pi/2} dw \cos^2 w \frac{d}{d\phi} (q^3 - 3q)$$

$$= \frac{1}{2\pi} \int_0^{\infty} dr \int_{-\infty}^{\infty} d\tau \frac{\cos^4 w}{r^2} \dot{q} \left( 3q^2 - 3 \right). \quad (2.4)$$

It follows from (2.3) that the equation of motion for the Euclidean Lüscher-Schechter Ansatz is:

$$\ddot{q} = -\frac{d}{dq} \left[ -\frac{1}{2} \left( q^2 - 1 \right)^2 \right], \quad (2.5)$$

so that $q(\phi)$ is the coordinate as a function of “time” $\phi$ of a particle moving in an inverted double-well potential

$$V(q) = -\frac{1}{2} \left( q^2 - 1 \right)^2. \quad (2.6)$$

Two extrema of the double-well, $q = 1$ and $q = -1$, correspond to vanishing field strengths, $F_{\mu\nu}$. The respective gauge potentials however differ by a large gauge transformation with winding number one.

Let us note that this Ansatz contains the BPST instanton [8]. It is given by the solution of (2.3)

$$q(\phi) = -\tanh \phi,$$

representing the motion of a particle which begins at $q = 1$ at time $\phi = -\infty$ and reaches $q = -1$ at time $\phi = +\infty$. It is easy to verify using (2.3) that the action is $8\pi^2/g^2$ and the topological charge (2.4) is unity.

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*4See [3.3] for explicit formulae relating $q$ to the gauge potentials.*
As stated previously, we will look only for solutions with a turning point at say, $\tau = 0$ for all $\vec{x}$. It follows from the explicit form of the mapping (2.2) that $\tau = 0$ is equivalent to $\phi = 0$ for all $r$. One can then easily verify, using the explicit formulae relating $q$ to gauge potentials (3.3), that the condition $\dot{\phi}(\phi = 0) = 0$ corresponds to a turning point of the gauge potentials at $\tau = 0$. A turning point of the gauge potentials requires $A^a_0(\tau = 0, \vec{x}) = 0$ and $\partial_r A^a_i(\tau = 0, \vec{x}) = 0$. Hence, the continuation of the gauge potentials to Minkowski time will be real as well.

A solution of (2.5) with such a turning point is easy to find by considering the one-dimensional double-well problem. It represents oscillatory motion in the well between $q = 1$ and $q = -1$ of the potential $V(q)$ (2.6). The turning point condition at $\phi = 0$ leaves one free parameter: the “energy” $\epsilon$ ($\epsilon < 1/2$), or equivalently the initial coordinate, $q_- = \sqrt{1 - \sqrt{2} \epsilon}$, of the particle in the well. This solution is explicitly given in terms of the Jacobian elliptic sine:

$$q(\phi(r, \tau)) = q_- \text{sn}(q_+ \phi(r, \tau) + K, k).$$

(2.7)

The two turning points in the well are:

$$q_\pm = \sqrt{1 \pm \sqrt{2} \epsilon},$$

and the modulus and primed-modulus of the elliptic sine are:

$$k^2 = \frac{q_-^2}{q_+^2} = \frac{1 - \sqrt{2} \epsilon}{1 + \sqrt{2} \epsilon}, \quad k'^2 \equiv 1 - k^2 = \frac{2 \sqrt{2} \epsilon}{1 + \sqrt{2} \epsilon}.$$

We shall be interested in what follows in the limit of small $\epsilon$ (or $k' \to 0$). This limit corresponds to solutions which are close to the vacuum $|q| = 1$ at the turning point. These solutions are of interest because they will be shown to describe transitions between initial and final coherent states containing a different number of particles [17]. In this limit, the

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5 The solution (2.7) can be shown to equal one of the Minkowski solutions given in [23, 24] by analytic continuation to Minkowski time, shift by half a period and use of the transformation formulae for elliptic functions [27].
periods of the elliptic sine have the following expansion [27]:

\[ K = \ln \frac{4}{k'} + \frac{k'^2}{4} \left( \ln \frac{4}{k'} - 1 \right) + O \left( k'^4 \ln k' \right), \quad K' = \frac{\pi}{2} \left( 1 + \frac{k'^2}{4} \right) + O \left( k'^4 \right). \quad (2.8) \]

Let us now turn to the calculation of the action and topological charge of the solution (2.7) on the complex time contour \( C_T \).

Both the imaginary part of the action and the topological charge are determined by the singularities of the solutions in the complex time plane, as will be clearly demonstrated below. The elliptic sine has only simple poles, whenever [27]

\[ q_+ \phi + K = 2nK + (2m + 1)iK', \quad n, m \in \mathbb{Z}. \]

Note that when analytically continued to complex \( \tau \), the imaginary part of \( \phi \) (2.2) obeys \( |\text{Im} \phi| \leq \pi \). Therefore, solutions of the above equation exist in the limit of small \( \epsilon \) only for \( m = -1 \) and \( m = 0 \). In \((r, \tau)\)-space, the singularities lie on the curves:

\[ \tau_{nm}(r) = q_{nm} \pm \sqrt{q_{nm}^2 - 1 - r^2}, \quad (2.9) \]

where

\[ q_{nm} = \coth \left( \frac{2n - 1}{q_+} K + (2m + 1)iK' \right). \]

There is also a “cross” of essential singularities of the mapping (2.2) \((r, \tau) \rightarrow (w, \phi)\):

\[ \tau = \pm(1 \pm ir), \quad r \neq 0, \]

where all four combinations of signs are allowed. The complex time contour \( C_T \) in fig. 1 should therefore be required to have \( \text{Re} \tau < 1 \) in order to avoid them. The equations of the first two singularity lines (2.9), for small \( \epsilon \) and \( r \gg 1 \), are:

\[ \tau_{1,-1}(r) = 1 - 2\frac{\sqrt{2\epsilon}}{8} - ir, \quad \tau_{2,-1}(r) = 1 - 2\left(\frac{\sqrt{2\epsilon}}{8}\right)^3 - ir, \]

after making use of the expansions (2.8). Note that \( m = -1 \) and \( m = 0 \) correspond to complex conjugation of \( \phi \). Since \( \phi \) is real for \( \text{Im} \tau = 0 \), it takes complex conjugate values at
points with $\text{Im} \tau \neq 0$, which are reflections of each other with respect to the Euclidean time axis. Hence, only the singularity lines with $m = -1$ lie in the $\text{Im} \tau = t < 0$ half-plane and are relevant to calculations on the contour $C_T$ in fig. 1.

These singularities are illustrated in fig. 2. As mentioned above, the parameter $T$ should be chosen to obey

$$\text{Re} \tau_{1,-1}(\infty) < T < \text{Re} \tau_{2,-1}(\infty)$$  \hspace{1cm} (2.10)

in order to avoid the singularity lines. Consider now the action (2.3) on the closed contour $C_T + C_M$, where $C_M$ runs along the Minkowski time axis for $-\infty < t < \infty$, and $C_T$ is the contour described above:

$$S_{C_T} + S_{C_M} = i \frac{12\pi}{g^2} \int_0^\infty dr 2\pi i \sum_{nm} \text{Res} \left\{ \frac{\cos^4 \omega}{r^2} \left[ \frac{1}{2} q^2 + \frac{1}{2} (q^2 - 1)^2 \right] \right\}_{\tau_{nm}(r)}.$$

The sum is over the singularity lines between the contour $C_T$ and the Minkowski time axis; the sum contains only one term for the contour in fig. 2.

![Diagram of contours](image)

Now, the imaginary part of $S_{C_T}$ is the quantity entering the WKB-exponent of a transition probability dominated by this solution. Since the solution is real on $C_M$, the contribution to the action from $C_M$ is purely real. So, the residue alone determines $\text{Im} S_{C_T}$.

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\[ \text{Im } S_{CT} = -\frac{24\pi^2}{g^2} \int_0^\infty dr \text{ Im } \sum_{nm} \text{Res} \left\{ \frac{\cos^4 w}{r^2} \left[ \frac{1}{2} q^2 + \frac{1}{2} \left( q^2 - 1 \right)^2 \right] \right\}_{\tau_{nm}(r)}. \] (2.12)

Consider now the topological charge \( Q \) (2.4) on the contour \( C_T + C_M \). Our solution (2.7) is an even function of \( \phi \), therefore the integral for \( Q \) on the Minkowski time axis vanishes. Thus, \( Q \) on \( C_T \) is determined by the residues at the singularity lines (2.9) as well:

\[ Q = 3i \int_0^\infty dr \sum_{nm} \text{Res} \left\{ \frac{\cos^4 w}{r^2} \frac{q}{(q^2 - 1)^2} \right\}_{\tau_{nm}(r)}. \]

Let us concentrate for simplicity on the case when our contour encloses only one singularity line, as illustrated in fig. 2. Using the Laurent expansion of the elliptic sine [27] at the pole at \( \phi_{1,-1} = \coth^{-1} q_{1,-1} \), we find for the action

\[ \text{Im } S_{CT} = -\frac{24\pi^2}{g^2} \int_0^\infty dr \times \]

\[ \text{Im } \text{Res} \left\{ \frac{\cos^4 w(r, \tau)}{r^2} \left[ \frac{1}{(\phi(\tau, r) - \phi_{1,-1})^4} - \frac{2}{3} \frac{1}{(\phi(\tau, r) - \phi_{1,-1})^2} + \cdots \right] \right\}_{\tau_{1,-1}(r)}, \] (2.13)

and for the topological charge

\[ Q = 3i \int_0^\infty dr \times \]

\[ \text{Res} \left\{ \frac{\cos^4 w(r, \tau)}{r^2} \left[ \frac{1}{(\phi(\tau, r) - \phi_{1,-1})^4} - \frac{2}{3} \frac{1}{(\phi(\tau, r) - \phi_{1,-1})^2} + \cdots \right] \right\}_{\tau_{1,-1}(r)}, \] (2.14)

where the ellipsis denotes terms regular as \( \phi \to \phi_{1,-1} \). Note that the singular terms of the Laurent expansion for \( \text{Im } S \) and \( Q \) are equal; the regular terms differ, however. Calculating the residue, we find:

\[ \int_0^\infty dr \text{ Res} \left\{ \frac{\cos^4 w(r, \tau)}{r^2} \left[ \frac{1}{(\phi(\tau, r) - \phi_{1,-1})^4} - \frac{2}{3} \frac{1}{(\phi(\tau, r) - \phi_{1,-1})^2} + \cdots \right] \right\}_{\tau_{1,-1}(r)}, \]

\[ = -\frac{5}{2} \int_0^\infty dr \left( q_{1,-1}^2 - 1 \right)^2 r^2 \sqrt{r^2 + 1 - q_{1,-1}^2} \]

\[ = -\frac{5}{2} \int_0^\infty \frac{y^2 dy}{(y^2 + 1)^{7/2}} = -\frac{i}{3}. \] (2.15)
Therefore, we have found that the imaginary part of the action and the topological charge are:

$$\text{Im} S = \frac{8\pi^2}{g^2}, \quad Q = 1.$$  (2.16)

Although the residue for a given $r$ depends on the number $n, m$ of the singularity line, the integral over $r$ does not. We have shown that these solutions have the remarkable property that the imaginary part of the action and the topological charge obey

$$\frac{g^2}{8\pi^2} \text{Im} S = Q = N, \quad (2.17)$$

where $N$ is the number of singularity lines between the complex time contour and the Minkowski time axis. Of course, this relation is identical to that obeyed by Euclidean multi-instanton configurations $[28, 29]$.

It should be stressed that the relation (2.17) is far from trivial on the complex time contour. The usual arguments for establishing the Bogomol’nyi bound do not seem to hold here, since the fields take complex values on the contour $[30]$. The turning point condition at $\tau = 0$ is crucial for (2.17) to hold. As was shown in $[24]$, the Minkowski time topological charge vanishes only for solutions with a turning point. The Lüscher-Schechter solutions without a turning point have fractional topological charge on the contour $C_T^F$.

Let us also note that the real part of the action of our solution on $C_T$ coincides, up to a minus sign, with the action on the Minkowski time contour $C_M$. This follows from the fact that the residue (2.15) is purely imaginary. As was shown in $[25]$, the action and the energy (See (4.19).) of the purely Minkowski solution are also finite.

### 3 Free Wave Asymptotics of the Gauge Field

In this section, we will find the free-wave asymptotics of the solution at the initial and final times, which determine the initial and final coherent states. We will explicitly demonstrate

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6The charge on the contour $C_T$ is in this case the sum of an (integer) residue and a (fractional $[24]$) Minkowski time contour contribution.
that the gauge field asymptotics at $t \to +\infty$ and $t \to -\infty$ belong to different topological sectors, confirming the calculation of the topological charge of the previous section.

The complex time contour $C_T$ provides a natural way of incorporating non-vacuum boundary conditions at the initial and final times. In the semiclassical approximation, the initial and final states are coherent states of the form:

$$|\{d(k)\}\rangle = \exp\left[\int d\mathbf{k} d(k) \hat{a}^\dagger(k) \right]|0\rangle \quad (3.1)$$

The creation operator is $\hat{a}^\dagger(k)$ and all color and polarization indices have been suppressed. The complex amplitudes $d(k)$ are determined by the free-field asymptotics of the solution at the ends of the contour $C_T$, in a manner to be discussed in the next section. In order to find them we need to know the Fourier transforms of the gauge potentials at the initial and final times.

Our strategy will be to start from the Minkowski part of the contour, at $t \to \infty$, and find the field asymptotics determining the final coherent state. Considering then the analytic properties of the solution (2.7) in the complex-$r$ plane, we will establish a simple relation between the Fourier transforms at the initial and final times, and will thus be able to determine the initial coherent state as well.

To begin, we note the formulae relating the solution of the one dimensional problem (2.3) and the four dimensional gauge potentials [25, 26]. In Minkowski space with metric $g_{\mu\nu} = (-, +, +, +)$, the gauge potentials are expressed through the solution $q(r, it)$ (2.7) as:

$$A_0^a (r, t) = \frac{4}{g} (q(r, it) - 1) \frac{tr_a}{(r^2 + 1 - t^2)^2 + 4t^2}, \quad (3.2)$$

$$A_l^a (r, t) = -\frac{4}{g} (q(r, it) - 1) \frac{1}{2} \frac{(1 + t^2 - r^2) \delta_{al} + \epsilon_{alm} r_m + r_a r_l}{(r^2 + 1 - t^2)^2 + 4t^2}. \quad (3.3)$$

Let us define a function

$$P(k, t) \equiv \frac{4}{g} \int d^3 r \frac{e^{ikr} (q(r, it) - 1)}{(r^2 + 1 - t^2)^2 + 4t^2} = \frac{8\pi}{ikg} \int_{-\infty}^{\infty} dr \frac{r e^{ikr} (q(r, it) - 1)}{(r^2 + 1 - t^2)^2 + 4t^2}. \quad (3.4)$$
Then, the Fourier transforms of the gauge potentials \((3.3)\) are easily expressed in terms of \(P(k,t)\) as:

\[
A_0^a(k,t) = -\frac{k^a}{k} t \frac{\partial P}{\partial k}, \quad (3.5)
\]

\[
A_i^a(k,t) = -\delta_{ai} \left( \frac{1}{2}(1+t^2)P + \frac{1}{2} \frac{\partial^2 P}{\partial k^2} \right) + i\epsilon_{aim} \frac{k_m}{k} \frac{\partial P}{\partial k} + \frac{k_i k_a}{k^2} \left( \frac{\partial^2 P}{\partial k^2} - \frac{1}{k} \frac{\partial P}{\partial k} \right). \quad (3.6)
\]

Recall that a pure gauge configuration with unit winding number, as explained in the previous section, is given by the potentials \((3.3)\) with \(q = -1\), the second extremum of the double-well potential \(V(q)\) \((2.6)\). For further use, let us denote the function \((3.4)\), corresponding to this configuration by \(\pi(k,t)\):

\[
\pi(k,t) = 4\pi \frac{i}{g} k t e^{ikt} - 4\pi \frac{i}{g} k t e^{-ikt}. \quad (3.7)
\]

In order to calculate the Fourier transforms of the gauge fields at large Minkowski time, we note that at large \(t\) the solution \((2.7)\) represents a thin shell of energy, expanding with the speed of light (see Fig. 3.).

For this configuration, the surface energy density decreases like \(1/r^2 \sim 1/t^2\) and we expect the nonlinear terms to become subdominant in the infinite time limit. Hence, as \(t \to \infty\), the solution reduces to a solution of the free equations of motion. From the Fourier series expansion of the elliptic sine \([27]\), we see that the only terms which solve the free equations are those proportional to \(1\) and \(\cos(2i\phi(r, it))\). (Note that free equations for \(A_\mu^a\) correspond to a harmonic approximation for \(q\) around one of the minima of the double well \((2.6)\).)

Therefore, in the limit of large time, the solution \((2.7)\) has the following representation:

\[
q(r, it) = 1 - \sqrt{\frac{\epsilon}{2}} \cos 2i\phi(r, it) = 1 - \sqrt{\frac{\epsilon}{2}} \left( \frac{r^2 + 1 - t^2}{r^2 + 1 - t^2} \right)^2 - 4t^2, \quad (3.8)
\]

The coefficient in front of the second term is fixed by the requirement that the energy at infinite time equals the exact energy of the classical solution. (See \((4.19)\) and fig. 3.) The corresponding function \(P_{\text{fin}}(k,t)\) is:

\[
P_{\text{fin}}(k,t) = \frac{\pi^2 i}{g(t - i)} e^{-ikt} - \frac{\pi^2 i}{g(t + i)} e^{ikt}. \quad (3.9)
\]
The Fourier transforms of the gauge potentials are then obtained in the form:

\[ A^a_0 (k, t) = -i \frac{\pi^2 \sqrt{2} \epsilon}{g} k^a k t e^{ikt} e^{-k} + \{ \text{h.c. and } k \to -k \}, \tag{3.10} \]

\[ A^a_l (k, t) = \frac{\pi^2 \sqrt{2} \epsilon}{g} \left[ \delta_{al} e^{ikt} e^{-k} + i \frac{\epsilon_{alm} k_m}{k} e^{ikt} e^{-k} \right. \]

\[ + \frac{k_a k_l}{k^2} \left( it - \frac{1}{k} \right) e^{ikt} e^{-k} + \{ \text{h.c. and } k \to -k \} \right]. \tag{3.11} \]

In order to see that they indeed obey the free equations of motion, it is convenient to represent them as a purely transverse part plus Abelian pure gauge:

\[ A^a_0 (k, t) = \frac{\partial}{\partial t} \omega^a (k, t), \tag{3.12} \]

\[ A^a_l (k, t) = \frac{\pi^2 \sqrt{2} \epsilon}{g} \left[ \left( \delta_{al} - \frac{k_a k_l}{k^2} \right) e^{ikt} e^{-k} + i \frac{\epsilon_{alm} k_m}{k} e^{ikt} e^{-k} \right. \]

\[ + \{ \text{h.c. and } k \to -k \} \right] - i k_l \omega^a (k, t) \]

\[ = \frac{1}{\sqrt{2k}} \sum_{i=1}^{2} \left[ e^i_l (k) g^a_{\ast i} (k) e^{ikt} + e^i_l (-k) g^a_i (-k) e^{-ikt} \right] - i k_l \omega^a (k, t), \tag{3.14} \]

with the Abelian gauge function

\[ \omega^a (k, t) = i \frac{\pi^2 \sqrt{2} \epsilon}{g} \frac{k^a}{k^2} \left( it - \frac{1}{k} \right) e^{ikt} e^{-k} + \{ \text{h.c. and } k \to -k \}. \tag{3.15} \]

Here \( e^i_l (k) \) are the two transverse polarization vectors, obeying \( e^i_l e^i_m = \delta_{lm} - k_l k_m / k^2 \), and

\[ g^a_i (k) = \frac{\pi^2 \sqrt{2} \epsilon}{g} \sqrt{2k} e^{-k} \left( e^i_a (k) - i \frac{\epsilon_{alm} k_m}{k} e^i_l (k) \right), \tag{3.16} \]

\[ g^a_{\ast i} (k) = \left[ g^a_i (k) \right]^\ast. \tag{3.17} \]

The calculation of the Fourier transforms of the fields at initial time, on the complex part of the contour \( C_T \), is less straightforward. Since at large early times the contour is trapped

\footnote{For a discussion of the asymptotic behaviour of classical Yang-Mills solutions in Minkowski space, see Lüscher \[31\]. The “radiation data” (3.12) are sufficient to determine the momentum distribution of the outgoing waves.}
between two singularity lines \((2.9)\), the approximation we used for the solution in Minkowski time cannot be justified. However, consideration of the analytic properties of the solution \((2.7)\) in the complex-\(r\) plane will allow us to relate the field asymptotics at the initial times to those at final times.

For the initial state, the formulae \((3.3)\), \((3.4)\), \((3.5)\) are applicable as well, up to the replacement \(t \to t + iT\). Hence, we need to calculate the function

\[
P_{\text{in}}(k, t) = \frac{8\pi}{ikg} \int_{-\infty}^{\infty} dr \frac{re^{ikr} (q(r, i(t + iT)) - 1)}{(r^2 + 1 - (t + iT)^2)^2 + 4(t + iT)^2}. \tag{3.18}
\]

The corresponding function for the final state is

\[
P_{\text{fin}}(k, t) = \frac{8\pi}{ikg} \int_{-\infty}^{\infty} dr \frac{re^{ikr} (q(r, it) - 1)}{(r^2 + t^2)^2 + 4t^2}. \tag{3.19}
\]

If no poles of the integrand in \((3.19)\) crossed the real-\(r\) axis when analytically continued to \(t + iT\), we would have \(P_{\text{in}}(k, t + iT) = P_{\text{fin}}(k, t + iT)\). The Fourier transform of the gauge field for the initial state would then be given by \((3.12)\), with the replacement \(t \to t + iT\), and the initial and final coherent states would be the same.

However, this is not the case for our contour \(C_T\). The solution \(q(r, it)\) \((2.7)\) has poles in the complex-\(r\) plane, the positions of which are given by the inversion of the equation of the singularity lines \((2.9)\). Let us define \(T \equiv 1 - \nu\), where, according to \((2.10)\)

\[
2\left(\frac{\sqrt{2\epsilon}}{8}\right)^3 < \nu < 2\frac{\sqrt{2\epsilon}}{8} \tag{3.20}
\]

In this case, it is easy to see that when continued from \(t\) to \(t + iT\), exactly two poles of the integrand in \((3.19)\) cross the real-\(r\) axis. The pole at

\[
r_0^-(t) = -t + i - 2i\alpha, \quad \alpha \equiv \frac{\sqrt{2\epsilon}}{8} - \frac{\epsilon}{16} \ln \epsilon \tag{3.21}
\]

crosses the real axis from above, and the one at

\[
r_0^+(t) = t - i + 2i\alpha \tag{3.22}
\]

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crosses from below, pushing thus the integration contour in (3.19) off the real axis. Hence, \( P_{\text{fin}}(k, t + iT) \) is given by

\[
P_{\text{fin}}(k, t + iT) = \frac{8\pi}{ikg} \int_{C_I} dr \left\{ \frac{r e^{ikr} (q(r, i(t + iT)) - 1)}{(r^2 + 1 - (t + iT)^2)^2 + 4(t + iT)^2} \right\},
\]

where the contour \( C_I \) is shown in fig. 4.

Cauchy’s theorem then gives:

\[
P_{\text{in}}(k, t + iT) = P_{\text{fin}}(k, t + iT) - 2\pi i \left\{ \text{Res}_{r_0^-(t + iT)} - \text{Res}_{r_0^+(t + iT)} \right\} \left\{ \frac{8\pi}{ikg} \frac{r e^{ikr} q(r, i(t + iT))}{(r^2 + 1 - (t + iT)^2)^2 + 4(t + iT)^2} \right\}.
\]

Calculating the residues is straightforward and yields:

\[
P_{\text{in}}(k, t + iT) = P_{\text{fin}}(k, t + iT) - \frac{4\pi^2 i}{gk(t + iT)} \left[ e^{ikr_0^-(t + iT)} - e^{ikr_0^+(t + iT)} \right].
\]

Substituting the equations for the poles (3.21), (3.22) we obtain:

\[
P_{\text{in}}(k, t + iT) = P_{\text{fin}}(k, t + iT) + \pi(k, t + iT) - \frac{8\pi^2 i e^{-ik(t + iT)}}{gk(t + iT)} e^{-k(1-\alpha)} \sinh k\alpha + \frac{8\pi^2 i e^{ik(t + iT)}}{gk(t + iT)} e^{-k\alpha} \sinh k(1 - \alpha).
\]

In this formula, the function \( \pi(k, t + iT) \) (3.7) corresponds to a topologically nontrivial
vacuum configuration with unit winding number. Its appearance provides an explicit confirmation of the fact that the initial and final states belong to different topological sectors.

We expect that this result is quite general. It is not difficult to show that for a complex time contour enclosing, say two singularity lines, a calculation similar to the previous one can be obtained. In this case, two poles of the integrand in (3.19) will cross the real- axis from above and two from below. The $\epsilon$-independent part of the residue at each pole gives a contribution to $P_{\text{in}}$ which corresponds to a topologically nontrivial vacuum configuration.

Now, after removing the $\pi(k, t + iT)$ piece by a large gauge transformation, and substituting our expression (3.9) for $P_{\text{fin}}(k, t + iT)$, we obtain for the negative frequency part of $P_{\text{in}}$:

$$P_{\text{in}}^{-}(k, t + iT) = \frac{i \pi^2}{g} e^{-ik(t+iT)} \left( \frac{\sqrt{2} \epsilon}{t - i + iT} - \frac{8}{t + iT} \frac{e^{k\alpha}}{k} \sinh k\alpha \right).$$

(3.27)

Expanding in $\epsilon$, we find

$$P_{\text{in}}^{-}(k, t + iT) = \frac{i \pi^2}{g} e^{-ik(t+iT)} \sqrt{2} \epsilon \left( \frac{1}{t - i + iT} - \frac{1}{t + iT} \right)$$

$$+ \frac{i \pi^2}{2g(t + iT)} \left( \epsilon \ln \epsilon - \frac{\epsilon k}{2} \right) e^{-ik(t+iT)} - k.$$  

(3.28)

Using (3.5), the negative frequency part of the gauge potentials can be represented, analogously to (3.12), as:

$$A_{a}^{-}(k, t + iT) = \frac{\partial}{\partial(t + iT)} \omega^{a-}(k, t + iT),$$

(3.29)

$$A_{l}^{-}(k, t + iT) = \frac{i \pi^2}{2g} \left( \epsilon \ln \epsilon - \frac{\epsilon k}{2} \right) \epsilon_{alm} \frac{k_{m}}{k} e^{-ik(t+iT)} e^{-k}$$

$$- \left( \delta_{al} - \frac{k_{a}k_{l}}{k^2} \right) \frac{\pi^2 \epsilon}{4g} (k - 1) e^{-ik(t+iT)} e^{-k} - i k_{l} \omega^{a-}(k, t + iT),$$

(3.30)

8More precisely, the residues appear with alternating signs, as a result of the fact that the conformally invariant Ansatz only distinguishes pure gauge configurations with unit difference of topological charge.
with the Abelian gauge function
\[
\omega^a(-k, t) = \frac{\pi^2 \epsilon \ln \epsilon}{2g} \frac{k^a}{k^2} (t - \frac{i}{k}) e^{-k e^{-ikt}} - \frac{\pi^2 \epsilon}{4g} \frac{k^a}{k} (t - i) e^{-ikt} e^{-k}.
\] (3.31)

The purely transverse negative frequency part of the gauge field at the initial time is therefore given by:
\[
A^a_{-t}(k, t + iT) = \frac{1}{\sqrt{2k}} \sum_{i=1}^{2} e^i(-k) f_i^a(-k) e^{-ikt},
\] (3.32)

with
\[
f_i^a(k) = \frac{i \pi^2}{\sqrt{2k}} \left( \frac{\epsilon k}{2} + \epsilon \ln \frac{1}{\epsilon} \right) \epsilon_{alm} \frac{k_m}{k} e^i(k) e^{k(T-1)} - \frac{\pi^2 \epsilon}{4g} \frac{\epsilon}{\sqrt{2k}} (k - 1) e^i(k) e^{k(T-1)}.
\]

The negative frequency components determine the initial coherent state. The calculation of the positive frequency part of the gauge potentials proceeds along the same lines, the result being:
\[
A^a_{0+}(k, t + iT) = \frac{\partial}{\partial(t + iT)} \omega^{a+}(k, t + iT),
\] (3.34)
\[
A^a_{i+}(k, t + iT) = \frac{1}{\sqrt{2k}} \sum_{i=1}^{2} e^i(k) \bar{f}^a_i(k) e^{ikt} - i k_l \omega^{a+}(k, t + iT),
\] (3.35)

with the Abelian gauge function
\[
\omega^{a+}(k, t) =
\]
\[
\frac{8\pi^2}{g} \frac{k^a}{k^2} \left[ (t + i\alpha) \frac{\sinh(1 - \alpha)}{k} - i \left( \frac{\sinh k(1 - \alpha)}{k} \right)' + i \frac{\sinh k(1 - \alpha)}{k^2} \right] e^{ikt - \alpha k},
\] (3.36)

and
\[
\bar{f}^a_i(k) = -\frac{8\pi^2}{g} e^a_i(k) \left[ \alpha \frac{\sinh k(1 - \alpha)}{k} - \left( \frac{\sinh k(1 - \alpha)}{k} \right)' \right] \sqrt{2k} e^{-kT - \alpha k} \]
\[
- i \frac{8\pi^2}{g} \epsilon_{alm} \frac{k_m}{k} e^i(k) \sinh k(1 - \alpha) \sqrt{2k} e^{-kT - \alpha k}.
\] (3.37)
This completes the calculation of the free field asymptotics of the solution. They are given by (3.12), (3.16) at $t \to \infty$ and (3.32), (3.33), (3.35), (3.37) at $\text{Re} t \to -\infty$. We saw explicitly that they belong to different topological sectors (3.26, 3.7). We also saw that they obey the free equations of motion and therefore determine the initial and final coherent states, as we will show in the next section.

4 Initial and Final States

In this section, the role of the gauge field configuration (2.7,3.5,3.6) in multiparticle scattering amplitudes will be explained. The gauge field configuration will be demonstrated to be the dominant contribution to an inclusive transition probability from a fixed initial state, in the saddle point approximation [19]. The initial state, and the most probable final state for transition from this initial state, will be characterized by the asymptotics found in the previous section.

The total transition probability from an initial coherent state, $| \{ a(k) \} \rangle$, projected onto fixed center-of-mass energy $E$, is:

$$
\sigma_E (\{ a(k) \}) = \sum_f | \langle f | \hat{S} P_Q P_E | \{ a(k) \} \rangle |^2.
$$

(4.1)

$P_E$ is a projection operator onto states of fixed center-of-mass energy, $E$. The probability is unity unless the initial state is projected also onto a subspace which does not commute with the Hamiltonian; a projection operator $P_Q$ onto states of fixed winding number $Q$ is implicit in our choice of a classical field with this property. Furthermore, the inclusive sum is over all final states built above a neighboring sector of the periodic vacuum.

This quantity is relevant to the study of multiparticle cross-sections for the following reason. When summed over all initial states,

$$
\sigma_E = \sum_a \sigma_E (\{ a(k) \})
$$

(4.2)
it gives the “microcanonical” transition probability in the one-instanton sector; the probabilities of transition from all states of energy \( E \) are equally weighted in this sum. When evaluated in the saddle point approximation, \( \sigma_E \) yields the maximal transition probability among all states with energy \( E \). It sets therefore an upper bound on the two-particle inclusive cross-section in the one-instanton sector [21].

The semiclassical approximation to (4.1) will be made clear by expressing it in an exponential form. The S-matrix in the interaction picture is

\[
\hat{S} = \lim_{t_i, t_f \to \pm \infty} e^{i\hat{H}_0 t_f} e^{-i\hat{H}(t_f - t_i)} e^{-i\hat{H}_0 t_i} .
\] (4.3)

Inserting a complete set of eigenstates of the gluon field operator \( A \) at initial and final times, we obtain:

\[
\sigma_E (\{a(k)\}) = \sum_f \left| \int dA_f dA_i \langle f | e^{i\hat{H}_0 t_f} | A_f \rangle \langle A_f | e^{-i\hat{H}(t_f - t_i)} P_Q | A_i \rangle \langle A_i | e^{-i\hat{H}_0 t_i} P_E | \{a(k)\} \rangle \right|^2 .
\] (4.4)

Each of the matrix elements in (4.4) may now be written in exponential form. The matrix element of the evolution operator between states of the field operator is the Feynman Path Integral:

\[
\langle A_f | e^{-i\hat{H}(t_f - t_i)} P_Q | A_i \rangle = \int_{A_i}^{A_f} [DA]_Q \exp [iS] ,
\] (4.5)

with boundary conditions \( A \to A_{i,f} \) as \( t \to t_{i,f} \), and the integral being taken over fields with topological charge \( Q \). The matrix element involving the initial state is the wavefunctional of the initial state. The projection onto states of fixed energy \( E \) may be expressed in an exponential form as follows:

\[
\langle A_i | e^{-i\hat{H}_0 t_i} P_E | \{a(k)\} \rangle = \int_{-\infty}^{+\infty} d\xi e^{-iE\xi} \langle A_i | e^{-i\hat{H}_0 t_i} e^{i\xi \hat{H}_0} | \{a(k)\} \rangle
\]

\[
= \int_{-\infty}^{+\infty} d\xi e^{-iE\xi} \langle A_i | e^{-i\hat{H}_0 t_i} | \{a(k)e^{i\xi k}\} \rangle
\]

\[
= \int_{-\infty}^{+\infty} d\xi e^{-iE\xi} \exp \left( B_i[a(k)e^{i\xi k}, A_f] \right) .
\]
The functional $B_i$ depends on the field asymptotics at early times

$$B_i[a(k), A_i] = -\frac{1}{2} \int dk \, a(k) \, a(-k) \, e^{-2ikt_i} - \frac{1}{2} \int dk \, k \, A_i(k) \, A_i(-k)$$

$$+ \int dk \, \sqrt{2k} \, a(k) \, A_i(k) \, e^{-ikt_i},$$

where color and polarization indices have been suppressed. $A_i(k)$ is the 3-dimensional Fourier transform of the field $A$, evaluated at initial time $t_i$.

The matrix element involving the final state may be put in a similar form by inserting the decomposition of unity in terms of coherent states

$$\sum_f \langle f | f \rangle = \int \mathcal{D}b^* \mathcal{D}b \, e^{-\int dk b^*(k)b(k)} = 1.$$  

Then, the transition probability $\sigma_E (\{a(k)\})$ becomes:

$$\sigma_E (\{a(k)\}) = \int \mathcal{D}b^* \mathcal{D}b \, e^{-\int dk b^*(k)b(k)} \times$$

$$\left[ \int dA_f(x) dA_i(x) \, \langle \{b^*(k)\} | e^{i\hat{H}_0 t_f} | A_f \rangle \, \langle A_f | e^{-i\hat{H}(t_f-t_i)} P_Q \, A_i \rangle \, \langle A_i | e^{-i\hat{H}_0 t_i} \, P_E | \{a(k)\} \rangle \right]^2.$$  

in terms of the wavefunctional of the final state, $\langle \{b^*(k)\} | A_f \rangle$. This will allow us to resolve the most probable final coherent state $\{b(k)\}$ from the inclusive sum.

The wavefunctional of the final coherent state is

$$\langle \{b^*(k)\} | e^{i\hat{H}_0 t_f} | A_f \rangle = \exp (B_f[b^*(k), A_f])$$

with a functional defined similarly to (4.7):

$$B_f[b^*(k), A_f] = -\frac{1}{2} \int dk \, b^*(k) \, b(\mathbf{k}) \, b^*(-\mathbf{k}) \, e^{2ikt_f} - \frac{1}{2} \int dk \, k \, A_f(k) \, A_f(-\mathbf{k})$$

$$+ \int dk \, \sqrt{2k} \, b^*(k) \, A_f(-\mathbf{k}) \, e^{ikt_f}.$$  

$A_f(k)$ is the 3-dimensional Fourier transform of the field $A$, evaluated at final time $t_f$.  

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The transition probability is now expressed as a path integral of an exponential by combining these factors:

\[ \sigma_E \{ \{ a \} \} = \int Db^* \mathcal{D}b \, d\xi d\xi' \mathcal{D}A \mathcal{D}A' \exp W, \quad (4.11) \]

\[ W = -\frac{1}{2} \int dk \, b^*(k) b(k) - i E \xi + B_i \{ a(k)e^{ik\xi}, A_i \} \]

\[ + B_f \{ b^*(k), A_f \} + iS(A) + \{ \text{h.c. and } \xi \to \xi', \, A \to A' \}. \quad (4.12) \]

This integral is dominated by its saddle point value if every term in the exponent is of order \(1/g^2\) as \(g \to 0\). The saddle point conditions have been derived in [19]:

1. Variation of \(W\) with respect to \(A\) and \(A'\) requires that the fields obey the Yang-Mills equations of motion on the complex time contour \(C_T\). The time contour is chosen so that the topological charge of the saddle point field configuration is \(Q\). The Lüscher-Schechter solution (2.7) with a turning point has these properties on the complex time contour \(C_T\) (fig. 2), as we showed in Section 2.

2. Variation with respect to \(b_k\) and \(b_k^\ast\) requires that \(A = A'\) everywhere in space-time. So, we need only consider a single solution to the equations of motion.

3. Variation with respect to the initial and final values of the fields \(A_i, A_f\) and \(A'_i, A'_f\) relates the saddle point values of \(b_k, b_k^\ast, a_k, a_k^\ast\) to the field asymptotics at \(|t| \to \infty\), found in the previous section. So, variation with respect to the field values at \(t \to +\infty\) yields the boundary condition, relating the field asymptotics at the final time to the complex amplitudes, determining the most probable final state:

\[ i\dot{A}_f(k) - k A_f(k) + \sqrt{2k} b^\ast(k) e^{ik\xi} = 0. \quad (4.13) \]

Here \(A_f(k)\) is the three-dimensional Fourier transform of the classical solution at the final time, given by equations (3.32, 3.35, 3.12). (Color and polarization indices are suppressed.)
Similarly, the condition, matching the complex amplitudes of the initial coherent state with the solution, is derived by varying $W$ with respect to the values of the fields at $\text{Re} \ t \to -\infty$:

$$-i\dot{A}_i(k) - k A_i(k) + \sqrt{2k} a(-k) e^{-ikt_i + ik\xi} = 0. \quad (4.14)$$

$A_i(k)$ is the three-dimensional Fourier transform of the classical solution at the initial time, given by (3.32, 3.35, 3.33, 3.37).

4. Variation with respect to $\xi$ and $\xi'$ gives a saddle point equation which determines the energy $E$ in terms of the asymptotics of the solution at the initial time.

Now, given the field asymptotics (3.32, 3.33, 3.12), we can find the initial coherent state $| \{ a(k) \} \rangle$ and the most probable final coherent state, which correspond to our solution. The saddle point conditions (4.13) at $t \to +\infty$ determine the most probable final coherent state in terms of the asymptotics of the solution (3.16) found in the previous section:

$$b^a_i(k) = g^a_i(k) = \frac{\pi^2 \sqrt{2\epsilon}}{g} \sqrt{2k} e^{-k} \left( e^a_i(k) - i \frac{\epsilon_{alm} k_m}{k} e^l_i(k) \right), \quad (4.15)$$

where the color $(a)$ and polarization $(i)$ indices have been restored. The final coherent state has then the form:

$$| \{ b(k) \} \rangle = \exp \left[ \int d\mathbf{k} \ b^a_i(k) \hat{a}^{*a}_i(k) \right] | 0 \rangle, \quad (4.16)$$

where $\hat{a}^{*a}_i(k)$ is a creation operator for a state with polarization $i$ ($i = 1, 2$), and color $a$ ($a = 1, 2, 3$) \footnote{The operators are normalized such that $[\hat{a}^a_i(p), \hat{a}^{*b}_j(k)] = \delta_{ij} \delta^{ab} \delta(p - k).$}. Then, the average number of particles with momentum $\mathbf{k}$ in the final state is

$$\bar{n}^\text{fin}_k = \sum_{i,a} b^a_i(k)^* b^a_i(k) = \frac{16 \epsilon \pi^4}{g^2} k e^{-2k}. \quad (4.17)$$

The total energy of the final coherent state is

$$E^\text{fin} = \int \frac{d\mathbf{k}}{(2\pi)^3} k \bar{n}^\text{fin}_k = \frac{6 \epsilon \pi^2}{g^2}. \quad (4.18)$$
As promised, the above expression for the energy exactly coincides with the energy of the classical solution. The latter is easiest calculated at \( t = \phi = 0 \) \([25]\):

\[
E_{\text{classical}} = \frac{12\pi}{g^2} \int_{-\pi/2}^{+\pi/2} dw \cos^2 w \epsilon = \frac{6 \epsilon \pi^2}{g^2}.
\] (4.19)

This correspondence is expected. For large \( t \), the solution represents a thin spherical shell of energy expanding with the speed of light and reduces to the superposition \([3.12]\) of plane waves. We find for the total average number of particles in the final state

\[
\bar{N}_{\text{fin}} = \int \frac{dk}{(2\pi)^3} n^\text{fin}_k = \frac{3 \epsilon \pi^2}{g^2}.
\] (4.20)

The saddle point conditions arising from the integration over the initial values of the fields \([4.14]\) determine the initial coherent state in terms of the asymptotics of the solution \([3.32, 3.33]\) \([17]\):

\[
a^\alpha_i(k) = f^\alpha_i(k) e^{-kt - ik\xi},
\] (4.21)

with \( f^\alpha_i(k) \) given by \([3.33]\).

Now, the real part of \( \xi \) can be removed by time translation. The imaginary part may be fixed by requiring the average energy of the initial state equal that of the final state \([17]\):

\[
\bar{E}_{\text{in}} = \int \frac{dk}{(2\pi)^3} k a^\alpha_i(k) a^\alpha_i(k) = \int \frac{dk}{(2\pi)^3} k f^\alpha_i(k) f^\alpha_i(k) e^{-2kt - 2k\text{Im}\xi} = \bar{E}_{\text{fin}} = \frac{6\epsilon \pi^2}{g^2}.
\] (4.22)

Substituting \([3.33]\) for \( f^\alpha_i \), we determine the value of \( \text{Im}\xi \):

\[
\text{Im}\xi = 1 - \left(\frac{45 \pi^3}{4} \epsilon^2\right)^{1/7}.
\]

We have omitted terms in this expression which are subdominant for \( \epsilon \ll 1 \). Now the average number of particles in the initial state is determined to be:

\[
\bar{N}_{\text{in}} = \int \frac{dk}{(2\pi)^3} f^\alpha_i(k) f^\alpha_i(k) e^{-2kt - 2k\text{Im}\xi} \sim \epsilon^{1/7} \bar{N}_{\text{fin}}.
\]
Our solution describes therefore a transition from a state with a smaller number of particles, $\bar{N}_{\text{in}}$, to a state with a larger number of particles, $\bar{N}_{\text{fin}}$, their ratio being controlled by the small parameter, $\epsilon^{1/7}$.

However, our solution does not maximize the microcanonical transition probability (4.2). It does not give the maximum transition probability at a given energy. The S-matrix element between our coherent states is an infinite sum of $n$-particle scattering amplitudes:

$$
\langle \{b\}|\hat{S}_Q|\{a\} \rangle \sim \sum_{n,m} \int \prod_{ij} d^3k_i \, d^3p_j \, c^*(k_1) \ldots c^*(k_n) \, d(p_1) \ldots d(p_m) \, \langle k_1, \ldots k_n | S_Q | p_1, \ldots p_m \rangle.
\tag{4.24}
$$

The above calculation does not allow the determination of any particular $n$-particle scattering amplitude entering the sum (4.24). It only gives an example of a semiclassically calculable multiparticle transition amplitude in the one-instanton sector.

## 5 Conclusions

In this paper, we investigated the role of a complex time solution in Yang-Mills theory in high-energy scattering processes. We argued that the complex time formalism is a natural one for describing the initial and final multiparticle states in different sectors of the periodic vacuum [18, 19]. The free-wave asymptotics of the solution at $|t| \to \infty$ define the initial and final coherent states, through the classical boundary value problem (eqs (4.14, 4.13)) discussed in the previous section.

In order to solve the boundary value problem however, we considered the case of a highly-symmetric solution of the Yang-Mills equations. The field equations were reduced to a quantum mechanical problem, by exploiting the $SO(4)$-conformal symmetry of the pure gauge theory [25, 26]. In particular, this simplification enabled us to analytically continue the Lüscher-Schechter solution to a complex time contour and provide some insight on the role of complex time singularities. The singularities were found to completely determine
the topological charge and the imaginary part of the action of the solution. Moreover, they turned out to obey a relation analogous to that obeyed by self-dual Euclidean solutions. This property is quite nontrivial on a complex time contour with complex valued fields where the usual Bogomol’nyi bound does not apply, and it makes the solution interesting for the problem of high energy fermion number or chirality violation.

We found that this solution, on a suitably chosen complex time contour, gives a saddle point contribution to a multiparticle scattering process, for which the average number of particles in the initial and final state are parametrically different. The ratio of the particle number in the initial and final coherent states is controlled by a small parameter of the solution. For reasons discussed in the Introduction, this solution may be relevant to the problem of initial state corrections [16, 18, 17]. The solution does not, however, maximize the transition probability in the one-instanton sector at a given energy. Thus, it can not be used to provide an upper bound on the $2 \to n$ process cross-section. It only gives an example of a semiclassically calculable multiparticle transition amplitude with fermion number or chirality violation.

The assumption of conformal symmetry may allow a straightforward extension of the ideas presented here to a few more complicated field equations, coupled to the Yang-Mills equations. Minkowski time solutions of the field equations for a scalar triplet [32] and fermion fields [33, 34] coupled to gauge fields have already appeared in the literature. It may be interesting to investigate the properties of these solutions on the complex time contour, with an eye towards incorporating the additional fields of the Standard Model in this formalism. In particular, it may be possible to understand the process of fermion number or chirality violation in the Dirac-Yang-Mills system on the complex time contour.

However, the high degree of symmetry assumed here clearly limits the scope of the results. The spherical symmetry $(SO(3)_{\text{rot}} \subset SO(4)_{\text{conf}})$ of the solution has led us astray from the problem of high-energy $2 \to n$ processes. The solution in Minkowski time has the form of a spherical shell of energy, which collapses from infinity, then, at $t = 0$, bounces back
and expands with the speed of light. Clearly such a classical field configuration is a poor approximation to an initial state of two highly energetic colliding particles. Physical intuition would lead one to believe that a solution with only a cylindrical symmetry might be a better candidate.

The assumption of conformal symmetry has also made less transparent an important application of this formalism: the electroweak theory. The mass scale $v \simeq 246$ GeV in the electroweak theory explicitly breaks the classical conformal invariance of the pure gauge theory. It is expected then that the arbitrary “scale size” $\rho$ of the classical solution, set to unity in the scale-invariant analysis above, will be fixed by the new mass scale. In the case that the center-of-mass energy $E$ greatly exceeds the symmetry breaking scale, the Yang-Mills theory considered here may correctly describe the classical behavior of the gauge sector of the electroweak theory. Then, our results have direct relevance to the behavior in this energy region \[19\].

Despite the explicit symmetry breaking parameter $v$ in the electroweak theory, a conformally-symmetric solution has played a major role in the Euclidean approach to the problem of multiparticle scattering. As reviewed in the Introduction, the BPST instanton forms the basis of a perturbative expansion of final state corrections to the leading semiclassical behavior of the multiparticle cross-section. In fact, the BPST instanton in this case represents only the “core” ($r \ll \rho \ll 1/M_w$) of the solution which dominates the cross-section for energies $E \ll E_0$, a complicated approximate solution to the electroweak gauge-Higgs field equations \[35\]. Corrections to the core behavior at large distances lead to higher order corrections in $E/E_0$ in the leading semiclassical behavior \[1, 1\] of the total cross-section \[3, 7\].

The complex time solution presented here may also be considered the “core” of a constrained solution in a spontaneously-broken gauge theory. At distances larger than $1/v$, the solution would have the exponentially-damped behavior characteristic of a massive gauge field. The complex time approach here differs however in at least one important respect. In the Euclidean approach, there are no exact finite-action solutions to the electroweak gauge-
Higgs field equations, as may be established by a simple scaling argument. In this case, the constrained expansion is a device to obtain the *approximate* solutions which provide the dominant semiclassical contribution to scattering amplitudes\[^{[3]}\]. In the present approach however, nothing prevents the existence of an *exact* solution to the Minkowski gauge-Higgs equations. Such a solution would represent an additional saddle point contribution to a transition amplitude. The effect of symmetry-breaking on the solution presented in this paper has yet to be explored.
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http://arxiv.org/ps/hep-ph/9305263v1
Figure 3: Energy density of the exact solution, in arbitrary units, for asymptotic values of time near the light cone. The solution describes a spherical shell, expanding at the speed of light.