Self-Affinity in the Gradient Percolation Problem

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We study the scaling properties of the solid-on-solid front of the infinite cluster in two-dimensional gradient percolation. We show that such an object is self affine with a Hurst exponent equal to $2/3$ up to a cutoff-length $\sim g^{-4/7}$, where $g$ is the gradient. Beyond this length scale, the front position has the character of uncorrelated noise. Importantly, the self-affine behavior is robust even after removing local jumps of the front. The previously observed multi affinity, due to the dominance of overhangs at small distances in the structure function. This is a crossover effect.

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Rough surfaces showing non-trivial scaling properties have been extensively studied theoretically, numerically and experimentally over the last couple of decades. Examples of such surfaces are those appearing during brittle fracture which were first characterized as being fractal but it was then realized that the concept of self affinity was more appropriate. The question of self affinity versus fractality has also been the focus of intense research on invasion fronts in porous media and on the dynamics of magnetic domain walls. It was recently reported that the displacement fronts in self-affine fractures are self affine. More recently, a possible explanation for the observed self affinity of fracture surfaces has been proposed and hinges on a clear understanding of the distinction between fractality and self affinity. It has also been suggested that brittle fracture surfaces are multi affine rather than simply self affine. Whether this is so remains an open question.

It is the aim of this Letter to study the question of fractality, self affinity and multi affinity of a front in a system which is simple enough to be tractable, namely that of the gradient percolation. There are already in the literature studies of this system in the present context. Furuberg et al. study the jumps in the position of the solid-on-solid (SOS) front of the infinite cluster, whereas Asikainen et al. conclude that this front is multi affine. We will in this Letter show that up to a given scale, the SOS front is self affine with a well-defined Hurst exponent, whereas on larger scales its position becomes uncorrelated. The self affinity is not caused by the jumps in the position of the front due to overhangs, but related to its fractal structure. The multi affinity seen by Asikainen et al. has its origin in the overhangs resulting from the definition of the SOS fronts and shows up in the structure function on small scales.

In gradient percolation, a spatial gradient in the occupation probability $p$ is introduced. A cartesian coordinate system $(i, j)$ is oriented with respect to the finite lattice of size $L_i \times L_j$ (assuming for the rest of this paper that the lattice is two dimensional), so that the $i$ axis runs perpendicular to the gradient (i.e. along the lower edge) and the $j$ axis along the gradient (i.e. the left edge). The gradient is introduced in the $j$ direction so that $p(j) = g_j$, where the gradient $g = 1/L_j$. However, the cluster connected to the lower edge will reach some average value, $j = j_p$, with an associated occupation probability $p_p = g_j$. The region around $j_p$ is critical and has a width $\xi$, spanning between $j_{p-} = j_p - \xi/2$, where $\xi$ is the correlation length associated with the critical region in the direction of the gradient. Defining $p_\pm = g_{j_{p-}}$ and setting $\xi = |p_{p-} - p_p|^{-\nu} = |g(j_{p-} - j_p)|^{-\nu}$, where $\nu$ is the correlation length exponent, Sapoval et al. found that $\xi \sim g^{-\nu/(1+\nu)} = g^{-4/7}$.
The infinite cluster has a fractal structure with an upper cutoff in length scale set by the width of the critical region, $\xi$. We now focus on the front of this infinite cluster and define precisely what we mean by this front $j(i)$ in the gradient percolation problem. Our starting point is the perimeter of the cluster of occupied sites that is attached to the $p = 1$ edge of the lattice. Since this perimeter contains overhangs and therefore is multivalued when interpreted as a function $j(i)$, we use the SOS method to extract a single-valued function for its position, see Fig. 1. For each $i$, we use either the $j$ value that is closest to the $p = 0$ edge (top side) or the $j$ value which is closest to the $p = 1$ side (bottom side) or the average over all the $j$ values attached to a given $i$ value (average front).

A trace $j(i)$ is statistically self affine if the probability density, $\pi(i,j)$, for it to have a value $j$ at $i$, given that $j = 0$ at $i = 0$, has the invariance

$$\lambda^\xi \pi(\lambda i, \lambda^\xi j) = \pi(i,j), \quad (1)$$

where $\xi$ is the Hurst exponent. This invariance must be caused by spatial correlations in $j$ along the $i$-axis. We note that a Lévy flight, which is an uncorrelated random walk whose step size $h$ is drawn from a power law distribution $N(h) \sim h^{-\beta-1}$, will satisfy Eq. (1) with an apparent Hurst exponent $\xi = 1/\beta$. However, in this example, satisfying Eq. (1) is due to the step size distribution and not to spatial correlations.

We have used the Average Wavelet Coefficient (AWC) method and 17 to analyse the structure of the SOS fronts. The AWC method consists of wavelet transforming $j(i)$, and averaging the wavelet coefficients $w(b,a)$ at each length scale $a$ over position $b$, $W(a) = \langle w(b,a) \rangle_b$. If $j(i)$ is self affine, the averaged wavelet coefficients will scale as

$$W(a) \sim a^{\xi+1/2}. \quad (2)$$

The data collapse of the averaged wavelet coefficients for the bottom side front based on lattice size $L_j = 64$ to 8192, while $L_i = 2048$. We have that $g = 1/L_j$. The straight line has a slope of $\xi + 1/2$, see Eq. 2.

We show in Fig. 2 the averaged wavelet coefficients based on the Daubechies-4 wavelets for the bottom-side fronts. The plots for the top and average fronts are comparable. The data are based on averages over 2201 samples for $L_j$ in the range 64 to 2048 and 200 samples for $L_i = 4096$ and 8192. $L_i$ was set to 2048 for all the different $L_j$. The gradient $g$ was set to $1/L_j$. There is a clear crossover between two regimes in these plots. At smaller length scales, one does indeed find the behavior of Eq. 2 indicating self affinity. On larger scales, the slope of the log-log plots are zero indicating $\xi = -1/2$, which corresponds to uncorrelated or white noise 10. Furthermore, we observe excellent data collapse when $W$ is scaled by $g^{-\beta}$ and the length scale, $a$, is scaled by $g^{-\alpha}$. We will show below that

$$\xi = 2 - D_e = \frac{2}{3}, \quad (3)$$

$$\alpha = \frac{\nu}{1+\nu} = \frac{4}{7}, \quad (4)$$

and

$$\beta = \frac{3}{2} \alpha = \frac{6}{7}. \quad (5)$$

where $D_e = 4/3$ is the fractal dimension of the external perimeter of the front 21.

The main goal of this Letter is to derive Eq. 3 and thus demonstrate that $\xi$ is a proper Hurst exponent and $j(i)$ a self-affine function. To this end, we need to demonstrate two things: First, $j(i)$ satisfies the scaling relation and, second, that this is not due to a power law tail in the step size distribution. We note that since the average wavelet coefficients obey Eq. 2, $j(i)$ automatically satisfies Eq. 1. Therefore, we now need only to identify the mechanism behind this scaling.
In order to derive Eq. (3), we start by noting that
the distribution of distances \( m \) between crossing points be-
tween a planar fractal curve with dimension \( D_e \) — e.g.,
the percolation perimeter — and a straight line follows
the power law \( \pi(m) \sim m^{-\zeta} \). Introducing a gradi-
ent in the \( j \) direction and placing the straight line in the
critical region interval, \([j_-, j_+]\), and parallel to the \( i \) axis,
the distribution of crossing point distances \( m \) remains
the same. A self-affine curve characterized by a Hurst exponent
\( \zeta \), leads to a distribution of crossing point distances
given by \( \pi(m) \sim m^{-(2-\zeta)} \). By comparing this ex-
pression to \( \pi(m) \sim m^{-D_e} \), Eq. (3) immediately follows.
However, we still need to show that \( j(i) \) is indeed self
affine, in other words the scaling relation Eq. (1) is not
caused by jumps.

First, we turn to deriving Eqs. (4) and (5). The cor-
relation length in the direction of the gradient, the \( j \) di-
rection, is \( \zeta \sim g^{-\nu/((1+\nu))} \). Since the perimeter is locally
isotropic, this is also the correlation length in the \( i \) direction.
The crossover length scale from self affinity to un-
 correlated noise is the correlation length \( \xi \). Hence, re-
scaling \( \alpha \to \alpha/\xi \sim \alpha g^{-\nu/((1+\nu))} \) gives data collapse along this
axis which demonstrates Eq. (4). Likewise, the crossover
length scale in the \( j \) direction is \( \xi \). This implies that the
normalized wavelet coefficient at this scale,

\[
W(\xi) = \frac{\sum_{j} w(j \xi - \nu) f(j \xi)}{\sum_{j} f(j \xi)}
\]

where \( j \) is the scale index. We have in particular that \( j_1(i) = j(i) \). It was shown in [11], that \( h \) is distributed according to

\[
N(h, g) = h^{-D_e-1} f(h g^\alpha),
\]

where \( D_e = 4/3 \) and \( f(z) \) approaches a constant as \( z \ll 1 \)
and falls off faster than any power law as \( z \to \infty \). The
step size distribution comes from the appearance of over-
hangs in the perimeter. An overhang is defined as the
jump made by the front from one position along the \( i \) axis
to the next due to a backwards turn [11, 13, 14]. In order to
confirm that the overhangs do not generate the Hurst exponent \( \zeta = 2/3 \), we analyse the filtered front \( j_0(i) \), de-

The scaling along the \( i \) axis is unchanged as no change
in the system has been made in that direction. However,
since all step sizes have been reset to unity in the trans-
formation \( j(i) \to j_0(i) \), the rescaling in the \( j \) direction is
no longer controlled by \( j_c \). In order to regain data col-
lapse for different \( g = 1/L_j \), we need to rescale the lattice
units in this direction by the Hurst exponent, \( \zeta = 2/3 \).
The straight line matching the small-\( n \) region of the fig-
ure has a slope \( 2/3 + 1/2 \), while the straight line matching
the large-\( n \) portion has a slope of \( 1/2 + 1/2 \) correspond-
ing to an uncorrelated random walk. This shows that,
for small scales, the \( \zeta = 2/3 \) is indeed a Hurst exponent.
On the other hand, for longer length scales, we expect
random walk behavior since white noise gives precisely
the exponent \( 1/2 \) in the transformation \( j(i) \to j_0(i) \).

In order to analyse the multi affinity that has been
reported in this problem [12], we construct the struc-
ture function \( C_k(n, g) = \langle |j(m+n) - j(m)|^k \rangle \). Multi
affinity occurs when \( C_k(n, g)^{1/k} \) does not scale with a
single \( k \)-independent exponent with respect to \( n \). Using
the overhang distribution [11], we find \( C_k(1, g) \sim g^{\alpha} \),
where \( s(k) = \min[0, \alpha(D_e - k)] = \min[0, (16/21 - 4k/7)] \). The
self-affine character of \( j(i) \) cannot be visible in the
structure function for \( n = 1 \) but will appear only gradu-
ally as \( n \) is increased. We may therefore analyse the
structure function based solely on the Lévy character
induced by the overhangs in the small-\( n \) limit. We will
call this the Lévy regime, whereas for larger \( n \) where the
self affinity dominates, we will refer to as the self-affine
regime. The scaling with respect to \( g \) for \( C_k(1, g) \) persists
for \( n > 1 \) in the Lévy regime since \( j(i+n) - j(i) \) follows a
Lévy distribution whose power law tail does not change
with increasing \( n \). Hence, we expect \( C_k(n, g) \sim g^{\alpha(k)} \)
in this regime. In order to derive its dependence on \( n \) in
the Lévy regime, we note that the distribution of dis-
tances \( l \) between overhangs follows the same power law
as the overhangs themselves. This can be seen as follows.
When there is a gradient present in the \( j \) direction, the
length of the perimeter scales as \( L_i^{D_e} \), when the gradi-
ent is kept fixed. Making a cut through the perimeter
with a straight line parallel to the \( i \) axis, the crossing
points of the perimeter with the line form a fractal set
with dimension \( D_e - 1 \). Hence, there are, in a given in-
terval \( l \), \( N_i \sim 1^{D_e - 1} \) overhangs [21]. These
overhangs give rise to an effective Hurst exponent \( 1/D_e \)
for the fractal set, see e.g. in the width of the trace,
\( \Delta_j \sim N_i^{1/D_e} \sim l(D_e - 1)/D_e \). Since the overhangs form a
fractal set, we will need \( N_b \sim l^{(D_e - 1)/D_e} \) boxes of size \( l \) to
cover it. Due to the averaging over position \( i \), there will
be yet another factor \( l \), see [22]. We may now assemble
these pieces to form the scaling of the structure function
in the Lévy regime, \( C_l(l, g) \sim N_i^{k} N_b \sim l^{k \zeta L} \) where

\[
\zeta_L = \left[ \frac{1 - \frac{1}{D_e}}{\frac{2 - D_e}{k}} \right] = \frac{1}{4} + \frac{2}{3k}.
\]

Therefore, in the Lévy regime, i.e. for small \( n \), there is
multi affinity. A similar analysis in the self-affine regime,
\( n \) yields

\[
\zeta_k^{SA} = \zeta = 2/3.
\]

Therefore there is no multi affinity in this regime. The
n for which there is the crossover between the Lévy and the self-affine regime will depend on k and is governed by prefactors that the scaling analysis presented here cannot access. For n beyond ξ, the front decorrelates and the structure function becomes independent of n. We show in Fig. 4 the k = 1, 2 and 3 structure functions. Their behavior is in accordance with our predictions. However, note that for k = 2, ζ L = 7/12 = 0.58 which is close to ζ = 2/3. Furthermore, the self-affine regime is close to the decorrelated flat regime. Hence, it is hard to distinguish between the Lévy and the self-affine regime for this value of k. As k increases, the Lévy regime grows, as the overhangs are emphasized for larger k.

To conclude, we have shown that the structure of the interface in a gradient percolation problem combines fractal and self-affine properties. The perimeter that includes numerous overhangs has the classical fractal structure. However, Solid-on-Solid fronts that are extracted from the perimeter, have a clear self-affine property up to a crossover length scale ξ even if local jumps, inherited from overhangs, are removed. On larger scales it shows an uncorrelated noise behavior. The structure function is, however, sensitive to the overhangs on smaller scales and this implies a multi-affine scaling behavior in this regime. Implications of our results for physical interpretations of analogical and numerical experiments are important.

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