CRYSTAL BASES AND CANONICAL BASES FOR QUANTUM BORCHERDS-BOZEC ALGEBRAS

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Abstract. Let $U^-_{q}(g)$ be the negative half of a quantum Borcherds-Bozec algebra $U_{q}(g)$ and $V(\lambda)$ be the irreducible highest weight module with $\lambda \in P^{+}$. In this paper, we investigate the structures, properties and their close connections between crystal bases and canonical bases of $U^-_{q}(g)$ and $V(\lambda)$. We first re-construct crystal basis theory with modified Kashiwara operators. While going through Kashiwara’s grand-loop argument, we prove several important lemmas, which play crucial roles in the later developments of the paper. Next, based on the theory of canonical bases on quantum Borcherds-Bozec algebras, we introduce the notion of primitive canonical bases and prove that primitive canonical bases coincide with lower global bases.

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1. Introduction

1.1. Background. In representation theory, it is always an important task to construct explicit bases of algebraic objects because those bases provide a deep insight in studying the various features and properties of these algebraic objects. The quantum groups, as a new class of non-commutative, non-cocommutative Hopf algebras, were discovered independently by Drinfeld and Jimbo in their study of quantum Yang-Baxter equation and 2-dimensional solvable lattice model [4, 10]. For the past forty years, the quantum groups have attracted a lot of research activities due to their close connection with representation theory, combinatorics, knot theory, mathematical physics, etc. Among others, Lusztig’s canonical basis theory and Kashiwara’s crystal basis theory are regarded as one of the most prominent achievements in the representation theory of quantum groups [17, 18, 14, 15]. The canonical basis theory was developed in a geometric way, while the crystal basis theory was constructed using algebraic methods.

From geometric point of view, Lusztig’s canonical basis theory is closely related to the theory of perverse sheaves on the representation variety of quivers without loops. In [2, 3], Bozec extended Lusztig’s theory to the study of perverse sheaves for the quivers with loops, thereby introduced the notion of quantum Borcherds-Bozec algebras. From algebraic point of view, the quantum Borcherds-Bozec algebras can be regarded as a huge generalization of quantum groups and quantum Borcherds algebras [4, 10, 11].

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The theory of canonical bases, crystal bases and global bases for quantum Borcherds-Bozec algebras have been developed and investigated in [2, 3, 6]. For the case of quantum groups associated with symmetric Cartan matrices, Grojnowski and Lusztig discovered that the canonical bases coincide with global bases [7]. Moreover, for the case of quantum Borcherds algebras associated with symmetric Borcherds-Cartan matrices without isotropic simple roots, Kang and Schiffmann showed that the canonical bases coincide with the global bases [13].

The aim of this paper is to investigate the deep connections between most significant bases for quantum Borcherds-Bozec algebras: canonical bases and crystal/global bases. We will show that the canonical bases coincide with global bases. Moreover, we expect there are much more to be explored in the theory of quantum Borcherds-Bozec algebras from various points of view.

1.2. New crystal basis theory. Let $U_q^-(g)$ be the negative half of a quantum Borcherds-Bozec algebra $U_q(g)$ associated with a Borcherds-Cartan datum $(A, P, P', \Pi, \Pi')$ and let $V(\lambda)$ be the irreducible highest weight module with $\lambda \in P^+$. For our purpose, we re-construct the crystal basis theory for $V(\lambda)$ and $U_q^-(g)$. More precisely, we first define a new class of Kashiwara operators on $V(\lambda)$ and $U_q^-(g)$ which is a modified version of the ones given in [3]. The main difference from Bozec’s definition is the case of $i \in I^{iso}$, where we define the Kashiwara operators as follows (Definition 3.1, Definition 3.7):

$$\tilde{e}_i u = \sum_{c \in C_i} c b_i, c \setminus \{i\} u_c,$$

$$\tilde{f}_i u = \sum_{c \in C_i} \frac{1}{c_i + 1} b_i, c \cup \{i\} u_c.$$

We use these new Kashiwara operators to define the pairs $(L(\lambda), B(\lambda))$ and $(L(\infty), B(\infty))$ for $V(\lambda)$ and $U_q^-(g)$, respectively. Then we prove that all the interlocking, inductive statements in Kashiwara’s grand-loop argument are true, thereby proving the existence and uniqueness of these crystal bases:

**Theorem A** (Theorem 3.5, Theorem 3.10).

1. The pair $(L(\lambda), B(\lambda))$ is a crystal basis of $V(\lambda)$.

2. The pair $(L(\infty), B(\infty))$ is a crystal basis of $U_q^-(g)$.

We further use these new crystal bases to construct global bases for $V(\lambda)$ and $U_q^-(g)$ and then verify that the global basis theory developed in [6] remains true with an appropriate modification.

1.3. Canonical bases and global bases. In order to study the connection between canonical bases and global bases, we define the notion of **primitive canonical bases**. Recall that in [6], we gave an alternative presentation of $U_q(g)$ in terms of primitive generators which arise naturally from Bozec’s algebra isomorphism $\phi: U_q^-(g) \rightarrow U_q(g)$ [2, 3] (See Proposition 2.3 in this paper). The primitive canonical bases are defined as the image of canonical bases under the isomorphism $\phi$.

In Proposition 6.14, we recall Bozec’s geometric results on canonical basis $B$ and in Corollary 6.15, we rewrite them in an algebraic way. Thus in Corollary 6.16, we obtain an interpretation of Bozec’s results on the primitive canonical basis $B_Q$. Using some critical properties of Lusztig’s bilinear form $( , )_L$ and Kashiwara’s bilinear form $( , )_K$, we prove the following Propositions which play an important role in the later development.

**Proposition B** (Proposition 7.5). For all $x, y \in U_q^-(g)$, we have

$$(x, y)_L = (x, y)_K \mod q A_0.$$

**Proposition C** (Proposition 7.6). For all $x, y \in U_q^-(g)$, we have

$$(\phi(x), \phi(y))_L = (x, y)_L.$$
1.4. Organization. This paper is organized as follows.

In the first part, we focus on the reconstruction of crystal basis theory for quantum Borcherds-Bozec algebras. More precisely, in Section 2, we recall the original definition of quantum Borcherds-Bozec algebras and their alternative presentation in terms of primitive generators. In Section 3, we define a new class of Kashiwara operators and construct the crystal bases \((L(\lambda), B(\lambda))\) for \(V(\lambda)\) and \((L(\infty), B(\infty))\) for \(U_q^-(g)\). We also review some of the basic theory of abstract crystals and give a simplified description of tensor product rule for quantum Borcherds-Bozec algebras. In Section 4, with the new class of Kashiwara operators, we go through all the interlocking, inductive statements in Kashiwara’s grand-loop argument and show that all of them are still true in our much more general setting. Hence we prove the existence and uniqueness of the crystal bases \((L(\lambda), B(\lambda))\) and \((L(\infty), B(\infty))\). As by-products, we obtain several important lemmas which will be used in later parts of this work in a critical way (for example, Lemma 4.23). In Section 5, we study the lower global bases \(B(\lambda)\) and \(B(\infty)\) following the outline given in [6].

The second part of this paper is devoted to the study of relations between canonical bases and global bases. More precisely, in Section 6, we recall the geometric construction of canonical basis \(\mathcal{B}\) and define the notion of primitive canonical basis \(\mathcal{B}_Q\). We then give a very brief review of some homological formulas, which leads to defining geometric bilinear form \((\cdot, \cdot)_G\) on perverse sheaves [19]. The geometric results proved by Bozec [2, 3] are expressed in algebraic language and then translated to the corresponding statements for primitive canonical bases. We close this section with several important key lemmas on global bases which are necessary to apply Grojnowski-Lusztig’s argument.

In Section 7, we first identify the geometric bilinear form and Lusztig’s bilinear form using the fact that both of them are Hopf pairings. We then show that Lusztig’s bilinear form and Kashiwara’s bilinear form are equivalent to each other up to mod \(q\mathfrak{A}_0\). Using the key lemmas proved in Section 6, we can apply Grojnowski-Lusztig’s argument to conclude the primitive canonical basis \(\mathcal{B}_Q\) coincides with the lower global basis \(\mathcal{B}(\infty)\). It follows immediately that the primitive canonical basis \(\mathcal{B}_Q^\lambda\) is identical to the lower global basis \(\mathcal{B}(\lambda)\).

Acknowledgements. Z. Fan was partially supported by the NSF of China grant 12271120, the NSF of Heilongjiang Province grant JQ2020A001, and the Fundamental Research Funds for the central universities. S.-J. Kang was supported by China grant YZ2260010601. Young Rock Kim was supported by the National Research Foundation of Korea grant 2021R1A2C1011467 and Hankuk University of Foreign Studies Research Fund.

2. Quantum Borcherds-Bozec Algebras

Let \(I\) be an index set which can be countably infinite. An integer-valued matrix \(A = (a_{ij})_{i,j \in I}\) is called an even symmetrizable Borcherds-Cartan matrix if it satisfies the following conditions:

(i) \(a_{ii} = 2, 0, -2, -4, ...\),
(ii) \(a_{ij} \leq 0\) for \(i \neq j\),
(iii) there exists a diagonal matrix \(D = \text{diag}(s_i \in \mathbb{Z}_{>0} \mid i \in I)\) such that \(DA\) is symmetric.

Set \(I^\text{re} = \{i \in I \mid a_{ii} = 2\}\), \(I^\text{im} = \{i \in I \mid a_{ii} \leq 0\}\) and \(I^\text{iso} = \{i \in I \mid a_{ii} = 0\}\).

A Borcherds-Cartan datum consists of:

(a) an even symmetrizable Borcherds-Cartan matrix \(A = (a_{ij})_{i,j \in I}\),
(b) a free abelian group \(P\), the weight lattice,
(c) \(P^\vee := \text{Hom}(P, \mathbb{Z})\), the dual weight lattice,
(d) \(\Pi = \{\alpha_i \in P \mid i \in I\}\), the set of simple roots,
(e) \(\Pi^\vee = \{h_i \in P^\vee \mid i \in I\}\), the set of simple coroots

satisfying the following conditions:

(i) \(\langle h_i, a_j \rangle = a_{ij}\) for all \(i, j \in I\),
(ii) \(\Pi\) is linearly independent over \(\mathbb{Q}\).
for each \( i \in I \), there exists an element \( \Lambda_i \in P \) such that
\[ \langle h_j, \Lambda_i \rangle = \delta_{ij} \quad \text{for all } i, j \in I. \]
The elements \( \Lambda_i \) \((i \in I)\) are called the fundamental weights.

Given an even symmetrizable Borcherds-Cartan matrix, it can be shown that such a Borcherds-Cartan datum always exists, which is not necessarily unique.

We denote by
\[ P^+ := \{ \lambda \in P \mid \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I \}, \]
the set of dominant integral weights. The free abelian group \( R := \bigoplus_{i \in I} \mathbb{Z} \alpha_i \) is called the root lattice. Set \( R_+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \) and \( R_- := -R_+ \). Let \( \mathfrak{h} := Q \otimes \mathbb{Z} P^\vee \) be the Cartan subalgebra.

Since \( A \) is symmetrizable and \( \Pi \) is linearly independent over \( Q \), there exists a non-degenerate symmetric bilinear form \( (\ , \ ) \) on \( \mathfrak{h}^* \) satisfying
\[ (\alpha_i, \lambda) = s_i \langle h_i, \lambda \rangle \quad \text{for all } \lambda \in \mathfrak{h}^*. \]

For each \( i \in I^\circ \), we define the simple reflection \( r_i \in GL(\mathfrak{h}^*) \) by
\[ r_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i \quad \text{for } \lambda \in \mathfrak{h}^*. \]
The subgroup \( W \) of \( GL(\mathfrak{h}^*) \) generated by the simple reflections \( r_i \) \((i \in I^\circ)\) is called the Weyl group of the Borcherds-Cartan datum given above. It is easy to check that \((\ , \ )\) is \( W \)-invariant.

Let \( q \) be an indeterminate. For \( i \in I \) and \( n \in \mathbb{Z}_{>0} \), we define
\[ q_i = q^{\alpha_i}, \quad q_{(i)} = q^{\frac{n \alpha_i}{q_i}}, \quad [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]! = [n][n - 1] \cdots [1]. \]

Set \( I^\infty := I^\circ \cup (I^{im} \times \mathbb{Z}_{>0}) \) and let \( \mathcal{F} = \mathbb{Q}(q) \langle f_{il} \mid (i, l) \in I^\infty \rangle \) be the free associative algebra generated by the formal symbols \( f_{il} \) with \( (i, l) \in I^\infty \). By setting \( \deg f_{il} = -l \alpha_i \), then \( \mathcal{F} \) becomes a \( R_- \)-graded algebra. For a homogeneous element \( x \in \mathcal{F} \), we denote by \( |x| \) the degree of \( x \) and for a subset \( A \subset R_- \), we define
\[ \mathcal{F}_A = \{ x \in \mathcal{F} \mid |x| \in A \}. \]

Following [20], we define a twisted multiplication on \( \mathcal{F} \otimes \mathcal{F} \) by
\[ (x_1 \otimes x_2)(y_1 \otimes y_2) = q^{-(|x_2|, |y_1|)} x_1 y_1 \otimes x_2 y_2 \]
for all homogeneous elements \( x_1, x_2, y_1, y_2 \in \mathcal{F} \).

We also define a \( \mathbb{Q}(q) \)-algebra homomorphism \( \delta : \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F} \) given by
\[ \delta(f_{il}) = \sum_{m+n-l} q_{(i)}^{-mn} f_{im} \otimes f_{in} \quad \text{for } (i, l) \in I^\infty, \]
where we understand \( f_{0l} = 1 \) and \( f_{il} = 0 \) for \( l < 0 \). Then \( \mathcal{F} \) becomes a \( \mathbb{Q}(q) \)-bialgebra.

**Proposition 2.1.** [19, 2, 3] Let \( \nu = (\nu_{il})_{(i, l) \in I^\infty} \) be a family of non-zero elements in \( \mathbb{Q}(q) \). Then there exists a symmetric bilinear form \( (\ , \ ) : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{Q}(q) \) such that
\begin{itemize}
\item[(a)] \( (1, 1)_L = 1 \),
\item[(b)] \( (f_{il}, f_{kl})_L = \nu_{il} \) for \( (i, l) \in I^\infty \),
\item[(c)] \( (x, y)_L = 0 \) if \( |x| \neq |y| \),
\item[(d)] \( (x, yz)_L = (\delta(x), y \otimes z)_L \) for all \( x, y, z \in \mathcal{F} \).
\end{itemize}

Let \( \mathcal{R} \) be the radical of \( (\ , \ )_L \) on \( \mathcal{F} \). Assume that
\[ \nu_{il} \equiv 1 \mod q \mathbb{Z}_{\geq 0}[q] \quad \text{for all } i \in I^{im} \setminus I^{iso} \text{ and } l > 0. \]
Then it was shown in [2, 3] that the radical $\mathcal{R}$ is generated by the elements

\begin{equation}
\sum_{r+s=1-l_{a_{ij}}} (-1)^r f_i^{(r)} f_j f_i^{(s)} \text{ for } i \in \mathbb{I}^e, \; i \neq (j, l) \in \mathbb{I}^\infty, \tag{2.3}
\end{equation}

$f_{il} f_{jk} - f_{jk} f_{il}$ for all $(i, l), (j, k) \in \mathbb{I}^\infty$ and $a_{ij} = 0$.

where $f_i^{(n)} = f^n_i/[n]!$ for $i \in \mathbb{I}^e$.

Given a Borcherds-Cartan datum $(A, P, P^\vee, \Pi, \Pi^\vee)$, we define $\hat{U}$ to be the associative algebra over $\mathbb{Q}(q)$ with 1, generated by the elements $q^h (h \in P^\vee)$ and $e_{il}, f_{il} ((i, l) \in \mathbb{I}^\infty)$ with defining relations

\begin{align*}
q^0 &= 1, \quad q^h q^{h'} = q^{h+h'} \text{ for } h, h' \in P^\vee, \\
q^h e_{jl} q^{-h} &= q^{l(h, \alpha_j)} e_{jl}, \quad q^h f_{jl} q^{-h} = q^{-l(h, \alpha_j)} f_{jl} \text{ for } h \in P^\vee, \; (j, l) \in \mathbb{I}^\infty, \\
\sum_{r+s=1-l_{a_{ij}}} (-1)^r e_i^{(r)} e_j e_i^{(s)} &= 0 \text{ for } i \in \mathbb{I}^e, \; (j, l) \in \mathbb{I}^\infty \text{ and } i \neq (j, l), \\
\sum_{r+s=1-l_{a_{ij}}} (-1)^r e_i^{(r)} f_j f_i^{(s)} &= 0 \text{ for } i \in \mathbb{I}^e, \; (j, l) \in \mathbb{I}^\infty \text{ and } i \neq (j, l), \\
e_{ik} e_{jl} - e_{jl} e_{ik} &= f_{ik} f_{jl} - f_{jl} f_{ik} = 0 \text{ for } a_{ij} = 0.
\end{align*}

We extend the grading on $\hat{U}$ by setting $|q^h| = 0$ and $|e_{il}| = \lambda \alpha_i$.

The algebra $\hat{U}$ is endowed with a comultiplication $\Delta: \hat{U} \to \hat{U} \otimes \hat{U}$ given by

\begin{align*}
\Delta(q^h) &= q^h \otimes q^h, \\
\Delta(e_{il}) &= \sum_{m+n=l} q_{ij}^{mn} e_{im} \otimes K^m_i e_{in}, \\
\Delta(f_{il}) &= \sum_{m+n=l} q_{ij}^{-mn} f_{im} K^n_i \otimes f_{in},
\end{align*}

where $K_i = q_i^h = q_i^{s_i h_i}$ ($i \in I$).

Let $\hat{U}^+$ (resp. $\hat{U}^-$) be the subalgebra of $\hat{U}$ generated by $e_{il}$ (resp. $f_{il}$) for all $(i, l) \in \mathbb{I}^\infty$. In particular, $\hat{U}^- \cong \mathcal{R}/\mathcal{R}$.

We denote by $\hat{U}^{\leq 0}$ be the subalgebra of $\hat{U}$ generated by $q^h (h \in P^\vee)$ and $f_{il} ((i, l) \in \mathbb{I}^\infty)$. We extend $(\; , \; )_L$ to a symmetric bilinear form $(\; , \; )_L$ on $\hat{U}^{\leq 0}$ by setting

\begin{align*}
(q^h, 1)_L &= 1, \quad (q^h, f_{il})_L = 0, \\
(q^h, K_j)_L &= q^{-(h, \alpha_j)}.
\end{align*}

Moreover, we define $(\; , \; )_L$ on $\hat{U}^+$ by

\begin{equation}
(x, y)_L = (\omega(x), \omega(y))_L \text{ for } x, y \in \hat{U}^+, \tag{2.7}
\end{equation}

where $\omega: \hat{U} \to \hat{U}$ is the involution defined by

$\omega(q^h) = q^{-h}, \; \omega(e_{il}) = f_{il}, \; \omega(f_{il}) = e_{il} \text{ for } h \in P^\vee, \; (i, l) \in \mathbb{I}^\infty$.

For any $x \in \hat{U}$, we will use the Sweedler notation to write

$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$.

Following the Drinfeld double construction, the quantum Borcherds-Bozec algebra is defined as follows.
Definition 2.2. The quantum Borcherds-Bozec algebra $U_q(\mathfrak{g})$ associated with a Borcherds-Cartan datum $(A, P, P^\vee, \Pi, \Pi^\vee)$ is the quotient algebra of $\hat{U}$ defined by relations
\begin{equation}
(2.8) \quad \sum (a_{(1)}, b_{(2)}) L \omega(b_{(1)}) a_{(2)} = \sum (a_{(2)}, b_{(1)}) L \omega(b_{(2)}) \quad \text{for all } a, b \in \hat{U}^{\leq 0}.
\end{equation}

Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by $e_{il}$ (resp. $f_{il}$) for $(i, l) \in I^\infty$ and let $U_q^0(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $q^h$ ($h \in P^\vee$). Then we have the triangular decomposition [12]
\begin{equation}
U_q(\mathfrak{g}) \cong U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}).
\end{equation}

For simplicity, we often write $U$ (resp. $U^+$ and $U^-$) for $U_q(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$).

Let $\tau : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ be the $\mathbb{Q}$-linear involution given by
\begin{equation}
(2.9) \quad \tau e_{il} = e_{il}, \quad \tau f_{il} = f_{il}, \quad \tau K_i = K_i^{-1}, \quad \tau = q^{-1}
\end{equation}
for $(i, l) \in I^\infty$ and $i \in I$.

The following proposition will play an extremely important role in our work.

Proposition 2.3. [2, 3] For each $i \in I^m$ and $l > 0$, there exist unique elements $a_{il}, b_{il} = \omega(a_{il})$ satisfying the following conditions.
(a) $Q(q)(e_{i1}, e_{i2}, \ldots, e_{il}) = Q(q)(a_{i1}, a_{i2}, \ldots, a_{il}),$
(b) $(a_{il}, u)_{L} = 0$ for all $u \in Q(q)(e_{ik} | k < l),$
(c) $a_{il} - e_{il} \in Q(q)(e_{ik} | k < l),$
(d) $\overline{a_{il}} = a_{il}, \overline{b_{il}} = b_{il},$

Let $\tau_{il} = (a_{il}, a_{il})_{L} = (b_{il}, b_{il})_{L}$. In [6], we obtain a new presentation of the quantum Borcherds-Bozec algebra $U_q(\mathfrak{g})$ in terms of primitive generators $q^h$ ($h \in P^\vee$), $a_{il}, b_{il}$ ($i, l \in I^\infty$).

Theorem 2.4. [6, Theorem 2.5] The quantum Borcherds-Bozec algebra $U_q(\mathfrak{g})$ is equal to the associative algebra over $Q(q)$ with 1 generated by $q^h$ ($h \in P^\vee$), $a_{il}, b_{il}$ ($i, l \in I^\infty$) with the defining relations
\begin{equation}
q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P^\vee,
q^h a_{ij} q^{-h} = q^{l(h, \alpha_j)} a_{ij}, \quad q^h b_{ij} q^{-h} = q^{-l(h, \alpha_j)} b_{ij} \quad \text{for } h \in P^\vee \text{ and } (j, l) \in I^\infty,
\end{equation}
\begin{equation}
(2.10) \quad \sum_{r+s=1-l_{ij}} (-1)^{r} a_{ij}^{(r)} a_{ij}^{(s)} = 0 \quad \text{for } i \in I^e, (j, l) \in I^\infty \text{ and } i \neq (j, l),
\end{equation}
\begin{equation}
\sum_{r+s=1-l_{ij}} (-1)^{r} b_{ij}^{(r)} b_{ij}^{(s)} = 0 \quad \text{for } i \in I^e, (j, l) \in I^\infty \text{ and } i \neq (j, l),
\end{equation}
\begin{equation}
a_{il} b_{jk} - b_{kj} a_{il} = \delta_{ij} \delta_{kl} \tau_{il} (K_i^{l} - K_i^{-l}),
a_{il} a_{jk} - a_{jk} a_{il} = b_{il} b_{jk} - b_{jk} b_{il} = 0 \quad \text{for } a_{ij} = 0.
\end{equation}
Note that $U^+ = \langle a_{il} \mid (i, l) \in I^\infty \rangle$ and $U^- = \langle b_{il} \mid (i, l) \in I^\infty \rangle$. 
The algebra $U_q(\mathfrak{g})$ has a comultiplication induced by (2.1) and Proposition 2.3.

$$\Delta(q^h) = q^h \otimes q^h,$$
(2.11)

$$\Delta(a_{il}) = a_{il} \otimes K_{i}^{-l} + 1 \otimes a_{il},$$

$$\Delta(b_{il}) = b_{il} \otimes 1 + K_{i}^{l} \otimes b_{il}.$$ Moreover, we define the counit and antipode by

$$\epsilon(q^h) = 1, \quad \epsilon(a_{il}) = \epsilon(b_{il}) = 0,$$
(2.12)

$$S(a_{il}) = -a_{il}K_{i}^{l}, \quad S(b_{il}) = -K_{i}^{-l}b_{il},$$

then the quantum Borcherds-Bozec algebra $U_q(\mathfrak{g})$ becomes a Hopf algebra.

From now on, we will take

$$\tau_{il} = (1 - q_{il}^2)^{-1} \quad \text{for} \ (i, l) \in I^\infty.$$

Set $A_{il} := -q_{il}^2a_{il}$ and $E_{il} := -K_{i}^{l}a_{il}$. Then we have

$$A_{ij}b_{jk}b_{il} = \delta_{ij}\delta_{kl}K_{i}^{l} - K_{i}^{-l},$$
(2.13)

$$E_{il}b_{jk} - q_{il}^{-kl}a_{ij}b_{jk}E_{il} = \delta_{ij}\delta_{kl}\frac{1 - K_{i}^{2l}}{1 - q_{il}^2}.$$}

We now briefly review some of the basic properties of the category $\mathcal{O}_{\text{int}}$. Let $U_q(\mathfrak{g})$ be a quantum Borcherds-Bozec algebra and let $M$ be a $U_q(\mathfrak{g})$-module. We say that $M$ has a weight space decomposition if

$$M = \bigoplus_{\mu \in P} M_{\mu}, \quad \text{where} \ M_{\mu} = \{m \in M \mid q^h m = q^{(h, \mu)} m \text{ for all } h \in P^{\vee}\}.$$

We denote $\text{wt}(M) := \{\mu \in \mathfrak{h}^* \mid M_{\mu} \neq 0\}$.

A $U_q(\mathfrak{g})$-module $V$ is called a highest weight module with highest weight $\lambda$ if there is a non-zero vector $v_\lambda$ in $V$ such that

(i) $q^h v_\lambda = q^{(h, \lambda)} v_\lambda$ for all $h \in P^{\vee}$,
(ii) $e_{il} v_\lambda = 0$ for all $(i, l) \in I^\infty$,
(iii) $V = U_q(\mathfrak{g})v_\lambda$.

Such a vector $v_\lambda$ is called a highest weight vector with highest weight $\lambda$. Note that $V_\lambda = \mathcal{Q}(q)v_\lambda$ and $V$ has a weight space decomposition $V = \bigoplus_{\mu \leq \lambda} V_{\mu}$, where $\mu \leq \lambda$ means $\lambda - \mu \in R_+$. For each $\lambda \in P$, there exists a unique irreducible highest weight module, which is denoted by $V(\lambda)$.

**Proposition 2.5.** [12] Let $\lambda \in P^+$ be a dominant integral weight and let $V(\lambda) = U_q(\mathfrak{g})v_\lambda$ be the irreducible highest weight module with highest weight $\lambda$ and highest weight vector $v_\lambda$. Then the following statements hold.

(a) If $i \in I^e$, then $b_i^{(h_i, \lambda)+1} v_\lambda = 0$.
(b) If $i \in I^m$ and $(h_i, \lambda) = 0$, then $b_{il} v_\lambda = 0$ for all $l > 0$.

Moreover, if $i \in I^m$ and $\mu \in \text{wt}(V(\lambda))$, we have

(i) $(h_i, \mu) \geq 0$,
(ii) if $(h_i, \mu) = 0$, then $V(\lambda)_{\mu - la_i} = 0$ for all $l > 0$,
(iii) if $(h_i, \mu) = 0$, then $f_{il}(V(\lambda)_{\mu}) = 0$,
(iv) if $(h_i, \mu) \leq -la_{ii}$, then $e_{il}(V(\lambda)_{\mu}) = 0$. 
Motivated by Proposition 2.5, we define the category $\mathcal{O}_{\text{int}}$ as follows.

**Definition 2.6.** The category $\mathcal{O}_{\text{int}}$ consists of $U_q(\mathfrak{g})$-modules $M$ such that

(a) $M$ has a weight space decomposition $M = \oplus_{\mu \in P} M_\mu$ with $\dim M_\mu < \infty$ for all $\mu \in \text{wt}(M)$,

(b) there exist finitely many weights $\lambda_1, \ldots, \lambda_k \in P$ such that

$$\text{wt}(M) \subset \bigcup_{j=1}^k (\lambda_j - R_+),$$

(c) if $i \in I^{re}$, $b_i$ is locally nilpotent on $M$,

(d) if $i \in I^{im}$, we have $\langle h_i, \mu \rangle \geq 0$ for all $\mu \in \text{wt}(M)$,

(e) if $i \in I^{im}$ and $\langle h_i, \mu \rangle = 0$, then $b_i(M_\mu) = 0$,

(f) if $i \in I^{im}$ and $\langle h_i, \mu \rangle \leq -la_{ii}$, then $a_i(M_\mu) = 0$.

**Remark 2.7.**

(i) By (b), $a_i$ is locally nilpotent on $M$.

(ii) If $i \in I^{im}$, then $b_i$ are not necessarily locally nilpotent.

(iii) The irreducible highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ with $\lambda \in P^+$ is an object of the category $\mathcal{O}_{\text{int}}$.

(iv) A submodule or a quotient module of a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{\text{int}}$ is again an object of $\mathcal{O}_{\text{int}}$.

(v) A finite number of direct sums or a finite number of tensor products of $U_q(\mathfrak{g})$-modules in the category $\mathcal{O}_{\text{int}}$ is again an object of $\mathcal{O}_{\text{int}}$.

The fundamental properties of the category $\mathcal{O}_{\text{int}}$ are given below.

**Proposition 2.8.**

(a) If a highest weight module $V = U_q(\mathfrak{g})v_\lambda$ satisfies the conditions (a) and (b) in Proposition 2.5, then $V \cong V(\lambda)$ with $\lambda \in P^+$.

(b) The category $\mathcal{O}_{\text{int}}$ is semisimple.

(c) Every simple object in the category $\mathcal{O}_{\text{int}}$ has the form $V(\lambda)$ for some $\lambda \in P^+$.

3. Crystal bases

Let $\mathbf{c} = (c_1, \ldots, c_r) \in \mathbb{Z}_{\geq 0}^r$ be a sequence of non-negative integers. We define $|\mathbf{c}| := c_1 + \cdots + c_r$. We say that $\mathbf{c}$ is a **composition** of $l$, denoted by $\mathbf{c} \vdash l$, if $|\mathbf{c}| = l$. If $c_1 \geq c_2 \geq \ldots \geq c_r$, we say that $\mathbf{c}$ is a **partition** of $l$. For each $i \in I^{im} \setminus I^{iso}$ (resp. $i \in I^{iso}$), we denote by $\mathcal{C}_{i,l}$ the set of compositions (resp. partitions) of $l$ and set $\mathcal{C}_l = \bigsqcup_{l \geq 0} \mathcal{C}_{i,l}$. For $i \in I^{re}$, we define $\mathcal{C}_{i,l} = \{\emptyset\}$.

For $\mathbf{c} = (c_1, \ldots, c_r)$, we define

$$a_{i,c} = a_{ic_1}a_{ic_2} \cdots a_{ic_r}, \quad b_{i,c} = b_{ic_1}b_{ic_2} \cdots b_{ic_r}.$$  

Note that $\{a_{i,c} \mid \mathbf{c} \vdash l\}$ (resp. $\{b_{i,c} \mid \mathbf{c} \vdash l\}$) forms a basis of $U_q(\mathfrak{g})_{la_1}$ (resp. $U_q(\mathfrak{g})_{-la_1}$).

3.1. Crystal bases for $V(\lambda)$.

Let $M = \oplus_{\mu \in P} M_\mu$ be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{\text{int}}$ and let $u \in M_\mu$ for $\mu \in \text{wt}(M)$.

For $i \in I^{re}$, by [15], the vector $u$ can be written uniquely as

$$u = \sum_{k \geq 0} b^{(k)}_i u_k$$  

such that
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(i) \( a_i u_k = 0 \) for all \( k \geq 0 \),
(ii) \( u_k \in M_{\mu+k\alpha_i} \),
(iii) \( u_k = 0 \) if \( \langle h_i, \mu + k\alpha_i \rangle = 0 \).

For \( i \in I_{im} \), by \([2, 3]\), the vector \( u \) can be written uniquely as

(iii)
\[
\sum_{c \in C} b_{i,c} u_c
\]

such that
(i) \( a_{ik} u_c = 0 \) for all \( k > 0 \),
(ii) \( u_c \in M_{\mu + |c|\alpha_i} \),
(iii) \( u_c = 0 \) if \( \langle h_i, \mu + |c|\alpha_i \rangle = 0 \).

The expressions (3.1) and (3.2) are called the \( i \)-string decomposition of \( u \). Note that (i) is equivalent to saying that \( A_{ik} u_c = E_{ik} u_c = 0 \) for all \( k > 0 \).

Given the \( i \)-string decompositions (3.1) and (3.2), we define the Kashiwara operators on \( M \) as follows.

**Definition 3.1.**

(a) For \( i \in I^{re} \), we define

\[
\tilde{e}_i u = \sum_{k \geq 1} b_{i,1}^{(k-1)} u_{k-1},
\]

\[
\tilde{f}_i u = \sum_{k \geq 0} b_{i,1}^{(k+1)} u_{k+1}.
\]

(b) For \( i \in I_{im} \setminus I^{iso} \) and \( l > 0 \), we define

\[
\tilde{e}_i(l) u = \sum_{c \in C_{i;1} = l} b_{i,c-1} u_c,
\]

\[
\tilde{f}_i(l) u = \sum_{c \in C_{i}} b_{i,l} u_c.
\]

(c) For \( i \in I^{iso} \) and \( l > 0 \), we define

\[
\tilde{e}_i(l) u = \sum_{c \in C_{i}} c_l \, b_{i,c} u_c,
\]

\[
\tilde{f}_i(l) u = \sum_{c \in C_{i}} \frac{1}{c_l+1} b_{i,c} u_c,
\]

where \( c_l \) denotes the number of \( l \) in \( c \).

It is easy to see that \( \tilde{e}_i(l) \circ \tilde{f}_i(l) = \text{id}_{M_{\mu}} \) for \( (i, l) \in I_{\infty} \) and \( \langle h_i, \mu \rangle > 0 \).

Let \( A_0 = \{ f \in Q(q) \mid f \text{ is regular at } q = 0 \} \). Then we have an isomorphism

\[
A_0/qA_0 \cong Q, \quad f + qA_0 \mapsto f(0).
\]

**Definition 3.2.**

Let \( M \) be a \( U_q(\mathfrak{g}) \)-module in the category \( O_{\text{int}} \) and let \( L \) be a free \( A_0 \)-submodule of \( M \). The submodule \( L \) is called a crystal lattice of \( M \) if the following conditions hold.

(a) \( Q \otimes_{A_0} L \cong M \),
(b) \( L = \oplus_{\mu \in P} L_\mu \), where \( L_\mu = L \cap M_\mu \).

(c) \( \tilde{e}_i L \subset L \), \( \tilde{f}_i L \subset L \) for \((i, l) \in I^\infty \).

Since the operators \( \tilde{e}_i, \tilde{f}_i \) preserve \( L \), they induce the operators

\[
\tilde{e}_i, \tilde{f}_i : L/qL \to L/qL.
\]

**Definition 3.3.**

Let \( M \) be a \( U_q(\mathfrak{g}) \)-module in the category \( \mathcal{O}_{\text{int}} \). A crystal basis of \( M \) is a pair \((L, B)\) such that

(a) \( L \) is a crystal lattice of \( M \),

(b) \( B \) is a \( \mathbb{Q} \)-basis of \( L/qL \),

(c) \( B = \sqcup_{\mu \in P} B_\mu \), where \( B_\mu = B \cap (L/qL)_\mu \),

(d) \( \tilde{e}_i B \subset B \cup \{0\} \), \( \tilde{f}_i B \subset B \cup \{0\} \) for \((i, l) \in I^\infty \),

(e) for any \( b, b' \in B \) and \((i, l) \in I^\infty \), we have \( \tilde{f}_i b = b' \) if and only if \( b = \tilde{e}_i b' \).

**Lemma 3.4.** Let \( M \) be a \( U_q(\mathfrak{g}) \)-module in the category \( \mathcal{O}_{\text{int}} \) and \((L, B)\) be a crystal basis of \( M \). For any \( u \in M_\mu \), we have

\[
\tilde{e}_i u \equiv E_i u \mod qL \text{ for } (i, l) \in I^\infty.
\]

**Proof.** Let \( u = b_{i, c} u_0 \) such that \( E_i k u_0 = 0 \) for any \( k > 0 \). Let \( m := \langle h_i, \text{wt}(u_0) \rangle \).

(a) Suppose \( i \not\in I^{\text{iso}} \) and let \( c = (c_1, \ldots, c_r) \in C_{i,l} \).

(i) If \( c_1 = l \), by (2.14), we have

\[
E_i(u) = E_i(b_{i, c} u_0) = E_i b_{i,l} (b_{i,c'} u_0) = (q_i^{l^2 a_{ii,l}} E_{il} + \frac{1 - K_i^{2l}}{1 - q_i^{2l}}) b_{i,c'} u_0 \\
\equiv b_{i,c'} u_0 \equiv \tilde{e}_i u \mod qL.
\]

(ii) If \( c_1 = k \neq l \), we have

\[
E_i(u) = E_i(b_{i, c} u_0) = E_i b_{i,k} (b_{i,c'} u_0) = q_i^{kl a_{ii,k}} b_{i,k} E_{il}(b_{i,c'} u_0) \equiv 0 \equiv \tilde{e}_i u \mod qL.
\]

(b) If \( i \in I^{\text{iso}} \), we have

\[
\langle h_i, \text{wt}(u_0) - \alpha \rangle = m \text{ for any } \alpha \in R_+,
\]

\[
(3.6)
\]

\[
E_i b_{i,l} - b_{i,l} E_{il} = \frac{1 - K_i^{2l}}{1 - q_i^{2l}},
\]

\[
E_i b_{i,k} - b_{i,k} E_{il} = 0 \text{ if } k \neq l.
\]

(iii) By induction on (3.6), one can prove:

\[
E_i(b_{i,l}^k u_0) = k \frac{1 - q_i^{2l m}}{1 - q_i^{2l}} b_{i,l}^{k-1} u_0 \equiv k b_{i,l}^{k-1} u_0 \equiv \tilde{e}_i (b_{i,l}^k u_0) \mod qL.
\]

(iv) We may write

\[
u = b_{i,c} u_0 = b_{i, c_{i1}}^{a_{i1}} b_{i, c_{i2}}^{a_{i2}} \cdots b_{i, c_{ir}}^{a_{ir}} u_0,
\]

where \( c_1 > c_2 > \cdots > l > \cdots > c_r \). Then we have

\[
E_i u = b_{i, c_{i1}}^{a_{i1}} \cdots E_i (b_{i,l}^k) \cdots b_{i, c_{ir}}^{a_{ir}} u_0.
\]
Let \( u' = b_{il}^a \cdots b_{ic}^a u_0 \). By the same argument as that in (iii), we can show that
\[
E_{il}(b_{il}^k u') \equiv k b_{il}^{k-1} u' \mod qL.
\]
Hence we have
\[
E_{il}(u) = c_i b_{i,e\setminus\{l\}} u_0 \equiv \tilde{e}_{il}(u) \mod qL.
\]

Let \( V(\lambda) = U_q(\mathfrak{g})v_{\lambda} \) be the irreducible highest weight \( U_q(\mathfrak{g}) \)-module with highest weight \( \lambda \in P^+ \). Let \( L(\lambda) \) be the free \( A_0 \)-submodule of \( V(\lambda) \) spanned by \( \tilde{f}_{i_1 l_1} \cdots \tilde{f}_{i_r l_r} v_{\lambda} \) \((r \geq 0, (i_k, l_k) \in I^\infty)\) and let
\[
B(\lambda) := \{ \tilde{f}_{i_1 l_1} \cdots \tilde{f}_{i_r l_r} v_{\lambda} + qL(\lambda) \} \setminus \{0\}.
\]

**Theorem 3.5.** The pair \((L(\lambda), B(\lambda))\) is a crystal basis of \( V(\lambda) \).

We will prove this theorem in Section 4.

**Example 3.6.** Let \( I = I^\text{im} = \{i\} \) and
\[
U = \mathbb{Q}(q) \langle a_{il}, b_{il}, K_i^{\pm l} \mid l > 0 \rangle = \mathbb{Q}(q) \langle E_{il}, b_{il}, K_i^{\pm l} \mid l > 0 \rangle.
\]
Let \( V = \bigoplus_{c \in C_i} \mathbb{Q}(q) b_{i,c} u_0 \) such that
\[
V = U u_0, \quad \langle h_i, \text{wt}(u_0) \rangle = m, \quad K_i^{\pm l} u_0 = q_i^{\pm lm} u_0, \quad E_{ik} u_0 = 0 \text{ for any } k > 0,
\]
and \( L = \bigoplus_{c \in C_i} A_0(b_{i,c} u_0) \).

If \( i \in I^\text{im} \setminus I^\text{iso} \), for \( c \in C_i \), let \( B_{i,c} = \{ b_{i,c} u_0 \} \) and \( B = \bigsqcup_{c \in C_i} B_{i,c} \). Define
\[
\tilde{e}_{il}(b_{i,c} u_0) = \begin{cases} b_{i,c \setminus \{l\}} u_0, & \text{if } c_1 = l, \\ 0, & \text{otherwise}, \end{cases}
\]
\[
\tilde{f}_{il}(b_{i,c} u_0) = b_{i,(l,c)} u_0.
\]

If \( i \in I^\text{iso} \), for \( c \in C_i \), let \( B_{i,c} = \{ \frac{1}{c_1!} b_{i,c} u_0 \} \) and set \( B = \bigsqcup_{c \in C_i} B_{i,c} \). Define
\[
\tilde{e}_{il} \left( \frac{1}{c_1!} b_{i,c} u_0 \right) = \frac{1}{(c_l - 1)!} b_{i,c \setminus \{l\}} u_0, \\
\tilde{f}_{il} \left( \frac{1}{c_1!} b_{i,c} u_0 \right) = \frac{1}{(c_l + 1)!} b_{i,c \setminus \{l\}} u_0.
\]

We can verify that the pair \((L, B)\) is a crystal basis of \( V \).

### 3.2. Crystal bases for \( U_q^-(\mathfrak{g}) \).

Now we will discuss the crystal basis for \( U_q^-(\mathfrak{g}) \).

Let \((i, l) \in I^\infty \) and \( S, T, W \in U_q^-\mathfrak{g} \) such that
\[
a_{il} S - S a_{il} = \frac{K_i^l T - K_i^{-l} W}{1 - q_i^l}.
\]
Equivalently, there are uniquely determined elements \( T, W \in U_q^-\mathfrak{g} \) such that
\[
A_{il} S - S A_{il} = \frac{K_i^l T - K_i^{-l} W}{q_i^l - q_i^{-l}}.
\]

We define the operators \( e'_{il}, e''_{il} : U_q^-\mathfrak{g} \rightarrow U_q^-\mathfrak{g} \) by
\[
e'_{il}(S) = W, \quad e''_{il}(S) = T.
\]
By \( (3.7) \) and \( (3.8) \), we have

\[
A_{il} S - S A_{il} = \frac{K_i^l(e''_{il}(S)) - K_i^{-l}(e'_{il}(S))}{q_i^l - q_i^{-l}}.
\]

Therefore we obtain

\[
\begin{align*}
\epsilon'_{il} b_{jk} &= \delta_{ij} \delta_{kl} + q_i^{-k q_{il}} b_{jk} \epsilon_{il}', \\
\epsilon''_{il} b_{jk} &= \delta_{ij} \delta_{kl} + q_i^{k q_{il}} b_{jk} \epsilon_{il}'', \\
\epsilon'_{il} \epsilon''_{jk} &= q_i^{k q_{il} e_{jk} \epsilon_{il}'},
\end{align*}
\]

Let \(* : U_q(g) \rightarrow U_q(g)\) be the \(Q(q)\)-linear anti-involution given by

\[
(q^h)^* = q^{-h}, \quad a^*_l = a_{il}, \quad b^*_l = b_{il}.
\]

By \( (2.9) \) and Proposition 2.3, we have \(** = id, - = -id\) and \(* - = -*\).

By \( (3.9) \), we have

\[
S^* A_{il} - A_{il} S^* = \frac{(e''_{il}(S))^* K_i^{-l} - (e'_{il}(S))^* K_i^l}{q_i^l - q_i^{-l}}.
\]

Therefore we obtain

\[
\begin{align*}
\epsilon'_{il}(S^*) &= K_i^l(e''_{il}(S))^* K_i^{-l}, \\
\epsilon''_{il}(S^*) &= K_i^{-l}(e'_{il}(S))^* K_i^l.
\end{align*}
\]

Let \( u \in U_q^{-}(g)_{-\alpha} \) with \( \alpha \in R_+ \). For \( i \in I^{re} \), by [15], the vector \( u \) can be written uniquely as

\[
u = \sum_{k \geq 0} b_{il}^{(k)} u_k
\]
such that

(i) \( \epsilon'_{il} u_k = 0 \) for all \( k \geq 0 \),
(ii) \( u_k \in U_q^{-}(g)_{-\alpha + k \alpha_i} \),
(iii) \( u_k = 0 \) if \( \langle h_i, -\alpha + k \alpha_i \rangle = 0 \).

For \( i \in I^{im} \), by [2, 3], the vector \( u \) can be written uniquely as

\[
u = \sum_{c \in C_i} b_{i,c} u_c
\]
such that

(i) \( \epsilon'_{ik} u_c = 0 \) for all \( k > 0 \),
(ii) \( u_c \in U_q^{-}(g)_{-\alpha + |c| \alpha_i} \),
(iii) \( u_c = 0 \) if \( \langle h_i, -\alpha + |c| \alpha_i \rangle = 0 \).

The expressions \((3.14)\) and \((3.15)\) are called the \(i\)-string decomposition of \( u \).

Given the \(i\)-string decompositions \((3.14)\) and \((3.15)\), we define the Kashiwara operators on \( U^-_q(g) \) as follows.

**Definition 3.7.**
(a) For $i \in I^\text{re}$, we define
\begin{equation}
\tilde{e}_i u = \sum_{k \geq 1} b_i^{(k-1)} u_k,
\end{equation}
\begin{equation}
\tilde{f}_i u = \sum_{k \geq 0} b_i^{(k+1)} u_k.
\end{equation}

(b) For $i \in I^\text{im} \setminus I^\text{iso}$ and $l > 0$, we define
\begin{equation}
\tilde{e}_{il} u = \sum_{c \in C_i : c_1 = l} c_i b_{i; c \setminus c_1} u_c,
\end{equation}
\begin{equation}
\tilde{f}_{il} u = \sum_{c \in C_i} b_{i; (l,c) \cup c} u_c.
\end{equation}

(c) For $i \in I^\text{iso}$ and $l > 0$, we define
\begin{equation}
\tilde{e}_{il} u = \sum_{c \in C_i} c_l b_{i; (l,c)} u_c,
\end{equation}
\begin{equation}
\tilde{f}_{il} u = \sum_{c \in C_i} \frac{1}{c_l + 1} b_{i; (l,c) \cup c} u_c,
\end{equation}
where $c_l$ denotes the number of $l$ in $c$.

It is easy to see that $\tilde{e}_{il} \circ \tilde{f}_{il} = \text{id}_{U_q^- (g)} - \alpha$ for $(i, l) \in I^\infty$ and $\langle h_i, -\alpha \rangle > 0$.

**Definition 3.8.** A free $A_0$-submodule $L$ of $U_q^- (g)$ is called a **crystal lattice** if the following conditions hold.

(a) $Q(q) \otimes_{A_0} L \cong U_q^- (g)$,
(b) $L = \oplus_{\alpha \in R_+} L_{-\alpha}$, where $L_{-\alpha} = L \cap U_q^- (g)_{-\alpha}$,
(c) $\tilde{e}_{il} L \subset L$, $\tilde{f}_{il} L \subset L$ for all $(i, l) \in I^\infty$.

The condition (c) yields the $Q$-linear maps $\tilde{e}_{il}, \tilde{f}_{il} : L/qL \to L/qL$.

**Definition 3.9.** A **crystal basis** of $U_q^- (g)$ is a pair $(L, B)$ such that

(a) $L$ is a crystal lattice of $U_q^- (g)$,
(b) $B$ is a $Q$-basis of $L/qL$,
(c) $B = \sqcup_{\alpha \in R_+} B_{-\alpha}$, where $B_{-\alpha} = B \cap (L/qL)_{-\alpha}$,
(d) $\tilde{e}_{il} B \subset B \cup \{0\}$, $\tilde{f}_{il} B \subset B \cup \{0\}$ for $(i, l) \in I^\infty$,
(e) for any $b, b' \in B$ and $(i, l) \in I^\infty$, we have $\tilde{f}_{il} b = b'$ if and only if $b = \tilde{e}_{il} b'$.

Let $L(\infty)$ be the $A_0$-submodule of $U_q^- (g)$ spanned by $\tilde{f}_{il_1} \cdots \tilde{f}_{ir} 1$ ($r \geq 0, (i_j, l_j) \in I^\infty$), and $B(\infty) = \{ \tilde{f}_{il_1} \cdots \tilde{f}_{ir} 1 + qL(\infty) \}$.

**Theorem 3.10.** The pair $(L(\infty), B(\infty))$ is a crystal basis of $U_q^- (g)$.

We will prove this theorem in Section 4.
Example 3.11. Let $I = I^\text{im} = \{i\}$ and let
\[ U^- = Q(q)(b_{il} \mid l > 0), \quad L := \bigoplus_{c \in \mathcal{C}} A_0(b_{i,c}1). \]

If $i \notin I_\text{iso}$, for $c \in \mathcal{C}$, define $B_{i,c} := \{b_{i,c}1\}$ and set $B = \prod_{c \in \mathcal{C}} B_{i,c}$. Define
\[ \tilde{c}_{il}(b_{i,c}1) = \begin{cases} b_{i,c}1, & \text{if } c_1 = l, \\ 0, & \text{otherwise}, \end{cases} \]
and
\[ \tilde{f}_{il}(b_{i,c}1) = b_{i,(l,c)}1. \]

If $i \in I_\text{iso}$, for $c \in \mathcal{C}$, define $B_{i,c} := \{\frac{1}{c!}b_{i,c}1\}$ and set $B = \prod_{c \in \mathcal{C}} B_{i,c}$. Define
\[ \tilde{c}_{il}(\frac{1}{c!}b_{i,c}1) = \frac{1}{(c_1 - 1)!} b_{i,c\setminus\{l\}}1, \]
and
\[ \tilde{f}_{il}(\frac{1}{c!}b_{i,c}1) = \frac{1}{(c_1 + 1)!} b_{i,c\setminus\{l\}}1. \]

We can verify that the pair $(L, B)$ is a crystal basis of $U^-.$

3.3. Abstract crystals.

By extracting the fundamental properties of the crystal bases of $V(\lambda)$ and $U_q^{-}(\mathfrak{g})$, we define the notion of abstract crystals as follows.

Definition 3.12. [5, Definition 2.1]

An abstract crystal is a set $B$ together with the maps $\text{wt}: B \to P$, $\varphi_i: B \to Z \cup \{-\infty\}$ ($i \in I$) and $\tilde{c}_{il}, \tilde{f}_{il}: B \to B \cup \{0\}$ ($(i, l) \in I^\text{r.e.}$) satisfying the following conditions:

(a) $\text{wt}(\tilde{f}_{il}b) = \text{wt}(b) - l\alpha_i$ if $\tilde{f}_{il}b \neq 0$, $\text{wt}(\tilde{c}_{il}b) = \text{wt}(b) + l\alpha_i$ if $\tilde{c}_{il}b \neq 0$.

(b) $\varphi_i(b) = \langle h_i, \text{wt}(b) \rangle + \varepsilon_i(b)$ for $i \in I$ and $b \in B$.

(c) $\tilde{f}_{il}b = b'$ if and only if $b = \tilde{c}_{il}b'$ for $(i, l) \in I^\text{r.e.}$ and $b, b' \in B$.

(d) For any $i \in I^\text{im}$ and $l > 0$ and $b \in B$, we have
\[
(1) \quad \varepsilon_i(\tilde{f}_{il}b) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_{il}b) = \varphi_i(b) - 1 \text{ if } \tilde{f}_{il}b \neq 0,
\]
\[
(2) \quad \varepsilon_i(\tilde{c}_{il}b) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{c}_{il}b) = \varphi_i(b) + 1 \text{ if } \tilde{c}_{il}b \neq 0.
\]

(e) For any $i \in I^\text{im}$, $l > 0$ and $b \in B$, we have
\[
(1') \quad \varepsilon_i(\tilde{f}_{il}b) = \varepsilon_i(b), \quad \varphi_i(\tilde{f}_{il}b) = \varphi_i(b) - l\alpha_i \text{ if } \tilde{f}_{il}b \neq 0,
\]
\[
(2') \quad \varepsilon_i(\tilde{c}_{il}b) = \varepsilon_i(b), \quad \varphi_i(\tilde{c}_{il}b) = \varphi_i(b) + l\alpha_i \text{ if } \tilde{c}_{il}b \neq 0.
\]

(f) For any $(i, l) \in I^\text{r.e.}$ and $b \in B$ such that $\varphi_i(b) = -\infty$, we have $\tilde{c}_{il}b = \tilde{f}_{il}b = 0.$

Remark 3.13.

(a) In Example 3.6, define
\[
\text{wt}(b_{i,c}u_0) = \text{wt}(u_0) - |c|\alpha_i, \quad \varepsilon_i(b_{i,c}u_0) = 0,
\]
\[
\varphi_i(b_{i,c}u_0) = \langle h_i, \text{wt}(b_{i,c}u_0) \rangle = \langle h_i, \text{wt}(u_0) - |c|\alpha_i \rangle = m - |c|a_{ii}.
\]
Then the set $B$ together with the maps $\tilde{c}_{il}, \tilde{f}_{il}$, wt, $\varepsilon_i, \varphi_i$ is an abstract crystal.

(b) In Example 3.11, define
\[
\text{wt}(b_{i,c}1) = -|c|\alpha_i, \quad \varepsilon_i(b_{i,c}1) = 0, \quad \varphi_i(b_{i,c}1) = -|c|a_{ii}.
\]
Then the set $B$ together with the maps $\tilde{c}_{il}, \tilde{f}_{il}$, wt, $\varepsilon_i, \varphi_i$ is an abstract crystal.
Definition 3.14.

(a) A crystal morphism \( \psi \) between two abstract crystals \( B_1 \) and \( B_2 \) is a map from \( B_1 \) to \( B_2 \cup \{0\} \) satisfying the following conditions:

(i) for \( b \in B_1 \) and \( i \in I \), we have \( \text{wt}(\psi(b)) = \text{wt}(b) \), \( \varepsilon_i(\psi(b)) = \varepsilon_i(b) \), \( \varphi_i(\psi(b)) = \varphi_i(b) \),
(ii) for \( b \in B_1 \) and \( (i,l) \in I^\infty \) satisfying \( f_{il}b \in B_1 \), we have \( \psi(f_{il}b) = f_{il}\psi(b) \).

(b) A crystal morphism \( \psi : B_1 \to B_2 \) is called strict if

\[
\psi(\tilde{e}_{il}b) = \tilde{e}_{il}(\psi(b)), \quad \psi(\tilde{f}_{il}b) = \tilde{f}_{il}(\psi(b))
\]

for all \( (i,l) \in I^\infty \) and \( b \in B_1 \).

We recall the tensor product rule from [5, Section 3]. Let \( B_1 \) and \( B_2 \) be abstract crystals and let \( B_1 \otimes B_2 = \{ b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2 \} \). Define the maps \( \text{wt}, \varepsilon_i, \varphi_i \) (\( i \in I \)), \( \tilde{e}_{il}, \tilde{f}_{il} \) ((\( i,l \) \( \in I^\infty \)) as follows.

\[
\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2),
\]

\[
\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \text{wt}(b_1) \rangle),
\]

\[
\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_1) + \langle h_i, \text{wt}(b_2) \rangle, \varphi_i(b_2)).
\]

If \( i \in I^e \),

\[
\tilde{e}_{il}(b_1 \otimes b_2) = \begin{cases} 
\tilde{e}_{il}b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\
b_1 \otimes \tilde{e}_{il}b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), 
\end{cases}
\]

\[
\tilde{f}_{il}(b_1 \otimes b_2) = \begin{cases} 
\tilde{f}_{il}b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\
b_1 \otimes \tilde{f}_{il}b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). 
\end{cases}
\]

If \( i \in I^m \),

\[
\tilde{e}_{il}(b_1 \otimes b_2) = \begin{cases} 
\tilde{e}_{il}b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) - l\alpha_{ii}, \\
b_1 \otimes \tilde{e}_{il}b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). 
\end{cases}
\]

\[
\tilde{f}_{il}(b_1 \otimes b_2) = \begin{cases} 
\tilde{f}_{il}b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\
b_1 \otimes \tilde{f}_{il}b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). 
\end{cases}
\]

Proposition 3.15. [5, Proposition 3.1]

If \( B_1 \) and \( B_2 \) are abstract crystals, then \( B_1 \otimes B_2 \) defined in (3.19)–(3.21) is also an abstract crystal.

From now on, we shall only consider the case with \( i \in I^m \), because the case with \( i \in I^e \) has already been studied in [15].

Let \( M \) be an object in \( \mathcal{O}_{int} \) and let \( (L, B) \) be a crystal basis of \( M \). We already have the maps

\[
\text{wt} : B \to P, \quad \tilde{e}_{il}, \tilde{f}_{il} : B \to B \cup \{0\}.
\]

Define

\[
\varepsilon_i(b) = 0, \quad \varphi_i(b) = \langle h_i, \text{wt}(b) \rangle \text{ for any } b \in B.
\]

Lemma 3.16. The set \( B \) together with the maps defined in (3.22)–(3.23) is an abstract crystal.

Proof. By Definition 3.1 and (3.23), we have

\[
\varepsilon_i(\tilde{e}_{il}b) = \varepsilon_i(b) = 0 \text{ and } \varepsilon_i(\tilde{f}_{il}b) = \varepsilon_i(b) = 0,
\]

and

\[
\varphi_i(\tilde{f}_{il}b) = \langle h_i, \text{wt}(\tilde{f}_{il}b) \rangle = \langle h_i, \text{wt}(b) - l\alpha_i \rangle = \varphi_i(b) - l\alpha_i,
\]

\[
\varphi_i(\tilde{e}_{il}b) = \langle h_i, \text{wt}(\tilde{e}_{il}b) \rangle = \langle h_i, \text{wt}(b) + l\alpha_i \rangle = \varphi_i(b) + l\alpha_i.
\]
Thus our assertion follows. □

Let $M_1, M_2 \in \mathcal{O}_{int}$ and $(L_1, B_1), (L_2, B_2)$ be their crystal bases, respectively. Set

$$M = M_1 \otimes_{Q(q)} M_2, \quad L = L_1 \otimes_{A_0} L_2, \quad B = B_1 \otimes B_2.$$ 

By Proposition 3.15, $B_1 \otimes B_2$ is an abstract crystal. The tensor product rule on $B_1 \otimes B_2$ can be simplified as follows.

Set $m_1 := \langle h_i, \text{wt}(b_1) \rangle$ and $m_2 := \langle h_i, \text{wt}(b_2) \rangle$. Then we have

$$\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2),$$

$$\varepsilon_i(b_1 \otimes b_2) = 0, \quad \varphi_i(b_1 \otimes b_2) = m_1 + m_2,$$

and

$$\tilde{f}_{il}(b_1 \otimes b_2) = \begin{cases} \tilde{f}_{il}b_1 \otimes b_2, & \text{if } m_1 > 0, \\ b_1 \otimes \tilde{f}_{il}b_2, & \text{if } m_1 = 0, \end{cases}$$

$$\tilde{e}_{il}(b_1 \otimes b_2) = \begin{cases} \tilde{e}_{il}b_1 \otimes b_2, & \text{if } m_1 > -l_{aii}, \\ 0 & \text{if } 0 < m_1 \leq -l_{aii}, \\ b_1 \otimes \tilde{e}_{il}b_2, & \text{if } m_1 = 0. \end{cases}$$

(3.24)

Note that $m_1 \geq 0$ because $\text{wt}(b_1) \in \mathbb{P}^+.$

Let $V, V'$ be $U$-modules as in Example 3.6 and let $(L, B), (L', B')$ be their crystal bases, respectively. Then $B \otimes B'$ is an abstract crystal under the simplified tensor product rule given in (3.24).

4. Grand-loop argument

In this section, we will give the proofs of Theorem 3.5 and Theorem 3.10 following the frame work of Kashiwara’s grand-loop argument [9, 15]. For this purpose, we need to prove the statements given below.

$$\tilde{e}_{il}L(\lambda) \subset L(\lambda), \quad \tilde{e}_{il}B(\lambda) \subset B(\lambda) \cup \{0\},$$

(4.1)

$$\tilde{f}_{il}b = b' \text{ if and only if } \tilde{e}_{il}b' = b \text{ for any } b, b' \in B(\lambda),$$

$$B(\lambda) \text{ is a } Q\text{-basis of } L(\lambda)/qL(\lambda),$$

and

$$\tilde{e}_{il}L(\infty) \subset L(\infty), \quad \tilde{e}_{il}B(\infty) \subset B(\infty) \cup \{0\},$$

(4.2)

$$\tilde{f}_{il}b = b' \text{ if and only if } \tilde{e}_{il}b' = b \text{ for any } b, b' \in B(\infty),$$

$$B(\infty) \text{ is a } Q\text{-basis of } L(\infty)/qL(\infty).$$

To apply the grand-loop argument, we need Kashiwara’s bilinear forms $(, )_K$ defined as follows.

Let $V(\lambda) = U_q(g)v_\lambda$ be an irreducible highest weight module with $\lambda \in \mathbb{P}^+$. By a standard argument, one can show that there exists a unique non-degenerate symmetric bilinear form $(, )_K$ on $V(\lambda)$ given by

$$(v_\lambda, v_\lambda)_K = 1, \quad (q^h u, v)_K = (u, q^h v)_K,$$

(4.3)

$$(b_{il}u, v)_K = -(u, K_i^l b_{il}v)_K,$$

$$(a_{il}u, v)_K = -(u, K_i^{-l} b_{il}v)_K,$$

where $u, v \in V(\lambda)$ and $h \in \mathbb{P}^\vee$.

Similarly, there exists a unique non-degenerate symmetric bilinear form $(, )_K$ on $U_q^{-1}(g)$ satisfying

$$(1, 1)_K = 1, \quad (b_{il}S, T)_K = (S, c_{il}T)_K \text{ for } S, T \in U_q^{-1}(g).$$

(4.4)

Now we begin to follow the grand-loop argument.
For \( \lambda \in P^+ \), we define a \( U_q^- (\mathfrak{g}) \)-module homomorphism given by
\[
(4.5) \quad \pi_\lambda : U_q^- (\mathfrak{g}) \to V (\lambda), \quad 1 \mapsto v_\lambda.
\]
Then we obtain \( \pi_\lambda (L(\infty)) = L(\lambda) \). The map \( \pi_\lambda \) induces a homomorphism
\[
(4.6) \quad \overline{\pi}_\lambda : L(\infty)/qL(\infty) \to L(\lambda)/qL(\lambda), \quad 1 + qL(\infty) \mapsto v_\lambda + qL(\lambda).
\]
For \( \lambda, \mu \in P^+ \), there exist unique \( U_q(\mathfrak{g}) \)-module homomorphisms
\[
\Phi_{\lambda, \mu} : V(\lambda + \mu) \to V(\lambda) \otimes V(\mu), \quad v_{\lambda+\mu} \mapsto v_\lambda \otimes v_\mu,
\]
\[
\Psi_{\lambda, \mu} : V(\lambda) \otimes V(\mu) \to V(\lambda + \mu), \quad v_\lambda \otimes v_\mu \mapsto v_{\lambda+\mu}.
\]
It is easy to verify that \( \Psi_{\lambda, \mu} \circ \Phi_{\lambda, \mu} = \text{id}_{V(\lambda+\mu)} \).

On \( V(\lambda) \otimes V(\mu) \), we define
\[
(u_1 \otimes u_2, v_1 \otimes v_2)_K = (u_1, v_1)_K (u_2, v_2)_K,
\]
where \((, )_K\) is the non-degenerate symmetric bilinear form defined in (4.3). It is straightforward to verify that
\[
(\Psi_{\lambda, \mu}(u), v)_K = (u, \Phi_{\lambda, \mu}(v))_K \quad \text{for } u \in V(\lambda) \otimes V(\mu), \ v \in V(\lambda + \mu).
\]

We now prove Theorem 3.5 and Theorem 3.10 using Kashiwara’s grand-loop argument as follows.

Let \((i,l) \in I^\infty\), \(\lambda, \mu \in P^+\) and \(\alpha \in R_+(r)\), where \(R_+(r) = \{\alpha \in R_+ \mid |\alpha| \leq r\}\).

**A(r):** \(\tilde{e}_\lambda L(\lambda)_{\lambda-\alpha} \subset L(\lambda), \ \tilde{e}_\lambda B(\lambda)_{\lambda-\alpha} \subset B(\lambda) \cup \{0\}\).

**B(r):** For \(b \in B(\lambda)_{\lambda-\alpha+\lambda_\alpha}, \ b' \in B(\lambda)_{\lambda-\alpha}, \ \tilde{f}_\lambda b = b'\) if and only if \(\tilde{e}_\lambda b' = b\).

**C(r):** \(\Phi_{\lambda, \mu}(L(\lambda + \mu)_{\lambda+\mu-\alpha}) \subset L(\lambda) \otimes L(\mu)\).

**D(r):** \(\Psi_{\lambda, \mu}((L(\lambda) \otimes L(\mu))_{\lambda+\mu-\alpha}) \subset L(\lambda + \mu), \ \Psi_{\lambda, \mu}((B(\lambda) \otimes B(\mu))_{\lambda+\mu-\alpha}) \subset B(\lambda + \mu) \cup \{0\}\).

**E(r):** \(\tilde{e}_\lambda B(\infty)_{\infty-\alpha} \subset B(\infty) \cup \{0\}\).

**F(r):** For \(b \in B(\infty-\alpha+\lambda_\alpha), \ b' \in B(\infty-\alpha), \ \tilde{f}_\lambda b = b'\) if and only if \(\tilde{e}_\lambda b' = b\).

**G(r):** \(B(\lambda)_{\lambda-\alpha}\) is a \(Q\)-basis of \((L(\lambda)/qL(\lambda))_{\lambda-\alpha}\), \(B(\infty)_{\infty-\alpha}\) is a \(Q\)-basis of \((L(\infty)/qL(\infty))_{\infty-\alpha}\).

**H(r):** \(\overline{\pi}_\lambda (L(\infty)_{\infty-\alpha}) = L(\lambda)_{\lambda-\alpha}\).

**I(r):** For \(S \in L(\infty)_{\infty+\lambda_\alpha}, \ \tilde{f}_\lambda (S v_\lambda) \equiv (\tilde{f}_\lambda S) v_\lambda \mod qL(\lambda)\).

**J(r):** If \(B^\lambda_{\lambda-\alpha} := \{b \in B(\infty)_{\infty-\alpha} \mid \overline{\pi}_\lambda (b) \neq 0\}\), then \(B^\lambda_{\lambda-\alpha} \cong B(\lambda)_{\lambda-\alpha}\).

**K(r):** If \(b \in B^\lambda_{\lambda-\alpha}\), then \(\tilde{e}_\lambda \overline{\pi}_\lambda (b) = \overline{\pi}_\lambda \tilde{e}_\lambda (b)\).

We shall prove the statements \(A(r), \ldots, K(r)\) by induction.

When \(r = 0, \ r = 1\), our assertions are true. We now assume that \(A(r-1), \ldots, K(r-1)\) are true.

**Lemma 4.1.** Let \(\alpha \in R_+(r-1)\) and \(b \in B(\lambda)_{\lambda-\alpha}\). If \(\tilde{e}_\lambda b = 0\) for any \((i,l) \in I^\infty\), then we have \(\alpha = 0\) and \(b = v_\lambda\).

**Proof.** The same argument in [9, Lemma 7.2], gives our claim. \(\square\)

**Lemma 4.2.** Let \(\alpha \in R_+(r-1), \ i \in I^{im}\), and \(u = \sum_{e \in C_i} b_{i,c} u_e \in V(\lambda)_{\lambda-\alpha}\) be the \(i\)-string decomposition of \(u\). If \(u \in L(\lambda)\), then \(u_e \in L(\lambda)\) for any \(c \in C_i\).

**Proof.** Suppose \(u = \sum_{e \in C_i} b_{i,e} u_e \in L(\lambda)\). We shall use the induction on \(|c|\). If \(|c| = 0\), the assertion follows naturally. If \(|c| > 0\), by \(A(r-1)\), we have \(\tilde{e}_\lambda u \in L(\lambda)\) for any \(l > 0\). By Definition 3.1, we have
\[
\tilde{e}_\lambda u = \begin{cases} \sum_{e \in C_i} b_{i,c} u_e \in L(\lambda), & \text{if } i \in I^{im} \setminus I^{iso}, \\ \sum_{e \in C_i} c_l b_{i,c} u_e \in L(\lambda), & \text{if } i \in I^{iso}. \end{cases}
\]
Hence $u_c \in L(\lambda)$ for any $c \neq 0$.

Set $u_1 := \sum_{c \neq 0} b_{i,c} u_c$. It follows that $u_1 \in L(\lambda)$. Hence $u_0 := u - u_1 \in L(\lambda)$, which proves our conclusion. \hfill \Box

**Lemma 4.3.** Let $\alpha \in R_+ (r - 1)$, $i \in I^m$ and let $u = \sum_{c \in C_i} b_{i,c} u_c \in V(\lambda)_{\lambda - \alpha}$ be the $i$-string decomposition of $u$. If $u + qL(\lambda) \in B(\lambda)$, then there exists $c \in C_i$ such that

(a) $u \equiv \tilde{f}_i,c u_c \mod qL(\lambda)$,

(b) $u_{c'} \equiv 0 \mod qL(\lambda)$ for any $c' \neq c$.

**Proof.** The case for $|c| = 0$ is trivial. For $|c| > 0$, by $A(r - 1)$, we have $\tilde{e}_i b \in B(\lambda) \cup \{0\}$ for any $l > 0$. If $\tilde{e}_i b = 0$ for any $l > 0$, by Lemma 4.2, we have $u_c \in qL(\lambda)$ for any $c \neq 0$. Then $u \equiv u_0 \mod qL(\lambda)$.

Setting $c = 0$, our assertion follows trivially.

Suppose $\tilde{e}_i b \neq 0$ for some $l > 0$. By induction, there exists $c_0 \in C_i$ such that

$$
\tilde{e}_i u = \begin{cases} 
\tilde{f}_i,c_0 u_{c_0} \mod qL(\lambda), \\
0 & \text{for any } c' \neq c_0.
\end{cases}
$$

Set $c = (l, c_0)$ or $c = c_0 \cup \{l\}$. By $B(r - 1)$, we obtain

$$
u \equiv \tilde{f}_i \tilde{e}_i u \equiv \tilde{f}_i \tilde{f}_i,c_0 u_{c_0} \equiv \tilde{f}_i,c u_c \mod qL(\lambda).
$$

If $c' \neq c$, then $c_1 \neq l$ or $c_1 = l$, $c'_0 \neq c_0$. It follows that $\tilde{e}_i \tilde{f}_i,c' u_{c_0} = 0$. \hfill \Box

By the same approach as that for Lemma 4.2 and Lemma 4.3, we have the following lemma.

**Lemma 4.4.** Let $\alpha \in R_+ (r - 1)$, $i \in I^m$ and let $u = \sum_{c \in C_i} b_{i,c} u_c \in U_q(g)_{\alpha}$ be the $i$-string decomposition of $u$.

(a) If $u \in L(\infty)$, then $u_c \in L(\infty)$ for any $c$.

(b) If $u + qL(\infty) \in B(\infty)$, then there exists $c \in C_i$ such that

(1) $u \equiv \tilde{f}_i,c u_c \mod qL(\infty)$,

(2) $u_{c'} \equiv 0 \mod qL(\infty)$ for any $c' \neq c$.

The following lemma plays an important role in our proofs.

**Lemma 4.5.** Let $\alpha, \beta \in R_+ (r - 1)$ and $i \in I^m$.

(a) For all $l > 0$, we have

$$
\tilde{e}_i (L(\lambda)_{\lambda - \alpha} \otimes L(\mu)_{\mu - \beta}) \subset L(\lambda) \otimes L(\mu),
$$

$$
\tilde{f}_i (L(\lambda)_{\lambda - \alpha} \otimes L(\mu)_{\mu - \beta}) \subset L(\lambda) \otimes L(\mu).
$$

(b) For all $l > 0$, we have

$$
\tilde{e}_i (B(\lambda)_{\lambda - \alpha} \otimes B(\mu)_{\mu - \beta}) \subset (B(\lambda) \otimes B(\mu)) \cup \{0\},
$$

$$
\tilde{f}_i (B(\lambda)_{\lambda - \alpha} \otimes B(\mu)_{\mu - \beta}) \subset (B(\lambda) \otimes B(\mu)) \cup \{0\}.
$$

(c) If $\tilde{e}_i (b \otimes b') \neq 0$, then $b \otimes b' = \tilde{f}_i \tilde{e}_i (b \otimes b')$.

(d) If $\tilde{e}_i (b \otimes b') = 0$ for all $l > 0$, then $b = v_\lambda$.

(e) For any $(i,l) \in I^\infty$, we have $\tilde{f}_i (b \otimes v_\mu) = \tilde{f}_i b \otimes v_\mu$, $0$.

(f) For any $(i_1, l_1), \ldots, (i_r, l_r) \in I^\infty$, we have

$$
\tilde{f}_{i_1} \tilde{f}_{i_r} (v_\lambda \otimes v_\mu) \equiv \tilde{f}_{i_1} \ldots \tilde{f}_{i_r} v_\lambda \otimes v_\mu \mod q(L(\lambda) \otimes L(\mu))
$$

or $\tilde{f}_{i_1} \ldots \tilde{f}_{i_r} v_\lambda \equiv 0 \mod qL(\lambda)$.
Proof. The proofs for (a), (b), (c), (e) and (f) are similar to the ones given in [9, Lemma 7.5]. So we shall only show the proof for (d).

Suppose $\bar{e}_{ii}(b \otimes b') = 0$ for any $l > 0$. If $m = \langle h_i, \text{wt}(b) \rangle > 0$, then there exists $l > 0$ such that

$$0 \leq -a_{ii} \leq \cdots \leq -la_{ii} \leq m \leq -(l + 1)a_{ii} \leq \cdots.$$ 

For $0 < k \leq l$, we have $\bar{e}_{ik}(b \otimes b') = \bar{e}_{ik}b \otimes b' = 0$, then $\bar{e}_{ik}b = 0$. By [12, Proposition 4.4], we have $\bar{e}_{ik}b = 0$ for any $k \geq l + 1$. It follows that $\bar{e}_{il}b = 0$ for any $l > 0$.

If $m = 0$, then $m \leq -la_{ii}$ for all $l > 0$. Hence by [12, Proposition 4.4], we have $\bar{e}_{il}b = 0$ for all $l > 0$. Therefore, by Lemma 4.1, we have $b = v_\lambda$. □

Proposition 4.6. (C(r)) For any $\alpha \in R_+(r)$, we have

$$\Phi_{\lambda, \mu}(L(\lambda + \mu)_{\lambda+\alpha}) \subset L(\lambda) \otimes L(\mu).$$

Proof. Note that

$$L(\lambda + \mu)_{\lambda+\alpha} = \sum_{(i,j) \in I_\infty} \tilde{f}_{ij}(L(\lambda + \mu)_{\lambda+\alpha+la_i}).$$

Then our assertion follows from C(r − 1) and Lemma 4.5 (a). □

Lemma 4.7. Let $(i_1, l_1), \ldots, (i_r, l_r) \in I^\infty$. Suppose that there exists $t$ with $t < r$ satisfying $i_t \neq i_{t+1} = \cdots = i_r$. Then for any $\mu \in P^+$ and $\lambda = \Lambda_{i_t}$, we have

$$\tilde{f}_{i_1i_2} \cdots \tilde{f}_{i_r}v_\lambda \equiv b \otimes b' \mod q(L(\lambda) \otimes L(\mu))$$

for some $b \in B(\lambda)_{\lambda-\alpha} \cup \{0\}$, $b' \in B(\mu)_{\mu-\beta} \cup \{0\}$ and $\alpha, \beta \in R_+(r−1)$.

Proof. The condition $\Lambda_{i_t}(h_r) = 0$ implies $b_{i_t}v_\lambda = 0$. Thus for any $v \in V(\mu)$, we have

$$b_{i_t}v_\lambda \equiv b_{i_t}v_\lambda \otimes v + K_{i_t}v_\lambda \otimes b_{i_t}v \equiv b_{i_t}v_\lambda \otimes v.$$ 

Set $v = b_{i_t+1}v_{\lambda+1} \cdots b_{i_r}v_{\mu}$. We have

$$b_{i_t}v_\lambda \equiv b_{i_t}v_{\lambda+1} \cdots b_{i_r}v_{\mu} = b_{i_t}v_\lambda \otimes b_{i_t}v_{\mu} + K_{i_t}v_\lambda \otimes b_{i_t}v_{\mu}.$$ 

where $\tilde{f}_{i_1i_2}v_\lambda \in B(\lambda)_{\lambda-\alpha} \cup \{0\}$ and $\tilde{f}_{i_t}v_\lambda, v_{\lambda+1} \cdots v_{i_t}v \in B(\mu)_{\mu-\beta}$. Then the lemma follows from the tensor product rule (3.21). □

By a similar argument as that for [9, Lemma 7.8], we have the following lemma.

Lemma 4.8. For any $\alpha \in R_+(r)$, we have

$$(L(\lambda) \otimes L(\mu))_{\lambda+\mu-\alpha} = \sum_{(i,j) \in I_\infty} b_{ij}(L(\lambda) \otimes L(\mu))_{\lambda+\mu-\alpha+la_i} + v_\lambda \otimes L(\mu)_{\mu-\alpha}.$$ 

For $\lambda, \mu \in P^+$, define a $U_q^- (\mathfrak{g})$-module homomorphism

$$S_{\lambda, \mu} : V(\lambda) \otimes V(\mu) \to V(\lambda), \quad u \otimes v \mapsto u,$$

$$V(\lambda) \otimes \sum_{(i,j) \in I_\infty} \tilde{f}_{ij}V(\mu) \mapsto 0.$$

Hence $u \otimes v \mapsto 0$ unless $v = \alpha v_\mu$ for some $\alpha \in Q(q)$.

Lemma 4.9. Let $\lambda, \mu \in P^+$.

(a) $S_{\lambda, \mu}(L(\lambda) \otimes L(\mu)) = L(\lambda)$.

(b) For any $\alpha \in R_+(r−1)$ and $w \in (L(\lambda) \otimes L(\mu))_{\lambda+\mu-\alpha}$, we have

$$S_{\lambda, \mu} \circ \tilde{f}_d(w) \equiv \tilde{f}_d \circ S_{\lambda, \mu}(w) \mod qL(\lambda).$$
Proof. (a) is obvious. For (b), we may assume that
\[ w = u \otimes u' = b_{i,c} u_c \otimes b_{i,c'} u_{c'}, \]
where \( u_c \in L(\lambda) \), \( u_{c'} \in L(\mu) \) and \( a_{ik} u_c = a_{ik} u_{c'} = 0 \) for any \( k > 0 \).

Let \( L \) be the \( A_0 \)-submodule of \( V(\lambda) \otimes V(\mu) \) generated by \( b_{i,c} u_c \otimes b_{i,c'} u_{c'} \) for all \( c \) and \( c' \). Thus \( L \subset L(\lambda) \otimes L(\mu) \). By the tensor product rule, we have
\[
\tilde{f}_{il}(w) = \tilde{f}_{il}(u \otimes u') = \begin{cases} 
\tilde{f}_{il}u \otimes u', & \text{if } \varphi_i(u) > 0, \\
\tilde{f}_{il}u' \otimes u, & \text{if } \varphi_i(u) = 0.
\end{cases}
\]

If \( \varphi_i(u) > 0 \), then we have \( \tilde{f}_{il}(w) = \tilde{f}_{il}u \otimes u' \) and
\[
S_{\lambda,\mu} \circ \tilde{f}_{il}(w) = \begin{cases} 
\tilde{f}_{il}u, & \text{if } c' = 0, \\
0, & \text{otherwise},
\end{cases}
\]

\[
S_{\lambda,\mu}(w) = \begin{cases} 
u, & \text{if } c' = 0, \\
0, & \text{otherwise}.
\end{cases}
\]

Hence we have
\[
\tilde{f}_{il} \circ S_{\lambda,\mu}(w) = \begin{cases} 
\tilde{f}_{il}u, & \text{if } c' = 0, \\
0, & \text{otherwise}.
\end{cases}
\]

If \( \varphi_i(u) = 0 \), then we have
\[
\varphi_i(b) = 0 \Rightarrow u = u_0,
\]
\[
c' = 0 \Rightarrow u' = u_0.
\]

By [12, Proposition 4.4], we have \( \tilde{f}_{il}(w) = u_0 \otimes \tilde{f}_{il}u_0' = 0 \). Hence \( S_{\lambda,\mu} \circ \tilde{f}_{il}(w) = 0 \). On the other hand, by [12, Proposition 4.4] again, we have \( \tilde{f}_{il} \circ S_{\lambda,\mu}(u \otimes u') = \tilde{f}_{il}(u) = 0 \).

**Lemma 4.10.** Let \( \alpha \in \mathbb{R}_+ \) and \( S \in U_q^{-}(g)_{-\alpha} \). For any \( \lambda \gg 0 \), we have
\[
(f_{il}S)v_{\lambda} \equiv f_{il}(sv_{\lambda}) \mod qL(\lambda),
\]
\[
(e_{il}S)v_{\lambda} \equiv e_{il}(sv_{\lambda}) \mod qL(\lambda).
\]

**Proof.** We may assume that \( S = b_{i,c} T \) and \( e'_{ik} T = 0 \) for any \( k > 0 \). Then we have \( E_{ik} T = 0 \) for any \( k > 0 \). Note that
\[
E_{ik}(Tv_{\lambda}) = q_{ik}^{-k(h_i,wt(T))} T(E_{ik}v_{\lambda}) + \frac{q_{ik}^{2k}(e'_{ik}(T) - K_{ik}^{2k} e'_{ik}(T))v_{\lambda}}{1 - q_{ik}^{2k}} v_{\lambda}
\]
\[
= \frac{q_{ik}^{2k}(h_i,\lambda + k\alpha_i + (h_i,wt(T)))}{1 - q_{ik}^{2k}} v_{\lambda}.
\]

Since \( \lambda \gg 0 \), we have \( E_{ik}(Tv_{\lambda}) \equiv 0 \mod qL(\lambda) \) for any \( k > 0 \).

1. If \( i \notin I_{iso} \), we have
\[
(f_{il}S)v_{\lambda} = (f_{il}(b_{i,c} T))v_{\lambda} = (b_{il}(b_{i,c} T))v_{\lambda} = b_{il}(b_{i,c}(Tv_{\lambda})).
\]
Since \( E_{ik}(Tv_{\lambda}) \equiv 0 \mod qL(\lambda) \) for any \( k > 0 \), we have
\[
b_{il}(b_{i,c}(Tv_{\lambda})) = f_{il}(b_{i,c} T v_{\lambda}) = f_{il}(sv_{\lambda}) \mod qL(\lambda),
\]
and
\[
(e_{il}S)v_{\lambda} = (e_{il}(b_{i,c} T))v_{\lambda} = (b_{i,c}'(T))v_{\lambda} = b_{i,c}'(Tv_{\lambda}) \equiv e_{il}(sv_{\lambda}) \mod qL(\lambda).
\]
(2) If \( i \in I_{\text{iso}} \), we have
\[
(f_i d) v_\lambda = (\tilde{f}_i d b_i e T) v_\lambda = \frac{1}{c_i + 1} (b_i d b_i e T) v_\lambda
\]
\[
= \frac{1}{c_i + 1} b_i d b_i e T v_\lambda = \tilde{f}_i d (b_i e T v_\lambda) = \tilde{f}_i d (S v_\lambda) \mod q L(\lambda),
\]
\[
(\tilde{c}_i d) v_\lambda = (\tilde{\epsilon}_i b_i e T) v_\lambda = c_i (b_i e T v_\lambda) = c_i (b_i e T v_\lambda) \mod q L(\lambda).
\]

\[\square\]

**Proposition 4.11.** (I(\( r \))) For \( \lambda \in P^+, \alpha \in R_+(r - 1) \) and \( S \in L(\infty)_{-\alpha} \), we have
\[
(f_i d) v_\lambda \equiv f_i d (S v_\lambda) \mod q L(\lambda).
\]

In particular, we have
\[
(f_i d_1 \cdots f_i d_t 1) v_\lambda \equiv f_i d_1 \cdots f_i d_t v_\lambda \mod q L(\lambda).
\]

**Proof.** Take \( \mu \gg 0 \) such that \( \lambda + \mu \gg 0 \). By Lemma 4.10, we have
\[
(f_i d) v_{\lambda + \mu} \equiv f_i d (S v_{\lambda + \mu}) \mod q L(\lambda + \mu).
\]

By Proposition 4.6, \( \Phi_{\lambda, \mu} \) gives
\[
(f_i d) (v_\lambda \otimes v_\mu) \equiv f_i d (S (v_\lambda \otimes v_\mu)) \mod q (L(\lambda) \otimes L(\mu)).
\]

On the other hand, by \( H(r - 1) \) and \( C(r - 1) \), we have
\[
S(v_\lambda \otimes v_\mu) = \Phi_{\lambda, \mu}(S v_{\lambda + \mu}) \in L(\lambda) \otimes L(\mu).
\]

Applying \( S_{\lambda, \mu} \) to (4.7), then Lemma 4.9 yields
\[
(f_i d) v_\lambda \equiv f_i d (S v_\lambda) \mod q L(\lambda).
\]

\[\square\]

By a similar argument as that for [9, Proposition 7.13], we have the following proposition.

**Proposition 4.12.** (H(\( r \))) For any \( \lambda \in P^+ \) and \( \alpha \in R_+(r) \), we have
\[
\pi_\lambda (L(\infty)_{-\alpha}) = L(\lambda)_{\lambda - \alpha}.
\]

**Corollary 4.13.** Consider the \( \mathbb{Q} \)-linear map
\[
\pi_\lambda : L(\infty)_{-\alpha} / q L(\infty)_{-\alpha} \rightarrow L(\lambda)_{\lambda - \alpha} / q L(\lambda)_{\lambda - \alpha}.
\]

(a) For any \( \beta \in R_+(r - 1) \) and \( b \in B(\infty)_{-\beta} \), we have
\[
\pi_\lambda (f_i d b) = f_i d (\pi_\lambda (b)).
\]

(b) For any \( \alpha \in R_+(r) \) and \( \lambda \in P^+ \), we have
\[
\pi_\lambda (B(\infty)_{-\alpha}) = B(\lambda)_{\lambda - \alpha} \cup \{0\}.
\]

(c) For any \( \alpha \in R_+(r) \) and \( \lambda \gg 0 \), the map \( \pi_\lambda \) induces the isomorphisms
\[
L(\infty)_{-\alpha} \sim L(\lambda)_{\lambda - \alpha}, \quad B(\infty)_{-\alpha} \sim B(\lambda)_{\lambda - \alpha}.
\]

Fix \( \lambda \in P^+, i \in I_{\text{fin}}, l_1, \ldots, l_r > 0 \) and \( \alpha = \sum_{j=1}^r l_j a_{ij} \). Take a finite set \( T \) containing \( \Lambda_{i_1}, \ldots, \Lambda_{i_r} \).

i) Since \( T \) is a finite set, we can take a sufficient large \( N_1 \geq 0 \) such that
\[
\tilde{c}_i d L(\tau)_{\tau - \alpha} \subset q^{-N_1} L(\tau) \text{ for all } \tau \in T.
\]
ii) Choose $N_2 \geq 0$ such that $\tilde{c}_d L(\infty)_{-\alpha} \subset q^{-N_2} L(\infty)$.

Then for any $\mu \gg 0$, Lemma 4.10 and Proposition 4.12 yield
\[
\tilde{c}_d L(\mu)_{-\mu-\alpha} = \tilde{c}_d L(\infty)_{-\alpha} v_\mu + qL(\mu)_{-\alpha-\tau} \subset q^{-N_2} L(\infty)_{-\alpha} v_\mu + qL(\mu)_{-\alpha-\tau} \subset q^{-N_2} L(\mu).
\]

Therefore, for any $\alpha \in R_+(r)$, there exists $N \geq 0$ such that
\[
\tilde{c}_d L(\mu)_{-\alpha} \subset q^{-N} L(\mu) \quad \text{for all } \mu \gg 0,
\]

\[(4.8)\]
\[\tilde{c}_d L(\tau)_{-\alpha} \subset q^{-N} L(\tau) \quad \text{for all } \tau \in T,
\]
\[\tilde{c}_d L(\infty)_{-\alpha} \subset q^{-N} L(\infty).
\]

**Lemma 4.14.** For any $\alpha \in R_+$, let $N \geq 0$ be a non-negative integer satisfying (4.8). For any $\mu \gg 0$ and $\tau \in T$, we have
\[
\tilde{c}_d (L(\tau) \otimes L(\mu))_{-\alpha} \subset q^{-N} L(\tau \otimes L(\mu)).
\]

**Proof.** Let $u \in L(\tau)_{-\beta}$ and $v \in L(\mu)_{-\gamma}$ such that $\alpha = \beta + \gamma$.

**Claim:** $\tilde{c}_d (u \otimes v) \in q^{-N} (L(\tau) \otimes L(\mu))$.

If $\beta \neq 0$ and $\gamma \neq 0$, the claim is exactly the one in Lemma 4.5 (a).

If $\beta = 0$, then $\gamma = \alpha$, we may assume that $u = \psi$. Let $v = \sum_{e \in \mathcal{C}} b_i, c_0 v_e$ be the i-string decomposition of $v$. By (4.8), we have
\[
\tilde{c}_d v = \begin{cases} 
\sum_{e \neq 0} b_i, c_1 v_e \in q^{-N} L(\mu), & \text{if } i \notin I_{iso}, \\
\sum_{e \neq 0} c_l b_i, c_1 (i) v_e \in q^{-N} L(\mu), & \text{if } i \in I_{iso}.
\end{cases}
\]

Hence by Lemma 4.2, we obtain
\[
v_c \in q^{-N} L(\mu) \quad \text{for any } c \neq 0.
\]

Let $L$ be the $A_0$-submodule of $L(\tau) \otimes L(\mu)$ generated by $b_i, c_1 v_\psi \otimes b_i, c_2 v_\sigma$ for $c_1, c_2, c \neq 0$. Then $\tilde{c}_d L \subset L$. It follows that
\[
\tilde{c}_d (v_\psi \otimes v) = \sum_{\mathbf{c} \neq 0} \tilde{c}_d (v_\psi \otimes b_i, c_0 v_c) \in L \subset q^{-N} (L(\tau) \otimes L(\mu)).
\]

Similarly, the claim can be shown for the case $\beta = \alpha, \gamma = 0$. \hfill \Box

**Lemma 4.15.** Let $\alpha \in R_+(r)$ and let $N > 0$ be the positive integer satisfying (4.8). Then we have:

(a) $\tilde{c}_d L(\mu)_{-\alpha} \subset q^{-1-N} L(\mu)$ for all $\mu \gg 0$,

(b) $\tilde{c}_d L(\tau)_{-\alpha} \subset q^{-1-N} L(\tau)$ for all $\tau \in T$,

(c) $\tilde{c}_d L(\infty)_{-\alpha} \subset q^{-1-N} L(\infty)$.

**Proof.** (a) Let $u = \tilde{f}_{i_1 \ldots i_t} \cdot \tilde{f}_{i_1 \ldots i_t} v_\mu \in L(\mu)_{-\alpha}$. Suppose $i_1 = i_2 = \cdots = i_l$. If $i = i_1$, then
\[
u = b_i, c_0 v_\mu, \quad \mathbf{c} = (i_1, \cdots, i_l).
\]

Hence
\[
\tilde{c}_d = \begin{cases} 
b_i, c_1 v_\mu, & \text{if } i \notin I_{iso}, \quad c_1 = l, \\
b_i, c_1 (i) v_\mu, & \text{if } i \in I_{iso}, \quad l \in \mathbf{c}, \\
0, & \text{otherwise}.
\end{cases}
\]

Therefore, we have $\tilde{c}_d u \in L(\mu)$.

If $i \neq i_1$, then $\tilde{c}_d u = 0$. Thus we may assume that there exists $s$ with $1 \leq s < t$ such that $i_s \neq i_{s+1} = \cdots = i_l$. Suppose $\mu \gg 0$ and set $\lambda_0 = \Lambda_{i_s}$. Then $\mu' = \mu - \lambda_0 \gg 0$. Set
\[
w := \tilde{f}_{i_1 \ldots i_t} (v_{\lambda_0} \otimes v_{\mu'}).
By Lemma 4.7, we have
\[ w \equiv v \otimes v' \mod qL(\lambda_0) \otimes L(\mu') \]
for some \( v \in L(\lambda_0)_{\lambda_0 - \beta}, \ v' \in L(\mu')_{\mu' - \gamma}, \ \alpha = \beta + \gamma \) and \( \beta, \gamma \in R_+(r - 1). \)

Then Lemma 4.5 (a) and Lemma 4.14 imply
\[ \tilde{e}_{il}w \in L(\lambda_0) \otimes L(\mu') + q\tilde{e}_{il}(L(\lambda_0) \otimes L(\mu'))_{\lambda_0 + \mu' - \alpha} \]
\[ \subset L(\lambda_0) \otimes L(\mu') + q^{1-N}L(\lambda_0) \otimes L(\mu') = q^{1-N}L(\lambda_0) \otimes L(\mu'). \]

Thus we have
\[ \tilde{e}_{il}w \in q^{1-N}(L(\lambda_0) \otimes L(\mu'))_{\lambda_0 + \mu' - \alpha + P}, \]
Applying \( \Psi_{\lambda_0, \mu'} \) to \( D(r - 1) \), we have
\[ \tilde{e}_{il}u = \tilde{e}_{il}\tilde{f}_{i_1l_1} \cdots \tilde{f}_{i_nl_n}v_{\mu} \in q^{1-N}L(\mu). \]

(b) Let \( \tau \in T \) and set \( u = \tilde{f}_{i_1l_1} \cdots \tilde{f}_{i_nl_n}v_{\tau} \in L(\tau)_{\tau - \alpha}. \) If \( u \in qL(\tau) \), our assertion follows from (4.8). If \( u \notin qL(\tau) \), for any \( \mu \in P^+ \), Lemma 4.5 (f) gives
\[ \tilde{f}_{i_1l_1} \cdots \tilde{f}_{i_nl_n}(v_{\tau} \otimes v_{\mu}) \equiv u \otimes v_{\mu} \mod qL(\tau) \otimes L(\mu). \]

If \( \mu \gg 0 \), (a) implies
\[ \tilde{e}_{il}\tilde{f}_{i_1l_1} \cdots \tilde{f}_{i_nl_n}v_{\tau + \mu} \in q^{1-N}L(\tau + \mu). \]

Applying \( \Phi_{\tau, \mu} \) and \( B(r - 1) \), we obtain
\[ \tilde{e}_{il}\tilde{f}_{i_1l_1} \cdots \tilde{f}_{i_nl_n}(v_{\tau} \otimes v_{\mu}) \in q^{1-N}L(\tau) \otimes L(\mu). \]

By (4.9) and Lemma 4.5, we have
\[ \tilde{e}_{il}(v \otimes v_{\mu}) \in q^{1-N}L(\tau) \otimes L(\mu') + q\tilde{e}_{il}(L(\tau) \otimes L(\mu)) \subset q^{1-N}L(\tau) \otimes L(\mu). \]

Let \( u = \sum_{e \in C} b_{i, c}u_e \) be the i-string decomposition of \( u \). By (4.8), we have \( \tilde{e}_{il}u \in q^{-N}L(\tau). \)
Recall
\[ \tilde{e}_{il}u = \begin{cases} \sum_{e \in C} b_{i, c}e_1u_e, & \text{if } i \notin \tilde{f}_{i}, \ c_1 = 0, \\ \sum_{e \in C} q_{i, c}b_{i, c}\tilde{f}_{i}u_e, & \text{if } i \notin \tilde{f}_{i}, \ c \neq 0, \\ 0 & \text{otherwise}. \end{cases} \]

By Lemma 4.2, we have \( u_e \in q^{-N}L(\tau) \). Let \( L \) be the \( A_0 \)-submodule of \( V(\tau) \otimes V(\mu) \) generated by \( b_{i, c}u_e \otimes b_{i, c}v_{\mu} \ (c_1 = l \text{ or } l \in \mathfrak{c}) \). Then we have \( L \subset q^{-N}L(\tau) \otimes L(\mu). \)

The tensor product rule gives
\[ \tilde{e}_{il}(u \otimes v_{\mu}) \equiv \tilde{e}_{il}u \otimes v_{\mu} \mod qL. \]
By (4.10), we have
\[ \tilde{e}_{il}u \otimes v_{\mu} \equiv \tilde{e}_{il}(u \otimes v_{\mu}) \in q^{1-N}L(\tau) \otimes L(\mu). \]
Hence \( \tilde{e}_{il}u \in q^{1-N}L(\tau). \)

(c) Let \( S \in L(\infty)_{\lambda_0} \) and take \( \mu \gg 0 \). By Lemma 4.10, we have \( (\tilde{e}_{il}S)v_{\mu} \equiv \tilde{e}_{il}(Sv_{\mu}) \mod qL(\mu). \)
Thus Proposition 4.12 implies
\[ (\tilde{e}_{il}S)v_{\mu} = \tilde{e}_{il}(Sv_{\mu}) \in \tilde{e}_{il}L(\mu)_{\mu - \alpha} \subset q^{1-N}L(\mu). \]
Hence by Corollary 4.13 (c), we have
\[ \tilde{e}_{il}S \in q^{1-N}L(\infty). \]

\[ \square \]

**Corollary 4.16.** For \( \alpha \in R_+(r) \), we have \( 0 \notin B(\infty)_{-\alpha}. \)
Proof. If \( b \in B(\infty)_{-\alpha} \), then there exist \((i,l) \in I_\infty \) and \( b' \in B(\infty)_{-\alpha+l\alpha_i} \) such that \( b = \tilde{f}_{i,l}b' \). By \( G(r-1) \), the set \( B(\infty)_{-\alpha+l\alpha_i} \) forms a \( Q \)-basis of \( L(\infty)_{-\alpha+l\alpha_i}/qL(\infty)_{-\alpha+l\alpha_i} \). Then we have \( b' \neq 0 \). Hence \( b \neq 0 \). \( \square \)

**Lemma 4.17.** Let \( \alpha \in R_+(r) \), \((i,l) \in I_\infty \), \( \lambda > 0 \) and \( b \in B(\infty)_{-\alpha} \). Then we have
\[
\pi_\lambda(\tilde{e}_d b) = \tilde{e}_d \pi_\lambda(b).
\]

**Proof.** The assertion follows directly from Lemma 4.10. \( \square \)

**Corollary 4.18.** Let \( \lambda, \mu \in P^+ \) and \( \alpha, \beta \in R_+(r) \).
(a) For the \( i \)-string decomposition \( u = \sum_{c \in C_i} b_i,c u_c \in L(\lambda)_{\lambda-\alpha} \), we have \( u_c \in L(\lambda) \) for all \( c \in C_i \).
(b) For any \((i,l) \in I_\infty \), we have
\[
\tilde{f}_d(L(\lambda)_{\lambda-\alpha} \otimes L(\mu)_{\mu-\beta}) \subset L(\lambda) \otimes L(\mu),
\]
\[
\tilde{e}_d(L(\lambda)_{\lambda-\alpha} \otimes L(\mu)_{\mu-\beta}) \subset L(\lambda) \otimes L(\mu).
\]

**Proof.** Since Lemma 4.5 depends only on \( A(r-1) \), the corollary follows from the proof of Lemma 4.2. \( \square \)

**Lemma 4.19.** Let \( \lambda, \mu \in P^+ \) and \( \alpha \in R_+(r) \). For any \( u \in L(\lambda)_{\lambda-\alpha} \), we have
\[
\tilde{e}_d(u \otimes v_\mu) = \tilde{e}_d u \otimes v_\mu \mod q(L(\lambda) \otimes L(\mu)).
\]

**Proof.** The lemma follows from the fact \( \tilde{e}_d v_\mu = 0 \). \( \square \)

**Proposition 4.20.** \( (K(r)) \) Let \( \lambda \in P^+ \) and \( \alpha \in R_+(r) \). If \( b \in B(\infty)_{-\alpha} \) and \( \pi_\lambda(b) \neq 0 \), then we have
\[
\tilde{e}_d \pi_\lambda(b) = \pi_\lambda(\tilde{e}_d b).
\]

**Proof.** We set
\[
S = \tilde{f}_{i_1}t_{i_1} \cdots \tilde{f}_{i_\ell}t_{i_\ell} 1 \in L(\infty)_{-\alpha},
\]
\[
b = S + qL(\infty)_{-\alpha} \in B(\infty)_{-\alpha},
\]
\[
u = \tilde{f}_{i_1}t_{i_1} \cdots \tilde{f}_{i_\ell}t_{i_\ell} v_\lambda.
\]

By Proposition 4.11, we have
\[
u = (\tilde{f}_{i_1}t_{i_1} \cdots \tilde{f}_{i_\ell}t_{i_\ell} v_\lambda) \equiv (\tilde{f}_{i_1}t_{i_1} \cdots \tilde{f}_{i_\ell}t_{i_\ell} v_\lambda) \mod q(L(\lambda)).
\]
Since \( \pi_\lambda(b) \neq 0 \) and \( u \notin qL(\lambda) \). By Lemma 4.5 (f), for any \( \mu \in P^+ \), we have
\[
\tilde{f}_{i_1}t_{i_1} \cdots \tilde{f}_{i_\ell}t_{i_\ell} (v_\lambda \otimes v_\mu) = \tilde{f}_{i_1}t_{i_1} \cdots \tilde{f}_{i_\ell}t_{i_\ell} v_\lambda \otimes v_\mu = u \otimes v_\mu \mod q(L(\lambda) \otimes L(\mu)).
\]
Hence by Lemma 4.19, we have
\[
(4.11) \quad \tilde{e}_d(\tilde{f}_{i_1}t_{i_1} \cdots \tilde{f}_{i_\ell}t_{i_\ell} (v_\lambda \otimes v_\mu)) \equiv (\tilde{e}_d u \otimes v_\mu) \equiv (\tilde{e}_d u \otimes v_\mu) \mod q(L(\lambda) \otimes L(\mu)).
\]

On the other hand, for \( \mu \gg 0 \), by Lemma 4.17, we have
\[
\tilde{e}_d(\tilde{f}_{i_1}t_{i_1} \cdots \tilde{f}_{i_\ell}t_{i_\ell} v_{\lambda+\mu}) \equiv \tilde{e}_d(Sv_{\lambda+\mu}) \equiv (\tilde{e}_d S)v_{\lambda+\mu} \mod q(L(\lambda + \mu)).
\]
Applying \( \Phi_{\lambda, \mu} \) and Proposition 4.6, we obtain
\[
(4.12) \quad \tilde{e}_d(\tilde{f}_{i_1}t_{i_1} \cdots \tilde{f}_{i_\ell}t_{i_\ell} (v_\lambda \otimes v_\mu)) \equiv (\tilde{e}_d S)(v_\lambda \otimes v_\mu) \mod q(L(\lambda) \otimes L(\mu)).
\]
Then (4.11) and (4.12) yield
\[
\tilde{e}_d u \otimes v_\mu \equiv (\tilde{e}_d S)(v_\lambda \otimes v_\mu) \mod q(L(\lambda) \otimes L(\mu)).
\]
Applying \( S_{\lambda, \mu} \), we conclude
\[
\tilde{e}_d u \equiv (\tilde{e}_d S)v_\lambda \mod qL(\lambda).
\]
Hence \( \tilde{e}_d \pi_\lambda(b) = \pi_\lambda(\tilde{e}_d b) \). \( \square \)
Proposition 4.21. \((E(r))\) For every \(\alpha \in R_+(r)\), we have
\[
\tilde{e}_i L(\infty)_{-\alpha} \subset L(\infty), \quad \tilde{e}_i B(\infty)_{-\alpha} \subset B(\infty) \cup \{0\}.
\]

Proof. Applying Lemma 4.15 (c) repeatedly, the first assertion holds. For the second assertion, let
\[
S = \tilde{f}_i \tilde{l}_i \cdots \tilde{f}_i \tilde{l}_i 1 \in L(\infty)_{-\alpha}, \quad b = S + qL(\infty)_{-\alpha} \in B(\infty)_{-\alpha}.
\]
If \(i_1 = i_2 = \cdots = i_t\), our assertion is true as we have seen in the proof of Lemma 4.15 (a). Here, we may assume that there exists \(s\) with \(1 \leq s < t\) such that \(i_s \neq i_{s+1} = \cdots = i_t\). Take \(\mu \gg 0\) and set \(\lambda_0 = \Lambda_{i_s}, \lambda = \lambda_0 + \mu \gg 0\). Then Lemma 4.7 yields
\[
S(v_{\lambda_0} \otimes v_\mu) = \tilde{f}_i \tilde{l}_i \cdots \tilde{f}_i \tilde{l}_i (v_{\lambda_0} \otimes v_\mu) \equiv v \otimes v' \mod q(L(\lambda_0) \otimes L(\mu))
\]
for some \(v \in L(\lambda_0)_{\lambda_0 - \beta}, v' \in L(\mu)_{\mu - \gamma}, \beta, \gamma \in R_+(r-1) \setminus \{0\}\) and \(\alpha = \beta + \gamma\) such that
\[
v + qL(\lambda_0) \in B(\lambda_0) \cup \{0\}, \quad v' + qL(\mu) \in B(\mu) \cup \{0\}.
\]
Therefore we have
\[
\tilde{e}_i (\tilde{f}_i \tilde{l}_i \cdots \tilde{f}_i \tilde{l}_i (v_{\lambda_0} \otimes v_\mu)) \equiv \tilde{e}_i (v \otimes v') \equiv \tilde{e}_i v \otimes v' \mod q(L(\lambda_0) \otimes L(\mu)).
\]
By \(A(r-1)\), we have
\[
\tilde{e}_i (\tilde{f}_i \tilde{l}_i \cdots \tilde{f}_i \tilde{l}_i (v_{\lambda_0} \otimes v_\mu)) + q(L(\lambda_0) \otimes L(\mu)) \in (B(\lambda_0) \otimes B(\mu)) \cup \{0\}.
\]
The map \(\Psi_{\lambda_0,b}\) and \(D(r-1)\) yield
\[
\tilde{e}_i b = \tilde{e}_i (\tilde{f}_i \tilde{l}_i \cdots \tilde{f}_i \tilde{l}_i 1 + qL(\infty)) \in B(\infty) \cup \{0\}.
\]
\(\square\)

Proposition 4.22. \((A(r))\) For any \(\lambda \in P^+\) and \(\alpha \in R_+(r)\), we have
\[
\tilde{e}_i L(\lambda)_{\lambda - \alpha} \subset L(\lambda), \quad \tilde{e}_i B(\lambda)_{\lambda - \alpha} \subset B(\lambda) \cup \{0\}.
\]

Proof. Proposition follows from Lemma 4.15, Proposition 4.20, Proposition 4.21, Corollary 4.13 (b) and Proposition 4.12. \(\square\)

For \((i, l) \in I^\infty\), let \(u = b_{il}^{\infty} u_0\) such that \(E_{ik} u_0 = 0\) for all \(k \geq 0\). Define an operator \(Q_{il} : V(\lambda) \to V(\lambda)\) by
\[
Q_{il}(u) = \begin{cases} 
(m + 1)u & \text{if } i \in I^{iso}, \\
u & \text{otherwise}
\end{cases}
\]
(4.13)

Lemma 4.23. Let \(\lambda \in P^+\) and \(\alpha \in R_+(r)\).

(a) For any \(u \in L(\lambda)_{\lambda - \alpha + l_{\alpha}}\) and \(v \in L(\lambda)_{\lambda - \alpha}\), we have
\[
(\tilde{f}_i Q_{il} u, v)_K \equiv (u, \tilde{e}_i v)_K \mod qA_0.
\]

(b) \((L(\lambda)_{\lambda - \alpha}, L(\lambda)_{\lambda - \alpha}))_K \subset A_0.

Proof. (a) By (4.3), we have
\[
(b_{il} u, v)_K = (u, E_{il} v)_K \equiv (u, \tilde{e}_i v)_K \mod qL(\lambda).
\]
Therefore, if \(i \notin I^{iso}\), the conclusion holds naturally.

If \(i \in I^{iso}\), we may assume that \(u = b_{il}^{\infty} u_0\) and \(E_{ik} u_0 = 0\) for any \(k \geq 0\). Then we have
\[
(\tilde{f}_i Q_{il} u, v)_K = (c_{il} + 1) (\tilde{f}_i u, v)_K = (b_{il} c_{il} u_0, v)_K = (b_{il} c_{il} u_0, E_{il} v)_K \equiv (u, \tilde{e}_i v)_K \mod qA_0,
\]
which gives our assertion.
(b) By induction, we have \((u, \tilde{f}_l v)_K \in A_0\). Hence \((\tilde{f}_l u, v)_K \in A_0\), which proves the claim. \(\square\)

**Lemma 4.24.** Let \(\alpha = l_1\alpha_i + \cdots + l_t\alpha_i \in R_+\), \(S, T \in U_q^{-}(\mathfrak{g})_{-\alpha}\) and \(m \in \mathbb{Z}\). For any \(\lambda \gg 0\), we have
\[
(S, T)_K = \prod_{k=1}^{l} (1 - q_i^{2l})^{-1} (Sv_{\lambda}, Tv_{\lambda})_K \mod q^m A_0.
\]

**Proof.** If \(S = 1\), then \(\alpha = 0\), and \((S, T)_K = (v_{\lambda}, v_{\lambda})_K = 1\).

We shall prove the assertion by induction on \(\text{ht}(\alpha)\). Assume that \(S = b_dW\) for some \(W \in U_q^{-}(\mathfrak{g})_{-\alpha + \lambda_a}\). Then we have
\[
(Sv_{\lambda}, Tv_{\lambda})_K = (Wv_{\lambda}, E_{di}(Tv_{\lambda}))_K
= (Wv_{\lambda}, q_i^{-(h_i, \text{wt}(T))} T(E_{di}v_{\lambda}))_K + (Wv_{\lambda}, e_{dl} T - K^2 e_{dl} T - 1 e_{dl} v_{\lambda})_K
= (1 - q_i^{2l})^{-1} (Wv_{\lambda}, (e_{dl} T)v_{\lambda}) - (1 - q_i^{2l})^{-1} q_i^{2(h_i, \lambda - \alpha)} + 2q_{2ai}(Wv_{\lambda}, (e_{dl} T)v_{\lambda})_K.
\]

Since \(\lambda \gg 0\), we have \(q_i^{2(h_i, \lambda - \alpha)} + 2q_{2ai} \equiv 0 \mod q^m A_0\). Hence by induction, we obtain
\[
(Sv_{\lambda}, Tv_{\lambda})_K \equiv (1 - q_i^{2l})^{-1} (Wv_{\lambda}, (e_{dl} T)v_{\lambda})_K
\equiv (1 - q_i^{2l})^{-1} (W, e_{dl} T)_K \equiv (1 - q_i^{2l})^{-1} (S, T)_K \mod q^m A_0.
\]

\(\square\)

Let \(L\) be a finitely generated \(A_0\)-submodule of \(V(\lambda)_{\lambda - \alpha}\) and set
\[
L^\gamma := \{u \in V(\lambda)_{\lambda - \alpha} \mid (u, L)_K \subset A_0\}.
\]

Similarly, let \(L\) be a finitely generated \(A_0\)-submodule of \(L(\infty)_{-\alpha}\) and set
\[
L^\gamma := \{u \in U_q^{-}(\mathfrak{g})_{-\alpha} \mid (u, L)_K \subset A_0\}.
\]

Then \((L^\gamma)^\gamma = L\) and we obtain

**Lemma 4.25.** If \(\lambda \gg 0\) and \(\alpha \in R_+(r)\), we have \(\pi_\lambda(L(\infty)_{-\alpha}) = L(\lambda)_{\lambda - \alpha}\).

**Proof.** Let \(\{S_k\}_{k \in I}\) be an \(A_0\)-basis of \(L(\infty)_{-\alpha}\) and let \(\{T_k\}_{k \in I}\) be its dual basis with respect to the bilinear form \((\ , \ )_K\), i.e., \((S_i, T_j)_K = \delta_{ij}\). Then \(L(\infty)_{-\alpha} = \sum_{j \in I} A_0 T_j\).

By Proposition 4.12, we have \(L(\lambda) = \sum_{k \in I} A_0 (S_k v_{\lambda})\). By Lemma 4.24, for \(\lambda \gg 0\), we have
\[
(S_k v_{\lambda}, T_j v_{\lambda})_K \equiv \delta_{kj} \mod q A_0.
\]

Hence we conclude
\[
L(\lambda)_{\lambda - \alpha} = \sum_{j \in I} A_0 T_j v_{\lambda} = \pi_\lambda(L(\infty)_{-\alpha}) \text{ for } \lambda \gg 0.
\]

\(\square\)

**Lemma 4.26.** Let \(\lambda \in P^+, \mu \gg 0\) and \(\alpha \in R_+(r)\). Then we have
\[
\Psi_{\lambda, \mu}((L(\lambda) \otimes L(\mu))_{\lambda + \mu - \alpha}) \subset L(\lambda + \mu + \lambda - \alpha).
\]

**Proof.** By Lemma 4.8, we have
\[
(L(\lambda) \otimes L(\mu))_{\lambda + \mu - \alpha} = \sum_{(i, l) \in I_\infty} \tilde{f}_l ((L(\lambda) \otimes L(\mu))_{\lambda + \mu - \alpha + l_a}) + v_{\lambda} \otimes L(\mu)_{\mu - \alpha}.
\]

By induction hypothesis \(D(r - 1)\), we get
\[
\Psi_{\lambda, \mu}(\sum_{(i, l) \in I_\infty} \tilde{f}_l ((L(\lambda) \otimes L(\mu))_{\lambda + \mu - \alpha + l_a})
\]
implies that \(4.18\), we have \(c\), we have

\[
\Psi_{\lambda,\mu}(v_{\lambda} \otimes L(\mu)_{\mu-\alpha}) \subset L(\lambda + \mu)_{\lambda + \mu - \alpha}.
\]

It remains to show

\[
\Psi_{\lambda,\mu}(v_{\lambda} \otimes L(\mu)_{\mu-\alpha}) \subset L(\lambda + \mu)_{\lambda + \mu - \alpha}.
\]

Let \(u \in L(\lambda + \mu)_{\lambda + \mu - \alpha}\). By Lemma 4.25, we have \(u = Sv_{\lambda+\mu}\) for some \(S \in L(\infty)_{\alpha}\). Note that

\[
\Delta(S) = S \otimes 1 + \text{(intermediate terms)} + K_\alpha \otimes S.
\]

Then we have

\[
\begin{align*}
&(\Phi_{\lambda,\mu}(u), v_{\lambda} \otimes L(\mu)_{\mu-\alpha}) = (\Delta(S)(v_{\lambda} \otimes v_{\mu}), v_{\lambda} \otimes L(\mu)_{\mu-\alpha}) \\
&= (Sv_{\lambda} \otimes v_{\mu} + \text{(intermediate terms)} + K_\mu v_{\lambda} \otimes v_{\mu}, v_{\lambda} \otimes L(\mu)_{\mu-\alpha}) \\
&= (Sv_{\lambda}, v_{\lambda})(v_{\mu}, L(\mu)_{\mu-\alpha}) + \text{(intermediate terms)} + K_\mu(v_{\lambda}, v_{\lambda})(Sv_{\mu}, L(\mu)_{\mu-\alpha}) \\
&= q^{(\alpha,\lambda)}(Sv_{\mu}, L(\mu)_{\mu-\alpha}).
\end{align*}
\]

Since \(\mu \gg 0\), Lemma 4.25 implies that \(Sv_{\mu} \in L(\mu)^{\vee}\). Thus

\[
(u, \Psi_{\lambda,\mu}(v_{\lambda} \otimes L(\mu)_{\mu-\alpha})) = (\Phi_{\lambda,\mu}(u), v_{\lambda} \otimes L(\mu)_{\mu-\alpha}) = q^{(\alpha,\lambda)}(Sv_{\mu}, L(\mu)_{\mu-\alpha}) \subset A_0.
\]

Hence \(\Psi_{\lambda,\mu}(v_{\lambda} \otimes L(\mu)_{\mu-\alpha}) \subset (L(\lambda + \mu)_{\lambda + \mu - \alpha})^{\vee} = L(\lambda + \mu)_{\lambda + \mu - \alpha}\). □

**Proposition 4.27.** (\(F(r)\)) Let \(\alpha \in B_+^*\) and \(b \in B(\infty)_{-\alpha}\). If \(\bar{e}_ib \neq 0\), then \(b = \bar{f}_i\bar{e}_ib\).

**Proof.** Let \(b = \bar{f}_{i_1} \cdots \bar{f}_{i_t} 1 \in B(\infty)_{-\alpha}\). We assume \(\bar{e}_ib \neq 0\). If \(i_1 = \cdots = i_t\) and \(i \neq i_1\), then

\[
\bar{e}_ib = \bar{e}_i\bar{f}_{i_1} \cdots \bar{f}_{i_t} 1 = \cdots = \bar{f}_{i_{i_1}} \cdots \bar{f}_{i_t}\bar{e}_ib 1 = 0.
\]

Hence we must have \(i = i_1 = \cdots = i_t\). In this case, our assertion follows easily.

Assume that there exists \(s\) with \(1 \leq s < t\) such that \(i_s \neq i_{s+1} = \cdots = i_t\). Take \(u \gg 0\) and set \(\lambda_0 = \Lambda_{i_s}, \lambda = \lambda_0 + \mu\).

Then Lemma 4.7 yields

\[
\bar{f}_{i_1} \cdots \bar{f}_{i_t}(v_{\lambda_0} \otimes v_{\mu}) \equiv v \otimes v' \mod q(L(\lambda_0) \otimes L(\mu))
\]

for some \(v \in L(\lambda_0)_{\lambda_0 - \beta}, v' \in L(\mu)_{\mu - \gamma}\), \(\beta, \gamma \in Q_+(r - 1) \backslash \{0\}\) and \(\alpha = \beta + \gamma\) such that

\[
v + qL(\lambda_0) \in B(\lambda_0) \cup \{0\}, \quad v' + qL(\mu) \in B(\mu) \cup \{0\}.
\]

By Corollary 4.18 (b), we have

\[
\bar{e}_i\bar{f}_{i_1} \cdots \bar{f}_{i_t}(v_{\lambda_0} \otimes v_{\mu}) \equiv \bar{e}_i(v \otimes v') \mod q(L(\lambda_0) \otimes L(\mu)).
\]

Then \(\Psi_{\lambda_0,\mu}\) and \(H(r - 1)\) yield

\[
\pi_\lambda(\bar{e}_i\bar{f}_{i_1} \cdots \bar{f}_{i_t} 1) = \bar{e}_i\bar{f}_{i_1} \cdots \bar{f}_{i_t} v_{\lambda_0 + \mu} \equiv \Psi_{\lambda_0,\mu}(\bar{e}_i(v \otimes v')) \mod q(L(\lambda)).
\]

Since \(\mu \gg 0\), we have \(\bar{e}_i(v \otimes v') \notin q(L(\lambda_0) \otimes L(\mu))\).

By Lemma 4.5 (c), we have

\[
\bar{f}_{i_1} \cdots \bar{f}_{i_t}(v_{\lambda_0} \otimes v_{\mu}) \equiv v \otimes v' \equiv \bar{f}_i\bar{e}_i(v \otimes v')
\]

\[
\equiv \bar{f}_i\bar{e}_i(\bar{f}_{i_1} \cdots \bar{f}_{i_t}(v_{\lambda_0} \otimes v_{\mu})) \mod q(L(\lambda_0) \otimes L(\mu)).
\]

Applying \(\Psi_{\lambda_0,\mu}\) and Lemma 4.26, we obtain

\[
\bar{f}_{i_1} \cdots \bar{f}_{i_t}v_{\lambda_0 + \mu} = \bar{f}_{i_1} \cdots \bar{f}_{i_t}v_{\mu} = \bar{f}_i\bar{e}_i(\bar{f}_{i_1} \cdots \bar{f}_{i_t}v_{\lambda}) \mod q(L(\lambda)).
\]

Since \(\lambda \gg 0\), we get \(b = \bar{f}_i\bar{e}_ib \mod q(L(\infty))\). □

**Proposition 4.28.** (\(B(r)\)) Let \(\lambda \in P^+\) and \(\alpha \in B_+^*\). For \(b \in B(\lambda)_{\lambda - \alpha + i\alpha_i}\) and \(b' \in B(\lambda)_{\lambda - \alpha}\), we have \(\bar{f}_ib = b'\) if and only if \(b = \bar{e}_ib'\).
Proof. Suppose $f_{il}b = b'$. By Lemma 4.3, there exists $c \in C_i$ with $|c| \geq l$, such that $b \equiv b_{l,e}u_0$, $E_{ik}u_0 = 0$ for all $k > 0$.

If $i \notin I_{iso}$, we have

\[
\tilde{e}_{il}b = b_{l,(i,c)}u_0 = b',
\]

\[
\tilde{f}_{il}b = \frac{1}{c+i+1} b_{l,e}u_0 = b'.
\]

Hence

\[
\tilde{e}_{il}b' = \frac{c_i + 1}{c_i + 1} b_{l,e}u_0 = b.
\]

Conversely, suppose $b' \in B(\lambda)_{\lambda + a}$ and $b = \tilde{e}_{il}b' \in B(\lambda)_{\lambda + a}$. By Corollary 4.13 (b), we have $b' = \pi(\lambda(b'_0))$ for some $b'_0 \in B(\infty)$. Proposition 4.20 implies that

\[
\pi(\lambda(\tilde{e}_{il}b'_0)) = \tilde{f}_{il}(\pi(\lambda(b'_0))) = \tilde{f}_{il}b' \neq 0.
\]

Hence $\tilde{e}_{il}b'_0 \neq 0$ in $B(\infty)$. By Proposition 4.27, we have $b'_0 = \tilde{f}_{il}\tilde{e}_{il}b'_0$.

Applying $\pi(\lambda)$, we obtain

\[
\tilde{f}_{il}b = \tilde{f}_{il}(\tilde{e}_{il}b') = \tilde{f}_{il}(\pi(\lambda(\tilde{e}_{il}b'_0))) = \pi(\lambda(\tilde{f}_{il}\tilde{e}_{il}b'_0)) = \pi(\lambda(b'_0)) = b'.
\]

\[
\square
\]

**Proposition 4.29.** (G(r)) Let $\lambda \in P^+$ and $\alpha \in R_+(r)$. We have the following facts.

(a) $B(\lambda)_{\lambda - \alpha}$ is a $Q$-basis of $L(\lambda)_{\lambda - \alpha}/qL(\lambda)_{\lambda - \alpha}$.

(b) $B(\infty)_{\lambda - \alpha}$ is a $Q$-basis of $L(\infty)_{\lambda - \alpha}/qL(\infty)_{\lambda - \alpha}$.

**Proof.** Suppose $\sum b \in B(\lambda)_{\lambda - \alpha} a_b b = 0$ for $a_b \in Q$.

By Proposition 4.22, we have $\tilde{e}_{il}B(\lambda)_{\lambda - \alpha} \subset B(\lambda) \cup \{0\}$ for any $(i, l) \in I^\infty$, which implies that

\[
\tilde{e}_{il}(\sum b \in B(\lambda)_{\lambda - \alpha} a_b b) = \sum_{b \in B(\lambda), \tilde{e}_{il}b \neq 0} a_b(\tilde{e}_{il}b) = 0.
\]

By G(r - 1) and Proposition 4.28, we have $a_b = 0$ whenever $\tilde{e}_{il}b \neq 0$. But for each $b \in B(\lambda)_{\lambda - \alpha}$, there exists $(i, l) \in I^\infty$ such that $\tilde{e}_{il}b \neq 0$. Thus $a_b = 0$ for any $b \in B(\lambda)_{\lambda - \alpha}$. Hence, the proposition holds.

\[
\square
\]

**Lemma 4.30.** Let $\lambda \in P^+$ and $\alpha \in Q_+(r)\backslash \{0\}$.

(a) If $u \in L(\lambda)_{\lambda - \alpha}/qL(\lambda)_{\lambda - \alpha}$ and $\tilde{e}_{il}u = 0$ for any $(i, l) \in I^\infty$, then $u = 0$.

(b) If $u \in V(\lambda)_{\lambda - \alpha}$ and $\tilde{e}_{il}u \in L(\lambda)$ for any $(i, l) \in I^\infty$, then $u \in L(\lambda)_{\lambda - \alpha}$.

(c) If $u \in L(\infty)_{\lambda - \alpha}/qL(\infty)_{\lambda - \alpha}$ and $\tilde{e}_{il}u = 0$ for any $(i, l) \in I^\infty$, then $u = 0$.

(d) If $u \in U_q(g)_{\lambda - \alpha}$ and $\tilde{e}_{il}u \in L(\infty)$ for any $(i, l) \in I^\infty$, then $u \in L(\infty)_{\lambda - \alpha}$.

**Proof.** (a) Let $u = \sum b \in B(\lambda)_{\lambda - \alpha} a_b b \ (a_b \in Q)$. For any $(i, l) \in I^\infty$, we have

\[
\tilde{e}_{il}u = \sum_{b \in B(\lambda)_{\lambda - \alpha}, \tilde{e}_{il}b \neq 0} a_b(\tilde{e}_{il}b) = 0.
\]

It follows from the proof of Proposition 4.29 that all $a_b = 0$. Hence $u = 0$.

(b) Choose the smallest $N \geq 0$ such that $q^N u \in L(\lambda)$. If $N > 0$, we have

\[
\tilde{e}_{il}(q^N u) = q^N(\tilde{e}_{il}u) \in qL(\lambda)
\]

for all $(i, l) \in I^\infty$. By (a), we have $q^N u \in qL(\lambda)$, i.e., $q^{N-1} u \in L(\lambda)$ which contradicts to the minimality of $N$. Hence $N = 0$ and $u \in L(\lambda)$. The proofs of (c) and (d) are similar.

\[
\square
\]
By a similar argument as that for [9, Proposition 7.34], we have the following proposition.

**Proposition 4.31.** (J(r)) Let \( \lambda \in P^+ \) and \( \alpha \in R_+(r) \), then we have
\[
B^\lambda_{-\alpha} := \{ b \in B(\infty)_{-\alpha} \mid \pi(\lambda) b \neq 0 \} \cong B(\lambda)_{\lambda-\alpha}.
\]

Using all the statements we have proved so far, we can show that Lemma 4.5 holds for all \( \alpha \in R_+(r) \).

In particular, we have

**Lemma 4.32.** Let \( \lambda, \mu \in P^+ \) and \( \alpha \in R_+(r) \).

(a) For all \((i, l) \in I^\infty\), we have
\[
\bar{e}_i(B(\lambda) \otimes B(\mu))_{\lambda+\mu-\alpha} \subset (B(\lambda) \otimes B(\mu)) \cup \{0\}.
\]
(b) If \( b \otimes b' \in (B(\lambda) \otimes B(\mu))_{\lambda+\mu-\alpha} \) and \( \bar{e}_i(b \otimes b') \neq 0 \), then we have
\[
b \otimes b' = \tilde{f}_i \bar{e}_i(b \otimes b').
\]

**Proposition 4.33.** (D(r)) For every \( \lambda, \mu \in P^+ \) and \( \alpha \in R_+(r) \), we have
(a) \( \Psi_{\lambda, \mu}(L(\lambda) \otimes L(\mu))_{\lambda+\mu-\alpha} \subset L(\lambda + \mu) \),
(b) \( \Psi_{\lambda, \mu}(B(\lambda) \otimes B(\mu))_{\lambda+\mu-\alpha} \subset B(\lambda + \mu) \cup \{0\} \).

**Proof.** Proposition follows by Lemma 4.26, Lemma 4.30, Lemma 4.32 and [9, Proposition 7.36]. \( \square \)

Thus we have completed the proofs of all the statements in Kashiwara’s grand-loop argument, which proves Theorem 3.5 and Theorem 3.10.

Let \((\, , )^0_K\) denote the \( \mathbb{Q} \)-valued inner product on \( L(\lambda)/qL(\lambda) \) (resp. \( L(\infty)/qL(\infty) \)) by taking crystal limit of \((\, , )_K\) on \( L(\lambda) \) (resp. \( L(\infty) \)).

**Lemma 4.34.** The crystal \( B(\lambda) \) (resp. \( B(\infty) \)) forms an orthogonal basis of \( L(\lambda)/qL(\lambda) \) (resp. \( L(\infty)/qL(\infty) \)) with respect to \((\, , )^0_K\).

**Proof.** We first consider the crystal \( B(\lambda) \). For all \( b, b' \in B(\lambda)_{-\lambda-\alpha} \), we shall prove \((b, b')^0_K = \delta_{b,b'} \mathbb{Z}_{>0} \) by using induction on \( \text{ht}(\alpha) \), where \( \alpha \in R_+(r) \).

If \( \text{ht}(\alpha) = 0 \), then our conclusion is trivial.

If \( \text{ht}(\alpha) > 0 \), we choose \((i, l) \in I^\infty\) such that \( \bar{e}_i b \neq 0 \). By \( B(r) \) and Lemma 4.23, we have
\[
(b, b')^0_K = (\tilde{f}_i Q_i^\alpha \bar{e}_i b, b')^0_K = (\bar{e}_i b, \bar{e}_i b')^0_K \in \delta_{\bar{e}_i b, \bar{e}_i b'} \mathbb{Z}_{>0} = \delta_{b,b'} \mathbb{Z}_{>0}.
\]

By Lemma 4.24 and a similar approach above, it is easy to show that the crystal \( B(\infty) \) is an orthogonal basis of \( L(\infty)/qL(\infty) \) with respect to \((\, , )^0_K\). \( \square \)

5. **Global bases**

Let \( \mathbb{A} = \mathbb{Z}[q, q^{-1}] \), \( \mathbb{A}_Q = \mathbb{Q}[q, q^{-1}] \) and \( \mathbb{A}_\infty \) be the subring of \( \mathbb{Q}(q) \) consisting of rational functions which are regular at \( q = \infty \).

**Definition 5.1.** Let \( V \) be a \( \mathbb{Q}(q) \)-vector space. Let \( V_Q, L_0 \) and \( L_\infty \) be an \( \mathbb{A}_Q \)-lattice, \( \mathbb{A}_0 \)-lattice and \( \mathbb{A}_\infty \)-lattice, respectively. We say that \((V_Q, L_0, L_\infty)\) is a balanced triple for \( V \) if the following conditions hold:

(a) \( \mathbb{Q} \)-vector space \( V_Q \cap L_0 \cap L_\infty \) is a free \( \mathbb{Q} \)-lattice of the \( \mathbb{A}_0 \)-module \( L_0 \).
(b) \( \mathbb{Q} \)-vector space \( V_Q \cap L_0 \cap L_\infty \) is a free \( \mathbb{Q} \)-lattice of the \( \mathbb{A}_\infty \)-module \( L_\infty \).
(c) \( \mathbb{Q} \)-vector space \( V_Q \cap L_0 \cap L_\infty \) is a free \( \mathbb{Q} \)-lattice of the \( \mathbb{A}_Q \)-module \( V_Q \).

**Theorem 5.2.** [8, 15] The following statements are equivalent.

(a) \((V_Q, L_0, L_\infty)\) is a balanced triple.
(b) The canonical map \( V_Q \cap L_0 \cap L_\infty \to L_0/qL_0 \) is an isomorphism.

(c) The canonical map \( V_Q \cap L_0 \cap L_\infty \to L_\infty/qL_\infty \) is an isomorphism.

Let \( (V_Q, L_0, L_\infty) \) be a balanced triple and let 
\[
G : L_0/qL_0 \to V_Q \cap L_0 \cap L_\infty
\]
be the inverse of the canonical isomorphism \( V_Q \cap L_0 \cap L_\infty \sim L_0/qL_0 \).

**Proposition 5.3.** [8, 15]
If \( B \) is a \( Q \)-basis of \( L_0/qL_0 \), then \( B := \{ G(b) \mid b \in B \} \) is an \( A_Q \)-basis of \( V_Q \).

**Definition 5.4.** Let \( (V_Q, L_0, L_\infty) \) be a balanced triple for a \( Q(q) \)-vector space \( V \).

(a) A \( Q \)-basis \( B \) of \( L_0/qL_0 \) is called a local basis of \( V \) at \( q = 0 \).

(b) The \( A_Q \)-basis \( B = \{ G(b) \mid b \in B \} \) is called the lower global basis of \( V \) corresponding to the local basis \( B \).

We define \( U_Z^{-}(g) \) (resp. \( U_Z^{-}(g) \)) to be the \( A \)-subalgebra (resp. \( A_Q \)-subalgebra) of \( U_q^{-}(g) \) generated by \( b_i^{(n)} \) (\( i \in I^e, n \geq 0 \)) and \( b_i^l \) (\( i \in I^m, l > 0 \)).

Let \( V(\lambda) = U_q^{-}(g)v_\lambda \) be the irreducible highest weight module with highest weight \( \lambda \in P^+ \). We define \( V(\lambda)_Z = U_Z^{-}(g)v_\lambda \) and \( V(\lambda)_Q = U_Q^{-}(g)v_\lambda \).

**Lemma 5.5.** For any \( S, T \in U_q^{-}(g) \), we have
\[
\begin{align*}
(Sb_i^l, T)_K &= (S, K_i^l)_{F_K}T_{K_i^{-l}}), \\
\end{align*}
\]
\[
(S, T)_K &= (S^*, T^*)_K.
\]

**Proof.** For (5.1), we shall use induction on \( |S| \). We write \( S = b_{jK}S_0 \). By (3.10), we have
\[
(Sb_i^l, T)_K = (b_{jK}S_0b_i^l, T)_K = (S_0b_i^l, e_i^lT)_K = (S_0, K_i^l)e_i^lT_{K_i^{-l}})_K = (S_0, e_{il}^lK_i^l)_{F_K}T_{K_i^{-l}})_K = (b_{jK}S_0, K_i^l)e_i^lT_{K_i^{-l}})_K = (S, K_i^l)e_i^lT_{K_i^{-l}})_K.
\]

For (5.2), it is enough to prove the following claim.
\[
((Sb_i^l)^*, T^*)_K = (Sb_i^l, T)_K.
\]

By (5.1) and (3.13), we have
\[
((Sb_i^l)^*, T^*)_K = (b_i^lS^*, T^*)_K = (S^*, e_i^lT^*)_K = (S^*, K_i^l(e_i^lT)^*_{K_i^{-l}})_K = (S, K_i^l(e_i^lT)_{K_i^{-l}})_K = (Sb_i^l, T)_K,
\]
which proves our assertion. \( \square \)

Combining Lemma 5.5, Lemma 4.10, Proposition 4.12, Corollary 4.13, Lemma 4.17 and Proposition 4.20 and using the same arguments in [6, Section 5], we obtain

**Theorem 5.6.** [6, Theorem 5.9]

There exist \( Q \)-linear canonical isomorphisms
\[
\begin{align*}
&\text{(a) } U_q^{-}(g)\cap L(\infty)\cap L(\infty) \sim L(\infty)/qL(\infty), \text{ where } - : U_q^{-}(g) \to U_q^{-}(g) \text{ is the } Q \text{-linear bar involution defined by (2.9)}, \\
&\text{(b) } V(\lambda)_Q \cap L(\lambda) \cap L(\lambda) \sim L(\lambda)/qL(\lambda), \text{ where } - \text{ is the } Q \text{-linear automorphism on } V(\lambda) \text{ defined by } P v_\lambda \mapsto \overline{P} v_\lambda \text{ for } P \in U_q^{-}(g).
\end{align*}
\]
Therefore we obtain:

**Proposition 5.7.** Let $G$ denote the inverse of the above isomorphisms.
(a) $\mathbf{B}(\infty) := \{G(b) \mid b \in B(\infty)\}$ is a lower global basis of $U_Q(-)\otimes V$.
(b) $\mathbf{B}(\lambda) := \{G(b) \mid b \in B(\lambda)\}$ is a lower global basis of $V_Q(\lambda)$.

6. Primitive Canonical Bases

For clarity and simplicity, we fix the notations for some of basic concepts in the theory of perverse sheaves.
(a) $X$: algebraic variety over $\mathbb{C}$
(b) $1 = 1_X$: constant sheaf on $X$
(c) $Sh(X)$: abelian category of sheaves on $X$ of $\mathbb{C}$-vector spaces
(d) $\mathcal{D}(X)$: derived category of complexes of sheaves on $X$
(e) $\mathcal{D}^b(X)$: full subcategory of $\mathcal{D}(X)$ consisting of bounded complexes on $X$
(f) $\mathcal{Perv}(X)$: abelian category of perverse sheaves on $X$
(h) For a complex $K$, let $D(K)$ denotes the Verdier dual of $K$.

6.1. Quiver with loops.

Let $Q = (I, \Omega)$ be a quiver, where $I$ is the set of vertices and $\Omega = \{h \mid s(h) \rightarrow t(h)\}$ is the set of arrows, where $s(h)$ and $t(h)$ are starting vertex and target vertex of $h$, respectively. Let $\Omega(i)$ denote the set of loops at $i$ and let $\omega_i = |\Omega(i)|$, the number of loops at $i$.

Let $h_{ij}$ denote the number of arrows $h : i \rightarrow j$. We define

$$a_{ij} = \begin{cases} 2(1 - \omega_i), & \text{if } i = j, \\ -h_{ij} - h_{ji}, & \text{if } i \neq j. \end{cases}$$

Then $A = A_Q := (a_{ij})_{i,j \in I}$ is a symmetric Borel-Berends-Cartan matrix. We will denote by $(A, P, P^\vee, R, R^\vee)$ the Borel-Berends-Cartan datum associated with $A$. Using the same notations as in Section 2, we write $R := \bigoplus_{i \in I} Z \alpha_i$, $R_+ := \sum_{i \in I} Z_{\geq 0} \alpha_i$ and $R_- = -R_+$.

Let $\alpha = \sum_{i \in I} d_i \alpha_i \in R_+$ and let $V_\alpha = \bigoplus_{i \in I} V_i$ be an $I$-graded vector space with dim$V_i = d_i$. Then the graded dimension of $V_\alpha$ is given by dim$V_\alpha = \sum_{i \in I} (\text{dim} V_i) \alpha_i$.

For every $I$-graded vector space $X$, we define

$$E_X = \bigoplus_{h \in \Omega} \text{Hom}(X_{s(h)}, X_{t(h)}),$$

and set $E(\alpha) = E_{V_\alpha}$, $G_\alpha = \prod_{i \in I} GL(V_i)$. Then $G_\alpha$ acts on $E(\alpha)$ by conjugation; i.e.,

$$(g.x)_h = g_{t(h)}x_h g_{s(h)}^{-1} \text{ for } h \in \Omega.$$

Let $i = (i_1, \ldots, i_r) \in I^r$ and $a = (a_1, \ldots, a_r) \in Z^r_{\geq 0}$. We say that $(i, a)$ is a *composition* of $\alpha$, denoted by $(i, a) \vdash \alpha$, if $a_1 \alpha_{i_1} + \cdots + a_r \alpha_{i_r} = \alpha$.

**Definition 6.1.** A flag $W = \{0\} = W_0 \subset \cdots \subset W_r = V_\alpha$ is called a flag of type $(i, a)$ if dim$(W_k/W_{k-1}) = a_k \alpha_{i_k}$ for all $1 \leq k \leq r$. 
Let $\mathcal{F}_{i,a}$ be the variety consisting of all flags of type $(i, a)$. Then we have
\begin{equation}
\dim(\mathcal{F}_{i,a}) = \sum_{i_k=i, \ k<i} a_k a_l.
\end{equation}

**Definition 6.2.** For $x = (x_h)_{h \in \Omega} \in E(\alpha)$, we say that a flag $W$ is $x$-stable if $x_h(W_k \cap V_{\alpha(h)}) \subset W_k \cap V_{\alpha(h)}$ for all $h \in \Omega$ and $k = 0, 1, \cdots, r$.

Let
\[ \tilde{\mathcal{F}}_{i,a} = \{(x, W) \mid x \in E(\alpha), W \in \mathcal{F}_{i,a}, W \text{ is } x\text{-stable}\} \subseteq E(\alpha) \times \mathcal{F}_{i,a}. \]

By (6.1), we have
\begin{equation}
\dim(\tilde{\mathcal{F}}_{i,a}) = \sum_{h \in \Omega} \sum_{i_k=i(h), \ k<i} a_k a_l + \sum_{i_k=i, \ k<i} a_k a_l.
\end{equation}

Consider the natural projection
\[ \pi_{i,a} : \tilde{\mathcal{F}}_{i,a} \to E(\alpha), \quad (x, W) \mapsto x. \]

Let $1 = 1_{\tilde{\mathcal{F}}_{i,a}}$ be the constant sheaf on $\tilde{\mathcal{F}}_{i,a}$. We define
\[ \tilde{\mathcal{L}}_{i,a} = (\pi_{i,a})_!(1) \text{ and } L_{i,a} = \tilde{\mathcal{L}}_{i,a}[\dim(\tilde{\mathcal{F}}_{i,a})]. \]

By [1], $L_{i,a}$ is semisimple and stable under the Verdier duality; i.e., $D(L_{i,a}) = L_{i,a}$.

Suppose $(i, a) \vdash \alpha$. Let $\mathcal{P}_{i,a}$ be the set of simple perverse sheaves possibly with some shifts appearing in the decomposition of $L_{i,a}$.

We define $\mathcal{P}_\alpha$ to be the full subcategory of $\mathcal{Perv}(E(\alpha))$ consisting of $P = \sum L$, where

(i) $L$ is a simple perverse sheaf,

(ii) $L[d]$ appears as a direct summand of $L_{i,a}$ for some $(i, a) \vdash \alpha$ and $d \in \mathbb{Z}$.

Now we define $\mathcal{Q}_\alpha$ to be the full subcategory of $\mathcal{D}(E(\alpha))$ consisting of complexes $K$ such that $K \cong \oplus_{L, d} L[d]$, where $L \in \mathcal{P}_\alpha$ and $d \in \mathbb{Z}$.

**Example 6.3.** Let $i \in I^\text{lm}$, $I = \{i\}$, $l > 0$ and $\alpha = l \alpha_i$. Then $(i, a) \vdash \alpha$ implies $i = (i, \cdots, i)$, $a = (a_1, \cdots, a_r)$ and $a_1 + \cdots + a_r = l$. Thus $a$ is a composition (or a partition) of $l$.

Let $V = V_{\alpha_i}$ with $\dim V = l \alpha_i$. Then $V \cong \mathbb{C}^l$, $G_\alpha \cong GL(\mathbb{C}^l)$ and
\[ E(\alpha) \cong \text{Hom}(V, V)^{\oplus \omega} \cong M_l \times \mathbb{C}^{\oplus \omega} \cong \mathbb{C}^{\oplus \omega l^2}, \]
where $\omega = \omega_i$, the number of loops at $i$.

In this special case, for simplicity, we will write $i$ for $i$. By (6.1) and (6.2), we have
\begin{equation}
\dim(\mathcal{F}_{i,a}) = \sum_{k<l} a_k a_l,
\end{equation}
\begin{equation}
\dim(\tilde{\mathcal{F}}_{i,a}) = d_{i,a} := \omega(\sum_{k<l} a_k a_l) + \sum_{k<l} a_k a_l = (\omega + 1) \sum_{k<l} a_k a_l.
\end{equation}

Then we have
\[ L_{i,a} = (\pi_{i,a})_!(1_{\tilde{\mathcal{F}}_{i,a}})[d_{i,a}]. \]

From now on, we will write $1_{i,a} := L_{i,a}$ for $a \vdash l$. In particular, when $a = (l)$, the trivial composition, we will write $1_{i,l}$ for $1_{i,(l)}$. 

6.2. Canonical bases.

Recall that $A = \mathbb{Z}[q, q^{-1}]$. We define $U^\ast_A(g)$ to be the $A$-subalgebra of $U_q(g)$ generated by $f_i^{(n)} (i \in I^n, n \geq 0)$ and $f_i (i \in I^m, l > 0)$.

Let $K(\alpha)$ be the Grothendieck group of $Q_\alpha$. Then $A$ acts on $K_\alpha$ via

$$q^{\pm 1}[P] = [P[\pm 1]],$$

where $[P]$ is the isomorphism class of a perverse sheaf $P$. Let $B_\alpha$ be the set of isomorphic class of simple perverse sheaves in $P_\alpha$. Then $B_\alpha$ is an $A$-basis of $K(\alpha)$. In particular, for $i \in I_\text{im}$ and $l > 0$, we have $B_{l\alpha_i} = \{[1_{i, a}] \mid a \vdash l\}$ and it is an $A$-basis of $K_{l\alpha_i}$.

Set

$$K = \bigoplus_{\alpha \in R_+} K(\alpha) \quad \text{and} \quad B = \bigsqcup_{\alpha \in R_+} B_\alpha.$$

Then $B$ is an $A$-basis of $K$.

Let $\gamma = \alpha + \beta$, $V = V_\gamma$ and $W \subset V$ such that $\dim(W) = \alpha$. Then we have $\dim(V/W) = \beta$.
Consider the natural isomorphisms

$$p : W \simto V_\alpha, \quad q : V/W \simto V_\beta,$$

which yields a diagram

$$E(\alpha) \times E(\beta) \xleftarrow{\kappa} E_\gamma(W) \xrightarrow{\iota} E(\gamma),$$

where

(a) $E_\gamma(W) = \{x \in E(\gamma) \mid x(W) \subset W\}$,
(b) $\iota$ is the canonical embedding,
(c) $\kappa(x) = (p_\ast(x|_W), q_\ast(x|_{V/W}))$.

We define

$$E(\alpha, \beta) = \{(x, W) \mid x \in E(\gamma), \ W \subset V, \ dim(W) = \alpha, \ x(W) \subset W\},$$

and

$$E(\alpha, \beta)^+ = \{(x, W, \sigma, \tau) \mid (x, W) \in E(\alpha, \beta), \ \sigma : W \simto V_\alpha, \ \tau : V/W \simto V_\beta\}.$$

Thus we obtain

$$E(\alpha) \times E(\beta) \xleftarrow{p_1} E(\alpha, \beta)^+ \xrightarrow{p_2} E(\alpha, \beta) \xrightarrow{p_3} E(\gamma),$$

where

$$p_1(x, W, \sigma, \tau) = (p_\ast(x|_W), q_\ast(x|_{V/W})), \quad p_2(x, W, \sigma, \tau) = (x, W), \quad p_3(x, W) = x.$$ 

Define the functors

$$\widehat{\text{Res}}_{\alpha, \beta} := \kappa \iota^* : Q(\gamma) \to Q(\alpha) \boxtimes Q(\beta),$$

$$\widehat{\text{Ind}}_{\alpha, \beta} := p_3 p_2 p_1^* : Q(\alpha) \boxtimes Q(\beta) \to Q(\gamma).$$

Remark 6.4. It is highly non-trivial to prove

$$\text{Im}(\widehat{\text{Res}}_{\alpha, \beta}) \subset Q(\alpha) \boxtimes Q(\beta), \quad \text{Im}(\widehat{\text{Ind}}_{\alpha, \beta}) \subset Q(\gamma).$$

In [19, Section 9.2], Lusztig gave a proof.

Assume that $\alpha = \sum d_i \alpha_i$ and $\beta = \sum d'_i \alpha_i$. Set $(\alpha, \beta) := \sum d_id'_i$ and denote by $l_1$ (resp. $l_2$) the dimension of fibers of $p_1$ (resp. $p_2$). Define the functors

$$\text{Res}_{\alpha, \beta} := \widehat{\text{Res}}_{\alpha, \beta}[l_1 - l_2 - 2(\alpha, \beta)], \quad \text{Ind}_{\alpha, \beta} := \widehat{\text{Ind}}_{\alpha, \beta}[l_1 - l_2].$$

These functors commute with Verdier duality.
Hence we obtain

\[ \text{Ind}_{\alpha,\beta} : \mathcal{K}(\alpha) \otimes \mathcal{K}(\beta) \rightarrow \mathcal{K}(\gamma), \]
\[ \text{Res}_{\alpha,\beta} : \mathcal{K}(\gamma) \rightarrow \mathcal{K}(\alpha) \otimes \mathcal{K}(\beta). \]

Since \( \mathcal{K} = \oplus_{\alpha \in R_+} \mathcal{K}_\alpha \), we obtain the \( A \)-algebra homomorphisms

\[ \mu : \mathcal{K} \otimes \mathcal{K} \rightarrow \mathcal{K}, \]
\[ \delta : \mathcal{K} \rightarrow \mathcal{K} \otimes \mathcal{K}, \]

induced by \( \text{Ind}_{\alpha,\beta} \) and \( \text{Res}_{\alpha,\beta} \). In this way, \( \mathcal{K} \) becomes an \( A \)-bialgebra.

The following theorem is one of the main results in [2].

**Theorem 6.5.** [2, Proposition 5, Theorem 1]

(a) The algebra \( \mathcal{K} \) is generated by \( [\mathbb{1}_d] \) \( ((i, l) \in I^\infty) \).

(b) There exists an isomorphism of \( A \)-bialgebras

\[ (6.4) \quad \Psi : U_\mathcal{A}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{K} \quad \text{given by} \quad f_{il} \mapsto [\mathbb{1}_d]. \]

**Definition 6.6.** The \( A \)-basis \( B := \Psi^{-1}(\mathcal{B}) \) of \( U_\mathcal{A}^-(\mathfrak{g}) \) is called the **canonical basis** of \( U_\mathcal{A}^-(\mathfrak{g}) \).

Let \( V(\lambda) = U_\mathcal{A}(\mathfrak{g}) v_\lambda \) be the irreducible highest weight module with highest weight \( \lambda \in P^+ \). We define \( V(\lambda)_A := U_\mathcal{A}(\mathfrak{g}) v_\lambda \). Then \( B^\lambda := B v_\lambda \) is an \( A \)-basis of \( V(\lambda)_A \) [19].

**Definition 6.7.** The \( A \)-basis \( B^\lambda \) of \( V(\lambda)_A \) is called the **canonical basis** of \( V(\lambda) \).

Unfortunately, the canonical bases \( B \) and \( B^\lambda \) do not coincide with the lower global bases \( B(\infty) \) and \( B(\lambda) \). To fix this situation, we introduce the notion of **primitive canonical bases**.

Recall that there is a \( Q(q) \)-algebra automorphism

\[ \phi : U^-_q(\mathfrak{g}) \rightarrow U^-_q(\mathfrak{g}) \quad \text{given by} \quad f_{il} \mapsto b_{il} \quad \text{for} \quad (i, l) \in I^\infty \]

defined in Proposition 2.3. By the definition of \( U_\mathcal{A}^-(\mathfrak{g}) \) and \( U_q^-(\mathfrak{g}) \), it is straightforward to see that \( \phi \) restricts down to the \( A_Q \)-algebra isomorphism

\[ (6.5) \quad \phi : Q \otimes U_\mathcal{A}(\mathfrak{g}) \rightarrow U_Q^-(\mathfrak{g}), \quad f_{il} \mapsto b_{il} \quad \text{for} \quad (i, l) \in I^\infty. \]

**Definition 6.8.** The \( A_Q \)-basis \( B_Q := \phi(B) \) of \( U_q(\mathfrak{g}) \) is called the **primitive canonical basis** of \( U_q(\mathfrak{g}) \).

For the irreducible highest weight module \( V(\lambda) \) with \( \lambda \in P^+ \), recall that \( V(\lambda)_Q := U_Q^- (\mathfrak{g}) v_\lambda \). Then \( B_Q^\lambda := \phi(B) v_\lambda \) is an \( A_Q \)-basis of \( V(\lambda)_Q \).

**Definition 6.9.** The \( A_Q \)-basis \( B_Q^\lambda \) of \( V(\lambda)_Q \) is called the **primitive canonical basis** of \( V(\lambda) \).

In later sections, we will prove that the primitive canonical bases \( B_Q \) and \( B_Q^\lambda \) coincide with the lower global bases \( B(\infty) \) and \( B(\lambda) \), respectively. Actually, \( \phi \) restricts down to the \( A \)-algebra isomorphism between \( U_\mathcal{A}(\mathfrak{g}) \) and \( U^-_Z(\mathfrak{g}) \). But to deal with the lower global bases, we need to consider \( Q \)-extensions, because the lower global bases are \( A_Q \)-bases for \( U_Q(\mathfrak{g}) \) and \( V(\lambda)_Q \).

### 6.3. Geometric bilinear forms

In this subsection, we recall some of basic parts of Lusztig’s theory on perverse sheaves.

Let \( X \) be an algebraic variety over \( \mathbb{C} \) and let \( G \) be a connected algebraic group. Let \( A, B \) be two \( G \)-equivariant semisimple complexes on \( X \) with \( G \)-action.
We choose

i) an integer \( m > 0 \),

ii) a smooth irreducible algebraic variety \( \Gamma \)
such that

a) \( G \) acts freely on \( \Gamma \),

b) \( H^k(\Gamma, \mathbb{C}) = 0 \) for \( k = 1, \ldots, m \).

Let \( G \) act diagonally on \( \Gamma \times X \) and set \( \Gamma X := G \setminus (\Gamma \times X) \). Consider the diagram

\[
X \xleftarrow{a} \Gamma \times X \xrightarrow{b} \Gamma X.
\]

Then \( \Gamma A, \Gamma B \) are well-defined semisimple complexes on \( \Gamma X \) and \( a^* A = b^* \Gamma A, a^* B = b^* \Gamma B \).

**Proposition 6.10.** [7, 19]

If \( m \) is sufficiently large, then we have

\[
\dim H^j_{\mathcal{C}}(\Gamma X, \Gamma A \otimes \Gamma B) = \dim H^j_{\mathcal{C}}(\Gamma X, \Gamma A[\dim G \setminus \Gamma] \otimes \Gamma B[\dim G \setminus \Gamma]).
\]

Let \( d_j(X, G; A, B) \) denote the equation (6.6). Then we obtain a series of properties of \( d_j(X, G; A, B) \).

**Lemma 6.11.** [7, 19]

(a) \( d_j(X, G; A, B) = d_j(X, G; B, A) \),

(b) \( d_j(X, G; A[m], B[n]) = d_{j+m+n}(X, G; A, B) \),

(c) \( d_j(X, G; A \oplus A', B) = d_j(X, G; A, B) + d_j(X, G; A', B) \).

**Lemma 6.12.** [7, 19]

(a) If \( A, B \) are perverse sheaves, then \( d_j(X, G; A, B) = 0 \) for \( j > 0 \).

(b) If \( A, B \) are simple perverse sheaves, then

\[
d_0(X, G; A, B) = \begin{cases} 
1, & \text{if } A \cong D(B), \\
0, & \text{otherwise}.
\end{cases}
\]

Let \( \alpha = \sum d_i \alpha_i \in R_+ \) and \( V = \oplus_{i \in I} V_i \) with \( \dim V = \alpha \). Let \( X = E(\alpha), G = G_\alpha \) and \( P, P' \) be simple perverse sheaves in \( \mathcal{P}_{-\alpha} \). We denote by \( B = [P], B' = [P'] \). Then we have \( B = [D(P)] = [P] = B \) and \( B' = B' \).

For \( A, B \in \mathcal{Q}_{-\alpha} \), we define

\[
(A, B)_G := \sum_{j \in \mathbb{Z}} d_j(E(\alpha), G_\alpha; A, B)q^{-j} \in \mathbb{Z}[[q]].
\]

**Proposition 6.13.** [7, 19]

(a) If \( P, P' \) are simple perverse sheaves, then we have

\[
(B, B')_G \in \delta_{B, B'} + q\mathbb{Z}_{\geq 0}[[q]].
\]

(b) \( (, )_G \) is a Hopf pairing, i.e.

\[
(B, B'B'')_G = (\delta(B), B' \otimes B'')_G,
\]

where \( \delta : \mathcal{K} \rightarrow \mathcal{K} \otimes \mathcal{K} \) is induced by the Res functor.

Since the map \( \Psi \) in (6.4) is an isomorphism of bialgebras, we can identify \( (, )_L \) with \( (, )_G \) by setting \((x, y)_L = (\Psi(x), \Psi(y))_G\).
For convenience, we will write \( B \in B \) for \( \Psi^{-1}(B) \). Thus we have

\[
(B, B')_L \in \delta_{B,B'} + qZ_{\geq 0}[[q]] \quad \text{for } B, B' \in B.
\]

In the sequel, we use \((\ , \ )G\) or \((\ , \ )L\) if there is no danger of confusion.

6.4. Bozec's results on perverse sheaves.

For \( x \in E(\alpha) \), we define \( V_i^\alpha = \bigoplus_{j \neq i} V_j \) and \( 3_i(x) = C(x)V_i^\alpha \). There exists a stratification \( E(\alpha) = \bigcup_{l \geq 0} E_{\alpha;i,l} \), where

\[
E_{\alpha;i,l} := \{ x \in E(\alpha) \mid \text{codim}_{\alpha} 3_i(x) = l \alpha_i \}.
\]

Set \( E_{\alpha;i,\geq 1} = \bigcup_{l \geq 1} E_{\alpha;i,l} \). Let \( P_{-\alpha;i,\geq 1} \) be the set of perverse sheaves in \( P_\alpha \) supported on \( E_{\alpha;i,\geq 1} \) and let \( P_{-\alpha;i,l} = P_{-\alpha;i,\geq 1} \setminus P_{-\alpha;i,\geq l+1} \).

**Proposition 6.14.** ([2, Proposition 4])

Let \((i, l) \in I^{\infty}\).

(a) For any simple perverse sheaf \( P \in P_{-\alpha;i,l} \), there exist a simple perverse sheaf \( P_0 \in P_{-\alpha + l \alpha_i;0} \) and a simple perverse sheaf \( P_{i,e} \in P_{-\alpha - i; l} \) such that

\[
[P_{i,e}|P_0] - [P] \in \bigoplus_{P' \in P_{-\alpha;i,\geq l+1}} \mathbb{A}[P'].
\]

(b) Conversely, for any simple perverse sheaf \( P_0 \in P_{-\alpha + l \alpha_i;0} \) and a simple perverse sheaf \( P_{i,e} \in P_{-\alpha - i; l} \), there exists a simple perverse sheaf \( P \in P_{-\alpha;i,l} \) such that

\[
[P_{i,e}|P_0] - [P] \in \bigoplus_{P' \in P_{-\alpha;i,\geq l+1}} \mathbb{A}[P'].
\]

Define

\[
B_{-\alpha;i,\geq 1} := \{ \Psi^{-1}([P]) \mid P \in P_{-\alpha;i,\geq 1} \},
\]

\[
B_{-\alpha;i,l} := B_{-\alpha;i,\geq 1} \setminus B_{-\alpha;i,\geq l+1} = \{ \Psi^{-1}([P]) \mid P \in P_{-\alpha;i,l} \}.
\]

It is straightforward to see that Proposition 6.14 can be rephrased as

**Corollary 6.15.** Let \((i, l) \in I^{\infty}\).

(a) For any \( B \in B_{-\alpha;i,l} \), there exist \( B_0 \in B_{-\alpha + l \alpha_i;0} \) and \( B_{i,e} \in B_{-\alpha - i; l} \) such that

\[
B_{i,e} B_0 - B \in \bigoplus_{B' \in B_{-\alpha;i,\geq l+1}} \mathbb{A} B'.
\]

(b) Conversely, for any \( B_0 \in B_{-\alpha + l \alpha_i;0} \) and \( B_{i,e} \in B_{-\alpha - i; l} \), there exists \( B \in B_{-\alpha;i,l} \) such that

\[
B_{i,e} B_0 - B \in \bigoplus_{B' \in B_{-\alpha;i,\geq l+1}} \mathbb{A} B'.
\]

Recall that the primitive canonical basis is defined by \( B_Q = \phi(B) \). Set

\[
(B_Q)_{-\alpha;i,\geq 1} = \phi(B_{-\alpha;i,\geq 1}), \quad (B_Q)_{-\alpha;i,l} = \phi(B_{-\alpha;i,l}).
\]

Actually, the above second equation can be rewritten by

\[
(B_Q)_{-\alpha;i,l} = (B_Q)_{-\alpha;i,\geq 1} \setminus (B_Q)_{-\alpha;i,\geq l+1}.
\]

Since the map \( \phi \) in (6.5) is an \( A_Q \)-algebra isomorphism, we obtain

**Corollary 6.16.** Let \((i, l) \in I^{\infty}\).
(a) For any \( \beta \in (B_{\mathbf{Q}})_{-\alpha; i, l} \), there exist \( \beta_0 \in (B_{\mathbf{Q}})_{-\alpha + l \alpha_i; 0} \) and \( \beta_{i,c} \in (B_{\mathbf{Q}})_{-l \alpha_i} (c \vdash l) \) such that
\[
\beta_{i,c} \beta_0 - \beta \in \bigoplus_{\beta' \in (B_{\mathbf{Q}})_{-\alpha_i, \geq i+1}} \mathbf{A} \beta'.
\]

(b) Conversely, for any \( \beta_0 \in (B_{\mathbf{Q}})_{-\alpha + l \alpha_i; 0} \) and \( \beta_{i,c} \in (B_{\mathbf{Q}})_{-l \alpha_i} (c \vdash l) \), there exists \( \beta \in (B_{\mathbf{Q}})_{-\alpha; i, l} \) such that
\[
\beta_{i,c} \beta_0 - \beta \in \bigoplus_{\beta' \in (B_{\mathbf{Q}})_{-\alpha_i, \geq i+1}} \mathbf{A} \beta'.
\]

6.5. Key lemmas on global bases.

Now we will prove some of key lemmas on lower global bases which will play important roles in later discussions.

**Proposition 6.17.** [16, Proposition 5.3.1]

Let \( i \in I^{re}, l \geq 0 \).

(a) For any \( b \in B(\infty)_{-\alpha; i, l} \), there exists \( b_0 \in B(\infty)_{-\alpha + l \alpha_i; 0} \) such that
\[
f_i^{(l)} G(b_0) - G(b) \in \bigoplus_{b' \in f_i^{(l+1)} B(\infty)} \mathbf{A} G(b').
\]

(b) For any \( b_0 \in B(\infty)_{-\alpha + l \alpha_i; 0} \), there exists \( b \in B(\infty)_{-\alpha; i, l} \) such that
\[
f_i^{(l)} G(b_0) - G(b) \in \bigoplus_{b' \in f_i^{(l+1)} B(\infty)} \mathbf{A} G(b').
\]

Let \( i \in I^{im} \) and \( l > 0 \). Define
\[
B(\infty)_{-\alpha; i, \geq l} := \bigcup_{c \vdash l} \tilde{f}_{i,c}(B(\infty)_{-\alpha}),
\]
\[
B(\infty)_{-\alpha; i, \leq l} := B(\infty)_{-\alpha; i, \geq l} \setminus B(\infty)_{-\alpha; i, \geq l+1}.
\]

**Lemma 6.18.** For any \( b \in B(\infty)_{-\alpha; i, l} \), there exist \( b_0 \in B(\infty)_{-\alpha + l \alpha_i; 0} \) and \( c \vdash l \) and \( C \in \mathbb{Z}_{>0} \) such that
\[
C b_{i,c} G(b_0) - G(b) \in \bigoplus_{b' \in B(\infty)_{-\alpha; i, \geq l+1}} \mathbf{A}_q G(b').
\]

Here, \( C = 1 \) for \( i \notin I^{iso} \).

**Proof.** Let \( b \in B(\infty)_{-\alpha; i, l} \). There exist \( b_0 \in B(\infty)_{-\alpha + l \alpha_i; 0} \) and \( c \vdash l \) such that \( \tilde{f}_{i,c} b_0 = b \); i.e.,
\[
\tilde{f}_{i,c} G(b_0) = G(b) \mod qL(\infty).
\]

If \( i \notin I^{iso} \), we have \( \tilde{f}_{i,c} = b_{i,c} \). Hence
\[
\tilde{f}_{i,c} G(b_0) = b_{i,c} G(b_0) = a_0 G(b) + \sum_{j=1}^r a_j G(b_j) \mod \bigoplus_{b' \in B(\infty)_{-\alpha; i, \geq l+1}} \mathbf{A}_q G(b'),
\]
where \( a_0, a_1, \ldots, a_r \in \mathbf{A}_q \), \( b_1, b_2, \ldots, b_r \in B(\infty)_{-\alpha; i, l} \).

Since \( b_{i,c} G(b_0) = b_{i,c} G(b_0) \), we must have
\[
(6.8) \quad \overline{a_0} = a_0, \quad \overline{a_1} = a_1, \quad \overline{a_r} = a_r.
\]

On the other hand, we have
\[
\tilde{f}_{i,c} G(b_0) = b_{i,c} G(b_0) = G(b) \mod qL(\infty).
\]
By taking $q \to 0$, we obtain

$$b = a_0 b + \sum_{j=1}^{r} a_j b_j \in L(\infty)/qL(\infty).$$

Thus $a_0 = 1, a_1 = \cdots a_r = 0 \mod qA_0$. Hence, by (6.8), we have $a_1 = \cdots = a_r = 0$, which proves our claim.

If $i \in I^{iso}$, then we have $\tilde{f}_{i,c} \neq b_{i,c}$. But since $b_0 \in B(\infty)_{-\alpha+i\alpha;i,0}$, we have $\tilde{e}_{ik} b_0 = 0$ for any $k > 0$. Thus $\tilde{e}_{ik} G(b_0) = 0 \mod qL(\infty)$ for any $k > 0$. Hence, we have

$$\tilde{f}_{i,c} G(b_0) = C b_{i,c} G(b_0) = a_0 G(b) + \sum_{j=1}^{r} a_j G(b_j) \mod \bigoplus_{\nu' \in B(\infty)_{-\alpha;i,\geq l+1}} A_{\nu'} G(b'),$$

where $a_0', a_1', \ldots, a_r' \in A_{\nu'}$, $b_1', \ldots, b_r' \in B(\infty)_{-\alpha;i,l}$.

By taking $q \to 0$, we obtain

$$\tilde{f}_{i,c} G(b_0) = C b_{i,c} G(b_0) = G(b) \mod qL(\infty).$$

Hence, we have $b = a_0' b + a_1' b_1' + \cdots + a_r' b_r' \in L(\infty)/qL(\infty)$. It follows that $a_0' = 1, a_1' = \cdots = a_r' = 0 \mod qA_0$. By (6.9), we get $a_0' = 1, a_1' = \cdots = a_r' = 0$, which proves our claim. \hfill \Box

**Lemma 6.19.** For any $b_0 \in B(\infty)_{-\alpha+i\alpha;i,0}$ and $c \vdash l$, there exist $b \in B(\infty)_{-\alpha;i,l}$ and a positive integer $C > 0$ such that

$$C b_{i,c} G(b_0) - G(b) \in \bigoplus_{\nu' \in B(\infty)_{-\alpha;i,\geq l+1}} A_{\nu'} G(b').$$

Here, $C = 1$ for $i \notin I^{iso}$.

**Proof.** Clearly, $\tilde{f}_{i,c} b_0 = b$ for some $b \in B(\infty)_{-\alpha;i,l}$. Hence $\tilde{f}_{i,c} G(b_0) = G(b) \mod qL(\infty)$.

If $i \notin I^{iso}$, then we have $\tilde{f}_{i,c} = b_{i,c}$. In this case, the conclusion naturally holds.

If $i \in I^{iso}$, then we have $\tilde{e}_{ik} G(b_0) = 0 \mod qL(\infty)$ for any $k > 0$, which yields

$$\tilde{f}_{i,c} G(b_0) = C b_{i,c} G(b_0) \mod qL(\infty)$$

for some $C \in Z_{>0}$.

Thus our claim follows naturally. \hfill \Box

7. Primitive canonical bases and global bases

In this section, we will show that the primitive canonical bases coincide with lower global bases.

7.1. Lusztig’s and Kashiwara’s bilinear forms.

We first compare Lusztig’s bilinear form and Kashiwara’s bilinear form defined in Proposition 2.1, (4.3) and (4.4).

**Lemma 7.1.** Let $b_k = b_{i_k l_k}$ $(1 \leq k \leq r)$. Then we have

$$\delta(b_1 \cdots b_r) = 1 \otimes (b_1 \cdots b_r) + b_1 \otimes (b_2 \cdots b_r) + \sum q^{-\|b_k\|} b_k \otimes (b_1 \cdots \hat{b_k} \cdots b_r) + \sum x_i \otimes y_i,$$
Lemma 7.3.

Corollary 7.2.

Proof. We will use induction on $r$. If $r = 1$, there is nothing to prove.

Assume that $r \geq 2$. Then we have

$$\delta(b_1 \cdots b_r) = \delta(b_1 \cdots b_{r-1}) \delta(b_r) = \delta(b_1 \cdots b_{r-1}) (1 \otimes b_r + b_r \otimes 1)$$

$$= 1 \otimes (b_1 \cdots b_{r-1} b_r) + b_1 \otimes (b_2 \cdots b_{r-1} b_r)$$

$$+ q^{-([b_r], \Sigma_{p=1}^{r-1} [b_p])} b_r \otimes (b_1 \cdots b_{r-1}) + q^{-([b_r], \Sigma_{p=1}^{r-1} [b_p])} b_1 b_r \otimes (b_2 \cdots b_{r-1})$$

$$+ \sum_{k=2}^{r-1} q^{-([b_k], \Sigma_{p=1}^{k-1} [b_p])} b_k \otimes (b_1 \cdots b_{k-1} b_k \cdots b_r)$$

$$+ \sum_{k=2}^{r-1} q^{-([b_k], \Sigma_{p=1}^{k-1} [b_p])} q^{-([b_r], \Sigma_{p=1, p \neq k}^{r-1} [b_p])} (b_k b_r \otimes b_1 \cdots b_{k-1} b_r)$$

$$+ \sum x_i \otimes y_i b_r + q^{-([y_i], [b_r])} x_i b_r \otimes y_i$$

$$= 1 \otimes (b_1 \cdots b_r) + b_1 \otimes (b_2 \cdots b_r)$$

$$+ \sum_{k=2}^{r} q^{-([b_k], \Sigma_{p=1}^{k-1} [b_p])} b_k \otimes (b_1 \cdots b_{k-1} b_k \cdots b_r) + \sum x_i' \otimes y_i'$$,

where $\deg x_i' \geq 2$ and our assertion follows. \hfill \Box

Corollary 7.2.

Let $a_k = b_{ik} l_k$ and $b_k = b_{jk} m_k$ ($1 \leq k \leq r$). Then we have

$$(a_1 \cdots a_r, b_1 \cdots b_r)_L = (a_1, b_1)_L (a_2 \cdots a_r, b_2 \cdots b_r)_L$$

$$+ \sum_{k=2}^{r-1} q^{-([b_k], \Sigma_{p=1}^{k-1} [b_p])} (a_1, b_k)_L (a_2 \cdots a_r, b_1 \cdots \hat{b}_k \cdots b_r)_L.$$ 

Proof. Our assertion follows immediately from Lemma 7.1.

$$(a_1 \cdots a_r, b_1 \cdots b_r)_L = (a_1 \otimes a_2 \cdots a_r, \delta(b_1 \cdots b_r))_L$$

$$= (a_1 \otimes a_2 \cdots a_r, 1 \otimes b_1 \cdots b_r)_L + (a_1 \otimes a_2 \cdots a_r, b_1 \otimes b_2 \cdots b_r)_L$$

$$+ (a_1 \otimes a_2 \cdots a_r, \sum_{k=2}^{r} q^{-([b_k], \Sigma_{p=1}^{k-1} [b_p])} b_k \otimes b_1 \cdots \hat{b}_k \cdots b_r)_L$$

$$+ (a_1 \otimes a_2 \cdots a_r, \sum x_i \otimes y_i)_L$$

$$= (a_1, b_1)_L (a_2 \cdots a_r, b_2 \cdots b_r)_L$$

$$+ \sum_{k=2}^{r} q^{-([b_k], \Sigma_{p=1}^{k-1} [b_p])} (a_1, b_k)_L (a_2 \cdots a_r, b_1 \cdots \hat{b}_k \cdots b_r)_L.$$ \hfill \Box

Next, we will show that Lusztig’s bilinear form and Kashiwara’s bilinear form are equivalent up to $q A_0$.

Lemma 7.3.
For $k = 1, 2, \ldots, r$, we have

$$
e^{r}_{i,l_1}(b_{j_1 m_1} \cdots b_{j_r m_r}) = \delta_{i,j_1} \delta_{l_1, m_1} (b_{j_2 m_2} \cdots b_{j_r m_r}) e^{r}_{i,l_1} + q_i^{-\sum_{p=1}^{r} l_1 m_p a_{i_1, p}} (b_{j_1 m_1} \cdots b_{j_r m_r}) e^{r}_{i,l_1} + \sum_{k=2}^{r} \delta_{i,j_k} \delta_{l_1, m_k} q_i^{-\sum_{p=1}^{k-1} l_1 m_p a_{i_1, p}} (b_{j_1 m_1} \cdots b_{j_k m_k} \cdots b_{j_r m_r}).$$

(7.1)

**Proof.** We will use induction on $r$. When $r = 1$, by (3.10), our assertion follows immediately.

Assume that $r \geq 2$. Using the induction hypothesis, we have

$$e^{r}_{i,l_1}(b_{j_1 m_1} \cdots b_{j_{r-1} m_{r-1}} b_{j_r m_r}) = (e^{r}_{i,l_1} b_{j_1 m_1} \cdots b_{j_{r-1} m_{r-1}}) b_{j_r m_r}$$

$$= \delta_{i,j_1} \delta_{l_1, m_1} (b_{j_2 m_2} \cdots b_{j_{r-1} m_{r-1}} b_{j_r m_r})$$

$$+ q_i^{-\sum_{p=1}^{r-1} l_1 m_p a_{i_1, p}} (b_{j_1 m_1} \cdots b_{j_{r-1} m_{r-1}}) (e^{r}_{i,l_1} b_{j_r m_r})$$

$$+ \sum_{k=2}^{r-1} \delta_{i,j_k} \delta_{l_1, m_k} q_i^{-\sum_{p=1}^{k-1} l_1 m_p a_{i_1, p}} (b_{j_1 m_1} \cdots b_{j_k m_k} \cdots b_{j_{r-1} m_{r-1}}) b_{j_r m_r}$$

$$= \delta_{i,j_1} \delta_{l_1, m_1} (b_{j_2 m_2} \cdots b_{j_{r-1} m_{r-1}} b_{j_r m_r})$$

$$+ q_i^{-\sum_{p=1}^{r-1} l_1 m_p a_{i_1, p}} \delta_{i,j_r} \delta_{l_1, m_r} (b_{j_1 m_1} \cdots b_{j_{r-1} m_{r-1}})$$

$$+ q_i^{-\sum_{p=1}^{r-1} l_1 m_p a_{i_1, p} + l_1 m_r a_{i_1, r}} (b_{j_1 m_1} \cdots b_{j_r m_r} e^{r}_{i,l_1})$$

$$+ \sum_{k=2}^{r-1} \delta_{i,j_k} \delta_{l_1, m_k} q_i^{-\sum_{p=1}^{k-1} l_1 m_p a_{i_1, p}} (b_{j_1 m_1} \cdots b_{j_k m_k} \cdots b_{j_{r-1} m_{r-1}} b_{j_r m_r})$$

$$= \delta_{i,j_1} \delta_{l_1, m_1} (b_{j_2 m_2} \cdots b_{j_{r-1} m_{r-1}} b_{j_r m_r}) + q_i^{-\sum_{p=1}^{r} l_1 m_p a_{i_1, p}} (b_{j_1 m_1} \cdots b_{j_r m_r} e^{r}_{i,l_1})$$

$$+ \sum_{k=2}^{r} \delta_{i,j_k} \delta_{l_1, m_k} q_i^{-\sum_{p=1}^{k-1} l_1 m_p a_{i_1, p}} (b_{j_1 m_1} \cdots b_{j_k m_k} \cdots b_{j_r m_r}),$$

as desired. \[ \square \]

**Corollary 7.4.**

Let $b_{i_k l_k}, b_{j_k m_k} \in U_q^{-}(\mathfrak{g})$ $(k = 1, 2, \ldots, r)$. Then Kashiwara’s bilinear form is given by

$$(b_{i_1 l_1} \cdots b_{i_r l_r}, b_{j_1 m_1} \cdots b_{j_r m_r})_K = \delta_{i_1,j_1} \delta_{l_1,m_1} (b_{i_2 l_2} \cdots b_{i_r l_r}, b_{j_2 m_2} \cdots b_{j_r m_r})_K$$

$$+ \sum_{k=2}^{r} \delta_{i_1,j_k} \delta_{l_1,m_k} q_i^{-\sum_{p=1}^{k-1} l_1 m_p a_{i_1, p}} (b_{i_2 l_2} \cdots b_{i_k l_k}, b_{j_1 m_1} \cdots b_{j_k m_k} \cdots b_{j_r m_r})_K.$$  

(7.2)

As we can see in the following proposition, Lusztig’s bilinear form and Kashiwara’s bilinear form are closely related.

**Proposition 7.5.**

Let $b_{i_k l_k}, b_{j_k m_k} \in U_q^{-}(\mathfrak{g})$ $(k = 1, 2, \ldots, r)$. Then we have

$$(b_{i_1 l_1} \cdots b_{i_r l_r}, b_{j_1 m_1} \cdots b_{j_r m_r})_L = \prod_{s=1}^{r} (1 - q_i^{e_{i_s}^2})^{-1} (b_{i_1 l_1} \cdots b_{i_r l_r}, b_{j_1 m_1} \cdots b_{j_r m_r})_K.$$  

Therefore, we have

$$(x, y)_L = (x, y)_K \mod q A_0$$  

for all $x, y \in U_q^{-}(\mathfrak{g}).$
Proof. We will use induction on $r$. If $r = 1$, our assertion follows from the definition of these bilinear forms.

Assume that $r \geq 2$. By Corollary 7.2 and induction hypothesis, we have

\[
(b_{i_1l_1} \cdots b_{i_rl_r}, b_{j_1m_1} \cdots b_{j_rm_r})_L
= (b_{i_1l_1}, b_{j_1m_1})_L (b_{i_2l_2} \cdots b_{i_rm_r})_L
+ \sum_{k=2}^{r} \left[ q^{-\sum_{p=1}^{k-1} (m_k \alpha_{j_k} \cdot m_p \alpha_{j_p})} (b_{i_1l_1}, b_{j_km_k})_L \right.
\times (b_{i_2l_2} \cdots b_{i_km_k}, b_{j_1m_1} \cdots b_{j_km_k} \cdots b_{j_rm_r})_L
\]

(7.3)

\[
= \delta_{i_1j_1} \delta_{i_1m_1} (1 - q_{i_1}^{2l_1})^{-1} \prod_{s=2}^{r} (1 - q_{i_s}^{2l_s})^{-1}
\times (b_{i_2l_2} \cdots b_{i_rm_r}, b_{j_1m_1} \cdots b_{j_rm_r})_K
+ \sum_{k=2}^{r} q^{-\sum_{p=1}^{k-1} (m_k \alpha_{j_k} \cdot m_p \alpha_{j_p})} \delta_{i_1j_k} \delta_{i_1m_k} (1 - q_{i_1}^{2l_1})^{-1}
\times (b_{i_2l_2} \cdots b_{i_km_k}, b_{j_1m_1} \cdots b_{j_km_k} \cdots b_{j_rm_r})_L.
\]

If $\delta_{i_1j_k} \delta_{i_1m_k} = 0$ for some $k \in \{2, \ldots, r\}$, then the corresponding summand of formula (7.3) will disappear. Therefore, we only need to consider the case of $\delta_{i_1j_k} \delta_{i_1m_k} = 1$. Then we must have $j_k = i_1$, $m_k = l_1$, which implies

\[
\sum_{k=2}^{r} q^{-\sum_{p=1}^{k-1} (m_k \alpha_{j_k} \cdot m_p \alpha_{j_p})} = \sum_{k=2}^{r} q^{-\sum_{p=1}^{k-1} l_1 m_p s_1 a_{1jp}} = \sum_{k=2}^{r} q^{-\sum_{p=1}^{k-1} l_1 m_p a_{1jp}}.
\]

It follows from Corollary 7.4 that

\[
(b_{i_1l_1} \cdots b_{i_rm_r}, b_{j_1m_1} \cdots b_{j_rm_r})_L
= \delta_{i_1j_1} \delta_{i_1m_1} \prod_{s=1}^{r} (1 - q_{i_s}^{2l_s})^{-1} (b_{i_2l_2} \cdots b_{i_rm_r}, b_{j_1m_1} \cdots b_{j_rm_r})_K
+ \sum_{k=2}^{r} q_{i_1}^{-\sum_{p=1}^{k-1} l_1 m_p a_{1jp}} \delta_{i_1j_k} \delta_{i_1m_k}
\times \prod_{s=1}^{r} (1 - q_{i_s}^{2l_s})^{-1} (b_{i_2l_2} \cdots b_{i_rm_r}, b_{j_1m_1} \cdots b_{j_km_k} \cdots b_{j_rm_r})_K
\]

\[
= \prod_{s=1}^{r} (1 - q_{i_s}^{2l_s})^{-1} (b_{i_1l_1} \cdots b_{i_rl_r}, b_{j_1m_1} \cdots b_{j_rm_r})_K
+ \prod_{s=1}^{r} (1 - q_{i_s}^{2l_s})^{-1} \sum_{k=2}^{r} q_{i_1}^{-\sum_{p=1}^{k-1} l_1 m_p a_{1jp}} \delta_{i_1j_k} \delta_{i_1m_k}
\times (b_{i_2l_2} \cdots b_{i_rm_r}, b_{j_1m_1} \cdots b_{j_km_k} \cdots b_{j_rm_r})_K
\]

\[
= \prod_{s=1}^{r} (1 - q_{i_s}^{2l_s})^{-1} (b_{i_1l_1} \cdots b_{i_rl_r}, b_{j_1m_1} \cdots b_{j_rm_r})_K.
\]
Proposition 7.6. For all \( x, y \in U_q(\mathfrak{g}) \), we have
\[ (\phi(x), \phi(y))_L = (x, y)_L. \]

Proof. It suffices to prove our assertion for monomials only. Let
\[ x = f_{i_1 l_1} \cdots f_{i_r l_r} \quad \text{and} \quad y = f_{j_1 m_1} \cdots f_{j_r m_r}. \]

By (2.1), we have
\[ \delta(y) = \sum_{a_1+b_1=m_1} \cdots \sum_{a_r+b_r=m_r} \left( \prod_{k=1}^r q_{(j_k)} \prod_{k=2}^r q^{-(a_k \alpha_j \sum_{p=1}^{k-1} b_p \alpha_j)}} \right) \times \left( \prod_{s=1}^r f_{j_s a_s} \right) \otimes \left( \prod_{t=1}^r f_{j_t b_t} \right). \]

It follows that
\[
(x, y)_L = (f_{i_1 l_1} \otimes f_{i_2 l_2} \cdots f_{i_r l_r}, \delta(y))_L \\
= \sum_{a_1+b_1=m_1} \cdots \sum_{a_r+b_r=m_r} \left( \prod_{k=1}^r q_{(j_k)} \prod_{k=2}^r q^{-(a_k \alpha_j \sum_{p=1}^{k-1} b_p \alpha_j)}} \right) \times \left( \prod_{s=1}^r f_{j_s a_s} \right) L (f_{i_1 l_1} \cdots f_{i_r l_r}) L.
\]

Let
\[ A = (f_{i_1 l_1} \cdots f_{i_r l_r}) L, \quad B = (f_{i_1 l_1} \cdots f_{i_r l_r}) L. \]

If \( AB \neq 0 \), then we have \( A \neq 0 \) and \( B \neq 0 \). Thus, there exists a positive integer \( k > 0 \) such that
(i) \( i_1 = j_k, l_1 = a_k \),
(ii) \( a_p = 0 \) for all \( p \neq k \).

Hence \( a_k = m_k, b_k = 0, b_p = m_p \) for all \( p \neq k \), which implies
\[ B = (f_{i_2 l_2} \cdots f_{i_r l_r} \cdots f_{i_1 l_1} \cdots \widetilde{f}_{j_k m_k} \cdots f_{j_r m_r}) L. \]

Note that \( \prod_{k=1}^r q_{(j_k)}^{-a_k b_k} = 1 \) because \( a_p = 0 \) for all \( p \neq k \) and \( b_k = 0 \).

Thus we have
\[
(x, y)_L = (f_{i_1 l_1}, f_{j_1 m_1}) L (f_{i_2 l_2} \cdots f_{i_r l_r}, f_{j_2 m_2} \cdots f_{j_r m_r}) L \\
+ \sum_{k=2}^r q^{-(m_k \alpha_j \sum_{p=1}^{k-1} m_p \alpha_j)}} (f_{i_1 l_1}, f_{j_k m_k}) L \times (f_{i_2 l_2} \cdots f_{i_r l_r}, f_{j_1 m_1} \cdots \widetilde{f}_{j_k m_k} \cdots f_{j_r m_r}) L.
\]

By induction hypothesis and Corollary 7.2, we obtain
\[
(x, y)_L = (b_{i_1 l_1}, b_{j_1 m_1}) L (b_{i_2 l_2} \cdots b_{i_r l_r}, b_{j_2 m_2} \cdots b_{j_r m_r}) L \\
+ \sum_{k=2}^r q^{-(m_k \alpha_j \sum_{p=1}^{k-1} m_p \alpha_j)}} (b_{i_1 l_1}, b_{j_k m_k}) L \times (b_{i_2 l_2} \cdots b_{i_r l_r}, b_{j_1 m_1} \cdots \widetilde{b}_{j_k m_k} \cdots b_{j_r m_r}) L \\
= (\phi(x), \phi(y))_L
\]
as desired. \( \square \)
To summarize, combining Proposition 7.5, Proposition 7.6 and Lemma 4.34, we obtain the following proposition.

**Proposition 7.7.**

Let $\mathbf{B}, \mathbf{B}_Q$ and $\mathbf{B}(\infty)$ be the canonical basis, primitive canonical basis and lower global basis of $U_q^-(\mathfrak{g})$, respectively. Then the following orthogonality statements hold.

(a) For all $B, B' \in \mathbf{B}$, $(B, B')_L = \delta_{B, B'} \mod q \mathbf{A}_0$.

(b) For all $\beta, \beta' \in \mathbf{B}_Q$, $(\beta, \beta')_L = (\beta, \beta')_K = \delta_{\beta, \beta'} \mod q \mathbf{A}_0$.

(c) For all $b, b' \in B(\infty)$, $(G(b), G(b'))_K = C \delta_{b, b'} \mod q \mathbf{A}_0$ for some $C \in \mathbb{Z}_{>0}$.

Similarly, we also have

**Proposition 7.8.**

Let $\mathbf{B}^\lambda, \mathbf{B}_Q^\lambda$ and $\mathbf{B}(\lambda)$ be the canonical basis, primitive canonical basis and lower global basis of $V(\lambda)$, respectively. Then the following orthogonality statements hold.

(a) For all $B, B' \in \mathbf{B}^\lambda$, $(B, B')_L = \delta_{B, B'} \mod q \mathbf{A}_0$.

(b) For all $\beta, \beta' \in \mathbf{B}_Q^\lambda$, $(\beta, \beta')_L = (\beta, \beta')_K = \delta_{\beta, \beta'} \mod q \mathbf{A}_0$.

(c) For all $b, b' \in B(\lambda)$, $(G(b), G(b'))_K = C \delta_{b, b'} \mod q \mathbf{A}_0$ for some $C \in \mathbb{Z}_{>0}$.

7.2. Grojnowski-Lusztig’s argument.

Now we are ready to prove that the primitive canonical bases coincide with lower global bases.

Let $\mathbf{B}_Q$ be the primitive canonical basis of $U_q^-(\mathfrak{g})$ and let $\beta$ be an element of $\mathbf{B}_Q$. Since the lower global basis $\mathbf{B}(\infty)$ is an $\mathbf{A}_Q$-basis of $U_q^-(\mathfrak{g})$, we may write

$$\beta = \sum_{b \in B(\infty), j \in \mathbb{Z}} a_{b,j} q^j G(b) \quad \text{for} \quad a_{b,j} \in \mathbb{Q}.$$

Since $(\ , \ )_L = (\ , \ )_K \mod q \mathbf{A}_0$, we will just use $(\ , \ )$ for both of them.

Let $j_0$ be the smallest integer such that $a_{b,j} \neq 0$ for some $b \in B(\infty)$. Since $(G(b), G(b')) = 0$ for $b \neq b'$, we have

$$(\beta, \beta) = \sum_{b \in B(\infty), j_0} a_{b,j_0}^2 q^{2j_0} (G(b), G(b)) + q^{2j_0+1} \mathbb{Q}[[q]],$$

which implies

$$(\beta, \beta) = \sum_{b \in B(\infty), j_0} a_{b,j_0}^2 q^{2j_0} (G(b), G(b)) \mod q \mathbf{A}_0.$$

By Proposition 7.7, we have $(\beta, \beta) = 1 \mod q \mathbf{A}_0$. Hence we must have

$$j_0 = 0, \quad a_{b,j} = 0 \quad \text{for} \quad j < 0, \quad b \in B(\infty).$$

Moreover, there exists $b \in B(\infty)$ such that

$$a_{b,0} = \pm 1, \quad (G(b), G(b)) = 1, \quad a_{b',0} = 0 \quad \text{for} \quad b' \neq b.$$

Hence $\beta - a_{b,0} G(b)$ is a linear combination of elements in $\mathbf{B}(\infty)$ with coefficients in $q \mathbf{A}_0$. Since $\beta - a_{b,0} G(b)$ is invariant under the bar involution, these coefficients are all 0, which implies $\beta = a_{b,0} G(b) = \pm G(b)$. That is, we may write

$$\beta = \epsilon_\beta G(b_\beta), \quad \text{where} \quad \epsilon_\beta = \pm 1.$$
Theorem 7.9.

The primitive canonical basis $B_Q$ coincides with the lower global basis $B(\infty)$.

Proof. We would like to show that $\epsilon_\beta = 1$ for all $\beta \in B_Q$.

Let $\beta \in (B_Q)_-\alpha$ for $\alpha \in R_+$. If $\alpha = 0$, our assertion is trivial. Hence we assume that $\alpha \neq 0$. Then there exist $i \in I$ and $l > 0$ such that $\beta \in (B_Q)_{-\alpha_i,i}$. (a) If $i \in I^m$, our assertion was proved in [7].

(b) If $i \in I^{im} \setminus I^{iso}$, by Corollary 6.16 (a), there exist $\beta_0 \in (B_Q)_{-\alpha_i,i,0}$ and $c \perp l$ such that

\[
\text{By induction hypothesis, we obtain } \epsilon_{\beta_0} = 1; \text{ i.e., } \beta_0 = G(b_0), \text{ where } b_0 = b_{\beta_0}. \text{ Note that } e'_{ik} \beta_0 = e'_{ik} G(b_0) = 0 \text{ for all } k > 0. \text{ Since } \tilde{f}_i = b_{d_i}, \text{ there exist } b \in B(\infty)_{-\alpha_i,i} \text{ and } c \perp l \text{ such that}
\]

\[
(7.5) \quad b_{i,c} G(b_0) - G(b) \in \bigoplus_{b' \in B(\infty)_{-\alpha_i,i,0}} A_Q G(b') \subset \sum_{|c'| \geq |l| + 1} b_{i,c} U^{-}(Q(g)).
\]

Comparing (7.4) and (7.5), we conclude $G(b) = \beta$, which yields $G(b) = \beta = \epsilon_\beta G(\beta) \in B(\infty)$. Since both $G(b)$ and $\epsilon_\beta G(\beta)$ belong to the lower global basis $B(\infty)$, we must have $\epsilon_\beta = 1$ and $b = b_\beta$.

(c) If $i \in I^{iso}$, by Corollary 6.16 (a), there exist $\beta_0 \in (B_Q)_{-\alpha_i,i,0}$ and $c \perp l$ such that

\[
(7.6) \quad b_{i,c} \beta_0 - \beta \in \bigoplus_{\beta' \in (B_Q)_{-\alpha_i,i,0}} A_Q \beta' \subset \sum_{|c'| \geq |l| + 1} b_{i,c} U^{-}(Q(g)).
\]

By induction hypothesis, we obtain $\epsilon_{\beta_0} = 1$; i.e., $\beta_0 = G(b_0)$, where $b_0 = b_{\beta_0}$.

By Lemma 6.19, there exist $b \in B(\infty)_{-\alpha_i,i}$ and a positive integer $C > 0$ such that

\[
(7.7) \quad C b_{i,c} G(b_0) - G(b) \in \bigoplus_{b' \in B(\infty)_{-\alpha_i,i,0}} A_Q G(b') \subset \sum_{|c'| \geq |l| + 1} b_{i,c} U^{-}(Q(g)).
\]

By (7.6) and (7.7), we obtain $G(b) = C \beta = C \epsilon_\beta G(\beta) \in B(\infty)$. Since both $G(b)$ and $C \epsilon_\beta G(\beta)$ are elements of $B(\infty)$, we must have $C \epsilon_\beta = 1$. Since $C$ is a positive integer and $\epsilon_\beta = \pm 1$, we must have $C = \epsilon_\beta = 1$. \qed

As an immediate consequence, we obtain

Corollary 7.10.

The primitive canonical basis $B_Q^\lambda$ coincides with the lower global basis $B(\lambda)$.

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