POST-LAST EXIT TIME PROCESS AND ITS APPLICATION TO LOSS-GIVEN-DEFAULT DISTRIBUTION

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ABSTRACT. We study a linear diffusion process after its last exit time from a certain regular point. Rather than treating the process as newly born at the last exit time, we view the whole path and separate the original process before and after the last exit time. This enables us not only to identify the transition semigroup, boundary behavior, entrance law, and reverse of the post-last exit time process, but also to establish a financial model for estimating the loss-given-default distribution of corporate debt (an all-time important open problem).

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1. INTRODUCTION

Let $X = \{X_t, t \geq 0\}$ be a linear diffusion. We study the process after its last exit time from a certain regular point: $(X_t)_{t > L_\alpha}$ where $L_\alpha$ is the last exit time from state $\{\alpha\}$. We call this $(X_t)_{t > L_\alpha}$ the post-last exit time process. The last exit time is an important subject in the probability literature due to being closely related to the concepts of transience/recurrence, Doob's h-transform, time-reversed processes, and the Martin boundary theory. For these interrelated topics, we refer the reader to Doob (1957), Nagasawa (1964), Kunita and Watanabe (1966), Salminen (1984), Rogers and Williams (1994a), Borodin and Salminen (2002), Chung and Walsh (2004), Revuz and Yor (2005) as well as the literature referenced therein.

While the last exit time is not a stopping time, it has been utilized widely in financial applications as discussed in Nikeghbali and Platen (2013). These applications include credit risk analysis (Egami and Kevkhishvili (2020)), valuation of defaultable claims (Elliott et al. (2000), Jeanblanc and Rutkowski (2000), Coculescu and Nikeghbali (2012), Jeanblanc et al. (2009)), option valuation (Profeta et al. (2010), Cheridito et al. (2012)), and insider trading (Imkeller (2002)).

The contribution of this paper is twofold: mathematical and practical. We study the post-last exit time process in Section 2. It is, to our knowledge, first studied by Meyer et al. (1972) where the process $(X_{L_\alpha+t}, \mathcal{F}_{L_\alpha+t})$ is proved to be a strong Markov process and its transition semigroup is identified. While it is treated as a newborn process at time $L_\alpha$ in Meyer et al. (1972), we treat the post-last exit time process \textit{in a continuum} of the original process.

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we look at the entire path. This approach allows us to identify and prove, as to the post-last exit time process, its transition semigroup (Proposition 2.1), the state \{\alpha\} being an entrance boundary (Proposition 2.2), its entrance law (Proposition 2.3), and its reversed process (Proposition 2.4). These results are summarized in Table 1, which is rather a comprehensive study.

As a practical contribution, we have a significant financial application: estimating loss-given-default (LGD, hereafter) distribution of corporate debt. This subject has long been an open issue. The model is set up in Section 3 and an empirical study is done in Section 4. The essential points are as follows. While the firm-value approach for credit risk analysis proposed in Merton (1974) is a fundamental method (as is the intensity-based approach) in estimating the default probability, one has some difficulties in calculating the LGD under this approach. For example, if the firm’s default is defined as the first hitting time of the firm value to a certain point, say \(c\), then the value upon default is necessarily equal to the deterministic number \(c\). To overcome this discrepancy, we consider the *last* exit time of the firm value (measured by its leverage ratio) to another threshold level \(\alpha > c\). This approach has a useful feature. Since the last exit time is not a stopping time, it is convenient when modeling the reality: during the period while the observer does not know whether the firm value would recover to \(\alpha\), a default occurs as a surprise. The post-last exit time process well captures this reality and is suitable for estimating LGD distribution. In fact, we compute LGD distribution through the intensity-based approach applied to the post-last exit time process. In this modeling, due to the mathematical contribution mentioned above, we can deal with the firm value process consistently and continuously: the firm-value approach before time \(L_\alpha\) and intensity-based approach after \(L_\alpha\).

With minimal and standard assumptions, the model to be presented allows us to obtain the LGD distribution implied in the current CDS market: we compute (1) the model-driven LGD distribution and (2) CDS spreads implied by the estimated LGD distribution. When checking the consistency of the model-implied LGD distribution with the market quotes, we compare “spreads per 1% loss given default” because the market CDS spreads are quoted under the assumption that the LGD is fixed at 60% for all firms. In the end, we emphasize that one should not confuse our loss right upon the default with the final loss after liquidation and/or bankruptcy procedure. For our analysis, the risk-free rate was obtained from the website of the U.S. Department of the Treasury. The remaining data was obtained from Refinitiv Eikon. We refer the interested reader to an extended literature review of LGD distribution in Appendix A.3.

2. A STUDY OF POST-LAST EXIT TIME PROCESS

Fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\). Let \(X = \{\omega(t), t \geq 0; \mathbb{P}_x\}\) be a linear regular canonical diffusion starting at \(x \in \mathbb{R}\): that is,

\[
\omega(t) = X_t(\omega), \quad t \geq 0.
\]

Its state space is given by \(\mathcal{S} = (\ell, r) \subset \mathbb{R}\). Let the last exit time from a regular point \(\alpha \in \mathcal{S}\) be denoted by

\[
L_\alpha := \sup\{t : \omega(t) = \alpha\}
\]

with \(\sup \emptyset = 0\). The last exit time is an example of co-optional time satisfying

\[
L_\alpha \circ \theta_t = (L_\alpha - t)^+, \quad t \geq 0,
\] (2.1)
where \( \{ \theta, t \geq 0 \} \) is the shift operator in the following manner. For each \( t, \theta \), maps \( \Omega \) into \( \Omega \) such that
\[
\forall t : (X_t \circ \theta_t)(\omega) = X_t(\theta_t \omega) = X_{t+s}(\omega). \tag{2.2}
\]

2.1. **Elements of Linear Diffusion.** We refer the reader to Chapter II in Borodin and Salminen (2002) for the basic facts regarding linear diffusions. The scale function and the speed measure of \( X \) are denoted by \( s(\cdot) \) and \( m(\cdot) \), respectively. We assume that \( s \) and \( m \) are absolutely continuous with respect to the Lebesgue measure and have smooth derivatives:
\[
m(dx) = m(x)dx \quad \text{and} \quad s(x) = \int^x s'(y)dy,
\]
where \( m \) and \( s' \) are continuous and positive. If \( X \) hits \( \ell \) or \( r \), it is killed and transferred immediately to the cemetery \( \Delta \notin \mathcal{F} \). The lifetime of \( X \) is given by
\[
\zeta = \inf \{ t : \omega(t-) = \ell \text{ or } r \}.
\]
We assume that \( X \) is transient. In this paper, we call a diffusion transient (following Salminen (1984)) if
\[
\forall x \in \mathcal{F}, A \text{ is compact in } \mathcal{B}(\mathcal{F}) : \mathbb{P}_x (L_\alpha < \infty) = 1.
\]
The transience in the above sense is equivalent to one or both of the boundaries being attracting; that is, \( s(\ell) > -\infty \) and/or \( s(r) < +\infty \). It is possible to obtain transient diffusion from originally recurrent diffusion by including an absorbing boundary in its state space. Such approach is often employed in engineering, finance, and other scientific fields when handling real-life problems. Thus, transient diffusions have a wide range of applications. To obtain concrete results, we set a specific assumption:

**Assumption 1.**
\[
s(\ell) > -\infty \quad \text{and} \quad s(r) = +\infty.
\]

Then, it holds that
\[
\mathbb{P}_x \left( \lim_{t \to \zeta} \omega(t) = \ell \right) = 1, \quad \forall x \in \mathcal{F}.
\]
The infinitesimal drift and diffusion parameters are given by \( \mu(\cdot) \) and \( \sigma(\cdot) \), respectively. We let \( \mathcal{I} \) denote the second-order differential operator
\[
\mathcal{I} f(x) = \frac{1}{2} \sigma^2(x)f''(x) + \mu(x)f'(x), \quad x \in \mathcal{I}.
\]
For each \( t \geq 0 \), the transition function is given by \( P_t : \mathcal{I} \times \mathcal{B}(\mathcal{I}) \mapsto [0, 1] \) such that for all \( t, s \geq 0 \)
\[
\mathbb{P}(X_{t+s} \in A | \mathcal{F}_s) = P_t(X_s, A), \quad \forall A \in \mathcal{B}(\mathcal{I}) \quad \text{a.s.}
\]
For each \( t > 0 \) and \( x \in \mathcal{I} \), \( P_t(x, \cdot) : A \mapsto P_t(x, A) \) is absolutely continuous with respect to the speed measure \( m \):
\[
P_t(x, A) = \int_A p(t;x,y)m(dy), \quad A \in \mathcal{B}(\mathcal{I}).
\]
The transition density \( p \) may be taken to be jointly continuous in all variables and symmetric such that \( p(t;x,y) = p(t;y,x) \).

The Laplace transform of the hitting time \( H_z := \inf \{ t : \omega(t) = z \} \) for \( z \in \mathcal{I} \) is given by
\[
\mathbb{E}_x \left[ e^{-qH_z} \right] = \begin{cases} 
\frac{\psi_0(x)}{\psi_0(z)}, & x \leq z, \\
\frac{\psi_0(x)}{\psi_0(z)}, & x \geq z,
\end{cases}
\tag{2.3}
\]
Then, we have is to identify the transition semigroup of the post-last exit time process and for this purpose, let us define

\[
G_q(x,y) := \begin{cases} 
\frac{\psi_q(x)\phi_q(y)}{w_q}, & x \leq y, \\
\frac{\psi_q(y)\phi_q(x)}{w_q}, & x \geq y
\end{cases}
\]  

(2.4)

with the Wronskian \( w_q := \psi_q^+(x)\phi_q(x) - \psi_q(x)\phi_q^+(x) = \psi_q^-(x)\phi_q(x) - \psi_q(x)\phi_q^-(x) \). We have \( G_q(x,y) = \int_0^\infty e^{-qf} p(t;x,y)dt \) for \( x,y \in \mathcal{S} \).

2.2. Post-last exit time process. We are interested in the distribution of a transient diffusion \( X \) after the last exit time from \( \alpha \); i.e., \( \mathbb{P}_x(X_t \in dy, L_\alpha < t) \). Recall that for transient diffusion \( L_\alpha < \infty \) a.s. Let us define

\[
h_\alpha(x) := \mathbb{P}_x(L_\alpha = 0)
\]

and a transform of the transition density of \( X \) by

\[
p^{h_\alpha}(t;x,y) := \frac{h_\alpha(y)}{h_\alpha(x)} p(t;x,y), \quad h_\alpha(x) \neq 0
\]

(2.5)

with respect to the speed measure \( m(dy) \).

Proposition 2.1. Under Assumption 1, the transition density (with respect to the speed measure \( m(dy) \)) of the post-last exit time process \( (X_t)_{t \geq L_\alpha} \) is given by (2.5). More precisely,

\[
p^{h_\alpha}(t-u;z,y) = \frac{h_\alpha(y)}{h_\alpha(z)} p(t-u;z,y), \quad y < z < \alpha, \quad u \in (L_\alpha, t)
\]

(2.6)

for \( u \) arbitrarily close to \( L_\alpha \); \( z \) is arbitrarily close to \( \alpha \).

Proof. Let us fix any two points \( x,y \in \mathcal{S} \setminus \{\alpha\} \) such that \( y < \alpha \) and consider \( \mathbb{P}_x(X_t \in dy, L_\alpha < t) \) for an arbitrary \( t > 0 \). By Bayes’ rule, we have

\[
\mathbb{P}_x(X_t \in dy, L_\alpha < t) = \mathbb{P}_x(L_\alpha < t | X_t \in dy) \mathbb{P}_x(X_t \in dy)
\]

\[
= \mathbb{P}_x(L_\alpha \circ \theta_t = 0 | X_t \in dy) \mathbb{P}_x(X_t \in dy)
\]

\[
= \mathbb{P}_x(L_\alpha = 0) \mathbb{P}_x(T_\alpha \geq t) m(dy)
\]

(2.7)

where the second line is due to (2.1) and the third line is due to the Markov property at time \( t \). Our objective here is to identify the transition semigroup of the post-last exit time process and for this purpose, let us define

\[
T_\alpha := \inf\{s > L_\alpha : X_s = \alpha\}
\]

(2.8)

Then, we have

\[
\mathbb{P}_x(X_t \in dy, L_\alpha < t) = \mathbb{P}_y(L_\alpha = 0) \mathbb{E}_x[p(t;x,y) \mathbb{1}_{\{T_\alpha < t\}} + p(t;x,y) \mathbb{1}_{\{T_\alpha \geq t\}}] m(dy)
\]

\[
= \mathbb{P}_y(L_\alpha = 0) \mathbb{E}_x[p(t;x,y) \mathbb{1}_{\{T_\alpha < t\}}] m(dy) + \mathbb{P}_y(L_\alpha = 0) p(t;x,y) \mathbb{P}_x(T_\alpha \geq t) m(dy)
\]

This result with (2.7) yields

\[
\mathbb{P}_x(X_t \in dy, L_\alpha < t) \mathbb{P}_x(T_\alpha < t) = \mathbb{P}_y(L_\alpha = 0) \mathbb{E}_x[p(t;x,y) \mathbb{1}_{\{T_\alpha < t\}}] m(dy).
\]

(2.9)

Now we rewrite (2.9) with a view from time \( T_\alpha \) to \( t \) instead of that from time zero to \( t \). Let us first undertake the second term of the right-hand side by observing that \( p(t;x,y) = \int_{\mathcal{S}} p(s;x,u) p(t-s;u,y) m(du) \) for \( 0 < s < t \).
For an arbitrary random time $T < \infty$ a.s., define a random measure $\tilde{\mu}(T; u, v)$ such that $\mathbb{E}_x[\tilde{\mu}(T; u, v)m(dy)] = \mathbb{P}_x(X_T \in dy)$ for $u, v \in \mathcal{F}$. Note that $\tilde{\mu}(T; x, v)m(dy) = 0$ a.s. when $v \neq z$. This is because by Fubini’s theorem

$$E_x\left[\int_{\mathcal{F}} \tilde{\mu}(T; x, y)m(dy)\right] = \int_{\mathcal{F}} \mathbb{P}_x(X_T \in dy) = \int_{\mathcal{F}} \int_{\mathcal{F}} \mathbb{P}_x(x \in dy, T_z \in ds)$$

$$= \int_0^\infty \mathbb{P}_x(x \in dz, T_z \in ds) = \mathbb{P}_x(X_T \in dz) = \mathbb{E}_x[\tilde{\mu}(T; x, v)m(dy)\delta_z(dy)]$$

and $\mathbb{P}_x(X_T \in dz) = 1$ by the definition of $T_z$. Hence the random measure $\tilde{\mu}(T; x, v)m(dy)$ has a point mass at $z$ almost surely. Then, together with (2.5), the second term on the right-hand side of (2.9) becomes

$$\mathbb{E}_x[p(t; x, y)\mathbbm{1}_{\{T_z < t\}}]m(dy) = \mathbb{E}_x\left[\int_0^t \int_{\mathcal{F}} p(s; x, u)p(t-s; u, y)m(du)\mathbbm{1}_{\{T_z < t\}}\right]m(dy)$$

$$= \mathbb{E}_x\left[\tilde{\mu}(T; x, y)\mathbbm{1}_{\{T_z < t\}}\right]m(dy)$$

$$= \mathbb{E}_x\left[\tilde{\mu}(T; x, y)\mathbbm{1}_{\{T_z < t\}}\right]m(dy)$$

$$= \mathbb{E}_x\left[\frac{h_{\alpha}(X_T)}{h_\alpha(y)} \tilde{\mu}(T; x, y)\mathbbm{1}_{\{T_z < t\}}\right]m(dy)$$

$$= \mathbb{E}_x\left[\frac{1}{h_\alpha(y)} \tilde{\mu}(T; x, y)\mathbbm{1}_{\{T_z < t\}}\right]m(dy)$$

where $\tilde{\mu}(T; u, v)m(dy) := \frac{h_{\alpha}(u)}{h_\alpha(v)} \tilde{\mu}(T; u, v)m(dy)$ for an arbitrary random time $T < \infty$ a.s. and $u < \alpha$ similarly to (2.5). Note that in the last equality we used $h_{\alpha}(X_T) = \mathbb{P}_x(X_T(L_\alpha = 0) = 1$ by the definition of $T_z$. This result simplifies (2.9) to

$$\mathbb{P}_x(X_t \in dy, L_\alpha < t)\mathbb{P}_x(T_z < t) = \int_0^t \int_\mathcal{F} p^{h_\alpha}(t-s; z^*, y)\mathbb{P}_x(T_z \in dz)m(dy)$$

(2.10)

where we use $z^*$ in $p^{h_\alpha}(\cdot, \cdot, \cdot)$ rather than a mere $z$ to emphasize and recall that this is the point attained by $X$ at the time $T_z$ defined in (2.8): $z^* := X_{T_z} = z$.

We have so far shown the following: in our process of rewriting the second term on the right-hand side of (2.9), we are able to use (2.5) to obtain (2.10). It follows that the remaining task is to rewrite the left-hand side of (2.10) with a view from time $T_z$. The corresponding expression is obtained by looking at the shifted path $(X_{\theta T})(\omega) = X_u(\theta T, \omega) = X_{T_z+u}(\omega)$. Recall the shift operator in (2.2). We have

$$\mathbb{P}_x(X_t \in dy, L_\alpha < t) = \mathbb{P}_x(X_t \in dy, T_z < t) = \mathbb{E}_x[\mathbb{E}_x[\mathbbm{1}_{\{X_t \in dy, T_z < t\}} | \mathcal{F}_T] | \mathcal{F}_T]$$

$$= \mathbb{E}_x[\mathbbm{1}_{\{T_z < t\}} \mathbb{E}_x[X_T - T_z \circ \theta_T \in dy | \mathcal{F}_T]] = \mathbb{E}_x[X_T - T_z \circ \theta_T \in dy, T_z < t].$$

(2.11)

Finally, division by $\mathbb{P}_x(T_z < t)$ together with (2.11) transforms (2.10) into

$$\mathbb{P}_x(X_t - T_z \circ \theta_T \in dy, T_z < t) = \left(\int_0^t \int_\mathcal{F} p^{h_\alpha}(t-s; z^*, y)\mathbb{P}_x(T_z \in dz, T_z < t)\right)\mathbb{P}_x(T_z < t)$$

$$= \left(\int_0^t \int_\mathcal{F} p^{h_\alpha}(t-s; z^*, y)\mathbb{P}_x(T_z \in dz | T_z < t)\right) \mathbb{P}_x(T_z < t)$$

(2.12)

where the randomness comes only from $T_z$. But again, by the path continuity, we can make $z$ as close to $\alpha$ as we please. Therefore, for any time $L_\alpha + s, s > 0$, the post- last exit time process is governed by (2.6) for $s$ arbitrarily close to zero.
Although \( h_\alpha(x) \) is not an excessive function, the transform \((2.6)\) has the same form as if it were an excessive function. Therefore, we can compute the infinitesimal drift and diffusion parameters of the transform \((2.5)\) in the same way as if it is an \( h \)-transform. But there is a caveat that \( x \neq \alpha \).

Suppose that \( Y \) is a diffusion (does not have to be transient) and let \( h \) be an excessive function. Let us denote the infinitesimal drift and diffusion parameters of the \( h \)-transform by \( \mu^h(\cdot) \) and \( \sigma^h(\cdot) \), respectively. The following result slightly generalizes the argument in Section 15.9 of Karlin and Taylor (1981).

**Lemma 2.1.** Assume \( h \) is an excessive function and its derivative \( h'(y) \) exists for \( y \in \mathcal{F} \). The infinitesimal drift \( \mu^h(y) \) and \( \sigma^h(y) \) of the \( h \)-transform of \( Y \) are

\[
\mu^h(y) = \mu(y) + \frac{h'(y)}{h(y)} \sigma^2(y) \quad \text{and} \quad \sigma^h(y) = \sigma(y).
\]  

(2.13)

**Proof.** See Appendix A.1. \( \square \)

Turning to our case,

\[
h_\alpha(x) = \mathbb{P}_x(L_\alpha = 0) = \frac{s(\alpha) - s(x)}{s(\alpha) - s(\ell)}, \quad x \leq \alpha.
\]  

(2.14)

We have \( \frac{h'_\alpha(x)}{h_\alpha(x)} = -\frac{s'(x)}{s(\alpha) - s(x)} \) so that

\[
\mu^\alpha(x) = \mu(x) - \frac{s'(x)}{s(\alpha) - s(x)} \sigma^2(x),
\]

(2.15)

\[
\sigma^\alpha(x) = \sigma(x),
\]

where we write \( \mu^\alpha \) and \( \sigma^\alpha \) instead of \( \mu^{h_\alpha} \) and \( \sigma^{h_\alpha} \) for simplicity.

**Proposition 2.2.** The point \( \alpha \) is an entrance boundary for the post-exit time process \( X^\alpha(t) := \{ \omega(t) : L_\alpha < T \wedge L_\alpha \} \).

**Proof.** For the post-exit time process, we need only to consider a point \( y < \alpha \). In view of \((2.14), (2.5)\) becomes

\[
p^{h_\alpha}(t;x,y) = \frac{s(\alpha) - s(y)}{s(\alpha) - s(x)} p(t;x,y).
\]

This means that the post-last exit time process is obtained equivalently via the transform \( g(\cdot) := s(\alpha) - s(\cdot) \). Call this transform also \( X^{h_\alpha} \) and its scale function and speed measure \( s^\circ \) and \( m^\circ \), respectively. By the general result for the transform of this kind (see Appendix A.1), we have

\[
m^\circ(dy) = g^2(y)m(dy) \quad \text{and} \quad (s^\circ)'(y) = \frac{1}{g^2(y)} s'(y),
\]

which gives us \( s^\circ(y) = \frac{1}{s(\alpha) - s(y)} \). Then, \( s^\circ'(\alpha) - s^\circ'(y) = \infty \) for \( y < \alpha \) and

\[
\int_y^\alpha (s^\circ(\eta) - s^\circ(y)) m^\circ(d\eta) = \int_y^\alpha \left( \frac{1}{s(\alpha) - s(\eta)} - \frac{1}{s(\alpha) - s(y)} \right) (s(\alpha) - s(\eta))^2 m(d\eta)
\]

\[
< \int_y^\alpha \frac{1}{s(\alpha) - s(\eta)} (s(\alpha) - s(\eta))^2 m(d\eta)
\]

\[
= \int_y^\alpha (s(\alpha) - s(\eta)) m(d\eta) < \infty.
\]
The last finiteness result is due to \( \alpha \) being a regular point for \( X \). This proves that \( \alpha \) is an entrance boundary for the \( g \)-transform, which is equivalent to the post-exit time process \( X^g_{\alpha} \) (see [Karlin and Taylor (1981), Chap.15, Table 6.2]).

2.3. **Excursion-theoretic view.** We can view \( \{X_t, t > L_\alpha\} \) as an excursion from \( \alpha \) of infinite length. To pursue this point of view, we review some of the excursion theory based on Getoor (1979) and Rogers and Williams (1994b), in particular its Sections VI. 50, 54 and 55.

Let us return to the original process \( X \) and consider an excursion from a regular point \( \alpha \). Let the local time at \( y \in \mathcal{I} \) be \( l_t^\alpha, t \geq 0 \). Set \( l_t = l_t^\alpha \) and define the right inverse \( \rho(t, \alpha) := \inf\{s : l_s > t\} \). It is well known that \( \rho \) is a right-continuous and nondecreasing subordinator under \( \mathbb{P}_\alpha \) and hence

\[
E_\alpha\left[e^{-q\rho(t,\alpha)}\right] = e^{-t g(q)}
\]  
(2.16)

where

\[
g(q) = m\{\alpha\} q + \int_{(0,\infty)} (1 - e^{-qs}) \nu(ds)
\]  
(2.17)

with \( m\{\alpha\} = 0 \) in our case and \( \nu \) being a measure on \((0,\infty)\) satisfying \( \int (1 \wedge s) \nu(ds) < \infty \).

By using the tail distribution

\[c(u) := \nu((u, \infty)],\]

the Laplace exponent is rewritten as

\[
g(q) = m\{\alpha\} q + q \int_0^\infty e^{-qs} c(s) ds.
\]  
(2.18)

Essentially, \( \rho(t, \alpha) \) corresponds to the length of excursion from point \( \alpha \) that occurs at local time \( t \), so that \( \lim_{u \to \infty} c(u) = \nu(\{+\infty\}) \) corresponds to an excursion of infinite length.

We define

\[r_q(x, y) := E_x \left( \int_0^\infty e^{-qs} dl_t^x \right), \quad x, y \in \mathcal{I}, q > 0.\]

Then, by a change of variable and (2.16),

\[r_q(\alpha, \alpha) = E_\alpha \left( \int_0^\infty e^{-qs} dl_t \right) = E_\alpha \left( \int_0^\infty e^{-q\rho(t,\alpha)} dt \right) = \frac{1}{g(q)}.\]  
(2.19)

But if we define the \( q \)-potential \( U^q f(x) := E_x \left( \int_0^\infty e^{-qs} f(X_t) dt \right) \) of a bounded function \( f \) with \( x \in \mathcal{I} \), as is written in the proof of Theorem V.50.7 in Rogers and Williams (1994b),

\[U^q f(x) = \int_\mathcal{I} m(dy) f(y) E_x \left( \int_0^\infty e^{-qs} dl_t^x \right)\]

which is also written as

\[U^q f(x) = \int_\mathcal{I} m(dy) f(y) G_q(x, y)\]

by using \( G_q(x, y) = \int_0^\infty e^{-qs} p(t;x,y) dt \). By comparison, \( G_q(x, y) = E_x \left( \int_0^\infty e^{-qs} dl_t^x \right) \) a.e. \( y \). But by the right continuity in \( y \), we have the equality for every \( y \in \mathcal{I} \). We obtain from (2.19) that \( g(q) = \frac{1}{r_q(\alpha, \alpha)} \) and hence

\[E_\alpha (e^{-q\rho(t,\alpha)}) = \exp \left( -\frac{t}{G_q(\alpha, \alpha)} \right).\]
Remark 2.1. Let $X$ be a linear diffusion with regular point $\alpha \in \mathcal{S}$. We consider the local time at $\alpha$ and excursions from this point. With $g(q) = \frac{1}{G_0(y, \alpha)}$, (2.18) becomes

$$
\int_0^\infty e^{-qs} c(s) ds = \frac{1}{q G_0(y, \alpha)},
$$

which can be used to obtain $c(u) = v([u, \infty))$ through the inverse Laplace transform. This $c(u)$ corresponds to the rate at which the first excursion of length greater than $u$ occurs. To consider an excursion of infinite length, which is closely related to a post-last exit time process discussed in Section 2.2, let us take $q \downarrow 0$ in (2.16) to find

$$
\mathbb{P}_\alpha(p(t, \alpha) < \infty) = e^{-tv([0, \infty))}
$$

by (2.17). Moreover, from $g(q) = \frac{1}{G_0(y, \alpha)}$, we have

$$
\nu(\{+\infty\}) = \lim_{q \downarrow 0} \frac{1}{G_0(y, \alpha)} := \frac{1}{G_0(y, \alpha)},
$$

from which we define

$$
A(x) := \mathbb{P}_\alpha(l_{IA} \leq x) = 1 - e^{-\frac{x}{G_0(y, \alpha)}}.
$$

Hence with 95% probability, $l_{IA}$ occurs before $A^{-1}(0.95)$. \hfill \Box

Note that the quantity $G_0(y, z)$ for $y, z \in \mathcal{S}$ has a representation in terms of the scale function:

$$
G_0(y, z) = \int_0^\infty p(t; y, z) dt = \begin{cases} 
\lim_{a \uparrow [b, \ell]\cap [r, \infty]} \frac{(s(y) - s(a))(s(b) - s(z))}{s(b) - s(a)}, & \ell < y \leq z < r, \\
\lim_{a \uparrow [b, \ell]\cap [r, \infty]} \frac{(s(z) - s(a))(s(b) - s(y))}{s(b) - s(a)}, & \ell < z \leq y < r.
\end{cases}
$$

Another important element of the excursion is the entrance law of the excursion measure,

$$
n^\alpha_t(\cdot) = n^\alpha(\{\varepsilon : \varepsilon(t) \in \cdot, t < \zeta'(\varepsilon)\}),
$$

where $n^\alpha$ is the characteristic (or Itô) measure of the excursion process, $\varepsilon(t)$ is a generic excursion process from point $\alpha$, and $\zeta'$ is lifetime of the excursion. The following is a simple but, to the best of our knowledge, a general result since it does not resort to the symmetry of transition densities (see Csáki et al. (1987)):

Proposition 2.3. Let $X$ be a linear diffusion with regular points $\alpha, y \in \mathcal{S}$ and $m(dy) = m(y)dy$ be its speed measure. Assume $\alpha > y$. Then, the entrance law $n^\alpha_t(dy) = n^\alpha_t(y)dy$ is given by

$$
n^\alpha_t(y) = q(t; \alpha, y)m(y), \quad a.e. \ y \in (\ell, \alpha)
$$

where $q(t; \alpha, y)$ is the density of the first passage time distribution from $y$ to $\alpha$: $\mathbb{P}_y(H_\alpha \in dt) = q(t; \alpha, y)dt$.

Proof. The entrance law has the characterization (see e.g. VI.(50.3) in Rogers and Williams (1994b))

$$
\int_0^\infty e^{-qt} n^\alpha_t(\Gamma) dt = \frac{U^\alpha \mathbb{I}_\Gamma(\alpha)}{\mathbb{P}_\alpha(0 = e^{-qt} d\ell)}
$$

where $\Gamma$ is a Borel set not containing $\{\alpha\}$. By (2.4) and (2.19),

$$
\int_0^\infty e^{-qt} \int_\Gamma n^\alpha_t(y) dy dt = \int_0^\infty e^{-qt} n^\alpha_t(\Gamma) dt = \int_\Gamma G_q(\alpha, y)m(dy) = \int_\Gamma \phi_q(\alpha) \psi_q(y) m(dy).
$$
Hence by Fubini, we have for almost every $y$ in $(\ell, \alpha)$,
\[
\int_0^\infty e^{-qt} n_t^\alpha(y) dt = \frac{\psi_q(y)}{\psi_q(\alpha)} m(y). \tag{2.23}
\]
Now by (2.3) in the second equality below,
\[
\int_0^\infty e^{-qt} n_t^\alpha(y) dt = \frac{\psi_q(y)}{\psi_q(\alpha)} m(y) = \mathbb{E}_y[e^{-qH_\alpha}] m(y) = \int_0^\infty e^{-qt} \mathbb{P}_y(H_\alpha \in dt) m(y),
\]
from which we obtain (2.22).

\[\square\]

**Remark 2.2.** Note that for the computation of the entrance law, the inverse Laplace transform based on (2.23) also works.

The next object is the entrance law associated with an excursion of infinite length: $n^\alpha(\{\varepsilon : \varepsilon(t) \in \cdot, |\zeta'(\varepsilon)| = \infty\})$.

**Corollary 2.1.** The entrance law of excursions below $\alpha$ of the post-last exit time process $(X_t)_{t \geq L_\alpha}$ is given by
\[
v_t^\alpha(y) = G_0(\alpha, \alpha) \cdot h_\alpha(y) q(t; \alpha, y) m(y) dy, \quad y < \alpha.
\]

**Proof.** By setting $E_\infty := \{\varepsilon : |\zeta'(\varepsilon)| = \infty\}$, it follows that
\[
n^\alpha(\{\varepsilon : \varepsilon(t) \in \cdot, |\zeta'(\varepsilon)| = \infty\}) = \frac{n^\alpha(\{\varepsilon(t) \in dy, |\zeta'(\varepsilon)| = \infty\})}{n^\alpha(E_\infty)} = \frac{n^\alpha(\{\varepsilon(t) \in dy, |\zeta'(\varepsilon)| = \infty\})}{n^\alpha(E_\infty)} \frac{n^\alpha(\varepsilon(t) \in dy)}{E_\infty} = \frac{\mathbb{P}_y(L_\alpha = 0)}{\nu(\{+\infty\})} n_t^\alpha(dy) = G_0(\alpha, \alpha) \cdot h_\alpha(y) q(t; \alpha, y) m(y) dy
\]
by (2.20) and (2.22).

\[\square\]

**Figure 1.** A typical path of the original process $X$
2.4. **Reversed process.** Assumption 1 is still in place. Figure 1 shows a typical path of the original process $X$. We wish to identify the reversed process of the $h_{\alpha}$-transform. Recall that $r$ is the right-end point of the state space. A minimal excessive function $k_r(x) := s(x) - s(\ell)$ produces the $k_r$-transform $X^r: \mathbb{P}^k_\alpha \left( \lim_{t \to \infty} \omega(t) = r \right) = 1$.

**Proposition 2.4.** Under Assumption 1 and $H_\ell < \infty$ almost surely, the reversed process of the $h_{\alpha}$-transform has the same distribution as the process starting at $\ell$, conditioned to hit $r$, and killed at the first hitting time to $\alpha$:

$$\{ \omega(H_\ell - t), L_\alpha < t < H_\ell, \mathbb{P}_\alpha \} \quad \text{and} \quad \{ \omega(t), 0 < t < H_\alpha, \mathbb{P}^k \}$$

are identical in law.

**Proof.** Let us first concentrate on the entire path: from time 0 via $L_\alpha$ to $H_\ell$. It is shown (see [Egami and Kevkhisvili (2020), Proposition 3.6]) that

$$A := \{ \omega(H_\ell - t), 0 < t < H_\ell, \mathbb{P}_x \} \quad \text{and} \quad B := \{ \omega(t), 0 < t < L_\alpha, \mathbb{P}^k_\ell \} \quad (2.24)$$

are identical in law and $\ell$ is an entrance boundary for $\mathbb{P}^k_\ell$.

Next we concentrate on the dashed part of the path in Figure 1: from time 0 to $L_\alpha$. Recall that $k_\alpha(y) := \mathbb{P}_y(L_\alpha > 0)$ is another minimal excessive function and the original process $X$ transformed by this function $k_\alpha$ (called the $k_\alpha$-transform) is identical in law with $X$ conditioned to hit $\alpha$ and killed at the last exit time from $\alpha$. It is known (see [Salminen (1984), Remark 9 (ii)]) that

$$a := \{ \omega(L_\alpha - t), 0 < t < L_\alpha, \mathbb{P}_x \} \quad \text{and} \quad b := \{ \omega(t), 0 < t < L_\alpha, \mathbb{P}^k_\alpha \} \quad (2.25)$$

are identical in law. Hence the reversed process of the $k_\alpha$-transform has the same distribution as the process starting at $\alpha$, conditioned to hit $r$, and killed at the last exit time from $\alpha$.

Now we turn to the red part of the path in Figure 1: from time $L_\alpha$ to $H_\ell$. Namely, the path of the post-last exit time process $X^{h_\alpha}(\omega) = \{ \omega(t) : L_\alpha < t < H_\ell, \mathbb{P}^k_\alpha \}$. Let us denote the processes in the statement of this proposition by

$$a' := \{ \omega(H_\ell - t), L_\alpha < t < H_\ell, \mathbb{P}_\alpha \} \quad \text{and} \quad b' := \{ \omega(t), 0 < t < H_\alpha, \mathbb{P}^k \}.$$

By Corollary 11.30 of Chung and Walsh (2004), the laws of $A$ and $a$ are the same. Since a sample path of $a'$ is part of a sample path of $A$ (we are talking about the original path, not the transforms), the laws of $a$ and $a'$ are the same. Moreover, by (2.24) and (2.25), so are the laws of $B$ and $b$ and hence of $b$ and $b'$. This implies the equivalence of the laws of $a'$ and $b'$.

**Remark 2.3.** (i) Note that unlike (2.24) or (2.25), neither the law of $a'$ nor $b'$ involves time reversal from co-optional time. Therefore, the celebrated Nagasawa’s theorem employed in Sharpe (1980) cannot be used here for the purpose of identifying a reversed process: a proof as in Proposition 2.4 is necessary.

(ii) We have seen that $\alpha$ is an entrance boundary for the $h_{\alpha}$-transform (Proposition 2.2) and $\ell$ is for the $k_\ell$-transform that starts at $\ell$. This is well reflected in the fact that

$$s^{h_{\alpha}}(x) = s^k(x) = \frac{1}{s(\alpha) - s(x)} \quad \text{and} \quad s'(x) = -\frac{1}{s(x) - s(\ell)},$$

where we notice that $\alpha$ and $\ell$ are interchanged. It is a natural result in view of Proposition 2.4.

We summarize this section in Table 1:
Table 1 Let $X$ be a transient linear diffusion with $s(\ell) > -\infty, s(r) = +\infty$. For a regular point $\alpha$, set $h_{\alpha}(x) := \mathbb{P}_x(L_{\alpha} = 0)$. The entries are the transition density with respect to $m(\cdot)$, the entrance law of excursions below $\alpha$, and the diffusion having the same distribution as the respective reversed process.

| Transition density | Entrance law | Reversed process |
|--------------------|--------------|------------------|
| Original process   | $p(t;x,y)$   | $n^\alpha(y) = q(t;\alpha,y)m(y)$ $\{\omega(t), 0 < t < L_{\alpha}, \mathbb{P}^{\alpha}_{\omega_{\ell}}\}$ |
| Post-last exit time process | $\frac{h_{\alpha}(y)}{h_{\alpha}(x)}p(t;x,y)$ | $G_0(\alpha, \alpha)h_{\alpha}(y)n^\alpha(y)$ $\{\omega(t), 0 < t < H_{\alpha}, \mathbb{P}^{\alpha}_{\omega_{\ell}}\}$ |

3. APPLICATION TO LOSS-GIVEN-DEFAULT DISTRIBUTION

We propose, as a financial application of post-last exit time process, a model to estimate loss-given-default (LGD) distribution of corporate debt. More specifically, we derive the distribution of the ratio of the firm’s total assets over its total debt upon default. We shall hereafter refer to this ratio as the leverage ratio and to the dynamics of this ratio as the leverage ratio process, denoted by $Y$. \(^1\)

In the firm-value (or Merton (1974)) model of credit risk, insolvency occurs upon the leverage ratio being equal to some level $c$. In this case, if we set $\alpha > c$, then there must exist the last exit time $L_{\alpha}$ recorded by the leverage ratio process. The idea is summarized as follows:

1. We choose $\alpha$ so that the default (insolvency to be exact) probability implied by our model is consistent with that implied by the CDS market. Hence the level $\alpha$ is calibrated to the market: there is no arbitrariness in the selection of $\alpha$.
2. Since $L_{\alpha}$ is not a stopping time, it is appropriate to model the interval, denoted by $\tau$, between $L_{\alpha}$ and default time $\xi$ by the intensity-based model:

$$\xi(\omega) := L_{\alpha}(\omega) + \tau \circ \theta(L_{\alpha}(\omega)).$$

In fact, this well explains the reality that a default occurs surprisingly during the period when non-insider investors do not have enough information about credit quality. The $L_{\alpha}$, not a stopping time, corresponds to the time when creditworthiness becomes hard to grasp, and $\xi$ occurs as a surprise after non-predicting $\tau$ has elapsed.

3. The company’s leverage ratio at $\xi$ is then computed as the distribution of $Y(\tau \circ \theta(L_{\alpha}))$, from which the LGD distribution at the corporate level is obtained. For this purpose, we use results from Section 2 including (2.13).

We explicitly derive the distribution of default time $\xi$ (Remark 3.1) and LGD (Corollary 3.2). These are computationally simple formulas. Furthermore, the model check conducted in Section 4 demonstrates that the model-implied default time and LGD distribution are consistent with the credit market and company-specific financial conditions. Specifically, the CDS spread implied from the estimated LGD distribution is consistent with the quoted spread in the market.

---

\(^1\)In our framework, we are focusing on the loss rate of the debt of the whole company, not each individual debt obligation. Based on this corporate-level loss distribution, credit managers can calculate loss rates for individual debt obligations having distinct characteristics; for example, senior or subordinated, and with or without collateral.
3.1. **Leverage ratio process.** We define the leverage ratio process and compute its infinitesimal drift and diffusion parameters by Lemma 2.1. Let us assume that the firm value (market value of total assets) $V$ follows a geometric Brownian motion:

$$dV_t = \mu V_t dt + \sigma V_t dW_t$$

with $\mu \in \mathbb{R}$ and $\sigma > 0$. The debt process is given by

$$B_t = B_0 e^{rt}, \quad B_0 \in \mathcal{F}_0, \quad t \geq 0,$$

where $r > 0$ denotes a constant rate. As announced above, we define the leverage ratio process $Y_t = V_t / B_t$:

$$Y_t = Y_0 e^{\left(\mu - \frac{1}{2} \sigma^2 - r\right)t + \sigma W_t}, \quad \ln(Y_t) = \ln(Y_0) + \left(\mu - \frac{1}{2} \sigma^2 - r\right)t + \sigma W_t.$$ 

The parameters $\mu, \sigma$ are such that the left boundary 0 is attracting and the right boundary $+\infty$ is non-attracting, which comply with Assumption 1. In particular, we assume that $\mu - \frac{1}{2} \sigma^2 - r < 0$. This ensures that the default occurs eventually with probability one. By fixing a certain level $\alpha \in \mathbb{R}^+$, we define $L_\alpha := \sup\{t \geq 0 : Y_t = \alpha\}$.

We have the normalized process

$$\frac{1}{\sigma}d\ln(Y_t) = M + dW_t,$$

by setting

$$M := \frac{\mu - \frac{1}{2} \sigma^2 - r}{\sigma}.$$ 

Now let us introduce the post-last exit time process

$$Z_t := \left(\frac{1}{\sigma} \ln(Y_t)\right) \circ \theta(L_\alpha), \quad t > 0.$$

Using (2.15) and the scale function $s(y) = \frac{1}{2M}(1 - e^{-2My})$ of the Brownian motion with drift $M$, we see that the process $Z_t$ follows the dynamics

$$dZ_t = M \coth(M(Z_t - \alpha^*)) dt + dW_t, \quad Z_0 = \alpha^* := \frac{1}{\sigma} \ln(\alpha), \quad t > 0.$$ 

The starting point of $Z$ is $\alpha^*$. Note that since $\lim_{y \to 0} \coth(y) = -\infty$, when the process $Z$ approaches $\alpha^*$ from below, the drift approaches negative infinity so that the process shall never reach the point $\alpha^*$ from the region $(-\infty, \alpha^*)$. Mathematically, as proved in Proposition 2.2, $\alpha^*$ is an entrance boundary for $Z$ and financially, the firm’s leverage ratio process $Y$ shall never recover to the threshold level $\alpha$.

3.2. **Loss-given-default distribution.** We define a random time that is suitable for default time. As is seen in the intensity-based default modeling, we need an exponential random variable $J$ with rate $\eta = 1$, independent of everything else. We let $\lambda : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a nonnegative piecewise continuous function and define the moment inverse of integral functional by

$$v(J) := \min\left(s : \int_0^s \lambda(Z_u) du = J\right).$$

Following Bielecki and Rutkowski (2002, Sec.8.2),
**Assumption 2.** We assume that \( \lambda \) satisfies
\[
\int_0^\infty \lambda(Z_s)\,ds = \infty \quad \mathbb{P} - a.s. \tag{3.5}
\]
\( \Diamond \)

We can define random time \( \tau := \nu(J) \) and model the default time \( \xi \) as
\[
\xi(\omega) := L_\alpha(\omega) + \tau \circ \theta(L_\alpha(\omega)) \tag{3.6}
\]
where \( \theta(\cdot) \) denotes the shift operator. Note that the construction of the random time is similar to the canonical construction of the default time in the intensity-based approach (see Proposition 5.26 in Capiński and Zastawniak (2017)). It follows from (3.4) that \( \mathbb{P}(\tau = 0 \mid Z_0 = \alpha^*) = 0 \) and from (3.5) that \( \mathbb{P}(\tau < +\infty \mid Z_0 = \alpha^*) = 1 \). We also see that \( \mathbb{P}(\tau > s \mid Z_0 = \alpha^*) > 0 \) for every \( s \geq 0 \). Moreover, we have \( \mathbb{P}(L_\alpha < \xi < +\infty) = 1 \) with (3.6). We have constructed the time \( \tau \) using the process which never returns to \( \alpha^* \), this feature being represented in (3.3).

We define LGD on debt \( B \) as
\[
K^B(\xi) := 1 - Y_\xi = 1 - \frac{V_\xi}{B_\xi}. \tag{3.7}
\]
Due to \( Y_\xi = Y_\tau \circ \theta(L_\alpha) \), it suffices to compute the distribution of \( Z_\tau = \frac{1}{\sigma} \ln(Y_\tau \circ \theta(L_\alpha)) \). Thus, we are able to find the LGD distribution by focusing on \( Z_\tau \) only. This simplifies the analysis as we do not need to consider the shift operator \( \theta \) in the definition (3.6).

Our setup is summarized in Table 2. For the LGD modeling, we have made only Assumptions 1 and 2.

**Table 2** Comparison of our model with the conventional setting

| Default time | Distribution of the final value |
|--------------|---------------------------------|
| Conventional | \( T = \inf\{ t \geq 0 : Y_t = c \} \) | \( Y_T = c \) |
| Our approach | \( \xi = L_\alpha + \tau \circ \theta(L_\alpha) \) | \( Y_\xi \sim e^{\sigma Z_\xi} \) |

As shown in Proposition 2.2, \( \alpha^* \) is an entrance boundary for \( Z \). Hence we first consider an arbitrary starting position \( z < \alpha^* \) and then take a limit. Fix any \( Q < \alpha^* \). According to (3.3) and Theorem IV.5.1 in Borodin (2017),
\[
U(z) := \mathbb{E}_z\left[e^{-\gamma T} \mathbb{1}_{[Z_t \leq Q]}\right], \quad z < \alpha^*, \quad \gamma > 0 \tag{3.8}
\]
is the unique continuous solution to the ordinary differential equation
\[
\frac{1}{2} U''(z) + M \coth(M(z - \alpha^*))U'(z) - (\lambda(z) + \gamma)U(z) = -\lambda(z) \mathbb{1}_{[z < Q]}(z), \quad z < \alpha^*. \tag{3.9}
\]
Here \( \mathbb{E}_z[\cdot] \) denotes the conditional expectation \( \mathbb{E}[\cdot \mid Z_0 = z] \). We will provide an explicit form of \( U(z) \) in Proposition 3.1. Using the solution \( U(z) \), we obtain \( \mathbb{E}_{\alpha^*}\left[e^{-\gamma T} \mathbb{1}_{[Z_t \leq Q]}\right] = \lim_{\epsilon \downarrow 0} \mathbb{E}_{\alpha^*}\left[e^{-\gamma T} \mathbb{1}_{[Z_t \leq Q]}\right] \) produces the distribution of \( K^B(\xi) \) via (3.7).

To obtain a concrete result, we specify the function \( \lambda \) in (3.4) as
\[
\lambda(Z_t) := \mathbb{1}_{[Z_t < \alpha^*]} \tag{3.10}
\]
so that the integral in (3.4) represents the occupation time of the process \( Z \) under the level \( \alpha^* \).
Hence we identify the firm’s default once the leverage process (in its logarithmic form) spends a certain random time below level \( \alpha \) after time \( L_\alpha \). This setup is quite reasonable since, as in other intensity-based models, the market observers are unaware of the firm’s state after a certain time (represented by \( L_\alpha \) in our model) and receive a default as a sudden shock.

**Remark 3.1.** Since the dynamics (3.2) is simple enough, we can compute the distribution of default time \( \xi \) explicitly. From (3.4) and (3.10), we see that \( v(J) = J \) since \( Z \) does not hit the region \([\alpha^*, \infty)\): the occupation time of \(( -\infty, \alpha^*)\) after \( L_\alpha \) until \( \xi \) is equal to \( J \). Using the fact that \( J \) is distributed exponentially with rate 1 and independent of \( L_\alpha \), we have, for \( T \in \mathbb{R}_+ \)

\[
P_{Y_0}(\xi \leq T) = P_{Y_0}(L_\alpha \leq T - J) = \int_0^T P_{Y_0}(L_\alpha \leq T - x) e^{-x} \, dx
\]

\[
= \int_0^T P_{Y_0}(0 < L_\alpha \leq T - x) e^{-x} \, dx + \int_0^T P_{Y_0}(L_\alpha = 0) e^{-x} \, dx.
\]

From (3.2) and (3.3), \( L_\alpha = \sup \{ t : \frac{1}{\sigma} \ln(Y_t) = \alpha^* \} \). The last exit time distribution is known: Proposition 4 in Salminen (1984) and Proposition 3.1 in Egami and Kervikhvili (2020) provide

\[
P_y(L_\alpha \in dt) = \frac{p(u,y,\alpha)}{G_0(\alpha, \alpha)} \, du \quad \text{and} \quad P_y(L_\alpha = 0) = 1 - \frac{G_0(y, \alpha)}{G_0(\alpha, \alpha)}
\]

where \( G_0(\alpha, \alpha) \) is available in (2.21). It can be confirmed that \( P_{Y_0}(L_\alpha = 0) = 0 \) when \( Y_0 \geq \alpha \) by (2.21) and Assumption 1. Since the transition density \( p \) for the process \( \frac{1}{\sigma} \ln(Y_t) \), a Brownian motion with drift \( M \), is available, we can evaluate the integral in the above equation explicitly

\[
P_{Y_0}(\xi \leq T) = \int_0^T \left( \int_0^{T-x} \frac{p(u, \frac{1}{\sigma} \ln(Y_0), \alpha^*)}{G_0(\alpha^*, \alpha^*)} \, du \right) e^{-x} \, dx + \int_0^T \left( 1 - \frac{G_0(\frac{1}{\sigma} \ln(Y_0), \alpha^*)}{G_0(\alpha^*, \alpha^*)} \right) e^{-x} \, dx
\]

\[
= e^{-M(\frac{1}{\sigma} \ln(Y_0) + \alpha^*)} \frac{4M|G_0(\alpha^*, \alpha^*)|}{G_0(\alpha^*, \alpha^*)} e^{-(M(\frac{1}{\sigma} \ln(Y_0) - \alpha^*)/\alpha^*)} \left[ 1 + \text{Erf} \left( \frac{M(T-x)+\frac{1}{\sigma} \ln(Y_0) - \alpha^*}{\sqrt{2}(T-x)} \right) - e^{M(\frac{1}{\sigma} \ln(Y_0)-\alpha^*)/\alpha^*} \text{Erfc} \left( \frac{M(T-x)+\frac{1}{\sigma} \ln(Y_0) - \alpha^*}{\sqrt{2}(T-x)} \right) \right] \, dx
\]

\[+ \left( 1 - \frac{G_0(\frac{1}{\sigma} \ln(Y_0), \alpha^*)}{G_0(\alpha^*, \alpha^*)} \right) (1-e^{-T}),
\]

where

\[\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \quad \text{and} \quad \text{Erfc}(x) = 1 - \text{Erf}(x).\]

As announced above (right after the beginning of this subsection), we shall find \( \alpha^* \) that matches (3.11) with \( T = 5 \) with the market-quoted 5-year default probability.

\[\diamondsuit\]

**Proposition 3.1.** Under (3.10), fix \( \gamma > 0 \) and let \( b_1 = \sqrt{1 + \frac{2(1+\gamma)}{M^2}} \). For \( Q < \alpha^* \), we have

\[
\mathbb{E}\alpha^* \left[ e^{-\gamma T} \mathbb{1}_{(Z_t \leq Q)} \right] = \frac{1}{1 + \gamma} \left( \cosh(\gamma[M(\alpha^* - Q)]) + b_1 \sinh(\gamma[M(\alpha^* - Q)]) \right) e^{-b_1|M(\alpha^* - Q)|}. \tag{3.12}
\]

**Proof.** Define \( Z_t^* := \alpha^* - Z_t \). Then, by (3.3)

\[
dZ_t^* = -dZ_t = -M \coth(M(Z_t - \alpha^*)) \, dt - dW_t = M \coth(MZ_t^*) \, dt + dW_t^* = |M| \coth(|M|Z_t^*) \, dt + dW_t^*
\]

...
Thus, we obtain the desired result by taking the limit $\gamma$.

Note that

Finally, $\lim_{z \uparrow \alpha^* - Q} Z(z) = \alpha^* - Q$. Then (3.14) is reduced to

For $0 < z < \alpha^* - Q$, we have $U(z) = d \psi_1(z)$ with some constant $d$. We have eliminated the second fundamental solution because $U(z)$ is bounded when $z \to +\infty$.

(ii) Let $0 < z < \alpha^* - Q$. In this case, (3.14) is reduced to

As in the case (i) above, its fundamental solutions are given by $\psi_1(z)$ and $\phi_1(z)$ in (3.15). Therefore, in this case we have $U(z) = d \psi_1(z)$ with some constant $d$. We have eliminated the second fundamental solution because $U(z)$ is bounded when $z \to 0$.

Finally, from the continuity of $U(z)$ and $U'(z)$ at $\alpha^* - Q$, we determine the constants $c$ and $d$. In particular,

Finally, $\lim_{z \downarrow 0} U(z) = \lim_{z \downarrow 0} d \psi_1(z) = d \sqrt{\frac{2}{\pi}}$, which proves (3.12) in view of (3.13).

**Corollary 3.1.** The distribution of $Z_\tau$ is given by

$$P_{\alpha^*}(Z_\tau \leq Q) = \begin{cases} 
(cosh(|M|((\alpha^* - Q))) + b_2 \sinh(|M|((\alpha^* - Q)))) e^{-b_2|M|((\alpha^* - Q)} & Q \leq \alpha^* \\
1 & Q > \alpha^*
\end{cases},$$

where $b_2 = \sqrt{1 + \frac{2}{M^2}}$.

**Proof.** Note that $P_{\alpha^*}(Z_\tau \leq Q) = 1$ for $Q > \alpha^*$ due to (3.3). For $Q \leq \alpha^*$, we have

Thus, we obtain the desired result by taking the limit $\gamma \downarrow 0$ in (3.12).
Corollary 3.2. The probability density function of \( K^B(\xi) \) in (3.7) is given by
\[
\mathbb{P}_{\alpha^*}(K^B(\xi) \in dx) = \frac{2}{|M|} \sinh\left(|M|\left(\alpha^* - \ln(1-x)\right)\right) e^{-b_2 |M| (\alpha^* - \ln(1-x))} \frac{1}{\sigma(1-x)} dx, \quad 1 - e^{\sigma \alpha^*} < x < 1
\]
where \( b_2 = \sqrt{1 + \frac{2}{\pi^2}}. \) Moreover,
\[
\mathbb{E}_{\alpha^*}[K^B(\xi)] = 1 - \frac{e^{\sigma \alpha^*}}{1 + \frac{x^2}{2} + b_2 \sigma |M|}.
\] (3.17)

Proof. Due to \( Y_\xi \overset{d}{\sim} e^{\sigma Z_\tau} \), we have \( K^B(\xi) \overset{d}{\sim} 1 - e^{\sigma Z_\tau} \) by (3.7). The density function of \( Z_\tau \) is obtained by differentiating (3.16) with respect to \( Q \):
\[
f(Q) := \frac{\mathbb{P}_{\alpha^*}(Z_\tau \in dQ)}{dQ} = \frac{2}{|M|} \sinh(|M|(\alpha^* - Q)) e^{-b_2 |M|(\alpha^* - Q)} dQ, \quad Q < \alpha^*.
\]
Then, the density of the transform \( 1 - e^{\sigma Z_\tau} \) is given by
\[
\mathbb{P}_{\alpha^*}(K^B(\xi) \in dx) = f\left(\frac{\ln(1-x)}{\sigma}\right) \frac{1}{\sigma(1-x)} dx, \quad 1 - e^{\sigma \alpha^*} < x < 1.
\]
Finally, \( \int_{1 - e^{\sigma \alpha^*}}^1 x \mathbb{P}_{\alpha^*}(K^B(\xi) \in dx) \) provides (3.17). \( \square \)

3.3. Parameter estimation. Fix any point in time as the current time. The procedure outlined below estimates the LGD distribution implied in the 5-year CDS spread using the information available up to the current time. The parameters are estimated in a way that the default probability and default time distribution obtained from the model are consistent with the default probability provided by Eikon (consistent with the quoted CDS spread).

Step 1: Estimate the parameters \((\mu, \sigma)\) of the asset process in (3.1). Since it is a well-known procedure, detailed explanation is delegated to Appendix A.2. The initial value \( Y_0 \) of the leverage ratio process is retrieved using the estimated \( \sigma \). The debt process \( B \) represents the sum of short-term debt and a half of long-term debt (as in Moody’s KMV approach): the reason being due to the good fit to empirical evidence.

Step 2: Calibrate \( \alpha \) so that the default time \( \xi \) in (3.6) matches the market-quoted 5-year default probability by using the method described in Remark 3.1, especially (3.11).

Step 3: We obtain the LGD distribution of \( K^B(\xi) \) from Corollary 3.2. However, in the case of CDS, it is more appropriate to consider the LGD distribution of the total debt \( D = (D_t)_{t \geq 0} \), which is done as follows:
Letting the processes \((D^S_t)_{t \geq 0}\) and \((D^L_t)_{t \geq 0}\) represent short- and long-term debts respectively, we have
\( D_t = D^S_t + D^L_t \). For simplicity, we assume that the proportion of long-term debt \( w = \frac{D^L_t}{D_t} \) remains constant. The loss rate of total debt \( D \), denoted by \( K^D(\xi) \), is then given by
\[
K^D(\xi) = K^B(\xi) + \frac{1}{2} w(1 - K^B(\xi)).
\] (3.18)

3.4. Model Check. Finally, we make sure that the LGD distribution from our model is consistent with the CDS market. For this purpose, we shall calculate the 5-year CDS spread based on our model and compare it to the quoted spread. Recall that the underlying credit risk behind CDS spread refers to the risk of the whole company, not a particular debt obligation. (Therefore, this convention is consistent with our modeling.)

Note that the market CDS spread is calculated with the assumption of 60% loss rate. In contrast, the value of the CDS spread we shall obtain below from our model depends on the LGD distribution. Therefore, the two CDS
spreads cannot be compared directly. Instead we compare
\[
\rho := \frac{\text{CDS spread}}{\text{average loss given default}}
\]  
which is a spread per 1% loss given default.

We set the principal amount to $1 and assume the spread payments are made quarterly. We find spread value for which the present value of spread payments (premium leg) and the present value of the payment in case of default (default leg) are equal. We ignore the counterparty risk and use the constant rate \( r \) from Step 1 as the discount rate. The following procedure is for estimating \( \rho \).

Let \( \omega_i, i = 1, \cdots, N \) denote a simulation path. Simulate a uniform random variable \( U(\omega_i) \) on \([0, 1] \).

Step (i): We shall obtain samples of \( K^D(\xi(\omega_i)) \). First, simulate pairs \( (\tau(\omega_i), Z_{\tau(\omega_i)}(\omega_i)) \). Recall that \( \tau = J \) where \( J \) is an exponential (with rate 1) random variable, so that we set \( \tau(\omega_i) = -\ln(U(\omega_i)) \). By inverting the distribution function (3.16) at \( U(\omega_i) \), we obtain \( Z_{\tau(\omega_i)}(\omega_i) \) which provides \( K^B(\xi(\omega_i)) = 1 - e^{\sigma Z_{\tau(\omega_i)}(\omega_i)} \) by (3.7) and, in turn, \( K^D(\xi(\omega_i)) \), by (3.18).

Step (ii): We shall obtain samples of \( \xi(\omega_i) \) by using \( \tau(\omega_i) \) in Step (i). Simulate \( L_\alpha(\omega_i) \) by inverting its cumulative distribution function (which is also used in (3.11)) at \( U'(\omega_i) \) (uniformly distributed random variable on \([0, 1] \), independent of \( U(\omega_i) \)). We thus obtain \( \xi(\omega_i) = L_\alpha(\omega_i) + \tau(\omega_i) \).

Step (iii) We have obtained the pairs \( (\xi(\omega_i), K^D(\xi(\omega_i))) \) from which we can compute the present value of the recovery (on the dollar) upon default. Specifically, we calculate the value of \( e^{-r(\xi(\omega_i))} K^D(\xi(\omega_i)) \mathbb{1}_{\{\xi(\omega_i) \leq 5\}} \). By taking the average over \( N \) trials, we find the value of default leg.

Step (iv): Using the discount rate \( r \), calculate the present value of spread payments until \( \xi(\omega_i) \) (including accrued premium) or the maturity, whichever is earlier. By taking average over \( N \) trials, we obtain the premium leg.

Step (v): We match the premium leg to the default leg to obtain the model-implied CDS spread.

Step (iv): Compute \( \rho \) by (3.19) with \( \mathbb{E}_{\omega_i} [K^D(\xi) \mid \xi \leq 5] \) for its denominator. Compare this \( \rho \) to the corresponding number from the CDS market to see these two \( \rho \)'s are close enough.

4. Example of Tyson Foods Inc.

In this section, we consider an example of Tyson Foods Inc., hereafter denoted shortly as Tyson Foods. The S&P Global credit rating of this company was downgraded from BBB+/Negative to BBB/Stable on November 28, 2023 (S&P Global, 2023). The main purpose of presenting this example is to illustrate how to implement our method to obtain the LGD distribution implied in the current credit market and company-specific financial conditions. Specifically, we estimate the model parameters in a way that the model-implied default probability (which is based on default time distribution) is consistent with the default probability provided by Eikon. Based on those parameters, we obtain the LGD distribution.

We choose December 29, 2023 as the current time and use the information available up to this time to obtain the LGD distribution implied on that day. We use 1-year daily data from the period 2022/12/29~ 2023/12/29 for the parameter estimation. We use 1-year Treasury bill rate (bank discount rate) 4.55% (of December 29, 2023) as the constant rate \( r \). The current quoted 5-year CDS spread for Tyson Foods is 69.34 bps and 5-year default probability is 5.965%. The values of the equity \( E \) and debt \( B \) are displayed in Figure 2. We interpolated daily debt values using standardized quarterly balance sheets. We defined the debt \( B \) as the sum of the short-term debt (notes payables,
short-term debt, current portion of long-term debt and capital leases) and one half of the long-term debt (long-term debt and capital leases).

![Graph (a)](image1)

![Graph (b)](image2)

**Figure 2.** Daily values of Tyson Foods data during 2022/12/29~2023/12/29: (a) Market capitalization $E$ in million USD and (b) Debt $B$ in million USD.

Table 3 reports the estimated parameters which provide the 5-year default probability consistent with the 5-year default probability provided by Eikon. The value of $\alpha$ is obtained using (3.11) in Figure 3. Using Corollary 3.1, Figure 4 displays the density function of $K^B(\xi) \sim 1 - e^{\sigma Z}$ in (3.7) and $K^D(\xi)$ in (3.18) based on the estimated parameters. The average value of the ratio of long-term debt to total debt is $\bar{w} = 70.1037\%$.

| Parameter | Value | Standard Error |
|-----------|-------|----------------|
| $\sigma$  | 0.2499| 0.0112         |
| $\mu$     | -0.0704|
| $M$       | -0.5888|
| $Y_0$     | 3.2693|
| $\alpha$ | 0.9304|

**Table 3** Estimated parameters up to 4 decimal points. The standard error of $\sigma$ from the maximum likelihood estimation is given in parenthesis.

Finally, we check the consistency of the model-implied LGD distribution with the CDS market. We set the number of trials $N$ as 100,000. Following the procedure in Section 3.4, we calculate the model-implied 5-year CDS spread which is based on the model-implied LGD distribution $K^D(\xi)$. This spread is 57.8976 bps. While $\mathbb{E}_{\alpha^*} [K^D(\xi)] = 57.2669\%$ from (3.17) and (3.18), we shall go with the average LGD: $\mathbb{E}_{\alpha^*} [K^D(\xi) \mid \xi \leq 5]$ since we have samples of $K^D(\xi(\omega_i))$ and $\xi(\omega_i)$ in Step (i) and Step (ii), respectively. The estimated average LGD is 51.6195%. In the model’s case, $\hat{\rho} = 57.8976/51.6195 = 1.1216$bps, while the counterpart of the market quote is $\rho_{\text{quote}} := 69.34/60 = 1.1557$bps. We see that $\hat{\rho}$ and the market estimate $\rho_{\text{quote}}$ are very close. Therefore, our method is consistent with the market information.
Furthermore, to check the robustness of our results with respect to the ratio of long-term debt to total debt (denoted by $w$), we report the results for four additional values of $w$ together with $\bar{w} = 70.1037\%$ (obtained in Step 4 of Section 3.3) in Table 4. We see that all of our estimates of $\rho$ are within 3% range from the market estimate of $\rho_{\text{quote}} = 1.1557\text{bps}$. We also notice that these $\hat{\rho}$’s themselves are stable with respect to $w$. We conclude that the default time and LGD distribution in our framework are consistent with the credit market and company-specific financial conditions.

Figure 3. 5Y default probability (3.11) as a function of $\alpha$. Market-quoted 5Y default probability is 5.965\%.

Figure 4. Probability density function of (a) $K^B(\xi)$ in (3.7) and (b) total debt LGD $K^D(\xi)$ in (3.18) based on the estimated parameters.
**Table 4** Model-implied 5-year CDS spreads based on the distribution of $K^D(\xi)$, average LGD ($K^D(\xi)$), and their ratios (spread per 1% LGD, denoted by $\hat{\rho}$) for each $w$ (up to 4 decimal points).

Note that $\hat{w} = 70.1037\%$ and the quoted spread per 1% LGD is $\rho_{\text{quote}} := 69.34/60 = 1.1557$bps.

| w (%)   | 68     | 69     | 70.1037 | 71     | 72   |
|---------|--------|--------|---------|--------|------|
| CDS Spread (bps) | 57.0172 | 57.4357 | 57.8976 | 58.2727 | 58.6912 |
| Average LGD (%)   | 50.8359 | 51.2084 | 51.6195 | 51.9533 | 52.3258 |
| $\hat{\rho}$     | 1.1216 | 1.1216 | 1.1216 | 1.1216 | 1.1217 |

**APPENDIX A.**

A.1. **Proof of Lemma 2.1.** Define $B(x) = \int_x^{\infty} \frac{2}{\sigma^2(y)} \mu(y) dy$. It is well-known that

$$m(x) = \frac{2}{\sigma^2(x)} e^{B(x)} \quad \text{and} \quad s'(x) = e^{-B(x)}$$

and

$$m^h(dy) = h^2(y) m(dy) \quad \text{and} \quad s^h(dy) = \frac{1}{h^2(y)} s(dy),$$

where $m^h(\cdot)$ and $\sigma^h(\cdot)$ are the speed measure and scale function of the $h$-transform of $X$. See Section II.1 of Borodin and Salminen (2002). By simple comparison, we obtain

$$\frac{1}{(\sigma^h)^2(x)} e^{B_h(x)} = \frac{1}{\sigma^2(x)} e^{B(x)} \quad \text{and} \quad e^{-B_h(x)} = \frac{1}{h^2(x)} e^{-B(x)} \quad \tag{A.1}$$

where $B_h(x) = \int_x^{\infty} \frac{2}{(\sigma^h)^2(y)} \mu^h(y) dy$. From (A.1), we have $\sigma^h(x) = \sigma(x)$. This in turn gives

$$\exp \left( 2 \int_x^{\infty} \frac{\mu^h(y) - \mu(y)}{\sigma^2(y)} dy \right) = h^2(x),$$

so that $\mu(x)$ satisfies $\mu^h(x) = \mu(x) + \frac{\mu'(x)}{h(x)} \sigma^2(x)$.

A.2. **Detailed comments on Step 1 in Section 3.3.** We estimate the parameters of the asset process in (3.2) using the data of the firm’s equity $E$ and debt $B$ available up to the current time. The debt process $B$ represents a certain amount of debt to be repaid and we let it be the sum of short-term debt and a half of long-term debt (as in Moody’s KMV approach) since empirical evidence has demonstrated that such level is an appropriate default threshold for the firm’s assets.

We use the option-theoretic approach of Merton (1974) where the value of equity represents the value of the European call option written on the firm’s assets with the strike price equal to the future value of debt. We estimate the asset parameters using the maximum likelihood method of Duan (1994), Duan (2000), and Lehar (2005). We assume that debt $B$ grows at the risk-free rate $r$. Assuming that debt $B$ is exogenous, the log-likelihood function to be maximized is given by

$$L(E_1, \ldots, E_n; \mu, \sigma \mid B_1, \ldots B_n) = - \frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2 \Delta_t) - \sum_{t=2}^{n} \ln(\hat{\mu}_t) - \sum_{t=2}^{n} \ln(\Phi(\hat{d}_t))$$

$$- \frac{1}{2\sigma^2 \Delta_t} \sum_{t=2}^{n} \left( \ln \left( \frac{\hat{V}_t}{\hat{V}_{t-1}} \right) - \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta_t \right)^2$$
where $\Phi$ is a standard normal cumulative distribution function and $\Delta_t$ is the time interval between two consecutive data points. We use $n$ number of past data points for the estimation. In the above equation, $\hat{V}_t$ is the solution to

$$E_t = V_t \Phi(d_t(V_t)) - B_t \Phi(d_t(V_t) - \sigma \sqrt{T_m})$$

(A.2)

with respect to $V_t$ and $\hat{d}_t = d_t(\hat{V}_t)$ where

$$d_t(V_t) = \frac{\ln \left( \frac{V_t}{B_t} \right) + \frac{\sigma^2 T_m}{2}}{\sigma \sqrt{T_m}}.$$

As in Lehar (2005), we set the maturity $T_m$ of the call option to 1 year. By differentiating the log-likelihood function with respect to $\mu$ and setting this partial derivative to zero, we obtain

$$\mu = \frac{\sum_{t=2}^n \ln \left( \frac{V_t}{V_{t-1}} \right)}{(n-1)\Delta_t} + \frac{1}{2} \sigma^2.$$

We may plug this into the log-likelihood function which then becomes a function of only one parameter $\sigma$. By maximizing the log-likelihood, we obtain the estimate of $\sigma$ and as a byproduct, the estimate of $\mu$ as well.

A.3. Literature review of the study of loss-given default (LGD). There exists a large body of literature regarding the loss-given-default (hereafter LGD) distribution because the information about this distribution is important for credit risk management. One example of its use is the estimation of loan-loss reserves in the banking industry. The LGD distribution should reflect credit market conditions as well as company-specific financial conditions. However, as Doshi et al. (2018) and many other studies point out, it is a standard practice to assume, irrespective of company names, a constant recovery rate (approximately 40%) of the debt upon default. This 60% loss rate is arbitrarily determined, possibly from the empirical distribution. Since the market data is usually quoted based on the predetermined constant loss rate (60%) for all firms, it is hard to estimate the LGD distribution by just observing the market information such as CDS spreads, default time distribution, and default probability. In contrast to the standard assumption of the constant loss rate, the empirical literature has documented the evidence of time-varying realized loss rates. Hence we cannot overemphasize the effectiveness of a model that provides the LGD distribution implied in the credit market and company-specific financial conditions.

As the empirical literature has documented the evidence of time-varying realized loss rates, it is important to build a stochastic model for LGD. Gambetti et al. (2019) analyzes determinants of recovery rate distributions and finds economic uncertainty to be the most important systematic determinant of the mean and dispersion of the recovery rate distribution. For their analysis, the authors use post-default bond prices of 1831 American corporate defaults during 1990-2013. Even though recovery rates vary with a firm’s idiosyncratic factors, Gambetti et al. (2019) states that the impact of systematic factors related to economic cycles should not be underestimated. For the summary of studies related to cross-sectional and time variation of recovery rates, we refer the reader to Gambetti et al. (2019).

Altman et al. (2004) provides a detailed review of how recovery rate and its correlation with default probability had been treated in credit risk models. The authors also discuss the importance of modeling the correlation between recovery rate and default probability. Altman et al. (2005) analyzes and measures the relationship between default and recovery rates of corporate bonds over the period of 1982-2002. They confirm that default rate is a strong indicator of average recovery rate among corporate bonds. Acharya et al. (2007) analyzes data of defaulted firms in the U.S. during 1982-1999 and finds that the recovery rate is significantly lower when the industry of the defaulted
firm is in distress. The authors discover that industry conditions at the time of default are robust and important determinants of recovery rates. Their results suggest that recovery rates are lower during industry distress not only because of the decreased worth of a firm’s assets but also because of the financial constraints that other firms in the industry face. The latter reason is based on the idea that the prices, at which the assets of the defaulted firm can be sold, depend on the financial condition of other firms in the industry.

Doshi et al. (2018) uses information extracted from senior and subordinate credit default swaps to identify risk-neutral stochastic recovery rate dynamics of credit spreads and studies the term structure of expected recovery. Their study is related to Schläfer and Uhrig-Homburg (2014) which also uses the fact that debt instruments of different seniority face the same default risk but have different recovery rates given default. Doshi et al. (2018) uses 5-factor intensity-based model for CDS contracts allowing stochastic dynamics of LGD. The authors allow firm-specific factors to influence the stochastic recovery rate. Their empirical analysis of 46 firms through the time period of 2001-2012 shows that the recovery rate is time-varying and the term structure of expected recovery is on average downward-sloping. They also find that industry characteristics have significant impact on CDS-implied recovery rates; however, they do not find the evidence that firms’ credit ratings explain the cross-sectional differences in recovery rates. For the summary of the literature related to the relationship between default rates and realized recovery rates, refer to Doshi et al. (2018).

Yamashita and Yoshiba (2013) is an example of a study that uses stochastic collateral value process to incorporate stochastic recovery rate into the model which assumes that a constant portion of the collateral value is recovered upon default. Yamashita and Yoshiba (2013) uses a quadratic Gaussian process for the default intensity and discount interest rate and derives an analytical solution for the expected loss and higher moments of the discounted loss distribution for a collateralized loan. The authors assume that the default intensity, discount interest rate, and collateral value are correlated through Brownian motions driving Gaussian state variables.

Finally, Cohen and Costanzino (2017) is an example of a study that extends a structural credit risk model and incorporates stochastic dynamics of the recovery rate. In contrast to our approach which does not introduce an additional recovery risk factor and is based solely on the dynamics of the leverage process, Cohen and Costanzino (2017) models asset and recovery processes separately as correlated geometric Brownian motions. In their model, the asset risk driver serves as a default trigger and the recovery risk driver determines the amount recovered upon default. The authors explicitly compute the prices of bonds and CDS under this framework. See also Kijima et al. (2009).

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