ELIMINATION OF COMPONENTS IN A COUNTEREXAMPLE TO QUILLEN’S CONJECTURE VIA OUTER $p$-AUTOMORPHISMS

KEVIN IVÁN PITERMAN*
DEPARTAMENTO DE MATEMÁTICA
IMAS-CONICET, FCEYN
UNIVERSIDAD DE BUENOS AIRES
BUENOS AIRES ARGENTINA
E-MAIL: KPITERMAN@DM.UBA.AR

STEPHEN D. SMITH
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS AT CHICAGO
CHICAGO ILLINOIS USA
(HOME: 728 WISCONSIN, OAK PARK IL 60304)
E-MAIL: SMITHS@MATH.UIC.EDU

Abstract. We generalize an earlier result of Segev, which shows that some component in a minimal counterexample to Quillen’s conjecture must admit an outer automorphism. We show in fact that every component must admit an outer automorphism. Thus we transform his restriction on components to an elimination-result: namely excluding any component which does not admit an outer. Indeed we show the outer automorphisms admitted must contain $p$-outers, that is of order divisible by $p$. This gives stronger, specific eliminations: for example if $p$ is odd, this eliminates sporadic and alternating components—thus reducing to Lie-type components (and typically forcing $p$-outers of field type). For $p = 2$ we obtain similar but less restrictive results. We also provide some tools to help eliminate components admitting $p$-outers in a minimal counterexample.

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1. Introduction

This paper began life as a sequel to Segev’s article “Quillen’s conjecture and the kernel on components” [Seg96] and expanded from there. One consequence of Segev’s main result (see Theorem 1.4 below) can be stated in the form:

**Corollary 1.1** (Segev). If $G$ is a minimal counterexample to Quillen’s conjecture, then $G$ induces outer automorphisms on some component.

We call this kind of result on a counterexample to Quillen’s conjecture a restriction result, since it shows that at least one component must have certain Property $P_G$. Here $P_G$ would mean that the component admits an outer automorphisms induced from $G$.

A consequence of our extension of his work (see Theorem 1.6 below) takes the form:

**Corollary 1.2.** If $G$ is a minimal counterexample to Quillen’s conjecture, then $G$ induces outer automorphisms on all components.

We will say that this is an elimination result, since it establishes that every component in a minimal counterexample to Quillen’s conjecture must have certain Property $P_G$—so that a component without $P_G$ (in this case, with no outers) is eliminated from a minimal counterexample.

In fact, we can sharpen the above corollary and eliminate components with only $p'$-outers.

**Corollary 1.3.** If $G$ is a minimal counterexample to Quillen’s conjecture, then $G$ induces outer automorphisms of order $p$ on all components.

For example: assume $p$ is odd. This eliminates alternating and sporadic components $L$, since they have $\text{Out}(L)$ a 2-group. So by the Classification, all the components are Lie-type. These have $\text{Out}(L)$ given by diagonal/field/graph. We can have diagonal/graph for only some $p$, so “mostly” this forces field automorphisms to occur.

Let $G$ be a finite group and $p$ a prime number dividing its order. In [Qui78], Quillen introduced the poset $A_p(G)$ of nontrivial elementary abelian $p$-subgroups and studied its homotopical and topological properties via its order complex. He showed that if $G$ has a nontrivial normal $p$-subgroup then $A_p(G)$ is contractible. Quillen conjectured the converse, that is, if $A_p(G)$ is contractible then $G$ has a nontrivial normal $p$-subgroup. Contrapositively, if the largest normal $p$-subgroup of $G$ is trivial then $A_p(G)$ is not contractible. This is the well-known Quillen conjecture. This conjecture remains open, but there have been important advances [AK90, AS93, Pit20, Qui78, Seg96] (see [Smi11, Ch.8] for historical discussion).

In this article, we consider the following stronger homology-version of the conjecture. Recall that $O_p(G)$ denotes the largest normal $p$-subgroup of $G$.

(H-QC) If $O_p(G) = 1$ then $\tilde{H}_*(A_p(G), \mathbb{Q}) \neq 0$.

Here, $\tilde{H}_*(X, \mathbb{Q})$ denotes the rational homology of the finite poset $X$, which is the homology of its order complex. We will work with rational homology, so in general we will drop the coefficient notation and write $\tilde{H}_*(X)$ for $\tilde{H}_*(X, \mathbb{Q})$.

Quillen established (H-QC) for solvable groups and groups of $p$-rank at most 2. Later, various authors extended (H-QC) to $p$-solvable groups (see [Smi11 8.2.12]). In this direction, in [AS93 1.6] and [Pit20] it is proved that a minimal counterexample to (H-QC) has $O_p'(G) = 1$, so that it has simple components of order divisible by $p$ (see Lemma 2.2 and Theorem 2.4). Recall that $O_p'(G)$ is the largest normal $p'$-subgroup of $G$.

In [AK90], Aschbacher-Kleidman showed (H-QC) for almost simple groups. The major advance on the conjecture is due to Aschbacher-Smith [AS93]. They showed that (H-QC) holds for $p > 5$ under suitable constraints on the unitary components. For example, in [AS93 1.7] it is shown that every component in a minimal counterexample to (H-QC) must fail the QD-property (see [AS93 p.474]). This is an elimination result.
In \cite{Seg96}, Segev worked with the kernel on components of $G$. This subgroup of $G$ is the kernel of the conjugation action of $G$ on its set of components, which is the following subgroup:

$$H := \bigcap_{L} N_{G}(L).$$

Here the intersection runs over all the components $L$ of $G$, and $N_{G}(L)$ denotes the normalizer of $L$ in $G$. Segev established (H-QC) under certain conditions on the kernel on components $H$ and if $O_{p'}(G) = 1$. In particular, Segev’s theorem below shows Quillen’s conjecture if $H = F^*(G)$, which means that the kernel on components $H$ induces no outer automorphisms on any component, so that the corollary stated earlier is a clear consequence. Recall that $F^*(G)$ is the generalized Fitting subgroup of $G$.

**Theorem 1.4** (Segev, \cite{Seg96} Thms 2 & 3). Suppose that $O_{p'}(G) = 1$. Let $H = \bigcap_{L} N_{G}(L)$ be the kernel on components. If for each component $L$ of $G$ the map $\mathcal{A}_{p}(L) \rightarrow \mathcal{A}_{p}(\text{Aut}_{H}(L))$ is not the zero map in homology, then $G$ satisfies (H-QC).

In particular, if $H = F^*(G)$, then $G$ satisfies (H-QC). This also holds if $\text{Out}_{H}(L)$ or $\text{Out}_{G}(L)$ is a $p'$-group for every component $L$ of $G$.

Here, $\text{Aut}_{G}(L) = N_{G}(L)/C_{G}(L)$, where $C_{G}(L)$ is the centralizer of $L$ in $G$. Recall that $\text{Out}_{G}(L) = \text{Aut}_{G}(L)/L$. Note that $\text{Aut}_{H}(L) \leq \text{Aut}_{G}(L)$ and $\text{Out}_{H}(L) \leq \text{Out}_{G}(L)$.

Segev’s theorem requires a common behavior in all the components of $G$, and that $O_{p'}(G) = 1$.

In this article, we show that we can focus on the behavior of a single component. Therefore, we will also consider the following hypothesis. Let $H$ be a component of $G$ and $L_1, \ldots, L_t$ its $G$-orbit.

1. Proper subgroups and proper central quotients of $G$ satisfy (H-QC).

By a proper central quotient of $G$ we mean a quotient of $G$ by a nontrivial central subgroup $Z \leq Z(G)$, where $Z(G)$ is the center of $G$. In the context of a counterexample of minimal order to (H-QC), the above hypothesis holds. Hence, we may also consider the following alternative hypothesis: (H-QC) holds for all groups $H$ such that $|H| < |G|$.

We will also consider the following hypothesis. Let $L$ be a component of $G$ and $L_1, \ldots, L_t$ its $G$-orbit.

2. $L$ is not a $p'$-group and $C_{G}(L_1 \ldots L_t)$ satisfies (H-QC).

By Remark 2.0 under (H1), $G$ satisfies (H-QC) or (HL(p)) for every component $L$ of $G$. Therefore, in the context of (H-QC), the (HL(p)) hypothesis is less restrictive than (H1).

We sharpen the ideas of \cite{Seg96} to prove that, under (H1), if there is some component $L$ of $G$ such that no elementary abelian $p$-subgroup of $G$ induces outer automorphisms on $L$, then $G$ satisfies (H-QC). See Theorem 1.3 below.

As immediate application of this result shows that, if $p$ is odd, we can eliminate alternating and sporadic components from a minimal counterexample to (H-QC): by \cite{GLS98} for example, if $L = A_n$ is the alternating group on $n$ letters, then $\text{Out}(L) = C_2$ for $n \neq 6$, and $\text{Out}(A_6) = C_2 \times C_2$. On the other hand, if $L$ is a sporadic group, then $\text{Out}(L) \leq C_2$. Therefore, if $p$ is odd and $L$ is a component of $G$ of alternating or sporadic type, $\text{Out}_{G}(L)$ is a $p'$-group.

In this way, we can also eliminate Suzuki and Ree components for $p = 2$ since their outer automorphism are $2'$-groups (see for example \cite{GLS98}).

We also give some tools to look for conditions on the outer automorphisms of a simple group and their centralizers in order to guarantee the hypotheses of Theorem 1.5 and hence establish (H-QC) for $G$ (cf. Proposition 5.7). This allows us to eliminate some alternating and sporadic components from a minimal counterexample to (H-QC) also in the case $p = 2$. 

\[ 3 \]
Corollary 1.5. Suppose that $G$ satisfies $(H1)$. Then $(H\text{-}QC)$ holds for $G$ if it satisfies one of the following:

1. $p$ is odd and $G$ has an alternating or a sporadic component.
2. $G$ has a sporadic component $H S$.
3. $G$ has an alternating component $\kappa_6$ or $\kappa_8$.
4. $G$ has a Lie type component $Sz(q)$, $2F_4(q)$ or $Ree(q)$.

When $p = 2$ we can also treat some of these components:

1. $G$ has a sporadic component $H S$.

Some sporadic groups $L$ have $Out(L) = 1$, so we get similar eliminations from a counterexample for $p = 2$. The interested reader can consult Table 5.3 of [GLS98] for $Out(L)$.

We state now one of our main theorems.

Theorem 1.6. Let $L$ be a component of $G$ and $L_1, \ldots, L_t$ its $G$-orbit. Suppose that $G$ satisfies $(H1)$ or $(H\text{-}LP)$.

Then $G$ satisfies $(H\text{-}QC)$. In particular, if $H = L_1 \cdots L_t C_G(L_1 \cdots L_t)$ then $G$ satisfies $(H\text{-}QC)$. This also holds if $Out_H(L)$ or $Out_G(L)$ is a $p'$-group.

In contrast with Segev’s Theorem [1.3] the “In particular” part of this theorem applies for some component of $G$, so that negating it forces nontrivial $Out_G(L)$ for all $L$. Hence, Theorem 1.6 is an elimination result.

In our theorem, we work with the local kernel $H = \bigcap_{i=1}^t N_G(L_i)$, where $L_1, \ldots, L_t$ is the $G$-orbit of the component $L$. In this case, instead of requiring $H = F^*(G)$ as in the “In particular” part of Segev’s Theorem [1.3] we ask for $H = L_1 \cdots L_t C_G(L_1 \cdots L_t)$, which holds for example if $Out_H(L) = 1$. The behavior of the remaining components is hidden in $C_G(L_1 \cdots L_t)$, which is covered by the inductive assumption (H1) or (HL(p)). Moreover, the original requirement $O_{p'}(G) = 1$ in Segev’s theorems is relaxed to the $p'$-divisibility condition of the component $L$.

Furthermore, the above theorems are particular cases of the more general and technical Theorem [1.1] which has more flexible hypotheses.

2. Notations and preliminaries

In this section, we establish some notation and recall some fundamental constructions on finite groups that will be used throughout this article. For more details on the assertions on finite posets and their homotopy properties in relation with their order complexes, we refer the reader to [Qui78]. For the results on finite groups we refer to [Asc00]. We will follow the conventions of [GLS83] for finite simple groups.

All the posets and simplicial complex considered here are finite. If $X$ is a finite poset then $K(X)$ denotes its order complex. Recall that the simplices of $K(X)$ are the non-empty chains of $X$. We study the homotopy properties of $X$ by means of its order complex. If $f : X \to Y$ is an order-preserving map between finite posets, then $f$ induces a simplicial map $f : K(X) \to K(Y)$. If $f, g : X \to Y$ are two order-preserving maps between finite posets and $f \leq g$ (i.e. $f(x) \leq g(x)$ for all $x \in X$), then the induced simplicial maps $f, g : K(X) \to K(Y)$ are homotopic. Write $X \simeq Y$ for finite posets $X, Y$ if their order complexes $K(X)$ and $K(Y)$ are homotopy equivalent.

We denote by $X * Y$ the join of the posets $X$ and $Y$. The underlying set of this join is the disjoint union of $X$ and $Y$, and the order is given as follows. We keep the given order in $X$ and $Y$, and we put $x < y$ for $x \in X$ and $y \in Y$. It can be shown that $K(X * Y) = K(X) * K(Y)$, where the latter join is the join of simplicial complexes. Moreover, its geometric realization coincides with the classical join of topological spaces. That is, if $K, L$ are simplicial complexes and $|K|$ denotes the geometric realization of $K$, then we have a homeomorphism $|K * L| = |K| * |L|$. For more details see [Qui78]. If $f : X \to X'$ and $g : Y \to Y'$ are order-preserving maps, then we have an induced map $f * g : X * Y \to X' * Y'$ defined by $(f * g)(x) = f(x) \in X'$ if $x \in X$, and $(f * g)(y) = g(y) \in Y'$ if $y \in Y$. 


Below we recall a generalized version of Quillen’s fiber lemma (cf. [Qui78, Prop. 1.6]). Recall that an $n$-equivalence is a continuous function $f : X \to Y$ such that $f$ induces isomorphisms in the homotopy groups $f_* : \pi_i(X) \to \pi_i(Y)$ with $i < n$, and an epimorphism in the $n$-th homotopy group. By the Hurewicz theorem, an $n$-equivalence also induces isomorphisms in the homology groups of degree $< n$, and an epimorphism in degree $n$. The topological space $X$ is $n$-connected if its homotopy groups of degree at most $n$ vanish (and hence its homology groups of degree at most $n$ also vanish). By convention, $(-1)$-connected means non-empty, and every space is $(-2)$-connected.

**Proposition 2.1** (Quillen’s fiber lemma). Let $f : X \to Y$ be a map between finite posets. Let $n \geq 0$. Suppose that for all $y \in Y$, $f^{-1}(Y_{\leq y}) * Y_{> y}$ (resp. $f^{-1}(Y_{\geq y}) * Y_{< y}$) is $(n-1)$-connected. Then $f$ is an $n$-equivalence.

In particular, if for all $y \in G$, $f^{-1}(Y_{\leq y}) * Y_{> y}$ (resp. $f^{-1}(Y_{\geq y}) * Y_{< y}$) is contractible, then $f$ is a homotopy equivalence.

Let $X$ be a finite poset. We denote by $\tilde{H}_n(X, R)$ the homology of $X$ with coefficients in the ring $R$, which is the homology of its order complex $K(X)$. In general we will work with $R = \mathbb{Q}$ and we will just write $\tilde{H}_n(X)$. If $f : X \to Y$ is an order-preserving map between finite posets, then we denote by $f_*$ the map induced in homology.

If $R$ is a field, by a theorem of Eilenberg-Zilber [EZ53] and the Künneth formulas, the homology of a join of spaces is the tensor product of homologies. That is, we have that

$$\tilde{H}_n(X * Y, R) = \tilde{H}_n(X, R) \otimes_R \tilde{H}_n(Y, R),$$

$$\tilde{H}_n(X * Y, R) = \bigoplus_{i+j=n-1} \tilde{H}_i(X, R) \otimes_R \tilde{H}_j(Y, R).$$

Note that we have a dimension shift in the join. Roughly, it adds one degree of connectivity in the above sense. For example, if $X, Y$ are non-empty (i.e. $(-1)$-connected), then $X * Y$ is path-connected (i.e. $0$-connected), and if one of them is $0$-connected, then their join is simply connected (i.e. $1$-connected). More generally, if $X$ is $n$-connected and $Y$ is $m$-connected, then $X * Y$ is $(n + m + 2)$-connected. For more details, see [Mil56].

We set now some notation on finite groups and recall some useful facts. All the groups considered here are finite. By a simple group we will mean a non-abelian simple group. The alternating and symmetric group on $n$ letters are denoted by $A_n$ and $S_n$ respectively. We also write $C_n$ and $D_n$ for the cyclic group of order $n$ and the dihedral group of order $n$, respectively.

For subgroups $H, K \leq G$, we denote by $N_K(H)$ the normalizer of $H$ in $K$, and by $C_K(H)$ the centralizer of $H$ in $K$. We also write $[H, K]$ for the subgroup generated by the commutators between elements of $H$ and $K$. Recall that $K$ normalizes $H$ if and only if $[H, K] \leq H$, and that $K$ centralizes $H$ if and only if $[H, K] = 1$. The derived subgroup of $G$ is $G' = [G, G]$. We denote by $Z(G)$, $F(G)$, $O_p(G)$, $O_{p'}(G)$ the center, the Fitting subgroup, the largest normal $p$-subgroup and the largest normal $p'$-subgroup of $G$, respectively.

Recall that $F(G)$ is the direct product of the subgroups $O_p(G)$ for $p$ a prime dividing the order of $G$. For solvable groups $G$, we have that $C_G(F(G)) \leq F(G)$ is self-centralizing. However, this property does not hold for arbitrary groups $G$ and we have to replace the subgroup $F(G)$ with a natural larger subgroup $F^*(G) = F(G)E(G)$, to get the desired self-centralizing property, as follows. The **generalized Fitting subgroup** of $G$ is the subgroup $F^*(G)$, which is the central product of the subgroups $F(G)$ and $E(G)$. The subgroup $E(G)$ is the **layer** of $G$ and it is defined as follows. A quasisimple group is a perfect group $L$ such that $L/Z(L)$ is simple. A **component** of $G$ is a subnormal quasisimple subgroup, and $E(G)$ is generated by all the components of $G$. Note that $G$ permutes its components via the natural conjugation action. If $L_1$ and $L_2$ are two different components of $G$, then they commute and $L_1 \cap L_2 \leq Z(L_1) \cap Z(L_2)$. Therefore $E(G)$ is a central product of quasisimple groups. We also have that $[F(G), E(G)] = 1$, and hence $F^*(G)$ is the central product of $F(G)$ and $E(G)$. The generalized Fitting subgroup is self-centralizing, that is,
$C_G(F^*(G)) \leq F^*(G)$, and when $G$ is solvable $F^*(G) = F(G)$. Moreover, $Z(F^*(G)) = Z(F(G))$ and $Z(E(G)) \leq Z(F(G))$. See Aschbacher [Asc00, Sec.37] for fuller reference.

Note that we always have $F(G) \leq O_p(G)O_p'(G)$. We will usually work under the assumption that $O_p(G) = 1 = O_p'(G)$, so that $F(G) = 1$ and hence, $Z(E(G)) = 1$. Since $Z(E(G))$ equals the product of the centers of the components of $G$, in this case we see that the components of $G$ are simple groups. We summarize these standard observations (compare [AS93, 1.6]) in the following lemma.

**Lemma 2.2.** Suppose that $O_p(G) = 1 = O_p'(G)$. Then $F(G) = 1$ and $F^*(G) = E(G)$ is the direct product of the components of $G$, which are all simple of order divisible by $p$. That is, $F^*(G) = L_1 \ldots L_t$ and each $L_i$ is a simple component of $G$. Moreover, since $C_G(F^*(G)) = Z(F(G)) = 1$, we have a natural inclusion

\[ F^*(G) \leq G \leq \text{Aut}(F^*(G)). \]

Recall that the outer automorphism group of $G$ is the group $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$, where $\text{Inn}(G) = G/Z(G)$ is the group of inner automorphisms of $G$. If $H \leq G$, recall that $\text{Out}_G(H) = N_G(H)/C_G(H)$ is the group of automorphisms of $H$ induced by $G$, and $\text{Out}_G(H) = N_G(H)/(HC_G(H))$ is the group of outer automorphisms of $H$ induced by $G$. The subgroup $HC_G(H)$ can be regarded as the subgroup of $G$ whose elements induce inner automorphisms on $H$. We will say that a subgroup $K \leq G$ induces (or acts by) outer automorphisms on $H$ if $K$ normalizes $H$ and $K$ contains no inner automorphism of $H$. That is, $K$ induces outer automorphisms on $H$ if and only if $K \cap (HC_G(H)) = 1$.

In the following lemma we relate the subgroups of inner automorphisms for different commuting subgroups of a given group. The proof is a straightforward use of the Dedekind modular law.

**Lemma 2.3.** Let $A, B \leq G$ such that $[A, B] = 1$. Then $(AC_G(A)) \cap (BC_G(B)) = ABC_G(AB)$. More generally, if $A_1, \ldots, A_r \leq G$ are such that $[A_i, A_j] = 1$ if $i \neq j$ then $\bigcap_i (A_iC_G(A_i)) = (A_1 \ldots A_r)C_G(A_1 \ldots A_r)$.

In Proposition 1.6 of [AS93] it is shown that if $G$ satisfies (H1), $p$ is odd and $G$ does not contain components of type $L_2(8)$, $U_3(8)$ or $Sz(32)$ for $p = 3, 3, 5$ resp., then $O_p'(G) \neq 1$ implies that (H-QC) holds for $G$. Inspired by this result, in Pit02 it is shown that the restriction on $p$ and the components is unnecessary. That is, if $G$ satisfies (H1) and $O_p'(G) \neq 1$, then $G$ satisfies (H-QC). We recall this result below.

**Theorem 2.4.** Suppose that $G$ satisfies (H1). If $O_p'(G) \neq 1$ then $G$ satisfies (H-QC).

**Remark 2.5.** Suppose that $G$ satisfies (H1). If $O_p'(G) \neq 1$, then $G$ satisfies (H-QC). If $O_p'(G) = 1$ and $O_p(G) = 1$, then by Lemma 2.2 every component $L$ of $G$ has order divisible by $p$. Moreover, if $L_1, \ldots, L_t$ is the $G$-orbit of $L$, then $C_G(L_1 \ldots L_t)$ is a proper subgroup of $G$ and hence it satisfies (H-QC) by (H1).

We haven shown that under (H1), then $G$ satisfies (H-QC) or (HL($p$)) for every component $L$.

To finish this section, we recall the definition of the inflated subposet $N_G(H)$ consisting of the elements of $A_p(G)$ which intersect $H$, where $H$ is a subgroup of $G$. This poset had been considered before in, for example, Seg96, SW94, and more recently in Pit20.

**Definition 2.6.** If $H \leq G$ and $B \subseteq A_p(G)$, let

\[ N_B(H) := \{ B \in B : B \cap H \neq 1 \}. \]

If $B = A_p(K)$ for some subgroup $K \leq G$, then we also write $N_B(H) = N_K(H)$.

In the following remark we recall some special features of this subposet.

**Remark 2.7.** We have that $N_G(H) \simeq A_p(H)$ via the retraction $E \mapsto r(E) = E \cap H$ with inverse given by the inclusion. Therefore, we can regard the subposet $N_G(H)$ as the inflation of $A_p(H)$ in $G$. 
Moreover, if $E \in A_p(G)$ and $E \cap H = 1$, then $r$ restricts to a homotopy equivalence $N_G(H)_{>E} \simeq A_p(C_H(E))$ with inverse given by $A \mapsto AE$. In particular, if we study the fibers of the inclusion $i : N_G(H) \to A_p(G)$, then for $E \in A_p(G) - N_G(H)$ we have that

$$i^{-1}(A_p(G)_{\geq E}) = N_G(H)_{>E} \simeq A_p(C_H(E)).$$

This observation connects Quillen’s fiber lemma (Proposition 2.1) with the study of centralizers.

3. Discussion: Segev’s Method, via a Mayer-Vietoris Argument

The idea of the proofs of Theorem 4.1 and then of Theorems 1.6 and 5.2 generalizes Segev’s argument [Seg96]. In this section we discuss the crucial points of Segev’s proof with an eye to alternative arguments.

Under $O_p(G) = 1$ for (H-QC) we want to prove:

(Goal) $\tilde{H}_*(A_p(G)) \neq 0$.

To this end, we look for a particular subgroup $H \leq G$ and consider its inflation $Y := N_G(H)$ along with the complement $Z := \{ E \in A_p(G) : E \not\leq H \}$. We can also regard $Y$ as the “neighborhood of $A_p(H)^*$”, and $Y \cap Z$ as its “boundary”. In this view, the effect of a condition like (A) below is to use restrictions on the “local” relation of the boundary $Y \cap Z$ to the neighborhood $Y$ to establish results for “global” $Y \cup Z$.

The Mayer-Vietoris sequence applied to the decomposition $A_p(G) = Y \cup Z$ yields

$$\ldots \to \tilde{H}_{k+1}(A_p(G)) \to \tilde{H}_{k}(Y \cap Z) \to \tilde{H}_{k}(Y) \oplus \tilde{H}_{k}(Z) \to \tilde{H}_{k}(A_p(G)) \to \ldots$$

Therefore, we see that $\tilde{H}_*(A_p(G)) \neq 0$ if the following condition holds:

(A) The induced map $\tilde{H}_*(Y \cap Z) \to \tilde{H}_*(Y)$ is not surjective.

Since $Y$ is homotopy equivalent to $A_p(H)$ via the retraction $r(E) = E \cap H$ (see Remark 2.7), condition (A) is equivalent to the following condition:

(A') The composition map $\tilde{H}_*(Y \cap Z) \to \tilde{H}_*(Y) \to \tilde{H}_*(A_p(H))$ is not surjective.

If we write $X_0$ for the image of $Y_0 = Y \cap Z$ under the map $r$, then condition (A') holds if the following stronger condition holds:

(B) $\tilde{H}_*(X_0) \to \tilde{H}_*(A_p(H))$ is not surjective.

That is, (B) implies (A'), which is equivalent to (A), and (A) implies (Goal). See diagram (3.1). Now, to show (B) we can explore the properties of the subgroup $H$. To show that the inclusion $X_0 \hookrightarrow A_p(H)$ is not surjective, Segev passed through the diagonal subposet $D_p(H)$ (see [Seg96] p.956]), where $H$ is the kernel on components of $G$. The subposet $D_p(H)$ consists of the elements $A \in A_p(H)$ such that there exist components $L_1, \ldots, L_t$ of $G$, $t \geq 2$, with $C_A(L_1 \ldots L_t) = C_A(L_i)$ for all $i$. In fact, Theorem 1 of [Seg96] requires that the inclusion $D_p(H) \to A_p(H)$ is not surjective in homology (and $O_p(G) = 1$), and he concludes (Goal). In the proof of his theorem, Segev shows that (B) holds since $X_0 \subseteq D_p(H)$.

For Theorems 2 and 3 of [Seg96], Segev passed through a poset $X$ obtained by breaking up $A_p(H)$ into a join of posets $X_i$, $i = 0, \ldots, s$, where the $X_i$ are posets closely related to very special subgroups of $H$ and $G$. Along with these posets we have projection maps $\pi_i : A_p(H) \to X_i \cup \{1\}$ such that if $\pi_i(A) = 1$ for all $i$ then $A = 1$. Then $X$ is defined as the join of the posets $X_i$ (in a fixed order) and $\psi : A_p(H) \to X$ is the poset map sending $A$ to $\pi_i(A)$, where $i$ is the maximum index with the property that $\pi_i(A) \neq 1$.

For the complex $K(X)$, there is a characteristic subcomplex $K_0$ such that the inclusion map $K_0 \hookrightarrow K(X)$ is the zero map in homology. At this point, in order to establish (B), Segev proposed to prove the following:

(C) $\psi(K(X_0)) \subseteq K_0$.

(D) $\tilde{H}_*(A_p(H)) \not\to \tilde{H}_*(X)$ is not the zero map in homology.
\[(E) \quad c : \tilde{H}_*(K_0) \to \tilde{H}_*(X) \text{ is the zero map in homology.}\]

We have that
\[(C) + (D) + (E) \Rightarrow (B) \Rightarrow (A') = (A) \Rightarrow (\text{Goal}).\]

See diagram \[3.1\]. To establish (C), Segev proved the containment \(X_0 \subseteq D_p(H)\) and \(K(D_p(H)) \subseteq K_0\) (see [Seg96, p.960]), where \(H\) is the kernel on components. Condition (D) is attained by the constraints on the components in the statements of Theorems 2 and 3 [Seg96], while condition (E) holds by the choice of \(K_0\) and \(X\), and the hypothesis \(O_p'(G) = 1\).

In this paper, we will mainly work with the local version of the kernel on components (that is, with the kernel of the permutation action on the \(G\)-orbit of a single component). Hence, we will not pass through the subposet \(D_p(H)\) and instead, we will establish (C) directly.

This discussion can be outlined in the following diagram:

\[
\begin{array}{ccc}
\tilde{H}_*(Y_0) & \xrightarrow{r|_{Y_0}} & \tilde{H}_*(X_0) \\
\downarrow a & & \downarrow b \\
\tilde{H}_*(Y) & \xrightarrow{r} & \tilde{H}_*(A_p(H)) \\
\downarrow c & & \downarrow \psi \\
\tilde{H}_*(X) & & \tilde{H}_*(X)
\end{array}
\]

Therefore, to get (Goal), we will look for a suitable choice of \(H\) which allows us to prove (D). Conditions (C) and (E) will follow easily by the construction of \(X\). To summarize, we can show (Goal) if we can establish one of the following:

- Prove (A) (or (A')) by showing that \(a : \tilde{H}_*(Y_0) \to \tilde{H}_*(Y)\) is the zero map and \(\tilde{H}_*(Y) \neq 0\) (or more generally, showing that \(a\) is not surjective).
- Prove (B) by showing that \(b\) is the zero map and \(\tilde{H}_*(A_p(H)) \neq 0\) (or more generally, showing that \(b\) is not surjective).
- Prove the following:
  - (C) by a suitable choice of \(X\) and \(K_0\),
  - (D) by showing that \(\psi\) is not the zero map in homology, and
  - (E), that is, \(c = 0\).

In Theorem 4.1, we will prove (C) and (E) by a proper choice of \(X\) and \(K_0\) as above. The construction of these spaces is inspired by Segev’s work, but with a slight variation. Then we will establish (D) by using one of the hypotheses (1), (2) and (3) of the statement of that theorem. Condition (D) is the key point which needs some extra hypotheses on the automorphisms of the components to hold.

\textbf{Remark 3.1.} When the map \(\psi\) is nonzero in homology we can regard this as the \textit{good behavior}. Otherwise, if \(\psi_* = 0\) in homology, we will say that this is a \textit{bad behavior}.

On the other hand, the map \(\psi_* \circ b \) or \(b\) has a \textit{good behavior} when it is the zero map in homology. Otherwise, when it is not the zero map in homology, we will say that this is a \textit{bad behavior}.

\section*{4. The Generalized Method, Leading to Elimination Results}

In this section, we provide an alternative version of Segev’s results [Seg96]. While Segev’s Theorem 1.4 requires a common behavior for all the components of \(G\), we show that in general we can focus on the behavior of a single component if we assume an additional inductive hypothesis, such as (H1).

The main result of this section is Theorem 4.1 below. Later, in Proposition 4.8 we give some sufficient conditions to guarantee the hypotheses of Theorem 4.1 in terms of the filtration given by the subgroups \(C_i(H)\) (see Definition 4.3 below). Finally, we close this section with Theorem 4.9 which is a generalization of [Seg96, Theorem 1].
Theorem 4.1. Let $L \leq G$ be a component of order divisible by $p$, and $\{L_1, \ldots, L_t\}$ its $G$-orbit by the conjugation action. Let $H = \bigcap_i N_G(L_i)$.

Suppose that there exists a subposet $\mathcal{B} \subseteq \mathcal{A}_r(H)$ such that $\psi|_\mathcal{B} : \mathcal{B} \to \mathcal{A}_p(C_H(N)) \ast \mathcal{A}_1 \ast \ldots \ast \mathcal{A}_t$ does not induce the zero map in homology. Then $G$ satisfies (H-QC). Hence, such an $L$ is eliminated from a counterexample.

In the original statement of Segev’s theorem, the hypothesis $O_p'(G) = 1$ is required. However, in our theorem it can be relaxed by only requiring that the components of $G$ have order divisible by $p$, as we emphasize in its proof below. Moreover, under (H1) either (H-QC) holds for $G$ in view of Theorem 2.4, or else $O_p'(G) = 1 = O_p(G)$. In the latter situation, the components of $G$ have order divisible by $p$ (see Lemma 2.2 and Remark 2.5).

If $X = \mathcal{A}_p(C_H(N)) \ast \mathcal{A}_1 \ast \ldots \ast \mathcal{A}_t$, then the hypotheses of the above theorem guarantee condition (D) (see the discussion in Section 3). That is, $\psi|_\mathcal{B}$ does not induce the zero in homology, which is the good behavior that we look for.

Now we explain the terminology of the above theorem, which we will use throughout this and the remaining sections. We begin by defining the image posets $\mathcal{A}_{G,L}$ and $\mathcal{A}_i$, that will play a fundamental role in our constructions.

Definition 4.2. For a subgroup $L$ of $G$, define the image poset $\mathcal{A}_{G,L}$ to be the image of the map

$$\pi : \mathcal{A}_p(N_G(L)) \to \mathcal{A}_p(C_G(L)) \to \mathcal{A}_p(\text{Aut}_G(L)).$$

Equivalently,

$$\mathcal{A}_{G,L} := \text{Im}(\pi : \mathcal{A}_p(N_G(L)) \to \mathcal{A}_p(\text{Aut}_G(L)) \cup \{1\}) - \{1\}.$$

Notation. We will write $\overline{A}$ to indicate the image of a subgroup $A$ under a fixed quotient map, when the latter is implicit.

In Lemma 4.2, we give some descriptions of the image poset $\mathcal{A}_{G,L}$ for $L$ any subgroup of $G$ with $p'$-center. In addition, if $L$ is quasisimple, we show that $H_\ast(\mathcal{A}_{G,L}) \neq 0$ (see Theorem 5.3).

We give now the definition of the posets $\mathcal{A}_i$ and the subgroups $C_i(H)$. The definition of these objects depends on the choice of the ordering of a $G$-orbit of components $L_1, \ldots, L_t$ of $G$.

Definition 4.3. Let $L_1, \ldots, L_t$ be a $G$-orbit of components of $G$. We define:

- $H := \bigcap_i N_G(L_i)$ and $N := L_1 \ldots L_t$.
- For $0 \leq i \leq t$, let $C_i(H) := C_H(L_{i+1} \ldots L_t)$, with $C_i(H) = C_H(1) = H$.
- $A_0 := \mathcal{A}_p(C_G(N))$ and $\pi_0$ is the identity map $\mathcal{A}_p(C_G(N)) \to \mathcal{A}_p(C_G(N))$.
- For $i \geq 1$, let $\pi_i$ be the map $\pi_i : \mathcal{A}_p(C_i(H)) \to \mathcal{A}_p(\text{Aut}_{C_i(H)}(L_i)) \cup \{1\}$ induced by taking quotient by $C_{C_i}(L_i) = C_{i-1}(L_i)$.
- For $i \geq 1$, the image poset $\mathcal{A}_i$ is the poset $\mathcal{A}_i := \mathcal{A}_{C_i(H),L_i}$.

From now on, we fix a component $L$ of $G$ and an ordering of its $G$-orbit $L_1, \ldots, L_t$. We may take $L_t = L$. Let $H, N, C_i(H)$ and $\mathcal{A}_i$ be as in the definition above. In the remark below, we discuss some immediate properties of these objects.

Remark 4.4. Note that

$$C_0(H) = C_H(L_1 \ldots L_t) = C_H(N) = C_G(N),$$

and by convention

$$C_t(H) = C_H(1) = H.$$

Moreover, we have a normal series of $H$ given by:

$$C_G(N) = C_0(H) \trianglelefteq C_1(H) \trianglelefteq \ldots \trianglelefteq C_t(H) = H.$$
Note that $H_i$ and $C_i(N)$ are normal subgroups of $G$.

On the other hand, note that $L_i \leq C_i(H)$ and $C_{i,j}(H)(L_i) = C_{i-1}(H)$, so Aut$_{C_i(L_i)}(L_i) = C_i(H)/C_{i-1}(H)$ and $A_i$ is the image of the $\pi_i$ map restricted to $A_p(C_i(H)) - A_p(C_{i-1}(H))$, since

$\pi_i^{-1}(1) = A_p(C_{i-1}(H))$.

Finally, if $0 \leq i \leq k \leq t$, then it is easy to verify that $C_i(C_k(i)) = C_i(H)$.

With this notation established, we construct a variant of the $\psi$ map given in Seg96. For $0 \leq i \leq t$, let $W_i = A_0 * A_1 * \ldots * A_i$. Define $\psi_{C_i(H)} : A_p(C_i(H)) \to W_i$ by

$\psi_{C_i(H)}(E) = \pi_k(E), \quad k = \max\{j \leq i : \pi_j(E) \neq 1\}$

$\psi_{C_i(H)}(E) = \pi_k(E), \quad k = \min\{j \leq i : E \subseteq C_j(E)\}$

The following lemma is an immediate consequence of the definition of these maps and Remark 1.4.

**Lemma 4.5.** The maps $\psi_{C_i(H)}$ are well-defined and order-preserving. Moreover, we have a commuting diagram for all $0 \leq i \leq k \leq t$:

\[
\begin{array}{ccc}
A_p(C_i(H)) & \xrightarrow[\psi_{C_i(H)}]{\psi_{C_i(H)}} & W_i \\
\downarrow & & \downarrow \\
A_p(C_k(H)) & \xrightarrow[\psi_{C_k(H)}]{\psi_{C_k(H)}} & W_k
\end{array}
\]

**Notation.** We will write $\psi_H$ or $\psi$ for $\psi_{C_i(H)}$, and $\psi_i$ for $\psi_{C_i(H)}$.

Now we can prove Theorem 4.1.

**Proof of Theorem 4.1.** Suppose that $O_p(G) = 1$. We show that, under the hypotheses of this theorem, $H_n(A_p(G)) \neq 0$. To that end, we use Segv’s argument following the discussion in Section 3. We will prove that conditions (C), (D) and (E) hold. We will see that (C) and (E) follow from the appropriate definition of $X$, $X_0$ and $K_0$. Condition (D) will follow immediately from the hypotheses.

Let $Y = N_G(H)$, $Z = A_p(G) - A_p(H)$ and $Y_0 = Y \cap Z = N_G(H) - A_p(H)$. See Definition 2.6. Consider the retraction $r : Y \to A_p(H)$ given by $r(E) = E \cap H$, which is a homotopy equivalence by Remark 2.7. By Mayer-Vietoris applied to the decomposition $A_p(G) = Y \cup Z$, we have that

$\ldots \to \tilde{H}_{k+1}(A_p(G)) \to \tilde{H}_k(Y_0) \to \tilde{H}_k(Y) \oplus \tilde{H}_k(Z) \to \tilde{H}_k(A_p(G)) \to \ldots$

Hence, $\tilde{H}_n(A_p(G)) \neq 0$ provided that the map $H_n(Y_0) \to \tilde{H}_n(Y)$ is not surjective (cf. condition (A) of Section 3). Let $X_0 = r(Y_0)$. In view of the commuting diagram 4.1, $\tilde{H}_n(A_p(G)) \neq 0$ if $H_n(X_0) \to \tilde{H}_n(A_p(G))$ is not surjective (cf. condition (B) of Section 3).

Let $\psi_H : A_p(H) \to A_0 * A_1 * \ldots * A_t$ be the map defined above. Let $X = A_0 * A_1 * \ldots * A_t$ and $K_0 = \bigcup_{i=0}^t \mathcal{K}(X - A_i)$. In the case $A_0 = \emptyset$, we exclude it from all the constructions above and simply take $X = A_1 * \ldots * A_t$.

In view of the discussion in Section 3 and diagram 4.1 (cf. diagram 4.1 below), we establish conditions (C), (D) and (E).

\begin{align*}
\tilde{H}_n(Y_0) & \xrightarrow{r|_{Y_0}} \tilde{H}_n(X_0) \xrightarrow{\psi_H|_{X_0}} \tilde{H}_n(K_0) \\
\tilde{H}_n(Y) & \xrightarrow{r_*} \tilde{H}_n(A_p(H)) \xrightarrow{\psi_H \neq 0} \tilde{H}_n(X)
\end{align*}

We show first that $\psi_H$ maps $\mathcal{K}(X_0)$ into $K_0$ (cf. condition (C) of Section 3).

**Claim:** (C) holds, that is, $\psi_H(\mathcal{K}(X_0)) \subseteq K_0$. 

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If $\sigma \in K(X_0)$, then $\sigma = (A_0 < \ldots < A_t)$ and there exists $E \in A_p(G)$ with $E \not\in H$ such that $A_t = E \cap H$. In particular $A_k \leq A_t \leq E$ for all $k$. The elements of $E - H$ induce nontrivial permutations on the set $\{L_1, \ldots, L_t\}$. Therefore, there exist $1 \leq i < j \leq t$ and $e \in E - H$ such that $L_i^e = L_j$. Suppose that $\psi_H(A_k) = \pi_i(A_k)$, so $A_k \leq C_i(H)$ and $A_k \not\leq C_{i-1}(H)$. These conditions say that $1 \neq [A_k, L_i]$ and $1 = [A_k, L_j]$ since $i < j$. Conjugating by $e$ gives that $1 \neq [A_k, L_i] = [A_k^e, L_i^e] = [A_k, L_j] = 1$, a contradiction. Hence, $\psi_H(A_k) \not\in A_i$ for all $k$ and we have that $\sigma \in K(X - A_t) \subseteq K_0$.

Condition (D) of Section 3 follows immediately since the restriction $\psi_H|B$ is not the zero map in homology by hypothesis. Hence, it remains to establish condition (E).

**Claim:** (E) holds, that is, $K_0 \hookrightarrow K(X)$ induces the zero map in homology.

Following Segev, we exhibit a contractible simplicial complex $K_0'$ such that $K_0 \subseteq K_0' \subseteq K(X)$. We take $K_0' = \bigcup_i St_{K(X)}(v_i)$, where $v_i \in A_i$ are fixed vertices and $St_{K(X)}(v_i)$ is the star of $v_i$ in $K(X)$. This complex is contractible by [AS92] (5.1). Note that at this point we are using that $A_i \neq \emptyset$ for all $i$, so that we can actually take these vertices $v_i$. To assure that $A_i \neq \emptyset$, we use that $L_i$ is not a $p'$-group and hence that $A_i \supseteq A_p(L_i) \neq \emptyset$ (see Lemma 6.2). If $A_i = \emptyset$, then $K_0 = K(X)$ and the claim does not hold unless $\tilde{H}_*(X) = 0$. Recall that if $A_0 = \emptyset$ then we exclude it from the definitions of $X$ and $K_0$. We take $\tilde{H}_*(X_0) \rightarrow \tilde{H}_*(A_p(H))$ cannot be surjective. In consequence, we get $\tilde{H}_*(A_p(G)) \neq 0$.

**Remark 4.6.** Note that in the above proof, if $G$ is almost simple, then $H = G, Y = A_p(G) = A_p(H), Y_0 = \emptyset$ and $X = A_t = A_p(Aut_G(F^n(G))) = A_p(G)$. Moreover, $\psi_H$ is the identity map of $A_p(G)$.

In particular, the condition on $\psi_H|B$ of Theorem 4.4 states that, if $G$ is almost simple, then there exists a subposet $B$ such that the inclusion $B \hookrightarrow A_p(G)$ is not the zero map in homology.

We give now some definitions and technical properties that may be useful to establish the conditions of Theorem 4.4. These are inspired by the inductive construction of the $A_p(H)$ poset via the filtration given by the $A_p(C_i(H))$ subposets, in the search for homology propagation and good behavior of the maps in homology.

For $1 \leq i \leq j \leq t$, we define the following maps:

$$\varphi_i : A_p(C_i(H)) \rightarrow A_p(C_{i-1}(H)) \ast A_i,$$

$$\varphi_i(E) := \begin{cases} E \in A_p(C_{i-1}(H)) & \text{if } E \leq C_{i-1}(H) \\ \pi_i(E) \in A_i & \text{if } E \not\leq C_{i-1}(H). \end{cases}$$

$$\Phi_{i,j} : A_p(C_i(H)) \ast A_{i+1} \ast \ldots \ast A_j \rightarrow A_p(C_{i-1}(H)) \ast A_i \ast A_{i+1} \ast \ldots \ast A_j$$

$$\Phi_{i,j}(E) := \begin{cases} \varphi_i(E) \in A_p(C_{i-1}(H)) \ast A_i & \text{if } E \in A_p(C_i(H)), \\ E \in A_k & \text{if } E \in A_k, i+1 \leq k \leq j. \end{cases}$$

The following lemma summarizes the relations between the maps $\Phi, \varphi$ and $\psi$.

**Lemma 4.7 (Relations $\Phi - \psi$).** The following hold:

1. $\Phi_{i,j} = \varphi_i \ast Id_{A_{i+1}} \ast \ldots \ast Id_{A_j}$.
2. $\Phi_{i,i} = \varphi_i$.
3. $\psi_i = \Phi_{i+1,i} \circ \ldots \circ \Phi_{2,i} \circ \Phi_{1,i}$.
4. If $\varphi_i$ induces an epimorphism in all the homology groups of degree $\leq n$, then so does $\Phi_{i,j}$ in all the homology groups of degree $\leq n + (j - i)$, for all $j \geq i$.
5. If $\varphi_i$ induces a monomorphism in all the homology groups of degree $\leq n$, then so does $\Phi_{i,j}$ in all the homology groups of degree $\leq n + (j - i)$, for all $j \geq i$. 


Proof. Parts (1), (2) and (3) are straightforward. For (4) and (5), note that the tensor product is exact over \( \mathbb{Q} \) and that \( \Phi_{i,j} = \varphi_i \circ \text{Id}_{A_{i+1}} \circ \cdots \circ \text{Id}_{A_i} \) induces the tensor product map in homology \( (\varphi_i)_* \circ (\text{Id}_{A_{i+1}})_* \circ \cdots \circ (\text{Id}_{A_i})_* \) (see equation 2.1).

We consider now the following two properties that will encode sufficient conditions to propagate homology from the posets \( A_p(C_i(H)) \) to \( A_p(H) \). We show then how to guarantee the conditions of Theorem 4.1 from these properties.

**Property E.** Let \( L \subseteq G \) be a normal component of \( G \) and denote by
\[
\pi : A_p(G) \rightarrow A_p(A_{\text{Aut}_G(L)}(L)) \cup \{1\} \text{ the map induced by the quotient. Let } A = A_{G,L} \text{ and define } \varphi : A_p(G) \rightarrow A_p(C_G(L)) * A \text{ by }
\]
\[
\varphi(E) = \begin{cases} 
E \in A_p(C_G(L)) & \text{if } E \leq C_G(L) \\
\pi(E) \in A & \text{if } E \not\leq C_G(L).
\end{cases}
\]
Fix \( n \geq 0 \). Then \( \varphi_* : \tilde{H}_m(A_p(G)) \rightarrow \tilde{H}_m(A_p(C_G(L)) * A) \) is an epimorphism for all \( m \leq n \).

**Property M.** Let \( L \subseteq G \) be a normal component of \( G \) and denote by
\[
\pi : A_p(G) \rightarrow A_p(A_{\text{Aut}_G(L)}(L)) \cup \{1\} \text{ the map induced by the quotient. Let } A = A_{G,L} \text{ and define } \varphi : A_p(G) \rightarrow A_p(C_G(L)) * A \text{ by }
\]
\[
\varphi(E) = \begin{cases} 
E \in A_p(C_G(L)) & \text{if } E \leq C_G(L) \\
\pi(E) \in A & \text{if } E \not\leq C_G(L).
\end{cases}
\]
Fix \( n \geq 0 \). Then \( \varphi_* : \tilde{H}_m(A_p(G)) \rightarrow \tilde{H}_m(A_p(C_G(L)) * A) \) is a monomorphism for all \( m \leq n \).

The following proposition gives sufficient conditions in terms of the above properties to establish the conditions of Theorem 4.1. We do not ask for (H1) nor (HL(p)), but we do require a special inductive assumption on the homology of certain \( p \)-subgroup posets.

**Proposition 4.8.** Let \( L \) be a component of \( G \) of order divisible by \( p \), and \( \{L_1, \ldots, L_t\} \) its \( G \)-orbit by the conjugation action. Let \( H = \bigcap_i N_G(L_i) \). Suppose that there exists \( n \geq 0 \) such that one of the following holds:

1. \( \tilde{H}_n(A_p(H)) \neq 0 \) and for all \( 1 \leq i \leq t \), \( (L_i, C_1(H)) \) has Property M for \( n-t+i \).
2. \( \tilde{H}_n(A_p(C_H(N)) * A_1 * \cdots * A_t) \neq 0 \) and for all \( 1 \leq i \leq t \), \( (L_i, C_1(H)) \) has Property E for \( n-t+i \).

Then the hypotheses of Theorem 4.1 hold, so that \( G \) satisfies (H-QC).

Proof. We show that (1) (resp. (2)) fulfills the conditions of Theorem 4.1 with \( B = A_p(H) \). By Lemma 4.7, \( \psi_H = \psi_i = \Phi_{1,t} \circ \cdots \circ \Phi_{1,i} \). By hypothesis, Property M (resp. Property E) holds for all the pairs \( (L_i, C_1(H)) \) and \( C_{C_1(H)}(L_i) = C_{i-1}(H) \), so each \( \varphi_i : A_p(C_1(H)) \rightarrow A_p(C_{i-1}(H)) * A_i \) induces a monomorphism (resp. an epimorphism) in all homology groups of degree \( \leq n-t+i \).

By Lemma 4.7 for all \( 1 \leq i \leq t \), \( \Phi_{i,t} \) induces monomorphisms (resp. epimorphisms) in all the homology groups of degree \( \leq (n-t+i) + (t-i) = n \). Therefore, \( \psi_H \) induces a monomorphism (resp. an epimorphism) in the \( n \)-th homology. Since the \( n \)-th homology group of the domain (resp. the codomain) of \( \psi_H \) is nontrivial by hypothesis, \( \psi_H \) does not induce the zero map in the \( n \)-th homology group.

To finish this section, we propose a local version of Theorem 1 of [Seg96]. For that purpose, we define a diagonal poset \( D_p(H) \) similarly as in [Seg96]. Let \( L_1, \ldots, L_t \) be an orbit of components of \( G \), and let \( H = \bigcap_i N_G(L_i) \) be the local kernel on these components. Let \( D_p(H) \) be the subposet of elements \( A \in A_p(H) \) such that there exists \( J \subseteq \{1, \ldots, t\} \) with \( |J| \geq 2 \) and \( C_A(L_j) = C_A(L_i) \) for all \( i, j \in J \).

We state our version of Theorem 1 of Segev. In contrast with the original theorem of Segev, we do not require \( O_p(p) = 1 \) nor some extra inductive hypothesis on the components.
Theorem 4.9. Let $L$ be a component of $G$ and $L_1, \ldots, L_t$ its $G$-orbit. Let $H$ be the local kernel $\bigcap_i N_G(L_i)$. If $D_p(H) \to A_p(H)$ is not surjective in homology, then $G$ satisfies (H-QC).

Proof. Assume that $O_p(G) = 1$. In view of the discussion in Section 3 we prove that (B) holds. That is, we need to show that $X_0 \to A_p(H)$ is not surjective in homology, where $X_0 = \{E \cap H : E \cap H \neq 1, E \notin H\}$. Following Segev, we show that $X_0 \subseteq D_p(H)$. In that case, we have a commuting diagram

\[
\begin{array}{ccc}
\hat{H}_*(X_0) & \xrightarrow{i} & \hat{H}_*(D_p(H)) \\
\downarrow b & & \downarrow d \\
\hat{H}_*(A_p(H)) & & \\
\end{array}
\]

where the maps are induced by the inclusions. Since $d$ is not surjective by hypothesis, $b = di$ is not surjective and hence (B) holds. Therefore $\hat{H}_*(A_p(G)) \neq 0$.

Now we prove that $X_0 \subseteq D_p(H)$. The proof follows the same idea of Claim (E) in the proof of Theorem 4.1 which is based on Segev’s original proof. Let $A \in X_0$, so that $A = E \cap H$, where $E \in A_p(G) - A_p(H)$ and $E \cap H \neq 1$. Then $E$ induces a nontrivial action on the $G$-orbit $\{L_1, \ldots, L_t\}$. Let $O$ be an orbit of this $E$-action, with $|O| \geq 2$, and take

\[J = \{i \in \{1, \ldots, t\} : L_i \in O\}.\]

By construction, $|J| = |O| \geq 2$. Let $i, j \in J$ and take $e \in E$ such that $L_i^e = L_j$. Since $e$ commutes with $A$, $\chi(A(L_i)) = \chi(A(L_i)^e) = \chi(A(L_j))$. Hence $A \in D_p(H)$, as we wanted. \qed

5. CONSEQUENCES OF THE GENERAL METHOD

In this section we establish some corollaries of the technical Theorem 4.1.

The local version of Segre’s Theorem 1.4 follows from Theorem 4.1 after assuming one of the inductive hypotheses (H1) or (HL(p)).

Corollary 5.1. Let $L$ be a component of $G$ and let $L_1, \ldots, L_t$ be its $G$-orbit. Suppose that $G$ satisfies (H1) or (HL(P)).

If $A_p(L_i) \to A_i$ is nonzero in homology for each $i$, then $G$ satisfies (H-QC). In particular, this holds if $A_p(L) \to A$ is not the zero map in homology, where $A$ is one of the following posets

\[A_{H,L}, A_{G,L}, A_p(Aut_H(L)), A_p(Aut_G(L)), A_p(Aut(L)).\]

Proof. By Remark 2.5 we can assume that $O_p(G) = 1$ and that (HL(p)) holds.

First, we show the “In particular” part. Fix $i$ and let $g \in G$ with $L_i = L^g$. Let $A$ be one of the following posets

\[A_{H,L}, A_{G,L}, A_p(Aut_H(L)), A_p(Aut_G(L)), A_p(Aut(L)).\]

Therefore $A_p(L_i) \subseteq A_i \subseteq A^g$, where $A^g$ is the corresponding $A$ poset for $L^g$ obtained via the conjugation action. If $A_p(L) \to A$ is not the zero map in homology, then neither is $A_p(L_i) = A_p(L^g) \to A^g$. Hence, $A_p(L_i) \to A_i$ is not the zero map in homology.

Now suppose that $A_p(L) \to A_i$ is not the zero map in homology. We prove that condition (3) of Theorem 4.1 holds for $B = A_p(H_0)$, where $H_0$ is the subgroup $C_G(N)L_1 \ldots L_t$. This shows that condition (D) of Section 3 holds. Note that $\psi_H|_B$ can be written as the composition of an inclusion $j$ with $\psi_{H_0}$ as follows:

\[A_p(C_G(N)L_1 \ldots L_t) \xrightarrow{\psi_{H_0}} A_p(C_G(N)) \ast A_p(L_1) \ast \cdots \ast A_p(L_t) \xrightarrow{j} A_p(C_H(N)) \ast A_1 \ast \cdots \ast A_t.\]

Here $\psi_{H_0}$ is obtained by restricting the map $\psi_H$, so that $\psi_H|_B = j \circ \psi_{H_0}$. We know that $\psi_{H_0}$ is a homotopy equivalence (cf. Theorem 1.4 or Proposition 2.1). On the other hand, if $j_i : A_p(L_i) \to A_i$, denotes the inclusion, with $j_0 : A_p(C_H(N)) \to A_p(C_H(N))$ the identity map, then $j = j_0 \ast j_1 \ast \cdots \ast j_t$ is the induced map in the join of the posets. Hence, the map induced in homology is $j_\ast =
(j_0)_* \otimes (j_1)_* \otimes \ldots \otimes (j_t)_*. The map j_* is nonzero since each map (j_i)_* is nonzero by hypothesis and we are taking tensor product over a field. Hence, the composition \((\psi H | B)_* = (j \circ \psi_{H_0})_* = j_* \circ (\psi_{H_0})_*\) is a nonzero map.

Note that Theorem 5.5 is just a particular case of Corollary 6.1.

We give below an alternative corollary of Theorem 4.1.

**Corollary 5.2.** Let \(L\) be a component of \(G\) and \(L_1, \ldots, L_t\) its \(G\)-orbit. Suppose that \(G\) satisfies \((H_1)\) or \((HL(p))\), and that:

(i) For each \(1 \leq i \leq t\), there exists a subgroup \(F_i\) such that \(L_i \leq F_i \leq N_G(L_i), [F_i, C_G(L_i)] = 1\) and \(F_i \cap C_G(L_i)\) is a \(p'\)-group.

(ii) \(A_p(N_G(L_i)) = A_p(F_i C_G(L_i)) \text{ for all } i\).

(iii) If \(i \neq j\) then \([F_i, F_j] = 1\).

Then \(G\) satisfies \((H-QC)\).

**Proof.** In view of Remark 2.5 since we are trying to prove \((H-QC)\), we can work directly under \((HL(p))\). Suppose that \(O_p(G) = 1\).

We prove that the special conditions on the \(F_i\) subgroups yield a good behavior for the map \(\psi_H\), giving the hypotheses of Theorem 4.1 and hence that \(H_*(A_p(G)) \neq 0\).

By (i), \(F_i \leq C_G(C_G(L_i))\), so \(C_G(L_i) \leq C_G(F_i)\) and equality holds. In this way, \(Z(F_i) = F_i \cap C_G(F_i)\) is a \(p'\)-group also by (i).

Suppose that \(i \neq j\). Then we have that \([F_i, F_j] = 1\) by (iii), so \(F_i \leq C_G(F_j)\) and \(F_i \cap F_j \leq Z(F_i) \cap Z(F_j)\) is a \(p'\)-group. Together with (ii) and Lemma 2.3 these observations allow us to conclude that

\[
A_p(H) = A_p(F_1 \ldots F_t C_H(F_1 \ldots F_t))
\]

and \(C_H(F_1 \ldots F_t) = C_G(F_1 \ldots F_t) = C_G(L_1 \ldots L_t)\). Moreover, \(F_1 \ldots F_t C_H(F_1 \ldots F_t)\) is a central product by \(p'\)-centers. Let \(N = L_1 \ldots L_t, \) so \(C_G(N) = C_H(N) = C_H(F_1 \ldots F_t)\).

Now we check the conditions of Theorem 4.1. We prove that the map \(\psi_H : A_p(H) \to A_p(C_H(N)) A_1 \times \ldots \times A_t\) is not zero in homology because it is a homotopy equivalence with non-cyclic codomain. Recall that \(A_i = \operatorname{Im}(A_p(C_i(H))) \to A_p(\operatorname{Aut}_{C_i(H)}(L_i) \cup \{1\}) - \{1\},\) where \(C_i(H) = C_H(L_{i+1} \ldots L_t)\) (see Definitions 4.2 and 4.3).

First, we claim that \(A_i = A_p(F_i / Z(F_i))\). By (iii) we have that

\[
F_i \leq \bigcap_{j \neq i} C_G(F_j) = \bigcap_{j \neq i} C_G(L_j) \leq C_i(H).
\]

Moreover, \([F_i, C_{i-1}(H)] \leq [F_i, C_G(L_i)] = 1\) by (i), and the intersection \(F_i \cap C_{i-1}(H) \leq F_i \cap C_G(L_i)\) is a \(p'\)-group. Therefore, if \(A, B \in A_p(F_i)\) and \(AC_{i-1}(H) = BC_{i-1}(H)\), then it is not hard to see that \(A = B\). Now, if \(A \in A_p(N_G(L_i))\), by (ii) \(A \leq A_0 \times A_1\), where \(A_0\) is the projection of \(A\) onto \(F_i\) and \(A_1\) is the projection of \(A\) onto \(C_G(L_i)\). If we quotient by \(C_i(H)\), then \(A_{C_i(H)} / C_{i-1}(H) = A_0\).

Hence, the image of the map that goes from \(A_p(C_i(H))\) into \(A_p(\operatorname{Aut}_{C_i(H)}(L_i) \cup \{1\}) \) is exactly \(A_p(F_i / Z(F_i)) \cup \{1\}\), so \(A_i = A_p(F_i / Z(F_i))\).

Second, we prove that the map \(\varphi : A_p(C_i(H)) \to A_p(C_{i-1}(H)) A_i\) is a homotopy equivalence for each \(i\). If \(A \in A_p(F_i / Z(F_i))\), then \(A\) can be viewed in \(A_p(F_i)\) since \(Z(F_i) = 1\). Hence,

\[
\varphi^{-1}(A_p(C_{i-1}(H)) A_i) = A_p(A C_{i-1}(H)),
\]

which is contractible since \(O_p(A C_{i-1}(H)) \geq A\). On the other hand, of \(A \in A_p(C_{i-1}(H))\) then

\[
\varphi^{-1}(A_p(A C_{i-1}(H)) A_i) = A_p(A)
\]

is also contractible. Therefore, by Quillen’s fiber lemma (cf. Proposition 2.1), \(\varphi\) is a homotopy equivalence. At this point, we note that \(\psi_H\) is a homotopy equivalence since \(\psi_H = \Phi_{1,t} \circ \ldots \circ \Phi_{t,t}\) is the composition of homotopy equivalences by Lemma 4.7 (each \(\Phi_{t,t}\) is a homotopy equivalence since \(\Phi_{t,t} = \varphi \circ \operatorname{Id} A_{i+1} \times \ldots \times \operatorname{Id} A_t\)).
It remains to show that \( \psi_H \) is not the zero map in homology. By (HL\( p \)), \( C_G(N) \) satisfies (H-QC), and \( O_p(C_G(N)) = 1 \) since \( C_G(N) \) is normal in \( G \) and \( O_p(G) = 1 \). Therefore \( \hat{H}_c(A_p(C_G(N))) \neq 0 \). Now, for each \( i \), the quotient \( F_i/Z(F_i) \) is an almost simple group, so by Theorem \ref{thm:hqc}, we have that \( \hat{H}_c(A_p(F_i/Z(F_i))) \neq 0 \). Finally, by the homology decomposition of the join of spaces (see equation \ref{eq:join}), \( A_p(C_G(N)) \ast A_p(F_1/Z(F_1)) \ast \ldots \ast A_p(F_i/Z(F_i)) \) has nontrivial homology, and since \( \psi_H \) is a homotopy equivalence, it is not the zero map in homology. \( \square \)

6. Properties of the image poset

In this section we provide some properties of the image poset \( A_{G,L} \). We show that \( \hat{H}_c(A_{G,L}) \neq 0 \) when \( L \) is quasisimple, extending the almost simple case of (H-QC) to posets of this form. We also give specific conditions on the centralizers of a simple group \( L \) such that \( G \subseteq A_{p} \) and \( \hat{H}_c(A_p) \neq 0 \).

**Definition 6.1.** Let \( L \) be a subgroup of \( G \). If \( A \in A_p(N_G(L)) \) is an elementary abelian \( p \)-subgroup such that \( A \cap (LC_G(L)) = 1 \), then we say that \( A \) is a \( p \)-outer of \( L \) in \( G \). We define the \( p \)-outer posets

\[
I_G(L) = \{ A \in A_p(N_G(L)) : A \text{ is } p\text{-outer on } L \}
\]

\[
\hat{I}_G(L) = I_G(L) \cup \{1\}.
\]

If \( P \subseteq A_p(G) \), let \( I_P(L) = P \cap I_G(L) \) and \( \hat{I}_P(L) = I_P(L) \cup \{1\} \).

We say that \( L \) admits only cyclic \( p \)-outers in \( G \) if \( I_G(L) \) is non-empty and its elements are cyclic of order \( p \) (that is, the \( p \)-outers of \( L \) are cyclic).

We study the properties of the poset image poset \( A_{G,L} \) defined in the previous section. Recall from Definition \ref{def:image-poset} that if \( L \leq G \) is a subgroup and \( \pi : A_p(N_G(L)) \to A_p(Aut_G(L)) \cup \{1\} \) is the map induced by the quotient \( N_G(L) \to Aut_G(L) \), then

\[
A_{G,L} = \pi(A_p(N_G(L))) - \{1\} = \pi(A_p(N_G(L)) - A_p(C_G(L))).
\]

We use the bar notation \( \bar{E} \) to denote the image of an element \( E \) under the map \( \pi \). We will characterize this poset in terms of the \( p \)-outers of \( L \) and prove that \( \hat{H}_c(A_{G,L}) \neq 0 \) if \( L \) is a quasisimple subgroup of \( G \) by using \cite{AK90}. This can be seen as a small improvement of the almost simple case of Quillen’s conjecture since we show that the conjecture also holds for the poset \( A_{G,L} \) and not only for posets \( A_p(T) \) with \( F^*(T) = L \).

In the following lemma we provide different descriptions of the image poset.

**Lemma 6.2 (Description of \( A_{G,L} \)).** Let \( L \leq G \) be a subgroup with \( p' \)-center, and let

\[
\pi : A_p(N_G(L)) \to A_p(Aut_G(L)) \cup \{1\}
\]

be the map induced by the quotient \( N_G(L) \to Aut_G(L) \). Then \( \pi \) embeds \( A_p(L) \) into \( A_p(Aut_G(L)) \), so we identify \( A_p(L) \) with its image via \( \pi \) and we suppress the bar notation on the elements of \( L \).

We have that

\[
A_{G,L} = \bigcup_{E \in I_G(L)} A_p(L\bar{E}).
\]

Equivalently, \( A_{G,L} \) is isomorphic to the poset

\[
\{ EC_G(L) : E \in A_p(N_G(L)) - N_G(C_G(L)) \} = \{ EC_G(L) : E \in A_p(N_G(L)), C_E(L) = 1 \}.
\]

In particular, we have that

\[
I_{A_{G,L}}(L) = \{ E : E \in I_G(L) \},
\]

and if \( I_G(L) = \emptyset \) (i.e. there are no \( p \)-outers), then \( A_{G,L} = A_p(L) \).
Proof. We have a natural poset isomorphism $\mathcal{A}_p(L) \to \mathcal{A}_p(L/Z(L))$ induced by the quotient map $L \to L/Z(L)$ since $Z(L)$ is a $p'$-group by hypothesis. Therefore, without loss of generality, we can assume that $Z(L) = 1$.

Let $A, B \in \mathcal{A}_p(L)$ such that $A \leq B$. Then $AC_G(L) \leq BC_G(L)$. Since $A$ and $B$ commute with $C_G(L)$ and $A \cap C_G(L) = 1 = B \cap C_G(L)$ by the condition $Z(L) = 1$, we get $A \leq B$. This shows that $\mathcal{A}_p(L)$ embeds into $\mathcal{A}_p(\text{Aut}_G(L))$ naturally via $\pi$.

Now we show that $\mathcal{A}_{G,L} = \bigcup_{E \in \mathcal{A}_p(N_G(L))} \mathcal{A}_p(L/E)$. If $E \in \mathcal{A}_p(N_G(L))$, then $E = (E \cap LC_G(L))E_1$ for some complement $E_1$ to $E \cap (LC_G(L))$. Then $E_1 \in \mathcal{I}_G(L)$. Write $E_0$ for the projection of $E \cap (LC_G(L))$ into $L$. Note that $E_0$ and $E_1$ commute and $(E_0E_1) \cap C_G(L) = 1$. Then,

$$E = C_G(L)E/C_G(L) = C_G(L)E_0E_1/C_G(L) \cong E_0E_1.$$  

Finally, since $E_0 \leq L$, we have that $E \in \mathcal{A}_p(LE_1)$.

By the isomorphism theorems, the poset $\mathcal{A}_{G,L}$ is isomorphic to the poset of subgroups

$$\{EC_G(L) : E \in \mathcal{A}_p(N_G(L)) - \mathcal{A}_p(C_G(L))\}.$$  

If $E \in \mathcal{A}_p(N_G(L)) - \mathcal{A}_p(C_G(L))$, write $E = E_0C_E(N)$ for some complement $E_0$ to $C_E(L)$ in $E$. In this case note that $E_0 \neq 1$. Then $EC_G(L) = E_0C_G(L)$, and $E_0 \notin N_G(C_G(L))$. Therefore,

$$\mathcal{A}_{G,L} \equiv \{EC_G(L) : E \in \mathcal{A}_p(N_G(L)) - \mathcal{A}_p(C_G(L))\} \equiv \{EC_G(L) : E \in \mathcal{A}_p(N_G(L)) - N_G(C_G(L))\}.$$  

The “in particular” part is clear from these descriptions of $\mathcal{A}_{G,L}$. □

By Lemma [52] if $L$ is simple, then $\mathcal{A}_{G,L} = \bigcup_{\bar{A} \in \mathcal{A}_p(L)} \mathcal{A}_p(\bar{L}A) \subseteq \mathcal{A}_p(\text{Aut}_G(L))$. This subposet may not be a poset of the form $\mathcal{A}_p(T)$ for some almost simple group $T \leq \text{Aut}_G(L)$, but it is somewhat analogous. Our aim now is to show that $\mathcal{A}_{G,L}$ behaves like the Quillen poset of an almost simple group, so that $\hat{H}_\ast(\mathcal{A}_{G,L}) \neq 0$. We use the generalized version of Robinson’s Lemma due to Aschbacher-Smith [AS93, Section 5], together with the proofs of [AK90] on the almost simple case of the conjecture.

Recall that a $q$-hyperelementary group is a group $H$ such that $O^q(H)$ is cyclic.

Lemma 6.3 (AS93, Lemma 0.14). Suppose that a $q$-hyperelementary $H$ acts on a poset $Y$ with $\hat{H}_\ast(Y) = 0$. Then $\bar{\chi}(Y^H) \equiv 0 \mod q$. In particular, $Y^H$ is non-empty.

The almost simple case of (H-QC) is a consequence of the above lemma and the main theorems of [AK90]. We summarize the results that we will need from [AK90] in the theorem below.

Theorem 6.4 (Aschbacher-Kleidman). Let $T$ be an almost simple group. The following are equivalent:

1. For all hyperelementary nilpotent $p'$-subgroup $H$ of $F^*(T)$, the fixed point set $S_p(T)^H$ is not empty.
2. $p = 2$, $F^*(T) = L_3(A)$ and $4 \mid |T : F^*(T)|$.

Moreover, (H-QC) holds for almost simple groups.

Theorem 3 of [AK90] establishes (H-QC) for almost simple groups by using the equivalence of the above theorem. However, the proof of that Theorem 3 has a small gap which can be easily fixed as we show in the proof below.

Theorem 6.5 (Homology for $\mathcal{A}_{G,L}$). Suppose that $L \leq G$ is a quasisimple subgroup with $p'$-center, and let $\mathcal{A}_{G,L}$ be the image poset. Then $\hat{H}_\ast(\mathcal{A}_{G,L}) \neq 0$.

Proof. Without loss of generality we can assume that $Z(L) = 1$. Also write $\mathcal{A} = \mathcal{A}_{G,L}$. Recall from Lemma [6.2] that $\mathcal{A}_p(L) \subseteq \mathcal{A}$.

If part (2) of Theorem 6.4 does not hold for $T = \langle \mathcal{A} \rangle \leq \text{Aut}_G(L)$. Then there exists a $q$-hyperelementary $p'$-subgroup $H \leq L$ such that $S_p(T)^H$ is the empty set. Note that $H$ acts on $\mathcal{A}$, and in particular, $\mathcal{A}^H \subseteq S_p(T)^H$ is empty. By Lemma [6.3] $\hat{H}_\ast(\mathcal{A}) \neq 0$.  

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Now suppose that part (2) of Theorem 6.3 does hold, that is, \( L = L_3(4) \) and \( 4 \mid [T : L] \). In this case, following the original proof of the almost simple case in page 211 of [AK90], we take \( P \in \text{Syl}_5(L) \), which is cyclic of order 5. In the original proof of Aschbacher-Kleidman it is stated that \( \mathcal{A}_p(\text{Aut}(L))^P \) has exactly three points: a subgroup of order 2 and two subgroups of order 16. But this is not correct, since we have in fact five points in \( \mathcal{A}_p(\text{Aut}(L))^P \). Namely, \( \mathcal{A}_p(L)^P \) is discrete with two points (two subgroups of order 16), and \( \mathcal{A}_p(\text{Aut}(L))^P \) is discrete with five points (two subgroups of order 16 and three subgroups of order 2). In any case, if a subposet \( Y \subseteq \mathcal{A}_p(\text{Aut}(L)) \) containing \( \mathcal{A}_p(L) \) is stable under the action of \( P \), then \( Y^P \) is a discrete poset of cardinality between 2 and 5. Therefore, \( \chi(Y^P) \) cannot be 0 mod 5. By Lemma 6.3 \( \hat{H}_*(Y) \neq 0 \). In particular, this holds if \( Y = \mathcal{A} \), since \( P \leq L \) and \( L \) acts on \( \mathcal{A} \), or if \( Y = \mathcal{A}_p(T) \) for some almost simple group \( T \) with \( F^*(T) = L \). This fixes the small gap in [AK90] and finishes our proof. □

Remark 6.6. The above theorem can be strengthened (details will appear elsewhere) to show that \( \mathcal{A}_{G,L} \) in fact satisfies the Lefschetz version of Quillen’s conjecture. That is, we have the stronger conclusion that \( \hat{\chi}(\mathcal{A}_{G,L}^g) \neq 0 \) for some \( g \in L \).

We close this section with a useful proposition that gives a sufficient condition to show that \( \mathcal{A}_p(L) \to \mathcal{A}_{G,L} \) is not the zero map in homology in terms of the centralizers of the \( p \)-outers of \( L \). It establishes (H-QC) for \( G \) in view of Corollary 5.1.

**Proposition 6.7.** Suppose that \( G \) satisfies (H1). Let \( L \) be a component of \( G \) such that the following conditions hold:

1. \( I_G(L) \) is either empty, or it consists only of cyclic \( p \)-outers,
2. There exists \( k \geq 0 \) such that the induced map \( \hat{H}_k(\mathcal{A}_p(C_L(E))) \to \hat{H}_k(\mathcal{A}_p(L)) \) is the zero map for all \( E \in I_G(L) \).
3. \( \hat{H}_k(\mathcal{A}_p(L)) \neq 0 \).

Then \( G \) satisfies (H-QC).

**Proof.** Suppose that \( O_p(G) = 1 \). By Theorem 2.3 we can assume that \( O_p'(G) = 1 \). Let \( \mathcal{A} = \mathcal{A}_{G,L} \).

In view of Corollary 5.1 we show that \( \mathcal{A}_p(L) \to \mathcal{A} \) is not the zero map in homology, where

\[
\mathcal{A} = \bigcup_{E \in I_G(L)} \mathcal{A}_p(L(E)) \subseteq \mathcal{A}_p(\text{Aut}_G(L))
\]

as in Lemma 6.2. Recall also that \( I_A(L) = \{ E : E \in I_G(L) \} \). Therefore, \( I_A(L) \) is either empty or it consists only of cyclic \( p \)-outers by hypothesis (1). Let \( N = N_A(L) \). If \( E \in I_A(L) \), then \( N_{E} \cong \mathcal{A}_p(C_L(E)) \) via the retraction \( A \to A \cap L \) with inverse \( A \to AE \) (cf. Remark 2.7).

Put \( I_A(L) = \{ E_1, \ldots, E_r \} \), with \( r \geq 0 \), and \( X_i = N \cup \{ E_1, \ldots, E_i \} \). Then \( X_{i+1} = X_i \cup N_{E_{i+1}} \) and \( X_i \cap N_{E_{i+1}} \). The Mayer-Vietoris sequence in degree \( k \) is:

\[
\cdots \to \hat{H}_{k+1}(X_{i+1}) \to \hat{H}_k(N_{E_{i+1}}) \to \hat{H}_k(X_i) \to \hat{H}_k(X_{i+1}) \to \cdots
\]

Therefore, the map \( \hat{H}_k(N_{E_i}(E_{i+1})) \to \hat{H}_k(X_i) \) is trivial and we have a monomorphism

\[
\hat{H}_k(X_i) \to \hat{H}_k(X_{i+1})
\]

for all \( i \). In particular, we have a monomorphism

\[
\hat{H}_k(\mathcal{A}_p(L)) \xrightarrow{\cong} \hat{H}_k(\mathcal{A}_p(L)) = \hat{H}_k(X_0) \to \hat{H}_k(X_r) = \hat{H}_k(\mathcal{A}).
\]

Note that this map is induced by the inclusion \( \mathcal{A}_p(L) \to \mathcal{A} \), and it is nontrivial if \( \hat{H}_k(\mathcal{A}_p(L)) \neq 0 \). By Corollary 5.1 we conclude that \( \hat{H}_*(A_p(G)) \neq 0 \). □
7. Elimination of components of sporadic type HS

In this section, we prove part (2) of Corollary 5.1. That is, we show that under (H1), if $G$ has a component isomorphic to $HS$ (the Higman-Sims sporadic group), then $G$ satisfies (H-QC). In view of Corollary 5.1, we prove that $A_2(HS) \to A_2(Aut(HS))$ is not the zero map in homology. To that end, we inspect the second homology groups of these posets by passing through their Euler characteristic, and the structure of the centralizers of the 2-outers. We performed some computations in GAP [GAP] with the package [EPSC]. Recall that for odd $p$ we have $A_p(HS) = A_p(Aut(HS))$, so that we can directly apply Corollary 5.1.

The main result of this section is the following theorem.

**Theorem 7.1.** Let $L = HS$ be the Higman-Sims sporadic group and let $A = Aut(HS)$. Then $\tilde{\chi}(A_2(L)) = 1767424$ and $\tilde{\chi}(A_2(A)) = 1204224$, and the following assertions hold:

1. $m_2(L) = 4$ and $m_2(A) = 5$.
2. $A_2(L) \to A_2(A)$ is a 2-equivalence.
3. $H_n(A_2(A)) = 0$ for $n \geq 4$.
4. $H_3(A_2(L)) \leq H_3(A_2(A))$.

In particular, if $G$ satisfies (H1) and $L$ is a component of $G$, then $G$ satisfies (H-QC).

For the “In particular” part of the above theorem, we show that, even though condition (2) of Proposition 5.1 does not hold, the same Mayer-Vietoris sequence of its proof shows that $A_2(HS) \to A_2(Aut(HS))$ is not the zero map in homology.

Recall from Definition 2.6 that if $H \leq G$, then $N_G(H) = \{A \in A_p(G) : A \cap H \neq 1\}$. This poset is homotopy equivalent to $A_p(H)$ via the retraction $r : N_G(H) \to A_p(H)$ defined by $A \mapsto A \cap H$, with inverse given by the inclusion $A_p(H) \hookrightarrow N_G(H)$. Moreover, if $E \notin N_G(H)$, then $N_G(H) > E$ is homotopy equivalent to $A_p(C_H(E))$ via the same retraction $r(A) = A \cap H$ with inverse $i : A_p(C_H(E)) \to N_G(H)$ given by $i(A) = AE$.

Also recall that

$$\tilde{\chi}(A_p(G)) = \sum_{E \in A_p(G) \cup \{1\}} (-1)^{m_p(E)-1}p^{m_p(E)(m_p(E)-1)/2}.$$

See for example [JM12].

**Proof of Theorem 7.1.** The values of the Euler characteristic follow from the above formula and direct computation.

For the following assertions on the structure of the subgroups of Aut(HS), we refer to Table 5.3m of [GLS98]. Since $L$ has index 2 in $A$, $I_A(L)$ consists only of cyclic 2-outers. Indeed, if $E \in I_A(L)$, then $E$ is of type 2C or 2D, in view of Table 5.3m. We recall below the structure of the centralizers of these involutions.

The centralizer $C_L(2C)$ is a non-split extension of an elementary abelian 2-group of 2-rank 4 by $O_2^-(2)$. In particular, $O_2^-(C_L(2C)) \neq 1$ and $A_2(C_L(2C))$ is contractible. Part (1) follows from Table 5.6.1 of [GLS98] and the fact that $m_2(C_L(2C)2C) = 5$.

On the other hand, $C_L(2D) \cong S_8$, and $A_2(S_8)$ is homotopy equivalent to a wedge of 512 spheres of dimension 2. This holds since $A_2(S_8)$ is simply connected, the Bouc poset $B_2(S_8)$ of non-trivial radical 2-subgroups has dimension 2 (which is homotopy equivalent to $A_2(S_8)$), and $\tilde{\chi}(A_2(S_8)) = 512$. These assertions can be directly computed or else checked with GAP and the package [EPSC].

Let $N = N_A(L)$, so that $A_2(A) = N = I_A(L)$. Moreover, if $E \in I_A(L)$ and $E$ is of type 2C, then $N_{>E} \cong A_2(C_L(2C))$ is contractible. Let $J_A(L) = \{E \in A_2(A) : E \text{ is of type 2D}\}$. Then $N \cup J_A(L) \hookrightarrow A_2(A)$ is a homotopy equivalence in view of Proposition 2.1. Further, the inclusion $N \hookrightarrow N \cup J_A(L)$ is a 2-equivalence also by Proposition 2.1 since if $E \in J_A(L)$ then

$$N_{>E} \cong A_2(C_L(E)) \equiv A_2(C_L(2D)).$$
is a wedge of 2-spheres, and so 1-connected. Since \( \mathcal{A}_2(L) \hookrightarrow \mathcal{N} \) is a homotopy equivalence, we conclude that the inclusion \( \mathcal{A}_2(L) \hookrightarrow \mathcal{A}_3(A) \) is a 2-equivalence, showing part (2).

For parts (3) and (4), note that \( \mathcal{A}_2(A) \simeq \mathcal{N} \cup J_4(L) \) is obtained from \( \mathcal{N} \) by adding the conjugates of 2D, glued through 2-spheres. Therefore, it does not change the homology groups of degree \( \geq 4 \) and hence we have that \( H_n(\mathcal{A}_2(A)) = H_2(\mathcal{A}_2(L)) = 0 \) for all \( n \geq 4 \). More precisely, let \( J_4(L) = \{ E_1, \ldots, E_r \} \) and \( X_i = \mathcal{N} \cup \{ E_1, \ldots, E_i \} \). Hence \( X_0 = \mathcal{N}, X_r = \mathcal{N} \cup J_4(L), \) (7.1) \[ X_{i+1} = X_i \cup \{ E_{i+1} \} = \mathcal{N} \cup \mathcal{A}_2(A)_{\geq E_{i+1}}, \] and \( \mathcal{N} \cap \mathcal{A}_2(A)_{\geq E_{i+1}} = \mathcal{N}_{E_{i+1}} \simeq \mathcal{A}_2(C_L(E_{i+1})) \equiv \mathcal{A}_2(C_L(2D)) \) is a wedge of 2-spheres. We apply the Mayer-Vietoris sequence to the decomposition of \( X_{i+1} \) given in the right hand of equation (7.1). Below we describe the relevant terms of this long exact sequence.

\[
\begin{array}{cccccccc}
0 & \rightarrow & H_n(X_i) & \rightarrow & H_n(X_{i+1}) & \rightarrow & 0 & \quad n \geq 4 \\
0 & \rightarrow & H_3(X_i) & \rightarrow & H_3(X_{i+1}) & \rightarrow & H_2(\mathcal{N}_{E_{i+1}}) & \rightarrow & H_2(X_i) & \rightarrow & H_2(X_{i+1}) & \rightarrow & 0 \\
& & & & & \uparrow i_* & \cong & \uparrow & \uparrow \\
& & & & & H_2(C_L(E_{i+1})) & \rightarrow & H_2(\mathcal{A}_2(L)) & \\
0 & \rightarrow & H_1(X_i) & \rightarrow & H_1(X_{i+1}) & \rightarrow & 0
\end{array}
\]

The above sequence shows that parts (3) and (4) holds.

Finally, we show that with the above computation, we can prove the “In particular” part of the theorem. If \( H_n(\mathcal{A}_2(L)) \neq 0 \) for \( n = 1 \) or 3, then we are done by Corollary 5.1 and parts (2) and (4). Hence, suppose that \( H_n(\mathcal{A}_2(L)) = 0 \) for \( n = 1, 3 \). In particular this shows that \( \dim H_2(\mathcal{A}_2(L)) = \tilde{\chi}(\mathcal{A}_2(L)) = 1767424 \) and \( H_1(\mathcal{A}_2(A)) = 0 \). Therefore, \( \tilde{\chi}(\mathcal{A}_2(A)) = \dim H_2(\mathcal{A}_2(A)) - \dim H_3(\mathcal{A}_2(A)) = 1204224 \) is a positive number. Since \( H_2(\mathcal{A}_2(L)) \rightarrow H_2(\mathcal{A}_2(A)) \) is surjective by part (2), this is not the zero map. Hence, Corollary 5.1 applies.

Alternatively, since both \( \tilde{\chi}(\mathcal{A}_2(L)) \) and \( \tilde{\chi}(\mathcal{A}_2(A)) \) are positive, and \( H_4(\mathcal{A}_2(A)) = 0 \), we have that both \( H_2(\mathcal{A}_2(L)) \) and \( H_2(\mathcal{A}_2(A)) \) are nonzero. Since also \( H_1(\mathcal{A}_2(L, C_L(2D))) = 0, H_2(\mathcal{A}_2(L)) \rightarrow H_2(\mathcal{A}_2(A)) \) is surjective. \( \square \)

8. Elimination of certain alternating components

We apply the results of the previous sections to establish some results on (H-QC) when \( G \) has an alternating component of type \( \alpha_6 \) or \( \alpha_5 \), and \( p = 2 \). This is part (3) of Corollary 1.5.

We discuss then how to generalize these results to arbitrary alternating groups \( \alpha_n \) arising as components of the group \( G \). We focus on the case \( p = 2 \), since for odd \( p \), \( \text{Out}(\alpha_n) \) is a \( p' \)-group and hence we can establish (H-QC) directly via Theorem 1.8. Finally, we discuss (H-QC) for a group having exactly two components, both isomorphic to \( \alpha_5 \) and permuted regularly. We show that (H-QC) holds for this group by using our methods, and also by Segre’s Theorem 1 [Seg96]. This example shows that the hypotheses of our theorems may be easier to check.

We begin with a direct application of Proposition 5.7 to some alternating components. We refer to [GLS98] for the assertions on the centralizers in alternating groups.

**Corollary 8.1.** Suppose that \( O_{p'}(G) = 1 \), or that \( G \) satisfies (H1). If \( G \) has a component \( L \) isomorphic to \( \alpha_6 \) with \( p = 2 \), then \( G \) satisfies (H-QC).

**Proof.** In view of Theorem 2.4, we can assume that \( O_2(G) = 1 = O_{p'}(G) \). We check the hypotheses of Proposition 5.7 for the component \( L \) of \( G \) isomorphic to \( \alpha_6 \).

Note that \( I_{\text{Out}(\alpha_6)}(\alpha_6) \) contains only cyclic \( p' \)-outers. That is, if \( E \in \mathcal{A}_2(\text{Aut}(\alpha_6)) \) and \( E \cap \alpha_6 = 1 \) then \( |E| = 2 \). Moreover, \( C_{\alpha_6}(E) \) is isomorphic to either \( D_{10} \) or \( S_4 \). In the former case, \( \mathcal{A}_2(D_{10}) \) is discrete with 5 points, and in the latter case \( O_2(S_4) \neq 1 \), so \( \mathcal{A}_2(S_4) \) is contractible. It remains to
show that for some \( k \geq 0 \), \( \mathcal{A}_2(C_{\mathfrak{A}_n}(E)) \hookrightarrow \mathcal{A}_2(\mathfrak{A}_n) \) is the zero map in the \( k \)-th homology group for all \( E \in \mathcal{I}_{\text{Aut}(\mathfrak{A}_n)}(\mathfrak{A}_n) \). Since \( \mathcal{A}_2(\mathfrak{A}_n) \) is connected (see [Qui78]), this holds for \( k = 1 \).

Finally, by Proposition 6.7, \( \tilde{H}_i(\mathcal{A}_p(G)) \neq 0 \).

\( \square \)

**Corollary 8.2.** Suppose that \( O_p'(G) = 1 \), or that \( G \) satisfies (H1). If \( G \) has a component \( L \) isomorphic to \( \mathfrak{A}_8 \) with \( p = 2 \), then \( G \) satisfies (H-QC).

**Proof.** We proceed in a similar way to the previous corollary and check the hypotheses of Proposition 6.7. Assume that \( O_p(G) = 1 = O_p'(G) \). If \( E \in \text{Aut}(\mathfrak{A}_8) \) induces outer automorphisms on \( \mathfrak{A}_8 \), then \( C_{\mathfrak{A}_8}(E) \cong S_n \). Since the homology of \( \mathcal{A}_2(S_n) \) is concentrated in degree 1 (in fact this poset is homotopy equivalent to a wedge of 16 1-spheres), and \( \mathcal{A}_2(\mathfrak{A}_8) \) is simply connected by [MP20, Proposition 6.11], we conclude that \( \mathcal{A}_2(C_{\mathfrak{A}_8}(E)) \hookrightarrow \mathcal{A}_2(\mathfrak{A}_8) \) induces the zero map in homology.

By Proposition 6.7, we conclude that \( \tilde{H}_i(\mathcal{A}_p(G)) \neq 0 \).

The above arguments fail for \( \mathfrak{A}_n \) with odd \( n \). For example, if \( n = 5 \), then \( \mathcal{A}_2(\mathfrak{A}_5) \) is discrete with 5 points while \( \mathcal{A}_2(S_5) \) is a connected wedge of 1-spheres. Therefore the inclusion \( \mathcal{A}_2(\mathfrak{A}_5) \hookrightarrow \mathcal{A}_2(S_5) \) induces the zero map in homology. One of the reasons of this behavior in homology is that the centralizers of outer 2-subgroups \( E \leq S_5 \) have centralizer in \( S_5 \) which is discrete with 3 points. That is, the centralizer have the same homology dimension than \( \mathfrak{A}_5 \). A similar (but more complex) situation arises with \( \mathfrak{A}_7 \) and \( S_7 \). The problem for odd \( n \) is that we have a “leap” of the dimension of the largest nontrivial homology group from \( \mathfrak{A}_n \) to \( S_n \) (at least for small \( n \)). It would be interesting to study these behaviors for \( n \geq 9 \) (both even and odd).

We propose the following problem. If \( r \geq 0 \) is a real number, denote by \( \lfloor r \rfloor \) the largest integer \( n \) with \( n \leq r \) (the floor function).

**Problem.** Let \( m_a = \lfloor n/2 \rfloor - 2 \) for \( n \geq 5 \), and \( m_s = \lfloor (n+1)/2 \rfloor - 2 \) for \( n \neq 4, 2 \). Show that \( m_a \) (resp. \( m_s \)) is the largest integer such that \( \mathcal{A}_2(\mathfrak{A}_n) \) (resp. \( \mathcal{A}_2(S_n) \)) has nontrivial homology in degree \( m_a \) (resp. \( m_s \)).

Suppose that the above problem holds since \( C_{\mathfrak{A}_n}(E) \cong S_{n-2} \) is the centralizer of an outer involution of \( \mathfrak{A}_n \), we see that \( \mathcal{A}_2(C_{\mathfrak{A}_n}(E)) \) and \( \mathcal{A}_2(\mathfrak{A}_n) \) share homology in the top nontrivial homology group if and only if

\[ \lfloor (n-2)/2 \rfloor - 2 = \lfloor n/2 \rfloor - 2, \]

that is, if and only if

\[ \lfloor (n-1)/2 \rfloor = \lfloor n/2 \rfloor. \]

This equality holds if and only if \( n \) is odd. Therefore, modulo the above problem, for even \( n \) we can proceed as in Corollary 8.2 and deduce that (H-QC) holds for \( G \) if it satisfies (H1) and has a component of type \( \mathfrak{A}_n \).

We conclude this section with an application of Theorem 4.1 to establish (H-QC) for certain group with \( \mathfrak{A}_5 \) components. We show how to check condition (2) of that theorem, which requires Property E. It would be interesting to investigate if this kind of proof can show (H-QC) in groups with \( \mathfrak{A}_n \) components for odd \( n \).

**Example 8.3.** Let \( G = (\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes (E \times R) \), where \( E \cong C_2 \) acts diagonally by outer involutions on both copies of \( \mathfrak{A}_5 \), and \( R \cong C_2 \) permutes both components. Note that the kernel on the components is \( H = (\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes E \). We apply Theorem 4.1 to show (H-QC) for \( G \) with \( p = 2 \). We will prove condition (D) via condition (2) of Theorem 4.1. That is, we will show that \( \psi_H \) is not the zero map in homology (the good behavior) by appealing to Property E of the corresponding \( \varphi \) maps.

Let \( L_1 \) and \( L_2 \) be the copies of \( \mathfrak{A}_5 \) as components of \( G \), and \( N = L_1L_2 = F^*(G) \). Note that \( G \) permutes \( L_1 \) with \( L_2 \), and that \( O_p(G) = 1 = O_p'(G) \). With the notation of Section 4,

\[
C_0(H) = C_H(L_1L_2) = 1, \quad C_1(H) = C_H(L_2) = L_1, \quad C_2(H) = H.
\]
We establish Property E for the pairs $(L_i,C_i(H))$ and $n = 2$ in order to invoke Theorem 4.3 with condition (2). We will show that $\varphi_i$ induces an epimorphism in all the homology groups, for $i = 1,2$.

If $i = 1$, then $\varphi_1 : A_p(L_1) \to A_p(1) \ast A_1$. Since $A_1 = A_p(L_1)$, $\varphi_1$ is the identity map.

If $i = 2$, then $\varphi_2 : A_p(H) \to A_p(L_1) \ast A_2$. In view of Lemma 6.2, regard $A_2$ as the set $\{L_1A : A \in A_p(H), C_A(L_1) = 1\}$. We show that $\varphi_2$ induces an epimorphism in homology. Moreover, we prove first that $\varphi_2$ is a 2-equivalence by applying the variant of Quillen’s fiber lemma, Proposition 2.1. In view of this proposition, it is enough to show that $W_y = \varphi_2^{-1}(Y_{\leq y}) \ast Y_{> y}$ is 1-connected, for all $y \in A_p(L_1) \ast A_2$. If $y \in A_p(L_1)$, then $\varphi_2^{-1}(Y_{\leq y}) = A_p(L_1)_{\leq y}$ is contractible, so $W_y$ is contractible and in particular 1-connected. If $y \in A_2$, then regard $y$ as $L_1F_0$ for some $F_0 \in A_p(H)$ such that $F_0 \cap L_1 = 1$. Therefore,

$$\varphi_2^{-1}(Y_{\leq y}) = A_p(L_1) \cup \{F \in A_p(H) - A_p(L_1) : L_1F \leq L_1F_0\} = A_p(L_1F_0).$$

Now, the poset $A_p(L_1F_0)$ is contractible unless $F_0$ acts faithfully on $L_1$ by outer automorphisms. Hence, $W_y$ is contractible in all cases except perhaps those for which $y = L_1F_0$, with $F_0$ inducing outer automorphisms on $L_1$. In the latter case, $F_0$ has order 2 and $A_p(L_1F_0) = A_p(S_5)$ is connected. On the other hand, $Y_{\leq y} = A_2 \ast y$. If we regard $A_2$ as $A_p(L_2E) = A_p(S_5)$, then this element $y$ is an elementary abelian 2-outer automorphisms on $A_2$. Now, the poset $A_p(L_1F_0)$ is contractible, and in particular, it is an epimorphism in all homology groups of degree at most 2. Since the homology groups of $Y$ vanish in degree $> 2$, $\varphi_2$ is an epimorphism in all the homology groups.

Now, in view of Theorem 4.3, it remains to show that $\tilde{H}_2(A_p(C_H(N)) \ast A_1 \ast A_2) \neq 0$. In this case, $C_H(N) = 1$ since there are no other components, $A_1 = A_p(L_1)$ and $A_2 = A_p(S_5)$. Hence, $H_2(A_p(C_H(N)) \ast A_1 \ast A_2) = H_2(A_p(S_5)) \neq 0$ either by the almost simple case Theorem 6.4 or by direct computation since $A_p(S_5)$ is a wedge of 4 0-spheres and $A_p(S_5)$ is a wedge of 16 1-spheres. Hence (H-QC) holds for $G$. This computation also shows that (H-QC) holds for $H$.

Finally, note that Segre's Theorem 1.3 does not apply here. We have that $H$ is the kernel on components and $H \neq F^*(H) = S_5 \times S_5$. Moreover, if $L$ is a component of $H$, then $\text{Aut}_H(L) = S_5$ and $A_p(L) \to A_p(\text{Aut}_H(L))$ is the zero map in homology (since $A_p(L)$ is discrete with 5 points and $A_p(\text{Aut}_H(L)) = A_p(S_5)$ is connected). Nevertheless, we can apply Theorem 1 of Segre by noting that $D_p(H) = A_p(H) - (A_p(L_1) \cup A_p(L_2))$. By computations in GAP, $D_p(H)$ is not surjective in homology, since $H_2(A_p(H))$ has dimension 384 while $H_2(D_p(H))$ has dimension 36. Hence, we get $\tilde{H}_*(A_p(G)) \neq 0$ by Theorem 1 of Segre.

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