Embedded $\mathcal{H}$–Holomorphic Maps and
Open Book Decompositions

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Abstract

We investigate nicely embedded $\mathcal{H}$–holomorphic maps into stable Hamiltonian three–manifolds. In particular we prove that such maps locally foliate and satisfy a no–first–intersection property. Using the compactness results of [vB09] we show that connected components of the space of such maps can be compactified if they contain a global surface of section. As an application we prove that any contact structure on a 3–manifold admits an $\mathcal{H}$–holomorphic open book decomposition. This work is motivated by the program laid out by Abbas, Cieliebak and Hofer in [ACH05] to give a proof to the Weinstein conjecture using holomorphic curves. The results in this paper, with the exception of the compactness statement, have been independently obtained by C. Abbas [Abb09].

1 Introduction

Let $(Z, \alpha, \omega, J)$ be a stable Hamiltonian structure on an oriented closed 3–manifold $Z$, i.e. $\alpha$ is a 1–form and $\omega$ is a closed 2–form such that $\alpha \wedge \omega > 0$ and $\ker \omega \subset \ker d\alpha$. This induces a splitting $TZ = L \oplus F$ where $L = \ker(\omega)$ is called the characteristic foliation and $F = \ker(\alpha)$ is the almost contact plane field. $L$ has a distinguished section $R$ defined by $\alpha(R) = 1$ called the characteristic vector field, and $J \in \text{End}(F, \omega)$ is a choice of $\omega$–compatible almost complex structure on $F$. We extend $J$ to all of $TZ$ by precomposing with the projection $\pi_F = \text{Id} - R \otimes \alpha$ along $R$ onto $F$. For more details we refer to [vB09].

**Definition 1.1 ($\mathcal{H}$–Holomorphic Maps).** Let $(\hat{\Sigma}, j)$ be a punctured Riemann surface. A map $v : \hat{\Sigma} \to Z$ is called $\mathcal{H}$–holomorphic if

$$\bar{\partial}_F^j v = 0, \quad \bar{\partial}_j F^F = \frac{1}{2} (\pi_F dv + J \pi_F dv j) \quad (1.1)$$

$$d(v^* \alpha \circ j) = 0, \quad (1.2)$$

$$\int_{\partial B_p(\varepsilon)} v^* \alpha \circ j = 0 \quad \forall \ p \in \Sigma \setminus \hat{\Sigma} \quad \text{and} \quad \varepsilon \ \text{small enough}. \quad (1.3)$$
\( \mathcal{H} \)-holomorphic maps were introduced by Abbas, Cieliebak and Hofer [ACH05] with the purpose to produce \( \mathcal{H} \)-holomorphic open book decompositions and use these to prove the Weinstein Conjecture in three dimensions, which has subsequently been proved by Taubes [Tau07]. The purpose of this work is to prove some of the steps in the direction indicated in [ACH05]. In Theorem 5.1 we show that indeed every contact structure is supported by an \( \mathcal{H} \)-holomorphic open book decomposition. We moreover prove that nicely embedded \( \mathcal{H} \)-holomorphic maps locally foliate stable Hamiltonian three–manifolds (Theorem 3.3), and that connected components of the space of nicely embedded \( \mathcal{H} \)-holomorphic maps can be compactified if one of the maps is a global surface of section (Theorem 4.4).

Compactness of the space of \( \mathcal{H} \)-holomorphic maps from a genus \( g \) surface \( \Sigma \) is a delicate issue. \( \mathcal{H} \)-holomorphic maps can be viewed as a parameter version of \( J \)-holomorphic maps with \( 2g \)-dimensional parameter space given by \( H^1(\Sigma; \mathbb{R}) \). The parameter space is not compact, causing connected components of the space of \( \mathcal{H} \)-holomorphic maps to be in general not compact. However, under certain topological and geometric assumptions, the space of \( \mathcal{H} \)-holomorphic maps does have a natural compactification. These results were established in [vB09], where a general criterion for compactness of \( \mathcal{H} \)-holomorphic maps was given that we will make use of and restate here for the convenience of the reader.

**Theorem 1.2 ([vB09]).** Let \((Z, \alpha, \omega, J)\) be a stable Hamiltonian structure so that all closed characteristics are Morse or Morse–Bott. The space of smooth \( \mathcal{H} \)-holomorphic maps into \( Z \) with uniformly bounded \( \omega \) and \( \alpha \)-energies with uniformly bounded periods has compact closure in the space of neck–nodal \( \mathcal{H} \)-holomorphic maps.

Bounded periods means that for each free homotopy class of simple closed loops there exists a constant \( C > 0 \) so that for each \( \mathcal{H} \)-holomorphic map \( v : (\hat{\Sigma}, j) \longrightarrow Z \) in the family

\[
P_{[\gamma]}(v) = \sup_{\phi \in \Phi([j, \gamma])} \sup_{s \in (0, 1)} \left| \int_{\sigma_s(\phi)} v^* \alpha \right| < C,
\]

where \( \Phi([j, \gamma]) \) is the set of 1–cylinder Strebel–differentials associated to \( j \) and the free homotopy class of a simple closed loop \( \gamma \) and \( \sigma_s \) denote the closed leaves of \( \phi \). The period integrals are essentially controlled by the harmonic part of \( v^* \alpha \). For more details see [vB09].

An \( \mathcal{H} \)-holomorphic map \( v : \hat{\Sigma} \longrightarrow Z \) is locally \( J \)-holomorphic, so \( v \) has the same local properties as a \( J \)-holomorphic map into \( Z \). In particular, in local conformal coordinates \( C = [0, \infty) \times S^1 \) near each puncture \( p \) the map \( v \) is asymptotic to a closed characteristic \( x : S^1 \longrightarrow Z \) in \( Z \) and has transverse approach governed by an eigenvector \( e \) of the asymptotic operator (see Equation (3.9)) with eigenvalue \( \lambda < 0 \). We say that the transverse asymptotic approach is *maximal* if \( \lambda \) is the largest negative eigenvalue of the asymptotic operator, and \(-\lambda < 2\pi\). For simplicity we will assume throughout that the transverse approach of the maps in question is maximal at all punctures. Throughout we work with weighted Sobolev spaces with weights \( \delta > 0 \) chosen small enough so that \( \delta < -\lambda \) at all punctures. The importance of the assumption that \(-\lambda < 2\pi\) is important for the interplay between the “tangent” and
“normal” operators in the asymptotic analysis and will become clear in Section \ref{sec:linearized}. For more details on the asymptotic operators and the choice of Sobolev spaces we refer to \[Dra04\].

In this work we will focus on nicely embedded $\mathcal{H}$–holomorphic maps into stable Hamiltonian three–manifolds.

**Definition 1.3.** An immersed $\mathcal{H}$–holomorphic map $v : \hat{\Sigma} \to Z$ is called nicely immersed if the transverse approach at all punctures is maximal and the Fredholm index of $v$ is 1.

An embedded nicely immersed map is called nicely embedded if the map extends to an embedding over the radial compactification; in particular all asymptotic orbits are covered once.

This definition is roughly modeled on the definition of nicely embedded $J$–holomorphic spheres given in \[Wen\].

It follows from standard index formulas (see e.g. \[Wen\], remembering that the index for $\mathcal{H}$–holomorphic maps is $2g - 1$ higher than the index of a $J$–holomorphic map into the symplectization) it means for an immersed $\mathcal{H}$–holomorphic map $v$ to be nicely immersed is that the all asymptotic orbits have odd Conley–Zehnder index. In other words, if $\lambda$ is the eigenvalue of the eigenvector of the asymptotic operator governing the transverse approach at any of the punctures, then the other eigenvector with the same winding has eigenvalue $\hat{\lambda} \leq \lambda$.

The argument in this paper can be roughly split into three parts. The first part considers the local Fredholm theory yielding transversality and a strong implicit function theorem and thus showing that nicely embedded maps locally foliate. The second part proves a weak version of global intersection theory recently developed in more generality by R. Siefring in \[Sie09\] yielding a no–first-intersections result. We combines these results with the compactness result from Theorem \[L2\] to prove that the space of nicely embedded maps can be compactified. The third part is an application showing that every contact structure is supported by an $\mathcal{H}$–holomorphic open book decomposition.

With the exception of the compactness results, these results have been independently established by C. Abbas \[Abb09\].

2 The Linearized Equation

Suppose $v : \hat{\Sigma} \to Z$ is a smooth embedded $\mathcal{H}$–holomorphic map. We want to linearize the equations at $v$. We will work with the (generalized) Tanaka–Webster connection $\tilde{\nabla}$ and we quickly recall the properties of $\tilde{\nabla}$ that we will make use of. For details we refer to \[Tan89\].

Let $X$ and $Y$ be sections of $F$. Then

\[
\begin{align*}
\tilde{\nabla} \alpha &= 0, \quad \tilde{\nabla} R = 0, \quad \tilde{\nabla} g = 0 \\
(\tilde{\nabla} R J) &= 0, \quad (\tilde{\nabla} X J) R = -X - \phi X, \\
\tilde{T}(R, R) &= 0, \quad \tilde{T}(R, JX) = -J \tilde{T}(R, X) = -\phi X,
\end{align*}
\]
where $\phi = \frac{1}{2} \mathcal{L}_{RJ}$. \[3\]
Given $\xi \in \Omega^0(\hat{\Sigma}, v^*TZ)$ we write $\xi = \zeta R + \chi$ with $\chi$ a section of $v^*F$ and set

$$\Phi_v : v^*TZ \longrightarrow \exp_v(\xi)^*TZ$$

denote parallel transport along the geodesic $s \mapsto \exp_v(s\xi(z))$ with respect to $\nabla$. The space of diffeomorphisms of $\hat{\Sigma}$ with $N$ (fixed) punctures acts on the space of maps in the usual way. Locally at a map $(v, j)$ consider we fix a $6g - 6 + 2N$–dimensional slice $T_J\mathcal{T}$ of infinitesimal variations of complex structure $h$. We will assume that the slice is chosen so that the variations of complex structure have support away from the punctures.

Then define

$$\mathcal{F}_v^F : \Omega^0(\hat{\Sigma}, v^*TZ) \times T_J\mathcal{T} \longrightarrow \Omega^0(\hat{\Sigma}, v^*TZ)$$

$$\mathcal{F}_v^F(\xi, h) = \Phi_v(\xi)^{-1}\frac{1}{2}(\pi_F d\exp_v(\xi) + J\pi_F d\exp_v(\xi) \circ j_h)$$

and

$$\mathcal{F}_v^L : \Omega^0(\hat{\Sigma}, v^*TZ) \times T_J\mathcal{T} \longrightarrow \Omega^2(\hat{\Sigma})$$

$$\mathcal{F}_v^L(\xi, h) = d(\Phi_v(\xi)^*\alpha \circ j_h)$$

where $j_h$ is a variation of complex structure satisfying $\frac{d}{ds}j_h|_{s=0} = h$. Such a family of variation can be chosen canonically, see e.g. Section 3.2 of [Wen].

**Lemma 2.1.** For any smooth map $v : \hat{\Sigma} \longrightarrow M$ asymptotic to closed characteristics at the punctures, define the operators

$$D_v^F(\xi, h) = d\mathcal{F}_v^F(0)(\xi, h), \quad D_v^L(\xi, h) = d\mathcal{F}_v^L(0)(\xi, h).$$

Then

$$D_v^F(\xi, h) = \tilde{\nabla}^{0,1} \chi + \frac{1}{2}(\tilde{\nabla} J)_{\pi_F dv \circ j} + \frac{1}{2}J_{\pi_F dv \circ h} + (J\phi \chi \otimes v^*\alpha)^{0,1} - \zeta J\phi_{\pi_F dv}(\xi)$$

$$D_v^L(\xi, h) = d[\zeta \circ j + v^*(\iota_x d\alpha) \circ j + v^*\alpha \circ h].$$

**Proof.** Consider the path $\mathbb{R} \longrightarrow C^\infty(\hat{\Sigma}, M)$ given by $s \mapsto \exp_v(s\xi)$ and a path $j_s$ with $\frac{d}{ds}j_s|_{s=0} = h$. Then

$$D_v^F(\xi, h) = \frac{d}{ds}\mathcal{F}_v^F(s\xi, sh)|_{s=0}$$

$$= \frac{d}{ds}\Phi_v(s\xi)^{-1}\frac{1}{2}(\pi_F dv_s + J\pi_F dv_s \circ j_s)|_{s=0}$$

$$= \frac{1}{2}\tilde{\nabla}_s(\pi_F dv_s + J\pi_F dv_s \circ j_s)|_{s=0}$$

$$= \frac{1}{2}\left(\pi_F(\tilde{\nabla}_s dv_s) + (\tilde{\nabla}_s J)_{\pi_F dv \circ j} + J\pi_F(\tilde{\nabla}_s dv_s) \circ j + J\pi_F dv \circ h\right)|_{s=0}$$
\[
\begin{align*}
&= \frac{1}{2} \left( \pi_F (\bar{\nabla}_\xi + \bar{T}(\xi, dv)) + (\bar{\nabla}_\xi J) \pi_F dv \circ j + J\pi_F (\bar{\nabla}_\xi + \bar{T}(\xi, dv)) \circ j \right. \\
&\hspace{1cm} + J\pi_F dv \circ h \big) \\
&= \bar{\nabla}^{0,1} \chi + (\pi_F \bar{T}(\xi, dv))^{0,1} + \frac{1}{2} (\bar{\nabla}_\chi J) \pi_F dv \circ j + \frac{1}{2} J\pi_F dv \circ h \\
&= \bar{\nabla}^{0,1} \chi + \frac{1}{2} (\bar{\nabla}_\chi J) \pi_F dv \circ j + \frac{1}{2} J\pi_F dv \circ h \\
&\hspace{1cm} + (\bar{T}(\xi, R) \otimes v^* \alpha)^{0,1} + \zeta \bar{T}(R, \pi_F dv) \\
&= \bar{\nabla}^{0,1} \chi + \frac{1}{2} (\bar{\nabla}_\chi J) \pi_F dv \circ j + \frac{1}{2} J\pi_F dv \circ h + (J\phi \otimes v^* \alpha)^{0,1} - \zeta J\phi \pi_F dv.
\end{align*}
\]

Also
\[
D^L_v(\xi, h) = \frac{d}{ds} F^L_v(s\xi, sh)|_{s=0} = \frac{d}{ds} d(l^v_\xi(s\xi^* \alpha \circ j_s)|_{s=0} = d((L_\xi \alpha) \circ j + v^* \alpha \circ h) \\
= d [d\zeta \circ j + v^*(\iota_\chi d\alpha) \circ j + v^* \alpha \circ h].
\]

We choose the usual functional analytic setup for the spaces of maps, i.e. at an \(H\)-holomorphic map \(v: \dot{\Sigma} \rightarrow Z\) we consider the linearized operator to act on
\[
\begin{align*}
D^F_v : W^{k,p}_\delta \left( \Omega^0(\dot{\Sigma}, v^* T Z) \right) \times T^J T &\longrightarrow W^{k-1,p}_\delta \left( \Omega^0(\dot{\Sigma}, v^* T Z) \right) \\
D^L_v : W^{k,p}_\delta \left( \Omega^0(\dot{\Sigma}, v^* T Z) \right) \times \mathbb{R}^N \times T^J T &\longrightarrow W^{k-2,p}_\delta \Omega^2(\dot{\Sigma})
\end{align*}
\]
where the weight \(\delta > 0\) is not in the spectrum of the asymptotic operator at the puncture and \(N\) is the number of punctures. The \(\mathbb{R}^N\)-factor in the linearization allows for functions that are asymptotically constant near each puncture. For a precise definition of this and the accompanying space of maps we refer the reader to [Dra04] or [vB07].

Usually we view the space of gauge transformations to be infinitesimal diffeomorphism extending smoothly over the punctures, or vanishing at the punctures. In our case it is at times convenient to alternatively consider more general gauge transformations that don’t necessarily extend over the punctures, but remain bounded in the cylindrical metric.

Let \(v : C_0 = [0, \infty) \times S^1 \rightarrow Z\) be a \(J\)-holomorphic map, with maximal transverse approach given by an eigenvector of the asymptotic operator with eigenvalue \(-2\pi < \lambda < 0\) and choose \(\delta > 0\) so that \(-\delta > \lambda\). Let \(v\) be a vector field on the domain. Then \(\pi_F dv(v) \in W^{k,p}_\delta(v^* F)\) if and only if \(v \in W^{k,p}_\delta(TC_0)\), where
\[
\delta = \delta + \lambda < 0
\]
satisfies \(-\tilde{\delta} < 2\pi\). We will use this notation for the weights \(\delta\) and \(\tilde{\delta}\) related via the maximal eigenvalue \(\lambda\) for the remaining of this and the following section.

The following Lemma gives an equivalent description to solutions of the linearized equation at a nicely immersed map.
Lemma 2.2. Let $v : \hat{\Sigma} \to Z$ be a nicely immersed $\mathcal{H}$–holomorphic map. Then $(\xi, h)$, where $\xi \in W^{k,p}_\delta(v^*TZ) \times \mathbb{R}^N$ and $h$ is a smooth deformation of complex structure, solves the linearized equation if and only if there exists an infinitesimal gauge transformation $\nu \in W^{k,p}_\delta(T\Sigma)$ and $f \in W^{k,p}_{\delta+\lambda}(\hat{\Sigma}, \mathbb{R})$ so that $\xi = f R + dv(\nu)$ and

$$\tilde{D} f = d(df \circ j + 2f v^*\alpha \circ \tilde{\phi}) = 0$$

(2.7)

where

$$\tilde{\phi} = (\pi_F dv)^*\phi = (\pi_F dv)^{-1} \circ \frac{1}{2} L_R J \circ (\pi_F dv).$$

Proof. Since $v$ is immersed we can rewrite the linearized Equation (2.4) with $\xi = dv(\nu) + f R$, where $\nu$ is a section of $T\hat{\Sigma}$ and $f : \hat{\Sigma} \to \mathbb{R}$. If $(\xi, h)$ solves the linearized equations, then

$$\nabla^{0,1} \nu - f j \tilde{\phi} + \frac{1}{2} j h = 0.$$  

(2.8)

Solving this for $h$

$$h = 2(j \nabla^{0,1} \nu + f \tilde{\phi}) = L_{\nu} j + 2f \tilde{\phi}$$

and plugging this into Equation (2.5) we obtain

$$d(df \circ j + 2f v^*\alpha \circ \tilde{\phi}) = 0,$$

after noting that

$$d(v^*\alpha(\nu)) \circ j + (\iota_\nu v^*d\alpha) \circ j + v^*\alpha(L_{\nu} j) = L_{\nu}(v^*\alpha \circ j)$$

is closed and that $\zeta$ in Equation (2.5) corresponds to $\zeta = f + v^*\alpha(\nu)$.

If $\xi \in W^{k,p}_\delta(v^*TZ) \times \mathbb{R}^N$, then $\nu \in W^{k,p}_\delta(T\hat{\Sigma})$ and $f \in W^{k,p}_{\delta+\lambda}(\hat{\Sigma}, \mathbb{R})$.

Conversely, if $\nu \in W^{k,p}_\delta(T\hat{\Sigma})$ and $f \in W^{k,p}_{\delta+\lambda}(\hat{\Sigma}, \mathbb{R})$, then $\xi = dv(\nu) + f R \in W^{k,p}_\delta(v^*TF) \times W^{k,p}_{\delta+\lambda}(v^*L)$, and $(\xi, h)$ satisfies the linearized equations. Noting that $-\delta < 2\pi$ by assumption we employ the standard asymptotic analysis for $J$–holomorphic curves near the puncture to split the asymptotic operator (see Section 3.4 of [Wen]), which is in upper triangular form, and see that $\xi$ is automatically of class $W^{k,p}_\delta(v^*TZ) \times \mathbb{R}^N$. \qed

3 Local Theory for Nicely Immersed Maps

In this section we discuss the properties of nicely immersed $\mathcal{H}$–holomorphic maps into stable Hamiltonian 3–manifolds. As we will see, these curves have very nice properties and are well–suited to yield finite energy foliations. The key result in this section is the observation that a strong version of the implicit function theorem \textnormal{3.3} holds for nicely immersed $\mathcal{H}$–holomorphic maps.
We need an expression for the asymptotic operator $A_\infty$ associated to an $\mathcal{H}$–holomorphic map $v$ asymptotic to a closed characteristic $x : S^1 \to \mathbb{Z}$ of period $T$ in the trivialization given by the eigenvector $e$ with eigenvector $\lambda < 0$ associated to $v$. Acting on sections $\eta$ of $x^*F$ we have

$$A_\infty \eta = -J(\nabla_t \eta - T \nabla_\eta R) = -J \nabla_t \eta + T(\eta + \phi \eta),$$

where $\nabla$ denotes the Levi–Civita connection and we used that $\nabla_X R = -J X - J \phi X$ and $\eta \in F$, where $\phi = \frac{1}{2} \mathcal{L}_R J$. Then with $\eta = z e$, $z = x + J y$ we get

$$A_\infty (z e) = -J z \nabla_t e - J \nabla_t (z e) + T z e + T \bar{z} \phi e$$

$$= z(-J \nabla_t e + T(e + \phi e)) - J \dot{z} e - T(z - \bar{z}) \phi e$$

$$= (-J \dot{z} + \lambda z - T(z - \bar{z}) \phi) e,$$

where we used that $\nabla_R J = 0$ and that $\phi$ is $J$ anti–linear. Using also that $\phi$ is symmetric (see e.g. Section 6.2 in [Bla02]) we write $\phi$ in matrix form with respect to the trivialization given by $e$ and $Je$

$$\hat{\phi} = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_2 & -\mu_1 \end{bmatrix}.$$ (3.10)

Set $\tilde{\mu} = \mu_1 + i \mu_2$. Then we can express the operator $A_\infty$ in the trivialization given by $e$ as

$$\tilde{A}_\infty z = -i \dot{z} + \lambda z - T(z - \bar{z}) \tilde{\mu}.$$ (3.11)

**Lemma 3.1.** Let $v : \tilde{\Sigma} \to \mathbb{Z}$ be a nicely immersed $\mathcal{H}$–holomorphic map and let $f \in W^{k,p}_\delta(\tilde{\Sigma}, \mathbb{R})$ satisfy $\tilde{D} f = 0$, where $\tilde{D}$ is the operator from Equation (2.7).

With the same notation for $e$ and $\lambda$ as above, at each puncture $f$ is given by the imaginary part (expressed in the trivialization given by $e$ and $Je$) of an eigenvector $\tilde{e}$ of the asymptotic operator $A_\infty$ with eigenvalue $\lambda \leq \lambda$, or $f \equiv 0$.

**Proof.** We need to understand the asymptotics of the operator $\tilde{D}$ in terms of the asymptotic operator $A_\infty$. Let $\kappa = v^* \alpha \circ \tilde{\phi}$. In the neck regions adjacent to the punctures write

$$df \circ j + 2 f \kappa = da.$$

evaluating this on $\partial_s$ and $\partial_t$ we get

$$f_t + 2 f \kappa(\partial_s) = a_s, \quad -f_s + 2 f \kappa(\partial_t) = a_t.$$

with $z = a + i f$ and $\tilde{\kappa} = \kappa(\partial_t) - i \kappa(\partial_s)$ this gives

$$z_s = -iz_t + 2 f (\kappa(\partial_s) + i \kappa(\partial_t)) = -iz_t - i(z - \bar{z})(\kappa(\partial_s) + i \kappa(\partial_t)) = -iz_t + (z - \bar{z})\tilde{\kappa}.$$ (3.12)
We wish to understand solutions of this equation as $z \to \infty$. Following the usual asymptotic analysis [HWZ96], denote the $L^2(x^*F)$ inner product, and the induced inner product on the trivialization, by $\langle \cdot, \cdot \rangle$, i.e. for $u, v$ sections of $x^*F$

$$
\langle u, v \rangle = \int_{S^1} g(u, v) d\theta
$$

and in the trivialization for $z, w$ complex values functions on $S^1$

$$
\langle z, w \rangle = \int_{S^1} g(z e, w e) d\theta.
$$

Denote the associated norm by $\| \cdot \|$ and set $\zeta = \frac{z}{\| z \|}$. Then

$$
\zeta_t = \frac{z_t}{\| z \|}, \quad \zeta_s = \frac{z_s}{\| z \|} - \alpha \zeta, \quad \text{where } \alpha(s) = \frac{\langle z_s, z \rangle}{\| z \|^2}.
$$

Then $\zeta$ satisfies

$$
\zeta_s = B \zeta + \alpha \zeta, \quad B \zeta = -i \zeta_t + (z - \bar{z}) \tilde{\kappa}
$$

Now consider the behavior of the terms in $B$ as $s \to \infty$. We have $v^*\alpha \to T dt$. To understand the behavior of $\phi$, recall that (see Section 4 in [HWZ99])

$$
\frac{\pi_F dv(\partial_s)}{\| \pi_F dv(\partial_s) \|} \to \frac{e}{\| e \|}, \quad \text{as } s \to \infty
$$

where $e$ is the eigenvector of $A_{\infty}$ with eigenvalue $\lambda$ governing the transverse approach of $v$ at the puncture. So in the basis $\partial_s$ and $\partial_t$ we have $\phi \to \hat{\phi}$, the matrix from Equation (3.10).

Thus $\kappa(\partial_s) \to T \mu_2$ and $\kappa(\partial_t) \to -T \mu_1$, so $\hat{\kappa} \to -T(\mu_1 + i \mu_2) = -T \hat{\mu}$. Then the operator $B$ approaches

$$
B_{\infty} \zeta = -i \zeta_t - T(\zeta - \bar{\zeta}) \hat{\mu},
$$

and the same asymptotic analysis as in [HWZ96] shows that solutions to Equation (3.12) converge to eigenvectors of $B_{\infty}$ exponentially, by the corresponding eigenvalue. Comparing with Equation (3.11) we see that $B_{\infty} = A_{\infty} - \lambda \text{Id}$, so the spectrum of $B_{\infty}$ is the spectrum of $A_{\infty}$ shifted by $-\lambda$ with identical eigenvectors.

Since $A_{\infty}$ has no eigenvalue in the interval $(\lambda, -\delta]$, and has one eigenvalue $\lambda$ we have that $B_{\infty}$ has no eigenvalue in the interval $(0, -\delta]$, and has an eigenvalue 0.

Let $z$ be a solution to Equation (3.12) of class $W^{k,p}_\delta$. Write $z = c + \tilde{z}$, where $c$ is a real constant and $\tilde{z}$ has vanishing average real part at the puncture. Then $\tilde{z}$ either vanishes identically or is an eigenvector of $A_{\infty}$ with nonpositive eigenvalue and the imaginary part of $\tilde{z}$ does not vanish identically.  

\[\square\]
After understanding the asymptotic behavior of elements in the kernel of $\tilde{D}$ we prove a transversality and non–vanishing result for nicely immersed curves. Recall our notation that $\tilde{\delta} = \delta + \lambda < 0$, where $\delta$ is the weight at the punctures and $\lambda$ is the eigenvalue governing the asymptotic approach.

**Lemma 3.2.** Let $v$ be a nicely immersed $\mathcal{H}$–holomorphic map. Then the operator $\tilde{D}$ from Equation (2.7) has index 1, is surjective and any non–trivial solution $f$ to $\tilde{D}f = 0$ has no zeros.

**Proof.** First we consider a neighborhood of each puncture. Let $\lambda$ be the eigenvalue of the eigenvector $e$ governing the transverse approach at the puncture. The eigenvector $\hat{e}$ of the asymptotic operator with maximal eigenvalue $\hat{\lambda} \leq \lambda$ has the same winding as $e$. Recall that $\hat{e}$ and $e$ are pointwise linearly independent (see Lemma 3.5 of [HWZ95]).

We aim to show that solutions to Equation (3.12) are necessarily asymptotic to the imaginary part $\hat{e}$ (in the trivialization given by $e$ and $Je$), after subtracting off a constant real part. We will do this by relating the kernel of $\tilde{D}$ to the kernel of the associated system of first order equations (in complex notation)

$$L : W^{k,p}_\delta(\hat{\Sigma}, \mathbb{C}) \times \mathcal{H}^{0,1}_C \longrightarrow W^{k-1,p}_\delta(T^0,1\hat{\Sigma}), \quad L(z, \eta) = \bar{\partial}z - (z - \bar{z})\kappa^{0,1} + \eta.$$ 

First recall that $\tilde{D} : W^{k,p}_\delta(\hat{\Sigma}, \mathbb{R}) \longrightarrow W^{k-2,p}_\delta(\hat{\Sigma})$, $\tilde{D}f = \ast d(df \circ j + 2f v^*\alpha \circ \tilde{\phi})$.

is Fredholm if and only if

$$D_\infty(\lambda) : W^{k,p}_\delta(S^1, \mathbb{R}) \longrightarrow W^{k-2,p}_\delta(S^1, \mathbb{R}), \quad D_\infty(\lambda) = -\lambda^2 f - f_{tt} + 2T(\lambda \mu_2 f + (f \mu_1)_{tt})$$

is an isomorphism (c.f. [LM85]), and that the Fredholm index is constant on connected components of $\mathbb{R}$ on which $D_\infty$ is an isomorphism.

We claim that the set of $\mu$ for which $D_\infty$ is an isomorphism coincides with the spectrum of $B_\infty$, with the exception of the eigenvalue 0 of $B_\infty$. If $\mu$ is an eigenvalue of $B_\infty$ with eigenvector $\zeta$, then a straightforward calculation shows that $D_\infty(\mu)f = 0$, where $f = \Im(\zeta)$. If $\mu \neq 0$, then $f$ is non–trivial (by Lemma 3.5 of [HWZ95]), and if $\mu = 0$, then $D_\infty$ is an isomorphism since by assumption the eigenvalue of $B_\infty$ has multiplicity 1 and the eigenvector is purely real.

Conversely, if $\mu \neq 0$ and $D_\infty(\mu)f = 0$ then $f \neq 0$ and

$$\zeta = a + if, \quad a = \frac{1}{\mu}(f_t + 2T\mu_2 f)$$

is an eigenvector of $B_\infty$ with eigenvalue $\mu$.

The adjoint $\tilde{D}^*$ of $\tilde{D}$ is given by

$$\tilde{D}^* : W^{-k+2,p}_\delta(T^*\hat{\Sigma}) \longrightarrow W^{-k,p}_\delta(\Lambda^2\hat{\Sigma}), \quad \tilde{D}^*g = \ast d(df \circ j) - 2dg \wedge v^*\alpha \circ \tilde{\phi},$$
where $\ast$ is taken with respect to the cylindrical metric. By elliptic regularity the kernel of $\tilde{D}^*$ is the same as the kernel of

$$
\tilde{D}^* : W^{l,p}_{-\tilde{\delta}}(T^*\tilde{\Sigma}) \to W^{l-2,p}_{-\tilde{\delta}}(\Lambda^2\tilde{\Sigma})
$$

for any $l$, which is trivial, as $\tilde{D}$ satisfies a maximum principle and the weight $-\tilde{\delta} > 0$. Thus $\tilde{D}$ is surjective. In particular, any non–trivial solution of $\tilde{D}f = 0$ is asymptotic to the eigenvector $\hat{\epsilon}$ of eigenvalue $\hat{\mu}$ at each puncture.

The same argument shows that $\tilde{D}_r f = \ast d(df \circ j + 2f \kappa_r)$ is surjective for any $r$, where $\kappa_r$ is a family of 1–forms so that the asymptotic operator of the associated first order equation has fixed eigenvalues $0$ and $\hat{\mu} = \hat{\lambda} - \lambda \leq 0$ for the eigenvectors with winding zero. By standard theory (see [HWZ95]) no other eigenvalues of the operator in this family can enter the interval $[\hat{\mu}, 0]$.

Thus solutions to $\tilde{D}_r f = 0$ are non–zero in some neighborhood of the punctures, where the neighborhood may depend on $r$. We now show that non–trivial solutions $f$ have no zeros.

Let $\kappa_r, r \in [0,1]$ be a family of 1–forms with $\kappa_1 = \kappa$ and $\kappa_0 = \beta(s)\frac{\hat{\mu}}{2} dt$, where $\beta : \tilde{\Sigma} \to [0,1]$ is a cutoff function supported on the neck regions adjacent to the punctures and equal to $1$ in some neighborhood of the punctures and monotone on each neck. Then

$$
\tilde{D}_0 f = -\Delta f + \hat{\mu} \partial_s (\beta f)
$$

has kernel given by functions that are equal to some constant $c$ on the thick part and equal to

$$
f(s,t) = c e^{\hat{\mu} \int_0^s \beta(s')ds'}
$$

on the necks, remembering that $f \in W^{k,p}_{-\delta}$. In particular, the kernel is 1–dimensional so $\tilde{D}$ has index 1, and non–trivial solutions to $\tilde{D}_0 f = 0$ have no zeros. Now fix a point $p$ in the tick part and consider the family of solutions $f_r$ of $\tilde{D}_r f_r = 0$ with $f_r(p) = 1$. All $f_r$ are positive in some neighborhood (depending on $r$) of the punctures. We claim that $f_r \neq 0$ for any $r \in [0,1]$. If not, there exists a smallest $r > 0$ and so that $f_r > 0$ and $f_r(z) = 0$ for some $z \in \tilde{\Sigma}$. But this is impossible by Harnack’s inequality. Thus $f_r > 0$ on $\tilde{\Sigma}$ for all $r \in [0,1]$.

Lemma 3.2 gives the local model for families of nicely immersed curves. The following Theorem makes this precise.

**Theorem 3.3.** Let $v : \tilde{\Sigma} \to Z$ be a nicely immersed $H$–holomorphic map. Then there exists a smooth 1–dimensional family of $H$–holomorphic maps which is, up to gauge transformations, given by translation along the characteristic direction by a non–zero function. If $v$ is nicely embedded and asymptotic to a collection of closed characteristics $B$ then the family locally foliates $Z \setminus B$. 

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Proof. Let $(\xi, h)$ be in the kernel of the linearized operator. By Lemma 2.2 we can choose an infinitesimal gauge transformation $\nu$ so that $\xi = d\nu + f R$, where $f$ satisfies $\tilde{D}f = 0$ and $\tilde{D}$ is the operator from Equation (2.7).

By Lemma 3.2 the operator $\tilde{D}$ is surjective and elements in the kernel extend continuously to the radial compactification at the punctures, they are unique up to scaling, and non-trivial solutions have no zeros.

By Lemma 2.2 there is a bijective correspondence between elements in the kernel of the system linearized equations from Lemma 2.1 and elements in the kernel of $\tilde{D}$. The surjectivity of $\tilde{D}$ then implies that the system of linearized equations is surjective and $v$ lives in a 1-parameter family. By Lemma 3.2 we see that the 1-dimensional space of solution is infinitesimally obtained from one another by “translating” in the characteristic direction by the nowhere zero bounded function $f$, up to gauge transformation.

Let $v_t, t \in I$ denote such a local family of solutions with $v_0 = v$. If $v$ is embedded, then so are nearby curves in this family, and since the function $f$ has no zeros there exists $\varepsilon > 0$ so that $v_t$ are mutually disjoint for $t \in (-\varepsilon, \varepsilon)$. Thus the map $\hat{v} : (-\varepsilon, \varepsilon) \times \hat{\Sigma} \rightarrow Z \setminus B$ is a diffeomorphism onto its image, so solutions foliate a tubular neighborhood of the image curve $v$ in $Z \setminus B$.

\section{Global Theory and Compactness}

The local foliation results from the previous section lead to no-first-intersection results that in turn allow to show that the periods of families of nicely embedded curves are uniformly bounded and thus yields compactification of connected components of the moduli space of nicely embedded curves \[4.4\].

When analyzing nicely embedded maps we will frequently choose a complement $\hat{\mathcal{H}}$ of the coexact forms in the coclosed forms that is supported away from the punctures and use this to lift maps to the symplectization. The definition of $\hat{\mathcal{H}}$ depends on the choice of domain complex structure $j$, so when considering families of maps and complex structures $j_t$ we may consider families of complements $\hat{\mathcal{H}}_t$ that are supported away from the punctures and sometimes also away from neighborhoods of other points in the interior of $\hat{\Sigma}$. The significance of choosing the support away from the punctures or other points is so that the lifts are then locally $J$–holomorphic and the standard theory applies there.

Similar to the $J$–holomorphic case, we can prove a “no first intersections”–type result for families of nicely embedded $\mathcal{H}$–holomorphic maps.

\textbf{Lemma 4.1.} For $t \in I = (a, b) \subset \mathbb{R}$, let $v_t : (\hat{\Sigma}, j_t) \rightarrow Z$ be a family of somewhere injective index 1 $\mathcal{H}$–holomorphic maps with smooth domain complex structures $j_t$ and assume $v_t$ is nicely embedded for some $t \in I$. Then all $v_t$ are nicely embedded for $t \in I$, and if $v_{t_0}(z_0) = v_{t_1}(z_1)$ for some $t_0, t_1 \in I$ and $z_0, z_1 \in \hat{\Sigma}$, then the images of $v_0$ and $v_1$ coincide.

\textbf{Proof.} First we show that all maps $v_t$ are embedded. Suppose $I_0 \subset I$ is a maximal interval so that all maps in $I_0$ are embedded. The interval $I_0$ is open and by assumption not empty. If
$I = I_0$, then all maps are embedded and there is nothing to prove. Otherwise let $s \in \partial I_0 \cap I$.

By possibly perturbing the almost complex structures in the neighborhood of the image of an injective point of $v_s$ and considering the corresponding family of maps that are $\mathcal{H}$–holomorphic with respect to that almost complex structure, we may assume that the self–intersections of $v = v_s$ are occurring in the union of some disjoint open balls in $\hat{\Sigma}$.

Let $\hat{U} \subset \hat{\Sigma}$ be the union of disjoint open balls containing the self–intersection locus and the punctures and denote the union of slightly bigger disjoint open balls containing $\hat{U}$ by $U$. Choose a complement $\mathcal{H}$ of the coexact forms in the coclosed forms so that its elements are supported off $U$. Let $(a, v)$ be a lift of $v$ to the symplectization with respect to $\mathcal{H}$, i.e. $v^*\alpha + da \circ j \in \mathcal{H}$. By possibly modifying $\mathcal{H}$ by adding a function that is supported in the $U$ we may assume that $\mathcal{H}$ is supported off $\hat{U}$ and the lift $(a, v) = (a_s, v_s)$ has at least one self–intersection.

Now extend $\mathcal{H} = \mathcal{H}_s$ to a family $\mathcal{H}_t$ of complements of the coexact forms in the coclosed forms (w.r.t. $j_t$) that is supported off of $\hat{U}$. We use $\mathcal{H}_t$ to lift the entire family $v_t$ to the symplectization by choosing a family of functions $a_t : \hat{\Sigma} \rightarrow \mathbb{Z}$ so that $v_t^*\alpha - da_t \circ j \in \mathcal{H}_t$ that extend the lift $(a_s, v_s)$. The maps $\tilde{v}_t = (a_t, v_t)$ are $J$–holomorphic on $\hat{U}$, so the self–intersection number of $\tilde{v}_s$ is positive. But $\tilde{v}_t$ has self–intersection number 0 for $t \in I_0$. Moreover, all maps $\tilde{v}_t$ are $J$–holomorphic in a neighborhood of the punctures and the Sobolev weights have been chosen to be maximal. But this contradicts that in this case the intersection number is a topological invariant by $[\text{Sie08}]$. So $v_t$ is embedded for all $t \in I$.

Next suppose that there are $t_0, t_1 \in I$ and $z_0, z_1 \in \hat{\Sigma}$ so that $v_{t_0}(z_0) = v_{t_1}(z_1)$ but the images of $v_{t_0}$ and $v_{t_1}$ don’t coincide. Let $s \in (t_0, t_1]$ so that $v_{t_0}$ is disjoint from $v_{t_1}$ for all $t \in (t_0, s)$ but $v_{t_0}$ and $v_s$ intersect. If $v_{t_0}$ and $v_s$ have the same image we are done. If not we may again assume, by possibly perturbing the almost complex structure in a neighborhood of a point on the image of $v_s$ that is disjoint from $v_{t_0}$, that all intersections of $v_{t_0}$ and $v_s$ occur in a union of disjoint balls. Just as above we may choose complements of the coexact forms in the coclosed forms $\mathcal{H}_{t_0} = \mathcal{H}_{t_0}(j_{t_0})$ and $\mathcal{H}_s = \mathcal{H}_s(j_s)$ so that the corresponding lifts to the symplectization $\tilde{v}_{t_0}$ and $\tilde{v}_s$ are $J$–holomorphic in a neighborhood of the punctures and the points where $v_{t_0}$ and $v_s$ intersect. Extend these complements to a smooth family $\mathcal{H}_t = \mathcal{H}_0(j_t)$ for all $t \in I$. It follows from $[\text{Sie08}]$ that the intersection number of maps in this family is a topological invariant since the weights have been chosen to be maximal. For $t \in (t_0, s)$ the intersection number of $\tilde{v}_t$ with $\tilde{v}_{t_0}$ is zero, contradiction that $\tilde{v}_{t_0}$ intersects $\tilde{v}_t$ positively, after possibly adding a constant function to the $\mathbb{R}$–factor of the map.

In conjunction we conclude that all maps $v_t$ are embedded and disjoint.

The Lemma explicitly excludes the cases that the complex structure becomes degenerate, or that maps in the family are multiply covered. The Lemma follows more easily from recent work of R. Siefring (Theorem 2.10 of $[\text{Sie09}]$).

We need the following standard definition.

**Definition 4.2.** An $\mathcal{H}$–holomorphic map $v : \hat{\Sigma} \rightarrow Z$ asymptotic to a collection of non–degenerate closed characteristics $B$ is called a surface of section if $v$ is embedded and every
characteristic flow line in $Z$ is either an asymptotic orbit for $v$ or intersects the image of $v$ in forward and backward time.

Lemma 4.3. Let for $t \in I = (a, b) \subset \mathbb{R}$, let $v_t : \hat{\Sigma} \rightarrow Z$ be a smooth family of nicely embedded $\mathcal{H}$–holomorphic maps so that for some $c \in I$ $v_c$ is a surface of section. Then all $v_t$, $t \in I$ are surfaces of section and the periods of the family $v_t$ are uniformly bounded.

Proof. By Lemma 3.2 the family $f_t$ is obtained from $v_c$ by shifting in the characteristic direction, so all $v_t$ are surfaces of section.

We are left to show that the periods are uniformly bounded. Since $v_c$ is a surface of section we can define the “characteristic width” $\hat{T}$ of $Z \setminus v_c$ as

$$\hat{T} = \hat{T}(v_c) = \sup_{z \in Z} \inf_{t \in (0, \infty)} \left\{ t \mid \phi_t(z) \in \text{image}(v_c) \right\}.$$

Now suppose that the periods of the $v_{n} = v_{t_n}$ are unbounded for some sequence $t_n \in I$. Choose a subsequence of $v_n$ so that the corresponding domain complex structures converge in some $\overline{\mathcal{M}}_{g,n}$. By the usual bubbling off analysis we may assume without loss of generality that there exists a constant $C > 0$ so that

$$|\pi_F dv_n(z)| \leq C(1 + |v_n^* \alpha(z)|) \leq C^2(1 + |\eta_n(z)|), \quad \forall z \in \hat{\Sigma}.$$

If the first inequality were not true there would be a sequence of points $p_n$ and a subsequence with $|\pi_F dv_n(p_n)| > n(1 + |v_n^* \alpha(p_n)|)$, which, after rescaling, yields a bubble with non–trivial $\omega$–energy, which can only happen finitely many times. If the first inequality holds true, but the second inequality does not, so $|\eta_n(p_n)|$ grows slower than the coexact part $|d\alpha_n(p_n)|$ and by the same argument we obtain a bubble map with non–trivial $\omega$–energy, which again can only happen finitely many times.

We split the remaining argument into two cases. Either $dv_n$, and thus $\eta_n$, becomes unbounded in the cylindrical metric or $v_n$ remains bounded but the twist of some neck becomes unbounded. In the former case we have that $\eta_n(z)$ becomes unbounded in some open set $U \subset \hat{\Sigma}$ by Harnack’s inequality. Then we may assume that there exists a sequence of points $p_n \in U$ so that $v_n^* \alpha(p_n)$ grows faster than $\pi_F dv_n(p_n)$, otherwise the $\omega$–energy would become unbounded. Extracting a convergent subsequence of maps centered at $p_n$ and rescaled so that $v_n^* \alpha$ has norm 1 we see that this subsequence converges uniformly on compact subsets of $\mathbb{C}$ to a characteristic flow line of unit speed. In particular, this subsequence must intersect $v_c$ since $\hat{T}$ is finite. But this is impossible by Lemma 4.1. Similarly, if $dv_n$ remains uniformly bounded, but the twist becomes unbounded, we may extract a subsequence so that the images of subsets of the necks $[-R_n, R_n] \times \{1\}$ converge uniformly to a characteristic flow line of unbounded length, again forcing an intersection with $v_c$, contradicting Lemma 4.1. \[Q.E.D.\]

Theorem 4.4. Suppose all closed characteristics of $(Z, \alpha, \omega)$ are non–degenerate and let $v : \hat{\Sigma} \rightarrow Z$ be an $\mathcal{H}$–holomorphic nicely embedded surface of section asymptotic to a collection of closed characteristics $B$. 

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Let \( M_1^v = M_1^v(\alpha, \omega, J) \) be the connected component of the moduli space of embedded \( \mathcal{H} \)-holomorphic maps with one marked point containing \( v \).

Then \( M_1^v \) has a natural compactification \( \overline{M}_1^v \). If \( M_1^v \) does not have boundary, then the evaluation map gives a diffeomorphism \( \text{ev} : M_1^v \to Z \setminus B \).

**Proof.** \( M_1^v \) has a natural compactification \( \overline{M}_1^v \) by Lemma 4.3 and Theorem 1.2.

Let \( f : \overline{M}_1^v \to \overline{M}_v \) be the forgetful map. Then \( \overline{M}_v \) is a closed, connected 1–dimensional manifold.

Assume it does not have boundary, then \( M_1^v = \overline{M}_1^v \). Let \( u \in M_v \). By Theorem 3.3 \( u \) is part of a unique local 1–parameter family \( u_t \) of maps in \( M_v \) with \( u = u_0 \) that are all embedded and that locally foliates \( Z \setminus B \). By Lemma 4.1 no two maps in \( M_v \) can intersect each other, so the image of the evaluation map is injective. Since \( v \) is a surface of section, every characteristic flow line intersects \( v \) in finite time. Then the fact that \( M_v \) is compact, together with the strong version of the implicit function theorem Theorem 3.3 implies that the image of the evaluation map from \( M_1^v \) to \( Z \setminus B \) is surjective.

This theorem suggests the existence of a finite energy foliation in the presence of a nicely embedded surface of section \( v \). The proof of this requires a gluing theorem to continue the moduli space \( \overline{M}_v \) past the boundary. This requires a slightly different pre–gluing spaces compared to the \( J \)–holomorphic case, due to the different behavior of neck–maps and is work in progress.

## 5 \( \mathcal{H} \)–Holomorphic Open Book Decompositions

As an application we now prove the existence of \( \mathcal{H} \)–holomorphic open book decomposition supported by any given contact structure. Armed with the results of Section 3 this is a straightforward extension of the work of Wendl in [Wen08].

**Theorem 5.1.** Suppose \( (Z, F) \) is a closed 3–manifold with positive, co-oriented contact structure \( F \) and \( \pi : Z \setminus B \to S^1 \) is an open book decomposition that supports \( F \). Then, after an isotopy of \( \pi \), there exists a nondegenerate contact form \( \alpha \) with \( \ker \alpha = F \), a compatible almost complex structure \( J \) on \( F \) and a smooth \( S^1 \)-family of nicely embedded \( \mathcal{H} \)-holomorphic maps parametrizing the pages of \( \pi \).

**Proof.** The proof follows the proof of the planar case given in [Wen08]. As explained in the first part of Section 3 of [Wen08] there exists a stable Hamiltonian structure \( (F, \alpha_0, \omega_0, J_0) \), and a small isotopy of \( \pi \) so that the pages of the open book decomposition are \( J_0 \)-holomorphic.

We now wish to deform this stable Hamiltonian structure to one arising from a contact form \( \alpha \) with contact structure \( F \) and some compatible almost complex structure \( J \), and we wish to deform the foliation to an \( \mathcal{H} \)-holomorphic foliation of \( (Z \setminus B) \) with respect to the almost complex structure \( J \) on \( F \). Any \( J_0 \)-holomorphic map is automatically \( \mathcal{H} \)-holomorphic (w.r.t. \( (\alpha_0, \omega_0, J_0) \)) and that all pages are nicely embedded. Denote this connected component of the
moduli space of $\mathcal{H}$–holomorphic maps by $\mathcal{M}_0$ and the space of maps with one marked point on the domain by $\mathcal{M}_0^1$.

Using the implicit function theorem on the compact family of curves, for a sufficiently small perturbation of stable Hamiltonian structure that leaves a neighborhood of the binding invariant, we can find nearby $\mathcal{H}$–holomorphic maps. More precisely, we obtain families $\{\mathcal{M}_t\}_{t \in [0,\varepsilon)}$ of moduli spaces of $\mathcal{H}$–holomorphic maps, where $\mathcal{M}_t$ consists of $\mathcal{H}$–holomorphic maps with respect the stable Hamiltonian structure $(\alpha_t, \omega_t, J_t)$, $t \in [0,\varepsilon]$ from [Wen08] in the connected component containing $\mathcal{M}_0$.

By possibly shrinking $\varepsilon$ we may assume that all elements of $\mathcal{M}_t$ are nicely embedded, and thus locally foliating by Theorem 3.3. Thus $\mathcal{M}_t$ foliates $Z \setminus B$ for $\lambda \in [0,\varepsilon]$ by Theorem 4.3.

By Theorem 4.4 we see that each $\mathcal{M}_t^1$ is diffeomorphic to $Z \setminus B$ via the evaluation map and that $\mathcal{M}_t$ is diffeomorphic to $S^1$. The characteristic vector field $R_t$ is transverse to each map in $\mathcal{M}_t^1$ since the maps are embedded. Thus the forgetful map $\pi_t^1 : \mathcal{M}_t^1 \to \mathcal{M}_t \approx S^1$ give the desired isotopy of open book projections.

By Giroux’s classification, the contact structures $F_t = \ker \alpha_t$ are contactomorphic for any $0 < t \leq \varepsilon$ since they are supported by the same open book decomposition. In particular, all are contactomorphic to $F$.

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