Exact Renormalization Flow and Domain Walls from Holography

Sergei V. Ketov

Fachbereich Physik
Universität Kaiserslautern
Erwin Schrödinger Strasse
67653 Kaiserslautern, Germany

ketov@physik.uni-kl.de

Abstract

The holographic correspondence between 2d, N=2 quantum field theories and classical 4d, N=2 supergravity coupled to hypermultiplet matter is proposed. The geometrical constraints on the target space of the 4d, N=2 non-linear sigma-models in N=2 supergravity background are interpreted as the exact renormalization group flow equations in two dimensions. Our geometrical description of the renormalization flow is manifestly covariant under general reparametrization of the 2d coupling constants. An explicit exact solution to the 2d renormalization flow, based on its dual holographic description in terms of the Zamolodchikov metric, is considered in the particular case of the four-dimensional NLSM target space described by the SU(2)-invariant (Weyl) anti-self-dual Einstein metrics. The exact regular (Tod-Hitchin) solutions to these metrics are governed by the Painlevé VI equation, and describe domain walls.

1Supported in part by the ‘Deutsche Forschungsgemeinschaft’
1 Introduction

A holographic correspondence was first proposed by ’t Hooft [1] and Susskind [2] in the particular context of black hole physics. According to the original formulation [1, 2], information on degrees of freedom inside a volume can be encoded in a surface enclosing this volume. There are many reasons to believe that the holographic principle may be valid far beyond its original framework, though a precise formulation of this principle is still lacking. The essence of the contemporary use of the holographic principle amounts to the assertion that a classical field theory (with gravity) in a volume is equivalent to certain Quantum Field Theory (QFT) (without gravity) defined on the boundary of this volume. This idea may be elevated to the existence of an equivalent description of QFT in terms of classical gravity in higher dimensions, or it may also be considered as the manifestation of the fundamental equivalence between quantum gauge field theories and strings [3]. For example, the very popular AdS/CFT correspondence in its original formulation [4] relates the type-IIB superstring theory on \( AdS_5 \times S^5 \) with the four-dimensional (4d), N=4 supersymmetric \( SU(N) \) Yang-Mills theory on the boundary of the \( AdS_5 \) space. The string loop corrections are proportional to \( N^{-2} \), whereas the \( \alpha' \)-corrections are proportional to \( \lambda^{-1/2} \), where \( \lambda = g_{\text{YM}}^2 N \) is the ’t Hooft coupling, so that the large-N and large-\( \lambda \) (strong coupling) limit can be investigated in the five-dimensional AdS-supergravity approximation (see ref. [5] for a review).

The Maldacena conjecture [4] relates two field theories in different dimensions, both having very high amount of symmetry (e.g., conformal symmetry, extended supersymmetry, electric-magnetic self-duality). Nevertheless, the holographic correspondence appears to be valid even if many of these symmetries are broken. For example, a massive deformation of the N=4 SYM theory breaks down both N=4 supersymmetry and conformal invariance, which result in a non-trivial Renormalization Group (RG) flow. The radial coordinate of the \( AdS_5 \) gives the natural scale to the 4d quantum gauge theory. The holographic duality in this context means the identification of the (classical) five-dimensional supergravity equations of motion in the bulk with the (quantum) RG-flow equations in the dual (large-N) quantum gauge theory on the four-dimensional boundary [4]. A specific example of the RG flow from the N=4 (superconformal) Yang-Mills theory in the UV to an N=1 (superconformal) gauge theory in the IR was given in ref. [7], where this flow was identified with a domain wall (BPS) solution to the five-dimensional (gauged) N=8 supergravity connecting two \( AdS_5 \) vacua. Away from a few known and highly symmetric examples of the holographic correspondence, it is far from being clear why does the holography
exist, and, perhaps, most importantly, where does the holographic principle apply.

To test the holographic principle and determine the area of its applicability, it is worthy to investigate more explicit examples of the holographic correspondence under the circumstances that are different from the standard AdS/CFT, e.g. in lower dimensions where many deep theorems about generic QFT and gravity are available, thus leaving less room for speculations. So we replace the 4d quantum Yang-Mills theories, playing the key role in the AdS/CFT correspondence, by the 2d quantum Non-Linear Sigma-Models (NLSM) with torsion and scalar potential. The 2d NLSM are known to share many key features with the 4d gauge theories, such as conformal invariance, renormalizability, solitons, asymptotic freedom, etc. The similarity between the RG flows in 4d gauge theories and the RG flows in 2d NLSM, in the context of the AdS/CFT correspondence, was also noticed in ref. [9]. To get control over quantum non-perturbative issues, we also impose N=2 extended supersymmetry in the 2d QFT under consideration, and require its integrability (i.e. the infinitely many conservation laws, or a factorizability of the S-matrix). We find that the RG flow in the 2d, N=2 QFT admits the most natural description in terms of the effective four-dimensional N=2 supergravity coupled to hypermultiplet matter. The hypermultiplet scalars represent the couplings of the 2d, N=2 QFT under consideration, whereas the four-dimensional N=2 supergravity serves as the geometrical background for the hypermultiplet NLSM (or as the source for the stress-energy tensor), as usual. The metric of the 4d, N=2 NLSM is identified with the Zamolodchikov metric of the 2d, N=2 QFT. This correspondence is nothing but the off-critical extension of the well-known world-sheet/spacetime correspondence [10] between the 2d, N=2 superconformal field theories and the effective four-dimensional N=2 supergravity theories, which is known in the type-II superstring compactification on Calabi-Yau spaces, under the preservation of integrability and supersymmetry. We find that the RG flow in the 2d, N=2 QFT can be naturally described by the radial dependence of the NLSM metric solution. The NLSM dual description automatically accommodates the general coordinate invariance in the space of QFT couplings [11], since it amounts to the reparametrizational invariance of the NLSM target space, while the existence of the natural radial coordinate in the NLSM target space is guaranteed by the quaternionic nature of the NLSM metric. Taken together, this means the existence of a holographic correspondence between certain two-dimensional (2d) N=2 supersymmetric QFT and classical four-dimensional (4d) N=2 supergravity with hypermultiplet matter.

Yet another simplification of our setup, in comparison to the higher-dimensional AdS/CFT correspondence, is the absence of a non-trivial four-form flux (or RR back-
(ground) that must be present in the \( AdS_5 \) space where it results in the gauging of an abelian isometry of the hypermultiplet NLSM — see ref. [12] where the domain-wall solutions from the M-theory compactification on Calabi-Yau threefolds were discussed. The use of holography in application to the 2d QFT versus 4d QFT apparently leads to the effective description in terms of the 4d supergravity versus 5d supergravity, respectively.

As an explicit example, we examine in detail the simplest non-trivial case of a single matter hypermultiplet in 4d, \( N=2 \) supergravity background from the viewpoint of holography. The Zamolodchikov metric then appears to be a four-dimensional Einstein-Weyl metric, whose exact regular solutions are given by Tod-Hitchin metrics [13, 14]. We demonstrate that the Einstein-Weyl gravity equations can be interpreted as the RG flow equations, whereas their regular solutions describe the domain walls relating two (UV and IR) fixed points. A derivation of any exact RG domain wall solution is known to be a formidable problem in physics. To the best of our knowledge, the Tod-Hitchin metrics were never considered in the context of the holographic correspondence, while they also give the remarkable connection between the RG flow in the integrable 2d, \( N=2 \) QFT and the standard integrable non-linear equations of mathematical physics, such as the Painlevé VI equation.

The paper is organized as follows. In sect. 2 generic 2d CFT and QFT are discussed from the viewpoint of the holographic correspondence. In sect. 3 we review the special features of this correspondence after adding \( N=2 \) supersymmetry. Sect. 4 is devoted to the covariant differential equations on the Zamolodchikov metric and their homogeneous solutions in the case of the 2d, \( N=2 \) superconformal field theories whose 4d, \( N=2 \) supergravity duals are described in terms of a single hypermultiplet. The relation between the Weyl self-duality and RG flow in this particular case is given in sect. 5. The explicit exact solutions to the RG flow, based on the regular Tod-Hitchin metrics, are given in sect. 6 where their relation to the Painlevé VI equation is explained. Sect. 7 is our conclusion. Basic facts about theta functions are summarized in Appendix.

## 2 QFT and Zamolodchikov theorems in 2d

Let’s consider an abstract two-dimensional (2d) Conformal Field Theory (CFT) of central charge \( c \). Let \( \mathcal{L}_{\text{CFT}} \) be its Lagrangian. A 2d QFT is defined by the perturbed Lagrangian

\[
\mathcal{L}_{\text{QFT}} = \mathcal{L}_{\text{CFT}} + \sum_i \lambda_i \mathcal{O}_i ,
\]

(2.1)
where $\mathcal{O}_i$ are some local (normally ordered) composite operators, and $\lambda_i$ are the associated (finite, not infinitesimal) coupling constants. We always assume that the 2d QFT (2.1) is renormalizable; in this case the perturbations $\mathcal{O}_i$ are also called renormalizable.

The 2d QFT (2.1) may define another 2d CFT provided the operators $\mathcal{O}_i$ are chosen to be of conformal weights $(1,1)$. In this case, the associated perturbations are called marginal, while they are particularly relevant for string theory [10]. In fact, all 2d CFT can be described by the use of marginal perturbations (see, e.g., ref. [14] for a review), so that a choice of the CFT to begin with is at our disposal. Though a generic CFT does not have a convenient Lagrangian description [15], the details of its Lagrangian are not going to be relevant for our purposes. As is well known, the 2d CFT are most efficiently described by the use of their symmetries. The initial CFT can be chosen to be maximally symmetric. Any 2d CFT defines an integrable field theory in the sense that it has the infinitely many conservation laws. This property can be naturally generalized: a 2d QFT is called integrable provided that it possesses the infinitely many conservation laws or, equivalently, if it has a factorizable S-matrix [16]. The Zamolodchikov techniques of derivation of the integrable (massive) deformations of CFT are essentially based on demanding the infinite number of conserved currents to survive the perturbations away from the criticality [16].

The fundamental difference between CFT and QFT is due to the fact that the latter has an (energy) scale $\mu$, whereas the former is scale invariant by definition. It is therefore natural to introduce the (one-parameter) family of QFT related to each other by a change of scale. Changing the scale in the parameter space $\mathcal{M} = \{\lambda_i\}$ defines a ‘flow’ known as the RG flow. The fixed points of this flow are CFT’s. A QFT is characterized by a point in $\mathcal{M}$. The RG trajectory through this point allows us to define the UV and IR limits of a given QFT, by taking $\mu \to \infty$ and $\mu \to 0$, respectively. The CFT we started with can then be naturally identified with the UV fixed point of QFT. We identify the scale $\mu$ of 2d QFT with one of its coupling constants $\lambda_0$.

As regards the IR-limit of a QFT, it may be either a massive (non-conformal) field theory or yet another CFT [17]. In the latter case, the RG flow interpolates between the UV- and IR-fixed points, so that it can be interpreted as a domain wall in $\mathcal{M}$. This is precisely the case that we would like to investigate in this paper, from the holographic point of view. Since we are interested in 2d QFT’s, we have the advantage of having two very powerful tools due to Zamolodchikov [17, 18]. The famous Zamolodchikov theorems imply (i) the existence of a metric in $\mathcal{M}$, which is
defined by the two-point function of perturbing operators on the plane at a fixed distance,

\[ G_{ij} = \langle O_i O_j \rangle , \]  

and (ii) the existence of a function (called \( c \)-function) in \( \mathcal{M} \), which monotonically decreases along the RG flow,

\[ \dot{c} \equiv \frac{dc}{dt} \leq 0 , \quad c = c(t) , \quad t = \ln \mu , \]  

and whose fixed points (\( \dot{c} = 0 \)) correspond to the RG-fixed points, i.e. the CFT’s. The RG beta-functions in QFT are defined by

\[ \dot{\lambda}_i = \beta_i (\lambda) = \Delta_{ij} \lambda_j + C_{ijk} \lambda_j \lambda_k + \ldots , \]  

where \( \Delta_{ij} \) represent the anomalous scaling dimensions, \( C_{ijk} \) are the Operator Product Expansion (OPE) coefficients of the operators \( \vec{O} \) in 2d QFT, and the dots stand for the higher order terms (in \( \vec{\lambda} \)) of the expansion of the beta-functions in power series near a fixed point. The OPE coefficients \( C_{ijk} \) are universal provided that \( \Delta_{ij} = 0 \).

The \( c \)-function of Zamolodchikov can be considered as a function of the coupling constants \( \lambda_i \), whose stationarity implies criticality, i.e.

\[ \frac{\partial c}{\partial \lambda_i} = 0 \text{ is equivalent to } \beta_i = 0 , \]  

while the critical value of the \( c \)-function at a fixed point can be identified with the central charge \( c \) of the corresponding CFT,

\[ c(\lambda_{\text{crit}}) = c , \quad \beta_i(\lambda_{\text{crit}}) = 0 . \]  

It should be stressed that eq. (2.5) is merely an on-shell relation. A stronger off-shell conjecture in the form

\[ \frac{\partial c}{\partial \lambda_i} = K_{ij} \beta_j \]  

with some invertible matrix \( K_{ij} \) often appears in the literature (together with yet another proposal that \( K_{ij} \) may even be proportional to Zamolodchikov metric \( G_{ij} \)), despite of the fact that explicit multi-loop calculations in the 2d NLSM with torsion do not support eq. (2.7) — see refs. \[19, 8\] for details.

In string theory, the massless physical modes (associated with a spacetime metric \( g_{\mu\nu} \), an antisymmetric tensor \( b_{\mu\nu} \), and a dilaton \( \Phi \)) of a string are described by marginal deformations of 2d CFT, while the low-energy string effective action in

\( ^2 \)A summation over repeated indices is always assumed.
spacetime is determined by the vanishing RG beta-functions of the 2d NLSM describing string propagation in the background of its massless modes [10, 8]. In this context, Zamolodchikov’s c-theorem just guarantees the existence of the string effective action. In fact, since the Zamolodchikov c-function also makes sense off criticality, it is always possible to promote the coupling constants $\lambda_i$ to the scalar fields $\phi_i(x)$ in spacetime $(x^\mu)$, whose low-energy effective action is given by the NLSM with Zamolodchikov metric $G_{ij}(\phi)$, and whose vacuum expectation values are $\langle\phi_i\rangle = \lambda_i$. The scalar fields $\phi_i$ then develop a non-trivial scalar potential $V(\phi)$ in the effective Lagrangian. The RG flow interpolates between two different extrema of $V(\phi)$ — here is the name ‘domain wall’ comes from (cf. refs. [6, 7]),

$$\frac{\partial V}{\partial \phi_i} = 0 \iff \dot{c} = 0.$$ (2.8)

We are thus led to the following (bosonic) low-energy effective Lagrangian [11]:

$$\mathcal{L}(\phi, g) = \frac{1}{2}G_{ij}(\phi)\partial^\mu \phi^i \partial_\mu \phi^j - V(\phi) - \frac{1}{2}R,$$ (2.9)

which is nothing but the minimal coupling of the NLSM (with the NLSM target space metric equal to the Zamolodchikov metric $G$, and the scalar potential $V$) to background gravity $g_{\mu\nu}$. From the 2d perspective, the spacetime metric $g_{\mu\nu}$ in the effective Lagrangian (2.9) represents marginal deformations, while the NLSM scalars $\phi_i$ appears as the sources for the local operators $\mathcal{O}_i$. Given the energy scale $\mu$ that is much smaller than the cut-off scale $\mu_c \sim e^{kr}$ of QFT, the Lagrangian (2.9) can be trusted. The ‘low-energy’ approximation means that we are only interested in the local part of the QFT effective action, or the two-point correlators of eq. (2.2). As was demonstrated in ref. [11], in the context of the AdS/CFT correspondence, the identification of the five-dimensional low-energy effective Lagrangian with eq. (2.9) indeed leads to the standard Callan-Symanzik RG equations for the holographic RG flow in the dual four-dimensional QFT. At very low energies, where merely the potential term in eq. (2.9) is relevant, its IR value determines the 4d central charge $c$ (i.e. the holographic Weyl anomaly, or the critical exponent $k$) of the CFT on the $AdS_5$ boundary, $\lim_{\mu \to 0} V = \frac{3}{2}k^{-2} = \frac{3}{16}e^{-2/3}$ [20]. We would like to describe the RG evolution in 2d QFT by identifying its parameter space $\mathcal{M}$ with the NLSM target space in eq. (2.9). A derivation of the exact Zamolodchikov metric (not just in the vicinity of a fixed point) is the much more complicated problem than a calculation of the central charge, and, to our knowledge, in the context of the holographic correspondence, it was never addressed elsewhere.

3The dimensional NLSM coupling constant in front of the NLSM action and the gravitational constant in front of the Einstein-Hilbert action are both set to be one in eq. (2.9).
3 Adding N=2 supersymmetry

The holographic correspondence formulated in the preceding section is very general, but it cannot be made more specific unless we add more structure to the dual QFT/gravity pairs. N=2 extended supersymmetry (eight supercharges) is the natural symmetry that puts both QFT and gravity under control (the ‘N=2 wonderland’). It is worth mentioning here that the 5d, N=1 AdS superalgebra SU(2, 2|1) corresponds to N=2 supersymmetry in 4d, or N=(2,2) superconformal symmetry in 2d. The supersymmetric world-sheet/spacetime correspondence is well known in the standard superstring compactification, with the 2d (world-sheet) N=2 superconformal models being used as the building blocks of the spacetime supersymmetric superstring vacua [10]. For example, as regards the closed type-II superstrings (compactified on a Calabi-Yau space \(\mathcal{Y}\)), their low-energy effective action is given by the four-dimensional N=2 supergravity theory, whose matter couplings have the ‘special’ Kähler geometry in the sector of N=2 vector multiplets [21] and the quaternionic-Kähler geometry in the sector of hypermultiplets [22]. We are going to concentrate here on hypermultiplets, each containing four real scalars and a Dirac hyperino. The corresponding (unique) action is given by the N=2 (locally) supersymmetric extension of eq. (2.9) in four spacetime dimensions.\(^4\) In the context of the superstring compactification, the Zamolodchikov metric \(G_{ij}\) of the underlying 2d, N=2 Superconformal Field Theory (SCFT) is identified with the \(4(h_1,2 + 1)\)-dimensional quaternionic metric on the moduli space \(\mathcal{M}\) of the Calabi-Yau threefold \(\mathcal{Y}\), whose Hodge number \(h_{1,2}\) is just the number of the harmonic (1,2)-forms on the threefold \(\mathcal{Y}\). Taken together, these facts amount to the existence of the holographic correspondence between the 2d, N=2 SCFT and the 4d, N=2 supergravities arising as the low-energy effective field theories of type-II superstrings.

This N=2 CFT/supergravity correspondence can be extended to the off-critical holographic correspondence between N=2 supersymmetric QFT and certain 4d, N=2 supergravities, since the Zamolodchikov metric still makes sense off criticality. In the early nineties, Cecotti and Vafa [23] applied topological methods to study integrable (massive) deformations of 2d, N=2 CFT (see ref. [24] too). In particular, the effective NLSM metrics for some 2d, N=2 CFT and QFT were calculated by identifying the Zamolodchikov metric with the metric of the N=2 supersymmetric ground states. The ground state metric, as the function of perturbation parameters \(\vec{\lambda}\), in the integrable case obeys the classical Toda (or affine Toda) equations, which arise as the flatness\(^4\)The number of spacetime dimensions could also be five or six, as long as it does not affect the NLSM target space.
conditions for certain holomorphic and anti-holomorphic connections in the vacuum bundle over $\mathcal{M}$ \cite{23, 24}. The coefficients of the Toda-like integrable equations are just given by the topological correlation functions \cite{23}. In the critical (N=2 SCFT) case, the exact solutions to these equations are governed by some holomorphic data of moduli, whereas the underlying data in the case of an integrable N=2 QFT is not holomorphic. Unfortunately, all examples of the massive (integrable) deformations of the N=2 supersymmetric Landau-Ginzburg models, considered in refs. \cite{23, 24}, lead to the massive field theories in the infra-red limit. In particular, the Cecotti-Vafa solutions associated with generic deformations of N=2 SCFT, are not the domain-wall solutions. The letter are associated with the special deformations whose choice is not apparent in 2d. The use of holography and R-invariance makes it possible to identify the relevant deformations in the dual picture and, sometimes, explicitly derive the corresponding regular Zamolodchikov metrics describing the domain-walls preserving both N=2 supersymmetry and R-symmetry (sects. 5 and 6).

It is worth mentioning that the presence of a non-trivial scalar potential in the N=2 supersymmetric extension of eq. (2.9) is not in conflict with N=2 supersymmetry, which prohibits a superpotential in renormalizable 4d, N=2 QFT. In fact, any non-trivial N=2 NLSM kinetic terms with non-vanishing central charges give rise to a (unique) non-trivial scalar potential — see, e.g., ref. \cite{25} for some explicit examples. Gauging the NLSM isometries also gives rise to a non-trivial scalar potential. We didn’t attempt to calculate the scalar potential explicitly.

4 Universal hypermultiplet and Wolf spaces

As is well known, the N=2 scalar (hypermultiplet) couplings in the four-dimensional N=2 supergravity are described by the NLSM with quaternionic-Kähler target spaces of negative scalar curvature \cite{22}. In our context (sects. 2 and 3) this result implies that the Zamolodchikov metric in a 2d, N=2 QFT is also quaternionic-Kähler. Moreover, any Einstein space of a constant negative scalar curvature admits a natural radial coordinate \cite{26}, whose existence is the central element of the holographic correspondence.

We shall only consider in detail the simplest non-trivial case of a single hypermultiplet, which corresponds to four coupling constants. One of the coupling constants is identified with the RG parameter $\mu$, so that we are going to deal with the RG flow of three couplings. In the four-dimensional type-II superstring models, the dilaton

\footnote{The 4d NLSM are non-renormalizable.}
belongs to a hypermultiplet, together with an axion and a complex RR scalar. In the literature this hypermultiplet is called the universal hypermultiplet \cite{27}, even though there seem to be nothing ‘universal’ in its coupling to either N=2 supergravity or N=2 matter multiplets, from the viewpoint of N=2 supersymmetry, when compared to other hypermultiplets.

All homogeneous quaternionic-Kähler spaces are classified \cite{28}, while they are naturally associated with the 2d, N=2 or N=4 SCFT via the standard Kazama-Suzuki construction \cite{13}. The four-dimensional target space of the 4d, N=2 NLSM in N=2 supergravity background is (Weyl) Anti-Self-Dual (ASD) and Einstein (see ref. \cite{29} for mathematical details). The ASD Einstein metric should, therefore, obey the differential equations

\[ W_{abcd} = 0 \quad \text{and} \quad R_{ab} = \frac{1}{2} \Lambda g_{ab}, \quad \Lambda = -24 \kappa^2, \quad \tag{4.1} \]

where \( W = W^- + W^+ \) is the Weyl tensor, \( R_{ab} \) is the Ricci tensor, \( a, b, c, d = 1, 2, 3, 4 \), and \( \kappa \) is the gravitational coupling constant. It is worth mentioning that eqs. (4.1) are manifestly covariant under general reparametrizations of the coupling constants. In other words, the general coordinate invariance of the RG flow in the space of couplings is guaranteed in the dual (NLSM) description.

Given a simple Lie group \( G \), the associated quaternionic symmetric (homogeneous) space is unique; it is called the Wolf space \cite{30},

\[ \frac{G}{H \times SU(2)_{\chi}} \quad \tag{4.2} \]

where \( \chi \) is the highest root of \( G \), the \( SU(2)_\chi \) is the subalgebra of \( G \), associated with the root \( \psi \), and \( H \) is the centralizer of \( SU(2)_\chi \) in \( G \). There are only two four-dimensional Wolf cosets of negative scalar curvature, \( SO(4,1)/SO(4) \) and \( SU(2,1)/U(2) \). Both Wolf spaces have the \( SU(2) \) isometry that can be identified with the automorphisms (R-symmetry) \( SU(2) \) of the rigid N=2 supersymmetry algebra.

A generic metric, possessing the \( SU(2) \) isometry, is most conveniently described (like in general relativity) by the Bianchi IX formalism where the \( SU(2) \) symmetry is manifest. Given a ‘radial’ coordinate \( r \) and ‘Euler angles’ \( (\theta, \psi, \phi) \), one introduces the \( SU(2) \)-covariant one-forms,

\[ \sigma_1 = \frac{1}{r}(\sin \psi d\theta - \sin \theta \cos \psi d\phi), \]
\[ \sigma_2 = -\frac{1}{r}(\cos \psi d\theta + \sin \theta \sin \psi d\phi), \]
\[ \sigma_3 = \frac{1}{r}(d\psi + \cos \theta d\phi), \quad \tag{4.3} \]
which satisfy the relation $\sigma_i \wedge \sigma_j = \frac{1}{2} \varepsilon_{ijk} d\sigma_k$. The standard metric, associated with the symmetric (Euclidean $AdS_4$) space $SO(4,1)/SO(4)$, is conformally flat,

$$ds^2 = \frac{1}{(1-r^2)^2} \left[ dr^2 + r^2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \right]. \quad (4.4)$$

The boundary of $AdS_4$ at $(r \to 1^-)$ is given by a three-dimensional sphere $S^3$, while the four-dimensional conformal structure in the ball $r^2 < 1$ induces the conformal structure on $S^3$. Unfortunately, the non-topological (non-trivial) QFT in three dimensions are not conformally invariant, and a conformal anomaly does not exist in three dimensions (like in any other odd-dimensional space). Therefore, we should not expect the existence of a non-trivial IR-fixed point, when starting from the $AdS_4$ and applying the RG flow.

The remaining symmetric Wolf space $SU(2,1)/U(2)$ is perfectly suitable for our purposes. The natural metric in this space is given by the so-called Bergmann metric which is dual to the standard Fubini-Study metric [29],

$$ds^2 = \frac{dr^2}{(1-r^2)^2} + \frac{r^2}{(1-r^2)^2} \sigma_2^2 + \frac{r^2}{(1-r^2)} (\sigma_1^2 + \sigma_3^2). \quad (4.5)$$

The conformal structure, associated with the metric (4.5) inside the unit ball in $\mathbb{C}^2$, does not extend across the boundary since the coefficient at $\sigma_2^2$ decays faster than the coefficients at $\sigma_1^2$ and $\sigma_3^2$. However, the conformal structure survives in the two-dimensional (2d) subspace of $S^3$, which is annihilated by $\sigma_2$, because it is protected by the Kähler nature of the metric (4.5).

The Bergmann metric of the symmetric space $SU(2,1)/U(1) \times SU(2)$ can be identified with the Zamolodchikov metric of certain 2d, N=2 SCFT (sect. 5), which may serve as the UV fixed point for the RG flow. This equally applies to any Wolf space, while the associated 2d, N=2 SCFT to be defined via the Kazama-Suzuki construction, in fact, possesses 2d, N=4 superconformal symmetry, albeit with the quadratically generated (Bershadsky-Knizhnik) algebra [31]. The formal central charge of the 2d SCFT, associated with $SU(2,1)/U(1) \times SU(2)$, is given by $c = 3(3p + 1)/(p + 3)$ [32]. Note that $c \to 9$ when $|p| \to \infty$.

The hypermultiplet moduli space, arising in the type-IIA superstring compactification on Calabi-Yau threefolds, also obeys eq. (4.1), while in the tree approximation (without quantum corrections) it is known to be described by the quaternionic manifold $Q_{\text{tree}}$ similar to $SU(2,1)/U(1) \times SU(2)$ [32]. However, unlike the $SU(2,1)/U(1) \times SU(2)$, the $Q_{\text{tree}}$ has the Heisenberg (isotropy) symmetry group instead of $SU(2)$, and thus belongs to the Bianchi II type.
5 Weyl self-duality and RG flow

The non-homogeneous solutions to eq. (4.1), which can be interpreted as the RG flow, are supposed to share the basic features of the latter, namely,

- they have to obey the first-order differential equations,
- there should be a well-defined RG flow parameter.

The second condition is most naturally ensured by the $SU(2)$ isometry of the four-dimensional Zamolodchikov metric because the non-degenerate action of this isometry leads to the well-defined three-dimensional orbits that can be parametrized by the ‘radial’ coordinate ($t$) to be identified with the RG parameter. In the context of N=2 supersymmetry, the $SU(2)$ isometry has its origin in the unbroken R-symmetry $SU(2)$. The (Weyl) ASD equations on the metric take the form of a first-order system of Ordinary Differential Equations (ODE), so that the first condition above is automatic. In other words, in the context of the holographic duality, we should add the $SU(2)$ symmetry to the general requirements of eq. (4.1), all dictated by the four-dimensional local N=2 supersymmetry alone.

We are thus led to a study of the $SU(2)$-invariant deformations of the Bergmann metric (4.5) subject to the differential constraints (4.1). This well-defined mathematical problem was addressed by Tod [13] and Hitchin [14]. A generic $SU(2)$ invariant metric in the Bianchi IX formalism reads

$$ds^2 = w_1 w_2 w_3 dt^2 + \frac{w_2 w_3}{w_1} \sigma_1^2 + \frac{w_3 w_1}{w_2} \sigma_2^2 + \frac{w_1 w_2}{w_3} \sigma_3^2 ,$$

(5.1)

where we have taken it in the diagonal form with respect to the $\sigma_i$ of eq. (4.3), without loss of generality. As was demonstrated, e.g., in ref. [13], the Weyl ASD conditions (4.1) applied to the Ansatz (5.1) result in the classical Halphen system of ODE [33],

$$\begin{align*}
\dot{A}_1 &= - A_2 A_3 + A_1 (A_2 + A_3) , \\
\dot{A}_2 &= - A_3 A_1 + A_2 (A_3 + A_1) , \\
\dot{A}_3 &= - A_1 A_2 + A_3 (A_1 + A_2) ,
\end{align*}$$

(5.2)

where the dots denote differentiation with respect to $t$, and the functions $A_i$, $i = 1, 2, 3$, are defined by the auxiliary system of ODE,

$$\begin{align*}
\dot{w}_1 &= - w_2 w_3 + w_1 (A_2 + A_3) , \\
\dot{w}_2 &= - w_3 w_1 + w_2 (A_3 + A_1) , \\
\dot{w}_3 &= - w_1 w_2 + w_3 (A_1 + A_2) .
\end{align*}$$

(5.3)
The Bergmann metric corresponds to the dual 2d SCFT since all its $A_i$ vanish, as expected: this follows from a comparison of eqs. (4.5), (5.1) and (5.3). Being considered as the SCFT Zamolodchikov metric, the metric (4.5) is non-trivial (i.e. non-flat), even though all $A_i = 0$.

Being of the form
\[ \dot{\lambda}_i = C_{ijk} \lambda_j \lambda_k , \] (5.4)
the ODE system (5.2) is the particular case of the RG flow equations (2.4) in the dual 2d, N=2 QFT, whose coefficients $C_{ijk}$ represent the normalized (and universal) OPE coefficients of the underlying 2d, N=2 SCFT at the UV fixed point of the 2d, N=2 QFT. The Zamolodchikov c-function defined by
\[ c(\lambda) = c - \frac{1}{3} C_{ijk} \lambda_i \lambda_j \lambda_k \] (5.5)
satisfies the Zamolodchikov condition,
\[ \dot{c} = -C_{ijk} \lambda_i \lambda_j \lambda_k = - \sum_i (\lambda_i)^2 \leq 0 , \] (5.6)
for any choice of the totally symmetric coefficients $C_{ijk}$ in eq. (5.4).

The second (Einstein) condition in eq. (4.1) can be easily satisfied by conformal rescaling of a solution to the (Weyl) ASD metric provided by the ODE systems (5.2) and (5.3), because any local Weyl transformation does not affect the vanishing Weyl tensor (see sect. 6 for details). Having obtained an explicit solution to the Halphen system (5.2), it may be substituted into eq. (5.3). To solve eq. (5.3), it is convenient to change variables as \[ w_i = \Omega_i \frac{\dot{x}}{\sqrt{x(1-x)}} , \] (5.7)
where the new variables $\Omega_i(x)$, $i = 1, 2, 3$, are constrained by the algebraic condition
\[ \Omega_2^2 + \Omega_3^2 - \Omega_1^2 = \frac{1}{4} \] (5.8)
that reduces the number of the newly introduced functions in eq. (5.7) from four to three, as it should.
Equations (5.3) in terms of the new variables take the form

\begin{align*}
\Omega_1' &= - \frac{\Omega_2\Omega_3}{x(1-x)}, \\
\Omega_2' &= - \frac{\Omega_3\Omega_1}{x}, \\
\Omega_3' &= - \frac{\Omega_1\Omega_2}{1-x},
\end{align*}

where the primes denote differentiation with respect to $x$. It is not difficult to verify that the algebraic constraint (5.8) is preserved under the flow (5.9), so that the transformation (5.7) is fully consistent. In terms of the new variables $(x, \Omega_i)$, the Einstein condition of eq. (4.1) on the metric (5.1), having the form

\begin{equation}
\begin{aligned}
ds^2 &= e^{2u} \left[ \frac{dx^2}{x(1-x)} + \frac{\sigma_1^2}{\Omega_1^2} + \frac{(1-x)\sigma_2^2}{\Omega_2^2} + \frac{x\sigma_3^2}{\Omega_3^2} \right],
\end{aligned}
\end{equation}

amounts to the algebraic relation

\begin{equation}
96\kappa^2 e^{2u} = \frac{8x\Omega_1^2\Omega_2^2\Omega_3^2 + 2\Omega_1\Omega_2\Omega_3(x(\Omega_1^2 + \Omega_2^2) - (1 - 4\Omega_3^2)(\Omega_2^2 - (1-x)\Omega_1^2))}{(x\Omega_1\Omega_2 + 2\Omega_3(\Omega_2^2 - (1-x)\Omega_1^2))^2}.
\end{equation}

Having interpreted the ODE system (5.4) as the RG flow equations in the (non-conformal) dual 2d, N=2 QFT originating from 2d, N=4 SCFT, the crucial question is, of course, about the behaviour of the RG flow in the IR limit, $x \to 1^-$. The holographic interpretation of an exact metric solution to eq. (4.1) requires it to be regular (or complete) in the bulk, so that all of its pole singularities (in a particular parametrization) have to be removable by coordinate transformations. The existence of an IR fixed point implies that the regular metric should have the asymptotical behaviour similar to that of the Berman metric, i.e. the metric coefficient at $\sigma_2^2$ in eq. (5.1) should decay faster than the others. This would mean that the boundary annihilated by $\sigma_2$ is two-dimensional, while it has a conformal structure and a conformal anomaly. The Kähler structure in the bulk is going to be extendable to the boundary, which implies that the 2d CFT on the boundary should be N=2 supersymmetric, at least. The explicit regular metric solutions (sect. 6) confirm these expectations.

### 6 Painlevé VI equation and complete solution

The ODE system (5.2) has a long history [34]. Perhaps, its most natural (manifestly integrable) derivation is provided via a reduction of the $SL(2, \mathbb{C})$ anti-self-dual Yang-Mills equations from four Euclidean dimensions to one [33]. A classification of all
possible reductions is known in terms of the so-called Painlevé groups that give rise to six different types of integrable Painlevé equations \[35\]. It remains to identify those of them that lay behind the Weyl-ASD (quaternionic-Kähler) geometry with $SU(2)$ symmetry. There are only two natural (or nilpotent, in the terminology of ref. \[35\]) types (III and VI) that give rise to a single non-linear integrable equation. In the geometrical terms, it is the Painlevé III equation that lays behind the four-dimensional Kähler spaces with vanishing scalar curvature \[36\], whereas the Painlevé VI equation is known to be behind the Weyl-ASD geometries having the $SU(2)$ symmetry \[13, 14, 37\]. A generic Painlevé VI equation has four real parameters \[35\], but they are all fixed by the quaternionic-Kähler property \[13, 14\]. This results in the following particular Painlevé VI equation:

$$
y'' = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) (y')^2 - \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) y' + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[ \frac{1}{8} - \frac{x}{8y^2} + \frac{x-1}{8(y-1)^2} + \frac{3x(x-1)}{8(y-x)^2} \right],$$  \hspace{1cm} (6.1)

where $y = y(x)$, and the primes denote differentiation with respect to $x$.

The equivalence between eqs. (5.2) and (6.1) via eq. (5.9) is well known to mathematicians \[13, 14, 37\]. Explicitly, in the Einstein case, it reads

$$
\Omega_1^2 = \frac{(y-x)^2y(y-1)}{x(1-x)} \left( v - \frac{1}{2(y-1)} \right) \left( v - \frac{1}{2y} \right),
\Omega_2^2 = \frac{(y-x)y^2(y-1)}{x} \left( v - \frac{1}{2(y-x)} \right) \left( v - \frac{1}{2y} \right),
\Omega_3^2 = \frac{(y-x)y(y-1)^2}{(1-x)} \left( v - \frac{1}{2y} \right) \left( v - \frac{1}{2(y-x)} \right),$$  \hspace{1cm} (6.2)

where the auxiliary variable $v$ is defined by the equation

$$
y' = \frac{y(y-1)(y-x)}{x(x-1)} \left( 2v - \frac{1}{2y} - \frac{1}{2(y-1)} + \frac{1}{2(y-x)} \right).$$  \hspace{1cm} (6.3)

An exact solution to the Painlevé equation (6.1), which leads to a complete (regular) metric, is known to be unique, while it can be expressed in terms of the standard theta-functions $\vartheta_\alpha(z|\tau)$ where $\alpha = 1, 2, 3, 4$. We use the standard definitions and notation for the theta functions \[38\] — see Appendix. In order to write down the relevant solution to eq. (6.1), the theta-function arguments should be related by $z = \frac{1}{2}(\tau - k)$, where $k$ is considered to be an arbitrary (real and positive) parameter. The relation to the $x$-variable of eq. (6.1) is given by $x = \vartheta_3^4(0)/\vartheta_4^4(0)$, where the value of $z$ is
explicitly indicated, as usual. One finds \[39, 14\]

\[
y(x) = \frac{\varphi''(0)}{3\pi^2 \varphi'(0) \varphi_1(0)} + \frac{1}{3} \left[ 1 + \frac{\varphi_1'(0)}{\varphi_1(0)} \right] + \frac{\varphi''(z) \varphi_1(z) - 2 \varphi''(0) \varphi_1'(z) + 2 \pi i (\varphi''(z) \varphi_1(z) - \varphi''(0))}{2 \pi^2 \varphi_1'(0) \varphi_1(z) (\varphi_1'(z) + \pi i \varphi_1(z))}.
\] (6.4)

The parameter \(k > 0\) describes the monodromy of the solution (6.4) around its essential singularities (branch points) \(x = 0, 1, \infty\). This (non-abelian) monodromy is generated by the matrices (with the eigenvalues \(\pm i\)) \[14\]

\[
M_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & i^{1-k} \\ i^{1+k} & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & i^{-k} \\ -i^k & 0 \end{pmatrix}.
\] (6.5)

The explicit (equivalent) form of an exact solution to the metric coefficients \(w_i\) in eq. (5.3) was derived in ref. \[40\], in terms of the theta functions with characteristics, by the use of the fundamental Schlesinger system and the isomonodromic deformation techniques.

The function (6.4) is meromorphic outside \(x = 0, 1, \infty\), with the simple poles at \(\bar{x}_1, \bar{x}_2, \ldots\), where \(\bar{x}_n \in (x_n, x_{n+1})\) and \(x_n = x(nk/(2n-1))\) for each positive integer \(n\). Accordingly, the metric is well-defined (complete) for \(x \in (\bar{x}_n, x_{n+1})\), i.e. in the unit ball with the origin at \(x = x_{n+1}\) and the boundary at \(x = \bar{x}_n\) \[14\]. Near the boundary the metric (11) has the asymptotical behaviour

\[
ds^2 = \frac{dx^2}{(1-x)^2} + \frac{4}{(1-x) \cosh^2(\pi k/2)} \sigma_1^2 + \frac{16}{(1-x)^2 \sinh^2(\pi k/2) \cosh^2(\pi k/2)} \sigma_2^2 + \frac{4}{(1-x) \sinh^2(\pi k/2)} \sigma_3^2 + \text{regular terms}.
\] (6.6)

As is clear from eq. (6.6), the coefficient at \(\sigma_2^2\) vanishes faster than the coefficients at \(\sigma_1^2\) and \(\sigma_3^2\) when approaching the boundary, \(x \to 1^-\), similarly to eq. (4.5), so that there is the natural conformal structure,

\[
\sinh^2(\pi k/2) \sigma_1^2 + \cosh^2(\pi k/2) \sigma_3^2,
\] (6.7)

on the two-dimensional boundary annihilated by \(\sigma_2\) \[13, 14\]. The only relevant parameter \(\tanh^2(\pi k/2)\) in eq. (6.7) represents the central charge (the conformal anomaly, or the critical exponent) of the two-dimensional superconformal field theory on the boundary. In the interior of the ball we have the spectral flow, with the monotonically decreasing ‘effective’ central charge (i.e. the \(c\)-function), in full accord with the \(c\)-theorem \[17\]. The RG evolution ends at another (IR) fixed point where the solution (6.4) has a removable pole. This IR-fixed point thus may be called a supersymmetric attractor.
7 Conclusion

The proposed holographic duality gives the simple and natural description of the 2d RG-flow in 2d, N=2 QFT in terms of the effective (internal ‘gravity’) NLSM in the background of N=2 supergravity in four spacetime dimensions. The local 4d, N=2 supersymmetry appears to be the sole source of the fundamental constraints on the NLSM (Zamolodchikov) metric. The regular SU(2)-invariant four-dimensional solutions to the Zamolodchikov metric are unique, being parametrized by the SCFT central charge describing the monodromy of the ‘master’ solution to the underlying Painlevé VI equation. Our geometrical description of the RG flow by the quaternionic-Kähler geometry or eq. (4.1) is manifestly covariant with respect to arbitrary reparametrizations of the 2d QFT coupling constants — cf. ref. [41].

Cecotti and Vafa [23] studied the integrable deformations of the 2d, N=2 superconformal Landau-Ginzburg models by the most relevant operators (see ref. [24] also). They identified the Zamolodchikov metric with the metric of the supersymmetric ground states, and found that it satisfies the classical Toda-like equations whose solutions are governed by the Painlevé III equation. The Cecotti-Vafa solutions are apparently associated with the Kähler metrics of vanishing scalar curvature [36] when the background gravity decouples, \( \kappa = 0 \). In our explicit example, the RG flow in a 2d, N=2 QFT is described by the ODE system (5.2) whose coefficients are the universal (normalized) OPE coefficients of the underlying CFT at the UV-fixed point of the QFT. Unlike the 2d, N=2 supersymmetric RG flow solutions found by Cecotti and Vafa [23], our RG flow has an IR fixed point and, therefore, it can be interpreted as a domain-wall solution.

The constraints (4.1) do not seem to imply any quantization condition on the monodromy parameter \( k \) since the regular metric solutions exist for any \( k > 0 \), whereas the related central charge (or the critical exponent \( k \) in eq. (6.7)) on the two-dimensional boundary is usually quantized in solvable 2d, N=2 SCFT like, e.g., the minimal N=2 superconformal models associated with compact (simply-laced) Lie groups. A resolution of this puzzle may be related to the negative curvature of the metrics. The ASD Einstein metrics of positive curvature take the similar form given by eqs. (5.1) or (5.10), while they are known to be related to the so-called Poncelet \( n \)-polygons that give rise to the quantization condition \( k = 2/n \), where \( n \in \mathbb{Z} \) [42]. So, it seems that the absence of quantization may be explained by the non-compact nature of the Lie group SU(2,1). Perhaps, the Tod-Hitchin metrics may also be interpreted as the kink-type solitons preserving some supersymmetry, i.e. as the BPS-type solutions in the context of higher-dimensional supergravity (cf. ref. [8]). It
would also be interesting to investigate their possible connections to matrix models, 2d gravity and non-commutative geometry.

A supersymmetric version of the Randall-Sundrum scenario [43] with a gravity localized near the wall under the exponential suppression, recently attracted much attention, partly because it appears to be impossible without the use of hypermultiplets [44, 43, 10]. Though we didn’t discuss here any solutions to the spacetime metric $g_{\mu\nu}$, the existence of the exact domain wall solutions to the RG flow associated with the NLSM in eq. (2.9) may be related to the domain-walls in spacetime via the Einstein equations. In particular, the need of IR fixed points for the existence of regular and supersymmetric Randall-Sundrum type domain-wall solutions in spacetime was emphasized in ref. [44].

Acknowledgements

I would like to thank Ioannis Bakas and Elias Kiritsis, the Organizers of the EU-RESCO Conference ‘Quantum Fields and Strings’, for a kind hospitality extended to me in Crete, where a part of this work was done. I am also grateful to all participants of the Conference for stimulating atmosphere and illuminating discussions.

Appendix: Basic facts about theta-functions

The first theta-function $\vartheta_1(z|\tau)$ is defined by the series [38]

$$\vartheta_1(z) \equiv \vartheta_1(z|\tau) = -i \sum_{n=-\infty}^{+\infty} (-1)^n \exp i \left\{ \left( n + \frac{1}{2} \right)^2 \pi \tau + (2n + 1)z \right\}$$

$$= 2 \sum_{n=0}^{+\infty} (-1)^n q^{(n+1/2)^2} \sin(2n + 1)z , \quad q = e^{i\pi \tau} , \quad (A.1)$$

where $\tau$ is regarded as the fundamental complex parameter, whose imaginary part must be positive, $q$ is called the nome of the theta-function, $|q| < 1$, and $z$ is the complex variable. The other theta-functions are defined by [38]

$$\vartheta_2(z|\tau) = \vartheta_1(z + \frac{1}{2}\tau)|\tau) = \sum_{n=-\infty}^{+\infty} q^{(n+1/2)^2} e^{i(2n+1)z}$$

$$= 2 \sum_{n=0}^{+\infty} q^{(n+1/2)^2} \cos(2n + 1)z , \quad (A.2)$$
\begin{align}
\vartheta_3(z|\tau) &= \vartheta_4(z + \frac{1}{2\pi}|\tau) = \sum_{n=-\infty}^{+\infty} q^{n^2} e^{2in\pi z} \\
&= 1 + 2 \sum_{n=1}^{+\infty} q^{n^2} \cos 2nz ,
\end{align}

(A.3)

and

\begin{align}
\vartheta_4(z|\tau) &= \sum_{n=0}^{+\infty} (-1)^n q^{n^2} e^{2in\pi z} = 1 + 2 \sum_{n=1}^{+\infty} (-1)^n q^{n^2} \cos 2nz .
\end{align}

(A.4)

The identities \[38\]
\begin{align}
\vartheta_3^4(0) &= \vartheta_2^4(0) + \vartheta_1^4(0) , \\
\vartheta_1^4(0) &= \vartheta_2(0) \vartheta_3(0) \vartheta_4(0) ,
\end{align}

(A.5) and

\begin{align}
\frac{\vartheta_4^{(n)}(0)}{\vartheta_4'(0)} = \frac{\vartheta_2^{(n)}(0)}{\vartheta_2'(0)} + \frac{\vartheta_3^{(n)}(0)}{\vartheta_3'(0)} + \frac{\vartheta_4^{(n)}(0)}{\vartheta_4'(0)} ,
\end{align}

(A.7)

where the primes denote differentiation with respect to \(z\), may be used to rewrite eq. (6.4) to other equivalent forms (cf. \[14, 39, 40\]).

References

[1] G. ‘t Hooft, *Dimensional Reduction in Quantum Gravity*, in ‘Salamfestschrift’, edited by A. Ali et al., World Scientific, 1993, p. 284; gr-qc/9310006

[2] L. Susskind, J. Math. Phys. 36 (1995) 6377

[3] A. M. Polyakov, *Gauge Fields and Strings*, Harwood Academic Publishers, 1987

[4] J. M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231;
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. 428B (1998) 105;
E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253;
L. Susskind and E. Witten, *The holographic bound in anti-de-Sitter space*, hep-th/9805114

[5] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Rep. 323 (2000) 183

[6] E. T. Akhmedov, Phys. Lett. 442B (1998) 152;
N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, Phys. Rev. D58 (1998) 046004;
L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, JHEP 9812 (1998) 022
E. Alvarez and C. Gomez, Nucl. Phys. B541 (1999) 441;
V. Balasubramanian and P. Kraus, Phys. Rev. Lett. 83 (1999) 3605;
V. Sahakian, Holography, a covariant c-function, and the geometry of the renormalization group, hep-th/9910099

[7] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, Renormalization group flows from holography: supersymmetry and c-theorem, hep-th/9904017

[8] S. V. Ketov, Quantum Non-linear Sigma-Models, Springer-Verlag, 2000

[9] C. Schmidhuber, Nucl. Phys. B580 (2000) 121

[10] M. B. Green, J. H. Schwarz and E. Witten, Superstring Theory, in two Volumes, Cambridge University Press, 1987;
    J. Polchinski, String Theory, in two Volumes, Cambridge University Press, 1998;
    S. V. Ketov, Introduction to Quantum Field Theory of Strings and Superstrings, Nauka, 1990 (in Russian)

[11] J. de Boer, E. Verlinde and H. Verlinde, JHEP 0008 (2000) 003

[12] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram, Nucl. Phys. B552 (1999) 246

[13] K. P. Tod, Phys. Lett. 190A (1994)

[14] N. J. Hitchin, J. Diff. Geom. 42 (1995) 30

[15] S. V. Ketov, Conformal Field Theory, World Scientific, 1995

[16] A. B. Zamolodchikov, JETP Lett. 46 (1987) 160

[17] A. B. Zamolodchikov, JETP Lett. 43 (1986) 731

[18] A. B. Zamolodchikov, Adv. Studies in Pure Math. 19 (1989) 641

[19] I. Jack and D. R. T. Jones, Nucl. Phys. B303 (1988) 260; ibid. 307 (1988) 531

[20] M. Henningson and K. Skenderis, JHEP 9807 (1998) 023

[21] B. de Wit, P. D. Lauwers and A. van Proeyen, Nucl. Phys. B255 (1985) 569

[22] J. Bagger and E. Witten Nucl. Phys. B222 (1983) 1

[23] S. Cecotti and C. Vafa, Nucl. Phys. B367 (1991) 359
[24] P. Fendley, S. D. Mathur, C. Vafa and N. P. Warner, Phys. Lett. \textbf{243B} (1990) 257;
    P. Fendley, W. Lerche, S. D. Mathur, and N. P. Warner, Nucl. Phys. \textbf{B348} (1991) 66;
    W. Lerche and N. P. Warner, Nucl. Phys. \textbf{B358} (1991) 571

[25] S. V. Ketov and Ch. Unkmeir, Phys. Lett. \textbf{422B} (1998) 179

[26] C. Fefferman and C. R. Graham, \textit{Conformal Invariants}, in ‘Elie Cartan et les Mathématiques d’àjourd’hui Astérisque, 1985, p. 95;
    C. R. Graham and J. M. Lee, Adv. Math. \textbf{87} (1991) 186

[27] S. Cecotti, S. Ferrara and L. Girardello, Int. J. Mod. Phys. \textbf{A4} (1989) 2475

[28] D. V. Alekseevsky, Math. USSR Izv. \textbf{9} (1975) 297

[29] A. L. Besse, \textit{Einstein Manifolds}, Springer-Verlag, 1980

[30] J. A. Wolf, J. Math. Mech. \textbf{14} (1965) 1033

[31] S. J. Gates, Jr., and S. V. Ketov, Phys. Rev. \textbf{D52} (1995) 2278

[32] S. Ferrara and S. Sabharwal, Class. and Quantum Grav. \textbf{6} (1989) L77;
    B. de Wit and A. Van Proeyen, Phys. Lett. \textbf{252B} (1990) 221;
    A. Strominger, Phys. Lett. \textbf{421B} (1998) 139

[33] G.-H. Halphen, Sur un système d’équations différentielles, C. R. Acad. Sci. Paris \textbf{92} (1881) 1101

[34] M. J. Ablowitz and P. A. Clarkson, \textit{Solitons, Non-Linear Evolution Equations and Inverse Scattering}, Cambridge University Press, 1991

[35] L. J. Mason and N. M. J. Woodhouse, \textit{Integrability, Self-Duality, and Twistor Theory}, Clarendon Press, 1996

[36] H. Pedersen and Y. S. Poon, Class. and Quantum Grav. \textbf{7} (1990) 1707

[37] R. Maszczyk, L. J. Mason and N. M. J. Woodhouse, Class. and Quantum Grav. \textbf{11} (1994) 65

[38] D. F. Lawden, \textit{Elliptic Functions and Applications}, Springer-Verlag, 1980

[39] B. A. Dubrovin, Funct. Anal. Appl. \textbf{24} (1990) 280
[40] M. V. Babich and D. A. Korotkin, Lett. Math. Phys. 46 (1998) 323

[41] R. Bousso, JHEP 07 (1999) 004

[42] N. J. Hitchin, A new family of Einstein metrics, in the Proceedings of the Pisa Conference in Honour of E. Calabi, Cambridge University Press, 1995

[43] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370; ibid. 83 (1999) 4690

[44] K. Behrndt and M. Cvetic, Phys. Lett. B475 (2000) 253; Phys. Rev. D61 (2000) 101901

[45] R. Kallosh and A. Linde, JHEP 02 (2000) 005

[46] A. Ceresole and D. Dall’Agata, General matter coupled N=2, D=5 gauged supergravity, hep-th/0004111;
K. Behrndt, C. Hermann, J. Louis and S. Thomas, Domain walls in five dimensional supergravity with non-trivial hypermultiplets, hep-th/0008112.