BLOWING UP AND DESINGULARIZING CONSTANT SCALAR CURVATURE KÄHLER MANIFOLDS

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1991 Math. Subject Classification: 58E11, 32C17.

1. Introduction

Abstract. This paper is concerned with the existence of constant scalar curvature Kähler metrics on blow ups at finitely many points of compact manifolds which already carry constant scalar curvature Kähler metrics. We also consider the desingularization of isolated quotient singularities of compact orbifolds which already carry constant scalar curvature Kähler metrics.

Let $(M,\omega)$ be either a $m$-dimensional compact Kähler manifold or a $m$-dimensional compact Kähler orbifold with isolated singularities. By definition, any point $p \in M$ has a neighborhood biholomorphic to a neighborhood of the origin in $\mathbb{C}^m/\Gamma$, where $\Gamma$ is a finite subgroup of $U(m)$ (this last fact is a consequence of the Kähler property) acting freely on $\mathbb{C}^m - \{0\}$. Observe that, when $p$ is a smooth point of $M$, the group $\Gamma$ reduces to the identity. In the case where $M$ is an orbifold, the Kähler form $\omega$ lifts, near any of the singularities of $M$, to a Kähler form $\tilde{\omega}$ on a punctured neighborhood of 0 in $\mathbb{C}^m$. We will always assume that $\tilde{\omega}$ can be smoothly extended through the origin, i.e. that $\omega$ is an orbifold metric.

If we further assume that the Kähler form $\omega$ has constant scalar curvature and if we are given $n$ distinct (smooth) points $p_1,\ldots,p_n \in M$, one of the questions we would like to address in this paper is whether the blow up of $M$ at the points $p_1,\ldots,p_n$ can still be endowed with a constant scalar curvature Kähler form. In this direction, we have obtained the following positive answer:

**Theorem 1.1.** Let $(M,\omega)$ be a constant scalar curvature compact Kähler manifold or Kähler orbifold with isolated singularities. Assume that there is no nonzero holomorphic vector field vanishing somewhere on $M$. Then, given finitely many smooth points $p_1,\ldots,p_n \in M$ and positive numbers $a_1,\ldots,a_n > 0$, there exists $\varepsilon_0 > 0$ such that the blow up of $M$ at $p_1,\ldots,p_n$ carries constant scalar curvature Kähler forms

$$\omega_{\varepsilon} \in \pi^*[\omega] - \varepsilon^2 (a_1 PD[E_1] + \ldots + a_n PD[E_n]),$$

where the $PD[E_j]$ are the Poincaré dual of the $(2m-2)$-homology classes of the exceptional divisors of the blow up at $p_j$ and $\varepsilon \in (0,\varepsilon_0)$.

If the scalar curvature of $\omega$ is not zero then the scalar curvatures of $\omega_{\varepsilon}$ and of $\omega$ have the same signs.
Following a suggestion of C. LeBrun, we also show that the proof of Theorem 1.1 can be used to produce zero scalar curvature Kähler metrics provided the Kähler form $\omega$ we start with has zero scalar curvature and the first Chern class of $M$ is not zero.

**Corollary 1.1.** Let $(M, \omega)$ be a zero scalar curvature compact Kähler manifold or orbifold with isolated singularities. Assume that there is no nonzero holomorphic vector field vanishing somewhere on $M$ and that the first Chern class of $M$ is non zero. Then the blow up of $M$ at finitely many (smooth) points carries zero scalar curvature Kähler forms.

Observe that, on manifolds (or orbifolds with isolated singularities) with discrete automorphism group, there are no nontrivial holomorphic vector fields. Hence, if these carry a constant scalar curvature Kähler form, they are examples to whom our results do apply.

On the other hand the assumption is verified also by some manifold with a continuous family of automorphisms. For example Kähler flat tori can be used as base manifolds in Theorem 1.1 (but not in Corollary 1.1 since their first Chern class vanish).

Theorem 1.1 and Corollary 1.1 are consequences of a more general construction which also allows one to desingularize isolated singularities of orbifolds. This desingularization procedure combined with Theorem 1.1 is enough to prove the following:

**Theorem 1.2.** Any compact complex surface of general type admits constant scalar curvature Kähler forms.

It is worth pointing out that some assumption on the initial manifold $(M, \omega)$ is indeed necessary for either the desingularization or the blow up procedure to be successful. In first place we know from the work of Matsushima [42] and Lichnerowicz [39] that the automorphism group of a manifold with a Kähler constant scalar curvature metric must be reductive, hence, for example, the projective plane blown up at one or two points does not to admit any constant scalar curvature Kähler metric (see e.g. [8] page 331). In the same spirit let us recall that, given a compact complex orbifold $M$ and a fixed Kähler class $\left[\omega\right]$, there is another obstruction for the existence of a constant scalar curvature Kähler metric in the class $\left[\omega\right]$. This obstruction was discovered by Futaki in the eighties [20], [21], [22] for smooth metrics and was extended to singular varieties by Ding and Tian [16] and to Kähler constant scalar curvature metrics by Bando, Calabi [12] and Futaki. This obstruction will be briefly described in Section 4, since it will play some rôle in our construction. The nature of this obstruction (being a character of the Lie algebra of the automorphism group) singles out two different types of Kähler manifolds or Kähler orbifolds with isolated singularities where to look for constant scalar curvature metrics: those with no nonzero holomorphic vector fields vanishing somewhere, where the obstruction is vacuous since they do not have any nontrivial holomorphic vector field, and the others where the Futaki invariant has to vanish for all holomorphic vector fields.

We will say that $(M, \omega)$, a constant scalar curvature Kähler manifold or Kähler orbifold with isolated singularities, is nondegenerate if it does not carry any nontrivial holomorphic vector field vanishing somewhere (note that this definition does not depend on the particular Kähler class on $M$), and we will say that $(M, \omega)$ is Futaki nondegenerate if the differential of the Futaki invariant...
Note that, thanks to Matsushima-Lichnerwicz’s decomposition of the Lie algebra of holomorphic vector fields on a manifold which admits a constant scalar curvature Kähler metric (see e.g. [8] and [22]), a constant scalar curvature Kähler manifold is nondegenerate if and only if every holomorphic vector field is parallel.

Our construction gives a quite precise description of the Kähler forms we obtain on the blown up manifold or on the desingularized orbifold. We shall now describe more carefully the general construction and some of its consequences, but also we shall give more details about the Kähler forms we construct.

The construction is obtained by choosing finitely many points \( p_1, \ldots, p_n \in M \) and replacing a small neighborhood of each point \( p_j \), biholomorphic to a neighborhood of the origin in \( \mathbb{C}^m / \Gamma_j \), by a (suitably scaled down by a small factor \( \varepsilon \)) piece of a Kähler manifold or a Kähler orbifold with isolated singularities \( (N_j, \eta_j) \), biholomorphic to \( \mathbb{C}^m / \Gamma_j \) away from a compact subset. This generalized connected sum yields a Kähler manifold or a Kähler orbifold with isolated singularities that we call

\[
M \sqcup_{\varepsilon,p_1} N_1 \sqcup \cdots \sqcup_{\varepsilon,p_n} N_n
\]

and whose complex structure does not depend on \( \varepsilon \neq 0 \). We proceed to perturb the Kähler forms \( \omega \) and \( \varepsilon^2 \eta_j \) on the various summands, analyzing in Section 5 the linear and in Section 6 the non linear part of the constant scalar curvature equation in a given Kähler class. This leads to a study of nonlinear fourth order elliptic partial differential equations on the Kähler potentials. Then, at the end of Section 6, we “glue” the Kähler potentials of the perturbed Kähler forms on the different summands to get a Kähler form whose scalar curvature is constant. The most important condition that ensures this program to be successful is the following: Each \( (N_j, \eta_j) \) is an "Asymptotically Locally Euclidean" (ALE) space and \( \eta_j \) is a zero scalar curvature Kähler form.

Since the term ALE has often been used with slightly different meanings, we make precise this definition. In this paper, an ALE space \( (N, \eta) \) is a \( m \)-dimensional Kähler manifold or Kähler orbifold with isolated singularities, which is biholomorphic to \( \mathbb{C}^m / \Gamma \) outside a compact set, where \( \Gamma \) is a finite subgroup of \( U(m) \) acting freely on \( \mathbb{C}^m - \{0\} \), and which is equipped with a Kähler metric \( \eta \) which converges to the Euclidean metric at infinity. In the case where \( N \) is an orbifold with isolated singularities, we assume that, near any singularity modeled after a neighborhood of 0 in \( \mathbb{C}^m / \Gamma \), the Kähler form \( \eta \) lifts smoothly to a neighborhood of 0 in \( \mathbb{C}^m \). In addition, we will always assume that there exist complex coordinates \( (u^1, \ldots, u^m) \) parameterizing \( N \) away from a compact set, in which the Kähler form \( \eta \) can be expanded as

\[
\eta = i \partial \bar{\partial} \left( \frac{1}{2} |u|^2 + \hat{\varphi}(u) \right)
\]

at infinity, where the potential \( \hat{\varphi} \) satisfies

\[
\hat{\varphi}(u) = a |u|^{4-2m} + O(|u|^{3-2m}),
\]
when $m \geq 3$ and
\begin{equation}
\hat{\varphi}(u) = a \log |u| + O(|u|^{-1}).
\end{equation}
when $m = 2$. Here $a \in \mathbb{R}$ and we agree that $O(|u|^q)$ is a smooth function whose $k$-th partial derivatives are bounded by a constant times $|u|^{q-k}$, for all $k \geq 2$. The growth (or decay) of the Kähler potentials for these models is a subtle problem of independent interest (see e.g. [4]). In given examples, we will see in Section 7 that these potentials can arise with various orders and decays, and we will show (Lemma 7.2) that, for zero scalar curvature metrics and under reasonable growth assumptions on the potential, one can change suitably the potential in order to get a potential for which (2)-(3) hold.

Let us now summarize the assumptions under which our general construction works. We will assume that :

(i) $(M, \omega)$ is a $m$-dimensional compact Kähler manifold or orbifold with isolated singularities.

(ii) The scalar curvature of $\omega$ is constant.

(iii) $(M, \omega)$ is either nondegenerate or is Futaki nondegenerate.

(iv) Given points $p_1, \ldots, p_n \in M$ which might be either singular or regular points of $M$, let $\Gamma_j$ be the finite subgroup of $U(m)$ acting freely on $\mathbb{C}^m - \{0\}$ such that a neighborhood of $p_j$ is biholomorphic to a neighborhood of the origin in $\mathbb{C}^m / \Gamma_j$. Each $\mathbb{C}^m / \Gamma_j$ has an ALE resolution $(N_j, \eta_j)$ (which might either be a manifold or an orbifold with isolated singularities) endowed with a zero scalar curvature Kähler form $\eta_j$. Furthermore, we assume that, away from a compact set, the Kähler form $\eta_j$ can be expanded as in (1) with a potential satisfying (2)-(3).

Our main result reads :

**Theorem 1.3.** Assume that assumptions (i)-(ii)-(iii) and (iv) are satisfied. Then, there exists $\varepsilon_0 > 0$ and, for all $\varepsilon \in (0, \varepsilon_0)$, there exists a constant scalar curvature Kähler form $\tilde{\omega}_\varepsilon$ defined on $M \sqcup p_{1,\varepsilon} N_1 \sqcup p_{2,\varepsilon} \cdots \sqcup p_{n,\varepsilon} N_n$.

As $\varepsilon$ tends to 0, the sequence of Kähler forms $\tilde{\omega}_\varepsilon$ converges (in $C^\infty$ topology) to the Kähler metric $\omega$, away from the points $p_j$ and the sequence of Kähler forms $\varepsilon^{-2} \tilde{\omega}_\varepsilon$ converges (in $C^\infty$ topology) to the Kähler form $\eta_j$, on compact subsets of $N_j$.

If $\omega$ has positive (resp. negative) scalar curvature then the Kähler forms $\tilde{\omega}_\varepsilon$ have positive (resp. negative) scalar curvature.

Moreover, if $(M, \omega)$ is nondegenerate
\begin{equation}
[\omega_\varepsilon] = [\omega] + \varepsilon^2 ([\eta_1] + \cdots + [\eta_n])
\end{equation}

Note that, when $(M, \omega)$ is Futaki nondegenerate, we cannot control the Kähler class where we find the constant scalar curvature Kähler metric.

All the previous results are consequences of this Theorem.
For example the blow up at smooth points is obtained by our generalized connected sum construction taking \( N_j \) to be the total space of the line bundle \( O(-1) \) over \( \mathbb{P}^{m-1} \) (in this case \( \Gamma_j = \{id\} \)). The key property (iv) asks for an ALE zero scalar curvature metric \( \eta_j \) on \( O(-1) \) such that \( [\eta_j] = -PD[E_j] \) and with appropriate decay at infinity. These Kähler forms have been obtained by Calabi [11]. When \( m = 2 \), \( \eta \) is usually referred to in the literature as Burns metric and it has been described (and generalized) in a very detailed way by LeBrun [34]. In higher dimensions, \( m \geq 3 \), these metrics have been generalized by Simanca [53]. The ALE property and the issue of the rate of decay of these metrics towards the Euclidean metric can be easily derived from these papers. In the 2 dimensional case, the Kähler form \( \eta \) is explicit and these properties follow at once, while, in higher dimensions, it can be shown that these metrics have a potential for which \( m \) and (2-3) are satisfied. The analysis of these asymptotic properties will be done in Lemma 7.1 (Raza [49] has given an alternative proof using toric geometry). In any case, assumption (iv) is fulfilled and, given smooth points \( p_1, \ldots, p_n \in M \) and positive constants \( a_1, \ldots, a_n \), the existence of such models can be plugged into Theorem 1.3 with all ALE spaces equal to \( N = O(-1) \) over \( \mathbb{P}^{m-1} \) with the Burns-Calabi-Simanca form \( \eta_j = a_j \eta \). This leads to the results of Theorem 1.1, which then also holds for Futaki nondegenerate manifolds \((M, \omega)\), only losing control on the Kähler class to represent.

In Section 8 we will observe that our gluing procedure decreases the starting scalar curvature. Therefore if \((M, \omega)\) has zero scalar curvature Theorem 1.3 gives (small) negative scalar curvature metrics. Nonetheless, if the first Chern class is not zero, Lebrun-Simanca [37] have shown that there exist nearby Kähler metrics \( \omega_+ \) and \( \omega_- \) of (small) positive and negative constant scalar curvature respectively. We can then apply Theorem 1.3 to \((M, \omega_+)\) and \((M, \omega_-)\) to get positive and negative Kähler metrics of constant scalar curvature on the blow up. We will show in Section 8 how this implies Corollary 1.1 a result which also extends to Futaki nondegenerate manifolds with nonzero first Chern class. A similar result had been previously proved in complex dimension 2 by Rollin-Singer [50] who have shown that one can desingularize compact orbifolds of zero scalar curvature with cyclic orbifold groups, keeping the scalar curvature zero, by solving on the desingularization the hermitian anti-selfdual equation.

To prove Theorem 1.2 we need to apply Theorem 1.3 more than once. The idea, which comes directly from algebraic geometry, is to associate to \( M \) a (possibly) singular complex surface \( \bar{M} \), such that \( M \) is obtained form \( \bar{M} \) by desingularizing and blowing up smooth points a finite number of times. Algebraic geometry (see e.g. [7]) tells us that if \( M \) is a surface of general type then \( \bar{M} \) (which is called the pluricanonical model of the minimal model of \( M \))

(i) is again a complex surface [30],

(ii) it has only isolated singular points whose local structure groups \( \Gamma_j \) are in \( SU(2) \) [9],

(iii) the first Chern class of \( \bar{M} \) is negative, hence it has only a discrete group of automorphisms ([7], Theorem 2.1 pag. 82),
(iv) $\tilde{M}$ admits a Kähler-Einstein orbifold metric \cite{29}.

We will explain below how Theorem 1.3 can be used to resolve $SU(2)$ singularities. Granted this, $M$ is then reobtained form this desingularized manifold after a finite number of blow ups at smooth points, and the constant scalar curvature Kähler metric is then given by Theorem 1.1.

If $p_j$ is a singular point (and hence $\Gamma_j$ is not the identity group), there is no unique way to resolve the singularity, and in fact this is an extremely rich area of algebraic geometry. Once again, whether constant scalar curvature metrics exist or not on such resolutions depends, according to Theorem 1.3 on the existence of ALE scalar flat Kähler resolution of $\mathbb{C}^m/\Gamma$. For a general finite subgroup $\Gamma \subset U(m)$, the existence of such a resolution is unknown and this prevents us to state general existence results for constant scalar curvature Kähler metrics. Nonetheless there are large classes of discrete nontrivial groups for which a good local model is known to exist, looking at Ricci-flat metrics, very much in the spirit of non compact versions of the Calabi conjecture.

This is the line started by Tian-Yau \cite{57} and Bando-Kobayashi \cite{4}, culminating in Joyce’s proof of the ALE Calabi conjecture \cite{26}. Joyce himself used this approach to have good local models for his well known special holonomy desingularization result. Joyce’s theorem, recalled in Theorem 7.1 implies that given a $\mathbb{C}^m/\Gamma$ such that a Kähler crepant resolution $N$ exists (i.e. a Kähler resolution with $c_1(N) = 0$), then $N$ has a Ricci-flat Kähler metric $\eta$ which, at infinity, can be expanded as

$$\eta = i \partial \bar{\partial} \left( \frac{1}{2} |u|^2 + \hat{\varphi}(u) \right)$$

for some potential $\hat{\varphi}$ satisfying $\hat{\varphi} = \mathcal{O}(|u|^{2-2m})$, so well inside the range of application of Theorem 1.3. This approach works for example when $\Gamma = \mathbb{Z}_m$, acting diagonally on $\mathbb{C}^m$ \cite{11}, and for any finite subgroup of $SU(2)$ and $SU(3)$, since in this case we know that $\mathbb{C}^m/\Gamma$ has a smooth Kähler crepant resolution (see \cite{52} and \cite{26}, Chapter 6.4 and Chapter 8 for these results). The 2-dimensional case can be handled directly also relying on Kronheimer’s result \cite{33}.

In light of these results, and when $m = 2$, we can apply Theorem 1.3 when $(M, \omega)$ is a 2-dimensional nondegenerate or Futaki nondegenerate Kähler orbifold with isolated singularities and $p_1, \ldots, p_n \in M$ is any set of points with a neighborhood biholomorphic to a neighborhood of the origin in $\mathbb{C}^2/\Gamma_j$, where $\Gamma_j$ is a finite subgroup of $SU(2)$ acting freely on $\mathbb{C}^2 - \{0\}$. As explained above, this is enough to prove Theorem 1.2.

Let us mention that other local models are known, for example when $\Gamma = \mathbb{Z}_k$ is acting on $\mathbb{C}^m$ by multiplication by a $k$-th root of unity. In this case $N$ is the total space of the line bundle $\mathcal{O}(-k)$ over $\mathbb{P}^m$ and the metric has zero scalar curvature but, in general, is not Ricci-flat. Rollin-Singer \cite{51} have proved the required decay properties on the corresponding Kähler potential.

Other examples should come from the work of Calderbank-Singer \cite{13}. They have in fact shown the existence of ALE zero scalar curvature resolutions of all $U(2)$ cyclic isolated singularities. The only piece of information missing at the moment to use them in our construction is the behavior at infinity of a Kähler potential associated to these ALE metrics.
One of the main sources of interest in Kähler metrics with constant scalar curvature lies in its relation with algebraic geometric properties of the underlying manifold, such as Chow-Mumford, Tian, or asymptotic Hilbert-Mumford stability (see for example [55], [17], [48], [46], [40]). We know for example, by Mabuchi’s extension of Donaldson’s work [40], [17], that if an integral class is represented by an extremal Kähler metric, then the underlying algebraic manifold is asymptotically stable in a sense which depends on the structure of the automorphism group which preserves the class. In particular if we have a Kähler manifold with discrete automorphism group, this stability reduces to the classical Chow stability. Rescaling the Kähler class $[\omega]$ by a factor $k$ to make it integral, our results have the following corollary:

**Corollary 1.2.** Let $(M, L)$ be a polarized compact algebraic manifold of complex dimension $m \geq 2$, with discrete automorphisms group, and $\omega$ a Kähler form with constant scalar curvature in an integral class. Then all the manifolds obtained by the blowing up any finite set of points are asymptotically Chow stable relative to the polarizing class $k\pi^*[\omega] - (PD[E_1] + \cdots + PD[E_n])$, where the $E_j$ are the exceptional divisors of the blow up and $k$ is sufficiently big.

Note that playing with the weights $a_j$ in Theorem 1.1 one gets abundance of different polarizing classes for which the above corollary holds (in the above statement we have used $a_1 = \cdots = a_n$). Moreover similar results for the different versions of stability which are known to be implied by the existence of constant scalar curvature Kähler metrics follow from our theorems. In this setting it is interesting to observe that Mabuchi and Donaldson results do not apply to Kähler orbifolds, due to the failure of Tian-Catling-Zelditch expansion [54]. Nonetheless if a full desingularizing process were to go through, then we would get the stability of the smooth polarized manifold obtained.

Another important phenomenon concerning constant scalar curvature Kähler metrics is that they are unique in their class, up to automorphisms. This result was proved for Kähler-Einstein metrics thanks to the work of Calabi [10] when $c_1 \leq 0$ and Bando-Mabuchi [5] when $c_1 > 0$. Uniqueness of constant scalar curvature Kähler metrics was then proved by Donaldson and Mabuchi [41] (for extremal metrics) in integral classes (with either discrete or continuous automorphism group), and by Chen [14] for any Kähler class on manifolds with $c_1 \leq 0$. Recently Chen and Tian [15] have proved it for any Kähler class even for extremal metrics. This implies that all the constant scalar curvature Kähler metrics produced in this paper are the unique such representative of their Kähler class, up to automorphisms.

Despite this monumental work, our knowledge of concrete examples is still limited and mainly confined in complex dimension 2. For example, Hong [24, 25] has proved the existence of such metrics in some Kähler classes of ruled manifolds, and Fine [19] has studied this problem for complex surfaces projecting over Riemann surfaces with fibres of genus at least two. The only case completely understood, and giving a rich source of examples, is the one of zero scalar curvature Kähler surfaces thanks to the work of Kim, LeBrun, Pontecorvo and Singer [38, 35, 28, 27], and the recent results of Rollin-Singer [50]. Our construction allows one to produce many new constant scalar curvature Kähler manifolds.

In our construction, it is possible to keep track of the geometric meaning of the parameter $\varepsilon$. Indeed, for the blow up construction $a_j^{m-1}\varepsilon^2m^{-2}$ gives the volume of the exceptional divisor $E_j$.
(up to a universal constant depending only on the dimension). The rôle of \( \varepsilon \) in our results has a direct analogue in Fine-Hong’s papers [19], [24], [25], replacing the exceptional divisor with the fiber of the projection onto the Riemann surface or the projectivized fiber of the vector bundle.

The last source of interest in our results we would like to mention is that they give a reverse picture to Tian-Viaclovsky [56] and Anderson [1] study of degenerations of critical metrics, in the special case of constant scalar curvature Kähler metrics in (real) dimension 4. If we look at the sequence of Kähler forms \( \omega_{\varepsilon} \) on \( M \sqcup \mathbb{RP}_1 \sqcup \mathbb{RP}_2 \sqcup \cdots \sqcup \mathbb{RP}_n \) seen as a fixed smooth manifold, we get from our analysis that they degenerate in the Gromov-Hausdorff sense to the orbifold \((M, \omega)\) so they provide many examples of the phenomenon studied in these works.

A natural generalization of these results is to look for gluing theorems for Kähler metrics with nontrivial automorphisms. The technical difficulties of this extension give rise to some new interesting phenomena and will be the subject of a forthcoming paper [3].

After the first version of these results were posted in electronic form, Claude LeBrun has indicated some implications of our main theorem we first missed (notably Corollary 1.1 and Theorem 1.2). We wish to thank him for his suggestions.

2. Gluing the orbifold and the ALE spaces together

We start by describing the Kähler orbifold near each of its singularities and we proceed with a description of the ALE spaces near infinity.

Let \((M, \omega)\) be a \( m \)-dimensional Kähler manifold or Kähler orbifold with isolated singularities. We choose points \( p_1, \ldots, p_n \in M \). By assumption, near \( p_j \), the orbifold \( M \) is biholomorphic to a neighborhood of 0 in \( \mathbb{C}^m/\Gamma_j \) where \( \Gamma_j \) is a finite subgroup of \( U(m) \) acting freely on \( \mathbb{C}^m - \{0\} \).

The group \( \Gamma_j \) depends on the point \( p_j \) as the subscript is meant to remind the reader. In the particular case where \( p_j \) is a regular point of \( M \), the group \( \Gamma_j \) reduces to the identity.

We can choose complex coordinates \( z := (z^1, \ldots, z^m) \) in a neighborhood of 0 in \( \mathbb{C}^m \) to parameterize a neighborhood of \( p_j \) in \( M \) and, in these coordinates, the Kähler form \( \omega \) can be expanded as

\[
\omega = \frac{i}{2} \sum_a d\bar{z}^a \wedge dz^a + \sum_{a,b} \mathcal{O}_{j,a,b}(\lvert z \rvert^2) \, d\bar{z}^a \wedge dz^b.
\]

near 0 \( \in \mathbb{C}^m \) (see [23]). The complex valued functions \( \mathcal{O}_{j,a,b}(\lvert z \rvert^2) \) are smooth functions which depend on \( j, a \) and \( b \), vanish at the origin and whose first order partial derivatives also vanish at the origin. Even though the coordinates \( z \) do depend on \( p_j \), we shall not make this dependence apparent in the notation and we hope that the meaning will be clear from the context.

It will be convenient to denote by

\[
B_{j,r} := \{ z \in \mathbb{C}^m/\Gamma_j : \lvert z \rvert < r \},
\]

\[
B^*_{j,r} := \{ z \in \mathbb{C}^m/\Gamma_j : 0 < \lvert z \rvert < r \},
\]

\[
\bar{B}^*_{j,r} := \{ z \in \mathbb{C}^m/\Gamma_j : 0 < \lvert z \rvert \leq r \},
\]
the open ball, the punctured open ball and the punctured closed ball of radius \( r \) centered at \( p_j \) (in the above defined coordinates which parameterize a neighborhood of \( p_j \) in \( M \)). We define, for all \( r > 0 \) small enough (say \( r \in (0, r_0) \))

\[
M_r := M - \bigcup_j B_{j,r}.
\]

In other words, \( M_r \) is obtained from \( M \) by excising small ball centered at the points \( p_j \). The boundaries of \( M_r \) will be denoted by \( \partial B_{1,r}, \ldots, \partial B_{n,r} \).

As promised, we now turn to the description of the ALE spaces near infinity. We assume that, for each \( j = 1, \ldots, n \), we are given a \( m \)-dimensional Kähler manifold or Kähler orbifold with isolated singularities \((N_j, \eta_j)\), with one end biholomorphic to a neighborhood of infinity in \( \mathbb{C}^m / \Gamma_j \). We further assume that the Kähler metric \( g_j \), which is associated to Kähler form \( \eta_j \), converges at order \( 2 - 2m \) towards the Euclidean metric. These assumptions imply that one can choose complex coordinates \( u := (u^1, \ldots, u^m) \) defined away from a neighborhood of 0 in \( \mathbb{C}^m \) to parameterize a neighborhood of infinity in \( N_j \) and, in these coordinates, the Kähler form \( \eta_j \) can be expanded as

\[
\eta_j = i \sum_a \frac{1}{2} du^a \wedge d\bar{u}^a + \sum_{a,b} O_{j,a,b}(|u|^{2-2m}) du^a \wedge d\bar{u}^b,
\]

away from a fixed neighborhood of the origin in \( \mathbb{C}^m \). Here, the complex valued function \( O_{j,a,b}(|u|^{2-2m}) \) is a smooth function which depends on \( j, a \) and \( b \), is bounded by a constant times \( |u|^{2-2m} \) and whose \( k \)-th partial derivatives are bounded by a constant (depending on \( k \)) times \( |u|^{2-2m-k} \). As will be explained in Section 7, this decay assumption is a natural one and, under some mild assumption, one can prove that this rate of decay in indeed achieved.

It will be convenient to denote by

\[
C_{j,R} := \{ u \in \mathbb{C}^n / \Gamma_j : |u| > R \},
\]

\[
\bar{C}_{j,R} := \{ u \in \mathbb{C}^n / \Gamma_j : |u| \geq R \},
\]

the complement of a close large ball and the complement of an open large ball in \( N_j \) (in the coordinates which parameterize a neighborhood of infinity in \( N_j \)). We define, for all \( R > 0 \) large enough (say \( R > R_0 \))

\[
N_{j,R} := N_j - C_{j,R}.
\]

which corresponds to the manifold \( N_j \) whose end has been truncated. The boundary of \( N_{j,R} \) is denoted by \( \partial C_{j,R} \).

We are now in a position to describe the generalized connected sum construction. For all \( \varepsilon > 0 \) small enough, we define a complex manifold by removing from \( M \) small balls centered at the points \( p_j \), for \( j = 1, \ldots, n \), and by replacing them by properly rescaled versions of the ALE space \( N_j \). More precisely, for all \( \varepsilon \in (0, r_0/R_0) \), we choose \( r_\varepsilon \in (\varepsilon R_0, r_0) \) and define

\[
R_\varepsilon := \frac{r_\varepsilon}{\varepsilon}.
\]
By construction
\[ M_\varepsilon := M \sqcup_{p_1,\varepsilon} N_1 \sqcup_{p_2,\varepsilon} \cdots \sqcup_{p_n,\varepsilon} N_n, \]
is obtained by connecting \( M_{r_\varepsilon} \) with the truncated ALE spaces \( N_{1,R_\varepsilon}, \ldots, N_{n,R_\varepsilon} \). The identification of the boundary \( \partial B_{j,r_\varepsilon} \) in \( M_{r_\varepsilon} \) with the boundary \( \partial C_{j,R_\varepsilon} \) of \( N_{j,R_\varepsilon} \) is performed using the change of variables
\[ (z^1, \ldots, z^m) = \varepsilon (u^1, \ldots, u^m), \]
where \( (z^1, \ldots, z^m) \) are the coordinates in \( B_{j,r_\varepsilon} \) and \( (u^1, \ldots, u^m) \) are the coordinates in \( C_{j,R_\varepsilon} \).

Observe that, when all singularities of \( M \) are in the set \( \{p_1, \ldots, p_n\} \) and the \( N_j \) are all smooth manifolds, then \( M_\varepsilon \) is a manifold, otherwise \( M_\varepsilon \) is still an orbifold.

3. Weighted spaces

In this section, we describe weighted spaces on \((M^*, \omega)\) where
\( M^* := M \setminus \{p_j, \mid j = 1, \ldots, n\}, \)
as well as weighted spaces on each \((N_j, \eta_j)\).

To begin with, we define the weighted space on \((M^*, \omega)\). These weighted spaces are by now well known and have been extensively used in many connected sum constructions. Roughly speaking, we are interested in functions whose rate of decay or blow up near any of the points \( p_j \) is controlled by a power of the distance to \( p_j \). To make this definition precise, we first need to define :

**Definition 3.1.** Given \( \bar{r} > 0, k \in \mathbb{N}, \alpha \in (0,1) \) and \( \delta \in \mathbb{R} \), the space \( C^{k,\alpha}_{\delta}(\bar{B}_j^{*}, \bar{r}) \) is defined to be the space of functions \( \varphi \in C^{k,\alpha}_{loc}(\bar{B}_j^{*}) \) for which the norm
\[ \|\varphi\|_{C^{k,\alpha}_{\delta}(\bar{B}_j^{*}, \bar{r})} := \sup_{0 < r \leq \bar{r}} r^{-\delta} \|\varphi(r \cdot)\|_{C^{k,\alpha}(\bar{B}_j^{*}-\bar{B}_j^{*}/2)} \]
is finite.

Observe that the function
\[ z \mapsto |z|^\delta', \]
belongs to \( C^{k,\alpha}_{\delta'}(\bar{B}_j^{*}) \) if and only if \( \delta' \leq \delta \). This being understood, we have the :

**Definition 3.2.** Given \( k \in \mathbb{N}, \alpha \in (0,1) \) and \( \delta \in \mathbb{R} \), the weighted space \( C^{k,\alpha}_{\delta}(M^*) \) is defined to be the space of functions \( \varphi \in C^{k,\alpha}_{loc}(M^*) \) for which the following norm
\[ \|\varphi\|_{C^{k,\alpha}_{\delta}(M^*)} := \|u\|_{C^{k,\alpha}(M_{r_\varepsilon}/2)} + \sum_j \|\varphi|_{B_{j,r_\varepsilon}}\|_{C^{k,\alpha}_{\delta}(\bar{B}_{j,r_\varepsilon})} \]
is finite.

With this definition in mind, we can now give a quantitative statement about the rate of convergence of potential associated to \( \omega \) toward the potential associated to the standard Kähler form on \( \mathbb{C}^m \), at any of the the points \( p_j \). More precisely, near \( p_j \) we can write
\( \omega = i \partial \bar{\partial} \left( \frac{1}{2} |z|^2 + \varphi_j \right), \)
where \( \varphi_j \) is a function which lifts smoothly to a neighborhood of \( 0 \) in \( \mathbb{C}^m \).
We claim that, without loss of generality, it is possible to choose the potential $\varphi_j$ in such a way that $\varphi_j \in C^4_{\partial}(B_{\rho_j}, \eta_j)$ (more precisely, $\varphi_j \in C^{4,\alpha}(B_{j,\rho_0})$ and has all its partial derivatives up to order 3 which vanish at 0). Indeed, the potential $\varphi_j$ lifts to a smooth potential defined on a neighborhood of 0 in $\C^n$. We can then perform the Taylor expansion of this potential at 0, namely

$$\varphi_j = \sum_{k=0}^{3} \varphi_j^{(k)} + \varphi_j'$$

where the polynomial $\varphi_j^{(k)}$ is homogeneous of degree $k$ and $\varphi_j'$, together with its partial derivatives up to order 3, vanish at 0. Obviously $\varphi_j^{(0)}$ and $\varphi_j^{(1)}$ are not relevant for the computation of the Kähler form $\omega$ since $\partial \bar{\partial} (\varphi_j^{(0)} + \varphi_j^{(1)}) = 0$, hence we might as well assume that $\varphi_j^{(0)} \equiv 0$ and $\varphi_j^{(1)} \equiv 0$. Next, making use of the fact that the coordinates $(z_1, \ldots, z_m)$ are chosen so that $\Box$ holds, we see that

$$\partial \bar{\partial} (\varphi_j^{(2)} + \varphi_j^{(3)}) = O(|z|^2)$$

but $\partial \bar{\partial} \varphi_j^{(2)}$ and $\partial \bar{\partial} \varphi_j^{(3)}$ being homogeneous polynomial of degree 0 and 1 respectively, we conclude that $\partial \bar{\partial} (\varphi_j^{(2)} + \varphi_j^{(3)}) \equiv 0$. Considering $\varphi_j'$ instead of $\varphi_j$, we have found a potential which satisfies the desired property.

Similarly, we define weighted spaces on the ALE spaces $(N_j, \eta_j)$. This time we are interested in functions which decay or blow up near the end of $N_j$ at a rate which is controlled by a power of the distance from a fixed point in $N_j$. To be more specific, we first define:

**Definition 3.3.** Given $R > 0$, $k \in \N$, $\alpha \in (0,1)$ and $\delta \in \R$, the space $C_{\theta}^{k,\alpha}(\bar{\C}, R)$ is defined to be the space of functions $\varphi \in C_{loc}^{k,\alpha}(\bar{\C}, R)$ such that the following norm

$$\|\varphi\|_{C_{\theta}^{k,\alpha}(\bar{\C}, R)} := \sup_{n \geq R} R^{-\delta} \|\varphi(R \cdot)\|_{C^{k,\alpha}(B_{n,2} - B_{n,1})},$$

is finite.

Again, observe that the function

$$u \mapsto |u|^\delta'$$

belongs to $C_{\theta}^{k,\alpha}(\bar{\C}, R)$ if and only if $\delta' \leq \delta$. We can now define:

**Definition 3.4.** Given $k \in \N$, $\alpha \in (0,1)$ and $\delta \in \R$, the weighted space $C_{\theta}^{k,\alpha}(N_j)$ is defined to be the space of functions $\varphi \in C_{loc}^{k,\alpha}(N_j)$ for which the following norm

$$\|\varphi\|_{C_{\theta}^{k,\alpha}(N_j)} := \|\varphi\|_{C_{\partial}^{k,\alpha}(N_j,2) + \|\varphi\|_{C_{\partial}^{k,\alpha}(N_j,2 \eta_0)} + \|\varphi\|_{C_{\partial}^{k,\alpha}(N_j,\eta_0)}$$

is finite.

We can now explain the assumption on the rate of convergence at infinity of the Kähler form $\eta_j$ toward the standard Kähler form on $\C^n$. We will assume that, away from a compact set in $N_j$

$$\eta_j = i \partial \bar{\partial} (\frac{1}{2} |u|^2 + \varphi_j),$$

(13)
for some potential $\tilde{\varphi}_j$ which satisfies

$$(14) \quad \tilde{\varphi}_j - a_j \cdot | \cdot |^{4-2m} \in C^{4,\alpha}_{3-2m}(\bar{C},R_0),$$

when $m \geq 3$ and

$$(15) \quad \tilde{\varphi}_j(u) + a_j \log | \cdot | \in C^{4,\alpha}(\bar{C},R_0),$$

when $m = 2$, for some $a_j \in \mathbb{R}$. As already mentioned in the introduction, this is a rather natural assumption which is fulfilled in many important examples.

**Remark 3.1.** We will show in Section 7 that, if one simply assumes that the potential $\tilde{\varphi}_j$ associated to $\eta_j$ satisfies

$$\tilde{\varphi}_j \in C^{4,\alpha}_{2-\gamma}(\bar{C},R_0)$$

for some $\gamma > 0$, then one can always replace $\tilde{\varphi}_j$ by some potential $\tilde{\varphi}'_j$ satisfying (14)-(15).

### 4. The geometry of the equation

The material contained in this section is well known (see for example [22]); we include it for completeness and to introduce the reader to the objects entering into the proofs of our results. Recall that $(M,\omega)$ is a $m$-dimensional compact Kähler manifold or a Kähler orbifold with isolated singularities. We will indicate by $g$ the Riemannian metric associated to $\omega$, $\text{Ric}_g$ its Ricci tensor, $\rho_g$ the Ricci form, and $s(\omega)$ its scalar curvature.

Following [36] and [8], we want to understand the behavior of the scalar curvature under deformations of the Kähler form $\omega$ of the form

$$\tilde{\omega} := \omega + i \partial \bar{\partial} \varphi + \beta,$$

where $\beta$ is a closed $(1,1)$ form and $\varphi$ a function defined on $M$. In local coordinates $(v^1, \ldots, v^m)$, if we write

$$\tilde{\omega} = \frac{i}{2} \sum_{a,b} \tilde{g}^{ab} dv^a \wedge d\bar{v}^b$$

then the scalar curvature of $\tilde{\omega}$ is given by

$$(16) \quad s(\tilde{\omega}) = -\sum_{a,b} \tilde{g}^{ab} \partial_{v^a} \partial_{\bar{v}^b} \log (\det(\tilde{g})).$$

The following result is proven in [36], [8] Lemma 2.158:

**Proposition 4.1.** The scalar curvature of $\tilde{\omega}$ can be expanded in terms of $\beta$ and $\varphi$ as

$$s(\tilde{\omega}) = s(\omega) - \left( \frac{1}{2} \Delta_g^2 \varphi + \text{Ric}_g \cdot \nabla_g^2 \varphi + \Delta_g(s(\omega), \beta) + 2 (\rho_g, \beta) \right) + Q_g(\nabla^2 \varphi, \beta),$$

where $Q_g$ collects all the nonlinear terms and where all operators on the right hand side of this identity are computed with respect to the Kähler metric $g$. 
Being a local calculation this formula holds for orbifolds with isolated singularities too. Of crucial importance will be the two linear operators which appear in this formula. First, we set 
\[ L_g := \Delta_g (\omega, \cdot) + 2 (\rho_g, \cdot), \]
which is a linear operator acting on closed $(1,1)$-forms and we also define the operator 
\[ L_g := \frac{1}{2} \Delta_g^2 + \text{Ric}_g \cdot \nabla_g^2, \]
which acts on functions.

For a general Kähler metric it can be very difficult to analyze these operators. Nevertheless, geometry comes to the rescue at a constant scalar curvature metric. Indeed, in this case we have 
\[ L_g = 2 (\bar{\partial} \partial^#) (\bar{\partial} \partial^#), \]
where $\partial^# \varphi$ denotes the $(1,0)$-part of the $g$-gradient of $\varphi$. In other words, 
\[ \partial^# := (\bar{\partial} \cdot) g^#, \]
where $#$ is the inverse of
\[ b : TM \otimes \mathbb{C} \rightarrow T^* M \otimes \mathbb{C} \\
\Xi \mapsto g(\Xi, \cdot). \]
Using (19) one observes that to any element $\varphi \in \text{Ker} L_g$ one can associate a holomorphic vector field, namely $\partial^# \varphi$, which vanishes somewhere on $M$. Indeed, just multiply $L_g \varphi = 0$ by $\varphi$ and integrate the result over $M$ using (19) to conclude that 
\[ \int_M |\bar{\partial} \partial^# \varphi|^2 dv_g = 0, \]
where $dv_g$ denotes the volume form associated to $g$. Therefore, $\bar{\partial} (\partial^# \varphi) = 0$. Conversely, given $\Xi$, a holomorphic vector field vanishing somewhere on $M$, then $\Xi = J v + i v$, where $v$ is a Killing vector field, which also must vanish somewhere. By Proposition 1 in [36], $v = \text{Im} (\partial^# \xi)$ for some real valued function $\xi$, and hence $\Xi = \partial^# \xi$. Since $\Xi$ is holomorphic and $g$ is assumed to have constant scalar curvature, we conclude that $\xi \in \text{Ker} L_g$.

For constant scalar curvature Kähler metrics, $\mathfrak{h}_0(M)$ the space of holomorphic vector fields vanishing somewhere on $M$ is therefore in one to one correspondence with the nontrivial kernel of the operator $L_g$ (i.e. elements of the kernel of $L_g$ whose mean over $M$ is 0).

We define the nonlinear mapping 
\[ S_\omega : \mathcal{C}^{4,\alpha}(M) \rightarrow \mathcal{C}^{0,\alpha}(M)/\mathbb{R} \\
\varphi \mapsto s(\omega + i \partial \bar{\partial} \varphi) \mod \text{cte} \]
This is the set-up for LeBrun-Simanca’s implicit function theorem applied to the mapping $S_\omega$. The application of the implicit function theorem is based on the result which extends immediately to orbifolds with isolated singularities.
Proposition 4.2. Assume that \((M, \omega)\) is nondegenerate and further assume that its scalar curvature \(s(\omega)\) is constant. Then, the operator

\[ \varphi \mapsto DS_\omega|_0 \varphi = -L_g \varphi \]

is surjective and has a kernel which is spanned by constant functions.

Indicating by \(\psi_g\) the function (up to constants) which gives \(\rho_g = \rho^h_g + i \partial \bar{\partial} \psi_g\) (where \(\rho^h_g\) is the harmonic representative for \([\rho_g]\)) and by \(\Xi \in \mathfrak{h}(M)\) a holomorphic vector field, we can define the Futaki invariant

\[ F(\Xi,[\omega]) := \int_M \Xi \psi_g dv_g, \]

where \(dv_g\) denotes the volume form associated to \(g\). Let us denote by \(\mathfrak{h}_0(M)\), the space of holomorphic vector fields which vanish somewhere in \(M\) (which is by the above discussion a linear subspace of \(\mathfrak{h}(M)\)). By definition, \((M, \omega)\) is Futaki nondegenerate if the “linearization” of the Futaki invariant

\[ DF_{[\omega]} : \mathfrak{h}_0(M) \longrightarrow (H^{(1,1)}(M, \mathbb{C}))^* \]

is injective. It is a standard fact, though non obvious, that \(F(\Xi,[\omega])\) only depends on the Kähler class and does not depend on its representative. On the other hand, if \([\omega]\) has a representative with constant scalar curvature, then \(F(\Xi,[\omega])\) vanishes for any \(\Xi \in \mathfrak{h}(M)\).

Now define the nonlinear mapping

\[ \hat{S}_\omega : C^{4,\alpha}(M) \times \mathcal{H}^{1,1}(M, \mathbb{C}) \longrightarrow C^{0,\alpha}(M)/\mathbb{R} \]

\[ (\varphi, \beta) \mapsto s(\omega + i \partial \bar{\partial} \varphi + \beta) \mod cte \]

where \(\mathcal{H}^{1,1}(M, \mathbb{C})\) is the space of \(\omega\)-harmonic \((1,1)\) forms. The result of [36] again extends to orbifolds with isolated singularities, and we have the :

Proposition 4.3. Assume that \((M, \omega)\) is Futaki nondegenerate and further assume that its scalar curvature \(s(\omega)\) is constant. Then, the operator

\[ (\varphi, \beta) \mapsto DS_\omega|_{(0,0)}(\varphi, \beta) = -(L_g \varphi + L_g \beta) \]

is surjective.

5. Mapping properties

We construct right inverses for the operator \(L_g\) which has been defined in the previous section. We first explain this construction in the case where the bounded kernel of \(L_g\) is spanned by constant functions (this is the case if \((M, \omega)\) is nondegenerate, i.e. when there are no nontrivial holomorphic vector field vanishing somewhere on \(M\)) and then we will explain the modifications which are needed to handle the case where \((M, \omega)\) is Futaki nondegenerate. We end this section by defining right inverses for the corresponding operator on \((N_j, \eta_j)\).
5.1. **Analysis of the operators defined on** \((M^\ast, \omega)\). Assume that \((M, \omega)\) is a compact Kähler manifold or Kähler orbifold with isolated singularities and further assume that \(\omega\) has constant scalar curvature. We first construct a right inverse for the operator \(\mathbb{L}_g\) when \(m \geq 3\) and when \((M, \omega)\) is nondegenerate. Next, we proceed with the proof of the corresponding result when \(m = 2\) and when the kernel of \(\mathbb{L}_g\) is nontrivial but \((M, \omega)\) is Futaki nondegenerate, i.e. when the linearized Futaki invariant is nondegenerate.

The mapping properties of \(\mathbb{L}_g\), when defined between weighted spaces, depends heavily on the choice of the weight parameter. Recall that, by definition, \(\zeta \in \mathbb{R}\) is an indicial root of \(\mathbb{L}_g\) at \(p_j\) if there exists some nontrivial function \(v \in C^\infty(\partial B_{j,1})\) such
\[
\mathbb{L}_g (|z|^\zeta v) = O(|z|^{\zeta-3})
\]
near 0 (here we have implicitly used the coordinates defined in Section 2 to parameterize \(M\) close to the point \(p_j\)).

Let \(\Delta_0\) denote the Laplacian in \(\mathbb{C}^m\) with its standard Kähler form. Using \([4]\), it is easy to check that, near each \(p_j\), \((20)\) holds for some function \(v\) if and only if
\[
\Delta_0^2 (|z|^\zeta v) = O(|z|^{\zeta-3})
\]
Therefore, the set of indicial roots of \(\mathbb{L}_g\) at \(p_j\) is equal to the set of indicial roots at the origin for the operator \(\Delta_0^2\) defined on \(\mathbb{C}^m/\Gamma_j\). This later turns out to be included in \(\mathbb{Z} - \{5 - 2m, \ldots, -1\}\) when \(m \geq 3\) and is included in \(\mathbb{Z}\) when \(m = 2\) (observe that the set of indicial roots depends on the group \(\Gamma_j\)). Indeed, let \(e\) be an eigenfunction of \(\Delta_{S^{2m-1}}\) which is invariant under the action of \(\Gamma_j\) and is associated to the eigenvalue \(\gamma (2m - 2 + \gamma)\), where \(\gamma \in \mathbb{N}\), hence
\[
\Delta_{S^{2m-1}} e = -\gamma (2m - 2 + \gamma) e.
\]
Here we identify \(S^{2m-1}\) with the unit sphere in \(\mathbb{C}^m\). Then
\[
\Delta_0^2 (|z|^\zeta e) = (\zeta - k)(\zeta - k - 2)(\zeta - 2 + 2m + k)(\zeta - 4 + 2m + k)|z|^{\zeta-4} e.
\]
Therefore, we find that \(k, k + 2, 2 - 2m - k\) and \(4 - 2m - k\) are indicial roots of \(\Delta_0^2\) at 0. Since the eigenfunctions of the Laplacian on the sphere constitute a Hilbert basis of \(L^2(S^{2m-1})\), we have obtained all the indicial roots of \(\Delta_0^2\) at the origin.

It is clear that the operator
\[
L_\delta' : \mathcal{C}_d^{4,\alpha}(M^\ast) \longrightarrow \mathcal{C}_d^{0,\alpha}(M^\ast)
\]
\[
\varphi \longmapsto \mathbb{L}_g \varphi,
\]
is well defined. It follows from the general theory in \([44]\), where weighted Sobolev spaces are considered instead of weighted Hölder spaces, and in \([43]\), where the corresponding analysis in weighted Hölder spaces is performed (see also \([47]\) that the operator \(L_\delta'\) has closed range and is Fredholm, provided \(\delta\) is not an indicial root of \(\mathbb{L}_g\) at the points \(p_1, \ldots, p_n\). Under this condition, some duality argument (in weighted Sobolev spaces) shows that the operator \(L_\delta'\) is surjective if and only if the operator \(L_{4-2m-\delta}'\) is injective. And, still under this assumption
\[
\dim \text{Ker} L_\delta' = \dim \text{Coker} L_{4-2m-\delta}'.
\]
Using these, we obtain the :

**Proposition 5.1.** Assume that \( m \geq 3 \), \( \delta \in (4 - 2m, 0) \) and assume that \((M, \omega)\) is nondegenerate so that the kernel of \( L_g \) is spanned by the constant function. Then, the operator

\[
L_{\delta} : \quad C^{4, \alpha}_{\delta}(M^*) \times \mathbb{R} \to C_{\delta-4}^{0, \alpha}(M^*)
\]

\[
(\varphi, \nu) \mapsto L_g \varphi + \nu
\]

is surjective and has a one dimensional kernel spanned by the constant function.

**Proof:** We claim that, when \( \delta \in (4 - 2m, 0) \), the operator \( L'_{\delta} \) has a one dimensional kernel spanned by the constant function. Indeed, when \( \delta \in (4 - 2m, 0) \), standard regularity theory implies that the isolated singularities of any element of the kernel of \( L'_{\delta} \) are removable and hence, the elements of the kernel of \( L'_{\delta} \) are in fact a smooth functions in \( M \). Therefore, it follows from our assumption that the kernel of \( L'_{\delta} \) reduces to the constant functions. It follows from (21) that the operator \( L'_{\delta} \) also has a one dimensional cokernel, which is easily seen to be spanned by the constant function since (thanks to (19))

\[
\int_M L_g \varphi dv_g = 0,
\]

for any \( \varphi \in C^{4, \alpha}_{\delta}(M^*) \). This completes the proof of the result. \( \square \)

When \( m = 2 \), the above result has to be modified (since \( 4 - 2m = 0 \) in this case !). We set

\[
D := \text{Span}\{\chi_1, \ldots, \chi_n\},
\]

where \( \chi_j \) is a cutoff function which is identically equal to 1 in \( B_{j, r_0/2} \) and identically equal to 0 in \( M - B_{j, r_0} \). This time, we have the :

**Proposition 5.2.** Assume that \( m = 2 \), \( \delta \in (0, 1) \) and assume that \((M, \omega)\) is nondegenerate so that the kernel of \( L_g \) is spanned by the constant function. Then

\[
L_{\delta} : \quad \left(C^{4, \alpha}_{\delta}(M^*) \oplus D\right) \times \mathbb{R} \to C_{\delta-4}^{0, \alpha}(M^*)
\]

\[
(\varphi, \nu) \mapsto L_g \varphi + \nu
\]

is surjective and has a one dimensional kernel spanned by the constant function.

**Proof:** We keep the notations of the previous proof. Assume that \( \delta > 0 \). Then the operator \( L'_{\delta} \) is injective (since we have assumed that the kernel of \( L_g \) is spanned by the constant function and a nonzero function does not belong to \( C_{\delta}^{4, \alpha}(M^*) \) when \( \delta > 0 \)). Therefore, when \( \delta > 0 \), \( \delta \notin \mathbb{N} \), the operator \( L'_{\delta} \) is surjective and admits a right inverse, which, unfortunately, is not unique.

Moreover, when \( \delta \in (0, 1) \), a relative index argument \cite{44} shows that the dimension of the kernel of \( L'_{\delta} \) and the dimension of the cokernel of \( L'_{\delta} \) are both equal to \( n \). The kernel of \( L'_{\delta} \) is rather explicit since it is spanned by the constant function and, for \( j = 1, \ldots, n - 1 \), the unique function \( \gamma_j \) solution (in the sense of distributions) of

\[
L_g \gamma_j = \delta_{p_{j+1}} - \delta_{p_j}
\]
and whose mean value over \(M\) is 0.

Let us now assume that \(\delta \in (0, 1)\). Given \(\psi \in C^0_{\delta}(M^*)\), we choose \(\nu \in \mathbb{R}\) to be equal to the mean value of the function \(\psi\). Since \(L'_{-\delta}\) is surjective, we have the existence of a solution of
\[
\mathbb{L}_g \varphi = \psi - \nu,
\]
which belongs to \(C^{4,\alpha}_{-\delta}(M^*)\) (this solution is for example obtained by applying to \(\psi - \nu\) a given right inverse for \(L'_{-\delta}\)). It follows from elliptic regularity theory that, near any \(p_j\), the function \(\varphi\) can be expanded as
\[
\varphi(z) = a_j + b_j \log |z| + \tilde{\varphi}_j(z)
\]
where \(a_j, b_j \in \mathbb{R}\) and \(\tilde{\varphi}_j \in C^{4,\alpha}_{\delta}(B^*_j, \mathcal{C})\). This implies that the function \(\varphi\) is a solution (in the sense of distributions) of
\[
(22) \quad \mathbb{L}_g \varphi + \nu = \psi - c_2 \sum_j b_j \delta_{p_j}
\]
where \(c_2 = 2 |S^3| \neq 0\). Using the fact that the functions \(\gamma_j\) are in the kernel of \(L'_{-\delta}\), we can assume without loss of generality that the \(b_j\) at the different points \(p_j\) are all equal, by adding to \(\varphi\) a suitable linear combination of the functions \(\gamma_j\) (this amounts to choose a particular right inverse of \(L'_{-\delta}\)). Integration of (22) over \(M\) implies that
\[
0 = -c_2 \sum_j b_j.
\]
Hence all \(b_j\) are equal to 0 and, near \(p_i\), the function \(\varphi\) can be expanded as
\[
\varphi(z) = a_j + \tilde{\varphi}_j(z).
\]
This shows that there exists a choice of the right inverse \(G'_{-\delta}\) of \(L'_{-\delta}\) such that, if \(\psi \in C^0_{\delta-4}(M^*)\) and if \(\nu\) is the mean value of \(\psi\), then
\[
G'_{-\delta}(\psi - \nu) \in C^{4,\alpha}_{\delta}(M^*) \oplus \mathcal{D}.
\]
This completes the proof of the result. \(\square\)

**Remark 5.1.** Observe that, given \(\psi \in C^0_{\delta-4}(M^*)\), the constant \(\nu \in \mathbb{R}\) in the equation \(\mathbb{L}_g \varphi + \nu = \psi\) is equal to the mean value of \(\psi\) so that \(\psi - \nu\) is \(L^2\)-orthogonal to the kernel of \(\mathbb{L}_g\) which is spanned by the constant function.

We turn to the case where the kernel of \(\mathbb{L}_g\) is not only spanned by the constant function and we now assume that \((M, \omega)\) is Futaki nondegenerate. The proof relies on the following result which replaces Proposition 5.1 and whose proof is identical.

**Proposition 5.3.** Assume that \(m \geq 3\) and that \((M, \omega)\) is Futaki nondegenerate. Then, for all \(\delta \in (4 - 2m, 0)\) the operator
\[
L_\delta: \quad C^4_{\delta}(M^*) \times \mathcal{H}^{1,1}(M, \mathbb{C}) \times \mathbb{R} \longrightarrow \quad C^0_{\delta-4}(M^*)
\]
\[
(\varphi, \beta, \nu) \quad \longrightarrow \quad \mathbb{L}_g \varphi + L_g \beta + \nu,
\]
is surjective and has a kernel which is equal to the kernel of \(\mathbb{L}_g\).
Given $\psi \in C^{0,\alpha}_{\delta-4}(M^*)$, the $(1,1)$ form $\beta \in \mathcal{H}^{1,1}(M,\mathbb{C})$ and the constant $\nu \in \mathbb{R}$ in the equation

$$\mathbb{L}_g \phi + \mathcal{L}_g \beta + \nu = \psi$$

are chosen in such a way that $\psi - \mathcal{L}_M \beta - \nu$ is $L^2$-orthogonal to the elements of the kernel of $\mathbb{L}_g$.

Since the space of holomorphic vector fields is finite dimensional so is $\text{Ker} \mathbb{L}_g$ and one can replace in the above statement the space $\mathcal{H}^{1,1}(M,\mathbb{C})$ by a finite dimensional subspace $D \subset \mathcal{H}^{1,1}(M,\mathbb{C})$. We claim that this subspace can in turn be replaced by $D_{\bar{r}_0}$, a finite dimensional space of closed $(1,1)$ forms which are supported in $M_{\bar{r}_0}$, provided $\bar{r}_0$ is fixed small enough. Indeed, near each $p_j$, any element of $\beta \in D \subset \mathcal{H}^{1,1}(M,\mathbb{C})$ can be decomposed as

$$\beta = d\pi_j$$

We truncate the potential $\pi_j$ between $2\bar{r}_0$ and $\bar{r}_0$ and define

$$\beta_{\bar{r}_0} := d((1 - \chi_{2\bar{r}_0}) \pi_j),$$

where $\chi_{2\bar{r}_0}$ is a cutoff function identically equal to 0 in $M_{2\bar{r}_0}$ and identically equal to 1 in each $B_j, \bar{r}_0$. We set

$$D_{\bar{r}_0} := \text{Span} \{\beta_{\bar{r}_0} : \beta \in D\}.$$

When $m \geq 3$, it is easy to check that, given $\delta \in (4 - 2m, 0)$, the operator

$$L'_{\delta} : \mathcal{C}^{1,\alpha}_{\delta}(M^*) \times D_{\bar{r}_0} \times \mathbb{R} \rightarrow \mathcal{C}^{0,\alpha}_{\delta-4}(M^*)$$

$$(\phi, \beta, \nu) \mapsto \mathbb{L}_M \phi + \mathcal{L}_M \beta + \nu$$

is surjective provided $\bar{r}_0$ is chosen small enough.

In dimension $m = 2$, this result has to be modified. As above we find that, given $\delta \in (0,1)$, the operator

$$L'_{\delta} : (\mathcal{C}^{1,\alpha}_{\delta}(M^*) \oplus \mathcal{D}) \times D_{\bar{r}_0} \times \mathbb{R} \rightarrow \mathcal{C}^{0,\alpha}_{\delta-4}(M^*)$$

$$(\phi, \beta, \nu) \mapsto \mathbb{L}_g \phi + \mathcal{L}_g \beta + \nu$$

is surjective and has a kernel which is equal to the kernel of $\mathbb{L}_g$.

5.2. Operators defined on $(N_j, \eta_j)$. Assume that $(N_j, \eta_j)$ is an ALE space with zero scalar curvature Kähler metric $\eta_j$. Further assume that, at infinity, the Kähler form $\eta_j$ can be expanded as

$$\eta_j = i\partial \bar{\partial} (\frac{1}{2} |u|^2 + \tilde{\varphi}_j(u)),$$

where $\tilde{\varphi}_j$ satisfies

$$\nabla^2 \tilde{\varphi}_j \in \mathcal{C}^{2,\alpha}_{2-2m}(C_j, R_0).$$

We denote by $g_j$ the metric associated to the Kähler form $\eta_j$. Again the analysis of $\mathbb{L}_{g_j}$ when defined between weighted spaces follows from the general theory developed in [43] and [44] (see also [47]) and the mapping properties of $\mathbb{L}_{g_j}$ when defined between weighted spaces depends heavily on the choice of the weight parameter.

Recall that $\zeta \in \mathbb{R}$ is an indicial root of $\mathbb{L}_{g_j}$ at infinity if there exists some nontrivial function $v \in \mathcal{C}^{\infty}(\partial B_{j,1})$ such that

$$\mathbb{L}_{g_j} (|u|^\zeta v) = O(|u|^{\zeta-5})$$

(23)
near $\infty$ (we have implicitly used the coordinates defined in Section 2 to parameterize $N_j$ near its end).

Again, it is easy to check that, (23) holds for some function $v$ if and only if
\[
\Delta_0^2 (|u|^{-5}) = O(|u|^{-5})
\]
(here one uses the fact that $g_j = g_{\text{eucl}} + O(|z|^{2-2m})$ at infinity and hence the coefficients of the Ricci tensor at infinity are bounded by a constant times $|u|^{-2m}$). Therefore, the set of indicial roots of $L_{g_j}$ at infinity is equal to the set of indicial roots at infinity for the operator $\Delta_0^2$ defined on $\mathbb{C}^m/\Gamma_j$. Again, this set is included in $\mathbb{Z} - \{5 - 2m, \ldots, -1\}$ when $m \geq 3$ and is included in $\mathbb{Z}$ when $m = 2$ (the set of indicial roots depends on the group $\Gamma_j$). The proof of this fact follows the analysis done in Section 5.1.

The operator
\[
\tilde{L}_\delta : C_{4,\alpha}^4(N_j) \longrightarrow C_{4-\delta}^0(N_j)
\]
\[
\varphi \longmapsto \mathbb{L}_{g_j} \varphi,
\]
is well defined (again one uses the fact that $g_j = g_{\text{eucl}} + O(|z|^{2-2m})$ at infinity). Moreover, according to [44] and [43] (see also [47]), this operator has closed range and is Fredholm, provided $\delta$ is not an indicial root of $L_{g_j}$ at infinity. Under this condition, some duality argument (in weighted Sobolev spaces) shows that the operator $\tilde{L}_\delta$ is surjective if and only if the operator $\tilde{L}_{4-2m-\delta}$ is injective. And, still under this assumption (24)
\[
\dim \text{Ker} \tilde{L}_\delta = \dim \text{Coker} \tilde{L}_{4-2m-\delta}
\]

The construction of a right inverse for the operator $L_{g_j}$ relies on the following result whose proof is essentially borrowed from [31]:

**Proposition 5.4.** Assume that $(N_j, \eta_j)$ is a constant scalar curvature ALE Kähler manifold or Kähler orbifold with isolated singularities. Then, there is no nontrivial solution of $\mathbb{L}_{g_j} \varphi = 0$, which belongs to $C_{4,\alpha}^4(N_j)$, for some $\delta < 0$.

**Proof:** Assume that $\mathbb{L}_{g_j} \varphi = 0$ and that $\varphi \in C_{4,\delta}^4(N_j)$, for some $\delta < 0$. Then, as explained in Section 4, the vector field $\partial_{\bar{g_j}} \varphi$ is a holomorphic vector field which tends to 0 at infinity. Indeed, we have
\[
\mathbb{L}_{g_j} = 2 (\tilde{\partial} \partial_{\bar{g_j}}^\#) \ast (\tilde{\partial} \partial_{g_j}^\#)
\]
and, multiplying $\mathbb{L}_{g_j} \varphi = 0$ by $\varphi$ and integrating by parts, we get
\[
\int_{N_j} |\tilde{\partial} \partial_{g_j}^\# \varphi|^2 dv_{g_j} = 0
\]
All integrations are justified because of the decaying behavior of $\varphi$ at infinity which implies that $\varphi \in C_{4-2m}^4(N_j)$ when $m \geq 3$. Therefore $\partial_{\bar{g_j}} \varphi = 0$. Using Hartogs’ Theorem, the restriction of $\partial_{\bar{g_j}} \varphi$ to $C_{j,R_0}$ can be extended to a holomorphic vector field on $\mathbb{C}^m$. Since this vector field decays at infinity, it has to be identically equal to 0. This implies that $\partial_{\bar{g_j}} \varphi$ is identically equal to 0 on $C_{j,R_0}$. However $\varphi$ being a real valued function, this implies that $\partial \varphi = \bar{\partial} \varphi = 0$ in $C_{j,R_0}$. Hence
the function \( \phi \) is constant in \( C_{j,R_0} \) and decays at infinity. This implies that \( \phi \) is identically equal to 0 in \( C_{j,R_0} \) and satisfies

\[
L_g j \phi = 0 \quad \text{in} \quad N_j.
\]

Now, we use the unique continuation theorem for solutions of linear elliptic equations to conclude that \( \phi \) is identically equal to 0 in \( N_j \).

This being understood, we have the:

**Proposition 5.5.** Assume that \( \delta \in (0,1) \). Then

\[
\tilde{L}_\delta : \quad C^{4,\alpha}_0(N_j) \quad \longrightarrow \quad C^{0,\alpha}_{4-2m}(N_j)
\]

\[
\varphi \quad \longmapsto \quad L_g j \varphi
\]

is surjective and has a one dimensional kernel spanned by the constant function.

**Proof:** It follows from Proposition 5.4 that, when \( \delta' < 0 \) the operator \( \tilde{L}_{\delta'} \) is injective and this implies that \( \tilde{L}_\delta \) is surjective whenever \( \delta > 4 - 2m \) is not an indicial root of \( L_g j \) at infinity.

\[\square\]

### 5.3. Bi-harmonic extensions.

We end up this section by the following simple result whose proof follows at once from the application of the maximum principle. Here, as usual, \( \Gamma \) is a finite subgroup of \( U(n) \) acting freely on \( \mathbb{C}^n \setminus \{0\} \). We define

\[
\bar{B}_\Gamma := \{ z \in \mathbb{C}^n / \Gamma : |z| \leq 1 \},
\]

\[
\bar{B}_\Gamma^* := \{ z \in \mathbb{C}^n / \Gamma : |z| \leq 1 \},
\]

\[
\bar{C}_\Gamma := \{ z \in \mathbb{C}^n / \Gamma : |z| \geq 1 \}.
\]

Therefore, when \( \Gamma = \Gamma_j \), we have \( \bar{B}_{\Gamma_j} = \bar{B}_{j,1} \), \( \bar{B}_{\Gamma_j}^* = \bar{B}_{j,1}^* \), and \( \bar{C}_{\Gamma_j} = \bar{C}_{j,1} \). Recall that \( \Delta_0 \) denotes the Laplacian in \( \mathbb{C}^m \) with the standard Kähler form. With these notations in mind, we have

**Proposition 5.6.** Assume that \( m \geq 3 \). Given \( h \in C^{4,\alpha}(\partial B_\Gamma) \) and \( k \in C^{2,\alpha}(\partial B_\Gamma) \) there exist bi-harmonic functions \( H_{h,k}^i \in C^{4,\alpha}(\bar{B}_\Gamma) \) and \( H_{h,k}^o \in C^{4,\alpha}_{4-2m}(\bar{C}_\Gamma) \) such that

\[
\Delta_0^2 H_{h,k}^i = 0 \quad \text{in} \quad B_\Gamma
\]

\[
\Delta_0^2 H_{h,k}^o = 0 \quad \text{in} \quad C_\Gamma,
\]

with

\[
H_{h,k}^i = H_{h,k}^o = h \quad \text{and} \quad \Delta_0 H_{h,k}^i = \Delta_0 H_{h,k}^o = k \quad \text{on} \quad \partial B_\Gamma.
\]

Moreover,

\[
\| H_{h,k}^i \|_{C^{4,\alpha}(B_\Gamma)} + \| H_{h,k}^o \|_{C^{4-2m,\alpha}(C_\Gamma)} \leq c \left( \| h \|_{C^{4,\alpha}(\partial B_\Gamma)} + \| k \|_{C^{2,\alpha}(\partial B_\Gamma)} \right).
\]

For later use, it will be convenient to get explicit formulas for \( H_{h,k}^i \) and \( H_{h,k}^o \). We decompose both functions \( h \) and \( k \) over eigenfunctions of the Laplacian on the sphere. Namely

\[
h = \sum_{\gamma=0}^{\infty} h^{(\gamma)} e_\gamma \quad \text{and} \quad k = \sum_{\gamma=0}^{\infty} k^{(\gamma)} e_\gamma
\]
where the function $e_\gamma$ satisfies
\[ \Delta_{S^{2m-1}} e_\gamma = -\gamma (2m - 2 + \gamma) e_\gamma \]
and is normalized so that $\|e_\gamma\|_{L^2} = 1$. Observe that we only have to consider the eigenvalues corresponding to eigenfunctions which are invariant under the action of $\Gamma$. Then, the functions $H_{h,k}^i$ and $H_{h,k}^0$ are explicitly given by
\[ H_{h,k}^i(z) = \sum_{\gamma=0}^{\infty} \left( \left( h^{(\gamma)} - \frac{k^{(\gamma)}}{4(m + \gamma)} \right) |z|^\gamma + \frac{k^{(\gamma)}}{4(m + \gamma)} |z|^{\gamma + 2} \right) e_\gamma \]
and
\[ H_{h,k}^0(z) = \sum_{\gamma=0}^{\infty} \left( \left( h^{(\gamma)} + \frac{k^{(\gamma)}}{4(m + \gamma - 2m - \gamma)} \right) |z|^{2-2m-\gamma} - \frac{k^{(\gamma)}}{4(m + \gamma - 2m - \gamma)} |z|^{3-2m-\gamma} \right) e_\gamma \]

**Proof of Proposition 5.6**: The existence of $H_{h,k}^i$ is clear and the estimate follows at once. The explicit expression of $H_{h,k}^0$ provides a direct proof of the estimate of this function. First observe that elliptic regularity implies that, there exists $c = c(m) > 0$ and $N = N(m) \in \mathbb{N}$ such that
\[ \|e_\gamma\|_{L^\infty} \leq c (1 + \gamma)^N \|e_\gamma\|_{L^2}^2 = c (1 + \gamma)^N \]
since we have normalized the functions $e_\gamma$ to have $L^2$-norm equal to 1. In addition, Cauchy-Schwarz inequality yields
\[ |h^{(\gamma)}| + |k^{(\gamma)}| \leq c (\|h\|_{C^4,\alpha} + \|k\|_{C^2,\alpha}) \]
for some constant which does not depend on $\gamma$. Using these two information together with (25) we conclude that
\[ \sup_{|z| \geq 2} \left( |z|^{2m-4} |H_{h,k}^0| + |z|^{2m-2} |\Delta_0 H_{h,k}^0| \right) \leq c (\|h\|_{C^4,\alpha} + \|k\|_{C^2,\alpha}) \]
since the series are absolutely convergent for $|z|$ larger than 1. The maximum principle applied in \{ $z \in C_\Gamma : |z| \in [1,2]$ \} then allows to fill in the gap in the estimate and we conclude that
\[ \sup_{|z| \geq 1} \left( |z|^{2m-4} |H_{h,k}^0| + |z|^{2m-2} |\Delta_0 H_{h,k}^0| \right) \leq c (\|h\|_{C^4,\alpha} + \|k\|_{C^2,\alpha}) \]
The estimates for the derivatives of $H_{h,k}^0$ follow from Schauder’s estimates. \qed

When $m = 2$, the result has to be slightly modified since in this case we can choose
\[ H_{h,k}^0(z) = h^{(0)} |z|^{-2} + \frac{k^{(0)}}{2} \log |z| + \sum_{\gamma=1}^{\infty} \left( \left( h^{(\gamma)} + \frac{k^{(\gamma)}}{4\gamma} \right) |z|^{-2-\gamma} - \frac{k^{(\gamma)}}{4\gamma} |z|^{-\gamma} \right) e_\gamma \]
This time, one can check that
\[ H_{h,k}^0 \in C^{4,\alpha}_{-1}(\bar{C}_\Gamma) \oplus \text{Span} \{ \log |z| \} \]
and that
\[ \|H_{h,k}^0\|_{C^{4,\alpha}_{-1}(\bar{C}_\Gamma) \oplus \text{Span} \{ \log |z| \}} \leq c (\|h\|_{C^4,\alpha(\partial B_\Gamma)} + \|k\|_{C^2,\alpha(\partial B_\Gamma)}). \]
6. CONSTANT SCALAR CURVATURE KÄHLER METRICS

We set
\[ r_\varepsilon := \varepsilon^{-\frac{1}{n}} \quad \text{and} \quad R_\varepsilon := \frac{r_\varepsilon}{\varepsilon} = \varepsilon^{-\frac{1}{n}}. \]

6.1. Perturbation of \( \omega \). We will now use the result of the previous sections to perturb \( \omega \), the Kähler form on \( M_{r_\varepsilon} \), into infinite families of constant scalar curvature Kähler forms which are defined on \( M_{r_\varepsilon} \) and which are parameterized by the boundary data of their potentials. We carry this analysis when \( (M, \omega) \) is Futaki nondegenerate since this is the most complete case. We consider the perturbed Kähler form
\[ (30) \tilde{\omega} = \omega + i \partial \bar{\partial} \varphi + \beta. \]
where \( \beta \) is a closed \((1,1)\) form and \( \varphi \) is a function defined on \( M_{r_\varepsilon} \). The scalar curvature of \( \tilde{\omega} \) is given by
\[ (31) s(\tilde{\omega}) = s(\omega) - (L_g \varphi + L_g \beta) + Q_g(\nabla^2 \varphi, \beta), \]

where the operators \( L_g \) and \( L_g \) have been defined in (17) and (18) and where \( Q_g \) collects all the nonlinear terms. The structure of \( Q_g \) is quite complicated however, away from the support of the elements of \( D_{\gamma_0} \) (i.e. in each \( \bar{B}_{j,\gamma_0} \)), we have \( Q_g(\nabla^2 \varphi, \beta) = Q_g(\nabla^2 \varphi, 0) \) and this operator, only acting on the function \( \varphi \), enjoys the following decomposition
\[ (32) Q_g(\nabla^2 \varphi, 0) = \sum q B_{q,4,2}(\nabla^4 \varphi, \nabla^2 \varphi) C_{q,4,2}(\nabla^2 \varphi) + \sum q B_{q,3,3}(\nabla^3 \varphi, \nabla^3 \varphi) C_{q,3,3}(\nabla^2 \varphi) \]
\[ + |z| \sum q B_{q,2,2}(\nabla^2 \varphi, \nabla^2 \varphi) C_{q,2,2}(\nabla^2 \varphi) \]

where the sum over \( q \) is finite, the operators \((U,V) \rightarrow B_{q,a,b}(U,V)\) are bilinear in the entries and have coefficients which are smooth functions on \( \bar{B}_{j,\gamma_0} \). The nonlinear operators \( W \rightarrow C_{q,a,b}(W) \) have Taylor expansions (with respect to \( W \)) whose coefficients are smooth functions on \( \bar{B}_{j,\gamma_0} \).

These facts follow at once from the expression of the scalar curvature of \( \tilde{\omega} \) in local coordinates as given in (16).

We would like to solve the equation
\[ (33) s(\tilde{\omega}) = s(\omega) + \nu \]
in \( M_{r_\varepsilon} \), where \( \nu \in \mathbb{R} \).

We fix a constant \( \kappa > 0 \) (large enough). Assume that we are given boundary data \( h_j \in C^{4,\alpha}(\partial B_{\Gamma_j}) \) and \( k_j \in C^{2,\alpha}(\partial B_{\Gamma_j}) \), for \( j = 1, \ldots, n \), satisfying
\[ (34) \|h_j\|_{C^{4,\alpha}(\partial B_{\Gamma_j})} \leq \kappa r_\varepsilon^4 \quad \text{and} \quad \|k_j\|_{C^{2,\alpha}(\partial B_{\Gamma_j})} \leq \kappa r_\varepsilon^4. \]

When \( m \geq 3 \), we define
\[ (35) H_{h,k} := \sum_j \chi_j H_{h_j,k_j} (\cdot / r_\varepsilon), \]
where we have set
\[ h := (h_1, \ldots, h_n) \quad \text{and} \quad k := (k_1, \ldots, k_n), \]
and where we recall that the cutoff functions \( \chi_j \) are identically equal to 1 in \( B_{j, r_0/2} \) and identically equal to 0 in \( M - B_{j, r_0} \).

When \( m = 2 \), some modifications are necessary. We decompose each \( k_j \) as
\[ k_j = k_j^{(0)} + k_j^\perp, \]
where \( k_j^{(0)} \) is a constant function and \( k_j^\perp \) has mean 0 on \( \partial B_{\Gamma_j} \).

With this decomposition in mind, we define
\[ H_{h, k} := \sum_j \chi_j \left( H_{h, k_j^\perp} / r \right) + k_j^{(0)} \frac{1}{2} \log | \cdot |. \]

We replace in (30) the function \( \phi \) by \( H_{h, k} + \phi \). Then, (33) leads to the equation
\[ L_g (H_{h, k} + \phi) + L_g \beta + \nu = Q_g (H_{h, k} + \phi, \beta), \]
which we would like to solve in \( M_{r_\varepsilon} \).

**Definition 6.1.** Given \( \tilde{r} \in (0, r_0/2) \), \( k \in \mathbb{N} \), \( \alpha \in (0, 1) \) and \( \delta \in \mathbb{R} \), the weighted space \( C^{k, \alpha}_{\tilde{r}}(M_{\tilde{r}}) \) is defined to be the space of functions \( \varphi \in C^{k, \alpha}(M_{\tilde{r}}) \) endowed with the norm
\[ \| \varphi \|_{C^{k, \alpha}_{\tilde{r}}(M_{\tilde{r}})} := \| \varphi \|_{C^{k, \alpha}(M_{\tilde{r}}/2)} + \sum_j \sup_{2\tilde{r} \leq r \leq r_0} r^{-\delta} \| \varphi \|_{C^{k, \alpha}(B_{\tilde{r}/2} - B_{\tilde{r}}/2)} \]
where $\varphi \in C^{1,\alpha}_\delta(M^*)$, when \( m \geq 3 \) and $\varphi \in C^{1,\alpha}_\delta(M^*) \oplus D$ when \( m = 2 \), $\beta \in D_{\bar{r}_0}$ and $\nu \in \mathbb{R}$ have to be determined. Observe that any solution of (43) is a solution of (37). The advantage of the latter versus the former is that we can now make use of the analysis of Section 6.1 which allows us to find $G_\delta$ a right inverse for the operator $L_\delta$ and rephrase the solvability of (39) as a fixed point problem

$$(\varphi, \beta, \nu) = \mathcal{N}(\varepsilon, h, k; \varphi, \beta)$$

where the nonlinear operator $\mathcal{N}$ is defined by

$$\mathcal{N}(\varepsilon, h, k; \varphi, \beta) := G_\delta(\mathcal{E}_{\varepsilon}(Q_g(H_{h,k} + \varphi, \beta) - L_{\varepsilon} H_{h,k})).$$

It will be convenient to denote

$$\mathcal{F} := C^{1,\alpha}_\delta(M^*) \times D_{\bar{r}_0} \times \mathbb{R}$$

when \( m \geq 3 \) and

$$\mathcal{F} := (C^{1,\alpha}_\delta(M^*) \oplus D) \times D_{\bar{r}_0} \times \mathbb{R}$$

when \( m = 2 \). This space is naturally endowed with the product norm.

We first estimate the terms on the right hand side of (39) when $\varphi = 0$ and $\beta = 0$ and next show that $\mathcal{N}$ is a contraction from a suitable small ball in $\mathcal{F}$. This is the content of the:

**Lemma 6.1.** There exists $c_\kappa = c(\kappa) > 0$, $\bar{c}_\kappa = \bar{c}(\kappa) > 0$ and there exists $\varepsilon_\kappa = \varepsilon(\kappa) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa)$

$$\|\mathcal{N}(\varepsilon, h, k; 0, 0)\|_F \leq c_\kappa r^{2m}_\varepsilon.$$ (40)

In addition,

$$\|\mathcal{N}(\varepsilon, h, k; \varphi, \beta) - \mathcal{N}(\varepsilon, h, k; \varphi', \beta')\|_F \leq \bar{c}_\kappa r^{2}\varepsilon \|(\varphi - \varphi', \beta - \beta')\|_F$$ (41)

and

$$\|\mathcal{N}(\varepsilon, h, k; \varphi, \beta) - \mathcal{N}(\varepsilon, h', k'; \varphi, \beta)\|_F \leq \bar{c}_\kappa r^{2m-4}\varepsilon \| (h - h', k - k')\|((C^{1,\alpha})^n \times (C^{2,\alpha})^n)$$ (42)

provided $(\varphi, \beta, 0), (\varphi', \beta', 0) \in \mathcal{F}$ satisfy

$$\|(\varphi, \beta, 0)\|_F \leq 2 c_\kappa r^{2m}_\varepsilon \quad \text{and} \quad \|(\varphi', \beta', 0)\|_F \leq 2 c_\kappa r^{2m}_\varepsilon,$$

and $h := (h_1, \ldots, h_n), h' := (h'_1, \ldots, h'_n), k := (k_1, \ldots, k_n)$ and $k' := (k'_1, \ldots, k'_n)$ satisfy (34).

**Proof:** We give a precise proof of the first estimate. The other estimates follow from similar considerations. In the proof, the constants $c_\kappa^{(l)} > 0$ only depend on $\kappa$.

**Step 1** We first estimate $\mathbb{L}_g H_{h,k}$. Using the result of Proposition 5.6 together with (34), we obtain

$$\|\nabla^2 H_{h,k}\|_{C^{2,\alpha}_{2-2m}(M_{\varepsilon})} \leq c_\kappa^{(l)} r^{2m}_\varepsilon.$$ (43)

Now observe that, by construction, $\nabla^2 H_{h,k} = 0$ in $M_{\varepsilon}$ and hence $\mathbb{L}_g H_{h,k} = 0$ in this set. Next,

$$\Delta^2_0 H_{h,k} = 0$$
in each $B_{j,r_0/2} - B_{j,r_1}$, hence
\[ \| - H_{h,k} = \left( L_g - \frac{1}{2} \Delta_g \right) - H_{h,k} \]
in this set. Using (4), we conclude that
\[ \| L_g H_{h,k} \|_{C^0(M_{b,r_0})} \leq c(2)_k r^{2m}_\varepsilon \]
and
\[ \int_M | \mathcal{E}_k (L_g H_{h,k}) | \, dv_g \leq c(2)_k r^{2m}_\varepsilon \]
These two estimates together with the properties of $G_{b,\varepsilon}$ immediately imply that
\[ \| G_{b,\varepsilon} (E_{k_1} H_{h,k}) \|_F \leq c_{b,\varepsilon}^2. \]

**Step 2** We turn to the derivation of the second estimate. Again, we use the structure of $Q_g$ as described in (32) together with (43) to get
\[ \| Q_g \|_{C^0,\varepsilon(M_{b,r_0/2})} \leq c(4)_k r^{2m}_\varepsilon, \]
and
\[ \| E_k (Q_g (E_{k_1} H_{h,k})) \|_{C^{0,\varepsilon}(\tilde{B}_{b,r_0})} \leq c(4)_k r^{2m}_\varepsilon. \]
Therefore, we conclude that
\[ \| E_k (Q_g (E_{k_1} H_{h,k})) \|_{C^{0,\varepsilon}(\tilde{B}_{b,r_0})} \leq c(6)_k r^{2m}_\varepsilon \]
as well as
\[ \int_M | E_k (Q_g (E_{k_1} H_{h,k})) | \, dv_g \leq c(6)_k r^{2m+2}_\varepsilon \]
The properties of $G_{b,\varepsilon}$ yield
\[ \| G_{b,\varepsilon} (E_k (Q_g (E_{k_1} H_{h,k}))) \|_F \leq c(6)_k r^{2m+2}_\varepsilon \]
This completes the proof of the first estimate.

**Step 3** We now turn to the derivation of the second estimate. Again, we use the structure of $Q_g$ as described in (32) to get
\[ \| Q_g (E_{k_1} H_{h,k} + \varphi, \beta - Q_g (E_{k_1} H_{h,k} + \varphi', \beta')) \|_{C^{0,\varepsilon}(M_{b,r_0})} \leq c(7)_k r^{2m}_\varepsilon \| (\varphi - \varphi', \beta - \beta', 0) \|_F, \]
and, arguing as above, we find that
\[ \| E_k (Q_g (E_{k_1} H_{h,k} + \varphi, \beta - Q_g (E_{k_1} H_{h,k} + \varphi', \beta')) \|_{C^{0,\varepsilon}(\tilde{B}_{b,r_0})} \leq c(7)_k r^{2m}_\varepsilon \| (\varphi - \varphi', \beta - \beta', 0) \|_F \]
Therefore, we conclude that
\[ \| E_k (Q_g (E_{k_1} H_{h,k} + \varphi, \beta - Q_g (E_{k_1} H_{h,k} + \varphi', \beta')) \|_{C^{0,\varepsilon}(\tilde{B}_{b,r_0})} \leq c(8)_k r^{2m-2}_\varepsilon \| (\varphi - \varphi', \beta - \beta', 0) \|_F \]
as well as
\[ \int_M | E_k (Q_g (E_{k_1} H_{h,k} + \varphi, \beta - Q_g (E_{k_1} H_{h,k} + \varphi', \beta')) | \, dv_g \leq c(8)_k r^{2m-2+\delta}_\varepsilon \| (\varphi - \varphi', \beta - \beta', 0) \|_F \]
Observe that, in order to derive the second estimate, we have implicitly used the fact that the computation of the scalar curvature only involves second and higher partial differential of the functions \( \varphi \) and \( \varphi' \) and hence, in dimension \( m = 2 \), the effect of the elements of \( \mathcal{D} \) have no influence in \( \bar{B}_{j,r_0} - B_{j,r_*} \). The estimate then follows from the boundedness of \( G_\delta \).

**Step 4** In order to prove the third estimate, we first observe that
\[
\|L_g (H_{h,k} - H_{h',k'})\|_{C^{2,\alpha}(M_r,\alpha)} \leq c_\kappa(9) r_\varepsilon^{2m-4} \|(h - h', k - k')\|_{(C^4,\alpha)^n \times (C^{2,\alpha})^n}
\]
and
\[
\int_M |\mathcal{E}_r (L_g (H_{h,k} - H_{h',k'}))| \, dv_g \leq c_\kappa(9) r_\varepsilon^{2m-4} \|(h - h', k - k')\|_{(C^4,\alpha)^n \times (C^{2,\alpha})^n}
\]
Next, we have
\[
\|\mathcal{E}_r (Q_g(\nabla^2 H_{h,k} + \varphi, \beta) - Q_g(\nabla^2 H_{h',k'} + \varphi, \beta))\|_{C^{0,\alpha}(M_\ast)} \leq c_\kappa(10) r_\varepsilon^{-2-\delta} \|(h - h', k - k')\|_{(C^4,\alpha)^n \times (C^{2,\alpha})^n}
\]
as well as
\[
\int_M |\mathcal{E}_r (Q_g(\nabla^2 H_{h,k} + \varphi, \beta) - Q_g(\nabla^2 H_{h',k'} + \varphi, \beta))| \, dv_g \leq c_\kappa(10) r_\varepsilon^{2m-2} \|(h - h', k - k')\|_{(C^4,\alpha)^n \times (C^{2,\alpha})^n}
\]
The third estimate now follows from the boundedness of \( G_\delta \).

This completes the proof of the result.

Reducing \( \varepsilon_\kappa > 0 \) if necessary, we can assume that,
\[
(44) \quad \hat{\varepsilon}_\kappa r_\varepsilon^2 \leq \frac{1}{2}
\]
for all \( \varepsilon \in (0, \varepsilon_\kappa) \). Then, the estimates (40) and (41) in the above Lemma are enough to show that
\[
(\varphi, \beta, \nu) \mapsto \mathcal{N}(\varepsilon, h, k; \varphi, \beta)
\]
is a contraction from
\[
\{(\varphi, \beta, \nu) \in \mathcal{F} : \|((\varphi, \beta, \nu)\|_\mathcal{F} \leq 2 c_\kappa r_\varepsilon^{2m}\}
\]
to itself and hence has a unique fixed point \((\varphi_\varepsilon, h, k, \beta_\varepsilon, h, k, \nu_\varepsilon, h, k)\) in this set. This fixed point is a solution of (37) on \( M_{r_\varepsilon} \) and hence provides a constant scalar curvature Kähler form on \( M_{r_\varepsilon} \).

To summarize, we have obtained the :

**Proposition 6.1.** Given \( \kappa > 0 \), there exists \( \hat{\varepsilon}_\kappa > 0 \) and \( \varepsilon_\kappa > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_\kappa) \), for all \( h_j \in C^{4,\alpha}(\partial B_{r_j}) \) and all \( k_j \in C^{2,\alpha}(\partial B_{r_j}) \) satisfying (32), the Kähler form
\[
\omega_{\varepsilon, h, k} := \omega + i \partial \bar{\partial} \varphi_{\varepsilon, h, k} + \beta_{\varepsilon, h, k},
\]
defined on \( M_{r_\varepsilon} \), has constant scalar curvature equal to
\[
\mathbf{s}(\omega_{\varepsilon, h, k}) = \mathbf{s}(\omega) + \nu_{\varepsilon, h, k}.
\]
Moreover, \( \beta_{\varepsilon, h, k} \in D_{\rho_0} \),

\[
\| \varphi_{\varepsilon, h, k} \|_{B_{2 R_{r_\varepsilon}} - B_{r_\varepsilon}} (r_{\varepsilon}^{-r}) - H^0_{h_1, k_1} \|_{C^{4,\alpha}(B_{2 R_{r_\varepsilon}} - B_{r_\varepsilon},1)} \leq \hat{c}_\varepsilon r_{\varepsilon}^{2m+\delta},
\]

and

\[
|\nu_{\varepsilon, h, k}| \leq \hat{c}_\varepsilon r_{\varepsilon}^{2m}.\]

Using (41) and (42), and increasing \( \hat{c}_\varepsilon \) if this is necessary, one can check that

\[
\| (\varphi_{\varepsilon, h, k} - \varphi_{\varepsilon, h', k'}) \|_{B_{2 R_{r_\varepsilon}} - B_{r_\varepsilon}} (r_{\varepsilon}^{-r}) - H^0_{h_j - h'_j, k_j - k'_j} \|_{C^{4,\alpha}(B_{2 R_{r_\varepsilon}} - B_{r_\varepsilon},1)} \leq \hat{c}_\varepsilon r_{\varepsilon}^{2m+\delta} \| (h - h', k - k') \|_{(C^{4,\alpha})^n \times (C^{2,\alpha})^n},
\]

and

\[
|\nu_{\varepsilon, h, k} - \nu_{\varepsilon, h', k'}| \leq \hat{c}_\varepsilon r_{\varepsilon}^{2m-4} \| (h - h', k - k') \|_{(C^{4,\alpha})^n \times (C^{2,\alpha})^n},
\]

Indeed, if

\[
(\varphi, \beta, \nu) = N(\varepsilon, h, k; \varphi, \beta) \quad \text{and} \quad (\varphi', \beta', \nu') = N(\varepsilon, h', k'; \varphi', \beta'),
\]

we can write

\[
(\varphi' - \varphi, \beta' - \beta, \nu' - \nu) = (N(\varepsilon, h', k'; \varphi', \beta') - N(\varepsilon, h', k; \varphi, \beta)) + (N(\varepsilon, h', k'; \varphi, \beta) - N(\varepsilon, h, k; \varphi, \beta))
\]

Using (41) we get

\[
\| (\varphi' - \varphi, \beta' - \beta, \nu' - \nu) \|_F \leq 2 \| (N(\varepsilon, h', k'; \varphi, \beta) - N(\varepsilon, h, k; \varphi, \beta)) \|_F
\]

and the result follows from (42).

6.2. **Perturbation of \( \eta_j \).** Now, we would like to perturb the Kähler form on \( N_{j,R_0} \) into some infinite dimensional family of constant scalar curvature Kähler forms which are parameterized by their scalar curvature and the boundary data of their potentials.

We consider the perturbed Kähler form

\[
\tilde{\eta}_j = \eta_j + i \partial \bar{\partial} \varphi,
\]

The scalar curvature of \( \tilde{\eta}_j \) is given by

\[
s(\tilde{\eta}_j) = -L_{\eta_j} \varphi + Q_{\eta_j}(\nabla^2 \varphi),
\]

since the scalar curvature of \( \eta_j \) is identically equal to 0. Again, the structure of the nonlinear operator \( Q_{\eta_j} \) is quite complicated but, in \( C_{j,R_0} \), it enjoys a decomposition similar to the one described in (42). Indeed, using (38), (39), (40) and (41), we see that we can decompose

\[
Q_{\eta_j}(\nabla^2 \varphi) = \sum_q B_{q,4,2}(\nabla^4 \varphi, \nabla^2 \varphi) C_{q,4,2}(\nabla^2 \varphi) + \sum_q B_{q,3,3}(\nabla^3 \varphi, \nabla^3 \varphi) C_{q,3,3}(\nabla^2 \varphi) + \sum_q |u|^{-2m} B_{q,3,2}(\nabla^3 \varphi, \nabla^2 \varphi) C_{q,3,2}(\nabla^2 \varphi) + \sum_q |u|^{-2m} B_{q,2,2}(\nabla^2 \varphi, \nabla^2 \varphi) C_{q,2,2}(\nabla^2 \varphi)
\]
where the sum over \( q \) is finite, the operators \((U, V) \rightarrow B_{q,a,b}(U, V)\) are bilinear in the entries and have coefficients which are bounded functions in \(C^{0,\alpha}(\tilde{C}_j,R_0)\). The nonlinear operators \(W \rightarrow C_{q,a,b}(W)\) have Taylor expansion (with respect to \(W\)) whose coefficients are bounded functions on \(C^{0,\alpha}(\tilde{C}_j,R_0)\). Even though these operators do depend on \(j\) we have not made this dependence apparent in the notation.

We would like to solve the equation
\[
s(\tilde{\eta}_j) = \varepsilon^2 \nu
\]
in \(N_{j,R_\varepsilon}\), where \(\nu \in \mathbb{R}\) and where we recall that
\[
R_\varepsilon := \frac{r_\varepsilon}{\varepsilon}.
\]

We fix a constant \(\kappa > 0\) large enough and assume that we are given \(\nu \in \mathbb{R}\) and boundary data \(h \in C^{4,\alpha}(\partial B_{\Gamma_j})\) and \(k \in C^{2,\alpha}(\partial B_{\Gamma_j})\) satisfying
\[
|\nu| \leq |s(\omega)| + 1, \quad \|h\|_{C^{4,\alpha}(\partial B_{\Gamma_j})} \leq \kappa R_\varepsilon^{4-2m}, \quad \|k\|_{C^{2,\alpha}(\partial B_{\Gamma_j})} \leq \kappa R_\varepsilon^{4-2m}.
\]
We decompose \(h = h(0) + h^\perp\) where \(h(0)\) is a constant function and \(h^\perp\) has mean 0 on \(\partial B_{\Gamma_j}\), and we define
\[
\tilde{H}_{h,k} := \tilde{\chi}_j H_{h,k}(\cdot/\varepsilon) + h(0)
\]
(51)
where \(\tilde{\chi}_j\) is a cutoff function which is identically equal to 1 in \(C^{2,\alpha}(\tilde{C}_j,2R_0)\) and identically equal to 0 in \(N_{j,R_0}\).

Replacing in (47) the function \(\varphi\) by \(\tilde{H}_{h,k} + \varphi\), we see that (46) can be written as
\[
\mathbb{L}_{g_j}(\tilde{H}_{h,k} + \varphi) = Q_{g_j}(\nabla^2 \tilde{H}_{h,k} + \varphi) - \varepsilon^2 \nu,
\]
which we would like to solve in \(N_{j,R_\varepsilon}\). Here \(\varphi \in C^{4,\alpha}_\delta(N_j)\) for some \(\delta \in \mathbb{R}\) has to be determined.

**Definition 6.2.** Given \(\bar{R} > 2R_0\), \(k \in \mathbb{N}\), \(\alpha \in (0,1)\) and \(\delta \in \mathbb{R}\), the weighted space \(C^{k,\alpha}_\delta(N_j,\bar{R})\) is defined to be the space of functions \(\varphi \in C^{k,\alpha}(N_j,\bar{R})\) endowed with the norm
\[
\|\varphi\|_{C^{k,\alpha}_\delta(N_j,\bar{R})} := \|\varphi\|_{C^{k,\alpha}(N_j,2R_0)} + \sup_{2R_0 \leq R \leq \bar{R}} R^{-\delta} \|\varphi\|_{C^{k,\alpha}(\tilde{C}_j,R_0)\setminus C_j,R_0}(R)\|_{C^{k,\alpha}(\tilde{B}_{j,1/2} - B_{j,1/2})}
\]
For each \(\bar{R} \geq 2R_0\), will be convenient to define an "extension" (linear) operator \(\tilde{\mathcal{E}}_{\bar{R}} : C^{0,\alpha}_{\delta'}(N_j,\bar{R}) \rightarrow C^{0,\alpha}_{\delta'}(N_j)\), as follows:

(i) In \(N_{j,R_0}\), \(\tilde{\mathcal{E}}_{\bar{R}}(\psi) = \psi\),
(ii) in \( C_j, \tilde{R} \rightarrow C_{j,2\tilde{R}} \)

\[
\tilde{E}_R(\psi)(u) = \frac{2 \tilde{R} - |u|}{R} \psi \left( \frac{\tilde{R}}{|u|} u \right),
\]

(iii) in \( C_{2\tilde{R}}, \tilde{E}_R(\psi) = 0. \)

It is easy to check that there exists a constant \( c = c(\delta') > 0, \) independent of \( \tilde{R} \geq 2R_0, \) such that

\[
\| \tilde{E}_R(\psi) \|_{C^0,\alpha(N_j)} \leq c \| \psi \|_{C^0,\alpha(N_j,\tilde{R})}.
\]

We fix \( \delta \in (0,1) \)

The equation we would like to solve can be rewritten as

\[
\tilde{L}_\delta \varphi = \tilde{E}_R \left( Q_{g_j}(\tilde{H}_{h,k} + \varphi) - \mathbb{L}_{g_j} \tilde{H}_{h,k} - \varepsilon^2 \nu \right).
\]

where \( \varphi \in C^4,\alpha(N_j) \) has to be determined. Observe that any solution of (54) is a solution of (51). Again, we make use of the analysis of Section 6.2 in order to find \( \tilde{G}_\delta \) a right inverse for the operator \( \tilde{L}_\delta \) and rephrase the solvability of (54) as a fixed point problem.

\[
\varphi = \tilde{N}_j(\varepsilon, h, k, \nu; \varphi)
\]

where the nonlinear operator \( \tilde{N} \) is defined by

\[
\tilde{N}(\varepsilon, h, k, \nu; \varphi) := \tilde{G}_\delta \left( \tilde{E}_R \left( Q_{g_j}(\tilde{H}_{h,k}) - \mathbb{L}_{g_j} \tilde{H}_{h,k} - \varepsilon^2 \nu \right) \right)
\]

To keep notations short, it will be convenient to define

\[
\tilde{F} := C^4,\alpha(N_j)
\]

We first estimate the terms on the right hand side of (55) when \( \varphi = 0 \) and next show that \( \tilde{N} \) is a contraction from a suitable small ball in \( \tilde{F}. \) This is the content of the:

**Lemma 6.2.** There exists \( c > 0 \) (independent of \( \kappa \)), \( \tilde{c}_\kappa = \tilde{c}(\kappa) > 0 \) and there exists \( \varepsilon_\kappa = \varepsilon(\kappa) > 0 \) such that, for all \( \varepsilon \in (0,\varepsilon_\kappa) \)

\[
\| \tilde{N}(\varepsilon, h, k, \nu; 0) \|_{\tilde{F}} \leq c R^{4-2m-\delta}_\varepsilon,
\]

Moreover, for all \( \varphi, \varphi' \in \tilde{F}, \) satisfying

\[
\| \varphi \|_{\tilde{F}} \leq 2 c R^{4-2m-\delta}_\varepsilon \quad \| \varphi' \|_{\tilde{F}} \leq 2 c R^{4-2m-\delta}_\varepsilon,
\]

we have

\[
\| \tilde{N}(\varepsilon, h, k, \nu; \varphi) - \tilde{N}(\varepsilon, h, k, \nu; \varphi') \|_{\tilde{F}} \leq \tilde{c}_\kappa R^{4-2m-\delta}_\varepsilon \| \varphi - \varphi' \|_{\tilde{F}},
\]

and

\[
\| \tilde{N}(\varepsilon, h, k, \nu; \varphi) - \tilde{N}(\varepsilon, h', k', \nu'; \varphi) \|_{\tilde{F}} \leq \tilde{c}_\kappa (R^{-1}_\varepsilon \| (h-h', k-k') \|_{C^4,\alpha \times C^2,\alpha} + R^{4-2m-\delta}_\varepsilon |\nu' - \nu|)
\]

provided \( h, h' \) and \( k, k' \) satisfy (50).
**Proof:** The proof is identical to the proof of Lemma 6.1. We give details about the derivation of the first estimate and leave the two other estimates to the reader.

It follows from the analysis of Section 5.3, together with (50) that

\[(59) \| \nabla^2 \tilde{H}_{h,k} \|_{C_2,\alpha}(N_j, R_\epsilon) \leq c(1) \kappa R_\epsilon^2 - 2m \epsilon \]

and also that

\[(60) \| \nabla^2 \tilde{H}_{h,k} \|_{C_2,\alpha}(\bar{C}_j, 2R_\epsilon - C_j, R_\epsilon) \leq c(1) \kappa R_\epsilon^3 - 2m \epsilon \]

We use the fact that, in \(C_j, 2R_\epsilon - C_j, R_\epsilon\), we can write

\(L_{g_j} H_{h,k} = (L_{g_j} - \frac{1}{2} \Delta_0^2) \tilde{H}_{h,k}\).

Then, (14)-(15) together with (59) yields

\(\| L_{g_j} \tilde{H}_{h,k} \|_{C_0,\alpha}(\bar{N}_j, R_\epsilon) \leq cR_\epsilon^3 - 2m \epsilon \)

Next, we use the structure of \(Q_{g_j}\) together with (59) to estimate

\(\| \tilde{E}_{R_\epsilon}(Q_{g_j}(\tilde{H}_{h,k})) \|_{C_4,\alpha}(N_j) \leq c(2) R_\epsilon^6 - 4m \epsilon \)

Finally, we estimate

\(\| \tilde{E}_{R_\epsilon}(\epsilon^2 \nu) \|_{C_4,\alpha}(N_j) \leq \tilde{c} R_\epsilon^{4-2m-\delta} \)

for some constant \(\tilde{c} > 0\) which does not depend on \(\epsilon\) since \(|\nu| \leq 1 + |s(\omega)|\). This completes the proof of the estimate. \(\square\)

Reducing \(\epsilon_\kappa > 0\) if necessary, we can assume that,

\[(61) \tilde{c}_\kappa R_\epsilon^{4-2m-\delta} \leq \frac{1}{2} \]

for all \(\epsilon \in (0, \epsilon_\kappa)\). Then, the estimates (59) and (57) in the above Lemma are enough to show that

\[\varphi \mapsto \tilde{N}(\epsilon, h, k, \nu; \varphi)\]

is a contraction from

\[\{ \varphi \in \tilde{F} : \| \varphi \|_{\tilde{F}} \leq 2c R_\epsilon^{4-2m-\delta} \},\]

into itself and hence has a unique fixed point \(\tilde{\varphi}_{\epsilon, h, k, \nu}\) in this set. This fixed point is a solution of (52) in \(N_j, R_\epsilon\) and hence provides a constant scalar curvature Kähler form on \(N_j, R_\epsilon\).

We have obtained the:

**Proposition 6.2.** There exist \(c > 0\) (independent of \(\kappa\)) and \(\epsilon_\kappa = \epsilon(\kappa) > 0\) such that, for all \(\epsilon \in (0, \epsilon_\kappa)\), for all \(h \in C^4,\alpha(\partial B_{j,1})\) and \(k \in C^2,\alpha(\partial B_{j,1})\) and \(\nu \in \mathbb{R}\) satisfying (44), the Kähler form

\[\eta_{h,k,\nu} := \eta_j + i \partial \bar{\partial} \varphi_{h,k,\nu}\]

defined on \(N_j, R_\epsilon\), has constant scalar curvature equal to \(\epsilon^2 \nu\). Moreover

\[\| \varphi_{h,k,\nu} C_{j, R_\epsilon/2} - C_{j, R_\epsilon} (R_\epsilon) - H_{h,k} \|_{C_4,\alpha(B_{j,1}-B_{j,1/2})} \leq c R_\epsilon^{4-2m},\]
for some constant $c > 0$ independent of $\kappa$ and $\nu$.

The important fact is that the last estimate involves a constant times $R_c^{4-2m}$ where the constant does not depend on $\kappa$ provided $\varepsilon \in (0, \varepsilon_\kappa)$.

Using (57) and (58) and increasing $\hat{c}_\kappa$ if necessary, one checks that
\begin{equation}
\|(\hat{\varphi}_{h,k,\nu} - \hat{\varphi}_{h',k',\nu'})|\partial B_{t,1/2} - c_{1,1,\varepsilon} (R_{t,1} - H^t_{h,k,\nu} - H^t_{h',k',\nu})\|_{C^{4,\alpha}(\bar{B}_{j,1} - B_{j,1/2})} 
\leq \hat{c}_\kappa (R_c^{4-1} \|(h - h', k - k')\|_{C^{4,\alpha} \times C^{2,\alpha}} + R_c^{4-2m} |\nu - \nu'|).
\end{equation}

6.3. Cauchy data matching: the proof of Theorem 1.3

Building on the analysis of the previous sections we complete the proof of Theorem 1.3.

Granted the results of Proposition 6.1 and Proposition 6.2 it remains to explain how to choose $h := (h_1, \ldots, h_n), \ k := (k_0, \ldots, k_n)$ satisfying (54) and $\hat{h} := (\hat{h}_1, \ldots, \hat{h}_n), \ \hat{k} := (\hat{k}_1, \ldots, \hat{k}_n)$ satisfying (53) in such a way that, for each $j = 1, \ldots, n$, the function $\psi_j := (\varphi_j + \hat{\varphi}_{h,k}) (R_{t,1})$, defined in $\bar{B}_{j,2} - B_{j,1}$ (see Proposition 6.1) on the one hand, and for $\nu := s(\omega_{t,h,k})$ the function $\psi^j := e^2 (\hat{\varphi}_j + \hat{\varphi}_{\hat{h}_j,\hat{k}_j,\alpha(\omega_{t,h,k})}) (R_{t,1})$, defined in $\bar{B}_{j,1} - B_{j,1/2}$ (see Proposition 6.1) on the other hand, have their partial derivatives up to order $3$ which coincide on $\partial B_{j,1}$.

In any case, our aim is now to solve the following system of equations
\begin{equation}
\psi_j^0 = \psi_j^1, \quad \partial_r \psi_j^0 = \partial_r \psi_j^1, \quad \Delta_0 \psi_j^0 = \Delta_0 \psi_j^1, \quad \partial_r \Delta_0 \psi_j^0 = \partial_r \Delta_0 \psi_j^1,
\end{equation}
on $\partial B_{j,1}$ where $r = |v|$ and $v = (v^1, \ldots, v^m)$ are coordinates in $B_{j,2}$.

Let us assume that we have already solved this problem. The first identity in (63) implies that $\psi_j^0$ and $\psi_j^1$ as well as all their $k$-th order partial derivatives with respect any vector field tangent to $\partial B_{j,1}$, with $k \leq 4$, agree on $\partial B_{j,1}$. The second identity in (63) then shows that $\partial_r \psi_j^0$ and $\partial_r \psi_j^1$ as well as all their $k$-th order partial derivatives with respect any vector field tangent to $\partial B_{j,1}$, with $k \leq 3$, agree on $\partial B_{j,1}$. Using the decomposition of the Laplacian in polar coordinates, it is easy to check that the third identity implies that $\partial_r^2 \psi_j^0$ and $\partial_r^2 \psi_j^1$ as well as all their $k$-th order partial derivatives with respect any vector field tangent to $\partial B_{j,1}$, with $k \leq 2$, agree on $\partial B_{j,1}$. And finally, the last identity in (63) implies that $\partial_r^3 \psi_j^0$ and $\partial_r^3 \psi_j^1$ as well as all their first order partial derivative with respect any vector field tangent to $\partial B_{j,1}$, agree on $\partial B_{j,1}$.

Moreover, the Kähler form
$$i \partial \bar{\partial} (\frac{1}{2} |v|^2 + \psi_j^0),$$
defined in $B_{j,2} - B_{j,1}$ and the Kähler form

$$i \partial \overline{\partial} \left( \frac{1}{2} |v|^2 + \psi_j \right),$$

defined in $B_{j,1} - B_{j,1/2}$, both have the same constant scalar curvature equal to $s(\omega_{\epsilon,h,k})$. This then implies that any $k$-th order partial derivatives of the functions $\psi_o^j$ and $\psi_i^j$, with $k \leq 4$, coincide on $\partial B_{j,1}$.

Therefore, we conclude that the function $\psi$ defined by $\psi := \psi_o^j$ in $B_{j,2} - B_{j,1}$ and $\psi := \psi_i^j$ in $B_{j,1} - B_{j,1/2}$ is $C^4$ in $B_{j,2} - B_{j,1/2}$ and is a solution of the nonlinear elliptic partial differential equation

$$s \left( i \partial \overline{\partial} \left( \frac{1}{2} |v|^2 + \psi \right) \right) = s(\omega_{\epsilon,h,k}) = cte.$$

It then follows from elliptic regularity theory together with a bootstrap argument that the function $\psi$ is in fact smooth. Hence, by gluing the Kähler metrics $\omega_{h,k}$ and $\omega_{h_j,k_j}$ on the different pieces constituting $M_{\epsilon}$, we have produced a Kähler metric on $M_{\epsilon}$ which has constant scalar curvature. This will end the proof of the Theorem 1.3.

**Remark 6.1.** In dimension 2, a slight modification is due since the functions involve some log terms. In view of (15) and (36), we consider the function $\tilde{\psi}_i^j$ defined by

$$\tilde{\psi}_i^j := \epsilon^2 \left( \tilde{\phi}_j + \tilde{\phi}_{h_j,k_j}(\omega_{h,k}) \right) (R_{\epsilon}^{-1}) - \epsilon^2 a_j \log R_{\epsilon} + \frac{k(0)}{2} \log r_{\epsilon}$$

There is no loss of generality in doing so since changing the potential by some constant function does not alter the corresponding Kähler forms.

It remains to explain how to find the boundary data

$$h = (h_1, \ldots, h_n), \; \; k = (k_1, \ldots, k_n), \; \; \tilde{h} = (\tilde{h}_1, \ldots, \tilde{h}_n) \; \; \text{and} \; \; \tilde{k} = (\tilde{k}_1, \ldots, \tilde{k}_n)$$

We will make use of the following result:

**Lemma 6.3.** Assume that $\Gamma$ is a discrete subgroup of $U(m)$ acting freely on $\mathbb{C}^m - \{0\}$. The mapping

$$\mathcal{P} : \mathcal{C}^{4,\alpha}(\partial B^r) \times \mathcal{C}^{2,\alpha}(\partial B^r) \rightarrow \mathcal{C}^{3,\alpha}(\partial B^r) \times \mathcal{C}^{1,\alpha}(\partial B^r)$$

$$(h,k) \rightarrow (\partial_r (H^o_{h,k} - H^o_{h,k}), \partial_r \Delta_0 (H^i_{h,k} - H^i_{h,k})),$$

is an isomorphism.

**Proof:** There are many different ways to prove this result for example. Let us concentrate on the case where $m \geq 3$ since the case $m = 2$ is essentially the same. We use the formulas (25) and (26) to compute

$$\partial_r (H^i_{h,k} - H^o_{h,k}) = \sum_{\gamma=0}^{\infty} 2(\gamma + m - 1) \left( h^{(\gamma)} + \frac{k^{(\gamma)}}{m(\gamma + m - 2)} \right) e_{\gamma}$$
and
\[ \partial \tau \Delta_0(H^t_{h,k} - H^o_{h,k}) = \sum_{\gamma=0}^{\infty} 2(\gamma + m - 1)k^{(\gamma)} e_\gamma \]

It is then easy to see that
\[
P : W^{4,2}(\partial B_T) \times W^{2,2}(\partial B_T) \rightarrow W^{3,2}(\partial B_T) \times W^{1,2}(\partial B_T)
\]
\[ (h, k) \mapsto (\partial \tau (H^t_{h,k} - H^o_{h,k}), \partial \tau \Delta_0(H^t_{h,k} - H^o_{h,k})) \]
is well defined and invertible. Recall that the norm in \( W^{\ell,2}(\partial B_T) \) can be taken to be
\[
\|f\|_{W^{\ell,2}} = \left( \sum_{\gamma=0}^{\infty} (1 + \gamma)^{2\ell} |f(\gamma)|^2 \right)^{1/2}
\]
whenever the function \( f \) is decomposed as
\[
f = \sum_{\gamma=1}^{\infty} f(\gamma)e_\gamma.
\]
Elliptic regularity theory then implies that the same result is true when the operator is defined between Hölder spaces.

It will be convenient to observe that \( \psi^o \) satisfies
\[
\|\psi^o_j - H_{h_j,k_j}\|_{C^4(B_{r_j} - B_{r_j/2})} \leq c r_\epsilon^4,
\]
and also that
\[
\|\psi^i - \varepsilon^2 \tilde{H}_{h_j,k_j}\|_{C^4(B_{r_j} - B_{r_j/2})} \leq c \varepsilon^2 P_{\epsilon}^{4-2n} = c r_\epsilon^4,
\]
for some constant \( c > 0 \) which does not depend on \( \kappa \), provided \( \varepsilon \) is chosen small enough, say \( \varepsilon \in (0, \varepsilon_\kappa) \). These two estimates follow at once from the estimates in Proposition 6.1, Proposition 6.2 and also from the choice of \( r_\epsilon \).

We use the following notations for the rescaled boundary data
\[
(h', k', \tilde{h'}, \tilde{k}') := (h, k, \varepsilon^2 \tilde{h}, \varepsilon^2 \tilde{k}).
\]
Using Lemma 6.3 the solvability of (63) reduces to a fixed point problem which can be written as
\[
(h', \tilde{h'}, k', \tilde{k'}) = S_\epsilon(h', \tilde{h'}, k', \tilde{k'}),
\]
and we know from (64) and (65) that the nonlinear operator \( S_\epsilon \) satisfies
\[
\|S_\epsilon(h', \tilde{h'}, k', \tilde{k'})\|_{(C^{4,\alpha})^n \times (C^{2,\alpha})^n} \leq c_0 r_\epsilon^4,
\]
for some constant \( c_0 > 0 \) which does not depend on \( \kappa \), provided \( \varepsilon \in (0, \varepsilon_\kappa) \). We finally choose
\[
\kappa = 2 c_0,
\]
and \( \varepsilon \in (0, \varepsilon_\kappa) \). We have therefore proved that \( S_\epsilon \) is a map from
\[
A_\epsilon := \left\{ (h', \tilde{h'}, k', \tilde{k'}) \in (C^{4,\alpha})^n \times (C^{2,\alpha})^n \ : \ \|(h', \tilde{h'}, k', \tilde{k'})\|_{(C^{4,\alpha})^n \times (C^{2,\alpha})^n} \leq \kappa r_\epsilon^4 \right\},
\]
into itself. It follows from (45), (46) and (62) that, reducing $\varepsilon$ if this is necessary, $S_\varepsilon$ is a contraction mapping from $A_\varepsilon$ into itself for all $\varepsilon \in (0, \varepsilon_\kappa)$. Therefore, $S_\varepsilon$ has a fixed point in this set. This completes the proof of the existence of a solution of (63).

The proof of the existence on $M_{r_\varepsilon}$ of a Kähler form $\omega_\varepsilon$ which has constant scalar curvature is therefore complete. Observe that the scalar curvature of $\omega$ and $\omega_\varepsilon$ are close since the estimate $|s(\omega_\varepsilon) - s(\omega)| \leq cr_\varepsilon^2 m$ follows directly from the construction.

7. Refined asymptotics for ALE spaces

Let us now describe in detail $(N, \eta)$, the blow up at the origin of $\mathbb{C}^m$ endowed with the Burns-Calabi-Simanca metric. Away from the exceptional divisor, the Kähler form $\eta$ is given by

$$\eta = i \partial \bar{\partial} A_m(|v|^2)$$

where $v = (v_1, \ldots, v_n)$ are complex coordinates in $\mathbb{C}^m - \{0\}$ and where the function $s \mapsto A_m(s)$ is a solution of the ordinary differential equation

$$s^2 (s \partial_s A_m)^{m-1} \partial_s^2 A_m + (m-1) s \partial_s A_m - (m-2) = 0$$

which satisfies $A_m \sim \log s$ near 0. We refer to [53] for a derivation of this equation. It turns out that, when $m = 2$, the function $A_2$ is explicitly given by

$$A_2(s) = \log s + \lambda s$$

where $\lambda > 0$, while in dimension $m \geq 3$, even though there is no explicit formula for $A_m$ we have the simple:

**Lemma 7.1.** Assume that $m \geq 3$. Then the function $A_m$ can be expanded as

$$A_m(s) = \lambda s - \lambda^{2-m} s^{2-m} + O(s^{1-m})$$

for $s > 1$, where $\lambda > 0$.

**Proof:** Define the function $\zeta$ by $s \zeta := s \partial_s A_m - 1$. A direct computation shows that $\zeta$ solves

$$(1 + s \zeta)^{m-1} s^2 \partial_s^2 \zeta = (1 + s \zeta)^{m-1} - 1 - (m-1) s \zeta$$

If in addition we take $\zeta(0) = 1$, then $\partial_s \zeta$ remains positive and one can check that $\zeta$ is well defined for all time and converges to some positive constant $\lambda$, as $s$ tends to $\infty$. This immediately implies that $s \partial_s A_m = \lambda s + O(1)$ at infinity. The expansion then follows easily.

Changing variables $u := \sqrt{2\lambda} v$, we see from the previous Lemma that the Kähler form $\eta$ can be expanded near infinity as

$$(66) \quad \eta = i \partial \bar{\partial} (\frac{1}{2} |u|^2 + \log |u|^2),$$

in dimension $m = 2$ and as

$$(67) \quad \eta = i \partial \bar{\partial} (\frac{1}{2} |u|^2 - 2^{m-2} |u|^{4-2m} + O(|u|^{3-2m})), $$

in dimension $m \geq 3$. 
We now recall the following results of Joyce [26] (which is a Corollary of his Theorem 8.2.3 in our notations):

**Theorem 7.1.** Given \( \Gamma \) a finite subgroup of \( SU(m) \) acting freely on \( \mathbb{C}^m - \{0\} \) and \( \pi: X \to \mathbb{C}^m / \Gamma \) a Kähler crepant resolution of \( \mathbb{C}^m / \Gamma \). Then there exists a Ricci-flat Kähler metric \( \omega \) such that

\[
(\pi^{-1})^* \omega = i \partial \bar{\partial} \left( \frac{1}{2} |u|^2 + \tilde{\varphi}(u) \right)
\]

outside a compact neighborhood of \( \pi^{-1}(0) \).

Moreover

\[
\tilde{\varphi}(u) = |u|^{2-2m} + O(|u|^\gamma)
\]

for some \( \gamma \in (1 - 2m, 2 - 2m) \).

We end this section by a proof of Remark 3.1.

**Lemma 7.2.** Assume we are given a potential \( \varphi \) defined on \( C_\Gamma \) such that \( \varphi \in C^4_{2-\gamma}(\mathcal{C}_\Gamma) \), for some \( \gamma > 0 \). Further assume that

\[
\eta := i \partial \bar{\partial} \left( \frac{1}{2} |u|^2 + \varphi \right)
\]

is a zero scalar curvature Kähler form. Then, the function \( \varphi \) can be expanded as

\[
\varphi = a \cdot u + b + c |u|^{4-2m} + O(|u|^{3-2m}),
\]

when \( m \geq 3 \) and as

\[
\varphi = a \cdot u + b + c \log |u| + O(|u|^{-1}),
\]

when \( m = 2 \). Here \( a \in \mathbb{C} \) and \( b \in \mathbb{R} \). In particular, the potential \( \tilde{\varphi} := \varphi - a \cdot u - b \) satisfies

\[
\eta := i \partial \bar{\partial} \left( \frac{1}{2} |u|^2 + \tilde{\varphi} \right).
\]

**Proof:** The key point is that, since \( \eta \) has zero scalar curvature, the potential \( \varphi \) is a solution of some nonlinear fourth order elliptic differential equation and satisfies some \( a \) priori bound. It is then possible to get "refined asymptotics" for the potential \( \varphi \) in the spirit of what has been done in [33] for constant scalar curvature metrics. These refined asymptotics are obtained by using a bootstrap argument in Hölder weighted spaces.

Using (19), we see that the scalar curvature of \( \eta \) can be expanded in powers of \( \varphi \) as

\[
s(\omega) = \frac{1}{2} \Delta_0^2 \varphi + Q_{g_{ewc}}(\nabla^2 \varphi),
\]

where the nonlinear operator \( Q_{g_{ewc}} \) collects all the nonlinear terms. We shall now be more specific about the structure of \( Q_{g_{ewc}} \). Indeed, it follows from the explicit computation of the Ricci curvature that the nonlinear operator \( Q_{g_{ewc}} \) can be decomposed as

\[
Q_{g_{ewc}}(\nabla^2 \varphi) = \sum_q B_{q,4,2}(\nabla^4 \varphi) C_{q,4,2}(\nabla^2 \varphi) + \sum_q B_{q,3,3}(\nabla^3 \varphi) C_{q,3,3}(\nabla^2 \varphi),
\]

where the sum over \( q \) is finite, the operators \( (U, V) \to B_{g,a,b}(U, V) \) are bilinear in the entries and have coefficients which are bounded functions in \( C^{0,\alpha}(\mathcal{C}_\Gamma) \). The nonlinear operators \( W \to \)
$C_{q,a,b}(W)$ have Taylor expansion (with respect to $W$) whose coefficients are bounded functions on $C^{0,\alpha}(\mathring{C}_T)$.

If we assume that $\varphi \in C^{4,\alpha}_{2-\gamma}(\mathring{C}_T)$, then we see that

$$Q_{g_{eucl}}(\nabla^2 \varphi) \in C^{0,\alpha}_{-2-2\gamma}(\mathring{C}_T)$$

Therefore, $\Delta^2 \varphi \in C^{0,\alpha}_{-2-2\gamma}(\mathring{C}_T)$.

Now, if $\Delta^2 \varphi \in C^{0,\alpha}_{-4-4}(\mathring{C}_T)$ and $\varphi \in C^{4,\alpha}_{2-\gamma}(\mathring{C}_T)$ for some $\gamma > 0$ then, depending on the value of $\gamma'$, the following alternative hold [43] :

(i) If $\gamma' \in (1,2)$, then $\varphi \in C^{4,\alpha}_{\gamma}(\mathring{C}_T)$.

(ii) If $\gamma' \in (0,1)$, then $\varphi \in C^{4,\alpha}_{\gamma}(\mathring{C}_T) \oplus \{ u \mapsto a \cdot u : a \in \mathbb{C} \}$.

(iii) If $m \geq 3$ and $\gamma' \in (4-2m,0)$, then $\varphi \in C^{4,\alpha}_{\gamma}(\mathring{C}_T) \oplus \{ u \mapsto a \cdot u : a \in \mathbb{C} \} \oplus \mathbb{R}$.

(iv) If $m \geq 3$ and $\gamma' \in (3-2m,4-2m)$, then $\varphi \in C^{4,\alpha}_{\gamma}(\mathring{C}_T) \oplus \{ u \mapsto a \cdot u : a \in \mathbb{C} \} \oplus \mathbb{R} \oplus \text{Span}\{ u \mapsto |u|^{4-2m} \}$.

(v) If $m = 2$ and $\gamma' \in (-1,0)$, then $\varphi \in C^{4,\alpha}_{\gamma}(\mathring{C}_T) \oplus \{ u \mapsto a \cdot u : a \in \mathbb{C} \} \oplus \mathbb{R} \oplus \text{Span}\{ u \mapsto \log |u| \}$.

Using these together with a bootstrap argument, we conclude that (69) and (70) hold. The result then follows by taking $\tilde{\varphi} := \varphi - a \cdot u - b$. \qed

8. Applications, examples and comments

Blow up of smooth manifolds: Theorem 1.1 follows at once from Theorem 1.3 and the analysis of Lemma 7.1 by taking $(N_j,\eta_j) = (N,a_j \eta)$ where $(N,\eta)$ is the Blow up at the origine of $\mathbb{C}^m$ endowed with the Burns-Calabi-Simanca metric and $a_j > 0$. Observe that the points of blow up $p_1, \ldots, p_n$ and the coefficients $a_1, \ldots, a_n$ are parameters of our construction.

A first natural question is to which base smooth manifolds can Theorem 1.1 be applied! Here, we do not make a comprehensive list but we highlight some large class of manifolds:

(i) All the Kähler-Einstein manifold with discrete automorphism group. This means any manifold with negative first Chern class and many families of examples of positive first Chern class we know of [35], [15], [4]. We should note that there are no Kähler-Einstein manifolds Futaki nondegenerate except the ones with discrete automorphisms as observed by LeBrun-Simanca [30].

(ii) Most of the zero scalar curvature Kähler surfaces which have been proved by Kim, LeBrun Pontecorvo, Rollin and Singer [38], [33], [28], [27] to admit such constant scalar curvature metric. In particular any blow up of a non Ricci-flat Kähler surface whose integral of the scalar curvature is non-negative has blow ups which admit zero scalar curvature Kähler
metrics. Of course if the number of blow ups is sufficiently large no continuous families of automorphisms survive and we can then apply Theorem 1.3.

(iii) Note that also flat tori of any dimension can be used as base manifolds, since, despite the presence of continuous automorphisms, there are no nonzero holomorphic vector fields vanishing somewhere. Their first Chern class being zero, Corollary 1.2 does not apply.

(iv) Some important classes of manifolds on which there are constant scalar curvature Kähler metrics have been provided by Fine [19]. Indeed, he has proved existence of Kähler constant scalar curvature metrics on complex surfaces with a holomorphic submersion onto a Riemann surface Σ with smooth fibres of genus at least two. If the genus of Σ is larger than or equal to 2, the automorphism group is indeed discrete.

(v) Another family of examples of constant scalar curvature Kähler manifolds with discrete automorphism group has been given by Hong [24], [25]. These are ruled manifolds given by the projectivization of some vector bundles over constant scalar curvature Kähler manifolds.

(vi) In [30], LeBrun and Simanca gave examples (and strategies to construct new ones) of Futaki nondegenerate manifolds with constant scalar curvature Kähler metrics.

(vii) Recall that the space of holomorphic vector fields on a blow manifold is isomorphic to the space of those holomorphic vector fields on the base manifold vanishing at the blow up points. Hence our procedure applied to any of the nondegenerate manifold above gives new nondegenerate manifolds (with constant scalar curvature by our result), so our procedure can be iterated.

(viii) Riemannian products of nondegenerate Kähler manifolds of constant scalar curvature is again a nondegenerate Kähler manifold of constant scalar curvature. By taking factors with scalar curvature of different signs and scaling one can then produce also on the blow ups Kähler metrics of any nonzero scalar curvature.

In addition, to any of the above examples, one can apply LeBrun-Simanca’s implicit function argument [30] to get open subset of the Kähler cone of fixed complex manifolds and also open subset of moduli of variations of complex structures for which constant scalar curvature Kähler metrics exist, providing a wealth of new examples.

**Zero scalar curvature examples. Proof of Corollary 1.1:** Let us now focus on the effect of our construction on the size of the scalar curvature when we blow up smooth points. Let us denote by π the standard projection from the blow up manifold \( \tilde{M} \) to the base manifold \( M \). To this aim let us recall that the average of the scalar curvature of a Kähler metric is a cohomological number given by

\[
s(\omega) = \frac{mc_1(M) \cup [\omega]^{m-1}([M])}{[\omega]^m ([M])}.
\]
Our gluing procedure constructs on \( \tilde{M} \) metrics in the Kähler classes
\[
[\omega_\varepsilon] = \pi^* [\omega] - \varepsilon^2 (a_1 PD[E_1] + \ldots + a_n PD[E_n])
\]
while the first Chern class behaves like
\[
c_1(\tilde{M}) = \pi^*(c_1(M)) - (m-1)(PD[E_1] + \ldots + PD[E_n]).
\]
Recalling (see [23] page 475) that for any \( j = 1, \ldots, n, \)
\[
(PD[E_j])^m[\tilde{M}] = (-1)^{m-1},
\]
we get
\[
(c_1(\tilde{M})) \cup ([\omega_\varepsilon])^{m-1}([\tilde{M}]) = (c_1(M) \cup [\omega]^{m-1})[M] - \varepsilon^{2m-2}(m-1)(\sum_{j=1}^n a_j)
\]
and
\[
[\omega_\varepsilon]^m([\tilde{M}]) = [\omega]^m([M]) + (-1)^{m-1}\varepsilon^{2m}(\sum_{j=1}^n a_j)
\]
The scalar curvature of this metric is hence given by
\[
s(\omega_\varepsilon) = m\frac{(c_1(M) \cup [\omega]^{m-1})([M]) - \varepsilon^{2m-2}(m-1)(\sum_{j=1}^n a_j)}{[\omega]^m([M]) + (-1)^{m-1}\varepsilon^{2m}(\sum_{j=1}^n a_j)}.
\]
It is easily seen that, since \( a_j > 0 \), this gives a decreasing function of \( \varepsilon \), for \( \varepsilon \) close to 0 (and of course it gives the old scalar curvature for \( \varepsilon = 0 \)).

The direct application of Theorem 1.1 would then give small negative scalar curvature if \( (M, \omega) \) had zero scalar curvature. Nonetheless changing the Kähler class we can bypass this problem provided the first Chern class of the base orbifold is nonzero, prescribing the scalar curvature to vanish in the gluing procedure.

**Corollary 8.1.** Any blow up (at a finite set of smooth points) of a compact smooth Kähler manifold (or orbifold) of zero scalar curvature of discrete type with nonzero first Chern class, has a Kähler metric of zero constant scalar curvature.

**Proof:** Let us denote by \( \omega(0) \) the zero scalar curvature Kähler metric on the base manifold \( M \), and by \( \rho \) the harmonic representative of the first Chern class \( c_1(M) \) (hence non zero by our assumption). LeBrun-Simanca have proved ([27], Corollary 1) that, if the first Chern class is nonzero, the automorphism group is discrete and for \( |t| \) sufficiently small (say \( t \in [-t_0, t_0] \)), each Kähler class \( [\omega(0)] - t\rho \) contains a metric \( \omega(t) \) of constant scalar curvature and this constant is positive for \( t > 0 \) and negative for \( t < 0 \). Moreover \( \omega(t) \) depends continuously on \( t \).

We can apply Theorem 1.3 to the continuous family of Kähler forms \( \omega(t) \). Given \( t \in [-t_0, t_0] \), this yields the existence of \( \varepsilon_0(t) > 0 \) and a family of Kähler metrics \( \omega(t, \varepsilon) \) of constant scalar curvature for all \( \varepsilon \in (0, \varepsilon_0(t)) \). It turns out that, the constant \( \varepsilon_0(t) \) are uniformly bounded from below by some positive constant \( \varepsilon_0 > 0 \) since \( \varepsilon_0(t) \) only depends on the \( C^{2,\alpha} \) norm of the coefficients of the Kähler form \( \omega(t) \) and these are uniformly bounded as \( t \in [-t_0, t_0] \). We claim that, reducing \( \varepsilon_0 \) if this is necessary, \( \omega(t, \varepsilon) \) depends continuously on \( t \). This follows easily from the
fact that the Kähler forms on the blown up manifold are obtained by solving nonlinear problems using a fixed point theorems for a contraction mapping. Therefore, they depend continuously on any of the parameters of our construction such as the Kähler class, the parameter \(\varepsilon\), the points which are blown up, the coefficients \(a_j > 0\), \ldots

Let us then look at the family of constant scalar curvature metrics \(\omega(t, \varepsilon)\). We known that, reducing \(\varepsilon_0\) if necessary, \(\omega(-t_0, \varepsilon)\) has constant negative scalar curvature while \(\omega(t_0, \varepsilon)\) has positive scalar curvature, for all \(\varepsilon \in (0, \varepsilon_0)\). Moreover \(s(\omega(t, \varepsilon))\) depends continuously on \(t\). Therefore, for each \(\varepsilon \in (0, \varepsilon_0)\), there exists \(t_\varepsilon \in [-t_0, t_0]\) such that \(s(\omega(t_\varepsilon, \varepsilon)) = 0\) as claimed. \(\square\)

Note that the above Corollary can be applied to most of the examples described above in (ii) and (viii).

Desingularization of orbifolds: More delicate is the situation for singular orbifolds since few examples even of Kähler-Einstein orbifolds are known. As mentioned in the introduction, the clearest picture is in the complex dimension 2 and 3, where, thanks to the work of Kronheimer \[33\] and Joyce \[26\] we know how to handle SU\(^m\) singular points. We summarize this in the following

**Corollary 8.2.** Let \((M, \omega)\) be a nondegenerate compact \(m\)-dimensional constant scalar curvature Kähler orbifold with \(m = 2\) or \(3\) and isolated singularities. Let \(p_1, \ldots, p_n \in M\) be any set of points with a neighborhood biholomorphic to a neighborhood of the origin in \(\mathbb{C}^m/\Gamma_j\), where \(\Gamma_j\) is a finite subgroup of SU\((m)\). Let further \(N_j\) be a Kähler crepant resolution of \(\mathbb{C}^m/\Gamma_j\) (which always exists, see \[7\] for \(m = 2\) and \[52\] for \(m = 3\)), and \(\eta_j\) given by Theorem 7.1.

Then there exists \(\varepsilon_0 > 0\), such that, for all \(\varepsilon \in (0, \varepsilon_0)\), there exists a constant scalar curvature Kähler form \(\omega_\varepsilon\) on \(M \sqcup p_1, \varepsilon N_1 \sqcup p_2, \varepsilon N_2 \cdots \sqcup p_n, \varepsilon N_n\).

Moreover,

(i) if \(\omega\) had positive (resp. negative) scalar curvature then \(\omega_\varepsilon\) has positive (resp. negative) scalar curvature,

(ii) if \(c_1(M) \neq 0\) and \(\omega_M\) has zero scalar curvature then \(\omega_\varepsilon\) can be chosen to have zero scalar curvature too.

The range of applicability of Corollary 8.2 is very large, even if we look just at Kähler-Einstein orbifolds of non positive scalar curvature thanks to Aubin-Yau’s solution of the Calabi conjecture (which holds in the orbifold category). In fact we can use it to prove the following general result mentioned in the introduction

**Corollary 8.3.** Any compact complex surface of general type admits constant scalar curvature Kähler metrics.

The proof of the above result requires some notions from algebraic geometry which can be found for example in \[7\] and which we quickly recall for reader’s convenience.
First of all a complex surface $M$ is called minimal if it does not contain a smooth rational curve of self-intersection $-1$. A fundamental result in complex surface theory (the Enriques-Castelnuovo Criterion, see e.g. [23] page 476) says that any such curve is in fact the exceptional divisor of a blow up at a smooth point of a smooth surface. Moreover one can apply the above procedure (“blowing down”) a finite number of times to be left with a minimal surface uniquely defined, hence called the minimal model $\bar{M}$ of $M$.

From a different perspective one can study an algebraic surface by looking at its images into projective spaces, via maps given by evaluating holomorphic sections of line bundles as in the celebrated Kodaira’s embedding theorem. In particular, if $K_M$ is the canonical line bundle of $M$, one has (rational) maps $\phi_{K_M^\otimes k}$ from $M$ into $\mathbb{P}(H^0(M, K_M^\otimes k))$. These in general may not be defined at points which annihilates all holomorphic sections of $K_M^\otimes k$, but for minimal surfaces of general type they are indeed globally defined holomorphic maps for $k \geq 5$ (see e.g. [7], page 220).

A complex surface $M$ is said to be of general type if $\dim(\phi_{K_M^\otimes k}(M)) = 2$ for $k$ large enough. If $M$ is a minimal surface of general type Kodaira [30] proved that $\phi_{K_M^\otimes k}$ is an embedding away from smooth rational curves of self-intersection $-2$, and Brieskorn [9] has proved that the image of these curves are isolated singular points of the image surface with local structure groups $\Gamma_j$, with $\Gamma_j \subset SU(2)$. We are now in position to give the proof of the above corollary.

Proof of Corollary 8.3: Let us first assume $M$ is a minimal complex surface of general type and suppose $k$ is chosen big enough to guarantee that the image of the pluricanonical rational map $\phi_{K_M^\otimes k}$ is an embedding away from the set of $(-2)$-curves of $M$, which get collapsed to points, giving the singularities of $\phi_{K_M^\otimes k}$.

Kobayashi [29] has proved that $\phi_{K_M^\otimes k}$ has a Kähler-Einstein orbifold metric of negative scalar curvature, extending Aubin’s proof of the Calabi conjecture. Moreover $c_1(M) < 0$ implies, as in the smooth case, the existence of only a discrete group of automorphisms.

As already observed, being the structure groups of the singularities in $SU(2)$, we have an ALE local model with the required decay at infinity. We can then apply Theorem 1.3. The complex manifold produced by our gluing construction is easily seen to be minimal, hence getting a constant negative scalar curvature Kähler metric on the minimal resolution $\bar{Y}$ of $\bar{M}$. But $\bar{M}$ is already a minimal model of $\bar{M}$, therefore the minimal model of $M$, and so $M$ is in fact $\bar{Y}$ proving our result.

If $M$ is not minimal, we apply the previous discussion to its minimal model $\bar{Y}$, which is a complex surface with discrete automorphism group to get a Kähler constant negative scalar curvature metric. Recalling that $\bar{M}$ is obtained from $\bar{Y}$ applying a finite number of blow ups, Theorem 1.3 (possibly applied more than once in case one needs to blow up at a point on the exceptional divisor of the previous blow up, and of course blowing up preserves the property of having only discrete automorphism groups) gives the conclusion.

Going back to the problem of resolving singularities in the Kähler constant scalar curvature setting, in dimension greater than 3 only few examples can be dealt at the moment.
Other types of singularities which can be dealt with are, for example, those locally modeled on $\mathbb{C}^m/\mathbb{Z}_m$, where $\mathbb{Z}_m$ acts diagonally on $\mathbb{C}^m$, by multiplication by a fixed the $m$-th root of unity $\zeta = e^{\frac{2\pi i}{m}}$. Putting $r = (|z|^2 + \cdots + |z|^m)^{1/2}$, Calabi \[11\] defined a Kähler potential on $X \setminus \{\text{exceptional divisor}\}$ by

$$\varphi = (r^{2m} + 1)^{\frac{1}{m}} + \frac{1}{m} \sum_{j=0}^{m-1} \zeta^j \log((r^{2m} + 1)^{\frac{1}{m}} - \zeta^j) .$$

We can then observe that $\eta = \frac{i}{2} \partial \bar{\partial} \varphi$ is indeed a Kähler form which extends through the exceptional divisor and is ALE, Ricci flat, asymptotic to $\mathbb{C}^m/\mathbb{Z}_m$. We can then glue $(X, \eta)$ to any smooth Kähler orbifold $(M, \omega)$ of constant scalar curvature provided the Futaki obstruction $s$ vanishes as described in Section 4.

The above example has been recently generalized by Rollin-Singer \[51\]. They have shown that if $G = \{1, \lambda, \ldots, \lambda^{k-1}\}$, $\lambda = e^{\frac{2\pi i}{k}}$, then $\mathbb{C}^m/G$ has an ALE scalar flat (in general not Ricci-flat) Kähler resolution whose metric decays at infinity of order $2 - 2m$.

These last examples can be used to produce compact orbifolds by taking global quotients of some of the smooth manifolds described in the first section (e.g. tori or Kähler-Einstein manifolds with negative first Chern class or with positive first Chern class and discrete automorphism group containing a group as above).

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