Cayley automatic representations of wreath products

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We construct the representations of Cayley graphs of wreath products using finite automata, pushdown automata and nested stack automata. These representations are in accordance with the notion of Cayley automatic groups introduced by Kharlampovich, Khoussainov and Miasnikov and its extensions introduced by Elder and Taback. We obtain the upper and lower bounds for a length of an element of a wreath product in terms of the representations constructed.

Keywords: Finite automata; pushdown automata; nested stack automata; Cayley graphs; wreath products.

1. Introduction

In this paper we study representations of Cayley graphs of wreath products of groups using finite automata, pushdown automata and nested stack automata. The representations considered in this paper are related to the notion of Cayley automatic groups that was introduced in [16]. The notion of Cayley automatic groups was introduced as a natural generalization of automatic groups in the sense of Thurston [9].

The set of Cayley automatic groups properly contains the set of automatic groups. In addition, the set of Cayley automatic groups includes finitely generated nilpotent groups of class at most two, the lamplighter group [16], and the Baumslag–Solitar groups [3]. Cayley automatic groups retain some nice properties of automatic groups. They are closed under direct product, free product and finite extensions. The word problem in Cayley automatic groups is decidable in quadratic time.

We assume that the reader is familiar with the notions of finite automaton and regular language. Let $\Sigma$ be a finite alphabet. Put $\Sigma_0 = \Sigma \cup \{\diamond\}$, where $\diamond \notin \Sigma$. The convolution of two words $w_1, w_2 \in \Sigma^*$ is the string $w_1 \odot w_2$ of length $\max\{|w_1|, |w_2|\}$ over the alphabet $\Sigma_0 \times \Sigma_0$ defined as follows. The $k$th symbol of the string is $(\sigma_1, \sigma_2)$, where $\sigma_i, i = 1, 2$ is the $k$th symbol of $w_i$ if $k \leq |w_i|$ and $\diamond$ otherwise. The convolution $\odot R$ of a binary relation $R \subseteq \Sigma^* \times \Sigma^*$ is $\odot R = \{w_1 \odot w_2 | (w_1, w_2) \in R\}$. We say that a binary relation $R \subseteq \Sigma_0 \times \Sigma_0$ is automatic if there exists a finite automaton in the alphabet $\Sigma_0 \times \Sigma_0$ that accepts $\odot R$. Such an automaton is called two–tape synchronous finite automaton.
Let $G$ be a group generated by a subset $S \subset G$. Consider the labeled and directed Cayley graph $\Gamma(G, S)$. We say that $G$ is Cayley automatic if some regular language $L \subset \Sigma^*$ over some alphabet $\Sigma$ uniquely represents elements of $G$ such that for every $s \in S$ the binary relation which is the set of directed edges of $\Gamma(G, S)$ labeled by $s$ is automatic. That is to say, the group $G$ is Cayley automatic if the labeled digraph $\Gamma(G, S)$ is an automatic structure in terms of [17].

In [7], Elder and Taback considered the extensions of the notion of Cayley automatic groups replacing the regular languages by more powerful languages. We denote by $C$ a class of languages; for example, it can be the class of regular languages, context–free languages or context–sensitive languages. The notion of Cayley automatic groups can be extended as follows.

**Definition 1.** Let $G$ be a group. We say that $G$ is $C$–Cayley automatic if there exists a subset $S \subset G$ generating $G$ for which the following properties hold:

- There exists a bijection $\psi : L \rightarrow G$ between a language $L \subset \Sigma^*$ from the class $C$ and the group $G$;
- For each $h \in S$ the language $L_h = \{w_1 \otimes w_2 | w_1, w_2 \in L, \psi(w_1)h = \psi(w_2)\}$ is in the class $C$.

In this paper all groups $G$ are finitely generated and generating sets $S$ are finite. If $C$ is the class of regular languages then $C$–Cayley automatic groups are Cayley automatic groups. The notion of Cayley automatic groups is invariant under the choice of generators. Recall that for context–free and indexed languages are the ones that are recognizable by the pushdown automata and nested stack automata, respectively. Notice that for context–free and indexed Cayley automatic groups Definition [3] depends on the choice of generators. In terms of [7], the group $G$, the subset $S$ and the finite alphabet $\Sigma$ in Definition [3] form a $C$–graph automatic triple.

The known results on $C$ Cayley automatic representations of groups are as follows. In [16], Kharlampovich, Khoussainov and Miasnikov constructed the Cayley automatic representations for finitely generated nilpotent groups of class at most two. In [3], we constructed the Cayley automatic representations for all Baumslag–Solitar groups. In [7], Elder and Taback constructed the deterministic non–blind 2–counter Cayley automatic representation for the countably generated free group. In [8], they constructed the deterministic non–blind 1–counter Cayley automatic representation for Thompson’s group $F$.

In this paper we consider the cases when $C$ is the class of regular languages, context–free languages and indexed languages. We assume that the reader is familiar with the notion of pushdown automata and context–free languages, and also nested stack automata and indexed languages. For the corresponding definitions we refer to [12],[14].

In this paper we construct the Cayley automatic representation for wreath products $G \wr \mathbb{Z}$, the context–free Cayley automatic representation for wreath products $G \wr F_n$ and the indexed Cayley automatic representation for the wreath product
In each case we specify the set of generators for wreath products with respect to which we consider their Cayley graphs. For the representations constructed we prove the inequalities of the form:

$$\lambda|w| + \mu \leq |g| \leq \xi|w| + \delta,$$

where $|g|$ is the length of a group element $g$ with respect to chosen set of generators and $|w|$ is the length of the word $w$ which is the representative of $g$, i.e., $\psi(w) = g$.

In [2], Baumslag, Shapiro and Short introduced the notion of parallel poly-pushdown groups. The definition of poly-pushdown groups uses two-tape asynchronous automata instead of synchronous ones used in Definition 1. They have shown that the set of poly-pushdown groups is closed under wreath product. This implies that all wreath products considered in this paper are parallel poly-pushdown groups.

For wreath products there is an abundance of results on quantitative characteristics such as growth rate [18], isoperimetric profiles [11] and drift of simple random walks [6, 10]. This makes studying representations of Cayley graphs of wreath products relevant to seeking connections between characteristics of groups and the computational power of automata that are sufficient to represent their Cayley graphs. In this paper we focus on the aforementioned classes of languages, i.e., regular, context-free and indexed.

The paper is organized as follows. In Section 2 we briefly recall the definitions and notation for wreath products of groups. In Section 3 we construct the Cayley automatic representations for wreath products $G \wr \mathbb{Z}$ and show the inequalities of the form (1) for them. In Section 4 we construct the context-free Cayley automatic representation for wreath products $G \wr F_n$ and show the inequalities of the form (1) for them. In Section 5 we construct the indexed Cayley automatic representation for the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}^2$.

2. Wreath products of groups: definitions and notation

Recall the definition of the restricted wreath product $A \wr B$. For more details on wreath products see, e.g., [13]. Given two groups $A$ and $B$, we denote by $A(B)$ the set of all functions $B \to A$ having finite supports. Recall that a function $f : B \to A$ has finite support if $f(x) \neq e$ for only finite number of $x \in B$, where $e$ is the identity of $A$. Given $f \in A(B)$ and $b \in B$, we define $f^b \in A(B)$ as follows. Put $f^b(x) = f(bx)$ for all $x \in B$. The group $A \wr B$ is the set product $B \times A(B)$ with the group multiplication given by $(b, f) \cdot (b', f') = (bb', ff')$.

For our purposes we use the converse order for representing the elements of a wreath product. Namely, we represent an element of $A \wr B$ as a pair $(f, b)$, where $f \in A(B)$ and $b \in B$. For such a representation, the group multiplication is given by $(f, b) \cdot (f', b') = (ff'^{-1}, bb')$.

There exist natural embeddings $B \to A \wr B$ and $A(B) \to A \wr B$ mapping $b$ to $(e, b)$ and $f$ to $(f, e)$ respectively, where $e$ is the identity of $A(B)$ and $e$ is the identity...
of $B$. For the sake of simplicity, we will identify $B$ and $A^{(B)}$ with the corresponding subgroups of $A \wr B$.

Recall that, according to [4], the wreath product $A \wr B$ of two finitely presented groups $A$ and $B$ is finitely presented iff either $A$ is the trivial group or $B$ is finite. Therefore, the wreath products $G \wr \mathbb{Z}$, $G \wr F_n$ and $\mathbb{Z}_2 \wr \mathbb{Z}^2$ considered here are not finitely presented. In particular, these groups are not automatic [9].

3. The wreath products of groups with the infinite cyclic group

We denote by $a$ the generator of $\mathbb{Z} = \langle a \rangle$, and by $h$ the nontrivial element of $\mathbb{Z}_2$. We consider the Cayley graph of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ with respect to the generators $a$ and $h$. The automatic presentations for the Cayley graph of $\mathbb{Z}_2 \wr \mathbb{Z}$ with respect to the generators $a$ and $h$ were constructed in [10] and [3]. In Theorem 2 below we modify the Cayley automatic representation used in [3] (see Theorem 4). This will enable us to get a simple proof of the inequalities of the form (1).

**Theorem 2.** There exists a Cayley automatic representation $\psi : L \rightarrow \mathbb{Z}_2 \wr \mathbb{Z}$ of the lamplighter group such that for every $g \in \mathbb{Z}_2 \wr \mathbb{Z}$ and its representative $w = \psi^{-1}(g)$ the inequalities $|w| - 1 \leq |g| \leq 3|w| - 2$ hold.

**Proof.** Recall that an element of $\mathbb{Z}_2 \wr \mathbb{Z}$ is a pair $(f, z)$, where $f$ is a function $f : \mathbb{Z} \rightarrow \mathbb{Z}_2$ that has finite support and $z \in \mathbb{Z}$ is the position of the lamplighter. In order to present $f(i) \in \mathbb{Z}_2$ we use the symbols 0 and 1: 0 means that a lamp in the position $z = i$ is unlit, i.e., $f(i) = e$; 1 means that the lamp is lit, i.e., $f(i) = h$. By abuse of notation we will write $f(i) = 0$ instead of $f(i) = e$ and $f(i) = 1$ instead of $f(i) = h$. Recall that only a finite number of lamps are lit for each element of the group $\mathbb{Z}_2 \wr \mathbb{Z}$. To show the position of the origin $0 \in \mathbb{Z}$ we use the symbols $A_0$ and $A_1$ if the lamp in the origin is unlit and lit respectively. To show the position of the lamplighter we use the symbols $C_0$ and $C_1$ if the lamp in the position of the lamplighter is unlit and lit respectively. In the case the lamplighter is at the origin we use the symbols $B_0$ and $B_1$.

Given an element $(f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}$, let $m$ be the smallest $i \in \mathbb{Z}$ such that $f(i) = 1$; if $f(i) = 0$ for all $i \in \mathbb{Z}$, then put $m = 0$. Put $\ell = \min\{m, z, 0\}$. Let $n$ be the largest $j$ such that $f(j) = 1$; if $f(j) = 0$ for all $j \in \mathbb{Z}$, then put $n = 0$. Put $r = \max\{n, z, 0\}$. Let us represent $(f, z)$ as follows:

$$f(\ell)f(\ell + 1) \ldots f(-1)f(0)f(1) \ldots f(z - 1)f(z + 1) \ldots f(r - 1)f(r),$$

here $z > 0$ is assumed; if $z < 0$, then $C_{f(z)}$ will appear on the left of $A_{f(0)}$. In the case $z = 0$ the word representing an element $(f, z)$ is

$$f(\ell)f(\ell + 1) \ldots f(-1)f(0)f(1) \ldots f(r - 1)f(r).$$

It can be observed that the language of the words representing all elements $(f, z)$ of the group $\mathbb{Z}_2 \wr \mathbb{Z}$ is regular. Let an element $g = (f, z)$ be represented by a word [2].
Writing the words representing \( g \) and \( ga = (f, z + 1) \) one under another we have
\[
\begin{align*}
    f(\ell) \cdots A_{f(0)} \cdots f(z-1) C_{f(z)} f(z+1) \cdots f(r) \\
    f(\ell) \cdots A_{f(0)} \cdots f(z-1) f(z) C_{f(z+1)} \cdots f(r).
\end{align*}
\]
(4)
The other cases are considered similarly. From (4) it is clear that the relation \( \{(g, ga) | g \in \mathbb{Z}_2 \} \) is recognized by a synchronous two-tape finite automaton.

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\end{align*}
\]
(5)
where \( \overline{f(z)} = 1 - f(z) \). The other cases are considered similarly. From (5) it is clear that the relation \( \{(g, gh) | g \in \mathbb{Z}_2 \} \) is recognized by a synchronous two-tape finite automaton.

Let us prove the inequalities \( |w| - 1 \leq |g| \leq 3|w| - 2 \). For the automatic representation constructed the length of a word \( w \) representing an element \( g = (f, z) \) is
\[
|w| = |r - \ell| + 1 = |\max\{n, z, 0\} - \min\{m, z, 0\}| + 1 = \max\{|n - m|, |n|, |m|, |n - z|, |m - z|, |z|\} + 1.
\]
(6)
For a given \( f : \mathbb{Z} \to \mathbb{Z}_2 \), we denote by \( \#\text{supp} f \) the cardinality of the set \( \text{supp} f = \{j | f(j) = h\} \). First we show that the word length of \( g \) with respect to the generators \( a \) and \( h \) is
\[
|g| = \#\text{supp} f + \min\{2 \max\{-m, 0\} + \max\{n, 0\} + |z - \max\{n, 0\}|,
    2 \max\{n, 0\} + \max\{-m, 0\} + |z + \max\{-m, 0\}|\}.
\]
(7)
By (5), the left–first and the right–first normal forms of \( g \) are
\[
a_{i_1} \cdots a_{i_p} a_{-j_1} \cdots a_{-j_q} a^z,
\]

\[
a_{-j_1} \cdots a_{-j_q} a_{i_1} \cdots a_{i_p} a^z,
\]
where \( i_p = n \) (if \( n \geq 0 \)), \( j_q = -m \) (if \( m \leq -1 \)), \( i_p > \cdots > i_1 \geq 0 \), \( j_q > \cdots > j_1 > 0 \) and \( a_k = a^k ha^{-k} \). It is proved [5, Proposition 3.6] that the word length of \( g \) with respect to the generators \( a \) and \( h \) is
\[
|g| = p + q + \min\{2j_q + i_p + |z - i_p|, 2i_p + j_q + |z + j_q|\}.
\]
Let us express \( |g| \) in terms of \( m \leq n \) for the three different cases:

- \( m \leq -1 \) and \( n \geq 0 \): \( |g| = p + q + \min\{-2m + n + |z - n|, 2n - m + |z - n|\} \),
- \( m \geq 0 \): \( |g| = p + q + n + |z - n| \),
- \( n \leq -1 \): \( |g| = p + q + m + |z - m| \).

It can be seen that \( \#\text{supp} f = p + q \). Therefore, we obtain (7).

Let us prove the inequality \( |g| \leq 3|w| - 2 \). By (6), \( |w| \geq n - m + 1 \). Therefore, \( |w| \geq \#\text{supp} f \). Consider each of the three cases: \( m \leq -1 < 0 \leq n \), \( n \leq -1 \) and \( 0 \leq m \) separately.
• The case \( m \leq -1 < 0 \leq n \). If \( z \geq n \), then we have: \(-2m + n + |z - n| = -2m + z \leq 2(z - m)\). If \( z \leq m \), then we have: \(2n - m + |z - m| = 2n - m + m - z \leq 2(n - z)\). If \( m < z < n \), then we have: \(-2m + n + |z - n| = 2(n - m) - z \) and \( 2n - m + |z - m| = 2(n - m) + z \). Therefore, by (1):

\[
\min\{-2m+n+|z-n|,2n-m+|z-m|\} \leq 2(|w|-1).
\]

Therefore, \(|g| \leq 3|w| - 2\).

• The case \( m \geq 0 \). By (6) we have: \( n + |z - n| \leq 2(|w| - 1) \). Thus, \(|g| \leq 3|w| - 2\).

• The case \( n \leq -1 \). By (6) we have: \(-m + |z - m| \leq 2(|w| - 1) \). Therefore, \(|g| \leq 3|w| - 2\).

Let us prove the inequality \(|w| - 1 \leq |g|\). The identity \( e \in \mathbb{Z}_2 \wr \mathbb{Z} \) is represented by the word \( B_0 \). Therefore, the inequality holds for \( g = e \). Suppose that the inequality holds for some \( g \in \mathbb{Z}_2 \wr \mathbb{Z} \). For the Cayley automatic representation constructed the length of the word representing \( gh \) equals \(|w|\), and the lengths of the words representing \( ga \) and \( ga^{-1} \) are equal to either \(|w|\), \(|w| + 1\) or \(|w| - 1\). This implies that the inequality holds for the elements \( gh, ga \) and \( ga^{-1} \). Thus, it holds for all \( g \in \mathbb{Z}_2 \wr \mathbb{Z} \). It can be verified that both bounds \(|w| - 1 \leq |g| \leq 3|w| - 2\) are reached for an infinite number of elements \( g \in \mathbb{Z}_2 \wr \mathbb{Z} \).

Let \( G \) be a Cayley automatic group and \( \psi_G : L_G \to G \) be a Cayley automatic representation of \( G \). We assume that the empty word \( e \notin L_G \). Let \( \{g_1, \ldots, g_n\} \subset G \) be a finite set generating \( G \). We consider the Cayley graph of \( G \wr \mathbb{Z} \) with respect to the generators \( g_1, \ldots, g_n \) and \( a \). Put \( d_j, j = 1, \ldots, n \) to be the maximum number of the padding symbols \( \circ \) in the convolutions \( \psi_G^{-1}(g) \otimes \psi_G^{-1}(gg_j) \). Put \( K_0 = |\psi_G^{-1}(e)| \), where \( e \in G \) is the identity. Put \( K = \max\{K_0, d_j \mid j \in [1, n]\} \). Theorem 2 can be generalized, using essentially the same technique, to the following result.

**Theorem 3.** There exists a Cayley automatic representation \( \psi : L \to G \wr \mathbb{Z} \) of the group \( G \wr \mathbb{Z} \) such that for every \( g \in G \wr \mathbb{Z} \) and its representative \( w = \psi^{-1}(g) \) the inequality \( \frac{1}{K}|w| - \frac{K^2}{K} \leq |g| \) holds. Suppose that the inequality \(|g| \leq C|\psi_G^{-1}(g)| + D \) holds for all \( g \in G \), where \( C > 0 \) and \( D \geq 0 \). Then for every \( g \in G \wr \mathbb{Z} \) and its representative \( w \) the inequality \(|g| \leq (C + D + 2)|w| - 2 \) holds.

**Proof.** For simplicity we may always assume that \( L_G \subset \{0, 1\}^* \). We introduce two counterparts of the symbols 0 and 1: \( \mathbb{0} \) and \( \mathbb{1} \), respectively, which specify the beginning of a word. In order to specify the position of the origin \( z = 0 \) we use the symbols \( A_0 \) and \( A_1 \) depending on whether the word that represents the element of \( G \) at \( z = 0 \) has 0 and 1 as the first letter. Similarly, we use the symbols \( C_0 \) and \( C_1 \) to specify the position of the lamplighter \( z \). The symbols \( B_0 \) and \( B_1 \) are used if \( z = 0 \).

Let us show two simple examples. Take an element \((f, 1) \in G \wr \mathbb{Z} \) such that \( f(j) = e \) for \( j \notin [-1, 2] \) and \( f(-1) \neq e \), \( f(0) \), \( f(1) \) and \( f(2) \neq e \) are represented by the words \( 011, 1001, 01 \) and \( 1111 \) respectively. Then the element \((f, 1)\) is represented by the word \( Q114_1001C_01111 \). Take an element \((f, 0) \in G \wr \mathbb{Z} \) such that \( f(j) = e \) for \( j \notin [-1, 1] \) and \( f(-1) \neq e \), \( f(0) \) and \( f(1) \neq e \) are represented by the words \( 111, 000 \).
and 01 respectively. Then the element \((f, 0)\) is represented by the word \(111B_00001\).

By abuse of notation, we denote by \(f(j)\) the word representing the group element \(f(j) \in G\) for which the first letter \(\sigma\) is changed to the underlined one \(\sigma\). We denote by \(A_{f(0)}\), \(B_{f(0)}\) and \(C_{f(z)}\) the corresponding words for which the first letter \(\sigma\) is changed to \(A_\sigma\), \(B_\sigma\) and \(C_\sigma\), respectively.

Let \((f, z) \in G \wr \mathbb{Z}\). The numbers \(\ell\) and \(r\) have the same meaning as in the proof of Theorem 2. Similar to (2) and (3) an element \((f, z)\) is represented by the word \(f(\ell) \ldots A_{f(0)} \ldots C_{f(z)} \ldots f(r)\) and an element \((f, 0)\) is represented by the word \(f(\ell) \ldots B_{f(0)} \ldots \) \(f(r)\). It is clear that the relation \(\{(g, g0) \mid g \in G \wr \mathbb{Z}\}\) is recognizable by a two-tape synchronous finite automaton. Since \(G\) is Cayley automatic, for every \(j = 1, \ldots, n\) the relation \(\{(g, gg_j) \mid g \in G \wr \mathbb{Z}\}\) is recognizable by a two-tape synchronous finite automaton. The inequalities \(\frac{1}{K} |w| - \frac{1}{3} \leq |g| / |g| \leq (C + D + 2)|w| - 2\) can be obtained using the same technique as in Theorem 2.

**Remark 4.** Suppose that the representation \(\psi_G : L_G \to G\) is Cayley biautomatic (see [10] for the definition of Cayley biautomatic groups). It can be verified that then the representation \(\psi : L \to G \wr \mathbb{Z}\) constructed in Theorem 3 is Cayley biautomatic.

4. The wreath products of groups with a free group

We denote by \(a\) and \(b\) the generators of the free group \(F_2 = \langle a, b \rangle\), and by \(h\) the nontrivial element of \(\mathbb{Z}_2\). We consider the Cayley graph of the wreath product \(\mathbb{Z}_2 \wr F_2\) with respect to the generators \(a, b\) and \(h\). Recall that an element of \(\mathbb{Z}_2 \wr F_2\) is a pair \((f, z)\), where \(f\) is a function \(f : F_2 \to \mathbb{Z}_2\) that has finite support and \(z \in F_2\) is the position of the lamplighter. We have the following theorem.

**Theorem 5.** There exists a context-free Cayley automatic representation \(\psi : L \to \mathbb{Z}_2 \wr F_2\) of the group \(\mathbb{Z}_2 \wr F_2\) such that for every \(g \in \mathbb{Z}_2 \wr F_2\) and its representative \(w = \psi^{-1}(g)\) the inequalities \(\frac{1}{3} |w| - \frac{1}{3} \leq |g| \leq 3|w| - 2\) hold.

**Proof.** In order to construct a context-free Cayley automatic representation of \(\mathbb{Z}_2 \wr F_2\) we extend the Cayley automatic representation of \(\mathbb{Z}_2 \wr \mathbb{Z}\) obtained in Theorem 2. In the Cayley automatic representation of \(\mathbb{Z}_2 \wr \mathbb{Z}\) we use the symbols \(0, 1, A_0, A_1, B_0, B_1, C_0, C_1\). In the context-free Cayley automatic representation of \(\mathbb{Z}_2 \wr F_2\) we use the brackets \((, )\) and \([, ]\). Along with the symbols \(0\) and \(1\) we use \(D_0, E_0\) and \(D_1, E_1\). Along with the symbols \(A_0, A_1, B_0, B_1, C_0, C_1\) we use \(D_0^A, D_1^A, D_0^B, D_1^B, D_0^C, D_1^C, E_0^C, E_1^C\). The symbols \(A_0, A_1, D_0^A, D_1^A\) are used to show the position of the origin \(e \in F_2\). The symbols \(C_0, C_1, D_0^C, D_1^C, E_0^C, E_1^C\) are used to show the position of the lamplighter \(z \in F_2\). The symbols \(B_0, B_1, D_0^B, D_1^B\) are used if the lamplighter is at the origin. See also the meaning of the symbols \(A_0, A_1, B_0, B_1, C_0, C_1\) in Theorem 2.

We say that a symbol is an \(A\), \(B\), \(C\), \(D\) and \(E\)-symbol if it belongs to the set \(\{A_0, A_1, D_0^A, D_1^A\}, \{B_0, B_1, D_0^B, D_1^B\}, \{C_0, C_1, D_0^C, D_1^C, E_0^C, E_1^C\}, \{1, 0\}, \)
\{D_0, D_1, D_0^A, D_1^A, D_0^B, D_1^B, D_0^C, D_1^C\} and \{E_0, E_1, E_0^C, E_1^C\}, respectively. We say that a symbol is basic if it belongs to the set \{0, 1, A_0, A_1, B_0, B_1, C_0, C_1\}.

For a given \(s \in F_2\), denote by \(r(s) \in \{a, a^{-1}, b, b^{-1}\}^*\) the reduced word representing \(s\). We denote by \(F_a\) the set of all group elements \(s \in F_2\) for which \(r(s) = aw\) or \(r(s) = a^{-1}w, w \in \{a, a^{-1}, b, b^{-1}\}^*\). We denote by \(F_b\) the set of all group elements \(s \in F_2\) for which \(r(s) = bw\) or \(r(s) = b^{-1}w, w \in \{a, a^{-1}, b, b^{-1}\}^*\). It is clear that \(F_2 = F_a \cup F_b \cup \{e\}\). We denote by \(H\) the subgroup of \(\mathbb{Z}_2 \wr F_2\) generated by \(a\) and \(h\).

It is clear that \(H\) is isomorphic to \(\mathbb{Z}_2 \wr \mathbb{Z}\).

For a given \((f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}\), depict it in a way shown Fig. 1. Let us consider the horizontal line going through the identity \(e \in F_2\). For a vertex \(s \in F_2\) on this line, put \(V_s = \{p \in F_b \mid f(sp) = h \vee z = sp\}\). Scan this line from the left to the right. If \(V_s = \emptyset\), then write the corresponding basic symbol (see Theorem 2). If \(V_s \neq \emptyset\), then write the corresponding \(D\)-symbol. For the element in Fig. 1 (left) we get \(11D_0^1D_1^1\). For the element in Fig. 1 (right) we get \(D_0A_1D_0^1\). If \((f, z) \in H\), then we obtain the same representative as in Theorem 2. If \((f, z) \notin H\), then \(D\)-symbols occur. In this case we continue as follows.

Take any occurrence of \(D\)-symbol. This occurrence corresponds to some vertex \(s \in F_2\). Let us consider a vertical line going through \(s\). For a vertex \(t \in F_2, t \neq s\) on this line, put \(H_t = \{p \in F_a \mid f(tp) = h \vee z = tp\}\). Insert the brackets \(\langle\rangle\) around the occurrence of a \(D\)-symbol. Scan, omitting \(s\), this line from the bottom to the top. If \(H_t = \emptyset\), then write the corresponding basic symbol inside the brackets. If \(H_t \neq \emptyset\), then write the corresponding \(E\)-symbol inside the brackets. Do it for every occurrence of a \(D\)-symbol. For the element in Fig. 1 (left) we get \(11(1E_0D_0^A)(E_0D_0E_0^1)1\). For the element in Fig. 1 (right) we get \((D_01)A_1(E_0D_0E_0^1)1\). If no \(E\)-symbols occur then we stop. If \(E\)-symbols occur, we insert the brackets \(\langle\rangle\) and \(\langle\rangle\) around each occurrence and repeat the step above for horizontal lines.

We continue this procedure until no new \(D\)- or \(E\)-symbols occur. After the procedure is finished, the result is the representative \(w = \psi^{-1}(g)\) of \(g = (f, z)\). For the element in Fig. 1 (left) the procedure of constructing the representative is \(11D_0^1D_1^1 \rightarrow 11(1E_0D_0^A)(E_0D_0E_0^1)1 \rightarrow 11(1E_0D_0^A)(E_0D_0E_0^1)1 \rightarrow 11(1E_0D_0^A)(E_0D_0E_0^1)1 \rightarrow 11(1E_0D_0^A)(E_0D_0E_0^1)1\). For the element in Fig. 1 (right) it is \(D_0A_1D_0^1 \rightarrow (D_01)A_1(E_0D_0E_0^1)1 \rightarrow (D_01)A_1(E_0D_0E_0^1)1\).

Let \(\Sigma = \{0, 1, D_0, D_1, E_0, E_1, \langle\rangle, \rangle\rangle, A_0, A_1, B_0, B_1, C_0, C_1, D_0^A, D_1^A, D_0^B, D_1^B, D_0^C, D_1^C, E_0^C, E_1^C\}\). Let \(L \subseteq \Sigma^*\) be the language of representatives \(w\) of all elements \(g \in \mathbb{Z}_2 \wr F_2\). It can be seen that the language \(L\) consists of the words satisfying the following properties.

- The configuration of brackets \(\langle\rangle, \rangle\rangle\) is balanced and, moreover, generated by the context-free grammar \(S \rightarrow SS \langle\rangle \mid \langle\rangle \mid \{T\} \mid T \rightarrow TT \mid \langle\rangle \mid \epsilon\) with the axiom \(S\).
- Each pair of matched brackets \(\langle\rangle\) is associated with a \(D\)-symbol which is placed inside these brackets, but not inside any other pair of matched
brackets between them. That is, the configuration of the subword between any two matched brackets ( and ) is \((p \ldots q \ldots r \ldots s \sigma t \ldots u \ldots v \ldots w)\), where \(\sigma\) is a \(D\)-symbol and \(p, q, r, s, t, u, v, w \in \{0, 1, C_0, C_1\}\).

- The \(D\)-symbols \(D_0^A, D_1^A, D_0^B, D_1^B\) are allowed to be associated only with a matched pair of brackets ( and ) of the first level.
- Each pair of matched brackets \(\{\text{and}\}\) is associated with an \(E\)-symbol which is placed inside these brackets but not inside any other pair of matched brackets between them. That is, the configuration of the subword between any two matched brackets \(\{\text{and}\}\) is \([p \ldots q \ldots r \ldots s \sigma t \ldots u \ldots v \ldots w]\), where \(\sigma\) is an \(E\)-symbol and \(p, q, r, s, t, u, v, w \in \{0, 1, C_0, C_1\}\).
- Each pair of matched brackets is separated by at least two symbols.
- The subwords \((0, 0), [0 \text{ and } 0]\) are not allowed.
- The symbol 0 is not allowed to be the first or the last one of a word.
- Among the symbols \(A_0, A_1, B_0, B_1, C_0, C_1, D_0^A, D_1^A, D_0^B, D_1^B, D_0^C, D_1^C, E_0^C, E_1^C\) each word of \(L\) contains either exactly one occurrence of a \(B\)-symbol and no \(A\)-symbols and \(C\)-symbols, or exactly one occurrence of an \(A\)-symbol and of a \(C\)-symbol, and no \(B\)-symbols.

It can be seen that \(L\) is recognizable by a deterministic pushdown automaton. The right multiplication by \(h\) either interchanges \(C_0\) and \(C_1\), \(D_0^B\) and \(D_1^B\), \(D_0^C\) and \(D_1^C\), or \(E_0^C\) and \(E_1^C\). Therefore, the language \(L_h = \{u \otimes v | u, v \in L, \psi(v) = \psi(u)h\}\) is clearly context-free. The right multiplication by \(a\) (or, \(b\)) moves the lamplighter by one step to the right (or, up). It is can be verified that the languages \(L_a = \)

![Fig. 1. A white box means that the value of a function \(f : F_2 \to Z_2\) is \(e \in Z_2\), a black box means that it is \(h \in Z_2\), a black disk specifies the position of the lamplighter \(z\) and tells us that the value of \(f\) is \(h\). For the element to the left the word representing it is \(11(11E_011D_0^C_0E_0^C_011D_0^C11E_1^C_0)1\), for the element to the right it is \((D_01)A_1(11E_011D_0[11E_1^C_1])1\).](image)
\{u \otimes v|u,v \in L, \psi(v) = \psi(u)a\} and \ L_b = \{u \otimes v|u,v \in L, \psi(v) = \psi(u)b\} are context–free as well.

Let us prove the inequality \(\frac{1}{2}|w| - \frac{1}{2} \leq |g|\). It is equivalent to \(|w| \leq 3|g| + 1\). The representative of \(e \in \mathbb{Z}_2 \wr \mathbb{F}_2\) is the word \(A_0\) of length 1, so the inequality holds for \(g = e\). For every \(g \in \mathbb{Z}_2 \wr \mathbb{F}_2\), the lengths of the representatives for \(g\) and \(ga\) (or \(gb\)) differ by at most 3. Therefore, the inequality \(|w| \leq 3|g| + 1\) holds for all \(g \in \mathbb{Z}_2 \wr \mathbb{F}_2\).

Let us prove by induction the inequality \(|g| \leq 3|w| - 2\). For the elements of the subgroup \(H\) the representatives are the same as in Theorem 2, so the inequality \(|g| \leq 3|w| - 2\) is satisfied for them. For the inductive step, we observe that a word \(w\) that represents any element \(g \in \mathbb{Z}_2 \wr \mathbb{F}_2\) has the form \(v_0(w_1)v_1(w_2)v_2...v_{n-1}(w_n)v_n\), where the words \(v_0, v_1, ..., v_n\) do not contain brackets. We obtain that \(|g| \leq \sum_{i=1}^{n}(3|w_i| - 2) + 3(|v_0 \ldots v_n| + n) - 2 \leq 3|w| - 2\). Therefore, the inequality \(|g| \leq 3|w| - 2\) holds for all \(g \in \mathbb{Z}_2 \wr \mathbb{F}_2\). It can be verified that both bounds \(\frac{1}{3}|w| - \frac{1}{3} \leq |g| \leq 3|w| - 2\) are reached for an infinite number of elements \(g \in \mathbb{Z}_2 \wr \mathbb{F}_2\).

Theorem 5 can be generalized, using essentially the same technique, to the following result.

**Theorem 6.** There exists a context–free Cayley automatic representation \(\psi : L \to \mathbb{Z}_2 \wr \mathbb{F}_n\) of the group \(\mathbb{Z}_2 \wr \mathbb{F}_n\), \(n \geq 2\) such that for every \(g \in \mathbb{Z}_2 \wr \mathbb{F}_n\), and its representative \(w = \psi^{-1}(g)\) the inequalities \(\frac{1}{2^n-1}|w| - \frac{1}{2^n-1} \leq |g| \leq 3|w| - 2\) hold.

Let \(\psi_G : L_G \to G\) be a context–free Cayley automaton representation of a group \(G\) with respect to a set of generators \(\{g_1, ..., g_m\} \subset G\) such that the following holds for some constant \(N\): for all \(u, v \in L_G\) such that \(\psi_G(u)g_j = \psi_G(v)\), \(j = 1, ..., m\) the inequality \(|u| - |v| | \leq N\) holds. We assume that the empty word \(\varepsilon \notin L_G\). Let \(a_1, ..., a_n\) be the generators of the free group \(F_n\). We consider the Cayley graph of \(G \wr F_n\) with respect to the generators \(g_1, ..., g_m, a_1, ..., a_n\). Put \(K_0 = |\psi_G^{-1}(e)|\), where \(e \in G\) is the identity. Put \(K = \max\{K_0 + 2(n-1), N\}\). Theorem 6 can be generalized to the following result (cf. Theorem 3).

**Theorem 7.** There exists a context–free Cayley automatic representation \(\psi : L \to G \wr F_n\) of the group \(G \wr F_n\) such that for every \(g \in G \wr F_n\), and its representative \(w = \psi^{-1}(g)\) the inequality \(\frac{1}{K}|w| - \frac{K}{K} \leq |g| \) holds. Suppose that the inequality \(|g| \leq C|\psi_G^{-1}(g)| + D\) holds for all \(g \in G\), where \(C > 0\) and \(D \geq 0\). Then for every \(g \in G \wr F_n\) and its representative \(w\) the inequality \(|g| \leq (C + D + 2)|w| - 2\) holds.

5. The wreath product \(\mathbb{Z}_2 \wr \mathbb{Z}^2\)

We denote by \(x\) and \(y\) the standard generators of \(\mathbb{Z}^2\), and by \(h\) the nontrivial element of \(\mathbb{Z}_2\). Let us consider the Cayley graph of the wreath product \(\mathbb{Z}_2 \wr \mathbb{Z}^2\) with respect to the generators \(x, y\) and \(h\). Recall that an element of \(\mathbb{Z}_2 \wr \mathbb{Z}^2\) is a pair \((f, z)\), where
$f$ is a function $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}_2$ that has finite support and $z \in \mathbb{Z}^2$ is the position of the lamplighter. We have the following theorem.

**Theorem 8.** There exists an indexed Cayley automatic representation $\psi : L \rightarrow \mathbb{Z}_2 \wr \mathbb{Z}_2$ for the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}_2$ such that $L$ is a regular language.

**Proof.** Put $\Sigma = \{0, 1, C_0, C_1\}$. Let us consider the map $t : \mathbb{N} \rightarrow \mathbb{Z}_2^2$ such that:

- $t(1) = (0, 0)$, $t(2) = (1, 0)$, $t(3) = (1, 1)$, $t(4) = (0, 1)$, $t(5) = (-1, 1)$, $t(6) = (-1, 0)$, $t(7) = (-1, -1)$, $t(8) = (0, -1)$, and etc.; the map $t : \mathbb{N} \rightarrow \mathbb{Z}_2^2$ is shown in Fig. 2.

For a given element $(f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}_2$, represent it as the word for which the $k$th symbol is 0 if $f(t(k)) = e \in \mathbb{Z}_2$, 1 if $f(t(k)) = h \in \mathbb{Z}_2$, $C_0$ if $f(t(k)) = e$ and $z = t(k)$, and $C_1$ if $f(t(k)) = h$ and $z = t(k)$. The last symbol of a word is not allowed to be 0, i.e., it should be either 1, $C_0$ or $C_1$.

It can be seen that the language $L \subseteq \Sigma^*$ of representatives of all elements $g \in \mathbb{Z}_2 \wr \mathbb{Z}_2$ is regular. Also, the language $L_h = \{w_1 \otimes w_2 | w_1, w_2 \in L, \psi(w_1)h = \psi(w_2)\}$ is regular. Let us consider the languages $L_x = \{w_1 \otimes w_2 | w_1, w_2 \in L, \psi(w_1)x = \psi(w_2)\}$ and $L_y = \{w_1 \otimes w_2 | w_1, w_2 \in L, \psi(w_1)y = \psi(w_2)\}$. We will show that $L_x$ and $L_y$ are indexed languages. Put the stack alphabet $\Xi = \{I, B, T\}$. The symbols $B$ and $T$ denote the bottom and the top of the stack, respectively. The symbol $I$ is used for all intermediate positions. Let $w \in L_x$. Consider the stack automaton $M_x$ that works as follows until it meets for the first time the letter that contains $C_0$ or $C_1$.

- Initially the stack is empty.
- $M_x$ reads off the first letter of $w$ and pushes $B$ onto the stack.

![Fig. 2](image-url)
• $M_x$ reads off the second letter of $w$ and pushes $T$ onto the stack.
• $M_x$ reads off the third letter of $w$ and goes one step down.
• $M_x$ reads off the fourth letter of $w$ and goes one step up.
• $M_x$ makes two silent moves popping $T$ and pushing $I$.
• $M_x$ reads off the fifth letter of $w$ and pushes $T$ onto the stack.
• $M_x$ reads off the sixth letter of $w$ and goes one step down.
• $M_x$ reads off the seventh letter of $w$ and goes one step down.
• $M_x$ reads off the eight letter of $w$ and goes one step up.
• The process continues by going up and down along the stack between $B$ and $T$. Each time the top is reached, it is raised up by one. This process is shown in Fig. 3.

It is easy to see the following. If a letter being read of a word $w$ at a position $m$ contains $C_0$ or $C_1$ for the first time, then either the next letter at the position $m + 1$ contains $C_0$ or $C_1$, or the letter at the position $m + (4n + 1)$ contains $C_0$ or $C_1$, where $n$ is the current height of the stack. In order to verify the latter case, the automaton $M_x$, after meeting the letter containing $C_0$ or $C_1$ for the first time, works as follows.

• $M_x$ makes silent moves going up until it reaches the top of the stack.
• Then the automaton $M_x$ pops a symbol out of stack each time it reads off four consecutive letters.
• After the stack is emptied the automaton reads off the next letter which should contain $C_0$ or $C_1$.
• After meeting the letter containing $C_0$ or $C_1$ for the second time, the automaton keeps working without using the stack.

Fig. 3. The content of the stack is shown for the first several iterations of $M_x$. In general situation, the content of the stack is shown to the right.
It is easy to see that the automaton $M_x$ recognizes the language $L_x$. In a similar way one can obtain the nested stack automaton $M_y$ that recognizes the language $L_y$.

**Remark 9.** For a given $g \in \mathbb{Z}_2 \wr \mathbb{Z}_2$, let $w$ be the representative of $g$ for the indexed Cayley automatic representations of $\mathbb{Z}_2 \wr \mathbb{Z}_2$ constructed in Theorem 8. Then the inequality $|g| \leq 2|w| - 1$ holds. However, no inequality of the form $\lambda|w| + \mu \leq |g|$ is satisfied for all $g \in \mathbb{Z}_2 \wr \mathbb{Z}_2$.

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