Interior Solutions of Fluid Sphere in $f(R,T)$ Gravity Admitting Conformal Killing Vectors

M. Zubair *(a)*, I. H. Sardar †(b), F. Rahaman(b) ‡ and G. Abbas(c) §

(a) Department of Mathematics,
COMSATS Institute of Information Technology, Lahore, Pakistan.
(b) Department of Mathematics,
Jadavpur University, Kolkata - 700032, India.
(c) Department of Mathematics,
The Islamia University of Bahawalpur, Bahawalpur, Pakistan.

Abstract

We discuss the interior solutions of fluid Sphere in $f(R,T)$ gravity admitting conformal killing vectors, where $R$ is Ricci scalar and $T$ is trace of energy momentum tensor. The solutions corresponding to isotropic and anisotropic configurations have been investigated explicitly. Further, the anisotropic case has been dealt by the utilization of linear equation of state. The results for both cases have been interpreted graphically. The equation of state parameter, integration constants and other parameters of the theory have been chosen to find the central density equal to standard value of central density of the compact objects. The energy conditions as well as stability of the solutions have been investigated in the background of $f(R,T)$ gravity.

*mzubairkk@gmail.com; drmzubair@ciitlahore.edu.pk
†iftikar.spm@gmail.com
‡rahaman@associates.iucaa.in
§abbassg91@yahoo.com
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1 Introduction

It is an admitted fact that the accelerated expansion of our Universe and the existence of dark matter are two such important aspects of modern cosmology that have been accepted on the background of observational data (Riess (2007), Perlmutter (1999), Hanany (2000), Peebles and Ratra (2003)). These findings have imposed some additional challenges to theories of gravitation. The most significant way to explain the observational data is by admitting that the Einstein theory of gravitation breaks down at large scales, and a more generalized form of action is required to describe the gravitational field at large scales. During the last decades the most general theoretical models of $f(R)$ gravity, where $R$ being Ricci scalar, have been extensively used to explain the cosmological results. The accelerated expansion of universe and the conditions for the presence of dark energy have been have studied in $f(R)$ gravity (Padmanabhan (2003)). The physical conditions for the viable cosmological models have been found in $f(R)$ (Nojiri and Odintsov (2011), Bamba et al(2012), and satisfy the weak field limit obtained from the classical tests of general relativity. The $f(R)$ models that satisfy the solar system tests of general relativity and provides the unification of inflation and dark energy were investigated in (Nojiri and Odintsov (2007), Cognola et al, (2008)). In $f(R)$ gravity, it has been proved that the galactic dynamic of massive test particles can be explained by excluding the possibility of dark matter ([Capozziello et al (2006), Borowiec et al (2007), Martins and Salucci (2007), Boehmer et al, (2008)). Further, investigations in $f(R)$ gravity can be found in detail in ([Sotiriou and Faraoni (2010) Lobo (2008), Capozziello and V. Faraoni (2010)).

A most general form of $f(R)$ theory of gravity was proposed in (Bertolami et al (2007)), by including an arbitrary function of the Ricci scalar $R$ with the matter Lagrangian density $L_m$ in the action of the theory. As a consequence of such modification the motion of massive particles is non-geodesic and there exists an extra-force The astronomical implication of non-minimal matter-geometry coupling were explored in (Nojiri and Odintsov (2004), Harko (2010)) and Palatini approach of non-minimal geometry-coupling models.
was discussed in (Harko and Lobo (2010)). In this coupling, a maximal extension of the Hilbert-Einstein action was performed in (Koivisto (2006)) by taking the gravitational Lagrangian as an arbitrary function of Ricci scalar $R$ and matter Lagrangian density $L_m$.

The field equations as well as the equations of motion for test particles have been formulated in the metric formalism, which is the covariant divergence of the stress-energy tensor. A specific form of above coupling was considered as another extension of general relativity as $f(R,T)$, modified theories of gravity, where action is given by an arbitrary function of the Ricci scalar $R$ and trace of the stress-energy tensor $T$ (Nesseris (2009)). Firstly, Lobo, et. al. (Harko et al.(2011)) introduced such modifications to obtain some specific results of cosmology, the more general aspects such as reconstruction of cosmological models and late time acceleration of universe was first studied in (Houndjo (2012)). Further, the energy conditions and thermodynamics in $f(R,T)$, theories have been investigated by Sharif and Zubair (Sharif and Zubair (2012), Sharif, M. and Zubair (2012)).

In general conformal Killing vectors (CKVs) explain the mathematical relation between the geometry and contents of matter in the spacetime via Einstein set of field equations. The CKVs are used to generate the exact solution of the Einstein field equation in more convenient form as compared to other analytical approaches. Further these are used to discover the conservation laws in any spacetime. The Einstein field equations being the highly non-linear partial differential equations can be reduced to a set of ordinary differential equations by using CKVs. A lot of astrophysical phenomena have been explored on the theoretical background using the CKVs approach (see (Ray et al (2008), Rahaman et al (2014), Rahaman et al (2015a,b,c))). The interior anisotropic fluid spheres admitting conformal motion have been studied during the last stages by Herrera and his collaborators (Herrera (1992), Herrera et al (1984), Herrera and Ponce de Leon (1985), Herrera and Ponce de Leon (1985a,b)).

In the present paper, our main motivation is to find the exact solution for static anisotropic spheres preserving the conformal motion in $f(R,T)$ gravity. Section 2 deals with formulation of field equations in $f(R,T)$ gravity. The exact solutions with isotropic and anisotropic configurations have been investigated in section 3. The last section summaries the results of the paper.
2 Interior Matter Distribution in $f(R, T)$ Gravity

The modified action in $f(R, T)$ is as follows
\[
\int dx^4 \sqrt{-g}[\frac{f(R,T)}{16\pi G} + \mathcal{L}_{(m)}],
\]
where $\mathcal{L}_{(m)}$ is matter Lagrangian and $g$ denote the metric tensor. Different choices of $\mathcal{L}_{(m)}$ can be considered, each of which directs to a specific form of fluid. The line element for general spherically symmetric metric describing the compact star stellar configuration is
\[
ds^2 = e^{a(r)}dt^2 - e^{b(r)}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2).
\]

Taking $8\pi G = 1$ and upon variation of modified EH action in $f(R, T)$ with respect to metric tensor $g_{uv}$, the following modified field equations are formed as
\[
G_{uv} = \frac{1}{f_R} \left[ (f_T + 1)T_{uv}^{(m)} - \rho g_{uv}f_T + \frac{f - Rf_R}{2}g_{uv} \right. \\
+ \left. (\nabla_u \nabla_v - g_{uv}\Box) f_R \right],
\]
where $T_{uv}^{(m)}$ denotes the usual matter energy momentum tensor that is considered to be anisotropic, is given by
\[
T_{uv}^{(m)} = (\rho + p_t)V_u V_v - p_r g_{uv} + (p_r - p_t)\chi_u \chi_v,
\]
where $\rho$, $p_r$ and $p_t$ denote energy density, radial and transverse stresses respectively. The four velocity is denoted by $V_u$ and $\chi_u$ to be the radial four vector satisfying
\[
V^u = e^{\frac{a}{2}} \delta^u_0, \quad V^u V_u = 1, \quad \chi^u = e^{\frac{b}{2}} \delta^u_1, \quad \chi^u \chi_u = -1.
\]

The conformal Killing vector is defined through the relation
\[
\mathcal{L}_\xi g_{\mu\nu} = g_{\eta\nu} \xi^\eta_{\mu} + g_{\mu\eta} \xi^\eta_{\nu} = \psi(r)g_{\mu\nu},
\]
where $\mathcal{L}$ represents the Lie derivative of metric tensor and $\psi(r)$ is the conformal vector.
Using Eq. (2) in (6), one can find [30]
\[ \xi^1 a' = \psi, \]
\[ \xi^1 = \frac{\xi r}{2}, \]
\[ \xi^1 b' + 2\xi^1 = \psi, \]
These results imply
\[ e^a = C_1 r^2, \]
\[ e^b = \left( \frac{C_2}{\psi} \right)^2, \]  \hspace{1cm} (7)
where \( C_1 \) and \( C_2 \) are integration constants.
When \( f(R, T) = f_1(R) + \lambda T \), the expression for \( \rho \), \( p_r \) and \( p_t \) can be extracted from modified field equations as follows
\[ \rho = \frac{e^{-b(r)}}{4r^2(1 + \lambda)(1 + 2\lambda)} \{ -2(2f_{1R}(-2 + (-5 + e^{b(r)}\lambda) + r(f'_1R(4 + 3\lambda) \\
+ r(e^{b(r)} f(1 + \lambda) + f''_{1R}(2 + 3\lambda))) + r(-2f_{1R} + f'_{1R}r(2 + 3\lambda))b'(r)) \}, \]  \hspace{1cm} (8)
\[ p_r = \frac{e^{-b(r)}}{4r^2(1 + \lambda)(1 + 2\lambda)} \{ 2(-2f_{1R} + f''_{1R}r^2)\lambda + f'_{1R}r(6 + 7\lambda) + e^{b(r)}(2f_{1R}\lambda \\
+ f'^2(1 + \lambda)) + r(f'_{1R}r\lambda + f_{1R}(6 + 8\lambda))b'(r)) \}, \]  \hspace{1cm} (9)
\[ p_t = \frac{e^{-b(r)}}{4r^2(1 + \lambda)(1 + 2\lambda)} \{ 2(2f_{1R}(-2 - 3\lambda + e^{b(r)}(1 + \lambda)) + r(f'_{1R}(4 + 9\lambda) \\
+ r(e^{b(r)} f(1 + 2\lambda) + f''_{1R}(2 + 3\lambda))) + r(-f'_{1R}r(2 + 3\lambda) + f_{1R}(2 \\
+ 6\lambda))b'(r)) \}. \]  \hspace{1cm} (10)
Here \( f_{1R} = \frac{df}{dR} \) and prime denotes the derivatives with respect to radial coordinate. Eqs. (8)-(10) are highly non-linear to find the \( e^{b(r)} \). Therefore, we consider the simplest case \( f(R, T) = R + \lambda T \) which represents the \( \Lambda \)CDM model in \( f(R, T) \) gravity. For this choice we can find the results for \( \rho \), \( p_r \) and \( p_t \) in the following form
\[ \rho = \frac{e^{-b(r)}}{2r^2(1 + \lambda)(1 + 2\lambda)} \{ 2(-1 + e^{b(R)} + 2\lambda) + r(2 + 3\lambda)b'(r)) \}, \]  \hspace{1cm} (11)
\[ p_r = \frac{e^{-b(r)}}{4r^2(1 + \lambda)(1 + 2\lambda)} \{ 6 - 2e^{b(r)} + 4\lambda + r\lambda b'(r)) \}, \]  \hspace{1cm} (12)
\[ p_t = \frac{e^{-b(r)}}{4r^2(1 + \lambda)(1 + 2\lambda)} \{ 2 - 2(-3 + e^{b(r)}\lambda - r(2 + 3\lambda)b'(r)) \}. \]  \hspace{1cm} (13)
3 Solutions

Now we are seeking solutions for two different physical situations. At first, we assume the isotropic case and secondly we will consider anisotropic model of the Fluid Sphere.

3.1 Isotropic case

For isotropic model of the Fluid Sphere, it is assumed that \( p_r = p_t = p \).

Using the isotropic pressures and solving the equations (11)-(13), we get

\[
e^{-b(r)} = \frac{1 - \lambda}{2 - \lambda} + C_3 r^{2-\lambda/2\lambda+1},
\]

\[
R = \frac{1}{r^2} \left\{ 4 + \frac{6}{\lambda - 2} + \frac{3C_3(4 + 3\lambda)r^{2-\lambda/2\lambda+1}}{1 + 2\lambda} \right\},
\]

\[
\rho = \frac{r^{-2-\frac{\lambda}{1 + 2\lambda}}}{2(\lambda - 2)(\lambda + 1)(2\lambda + 1)^2} \left\{ 2r^{\lambda/1+2\lambda}(-1 - 4\lambda - 2\lambda^2 + 4\lambda^3) \right. \\
+ \left. C_3 r^{2+2\lambda}(12 + 2\lambda - 26\lambda^2 + 11\lambda^3) \right\},
\]

\[
p = \frac{r^{-2-\frac{\lambda}{1 + 2\lambda}}}{2(\lambda - 2)(\lambda + 1)(2\lambda + 1)^2} \left\{ 2r^{\lambda/1+2\lambda}(-1 - 2\lambda + 2\lambda^2 + 4\lambda^3) \right. \\
+ \left. C_3 r^{2+2\lambda}(-12 - 22\lambda - 4\lambda^2 + 9\lambda^3) \right\},
\]

\[
\psi = C_2 \left( \frac{1 - \lambda}{2 - \lambda} + C_3 r^{2-\lambda/2\lambda+1} \right)^{1/2},
\]

where \( C_3 \) is an arbitrary constant.

To search the physical properties of the interior of the fluid sphere, we draw the profile of matter density and pressure in fig.1(left) and fig.1 (middle) respectively. The profile indicates that matter density and pressure all are positive inside the fluid Sphere. It is to be noted that density and radial pressure are decreasing with the radial coordinate \( r \) which are the common features. Obviously all energy conditions are satisfied see fig.1 (right). Here, the model indicates equation of state parameter as well as sound velocity are less than unity, see fig 2. Thus our solutions satisfy all criteria for physically valid solution of a fluid sphere.
Figure 1: (left) Density is plotted against \( r \). (middle) Pressure is plotted against \( r \). (right) Variation of \( p + \rho \) is shown against \( r \).

Figure 2: (left) EoS is plotted against \( r \). (right) Variation of sound speed is shown against \( r \).
3.2 Anisotropic case

Our objective in this section is to develop a model for anisotropic fluid and, therefore, we assume \( p_r \neq p_t \). The simplest form of the fluid sphere EoS having the form

\[
P_r = \omega \rho. \tag{19}
\]

Therefore the solutions are obtained in the following form

\[
e^{-b(r)} = \frac{2(1 + \omega)}{(6 + 4\lambda) + \omega(2 - 4\lambda)} + C_4 r^{-\frac{2(1 + \omega)}{\lambda - \omega(2 + 3\lambda)}}, \tag{20}
\]

\[
R = \frac{4((1 + \lambda)\omega - \lambda)}{r^2(3 + \omega + (1 - \omega)2\lambda) + C_4 6((1 + 3\lambda)\omega - (1 + \lambda))r^{-2 - \frac{2(1 + \omega)}{\lambda - \omega(2 + 3\lambda)}}}, \tag{21}
\]

\[
\rho = \frac{1}{r^2(1 + \lambda)(1 + 2\lambda)} \left\{ \frac{2C_4(1 + \lambda(2 - \lambda + \omega(2 + 3\lambda)))r^{-2 - \frac{2(1 + \omega)}{\lambda - \omega(2 + 3\lambda)}}}{\lambda - \omega(2 + 3\lambda)} \right. + \left. \frac{2 + 4\lambda}{3 + \omega + 2\lambda(1 - \omega)} \right\}, \tag{22}
\]

\[
p_r = \frac{1}{r^2(1 + \lambda)(1 + 2\lambda)} \left\{ \frac{2C_4(-\lambda(1 + \lambda) + \omega(3 + \lambda(7 + 3\lambda)))r^{-2 - \frac{2(1 + \omega)}{\lambda - \omega(2 + 3\lambda)}}}{\lambda - \omega(2 + 3\lambda)} \right. + \left. \frac{2\omega(1 + 2\lambda)}{3 + \omega + 2\lambda(1 - \omega)} \right\}, \tag{23}
\]

\[
p_r = \frac{1}{r^2(1 + \lambda)(1 + 2\lambda)} \left\{ \frac{C_4(-2 + \lambda(-4 - 3\lambda + \omega(6 + 9\lambda)))r^{-2 - \frac{2(1 + \omega)}{\lambda - \omega(2 + 3\lambda)}}}{\lambda - \omega(2 + 3\lambda)} \right. + \left. \frac{1 + \omega + 2\omega\lambda + 2(-1 + \omega)\lambda^2}{3 + \omega + 2\lambda(1 - \omega)} \right\}, \tag{24}
\]

\[
\psi = C_2 \left( \frac{2(1 + \omega)}{(6 + 4\lambda) + \omega(2 - 4\lambda)} + C_4 r^{-\frac{2(1 + \omega)}{\lambda - \omega(2 + 3\lambda)}} \right)^{1/2}. \tag{25}
\]

4 Concluding Remarks

In this paper, we have developed a compact star model in \( f(R, T) \) gravity which satisfy the conformal Killing vectors equations. In this setting, we have studied in detail the \( f(R, T) \) gravity for the case \( f(R, T) = R + \lambda T \) with isotropic pressure \( (p_r = p_t = p) \) as well as anisotropic pressure \( (p_r \neq p_t) \). Further, we would like to mention that a linear equation of state for the
Figure 3: (left) Density is plotted against $r$. (middle) Radial pressure is plotted against $r$. (right) Transverse pressure is plotted against $r$.

Figure 4: (left) Variation of $\rho + p_r$ is shown against $r$. (middle) Variation of $\rho + p_t$ is shown against $r$. (right) Variation of $\rho + p_r + 2p_t$ is shown against $r$. 
Figure 5:  ( left) Variation of $\rho + p_r$ is shown against $r$.  ( middle) Variation of $\rho + p_t$ is shown against $r$.  ( right) Variation of $\rho + p_r + 2p_t$ is shown against $r$.

Figure 6:  ( left) Variation of radial sound speed is shown against $r$.  ( middle) Variation of transverse sound speed is shown against $r$.  ( right) Variation of $v_t^2 - v_r^2$ is shown against $r$.  
anisotropic case has been employed. The equation of state parameter, integration constants and parameter of the theory $\lambda$ have been chosen arbitrarily, so that in the present background the value of the central energy density becomes approximately equal to the standard value of energy density for the compact stars. The regularity as well as energy conditions for the both solutions have been discussed in detail.

It has been found that the energy density and pressure are positive and finite throughout interior of the stars. The constraint on the equation of state parameters are given by $0 < \omega_r < 1$ and $0 < \omega_t < 1$, (as shown in figure 5) which are in agreement with the normal matter distribution in $f(R, T)$ gravity. In 1992 Herrera proposed the cracking concept (also known as overturning) which determine the stability of anisotropic star. In our model, we have shown from figure (6) that radial speed of sound is always greater than the transverse speed of sound everywhere inside the stars due to same sign of $v_r^2 - v_t^2$. Therefore according to cracking concept our star model is stable in $f(R, T)$ gravity. This work can be extended by taking more general form of the $f(R, T)$ gravity model, to discuss some other physical properties like anisotropic parameter $\Delta$, optimality of density and pressure and surface red-shift.

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