Non-smooth convex caustics for Birkhoff billiard.

Maxim Arnold*    Misha Bialy†

March 26, 2018

Abstract

This paper is devoted to the examination of the properties of the string construction for the Birkhoff billiard. Based on purely geometric considerations, string construction is suited to provide a table for the Birkhoff billiard, having the prescribed caustic. Exploiting this framework together with the properties of convex caustics, we give a geometric proof of a result by Innami first proved in 2002 by means of Aubry-Mather theory. In the second part of the paper we show that applying the string construction one can find a new collection of examples of $C^2$-smooth convex billiard tables with a non-smooth convex caustic.

1 Introduction.

Let $\Gamma$ be a simple closed $C^1$-smooth convex curve in the Euclidean plane. We consider Birkhoff billiard inside $\Gamma$. This simple dynamical system creates many geometric and dynamical questions and reflects many difficulties appearing in general Hamiltonian systems. Reader may refer to any textbook among the wide variety written on the subject (e.g. [11], [13], [16], [18]).

In the present paper we will use the following non-standard notations: the interior of the set bounded by simple closed curve $\gamma$ will be denoted by $\gamma^\circ$, while $\overline{\gamma}$ denotes the compact $\gamma^\circ \cup \gamma$. Length of the curve is denoted by $\text{Length}(\gamma)$. Convex hull of $\gamma$ is denoted by $\text{Conv}(\gamma)$. The following definition of convex caustics is used in this paper:

**Definition 1.** Simple closed curve $\gamma \subset \Gamma^\circ$ is called convex caustic for $\Gamma$ if $\overline{\gamma}$ is a convex set and any supporting line to $\overline{\gamma}$ remains a supporting line to $\overline{\gamma}$ after the billiard reflection in $\Gamma$.

Every convex caustic $\gamma$ corresponds to the invariant curve $r_\gamma$ of the billiard ball map. Curve $r_\gamma \subset \mathbb{R}_+ \times S^1$ consists of all supporting lines to $\gamma$. This curve winds once around the phase cylinder and therefore is called rotational. We shall denote its rotation number by $\rho_\gamma$.

In the original Birkhoff paper [4] there was posed a conjecture that the existence of a continuous set of caustics, being very restrictive property, actually provide an extreme rigidity on the shape of curve $\Gamma$. First result in this direction was achieved in [3]. Our paper is motivated by recent progress in the Birkhoff conjecture solution achieved in [2, 9]. The crucial assumption in these papers consists in the existence of convex caustics such that the rotation numbers of the corresponding invariant curves form a rational sequence in the interval $(0; \frac{1}{3}]$, converging to 0. It seems natural to compare such result with one proved by N. Innami [8].

*University of Texas at Dallas, Richardson, TX, USA
†Tel Aviv University, Tel Aviv, Israel
Theorem 1 (Innami (2002), [8]). Assume that there exists a sequence of convex caustics \( \gamma_n \) inside \( \Gamma \) such that the rotation numbers \( \rho_n \) of the corresponding invariant curves tend to \( \frac{1}{2} \). Then \( \Gamma \) is an ellipse.

Originally, Innami’s arguments were based on the Aubry-Mather variational theory. In the next section we present a simple geometric proof using string construction. Yet, it remains a challenging question if one can prove more general statement relaxing the requirement of convexity of the caustics.

Let us remind the string construction framework. Given a convex compact set \( \bar{\gamma} \) bounded by \( \gamma \), and a number \( S > \text{Length}(\gamma) \) define the curve \( \Gamma \) as a union of those points \( P \) that the cap-body \( \text{Conv}(P \cup \bar{\gamma}) \) has the boundary of the length \( S \). Geometrically such a construction gives the set of all points traversed by the tip of non-elastic string of length \( S > \text{Length}(\gamma) \) wrapped around \( \gamma \) and stretched to the very extent. Curve \( \Gamma \) provided by such construction has \( \gamma \) as its billiard caustic. We shall refer to \( S \) as a string parameter of the caustic. A closely related so-called Lazutkin parameter is defined as \( L = S - \text{Length}(\gamma) \).

The string construction is widely known and can be easily proved to provide \( \Gamma \) for smooth enough \( \gamma \). In fact it remains valid also in more general case as it is stated in the following theorem.

Theorem 2 (Stall (1930), [17]; Turner (1982), [19]).

1. For a given compact convex set \( \bar{\gamma} \) and for every \( S > \text{Length}(\gamma) \) the string construction determines a \( C^1 \)-smooth convex closed curve \( \Gamma \) such that \( \gamma \) is a billiard caustic for \( \Gamma \).

2. If \( \gamma \) is a convex billiard caustic for \( C^1 \) curve \( \Gamma \) then \( \Gamma \) can be obtained from \( \gamma \) by the string construction for some \( S \).

Let us emphasize that the string construction is highly non-explicit and difficult for calculations. Very important consequence of KAM theory, proved by Lazutkin [14, 15] and Douady [5], states the existence of convex caustics near the boundary of sufficiently smooth (at least \( C^6 \)) billiard table. On the other hand, applying string construction to the triangle, one gets billiard table which is piecewise \( C^2 \) with jumps of the curvature and hence by [7] can not have caustics near the boundary.

The scenario of destruction of caustics when one moves away from the boundary towards the interior could be understood in principle by the analogy with wave front propagation inside a convex curve ([16]). Take for example the ellipse and consider the wave fronts as on the famous picture ([1, Fig.36]). For small distances the fronts remain smooth, but starting from some critical value they start to develop singularities. However, nobody saw such a bifurcation in practice for caustics of convex billiards due to the lack of integrable examples. On the other hand, non-convex caustics exist for instance for convex bodies of constant width, and were studied in [12].

Motivated by the above discussion, the natural question about the existence of non-smooth convex caustics arises. More generally, it is natural to study how irregular the convex caustic can be. In [6] a billiard table of class \( C^2 \) was constructed which has a caustic of regular hexagon. In the present paper we were able to construct whole functional family of the examples of \( C^2 \) billiard tables having non-smooth convex caustics.

Theorem 3. There exist a one-parametric family of strictly convex non-smooth compact sets \( \bar{\gamma} \) and the values of the string parameter \( S \) such that the curves \( \Gamma \) obtained by the string construction are \( C^2 \)-smooth.
We will use the following geometric idea (we will use complex notations \( x + iy \) for points \((x, y)\) in the plane). Start with a curve \( \gamma_0(t) : [-1, 1] \rightarrow \mathbb{C} \) such that \( \gamma_0(-1) = A = -1 - i \), \( \gamma_0(1) = iA = 1 - i \) and \( \gamma_0(t) \) is symmetric with respect to vertical axis (i.e. \( i\gamma_0(-t) = i\gamma_0(t) \)) (see Fig. 1). Construct \( \gamma \) as a concatenation of \( \{i^k\gamma_0\}_{k=0}^{3} \). Parametrize \( \gamma \) by the arc-length parameter \( s \) and choose the initial point in such a way that \( \gamma(0) = A \). We will denote the total length of \( \gamma \) by \( 4S \). Then \( \gamma(S) = iA \).

![Figure 1: Switched caustic string construction.](image)

Main idea is to choose the curve \( \gamma \) and string parameter \( S \) in such a way that the string construction will have the following properties:

- At the beginning (point \( P \) on Fig.1), left part \( AP \) of the string remains fixed at point \( A \) while the right part of the string unwind from the arc \( (iA, i^2A) \).

- At the moment when the left part of the string became tangent to \( \gamma \) at the point \( A \) (this corresponds to the point \( P \) on \( \Gamma \)) right part reaches the point \( i^2A \) and remains fixed after that. We will call this moment the *switching of the first kind*.

- While the left part of the string winds around the arc \( (A, iA) \) the right part remains fixed at \( i^2A \) (see Fig. 1) till the moment when the vertex of the string reaches the point \( iP \). We will call this *switching of the second kind*.

- \( D_4 \) symmetry provides the whole picture.

Let us reemphasize, that the string construction being non-explicit procedure, typically does not provide any analytic expression for the table \( \Gamma \) from given \( \gamma \). In the example [6], the construction is made explicit by fixing two end-points on the string. Disadvantage of such situation is the complete loss of any flexibility, since the corresponding table may consist only of the elliptic arcs. In the current paper we propose another, more flexible yet explicit construction, fixing only one end-point of the string and allowing another point to slide along the given curve \( \gamma \).
Structure of the paper.

In the next section we will provide geometric arguments for the proof of the Theorem 1. Section 3 is devoted to the construction of the $C^2$ tables with non-smooth caustics. In Section 4 we will pose some open questions arising in our considerations.

Acknowledgments.

MB is thankful to the participants of the course "Billiards" given in Tel Aviv University for very useful discussions and ideas. MB was supported by ISF 162/15.

2 Geometric proof of Innami’s result.

We will start with the following simple remarks.

Remark 1. If billiard in $\Gamma$ has a convex caustic $\gamma$ with $\gamma^\circ = \emptyset$ then $\Gamma$ is either an ellipse or a circle.

Indeed, condition $\gamma^\circ = \emptyset$ for convex $\gamma$ means that $\gamma$ is either a point or a segment. The rest follows from the string construction.

Remark 2. If convex caustic $\gamma$ has non-empty interior, then every supporting line to $\bar{\gamma}$ after reflection in $\Gamma$ at point $P$ becomes second supporting line to $\bar{\gamma}$ from $P$. Recall that for any point $P$ and for any convex body $C$ there exist exactly two supporting lines to $C$ passing through $P$.

Assume that there exists a supporting line to $\bar{\gamma}$ which is reflected to itself. Then, by continuity, since $\gamma$ has non-empty interior, all lines must behave like this. Therefore, all lines tangent to $\bar{\gamma}$ are diameters, but then for any point $P \in \Gamma$ there are two diameters passing through $P$ which is not possible. Finally, we get:

Lemma 1. Let $\gamma$ be a convex caustic for $\Gamma$. Then $\gamma^\circ \neq \emptyset$ if and only if the rotation number of the corresponding invariant curve is strictly less than $\frac{1}{2}$.

Proof. If a convex caustic $\gamma$ has empty interior then, by Remark 1, $\Gamma$ is necessarily an ellipse and the invariant curve corresponding to $\gamma$ has rotation number $\frac{1}{2}$ since it contains a diameter. On the other hand, if $\gamma^\circ$ contains some ball of radius $\delta$, then every reflection produces an angle deficit which can be bounded from below by $\delta$ (see Fig.2). Therefore the average number of turns of the billiard trajectory tangent to $\gamma$ is bounded away from $\frac{1}{2}$. Hence the rotation number is strictly less than $\frac{1}{2}$. \hfill $\Box$

Let $\gamma_n$ be a sequence of convex caustics with the rotation numbers of corresponding invariant curves $\rho_n \in (0; \frac{1}{2}]$. By Lemma 1 we may assume that $\rho_n < \frac{1}{2}$ since otherwise $\gamma_n$ has empty interior and then must be an ellipse by Remark 1. Passing to a subsequence we can assume with no loss of generality that $\rho_n$ is strictly increasing, $\rho_n \nearrow \frac{1}{2}$.

Lemma 2. Let $\gamma_1$ and $\gamma_2$ be two convex caustics for $\Gamma$. If the corresponding invariant curves have rotation numbers $\rho_1 < \rho_2$, then $\bar{\gamma}_2 \subset \gamma_1^\circ$.
Figure 2: Left: Family of nested convex caustics with decreasing string parameter. Right: Rotation number 1/2 could not correspond to a convex caustic with non empty interior.

Proof. Assume that $\bar{\gamma}_2$ is not a subset of $\gamma_1^o$. Then there are three possibilities: (1) $\bar{\gamma}_1 \cap \bar{\gamma}_2 = \emptyset$; (2) $\gamma_1 \cap \gamma_2 \neq \emptyset$ or (3) $\bar{\gamma}_1 \subset \gamma_2^o$.

In the third case one obviously has $\rho_1 \geq \rho_2$ contrary to the assumption of the Lemma. In the first and the second cases there necessarily exists a line supporting to both $\bar{\gamma}_1$ and $\bar{\gamma}_2$. Therefore, all billiard reflections in $\Gamma$ of this line are also supporting lines for both $\bar{\gamma}_1$ and $\bar{\gamma}_2$. This means that there exist a whole infinite orbit lying in the intersection of the two invariant curves corresponding to $\gamma_1$ and $\gamma_2$. But then $\rho_1$ must be equal to $\rho_2$, since they are completely determined by one orbit.

Remark 3. The statement of Lemma 1 holds true also in the opposite direction which will not be used below. Namely, $\bar{\gamma}_2 \subset \gamma_1^o$ implies $\rho_1 < \rho_2$. As we already mentioned in the proof it is obvious that $\rho_1 \leq \rho_2$. In addition $\rho_1$ can not be equal to $\rho_2$. Indeed, otherwise there exist two disjoint graphs of $r_1$ and $r_2$ with the same rotation number, invariant under the billiard map of the cylinder, which is impossible since billiard map is a twist map (see for example [10, p.428]).

Let $\{S_n\}$ be the sequence of string parameters corresponding to the caustics $\gamma_n$. Then by Lemma 2, $S_n$ is decreasing. Denote $S = \lim_{n \to \infty} S_n$.

Lemma 3. Boundary of the intersection set

$$C = \bigcap_{n=1}^{\infty} \bar{\gamma}_n$$

is a convex caustic for $\Gamma$ with string parameter $S$.

Proof. The intersection set $C$ is compact and convex. Moreover, it is easy to see that $\partial C$ is also a caustic with string parameter $S$. Indeed, this follows from the following geometric consideration (see
Left part of Fig.2). Fix a point $P$ on $\Gamma$ and consider the cap-bodies

$$K_n = \text{Conv}(P \cup \bar{\gamma}_n), \quad K = \text{Conv}(P \cup C).$$

Then obviously

$$K_n \subseteq K, \quad K = \bigcap_n K_n,$$

and moreover

$$\text{Length}(\partial K_n) = S_n \to S = \text{Length}(\partial K).$$

In addition, since $\gamma_n$ is a caustic then $S_n$ does not depend on $P \in \Gamma$ (by Theorem 1). Therefore, $S$ also does not depend on $P$, and hence $C$ reconstructs $\Gamma$ via string construction. Thus $\partial C$ is a caustic by Theorem 1.

The last step in the proof of the Theorem 2 consists in the following Lemma.

**Lemma 4.** The limit caustic $\partial C$ has empty interior.

**Proof.** First notice that it follows from continuity of invariant curves and their rotation numbers that the invariant curve corresponding to $C$ has rotation number $\frac{1}{2}$. Then from Lemma 1 we conclude that $\partial C$ has empty interior.

3 Non-smooth caustic.

Main idea of the proof of our result is to carefully choose the Lazutkin parameter and the germ of function $\gamma$ at the point $A$. While vertex of the string slides in the regime corresponding to the unwinding from $\gamma(s)$, its trajectory corresponds to the smooth curve. Thus we have to take care of the smoothness of $\Gamma$ near only two points corresponding to the switching moments of the first and second kind respectively. We will denote by $\Gamma(s)$ the part of $\Gamma$ corresponding to the switching of the second kind about the point $A$. $\hat{\Gamma}$ will denote the part of $\Gamma$ corresponding to the switching of the
first kind about the point $A$. Then the smoothness conditions read as follows: all odd terms in the germs of $\Gamma$ and $\hat{\Gamma}$ have to be orthogonal to the axis of the symmetry while all the even terms has to be collinear with the axis of symmetry.

**Coordinate formulation.** Parametrize the curve $\gamma$ by the arc-length parameter $s$, so that $|\gamma'(s)| = 1$. Choose the initial point such that $\gamma(0) = A$. Denote by $\alpha$ the angle between $\gamma'(0)$ and horizontal axis. Then one easily obtains a parametrization for $\Gamma$ and $\hat{\Gamma}$ (see Fig. 3):

$$\Gamma(s) = \gamma(s) - t(s)\gamma'(s), \quad \hat{\Gamma}(s) = \gamma(s) + \hat{\ell}(s)\gamma'(s)$$

(1)

where $t(s)$ and $\hat{\ell}(s)$ are some functions of $s$ denoting the length of the right part of the string near point $\Gamma(s)$ and left part of the string near point $\hat{\Gamma}(s)$ correspondingly. Functions $t$ and $\hat{\ell}$ can be found from the condition of the string to be unstretchable. We will denote $iA = B$.

\[
|\Gamma(s) + B| + |t\gamma'(s)| - s = 2\ell \\
|\hat{\Gamma}(s) + A| + |\hat{\ell}\gamma'(s)| + s = 2\hat{\ell}
\]

(2)

where $\ell = \frac{1}{\sin\alpha}$ and $\hat{\ell} = \frac{\sqrt{2}}{\sin(\pi/4 - \alpha)}$. Simple computations yield for $t(s)$ and $\hat{\ell}(s)$:

$$t(s) = \frac{p(s)}{p'(s)}, \quad \text{with} \quad p(s) = \frac{1}{2} \left( (s + 2\ell)^2 - |\gamma(s) + B|^2 \right),$$

$$\hat{\ell}(s) = -\frac{\hat{p}(s)}{\hat{p}'(s)}, \quad \text{with} \quad \hat{p}(s) = \frac{1}{2} \left( (s - 2\hat{\ell})^2 - |\gamma(s) + A|^2 \right).$$

(3)

Finally, introducing (3) into (1) we get

$$\Gamma(s) = \gamma(s) - \frac{p(s)}{p'(s)}\gamma'(s), \quad \hat{\Gamma}(s) = \gamma(s) - \frac{\hat{p}(s)}{\hat{p}'(s)}\gamma'(s).$$

(4)

Orient curve $\gamma$ as it is shown on Fig. 3. We will use complex notation for the coordinates of the points. Then smoothness conditions for the $n$-th derivative of $\Gamma$ read

$$\Re \left( i^{n-1} \Gamma^{(n)}(0) \right) = 0, \quad \Re \left( i^{n-1} \hat{\Gamma}^{(n)}(0) \right) = \Im \left( i^{n-1} \hat{\Gamma}^{(n)}(0) \right).$$

(5)

For the curve $\gamma(s)$ we get the following parametrization:

$$\gamma(s) = A + \int_0^s \exp \{i(\varphi(t) - \alpha)\} \, dt, \quad \text{where} \quad \varphi(t) = \sum_{n=0}^\infty \varphi_n t^n$$

(6)

Thus $\varphi_0 = 0$, and $\varphi_n$ corresponds to the $(n-1)$-st derivative of the curvature $\kappa$.

**Lemma 5.** Smoothness conditions (5) for $n = 1$ are always satisfied.

The statement of this lemma follows from the fact that any $C^0$ caustic produces $C^1$ table via string construction. However, we present more analytic proof of this result for a sake of completeness.

**Proof.**
1. **Switching of the second kind.** From (4) we get

\[ \Gamma' = \left(1 - \left(\frac{p}{p'}\right)^2 \right) \gamma' - \frac{p}{p'} \gamma'' \]

therefore conditions (5) read

\[ \Re(p'' \gamma' - p' \gamma'') = 0 \]

We will denote \( z_1 \cdot z_2 := \frac{1}{2} \Re(z_1 \bar{z}_2) \). Using expressions (3) we get

\[ p' = -(A + B) \cdot \gamma' + 2 \ell, \quad p'' = -(A + B) \cdot \gamma'' \]

From (6) it follows that \( \gamma'' = i \kappa \gamma' \) thus \( p'' \gamma' - p' \gamma'' \) can be written as

\[ p'' \gamma' - p' \gamma'' = \frac{1}{2} \left( 4 \Re((A + B)i \kappa \gamma') \gamma' + \Re((A + B)\gamma')(i \kappa \gamma') - 4 \ell i \kappa \gamma' \right) = i \kappa (A + B - 2 \ell \gamma'). \]

Thus

\[ \Re(p'' \gamma' - p' \gamma'') = \kappa \Im(A + B - 2 \ell \gamma') \]

The latter is identically zero since \( \ell \gamma'(0) = \Gamma(0) - \gamma(0) \) and so \( \Im(\ell \gamma') = \Im(A) \) (see Fig. 3).

2. **Switching of the first kind.** Similarly, smoothness conditions (5) reads

\[ \Re(\hat{p}'' \gamma' - \hat{p}' \gamma'') = \Im(\hat{p}'' \gamma' - \hat{p}' \gamma'') \]

where

\[ \hat{p}' = -(2A) \cdot \gamma' - 2 \hat{\ell}, \quad \hat{p}'' = -(2A) \cdot \gamma'' \]

and so

\[ \hat{p}'' \gamma' - \hat{p}' \gamma'' = \left( \Re(A i \kappa \gamma') \gamma' + \Re(A \gamma')(i \kappa \gamma') + 2 \hat{\ell} i \kappa \gamma' \right) = 2 i \kappa (A + \hat{\ell} \gamma') \]

Real part of the right-hand side of the latter is always equal to the imaginary part by the definition of \( \hat{\ell} \).

Two conditions (5) for \( n = 2 \) provide, via the computations similar to the above, two equations for parameters \( \varphi_1 \) and \( \varphi_2 \) with coefficients depending on \( \alpha \).

\[
\begin{align*}
\varphi_1^2 \sin \alpha - \varphi_1 \sin \alpha \cos \alpha - \varphi_2 \cos \alpha &= 0, \\
\varphi_1 \cos 2\alpha + 2(\sin \alpha - \cos \alpha) \varphi_1 - 2(\cos \alpha + \sin \alpha) \varphi_2 &= 0.
\end{align*}
\]

The latter system has a solution

\[ \varphi_1 = \frac{1}{2} \cos \alpha(1 + \sin 2\alpha), \quad \varphi_2 = -\frac{1}{8} \cos^2 2\alpha \sin 2\alpha, \]

which provides a family of germs for \( \gamma \), depending on parameter \( \alpha \), guaranteeing the \( C^2 \)-smoothness for the table \( \Gamma \).
Next we will need to construct the whole curve \( \gamma \) providing the needed phenomenon in the string construction. Recall that our geometric idea was based on the construction of the curve \( \gamma_0 \) (see Fig.1). Thus we need to present a convex curve of length \( S \), starting at \( A \) and ending at \( iA \), having tangent slope \( -\alpha \) at the left end and being symmetric with respect to the vertical axis. We define \( \gamma \) from \( \varphi \) through (6). In order to finish the construction we have to prove the following theorem.

**Theorem 4.** There exists a strictly monotonically increasing function \( \varphi(s) \) satisfying the following conditions

1. \( \varphi(s) \) has the given germ (7) at \( s = 0 \).
2. \( \varphi_0(S/2) = \alpha \) and \( \varphi_{2n}(S/2) = 0 \) for \( n \geq 1 \).
3. \( \int_0^{S/2} \cos \varphi(s) ds = 1 \).

**Proof.** Thanks to Borel theorem there exist a set \( \Psi \) of \( C^\infty \) functions having given germs at \( s = 0 \) and \( s = S/2 \). Since for \( \alpha < \frac{\pi}{2} \) term \( \varphi_1 \) in (7) is positive, one may assume without loss of generality that \( \Psi \) consists of strictly monotonically increasing functions. Therefore the only condition which has to be satisfied is the condition 3. Taking small enough \( \varepsilon \)-step in \( s \) we can assure \( \psi(\varepsilon) < \frac{\alpha}{100} \) for all \( \psi \in \Psi \).

Next we choose two functions \( \psi_- \) and \( \psi_+ \) from the set \( \Psi \) as on Fig. 4. That is \( \psi_+(s) \) almost equals to \( \alpha \) for \( s \in (\varepsilon + \delta, S/2 - \delta) \) and \( \psi_-(s) \) is almost equal to \( \psi(\varepsilon) \) for \( s \in (\varepsilon, S/2 - \delta) \) for small enough \( \delta \). We will look for \( \varphi \) as a convex combination \( \varphi(s) = l \psi_-(s) + (1 - l) \psi_+(s) \). Therefore \( \varphi(s) \) obviously satisfies conditions 1 and 2.

If we may choose \( \psi_\pm \) in such a way that

\[
(S/2) \cos \alpha < \int_0^{S/2} \cos(\psi_-(s) - \alpha) ds < 1 \quad \text{and} \quad S/2 > \int_0^{S/2} \cos(\psi_+(s) - \alpha) ds > 1
\]

than there exists such \( l \) that \( \int_0^{S/2} \cos(\varphi(s)) ds = 1 \), thus satisfying condition 3. Hence it is sufficient to check that conditions (8) can be satisfied for an open set of parameters \( \alpha \). Recall, that by the construction \( S = 2\ell - 2\ell \). From the first inequality (8) we obtain, since \( \alpha < \frac{\pi}{4} \),

\[
\ell - \ell = \frac{2}{\cos \alpha - \sin \alpha} - \frac{1}{\sin \alpha} < \frac{1}{\cos \alpha}. 
\]
This condition can be interpreted as follows: length of the curve $\gamma$ could not exceed the sum of lengths of the segments of two tangent lines from point $P$ to $\gamma$ (see Fig. 1). The latter inequality is satisfied whenever $\tan 2\alpha < 1$ or

$$\alpha < \frac{\pi}{8} \quad (9)$$

Second condition in (8) has the following geometric interpretation: length of $\gamma$ could not be less than the distance between points $A$ and $B$. This yields:

$$3 \sin \alpha - \cos \alpha > \cos \alpha \sin \alpha - \sin^2 \alpha$$

Since the latter is satisfied for $\alpha = \frac{\pi}{8}$ we have found an open set of $\alpha$ for which one could find appropriate functions $\psi_-$ and $\psi_+$ shown on Fig.4.

Remark 4. Since conditions (5) provides two conditions on $\varphi_n$ to obtain $C^3$ of $\Gamma$ one gets four equations for $\varphi_1, \varphi_2, \varphi_3$ and $\alpha$. Yet the number of parameters match the number of equations, the corresponding value of $\alpha$ violates inequality (9). Since inequality (9) arise from the construction based on square symmetry, there is a hope that starting from other regular polygons one can obtain inequality which can be satisfied. However, we haven’t found such examples.

4 Open problems.

Here we want to stress the general questions which are ultimately related to the string construction. Since the string construction is very implicit these questions turn out to be non-trivial.

Question 1. Is it possible to have two convex caustics $\gamma_1$ and $\gamma_2$ of $\Gamma$ such that none of them is a subset of the interior of the other?

In such a case $\gamma_1$ and $\gamma_2$ must have the same rotation number since there is a line tangent to both of the caustics. Moreover it is obvious that $\overline{\gamma_1}$ and $\overline{\gamma_2}$ cannot be disjoint. So the question is if it is possible that two convex caustics have non-trivial intersection. In such a case also their convex hull is also a caustic. One can strengthen the question:

Question 2. Is it possible for $\Gamma$ which is symmetric with respect to certain axis to have a convex caustic $C$ which is not symmetric with respect to this axis.

For example one could imagine two caustics forming rounded David Star (Fig 5). Let us remark that the answer to the quantum analog of this question is positive: for symmetric domain Dirichlet eigenfunction can be non-symmetric. We could not however decide if such a counterexample would be possible in the original setting.

Question 3. How irregular a convex caustic can be versus regular boundary curve $\Gamma$?

Question 4. Let $\Gamma$ be a billiard table different from circle having a convex caustic $\gamma$. For every point $P \in \Gamma$ denote $P_-, P_+$ the points on the caustic $\gamma$ which are tangency points on the tangent lines to $\gamma$ passing through $P$. Is it possible that the length of the arc of $\gamma$ between $P_-, P_+$ is constant not depending on $P$?
Figure 5: Convex hull of two intersecting caustics is also a caustic.

References

[1] V. I. Arnold. *Singularities of caustics and wave fronts*, volume 62 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1990.

[2] Artur Avila, Jacopo De Simoi, and Vadim Kaloshin. An integrable deformation of an ellipse of small eccentricity is an ellipse. *Ann. of Math. (2)*, 184(2):527–558, 2016.

[3] Misha Bialy. Convex billiards and a theorem by E. Hopf. *Math. Z.*, 214(1):147–154, 1993.

[4] George D. Birkhoff. Dynamical systems with two degrees of freedom. *Trans. Amer. Math. Soc.*, 18(2):199–300, 1917.

[5] R. Douady. Applications du théorème des tores invariants. *These, Université Paris VII*, 1982.

[6] H.L. Fetter. Numerical exploration of a hexagonal string billiard. *Physica D*, 241(8):830–846, 2012.

[7] Andrea Hubacher. Instability of the boundary in the billiard ball problem. *Comm. Math. Phys.*, 108(3):483–488, 1987.

[8] N. Innami. Geometry of geodesics for convex billiards and circular billiards. *Nihonkai Math. J.*, 13(1):73–120, 2002.

[9] V. Kaloshin and A. Sorrentino. On Local Birkhoff Conjecture for Convex Billiards. *ArXiv e-prints*, December 2016.

[10] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.
[11] A. Katok, J.-M. Strelcyn, F. Ledrappier, and F. Przytycki. Invariant manifolds, entropy and billiards; smooth maps with singularities, volume 1222 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986.

[12] O. Knill. On nonconvex caustics of convex billiards. Elem. Math., 53(3):89–106, 1998.

[13] V.V. Kozlov and D.V. Treshchev. Billiards, volume 89 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1991. A genetic introduction to the dynamics of systems with impacts, Translated from the Russian by J. R. Schulenberger.

[14] V. F. Lazutkin. Existence of caustics for the billiard problem in a convex domain. Izv. Akad. Nauk SSSR Ser. Mat., 37:186–216, 1973.

[15] V. F. Lazutkin. Vypuklyi billiard i sobstvennye funktsii operatora Laplasa. Leningrad. Univ., Leningrad, 1981.

[16] J.N. Mather and G. Forni. Action minimizing orbits in Hamiltonian systems. In Transition to chaos in classical and quantum mechanics (Montecatini Terme, 1991), volume 1589 of Lecture Notes in Math., pages 92–186. Springer, Berlin, 1994.

[17] A. Stall. Ueber den Kappenkörper eines konvexen Körpers. Commentarii Mathematici Helvetici, 2:35–68, 1930.

[18] Serge Tabachnikov. Geometry and billiards, volume 30 of Student Mathematical Library. American Mathematical Society, Providence, RI; Mathematics Advanced Study Semesters, University Park, PA, 2005.

[19] Philip H. Turner. Convex caustics for billiards in $\mathbb{R}^2$ and $\mathbb{R}^3$. In Convexity and related combinatorial geometry (Norman, Okla., 1980), volume 76 of Lecture Notes in Pure and Appl. Math., pages 85–106. Dekker, New York, 1982.