The PDF of fluid particle acceleration in turbulent flow with underlying normal distribution of velocity fluctuations

A.K. Aringazin and M.I. Mazhitov

Department of Theoretical Physics, Institute for Basic Research, Eurasian National University, Astana 473021 Kazakhstan
(Dated: 15 May 2003)

We describe a formal procedure to obtain and specify the general form of a marginal distribution for the Lagrangian acceleration of fluid particle in developed turbulent flow using Langevin type equation and the assumption that velocity fluctuation \( u \) follows a normal distribution with zero mean, in accord to the Heisenberg-Yaglom picture. For a particular representation, \( \beta = \exp(\eta) \), of the fluctuating parameter \( \beta \), we reproduce the underlying log-normal distribution and the associated marginal distribution, which was found to be in a very good agreement with the new experimental data by Crawford, Mordant, and Bodenschatz on the acceleration statistics. We discuss on arising possibilities to make refinements of the log-normal model.

PACS numbers: 05.20.Jj, 47.27.Jv

I. INTRODUCTION

In the context of Tsallis formalism and generalized statistics approach, C. Beck used the underlying log-normal distribution, \( f(\beta) \), to describe fluctuation of the parameter \( \beta \) entering some Langevin equation for the acceleration \( a \) of fluid particle in the Lagrangian frame; see also. For the case of a linear drift force, the resulting marginal probability density function of the acceleration is \( P(a) \)

\[
P(a) = \frac{1}{2\pi s} \int_{-\infty}^{\infty} d\beta \beta^{-1/2} \exp \left[ -\frac{(\ln s^2)^2}{2s^2} \right] \exp \left[ -\frac{1}{4} \beta a^2 \right], \tag{1}
\]

where \( m = \exp[s^2] \) provides a unit variance and \( s \) is a fitting parameter, \( s^2 = 3.0 \). This distribution was found to be in a very good agreement with the recent Lagrangian experimental data by Porta, Voth, Crawford, and Bodenschatz, the new data by Crawford, Mordant, and Bodenschatz (the Taylor microscale Reynolds number is \( R_\lambda = 690 \), the normalized acceleration range is \([-60, 60]\) \( \geq a \), the Kolmogorov timescale is resolved) \( \mathbf{8} \), Mordant, Delour, Leveque, Arneodo, and Pinton (\( R_\lambda = 740, a \in [-20, 20], \tau_n \) is not resolved) \( \mathbf{10} \), and direct numerical simulations of the Navier-Stokes equations by Kraichnan and Gotoh (\( R_\lambda = 380, a \in [-150, 150] \)) \( \mathbf{11} \). The same approach with the underlying \( \chi^2 \) distribution, \( f(\beta) \), of fluctuations resulted in an analytically explicit distribution \( \mathbf{12} \).

\[
P(a) = \frac{C \exp[-a^2/a_c^2]}{(1 + \frac{1}{2}V_0 (q-1)a_c^2)^{1/(a-1)}}, \tag{2}
\]

where \( C \) is a normalization constant, \( V_0 = 4 \), and \( q = 3/2 \) (Tsallis entropic index) are due to the theory, and \( a_c \) is a free parameter used for a fitting, \( a_c = 36.0 \). This distribution was found to be in a good agreement with the experiments \( \mathbf{9} \). The result \( \mathbf{2} \) is based on the consideration of Ref. \( \mathbf{13} \).

In general, the marginal distribution is defined as

\[
P(a) = \int_0^\infty d\beta P(\beta|a)f(\beta), \tag{3}
\]

where \( P(\beta|a) \) is a probability density function associated to a surrogate dynamical equation, the Langevin equation for the acceleration \( a \).

\[
\partial_t a = \gamma F(a) + \sigma L(t). \tag{4}
\]

Here, \( F(a) \) is a "drift force", \( L(t) \) is delta-correlated Langevin source (Gaussian white noise, \( \langle L(t) \rangle = 0, \langle L(t)L(t') \rangle = 2\delta(t-t') \), \( \beta = \gamma/\sigma^2 \) is assumed to be a fluctuating real positive parameter. For constant parameters \( \gamma \) and \( \sigma \), this model assumes that the stochastic process \( \mathbf{14} \) is Markovian, and \( P(\beta|a) \) is found as a stationary solution of the associated Fokker-Planck equation. For a linear drift force, \( F(a) = -a \), the stationary conditional distribution, \( P(\beta|a) \), is found to be

\[
P(\beta|a) = C(\beta) \exp[-\beta a^2/2], \tag{5}
\]

where \( C(\beta) \) is a normalization constant, \( a \in [-\infty, \infty] \).

The function \( f(\beta) \) entering Eq. (3) is probability density function arising from the assumption that the parameter \( \beta \) is a stochastic variable. For a constant \( \beta \), the Gaussian solution \( \mathbf{5} \) meets the large scale (large time increment) statistics but it fails to describe observed Reynolds number dependent stretched exponential tails of the experimental small scale \( P(a) \) \( \mathbf{8} \) that correspond to high probability to find extremely big values of the fluid particle acceleration in the developed turbulent flow.

The interest in studying Langevin type equations to describe developed turbulence is motivated by the recent high precision Lagrangian experiments \( \mathbf{8} \), \( \mathbf{10} \), \( \mathbf{11} \).
which give an important dynamical information and new look to the intermittency in fluid turbulence. Time response characteristics of the tracer polystyrene particle and the precision in the experiments \cite{1,3} allow to resolve about 1/20 of the Kolmogorov time and 1/40 of the Kolmogorov length ($R_\lambda = 970$) so that the acceleration is really resolved and particle follows rare violent events (within 7% of the ideal value of acceleration even at the high Reynolds numbers studied there), with the collected statistical data sufficient to establish convergence of the fourth moment of acceleration.

In Sec. 2, we construct a simple model that allows one to derive the marginal probability density function, $P(a)$, of the acceleration of fluid particle in turbulent flow, with underlying normally distributed velocity fluctuations. In Sec. 3, we reproduce the result of Ref. \cite{5} as a particular case, and propose a generalization of the log-normal model. In Sec. 4, we briefly summarize the results and make a few remarks.

II. THE MARGINAL DISTRIBUTION WITH UNDERLYING NORMALLY DISTRIBUTED VELOCITY FLUCTUATIONS

While the conditional distribution $P(a|\beta)$ can be found starting from the Langevin equation \cite{4}, we do not have sufficiently strong theoretical requirements to determine a unique form of the distribution $f(\beta)$ except for that it should obey general conditions, such as that the integral in Eq. \cite{4} should converge and the resulting marginal distribution $P(a)$ should be normalizable. To select a particular form of the distribution $f(\beta)$, one can use ad hoc statistical distributions or make assumptions stemming from considerations of the developed turbulence.

As a first step, one can associate the parameter $\beta$ with the mean energy dissipation rate $\epsilon$, and use the relationship of $\epsilon$ with the fluid velocity $v_i$, within the framework of Kolmogorov scaling theory of developed turbulence.

Second, as shown by Renner, Peinke, and Friedrich \cite{14}, the small scale intermittency can be traced back to a stochastic nature of the averaged energy dissipation rate \cite{15}, and the Markovian condition for the stochastic process is fulfilled at scales larger than the Taylor microscale $\lambda$. Thus, a stochastic nature of $\beta$ can be related to the stochastic energy dissipation rate (measured in Eulerian experiments), which we denote by $\tilde{\epsilon}$.

For example, the assumptions $\beta \sim \tilde{\epsilon}$ and $\epsilon \sim \sum u_i^2$ discussed at length by Beck \cite{13} (we do not repeat it here for brevity), for the fluctuating, averaged over a ball of radius $r$, energy dissipation $\epsilon$ and velocity fluctuations $u_i$ at the Kolmogorov scale were used to propose $\beta \sim \sum u_i^2$ and to select the $\chi^2$ distribution of $\beta$. Three-dimensionality of the space was used explicitly to determine the value of the free parameter (Tsallis entropic index) in this model. A comparison of the formalism based on Eq. \cite{4} with the Sawford model \cite{16} implies $\beta = 2C_0^{-2} u_0 v^{1/2} \epsilon^{-3/2}$ \cite{4} so that with the replacement of $\epsilon$ by the fluctuating $\tilde{\epsilon}$ and using again $\tilde{\epsilon} \sim \sum u_i^2$, one can obtain a different relation, $\beta \sim \sum u_i^{-3}$. The Kolmogorov’s 1941 approach implies $\epsilon \sim \bar{u}^3$, where $\bar{u}$ stands for the rms velocity, which can also be used to try to relate $\tilde{\epsilon}$ to the velocity fluctuations. Although successful in capturing main features of the experimental data, different models use different powers of $\epsilon$ to represent $\beta$ that makes a theoretical problem when selecting an appropriate distribution of $\beta$.

It is however common to the above representations that velocity fluctuations, $u_i$, are assumed to be normally distributed with zero mean, and that statistics of $\beta$, $\tilde{\epsilon}$, and $u_i$ are related to each other due to some functional dependencies between their characteristics holding in the inertial range. A direct dependence of statistical properties of the acceleration $a$ on velocity fluctuations was established in the well-known Heisenberg-Yaglom theory. This gives a different look to the problem since the Lagrangian description refers to individual trajectories of fluid particles as compared with the well established Eulerian framework, in which the intermittency is understood in terms of anomalous scaling of the moments of velocity increments in space related to the stochastic nature of the energy dissipation rate (non-dynamical description).

From a simplified dynamical point of view, one can think of that the tracer particle trajectory extends to a large region (few integral length scales, in the experiments) crossing during the course subregions characterized by different local amplitudes of velocity fluctuations which are randomly distributed in space and varies with time (Brownian like motion of the particle driven by the stochastic delta-correlated force with stochastically slow varying intensity). The Lagrangian velocity autocorrelation function is known to decay very slowly, to vanish at times bigger than the integral time scale. This view is valid for sufficiently high Reynolds numbers and meets that provided by the stochastic energy dissipation rate argued earlier \cite{16}. This is also in an agreement with the remark made in Ref. \cite{4} that extremely high accelerations seem to be associated with coherent structures (geometrically these may be thought of as very thin tornadoes) which persist for many Kolmogorov times, substantially longer than the correlation time of the acceleration components. At lower Reynolds numbers, $R_\lambda < 500$, the acceleration is increasingly coupled to large scales of the flow in a dynamical way so that a different approach may be required to describe such a situation to a good accuracy.

In the Lagrangian frame, it is then natural to relate parameters of the stochastic dynamical equation \cite{4}, which describes the acceleration in the context of a generalized Brownian motion, to velocity fluctuations due to the Heisenberg-Yaglom picture of developed turbulence. Ultimately of course we deal with a statistical description of the acceleration and the velocity fluctuations within the framework of Fokker-Planck equation associated to Eq. \cite{4}. Guided by this observation, one can develop a general consideration along this line of reasoning.

In the present paper, we formulate some general re-
requirements for \( f(\beta) \) using the assumption that the velocity fluctuations \( u_i \) are Gaussian distributed random variables with zero means, \( \langle u_i \rangle = 0 \), and variances \( s_i \),

\[
g(u_i) = C(s_i) \exp \left[ -u_i^2/(2s_i^2) \right], \quad (6)
\]

where \( C(s_i) \) is normalization constant; \( u_i \in [-\infty, \infty], \ i = 1, 2, 3 \). This assumption meets the Lagrangian experimental data \( 8 \), for each component of the velocity fluctuations.

It should be stressed that our treatment is made regardless to a particular functional dependence of \( \beta \) on \( \dot{\beta} \), or a particular form of \( f(\beta) \), and provides a simple way to formalize the general model in accord to the Heisenberg-Yaglom picture, which relates some statistical properties of the acceleration to that of the velocity fluctuations.

We assume that \( \beta \) depends on the velocity \( v_t \), the dynamics of which is decoupled from that of the acceleration so that the well known fluctuating character of \( v_t \) for the turbulent flow determines fluctuations of \( \beta \). Below we drop the index \( i \) to simplify notation. Also, we treat the developed turbulent flow to be statistically isotropic and translation invariant thus discarding skewness effects; the statistical anisotropy is known to become very small with the increase of the Reynolds number, e.g., at \( R_L > 900 \).

In the mean field approximation, we can represent \( v = \langle v \rangle + u \), where the fluctuation \( u \) is characterized by zero mean, \( \langle u \rangle = 0 \). Hence, we can write

\[
\beta = \beta_c(\langle v \rangle) + B(v, u), \quad (7)
\]

where we formally separated out \( \beta_c \) which depends only on \( \langle v \rangle \).

We suppose that the function \( B(u) \equiv B(\langle v \rangle, u) \) satisfies the following natural requirements: (i) It is a sufficiently smooth invertible function with respect to \( u \) so that the inverse function \( u(\beta - \beta_c) \) can be found; (ii) it maps the interval \([-\infty, \infty] \) with \([-\beta_c, \infty] \) to provide positivity of \( \beta \); also one can restrict consideration by the requirement that (iii) there is one-to-one correspondence between \( \beta \) and \( u \).

In general, for \( \beta \) to be a stochastic variable the function \( B(u) \) should be Borel function of the stochastic variable \( u \). We assume that \( u \) has an absolutely continuous distribution function and the associated probability density function, in particular, that given by Eq. \( 6 \). Not any Borel function \( B(u) \) provides existence of a probability density function for \( \beta \). However, a probability density function for \( \beta \) a priori exists if \( B(u) \) is a monotonic function.

Using the general relation

\[
g(u)du = g(u(\beta - \beta_c)) \left| \frac{du}{d\beta} \right| d\beta \quad (8)
\]

and Eq. \( 6 \), we can make the identification \( f(\beta) = g(u(\beta - \beta_c))|du/d\beta|, \ i.e.,

\[
f(\beta) = C(s) \exp \left[ -\frac{(u(\beta - \beta_c))^2}{2s^2} \right] \left| \frac{du}{d\beta} \right|, \quad (9)
\]

where \( C(s) \) is a normalization constant. The inverse function \( u(\beta - \beta_c) \) should provide positive definiteness and normalizability of the above function \( f(\beta) \). From Eq. \( 8 \) we then finally have

\[
P(a) = C(s) \int_0^{\infty} d\beta P(a|\beta) \exp \left[ -\frac{|u(\beta - \beta_c)|^2}{2s^2} \right] \left| \frac{du}{d\beta} \right|. \quad (10)
\]

This equation allows one to calculate the distribution \( P(a) \) for given stationary solution \( P(a|\beta) \) associated to the Langevin equation \( 9 \) and the invertible (monotonic) function \( B(u) \). By construction, the structure of this model is such that the velocity fluctuation \( u \), underlying the dynamics, is normally distributed with zero mean and variance \( s \). Note that only the amplitude of \( u \) contributes \( P(a) \). Eq. \( 10 \) can be viewed as a specific case of the general equation \( 6 \) which captures one of the well known features of the turbulent flow, and thus enables to rule out some ad hoc types of distribution of \( \beta \).

Note that in Eq. \( 10 \) we did not change the variable over which the integral is performed as compared to Eq. \( 6 \). The main idea was to introduce the variable \( u \) which follows Gaussian distribution and encode various underlying distributions in the functional dependence \( \beta = B(u) \). This significantly reduces the class of admissible probability density functions \( f(\beta) \) that we consider as one of the main results of our approach.

In the next Section, we study a specific choice of \( B(u) \) which is relevant both from the mathematical and physical points of view.

III. THE LOG-NORMAL MODEL AND BEYOND

As mentioned by Beck \( 3 \), selection of the log-normal distribution \( f(\beta) \) provides that for any power law, \( \beta \sim \dot{\epsilon}^\kappa \), the function \( \ln \beta = \kappa \ln \dot{\epsilon} \) guarantees that the functional form of the log-normal distribution does not change, which is viewed as a hint for a physical relevance of a log-normally distributed \( \beta \). Also, it is known that the probability density function of the averaged energy dissipation rate is log-normal in agreement with Kolmogorov’s assumption in K62 theory.

The exponential dependence,

\[
B(u) = e^{u/u_0}, \quad (11)
\]

where \( u_0 \) is constant, and \( \beta_c = 0 \), provides a relevant example for which we get

\[
u = u_0 \ln \beta, \quad \left| \frac{du}{d\beta} \right| = \frac{|u_0|}{\beta}. \quad (12)
\]

The distribution \( 9 \) then becomes the log-normal distribution, and Eq. \( 10 \) for \( P(a|\beta) \) given by Eq. \( 5 \) reproduces the marginal distribution \( 11 \), with \( u_0 = 1 \).

Also, supposing \( \beta \sim \dot{\epsilon}^\kappa \), we obtain the relation

\[
\dot{\epsilon} \sim e^{u/u_0}, \quad (13)
\]
where we put $u_0 = 1$ for simplicity. Again, one can see that the only measurable effect of the parameter $\kappa$ is that it scales the variance, $s \rightarrow s/\kappa$, in Eq. (10).

One can make the following refinement of the dynamical equation (4) using the relation (11), which corresponds to the log-normal model (1). The defining feature of the considered model is the assumption of slow fluctuating character of the composite parameter $\beta = \gamma/\sigma^2$. It can be easily shown that a noisy character of the drift coefficient $\gamma$, with $F(a) = -a$ and delta-correlated Gaussian white noise $\gamma$ with non-zero mean, can be used to describe intermittency effects (power law tails) because it is related to well-known random multiplicative processes extensively studied in the literature. Below, we focus on a stochastic nature of the additive noise amplitude $\sigma$ and study its contribution to the intermittency separately. A joint effect of both the noisy $\gamma$ and random intensity of the additive noise $\sigma$ is out of scope of the present paper and can be studied elsewhere.

From Eq. (11) it follows $\sigma^2 = e^{-u/u_0}$ so that the Langevin equation (4) becomes
\[
\partial_t a = \gamma F(a) + \exp\left[-u/(2u_0)\right] L(t),
\]
(14)
where the velocity fluctuation $u$ is treated statistically and follows Gaussian distribution with zero mean, $F(a) = -a$, and $\gamma$ is taken to be constant to simplify consideration of the contribution of the additive noise intensity. The Langevin model (14) meets the Lagrangian experiments to a very good accuracy as it implies the probability density function of the form (1).

Equation (14) assumes that the dynamics of Lagrangian acceleration $a$ (the acceleration in the comoving frame, $a_i \equiv \partial_t v_i + v_j \partial_j v_i$), which comes mainly from small scales, and that of the fluctuating energy dissipation rate $\dot{\epsilon}$, or the related amplitude of velocity fluctuations, $u = v_i - \langle v_i \rangle$, which is associated mainly to large scales, are weakly related to each other at high Reynolds numbers, $R_\lambda > 500$, so that in the lowest approximation it is natural to take them independent. At small time scale, the velocity fluctuation is characterized by approximately constant amplitude while its directional part changes wildly (cf. [8]). However, in the statistical context, which reflects Lagrangian dynamics at all the time scales, the acceleration is related to the amplitude of velocity fluctuation as it follows, for example, from the Heisenberg-Yaglom scaling, $\langle a^2 \rangle \approx a_0 u^{5/2} u^{-1/2} L^{-3/2}$, where $\bar{u}$ is the rms velocity and $L$ is the integral scale length. This longstanding universal $\bar{u}^{5/2}$ scaling was confirmed by the recent Lagrangian experiments [8] to a very high accuracy, for $R_\lambda > 500$. The large scale origin of the additive noise stochastic intensity, slowly varying in time, is precisely the reason of the stretching of exponential tails of the acceleration probability density function $P(a)$. The observed Reynolds number dependence of the stretching can be thus qualitatively understood as the result of increasing coupling of the acceleration dynamics to large scale dynamics at lower Reynolds numbers.

In summary, the dynamical equation (14) reflects the above point of view by incorporating exponential of the dynamically independent $u$, with the prescribed external statistics, as the intensity of the additive noise. A full dynamical treatment of Eq. (14) with $u$ viewed as a random function of time can be made elsewhere. Here, we note only that Langevin type equation with the additive noise of the form $\omega(t) L(t)$, with $\omega(t)$ a longtime correlated stochastic process, was recently proposed [10] to describe the Lagrangian intermittency in the context of new multifractal random walk model by Bacry and Muzy, a continuous extension of discrete cascade models. Our approach (14) meets this result. The difference is that we approximate the random process $\omega$ to be trivially stationary and it obeys normal statistics.

In accord to Eqs. (15) and (16), the associated stationary distribution for $F(a) = -a$ is given by
\[
P(a|u) = C(u) \exp\left[-e^{u/u_0}a^2/2\right],
\]
(15)
where $C(u)$ is a normalization constant. From this point of view, small deviations of the theoretical $P(a)$ given by Eq. (1) from the experimental data [9] (see [12] for details) could be attributed to small deviations from Gaussian distribution of the velocity fluctuation $u$; the flatness was measured to be approximately 2.8 for the axial and 3.2 for the longitudinal velocity component [5] compared with the flatness value 3 for an exact Gaussian distribution. Eq. (15) can be used to account for such deviations in a phenomenological way,
\[
P(a) = \int_{-\infty}^{\infty} du \ C(u) \exp\left[-e^{u/u_0}a^2/2\right]g_1(u),
\]
(16)
where $g_1(u)$ is the distribution which is approximately Gaussian one. However, it should be noted that small departure from the experimental data can also arise from shortcomings of the considered one-dimensional Langevin model of developed turbulence.

While it is evident that the three-dimensional Navier-Stokes equation with a Gaussian random forcing belongs to a class of non-linear stochastic equations for which one can associate some generalized Fokker-Planck equations, it is a theoretical challenge to make a link between the Navier-Stokes equation and such a surrogate one-dimensional Langevin model, which is, of course, far from being a full model of the Lagrangian dynamics of fluid in the turbulent regime. This problem is addressed in the forthcoming paper [17]. Here, we note only that the Lagrangian description simplifies the problem of finding the stationary probability measure since the advection term, which dominates in the developed turbulent flow, is incorporated to the definition of Lagrangian acceleration. Also, large scale dynamics and an interaction between the large scale and small scale dynamics within the inertial range are important to provide understanding of the origin of noises (multiplicative and additive ones) used in such Langevin models and a weak dependence of the noise statistics on the small scale dynamics at high Reynolds numbers.
IV. CONCLUSIONS

(i) The presented approach implies a specific form of the probability density function \( f(\beta) \), given by Eq. [9], which stems from the assumption that velocity fluctuations are normally distributed with zero mean, Eq. [6], as it is confirmed by the experimental data \([5, 6]\) for high-Reynolds-number turbulent flows, to a good accuracy.

(ii) Given the exponential dependence \([11]\) the presented formalism leads to a consideration of the log-normal distribution \( f(\beta) \), which was proved to be relevant from the experimental point of view \([3]\). Also, such a distribution is in agreement with the log-normal distribution of mean energy dissipation in Kolmogorov 1962 picture. An exponential relation of the type \([11]\) requires a physical treatment in the context of Langevin, or Fokker-Planck, equation which can be made elsewhere.

Here, we note that the exponential dependence on the velocity fluctuation, \( \exp[u] \), indicates a specific variance of the absolute value of velocity increment during the time-evolution, and can be thought of as a kind of Lyapunov instability in the velocity space.

(iii) One may also be interested in using other distributions, instead of the normal one \([8]\), to derive \( f(\beta) \) and the associated theoretical distribution \( P(u) \).

(iv) Finally, one can try other appropriate functions \( B(u) \), for instance,

\[
B(u) = e^{u/u_0} \sum c_k u^k,
\]

where \( c_k \) are constants, instead of \([11]\), to reproduce the experimental data on the acceleration statistics of fluid particle in turbulent flow. In particular, for \( \beta_c = 0 \) and the function \( B(u) = u^2 \) the map doubly covers the interval \([0, \infty] \ni \beta \), with each covering being a monotonic map, and we obtain \( u = \pm \sqrt{\beta} \) and \( |du/d\beta| = \beta^{-1/2} \). The basic equation \([9]\) yields \( f(\beta) \sim \beta^{-1/2} \exp[-\beta/(2s^2)] \), which is \( \chi^2 \) probability density function of order one.

Acknowledgement. The authors are grateful to referee for valuable comments, which allowed to improve physical argumentation in the revised version of the paper.

\[\text{(17)}\]

\[\text{IV. CONCLUSIONS}\]

(i) The presented approach implies a specific form of the probability density function \( f(\beta) \), given by Eq. [9], which stems from the assumption that velocity fluctuations are normally distributed with zero mean, Eq. [6], as it is confirmed by the experimental data [5, 6] for high-Reynolds-number turbulent flows, to a good accuracy.

(ii) Given the exponential dependence [11] the presented formalism leads to a consideration of the log-normal distribution \( f(\beta) \), which was proved to be relevant from the experimental point of view [3]. Also, such a distribution is in agreement with the log-normal distribution of mean energy dissipation in Kolmogorov 1962 picture. An exponential relation of the type [11] requires a physical treatment in the context of Langevin, or Fokker-Planck, equation which can be made elsewhere.

Here, we note that the exponential dependence on the velocity fluctuation, \( \exp[u] \), indicates a specific variance of the absolute value of velocity increment during the time-evolution, and can be thought of as a kind of Lyapunov instability in the velocity space.

(iii) One may also be interested in using other distributions, instead of the normal one [8], to derive \( f(\beta) \) and the associated theoretical distribution \( P(u) \).

(iv) Finally, one can try other appropriate functions \( B(u) \), for instance,

\[
B(u) = e^{u/u_0} \sum c_k u^k,
\]

where \( c_k \) are constants, instead of [11], to reproduce the experimental data on the acceleration statistics of fluid particle in turbulent flow. In particular, for \( \beta_c = 0 \) and the function \( B(u) = u^2 \) the map doubly covers the interval \([0, \infty] \ni \beta \), with each covering being a monotonic map, and we obtain \( u = \pm \sqrt{\beta} \) and \( |du/d\beta| = \beta^{-1/2} \). The basic equation [9] yields \( f(\beta) \sim \beta^{-1/2} \exp[-\beta/(2s^2)] \), which is \( \chi^2 \) probability density function of order one.

Acknowledgement. The authors are grateful to referee for valuable comments, which allowed to improve physical argumentation in the revised version of the paper.

\[\text{(17)}\]