EXCEPTIONAL HOLONOMY ON VECTOR BUNDLES WITH TWO-DIMENSIONAL FIBERS

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Abstract. An $SU(3)$- or $SU(1,2)$-structure on a 6-dimensional manifold $N^6$ can be defined as a pair of a 2-form $\omega$ and a 3-form $\rho$. We prove that any analytic $SU(3)$- or $SU(1,2)$-structure on $N^6$ with $d\omega \wedge \omega = 0$ can be extended to a parallel Spin(7)- or Spin$_0(3,4)$-structure $\Phi$ that is defined on the trivial disc bundle $N^6 \times B_\epsilon(0)$ for a sufficiently small $\epsilon > 0$. Furthermore, we show by an example that $\Phi$ is not uniquely determined by $(\omega, \rho)$ and discuss if our result can be generalized to non-trivial bundles.

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1. Introduction

In his article on stable forms, Hitchin [9] proposed a new method to construct manifolds with exceptional holonomy. The starting point of his construction is a 7-dimensional manifold $M$ with a $G_2$-structure $\phi$ that satisfies $d^* \phi = 0$. We can take $\phi$ as an initial value for a certain flow equation such that the solution of the initial value problem yields a parallel Spin(7)-structure on $M \times (-\epsilon, \epsilon)$ for an $\epsilon > 0$. This idea can be generalized to the semi-Riemannian case where we obtain a parallel Spin$_0(3,4)$-structure [6].

Many of the known complete metrics with holonomy Spin(7) are not defined on a manifold of type $M \times (-\epsilon, \epsilon)$ but on a disc bundle over a lower-dimensional manifold [11 2 5 7 10 11 14]. The reason behind this is that
those metrics are of cohomogeneity one and that the cohomogeneity-one
manifolds of this type are the only ones that admit complete metrics with
holonomy Spin(7) \cite{14}.

Bielawski \cite{3} proves another result that fits into this context. Let \( X \) be a
real analytic Kähler manifold. We identify \( X \) with the zero section of its
canonical bundle. The Kähler metric on \( X \) can be uniquely extended to a
Ricci-flat Kähler metric on a neighborhood of \( X \) such that the \( U(1) \)-action
on the bundle is isometric and Hamiltonian. We thus have extended the
\( U(n) \)-structure on the base to an \( SU(n+1) \)-structure on the bundle.

Motivated by these facts, we attempt to construct parallel Spin(7)- or
Spin\(_0\)\((3,4)\)-structures on \( \mathbb{R}^2 \)-bundles. More precisely, let \( (\omega, \rho) \) be a pair
of a 2-form and a 3-form on a 6-dimensional manifold \( N^6 \) that defines an
\( SU(3) \)- or \( SU(1,2) \)-structure. We search for conditions on \( (\omega, \rho) \) such that
on \( N^6 \times B_\epsilon(0) \), where \( B_\epsilon(0) \) is a ball of radius \( \epsilon > 0 \) in \( \mathbb{R}^2 \), there exists a
parallel Spin(7)- or Spin\(_0\)\((3,4)\)-structure that extends in a suitable sense the
\( G \)-structure \( (\omega, \rho) \).

The article is organized as follows. In Section 2 and 3 we give an introduc-
tion to the \( G \)-structures that we need and to Hitchin’s flow equation. We
set up our initial value problem and prove that it has a local solution in the
following section. After that we show with help of an example that our solu-
tion can be non-unique. In the sixth section, we finally discuss if our result
can be generalized to non-trivial bundles over 6-dimensional manifolds.

2. \( G \)-structures

2.1. \( G \) is \( SU(3) \) or \( SU(1,2) \). In order to prove our theorem we have to
introduce several \( G \)-structures. We start with \( G \)-structures on 6-dimensional
manifolds and then proceed to the 7- and 8-dimensional case. A well written
introduction to all of these \( G \)-structures can be found in Cortés et al. \cite{6}.

We use similar conventions as \cite{6} and only recapitulate the facts that we
need for our considerations. Although a \( G \)-structure is in general defined as
a principal bundle, all \( G \)-structures in this section can be described with help
of certain differential forms. Throughout this article we use the following
convention.

\textbf{Convention 2.1.} Let \( (v_i)_{i \in I} \) be a basis of a vector space \( V \). We denote its
dual basis by \( (v^i)_{i \in I} \) and abbreviate \( v^{i_1} \wedge \ldots \wedge v^{i_k} \) by \( v^{i_1 \ldots i_k} \).

Let \( (e_i)_{i=1,\ldots,6} \) be the canonical basis of \( \mathbb{R}^6 \). We define the 2-forms

\begin{equation}
\omega_{SU(3)} := e^{12} + e^{34} + e^{56}
\end{equation}

and
Moreover, we introduce the canonical 3-form

(3) \[ \rho_{\text{can.}} := e^{135} - e^{146} - e^{236} - e^{245}. \]

The following lemma is proven in [6].

**Lemma 2.2.** Let \( G \in \{ SU(3), SU(1, 2) \} \). The subgroup of all \( A \in GL(6, \mathbb{R}) \) that stabilize \( \omega_G \) and \( \rho_{\text{can.}} \) simultaneously is isomorphic to \( G \).

This motivates the following definition.

**Definition 2.3.** Let \( G \in \{ SU(3), SU(1, 2) \} \), \( V \) be a 6-dimensional real vector space and \( (\omega, \rho) \) be a pair of a 2-form and a 3-form on \( V \). If there exists a basis \( (v_i)_{i=1,...,6} \) of \( V \) such that with respect to this basis \( \omega \) can be identified with \( \omega_G \) and \( \rho \) with \( \rho_{\text{can.}} \), \( (\omega, \rho) \) is called a \( G \)-structure.

Hitchin [9] has introduced the notion of a stable form.

**Definition 2.4.** Let \( V \) be a real or complex vector space and \( \beta \in \bigwedge^k V^* \) with \( k \in \{0, \ldots, \dim V\} \) be a \( k \)-form. \( \beta \) is called stable if the \( GL(V) \)-orbit of \( \beta \) is an open subset of \( \bigwedge^k V^* \).

**Lemma 2.5.** Let \( (\omega, \rho) \) be a \( G \)-structure where \( G \in \{ SU(3), SU(1, 2) \} \). In this situation, \( \omega \) and \( \rho \) are both stable forms.

**Remark 2.6.** The stable forms are an open dense subset of \( \bigwedge^2 \mathbb{R}^{6*} \) and of \( \bigwedge^3 \mathbb{R}^{6*} \). There is exactly one open \( GL(6, \mathbb{R}) \)-orbit in \( \bigwedge^2 \mathbb{R}^{6*} \) and two open orbits in \( \bigwedge^3 \mathbb{R}^{6*} \). One of them is the orbit of \( \rho_{\text{can.}} \). The other one can be used to define the notion of an \( SL(3, \mathbb{R}) \)-structure, which we will not consider in this article.

Let \( V \) be a 6-dimensional real vector space and \( \bigwedge^k V^* \) be the set of all stable \( k \)-forms on \( V \). We can assign to any \( \rho \in \bigwedge^3 V^* \) a certain endomorphism \( J_\rho \) by a map

(4) \[ i : \bigwedge^3 V^* \to V \otimes V^*. \]

\( i \) is a rational \( GL(6, \mathbb{R}) \)-equivariant map and is described in detail in [3]. \( i(\rho_{\text{can.}}) \) is the canonical complex structure on \( \mathbb{R}^6 \) which maps \( e_{2i-1} \) to \( -e_{2i} \) and \( e_{2i} \) to \( e_{2i-1} \) for all \( i \in \{1, 2, 3\} \). If \( (\omega, \rho) \) is an \( SU(3) \)- or an \( SU(1, 2) \)-structure, \( J_\rho \) is a complex structure, too. With help of another map

(5) \[ j : \bigwedge^2 V^* \times \bigwedge^3 V^* \to S^2(V^*) \]
we can assign to \((\omega, \rho)\) a symmetric non-degenerate bilinear form. \(j\) is also a rational \(GL(6,\mathbb{R})\)-equivariant map that is described explicitly in \([6]\). If \((\omega, \rho)\) is an

1. \(SU(3)\)-structure, \(j(\omega, \rho)\) is a metric with signature \((6, 0)\). In particular, \(j(\omega_{SU(3)}, \rho_{\text{can}})\) is the Euclidean metric on \(\mathbb{R}^6\).
2. \(SU(1,2)\)-structure, \(j(\omega, \rho)\) is a metric with signature \((2, 4)\). In particular,

\[
j(\omega_{SU(1,2)}, \rho_{\text{can}}) = -e^1 \otimes e^1 - e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4 + e^5 \otimes e^5 + e^6 \otimes e^6.
\]

**Convention 2.7.**

1. We call \(J\rho\) the complex structure that is associated to \(\rho\) or shortly the associated complex structure.
2. We call \(j(\omega, \rho)\) the metric that is associated to \((\omega, \rho)\) or shortly the associated metric. We denote it by \(g_6\), since we will also work with metrics on 7- or 8-dimensional spaces.

We remark that the objects that we have defined are related by the formula

\[
\omega(v, w) := g_6(v, J_\rho(w)).
\]

We can decide if a pair \((\omega, \rho)\) determines an \(SU(3)\)- or \(SU(1,2)\)-structure without referring to a special basis.

**Theorem 2.8.** Let \(V\) be a 6-dimensional real vector space and let \(\omega \in \bigwedge^2 V^*\) and \(\rho \in \bigwedge^3 V^*\) be stable. Moreover, let \(J_\rho\) and \(g_6\) be defined as above. We assume that \(\omega\) and \(\rho\) satisfy the equations

1. \(\omega \wedge \rho = 0\),
2. \(J_\rho^* \rho \wedge \rho = \frac{2}{3} \omega \wedge \omega \wedge \omega\).

If in this situation

1. \(g_6\) has signature \((6, 0)\) and \(J_\rho\) is a complex structure, \((\omega, \rho)\) is an \(SU(3)\)-structure.
2. \(g_6\) has signature \((2, 4)\) and \(J_\rho\) is a complex structure, \((\omega, \rho)\) is an \(SU(1,2)\)-structure.

**Remark 2.9.**

1. Since \(J_\rho^* \rho \wedge \rho\) and \(\frac{2}{3} \omega \wedge \omega \wedge \omega\) are both 6-forms, the second condition from the theorem is a normalization of the pair \((\omega, \rho)\).
2. If \((\omega, \rho)\) is a pair of stable forms satisfying \(\omega \wedge \rho = 0\) and \(J_\rho^* \rho \wedge \rho = \frac{2}{3} \omega \wedge \omega \wedge \omega\) and it is not an \(SU(3)\)- or \(SU(1,2)\)-structure, \(J_\rho\) is a para-complex structure and \((\omega, \rho)\) is an \(SL(3,\mathbb{R})\)-structure.

The reason for the above considerations is to define \(G\)-structures on manifolds.
Definition 2.10. Let $M$ be a 6-dimensional manifold, $\omega \in \bigwedge^2 T^* M$, and $\rho \in \bigwedge^3 T^* M$. Moreover, let $G \in \{ SU(3), SU(1, 2) \}$. $(\omega, \rho)$ is called a $G$-structure on $M$ if for all $p \in M$, $(\omega_p, \rho_p)$ is a $G$-structure on $T_p M$.

Convention 2.11. Since the endomorphism field $J_{\rho}$ in general has torsion, we call it the almost complex structure on $M$.

2.2. $G$ is $G_2$ or $G_2^*$. With help of the concepts from the previous subsection we are able to define $G_2$- and $G_2^*$-structures.

Definition and Lemma 2.12. We supplement the basis $(e_i)_{i=1,\ldots,6}$ of $\mathbb{R}^6$ with $e_7$ to a basis of $\mathbb{R}^7$. The form
\[
(1) \quad \phi_{G_2} := \omega_{SU(3)} \wedge e^7 + \rho_{\text{can.}} \text{ is stabilized by } G_2, \\
(2) \quad \phi_{G_2^*} := \omega_{SU(1,2)} \wedge e^7 + \rho_{\text{can.}} \text{ is stabilized by } G_2^*.
\]

$G_2$ denotes the compact real form of the complex Lie group $G_2^C$ and $G_2^*$ denotes the split real form. Let $V$ be a 7-dimensional real vector space and $\phi$ be a 3-form on $V$. If there exists a basis $(v_i)_{i=1,\ldots,7}$ of $V$ such that with respect to $(v_i)_{i=1,\ldots,7}$
\[
(1) \quad \phi \text{ can be identified with } \phi_{G_2}, \ \phi \text{ is called a } G_2\text{-structure.} \\
(2) \quad \phi \text{ can be identified with } \phi_{G_2^*}, \ \phi \text{ is called a } G_2^*\text{-structure.}
\]

Remark 2.13. There are exactly two open orbits of the action of $GL(7, \mathbb{R})$ on $\bigwedge^3 \mathbb{R}^7$. [13, 15]. Their union is a dense subset of $\bigwedge^3 \mathbb{R}^7$. One orbit consists of all 3-forms that are stabilized by $G_2$ and the other one consists of all 3-forms that are stabilized by $G_2^*$.

Any $G_2$- or $G_2^*$-structure on a vector space $V$ determines a symmetric non-degenerate bilinear form $g_7$ and a volume form $\text{vol}_7$. As in the previous subsection, there are explicit rational $GL(7, \mathbb{R})$-equivariant maps $\bigwedge^3 V^* \to S^2(V^*)$ and $\bigwedge^3 V^* \to \bigwedge^7 V^*$ that assign $g_7$ and $\text{vol}_7$ to $\phi$. The explicit definition of these maps can be found in [9]. The tensors $\phi, g_7,$ and $\text{vol}_7$ are related by the formula
\[
g_7(v, w) \text{ vol}_7 = \frac{1}{6} (v \wedge \phi) \wedge (w \wedge \phi) \wedge \phi \quad \forall v, w \in V.
\]

Analogously to Subsection 2.1 we have

Lemma 2.14. Let $V$ be a 7-dimensional real vector space and $\phi$ be a stable 3-form on $V$.
\[
(1) \quad \text{If } \phi \text{ is a } G_2\text{-structure, } g_7 \text{ has signature } (7, 0). \text{ In particular, } g_7 \text{ is the Euclidean metric on } \mathbb{R}^7 \text{ if } \phi \text{ coincides with } \phi_{G_2}. \\
(2) \quad \text{If } \phi \text{ is a } G_2^*\text{-structure, } g_7 \text{ has signature } (3, 4). \text{ In particular, } g_7 = g_6 + e^7 \otimes e^7 \text{ if } \phi \text{ coincides with } \phi_{G_2^*}.
\]
We can relate $\text{vol}_7$ to the 3-forms on the 6-dimensional subspace $\text{span}(v_i)_{i=1,...,6}$.

**Lemma 2.15.** Let $\phi$ be a $G_2$- or $G^*_2$-structure on a vector space $V$ and $(v_i)_{i=1,...,7}$ be a basis of $V$ with the properties from Definition and Lemma 2.12. On $\text{span}(v_i)_{i=1,...,6}$ there exists a canonical $SU(3)$- or $SU(1,2)$-structure $(\omega, \rho)$ and we have

\begin{equation}
\text{vol}_7 = \frac{1}{4} J_\rho^* \rho \wedge \rho \wedge v^7.
\end{equation}

In particular, $\text{vol}_7$ is $e^{1234567}$ if $\phi$ is $\phi_{G_2}$ or $\phi_{G^*_2}$.

g_7 and $\text{vol}_7$ determine a Hodge-star operator $\ast$ on $\bigwedge^* V^*$.

**Lemma 2.16.** Let $\phi$ be a $G_2$- or $G^*_2$-structure. The 4-form $\ast \phi$ is stable and can be described as

\begin{equation}
v^7 \wedge J_\rho^* \rho + \frac{1}{2} \omega \wedge \omega.
\end{equation}

**Convention 2.17.** We call $g_7$ ($\text{vol}_7$, $\ast \phi$) the metric (volume form, 4-form) that is associated to $\phi$.

We define $G_2$- and $G^*_2$-structures on manifolds as in the previous subsection.

**Definition 2.18.** Let $M$ be a 7-dimensional manifold and $\phi \in \bigwedge^3 T^* M$. Moreover, let $G \in \{G_2, G^*_2\}$. $\phi$ is called a $G$-structure on $M$ if for all $p \in M$ $\phi_p$ is a $G$-structure on $T_p M$.

2.3. $G$ is Spin(7) or Spin$_0(3,4)$. In this final subsection, we introduce Spin(7)- and Spin$_0(3,4)$-structures.

**Definition and Lemma 2.19.** We supplement the basis $(e_i)_{i=1,...,7}$ of $\mathbb{R}^7$ with $e_8$ to a basis of $\mathbb{R}^8$. The form

1. $\Phi_{\text{Spin}(7)} := e^8 \wedge \phi_{G_2} + *\phi_{G_2}$ is stabilized by Spin(7).
2. $\Phi_{\text{Spin}_0(3,4)} := e^8 \wedge \phi_{G^*_2} + *\phi_{G^*_2}$ is stabilized by the identity component Spin$_0(3,4)$ of Spin(3, 4).

Let $V$ be an 8-dimensional real vector space and $\Phi$ be a 4-form on $V$. If there exists a basis $(v_i)_{i=1,...,8}$ of $V$ such that with respect to $(v_i)_{i=1,...,8}$

1. $\Phi$ can be identified with $\Phi_{\text{Spin}(7)}$, $\Phi$ is called a Spin(7)-structure.
2. $\Phi$ can be identified with $\Phi_{\text{Spin}_0(3,4)}$, $\Phi$ is called a Spin$_0(3,4)$-structure.

Analogously to Subsection 2.1 and 2.2 any Spin(7)- or Spin$_0(3,4)$-structure determines a symmetric non-degenerate bilinear form $g_8$ and a volume form $\text{vol}_8$. $\text{vol}_8$ is given by $\frac{1}{14} \Phi \wedge \Phi$ and $g_8$ satisfies a slightly more complicated relation as in [12], which can be found in Karigiannis [12].
Unlike $\omega$, $\rho$, and $\phi$, $\Phi$ is not a stable form. Nevertheless, we have similar results as in the previous two subsections.

**Lemma 2.20.** Let $\Phi$ be a Spin(7)- or Spin$_0(3,4)$-structure. In the first case $g_8$ has signature $(8,0)$ and in the second case it has signature $(4,4)$. In particular, $g_8$ is the Euclidean metric on $\mathbb{R}^8$ if $\Phi$ coincides with $\Phi_{\text{Spin}(7)}$ and $g_8 = g_7 + e^8 \otimes e^8$ if $\Phi$ coincides with $\Phi_{\text{Spin}_0(3,4)}$. In both cases, we have

$$\text{(11)} \quad \text{vol}_8 = \text{vol}_7 \land v^8.$$  

**Convention 2.21.** As in the previous subsections, we call $g_8$ the associated metric and $\text{vol}_8$ the associated volume form.

**Remark 2.22.**
1. $\Phi$ is self-dual with respect to $g_8$ and $\text{vol}_8$.
2. Any 4-form on an 8-dimensional real vector space that is stabilized by Spin(7) or Spin$_0(3,4)$ is a Spin(7)- or Spin$_0(3,4)$-structure. However, there is no simple criterion like Theorem 2.8 that decides if a given 4-form is a Spin(7)- or Spin$_0(3,4)$-structure.

The notion of a Spin(7)- or a Spin$_0(3,4)$-structure on an 8-dimensional manifold can be defined completely analogously to Definition 2.10 and 2.18.

### 3. Hitchin’s flow equation

One motivation to study $G$-structures is their relation to metrics with special holonomy.

**Definition 3.1.** Let $G \in \{\text{Spin}(7), \text{Spin}_0(3,4)\}$ and let $\Phi$ be a $G$-structure on an 8-dimensional manifold. $\Phi$ is called torsion-free if $d\Phi = 0$.

**Lemma 3.2.** Let $G$ be as above. The holonomy group of the metric that is associated to a torsion-free $G$-structure is a subgroup of $G$. Conversely, let $(M,g)$ be a semi-Riemannian manifold whose holonomy is contained in $G$. Then there exists a torsion-free $G$-structure on $M$ whose associated metric is $g$.

**Proof.** See [8] for $G = \text{Spin}(7)$ and [4] for $G = \text{Spin}_0(3,4)$.

**Remark 3.3.** There are analogous results for $G \in \{SU(3), SU(1,2), G_2, G_2^*\}$.

We also need the following $G$-structures with torsion.

**Definition 3.4.** Let $\phi$ be a $G_2$- or $G_2^*$-structure on a 7-dimensional manifold. $\phi$ is called cocalibrated if $d^* \phi = 0$.

Compact Riemannian manifolds with holonomy Spin(7) are hard to construct. However, many non-compact examples with cohomogeneity one are known [1, 2, 5, 7, 10, 11, 14]. All of these metrics can be obtained by a method that was developed by Hitchin [9]. As in the previous section, our presentation of the issue is similar as in [6].
Theorem 3.5. (See [6, 9]) Let $N^7$ be a 7-dimensional manifold and $U \subset N^7 \times \mathbb{R}$ be an open neighborhood of $N^7 \times \{0\}$. Furthermore, let $G \in \{G_2, G_2^*\}$ and $\phi$ be a cocalibrated $G$-structure on $N^7$. Finally, let $\phi_t$ be a one-parameter family of 3-forms such that $\phi_t$ is defined on $U \cap (N^7 \times \{t\})$. We assume that $\phi_t$ is a solution of the initial value problem

\begin{equation}
\frac{\partial}{\partial t} \ast_7 \phi_t = d_7 \phi_t
\end{equation}

\begin{equation}
\phi_0 = \phi
\end{equation}

The index "7" emphasizes that we consider $\ast$ and $d$ as operators on $U \cap (N^7 \times \{t\})$ instead of $U$. If $U$ is sufficiently small, $\phi_t$ is a $G$-structure for all $t$ with $U \cap (N^7 \times \{t\}) \neq \emptyset$. Moreover, it is cocalibrated for all $t$. The 4-form

\begin{equation}
\Phi := dt \wedge \phi_t + \ast_7 \phi_t
\end{equation}

is a torsion-free Spin(7)-structure if $G = G_2$ and a torsion-free Spin$_0(3,4)$-structure if $G = G_2^*$. Let $g_8$ be the metric that is associated to $\Phi$ and $g_t$ be the metric on $N^7 \times \{t\}$ that is associated to $\phi_t$. With this notation we have

\begin{equation}
g_8 = g_t + dt^2.
\end{equation}

Remark 3.6. (1) The equation $\frac{\partial}{\partial t} \ast_7 \phi_t = d_7 \phi_t$ is called Hitchin’s flow equation. Since $\ast_7$ depends non-linearly on $\phi_t$, it is a non-linear partial differential equation.

(2) If $N^7$ and $\phi_0$ are real analytic, the system [12] has a unique maximal solution that is defined on an open neighborhood of $N^7 \times \{0\}$ [6]. This is a consequence of the Cauchy-Kovalevskaya Theorem. We assume from now that all initial data are analytic.

(3) If $N^7$ is in addition compact, there exists a unique maximal open interval $I$ with $0 \in I$ such that the solution is defined on $N^7 \times I$.

(4) Let $f : N^7 \to N^7$ be a diffeomorphism, $I$ an interval with $0 \in I$, $U = N^7 \times I$, and $\phi_t$ be a solution of Hitchin’s flow equation on $U$. In this situation, the pull-back $f^* \phi_t$ is also a solution with the initial value $f^* \phi_0$.

4. Proof of the main theorem

In this section, we consider a 6-dimensional manifold $N^6$ that carries an $SU(3)$- or $SU(1,2)$-structure $(\omega_0, \rho_0)$. Our aim is to construct a parallel Spin(7)- or Spin$_0(3,4)$-structure $\Phi$ on a tubular neighborhood of the zero section of the trivial bundle $N^6 \times \mathbb{R}^2$ such that the restriction of $\Phi$ to $N^6$ is $(\omega_0, \rho_0)$ in a suitable sense. More precisely, let $\epsilon > 0$ be sufficiently small and
We denote $N^6 \times \{0\} \subset N^6 \times B_{\epsilon}(0)$ shortly by $N^6$. On that submanifold we want to have

$$\Phi = \frac{1}{2}\omega_0 \wedge \omega_0 + dx \wedge \rho_0 + dy \wedge J^*_\rho \rho_0 + dx \wedge dy \wedge \omega_0$$
or equivalently

$$\frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \Phi \right) = \omega_0$$
$$\frac{\partial}{\partial x} \Phi - dy \wedge \omega_0 = \rho_0$$

Our first step is to construct a $G_2$- or $G_2^*$-structure $\phi$ on

$$V_\epsilon := N^6 \times \{(0, y) \in \mathbb{R}^2 | y^2 < \epsilon\}$$

that satisfies

$$\phi = \rho + dy \wedge \omega \quad \text{and} \quad d * \phi = 0$$

for a $y$-dependent $SU(3)$- or $SU(1, 2)$-structure $(\omega, \rho)$ on $N^6$. Next, we insert $\phi$ as initial condition into Hitchin’s flow equation, where $x$ plays the role of the coordinate $t$ in Theorem 3.5. After that, we have finally found our $\Phi$. We describe how to construct the 3-form on $V_\epsilon$. The Hodge dual of $\phi$ is

$$* \phi = \frac{1}{2}\omega \wedge \omega + dy \wedge J^*_\rho \rho.$$

$\phi$ is thus cocalibrated if and only if

$$\left( \frac{\partial}{\partial y} \omega \right) \wedge \omega = dJ^*_\rho \rho$$
$$d\omega \wedge \omega = 0$$

for all $y$. In the above equation, $d$ denotes the exterior derivative on the 6-dimensional manifold $N^6 \times \{(0, y)\}$. We see that any choice of $\rho$ satisfies the system (20). Since

$$\omega \wedge \omega) = \omega_0 \wedge \omega_0 + 2 \int_0^y dJ^*_\rho \rho d\tilde{y}$$
and \( d^2 = 0, d\omega \wedge \omega = 0 \) is satisfied for all \( y \) if it is satisfied for \( y = 0 \). Of course, \( (\omega, \rho) \) shall be an \( SU(3) \)- or \( SU(1,2) \)-structure for all \( y \in (\epsilon, \epsilon) \). Therefore, the system that \( (\omega, \rho) \) has to satisfy is in fact

\[
\left( \frac{\partial}{\partial y} \omega \right) \wedge \omega = dJ^*_\rho \rho \\
\omega \wedge \rho = 0 \\
2\omega^3 = 3 \rho \wedge J^*_\rho \rho 
\]

If we take the derivative of the last two equations with respect to \( y \), we obtain the following system of first order differential equations

\[
\left( \frac{\partial}{\partial y} \omega \right) \wedge \omega = dJ^*_\rho \rho \\
3 \left( \frac{\partial}{\partial y} \rho \right) \wedge J^*_\rho \rho + 3 \rho \wedge \left( \frac{\partial}{\partial y} J^*_\rho \rho \right) - 6 \left( \frac{\partial}{\partial y} \omega \right) \wedge \omega^2 = 0 
\]

with the initial conditions

\[
d\omega_0 \wedge \omega_0 = 0 \\
\omega_0 \wedge \rho_0 = 0 \\
2\omega_0^3 = 3 \rho_0 \wedge J^*_{\rho_0} \rho_0 
\]

Since all forms in a neighborhood of \( \omega_0 \) or \( \rho_0 \) are stable, any solution of \( (23) \) and \( (24) \) describes a \( G_2 \)- or \( G_2^* \)-structure if \( \epsilon \) is sufficiently small. Let \( z^1, \ldots, z^6 \) be coordinates on an open subset \( U \subset \mathbb{C}^6 \). The system \( (23) \) consists of 22 equations for the 35 coefficient functions of \( \omega \) and \( \rho \). It can be written as

\[
F \left( \omega, \rho, \frac{\partial \omega}{\partial \overline{z}^1}, \ldots, \frac{\partial \omega}{\partial \overline{z}^6}, \frac{\partial \rho}{\partial \overline{z}^1}, \ldots, \frac{\partial \rho}{\partial \overline{z}^6}, \frac{\partial \omega}{\partial y}, \frac{\partial \rho}{\partial y} \right) = 0 .
\]

\( \omega \) is up to the sign uniquely determined by \( \omega^2 \) \([6, 9]\). The first equation of \( (23) \) thus fixes the value of \( \frac{\partial \omega}{\partial y} \). The second and third equation restrict \( \rho \) at each \( p \in U \) to a submanifold of \( \wedge^3 T^*_p U \) of codimension 7. \( (dF)(\frac{\partial \omega}{\partial y}, \frac{\partial \rho}{\partial y}) \) therefore has maximal rank. The metric that is associated to \( (\omega, \rho) \) induces a metric on \( \wedge^3 T^*_p U \). We denote the orthogonal projection of a 3-form to the tangent space of the set of all \( \rho \) that satisfy \( \omega \wedge \rho = 0 \) and \( 2\omega^3 = 3 \rho \wedge J^*_\rho \rho \) by \( \pi_\omega \). We add the equation

\[
\pi_\omega \left( \frac{\partial \rho}{\partial y} \right) = 0 
\]
to \((23)\) and obtain a system of type \((25)\), where \(F\) is replaced by a \(\tilde{F}\) that satisfies

\[
\text{rk}(d\tilde{F}(\frac{\partial\omega}{\partial t}, \frac{\partial\rho}{\partial t})) = 35.
\]

With help of the implicit function theorem, the extended system can be rewritten to

\[
\begin{align*}
\frac{\partial\omega}{\partial y} &= F_1(\omega, \rho, \frac{\partial\omega}{\partial x_1}, \ldots, \frac{\partial\omega}{\partial x_6}, \frac{\partial\rho}{\partial x_1}, \ldots, \frac{\partial\rho}{\partial x_6}) \\
\frac{\partial\rho}{\partial y} &= F_2(\omega, \rho, \frac{\partial\omega}{\partial x_1}, \ldots, \frac{\partial\omega}{\partial x_6}, \frac{\partial\rho}{\partial x_1}, \ldots, \frac{\partial\rho}{\partial x_6})
\end{align*}
\]

Since \(N^6\) is a real analytic manifold, \(F_1\) and \(F_2\) are analytic, too. As in [6], the Cauchy-Kovalevskaya theorem guarantees that the extended system has a unique solution on an open neighbourhood of \(N^6 \subset N^6 \times \mathbb{R}\). Thus, \((23)\) has at least one solution on the same open set. If \(N^6\) is compact, the solution exists on \(V_\epsilon\) for a sufficiently small \(\epsilon > 0\). With help of Theorem 3.5, we are finally able to prove our main theorem.

**Theorem 4.1.** Let \(N^6\) be an analytic compact 6-manifold and let \((\omega_0, \rho_0)\) be an analytic \(SU(3)\)- or \(SU(1, 2)\)-structure with \(d\omega_0 \wedge \omega_0 = 0\) on \(N^6\). Then, there exists an \(\epsilon > 0\) and a parallel \(Spin(7)\)- or \(Spin_0(3, 4)\)-structure \(\Phi\) on \(N^6 \times B_\epsilon(0)\) such that on \(N^6 \times \{0\}\) we have

\[
\begin{align*}
\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x} \Phi &= \omega_0 \\
\frac{\partial}{\partial x} \Phi - dy \wedge \omega_0 &= \rho_0
\end{align*}
\]

where \(x\) and \(y\) are the standard coordinates on \(B_\epsilon(0)\).

### 5. An example

In this section, we show that the 4-form \(\Phi\) from Theorem 4.1 is not uniquely determined by the initial value \((\omega_0, \rho_0)\). Before we start, we define what we mean by uniqueness in this situation.

**Definition 5.1.** Let \(\Phi_1\) and \(\Phi_2\) be two \(Spin(7)\)- or \(Spin_0(3, 4)\)-structures on \(N^6 \times B_\epsilon(0)\) such that on \(N^6 \times \{0\}\) we have

\[
\begin{align*}
\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x} \Phi_1 &= \omega_0 \\
\frac{\partial}{\partial x} \Phi_1 - dy \wedge \omega_0 &= \rho_0
\end{align*}
\]

We call \(\Phi_1\) and \(\Phi_2\) equivalent if there exists a diffeomorphism \(f\) of \(N^6 \times B_\epsilon(0)\) that is the identity on \(N^6 \times \{0\}\) and satisfies \(f^*\Phi_1 = \Phi_2\). Analogously, let
\( \phi_1 \) and \( \phi_2 \) be \( G_2 \)- or \( G_2^* \)-structures on \( N^6 \times (-\epsilon, \epsilon) \) such that on \( N^6 \times \{0\} \) we have

\[
\frac{\partial}{\partial y} \phi_1 = \frac{\partial}{\partial y} \phi_2 =: \omega_0
\]

\( \phi_1 \) and \( \phi_2 \) are called equivalent if there exists a diffeomorphism of \( N^6 \times (-\epsilon, \epsilon) \) with the same properties as above.

We restrict ourselves to the Riemannian case. For our example, \((\omega_0, \rho_0)\) shall be torsion-free. In other words, \( N^6 \) together with the initial \( SU(3) \)-structure is a Calabi-Yau manifold. Our strategy is to construct a one-parameter family of \( G_2 \)-structures \( \phi_\delta \) on \( N^6 \times S^1 \) such that the standard coordinate \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \) of \( S^1 \) plays the role of \( y \). After that, we consider Hitchin’s flow equation with initial value \( \phi_\delta \) in order to obtain 4-forms \( \Phi_\delta \). Let \( \alpha \) be a closed 3-form on \( N^6 \). We define a \( G_2 \)-structure \( \phi_\delta \) on \( N^6 \times S^1 \) by

\[
\phi_\delta = \omega_0 \wedge d\theta - J_{\rho_0} \rho_0 + \delta \cdot \sin \theta \cdot *_6 \alpha
\]

where \(*_6\) is the Hodge-star on \( N^6 \). We have

\[
* \phi_\delta = d\theta \wedge (\rho_0 + \delta \cdot \sin \theta \cdot \alpha) + \frac{1}{2} \omega_0 \wedge \omega_0.
\]

Since \( \phi_0 \) is a \( G_2 \)-structure, \( \phi_\delta \) is also a \( G_2 \)-structure if \( \delta \) is sufficiently small. Moreover, \( \phi_\delta \) is cocalibrated and at \( \theta = 0 \) each term of (31) is independent of \( \delta \). Let \( g_6 \) be the metric on \( N^6 \) that is associated to \((\omega_0, \rho_0)\) and \( g_{8,\delta} \) be the metric on \( N^6 \times S^1 \times (-\epsilon, \epsilon) \) that is associated to \( \Phi_\delta \). Since \( \phi_0 \) and \( \Phi_0 \) are both torsion-free, we have \( g_{8,0} = g_6 + d\theta^2 + dx^2 \) and the second fundamental form \( II \) of \( N^6 \times \{(0,0)\} \) vanishes. If we find an \( \alpha \) such that \( II \neq 0 \), \( \Phi_0 \) and \( \Phi_\delta \) are non-equivalent.

Let \( X \) be a unit vector field on \( N^6 \). \( X \) can be lifted to a vector field on the product \( N^6 \times S^1 \times (-\epsilon, \epsilon) \). Outside of \( N^6 \times \{(0,0)\} \), \( X \) is in general not a unit vector field anymore. For all \( \alpha \), \( \frac{\partial}{\partial \theta} \) is a unit normal field of \( N^6 \times \{(0,0)\} \). Since \( [X, \frac{\partial}{\partial \theta}] = 0 \), we have on \( N^6 \times \{(0,0)\} \)

\[
g \left( II(X, X), \frac{\partial}{\partial \theta} \right) = g \left( \nabla_X X, \frac{\partial}{\partial \theta} \right)
\]

\[
= \frac{1}{2} \left( X g(X, \frac{\partial}{\partial \theta}) + X g(\frac{\partial}{\partial \theta}, X) - \frac{\partial}{\partial \theta} g(X, X) \right)
\]

\[
= -\frac{1}{2} \frac{\partial}{\partial \theta} g(X, X).
\]

Since we can prescribe the value of a closed 3-form at a fixed point arbitrarily, there exists an \( \alpha \) such that the last term of the above equation does not
vanish globally if $\delta > 0$. We thus have proven that $\Phi_0$ and $\Phi_\delta$ are non-equivalent, although they share the same initial values.

6. OutLook

Let $N^6$ be a 6-dimensional manifold and $M^8$ be an arbitrary $\mathbb{R}^2$-bundle over $N^6$. For reasons of brevity, we denote the zero section of $M^8$ also by $N^6$. We check under which conditions $M^8$ admits a not necessarily parallel Spin(7)- or Spin$_0(3,4)$-structure $\Phi$.

First, we assume that a Spin(7)-structure $\Phi$ exists on $M^8$. Let $\pi : M^8 \to N^6$ be the projection map and $\pi^{-1}(U)$ with $U \subset N^6$ be the image of a local trivialization. Moreover, let $e_x$ and $e_y$ be orthonormal vertical vector fields on $\pi^{-1}(U)$ and $(e^x, e^y)$ be the duals of $(e_x, e_y)$ with respect to the metric. If we replace in equation (17) $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ by $(e_x, e_y)$ and $dy$ by $e^y$, we obtain an $SU(3)$-structure $(\omega, \rho)$ on $U$. However, the $SU(3)$-structure can in general not be extended to all of $N^6$, since the bundle may not admit two global linearly independent sections.

Spin(7) acts transitively on the set of all oriented 6-dimensional subspaces of $\mathbb{R}^8$. The subgroup that fixes a subspace is isomorphic to $U(3)$. Therefore, any 6-dimensional oriented submanifold of a Spin(7)-manifold carries a canonical $U(3)$-structure and this is the most natural kind of geometry to suppose on $N^6$. In terms of tensor fields, a $U(3)$-structure is defined by a non-degenerate 2-form $\omega$, a Riemannian metric $g$ and an almost complex structure $J$ such that $\omega(X,Y) = g(X,J(Y))$ for all vector fields $X$ and $Y$. In our situation, the $U(3)$-structure is determined by $\omega := e_y \wedge e_x \wedge \Phi$ and the restriction of the associated metric to the tangent space of $N^6$. Our definition of $\omega$ is independent of the choice of $(e_x, e_y)$ and $\omega$ is thus globally defined. The Spin$_0(3,4)$-case is completely analogous, since Spin$_0(3,4)/U(1,2)$ is the Grassmannian of all positive oriented planes in $\mathbb{R}^{4,4}$.

We return to the local situation. The restriction of the 4-form to the subset $U$ of the zero section can be written as

$$\Phi = \frac{1}{2} \omega \wedge \omega + e^x \wedge \rho + e^y \wedge J^*_\rho \rho + e^x \wedge e^y \wedge \omega.$$  

We choose another $\pi^{-1}(\tilde{U})$ and vertical vector fields $\tilde{e}_x$ and $\tilde{e}_y$ on $\tilde{U}$ with the same properties as above. Moreover, we assume that $U \cap \tilde{U} \neq \emptyset$. On $\tilde{U}$ we have

$$\Phi = \frac{1}{2} \tilde{\omega} \wedge \tilde{\omega} + \tilde{e}^x \wedge \tilde{\rho} + \tilde{e}^y \wedge J^*_\tilde{\rho} \tilde{\rho} + \tilde{e}^x \wedge \tilde{e}^y \wedge \tilde{\omega}$$

for another $SU(3)$- or $SU(1,2)$-structure $(\tilde{\omega}, \tilde{\rho})$. On the intersection $\pi^{-1}(U \cap \tilde{U})$ we have
(37) \[
\begin{align*}
\tilde{e}_x &= \cos \theta e_x + \sin \theta e_y \\
\tilde{e}_y &= -\sin \theta e_x + \cos \theta e_y
\end{align*}
\]
for a function \(\theta : U \cap \tilde{U} \to \mathbb{R}\). Both terms for \(\Phi\) coincide only if

(38) \[
\begin{align*}
\tilde{\rho} &= \cos \theta \rho + \sin \theta J^* \rho \\
J^* \tilde{\rho} &= -\sin \theta \rho + \cos \theta J^* \rho
\end{align*}
\]

The transition functions for the bundle \(M^8\) thus have to be transition functions for the bundle \(\bigwedge^{3,0} T^* N^6\), too. In other words, \(M^8\) has to be isomorphic to the canonical bundle of \(N^6\) with respect to the almost complex structure \(J\).

Conversely, we assume that \(M^8\) is isomorphic to \(\bigwedge^{3,0} T^* N^6\) and that \(N^6\) carries a \(U(3)\)- or \(U(1,2)\)-structure \((\omega, g, J)\). We choose local trivializations \(\varphi_\alpha : U_\alpha \times \mathbb{R}^2 \to \pi^{-1}(U_\alpha)\) such that the transition functions have values in \(SO(2)\). Let \(x\) and \(y\) be the standard coordinates of \(\mathbb{R}^2\). There exist unique one-forms \(e^1\) and \(e^2\) such that \(\varphi_\alpha^*(e^1) = dx\) and \(\varphi_\alpha^*(e^2) = dy\). If the \(U_\alpha\) are sufficiently small, there exists a \((3,0)\)-form \(\rho\) on \(U_\alpha\) such that \((\omega, \rho)\) is an \(SU(3)\)- or \(SU(1,2)\)-structure whose associated metric and almost complex structure coincide with \(g\) and \(J\). Any other \((3,0)\)-form with the same properties as \(\rho\) can be written as

(39) \[
\cos \theta_\alpha \rho + \sin \theta_\alpha J^* \rho
\]
for a function \(\theta_\alpha : U_\alpha \to \mathbb{R}\). We define a 4-form

(40) \[
\Phi = \frac{1}{2} \pi^* \omega \wedge \pi^* \omega + e^1 \wedge \pi^* \rho + e^2 \wedge \pi^* J^* \rho + e^1 \wedge e^2 \wedge \pi^* \omega
\]
on \(\pi^{-1}(U_\alpha)\). \(\Phi\) is a \(Spin(7)\)- or \(Spin_0(3,4)\)-structure. By a similar argument as above, we can prove that \(\Phi\) is globally defined. The above observations yield the following lemma.

**Lemma 6.1.** Let \(M^8\) be an \(\mathbb{R}^2\)-bundle over a manifold \(N^6\) that admits a \(U(3)\)- or \(U(1,2)\)-structure \((\omega, g, J)\). \(M^8\) admits a \(Spin(7)\)- or \(Spin_0(3,4)\)-structure if and only if \(M^8\) is isomorphic to the canonical bundle of \(N^6\).

We therefore propose the following conjecture.

**Conjecture 6.2.** Let \(N^6\) be a 6-dimensional manifold with a \(U(3)\)- or \(U(1,2)\)-structure \((\omega, g, J)\) that satisfies \(d \omega \wedge \omega = 0\). Then there exists a parallel \(Spin(7)\)- or \(Spin_0(3,4)\)-structure \(\Phi\) on a tubular neighborhood of the zero section of the canonical bundle of \(N^6\) such that

1. the restriction of the associated metric to \(N^6\) coincides with \(g\) and
(2) \( e_y J(e_x \Phi) = \omega \) for any two orthonormal vertical vector fields \( e_x \) and \( e_y \) along \( N^6 \).

We finally remark that unlike in [3] we cannot make \( \Phi \) unique by assuming that the standard \( U(1) \)-action on the canonical bundle leaves \( \Phi \) invariant. Any \( U(1) \)-action that acts as the identity on the base has a differential of type \( A_\theta := \text{diag}(e^{i\theta}, 1, 1, 1) \). The fact that this matrix commutes with \( SU(4) \) allows the existence of a \( U(1) \)-invariant \( SU(4) \)-structure on the bundle. It is essential for the construction of Bielawski [3] that this works in any complex dimension. Unfortunately, \( A_\theta \) does not commute with \( \text{Spin}(7) \) or \( \text{Spin}_Q(3, 4) \) if we interpret it as a real \( 8 \times 8 \)-matrix. Therefore, the \( U(3) \)- or \( U(1, 2) \)-structure can in general not be extended to a \( U(1) \)-invariant parallel \( \text{Spin}(7) \)- or \( \text{Spin}_Q(3, 4) \)-structure.

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