ON SOME FRICITIONAL CONTACT PROBLEMS WITH VELOCITY CONDITION FOR ELASTIC AND VISCO-ELASTIC MATERIALS

KHALID ADDI
University of La Réunion
PIMENT EA4518
97715 Saint-Denis Messag cedex 9 La Réunion, France

OANH CHAU AND DANIEL GOELEVEN
University of La Réunion
PIMENT EA4518
97715 Saint-Denis Messag cedex 9 La Réunion, France

Abstract. We study the evolution of a class of quasistatic problems, which describe frictional contact between a body and a foundation. The constitutive law of the materials is elastic, or visco-elastic: with short or long memory, and the contact is modelled by a general subdifferential condition on the velocity. We derive weak formulations for the models and establish existence and uniqueness results. The proofs are based on evolution variational inequalities, in the framework of monotone operators and fixed point methods. We show the approach of the viscoelastic solutions to the corresponding elastic solutions, when the viscosity tends to zero. Finally we also study the approach to short memory visco-elasticity when the long memory relaxation coefficients vanish.

1. Introduction. Situations involving contact between deformable bodies take place frequently in industry and everyday life, such as impact problems, the tire with the road, or a shoe with the floor, which are just but three simple examples. The contact conditions could be very various, thus the modelling of the phenomena may be very complex, which is a source of richness of the domain of research, and the mathematical as well as engineering literature concerning this topic is rather extensive. An early attempt to study contact problems for elastic and viscoelastic materials within the framework of variational inequalities was made in the pioneering reference works [11, 13, 23]. Excellent references on analysis and numerical approximation of variational inequalities arising from contact problems are [20] and [17]. The mathematical, mechanical and numerical state of the art can be found in the proceedings [26].

Quasistatic processes arise when the forces applied to a system vary slowly in time so that acceleration is negligible. A number of papers investigating quasistatic frictional contact problems with viscoelastic materials have been published in [2, 9, 1, 27]. In [9] the frictional contact was modelled by a general velocity dependent dissipation functional. In [2], frictional contact with normal damped

2000 Mathematics Subject Classification. Primary: 74M15, 74M10; Secondary: 34G25.

Key words and phrases. Elastic material, visco-elastic material, long and short memory, subdifferential contact condition, quasistatic process, fixed point, variational inequality, weak solution, approaches to elasticity and short memory viscoelasticity.
response was considered. The variational analysis of some quasistatic frictional contact problems with normal compliance and friction can be found for instance in [1, 27], within nonlinear visco-elasticity. There, the problems were formulated as evolution variational inequalities for which existence and uniqueness results were obtained. The existence of a weak solution to the quasistatic Signorini’s contact problem with friction for elastic materials has been established in [12].

Existence and uniqueness results for dynamic unilateral frictional contact problems involving viscoelastic materials have been obtained in [19]. There, one of the major mathematical difficulty of the problems consists in the Signorini’s boundary conditions formulated in displacements. Dynamic contact problems with normal compliance were considered in [22].

Dynamic contact problems in visco-elasticity involving thermal effects were analyzed in [8], where abstract error analysis of a fully discrete numerical scheme has been established. Unilateral dynamic contact problems formulated in velocities for thermo-viscoelastic materials with small Coulomb friction were studied in [14]. Further extensions to non-convex contact conditions, with non-monotone and possibly multivalued constitution laws led to the fascinating active domain of non-smooth mechanic, formulated by the so-called hemivariational inequalities. For theoretical considerations as well as concrete applications in this domain, we refer to the self-contained books of [16, 25].

This paper is a companion work and an extension of [9], where a class of quasistatic contact problems have been studied for viscoelastic body of short memory, with subdifferential contact conditions. Here we investigate a model for the quasistatic process of frictional contact, moreover the body is supposed to be viscoelastic with long memory, defined by a relaxation operator, which extends the case of short memory when the relaxation coefficients vanish. This leads to a new and non standard evolution inequality. Then we prove the existence and uniqueness of a weak solution, by Banach contraction principle. Finally we study the convergence of the solutions of long memory material to the solution of the corresponding problem with short memory, as the relaxation coefficients tend to zero.

The plan of the paper is as follows. In Section 2 we introduce the notation and preliminary material. In Section 3 we describe the mechanical problems for three different types of constitution laws, and with a velocity frictional contact condition. Then, after specifying the assumptions on the data, we derive weak formulations for the problems, and we prove three existence and uniqueness results. The proofs are based on evolution variational inequalities, different choices of fixed points arguments, and Banach fixed point principles. In Section 4 we prove a convergence result which shows that the solutions of the viscoelastic problems converge to the solution of the elastic problem, when the viscosity tends to zero. Finally, in Section 5 we will show the approach of long memory viscoelastic solutions to the corresponding short memory viscoelastic solution, as the relaxation decreases.

2. Notation and preliminaries. Here we present the notation we shall use and some preliminary materials. For further details we refer the reader to [13, 24].

We denote by $S_d$ the space of second order order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$); “.” and $|\cdot|$ will represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and $S_d$, respectively. Thus,

$$u \cdot v = u_i v_i, \quad |v| = (v \cdot v)^{1/2}, \quad \forall u, v \in \mathbb{R}^d,$$

$$\sigma \cdot \tau = \sigma_{ij} \tau_{ij}, \quad |\tau| = (\tau \cdot \tau)^{1/2}, \quad \forall \sigma, \tau \in S_d.$$
Here and below, the indices \( i \) and \( j \) run between 1 and \( d \), the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable.

Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain with a Lipschitz continuous boundary \( \Gamma \), which will represent the body in contact with a foundation. We need also the following notations of spaces and operator, to describe the displacement field \( u = (u_i) \), the stress field \( \sigma = (\sigma_{ij}) \) and the linearized strain tensor \( \varepsilon(u) = (\varepsilon_{ij}(u)) \), in the framework of small deformations.

\[
H = \{ u = (u_i) \mid u_i \in L^2(\Omega) \}, \\
\mathcal{H} = \{ \sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \}, \\
H_1 = \{ u = (u_i) \mid u_i \in H^1(\Omega) \}, \\
H_1 = \{ \sigma \in \mathcal{H} \mid \sigma_{ij,j} \in H^1 \}.
\]

The spaces \( H, \mathcal{H}, H_1 \) and \( H_1 \) are real Hilbert spaces endowed with the inner products given by

\[
(u, v)_H = \int_\Omega u_i v_i \, dx, \\
(\sigma, \tau)_\mathcal{H} = \int_\Omega \sigma_{ij} \tau_{ij} \, dx, \\
(u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_\mathcal{H}, \\
(\sigma, \tau)_{H_1} = (\sigma, \tau)_\mathcal{H} + (\text{Div} \sigma, \text{Div} \tau)_H,
\]

respectively, where

\[
\varepsilon: H_1 \to \mathcal{H}, \\
\text{Div}: H_1 \to H
\]

are the deformation and the divergence operators, respectively, defined by

\[
\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{Div} \sigma = (\sigma_{ij,j}).
\]

The associated norms on the spaces \( H, \mathcal{H}, H_1 \) and \( H_1 \) are denoted by \( |\cdot|_H, |\cdot|_\mathcal{H}, |\cdot|_{H_1} \) and \( |\cdot|_{H_1} \), respectively.

Since the boundary \( \Gamma \) is Lipschitz continuous, the unit outward normal vector \( \nu \) on the boundary is defined a.e. For every vector field \( v \in H_1 \), we use the notation \( v \) to denote the trace \( \gamma v \) of \( v \) on \( \Gamma \) and we denote by \( v_\nu \) and \( v_\tau \) the normal and the tangential components of \( v \) on the boundary given by

\[
v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu \nu.
\]

For a regular (say \( C^1 \)) stress field \( \sigma \), the application of its trace on the boundary to \( \nu \) is the Cauchy stress vector \( \sigma \nu \). We define, similarly, the normal and tangential components of the stress on the boundary by the formulas

\[
\sigma_\nu = (\sigma \nu) \cdot \nu, \quad \sigma_\tau = \sigma \nu - \sigma_\nu \nu,
\]

and we recall that the following Green’s formula holds:

\[
(\sigma, \varepsilon(v))_\mathcal{H} + (\text{Div} \sigma, v)_H = \int_\Gamma \sigma \nu \cdot v \, da \quad \forall v \in H_1.
\]

Finally, for every real Hilbert space \( X \) we use the classical notation for the spaces \( L^p(0, T, X) \) and \( W^{k,p}(0, T, X) \), \( 1 \leq p \leq +\infty, \ k = 1, 2, \ldots \).
3. Mechanical problems and weak formulations.

3.1. The mechanical problems. Here we describe three mathematical models for the quasistatic process of frictional contact between a deformable body and an obstacle, the so-called foundation. In the first problem $QP^{ve}$ we assume that the body has a viscoelastic behavior of short memory. In the second one $QP^{ver}$, we assume that the body is viscoelastic of long memory. In the third one $QP^{e}$, the body is supposed to be purely elastic.

For the viscoelastic body we use a Kelvin-Voigt constitutive law, i.e. of the type

$$\sigma = A(\varepsilon(\dot{u})) + G(\varepsilon(u)) \quad \text{in} \quad \Omega \times (0, T);$$

where $G$ is the elasticity operator and $A$ is the viscosity operator. Here and everywhere in the sequel the dot above represents the time derivative.

For technical reason, we model the behavior of the elastic body with the linear elastic constitutive law

$$\sigma_{ij} = g_{ijkl} \varepsilon_{kl}(u).$$

In other words, the operator $G$ is defined as an elastic tensor.

Finally, we model the frictional contact with a velocity boundary condition into the following general form

$$u \in U, \quad \varphi(v) - \varphi(\dot{u}) \geq -\sigma \nu (v - \dot{u}) \quad \forall v \in U.$$ 

Here $U$ represents the set of contact admissible test functions, $\sigma \nu$ denotes the Cauchy stress vector on the contact boundary, and $\varphi$ is a given convex function. The above inequality holds almost everywhere on the contact surface. Examples and detailed explanations of inequality problems in contact mechanics which lead to boundary conditions of this form can be found in the monograph [24].

The physical setting is as follows. A deformable body occupies the domain $\Omega \subset \mathbb{R}^d$ and is acted upon by given forces and tractions and, as a result, its mechanical state evolves over the time interval $[0, T], \ T > 0$. We assume that the boundary $\Gamma$ of $\Omega$ is partitioned into three disjoint measurable parts $\Gamma_1, \Gamma_2$ and $\Gamma_3$, such that $\text{meas} (\Gamma_1) > 0$, where $\text{meas}$ denotes the $(d - 1)$-dimensional Lebesgue measure in $\mathbb{R}^d$. The body is clamped on $\Gamma_1 \times (0, T)$ and surface tractions of density $f_2$ act on $\Gamma_2 \times (0, T)$. The solid is in frictional contact with a rigid obstacle on $\Gamma_3 \times (0, T)$ and this is where our main interest lies. Moreover, a volume force of density $f_0$ acts on the body in $\Omega \times (0, T)$. We assume a quasistatic process and use a boundary velocity contact conditions. With these assumptions, our three initial mechanical problems may be stated as follows.

Problems $QP$. Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0, T] \to S_d$ such that :

$QP^{ve}$

$$\sigma = A(\varepsilon(\dot{u})) + G(\varepsilon(u)) \quad \text{in} \quad \Omega \times (0, T)$$

or $QP^{ver}$

For a.e. $t \in (0, T)$

$$\sigma(t) = A\varepsilon(\dot{u}(t)) + G\varepsilon(u(t)) + \int_0^t B(t - s) \varepsilon(u(s)) \, ds \quad \text{in} \quad \Omega$$

or $QP^{e}$

$$\sigma = G(\varepsilon(u)) \quad \text{in} \quad \Omega \times (0, T)$$

and

$$\text{Div} \sigma + f_0 = 0 \quad \text{in} \quad \Omega \times (0, T)$$
\[ u = 0 \quad \text{on} \quad \Gamma_1 \times (0, T) \]
\[ \sigma \nu = f_2 \quad \text{on} \quad \Gamma_2 \times (0, T) \]
\[ u(t) \in U, \quad \varphi(w) - \varphi(\dot{u}(t)) \geq -\sigma(t) \nu \cdot (w - \dot{u}(t)) \quad \forall w \in U \quad \text{on} \quad \Gamma_3 \times (0, T) \]
\[ u(0) = u_0 \quad \text{in} \quad \Omega. \]

3.2. Functional framework. To obtain variational formulations of these mechanical problems \( QP \), we suppose that
\[ U \subset H_1, \quad \mathcal{D}(\Omega)^d \subset U, \]
and
\[ \varphi : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}. \]
We define the space of admissible displacements
\[ V = \{ v \in H_1 \mid v = 0 \text{ on } \Gamma_1 \} \cap U. \]
Let \( j : V \to \mathbb{R} \) be the contact functional defined by
\[ j(v) = \begin{cases} \int_{\Gamma_3} \varphi(v) \, da & \text{if } \varphi(v) \in L^1(\Gamma_3), \\ +\infty & \text{otherwise.} \end{cases} \]
We suppose everywhere in the sequel that
\[ V \] is a closed subspace of \( H_1, \]
and
\[ j \] is a convex, proper and lower semicontinuous function on \( V. \)

3.3. Assumptions. We suppose that the viscosity nonlinear operator to be Lipschitz continuous and strongly monotone:
\[ A : \Omega \times S_d \to S_d, \]
\[ (a) \text{ there exists } L_A > 0 \text{ such that } \]
\[ |A(x, \varepsilon_1) - A(x, \varepsilon_2)| \leq L_A |\varepsilon_1 - \varepsilon_2| \quad \forall \varepsilon_1, \varepsilon_2 \in S_d \text{, a.e. } x \in \Omega; \]
\[ (b) \text{ there exists } m_A > 0 \text{ such that } \]
\[ (A(x, \varepsilon_1) - A(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_A |\varepsilon_1 - \varepsilon_2|^2 \quad (1) \]
\[ \forall \varepsilon_1, \varepsilon_2 \in S_d \text{, a.e. } x \in \Omega; \]
\[ (c) x \mapsto A(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \]
\[ \forall \varepsilon \in S_d; \]
\[ (d) \text{ the mapping } x \mapsto A(x, 0) \in \mathcal{H}. \]
The elastic operator is linear and coercive:
\[ G : \Omega \times S_d \to S_d; \]
\[ G(\cdot, \tau) = (g_{ijkh} \tau_{kh}); \]
\[ (a) g_{ijkh} \in L^\infty(\Omega); \]
\[ (b) G(\sigma \cdot \tau) = \sigma \cdot G\tau \forall \sigma, \tau \in S_d, \text{ a.e. in } \Omega; \]
\[ (c) \text{ There exists } m_G > 0 \text{ such that } \]
\[ G\tau \cdot \tau \geq m_G |\tau|^2 \quad \forall \tau \in S_d, \text{ a.e. in } \Omega. \]
Then we could define the inner product on the space of admissible displacement as follows:
\[ (u, v)_V := (G\varepsilon(u), \varepsilon(v))_\mathcal{H}; \quad |u|_V := (u, v)_V^{1/2} \]
The relaxation operator satisfies:

\[
\begin{cases}
B : (0,T) \times \Omega \times S_d \rightarrow S_d; \quad B(\cdot, \cdot, \tau) = (B_{ijkh} \tau_{kh}) \\
(a) \quad B_{ijkh} \in W^{1,\infty}(0,T; L^\infty(\Omega)); \\
(b) \quad B(t) \sigma \cdot \tau = \sigma \cdot B(t) \tau \\
\forall \sigma, \tau \in S_d, \text{ a.e. } t \in (0,T), \text{ a.e. in } \Omega.
\end{cases}
\]  

The volume and surface densities of the body verify:

\[
f_0 \in W^{1,2}(0,T; H) ; \quad f_2 \in W^{1,2}(0,T ; L^2(\Gamma^2)^d).
\]  

This previous assumption allows us to define the following vector function:

\[
(f(t), w)_V = (f_0(t), w)_H + (f_2(t), w)_{L^2(\Gamma^2)^d}, \quad \forall w \in V, \ t \in [0,T];
\]

and verifying the regularity:

\[
f \in W^{1,2}(0,T; V).
\]

Finally the initial displacement satisfies

\[
u_0 \in V;
\]

and

\[
(u_0, w)_V + j(w) \geq (f(0), w)_V, \quad \forall w \in V.
\]

Using then Green’s formula, we obtain the following weak formulations for our different mechanical problems.

3.4. Variational formulations. Find a displacement field \(u: [0,T] \rightarrow V\) and a stress field \(\sigma: [0,T] \rightarrow \mathcal{H}_1\) such that \(QP\)

\[
\sigma = \mathcal{A}(\varepsilon(\dot{u})) + \mathcal{G}(\varepsilon(u)) \quad \text{in} \quad \Omega \times (0,T)
\]

or \(QP^{ve}\) For a.e. \(t \in (0,T)\)

\[
\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{G}(\varepsilon(u(t)) + \int_0^t \mathcal{B}(t-s) \varepsilon(u(s)) \, ds \quad \text{in} \quad \Omega
\]

or \(QP^{ee}\)

\[
\sigma = \mathcal{G}(\varepsilon(u)) \quad \text{in} \quad \Omega \times (0,T)
\]

and

\[
(\sigma(t), \varepsilon(w) - \varepsilon(\dot{u}(t)))_H + j(w) - j(\dot{u}(t)) \geq (f(t), w - \dot{u}(t))_V \\
\forall w \in V, \text{ a.e. } t \in (0,T)
\]

\[
u(0) = u_0
\]

4. Existence and uniqueness results in elasticity and visco-elasticity. Now we are going to prove an existence and uniqueness result for each variational formulation.

**Theorem 4.1.** Assume that assumptions (1)-(2), (4)-(5) hold. Then there exists a unique solution \(\{u, \sigma\}\) of the problem \(QP^{ve}\), with the regularity:

\[
u \in W^{2,2}(0,T; V), \quad \sigma \in W^{1,2}(0,T; \mathcal{H}_1)
\]

**Proof of Theorem 4.1.** The proof is based on some preliminary lemmas we are going to precise.
Lemma 4.2. For all \( \eta \in W^{1,2}(0, T; \mathcal{H}) \), there exists a unique solution \( v_\eta \in W^{1,2}(0, T; V) \), such that
\[
(A \varepsilon(v_\eta(t)), \varepsilon(v) - \varepsilon(v_\eta(t)))_H + (\eta(t), \varepsilon(v) - \varepsilon(v_\eta(t)))_H \\
+j(v) - j(v_\eta(t)) \geq (f(t), v - v_\eta(t))_V \quad \forall v \in V, \ t \in [0, T].
\]

Proof. From (1) we can define \( v_T \)

Then \( A \) is Lipschitz continuous and strongly monotone. For any fixed \( \eta \in W^{1,2}(0, T; \mathcal{H}) \) and \( t \in [0, T] \), we deduce by general result on variational inequality of second kind the existence of \( v_\eta(t) \) (see e.g. [18, 21]).

To prove the regularity of \( v_\eta \), we verify that for any \( t_1, t_2 \in [0, T] \):
\[
(A \varepsilon(v_\eta(t_1)) - A \varepsilon(v_\eta(t_2)), \varepsilon(v_\eta(t_1)) - \varepsilon(v_\eta(t_2)))_H \\
\leq (f(t_1) - f(t_2), v_\eta(t_1) - v_\eta(t_2))_V \\
+ (\eta(t_1) - \eta(t_2), \varepsilon(v_\eta(t_1)) - \varepsilon(v_\eta(t_2)))_H.
\]

Using then the strong monotonicity of \( A \), we deduce that there exists some constant \( c > 0 \)
\[
|v_\eta(t_1) - v_\eta(t_2)|_V \leq c \left( |f(t_1) - f(t_2)|_V + |\eta(t_1) - \eta(t_2)|_H \right).
\]

Then \( v_\eta \) is a.e. differentiable, and the regularity \( \eta \in W^{1,2}(0, T; \mathcal{H}) \) follows from those of \( f \) and \( \eta \).

Now consider the set \( \mathcal{W} = \{ \eta \in W^{1,2}(0, T; \mathcal{H}) \mid \eta(0) = \mathcal{G} \varepsilon(u_0) \} \), and the operator \( \Lambda : \mathcal{W} \to \mathcal{W} \) defined by:
\[
\begin{aligned}
\Lambda \eta(t) &= \mathcal{G} \varepsilon(u_\eta(t)) \\
u_\eta(t) &= \int_0^t \varepsilon_\eta(s) \, ds + u_0, \quad \forall t \in [0, T], \ \forall \eta \in W^{1,2}(0, T; \mathcal{H}).
\end{aligned}
\]

We claim that:

Lemma 4.3. The operator \( \Lambda \) has a unique fixed point \( \eta^* \in \mathcal{W} \).

Proof. Let \( \eta_1, \eta_2 \in \mathcal{W}, \ t \in [0, T] \). By the definition of \( \cdot |_V \) and the linearity of \( \mathcal{G} \), we have:
\[
|\Lambda \eta_1(t) - \Lambda \eta_2(t)|_H \leq c \int_0^t |v_{\eta_1}(s) - v_{\eta_2}(s)|_V \, ds;
\]
and
\[
\left| \frac{d}{dt}(\Lambda \eta_1(t) - \Lambda \eta_2(t)) \right|_H \leq c |v_{\eta_1}(t) - v_{\eta_2}(t)|_V.
\]
As
\[
|v_{\eta_1}(t) - v_{\eta_2}(t)|_V \leq c |\eta_1(t) - \eta_2(t)|_H;
\]
we deduce that
\[
|\Lambda \eta_1(t) - \Lambda \eta_2(t)|_H^2 \leq c \int_0^t |\eta_1(s) - \eta_2(s)|_H^2 \, ds
\]
and
\[
\left| \frac{d}{dt}(\Lambda \eta_1(t) - \Lambda \eta_2(t)) \right|_H^2 \leq c \int_0^t |\dot{\eta}_1(s) - \dot{\eta}_2(s)|_H^2 \, ds.
\]
To continue, we recall that \( \mathcal{W} \) is a Banach space endowed with the norm
\[
\forall \eta \in \mathcal{W}, \ \| \eta \|_{\mathcal{W}}^2 = \int_0^T |\eta(s)|_H^2 \, ds + \int_0^T |\dot{\eta}(s)|_H^2 \, ds.
\]
Then from the last two inequalities we verify that for any $\eta_1, \eta_2 \in \mathcal{W}$, $n \in \mathbb{N}$, we have
\[
|\Lambda^n \eta_1 - \Lambda^n \eta_2|_{\mathcal{W}} \leq \frac{(cT)^n}{n!} |\eta_1 - \eta_2|_{\mathcal{W}}.
\]

This shows that for some $n \in \mathbb{N}$, $\Lambda^n$ is a contracting operator. Then by Banach fixed point Theorem, $\Lambda^n$ has a unique fixed point which is also the unique fixed point of $\Lambda$ (see e.g. Tome 1 page 17 in [28]).

We have now all the ingredients to prove the Theorem 4.1. Let $\eta^* \in \mathcal{W}$ be the fixed point of $\Lambda$. Define
\[
\left\{
\begin{array}{l}
u(t) := u_0 + \int_0^t v_{\eta^*}(s) \, ds \\
\sigma(t) := A\varepsilon(v_{\eta^*}(t)) + G\varepsilon(u(t)), \quad \forall t \in [0, T].
\end{array}
\right.
\]

Using $\eta = \eta^*$ in Lemma 4.2, we see that $\{u, \sigma\}$ is solution of the variational inequality in $QVP^{we}$. Putting then
\[
w = \dot{u}(t) \pm \psi \quad \text{where} \quad \psi \in \mathcal{D}(\Omega)^d,
\]
into the variational inequality, we obtain
\[
\text{Div } \sigma(t) + f_0(t) = 0, \quad \forall t \in [0, T].
\]
We conclude that $\{u, \sigma\}$ is the unique solution to Problem $QVP^{we}$ with the corresponding regularity. The uniqueness comes from the uniqueness in Lemma 4.2 and in Lemma 4.3. \hfill \square

**Theorem 4.4.** Assume that assumptions (1)-(5) hold. Then there exists a unique solution $\{u, \sigma\}$ of the problem $QVP^{we}$, with the regularity:
\[
\begin{align*}
\frac{\partial}{\partial t} & u(t) \in W^{2,2}(0, T; V), & \sigma(t) & \in W^{1,2}(0, T; H_1).
\end{align*}
\]

**Proof.** We proceed closely as in the proof of Theorem 4.1.

Let consider the set $\mathcal{W} = \{\eta \in W^{1,2}(0, T; H) \mid \eta(0) = G\varepsilon(u_0)\}$ and the operator
\[
\Lambda : \mathcal{W} \to \mathcal{W}
\]
\[
\left\{
\begin{array}{l}
\Lambda \eta(t) = G\varepsilon(u(t)) + \int_0^t B(t - s) \varepsilon(u(s)) \, ds \\
u_\eta(t) = \int_0^t v_\eta(s) \, ds + u_0, \quad \forall t \in [0, T], \forall \eta \in \mathcal{W}.
\end{array}
\right.
\]

After some algebraic manipulations, it is not difficult to see that there exists some constant $c > 0$, such that for all $\eta_1, \eta_2 \in \mathcal{W}$, and for all $t \in [0, T]$, we have:
\[

|\Lambda \eta_1(t) - \Lambda \eta_2(t)|_{\mathcal{H}}^2 + \left| \frac{d}{dt} (\Lambda \eta_1(t) - \Lambda \eta_2(t)) \right|_{\mathcal{H}}^2 \\
\leq c \int_0^t |\eta_1(s) - \eta_2(s)|_{\mathcal{H}}^2 \, ds + \leq c \int_0^t |\dot{\eta}_1(s) - \dot{\eta}_2(s)|_{\mathcal{H}}^2 \, ds
\]

Then by Banach contraction principle, we know that there exists a unique fixed point of $\Lambda$, and we conclude as in the proof of Theorem 4.1. \hfill \square

**Theorem 4.5.** Assume that assumptions (2), (4)-(6) hold. Then there exists a unique solution $\{u, \sigma\}$ of the problem $QVP^{we}$, with the regularity:
\[
\begin{align*}
\frac{\partial}{\partial t} & u(t) \in W^{1,2}(0, T; V), & \sigma(t) & \in W^{1,2}(0, T; H_1).
\end{align*}
\]
Proof. The result in Theorem 4.5 is a direct consequence of the following general result.

**Theorem (Brézis)** Let $(V, (\cdot, \cdot)_V)$ be a real Hilbert space and let $j : V \to (-\infty, +\infty]$ be a convex proper lower semi-continuous functional. Let $f \in W^{1,2}(0,T;V)$ and $u_0 \in V$ be such that

$$\sup_{v \in D(j)} \{(f(0), v)_V - (u_0, v)_V - j(v)\} < +\infty.$$ 

Then, there exists a unique element $u \in W^{1,2}(0,T;V)$ which satisfies

$$(u(t), v - \dot{u}(t))_V + j(v) - j(\dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V,$$

for all $v \in V$, a.e. $t \in (0,T)$, and

$$u(0) = u_0.$$

Theorem (Brézis) has been proved in [6], p.117, using arguments of evolution equations in the framework of maximal monotone operators. A version of this theorem has been considered in [18], where the proof was based on a time-discretization method.

Note here that $V$ is our admissible displacement space, endowed with the inner product defined by the linear tensor $\mathcal{G}$, which verifies (2). The vector function $f$ from densities of applied forces has been defined after the hypothesis (4). Finally the conditions (5) and (6) on the initial displacement allow us to use Theorem (Brézis), and to conclude the existence of displacement solution field $u$ in the problem $QVP^e$, with $u \in W^{1,2}(0,T;V)$. We use now an argument similar to that used at the end of the proof of Theorem 1 in $QP^e$, to obtain $\sigma \in W^{1,2}(0,T;\mathcal{H}_1)$. This concludes the existence part of Theorem 4.5. The uniqueness part results from the uniqueness guaranteed by Theorem (Brézis).

**Remark 1.** We remark that Theorem 4.1 for short memory visco-elastic material is a particular case of Theorem 4.4 for long memory visco-elasticity, where the relaxation coefficients vanish. Thus Theorem 4.4 extends the result concerning the short memory visco-elasticity obtained in [9].

5. Approach to elasticity when the viscosity tends to zero. Let

$$u_0^\theta \in V, \quad \forall \theta > 0$$

**Family of visco-elastic problems** $QVP^e_\theta$. Let any $\theta > 0$. Define the displacement field $u_\theta : [0,T] \to V$ and the stress field $\sigma_\theta : [0,T] \to \mathcal{H}_1$, such that

$$\sigma_\theta(t) = \theta A\varepsilon(u_\theta) + G\varepsilon(\sigma_\theta), \quad \forall t \in [0,T],$$

$$(\sigma_\theta(t), \varepsilon(w) - \varepsilon(u_\theta(t)))_V + j(w) - j(\dot{u}_\theta(t))$$

$$\geq (f(t), w - \dot{u}_\theta(t))_V \quad \forall w \in V, \quad t \in [0,T],$$

$$u_\theta(0) = u_0^\theta.$$ 

and satisfying the regularity :

$$u_\theta \in W^{2,2}(0,T;V), \quad \sigma_\theta \in W^{1,2}(0,T;\mathcal{H}_1).$$

**Elastic problem.** Let $u \in W^{1,2}(0,T;V), \quad \sigma \in W^{1,2}(0,T;\mathcal{H}_1)$ be the unique solution of the elastic problem $QVP^e$. 


Theorem 5.1. There exists some constant \( c > 0 \) such that:
\[
| u_\theta(t) - u(t) |^2_V \leq c \theta + c | u_\theta^0 - u_0 |^2_V \\
\forall t \in [0, T], \forall \theta > 0.
\]
\[
| \sigma_\theta(t) - \sigma(t) |^2_{H_t} \leq c | u_\theta(t) - u(t) |^2_V + c \theta^2 | \dot{u}(t) |^2_V \\
a.e. t \in (0, T), \forall \theta > 0.
\]

We deduce then immediately the following consequence.

**Corollary 1.** Assume:
\[
u_0^\theta \longrightarrow u_0 \quad \text{in} \quad V \quad \text{as} \quad \theta \longrightarrow 0 + .
\]
Then
\[
\max_{t \in [0, T]} | u_\theta(t) - u(t) |_V \longrightarrow 0 \quad \text{as} \quad \theta \longrightarrow 0 + .
\]
\[
| \sigma_\theta - \sigma |_{L^2(0, T; H)} \longrightarrow 0 \quad \text{as} \quad \theta \longrightarrow 0 + .
\]

**Proof of Theorem 5.1.** Put \( w = \dot{u}(t) \) in \( QV P^c_\theta \) and \( w = \dot{u}_\theta(t) \) in \( QV P^c \). We have for all \( t \in [0, T] \):
\[
\theta ( A \varepsilon(\dot{u}_\theta), \varepsilon(\dot{u}) - \varepsilon(\dot{u}_\theta) )_{H_t} + ( G \varepsilon(\dot{u}_\theta), \varepsilon(\dot{u}) - \varepsilon(\dot{u}_\theta) )_{H_t} \\
+ j(\dot{u}) - j(\dot{u}_\theta) \geq ( f, \ddot{u} - \ddot{u}_\theta )_V .
\]
and
\[
( G \varepsilon(u), \varepsilon(\dot{u}_\theta) - \varepsilon(\dot{u}) )_{H_t} + j(\dot{u}_\theta) - j(\dot{u}) \\
\geq ( f, \ddot{u}_\theta - \ddot{u} )_V .
\]

We then add the two inequalities to obtain:
\[
\theta ( A \varepsilon(\dot{u}_\theta) - A \varepsilon(\dot{u}), \varepsilon(\dot{u}_\theta) - \varepsilon(\dot{u}) )_{H_t} + ( u_\theta - u, \dot{u}_\theta - \dot{u} )_V \\
\leq \theta ( A \varepsilon(\dot{u}), \varepsilon(\dot{u}) - \varepsilon(\dot{u}_\theta) )_{H_t}
\]
Then we integrate the last inequality by using the strong monotonicity of the viscosity operator:
\[
\theta m_A \int_0^t | \varepsilon(\dot{u}_\theta) - \varepsilon(\dot{u}) |^2_{H_t} \, ds + \frac{1}{2} | u_\theta(t) - u(t) |^2_V \\
\leq \theta \int_0^t | A \varepsilon(\dot{u}) |_{H_t} | \varepsilon(\dot{u}_\theta) - \varepsilon(\dot{u}) |_{H_t} \, ds + \frac{1}{2} | u_\theta^0 - u_0 |^2_V .
\]

We deduce then
\[
| u_\theta(t) - u(t) |^2_V \leq c \theta + c | u_\theta^0 - u_0 |^2_V 
\]
and
\[
| \sigma_\theta(t) - \sigma(t) |_{H_t} \leq c \theta | \dot{u}_\theta(t) |_V + c | u_\theta(t) - u(t) |_V .
\]
As
\[
\text{Div} \, \sigma_\theta(t) = \text{Div} \, \sigma(t) = - f_0(t);
\]
we see that
\[
| \sigma_\theta(t) - \sigma(t) |_{H_t} \leq c \theta | \dot{u}_\theta(t) |_V + c | u_\theta(t) - u(t) |_V .
\]
As we have
\[
\theta | \dot{u}_\theta(t) - \dot{u}(t) |_V \leq c \left( | u_\theta(t) - u(t) |_V + \theta | \dot{u}(t) |_V \right),
\]
we deduce that
\[
| \sigma_\theta(t) - \sigma(t) |_{H_t} \leq c | u_\theta(t) - u(t) |_V + c \theta | \dot{u}(t) |_V .
\]
This gives the results stated in Theorem 5.1. \( \square \)
6. **Approach to short memory visco-elasticity when the relaxation tends to zero.** Let a family of bounded initial displacements
\[ u_0^\lambda \in V, \quad \forall \lambda > 0 \]

**Family of long memory visco-elastic problems.** \( QVP^\text{ver}_\lambda \). Let any \( \lambda > 0 \).

Define the displacement field
\[ u_\lambda \in W^{2,2}(0,T;V) \]
such that
\[ \sigma_\lambda(t) = \mathcal{A} \varepsilon(\dot{u}_\lambda) + \mathcal{G} \varepsilon(u_\lambda) + \lambda \int_0^t B(t-s) \varepsilon(u_\lambda(s)) \, ds, \quad \forall t \in [0,T]; \]
\[ (\sigma_\lambda(t), \varepsilon(w) - \varepsilon(\dot{u}_\lambda(t)))_H + j(w) - j(\dot{u}_\lambda(t)) \geq (f(t), w - \dot{u}_\lambda(t))_V \quad \forall w \in V, \ t \in [0,T]; \]
\[ u_\lambda(0) = u_0^\lambda. \]

**Short memory visco-elastic problem.** Let \( u \in W^{2,2}(0,T;V) \) be the unique displacement solution field of the problem \( QVP^\text{ver}_\lambda \).

**Theorem 6.1.** There exists some constant \( c > 0 \) such that :
\[ |u_\lambda(t) - u(t)|_V^2 \leq c \lambda + c |u_0^\lambda - u_0|_V^2 \]
\[ \forall t \in [0,T], \ \forall \lambda > 0. \]

We obtain then:

**Corollary 2.** Assume :
\[ u_0^\lambda \longrightarrow u_0 \text{ in } V \text{ as } \lambda \longrightarrow 0+. \]

Then
\[ \max_{t \in [0,T]} |u_\lambda(t) - u(t)|_V \longrightarrow 0 \text{ as } \lambda \longrightarrow 0+. \]

**Proof of Theorem 6.1.** Put \( w = \dot{w}(t) \) in \( QVP^\text{ver}_\lambda \) and \( w = \dot{u}_\lambda(t) \) in \( QVP^\text{ver} \). Add the inequalities and integrate by using again the strong monotonicity of the viscosity operator. We obtain for all \( t \in [0,T] : \)
\[ m_A \int_0^t |\dot{u}_\lambda - \dot{u}|_V^2 \leq - \int_0^t (\mathcal{G} \varepsilon(\dot{u}_\lambda) - \mathcal{G} \varepsilon(u) - \varepsilon(\dot{u}))_H + \lambda \int_0^t \left( \int_0^r B(\tau-s) \varepsilon(u_\lambda(s)) \, ds, \varepsilon(\dot{u}(\tau)) - \varepsilon(\dot{u}_\lambda(\tau)) \right)_H \, d\tau. \]

From which we deduce that
\[ \int_0^t |\dot{u}_\lambda - \dot{u}|_V^2 \leq c \lambda \int_0^t |u_\lambda|_V^2 + c \int_0^t |u_\lambda - u|_V^2, \quad \forall \lambda > 0. \]

Then we have
\[ \int_0^t |\dot{u}_\lambda - \dot{u}|_V^2 \leq c \lambda + c \int_0^t |u_\lambda - u|_V^2, \quad \forall \lambda > 0. \]

As
\[ u_\lambda(t) - u(t) = u_0^\lambda - u_0 + \int_0^t (\dot{u}_\lambda - \dot{u}), \quad \forall t \in [0,T]. \]
We deduce that
\[ |u_\lambda(t) - u(t)|_V^2 \leq c |u_\lambda^0 - u_0|^2_V + c \int_0^t |\dot{u}_\lambda - \dot{u}|_V^2, \quad \forall t \in [0, T]. \]
The conclusion follows then by using the following version of Gronwall’s inequality (see e.g. Lemma 2 page 10 in [4]), that we recall.

**Theorem 6.2. (Gronwall).** Let \( T > 0, c_1 \geq 0, c_2 \geq 0 \) and \( \xi : [0, T] \rightarrow \mathbb{R} \) a continuous mapping such that :
\[ \forall t \in [0, T], \quad \xi(t) \leq c_1 + c_2 \int_0^t \xi(s) ds. \]
Then we have
\[ \forall t \in [0, T], \quad \xi(t) \leq c_1 e^{c_2 T}. \]
[19] J. Jarušek, Dynamic contact problems with given friction for viscoelastic bodies, Czechoslovak Mathematical Journal, 46 (1996), 475–487.
[20] N. Kikuchi and J. T. Oden, “Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods,” SIAM Studies in Applied Mathematics, 8, SIAM, Philadelphia, PA, 1988.
[21] J. L. Lions and G. Stampacchia, Variational inequalities, Commun. Pure Appl. Math., 20 (1967), 493–519.
[22] J. A. C. Martins and J. T. Oden, Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws, Nonlin. Anal., 11 (1987), 407–428.
[23] J. Nečas and I. Hlavaček, “Mathematical Theory of Elastic and Elastoplastic Bodies: An Introduction,” Elsevier, Amsterdam, 1981.
[24] P. D. Panagiotopoulos, “Inequality Problems in Meechical and Applications. Convex and Nonconvex Energy Functions,” Birkhäuser Boston, Inc., Boston, MA, 1985.
[25] P. D. Panagiotopoulos, “Hemivariational Inequalities, Applications in Mechanics and Engineering,” Springer-Verlag, Berlin, 1993.
[26] M. Raous, M. Jean and J. J. Moreau, eds., “Contact Mechanics,” Plenum Press, New York, 1995.
[27] M. Rochdi, M. Shillor and M. Sofonea, Quasistatic viscoelastic contact with normal compli-
ance and friction, Journal of Elasticity, 51 (1998), 105–126.
[28] E. Zeidler, “Nonlinear Functional Analysis and its Applications,” Springer-Verlag, 1997.

Received October 2009; revised May 2010.

E-mail address: khalid.addi@univ-reunion.fr
E-mail address: oanh.chau@univ-reunion.fr
E-mail address: daniel.goeleven@univ-reunion.fr