COMPUTING THE CASSELS-TATE PAIRING IN THE CASE OF A RICHELOT ISOGENY

JIALI YAN

ABSTRACT. In this paper, we study the Cassels-Tate pairing on Jacobians of genus two curves admitting a special type of isogenies called Richelot isogenies. Let \( \phi : J \to \tilde{J} \) be a Richelot isogeny between two Jacobians of genus two curves. We give an explicit formula as well as a practical algorithm to compute the Cassels-Tate pairing on \( \text{Sel}^\phi(\tilde{J}) \times \text{Sel}^\phi(\tilde{J}) \) where \( \tilde{\phi} \) is the dual isogeny of \( \phi \). The formula and algorithm are under the simplifying assumption that all two torsion points on \( J \) are defined over \( K \). We also include a worked example demonstrating we can turn the descent by Richelot isogeny into a 2-descent via computing the Cassels-Tate pairing.

1. Introduction

For any principally polarized abelian variety \( A \) defined over a number field \( K \), Cassels and Tate [Cas59] [Cas62] and [Tat62] constructed a pairing
\[
\text{III}(A) \times \text{III}(A) \to \mathbb{Q}/\mathbb{Z},
\]
that is nondegenerate after quotienting out the maximal divisible subgroup of \( \text{III}(A) \). This pairing is called the Cassels-Tate pairing and it naturally lifts to a pairing on Selmer groups. One application of this pairing is in improving the bound on the Mordell-Weil rank \( r(A) \) obtained by performing a standard descent calculation. Suppose \( \text{III}(A) \) is finite, then carrying out an \( n \)-descent and compute the Cassels-Tate pairing on \( \text{Sel}^n(A) \times \text{Sel}^n(A) \) gives the same bound as obtained from the \( n^2 \)-descent where \( \text{Sel}^n(A) \) needs to be computed as shown in [Yan, Proposition 1.9.3].

There have been many results on computing the Cassels-Tate pairing in the case of elliptic curves. For example, in addition to defining the pairing, Cassels also described a method for computing the pairing on \( \text{Sel}^2(E) \times \text{Sel}^2(E) \) in [Cas98] by solving conics over the field of definition of a two-torsion point. Donnelly [Don15] then described a method that only requires solving conics over \( K \) and Fisher [Fis16] used the invariant theory of binary quartics to give a new formula for the Cassels-Tate pairing on \( \text{Sel}^2(E) \times \text{Sel}^2(E) \) without solving any conics. In [vB] [vBF18], van Beek and Fisher computed the Cassels-Tate pairing on the 3-isogeny Selmer group of an elliptic curve. For \( p = 3 \) or 5, Fisher computed the Cassels-Tate pairing on the \( p \)-isogeny Selmer group of an elliptic curve in a special case in [Fis03]. In [FN14], Fisher and Newton computed the Cassels-Tate pairing on \( \text{Sel}^3(E) \times \text{Sel}^3(E) \). We are interested in the natural problem of generalizing the different algorithms for computing the Cassels-Tate pairing for elliptic curves to compute the pairing for abelian varieties of higher dimension.
In this paper, we study the Cassels-Tate pairing on Jacobians of genus two curves admitting a special type of isogeny called a Richelot isogeny. Let $\phi : J \rightarrow \hat{J}$ be a Richelot isogeny between Jacobians of two genus two curves. We will be working under the simplifying assumption that all two-torsion points on $J$ are defined over $K$. Consider the following long exact sequence

$$(1.1) \ 0 \rightarrow J[\phi](\mathbb{Q}) \rightarrow J[2](\mathbb{Q}) \rightarrow \hat{J}[\hat{\phi}](\mathbb{Q}) \rightarrow \text{Sel}^\phi(J) \rightarrow \text{Sel}^2(J) \xrightarrow{\alpha} \text{Sel}^\hat{\phi}(\hat{J}).$$

Let $\langle , \rangle_{CT}$ denote the Cassels-Tate pairing on $\text{Sel}^\hat{\phi}(\hat{J})$. It is shown in Remark 3.4 that we can replace $\text{Sel}^\hat{\phi}(\hat{J})$ with $\ker\langle , \rangle_{CT}$ and (1.1) remains exact. Thus computing the pairing $\langle , \rangle_{CT}$ potentially improves the rank bound given by carrying out a descent by Richelot isogeny.

In Section 2, we give some background results needed for the later sections and we define a pairing on $\text{Sel}^\phi(J) \times \text{Sel}^\phi(J)$ following the Weil pairing definition of the Cassels-Tate pairing for the Richelot isogeny $\phi$. In Section 3, we then give an explicit formula as well as a practical algorithm to compute the Cassels-Tate pairing on $\text{Sel}^\phi(\hat{J}) \times \text{Sel}^\hat{\phi}(\hat{J})$ where $\hat{\phi}$ is the dual isogeny of $\phi$ and also a Richelot isogeny. In Section 4, we give some details of the explicit computation and show directly that the formula for the Cassels-Tate pairing is always a finite product with a computable bound. In Section 5, we include a worked example demonstrating we can turn the descent by Richelot isogeny into a 2-descent via computing the Cassels-Tate pairing. The content of this paper is based on Chapter 2 of the thesis of the author [Yan].

Acknowledgements. I express my sincere and deepest gratitude to my PhD supervisor, Dr. Tom Fisher, for his patient guidance and insightful comments at every stage during my research.

2. Preliminary Results

2.1. The set-up. In this paper, we are working over a number field $K$. For any field $k$, we let $\overline{k}$ denote its algebraic closure and let $\mu_n \subset \overline{k}$ denote the $n^{th}$ roots of unity in $\overline{k}$. We let $G_k$ denote the absolute Galois group $\text{Gal}(\overline{k}/k)$.

Let $C$ be a general genus two curve defined over $K$ with all Weierstrass points defined over $K$, which is a smooth projective curve and it can be given in the following hyperelliptic form:

$$(2.1) \quad C : y^2 = f(x) = G_1(x)G_2(x)G_3(x),$$

where $G_1(x) = \lambda(x - \omega_1); G_2(x) = (x - \omega_2)(x - \omega_3); G_3(x) = (x - \omega_4)(x - \omega_5)$ with $\lambda, \omega_i \in K$ and $\lambda \neq 0$.

We let $J$ denote the Jacobian variety of $C$, which is an abelian variety of dimension 2 defined over $K$ that can be identified with $\text{Pic}^0(C)$. We denote the identity element of $J$ by $O_J$. Via the natural isomorphism $\text{Pic}^2(C) \rightarrow \text{Pic}^0(C)$ sending $[P_1 + P_2] \mapsto [P_1 + P_2 - \infty^+ - \infty^-]$, a point $P \in J$ can be identified with an unordered pair of points of $C$. 

\{P_1, P_2\}. This identification is unique unless \(P = O_J\), in which case it can be represented by any pair of points on \(C\) in the form \((x, y), (x, -y)\) or \(\{\infty^+, \infty^-\}\). Moreover, \(J[2] = \{O_J, \{(\omega_i, 0), (\omega_j, 0)\} \mbox{ for } i \neq j, \{(\omega_i, 0), \infty\}\}\). Let \(e_2 : J[2] \times J[2] \to \mu_2\) denote the Weil pairing on \(J[2]\). As described in [CF96, Chapter 3, Section 3], suppose \(\{P_1, P_2\}\) and \(\{Q_1, Q_2\}\) represent \(P, Q \in J[2]\) where \(P_1, P_2, Q_1, Q_2\) are Weierstrass points, then
\[
eq (1)^{(P_1, (P_2) \cap (Q_1, Q_2))}.\]

2.2. Richelot isogenies. A Richelot isogeny is a polarized \((2, 2)\)-isogeny between Jacobians of genus 2 curves. Equivalently, it is an isogeny \(\phi : J \to \tilde{J}\) such that \(\tilde{J}[\phi] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) and \(J, \tilde{J}\) are Jacobians of genus two curves.

A special case of [Mil86, Proposition 16.8] and [BD11, Lemma 2.4] shows that the kernel of a Richelot isogeny is actually a maximal isotropic subgroup of \(J[2]\) with respect to the Weil pairing \(e_2\) on \(J[2] \times J[2]\). We have the following general proposition on Richelot isogenies from [CF96, Chapter 9 Section 2] and [Fly18, Section 3]. In Remark 2.2, we gave the extra details for the case where the hyperelliptic form of the underlying curve is degree 5.

**Proposition 2.1.** Suppose the curve \(C\) is of the form
\[
y^2 = f(x) = G_1(x)G_2(x)G_3(x),\]
where \(G_j(x) = g_{j2}x^2 + g_{j1}x + g_{j0}\), and each \(g_{ji} \in K\). Then there is a Richelot isogeny \(\phi\) from \(J\), the Jacobian of \(C\), to \(\tilde{J}\), the Jacobian of the following genus two curve:
\[
\tilde{C} : \Delta y^2 = L_1(x)L_2(x)L_3(x),\]
where each \(L_i(x) = G'_1(x)G_k(x) - G_j(x)G'_k(x)\), for \([i, j, k] = [1, 2, 3], [2, 3, 1], [3, 1, 2]\), and \(\Delta = \det(g_{ij})\) which we assume to be non-zero.

In addition, the kernel of \(\phi\) consists of the identity \(O_J\) and the 3 divisors of order 2 given by \(G_i = 0\). We have the similar result for the dual isogeny \(\tilde{\phi}\).

Moreover, any genus two curve \(C\) that admits a Richelot isogeny with all the elements of the kernel \(K\)-rational is of the form \(y^2 = f(x) = G_1(x)G_2(x)G_3(x)\) as above.

**Remark 2.2.** We exclude the case \(\Delta = 0\) in the above proposition. In fact, by [CF96, Chapter 14], \(\Delta = 0\) implies that the Jacobian of \(C\) is the product of elliptic curves. It can be checked that the analogue of \(\Delta\) for \(\tilde{C}\) is \(2\Delta^2\), so no need to have further condition like \(\Delta \neq 0\) from \(\tilde{C}\). Also, in the case where \(G_i\) is linear, say \(G_i = a(x - b)\), then we say \(\{(b, 0), \infty\}\) is the divisor given by \(G_i = 0\) which gives an element in \(\ker \phi\).

We use the notation in Proposition 2.1 and denote the nontrivial elements in the kernel of \(\phi\) by \(P_i\) corresponding to the divisors of order 2 given by \(G_i = 0\) as well as denote
the nontrivial elements in the kernel of $\hat{\phi}$ by $P'$. From [CF96, Chapter 9, Section 2] and [TY09, Section 3.2], we have the following description of the Richelot isogeny $\phi$. Associated with a Weierstrass point $P = (\omega_1, 0)$ with $G_1(\omega_1) = 0$, $\phi : J \to \hat{J}$ is given explicitly as
\[ \{(x, y), P\} \mapsto \{(z_1, t_1), (z_2, t_2)\}, \]
where $z_1, z_2$ satisfy
\[ G_2(x)L_2(z) + G_3(x)L_3(z) = 0; \]
and $(z_i, t_i)$ satisfies
\[ yt_i = G_2(x)L_2(z_i)(x - z_i). \]

Denote the set of two points on $C$ given by $G_i = 0$ by $S_i$ for $i = 1, 2, 3$. From the explicit description above, we know that the preimages of $P'_i$ under $\phi$ are precisely $\{(Q_1, Q_2) \in J[2] \mid Q_1 \in S_2, Q_2 \in S_3\}$. Similarly we know the preimages of $P'_2$ and $P'_3$.

2.3. The Weil pairing for the Richelot isogeny. Let $J$ and $\hat{J}$ be Jacobian varieties of genus two curves defined over $K$. Assume there is a Richelot isogeny $\phi : J \to \hat{J}$ with $\hat{\phi}$ being its dual, i.e. $\phi \circ \hat{\phi} = [2]$. Then we have the Weil pairing
\[ e_\phi : J[\phi] \times \hat{J}[\hat{\phi}] \to \hat{K}^*, \]
where $e_\phi(P, Q) = e_{2,J}(P, Q')$ for any $Q' \in J[2]$ such that $\phi(Q') = Q$. The image of $e_\phi$ is in fact $\mu_2(\hat{K}^*) \subset \hat{K}^*$. Recall $J[\phi]$ is isotropic with respect to $e_{2,J}$ as discussed in 2.2. This implies that $e_{2,J}(P, Q') = e_{2,J}(P, Q'')$ if $\phi(Q') = \phi(Q'')$ and hence $e_\phi$ is well-defined. By (2.2) and the end of Section 2.2, $e_\phi(P_i, P'_i) = 1$ for any $i = 1, 2, 3$ and $e_\phi(P_i, P'_j) = -1$ for any $i \neq j$.

2.4. Definition of the Cassels-Tate pairing in the case of a Richelot isogeny. In this section, we give the definition of the Cassels-Tate pairing in the case of a Richelot isogeny. There are four equivalent definitions of the Cassels-Tate pairing stated and proved in [PS99]. The compatibility of the definition below with the Weil Pairing definition of the Cassels-Tate pairing on $\text{Sel}^2(J) \times \text{Sel}^2(J)$ is shown in [Yan, Proposition 2.1.6].

Let $(J, \lambda_1)$ and $(\hat{J}, \lambda_2)$ be Jacobian varieties of genus two curves defined over a number field $K$ such that there exists a Richelot isogeny $\phi : J \to \hat{J}$ with $\hat{\phi} : \hat{J} \to J$ being its dual isogeny. It is shown in [Yan, Lemma 2.1.4] that for any $b \in \text{Sel}^\phi(J)$, there exists $b_1 \in H^1(G_K, \hat{J}[2])$ mapping to $b$ under the map induced by $\hat{J}[2] \xrightarrow{\hat{\phi}} J[\phi]$.

The definition of the pairing

Let $a, a' \in \text{Sel}^\phi(J)$. Suppose $a_1 \in H^1(G_K, \hat{J}[2])$ maps to $a \in \text{Sel}^\phi(J) \subset H^1(G_K, J[\phi])$ under the map induced by $\hat{J}[2] \xrightarrow{\hat{\phi}} J[\phi]$. 

Consider the commutative diagram below.

\[
\begin{array}{ccc}
\hat{J}(K_v) & \xrightarrow{\delta} & J(K_v) \\
\downarrow & & \downarrow \\
\hat{J}(K_v) & \xrightarrow{\delta_2} & H^1(G_{K_v}, \hat{J}[^2]) \\
\downarrow & & \downarrow \\
J(K_v) & \xrightarrow{P_v \mapsto a_v} & H^1(G_{K_v}, J[\phi])
\end{array}
\]

Let \( v \) be a place of \( K \). Let \( P_v \in \hat{J}(K_v) \) be a lift of \( a_v \in H^1(G_{K_v}, J[\phi]) \). Then \( \delta_2(P_v) \) and \( a_{1,v} \) in \( H^1(G_{K_v}, J[\phi]) \) both map to \( a_v \). Hence, we may choose \( \rho_v \in H^1(G_{K_v}, \hat{J}[^2]) \) a lift of \( \delta_2(P_v) - a_{1,v} \) and define \( \eta_v = \rho_v \cup_{\hat{\phi},v} a_v \in H^2(G_{K_v}, \hat{K}_v^*) \). Here \( \cup_{\hat{\phi},v} \) denotes the cup product \( H^1(G_{K_v}, \hat{J}[^2]) \times H^1(G_{K_v}, J[\phi]) \to H^2(G_{K_v}, \hat{K}_v^*) \) associated to \( e_{\hat{\phi}} \). The Cassels-Tate pairing is defined by

\[
\langle a, a' \rangle_{CT} := \sum_v \text{inv}_v(\eta_v).
\]

We sometimes refer to \( \text{inv}_v(\eta_v) \) above as the local Cassels-Tate pairing between \( a, a' \in \text{Sel}^0(J) \) for a place \( v \) of \( K \), noting that this depends on the choice of the global lift \( a_1 \).

**Remark 2.3.** In [Yan, Proposition 1.8.4], the Weil pairing definition of the Cassels-Tate pairing is proved to be independent of the choices made in the definition. Since the above pairing is compatible with the Weil pairing definition as in [Yan, Proposition 2.1.6], we know it is also independent of the choices we make.

### 3. Computation of the Cassels-Tate Pairing

Recall that we are working with a genus curve \( C \) in the form (2.1) and we fix a choice of Richelot isogeny \( \phi : J \to \hat{J} \) where \( J \) is the Jacobian of \( C \) and \( \hat{J} \) is the Jacobian of the genus two curve defined by (2.3). We write \( \hat{\phi} \) for the dual of \( \phi \). This implies that all points in \( J[2] \) are defined over \( K \) and all points in \( \hat{J}[\hat{\phi}] \) are defined over \( K \) by Proposition 2.1. Recall, we denote the nontrivial elements in \( J[\phi] \) by \( P_1, P_2, P_3 \) where \( P_i \) corresponds to the divisor given by \( G_i = 0 \) and the nontrivial elements in \( \hat{J}[\hat{\phi}] \) by \( P'_1, P'_2, P'_3 \) where \( P'_i \) corresponds to the divisor given by \( L_i = 0 \) as in the same Proposition. In this section, we will give a practical formula for the explicit computation for the Cassels-Tate pairing in the case of Richelot isogenies.

#### 3.1. Explicit embeddings of \( H^1(G_K, J[\phi]) \) and \( H^1(G_K, J[2]) \)

In order to give the formula for the Cassels-Tate pairing, we first describe some well-known embeddings that are useful for the explicit computation.

---

This document provides a detailed exposition on the computation of the Cassels-Tate pairing in the context of Richelot isogenies. It builds on foundational results from [Yan, Proposition 1.8.4] and [Yan, Proposition 2.1.6] to establish the independence of the pairing definition from choices made in its formulation. The core of the discussion focuses on a commutative diagram illustrating the relationship between lifts of the Jacobian and the Cassels-Tate pairing definition. The process involves choosing lifts of elements in cohomology and computing their cup products to define the local Cassels-Tate pairing. The final formula is given as a sum over all places, capturing the dependence on the global lift used. The remark highlights the independence of this pairing from the choices made, a critical property for practical applications. The section concludes with a discussion on the computation, emphasizing the need for explicit embeddings of cohomology classes to facilitate practical computation.
Recall all points in \( J[2] \) and \( \hat{J}[\hat{\phi}] \) are defined over \( K \). From the exact sequence
\[
0 \to J[\phi] \xrightarrow{w_{\phi}} (\mu_2)^3 \xrightarrow{N} \mu_2 \to 0,
\]
where \( w_{\phi} : P \mapsto (e_{\phi}(P, P'_1), e_{\phi}(P, P'_2), e_{\phi}(P, P'_3)) \) and \( N : (a, b, c) \mapsto abc \), we get
\[
H^1(G_K, J[\phi]) \xrightarrow{\text{inj}} H^1(G_K, (\mu_2)^3) \cong (K^*/(K^*)^2)^3 \xrightarrow{N} H^1(G_K, \mu_2) \cong K^*/(K^*)^2,
\]
where \( \cong \) denotes the Kummer isomorphism derived from Hilbert’s Theorem 90 and \( N \) is induced by \( N \). The induced map \( H^1(G_K, J[\phi]) \to H^1(G_K, (\mu_2)^3) \) is injective as the map \( (\mu_2)^3 \xrightarrow{N} \mu_2 \) is surjective. Furthermore, the image of this injection contains precisely all the elements with norm a square from the exactness of the sequence above, i.e. \( H^1(G_K, J[\phi]) \cong \ker((K^*/(K^*)^2)^3 \xrightarrow{N} K^*/(K^*)^2) \). We have a similar embedding for \( H^1(G_K, \hat{J}[\hat{\phi}]) \).

Also, from the exact sequence
\[
0 \to J[2] \xrightarrow{w_2} (\mu_2)^5 \xrightarrow{N} \mu_2 \to 0,
\]
where \( w_2 : P \mapsto (e_2(P, \{\omega_1, 0\}, \infty)), ..., e_2(P, \{\omega_5, 0\}, \infty)) \) and \( N : (a, b, c, d, e) \mapsto abcde \), we get
\[
H^1(G_K, J[2]) \xrightarrow{\text{inj}} H^1(G_K, (\mu_2)^5) \cong (K^*/(K^*)^2)^5 \xrightarrow{N} H^1(G_K, \mu_2) \cong K^*/(K^*)^2,
\]
where \( \cong \) denotes the Kummer isomorphism derived from Hilbert’s Theorem 90 and \( N \) is induced by \( N \). Again the induced map \( H^1(G_K, J[2]) \to H^1(G_K, (\mu_2)^5) \) is injective as the map \( (\mu_2)^5 \xrightarrow{N} \mu_2 \) is surjective. Furthermore, the image of this injection also contains precisely all the elements with norm a square from the exact sequence above. In particular, we have
\[
H^1(G_K, J[2]) \cong (K^*/(K^*)^2)^4.
\]

3.2. Explicit Formula. Using the embeddings described in Section 3.1, we can now state and prove the explicit formula for the Cassels-Tate pairing in the case of a Richelot isogeny.

**Proposition 3.1.** Under the embeddings of \( H^1(G_K, J[\phi]) \) and \( H^1(G_K, \hat{J}[\hat{\phi}]) \) in \((K^*/(K^*)^2)^3\) as described in Section 3.1, we get that the cup product \( \cup_{\phi} \) induced by \( e_{\phi} \) is
\[
H^1(G_K, J[\phi]) \times H^1(G_K, \hat{J}[\hat{\phi}]) \to \text{Br}(K)[2]
\]
\[
((a_1, b_1, c_1), (a_2, b_2, c_2)) \mapsto (a_1, a_2) + (b_1, b_2) + (c_1, c_2),
\]
where \((, ,)\) represents the quaternion algebra and also its equivalence class in \( \text{Br}(K)[2] \).

**Proof.** Recall that the embedding \( J[\phi] \to (\mu_2)^3 \) is given by sending \( P \in J[\phi] \) to
\[
(e_{\phi}(P, P'_1), e_{\phi}(P, P'_2), e_{\phi}(P, P'_3))
\]
and the embedding \( \hat{J}[\phi] \to (\mu_2)^3 \) is given by sending \( Q \in \hat{J}[\phi] \) to 
\[(e_\phi(P_1, Q), e_\phi(P_2, Q), e_\phi(P_3, Q)).\]

It can be checked, via the end of discussion of Section 2.2, that we have the following commutative diagram:
\[
\begin{array}{ccc}
J[\phi] \times \hat{J}[\phi] & \xrightarrow{\text{inj}} & (\mu_2)^3 \times (\mu_2)^3 \\
\mu_2 \downarrow & & \downarrow f \\
\mu_2 & \to & \mu_2,
\end{array}
\]
where \( f \) sends \((-1)^a, (-1)^b, (-1)^c\), \((-1)^{a'}, (-1)^{b'}, (-1)^{c'}\) to \((-1)^{aa'+bb'+cc'}\) with \(a, b, c \in \{0, 1\}\).

Consider the natural pairing \( \phi : \mu_2 \times \mu_2 \to \mu_2 \) sending \((-1)^a, (-1)^b\) to \((-1)^{ab}\). This gives a cup product pairing
\[H^1(G_K, \mu_2) \times H^1(G_K, \mu_2) \to H^2(G_K, \mu_2) \cong \text{Br}(K)[2],\]
where \(\phi(a_\sigma, b_\tau) = \phi(a_\sigma, b_\tau)\).

By Hilbert’s Theorem 90, we can identify \(H^1(G_K, \mu_2)\) with \(K^*/(K^*)^2\). Under this identification, the image of \((a, b) \in K^*/(K^*)^2 \times K^*/(K^*)^2\) is precisely the equivalence class of the quaternion algebra \((a, b)\) by [Ser79, Chapter XIV, Section 2, Proposition 5] and [GS06, Corollary 2.5.5(1), Proposition 4.7.3].

Therefore, we get that the induced cup product is
\[H^1(K, J[\phi]) \times H^1(K, \hat{J}[\phi]) \to \text{Br}(K)[2],\]
\[((a_1, b_1, c_1), (a_2, b_2, c_2)) \mapsto (a_1, a_2) + (b_1, b_2) + (c_1, c_2).\]

\[\square\]

**Proposition 3.2.** Under the embeddings of \(H^1(G_K, J[\phi])\) and \(H^1(G_K, J[2])\) in \((K^*/(K^*)^2)^3\) and \((K^*/(K^*)^2)^5\) respectively as described in Section 3.1, the map \(\Psi : H^1(G_K, J[\phi]) \to H^1(G_K, J[2])\) induced from the inclusion \(J[\phi] \to J[2]\) is given by
\[(a, b, c) \mapsto (1, c, b, b).\]

**Proof.** Recall the embedding of \(H^1(G_K, J[2])\) in \((K^*/(K^*)^2)^5\), and the embedding of \(H^1(G_K, J[\phi])\) in \((K^*/(K^*)^2)^3\) are induced from the short exact sequences in the following commutative diagram:
\[
\begin{array}{ccccccc}
0 & \to & J[\phi] & \xrightarrow{w_\phi} & (\mu_2)^3 & \xrightarrow{N} & \mu_2 & \to & 0 \\
& & \downarrow \text{inc} & & \downarrow \Psi & & \downarrow = & & \\
0 & \to & J[2] & \xrightarrow{w_2} & (\mu_2)^5 & \xrightarrow{N} & \mu_2 & \to & 0.
\end{array}
\]

Suppose \(P \in J[\phi]\) maps to \((\alpha, \beta, \gamma)\) via \(w_\phi\). Then \(e_\phi(P, P_1') = \alpha, e_\phi(P, P_2') = \beta, e_\phi(P, P_3') = \gamma\). By definition, \(e_\phi(P, \phi(Q)) = e_2(P, Q)\) for any \(Q \in J[2]\). From the explicit description
Proposition 3.3. Under the embeddings of \( H^1(G_K, J[\phi]) \) and \( H^1(G_K, J[2]) \) in \((K^*/(K^*)^2)^3\) and \((K^*/(K^*)^2)^5\) respectively as described in Section 3.1, the map \( \Psi : H^1(G_K, J[2]) \to H^1(G_K, J[\phi]) \) induced from \( J[2] \xrightarrow{\phi} J[\phi] \) is given by \((a_1, a_2, a_3, a_4, a_5) \mapsto (a_1, a_2a_3, a_4a_5)\).

Proof. Consider the following commutative diagram whose rows are exact sequences

\[
\begin{array}{cccccc}
0 & \longrightarrow & J[2] & \xrightarrow{w_2} & (\mu_2)^5 & \xrightarrow{N} & \mu_2 & \longrightarrow & 0 \\
0 & \xrightarrow{\phi} & \J[\phi] & \xrightarrow{w_\phi} & (\mu_2)^3 & \xrightarrow{N} & \mu_2 & \longrightarrow & 0.
\end{array}
\]

Suppose \( P \in J[2] \) maps to \((\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)\) via \( w_2 \). Then \( \alpha_i = e_2(P, (\omega_i, 0)) \). Recall that \( e_2(\phi(P), P_1) = e_2(P, P_1) \) by the discussion at the end of Section 2.3. This implies that \( \phi(P) \) maps to \((\alpha_1, \alpha_2a_3, \alpha_4a_5)\) via \( w_\phi \). Therefore, we can verify that the induced map \( \Psi : H^1(G_K, J[2]) \to H^1(G_K, \J[\phi]) \) under the embeddings in Section 3.1 is given by

\((a_1, a_2, a_3, a_4, a_5) \mapsto (a_1, a_2a_3, a_4a_5)\).

Remark 3.4. We observe that, under the assumption of this section, we have the following short exact sequence:

\[0 \to H^1(G_K, J[\phi]) \to H^1(G_K, J[2]) \to H^1(G_K, \J[\phi]) \to 0.\]

The injectivity of the map \( H^1(G_K, J[\phi]) \to H^1(G_K, J[2]) \) is due to the surjectivity of \( J(K)[2] \xrightarrow{\phi} \J(K)[\phi] \). Observe that the element in \( H^1(G_K, \J[\phi]) \) represented by \((a, b, c)\) has a preimage in \( H^1(G_K, J[2]) \) represented by \((a, 1, b, 1, c)\) by Proposition 3.3. This implies that \( H^1(G_K, J[2]) \to H^1(G_K, \J[\phi]) \) is surjective.

Remark 3.5. Let \( v \) be a place of \( K \). We also have the explicit embeddings of \( H^1(G_K, J[\phi]) \) and \( H^1(G_K, J[2]) \) described in Section 3.1 as well as the explicit maps given in this section.
if we replace $K$ with $K_v$ or $K_v^{ur}$.

Using the above three propositions, we now have the explicit formula for the Cassels-Tate pairing in the case of a Richelot isogeny.

**Theorem 3.6.** Let $J$ be the Jacobian variety of a genus two curve defined over a number field $K$. Suppose all points in $J[2]$ are defined over $K$ and there exists a Richelot isogeny $\phi : J \to \widehat{J}$ where $\widehat{J}$ is the Jacobian variety of another genus two curve. Let $\widehat{\phi}$ be the dual isogeny of $\phi$. Consider $a, a' \in \text{Sel}^1(\widehat{J})$. Suppose $(\alpha'_1, \alpha'_2, \alpha'_3) \in (K^*/(K^*)^2)^3$ represents $a'$. For any place $v$, we let $P_v \in J(K_v)$ denote a lift of $a_v \in H^1(G_{K_v}, \widehat{J}(\overline{\phi}))$ and suppose $\delta_2(P_v) \in H^1(G_{K_v}, J[2])$ is represented by $(x_1, v, x_2, v, x_3, v, x_4, v, x_5, v) \in (K^*/(K^*)^2)^5$. Then we have

$$\langle a, a' \rangle_{CT} = \prod_v (x_2, v, x_4, v, \alpha'_1) \cdot (x_4, v, \alpha'_2) \cdot (x_2, v, \alpha'_3)_v,$$

where $(\ , \ )_v$ represents the Hilbert symbol.

**Proof.** Suppose $a$ is represented by $(\alpha_1, \alpha_2, \alpha_3) \in (K^*/(K^*)^2)^3$. Then it has a preimage $a_1 \in H^1(G_K, J[2])$ represented by $(\alpha_1, 1, \alpha_2, 1, \alpha_3)$ by Proposition 3.3. So following the definition of $\langle a, a' \rangle_{CT}$, we need to compute $\rho_v \cup_{\phi, v} \eta \in H^2(G_{K_v}, \overline{K_v}^*)$ where $\rho_v \in H^1(G_{K_v}, J[\phi])$ is a lift of $\delta_2(P_v) - a_1, v$ and $\cup_{\phi, v}$ is the cup product induced by $\epsilon_\phi$. Since $\delta_2(P_v) - a_1, v$ is represented by $(x_1, v, a_1, x_2, v, x_3, v, a_2, x_4, v, x_5, v, a_3)$, by Proposition 3.2, $\rho_v$ is represented by $(x_2, v, x_4, v, x_4, v, x_2, v)$. Hence, by Proposition 3.1, we know $\langle \epsilon, \eta \rangle_{CT} = \sum_v \text{inv}_v((x_2, v, x_4, v, \alpha'_1) + (x_4, v, \alpha'_2) + (x_2, v, \alpha'_3)) = \prod_v (x_2, v, x_4, v, \alpha'_1) \cdot (x_4, v, \alpha'_2) \cdot (x_2, v, \alpha'_3)_v$.

\[\square\]

### 4. Computational details

In this section, we will describe some further details for the explicit computation for the Cassels-Tate pairing using the formula in Theorem 3.6.

#### 4.1. Embedding of $\widehat{J}(K)/\phi(J(K))$ and $J(K)/2J(K)$

General results show that we have the injection, which is the composition of the connecting map $\delta_\phi : \widehat{J}(K)/\phi(J(K)) \to H^1(G_K, J[\phi])$ and the embedding described above $H^1(G_K, J[\phi]) \to (K^*/(K^*)^2)^3$. This is discussed in [Fly18, Section 3] [CF96, Chapter 10 Section 2]. More explicitly, we have

$$\mu^\phi : \widehat{J}(K)/\phi(J(K)) \to K^*/(K^*)^2 \times K^*/(K^*)^2 \times K^*/(K^*)^2 \times \left\{(x_1, y_1), (x_2, y_2)\right\} \mapsto (L_1(x_1)L_1(x_2), L_2(x_1)L_2(x_2), L_3(x_1)L_3(x_2)).$$

Similarly we have the injection:

$$\mu^\phi : J(K)/\phi(J(K)) \to \left\{(x_1, y_1), (x_2, y_2)\right\} \mapsto (G_1(x_1)G_1(x_2), G_2(x_1)G_2(x_2), G_3(x_1)G_3(x_2)).$$

Note the following special cases. When $x_j$ is a root of $G_i$, then $G_i(x_j)$ should be taken to be $\prod_{l \in \{1, 2, 3\} \setminus \{i\}} G_l(x_j)$. We have a similar solution when $x_j$ is a root of $L_i$, which
is replacing \( L_i(x_j) \) with \( \Delta \prod_{i \in \{1,2,3\} \setminus \{i\}} L_i(x_j) \). When \((x_j, y_j) = \infty\), then \( G_i(x_j) \) is taken to be 1. In the case where one of \( L_i \) is linear and \((x_j, y_j) = \infty\), then \( L_i(x_j) \) is taken to be 1.

On the other hand, we have a standard injection, which is the composition of the connecting map \( \delta_2 : J(K)/2J(K) \to H^1(G_K, J[2]) \) and the embedding described above \( H^1(G_K, J[2]) \to (K^*/(K^*)^2)^5 \). This can also be found in [Fly18, Section 3] [CF96, Chapter 10 Section 2].

\[
\mu : \frac{J(K)}{2J(K)} \to (K^*/(K^*)^2)^5 \quad \{(x_1, y_1), (x_2, y_2)\} \mapsto ((x_1 - \omega_1)(x_2 - \omega_1), \ldots, (x_1 - \omega_5)(x_2 - \omega_5)) .
\]

Note the following special cases. When \((x_j, y_j) = (\omega_i, 0)\), then \( x_j - \omega_i \) should be taken to be \( \lambda \prod_{i \in \{1,2,3,4,5\} \setminus \{i\}} (\omega_i - \omega_j) \). When \((x_j, y_j) = \infty\), then \( x_j - \omega_i \) is taken to be \( \lambda \).

Observe that the image of the maps \( \mu^\phi \) and \( \mu^\hat{\phi} \) are both contained in the kernel of \((K^*/(K^*)^2)^3 \xrightarrow{\varphi} K^*/(K^*)^2\). Similarly, the image of \( \mu \) is contained in the kernel of \((K^*/(K^*)^2)^5 \xrightarrow{\varphi} K^*/(K^*)^2\).

### 4.2. Bounding the set of bad primes

The formula for the local Cassels-Tate pairing in Theorem 3.6 is trivial outside the finite set of places \( S = \{\text{places of bad reduction for } C\} \cup \{\text{places dividing } 2\} \cup \{\text{finite places}\} \). This is explained as follows.

By [Mil06, Chapter I, Section 6] [Sch95, Section 3], we have \( \text{Sel}^\phi(J) \subset H^1(G_K, J\phi); S = \ker (H^1(G_K, J\phi) \to \prod_{v \in S} H^1(G_{K_v^nr}, J\phi)) \). Similarly, \( \text{Sel}^\hat{\phi}(\hat{J}) \subset H^1(G_K, \hat{J}\hat{\phi}); S \) and \( \text{Sel}^2(J) \subset H^1(G_K, J[2]); S \) . It can be shown that \( \ker (K^*/(K^*)^2 \to \prod_{v \notin S} K_v^nr^*/(K_v^nr^*)^2) = K(S,2) \), where \( K(S,2) \) is defined to be \( \{x \in K^*/(K^*)^2 : \text{ord}_v(x) \text{ is even for all } v \notin S\} \). So, we have \( \alpha_i, \alpha'_i \in K(S,2) \) for all \( i \). Suppose \( v \notin S \). We know there exists a representation of the image of \( a_{i,x} \) in \((K_v^*/(K_v^*)^2)^5\) such that all its coordinates have valuation 0. Since \( J(K_v^nr) \xrightarrow{\varphi} J(K_v^nr) \) is surjective by [AS02, Lemma 3.4], the map \( H^0(G_{K_v^nr}, J) \to H^1(G_{K_v^nr}, J[2]) \) is the zero map and hence the image of \( P_v \) is trivial in \( H^1(G_{K_v^nr}, J[2]) \). This implies that \( \delta_2(P_v) \subset (K_{v^nr}^*/(K_{v^nr}^*)^2)^5 \) has a representation such that all its coordinates have valuation 0. This implies that \( \delta_2(P_v) - a_{i,v} \subset H^1(G_K, J[2]) \subset (K_{v^nr}^*/(K_{v^nr}^*)^2)^5 \) has a representation such that all its coordinates have valuation 0. Then, by the formula in Proposition 3.2, \( \rho_v \in H^1(G_K, \hat{J}(\hat{\phi})) \subset (K_v^*/(K_v^*)^2)^3 \) also has a representation such that all its coordinates have valuation 0. From the first part of the theorem, we know computing \( \langle a, a' \rangle_{CT} \) requires computing the Hilbert symbol. It can be checked that the Hilbert symbol between \( x \) and \( y \) is trivial when the valuations of \( x, y \) are both 0 and the local field has odd residue characteristic. Hence, the local Cassels-Tate pairing is trivial for all but finitely many places contained in the set \( S \).

### 5. Worked Example

We explicitly compute the Cassels-Tate pairing in an example where this improves the rank bound obtained via descent by Richelot isogeny. We will be using the same notation as in Section 2.4 to compute \( \langle \cdot, \cdot \rangle_{CT} \) on \( \text{Sel}^\phi(\hat{J}) \times \text{Sel}^\phi(\hat{J}) \). Our base field \( K \) is the field of
the rationals, \( \mathbb{Q} \).

Let us consider the following genus two curve which is obtained by taking \( k = 113 \) in [Fly18, Theorem 1]

\[
\mathcal{C} : y^2 = (x + 2 \cdot 113)x(x - 6 \cdot 113)(x + 113)(x - 7 \cdot 113),
\]

with \( G_1 = (x + 2 \cdot 113), G_2 = x(x - 6 \cdot 113), G_3 = (x + 113)(x - 7 \cdot 113) \) and

\[
\begin{bmatrix}
2 \cdot 113 & 1 & 0 \\
0 & -6 \cdot 113 & 1 \\
-7 \cdot 113^2 & -6 \cdot 113 & 1
\end{bmatrix} = -7 \cdot 113^2,
\]

\[
L_1 = G_2^*G_3 - G_3^*G_2 = -14 \cdot 113^2(x - 3 \cdot 113),
\]

\[
L_2 = G_3^*G_1 - G_1^*G_3 = (x + 5 \cdot 113)(x - 113),
\]

\[
L_3 = G_1^*G_2 - G_2^*G_1 = -(x + 6 \cdot 113)(x - 2 \cdot 113).
\]

So we have a Richelot isogeny \( \phi \) from \( J \), the Jacobian variety of \( \mathcal{C} \), to \( \hat{J} \), the Jacobian variety of the following curve.

\[
\hat{\mathcal{C}} : y^2 = -2(x - 3 \cdot 113)(x + 5 \cdot 113)(x - 113)(x + 6 \cdot 113)(x - 2 \cdot 113)
\]

It can be shown, using MAGMA [BCP97], that:

(5.1)
\[
\operatorname{Sel}^\phi(\hat{J}) = \langle (2 \cdot 113, -14 \cdot 113, -7), (113, 7, 7 \cdot 113), (113, 113, 1), (2, 2, 1), (1, 7, 7) \rangle \subset (\mathbb{Q}^*/(\mathbb{Q}^*)^2)^3,
\]

(5.2)
\[
\operatorname{Sel}^\phi(J) = \langle (113, -7 \cdot 113, -7), (2 \cdot 113, 7, 14 \cdot 113), (113, 1, 113) \rangle \subset (\mathbb{Q}^*/(\mathbb{Q}^*)^2)^3.
\]

Now we will compute the Cassels-Tate pairing matrix on \( \operatorname{Sel}^\phi(\hat{J}) \times \operatorname{Sel}^\phi(\hat{J}) \). Since \((2 \cdot 113, -14 \cdot 113, -7), (113, 7, 7 \cdot 113)\) are images of elements \{(0, 0), (-2 \cdot 113, 0)\} and \{(-2 \cdot 113, 0), (-113, 0)\} in \( J(\mathbb{Q})/\overline{\phi}(\hat{J}(\mathbb{Q})) \) via \( \mu^\phi \) in (4.1), they are in the kernel of the Cassels-Tate pairing. So it is sufficient to look at the pairing on \( \langle (113, 113, 1), (2, 2, 1), (1, 7, 7) \rangle \times \langle (113, 113, 1), (2, 2, 1), (1, 7, 7) \rangle \).

Since the primes of bad reduction are \( \{2, 3, 7, 113\} \), we know these are the only primes for which we need to consider by Theorem 3.6. The tables below give details of the local computations at these primes.

Let \( a = (113, 113, 1) \in \operatorname{Sel}^\phi(\hat{J}) \). By the formula given in Proposition 3.3, it has a lift \( a_1 = (113, 1, 113, 1, 1) \in H^1(G_K, J[2]) \). Then for the local calculation, we have the following table:
JIALI YAN

| place $v$ | $\infty$ | 2 | 3 | 7 | 113 |
|-----------|----------|---|---|---|-----|
| $P_v$     | id       | id | \{(0,0),(-113,0)\} | id | \{(0,0),(-2\cdot113,0)\} |
| $\delta_2(P_v)$ | id       | id | (-1,3,-3,-1,-1) | id | (113,3\cdot113,3,1,1) |
| $a_{1,v}$ | id       | id | (-1,1,-1,1,1)   | id | (113,1,113,1,1)    |
| $\delta_2(P_v) - a_{1,v}$ | id       | id | (1,3,3,-1,-1)   | id | (1,3\cdot113,3\cdot113,1,1) |
| $\rho_v$  | id       | id | (-3,-1,3)       | id | (3\cdot113,1,3\cdot113) |

Now let $a = (2,2,1) \in \text{Sel}^\hat{\phi}(J)$. By the formula given in Proposition 3.3, it has a lift $a_1 = (2,1,2,1,1) \in H^1(G_K,J[2])$. Then for the local calculation, we have the following table:

| place $v$ | $\infty$ | 2 | 3 | 7 | 113 |
|-----------|----------|---|---|---|-----|
| $P_v$     | id       | \{(0,0),(-2\cdot113,0)\} | \{(0,0),(-113,0)\} | id | id |
| $\delta_2(P_v)$ | id       | (2,6,3,-1,-1) | (-1,3,-3,-1,-1) | id | id |
| $a_{1,v}$ | id       | (2,1,2,1,1) | (-1,1,-1,1,1) | id | id |
| $\delta_2(P_v) - a_{1,v}$ | id       | (1,6,6,-1,-1) | (1,3,3,-1,-1) | id | id |
| $\rho_v$  | id       | (-6,-1,6)   | (-3,-1,3)      | id | id |

Lastly let $a = (1,7,7) \in \text{Sel}^\hat{\phi}(J)$. By the formula given in Proposition 3.3, it has a lift $a_1 = (1,1,7,1,7) \in H^1(G_K,J[2])$. Then for the local calculation, we have the following table:

| place $v$ | $\infty$ | 2 | 3 | 7 | 113 |
|-----------|----------|---|---|---|-----|
| $P_v$     | id       | \{(-2\cdot113,0),(-113,0)\} | \{(-2\cdot113,0),(-113,0)\} | id | id |
| $\delta_2(P_v)$ | id       | (1,2,-2,-2,2) | (1,1,7,7,1) | id | id |
| $a_{1,v}$ | id       | (1,1,-1,1,-1) | (1,1,7,1,7) | id | id |
| $\delta_2(P_v) - a_{1,v}$ | id       | (1,2,2,-2,-2) | (1,1,1,7,7) | id | id |
| $\rho_v$  | id       | (-1,-2,2)   | (7,7,1)       | id | id |

Following the explicit algorithm for computing the Cassels-Tate pairing, we get that the Cassels-Tate pairing between (113,113,1) and (2,2,1) is the only nontrivial one.

Therefore, we get the $5 \times 5$ Cassels-Tate pairing matrix from the 5 generators of $\text{Sel}^\hat{\phi}(J)$. More specifically, the $ij^{th}$ entry of the matrix is the Cassels-Tate pairing between the $i^{th}$ and the $j^{th}$ generators of $\text{Sel}^\hat{\phi}(J)$, where the generators are in the same order as listed in the Selmer group $\text{Sel}^\hat{\phi}(J)$ (5.1).

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]
Remark 5.1. From the computation above, we have shown that the kernel of the Cassels-Tate pairing has dimension 3. We make the following observations:

- We bound the rank of $J(\mathbb{Q})$ via bounding $|J(\mathbb{Q})/\hat{\phi}(\hat{J}(\mathbb{Q}))|$ by $|\ker(\cdot, \cdot_{CT})| = 2^4$ instead of $|\text{Sel}^\phi(\hat{J})| = 2^5$. This improves the rank bound of $J(\mathbb{Q})$ from 4 to 2.

- Consider the exact sequence (1.1). It can be shown that $\text{Im}\, \alpha$ is contained inside $\ker(\cdot, \cdot_{CT})$, the kernel of the Cassels-Tate pairing on $\text{Sel}^\phi(\hat{J}) \times \text{Sel}^\phi(\hat{J})$. Indeed, if $a \in \text{Sel}^\phi(\hat{J})$ is equal to $\alpha(b)$, where $b \in \text{Sel}^\phi(\hat{J})$, then following the earlier notations, we can let $a_1 = b$. Then we can pick $P_v \in J(\mathbb{Q}_v)$ to be the lift of $a_{1,v}$. Therefore, $\delta_2(P_v) - a_{1,v} = 0 \in H^1(G_{\mathbb{Q}_v}, J[2])$ which implies, $a \in \ker(\cdot, \cdot_{CT})$. Hence, we can always bound $|\text{Sel}^\phi(J)|$ and this bound will be sharp when $\text{Im}\, \alpha = \ker(\cdot, \cdot_{CT})$.

We used MAGMA to compute the size of $\text{Sel}^\phi(J)$, which is equal to $2^6$, and we have the exact sequence:

$$0 \to J[\phi](\mathbb{Q}) \to J[2](\mathbb{Q}) \to \hat{J}[\hat{\phi}](\mathbb{Q}) \to \text{Sel}^\phi(J) \to \text{Sel}^\phi(J) \xrightarrow{\alpha} \ker(\cdot, \cdot_{CT}) \to 0.$$ 

So for this example, we have turned the descent by Richelot isogeny into a 2-descent via computing the Cassels-Tate pairing.

References

[AS02] A. Agashe and W. Stein, Visibility of Shafarevich–Tate Groups of Abelian Varieties, Journal of Number Theory 97 (2002), no. 1, 171-185, DOI 10.1006/jnth.2002.2810. ↑10

[vB] M. van Beek, Computing the Cassels-Tate Pairing. Doctoral Dissertation. University of Cambridge, 2015. ↑11

[vBF18] M. van Beek and T. A. Fisher, Computing the Cassels-Tate pairing on 3-isogeny Selmer groups via cubic norm equations, Acta Arithmetica 185 (2018), no. 4, 367-396, DOI 10.4064/AA171108-11-4. ↑1

[BCP97] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997), no. 3-4, 235-265, DOI 10.1006/jsco.1996.0125. ↑11

[BD11] N. Bruin and K. Doerksen, The Arithmetic of Genus Two Curves with $(4, 4)$-Split Jacobians, Canadian Journal of Mathematics 63 (2011), no. 5, 992-1024, DOI 10.4153/CJM-2011-039-3. ↑13

[Cas59] J. W. S. Cassels, Arithmetic on Curves of Genus 1. I. On a conjecture of Selmer., Journal für die reine und angewandte Mathematik 202 (1959), 52-99. ↑1

[Cas62] ______, J. W. S. Cassels, Arithmetic on curves of genus 1, IV. Proof of the Hauptvermutung., Journal für die reine und angewandte Mathematik 211 (1962), 95-112, DOI 10.1515/crll.1962.211.95. ↑1

[Cas98] ______, Second Descents for Elliptic Curves, Journal für die reine und angewandte Mathematik 494 (1998), 101-127, DOI 10.1515/crll.1998.001. ↑1

[CF96] J. W. S. Cassels and E. V. Flynn, Prolegomena to a MiddleBrow Arithmetic of Curves of Genus 2, London Mathematical Society Lecture Note Series, vol. 230, Cambridge University Press, 1996. ↑13, 4, 9, 10

[CF67] J. W. S. Cassels and A. Frohlich, Algebraic Number Theory, Proceedings of an instructional conference organized by the London Mathematical Society (a nano advanced study institute) with the support of the international union, Academic Press Inc. (London) LTD., 1967. ↑1

[Don15] S. Donnelly, Algorithms for the Cassels-Tate pairing (2015). preprint. ↑1
[Fis03] T. A. Fisher, *The Cassels–Tate pairing and the Platonic solids*, Journal of Number Theory 98 (2003), no. 1, 105-155, DOI 10.1016/S0022-314X(02)00038-0. ↑1

[Fis16] ______, *On binary quartics and the Cassels-Tate pairing* (2016). preprint. ↑1

[FN14] T. A. Fisher and R. Newton, *Computing the Cassels-Tate pairing on the 3-Selmer group of an elliptic curve*, Journal of Number Theory 10 (2014), no. 7, 18811907, DOI 10.1142/S1793042114500602. ↑1

[Fly18] E. V. Flynn, *Arbitrarily Large Tate–Shafarevich Group on Abelian Surfaces*, Journal of Number Theory 186 (2018), 248-258, DOI 10.1016/j.jnt.2017.10.004. ↑3, 9, 10, 11

[GS06] P. Gille and T. Szamuely, *Central Simple Algebras and Galois Cohomology*, Cambridge Studies in Advanced Mathematics, vol. 101, Cambridge University Press, 2006. ↑7

[Mi86] J.S. Milne, *Abelian Varieties*, Springer, New York, 1986. pp 103-150 of Arithmetic Geometry. ↑3

[Mil06] ______, *Arithmetic Duality Theorems, Second Edition*, BookSurge, LLC, 2006. ↑10

[Mil08] ______, *Abelian Varieties, Second Edition*, 2008. Available at www.jmilne.org/math/. ↑

[PS99] B. Poonen and M. Stoll, *The Cassels-Tate pairing on polarized abelian varieties*, Annals of Mathematics 150 (1999), no. 3, 1109-1149, DOI 10.2307/121064¡. MR1740984 ↑4

[Sch95] E. F. Schaefer, *2-Descent on the Jacobians of Hyperelliptic Curves*, Journal of Number Theory 51 (1995), no. 2, 219-232, DOI 10.1006/jnth.1995.1044. ↑10

[Ser79] Jean-Pierre Serre, *Local Fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag New York, 1979. ↑7

[Sko] A. Skorobogatov, *Abelian varieties over local and global fields*. TCC course, Spring 2016. ↑

[TY09] K. Takashima and R. Yoshida, *An algorithm for computing a sequence of Richelot isogenies*, Bulletin of the Korean Mathematical Society 46 (2009), no. 4, 789-802, DOI 10.4134/BKMS.2009.46.4.789. ↑4

[Tat62] J. Tate, *Duality theorems in Galois cohomology over number fields*, Proc. Internat. Congr. Mathematicians (Stockholm) (1962), 288–295. Inst. Mittag-Leffler, Djursholm (1963). ↑1

[Yan] J. Yan, *Computing the Cassels-Tate Pairing for Jacobian Varieties of Genus Two Curves*. Doctoral Dissertation. University of Cambridge, 2021. ↑1, 2, 4, 5