ANALYSIS OF THE LERAY-$\alpha$ MODEL WITH NAVIER SLIP BOUNDARY CONDITION

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ABSTRACT. In this paper, we establish the existence and the regularity of a unique weak solution to turbulent flows in a bounded domain $\Omega \subset \mathbb{R}^3$ governed by the so-called Leray-$\alpha$ model. We consider the Navier slip boundary conditions for the velocity. Furthermore, we show that, when the filter coefficient $\alpha$ tends to zero, the weak solution constructed converges to a suitable weak solution to the incompressible Navier Stokes equations subject to the Navier boundary condition. Similarly, if $\lambda \to 1$ we recover a solution to the Leray-$\alpha$ model with the homogeneous Dirichlet boundary conditions.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^\infty$ boundary, $T \in (0, \infty)$, and $\alpha > 0$. Our goal is to study properties of the Leray-$\alpha$ model ($\mathcal{L}(\alpha)$)

\begin{align*}
\text{div} \, v &= 0, \\
v_t + \text{div}(\nabla \otimes v) - 2\nu \text{div}D(v) &= -\nabla p + f, \\
-\alpha^2 \text{div}D(\nabla v) + \nabla \pi &= 0, \quad \text{div} v = 0.
\end{align*}

considered in $(0, T) \times \Omega$. Here, all appearing quantities are smoothed. The unknown functions are the fluid velocity field $v$ and the pressure $p$. The external body force $f$ and the viscosity $\nu > 0$ are given.

The system is completed by an initial condition

\begin{equation}
(1.4) \quad v(0, x) = v_0(x) \quad \text{in } \Omega,
\end{equation}

and a boundary condition

\begin{align*}
(1.5) \quad v \cdot n &= 0, \quad \lambda v_\tau + (1 - \lambda)(D(v)n)_\tau = 0 \quad \text{on } (0, T) \times \partial \Omega, \\
(1.6) \quad \nabla \cdot n &= 0, \quad \lambda \pi_\tau + (1 - \lambda)(D(\nabla v)n)_\tau = 0 \quad \text{on } (0, T) \times \partial \Omega.
\end{align*}

Here, $n = n(x)$ is the outer normal located at $x \in \partial \Omega$ to the boundary, $w_\tau := w - (w \cdot n)n$ is the projection of a vector $w = w(x)$ to the tangent plane of the boundary at $x$, and the parameter $\lambda \in [0, 1]$ homotopically connects perfect slip boundary condition when $\lambda = 0$ with no-slip boundary conditions when $\lambda = 1$. If $0 < \lambda < 1$, then (1.5) is called Navier slip boundary conditions. In this paper we assume that $\lambda$ is any number from $[0, 1)$.

We start our investigation showing that the problem (1.1)-(1.6) possesses a unique weak solution. Since the existence and regularity theory of the problem

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\((1.3)\) with boundary condition \((1.6)\) is well known, compare Lemma 2.1 and Corollary 2.1. \(v\) can be always uniquely reconstructed from \(v\). In this sense we understand \(\overline{v}\) in the whole article and we concentrate only on properties of \((v,p)\).

**Theorem 1.1.** Let \(f \in L^2(0,T;W^{-1,2}_n), \ v_0 \in L^2_{n,\text{div}}.\) Then there exists a unique solution \((v,p)\) to the system \((1.1)-(1.3)\) such that

\[
\begin{align*}
(1.7) & \quad v \in C(0,T;L^2_{n,\text{div}}) \cap L^2(0,T;W^{1,2}_n,\text{div}), \\
(1.8) & \quad v_t \in L^2(0,T;W^{-1,2}_n), \\
(1.9) & \quad p \in L^2(0,T;L^2)
\end{align*}
\]

\[(1.10) \quad \int_\Omega p dx = 0 \quad \text{for a.e.} \ t \in (0,T).\]

and

\[
\int_0^T \langle v_t, w \rangle - (\overline{v} \otimes v, \nabla w) + \frac{2\nu\lambda}{1-\lambda}(v, w) d\Omega + 2
\nu(D(v), D(w)) dt
\]

\[(1.11) = \int_0^T (p, \text{div} w) + \langle f, w \rangle dt \quad \text{for all} \ w \in L^2(0,T;W^{1,2}_n).\]

The initial conditions are attained in the following sense

\[(1.12) \quad \lim_{t \to 0^+} \|v(t) - v_0\|^2_2 = 0.\]

Moreover, the solution \((v,p)\) satisfies the local energy equality

\[
\begin{align*}
(1.13) & \quad \frac{1}{2} \int_\Omega |v|^2 \phi(t,x) dx + \nu \int_0^T \int_\Omega |\nabla v|^2 \phi dt dx \\
= & \quad \frac{1}{2} \int_\Omega |v_0|^2 \phi(0,x) dx + \int_0^T \int_\Omega \left( \frac{|v|^2}{2} \phi_t + \nu \Delta \phi \right) dx dt \\
& \quad + \int_0^T \int_\Omega \left( \frac{|v|^2}{2} \overline{v} + pv \right) \cdot \nabla \phi dx dt + \int_0^T \langle f, \phi \rangle dt, 
\end{align*}
\]

for all \(t \in (0,T)\) and for all non negative functions \(\phi \in C^\infty(\Omega \times \mathbb{R})\) and \(\text{spt} \phi \subset \subset \mathbb{R} \times \Omega.\)

In the next theorem we focus our attention to the regularity of the unique weak solution of \((1.1)-(1.6)\). First, we define the spaces of initial conditions. We follow [25]. We set for \(q \geq 2\)

\[
D_q := \{ \phi \in B_q^{2(1-\frac{\alpha}{q})} \cap \mathcal{L}_{n,\text{div}}^q : (1.5) \ \text{holds if} \ q > 3 \}.
\]

Here the spaces \(B_q^{a,p}\) are the standard Besov spaces, see [25] Section 2.2. Note that \(D_2 = W^{1,2}_{n,\text{div}}.\)

Now we can formulate the maximal regularity result

**Theorem 1.2.** Assume \(q \geq 2, q \neq 3, f \in L^q(0,T;\mathcal{L}_{n,\text{div}})\) and \(v_0 \in D_q.\) Then the unique weak solution of the problem \(L(\alpha)\) with initial boundary condition \((1.3)\) and boundary condition \((1.5)-(1.6)\) is regular, i.e. \(v \in L^q(0,T;W^{2,q}_{n,\text{div}}), v_t \in L^q(0,T;W^{1,q}_{n,\text{div}})\) and \(p \in L^q(0,T;W^{1,q}).\)

Further we are interested in behavior of the unique weak solution to \((1.1)-(1.6)\) if \(\alpha \to 0+\), see Theorem 4.1, \(\lambda \to 1-\), see Theorem 5.1 or \(\lambda \to 1-\) and \(\alpha \to 0+\) simultaneously in Theorem 6.1.
Leray \cite{19} was the first one who regularized the Navier Stokes equations by smoothing the convective velocity with regularization made by convolution. The $\alpha$ models are based on a smoothing obtained with the application of the inverse of the Helmholtz operator $I - \alpha^2 \Delta$. There exists a large family of the $\alpha$ models, see for example \cite{11,15,7,9,12,13,17,2}. One of the first $\alpha$ model is the Lagrangian averaged Navier Stokes equations (LANS-$\alpha$) \cite{8} that was introduced as a sub-grid scale turbulence model. In \cite{12} the authors suggest the LANS-$\alpha$ as a closure model for the Reynolds averaged equations. The Leray-$\alpha$ model \cite{9}, as the other family of the $\alpha$ models, enjoys the same results of existence and uniqueness of the solutions and was also used as a closure models for the Reynolds averaged equations. The Leray-$\alpha$ was tested numerically in \cite{9,15}. In the numerical simulation the authors showed that the large scales of motion bigger than $\alpha$ in flow are captured. It was shown also that for scales of motion smaller than $\alpha$, the energy spectra decays faster in comparison to that of Navier Stokes equations. In \cite{9}, the convergence of a weak solution of the Leray-$\alpha$ to a weak solution of the Navier-Stokes equations as $\alpha \to 0$ was established. It is shown in \cite{2} that the Leray-$\alpha$ equations give rise to a suitable weak solution to the Navier-Stokes equations. All previously mentioned results were derived under the periodic boundary conditions.

There is only a few studies on the $\alpha$ models on bounded domains. The global existence and the uniqueness of weak solutions to the LANS-$\alpha$ on bounded domain with no-slip boundary condition is given in \cite{10}. The fact that we are able to establish such results of existence, uniqueness and convergence with Navier slip boundary conditions to the $L(\alpha)$ model is a novel feature of the the present study. The use of the $\alpha$-equations as a model of turbulent flows in more complicated geometries remains to be studied.

Finally, one may ask questions about other closure model of turbulence on bounded domains with usual boundary conditions, such as the Navier slip conditions. This is an crucial problem, because the filter in this case does not commutes with the differential operators \cite{4,14,18,11}.

The paper is organized as follows. In Sect. 2 we introduce relevant function spaces and we recall some preliminary results concerning solutions of elliptic equations with Navier boundary conditions. Then, in Sect. 3, inspired by result in \cite{5}, we give the proofs of Theorems \cite{1.1} and \cite{1.2}. In Sect. 4 we concentrate on analysis of the behavior of the solutions $(v^\alpha, p^\alpha)$; as $\alpha \to 0+$, where we show that the $\alpha$ regularization gives rise to a suitable weak solution to the Navier-Stokes equations. In Sect. 5 we take care of the dependence of the solution of the parameter $\lambda$ in order to pass to the limit as $\lambda \to 1-$ and in the last section we pass to the limit as $\alpha \to 0+$ and $\lambda \to 1-$ simultaneously.

2. Notation and auxiliary results

2.1. Notation. We use standard notation for Lebesgue, Sobolev and Besov spaces on a domain $O$ and their norms, e.g. $L^2(O)$, $W^{1,2}(O)$, $B^{1,2}_2(O)$ ($= W^{1,2}(O)$ if $O$ is smooth). If $O = \Omega$ we drop $(\Omega)$, e.g. $L^{5/2}$. By $(\cdot, \cdot)_O$ we denote the inner product in $L^2(O)$, $(\cdot, \cdot)$ stand for a duality pairing. We do not distinguish the scalar and vectorial spaces. The correct meaning is always clear from the context. Next we
define relevant function spaces for the velocity field. Let $k \in \mathbb{N}$, $p, q \geq 1$, then
\begin{align*}
W^{k,p}_n := \{ v \in W^{k,p} :\ v \cdot n = 0 \text{ on } \partial \Omega \}, \\
W^{k,p}_{n,\text{div}} := \{ v \in W^{k,p} :\ \text{div} v = 0 \text{ in } \Omega \}, \\
W_{n,\text{div}}^{−k,p'} := (W^{k,p}_n)^*, \quad W^{−k,p'}_{n,\text{div}} := (W^{k,p}_{n,\text{div}})^*;
\end{align*}

Let $L^q_{n,\text{div}} := \overline{W^{1,q}_{n,\text{div}}}$. 

2.2. Stokes problem. In this subsection we collect some known results concerning properties of solutions to Stokes problem with Navier boundary condition (1.5).

Let us first consider the stationary Stokes problem for some fixed function $\nu$.

\begin{align*}
& (2.1) \quad −\alpha^2 \text{div} D(\nu) + \nu + \nabla \pi = v, \quad \text{div} \nu = 0 \quad \text{on } \Omega, \\
& (2.2) \quad \nu \cdot n = 0, \quad \lambda \nu + (1−\lambda)(D(\nu)n)_\nu = 0 \quad \text{on } \partial \Omega, \\
& (2.3) \quad \int_{\Omega} \pi dx = 0.
\end{align*}

We have the following lemma about existence and regularity of solutions.

**Lemma 2.1.** Assume that $\alpha_0 > 0$, $\alpha \in (0, \alpha_0)$, $q > 1$, $v \in L^q$. Then the unique solution $(\nu, \pi)$ of system (2.2)-(2.3) is in $W^{2,q}_{n,\text{div}} \times W^{1,q}$ and satisfies the estimates
\[
\|\nu\|_{2,q} + \|\pi\|_{1,q} \leq C(\alpha)\|v\|_q, \quad \|\pi\|_q \leq C(\alpha_0)\|v\|_q.
\]
The constant $C(\alpha) > 0$ may depend (and in a fact depends on $\alpha$) while $C(\alpha_0) > 0$ may depend on $\alpha$ only through $\alpha_0$.

If moreover $k \in \mathbb{N}$, $k > 1$ and $v \in W^{k,q}$. Then $(\nu, \pi) \in W^{k+2,q}_{n,\text{div}} \times W^{k+1,q}$ and the following estimate hold
\[
\|\nu\|_{k+2,q} + \|\pi\|_{k+1,q} \leq C(\alpha)(\|f\|_{k+q} + \|\nu\|_q + \|\pi\|_q).
\]

**Proof.** The first part of the lemma is proved in [21] Theorem 1.3, (1)]. The second part of the theorem follows from the result [1] Theorem 10.5], since the Stokes operator satisfies the ellipticity condition [1] Section 1.1] and Navier boundary condition is a complementary one, see [1] Section 1.2].

**Corollary 2.1.** Let $k \in \mathbb{N} \cup \{0\}$, $r \in [1, +\infty)$, $q > 1$. Assume that $v \in L^r(0, T; W^{k,q}_n)$. Then the unique solution $(\nu, \pi)$ of problem (1.3) with boundary conditions (1.4) and (2.3) satisfies $\nu \in L^r(0, T; W^{k+2,q}_{n,\text{div}})$, $\pi \in L^r(0, T; W^{k+1,q}_{n,\text{div}})$.

Now we turn our attention to the evolutionary variant of the problem (2.1).

\begin{align*}
(2.4) \quad \text{div} v = 0, \quad v_t - 2\nu \text{div} D(v) = −\nabla p + f.
\end{align*}

**Lemma 2.2.** Let $2 \leq q < +\infty$, $q \neq 3$. If $v_0 \in D_q$ and $f \in L^q(0, T; L^q)$ then problem (2.4) with (1.10), boundary condition (1.3) and initial condition (1.4) admits a unique solution $(v, p)$ such that
\[
v \in L^q(0, T; W^{−2,q}_n) \cap W^{1,q}(0, T; L^q), \quad p \in L^q(0, T; W^{1,q}).
\]

**Proof.** This Theorem is proved in [22] Theorem 1.2].
2.3. Auxiliary lemma. We finish this section by the following interpolation lemma.

**Lemma 2.3.** Let $\Omega \subset \mathbb{R}^n$ be bounded Lipschitz domain, $r > 1$, and $f \in L^\infty(0, T; L') \cap L'(0, T; W^{2,r})$. Then $\nabla f \in L^s(Q)$ for $s = r + r^2/(n + r)$.

**Proof.** First we realize that the inequality

$$\|\nabla f\|_s \leq C \|f\|_{1,r}^{1-\theta} \|f\|^\theta_{1,2},$$

with $\theta = (n + r)/(n + 2r)$ holds as a consequence of [26; 4.2.1/3], [26] 2.4.2/11 and 4.3.2/Theorem 2], [26] Theorem 4.6.2a]. Taking the $s$ power of this inequality the statement of the lemma then follows since $\theta s = r$. \hfill \Box

3. Proof of Theorems 1.1 and 1.2

3.1. **Proof of Theorem 1.1.** We prove the theorem using Schauder fixed point theorem. To this end we fix $r > 1$, $q > 1$ (the exact values of $r$ and $q$ will be determined later) and study properties of the mapping $M_2 : L^2(0, T; W^{1,2}_{n,\text{div}}) \cap L'(0, T; L^q) \to L^2(0, T; W^{1,2}_{n,\text{div}}) \cap L^\infty(0, T; L^2)$, $M_2(\varphi) = u$, where $u \in L^2(0, T; W^{1,2}_{n,\text{div}}) \cap L^\infty(0, T; L^2)$ is the unique solution of the problem

$$\text{div} u = 0, \quad u_t + \text{div}(\varphi \otimes u) - 2\nu \text{div} D(u) = -\nabla p + f,$$

with an initial condition $u(0, x) = v_0(x)$ in $\Omega$.

and a boundary condition

$$u \cdot n = 0, \quad \lambda u + (1 - \lambda)(D(u)n) = 0 \quad \text{on } (0, T) \times \partial \Omega.$$

Our first goal is to determine the constants $r$, $q$ such that the mapping $M_2$ is well defined and continuous. Since for any $\gamma > 2$

(3.1) \hspace{1cm} $L^\infty(0, T; L^2) \cap L^2(0, T; W^{1,2}) \hookrightarrow L^\gamma(0, T; L^{\frac{2q}{\gamma-2}})$

it is enough to assume for some $\gamma > 2$ that

(3.2) \hspace{1cm} $r \geq \frac{2\gamma}{\gamma - 2}, \quad q \geq \frac{3\gamma}{2}.$

Under this assumptions $|\varphi||u| \in L^2(0, T; L^2)$ and the correctness of the definition of $M_2$ and its continuity follow by standard technique. Moreover, it is also seen that there exists $C > 0$ independent of $\varphi$ that

(3.3) \hspace{1cm} $\|u\|_{L^\gamma(0, T; L^{\frac{2q}{\gamma-2}})} + \|u\|_{L^\infty(0, T; L^2)} + \|u\|_{L^2(0, T; W^{1,2})} \leq C.$

Condition (3.2) also assures that

$$u_t \in L^2(0, T; (W^{1,2}_{n,\text{div}})^*)$$

and Aubin-Lions compactness lemma provides that

(3.4) \hspace{1cm} $M_2 : L^2(0, T; W^{1,2}_{n,\text{div}}) \cap L'(0, T; L^q) \to L^\gamma(0, T; L^s)$

is compact for any $\gamma > 2$ and $s \in (1, 6\gamma/(3\gamma - 4))$. Compare (3.1).

For $s \in (1, 3/2)$ we introduce a mapping

$$M_1 : L^\gamma(0, T; L^s) \to L^\gamma(0, T; W^{2,s}), \quad M_1(v) = \varphi,$$
where $\mathfrak{v}$ is the unique solution of the problem (2.1)-(2.3). Its existence and regularity is assured by Corollary 2.1. Here $\gamma$, $s$, $r$ and $q$ are sought such that

\begin{equation}
L^s(0,T;W^{2,s}) \hookrightarrow L^r(0,T;L^q) \cap L^2(0,T;W^{1,2}).
\end{equation}

It is needed $\gamma \geq r$, $\gamma \geq 2$ and $3s/(3-2s) \geq q$, $3s/(3-s) \geq 2$.

Finally we want to apply Schauder fixed point theorem to $M = M_2 \circ M_1$. To this end we set $\gamma = r = q = 5$. In order to have $M$ well defined we need (3.3) which is verified if $s > 6/5$. The compactness of $M$ follows from (3.5) provided $s < 30/11$. It is seen that we can fix $s \in (6/5,3/2)$. Altogether we got that

$$M : L^5(0,T;L^s) \hookrightarrow L^5(0,T;L^s)$$

is continuous, compact mapping that maps a certain ball into itself, see (3.3). Schauder fixed point theorem gives a fixed point of $M$ which solves (1.1)-(1.6) in the weak sense and satisfies (1.7), (1.8) and (1.12). It remains to reconstruct its uniqueness. Let $(\nu, u^0)$. Compare [6, Section 2.3]. The procedure gives (1.9)-(1.11). Up to now we proved the existence of the weak solution. Now we concentrate to its uniqueness. Let $(\nu_1, p_1)$ and $(\nu_2, p_2)$ be any two solutions of $L(\alpha)$ on the interval $[0,T]$, with initial values $\nu_1(0)$ and $\nu_2(0)$. Let us denote by $w = \nu_1 - \nu_2$ and $\mathfrak{w} = \mathfrak{v}_1 - \mathfrak{v}_2$. We subtract the equation for $\nu_2$ from the equation for $\nu_1$ and test it with $w$. We get using successively Korn’s inequality, embedding theorem and Lemma 2.1

\begin{equation}
\|w_r\|_2^2 + 4\nu\|D(w)\|_2^2 \leq \frac{C}{\nu}\|\mathfrak{w}\|_2^2 + \nu\|w\|_2^2 + \|D(w)\|_2^2
\end{equation}

$$\leq \frac{C}{\nu}\|\mathfrak{w}\|_{H^2}^2\|u_1\|_{H^1}^2 + \nu\|w\|_2^2 + \|D(w)\|_2^2
\leq \|w\|_2^2\left(\frac{C}{\nu}\|u_1\|_{H^1}^2 + \nu\right) + \nu\|D(w)\|_2^2.$$

Using Gronwall’s inequality we conclude the continuous dependence of the solutions on the initial data in the $L^\infty(0,T, L^2_{n,\text{div}})$ norm. In particular, if $\mathfrak{w}_0 = 0$ then $w = 0$ and the solution $\nu$ is unique. Since the pressure part of the solution is uniquely determined by the velocity part and the condition (1.10), the proof of the uniqueness is finished.

It remains to prove that the unique solution $(\nu, p)$ verifies the local energy equality (1.13). To this end let us take $\phi \nu$ as test function in (1.11). We note that the regularity of $\mathfrak{v}$ ensure that all the terms are well defined. In particular the integral

$$\int_0^T \int_\Omega \mathfrak{v} \otimes \nu \cdot \nabla(\nu \phi) \ dx \ dt$$

is finite by using the fact that $\mathfrak{v} \otimes \nu \in L^2(0,T;L^2)$ at least and $\phi \nu \in L^2(0,T;W^{1,2})$. An integration by parts combined with the following identity

\begin{equation}
\int_\Omega \mathfrak{v} \otimes \nu \cdot \nabla(\nu \phi) \ dx = \frac{1}{2} \int_\Omega \|\mathfrak{v}^2 \cdot \nabla \phi \| \ dx
\end{equation}
yields that for all \( t \in (0, T) \) and for all non negative functions \( \phi \in C^\infty \) and \( \text{spt } \phi \subset \subset \Omega \times (0, T), (v, p) \) verifies

\[
\frac{1}{2} \int_\Omega |v(t)|^2 \phi(t, x) \, dx + \nu \int_0^t \int_\Omega |D(v)|^2 \, \phi \, dx \, dt
\]

\[
= \frac{1}{2} \int_\Omega |v_0|^2 \phi(0, x) \, dx + \int_0^t \int_\Omega |v|^2 \phi_t \, dx \, dt
\]

\[
+ \int_0^t \int_\Omega \left( \frac{|v|^2}{2} \nabla \cdot \nu \text{div}(D(v)) v\right) \cdot \nabla \phi \, dx \, dt + \int_0^t \langle f, \phi \rangle \, dt.
\]

Integrating by parts once more in the above equality, we obtain (1.13) and the proof of Theorem 1.1 is finished.

**Remark 3.1.** Since \( T > 0 \) was arbitrary the solution constructed in Theorem 1.1 may be uniquely extended for all time.

### 3.2. Proof of Theorem 1.2
First we realize that by Theorem 1.1 we know existence of a solution \( v \) of the problem \( \mathcal{L}(\alpha) \) such that \( v \in C(0, T; L^2_{\text{div}}(\Omega)) \cap L^2(0, T; W^{1,2}_{\text{div}}) \). By Corollary 2.1 we get that \( \overline{v} \in L^\infty(0, T; W^{2,2}_{\text{div}}) \cap L^2(0, T; W^{3,2}_{\text{div}}) \) and by embedding theorem \( \overline{v} \in L^\infty(Q) \).

We know that \( \nabla v \in L^2(Q) \). From the regularity of \( \overline{v} \) it follows that \( \text{div}(\overline{v} \otimes \nabla v) \in L^2(Q) \). Applying Lemma 2.2 we get \( v \in W^{1,2}(0, T; L^2_{\text{div}}(\Omega)) \cap L^2(0, T; W^{2,2}_{\text{div}}) \) and by Lemma 2.3 \( \nabla v \in L^{s(2)}(Q) \) with function \( s(r) := r + r^2/(3 + r) \).

Let us assume \( \nabla v \in L^r(Q) \) with \( r \in [2, q] \), then \( \text{div}(\overline{v} \otimes \nabla v) \in L^r(Q) \) and by Lemma 2.2 \( v \in W^{1,r}(0, T; L^2_{\text{div}}(\Omega)) \cap L^r(0, T; W^{2,r}_{\text{div}}) \). Lemma 2.3 gives \( \nabla v \in L^{s(r)}(Q) \). Since for all \( r \geq 2 \) it holds \( s(r) > r \), the statement of the theorem follows by iterating this procedure. \( \square \)

### 4. PASSAGE TO THE LIMIT AS \( \alpha \to 0^+ \)

If we set \( \alpha = 0 \) in \( \mathcal{L}(\alpha) \) we obtain the Navier Stokes system \( \mathcal{NS} \)

\[
(4.1) \quad \text{div } v = 0,
\]

\[
(4.2) \quad \partial_t v + \text{div}(v \otimes v) - 2\nu \text{div } D(v) = -\nabla p + f,
\]

\[
(4.3) \quad v(0, x) = v_0(x).
\]

Our aim here is to show that the solutions of \( \mathcal{L}(\alpha) \) from Theorem 1.1 with \( \alpha > 0 \) converge to a suitable weak solution to \( \mathcal{NS} \). The notion of a suitable weak solution of \( \mathcal{NS} \) was introduced by Scheffer [20]. It is related to the notion of the weak solution, however, in addition, a local energy inequality is required (see (4.10) below).

First we examine a connection between \( v \) and \( \overline{v} \).

**Lemma 4.1.** Assume that \( v \in W^{1,2}_{\text{div}} \) and \( \overline{v} \) is solution to (1.3) with boundary conditions (1.6). Then

\[
\alpha^2 \|D(v - \overline{v})\|^2 + \frac{\alpha^2}{1 - \lambda} \|v - \overline{v}\|^2_{2,\partial \Omega} + 2 \|\overline{v} - v\|^2_2
\]

\[
\leq \alpha^2 (\|D(v)\|^2 + \frac{\lambda}{1 - \lambda} (v, v)_{\partial \Omega}).
\]
Proof. Testing the weak formulation of (1.3) with \( v - \overline{v} \) yields
\[
\alpha^2 \| D(v) - D(\overline{v}) \|^2 + \alpha^2 \frac{\lambda}{1 - \lambda} (v - \overline{v}, v - \overline{v})_{\partial \Omega} + \| v - \overline{v} \|^2 \\
= \alpha^2 (D(v), D(v))_\Omega + \frac{\lambda}{1 - \lambda} (v, v)_{\partial \Omega} \\
\leq \frac{1}{2} \left( \alpha^2 \| D(v) \|^2 + \alpha^2 \| D(v) - D(\overline{v}) \|^2 \\
+ \alpha^2 \frac{\lambda}{1 - \lambda} (v, v)_{\partial \Omega} + \alpha^2 \frac{\lambda}{1 - \lambda} (v - \overline{v}, v - \overline{v})_{\partial \Omega} \right)
\]
and the result follows at once. \( \square \)

**Theorem 4.1.** Let \( \alpha_j \to 0^+ \) as \( j \to +\infty \), \( v_0 \in L^2_{\text{div}}, f \in L^2(0,T;W^{-1,2}_n) \). Let \( v^{\alpha_j} \) be the unique solution of \( L(\alpha) \) with (1.4), (1.5) and \( \alpha = \alpha_j \). Then there is a subsequence of \( \{ \alpha_j \} \), we denote it again \( \{ \alpha_j \} \), \( v \in C_{\text{weak}}(0,T;L^2_{\text{div}}) \cap L^2(0,T;W^{1,2}_n), p \in L^{5/3}(\Omega \times (0,T)) \) with \( v_t \in (L^{5/2}(0,T;W^{1,5/2}_n))^* \) and \( v(0) = v_0 \) such that as \( j \to +\infty \)

\begin{align*}
(4.5) & \quad v^{\alpha_j} \rightharpoonup v \quad \text{weakly in } L^2(0,T;W^{1,2}_n), \\
(4.6) & \quad v_t^{\alpha_j} \rightharpoonup v_t \quad \text{weakly in } (L^{5/2}(0,T;W^{1,5/2}_n))^*, \\
(4.7) & \quad v^{\alpha_j} \to v \quad \text{strongly in } L^q(0,T;L^q), \text{ for all } 1 \leq q < 10/3 \\
(4.8) & \quad p^{\alpha_j} \to p \quad \text{weakly in } L^{5/3}(0,T;L^{5/3}).
\end{align*}

Consequently, \( (v,p) \) is a weak dissipative solution of \( \mathcal{NS} \) with Navier boundary condition (1.6) and the initial condition (1.3), i.e.

\[
\int_0^T (v_t, w) - (v \otimes v, \nabla w) + \frac{2\nu\lambda}{1 - \lambda} (v, w)_{\partial \Omega} + 2\nu(D(v), D(w)) \, dt \\
= \int_0^T (p, \text{div } w) + \langle f, w \rangle \, dt \quad \text{for all } w \in L^2(0,T;W^{1/2}_n).
\]

Moreover, the solution \( (v,p) \) satisfies the following local energy inequality

\[
\frac{1}{2} \int_{\Omega} (|v|^2 \phi)(t,x) \, dx + \nu \int_0^t \left( \int_{\Omega} |\nabla v|^2 \phi \, dx \right) \, dt \\
\leq \frac{1}{2} \int_{\Omega} |v_0|^2 \phi(0,x) \, dx + \int_0^t \left( \int_{\Omega} |v|^2 (\phi_t + \nu \Delta \phi) \, dx \right) \, dt \\
+ \int_0^t \int_{\Omega} \left( \frac{|v|^2}{2} v + pv \right) \cdot \nabla \phi \, dx \, dt + \int_0^t \langle f, v \phi \rangle \, dt
\]

for a.e. \( t \in (0,T) \) and for all non negative function \( \phi \in C^\infty \) and supp \( \phi \subset \subset \Omega \times (0,T) \).

**Proof of Theorem 4.1.** In order to prove Theorem 4.1, we need to find estimates that are independent of \( \alpha \). In this proof constant \( C > 0 \) is independent of \( \alpha \).

First, we obtain, testing (1.11) by \( v^\alpha \), the existence of \( C > 0 \) that for all \( \alpha \) we have

\[
\frac{2\nu\lambda}{1 - \lambda} \| v^\alpha \|_{L^2(0,T;L^2(\partial \Omega))} + \| v^\alpha \|_{L^\infty(0,T;L^2)} + \| v^\alpha \|_{L^2(0,T;W^{1,2})} \leq C.
\]
By standard interpolation we get
\begin{equation}
\|v^\alpha\|_{L^{10/3}(0,T;L^{10/3})} \leq C.
\end{equation}

Lemma 2.1 gives
\begin{equation}
\|v^\alpha\|_{L^{10/3}(0,T;L^{10/3})} \leq C.
\end{equation}

Since we consider Navier boundary conditions and in \(W_{n}^{1.5/2}\) there holds Helmholtz decomposition, compare [6, Section 2.3], we can conclude from (4.11), (4.12) and (4.13) a uniform bound
\begin{equation}
\|v^\alpha\|_{(L^{5/2}(0,T;W_{n}^{1.5/2}))^n} \leq C.
\end{equation}

From [5, Remark 3.1] we know that for all \(h \in L^\infty\) and a.e. \(t \in (0,T)\)
\[-(p^\alpha(t),h) = - (\nabla^2(t) \otimes v^\alpha(t), \nabla^2 H) + \frac{2\nu\lambda}{1-\lambda} (v^\alpha(t), \nabla H)_{\partial\Omega} + \frac{2\nu}{1-\lambda} (D(v^\alpha(t)), \nabla H) - \langle f(t), \nabla H \rangle,
\]
holds, where \(H\) is solution of \(-\nabla H = h\) in \(\Omega\), \(\partial H/\partial n = 0\) on \(\partial\Omega\), \(\int_\Omega H = 0\). It is seen that integrability of the pressure follows from the integrability of \(\nabla \otimes v, D(v), f\) and \(v\). It is standard to show from (4.11), (4.12) and (4.13) that
\begin{equation}
\|p^\alpha\|_{L^2(0,T;L^2^*}) \leq C.
\end{equation}

It follows from (4.11), (4.14) and (4.15) that we can find subsequence of \(\{\alpha^j\}\) and \((v,p)\) such that \((4.6), (4.7), (4.8)\) hold and \(v \in L^\infty(0,T;L^2)\). An another subsequence can be extracted such that \((4.7)\) holds due to (4.11) and (4.14) by Aubin-Lions lemma.

To show that \((v,p)\) solves (4.9) and (4.10) it is necessary to pass to the limit \(\alpha^j \to 0\) as \(j \to +\infty\) in (4.11) and (4.13). This is standard if we realize that due to Lemma 2.1 and (4.11) we know that there exist \(C > 0\) such that
\begin{equation}
\|v^\alpha - v^\alpha\|_{L^2(0,T;L^2)} \leq C\alpha^2,
\end{equation}
and that this fact implies (together with (4.7) and (4.13)) that, up to a subsequence, \(\nabla v^\alpha \to v\) in \(L^q(0,T;L^q)\) for all \(q \in [2,\frac{6}{5}]\) as \(j \to +\infty\).

It remains to show weak continuity of \(v\), which however follows from the fact that \(v \in C(0,T;W_{n,div}^{1.5/2})\) by (5.2) and \(v \in L^\infty(0,T;L^2)\).

5. Passage to the limit as \(\lambda \to 1\)

Now we want to take care of dependence of the solution of the parameter \(\lambda\) from (1.5) and (1.6). We will denote this dependence by superscript \(\lambda\).

When \(\lambda \to 1\) in (1.5) we obtain the homogeneous Dirichlet boundary condition (i.e. the condition \(v = 0\) on \((0,T) \times \partial\Omega\)). In this case the problem \(\mathcal{L}(\alpha)\) with homogeneous Dirichlet boundary condition can be obtained as a limit from \(\mathcal{L}(\alpha)\) with Navier slip boundary conditions for any \(\alpha > 0\) by letting \(\lambda\) in (1.5) and (1.6) tend to 1-

**Theorem 5.1.** Let \(\lambda_j \to 1\) as \(j \to +\infty\), \(v_0 \in L^2_{n,div}, f \in L^2(0,T;W_{n,div}^{-1,2})\). Let \(v^{\lambda_j}\) be the unique solution of \(\mathcal{L}(\alpha)\) with (1.3)–(1.6) and \(\lambda = \lambda_j\).
Then there is a subsequence of \( \{ \lambda_j \} \), we denote it again \( \{ \lambda_j \} \), \( v \in C(0, T; L^2_{n, \text{div}}) \cap L^2(0, T; W^{1,2}_{0, \text{div}}) \) with \( v_t \in (L^2(0, T; W^{1,2}_{0, \text{div}}))^* \) and \( v(0) = v_0 \) such that as \( j \to +\infty \)

\[
(5.1) \quad v^{\lambda_j} \rightharpoonup v \quad \text{weakly in} \quad L^2(0, T; W^{1,2}),
\]

\[
(5.2) \quad v_t^{\lambda_j} \rightharpoonup v_t \quad \text{weakly in} \quad (L^2(0, T; W^{1,2}_{0, \text{div}}))^*,
\]

\[
(5.3) \quad v^{\lambda_j} \to v \quad \text{strongly in} \quad L^q(0, T; L^2), \quad \text{for all} \quad 1 \leq q < 10/3,
\]

\( v \) is the unique weak solution to \( L(\alpha) \) with homogeneous Dirichlet boundary condition and initial condition \( (1.2) \), i.e.

\[
(5.4) \quad \int_0^T (v_t, w) - (\nabla \cdot v, \nabla w) + 2\nu(D(v), D(w)) \, dt = \int_0^T (f, w) \, dt
\]

for all \( w \in L^2(0, T; W^{1,2}_{0, \text{div}}) \).

Let moreover \( f \in L^q(0, T; L^q_{n, \text{div}}) \) for some \( q \geq 2 \), \( v_0 \in W^{2-2/q, q} \) with \( v_0 = 0 \) on \( \Omega \) and \( \nabla v_0 = 0 \) is \( \Omega \). Then

\[
(5.5) \quad v \in L^q(0, T; W^{2, q}_{0, \text{div}}) \cap W^{1, q}(0, T; L^q_{n, \text{div}})
\]

and a pressure can be reconstructed in such a way that \( p \in L^q(0, T; W^{1, q}) \) and \( (1.10) \) holds.

**Proof.** Testing \( (1.1) \) with \( v^\lambda \) we know that

\[
(5.6) \quad \sup_{t \in (0, T)} \| v^\lambda(t) \|_{L^2}^2 + \nu \int_0^T \| v^\lambda(t) \|_{L^2}^2 \, dt + \nu \frac{\lambda}{1 - \lambda} \int_0^T (v^\lambda, v^\lambda)_{\partial \Omega} \leq C(v_0, f) < \infty.
\]

Testing \( (1.3) \) by \( v^\lambda \) we get using \( (5.6) \) the estimate

\[
(5.7) \quad \| v^\lambda \|_{L^\infty(0, T; W^{1,2})} + \| v^\lambda \|_{L^\infty(0, T; L^q)} \leq C(v_0, f).
\]

From \( (5.6) \) and \( (5.7) \) we get that

\[
\| v^\lambda \|_{L^3/2(Q)} \leq C(v_0, f),
\]

and consequently

\[
(5.8) \quad \| v_t^{\lambda_j} \|_{(L^2(0, T; W^{1,2}_{0, \text{div}}))^*} \leq C(v_0, f).
\]

Using \( (5.6) \) and \( (5.8) \) it is standard to find a subsequence \( \{ \lambda_j \} \) and \( v \) such that \( (5.1) \)–\( (5.3) \) and \( (5.4) \) hold. The equation \( (5.4) \) is obtained letting \( \lambda_j \to 1- \) in \( (1.11) \).

The boundary terms disappear since the test functions vanish on the boundary and the term with pressure is not present because the test functions are divergence-free.

Now we show that the trace of \( v \) is zero. It follows from \( (5.6) \) since

\[
\int_0^T \| v^\lambda \|_{L^2(0, T; W^{1,2}_{0, \text{div}})}^2 \leq C \frac{1 - \lambda}{\lambda} \to 0 \quad \text{as} \quad \lambda \to 0 + .
\]

Last, we need that \( v(0) = v_0 \). That follows from the initial condition for \( v^{\lambda_j}(0) = v(0) \) since \( v, v^{\lambda_j} \in C_{\text{weak}}(0, T; L^2_{n, \text{div}}) \). (The last statement follows from the fact that \( v, v^{\lambda_j} \in C(0, T; (W^{1,5/2}_{n, \text{div}})^*) \cap L^\infty(0, T; L^2) \to C_{\text{weak}}(0, T; L^2_{n, \text{div}}) \).)

In the situation when \( f \in (L^2(0, T; W^{1,2}_{0, \text{div}}))^* \) only it is not known how to construct pressure as a function \( p \in L^2((0, T) \times \Omega) \), compare [24, Section IV.2.6]. A different situation occurs if \( f \in L^q(Q), q \geq 2 \). Then the regularity \( (5.5) \) of the solution \( v \) can be shown as in Theorem \( (1.2) \) since Lemmas \( (2.1) \) and \( (2.2) \) hold also
under homogeneous Dirichlet boundary conditions, compare\cite{3, 16}. Having \cite{5.3}\cite{5.3}
the pressure can be reconstructed on a.e. time level by de Rham’s theorem and its
regularity can be read from the equation.

\section*{6. Passage to the Limit as $\lambda \to 1^-$ and $\alpha \to 0^+$}

When $\lambda \to 1^-$ and $\alpha \to 0^+$ a theorem similar to Theorem\\cite{5.1}\cite{5.1} can be proved.

\textbf{Theorem 6.1.} Let $\lambda_j \to 1^-$, $\alpha_j \to 0^+$, $v_0 \in L^2(N, \nabla \div)$, $f \in L^2(0, T; W^{-1,2})$. Let
$v_{\lambda_j, \alpha_j}$ be the unique solution of $L(\alpha)$ with \cite{1.4}, \cite{1.6}, $\lambda_j = \lambda$ and $\alpha = \alpha_j$. Then
there is a subsequence of $\{\lambda_j, \alpha_j\}$, we denote it again $\{\lambda_j, \alpha_j\}$, $v \in C_{\text{weak}}(0, T; L^2(0, T; W^{-1,2}) \cap
L^2(0, T; W^{1,2}_{0, \text{div}})$, with $v_j \in (L^2(0, T; W^{1,3}_{0, \text{div}}))^*$ and $v(0) = v_0$ such that as $j \to +\infty$

\begin{align}
(6.1) & v_{\lambda_j, \alpha_j} \rightharpoonup v \quad \text{weakly in } L^2(0, T; W^{-1,2}), \\
(6.2) & v_{\lambda_j, \alpha_j} \rightharpoonup v_j \quad \text{weakly in } (L^2(0, T; W^{1,3}_{0, \text{div}}))^*, \\
(6.3) & v_{\lambda_j, \alpha_j} \to v \quad \text{strongly in } L^q(0, T; L^q), \quad \text{for all } 1 \leq q < 10/3
\end{align}

Consequently, the velocity part $v$ is a weak dissipative solution of the Navier Stokes
equations with homogeneous Dirichlet boundary condition and the initial condition
$v_0$, i.e.

\begin{equation}
(6.4) \int_0^T \langle v_{\nabla}, w \rangle - (v \otimes v, \nabla w) + 2\nu(D(v))D(w) dt = \int_0^T \langle f, w \rangle dt \\
\text{for all } w \in L^2(0, T; W^{1,3}_{0, \text{div}}).
\end{equation}

\textbf{Proof.} The proof of this Theorem follows the lines of the proof of Theorem\\cite{1.1}\cite{1.1} and
Theorem\\cite{5.1}\cite{5.1}. First we obtain uniform estimates \cite{4.11}\cite{4.11} and \cite{4.12}\cite{4.12}. Now we need to
reconstruct a uniform estimates for $v_{\lambda_j, \alpha_j}$. Since in Lemma\\cite{2.1}\cite{2.1} the dependence of
constants on $\lambda$ is not addressed we cannot use it. Instead we test \cite{1.3}\cite{1.3} with $v_{\lambda_j, \alpha_j}$
and we test \\cite{4.11}\cite{4.11} and get a uniform estimate

\begin{equation}
(6.5) \|v_{\lambda_j, \alpha_j}\|_{L^q(0, T; L^q)} < C.
\end{equation}

It follows

\begin{equation}
\|v\|_{L^q(0, T; L^q)} < C \quad \text{and } \|v_{\lambda_j, \alpha_j}\|_{L^q(0, T; W^{1,3}_{0, \text{div}})} < C.
\end{equation}

Consequently we can extract a subsequence $(\lambda_j, \alpha_j)$ that \cite{6.3}\cite{6.3}\cite{6.3} hold. Combining Lemma\\cite{4.1}\cite{4.1} with the estimate \cite{4.11}\cite{4.11} we get
that $v_{\lambda_j, \alpha_j} \to v$ in $L^q(Q) \cap L^q(0, T; L^q)$ for all $s > 2$ as $j \to +\infty$.

The limit function $v$ must be traceless due to \cite{4.11}\cite{4.11}. With this information it is
standard to pass to the limit as $j \to +\infty$ in \cite{1.11}\cite{1.11} to get \cite{6.4}\cite{6.4}.

\textbf{Remark 6.1.} Generally with homogeneous Dirichlet boundary condition the existence
of the pressure term $p$ of the Navier Stokes equations is not obvious and the pressure
may not exist, compare \cite{23}.

\textbf{Remark 6.2.} Finally we would like to notice that the results reported here can be
extended to the Navier-Stokes-Voigt equations (see in \cite{7} and the references inside)
with Navier slip boundary condition.
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