The Witten-Reshetikhin-Turaev Invariants of Lens Spaces

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Abstract

We derive an explicit formula for the Witten-Reshetikhin-Turaev $SO(3)$-invariants of lens spaces. We use the representation of the mapping class group of the torus corresponding to the Witten-Reshetikhin-Turaev $SO(3)$-TQFT to give such formula.

Keywords: Lens spaces, TQFT, $SO(3)$-quantum invariant.

1. Introduction

We consider a variation of the $(2 + 1)$ cobordism category that was studied in [1]. The variation consists of replacing the $p_1$ structure by integers that are called weights. This notion was first introduced by [9, 10]. This weighted category can be described roughly as follows. The objects of this category $C$ are closed surfaces $\Sigma$ equipped with a Lagrangian subspace $\lambda \subset H_1(\Sigma, \mathbb{R})$. We will denote objects by pairs $(\Sigma, \lambda)$. A cobordism from $(\Sigma, \lambda)$ to $(\Sigma', \lambda')$ is a 3-manifold with an orientation preserving homeomorphism (called its boundary identification) from its boundary to $-\Sigma \sqcup \Sigma'$. Here and elsewhere, $-\Sigma$ denotes $\Sigma$ with the opposite orientation. Two cobordisms are equivalent if there is an orientation preserving homeomorphism between the underlying 3-manifolds that commutes with the boundary identifications. A morphism $M : (\Sigma, \lambda) \to (\Sigma', \lambda')$ is an equivalence class of cobordisms from $(\Sigma, \lambda)$ to $(\Sigma', \lambda')$ together with an integer weight. We denote morphisms by $(M, w(M))$, where $w(M)$ denotes the weight of $M$. The gluing of 3-manifolds represents the composition and the weight of the composed morphism is given by [3, Equation(1.6)] which was derived from [9, Thm.(4.1.1)].

The version of the WRT invariant that we study here is the invariant that is obtained from the $SO(3)$-TQFT-functor for $r \equiv 1 (\text{mod } 4)$ on $C$ over a commutative ring $K_r$ that was given in [1]. The $SO(3)$-TQFT-functor $(V_r, Z_r)$ is a functor from $C$ to the category of finitely generated free $k_r$-modules. It is defined as follows: $V_r(\Sigma)$ is a quotient of the $k_r$-module generated by all cobordisms with boundary $\Sigma$, and $Z_r(M)$ is the $k_r$-linear map from $V_r(\Sigma)$ to $V_r(\Sigma')$ (where $\partial M = -\Sigma \sqcup \Sigma'$) induced by gluing...
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representatives of elements of \( V_r(\Sigma) \) to \( M \) along \( \Sigma \) via the identification map. The ring is \( K_r = \mathbb{Q}[\zeta_{2r}, i] \) with \( A = \zeta_{2r} \), and \( \kappa = iA^{-1} \) where \( \zeta_{2r} = e^{\frac{2\pi i}{2r}} \). This invariant is recovered from the representation of the gluing map between the two parts of the Heegaard splitting of the 3-manifold on the \( SO(3) \)-TQFT-vector space \( V_r(\Sigma) \) of the boundary surface.

2. The \( SO(3) \)-TQFT-Representation of \( SL(2, \mathbb{Z}) \)

The \( SO(3) \)-TQFT associates to a surface \( \Sigma \), a representation of the mapping class group on the vector space \( V_r(\Sigma) \). We consider the case where the surface is the torus. The mapping class group of the torus is known to be \( SL(2, \mathbb{Z}) \). Hence, we obtain a representation of the group \( SL(2, \mathbb{Z}) \) on the vector space \( V_r(S^1 \times S^1) \).

The group \( SL(2, \mathbb{Z}) \) is specified in terms of two generators \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), and \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) with the relations \( S^4 = (ST)^6 = 1 \). So any representation is specified in terms of these two generators satisfying the relations.

We adopt the following notation of [5] that has been used also in [4]: \( e(\alpha) \overset{\mathrm{def}}{=} \exp(2\pi i \alpha) \), \( e_n(\alpha) \overset{\mathrm{def}}{=} \exp(2\pi i \frac{\alpha}{n}) \). Also, \( \zeta \overset{\mathrm{def}}{=} \exp \frac{i\pi}{4} \).

The representation is given explicitly by the assignments \( S \mapsto D^{-1}S \) and \( T \mapsto \kappa^{-1}T^{-1} \) as given in [9, Page 98]. This assignments define a projective representation because of the factor \( \Delta D^{-1} \in K \). This factor is expressed in terms of \( \kappa \) in the following lemma.

**Lemma 1** Let \( \kappa \in K \) be as above, then \( \Delta D^{-1} = \kappa^3 = -iA^{-3} \).

**Proof :-**

\[
\kappa^3 = \kappa^3 D \langle S^3, 0 \rangle = D \left( \kappa^3 \langle S^3, 0 \rangle \right) \\
= D \langle S^3, 1 \rangle \\
= D \left( D \Delta^{-1} \right) \langle S^3, 0 \rangle \\
= D \left( D \Delta^{-1} \right) D^{-1} = D \Delta^{-1}.
\]

Hence the assignments \( S \mapsto D^{-1}S \) and \( T \mapsto \kappa^{-1}T^{-1} \) give a linear representation as stated in the following proposition whose proof can be found in [9, §2.3].

**Proposition 2** The representation of \( SL(2, \mathbb{Z}) \) on the vector space \( V_r(S^1 \times S^1) \) obtained from the \( SO(3) \)-TQFT-representation is given by

\[
S_{jl} = \frac{1}{i\sqrt{r}} [e_r(jl) - e_r(-jl)] = \frac{2}{\sqrt{r}} \sin \left( \frac{jl\pi}{r} \right), \\
T_{jl} = \delta_{jl}T_j, \quad T_j = ie_{2r}(j^2),
\]

for \( 1 \leq j, l \leq \frac{r-1}{2} \).
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Remark 3 In the above two matrices, we should have \( j, l \in \{0, 2, 4, \ldots, r-3\} \) since the last set represents the set of simple objects in the \( SO(3) \)–modular category. However, the above two matrices and the actual matrices are similar by the same permutation matrix, i.e

\[
S = PS_aP^{-1}, \ T = PT_aP^{-1}
\]

where \( S_a, T_a \) are the actual matrices.

Lemma 4 The entries of \( S_{jl} \), and \( T_j \) satisfy the following symmetries:

\[
S_{jl} = S_j(r+l) = -S_j(r-l), \quad T_j = T_{j+r} = T_{j-r}.
\]

Proof: - It is clear for the entries of the \( S \)-matrix. For the entries of the \( T \)-matrix, we know that \( e_{2r}(1) = -e_r(n) \) for some integer \( n \). Therefore, \( T_j = -ie_r(nj^2) = -ie_r(n(r-j)^2) \).

Now, we wish to give an explicit formula for the representation of any element of \( SL(2, \mathbb{Z}) \) independent from the way we write that element in terms of the generators \( S \), and \( T \). But before we do that, we would like to quote the Gauss sum reciprocity formula in one dimension from [4].

Proposition 5 If \( \lambda, n, m \in \mathbb{Z} \) with \( nm \) is even and \( n\psi \in \mathbb{Z} \), then:

\[
\sum_{\lambda \mod n} e_{2n}(m\lambda^2)e(\psi\lambda) = \sqrt{in} \sum_{\lambda \mod m} e_{2m}(-n(\lambda + \psi)^2). \tag{1}
\]

2.1 The formula of the Representation

Let \( U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \), we want to give a formula for the representation of \( U \) in terms of its entries.

Definition 6 A continued fraction expansion for \( U \) is a tuple of integers \( C = (m_1, \ldots, m_t) \) such that

\[
U = T^{m_t}ST^{m_{t-1}} \cdots T^{m_1}.S.
\]

We would like to quote the following proposition from [4].

Proposition 7 [4, Prop.(2.5)] Suppose \( U \), and \( C \) as above. Then

(i) \[
a/c = m_t - \frac{1}{m_{t-1} - \frac{1}{\cdots - \frac{1}{1}}.}
\]

(ii) \[
b/a = -\left(\frac{1}{a_1} + \frac{1}{a_2a_1} + \cdots + \frac{1}{a_ta_{t-1}}\right).
\]

Moreover, define \( a_i, b_i, c_i, d_i \) by the partial evaluation of this product:
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\[
\left( \begin{array}{cc}
  a_t & b_t \\
  c_t & d_t
  \end{array} \right) = T^{m_1} S T^{m_{t-1}} \ldots T^{m_1} S,
\]

with the convention that:

\[ a_0 = d_0 = 1, \quad b_0 = c_0 = 0. \]

Then these satisfy the recurrence relations (for \( t \geq 2 \))

\[ (iii) \quad a_t = m_t a_{t-1} - c_{t-1}, \quad c_t = a_{t-1}; \]

\[ (iv) \quad b_t = m_t a_{t-1} - d_{t-1}, \quad d_t = b_{t-1}. \]

Lemma 8

Denote by \( \mathcal{I}_t \) the sum

\[
\mathcal{I}_t = \sum_{j_1, \ldots, j_t=1}^{r-1} S^{j_{t+1}} T^{m_1} S \ldots T^{m_{t-1}} T^{m_1} S^{j_{t} j_0}.
\]

Then in terms of the previous proposition

\[
\mathcal{I}_t = C_t \sum_{\gamma \equiv j_{t+1} (\text{mod } r)}^{(m_1, \ldots, m_t) \text{ (mod } 2r a_t)} e^{2\pi \alpha_t \left( -c_t \left( \gamma + \frac{j_0}{c_t} \right)^2 \right)} - e^{2\pi \alpha_t \left( -c_t \left( \gamma - \frac{j_0}{c_t} \right)^2 \right)},
\]

where

\[
C_t = -i^{(t-1)} \zeta^{t-1} \zeta^{\text{sign}(a_t)} \frac{i^{(m_1, \ldots, m_t)}}{\sqrt{r |a_t|}} e^{2\pi \left\{ - \left( \frac{1}{a_0 a_1} + \ldots + \frac{1}{a_{t-2} a_{t-1}} \right) \frac{j_0^2}{j_0} \right\}}.
\]

Proof: We observe that each of the indices \( j_1, \ldots, j_t \) appears twice for the entries of \( S \) in (2), so we may divide by 2 and replace the sum over \( j_1, \ldots, j_t \) from 1 to \( r \) (using the symmetries in Lemma (4)). To prove the result, we use induction on \( t \). For \( t = 1 \):

\[
\mathcal{I}_1 = \frac{1}{2} \sum_{j_1=1}^{r} S^{j_2 j_1} T^{m_1} S^{j_1 j_0}
\]

\[
= \frac{-1}{r} i^{m_1} \sum_{j_1=1}^{r} e^{2\pi \left( m_1 j_1 j_1 \right)} \left\{ e_{r} (j_2 j_1) - e_{r} (-j_2 j_1) \right\} \left\{ e_{r} (j_1 j_0) - e_{r} (-j_1 j_0) \right\}
\]

\[
= \frac{-1}{2r} \sum_{j_1=1}^{r} e^{2\pi \left( 2m_1 j_1 j_1 \right)} \left\{ e_{r} ((j_2 + j_0) j) - e_{r} ((j_2 - j_0) j) \right\}
\]

\[
= \frac{-1}{r} \sqrt{\frac{2\pi r}{2m_1}} \sum_{j_1=1}^{r} e^{2\pi \left( 2m_1 j_1 j_1 \right)} \left\{ e_{r} ((j_2 + j_0) j) - e_{r} ((j_2 - j_0) j) \right\}
\]

\[
= \sum_{\beta \equiv j_0 (\text{mod } 2m_1)}^{2m_1} e^{2\pi \left( -2\beta - \frac{j_2 + j_0}{r} \right)^2} - e^{2\pi \left( -2\beta - \frac{j_2 - j_0}{r} \right)^2}
\]
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\[-i^{m_1} \frac{\sign(a_1)}{\sqrt{r^*|a_1|}} \times \sum_{\beta \pmod{2a_1}} e_{2ra_1} \left(- (r\beta + j_2 + j_0)^2\right) - e_{2ra_1} \left(- (r\beta + j_2 - j_0)^2\right)\]

Two terms in the third equality corresponding to the complex conjugates of the terms shown have been removed, and the overall expression was multiplied by 2: this results from the substitution \( j \to -j \). The fourth equality was obtained by applying the reciprocity formula (1). We obtain the required result by substituting \( \gamma = r\beta + j_2 \). Hence, this confirms the first step of the induction. Now, we assume that the result holds inductively for \( t - 1 \). To prove the result for \( \mathfrak{S}_t \), we use the symmetries in Lemma (4) to expand the sum over \( j_t \) from 1 to \( r \).

\[\mathfrak{S}_t = \frac{1}{2} \sum_{j_t=1}^{r} S_{j_t+1} j_t T^{m_t}_{j_t} \mathfrak{S}_{t-1}\]

\[= \frac{i^{m_t}}{2i^{r}} C_{t-1} \sum_{j_t \pmod{r}} \sum_{\gamma \pmod{2r_{a_{t-1}}} \gamma \equiv j_t \pmod{r}} e_{2ra_{t-1}} (m_t a_{t-1} j_t^2) e_{2ra_{t-1}} (-c_{t-1} \gamma^2)\]

\[e_{2ra_{t-1} c_{t-1}} (-j_0^2) \left\{ e_{ra_{t-1}} (-\gamma j_0) - e_{ra_{t-1}} (\gamma j_0) \right\} \left\{ e_r (j_{t+1} j_t) - e_r (-j_{t+1} j_t) \right\} + e_{ra_{t-1}} (-\gamma j_0) - e_{ra_{t-1}} (\gamma j_0) \right\} \right].\]

Now, we replace \( j_t \) by \( \gamma \) and we combine the coefficients of the \( \gamma^2 \) factors using Proposition 7(iii), so we get:

\[\mathfrak{S}_t = \frac{i^{m_t}}{2i^{r}} C_{t-1} e_{2ra_{t-1} c_{t-1}} (-j_0^2) \sum_{\gamma \pmod{2r_{a_{t-1}}}} e_{2ra_{t-1}} (a_t \gamma^2)\]

\[\times \left\{ e_r (j_{t+1} \gamma) - e_r (-j_{t+1} \gamma) \right\} \left\{ e_{ra_{t-1}} (-\gamma j_0) - e_{ra_{t-1}} (\gamma j_0) \right\} \right].\]

Now the substitution \( \gamma \to -\gamma \) allows us to condense four terms to two and a factor of -2 in front of the sum, so this yields:

\[\mathfrak{S}_t = \frac{i^{m_t}}{2i^{r}} C_{t-1} e_{2ra_{t-1} c_{t-1}} (-j_0^2) \sum_{\gamma \pmod{2r_{a_{t-1}}}} e_{2ra_{t-1}} (a_t \gamma^2)\]

\[\times \left\{ e_{ra} (c_t j_{t+1} + j_0 \gamma) - e_{ra} (c_t j_{t+1} - j_0 \gamma) \right\} \right]\]

\[= \frac{i^{m_t}}{2i^{r}} C_{t-1} e_{2ra_{t-1} c_{t-1}} (-j_0^2) \sum_{\gamma \pmod{2r_{a_{t-1}}}} e_{4ra} (2a_t \gamma^2)\]

\[\times \left\{ e_{ra} (c_t j_{t+1} + j_0 \gamma) - e_{ra} (c_t j_{t+1} - j_0 \gamma) \right\} \right].\]
Hence, the formula for $\zeta_t$ is given by

$$\zeta_t = -\frac{i m_t}{\sqrt{r}} \left( C_{t-1} e_{2r a_{t-1} c_{t-1}} (-j_0^2) \right)^{\frac{2 r c_t}{2 d t}} \beta \sum_{\text{mod } 2a_t} \left\{ e_{2a_t} \left( -2 c_t \left( \beta + \frac{c_t j_{t+1} + j_0}{r c_t} \right)^2 \right) - e_{4a_t} \left( -2 c_t \left( \beta + \frac{c_t j_{t+1} - j_0}{r c_t} \right)^2 \right) \right\}$$

Now, we use the reciprocity formula (1) to obtain:

$$\zeta_t = -\frac{i m_t}{\sqrt{r}} C_{t-1} e_{2r a_{t-1} c_{t-1}} (-j_0^2) \sqrt{\frac{2 r c_t}{2 d t}} \beta \sum_{\text{mod } 2a_t} \left\{ e_{2a_t} \left( -c_t \left( r \beta + j_{t+1} + \frac{j_0}{c_t} \right)^2 \right) - e_{2a_t} \left( -c_t \left( r \beta + j_{t+1} - \frac{j_0}{c_t} \right)^2 \right) \right\}.$$

Now, the main formula holds using the substitution $\gamma = r \beta + j_{t+1}$. Finally, we use induction for $C_t$:

$$C_t = -\frac{i m_t}{\sqrt{r}} C_{t-1} e_{2r a_{t-1} c_{t-1}} (-j_0^2) \sqrt{\frac{2 r c_t}{2 d t}} \beta \sum_{\text{mod } 2a_t} \left\{ e_{2a_t} \left( -c_t \left( r \beta + j_{t+1} + \frac{j_0}{c_t} \right)^2 \right) - e_{2a_t} \left( -c_t \left( r \beta + j_{t+1} - \frac{j_0}{c_t} \right)^2 \right) \right\}.$$

Hence, the formula for $C_t$ holds from the induction hypothesis for $C_{t-1}$.

**Proposition 9** The representation of $SL(2, \mathbb{Z})$ on $V_r(S^1 \times S^1)$ is given by

$$\mathcal{R}(U)_{j l} = (-i K_t) \frac{1}{\sqrt{r |c|}} e_{2rc}(dl^2) \sum_{\gamma \equiv j \text{ (mod } 2r)} e_{2rc}(a \gamma^2) \left\{ e_{rc}(\gamma l) - e_{rc}(-\gamma l) \right\},$$

where $K_t = i^{(l-1)} \zeta^{l-2} \text{sign}(a_{t-1}) \frac{1}{2} (m_{t-1} + \cdots + m_1)$ for $t \geq 2$ and $K_1 = i^{m_1}$.

**Proof:** We prove the case for $t \geq 2$, we have

$$\mathcal{R}(U)_{j l j_0} = T_{j l}^{m_t} \zeta_{t-1}$$

$$= i^{m_t} C_{t-1} \sum_{\gamma \equiv j \text{ (mod } 2r)} e_{2r}(m_t \gamma^2)$$

$$\times \left\{ e_{2r a_{t-1}} \left( -c_{t-1} (\gamma + \frac{j_0}{c_{t-1}})^2 \right) - e_{2r a_{t-1}} \left( -c_{t-1} (\gamma - \frac{j_0}{c_{t-1}})^2 \right) \right\}$$

$$= i^{m_t} C_{t-1} e_{2r a_{t-1} c_{t-1}} (-j_0^2) \sum_{\gamma \equiv j \text{ (mod } 2r)} e_{2r a_{t-1}}(m_t a_{t-1} \gamma^2)$$

$$\times e_{2r a_{t-1}} (-c_{t-1}^2) \left\{ e_{ra_{t-1}} (-\gamma j_0) - e_{ra_{t-1}} (\gamma j_0) \right\}.$$
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We combine the coefficients of the $\gamma^2$ factors using Proposition 7(iii), to obtain:

$$R(U)_{j_0} = -i^m C_{t-1} e^{2r_{a_{t-1}} a_{t-2}} (-j_0^2) \sum_{\gamma \equiv j_0 \pmod{r}} e^{2r_{a_{t-1}} (a_t \gamma^2)} \left\{ e^{r_{a_{t-1}} (\gamma j_0) - e^{r_{a_{t-1}} (-\gamma j_0)}} \right\}$$

$$= -i^m C_{t-1} e^{2r_{a_{t-1}} a_{t-2}} (-j_0^2) e^{2r_{a_{t-1}} (-j_0^2)} \sum_{\gamma \equiv j_0 \pmod{r}} e^{2r_{a_{t-1}} \left( a_t \left( \gamma + \frac{j_0}{a_t} \right)^2 \right)} - e^{2r_{a_{t-1}} \left( a_t \left( \gamma - \frac{j_0}{a_t} \right)^2 \right)} \right\}.$$

We substitute the value of $C_{t-1}$ from the previous lemma, to get:

$$R(U)_{j_0} = i^m \zeta^{-2} \zeta^t \gamma^2 e^{2r_{a_{t-1}} a_{t-2}} (-j_0^2) \sum_{\gamma \equiv j_0 \pmod{r}} e^{2r_{a_{t-1}} \left( a_t \left( \gamma + \frac{j_0}{a_t} \right)^2 \right)} - e^{2r_{a_{t-1}} \left( a_t \left( \gamma - \frac{j_0}{a_t} \right)^2 \right)} \right\}.$$

Now, we use Proposition 7(ii) for the prefactor involving $j_0^2$ to obtain:

$$R(U)_{j_0} = -i K_t \left\{ e^{2r \left( b j_0^2 \right)} \sum_{\gamma \equiv j_0 \pmod{r}} e^{2r_{a_{t-1}} \left( a_t \left( \gamma + \frac{j_0}{a_t} \right)^2 \right)} - e^{2r_{a_{t-1}} \left( a_t \left( \gamma - \frac{j_0}{a_t} \right)^2 \right)} \right\}$$

$$= -i K_t \left\{ e^{2r \left( b c j_0^2 \right)} \sum_{\gamma \equiv j_0 \pmod{r}} e^{2r_{a_{t-1}} \left( a_t \gamma^2 \right)} \left\{ e^{r_{c} (\gamma j_0) - e^{r_{c} (-\gamma j_0)}} \right\} \right\}$$

$$= -i K_t \left\{ e^{2r \left( d j_0^2 \right)} \sum_{\gamma \equiv j_0 \pmod{r}} e^{2r_{a_{t-1}} \left( a_t \gamma^2 \right)} \left\{ e^{r_{c} (\gamma j_0) - e^{r_{c} (-\gamma j_0)}} \right\} \right\}.$$ (3)

Here $K_t = i^{(t-1) \zeta^{-2} \zeta^t \gamma^2 e^{2r_{a_{t-1}} a_{t-2}} (-j_0^2)}$. The last equality follows from the equation $bc + 1 = ad$. 431
The lens space \( L(p, q) \) is specified by a pair of coprime integers \( p, q \). We can assume that \( 0 < -q < p \) as it was shown in [8] that \( L(p, q) \) is diffeomorphic to \( L(p, q + np) \) for any integer \( n \). The above lens space is obtained by doing a rational surgery on \( S^3 \) along the unknot with coefficient \(-p/q\). Equivalently, it is obtained by an integer surgery on \( S^3 \) along a chain link \( L \) with successive framings by integers \( m_1, m_2, \ldots, m_{t-1} \) such that \( C = (m_1, m_2, \ldots, m_{t-1}) \) is a continued fraction of \(-p/q\) as in [7]. The rational surgery on \( S^3 \) means removing a solid torus around the unknot and gluing it back using the matrix:

\[
A = \begin{pmatrix} p & d \\ -q & -b \end{pmatrix} \in SL(2, \mathbb{Z}).
\]

Hence, the lens space \( L(p, q) \) is obtained by gluing two solid tori by \( U = SA = \begin{pmatrix} q & b \\ p & d \end{pmatrix} \).

Therefore, \( U \) has a continued fraction expansion \( C = (m_1, \ldots, m_{t-1}, m_t = 0) \). Thus, according to the TQFT axioms, the WRT invariant of \( L(p, q) \) with the weight obtained from the composition is given by

\[
\langle L(p, q), w \rangle_r = \mathcal{R}(U)_{11},
\]

where \( w = \sum_{i=2}^{t} \text{sign} \left( c_{i-1}c_i \right) \) as computed in [2, Page. 415]. As \( 0 < -q < p \), we have \(-p/q > 1\). We need to use a generalized version of [4, Lem. (3.1)].

**Lemma 10** \(-p/q\) has a unique continued fraction expansion with all \( m_i \geq 2\).

Proof :- We know that any rational number has a continued fraction expansion. The construction of the required fraction expansion is given as follows: set \( m_{t-1} = \left\lfloor -p/q \right\rfloor \geq 2 \), so \(-p/q = m_{t-1} - \left( p'/q' \right)^{-1} \) with \( 1 \leq q' < p' = -q < p \) and \( p'/q' > 1 \). We repeat this process for \( p'/q' \) and since the denominators continue to decrease, this process will terminate. This gives a continued fraction expansion with \( m_i \geq 2 \). The idea of this construction is due to Jeffery in the her proof of [4, Lem. (3.1)]. To prove the uniqueness, we assume that \( C' = (m_1, m_2, \ldots, m_{r-1}) \) is another continued fraction expansion of \(-p/q\) with all \( m_i \geq 2 \). Let \( k \) be the first integer such that \( m_{t-1-k} \neq m_{r-1-k} \). Hence, we would have \( s/u = m_{r-1-k} - \left( s'/u' \right)^{-1} \) with \( s'/u' < 1 \). Therefore, \( (m_1, m_2, \ldots, m_{r-2-k}) \) is a continued fraction expansion for \( (s'/u') < 1 \) with \( m_i \geq 2 \) which contradicts the result of the next lemma.

**Lemma 11** If \( s/u < 1 \), then there is no continued fraction expansion for \( s/u \) with all \( m_i \geq 2 \).
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Proof: Assume such a continued fraction expansion exists for \( s/u \). We consider first the case \( 0 < s/u < 1 \). If \( s/u = m_{v-1} - (s'/u')^{-1} \) with \( 1 \leq s < u = s' < u' \) then it implies \( 0 < s'/u' < 1 \). Now, if we repeat this process for \( s'/u' \) then it will never terminate as long as \( m_i \geq 2 \). For the second case, if \( s/u < 0 \) then it is enough to notice that \( s/u = m_{v-1} - (s'/u')^{-1} \) where \( 0 < s'/u' < 1 \). Now we can apply the first case to \( s'/u' \) to conclude that there is no continued fraction expansion with all \( m_i \geq 2 \).

Theorem 12 There is a unique continued fraction expansion \( C \) for any rational number \(|s/u| \neq 1\) with

1. \( m_i \geq 2, 1 \leq i \leq t \) if \( s/u > 1 \).
2. \( m_i \leq -2, 1 \leq i \leq t \) if \( s/u < -1 \).
3. \( m_i \leq -2, 1 \leq i \leq t - 1 \) and \( m_t = 0 \) if \( 0 \leq s/u < 1 \).
4. \( m_i \geq 2, 1 \leq i \leq t - 1 \) and \( m_t = 0 \) if \( -1 < s/u \leq 0 \).

We need to use the next lemma whose proof can be found in [4].

Lemma 13 (4, Lem. (3.2)) If \( A \) is given by the continued fraction with all \( m_i \geq 2 \) as above, then

\[
\Phi(U) = -3(t - 1) + \sum_{i=0}^{t-1} m_i.
\]

Here and elsewhere, \( \Phi(U) \) is the Rademacher phi function of \( U \) (see [4, 6] for more details about this function). Now, we have

\[
r^3 \text{Sign}(W_L) \langle L(p, q), 0 \rangle_r = \langle L(p, q), w \rangle_r = \kappa \text{Trace}(W_L) R(U)_{11}.
\]

where \( W_L \) is the linking matrix of the link \( L \). Hence, we have

\[
\langle L(p, q), 0 \rangle_r = \kappa \Phi(U) R(U)_{11}.
\]

Therefore, we can conclude the following theorem.

Theorem 14 The Witten-Reshetikhin-Turaev invariant of the lens space \( L(p, q) \) weighted zero is given by

\[
\langle L(p, q), 0 \rangle_r = \frac{i^{t-1}}{\sqrt{rp}} e^{2 \pi i s(q, p)} \sum_{\pm} \sum_{n=1}^{p} e_{rp}(\pm 1) e_{rp}(2qrn^2) e_{rp}(2n(q \pm 1)),
\]

where the Dedekind sum \( s(q, p) \) is defined in [4, Equation(2.16)].
Proof: Using equation (3), we have
\[ R(U)_{11} = -iK_1 \frac{1}{\sqrt{rp}} e_{2rq}(b) \]
\[ \sum_{\gamma \equiv 1 (mod \ 2rp)} \left\{ e_{2rp} \left( q \left( \gamma + \frac{1}{q} \right) \right) - e_{2rp} \left( q \left( \gamma - \frac{1}{q} \right) \right) \right\} \]
\[ = -iK_1 \frac{1}{\sqrt{rp}} e_{2rq}(b) \left\{ e_{2rp}(2rnq + q + 1)^2 - e_{2rp}(2rnq + q + 1)^2 \right\} \]
\[ = -iK_1 \frac{1}{\sqrt{rp}} e_{2rq}(b) \sum_{n=1}^{p} e_p(2qrn^2) \]
\[ \left\{ e_p(2n(q + 1))e_{2rpq}(q + 1)^2 - e_p(2n(q - 1))e_{2rpq}(q - 1)^2 \right\}. \]

Therefore by (5), we get:
\[ \langle L(p, q), 0 \rangle_r = -i \frac{\Phi(U)}{\gamma} e^{2 zig}(p) \left( i^{-1} e_{2r}(-1) \right)^{\Phi(U)} \frac{1}{\sqrt{rp}} e_{2rq}(b) \]
\[ \times \sum_{n=1}^{p} e_p(2qrn^2) \]
\[ \times \left\{ e_p(2n(q + 1))e_{2rpq}(q + 1)^2 - e_p(2n(q - 1))e_{2rpq}(q - 1)^2 \right\}. \]

Now,
\[ e_{2rq}(b)e_{2pq}(q \pm 1)^2 = e_{2pq}(bp + q^2 \pm 2q + 1) = e_{2pq}(d + q \pm 2), \]
as \( bp + 1 = dq \). Also, if we introduce the integer \( q^* \) solving \( qq^* \equiv 1 \ (mod \ p) \) then we have

\[ e_{2rp}(d + q \pm 2) (-1)^{\Phi(U)} = e_{2rp}(\pm 1)e_{2r}(12s(d, p)) = e_{2r}(\pm 1)e_{2r}(12s(q^*, p)), \]
as \( \Phi(U) = \frac{d+q}{p} - 12s(d, p) \) and \( q^* \equiv d \ (mod \ p) \). Finally, we obtain the result as \( s(q^*, p) = s(q, p). \)
Summary and Acknowledgments:

The main conclusion of this work is to obtain formula (5) that gives the WRT-invariant of lens spaces in terms of a finite sum. As generalized version of [4, Lem (3.1)] was needed, we proved Theorem (12) which gives more information than what we needed.

I would like to acknowledge that the idea behind this work was inspired by Jeffery's work [4]. The main difference between the two is the TQFT that was considered. In particular, her work was based on the TQFT associated to the group $SU(2)$ where our work is based on the TQFT associated to the group $SO(3)$. Also, I would like to thank my Ph.D. adviser P. Gilmer for motivating and encouraging me to do this work.

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