Stochastic Quantization of an Abelian Gauge Theory

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Abstract: We study the Langevin dynamics of a $U(1)$ lattice gauge theory on the two-dimensional torus, and prove that they converge for short time in a suitable gauge to a system of stochastic PDEs driven by space-time white noises. We fix gauge via a DeTurck trick. This also yields convergence of some gauge invariant observables on a short time interval. The proof relies on a discrete version of the theory of regularity structures.

Contents

1. Introduction ..................................... 1445
2. Lattice Gauge Theory and Observables ............ 1452
3. Gauge Transformations ........................... 1461
4. Discrete Regularity Structure Theory ................ 1468
5. Renormalization and Bounds on the Models ........... 1477
6. Convergence of Solutions and Observables ............ 1489
Appendix A Discussions and possible extensions ........... 1504
Appendix A Discussions and possible extensions ........... 1506

1. Introduction

Consider a $U(1)$ gauge theory defined by the following Hamiltonian, which is sometimes called the Higgs model or scalar quantum electromagnetic dynamics (scalar QED):

$$\mathcal{H}(A, \Phi) \overset{\text{def}}{=} \frac{1}{2} \int_{\mathbb{T}^2} (F_A(x))^2 + \sum_{j=1,2} |D_j^A \Phi(x)|^2 \ d^2x. \quad (1.1)$$

Here $A$ and $\Phi$ are both $\mathbb{R}^2$ valued functions on the unit torus $\mathbb{T}^2$, and by the standard convention we will call $A = (A_1, A_2)$ a vector or gauge field and $\Phi = \Phi_1 + i\Phi_2$ a
complex valued field \((i = \sqrt{-1})\). Moreover,

\[ F_A \overset{\text{def}}{=} \text{curl}A = \partial_1 A_2 - \partial_2 A_1 \]

is a scalar function on \(T^2\) and \(D_j^A\) is the \textit{gauge covariant} derivative\(^1\)

\[ D_j^A \Phi \overset{\text{def}}{=} \partial_j \Phi - i \lambda A_j \Phi \quad (j = 1, 2) \]

where \(\lambda \in \mathbb{R}\) and finally \(|D_j^A \Phi(x)|\) is the complex amplitude of \(D_j^A \Phi(x)\). Here gauge covariance means that for any real valued function \(f\), if we introduce a gauge transformation as

\[ \tilde{A} = A + \nabla f, \quad \tilde{\Phi} = e^{i \lambda f} \Phi, \quad (1.2) \]

then one can check that

\[ D_j^A \tilde{\Phi} = e^{i \lambda f} D_j^A \Phi. \quad (1.3) \]

Since the vector field \(A\) can be also naturally viewed as a differential one-form, the quantity \(F_A = dA\) where \(d\) is the exterior differential and the gauge transformation on \(A\) is simply adding an exact one-form \(df\). The gauge covariant derivative has an adjoint operator which is \((D_j^A)^* = -\partial_j + i \lambda A_j = -D_j^A\).\(^2\)

The model (1.1) exhibits gauge symmetry, namely, under the transformation (1.2) one can check that

\[ \mathcal{H}(\tilde{A}, \tilde{\Phi}) = \mathcal{H}(A, \Phi) \]

since both \(F_A(x)\) and \(|D_j^A \Phi(x)|\) remain unchanged, assuming that \(f \in C^2(T^2)\) such that \(\text{curl} \nabla f = 0\). This symmetry is called \(U(1)\) gauge invariance due to the nature of the transformation (1.2): the transformation \(e^{i \lambda f}\) is a function taking values in the Lie group \(U(1)\) and the term \(-i \lambda A_j\) which defines the gauge covariant derivative is a function taking values in the Lie algebra \(u(1)\).

Because of this invariance, the formal “Gibbs measure” (i.e. quantum gauge theory)

\[ e^{-\mathcal{H}(A, \Phi)} \, DA \, D\Phi \quad (1.4) \]

where \(DA \, D\Phi\) is the formal “Lebesgue measure” can not be naively defined as a probability measure, even if \(T^2\) is replaced by a finite lattice, since the measure would not be normalizable. (However, in a “compact” setting, the measure can be normalized on a \textit{fixed} finite lattice, see Remark 2.3.) To make sense of this probability measure one has to fix a gauge, which amounts to selecting a suitable representative element from each trajectory of the action by the gauge transformation (1.2).

The aim of this article is to study the Langevin dynamic associated with the potential \(\mathcal{H}(A, \Phi)\), which is \textit{formally} given by

\[ \partial_t A = -\text{curl}^* \text{curl}A - \frac{i \lambda}{2} \left( \Phi D^A \Phi - \Phi D^A \Phi \right) + \xi, \quad (1.5a) \]

\[ \partial_t \Phi = \sum_{j=1}^{2} D_j^A D_j^A \Phi + \zeta, \quad (1.5b) \]

\(^1\) The specific value of the coupling constant \(\lambda\) is not important. We prefer to have this parameter in the model for convenience so that a “perturbation theory” will be more organized.

\(^2\) If we view \(D^A\) as an operator which gives a 1-form, then for a 1-form \(B\), \((D^A)^* B = -\text{div} B + i A \cdot B\).
where $\xi = (\xi_1, \xi_2)$, $\zeta = \zeta_1 + i\zeta_2$ and $\xi_1, \xi_2, \zeta_1, \zeta_2$ are four independent space-time white noises, and $\sum_{j=1}^2 D_j^AD_j^A = - \sum_{j=1}^2 (D_j^A)^* D_j^A$ is called the gauge covariant Laplacian (which is nonlinear). For $\zeta$ as a complex valued space-time white noise one has $E(\zeta(t,x)\zeta(s,y)) = 2\delta(t-s)\delta(x-y)$. Equation (1.5) is often called the “stochastic quantization” of the quantum field theory defined by $\mathcal{H}(A, \Phi)$: formally, (1.4) is the invariant measure of (1.5). Stochastic quantization was first proposed by physicists Parisi and Wu in the paper “perturbation theory without gauge fixing” [PW81]; as a matter of fact their main motivation was to study gauge theory, especially the gauge-fixing issue in the dynamic setting; see “Appendix A.2” for more formal discussion.

The nonlinear equation (1.5) also exhibits $U(1)$ gauge invariance in the distributional sense. In fact, at the purely formal level, we can define the gauge transformed functions $\tilde{A}, \tilde{\Phi}$ as (1.2) with $f : T^2 \to \mathbb{R}$ (independent of time). One can formally check\(^3\) using (1.3) that $\tilde{A}$ and $\tilde{\Phi}$ still satisfy (1.5a), while (1.5b) becomes

$$\partial_t \tilde{\Phi} = e^{i\lambda f} \partial_t \Phi = e^{i\lambda f} \sum_j D_j^A D_j^A \Phi + e^{i\lambda f} \zeta = \sum_j D_j^A D_j^A \tilde{\Phi} + e^{i\lambda f} \zeta.$$  

Note that the gauge transformed noise $e^{i\lambda f} \zeta$ is still distributed as a complex valued Gaussian space-time white noise, in particular $E(e^{i\lambda f} \zeta(t,x)e^{i\lambda f} \zeta(s,y)) = 2\delta(t-s)\delta(x-y)$. Mathematically there are several reasons that the study of this system is interesting and nontrivial. First, note that the equations in (1.5) are completely formal. Indeed, in two spatial dimensions, the solution of the linearized equation $\partial_t \Phi = A \Phi + \zeta$ is distributional, so that the nonlinearities of the system lack a classical meaning. This is the general problem shared by a large class of singular stochastic PDEs that have been intensively studied recently, such as the KPZ equation [Hai13, GP17], generalized parabolic Anderson model [GIP15, Hai14], stochastic Navier–Stokes equation [DPD02, ZZ15], as well as some other equations which arise from the stochastic quantization procedure such as the $\Phi^4$ equation [DPD03, Hai14] and sine-Gordon equation [HS16, CHS18].

The theory of regularity structures [Hai14], the paracontrolled distribution method [GIP15, BB19] and the renormalization group method [Kup16] are developed to provide solution theories to equations with such ill-defined nonlinearities. Typically, these equations require renormalizations. For the formal equation (1.5), we will see that the only renormalization we will have is a “mass renormalization” for $\Phi$ in the second equation. This amounts to renormalizing the Hamiltonian by subtracting a term $\infty|\Phi|^2$ which is again invariant under the transformation (1.2).

Another difficulty comes from the linear part; note that the Eq. (1.5a) for $A$ is not parabolic. In fact, if we drop the field $\Phi$ (by setting $\lambda = 0$) Eq. (1.5a) then becomes a linear system

$$\begin{align*}
\partial_t A_1 &= -\partial_2 F_A + \xi_1 = \partial_2^2 A_1 - \partial_1 \partial_2 A_2 + \xi_1 \\
\partial_t A_2 &= \partial_1 F_A + \xi_2 = \partial_1^2 A_2 - \partial_1 \partial_2 A_1 + \xi_2.
\end{align*}$$

(1.6)

This is the dynamic for the potential

$$\frac{1}{2} \int_{T^2} F_A(x)^2 \, d^2x = \frac{1}{2} \int_{T^2} (\partial_1 A_2 - \partial_2 A_1)(x)^2 \, d^2x. \quad (1.7)$$

Obviously the system (1.6) is not parabolic. The loss of parabolicity is related with the $U(1)$ gauge symmetry: the quadratic form (1.7) annihilates all the gradients $A = \nabla f$.\(^3\)

\(^{3}\) Of course the initial condition will change under this transformation.
Before the discussion on the full system (1.5), we illustrate the main idea of gauge tuning which allows us to recover the parabolicity using the simple linear case (1.6). Consider the system (1.6) with an initial data $\hat{A} = (\hat{A}_1, \hat{A}_2)$. We apply a version of “DeTurck trick” (originally arising from the context of Ricci flow [DeT83]) to “gauge out the non-parabolic part” as follows. Let $B = (B_1, B_2)$ solve the parabolic equation

$$\partial_t B_j = \Delta B_j + \xi_j \quad (j = 1, 2)$$

with the same initial condition $\hat{B} = \hat{A}$, whose solution is a well-known Ornstein-Uhlenbeck process. Then, with $\text{div} B = \partial_1 B_1 + \partial_2 B_2$, and let

$$A_1(t) \overset{\text{def}}{=} B_1(t) - \int_0^t \partial_1 \text{div} B(s) \, ds = B_1(t) - \int_0^t \left( \partial_1 \partial_2 B_2(s) + \partial_2^2 B_1(s) \right) \, ds$$

$$A_2(t) \overset{\text{def}}{=} B_2(t) - \int_0^t \partial_2 \text{div} B(s) \, ds = B_2(t) - \int_0^t \left( \partial_1 \partial_2 B_1(s) + \partial_2^2 B_2(s) \right) \, ds$$

one can check that

$$\partial_t A_1 = \Delta B_1 + \xi_1 - \left( \partial_1 \partial_2 B_2 + \partial_2^2 B_1 \right) = -\partial_2 F_B + \xi_1, \quad \partial_t A_2 = \Delta B_2 + \xi_2 - \left( \partial_1 \partial_2 B_1 + \partial_2^2 B_2 \right) = \partial_1 F_B + \xi_2.$$

The important point is that with $A$ defined via $B$ as in (1.9), one has

$$F_A = \partial_1 A_2 - \partial_2 A_1 = \partial_1 B_2 - \partial_2 B_1 = F_B$$

so $A$ actually solves the original Eq. (1.6), with the initial condition $A(t = 0) = \hat{A}$ obviously satisfied.

The above manipulation can be rephrased as follows. For the Eq. (1.6) for $A$, there exists a time-dependent family of gauge transformations $G_t$, such that $G_0$ is identity transformation and the transformed process

$$B(t) = G_t A(t) \overset{\text{def}}{=} A(t) + \nabla f(t), \quad f(t) \overset{\text{def}}{=} \int_0^t \text{div} B(s) \, ds$$

satisfies a parabolic equation. Note that the transformations $G_t$ depend on the “target” (i.e. transformed) process $B$, for which one has to solve first from (1.8).

**Remark 1.1.** With differential form notation (1.6) can be written as $\partial_t A = -d^* dA + \xi$, and (1.8) can be written as $\partial_t B = -d^* dB - dd^* B + \xi$. So, in a sense we have inserted a “gauge fixing” term $-dd^* B = -\nabla \text{div} B$. Note that this term breaks the $U(1)$ gauge symmetry: unlike (1.5) or (1.6) which enjoy gauge invariance as discussed above, if $B$ satisfies (1.8) the function $B + \nabla g$ for a time-independent function $g$ will not satisfy (1.8) anymore ($\Delta \nabla g \neq 0$ in general). However (1.8) has a “gauge covariance” property in the sense that if $B$ and $\tilde{B}$ satisfy (1.8) with gauge equivalent initial conditions $B$ and $\tilde{B} + \nabla g_0$ respectively for some function $g_0$ then one has $\tilde{B} = B + \nabla g$ for any $t > 0$ as long as $g$ satisfies the PDE $\partial_t g = \Delta g$ with initial condition $g_0$, because

$$\partial_t \tilde{B} = \Delta B + \xi + \nabla \Delta g = \Delta \tilde{B} + \xi.$$
Remark 1.2. There is of course another natural way to fix gauge for our Abelian model. Define an equivalence relation \( \sim \) as: \( A \sim \bar{A} \) if and only if \( A = \bar{A} + \nabla f + (c_1, c_2) \). One can define a Gaussian measure formally given by \( \exp\left( -\frac{1}{2} \int F_A^2 \, dx \right) DA/Z \) on the space of equivalence classes, where \( DA \) is the formal Lebesgue measure and \( Z \) is the normalization factor. In fact, by Helmholtz decomposition a smooth vector field on \( T^2 \) can be uniquely decomposed into a divergence free part and a gradient, i.e.

\[
A = (-\partial_2 g, \partial_1 g) + \nabla f + (c_1, c_2) \tag{1.12}
\]

where the function \( g \) solves \( \Delta g = F_A \) and \( f \) solves \( \Delta f = \text{div} A \), and \( c_1, c_2 \) are constants. One can easily check that the first term on the right hand side is divergence free, and \( \nabla f \) is curl free. It seems natural to solve our equation by projecting \( A \) onto the first subspace of this decomposition, as in the case of stochastic Navier–Stokes equation [DPD02, ZZ15].

We instead choose the present approach in this paper mainly because we aim to develop methods here which are potentially able to study SPDEs with non-Abelian gauge symmetry. For non-Abelian gauge theories the gauge-fixing cannot be achieved by the above linear decomposition. On the other hand, DeTurck trick has been successful in the study of deterministic Yang–Mills flow with non-Abelian gauge groups (see for instance [DK90, Section 6.3] or the more recent monograph [Fee14] and many references therein). For stochastic quantization of non-Abelian gauge theories, the use of the gauge fixing method that is close to the DeTurck trick goes back to physicists [Zwa81, Sad87] (see [BHST87, Eq. (2.1)–(2.2)] which takes the viewpoint of inserting a “gauge fixing” term as in Remark 1.1). Recently [CG13] applies this gauge fixing to deterministic Yang–Mills flow to define a kind of nonlinear negative-index Sobolev space (the motivation is to find possible orbit spaces where non-Abelian gauge theoretic measures could live on) and they refer to this trick as Zwanziger–Donaldson–Sadun gauge fixing, see also [CG15, Gro16] for subsequential works along this line.

With the above linear example in mind, our strategy is then to apply the DeTurck trick together with the theory of regularity structures [Hai14] which provide the solution theory by regularizing the equation, renormalization, and passing to the limit. In the regularization step, we will consider the dynamics of lattice gauge theory, which was first proposed by Wilson [Wil74], see for instance the physics book [Rot12] on this extensively studied topic. Lattice gauge theory provides us with discretization of Hamiltonian (1.1) which preserves the exact gauge symmetry; this is discussed in Sect. 2. One main reason we are particularly interested in the lattice gauge theory is that one would hope to obtain gauge invariant continuum limit, so we start from regularizations that preserve the exact symmetry. Another reason we choose a gauge invariant regularization is that our arguments such as DeTurck trick and Ward identities rely on gauge invariance of the nonlinearities.

In applying this strategy, several issues arise and need to be addressed. The dynamic of our lattice gauge theory is slightly nonlocal so we will need to localize these terms and control the errors (see Sect. 4). In the renormalization step, the renormalization terms

\[
\exp\left( \frac{\epsilon^2}{\Delta A} \right)
\]

where \( \Delta A \) is the gauge covariant Laplacian. It would be very interesting to study this rigorously, in particular show that this regularization yields the same limit.
must also preserve the gauge invariance in order to make our gauge tuning argument work and more importantly in order to obtain a gauge invariant limiting dynamic; so we should prove that any “symmetry-breaking” renormalizations will actually cancel out (these are discussed in Sects. 3 and 5).

**Main results.** In Sect. 2 we will discretize the Hamiltonian (1.1) and the stochastic PDE, which preserves the exact gauge symmetry. Here assuming all the precise definitions of discretizations (for which we refer to Sect. 2) we state our main results. Let $\varepsilon$ be the lattice spacing. For each $t \geq 0$, the discretized field $A^\varepsilon$ will be a real valued function on the set of edges of the lattice (with component $A^\varepsilon_1$ on the horizontal edges and component $A^\varepsilon_2$ on the vertical edges), while the discretized field $\Phi^\varepsilon$ will be a complex valued function on the set of vertices of the lattice. With such a discretization at hand, we say that two pairs $(A^\varepsilon, \Phi^\varepsilon)$ and $(B^\varepsilon, \Psi^\varepsilon)$ are gauge equivalent if there exists a real valued function $f^\varepsilon$ on the vertices of the lattice such that

$$A^\varepsilon_j(e) = B^\varepsilon_j(e) - \nabla_j f^\varepsilon(e), \quad \Phi^\varepsilon(x) = e^{-i\lambda f^\varepsilon(x)} \Psi^\varepsilon(x), \quad (1.13)$$

for every edge $e$, and vertex $x$ and $j \in \{1, 2\}$. Here $\nabla_j f^\varepsilon$ is the finite difference of $f^\varepsilon$. Denote the set of vertices by $\Lambda^\varepsilon$, and the set of edges by $\mathcal{E}^\varepsilon$. The main result of the article is then the following. The spaces $C^\eta, C^\delta, \alpha$ and distances $\|\cdot;\cdot\|_{C^\eta}$ and $\|\cdot;\cdot\|_{C^\delta, \alpha_{\eta, T^*}}$ will be defined in (2.15), (2.19), (2.20) and (2.23).

**Theorem 1.3.** For every $\varepsilon = 2^{-N}$ with $N \in \mathbb{N}$, let $\mathcal{H}^\varepsilon$ be a gauge invariant discretization of (1.1) given by (2.7) below, and let $(A^\varepsilon(t), \Phi^\varepsilon(t))_{t \geq 0}$ be the solution to the Itô system parametrized by the set of lattice sites $\Lambda^\varepsilon$ and the set of edges $\mathcal{E}^\varepsilon$

$$dA^\varepsilon(e) = -\frac{\partial \mathcal{H}^\varepsilon}{\partial A^\varepsilon(e)}\varepsilon^{-2} dt + dW^\varepsilon_t(e) \quad (e \in \mathcal{E}^\varepsilon),$$

$$d\Phi^\varepsilon(x) = -\frac{\partial \mathcal{H}^\varepsilon}{\partial \Phi^\varepsilon(x)}\varepsilon^{-2} dt - C^\varepsilon(\Phi^\varepsilon) + dW^\varepsilon_t(x) \quad (x \in \Lambda^\varepsilon), \quad (1.14)$$

whose explicit form is given in (2.10), with initial data $(\hat{A}^\varepsilon, \hat{\Phi}^\varepsilon)$. Suppose that the initial data satisfy almost surely

$$\lim_{\varepsilon \to 0} \left( \|\hat{\Phi}; \hat{\Phi}^\varepsilon\|_{C^\eta} + \|\hat{A}; A^\varepsilon\|_{C^\eta} \right) = 0, \quad (1.15)$$

for $\hat{A}, \hat{\Phi} \in C^\eta$ with $\eta > -\frac{1}{2}$.

Then, there exists $(B^\varepsilon(t), \Psi^\varepsilon(t))$ which is gauge equivalent with $(A^\varepsilon(t), \Phi^\varepsilon(t))$ for each $t > 0$ and $(B^\varepsilon(0), \Psi^\varepsilon(0)) = (\hat{A}^\varepsilon, \hat{\Phi}^\varepsilon)$ such that the following holds: for every $\alpha < 0$ there is a sequence of renormalization constants $C^\varepsilon = O(\log \varepsilon)$, a sequence of stopping times $T^\varepsilon$ satisfying $\lim_{\varepsilon \to 0} T^\varepsilon = T^*$ in probability with $\mathbb{P}(T^* > 0) = 1$, and processes $B, \Psi \in C^\delta, \alpha_{\eta, T^*}$ such that, for every $\hat{\eta} < \eta \land \alpha$, and for any $\delta > 0$ small enough, one has the limit in distribution

$$\lim_{\varepsilon \to 0} \left( \|\Psi; \Psi^\varepsilon\|_{C^\delta, \alpha_{\hat{\eta}, T^*}} + \|B; B^\varepsilon\|_{C^\delta, \alpha_{\hat{\eta}, T^*}} \right) = 0. \quad (1.16)$$
The above theorem does not make any claim about convergence of the original fields \((A^\varepsilon, \Phi^\varepsilon)\). However, it is really the convergence of \textit{gauge invariant observables} that is of true interest. Once we obtain convergence of gauge and scalar fields in a suitable gauge, we then have convergence of local observables such as curvature, composite fields of \(\Phi^\varepsilon\) and its gauge covariant derivative, as well as some global observables such as “loop” observables. These are defined in Sect. 2 and their convergences are given in Theorem 2.7 below.

\textbf{Remark 1.4.} With the method developed in this article, it should be straightforward to obtain a local solution theory for such a \(U(1)\) theory in two spatial dimensions with “Ginzburg-Landau potentials”, that is, with polynomial terms \(P(|\Phi|^2)\) in the Hamiltonian \(\mathcal{H}\). These are important models in superconductivity, with the “vortices dynamics” of particular interest; to our best knowledge the stochastic dynamics of vortices have been only defined intuitively so far (see for instance [IS17]).

We conclude the introduction by discussing the earlier related literature. The Higgs model (i.e. the formal measure (1.4) with Hamiltonian (1.1), or its variants and generalizations) has been of great interest in constructive quantum field theory. Let us mention the previous works on this model, especially those related with our continuum limit question. In a series of papers [BFS79a, BFS79b, BFS80] Brydges, Fröhlich and Seiler showed several important properties of general lattice gauge theories such as diamagnetic inequalities, correlation inequalities, reflection positivity, etc. Focusing on two dimensional Higgs model, with an additional polynomial interaction \(P(|\Phi|^2)\) in the Hamiltonian, they then proved in [BFS81, Section 2] the continuum limit of (not necessarily gauge invariant) correlation functions, using the diamagnetic inequalities, Nelson / hypercontractive estimate and some power counting results for Feynman graphs; in [BFS81, Section 4], the infinite volume limit for generating functional of \(F_A\) and \(|\Phi|^2\) is obtained using the correlation inequalities and Osterwalder–Schrader axioms are verified (except for clustering) for correlations of these gauge invariant local fields. The series of papers by Balaban [Bal82a, Bal82b, Bal83] studied the quantum Higgs models of type (1.1) with \(|\Phi|^4\) interaction where \(\Phi\) is an \(N\)-component field, in both two and three space dimensions restricted to a finite volume, and proved upper and lower bounds uniform in lattice spacings on the partition function using renormalization group transformation techniques. Using these renormalization group techniques, working also in both two and three space dimensions, King [Kin86a, Theorem 2.1] proved continuum limit of generating functional of the fields \(F_A\) and \(|\Phi|^2\) and in [Kin86b] combining with correlation inequalities he proved infinite volume limit and verified some of the Osterwalder–Schrader axioms such as reflection positivity.\footnote{We take this chance to make a few remarks on reflection positivity. Jaffe [Jaf15] pointed out the stochastic quantization equation starting from a generic initial condition would generally break reflection positivity for \textit{finite} time. Therefore, it is unclear to us whether our approach to abelian gauge theory could restore reflection positivity in infinite time, since we haven’t obtained any global-in-time estimates. On the other hand, note that [GH18] used SPDE method to derive bounds which are strong enough to prove the tightness of the family of lattice \(\Phi^4\) measures which do have reflection positivity. It would be very interesting to see if the method of [GH18] could be applied to gauge theories.} Clustering properties and mass gap are discussed in [BBIJ84, BIJ85, BIJ88].

In the pure gauge case without the scalar field \(\Phi\) (i.e. \(\lambda = 0\) in the Hamiltonian \(\mathcal{H}\)), a lot more is known since the quantum theory is Gaussian after fixing a suitable gauge. For continuum limit results of these pure Abelian gauge models, see [Gro83] in three dimensional case and [Dri87] in four dimensional case. Note that some of the literature is concerned with gauge theory models in “compact” setting, for instance in...
2. Lattice Gauge Theory and Observables

2.1. Lattice gauge theory. We henceforth consider discretizations of (1.1) and (1.5). Let $T^2_\varepsilon$ be the dyadic discretization of the unit torus $T^2$ with mesh size $\varepsilon = 2^{-N}$ for $N \in \mathbb{N}$, and let $\Lambda_\varepsilon$, $E_\varepsilon$ and $P_\varepsilon$ be the sets of vertices, undirected edges and squares of the lattice $T^2_\varepsilon$. Let $E_\varepsilon^1$ and $E_\varepsilon^2$ be the set of horizontal and vertical undirected edges respectively. Let $e_1 = (\varepsilon, 0)$ and $e_2 = (0, \varepsilon)$.

For $j \in \{1, 2\}$, let $E_\varepsilon^j$ and $\tilde{E}_\varepsilon^j$ be two copies of $E_\varepsilon^j$, which consist of directed edges that are oriented in the positive and negative directions along $e_j$ respectively. Let $\tilde{E}_\varepsilon = E_\varepsilon^1 \cup \tilde{E}_\varepsilon^2$ and $\tilde{E}_\varepsilon = E_\varepsilon^1 \cup E_\varepsilon^2$. For a directed edge $e$, the edge $-e$ denotes the edge with the reversed direction. A directed edge is often denoted by $(x, y)$ while an undirected edge is often denoted by $\{x, y\}$, where $x, y \in \Lambda_\varepsilon$ are two nearest neighbor vertices. For any $e \in \tilde{E}_\varepsilon^j$, we will often denote by $e_-^j$ and $e_+^j$ the starting and terminating vertices of $e$, namely

$$e = (e_-, e_+) \in \tilde{E}_\varepsilon^j,$$

where $e_+^j = e_-^j + e_j$.

According to these definitions we have $|\Lambda_\varepsilon| = \varepsilon^{-2}$ and $|E_\varepsilon| = 2\varepsilon^{-2}$.

A vertex $x \in \Lambda_\varepsilon$ can be naturally viewed as a point in $T^2_\varepsilon$. For an edge $e$, we will also sometimes think of it as a point in $T^2$, that is, the point $\frac{e_+ + e_-}{2}$. For two edges $e, \tilde{e}$, we will sometimes write $e - \tilde{e} \overset{d}{=} \frac{e_+ + e_- - \tilde{e}_+ + \tilde{e}_-}{2} \in \mathbb{R}^2$; and for an edge $e$ and a grid point $x$ we write $e - x \overset{d}{=} \frac{e_+ + e_- - e}{2} - x$.

Discretized fields. A discretized complex field $\Phi^\varepsilon \in \mathbb{C}^\Lambda_\varepsilon$ is a function which assigns each vertex $x \in \Lambda_\varepsilon$ with a complex number $\Phi^\varepsilon(x) = \Phi^\varepsilon_1(x) + i\Phi^\varepsilon_2(x)$. A discretized

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7 Each square is simply an $\varepsilon \times \varepsilon$ area surrounded by four edges. The notation $\mathcal{P}_\varepsilon$ here should not be confused with the operator $\mathcal{P}^\varepsilon$ introduced in (4.20).
vector field (i.e. gauge field) \( A^\varepsilon = (A_1^\varepsilon, A_2^\varepsilon) \in \mathbb{R}^{E_\varepsilon} \) is a function which assigns each horizontal (resp. vertical) undirected edge \( e \in E_\varepsilon \) with a real number \( A_1^\varepsilon(e) \) (resp. \( A_2^\varepsilon(e) \)). When \( e \) is known in the context as horizontal or vertical, we sometimes simply write \( A^\varepsilon(e) \). The field \( A^\varepsilon \in \mathbb{R}^{E_\varepsilon} \) can be naturally extended to the directed edges: If \( (x, y) \in \tilde{E}_\varepsilon \), \( A^\varepsilon(x, y) := A^\varepsilon([x, y]) \); if \( (x, y) \in \tilde{E}_\varepsilon \), \( A^\varepsilon(x, y) := -A^\varepsilon([x, y]) \).

For \( \Phi^\varepsilon \in \mathbb{R}^{\Lambda_\varepsilon} \) or \( \mathcal{C}^{\Lambda_\varepsilon} \), \( j \in \{1, 2\}, x \in \Lambda_\varepsilon, e \in \{\pm e_j\} \) we define the discrete derivative

\[
(\nabla_e^\varepsilon \Phi^\varepsilon)(x) \overset{\text{def}}{=} \varepsilon^{-1}(\Phi^\varepsilon(x + e) - \Phi^\varepsilon(x))
\]

and set \( (\nabla_j^\varepsilon \Phi^\varepsilon)(x) \overset{\text{def}}{=} (\nabla_{e_j}^\varepsilon \Phi^\varepsilon)(x) \) so that one has \( (\nabla_{e_j}^\varepsilon \Phi^\varepsilon)(x) = -(\nabla_{j}^\varepsilon \Phi^\varepsilon)(x) = -(x - e_j) \).

Discrete derivatives are also defined on the fields on the edges. For \( j, k \in \{1, 2\}, e \in \mathcal{E}^\varepsilon, \ e \in \{\pm e_j\} \) and \( A_k^\varepsilon(e) \in \mathbb{R}^{E_\varepsilon} \),

\[
(\nabla_e^\varepsilon A_k^\varepsilon)(e) := \varepsilon^{-1}(A_k^\varepsilon(e + e) - A_k^\varepsilon(e)) \tag{2.1}
\]

where \( e + e = \{x + e, y + e\} \in \mathcal{E}^\varepsilon \) if \( e = \{x, y\} \), so that \( e + e \) and \( e \) are end-to-end if \( j = k \) and are parallel if \( j \neq k \). We also set \( (\nabla_j^\varepsilon A^\varepsilon)(x) \overset{\text{def}}{=} (\nabla_{e_j}^\varepsilon A^\varepsilon)(x) \).

**Remark 2.1.** For \( \Phi^\varepsilon \in \mathbb{R}^{\Lambda_\varepsilon} \) or \( \mathcal{C}^{\Lambda_\varepsilon} \), we will sometimes also write \( \nabla_j^\varepsilon \Phi^\varepsilon(e) \) instead of \( \nabla_j^\varepsilon \Phi^\varepsilon(x) \) where \( e = \{x, x + e_j\} \), meaning that we regard \( \nabla_j^\varepsilon \Phi^\varepsilon \) as a field on \( \mathcal{E}^\varepsilon \).

**Discretized curl.** Let \( A^\varepsilon \in \mathbb{R}^{E_\varepsilon} \). For each square \( p \in \mathcal{P}_\varepsilon \), summing over the values of \( A^\varepsilon \) along its four adjacent edges counter-clockwise then yields the value of \( F_{A^\varepsilon}^\varepsilon \) on \( p \):

\[
F_{A^\varepsilon}^\varepsilon(p) \overset{\text{def}}{=} \varepsilon^{-1}\left(\frac{A^\varepsilon(e_p^E) + A^\varepsilon(e_p^N) - A^\varepsilon(e_p^W) - A^\varepsilon(e_p^S)}{2}\right) \tag{2.2}
\]

where \( e_p^E, e_p^N, e_p^W, e_p^S \in \mathcal{E}_\varepsilon \) are the edges to the east, north, west and south of the square \( p \) as shown in the middle part of Fig. 1. \( F_{A^\varepsilon}^\varepsilon \) is often called the (discrete) curl or curvature of \( A^\varepsilon \). It is easy to see that \( F_{A^\varepsilon}^\varepsilon(p) \) is invariant under \( A^\varepsilon \to A^\varepsilon + \nabla e f^\varepsilon \) for any function \( f^\varepsilon \in \mathbb{R}^{\Lambda_\varepsilon} \), since summing a gradient over a loop gives zero.

**Discretized gauge covariant derivative.** We introduce a gauge-invariant discretization of the covariant derivative. For \( x \in \Lambda_\varepsilon \) and \( \Phi^\varepsilon \in \mathcal{C}^{\Lambda_\varepsilon} \), let

\[
(\nabla_A^\varepsilon \Phi^\varepsilon)(x) = \varepsilon^{-1}\left(e^{-2\pi i \lambda A^\varepsilon(x, x + e_j)} \Phi^\varepsilon(x + e_j) - \Phi^\varepsilon(x)\right). \tag{2.3}
\]
With this discretization, the covariance property (1.2) (1.3) remains true: for any \( f^ε \in R^Λ_ε\),
\[
(D_j^{A^ε + ∇^ε f^ε } (e^{iλ f^ε } \Phi^ε ))(x) = e^{-1} \left( e^{-iελA^ε(x,x+e_j)} - iλf^ε(x+e_j) \right) \left( e^{iλf^ε(x+e_j)} \Phi^ε(x+e_j) \right) - e^{-1} \left( e^{-iελA^ε(x,x-e_j)} - iλf^ε(x-e_j) \right) \Phi^ε(x).
\]
(2.4)

It is easy to check that its adjoint operator is given by
\[
((D_j^{A^ε } )^* \Phi^ε) (x) = e^{-1} \left( e^{-iελA^ε(x,x-e_j)} - iλf^ε(x-e_j) \right) - \Phi^ε(x).
\]
(2.5)

which is the discrete covariant derivative along the “negative direction” \(-e_j\). Here according to our aforementioned convention \((x, x - e_j) ∈ E^j_ε\) and \(A^ε(x, x - e_j) = -A^ε((x - e_j, x))\).

**Remark 2.2.** Note that a “naive” discretization of \(D_j^A\Phi\) such as
\[
e^{-1} \left( \Phi^ε(x + e_j) - \Phi^ε(x) \right) - iλA^ε(x, x + e_j) \Phi^ε(x)
\]
would violate the gauge covariance property, and hence the gauge invariance of the Hamiltonian. As discussed in the introduction, we choose gauge invariant discretization in order to implement the gauge tuning argument in Sect. 3, and also because we aim to obtain the “right” limit that inherits gauge symmetry. There is obviously another choice of discretization: \((\tilde{D}_j^{A^ε } \Phi^ε) (x) \overset{\text{def}}{=} e^{-1} \left( \Phi^ε(x + e_j) - e^{iελA^ε(x,x+e_j)} \Phi^ε(x) \right)\), which also satisfies the covariance property, but it only differs with (2.3) by a phase factor, which does not change the Hamiltonian (2.7). One might also think that the covariance derivative along the “negative” directions as defined in (2.5) should also be included in the Hamiltonian to look more “symmetric”, but obviously (2.5) is just equal to \(- (\tilde{D}_j^{A^ε } \Phi^ε)(x - e_j)\) so it does not matter under \(\sum_x |·|^2\).

**Discretized divergence.** The discrete divergence for a vector field \(A^ε \in R^{E^ε}\) gives a scalar field on \(Λ_ε\) and is defined as
\[
(\text{div}^ε A^ε)(x) \overset{\text{def}}{=} e^{-1} \left( A^ε(e_N^x) - A^ε(e_S^x) + A^ε(e_E^x) - A^ε(e_W^x) \right)
\]
(2.6)

where \(e_N^x, e_S^x, e_E^x, e_W^x \in E_ε\) are the edges to the north, south, east and west of the site \(x\) respectively, see the right picture in Fig. 1.

With the above discrete objects at hand, our discretized Hamiltonian is then written as
\[
\mathcal{H}^ε(A^ε, \Phi^ε) = \frac{ε^2}{2} \sum_{p \in P_ε} F_{A^ε}(p)^2 + \frac{ε^2}{2} \sum_{x \in Λ_ε} \sum_{j=1,2} |(D_j^{A^ε } \Phi^ε)(x)|^2
= \mathcal{H}^ε_F(A^ε) + \mathcal{H}^ε_D(A^ε, \Phi^ε)
\]
(2.7)

where \(\mathcal{H}^ε_F(A), \mathcal{H}^ε_D(A, \Phi)\) denote the two terms respectively. With all the aforementioned invariance / covariance properties enjoyed by the quantities on the right hand side of (2.7), one has that \(\mathcal{H}^ε(A^ε, \Phi^ε)\) is invariant under
\[
\tilde{A}^ε(e) = A^ε(e) + (∇^ε f^ε)(e), \quad \tilde{Φ}^ε(x) = e^{iλf^ε(x)}\Phi^ε(x),
\]
(2.8)
for any function $f \in \mathbb{R}^{\Lambda_{\varepsilon}}$. The fact that our discretization preserves gauge symmetry will be crucial for the rest of the paper, especially the gauge tuning arguments in Sect. 3.

Remark 2.3. Note that some literature is concerned with abelian lattice gauge theory models in “compact” setting, for instance in [Dri87, Section 3], the abelian lattice model is defined via a periodic function of period $2\pi$; typical choices of such models are the ones defined by Wilson energy (or its generalization) and Villain energy. The lattice fields can be viewed as taking values in the Lie group $U(1)$. On a fixed finite lattice (or more generally graph), the Yang–Mills–Higgs measure can be defined as a density multiplied by the product Haar measure on the compact Lie group, and in this case it can be normalized. The lattice model we consider here is in the “non-compact” setting where the lattice fields take values in the Lie algebra. It would be interesting to show that all these different discretizations of abelian gauge theory lead to the same limit, which should be feasible with the method developed in this paper.

2.2. Dynamics of lattice gauge theory. The (renormalized) Langevin dynamic for the discretized Hamiltonian (2.7), sped up by a factor $\varepsilon^{-2}$, is given by the following system of Itô stochastic differential equations parametrized by $\mathcal{E}_{\varepsilon} \sqcup \Lambda_{\varepsilon}$ (with $t$ understood as the macroscopic time variable)

$$
\begin{align*}
&dA^\varepsilon(e) = -\frac{\partial \mathcal{H}^\varepsilon}{\partial A^\varepsilon(e)} \varepsilon^{-2} dt + dW^\varepsilon_t(e) \quad (e \in \mathcal{E}_\varepsilon), \\
&d\Phi^\varepsilon(x) = -\frac{\partial \mathcal{H}^\varepsilon}{\partial \Phi^\varepsilon(x)} \varepsilon^{-2} dt - C^{(\varepsilon)} \Phi^\varepsilon(x) + dW^\varepsilon(x) \quad (x \in \Lambda_{\varepsilon}),
\end{align*}
$$

(2.9)

where $\{W^\varepsilon(e)\}_{e \in \mathcal{E}_\varepsilon}$, $\{\text{Re} W^\varepsilon(x), \text{Im} W^\varepsilon(x)\}_{x \in \Lambda_{\varepsilon}}$ are $|\mathcal{E}_\varepsilon| + 2|\Lambda_{\varepsilon}| = 4\varepsilon^{-2}$ independent real valued Brownian motions all with quadratic covariation

$$
[dW^\varepsilon(e)]_t = [\text{Re} W^\varepsilon(x)]_t = [\text{Im} W^\varepsilon(x)]_t = \varepsilon^{-2} dt.
$$

Note that we have inserted a renormalization term $-C^{(\varepsilon)} \Phi^\varepsilon$, as alluded in the Introduction.

Let $\xi^\varepsilon(e) = \dot{W}^\varepsilon(e), \xi^\varepsilon(x) = \dot{W}^\varepsilon(x)$ be the time derivatives of the Brownian motions, i.e. the discrete white noises. We will write $\xi^\varepsilon_j(e)$ with $j = 1$ (resp. $j = 2$) if $e$ is a horizontal (resp. vertical) edge.

Lemma 2.4. The Eq. (2.9) satisfied by $(A^\varepsilon, \Phi^\varepsilon)$ has the following explicit form

$$
\begin{align*}
&\partial_t A^\varepsilon_1(e) = \varepsilon^{-1} \left(F_{A^\varepsilon}(p^S_e) - F_{A^\varepsilon}(p^N_e)\right) + \varepsilon^{-1} \lambda \text{Im} \left(e^{-i\varepsilon \lambda A^\varepsilon_1(e)} \Phi^\varepsilon(e+) \Phi^\varepsilon(e-)\right) + \xi^\varepsilon_1(e) \\
&\partial_t A^\varepsilon_2(e) = \varepsilon^{-1} \left(F_{A^\varepsilon}(p^E_e) - F_{A^\varepsilon}(p^W_e)\right) + \varepsilon^{-1} \lambda \text{Im} \left(e^{-i\varepsilon \lambda A^\varepsilon_2(e)} \Phi^\varepsilon(e+) \Phi^\varepsilon(e-)\right) + \xi^\varepsilon_2(e) \\
&\partial_t \Phi^\varepsilon(x) = \Delta^\varepsilon_{A^\varepsilon} \Phi^\varepsilon(x) - C^{(\varepsilon)} \Phi^\varepsilon(x) + \xi^\varepsilon(x)
\end{align*}
$$

(2.10)

for every $x \in \Lambda_{\varepsilon}$ and $e \in \mathcal{E}_\varepsilon$, where

$$
\Delta^\varepsilon_{A^\varepsilon} \Phi^\varepsilon(x) \overset{\text{def}}{=} \varepsilon^{-2} \left(\sum_e e^{-i\varepsilon \lambda A^\varepsilon_1(x,e)} \Phi^\varepsilon(x + e) - 4 \Phi^\varepsilon(x)\right)
$$

(2.11)

with $e$ summed over $\{\pm e_1, \pm e_2\}$ is called the discrete gauge covariant Laplacian, and we have used the notation for directed edge $e = (e_-, e_+)$, and $p^S_e$, $p^W_e$ are the squares.
Fig. 2. The left picture illustrates the linear part of the evolution of $A^\varepsilon_1(e)$. The right picture shows that the evolution of the field $A^\varepsilon_1$ on the red edge depends on the noise on that edge and values of the fields surrounding that edge; the evolution of the field $\Phi^\varepsilon_1$ on the red vertex depends on the noise on that vertex and values of the fields surrounding that vertex. Dependence on $\varepsilon$ is dropped for cleaner picture immediately below and above the horizontal edge $e$ respectively (as shown in the left picture of Fig. 2), and $p^W_e$, $p^E_e$ are the squares immediately to the left and right of the vertical edge $e$ respectively.

The linear part of the equation for $A^\varepsilon_1$ has the form

$$
\varepsilon^{-1} \left( F^{\varepsilon}_A(p^S_e) - F^{\varepsilon}_A(p^N_e) \right)
= \varepsilon^{-2}(A^\varepsilon_1(\hat{e}) + A^\varepsilon_1(e) - 2A^\varepsilon_1(e)) - \varepsilon^{-2}(A^\varepsilon_2(e') + A^\varepsilon_2(e) - A^\varepsilon_2(e) - A^\varepsilon_2(e))
$$

for every horizontal edge $e$, where the dotted edges are the edges around $e$ as shown in the left picture of Fig. 2 (which also illustrates the coefficients in front of $A^\varepsilon_j$ on the right hand side of (2.12)), and it is clearly a discretization of $\partial_1 \partial_2 A_1 - \partial_1 \partial_2 A_2$. The linear part in the equation for $A^\varepsilon_2$ for every vertical edge $e$ is an analogous discretization of $\partial_1 \partial_2 A_2 - \partial_1 \partial_2 A_1$.

Proof of Lemma 2.4. The proof is by straightforward computations. Recall from (2.7) that $\mathcal{H}^\varepsilon = \mathcal{H}_F^\varepsilon + \mathcal{H}_D^\varepsilon$. For a horizontal edge $e$ as shown in the picture, differentiating $-\varepsilon^{-2}\mathcal{H}_F^\varepsilon$ w.r.t. $A^\varepsilon_1(e)$ yields precisely (2.12). Regarding $-\varepsilon^{-2}\frac{\partial \mathcal{H}_F^\varepsilon}{\partial A^\varepsilon_1(e)}$ the only term that contributes to this derivative is

$$
-\frac{1}{2} |(DA^\varepsilon_1(\Phi^\varepsilon))(x)|^2 = -\frac{\varepsilon^{-2}}{2} \left| e^{-i\varepsilon\lambda A^\varepsilon_1(e)} \Phi^\varepsilon(x + e_1) - \Phi^\varepsilon(x) \right|^2
$$

where $e = (x, x + e_1)$, and its derivative with respect to $A^\varepsilon_1(e)$ is

$$
\begin{align*}
&= -\varepsilon^{-2} \text{Re} \left( -i\varepsilon \lambda e^{-i\varepsilon\lambda A^\varepsilon_1(e)} \Phi^\varepsilon(x + e_1) \left( (e^{i\varepsilon\lambda A^\varepsilon_1(e)} \Phi^\varepsilon(x + e_1) - \Phi^\varepsilon(x)) \right) \right) \\
&= -\varepsilon^{-2} \text{Re} \left( (i\varepsilon \lambda) e^{-i\varepsilon\lambda A^\varepsilon_1(e)} \Phi^\varepsilon(x + e_1) \Phi^\varepsilon(x) \right) = -\varepsilon^{-1} \lambda \text{Im} \left( e^{-i\varepsilon\lambda A^\varepsilon_1(e)} \Phi^\varepsilon(x + e_1) \Phi^\varepsilon(x) \right).
\end{align*}
$$
which is the claimed nonlinear term in the equation for $A^ε_1$. The derivation for the
equation for $A^ε_2$ follows in the same way. Turning to the equation for $Φ^ε (x)$, note that
$-ε^{-2} \frac{∂H^ε_p}{∂Φ^ε(x)} = -(D^A_j)^*(D^A_jΦ^ε)(x)$ which equals
$-ε^{-2}\left[ e^{-iελA^ε(x,x-e_j)} \left( e^{-iελA^ε(x-e_j,x)}Φ^ε(x) - Φ^ε(x - e_j) \right) \right. 
- \left( e^{-iελA^ε(x,x+e_j)}Φ^ε(x + e_j) - Φ^ε(x) \right) \left] \right]
= ε^{-2}\left[ e^{-iελA^ε(x,x-e_j)}Φ^ε(x - e_j) + e^{-iελA^ε(x,x+e_j)}Φ^ε(x + e_j) - 2Φ^ε(x) \right]
$ and summing $j$ over 1, 2 gives the equation for $Φ^ε (x)$. □

Remark 2.5. By Taylor expanding the exponential factors into polynomials in the field
$A^ε$ and only retaining the leading terms one obtains the “approximate” equations
$\partial_t A^ε_1 = \partial^2_1 A^ε_1 - \partial_1 \partial_2 A^ε_2 - \frac{iλ}{2} \left( Φ^ε \partial_1 Φ^ε - Φ^ε \partial_1 \bar{Φ}^ε - 2iλA^ε_1 |Φ^ε|^2 \right) + \xi^ε_1$
$\partial_t A^ε_2 = \partial^2_1 A^ε_2 - \partial_1 \partial_2 A^ε_1 - \frac{iλ}{2} \left( Φ^ε \partial_2 Φ^ε - Φ^ε \partial_2 \bar{Φ}^ε - 2iλA^ε_2 |Φ^ε|^2 \right) + \xi^ε_2$
$\partial_t Φ^ε = \Delta Φ^ε - iλ \sum_j \partial_j (A^ε_j Φ^ε) - iλ \sum_j A^ε_j (∂_j Φ^ε) - λ^2 \sum_j (A^ε_j)^2 Φ^ε - Ψ^ε(Φ^ε) + ξ^ε . \tag{2.13}$

It is then clearer that this system is discretization of (1.5) (with a mass renormalization
term for $Φ^ε$). We will carry out such an expansion in the proof of Lemma 3.5 below. These
equations appear in the physics review on stochastic quantization [DH87, (4.32,4.38)],
except that in their convention there is not the factor $\frac{1}{2}$ in the equation for $A$. By simple
power counting as in [Hai14], is critical spatial dimension is four. Note that the Da
Prato-Debussche method [DPD03] would not work for this model, even it is in two
dimensions, because of the term $\frac{1}{2}$ in the equation for $A$.

2.3. Norms and distances. With the discretized functions and processes as above we
define norms and distances for them.

1. Hölder spaces and test function spaces. For $r > 0$, we denote by $C^r (\mathbb{R}^d)$ the usual
Hölder space on $\mathbb{R}^d$, and by $C^r_0 (\mathbb{R}^d)$ the space of compactly supported $C^r$-functions.
Denote by $B^r_0 (\mathbb{R}^d)$ the set of $C^r_0 (\mathbb{R}^d)$-functions supported in the unit ball centered at the
origin and with the $C^r$-norm bounded by 1. For $φ ∈ B^r_0 (\mathbb{R}^d)$, $λ > 0$ and $x, y ∈ \mathbb{R}^d$ we define
$φ^λ (y) \overset{def}{=} λ^{-d} φ(λ^{-1}(y - x)). \tag{2.14}$
For $α < 0$, we define the space $C^α (\mathbb{R}^d)$ to consist of $ζ ∈ Σ' (\mathbb{R}^d)$, belonging to the dual
space of the space of $C^r_0$-functions, with $r > - \lfloor α \rfloor$, and such that
$∥ζ∥_{C^α} \overset{def}{=} \sup_{φ ∈ B^r_0} \sup_{x ∈ \mathbb{R}^d} \sup_{λ ∈ (0,1]} λ^{-α} |(ξ, φ^λ_x)| < ∞. \tag{2.15}$

8 The scaling parameter $λ$ appearing in functions $φ^λ$ should be distinguished from the coupling constant
in our system (1.5). Since this is always clear from the context, we choose to use the same letter (which is a
slight abuse of notation).
We also define Hölder spaces over space-time $\mathbb{R}^{d+1}$ as follows. For $\varphi \in B_0^0(\mathbb{R}^{d+1})$, $\lambda > 0$ and $(t, x), (s, y) \in \mathbb{R} \times \mathbb{R}^d$ we define

$$
\varphi^\lambda_{(t,x)}(s, y) \overset{\text{def}}{=} \lambda^{-d-2} \varphi(\lambda^{-2}(s-t), \lambda^{-1}(y-x)).
$$

(2.16)

For $\alpha < 0$, we define the space $\hat{C}^\alpha(\mathbb{R}^{d+1})$ to consist of $\zeta \in S'(\mathbb{R}^{d+1})$, belonging to the dual space of the space of $C^\alpha_0$-functions, with $r > -|\alpha|$, and such that

$$\|\zeta\|_{\hat{C}^\alpha} \overset{\text{def}}{=} \sup_{\varphi \in B_0^0(\mathbb{R}^{d+1})} \sup_{t \in \mathbb{R}} \sup_{x \in \mathbb{R}^d} \lambda^{-\alpha} |\langle \zeta, \varphi^\lambda_{(t,x)} \rangle| < \infty.
$$

(2.17)

2. The space $C^{\delta, \alpha}_\eta([0, T], \mathbb{R}^d)$. For a function $\mathbb{R} \ni t \mapsto \zeta_t$ we define the operator $\delta^{s,t}$ by

$$\delta^{s,t} \zeta \overset{\text{def}}{=} \zeta_t - \zeta_s.
$$

(2.18)

and for $\delta > 0$, $\eta \leq 0$ and $T > 0$, we define the space $C^{\delta, \alpha}_\eta([0, T], \mathbb{R}^d)$ to consist of the functions $(0, T) \ni t \mapsto \zeta_t \in C^\alpha(\mathbb{R}^d)$, such that the following norm is finite

$$
\|\zeta\|_{C^{\delta, \alpha}_\eta} \overset{\text{def}}{=} \sup_{t \in (0, T)} |t|^{-\eta} \|\zeta_t\|_{C^\alpha} + \sup_{s \neq t \in (0, T)} |t, s|_0^{-\eta} \frac{\|\delta^{s,t} \zeta\|_{C^{\alpha-\delta}}}{|t-s|^\frac{\delta}{2}},
$$

(2.19)

where $|t|_0 \overset{\text{def}}{=} |t|^1 \wedge 1$ and $|t, s|_0 \overset{\text{def}}{=} |t|_0 \wedge |s|_0$.

3. Comparing functions on lattice and continuum. We will need to compare discrete functions on the lattice (either on the grid points or on the edges) with their continuous counterparts, and for this we first introduce the following convention. For a continuous function $\varphi : \mathbb{R}^d \to \mathbb{R}$, a site $x \in \Lambda_\varepsilon$ can be naturally identified with a point $x \in \mathbb{R}^d$, thus giving the meaning of $\varphi(x)$. For an edge $e = (e_-, e_+) \in \hat{E}_\varepsilon$, we define $\varphi(e) \overset{\text{def}}{=} \varphi(\frac{e_+ - e_-}{2})$.

The values of $\varphi$ on differences of grid points or edges are then defined in a similar way. For instance, the quantity $\varphi(e - \bar{e})$ for two edges $e, \bar{e}$ then stands for $\varphi(\frac{e_+-e_- - \bar{e}_+ + \bar{e}_-}{2})$, and the quantity $\varphi(e - x)$ for an edge $e$ and a grid point $x$ then stands for $\varphi(\frac{e_+ - e_- + x}{2} - x)$.

Given this convention, in order to compare discrete functions $\zeta^\varepsilon \in \mathbb{R}^{\Lambda_\varepsilon}$ on the dyadic grid with their continuous counterparts $\zeta \in C^\alpha(\mathbb{R}^d)$ with $\alpha \leq 0$, we introduce the following “distance”

$$
\|\zeta; \zeta^\varepsilon\|_{C^\alpha} \overset{\text{def}}{=} \sup_{\varphi \in B_0^0} \sup_{x \in \mathbb{R}^d} \lambda^{-\alpha} |\langle \zeta; \varphi^\lambda_x \rangle - \langle \zeta^\varepsilon; \varphi^\lambda_x \rangle|.
$$

(2.20)

where

$$
\langle \zeta^\varepsilon; \varphi^\lambda_x \rangle \overset{\text{def}}{=} \varepsilon^d \sum_{y \in \Lambda_\varepsilon} \zeta^\varepsilon(y) \varphi^\lambda_x(y).
$$

(2.21)

For discrete functions $\zeta^\varepsilon \in \mathbb{R}^{E^\varepsilon_j}$ with $j \in \{1, 2\}$ on the edges, we define $\|\zeta; \zeta^\varepsilon\|_{C^\alpha}$ in the same way but with $\langle \zeta^\varepsilon; \varphi^\lambda_x \rangle$ defined by

$$
\langle \zeta^\varepsilon; \varphi^\lambda_x \rangle \overset{\text{def}}{=} \varepsilon^d \sum_{e \in E^\varepsilon_j} \zeta^\varepsilon(e) \varphi^\lambda_x(e).
$$

(2.22)

Here $\varphi^\lambda_x(y)$ and $\varphi^\lambda_x(e)$ are defined according to the aforementioned convention as well as (2.14). For $\zeta^\varepsilon : \mathbb{R} \times \Lambda_\varepsilon \to \mathbb{R}$ and its continuous counterpart $\zeta \in \hat{C}^\alpha(\mathbb{R}^{d+1})$ with $\alpha \leq 0$,
we also define the “distance” \( \| \zeta; \xi \|_{C^a}^{(e)} \) in the analogous way with the parabolically scaled test function (2.16) and scalar product that is discrete in space and continuous in time.

4. Comparing processes on lattice and continuum. Assume that we are given a space-time distribution \( \zeta \), and a function \( \xi(e) \) on \( (0, T] \times \Lambda \) or on \( (0, T] \times \mathcal{E}_\delta \) with \( j \in \{1, 2\} \).

For \( \delta > 0 \) and \( \eta \leq 0 \), we define

\[
\| \zeta; \xi \|_{C^a,T}^{(e)} \overset{\text{def}}{=} \sup_{t \in (0, T]} |t|^{-\eta} \| \zeta_t; \xi_t \|_{C^a}^{(e)} + \sup_{s \neq t \in (0, T]} |s|^{-\eta} \| \delta^{s, t} \zeta; \delta^{s, t} \xi \|_{C^a}^{(e)},
\]

(2.23)

where \( |t|_\varepsilon \overset{\text{def}}{=} |t|_0 + \varepsilon \) and \( |s, t|_\varepsilon \overset{\text{def}}{=} |s|_\varepsilon \wedge |t|_\varepsilon \), and \( \| \zeta; \xi \|_{C^a}^{(e)} \) is defined as above with the discrete pairing defined as in (2.21) or (2.22), depending on the domain where \( \zeta(e) \) is defined.

Furthermore, we define the norms \( \| \zeta(e) \|_{C^a}^{(e)} \) and \( \| \zeta(e) \|_{C^a, T}^{(e)} \) in the same way as in (2.15) and (2.19), but using the discrete pairing (2.21) and (2.22).

If \( \zeta(e) = (\zeta_1, \zeta_2) \) is vector valued or \( \zeta(e) = \zeta_1 + i \zeta_2 \) is complex valued, and so is \( \zeta \), then \( \| \zeta; \xi \|_{C^a, T}^{(e)} = \sum_{k=1}^{2} \| \zeta_k; \xi_k \|_{C^a, T}^{(e)} \) with \( \zeta_k \) and \( \xi_k \). The other norms or distances for vector or complex fields are defined in the same way.

Notation. We will use \( *_\varepsilon \) to denote the space-time convolutions on \( R \times \Lambda \), and \( K^\varepsilon \) to denote the truncated discrete heat kernel as given in [HM18, Lemma 5.4] (that is, the discrete heat kernel on \( R \times \Lambda \) subtracting a compactly supported, non-anticipative function with \( C^r \) norm bounded uniformly in \( \varepsilon \).)

2.4. Gauge invariant observables. We now define a collection of discrete gauge invariant observables and state the convergence result for them, as mentioned in the introduction.

As discussed below (2.2), given a discrete vector field \( A^\varepsilon \), the curvature field \( F^\varepsilon \) is gauge invariant.

We can also define composite scalar fields \( |\Phi^\varepsilon|^{2n} \) as follows. Given a stationary complex valued Gaussian field \( \psi^\varepsilon \) on \( \Lambda \), recall that for processes \( \Phi^\varepsilon \) on \( \Lambda \) and its complex conjugate \( \Phi^\varepsilon \), the Wick power (with respect to the Gaussian measure of \( \psi^\varepsilon \)) is given by

\[
|\Phi^\varepsilon|^2 \overset{\text{def}}{=} \mathbb{E} \left( (t_1 \psi^\varepsilon(x) + t_2 \bar{\psi}^\varepsilon(x))^2 \right) \Bigg|_{t_1 = t_2 = 0} \quad (x \in \Lambda \varepsilon).
\]

(2.24)

From this it is easy to see that for any integer \( n \geq 1 \), the quantity \( |\Phi^\varepsilon|^{2n} \) is a polynomial in \( \Phi^\varepsilon \bar{\Phi}^\varepsilon \) of order \( n \). Therefore \( |\Phi^\varepsilon|^{2n} \) is gauge invariant since \( \Phi^\varepsilon \bar{\Phi}^\varepsilon \) is invariant under the gauge transformation (2.8). In the sequel \( \Phi^\varepsilon \) will be taken as the solution to our discrete equation and thus time dependent, and \( \psi^\varepsilon \overset{\text{def}}{=} K^\varepsilon *_\varepsilon \zeta^\varepsilon \).

Another type of composite field is given by

\[
\tilde{\Phi}^\varepsilon(e_-)(D^A_j \Phi^\varepsilon)(e) - C^\varepsilon_{\Phi D A \Phi} (2.25)
\]

for \( j \in \{1, 2\} \) and \( e = (e_-, e_+) \in \tilde{E}^j_\varepsilon \), where \( C^\varepsilon_{\Phi D A \Phi} \) is a renormalization constant. By (2.4), this field is also gauge invariant under the transformation (2.8).
Besides these local observables, we also define some global observables as follows. To start with, given a simple $C^2$ curve $C$ with a parametrization $r(\sigma)$ for $\sigma \in [0, 1]$ with $|\dot{r}(\sigma)| \neq 0$, and $\varepsilon > 0$ sufficiently small (depending on $C$), we introduce a notion of regular approximation of $C$ as follows. First, we find a collection $P_{\varepsilon}^C$ of squares in $\mathcal{P}_{\varepsilon}$ which covers the curve $C$, in an inductive way as follows. Letting $p_1 \in \mathcal{P}_{\varepsilon}$ be the square that contains $C(r(0))$, we impose that $p_1 \in P_{\varepsilon}^C$. Assuming that $p_i \in P_{\varepsilon}^C$ for some $i \geq 1$, if $C$ exits $p_i$ from edge $e$ (generically $C$ will not intersect with $p_i$ at the corner of $p_i$), then we add $p_{i+1} \in \mathcal{P}_{\varepsilon}$ into the collection $P_{\varepsilon}^C$ such that $p_{i+1} \cap p_i = e$, and repeat this procedure until $C$ is completely covered by such squares. By construction, provided that $\varepsilon > 0$ is sufficiently small, every vertex $x \in \Lambda_{1\varepsilon}$ is shared by at most 3 squares in the collection $P_{\varepsilon}^C$. As an example, in the following picture with a curve $C$ running from left to right as $\sigma$ increases, it exits $p_i$ from the right edge of $p_i$, so $p_{i+1}$ is the adjacent square on the right.

By construction every point $r(\sigma) \in C$ lies in a unique square in $P_{\varepsilon}^C$, therefore this collection $P_{\varepsilon}^C$ specifies a partition of $[0, 1]$ into subintervals:

$$0 = \sigma_0 < \sigma_1 < \cdots < \sigma_{|P_{\varepsilon}^C|} = 1$$

such that $\sigma \in [\sigma_i, \sigma_{i+1}]$ if and only if the point $r(\sigma)$ is in the corresponding square. With slight abuse of notation we will denote this partition of $[0, 1]$ also by $P_{\varepsilon}^C$.

We then select one edge from each $p \in P_{\varepsilon}^C$, such that these edges concatenate forming a discrete curve $C^\varepsilon$. We call $C^\varepsilon$ a regular approximation of $C$. In the above picture a regular approximation $C^\varepsilon$ is drawn in blue. By construction we have a one-to-one correspondence between edges in $C^\varepsilon$ and subintervals in the partition $P_{\varepsilon}^C$.

Remark 2.6. For a $C^2$ curve $C$, provided that $\varepsilon > 0$ is sufficiently small, for any point $r(\sigma) = (r_1(\sigma), r_2(\sigma)) \in C$ such that $r_2(\sigma) = 0$, namely the tangent line is horizontal, the edge $e(\sigma) \in C^\varepsilon$ is necessarily an horizontal edge. Indeed, in this situation a small section of $C$ containing $r(\sigma)$ must be covered by a few squares in $\mathcal{P}_{\varepsilon}$ concatenated horizontally, and by inspection (or simply looking at the above picture) the edges of $C^\varepsilon$ which are associated to these squares must also be horizontal, in order to ensure that each square in $\mathcal{P}_{\varepsilon}$ is associated with a unique edge in $C^\varepsilon$. The same holds near a point with a vertical tangent line.

Let $C \subset T^2$ be a given simple $C^2$ closed curve. Let $C^\varepsilon$ be a discrete loop which is a regular approximation of $C$. Let $A^\varepsilon \in \mathbb{R}^{E_{\varepsilon}}$ be a discrete vector field. One can then consider the loop observables

$$O_{C, \varepsilon} \overset{\text{def}}{=} \varepsilon \sum_{e \in C^\varepsilon} A^\varepsilon(e) \quad \text{and} \quad \hat{O}_{C, \varepsilon} \overset{\text{def}}{=} e^{i\varepsilon} \sum_{e \in C^\varepsilon} A^\varepsilon(e).$$

These are obviously gauge invariant since $\sum_{C^\varepsilon} \nabla f^\varepsilon = 0$ for any $f \in \mathbb{R}^{A_{\varepsilon}}$. 

\[\text{(2.26)}\]
**Theorem 2.7.** Let \((A^\varepsilon, \Phi^\varepsilon)\) be the local Itô solution to (2.10) with initial data \((\hat{A}^\varepsilon, \hat{\Phi}^\varepsilon)\). As \(\varepsilon \to 0\), the gauge invariant observables \(F_{A^\varepsilon}^\varepsilon\) and
\[
\tilde{\Phi}^\varepsilon(\cdot)(D_{\cdot}^\varepsilon A \Phi^\varepsilon)(\cdot) = C_{\Phi A^\varepsilon}\Phi
\]
for \(j \in \{1, 2\}\) and \(C_{\Phi A^\varepsilon} \Phi = O(\varepsilon^{-1})\) given in (6.12) below converge in distribution with respect to the distance \(\|\cdot\|_{C^\alpha(\bar{\Omega}, \alpha)}\). For any integer \(n \geq 1\), the gauge invariant observables \(|\Phi^\varepsilon|^{2n}\) converge in distribution with respect to the distance \(\|\cdot\|_{C^\alpha(\bar{\Omega}, \alpha)}\), for every \(\alpha < 0\) and \(\delta, \eta\) are as in Theorem 1.3.

Moreover, as \(\varepsilon \to 0\), the loop observables \(\mathcal{O}_{C, \varepsilon}\) defined in (2.26) via the solutions \((A^\varepsilon, \Phi^\varepsilon)\) converge in \(C([0, T], \mathbb{R})\) in distribution, and the loop observables \(\tilde{\mathcal{O}}_{C, \varepsilon}\) in (2.26) converge in \(C([0, T], \mathbb{C})\) in distribution.

**Remark 2.8.** There are other observables which could be constructed using our method. For instance, given a two-dimensional subdomain \(\Omega\) of the torus \(\mathbb{T}^2\), and let \(\Omega_\varepsilon\) be a suitable approximation to \(\Omega\), which is a union of a collection of squares in \(\mathbb{P}_\varepsilon\). One can consider the gauge invariance observable \(\varepsilon^2 \sum_{p \in \Omega_\varepsilon} F_{A^\varepsilon}\). In view of the above Corollary, the field \(F_{A^\varepsilon}\) converges in the space of regularity \(\alpha - 1\) for \(\alpha < 0\), so one would expect that the second moment of this observable behaves like \(\varepsilon^4 \sum_{p, \bar{p} \in \Omega_\varepsilon} |p - \bar{p}|^{2(\alpha - 1)}\) which seems to diverge. However, using the definition of \(F_{A^\varepsilon}\) together with a discrete Green’s theorem, this observable can be re-written in terms of (possibly a sum of, if the boundary of \(\Omega\) is not connected) the observable \(\mathcal{O}_{C, \varepsilon}\).

Another type of observables is the “string observables” which are formally written as \(\Phi(x)e^{-i\lambda \int_x^y A \Phi(y)}\) where \(\int_x^y A\) denotes the line integral of \(A\) along a given curve from \(x\) to \(y\). Since for fixed \(x, y\) even when \(\lambda = 0\), the formal expression \(\Phi(x)\Phi(y)\) would be meaningless in the continuum, we may need to define it in an averaged sense. Assume that \(C \subset \mathbb{T}^2\) is a given simple smooth open curve, and that \(C^\varepsilon\) is a discrete curve which is a regular approximation of \(C\). For smooth test functions \(\varphi_1, \varphi_2\) on \(\mathbb{T}^2\) we define
\[
\mathcal{O}^{\varphi_1, \varphi_2}_{C, \varepsilon} \overset{\text{def}}{=} \varepsilon^2 \sum_{x, y \in C^\varepsilon} \varphi_1(x) \left( \tilde{\Phi}^\varepsilon(x) e^{-i\lambda \int_x^y \Phi(y)} \right) \varphi_2(y)
\]
(2.27)
where \(C^\varepsilon(x \to y) \subset C^\varepsilon\) is the part of \(C^\varepsilon\) between the two sites \(x, y\). It is also gauge invariant since the quantity in the parenthesis is gauge invariant.

A variant of this is the “smeared string observable” (for instance [KK86]) formally written as
\[
\tilde{\Phi}(x)e^{-i\lambda \int_x^y A \nabla U} \Phi(y)
\]
where \(\int_x^y A \cdot \nabla U\) is the line integral of the scalar function \(A \cdot \nabla U\) and \(U\) is the Coulomb potential generated by a negative charge at \(x\) and a positive charge at \(y\), namely \(-\Delta U = -\delta_x + \delta_y\). We do not rigorously construct these observables in the present article.

### 3. Gauge Transformations

#### 3.1. DeTurck trick

In this section we apply the DeTurck trick, which amounts to introducing a time dependent family of gauge transformations that can turn the system (1.5)
or (2.10) into a parabolic system. The argument relies crucially on the gauge invariant discretization discussed in the previous section.

Fix $\varepsilon > 0$. Let $(B^\varepsilon, \Psi^\varepsilon)$ solve the following Itô system:

$$
\partial_t B_j^\varepsilon(e) = \Delta^\varepsilon B_j^\varepsilon(e) + \varepsilon^{-1}\lambda \text{Im} \left( e^{-i\varepsilon \lambda B_j^\varepsilon(e)} \Psi^\varepsilon(e_+) \bar{\Psi}^\varepsilon(e_-) \right) + \xi_j^\varepsilon(e)
$$

$$
\partial_t \Psi^\varepsilon(x) = \Delta^\varepsilon \Psi^\varepsilon(x) + i\lambda \text{div} B^\varepsilon(x) \Psi^\varepsilon(x) - C^\varepsilon \Psi^\varepsilon(x) + e^{i\lambda \int_0^t \text{div} B^\varepsilon(s,x) ds } \xi^\varepsilon(x)
$$

(3.1)

for $j = 1, 2$ with the same initial condition $(A^\varepsilon, \Phi^\varepsilon)$ as that of $(A^\varepsilon, \Phi^\varepsilon)$, where the directed edge $e = (e^-, e^+)$ and

$$
\Delta^\varepsilon B_j^\varepsilon(e) = \varepsilon^{-2} \left( B_j^\varepsilon(e^N) + B_j^\varepsilon(e^S) + B_j^\varepsilon(e^E) + B_j^\varepsilon(e^W) - 4B_j^\varepsilon(e) \right)
$$

(3.2)

where $e^N, e^S$ are the horizontal edges above ("north") and below ("south") $e$ respectively (they are parallel with $e$), and $e^W, e^E$ are the horizontal edges on the left ("west") and right ("east") of $e$ respectively (they are aligned and end-to-end with $e$). For a vertical edge $e$ the quantity $\Delta^\varepsilon B_j^\varepsilon(e)$ is defined in the analogous way. We emphasize that the last term in the equation for $\Psi^\varepsilon$ is understood as the Itô product of $\xi^\varepsilon$ and the process $e^{i\lambda \int_0^t \text{div} B^\varepsilon(s,x) ds }$ which is adapted to the filtration generated by the Brownian motions $W^\varepsilon$ introduced in (2.9), since the Itô solution $(B^\varepsilon, \Psi^\varepsilon)$ is adapted to this filtration.

For each $\varepsilon > 0, x \in \Lambda_\varepsilon, e \in \mathcal{E}_\varepsilon, t > 0$, we then define

$$
A_j^\varepsilon(t,e) \overset{\text{def}}{=} B_j^\varepsilon(t,e) - \int_0^t \nabla_j^\varepsilon \text{div} B^\varepsilon(s,e) ds
$$

$$
\Phi^\varepsilon(t,x) \overset{\text{def}}{=} e^{-i\lambda \int_0^t \text{div} B^\varepsilon(s,x) ds } \Psi^\varepsilon(t,x).
$$

(3.3)

The quantity $\nabla_j^\varepsilon \text{div} B^\varepsilon(e)$ has the following explicit form (by (2.6) and Remark 2.1):

$$
\nabla_j^\varepsilon \text{div} B_j^\varepsilon(e) = \varepsilon^{-2} \left( B_j^\varepsilon(e^+) + B_j^\varepsilon(e^-) - B_j^\varepsilon(e) \right)
$$

$$
- \varepsilon^{-2} \left( B_j^\varepsilon(e^+) + B_j^\varepsilon(e^-) - B_j^\varepsilon(e) \right)
$$

(3.4)

for a horizontal edge $e$, which is illustrated in the following picture, and similarly for a vertical edge.
Remark 3.1. The U(1) gauge transformation $g^\varepsilon(t, x) \overset{\text{def}}{=} e^{i\lambda\int_0^t \text{div}^\varepsilon B^\varepsilon(s, x) \, ds}$ satisfies
\[ \partial_t g^\varepsilon = (i\lambda \text{div}^\varepsilon B^\varepsilon) \, g^\varepsilon \quad \text{with} \quad g^{\varepsilon}(0) = 1. \]

In general gauge theories with non-Abelian gauge groups one should expect that the gauge transformation solves a differential equation when applying the DeTruck trick, see the book [DK90, Section 6.3.1].

Lemma 3.2. For $\varepsilon > 0$, let $(A^\varepsilon, \Phi^\varepsilon)$ be defined as in (3.3), where $(B^\varepsilon, \Psi^\varepsilon)$ is the solution to (3.1) with initial condition $(\tilde{A}^\varepsilon, \tilde{\Phi}^\varepsilon)$ with constant $C^{(\varepsilon)}$. Then $(A^\varepsilon, \Phi^\varepsilon)$ satisfy the system (2.10) with the same initial condition $(\tilde{A}^\varepsilon, \tilde{\Phi}^\varepsilon)$ and the same constant $C^{(\varepsilon)}$.

Proof. From the definition of the transformation (3.3) we can immediately see that $(A^\varepsilon, \Phi^\varepsilon)$ and $(B^\varepsilon, \Psi^\varepsilon)$ satisfy the same initial condition. 9

For $j = 1, 2$, by the transformation (3.3), one has
\[ \partial_t A_j^\varepsilon(e) = \partial_t B_j^\varepsilon(e) - \nabla_j \text{div}^\varepsilon B^\varepsilon(e) \]
\[ = \Delta^\varepsilon B_j^\varepsilon - \nabla_j \text{div}^\varepsilon B^\varepsilon(e) + \varepsilon^{-1}\lambda \text{Im}\left(e^{-i\varepsilon\lambda B_j^\varepsilon(e)}\Psi^\varepsilon(e_+)\tilde{\Phi}^\varepsilon(e_-)\right) + \xi_j^\varepsilon(e). \] (3.6)

By (2.12), (3.2) and (3.4), one has [this is actually best seen by comparing the left picture of Fig. 2 and the picture (3.5)]
\[ \Delta^\varepsilon B_1^\varepsilon - \nabla_j^\varepsilon \text{div}^\varepsilon B^\varepsilon(e) = \varepsilon^{-1}(F_{B^\varepsilon}(p^S_e) - F_{B^\varepsilon}(p^N_e)) \] (3.7)
and the same is true with all the subscripts 1, 2 swapped.

With (3.7) at hand we argue that one can replace $(B^\varepsilon, \Psi^\varepsilon)$ by $(A^\varepsilon, \Phi^\varepsilon)$ on the right hand side of (3.6) by invoking gauge invariance, and thus show that $A_j^\varepsilon$ satisfies (2.10). Indeed, by the fact that $F_{A^\varepsilon}(p)$ defined in (2.2) annihilate any gradient of a function on the lattice sites, together with the fact that the transformation (3.3) is precisely shifting the $B^\varepsilon$ by a gradient, one has $F_{A^\varepsilon} = F_{B^\varepsilon}$ and thus (3.7) equals $\varepsilon^{-1}(F_{A^\varepsilon}(p^S_e) - F_{A^\varepsilon}(p^N_e))$.

As for the nonlinear term on the right hand side of (3.6), by (3.3) we have
\[ e^{-i\varepsilon\lambda B_j^\varepsilon(e)}\Psi^\varepsilon(e_+)\tilde{\Phi}^\varepsilon(e_-) = e^{-i\varepsilon\lambda A_j^\varepsilon(e)} e^{-i\varepsilon\lambda\int_0^t \nabla_j^\varepsilon \text{div}^\varepsilon B^\varepsilon(s,e) \, ds} \cdot \Psi^\varepsilon(e_+)\tilde{\Phi}^\varepsilon(e_-) \]
\[ = e^{-i\varepsilon\lambda A_j^\varepsilon(e)}(e^{-i\lambda\int_0^t \text{div}^\varepsilon B^\varepsilon(s,e_+) \, ds} \Psi^\varepsilon(e_+)) \left( e^{i\lambda\int_0^t \text{div}^\varepsilon B^\varepsilon(s,e_-) \, ds} \tilde{\Phi}^\varepsilon(e_-) \right) \]
\[ = e^{-i\varepsilon\lambda A_j^\varepsilon(e)} \Phi^\varepsilon(e_+)\tilde{\Phi}^\varepsilon(e_-). \] (3.8)

Therefore $A^\varepsilon$ satisfies (2.10).

Regarding the equation for $\Phi^\varepsilon$, note that by (3.1) and (3.3) one has
\[ \partial_t \Phi^\varepsilon(x) = e^{-i\lambda\int_0^t \text{div}^\varepsilon B^\varepsilon(s,x) \, ds} \partial_t \Psi^\varepsilon(x) - i\lambda \text{div}^\varepsilon B^\varepsilon(x) \Phi^\varepsilon(x) \]
\[ = e^{-i\lambda\int_0^t \text{div}^\varepsilon B^\varepsilon(s,x) \, ds} \left( \Delta_{B^\varepsilon} \Psi^\varepsilon(x) - C^{(\varepsilon)}\Psi^\varepsilon(x) \right) + \xi^\varepsilon(x). \]

Note that when taking the time derivative of the product on the right hand side of (3.3), the usual product rule applies without Itô correction term. Also, note that the term $i\lambda(\text{div}^\varepsilon B^\varepsilon)\Psi^\varepsilon$ appearing in (3.1) has been used to cancel (after multiplying the

9 Here the fact that the transformation (3.3) is identity at $t = 0$ is only a matter of convenience. One could also change the initial condition to a gauge equivalent one.
exponential factor $e^{-i\lambda \int_0^t \text{div}^\varepsilon B^\varepsilon(s,x) \, ds}$ the term $-i\lambda (\text{div}^\varepsilon B^\varepsilon) \Phi^\varepsilon$ which arises from the time derivative of the exponential factor. We also have the gauge covariance of the discrete covariant Laplacian, namely, with $f^\varepsilon(x) \overset{\text{def}}{=} \int_0^t \text{div}^\varepsilon B^\varepsilon(s,x) \, ds$ and by (2.11), (3.3) we have

$$e^{-i\lambda f^\varepsilon(x)} \Delta \Phi^\varepsilon(x) = e^{-i\lambda f^\varepsilon(x)} \left( \sum \mathcal{E} \mathcal{E} e^{-i\varepsilon \lambda B^\varepsilon x^\varepsilon \varepsilon + \varepsilon} \Phi^\varepsilon(x + \varepsilon) - 4 \Phi^\varepsilon(x) \right)$$

$$= \sum \mathcal{E} \mathcal{E} e^{-i\varepsilon \lambda B^\varepsilon x^\varepsilon \varepsilon + \varepsilon} \Phi^\varepsilon(x + \varepsilon) - 4 \Phi^\varepsilon(x) \right)$$

$$= \sum \mathcal{E} \mathcal{E} e^{-i\varepsilon \lambda A^\varepsilon x^\varepsilon \varepsilon + \varepsilon} \Phi^\varepsilon(x + \varepsilon) - 4 \Phi^\varepsilon(x) = \Delta \Phi^\varepsilon$$

where $\varepsilon$ summed over $\{\pm \varepsilon_1, \pm \varepsilon_2\}$. Note also that $C(\varepsilon) e^{-i\lambda \int_0^t \text{div}^\varepsilon B^\varepsilon(s,x) \, ds \Psi^\varepsilon(x)} = C(\varepsilon) \Phi^\varepsilon(x)$. Therefore $\Phi^\varepsilon$ satisfies the second equation in (2.10).

At this stage, the factor $e^{i\lambda \int_0^t \text{div}^\varepsilon B^\varepsilon(s,x) \, ds}$ in front of $\xi^\varepsilon$ in (3.1) looks harsh to deal with since this factor depends on the solution $B^\varepsilon$ itself. However, we observe that the noise is also gauge invariant in a distributional sense, which is manifest in our Itô regularization: as a general fact, for a vector valued Itô equation $d\tilde{X}_t = M(t, X) \, dB_t$, where the matrix valued process $M$ is adapted to the filtration generated by $B_t$ and satisfies $MM^T = \text{Id}$, then $\tilde{X}_t$ has quadratic variation $t \text{Id}$ and thus $\tilde{X}_t$ is distributed as a Brownian motion. We write this as a lemma for our case.

**Lemma 3.3.** For each $j \in \{1, 2\}$ let $\xi_j$ and $\xi_j \xi_j$ be independent space-time white noises over $L^2(\mathbb{R} \times \mathbb{T}^2)$ which are all independent of the noises $\{W^\varepsilon(e)\}_{e \in \varepsilon}$ and $\{W^\varepsilon(x)\}_{x \in \Lambda}$ in (2.9), and define discretized white noises

$$\xi_j^\varepsilon(t, e) \overset{\text{def}}{=} \varepsilon^{-2} \langle \xi(t, \cdot), 1_{|t-e| \leq \varepsilon/2} \rangle, \quad \xi_j^\varepsilon(t, x) \overset{\text{def}}{=} \varepsilon^{-2} \langle \xi(t, \cdot), 1_{|t-x| \leq \varepsilon/2} \rangle,$$  

and $\xi^\varepsilon = \xi_1^\varepsilon + i \xi_2^\varepsilon$. The solution to the following Itô system

$$\partial_t B_j^\varepsilon(e) = \Delta \Phi^\varepsilon \Phi^\varepsilon + e^{-1} \lambda \text{Im} \left(e^{-i\lambda B_j^\varepsilon(e)} \Phi^\varepsilon(e) \Phi^\varepsilon(e_-) \right) + \xi_j^\varepsilon(e),$$

$$\partial_t \Phi^\varepsilon(x) = \Delta \Phi^\varepsilon \Phi^\varepsilon + i \lambda \text{div}^\varepsilon B^\varepsilon(x) \Phi^\varepsilon(x) - C(\varepsilon) \Phi^\varepsilon(x) + \xi^\varepsilon(x)$$

for $j \in \{1, 2\}$ where $x \in \Lambda$, $e = (e_-, e_+) \in \varepsilon_j$, with initial condition $(\hat{A}^\varepsilon, \Phi^\varepsilon)$ has the same law as the solution to the Itô system (3.1).

**Proof.** Since the process $e^{i\lambda \int_0^t \text{div}^\varepsilon B^\varepsilon(s,x) \, ds}$ is adapted to the filtration generated by the Brownian motions $\{W^\varepsilon(e), \text{Re} W^\varepsilon(x), \text{Im} W^\varepsilon(x)\}_{e \in \varepsilon, x \in \Lambda}$ introduced in (2.9), using the fact that the transformation $e^{i\lambda \int_0^t \text{div}^\varepsilon B^\varepsilon(s,x) \, ds}$ is orthogonal, one has that

$$\left\{W^\varepsilon(e), \text{Re} \int_0^t e^{i\lambda \int_0^t \text{div}^\varepsilon B^\varepsilon(s,x) \, ds} \, dW^\varepsilon(s,x), \text{Im} \int_0^t e^{i\lambda \int_0^t \text{div}^\varepsilon B^\varepsilon(s,x) \, ds} \, dW^\varepsilon(s,x) \right\}_{e \in \varepsilon, x \in \Lambda}$$

has $(\varepsilon^{-2} t)\text{Id}_{4\varepsilon-2}$ as its quadratic covariation, where $\text{Id}_{4\varepsilon-2}$ is a $4\varepsilon^{-2}$ by $4\varepsilon^{-2}$ identity matrix [recall that $\varepsilon^{-1} = 2^N$ is an integer]. Thus by Lévy’s characterization of Brownian motion the above process is a collection of $4\varepsilon^{-2}$ independent standard Brownian motions sped up by $\varepsilon^{-2}$. The statement of the lemma then immediately follows. □
Thanks to the above lemma, we will simply work with (3.11), without changing the distribution of our solution.

**Remark 3.4.** Another possible approach, at least in the Abelian case we are considering, is to project the field $A^\varepsilon$ into the divergence-free subspace as discussed in Remark 1.2. This would be more reminiscent of the procedure of fixing the “Coulomb gauge” in the corresponding quantum field theory. In the end the analysis might turn out to be similar, because setting $\text{div}^\varepsilon B^\varepsilon = 0$ Eq. (3.11) would have the same form as (2.10) since $\Delta^\varepsilon = \nabla^\varepsilon \text{div}^\varepsilon + \text{curl}^* \text{curl}$.

### 3.2. Preparation for regularity structures

To study (3.11) via regularity structures, we start by expanding the nonlinearities into polynomial forms with remainders.

**Lemma 3.5.** Equation (3.11) can be written in the following form:

\[
\partial_t B_j^\varepsilon (e) = \Delta^\varepsilon B_j^\varepsilon (e) + \lambda \text{Im} \left( \nabla_j^\varepsilon \Psi^\varepsilon (e_+) \tilde{\Psi}^\varepsilon (e_-) \right) \\
- \lambda^2 \text{Re} \left( B_j^\varepsilon (e) \Psi^\varepsilon (e_+) \tilde{\Psi}^\varepsilon (e_-) \right) + R_{B_j}^\varepsilon (e) + \xi_j^\varepsilon (e) \\
\partial_t \Psi^\varepsilon (x) = \Delta^\varepsilon \Psi^\varepsilon (x) - i\lambda \sum_{e} B^\varepsilon (x, x+e) \nabla_e^\varepsilon \Psi^\varepsilon (x) \\
- \frac{1}{2} \lambda^2 \sum_{e} B^\varepsilon (x, x+e)^2 \Psi^\varepsilon (x+e) - C^\varepsilon \Psi^\varepsilon (x) + R_{\Psi^\varepsilon}^\varepsilon (x) + \zeta^\varepsilon (x) \\
R_{B_j}^\varepsilon (e) \overset{\text{def}}{=} \varepsilon^{-1} \lambda \text{Im} \left( \tilde{F}_1 \left( -i\varepsilon \lambda B_j^\varepsilon (e) \right) \Psi^\varepsilon (e_+) \tilde{\Psi}^\varepsilon (e_-) \right) \\
R_{\Psi^\varepsilon}^\varepsilon (x) \overset{\text{def}}{=} \varepsilon^{-2} \sum_{e} \tilde{F}_2 \left( -i\varepsilon \lambda B^\varepsilon (x, x+e) \right) \Psi^\varepsilon (x+e)
\]

where $\tilde{F}_1 (z) \overset{\text{def}}{=} \varepsilon -1 - z$ and $\tilde{F}_2 (z) \overset{\text{def}}{=} \varepsilon^2 -1 - z^2 - 2/2$; the sums are over $e \in \{ \pm e_1, \pm e_2 \}$, and $e = (e_-, e_+) \in \tilde{E}_e^j$.

**Proof.** Taylor expanding the factor $e^{-i\varepsilon \lambda B_j^\varepsilon (e)}$ in the nonlinearity of (3.11a) yields

\[
\varepsilon^{-1} \lambda \text{Im} \left( e^{-i\varepsilon \lambda B_j^\varepsilon (e)} \Psi^\varepsilon (e_+) \tilde{\Psi}^\varepsilon (e_-) \right) \\
= \varepsilon^{-1} \lambda \text{Im} \left( \Psi^\varepsilon (e_+) \tilde{\Psi}^\varepsilon (e_-) \right) - \lambda^2 \text{Im} \left( i B_j^\varepsilon (e) \Psi^\varepsilon (e_+) \tilde{\Psi}^\varepsilon (e_-) \right) + R_{B_j}^\varepsilon (e).
\]

Writing $\Psi^\varepsilon (e_+) = \Psi^\varepsilon (e_-) + \varepsilon \nabla_j^\varepsilon \Psi^\varepsilon (e_-)$ and noting that $\text{Im} \left( \Psi^\varepsilon (e_-) \tilde{\Psi}^\varepsilon (e_-) \right) = 0$ one obtains the first equation in (3.12).

For each $j \in \{1, 2\}$, expanding $e^{-i\varepsilon \lambda B_j^\varepsilon (x, x+e)}$ in the term $\Delta^\varepsilon B_j^\varepsilon \Psi^\varepsilon (x)$ of (3.11b):

\[
\varepsilon^{-2} \left( \sum_{e=\pm e_j} e^{-i\varepsilon \lambda B_j^\varepsilon (x, x+e)} \Psi^\varepsilon (x+e) - 2 \Psi^\varepsilon (x) \right) = \Delta^\varepsilon B^\varepsilon (x) \\
- i\varepsilon^{-1} \lambda \sum_{e=\pm e_j} B_j^\varepsilon (x, x+e) \Psi^\varepsilon (x+e) - \frac{\lambda^2}{2} \sum_{e=\pm e_j} B_j^\varepsilon (x, x+e)^2 \Psi^\varepsilon (x+e) + R_{\Psi^\varepsilon}^\varepsilon (x).
\]

It is straightforward to check that the second term on the right hand side is equal to (recall that $\nabla_j^\varepsilon B$ is defined in (2.1))

\[
-i \lambda \sum_{e=\pm e_j} B^\varepsilon (x, x+e) \nabla_e^\varepsilon \Psi^\varepsilon (x) - i \lambda \nabla_j^\varepsilon B^\varepsilon (x - e_j, x) \Psi^\varepsilon (x).
\]
The second term in (3.14), upon summed over \(j \in \{1, 2\}, \) precisely cancel with the term
\[ i \lambda \text{div}^x B^\varepsilon (x) \Psi^\varepsilon (x) \] in (3.11b). The other terms when summing over \(j \in \{1, 2\} \) precisely
give the right hand side of the second equation in (3.12). \( \square \)

Recall that the definition of the spaces \(C_{\alpha,T_r}^{\delta,\alpha} \) depends on a choice of regularity \(r > 0 \) of test functions. In the following lemma, we explicitly record this dependence and write
the spaces as \(C_{\alpha,T_r}^{\delta,\alpha,r} \), and we turn the system into a slightly different one that only lives on the lattice sites \(\Lambda_\varepsilon.\) \(^{10}\)

Lemma 3.6. Let the real-valued processes \(\{(B^\varepsilon_j(t,x), \Psi^\varepsilon_j(t,x)) : x \in \Lambda_\varepsilon, j \in \{1, 2\}, t \in [0, T_\varepsilon]\} \) be the solution to

\[
\begin{align}
\partial_t B^\varepsilon_j (x) &= \Delta^x B^\varepsilon_j (x) + \lambda \left( \Psi_1^\varepsilon (x) \nabla^x \Psi_2^\varepsilon (x) - \Psi_2^\varepsilon (x) \nabla^x \Psi_1^\varepsilon (x) \right) \\
&\quad - \frac{1}{2} \lambda^2 \sum_{k=1,2} \Psi_1^\varepsilon (x + e_j) \Psi_2^\varepsilon (x) + R^\varepsilon_B (x) + \xi^\varepsilon_j (x) \\
&\quad + \nabla_{e_\varepsilon} B^\varepsilon_k (x) \Psi_1^\varepsilon (x) - \nabla_{e_\varepsilon} (B_k^\varepsilon \Psi_2^\varepsilon) (x) \\
&\quad - \frac{1}{2} \lambda^2 \sum_{k=1,2} (B_k^\varepsilon (x) \Psi_2^\varepsilon (x + e_k) + B_k^\varepsilon (x - e_k)^2 \Psi_1^\varepsilon (x - e_k)) \\
&\quad - C(\varepsilon) \Psi_2^\varepsilon (x) + R^\varepsilon_{\Psi_j} (x) + \xi^\varepsilon_j (x),
\end{align}
\]

where \(\xi^\varepsilon_j (x) \triangleq \xi^\varepsilon_j (x, x + e_j) \) and \(\xi^\varepsilon_j (x) \) as in (3.10), and \(R^\varepsilon_B (x) \triangleq R^\varepsilon_B (x, B^\varepsilon, \Psi^\varepsilon) \) and
\(R^\varepsilon_{\Psi_j} (x) \triangleq R^\varepsilon_{\Psi_j} (x, B^\varepsilon, \Psi^\varepsilon) \) are defined as

\[
R^\varepsilon_B (x, B^\varepsilon, \Psi^\varepsilon) \triangleq \exp \left( -\lambda \right) \text{Im} \left( F_1 \left( -i \varepsilon \lambda B^\varepsilon (x) \right) (\Psi_1^\varepsilon + i \Psi_2^\varepsilon) (x + e_j) (\Psi_1^\varepsilon - i \Psi_2^\varepsilon) (x) \right)
\]

\[
R^\varepsilon_{\Psi_j} (x, B^\varepsilon, \Psi^\varepsilon) \triangleq \exp \left( -\lambda \right) \left( -i \right)^{j-1} \text{Re} \left( \left( -i \varepsilon \lambda B^\varepsilon (x) \right) (\Psi_1^\varepsilon + i \Psi_2^\varepsilon) (x + e_j) \right),
\]

with \(x = x \) if \(e = +e_k \) and \(x = x + e \) if \(e = -e_k \), and \(F_1, F_2 \) are the functions defined below (3.13).

If these solutions converge in probability to a limiting process \((B, \Psi)\) on \([0, T^\ast]\) in \(C_{\alpha,T_r}^{\delta,\alpha,r-1} \) for every \(\alpha < \tilde{\alpha} \) as \(\varepsilon \to 0 \) where the stopping time \(T^\ast = \lim_{\varepsilon \to 0} T_\varepsilon \) (in probability), then the solutions to (3.12) converge in probability to the same limit \((B, \Psi)\) in \(C_{\alpha,T_r}^{\delta,\alpha,r} \) for every \(\alpha < \tilde{\alpha} \).

Proof. With \(\Psi^\varepsilon = \Psi_1^\varepsilon + i \Psi_2^\varepsilon \) we readily rewrite (3.12) into the following real form

\[
\partial_t B^\varepsilon (e) = \Delta^x B^\varepsilon (e) + \lambda \left( \Psi_1^\varepsilon (e_-) \nabla^x \Psi_2^\varepsilon (e_-) - \Psi_2^\varepsilon (e_-) \nabla^x \Psi_1^\varepsilon (e_-) \right)
\]

\(^{10}\) Alternatively one could introduce modeled distributions on both vertices and edges, and two reconstruction operators (one on \(\Lambda_\varepsilon \) and one on \(E_\varepsilon \)). This however would probably cause notational complication.
\[ -\lambda^2 B_j^\varepsilon (e) \sum_{k=1,2} \Psi_k^\varepsilon (e_+) \Psi_k^\varepsilon (e_-) + R_{B_j}^\varepsilon (e) + \xi_j^\varepsilon (e) \]

\[ \partial_t \Psi_j^\varepsilon (x) = \Delta^\varepsilon \Psi_j^\varepsilon (x) - (-1)^j \lambda \sum_e B_j^\varepsilon (x, x + e) \nabla_e^\varepsilon \Psi_{3-j}^\varepsilon (x) \]

\[ -\frac{1}{2} \lambda^2 \sum_e B_j^\varepsilon (x, x + e)^2 \Psi_j^\varepsilon (x + e) - C^\varepsilon \Psi_j^\varepsilon (x) + R_{\Psi_j}^\varepsilon (x) + \zeta_j^\varepsilon (x) \]

\[ (3.17) \]

where \( e \) is summed over \( \{ \pm e_1, \pm e_2 \} \). Here \( R_{B_j}^\varepsilon (e) \) is as in (3.13), and \( R_{\Psi_j}^\varepsilon (x) \) is such that \( R_{\Psi_j}^\varepsilon (x) \) in (3.13) equals \( R_{\Psi_j}^\varepsilon (x) + i R_{\Psi_j}^{\varepsilon} (x) \), where in (3.13) every incidence of \( \Psi^\varepsilon \) should be replaced by \( \Psi_1^\varepsilon + i \Psi_2^\varepsilon \).

Denote by \((\tilde{B}^\varepsilon, \tilde{\Psi}^\varepsilon)\) the solution to (3.15), and by \((B^\varepsilon, \Psi^\varepsilon)\) the solution to (3.17). Noting that

\[ -\nabla_e^\varepsilon (B_k^\varepsilon \Psi_{3-j}^\varepsilon (x)) + B_k^\varepsilon (x) \nabla_{e_k}^\varepsilon \Psi_{3-j}^\varepsilon (x) + \nabla_{e_k}^\varepsilon B_k^\varepsilon (x) \Psi_{3-j}^\varepsilon (x) \]

\[ = B_k^\varepsilon (x) \nabla_{e_k}^\varepsilon \Psi_{3-j}^\varepsilon (x) - B_k^\varepsilon (x - e_k) \nabla_{e_k}^\varepsilon \Psi_{3-j}^\varepsilon (x), \]

one can see that in going from (3.17) to (3.15) we have just re-indexed the collection of processes \( B^\varepsilon \) by

\[ \tilde{B}_j^\varepsilon (t, x) \equiv B_j^\varepsilon (t, x + e_j), \quad \text{i.e.} \quad B_j^\varepsilon (t, e) = \tilde{B}_j^\varepsilon (t, e^-) \quad \text{where} \ e = (e^-, e^+), \]

and \( \tilde{\Psi}_j^\varepsilon (t, x) \equiv \Psi_j^\varepsilon (t, x) \) for every \( t \) and every realization of the noises.

By assumption we have \( \lim_{\varepsilon \to 0} ||B; \tilde{B}^\varepsilon||_{C_{\varepsilon, \alpha, r}^0} = 0 \) in probability. To control

\[ ||B; \tilde{B}^\varepsilon||_{C_{\varepsilon, \alpha, r}^0}, \]

we first estimate the first term on the right hand side of (2.23). Write

\[ \left| \langle B_j^\varepsilon (t), \varphi_h^\varepsilon \rangle_\varepsilon - \langle B_j (t), \varphi_h \rangle \right| \leq \left| \langle B_j^\varepsilon (t), \varphi_h^\varepsilon \rangle_\varepsilon - \langle \tilde{B}_j^\varepsilon (t), \varphi_h^\varepsilon \rangle_\varepsilon \right| + \left| \langle \tilde{B}_j^\varepsilon (t), \varphi_h^\varepsilon \rangle_\varepsilon - \langle B_j (t), \varphi_h \rangle \right|. \]

By convergence of \( \tilde{B}^\varepsilon \) the second term multiplied by \( \lambda^{-\alpha} |t|^{-\eta} \) goes to zero uniformly in \( \varphi \in B_0^\alpha, x \in \mathbf{R}^d \) and \( \lambda \in [\varepsilon, 1] \). The first term equals (recall that \( \varphi_h^\varepsilon (e) = \varphi_h^\varepsilon (\varepsilon^+e^+ + \varepsilon^-e^-) \))

\[ \left| \varepsilon^d \sum_{e \in \mathcal{E}_\varepsilon^j} B_j^\varepsilon (t, e) \varphi_h^\varepsilon (e) - \varepsilon^d \sum_{y \in \Lambda_\varepsilon} \tilde{B}_j^\varepsilon (t, y) \varphi_h^\varepsilon (y) \right| \]

\[ = \left| \varepsilon^d \sum_{y \in \Lambda_\varepsilon} \tilde{B}_j^\varepsilon (t, y) \left( \varphi_h^\varepsilon (y + \frac{e}{2} e_j) - \varphi_h^\varepsilon (y) \right) \right|. \]

Define \( \psi (z) \equiv \varphi (2z + \frac{e}{2} e_j) - \varphi (2z) \). Since \( \varphi \) is supported in the unit ball centered at origin, so is \( \psi \) for \( \varepsilon \leq \lambda \). One can also prove that there exists a constant \( c_r \alpha \) depending on \( r \) such that \( ||\psi||_{C^{r-1}} \leq c_r \frac{\varepsilon}{\alpha} ||\varphi||_{C^r} \leq c_r \frac{\varepsilon}{r} \), namely \( c_r^{-1} \frac{\varepsilon}{\alpha} \psi \in B_0^{r-1} \). Thus the above quantity is bounded by

\[ \left| \varepsilon^d \sum_{y \in \Lambda_\varepsilon} \tilde{B}_j^\varepsilon (t, y) \lambda^{-d} \psi \left( \frac{y - x}{2x} \right) \right|. \]

Be cautious that product rule \( \partial (fg) = \partial f g + f \partial g \) does not exactly hold on lattice.
Remark 1.1 and the fact that nonlinear terms in the for any $\theta 
exists H. Shen

Proposition 3.7. There is a sequence of renormalisation constants $C^{(\varepsilon)} = O(\log \varepsilon)$ such that the following holds. Let $(B^\varepsilon(t), \Psi^\varepsilon(t))_{t \geq 0}$ be the solution to the system (3.15). Assume that the initial data $(B^\varepsilon(0), \Psi^\varepsilon(0)) = (\bar{A}^\varepsilon, \bar{\Phi}^\varepsilon)$ satisfies (1.15). Then for every $\alpha < 0$, there is a sequence of stopping times $T_\varepsilon$ satisfying $\lim_{\varepsilon \to 0} T_\varepsilon = T^*$ in probability, and $(B, \Psi)$ such that, for every $\bar{\eta} < \eta \wedge \alpha$, and for any $\delta > 0$ small enough, one has the following limit in probability:

$$\lim_{\varepsilon \to 0} \| (B^\varepsilon, \Psi^\varepsilon), (B, \Psi) \|_{C^{(\varepsilon)}_{\bar{\eta}, \alpha}, T^*} = 0. \quad (3.18)$$

Proof of Theorem 1.3. The proof of this main theorem is an immediate consequence of Proposition 3.7 together with Lemmas 3.2, 3.3, 3.5 and 3.6.

Remark 3.8. The limit $(B, \Psi)$ satisfies the “gauge covariance” property as described in Remark 1.1, with the time-dependent family of gauge transformations $g$ that satisfies the heat equation. In fact $(B^\varepsilon, \Psi^\varepsilon)$ has this property, due to the calculation (1.11) in Remark 1.1 and the fact that nonlinear terms in the $(B^\varepsilon, \Psi^\varepsilon)$ system is invariant under (time-dependent or time-independent) gauge transformations—thanks to the gauge invariant discretization. Since $g$ is smooth, one can pass to the limit and obtain the “gauge covariance” property for $(B, \Psi)$. We also remark that around the time that the present article was completed, [CCHS20] found a much better approach to construct the gauge covariant Langevin dynamic for non-abelian gauge theories, without having to regularize via lattice gauge theory.

The rest of the paper is devoted to proving Proposition 3.7 and convergence of observables.

4. Discrete Regularity Structure Theory

4.1. Regularity structure. From this section we aim to prove the limit claimed in Proposition 3.7 using the theory of regularity structures [Hai14] and its discrete version [HM18]. We will first build regularity structures for the “unrenormalized version” of Eq. (3.15), namely setting $C^{(\varepsilon)} = 0$, and then show that the renormalized equation given by the theory is indeed (3.15) with a divergent constant $C^{(\varepsilon)}$. We start by briefly recalling the framework.

To start with we consider a parabolic scaling $s = (2, 1, 1)$ over space-time $R^{2+1}$. We set $|s| \equiv 4$, denote by $|x|$ the $l^\infty$-norm of a point $x \in R^2$, and define $\|z\|_s \equiv |t|^{1/2} \vee |x|$ to be the $s$-scaled $l^\infty$-norm of $z = (t, x) \in R^3$. For a multiindex $k \in N^3$ we define $|k|_s \equiv 2k_0 + k_1 + k_2$, and for $k \in N^2$ with the scaling $(1, 1)$ we denote the respective

12 We thank the anonymous referee who suggested us to make this remark.
norm by $|k|$. Here $\mathbb{N} = \{0, 1, 2, \ldots \}$. Moreover, for $r > 0$, we denote by $C^r(\mathbb{R}^2)$ the usual Hölder space on $\mathbb{R}^2$, by $C^r_0(\mathbb{R}^2)$ we denote the space of compactly supported $C^r$-functions and by $B^r_0(\mathbb{R}^2)$ we denote the set of $C^r$-functions, compactly supported in $B(0, 1)$ (the unit ball centered at the origin) and with the $C^r$-norm bounded by 1.

Recall from [Hai14] that a regularity structure $\mathcal{T} = (T, G)$ consists of two objects: a graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$, where $A \subset \mathbb{R}$ is a finite set of “homogeneities”; and a structure group $G$ of linear transformations of $T$, such that for every $\Gamma \in G$, every $\alpha \in A$ and every $\tau \in T_\alpha$ one has $\Gamma \tau - \tau \in \bigoplus_{\beta < \alpha} T_\beta$. One example of a regularity structure denoted by $T_{\text{poly}}$ is given by the “abstract polynomials” in $2 + 1$ indeterminates $X_i$, with $i = 0, 1, 2$, where $A$ consists of the natural numbers $\alpha \leq r$ for some fixed $r > 0$, and $T_\alpha$ contains all monomials $X^k = X_0^{k_0} X_1^{k_1} X_2^{k_2}$ of scaled degree $|k|_s = \alpha$; the structure group $G_{\text{poly}}$ is then simply the group of translations in $\mathbb{R}^3$ acting on $X^k$ by $h \mapsto (X - h)^k$.

We build the regularity structure relevant for the analysis of (3.15). For now we ignore the slight non-locality of the nonlinear terms in (3.15) and will control the discrepancy later.

We write $U_{B_j}, U_{\Psi_j}, U_{B_k}, U_{\Psi_k}$ for $j, k \in \{1, 2\}$ for collections of formal expressions that will describe the solutions $B_j, \Psi_j, \nabla \cdot e_j B_k$ and $\nabla \cdot e_j \Psi_k$. (Here, the notation $\nabla \pm_j f$ should be thought of as discrete derivatives of $f$ along the direction $\pm e_j$ in the discrete setting, while in the continuum setting $\nabla \cdot e_j f$ should be simply thought of as $-\partial_j f$.) We write $V_{B_j}, V_{\Psi_j}$ for $j \in \{1, 2\}$ for a collection of formal expressions useful to describe the right hand side of the equation for $B_j$ and $\Psi_j$ respectively. We decree that all the above sets contain at least the polynomials of $X_0, X_1, X_2$. We then introduce additional symbols $\Xi_{\xi_j} \in V_{B_j}$ and $\Xi_{\zeta_j} \in V_{\Psi_j}$, with $j \in \{1, 2\}$, as well as $I, I'_j$ and $I''_j$, where $\Xi_{\xi_j}, \Xi_{\zeta_j}$ will be interpreted as an abstract representation of (discretization of) the driving noises $\xi_j$ and $\zeta_j$, and $I, I'_j$ and $I''_j$ will be interpreted as the operation of convolving with a truncation of the heat kernel and its spatial derivative along the direction $e_j$ and $-e_j$ respectively.

In view of the structure of the Eq. (3.15), to describe the fixed point problem we should at least also decree that

1. If $\tau_1 \in U_{\Psi_k}$ and $\tau_2 \in U_{\Psi_k}^j$, where $k \neq \ell$, then $\tau_1 \tau_2 \in V_{B_j}$.
2. If $\tau_1 \in U_{B_j}$ and $\tau_2, \tau_3 \in U_{\Psi_k}$, then $\tau_1 \tau_2 \tau_3 \in V_{B_j}$.
3. If $\tau \in V_{B_j}$ then $I(\tau) \in U_{B_j}$ and $I'_{-j}(\tau) \in U_{B_j}^k$.
4. If $\tau_1 \in U_{B_k}$ and $\tau_2 \in U_{\Psi_j}$, then $\tau_1 \tau_2 \in V_{\Psi_j}$ and $I''_{-j}(\tau_1 \tau_2) \in U_{\Psi_j}$, for $j \neq \ell$.
5. If $(\tau_1, \tau_2) \in (U_{B_k} \times U_{\Psi_j}) \cup (U_{B_k} \times U_{\Psi_j})$ then $\tau_1 \tau_2 \in V_{\Psi_j}$, $I(\tau_1 \tau_2) \in U_{\Psi_j}$ and $I''_{-j}(\tau_1 \tau_2) \in U_{\Psi_j}^m$ for any $m \in \{1, 2\}$.
6. If $\tau_1, \tau_2 \in U_{B_k}$ and $\tau_3 \in U_{\Psi_j}$, then $\tau_1 \tau_2 \tau_3 \in V_{\Psi_j}$, $I(\tau_1 \tau_2 \tau_3) \in U_{\Psi_j}$, and $I''_{-j}(\tau_1 \tau_2 \tau_3) \in U_{\Psi_j}^m$ for any $m \in \{1, 2\}$.

We furthermore decree that $\tau \bar{\tau} = \bar{\tau} \tau$, $I(X^k) = I'_{j}(X^k) = I''_{-j}(X^k) = 0$, and $\tau \mathbf{1} = \tau$. We then define

$$\mathcal{F} \equiv \bigcup_{j, k \in \{1, 2\}} \left( U_{B_j} \cup U_{\Psi_k} \cup U_{B_k}^j \cup U_{\Psi_k}^j \cup V_{B_j} \cup V_{\Psi_j} \right).$$

For each formal expression $\tau$, the homogeneity $|\tau| \in \mathbb{R}$ is assigned in the following way. Set

$$|\Xi_{\xi_j}| = |\Xi_{\zeta_j}| = -2 - \bar{k} \quad j \in \{1, 2\}$$

Stochastic Quantization 1469
where $\kappa > 0$ is a fixed small number, and we define the homogeneity for every formal expression by

$$|\tau \bar{\tau}| = |\tau| + |\bar{\tau}|, \quad |I(\tau)| = |\tau| + 2, \quad |I'_j(\tau)| = |I'_j(\bar{\tau})| = |\tau| + 1.$$  

The linear space $T$ is then defined as the linear span of $F$, and $T_\alpha$ is the subspace spanned by $\{\tau : |\tau| = \alpha\}$. The product is defined on the regularity structure by construction.

The sector $\langle U_{\Psi_k} \rangle$ comes with an abstract derivative $D_j : \langle U_{\Psi_k} \rangle \to \langle U_{\Psi_j} \rangle$ and likewise $D_-j : \langle U_{B_k} \rangle \to \langle U_{-jB_k} \rangle$ which decrease homogeneity by 1. By a simple power-counting argument, one finds that as long as $\kappa < 1/4$, the sets $\{\tau \in F : |\tau| < \gamma\}$ are finite for every $\gamma \in \mathbb{R}$, reflecting the fact that the Eq. (3.15) in two space dimensions is subcritical.

We will represent the relevant elements in graphic notations. We use $\circ$ and $\bullet$ to represent the expressions $/Xi_{1\xi}^j$ and $/Xi_{1\xi}^j$ respectively, and $\bigcirc$ and $\bigotimes$ to represent the expressions $/Xi_{1\xi}^j$ and $/Xi_{1\xi}^j$ respectively, and also use $-$ to represent the operator $I$, and $\bigcirc$ or $\bigotimes$ to represent the operator $I'_j$ or $I'_{-j}$. Joining of formal expressions by their roots denotes their product. For example, $\circ \bigotimes = I(\bigotimes_{1\xi}^j)I'_j(\bigotimes_{1\xi}^j)$.

The elements in $F$ are completely determined by the above recursive construction. We list some of these elements which have negative homogeneities other than $/Xi_{1\xi}^j, /Xi_{1\xi}^j$ that will be useful later.

At the homogeneity $-1 - 2\kappa$ we have

$$\begin{align*}
\begin{array}{c}
\bigotimes \\
\bigotimes
\end{array}
\end{align*} \quad \text{(where } k \neq \ell) \quad \begin{array}{c}
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\end{array} \quad \begin{array}{c}
\bigotimes \\
\bigotimes
\end{array} \quad (4.1)
\end{align*}$$

At the homogeneity $-1 - \kappa$ we have

$$\begin{array}{c}
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\bigotimes
\end{array} \quad \begin{array}{c}
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\end{array} \quad \begin{array}{c}
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\bigotimes
\end{array}
\end{array}$$

At the homogeneity $-3\kappa$ we have

$$\begin{array}{c}
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\end{array} \quad (4.2)
$$

One can find many other elements of negative homogeneities in $F$ in Sect. 5.1.

One can also build a structure group $G$ acting on $T$ as done in [Hai14] in such a way that the operation $I$ satisfies $I\Gamma \tau - \Gamma I \tau \in T_{\text{poly}}$ for every $\tau \in T$ and $\Gamma \in G$, and $\Gamma(\tau \bar{\tau}) = (\Gamma \tau)(\Gamma \bar{\tau})$ for every $\Gamma \in G$ and $\tau, \bar{\tau} \in T$ with $\tau \bar{\tau} \in T$, and $G$ acts on $T_{\text{poly}}$ by translations.

A problem with this regularity structure $T = (T, G)$ is that $\Xi_{\xi_j}, \Xi_{\xi_j}$ and the elements in (4.1) will not be realized by our model as processes which are continuous in time. To circumvent this problem we actually work with a smaller regularity structure $\hat{T} \subset T$ as in [HM18, Section 3.2]. Instead of generating the regularity structure from $\Xi_{\xi_j} \in V_{B_j}$ and $\Xi_{\xi_j} \in V_{\Psi_j}$ and the abstract polynomials in these sets, we actually define a set of generating elements $F^{\text{gen}}$ which consists of the abstract polynomials and the following symbols

$$\begin{align*}
\begin{array}{c}
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\bigotimes \\
\bigotimes \\
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\bigotimes \\
\bigotimes \\
\bigotimes
\end{array} \quad (4.2)
\end{align*}$$

for $j, k, \ell \in \{1, 2\}$ where in the last symbol $k \neq \ell$. We then define the set $\hat{F}$ as the subset of $F$ generated by $F^{\text{gen}}$ via the aforementioned procedure. The advantage of
this manipulation is that \( \Xi_{ij}, \Xi_{ij} \) and elements (4.1) are not included in \( \hat{\mathcal{F}} \). These elements could not be realized as processes continuous in time if they were included in our regularity structure. The symbols \( \otimes \) and \( \mathfrak{m} \) have homogeneity \( -\kappa \) and the other symbols have homogeneity \( 1 - 2\kappa \).

We then define the truncated regularity structure \( \hat{\mathcal{T}} = (\hat{T}, \mathcal{G}) \) with \( \hat{T} \overset{\text{def}}{=} \text{span}\{ \tau \in \hat{\mathcal{F}} : |\tau| \leq r \} \subset T \), which is closed under \( \mathcal{G} \) [HM18, Remark 3.5].

4.2. Inhomogeneous models. Given a regularity structure \( \mathcal{T} = (T, \mathcal{G}) \), recall from [HM18] that an inhomogeneous model \( (\Pi, \Gamma, \Sigma) \) consists of the following three elements. First, a collection of maps \( \Gamma^{t} : \mathbb{R}^{2} \times \mathbb{R}^{2} \to \mathcal{G} \), parametrized by \( t \in \mathbb{R} \), such that

\[
\Gamma_{xx}^{t} = 1, \quad \Gamma_{xy}^{t} = \Gamma_{yx}^{t}, \quad \forall x, y, z \in \mathbb{R}^{2}, \quad t \in \mathbb{R} \tag{4.3}
\]

and their action on \( \mathcal{F}_{\text{pol}} \) is given by \( \Gamma_{xy}^{t} X^{k} = (X - (0, y - x))^{k} \). Secondly, a collection of maps \( \Sigma_{x} : \mathbb{R} \times \mathbb{R}^{2} \to \mathcal{G} \), parametrized by \( x \in \mathbb{R}^{2} \), such that

\[
\Sigma_{xx}^{st} = 1, \quad \Sigma_{x} \Sigma_{x}^{st} = \Sigma_{x}^{st}, \quad \Sigma_{x} \Gamma_{xy}^{s} = \Gamma_{xy}^{r} \Sigma_{x}^{st}, \quad \forall x \in \mathbb{R}^{2}, \quad s, r, t \in \mathbb{R} \tag{4.4}
\]

and \( \Sigma_{x}^{st} X^{k} = (X - (t - s, 0))^{k} \). Finally, a collection of linear maps \( \Pi_{x}^{t} : \mathcal{T} \to S'(\mathbb{R}^{2}) \), such that

\[
\Pi_{x}^{t} = \Pi_{x}^{t} \Gamma_{xy}^{t}, \quad (\Pi_{x}^{t} X^{(0,\hat{\kappa})})(y) = (y - x)^{\hat{\kappa}}, \quad (\Pi_{x}^{t} X^{(k_{0},\hat{\kappa})})(y) = 0 \tag{4.5}
\]

for all \( x, y \in \mathbb{R}^{2}, t \in \mathbb{R}, \hat{\kappa} \in \mathbb{N}^{2}, k_{0} \in \mathbb{N} \) such that \( k_{0} > 0 \). Moreover, it requires that for any \( \gamma > 0 \) and every \( T > 0 \), there is a constant \( C \) such that

\[
|\langle \Pi_{x}^{t} \tau, \varphi_{x}^{\lambda} \rangle| \leq C \|\tau\| \lambda^{l}, \quad \|\Gamma_{xy}^{t} \tau\|_{m} \leq C \|\tau\||x - y||^{l-m}, \tag{4.6a}
\]

\[
\|\Sigma_{x}^{st} \tau\|_{m} \leq C \|\tau\| |t - s|^{(l-m)/2}, \tag{4.6b}
\]

uniformly over all \( \tau \in \mathcal{T}_{l} \), with \( l \in \mathcal{A} \) and \( l < \gamma \), all \( m \in \mathcal{A} \) such that \( m < l \), all \( \lambda \in (0, 1) \), all \( \varphi \in \mathcal{B}_{0}^{r}(\mathbb{R}^{2}) \) with \( r > -\|\min \mathcal{A}\| \), and all \( t, s \in [-T, T] \) and \( x, y \in \mathbb{R}^{2} \) such that \( |t - s| \leq 1 \) and \( |x - y| \leq 1 \). In addition, we say that the map \( \Pi \) has time regularity \( \delta > 0 \), if the bound

\[
|\langle (\Pi_{x}^{t} - \Pi_{x}^{s}) \tau, \varphi_{x}^{\lambda} \rangle| \leq C \|\tau\||t - s|^{\delta/2} \lambda^{l-\delta}, \tag{4.7}
\]

holds for all \( \tau \in \mathcal{T}_{l} \) and the other parameters as before. We can also define a discrete inhomogeneous model \( (\Pi^{e}, \Gamma^{e}, \Sigma^{e}) \) which consists of the collections of maps

\[
\Pi_{x}^{e} : \mathcal{T} \to \mathbb{R}^{\Lambda_{e}}, \quad \Gamma_{x}^{e} : \Lambda_{e} \times \Lambda_{e} \to \mathcal{G}, \quad \Sigma_{x}^{e} : \mathbb{R} \times \mathbb{R} \to \mathcal{G},
\]

which have all the algebraic properties of their continuous counterparts (4.3), (4.4) and (4.5) with the spatial variables restricted to the grid \( \Lambda_{e} \). Additionally,\(^{13}\) we require \( (\Pi_{x}^{e} \tau)(x) = 0 \), for all \( \tau \in \mathcal{T}_{l} \) with \( l > 0 \), and all \( x \in \Lambda_{e} \) and \( t \in \mathbb{R} \).

For a model \( Z = (\Pi, \Gamma, \Sigma) \), we denote by \( \|\Pi\|_{Y:T}, \|\Gamma\|_{Y:T} \) and \( \|\Sigma\|_{Y:T} \) the smallest constants \( C \) such that the bounds on \( \Pi, \Gamma \) and \( \Sigma \) in (4.6a) and (4.6b) hold. We then define \( \|Z\|_{Y:T} \overset{\text{def}}{=} \|\Pi\|_{Y:T} + \|\Gamma\|_{Y:T} + \|\Sigma\|_{Y:T} \) and we can define the “distance” \( \|Z; \hat{Z}\|_{Y:T} \)

\(^{13}\) This does not follow automatically from the discrete analogue of (4.6a) since these are only assumed to hold for test functions at scale \( \lambda \geq \varepsilon \).
between two models by taking the corresponding differences (in \(\|\cdot\|_{Y,T}\) norms) of \((\Pi, \Gamma, \Sigma)\). To take time regularity into account we also set \(\|\Pi\|_{\delta,Y,T} \equiv \|\Pi\|_{Y,T} + C\), where \(C\) is the smallest constant such that the bound (4.7) holds, and we define \(\|Z\|_{\delta,Y,T} \equiv \|\Pi\|_{\delta,Y,T} + \|\Pi\|_{Y,T} + \|\Sigma\|_{Y,T}\), and the “distance” \(\|Z; \tilde{Z}\|_{\delta,Y,T}\) is defined analogously, as in [HM18, Remark 2.5].

In the discrete setting we define the quantities \(\|\Pi^t\|_{Y,T}^{(e)}, \|\Gamma^t\|_{Y,T}^{(e)}\) and \(\|\Sigma^t\|_{Y,T}^{(e)}\) to be the smallest constants \(C\) such that the bounds (4.6a), (4.6b) hold uniformly in \(x, y \in \Lambda_{\varepsilon}\), \(\lambda \in [\varepsilon, 1]\) and the other parameters as in the continuum setting, and measure the time regularity of \(\Pi^t\) as in (4.7), with the discrete pairing (2.21) in place of the standard one, and \(|t - s|^2 \sim \varepsilon\) in place of \(|t - s|^2\). The other norms and distances \(\|\cdot\|_{Y,T}^{(e)}, \|\cdot\|_{\delta,Y,T}^{(e)}\) and \(\|\cdot\|_{\delta,Y,T}^{(e)}\) are defined by analogy with their continuous counterparts, as in [HM18, Section 4.1].

With the specific regularity structure defined above, we define a discrete model \(Z^t(\varepsilon) = (\Pi^t, \Gamma^t, \Sigma^t)\) on \(\mathcal{F}^{gen}\) as follows. Let \(K^t\) be the truncated discrete heat kernel as given in [HM18, Lemma 5.4]. Define

\[
\Pi^t_{x,y}(y) = (K^t *_{x,y} \xi^t_{x}(t, y)), \quad \Pi^t_{x,t}(y) = (K^t *_{x} \xi^t(t, y)),
\]

where recall that \(*_{x,y}\) denotes the convolutions on \(\mathbb{R} \times \Lambda_{\varepsilon}\); note that the expressions on the right hand sides do not depend on the base point \(x\). The group elements \(\Gamma^t_{xy}, \Sigma^t_{st}\) act on the above symbols as identities. Furthermore, we denote the functions on \(\mathbb{R} \times \Lambda_{\varepsilon}\)

\[
Y^t_{kj} = K^t *_{x} \left( (K^t *_{x} \xi^t_{k})(\nabla_{j}K^t *_{x} \xi^t_{k}) \right),
\]

\[
\tilde{Y}^t_{kj} = K^t *_{x} \left( (K^t *_{x} \xi^t_{k})(\nabla_{j}K^t *_{x} \xi^t_{k}) \right),
\]

\[
\tilde{Y}^t_{kj} = K^t *_{x} \left( (\nabla_{k}K^t *_{x} \xi^t_{k})(K^t *_{x} \xi^t_{k}) \right)
\]

and set for \(k \neq \ell\)

\[
(\Pi^t_{x,y}(y)) = Y^t_{kj}(t, y) - Y^t_{kj}(t, x), \quad \Gamma^t_{xy}(y) = \nabla_{y} - \left( Y^t_{kj}(t, y) - Y^t_{kj}(t, x) \right) 1,
\]

\[
\Sigma^t_{s,t} = \nabla_{s} - \left( Y^t_{kj}(t, s) - Y^t_{kj}(t, x) \right) 1.
\]

We define the action of \((\Pi^t, \Gamma^t, \Sigma^t)\) on \(\mathcal{F}^{gen}_{\delta,\varepsilon}\) in the same way with \(\tilde{Y}^t\) and \(\tilde{Y}^t\).

We then extend the model to the entire \(\mathcal{F}\) in the canonical way. For elements of the form \(\tau_{1}\tau_{2}\) we define

\[
(\Pi^t_{x,y}(\tau_{1}\tau_{2})) = (\Pi^t_{x,y}(\tau_{1}))(y) \cdot (\Pi^t_{x,y}(\tau_{2}))(y) \quad (4.10a)
\]

\[
\Gamma^t_{xy}(\tau_{1}\tau_{2}) = \left( \Gamma^t_{xy}(\tau_{1}) \right) \left( \Gamma^t_{xy}(\tau_{2}) \right), \quad \Sigma^t_{s,t}(\tau_{1}\tau_{2}) = \left( \Sigma^t_{s,t}(\tau_{1}) \right) \left( \Sigma^t_{s,t}(\tau_{2}) \right)
\]

(4.10b)

for all \(x, y, s, t \in \mathbb{R}\).

For the symbols of the form \(\mathcal{I}\tau\) with \(\tau \in \mathcal{I}_{\varepsilon}\) we define the actions by the models as follows.

\[
\Pi^t_{x,y}(\mathcal{I}\tau)(y) \equiv \int_{\mathcal{R}} \langle \Pi^t_{x,y} \Sigma^t_{x,\tau}, K^t_{I-} \rangle ds - \Pi^t_{x,y}(\mathcal{I}_{I_{\varepsilon}}^{\tau})(y),
\]
where \[ \mathcal{J}_{t,x} \overset{\text{def}}{=} \sum_{|k|_a < \alpha + 2} \frac{X^k}{k!} \int_{\mathbb{R}} (\Pi^e(x) \Sigma^e_{x,t} x, D^e K_{t-s}(x - \cdot)) ds. \] (4.11)

Here \( k \in \mathbb{N}^3 \) and the derivative \( D^k = \partial_{t}^k \left( \nabla^e_1 \right)^k \left( \nabla^e_2 \right)^k \). Moreover, we require that

\[
\begin{align*}
\Gamma^{e,t}_{x,y} (\mathcal{I} \tau) &= (\mathcal{I} + \mathcal{J}_{t,x}) \Gamma^{e,t}_{x,y} \tau - \Gamma^{e,t}_{x,y} (\mathcal{J}_{t,x} \tau), \\
\Sigma^{e,t}_{x} (\mathcal{I} \tau) &= (\mathcal{I} + \mathcal{J}_{s,x}) \Sigma^{e,t}_{x} \tau - \Sigma^{e,t}_{x} (\mathcal{J}_{t,x} \tau),
\end{align*}
\] (4.12)

for all \( s, t \in \mathbb{R} \) and \( x, y \in \Lambda_\varepsilon \). For elements of the form \( \mathcal{I} \tau \) the model action is defined in the same way as (4.11) with \( K^e \) replaced by \( \nabla^e_{\pm j} K^e \) and with \( k \) sums over \( |k|_a < \alpha + 1 \). It satisfies

\[
\Pi^{e,t}_{x} (\mathcal{D}_{\pm j} (\mathcal{I} \tau) ) (y) = \Pi^{e,t}_{x} (\mathcal{I} \tau) (y) = \nabla^e_{\pm j} (\Pi^{e,t}_{x} (\mathcal{I} \tau) ) (y) \quad \forall x, y \in \Lambda_\varepsilon, \ t \in \mathbb{R},
\] (4.13)

where \( \mathcal{D}_{\pm j} \) is defined in Sect. 4.1.

We call a discrete model \( Z^e = (\Pi^e, \Gamma^e, \Sigma^e) \) defined on \( \mathcal{T} \) admissible, if it satisfies the identities (4.10b), (4.11) and (4.12); the set of admissible models is denoted by \( \mathcal{M}_\varepsilon \). We also denote by \( \mathcal{M}_0 \) the set of admissible models defined in continuum with all the discrete maps and scalar products in (4.10b), (4.11) and (4.12) replaced by their continuous counterparts. For each \( \varepsilon > 0 \), \( (\Pi^e, \Gamma^e, \Sigma^e) \) defined above is an admissible discrete inhomogeneous model.

4.3. Modeled distributions. We recall from [HM18] the notion of modeled distributions, both in continuum and on a grid. Given a regularity structure \( \mathcal{T} = (\mathcal{T}, \mathcal{G}) \) with an inhomogeneous model \( Z = (\Pi, \Gamma, \Sigma), \gamma, \eta \in \mathbb{R} \) and \( T > 0 \), we consider maps \( H : (0, T) \times \mathbb{R}^d \to \mathcal{T}_{<\gamma} \) and define

\[
\| H \|_{\gamma, \eta; T} \overset{\text{def}}{=} \sup_{t \in (0,T)} \sup_{x \in \mathbb{R}^d} \sup_{l < \gamma} \| H_t(x) \|_l + \sup_{x \in \mathbb{R}^d} \sup_{l \leq \gamma} \sup_{i \in \mathcal{A}} \sup_{t \in (0,T)} \frac{\| H_t (x) - \nabla^e_{x,y} H_t (y) \|_l}{\| t \|_0^{\gamma} |x - y|^{-1}}.
\] (4.14)

where \( |t|_0 \overset{\text{def}}{=} |t|^{1/2} \wedge 1 \). The space of inhomogeneous modeled distributions \( \mathcal{D}^{\gamma, \eta}_T (Z) \) consists of all such functions \( H \) such that

\[
\| H \|_{\gamma, \eta; T} \overset{\text{def}}{=} \| H \|_{\gamma, \eta; T} + \sup_{s \neq t \in (0,T)} \sup_{x \in \mathbb{R}^d} \sup_{l \leq \gamma} \frac{\| H_t(x) - \Sigma^e_{x,s} H_s(x) \|_l}{|t - s|_{0}^{-\gamma} |t - s|^{(\gamma - 1)/2}} < \infty.
\] (4.15)

Here \( |t, s|_0 \overset{\text{def}}{=} |t|_0 \wedge |s|_0 \). For a discrete inhomogeneous model \( Z^e = (\Pi^e, \Gamma^e, \Sigma^e) \), and for a function \( H : (0, T) \times \Lambda_\varepsilon \to \mathcal{T}_{<\gamma} \), we define \( \| H \|_{\gamma, \eta; T} \) and \( \| H \|_{\gamma, \eta; T} \) the same way as (4.14) and (4.15) but with spatial variables restricted to the grid \( \Lambda_\varepsilon \), \( |t|_0 \) replaced by \( |t|_e \overset{\text{def}}{=} |t|_0 \wedge \varepsilon \) and \( |t, s|_0 \) replaced by \( |t, s|_e \overset{\text{def}}{=} |s|_e \wedge |t|_e \). We call such functions \( H \) discrete modeled distributions and denote the space for them by \( \mathcal{D}^{\gamma, \eta}_e, T (Z^e) \). Both \( \mathcal{D}^{\gamma, \eta}_e, T \) and \( \mathcal{D}^{\gamma, \eta}_e, T \) come with products that are defined pointwise. The space \( \mathcal{D}^{\gamma, \eta}_e, T \) comes with derivative operators still denoted by \( \mathcal{D}_j \) and \( \mathcal{D}_{-j} \) for \( j \in \{1, 2\} \) which act on modeled distributions pointwise. On the space \( \mathcal{D}^{\gamma, \eta}_e, T \), the operator \( \mathcal{D}_j \) also acts pointwise as before, while the
linear operator $D_j$ acts pointwise on all elements except on polynomials in a slightly nonlocal way: for $(\nabla \varphi, X) \overset{\text{def}}{=} \sum_{j=1}^{d} \nabla_j \varphi X_j$ with $\nabla \varphi = (\nabla_1 \varphi, \ldots, \nabla_d \varphi) : Z^d \to \mathbb{R}^d$, one has $(D_j(\nabla \varphi, X))(x) \overset{\text{def}}{=} -\nabla_j \varphi (x - e_j)$. We can check that

$$\Pi_x^{s,t} D_j(\nabla \varphi, X) = \nabla_j \Pi_x^{s,t} \nabla \varphi, \Pi_x^{s,t} = (\nabla_j \varphi)(x),$$

$$\Pi_x^{s,t} D_j(\nabla \varphi, X) = -\nabla_j \varphi(x - e_j).$$

We will also sometimes write $D^{\eta}_{0,T}$ which stands for $D^{\eta}_{T}$.

One can compare two modeled distributions $H \in D^{\eta}_{T}(Z)$ and $\tilde{H} \in D^{\eta}_{T}(\tilde{Z})$ by the distance $|||H; \tilde{H}|||_{\gamma;\eta;T}$, as well as two discrete modeled distributions $H \in D^{\eta}_{\epsilon,T}(Z^\epsilon)$ and $\tilde{H} \in D^{\eta}_{\epsilon,T}(\tilde{Z}^\epsilon)$ by the distance $|||H; \tilde{H}|||_{\gamma;\eta;T}$ defined in the natural way as in [HM18, Section 2.2 and 4.1].

The inhomogeneous modeled distributions can be “realized” as $C^\alpha$ distributions. By [HM18, Theorem 2.11] or [Hai14. Theorem 3.10], letting $\alpha \overset{\text{def}}{=} \min A < 0$, for every $\eta \in \mathbb{R}$, $\gamma > 0$ and $T > 0$, there is a unique family of linear operators $R_t : D^{\eta}_{T} \to C^\alpha(\mathbb{R}^d)$, parametrized by $t \in (0, T]$, called reconstruction operators, such that the bound

$$|||\gamma \eta|||_{\gamma;\eta;T} \leq \lambda^\gamma \|H\|_{\gamma;\eta;T} \|\Pi\|_{\gamma;T} \tag{4.16}$$

holds uniformly in $H \in D^{\eta}_{T}, t \in (0, T], x \in \mathbb{R}^d, \lambda \in (0, 1]$ and $\varphi \in \mathcal{B}_0(\mathbb{R}^d)$ with $r > -[\alpha]$. If furthermore the map $\Pi$ has time regularity $\delta > 0$, then by [HM18, Theorem 2.11], for any $\tilde{\delta} \in (0, \delta)$ such that $\tilde{\delta} \leq (m - \xi)$ for all $m, \xi, \gamma \in (-\infty, \gamma) \cap A \cup \{\gamma\}$ such that $\xi < m$, one has $|||\gamma \eta|||_{\gamma;\eta;T} \leq \|\Pi\|_{\gamma;T} (1 + \|\Pi\|_{\gamma^2;T}) |||H|||_{\gamma;\eta;T}$ and $\mathcal{R}$ is locally Lipschitz continuous in $H$ and the model.

**Definition 4.1.** Given a discrete model $Z^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon, \Sigma^\epsilon)$ and a discrete modeled distribution $H$ we define the discrete reconstruction map $R^\epsilon_H$ by $R^\epsilon_H = 0$ for $t \leq 0$, and

$$R^\epsilon_H(x, t) \overset{\text{def}}{=} \Pi^{\epsilon,t}(H(x))(x), \quad (t, x) \in (0, T] \times \Lambda_\epsilon. \tag{4.17}$$

For a discrete model $Z^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon, \Sigma^\epsilon)$ the analogous bound (4.16) holds with discrete pairing, $|||\gamma \eta|||_{\gamma;\eta;T}$ replaces the continuous one, uniformly in $x \in \Lambda_\epsilon, \lambda \in [\epsilon, 1]$ and other variables as in continuum. The map $R^\epsilon$ is also locally Lipschitz continuous in $H$ and the model.

Finally as in [HM18, Remark 4.6] one can compare a discrete model $Z^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon, \Sigma^\epsilon)$ with a continuous model $Z = (\Pi, \Gamma, \Sigma)$ by distance

$$|||\Pi; \Pi^\epsilon|||_{\delta,\gamma;T} \overset{\text{def}}{=} \sup_{\varphi, \lambda, t \in [-T, T]} \lambda^{-\delta} |\langle \Pi^{\epsilon,t}_\lambda \varphi, \varphi^{\epsilon}_\lambda \rangle - \langle \Pi^{\epsilon,t}_\lambda \varphi, \varphi^{\epsilon}_\lambda \rangle| \tag{4.18}$$

where

$$|||\Pi; \Pi^\epsilon|||_{\delta,\gamma;T} \overset{\text{def}}{=} \sup_{\varphi, \lambda, t \in [-T, T]} \lambda^{-\delta} \left(\int_{\gamma_1}^{\gamma_2} \frac{1}{|t-s|^{\frac{1}{2}}} \right)^{\gamma_1}.$$
where \( \varphi \in \mathcal{B}_0^\epsilon, x \in \Lambda_\epsilon, \lambda \in [\epsilon, 1], l < \gamma \) and \( \tau \in \mathcal{T}_l \) with \( \| \tau \| = 1 \), and \( \| \Gamma; \Gamma^\epsilon \|_{y, T} \) and \( \| \Sigma; \Sigma^\epsilon \|_{y, T} \) are defined in the analogous way (of course without the need of the second line here that measures time regularity). We can also compare discrete and continuous modeled distributions in the analogous way by \( \| H; H^\epsilon \|_{y, \eta, T} \). With these notions for \( H \in \mathcal{D}^\epsilon_T(Z) \) and a discrete modeled distribution \( H^\epsilon \) one can prove the estimate

\[
\| RH; \mathcal{R}^\epsilon H^\epsilon \|_{\ell_\delta, \alpha, \eta} \lesssim \| H; H^\epsilon \|_{y, \eta, T} + \| Z; Z^\epsilon \|_{y, \eta, T} + \epsilon^\theta, \tag{4.19}
\]

for \( \delta > 0 \) and \( \theta > 0 \) small enough.

We can define a nonlocal operator \( \mathcal{P}^\epsilon \) on the space \( \mathcal{D}^\epsilon_{y, T} \) such that

\[
\mathcal{R}^\epsilon_t \mathcal{P}^\epsilon = \mathcal{P}^\epsilon \ast \mathcal{R}^\epsilon_t \tag{4.20}
\]

where \( \mathcal{P}^\epsilon \) is the discrete heat kernel, see [HM18, Eq.(4.35)(5.18)] for definition of \( \mathcal{P}^\epsilon \) which utilizes a decomposition of \( P^\epsilon \) given in [HM18, Eq.(5.13)]. We can also define an operator \( \mathcal{P}^\epsilon_{-k} \equiv \mathcal{D}_k \mathcal{P}^\epsilon \).

Fixing \( \alpha \in (-\frac{4}{3}, -1), \gamma > |\alpha|, \eta \in (-\frac{1}{2}, 1) \), we now formulate the abstract fixed point problem in the space \( \mathcal{D}^\epsilon_{y, T} \) for the system of Eq. (3.15). Set

\[
\hat{F}_{B_j}^\epsilon(B^\epsilon, \Psi^\epsilon) \equiv \lambda \left( \Psi^\epsilon_1 \mathcal{D}_j \Psi_2^\epsilon - \Psi_2^\epsilon \mathcal{D}_j \Psi_1^\epsilon \right) - \lambda^2 \sum_{k=1,2} B^\epsilon_k(\Psi^\epsilon_k)^2 - \lambda \left( \Psi^\epsilon_1 \mathcal{D}_j \Psi_2^\epsilon - \Psi_2^\epsilon \mathcal{D}_j \Psi_1^\epsilon \right) \tag{4.19}
\]

\[
\hat{F}_{\Psi_j}^\epsilon(B^\epsilon, \Psi^\epsilon) \equiv -(1)^j \lambda \sum_{k=1,2} \left( B^\epsilon_k \mathcal{D}_k \Psi^\epsilon_\ell + \Psi^\epsilon_\ell \mathcal{D}_k B^\epsilon_k \right) - \lambda^2 \sum_{k=1,2} (B^\epsilon_k)^2 \Psi^\epsilon_\ell 
+ (1)^j \lambda \sum_{k \neq j, \ell \neq j} \left( \Psi^\epsilon_k \mathcal{D}_k \Psi^\epsilon_\ell + \Psi^\epsilon_\ell \mathcal{D}_k \Psi^\epsilon_k \right) \tag{4.21}
\]

for modeled distributions \( B^\epsilon_j \) taking values in \( \mathcal{U}_{B_j} \) and \( \Psi^\epsilon_j \) taking values in \( \mathcal{U}_{\Psi_j} \).

The abstract fixed point problem then reads

\[
B^\epsilon_j = \mathcal{P}^\epsilon \left( \hat{F}_{B_j}^\epsilon(B^\epsilon, \Psi^\epsilon) + R^\epsilon_{B_j} \right) + \mathcal{S}^\epsilon \hat{A}_j + \mathcal{I} + \lambda \left( \Psi^\epsilon_1 - \Psi^\epsilon_2 \right) + \tilde{R}_{B_j}^\epsilon \tag{4.22a}
\]

\[
\Psi^\epsilon_j = \mathcal{P}^\epsilon \left( \hat{F}_{\Psi_j}^\epsilon(B^\epsilon, \Psi^\epsilon) + R^\epsilon_{\Psi_j} \right) + \sum_k \mathcal{P}^\epsilon_{-k} \hat{F}^{(k)}_{\Psi_j}(B^\epsilon, \Psi^\epsilon) 
+ \mathcal{S}^\epsilon \hat{\Phi}_j^\epsilon + \mathcal{I} - (1)^j \lambda \sum_{k=1,2} \left( \Psi^\epsilon_{k\ell} - \Psi^\epsilon_{\ell k} \right) + \tilde{R}_{\Psi_j}^\epsilon \tag{4.22b}
\]
where $\mathcal{Y}, \tilde{\mathcal{Y}}, \tilde{\mathcal{Y}}$ are defined the same way as $Y, \tilde{Y}, \tilde{Y}$ in (4.8), but with a slight tweak that the left-most $K^\varepsilon$ appearing in each equation of (4.8) is replaced by the un-truncated heat kernel $P^\varepsilon$. Moreover $S^\varepsilon$ is the discrete semi-group so that $S^\varepsilon \hat{A}^e_j, S^\varepsilon \hat{\Phi}^e_j$ are solutions to the discrete heat equation with initial conditions $\hat{A}^e_j, \hat{\Phi}^e_j$ respectively, and they are naturally lifted into $D_{\varepsilon,T}^{Y,\eta}$ as [HM18, Lemma 3.6]. Finally the “remainder” terms are defined as

\[ R^\varepsilon_{B_j} (x) \overset{\text{def}}{=} R^\varepsilon_{B_j} (x; \mathcal{R}^\varepsilon E_i, \mathcal{R}^\varepsilon \Psi^\varepsilon) \mathbf{1}, \quad \text{(4.23a)} \]

\[ R^\varepsilon_{\Psi_j} (x) \overset{\text{def}}{=} R^\varepsilon_{\Psi_j} (x; \mathcal{R}^\varepsilon E_i, \mathcal{R}^\varepsilon \Psi^\varepsilon) \mathbf{1}, \quad \text{(4.23b)} \]

\[ \tilde{R}^\varepsilon_{B_j} \overset{\text{def}}{=} -\lambda^2 P^\varepsilon \ast \varepsilon \left( \sum_{k=1,2} (\mathcal{R}^\varepsilon B_k^\varepsilon) (\nabla^\varepsilon \Psi^\varepsilon_k) (\mathcal{R}^\varepsilon \Psi^\varepsilon_k) - 2c^\varepsilon B^\varepsilon \right) \circ \langle \mathbf{1}, \mathbf{X} \rangle, \quad \text{(4.23c)} \]

\[ \tilde{K}^\varepsilon_{\Psi_j} \overset{\text{def}}{=} -\lambda^2 P^\varepsilon \ast \varepsilon \left[ \sum_{k=1,2} \frac{\varepsilon}{2} \left( (\mathcal{R}^\varepsilon B_k^\varepsilon) (\nabla^\varepsilon \mathcal{R}^\varepsilon \Psi^\varepsilon_k) + (\mathcal{R}^\varepsilon B_k^\varepsilon) (\nabla^\varepsilon \mathcal{R}^\varepsilon \Psi^\varepsilon_k) \right) + (\mathcal{R}^\varepsilon B_k^\varepsilon) \right] \circ \langle \mathbf{1}, \mathbf{X} \rangle \quad \text{(4.23d)} \]

with $R^\varepsilon_{B_j}$ and $R^\varepsilon_{\Psi_j}$ defined in (3.16). We will specify the constants $c^\varepsilon_B$ and $c^\varepsilon_\Psi$ later but let’s immediately remark that they will converge to finite limits. Here we used a shorthand notation (where $\nabla$ denotes the gradient):

\[ F \circ \langle \mathbf{1}, \mathbf{X} \rangle \overset{\text{def}}{=} F \mathbf{1} + \langle \nabla F, \mathbf{X} \rangle. \quad \text{(4.24)} \]

We will renormalize the models in the next section. For now given that the abstract fixed point problem is locally wellposed we give the equation we obtain using the reconstruction operator (Definition 4.1) for the un-renormalized models:

**Lemma 4.2.** Suppose that $c^\varepsilon_\Psi = c^\varepsilon_B = 0$, and that $(B^\varepsilon, \Psi^\varepsilon)$ is a solution to the abstract fixed point problem (4.22). Then $(B^\varepsilon, \Psi^\varepsilon) \overset{\text{def}}{=} (\mathcal{R}^\varepsilon E_i B^\varepsilon, \mathcal{R}^\varepsilon \Psi^\varepsilon)$ satisfy (3.15) with $C^{(e)} = 0$.

We will turn on the constants $c^\varepsilon_\Psi, c^\varepsilon_B$ in Lemma 5.2 when we do renormalizations.

**Proof.** This can be readily checked by action of reconstruction $\mathcal{R}^\varepsilon$ for the above canonical model on both sides of (4.22), together with the following properties (1) using the property $\mathcal{R}^\varepsilon \mathcal{P}^\varepsilon = P^\varepsilon \ast \varepsilon \mathcal{R}^\varepsilon$ in (4.20) one can take $\mathcal{R}^\varepsilon$ inside $\mathcal{P}^\varepsilon$ in (4.22), (2) the definition of reconstruction operator $\mathcal{R}^\varepsilon_i$ in (4.17) then tells us that the action by $\mathcal{R}^\varepsilon_i$ at a lattice point $x$ is simply given by the map $\Pi^\varepsilon_{\mathcal{R}^\varepsilon_i}$ at $x$ (3) since $\Pi^\varepsilon_{\mathcal{R}^\varepsilon_i}$ is the canonical model we can apply (4.10) (4) and we have the compatibility of the abstract derivative (4.13).

This shows that $\mathcal{R}^\varepsilon_i$ acting on $\tilde{F}_B^\varepsilon_j$ gives

\[ \lambda \left( \Psi_1^\varepsilon (x) \nabla^\varepsilon \Psi_2^\varepsilon (x) - \Psi_2^\varepsilon (x) \nabla^\varepsilon \Psi_1^\varepsilon (x) \right) - \lambda^2 B_j^\varepsilon (x) \sum_{k=1,2} \Psi_k^\varepsilon (x)^2 \]
and similarly the term $\hat{F}_{\psi_j}^{(k)}$ and $\hat{F}_{\psi_j}$ gives
\[ -( -1)^j \lambda \sum_{k=1,2} \left( B_{\psi_j}^{(k)}(x) \nabla_{\psi_j}^{(k)}(x) + \nabla_{\psi_j}^{(k)} B_{\psi_j}^{(k)}(x) \Psi_{\psi_j}^{(k)}(x) - \nabla_{\psi_j}^{(k)}(B_{\psi_j}^{(k)}(x) \Psi_{\psi_j}^{(k)}(x)) \right) \]
\[ -\lambda^2 \sum_{k=1,2} B_{\psi_j}^{(k)}(x)^2 \Psi_{\psi_j}^{(k)}(x) \]
and these are the quadratic terms in (3.15) as well as the “localized” cubic terms therein.

Note then that the “remainder” terms (4.23c) and (4.23d) are defined in order to exactly obtain the slightly nonlocal cubic terms in (3.15) upon action by the reconstruction:
\[
\mathcal{R}^\varepsilon \mathcal{P}^\varepsilon \left( -\lambda^2 \sum_{k=1,2} B_{\psi_j}^{(k)}(x)^2 \right) + \mathcal{R}^\varepsilon \mathcal{R}_{B_{\psi_j}^{(k)}} = -\lambda^2 \sum_{k=1,2} P^\varepsilon \ast_{\varepsilon} \left( B_{\psi_j}^{(k)}(x)^2 + \varepsilon B_{\psi_j}^{(k)}(x) \Psi_{\psi_j}^{(k)}(x) \right)
\]
where the terms in the parenthesis equal to the cubic term $B_{\psi_j}^{(k)}(x) \Psi_{\psi_j}^{(k)}(x + e_j) \Psi_{\psi_j}^{(k)}(x)$ in (3.15a), and by straightforward computation
\[
\mathcal{R}^\varepsilon \mathcal{P}^\varepsilon \left( -\lambda^2 \sum_{k=1,2} (B_{\psi_j}^{(k)})^2 \right) + \mathcal{R}^\varepsilon \mathcal{R}_{\psi_j} = -\frac{\lambda^2}{2} \sum_{k=1,2} P^\varepsilon \ast_{\varepsilon} \left( (B_{\psi_j}^{(k)})^2 - \left( \frac{\partial B_{\psi_j}^{(k)}}{\partial \varepsilon} \cdot \left( \frac{\partial B_{\psi_j}^{(k)}}{\partial \varepsilon} \right) \right) \right)
\]
recovering the cubic terms in (3.15b). □

5. Renormalization and Bounds on the Models

5.1. Renormalization. The models defined in the previous section will not converge as $\varepsilon \to 0$, and to obtain a limit we need to renormalize them. For each $\varepsilon > 0$ let $M_\varepsilon : \mathcal{T} \to \mathcal{T}$ be the map
\[
M_\varepsilon = \exp \left( -C_1^{(k)} L_1^{(k)} - C_2^{(k)} L_2^{(k)} - C_3^{(k)} L_3^{(k)} - C_4^{(k)} L_4^{(k)} - C_5^{(k)} L_5^{(k)} - C_6^{(k)} L_6^{(k)} - C_7^{(k)} L_7^{(k)} \right)
\]
where Einstein’s notation is used (all the indices are summed over {1, 2}), and the renormalization constants are specified below; the nilpotent linear operators (i.e. generators) on $\mathcal{T}$ are determined by the following contraction rules: for $k, j, \ell \in \{1, 2\}$
\[
L_1^{(k)} : \begin{array}{c} \circ \end{array} \to 1, \quad L_2^{(k)} : \begin{array}{c} \circ \end{array} \to 1,
\]
\[
L_3^{(k)} : \begin{array}{c} \circ \end{array} \to 1, \quad L_3^{(-\ell)} : \begin{array}{c} \circ \end{array} \to 1,
\]
\[
L_4^{(k)} : \begin{array}{c} \circ \end{array} \to 1, \quad L_4^{(-\ell)} : \begin{array}{c} \circ \end{array} \to 1,
\]
\[
L_5^{(k)} : \begin{array}{c} \circ \end{array} \to 1, \quad L_5^{(-\ell)} : \begin{array}{c} \circ \end{array} \to 1,
\]
\[
L_6^{(k)} : \begin{array}{c} \circ \end{array} \to 1, \quad L_6^{(-\ell)} : \begin{array}{c} \circ \end{array} \to 1,
\]
\[
L_7^{(k)} : \begin{array}{c} \circ \end{array} \to 1, \quad L_7^{(-\ell)} : \begin{array}{c} \circ \end{array} \to 1.
\]
This should be understood in the sense that if $\tau$ is an arbitrary formal expression, then $L^{(k)}_1 \tau$ is the sum of all formal expressions obtained from $\tau$ by performing a substitution of the type $\otimes \otimes \rightarrow \cdot \cdot$; the specification of the other generators are understood in the same way.

To define the renormalization constants we introduce some graphical notations. Each arrow $\rightarrow$ represents the kernel $K^\varepsilon(y - x)$, and each arrow $\rightarrow_j$ represents the kernel $\nabla_{e_j} K^\varepsilon(y - x)$, with $x$ and $y$ being the starting and end space-time points of the arrow respectively. Each dot $\cdot$ represents a space-time point that is integrated out, and each green dot $\bullet$ represents the origin of space-time which is not integrated. Define the following renormalization constants:

$$
\begin{align*}
C_2^{(\varepsilon)} & \equiv \bullet, \\
C_{3,j}^{k(\varepsilon)} & \equiv \bullet, \\
C_{3,-k}^{k(\varepsilon)} & \equiv \bullet, \\
C_{4,j}^{k(\varepsilon)} & \equiv \bullet, \\
C_{4,-k}^{k(\varepsilon)} & \equiv \bullet, \\
C_{5,j}^{k(\varepsilon)} & \equiv \bullet, \\
C_{6,j}^{k(\varepsilon)} & \equiv \bullet, \\
C_{7,j}^{k(\varepsilon)} & \equiv \bullet.
\end{align*}
$$

Furthermore we define

$$
\begin{align*}
c_B^{\varepsilon} & \equiv \varepsilon \int_{\mathbb{R} \times \Lambda_\varepsilon} K^\varepsilon(-z) \nabla_{e_j} K^\varepsilon(-z) \, dz, \\
C_1^{\varepsilon} & \equiv \sum_{k=1,2} \left( C_{3,j}^{k(\varepsilon)} - C_{4,j}^{k(\varepsilon)} + C_{5,j}^{k(\varepsilon)} - C_{7,j}^{k(\varepsilon)} \right) - c_B^{\varepsilon}, \\
c_{\Psi}^{\varepsilon} & \equiv \varepsilon \left( -j - \varepsilon \right).
\end{align*}
$$

Note that the values of the last three constants do not depend on $j$ since the integrals remain the same values when the spatial coordinates are switched. The choice of $C_1^{\varepsilon}$ is such that in the renormalized equation for $B^\varepsilon$ derived below, all the renormalization constants will precisely cancel out (at the same time we can ensure that the models converge). These constants then determine for each $\varepsilon > 0$ an element $M_\varepsilon$ of the renormalization group. The maps $M_\varepsilon$ acting on the canonical models $Z^\varepsilon$ then yield a sequence of models $\hat{Z}^\varepsilon$ which are called renormalized models. This follows from the general result in [BHZ19].

In fact all the constants with two distinct indices $k \neq j$, converge to finite limits, which follows from the following lemma (“parity”). The other renormalization constants defined in (5.3), (5.4) diverge logarithmically. The constant $c_{\Psi}^{\varepsilon}$ and $c_{\Psi}^{\varepsilon}$ converge to finite but non-zero limits by Lemma 6.5.
Lemma 5.1. For \( k \neq j \), one has
\[
\lim_{\varepsilon \to 0} \int_{(\mathbb{R} \times \Lambda^\varepsilon)^2} K^\varepsilon(x - y) \nabla^\varepsilon_{\pm e_j} K^\varepsilon(x - z) \nabla^\varepsilon_{\pm e_k} K^\varepsilon(z - y) \, dz \, dy < \infty. \tag{5.5}
\]

The inequality also holds with the last factor of the integrand replaced by \( \nabla^\varepsilon_{\pm e_k} K^\varepsilon(y - z) \).

Proof. We only prove the case where the two discrete derivatives in (5.5) are \( \nabla^\varepsilon_{e_j} \) and \( \nabla^\varepsilon_{e_k} \); the other cases follow in the same way. Since the truncated heat kernel \( K^\varepsilon(x(z^{(0)}, x^{(1)}, x^{(2)}) \) for \( x \in \mathbb{R} \times \Lambda^\varepsilon \) is even in both spatial variables \( x^{(1)} \) and \( x^{(2)} \), simultaneously flipping the signs of the \( j \)th spatial variables of \( x \), \( y \), \( z \) in (5.5) we see that the integral in (5.5) is equal to the same integral with \( \nabla^\varepsilon_{e_j} \) replaced by \( \nabla^-_{e_j} \). Therefore (5.5) is equal to 1/2 times
\[
-\varepsilon \int_{(\mathbb{R} \times \Lambda^\varepsilon)^2} K^\varepsilon(x - y) \nabla^-_{e_j} \nabla^-_{e_j} K^\varepsilon(x - z) \nabla^\varepsilon_{e_k} K^\varepsilon(z - y) \, dz \, dy.
\]
It is easy to show that \( \nabla^-_{e_j} \nabla^-_{e_j} K^\varepsilon \) is a kernel of order \(-4\) (see (5.16) below). By Lemma 5.5 below and a bound \( \int_{\mathbb{R} \times \Lambda^\varepsilon} \varepsilon(\|z\|_8 \vee \varepsilon)^{-5} \, dz = O(1) \) we obtain (5.5). \( \square \)

In the previous section, we saw in Lemma 4.2 an “un-renormalized” equation by applying the reconstruction operator \( \mathcal{R}^\varepsilon \) for the model \( \mathcal{Z}^\varepsilon \) on the abstract Eq. (4.22). Its solution would not converge. In the sequel, we will consider the abstract Eq. (4.22) with every incidence of the reconstruction operator \( \mathcal{R}^\varepsilon \) in the remainder terms (4.23) replaced by the reconstruction operator denoted by \( \hat{\mathcal{R}}^\varepsilon \) for the renormalized model \( \hat{\mathcal{Z}}^\varepsilon \).

Lemma 5.2. Suppose that \( (\mathcal{B}^\varepsilon, \Psi^\varepsilon) \) is a solution to the abstract fixed point problem (4.22). Then \( (\hat{\mathcal{R}}^\varepsilon_1 \mathcal{B}^\varepsilon, \hat{\mathcal{R}}^\varepsilon_1 \Psi^\varepsilon) \) satisfy (3.15), with the constant \( C^{(\varepsilon)} \) in (3.15) given by
\[
C^{(\varepsilon)} = -2\lambda^2 C^{(\varepsilon)} + \lambda^2 \sum_{k=1,2} \left( C^{(\varepsilon)}_{3,k} + C^{(\varepsilon)}_{4,-k} \right) + \lambda^2 C^{(\varepsilon)}_{\Psi}. \tag{5.6}
\]

Proof. In view of the abstract Eq. (4.22) (with \( \mathcal{R}^\varepsilon \) in the remainder terms (4.23) replaced by \( \hat{\mathcal{R}}^\varepsilon \)), the solution \( (\mathcal{B}^\varepsilon, \Psi^\varepsilon) \) in \( D_{y,t}^{\mathcal{Y}^\varepsilon,0} \) must have the following expansion truncated at homogeneity 1
\[
\mathcal{B}^\varepsilon_j(x) = \underbrace{b_j(x) \mathbf{1}}_{\text{(1)}} + \lambda \varphi_j(x) \underbrace{\mathbf{1}}_{\text{(2)}} - \lambda \varphi_2(x) \underbrace{\mathbf{1}}_{\text{(3)}} + \lambda \underbrace{\mathbf{1}}_{\text{(4)}} - \lambda \underbrace{\mathbf{1}}_{\text{(5)}} \underbrace{\langle \nabla b_j(x), \mathbf{X} \rangle}_{\text{(6)}} \tag{5.7}
\]
\[
\Psi^\varepsilon_j(x) = \underbrace{\varphi_j(x) \mathbf{1}}_{\text{(7)}} + \langle \nabla \varphi_j(x), \mathbf{X} \rangle - (-1)^{i-1} \lambda \underbrace{\sum_{k=1,2, k \neq j} \left[ b_k(x) \underbrace{\mathbf{1}}_{\text{(8)}} + \varphi_k(x) \underbrace{\mathbf{1}}_{\text{(9)}} - \varphi_2(x) \underbrace{\mathbf{1}}_{\text{10}} \right] - b_k(x) \underbrace{\mathbf{1}}_{\text{11}} + \underbrace{\mathbf{1}}_{\text{12}} + \underbrace{\mathbf{1}}_{\text{13}} - \underbrace{\mathbf{1}}_{\text{14}} \right]}_{\text{15}}
\]
for functions \(^{16}\) \( b_j, \varphi_j : \Lambda^\varepsilon \times \mathbb{R} \to \mathbb{R} \) and \( \nabla b_j, \nabla \varphi_j : \Lambda^\varepsilon \times \mathbb{R} \to \mathbb{R}^2 \) and \( \nabla \varphi_j \) denotes the \( i \)th component of \( \nabla \varphi_j \). The modeled distributions \( \mathcal{D} \Psi^\varepsilon_j \) (resp. \( \mathcal{D}_i \Psi^\varepsilon_j \)) then have expansion derived directly: namely, every symbol gets an extra “derivative” decoration \( i \) (resp. \(-i\)) on the bottom line, and the abstract polynomial part is given by \( \nabla_i \varphi_j(x) \mathbf{1} \) (resp. \( -\nabla_i \varphi_j(x - e_i) \mathbf{1} \)). \( \mathcal{D}_j \mathcal{B}^\varepsilon \) is expanded in the same way.

\(^{16}\) We omit the dependence of \( b_j, \varphi_j, \nabla b_j, \nabla \varphi_j \) on \( \varepsilon \) in our notation.
It is then straightforward to check that the nonlinear term $\hat{F}_{B_j^o} = \hat{F}_{B_j^o}(B^o, \Psi^o)$ defined in (4.21) has the following expression, truncated at homogeneity 0:

$$\hat{F}_{B_j^o}(x) = \sum_k (-1)^k \lambda \left( \varphi_{3-k}(x) \frac{\partial}{\partial x} - \nabla_j \varphi_{3-k}(x) \right) + (\nabla \varphi_{3-k}(x), x) + \varphi_{3-k}(x) \nabla \varphi_k(x) 1 \right) + \lambda^2 \sum_{k, \ell} \left[ (M^o \cdots) \right]$$

$$+ \lambda^2 \sum_{k, \ell} b_k(x) \left[ (k \cdots) - (k \cdots) - (k \cdots) + (k \cdots) + (\cdots) \right] + \lambda^2 \sum_k \left[ (2 \cdots) + b_j(x) (2 \cdots) + \varphi_k(x) 1 + 2 \varphi_k(x) - b_j(x) \right] \right]$$

Here, $k, \ell$ sum over \{1, 2\}, and $(\cdots)$ denotes a linear combination of symbols on which $M^o$ acts trivially (and they arise from suitably modifying symbols in the line above “$(\cdots)$” by abstract polynomials); for instance, it includes a term $-\lambda^2 \varphi_k(x)$, which arises from replacing an incidence of $\varphi$ in $-\lambda^2 \lambda$ by $\varphi_k(x)$. We refrain from listing all these terms because they do not contribute any renormalization at all in the renormalized equation.

Applying the reconstruction operator $\hat{R}^o$ associated to the renormalized model $\hat{\Pi}^o$ on both sides of (4.22), using $(\hat{\Pi}^o \tau)(t, x) = (\hat{\Pi}^o \epsilon M^o \tau)(t, x)$, following the analogous arguments in proof of Lemma 4.2 and noting that

$$\hat{R}^o B_j^o = K^o * \xi_j^o + b_j \quad \hat{R}^o \Psi_j^o = K^o * \xi_j^o + \varphi_j$$

one has that $(\hat{R}^o B_j^o, \hat{R}^o \Psi_j^o)$ satisfy (3.15a) with the following extra terms on the right hand side:

$$2 \lambda^2 \sum_{k=1,2} \left( - C_{3,j}^o + C_{4,j}^o - C_{5,j}^o + C_{7,j}^o \right) \hat{R}^o B_j^o (\cdot) + 2 \lambda^2 c_B \hat{R}^o B_j^o$$

$$+ 2 \lambda^2 C_{1,j}^o \hat{R}^o B_j^o (\cdot) \right)$$

(5.8)

where the factor 2 comes from summing over an index, and the term $2 \lambda^2 c_B \hat{R}^o B_j^o$ arises from the “remainder” term $\hat{R}^o B_j^o$ given in (4.23c).

By definition of $C_{1,j}^o$ given in (5.4) we have

$$\text{(5.8)} = 0.$$

It is also straightforward to check that $\hat{F}_j^o(x) = \sum_{m=1}^4 \hat{F}_{j(m)}^o(x)$ with (indices sum over \{1, 2\} if not specified)
Applying the reconstruction operator where again (\(3.15b\)) with the following extra terms on the right hand side

\[
\begin{align*}
\hat{f}_{\psi_f}^{(1)}(x) & \equiv (-1)^j \lambda \sum_{k=1,\ell \neq j} \left[ \nabla_k \psi_\ell(x) \Psi_{1,\epsilon} + b_k(x) \frac{\partial}{\partial x} + \psi_\ell(x) \frac{\partial}{\partial x} + \nabla_x b_k(x) \frac{\partial}{\partial x} \right] \\
+ (b_k(x) \nabla_k \psi_\ell(x) - \nabla_k b_k(x) - e_k(x) \psi_\ell(x)) & 1 + \left( \nabla b_k(x) \right) \frac{\partial}{\partial x} \left( \nabla \psi_\ell(x) \right) \frac{\partial}{\partial x} \\
\hat{f}_{\psi_f}^{(2)}(x) & \equiv \lambda^2 \sum_{k,\ell} \left( \begin{array}{c}
\nabla_k \psi_\ell(x) \\
\psi_\ell(x)
\end{array} \right) + \left( \begin{array}{c}
\nabla_k \psi_\ell(x) \\
\psi_\ell(x)
\end{array} \right) + \left( \begin{array}{c}
\nabla_k \psi_\ell(x) \\
\psi_\ell(x)
\end{array} \right) - \left( \begin{array}{c}
\nabla_k \psi_\ell(x) \\
\psi_\ell(x)
\end{array} \right) + \left( \begin{array}{c}
\nabla_k \psi_\ell(x) \\
\psi_\ell(x)
\end{array} \right) + \left( \begin{array}{c}
\nabla_k \psi_\ell(x) \\
\psi_\ell(x)
\end{array} \right)
\right) + (\cdots)
\right)
\end{align*}
\]

\[
\begin{align*}
\hat{f}_{\psi_f}^{(3)}(x) & \equiv \lambda^2 \sum_{k,\ell} \left( \begin{array}{c}
\nabla_k \psi_\ell(x) \\
\psi_\ell(x)
\end{array} \right) - \left( \begin{array}{c}
\nabla_k \psi_\ell(x) \\
\psi_\ell(x)
\end{array} \right) + \left( \begin{array}{c}
\nabla_k \psi_\ell(x) \\
\psi_\ell(x)
\end{array} \right) - \left( \begin{array}{c}
\nabla_k \psi_\ell(x) \\
\psi_\ell(x)
\end{array} \right) + \left( \begin{array}{c}
\nabla_k \psi_\ell(x) \\
\psi_\ell(x)
\end{array} \right) + (\cdots)
\right)
\end{align*}
\]

where again (\(\cdots\)) denotes linear combination of symbols on which \(M_\epsilon\) acts trivially, which arise from suitably modifying symbols in the first line by abstract polynomials, and do not contribute any renormalization at all in the renormalized equation. One also has (recall (4.21))

\[
\hat{F}_{\psi_f}^{(k)}(x) = (-1)^j \lambda \sum_{\ell \neq j} \left( \begin{array}{c}
\nabla_\ell \psi_\ell(x) \\
\psi_\ell(x)
\end{array} \right) + b_\ell(x) \frac{\partial}{\partial x} + \left( b_\ell(x) \psi_\ell(x) \right) 1.
\]

Applying the reconstruction operator \(\hat{R}_\epsilon\) on (4.22) as before, we see that \((\hat{R}_\epsilon \Psi, \hat{R}_\epsilon \psi_\ell)\) satisfy (\(3.15b\)) with the following extra terms on the right hand side

\[
\begin{align*}
\lambda^2 \sum_{k=1,2} \left( -C_{4,k} - C_{3,-k} - C_{6,k} + C_{7,-k} - C_{4,3} - C_{5,-k} - C_{4,5} \right) \hat{R}_\epsilon \psi_f^j - \lambda^2 C_\psi \hat{R}_\epsilon \psi_f^j \\
+ 2\lambda^2 C_\epsilon \hat{R}_\epsilon \psi_f^j.
\end{align*}
\]

By simple summation by parts \(\sum_y \nabla_\epsilon f(x-y) g(y-z) = \sum_y f(x-y) \nabla_\epsilon g(y-z)\), one has

\[
-C_{4,k} - C_{6,k} = -C_{3,-k} + C_{7,-k} = 0.
\]

Therefore the renormalization constant \(C(\epsilon)\) on the right hand side is given by (5.6). □

Remark 5.3. By the argument in Remark 5.7 one can prove that \(C(\epsilon) = -\lambda^2 C_\epsilon^j + O(1) = O(|\log \epsilon|)\) which indeed diverges logarithmically. We omit the detail of this calculation since it is straightforward, but only remind the reader that the mass renormalization \(C(\epsilon) \Phi^j\) does not break gauge invariance.
5.2. Moment bounds. Our random models \( \hat{Z}^\varepsilon \) are stationary and Gaussian models (i.e. built from Gaussian noises). They belong to a finite inhomogeneous Wiener chaos: there exist kernels \( \mathcal{W}^{(\varepsilon;k)}_\tau \) such that \( (\mathcal{W}^{(\varepsilon;k)}_\tau)(z) \in H^\otimes_k \) for \( z \in \mathbb{R} \times \Lambda_\varepsilon \) where \( H^\otimes_k \equiv L^2([-T, T] \times \Lambda_\varepsilon) \), and

\[
\langle \hat{\Pi}^{\varepsilon,t}_0, \varphi, \varepsilon \rangle = \sum_{k \leq \| \tau \| } I^\varepsilon_k \left( \int_{\Lambda_\varepsilon} \varphi(y) \left( \mathcal{W}^{(\varepsilon;k)}_\tau(t, y) dy \right) \right),
\]

(5.9)

where \( \| \tau \| \) is the number of total occurrences of \( \Xi_{\varepsilon_j} \) or \( \Xi_{\xi_j} \) for \( j = 1 \) or \( 2 \) in \( \tau \), and \( I^\varepsilon_k \) is the \( k \)-th order Wiener integrals with respect to \( \varepsilon_j \) or \( \xi_j \) depending on the particular types of noise \( \Xi \) occurred in \( \tau \). Then we define

\[
(\mathcal{K}^{(\varepsilon;k)}_\tau)(z_1, z_2) \overset{def}{=} \langle (\mathcal{W}^{(\varepsilon;k)}_\tau)(z_1), (\mathcal{W}^{(\varepsilon;k)}_\tau)(z_2) \rangle_{H^\otimes_k},
\]

(5.10)

for \( z_1 \neq z_2 \in \mathbb{R} \times \Lambda_\varepsilon \). The functions \( \mathcal{K}^{(\varepsilon;k)}_\tau \) will depend on the time variables \( t_1 \) and \( t_2 \) only via \( t_1 - t_2 \), i.e. \( (\mathcal{K}^{(\varepsilon;k)}_\tau)_{t_1-t_2}(x_1, x_2) \overset{def}{=} (\mathcal{K}^{(\varepsilon;k)}_\tau)(z_1, z_2) \) where \( z_i = (t_i, x_i) \).

We will then apply the following result with \( \delta = 2 \). Recall that the norm \( \| Z^\gamma \|_{\delta, \gamma; \tau} \) on discrete inhomogeneous models \( Z \) is defined in Sect. 4.2, and the distance \( \| Z; Z^\gamma \|_{\delta, \gamma; \tau} \) defined in (4.18). We will also use the following set in the statement of the following result

\[
\hat{\mathcal{F}}^- \overset{def}{=} \{ \tau \in \hat{\mathcal{F}} : |\tau| < 0 \} \cup \mathcal{F}^{\text{gen}} \setminus \mathcal{F}^\text{poly}. \tag{5.11}
\]

Proposition 5.4. Let \( Z^\varepsilon \) be an admissible stationary Gaussian discrete model on the truncated regularity structure \( \hat{\mathcal{F}} = (\hat{T}, \hat{\mathcal{G}}) \). Let \( \kappa > 0 \) be such that the bounds

\[
\mathbb{E}\| \Gamma^\varepsilon_{x,y} \|_m^2 \lesssim |x - y|^{2(|\tau| - m) + \kappa}, \quad \mathbb{E}\| \Sigma^\varepsilon_{x,t} \|_m^2 \lesssim (|t - s|^2 + \varepsilon)^{2(|\tau| - m) + \kappa} \tag{5.12}
\]

hold for \( \tau \in \mathcal{F}^{\text{gen}} \setminus \mathcal{F}^\text{poly} \), for all \( s, t \in [-T, T] \), all \( x, y \in \Lambda_\varepsilon \), all \( m < |\tau| \). Assume that for some \( \delta > 0 \) and for each \( \tau \in \hat{\mathcal{F}}^- \) the bounds

\[
\mathbb{E}\| (\Pi^\varepsilon_{x,t} \varphi^\lambda_{x}) \|_{\varepsilon}|^2 \lesssim \lambda^{2|\tau| + \kappa}, 
\]

(5.13)

\[
\mathbb{E}\| (\Pi^\varepsilon_{x,t} - \Pi^\varepsilon_{x,s}) \varphi^\lambda_{x}) \|_{\varepsilon}|^2 \lesssim \lambda^{2(|\tau| - \delta) + \kappa}(|t - s|^2 + \varepsilon)^{2\delta},
\]

hold uniformly in all \( \lambda \in [\varepsilon, 1] \), all \( s, t \in [-T, T] \), all \( x \in \Lambda_\varepsilon \) and all \( \varphi \in \mathcal{B}_0^\varepsilon(\mathbb{R}^d) \) with \( r > -[\min \hat{\Lambda}] \). Then, one has \( \mathbb{E}(\| Z^\gamma \|_{\delta, \gamma; \tau}^{(\varepsilon)})^p \lesssim 1 \) for every \( \gamma > 0 \), \( p \geq 1 \) and \( \delta \in [0, \delta] \).

Furthermore, let \( Z = (\Pi, \Gamma, \Sigma) \) be an admissible stationary Gaussian discrete model on \( \hat{\mathcal{F}} \) such that for some \( \theta > 0 \), the maps \( \Gamma^\varepsilon - \Gamma \) and \( \Sigma^\varepsilon - \Sigma \) satisfy (5.12) with proportionality constants of order \( \varepsilon^{2\theta} \) and

\[
\mathbb{E}\| (\Pi^\varepsilon_{x,t} \varphi^\lambda_{x}) - (\Pi^\varepsilon_{x,t} \varphi^\lambda_{x}) \|_{\varepsilon}|^2 \lesssim \varepsilon^{2\theta}\lambda^{2|\tau| + \kappa}, 
\]

(5.14)

\[
\mathbb{E}\| (\Pi^\varepsilon_{x,t} - \Pi^\varepsilon_{x,s}) \varphi^\lambda_{x}) - (\Pi^\varepsilon_{x,t} - \Pi^\varepsilon_{x,s}) \varphi^\lambda_{x}) \|_{\varepsilon}|^2 \lesssim \varepsilon^{2\theta}\lambda^{2(|\tau| - \delta) + \kappa}(|t - s|^2 + \varepsilon)^{2\delta},
\]

uniformly in all parameters as above. Then, one has \( \mathbb{E}(\| Z; Z^\gamma \|_{\delta, \gamma; \tau}^{(0, \varepsilon)})^p \lesssim \varepsilon^{p\theta} \) for every \( \gamma > 0 \), \( p \geq 1 \) and \( \delta \in [0, \delta] \).
Proof. The bound on $\|Z^\varepsilon\|_{\delta;\gamma,T}^{(\varepsilon)}$ is precisely the content of [HM18, Theorem 6.1]. The bound on $\|Z; Z^\varepsilon\|_{\delta;\gamma,T}^{(0,\varepsilon)}$ follows in the same way as [Hai14, Theorem 10.7]. □

One useful way to verify conditions in the above proposition is, by [HM18, Prop 6.2], that it holds once we have that for every $\tau \in \mathcal{F}^{-}$ there are values $\alpha > |\tau| \vee (-d/2)$ and $\delta \in (0, \alpha + d/2)$ such that

$$
|\mathcal{K}^{(\varepsilon;k)}_0(x_1, x_2)| \lesssim \sum_{\zeta \geq 0} \left( |x_1| \vee |x_2| \right)^\zeta \left( |x_1 - x_2| \vee \varepsilon \right)^{2\alpha - \zeta},
$$

$$
|\delta^{0,t}(\mathcal{K}^{(\varepsilon;k)}_0(x_1, x_2)| \lesssim \sum_{\zeta \geq 0} \left( |x_1| \vee |x_2| \vee |t|^{1/2} \right)^\zeta \left( |x_1 - x_2| \vee \varepsilon \right)^{2\alpha - 2\delta - \zeta},
$$

(5.15)

for $x_1, x_2 \in \Lambda_\varepsilon$ and $k \leq \|\tau\|$, where the operator $\delta^{0,t}$ is defined in (2.18), and the sums run over finitely many values of $\zeta \in [0, 2\alpha - 2\delta + d)$. To verify (5.15), recall that for a function $\mathcal{K}$ on $\mathbb{R} \times \Lambda_\varepsilon$ supported in a ball around the origin, we say that it is of order $\alpha \in \mathbb{R}$ if for some $m \in \mathbb{N}$ the quantity

$$
\|\mathcal{K}\|_{\alpha; m}^{(\varepsilon)} \equiv \max_{|k|_s \leq m} \sup_{z \in \mathbb{R} \times \Lambda_\varepsilon} \frac{|D^k_\varepsilon \mathcal{K}(z)|}{(\|z\|_s \vee \varepsilon)^{\alpha - |k|_s}}
$$

is bounded uniformly in $\varepsilon$, where $k = (k_0, k_1, k_2) \in \mathbb{N}^3$, and $D^k_\varepsilon \equiv \partial^{k_0}_t (\nabla^{k_1}_x (\nabla^{k_2}_x)^2$. In particular, the truncated heat kernel $\mathcal{K}^{(\varepsilon)}$ in our case is of order $-2$. Often the kernels $\mathcal{K}^{(\varepsilon;k)}$ will depend on the spatial variables $x_1, x_2$ via the difference $x_1 - x_2$ and comes with an order of singularity i.e. $\mathcal{K}^{(\varepsilon;k)}_0$ is a function on $\mathbb{R} \times \Lambda_\varepsilon$, and supported in a ball around the origin; in this case one can easily prove that if $\mathcal{K}^{(\varepsilon;k)}_0$ is of order $2\alpha < 0$ then (5.15) holds with $\zeta = 0$ and $\delta \in (0, (\alpha + d/2) \wedge 1)$.

To obtain the orders of the kernels we will frequently use the following results from [HM18, Lemma 7.2 and 7.4] with $|s| = 4$ and $d = 2$.

**Lemma 5.5.** Let $K^{\varepsilon}_1$ and $K^{\varepsilon}_2$ be functions on $[-T, T] \times \Lambda_\varepsilon$ of order $\xi_1$ and $\xi_2 \in \mathbb{R}$ respectively. Let $\bar{\xi} \equiv \xi_1 + \xi_2 + |s|$. Then for any $m \in \mathbb{N}$ the following bounds hold uniformly in $\varepsilon$ for $\xi$ sufficiently small.

1. $\|K^{\varepsilon}_1 K^{\varepsilon}_2\|_{\xi_1 + \xi_2; m} \lesssim \|K^{\varepsilon}_1\|_{\xi_1; m} \|K^{\varepsilon}_2\|_{\xi_2; m}$

2. $\|K^{\varepsilon}_1 *_{\varepsilon} K^{\varepsilon}_2\|_{\xi_1 + \xi_2; m} \lesssim \|K^{\varepsilon}_1\|_{\xi_1; m} \|K^{\varepsilon}_2\|_{\xi_2; m}$ provided that $\xi_1 \wedge \xi_2 > |s|$ and $\bar{\xi} < 0$.

3. If $\bar{\xi} \in (0, 2) \setminus \mathbb{N}$, then $\|\widetilde{K}^{\varepsilon}\|_{\xi; m} \lesssim \|K^{\varepsilon}_1\|_{\xi_1; \bar{m}} \|K^{\varepsilon}_2\|_{\xi_2; \bar{m}}$ where $\bar{m} = m \vee (\lceil \bar{\xi} \rceil + 2)$ and

$$
\widetilde{K}^{\varepsilon}(z) \equiv K^{\varepsilon}_1 *_{\varepsilon} K^{\varepsilon}_2(z) - \sum_{|k|_s < \bar{\xi}} \frac{\zeta^k}{k!} D^k_\varepsilon (K^{\varepsilon}_1 *_{\varepsilon} K^{\varepsilon}_2)(0).
$$

4. If $\xi_1 \in (-|s| - 1, -|s|]$ and $\xi_2 \in (-2|s| - \xi_1, 0]$, then

$$
\|(*_{\varepsilon} K^{\varepsilon}_1) *_{\varepsilon} K^{\varepsilon}_2\|_{\xi; m} \lesssim \|K^{\varepsilon}_1\|_{\xi_1; m} \|K^{\varepsilon}_2\|_{\xi_2; m + 2}
$$

(5.17)

where $(\mathcal{A} K^{\varepsilon})(\varphi) \equiv \int_{\mathbb{R} \times \Lambda_\varepsilon} K^{\varepsilon}(z) (\varphi(z) - \varphi(0)) \, dz$ for any compactly supported test function $\varphi$ on $\mathbb{R}^{d+1}$.
Proposition 5.6. Let \( \hat{Z}^\varepsilon = (\hat{\Gamma}^\varepsilon, \hat{\Phi}^\varepsilon) \) be the renormalized inhomogeneous model defined above. There exists a sufficiently small \( \delta > 0 \) such that \( E[(\|Z^\varepsilon\|_{\delta, \gamma}^p)_{\delta, \gamma}] \lesssim 1 \) for every \( \gamma > 0, p \geq 1 \) and \( \delta \in (0, \delta) \), uniformly in \( \varepsilon \in (0, 1) \).

**Proof.** We will find the kernels \( K^\varepsilon \) for each of the symbols and verify the condition (5.15).

We will denote by \( \bigcirc \) a generic noise, and a thin line for a heat kernel and a thick line for a spatial derivative (in a generic direction) of the heat kernel. For the symbols \( \bigcirc \) the chaos expansion (5.9) only involves the first chaos (i.e. \( k = 1 \)), and one has for both cases

\[
(K^{(\varepsilon; 1)}_\tau)(z_1, z_2) = \langle K^\varepsilon(z_1 - \cdot), K^\varepsilon(z_2 - \cdot) \rangle_{H^\varepsilon}
\]

which is of order \( 0^- \) (i.e. order \( \alpha \) for any \( \alpha < 0 \)) as a function of \( z_1 - z_2 \) by item 2 of Lemma 5.5. By the discussion above Lemma 5.5, (5.15) holds. For symbol \( \bigcirc \) since \( \hat{\Pi}_x, \hat{\Phi}_x, \hat{\Gamma}^\varepsilon, \hat{\Sigma}^\varepsilon \) act on them trivially as they do for \( \bigcirc \), the bounds (4.6a), (4.6b) and (4.7) still almost surely hold uniformly in \( \varepsilon \). Regarding the symbols \( X \bigcirc \) again using item 2 of Lemma 5.5 one obtains a bound

\[
|\langle K^{(\varepsilon; 1)}_\tau \rangle_0(x_1, x_2)| \lesssim (|x_1| \vee |x_2|)^2(|x_1 - x_2| \vee \varepsilon)^{-2},
\]

as well as a similar bound on \( \delta^{0, t}(K^{(\varepsilon; 1)}_\tau) \). Successively applying item 2 of Lemma 5.5 and changes of variables also yields kernels \( K^{(\varepsilon; 1)}_\tau \) for \( \tau = \bigcirc \) which is of order \( 0^- \).

For the symbols of the type \( \bigcirc \bigcirc \) in the list (4.2), by conspersion for our models the two noises are always independent (recall that \( \xi^\varepsilon_1, \xi^\varepsilon_2, \xi^\varepsilon_1, \xi^\varepsilon_2 \) are independent). It is straightforward to check using Lemma 5.5 (especially item 3) that for both symbols we have

\[
|\langle K^{(\varepsilon; 2)}_\tau \rangle_0(x_1, x_2)| \lesssim (|x_1| \vee |x_2|)^{2-\kappa}, \quad \frac{|\delta^{0, t}(K^{(\varepsilon; 2)}_\tau)(x_1, x_2)|}{(|t|^{\frac{1}{2}} \vee \varepsilon)^{2\delta}} \lesssim (|x_1| \vee |x_2| \vee |t|^{\frac{1}{2}})^{2-\kappa-2\delta}
\]

for \( \kappa > 0 \), so (5.15) holds with \( \alpha = 1 - \frac{\kappa}{2} \). The desired bounds on \( \bigcirc \bigcirc \) then immediately follow.

We now focus on objects with two noises, three heat kernels with two spatial derivatives.

Let \( \tau = \bigcirc \bigcirc \bigcirc \). Recall the graphical notation in (5.3). Moreover, we will denote by \( \bigcirc \) for the noise \( \xi^\varepsilon \), and \( \bigcirc \) for the noise \( \xi^\varepsilon \), and a green arrow \( \rightarrow \) represents a rescaled test function \( \varphi^\lambda \). The entire graph with, say, \( m \) occurrences of \( \bigcirc \) or \( \bigcirc \) then represents
the $m$th Wiener integral against the noises represented by these $\square$ or $\bigcirc$. One then has

$$
\langle \hat{\Pi}_0^{\epsilon, j} \tau, \phi^\lambda \rangle_\epsilon = \ \bullet \bullet \bullet + \ \bullet \bullet \bullet - C^{k(\epsilon)}_{4,j} \ \nabla \ (5.18)
$$

For the first graph (i.e. second chaos), Lemma 5.5 then yields a kernel $K^{(\epsilon,2)}$ of order $0^-$. For the other two terms (i.e. zeroth chaos) the renormalization constant $C^{k(\epsilon)}_{4,j}$ is precisely defined such that the two zeroth chaos terms cancel. The symbols $\bigcirc \bigcirc \bigcirc$ and $\bigcirc \bigcirc \bigcirc$ are treated analogously with renormalization constants $C^{k(\epsilon)}_{4,-k}$ and $C^{\epsilon}_{4,\ell}$ respectively.

The symbols $\bigcirc \bigcirc \bigcirc$ and $\bigcirc \bigcirc \bigcirc$ are treated in the same way so we only consider the first one. We introduce one more graphical notation: a barred arrow $\overrightarrow{\text{---}}$ represents $K^\epsilon (z - w) - K^\epsilon (-w)$ where $w$ and $z$ are the coordinates of the starting and end point of the arrow respectively. One then has

$$
\langle \hat{\Pi}_0^{\epsilon, j} \tau, \phi^\lambda \rangle_\epsilon = \ \bullet \bullet \bullet + \ \bullet \bullet \bullet - \delta_{i, \ell} C^{k(\epsilon)}_{3,j} \ \nabla \ (5.19)
$$

For the first graph (i.e. second chaos) applying item 2 and 3 of Lemma 5.5 one obtains a bound

$$
|\langle (K^{(\epsilon,2)} \tau \rangle_0 (x_1, x_2)| \lesssim (|x_1| + |x_2|)^{2-2\delta} (|x_1 - x_2| + \epsilon)^{-2-2\delta}
$$

and a similar bound on $\delta^{0, j} (K^{(\epsilon,2)} \tau)$. Assuming now $i = \ell$ and considering the second graph, we bound the function represented by the barred arrow as a sum of two separate parts: a kernel $K^\epsilon$ represented by an arrow pointing towards the origin and a kernel $K^\epsilon$ represented by an arrow pointing towards the tip of the green arrow

$$
- \ \bullet \bullet \bullet + \ \bullet \bullet \bullet - C^{k(\epsilon)}_{3,j} \ \nabla
$$

By Lemma 5.5 the first term here is then deterministically bounded by $\lambda^\alpha$ for any $\alpha < 0$. Our choice of $C^{k(\epsilon)}_{3,j}$ precisely cancel the second term.

All the other objects with two independent noises, three heat kernels with two spatial derivatives such as $\bigcirc \bigcirc \bigcirc$ give $K^{(\epsilon,2)}$ of order $0^-$ by Lemma 5.5, thus the desired bounds (5.15) hold.
The symbol $\tau = \frac{\partial}{\partial x} \frac{\partial}{\partial y}$ is treated in a similar way as $\frac{\partial}{\partial x}$ above. In fact we have

$$
\langle \hat{\Pi}^{\epsilon,f}, \varphi^{\lambda} \rangle_{\epsilon} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} - C_{4,k}^{j(e)}
$$

(5.20)

Applying Lemma 5.5 to the first graph yields kernel $K^{(\epsilon,3)}$ of order $0^{-\kappa}$. Denoting by $\mathcal{R}$ the renormalized kernel $K^{\epsilon,3}$ where

$$
K^{\epsilon,3}(x-z) \stackrel{\text{def}}{=} \int_{\mathbb{R} \times \Lambda^{\epsilon}} K^{\epsilon}(x-y) \nabla_{k}^{\epsilon} K^{\epsilon}(x-z) \nabla_{j}^{\epsilon} K^{\epsilon}(z-y) dy,
$$

(5.21)

and $\mathcal{R}$ is defined below (5.17), the second graph is then equal to

$$
\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \delta_{i,k} - \delta_{i,k} C_{3,j}^{j(e)}
$$

(5.22)

By Lemma 5.5 (item 2 and item 4), the first graph yields a kernel $K^{0,-\kappa}$, and the other graph is then cancelled by the last term in (5.20).

Consider now $\tau = \frac{\partial}{\partial x} \frac{\partial}{\partial y}$ where $i \neq \ell$. One has

$$
\langle \hat{\Pi}^{\epsilon,f}, \varphi^{\lambda} \rangle_{\epsilon} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \delta_{i,k} - \delta_{i,k} C_{3,j}^{j(e)}
$$

(5.22)

The first term (i.e. the third chaos) results in a kernel with bound

$$
|\langle K^{(\epsilon;3)} \tau \rangle_{0}(x_{1}, x_{2})| \lesssim (|x_{1}| \vee |x_{2}|)^{2} (|x_{1} - x_{2}| \vee \epsilon)^{-2-\kappa}
$$

for $\kappa > 0$ and a similar bound on $\delta^{0,-}(K^{(\epsilon;3)} \tau)$. For the second term (i.e. the first chaos), we bound its second moment as follows. Denote by $y$ and $x$ the starting and end points of $\tau$ respectively, so that it represents $K^{\epsilon}(x-y) - K^{\epsilon}(-y)$. We bound the integral with $K^{\epsilon}(x-y)$ and the integral with $K^{\epsilon}(-y)$ separately. For the integral with $K^{\epsilon}(x-y)$, one obtains a kernel $K^{\epsilon}(x-y)$ of order $\alpha$ for any $\alpha < 0$. For the integral with the function $K^{\epsilon}(x-y)$, if $\|x-y\|_{\delta} \leq \frac{2}{\epsilon} \|y\|_{\delta}$, we bound $|K^{\epsilon}(x-y)| \lesssim (\|x-y\|_{\delta} \vee \epsilon)^{-2}$ and the bound follows in the same way as that for the integral with $K^{\epsilon}(x-y)$. On the other hand, assume that $\|x-y\|_{\delta} \geq \frac{2}{\epsilon} \|y\|_{\delta}$, then one necessarily has $\|x\|_{\delta} < \frac{3}{2} \|x-y\|_{\delta}$, so we can multiply the integrand by $\|x\|_{\delta}^{-\delta} \|x-y\|_{\delta}^{-\delta}$ for a sufficiently small $\delta > 0$ and bound $|K^{\epsilon}(x-y)| \|x-y\|_{\delta} \lesssim (\|y\|_{\delta} \vee \epsilon)^{-2+\delta}$. Note that now the integration over $y$ and the integration over $x$ factorize and thus can be computed independently. Applying Lemma 5.5, together with $\int \varphi^{\lambda}(x) \|x\|_{\delta}^{-\delta} dx \leq \lambda^{-\delta}$ we obtain the desired bound for the
second graph on the right hand side of (5.22). The last two terms on the right hand side of (5.22) are equal to \( \delta_{\ell,k} \) times

\[
\int_{\mathbb{R} \times \Lambda^\varepsilon} \nabla_j K^\varepsilon(x - y) K^\varepsilon(x - z) \nabla_j K^\varepsilon(z - y) \, dy.
\]

Lemma 5.5 (item 2 and item 4) immediately gives the desired bound for the first term by \( \lambda^\alpha \) for any \( \alpha < 0 \). The second term can be treated in the same way as the second graph on the right hand side of (5.22). The other similar symbols such as \( \frac{\partial}{\partial \varepsilon} \) are bounded in the analogous way.

Note that symbols of the form \( \delta_{\ell,k} \) (the top edge with derivative carries a noise that is independent from all other noises) have negative homogeneity but do not need renormalization. Indeed, the model yield a process with the third chaos and first chaos:

Using Lemma 5.5 we obtain kernels \( \mathcal{K}^{(e,3)} \) and \( \mathcal{K}^{(e,1)} \) both of order \( \alpha \) for \( \alpha < 0 \).

The model acting on \( \mathcal{K}^{(e,2)} \) yields processes that live in the second homogeneous chaos by independence of \( \xi^\varepsilon \) and \( \xi^\varepsilon \). By Lemma 5.5, \( \mathcal{K}^{(e,2)} \) is of order 0⁻, thus the desired bounds (5.15) hold. Turning to \( \tau = \mathcal{K}^{(e,2)} \), its second chaos component gives a kernel \( \mathcal{K}^{(e,2)} \tau \) of order 0⁻, and the zeroth chaos component of \( (\hat{\Pi}^{e,t}, \tau, \varphi)_\varepsilon \) (recall definition (5.3))

\[
\int \sum_{y,z \in \Lambda^\varepsilon} K^\varepsilon(t - s, y - z) \varphi(y) \, ds - C_2^{(e)} \sum_{y \in \Lambda^\varepsilon} \varphi(y) = 0.
\]

For the symbol \( \hat{\Pi}^{e,t} \), the second chaos component again results in a kernel \( \mathcal{K}^{(e,2)} \) of order 0⁻. The zeroth component is now equal to

\[
\int \sum_{y,z \in \Lambda^\varepsilon} K^\varepsilon(t - s, y - z) \varphi(y) \, ds - C_1^{(e)} \sum_{y \in \Lambda^\varepsilon} \varphi(y) = 0.
\]

and recall that we have defined \( C_1^{(e)} \) by (5.4). In fact, the above expression would vanish if \( C_1^{(e)} \) was replaced by \( C_2^{(e)} \); thus it remains to show that \( C_1^{(e)} - C_2^{(e)} \) converges to a
finite limit. Recall from [HM18, Lemma 5.4] that we have $P^\varepsilon = K^\varepsilon + R^\varepsilon$, where $P^\varepsilon$ is the discrete heat kernel, and $R^\varepsilon$ is compactly supported with norm $\|R^\varepsilon\|_{C^r}$ bounded uniformly in $\varepsilon$. Therefore, denoting by a thick blue arrow for $P^\varepsilon$, we have that $C_1^{(\varepsilon)} - C_2^{(\varepsilon)}$ equals

$$\sum_k \left( \begin{array}{c} k \\ j \end{array} \right) - \begin{array}{c} k \\ j \end{array} + \begin{array}{c} -k \\ j \end{array} - \begin{array}{c} -j \\ k \end{array} - \begin{array}{c} -j \\ k \end{array} - \varepsilon \begin{array}{c} k \\ j \end{array} \right) (5.24)$$

plus some terms which converge to finite limits. Note that $(5.24)$ remains the same with $\sum_k$ replaced by $\sum_j$: indeed $(5.24)$ are equal for $j \in \{1, 2\}$, so we can sum $(5.24)$ over $j$ and then by the same reason drop the sum over $k$. We can then rename $k$ by $\ell$ so that $(5.24)$ equals

$$\sum_j \left( \begin{array}{c} \ell \\ j \end{array} \right) - \begin{array}{c} \ell \\ j \end{array} + \begin{array}{c} -\ell, j \end{array} - \begin{array}{c} -\ell \\ j \end{array} - \begin{array}{c} -\ell \\ j \end{array} - \varepsilon \begin{array}{c} \ell \\ j \end{array} \right) (5.25)$$

for each $\ell \in \{1, 2\}$. We prove that $(5.25)$ has a finite limit using the “Ward identity” derived in Section B. With our graphical notation, the identity (B.7) with $t = x = 0$ can be represented as

$$-\begin{array}{c} k_e \end{array} - \sum_{j, e \in \{0, e\}} \begin{array}{c} \ell \\ j \end{array} + \sum_{j, e \in \{0, e\}} \begin{array}{c} \ell \\ j \end{array} = 0 (5.26)$$

for each $\ell \in \{1, 2\}$, where we dropped the summation over $k \in \{1, 2\}$ in (B.6) because (B.6) only depends on $k$ through the noise $\zeta^\varepsilon_k$ and this dependence is gone upon taking expectation. Here, a lattice vector $e$ (of length $\varepsilon$) near a tip of an arrow means that the kernel represented by the arrow is evaluated at the point $z + e$ if $z$ is the end point of the arrow; for instance $k_e$ represents the kernel $\nabla^\varepsilon_k P^\varepsilon(y + e - x)$, with $x$ and $y$ being the starting and end points of the arrow.

Now since $\sum_{y} \nabla^\varepsilon_l P^\varepsilon(x - y) P^\varepsilon(y - z) = -\sum_{y} P^\varepsilon(x - e_\ell - y) \nabla^\varepsilon_{e_\ell} P^\varepsilon(y - z)$ we see that the four triangular graphs in $(5.25)$ are exactly the same as the four triangular graphs in $(5.26)$. The sum of the two last terms of $(5.25)$ is then obviously equal to the first term of $(5.26)$.

The symbols $\begin{array}{c} k_e \end{array}$ and $\begin{array}{c} j_e \end{array}$ are then analyzed in the same way as $\begin{array}{c} k_e \end{array}$ and $\begin{array}{c} j_e \end{array}$ so we omit the details.  

\begin{remark}
In the previous proof we explained boundedness of $C_1^{(\varepsilon)} - C_2^{(\varepsilon)}$ by gauge symmetry, but of course one can also prove it via the more elementary arguments. Indeed, one can start by ignoring the terms in $(5.24)$ for $k \neq j$, because they converge to zero as $\varepsilon \to 0$ due to $x \to -x$ symmetry. One then has (for instance see [CST18, Lemma 4.2],
or [HQ18, Section 6] for one spatial dimension case in continuum)\(^\text{17}\)

\[
2 \sum_{x \in \mathbb{Z}^d} \sum_{j=1}^{d} \int_{-\infty}^{\infty} \nabla_j P^\varepsilon(t+s, x+y) \nabla_j P^\varepsilon(t+s', x+y') \, dt = P^\varepsilon(|s-s'|, y-y'), \quad (5.27)
\]

for all \(s, s' \in \mathbb{R}\) and \(y, y' \in \mathbb{Z}^d\). Therefore

\[
\begin{align*}
\sum_{x \in \mathbb{Z}^d} = \frac{1}{2} \sum_{k=1}^{2} j & \quad = \frac{1}{2} \sum_{k=1}^{2} k \quad = \frac{1}{4} \ni \\
\end{align*}
\]

The first equality here is by the fact that the value of the first graph is the same for \(j \in \{1, 2\}\), and in the last equality we used the identity (5.27) to compute the integration of the upper-right vertex. The other graphs can be manipulated in a similar way, for instance,

\[
\begin{align*}
\sum_{x \in \mathbb{Z}^d} = - j & \quad = - \frac{1}{4} \ni \\
\end{align*}
\]

where we applied integration by parts twice to shift the derivative in the \(j\)th coordinate from the arrow on the left to the arrow on the top and then to the arrow on the right. This also proves finiteness of (5.24).

6. Convergence of Solutions and Observables

In the following proposition we will assume that our models \(\hat{\Pi}^\varepsilon\) satisfy the following uniform bound (writing \(\tau = \hat{\Pi}^\varepsilon \tau = \hat{\Pi}^\varepsilon \tau\) for short here)

\[
|e^\varepsilon \hat{\Pi}^\varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right)\ni) + |e^\varepsilon \hat{\Pi}^\varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right)\ni)| \lesssim 1, \quad \text{and}
\]

\[
\begin{align*}
\varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni \varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni \varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni \varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni, \quad (\varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni (\varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni, (6.1a) & \\
\end{align*}
\]

\[
\begin{align*}
\varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni \varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni \varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni \varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni, \quad (\varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni (\varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni, (6.1b) & \\
\end{align*}
\]

\[
\begin{align*}
\varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni \varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni \varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni \varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni, \quad (\varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni (\varepsilon \left(\begin{array}{c} \hat{\nu}^\varepsilon \ni \\
\end{array}\right) \ni, (6.1c) & \\
\end{align*}
\]

are all uniformly bounded in the \(|\cdot|^{(\varepsilon)}_{\hat{\nu}^\varepsilon \ni}\) norm for some sufficiently small \(\kappa > 0\) and any \(j, k \in \{1, 2\}\) and \(e \in \{0, e_k\}\).

**Proposition 6.1.** Let \(\alpha \in (-\frac{4}{3}, -1), \gamma \in (|\alpha|, 2), \eta \in (-\frac{1}{2}, 0), \text{and } \varepsilon \in (0, 1)\). Then, for final time \(T > 0\) sufficiently small and for any model \(\hat{\Pi}^\varepsilon\) in \(\mathcal{M}_\varepsilon\) such that (6.1) hold, there exists a unique solution to the fixed point problem (4.22) in \(\mathcal{D}^\gamma_{\varepsilon, \eta}\). The existence time \(T\) can be chosen uniformly over \(\varepsilon \in (0, 1]\), over bounded sets of initial conditions in \(\mathcal{C}^\gamma_{\varepsilon, \eta}\), and over bounded sets in \(\mathcal{M}_\varepsilon\) satisfying the above uniform bounds (6.1).

\(^{17}\) The constant in front of the left hand side of [CST18, (4.9)] is \(\frac{1}{2}\) instead of 2 as here, because the heat kernel is defined as \((\partial_t - \frac{1}{\varepsilon^2} \Delta)^{-1}\) therein.
Proof. For the local polynomial nonlinear terms in fixed point problem (4.22), we can verify the assumption of Theorem 3.9 and Assumption 5.5 in [HM18]. Indeed, the maps $\mathcal{D}^{\nu, \eta}_{\varepsilon, \mathcal{T}}(V) \times \mathcal{D}^{\nu, \eta}_{\varepsilon, \mathcal{T}}(\tilde{V}) \rightarrow \mathcal{D}^{\nu+2\varepsilon, 3\eta}_{\varepsilon, \mathcal{T}}(\tilde{V})$ given by
\[
(\Psi^e_k, \Psi^e_j) \mapsto (B^e_k, \Psi^e_j) \quad (B^e_k, \Psi^e_j) \mapsto (B^e_k, \tilde{\Psi}^e_j)
\]
are locally Lipschitz where $V$ and $\tilde{V}$ are sectors with homogeneity $\varsigma \in (-\frac{1}{3}, 0)$ and $\tilde{V}$ is sector with homogeneity $2\varsigma - 1$. Moreover, the maps $\mathcal{D}^{\nu, \eta}_{\varepsilon, \mathcal{T}}(V) \times \mathcal{D}^{\nu, \eta}_{\varepsilon, \mathcal{T}}(\tilde{V}) \rightarrow \mathcal{D}^{\nu+2\varepsilon, 3\eta}_{\varepsilon, \mathcal{T}}(\tilde{V})$ given by $(B^e_j, \Psi^e_k) \mapsto B^e_j(\Psi^e_k)^2$ and $(B^e_j, \Psi^e_k) \mapsto (B^e_j)^2\Psi^e_k$ are also locally Lipschitz, uniformly in $\varepsilon$, where $V$ and $\tilde{V}$ are as above and $\tilde{V}$ is sector with homogeneity $3\varsigma$. The conditions in [HM18, Theorem 3.9]

$$\eta < \tilde{\eta} \land \alpha + 2, \quad \gamma < \tilde{\gamma} + 2, \quad \tilde{\eta} > -2$$

then hold with $\alpha, \eta, \gamma$ as above, $\tilde{\gamma} \overset{\text{def}}{=} (\gamma + \varsigma - 1) \land (\gamma + 2\varsigma)$, $\tilde{\eta} \overset{\text{def}}{=} (2\eta - 1) \land (3\eta)$ and $\beta = 2$ therein. Therefore if there was not the other “remainder” terms (4.23) in (4.22), [HM18, Theorem 5.7] would immediately apply yielding unique solution to the Eq. (4.22) over some interval $[0, T]$ that is jointly Lipschitz continuous in bounded sets of initial conditions and models uniformly in $\varepsilon$.

To control the other terms, note that if $B^e_j, \Psi^e_k$ solve the fixed point problem (4.22) one necessarily has
\[
B^e_j = \mathfrak{B}^e + \tilde{B}^e_j \quad \Psi^e_j = \mathfrak{B}^e + \tilde{\Psi}^e_j \quad (6.2)
\]
where $\tilde{B}^e_j, \tilde{\Psi}^e_j$ take values in the subspace of the regularity structure spanned by $\mathfrak{B}$ and elements with strictly positive homogeneity. Denote by $L^\infty_\varepsilon$ the space of functions on $[0, T] \times \Lambda_\varepsilon$ endowed with the supremum norm.

Consider the term $R^e_{B^e_j}$ as given by (4.23a). For any $\kappa > 0$, the maps $B^e_j \mapsto \varepsilon^\kappa R^e B^e_j$ and $\Psi^e_k \mapsto \varepsilon^\kappa R^e^2 \Psi^e_k$ are locally Lipschitz continuous from $\mathcal{D}^{\nu, \eta}_{\varepsilon, \mathcal{T}}$ to $L^\infty_\varepsilon$, uniformly over $\varepsilon \in (0, 1)$, over bounded balls in $\mathcal{D}^{\nu, \eta}_{\varepsilon, \mathcal{T}}$, and over models in $\mathcal{M}_\varepsilon$ with bounded norm and with (6.1a). Moreover, regarding the function $\tilde{F}_1(z) = e^z - 1 - z$ appearing in $R^e_{B^e_j}$ [defined below (3.13)] one has $|\tilde{F}_1(u) - \tilde{F}_1(v)| \overset{\leq}{\lesssim} |u - v|(|u| + |v|)$ for $u$ and $v$ bounded. So provided that $\kappa > 0$ is small enough ($\kappa < \frac{1}{4}$ suffices), the map $(B^e_j, \Psi^e_k, \Psi^e_j) \mapsto R^e_{B^e_j}$ is locally Lipschitz continuous from $(\mathcal{D}^{\nu, \eta}_{\varepsilon, \mathcal{T}})^3$ to $L^\infty_\varepsilon$, uniformly over the same data as above; in fact, the norm of $R^e_{B^e_j}$ and the Lipschitz constant are bounded by $\varepsilon^{1-4\kappa}$.

Similarly, regarding the term $R^e_{\Psi^e_j}$ as given by (4.23b), the map $(B^e_1, B^e_2, \Psi^e_1, \Psi^e_2) \mapsto R^e_{\Psi^e_j}$ is locally Lipschitz continuous from $(\mathcal{D}^{\nu, \eta}_{\varepsilon, \mathcal{T}})^4$ to $L^\infty_\varepsilon$, uniformly over the aforementioned data, with the norm of $R^e_{\Psi^e_j}$ and the Lipschitz constant bounded by $\varepsilon^{1-4\kappa}$. The argument for this follows in the same way as above except that for the function $\tilde{F}_2$ appearing in $R^e_{\Psi^e_j}$ [defined below (3.13)] one has $|\tilde{F}_2(u) - \tilde{F}_2(v)| \overset{\leq}{\lesssim} |u - v|(|u|^2 + |v|^2)$ for $u$ and $v$ bounded. The maps $R^e_{B^e_j} \mapsto \mathcal{P}^e(R^e_{B^e_j} 1)$ and $R^e_{\Psi^e_j} \mapsto \mathcal{P}^e(R^e_{\Psi^e_j} 1)$ on the right hand side of (4.22) then map $L^\infty_\varepsilon$ into $\mathcal{D}^{\nu, \eta}_{\varepsilon, \mathcal{T}}$ with norms bounded uniformly in $\varepsilon$ and behaves like $T^\theta$ for some $\theta > 0$. 


Next, consider the term $\tilde{R}_B^e$ as given by (4.23c). Write
\begin{align}
\mathcal{R}^e B_j^e &= b_j^e + w_j^e,
\mathcal{R}^e \Psi_j^e &= \psi_j^e + v_j^e,
\end{align}
where $b_j^e = \tilde{\Pi}^e \tilde{\Pi}^e$, $\psi_j^e = \tilde{\Pi}^e \tilde{\Pi}^e$. (6.3)

Using (6.2) we have the decomposition
\begin{align}
\varepsilon (\mathcal{R}^e B_j^e(\nabla^e_j \mathcal{R}^e \Psi_k^e)) + (\mathcal{R}^e \Psi_k^e) &= \varepsilon b_j^e \psi_k^e \nabla^e_j \psi_k^e + \varepsilon w_j^e \psi_k^e \nabla^e_j \psi_k^e + \varepsilon b_j^e v_k^e \nabla^e_j \psi_k^e + \varepsilon b_j^e \psi_k^e \nabla^e_j v_k^e \\
&+ \varepsilon \left( b_j^e v_k^e \nabla^e_j \nabla^e_k v_k^e + w_j^e v_k^e \nabla^e_j \psi_k^e + w_j^e \psi_k^e \nabla^e_j v_k^e + w_j^e \psi_k^e \nabla^e_j \nabla^e_k v_k^e \right),
\end{align}
(6.4)

By the bound (6.1b) the convolution of the first term on the right hand side with heat kernel can be lifted to the subspace of $D_{\varepsilon,T}^{\gamma,\eta}$ spanned by $I$ and $X$, with $D_{\varepsilon,T}^{\gamma,\eta}$ norm bounded uniformly in $\varepsilon > 0$. The same holds for the second and third terms in the first line using the bound (6.1c). All the terms on the second line have uniformly bounded norms, by the classical Young theorem, because for any $u^e$ with $\|u^e\|_{\tilde{C}_\alpha} \lesssim 1$ with $\alpha = 1 - \kappa$ or $\alpha = -\kappa$ here, one has the uniform bound $\|\varepsilon \nabla^e u^e\|_{\tilde{C}_\alpha} \lesssim 1$ with the same $\alpha$. The remaining term $\varepsilon b_j^e \psi_k^e \nabla^e_j \nabla^e_k v_k^e = (e^\kappa b_j^e) (e^\kappa \psi_k^e) (\varepsilon^{1-2\kappa} \nabla^e_j v_k^e)$ can be controlled again by (6.1a) and Young theorem.

The map $(B_j^e, \Psi_1^e, \Psi_2^e) \mapsto \tilde{R}_B^e$ is thus locally Lipschitz continuous from $(D_{\varepsilon,T}^{\gamma,\eta})^3$ to $D_{\varepsilon,T}^{\gamma,\eta}$, uniformly over $\varepsilon \in (0, 1)$, over bounded balls in $(D_{\varepsilon,T}^{\gamma,\eta})^3$, and over models in $\mathcal{M}_\varepsilon$ with bounded norm and satisfying the uniform bound (6.1).

Finally, for the term $\tilde{R}_\Psi^e$ given by (4.23d), each of the terms on the right hand side of (4.23d) is of the same form as the left hand side of (6.4), which is cubic with one derivative and one power of $e$. Decomposing it in the same way as (6.4), then using the bounds (6.1) we have that its convolution with heat kernel can be lifted to the subspace of $D_{\varepsilon,T}^{\gamma,\eta}$ spanned by $I$ and $X$, with $D_{\varepsilon,T}^{\gamma,\eta}$ norm bounded uniformly in $\varepsilon > 0$.

Summarizing the above bounds we then have $T > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the fixed point map is contractive in a ball of large enough radius provided the models are uniformly bounded as above and the initial conditions are uniformly bounded. The solutions can be continued uniquely until the explosion time as in [Hai14, Sec. 7]. □

We will consider the following fixed point problem in continuum. Recall from [Hai14, HM18] that the operator $\mathcal{P}$ and $\mathcal{P}'$ are such that $\mathcal{R}_t \mathcal{P} = P \ast \mathcal{R}_t$ and $\mathcal{P}_t \mathcal{P}' = P' \ast \mathcal{R}_t$.

**Proposition 6.2.** There exists a unique solution $B_j, \Psi_j \in D_{\varepsilon,T}^{\gamma,\eta}$ (for $j \in \{1, 2\}$) to the following fixed point problem with an admissible model $\hat{Z}$ up to some terminal time $T > 0$
\begin{align}
B_j &= \mathcal{P} \hat{F}_{B_j} (B, \Psi) + S \hat{A}_j + \left[ \left( \mathcal{Y}_{1j} - \mathcal{Y}_{2j} \right) \mathcal{I} \right] \\
\Psi_j &= \mathcal{P} \hat{F}_{\Psi_j} (B, \Psi) - \sum_k \mathcal{P}' \hat{F}_{(k)} (B, \Psi) + S \hat{\Phi}_j + \left[ (-1)^j \lambda \sum_{k=1,2} \left( \mathcal{Y}_{k} - \mathcal{Y}_{k} \right) \mathcal{I} \right]
\end{align}
(6.5)

where the modeled distributions $\hat{F}_{B_j}$, $\hat{F}_{\Psi_j}$ and $\hat{F}_{(k)}$ are defined in the same way as $\hat{F}_{B_j^e}$ and $\hat{F}_{\Psi_j^e}$ in Sect. 5.1 except that the spatial variable $x$ takes values in $T^2$, and $S$ is the semi-group so that $SA_j, S\hat{\Phi}_j$ are naturally lifted into $D_{T}^{\gamma,\eta}$.
Proof. Existence and uniqueness of the fixed point problem (6.5) follows in the same way as in the first part of proof to Proposition 6.1, with all the discrete spaces $D_{\varepsilon,T}^{Y,\eta}$ there replaced by their continuous counter-parts $D_{T}^{Y,\eta}$. □

Proof of Proposition 3.7. We prove the limit by a diagonal argument as in [MW17a] (and as followed later by [HS17,CS17,HM18,CM18] in regularity structures or [CGP17] in the context of paracontrolled distributions).

We take a function $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}$ which is smooth, compactly supported, symmetric under $x \rightarrow -x$ and integrates to 1, and for some $\bar{\varepsilon} \in [\varepsilon, 1]$ we define $\psi^{\bar{\varepsilon}}(t, x) \overset{\text{def}}{=} -4\varepsilon^2 \psi(\varepsilon^{-2} t, \varepsilon^{-1} x)$ and the mollified noises $\xi^{\bar{\varepsilon},0} \overset{\text{def}}{=} \xi * \psi^{\bar{\varepsilon}}$ and $\zeta^{\bar{\varepsilon},0} \overset{\text{def}}{=} \zeta * \psi^{\bar{\varepsilon}}$.

We define an inhomogeneous continuous model $\hat{Z}^{\bar{\varepsilon},0}$ for our truncated regularity structure $\hat{\mathcal{F}}$, in the same way as (4.8)–(4.13) except that we now use the kernel $K$ and the noises $(\hat{\xi}^{\bar{\varepsilon},0}, \hat{\zeta}^{\bar{\varepsilon},0})$, and sum over $\Lambda_{\bar{\varepsilon}}$ replaced by $T^2$, $\ast_{\bar{\varepsilon}}$ replaced by continuous convolution $\ast$, backward finite difference $\nabla_{\bar{\varepsilon}}$, replaced by $-\partial_k$. We then define the renormalized model $\hat{Z}^{\bar{\varepsilon},0}$ in the same way as (5.1)–(5.4), except that the constants (5.3), (5.4) (now depending on $\bar{\varepsilon} ps$) are defined by the singular kernels $K$ and its derivatives and then contracting the $\psi^{\bar{\varepsilon}}$-regularized noises in the symbols (5.2). For instance, $C_{4,k}^{\bar{\varepsilon}(\bar{\varepsilon} ps)}$ in (5.3) is replaced by

$$C_{4,k}^{\bar{\varepsilon}(\bar{\varepsilon} ps)} \overset{\text{def}}{=} \int_{\mathbb{R}^9} \partial_k K (z) (\nabla_{\bar{\varepsilon}} (\hat{Z}^{\bar{\varepsilon},0} - y) \psi^{\bar{\varepsilon},0} (x - y)) \, dx \, dy \, dz$$

where $\psi^{\bar{\varepsilon},0} (x - y) \overset{\text{def}}{=} \int \psi^{\bar{\varepsilon}}(x - w) \psi^{\bar{\varepsilon}}(y - w) \, dw$. Also, we set $c_B^{\bar{\varepsilon}} \overset{\text{def}}{=} 0$ and $c_{\psi}^{\bar{\varepsilon}} \overset{\text{def}}{=} 0$.

We then let $(\hat{B}^{\bar{\varepsilon},0}, \hat{\Psi}^{\bar{\varepsilon},0}) = \hat{\mathcal{R}}^{\bar{\varepsilon},0} (\hat{B}^{\bar{\varepsilon},0}, \hat{\Psi}^{\bar{\varepsilon},0})$ where $\hat{\mathcal{R}}^{\bar{\varepsilon},0}$ is the reconstruction with the model $\hat{Z}^{\bar{\varepsilon},0}$ and $(\hat{B}^{\bar{\varepsilon},0}, \hat{\Psi}^{\bar{\varepsilon},0})$ solves the fixed point problem (6.5). As the derivation in Lemma 5.2, they solve:

$$\partial_t \hat{B}_j^{\bar{\varepsilon},0} = \Delta \hat{B}_j^{\bar{\varepsilon},0} + \lambda \left( \psi_1^{\bar{\varepsilon},0} \partial_j \psi_2^{\bar{\varepsilon},0} - \psi_2^{\bar{\varepsilon},0} \partial_j \psi_1^{\bar{\varepsilon},0} \right) - \lambda^2 \sum_{k=1,2} B_j^{\bar{\varepsilon},0} (\bar{\varepsilon}^{\bar{\varepsilon}} ps)^2 + \xi_j^{\bar{\varepsilon} ps} $$

$$\partial_t \hat{\Psi}_j^{\bar{\varepsilon},0} = \Delta \hat{\Psi}_j^{\bar{\varepsilon},0} - 2(1) \lambda \sum_{k=1,2} \bar{\varepsilon}^{\bar{\varepsilon}} ps \partial_j \hat{B}_k^{\bar{\varepsilon},0} - \lambda^2 \sum_{k=1,2} (B_k^{\bar{\varepsilon},0})^2 \hat{\Psi}_j^{\bar{\varepsilon},0} - C^{(\bar{\varepsilon} ps)} \hat{\Psi}_j^{\bar{\varepsilon},0} + \xi_j^{\bar{\varepsilon} ps}. \quad (6.6)$$

Here $C^{(\bar{\varepsilon} ps)}$ is defined as in (5.6) with the constants replaced by the respective $\bar{\varepsilon}$-dependent ones.

By retracing the proof of Proposition 5.6, (or alternatively using the blackbox theorem from [CH16]), we can immediately prove that there exists a model $\hat{Z} \in \mathcal{M}_0$ such that one has

$$\lim_{\bar{\varepsilon} \rightarrow 0} E \| \hat{Z}^{\bar{\varepsilon},0} - \hat{Z} \|_{\delta, \gamma; T} = 0. \quad (6.7)$$

Indeed, the kernels $K$ and $K * \psi^{\bar{\varepsilon}}$ satisfy all the conditions required in the proof of Proposition 5.6 to yield the uniform moment bounds for the models $\hat{Z}^{\bar{\varepsilon},0}$. Moreover, it is easy to identify the limiting random variables $\langle \hat{\Pi}_0^T \tau, \varphi^{\lambda} \rangle$ for each relevant $\tau$, which we briefly explain now. For symbols where the renormalization constant exactly kills the zero-th chaos as in (5.18), $\langle \hat{\Pi}_0^T \tau, \varphi^{\lambda} \rangle$ is simply defined as the limit of the second chaos.
For instance, for \( \tau = \frac{j}{k} \) one has, in view of (5.18),

\[
\langle \hat{N}_0', \varphi \rangle = \langle \hat{N}_0', \varphi \rangle
\]

where now all arrows represent the singular kernel \( K \), decorations \( k, j \) represent its respective derivatives, and noises are the white noises \( \xi, \zeta \). For symbols \( \tau = \frac{j}{k} \) (and similarly for \( -\frac{j}{k} \)), by the analysis below (5.19), one has

\[
\langle \hat{N}_0', \varphi \rangle = \langle \hat{N}_0', \varphi \rangle - \delta_{i, \ell}
\]

For symbols with three noises such as \( \tau = \frac{j}{k} \) by the analysis below (5.20) one has

\[
\langle \hat{N}_0', \varphi \rangle = \langle \hat{N}_0', \varphi \rangle + \langle \hat{N}_0', \varphi \rangle
\]

where \( \hat{N}_0' \) is re-interpreted as the renormalized kernel \( \hat{K}_{3} \) where \( K_{3} \) is defined as in (5.21) with \( K_{\varepsilon} \) replaced by \( K \). One can then bound the moments of \( \langle \hat{N}_0', \tau - \hat{N}_0', \varphi \rangle \) by \( \tilde{\varepsilon}^{\kappa} \) by proceeding as in [Hai14] and using the bound

\[
\| K - K \ast \psi_{\tilde{\varepsilon}} \|_{-2 - \kappa; m} \lesssim \tilde{\varepsilon}^{\kappa} \| K \|_{-2; m + 2}
\]

for some sufficiently small \( \kappa > 0 \).

To conclude the proof of (6.7), the only subtlety lies in the zeroth chaos of \( \hat{N}_0' \tau \) for \( \tau = \frac{j}{k} \), which is given by

\[
\lim_{\tilde{\varepsilon} \to 0} \left( C_{2}^{(\tilde{\varepsilon})} - C_{1}^{(\tilde{\varepsilon})} \right).
\]

(6.8)

To show that it is finite, note that \( C_{2}^{(\tilde{\varepsilon})} - C_{1}^{(\tilde{\varepsilon})} \) is, up to a finite part due to the truncation of heat kernel, given by (5.24) without the last term there and with the discrete kernel \( P_{\varepsilon} \) replaced by the kernels in continuum. Since the last term in (5.24) i.e. \( -c_{B}^{(\tilde{\varepsilon})} \) converges to a finite limit, arguing as in Remark 5.7 yields finiteness of the zeroth chaos of this object.

As the next step, we discretize the noise \( \xi_{\tilde{\varepsilon}, 0} \). Define the function

\[
\psi_{\tilde{\varepsilon}, \xi}(t, x) \overset{\text{def}}{=} e^{-2} \int_{\mathbb{R}^{2}} \psi_{\tilde{\varepsilon}}(t, y) \, 1_{|y - x| \leq \varepsilon / 2} \, dy, \quad (t, x) \in \mathbb{R} \times \Lambda_{\tilde{\varepsilon}}^{2},
\]
and the discrete noises

\[ \xi_{\tilde{\varepsilon}, \varepsilon} \equiv \psi_{\tilde{\varepsilon}, \varepsilon} \ast_{\varepsilon} \xi, \quad \zeta_{\tilde{\varepsilon}, \varepsilon} \equiv \psi_{\tilde{\varepsilon}, \varepsilon} \ast_{\varepsilon} \zeta. \]

We define the discrete model \( \tilde{Z}_{\tilde{\varepsilon}, \varepsilon} \) with the noises \( \xi_{\tilde{\varepsilon}, \varepsilon}, \zeta_{\tilde{\varepsilon}, \varepsilon} \), and renormalization constants as in (5.3), (5.4) using the discrete kernels \( K_{\varepsilon} \), but then contracting the \( \psi_{\tilde{\varepsilon}} \)-regularized discrete noises in the symbols (5.2). For instance, \( C_{4, k}^{-\kappa(\tilde{\varepsilon}, \varepsilon)} \) in (5.3) is replaced by

\[ C_{4, k}^{-\kappa(\tilde{p}\varepsilon, \varepsilon)} \equiv \int_{(\mathbb{R} \times \Lambda_{\varepsilon})^{3}} \nabla_{\varepsilon} \psi_{\tilde{\varepsilon}, \varepsilon}(x) \nabla_{\varepsilon} K(z - x) \psi_{\tilde{\varepsilon}, \varepsilon}(y) \psi_{\tilde{\varepsilon}, \varepsilon}(z) \, dx \, dy \, dz \]

where \( \psi_{\tilde{\varepsilon}, \varepsilon}(x) \equiv \int_{\mathbb{R} \times \Lambda_{\varepsilon}} \psi_{\tilde{\varepsilon}, \varepsilon}(x - w) \psi_{\tilde{\varepsilon}, \varepsilon}(y - w) \, dw \). Let \( (\tilde{B}_{\tilde{\varepsilon}, \varepsilon}, \psi_{\tilde{\varepsilon}, \varepsilon}) = \tilde{R}_{\tilde{\varepsilon}, \varepsilon} (B_{\tilde{\varepsilon}, \varepsilon}, \Psi_{\tilde{\varepsilon}, \varepsilon}) \) where \( \tilde{R}_{\tilde{\varepsilon}, \varepsilon} \) is the reconstruction with the model \( \tilde{Z}_{\tilde{\varepsilon}, \varepsilon} \) and \( (B_{\tilde{\varepsilon}, \varepsilon}, \Psi_{\tilde{\varepsilon}, \varepsilon}) \) solves the fixed point problem (4.22).

For any fixed \( \tilde{\varepsilon} > 0 \), one has that the “remainder” terms \( R_{B_{j}}^{\varepsilon}, R_{\tilde{\varepsilon}}^{\varepsilon} \) as well as \( \tilde{R}_{\tilde{\varepsilon}, \varepsilon} \tilde{R}_{B_{j}}^{\varepsilon} \), \( \tilde{R}_{\tilde{\varepsilon}, \varepsilon} \tilde{R}_{\tilde{\varepsilon}, \varepsilon} \) converge to zero as \( \varepsilon \to 0 \) in \( C^{\infty} \) norms. The convergence of \( (B_{\tilde{\varepsilon}, \varepsilon}, \Psi_{\tilde{\varepsilon}, \varepsilon}) \) to \( (B^{\tilde{\varepsilon}, 0}, \Psi^{\tilde{\varepsilon}, 0}) \) as \( \varepsilon \to 0 \) for fixed \( \tilde{\varepsilon} \) is then standard in numerical analysis [Lui11].

It remains to bound the difference between \( (B_{\tilde{\varepsilon}, \varepsilon}, \Psi_{\tilde{\varepsilon}, \varepsilon}) \) and \( (B^{\tilde{\varepsilon}, \varepsilon}, \Psi^{\tilde{\varepsilon}, \varepsilon}) \).

It is standard to prove that for sufficiently small final time,

\[ \| B_{j}^{\tilde{\varepsilon}, \varepsilon} : B_{j}^{\varepsilon} \|_{Y, \eta} \lesssim \| \tilde{Z}_{\tilde{\varepsilon}, \varepsilon}^{(e)} : \tilde{Z}_{\tilde{\varepsilon}, \varepsilon}^{(e)} \|_{\delta, \eta; T} + \| \mathcal{P} : R_{B_{j}}^{\tilde{\varepsilon}, \varepsilon} : \mathcal{P} : R_{B_{j}}^{\varepsilon} \|_{Y, \eta} + \| \tilde{R}_{B_{j}}^{\tilde{\varepsilon}, \varepsilon} : \tilde{R}_{B_{j}}^{\varepsilon} \|_{Y, \eta} \quad (6.9) \]

and similarly for \( \| \Psi_{j}^{\tilde{\varepsilon}, \varepsilon} : \Psi_{j}^{\varepsilon} \|_{Y, \eta} \). By [HM18, Lemma 7.5] one has the bound uniformly in \( \varepsilon \leq \tilde{\varepsilon} \)

\[ \| K^{\tilde{\varepsilon}, \varepsilon} - K^{\varepsilon} : \psi^{\tilde{\varepsilon}, \varepsilon} \|_{-2 - \kappa; m} \lesssim \tilde{\varepsilon} \| \psi^{\tilde{\varepsilon}, \varepsilon} \|_{-2; m + 2} \]

From this one has the bound \( \| \tilde{Z}_{\tilde{\varepsilon}, \varepsilon}^{(e)} : \tilde{Z}_{\tilde{\varepsilon}, \varepsilon}^{(e)} \|_{\delta, \eta; T} \lesssim \tilde{\varepsilon} \kappa \) for some \( \kappa > 0 \) uniformly in \( \varepsilon < \tilde{\varepsilon} \).

Now consider

\[ \tilde{R}_{B_{j}}^{\tilde{\varepsilon}, \varepsilon} \equiv -\lambda^{2} P^{\varepsilon} : \ast_{\varepsilon} \left( \sum_{k=1,2} \varepsilon (B_{j}^{\tilde{\varepsilon}, \varepsilon}) (\nabla_{\varepsilon} \Psi_{k}^{\tilde{\varepsilon}, \varepsilon}) (\Psi_{k}^{\tilde{\varepsilon}, \varepsilon}) - c_{B}^{\tilde{\varepsilon}, \varepsilon} B_{j}^{\tilde{\varepsilon}, \varepsilon} \right) \circ (1, X). \]

Note that if \( c_{B}^{\tilde{\varepsilon}, \varepsilon} \) and \( c_{B}^{\tilde{\varepsilon}, \varepsilon} \) were zero, the difference between \( \tilde{R}_{B_{j}}^{\tilde{\varepsilon}, \varepsilon} \) and \( \tilde{R}_{B_{j}}^{\tilde{\varepsilon}, \varepsilon} \) would not be bounded by \( O(\tilde{\varepsilon}^{\kappa}) \) uniformly in \( \varepsilon \), because for fixed \( \tilde{\varepsilon} \) as \( \varepsilon \to 0 \), \( \tilde{R}_{B_{j}}^{\tilde{\varepsilon}, \varepsilon} \) would vanish while \( \tilde{R}_{B_{j}}^{\tilde{\varepsilon}, \varepsilon} \) would converge to a non-zero linear term. Now with \( c_{B}^{\tilde{\varepsilon}, \varepsilon} \) and \( c_{B}^{\tilde{\varepsilon}, \varepsilon} \) defined above, we can actually show a stronger statement that for any fixed \( \tilde{\varepsilon} > 0 \), one has \( \| \tilde{R}_{B_{j}}^{\tilde{\varepsilon}, \varepsilon} \|_{Y, \eta} + \| \tilde{R}_{B_{j}}^{\tilde{\varepsilon}, \varepsilon} \|_{Y, \eta} \lesssim \tilde{\varepsilon}^{\kappa} \) uniformly in \( \varepsilon \in (0, \varepsilon_{0}(\tilde{\varepsilon})) \). To this end, decompose \( B_{\tilde{\varepsilon}, \varepsilon}, \Psi_{\tilde{\varepsilon}, \varepsilon}, \tilde{B}_{\tilde{\varepsilon}, \varepsilon}, \tilde{\Psi}_{\tilde{\varepsilon}, \varepsilon} \) as in (6.3). It is easy to show that

\[ \varepsilon b_{j}^{\tilde{\varepsilon}, \varepsilon} : \psi_{k}^{\tilde{\varepsilon}, \varepsilon} : \nabla_{\varepsilon}^{\varepsilon} \psi_{k}^{\varepsilon} - c_{B}^{\tilde{\varepsilon}, \varepsilon} b_{j}^{\tilde{\varepsilon}, \varepsilon} \quad \text{and} \quad \varepsilon b_{j}^{\tilde{\varepsilon}, \varepsilon} : \psi_{k}^{\tilde{\varepsilon}, \varepsilon} : \nabla_{\varepsilon}^{\varepsilon} \psi_{k}^{\varepsilon} - c_{B}^{\tilde{\varepsilon}, \varepsilon} b_{j}^{\tilde{\varepsilon}, \varepsilon} \quad (6.10) \]

converge in \( L^{p} \) to zero in \( \tilde{C}^{\alpha} \) for \( \alpha < 0 \). Indeed, using similar arguments as in the proof of Proposition 5.6 and the ‘extra’ \( \varepsilon \), the third chaos of the left hand side of (6.10) vanishes.
in the limit; and the first chaos vanishes by definitions of \( c^\varepsilon_B, c^{\tilde{\varepsilon}}_B \). One can also show that

\[
\varepsilon w_j^\varepsilon \psi_k^\varepsilon \nabla_j^\varepsilon \psi_k^\varepsilon - c^\varepsilon_B w_j^\varepsilon \quad \text{and} \quad \varepsilon w_j^{\tilde{\varepsilon}} \psi_k^{\tilde{\varepsilon}} \nabla_j^{\tilde{\varepsilon}} \psi_k^{\tilde{\varepsilon}} - c^{\tilde{\varepsilon}}_B w_j^{\tilde{\varepsilon}}
\]

converge in \( L^p \) to zero in \( \hat{C}^\alpha \) for \( \alpha < 0 \). This follows from the fact that \( \varepsilon \psi_k^\varepsilon \nabla_j^\varepsilon \psi_k^\varepsilon - c^\varepsilon_B \) converges to zero in \( \hat{C}^\alpha \) with any \( \alpha < 0 \) and that \( w_j^\varepsilon \) converges in \( C^\alpha \) with any \( \alpha < 1 \), together with the classical Young’s theorem; same for the expression depending on \( (\tilde{\varepsilon}, \varepsilon) \). One also has convergences of the other terms such as \( \varepsilon b_j^\varepsilon \psi_k^\varepsilon \nabla_j^\varepsilon \psi_k^\varepsilon \rightarrow 0 \) and \( \varepsilon b_j^{\tilde{\varepsilon}} \psi_k^{\tilde{\varepsilon}} \nabla_j^{\tilde{\varepsilon}} \psi_k^{\tilde{\varepsilon}} \rightarrow 0 \) in \( \hat{C}^\alpha \) for \( \alpha < 0 \). Given all these, we conclude the claimed uniform bounds on \( \| \hat{R}_j^{\varepsilon,\tilde{\varepsilon}}, \|_{\gamma,\eta}^{(\varepsilon)} \) and \( \| \hat{R}_j^{\varepsilon,\tilde{\varepsilon}} \| \). The second term on the right hand side of (6.9) can be shown to be bounded by \( \tilde{\varepsilon}^\prec \) uniformly in \( \varepsilon \in (0, \varepsilon_0(\tilde{\varepsilon})) \) by similar (even simpler) arguments.

The difference \( \| \Psi_j^{\varepsilon}; \Psi_j^{\varepsilon,\tilde{\varepsilon}} \|_{\gamma,\eta}^{(\varepsilon)} \) can be bounded in the same way.

Summarizing the above estimates, the convergence of the discrete solutions in probability (3.18) as well as the convergence of the stopping times then follow in the same way as the diagonal arguments in [HM18, below (7.16)]. □

**Remark 6.3.** Equations (5.6) where the noises are continuously mollified do not enjoy the gauge invariance property, in particular Lemma 3.3. Here we know *a posteriori* that (5.6) without any mass renormalization in the \( B^\varepsilon,0 \) equation actually achieve the same limit; this seems to be due to the simplicity of this two-dimensional Abelian model. In fact, if \( c^\varepsilon_B \) were actually divergent (it is imaginable that this may happen in the more sophisticated gauge theory models discussed in “Appendix A”), then the argument below (6.8) would not be valid anymore and thus one would have to define \( C_1(\tilde{\varepsilon}) \) differently, which would lead to a nontrivial mass renormalization for \( B^\varepsilon,0 \) in the Eq. (5.6).

### 6.1. Convergence of observables

We start to prove Theorem 2.7. Since all the observables considered in the theorem are gauge invariant, it suffices to show their convergence with every incidence of \((A^\varepsilon, \Phi^\varepsilon)\) replaced by \((B^\varepsilon, \Psi^\varepsilon)\).

**Lemma 6.4.** Let \((A^\varepsilon, \Phi^\varepsilon)\) be the solutions to (2.9). The curvature \( F^\varepsilon_{A^\varepsilon} \) and the Wick powers \( :|\Phi^\varepsilon|^{2n} \) : converge in distribution with respect to the distance \( \| :|; :\|_{\varepsilon,\alpha,1}^{(\varepsilon)} \) and \( \| :|; :\|_{\varepsilon,\alpha,1}^{(\varepsilon)} \) respectively.

**Proof.** By definition (2.2), \( F^\varepsilon_{A^\varepsilon} = F^\varepsilon_{B^\varepsilon} \) is simply a finite difference discretization of the curl of \( B^\varepsilon \). Therefore, invoking the convergence of \( B^\varepsilon \) to its distributional limit \( B \) with respect to the distance \( \| :|; :\|_{\varepsilon,\alpha,1}^{(\varepsilon)} \), one immediately obtains the convergence of \( F^\varepsilon_{B^\varepsilon} \) to the distributional limit which is curl\( B \), with respect to the distance \( \| :|; :\|_{\varepsilon,\alpha,1}^{(\varepsilon)} \).

Consider the observable \( :|\Phi^\varepsilon|^{2n} := :|\Phi^\varepsilon|^2 = (\Phi^\varepsilon \bar{\Phi}^\varepsilon)^n \) for \( n \geq 1 \) where \( \bar{\Phi}^\varepsilon \) is the complex conjugate of \( \Phi^\varepsilon \). One has \( \Phi^\varepsilon = \psi^\varepsilon + v^\varepsilon \) where \( \psi^\varepsilon \overset{\text{def}}{=} K^\varepsilon \ast \xi^\varepsilon \) and \( v^\varepsilon \rightharpoonup v \in C([0,T], \hat{C}^\beta) \) for any \( \beta < 1 \). From the definition of Wick powers (2.24) with respect to the Gaussian measure of \( \psi^\varepsilon \), one can also deduce

\[
:|\Phi^\varepsilon \bar{\Phi}^\varepsilon|^n := \sum_{p,q=0}^n \binom{n}{p} \binom{n}{q} :\psi^p \bar{\psi}^q : v^{n-p} \bar{v}^{p} v^{-q} - q.
\]

(6.11)
Note that $\psi^p_j \psi^q_k$ is simply a Wick product of Gaussian process $(\psi_j, \bar{\psi}_j)$ which converge in probability in $C([0, T], C^\beta)$ for any $\beta < 0$. Using the convergence of the Wick products and the aforementioned $C^\beta$ regularity of $\nu$, by applying the classical Young’s theorem to the products in (6.12) we obtain the convergence in probability of $(\Psi^\epsilon \tilde{\Psi}^\epsilon)^n$ and thus the convergence in distribution of $C(\Phi^\epsilon \tilde{\Phi}^\epsilon)^n$. Note that the logarithmic renormalization constants in the Wick products, and thus the limit of $(\Phi^\epsilon \tilde{\Phi}^\epsilon)^n$, do not depend on the choice of the truncated heat kernel $K^\epsilon$; see for instance [HS16, Lemma 3.1]. □

Let

$$C^\epsilon_{\Phi D^A \Phi} \overset{\text{def}}{=} 2 \int_{\mathbb{R} \times \Lambda_\epsilon} P^\epsilon(-z) \nabla_j^\epsilon P^\epsilon(-z) \, dz. \quad (6.12)$$

Note that the “parity” symmetry $x \to -x$ would render a zero limit if the above expression were in continuum, but it does not apply here on the lattice. Instead:

**Lemma 6.5.** One has $C^\epsilon_{\Phi D^A \Phi} = -1/(4\epsilon)$.

**Proof.** As the argument in Lemma 5.1 one has

$$C^\epsilon_{\Phi D^A \Phi} = 2 \int_{\mathbb{R} \times \Lambda_\epsilon} P^\epsilon(-z) \nabla_j^\epsilon P^\epsilon(-z) \, dz$$

and adding this to (6.12) one has

$$C^\epsilon_{\Phi D^A \Phi} = \epsilon \int_{\mathbb{R} \times \Lambda_\epsilon} P^\epsilon(-z) \nabla_j^\epsilon \nabla_j^\epsilon P^\epsilon(-z) \, dz = \frac{\epsilon}{d} \int_{\mathbb{R} \times \Lambda_\epsilon} P^\epsilon(t, x) \Delta^\epsilon P(t, x) \, dt \, dx$$

$$= \frac{\epsilon}{2d} \int_{\mathbb{R} \times \Lambda_\epsilon} \partial_t (P^\epsilon(t, x)^2) \, dt \, dx = -\frac{\epsilon}{2d} \int_{\Lambda_\epsilon} P^\epsilon(0, x)^2 \, dx = -\frac{1}{2d \epsilon^{d-1}} = -\frac{1}{4\epsilon}.$$ □

**Lemma 6.6.** The composite field observables $\tilde{\Psi}^\epsilon(e-) D^A_j \Phi^\epsilon(e) - C^\epsilon_{\Phi D^A \Phi}$ as introduced in (2.25) with $e \in \mathcal{E}^j_\epsilon$ converge in distribution with respect to the distance $\|\cdot\|_{\epsilon^0, \mu, \tau_\epsilon}$.

**Proof.** By its gauge invariance it again suffices to consider $\tilde{\Psi}^\epsilon(e-) D^B_j \Psi^\epsilon(e)$. Recall that $\Psi^\epsilon = \Psi^\epsilon_1 + i \Psi^\epsilon_2$. By definition for $e \in \mathcal{E}^j_\epsilon$ one has

$$\tilde{\Psi}^\epsilon(e_-)(D^B_j \Psi^\epsilon)(e) - C^\epsilon_{\Phi D^A \Phi}$$

$$= \epsilon^{-1} \tilde{\Psi}^\epsilon(e_-) \left( e^{-i\epsilon \lambda B^\epsilon_j(e)} \Psi^\epsilon(e_+) - \Psi^\epsilon(e_-) \right) - C^\epsilon_{\Phi D^A \Phi}$$

$$= \sum_k \left( \Psi^\epsilon_k(e_-) \nabla_j^\epsilon \Psi^\epsilon_k(e_-) \right) + \lambda \Psi^\epsilon_1(e_-) B^\epsilon_j(e) \Psi^\epsilon_2(e_+) - \lambda \Psi^\epsilon_2(e_-) B^\epsilon_j(e) \Psi^\epsilon_1(e_+)$$

$$+ i \left( \Psi^\epsilon_1(e_-) \nabla_j^\epsilon \Psi^\epsilon_2(e_-) - \Psi^\epsilon_2(e_-) \nabla_j^\epsilon \Psi^\epsilon_1(e_-) - \lambda B^\epsilon_j(e) \sum_k \Psi^\epsilon_k(e_-) \Psi^\epsilon_k(e_+) \right)$$

$$- C^\epsilon_{\Phi D^A \Phi} + R^\epsilon_j(e) \quad (6.13)$$

where we have separated the real and imaginary parts of the process, and $R^\epsilon_j$ is a remainder of “order $O(\epsilon B^2 \Psi^2)$”

$$R^\epsilon_j(e) \overset{\text{def}}{=} \epsilon^{-1} \tilde{\Psi}^\epsilon(e_-) \left( e^{-i\epsilon \lambda B^\epsilon_j(e)} - 1 + i \epsilon \lambda B^\epsilon_j(e) \right) \Psi^\epsilon(e_+).$$
Since
\[ |e^{-i\varepsilon \lambda B_j^\varepsilon (e)} - 1 + i \varepsilon \lambda B_j^\varepsilon (e)| \leq \cos(\varepsilon \lambda B_j^\varepsilon (e)) - 1 + |\varepsilon \lambda B_j^\varepsilon (e) - \sin(\varepsilon \lambda B_j^\varepsilon (e))| \lesssim \varepsilon^2 B_j^\varepsilon (e)^2 \]
and \( B_j^\varepsilon, \Psi^\varepsilon \) converge in \( C([0, T], C^\beta) \) for any \( \beta < 0 \), we have \( \sup_e |R_j^\varepsilon (e)| \to 0 \).

We will now prove convergence of the real and imaginary parts of (6.13) separately, using the knowledge that the solution \((B^\varepsilon, \Psi^\varepsilon)\) is given by the reconstruction of the abstract solution which has the expansion (5.7), namely, with \( b_j^\varepsilon = \hat{\Pi}_j^\varepsilon \) and \( \psi_j^\varepsilon = \hat{\Pi}_j^\varepsilon \) as before, and \( \hat{\Pi}^\varepsilon = \hat{\Pi}_k^\varepsilon \) for short
\[
B_j^\varepsilon (e) = B_j^\varepsilon (e) = b_j^\varepsilon (e) + w_j^\varepsilon (e-) \quad (6.14a) \\
\Psi_j^\varepsilon (x) = \psi_j^\varepsilon (x) - (-1)^j \lambda \sum_{k=1,2} \left[ \hat{\Pi}_k^\varepsilon \hat{\Pi}_j^\varepsilon + \hat{\Pi}_k^\varepsilon \hat{\Pi}_j^\varepsilon - \hat{\Pi}_k^\varepsilon \hat{\Pi}_k^\varepsilon \right] (x) + \tilde{v}_j^\varepsilon (x). \quad (6.14b)
\]

Consider the real part. Let \( x = e_- \) and consider \( \psi_k^\varepsilon (x) \nabla_j^\varepsilon \psi_k^\varepsilon (x) \). By standard moment analysis as in Sect. 5.2, we get convergence of the following processes with respect to the distance \( \| \cdot \|_{\rho_\varepsilon^D, \sigma^{-1}} \)
\[
\psi_k^\varepsilon (x) \nabla_j^\varepsilon \psi_k^\varepsilon (x) - \frac{1}{2} C^\varepsilon_{\Phi D^A \Phi},
\]
noting that \( C^\varepsilon_{\Phi D^A \Phi} \) in (6.12) is exactly such that the above renormalized process is mean zero. One also needs to control the “cross terms” between \( \psi_k^\varepsilon \) and \( O(\lambda) \)-terms of (6.14b), namely:

where, it is important to notice that in all these terms, the indices \( \ell \neq k \). Thus by standard moment analysis and independence of \( \psi_k^\varepsilon \) and \( \psi_j^\varepsilon \) one obtains convergences of each of these processes. All the other terms in \( \Psi_k^\varepsilon (x) \nabla_j^\varepsilon \psi_k^\varepsilon (x) \) then fall into the scope of Young’s theorem and thus converge in the desired topologies. The convergence of the term \( \Psi_k^\varepsilon (e_-) B_j^\varepsilon (e) \Psi_{3-k}^\varepsilon (e+) \) is even simpler since one only needs the first order expansion in (6.14), the independence of \( \psi_k^\varepsilon, b_j^\varepsilon \), and Young’s theorem.

Regarding the imaginary part on the right side of (6.13), it turns out that we must treat all the terms in the parenthesis in the last line of (6.13) together, since each individual term does not converge!

Indeed, it is easy to see by independence of \( \psi_1^\varepsilon \) and \( \psi_2^\varepsilon \) that to the “leading order”, one has convergences of \( \psi_1^\varepsilon \nabla_j^\varepsilon \psi_2^\varepsilon \) and \( \psi_2^\varepsilon \nabla_j^\varepsilon \psi_1^\varepsilon \) (individually). However, the “next order” requires more careful analysis. Considering the “next order” objects from \( \Psi_1^\varepsilon (e_-) \nabla_j^\varepsilon \Psi_2^\varepsilon (e_-) \) and \( \Psi_2^\varepsilon (e_-) \nabla_j^\varepsilon \Psi_1^\varepsilon (e_-) \), we note that all terms of the form
\[
\nabla_j^\varepsilon \psi_j^\varepsilon \hat{\Pi}_k^\varepsilon \quad \text{or} \quad \psi_j^\varepsilon \nabla_j^\varepsilon \hat{\Pi}_k^\varepsilon
\]
will (individually) converge, because in the symbol \( \frac{d}{dx} \) the derivative lives on the edge \( k \), so the first chaos has no divergence. It remains to consider the following objects

\[
\sum_{k=1}^{2} \left( \begin{array}{c}
\begin{array}{c}
 k \\
 j
\end{array}
\end{array} \begin{array}{c}
 3 \\
 2
\end{array}
 \right) - \left( \begin{array}{c}
\begin{array}{c}
 k \\
 j
\end{array} \begin{array}{c}
 2 \\
 j
\end{array}
\end{array} \begin{array}{c}
 1 \\
 -k, j
\end{array}
 + \left( \begin{array}{c}
\begin{array}{c}
 -k \\
 j
\end{array} \begin{array}{c}
 2 \\
 j
\end{array}
\end{array} \begin{array}{c}
 1 \\
 -k, j
\end{array}
 + \left( \begin{array}{c}
\begin{array}{c}
 -k \\
 j
\end{array} \begin{array}{c}
 2 \\
 j
\end{array}
\end{array} \begin{array}{c}
 1 \\
 -k, j
\end{array}
(6.15)
\end{array} \right)
\]

where the graphs on the first line arise from \( \Psi_1^\varepsilon \nabla_j^\varepsilon \Psi_2^\varepsilon \), and the graphs on the second line arise from \( -\Psi_2^\varepsilon \nabla_j^\varepsilon \Psi_1^\varepsilon \). Standard moment analysis as in Sect. 5 shows that the third chaos of each graph converges. Apparently, the first chaos of each graph diverges when \( k = j \) and thus one needs a logarithmic renormalization constant for each graph in order to obtain a finite limit. However, the renormalizations needed for the terms in the first line of (6.15) are

\[
\sum_{k} \left( \begin{array}{c}
\begin{array}{c}
 k \\
 j
\end{array} \begin{array}{c}
 3 \\
 j
\end{array}
\end{array} - \left( \begin{array}{c}
\begin{array}{c}
 k \\
 j
\end{array} \begin{array}{c}
 2 \\
 j
\end{array}
\end{array} \begin{array}{c}
 1 \\
 -k, j
\end{array}
 + \left( \begin{array}{c}
\begin{array}{c}
 -k \\
 j
\end{array} \begin{array}{c}
 2 \\
 j
\end{array}
\end{array} \begin{array}{c}
 1 \\
 -k, j
\end{array}
 + \left( \begin{array}{c}
\begin{array}{c}
 -k \\
 j
\end{array} \begin{array}{c}
 2 \\
 j
\end{array}
\end{array} \begin{array}{c}
 1 \\
 -k, j
\end{array}
(6.15)
\end{array} \right)
\]

and the renormalizations needed for the terms in the second line of (6.15) are the same; and, these are exactly the first four terms in (5.24). These together with a constant

\[
\sum_{k} E(\psi_k^\varepsilon(e_-)\psi_k^\varepsilon(e_+))
\]

which renormalizes the term \( -B_\varepsilon^\Phi(e) \sum_k \Psi_k^\varepsilon(e_-)\Psi_k^\varepsilon(e_+) \) in (6.13) yields a finite limit, which is true due to the arguments (5.24), (5.25) and the graphic “Ward identity” (5.26).

\[\square\]

**Remark 6.7.** Note that any renormalization to the observable \( \tilde{\Phi}(e_-)D_j^A\Phi(e) \) other than subtracting a constant \( C_\varepsilon^\Phi \) would violate its gauge invariance; thanks to the above cancellation this is the only renormalization.

Turning to the loop observables (2.26), let \( C \) be a \( C^2 \) loop with \( C^2 \) parametrization \( r : [0, 1] \to \mathbb{T}^2 \) such that

\[|r(\sigma) - r(\tilde{\sigma})| \asymp |\sigma - \tilde{\sigma}|\]

for any \( \sigma, \tilde{\sigma} \in [0, 1] \) such that \( |\sigma - \tilde{\sigma}| \) is sufficiently small, where \( \asymp \) denotes being bounded from above and below up to multiplicative constants. Let \( C^\varepsilon \) be a regular approximation of \( C \), and \( P^C \) be the partition of \([0, 1] \) as in Sect. 2.4. Since the solution \( B^\varepsilon = K^\varepsilon \ast \varepsilon \xi^\varepsilon + w^\varepsilon \) where \( w^\varepsilon \to w \) in \( C^\delta((0, T], C^{\alpha-\delta}) \) for any \( \alpha < 1 \) and \( \delta > 0 \) sufficiently small, we show convergence of \( \varepsilon \sum_{e \in C^\varepsilon} (K^\varepsilon \ast \varepsilon \xi^\varepsilon)(e) \) and \( \varepsilon \sum_{e \in C^\varepsilon} w^\varepsilon(e) \) separately.
Fig. 3. A segment \([\sigma_i, \sigma_{i+1}]\). There are 6 distinct edges \(e \in \mathcal{E}_\varepsilon\) such that \(e \in [\sigma_i, \sigma_{i+1}]\), with \(n_{1}^{[\sigma_i, \sigma_{i+1}]} = 5\) and \(n_{2}^{[\sigma_i, \sigma_{i+1}]} = 1\)

Lemma 6.8. For each \(t \in (0, T)\),

\[
\lim_{\varepsilon \to 0} \varepsilon \sum_{e \in C^\varepsilon} w^\varepsilon(t, e) = \int_C w(t).
\]

Moreover this limit holds as processes in the space \(C([0, T], \mathbb{R})\).

This seems to be a classical result of approximating line integrals, but we did not find a reference for exactly such a statement, so we give a self-contained proof which also gives some hint on the next proof.

Proof. Assuming \(\varepsilon\) sufficiently small, we divide the interval \([0, 1]\) into subintervals \(0 = \sigma_0 < \sigma_1 < \cdots < \sigma_M = 1\) such that each of \([\sigma_i, \sigma_{i+1}]\) is a union of intervals in \(P^C\), and \(\frac{5}{7} \sqrt{\varepsilon} \leq |\sigma_{i+1} - \sigma_i| \leq \frac{8}{7} \sqrt{\varepsilon}\) (so \(M = O(\varepsilon^{-\frac{1}{2}})\)).

Fixing a time \(t\) and omitting it in our notation, one has

\[
\left| \int_C w - \varepsilon \sum_{e \in C^\varepsilon} w^\varepsilon(e) \right| \leq \sum_{i=1}^{M} \left| \int_{\sigma_i}^{\sigma_{i+1}} w(r(\sigma)) \cdot \dot{r}(\sigma) \, d\sigma - \int_{\sigma_i}^{\sigma_{i+1}} w(r(\sigma_i)) \cdot \dot{r}(\sigma_i) \, d\sigma \right|
\]

\[
+ \left| \int_{\sigma_i}^{\sigma_{i+1}} w(r(\sigma_i)) \cdot \dot{r}(\sigma_i) \, d\sigma - \varepsilon \sum_{e \in [\sigma_i, \sigma_{i+1}]} \left( w_1(\mathbf{r}(\sigma_i))\mathbf{1}_{e \in \mathcal{E}_1^\varepsilon} + w_2(\mathbf{r}(\sigma_i))\mathbf{1}_{e \in \mathcal{E}_2^\varepsilon} \right) \right|
\]

\[
+ \varepsilon \sum_{e \in [\sigma_i, \sigma_{i+1}]} \left( w_1(\mathbf{r}(\sigma_i))\mathbf{1}_{e \in \mathcal{E}_1^\varepsilon} + w_2(\mathbf{r}(\sigma_i))\mathbf{1}_{e \in \mathcal{E}_2^\varepsilon} \right) - \varepsilon \sum_{e \in [\sigma_i, \sigma_{i+1}]} w^\varepsilon(e), \quad (6.16)
\]

where \(\dot{r}\) is the derivative w.r.t. the parameter \(\sigma\), and \(e \subset [\sigma_i, \sigma_{i+1}]\) means that the edge \(e \in \mathcal{E}_\varepsilon\) corresponds to a subinterval of \([\sigma_i, \sigma_{i+1}]\). For the first term on RHS of (6.16), since \(w \in C^{1-}\) and \(\dot{r} \in C^2\), one has

\[
|\dot{r}(\sigma) - \dot{r}(\sigma_i)| \lesssim |\sigma - \sigma_i| \lesssim \varepsilon^{\frac{1}{2}} \quad \text{for } \sigma \in [\sigma_i, \sigma_{i+1}]
\]

\[
|w(r(\sigma)) - w(r(\sigma_i))| \lesssim |r(\sigma) - r(\sigma_i)|^{\alpha} \lesssim \varepsilon^{\alpha/2}
\]

for any \(\alpha < 1\), thus \(\sum_{i=1}^{M} \int_{\sigma_i}^{\sigma_{i+1}} O(\varepsilon^{\alpha/2}) \, d\sigma = O(\varepsilon^{-\frac{1}{2}})O(\varepsilon^{\frac{1}{2}})O(\varepsilon^{\alpha}) \to 0\).
Consider the second term on RHS of (6.16). One has that for $j \in \{1, 2\}$

$$
\sum_{e \subset [\sigma_i, \sigma_{i+1}]} \left( w_1(r(\sigma_i)) 1_{e \in \mathcal{E}_j^c} + w_2(r(\sigma_i)) 1_{e \in \mathcal{E}_j^c} \right)
\quad = (w_1(r(\sigma_i)), w_2(r(\sigma_i))) \cdot (n_1^{[\sigma_i, \sigma_{i+1}]}, n_2^{[\sigma_i, \sigma_{i+1}]})
$$

where $n_1^{[\sigma_i, \sigma_{i+1}]}$ (resp. $n_2^{[\sigma_i, \sigma_{i+1}]}$) is the number of horizontal (resp. vertical) edges in the discrete curve $C^e$ that correspond to subintervals of $[\sigma_i, \sigma_{i+1}]$, see Fig. 3.

Note that $(n_1^{[\sigma_i, \sigma_{i+1}]}, n_2^{[\sigma_i, \sigma_{i+1}]})$ approximately gives the tangent direction of $C$ at $r(\sigma_i)$. To be more precise, since $r \in C^2$ one has

$$
\dot{r}(\sigma_{i+1}) - \dot{r}(\sigma_i) = \ddot{r}(\sigma_i)(\sigma_{i+1} - \sigma_i) + O(\varepsilon).
$$

On the other hand when $\varepsilon$ is sufficiently small one has

$$
\dot{r}(\sigma_{i+1}) - \dot{r}(\sigma_i) = \varepsilon(n_1^{[\sigma_i, \sigma_{i+1}]}, n_2^{[\sigma_i, \sigma_{i+1}]}) + O(\varepsilon).
$$

These show that $\sum_{i=1}^M$ of the second term on RHS of (6.16) equals $\sum_{i=1}^M O(\varepsilon) = O(\varepsilon^{-\frac{1}{2}})O(\varepsilon)$ which vanishes in the limit.

Finally, since $w^e$ converges to $w$ in $C^{1-}$, and $|r(\sigma_i) - \varepsilon| \lesssim \varepsilon^{\frac{1}{2}}$, we immediately have that $\sum_{i=1}^M$ of the last term on RHS of (6.16) vanishes in the limit.

Finally, to prove continuity in time of this observable, we can bound $\left| \int_C \delta^{s,t} w - \varepsilon \sum_{e \in C^e} \delta^{s,t} w^e(\varepsilon) \right|$ in the same way as above, replacing $w$ and $w^e$ by $\delta^{s,t} w$ and $\delta^{s,t} w^e$ respectively, and using $w^e \to w$ in $C^\alpha([0, T], C^{\alpha-\delta})$. $\square$

We now turn to convergence of $\varepsilon \sum_{e \in C^e} K^e \ast e \xi^e(\varepsilon)$ and $\varepsilon^{\frac{1}{2}} \sum_{e \in C^e} K^e \ast e^2 \xi^e(\varepsilon)$. Fixing $t \in [0, T]$, let $r : [0, 1] \to T^2$ be a parametrization of the loop $C$. Write $r = (r_1, r_2)$.

We define $\int_C K \ast \xi$ to be the centered Gaussian random variable with variance

$$
\int_0^1 \int_0^1 \int_{T^2} K(t - s, r(\sigma) - y) K(t - s, \tilde{r}(\tilde{\sigma}) - y) \left( \tilde{r}(\sigma) \cdot \tilde{r}(\tilde{\sigma}) \right) \, ds \, dy \, d\sigma \, d\tilde{\sigma}
$$

which is finite since the integral over $(s, y)$ yields a function which behaves as $\log |r(\sigma) - \tilde{r}(\tilde{\sigma})|$ for $|\sigma - \tilde{\sigma}|$ small, and $|r(\sigma) - \tilde{r}(\tilde{\sigma})|$ is bounded from above and below by $|\sigma - \tilde{\sigma}|$.

**Lemma 6.9.** As $\varepsilon \to 0$, $\varepsilon \int_C K \ast \xi \xi^e(\varepsilon) \ast e$ converge in law to $\int_C K \ast \xi$ in $C([0, T], \mathbf{R})$.

**Proof.** Since with the fixed loop $C$ and fixed $t$, the sequence of random variables in consideration are centered Gaussians, to prove convergence for each fixed $t$ we only need to prove convergence of the variances. Omitting $t$ in our notation, for $x \neq y \in T^2$ let

$$
\hat{K}(x, y) = \hat{K}(x - y) \overset{\text{def}}{=} \int_{T^2} K(t - s, x - z) K(t - s, y - z) \, ds \, dz
$$

and for $e, \bar{e} \in \mathcal{E}_j^e$ for $j \in \{1, 2\}$ (meaning $e, \bar{e}$ are both horizontal or both vertical)

$$
\hat{K}^e(e, \bar{e}) = \hat{K}^e(e - \bar{e}) \overset{\text{def}}{=} \int_{R} \varepsilon^2 \sum_{e' \in \mathcal{E}_j^e} K^e(t - s, e - e') K^e(t - s, \bar{e} - e') \, ds.
$$

(6.17)
The variance of $\int C K * \xi$ is then equal to
\[
\int_0^1 \int_0^1 \hat{K} (r(\sigma) - r(\tilde{\sigma})) \, (\dot{r}(\sigma) \cdot \dot{r}(\tilde{\sigma})) \, d\sigma d\tilde{\sigma}
\] (6.18)
and the variance of $\varepsilon \sum_{e \in C^e} K^e * \xi^e(e)$ is equal to
\[
\varepsilon^2 \sum_{e, \tilde{e} \in C^e} \hat{K}^\varepsilon (e - \tilde{e}) (1_{e, \tilde{e} \in \mathcal{E}_1^e} + 1_{e, \tilde{e} \in \mathcal{E}_2^e}).
\] (6.19)

As in the proof above we divide the interval $[0, 1]$ into subintervals $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_M = 1$ such that each of $[\sigma_i, \sigma_{i+1})$ is a union of intervals in $P_{\varepsilon}^C$, so that $|\sigma_{i+1} - \sigma_i| = O(\sqrt{\varepsilon})$ and $M = O(\varepsilon^{-\frac{1}{2}})$. The difference between (6.18) and (6.19) is equal to $S_1 + S_2 + S_3 + S_4$ where
\[
S_1 \overset{\text{def}}{=} \sum_{|i - j| > 1} \int_{\sigma_i}^{\sigma_{i+1}} \int_{\sigma_j}^{\sigma_{j+1}} \left( \hat{K} (r(\sigma) - r(\tilde{\sigma})) \dot{r}(\sigma) \cdot \dot{r}(\tilde{\sigma}) - \hat{K} (r(\sigma_i) - r(\sigma_j)) \dot{r}(\sigma_i) \cdot \dot{r}(\sigma_j) \right) d\sigma d\tilde{\sigma}
\]
\[
S_2 \overset{\text{def}}{=} \sum_{|i - j| > 1} \left[ \int_{\sigma_i}^{\sigma_{i+1}} \int_{\sigma_j}^{\sigma_{j+1}} \hat{K} (r(\sigma) - r(\tilde{\sigma})) \dot{r}(\sigma) \cdot \dot{r}(\tilde{\sigma}) d\sigma d\tilde{\sigma} - \varepsilon^2 \sum_{e \in \sigma_i, \sigma_j+1} \hat{K}^\varepsilon (e - \tilde{e}) (1_{e, \tilde{e} \in \mathcal{E}_1^e} + 1_{e, \tilde{e} \in \mathcal{E}_2^e}) \right]
\]
\[
S_3 \overset{\text{def}}{=} \sum_{|i - j| > 1} \left[ \int_{\sigma_i}^{\sigma_{i+1}} \int_{\sigma_j}^{\sigma_{j+1}} \hat{K} (r(\sigma) - r(\tilde{\sigma})) \dot{r}(\sigma) \cdot \dot{r}(\tilde{\sigma}) d\sigma d\tilde{\sigma} - \varepsilon^2 \sum_{e \in \sigma_i, \sigma_j+1} \hat{K}^\varepsilon (e - \tilde{e}) (1_{e, \tilde{e} \in \mathcal{E}_1^e} + 1_{e, \tilde{e} \in \mathcal{E}_2^e}) \right]
\]
\[
S_4 \overset{\text{def}}{=} \sum_{|i - j| \leq 1} \left[ \int_{\sigma_i}^{\sigma_{i+1}} \int_{\sigma_j}^{\sigma_{j+1}} \hat{K} (r(\sigma) - r(\tilde{\sigma})) \dot{r}(\sigma) \cdot \dot{r}(\tilde{\sigma}) d\sigma d\tilde{\sigma} - \varepsilon^2 \sum_{e \in \sigma_i, \sigma_j+1} \hat{K}^\varepsilon (e - \tilde{e}) (1_{e, \tilde{e} \in \mathcal{E}_1^e} + 1_{e, \tilde{e} \in \mathcal{E}_2^e}) \right]
\]

Consider $S_1$. By definition of $\hat{K}$ one has $|\hat{K}'(x)| \lesssim |x|^{-1}$. One also has $|r(\sigma) - r(\sigma_i)| \lesssim \varepsilon^{\frac{1}{2}}$ and $|r(\tilde{\sigma}) - r(\sigma_j)| \lesssim \varepsilon^{\frac{1}{2}}$ as well as
\[
|r(\sigma) - r(\tilde{\sigma})|^{-1} \lesssim \frac{1}{|i - j| \sqrt{\varepsilon}} \quad \text{and} \quad |r(\sigma_i) - r(\sigma_j)|^{-1} \lesssim \frac{1}{|i - j| \sqrt{\varepsilon}} \quad \text{for } |i - j| > 1.
\]
So by a Taylor remainder theorem
\[
|\hat{K} (r(\sigma) - r(\tilde{\sigma})) - \hat{K} (r(\sigma_i) - r(\sigma_j))| \lesssim \frac{1}{|i - j|}. \quad (6.20)
\]
Noting that \( r \in C^2 \) one only needs to bound \( S'_1 \) which is the same as \( S_1 \) except that \( \hat{r}(\sigma) \cdot \hat{r}(\bar{\sigma}) \) is replaced by \( \hat{r}(\sigma_i) \cdot \hat{r}(\sigma_j) \). Since \(|\hat{r}(\sigma) \cdot \hat{r}(\bar{\sigma}) - \hat{r}(\sigma_i) \cdot \hat{r}(\sigma_j)| \lesssim \varepsilon^2 \) it is easy to see that \(|S_1 - S'_1| \rightarrow 0\). Invoking the decay (6.20) one can bound \( S'_1 \) by

\[
|S'_1| \lesssim \sum_{1 \leq i, j \leq M}^1 \frac{(\sigma_{i+1} - \sigma_{i})(\sigma_{j+1} - \sigma_{j})}{|i - j|} \lesssim \varepsilon \varepsilon M \log M \lesssim \varepsilon \varepsilon^{-\frac{1}{2}} |\log \varepsilon| \rightarrow 0.
\]

Consider \( S_2 \). As in the proof of Lemma 6.8, one has

\[
\hat{r}(\sigma_i)(\sigma_{i+1} - \sigma_i) = \epsilon \left(n_1^{|\sigma_{i+1} - \sigma_i|}, n_2^{|\sigma_{i+1} - \sigma_i|}\right) + O(\epsilon)
\]

and the same holds with \( i \) replaced by \( j \); thus

\[
\int_{\sigma_{i}}^{\sigma_{i+1}} \int_{\sigma_{j}}^{\sigma_{j+1}} \hat{r}(\sigma_i) \cdot \hat{r}(\sigma_j) d\sigma d\bar{\sigma} = \epsilon^2 \left(n_1^{|\sigma_{i+1} - \sigma_i|}, n_2^{|\sigma_{i+1} - \sigma_i|}\right) \cdot \left(n_1^{|\sigma_{j+1} - \sigma_j|}, n_2^{|\sigma_{j+1} - \sigma_j|}\right) + O(\epsilon^3).
\]

Since \(|\hat{K}(x)| \lesssim |x|^{-\frac{1}{2}} \) and \(|r(\sigma_i) - r(\sigma_j)| \geq \frac{1}{\sqrt{\epsilon}} \) for \(|i - j| > 1\), one has

\[
|S_2| \lesssim \sum_{i, j} \varepsilon^{-\frac{1}{2}} \varepsilon^3 = \epsilon^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}} \varepsilon^3 \rightarrow 0.
\]

Turning to \( S_3 \), as in the analysis of \( S_1 \) one has (recall from Sect. 2.1 that in the notation \( \hat{K}(e - \bar{e}), e, \bar{e} \) denote the mid-points of the edges)

\[
\sum_{|i - j| > 1} \varepsilon^2 \sum_{e \subset [\sigma_i, \sigma_{i+1}], \bar{e} \subset [\sigma_j, \sigma_{j+1}]} |\hat{K}(r(\sigma_i) - r(\sigma_j)) - \hat{K}(e - \bar{e})| (1_{e, \bar{e} \in \mathcal{E}_1} + 1_{e, \bar{e} \in \mathcal{E}_2}) \lesssim \varepsilon^2 \varepsilon^{-\frac{1}{2}} \sum_{|i - j| > 1} \frac{|r(\sigma_i) - e| + |r(\sigma_j) - \bar{e}|}{|i - j| \sqrt{\epsilon}} \lesssim \epsilon \epsilon^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}} \log \epsilon \rightarrow 0.
\]

To finish the estimate of \( S_3 \) we claim that for \(|e - \bar{e}| > \sqrt{\epsilon}\),

\[
|\hat{K}(e - \bar{e}) - \hat{K}_e(e - \bar{e})| \lesssim \epsilon^\kappa |e - \bar{e}|^{-\kappa}
\]

for some small \( \kappa > 0 \) so that

\[
\sum_{|i - j| > 1} \varepsilon^2 \sum_{e \subset [\sigma_i, \sigma_{i+1}], \bar{e} \subset [\sigma_j, \sigma_{j+1}]} |\hat{K}_e(e - \bar{e}) - \hat{K}(e - \bar{e})| (1_{e, \bar{e} \in \mathcal{E}_1} + 1_{e, \bar{e} \in \mathcal{E}_2}) \lesssim \sum_{|i - j| > 1} \varepsilon^2 \sum_{e \subset [\sigma_i, \sigma_{i+1}], \bar{e} \subset [\sigma_j, \sigma_{j+1}]} \epsilon^\kappa \epsilon^{-2\kappa} \lesssim \epsilon^{-1} \epsilon^2 \epsilon^{-1} \epsilon^\kappa \epsilon^{-2\kappa} \rightarrow 0,
\]

where we used the fact that as long as \(|i - j| > 1\), \(|e - \bar{e}|^{-\kappa} \lesssim \epsilon^{-2\kappa} \). To prove (6.21), we assume \(|e - \bar{e}| > \sqrt{\epsilon} \) and note that for any constant \( c > 0 \)

\[
\left| \int_{t - \epsilon}^t \int_{T^2} K(t - s, x - z) K(t - s, y - z) dsdz \right| \lesssim \epsilon^\kappa |x - y|^{-\kappa},
\]
and the same bound holds in the discrete case with $K$ replaced by $K^\varepsilon$ and spatial integral replaced by spatial summation times $\varepsilon^2$. Indeed, $K$ coincides with $P$ in an $O(1)$ neighborhood of the origin, and one has $\int_{T_2} P(t-s, x-z) P(t-s, y-z) dz = P(2t - 2s, x - y)$; then it suffices to realize that $f(x) \equiv \int_0^\varepsilon P(s, x) ds$ satisfies the equation $\Delta f(x) = P(\varepsilon, x)$ because $P$ satisfies the heat equation, namely $f = P(\varepsilon, \cdot)^*(\Delta)^{-1}$ from which the above bound follows.

Now to prove (6.21), using Lemma 5.5, item 2, it suffices to prove that for $t > c\varepsilon$ for some constant $c > 0$ one has

$$|K^\varepsilon(z) - K(z)| \lesssim \varepsilon^k \|z\|^{-k}_g$$

for some small $\kappa > 0$. Again it suffices to consider $|P^\varepsilon(z) - P(z)|$. We have a local central limit theorem for continuous time random walk: by [LL10, Theorem 2.5.6] translating into our notation, one has that as long as $\varepsilon|x| \leq t/2$ (which follows from $t > c\varepsilon$ for some constant $c > 0$ since $|x| = O(1)$) $|P^\varepsilon(t, x) - P(t, x)| \lesssim t^{-1} e^{-(\varepsilon/\sqrt{t}) + \varepsilon|x|^3/t^2)}$.

When $|x| \leq \sqrt{t}$, the dominant term in the parenthesis is $\varepsilon/\sqrt{t}$, and using $e^{-|x|^2/4t} \leq 1$ we obtain the desired bound. When $|x| > \sqrt{t}$, the dominant term in the parenthesis is $\varepsilon|x|^3/t^2$, using $e^{-|x|^2/4t} \lesssim t^3/|x|^6$ we again obtain the desired bound.

Finally, consider $S_4$. Now $\sum_{|i-j|\leq 1}$ is only a sum of $O(\varepsilon^{-\frac{1}{2}})$ terms. We can estimate the two terms in $S_4$ separately without using the minus sign. For the first term one can bound

$$|\hat{K}(r(\sigma) - r(\hat{\sigma})) \dot{r}(\sigma) \cdot \dot{r}(\hat{\sigma})| \lesssim |\sigma - \hat{\sigma}|^{-\frac{1}{8}}.$$}

So $\sum_{|i-j|\leq 1} \int_{\sigma_i}^{\sigma_{i+1}} \int_{\sigma_j}^{\sigma_{j+1}}$ of the above quantity is bounded by $\varepsilon^{-\frac{1}{2}}(\sqrt{\varepsilon})^{2-\frac{1}{2}}$ which vanishes in the limit. Since $|\hat{K}^\varepsilon(e - \varepsilon)| \lesssim \varepsilon^{-\frac{1}{8}}$ the same argument shows that the other term also vanishes in the limit.

The continuity in time essentially follows in the same way. To bound $E(\varepsilon^{\sum_c K^\varepsilon \star_x \xi^\varepsilon(t) - \varepsilon^{\sum_c K^\varepsilon \star_x \xi^\varepsilon(\tilde{t})})^2$ it boils down to (6.19) except that on the right hand side of (6.17) one of the kernel $K^\varepsilon$ is replaced by $\delta^{\tilde{t}} K^\varepsilon(z)$ which is bounded by $|t-\tilde{t}|^k \|z\|^{-k}_g$ for some sufficiently small $\kappa > 0$. Proceeding as the arguments above one gets a uniform bound by $|t-\tilde{t}|^k$; then by Gaussianity this yields higher moments bounds and the continuity in time then follows from Kolmogorov continuity theorem.

\[\square\]

**Proof of Theorem 2.7.** The statements follow from Lemmas 6.4–6.9. The convergence of loop observables $\hat{O}_C, \varepsilon$ defined in (2.26) simply follows by continuous map theorem.

\[\square\]

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\[\text{[LL10, Theorem 2.5.6] is a statement in one spatial dimension, but it holds in higher dimensions too as remarked in the beginning of [LL10, Section 2.5].}\]
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Appendix A Discussions and possible extensions

A.1 Review of blackbox theorems and possible extension to three dimensions. The model under consideration is also defined in three spatial dimensions, with fields $A = (A_1, A_2, A_3) = \sum_{j=1}^3 A_j \, dx_j$ and $\Phi : T^3 \rightarrow \mathbb{C}$:

$$\mathcal{H}(A, \Phi) \overset{\text{def}}{=} \frac{1}{2} \int_{T^3} \left( \sum_{j=1}^3 F_{A_j}(x)^2 + \sum_{j=1}^3 |D_j^A \Phi(x)|^2 \right) d^3 x .$$  \hspace{1cm} (A.1)

Here, the curvature field $F_A = dA$ with components $F_{A,j}$ defined by

$$F_{A,j} \, dx_1 \wedge dx_2 + F_{A,1} \, dx_2 \wedge dx_3 + F_{A,2} \, dx_1 \wedge dx_3$$

$$= (\partial_1 A_2 - \partial_2 A_1) \, dx_1 \wedge dx_2 + (\partial_2 A_3 - \partial_3 A_2) \, dx_2 \wedge dx_3 + (\partial_1 A_3 - \partial_3 A_1) \, dx_1 \wedge dx_3$$

and $D_j^A \Phi$ is defined the same way. The SPDE is a system for $(A_1, A_2, A_3, \Phi)$ driven by independent white noises $\xi_1, \xi_2, \xi_3$ and a complex valued white noise $\zeta$. The system is again not parabolic, for instance, the equation for $A_1$ is of the form

$$\partial_t A_1 = \partial_x^2 A_1 - \partial_1 \partial_x A_2 + \partial_x^2 A_1 - \partial_1 \partial_3 A_3 + (\cdots) + \xi_1$$

where $(\cdots)$ denotes the nonlinearities. Using the same gauge tuning trick, one obtains a parabolic system, and the regularity of its solutions is expected to be $\alpha$ for $\alpha < -\frac{1}{2}$. Since it is more singular than the situation in two spatial dimensions, to obtain a local solution theory via lattice approximations, certain “general tools” should then be in place.

We thus take this chance to review the fast-growing literature in the theory of regularity structures—particularly focusing on the “blackbox” theorems, and discuss the possibility and challenge of applying them to the three dimensional convergence problem of (Abelian) lattice gauge theory.

The key idea of the theory is to lift the noise input of an SPDE to a space of models denoted by $\mathcal{M}$. The seminal work [Hai14] constructed this space $\mathcal{M}$, and proved continuity of the solution map from $\mathcal{M}$ to the space of distributions. Discrete analogues of these results are then developed by [HM18] on which the present article crucially relies; soon later [EH19] developed a more general discrete framework. These results would constitute the analytical foundation if one were to study the 3D lattice gauge theory (A.1). In parallel, “universality results” in regularity structures are proved in a series of papers [HQ18, HX19, HX18, SX18]; one possible challenge for 3D Abelian lattice gauge theory would be to see whether the techniques developed in these papers—suitably implemented on lattice—are sufficient to control the “remainder terms”. In 2D, in some sense, the “remainder terms” (4.23) are in a situation that is similar as in the “intermediate disorder” regime in [HQ18].

The algebraic step was then developed in [BHZ19], which has a systematic description of a group action on the space $\mathcal{M}$, which allows for the transformation from the canonical model to the renormalized model. This algebraic “blackbox” result is very general and applies to 2D (as in the starting part of Sect. 5.1) and 3D gauge theories as well.
A probabilistic step was then achieved by [CH16] which gives us—in an automatic way under very general conditions—convergence in probability of the renormalized models (in continuous regularizations) to a limit model. In 3D, there would be much more elements in the regularity structure, and explicit moment calculations as done in Sect. 5.2 would be a formidable task, thus it would require a discrete version of the BPHZ theorem in [CH16] to prove convergence of the discrete models. Although it seems to me that the arguments in [CH16] should be adapted to discrete settings given substantial effort, a complete formulation and proof for such a discrete theorem is not available yet at the moment.

Finally, another component of the “blackbox” which is also algebraic (or combinatoric), is established in [BCCH17] to identify the renormalized equations satisfied by the sequence of solutions given by the solution map for the renormalized models.

Regarding observables, the loop or string observables would be more difficult (if possible) to construct since one has to integrate a more singular gauge field along a curve.

Remark A.1. It would also be interesting to construct long time solutions as done by [MW17b, MW20, MW18] for the dynamical $\Phi^4$ equation. To obtain some bound uniform in time, it might be helpful to have some damping terms, for instance adding more gauge invariant terms $\mu |\Phi|^2 - |\Phi|^4$ to (1.1). Let’s mention a very simple trick that the equation for $B$ in for instance (6.6) can obtain a term $-\lambda^2 c^2 B$ by considering the equations for $B$ and the shifted field $\Psi \mapsto \Psi + c$. This would be reminiscent to the “Higgs mechanism” for the gauge field to acquire a mass by shifting the scalar field to a point in the bottom of a mexican-hat potential. These heuristics still seem to be far from achieving any rigorous results.

A.2 Non-Abelian case. As hinted in Remark 1.2, the gauge tuning method being exploited in the present paper is very likely to be generalized into the non-Abelian case (whereas a gauge fixing with a linear decomposition by (1.12) seems not generalizable). We would like to have some further discussion on this point (informally), and also make a comparison between gauge fixing in stochastic PDE formulation and that in functional integral formulation of quantum gauge theories.

In general gauge theories, with only gauge field $A$, one has $\mathcal{F}_A \equiv dA + A \wedge A$, where $A$ is a 1-form taking values in an Lie algebra—for instance $\mathfrak{u}(N)$ for some $N \geq 1$. In the Abelian case, $N = 1$, and $A \wedge A = 0$ which is studied in this paper. In the functional integral approach, one is interested in the formal measure $e^{-\mathcal{H}(A)}DA$ where $\mathcal{H}(A) = \frac{1}{2} \int_T \text{tr}(F \wedge *F)$ is invariant under the gauge transformation $A \mapsto g^* A \equiv g^{-1} A g + g^{-1} dg$ for $g : T^d \to U(N)$. Note that in Abelian case $g = e^{if}$ and this is precisely the first transformation in (1.2). To make the “measure” normalizable, a Fadeev–Popov trick based on the identity $1 = \int \delta(g(x)) \det(\frac{dg}{dx}) dx$ is often used. In Abelian lattice gauge theory, writing $A_f = A + df$ so that $d^* A_f = d^* A + \Delta f$, one then writes the formal partition function as

$$
\int e^{-\mathcal{H}(A)}DA = \int e^{-\mathcal{H}(A)} \delta(d^* A_f - \omega) \det(\frac{\delta(d^* A_f - \omega)}{\delta f}) Df DA.
$$

The Jacobian factor $\det(\frac{\delta(d^* A_f - \omega)}{\delta f}) = \det(\Delta)$ can be factorized out, and using gauge invariance one can also replace $A_f$ by $A$ and factor out the “infinite integral” $\int Df$. If $\omega = 0$ this simply amounts to imposing the divergence free condition as mentioned
in Remark 1.2. (The only reason one usually introduces the field $\omega$ in the Abelian case being considered here is that upon integrating it against a Gaussian weight one obtains a factor $e^{-\frac{(d^*A)^2}{2}}$ that is slightly more convenient to analyze.)

The Fadeev–Popov trick gets much more complicated in non-Abelian case, because the determinant will generally depend on $A$ and thus does not factor out. This determinant however can be expressed by an integral over anti-commuting variables, which are called ghost fields. One then studies the model for gauge field $A$ coupled with ghost fields. Furthermore, fixing the gauge globally is not always possible in non-Abelian theories, due to topological obstructions, a phenomena usually referred to as Gribov ambiguity. The ghost fields would not show up in the stochastic PDE approach with gauge fixed by DeTurck trick; this seems to be already observed by physicists [Zwa81, Sad87, BHST87]. In fact, for the corresponding stochastic quantization equation

$$\frac{\partial A}{\partial t} + d_A^* FA = \xi$$

with a Lie algebra valued $d$-component space-time white noise $\xi$, where $d_A$ is the gauge covariant derivative, one can check by straightforward computation that $B \overset{def}{=} g^* A$ satisfies $\frac{\partial B}{\partial t} = g^{-1} \frac{\partial A}{\partial t} g + d_B \left( g^{-1} \frac{\partial g}{\partial t} \right)$, so by solving $g$ from the ODE $g^{-1} \frac{\partial g}{\partial t} = -d_B^* B$ and invoking gauge invariance of $d_A^* FA$ one obtains a parabolic equation $\frac{\partial B}{\partial t} = -d_B^* F_B - d_B d_B^* B + g^{-1} \xi g$ where $g^{-1} \xi g \overset{law}{=} \xi$. This is well-know when $\xi = 0$, see [DK90, Section 6.3]; and in the presence of the noise $\xi$, the aforementioned physics literature simply put a term $-d_A d_A^* A$ into the Eq. (A.2) and call this a gauge-fixing term.

To obtain a local solution theory to (A.2) via lattice approximations, one again needs some general tools as discussed in the three dimensional Abelian case.

Appendix B Ward identity

We derive an important identity that will be useful for cancellation of renormalization in Sect. 5.2 as well as showing convergence of certain observables in Sect. 6.1.

A reader familiar with renormalization would imagine that the equations (3.11) or (3.15) would need a mass renormalization for $B_\varepsilon$ (i.e. a term $\tilde{C}_\varepsilon B_\varepsilon$ for some constant $\tilde{C}_\varepsilon$). However, a mass renormalization for $B_\varepsilon$ would break gauge symmetry so many proofs such as Lemma 3.2 would break down. We will see that actually there will be several contributions to a mass renormalization for $B_\varepsilon$ and these contributions cancel. This cancellation is due to gauge symmetry. One can prove such cancellation by some elementary tricks (see Remark 5.7), but here we derive a version of “Ward identity” which seems more systematic, and could be more useful when certain tricks are not available (i.e. in $d = 3$). The idea is straightforward: renormalization constants arise from expansion of nonlinearities [see (B.4)] in $\lambda$ and taking expectations, and we can make use of gauge invariance of such nonlinearities to obtain useful identities.

Fix $\varepsilon > 0$. Consider Eq. (3.11), but now on $\Lambda_\varepsilon^M$ which is the discrete torus of length size $M \geq 1$ and lattice spacing $\varepsilon$, and with initial condition $B^\varepsilon(-T) = 0$ and $\Psi^\varepsilon(-T) = 0$ at time $-T$ for $T > 0$. Let $E_\varepsilon^{M,j}$ for $j \in \{1, 2\}$ denote the sets of horizontal and vertical edges of $\Lambda_\varepsilon^M$. With a slight abuse of notation we write $P_\varepsilon^{M,j}$ for the transition probability of continuous time random walk either on the vertices $\Lambda_\varepsilon^M$, or the edges $E_\varepsilon^{M,j}$ for some
$j \in \{1, 2\}$; they are essentially the same kernel since $\Lambda_{\varepsilon}^M$ and $\xi_{\varepsilon}^{M,j}$ only differ by a small translation, and it will be clear which case is under consideration in the context. Fix a function $f^\varepsilon$ on $\Lambda_{\varepsilon}^M$ which is independent of time. Let

$$B^\varepsilon_f = B^\varepsilon + \nabla^\varepsilon f^\varepsilon, \quad \hat{\Psi}^\varepsilon_f = \Psi^\varepsilon e^{i\lambda f^\varepsilon},$$

where $(B^\varepsilon, \Psi^\varepsilon)$ is the solution to the above initial value problem. One then has $B^\varepsilon_f (-T) = \nabla^\varepsilon f^\varepsilon$ and $\hat{\Psi}^\varepsilon_f (-T) = 0$. By gauge invariance of the nonlinearity as shown in (3.8), one has

$$B^\varepsilon_{f,j}(t, e) = B^\varepsilon_j(t, e) + \nabla^\varepsilon_j f^\varepsilon(e)$$

$$= P_M^{\ast_{\varepsilon, T}} \left( e^{-i\varepsilon \lambda B^\varepsilon_{f,j}(\cdot)} \hat{\Psi}^\varepsilon_f (\cdot) + \xi^\varepsilon_j (\cdot) \right)(t, e) + \nabla^\varepsilon_j f^\varepsilon(e).$$

Here, for $\Lambda \in \{\Lambda_{\varepsilon}^M, \xi_{\varepsilon}^{M,1}, \xi_{\varepsilon}^{M,2}\}$, $x \in \Lambda$, and any space-time function $g^\varepsilon \in C(\mathbb{R}, \mathbb{R}^\Lambda)$ we have introduced the convolution notation

$$P_M^{\ast_{\varepsilon, T}} g^\varepsilon(t, x) \overset{\text{def}}{=} \int_{-T}^{\infty} \sum_{y \in \Lambda} P_M^{\ast_{\varepsilon, T}}(t - s, x - y) g^\varepsilon(s, y) \, ds.$$

By the covariance property (3.9) of the covariant Laplacian one also has

$$\partial_t \hat{\Psi}^\varepsilon_f (x) = e^{i\lambda f^\varepsilon(x)} \partial_t \Psi^\varepsilon_f (x) = \Delta^\varepsilon_{B^\varepsilon_f} \Psi^\varepsilon_f (x) + i\lambda \text{div}^\varepsilon (B^\varepsilon_f - \nabla^\varepsilon f^\varepsilon)(x) \hat{\Psi}^\varepsilon_f (x) + \hat{\xi}^\varepsilon(x),$$

where $\hat{\xi}^\varepsilon(x) \overset{\text{def}}{=} e^{i\lambda f^\varepsilon(x)} \xi^\varepsilon(x)$. Let $\Psi^\varepsilon_f$ solve

$$\partial_t \Psi^\varepsilon_f (x) = \Delta^\varepsilon_{B^\varepsilon_f} \Psi^\varepsilon_f (x) + i\lambda \text{div}^\varepsilon (B^\varepsilon_f - \nabla^\varepsilon f^\varepsilon)(x) \Psi^\varepsilon_f (x) + \xi^\varepsilon(x), \quad (B.1)$$

with initial condition $\Psi^\varepsilon_f (-T) = 0$. We have $(B^\varepsilon_f, \Psi^\varepsilon_f) \overset{\text{law}}{=} (B^\varepsilon_j, \hat{\Psi}^\varepsilon_j)$. Define

$$b^\varepsilon_f = b^\varepsilon_j(f^\varepsilon) \overset{\text{def}}{=} B^\varepsilon_{f,j}|_{\lambda = 0} = P_M^{\ast_{\varepsilon, T}} \xi^\varepsilon_j + \nabla^\varepsilon_j f^\varepsilon,$$

$$\psi^\varepsilon \overset{\text{def}}{=} \Psi^\varepsilon_f |_{\lambda = 0} = P_M^{\ast_{\varepsilon, T}} \xi^\varepsilon. \quad (B.2)$$

(We drop the $M$ dependence in the notation $b^\varepsilon_j$ and $\psi^\varepsilon$ for simplicity.) Also, differentiating (B.1) with respect to $\lambda$ and by similar computation as in proof of Lemma 3.5, one can check that

$$\partial_t \left( \partial_\lambda \Psi^\varepsilon_f (x)|_{\lambda = 0} \right)$$

$$= \Delta^\varepsilon \partial_\lambda \Psi^\varepsilon_f (x)|_{\lambda = 0} - i \sum_e b^\varepsilon_e (x, x + e) \nabla^\varepsilon_e \Psi^\varepsilon_f (x)|_{\lambda = 0} - i \Delta^\varepsilon_f(x) \Psi^\varepsilon_f (x)|_{\lambda = 0}$$

where $e \in \{\pm e_1, \pm e_2\}$, namely

$$\partial_\lambda \Psi^\varepsilon_f |_{\lambda = 0} = -i P_M^{\ast_{\varepsilon, T}} \left( \sum_e b^\varepsilon_e (\cdot, \cdot + e) \nabla^\varepsilon_e \psi^\varepsilon + \Delta^\varepsilon f^\varepsilon \psi^\varepsilon \right). \quad (B.3)$$

Note that this quantity depends on $f^\varepsilon$ via $b^\varepsilon$ and $\Delta f^\varepsilon$.

For any $x \in \Lambda_{\varepsilon}^M$ and $t \geq -T$ consider the following observable

$$Z^\varepsilon_f(t, x) \overset{\text{def}}{=} \varepsilon^{-1} \text{Im} \left( e^{-i\varepsilon \lambda B^\varepsilon_{f,j}(t, e)} \Psi^\varepsilon_f (t, x + e_j) \hat{\Psi}^\varepsilon_f (t, x) \right). \quad (B.4)$$
where \( e = \{x, x + e_j\} \). Applying (B.2) (B.3) (dropping \( t \) for simplicity of notation)

\[
\partial_x Z_j^\varepsilon(t, x)|_{\lambda=0} = \text{Re} \left( b_j^\varepsilon(e) \psi^\varepsilon(x + e_j) \psi_t^\varepsilon(x) \right) \\
- \varepsilon^{-1} \text{Re} \left( P_M^\varepsilon *_{e, T} \left( \sum e b^\varepsilon(\cdot + e) \nabla_e \psi^\varepsilon + \Delta^\varepsilon f^\varepsilon \psi^\varepsilon \right)(x + e_j) \cdot \psi^\varepsilon_t(x) \right) \\
+ \varepsilon^{-1} \text{Re} \left( \psi^\varepsilon(x + e_j) \cdot P_M^\varepsilon *_{e, T} \left( \sum e b^\varepsilon(\cdot + e) \nabla_e \psi^\varepsilon + \Delta^\varepsilon f^\varepsilon \psi^\varepsilon \right)(x) \right) \\
= - \sum_k b_j^\varepsilon(e) \psi_k^\varepsilon(x + e_j) \psi^\varepsilon_k(x) \\
- \sum_k \nabla_j P_M^\varepsilon *_{e, T} \left( \sum e b^\varepsilon(\cdot + e) \nabla_e \psi_k^\varepsilon + \Delta^\varepsilon f^\varepsilon \psi_k^\varepsilon \right)(x) \cdot \psi^\varepsilon_k(x) \\
+ \sum_k \nabla_j \psi^\varepsilon_k(x) \cdot P_M^\varepsilon *_{e, T} \left( \sum e b^\varepsilon(\cdot + e) \nabla_e \psi_k^\varepsilon + \Delta^\varepsilon f^\varepsilon \psi_k^\varepsilon \right)(x)
\]

where \( k \) sums over \( \{1, 2\} \). \( \psi^\varepsilon = \psi_1^\varepsilon + i \psi_2^\varepsilon \). Here, we have expanded a factor in the third line as \( \psi^\varepsilon(x + e_j) = \psi^\varepsilon(x) + \varepsilon \nabla^\varepsilon \psi^\varepsilon(x) \), and expanded a factor in the second line as \( P_M^\varepsilon *_{e, T}(\cdots)(x + e_j) = P_M^\varepsilon *_{e, T}(\cdots)(x) + \varepsilon \nabla_j P_M^\varepsilon *_{e, T}(\cdots)(x) \); the two terms of order \( \varepsilon^{-1} \) cancel each other, therefore the factor \( \varepsilon^{-1} \) does not show up in the last two lines of the above equation.

It is important to note that \( Z_j^\varepsilon(t, x) \) is gauge-invariant in law, in the sense that although \( (B_j^\varepsilon, \Psi_j^\varepsilon) \) depends on \( f^\varepsilon \), the law of the quantity \( Z_j^\varepsilon(t, x) \) is independent of \( f^\varepsilon \). Indeed, \( Z_j^\varepsilon(t, x) \) would be deterministically gauge invariant by the calculation (3.8) if \( \Psi_j^\varepsilon \) was replaced by \( \hat{\Psi}_j^\varepsilon \); on the other hand \( (B_j^\varepsilon, \Psi_j^\varepsilon) \) law \( = (B_j^\varepsilon, \hat{\Psi}_j^\varepsilon) \). Therefore the expectation of \( \partial_x Z_j^\varepsilon(t, x)|_{\lambda=0} \) is independent of \( f^\varepsilon \).

Replacing \( f^\varepsilon \) by \( \alpha f^\varepsilon \) in the above arguments, the expectation of \( \partial_x \partial_t Z_j^\varepsilon(t, x)|_{\lambda=0} \) must then be zero, namely,

\[
\mathbb{E} \left[ - \left( \nabla_j f^\varepsilon, \sum_k \delta(\cdot - x) \psi_k^\varepsilon(\cdot + e_j) \psi^\varepsilon_k(\cdot) \right) \right]_{\Lambda^\varepsilon M} \\
- \sum_e \left[ \nabla_e f^\varepsilon, \sum_k \psi_k^\varepsilon(x) \int_{-T}^\infty \nabla_e P_M^\varepsilon(t - s, x - \cdot) \psi^\varepsilon_k(s, \cdot) ds \right]_{\Lambda^\varepsilon M} \\
+ \sum_e \left[ \nabla_e f^\varepsilon, \sum_k \nabla_j \psi^\varepsilon_k(x) \int_{-T}^\infty P_M^\varepsilon(t - s, x - \cdot) \psi^\varepsilon_k(s, \cdot) ds \right]_{\Lambda^\varepsilon M} \\
+ \left( \Delta^\varepsilon f^\varepsilon, \mathcal{U}_j^\varepsilon \right)_{\Lambda^\varepsilon M} = 0
\]

where all the functions without time variables are evaluated at \( t \), and the function \( \mathcal{U}_j^\varepsilon(y) \), whose precise form will not actually matter much in the sequel, is given by

\[
\sum_k \int_{-T}^\infty \left( P_M^\varepsilon(t - s, x - y) \mathbb{E} \left[ \psi_k^\varepsilon(s, y) \nabla_j \psi^\varepsilon_k(x) \right] \\
- \nabla_j P_M^\varepsilon(t - s, x - y) \mathbb{E} \left[ \psi_k^\varepsilon(s, y) \psi^\varepsilon_k(x) \right] \right) ds.
\]
Summing over \( j \in \{1, 2\} \) and integrating by parts, noting that \( f^\varepsilon \) is arbitrary, one obtains

\[
(\text{div}^\varepsilon V^\varepsilon)(t, y) + \Delta^\varepsilon U^\varepsilon(t, y) = 0 \quad \forall y \in \Lambda^M_\varepsilon \tag{B.5}
\]

where the vector field \( V^\varepsilon = (V_{1\varepsilon}^\varepsilon, V_{2\varepsilon}^\varepsilon) \) is defined as

\[
V^\varepsilon(y, y + e_\ell) \overset{\text{def}}{=} -\sum_k \delta(y - x) \mathbf{E}\left[ \psi_k^\varepsilon(t, y + e_\ell) \psi_k^\varepsilon(t, y) \right] - \sum_{e \in \{0, e_\ell\}} \sum_{k, j} \int_{-T}^\infty \nabla_j^\varepsilon P_M^\varepsilon(t - s, x - y - e) \mathbf{E}\left[ \nabla_k^\varepsilon \psi_k^\varepsilon(s, y) \psi_k^\varepsilon(t, x) \right] ds + \sum_{e \in \{0, e_\ell\}} \sum_{k, j} \int_{-T}^\infty P_M^\varepsilon(t - s, x - y - e) \mathbf{E}\left[ \nabla_k^\varepsilon \psi_k^\varepsilon(s, y) \nabla_j^\varepsilon \psi_k^\varepsilon(t, x) \right] ds \tag{B.6}
\]

for \( \ell \in \{1, 2\} \), and the function \( U^\varepsilon \overset{\text{def}}{=} \sum_{j=1}^2 U_j^\varepsilon \); we suppressed their dependence on \( M, T \) for cleaner notation. Note that the left hand side of (B.5) is analytic in \( t > -T \) so once we have obtained (B.5) for small range of \( T \) we have it for any \( t > -T \). We can now take \( T \to \infty \), because the quantities \( U^\varepsilon, V^\varepsilon \) are nothing but combinations of heat kernels, and are analytic in \( T \) so once (B.5) holds for a small range of \( T \) it can be extend for all \( T > 0 \). (In particular this is just a property of heat kernels and we are not using any long-time existence of the SPDE.) Also since (B.5) holds for any \( M \geq 1 \) we can send \( M \to \infty \). So (B.5) holds with \( P^\varepsilon \) in place of \( P_M^\varepsilon \) and the time integrals over entire \( \mathbf{R} \). We then have that there exists a function \( W^\varepsilon \) such that\(^{19}\)

\[
V^\varepsilon + \nabla^\varepsilon U^\varepsilon = \text{curl}^\varepsilon W^\varepsilon \quad \left( = \left(-\nabla_2^\varepsilon W^\varepsilon, \nabla_1^\varepsilon W^\varepsilon \right) \right).
\]

It is easy to see that \( U^\varepsilon \) and \( V^\varepsilon \) both decay to zero at infinity, so \( W^\varepsilon \) must go to a constant at infinity; we can well shift \( W^\varepsilon \) by constant so that \( W^\varepsilon \) also decays to zero at infinity. Summing the above identity over horizontal or vertical edges of \( \mathbf{Z}_2^2 \), one has

\[
\sum_{e \in \mathcal{E}^\ell(\mathbf{Z}_2^2)} \mathbf{V}^\varepsilon(e) = 0, \quad \ell \in \{1, 2\}. \tag{B.7}
\]

This is precisely what we use for cancellation of mass renormalization of the gauge field as well as the convergence of the composite field observable (see Lemma 6.6). (Alternatively this can be also done using Remark 5.7.)

**Remark B.1.** Identities of the type (B.5) are usually called Ward–Takahashi identities in QFT, see [BFS80, Eq. (A.3)] and [Bal83, Eq. (2.27)] similar identities in the context of Abelian gauge theory (i.e. Higgs model). Note that the “useless” term of the form \( \Delta^\varepsilon U^\varepsilon \) did not appear in these references; this is probably because their gauge fixing condition amounts to imposing the gauge field to be divergence free, and without the div term in (B.1) the term \( \Delta^\varepsilon U^\varepsilon \) would not appear. For a reader who is familiar with quantum electrodynamics (QED), (B.5) (without the term \( \Delta^\varepsilon U^\varepsilon \)) is reminiscent to the QED Ward identity \( \sum_j k_j \mathcal{M}^j(k) = 0 \) in Fourier space where \( \mathcal{M}(k) \) is an amplitude for some QED process involving an external photon (i.e. gauge field) with momentum \( k \).

\(^{19}\) This is by a discrete version of Helmholtz decomposition or Hodge decomposition (see e.g. [War13, Chapter 6]) and its proof is exactly the same as its continuous version and thus is omitted.
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