SOLUTION OF THE ODDERON PROBLEM

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Abstract

The intercept of the odderon trajectory is derived, by finding the spectrum of the second integral of motion of the three reggeon system in high energy QCD. When combined with earlier solution of the appropriate Baxter equation, this leads to the determination of the low lying states of that system. In particular, the energy of the lowest state gives the intercept of the odderon $\alpha_o(0) = 1 - 0.2472\alpha_s N_c/\pi$.

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One of the, still unsolved, problems of perturbative QCD is the behaviour of the theory in the Regge limit, characterized by high energies and fixed momentum transfers. This limit — the (Generalized) Leading Logarithmic Approximation — is described by the exchange of reggeized gluons. The intercept of the pomeron trajectory, the BFKL pomeron, has been derived in the classic works of Balitski, Fadin, Kuryaev and Lipatov \[1\]. The next natural step is to find the intercept of the odderon trajectory, which however, turned out to be very difficult \[2, 3, 4\].

An important progress was made by Lipatov, Faddeev and Korchemsky \[5, 6\] who reduced the problem to the solution of a functional equation — the Baxter equation — for physical values of the two relevant constants of motion \((q_2\) and \(q_3\)) of the system of three reggeized gluons. Various approximation techniques \[6, 7, 8\] for solving the Baxter equation have been used, and in our previous work \[10, 11\] an exact method of constructing a solution for general \(q_2\) and \(q_3\) was developed. However, while the eigenvalues of \(\hat{q}_2\) are known, the spectrum of \(\hat{q}_3\) remained unavailable, apart from asymptotic results of \[7, 9\]. In this letter we report on the solution of the eigenproblem of \(\hat{q}_3\) which removes the last obstacle in deriving the numerical value of \(\alpha_O\) in the Leading Logarithmic Approximation.

The intercept of the odderon trajectory is given by

\[
\alpha_O(0) = 1 + \frac{\alpha_s N_c}{4\pi} \left( \epsilon_3(h, q_3) + \bar{\epsilon}_3(\overline{h}, \overline{q}_3) \right),
\]

where \(\epsilon_3\) and \(\bar{\epsilon}_3\) are respectively the largest eigenvalues of the three reggeon hamiltonian and its antiholomorphic counterpart \[6\]. The conformal weight, \(h\), parameterizes the eigenvalues of the Casimir operator \(\hat{q}_2\)

\[
q_2 = h(1 - h), \quad h = \frac{1}{2} (1 + m) - i\nu, \quad m \in Z, \nu \in R.
\]

Analogous formulas hold for the antiholomorphic sector with \(\overline{h} = (1-m)/2 - i\nu \[8\]. After an explicit expression for the energies \(\epsilon_3(h, q_3)\) \((\bar{\epsilon}_3(\overline{h}, \overline{q}_3))\) \(q_2\) and \(q_3\) was derived \[10\] the only unknown ingredient was the quantization of \(q_3\).

The eigenproblem of the \(\hat{q}_3\) operator was formulated in general terms by Lipatov \[6\]. However the quantitative solution was lacking due to the complicated and indirect way in which the boundary conditions enter and fix the spectrum. To begin, we quote a general form of the wave function of the compound state of three reggeized gluons

\[
f(\rho, \rho) = \sum_{r,s} c_{rs} f_r(\rho) \overline{f}_s(\rho),
\]
where \( \rho = \{ \rho_1, \rho_2, \rho_3 \} \) denotes three transverse coordinates in the complex number representation \( \rho_k = x_k + iy_k, \ k = 1, 2, 3 \). All other quantum numbers are collectively denoted by index \( r \). Similarly in the anti-holomorphic sector \( \bar{\rho}_k = x_k - iy_k \). Since the integrals of motion \( \hat{q}_2, \hat{q}_3 \) commute with the hamiltonian, one chooses \( f(\rho) \) as a simultaneous eigenfunction of these operators.

With the conformally covariant Ansatz, \( z = \frac{\rho_1 \rho_3 \rho_2}{\rho_0 \rho_2^*} \), \( \mu = h/3 \),

\[
f_{\rho_0, q_2, q_3} = \left( \frac{\rho_1 \rho_3 \rho_2}{\rho_0 \rho_2^*} \right)^\mu \Phi^{(h,q_3)}(z), \quad (4)
\]
the eigenfunction, \( \hat{q}_3 f = q_3 f \), reads, in terms of \( \Phi \), and at fixed \( q_2 \) given by \( (2) \).

\[
a(z) \frac{d^3}{dz^3} \Phi(z) + b(z) \frac{d^2}{dz^2} \Phi(z) + c(z) \frac{d}{dz} \Phi(z) + d(z) \Phi(z) = 0, \quad (5)
\]
where

\[
a(z) = z^3(1-z)^3 = \sum_{i=0}^{3} a_i z^{i+3}, \quad b(z) = 2z^2(1-z)^2(1-2z) = \sum_{i=0}^{3} b_i z^{i+2},
\]

\[
c(z) = z(z-1)(z-1)(3\mu + 2)(\mu - 1) + 3\mu^2 - \mu = \sum_{i=0}^{3} c_i z^{i+1},
\]

\[
d(z) = \mu^2(1-\mu)(z+1)(z-2)(2z-1) - i q_3 z(1-z) = \sum_{i=0}^{3} d_i z^i.
\]

This is a third order linear differential equation with the three regular singular points at \( z = 0, 1 \) and \( \infty \), introduced by Lipatov in \( [3] \), and investigated in \( [12] \) in a slightly different form. We will solve this equation by standard methods and identify proper boundary conditions which lead to the quantization of \( q_3 \). To this end we first construct a fundamental set of three linearly independent solutions \( \Phi^{(0)}(z) = (u_1^{(0)}(z), u_2^{(0)}(z), u_3^{(0)}(z)) \), around \( z = 0 \).

\[
u^{(0)}_1(z) = z^{s_1} \sum_{n=0}^{\infty} f_n^{(1)} z^n, \quad u^{(0)}_2(z) = z^{s_2} \sum_{n=0}^{\infty} f_n^{(2)} z^n, \quad (6)
\]

\[
u^{(0)}_3(z) = \log z u^{(0)}_2(z) + z^{s_3} \sum_{n=0}^{\infty} r_n z^n,
\]

where \( s_1 = 2h/3, \ s_2 = -h/3 + 1, \ s_3 = -h/3 \) and the coefficients of the expansions are determined by the following recursion relations

\[
f_n^{(i)} = - \sum_{m=1}^{3} f_{n-m}^{(i)} \lambda_m (s_i + n)/\lambda_0 (s_i + n), \quad f_0^{(i)} = 1, \ f_{-j}^{(i)} = 0, \ i = 1, 2, j > 0, \quad (7)
\]
\[ \lambda_m(x) = a_m(x-m)(x-m-1)(x-m-2) + b_m(x-m)(x-m-1) + c_m(x-m) + d_m, \]

and for the logarithmic solution

\[ r_n = -\left(p_{n-1} + \sum_{m=1}^{3} r_{n-m}\lambda_m(s_3 + n)/\lambda_0(s_3 + n)\right), \]

\[ r_1 = 1, \quad r_0 = -p_0/\lambda_1(s_3 + 1), \quad r_{-1} = 0, \quad p_n = \sum_{m=0}^{3} f_{n-m}^{(2)} \gamma_m(s_2 + n), \quad \gamma_m(x) = a_m(3(x-m)(x-m+2) + 2) + b_m(2(x-m) - 1) + c_m. \]

There are two physical conditions which our solution should satisfy: a) the complete wave function of the compound system (3) must be single valued in the whole transverse plane of reggeon coordinates, and b) the wave function (3) must obey Bose symmetry. Together with the analyticity of the solution \( \Phi(z) \) these conditions unambiguously determine the spectrum of \( \hat{q}_3 \).

The series in Eq.(6) are convergent in the unit circle \( R_0 \) around \( z = 0 \), and therefore they determine uniquely the analytic continuation \( \vec{u}^{(0)}(z) \) to the cut complex plane. To achieve this continuation in practice we construct two other fundamental sets of solutions \( \vec{u}^{(1)}(z) \) and \( \vec{u}^{(\infty)}(z) \) around \( z = 1 \) and \( z = \infty \). This could be done analogously to Eqs.(6,7), for the transformed equation, however, because of the symmetry under permutations of reggeon coordinates, one can partly satisfy condition b) by a proper choice of these bases. We define

\[ \vec{u}^{(1)}(z) = \vec{u}^{(0)}(1 - \frac{1}{z}), \quad \vec{u}^{(\infty)}(z) = \vec{u}^{(0)}(\frac{1}{1 - z}). \]

These series are convergent in the regions \( R_1 : \text{Re}(z) > 1/2 \) and \( R_\infty : |1 - z| > 1 \) respectively. Analytic continuation is realized by the transition matrices

\[ u_i^{(0)}(z) = \Gamma_{ij} u_j^{(1)}(z), \quad u_i^{(1)}(z) = \Omega_{ij} u_j^{(\infty)}(z), \]

which depend on \( h \) and \( q_3 \) only. They contain the full information about the system, in particular about its spectrum. In practice we calculate transition matrices as the solutions of the systems of \( 3 \times 3 \) algebraic equations (10) written at some judiciously chosen point \( z = \zeta \). For example

\[ \Gamma_{ik} = \frac{W_{ik}}{W(\vec{u}^{(1)}(\zeta))}, \quad W_{ik} = W(u_k^{(1)} \rightarrow u_i^{(0)}), \]

4
where $W(\vec{u}^{(1)}(\zeta))$ is the Wronskian determinant of the fundamental solutions $\vec{u}^{(1)}(\zeta)$. As long as $\zeta$ is in the intersection of the convergence regions $R_0$ and $R_1$, the matrix elements obtained from Eq. (11) are independent of $\zeta$ provided enough terms in the series (6) are included.

Finally we implement the uniqueness constraints a). It is crucial to observe that requiring singlevaluedness in the holomorphic and antiholomorphic sectors separately gives a too strong condition and is in fact not necessary. Even though the hamiltonians in both sectors commute, proper boundary conditions should be formulated only for the wave function of the whole system (3). We therefore define a general bilinear form [4]

$$\Psi_{h,\bar{h},q_3,q_3}(z,\bar{z}) = \vec{u}^{(1)}(\bar{z})^T A^{(1)} \vec{u}^{(1)}(z),$$

and demand its uniqueness in the whole transverse plane. The compound function (12) has 9 free parameters [4]. By inspecting Eqs. (3) we see that the most general choice of coefficients, consistent with the uniqueness of the wave function in the neighbourhood of $z = 1$, is

$$A^{(1)} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & \gamma & 0 \end{pmatrix},$$

which has freedom of three parameters. Incidentally this form guarantees also the normalizability of the wave function. Rewriting the wave function in terms of other bases (9) around $z = 0$ and $z = \infty$, gives another coefficient matrices

$$A^{(0)} = (\bar{\Gamma}^{-1})^T A^{(1)} \Gamma^{-1}, \quad A^{(\infty)} = \bar{\Omega}^T A^{(1)} \Omega.$$  

Now, uniqueness in the whole transverse plane requires that the transformed matrices, Eq. (14) have the same form as (13), with possibly different coefficients. Therefore we require

$$A^{(0)}_{12} = A^{(0)}_{13} = A^{(0)}_{33} = A^{(0)}_{21} = A^{(0)}_{31} = 0,$$

$$A^{(\infty)}_{12} = A^{(\infty)}_{13} = A^{(\infty)}_{33} = A^{(\infty)}_{21} = A^{(\infty)}_{31} = 0.$$  

In fact only one of these sets is sufficient, as can be seen from the following topological argument. Any possible cut in the domain of the full wave function has to begin and end at the singular points of the equation, i.e. at 0, 1

\[\text{[1]}\] The power prefactors in (4) are irrelevant for this discussion.

\[\text{[2]}\] We thank Gregory Korchemsky for the discussion on that point.
or at $\infty$. Therefore eliminating two of these points guarantees that there is no cut beginning at the third one. Equations (15,16) are linear homogeneous equations for the coefficients $\alpha, \beta$ and $\gamma$. The condition of the existence of a nonzero solutions of Eqs (15) or (16) provides the quantization of $q_3$ and $\bar{q}_3$ we looked for. It can conveniently be written as

$$
\mathcal{B}_U \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0 \quad \text{and} \quad \mathcal{B}_L \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0.
$$

where the rows of the matrix $\mathcal{B}_U$ ($\mathcal{B}_L$) are the coefficients of $\alpha, \beta, \gamma$ in, e.g., $A_{12}^{(\infty)}$, $A_{13}^{(\infty)}$ and $A_{33}^{(\infty)}$. Explicitly

$$
\mathcal{B}_U = \begin{pmatrix}
\bar{\Omega}_{11} \Omega_{12} & \bar{\Omega}_{21} \Omega_{22} & \bar{\Omega}_{21} \Omega_{32} + \bar{\Omega}_{31} \Omega_{22} \\
\bar{\Omega}_{11} \Omega_{13} & \bar{\Omega}_{21} \Omega_{23} & \bar{\Omega}_{21} \Omega_{33} + \bar{\Omega}_{31} \Omega_{23} \\
\bar{\Omega}_{13} \Omega_{13} & \bar{\Omega}_{23} \Omega_{23} & \bar{\Omega}_{23} \Omega_{33} + \bar{\Omega}_{33} \Omega_{23}
\end{pmatrix}.
$$

Figure 1 shows the, suitably transformed, absolute value of the determinant of $\mathcal{B}_U$ as a function of $q_3$ along the imaginary axis in the complex $q_3$ plane. The eigenvalue of the first Casimir operator $q_2$ is fixed to $q_2 = \bar{q}_2 = 1/4$ which corresponds to the lowest representation of the $SL(2, C)$. All formulae in the antiholomorphic sector are the same with $\bar{q}_3 = q^*_3$, a $*$ denoting a complex conjugate [6]. Clearly a set of discrete $q_3$ values exists where the first condition (17) is satisfied. Imposing the second condition eliminates half of the candidates. In addition a discrete series of real solutions of (17) exists. Both groups lead to single valued wave functions, however the condition of Bose symmetry singles out the imaginary $q_3$ only. Namely, for purely imaginary eigenvalues, the matrices $A^{(0)}, A^{(1)}$ and $A^{(\infty)}$ coincide. This, together with the definition of the basis (9), guarantees the invariance of the wave function under even permutations. In order to implement a full Bose symmetry it suffices to take the complete wave function as

$$
\Psi(z, \bar{z}) = \Psi_{\frac{1}{2}^{\frac{1}{2}, q_3, \bar{q}_3}}(z, \bar{z}) + \Psi_{\frac{1}{2}^{\frac{1}{2}, q_3, \bar{q}_3}} \left(\frac{z}{z - 1}, \frac{\bar{z}}{\bar{z} - 1}\right)
$$

The second term is just the wave function in the $(-q_3, -\bar{q}_3)$ sector which, due to the degeneracy $q_3 \longleftrightarrow -q_3$, obeys the general structure (3). On the other hand, for real eigenvalues $q_3$, the matrices $A^{(0)}, A^{(1)}$ and $A^{(\infty)}$ differ by a phase factor $e^{2\pi i/3}$, and the only symmetric solution is identical with zero.
For \( q_3 \) lying outside the real and imaginary axes both constraints (17) cannot be satisfied simultaneously. Therefore we are led to conclude that the physical spectrum of \( q_3 \) for \( h = 1/2 \) lies on the imaginary axis. The first few levels are quoted in Table 1.

It is very instructive to superimpose this result on our earlier calculations, based on a different approach (Bethe Ansatz), which resulted in the analytic expression for the eigenenergy of the three reggeon system as a function of \( h \) and \( q_3 \) [10]. Figure 2 shows \( \epsilon_3(1/2, q_3) \) along the imaginary axis of \( q_3 \). Black dots and crosses mark values of \( q_3 \) quantized according to the first condition in (17). It turns out that the candidates which were eliminated by the second condition (crosses) are numerically very close to the poles of the \( \epsilon_3 \). They are, however, non physical since the corresponding wave functions are not single valued.

The intercept of the odderon trajectory is determined by the largest eigenvalue \( \epsilon_3(1/2, q_3^O) \). This corresponds to the first non-zero \( q_3^O = q_3^{(1)} \) on the imaginary axis, with the numerical value

\[
q_3^O = -0.20526i, \tag{20}
\]

which, together with our solution of the Baxter Equation, \( \epsilon_3(h, q_3) \) [10], gives for the energy of the odderon state

\[
\epsilon_3(1/2, q_3^O) = -0.49434. \tag{21}
\]

This translates for the intercept of the odderon trajectory, c.f. Eq.(11),

\[
\alpha_O(0) = 1 - 0.24717 \frac{\alpha_s N_c}{\pi}, \tag{22}
\]

which may solve the longstanding phenomenological puzzle why the odderon trajectory is so hard to observe experimentally. However any phenomenological consequences of this result should be taken with a great caution. Higher order and running \( \alpha_s \) corrections can change this number. Connection between hard and soft exchanges should be better understood. In any case the general assumptions behind the derivation of (22) are the same as those of the famous calculation of the pomeron trajectory in the classic work of Balitski, Fadin, Kuryaev and Lipatov[1].

In Table 1 we quote first few quantized values of \( q_3 \), together with corresponding energies. Indeed next states have a substantially smaller intercept
and consequently, their contribution to the high energy scattering is negligible. There exists also the consistent solution of Eqs.(17) with $q_3 = 0$ which has smaller $\epsilon_3$ but relatively close to the energy of the odderon solution. However this solution is highly pathological and will not be discussed here in detail.

Our method provides also explicit expressions for the wave functions of the compound states. At the eigenvalues of $\hat{q}_3$ the coefficients of the expansion (12) are given by the common eigenvector corresponding to the zero eigenvalue of $B_U$ or $B_L$. Therefore the wave functions are given explicitly, in terms of known bases, and can be used for various applications. For the odderon state we obtain

$$\alpha^O = 0.7096, \quad \beta^O = -0.6894, \quad \gamma^O = 0.1457.$$  \hspace{1cm} (23)

Note that the asymptotic form of the wave function at $z = 1$, implied by the uniqueness condition (13), agrees with that derived by Lipatov from the symmetry considerations [3]. Moreover, it follows from Eqs.(15,16) that the same asymptotics holds around other singular points.

A variational estimate of the lower bound for the odderon $\alpha_O(0) > 1 + 0.36\alpha_sN_c/\pi$ was derived in Ref.[4]. Recently this has been challenged by Braun, who gives the bound $\alpha_O(0) > 1 - 0.339\alpha_sN_c/\pi$ [13]. The latter estimate is consistent with our exact result. Note that this bound is rather close to the intercept of our degenerate solution of the Baxter equation for $q_3 = 0$. It would be interesting to repeat their variational calculation with our exact wave function, especially including more terms in the ‘logarithmic part’ of their wave function.

Very recently Korchemsky has studied dependence of the eigenvalues of $\hat{q}_3$ on $h$ using the sophisticated technique of Whitham dynamics [9]. His results, as based on the quasiclassical approximation, should agree with ours for higher states. In fact we have found earlier that his WKB formulae reproduce exact results quite well even at low values of $q_3$.

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References
| No. | $q_3$  | $\epsilon_3$ |
|------|--------|-------------|
| 0    | 0      | -0.73801    |
| 1    | 0.20526i | -0.49434   |
| 2    | 2.34392i | -5.16930   |
| 3    | 8.32635i | -7.70234   |

Table 1: Quantization of $q_3$ and corresponding eigenvalues of the holomorphic hamiltonian.

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Figure 1: Quantization of $\hat{q}_3$. Only half of the zeros shown constitutes the physical spectrum.
Figure 2: The holomorphic energy of the three reggeized gluons for imaginary $q_3$ (solid lines $[10]$). Dots and crosses show solutions of the first condition $[17]$. Solutions close to the poles of $\epsilon_3$ are eliminated by the second condition.