Colloquium: Nonlinear collective interactions in quantum plasmas with degenerate electron fluids

P. K. Shukla* and B. Eliasson

RUB International Chair, Fakultät für Physik und Astronomie, Ruhr-Universität Bochum, D-44780 Bochum, Germany

(Received 27 September 2010; Revised 18 February 2011; Accepted 6 May 2011, in Reviews of Modern Physics)

The current understanding of some important nonlinear collective processes in quantum plasmas with degenerate electrons is presented. After reviewing the basic properties of quantum plasmas, we present model equations (e.g. the quantum hydrodynamic and effective nonlinear Schrödinger-Poisson equations) that describe collective nonlinear phenomena at nanoscales. The effects of the electron degeneracy arise due to Heisenberg’s uncertainty principle and Pauli’s exclusion principle for overlapping electron wavefunctions that result in tunneling of electrons and the electron degeneracy pressure. Since electrons are Fermions (spin-1/2 quantum particles), there also appears an electron spin current and a spin force acting on electrons due to the Bohr magnetization. The quantum effects produce new aspects of electrostatic (ES) and electromagnetic (EM) waves in a quantum plasma that are summarized in here. Furthermore, we discuss nonlinear features of ES ion waves and electron plasma oscillations (ESOs), as well as the trapping of intense EM waves in quantum electron density cavities. Specifically, simulation studies of the coupled nonlinear Schrödinger (NLS) and Poisson equations reveal the formation and dynamics of localized ES structures at nanoscales in a quantum plasma. We also discuss the effect of an external magnetic field on the plasma wave spectra and develop quantum magnetohydrodynamic (Q-MHD) equations. The results are useful for understanding numerous collective phenomena in quantum plasmas, such as those in compact astrophysical objects (e.g. the cores of white dwarf stars and giant planets), as well as in plasma-assisted nanotechnology (e.g. quantum diodes, quantum free-electron lasers, nanophotonics and nanoplasmonics, metallic nanostructures, thin metal films, semiconductor quantum wells and quantum dots, etc.), and in the next-generation of intense laser-solid density plasma interaction experiments relevant for fast ignition in inertial confinement fusion schemes.
tainties of the position and momentum is equal to or larger than $\hbar/2$, where $\hbar = 1.0544 \times 10^{-27}$ erg sec) is Planck’s constant divided by $2\pi$. The position of an electron subjected to the influence of an atomic nucleus is very well defined (the force to which it is subjected is large). However, owing to Heisenberg’s uncertainty principle, the electron momentum is ill defined. An electron has a continuous motion around the position it occupies. This motion exerts pressure on the surrounding medium, exactly as the thermal agitation of the particles of a gas exerts its pressure. This pressure is called the electron degeneracy pressure. This pressure, since it is nonthermal in origin, is, of course, independent of the electron temperature; the pressure of degenerate electrons increases with increasing electron number density. It is, however, only at very high densities that the degeneracy pressure becomes comparable or larger to the thermal gas pressure. One then says that the plasma matter is in an exotic state, comprising degenerate electrons and positrons or holes.

Plasmas with degenerate electrons and positrons with number densities comparable with solids and temperatures of several electron volts fall under the category of dense matter (Fortov, 2009; Ichimaru, 1982) that appears in the core of giant planets (Chabrier et al., 2006; Chabrier, 2009; Horn, 1991) and the crusts of old stars (Guillot, 1999). Dense compressed plasmas are currently of wide interest due to their applications to astrophysical and cosmological environments (Buenvento and De Vito, 2005; Harding and Lai, 2006; Lai, 2001; Opher, 2001), as well as to inertial fusion science involving intense laser-solid density plasma interaction experiments (Andreev, 2000; Froula et al., 2011; Glenzer and Redmer, 2009; Glenzer et al., 2007; Hu and Keitel, 1999; Kritcher et al., 2008; Lee et al., 2009; Lindl, 1995; Malik et al., 2007; Marklund and Shukla, 2006; Mendonça, 2001; Neumayer et al., 2010; Salamin et al., 2006; Son and Fisch, 2005) for inertial confinement fusion (Azechi et al., 2006) based on the high-energy density plasma physics (Drake, 2009, 2010) (Norreys et al., 2009). Plasma-like collective behavior is well studied experimentally and theoretically in solid state physics (Kittel, 1996), in which metals and semi-conductors support both transverse optical modes, and longitudinal electrostatic modes, such as plasmons and phonons on electron and ion time-scales, and, in addition various lattice modes. Plasmons and phonons are usually probed by measuring the energy of electrons which have been passed through thin foils, or by laser scattering techniques. For example, the dispersion relation of collective electron plasma waves has been measured for several metal specimen by using an electron velocity analyzer of Möllenstedt type (Watanabe, 1956). Collective dispersive behavior of plasmons, including shifts in the plasmon frequency due to quantum effects, in solid-density plasmas have been observed by Glenzer et al. (2007) and Neumayer et al. (2010) using spectrally resolved x-ray scattering techniques (Kritcher et al., 2008; Lee et al., 2009). In these experiments, powerful x-ray sources are employed for accessing narrow bandwidth spectral lines via collective Thomson scattering of light off electron-density fluctuations. These experimental techniques also allow accurate measurements of the electron velocity distribution function, temperature, and ionization state in the dense matter regime. Gregori and Gericks (2009) also proposed future experiments to measure low-frequency oscillations in plasmas when keV free-electron lasers will become available. Froula et al. (2011) summarizes the measurement techniques using scattering of EM waves in plasmas, and recent experimental results from x-ray scattering experiments in dense plasmas reveal that quantum mechanical effects are indeed important (Glenzer and Redmer, 2009; Glenzer et al., 2007).

Furthermore, due to recent experimental progress in femtosecond pump-probe spectroscopy, the field of quantum plasmas is also gaining significant attention (Crouseilles et al., 2008) in connection with the collective dynamics of an ensemble of degenerate electrons in metallic nanostructures and thin metal films. The physics of quantum plasmas is also relevant in the context of quantum diodes (Ang et al., 2003, 2007; Shukla and Elias-Son, 2008b), nanophotonics and nanowires (Barnes et al., 2003; Chang et al., 2006; Shpatakovskaya, 2006), nanoplasmronics (Atwater, 2007; Maier, 2007; Marklund et al., 2008; Ozbay, 2006; Stockman, 2011), high-gain quantum free-electron lasers (Serbeto et al., 2008, 2009), microplasma systems (Becker et al., 2006), and small semiconductor devices (Hang and Koch, 2004, 2007; Manfredi and Hervieux, 2007; Markovich et al., 1999), such as quantum wells and piezomagnetic quantum dots (Adolfath et al., 2008). The latter can be used as nanoscale magnetic switches.

Collective interactions between an ensemble of degenerate electrons and positrons/holes give rise to novel waves and structures in quantum plasmas. Studies of linear waves in a non-relativistic unmagnetized quantum plasma with degenerate electrons begun with the pioneering theoretical works of Bohm (1953); Bohm and Pines (1953); Klimontovich and Silin (1952a,b, 1961) and Pines (1961), who studied the dispersion properties of high-frequency electron plasma oscillations (EPOs). The frequencies of the latter with an arbitrary electron degeneracy have been found by Maafa (1993). In the theoretical description of the EPOs, Klimontovich-Silin and Bohm-Pines used the Wigner distribution function (Wigner, 1932) and the density matrix approach to demonstrate that in a quantum plasma with a Fermi-Dirac equilibrium distribution function for degenerate electrons, the frequency of the EPOs is significantly different from the Bohm-Gross frequency in a classical electron-ion plasma with non-degenerate electrons obeying the Maxwell-Boltzmann distribution function. The dispersion to the EPOs appears through the electron Fermi pressure and electron tunneling effects (Gardner and Ringhofer, 1996; Jungel et al., 2006; Manfredi, 2003; Manfredi and Haas, 2001; Misra, 2009; Shukla, 2006).
The quantum effects are important for the dielectric and dispersive properties of a quantum plasma. The longitudinal and transverse dielectric constants of an isotropic quantum plasma were worked out by Lindhard (1954), Silin and Rukhadze (1961), and Kuzelev and Rukhadze (1999). Contributions of the electron spin and exchange interactions to the electromagnetic (EM) wave dispersion relations in an unmagnetized quantum plasma have been presented by Burt and Wahlquist (1962) by using a quantum kinetic theory. The quantum mechanical phase space distribution of Wigner (1932) has been further generalized by Brittin and Chappell (1962) for a system of charged particles including the quantized EM field and Green’s functions involving correlations of distribution functions and vector potentials. Kinetic models for spin-polarized plasmas have been developed by Cowley et al. (1986), Zhang and Balescu (1988), and Balescu and Zhang (1988). More recently, electron spin–1/2 effects in a quantum magnetoplasma have also been considered by Brodin et al. (2008a) and Zamanian et al. (2010). The gauge problem in quantum kinetics has been treated by Stratonovich (1956) and Serimaa et al. (1986), which is important whenever the fields are not electrostatic. In a quantum magnetoplasma, one finds that the external magnetic field significantly affects the dynamics of degenerate electrons, and that the thermodynamics and kinetics (Steinberg, 2000) in a quantum magnetoplasma are significantly different from those in an unmagnetized quantum plasma. Oberman and Ron (1963) derived the expression for the dielectric function for longitudinal waves in a non-relativistic magnetized quantum plasma and discussed applications of their work to heavily doped semiconductors. Kelly (1964) studied the dispersive properties of a magnetized quantum plasma by using the Wigner distribution function and the Maxwell equations. Finally, we mention that useful foundations for the theory of quantum plasmas are presented by de Groot and Suttorp (1972), while quantum kinetic models including the effects of spin are reviewed by Lee (1995).

During the last decade, there has been a surge in investigating new aspects of collective interactions in dense quantum plasmas by means of non-relativistic quantum hydrodynamic (Gardner and Ringhofer, 1996; Jungel et al., 2006; Manfredi, 2005; Manfredi and Haas, 2001; Shukla and Eliasson, 2010) and quantum kinetic (Bonitz, 1998; Kremp et al., 2010; Tsintsadze and Tsintsadze, 2009) equations. Models for non-ideal effects in a strongly coupled dense plasma have been presented by Carruthers and Zachariasen (1983), Kremp et al. (2005), and Redmer and Röpke (2010). The Wigner-Poisson (WP) model (Hillery et al., 1983) has been used to derive a set of QHD equations (Manfredi 2005; Manfredi and Haas, 2001) for ES waves in a quantum plasma. The relation between the QHD and kinetic models have been investigated by Haas et al. (2010b). The quantum nature (Manfredi and Haas, 2001; Shukla and Eliasson, 2010) is manifested in the non-relativistic electron momentum equation through the quantum statistical pressure, which requires the knowledge of the Wigner electron distribution function for a quantum mixture of electron wavefunctions, each characterized by an occupation probability. The quantum part of the electron pressure is also represented as a nonlinear quantum force (Gardner and Ringhofer, 1996; Manfredi and Haas, 2001; Wilhelm, 1971) −∇φB, where φB = −(ℏ2/2me√ne)∇2√ne is the Bohm potential, and me and ne are the electron mass and electron number density, respectively. Defining the effective wavefunction ψ = √ne(r,t)exp[iS(r,t)/ℏ], where ∇S(r,t) = meur(t) and ur(t) is the electron fluid velocity, the non-relativistic electron momentum equation can be cast into an effective nonlinear Schrödinger (NLS) equation (Manfredi, 2005; Manfredi and Haas, 2001; Shukla et al., 2006; Shukla and Eliasson, 2006, 2010), in which there appears a coupling between the electron wavefunction and the ES potential associated with the EPOs. The ES potential, in turn, is determined from Poisson’s equation. One thus has the coupled NLS and Poisson equations, governing the dynamics of nonlinearly interacting EPOs is a quantum plasma. Both non-relativistic QHD and NLS-Poisson equations exclude strong interactions among the quantum particles and electron exchange interactions (Hohenberg and Kohn, 1964; Kohn and Sham, 1965) between an electron and the background plasma particles (e.g. degenerate electrons and non-degenerate ions). However, it has turned out that the QHD and NLS-Poisson equations have been quite useful for studying linear and nonlinear plasma waves, as well as stability of quantum plasmas (Haas et al., 2005, 2007; Haas et al., 2003; Manfredi, 2005; Manfredi and Haas, 2001; Shukla et al., 2006; Shukla and Eliasson, 2006, 2010) at nanoscales involving the quantum force (Gardner and Ringhofer, 1996; Wilhelm, 1971) and the quantum statistical pressure law for an unmagnetized quantum plasma with degenerate electrons. New effects also appear when one accounts for the potential energy of the electron spin–1/2 in a magnetic field (Brodin and Marklund, 2007ab; Brodin et al., 2008ab; 2010; Marklund and Brodin, 2007; Misra; 2007; 2009; Misra and Samanta, 2007; 2009; 2010; 2011).
II. BASIC PROPERTIES OF QUANTUM PLASMAS

Let us first summarize some of the basic properties of quantum plasmas that are quite distinct from classical plasmas. While classical plasmas are composed of non-degenerate plasma particles with low number densities and relatively high electron and ion temperatures, quantum plasmas have degenerate electrons and/or positrons with extremely high number densities and relatively low temperatures. The ions can usually be treated as non-degenerate plasma particles. Figure 1 depicts the plasma parameter regimes (the electron temperature versus the ion temperature) under which quantum plasmas occur in different physical environments. Specifically, we discuss the formation and dynamics of nanostructures (e.g. 1D electron fluid turbulence at nanoscales. Also presented are nonlinear interactions between intense EM waves and ESOs, which reveal stimulated scattering of EM waves off quantum plasma oscillations and trapping of light into a quantum electron density cavity. The effects of an external magnetic field on linear and nonlinear wave phenomena in a quantum magnetoplasma are examined. Finally, we highlight possible applications, as well as future perspectives and outlook of nonlinear quantum plasma physics.

The relevant degeneracy parameter for the quantum plasma is

\[
\frac{T_F}{T} = \frac{1}{2} (3\pi^2)^{2/3} (n\lambda_B^3)^{2/3} \geq 1. 
\]  

For typical metallic densities of free electrons, \( n \sim 5 \times 10^{22} \text{ cm}^{-3} \), we have \( T_F \sim 6 \times 10^6 \text{ K} \), which should be compared with the usual temperature \( T \).

When the plasma particle temperature approaches \( T_F \), one can show, by using a density matrix formalism (Bransden and Joachain, 2000), that the equilibrium distribution function changes from the Maxwell–Boltzmann function to the Fermi-Dirac (FD) distribution function

\[
\mathcal{F}_{FD} = 2 \left( \frac{m}{2\pi\hbar^2} \right)^3 \left[ 1 + \exp \left( \frac{E - \mu}{k_B T} \right) \right]^{-1},
\]
where in the non-relativistic limit the energy is \( E = (m/2)v^2 = (m/2)(v_x^2 + v_y^2 + v_z^2) \). The chemical potential is denoted by \( \mu \). The parameter \( \mu/k_BT \) is large and negative in the non-degenerate limit, and is large and positive in the completely degenerate limit. The equilibrium electron number density associated with the FD distribution function is

\[
n_0 = \int F_{FD} \, d^3v = -\frac{1}{4} \left( \frac{2mk_BT}{\pi \hbar^2} \right)^{3/2} \text{Li}_{3/2}[-\exp(\xi_\mu)],
\]

where \( \text{Li}_{3/2} \) is the poly-logarithm function, and \( \xi_\mu = \mu/k_BT \). The completely degenerate limit corresponds to \( \mu \rightarrow k_BT \) and \( T_F \gg T \). The relation between \( T_F/T \) and \( \xi_\mu \) is \( -\text{Li}_{3/2}[-\exp(\xi_\mu)] = (4/3\sqrt{\pi})(T_F/T)^{3/2} \).

It is useful to define the quantum coupling parameters for electron-electron and ion-ion interactions. The electron-electron Coulomb coupling parameter is defined as the ratio between the electrostatic interaction energy \( E_{int} = e^2/a_\lambda \) between electrons and the electron Fermi energy \( E_{Fe} = k_BT_F \), where \( e \) is the magnitude of the electron charge and \( a_\lambda = (3/4\pi n_e)^{1/3} \) is the mean inter-electron distance. We have

\[
\Gamma_e = \frac{E_{int}}{E_{Fe}} \approx 0.3 \left( \frac{1}{n_e \lambda_F^3} \right)^{2/3} \approx 0.3 \left( \frac{\hbar \omega_{pe}}{k_BT_F} \right)^2,
\]

where \( \lambda_F = V_{Fe}/\omega_{pe} \), \( V_{Fe} = (2E_{Fe}/m_e)^{1/2} = (\hbar/m_e)(3\pi^2 n_e)^{1/3} \) is the electron Fermi speed, and \( \omega_{pe} = (4\pi n_e e^2/m_e)^{1/2} \) the electron plasma frequency. Furthermore, the ion-ion Coulomb coupling parameter is \( \Gamma_i = Z_i^2 e^2/a_i k_BT_i \), where \( Z_i \) is the ion charge state, \( a_i = (3/4\pi n_i)^{1/3} \) the mean ion-inter-distance, and \( T_i \) the ion temperature.

Since \( \Gamma_e \) for metallic plasmas could be larger than unity, it is of interest to enquire the role of inter-particle collisions on collective processes in a quantum plasma. It turns that the Pauli blocking reduces the collision rate for most practical purposes (Manfredi 2005, Son and Fisch 2005). Due to Pauli blocking, only electrons with a shell of thickness \( k_BT \) about the Fermi surface suffer collisions. For these electrons, the electron-electron collision frequency is proportional to \( k_BT/h \). The average collision frequency among all electrons turns out to be (Manfredi 2005)

\[
\nu_{ee} = \frac{k_BT^2}{hT_F},
\]

Typically, \( \nu_{ee} \ll \omega_{pe} \) when \( T < T_{Fe} \), which is relevant for metallic electrons. On the other hand, the typical timescale for electron-ion (lattice) collisions is \( \tau_{ei} \approx 10 \) fs, which is one order of magnitude greater than the electron plasma period. Accordingly, a collisionless quantum plasma regime is relevant for phenomena appearing on the timescale of the order of a femtosecond in a metallic plasma.

In compact astrophysical objects such as white dwarf stars, the mean distance \( n_e^{-1/3} \) between electrons become comparable to the Compton length \( \lambda_C = h/m_e c \), and accordingly the speed of an electron on the Fermi surface becomes comparable to the speed of light \( c \) in vacuum, so that one has to take relativistic effects into account. Relativistic degenerate electrons are found in the core of massive white dwarf stars (Koester and G. Chandrasekhar [1990], Shapiro [1983]), aptly named due to their very low luminosities yet high surface emissivities, which are compact bodies with radii \( \leq 10^{-2} R_\odot \) and masses typically \( \leq M_\odot \). Consequently, the average electron number densities are quite high (~10^{30} cm\(^{-3}\)). Since electrons are Fermions, only one electron can occupy a given quantum state (position, spin). In a simplified picture, each electron will on average occupy a volume \( 1/n_e \). Then, by Heisenberg’s uncertainty principle (Bransden and Joachain [2000]), \( \Delta x\Delta p \lesssim h/2 \), the mean moment of electrons can be estimated to be \( p_e \approx \hbar n_e^{1/3} \).

If electrons are non-relativistic, the velocity of the electron is \( \sim p_e/m_e = \hbar n_e^{1/3}/m_e \); however, if electrons are relativistic, their velocity will be close to \( c \). Now the electron pressure, as it is for a simple gas, is the momentum transfer per unit area, or \( P_e = (\text{momentum}) \times (\text{velocity}) \times (\text{number density}) \). For non-relativistic electrons, we have

\[
P_e = \hbar n_e^{1/3}(\hbar n_e^{1/3}/m_e)n_e = \hbar^2 n_e^{5/3}/m_e.
\]

On the other hand, when electrons are relativistic, the relativistic electron pressure is \( P_{e\gamma} = \hbar n_e^{1/3}n_e = \hbar n_e^{4/3} \). In the past, Chandrasekhar (1931a,b, 1935, 1939) and others presented a rigorous derivation of the electron pressure \( P_C \) for arbitrary relativistic electron degeneracy pressure in dense matter. It reads

\[
P_C = \frac{\pi}{3h^3} m_e^4 c^5 f(\xi_e),
\]

where \( f(\xi_e) = \xi_e(2\xi_e^2 - 3)(1 + \xi_e^2)^{1/2} + 3\sinh^{-1}(\xi_e) \), \( \xi_e = p_e c/m_e c \), and \( p_e = (3h^3 n_e/8\pi)^{1/3} \) is the momentum of an electron on the Fermi surface. In the non-relativistic limit \( \xi_e \ll 1 \), we have (Chandrasekhar 1935, 1939)

\[
P_n = \frac{(\pi)^{2/3}}{5m_e} \hbar^2 n_e^{5/3},
\]

while in the ultra-relativistic limit \( \xi_e \gg 1 \), the degenerate electron pressure reads (Chandrasekhar 1931a)

\[
P_u = \frac{(3\pi)^{1/3}}{4} \hbar cn_e^{4/3} = \frac{3}{4} \hbar cn_e^{4/3}.
\]

Thus, the intuitively obtained formulas of Gursky [1976] for non-relativistic and ultra-relativistic pressures for degenerate electrons are in agreement with those deduced from the pressure formula (8) for an arbitrary relativistic electron degeneracy pressure.

In his Nobel Prize winning papers on the structure of compact stars, Chandrasekhar (1931a,b) balanced the gradient of the ultra-relativistic electron degeneracy pressure \( P_u/R \) and the gravitational force \( G = \)
(GM/R^2)n_εm_ε, where G is the gravitational constant, M and R are the mass and radius of a star, respectively, m_ε the mass of the nuclei (n_εm_ε = M/R^3); to deduce the critical mass of a star M_c = (h^2/G)^{1/2}m_ν^-2 ≈ 1.4 M_☉, where M_☉ is the solar mass. The interior of white dwarf stars usually consists of fully ionized helium, carbon, and oxygen, which approximately consist of equal amounts of protons and neutrons. Hence, the effective mass of the nuclei can be taken to be the proton mass plus the neutron mass. Since M_ν is independent of density, it means that this mass is obtained independent of radius. This is the limiting mass; more massive stars cannot be supported by electron degeneracy pressure no matter how small they are. This was the discovery of Chandrasekhar; that the pressure dependence on density changed in going from nonrelativistic to relativistic conditions and, as a consequence, there arose a finite limit to the mass of a star with ultra-relativistic degenerate electrons.

III. MODEL EQUATIONS FOR QUANTUM PLASMAS

In quantum systems, the Dirac and Maxwell equations are often used to study the dynamics of a relativistic quantum particle/Fermion (electrons and positrons) in the presence of intense electromagnetic fields. Quantum particles have spin. For example, an electron spin s = 1/2 is an intrinsic property of electrons which have an intrinsic angular momentum characterized by quantum number 1/2, and a magnetic moment for individual electrons. In fact, the relativistic Dirac equation provides a description of quantum particles (spin with spin) under the action of the electromagnetic fields. The spin of electrons (and positrons)—which have the spin-1/2 has been introduced through Dirac’s Hamiltonian [Dirac 1981]

\[ \mathcal{H} = c \alpha \cdot \left( p_c + \frac{e}{c}A \right) - e\phi + \beta mc^2, \]  

(11)

where p_c = -ih\nabla is the momentum operator, and α and β are the Dirac matrices. The three Cartesian components α_j (j = 1, 2, 3) of α_ε are usually constructed with help of the Pauli spin matrices σ_ε, σ_ε and σ_ε. [Brazen and Joachain 2000]. The corresponding wave functions ψ are four-component spinors. The magnetic field is B = ∇ × A, where A and φ are the vector and scalar potentials, which are determined from the Maxwell equations.

In the non-relativistic limit, the Pauli equation [Berestetskii et al. 1999] in the presence of the electromagnetic fields describes the dynamics of a single quantum particle. It reads [Tsintsadze and Tsintsadze 2009]

\[ i\hbar \frac{\partial \psi_ε}{\partial t} = H_ε \psi_ε, \]  

(12)

where

\[ H_ε = -\frac{\hbar^2}{2m_ε} \nabla^2 - \frac{iq_ε\hbar}{2m_εc} (A \cdot \nabla + \nabla \cdot A) + \frac{q_ε^2A^2}{2m_εc^2} + q_ε\phi - \mu_ε \cdot B, \]  

(13)

is the Hamiltonian, and ψ_ε(r, t, σ) is the wavefunction of the single quantum particle species α with the spin s = (1/2)σ (|σ| = 1), and q_ε = -e (+e) for electrons (positrons). The last term in (13) is the potential energy of the magnetic dipole in the external magnetic field, the magnetic moment of which is μ_ε = (q_ε\hbar/2m_εc)σ = μ_Bσ, where μ_B = q_ε\hbar/2mc is the Bohr-Pauli magneton and σ the spin-operator of a single quantum particle [Landau and Lifshitz 1998a].

By using the Madelung representation [Madelung 1926] for the complex wavefunction ψ_ε, viz.

\[ ψ_ε(r, t, σ) = Ψ_ε(r, t, σ) \exp(iS_ε/\hbar), \]  

(14)

where Ψ_ε(r, t, σ) and S_ε(r, t, σ) are real, in the Pauli equation (12), we obtain the quantum Madelung fluid equations [Tsintsadze and Tsintsadze 2009]

\[ \frac{\partial n_ε}{\partial t} + \nabla \cdot (n_εp_ε/m_ε) = 0, \]  

(15)

and

\[ \frac{dp_ε}{dt} = q_ε \left( E + \frac{u_ε \times B}{c} \right) + F_Q + F_s, \]  

(16)

where we have denoted

\[ F_Q = \frac{\hbar^2}{2m_ε} \nabla \left( \sqrt{n_ε} \sqrt{n_ε} \right), \]  

(17)

and

\[ F_s = μ_B \nabla (σ \cdot B). \]  

(18)

Here n_ε = |Ψ_ε|^2 is the probability density of finding a single quantum particle with a spin s at some point in space, p_ε = \nabla S_ε - q_ε A_ε/c is the momentum operator of a quantum particle, d/dt = (\partial/\partial t) + u_ε \cdot \nabla, u_ε is the velocity of a quantum particle, and E = -\nabla φ - c^-1\partial A/\partial t and B = \nabla × A.

The spin force F_s in a quantum magnetoplasm can also be written as [Brodin and Marklund 2007a,b,c; Brodin et al. 2008b; Marklund and Brodin 2007]

\[ F_s = μ_B \tanh \left( \frac{μ_B B}{k_BT_α} \right) \nabla B, \]  

(19)

where B = |B| and tanh(ξ) = B_{1/2}(ξ), with the Brillouin function with argument "1/2" describing particles of spin-1/2. The Langevin parameter tanh(ξ) accounts for the macroscopic magnetization of electrons due to the electron thermal agitation and electron-electron collisions.
A. The Schrödinger and Wigner-Poisson Equations

The quantum $N$-body problem is governed by the Schrödinger equation for the $N$-particle wavefunction

$$\psi(q_1, q_2, \ldots, q_N),$$

where $q_j = (r_j, s_j)$ is the coordinate (space, spin) of the particle $j$, each particle associated with energy $E_j$. A drastic simplification occurs if one neglects the correlation between the particles at every order in $1/N$ and describes the full wavefunction as the product of the single particle wavefunctions. For identical quantum particles, the $N$-particle wavefunction is given by the Slater determinant (Bransden and Joachain 2000)

$$\psi(q_1, q_2, \ldots, q_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(q_1, t) & \psi_2(q_1) & \cdots & \psi_N(q_1) \\ \psi_1(q_2, t) & \psi_2(q_2) & \cdots & \psi_N(q_2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(q_N) & \psi_2(q_N) & \cdots & \psi_N(q_N) \end{vmatrix}, \quad (20)$$

which is antisymmetric with respect to an interchange of any two particle coordinates. This property is required by the Pauli exclusion principle under the second quantization procedure for a system of $N$ identical non-relativistic quantum particles. Accordingly, $\psi$ vanishes if two rows are identical, i.e. two identical quantum particles cannot occupy the same state. Example ($N = 2$): $\psi(q_1, q_2) = \frac{1}{\sqrt{2}} [\psi_1(q_1)\psi_2(q_2) - \psi_1(q_2)\psi_2(q_1)]$ so that $\psi(q_2, q_1) = -\psi(q_1, q_2)$ and $\psi(q_1, q_1) = 0$. In the zero temperature limit, all energy states up to the Fermi energy level are occupied, while no energy states above the Fermi level are occupied.

To capture collective effects in quantum plasmas, Haas et al. (2000) and Anderson et al. (2002) used the time-dependent Hartree model where electrons are described by a statistical mixture of $N$ pure states, where each wavefunction $\psi_j$, $j = 1, \ldots, N$ obeys the Schrödinger equation (Anderson et al. 2002)

$$i\hbar \frac{\partial \psi_j}{\partial t} + \frac{\hbar^2}{2m_e} \nabla^2 \psi_j + e\phi \psi_j = 0, \quad (21)$$

which is coupled with Poisson’s equation

$$\nabla^2 \phi = 4\pi e \left( \sum_{j=1}^{N} |\psi_j|^2 - Z_i n_i \right), \quad (22)$$

where $n_i$ is the ion number density (to be obtained from the hydrodynamic equations for non-degenerate ions, to be discussed later), and $\phi$ the electrostatic potential arising from the charge distribution of $N$ electrons. Equations (21) and (22) have been used to study streaming instabilities (Anderson et al. 2002) and other kinetic effects in a quantum system composed of an ensemble of electrons. Within the Hartree-Fock model, Eq. (21) can be further generalized by including the electron-exchange term resulting from the Pauli exclusion principle. The effect of exchange is for electrons of like-spin to avoid each other. Each electron of a given spin is consequently surrounded by an “exchange hole”, a small volume around the electron which like-spin electrons avoid. For the study of magnetic ordering in quantum dots (QDs) doped with magnetic impurities, Eqs. (21) and (22) must also be enlarged by including a 3D QD confining potential and a Vosko-Wilk-Nusair spin dependent exchange-correlation potential (Dharmandana and Perrot 1995). Hence, the self-consistent model will go far beyond the Kohn-Sham’s description (Kohn and Sham 1965) for treating the dynamics of correlated electrons in electron clusters, accounting for electron-exchange and electron-correlation effects. In the presence of time-dependent potentials, the properties and dynamics of many-electron systems can be investigated by using a time-dependent functional theory (Runge and Gross 1984).

However, in a non-relativistic quantum plasma with an ensemble of degenerate electrons, it is more appropriate to use the quantum statistical theory involving the Wigner distribution function (Wigner 1932)

$$f_w(r, v) = \left( \frac{m_e}{2\pi\hbar^2} \right)^3 \times \int \exp(i m_e v \cdot R / \hbar) \psi^* (r + R / 2) \psi (r - R / 2) d^3 R, \quad (23)$$

where the asterisk denotes the complex conjugate. Equation (23) has also been used by Moyal (Moyal 1949) for studying the dynamics of electrons in a quantum system.

For electrostatic interactions in a quantum plasma, the Wigner-Poisson equations, to a leading order (in the limit of weak quantum coupling parameter $\Gamma_e$), can be written as

$$\frac{\partial f_w}{\partial t} + v \cdot \nabla f_w = - \frac{i e m_e^3}{(2\pi)^3 \hbar^2} \int e^{i m_e (v - v') \cdot R / \hbar} \times \left[ \phi \left( x + \frac{R}{2}, t \right) - \phi \left( x - \frac{R}{2}, t \right) \right] f_w (x, v', t) d^3 R d^3 v', \quad (24)$$

and

$$\nabla^2 \phi = 4\pi e \left( \int f_w d^3 v - Z_i n_i \right). \quad (25)$$

B. The QHD Equations

The non-relativistic QHD equations (Wilhelm 1971) have been developed in condensed matter physics (Gardiner and Ringhofer 1996) and in plasma physics (Manfredi 2005, Manfredi and Has 2001). The non-relativistic QHD equations are composed of the electron continuity equation

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e u_e) = 0, \quad (26)$$
the electron momentum equation \cite{Wilhelm1971}
\[ m_e \left( \frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e \right) = e \nabla \phi - \frac{1}{n_e} \nabla P_e + \mathbf{F}_Q, \] (27)
and Poisson’s equation
\[ \nabla^2 \phi = 4\pi e(n_e - Z_i n_i). \] (28)

In a quantum plasma with non-relativistic degenerate electrons, the quantum statistical pressure in the zero electron temperature limit can be modeled as \cite{Crouseilles2008, Manfredi2001}
\[ P_e = \frac{m_e V_{Fe}^2 n_0}{3} \left( \frac{n_e}{n_0} \right)^{(D+2)/D}, \] (29)
where \( D \) is the number of space dimension of the system, and \( V_{Fe} = (\hbar/m_e)(3\pi^2 n_e)^{1/3} \) the electron Fermi speed.

C. The NLS-Poisson Equations

For investigating nonlinear properties of dense quantum plasmas, it is appropriate to work with a NLS equation. Hence, by introducing the wavefunction
\[ \psi(\mathbf{r}, t) = \sqrt{n_e(\mathbf{r}, t)} \exp \left( i \frac{S_e(\mathbf{r}, t)}{\hbar} \right), \] (30)
where \( S_e \) is defined according to \( m_e \mathbf{u}_e = \nabla S_e \) and \( n_e = |\psi|^2 \), it can be shown that (27) can be cast into a NLS equation \cite{Manfredi2005, Manfredi2001}
\[ i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m_e} \nabla^2 \psi + e\phi \psi - \frac{m_e V_{Fe}^2}{2n_0^2} |\psi|^4 \psi = 0, \] (31)
where the electrostatic field \( \phi \) is determined from Poisson’s equation
\[ \nabla^2 \phi = 4\pi e(|\psi|^2 - Z_i n_i). \] (32)

We note that the third and fourth terms in the left-hand side of Eq. (31) represent the nonlinearities associated with the nonlinear coupling between the electrostatic potential and the electron wavefunction and the nonlinear quantum statistical pressure, respectively.

IV. LINEAR WAVES IN QUANTUM PLASMAS

A. Electron Plasma Oscillations (EPOs)

Linearization of the NLS-Poisson Equations (31) and (32) around the equilibrium state and combining the resultant equations, we obtain the frequency \( \omega \) of the EPOs \cite{Bohm1953, Bohm1953, Klimontovich1952a, Lifshitz1981} and the refractive index for EM waves \cite{Burt1962} by using a quantum kinetic theory based on the Wigner and Poisson-Maxwell equations in a quantum plasma. In the following, we briefly discuss the well known results for the ES \cite{Bohm1953, Bohm1953, Klimontovich1952a, Lifshitz1981} and EM \cite{Burt1962} waves in a unmagnetized quantum plasma.

The dielectric constant for ES waves in a plasma with completely degenerate electrons reads \cite{Lifshitz1981}
\[ D_e(\omega, \mathbf{k}) = 1 + \frac{3e^2}{2k^2 V_{Fe}^2} \left[ 1 - g(\omega_+) + g(\omega_-) \right], \] (36)
where \( \omega_\pm = \omega \pm \hbar k^2/2m_e \), and
\[ g(\omega_\pm) = \frac{m_e (\omega_\pm^2 - k^2 V_{Fe}^2)}{2\hbar k V_{Fe}} \log \left( \frac{\omega_\pm + k V_{Fe}}{\omega_\pm - k V_{Fe}} \right). \] (37)

Assuming that the phase velocity \( (\omega/k) \) of the ES wave is much larger than \( V_{Fe} \), we obtain by setting \( D_e(\omega, \mathbf{k}) = 0 \) the frequency of the EPOs, given by (36). On the other hand, in the semi-classical limit, viz. \( \hbar |\mathbf{k}| \ll \rho_{Fe} = k V_{Fe} \),
\( h(3\pi^2 n_i)^{1/3} \), we have [Lifshitz and Pitaevskii 1981] from Eq. (36)

\[
D_e(\omega, k) = 1 + \frac{3\omega_{pe}^2}{k^2 V_F^2} \left( 1 - \frac{\omega}{2kV_F} \log \left| \frac{\omega + kV_F}{\omega - kV_F} \right| \right),
\]
which in the short wavelength limit, viz. \( kV_F \gg \omega_{pe} \), yields the so-called electron thermal quasi-mode (Klimontovich and Silin, 1952a,b, 1961)

\[
\omega = kV_F \left[ 1 + 2 \exp \left(-2k^2 \lambda_s^2 - 2 \right) \right],
\]
where \( \lambda_s = \lambda_{Fe}/\sqrt{3} \) is the Thomas-Fermi screening length.

Furthermore, when \( \omega = 0 \), the expression (38) as a function of \( k \) has a Kohn singularity at \( h\kappa = 2p_{Fe} \equiv \) the diameter of the Fermi sphere. Here we have

\[
D_e(0, k) = 1 + \frac{e^2}{2\pi \hbar E_F} \left[ 1 - \xi \log(1/|\xi|) \right],
\]
where \( \xi = (h\kappa - 2p_{Fe})/2p_{Fe} \) and \( |\xi| \ll 1 \). In a quantum plasma, with \( D(0, k) \) given by (38), the potential distribution \( \varphi(r) \) around a stationary test charge \( q_t \) is

\[
\varphi(r) = \frac{4\pi q_t}{(2\pi)^3} \int \exp(ik \cdot r)d^3k,
\]
which gives [Else et al., 2010]

\[
\varphi(r) \approx q_t \frac{12\lambda_{Fe}^2 \eta^4 \cos(2k_F r)}{(2 + 3\eta^2)^2 r^3},
\]
where \( \eta = \hbar \omega_{pe}/4kB_T \) and \( k_F = p_{Fe}/\hbar \). We note that (42), which is proportional to \( r^{-3} \cos(2k_F r) \), considerably differs from the Debye-Hückel shielding potential that is proportional to \( r^{-1} \exp(-r/\lambda_{De}) \) in a classical plasma with the Maxwell-Boltzmann electron distribution function. Here \( \lambda_{De} \) is the electron Debye radius. We further note that the shielding of a moving test charge in an unmagnetized quantum plasma has been investigated by Else et al. [2010] both analytically and numerically.

**B. Ion Plasma Oscillations (IPOs)**

We now focus our attention on the effect of the dynamics of non-relativistic and non-degenerate ions in an unmagnetized quantum plasma. The dynamics of strongly coupled ions is governed by the ion hydrodynamic equations composed of Poisson’s equation (28), and the continuity and momentum equations. The latter are

\[
\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0,
\]
and

\[
\left( 1 + \tau_m \frac{\partial}{\partial t} \right) \left[ \left( \begin{array}{c} \frac{\partial}{\partial t} + \mathbf{u}_i \cdot \nabla \end{array} \right) \mathbf{u}_i + \frac{Z_i e}{m_i} \nabla \phi - \frac{\gamma_i k_B T_i}{m_i n_i} \nabla n_i \right] - \frac{\eta}{\rho_i} \nabla^2 \mathbf{u}_i - \frac{(\xi + \frac{\eta}{3})}{\rho_i} \nabla(\nabla \cdot \mathbf{u}_i) = 0,
\]
where \( n_i \) is the ion number density, \( \mathbf{u}_i \) the ion fluid velocity, \( m_i \) the ion mass, \( \rho_i = n_i m_i \) the ion mass density, \( \gamma_i \) the adiabatic index for the ion fluid, \( \tau_m \) the viscoelastic relaxation time for ions, \( \eta \) and \( \xi \) the bulk ion viscosities. The viscoelastic equation (44) for strongly systems has been successfully used [Ichimaru and Tanaka, 1986; Kaw and Sex 1998] for investigating collective processes in classical plasmas with non-degenerate plasma particles.

The ions are coupled with degenerate electrons by the space charge electric field \( \mathbf{E} = -\nabla \phi \). For low-phase velocity (in comparison with the electron Fermi speed) ES waves, we can neglect the inertia of the electrons to obtain

\[
n_e \nabla \phi - \frac{9}{5} \frac{\hbar^2}{m_e} \nabla n_e^{5/3} + \frac{\hbar^2 n_e}{2m_e} \nabla \left( \frac{\nabla^2 n_e}{\nabla n_e} \right) = 0,
\]
for a quantum plasma with weakly relativistic degenerate electrons, while for a quantum plasma with ultra-relativistic degenerate electrons, we have

\[
n_e \nabla \phi - \frac{3}{4} \frac{\hbar c}{\nabla n_e^{3/3}} = 0.
\]

Due to the ion inertia, one has new dielectric constants for the low-frequency (in comparison with the electron plasma frequency) ES waves ([Eliaison and Shukla, 2008a; Mushtaq and Melrose 2009; Pines 1963; Pines and Nozieres 1989; Shukla and Eliaison, 2008b]). In a quantum plasma with non-relativistic degenerate electrons with \( \omega^2 \ll k^2 V_F^2 + \hbar^2 k^4/4m_e^2 \), we can linearize (28), (43), (44), and (45), Fourier transform them, and combine the resultant equations to obtain

\[
D_i(\omega, k) = 1 + \frac{3\omega_{pe}^2}{k^2 V_F^2 + \hbar^2 k^4/4m_e^2} \frac{-\omega_{pi}^2}{\Omega_i^2},
\]
where \( \omega_{pi} = (4\pi n_0 Z_i^2 e^2/m_i)^{1/2} \) is the ion plasma frequency, and \( \Omega_i^2 = \omega^2 - \gamma_i k^2 V_F^2 + i\omega k^2 \eta_i/(1 - i\omega \tau_m) \), with \( V_F = (k_B T_i/m_i) \) and \( \eta_i = (\xi + 4\eta/3)/m_i n_i \). On the other hand, in a quantum plasma with ultra-relativistic degenerate electrons, we have (28), (43), (44), and (46)

\[
D_i(\omega, k) = 1 + \frac{\omega_{pe}^2}{k^2 C_i^2} - \frac{\omega_{pi}^2}{\Omega_i^2},
\]
where we have denoted \( C_i^2 = c^2 \lambda_{Ci} c_0^{1/3} \), and \( \lambda_C = h/m_e c \) is the Compton length. By setting \( D_i(\omega, k) = 0 \), we obtain the frequencies of the IPOs. For the case with non-relativistic degenerate electrons we have

\[
\omega^2 = \gamma_i k^2 V_F^2 + \frac{k^2 \eta_i}{\tau_m} + \frac{\omega_{pi}^2 k^2 \lambda_i^2 \hbar^2}{(1 + k^2 \lambda_i^2 \hbar^2)},
\]
while for the case with ultra-relativistic degenerate electrons the result is

\[
\omega^2 = \gamma_i k^2 V_F^2 + \frac{k^2 \eta_i}{\tau_m} + \frac{\omega_{pi}^2 k^2 \lambda_i^2 \hbar^2}{(1 + k^2 \lambda_i^2 \hbar^2)},
\]
where we have assumed $\omega \tau_m \ll 1$ and denoted $\lambda_{TF} = (\lambda_C^2 + \hbar^2 k^2/4m^2 \omega_{pe}^2)^{1/2}$, and $\lambda_c = C_h/\omega_{pe}$. The domain of validity of the hydrodynamic description for the ions in the context of ion oscillations in a weakly relativistic dense plasma has also been recently discussed by [Mithen et al. 2011].

Melrose and Mushtaq (2009) and Mushtaq and Melrose (2009) have presented Landau damping rates for both electron and ion plasma waves in an unmagnetized dense quantum plasma. The imaginary parts of the dielectric constants can be used to calculate the structural form factor [Ichimaru 1982] in quantum plasma with degenerate electrons.

Shukla and Eliasson (2009) used the dielectric constant (47) without the quantum statistical pressure term (viz. the $V_F^2$-term) to investigate the screening and wake potentials around a test charge in an electron-ion plasma. They found a new screening potential

$$\phi_{se} = \frac{q_t}{r} \exp(-k_q r) \cos(k_q r),$$

and the wake potential

$$\phi_w = -\frac{q_t}{|z - u_0 t|} \cos \left( \frac{\omega_{pe}}{u_0} (z - u_0 t) \right),$$

where $k_q = \sqrt{2/\hbar/m_\omega \omega_{pe}}$ is the quantum wave number, and $r = (x^2 + y^2 + (z - u_0 t)^2)^{1/2}$ the distance from the test charge moving with the speed $u_0$ along the $z$ axis in a Cartesian co-ordinate system. The wake potential (52) behind a test charge arises due to collective interactions between a test charge and the ion oscillations with the frequency $\omega_k \approx \omega_{pe} k_\perp/(k_F^2 + k_\perp^2)^{1/2}$, with $k_\perp \ll k_q, k_\perp = (k_x^2 + k_y^2)^{1/2}$. We note that the Shukla-Eliasson (SE) exponential cosine-screened Coulomb potential $\phi_{se}$ has a minimum of $\phi_{se} \approx -0.02 q_t k_q$ at $r \approx 3k_q^{-1}$, similar to the Lennard-Jones potential for atoms. The SE screening potential $\phi_{se}$, which is independent of the test charge speed $u_0$, is different from the Yukawa screening potential $(q_t/r) \exp(-r/\lambda)$ that is valid in the limit $V_F \gg \hbar k/2m_\omega$. Recently, several authors [Ghoshal and Ho 2009a,b; Xia et al. 2010] have used the SE potential to study doubly excited resonance states of Helium and hydrogen atoms embedded in a quantum plasma [Ghoshal and Ho 2009a,b], and lattice waves in 2D hexagonal quantum plasma crystals [Xia et al. 2010].

Furthermore, by using $D_c$ from (48), one can deduce potential distributions around a moving test charge in a quantum plasma with ultra-relativistic electrons. We have

$$\phi(r, z) = \frac{q_t}{r} \exp \left( -\frac{r}{\Lambda_C} \right) + \frac{q_t}{|z - u_0 t|} \cos \left( \frac{z - u_0 t}{L_c} \right),$$

where $\Lambda_C = C_h/\omega_{pe}$ and $L_c = \lambda_c (M^2 - 1)^{1/2} > 0$, with $M = u_0/(C_h)$.  

**C. High-Frequency EM Waves**

Finally, we turn our attention to the high-frequency (HF) EM waves in an unmagnetized quantum plasma. Noting that HF-EM waves in the latter do not give rise to any density perturbations, we have the EM wave frequency

$$\omega = (k^2 c^2 + \omega_{pe}^2)^{1/2}.$$  

However, consideration of the electron spin current and electron exchange potential contributions in a quantum plasma gives rise to additional contributions to the refractive index $N$. We have [Burt and Wahlquist 1962]

$$\frac{k^2 c^2}{\omega^2} = N \approx 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pe}^2 \hbar^2 k^2}{m^2_\omega \omega^4} \left( \frac{1}{5} K_F^2 + \frac{1}{4} k^2 \right),$$

which includes the electron spin correction, and is valid at zero temperature. Here $\hbar K_F = (2m_\omega E_{Fe})^{1/2}$ is the momentum of degenerate electrons at the Fermi surface, the $(1/5)K_F^2$ term is related to the leading quantum term from the ordinary transverse current, and the $k^2/4$ term arises from the electron spin interactions. On the other hand, the EM wave dispersion relation, which accounts for the electron exchange potential and discards the spin correction, reads [Burt and Wahlquist 1962]

$$\frac{k^2 c^2}{\omega^2} = N \approx 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pe}^2 \hbar^2 k^2 K_F^2}{5m^2_\omega \omega^4} + \frac{3 \omega_{pe}^2 k^2}{40 \omega^4 K_F^2}.$$  

**V. QUANTUM DARK SOLITONS AND VORTICES**

Let us now discuss nonlinear properties and dynamics of 1D quantum dark solitons (characterized by the local electron density depletion associated with a positive potential) and 2D azimuthally symmetric electron vortices in an unmagnetized quantum plasma (Shukla and Eliasson 2006) with immobile ions. The assumption of stationary ions is justified because we are looking for the nonlinear phenomena on a timescale much shorter than the ion plasma period.

We use the normalized NLS-Poisson equations [Shukla 2006, Shukla and Eliasson 2006]

$$i \frac{\partial \Psi}{\partial t} + \mathcal{A} \nabla^2 \Psi + \varphi \Psi - |\Psi|^{4/D} \Psi = 0,$$

and

$$\nabla^2 \varphi = |\Psi|^2 - 1,$$

where the time and space variables are in units of $\hbar/k_B T_{Fe}$ and the electron Fermi-Thomas screening length $\lambda_{TF}$, respectively. Furthermore, we have denoted $\Psi = \psi/\sqrt{n_0}, \varphi = c_0 \phi/k_B T_{Fe}$, and $\mathcal{A} = 2 \pi n_0^{1/3} c^2 / k_B T_{Fe}$. The system (57) and (58) is supplemented by

$$\frac{\partial E_z}{\partial t} = i \mathcal{A} (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi),$$

where $E_z$ is the $z$ component of the electric field.
where \( \mathbf{E}_\varphi = -\nabla \varphi \). Equations (57)–(59) have the following conserved integrals (Shaikh and Shukla 2007; Shukla and Eliasson 2006): the number of electrons
\[
N = \int |\Psi| d^3x,
\]
the electron momentum
\[
P = -i \int \Psi^* \nabla \Psi d^3x,
\]
the electron angular momentum
\[
L = -i \int \Psi^* \mathbf{r} \times \nabla \Psi d^3x,
\]
and the total energy
\[
\mathcal{E} = \int \left( -A \Psi^\ast \nabla^2 \Psi + \frac{|\nabla \varphi|^2}{2} + \frac{D}{(2 + D)} |\Psi|^{(2 + 4/D)} \right) d^3x.
\]

A. Quantum Electron Cavity

For quasi-stationary, 1D nonlinear structures moving with a constant speed \( v_0 \), one can find solitary wave solutions of Eqs. (57) and (58) by introducing the ansatz \( \Psi = W(\xi) \exp(iK_s x - i\Omega_s t) \), where \( W \) is a complex-valued function of the argument \( \xi = x - v_0 t \), and \( K_s \) and \( \Omega_s \) are a constant wavenumber and frequency shift, respectively. By the choice \( K_s = v_0 / 2A \), the coupled system of equations (57) and (58) can then be written as
\[
\frac{d^2W}{d\xi^2} + \lambda W + \frac{\varphi W}{A} - \frac{|W|^4W}{A} = 0,
\]
and
\[
\frac{d^2\varphi}{d\xi^2} = |W|^2 - 1,
\]
where \( \lambda = (\Omega_s / A) - v_0^2 / 4A^2 \) is an eigenvalue of the system. From the boundary conditions \( |W| = 1 \) and \( \varphi = 0 \) at \( |\xi| = \infty \), we determine \( \lambda = 1 / A \) and \( \Omega_s = 1 + v_0^2 / 4A \). The system of Eqs. (64) and (65) admits a first integral in the form
\[
H_k = A \left| \frac{dW}{d\xi} \right|^2 - \frac{1}{2} \left( \frac{d\varphi}{d\xi} \right)^2 + \frac{|W|^2}{2},
\]
where the boundary conditions \( |W| = 1 \) and \( \varphi = 0 \) at \( |\xi| = \infty \) have been employed.

Figure 2 shows profiles of \( |W|^2 \) and \( \varphi \) obtained numerically from (64) and (65) for a few values of \( A \), where \( W \) was set to \(-1\) on the left boundary and to \(+1\) on the right boundary, i.e. the phase shift is 180 degrees between the two boundaries. The solutions are in the form of dark solitons, with a localized depletion of the electron density \( N_e = |W|^2 \), associated with a localized positive potential. Larger values of the quantum coupling parameter \( A \) give rise to larger-amplitude and wider dark solitons. The solitons localized “shoulders” on both sides of the density depletion.

A numerical solution of the time-dependent system of Eqs. (57) and (58) is displayed in Fig. 3 with initial conditions close (but not equal) to the ones in Fig. 2. Two very clear and long-lived dark solitons are visible, associated with a positive potential of \( \varphi \approx 3 \), in agreement with the quasi-stationary solution of Fig. 2 for \( A = 5 \). In addition there are oscillations and wave turbulence in the time-dependent solution presented in Fig. 3. Hence, the dark solitons seem to be robust structures that can withstand perturbations and turbulence during a considerable time.
FIG. 4 The electron density $|\Psi|^2$ (upper panel) and ES potential $\varphi$ (lower panel) associated with 2D electron vortices supported by the system (67) and (68), for the charge states $s = 1$ (solid lines), $s = 2$ (dashed lines) and $s = 3$ (dash-dotted lines), with $A = 5$ in all cases. After Shukla and Eliasson (2006).

B. Quantum Electron Vortices

For two-dimensional ($D = 2$) EPOs in quantum plasmas, one can look for quantum vortex structures of the form $\Psi = \psi(r) \exp(is\theta - i\Omega_v t)$, where $r$ and $\theta$ are the polar coordinates defined via $x = r \cos(\theta)$ and $y = r \sin(\theta)$, $\Omega_v$ is a constant frequency shift, and $s = 0, \pm 1, \pm 2, \ldots$ for different excited states (charge states). With this ansatz, Eqs. (57) and (58) can be written as, respectively,

\[
\Omega_v + A \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{s^2}{r^2} \right) \varphi = 0,
\]

and

\[
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \varphi = |\psi|^2 - 1,
\]

where the boundary conditions $\psi = 1$ and $\varphi = d\psi/dr = 0$ at $r = \infty$ determine the constant frequency $\Omega_v = 1$. Different signs of the charge state $s$ describe different rotation directions of the quantum vortex. For $s \neq 0$, one must have $\psi = 0$ at $r = 0$, and from symmetry considerations one has $d\varphi/dr = 0$ at $r = 0$. Figure 4 depicts numerical solutions of Eqs. (67) and (68) for different values of $s$ and for $A = 5$. Here a quantum vortex is characterized by a complete depletion of the electron density at the core of the vortex, and is associated with a positive ES potential.

A time-dependent solution of Eqs. (57) and (58) in two-space dimensions for singly charged ($s = \pm 1$) electron vortices is shown in Fig. 5, where, in the initial condition, four vortex-like structures were placed at some distance from each other. The initial conditions were such that the vortices are organized in two vortex pairs, with $s_1 = +1$, $s_2 = -1$, $s_3 = -1$, and $s_4 = +1$, seen in the upper panels of Fig. 5. The vortices in the pairs have opposite polarity on the electron fluid rotation, as seen in the upper right panel of Fig. 5. Interestingly, the “partners” in the vortex pairs attract each other and propagate together with a constant velocity, and in the collision and interaction of the vortex pairs (see the second and third pairs of panels in Fig. 5), the vortices keep their identities and change partners, resulting into two new vortex pairs which propagate obliquely to the original propagation direction. On the other hand, as shown in Fig. 6, vortices that are multiply charged ($|s| > 1$) are unstable. Here the system of Eqs. (57) and (58) was again solved numerically with the same initial condition as the one in Fig. 5, but with doubly charged vortices $s_1 = +2$, $s_2 = -2$, $s_3 = -2$, and $s_4 = +2$. The second row of panels in Fig. 6 reveals that the vortex pairs keep their identities for some time, while a quasi 1D density

FIG. 5 The electron density $|\Psi|^2$ (left panel) and an arrow plot of the electron current $i (\nabla \Psi^* - \Psi^* \nabla \Psi)$ (right panel) associated with singly charged ($|s| = 1$) 2D electron vortices, obtained from a simulation of the time-dependent system of equations (57) and (58), at times $t = 0, t = 3.3, t = 6.6$ and $t = 9.9$ (upper to lower panels), with $A = 5$. The singly charged vortices form pairs and keep their identities. After Shukla and Eliasson (2006).
The statistical properties of quantum electron fluid turbulence and its associated electron transport properties at nanoscales in a quantum plasma have been investigated in both 2D and 3D by using the coupled NLS and Poisson equations [Shaikh and Shukla, 2007, 2008]. It has been found that nonlinear couplings between the EPOs of different scale sizes give rise to small-scale electron density structures, while the ES potential cascades towards large-scales. The total energy associated with the quantum electron plasma wave turbulence processes a non-universal spectrum that depends on the quantum electron coupling parameter.

To investigate 3D quantum electron plasma wave turbulence, we use the NLS-Poisson equations [Manfredi and Haas, 2001, Shaikh and Shukla, 2008, Shukla, 2006, Shukla and Eliasson, 2006]

$$i\sqrt{2\hbar} \frac{\partial \Psi}{\partial t} + H \nabla^2 \Psi + \phi \Psi - |\Psi|^{4/3} \Psi = 0,$$  \hspace{1cm} (69)

and

$$\nabla^2 \phi = |\Psi|^2 - 1,$$  \hspace{1cm} (70)

were used, which govern the dynamics of nonlinearly interacting EPOs of different wavelengths. In Eqs. (69) and (70) the wavefunction is normalized by $\sqrt{n_0}$, the ES potential by $k_B T_{Fe}/e$, the time by the electron plasma period $\omega_{pe}^{-1}$, and the space by the electron Fermi-Thomas screening length $\lambda_{Fe} = V_{Fe}/\omega_{pe}$. Here $\sqrt{\hbar} = \hbar \omega_{pe}/\sqrt{2k_B T_{Fe}}$ was introduced.

The nonlinear wave-wave coupling studies have been performed to investigate the multi-scale evolution of a decaying 3D electron plasma wave turbulence, which is described by Eqs. (69) and (70). All fluctuations are initialized isotropically (no mean fields are assumed) with random phases and amplitudes in Fourier space, and are evolved in time by the integration of Eqs. (69) and (70) numerically. The initial isotropic turbulent spectrum was initially chosen close to $k^{-2}$, with random phases in all three directions. The choice of such (or even a flatter than $k^{-2}$) spectrum treats the turbulent fluctuations on an equal footing and avoids any influence on the dynamical evolution that may be due to the initial spectral non-symmetry.

The properties of 3D electron plasma wave turbulence, composed of nonlinearly interacting EPOs, were studied for two specific physical systems, corresponding to dense plasmas in the next generation of laser-based plasma compression (LBPC) schemes [Malkin et al., 2007], and in superdense astrophysical objects [Chabrier et al., 2002, 2006, Chabrier, 2009, Harding and Lai, 2006, Lai, 2001] (e.g. white dwarfs). It is expected that in LBPC schemes, the electron number density may reach $10^{27}$ cm$^{-3}$ and beyond. Hence, we have $\omega_{pe} = 1.76 \times 10^{18}$ s$^{-1}$, $T_{Fe} = 1.7 \times 10^{-9}$ erg, $\hbar \omega_{pe} = 1.7 \times 10^{-9}$ erg, and $H = 1$, and the electron Fermi-Thomas screening length $\lambda_{Fe} = 0.1$ A. On the other hand, in the core of white dwarf stars, we typically have $n_0 \sim 10^{30}$ cm$^{-3}$, yielding $\omega_{pe} = 5.64 \times 10^{19}$ s$^{-1}$, $T_{Fe} = 1.7 \times 10^{-7}$ erg (0.1 MeV), $\hbar \omega_{pe} = 5.64 \times 10^{-8}$ erg, $H \approx 0.3$, and $\lambda_{Fe} = 0.025$ A. The numerical solutions of Eqs. (69) and (70) for $H = 0.4$ and $H = 0.01$ (corresponding to $n_0 = 10^{27}$ cm$^{-3}$ and $n_0 = 10^{30}$ cm$^{-3}$, respectively) are displayed in Fig. 7 which shows the electron number density and ES potential distributions in the $(x, y, z)$-cube.

Figure 7 reveals that the electron density distribution has a tendency to generate smaller length-scale structures, while the ES potential cascades towards larger

FIG. 6 The electron density $|\Psi|^2$ (left panel) and an arrow plot of the electron current $i(\Psi \nabla \Psi^* - \Psi^* \nabla \Psi)$ (right panel) associated with double charged ($|s| = 2$) 2D electron vortices, obtained from a simulation of the time-dependent system of Eqs. (67) and (68), at times $t = 0$, $t = 3.3$, $t = 6.6$ and $t = 9.9$ (upper to lower panels), with $A = 5$. The doubly charged vortices dissolve into nonlinear structures and wave turbulence. After Shukla and Eliasson (2006).

cavity is formed between the two vortex pairs. At a later stage, the four vortices dissolve into complicated nonlinear structures and wave turbulence. Hence, the nonlinear dynamics is very different between singly and multiply charged solitons, where only singly charged vortices are long-lived and keep their identities.

VI. QUANTUM ELECTRON FLUID TURBULENCE
FIG. 7 Small-scale fluctuations in the electron density resulting from steady turbulence simulations, for $H = 0.4$ (top panels) and $H = 0.01$ (bottom panels). Forward cascades are responsible for the generation of small-scale fluctuations seen in panels (a) and (c). Large scale structures are present in the ES potential, seen in panels (b) and (d), essentially resulting from an inverse cascade. After Shaikh and Shukla (2008).

FIG. 8 Energy $\mathcal{E}_k$ per vector wavenumber of 3D EPOs in the forward cascade regime. A Kolmogorov-like spectrum $\mathcal{E}_k \sim k^{-11/3}$ is observed for $H = 0.4$. The spectral index changes as a function of $H$. After Shaikh and Shukla (2008).

scales. The co-existence of the small and larger scale structures in turbulence is a ubiquitous feature of various 3D turbulence systems. For example, in 3D hydrodynamic turbulence, the incompressible fluid admits two invariants, namely the energy and the mean squared vorticity. The two invariants, under the action of an external forcing, cascade simultaneously in turbulence, thereby leading to a dual cascade phenomena. In these processes, the energy cascades towards longer length-scales, while the fluid vorticity transfers spectral power towards shorter length-scales. Usually, a dual cascade is observed in a driven turbulence simulation, in which certain modes are excited externally through random turbulent forces in spectral space. The randomly excited Fourier modes transfer the spectral energy by conserving the constants of motion in $k$-space. On the other hand, in freely decaying turbulence, the energy contained in the large-scale eddies is transferred to the smaller scales, leading to a statistically stationary inertial regime associated with the forward cascades of one of the invariants. Decaying turbulence often leads to the formation of coherent structures as turbulence relaxes, thus making the nonlinear interactions rather inefficient when they are saturated.

The power spectrum exhibits an interesting feature in the 3D electron plasma system discussed here, unlike the 3D hydrodynamic turbulence (Frisch, 1995; Kolmogorov, 1941a,b; Lesieur, 1990). Figure 8 shows the energy $\mathcal{E}_k$ per vector wavenumber. For isotropic 3D turbulence, it is related to the energy $E_k$ per scalar wavenumber as $E_k = 4\pi k^2 \mathcal{E}_k$; see e.g. Knight and Sirovich (1990). For $H = 0.4$, the spectrum per vector wavenumber is close to $\mathcal{E}_k \sim k^{-11/3}$ and hence yields the standard Kolmogorov power spectrum (Kolmogorov, 1941a,b) $E_k \sim k^{-5/3}$. However, the spectrum is not universal but changes for different values of $H$. For 2D quantum electron fluid turbulence (Shaikh and Shukla, 2007) the spectral slope was more close to the Iroshnikov-Kraichnan power law (Iroshnikov, 1963; Kraichnan, 1965) $E_k \sim k^{-3/2}$. The origin of the differences in the observed spectral indices resides with the nonlinear character of the underlying plasma models, as nonlinear interactions in the 2D and 3D systems are governed typically by different nonlinear forces. The latter modify the spectral evolution of turbulent cascades to a significant degree. Physically, the flatness (or deviation from the $k^{-5/3}$ law), results from the short wavelength part of the EPO spectrum which is controlled by the quantum electron tunneling effect associated with the Bohm potential. The peak in the energy spectrum can be attributed to the higher turbulent power residing in the EPO potential, which eventually leads to the generation of larger scale structures, as the total energy encompasses both the electrostatic potential and electron density components. In the dual cascade process, there is a delicate competition between the EPO dispersions caused by the statistical pressure law (giving the $k^2 V_F^2$ term, which dominates at longer scales) and the quantum Bohm force (giving the $\hbar^2 k^4/4m_e$ term, which dominates at shorter scales).

The electron diffusion in the presence of small and large scale turbulent EPOs can be estimated in the following manner. An effective electron diffusion coefficient caused by the momentum transfer can be calculated from $D_{eff} = \int_0^\infty \langle \mathbf{P}(r,t) \cdot \mathbf{P}(r,t+t') \rangle dt'$, where $\mathbf{P}$ is electron momentum and the angular bracket denotes spatial averages and the ensemble averages are normalized to unit mass. The effective electron diffusion coefficient, $D_{eff}$, essentially relates the diffusion processes associated with random translational motions of the electrons in nonlinear plasmonic fields. To measure the turbulent electron transport that is associated with the turbulent
FIG. 9 Time evolution of the effective electron diffusion coefficient associated with the large-scale ES potential and the small-scale electron density, for $H = 0.4$, $H = 0.1$ and $H = 0.01$. Smaller values of $H$ corresponds to a small effective diffusion coefficient, which characterizes the presence of small-scale turbulent eddies that suppress the electron transport. After Shaikh and Shukla (2008).

structures, $D_{eff}$ is computed. It is observed that the effective electron diffusion is lower when the field perturbations are Gaussian. On the other hand, the electron diffusion increases rapidly with the eventual formation of large-scale structures, as shown in Fig. 9. The electron diffusion due to large scale potential distributions in a quantum plasma dominates substantially, as depicted by the solid-curve in Fig. 9. Furthermore, in the steady-state, nonlinearly coupled EPOs form stationary structures, and $D_{eff}$ eventually saturates. Thus, remarkably an enhanced electron diffusion results primarily due to the emergence of large-scale potential structures.

VII. NONLINEARLY COUPLED EM AND ES WAVES

We turn our attention to nonlinear interactions between large amplitude EM and ES waves in a quantum plasma. Shukla and Stenflo (2006) considered nonlinear couplings between large amplitude EM waves and finite amplitude electron and ion plasma waves, and presented nonlinear dispersion relations that exhibit stimulated Raman scattering (SRS), stimulated Brillouin scattering (SBS), and modulational instabilities. The work of Shukla and Stenflo (2006) has been further generalized by including thermal corrections to the ES waves (Stenflo and Shukla 2009) and relativistic electron mass variations (Shukla and Eliasson, 2007) caused by EM waves in an unmagnetized quantum plasma.

A. Stimulated Scattering Instabilities

First, we present the governing equations for the HF-EM waves and the EM wave driven modified EPOs and IPOs. We have (Stenflo and Shukla, 2009)

$$\left( \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + \omega_{pe}^2 \right) A + \omega_{pe}^2 \frac{n_1}{n_0} A \approx 0, \quad (71)$$

for the HF-EM wave,

$$\left( \frac{\partial^2}{\partial t^2} + \omega_{pe}^2 - \frac{3}{5} V_F^2 \nabla^2 + \frac{\hbar^2}{4m_e^2c^4} \right) n_1 \frac{n_0}{m_e} = \frac{e^2}{2m_e m_i c^2} \nabla|A|^2, \quad (72)$$

for the HF-EM wave pressure driven EPOs, and

$$\left( \frac{\partial^2}{\partial t^2} - C_{TF}^2 \nabla^2 + \frac{\hbar^2}{4m_e m_i} \nabla^4 \right) n_1 \frac{n_0}{m_e} = \frac{e^2}{2m_e m_i c^2} \nabla|A|^2, \quad (73)$$

for the EM wave pressure driven modified IPOs without the ion thermal, ion viscoelastic relaxation, and ion viscosity effects. Here $C_{TF} = (k_B T_F / m_e)^{1/2}$ and $n_1 (\ll n_0)$ is a small perturbation in the electron number density.

Following the standard procedure of the parametric instabilities (Murtaza and Shukla, 1984; Sharma and Shukla, 1983; Shukla, 2006; Shukla and Eliasson, 2004; Shukla et al., 1981; Yu et al., 1974), we can Fourier analyze (71)-(73) and combine the resultant equations to obtain the nonlinear dispersion relations

$$\omega^2 - \Omega_R^2 = - \frac{e^2 \omega_{pe}^2 k^2 |A_0|^2}{2m_e c^2} \left( \frac{1}{D_+} + \frac{1}{D_-} \right), \quad (74)$$

and

$$\omega^2 - \Omega_B^2 = \frac{e^2 \omega_{pe}^2 k^2 |A_0|^2}{2m_e m_i c^2} \left( \frac{1}{D_+} + \frac{1}{D_-} \right), \quad (75)$$

for the driven EPOs and IPOs, respectively, which admit SRS, SBS, and modulational instabilities of the HF-EM pump (with the amplitude $A_0$) in a quantum plasma. Here, $D_\pm = \pm 2\omega_0 (\omega - k \cdot V_g) - k^2 c^2$, where $V_g = kc^2 / 2\omega_0$ is the group velocity of the HF-EM pump wave with the frequency $\omega_0 = (k^2 c^2 + \omega_{pe}^2)^{1/2}$, and

$$\Omega_R^2 = \omega_{pe}^2 + \frac{3}{5} k^2 V_F^2 + \frac{\hbar^2 k^4}{4m_e^2}, \quad (76)$$

and

$$\Omega_B^2 = k^2 C_{TF}^2 + \frac{\hbar^2 k^4}{4m_e m_i}. \quad (77)$$

The growth rates of SRS and SBS instabilities (Shukla and Stenflo, 2006) are, respectively,

$$\gamma_R = \frac{\omega_{pe} e K |A_0|}{2\sqrt{2\omega_0 \Omega_R m_e c}}, \quad (78)$$

and

$$\gamma_B = \frac{\omega_{pe} e K |A_0|}{2\sqrt{2\omega_0 \Omega_B m_e m_i c}}, \quad (79)$$
The present results of SRS and SBS instabilities will help to identify the electrostatic spectral lines that are enhanced by the large amplitude HF-EM pump wave in a quantum plasma.

B. Nonlinearly Coupled Intense EM and EPOs

Let us now consider nonlinear interactions between an arbitrary large amplitude circularly polarized electromagnetic (CPEM) wave and nonlinear EPOs that are driven by the relativistically strong CPEM pump wave. Such an interaction gives rise to an envelope of the CPEM vector potential \( \mathbf{A}_\perp = A_\perp (\mathbf{x} + i \mathbf{y}) \exp(-i\omega_0 t + i k_0 z) \), which obeys the NLS equation \( \text{Shukla and Eliasson (2007)} \)

\[
2i \epsilon \frac{\partial}{\partial t} + U_g \frac{\partial}{\partial z} A_\perp + \frac{\partial^2 A_\perp}{\partial z^2} - \left( \frac{|\psi|^2}{\gamma} - 1 \right) A_\perp = 0, \tag{80}
\]

where \( \epsilon = \omega_0/\omega_{pe} \), and the normalized (by \( \sqrt{\gamma_0} \)) electron wavefunction \( \psi \) and the normalized (by \( m_0 c^2/\epsilon \)) scalar potential are governed by, respectively,

\[
i H_e \frac{\partial \psi}{\partial t} + \frac{H_e^2}{2} \frac{\partial^2 \psi}{\partial z^2} + (\phi - \gamma + 1) \psi = 0, \tag{81}
\]

and

\[
\frac{\partial^2 \phi}{\partial z^2} = |\psi|^2 - 1, \tag{82}
\]

where \( m_0 \) is the rest mass of the electrons, \( U_g = k_0 c/2 \omega_0 \), \( H_e = \hbar \omega_{pe}/m_0 c^2 \) is the ration between the plasmonic energy density to the rest electron energy, and \( \gamma = (1 + |A_\perp|^2)^{1/2} \) is the relativistic gamma factor due to the electron quiver velocity in the CPEM wave fields. The time and space variables are in units of the electron plasma period \( (\omega_{pe}^{-1}) \) and the electron skin depth \( \lambda_e = c/\omega_{pe} \). The electron density and \( A_\perp \) are in units of \( n_0 \) and \( m_0 c^2/\epsilon \) \( \text{Shukla and Eliasson (2007)} \). The nonlinear coupling between intense CPEM waves and EPOs comes about due to the nonlinear current density, which is represented by the term \( |\psi|^2 A_\perp / \gamma \) in Eq. (80). In Eq. (81), \( 1 - \gamma \) is the relativistic ponderomotive potential \( \text{Shukla and Yu (1984) Shukla et al. (1985)} \). The latter arises from the averaging (over the CPEM wave period \( 2\pi/\omega_0 \)) of the relativistic advection and the nonlinear Lorentz force involving the electron quiver velocity and the CPEM wave electric and magnetic fields.

A relativistically strong EM wave in a classical electron-ion plasma is subject to SRS and modulational instabilities \( \text{McKinstrie and Bingham (1992)} \). One can expect that these instabilities will be modified at quantum scale by the dispersion effects caused by the tunneling of electrons through the quantum Bohm potential. The growth rate of the relativistic parametric instabilities in a dense quantum plasma in the presence of a relativistically strong CPEM pump wave can be obtained in a standard manner \( \text{Shukla et al. (1985)} \) by letting \( \phi(z, t) = \phi_1(z, t), A_\perp(z, t) = [A_0 + A_1(z, t)] \exp(-i\omega_0 t) \) and \( \psi(z, t) = [1 + \psi_1(z, t)] \exp(-i\beta_0 t) \), where \( A_0 \) is the large-amplitude CPEM pump and \( A_1 \) is the small-amplitude perturbation of the CPEM wave amplitude due to the nonlinear coupling between the CPEM waves and EPOs, i.e. \( |A_1| < |A_0| \), and \( \psi_1 \ll 1 \) is the small-amplitude perturbation in the electron wave function. The constants \( \omega_0 \) and \( \beta_0 \) are constant frequency shifts, determined from Eqs. (80) and (81) to be \( \omega_0 = (1/\gamma_0 - 1)/(2e) \) and \( \beta_0 = (1 - \gamma_0)/H_e \), where \( \gamma_0 = (1 + |A_0|^2)^{1/2} \). The first-order perturbations in the electromagnetic vector potential and the electron wave function are expanded into their respective sidebands as \( A_1(z, t) = A_\perp \exp(iKz - i\Omega t) + A_\perp \exp(-iKz + i\Omega t) \) and \( \psi_1(z, t) = \psi_\perp \exp(iKz - i\Omega t) + \psi_\perp \exp(-iKz + i\Omega t) \), while the potential is expanded as \( \phi(z, t) = \tilde{\phi} \exp(iKz - i\Omega t) + \tilde{\phi}^* \exp(-iKz + i\Omega t) \), where \( \Omega \) and \( K \) are the normalized frequency and the normalized wave number of the EPOs, respectively. Inserting the above mentioned Fourier ansatz into Eqs. (80)–(82), linearizing the resultant system of equations, and sorting into equations for different Fourier modes, one obtains the nonlinear dispersion relation \( \text{Shukla and Eliasson (2007)} \)

\[
1 + \left( \frac{1}{D_+} + \frac{1}{D_-} \right) \left( 1 + \frac{K^2}{D_L} \right) |A_0|^2 \frac{2\gamma_0^3}{\lambda_0^3} = 0, \tag{83}
\]

where \( D_\pm = \pm 2\epsilon (\Omega - KU_g) - K^2 \) and \( D_L = 1 - \epsilon^2 + H_e^2 K^4/4 \). One notes that \( D_L = 0 \) yields the linear dispersion relation \( \omega^2 = 1 + H_e^2 K^4/4 \) for the EPOs in a dense quantum plasma \( \text{Pines (1961)} \). For \( H_e \rightarrow 0 \) we recover from \( \text{Shukla and Eliasson (1985)} \) the nonlinear dispersion relation for relativistically large amplitude EM waves in a classical electron plasma \( \text{McKinstrie and Bingham (1992)} \). The dispersion relation (83) governs stimulated Raman backward and forward scattering instabilities, as well as the modulational instability. In the long wavelength limit \( U_g \ll 1, \epsilon \approx 1 \) one can use the ansatz \( \Omega = i\Gamma \), where the normalized (by \( \omega_{pe} \)) growth rate \( \Gamma \ll 1 \), and obtain from Eq. (83) the growth rate \( \Gamma = (1/2|K| \{ |A_0|^2/\gamma_0^3 \} [1 + K^2/(1 + H_e^2 K^4/4)] - K^2 \})^{1/2} \) of the modulational instability. For \( |K| < 1 \) and \( H_e < 1 \), the linear growth rate is only weakly depending on the quantum parameter \( H_e \).

The quantum dispersion effects on nonlinearly coupled CPEM and EPOs can be studied by considering a steady state structure moving with a constant speed \( U_g \). Inserting the ansatz \( A_\perp = W(\xi) \exp(-i\Omega t), \psi = P(\xi) \exp(iKx - i\omega_0 t) \) and \( \phi = \phi(\xi) \) into Eqs. (80)–(82), where \( \xi = z - U_g t, k_e = U_g/H_e \) and \( \omega_0 = \omega_e^2/2H_e \), and where \( W(\xi) \) and \( P(\xi) \) are real, one obtains from (80)–(82) the coupled system of equations \( \text{Shukla and Eliasson (2007)} \)

\[
\frac{\partial^2 W}{\partial \xi^2} + \left( \lambda - \frac{P^2}{2 \gamma + 1} \right) W = 0, \tag{84}
\]
where $\gamma = (1 + W^2)^{1/2}$, and
\[
\frac{H^2}{2} \frac{\partial^2 P}{\partial \xi^2} + (\phi - \gamma + 1)P = 0, \tag{85}
\]
with the boundary conditions $W = \Phi = 0$ and $P^2 = 1$ at $|\xi| = \infty$. In Eq. \(84\), $\lambda = 2e\Omega$, represents a nonlinear frequency shift of the CPEM wave. In the limit $H_e \to 0$, one has from \(85\) $\phi = \gamma - 1$, where $P \neq 0$, and one recovers the classical (non-quantum) case of the relativistic solitary waves in a cold plasma \cite{Marburger and Tooper, 1972}.

The system of equations \(84\)–\(86\) admits a Hamiltonian
\[
Q_H = \frac{1}{2} \left( \frac{\partial W}{\partial \xi} \right)^2 + \frac{H^2}{2} \left( \frac{\partial P}{\partial \xi} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial \xi} \right)^2 + \frac{1}{2} (\lambda + 1)W^2 + P^2 - \gamma P^2 + \phi P^2 - \phi = 0, \tag{87}
\]
where the boundary conditions $\partial / \partial \xi = 0$, $W = \phi = 0$ and $|P| = 1$ at $|\xi| = \infty$ have been used.

Numerical solutions of the quasi-stationary system \(84\)–\(86\) are presented in Figs. 10 and 11 while time-dependent solutions of Eqs. \(80\)–\(82\) are displayed in Figs. 12 and 13. Here parameters were used that are representative of the next generation of laser-based plasma compression (LBPC) schemes \cite{Azzechi et al, 2006, Malkin et al, 2007}. The formula \cite{Shukla et al, 1985} $eA_\perp/mc^2 = 6 \times 10^{-10}\lambda_e \sqrt{I}$ will determine the normalized vector potential, provided that the CPEM wavelength $\lambda_e$ (in microns) and the CPEM wave intensity $I$ (in W/cm$^2$) are known. It is expected that in LBPC schemes, the electron number density $n_0$ may reach $10^{27}$ cm$^{-3}$ and beyond, and the peak values of $eA_\perp/mc^2$ may be in the range 1-2 (e.g. for focused EM pulses with $\lambda_e \sim 0.15$ nm and $I \sim 5 \times 10^{27}$ W/cm$^2$). For $\omega_{pe} = 1.76 \times 10^{18}$ s$^{-1}$, one has $\hbar \omega_{pe} = 1.76 \times 10^{-9}$ erg and $H_e = 0.002$, since $mc^2 = 8.1 \times 10^{-7}$ erg. The electron skin depth $\lambda_e \sim 1.7$ Å. On the other hand, a higher value of $H_e = 0.007$ is achieved for $\omega_{pe} = 5.64 \times 10^{18}$ s$^{-1}$. Thus, the numerical solutions below, based on these two values of $H_e$, have focused on scenarios that are relevant for the next generation intense laser-solid density plasma interaction experiments \cite{Malkin et al, 2007}.

Figures 10 and 11 exhibit numerical solutions of Eqs. \(84\)–\(86\) for $H_e = 0.002$ and $H_e = 0.007$. The nonlinear boundary value problem was solved with the boundary conditions $W = \phi = 0$ and $P = 1$ at the boundaries at $\xi = \pm 10$. The solitary envelope pulse is composed of a single maximum of the localized vector potential $W$ and a local depletion of the electron density $P^2$, and a localized positive potential $\phi$ at the center of the solitary pulse. The latter has a continuous spectrum in $\lambda$, where larger values of negative $\lambda$ are associated with larger amplitude solitary EM pulses. At the center of the solitary EM pulse, the electron density is partially depleted, as in panels a) of Fig. 10, and for larger amplitudes of the EM waves one has a stronger depletion of the electron density, as shown in panels b) and c) of Fig. 10. For cases where the electron density goes to almost zero in the classical case \cite{Marburger and Tooper, 1972}, one important quantum effect is that the electrons can tunnel through the depleted density region. This is seen in Fig. 11 where the electron density remains nonzero for $H_e = 0.007$ in panels a), while the density shrinks to zero for $H_e = 0.002$ in panel b).

![Fig. 10](image_url)  
**Fig. 10** The profiles of the CPEM vector potential $W$ (top row), the electron number density $P^2$ (middle row) and the scalar potential $\Phi$ (bottom row) for $\lambda = -0.3$ (left column), $\lambda = -0.34$ (middle column) and $\lambda = -0.4$ (right column), with $H_e = 0.002$. After \text{Shukla and Eliasson, 2007}.

Figures 12 and 13 depict numerical simulation results of Eqs. \(80\)–\(82\) for the long-wavelength limit characterized by $\omega_0 \approx 1$ and $V_g \approx 0$. As initial conditions, we used an EM pump with a constant amplitude $A_\perp = A_0 = 1$ and a uniform plasma density $\psi = 1$, together with a small amplitude noise (random numbers) of order $10^{-2}$ added to $A_\perp$ to give a seeding any instability. The numerical results are displayed in Figs. 12 and 13 for $H_e = 0.002$ and $H_e = 0.007$, respectively. In both cases, we see an initial linear growth phase and a wave collapse at $t \approx 70$, in which almost all the CPEM wave energy is contracted into a few well separated large-amplitude, localized CPEM envelopes, associated with an almost complete depletion of the electron density at the center of the CPEM wavepacket, and a large-amplitude positive electrostatic potential. One can see that there is a more complex dynamics of localized CPEM wavepackets for $H_e = 0.007$, shown in Fig. 13 in comparison with $H_e = 0.002$, shown in Fig. 12, where the wavepackets are almost stationary when they are fully developed.
FIG. 11 The profiles of the CPEM vector potential $W$ (top row), the electron number density $P_2$ (middle row) and the scalar potential $\Phi$ (bottom row) for $H_e = 0.007$ (left column) and $H_e = 0.002$ (right column), with $\lambda = -0.34$. After Shukla and Eliasson (2007).

FIG. 12 The dynamics of the CPEM vector potential $A_\perp$ and the electron number density $|\psi|^2$ (upper panels) and the electrostatic potential $\phi$ (lower panel) for $H_e = 0.007$. After Shukla and Eliasson (2007).

FIG. 13 The dynamics of the CPEM vector potential $A_\perp$ and the electron number density $|\psi|^2$ (upper panels) and the electrostatic potential $\phi$ (lower panel) for $H_e = 0.007$. After Shukla and Eliasson (2007).

VIII. MAGNETIZED QUANTUM PLASMAS

Magnetized quantum plasmas occur in white dwarf stars and on the surface of magnetized stars (e.g. magnetars) where degenerate electrons could be ultra-relativistic, but the ions are in a non-degenerate state. How strong magnetic fields in dense stars come about is still a mystery, although there are evidence of the strong magnetization of dense plasmas in astrophysical environments. In dense magnetized plasmas, one has to account for the Lorentz force and the Landau quantization effect (Landau and Lifshitz, 1998a), and develop the appropriate quantum Hall-magnetohydrodynamics (Q-HMHD) equations starting from the Wigner-Maxwell equations. We stress, however, that the Q-HMHD equations discussed here does not capture the particular physics of the quantized Hall resistance $R_k = \hbar/\nu e^2$ (Klitzing et al. 1980). In semiconductors with 2D electrons, the latter is associated with the quantized electron density $n_q = \nu e B_0 / \hbar c$ at high magnetic fields and low temperature, where $\nu$ is an integer, appearing in the electron current ($-en_u d$) flowing through a conductor. Here $u_u = (e/B_0^2) E \times B_0$ is the cross-field electron drift associated with the space charge electric field $E$ that results from the motion of electrons by the Lorentz force. The Ohm’s law, in turn, determines the von Klitzing resistance, which is independent of the magnetic field.

A. Landau Quantization

In a strong magnetic field $\hat{z}B_0$, where $\hat{z}$ is the unit vector along the $z$-axis in a Cartesian coordinate system, and $B_0$ the strength of the external magnetic field, the electron motion in a plane perpendicular to the magnetic field direction is quantized (Landau and Lifshitz, 1998b). The electron energy level is determined by the non-relativistic limit by the expression (Landau and Lifshitz 1998b; Tsintsadze 2010)

$$E_{l,\sigma} = \frac{p_z^2}{2m_e} + (2l + 1 + \sigma) \mu_H B_0,$$  \hspace{1cm} (88)$$

where $p_z$ is the electron momentum in the $z$-direction, $l$ the orbital angular number ($l = 0, 1, 2$), and $\sigma = \pm 1$ represents the spin orientation. For $\sigma = -1$, we have from (88)
where $\omega_{ce} = eB_0/m_e c$ is the electron gyrofrequency. Accordingly, the Fermi-Dirac electron distribution is degenerate electrons in a magnetized quantum plasma. 

\[ \langle \xi^2 \rangle = \frac{p_x^2}{2m_e} + l\hbar \omega_{ce}, \] (89)

where $\omega_{ce} = eB_0/m_e c$ is the electron gyrofrequency. Accordingly, the Fermi-Dirac electron distribution is degenerate electrons in a magnetized quantum plasma. (Tsintsadze, 2010)

\[ F_D(p_z, l) \propto \frac{1}{1 + \exp[(E_z + l\hbar \omega_{ce} - \mu_e)/k_B T_e]}, \] (90)

where $E_z = (m_e/2)v_z^2$ is the parallel (to $\hat{z}$) kinetic energy of degenerate electrons.

Assuming that $|l\omega_{ce} - \mu_e| \gg k_B T_e$, one can approximate the Fermi-Dirac distribution function by the Heaviside step function $H(\mu_e - \xi^2)$, which equal 1 for $\mu_e = E_{Fe} = k_B T_F e = (p_F^2/2m_e)^{1/2} > \xi^2$ and zero for $E_{Fe} < \xi^2$, where $p_F = m_e V_T F$. The equilibrium electron number density is (Tsintsadze, 2010)

\[ n_e = \frac{p_F^2}{2\pi^2 \hbar^3} \left[ \Gamma_B + \frac{2}{3}(1 - \Gamma_B)^{3/2} \right], \] (91)

where $\Gamma_B = \hbar \omega_{ce}/k_B T_F e$. The current carried by degenerate electrons in a magnetized quantum plasma is $-e_n u_z$, which yields the plasma resistivity $R_s = e_n e^2 c/B_0$.

### B. ESOs and EM Waves

In a magnetized quantum plasma, there are finite density perturbations associated with high-frequency electrostatic electron-Bernstein (EB) waves and elliptically polarized EM waves (EP-EM waves) that propagate across the magnetic field direction $\hat{z}$. Furthermore, the CPEM wave propagating along $\hat{z}$ are not associated with any density perturbation.

The dispersion relation for the EB waves in a Fermi-Dirac distributed plasma is in the ultra-cold limit (Eliasson and Shukla, 2008b)

\[ \omega^2 = \omega_{pe}^2 + \frac{3}{5} \frac{\omega_{pe}^2 k_L^2 V_T^2 F_e}{\omega^2 (4\omega_{pe}^2 + \omega_{ce}^2)^{1/2} \pm \omega_{ce}}, \] (93)

where $k_L$ is the perpendicular (to $\hat{z}$) component of the propagation wave vector. For $\omega \approx \omega_H$, Eq. (93) reveals that the propagating UH waves have positive (negative) group dispersion in plasmas with $\omega_{pe} > (\omega_{ce})^{1/2}$.

Furthermore, the refractive index $N_x$ for the EP-EM waves propagating along the $x$ axis (which is orthogonal to $\hat{z}$) is (Shukla, 2007)

\[ N_x = \frac{k_x^2 c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2} \frac{\omega_{pe}^2 \omega_{ce}^2 [1 + \eta(\alpha)k_x^2 \lambda_0^2]}{\omega^2 [\omega^2 - \omega_{pe}^2 + k_x^2 V_T^2 F_e (1 + k_x^2 \lambda_0^2)]}, \] (94)

where $k_x$ is the $x$ component of the propagation wave vector, $\lambda_0^2 = \hbar^2/4m_e V_T^2 F_e$, $\lambda_0 = \sqrt{\hbar/2m_e \omega_{ce}}$, $\eta(\alpha) = 2 \tanh(\alpha)$, $\alpha = \mu_B B_0/k_B T_F e$. Several comments are in order. First, we note that the electron spin-1/2 effect enhances the electron gyrofrequency by a factor $(1 + \eta k_x^2 \lambda_0^2)^{1/2}$ in the numerator of the third term in the right-hand side of (94). Second, the quantum Bohm force produces a dispersion term $\hbar k_x^2/4m_e^2$ in the denominator of the third term in (94). Third, in the limit of vanishing $\hbar$, Eq. (95) correctly reproduces the EP-EM wave dispersion relation. Furthermore, Eq. (94) reveals that the cut-off frequencies (at $k_x = 0$) in dense magnetoplasmas are

\[ \omega = \omega_\pm = \frac{1}{2} \left[ (4\omega_{pe}^2 + \omega_{ce}^2)^{1/2} \pm \omega_{ce} \right], \] (95)

which are the same as the cutoffs of the X (upper sign) and Z (lower sign) mode waves in a classical plasma (Chen, 2006). Short wavelength electromagnetic propagation in magnetized quantum plasma, including quantum electrodynamic effects, has also been considered by Lundin et al., 2007.

The vector representation of spinning quantum particles in the quantum theory was first introduced by...
Takabayasi (1955) who developed the QHD involving the evolution of the quantum particle spin. The idea of Takabayasi has been further elaborated by Brodin et al. (2010) in the context of the spin contribution to the ponderomotive force of the magnetic field-aligned CPEM waves in a quantum magnetoplasma. In fact, by using the non-relativistic electron momentum equation (Brodin et al. 2010)

\[
\frac{m_e}{2} \frac{\partial}{\partial t} \mathbf{u}_e + \mathbf{u}_e \cdot \nabla (\mathbf{u}_e) = -e \left( \mathbf{E} + \frac{1}{c} \mathbf{u}_e \times \mathbf{B} \right) - \frac{g}{\hbar} \mu_B \nabla (\mathbf{B} \cdot \mathbf{s}),
\]

and the spin evolution equation

\[
\frac{\partial}{\partial t} \mathbf{s} = \frac{g \mu_B}{\hbar} (\mathbf{B} \cdot \mathbf{s}),
\]

(96)

(97)

together with Ampère’s law and suitable Maxwell’s equation (incorporating the electron magnetization current, \(J_M = -\frac{(4\pi e)}{c} (g\mu_B / \hbar) \nabla \times (n_e \times \mathbf{s})\), due to the electron 1/2—spin effect), where \(\mathbf{s}\) is the spin angular momentum, with its absolute value \(|\mathbf{s}| = s_0 = \hbar/2\). The quantity \(g = 2.0023192\) is the electron Gaunt factor (sometimes called the \(g\) factor or spectroscopic splitting factor). The value \(g = 2\) is predicted from Dirac’s relativistic theory of the electron, while the correction to this value comes from the quantum electrodynamics (Bransden and Joachain 2000, Kittel 1996). Brodin et al. (2010) derived the spin ponderomotive force \(\mathbf{F}_s\) for the CPEM wave, where

\[
\mathbf{F}_s = \frac{g^2 \mu_B^2}{m_e^2 c^2} \frac{s_0}{(\omega \pm \omega_g)} \left[ \frac{\partial}{\partial \omega} - \frac{k}{(\omega \pm \omega_g)} \frac{\partial}{\partial t} \right] |B_w|^2.
\]

(98)

Here \(\omega_g = g\mu_B B_0 / \hbar\) the spin-precession frequency, and \(B_w\) is the CPEM wave magnetic field. The spin ponderomotive force comes from the averaging of the third term in (99) over the CPEM wave period \(2\pi/\omega\). The CPEM wave frequency \(\omega\) is determined from the dispersion relation

\[
\left[ 1 \mp \frac{\omega_{\mu}}{(\omega \pm \omega_g)} \right] N_z^2 = 1 - \frac{\omega_{pe}^2}{\omega (\omega \pm \omega_{ce})},
\]

(99)

where \(N_z = k_z c / \omega\), \(k_z\) is the component of the wave vector \(\mathbf{k}\) along the \(z\) axis, \(\omega_{\mu} = g^2 s_0 / 4 m_e \lambda_{e0}^2\), \(\lambda_e = c / \omega_{pe}\), and the \(+(-)\) represents the left- (right-) hand circular polarization. The \(\omega_{\mu}\)-term in (100) is associated with the electron spin evolution. It changes the dispersion properties of the magnetic field-aligned EM electron-cyclotron waves in a quantum magnetoplasma. Furthermore, the spin-ponderomotive force induces a strong spin-polarization of a quantum magnetoplasma.

It should be noted that there is also a standard non-stationary ponderomotive force \((\mathbf{2F}_c)\) (Karpman and Washimi 1977) of the CPEM waves arising from the averaging of the nonlinear Lorentz force term \(-e/(m_e c) \mathbf{z} \cdot (\mathbf{u}_e \times \mathbf{B}_w)\) over the CPEM wave period \(2\pi/\omega\), where

\[
F_c = -\frac{e^2}{2 m_e^2 \omega (\omega \pm \omega_{ce})} \left\{ \frac{\partial}{\partial \omega} + \frac{k_z \omega_{ce}}{\omega (\omega \pm \omega_{ce})} \frac{\partial}{\partial t} \right\} |\mathbf{E}_w|^2,
\]

(100)

where \(\mathbf{E}_w = (\omega / k_z c) \mathbf{B}_w\) is the CPEM wave electric field.

C. Q-HMHD Equations

To a first approximation, the dynamics of low phase speed (in comparison with the speed of light in vacuum) electromagnetic waves in dense magnetoplasmas is modeled by the Q-HMHD equations. The latter include the inertialess electron momentum equation

\[
0 = -e n_e \left( \mathbf{E} + \frac{1}{c} \mathbf{u}_e \times \mathbf{B} \right) - \nabla \mathbf{P}_e,
\]

(101)

where the quantum Bohm and quantum spin forces are supposed to be unimportant on the characteristic scale-length of present interest. The degenerate electrons are coupled with the non-degenerate ions through the EM forces. The ion dynamics is governed by the ion continuity equation (43) and the momentum equation

\[
m_i n_i \frac{d \mathbf{u}_i}{dt} = n_i e \left( \mathbf{E} + \frac{1}{c} \mathbf{u}_i \times \mathbf{B} \right),
\]

(102)

where \(d/dt = (\partial/\partial t) + \mathbf{u}_i \cdot \nabla\). For the sake of simplicity, we have here assumed that \(\tau_m \partial / \partial t \ll 1\) and \(\partial \mathbf{u}_i / \partial t \gg (\eta / \rho_i) \nabla \cdot \nabla \mathbf{u}_i + \rho_{i}^{-1} (\xi + \eta / 3) \nabla (\nabla \cdot \mathbf{u}_i)\). The EM fields are given by Ampère’s law

\[
\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E},
\]

(103)

and Maxwell’s equation

\[
\nabla \times \mathbf{B} = \frac{4\pi e}{c} (n_e \mathbf{u}_e - n_i \mathbf{u}_i) + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}.
\]

(104)

By using (102), we can eliminate the electric field \(\mathbf{E}\) from (103), obtaining for a quasi-neutral \((n_e = n_i = n)\) quantum magnetoplasma

\[
m_i n \frac{d \mathbf{u}_i}{dt} = -\nabla \mathbf{P}_C - \frac{1}{8\pi} \nabla \mathbf{B}^2 + \frac{(\mathbf{B} \cdot \nabla) \mathbf{B}}{4\pi},
\]

(105)

where we have used (104) without the displacement current (the last term on the right-hand side) for the low-phase speed (in comparison with \(c\)) EM wave phenomena. By using the electric field from (102), we can write (103) as
\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{m_e c d u}{e dt},
\]

Equations (43), (105) and (106) are the desired Q-HMHD equations for studying the linear and nonlinear dispersive EM waves, as well as new aspects of 3D quantum fluid turbulence in a quantum magnetoplasma with degenerate electrons having Chandrasekhar’s pressure law. However, when the Landau quantization effect in a very strong magnetic field is accounted for, one can replace \( P_C \) by the appropriate pressure law \( \text{(Eliezer et al. 2005)} \)

\[
P_L = \frac{4e B_0 (2m_e)^{1/2} E^{3/2}_F}{3(2\pi^2\hbar^2 c)} \times \left[ 1 + 2 \sum_{l=1}^{l_m} \left( 1 - \frac{l \hbar \omega_{ce}}{k_B T_{Fe}} \right)^{3/2} \right],
\]

where the value of \( l_m \) is fixed by the largest integer that satisfies \( k_B T_{Fe} - l \hbar \omega_{ce} \leq 0 \).

**IX. SUMMARY AND OUTLOOK**

In this Colloquium paper, we have described the essential physics of quantum plasmas with degenerate electron fluids. We have reviewed the properties of quantum plasmas and quantum models that describe the salient features of linear and nonlinear ES and EM waves. Specifically, the focus of the present colloquium article has been on developing the model nonlinear equations that depict new features of nonlinear waves and quantum electron fluid turbulence at nanoscales. Numerical simulations of the nonlinear Schrödinger (NLS)-Poisson equations reveal quasi-stationary, localized structures in the form of one-dimensional electron density holes (dark solitons) and 2D quantum electron vortices. These localized quantum structures, which are associated with a local depletion of the electron density and a positive electrostatic potential, arise due to a balance between the nonlinear and dispersion effects involved in the dynamics of nonlinearly interacting EPOs. In 2D, there exist a class of quantum electron vortices of different excited states (charge states). Furthermore, numerical simulations also depict that the time-dependent NLS-Poisson equations exhibit stability of a dark soliton in one-space dimension. In 2D the dark solitons of the first excited state are stable and the preferred nonlinear state is in the form of quantum vortex pairs of different polarities. The one-dimensional dark soliton and singly charged 2D quantum vortices are thus long-lived nonlinear structures at nanoscales. Also presented are theoretical and computer simulation studies of nonlinearly coupled intense EM waves and EPOs in an unmagnetized quantum plasma. We have reported new classes of stimulated scattering instabilities of EM waves and trapping of intense EM waves in a quantum electron density hole.

It should be noted that inclusion of non-degenerate ion dynamics gives rise to new features to linear and nonlinear IPOS \( \text{(Eliasson and Shukla 2008a; Haas et al. 2003)} \). Furthermore, nonlinear equations governing the coupling between the dispersive Langmuir and ion-acoustic waves, which are known as the quantum Zakharov equations \( \text{(Garcia et al. 2005; Haas and Shukla 2009; Misra et al. 2005; Simpson et al. 2009)} \), admit periodic, quasi-periodic, chaotic and hyper-chaotic states \( \text{(Misra et al. 2008)} \), in addition to arresting the Langmuir wave collapse \( \text{(Haas and Shukla 2009; Simpson et al. 2009)} \) due to quantum dispersion effects. There may also emerge new aspects of nonlinear EPOs and IPOs when the particle trapping \( \text{(Jovanovic and Fedele 2007)} \) in the strong wave potential is included. Here one has to obtain nonlinear solutions of non-stationary Wigner-Poisson equations, which might reveal a modified (by the electrostatic wave potential) Fermi-Dirac electron distribution function. Furthermore, there is a scope for studying the collective nonlinear response of correlated Coulomb electron systems at finite temperatures by means of kinetic theory concepts \( \text{(Domp s et al. 1997)} \) to incorporate collisions and Green’s function methods originally developed by Baym and Kadanoff (1961). We note that the Baym-Kadanoff approach has been used by Kwong and Bonitz (2000) to investigate the dielectric properties (viz. inverse dielectric function and dynamic structure factor) of linear EPOs in a correlated electron gas. Furthermore, the ion-ion dynamic structure factor, which contains a wealth of information about ions including structure and low-frequency collective modes in a dense quantum plasma, has been studied by Murillo (2010).

The field of the nonlinear quantum plasma physics is vibrant, and its potential applications rest on our complete understanding of numerous collective processes in compact astrophysical objects, as well as in the next-generation of intense laser-solid density plasma experiments and in the plasma assisted nanotechnology \( \text{(e.g. quantum free electron laser devices, quantum-diodes, metallic nanostructures, nanowires, nanotubes, etc.)} \). However, nonlinear quantum models presented in this Colloquium paper have to be further improved and generalized by including the effects of the electron exchange interactions, strong electron-electron correlations, equilibrium inhomogeneities of the magnetic field and the plasma density, as well as fully relativistic and Landau quantization effects in a nonuniform quantum magnetoplasma. We have also to understand the features of quantum oscillations of electrons and possible formation of bound states of electrons in the presence of an external magnetic field. For this purpose, we have to calculate the interaction potential among highly correlated electrons and use molecular dynamic simulations to demonstrate attraction among electrons due to collective wave-quantum particle interactions that give rise to Cooper’s pairing of degenerate electrons. Cooper’s pair-
ing of electrons could possibly provide a scenario of superconducting behavior of a quantum plasma. Furthermore, 2D system composed of electron clusters at finite temperature exhibits Wigner Coulomb crystallization [Egger et al. 1999] [Filinov et al. 2001]. The latter has been investigated by means of Monte Carlo simulations based on a quantum Hamiltonian with parabolic confining and Coulomb interaction potentials. In a nonuniform quantum magnetoplasma, we have ES drift waves [Ali et al. 2007] [Saleem et al. 2008] [Shokri and Ruhadze 1999] which can drastically affect the cross-field electron transport. For applications to plasma assisted nano-technology devices (e.g. nonlinear electrostatic and electromagnetic surface waves in metallic nanostructure-devices, photonic band gap and x-ray optical systems, quantum X-ray free-electron laser systems), one must also study nonlinear collective processes by including both the electron spin-$1/2$ and quantum electron tunneling effects on an equal footing. Finally, the localization of coupled ES and EM waves due to nonlinear quantum effects in a nonuniform quantum magnetoplasma with an arbitrary electron pressure degeneracy should provide clues to the origin of very intense X-rays (Coe et al. 1978) and gamma rays (Hurley et al. 2005) from both astrophysical and laboratory plasmas.

Acknowledgments

This research was supported by the Deutsche Forschungsgemeinschaft through the project SH21/3-1 of the Research Unit 1048, and by the Swedish Research Council (VR).

References

Ali, S., N. Shukla, and P. K. Shukla, 2007, Europhys. J. Lett. 78, 45001.

Adolphat, R. M., A. G. Petukhov, and I. Zutic, 2008, Phys. Rev. Lett. 101, 207202.

Anderson, D., B. Hall, M. Lisak, and M. Marklund, 2002, Phys. Rev. E 65, 046417.

Andreev, A. V., 2000, JETP Lett. 72, 238.

Ang, L. K., T. J. T. Kwan, and Y. Y. Lau, 2003, Phys. Rev. Lett. 91, 208303.

Ang, L. K., and P. Zhang, 2007, Phys. Rev. Lett. 98, 164802.

Ancona, M. G., and G. J. Iafrate, 1989, Phys. Rev. B 39, 9536.

Atwater, H. A., 2007, Sci. Am. 296, 56.

Azechi, H., et al., 2006, Plasma Phys. Control. Fusion 48, B267.

Balescu, R., and W. Zhang, 1988, J. Plasma Phys. 40, 215.

Barnes, W., A. Dereux, and T. Ebbesen, 2003, Nature (London) 424, 824.

Baym, G., and L. P. Kadanoff, 1961, Phys. Rev. 124, 287.

Beker, K. H., K.H. Schoenbach, and J.G. Eden, 2006, J. Phys. D: Appl. Phys. 39, R55.

Benvenuto O. G., and M. A. De Vito, 2005, Mon. Not. R. Astron. 362, 891.

Berestetskii, B., E. M. Lifshitz, and L. P. Pitaevskii, 1999, Quantum Electrodynamics (Butterworth-Heinemann, Oxford), p. 123.

Bohm, D., 1952, Phys. Rev. 85, 166.

Bohm, D., 1953, Phys. Rev. 92, 626.

Bohm, D., and D. Pines, 1953, Phys. Rev. 92, 609.

Bonitz, M., 1998, Quantum Kinetic Theory (Teubner, Stuttgart).

Bransden, B. H., and C. J. Joachain, 2000, Quantum Mechanics (2nd Edition) (Pearson Education Limited, Essex, England).

Brittin, W. E., and W. R. Chappell, 1962, Rev. Mod. Phys. 34, 620.

Brodin, G., and M. Marklund, 2007a, New J. Phys. 9, 277.

Brodin, G., and M. Marklund, 2007b, Phys. Plasmas 14, 11207.

Brodin, G., and M. Marklund, 2007c, Phys. Rev. E 76, 055403(R).

Brodin, G., M. Marklund and G. Manfredi, 2008a, Phys. Rev. Lett. 100, 175001.

Brodin, G., M. Marklund, J. Zamanian, A. Ericsson, and P. L. Mana, 2008b, Phys. Rev. Lett. 101, 245002.

Brodin, G., A. P. Misra, and M. Marklund, 2010, Phys. Rev. Lett. 105, 105004.

Burt, P., and D. Wahlquist, 1962, Phys. Rev. 125, 1785.

Carruthers, P., and F. Zachariasen, 1983, Rev. Mod. Phys. 55, 245.

Chabrier, G., et al., 2002, J. Phys.: Condens. Matter 14, 9133.

Chabrier, G., D. Saumon, and A. Y. Potekhin, 2006, J. Phys. A: Math. Gen. 39, 4411.

Chabrier, G., 2009, Plasma Phys. Control. Fusion 51, 124014.

Chandrasekhar, S., 1931a, Astrophys. J. 74, 81.

Chandrasekhar, S., 1931b, Phil. Mag. 74, 85.

Chandrasekhar, S., 1935, Mon. Not. R. Astron. Soc. 170, 405.

Chandrasekhar, S., 1939, An Introduction to the Study of Stellar Structure (University of Chicago Press, Chicago), p. 360.

Chang, D. E., A. S. Sørensen, P. R. Hemmer, and M. D. Lukin, 2006, Phys. Rev. Lett. 97, 053002.

Chen, F. F., 2006, Introduction to Plasma Physics and Controlled Fusion. Volume 1, Plasma Physics. (Springer, New York).

Coe, M. J., A. R. Engel, and J. J. Quenby, 1978, Nature (London) 272, 37.

Cowley, S. C., R. M. Kulsrud, and E. Valeo, 1986, Phys. Fluids 29, 430.

Crouseilles, N., P. A. Hervieux, and G. Manfredi, 2008, Phys. Rev. B 78, 155412.

Dharma-wardana, C., and F. Perrot, 1995, in Density Functional Theory, Eds. E. K. U. Gross and R. M. Dreizler (Plenum Press, New York).

Dirac, P. A. M., 1981, Principles of Quantum Mechanics (Oxford University Press, Oxford).

Domps, A., P.-G. Reinhard, and E. Suraud, 1997, Ann. Phys. 26, 171.

Dreike, R., P. R., 2010, Phys. Today 63, 28.

Egger, R., W. Häusler, C. H. Mak, and H. Grabert, 1999, Phys. Rev. Lett. 82, 3320.

Eliasson, B., and P. K. Shukla, 2010, Relativistic Laser-Plasma Interactions in the Quantum Regime, arXiv:1011.5801v1 [physics.plasm-ph].

Eliasson, B., and P. K. Shukla, 2008a, J. Plasma Phys. 64,
Haas, F., 2005, Phys. Plasmas 12, 062102.
Haas, F., 2005, Phys. Plasmas 12, 062117.
Haas, F., 2007, Europhys. Lett. 44, 45004.
Haas, F., L. G. Garcia, J. Goedert, and G. Manfredi, 2003, Phys. Plasmas 10, 3858.
Haas, F., and P. K. Shukla, 2009, Phys. Rev. E 79, 066402.
Haas, F., M. Marklund, G. Brodin, and J. Zamanian, 2010a, Phys. Lett. A 374, 481.
Haas, F., J. Zamanian, M. Marklund, and G. Brodin, New. J. Phys. 12, 073027 (2010b).
Hakim, R., and J. Heyvaerts, 1978, Phys. Rev. A 18, 1250.
Harding, A. K., and D. Lai, 2006, Rep. Prog. Phys. 69, 2631.
Haug, H., and S. W. Koch, 2004, Quantum Theory of Optical and Electronic Properties of Semiconductors (World Scientific, Singapore).
Haug, H., and A.-P. Jauho, 2007, Quantum Kinetics in Transport and Optics of Semiconductors, Springer Series in Solid-State Sciences, Vol. 123 (Springer, Berlin).
Hillery, M., et al., 1984, Phys. Rep. 106, 121.
Hohenberg, P., and W. Kohn, 1964, Phys. Rev. 136, B864.
Holland, P. R., 1993, The Quantum Theory of Motion (Cambridge University Press, Cambridge).
Horn, H. M., 1991, Science 252, 381.
Hu, S. X., and C. H. Keitel, 1999, Phys. Rev. Lett. 83, 4709.
Hurley, K., S. E. Boggs, D. M. Smith et al., 2005, Nature (London) 434.
Iafrate, G. J., H. L. Grubin, and D. K. Ferry, 1981, J. Phys. (Paris) (Colloq.) 42, C10-307.
Ichimaru, S., 1982, Rev. Mod. Phys. 54, 1017.
Ichimaru, S., and S. Tanaka, 1986, Phys. Rev. Lett. 56, 2815.
Iroshnikov, P., 1963, Sov. Astron. 7, 566.
Jovanovic, D., and R. Fedele, 2007, Phys. Lett. A 364, 304.
Jüngel, A., D. Matthes, and J. F. Milisic, 2006, SIAM 67, 46.
Hohenberg, P., and W. Kohn, 1964, Phys. Rev. 136, 684.
Kaw, P. K., and A. Sen, 1998, Phys. Plasmas 5, 3552.
Karpman, V. I., and H. Washimi, 1977, J. Plasma Phys. 18, 173.
Kelly, D. C., 1964, Phys. Rev. 134, A641.
Kittel, C., 1996 Introduction to Solid State Physics, Seventh Edition (John Wiley & Sons, Inc., New York)
Kleinert, H., 1986, Phys. Lett. B 181, 324.
Klimontovich, Y. L., and V. P. Silin, 1952a, Doklady Akad. Nauk. SSSR 82, 361.
Klimontovich, Y. L., and V. P. Silin, 1952b, Zh. Eksp. Teor. Fiz. 23, 151.
Klimontovich, Y. L., and V. P. Silin, 1961, in Plasma Physics, Ed. J. E. Drummond (McGraw Hill, New York), pp. 35-87.
Klimontovich, Y. L., and V. P. Silin, 1961, in Plasma Physics, Ed. J. E. Drummond (McGraw Hill, New York), pp. 35-87.

Ghoshal, A., and Y. K. Ho, 2009a, Phys. Rev. A 79, 062514.
Ghoshal, A., and Y. K. Ho, 2009b, J. Phys. B: At. Mol. Opt. Phys. 43, 175006.
Glenzer, S. H., and R. Redmer, 2009, Rev. Mod. Phys 81, 1625.
Glenzer, S. H., et al., 2007, Phys. Rev. Lett. 98, 065002.
Gottlieb, D., and S. A. Orszag, 1977, Numerical Analysis of Spectral Methods (SIAM, Philadelphia).
Gregori, G., and D. O. Gericke, 2009, Phys. Plasmas 16, 056306.
de Groot, S. R., and L. G. Suttorp, 1972, Foundations of Electrodynamics (North-Holland, Amsterdam).
Guillot, T., 1999, Science 286, 72.
Gursky, H., 1976, in Frontiers of Astrophysics, Ed. E. H. Avrett (Harvard University Press, Cambridge, Massachusetts), Chap. 5, pp. 152,153.
Haas, F., G. Manfredi, and M. R. Feix, 2000, Phys. Rev. E 62, 2763.
Haas, F., 2005, Phys. Plasmas 12, 062117.
Haas, F., 2007, Europhys. Lett. 44, 45004.
Haas, F., L. G. Garcia, J. Goedert, and G. Manfredi, 2003, Phys. Plasmas 10, 3858.
Haas, F., and P. K. Shukla, 2009, Phys. Rev. E 79, 066402.
Haas, F., M. Marklund, G. Brodin, and J. Zamanian, 2010a, Phys. Lett. A 374, 481.
Haas, F., J. Zamanian, M. Marklund, and G. Brodin, New. J. Phys. 12, 073027 (2010b).
Hakim, R., and J. Heyvaerts, 1978, Phys. Rev. A 18, 1250.
Harding, A. K., and D. Lai, 2006, Rep. Prog. Phys. 69, 2631.
Haug, H., and S. W. Koch, 2004, Quantum Theory of Optical and Electronic Properties of Semiconductors (World Scientific, Singapore).
Haug, H., and A.-P. Jauho, 2007, Quantum Kinetics in Transport and Optics of Semiconductors, Springer Series in Solid-State Sciences, Vol. 123 (Springer, Berlin).
Hillery, M., et al., 1984, Phys. Rep. 106, 121.
Hohenberg, P., and W. Kohn, 1964, Phys. Rev. 136, B864.
Holland, P. R., 1993, The Quantum Theory of Motion (Cambridge University Press, Cambridge).
Horn, H. M., 1991, Science 252, 381.
Hu, S. X., and C. H. Keitel, 1999, Phys. Rev. Lett. 83, 4709.
Manfredi, G., and P.-A. Hervieux, 2007, Appl. Phys. Lett.
Manfredi, G., and F. Haas, 2001, Phys. Rev. B 64, 075316.
Manfredi, G., and P.-A. Hervieux, 2007, Appl. Phys. Lett. 91, 061108.
Marburger, J. H., and R. F. Tooper, 1975, Phys. Rev. Lett. 35, 1001.
Marklund, M., and P. K. Shukla, 2006, Rev. Mod. Phys. 78, 591.
Marklund, M., and G. Brodin, 2007, Phys. Rev. Lett. 98, 025001.
Marklund, M., G. Brodin, L. Stenflo, and C. S. Liu, 2008, Europhys. Lett. 84, 17006.
Markovich, P. A., et al., 1990, Semiconductor Equations (Springer, Berlin).
McKinstrie, C. J., and R. Bingham, 1992, Phys. Fluids B 4, 2626.
Massoud, W., B. Eliasson, and P. K. Shukla, Phys. Rev. E 81, 066401 (2010).
Mayor, F. S., A. Askar, and H. A. Rabitz, 1999, J. Chem. Phys. 111, 2423.
Melrose, D. B., 2008, Quantum Plasmadynamics: Unmagnetized Plasmas (Springer, New York). Lecture Notes Phys. 735.
Melrose, D. B., and A. Mushtaq, 2009, Phys. Plasmas 16, 094508.
Mendonça, J. T., 2001, Theory of Photon Acceleration (Institute of Physics, Bristol).
Misra, A. P., 2007, Phys. Plasmas 14, 064501.
Misra, A. P., D. Ghosh, and A. R. Chowdhury, 2008, Phys. Lett. A 372, 1469.
Misra, A. P., 2009, Phys. Plasmas 16, 033702.
Misra, A. P., and S. Samanta, 2010, Phys. Rev. E 82, 037401.
Mithen, J. P., J. Daligault, and G. Gregori, 2011, Phys. Rev. E 83, 015401(R).
Moyal, J. E., 1949, Proc. Cambridge Philos. Soc. 45, 99.
Murillo, M. S., 2010, Phys. Rev. E 81, 036403 (2010).
Murtaza, G. M., and P. K. Shukla, 1984, J. Plasma Phys. 31, 423.
Mushtaq, A., and D. B. Melrose, 2009, Phys. Plasmas 16, 102110.
National Research Council, 1995, Plasma Science: From Fundamental Research to Technological Applications (National Academy Press, Washington DC).
Neumayer, P., C. Fortmann, T. Döpner, et al., 2010, Phys. Rev. Lett. 105, 075003.
Norreys, P. A., F. N. Beg, Y. Sentoku, et al., 2009, Phys. Plasmas 16, 041002.
Oberman, C., and A. Ron, 1963, Phys. Rev. 130, 1291.
Opher, M., et al., 2001, Phys. Plasmas 8, 2454.
Ozbay, E., 2006, Science 311, 189.
Pines, D., 1961, J. Nucl. Energy: Part C: Plasma Phys. 2, 5.
Pines, D., 1983, Elementary Excitations in Solids (Benjamin, Massachusetts).
Pines, D., and P. Nozieres, 1989, The Theory of Quantum Liquids (Benjamin, New York).
Redmer, R., and G. Röpke, 2010, Contrib. Plasma Phys. 50, 970.
Runge, E., and E. K. U. Gross, 1984, Phys. Rev. Lett. 52, 997.
Salamin Y. A., et al., 2006, Phys. Rep. 427, 41.
Saleem, H., A. Ahmad, and S. A. Khan, 2008, Phys. Plasmas 15, 014503.
Scott R. K., 2007, Phys. Rev. E 75 046301.
Serbeto, A., J. T. Mendonça, K. H. Tsui, and R. Bonifacio, 2008, Phys. Plasmas 15, 013110.
Serbeto, A., L. F. Monteiro, K. H. Tsui, and J. T. Mendonça, 2009, Plasma Phys. Control. Fusion 51, 124024.
Serimaa, O. T., J. Javanainen, and S. Varro, 1986, Phys. Rev. A 33, 2913.
Shaikh, D., and P. K. Shukla, 2007, Phys. Rev. Lett. 99, 125002.
Shaikh, D., and P. K. Shukla, 2008, New J. Phys. 10, 083007.
Shapiro, S. L., and S. L. Teukolsky, 1983, Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects (John Wiley & Sons, New York).
Sharma, R. P., and P. K. Shukla, 1983, Phys. Fluids 26, 83.
Shukla, P. K., 2006, Phys. Lett. A 352, 242.
Shukla, P. K., 2007, Phys. Lett. A 369, 312.
Shukla, P. K., 2009, Nature Phys. 5, 92.
Shukla, P. K., 2010, Phys. Lett A 374, 2897.
Shukla, P. K., and B. Eliasson, 2006, Phys. Rev. Lett. 96, 245001.
Shukla, P. K., and B. Eliasson, 2007, Phys. Rev. Lett. 99, 096401.
Shukla, P. K., and B. Eliasson, 2008a Phys. Rev. Lett. A 372, 2897.
Shukla, P. K., and B. Eliasson, 2008b, Phys. Rev. Lett. 100, 036401.
Shukla, P. K., and B. Eliasson, 2010, Phys. Usp. 53, 51.
Shukla, P. K., and L. Stenflo, 2006, Phys. Plasmas 13, 044505.
Shukla, P. K., and M. Y. Yu, 1984, Plasma Phys. Controll. Fusion 26, 841.
Shukla, P. K., M. Y. Yu, H. U. Rahman, and K. H. Spatschek, 1981, Phys. Rev. A 23, 321.
Shukla, P. K., N. N. Rao, M. Y. Yu, and N. L. Tsintsadze, 1986, Phys. Rep. 138, 1.
Shokri, B., and A. A. Rukhadze, 1999, Phys. Plasmas 6, 3450.
Shpatakovskaya, G., 2006, J. Exp. Teor. Phys. 102, 466.
Silin, V. P., and A. A. Rukhadze, 1961, Electromagnetic Properties of Plasmas and Plasma-like Media (Gosatomizdat, Moscow).
Simpson, G., C. Sulem, and P. L. Sulem, 2009, Phys. Rev. E 80, 056405.
Son, S., and N. J. Fisch, 2005, Phys. Rev. Lett. 95, 225002.
Steinberg, M., 2000, Thermodynamics and Kinetics of a Magnetized Quantum Plasma (Logos, Berlin).
Stenflo, L., and P. K. Shukla, 2009, in From Leonardo to ITER: Nonlinear and Coherence Aspects, Ed. J. Weiland. (AIP Conf. Proc., Vol 1177, New York), pp. 4-9.
Stockman, M. I., 2011, Phys. Today 64, 39.
Stratonovich, R. L., 1956, Sov. Phys. D 1, 414.
Takabayasi, T., 1953, Prog. Theor. Phys. 9, 187.
Takabayasi, T., 1955, Prog. Theor. Phys. 14, 283.
Thiele, R., T. Bornath, C. Fortmann, A. Höll, R. Redmer, H. Reinholz, G. Röpke, A. Wierling, S. H. Glenzer, and G. Gregori, 2008, Phys. Rev. E 78, 026411.
Tsintsadze, L. N., 2010, in New Frontiers in Advanced Plasma Physics, Eds. B. Eliasson and P. K. Shukla, (AIP Conf. Proc. #1306, AIP, New York), pp. 89-102.
Tsintsadze, N. L., and L. N. Tsintsadze, 2009, Europhys. Lett. 88, 35001.
Watanabe, H., 1956, J. Phys. Soc. Jpn. 11, 112.
Wigner, E., 1932, Phys. Rev. 40, 749.
Wilhelm, H. E., 1971, Z. Physik 241, 1.
Wyatt, R. E., 2005, Quantum Dynamics with Trajectories:
Introduction to Quantum Hydrodynamics (Springer Science, New York).
Xia, S. X., W. C. Hua, and G. Feng, 2010, Chin. Phys. Lett. 27, 025204.
Yu, M. Y., et al., 1974, Z. Naturforsch 29a, 1736.
Zamanian, J., M. Marklund, and G. Brodin, 2010, New J. Phys. 12, 043019.
Zhang, W., and R. Balescu, 1988, J. Plasma Phys. 40, 199.
Zhu, J., and P. Ji, 2010, Phys. Rev. E 81, 036406.