A new search direction for full-Newton step infeasible interior-point method in linear optimization

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Abstract
In this paper, we study an infeasible interior-point method for linear optimization with full-Newton step. The introduced method uses an algebraic equivalent transformation on the centering equation of the system which defines the central path. We prove that the method finds an ε-optimal solution of the underlying problem in polynomial time.

keywords: Linear optimization, infeasible interior-point methods, new search directions, polynomial complexity.

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1 Introduction
Interior-point methods (IPMs) for linear optimization (LO) began when Karmarkar [6] published his exceptional paper in 1984. After that, several variants of this algorithm were presented. In the meantime, we can talk about feasible and infeasible IPMs. In feasible IPMs we presume that a strictly feasible point is at hand which the algorithm can be immediately beginning. Usually find such a starting point is not simple. In that case an infeasible IPM (IIPM) should be used. These methods begin from an arbitrary positive point and try to reach both feasibility and optimality. IIPMs were first introduced by Lustig [18] and Tanabe [22]. The first feasible IPM with full-Newton step for LO was presented by Roos et al. [21]. Determining the search directions plays a very important role in IPMs. In 2003, Darvay [2] utilizes the AET technique on the centering equation of the system defining the central path for LO. He uses the square root function in the AET strategy and then applies the Newton method for obtain the search directions. This method is extended in [1, 23, 24, 25], respectively, to convex quadratic optimization (CQO), second-order cone optimization
(SOCO), symmetric optimization (SO) and the Cartesian $P_*(\kappa)$ linear complementarity problem (LCP). Kheirfam and Haghighi [16] have proposed an IPM for $P_*(\kappa)$-LCP which uses the function $\psi(t) = \sqrt{t}/(1 + \sqrt{t})$ in the AET technique. An infeasible version of the method proposed in [21] has presented by Roos in [19] which needs a feasibility step and three centering steps in each main iteration. Some generalizations and versions of the method can be seen in [15, 8, 9, 17, 27, 10]. The author is improved this algorithm so that the algorithm performs only one feasibility step in each iteration and does not need centering steps [20]. Kheirfam [11, 12, 13, 14] extended the algorithm proposed in [20] to HLCP, the Cartesian $P_*(\kappa)$-LCP, the convex quadratic symmetric cone optimization (CQSCO) and SO. By considering the AET technique based on the function $\psi(t) = t - \sqrt{t}$, Darvay et al. [3] have introduced a full-Newton step IPM for LO. Kheirfam [7] has presented an infeasible version of this algorithm for SDLCP. Darvay et al. [5] published a corrector-predictor IPM (CP-IPM) for LO using the function $\psi(t) = t - \sqrt{t}$ for AET. Darvay and Takács [4] proposed an IPM for LO based on a new type of AET on the centering equation of the central path.

Motivated by the aforementioned works, in this paper we aim to present a full-Newton step IIPM for LO using the AET $\psi(t) = \psi(\sqrt{t})$ for the centering equation of the central path. The method uses the function $\psi(t) = t^2$ in order to determine the new search directions and performs only one feasibility step in a main iteration. In fact, our method is an infeasible version of the method proposed in [3]. We prove that the proposed algorithm enjoys the best-known iteration complexity for IIPMs.

The paper is organized in the following way. In the next section, we remember the problem pair (P) and (D). We state the perturbed problems corresponding to (P) and (D) and then provided the central path. In Sect. 3, the new search directions based on the new type of AET using the function $\psi(t) = t^2$ is discussed, and finally the algorithm is presented. Section 4 consists of the complexity analysis of the introduced IIPM with the new search directions. In Section 5, some concluding remarks are followed.

## 2 Preliminaries

Let us consider the LO problem in the standard form

$$(P) \quad \min \ \{c^T x : Ax = b, \ x \geq 0\},$$

where $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = m, b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. The dual of this problem can be written in the following standard form:

$$(D) \quad \max \ \{b^T y : A^T y + s = c, \ s \geq 0\}.$$ 

In accordance with the routine of IIPMs, we consider the starting point $(x^0, y^0, s^0) = \xi(e, 0, e)$ such that $\|(x^*; s^*)\|_{\infty} \leq \xi$ for some primal-dual optimal solution $(x^*, y^*, s^*)$, where $e$ is the all-one vector and $\xi$ is a positive scalar. It should be noted that for the optimal solution $(x^*, y^*, s^*)$ the inequality $\|(x^*; s^*)\|_{\infty} \leq \xi$ is true if and only if

$$0 \leq x^* \leq \xi e, \ 0 \leq s^* \leq \xi e. \quad (1)$$
For an IIPM, a triple \((x, y, s)\) is called an \(\varepsilon\)-solution of \((P)\) and \((D)\) if
\[
\max \left\{ x^T s, \|b - Ax\|, \|c - A^T y - s\| \right\} \leq \varepsilon,
\]
where \(\varepsilon\) is a accuracy parameter. Following [19], for any \(0 < \nu \leq 1\) we consider the perturbed problem pair \((P_\nu)\) and \((D_\nu)\) as follows:
\[
(P_\nu) \quad \min \left\{ (c - \nu r_0^c)^T x : b - Ax = \nu r_0^b, \ x \geq 0 \right\},
\]
\[
(D_\nu) \quad \max \left\{ (b - \nu r_0^b)^T y : c - A^T y - s = \nu r_0^c, \ s \geq 0 \right\},
\]
where \(r_0^b := b - A\xi e\) and \(r_0^c := c - \xi e\). It is simply seen that \((x^0, y^0, s^0) = \xi(e, 0, e)\) is a feasible solution of the problem pair \((P_\nu)\) and \((D_\nu)\) if \(\nu = 1\). We conclude that if \(\nu = 1\), then \((P_\nu)\) and \((D_\nu)\) satisfy the interior point condition (IPC). We recall the following lemma.

**Lemma 1.** (Theorem 5.13 in [26]) The original problems, \((P)\) and \((D)\) are feasible if and only if for each \(\nu\) satisfying \(0 < \nu \leq 1\) the perturbed problems \((P_\nu)\) and \((D_\nu)\) satisfy the IPC.

In the view of Lemma 1 we assume that the original problem pair \((P)\) and \((D)\) is feasible and \(\nu \in (0, 1]\), the central path of the perturbed pair \((P_\nu)\) and \((D_\nu)\) exists;
\[
\begin{align*}
 b - Ax &= \nu r_0^b, \quad x \geq 0, \\
 c - A^T y - s &= \nu r_0^c, \quad s \geq 0, \\
 xs &= \mu e,
\end{align*}
\]
has a unique solution \((x(\mu, \nu), y(\mu, \nu), s(\mu, \nu))\) for every \(\mu > 0\). This solution consists of the \(\mu\)-centers of the perturbed problems \((P_\nu)\) and \((D_\nu)\). Note that for \(x, s > 0\) and \(\mu > 0\) from the third equation of system (2) we deduce that
\[
x \frac{s}{\mu} = \mu e \iff \frac{x}{\mu} = \frac{s}{\mu} = e \iff \sqrt{\frac{x}{\mu}} = e \iff \frac{x}{\mu} = \sqrt{\frac{x}{\mu}}.
\]
Now the perturbed central path can be equivalently stated as follows:
\[
\begin{align*}
 b - Ax &= \nu r_0^b, \quad x \geq 0, \\
 c - A^T y - s &= \nu r_0^c, \quad s \geq 0, \\
 x \frac{s}{\mu} &= \sqrt{x} \frac{s}{\mu}.
\end{align*}
\]
In the sequel, the parameters \(\mu\) and \(\nu\) always satisfy the relation \(\mu = \nu \mu^0 = \nu \xi^2\).

**3 New search directions**

In accordance with the Darvay’s idea, we consider the function \(\psi\) defined and continuously differentiable on the interval \((k^2, \infty)\), where \(0 \leq k < 1\), such that \(2t \psi'(t^2) - \)
ψ'(t) > 0, ∀t > k^2. Now, if we apply the AET method to (4), then we get
\begin{align*}
b - Ax &= νr^0_b, \quad x \geq 0, \quad (5) \\
c - A^T y - s &= νr^0_c, \quad s \geq 0, \quad (6) \\
ψ\left(\frac{xS}{μ}\right) &= ψ\left(\sqrt{\frac{xS}{μ}}\right). \quad (7)
\end{align*}

Let \((x, y, s)\) be a feasible solution of the perturbed pair \((P_ν)\) and \((D_ν)\). We consider the notation
\[f(x, y, s) = \begin{bmatrix}
ν^+ r^0_b - b - Ax \\
ν^+ r^0_c - c + A^T y + s \\
ψ\left(\frac{xS}{μ}\right) - ψ\left(\sqrt{\frac{xS}{μ}}\right)
\end{bmatrix} = 0,
\]
where \(ν^+ = (1 - \theta)ν\) and \(θ ∈ (0, 1)\). Applying Newton’s method to this system, we get
\[J_f(x, y, s) \begin{bmatrix}
Δx \\
Δy \\
Δs
\end{bmatrix} = -f(x, y, s),
\]
where \(J_f(x, y, s)\) denotes the Jacobian matrix of \(f\) at \((x, y, s)\). After some computations, we obtain the following system:
\begin{align*}
AΔx &= θνr^0_b, \\
A^T Δy + Δs &= θνr^0_c, \\
\frac{1}{μ}(sΔx + xΔs) &= \frac{-ψ(\frac{xS}{μ}) + ψ\left(\sqrt{\frac{xS}{μ}}\right)}{ψ'\left(\frac{xS}{μ}\right) - \frac{1}{2\sqrt{μ}}ψ'\left(\sqrt{\frac{xS}{μ}}\right)}. \quad (8)
\end{align*}

Defining the scaled search directions
\[d_x := \frac{vΔx}{x}, \quad d_s := \frac{vΔs}{s}, \quad \text{where} \quad v = \sqrt{\frac{xS}{μ}}, \quad (9)
\]
we can give the scaled form of system (8):
\begin{align*}
\bar{Α}d_x &= θνr^0_b, \\
Α^T \bar{δ}u + d_s &= θνμs^{-1}r^0_c, \\
d_x + d_s &= p_v, \quad (10)
\end{align*}
where

\[p_v := \frac{2ψ(v) - 2ψ(v^2)}{2vψ'(v^2) - ψ'(v)}, \quad \text{and} \quad \bar{Α} := \text{Adiag}(\frac{x}{μ}).
\]

If we use the function \(ψ : (\frac{1}{\sqrt{2}}, \infty) → \mathbb{R}, ψ(t) = t^2\) introduced in [4], then we obtain
\[p_v = \frac{v - v^3}{2v^2 - c}. \quad (11)
\]
After a full-Newton step, the new iterate is given by

\[ x_+ := x + \Delta x, \quad y_+ := y + \Delta y, \quad s_+ := s + \Delta s. \]  

(12)

Furthermore, in each iteration of the algorithm, a quantity is needed to measure how far an iterate is from the central path. We consider the proximity measure defined by

\[ \delta(v) := \delta(x, s; \mu) = \frac{1}{2} \frac{\| v - v^3 \|}{2v^2 - e}, \]  

(13)

which was first suggested for a feasible IPM in [4].

Let \( q_v = d_x - d_s \). Then

\[ d_x = \frac{p_v + q_v}{2}, \quad d_s = \frac{p_v - q_v}{2}, \quad d_x d_s = \frac{p_v^2 - q_v^2}{4}, \]  

(14)

and

\[ \frac{\| q_v \|^2}{4} - \frac{\| d_x - d_s \|^2}{4} = \frac{\| d_x + d_s \|^2}{4} - d_x^T d_s = \frac{\| p_v \|^2}{4} - d_x^T d_s. \]  

(15)

Suppose that for some \( \mu \in (0, \mu^0] \), our algorithm begins from a feasible solution \((x, y, s)\) of the problem pair \((P, \nu)\) and \((D, \nu)\) with \( \nu = \frac{\mu}{\mu^0} \), and such that \( \delta(x, s; \mu) \leq \tau, \tau \in (0, 1) \). Then, the algorithm finds a feasible solution \((x_+, y_+, s_+)\) of \((P, \nu)\) and \((D, \nu)\), where \( \nu = (1 - \theta) \nu, \theta \in (0, 1) \). In this case, \( \mu \) is decreased to \( \mu = (1 - \theta) \mu \) and such that \( \delta(x_+, s_+; \mu) \leq \tau \). This procedure is repeated until an \( \varepsilon \)-solution is found. We are now in a position to state the theoretical framework of the infeasible interior-point algorithm as follows:

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**Algorithm1**: an infeasible interior-point algorithm

**Input** :
- Accuracy parameter \( \varepsilon > 0 \);
- barrier update parameter \( \theta, \ 0 < \theta < 1 \);
- threshold parameter \( \tau > 0 \).

**begin**
- \( x := \xi e; \ y := 0; \ s := \xi e; \ \mu := \nu \xi^2; \ \nu = 1 \);

**while** \( \max(x^T s, \| r_b \|, \| r_c \|) > \varepsilon \) **do**

**begin**
- solve the system (10) and use (9) to obtain \((\Delta x, \Delta y, \Delta s)\);
- \( (x, y, s) := (x, y, s) + (\Delta x, \Delta y, \Delta s) \);
- update of \( \mu \) and \( \nu \):
  - \( \mu := (1 - \theta) \mu \);
  - \( \nu := (1 - \theta) \nu \);

**end**

**end.**
4 Analysis of the algorithm

Here, we will prove that Algorithm 1 is well-defined. The main goal of our analysis is to find some values for the parameters \( \tau \) and \( \theta \) such that \( x_+ > 0 \) and \( s_+ > 0 \), and we have \( \delta(x_+, s_+; \mu^+) \leq \tau \). In the following section, we obtain an upper bound for the proximity measure after an iteration of the algorithm.

4.1 Upper bound for \( \delta(v^+) \)

In the next lemma, we give a condition on the proximity measure which ensures the feasibility of a full-Newton step. In what follows, we use the notation \( \omega = \frac{1}{2}(\|d_x\|^2 + \|d_s\|^2) \).

Lemma 2. The iterate \( (x_+, y_+, s_+) \) with \( v > \frac{1}{\sqrt{2}} \) is strictly feasible if \( \delta(v)^2 + \omega < 1 \).

Proof. Let \( 0 \leq \alpha \leq 1 \). We define \( x(\alpha) := x + \alpha \Delta x \) and \( s(\alpha) := s + \alpha \Delta s \). Using (9), the third equation of (10) and (14) one can find

\[
\frac{x(\alpha)s(\alpha)}{\mu} = xs(v + \alpha d_x)(v + \alpha d_s) = v^2 + \alpha v(d_x + d_s) + \alpha^2 d_x d_s
\]

\[
= (1 - \alpha)v^2 + \alpha(v^2 + vp_v) + \alpha^2 \left( \frac{P_v^2 - q_v^2}{4} \right)
\]

\[
\geq (1 - \alpha)v^2 + \alpha^2 e + \alpha^2 \frac{P_v^2}{4} - \alpha^2 q_v^2
\]

where the inequality is due to \( \alpha \geq \alpha^2 \) and the following inequality:

\[
v^2 + vp_v - e = v^2 + \frac{v^2 - v^4}{2v^2 - e} - e = \frac{v^4}{2v^2 - e} - e = \frac{(v^2 - e)^2}{2v^2 - e} \geq 0.
\]

The inequality \( x(\alpha)s(\alpha) > 0 \) holds if

\[
\left\| - \frac{P_v^2}{4} + \frac{q_v^2}{4} \right\|_\infty \leq \left\| \frac{P_v^2}{4} \right\|_\infty + \left\| \frac{q_v^2}{4} \right\|_\infty \leq \frac{\|P_v\|^2}{4} + \frac{\|q_v\|^2}{4}
\]

\[
= \delta(v)^2 - \|d^T_x d_s\| \leq \delta(v)^2 + \|d_x\| \|d_s\| \leq \delta(v)^2 + \omega < 1,
\]

where the equality is due to (15), the third inequality uses from the Cauchy-Schwarz inequality and the last inequality holds due to the assumption of the lemma. Thus, \( x(\alpha)s(\alpha) > 0 \), for \( 0 \leq \alpha \leq 1 \); \( x(\alpha) \) and \( s(\alpha) \) do not change sign on the interval \([0, 1]\). Consequently, \( x(0) = x > 0 \) and \( s(0) = s > 0 \) yields \( x(1) = x_+ > 0 \) and \( s(1) = s_+ > 0 \). Thus, the proof is completed.

In correspondence to the definition (13), we have

\[
\delta(v_+) = \delta(x_+, s_+; \mu^+) = \frac{1}{2} \| \frac{v_+ - v_+^3}{2v_+^2 - e} \|, \text{ where } v_+ = \sqrt{\frac{x_+ s_+}{\mu^+}}.
\]
Lemma 3. Let $\delta(v)^2 + \omega < \frac{1}{2}(1 - \theta)$ and $v > \frac{1}{\sqrt{2}}e$. Then, $v_+ > \frac{1}{\sqrt{2}}e$ and

$$
\delta(v_+) \leq \frac{\sqrt{1 - \delta(v)^2 - \omega (\theta \sqrt{n} + 10\delta(v)^2 + \omega)}}{2\sqrt{1 - \theta(2(1 - \delta(v)^2 - \omega) - (1 - \theta))}}.
$$

Proof. Let $\alpha = 1$. Then from (16) it follows that

$$
v^2 = \frac{x^2}{\mu^+} = \nu^2 + vp_\nu + \frac{p_\nu^2}{4} - \frac{q_\nu^2}{4} = \frac{e + (\nu^2 - e)^2}{1 - \theta} + \frac{e - \frac{q_\nu^2}{4}}{1 - \theta} \geq \frac{e - \frac{q_\nu^2}{4}}{1 - \theta},
$$

where the second equality is due to (17) and the inequality follows from the fact that $9\nu^2 - 4e \geq 0.5e > 0$. Consequently, we have

$$
\min(v_+) \geq \frac{\sqrt{\frac{1 - \frac{1}{4}\|q_v\|_\infty^2}{1 - \theta}}} \geq \sqrt{\frac{1 - \frac{1}{4}\|q_v\|^2}{1 - \theta}} \geq \sqrt{\frac{1 - \delta(v)^2 - \omega}{1 - \theta}},
$$

(18)

where the last inequality follows from (15) and the Cauchy-Schwarz inequality.

From $\delta(v)^2 + \omega < \frac{1}{2}(1 - \theta)$ it follows that $\min(v_+) > \frac{1}{\sqrt{2}}$, hence $v_+ > \frac{1}{\sqrt{2}}e$. Now, we have

$$
\delta(v_+) = \frac{1}{2} \frac{\|v_+ - v_+^3\|_2}{2v_+^2 - e} = \frac{1}{2} \frac{\|v_+ - (e - v_+^2)\|}{2v_+^2 - e - (e - v_+^2)}
\leq \frac{\min(v_+)}{2(2\min(v_+)^2 - 1)} \|e - v_+^2\|
\leq \frac{\sqrt{(1 - \theta)(1 - \delta(v)^2 - \omega)}}{2(1 - \delta(v)^2 - \omega) - (1 - \theta))} \|e - v_+^2\|.
$$

(19)

On the other hand, one has

$$
\|e - v_+^2\| = \left\| \frac{e + \left(9\nu^2 - 4e\right)p_\nu^2}{1 - \theta} - \frac{q_\nu^2}{4} \right\|
\leq \frac{1}{1 - \theta} \left( \|\theta e\| + \left\| \left(9\nu^2 - 4e\right)p_\nu^2 - \frac{q_\nu^2}{4} \right\| \right)
\leq \frac{1}{1 - \theta} \left( \theta \sqrt{n} + 9\frac{\|p_v\|^2}{4} + \frac{\|q_v\|^2}{4} \right)
= \frac{1}{1 - \theta} \left( \theta \sqrt{n} + 10\delta(v)^2 + \omega \right).
$$

Substituting this bound into (19) gives us exactly the desired result. Thus, the proof is completed.
4.2 Upper bound for $\omega$

Following [20], let $\mathcal{N} := \{\zeta : \bar{A}\zeta = 0\}$ denote the null space of the matrix $\bar{A}$. Then, the $\{\zeta : \bar{A}\zeta = \theta v^0\}$ affine space equals $\mathcal{N} + d_x$. Since the row space of $\bar{A}$ is the orthogonal complement $\mathcal{N}^\perp$ of $\mathcal{N}$, thus $d_s \in \theta v^{-1} v_0^0 + \mathcal{N}^\perp$. Also note that $\mathcal{N} \cap \mathcal{N}^\perp = \{0\}$, and the affine spaces $\mathcal{N} + d_x$ and $\mathcal{N}^\perp + d_s$ meet in a unique point $q$. Applying a similar argument to Lemma 3.4 in [20], we can find

$$2\omega \leq \|q\|^2 + \left(\|q\| + \left\|\frac{v - v^3}{2v^2 - e}\right\|\right)^2 = \|q\|^2 + (\|q\| + 2\delta(v))^2. \tag{20}$$

Again from [20], we have

$$\|q\| \leq \frac{\theta(n + \|v\|^2)}{\min(v)}. \tag{21}$$

By definition [13], we have

$$2\delta(v) = \left\|\frac{v - v^3}{2v^2 - e}\right\| = \left\|\frac{v^2 + v}{2v^2 - e}(e - v)\right\| \geq \frac{1}{2}\|e - v\| \geq \frac{1}{2}(\|v\| - \|e\|),$$

which implies

$$\|v\| \leq \|e\| + 4\delta(v) = \sqrt{n} + 4\delta(v).$$

Furthermore, we have

$$4\delta(v) \geq \|e - v\| \geq |1 - v_i|, i = 1, \ldots, n.$$ 

This gives $\min(v) \geq 1 - 4\delta(v)$. Combining these two inequalities with (21), we will get

$$\|q\| \leq \frac{\theta\left(n + (\sqrt{n} + 4\delta(v))^2\right)}{1 - 4\delta(v)}. \tag{22}$$

4.3 Values for $\theta$ and $\tau$

In this section, we require finding values $\theta$ and $\tau$ such that if $\delta(v) \leq \tau$ holds, then $\delta(v+) \leq \tau$. From Lemma 3 it suffices to have

$$\frac{\sqrt{1 - \delta(v)^2} - \omega(\theta\sqrt{n} + 10\delta(v)^2 + \omega)}{2\sqrt{1 - \theta(2(1 - \delta(v)^2 - \omega) - (1 - \theta))}} \leq \tau, \tag{23}$$

provided that $\delta(v)^2 + \omega < \frac{1}{2}(1 - \theta)$. One can easily see the right-hand-side of (22) is monotonically increasing with respect to $\delta(v) < 1$. Hence, invoking $\delta(v) \leq \tau$, we have

$$\|q\| \leq \frac{\theta\left(n + (\sqrt{n} + 4\tau)^2\right)}{1 - 4\tau}.$$
By substituting the above result into (20) and using again $\delta(v) \leq \tau$, we obtain
\[
\omega \leq \frac{1}{2} \left[ \left( \frac{\theta(n + \sqrt{n} + 4\tau)}{1 - 4\tau} \right)^2 + \left( \frac{\theta(n + \sqrt{n} + 4\tau)}{1 - 4\tau} + 2\tau \right)^2 \right] =: f(\tau).
\]

We claim that
\[
\chi(t) := \frac{\sqrt{1-t}}{2(1-t) - (1-\theta)}, \quad 0 \leq t \leq \frac{1}{2}(1-\theta), \tag{24}
\]
is increasing. Hence, $0 \leq \delta(v)^2 + \omega \leq \tau^2 + f(\tau)$ implies $\chi(\delta(v)^2 + \omega) \leq \chi(\tau^2 + f(\tau))$. Therefore, $\delta(v)^2 + \omega \leq \frac{1}{2}(1-\theta)$ and (23) will certainly hold if
\[
\tau^2 + f(\tau) \leq \frac{1}{2}(1-\theta), \quad y(\tau) := \frac{\chi(\tau^2 + f(\tau))(\theta \sqrt{n} + 10\tau^2 + f(\tau))}{2\sqrt{1-\theta}} \leq \tau.
\]
If we take $\tau = \frac{1}{12}$ and $\theta = \frac{1}{22n}$, $n \geq 4$, then $\tau^2 + f(\tau) = 0.0645 < 0.4773 \leq \frac{1}{2}(1-\theta)$ and $y(\tau) \leq 0.0827 < \frac{1}{12}$. Hence, we may state the following result.

Lemma 4. If $\tau = \frac{1}{12}$ and $\theta = \frac{1}{22n}$, $n \geq 4$, then $\delta(v) \leq \tau$ implies $\delta(v_+) \leq \tau$.

4.4 Complexity analysis

Lemma 4 establishes the proposed algorithm is well-defined, in the sense that the property $\delta(x, s; \mu) := \delta(v) \leq \tau$ is maintained in all iterations.

In each main iteration, both the barrier parameter $\mu$ and the norms of the residual vectors are reduced by the factor $1 - \theta$. Hence, the total number of main iterations is bounded above by
\[
\frac{1}{\theta} \log \max \left\{ n\xi^2, \|r_b^0\|, \|r_c^0\| \right\} \varepsilon.
\]

Now, we state our main result.

Theorem 5. If (P) and (D) are feasible and $\xi > 0$ such that $\|(x^*; s^*)\|_{\infty} \leq \xi$ for some optimal solutions $x^*$ of (P) and $(y^*, s^*)$ of (D), then after at most
\[
22n \log \max \left\{ n\xi^2, \|r_b^0\|, \|r_c^0\| \right\} \varepsilon
\]
iterations, the algorithm finds an $\varepsilon$-optimal solution of (P) and (D).

5 Conclusions

The method presented in this paper is a full-Newton step IIPM for LO based on the AET proposed in [4]. The method is used in each iteration only one feasibility step. Our method analysis is different from the existing IIPMs based on the AET because it uses a different AET. The obtained complexity bound coincides with the current best-known theoretical iteration bound for IIPMs.
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