Ordinary differential equations defined by a trigonometric polynomial field: behaviour of the solutions

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ABSTRACT
We consider the ordinary differential equations defined by a trigonometric polynomial field and we prove that any solution \( x \) admits a rotation vector \( \rho \in \mathbb{R}^n \). More precisely, the function \( t \mapsto x(t) - \rho t \) is bounded on time and it is a weak almost periodic function of slope \( \rho \).

1. Introduction
In this article, we study the asymptotic behaviour of solutions for ordinary differential equations (ODE) defined by a trigonometric polynomial field. The idea comes from the scalar case, where in this case H. Poincaré defined the rotation number for circle homeomorphisms [7]. The simple example is a scalar differential equation

\[
\dot{x} = f(x), \quad x(0) \in \mathbb{R}, \quad t \in \mathbb{R},
\]

where \( f : \mathbb{R} \to \mathbb{R} \) is lipschitz, 1-periodic and \( t \mapsto x(t) \) is the state of the system. There exists a rotation number \( \lambda \in \mathbb{R} \) for which the function \( t \mapsto x(t) - \lambda t \) is bounded (periodic). We know that any non-autonomous system can be written as an autonomous system. Our result is a generalization of this asymptotic behaviour to any dimension. In this case, \( \lambda \) is a vector and called a rotation vector or rotation set as it is defined in [5]. Under some assumptions of stability [8], [2] proved the existence of the rotation vector. Some biological works use the ODE defined by a trigonometric polynomial field and study the rotation vector components as in [1], [3], [4], [10]. Our contribution to this biological works has two key points, the mathematical proof of existence of the rotation vector and the study of the behaviour of solutions.
2. Definition and main result

We study in this article the following system:

\[ \dot{x} = f(x), \quad t \in \mathbb{R}, \quad x(0) = x_0 \in \mathbb{R}^n, \]

where \( t \mapsto x(t) := (x_j(t))_{j=1}^n \) is the state of the system and \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a trigonometric polynomial in the following sense

**Definition 2.1 (Trigonometric polynomial function):** A function \( g : \mathbb{R}^n \to \mathbb{R} \) is called a trigonometric polynomial if there exists a finite sequence \((c_p)_{p \in 2\pi \mathbb{Z}^n} \subset \mathbb{C} \) such that

\[ \forall x \in \mathbb{R}^n : \quad g(x) = \sum_{p \in 2\pi \mathbb{Z}^n} c_p \exp(i \langle x, p \rangle), \]

where \( \langle . . . \rangle \) is the usual scalar product on \( \mathbb{R}^n \). A function \( g : \mathbb{R}^n \to \mathbb{R}^n \) is a trigonometric polynomial if each component is a trigonometric polynomial function.

To formulate the Main results, let us introduce the following definitions. We use the usual norm \( \|y\| := \max_{1 \leq j \leq n} \|y_j\| \) for every \( y := (y_j)_{j=1}^n \in \mathbb{C}^n \).

**Definition 2.2 (Rotation vector):** Let \( \lambda \in \mathbb{R}^n \) and \( \phi : \mathbb{R} \to \mathbb{R}^n \) be a function. We say that \( \phi \) admits \( \lambda \) as the rotation vector if

\[ \sup_{t \in \mathbb{R}} \|\phi(t) - \lambda t\| < \infty. \]

For more information about the behaviour of solutions, we introduce the following definitions.

**Definition 2.3 (Periodic modulo \( \mathbb{Z}^n \) function):** A function \( g : \mathbb{R}^n \to \mathbb{R}^n \) is called periodic modulo \( \mathbb{Z}^n \), if

\[ g(z_1 + k_1, \ldots, z_n + k_n) = g(z_1, \ldots, z_n), \quad \forall (k_j)_{j=1}^n \in \mathbb{Z}^n, \quad \forall (z_j)_{j=1}^n \in \mathbb{R}^n. \]

**Definition 2.4 (Weakly almost-periodic function):** Let be \( r \in \mathbb{R}^n \). A function \( h : \mathbb{R} \to \mathbb{R}^n \) is weakly almost periodic of slope \( r \) if it is \( C^\infty \) and if there exists a uniformly bounded sequence for the sup-norm of \( C^\infty \) functions \( (g_k : \mathbb{R}^n \to \mathbb{R}^n)_{k \in \mathbb{N}} \) that are periodic modulo \( \mathbb{Z}^n \) and there exists a sequence \( (r_k)_{k \in \mathbb{N}} \subset \mathbb{Q}^n \) such that

\[ \lim_{k \to \infty} r_k = r \quad \text{and} \quad \forall t > 0 : \lim_{k \to \infty} \sup_{s \in [-t, t]} \|g_k(r ks) - h(s)\| = 0. \]

We call the sequence \( (g_k)_{k} \) the \( \mathbb{Z}^n \)-periodic sequence of the function \( h \).

**Remark 2.5:** Remark that for every \( k \in \mathbb{N} \) the function \( s \mapsto g_k(r ks) \) is a periodic function.
Main Result Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a trigonometric polynomial function. For every \( x_0 \in \mathbb{R}^n \) the unique solution \( x : \mathbb{R} \to \mathbb{R}^n \) of the differential equation
\[
\dot{x} = f(x), \quad t \in \mathbb{R}, \quad x(0) = x_0,
\]
admits a unique rotation vector \( \rho \in \mathbb{R}^n \). In addition, the function
\[
t \mapsto x(t) - \rho t,
\]
is weakly almost periodic of slope \( \rho \).

3. Space of \( C^\infty \) periodic modulo \( \mathbb{Z}^n \) functions

We define in this section the space and the norm used to prove the main result. The proofs of Lemmas for this Section are left in Appendix. To use the Fourier development, let us introduce the following notation.

Notation 3.1: For every continuous function \( g : \mathbb{R}^n \to \mathbb{R}^n \) and every \( p \in 2\pi \mathbb{Z}^n \) we denote \( a_p[g] \in \mathbb{C}^n \) the following limit if it exists
\[
a_p[g] := \lim_{t \to +\infty} \frac{1}{(2t)^n} \int_{-t}^{t} \cdots \int_{-t}^{t} g(z) \exp(-i(z,p)) dz_1 \cdots dz_n.
\]
In this section and Section 4, for every function \( g : \mathbb{R}^n \to \mathbb{R}^n \) and every \( \alpha \in \mathbb{R} \), we denote \( g_\alpha \) the function defined as \( g_\alpha(z) := g(\alpha z) \) for all \( z \in \mathbb{R}^n \). The following constant \( \omega \) will be used as change of variable to find a contraction in Lemmas 4.5 and 4.6. For every \( \omega \in \mathbb{N}^* \), we denote \( E_\omega(\mathbb{R}^n) \) the set of \( C^\infty \) function \( g : \mathbb{R}^n \to \mathbb{R}^n \) such that \( g_\omega \) is a periodic modulo \( \mathbb{Z}^n \) function. We remark, for every \( \omega \in \mathbb{N}^* \) and \( g \in E_\omega(\mathbb{R}^n) \) that
\[
a_p[g_\omega] = \int_{0}^{1} \cdots \int_{0}^{1} g_\omega(z) \exp(-i(z,p)) dz_1 \cdots dz_n,
\]
which is the Fourier coefficient of the function \( g_\omega \). Since \( g \in E_\omega(\mathbb{R}^n) \) is \( C^\infty \) and periodic relatively to each variable, by Dirichlet Theorem [[9], Corollary 2.4] applied to each variable and for the \( q^th \) derivative of \( g \), we get
\[
g(z) = \sum_{p \in 2\pi \mathbb{Z}^n} a_p[g_\omega] \exp \left( \frac{i}{\omega} \langle z, p \rangle \right),
\]
\[
\forall q \geq 0 : \sum_{p \in 2\pi \mathbb{Z}^n} \|a_p[g_\omega]\| \left\| \frac{P}{\omega} \right\|^q < +\infty.
\]
We are now in position to define the following seminorm in \( E_\omega(\mathbb{R}^n) \): Let \( \omega \in \mathbb{N}^* \) and \( g \in E_\omega(\mathbb{R}^n) \), we denote for every \( \omega \in \mathbb{N}^* \) and \( q \geq 0 \)
\[
\|g\|_{\omega,q} := 2 \sum_{p \in 2\pi \mathbb{Z}^n / \{0\}} \|a_p[g_\omega]\| \left\| \frac{P}{\omega} \right\|^q,
\]
where \( 0 := (0, \ldots, 0) \in \mathbb{R}^n \) and where we recall that \( \|y\| := \max_{1 \leq j \leq n} \|y_j\| \) for every \( y := (y_j)_{j=1}^{n} \in \mathbb{C}^n \). We prove in the following Lemma that a periodic modulo \( \mathbb{Z}^n \) function \( g \) is
$C^\infty$ if it is uniformly bounded for the seminorm, i.e.
\[ \forall q \geq 0 : \|g\|_{\omega, q} < +\infty. \]

In other words, the set $E_\omega(\mathbb{R}^n)$ is included in the set of the periodic modulo $\mathbb{Z}^n$ functions uniformly bounded for the seminorm.

**Lemma 3.2:** Let $\omega \in \mathbb{N}^*$. Let $(c_p)_p$ be a complex-valued family such that
\[ \forall q \geq 0 : \sum_{p \in 2\pi \mathbb{Z}^n/\{0\}} \|c_p\| \left\| \frac{p}{\omega} \right\|^q < +\infty. \]

Then the following series is normally convergent:
\[ g(z) := \sum_{p \in 2\pi \mathbb{Z}^n} c_p \exp \left( i \frac{1}{\omega} \langle z, p \rangle \right), \quad c_p \in \mathbb{C}^n, \]
and $c_p = a_p[g_\omega]$ for every $p \in 2\pi \mathbb{Z}^n$. Further, $g \in E_\omega(\mathbb{R}^n)$.

**Proof:** Appendix 1.

In the following Lemma, we prove that the seminorm $\|\cdot\|_{\omega, 0}$ is a norm on the space $\{g \in E_\omega(\mathbb{R}^n) : g(0) = 0\}$ and we compare it to the uniform norm topology.

**Lemma 3.3:** Let be $\omega \in \mathbb{N}^*$ and $g \in E_\omega(\mathbb{R}^n)$ such that $g(0) = 0$ then
\[ \|a_0[g_\omega]\| \leq \frac{1}{2} \|g\|_{\omega, 0} \quad \text{and} \quad \|g\|_{\infty} \leq \|g\|_{\omega, 0}. \]

**Proof:** Appendix 2.

We denote $d^k g$ the $k$th differential of a function $g : \mathbb{R}^n \to \mathbb{R}^n$. The following Lemma gives an upper bound of the quantity $\|d^k g\|_{\omega, q}$ when $g$ is a trigonometric polynomial. We recall that $g_{\omega}^1(z) := g(\frac{z}{\omega})$.

**Lemma 3.4:** Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a trigonometric polynomial function. Then there exists $\beta := \beta(g) > 0$ such that for every $\omega \in \mathbb{N}^*$ we have
\[ \|d^k g_{\omega}^1\|_{\omega, q} < n^k \beta \left( \frac{\beta}{\omega} \right)^{q+k}, \quad \forall q, k \geq 0. \]

**Proof:** Appendix 3.

We end this section by the following inequality.

**Lemma 3.5:** Let $\omega \in \mathbb{N}^*$ and $(h_j \in E_\omega(\mathbb{R}^n))_{j=1}^k$. Then
\[ \forall q \in \mathbb{N}, \quad \forall k \in \mathbb{N}^* : \left\| \prod_{j=1}^k h_j \right\|_{\omega, q} \leq \frac{(k \omega^{k-1})^q}{2^{k-1}} \prod_{j=1}^k \left[ \|2 a_0[h_{j, \omega}]\| + \|h_j\|_{\omega, q} \right]. \]

**Proof:** Appendix 4.
4. Main proposition

The main result affirms that the solution $x$ of Equation (1) is a sum of a linear part and a bounded part. The strategy to prove the main result is to approximate the bounded part of $x$ by a $\mathbb{Z}^n$-periodic sequence. Using the Fourier development and Equation (3), remark, for every $C^\infty$ periodic modulo $\mathbb{Z}^n$ function $g$ that

$$f(z + g(z)) = \sum_{p \in 2\pi \mathbb{Z}^n} a_p[H[g]] \exp(i \langle z, p \rangle), \quad H[g](z) := f(z + g(z)),$$

under some convergence assumption of the series, by integration we get

$$\forall v \in \mathbb{R}^n : \int_0^t f(vs + g(vs))ds = t \sum_{p \in 2\pi \mathbb{Z}^n, \langle v, p \rangle = 0} a_p[H[g]] + \sum_{p \in 2\pi \mathbb{Z}^n, \langle v, p \rangle \neq 0} \frac{a_p[H[g]]}{i\langle v, p \rangle} \left(\exp(i \langle v, p \rangle t) - 1\right).$$

The last term of the right member of the last equality will play the role of $\mathbb{Z}^n$-periodic sequence of the bounded part of the solution $x$ of Equation (1). To find an upper bound of the bounded part, let us introduce the following notations.

**Notation 4.1**: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a trigonometric polynomial. We denote the finite subset $\Lambda_f \subset 2\pi \mathbb{Z}^n$ as

$$\Lambda_f := \{p \in 2\pi \mathbb{Z}^n : \|a_p[f]\| \neq 0\},$$

and we denote

$$|\Lambda_f| := \max\{\|p\|, \quad p \in \Lambda_f\}.$$  

Let be $y \in \mathbb{R}^n/\{0\}$. Define

$$\Lambda(f, y) := \{p \in 2\pi \mathbb{Z}^n : \|p\| \leq 2\pi + |\Lambda_f|, \langle y, p \rangle \neq 0\}.$$  

Remark that $\Lambda(f, y) \neq \emptyset$. We denote

$$\tau(f, y) := \max \left\{ \frac{1}{|\langle y, p \rangle|} : \quad p \in \Lambda(f, y) \right\}.$$  

Let be $y \in \mathbb{Q}^n/\{0\}$, we denote

$$\tau(y) = \max \left\{ \frac{1}{|\langle y, p \rangle|} : \quad p \in 2\pi \mathbb{Z}^n, \langle y, p \rangle \neq 0 \right\}.$$  

We denote $\beta$ the constant $\beta(f)$ of the function $f$ defined in Lemma 3.4.

In the following proposition, we prove that the bounded part of the solution $x$ of Equation (1) can be approximated by a $\mathbb{Z}^n$-periodic functions and we find an appropriate upper bound.
**Proposition 4.2 (Main proposition):** Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a trigonometric polynomial function. Then for every $r \in \mathbb{Q}^n / \{0\}$ and every $\epsilon > 0$ there exists a $C^\infty$ periodic modulo $\mathbb{Z}^n$ function $\phi_{r,\epsilon} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\|\phi_{r,\epsilon}\|_\infty < 2\beta \tau(f, r)$ and such that

$$
\sup_{z \in \mathbb{R}^n} \left\| \phi_{r,\epsilon}(z) - \sum_{p \in 2\pi \mathbb{Z}^n, (r,p) \neq 0} \frac{a_p[H[\phi_{r,\epsilon}]]}{i(r,p)} (\exp(i(z,p)) - 1) \right\| < \epsilon,
$$

where

$$H[\phi_{r,\epsilon}](z) := f(z + \phi_{r,\epsilon}(z)), \quad \forall z \in \mathbb{R}^n.$$ 

As is stated in the above section, the following constant $\omega$ is used as change of variable to find a contraction.

**Definition 4.3:** For every $r \in \mathbb{R}^n$ and every $\omega \in \mathbb{N}^*$, define the set $K_{r,\omega}$ as $g \in K_{r,\omega}$ if

- there exists a complex-valued family $(c_p)_{p \in 2\pi \mathbb{Z}^n}$ such that

$$g(z) = \sum_{p \in 2\pi \mathbb{Z}^n} c_p \left( \exp\left(\frac{1}{\omega} i\langle z, p \rangle\right) - 1 \right), \quad \forall z \in \mathbb{R}^n,$$

- $\|g\|_{\omega,0} \leq 2\beta \tau(f, r)$,
- $\|g\|_{\omega,q} < \infty$ for every $q \geq 1$.

**Lemma 4.4:** The set $K_{r,\omega}$ is a nonempty subset of $E_\omega(\mathbb{R}^n)$.

**Proof:** The set $K_{r,\omega} \neq \emptyset$ because it contains the function $z \mapsto g(z) = 0$. By definition of $K_{r,\omega}$ and by Lemma 3.2, the function $g$ is $C^\infty$. \hfill \blacksquare

For every $r \in \mathbb{R}^n$, for every $\omega \in \mathbb{N}^*$, and every $g \in K_{r,\omega}$ let be $\Psi[r, \omega, g]$ the function defined by the following series in its convergence domain:

$$\forall z \in \mathbb{R}^n : \quad \Psi[r, \omega, g](z) := \sum_{p \in 2\pi \mathbb{Z}^n, (r,p) \neq 0} \frac{a_p[H[\omega, g]]}{i(r,p)} \left( \exp\left(\frac{1}{\omega} i\langle z, p \rangle\right) - 1 \right),$$

where

$$H[\omega, g](z) := f(z + g(\omega z)).$$

Since $f$ is a real polynomial trigonometric function then $\Psi[r, \omega, g](z) \in \mathbb{R}^n$ for every $z \in \mathbb{R}^n$ such that the series converge. In the following Lemma, we prove that $K_{r,\omega}$ is invariant under the operator $\Psi[r, \omega, .]$. We deduce that $\Psi[r, \omega, g](z)$ is defined for every $g \in K_{r,\omega}$ and $z \in \mathbb{R}^n$.

**Lemma 4.5:** Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a trigonometric polynomial function. For every $r \in \mathbb{Q}^n / \{0\}$, there exists $\omega_r > 0$ such that for every $\omega > \omega_r$ we have

$$g \in K_{r,\omega} \implies \Psi[r, \omega, g] \in K_{r,\omega}.$$

In addition, $\Psi[r, \omega, g]$ is defined for every $z \in \mathbb{R}^n$. 
Proof: Prove that
\[ \forall g \in K_{r,\omega} : \| \Psi[r, \omega, g] \|_{\omega, q} < \infty, \quad \forall q \geq 0. \]

Let be \( g \in K_{r,\omega} \) and denote
\[ \tilde{H}[\omega, g](z) := f\left( z + g(\omega z) \right) - f(z). \]

We have
\[ a_p[H[\omega, g]] = a_p[f] + a_p[\tilde{H}[\omega, g]]. \]

By definition of \( \Psi[r, \omega, .] \) for every \( g \in K_{r,\omega} \),
\[ \Psi[r, \omega, g](z) = \sum_{p \in 2\pi \mathbb{Z}^n, \langle r, p \rangle \neq 0} \frac{a_p[f]}{i\langle r, p \rangle} \left( \exp \left( i\omega \langle z, p \rangle \right) - 1 \right) \]
\[ + \sum_{p \in 2\pi \mathbb{Z}^n, \langle r, p \rangle \neq 0} \frac{a_p[\tilde{H}[\omega, g]]}{i\langle r, p \rangle} \left( \exp \left( i\omega \langle z, p \rangle \right) - 1 \right). \]

Recall that \( |\Lambda_f| \) is defined on Notation 4.1: For every \( p \in 2\pi \mathbb{Z}^n \) such that \( \langle r, p \rangle \neq 0 \) and such that \( \|p\| > |\Lambda_f| \) we have \( a_p[f] = 0 \). By definition of \( \tau(f, r) \) and \( \tau(r) \) in Notation 4.1, we get
\[ \| \Psi[r, \omega, g] \|_{\omega, q} \leq 2\tau(f, r) \sum_{p \in 2\pi \mathbb{Z}^n, \langle r, p \rangle \neq 0} \| a_p[f] \|_q \| \frac{p}{\omega} \|^q \]
\[ + 2\tau(r) \sum_{p \in 2\pi \mathbb{Z}^n, \langle r, p \rangle \neq 0} \| a_p[\tilde{H}[\omega, g]] \|_q \| \frac{p}{\omega} \|^q. \]

By definition of the seminorm given by Equation (4), we obtain
\[ \| \Psi[r, \omega, g] \|_{\omega, q} \leq \tau(f, r) \| f_{\frac{1}{\omega}} \|_{\omega, q} + \tau(r) \| (\tilde{H}[\omega, g])_{\frac{1}{\omega}} \|_{\omega, q}. \quad (5) \]

By Lemma 3.4, we have
\[ \| f_{\frac{1}{\omega}} \|_{\omega, q} \leq \frac{\beta^{q+1}}{\omega^q}, \quad \forall q \geq 0. \quad (6) \]

Now, estimate the quantity \( \| (\tilde{H}[\omega, g])_{\frac{1}{\omega}} \|_{\omega, q} \). By definition
\[ \left( \tilde{H}[\omega, g] \right)_{\frac{1}{\omega}}(z) = f\left( \frac{1}{\omega} z + g(z) \right) - f\left( \frac{1}{\omega} z \right) = \sum_{k=1}^{\infty} \frac{d^k f(z)}{k!} (g(z))^{(k)}, \]
where
\[ d^k f(z)(g(z))^{(k)} := \sum_{i_1, \ldots, i_k=1}^n \frac{\partial f}{\partial z_{i_1} \ldots \partial z_{i_k}} \left( \frac{z}{\omega} \right) g_{i_1}(z) \ldots g_{i_k}(z). \]
Since $f$ is polynomial trigonometric function, then

$$\forall z \in \mathbb{R}^n : f_\omega(z) = f\left(\frac{z}{\omega}\right) = a_0[f] + \sum_{p \in 2\pi \mathbb{Z}^n, p \neq 0} a_p[f] \exp\left(i\frac{1}{\omega}(z, p)\right),$$

then for every $k \geq 1$ we get

$$d^k f_\omega(z) = \sum_{p \in 2\pi \mathbb{Z}^n, p \neq 0} a_p[f] d^k \left[ \exp\left(i\frac{1}{\omega}(z, p)\right) \right] = \sum_{p \in 2\pi \mathbb{Z}^n, p \neq 0} \left( a_p[f] \sum_{s_1=1}^n \cdots \sum_{s_k=1}^n i^k p_{s_1} \frac{p_{s_2}}{\omega} \cdots \frac{p_{s_k}}{\omega} \right) \exp\left(i\frac{1}{\omega}(z, p)\right).$$

By Notation 3.1

$$a_p[d^k f_\omega] := \lim_{t \to +\infty} \frac{1}{(2t)^n} \sum_{p \in 2\pi \mathbb{Z}^n, p \neq 0} \left( a_p[f] \sum_{s_1=1}^n \cdots \sum_{s_k=1}^n i^k p_{s_1} \frac{p_{s_2}}{\omega} \cdots \frac{p_{s_k}}{\omega} \right) \theta(\omega, -t, t),$$

where

$$\forall t \in \mathbb{R} : \theta(\omega, t, -t) := \int_{-t}^t \cdots \int_{-t}^t \exp\left(i\frac{1}{\omega}(z, p)\right) dz_1 \cdots dz_n.$$ 

Since $p \neq 0$, then

$$\lim_{t \to +\infty} \frac{1}{(2t)^n} \theta(\omega, t, -t) = 0,$$

we deduce that

$$\|a_0[d^k f_\omega]\| = 0, \quad \forall k \geq 1.$$ 

Since $g \in K_{r, \omega}$ then $g(0) = 0$. Thanks to Lemma 3.3, we obtain

$$\|a_0[g]\| \leq \|g\|_{\omega, 0} \leq \alpha_0 := 2\beta \tau(f, r).$$

By Lemma 3.5, we have for all $q \geq 0$

$$\|\left(\hat{H}[\omega, g]\right)_{\omega}^1\|_{\omega, q} \leq \sum_{k=1}^{\infty} \frac{1}{k!} \|d^k f_\omega(g)^{(k)}\|_{\omega, q} \leq \sum_{k=1}^{\infty} \frac{(k + 1)q_{\omega}^{kq}}{k!2^k} \|d^k f_\omega\|_{\omega, q} (\|g\|_{\omega, q} + \alpha_0)^k.$$
By Lemma 3.4, we find
\[ \forall q \geq 0 : \| \tilde{H}(\omega, g) \|_{\omega, q} \leq \sum_{k=1}^{\infty} \frac{\exp(qk)\omega^k}{k!2^k} n^k \frac{\beta^{k+q+1}}{\omega^{k+q}} \left( \alpha_0 + \|g\|_{\omega, q} \right)^k. \]

By hypothesis \( g \in K_{r,\omega} \), then
\[ \forall q \geq 0, \exists \alpha_q > 0 : \|g\|_{\omega, q} \leq \alpha_q \] where \( \alpha_0 : = 2\beta \tau(f, r) \), we deduce that for all \( q \geq 0 \),
\[ \| \tilde{H}(\omega, g) \|_{\omega, q} \leq \sum_{k=1}^{\infty} \exp(qk)\omega^k \frac{\beta^{k+q+1}}{\omega^{k+q}} \left( \alpha_0 + \alpha_q \right)^k \]
\[ = \frac{\beta^{q+1}}{\omega^q} \sum_{k=1}^{\infty} \frac{1}{k!} \left( n \exp(q)\omega^q \frac{\beta}{2\omega} (\alpha_0 + \alpha_q) \right)^k \]
\[ \leq \frac{\beta^{q+1}}{\omega^q} \left[ \exp \left( n \exp(q)\omega^q \frac{\beta}{2\omega} (\alpha_0 + \alpha_q) - 1 \right) \right] < \infty. \]

By Equations (6) and (5), we obtain
\[ \forall g \in K_{r,\omega} : \|\Psi[r, \omega, g]\|_{\omega, q} < \infty, \forall q \geq 0. \]

Choose \( \omega > \omega_r > 0 \), where \( \omega_r \in \mathbb{N}^* \) satisfies
\[ \beta \left[ \exp \left( n \frac{\beta}{\omega_r} \alpha_0 \right) - 1 \right] < \frac{\beta \tau(f, r)}{\tau(r)}. \]

We obtain
\[ \forall \omega > \omega_r : \| H(\omega, g) \|_{\omega, 0} < \frac{\beta \tau(f, r)}{\tau(r)}. \] (7)

Replace both Equations (6) and (7) on Equation (5), we obtain
\[ \forall \omega > \omega_r : \|\Psi[r, \omega, g]\|_{\omega, 0} < \beta \tau(f, r) + \beta \tau(f, r) = 2\beta \tau(f, r) = \alpha_0. \]

**Lemma 4.6:** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a trigonometric polynomial function. For every \( r \in \mathbb{Q}^n / \{0\} \) there exists \( \omega := \omega(r) > 0 \) such that for every \( \epsilon > 0 \), there exists \( \phi_{r,\omega,\epsilon} \in K_{r,\omega} \) satisfying
\[ \| \phi_{r,\omega,\epsilon} - \Psi[r, \omega, \phi_{r,\omega,\epsilon}] \|_{\infty} < \epsilon. \]

**Proof:** Let be \( \omega > \beta \) and \( h, g \in K_{r,\omega} \). For every fixed \( s \in [0, 1] \), define the function \( V_s \in K_{r,\omega} \) as
\[ V_s(z) := sh(z) + (1-s)g(z), \quad \forall z \in \mathbb{Z}^n. \]

\[ \forall z \in \mathbb{R}^n : \Psi[r, \omega, h](z) - \Psi[r, \omega, g](z) = \Psi[r, \omega, V_1(z)] - \Psi[r, \omega, V_0(z)] \]
\[
\int_0^1 \frac{d}{ds} \Psi[r, \omega, V_s(z)] ds.
\]

By definition of \( \Psi \), we get

\[
\Psi[r, \omega, h](z) - \Psi[r, \omega, g](z) = \int_0^1 \sum_{p \in 2\pi \mathbb{Z}^n, (r,p) \neq 0} \frac{d}{ds} a_p[H[\omega, V_s]] \left( \exp \left( i \frac{1}{\omega} (z, p) \right) - 1 \right) ds.
\]

We have

\[
\frac{d}{ds} a_p[H[\omega, V_s]] = a_p \left[ \frac{d}{ds} H[\omega, V_s] \right],
\]

Since \( \frac{d}{ds} V_s = h - g \), then

\[
\frac{d}{ds} H[\omega, V_s](z) = df \left( z + V_s(\omega z) \right) \frac{d}{ds} V_s(\omega z).
\]

Then

\[
\frac{d}{ds} H[\omega, V_s](z) = df \left( z + V_s(\omega z) \right) [h - g](\omega z).
\]

Then

\[
a_p \left[ \frac{d}{ds} H[\omega, V_s] \right] = a_p \left[ \phi[\omega, V_s](h - g) \right],
\]

where

\[
\phi[\omega, V_s](z) := df \left( z + V_s(\omega z) \right).
\]

For every fixed \( s \in [0,1] \) we have \( \phi[\omega, V_s] \in K_{r,\omega} \). By Lemma 3.5, for every fixed \( s \in [0,1] \) we have

\[
\left\| \phi[\omega, V_s] h - g \right\|_{\omega,0} \leq \left\| \phi[\omega, V_s] \right\|_{\omega,0} \| h - g \|_{\omega,0}.
\]

Then

\[
\left\| \Psi[r, \omega, h] - \Psi[r, \omega, g] \right\|_{\omega,0} \leq \tau(r) \left( \sup_{s \in [0,1]} \left\| \phi[\omega, V_s] \right\|_{\omega,0} \right) \| h - g \|_{\omega,0}.
\]

Prove that there exists \( \omega_r > 0 \) such that for every \( \omega > \omega_r \) we have

\[
\forall g, h \in K_{r,\omega} : \tau(r) \left( \sup_{s \in [0,1]} \left\| \phi[\omega, V_s] \right\|_{\omega,0} \right) < \frac{1}{2}.
\]

As in Proof of Lemma 4.5: By Equation (8) we have for every \( s \in [0,1] \),

\[
\left( \phi[\omega, V_s] \right)_{\omega,0} = \sum_{k=0}^{\infty} \frac{d^{k+1} f_1(z)}{k!} (V_s(s))^{(k)}.
\]
By Lemma 3.5

$$\forall s \in [0, 1]: \left\| \left( \phi[\omega, V_s] \right)_{1 \over \omega} \right\|_{\omega,0} \leq \sum_{k=0}^{\infty} \frac{1}{k! 2^k} \frac{n^{k+1} \beta^{k+2}}{\omega^{k+1}} \frac{1}{\omega} (4\beta \tau(f, r))^k$$

By hypothesis $g, h \in K_{r, \omega}$, by consequence

$$\sup_{s \in [0, 1]} \| V_s \|_{\omega,0} \leq \| g \|_{\omega,0} + \| h \|_{\omega,0} \leq 4\beta \tau(f, r).$$

Using Lemma 3.4, we get

$$\sup_{s \in [0, 1]} \left\| \left( \phi[\omega, V_s] \right)_{1 \over \omega} \right\|_{\omega,0} \leq \sum_{k=0}^{\infty} \frac{1}{k! 2^k} \frac{n^{k+1} \beta^{k+2}}{\omega^{k+1}} \left( \frac{2n\beta^2}{\omega} \beta \tau(f, r) \right)^k$$

$$= \frac{n\beta^2}{\omega} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{2n\beta^2}{\omega} \beta \tau(f, r) \right)^k$$

$$= \frac{n\beta^2}{\omega} \exp \left( \frac{2n\beta^2}{\omega} \beta \tau(f, r) \right).$$

Choose $\omega_r > 0$ large such that

$$\tau(r) \frac{n\beta^2}{\omega_r} \exp \left( \frac{2n\beta^2}{\omega_r} \beta \tau(f, r) \right) < \frac{1}{2}.$$  

We have proved Equation (10). Thanks to Equation (9), for every $\omega > \omega_r$ we have

$$\left\| \left[ \Psi[r, \omega, h] \right] \right\|_{\omega,0} \leq \frac{1}{2} \| h - g \|_{\omega,0}.$$ 

Now, choose $\omega > 0$ fixed and large. Let be $g \in K_{r, \omega}$, by the last inequality, for every $\epsilon > 0$ there exists $k_\epsilon \geq 0$ such that

$$\left\| \left[\Psi^{k_\epsilon+1}[r, \omega, g] - \Psi^{k_\epsilon}[r, \omega, g] \right] \right\|_{\omega,0} < \epsilon.$$ 

Denote

$$\phi_{r, \omega, \epsilon} := \Psi^{k_\epsilon}[r, \omega, g].$$

By Lemma 4.5, we have $\phi_{r, \omega, \epsilon} \in K_{r, \omega}$. By Lemma 3.3, we obtain

$$\left\| \left[\Psi[r, \omega, \phi_{r, \omega, \epsilon}] \right] \right\|_{\infty} < \epsilon.$$  

$\blacksquare$
Proof of Proposition 4.2: Let be $r \in \mathbb{Q}^n \setminus \{0\}$. By Lemma 4.6, there exists $\omega := \omega_r > 0$ such that for every $\epsilon > 0$ there exists $\phi_{r,\omega,\epsilon} \in K_{r,\omega}$ satisfying

$$\|\Psi[r, \omega, \phi_{r,\omega,\epsilon}] - \phi_{r,\omega,\epsilon}\|_\infty < \epsilon.$$  \hspace{1cm} (11)

Define the functions,

$$\tilde{\phi}_{r,\epsilon}(z) := \phi_{r,\omega,\epsilon}(\omega z), \text{ and } H[g](z) := f(z + g(z)).$$

We recall that

$$H[\omega, \phi_{r,\omega,\epsilon}](z) = f\left(z + \phi_{r,\omega,\epsilon}(\omega z)\right).$$

By Equation (2),

$$a_p[H[\omega, \phi_{r,\omega,\epsilon}]] = a_p[H[\tilde{\phi}_{r,\epsilon}]].$$

Using the definition of $\Psi[r, \omega, \tilde{\phi}_{r,\epsilon}]$ and replace on Equation (11), the function $\tilde{\phi}_{r,\epsilon}$ satisfies

$$\sup_{z \in \mathbb{R}^n} \|\tilde{\phi}_{r,\epsilon}(z) - \sum_{p \in 2\pi \mathbb{Z}^n, \langle r, p \rangle \neq 0} \frac{a_p[H[\tilde{\phi}_{r,\epsilon}]]}{i\langle r, p \rangle} \exp(i\langle z, p \rangle) - 1\| < \epsilon.$$ 

By Lemma 4.6, $\phi_{r,\omega,\epsilon} \in K_{r,\omega}$ then

$$\|\tilde{\phi}_{r,\epsilon}\|_\infty \leq \|\phi_{r,\omega,\epsilon}\|_\infty < 2\beta \tau(f, r).$$

By Lemma 4.4, the set $K_{r,\omega}$ is a subset of $E_\omega(\mathbb{R}^n)$. Then $\tilde{\phi}_{r,\epsilon}$ is a $C^\infty$ periodic modulo $\mathbb{Z}^n$ function.

5. Proof of the main result

Proof of main results: Consider System (1) where $f$ is a polynomial function. There exists $q \in \mathbb{N}^*$ such that $\|f\|_\infty < q$. Use the change of variables

$$x_q(t) = x(t) - x_0 + qt, \quad \forall t \in \mathbb{R},$$

we get

$$\frac{dx_q}{dt}(t) = f(x_q(t) + x_0 - qt) + q, \quad t \in \mathbb{R}, \quad x_q(0) = 0. \hspace{1cm} (12)$$

where $1 := (1, \ldots, 1) \in \mathbb{R}^n$. Now, transform the last system to an autonomous systems. Define the functions $x_{n+1} : \mathbb{R} \to \mathbb{R}$ as the identity function: $x_{n+1}(t) := t$ for every $t \in \mathbb{R}$, the system (12) can be written as

$$\begin{align*}
\dot{x}_q &= f(x_q + x_0 - qx_{n+1}) + q, \quad t \in \mathbb{R}, \quad x_q(0) = 0, \\
\dot{x}_{n+1} &= 1, \quad t \in \mathbb{R}, \quad x_{n+1}(0) = 0.
\end{align*}$$
In other words,

\[ \hat{x} = f_q(\hat{x}), \quad t \in \mathbb{R}, \quad \hat{x} = (x_q, x_{n+1}), \quad \hat{x}(0) = (0, 0), \]

where \( f_q : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) satisfies

\[ f_q(z) := \left( f(z + x_0 - qz_{n+1}) + 1 \right), \quad \forall z := (z, z_{n+1}) \in \mathbb{R}^{n+1}. \]

Since \( q \in \mathbb{N}^* \) then \( f_q \) is a polynomial trigonometric function. In addition, \( \min_{z \in \mathbb{R}^{n+1}} f_q(z) \geq \min\{q - \|f\|_{\infty}, 1\} > 0 \). Without loss of generality, we consider the system (1) by supposing that \( x(0) = 0 \) and \( f(z) \neq 0 \) for all \( z \in \mathbb{R}^n \). Let \( (\epsilon_k)_k \subset (0, 1) \) be a sequence satisfying \( \lim_{k \to \infty} \epsilon_k = 0 \). For every \( k \geq 1 \) let \( \gamma_k \in \mathbb{R}^n \to \mathbb{Q}^n/\{0\} \) be a function satisfying

\[ \forall y \in \mathbb{R}^n : \quad \|y - \gamma_k(y)\| < \epsilon_k. \]

We have \( \gamma_k(y) \in \mathbb{Q}^n/\{0\} \). For every \( y \in \mathbb{R}^n \) and every \( k \geq 1 \) consider the function \( \phi_{\gamma_k(y)} \) satisfying the main proposition such that

\[ \sup_{z \in \mathbb{R}^n} \left\| \phi_{\gamma_k(y)}(z) - \sum_{p \in 2\pi \mathbb{Z}^n, (\gamma_k(y), p) \neq 0} \frac{a_p[H[\phi_{\gamma_k(y)}]]}{i(\gamma_k(y), p)} (\exp(i(z, p)) - 1) \right\| < \frac{1}{k}, \]

Define the recurrent sequence \( (\rho_k)_k \subset \mathbb{R}^n \) as

\[ \rho_0 = 0, \quad \rho_{k+1} := \sum_{p \in 2\pi \mathbb{Z}^n, (\gamma_k(\rho_k), p) = 0} a_p[H[\phi_{\gamma_k(\rho_k)}]], \quad \forall k \geq 0. \]

Prove that the sequence \( (\rho_k)_k \) is bounded. Let be \( \psi_k : \mathbb{R} \to \mathbb{R}^n \) the function defined by \( t \mapsto \psi_k(t) := \phi_{\gamma_k(\rho_k)(\gamma_k(\rho_k))} \). By the main proposition, we get

\[ \sup_{t \in \mathbb{R}} \left\| \psi_k(t) - \sum_{p \in 2\pi \mathbb{Z}^n, (\gamma_k(\rho_k), p) \neq 0} \frac{a_p[H[\phi_{\gamma_k(\rho_k)}]]}{i(\gamma_k(\rho_k), p)} (\exp(i(\gamma_k(\rho_k), p)) - 1) \right\| < \frac{1}{k}, \]

since the sum is normally convergent that implies

\[ \sup_{t \in \mathbb{R}} \left\| \psi_k(t) - \int_0^t \sum_{p \in 2\pi \mathbb{Z}^n, (\gamma_k(\rho_k), p) \neq 0} \frac{a_p[H[\phi_{\gamma_k(\rho_k)}]]}{i(\gamma_k(\rho_k), p)} \exp(i(\gamma_k(\rho_k), p)) ds \right\| < \frac{1}{k}. \]

By Equation (3), we have the following Fourier development:

\[ f(z + \phi_{\gamma_k(\rho_k)}(z)) = \sum_{p \in 2\pi \mathbb{Z}^n} a_p[H[\phi_{\gamma_k(\rho_k)}]] \exp(i(z, p)), \]

then

\[ \sup_{t \in \mathbb{R}} \left\| \psi_k(t) - \int_0^t f(\gamma_k(s)) ds - t\rho_{k+1} \right\| < \frac{1}{k}. \quad (13) \]

Since \( \|\psi_k\|_{\infty} < \infty \) then

\[ \|\rho_{k+1} - \lim_{t \to \infty} \frac{1}{t} \int_0^t f(\gamma_k(s)) ds\| = 0. \]

we deduce that \( \limsup_{k \to \infty} \|\rho_k\| \leq \|f\|. \) There exists \( \rho \in \mathbb{R}^n \) and a sub-sequence \( (\rho_{k_i})_i \) which converge to \( \rho \). To simplify the notation, we suppose that \( (\rho_k)_k \) converge to \( \rho \). Since
\[ \epsilon_k \to 0 \text{ then } \lim_{k \to \infty} \rho_k = \lim_{k \to \infty} \gamma_k(\rho_k) = \rho. \]

We have supposed that \( f(z) \neq 0 \) for every \( z \in \mathbb{R}^n \), then \( \rho \neq 0 \). There exists \( c > 0 \) and \( k_0 \geq 0 \) such that

\[ \tau(f, \gamma_k(\rho_k)) < c, \quad \forall k \geq k_0. \]

By the main proposition, we obtain

\[ \sup_{k \geq k_0} \| \phi_{\gamma_k(\rho_k)} \|_{\infty} \leq 2\beta \sup_{k \geq k_0} \tau(f, \gamma_k(\rho_k)) < 2\beta c. \]

Now, prove that the sequence functions \( (\psi_k) \) converge uniformly on every interval \([0, T]\).

Since \( f \) is a polynomial trigonometric function, then there exist \( \eta > 0 \) such that \( f \) is uniformly \( \eta \)-Lipschitz function. For every \( T > 0 \) we have

\[ \sup_{t \in [0,T]} \| \psi_{k_2}(t) - \psi_{k_1}(t) \| = \sup_{t \in [0,T]} \| \exp(2\eta t) \exp(-2\eta t) \psi_{k_2}(t) - \psi_{k_1}(t) \| \leq \exp(2\eta T) \| \psi_{k_2} - \psi_{k_1} \|_T, \quad \forall k_1, k_2 \in \mathbb{N}, \]

where

\[ \| \psi_{k_2} - \psi_{k_1} \|_T := \sup_{t \in [0,T]} \| \exp(-2\eta t) [\psi_{k_2}(t) - \psi_{k_1}(t)] \|. \]

It is sufficient to prove that

\[ \lim_{k_2, k_1 \to +\infty} \| \psi_{k_2} - \psi_{k_1} \|_T = 0. \]

By Equation (13)

\[ \| \psi_{k_2} - \psi_{k_1} \|_T \leq \eta \sup_{t \in [0,T]} \left[ \exp(-2\eta t) \int_0^t s \| \gamma_{k_2}(\rho_{k_2}) - \gamma_{k_1}(\rho_{k_1}) \| ds \right] \]

\[ \quad + \eta \sup_{t \in [0,T]} \left[ \exp(-2\eta t) \int_0^t \| \psi_{k_2}(s) - \psi_{k_1}(s) \| ds \right] \]

\[ \quad + \sup_{t \in [0,T]} \left[ \exp(-2\eta t) t \| \rho_{k_2+1} - \rho_{k_1+1} \| + \frac{1}{k_2} + \frac{1}{k_1} \right] \]

\[ \leq \eta T^2 \| \gamma_{k_2}(\rho_{k_2}) - \gamma_{k_1}(\rho_{k_1}) \| + \frac{1}{2} \| \psi_{k_2} - \psi_{k_1} \|_T \]

\[ + T \| \rho_{k_2+1} - \rho_{k_1+1} \| + \frac{1}{k_2} + \frac{1}{k_1}. \]

Then

\[ \frac{1}{2} \| \psi_{k_2} - \psi_{k_1} \|_T \leq \eta T^2 \| \gamma_{k_2}(\rho_{k_2}) - \gamma_{k_1}(\rho_{k_1}) \| \]

\[ + T \| \rho_{k_2+1} - \rho_{k_1+1} \| + \frac{1}{k_2} + \frac{1}{k_1} \to 0, \text{ when } k_2, k_1 \to +\infty. \]
We deduce that the sequence function \((\psi_k)_k\) is a Picard iteration \([6], \text{Chapter 4}\) for the solution of the differential equation

\[
\dot{x} = f(x), \quad x(0) = 0,
\]

there exists a weakly almost periodic function \(\psi^*_\rho : \mathbb{R} \to \mathbb{R}^n\) of slope \(\rho\) such that

\[
\psi^*_\rho(t) = \lim_{k \to \infty} \psi_k(t) = \int_0^t \lim_{k \to \infty} f(y_k(\rho_k)s + \psi_k(t)) \, ds - t \lim_{k \to \infty} \rho_k
\]

\[
= \int_0^t f(\rho s + \psi^*_\rho(s)) \, ds - t \rho, \quad \forall t \in \mathbb{R}.
\]

By uniqueness of solution of differential equation, we have proved that

\[
x(t) = \rho t + \psi^*_\rho(t), \quad \forall t \in \mathbb{R}. \tag{14}
\]

The rotation vector \(\rho\) of the solution \(x\) is unique. In fact, suppose that there exists \(\tilde{\rho} \in \mathbb{R}^n\) such that \(\tilde{\rho} \neq \rho\) and such that

\[
\sup_{t \geq 0} \|x(t) - \tilde{\rho} t\| < +\infty.
\]

Denote

\[
y_\rho(t) := x(t) - \tilde{\rho} t, \quad \forall t \geq 0.
\]

By Equation (14), we get

\[
\psi^*_\rho(t) - y_\rho(t) = -(\rho - \tilde{\rho}) t, \quad \forall t \geq 0.
\]

Implies

\[
+\infty = \sup_{t \geq 0} \|(\rho - \tilde{\rho}) t\| \leq \sup_{t \geq 0} \|\psi^*_\rho(t)\| + \sup_{t \geq 0} \|y_\rho(t)\| < +\infty.
\]

We obtain a contradiction. We deduce that the rotation vector \(\rho\) of the solution \(x\) is unique. \(\blacksquare\)

6. Conclusion

We have proved that any solution \(x\) of ODE defined by a trigonometric polynomial field can be approximated by a sequence functions \(t \mapsto \rho_k t + \psi_k(t)\) where \((\rho_k)_k \subset \mathbb{Q}^n\) and converge to the rotation vector of \(x\). The functions \(\psi_k : \mathbb{R} \to \mathbb{R}^n\) are periodic on \(t\) and uniformly bounded.

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Appendices

Appendix 1

**Proof of Lemma 3.2:** By hypothesis, for $q = 0$ we have
\[ \sum_{p \in 2\pi \mathbb{Z}^n / \{0\}} \|c_p\| < +\infty, \]
the series is normally convergent and we have
\[ \forall p \in 2\pi \mathbb{Z}^n : g(\omega z) \exp(-i(z, p)) = \sum_{q \in 2\pi \mathbb{Z}^n} c_q \exp(i(z, q - p)), \quad c_q \in \mathbb{C}^n, \]
implies
\[ a_p[g_\omega] = \int_0^1 \ldots \int_0^1 g(\omega z) \exp(-i(z, p)) dz_1 \ldots dz_n \]
\[ = \sum_{q \in 2\pi \mathbb{Z}^n} c_q \int_0^1 \ldots \int_0^1 \exp(i(z, q - p)) dz_1 \ldots dz_n = c_p. \quad (A1) \]
Now, prove that $g \in E_\omega(\mathbb{R}^n)$. Denote
\[ \theta_p(z) := \exp \left( i \frac{1}{\omega} (z, p) \right), \quad \forall z \in \mathbb{R}^n, \]
It is sufficient to prove that for every $q \geq 1$ we have
\[ S_q := \sum_{p \in 2\pi \mathbb{Z}^n / \{0\}} \|c_p\| \|d^q \theta_p\|_\infty < +\infty, \]
where \( \frac{dq}{g} \) is the \( q \)th differential of \( g \). The function \( \frac{dq}{g} \) is defined as

\[
\frac{dq}{g} = \sum_{p \in 2\pi \mathbb{Z}^n} c_p \frac{1}{\omega q} p_{k_1} \ldots p_{k_q} \exp(i \langle z, p \rangle),
\]

We have

\[
\forall p := (p_j)_{j=1}^n \in 2\pi \mathbb{Z}^n:
\]

\[
d_{\omega} \theta_p = \sum_{k_1=1}^n \ldots \sum_{k_q=1}^n \frac{1}{\omega q} p_{k_1} \ldots p_{k_q} \exp(i \langle z, p \rangle),
\]

By consequence,

\[
\|d_{\omega} \theta_p\|_{\infty} = \left( \sum_{k=1}^n \frac{|p_k|}{\omega} \right)^q \leq n^q \left( \frac{\|p\|}{\omega} \right)^q.
\]

Thanks to Equation (A1), we get

\[
\forall q \geq 1: S_q \leq n^q \sum_{p \in 2\pi \mathbb{Z}^n/\{0\}} \|c_p\| \left( \frac{\|p\|}{\omega} \right)^q = n^q \sum_{p \in 2\pi \mathbb{Z}^n/\{0\}} \|a_p[g_\omega]\| \left( \frac{\|p\|}{\omega} \right)^q = \frac{1}{n^q} \|g\|_{\omega,q} < +\infty,
\]

which implies that \( g \in E_\omega(\mathbb{R}^n) \).

**Appendix 2**

**Proof of Lemma 3.3:** By Equation (3), we have

\[
g(z) = \sum_{p \in 2\pi \mathbb{Z}^n} a_p[g_\omega] \exp(i \frac{1}{\omega} \langle z, p \rangle),
\]

Since \( g(0) = 0 \), then

\[
a_0[g_\omega] = - \sum_{p \in 2\pi \mathbb{Z}^n/\{0\}} a_p[g_\omega],
\]

implies

\[
\|a_0[g_\omega]\| \leq \sum_{p \in 2\pi \mathbb{Z}^n/\{0\}} \|a_p[g_\omega]\| = \frac{1}{2} \|g\|_{\omega,0}.
\]

Since

\[
\|g\|_{\infty} \leq \sum_{p \in 2\pi \mathbb{Z}^n} \|a_p[g_\omega]\|.
\]

We deduce that

\[
\|g\|_{\infty} \leq \|g\|_{\omega,0}.
\]

**Appendix 3**

**Proof of Lemma 3.4:** Since \( g : \mathbb{R}^n \to \mathbb{R}^n \) is a trigonometric polynomial, then it is \( C^\infty \) and there exists \( m \in \mathbb{N} \) such that

\[
g(z) = \sum_{p \in 2\pi \mathbb{Z}^n, \|p\| \leq m} a_p[g] \exp(i \langle z, p \rangle), \quad \forall z \in \mathbb{R}^n.
\]
Implies
\[ g_\omega^{1/2}(z) = g\left(\frac{z}{\omega}\right) = \sum_{p \in 2\pi \mathbb{Z}^n, \|p\| \leq m} a_p [g] \exp\left(i \frac{1}{\omega} \langle z, p \rangle \right). \]

Denote
\[ \theta_p(z) := \exp\left(i \frac{1}{\omega} \langle z, p \rangle \right), \quad \forall z \in \mathbb{R}^n. \]

Denote \( d^k g \) is \( k^{th} \) differential of \( g \), which implies
\[ d^k g_\omega^{1/2}(z) = \sum_{p \in 2\pi \mathbb{Z}^n, \|p\| \leq m} a_p [g] d^k \theta_p. \]

We have
\[ \forall p := (p_j)_{j=1}^n \in 2\pi \mathbb{Z}^n : d^k \theta_p = \sum_{s_1=1}^n \ldots \sum_{s_k=1}^n \prod_{j=1}^k p_{s_j} \frac{1}{\omega} \exp\left(i \frac{1}{\omega} \langle z, p \rangle \right), \]

then
\[ \forall p \in 2\pi \mathbb{Z}^n, \|p\| \leq m : \|d^k \theta_p\|_\infty = \left(\sum_{s=1}^n p_{s} \frac{1}{\omega}\right)^k \leq n^k \|m\| \omega^k. \]

Since \((g_\omega^{1/2})_\omega = g\), we obtain
\[ \|d^k g_\omega^{1/2}\|_{\omega, q} \leq 2n^k \left(\frac{m}{\omega}\right)^k \sum_{p \in 2\pi \mathbb{Z}^n, \|p\| \leq m} \|a_p [g]\| \frac{p}{\omega}^q \]
\[ \leq 2n^k \left(\frac{m}{\omega}\right)^{k+q} \sum_{p \in 2\pi \mathbb{Z}^n, \|p\| \leq m} \|a_p [g]\|. \]

It is sufficient to choose \( \beta := 2 \max\{\sum_{p \in 2\pi \mathbb{Z}^n, \|p\| \leq m} \|a_p [g]\|, m\}. \)

\section*{Appendix: 4}

\textbf{Proof of Lemma 3.5:} Since \((h_j)_{j=1}^k \subset E_\omega(\mathbb{R}^n), \) by Equation (3) we can write
\[ h_j(z) = \sum_{p_j \in 2\pi \mathbb{Z}^n} a_{p_j} [h_{j,\omega}] \exp\left(i \frac{1}{\omega} \langle z, p_j \rangle \right). \]

By definition of the seminorm,
\[ \|h_j\|_{\omega, q} = 2 \sum_{p_j \in 2\pi \mathbb{Z}^n / \{0\}} \|a_{p_j} [h_{j,\omega}]\| \left\| \frac{p_j}{\omega} \right\|_q. \quad (A2) \]

We have
\[ \Pi_{j=1}^k h_j(z) = \sum_{p_1, \ldots, p_k \in 2\pi \mathbb{Z}^n} \Pi_{j=1}^k a_{p_j} [h_{j,\omega}] \exp\left(i \frac{1}{\omega} \left( z, \sum_{j=1}^k p_j \right) \right) \]
\[ = \sum_{v \in 2\pi \mathbb{Z}^n} \sum_{\sum_{j=1}^k h_j = v} \Pi_{j=1}^k a_{p_j} [h_{j,\omega}] \exp\left(i \frac{1}{\omega} \langle z, v \rangle \right). \]
Then
\[
\|
\Pi_{j=1}^k h_j\|_{\omega,q} = 2 \sum_{v \in 2\pi \mathbb{Z}^n / \{0\}} \left| \sum_{\sum_{j=1}^k p_j = v} \Pi_{j=1}^k a_{p_j} [h_{j,\omega}] \right| \| v \|_\omega^q
\]
\[
\leq 2 \sum_{v \in 2\pi \mathbb{Z}^n} \left| \sum_{\sum_{j=1}^k p_j = v} \Pi_{j=1}^k a_{p_j} [h_{j,\omega}] \right| \| v \|_\omega^q.
\]
Using the triangular inequality, we obtain
\[
\|
\Pi_{j=1}^k h_j\|_{\omega,q} \leq 2 \sum_{p_1, \ldots, p_k \in 2\pi \mathbb{Z}^n} \|
\Pi_{j=1}^k a_{p_j} [h_{j,\omega}]\| \left| \frac{1}{\omega} \sum_{j=1}^k p_j \right|^q.
\]
Since
\[
\left| \sum_{j=1}^k p_j \right| \leq k \Pi_{\| p_j \| \neq 0} \| p_j \|, \quad \forall p_j, \in 2\pi \mathbb{Z}^n,
\]
then
\[
\left| \frac{1}{\omega} \sum_{j=1}^k p_j \right|^q \leq k^q \omega^{(k-1)q} \Pi_{\| p_j \| \neq 0} \| p_j \|_\omega^q, \quad \forall p_j, \in 2\pi \mathbb{Z}^n.
\]
We deduce that
\[
\|
\Pi_{j=1}^k h_j\|_{\omega,q} \leq 2k^q \omega^{(k-1)q} \sum_{p_1, \ldots, p_k \in 2\pi \mathbb{Z}^n} \left( \Pi_{\| p_j \| \neq 0} \| p_j \|_\omega^q \right) \left( \Pi_{\| p_j \| \neq 0} \| a_{p_j} [h_{j,\omega}] \| \right)
\]
\[
= 2k^q \omega^{(k-1)q} \sum_{p_1, \ldots, p_k \in 2\pi \mathbb{Z}^n} \left( \Pi_{\| p_j \| = 0} \| a_{p_j} [h_{j,\omega}] \| \right) \Pi_{\| p_j \| \neq 0} \left[ \left| \frac{p_j}{\omega} \right|^q a_{p_j} [h_{j,\omega}] \| \right].
\]
By Equation (A2),
\[
\|
\Pi_{j=1}^k h_j\|_{\omega,q} \leq \frac{1}{2^{k-1}} k^q \omega^{(k-1)q} \Pi_{j=1}^k \left[ 2 \left( \| a_0 [h_{j,\omega}] \| + \sum_{p_j \in 2\pi \mathbb{Z}^n / \{0\}} \| p_j \|_\omega^q a_{p_j} [h_{j,\omega}] \| \right) \right].
\]