Transport properties of a quantum wire in the presence of impurities and long-range Coulomb forces

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Abstract

One-dimensional electron systems interacting with long-range Coulomb forces (quantum wires) show a Wigner crystal structure. We investigate in this paper the transport properties of such a Wigner crystal in the presence of impurities. Contrary to what happens when only short-range interactions are included, the system is dominated by $4k_F$ scattering on the impurities. There are two important length scales in such a problem: one is the pinning length above which the (quasi-)long-range order of the Wigner crystal is destroyed by disorder. The other length $\xi_{cr}$ is the length below which Coulomb interactions are not important and the system is behaving as a standard Luttinger liquid with short-range interactions. We obtain the frequency and temperature dependence of the conductivity. We show that such a system is very similar to a classical charge density wave pinned by impurities, but with important differences due to quantum fluctuations and long-range Coulomb interactions. Finally we discuss our results in comparison with experimental systems.
I. INTRODUCTION

Recently nanostructure technology has made it possible to create quasi one-dimensional electronic structures, the so-called quantum wires. Experimentally situations have been reached where the width of such a wire is of the order of the Fermi wavelength of the conduction electrons, which makes it a good realization of a one-dimensional electron gas.

In such a system one expects the electron-electron interactions to play an important role. In particular, at variance with what happens in other one-dimensional conductors, the long-range Coulomb interaction in a quantum wire is not screened since the wire contains only one channel of electrons. One can therefore expect very different physical properties than those of a Luttinger liquid with short-range interactions. It has been proposed that due to these long-range interactions, the electrons in a quantum wire will form a Wigner crystal. The formation of a Wigner crystal can be described as a modulation of the charge density \( \rho(x) \approx \rho_0 \cos(Qx + 2\sqrt{2}\Phi) \) where \( \rho_0 \) is the uniform amplitude of the charge density, \( Q = 4k_F \) its wave vector and \( \Phi \) describes the location and the motion of the Wigner crystal. The existence of such a Wigner crystal should have observable consequences on the transport properties of the system. Indeed, in the presence of impurities the Wigner crystal will be pinned: the phase \( \Phi(x) \) adjusts to the impurity potential on a scale given by \( L_0 \) called the pinning length. This process of pinning is analogous to what happens in charge density waves in the presence of impurities. Since, in the presence of long-range Coulomb interactions, the most divergent fluctuation is now \( 4k_F \) density fluctuation, the transport properties are dominated by \( 4k_F \) scattering on impurities, and not the usual \( 2k_F \) scattering, as was assumed previously. Due to the \( 4k_F \) scattering, one can also expect different transport properties than those of a Luttinger liquid with short-range interactions where \( 2k_F \) fluctuations are the dominant one.

Non linear \( I - V \) curves have been observed experimentally which could be interpreted as the result of pinning. Up to now only short wires have been made, for which only few impurities are in the wire and dominate the transport. Even in that case what has mainly been focussed on theoretically is a system with short-range interactions. For long wires, it is important to consider the case of a uniform disorder, e.g. a thermodynamic number of impurities, as well as the long-range Coulomb forces.

In this paper we study the effects of disorder on the transport properties of a quantum wire. Although the problem is very close to that of a charge density wave pinned by impurities, there are important differences that are worth investigating. Due to the long-range nature of the forces one can expect modifications of the pinning length and frequency dependence of the conductivity. In addition quantum fluctuations have to be taken into account, and for the case of short-range forces are known to drastically affect the transport properties compared to a classical situation.

The plan of the paper is as follows. The model is derived in section II. Effects of the pinning and the pinning length are studied in section III. The frequency dependence of the conductivity is computed in section IV and the temperature dependence of the conductivity (and conductance) is discussed in section V. Discussion of the comparison with experiments and conclusions can be found in section VI. Some technical details about the treatment of quantum fluctuations can be found in the appendices.
II. MODEL

We consider a gas of electrons confined in a channel of length $L$, with a width $d \ll L$ and a thickness $e \ll d \ll L$. We will assume that both $d$ and $e$ are small enough for the system to be regarded as one-dimensional, meaning that only one band is filled in the energy-spectrum of the electrons. Such a situation will be realized when $d$ and $e$ become comparable to the Fermi wavelength. In the following we will therefore keep only the degrees of freedom along the wire. Since we are interested only in low energy excitations we can linearize the spectrum around the Fermi points and take for the free part of the Hamiltonian:

$$H_0 = v_F \sum_k (k - k_F) a_{+,k}^\dagger a_{+,k} + (-k - k_F) a_{-,k}^\dagger a_{-,k}$$

(1)

where $v_F$ is the Fermi velocity and $a_{+,k}^\dagger$ ($a_{-,k}^\dagger$) is the creation operator of an electron on the right(left)-going branch with wave-vector $k$. In addition we assume that the electrons interact through the Coulomb interaction

$$H_c = \frac{1}{2} \int_0^L \int_0^L dx dx' V(x - x') \rho(x) \rho(x') = \frac{1}{2L} \sum_k V_k \rho_k \rho_{-k}$$

(2)

In a strictly one-dimensional theory, a $\frac{1}{r}$ Coulomb potential has no Fourier transform because of the divergence for $r \to 0$. In the real system such a divergence does not exist owing to the finite width $d$ of the wire. We will use for $V(r)$ the following approximate form\cite{2} which cuts the singularity at $r \approx d$, and gives the correct asymptotic behavior at large $r$

$$V(r) = \frac{e^2}{\sqrt{r^2 + d^2}}$$

(3)

the Fourier transform of which is

$$V(q) = \int_{-L/2}^{L/2} dr V(r) e^{iqr} \approx 2e^2 K_0(qd)$$

(4)

where $K_0$ is a Bessel function, and one has assumed the wire to be long enough $L \to \infty$. In the following we shall frequently use the asymptotic expression

$$K_0(qd) \approx -\ln(d) \quad \text{when} \quad qd \ll 1$$

(5)

The model (1) plus (2) has been studied by Schulz\cite{3}, who showed that the system is dominated by $4k_F$ charge density wave fluctuations, which decay as

$$\langle \rho_{4k_F} (x) \rho_{4k_F} (0) \rangle \sim e^{-\ln^{1/2}(x)}$$

(6)

The presence of such a $4k_F$ charge fluctuation can be viewed as the formation of a Wigner crystal. In order to describe the pinning of such a Wigner crystal we add to the hamiltonian (1) and (2) the contribution due to impurities. We assume that impurities are located in the wire at random sites $X_j$, and that each impurity acts on the electrons with a potential $V_{imp}$. We will assume in the following that the potential due to the impurities is short-ranged, and will replace it by a delta function.
\[ V_{\text{imp}}(x - X_j) = V_0 \delta(x - X_j) \] (7)

The part of the Hamiltonian stemming from a particular configuration of the impurities is then

\[ H_{\text{imp}} = \sum_j \int_0^L V_0 \delta(x - X_j) \rho(x) = \sum_j V_0 \rho(X_j) \] (8)

In order to treat the problem we use the representation of fermion operators in term of boson operators.\(^{21,22}\) One introduces the phase field

\[ \Phi(x) = -\frac{i \pi}{L} \sum_{k \neq 0} \frac{1}{k} e^{-ikx}(\rho_{+k} + \rho_{-k}) \] (9)

where \(\rho_{+k}(\rho_{-k})\) are the charge density operators for right(left)-moving electrons, and \(\Pi\), the momentum density conjugate to \(\Phi\). The boson form for (1) plus (2) is

\[ H_0 + H_c = u \frac{2}{2\pi} \int_0^L dx [K(\pi \Pi)^2 + \frac{1}{K}(\partial_x \Phi)^2] + \frac{1}{\pi^2} \int_0^L \int_0^L dxdx' V(x - x')(\partial_x \Phi(x))(\partial_x \Phi(x')) \] (10)

\(K\) is a number containing the backscattering effects due to the Fourier components of the interaction close to \(2k_F\) and \(u\) is the renormalized Fermi velocity due to the same interactions.\(^{21,22}\) We have taken \(\hbar = 1\) in (10). The long-range nature of the Coulomb interaction manifests itself in the last term of (10). As we shall precise in the following, both \(K\) and the Coulomb potential \(V\) control the strength of quantum effects.

Since for (10) the most divergent fluctuation corresponds to a \(4k_F\) charge modulation,\(^8\) we will consider only the coupling of the impurities with this mode and ignore the \(2k_F\) part of the charge density. The range of validity of such an approximation will be discussed in the following. Using the boson representation of the density\(^{21,22}\) and impurity Hamiltonian (8), the total Hamiltonian becomes

\[ H = u \frac{2}{2\pi} \int_0^L dx [K(\pi \Pi)^2 + \frac{1}{K}(\partial_x \Phi)^2] + \sum_j V_0 \rho_0 \cos(4k_F X_j + 2\sqrt{2} \Phi(X_j)) \\
+ \frac{1}{\pi^2} \int_0^L \int_0^L dxdx' V(x - x')(\partial_x \Phi(x))(\partial_x \Phi(x')) \] (11)

where \(\rho_0\) is the average density of electrons. The Hamiltonian (11) has similarities with the phase Hamiltonian of a pinned charge density wave.\(^{10}\) Similarly to the CDW case one can expect the phase to distort to take advantage of the impurity potential, leading to the pinning of the Wigner crystal. As for standard CDW, one has to distinguish between strong and weak pinning on the impurities.\(^{10}\) In the first case the phase adjusts itself on each impurity site. This corresponds to a strong impurity potential or dilute impurities. In the weak pinning case, the impurity potential is too weak or the impurities too close for the phase to be able to adjust on each impurity site, due to the cost in elastic energy. Although the problem has similarities with the CDW problem, there are two important a priori physical differences that have to be taken into account: compared to the CDW case, one has to take
into account the long-range Coulomb interaction. One can expect such an interaction to make the Wigner crystal more rigid than a CDW and therefore more difficult to pin. In addition, for the Wigner crystal, one cannot neglect the quantum term \((\Pi^2)\) as is usually done for the CDW problem owing to the large effective mass of the CDW. In the absence of long-range interactions such a term is known to give important quantum corrections on both the pinning length and the conductivity.

In the following sections we will examine both cases of strong and weak pinning.

III. CALCULATION OF THE PINNING LENGTH

Let us first compute the pinning length \(L_0\) over which the phase \(\Phi(x)\) in the ground state varies in order to take advantage of the impurity potential. If the impurities are dilute enough, or the impurity potential strong enough, the phase \(\Phi(x)\) adjusts on each impurity site such that \(\cos(4k_F x_j + 2\sqrt{2} \Phi(X_j)) = -1\). This is the so-called strong pinning regime where the pinning length is the distance between impurities \(L_0 = n_i^{-1}\). If the impurities are dense enough, or their potential weak enough then the cost of elastic and Coulomb energy in distorting the phase has to be balanced with the gain in potential energy. One is in the weak pinning regime where the pinning length can be much larger than the distance between impurities. In this regime, we calculate \(L_0\) using Fukuyama and Lee’s method developed for the CDW. This method neglects the quantum fluctuations of the phase, and the effect of such fluctuations will be discussed at the end of this section. One assumes that the phase \(\Phi\) varies on a scale \(L_0\). One can therefore divide the wire in segments of size \(L_0\) where the phase is roughly constant and takes the optimal value to gain the maximum pinning energy. \(L_0\) is determined by optimizing the total gain in energy, equal to the gain in potential energy minus the cost in elastic and Coulomb energy. If one assumes that the phase varies of a quantity of order \(2\pi\) over a length \(L_0\), the cost of elastic energy per unit length is

\[
\mathcal{E}_{el} = \frac{u}{2\pi K \alpha L_0^2} \tag{12}
\]

where \(\alpha\) is a number of order unity depending on the precise variation of the phase. Since the impurity potential varies randomly in segments of length \(L_0\), the gain per unit length due to pinning is

\[
\mathcal{E}_{imp}(L_0) = -V_0 \rho_0 \left(\frac{n_i}{L_0}\right)^\frac{1}{2} \tag{13}
\]

In our case we also have to consider the cost in Coulomb energy.

\[
\mathcal{E}_{coul} = \frac{1}{L \pi^2} \int_0^L dx \int_0^L dx' V(x-x') \langle \partial_x \Phi \rangle_{av} \langle \partial_{x'} \Phi \rangle_{av} \tag{14}
\]

where the subscript \(av\) indicates that the quantity is averaged over all impurity configurations. Since one assumes that the phase varies of a quantity of order \(2\pi\) over a length \(L_0\), the phases for electrons distant of more than \(L_0\) are uncorrelated, so that the interactions between such pairs of electrons do not contribute to the energy. The calculation can thus be reduced to the evaluation of the energy for a segment of length \(L_0\)
\[ E_{\text{coul}} \approx \frac{1}{\pi^2} \frac{1}{L_0} L_0 \int_{-L_0}^{L_0} du V(u) \frac{\langle \Phi^2(x) \rangle_{av}}{L_0^2} = \frac{2e^2}{\pi^2 \alpha L_0^2} \ln \frac{L_0}{d} \] (15)

where \( \alpha \) is the constant introduced in (12).

The minimization of the total energy provides a self-consistent expression for \( L_0 \):

\[ L_0 = \left( \frac{8e^2}{\alpha \pi^2 V_0 \rho_0 n_i^2} \right)^{\frac{1}{3}} \ln^{\frac{1}{3}} \left( \frac{C L_0}{d} \right) \] (16)

where \( C \) is a constant of order one

\[ C = e^{\left( \frac{\pi u}{4K} \right)^{\frac{1}{2}}} \] (17)

Taking typical values \( u = 3 \times 10^7 \text{cm.s}^{-1} \) so that \( \hbar u = 3.15 \times 10^{-20} \text{ e.s.u.} \) and \( K = 0.5 \), one gets \( C \approx 0.75 \). For these typical values of the parameters the contribution of the elastic (short-range) part of the hamiltonian to that result is negligible compared to that of the Coulomb term. In the following, since we expect \( \frac{L_0}{d} \gg 1 \) we approximate \( \ln CL_0 \sim \ln L_0 \).

Neglecting log(log) corrections, one can solve (16) to get

\[ L_0 = \left( \frac{8e^2}{\alpha \pi^2 V_0 \rho_0 n_i^2} \right)^{\frac{1}{3}} \ln^{\frac{1}{3}} \left( \frac{8e^2}{\alpha \pi^2 V_0 \rho_0 n_i^{1/2}} \right)^{\frac{2}{3}} \] (18)

Compared to the pinning length of a CDW, \( L_0 \approx (\frac{u^2}{\alpha \pi V_0 \rho_0 n_i^2})^{2/3} \), the pinning length (18), is enhanced by a logarithmic factor. This is due to the Coulomb interaction which enhances the rigidity of the system and makes it more difficult to pin than a classical CDW.

The expression (18) has been derived for the weak pinning case where \( L_0 \gg n_i^{-1} \). The crossover to the strong pinning regime occurs when the phase can adjust itself on each impurity site and \( L_0 = n_i^{-1} \). One can introduce a dimensionless quantity \( \epsilon_0 \) characterizing the two regimes

\[ \epsilon_0 = \frac{\alpha \pi^2 V_0 \rho_0}{8n_i e^2} \ln^{-1} \left( \frac{1}{d} \left( \frac{8e^2}{\alpha \pi^2 V_0 \rho_0 n_i^{1/2}} \right)^{\frac{2}{3}} \right) \] (19)

The weak pinning corresponds to \( \epsilon_0 \ll 1 \), and the strong starts at \( \epsilon_0 \simeq 1 \). Compared to a CDW where \( \epsilon_0 = \frac{V_0 \rho_0}{n_i e^2} \), the domain of weak pinning is larger due to the Coulomb interaction. This is again a consequence of the enhanced rigidity of the system that makes it more difficult to pin. To study the conductivity it is also convenient to introduce

\[ \epsilon = \frac{V_0 \rho_0}{n_i e^2} \] (20)

Indeed we have evaluated, using typical values \( d = 10^{-8} \text{m} \) and \( L_0 \simeq 10^{-6} \text{m} \), (estimated for typical wires in section [VI]), that \( \epsilon_0 \simeq 2\epsilon \) so that \( \epsilon \) can also be used as criterion to distinguish the two regimes of pinning.

Expressions (18) and (19) do not take into account the effects of quantum fluctuations. In the absence of Coulomb interactions, the quantum fluctuations drastically increase the
pinning length compared to the classical case giving a pinning length (for a $4k_F$ dominant scattering)

$$L_0 \sim (1/V_0)^{2/(3-4K)}$$  \hspace{1cm} (21)

To compute the effect of the quantum fluctuations in the presence of the Coulomb interaction we use the self-consistent harmonic approximation for the cosine term in (10)

$$\cos(Qx + 2\sqrt{2}(\Phi_{cl} + \hat{\Phi})) = e^{-4\langle\hat{\Phi}^2(x)\rangle} \cos(Qx + 2\sqrt{2}\Phi_{cl})(1 - 4\langle\hat{\Phi}^2(x)\rangle - \langle\hat{\Phi}^2(x)\rangle)$$  \hspace{1cm} (22)

where $\Phi = \Phi_{cl} + \hat{\Phi}$ and $\hat{\Phi}$ represents the quantum fluctuations around the classical solution $\Phi_{cl}$. The average $\langle\hat{\Phi}\rangle$ has to be done self consistently. Such a calculation is performed in appendix A and one obtains for the pinning length

$$L_0 = \left(\frac{8e^2}{\alpha\pi^2V_0\rho_0\gamma n_i^{1/2}}\right)^{1/3} \ln^{2/3}(\frac{CL_0}{d})$$  \hspace{1cm} (23)

where $\gamma = e^{-4\langle\Phi^2\rangle} \approx e^{\frac{-2\tilde{K}}{\sqrt{2\pi}}}\ln^{1/2}V_0$ and $\tilde{K} = \frac{\sqrt{\pi}uK}{2\sqrt{2\pi}}$ instead of (18). The quantum fluctuations can thus be taken into account by replacing $V_0$ by the effective impurity potential $V_0\gamma$. There is an increase of the pinning length due to the quantum fluctuations which can be considerable since $\gamma \ll 1$. Opposite to what happens for the case of short range interactions, there is no correction in the exponent for the pinning length. This can be traced back to the fact that the correlation functions decay much more slowly ($e^{-\ln^{1/2}(r)}$ instead of a power law), therefore the system is much more ordered and the fluctuations around the ground state are much less important. As a consequence even if one is dealing with a system of electrons, and not a classical CDW, the Coulomb interactions push the system to the classical limit where quantum fluctuations can be neglected except for the redefinition of the impurity potential $V_0 \rightarrow V_0\gamma$. Note that this effect can be very important quantitatively, since $L_0$ is very large for dilute impurities. Such a fluctuation effect also contributes to make the system more likely to be in the weak pinning regime.

We will in the following make the assumption that all quantum fluctuation effects have been absorbed in the proper redefinition of the pinning length. Such an approximation will be valid as long as one is dealing with properties at low enough frequencies. At high frequencies the effect of quantum fluctuations will again be important and will be examined in section IV C.

IV. CALCULATION OF THE CONDUCTIVITY

In order to study the transport properties, one makes an expansion around the static solution $\Phi_0(x)$ studied in section II that minimizes the total energy, assuming that the deviations $\Psi(x,t)$ are small

$$\Psi(x,t) = \Phi(x,t) - \Phi_0(x)$$  \hspace{1cm} (24)

One can expand the Hamiltonian in $\Psi(x,t)$ to quadratic order
\[
\mathcal{H}_\Psi = \frac{u}{2\pi} \int_0^L dx K (\pi \Phi)^2 + \frac{1}{K} (\partial_x \Psi)^2 + \frac{4}{\pi^2} \int_0^L \int_0^L dx dx' V(x-x') \partial_x \Psi \partial_{x'} \Psi \\
-4 \sum_j V_0 \rho_0 \cos(4k_F X_j + 2\sqrt{2} \Phi_0(X_j)) (\Psi(X_j))^2 
\]

(25)

This expansion is valid in the classical case. We assume that for the quantum problem all quantum corrections are absorbed in the proper redefinition of the pinning length \(L_0\), as explained in section III and appendix A. Such corrections do not affect the frequency dependence of the conductivity. From Kubo formula and the representation of the current in terms of the field \(\Psi\), the conductivity takes the form

\[
\sigma(\omega) = 2i\omega \left( \frac{\epsilon}{n} \right)^2 D(0,0;i\omega_n) \bigg|_{i\omega_n \rightarrow \omega+i0^+} 
\]

(26)

where \(D(q,q';i\omega_n)\) is the Green’s function of the field \(\Phi\)

\[
D(q,q';i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} \langle T_\tau \Psi_q(\tau) \Psi_{-q'}(0) \rangle 
\]

(27)

with \(\Psi_q(\tau) = e^{H\tau} \Psi_q e^{-H\tau}\), \(\beta = T^{-1}(k_B = 1)\) and \(\omega_n = 2\pi n T\), where \(T\) is the temperature, and \(T_\tau\) is the time-ordering operator. Our problem is then reduced to the evaluation of this Green function. From (25) one gets the Dyson equation

\[
D(q,q';i\omega_n) = D_0(q,i\omega_n) [\delta_{q,q'} + 8V_0 \rho_0 \sum_{q''} S(q''-q) D(q'',q';i\omega_n)] 
\]

(28)

where

\[
S(q) = \frac{1}{L} \sum_j e^{iqX_j} \cos(QX_j + 2\sqrt{2} \Phi_0(X_j)) 
\]

(29)

After averaging over all impurity configurations (28) becomes

\[
\langle D(q,q';i\omega_n) \rangle_{av} = \delta_{q,q'} D(q,i\omega_n) = \frac{1}{D_0(q,i\omega_n)^{-1} - \Sigma(q,i\omega_n)} 
\]

(30)

where the self-energy term \(\Sigma\) contains all connected contributions to \(D\), and \(D_0\) is the free Green Function

\[
D_0(q,i\omega_n) = \frac{\pi u K}{\omega_n^2 + q^2 u^2 (1 + \frac{2KV_0}{\pi u})} 
\]

(31)

In a similar fashion than for CDW we will compute the self-energy, using a self-consistent Born approximation\footnote{\[\text{ref}\]}, for the two limiting cases of strong and weak pinning.

A. Weak pinning case \((\epsilon \ll 1)\)

In that case, as for standard CDW\footnote{\[\text{ref}\]}, the self-energy can be expanded to second-order in perturbation, \(\Sigma \approx \Sigma_1 + \Sigma_2\). Indeed we easily verify that in the weak pinning case
\[ \Sigma_1 \sim \Sigma_2 \sim n_i^2 (V_0 \rho_0 / n_i)^{4/3} \], whereas for \( n \geq 1 \), \( \Sigma_{2n+1} = 0 \) and \( \Sigma_{2n} \sim n_i^2 (\frac{V_0 \rho_0}{n_i})^{2+2n} \). Since \( \frac{V_0 \rho_0}{n_i} \sim \varepsilon e^2 \ll 1 \), self-energy terms of higher order than \( \Sigma_2 \) are negligible. \( \Sigma_1 \) is easily computed as
\[ \Sigma_1 = 8V_0 \rho_0 \langle S(0) \rangle_{av} = -8V_0 \rho_0 (\frac{n_i}{L})^{1/3} \tag{32} \]
since again one can divide the wire into \( L/L_0 \) segments of length \( L_0 \), and use, as for equation \( (13) \), the random-walk argument of reference \( 10 \) which gives
\[ \frac{1}{L} \langle \sum_j \cos(QX_j + 2\sqrt{2} \Phi_0(X_j)) \rangle_{av} \approx \sqrt{\frac{n_i}{L_0}} \tag{33} \]
\( \Sigma_2 \) is given by
\[ \Sigma_2 = (8V_0 \rho_0)^2 \sum_{q''} D_0(q'', i\omega_n) \langle S(q'' - q)S(q - q'') \rangle_{av} \tag{34} \]
If one assumes that there is no interference between scattering on different impurities (single site approximation), then the exponentials in \( \langle S(q'' - q)S(q - q'') \rangle_{av} \) cancel and we find
\[ \Sigma_2 = \left( \frac{8V_0 \rho_0}{L} \right)^2 \sum_{q''} D_0(q'', i\omega_n) \langle \sum_j \cos^2(QX_j + 2\sqrt{2} \Phi_0(X_j)) \rangle_{av} \tag{35} \]
\[ = 64\frac{n_i}{2} (V_0 \rho_0)^2 \frac{1}{L} \sum_{q''} D_0(q'', i\omega_n) \]
The approximation \( \frac{1}{L} \langle \sum_j \cos^2(QX_j + 2\sqrt{2} \Phi_0(X_j)) \rangle_{av} \approx \frac{1}{2} n_i \) is valid in the weak pinning case only. A more general result is
\[ \frac{1}{L} \langle \sum_j \cos^2(QX_j + 2\sqrt{2} \Phi_0(X_j)) \rangle_{av} = \frac{1}{L} \langle \sum_j \frac{1}{2} (1 + \cos 2(QX_j + 2\sqrt{2} \Phi_0(X_j)) \rangle_{av} \]
\[ \approx \frac{1}{2} \left( n_i + \sqrt{\frac{n_i}{L_0}} \right) \tag{36} \]
but in the weak pinning case it can be simplified using \( n_i L_0 \gg 1 \).
\( \Sigma_2 \) given by \( (35) \) diverges as \( \frac{1}{|\omega_n|} \ln \frac{1}{|\omega_n|} \) when \( |\omega_n| \to 0 \), so one has to compute \( \Sigma \) self-consistently, and replace \( D_0 \) by \( D \) in the calculation of \( \Sigma_2 \). \( (35) \) is replaced by
\[ \Sigma_2 = 32n_i (V_0 \rho_0)^2 \frac{1}{L} \sum_{q''} D(q'', i\omega_n) = 32n_i (V_0 \rho_0)^2 \frac{1}{L} \sum_{q''} [D_0^{-1}(q'', i\omega_n) - \Sigma(q'', i\omega_n)]^{-1} \tag{37} \]
giving the self-consistent equation for \( \Sigma \)
\[ \Sigma = \Sigma_1 + \Sigma'_2 = -8V_0 \rho_0 \sqrt{\frac{n_i}{L_0}} + 16n_i (V_0 \rho_0)^2 (\pi u K)^2 \frac{2}{\pi} \int_0^\infty dq \int_0^\infty dq' \frac{dq}{-\omega^2 - \pi u K + q^2 u^2 (1 + \alpha \varepsilon K_0(qd))} \tag{38} \]
where we have done the analytic continuation $i\omega_n \to \omega + i0^+$ and we have noted $\alpha_c = \frac{4Kn_e^2}{\pi a}$. It is convenient to rescale (38) by the pinning frequency $\omega^*$ defined by

$$\omega^* \ln^{1/2} \frac{\tilde{u}}{\omega^*} = 16n_i u(V_0\rho_0 K)^2 \frac{1}{\sqrt{\alpha_c}} = \frac{8\pi^{5/2}(uK)^{3/2}}{e}n_i(V_0\rho_0)^2$$

(39)

where $\tilde{u} = u\sqrt{\alpha_c}$. In terms of $L_0$, (39) can be rewritten as

$$\omega^* \ln^{1/6} \frac{\tilde{u}}{\omega^*d} = 4\alpha^{-2/3}\tilde{u}L_0^{-1} \ln^{2/3} \left(\frac{L_0}{d}\right)$$

(40)

Neglecting log(log) factors, and in the limit $L_0 \gg d$ allowing to discard the constants in the logarithm ($\ln \frac{\tilde{u}}{\omega^*d} \approx \ln \frac{L_0}{d}$) we obtain

$$\omega^* \approx 4\alpha^{-2/3}\tilde{u}L_0^{-1} \ln^{1/2} \left(\frac{L_0}{d}\right)$$

(41)

Leaving aside a factor $4\alpha^{-2/3}$, $\omega^*$ given by (11) is the characteristic frequency of a segment of the wire of length $L_0$. Indeed if we modelize the wire as a collection of independent oscillators of typical length $L_0$ and use the dispersion law $\omega \sim q \ln^{1/2} q$ of the Wigner Crystal, those oscillators have the frequency $\omega_0 = \tilde{u}L_0^{-1} \ln^{1/2} \left(\frac{L_0}{d}\right)$. Numerically we find $4\alpha^{-2/3} \approx 1$ so that actually $\omega^* \approx \omega_0$. Introducing the rescaled quantities $y = \frac{\omega}{\omega^*}$ and $G = \frac{\pi K\Sigma}{\omega^*2}$, we rewrite (38) as

$$G = G_1 + G'_2$$

(42)

with $G_1 = -\alpha_1^{1/2}$ and

$$G'_2 = \ln^{1/2} \frac{\tilde{u}}{\omega^*} \sqrt{\alpha_c} \frac{2}{\pi} \int_0^\infty dt \frac{dt}{-y^2 - G + t^2(1 + \alpha_c K_0(\omega^*d t))}$$

(43)

The rescaled conductivity is:

$$\omega^* \Re \sigma(y) = -\frac{2uK_e^2}{\pi}y \Im m \left( \frac{1}{-y^2 - G} \right)$$

(44)

The full solution of (12) has to be obtained numerically, but it is possible to obtain analytically the asymptotic expressions at small and large frequencies. To evaluate the integral $G'_2$, one notices that there is a frequency $\omega_{cr}$ above which the Coulomb term will be negligible compared to the kinetic (short-range) term. $\omega_{cr}$ defines a crossover length $\xi_{cr} \sim u/\omega_{cr}$ which is roughly given by

$$\xi_{cr} \sim d e^{1/\alpha_c} = d e^{-\frac{\pi}{4Ke^2}}$$

(45)

Using a numerical estimate for $\alpha_c$ and the values of the Bessel function one gets $\alpha_c K_0(x) \sim 1$ for $x \sim 1.5$, giving a crossover frequency $\omega_{cr} \sim 1.5 \tilde{u} d \sim 10^{14} Hz$. Such a frequency is two order of magnitude larger than the pinning frequency $\omega^*$. For frequencies above $\omega_{cr}$ the system is dominated by short-range interactions: in that case the dominant fluctuations are
always the $2k_F$ charge fluctuations and not the $4k_F$ ones, and therefore the model (11) is not applicable. One has to take into account the pinning on a $2k_F$ fluctuation as done in reference [13][16]. Note that it makes sense to use a one-dimensional model to describe the behavior above $\omega_{cr}$ only if $\xi_{cr} \gg d$. This can occur for example if the short-range interactions are strong enough so that $K$ is very small. With the numerical values of $u$ that seem relevant for experimental quantum wires and assuming that $K$ is not too small $K \sim 0.5$, one gets $\xi_{cr} \sim d$. Therefore one can assert that in all the range of frequencies for which the problem can be considered as one-dimensional, Coulomb interactions will dominate. Consequently the result of the integration (43) is, when $\omega^* \sqrt{-y^2 - G} \ll \omega_{cr}$

$$G'_2 = \frac{1}{\sqrt{-y^2 - G}} \ln^{-1/2} \frac{\tilde{u}}{d\omega^* \sqrt{-y^2 - G}} \ln^{1/2} \frac{\tilde{u}}{d\omega^*}$$ (46)

Let us focus on small frequencies $\omega \ll \omega^*$. We will show that in that limit $\omega^* \sqrt{-y^2 - G} \sim \omega^* \ll \omega_{cr}$, so that we use (46) and replace in (42)

$$G = G_1 + \frac{1}{\sqrt{-y^2 - G}} \ln^{-1/2} \frac{\tilde{u}}{\omega^* d \sqrt{-y^2 - G}} \ln^{1/2} \frac{\tilde{u}}{d\omega^*} \text{ when } y \ll 1$$ (47)

$G$ tends to a limit $G_0$ verifying

$$G_0 = G_1 + (-G_0)^{-1/2} \ln^{-1/2} \left( \frac{\tilde{u}}{d\omega^*} (-G_0)^{-1/2} \right) \ln^{1/2} \frac{\tilde{u}}{d\omega^*}$$ (48)

Equation (48) has different classes of solutions depending on the value of $\alpha$ (zero, one or two roots), but the only physically relevant situation is the case of a single solution (for a discussion, see reference [10]). The corresponding value of $G_0$ is

$$G_0 \approx -2^{-2/3}$$ (49)

Expanding (48) in terms of $y$ and of $G - G_0$ around that solution, we find

$$G - G_0 = \pm i \frac{2}{\sqrt{3}} (-G_0)^{1/2} y \text{ for } y \ll 1$$ (50)

We assumed in deriving (49) and (50) that $\ln^{-1}(u/d\omega^*)$ is small compared to 1. This can be verified numerically for the parameters we are taking. Using $\omega^* \sim 1 \times 10^{12} Hz$ (as estimated in section (11)) and taking $d \sim 10^{-8} m$ one obtains $\ln^{-1}(u/d\omega^*) \sim 0.2$. Replacing in (44), we find for the conductivity

$$\omega^* \Re \sigma(y) = \frac{uKe^2}{\pi} \frac{8}{\sqrt{3}} y^2 \text{ (} y \to 0)$$ (51)

One can now check that the hypothesis $\omega^* \sqrt{-y^2 - G} \sim \omega^* \ll \omega_{cr}$ is indeed verified. $\sqrt{-y^2 - G}$ is well defined for $y \ll 1$ since $-G_0$ is positive, and $\sqrt{-y^2 - G} \sim 1$ since $\sqrt{-G_0}$ is of the order of 1.

We have plotted in figure 1, the full frequency behavior of the conductivity, together with the analytic estimate at small frequencies. The small $\omega$ behavior as well as the general shape
of the conductivity is very similar to the one of a classical charge density wave: the small $\omega$ conductivity is behaving as $\omega^2$, there is a maximum at the pinning frequency $\omega^*$ followed by a decrease in $1/\omega^4$. As shown in appendix A, the quantum fluctuations do not change the frequency dependence for frequencies lower than $\omega^*$. The large frequency behavior will be analyzed in details in section IV C.

The low frequency conductivity obtained in our approximation is to be contrasted with the previous result of Efros and Shklovskii who find that the low frequency conductivity of a one-dimensional electron gas in the presence of Coulomb interactions should behave as $\omega$. This result is derived in a very different physical limit where the localization length is much smaller than the interparticle distance, whereas the implicit assumption to derive the model is that the localization length is much larger than the interparticle distance $k_F^{-1}$. In the limit that was considered in the phase $\phi$ would consist of a series of kinks of width $l$ the localization length and located at random positions (with an average spacing $k_F^{-1} \gg l$). The low-energy excitations that are taken into account in would correspond to soliton-like excitations for the phase $\phi$, where the phase jumps by $2\pi$ between two distant kinks. In the physical limit we are considering $k_F^{-1} \ll L_0$, the phase $\phi$ has no kink-like structure but rather smooth distortions between random values at a scale of order $L_0$. To get the dynamics, the approximation we are using only retains the small “phonon” like displacements of the phase $\phi$ relative to the equilibrium position and no “soliton” like excitations are taken into account. In the absence of Coulomb interactions the phonon-like excitations alone, when treated exactly in the classical limit $K \to 0$ are known to give the correct frequency dependence of the conductivity $\omega^2 \ln^2(1/\omega)$ (the self-consistent Born approximation only gets the $\omega^2$ and misses the log correction). When Coulomb interactions are included and one is in the limit where the localization length is much larger than the interparticle distance, it is not clear whether soliton-like interactions similar to those considered by Efros and Shklovskii have to be taken into account. From the solution of a uniform sine-Gordon equation, one could naively say that solitons are only important when the quantum effects are large $K \sim 1$. In the classical limit $K \to 0$, the phonon modes have a much lower energy than the soliton excitations, and the physical behavior of the system should be dominated by such modes. We would therefore argue that the conductivity is given correctly by our result (up to possible log corrections) and to behave in $\omega^2$, and not $\omega$, at least if the system is classical enough ($K$ small) thanks to the short-range part of the interaction. If our assumption is correct the crossover towards the Efros and Shklovskii result when the disorder becomes stronger would be very interesting to study.

B. Strong pinning case $\epsilon > 1$

Let us now look at the other limit case of strong pinning. In that case one cannot expand the self-energy $\Sigma$, all the single-site contributions have to be summed. The result of that summation is

$$\Sigma = (-8V_0\rho_0n_i) \frac{1}{1 + 8V_0\rho_0A}$$

(52)

where $A$ is defined by
Here we rescale the conductivity by the characteristic frequency
\[
\omega_0 = n_i \bar{u} \ln^{1/2} \left( \frac{1}{dn_i} \right)
\]  
(54)
corresponding to a pinning length \( L_0 \sim n_i^{-1} \). It is thus the analog of \( \omega^* \), to a factor \( 4\alpha^{-2/3} \approx 1 \). We use as rescaled parameters \( \bar{y} = \frac{y}{\omega_0} \) and \( \bar{G} = \frac{\pi u K_0}{\omega_0^2} \), in which terms the expression of the conductivity is similar to (44), where we replace \( y, G \) and \( \omega^* \) respectively by \( \bar{y}, \bar{G}, \) and \( \omega_0 \). The resolution is quite similar to what was done for the CDW, so that we give only the main results.

The exact equation on the rescaled self-energy \( \bar{G} \) is
\[
\bar{G} = -\left[ \frac{1}{2\pi^2} \ln \frac{\bar{u}}{\omega_0 d} \epsilon + \sqrt{\alpha_c} \ln^{1/2} \frac{\bar{u}}{\omega_0 d} \pi \int_0^\infty \frac{dt}{-(-y^2 - \bar{G}) + t^2(1 + \alpha_c K_0(\frac{\omega_0 d}{t}))} \right]^{-1}
\]  
(55)
where \( \epsilon \), strength of the pinning, was defined in (20). The numerical resolution of this equation gives the conductivity plotted on Fig. 2, for different values of \( \epsilon \). There is a gap below a frequency \( \omega_{\text{lim}} < \omega_0 \) close to the pinning frequency and tending to it as \( \epsilon \) gets bigger.

In the extremely strong pinning limit \( \epsilon \gg 1 \) one can obtain analytically the conductivity. The equation for the self energy (55), after replacement of the integral by its analytical approximate which can be taken from (46) since \( \omega_0 \ll \omega_{\text{cr}} \), is:
\[
\bar{G} = -2 \left( \ln^{-1/2} \frac{\bar{u}}{\omega_0 d} \right) \sqrt{-y^2 - \bar{G}} \ln^{1/2} \frac{\bar{u}}{\omega_0 d \sqrt{-y^2 - \bar{G}}} \]  
(56)
where the integral is given by (IV A) since \( \omega_0 \ll \omega_{\text{cr}} \). In this limit \( \omega_{\text{lim}} = \omega_0 \) and the conductivity is given near the threshold by
\[
\omega_0 \text{Re} \sigma(\bar{y}) \approx \frac{4\sqrt{2} u K e^2}{\pi} \sqrt{\bar{y} - 1}
\]  
(57)
This gap below \( \omega_{\text{lim}} \) is not physical and is an artifact of considering only the mean distance \( n_i^{-1} \) between impurities. In the real system there is a finite probability of finding neighboring impurities farther apart than \( n_i^{-1} \). Such configurations will give contributions at frequencies smaller than \( \omega_{\text{lim}} \). An estimation of those contributions can be done in a similar way than for a CDW. The probability of finding two neighboring impurities at a distance \( l \) is \( \frac{n_i}{e^{-l/d}} \). In the strong pinning case where we model our pinned CDW by a collection of independent oscillators with frequencies \( \frac{u \pi}{l} \ln^{1/2} (l/d) \), the conductivity for \( \omega < \omega_{\text{lim}} \) will then be proportional to the sum of the contributions over all possible \( l \)
\[
\text{Re} \sigma(\omega) \sim \int_0^\infty dl \frac{n_i e^{-n_i l}}{l} \delta(\omega - \frac{u \pi}{l} \ln^{1/2} \frac{l}{d})
\sim \omega^{-2} \ln^{1/2} \frac{1}{\omega d} e^{-\pi n_i \frac{u \pi}{d} \ln^{1/2} \frac{u \pi}{d}}
\]  
(58)
Compared to a CDW, the conductivity in the pseudo gap is lowered in the presence of Coulomb interactions. This can again be related to the fact that the long-range forces make the Wigner crystal more rigid.
C. Large frequency conductivity

We focus now on large frequencies \( \omega \gg \omega^*, (\omega_0) \), where we expect the physics to be determined over segments of typical size \( l_\omega \sim \frac{n_i}{\omega} \ll L_0, (n_i^{-1}) \), so that intuitively the behavior of the conductivity should be independent of whether we are in the strong or weak pinning regime. And indeed at high \( \omega \) the conductivity can always be computed using the approximation \( \Sigma \approx \Sigma_1 + \Sigma_2 \), whatever the pinning is, since the self-energy terms \( \Sigma_n \)'s are of order \( \left( \frac{1}{\omega} \right)^{n-1} \). But we recall that we made drastic assumptions on the phase \( \Phi \), depending on the pinning regime. To be consistent, they should give similar results at high frequencies. Let’s start first from a weak pinning regime: at \( \omega \ll \omega^* \) we supposed the physics to be determined on domains of length \( L_0 \) on which the phase \( \Phi \) is roughly constant. If we now increase \( \omega \) above \( \omega^* \) we simply replace \( L_0 \) by \( l_\omega \) in the evaluation of \( \Sigma_1 \) and \( \Sigma_2 \). More precisely (33) and (36) are replaced by:

\[
\frac{1}{L} \left( \sum_j \cos(QX_j + 2\sqrt{2}\Phi_0(X_j)) \right)_{av} = \frac{n_i}{l_\omega} (59)
\]
\[
\frac{1}{L} \left( \sum_j \cos^2(QX_j + 2\sqrt{2}\Phi_0(X_j)) \right)_{av} = \frac{1}{2} n_i \left( 1 + \frac{1}{\sqrt{n_i} l_\omega} \right) (60)
\]

This is valid of course as long as \( l_\omega \gg n_i^{-1} \), above which those averages saturate at values:

\[
\frac{1}{L} \left( \sum_j \cos(QX_j + 2\sqrt{2}\Phi_0(X_j)) \right)_{av} = n_i (61)
\]
\[
\frac{1}{L} \left( \sum_j \cos^2(QX_j + 2\sqrt{2}\Phi_0(X_j)) \right)_{av} = n_i (62)
\]

Starting from the strong pinning case and keeping the picture of the phase being adjusted on each impurity site we find expressions identical to (61) and (62), regardless of the frequency.

In the end, using results of section IV A we compute the conductivity to be

\[
\omega^* \Re \sigma(y) = \frac{c_\Phi 4 n_0 K e^2}{\pi} y^{-4} \ln^{-1/2} \frac{\tilde{u}}{d\omega^* y} \ln^{1/2} \frac{\tilde{u}}{d\omega^*} (63)
\]

when \( \omega^*, \omega_0 \ll \omega \ll \omega_{cr} \), and where \( c_\Phi \) is a numerical coefficient between \( \frac{1}{2} \) and 1. More precisely

\[
c_\Phi = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{n_i} l_\omega} \right) \quad \text{for} \quad l_\omega \geq n_i^{-1} (64)
\]
\[
c_\Phi = 1 \quad \text{for} \quad l_\omega \leq n_i^{-1}
\]

which sums up both weak and strong pinning results.

The result \( (63) \) does not take into account the effect of quantum fluctuations. Such effects are expected to become important for frequencies larger than the pinning frequency. For short-range interactions, using renormalization group techniques\(^{16, 28}\), one can show that if it is possible to neglect the renormalization of the interactions by disorder (for example for very weak disorder) the conductivity becomes (for a \( 4k_F \) pinning) \( \sigma(\omega) \sim \omega^{4K-4} \) instead of \( \omega^{-4} \).
due to quantum effects, and would be \( \sigma(\omega) \sim \omega^{K-3} \) for \( 2k_F \) scattering. Although one can derive these results and get the conductivity at high frequency for long-range interactions, using the memory function formalism in a way similar to \cite{28,29}, we will show here how to use the SCHA to get the high frequency conductivity. A naive way to take the frequency into account in the SCHA is to divide the system into segments of length \( u/\omega \), and look at the system on scale of such a segment. Using this method it is possible to rederive the results for the short-range interactions and tackle the case of long-range interactions in which we are interested. Such a calculation is performed in appendix B. Instead of (63), one gets

\[
\omega^{*} \Re \sigma(y) = \frac{c_{\Phi} 4 u K e^2}{\pi} y^{-4} \ln^{-1/2} \frac{\tilde{u}}{d\omega^* y} \ln^{1/2} \frac{\tilde{u}}{d\omega^*} e^{-8\sqrt{2} \ln^{1/2} \frac{\tilde{u}}{d\omega^*}} (65)
\]

From (65) one sees that, as far as exponents are concerned, the conductivity still decays as \( 1/\omega^4 \). This would correspond to a nearly classical \((K \sim 0)\) system with short-range interactions. Note that in this limit the \( 4k_F \) scattering is indeed dominant over the \( 2k_F \) one since the latter would only give a conductivity in \( 1/\omega^3 \) for \( K \to 0 \) (in the above power laws the frequency is normalized by the bandwidth so that \( \omega \ll 1 \)).

V. TEMPERATURE DEPENDENCE OF THE CONDUCTIVITY

One can use arguments similar to the one introduced in section IV to obtain the temperature dependence of the conductivity. Instead of having a cutoff length imposed by the frequency \( \omega \sim u/l_w \), or more precisely as in (17) when Coulomb interactions dominate, one can introduce a thermal length \( \xi_T \) such that \( T \sim u/\xi_T \), which will act as a similar cutoff. Instead of rederiving all the expressions as a function of the temperature, it is simpler to use the following relation for the conductivity

\[
\sigma(\omega, T = 0) \sim M(\omega, T = 0)/\omega^2
\]

\[
\rho(\omega = 0, T) \sim M(\omega = 0, T)
\]

(66)

where \( M \) is the so-called memory function. \( M \) has the same functional form depending on the lowest cutoff in the problem. Therefore \( M(\omega, T = 0) \) and \( M(\omega = 0, T) \) have identical form provided one replaces \( \omega \) by \( T \). From (66), one sees that it is possible to obtain the temperature dependence of the resistivity by multiplying the frequency form obtained in section IV, and then substituting \( \omega \) by \( T \). Such a procedure will be valid as long as one can have a perturbation expansion in the scattering potential, so that (66) is valid\cite{28,29}. This will be the case as long as the thermal length \( \xi_T \) is smaller than the pinning length \( L_0 \). Let us examine the various regimes

A. \( \xi_T \ll \xi_{cr} \)

As discussed in section IV A for quantum wires with unscreened long-range Coulomb interactions, in such a regime a one-dimensional model is probably not applicable. However, it can have application either if the long-range interactions are screened or if the short-range interactions are strong enough \((K \text{ small})\) so that \( \xi_{cr} \gg d \). In that case, as discussed in
section IV C, the short-range interactions dominate. One is back to the situation of $2k_F$ scattering in a Luttinger liquid for which the temperature dependence of the conductivity was computed in reference 13,14. Let us briefly recall the results (for a complete discussion see 15,16): for repulsive interactions the conductivity is roughly given by

$$\sigma(T) \sim T^{\frac{5}{2} - K(T) - \frac{3}{2}K_\sigma(T)}$$

(67)

where $K(T)$ and $K_\sigma(T)$ are the renormalized Luttinger liquid parameters for charge and spin at the length scale $\xi_T$. The $K$ are renormalized by the disorder and decrease when the temperature is lowered. Such a decrease of the exponents is a signature of the tendency of the system to localize 16. As a result the conductivity has no simple power law form since the exponents themselves depend on the temperature. If the disorder is weak enough so that one can neglect the renormalization of the exponents, one gets the approximate expression for the conductivity 13,16 (see also appendix B for a rederivation of this result using SCHA)

$$\sigma(T) \sim T^{1-K}$$

(68)

since (in the absence of renormalization by disorder) $K_\sigma = 1$ due to spin symmetry. The expression (68) coincides with the one obtained subsequently for the conductance of a single impurity 17,18. For one single impurity there is no renormalization of the exponents and the conductance is given by

$$G_0 \sim T^{1-K}$$

(69)

at all temperatures. If one assumes that there are $N_i$ impurities in a wire of length $L$ and that the impurities act as independent scatterers, then the conductivity would be, if $G$ is the conductance of the wire

$$\sigma(T) = LG = \frac{L}{N_i}G_0 = \frac{1}{n_i}G_0$$

(70)

and one recovers (68) (the impurity density $n_i$ is included in the disorder in (68)). When many impurities are present the assumption that their contributions can be added independently is of course incorrect. The collective effects of many impurities leads to the renormalization of the Luttinger liquid parameters (and in particular to localization) and to the formula (67) for the conductivity instead of (68).

B. $\xi_{cr} \ll \xi_T \ll L_0$

In this regime Coulomb interactions dominate and the $4k_F$ scattering is the dominant process. Using (63) one gets

$$\rho(T) \sim \frac{1}{T^2} \ln^{-1/2} \frac{\tilde{u}}{d} e^{-\sqrt{2}K \ln \frac{\tilde{u}}{d (T d)}}$$

(71)

Provided the wire is long enough (71) gives also the temperature dependence of the conductance of the wire. In this regime the $2k_F$ scattering would give $\rho_{2k_F}(T) \sim 1/T$ and is subdominant. Due to the long-range interactions the renormalization of the exponents of the conductivity that took place for short-range interactions does not take place. Such a change of exponent with temperature is replaced by sub-leading corrections. This is due to the fact that the correlation functions decay much more slowly than a power-law.
C. $L_0 \ll \xi_T$

This is the asymptotic regime for which the system is pinned and no expansion like (66) is available. In this regime the temperature dependence is much less clear. In analogy with the collective pinning of vortex lattices, one could expect a glassy-type nonlinear $I-V$ characteristic of the form

$$I \sim e^{-\beta(1/E)^\mu}$$

(72)

Such an $I-V$ characteristic would correspond to diverging barrier between metastable states as the voltage goes to zero. (72) implies that the linear conductivity vanishes at a finite temperature. Since this could be viewed as a phase transition (with the linear conductivity as an order parameter), it is forbidden in a strictly one dimensional system. In fact, in a purely one dimensional system (in principle for $d < 2$), the barriers should remain finite. In that case one gets a finite linear conductivity, going to zero when $T \to 0$. A possible form being

$$\sigma(T) \sim e^{-E_B/T}$$

(73)

where $E_B \sim 1/L_0$ is a typical energy scale for the barriers. However, no definite theoretical method exists to decide the issue, and an experimental determination of the low temperature conductivity would prove extremely interesting.

VI. DISCUSSION AND CONCLUSIONS

We have looked in this paper at the conductivity of a one-dimensional electron gas in the presence of both disorder and long-range Coulomb interactions. Due to long-range interactions, the electron gas forms a Wigner crystal which will be pinned by impurities. As a result, conversely to what happens in a Luttinger liquid, the dominant scattering corresponds to $4k_F$ scattering on the impurities and not $2k_F$ scattering. Such a pinned Wigner crystal is close to classical charge density waves but important a priori differences lie in the presence of long-range interactions and non-negligible quantum fluctuations.

We have computed the pinning length above which the (quasi) long-range crystalline order is destroyed by the disorder. Compared to the standard CDW case the pinning length is increased both by the Coulomb interactions that makes the system more rigid and therefore more difficult to pin, and by the quantum fluctuations that make the pinning less effective. These effects make the system more likely to be in the weak pinning regime. We have also computed the frequency dependence of the conductivity of such a system. At low frequencies, the conductivity varies as $\omega^2$ if the pinning on impurities is weak. This is to be contrasted to the result of Efros and Shklovskii $\sigma \sim \omega$. We believe that this difference is due to the fact that their result was derived in a different physical limit, namely when the pinning length is much shorter than the interparticle distance. However since the method we use is approximate, it could also be the consequence of having neglected soliton-like excitations of the phase field. Although we do expect such excitations to play little role at least when the short-range repulsion is strong enough ($K$ small). More theoretical,
and especially more experimental investigations would prove extremely interesting to settle this important issue. For the case of strong pinning there is a pseudo-gap in the optical conductivity up to the pinning frequency. In the pseudo gap the conductivity behaves as $\frac{1}{\omega} \ln^{1/2} \frac{K}{\omega} e^{-1/\omega \ln^{1/2} \frac{d}{2}}$. Above the pinning frequency, for both regimes, the conductivity decreases as $1/\omega^4 \ln^{-1/2} (\omega_{cr}/\omega) e^{-C \omega^{4/2} (\omega_{cr}/\omega)}$ up to the crossover frequency $\omega_{cr}$ above which the long-range Coulomb interactions become unimportant. For the parameter we took here, $\omega_{cr}$ is also the limit of applicability of a one-dimensional system since $\xi_{cr} \sim d$, the width of the wire. However if the short-range interactions are strong enough ($K$ small), so that $\xi_{cr} \gg d$, then above $\omega_{cr}$ a one-dimensional description will still be valid. One is back to the situation of $2k_F$ scattering in a Luttinger liquid which was studied in detail in 16. The conductivity then behaves as $(1/\omega)^{\mu(\omega)}$, where $\mu(\omega)$ is a non universal exponent depending on the short-range part of the interactions, and due to the renormalization of Luttinger liquid parameters by disorder, also dependent on the frequency 15. If one can neglect such a renormalization of the exponents (e.g. for very weak disorder) then $\mu = 3 - K$.

The temperature dependence can be obtained by similar methods. One can define a thermal length $\xi_T \sim u/T$. When $\xi_T < L_0$, the frequency and temperature dependence of the conductivity are simply related by $\rho(\omega = 0,T) \sim T^2 \sigma(\omega \rightarrow T,T = 0)$, giving $\rho(T) \sim 1/T^2 \ln^{-1/2}(1/T) e^{-C \omega^{4/2}}(1/T)$. Above the pinning length, frequency and temperature can no longer be treated as equivalent cutoffs, and the conductivity is much more difficult to compute. On can expect an exponentially vanishing linear conductivity, provided that the barriers between metastable states remain finite. If it is not the case, one should get a non-linear characteristic of the form $I \sim \exp[-\beta (1/E)^{\alpha}]$, where $\beta = 1/T$, and $\alpha$ is an exponent. Again an experimental determination of $\sigma(T)$ would prove extremely useful. Note that although we considered here the conductivity, most of the results can be applied to the conductance of a finite wire, provided that the size of the wire $L$ is larger than the thermal length $L_T$.

We know that under the application of strong enough electric fields, a classical CDW can be depinned 10,11. Similarly we expect for a Wigner crystal the existence of a threshold electric field $E_{th}$ above which a finite static conductivity appears. We can make a crude estimation of this threshold field, made on the simple assertion that the electrical energy at threshold must be of the order of the pinning energy $E_{pin} \sim \frac{\tilde{u}}{L_0} \ln^{1/2} \frac{L_0}{d}$. This energy can be written as $\epsilon U$, where $U$ is the electrical potential corresponding to a segment of the wire of length $L_0$, that is to say $U = E_{th} L_0$. Thus the threshold field is estimated as

$$E_{th} \sim \frac{\tilde{u}}{e L_0^2} \ln^{1/2} \frac{L_0}{d} \quad (74)$$

From this we can extract an estimation of the pinning frequency $\omega^{*}$. Indeed experimental values of such threshold fields can be found in the literature 10. The latter reference gives a value of threshold field, for a wire of length of about 10$\mu$m, of $E_{th} = 5 \times 10^2 V.m^{-1}$. Thus (74) gives $L_0 \sim 1.4 \mu m$, which seems quite reasonable compared to the length of the wire, and gives for the pinning frequency the estimation $\omega^{*} \sim 1 \times 10^{12} Hz$ (since the wires of reference 10 contain typically two or three impurities, one is probably here in a strong pinning regime). Data reported here are just meant as typical values. The system studied in 10 is at the limit of applicability of our study at low temperatures, since the wire is so
short that it contains only few impurities. However, regardless of the pinning mechanism and number of impurities, our theory should give correctly the temperature dependence of the conductivity (or conductance) at temperatures such that $\xi_T < n_i^{-1}$, since in this regime the impurities act as independent scatterers. To make a study of the low temperature/low frequencies properties longer wires would be needed.

The above estimates, although very crude, show that typical frequencies or temperatures for such systems are in the range of experimentally realizable values, which gives hope for more experimental evidence for the existence of such pinned Wigner crystals. In particular, measurements of the temperature dependence of the conductivity/conductance would prove decisive. At low temperature they would provide evidence for a pinned Wigner crystal, and at higher temperature test for the scattering on impurities in the presence of long-range interactions ($\sim 1/T^2$ behavior). A possible crossover between a Luttinger liquid (dominated by short-range interactions) and the Wigner Crystal (dominated by long-range interactions) could also in principle be seen on the temperature dependence of the conductivity. Frequency dependent conductivity measurement are probably much more difficult to carry out, but would be also of high importance. At low frequency they could serve as tests both on the nature of the pinning mechanism and on the effects on long-range Coulomb interactions on the frequency dependence of the conductivity. For these purposes, quantum wires would constitute a much cleaner system than the standard CDW compounds.

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**APPENDIX A: EFFECT OF QUANTUM FLUCTUATIONS AT LOW FREQUENCIES**

In this section we calculate the corrections to the pinning length $L_0$ (in the weak pinning case) introduced by the quantum effect, at zero temperature. We shall use the SCHA method to take it into account. We write

$$\Phi = \Phi_{cl} + \hat{\Phi}$$

(A1)

where $\Phi_{cl}$ is the classical solution studied in section and $\hat{\Phi}$ represents the quantum fluctuations around $\Phi_{cl}$. We then use the relation

$$\cos(Qx + \Phi_{cl} + 2\sqrt{2}\hat{\Phi}) = e^{-4\langle \hat{\Phi}^2(x) \rangle} \cos(Qx + 2\sqrt{2}\Phi_{cl})(1 - 4\langle \hat{\Phi}^2(x) \rangle)$$

(A2)

instead of the classical expansion done in (25).

Thus the hamiltonian $H_\Phi$ in terms of $\Phi$ is similar to (25), but has an additional constant part

$$H_{pot}' = \sum_j V_0 \rho_0 e^{-4\langle \hat{\Phi}^2(x) \rangle} \cos(QX_j + 2\sqrt{2}\Phi_{cl})(1 + 4\langle \hat{\Phi}^2(x) \rangle)$$

(A3)
We note $\gamma \equiv e^{-4(\hat{\Phi}^2(x))}$ and $\langle \hat{\Phi}^2 \rangle = \langle \hat{\Phi}^2(x) \rangle$. $\langle \hat{\Phi}^2 \rangle$ has to be determined self-consistently from

$$
\langle \hat{\Phi}^2 \rangle = \frac{K}{2} \int_0^{\Lambda^{-1}} dq \frac{1}{\sqrt{q^2(1 + \frac{2KV(q)}{\pi u}) + q_c^2}}
$$

(A4)

where $\Lambda^{-1}$ is a cut-off the choice of which is discussed in the following and

$$
\frac{1}{2\pi K} q_c^2 = -4V_0\rho_0 e^{-4(\hat{\Phi}^2)} \frac{1}{L} \langle \sum j \cos(QX_j + 2\sqrt{2}\hat{\Phi}_c) \rangle = 4V_0\rho_0 \frac{(\pi i L_0)}{L_0}^{1/2} \gamma \equiv 4V\gamma
$$

(A5)

Indeed the potential of impurities results in a change of the spectrum $\omega(q)$:

$$
\omega(q)_{V_0 \neq 0} = u \sqrt{q^2(1 + \frac{2KV(q)}{\pi u}) + q_c^2} \approx u \sqrt{q^2(1 - \frac{4K\epsilon^2}{\pi u} \ln(qd)) + q_c^2}
$$

(A6)

Replacing $q_c$ by its value, we find a square equation on $\langle \hat{\Phi}^2 \rangle$. The result is simplified using $\ln \frac{1}{qd} \gg 1$. Thus we find

$$
\langle \hat{\Phi}^2 \rangle \approx \tilde{K} \ln^{1/2} \frac{\alpha_c}{q_c^2 d^2}
$$

(A7)

where $\tilde{K} = \frac{\sqrt{\pi u K}}{2\sqrt{2}c}$.

Now we can calculate $L_0$ by estimating the cost in energy of a variation of the phase $\Phi_c$ due to the impurity potential $V_0\rho_0$. The costs in elastic and Coulomb energy are again given by (12) and (15) respectively. The cost in potential energy slightly differs from (13):

$$
\mathcal{E}_{\text{imp}}(L_0) = -\nabla \gamma (1 + 4(\hat{\Phi}^2))
$$

(A9)

The new contribution we have to add is the variation of the zero-point energy due to the fact that the spectrum is modified at $V_0 \neq 0$. It is, by unit length

$$
\delta \mathcal{E} = \frac{1}{L} \frac{1}{2} \sum_q \omega(q)_{V_0 \neq 0} - \omega(q)_{V_0 = 0}
$$

$$
= u \int_0^{\Lambda^{-1}} dq \frac{1}{2\pi} \sqrt{q_c^2 + q^2(1 + 4K\epsilon^2/\pi u K_0(qd))} - \sqrt{q^2(1 + 4K\epsilon^2/\pi u K_0(qd))}
$$

(A10)

We can approximate
$$\delta E \approx u \int_{0}^{\Lambda^{-1}} dq \frac{\sqrt{q^2 - \frac{4Ke^2}{\pi u} q^2 \ln q} - \sqrt{-\frac{4Ke^2}{\pi u} q^2 \ln q}}{2\pi} \tag{A11}$$

In a first step we find

$$\delta E = \frac{1}{2} uq_c^2 \int_{0}^{\Lambda^{-1}} dq \frac{1}{\sqrt{q^2 - \frac{4Ke^2}{\pi u} q^2 \ln q}} + \frac{1}{4} \frac{uq_c^2}{2\pi} \sqrt{\frac{\pi u}{4Ke^2}} \ln^{-1/2} \Lambda \tag{A12}$$

Using (A4) it is rewritten:

$$\delta E = \nabla \gamma (4\langle \hat{\Phi}^2 \rangle + \sqrt{2} \hat{K} \ln^{-1/2} \frac{\Lambda}{d}) \tag{A13}$$

We remark that the terms in $\langle \hat{\Phi}^2 \rangle$ annihilate when summing (A9) and (A13).

$$E_{imp} + \delta E = -V_0 \rho_0 \left( \frac{n_i}{L_0} \right)^{\frac{1}{2}} e^{-4\langle \hat{\Phi}^2 \rangle} \left( 1 - \sqrt{2} \hat{K} \ln^{-1/2} \frac{\Lambda}{d} \right) \approx -\nabla \gamma \tag{A14}$$

The previous approximation follows from $\ln \frac{\Lambda}{d} \gg 1$. In the case of short-range interactions (1 - $\sqrt{2} \hat{K} \ln^{-1/2} \frac{\Lambda}{d}$) would be replaced by (1 - 2$\hat{K}$) and one could not neglect that correction. Here the calculation finally reduces to the replacement of $V_0 \rho_0$ by $V_0 \rho_0 e^{-4\langle \hat{\Phi}^2 \rangle} = V_0 \rho_0 \gamma$, which does not change the exponents since $\gamma \approx e^{-4\hat{K} \ln^{1/2} \frac{1}{V}}$.

We find instead of (18)

$$L_0 = \left( \frac{8e^2}{\alpha \pi^2 V_0 \rho_0 \gamma n_i^{\frac{1}{2}}} \right)^{\frac{1}{2}} \ln^{\frac{1}{2}} \left( \frac{C L_0}{d} \right) \tag{A15}$$

where $\gamma \approx e^{-\frac{2\hat{K}}{\sqrt{3} \ln^{1/2} \frac{1}{V}} \rho_0}$.

**APPENDIX B: EFFECT OF QUANTUM FLUCTUATIONS AT HIGH FREQUENCIES**

As explained in (IV C), we handle the quantum fluctuations at high frequencies by a method analogous to SCHA but where we assume that the system is cut into subparts of length $L_\omega \sim \frac{\tilde{u}}{\omega}$. Such a description is expected to be valid as long as $L_\omega < L_0$. This provides, instead of the effective cut-off $q_c$ due to disorder as in (A4), a natural cut-off $\tilde{u}$ which appears at frequencies $\omega > \tilde{u}q_c$. Thus for high frequencies we use, instead of (A4)

$$\langle \hat{\Phi}^2 \rangle = \frac{K}{2} \int_{\tilde{u}}^{\Lambda^{-1}} dq \frac{1}{\sqrt{q^2(1 + \frac{2KV(q)}{\pi u}) + q_c^2}} \approx \frac{K}{2} \int_{\tilde{u}}^{\Lambda^{-1}} dq \frac{1}{\sqrt{q^2(1 + \frac{2KV(q)}{\pi u})}} \tag{B1}$$

In the first subsection we make that calculation for our problem, at $\omega \ll \omega_{cr}$ and in the second one we derive as a comparison what the corrections would be in the absence of the Coulomb term. In the following we can take $\Lambda = d$ since we stay in the one-dimensional regime $\omega \ll \frac{\omega_{cr}}{d}$.
1. long-range interactions

In the range $\omega^* \ll \omega \ll \omega_{cr}$ we have

$$\langle \hat{\Phi}^2 \rangle \approx \frac{K}{2} \int_{-\frac{1}{\omega}}^{d-1} dq \frac{1}{q} \sqrt{\frac{2KV(q)}{\pi u}} \approx \sqrt{2K} \ln \frac{1}{\omega d}$$  \hspace{1cm} (B2)

Since we are in the limit where $l_\omega < L_0$, and are dealing with segments of size $l_\omega$, we can replace the pinning length by $l_\omega$ in the calculation of the conductivity, computed using

$$\Sigma \approx \Sigma_1 + \Sigma_2.$$  

We have

$$\Sigma_1 = -8V_0\rho_0 \left( \frac{n_i}{\omega} \right)^{\frac{1}{2}} e^{-4\langle \hat{\Phi}^2 \rangle} \text{ for } l_\omega \geq n_i^{-1}$$

$$= -8V_0\rho_0 n_i e^{-4\langle \hat{\Phi}^2 \rangle} \text{ for } l_\omega \leq n_i^{-1}$$  \hspace{1cm} (B3)

and

$$\Sigma_2 = c_\Phi n_i (8V_0\rho_0\gamma)^2 \int_{-\frac{1}{\omega}}^{d-1} dq \frac{\pi u K}{2\pi \omega_n^2 + q^2 u^2 (1 + \frac{2KV(q)}{\pi u})}$$  \hspace{1cm} (B4)

where $c_\Phi$ is defined in (64). Since $\Sigma_2$ is multiplied by $\gamma^2$ compared to the classical case, the conductivity becomes instead of (63)

$$\omega^* \Re \sigma(y) = \frac{c_\Phi 4uK e^2}{\pi} y^{-4} \ln^{-1/2} \frac{\bar{u}}{d\omega^*y} \ln^{1/2} \frac{\bar{u}}{d\omega^*} e^{-8\sqrt{2K} \ln \frac{1}{\omega y} \ln \frac{1}{\omega y}}$$  \hspace{1cm} (B5)

2. short-range interactions

Here we rederive the results for short-range interactions obtained by the renormalization group. As pointed out in [15], the SCCHA does not correctly reproduce the renormalization of the exponents with disorder and is therefore limited, for the case of short-range interactions, to describe infinitesimal disorder. For finite disorder the exponent themselves are renormalized by the disorder and are functions of frequency or temperature [15]. The derivation is given for simplicity for the case of a $4k_F$ scattering. In the case of short-range interactions one has to consider the $2k_F$ scattering (which is dominant). Since one has

$$\rho_{4k_F}(x) \sim e^{i2\sqrt{2}\Phi(x)}$$

$$\rho_{2k_F}(x) \sim e^{i\sqrt{2}\Phi(x)} \cos(\sqrt{2}\Phi_\sigma(x))$$

where $\Phi$ and $\Phi_\sigma$ are two independent bosons for charge and spin excitations, one can obtain the conductivity for the $2k_F$ scattering by the substitution $4K \rightarrow K + 1$ (see [16] for a complete derivation of the conductivity for $2k_F$ scattering).

For short-range interactions

$$\langle \hat{\Phi}^2 \rangle = \frac{K}{2} \int_{-\frac{1}{\omega}}^{d-1} dq \frac{1}{q} = \frac{K}{2} \ln \frac{ud^{-1}}{\omega}$$  \hspace{1cm} (B7)
which gives

\[ \gamma = e^{-4\langle \Phi_0^2 \rangle} = \left( \frac{ud^{-1}}{\omega} \right)^{-2K} \]  

(B8)

We have

\[ \Sigma_1 = -8V_0 \rho_0 \left( \frac{n_i}{l_\omega} \right)^{\frac{3}{2}} e^{-4\langle \Phi_0^2 \rangle} \sim -\left( \frac{\omega d}{u} \right)^{2K+\frac{1}{2}} \text{ for } l_\omega \geq n_i^{-1} \]

\[ = -8V_0 \rho_0 n_i e^{-4\langle \Phi_0^2 \rangle} \sim -\left( \frac{\omega d}{u} \right)^{2K} \text{ for } l_\omega \leq n_i^{-1} \]  

(B9)

and

\[ \Sigma_2 = c_\Phi n_i (8V_0 \rho_0 \gamma)^2 \int_{\omega_n}^{\omega_n - 1} dq \frac{\pi u K}{2 \pi \omega_n^2 + q^2 u^2} \]  

(B10)

The previous integral equals a constant times \( \frac{1}{|\omega_n|} \). Taking the limit \( i\omega_n \to \omega - i0^+ \),

\[ \Sigma_2 \sim -i\left( \frac{\omega d}{u} \right)^{4K} (\omega)^{-1} \]  

(B11)

We now compute the frequency dependence of the conductivity. From (B9), one sees that \( \Sigma_1 \ll \omega^2 \) at high frequencies. From (B4) one finds

\[ \Re \sigma_{4k_F}(\omega) \sim \left( \frac{\omega d}{u} \right)^{4K} \omega^{-4} \]  

(B12)

Similarly one would get \( \sigma_{2k_F}(\omega) \sim \omega^{K-3} \)
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FIGURES

FIG. 1. Frequency dependence of the rescaled conductivity in the weak pinning case versus the rescaled frequency $y = \frac{\omega}{\omega^*}$. We have noted $\zeta = \frac{2uK_e^2}{\pi\omega}$. The solid line was obtained by numerical resolution of equation (38), where we have taken typical parameters $\frac{\omega^* d}{u} = 3.33 \times 10^{-2}$ and $\frac{4K_e^2}{\pi u} = 4.7$. The dash-dotted curve represents the analytical expression (51) computed for low frequencies.

FIG. 2. Frequency dependence of the rescaled conductivity in the strong pinning case versus the rescaled frequency $y = \frac{\omega}{\omega_0}$. We have noted $\zeta = \frac{2uK_e^2}{\pi\omega_0}$. The three curves were obtained by solving numerically equation (53) for different values of the strength of the pinning $\epsilon$: $\epsilon = 1$ (dashed line), $\epsilon = 10$ (dash-dotted line), and $\epsilon = 1000$ (solid line). We have taken $\frac{\omega_0 d}{u} = 3.33 \times 10^{-2}$ and $\frac{4K_e^2}{\pi u} = 4.7$. 