The Dynamics of Rank-Maximal and Popular Matchings

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Abstract

Given a bipartite graph, where the two sets of vertices are applicants and posts and ranks on the edges represent preferences of applicants over posts, a rank-maximal matching is one in which the maximum number of applicants is matched to their rank one posts and subject to this condition, the maximum number of applicants is matched to their rank two posts, and so on. A rank-maximal matching can be computed in \(O(\min(c\sqrt{n},n)m)\) time, where \(n\) denotes the number of applicants, \(m\) the number of edges and \(c\) the maximum rank of an edge in an optimal solution [9].

We study the dynamic version of the problem in which a new applicant or post may be added to the graph and we would like to maintain a rank-maximal matching. We show that after the arrival of one vertex, we are always able to update the existing rank-maximal matching in \(O(\min(cn,n^2) + m)\) time; moreover, by the application of just one alternating path. The time bound can be considered optimal under the circumstances, as improving it would imply a better running time for the rank-maximal matching problem. Additionally, our solution has the property that enables to minimize the number of needed changes.

As a by-product we also get an analogous \(O(m)\) result for the dynamic version of the (one-sided) popular matching problem.

1 Introduction

We consider the dynamic version of the rank-maximal matching problem. In the rank-maximal matching problem we are given a bipartite graph \(G = (A \cup P, E)\), where \(A\) is a set of applicants, \(P\) is a set of posts and edges have ranks. An edge \((a,p)\) has rank \(i\) if the post \(p\) is one of the applicant \(a\)'s \(i\)th choices. A matching of the graph \(G\) is said to be rank-maximal if it matches the maximum number of applicants to their rank one posts and subject to this condition, it matches the maximum number of applicants to their rank two posts, and so on. A rank-maximal matching can be computed in \(O(\min(c\sqrt{n},n)m)\) time, where \(n\) denotes the number of applicants, \(m\) the number of edges and \(c\) the maximum rank of an edge in an optimal solution [9].

In the dynamic variant of the problem a new vertex may be added to the graph and we would like to maintain a rank-maximal matching. When the new vertex \(v\) is added to the graph \(G\) we assume that the graph \(G\) itself does not change. In particular, if a new post \(p\) arrives, the applicants of \(G\) cannot change their preferences over posts that are already included in \(G\). Let us call the graph \(G\) extended by \(v\) and the edges incident to \(v\) as the graph \(H\). In order to have a rank-maximal matching of \(H\), we would like to be able to transform a rank-maximal matching \(M\) of \(G\) into a
rank-maximal matching $N$ of $H$, making only the smallest needed number of changes. In some cases a rank-maximal matching of $G$ is also rank-maximal in $H$. We design an algorithm that updates $M$ by an application of just one alternating path $P$, i.e., $P$ is such that $M \oplus P = (M \setminus P) \cup (P \setminus M)$ is a rank-maximal matching of $H$. To be able to compute $P$ efficiently, we need access to the reduced graphs $G_1', G_2', \ldots, G_n'$ of $G$ (the notion is defined in [9] and also recalled in Section 2) and their Edmonds-Gallai decompositions. We show that we can compute a required alternating path $P$ as well as update the reduced graphs $H_1', H_2', \ldots, H_n'$ of $H$ and their Edmonds-Gallai decompositions in $O(\min(cn, n^2) + m)$ time. The result may seem rather surprising. For comparison, let us note that it is much easier to update the matching gradually - separately in each of the graphs $H_i$ that consists of edges of rank at most $i$. In such approach, however, it is required to compute and apply $r$ alternating paths. Each such computation and update of the reduced graph $H_i'$ can be carried out in $O(n + m)$ time and thus the overall running time is $O(r(n + m))$. This is how the problem is dealt with in a recent paper by Nimbhorkar and Rameshvar [12]. Observe that our algorithm is significantly faster than the one in [12] and the improvement may be of the order $\Omega(m)$ if $r$ is of the order $\Omega(m)$. Additionally, our solution has the property of practical relevance that enables to minimize the number of needed changes - namely from all alternating paths such that $M \oplus P$ is a rank-maximal matching of the new graph, we are able to find the shortest one. The construction of our algorithm requires a good grasp of the properties of the Edmonds-Gallai decomposition and a knowledge of the structure of rank-maximal matchings.

The popular matching problem in the one-sided version is defined as follows. The input is the same as in the rank-maximal matching problem - we are given a bipartite graph $G$, in which the vertices of one side of the graph express their preferences over the vertices of the other side. The goal is to find a popular matching in $G$, if it exists. A matching $M$ is said to popular if there exists no other matching $M'$ such that $M'$ is more popular than $M$. A matching $M'$ is more popular than $M$ if the number of applicants preferring $M'$ to $M$ is greater than the number of applicants preferring $M$ to $M'$ and an applicant $a$ prefers $M'$ to $M'$ if (i) he is matched in $M'$ and unmatched in $M$ or (ii) he prefers the post $p$ assigned to him in $M'$ to the one he gets in $M$. Not every instance of the problem admits a popular matching. Nevertheless, Abraham et al. [2] gave an $O(\sqrt{nm})$ algorithm that computes a popular matching, if it exists. The algorithm is in a certain sense similar to the one computing a rank-maximal matching. It consists of two phases that are the same as in the algorithm for rank-maximal matchings, but the edges participating in the second phase are selected in some special way. To obtain a solution for a dynamic version of the popular matching problem, in which new vertices may be added to the graph, we can directly use the algorithm for the dynamic version of the rank-maximal matching by redefining appropriately rank two edges. Thus we are able to update a popular matching in $O(m)$ time after the arrival of a new vertex.

The algorithm for updating a rank-maximal matching can be also used for updating a bounded unpopularity matching in the same time bound of $O(\min(cn, n^2) + m)$ [7].

There are many real world applications of this algorithm. Let us assume that there is a company that recruits new employees each year and each recruited applicant has a preference list on the posts. A central authority matches each applicant with a post. There is already a existing matching from the previous year. It is not efficient to calculate the matching from scratch every year. It is efficient if we can update the existing matching from the previous year when we calculate the new matching. This is the motivation behind the dynamic matching problem. The rank maximal matching and the popular matching are two well known optimality criterion when we have a bipartite graph with one sided preference list. We combine these two properties and give a dynamic matching algorithm that works for both rank maximal and popular matching algorithm. This is the main motivation behind our problem.

**Previous work** A rank-maximal matching can be found via a relatively straightforward
reduction to a maximum weight matching. The running time of the resulting algorithm is \( O(r^2 \sqrt{nm \log n}) \), where \( r \) denotes the maximal value of a rank, if we use the Gabow-Tarjan algorithm, or \( O(rn(m + n \log n)) \) for the Fredman-Tarjan algorithm. The first algorithm for rank-maximal matchings was given by Irving in [8] for the version without ties and with running time of \( O(d^2n^3) \), where \( d \) denotes the maximum degree of an applicant in the graph (thus \( d = c \)). The already mentioned [9] gives a combinatorial algorithm that runs in \( O(\min(n,c\sqrt{n})m) \) time (and not \( O(\min(n+r,c\sqrt{n})m) \)). The capacitated and weighted versions were considered, respectively, in [13] and [10]. A switching graph characterisation of the set of all rank-maximal matchings is described in [6]. Independently of our work, in a recent paper [12] Nimbhorkar and Rameshvar also study the dynamic version of the rank-maximal matching problem and develop an \( O(r(n+m)) \) algorithm for updating a rank-maximal matching after the addition or deletion of a vertex or edge.

**Related work** Matchings under preferences in the dynamic setting have been studied under different notions of optimality.

In [11] McCutchen introduced the notion of unpopularity factor and showed that it is NP-hard to compute a least unpopular matching in one-sided instances. Bhattacharya et. al. [4] gave an algorithm to maintain matchings with unpopularity factor \( (\Delta + k) \) by making an amortized number of \( O(\Delta + \Delta^2/k) \) changes per round, for any \( k > 0 \) where \( \Delta \) denotes the maximum degree of any agent in any round. Note that this is the number of changes made to the matching and not the update time, which is much higher and requires a series of computations of a maximum weight matching.

In [3] Abraham and Kavitha describe the notion of a so called *voting path*. A voting path is a sequence of matchings which starts from an arbitrary matching, and ends at a popular matching and each matching in the sequence is more popular than the previous one. The authors showed that in a one-sided setting with ties there always exists a voting path of length at most two. They also show how to compute such paths in linear time, given a popular matching in the graph, what allows them to maintain a popular matching under a sequence of deletions and additions of vertices to the graph, however, in \( O(\sqrt{nm}) \) time per each update.

Another example is the notion of Pareto optimality. In [1] authors gave an \( O(\sqrt{nm}) \) for computing Pareto optimal matchings. In [5] Fleischer and Wang studied Pareto optimal matchings in the dynamic setting. The authors gave a linear time algorithm to maintain a maximum size Pareto matching under a sequence of deletions and additions of vertices.

### 2 Preliminaries

Let \( G = (A \cup P, E) \) be a bipartite graph and let \( M \) be a maximum matching of \( G \). We say that a path \( S \) is \( M \)-alternating if its edges belong alternately to \( M \) and \( E \setminus M \). We say that a vertex \( v \) is *free* or *unmatched* in \( M \) if no edge of \( M \) is incident to \( v \). An \( M \)-alternating path is said to be \( M \)-augmenting (or augmenting if the matching is clear from the context) if it starts and ends at an unmatched vertex.

Given a maximum matching \( M \) we can partition the vertex set of \( G \) into three disjoint sets \( E \), \( O \) and \( U \). Nodes in \( E \), \( O \) and \( U \) are called *even*, *odd* and *unreachable* respectively and are defined as follows. A vertex \( v \in V(G) \) is *even* (resp. *odd*) if there is an even (resp. odd) length alternating path in \( G \) with respect to \( M \) from an unmatched vertex to \( v \). A vertex \( v \in V(G) \) is unreachable if there is no alternating path in \( G \) with respect to \( M \) from an unmatched vertex to \( v \). For vertex sets \( A \) and \( B \), we call an edge connecting a vertex in \( A \) with a vertex in \( B \) an *AB* edge. The following lemma is well-known in matching theory.

**Lemma 1.** *Edmonds-Gallai decomposition (EG-decomposition)* [14, 9]
Let $M$ be a maximum matching in $G$ and let $E$, $O$ and $U$ be defined as above.

1. The sets $E$, $O$, $U$ are pairwise disjoint

2. Let $N$ be any maximum matching in $G$

   (a) $N$ defines the same sets $E$, $O$ and $U$
   (b) $N$ contains only $UU$ and $OE$ edges
   (c) Every vertex in $O$ and every vertex in $U$ is matched by $N$
   (d) $|N| = |O| + |U|/2$

3. There is no $EU$ and no $EE$ edge in $G$

Throughout the paper we consider many graphs at once, thus to avoid confusion, for a given graph $G$ we denote the sets of even, odd and unreachable vertices as $E(G)$, $O(G)$ and $U(G)$ respectively.

### 2.1 Rank-Maximal Matchings

In this section we briefly recall known facts about rank-maximal matchings (see [9] for more details). An instance of the rank-maximal matching problem is a bipartite graph $G = (A \cup P, E)$, where $A$ is a set of applicants, $P$ is a set of posts, and $E$ can be partitioned as $E = E_1 \cup E_2 \cup \ldots \cup E_r$. For each $i \leq r$ the set $E_i$ consists of edges of rank $i$ and $r$ is a maximum rank any applicant assigns to a post.

**Definition 1.** The signature of a matching is defined as an $r$-tuple $\rho(M) = (x_1, x_2, \ldots, x_r)$, where for each $1 \leq i \leq r$, $x_i$ is the number of applicants $a \in A$ such that $(a, M(a)) \in E_i$.

We define an ordering of matchings imposed by the lexicographical order of their signatures. Let $M, M'$ be two matchings in $G$. We denote $M > M'$ if and only if $\rho(M) >_{\text{lex}} \rho(M')$. A matching $M$ in $G$ is rank-maximal if $M$ has the maximum signature under the ordering $>$. The pseudocode of Irving et al.’s algorithm [9] is denoted as Algorithm 1.

**Theorem 1.** [9] Algorithm 1 computes a rank-maximal matching in $O(\min\{c\sqrt{n}, n\}m)$ time, where $c \leq r$ denotes a maximal-rank in the optimal solution.

**Algorithm 1** for computing a rank-maximal matching

1: procedure RANKMAXIMALMATCHING(G)
2: $G'_1 \leftarrow G_1$
3: Let $M_1$ be any maximum matching of $G'_1$
4: for $i = 1, 2, \ldots, r$ do
5:     Determine a partition of the vertices of $G'_i$ into the sets $E(G'_i)$, $O(G'_i)$ and $U(G'_i)$
6:     Delete all edges in $E_j$ (for $j > i$) which are incident on nodes in $O(G'_i) \cup U(G'_i)$. These are the nodes that are matched by every maximum matching in $G'_i$.
7:     Delete all $O(G'_i)O(G'_i)$ and $O(G'_i)U(G'_i)$ edges from $G'_i$. These are the edges that are not used by any maximum matching in $G'_i$.
8:     Add the edges in $E_{i+1}$ and call the resulting graph $G'_{i+1}$.
9:     Determine a maximum matching $M_{i+1}$ in $G'_{i+1}$ by augmenting $M_i$.
10: return $M_r$

The following invariants of Algorithm 1 are proven in [9].
1. For every $1 \leq i \leq r$, every rank maximal matching in $G_i$ is contained in $G'_i$.

2. The matching $M_i$ is rank-maximal of $G_i$, and is a maximum matching in $G'_i$.

3. If a rank-maximal matching in $G$ has signature $(s_1, s_2, \ldots, s_i, \ldots, s_r)$ then $M_i$ has signature $(s_1, s_2, \ldots, s_i)$.

4. The graphs $G'_i$ ($1 \leq i \leq r$) constructed during the execution of Algorithm 1 are independent of the rank-maximal matching computed by the algorithm.

2.2 The Dynamic Rank-Maximal Matching Problem

We assume that we have already executed Algorithm 1 on $G$ and computed a rank-maximal matching $M$. Additionally we store all the structures computed by the algorithm i.e. reduced graphs $G'_i$ along with their EG-decompositions and matchings $M_i$. At some point a new applicant $a \notin A$ arrives. Let $H = (A \cup \{a\} \cup P, \mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots \cup \mathcal{F}_r)$ be a modified graph obtained by adding an applicant $a$ along with the information about his post preferences to $G$. Our goal in the dynamic version of the rank-maximal matching problem is to compute a rank-maximal matching in $H$ along with all the auxiliary structures that Algorithm 1 normally computes when it is executed on $H$. We would like to refrain from executing Algorithm 1 on $H$ and solve the problem more efficiently. Let us for each $1 \leq i \leq r$ denote: $H_i = (A \cup \{a\} \cup P, \mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots \cup \mathcal{F}_i)$, $H'_i = (A \cup \{a\} \cup P, \mathcal{F}'_1 \cup \mathcal{F}'_2 \cup \ldots \cup \mathcal{F}'_i)$ - reduced graphs computed by Algorithm and $N_i$ - a rank-maximal matching of $H_i$.

Our goal is to compute graphs $H'_i$ along with their EG-decompositions and matchings $N_i$, assuming that we already have graphs $G'_i$, their decompositions and matchings $M_i$. In order to simplify the notation, we are going to assume that the newly arriving applicant $a$ is already present in $V(G)$ but it is isolated in $G$.

3 Is $M$ a Rank-Maximal Matching of $H$?

Before we describe our algorithm for the dynamic rank-maximal matching problem we first introduce and solve a simplified variant of this problem, in order to build some intuition.

Our first assumption is that a newly arriving vertex $a$ has only one edge incident on it. We also slightly change our goal. Instead of computing a rank-maximal matching in $H$ we only determine if $M$ is a rank-maximal matching of $H$. Our goal is to solve this problem in $O(m)$ time. The following is the main theorem of this section:

**Theorem 2.** Assume that we are given reduced graphs $G'_1, G'_2, \ldots, G'_{r+1}$, their EG-decompositions and matchings $M_1, M_2, \ldots, M_r$. There is an $O(m)$ time algorithm to determine if $M_r$ is a rank-maximal matching of $H$.

In the following lemma we examine how the maximum matching $M$ in a bipartite graph $G$ and its EG-decomposition change if we add one edge to $G$.

**Lemma 2.** Let $G = (A \cup P, \mathcal{E})$ be a bipartite graph, $M$ a maximum matching of $G$ and $a \in A$ and $p \in P$ two vertices of $G$ such that $(a, p) \notin \mathcal{E}$ and $a$ has type $E$ in the EG-decomposition of $G$ ($a \in E(G)$). Then the graph $J = (A \cup P, \mathcal{E} \cup (a, p))$ has the following properties:

1. If $p \in E(G)$, then the edge $(a, p)$ belongs to every maximum matching of $J$. A maximum matching of $J$ is of size $|M| + 1$. 

2. If \( p \in O(G) \), then the edge \((a, p)\) belongs to some maximum matching of \( J \) but not to every one and \( M \) remains a maximum matching of \( J \). Additionally, the EG-decomposition of the graph \( J \) is the same as that of \( G \).

3. If \( p \in U(G) \), then the edge \((a, p)\) belongs to some maximum matching of \( J \) but not to every one and \( M \) remains a maximum matching of \( J \). Additionally, the EG-decomposition of the graph \( J \) is different from that of \( G \) in the following way. A vertex \( v \in U(G) \) belongs to \( E(J) \) (respectively, \( O(J) \)) if there exists an even-length (corr., odd-length) alternating path starting with the edge \((a, p)\) and ending at \( v \). Apart from this every vertex has the same type in the EG-decomposition of \( G \) and \( J \), i.e., if \( v \in X(G) \), then \( v \in X(J) \), where \( X \in \{E, O, U\} \).

We can use the property (3) of Lemma 2 in order to efficiently update the decomposition of the graph \( G \) once the edge \((a, p)\), where \( a \in E(G) \) and \( p \in U(G) \) is added to \( G \).

Below we describe the intuitions behind Algorithm 2. We assume that the newly added edge \((a, x)\) is of rank \( k \). We will show that for each \( i \) it is possible to efficiently deduce the structure of \( H'_i \) from the structure of \( G'_i \) without actually executing Algorithm 1 on \( H \).

From the pseudocode of Algorithm 1 we can easily see that for each \( i \) such that \( 1 \leq i < k \) we have \( G'_i = H'_i \), and that \( M_i \) is a rank-maximal matching of \( H_i \). How do graphs \( G'_k \) and \( H'_k \) differ when we enter the line \( 4 \) of the algorithm during phase \( k \)? One can easily see that either \( G'_k + (a, x) = H'_k \) or \( G'_k = H'_k \) holds. The latter case happens when the edge \((a, x)\) is removed from \( F_k \) if in some iteration \( j < k \) we have \( x \notin E(G'_j) \).

From now on we assume that \( x \in \bigcap_{i=1}^{k-1} E(G'_i) \). One can easily check that when we enter the loop \( for \) in the line 4 of Algorithm 1 we have \( G'_k + (a, x) = H'_k \). We can use Lemma 2 to obtain the information about the EG-decomposition of \( H'_k \) from the decomposition of \( G'_k \). There are three cases depending on the type of \( x \) in \( G'_k \):

1. If \( x \in E(G'_k) \), then \((a, x)\) belongs to every maximum matching of \( H'_k \). From invariants \((1 - 4)\) of the Algorithm 1 \((a, x)\) belongs to every rank-maximal matching of \( H \).

2. If \( x \in U(G'_k) \), then the EG-decomposition of \( H'_k \) can be easily inferred from the EG-decomposition of \( G'_k \).

3. If \( x \in O(G'_k) \), then the EG-decomposition of \( H'_k \) is the same as that of \( G'_k \).

In case (1) we can simply halt the algorithm and claim that \( M \) is not a rank-maximal matching of \( H \).

Let us consider the case (2). From Lemma 2 we can see that some vertices may belong to \( U(G'_k) \cap E(H'_k) \) or \( U(G'_k) \cap O(H'_k) \). If a vertex \( v \in U(G'_k) \cap E(H'_k) \) (resp. \( v \in U(G'_k) \cap O(H'_k) \)) then we say that \( v \) changes its type from \( U \) to \( E \) (resp. \( O \)) in the phase \( k \). What implications does this fact have on the execution of Algorithm 1 on \( H \)? Note that in lines 6 and 7 of Algorithm 1 we remove some edges incident to vertices of types \( O \) and \( U \). If \( v \) changes its type from \( U \) to \( E \) in phase \( k \) then some edges incident to \( v \) are deleted in the phase \( k \) during the execution of Algorithm 1 on \( G \), but these edges are not deleted in the phase \( k \) during the execution of Algorithm 1 on \( H \). We call such edges activated edges and in the pseudocode we denote the set of these edges as \( AE_a \). Additionally vertices which change type from either \( U \) or \( O \) to \( E \) are called activated vertices. The set of such vertices is denoted as \( AV \).

It remains to consider the case (3). We already know from Lemma 2 that the presence of \((a, x)\) in \( H'_k \) does not affect its EG-decomposition. It turns out however that if for some \( k' > k \) we have \( x \in U(G'_{k'}) \) but \( x \in O(G'_{k'-1}) \) then the presence of \((a, x)\) in \( H \) might affect the EG-decomposition
of \(H'_{k'}\), but will not have any impact on the decompositions of graphs \(H'_l\) for \(k < l < k'\). Such edges are also marked as activated edges and added to \(AE_a\).

The main idea behind the remaining part of the algorithm is to maintain sets of activated edges so that in any phase \(k' > k\) a reduced graph \(H'_{k'}\) is obtained from \(G'_{k'}\) by adding to this graph a certain subset of edges activated in phases preceding the phase \(k\). The EG-decomposition of \(H'_{k'}\) is then computed with the aid of decompositions of \(G'_{k'}\) and \(H'_{k'-1}\). The main difficulty comes from the fact that in the phase \(k\) graphs \(G'_k\) and \(H'_k\) differ by exactly one edge, whereas in a phase \(k'\) \((k' > k)\) \(H'_{k'}\) may potentially contain multiple activated edges. A suitable generalisation of Lemma 2 allows us to deal with this problem.

**Lemma 3.** Let \(G = (A \cup P, \mathcal{E})\) be a bipartite graph, \(M\) a maximum matching of \(G\) and \(\mathcal{E}' = \{(a_1, p_1), (a_2, p_2), \ldots, (a_s, p_s)\}\) the set of edges none of which belongs to \(\mathcal{E}\). Additionally, each edge \(a_i\) belongs to \(A \cap E(G)\), each edge \(p_i\) belongs to \(P\). The edges in \(\mathcal{E}'\) need not be vertex disjoint. Let \(G' = (A \cup P, \mathcal{E} \cup \mathcal{E}')\).

There is an algorithm which can correctly distinguish between the following two situations:

(A) Maximum matching of \(G'\) is of size at least \(|M| + 1\).

(B) \(M\) remains a maximum matching of \(G'\).

If (A) occurs then the algorithm runs in \(O(m)\) time. If (B) occurs then the algorithm additionally determines the EG-decomposition of \(G'\) and runs in time \(O(|X|)\), where \(X = \{(x, y) \in \mathcal{E} \cup \mathcal{E}' : x \in U(G), x \notin U(G')\}\).

Note that we do not prove any theorems from this section, as the correctness of Algorithm 2 is implied by the correctness of Algorithm 3 which is proven in the paper.

Alive\((i)\) denotes the set of vertices that are alive in \(G'_i\) at the beginning of phase \(i\), i.e., \(v \in \text{Alive}(i)\) iff \(v \in \bigcap_{j=1}^{i-1} E(G'_j)\).

## 4 Augmentation

In this section we want to look closer at the situation when a rank-maximal matching of \(G\) is not a rank-maximal matching of \(H\). This happens when at some point Algorithm 2 in line 9 encounters an edge \((v, w)\) such that \(v\) belongs to \(U(G'_i) \cap E(H'_i)\) and \(w\) belongs to \(E(G'_i)\) and thus it executes line 10 and outputs "\(M_r\) is not a rank-maximal matching of \(H\)". In other words, it occurs when the maximum matching in the reduced graph \(H'_i\) is larger by one than the maximum matching in the reduced graph \(G'_i\).

Let us examine two such examples depicted in Figure 1. Here the edge \((a, x)\) is of rank 1 and the edge \((a_1, p_1)\) of rank 2. Vertex \(a_1\) belongs to \(U(G'_2) \cap E(H'_2)\) and \(p_1\) belongs to \(E(G'_2)\) - thus Algorithm 2 outputs the answer "\(M_r\) is not a rank-maximal matching of \(H\)". This means that in \(H'_2\), there exists an \(M_r\)-augmenting path containing the edges \((a, x)\) and \((a_1, p_1)\). However, in order to obtain a rank-maximal matching \(N\) of \(H\), we would like to carry out the augmentation as late as possible. In fact, once we know that a rank-maximal matching of \(G\) is not a rank-maximal matching of \(H\) and the augmentation in some phase \(i\) is inevitable, we want to postpone augmenting \(N_i\) and thus changing \(N\) till the last phase.

In the first example of Figure 1, we can notice that the vertex \(p_1\) belongs to \(E(G'_i)\) for every \(i\) such that \(2 \leq i \leq 4\) and that the graph \(G'_4\) together with the edges \((a, x)\) and \((a_1, p_1)\) still contains an augmenting path containing \((a, x)\) and \((a_1, p_1)\). What is more, this augmenting path
Algorithm 2 for checking if $M_r$ is a rank-maximal matching of $H$

1: $C \leftarrow \{a_0\}$ (a new vertex)
2: $AV \leftarrow \{a_0\}$ activated vertices
3: $AE \leftarrow \emptyset$ activated edges
4: $i \leftarrow 1$
5: while $i \leq r$ do
6: \hspace{1em} $R \leftarrow G_i' \setminus C$
7: \hspace{1em} for all $a \in AV$ do
8: \hspace{2em} $AE \leftarrow AE \cup \{(a,p) \in \tilde{E}_i : a \in AV \land p \in Alive(i)\}$
9: \hspace{1em} if there exists $(a,p) \in AE$ such that $p \in E(G_i')$ then
10: \hspace{2em} return $M_r$ is not a rank-maximal matching of $H$
11: \hspace{1em} else
12: \hspace{2em} $\tilde{H}_i \leftarrow C \cup R \cup AE$
13: \hspace{2em} $AE_u \leftarrow \{(a,p) \in AE : p \in U(G_i')\}$
14: \hspace{2em} for all $S : S$ is an even-length $M_i$-alt. path in $\tilde{H}_i$ between $a_0$ and $a \in U(G_i')$ do
15: \hspace{3em} $C \leftarrow C \cup S$
16: \hspace{3em} $AV \leftarrow AV \cup \{a\}$
17: \hspace{2em} $AE_o \leftarrow \{(a,p) \in AE \cup G_i' : a \in C \land p \in O(G_i')\}$
18: \hspace{2em} $AE \leftarrow AE \cup AE_o \setminus AE_u$
19: \hspace{1em} $i \leftarrow i + 1$
20: return $M_r$ is a rank-maximal matching of $H$

was already present in the graph $G_2' \cup \{(a,x),(a_1,p_1)\}$. We can check that after applying it we obtain a rank-maximal matching of $H_4$.

In the second example of Figure 3, the vertex $p_1$ belongs to $E(G_i')$ for every $i$ such that $2 \leq i \leq 4$ and it belongs to $U(G_5')$. Thus $G_5' \cup \{(a,x),(a_1,p_1)\}$ does not contain any augmenting paths and we are stuck with a matching $M_5$ which is not rank-maximal in $H_5$. We observe that if we had augmented $M_4$ in the graph $G_4' \cup \{(a,x),(a_1,p_1)\}$ obtaining a rank-maximal matching $N_4$ of $H_4$, then one of the edges of rank 5 would not be present in the maximum matching of $G_5' \cup \{(a,x),(a_1,p_1)\}$ if we computed it by augmenting $N_4$. So, in order to get a rank-maximal matching of $H_5$ from $M_5$ we should ”undo” one of the augmentations that was carried out in phase 5. Using matching terminology we should apply any even length $M_5$-alternating path starting at $a$ and containing $(a_1,p_1)$ and one of the edges of rank 5 belonging to $M_5$.

Later on, in Theorem 3 we prove that in such two types of scenarios the described approach is correct, i.e. although we discover that $M$ is not a rank-maximal matching of $H$ already in some phase $i$, we can afford to wait till some later phase $j$ and apply either an augmenting path or an even length alternating path in the graph $G_j' \cup F_i'$. The correctness of this approach is based on the following two lemmas.

Lemma 4. Let $G = (A \cup P, E)$ be a bipartite graph, $M$ its maximum matching and $C$ a connected component of $G$ that contains exactly one free vertex of $A$ in $M$.

Let $E_1 = \{(a_1,p_1),(a_2,p_2),\ldots,(a_r,p_r)\}$ denote a new set of edges such that each $a_i$ belongs to $C \cap E(G)$ and no $p_i$ belongs to $C$. Let $G_1$ denote the graph $G \cup E_1$. Let $n_0$ denote the number of edges of a maximum matching of $G$. Assume also that each maximum matching of $G_1$ is computed by augmenting a maximum matching of $G$. Then we have:

1. If there exists $i$ such that $p_i \in E(G)$, then:
(a) Every maximum matching of $G_1$ contains $n_0$ edges of $E$ and one edge of $E_1$.

(b) The edge $(a_i, p_i) \in E_1$ such that $p_i \in E(G)$ belongs to some maximum matching of $G_1$.

(c) No edge $(a_j, p_j) \in E_1$ such that $p_j \in O(G) \cup U(G)$ belongs to any maximum matching of $G_1$.

(d) (i) Each vertex of $G \setminus C$ either has the same type in $G_1$ or (ii) it belongs to $E(G) \cup O(G)$ and $U(G_1)$. Each vertex of $C$ belongs to $U(G_1)$ or $(E(G) \cap O(G_1)) \cup (O(G) \cap E(G_1))$.

(e) No edge $(a, p)$ of $G$ such that one of its endpoints belongs to $O(G)$ and the other to $O(G) \cup U(G)$ belongs to any maximum matching of $G_1$.

2. If there exists no $i$ such that $p_i \in E(G)$, then:

(a) (i) Each vertex of $G \setminus C$ either has the same type in $G$ and $G_1$ or (ii) it belongs to $U(G)$ and $E(G_1) \cup O(G_1)$. Each vertex of $U(G)$ that belongs also to $E(G_1) \cup O(G_1)$ is reachable in $G_1$ from $a_0$ by an even/odd length $M$-alternating path. Each vertex of $C$ has the same type in $G$ and $G_1$.

(b) An edge $(a, p)$ such that $a \in U(G) \cap E(G_1)$ and $p \in O(G)$ belongs to some maximum matching of $G_1$. Every other edge $(a, p)$ of $G$ such that one of its endpoints belongs to $O(G)$ and the other to $O(G) \cup U(G)$ belongs to no maximum matching of $G_1$.

We say that a graph $G$ is reduced if it does not contain any edge $(u, v)$ such that either both $u$ and $v$ belong to $O(G)$ or exactly one of the vertices belongs to $U(G)$ and the other one to $O(G)$.

Lemma 5. Let $G = (A \cup P, E)$ be a reduced bipartite graph and $C$ a connected component of $G$ that contains exactly one free vertex of $A$ in a maximum matching of $G$. 

Figure 1: Thick edges belong to the matching.
Let $E_1 = \{(a_1,p_1),(a_2,p_2),\ldots,(a_r,p_r)\}$ and $E_2$ denote two new sets of edges such that each endpoint of a new edge belongs to $E(G)$. Also, each edge of $E_1$ connects a vertex of $C \cap A$ with a vertex not contained in $C$ and each edge of $E_2$ connects two vertices not belonging to $C$. Let $G_1,G_2$ and $H$ denote respectively $G \cup E_1,G \cup E_2$ and $G \cup E_1 \cup E_2$. We assume that we calculate maximum matchings of $G_1$ and $G_2$ by augmenting a maximum matching of $G$. A maximum matching of $H$ is calculated by augmenting a maximum matching of $G_1$ or $G_2$. $M_{12}$ is defined as the set of maximum matchings of $H$ that we get from augmenting a maximum matching of $G_1$. Similarly, $M_{21}$ denotes the set of maximum matching of $H$ that we get from augmenting a maximum matching of $G_2$. Let $n_0$ denote the number of edges of a maximum matching of $G$ and $n_2$ the number of edges of $E_2$ contained in a maximum matching of $G_2$ after we augment the maximum matching of $G$ to get a maximum matching of $G_2$. Then we have:

1. If there exists $i$ such that $p_i \in E(G_2)$, then every matching of $M_{12} \cup M_{21}$ contains $n_0$ edges of $E$, $n_2$ edges of $E_2$ and one edge of $E_1$.

2. If there exists no $i$ such that $p_i \in E(G_2)$, then:
   
   (a) A matching of $M_{12}$ contains $n_0$ edges of $E$, $n_2 - 1$ edges of $E_2$ and one edge of $E_1$. On the other hand, every matching of $M_{21}$ is a maximum matching of $G_2$.
   
   (b) A matching of $M_{12}$ that contains $(a_j,p_j)$ can be obtained from any matching of $M_{21}$ by applying any even length alternating path in $H$ starting at a free vertex of $C$, containing $(a_j,p_j)$ and an odd number of edges from $E_2$.
   
   (c) Additionally, if $p_j \in O(G_2)$, then the even length alternating path ends at a vertex $v \in E(G) \cap E(G_2)$. Also the endpoint that alternating path belongs to $E(H) \cap E(G_1) \cap E(G)$. Conversely, for every alternating path in $H$ that starts at a free vertex of $C$, contains $(a_j,p_j)$ and ends at a vertex $v \in E(G) \cap E(G_2)$, it holds that it contains an odd number of edges from $E_2$ and has even length.

5 Algorithm for updating a rank-maximal matching

In this section we present the algorithm for computing a rank-maximal matching of $H$. Its pseudocode is written as Algorithm 3. In Algorithm 3 for each $1 \leq i \leq r$, a matching $M_i$ denotes a rank-maximal
matching of \( G_i \). Also, for each \( r \geq j > i \) a matching \( M_j \) is contained in \( M_j \).

By phase \( i \) of Algorithm 3 we mean an \( i \)-ith iteration of the loop for from line 4. By \( C_i \) and \( R_i \) we denote \( C \) or \( R \), respectively, at the beginning of phase \( i \). By phase 0 we denote the part of Algorithm 3 before the start of phase 1. Depending on whether during phase \( i \) lines 9-14 or 16-18 are carried out, the phase is either called augmenting or non-augmenting.

We say that a vertex \( v \) is active in \( G_i' \) if \( v \in \mathcal{O}(G_i') \cup E(G_i') \) and not active or inactive (or unreachable) otherwise.

In Algorithm 3 the subgraph \( C_i \) may be viewed as the subgraph that encompasses the ("positive") changes between graphs \( H_i' \) and \( G_i' \). \( C_i \) always contains a new vertex \( a_0 \) that belongs to \( H \) and not to \( G \) as well as vertices that are at this point unreachable in \( G \), i.e., they belong to \( U(G_i') \). If there exist vertices that are active in \( H_i' \) but inactive in \( G_i' \), then they belong to \( C_i \). Also, any vertex of \( C_i \) is active in some graph \( H_i' \) such that \( j \leq i \) but inactive in \( G_j' \). Any edge that belongs to \( H_i' \setminus G_i' \) is either contained in \( C_i \) or one of its endpoints belongs to \( C_i \). Each such edge belongs to the set \( AE \) at some point and is called an activated edge. The subgraph \( C_i \) does not encompass all changes in particular, it may happen that some vertex \( v \notin C_i \) is active in \( G_i' \) but unreachable in \( H_i' \) or that some edge belongs to \( G_i' \setminus C_i \) but not to \( H_i' \).

During phase \( i \), we construct a graph \( \tilde{H}_i \) that contains every edge belonging to some rank-maximal matching of \( H_i \). At the beginning of phase \( i \), the set \( AV \) contains activated vertices, each of which is alive in \( H_i' \) but not in \( G_i' \).

We later show in Theorem 3 that a rank-maximal matching of \( H_i \) may be obtained from a rank-maximal matching \( M_i \) by the application of any alternating path \( s_i \in S_i \) and that every matching obtained in this way is rank-maximal. The paths \( s_i \) of \( S_i \) are defined as follows.

**Definition 2.** For each \( i \in \{1, 2, \ldots, r\} \) we define the set \( S_i \) of \( M_i \)-alternating paths contained in \( \tilde{H}_i \), each of which starts at \( a_0 \). Let \( AV \) denote the set \( AV \) at the end of a respective phase \( i \).

If at the end of phase \( i \) of Algorithm 3 there exists an edge of \( AE \) that connects a vertex of \( C \) with a vertex of \( E(G_i') \), then \( s_i \in S_i \) iff it is an \( M_i \)-augmenting path ending at any free vertex in \( M_i \). (Each such path contains exactly one edge of \( AE \).)

Otherwise, \( s_i \in S_i \) iff (i) it is an \( M_i \)-alternating path ending at any vertex of \( AV \) (each such path contains exactly one edge of \( AE \) or is a path of length 0) or (ii) it is an \( M_i \)-alternating path ending at any vertex of \( Alive(i) \) (each such path contains exactly one edge of \( AE \).

In order to obtain a rank-maximal matching \( N_r \) that differs from \( M_r \) in the smallest possible way, we choose that path \( s_r \in S_r \) which is shortest.

**Theorem 3.** For each \( i \in \{1, 2, \ldots, r\} \) it holds:

1. \( \tilde{H}_i \) contains every edge belonging to some rank-maximal matching of \( H_i \).  
2. \( C_i \) has the properties of \( C \) from Lemma 2 with respect to the matching \( M_i \). Also, \( C_i \) contains no vertex that is active in \( G_i' \) and at the beginning of phase \( i \) but after the execution of line 6 each edge of \( AE \) connects a vertex of \( C_i \) with a vertex of \( R_i \).  
3. For each \( s_i \in S_i \) a matching \( M_i \oplus s_i \) is a rank-maximal matching of \( H_i \) and a maximum matching of \( H_i \).  
4. Each rank-maximal matching \( N_i \) of \( H_i \) is such that \( M_i \oplus N_i \) contains one of the paths \( s_i \in S_i \).  
5. At the end of phase \( i - 1 \) the set \( AV \) contains all vertices that are active in \( H_i' \) and not active in \( G_i' \).
Case 1:

Phase $i$ of rank-maximal matching of $H_i$

Algorithm 3 for computing a rank-maximal matching of $H_i$

1: $C \leftarrow \{a_0\}$ (a new vertex)
2: $AV \leftarrow \{a_0\}$ activated vertices
3: $AE \leftarrow \emptyset$ activated edges
4: for $i = 1, 2, \ldots, r$ do
5: \hspace{1em} $R \leftarrow G'_i \setminus C$
6: \hspace{1em} $AE \leftarrow AE \cup \{(a, p) \in \tilde{E}_i : a \in AV \land p \in Alive(i)\}$
7: \hspace{1em} $\tilde{H}_i \leftarrow C \cup R \cup AE$
8: \hspace{1em} if each $(a, p) \in AE$ is such that $p \in U(G'_i) \cup O(G'_i)$ then
9: \hspace{2em} $AE_u \leftarrow \{(a, p) \in AE : p \in U(G'_i)\}$
10: \hspace{2em} for all every even-length $M_i$-alternating path $S$ in $\tilde{H}_i$ between $a_0$ and $a \in U(G'_i)$ that contains an odd number of edges of $(R \cap E_i) \setminus AE$ do
11: \hspace{3em} $C \leftarrow C \cup S$
12: \hspace{3em} $AV \leftarrow AV \cup \{a\}$
13: \hspace{3em} $AE_o \leftarrow \{(a, p) \in AE \cup G'_i : a \in C \land p \in O(G'_i)\}$
14: \hspace{3em} $AE \leftarrow AE \cup AE_o \setminus AE_u$
15: \hspace{1em} else (there exists $(a, p) \in AE$ such that $p \in E(G'_i)$)
16: \hspace{2em} for all $(a, p) \in AE$ such that $p \in O(G'_i) \cup U(G'_i)$ do
17: \hspace{3em} $AE \leftarrow AE \setminus \{(a, p)\}$
18: \hspace{1em} $AV \leftarrow \emptyset$
19: $N_r \leftarrow M_r \oplus s_r$, where $s_r$ - any path belonging to $S_r$
20: return $N_r$

Proof.

Claim 1. Suppose that $i > 1$ and $\tilde{H}_i$ contains every edge belonging to some rank-maximal matching of $H_i$. Then a maximum matching of $\tilde{H}_i$ that contains a rank-maximal matching of $H_{i-1}$ is a rank-maximal matching of $H_i$.

At the end of phase 0, the subgraph $C_0$ as well as the set $AV$ contains exactly one vertex $a_0$. We can notice that $H_1 = \tilde{H}_1 = H'_1$. Using Lemma 4 it is easy to check that the theorem holds for $i = 1$.

Suppose now that $i > 1$. We first argue that every edge of $H'_i \setminus H'_{i-1}$ is present in $\tilde{H}_i$. Every edge of $H'_i \setminus H'_{i-1}$ has both endpoints in the set of alive vertices of $H'_i$. If both endpoints of such an edge belong to $R_i$, then they are also alive in $G'_i$ (a vertex that is not alive in $G'_i \setminus C_i$ is also not alive in $H'_i$) and thus such an edge is included in $\tilde{H}_i$. If one of the endpoints $v$ of such edge $(v, w)$ belongs to $C_i$, then $v$ must belong to the set $AV$ - by the induction hypothesis point 4 and hence edge $(v, w)$ is added to $\tilde{H}_i$ in line 6.

The edges belonging to $H_{i-1} \setminus \tilde{H}_i$ are either those belonging to $G'_{i-1} \setminus G'_i$ or those removed in line 17 during phase $i - 1$. By Lemma 4 1c,e and 2e no such edge belongs to a maximum matching of $H_{i-1}$ and therefore by the induction hypothesis point 3, no such edge belongs to a rank-maximal matching of $H_{i-1}$.

We have thus proved that $\tilde{H}_i$ contains every edge belonging to some rank-maximal matching of $H_i$.

It is easy to see that $C_i$ satisfies all the properties stated in point 2 of the theorem.

We will show now that every matching $M_i \oplus s_i$ is a maximum matching of $\tilde{H}_i$ that contains a rank-maximal matching of $H_{i-1}$.

Suppose first that phase $(i - 1)$ is an augmenting phase.

Case 1: Phase $i$ is also an augmenting phase. There exists then an edge $(a, p) \in AE$ such that
$p \in E(G'_i)$. We claim that $p \in E(G'_{i-1})$. In phase $i - 1$ the edge $(a, p)$ also belongs to $AE$. If it were the case that $p \notin E(G'_{i-1})$, then $(a, p)$ would have been deleted during phase $i - 1$ in line 17. Since $(a, p)$ is such that $p \in E(G'_i)$, there exists in $\tilde{H}_i$ an $M_i$-augmenting path $T$ containing $(a, p)$ that ends at some free vertex $p'$ in $G'_i \setminus C_i$. The vertex $p'$ is also free in $G'_{i-1} \setminus C_{i-1}$. If $T$ is contained in $\tilde{H}_{i-1}$, then by definition of the set $S_{i-1}$ the path $T$ belongs to $S_{i-1}$ and thus by induction hypothesis, $M_{i-1} \oplus T$ is a rank-maximal matching of $H_{i-1}$. This then would mean that $M_i \oplus T$ is a rank-maximal matching of $H_i$, because $M_i \oplus T$ contains the same number of rank $i$ edges as $M_i$.

Assume then that the path $T$ is not contained in $\tilde{H}_i$. Let $T'$ denote the maximal subpath of $T$ that starts at $a_0$ and is contained in $\tilde{H}_{i-1}$. It must end at a vertex $p''$ that is free in $M_{i-1}$ and matched in $M_i$. We know that $M_{i-1} \oplus T'$ is a rank-maximal matching of $H_{i-1}$. The path $T'' = T \setminus T'$ connects two vertices that are alive in $G'_i$, one of which is free in $M_i$. Also, $T''$ is contained in $G'_i \setminus C_i$. Therefore, by Theorem 1 of \cite{6}, the matching $M_i \oplus T''$ is a rank-maximal matching of $G_i$. This means that $M_i \oplus T$ is a rank-maximal matching of $H_i$.

Case 2: Phase $i$ is non-augmenting. Let $n_0$ denote the number of edges of $M_i$ that have rank smaller than $i$ and $n_2$ the number of edges of $M_i$ with rank $i$. Let us note, that since phase $i - 1$ is augmenting, each matching $M_{i-1} \oplus s_{i-1}$ contains one edge more than $M_{i-1}$. A maximum matching of $\tilde{H}_i$ has the same cardinality as $M_i$. Therefore, a rank-maximal matching of $H_i$ contains at most $n_2 - 1$ edges of rank $i$.

Each $s_i$ from $S_i$ is an even length $M_i$-alternating path that contains an odd number of edges of rank $i$. By Lemma \ref{maximal_matching}, the matching $M_i \oplus s_i$ is a maximum matching of $\tilde{H}_i$ that has one edge of $AE$, $n_0$ edges not belonging to $AE$ and of rank strictly smaller than $i$ and $n_2 - 1$ edges of rank $i$. Each of the paths $s_i \in S_i$ contains some path $s_{i-1} \in S_{i-1}$. Also each edge of $AE$ belongs to $S_i \cap S_{i-1}$. It is so because each edge $(a, p)$ that belongs to $AE$ at the beginning of phase $i$ is such that $p \in E(G'_{i-1})$ and thus $(a, p)$ is contained in some $s_{i-1}$ as well as some $s_i$. Hence, $M_i \oplus s_i$ is a rank maximal matching of $H_i$.

Suppose now that phase $i - 1$ is non-augmenting and phase $i$ is augmenting. It means that there exists an edge $e = (a, p) \in AE$ such that $p \in E(G'_i)$. If the edge $e$ did not belong to $AE$ in phase $i - 1$, it means that $a \in AV$ and thus there exists a path $s_{i-1}$ ending at $a$, whose application to $M_{i-1}$ yields a rank-maximal matching of $H_{i-1}$. Also, in $G'_i \setminus C_i$ there exists an even length $M_i$-alternating path $T$ that starts at $p$ and ends at a free vertex $p'$. By Theorem 1 of \cite{6}, the matching $M_i \oplus T$ is rank-maximal in $G_i$. The edge $e$ clearly has rank $i$. Therefore $M_i \oplus (\{e\} \cup T \cup s_{i-1})$ is a maximum matching of $\tilde{H}_i$ that contains a rank-maximal matching of $H_{i-1}$ and has one edge of rank $i$ more than $M_i$ - therefore it is rank-maximal in $H_i$.

If the edge $e$ did belong to $AE$ in phase $i - 1$, then $p \in O(G'_{i-1})$. In $G'_i \setminus C_i$ there exists an even length $M_i$-alternating path $T$ that starts at $p$ and ends at a free vertex $p'$. Let $T'$ denote the maximal subpath of $T$ starting at $p$ and contained in $G'_{i-1} \setminus C_i$ and let $T''$ denote $T \setminus T'$. Let also $s$ denote an $M_i$-alternating path from $a_0$ to $p$. We notice that $T''$ ends at a vertex $p''$ alive in $G'_{i-1}$ and thus $M_{i-1} \oplus (s \cup T')$ is rank-maximal in $H_{i-1}$. Let us note that the edge $e' = (p', a'')$ is of rank $i$. Also, $M_i \oplus (T'' \setminus e'')$ is rank-maximal in $G_i$. Therefore $M_i \oplus (s \cup T)$ is rank-maximal in $H_i$.

The analysis of the case when both phases $i - 1$ and $i$ are non-augmenting is similar and is left for the reader.

If phase $i$ is augmenting, then we set the set $AV$ as empty. This is because any vertex $a$ that is alive in $H'_i$ and thus belongs to $E(H'_j)$ for every $j < i$ will belong to $O(H'_j) \cup U(H'_i)$ - this follows from Lemma \ref{maximal_matching} 1d.

\textbf{Theorem 4.} Algorithm 3 runs in $O(\min(rn, n^2) + m)$ time.
Figure 3: On the right-hand side there is the graph $H$ obtained by adding the vertex $a_0$ and the edge $(a_0, p_0)$ to $G$. The graph $\tilde{H}_2$ is presented on the left-hand side. In this particular example the graph on the right-hand side is also equal to $\tilde{H}_3$ (i.e. $H = \tilde{H}_3$). The vertices inside the ellipse form the set $C$. Label of each edge is equal to its rank.

**Proof.** Without any additional assumptions the running time of Algorithm 3 is $O(rn + m)$. The executions of line 8 and 15 may require $O(rn)$ time as every edge $(a, p)$ may belong to $AE$ for a number of phases and in each one of them we need to check to which of the sets $E(G_i'), U(G_i'), O(G_i')$ the endpoint $p$ belongs. The rest of the time the algorithm takes is $O(m)$.

The runtime of the algorithm may be reduced if we assume that for each vertex $v$ we maintain a list $i_1 < i_2 < \ldots < i_k$ which denotes the changes of type among reduced graphs $G_i'$, i.e., $v$ belongs to $O(G_{i_1}'), E(G_{i_2}'), \ldots, U(G_{i_k}')$. Each such list has length at most the number of augmentations in $G$, which is at most $n$.

**Theorem 5.** Assuming we are given the reduced graphs $G_1', \ldots, G_r'$, the reduced graphs of $H$ can be computed $O(\min(rn, n^2) + m)$ time.

### 6 Example of how Algorithm 3 works

Let us take a look at what Algorithm 3 does when executed on the graph presented on Figure 3. At the beginning of the first iteration $C$ contains only vertex $a_0$, $R$ contains the rest of the graph. The
Figure 4: The figure presents the graph $H$ which is obtained by adding the edge $(a_0,p_1)$ to $G$. Thick edges belong to a rank-maximal matching of $G$.

vertex $a_0$ is added to the set $AV$. The edge $(a_0,p_0)$ is added to the set $AE$ of activated edges. Since $p_0 \in U(G_1')$, sets $C$ and $AV$ are updated. The set $C$ from now on contains the subgraph inside the ellipse on the left-hand side of Figure 3 and $AV$ contains the vertices $a_0$ and $a_1$. In order to get a rank maximal matching of $H_1$, we may apply one of the two alternating paths - each one starts at $a_0$ and one of them finishes at $a_1$ and the other at $a_0$ (a zero-length path).

In the second iteration the algorithm first updates the set of activated edges. The updated graph is presented on the left-hand side of Figure 3. At this point the set $AE$ contains two edges incident to $a_1$ and crossing the ellipse. Since both endpoints of edges of $AE$ at this point belong to $E(G_2')$, the algorithm enters an augmenting phase. The set $AV$ is reset to empty. In order to obtain a rank-maximal matching of $H_2$ any augmenting path of $H_2$ may be applied. Sets $C$ and $R$ remain the same.

In the third iteration the algorithm first updates the sets $AV$ and $C$. The updated graph is presented on the right-hand side of Figure 3. Since there are no edges in $AE$ such that both endpoints belong to $E(G_3')$, the algorithm enters a non-augmenting phase. The set $AE$ contains an edge of rank 2 crossing the ellipse. Vertices which are alive and reachable from $a_0$ after this iteration are denoted as $a_4$ and $a_5$. In order to obtain a rank-maximal matching of $H_3$ we can either apply an even length alternating path starting at $a_0$ and ending at an alive vertex or apply an even length alternating path starting at $a_0$ and ending at a vertex of $AV$.

Let us now take a look at the example presented on Figure 4. This example gives us an idea about changing from an augmenting phase to a non-augmenting phase and vice versa. After the first iteration, the vertex $a_0$ is an activated vertex and $(a_0,p_1)$ is an activated edge and the algorithm enters a non-augmenting phase. In the second iteration the edge $(a_1,p_2)$ of rank 2 becomes activated. Also $p_2$ is an even vertex in $G_2'$, thus the algorithm enters an augmenting phase and a rank-maximum
matching in \( H_2 \) can be obtained by applying the augmenting path containing the edge \((a_1, p_2)\). The vertex \( p_2 \) is still reachable from \( p_7 \) by an even length alternating path in the third iteration. Thus \( p_2 \) remains even in \( G'_5 \) and in the third iteration algorithm remains in an augmenting phase.

In the fourth iteration \( p_2 \) becomes unreachable in the \( G'_4 \), so the algorithm enters a non-augmenting phase. Since \( p_2 \) is unreachable in the \( G'_4 \) graph, the algorithm updates the set \( C \). At this point the set \( C \) consists of vertices belonging to two alternating paths 1) a path from \( a_0 \) to \( a_5 \) and 2) a path from \( a_0 \) to \( a_9 \). The edge \((a_9, p_9)\) becomes activated in the 5th iteration. Since \( p_9 \) is an even vertex in \( G'_5 \), the algorithm again enters an augmenting phase. In sixth iteration \( p_9 \) becomes unreachable thus the algorithm is again in a non-augmenting phase and the set \( C \) is again updated. Updated set \( C \) at this point contains all the vertices of the graph with the exception of \( a_{12} \) and \( p_{12} \). Finally, in order to get a rank maximal matching in \( H \), we can apply an alternating path from \( a_0 \) to \( a_{10} \) as this vertex is alive in the sixth iteration.

7 Updates of reduced graphs \( H'_i \)

Algorithm 3 computes graphs \( \tilde{H}_i \) such that \( \tilde{H}_i \) contains every edge that belongs to some rank-maximal matching of \( H_i \). Note that it is possible that \( H'_i \) contains an edge that does not belong to \( \tilde{H}_i \). Each such edge does not belong to any rank-maximal matching of \( H_i \). Thus both of its endpoints belong to \( U(H'_i) \). We will deal with such edges at the end. Our more important goal is to determine which edges of \( \tilde{H}_i \) do not belong to \( H'_i \).

By Theorem 3 point 4 we know that any rank-maximal matching \( N_i \) of \( H_i \) is such that \( M_i \oplus N_i \) contains some path \( s_i \in S_i \) is a good path.

**Lemma 6.** A vertex \( v \) belongs to \( E(H'_i) \) (resp. \( O(H'_i) \)) iff in the graph \( \tilde{H}_i \) there exists an \( N_i \)-alternating path \( P \) from a free vertex \( u \) to a vertex \( w \in \text{Alive}(i) \cup AV_i \) such that \( v \) lies on \( P \) and its distance on \( P \) from \( u \) is even (respectively, odd).

**Proof.** Suppose first that \( v \in E(H'_i) \). Then in \( H'_i \) there exists an \( N_i \)-alternating path \( P \) from a free vertex \( u \) to a vertex \( w \in \bigcap_{j=1}^{i} E(H'_j) \) such that \( v \) lies on \( P \) and its distance on \( P \) from \( u \) is even. By Theorem 3 point 4 \( N_i \oplus P \) is also a rank-maximal matching of \( H_i \). This means that every edge of \( P \) belongs to some rank-maximal matching of \( H_i \) and therefore by Theorem 3 point 1 the graph \( \tilde{H}_i \) contains each edge of \( P \). By Theorem 3 point 5 the set of vertices \( \bigcap_{j=1}^{i} E(H'_j) \) is equal to \( \text{Alive}(i) \cup AV_i \).

The case when \( v \in O(H'_i) \) is analogous.

Let us assume now that in the graph \( \tilde{H}_i \) there exists an \( N_i \)-alternating path \( P \) from a free vertex \( u \) to a vertex \( w \in \text{Alive}(i) \cup AV_i \) such that \( v \) lies on \( P \) and its distance on \( P \) from \( u \) is even. It suffices to show that every edge of \( P \) belongs to some rank-maximal matching of \( H_i \), because it will mean that every edge of \( P \) belongs to \( H'_i \).

If every edge of \( N_i \cap P \) belongs also to \( M_i \), then the whole path \( P \) lies in \( G'_i \setminus C_i \) and thus by Lemma 4 from [9] \( N_i \oplus P \) is also a rank-maximal matching of \( H_i \) and we are done.

In order to show this it suffices to use the fact that \((M_i \oplus s_i) \oplus P = M_i \oplus (s_i \oplus P)\).

Hence, the matching \( N'_i = M'_i \oplus s' \) is a rank-maximal matching of \( H_i \) and thus we have shown that every edge of \( P \) either belongs to \( N_i \) or to \( N'_i \).

\[ \square \]

It remains to show how to efficiently compute \( \text{EG} \)-decompositions of graphs \( H'_i \) given \( \tilde{H}_i \). Note that we cannot simply apply Lemma 6 multiple times for each of the graphs \( \tilde{H}_i \), as such an approach would lead to an algorithm of complexity \( O(rm) \). Below we describe a general idea behind Algorithm 3 for computing \( \text{EG} \)-decompositions of \( H'_i \), then in Lemma 7 we show that it is possible to implement this algorithm to achieve an \( O(m + \min(rn, n^2)) \) runtime.
From Lemma 8 we know that if $P$ is a path in $\tilde{H}_i$ from a free vertex $u$ to a vertex $w \in \text{Alive}(i) \cup AV_i$, then we can determine types of all vertices of $P$ in $H'_i$. Additionally from Lemma 8 it is possible to determine types of such vertices in graphs $H'_j$ for each $j < i$. The above observations are a basis of Algorithm 4. We start with $v \notin X_i$ and determine the set $Z_i$ of all vertices belonging to paths as described above (i.e. from a free vertex $u$ to a vertex $w \in \text{Alive}(i) \cup AV_i$). Then we update the type of each vertex from $Z_i$ using Lemma 8, set $i \leftarrow i - 1$ and repeat the process for the new graph $\tilde{H}_i$. We continue iterating over $i$ until we reach $i = 0$. Note that if for some vertex $v$ we have $v \in Z_i$ and $v \notin Z_j$ for each $j > i$, then we have $v \notin U(H'_j)$ for each $j > i$. Thus we can correctly determine types of all the vertices using this approach.

Of course a naive implementation of the above idea does not achieve an $O(m + \min(rn, n^2))$ runtime. Additional observations are needed. Let $v$ be any vertex. First we note that if $i$ is maximal such that $v$ belongs to $Z_i$ then types of $v$ in graphs $H'_1, H'_2, \ldots, H'_i$ can be correctly determined. There is no need to update the type of $v$ anymore even if it belongs to some $Z_j$ for $j < i$. Thus throught the execution of the algorithm we maintain the set $Z$ of vertices for which we have already computed types and make sure to only update the types of vertices belonging to $Z_i \setminus Z$. In order to speed up the algorithm we need to show how to efficiently compute sets $Z_i \setminus Z$.

Let us first show how to find vertices belonging to $Z_i$. We first build an $N_i$-alternating forest of vertices reachable from the set $F_i$ of free vertices with respect to $N_i$. Then we determine the set $X$ of vertices belonging to $T_i$ and $\text{Alive}(i) \cup AV$. Next we consider a graph $(V(T_i), W_i)$ where $W_i$ is the set of edges with both endpoints in $T_i$. It is easy to see that all vertices reachable by alternating paths from $X$ in this graph form the set $Z_i$. Note that from Lemma 8 it follows that $V(T_i) \subseteq V(T_j)$ for $j < i$, hence we do not have to build the alternating forest from scratch in each iteration. Instead for each $i$ we simply determine $T_i$ using the forest $T_{i+1}$. The set $Z_i \setminus Z$ can be determined similarly to the set $Z_i$. Instead of considering a graph $(V(T_i), W_i)$ we simply consider a graph $(V(T_i) \setminus Z, W_i)$ and claim that $Z_i \setminus Z$ is equal to the set of vertices reachable from $X$ in this graph.

Computations of forests $T_i$ take $O(m + \min(nc, n^2))$ time in total. It is a straightforward consequence of the fact that $V(T_i) \subseteq V(T_{i-1})$. Similarly we can see that the time needed to compute all vertices reachable from $X$ in graphs $(V(T_i) \setminus Z, W_i)$ over the duration of the algorithm is also bounded by $O(m + \min(nc, n^2))$. It is a consequence of the fact that once a vertex $v$ is detected to be in $Z_i \setminus Z$ it is added to $Z$ and none of the edges incident to such a vertex is visited in any of the following iterations.

From the above discussion we obtain the correctness of the following lemma.

**Lemma 7.** Algorithm 4 computes $\text{EG}$-decompositions of graphs $H'_i$ in $O(m + \min(rn, n^2))$ time.

**Lemma 8.** Let $i_1 < i_2 < \ldots < i_k$ denote the list of numbers of augmenting phases of Algorithm 4. Let $n_A$ denote the number of augmenting phases of Algorithm 4 and $n_A(i)$ the number of augmenting phases till phase $i$ and including phase $i$.

1. Free vertices in $N_i$ are the same in $H'_i$ and $\tilde{H}_i$.

2. If a vertex $v$ is reachable in $\tilde{H}_i$ from a free vertex in $N_i$ via an $N_i$-alternating path and ending at an alive vertex, then:

   (a) $v$ is reachable in $\tilde{H}_j$ from a free vertex in $N_j$ via an $N_j$-alternating path ending at an alive vertex for every $j < i$.

   (b) for every $j \leq i$ such that $v \in E(G'_j) \cup O(G'_j)$ $v$ has the same type in $H'_j$ and $G'_j$, i.e., $(v \in E(G'_j) \iff v \in E(H'_j))$ and $(v \in O(G'_j) \iff v \in O(H'_j))$. 

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Algorithm 4 for computing $EG$-decompositions of graphs $H'_i$

1: $Z \leftarrow \emptyset$
2: $F \leftarrow \emptyset$
3: $N_{r+1} \leftarrow N_r$
4: for $i = r, r-1, \ldots, 1$ do
5:    $N_i \leftarrow N_{i+1} \setminus F_{i+1}$
6:    $F_i \leftarrow$ free vertices with respect to $N_i$
7:    $T_i \leftarrow$ $N_i$-alternating forest in $H_i$ starting from vertices of $F_i$
8:    $X \leftarrow$ vertices belonging to $T_i$ and $\text{Alive}(i) \cup AV$
9:    $W_i \leftarrow$ edges of $H_i$ with both endpoints in $T_i$
10:   $Z_i \leftarrow$ all vertices reachable in $(V(T_i) \setminus Z, W_i)$ from $X$ via an $N_i$-alternating path
11:   for all $v \in Z_i \setminus Z$ do
12:       for every $j > i$, $v \in U(H'_j)$
13:          $v \in E(H'_j)$ (resp. $O(H'_j)$) if $v$ is reachable via an even (resp. odd) length $N_i$-alternating path
14:       if $v \in E(G'_i) \cup O(G'_i)$ then
15:          for every $j \leq i$, $v$’s type in $H'_j$ is the same type as in $G'_j$
16:       else
17:          for every $j < i$ such that $v \in U(G'_j)$, $v$’s type in $H'_j$ is determined as in Lemma 8.2c
18:          for every $j < i$ such that $v \in E(G'_j) \cup O(G'_j)$, $v$’s type in $H'_j$ is the same type as in $G'_j$
19:     $Z \leftarrow Z \cup Z_i$
20: for all $v \in V \setminus Z$ do
21:    $v \in U(H'_i)$ for every $i$
(c) for every $v \in U(G'_j)$ and every $j < i$, $v$ has the same type in $H'_i$ as in $H'_j$ if $n_A(i)$ has the same parity as $n_A(j)$ and otherwise, $v$ belongs to $O(H'_i)$ iff it belongs to $E(H'_i)$ and $v$ belongs to $E(H'_i)$ iff it belongs to $O(H'_i)$.

Proof. We first prove (1). From Theorem 3 point 3 we know that $N_i$ is a rank maximal matching of $H_i$ and also $N_i$ is a maximum matching of $H'_i$. Theorem 3 point 3 implies that $N_i$ is a maximum matching of $\tilde{H}_i$. Both $H'_i$ and $\tilde{H}_i$ have the same set of vertices, thus free vertices with respect to $N_i$ are the same in $H'_i$ and $\tilde{H}_i$ and (1) holds.

2(a) This part directly follows from the lemma 6.

2(b) Let $v \in E(G'_j) \cup O(G'_j)$. This assumption implies that $v \notin C$. From Lemma 4 points 1d and 2a we know that $v$ has the same type in $G'_j$ and $\tilde{H}_j$. Additionally $v$ is reachable from a free vertex by an alternating path ending at an alive vertex in the graph $\tilde{H}_j$. From Lemma 6 we know that $v \in E(\tilde{H}_j)$ if and only if $v \in E(H'_j)$. Similarly we have $v \in O(\tilde{H}_j)$ if and only if $v \in O(H'_j)$, hence 2(b) is proven.

2(c) Let $v \in U(G'_j)$ for some $j < i$. We can assume that $AE$ contains more than one edge. There is a path from the free vertex in $C$ to $v$ in $\tilde{H}_j$, so $v \in C$. Without loss of generality, we assume that the alternating path from the free vertex in $C$ and $v$ is of even length. Therefore in each phase, $v$ is an even vertex before we apply the alternating path starting from $C$. Since, the free vertex in $C$ belongs to $A$, $v$ belongs to $A$. Let $j'$ be the next augmenting phase after $j$ ($j < j' < i$). In phase $j'$ there exists an edge in $AE$ such that both endpoints are even vertices in $\tilde{H}_{j'}$. Then by Lemma 4 point 1e the type of the vertex $v$ changes, thus $v \in O(\tilde{H}_{j'})$ after the augmentation. Therefore, for each augmenting phase, the type of $v$ changes from even to odd. Let us assume that $j''$ is a non-augmenting phase such that $j < j'' < i$. In the iteration $j''$ there is no edge in $AE$ such that both endpoints are even vertices. So by Lemma 4 point 2a the vertex $v$ retains its type, so $v$ is an even vertex in $\tilde{H}_{j''}$. Thus, for every non-augmenting phase, $v$ is an even vertex. Also each path containing $v$ starts from a free vertex and ends at an alive vertex. Hence the vertex $v$ has the same type in both $\tilde{H}_j$ and $H'_j$ for each $j < i$ and from the above it easily follows that 2(c) holds.

$$\square$$

8 Proofs of Lemmas 4 and 5

Proof of Lemma 4

Proof. We first prove 1(a) − 1(e). Let us assume that there exists $i$ such that $p_i \in E(G)$.

Consider a maximum matching $M$ in the graph $G$. From the fact that $a_i, p_i \in E(G)$ and Lemma 1 we can see that in $G$ there exist an $M$-alternating path $P_1$ from a free vertex $f_1$ to $a_i$ and an $M$-alternating path $P_2$ from a free vertex $f_2$ to $p_i$. Thus paths $P_1$ and $P_2$ along with the edge $(a_i, p_i)$ form an $M$ augmenting path $P^e$ in $G_1$. From the fact that $C$ is a connected component one can easily see that vertices of $P_1$ are contained in $C$ (i.e., $V(P_1) \subseteq C$). Similarly we have $V(P_2) \subseteq A \cup P \setminus C$. Let $M' = M \oplus P^e$. Obviously $M'$ contains $n_0$ edges of $E$ and one edge of $E_1$.

We now prove that $M'$ is a maximum matching of $G_1$. Assume by contradiction that there exists a matching $M''$ such that $|M''| = |M| + 2$. The symmetric difference $M'' \oplus M$ contains two $M$-augmenting paths in $G_1$. At least one of these paths has both endpoints in $A \cup P \setminus C$. Let this path be $X$. The path $X$ has to contain at least two edges of $E_1$ as otherwise it would be contained in the graph $G$, contradicting the maximality of $M$. Let $x_1$ and $x_2$ be respectively first and last edge of $E_1$ on the path $X$. The path $X$ can be split into the following subpaths $(X_1, x_1, Y, x_2, X_2)$. 

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Note that $X_1$ starts with an unmatched vertex and ends with a matched edge, thus the length of $X_1$ is even. Similarly the length of $X_2$ is even. Additionally both endpoints of $Y$ belong to $A$, hence $Y$ is also of even length. It is a contradiction as the length of $X$ is clearly even despite the fact that we assumed that $X$ is an augmenting path.

Note that from the above discussion it already follows that 1(b) holds. Let us now prove 1(a). We are going to show that every maximum matching of $G_1$ contains at most one edge of $E_1$. Assume by contradiction that there exists a maximum matching $M_1$ of $G_1$ containing at least two edges of $E_1$. Since $|M_1| = |M| + 1$ there is an $M$-augmenting path $P$ in $G_1$ containing at least two edges of $E_1$. Consider a subpath $X$ of $P$ between any two consecutive edges of $E_1$. We have $E_1 \cap M = \emptyset$ thus $X$ must begin and end with matched edges. Also note that both endpoints of $X$ belong to the same side of the bipartition of $G_1$, thus $X$ is an alternating path of even length. It is a contradiction, hence 1(a) follows.

Now we prove 1(c). Assume by contradiction that some $(a_j, p_j)$ belongs to a maximum matching $M_1$ and that $p_j \notin O(G) \cup U(G)$. Consider an $M$-augmenting path $P$ belonging to $M \oplus M_1$. From 1(a) it follows that the path $P$ can be split into subpaths $(P_1, (a_j, p_j), P_2)$, where $P_1$ is contained in $C$ and $P_2$ in $A \cup P \setminus C$. The path $P_2$ is an even length $M$-alternating path in $G$ from a free vertex, ending at $p_j$, contradicting the fact that $(a_j, p_j) \notin M_1$. Assume by contradiction that $(x, y) \in M_1$. Let $P'$ be a subpath of $P$ from a free vertex $v_0$ to $y$. Note that $M_1 \oplus P'$ is a maximum matching in $G$ of size $|M| + 1$ - a contradiction. Thus $(x, y) \notin M_1$ and we have $y \in O(G)$. From the definition of $E_1$ we also have $y \in E(G)$. One can easily see that if a vertex $x \in P \cap C$ belongs to $O(G_1)$ then it belongs to $E(G)$ and also that if $x \in C \cap E(G_1)$ then $x \in O(G)$. The case $v \in C$ of 1(d) easily follows from the fact that the choice of $P$ is arbitrary.

Let us now consider the case $v \in A \cup P \setminus C$. It suffices to show that $v \in E(G_1)$ implies $v \in O(G)$ and that $v \in O(G)$ implies $v \in E(G)$. Let us consider a maximum matching $M_1$ in $G_1$ and any alternating path $P$ from a free vertex $v_0 \in A \cup P \setminus C$ crossing the set $C$. Let $(x, y)$ be the first edge of $E_1$ on this path and $x \notin C$, $y \in C$. Our goal is to show that $(x, y) \notin M_1$. Assume by contradiction that $(x, y) \in M_1$. Let $P'$ be a subpath of $P$ from a free vertex $v_0$ to $y$. Note that $M_1 \oplus P'$ is a maximum matching in $G$ of size $|M| + 1$ - a contradiction. Thus $(x, y) \notin M_1$ and we have $y \in O(G)$. From the definition of $E_1$ we also have $y \in E(G)$. One can easily see that if a vertex $x \in P \cap C$ belongs to $O(G_1)$ then it belongs to $E(G)$ and also that if $x \in C \cap E(G_1)$ then $x \in O(G)$. The case $v \in C$ of 1(d) easily follows from the fact that the choice of $P$ is arbitrary.

Let us now consider the case $v \in A \cup P \setminus C$. It suffices to show that $v \in E(G_1)$ implies $v \in E(G)$ and that $v \in O(G_1)$ implies $v \in O(G)$. Let $v \in E(G_1)$. From Lemma 1 there exists a maximum matching $M_1$ in $G_1$ such that $v$ is unmatched. One edge of $M_1$ belongs to $E_1$. If we remove this edge we obtain a matching $M_2 = M_1 \cap E$ which is maximal in $G$. Note that $v$ is still unmatched in this matching, hence we have $v \in E(G)$. Assume now that $v \in O(G_1)$. We also consider a maximum matching $M_1$ in $G_1$. From Lemma 1 the vertex $v$ is matched to a vertex $w$ such that $w \in E(G_1)$. From the above discussion it follows that $w \in E(G)$. If $(v, w) \notin E_1$ then obviously we have $v \in O(G)$. Assume that $(v, w) \in E_1$. From the assumption that $v \in A \cup P \setminus C$ it follows that $w \in C$. Recall that we have $w \in E(G) \cap E(G_1)$ - a contradiction with the previously proven part of 1(d). Thus 1(d) is proven.

1(e). Assume by contradiction that there is an edge $(a, p)$ of $G$ such that $a \in O(G)$, $p \in O(G) \cup U(G)$ and $(a, p)$ belongs to a maximum matching $M_1$ of $G_1$. Consider a matching $M' = M \cap E$. From 1(a) we have $(a, p) \in M'$, $|M'| = |M_1| - 1$ and $M'$ is a maximum matching of $G$. However from Lemma 1 it follows that $(a, p)$ does not belong to any maximum matching of $G$ - a contradiction.

Let us now prove that 2(a) - 2(d) hold. Assume that there is no $i$ such that $p_i \in E(G)$.

2(a) We first show that $M$ remains a maximum matching in $G_1$. Assume by contradiction that there is an $M$-augmenting path $P$ in $G_1$. Let $x$ be an endpoint of the path not belonging to $C$ and let $(a_j, p_j)$ be the first edge of $P$ going from $x$ belonging to $E_1$. The existence of a subpath from $x$ to $p_j$ implies that $p_j \in E(G)$ - a contradiction. Thus every maximum matching of $G$ is also a maximum matching of $G_1$. We now show that a maximum matching of $G_1$ can contain at most one edge of $E_1$. Let us assume by contradiction that there exists a matching $M'$ containing at least two such edges $e_1$ and $e_2$. Consider a symmetric difference $M \oplus M'$. First note that the fact that
$G_1$ is bipartite implies that neither $e_1$ nor $e_2$ belong to an alternating cycle of $M \oplus M'$. Thus at least one of these two edges belongs to an alternating path starting from a free vertex $x' \notin C$. This fact implies that the endpoint of this edge belonging to $P$ also belongs to $E(G)$ - contrary to our initial assumption. Thus $2(a)$ holds.

2(b) Let $(a_j, p_j)$ be any edge of $E_1$. Since $a_j \in E(G)$, there is a an $M$-alternating path $P$ in $G$ from a free vertex in $C$ to $a_j$. Note that $p_j \notin E(G)$ and Lemma 1 imply that $p_j$ is matched in $M$ to some $M(p_j) \neq a_j$. The path $P$ together with edges $(a_j, p_j)$ and $(p_j, M(p_j))$ form an $M$-alternating path $P'$ in $G_1$. Clearly $M \oplus P'$ is a maximum matching in $G_1$ containing the edge $(a_j, p_j)$, thus $2(b)$ holds.

2(c) Recall that $M$ is a maximum matching in both $G$ and $G_1$. Every vertex of $C$ is from definition reachable in $G$ from a free vertex $x$ belonging to $C$ by an $M$ alternating path. Obviously each such a path is also an $M$-alternating path in the graph $G_1$. This fact and Lemma 1 imply that types of vertices belonging to $C$ are identical in both $G$ and $G_1$. Let us assume that there is a vertex $v \notin C$ such that its types in $G$ and $G_1$ differ. Note that if it was $v \in E(G) \cup O(G)$ then there would be an $M$-alternating path in $G$ from a free vertex belonging to $A \cup P \setminus C$ to $v$. Such a path would also remain an $M$-alternating path in $G_1$ contradicting the assumption that types of $v$ in $G$ and $G_1$ differ. Thus there is $v \in U(G)$ and $v \in E(G_1) \cup O(G_1)$. From Lemma 1 there exists an $M$-alternating path $P$ from a free vertex $x$ to $v$. We know that $v \in U(G)$ thus at least one edge of $E_1$ belongs to $P$. From the fact that $G_1$ is bipartite it easily follows that exactly one edge of $E_1$ belongs to $P$. Note that this fact also implies that $P$ starts in the only free vertex of $C$, thus $2(c)$ holds.

2(d) Let $(a, p)$ be an edge such that $a \in U(G) \cup E(G_1)$ and $p \in O(G)$. So $a$ is reachable from a free vertex $a_0$ in $C$ by an even length $M$-alternating path. Therefore, $p$ is reachable from $a_0$ by an odd length $M$-alternating path in $G_1$. If we apply the alternating path, then $p$ is matched in some maximum matching of $G_1$.

Now let $(a, p)$ be an edge such that $a \in U(G) \cup O(G)$ and $p \in O(G)$. So $p \in O(G_1)$. Therefore the edge $(a, p)$ is an $OO$ or $OU$ type edge in $G_1$. So the edge $(a, p)$ is never matched in any maximum matching of $G_1$.

Proof of Lemma 5

**Proof.** Let $G$ be a bipartite graph and $G_1$, $G_2$ and $H$ denote respectively $G \cup E_1, G \cup E_2$ and $G \cup E_1 \cup E_2$.

(1) Let us assume that $M_2$ be a maximum matching of $G_2$. Let both endpoints of the edge $(a_i, p_i) \in E_1$ belong to $E(G_2)$. When we add $E_1$ to $G_2$, we have an $M_2$-augmenting path in $H$. First, we show that the $M_2$-augmenting path contains even number of edges from $E_2$. Every augmenting path contains odd number of edges. Here, the $M_2$-augmenting path contains one edge from $E_1$, some even length path segments consisting of the edges from $E$ and some edges from $E_2$. So the number of edges from $E_2$ is even. Also two edges from $E_2$ are separated by an even length path segment consisting of the edges from $E$. So we can say that in the $M_2$-augmenting path, between every two matched edges from $E_2$, there is an unmatched edge from $E_2$ and vice versa. So half of the edges of $E_2$ in the $M_2$-augmenting path are matched. Therefore, when we apply the $M_2$-augmenting path, the number of matched edges of $E$ and $E_2$ remains the same. The number of matched edge from $E_1$ becomes 1. Also by the construction, the matching we get belongs to $M_{21}$. So every matching of $M_{21}$ contains $n_0$ edges of $E$, $n_2$ edges of $E_2$ and one edge of $E_1$. 

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Next we show that $M_{21}$ and $M_{12}$ are the same sets. This will prove that every matching of $M_{12}$ also contains $n_0$ edges of $E$, $n_2$ edges of $E_2$ and one edge of $E_1$.

Let us consider a matching $M \in M_{21}$. First we construct a maximum matching $G$. Let us consider the matching $M$ of $M_{21}$ induced in $G$. This matching contains $n_0$ edges. We know that a maximum matching of $G$ contains $n_0$ edges. So this matching is a maximum matching of $G$ and we denote is as $M'$.

Next we show that $M \in M_{12}$. First we take the symmetric difference $M \oplus M'$. The symmetric difference only contains one edge from $E_1$ and some edges from $E_2$. These edges are vertex disjoint. First we consider the graph $G$ and its maximum matching $M'$. Next we add $E_1$ to $G$ and match the edge from $E_1$, that is matched in $M$. So we get a maximum matching of $G_1$. Next we add $E_2$ to $G_1$ and apply rest of the length 1 augmenting paths. Finally we get the matching $M$. But we get this matching by augmenting a maximum matching of $G_1$. So $M \in M_{12}$. Therefore, $M_{21} \subseteq M_{12}$.

Similarly, it can be shown that $M_{12} \subseteq M_{21}$. Thus we can conclude that $M_{12} = M_{21}$.

2(a) There are two ways to get a maximum matching $H$ from a maximum matching $M$ of $G$. In the first case, we add $E_2$ to the graph $G$ to get a maximum matching $M_2$ of the graph $G_2$. There does not exist any edge $(a_i, p_i) \in E_1$ such that $p_i \in E(G_2)$. When we add the edges from $E_1$ to $G_2$, we get the graph $H$. Since we do not get any $M_2$-augmenting path in $H$, the maximum matching of $G_2$ is a maximum matching of $H$. By the definition in the statement of the lemma, this maximum matching of $H$ belongs to $M_{21}$. So, a matching of $M_{21}$ is a maximum matching of $G_2$.

Next we prove that the matching that is constructed by the other case, is a matching from $M_{12}$. First we add $E_1$ to $G$ to get the graph $G_1$. Since both endpoints of the edges from $E_1$ belong to $E(G)$, we get some $M$-augmenting paths in $G_1$. If we apply any $M$-augmenting path we get a maximum matching $M_1$ of $G_1$. $M_1$ contains 1 edge from $E_1$ and $n_0$ edges from $E$. When we add $E_2$ to $G_1$, we get the graph $H$. We know that none of the vertices of $C$ is free in $M_1$. Therefore there does not exist any $M_1$-augmenting path in $H$ finishing at a vertex from $C$. So we can get augmenting paths of two types. The first possibility is that the augmenting path is totally contained in $G_2$. The other possibility is that the augmenting path contains one matched edge and one unmatched edge from $E_1$. So each $M_1$-augmenting path of $H$ only increases the number of the matched edges from $E_2$ by 1.

We know the size of a maximum matching of $H$ is $n_0 + n_2$. Also this matching is a matching from $M_{12}$. So a matching of $M_{12}$ contains 1 edge from $E_1$, $n_0$ edges from $E$ and $n_2 - 1$ edges from $E_2$.

$2(b)$ Let us consider a matching $M$ from $M_{21}$. Let us have an $M$-alternating path starting from a free vertex of $C$, containing the edge $(a_j, p_j)$ and an odd number of edges from $E_2$. So we can assume that $M$-alternating path contains $2k - 1$ edges from $E_2$ and 1 edge from $E_1$. In the alternating path, any two edges not belonging to $G$ are separated by an even length path segment consisting of edges from $E$. Also the edge $(a_j, p_j)$ is unmatched in the alternating path. So the number of matched edges from $E_2$ in the $M$-alternating path is $k$. When we apply the $M$-alternating path, the matching now contains the edge $(a_j, p_j)$. Also the number of edges of $E_2$ in the alternating path is $k - 1$. So the number of edges from $E$ and $E_2$ are $n_0$ and $n_2 - 1$ respectively. So the resultant maximum matching is a valid matching of $M_{12}$.

$2(c)$ In the next part we have $p_j \in O(G_2)$.

Let us consider a matching $M_2 \in M_{21}$. We assume that the $M_2$-alternating path starts from a free vertex of $C$ in the graph $H$, contains $(a_j, p_j)$ from $E_1$ and odd number of edges from $E_2$. Since the graph is a bipartite graph, the other endpoint of the alternating path also belongs to $A$. Let the other endpoint of the alternating path be $a'$. We have assumed that $p_j \in O(G_2)$. Since $M_2 \in M_{21}$, a maximum matching of $G_2$ is a maximum matching of $H$. So when we add $E_1$ to $G_2$, we get an alternating path from the free vertex of $C$ to $p_j$. The alternating path contains the edges from $C$ and one edge from $E_1$. Also $p_j \in O(G_2)$ implies that $p_j$ is reachable from a free vertex by an odd length alternating path contained in $G_2$. These two alternating paths are vertex disjoint except for
the vertex \( p_j \). So the vertices that are reachable from \( C \) via the edge \((a_j, p_j)\) in \( H \) is reachable from a free vertex via \( p_j \) in \( G_2 \). So \( a' \) is reachable from a free vertex via \( p_j \) in \( G_2 \). So \( a' \in E(G_2) \).

If there are two alternating components connected by an \( EE \) edge in a bipartite graph, and vertices from \( A \) are even in one component, then vertices from \( P \) would be even in other component. In our graph, there are odd number of edges from \( E_2 \) between \( p_j \) and \( a' \) with \( p_j \in E(G) \). Also both endpoints of the edges from \( E_2 \) belong to \( E(G) \). So \( a' \in E(G) \). Therefore, \( a' \in E(G) \cup E'(G_2) \cup E(H) \).

Next we show \( a' \in E(G_1) \). We know that \( a' \in E(G) \). When we add \( E_1 \) to \( G \), we have some augmenting paths in \( G_1 \). The even vertices from \( A \), that are part of any augmenting path in \( G_1 \), belong to \( C \). Hence none of the even vertices of \( A \) from \( G \setminus C \) belong to any augmenting path. Hence the type of the vertices of \( A \) from \( G \setminus C \) remains the same. Therefore, \( a' \in E(G_1) \).

Let \( M \) be a matching of \( M_{21} \) and \( M' \) be a matching of \( M_{12} \) containing the edge \((a_j, p_j)\). We prove that for any \( M \)-alternating path in \( H \) that starts from a free vertex of \( C_1 \), contains the edge \((a_j, p_j)\) and ends at \( a' \in E(G) \cap E(G_2) \) will always have an odd number of edges from \( \mathcal{E}_2 \) and has even length. The \( M \)-alternating path starts at a free vertex, and the path is not an \( M \)-augmenting path. So the path is of even length. In the next part, we show the \( M \)-alternating path contains odd number of edges from \( \mathcal{E}_2 \). The alternating path contains some subpaths from \( G \), 1 edge from \( \mathcal{E}_1 \) and some edges from \( \mathcal{E}_2 \). \( M \) contains \( n_2 \) edges from \( \mathcal{E}_2 \) and no edge from \( \mathcal{E}_1 \). \( M' \) contains an edge from \( \mathcal{E}_1 \) and \( n_2 - 1 \) edges from \( \mathcal{E}_2 \). If we take the symmetric difference \( M \oplus M' \), we claim that it contains odd number of edges from \( \mathcal{E}_2 \). If total number of common edges from \( \mathcal{E}_2 \) in the matchings \( M \) and \( M' \) is \( k' \), then the number of edges from \( \mathcal{E}_2 \) in the symmetric difference is \( n_2 + (n_2 - 1) + 2k' = 2(n_2 + k') - 1 \), which is odd. So alternating path contains odd number of edges from \( \mathcal{E}_2 \).

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