Notes on Abstract Argumentation Theory

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Abstract
This note reviews Section 2 of Dung’s seminal 1995 paper on abstract argumentation theory. In particular, we clarify and make explicit all of the proofs mentioned therein, and provide more examples to illustrate the definitions, with the aim to help readers approaching abstract argumentation theory for the first time. However, we provide minimal commentary and will refer the reader to Dung’s paper for the intuitions behind various concepts. The appropriate mathematical prerequisites are provided in the appendices.

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1 Introduction

Abstract argumentation theory [14] is concerned with the formalisation and implementation of methods that resolve disagreements rationally, based on the pattern of disagreements alone. Such a need typically arises when reasoning with incomplete and contradictory information from multiple sources, whether human or machine. It provides a general approach to modelling conflict between arguments and the agents putting forward those arguments. This is based on the commonsensical idea that the “winning” arguments are those that are collectively consistent and adequately responds to all counterarguments. It is assumed that arguments that have no counterarguments will win by default. Such ideas can be quite intuitive in its handling of conflict and justification [14, 19].

In this note, we review the mathematical background of abstract argumentation theory [14, Section 2], making explicit all steps in the proofs, and occasionally providing lemmas that can make the longer proofs easier to comprehend. We also illustrate many of the concepts with examples. Further, we briefly recap the relevant aspects of directed graphs and lattice theory in the appendices.

This note will focus on definitions and technical results with minimal commentary. We do not claim originality as many of these results, especially those not explicitly stated in [14], should be folklore. Our intention for writing this note is to collate all relevant results that may assist a reader coming to abstract argumentation theory, in particular [14], for the first time; this document can also serve as a reference for researchers. We will not cover further topics such as argument labellings (e.g. [8]), non-Dung semantics (e.g. [1, 2]), dialogical argumentation (e.g. [17]) and structured argumentation (e.g. [5, 18]).
2 Abstract Argumentation Frameworks

2.1 Definition and Basic Examples

An abstract argumentation framework is a directed graph (digraph) \((A, R)\) where the set of nodes \(A\) represent the set of arguments under consideration and the set of directed edges \(R\) denote when a given argument is a counterargument to another argument or itself, usually due to logical inconsistency or conflicting values. This representation of arguments and how they disagree abstracts away from the internal structure and content of arguments and the nature of such disagreements, hence the term “abstract” argumentation. This results in an external theory of justification [14], as opposed to an internal theory of justification concerned with whether individual arguments are valid or plausible. We assume the reader is familiar with graph theory, but have recapped the basic ideas, notation and definitions of graph theory in Appendix A (page 82).

Definition 2.1. [14, Definition 2] An (abstract) argumentation framework (AF) is a digraph \((A, R)\) where \(A\) is the set of arguments and \(R \subseteq A^2\) is the attack relation. For \(a, b \in A\), we say \(a\) attacks / is a counter-argument to / disagrees with \(b\) iff \((a, b) \in R\), denoted as \(R(a, b)\). Further, we denote \((a, b) \notin R\) with \(\neg R(a, b)\).

Example 2.2. [14, Example 9] The Nixon diamond is the AF whose digraph is isomorphic to the directed cycle graph on two nodes, denoted \(C_2\) (see Example A.3, page 82, for the notation), i.e. \(A = \{a, b\}\) and \(R = \{(a, b), (b, a)\}\). This is depicted in Figure 2.1.

![Figure 2.1: The AF depicting the Nixon diamond, from Example 2.2.](image)

Example 2.3. Simple reinstatement (e.g. [7, 19]) is the AF whose digraph is isomorphic to the directed path graph on three nodes, denoted \(P_3\) (see Example A.2, page 82, for the notation), i.e. \(A = \{a, b, c\}\) and \(R = \{(b, a), (c, b)\}\). This is depicted in Figure 2.2.

![Figure 2.2: The AF depicting simple reinstatement, from Example 2.3.](image)

\(^1\)This is also called a chain of three arguments [3].
Example 2.4. **Double reinstatement** is the AF where $A = \{a, b, c, e\}$ and $R = \{(b, a), (c, b), (e, b)\}$. This is depicted in Figure 2.3.

![AF depicting double reinstatement](image1)

**Example 2.5.** [14, Examples 1 and 3] Consider the AF with $A = \{a, b, c\}$ and $R = \{(a, b), (b, a), (c, b)\}$. This is depicted in Figure 2.4.

![AF depicting Example 3](image2)

**Example 2.6.** [4, Example 2.3.5] We can also have an AF whose underlying digraph is isomorphic to $P_4$, the directed path graph on four nodes (Example A.2, page 82), i.e. $A = \{a, b, c, e\}$, $R = \{(e, c), (c, b), (b, a)\}$. This is depicted in Figure 2.5.

![AF depicting simple reinstatement](image3)

**Example 2.7.** (See [9, Figure 1] and [19, Figure 2]) Floating reinstatement is the AF where $A = \{a, b, c, e\}$, $R = \{(a, b), (b, a), (a, c), (b, c), (c, e)\}$. This AF is depicted in Figure 2.0.

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2 This name may not be standard in the literature.

3 We will not use the letter “d” to denote arguments - see Section 4.1, page 29. 

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9
From now on we assume an arbitrary AF \( \langle A, R \rangle \). Note that \( R \) is not assumed to be a symmetric relation. This is to include cases where two arguments disagree in only one direction.

**Example 2.8.** (From [19]) Let the argument \( a \) represent “Mary does not limit her phone usage. Therefore, Mary has a large phone bill.” Let the argument \( b \) represent “Mary has a speech disorder. Therefore, Mary limits her phone usage.” In real-life dialogues, arguments support a conclusion, and it is the conclusion (instead of any other part of the argument) that is used to agree or disagree with other arguments that support various other conclusions [18, Section 2]. In this example, the argument \( b \) attacks \( a \) because the conclusion of \( b \) attacks an assumption of \( a \), and the attack is not symmetric because the conclusion of \( a \) does not disagree with anything \( b \) has concluded or assumed.

### 2.2 Types of Attacks and Examples

**Definition 2.9.** [14, Remark 4] We say \( S \subseteq A \) attacks \( a \in A \) iff \( a \in S^+ \). We say \( a \) attacks \( S \) iff \( a \in S^- \). We say \( S \) attacks \( T \subseteq A \) iff \( S^+ \cap T \neq \emptyset \).

Definition 2.9 generalises attacks between individual arguments to sets of arguments. By Corollary A.12 (page 83), the empty set \( \emptyset \) can never attack any argument, nor can it be attacked by any argument.

**Example 2.10.** (Example 2.2 continued) In the Nixon diamond, \( b \in \{a\}^+ = a^+ \), hence the set \( \{a\} \) attacks \( b \). By symmetry, \( a \in b^+ \).

**Example 2.11.** (Example 2.5 continued) Clearly, \( c \) attacks the set \( \{a, b\} \), i.e. \( c \in \{a, b\}^- \), because \( c \) attacks \( b \).

**Example 2.12.** (Example 2.4 continued) In double reinstatement, \( \{c, e\} \) attacks \( \{a, b\} \), because both \( c \) and \( e \) attack \( b \).

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4For recent experiments that investigate whether we can infer the direction of attacks from natural language, see [11].

5\( S^+ \) denotes the set of arguments attacked by some argument in \( S \) (see Definition A.7 page 83). This is also called the forward or successor set of \( S \) in a digraph.

6\( S^- \) denotes the set of arguments attacking some argument in \( S \) (see Definition A.7 page 83). This is also called the backward or predecessor set of \( S \) in a digraph.
Definition 2.13. We say \( a \in A \) is an **unattacked argument** iff \( a^- = \emptyset \).

When arguments are represented by nodes of a digraph, unattacked arguments correspond to source nodes (Definition A.6 page 82). Unattacked arguments are important because we will see that they always *win*. This formalises the idea that the person with the last word always wins the argument, because such arguments have no counter-argument represented in the AF [14, Section 1]. Another way of understanding this is that the claim of such an argument is seen as provisionally true until it is explicitly rebutted.

Definition 2.14. \( U := \{ a \in A | a^- = \emptyset \} \) is the set of all unattacked arguments.

Example 2.15. (Example 2.6 continued) We have \( U = \{ e \} \).

Example 2.16. (Example 2.7 continued) In floating reinstatement, we have \( U = \emptyset \) because every argument is being attacked.

Corollary 2.17. We have that \( a \in U \iff a \notin A^+ \).

Proof. \( a \in U \iff a^- = \emptyset \iff (\forall b \in A) b \notin a^- \iff a \notin A^+ \). \( \square \)

Definition 2.18. An argument \( a \in A \) is **self-attacking** iff \( a \in a^+ \), equivalently \( a \in a^- \) by Corollary A.5 (page 82).

Example 2.19. Consider \( A = \{ a, b \} \) and \( R = \{ (a, a), (a, b) \} \). This AF is depicted in Figure 2.7.

\[
\begin{tikzpicture}
\node (a) at (0,0) {a};
\node (b) at (1,0) {b};
\draw (a) edge (b);
\end{tikzpicture}
\]

Figure 2.7: An AF with a self-attacking argument, from Example 2.19.

In this AF, \( a \) is our self-attacking argument. Further, the set of unattacked arguments \( U = \emptyset \).

2.3 Basic Types of Abstract Argumentation Frameworks

Definition 2.20. An AF is **empty** iff \( A = \emptyset \).

Definition 2.21. An AF is **trivial** iff \( R = \emptyset \).

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7We have a slightly less general definition here compared to [11 Definition 2.9], where a set \( S \) of non-empty arguments is **unattacked** iff \( S^- = \emptyset \), i.e. there are no arguments outside of \( S \) that is attacking \( S \). In our case, \( U \) is the \( \subseteq \)-greatest such set.
Definition 2.22. An AF is **symmetric** iff \( R \) is a non-empty symmetric relation.\(^8\)

Definition 2.23. An AF is **finite** iff \( A \) is a finite set. Else, the AF is **infinite**.

Example 2.24. All of Examples 2.2 to 2.19 above are finite, non-trivial and non-empty AFs. Of these examples, only Example 2.22 is a symmetric AF.

In this note, we do not assume that the AFs we deal with are finite; they can be finite or infinite.\(^2\)

Example 2.25. The following AF is infinite. Let \( A = \{a_i, b_i\}_{i \in \mathbb{N}} \) and \( R = \{(b_i, a_i), (a_{i+1}, b_i)\}_{i \in \mathbb{N}} \). This AF is depicted in Figure 2.8.

![Figure 2.8: An infinite AF, from Example 2.25](image)

Example 2.26. The following AF is also infinite. Let \( A = \{a_i, b_i\}_{i \in \mathbb{Z}} \) and \( R = \{(b_i, a_i), (a_{i+1}, b_i)\}_{i \in \mathbb{Z}} \). This AF is depicted in Figure 2.9.

![Figure 2.9: An infinite AF, from Example 2.26](image)

The infinite\(^9\) AFs from Examples 2.25 and 2.26 are **locally finite** in that each argument only has one other argument attacking it. This motivates the following definition:

Definition 2.27. [14, Definition 27] An AF is **finitary** iff \((\forall a \in A) |a^-| < \aleph_0\).

Corollary 2.28. Finite AFs are finitary. The converse is not true.

Proof. If \( \langle A, R \rangle \) is finite, then for all \( a \in A \), the set \( a^- \subseteq A \) is also finite.

An example of an infinite finitary AF is depicted in Example 2.25 where every argument has exactly one attacker (finitary) but there are countably infinitely many arguments. Therefore, the converse is not true.\(\Box\)

\(^8\)Like [10], we exclude the empty relation as it is vacuously symmetric. Unlike [10], we do not restrict our attention to finite argumentation frameworks, i.e. when \( A \) of \( \langle A, R \rangle \) is a finite set. Further, we do not exclude the possibility of having self-attacking arguments, as \((a, a) \in R\) does not violate that \( R \) is a symmetric relation.

\(^9\)Clearly the AFs from Examples 2.25 and 2.26 have countably infinitely many arguments. It is also possible for AFs to have uncountably infinitely many arguments. See [14, Section 3.1] or [25, 26].
Example 2.29. The following is an example of a non-finitary AF. By the contrapositive of Corollary 2.28, the AF cannot be finite. Let \( A = \{a\} \cup \{b_i\}_{i \in \mathbb{N}} \) and \( R = \{(b_i,a)\}_{i \in \mathbb{N}} \). This AF is depicted in Figure 2.10.

![Figure 2.10: An infinite non-finitary AF, from Example 2.29.](image)

Then \( a^- = \{b_i\}_{i \in \mathbb{N}} \), which means it has infinitely many attackers. This \( \langle A, R \rangle \) is therefore not finitary.

Example 2.30. The following is another example of a non-finitary AF. Let \( A = \{a\} \cup \{b_i,c_i\}_{i \in \mathbb{N}} \) and \( R = \{(b_i,a),(c_i,b_i)\}_{i \in \mathbb{N}} \). We have \( a^- = \{b_i\}_{i \in \mathbb{N}} \) hence \( |a^-| = \aleph_0 \). This AF is depicted in Figure 2.11.

![Figure 2.11: An infinite non-finitary AF, from Example 2.30.](image)

Definition 2.31. Let \( B \subseteq A \). The (induced) sub-framework w.r.t. \( B \) is the AF \( \langle B, R \cap B^2 \rangle \).

Example 2.32. (Example 2.26 continued) Clearly, the AF from Example 2.26 is a sub-framework of this AF.

Example 2.33. (Example 2.30 continued) Clearly, the AF in Figure 2.10 is a sub-framework of the AF in Figure 2.11.
Example 2.34. (Example 2.3 continued, page 8) The idea of induced sub-frameworks allows us to model the course of a dialogue (e.g., such as online debates [6, 23, 24]). The set of arguments could be the arguments that have so far been mentioned during the dialogue. Consider a dialogue based on simple reinstatement (Example 2.3). We can imagine Agent 1 claiming $a$ and we have the induced sub-framework w.r.t. $\{a\}$. We then imagine Agent 2 claiming $b$, so the set of arguments mentioned so far is $\{a, b\}$, and the corresponding induced sub-framework also has the attack $R(b, a)$. Finally, Agent 1 responds by claiming $c$ and the dialogue ends, so the set of arguments mentioned so far is $\{a, b, c\}$ and we recover the full framework of simple reinstatement.

2.4 Cycles in Argumentation Frameworks

One could then imagine cycles in AFs which can represent “never-ending” courses of dialogue where agents can repeat the same arguments over and over again. We will see that this will make determining the winning arguments problematic [3].

Definition 2.35. We say that an AF is cyclic iff it contains a (directed) cycle, else it is acyclic.

Necessarily, the number of arguments in a cycle is finite.

Definition 2.36. Given a cycle in an AF, its parity is whether the number of arguments in the cycle is even or odd.

Example 2.37. (Example 2.19 continued, page 11) This is a cyclic AF with an odd cycle, specifically the 1-cycle where the argument $a$ is self-attacking.

Example 2.38. (Example 2.2 continued, page 8) The Nixon diamond is a cyclic AF with a 2-cycle.

Example 2.39. [4] Example 2.3.1] The following is a cyclic AF with a 3-cycle: $A = \{a, b, c, e\}$ and $R = \{(b, a), (c, b), (e, c), (b, e)\}$. This AF is depicted in Figure 2.12.

Figure 2.12: An AF containing a 3-cycle, from Example 2.39.

Definition 2.40. [14] Definition 2.9] An AF is well-founded iff there is no $A$-sequence $\{a_i\}_{i \in \mathbb{N}}$ such that $(\forall i \in \mathbb{N}) R(a_{i+1}, a_i)$. 

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Example 2.41. (Example 2.6 continued, page 9) This AF is well-founded because its attack sequence is finite, involving only four arguments.

Example 2.42. (Example 2.2 continued, page 8) This AF is not well-founded due to the $A$-sequence \{a, b, a, b, a, b, a, b, \ldots\}, because $R(a, b)$ and $R(b, a)$.

Corollary 2.43. If $\langle A, R \rangle$ is well-founded then $U \neq \emptyset$. The converse is not true in general.

Proof. (Contrapositive) If $U = \emptyset$ then $\forall a \in A \ a^- \neq \emptyset$. Let $a_0 \in A$, then there is some $a_1 \in a_0^-$. Similarly, there is some $a_2 \in a_1^-$. So for any $a_i \in A$ there exists some $a_{i+1} \in a_i^-$. This gives an infinite sequence \{a_i\}_{i \in \mathbb{N}} such that $\forall i \in \mathbb{N} \ R(a_{i+1}, a_i)$. Therefore, $\langle A, R \rangle$ is not well-founded.

For the converse, consider $A = \{a_i\}_{i \in \mathbb{N}} \cup \{b\}$ such that $\forall i \in \mathbb{N} \ R(a_{i+1}, a_i)$. This AF is depicted in Figure 2.13.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.13}
\caption{An example of an AF that satisfies $U \neq \emptyset$ and is not well-founded, from Corollary 2.43}
\end{figure}

This AF has an infinite sequence of arguments with an isolated argument $b$. Notice that $U = \{b\} \neq \emptyset$, but this AF is not well-founded. \hfill \square

Corollary 2.44. [10, Proposition 2] Cyclic AFs are not well-founded. The converse is not true in general.

Proof. If $\langle A, R \rangle$ has a cycle then denote that cycle as \{a_i\}_{i=1}^k such that

\[(\forall 1 \leq i \leq k-1) \ R(a_{i+1}, a_i) \text{ and } R(a_1, a_k).\]

This gives an infinite, periodic $A$-sequence

\[a_1, a_k, a_{k-1}, a_{k-2}, \ldots, a_2, a_1, a_k, a_{k-1}, \ldots\]

such that $R(a_1, a_k), R(a_k, a_{k-1}), \ldots$ etc. Therefore, $\langle A, R \rangle$ cannot be well-founded.

For the converse, Figure 2.13 from Corollary 2.43 is an example of a non-well-founded AF that is acyclic. \hfill \square

Corollary 2.45. [10, Proposition 3] If an AF is symmetric, then it is not well-founded. The converse is not true.

Proof. If an AF $\langle A, R \rangle$ is symmetric, then $R \neq \emptyset$ is a symmetric relation. We have some $(a, b) \in R$ and hence $(b, a) \in R$. This is a 2-cycle, hence by Corollary 2.44 this AF is not well-founded.

For the converse, Example 2.25 (page 12) is a non-well-founded AF that is not symmetric. \hfill \square
The following result gives an equivalent characterisation of non-well-foundedness for an AF.

**Corollary 2.46.** \( \langle A, R \rangle \) is not well-founded iff there exists some \( \emptyset \neq S \subseteq A \) such that \( S \subseteq S^+ \).

Proof. (\( \Leftarrow \)) To demonstrate that the underlying \( \langle A, R \rangle \) is not well-founded, we use induction to construct the desired sequence \( \{a_i\}_{i \in \mathbb{N}} \):

1. (Base) As \( \emptyset \neq S \), let \( a_0 \in S \).

2. (Inductive) Let \( a_i \in S \), then \( a_i \in S^+ \) by the hypothesis \( S \subseteq S^+ \) and hence there is some \( a_{i+1} \in S \) such that \( R(a_{i+1}, a_i) \).

By induction, \( \{a_i\}_{i \in \mathbb{N}} \) is the sequence satisfying \( (\forall i \in \mathbb{N}) R(a_{i+1}, a_i) \). This shows that the underlying \( \langle A, R \rangle \) is not well-founded.

(\( \Rightarrow \)) If \( \langle A, R \rangle \) is not well-founded, then there is a sequence \( \{a_i\}_{i \in \mathbb{N}} \) such that \( (\forall i \in \mathbb{N}) R(a_{i+1}, a_i) \). Let \( S := \{a_i\}_{i \in \mathbb{N}} \). Clearly, \( S \neq \emptyset \). For any \( a_i \in S \), there is some \( a_{i+1} \in S \) such that \( R(a_{i+1}, a_i) \), therefore \( a_i \in S^+ \) and hence \( S \subseteq S^+ \).

**Corollary 2.47.** A finite acyclic AF is well-founded. The converse is not true.

Proof. (Contrapositive) Let \( \langle A, R \rangle \) be finite and not well-founded. We seek to construct an attack cycle. By non-well-foundedness, there exists an \( A \)-sequence \( \{a_i\}_{i \in \mathbb{N}} \) such that \( (\forall i \in \mathbb{N}) R(a_{i+1}, a_i) \). As \( A \) is finite, \( \text{WLOG} \ |A| = N \in \mathbb{N} \), then the first \( N+1 \) terms of the sequence \( \{a_i\}_{i \in \mathbb{N}} \) must have some repeating argument by the pigeonhole principle. Let \( b \) be such an argument, then we can construct a cycle starting and ending with \( b \) through the property \( (\forall i \in \mathbb{N}) R(a_{i+1}, a_i) \). Therefore, \( AF \) is cyclic.\(^{10}\)

As for the converse, the AF where \( A = \{a_i, b_i\}_{i \in \mathbb{N}} \) and \( R = \{(a_i, b_i)\}_{i \in \mathbb{N}} \) is well-founded, acyclic but infinite. This AF is depicted in Figure 2.14.

![Figure 2.14: An infinite AF that is well-founded, from Corollary 2.47.](image)

Therefore, the converse is not true. The result follows.\( \square \)

\( ^{10} \)We cannot prove the contrapositive by assuming that the AF is cyclic and not well-founded because this contradicts Corollary 2.44.
2.5 Controversy in Argumentation Frameworks

Definition 2.48. We say $a \in A$ indirectly attacks $b \in A$ iff there is an odd-length path from $a$ to $b$ in $\langle A, R \rangle$.

Notice that direct attacks, i.e. paths of length 1, are special cases of indirect attacks.

Example 2.49. (Example 2.6 continued, page 9) The argument $e$ indirectly attacks the argument $a$, as there is a path of length 3 from $e$ to $a$.

Example 2.50. Self-attacking arguments both directly and indirectly attack themselves.

Definition 2.51. We say $a \in A$ indirectly defends $b \in A$ iff there is an even-length path from $a$ to $b$ in $\langle A, R \rangle$.

Example 2.52. (Example 2.2 continued, page 8) The argument $b$ indirectly defends itself as there is a path of length 2 from itself to itself.

Example 2.53. Self-attacking arguments also indirectly defend themselves, by going through their loop twice to obtain a path of length 2.

Corollary 2.54. Every non-self-attacking argument in an AF indirectly defends itself.

Proof. Every non self-attacking argument has a path length of 0 to itself, which is an even path. $\square$

Definition 2.55. We say $a \in A$ is controversial w.r.t. $b \in A$ iff $a \in A$ indirectly attacks and indirectly defends $b \in A$.

Example 2.56. [4, Example 2.1.2] Consider $A = \{a, b, c\}$ such that $R(a, b)$, $R(b, c)$ and $R(a, c)$. It is clear that $a$ is controversial w.r.t. $c$. This AF is depicted in Figure 2.15.

Example 2.57. Each self-attacking argument is controversial with respect to itself, because it both indirectly attacks and indirectly defends itself.

$^{11}$We would not call such “paths” cycles though.
Definition 2.58. We say $a \in A$ is controversial iff there is some $b \in A$ such that $a$ is controversial w.r.t. $b$.

Definition 2.59. [14, Definition 32(1)] An AF $\langle A, R \rangle$ is controversial iff there is some controversial argument in $A$. Else, the AF is uncontroversial.

Example 2.60. (Example 2.56 continued) This AF is controversial, as $a$ is a controversial argument.

Example 2.61. (Example 2.19, page 11 continued) This AF is controversial because it contains the self-attacking argument $a$.

Definition 2.62. [14, Definition 32(2)] An AF $\langle A, R \rangle$ is limited controversial iff there is no $A$-sequence $\{a_i\}_{i \in \mathbb{N}}$ such that $a_{i+1}$ is controversial with respect to $a_i$.

Corollary 2.63. If an AF is limited controversial then it cannot have self-attacking arguments.

Proof. (Contrapositive) If an AF has a self-attacking argument $a$, then it is controversial w.r.t. itself so the constant sequence $\{a\}_{i \in \mathbb{N}}$ renders the AF not limited controversial.

We will answer the converse of Corollary 2.63 in Corollary 2.66.

Corollary 2.64. Uncontroversial AFs are limited controversial. The converse is not true in general.

Proof. An uncontroversial AF has no controversial arguments and hence there is no infinite sequence of arguments controversial with respect to its predecessor. Therefore, such AFs are also limited controversial.

For the converse, there is no infinite sequence of arguments in the AF of Example 2.56 where each is controversial with respect to its predecessors. Therefore, in this example, $\langle A, R \rangle$ is limited controversial and controversial.

The property of being limited controversial is downwards inheritable.

Corollary 2.65. If $\langle A', R' \rangle \subseteq_g \langle A, R \rangle$ and $\langle A, R \rangle$ is limited controversial, then $\langle A', R' \rangle$ is also limited controversial. The converse is not true.

Proof. If $\langle A, R \rangle$ is limited controversial, then there is no infinite $A$-sequence of arguments such that each is controversial w.r.t. to its predecessor. Therefore, any sub-framework of $\langle A, R \rangle$ is also limited controversial.

For the converse, consider any AF $\langle A, R \rangle$ and take its disjoint union with the AF consisting of a single self-attacking argument. More precisely, for $x \notin A$, consider the AF $\langle A \cup \{x\}, R \cup \{(x, x)\} \rangle$. The first AF is an induced subgraph of the second, but the second is not limited controversial by the contrapositive of Corollary 2.63.

As self-attacking arguments are cycles of length 1 (hence odd), the following result generalises Corollary 2.63 and also shows that its converse is not true.
Corollary 2.66. If an AF is limited controversial, then it has no odd cycle. The converse is not true.

Proof. (Contrapositive, from [3]) Assume the AF has an odd cycle with arguments with \(a_1, a_2, \ldots, a_n\), where \((\forall 1 \leq i \leq n) R(a_i, a_{i+1})\) and \((\forall a_n, a_1)\), and \(n\) is odd. For \(1 \leq i \leq n\), there is a path of length \(i + nk\) from \(a_{i+1}\) to \(a_i\), where \(k \in \mathbb{N}\) is the number of times around the cycle. Depending on \(k\), \(j - i + nk\) is both even and odd. Therefore, \(a_{i+1}\) is controversial w.r.t. \(a_n\), down to \(a_1\) is controversial w.r.t. \(a_0\). We can do this infinitely many times by continuing that \(a_1\) is controversial w.r.t. \(a_n\) ... etc. This generates our infinite A-sequence such that each \(a_{i+1}\) is controversial w.r.t. \(a_i\).

For the converse, we construct an AF that is not limited controversial, but has no odd cycle. Consider an AF with arguments \(A = \{a_i, b_i\}_{i \in \mathbb{N}}\) and attacks \(R = \{(a_{i+1}, a_i)\}_{i \in \mathbb{N}} \cup \{(b_i, a_i)\}_{i \in \mathbb{N}} \cup \{(a_{i+1}, b_i)\}_{i \in \mathbb{N}}\). This is depicted in Figure 2.16.

![Figure 2.16: An example of an AF that is not limited controversial and has no odd cycle, from Corollary 2.66](image)

Clearly, \(a_{i+1}\) is controversial w.r.t. \(a_i\) for \(i \in \mathbb{N}\), therefore this AF is not limited controversial. However, there is no odd cycle. Therefore, the converse to this result is not true in general.

Corollary 2.67. A finite AF without any odd cycles is limited controversial.

Proof. (Contrapositive) Assume that the finite AF is not limited controversial. Then there exists an infinite sequence of arguments \(\{a_i\}_{i \in \mathbb{N}}\) such that \(a_{i+1}\) is controversial w.r.t. \(a_i\). But as the AF is finite, this sequence must be periodic, so we have \(a_0, a_1, \ldots, a_k, a_0\) for some \(k \in \mathbb{N}\) such that \(a_{i+1}\) has both an even and an odd path to \(a_i\). We construct our odd cycle as follows: we take all even paths from \(a_0\) to \(a_k\), \(a_k\) to \(a_{k-1}\), ... and \(a_2\) to \(a_1\), but take an odd path from \(a_1\) to \(a_0\). The result is an odd cycle.

2.6 Summary

- An (abstract) argumentation framework (AF) is a digraph \((A, R)\) where \(A\) is the set of arguments under consideration and \(R\) is the binary attack relation.

\[\text{We cannot assume that the AF is infinite because an infinite AF without any odd cycles does not have to be limited controversial by the converse of Corollary 2.66}\]
• For \( S \subseteq A \), \( S^+ \subseteq A \) is the set of arguments attacked by \( S \), and \( S^- \subseteq A \) is the set of arguments attacking \( S \). When \( S = \{a\} \) we write \( a^+ \) and \( a^- \) respectively.

• \( U \subseteq A \) is the set of unattacked arguments, i.e. \( a \in U \iff a^- = \emptyset \).

• A self-attacking argument \( a \) satisfies \( a \in a^+ \).

• An AF \( \langle A, R \rangle \) is empty iff \( A = \emptyset \), trivial iff \( R = \emptyset \), finite iff \( A \) is a finite set, and infinite iff \( A \) is an infinite set.

• An AF \( \langle A, R \rangle \) is symmetric if \( R \) is a non-empty symmetric relation. This does not exclude the possibility of there being self-attacking arguments.

• An AF is finitary iff all arguments has finitely many attackers. All finite AFs are finitary, but finite AFs can have infinitely many arguments.

• An induced argumentation sub-framework of \( \langle A, R \rangle \) is the induced digraph on a set \( B \subseteq A \).

• An AF is cyclic iff it contains a directed cycle, else the AF is acyclic. A cycle can be odd or even depending on how many arguments it contains.

• An AF is well-founded iff there exists no countably infinite backward chain of arguments. Cyclic AFs are not well-founded. Finite acyclic AFs are well-founded. Well-founded AFs satisfy \( U \neq \emptyset \).

• An argument \( a \) indirectly attacks an argument \( b \) iff there is an odd-length path in \( R \) from \( a \) to \( b \). An argument \( a \) indirectly defends an argument \( b \) iff there is an even-length path in \( R \) from \( a \) to \( b \). We say \( a \) is controversial w.r.t. \( b \) iff \( a \) both indirectly attacks and indirectly defends \( b \). An argument \( a \) is controversial iff there exists some argument \( b \) w.r.t. which it is controversial. An AF is controversial iff there is some controversial argument, else it is uncontroversial.

• An AF is limited controversial iff there is no countably infinite backward chain of controversial arguments. Uncontroversial AFs are trivially limited controversial. Limited controversial AFs have no odd cycles. A finite AF without any odd cycles is limited controversial.
3 Neutrality and Conflict-Freeness

3.1 The Neutrality Function

3.1.1 Definition

Definition 3.1. Given an AF, its \textit{neutrality function} is

\[
    n : \mathcal{P}(A) \rightarrow \mathcal{P}(A) \\
    S \mapsto n(S) := A - S^+.
\]  (3.1)

The neutrality function of \( S \subseteq A \) selects all arguments not attacked by \( S \), i.e. \( S \) is \textit{neutral} towards these arguments. If the underlying AF needs to be explicitly specified, we can write \( n_A \), for \( A = \langle A, R \rangle \)\(^{13}\). From now, we will reserve the letter “\( n \)” to denote the neutrality function, and nothing else\(^{14}\).

Corollary 3.2. \( n \) is well-defined as a function.

Proof. Given \( S \in \mathcal{P}(A) \), \( n(S) = A - S^+ \in \mathcal{P}(A) \) is well-defined. Further, for \( S = T \), \( n(S) = n(T) \) by Corollary A.8 (page 83). Therefore, \( n \) is a well-defined function. \( \square \)

Example 3.3. (Example 2.2 continued, page 8) For the Nixon diamond, the values of \( n \) are depicted in Table 3.1. Recall that in this case \( A := \{a, b\} \) and \( R = \{(a, b), (b, a)\} \).

| \( S \)   | \( \emptyset \) | \( \{a\} \) | \( \{b\} \) | \( A \) |
|----------|-----------------|-------------|-------------|------|
| \( n(S) \) | \( A \)          | \( \{a\} \) | \( \{b\} \) | \( \emptyset \) |

Table 3.1: The values of the neutrality function \( n \), for Example 2.2

Example 3.4. (Example 2.3 continued, page 8) For simple reinstatement, the values of \( n \) are depicted in Table 3.2. Recall that in this case \( A := \{a, b, c\} \) and \( R = \{(c, b), (b, a)\} \).

| \( S \)   | \( \emptyset \) | \( \{a\} \) | \( \{b\} \) | \( \{c\} \) | \( \{a, b\} \) | \( \{b, c\} \) | \( \{a, c\} \) | \( \{c\} \) |
|----------|-----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( n(S) \) | \( A \)          | \( \{a, b\} \) | \( \{a, c\} \) | \( \{b, c\} \) | \( \{c\} \) | \( \{a, c\} \) | \( \{c\} \) | \( \{c\} \) |

Table 3.2: The values of the neutrality function \( n \), for Example 2.3

3.1.2 Properties

For any AF, the following results hold.

\(^{13}\)This is denoted as \( PL_{AF} \) in \cite[Section 4.2]{14}.

\(^{14}\)E.g. we will not use \( n \) to denote arguments, or even indices in generalised operators such as unions and meets.
Corollary 3.5. We have that $n(\emptyset) = A$.

Proof. By Corollary A.12 (page 84), $\emptyset = \emptyset$ so $n(\emptyset) = A - \emptyset = A$. □

Corollary 3.6. We have that $n(A) = U$.

Proof. Let $a \in n(A)$ iff $a \in A - A^+$ iff $a \notin A^+$ iff $(\forall b \in A) \neg R(b, a)$ iff $a^- = \emptyset$. Therefore, $a \in U$ by Definition 2.14 (page 11). □

Corollary 3.7. $n$ is $\subseteq$-antitone.

Proof. If $S \subseteq T \subseteq A$, then $S^+ \subseteq T^+ \subseteq A$ by Corollary A.13 (page 84) and hence $A - T^+ \subseteq A - S^+ \subseteq A$. Therefore, $n(T) \subseteq n(S)$, so $n$ is $\subseteq$-antitone. □

Corollary 3.8. The square of the neutrality function, $n^2(S) := n(n(S))$, is $\subseteq$-monotone.

Proof. This follows from the fact that the composition of an antitone function with itself results in a monotone function. □

Corollary 3.9. Let $I$ be an index set and $\{S_i\}_{i \in I}$ be a family of subsets of $A$. We have that

$$n\left(\bigcup_{i \in I} S_i\right) = \bigcap_{i \in I} n(S_i)$$

(3.2) and

$$n\left(\bigcap_{i \in I} S_i\right) \supseteq \bigcup_{i \in I} n(S_i).$$

(3.3)

The reverse containment of Equation (3.3) is not true in general.

Notice that for $I = \emptyset$, the first equation reduces to Corollary 3.5 and the second equation reduces to $n(A) \supseteq \emptyset$, which is trivially true.

Proof. For the first result we apply Equation B.11 (page 87) and De Morgan’s laws.

$$n\left(\bigcup_{i \in I} S_i\right) = A - \left(\bigcup_{i \in I} S_i\right)^+ = A - \bigcup_{i \in I} S_i^+ = \bigcap_{i \in I} (A - S_i^+) = \bigcap_{i \in I} n(S_i).$$

For the second result we apply Equation B.12 (page 87) and De Morgan’s laws.

$$n\left(\bigcap_{i \in I} S_i\right) = A - \left(\bigcap_{i \in I} S_i\right)^+ \supseteq A - \bigcap_{i \in I} S_i^+ = \bigcup_{i \in I} (A - S_i^+) = \bigcup_{i \in I} n(S_i).$$

Now consider the AF whose underlying digraph is the same as that of Corollary 3.3, depicted in Figure B.1 (page 88). We have $n(S_1 \cap S_2) = n(\{b\}) = \{a, b, c, x\}$. However, $n(S_1) = n(S_2) = \{a, b, c\}$ and hence $n(S_1) \cup n(S_2) = \{a, b, c\}$. Clearly, $n(S_1 \cap S_2) \subseteq n(S_1) \cup n(S_2) = \{a, b, c\}$. Therefore, the converse of the second result does not hold in general. □

Note that Equation (3.2) means that $n$ is a *join antimorphism* [21, Definition 3.3.21] on the complete lattice $\langle P(A), \cap, \cup \rangle$ (see Appendix C page 89).
3.2 Conflict-Free Sets

We now use the neutrality function to define what it means for a set of arguments in an AF to be collectively consistent.

3.2.1 Definition

Theorem 3.10. For $S \subseteq A$, TFAE\(^{15}\)

1. $S \subseteq n(S)$, i.e. $S$ is a postfixed point of $n$ (Definition C.20, page 90),
2. $S \cap S^+ = \emptyset$ and
3. $S^2 \cap R = \emptyset$ and
4. $S \cap S^- = \emptyset$.

Proof. (1) and (2) are equivalent because

$$S \cap S^+ = \emptyset \iff S \subseteq A - S^+ \iff S \subseteq n(S).$$

(1) and (3) are equivalent because

$$S \not\subseteq n(S) \iff (\exists a \in S) a \notin n(S) \iff (\exists a \in S) a \in S^+ \iff (\exists a \in S) (\exists b \in S) R(b, a)$$

$$\iff (\exists a, b \in S) R(b, a) \iff (\exists (b, a) \in S^2) R(b, a) \iff S^2 \cap R \neq \emptyset.$$

(3) and (4) are equivalent because

$$a \in S \cap S^- \iff (\exists a, b \in S) R(a, b) \iff S^2 \cap R \neq \emptyset.$$  

This shows the result. □

Definition 3.11. [14, Definition 5] A set $S \subseteq A$ is conflict-free (cf) iff $S$ satisfies any of the four equivalent conditions in Theorem 3.10.

Intuitively, a cf set of arguments consists of arguments that do not disagree with (i.e. attack) each other. This denotes that the arguments are mutually consistent. Graph-theoretically, cf sets correspond to independent sets of the AF as a digraph.

Example 3.12. (Example 3.4 continued) From Table 3.2 (page 21), we can see that $\emptyset$, \{a\}, \{b\}, \{c\} and \{a, c\} are all cf sets.

Example 3.13. (Example 2.7 continued, page 9) For floating reinstatement, we have

$$CF = \{\{a\}, \{b\}, \{c\}, \{e\}, \{a, e\}, \{b, e\}\}.$$  \hspace{1cm} (3.4)

Example 3.14. (Example 2.26 continued, page 12 continued) For this AF, the cf sets are all subsets of $A$ that do not have $a_i$ and $b_i$ together for $i \in \mathbb{Z}$, because $R(b_i, a_i)$, and also all subsets that do not have $a_{i+1}$ and $b_i$ together, because $R(a_{i+1}, b_i)$. This would include $\emptyset$, all singleton sets (as no argument is self-attacking), all sets of two non-adjacent arguments, e.g. \{a_1, a_2\} or \{a_5, b_6\}... etc.

---

\(^{15}\)TFAE stands for “the following (statements) are (logically) equivalent”.
3.2.2 Existence

Definition 3.15. Given an underlying AF, let $\text{CF} \subseteq \mathcal{P}(A)$ denote its set of cf sets.

If the underlying AF $\mathcal{A} := \langle A, R \rangle$ needs to be explicitly specified, then we write $\text{CF}(\mathcal{A})$, or $\text{CF}(\langle A, R \rangle)$.

Corollary 3.16. $\emptyset \in \text{CF}$.

Proof. Trivially, $\emptyset \subseteq n(\emptyset) = A$ by Corollary 3.5 (page 22).

Therefore, for any AF, cf sets always exist; as $\emptyset$ is cf, so $\text{CF} \neq \emptyset$. Further:

Corollary 3.17. $U \in \text{CF}$.

Proof. As $U = \emptyset$, we have $U \cap U = \emptyset$ and hence $U \in \text{CF}$ by Theorem 3.10.

3.2.3 Lattice-Theoretic Structure

Corollary 3.18. $\text{CF}$ is $\subseteq$-downward closed.

Proof. Assume $S \in \text{CF}$ and $T \subseteq S$. As $S \subseteq n(S)$, then $T \subseteq S \subseteq n(S) \subseteq n(T)$ by Corollary 3.7. Therefore, $T \subseteq n(T)$ and hence $T \in \text{CF}$.

Corollary 3.19. If $S \subseteq U$ then $S \in \text{CF}$. The converse is not true.

Proof. This follows from Corollaries 3.17 and 3.18.

The converse is not true, e.g. Example 2.3 (page 8) where $U = \{c\}$ and $\{a, c\} \subseteq \text{CF}$, $U = \{c\}$ and $\{a, c\} \not\subseteq \{c\}$.

Corollary 3.20. If $S \notin \text{CF}$ and $S \subseteq T$, then $T \notin \text{CF}$.

Proof. Immediate by taking the contrapositive of Corollary 3.18.

Corollary 3.21. $\text{CF}$ is closed under arbitrary non-empty intersections.

Proof. Let $I \neq \emptyset$ be an index set. Let $\{S_i\}_{i \in I}$ be a family of cf sets, then for all $i \in I$, $S_i \subseteq n(S_i)$. Applying Corollaries 3.7 and 3.9 (page 22), starting from $(\forall i \in I) S_i \subseteq n(S_i)$,

$$\bigcap_{i \in I} S_i \subseteq \bigcap_{i \in I} n(S_i) = n \left( \bigcup_{i \in I} S_i \right) \subseteq n \left( \bigcap_{i \in I} S_i \right).$$ (3.5)

This shows that $\bigcap_{i \in I} S_i$ is also a cf set. The result follows.

Note that if the intersection is over the empty family of cf sets, we get $A$, which is not in general cf unless the AF is trivial.

Corollary 3.22. $\text{CF}$ is not in general closed under unions.
Proof. Consider the AF $\langle \{a, b\}, \{(a, b)\} \rangle$. We depict this in Figure 3.1.

![AF diagram](image-url)

Figure 3.1: The AF from Corollary 3.22

Clearly $n(\{a\}) = \{a\}$, $n(\{b\}) = \{a, b\}$ and $n(\{a, b\}) = \{a\}$. Therefore, $\{a\}$ and $\{b\}$ are cf sets, but $\{a\} \cup \{b\} = \{a, b\} \not\subseteq n(\{a, b\}) = \{a\}$ is not a cf set. □

We now give increasingly stronger completeness results for $CF$. Refer to Appendix C (page 89) for the definitions.

**Theorem 3.23.** $\langle CF, \subseteq \rangle$ is $\omega$-complete.

**Proof.** Let $\{S_i\}_{i \in \mathbb{N}}$ be an ascending $\omega$-chain in $CF$. Let $S := \bigcup_{i \in \mathbb{N}} S_i$. Assume for contradiction that $S^2 \cap R \neq \emptyset$. Then there are $a, b \in S$, $R(a, b)$. Therefore, $a \in S_i$ and $b \in S_j$ by definition of $S$, for some $i, j \in \mathbb{N}$. As $\{S_i\}_{i \in \mathbb{N}}$ is an ascending chain, WLOG assume $j \geq i$ hence $a, b \in S_j$ and hence $S^2_j \cap R \neq \emptyset$, so $S_j \notin CF$ – contradiction as $\{S_i\}_{i \in \mathbb{N}}$ is an ascending chain in $CF$. Therefore, $\bigcup_{i \in \mathbb{N}} S_i \in CF$ and hence $\langle CF, \subseteq \rangle$ is $\omega$-complete. □

**Corollary 3.24.** $\langle CF, \subseteq \rangle$ is chain complete.

**Proof.** Let $C$ be an ascending chain in $CF$ of arbitrary cardinality. Let $C := \bigcup C$. Assume for contradiction that there are $a, b \in C$ such that $R(a, b)$. Therefore, $a, b \in C$ and $b \in B$. As $C$ is a chain, WLOG let $A \subseteq B$ so $a, b \in B$. Therefore, $B \cap R \neq \emptyset$, meaning that $B \notin CF$ – contradiction. Therefore, $\bigcup C \in CF$ for all chains $C$. Therefore $CF$ is chain complete. □

**Corollary 3.25.** $\langle CF, \subseteq \rangle$ is directed complete.

**Proof.** Let $D$ be a directed set in $CF$. Let $D := \bigcup D$. Assume for contradiction that there are $a, b \in D$ such that $R(a, b)$. There exists $A, B \in D$ such that $a \in A$ and $b \in B$. As $D$ is a directed set, WLOG let $A \subseteq C$ for some $C \in D$, so $a, b \in C$. Therefore, $C \cap R \neq \emptyset$, meaning that $C \notin CF$ – contradiction. Therefore, $\bigcup D \in CF$ for all directed sets $D$. Therefore $CF$ is directed complete. □

**Theorem 3.26.** $\langle CF, \subseteq \rangle$ is a complete semilattice (Definition C.39, page 93).

**Proof.** Every non-empty subset of $\langle CF, \subseteq \rangle$ has an infimum by Corollary 3.24 which is calculated by set-theoretic intersection. Further, $\langle CF, \subseteq \rangle$ is chain complete by Corollary 3.24. □

The neutrality function is not closed on $CF$.

**Corollary 3.27.** If $S \in CF$ then it is not generally true that $n(S) \in CF$. 25
Proof. Consider the AF $A = \{a, b, c, e\}$ with $R = \{(c, e)\}$ only, and $S = \{a, b\}$. This AF is depicted in Figure 3.2.

![AF Diagram](image)

Figure 3.2: The AF from Corollary 3.27

Clearly, $n(S) = A$ and is not cf.  

3.3 Naive Extensions

We now begin to consider what it means for a set of arguments to be justified or winning. Let $\langle A, R \rangle$ denote an AF. One natural choice of justified arguments would be the $\subseteq$-maximal $CF$ subsets of this AF. This is a natural choice because as $R$ denotes inconsistency, consistent sets are by analogy $cf$ sets, and $\subseteq$-maximal such sets are akin to maximal consistent subsets, which in logic is one way of drawing sensible inferences from an inconsistent set of formulae (e.g. [12]).

3.3.1 Definition

**Definition 3.28.** The set $S \subseteq A$ is a naive extension iff $S \in \text{max}\subseteq CF$.

**Example 3.29.** (Example 2.7 continued, page 24) As $CF = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{b, e\}\}$, the naive extensions are $\{c\}$, $\{a, e\}$ and $\{b, e\}$.

**Example 3.30.** (Example 2.26 continued, page 12) Among the $\subseteq$-maximal cf sets are $S_a := \{a_i\}_{i \in \mathbb{Z}}$ and $S_b := \{b_i\}_{i \in \mathbb{Z}}$.

3.3.2 Existence and Lattice-Theoretic Structure

**Definition 3.31.** We denote the set of all naive extensions of an AF $\langle A, R \rangle$ as $NAI \subseteq \mathcal{P}(A)$, or $NAI(\langle A, R \rangle)$ if we need to make the underlying AF explicit.

Clearly, $NAI = \text{max}\subseteq CF$. We show that every AF has naive extensions.

**Theorem 3.32.** $NAI \neq \emptyset$.

**Proof.** By Corollary 3.24 (page 25), every chain $C$ in $CF$ has an upper bound $\bigcup C \in CF$. Therefore, by Zorn's lemma, $CF$ has at least one $\subseteq$-maximal element. The result follows.  

---

16Another example would be $\emptyset \in CF$ and $n(\emptyset) = A$. Unless the underlying AF is trivial, $A \notin CF$.  

---

26
Notice in the proof of Theorem 3.32, we have used Zorn’s lemma, which is equivalent to the axiom of choice. This means that the axiom of choice is sufficient to demonstrate that naive extensions exist for all AFs. It is also reasonable to ask whether it is necessary. We address this in Appendix D (page 96).

Clearly, NAI is not a singleton set, i.e. the naive extension does not have to be unique.

Example 3.33. (Example 2.3 continued, page 8) Clearly, NAI = \{\{a, c\}, \{b\}\}.

Example 3.34. (Example 2.2 continued, page 8) As CF = \{\emptyset, \{a\}, \{b\}\}, the naive extensions are \{a\} and \{b\}.

Unlike CF, the lattice-theoretic structure of NAI is trivial.

Corollary 3.35. \langle NAI, \subseteq \rangle \subseteq \langle \mathcal{P}(A), \subseteq \rangle is a \subseteq\text{-}antichain.

Proof. If NAI is singleton, then it is trivially an antichain. Else, as NAI \subseteq CF, let S_1, S_2 \in NAI be distinct. Then S_1 \nsubseteq S_2 and S_2 \nsubseteq S_1, by maximality of NAI in CF. Therefore, NAI is a \subseteq\text{-}antichain.

3.3.3 Criticism of Naive Extensions

The naive extensions provide one suggestion of what a set of winning arguments should be by using the analogy from logic of drawing inferences from maximal consistent subsets of an otherwise inconsistent set of propositions. Graph-theoretically, these are the \subseteq\text{-}maximal independent sets. However, this does not seem like a sensible suggestion for the sets of winning arguments. While in examples such as Example 3.34, NAI seems sensible in giving \{a\} and \{b\} as sets of winning arguments, examples such as Example 3.33 gives \{a, c\} and \{b\} as naive extensions. In this case, although \{a, c\} seems sensible as a set of winning arguments because c is unattacked, so it defeats b, which means that a is no longer defeated by b and hence a should be justified, the naive semantics also suggest that \{b\} should be winning, which does not seem to make sense as b is defeated by c, which is undefeated. It is counter-intuitive examples such as this that discourage people from using the naive semantics as a way of defining winning arguments.

Despite this, the naive semantics are simple to understand, and motivate questions such as existence, uniqueness and lattice-theoretic structure that we will consider when investigating other notions of winning arguments. Further, the naive extensions serve as a useful intermediate concept when investing other extensions, which we will use in Sections 4.4.3 (page 61) and 4.2.3 (page 67).

3.4 Summary

- Given an AF, its neutrality function is \( n : \mathcal{P}(A) \to \mathcal{P}(A), n(S) = A - S^+ \).
\begin{itemize}
  \item $n$ satisfies: $n(\emptyset) = A$, $n(A) = U$, $n$ is $\subseteq$-antitone, and for any index set $I$ and family of subsets $\{S_i\}_{i \in I}$ of $A$,
  \[
  n \left( \bigcup_{i \in I} S_i \right) = \bigcap_{i \in I} n(S_i) \quad \text{and} \quad n \left( \bigcap_{i \in I} S_i \right) \supseteq \bigcup_{i \in I} n(S_i).
  \]
  \hfill (3.6)

  \item We say $S \subseteq A$ is conflict-free iff $S \subseteq n(S)$. The set of all conflict-free sets of a given AF $A$ is $CF(A)$ or just $CF$.

  \item For all AFs, $\emptyset, U \in CF$. Furthermore, $\langle CF, \subseteq \rangle$ is $\subseteq$-downward closed, is not closed under unions, and is a complete semilattice that is also directed complete. Furthermore, the neutrality function $n$ is not closed on $CF$.

  \item We may consider the naive extensions, where the set of all naive extensions is $NAI = \max_{\subseteq} CF$, which is a $\subseteq$-antichain. Further, $NAI \neq \emptyset$, and generally not singleton. Unfortunately, $NAI$ should not define when arguments win.
\end{itemize}
4 Defence

4.1 The Defence Function

4.1.1 Definition

The defence function formalises how a set of arguments can defend another argument.

**Definition 4.1.** [Definition 6(1)] Given \( S \subseteq A \) and \( a \in A \), \( a^- \subseteq S^+ \) iff

- \( a \) is acceptable w.r.t. \( S \),
- \( S \) defends \( a \),
- \( S \) reinstates \( a \). [7]

All three terms are equivalent.

**Example 4.2.** (Example 2.3, page 8 continued) In simple reinstatement, we have \( \{c\} \) reinstating \( a \).

**Example 4.3.** (Example 2.4, page 9 continued) In double reinstatement, we have \( \{c\}, \{e\} \) and \( \{c, e\} \) reinstating \( a \).

**Example 4.4.** [Exercise 2] Consider the following AF [7, Figure 4], depicted in Figure 4.1.

![AF Diagram](image)

Figure 4.1: The AF depicting [7, Figure 4], from Example 4.4.

1. Does \( \{a\} \) defend \( c \)? Yes, because \( c^- = \{b\} \) and \( \{a\}^+ = \{b\} \), therefore \( c^- \subseteq \{a\}^+ \).

2. Does \( \{c\} \) defend \( a \)? Yes, because \( c^- = \{b\} \) and \( \{c\}^+ = \{b, e\} \), therefore \( c^- \subseteq \{c\}^+ \).

3. Does \( \{b\} \) defend \( c \)? No, because \( c^- = \{b\} \) and \( \{b\}^+ = \{b\} \), and \( c^- \nsubseteq \{b\}^+ \).

In order for a set of arguments \( S \) to defend \( a \), \( S \) attacks all of the attackers of \( a \). This motivates the following function:

**Definition 4.5.** [Definition 16] Given an AF, its defence function [14] is

\[
d : \mathcal{P}(A) \rightarrow \mathcal{P}(A)
\]

\[
S \mapsto d(S) := \{ a \in A | a^- \subseteq S^+ \}.
\]

This is called the characteristic function in [14] and denoted \( F \).
If the underlying AF \( \mathcal{A} = (A, R) \) needs to be explicitly specified, then we can write \( d_{\mathcal{A}} \) [Remark 17]. From now, we will reserve the letter \( d \) to denote the defence function only.

**Corollary 4.6.** \( d : \mathcal{P}(A) \to \mathcal{P}(A) \) is a well-defined function.

**Proof.** The set \( d(S) := \{ a \in A \mid a^- \subseteq S^+ \} \) is well-defined. Further, for \( S = T \), \( d(S) = d(T) \) by Corollary A.8 (page 83). Therefore, \( d \) is a well-defined function.

**Corollary 4.7.** \( a \in A \) is acceptable w.r.t. \( S \subseteq A \) iff \( a \in d(S) \).

**Proof.** This follows immediately from Definitions 4.1 and 4.5.

**Example 4.8.** (Example 2.2 continued, page 8) For the Nixon diamond, the values of \( d \) are depicted in Table 4.1. Recall that in this case \( A := \{a, b\} \) and \( R = \{(a, b), (b, a)\} \).

![Table 4.1: The values of the neutrality function \( n \), for Example 2.2.](image)

**Example 4.9.** (Example 2.3 continued, page 8) For simple reinstatement, the values of \( d \) are depicted in Table 4.2. Recall that in this case \( A := \{a, b, c\} \).

![Table 4.2: The values of the defence function \( d \) for Example 4.9.](image)

**Example 4.10.** Consider the AF \( \mathcal{A} = \{a, b\} \) and \( R = \{(a, a), (a, b), (b, a)\} \). Notice that this AF is symmetric (Definition 2.22, page 12). We depict this AF in Figure 4.2.

![Figure 4.2: The AF from Example 4.10.](image)

The values of \( d \) are depicted in Table 4.3. Recall that in this case \( A := \{a, b\} \).
Table 4.3: The values of the defence function \(d\) for Example 4.10.

| \(S\) | \(\emptyset\) | \(\{a\}\) | \(\{b\}\) | \(\{a, b\}\) | \(d(S)\) |
|-------|-------------|-----------|-----------|-------------|-----------|
| \(\emptyset\) | \(\emptyset\) | \(\{a\}\) | \(\{b\}\) | \(\{a, b\}\) | \(\emptyset\) |

**Example 4.11.** (Example 2.29, page 13) Consider the non-finitary AF \(A = \{a\} \cup \{b_i\}_{i \in \mathbb{N}}\) and \(R = \{(b_i, a)\}_{i \in \mathbb{N}}\). Let \(S \subseteq A\). If \(S = \emptyset\) or \(S = \{a\}\), then in both cases \(S^+ = \emptyset\), hence \(d(S) = \{x \in A | x^- \subseteq \emptyset\} = \{b_i\}_{i \in \mathbb{N}} = U\), the set of unattacked arguments.

If \(S\) is neither empty nor only \(\{a\}\), then \(S^+ = \{a\}\). Therefore, \(d(S) = \{x \in A | x^- \subseteq \{a\}\} = \{x \in A | x^- = \emptyset \text{ or } x^- = \{a\}\}\). However, for all \(x \in A\), \(x^- \neq \{a\}\). Therefore, \(d(S) = U\) as well.

In summary, for the AF depicted in Example 2.29, \(d(S)\) is a constant function, equal to \(U\), the set of all unattacked arguments.

**Example 4.12.** [7, Exercise 3] Consider the following AF [7, Figure 7], depicted in Figure 4.3.

We can calculate:

1. \(d(\{a\}) = \{x \in A | x^- \subseteq \{a\}^+\} = \{x \in A | x^- \subseteq \{b\}\} = \{a\}\), because only \(a^- = \{b\}\), while \(c^- = \{b, f\} \not\subseteq \{b\}\). Therefore, \(d(\{a\}) = \{a\}\).

2. \(d(\{b\}) = \{x \in A | x^- \subseteq \{b\}^+\} = \{x \in A | x^- \subseteq \{a, c\}\} = \{b, e\}\). Therefore, \(d(\{b\}) = \{b, e\}\).

3. \(d(\{b, e\}) = \{x \in A | x^- \subseteq \{b, e\}^+\} = \{x \in A | x^- \subseteq \{a, c, f\}\} = \{b, e\}\), as \(c^- = \{b, f\} \not\subseteq \{a, c, f\}\). Therefore, \(d(\{b, e\}) = \{b, e\}\).

**4.1.2 Properties**

**Corollary 4.13.** [14, Lemma 19] The defence function is \(\subseteq\)-monotone
Proof. Assume \( S \subseteq T \subseteq A \). If \( a \in d(S) \), then \( a^- \subseteq S^+ \subseteq T^+ \) by Definition 4.5 and Corollary A.13 (page 84), and hence \( a^- \subseteq T^+ \). Therefore, \( a \in d(T) \). As \( a \) is arbitrary, \( d(S) \subseteq d(T) \).

Definition 4.14. Let \( F_d := \{ S \subseteq A \mid d(S) = S \} \) be the set of fixed points of \( d \).

Corollary 4.15. \( \langle F_d, \subseteq \rangle \) is a complete lattice.

Proof. Clearly, \( \langle \mathcal{P}(A), \subseteq \rangle \) is a complete lattice and \( d : \mathcal{P}(A) \to \mathcal{P}(A) \) is a \( \subseteq \)-monotone by Corollary A.13. By the Knaster-Tarski theorem (Theorem C.32, page 92), the result follows.

Corollary 4.16. There exists a fixed point of \( d \).

Proof. Immediate as \( F_d \) is a complete lattice, so \( F_d \neq \emptyset \).

Further, as complete lattices are bounded, \( d \) will have a least fixed point and a greatest fixed point.

Corollary 4.17. For all \( a \in A \), \( a^- = \emptyset \) iff \( a \in d(\emptyset) \).

Proof. Recalling Corollary A.12 (page 84), we have that \( a \in d(\emptyset) \iff a^- \subseteq \emptyset^+ \iff a^- \subseteq \emptyset \iff a^- = \emptyset \).

Corollary 4.18. \( U = d(\emptyset) \).

Proof. Immediate from the Corollary 4.17 and Definition 2.14 (page 11).

This means that the unattacked arguments do not have to be defended by anything.

Corollary 4.19. For any \( S \subseteq A \), \( U \subseteq d(S) \).

Proof. Clearly \( \emptyset \subseteq S \) and hence \( d(\emptyset) = U \subseteq d(S) \) by Corollary A.13.

Intuitively, the unattacked arguments are amongst all defended arguments, because they do not need to be defended by anything.

The set of arguments defended by the set of all arguments \( A \) are exactly those arguments who are not attacked by an undefeated argument.

Corollary 4.20. We have that \( d(A) = \{ a \in A \mid a^- \cap U = \emptyset \} \).

Proof. We have that \( a \in d(A) \) iff \( a^- \subseteq A^+ \) iff \( \forall b \in a^- \) \( b \in A^+ \) iff \( \forall b \in a^- \) \( b \notin U \) (by Corollary 2.17, page 11), iff \( a^- \cap U = \emptyset \).

Corollary 4.21. We have that

\[
\bigcup_{i \in I} d(S_i) \subseteq d\left( \bigcup_{i \in I} S_i \right) \quad \text{and} \quad \bigcap_{i \in I} d(S_i) \supseteq d\left( \bigcap_{i \in I} S_i \right),
\] (4.2)

where in both cases the reverse containment may not be true.
Notice if $I = \emptyset$, Equation 4.21 reduces to $\emptyset \subseteq d(\emptyset)$ and $A \supseteq d(A)$, both of which are trivially true.

**Proof.** For the first result:

$$a \in \bigcup_{i \in I} d(S_i) \Leftrightarrow (\exists i \in I) a^- \subseteq S_i^+ \Rightarrow a^- \subseteq \bigcup_{i \in I} S_i^+ \Leftrightarrow a \in d\left(\bigcup_{i \in I} S_i\right).$$

The converse to the first result does not hold in general. Consider the argument framework $(\{a, b, c, e, f\}, \{(f, b), (e, c), (b, a), (c, a)\})$. This AF is depicted in Figure 4.3

![Figure 4.4: An AF that shows the converse to the first result of Corollary 4.21 is false.](image)

Let $S_1 = \{f\}$ and $S_2 = \{e\}$. We have $d(S_1 \cup S_2) = d(\{f, e\}) = \{f, e, a\}$. However, $d(S_1) = \{e, f\}$ and $d(S_2) = \{e, f\}$, as $a^- = \{b, c\}$. Therefore, $d(S_1) \cup d(S_2) = \{e, f\}$ while $d(S_1 \cup S_2) = \{f, e, a\}$, so $d(S_1 \cup S_2) \not\subseteq d(S_1) \cup d(S_2)$.

For the second result we apply Equation B.12 (page 97):

$$a \in d\left(\bigcap_{i \in I} S_i\right) \Leftrightarrow a^- \subseteq \left(\bigcap_{i \in I} S_i\right)^+ \subseteq \bigcap_{i \in I} S_i^+ \Rightarrow (\forall i \in I) a^- \subseteq S_i^+$$

$$\Leftrightarrow (\forall i \in I) a \in d(S_i) \Leftrightarrow a \in \bigcap_{i \in I} d(S_i).$$

The converse to the second result does not hold in general. Recall the AF from Example 2.3 (page 9), depicted in Figure 2.3. Let $S_1 = \{c\}$ and $S_2 = \{e\}$. We have that $d(S_1) = d(S_2) = \{a, c, e\}$ hence $d(S_1) \cap d(S_2) = \{a, c, e\}$. However, $d(S_1 \cap S_2) = d(\emptyset) = \{c, e\}$. Clearly, $\{c, e, a\} \not\subseteq \{c, e\}$ and hence $d(S_1) \cap d(S_2) \not\subseteq d(S_1 \cap S_2)$. □

**Theorem 4.22.** [14, Lemma 28] If $\langle A, R \rangle$ is finitary, then $d$ is $\omega$-continuous (Definition C.41, page 97). Else, $d$ may or may not be $\omega$-continuous.

**Proof.** Let \( \{S_i\}_{i \in \mathbb{N}} \) be an $\omega$-chain in $\mathcal{P}(A)$ with limit $S := \bigcup_{i \in \mathbb{N}} S_i$, where $i < j$ implies $S_i \subseteq S_j$. Assume $\langle A, R \rangle$ is finitary. Let $a \in d(S)$, then as $a^-$ is finite, let $a^- := \{b_1, b_2, \ldots, b_m\}$. For each such $b_j \in a^-$, we have $b_j \in S^+$, which means for each $1 \leq j \leq m$ there is some $b_j \in S^+_{i_j}$. Let $k := \max \{i_1, i_2, \ldots, i_m\}$, so
Given this, the $S_i$'s form a chain. Therefore, $(\exists k \in \mathbb{N}) a \in d(S_k)$ and hence $a \in \bigcup_{k \in \mathbb{N}} d(S_k)$. As $a$ is arbitrary,

$$d(S) = d \left( \bigcup_{i \in \mathbb{N}} S_i \right) \subseteq \bigcup_{i \in \mathbb{N}} d(S_i),$$

(4.3)

where we have applied Corollary 4.21 by choosing $I = \mathbb{N}$. By instantiating Definition C.41 (page 94), we conclude that $d$ is $\omega$-continuous.

The following example is a non-finitary AF where $d$ is not $\omega$-continuous. Consider Example 2.30 (page 13). This is not a finitary AF because $a^-$ is a countably infinite set. For $i \in \mathbb{N}$, let $S_i := \{c_j\}_{j=0}^i = \{c_0, c_1, \ldots, c_i\}$. Clearly, $\{S_i\}_{i \in \mathbb{N}}$ is an $\omega$-chain in $\mathcal{P}(A)$, with limit $U = \bigcup_{i \in \mathbb{N}} S_i = \{c_0, c_1, c_2, \ldots\}$. By Corollary 4.19, we can see that $d(S_i) = U$ for all $i \in \mathbb{N}$, hence $\bigcup_{i \in \mathbb{N}} d(S_i) = \bigcup_{i \in \mathbb{N}} U = U$. However, $d(U) = U \cup \{a\}$, so $d(\bigcup_{i \in \mathbb{N}} S_i) \not\subseteq \bigcup_{i \in \mathbb{N}} d(S_i)$. Therefore, $d$ for this non-finitary AF is not $\omega$-continuous.

The following example is a non-finitary AF where $d$ is $\omega$-continuous. (Example 2.30 page 93 continued) We have that $d \equiv \{b_i\}_{i \in \mathbb{N}} = U$ for this non-finitary AF; this can now be seen from the definition of $d$ and Corollary 4.19. Let $\{S_i\}_{i \in \mathbb{N}}$ be any $\omega$-chain of unattacked arguments with limit $U \subseteq A - \{a\}$. Clearly, $\bigcup_{i \in \mathbb{N}} d(S_i) = A - \{a\} = U$, and $d(\bigcup_{i \in \mathbb{N}} S_i) = U \subseteq U$, which is true. Therefore, $d$ is $\omega$-continuous.

### 4.2 Self-Defending Sets

Self-defending sets formalise the idea that a set of arguments can respond to all counterattacks.

#### 4.2.1 Definition

**Theorem 4.23.** Let $S \subseteq A$. $S \subseteq d(S)$ iff $S^- \subseteq S^+$.

**Proof.** Let $S \subseteq A$. $(\Rightarrow)$ If $a \in S \subseteq d(S)$, then $a \in d(S)$, iff $a^- \subseteq S^+$, which by taking the union of both sides over all $a \in S$ gives $\bigcup_{a \in S} a^- \subseteq S^+$, iff $S^- \subseteq S^+$.

$(\Leftarrow$, contrapositive) If $S \not\subseteq d(S)$, then there is some $a \in S$ such that $a^- \not\subseteq S^+$, so given this $a \in S$ there is some $b \in a^-$ such that $b \notin S^+$. As $a \in S$ this implies that $a^- \not\subseteq S^-$, hence there is a $b \in S^-$ such that $b \notin S^+$, therefore $S^- \not\subseteq S^+$. This means that $S^- \not\subseteq S^+$ implies $S \not\subseteq d(S)$. 

**Definition 4.24.** We say $S \subseteq A$ is **self-defending** iff it satisfies any one of the two equivalent properties in Theorem 4.23.

Intuitively, a self-defending set of arguments attacks all of its counterarguments. Formally, these sets are postfixed points of $d$ (Definition C.20 page 90).
4.2.2 Existence

**Definition 4.25.** Given an underlying AF, let $SD \subseteq \mathcal{P}(A)$ denote the set of self-defending sets.

**Example 4.26.** (Example 2.39 continued, page 14) For this AF, we have

$$SD = \{\emptyset, \{b, c, e\}, \{a, b, c, e\}\}. \quad (4.4)$$

If we need to make the underlying AF $A = \langle A, R \rangle$, explicit, then we may write $SD(A)$ or $SD(\langle A, R \rangle)$.

**Corollary 4.27.** $\emptyset \in SD$.  
*Proof.* We have that $\emptyset \subseteq d(\emptyset) = U$ (Corollary 4.18, page 32), trivially. Therefore, for any AF, self-defending sets always exist as $\emptyset$ is (vacuously) self-defending. So $SD \neq \emptyset$.

**Corollary 4.28.** $U \in SD$.  
*Proof.* As $\emptyset \in SD$, $\emptyset \subseteq d(\emptyset)$ and hence by Corollary 4.13 (page 31), $d(\emptyset) \subseteq d^2(\emptyset)$. By Corollary 4.18 (page 32), it follows that $U \subseteq d(U)$ and hence $U \in SD$. 

**Corollary 4.29.** If $S \subseteq U$, then $S \in SD$. The converse is false.  
*Proof.* We prove the contrapositive. If $S \notin SD$, then $S \nsubseteq d(S)$. This means there is some $a \in S$ such that $a \notin d(S)$. For this $a$, it means that $a^- \nsubseteq S^+$, i.e. there is some $b \in a^-$ such that $b \notin S^+$. But if there is some $b \in a^-$, then $a^- \neq \emptyset$ and hence $a \notin U$. Therefore, there exists an $a \in S$ such that $a \notin U$. Therefore, $S \notin U$.

For the converse, consider Example 2.39 (page 8) for $S = \{a, c\} \in SD$ but $\{a, c\} \nsubseteq \{c\} = U$. 

**Corollary 4.30.** $SD$ is closed under $d$.  
*Proof.* Let $S \in SD$, then $S \subseteq d(S)$. As $d$ is $\subseteq$-monotonic, then $d(S) \subseteq d[d(S)]$. Therefore, $d(S) \in SD$. As $S \in SD$ is arbitrary, $d(S) \in SD$ and hence $d : SD \rightarrow SD$.

**Corollary 4.31.** $SD$ is not closed under $n$.  
*Proof.* Consider the following AF: $A = \{a, b, c, e\}$ and $R = \{(a, b), (b, a), (e, c)\}$, depicted in Figure 4.5.

![Figure 4.5: The AF from Corollary 4.31](image-url)
Clearly, \( \{a\} \in SD \), because \( d(\{a\}) = \{x \in A | x <\{b\}\} = \{a, e\} \supseteq \{a\} \). Consider \( n(\{a\}) = \{c, e\} \). Is \( \{c, e\} \in SD \)? No, because \( \{c, e\}^+ = \{e\} \) and that \( d(\{c, e\}) = \{x \in A | x \subseteq \{e\}\} = \{e\} \not\supseteq \{c, e\} \). Therefore, \( \{a\} \in SD \) but \( n(\{a\}) \notin SD \).

### 4.2.3 Lattice-Theoretic Structure

**Corollary 4.32.** \( SD \) is not in general \( \subseteq \)-upward closed.

**Proof.** From Example 2.3 (page 8), clearly that \( \emptyset \in SD \), and \( \emptyset \subseteq \{b\} \), but \( \{b\} \notin SD \).

**Corollary 4.33.** \( A \in SD \) iff \( A \) is the \( \subseteq \)-largest fixed point of \( d \).

**Proof.** (\( \Rightarrow \)) For any set \( S \subseteq A \), \( d(S) \subseteq A \) hence \( d(A) \subseteq A \). As \( A \in SD \), then \( A \subseteq d(A) \). It follows that \( A \) is a fixed point of \( d \). For any set \( S \in \mathcal{P}(A) \), we have \( S \subseteq A \) hence \( A \) is the \( \subseteq \)-largest fixed point of \( d \).

(\( \Leftarrow \)) If \( d(A) = A \) then trivially \( A \subseteq d(A) \) and hence \( A \in SD \).

**Corollary 4.34.** \( SD \) is closed under arbitrary unions.

**Proof.** Let \( \{S_i\}_{i \in I} \) be a family of self-defending subsets of \( A \). For each \( i \in I \), \( S_i \subseteq d(S_i) \). We have by Corollary 4.21 (page 32),

\[
\bigcup_{i \in I} S_i \subseteq \bigcup_{i \in I} d(S_i) \subseteq d\left( \bigcup_{i \in I} S_i \right).
\]

Therefore, \( \bigcup_{i \in I} S_i \) is also self-defending.

Notice if we take the empty union in Corollary 4.34, then \( \emptyset \in SD \) which is also true. It follows that \( SD \) is \( \omega \)-complete, chain complete and directed complete when instantiating this arbitrary family of self-defending sets into any \( \omega \)-chain, chain or directed set, respectively.

**Corollary 4.35.** \( SD \) is not in general closed under intersections.

**Proof.** Consider Example 2.4 (page 9) where \( \langle \{a, b, c, e\}, \{(c, b), (e, b), (b, a)\} \rangle \).

Let \( S_1 = \{c, a\} \) and \( S_2 = \{e, a\} \). Clearly \( d(S_1) = \{c, a\} \) and \( d(S_2) = \{e, a\} \), therefore both \( S_1 \) and \( S_2 \) are self-defending. However, \( S_1 \cap S_2 = \{a\} \). Further, \( d(\{a\}) = \{c, e\} \), and \( \{a\} \not\subseteq d(\{a\}) \), hence \( S_1 \cap S_2 \) is not self-defending.

### 4.3 On the Interaction Between the Neutrality and Defence Functions

Assume a fixed underlying \( AF \) with neutrality and defence functions \( n \) and \( d \) respectively.
4.3.1 Composing $d$ and $n$

The first result shows that $d$ is the square of $n$, or that $n$ is the “square root” of $d$.

**Theorem 4.36.** [14, Lemma 45] For all $S \subseteq A$, we have that $d(S) = n^2(S)$.

**Proof.** Let $S \subseteq A$ be arbitrary. We have that

$$a \in n^2(S) \iff a \notin (A - S^+)$$

by Definition 3.1 (page 21),

$$\iff (\forall b \in A - S^+) \neg a \notin b^+$$

by Definition A.7 (page 83),

$$\iff (\forall b \in A - S^+) b \notin a^-$$

by Corollary A.5, page 82,

$$\iff (\forall b \in A) [b \notin S^+ \implies b \notin a^-]$$

by bounded quantifiers,

$$\iff (\forall b \in A) [b \in a^- \implies b \in S^+]$$

by the contrapositive,

$$\iff a^- \subseteq S^+.$$  

This shows the result. 

The second result shows that $d$ and $n$ commute when composed.

**Theorem 4.37.** For all $S \subseteq A$, we have $d(n(S)) = n(d(S))$.

**Proof.** From Theorem 4.36, $d = n^2$ and hence $n \circ d = n \circ (n \circ n) = (n \circ n) \circ n$ by associativity of composition, hence $n \circ d = d \circ n$. 

4.3.2 Preservation of CF by $d$

Theorem 4.37 has many consequences. Firstly, unlike $n$ (Corollary 3.27, page 25), $d$ preserves CF sets.

**Corollary 4.38.** For any AF, if $S \subset CF$ then $d(S) \subset CF$. The converse is false.

**Proof.** The result follows from the $\subseteq$-monotonicity of $d$ (Corollary 4.13, page 31) and Theorem 4.37. If $S \subset CF$, then $S \subseteq n(S)$, then $d(S) \subseteq d(n(S)) = n(d(S))$, therefore $d(S) \subset CF$.

The converse is false. The counter-example is as follows: consider $\langle A, R \rangle$ where $A = \{a, b, c\}$ and $R = \{(a, b), (b, a), (a, c), (c, b)\}$; this is depicted in Figure 4.6.

![Figure 4.6: The AF that is a counter-example to the converse of Corollary 4.38](image-url)
Consider the subset of arguments $S = \{a, c\}$. We see that $d(S) = \{a\}$. We can see that $d(S) \in CF$ and $S \notin CF$. Therefore, the converse of the result is not true.

By induction, any finite iteration of $d$ also preserves cf sets.

**Corollary 4.39.** For $S \in CF$, the $P(A)$-sequence $\{S_i\}_{i \in \mathbb{N}}$ where $S_0 := S$ and $S_{i+1} := d(S_i)$ is a $CF$-sequence.

**Proof.** We show that $(\forall i \in \mathbb{N}) S_i \in CF$ by induction on $i$.

1. (Base) By assumption $S_0 \in CF$.
2. (Inductive) If $S_i \in CF$ then $d(S_i) \in CF$ by Corollary 4.38.

Therefore, by induction, the result follows.

Not only that finite iterations of $d$ on $S \in CF$ is also in $CF$, but that the limit of the ascending $\omega$-chain $\{d^k(S)\}_{k \in \mathbb{N}}$ is also cf.

**Corollary 4.40.** Let $S \in CF$. The limit of the chain $\{d^i(S)\}_{i \in \mathbb{N}} \cup_{i \in \mathbb{N}} d^i(S)$, is also cf.

**Proof.** Immediate from Corollary 4.39 (page 38) and Theorem 3.23 (page 25).

Further, if $d$ is $\omega$-continuous then the supremum of iterating $d$ on a self-defending set is also a fixed point of $d$.

**Corollary 4.41.** If $d$ is $\omega$-continuous (Definition 4.41 page 24), then for $S \in SD$, the limit of the chain $\{d^i(S)\}_{i \in \mathbb{N}} \cup_{i \in \mathbb{N}} d^i(S)$, is a fixed point of $d$.

**Proof.** By $\omega$-continuity of $d$, we have that

$$d \left( \bigcup_{i \in \mathbb{N}} d^i(S) \right) = \bigcup_{i \in \mathbb{N}} d^{i+1}(S) = \bigcup_{i \in \mathbb{N}} d^i(S) = S \cup \bigcup_{i \in \mathbb{N}} d^{i+1}(S),$$

(4.5)

where the last equality is because $S = d^0(S) \subseteq d(S)$, hence $d(S) = S \cup d(S)$. This means

$$S \cup \bigcup_{i \in \mathbb{N}} d^{i+1}(S) = \bigcup_{i \in \mathbb{N}} d^i(S).$$

(4.6)

Therefore, the limit is a fixed point of $d$.

We can further strengthen this result via transfinite induction on ordinal-valued iterations of $d$. This is necessary as we do not assume the AFs we are dealing with are finite. For finite $|A|$ it is sufficient to have ordinary induction over $\mathbb{N}$ (as $\omega$). But if $|A|$ is any cardinal number then we need to perform induction over a sufficiently large ordinal number.

\footnote{This is an ascending chain as $d$ is $\subseteq$-monotone by Corollary 4.16 (page 31).}

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Lemma 4.42. Let $\alpha$ and $\beta$ be ordinal numbers and $S \subseteq A$. We have

$$\alpha < \beta \Rightarrow d^\alpha(S) \subseteq d^\beta(S). \tag{4.7}$$

Proof. Let $\alpha$ be a given ordinal. If $\beta = \alpha + 1$, then the result follows by the $\subseteq$-monotonicity of $d$. If $\beta$ is a limit ordinal larger than $\alpha$, then

$$d^\beta(S) = \bigcup_{\gamma < \beta} d^\gamma(S). \tag{4.8}$$

However, one of those terms in the union is $\gamma = \alpha$ and hence $d^\alpha(S) \subseteq d^\beta(S)$. Therefore, the result follows.

Theorem 4.43. Let $\beta$ be an ordinal number. If $S \in CF$ then $(\forall \alpha < \beta) d^\alpha(S) \in CF$.

Proof. We apply transfinite induction on $\alpha$.

1. (Base) If $\alpha = 0$ then $d^0(S) = S \in CF$.

2. (Successor) If $d^\alpha(S) \in CF$ then $d(d^\alpha(S)) = d^{\alpha+1}(S) \in CF$, by Corollary 4.38.

3. (Limit) Let $\gamma < \beta$ be a limit ordinal. Assume that $(\forall \alpha < \gamma) d^\alpha(S) \in CF$, then

$$T := d^\gamma(S) = \bigcup_{\alpha < \gamma} d^\alpha(S). \tag{4.9}$$

Assume for contradiction that $T \notin CF$, then there are some $a, b \in T$ such that $R(a, b)$. Therefore, there are ordinal numbers $\alpha_1, \alpha_2 < \gamma$ such that $a \in d^{\alpha_1}(S)$ and $b \in d^{\alpha_2}(S)$. By Lemma 4.42 we let $\delta := \max(\alpha_1, \alpha_2) < \gamma$ and hence $a, b \in d^\delta(S)$. As $R(a, b)$, this means $d^\delta(S) \notin CF$ – contradiction, as we had assumed for all $\gamma < \beta$, $d^\gamma(S) \in CF$. Therefore,

$$d^\gamma(S) = \bigcup_{\alpha < \gamma} d^\alpha(S) \in CF. \tag{4.10}$$

By transfinite induction, this shows the result.

Therefore, $d$ is closed on $CF$. This means $d : CF \rightarrow CF$ is well-defined.

4.3.3 Interaction of Fixed Points of $n$ and $d$

The following result shows that $d$ is closed on the set of fixed points of $n$.

Corollary 4.44. If $S \subseteq A$ is a fixed point of $n$, then $d(S)$ is also a fixed point of $n$. The converse is false.
Proof. If \( S = n(S) \) then \( d(S) = d(n(S)) = n(d(S)) \) by Theorem 4.37. The converse is false: consider \( S = \{ c \} \) for Example 2.3 (page 8). We have \( d(S) = \{ a, c \} \) which is a fixed point of \( n \). However, \( S \) is not a fixed point of \( n \). □

Further, any fixed point of \( n \) is also a fixed point of \( d \).

**Corollary 4.45.** If \( S \subseteq A \) is a fixed point of \( n \), then it is also a fixed point of \( d \). The converse is false.

Proof. If \( S = n(S) \), then \( n^2(S) = n(S) = S \), and hence \( d(S) = S \). The converse is false: consider \( S = \emptyset \) for Example 2.2 (page 8), then \( d(\emptyset) = \emptyset \) yet \( n(S) = A \neq \emptyset \). Therefore, \( S \) is a fixed point of \( d \) but not of \( n \). □

If \( S \) is a fixed point for \( d \), then so is \( n(S) \).

**Corollary 4.46.** If \( S \subseteq A \) is a fixed point of \( d \), then \( n(S) \) is also a fixed point of \( d \). The converse is false.

Proof. We have that \( S = d(S) \) means \( n(S) = n(d(S)) = d(n(S)) \). For the converse: consider Example 2.3 (page 8) again, where \( S = \{ c \} \) and \( n(S) = \{ a, c \} \) is a fixed point of \( d \), but \( \{ c \} \) is not a fixed point of \( d \). □

### 4.4 Summary

- Given an AF, its defence function \( d : \mathcal{P}(A) \rightarrow \mathcal{P}(A) \) is defined as \( a \in d(S) \iff a^- \subseteq S^+ \).
- \( d \) is \( \subseteq \)-monotone. Therefore, let \( F_d \) denote the set of all fixed points of \( d \), then \( \langle F_d, \subseteq \rangle \) is a complete lattice, so \( d \) has a fixed point.
- \( d \) satisfies the following properties: \( U = d(\emptyset) \), for any \( S \subseteq A, U \subseteq d(S) \), and 
  \[
  \bigcup_{i \in I} d(S_i) \subseteq d\left( \bigcup_{i \in I} S_i \right) \quad \text{and} \quad \bigcap_{i \in I} d(S_i) \supseteq d\left( \bigcap_{i \in I} S_i \right),  \tag{4.11}
  \]
- If the underlying AF is finitary, then \( d \) is \( \omega \)-continuous, else \( d \) may or may not be \( \omega \)-continuous.
- A set \( S \subseteq A \) is self-defending iff \( S \subseteq d(S) \). The set of all self-defending sets for an AF \( A \) is \( SD(A) \) or just \( SD \).
- For any AF, \( SD \) satisfies \( \emptyset, U \in SD \), and the restriction of the defence function \( d : SD \rightarrow SD \) is well-defined.
- \( SD \) is not \( \subseteq \)-upward closed, not closed under intersections, but closed under arbitrary unions.
- \( A \in SD \) iff \( A \) is the \( \subseteq \)-largest fixed point of \( d \).
\begin{itemize}
\item Given an AF with neutrality function $n$ and defence function $d$, both from $\mathcal{P}(A) \rightarrow \mathcal{P}(A)$, we have $d \circ n = n \circ d$ and $n^2 = d$.
\item $d : CF \rightarrow CF$ is well-defined. Further, for any $S \in CF$, one can generate a conflict-free-sequence $\{S_i\}_{i \in \mathbb{N}}$ where $S_0 = S$ and $S_{i+1} = d(S_i)$, whose limit is also conflict-free.
\item If $d$ is $\omega$-continuous, then for $S \in SD$, the limit of the ascending chain of iterates on $S$ under $d$ is a fixed point of $d$.
\item If $S$ is a fixed point of $n$, then $d(S)$ is also a fixed point of $n$.
\item If $S$ is a fixed point of $n$, then it is also a fixed point of $d$.
\item If $S$ is a fixed point of $d$, then $n(S)$ is also a fixed point of $d$.
\end{itemize}
5 Admissible Sets

Recall that conflict-freeness formalises a self-consistent set of arguments (Section 3.2, page 23), and self-defence formalises the idea that a set of arguments replies adequately to all external criticisms (Section 4.2, page 34). We now combine both these ideas to define when is it that an argument is “winning”.

5.1 Definition

**Definition 5.1.** [14, Definition 6(2), Lemma 18] The set $S \subseteq A$ is an admissible set iff $S \subseteq n(S)$ and $S \subseteq d(S)$.

Intuitively, admissible sets serve as the starting point for “winning” set of arguments, because these are the arguments that are mutually consistent and attacks all counterarguments. This addresses the main criticism against naive semantics (Section 3.3.3, page 27) by now requiring that conflict-free sets of arguments also defend themselves.

**Definition 5.2.** Given an underlying AF, let $\text{ADM} \subseteq \mathcal{P}(A)$ denote the set of admissible sets.

If the underlying AF $A = \langle A, R \rangle$ needs to be explicitly specified, we can write $\text{ADM}(A)$ or $\text{ADM}(\langle A, R \rangle)$.

**Example 5.3.** (Example 2.7, page 9) For floating reinstatement, we have

\[
\text{ADM} = \{ \emptyset, \{a\}, \{b\}, \{a, e\}, \{b, e\} \}.
\]

(5.1)

**Example 5.4.** [4, Example 2.2.1] Consider the AF $A = \{a, b, c, f, e\}$ and $R = \{(a, b), (c, b), (c, f), (f, c), (f, e), (e, e)\}$.

This is depicted in Figure 5.1.

![Figure 5.1: The AF from Example 5.4](image)

We have $\text{ADM} = \{ \emptyset, \{a\}, \{e\}, \{a, e\}, \{f\}, \{a, f\}\}$.

**Example 5.5.** (Example 2.26 continued, page 12) We claim for this AF the non-empty admissible sets take the form $S(a_i) := \{a_j \in A \mid j \geq i\}$ and $S(b_i) := \{b_j \in A \mid j \geq i\}$, for $i \in \mathbb{Z}$.

Recall the AF as illustrated in Figure 2.9 which we repeat here for convenience.
For example, we have $S(a_{-1}) = \{a_{-1}, a_0, a_1, a_2, \ldots\}$. Notice that $S(a_{-1})^- = \{b_j \in A \mid j \geq -1\} \subseteq S(a_{-1})^+$. 

More generally, it is easy to see that all sets of this form are cf. Given $S(a_i)$ as above, by inspecting the AF, we can see that $S(a_i)^- = \{b_j \in A \mid j \geq i\}$, because $b_i$ attacks $a_i$, $b_{i+1}$ attacks $a_{i+1} \in S(a_i)$... etc. Further, $S(a_i)^+ = \{b_j \in A \mid j \geq i - 1\}$, because $a_i$ attacks $b_{i-1}$, $a_{i+1}$ attacks $b_i$... etc. Therefore, we have that $S(a_i)^- \subseteq S(a_i)^+$ for all $i \in \mathbb{Z}$ and hence $S(a_i) \in SD$ by Theorem 4.23 (page 34). By a similar argument, $S(b_i) \in SD$ for all $i \in \mathbb{Z}$. Therefore, all sets of this form are admissible.

To show that these are the only non-empty, admissible sets, we can see that for any other conflict-free set, we need to defend against all attacks. If a cf set $S$ is finite, then it cannot be admissible as we choose the argument $a_i$ or $b_i \in A$ such that $i$ is the largest index and one of $a_i$ or $b_i$ has an attacker not defended by the set $S$, so no finite set of arguments is admissible. Further, no admissible set can have both $a_i$ and $b_j$ arguments for a given $i, j \in \mathbb{Z}$, because the attackers of $a_i$ or $b_i$ can only be defended against by, respectively, an $b_i$ or $a_i$ argument such that if you include all defenders, the resulting set cannot be cf.

Satisfying self-defence would mean that the set includes all defender arguments with indices larger indices $i$. Therefore,

$$ADM = \{\{a_j \in A \mid j \geq i\}, \{b_j \in A \mid j \geq i\} \mid i \in \mathbb{Z}\} \cup \{\emptyset\}. \tag{5.2}$$

**Example 5.6.** [7, Exercise 6(a)] Consider the AF in Figure 4.1 (page 29). Is $\{a\} \in ADM$? Yes, because $\{a\} \in CF$ and $a$ has no attackers so it is vacuously self defending.

**Example 5.7.** [7, Exercise 6(b)] Consider the following AF [7, Figure 5], depicted in Figure 5.2.

![Figure 5.2: The AF depicting [7, Figure 5], from Example 5.7](image)

Is $\{c\} \in ADM$? No, because $b$ attacks $c$, and $c$ does not attack $b$ back in turn. Therefore, $\{c\} \notin SD$.

**Example 5.8.** [7, Exercise 6(c)] Consider the following AF [7, Figure 6], depicted in Figure 5.3.

![Figure 5.3: The AF depicting [7, Figure 6], from Example 5.8](image)
Is \{a\} ∈ ADM? No, because \{a\} \notin CF as it is self-attacking.

**Example 5.9.** [7, Exercise 6(d)] Consider the AF in Figure 4.3 (page 31). Is \{a, c, e\} ∈ ADM? No, because c attacks e and hence \{a, c, e\} \notin CF.

### 5.2 Existence

**Corollary 5.10.** ADM = CF ∩ SD.

*Proof.* Immediate from Definition 5.2 (page 42). 

**Corollary 5.11.** ∅ ∈ ADM.

*Proof.* By Corollary 5.10 (page 44) and that ∅ ∈ CF (Corollary 3.11) and ∅ ∈ SD (Corollary 4.27) page 35).

Therefore, for any AF, admissible sets exist (ADM ≠ ∅), with ∅ being admissible.

The following result shows why the set of unattacked arguments can always be seen as winning.

**Corollary 5.12.** U ∈ ADM.

*Proof.* Immediate from Corollaries 3.17 and 4.28 (pages 24 and 35, respectively).

**Corollary 5.13.** If S ⊆ U then S ∈ ADM. The converse is not true.

*Proof.* As U ∈ CF, S ∈ CF. We apply Corollary 4.19 (page 32) to S, so U ⊆ d(S). Therefore, S ⊆ U ⊆ d(S) and hence S ∈ SD. Therefore, S ∈ ADM.

The converse is not true, e.g. Example 2.3 \{a, c\} ∈ ADM but \{a, c\} ∉ U = \{c\}.

Corollary 5.13 can be written succinctly as \(\mathcal{P}(U) \subseteq ADM\).

**Corollary 5.14.** ADM ⊆ CF, and the converse is generally not true.

*Proof.* ADM ⊆ CF is immediate from Corollary 5.10. For the converse, consider the AF from Corollary 3.22 (page 24). Clearly \{b\} ∈ CF but d(\{b\}) = \{a\} and \{b\} ⊈ \{a\}. Therefore, \{b\} \notin ADM.
Corollary 5.15. If $S \in \text{ADM}$ then $d(S) \in \text{ADM}$. The converse is not true.

Proof. If $S \in \text{ADM}$ then $S \in \text{CF}$ and $S \in \text{SD}$. By Corollary 4.38 (page 37), $d : \text{CF} \rightarrow \text{CF}$ and hence $d(S) \in \text{CF}$. By Corollary 4.30 (page 35), $d : \text{SD} \rightarrow \text{SD}$ and hence $d(S) \in \text{SD}$. Therefore, $d(S) \in \text{CF} \cap \text{SD} = \text{ADM}$.

For the converse, we can see from Example 5.4 that $\{a, f\} \in \text{ADM}$ but $\{a, f\} \not\subseteq U = \{a\}$. Therefore, $d : \text{ADM} \rightarrow \text{ADM}$ is well-defined.

Corollary 5.16. If $S \in \text{ADM}$ then $n(S) \not\in \text{ADM}$ in general.

Proof. In Example 2.2 (page 8), we have $n(\emptyset) = \{a, b\}$, where $\emptyset \in \text{ADM}$ but $\{a, b\} \not\in \text{CF}$ and hence $\{a, b\} \not\in \text{ADM}$.

5.3 Lattice Theoretic Properties

Lemma 5.17. ADM is in general not closed under intersections.

Proof. Example 2.7 (page 9), we have $\{a, d\}, \{b, d\} \in \text{ADM}$ but $\{a, d\} \cap \{b, d\} = \{d\} \not\in \text{ADM}$.

Lemma 5.18. ADM is in general not closed under unions.

Proof. Example 2.7 (page 9), we have $\{a, d\}, \{b, d\} \in \text{ADM}$ but $\{a, d\} \cup \{b, d\} = \{a, b, d\} \not\in \text{ADM}$.

But if the union of a family of admissible sets is cf, then that the union of that family is also admissible. The following result generalises [9, Lemma 1] to accommodate possibly infinite AFs and generalised unions.

Lemma 5.19. [9, Lemma 1] Let $I$ be an index set and $\{S_i\}_{i \in I} \subseteq \text{ADM}$. $\bigcup_{i \in I} S_i \in \text{CF}$ iff $\bigcup_{i \in I} S_i \in \text{ADM}$.

Proof. ($\Rightarrow$) If $I = \emptyset$, then $\{S_i\}_{i \in I} = \emptyset$ and the result follows by Corollary 5.11 as $\emptyset \in \text{ADM}$. Otherwise, we know that $(\forall i \in I) S_i \subseteq d(S_i)$. Therefore, $\bigcup_{i \in I} S_i \subseteq \bigcup_{i \in I} d(S_i)$. By Corollary 4.21 (page 32), $\bigcup_{i \in I} d(S_i) \subseteq d\left(\bigcup_{i \in I} S_i\right)$. Therefore, $\bigcup_{i \in I} S_i \in \text{SD}$ and hence $\bigcup_{i \in I} S_i \in \text{ADM}$.

($\Leftarrow$) Trivial, as $\text{ADM} \subseteq \text{CF}$.

Corollary 5.20. If $S \in \text{ADM}$, then the limit of the $\omega$-chain $\{d^k(S)\}_{k \in \mathbb{N}}$ is also in $\text{ADM}$.

Proof. As $d : \text{ADM} \rightarrow \text{ADM}$, this chain is in $\text{ADM}$ by induction on $k$. By Corollary 4.40 (page 38), the limit of this chain is in $\text{CF}$. By Lemma 5.19, the result follows.

After iterating $d$ a transfinite number of times on any $S \in \text{ADM}$, the result is still in $\text{ADM}$.

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Theorem 5.21. Let $S \in \text{ADM}$. Let $\beta$ be an ordinal number. We have that

\[(\forall \alpha < \beta) \ d^\alpha (S) \in \text{ADM}. \quad (5.3)\]

Proof. We apply transfinite induction on $\alpha$.

1. (Base) If $\alpha = 0$ then $d^0 (S) = S \in \text{ADM}$.

2. (Successor) If $d^\alpha (S) \in \text{ADM}$ then $d (d^\alpha (S)) \in \text{ADM}$, by Corollary 5.15.

3. (Limit) Let $\gamma < \beta$ be a limit ordinal. Assume that $(\forall \alpha < \gamma) \ d^\alpha (S) \in \text{ADM}$. Then

\[T := d^\gamma (S) = \bigcup_{\alpha<\gamma} d^\alpha (S). \quad (5.4)\]

From the limit case of Theorem 4.43, $T \in \text{CF}$. Now assume for some $a \in T$, $b \in a^-$. There is some $\alpha < \gamma$ such that $a \in d^\alpha (S)$. As $d^\alpha (S) \in \text{ADM}$ by assumption, we must have $b \in (d^\alpha (S))^+$. Therefore, $b \in T^+$ and hence $T \in \text{ADM}$.

By transfinite induction, this shows the result. \qed

Theorem 5.22. $\langle \text{ADM}, \subseteq \rangle$ is $\omega$-complete.

Proof. This follows from Theorem 3.23 (page 25) and Corollary 4.34 (page 36). Let $\{S_i\}_{i \in \mathbb{N}}$ be an ascending chain in $\text{ADM}$. Let $S = \bigcup_{i \in \mathbb{N}} S_i$. Clearly $S \in \text{CF}$. Further, if $a \in S$, then $(\exists i \in \mathbb{N}) a \in S_i \in \text{ADM}$ hence $(\exists i \in \mathbb{N}) a \in d (S_i)$. As $S_i \subseteq S$, we have $d (S_i) \subseteq d (S)$ and hence $a \in d (S)$. Therefore, $S \subseteq d (S)$. Therefore, $S \in \text{ADM}$.

Theorem 5.23. $\langle \text{ADM}, \subseteq \rangle$ is chain complete.

Proof. Let $Ch \subseteq \text{ADM}$ be an arbitrary $\subseteq$-chain. By Theorem 5.14 $Ch \subseteq \text{CF}$ and by Theorem 3.25 (page 25), $\bigcup Ch \in \text{CF}$. By Lemma 5.19 it follows that $\bigcup Ch \in \text{ADM}$. As $Ch$ is any chain in $\text{ADM}$, the result follows. \qed

Corollary 5.24. $\text{max}_{\subseteq} \text{ADM} \neq \emptyset$.

Proof. Every chain in $\text{ADM}$ has a least upper bound in $\text{ADM}$ because the poset $\langle \text{ADM}, \subseteq \rangle$ is chain complete from the preceding theorem. By Zorn’s lemma, $\langle \text{ADM}, \subseteq \rangle$ has at least one $\subseteq$-maximal element. \qed

Just as in Theorem 3.32 (page 26), we have used Zorn’s lemma to show that there exist maximal admissible sets. Therefore, Zorn’s lemma is sufficient for showing Corollary 5.24. However, we may ask whether Zorn’s lemma is necessary for Corollary 5.24. We answer this in Appendix D (page 96).

Corollary 5.25. Let $\mathcal{D} \subseteq \text{ADM}$ be a directed subset under $\subseteq$. Its supremum $\bigcup \mathcal{D} \in \text{ADM}$.
Proof. We have that $\mathcal{D} \subseteq \text{ADM} \subseteq \text{CF}$ where $\subseteq$ is the underlying partial order, so $\mathcal{D}$ is also a directed set in $\text{CF}$ and hence $\bigcup \mathcal{D} \in \text{CF}$. By choosing $\{S_i\}_{i \in I}$ from Lemma 5.19 (page 45) to $\mathcal{D}$, it follows that $\bigcup \mathcal{D} \in \text{ADM}$. □

Corollary 5.26. [14, Theorem 11(1)] The set $\langle \text{ADM}, \subseteq \rangle$ is a pointed directed complete partial order (dcpo, Definition C.44, page 94).

Proof. This is immediate from Corollaries 5.25 and 5.11 (page 44). □

Corollary 5.27. Every non-empty set of admissible sets has a $\subseteq$-glb.

Proof. Let $\emptyset \neq S \subseteq \text{ADM}$ be a non-empty set of admissible sets. We show that $S$ has a $\subseteq$-glb $S \in \text{ADM}$.

Let $LB := \{T \in \text{ADM} \mid (\forall S' \in S) T \subseteq S'\}$ be the set of admissible $\subseteq$-lower bounds of $S$. As $\emptyset \in LB$ by Corollary 5.11 we have $LB \neq \emptyset$. Let $S := \bigcup LB$. To show that $S \in \text{ADM}$, take any $S' \in S$, which is cf. As for all $T \in LB$, $T \subseteq S'$, it follows that $S \subseteq S'$. Therefore, $S \in \text{CF}$. By Lemma 5.19 $S \in \text{ADM}$. Clearly, $S$ is a lower bound of $S$. For any admissible $\subseteq$-lower bound of $S$, say $S''$, we have $S'' \in \text{ADM}$ and hence $S'' \subseteq S$, thus $S'' \subseteq S$. Therefore, $S \in \text{ADM}$ is the greatest $\subseteq$-lower bound of $S$. □

Corollary 5.28. The poset $\langle \text{ADM}, \subseteq \rangle$ is a complete semilattice.

Proof. By Corollary 5.26, $\langle \text{ADM}, \subseteq \rangle$ is directed complete and hence chain complete. By Corollary 5.27, every non-empty subset of $\langle \text{ADM}, \subseteq \rangle$ has a $\subseteq$-glb. The result follows. □

In summary, $\langle \text{ADM}, \subseteq \rangle$ is a complete semilattice that is also directed complete. This generalises [14, Theorem 25(3)] from complete extensions to admissible sets.

5.4 Dung’s Fundamental Lemma

The intuition behind Dung’s fundamental lemma is that whatever one can defend, one can also incorporate into one’s own knowledge in a consistent manner. This is formalised as follows:

Lemma 5.29. (Dung’s fundamental lemma [14, Lemma 10]) Let $S \in \text{ADM}$ and $a, b \in d(S)$, then

1. $S \cup \{a\} \in \text{ADM}$ and
2. $b \in d(S \cup \{a\})$.

Proof. In turn:

Note that $LB = \{\emptyset\} \neq \emptyset$ iff $S = \emptyset$. In such a case, $\emptyset \in \text{ADM}$ is the $\subseteq$-glb of $S$. 47
1. We have to show that $S \cup \{a\} \in CF \cap SD$. Note that $S \cup \{a\} \in SD$ is true because $a \in d(S) \Leftrightarrow \{a\} \subseteq d(S)$ and $d(S) \subseteq d(S \cup \{a\})$. As $S \in ADM$ and hence $S \subseteq SD$, so $S \subseteq d(S) \subseteq d(S \cup \{a\})$, it follows that $S \cup \{a\} \subseteq d(S \cup \{a\})$. Now we need to show $d(S \cup \{a\}) \in CF$. Assume for contradiction that $d(S \cup \{a\}) \notin CF$. There exists $x, y \in d(S \cup \{a\})$ such that $R(x, y)$. There are four cases:

(a) $x, y \in S$ – this is impossible because $S \in ADM$ means $S \in CF$.

(b) $x \in S$ and $y = a$ – it follows that $a \in S^+$ so $S \cap a^- \neq \emptyset$, but as $a \in d(S)$, $a^- \subseteq S^+$ and hence $S \cap S^+ \neq \emptyset$ – contradiction because $S \in ADM \subseteq CF$.

(c) $x = a$ and $y \in S$ – as $S \in ADM$, $a \in S^+$, which is impossible for the same reasons as the preceding case..

(d) $x = y = a$ – if $a$ is self-defeating, then as $a \in d(S)$, $a \in S^+$, which is impossible for the same reasons as the previous two cases.

Since all four cases lead to contradiction, it follows that $d(S \cup \{a\}) \in CF$. This means $d(S \cup \{a\}) \subseteq ADM$.

2. $b \in d(S) \subseteq d(S \cup \{a\})$ by Corollary 4.13 (page 31).

This shows the result.

We can generalise Dung’s fundamental lemma as follows.

**Lemma 5.30.** (Generalised fundamental lemma) If $S \in ADM$ and $W, V \subseteq d(S)$, then

1. $S' := S \cup W \in ADM$ and
2. $V \subseteq d(S')$.

**Proof.** In turn:

1. As $S \in ADM$, then $S \subseteq d(S)$. Similarly, $W \subseteq d(S)$ and hence $S' := S \cup W \subseteq d(S \cup W)$. Therefore, $S' \in SD$. Similarly, as $S \subseteq d(S)$, $S \cup d(S) = d(S) \in CF$ by Corollary 4.38 (page 31). As $W \subseteq d(S)$, we have that $S' = S \cup W \subseteq d(S)$ and hence $S' \in CF$ by Corollary 3.18 (page 24). Therefore, $S' \subseteq ADM$.

2. As $S \subseteq S \cup W = S'$, we have $d(S) \subseteq d(S')$ but as $V \subseteq d(S)$, we have $V \subseteq d(S')$.

This shows the result.

The fundamental lemma is thus recovered from the generalised fundamental lemma by choosing $W$ and $V$ to be singleton sets. Therefore, Lemma 5.29 and Lemma 5.30 are logically equivalent.
5.5 Symmetric Argumentation Frameworks

**Theorem 5.31.** An non-empty, non-trivial AF without self-attacking arguments is symmetric iff \( CF = ADM \).

**Proof.** \( (\Rightarrow) \) (From [10], Proposition 4) From Corollary [5.10] we have \( ADM \subseteq CF \) for all AFs. Therefore, it is sufficient to show \( CF \subseteq ADM \) for symmetric AFs.

Let \( S \in CF \). We need to show \( S \in SD \), which by Theorem 4.23 (page 34) is equivalent to \( S^- \subseteq S^+ \). Then \( S \in SD \) follows, because

\[
a \in S^- \iff (\exists b \in S) R(a, b) \Rightarrow (\exists b \in S) R(b, a) \iff a \in S^+,
\]

by symmetry of \( R \). Therefore, \( S \in SD \) and hence \( S \in ADM \). The result follows as \( S \) is arbitrary.

\( (\Leftarrow) \) (contrapositive) Assume our AF \( \langle A, R \rangle \) is not symmetric. As we have assumed our AF is neither empty nor trivial and has no self-attacking arguments, then \( R \) is a non-empty non-symmetric relation. There exists \( a, b \in A \) such that \( R(a, b) \) and \( \neg R(b, a) \). Clearly, \( \{b\} \in CF \) and \( \{b\} \notin ADM \), as it is not self-defending. Therefore, \( CF \neq ADM \). \( \square \)

5.6 Summary

- \( S \subseteq A \) is admissible iff it is both conflict-free and self-defending. We denote the set of all admissible sets of an AF \( A \) \( ADM(A) \) or just \( ADM \).
- For any AF, we have \( \emptyset, U \in ADM \), and \( ADM \subseteq CF \). Further, \( d : ADM \to ADM \) is well-defined.
- \( ADM \) is not closed under unions or intersections.
- For any family of sets \( S \subseteq ADM \), if \( \bigcup S \in CF \) then \( \bigcup S \in ADM \).
- If \( S \in ADM \) then the limit of the chain \( \{d^k(S)\}_{k \in \mathbb{N}} \) is also in \( ADM \).
- \( \langle ADM, \subseteq \rangle \) is a complete semilattice that is also directed complete, and \( \max \subseteq ADM \neq \emptyset \).
- Every non-empty family of admissible sets has a \( \subseteq \)-glb that is also admissible.
- If \( S \in ADM \) and \( W, V \subseteq d(S) \), then \( S \cup W \in ADM \) and \( V \subseteq d(S \cup W) \).
- An AF is symmetric iff \( CF = ADM \). This assumes the AF is non-empty, non-trivial and has no self-attacking arguments.
6 Complete Extensions

6.1 Definition

Definition 6.1. [14, Definition 23 and Lemma 24] $S \subseteq A$ is a complete extension iff $S \in CF \cap F_d$.

Intuitively, complete extensions are stronger than admissible sets because a complete extension demands that you believe everything that you can defend while still maintaining consistency.

Definition 6.2. Given an underlying AF, let $COMP$ denote the set of all complete extensions.

If the underlying AF $A$ needs to be explicitly specified, we can write $COMP(A)$.

Example 6.3. (Example 5.4 continued, page 42) Recall we have arguments $A = \{a, b, c, f, e\}$ and attacks $R = \{(a, b), (c, b), (c, f), (f, c), (f, e), (e, e)\}$. We have

$COMP = \{\{a\}, \{a, c\}, \{a, f\}\}$.

Example 6.4. (Example 2.7 continued, page 9) It can be shown that $COMP = \{\emptyset, \{a, e\}, \{b, e\}\}$.

Example 6.5. (Example 2.26 continued, page 12). We show that $COMP = \{\emptyset, \{a\}_{i \in \mathbb{Z}}, \{b\}_{i \in \mathbb{Z}}\}$. Clearly, $\emptyset \in COMP$ because there are no unattacked arguments. Also, $\{a\}_{i \in \mathbb{Z}}$ is a complete extension because it is in $ADM$ and every argument that is defended by $\{a\}_{i \in \mathbb{Z}}$ also belongs to $\{a\}_{i \in \mathbb{Z}}$. The same can be argued for $\{b\}_{i \in \mathbb{Z}}$.

6.2 Existence

6.2.1 Existence from Fixed Point Theory

For each AF, complete extensions exist. To prove that $COMP \neq \emptyset$, we first prove the following theorem.

Theorem 6.6. [14, Theorem 25(2)] Let $\langle A, R \rangle$ be an arbitrary AF with defence function $d : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$. The following two statements are equivalent.

1. $S$ is the $\subseteq$-least complete extension.

2. $S$ is the $\subseteq$-least element of $F_d$.

Proof. $(1 \Rightarrow 2,$ contrapositive) Assume that $S$ is not the $\subseteq$-least element of $F_d$. Either $S \notin F_d$, or $S \in F_d$ and $S$ is not the $\subseteq$-least element of $F_d$.

1. If $S \notin F_d$, then $d(S) \neq S$ and hence $S \notin COMP$. Therefore, $S$ cannot be the $\subseteq$-least complete extension.

2. If $S \in F_d$ and $S$ is not the $\subseteq$-least element of $F_d$, then $(\exists T \in F_d) T \subset S$ by definition of $\subseteq$-least. Either $T \in CF$ or $T \notin CF$.
(a) If $T \in CF$, then $T \in COMP$. As $T \subseteq S$, $S$ cannot be the $\subseteq$-least complete extension.

(b) If $T \notin CF$, then as $T \subseteq S$, $S \notin CF$ (Corollary 3.20, page 24) either so $S \notin COMP$. Therefore, $S$ cannot be the $\subseteq$-least complete extension.

In all cases, $S$ cannot be the $\subseteq$-least complete extension.

$2 \Rightarrow 1$, contrapositive) Assume $S$ is not the $\subseteq$-least complete extension, then either $S \notin COMP$, or $S \in COMP$ and $S$ is not $\subseteq$-least.

1. If $S \in COMP$ and $S$ is not $\subseteq$-least, then $(\exists T \in COMP) T \subseteq S$, but as $S, T \in COMP$, we have $S, T \in F_d$ and hence $S$ is not the $\subseteq$-least element of $F_d$.

2. If $S \notin COMP$, then either $S \notin CF$ or $S \notin F_d$.

(a) If $S \notin F_d$, then $S$ cannot be the $\subseteq$-least element of $F_d$.

(b) If $S \notin CF$, then assume for contradiction that $S$ is the $\subseteq$-least fixed point of $d$. Therefore, $S \in F_d$ and $(\forall T \in F_d) S \subseteq T$. It follows that $(\forall T \in F_d) T \notin CF$, because any superset of a non-cf set cannot be cf (Corollary 3.20, page 24). It follows that $F_d \cap CF = \emptyset$, which means $COMP = \emptyset$. However, we have assumed that the underlying AF is arbitrary. It cannot be true that $COMP = \emptyset$ for arbitrary AFs. For example, in Example 2.7 (page 9), we have an AF where $COMP \neq \emptyset$. Therefore, $S$ is not the $\subseteq$-least fixed point of $d$.

In all cases, $S$ is not the $\subseteq$-least fixed point of $d$.

The result follows. \hfill \square

**Corollary 6.7.** For any AF, $COMP \neq \emptyset$.

**Proof.** Given an AF, let $d$ be its defence function. The least fixed point of $d$ exists, call it $G$. By Theorem 6.6, $G$ is also the $\subseteq$-least complete extension. Therefore, $G \in COMP$, hence $COMP \neq \emptyset$. \hfill \square

**Definition 6.8.** For an AF $\langle A, R \rangle$, let $G \subseteq A$ denote the $\subseteq$-least complete extension.

**Corollary 6.9.** We have that $G = \bigcap COMP$.

**Proof.** As $G$ is the $\subseteq$-least complete extension, we have $(\forall C \in COMP) G \subseteq C$ and hence $G \subseteq \bigcap_{C \in COMP} C = \bigcap COMP$. Therefore, $G \subseteq \bigcap COMP$. Now let $a \in \bigcap COMP$, then for all $C \in COMP$, $a \in C$, in particular as $G \in COMP$, $a \in G$. Therefore, as $a$ is arbitrary, $\bigcap COMP \subseteq G$. The result follows. \hfill \square

**Corollary 6.10.** $U = \emptyset$ iff $G = \emptyset$.

**Proof.** ($\Rightarrow$) If $U = \emptyset$ then as $U = d(\emptyset)$, $\emptyset$ is the least fixed point of $d$ by definition hence $G = \emptyset$.

($\Leftarrow$) If $G = \emptyset$, then as $U \subseteq G = \emptyset$, we must have $U = \emptyset$. \hfill \square

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Corollary 6.11. Let $U$ be the set of undefeated arguments. For any $C \in \text{COMP}$, we have $U \subseteq C$.

Proof. Recall that for any $S \subseteq A$, $U \subseteq d(S)$. Let $C \in \text{COMP}$, then $d(C) = C$ and hence $U \subseteq C$.\hfill\Box

This result shows that every complete extension will have at least the unattacked arguments. The additional arguments are those that can be indirectly defended by the unattacked arguments.

Trivially, if there are unattacked arguments then the complete extensions cannot be empty.

Corollary 6.12. If $U \neq \emptyset$, then $\emptyset \notin \text{COMP}$.

Proof. If $U \neq \emptyset$, then as $U \subseteq C$ for all $C \in \text{COMP}$, $C \neq \emptyset$ for all $C \in \text{COMP}$. Hence, $\emptyset \notin \text{COMP}$.\hfill\Box

Example 6.13. [7, Exercise 4(a)] From Figure 4.1 (page 29), we show that $\text{COMP} = \{\{a, c\}\}$. This is because $U = \{a\}$ and $c$ is reinstated by $a$. There are no other complete extensions.

Example 6.14. [7, Exercise 4(b)] From Figure 5.2 (page 43) we can see that $\text{COMP} = \{\{a\}, \{a, c, f\}, \{a, e\}\}$. This is because $U = \{a\}$, and $d(\{a\}) = \{a\}$ because $c$ is not reinstated by $a$ due to the attack on $c$ from $e$. As $a$ is not self-attacking, we conclude that $\{a\} \in \text{COMP}$. Further, $\{a, e\}$ is a complete extension because it is conflict-free and contains all arguments it can defend by defending against the attack $c$ against $e$. Similarly, $\{a, c, f\}$ is a complete extension because it is conflict-free and it contains all arguments it defends by defending against the attacks $b$ against $c$ and $e$ against $f$.

Example 6.15. [7, Exercise 4(c)] From Figure 5.3 (page 44) we can see that $\text{COMP} = \{\{b, e\}\}$, because $b$ is unattacked and $e$ is reinstated, and there are no further arguments defended. Note $a$ is self-attacking and cannot be in any complete extension. This is the only complete extension.

Example 6.16. [7, Exercise 4(d)] From Figure 4.3 (page 31), we have that $\text{COMP} = \{\emptyset, \{a\}, \{b, e\}\}$. Firstly, $\emptyset \in \text{CF}$ and there are no unattacked arguments, so $d(\emptyset) = \emptyset$ and hence $\emptyset \in \text{COMP}$. The set $\{a\} \in \text{CF}$ and defends only itself as $c$ is not reinstated by $a$ due to the attack from $f$. Finally, $\{b, e\} \in \text{CF}$ and defends exactly itself by having $b$ attacking $a$ and $c$ and that $e$ attacks $f$.

6.2.2 Existence from Admissible Sets

Theorem 6.17. Let our AF be $(A, R)$ and let $S \in \text{ADM}$. Let $\beta$ be a sufficiently large ordinal number such that the ordinal-indexed sequence $\{d^\alpha(S)\}_{\alpha<\beta}$ has stabilised, for a given cardinal number $|A|$. We have that

$$\bigcup_{\alpha<\beta} d^\alpha(S) \in \text{COMP}. \quad (6.1)$$
Proof. By Theorem 5.21 (page 46), the limit \( L := \bigcup_{\alpha < \beta} d^\alpha (S) \in ADM \). Therefore we need to show this limit is in \( COMP \). It is sufficient to show \( d(L) \subseteq L \).

For \( a \in A \),
\[
a \in d(L) \iff a^- \subseteq L^+
\]
\[
\iff a^- \subseteq \left( \bigcup_{\alpha < \beta} d^\alpha (S) \right)^+ 
\]
\[
\iff a^- \subseteq \bigcup_{\alpha < \beta} [d^\alpha (S)]^+ \quad \text{by Corollary B.3 (page 87)},
\]
\[
\iff (\exists \alpha < \beta) a^- \subseteq [d^\alpha (S)]^+ , \quad \text{"\Rightarrow" follows as \( \{d^\alpha (S)\}_{\alpha < \beta} \) is a chain,}
\]
\[
\iff (\exists \alpha < \beta) a \subseteq d^{\alpha+1} (S) \quad \text{by definition of} \ d,
\]
\[
\iff (\exists \alpha < \beta) a \in d^\alpha (S) \quad \text{by definition of} \ \exists,
\]
\[
\iff a \in \bigcup_{\alpha < \beta} d^\alpha (S) = L,
\]
and therefore \( d(L) = L \), so \( L \in COMP \). Therefore, any transfinite iteration of an admissible set by \( d \) stabilises into a complete extension, for a suitably large ordinal.

\[ \square \]

Corollary 6.18. Let \( S \in ADM \), then there exists an \( L \in COMP \) such that \( S \subseteq L \).

Proof. Given \( S \), we iterate \( d \) on \( S \) a transfinite number of times until the sequence stabilises at some limit \( L \), which is guaranteed because \( d \) is \( \subseteq \)-monotonic and when the cardinality of the ordinal number denoting the iteration is at least \(|A|\). Clearly, \( S \subseteq L \), and by Theorem 6.17 \( L \in COMP \).

\[ \square \]

6.2.3 Existence of Non-Empty Complete Extensions from Limited Controversy

Definition 6.19. We say \( a \in A \) threatens \( S \subseteq A \) iff \( a \in S^- \setminus S^+ \).

In other words, the set of all threats to \( S \) are the set of arguments attacking \( S \) that \( S \) fails to defend against.

Corollary 6.20. \( S \in SD \) iff no argument threatens \( S \), i.e. \( S^- \setminus S^+ = \emptyset \).

Proof. We have \( S \in SD \) iff \( S^- \subseteq S^+ \) (Theorem 4.23 page 34) iff \( S^- \setminus S^+ = \emptyset \).

\[ \square \]

20The \( \Leftarrow \) direction is trivial because \( a^- \subseteq [d^\alpha (S)]^+ \subseteq \bigcup_{\alpha < \beta} [d^\alpha (S)]^+ \). The \( \Rightarrow \) direction is valid by the chain property because if the elements of \( a^- \) are spread out over different sets in the generalised union \( \bigcup_{\alpha < \beta} [d^\alpha (S)]^+ \), we can choose the largest of these sets which exist by the chain property, such that there is some sufficiently large \( \alpha \) strictly less than \( \beta \) (because \( d \) is \( \subseteq \)-monotone), which will contain the set \( a^- \).
Definition 6.21. We say $D \subseteq A$ is a defense of $S \subseteq A$ iff $S^- - S^\pm \subseteq D^\pm$.

Corollary 6.22. $S \in SD$ iff $\emptyset$ is a defense of $S$.

Proof. $\emptyset$ is a defense of $S$ iff $S^- - S^\pm \subseteq \emptyset^\pm$ iff $S^- - S^\pm \subseteq \emptyset$ iff $S^- \subseteq S^\pm$ iff $S \in SD$.

In other words, self-defending sets do not need anything else as a defense.

Corollary 6.23. If $D$ is a defense of $S$, then there exists an argument in $D$ that indirectly defends the arguments in $S$.

Proof. If $S^- - S^\pm \subseteq D^\pm$, then for $a \in S^- - S^\pm$, there is some $b \in S$ such that $R(a, b)$. However, as $a \in D^\pm$, then there is some $c \in D$ such that $R(c, a)$. Therefore, there is an even-length path from $c \in D$ to $b \in S$ and hence there is an argument in $D$ that indirectly defends arguments in $S$.

Corollary 6.24. $D$ is a defense of $S$ iff $S \subseteq d(S \cup D)$.

Proof. $(\Rightarrow)$ Let $a \in S$ and $b \in A$ be arbitrary such that $R(b, a)$, hence $b \in S^-$. Either $b \in S^\pm$ or $b \notin S^\pm$. In the former case, $b \in S^\pm$ would mean $a \in d(S) \subseteq d(S \cup D)$. In the latter case, $b \in S^- - S^\pm$ and hence $b \in D^\pm$, which means $b \in D^\pm \cup S^\pm = (S \cup D)^\pm$ by Corollary 13.3 (page 87), $a \in d(S \cup D)$. The result follows.

$(\Leftarrow)$ If $S \subseteq d(S \cup D)$, then $S^- \subseteq (S \cup D)^\pm = S^\pm \cup D^\pm$ by Corollary 13.3 (page 87). Now let $b \in S^- - S^\pm$ be arbitrary, then $b \in S^\pm$ and $b \notin S^\pm$. The first case implies that $b \in S^\pm \cup D^\pm$, but the second case means $b \in D^\pm - S^\pm \subseteq D^\pm$. Therefore, $S^- - S^\pm \subseteq D^\pm$, and hence $D$ is a defense of $S$.

Theorem 6.25. Lemma 34 \cite{14} If an AF is limited controversial then there exists a non-empty complete extension.

Proof. Recall that $G$ is the $\subseteq$-least complete extension (Definition 6.8, page 51). If $G \neq \emptyset$, then the result follows as $G \in COMP$.

If $G = \emptyset$, then $U = \emptyset$ by Corollary 6.10 (page 51), and the AF is not well-founded by the contrapositive of Corollary 2.44 (page 15). As the AF is limited controversial, there are no infinite sequences of arguments $\{a_i\}_{i \in \mathbb{N}}$ such that $a_{i+1}$ is controversial w.r.t. $a_i$. Therefore, all such sequences must terminate. Therefore, there is some argument $a \in A$ such that for all $b \in A$, $b$ is not controversial w.r.t. $a$.

Let $E_0 = \{a\}$. For $i \in \mathbb{N}$ let $E_{i+1} := E_i \cup D_i$, where $D_i \subseteq A$ is a $\subseteq$-minimal defense set of $E_i$ (Definition 6.21). We prove by strong induction on $i$ that $E_i \in CF$ and each argument in $E_i$ indirectly defends $a$.

1. (Base) As our AF is limited controversial, then there are no self-attacking arguments by Corollary 2.63 (page 18). Hence, $E_0 = \{a\} \in CF$. By Corollary 2.65 (page 17), $a$ indirectly defends itself.
2. (Inductive) Assume that $E_k \in CF$ and all arguments in $E_k$ defend $a$, for all $0 \leq k \leq i$. As $U = \emptyset$, every argument in $E_k$ is attacked by some other argument. We can construct a $\subseteq$-minimal defense set $D_k$ such that $E_k^- - E_k^+ \subseteq D_k^+$. By Corollary 6.23, there are arguments in $D_k$ that indirectly defend the arguments in $E_k$. As $D_k$ is $\subseteq$-minimal, we may assume that all arguments in $D_k$ indirectly defend the arguments in $E_k$.

In particular, there is an even-length path from all arguments in $D_k$ to $a$. By the inductive hypothesis, all arguments in $E_k \cup D_k =: E_{k+1}$ indirectly defends $a$.

To show that $E_{k+1} \in CF$, assume for contradiction that $E_{k+1} \notin CF$. There are some $b, c, \in E_{k+1}$ such that $R(b, c)$. But as each argument in $E_{k+1}$ indirectly defends $a$, then $b$ also indirectly attacks $a$ and hence $b$ is controversial w.r.t. $a$ − contradiction. Therefore, $E_{k+1} \in CF$.

As $\{E_i\}_{i \in \mathbb{N}}$ is a chain in $CF$, let $L := \bigcup_{i \in \mathbb{N}} E_i$ be its limit. As each $E_i \in CF$, by Theorem 6.23 (page 25), $L \in CF$. Let $b, c, \in L$ and assume $R(c, b)$ for some $c, \in A$. There is some $i \in \mathbb{N}^+$ such that $b \in E_i = E_{i-1} \cup D_{i-1} \subseteq d(E_i \cup D_i)$ by Corollary 6.24. Therefore, either $c$ is attacked by $E_i$, or $D_i$, which means $c$ is attacked by $L$. Therefore, $L$ is self-defending and hence $L \in ADM$. As $a, \in L$, $L \neq \emptyset$. By Theorem 6.17 (page 52), we iterate $d$ on $L$ a transfinite number of times to get our desired non-empty complete extension, $C$.

**Theorem 6.26.** [74] Lemma 35] Let $AF$ be uncontroroversial. Let $a \notin G \cup G^+$. There exists $C_1, C_2 \in COMP$ such that $a \in C_1 \cap C_2^+$.

**Proof.** Let $A' := A - (G \cup G^+) \subseteq A$. By assumption, $a \in A'$ so $A' \neq \emptyset$.

Consider the induced sub-framework $\mathcal{A}'$ on $A'$, which is also uncontroroversial (because $A$ does not have any controversial arguments). We repeat the proof of Theorem 6.25 on the uncontroroversial sub-framework $AF'$ to construct a non-empty complete extension $C$ of $AF'$ containing $a$. Consider $C_1 := G \cup C$. Clearly, $a \in C_1$. We show that $C_1$ is a complete extension of $AF$.

Now as $a \notin G$ and $U \subseteq G$, $a \notin U$ and hence $a$ is attacked by some argument $b \in A'$. As $AF'$ is uncontroroversial we repeat the proof of Theorem 6.25 on $b$ to construct a complete extension $C'$ of $AF'$ such that $b \in C'$ and $a \in C'^+$. Therefore, $C_2 := G \cup C'$ is our desired complete extension.

**Theorem 6.27.** COMP $\subseteq$ ADM. The converse is not true in general.

**Proof.** If $S \in COMP$ then $S$ is cf and $S = d(S)$, which implies that $S \subseteq d(S)$ and hence $S \in ADM$.

The converse is not true in general: from Example 6.4 (page 42), $\emptyset \in ADM$ but $\emptyset \notin COMP$.

**Corollary 6.28.** $d$ is closed on COMP.

**Proof.** Trivially, $S \in COMP = CF \cap F_d$ and hence $S \in F_d$ so $d(S) = S \in COMP$.  

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Therefore, \( d : \text{COMP} \to \text{COMP} \) is well-defined.

Now there are some situations where AFs can fail to infer anything.

**Corollary 6.29.** \( \text{ADM} = \{ \emptyset \} \) iff \( \text{COMP} = \{ \emptyset \} \).

**Proof.** (\( \Rightarrow \)) As \( \emptyset \neq \text{COMP} \subseteq \text{ADM} = \{ \emptyset \} \) by Corollary 6.7 and Theorem 6.27, the result follows.

(\( \Leftarrow \), contrapositive) If \( \text{ADM} \neq \{ \emptyset \} \), then as \( \text{ADM} \neq \emptyset \), there is some \( S \in \text{ADM} \) such that \( S \neq \emptyset \). By Corollary 6.18 (page 53), there is some \( L \in \text{COMP} \) such that \( S \subseteq L \). Clearly, \( L \neq \emptyset \) because \( S \neq \emptyset \). Therefore, \( \text{COMP} \neq \{ \emptyset \} \).

**Example 6.30.** Recall Example 2.19 (page 11). Clearly \( \text{CF} = \{ \emptyset, \{ b \} \} \) and \( \text{ADM} = \{ \emptyset \} \) because \( b \) cannot be self-defending. Therefore, \( \text{COMP} = \{ \emptyset \} \) as well. This is consistent with that \( U = \emptyset \) hence \( G = \emptyset \) (Corollary 6.10, page 52).

### 6.3 Lattice Theoretic Properties

We can generalise Theorem 6.17 (page 52) to arbitrary chains of complete extensions.

**Theorem 6.31.** \( \langle \text{COMP}, \subseteq \rangle \) is chain complete.

**Proof.** Let \( \langle \text{Ch}, \subseteq \rangle \subseteq \langle \text{COMP}, \subseteq \rangle \) be any chain. By Theorem 6.27, \( \langle \text{Ch}, \subseteq \rangle \) is also a chain (and hence a directed set) in \( \langle \text{ADM}, \subseteq \rangle \). By Corollary 5.23 (page 44), \( S := \bigcup \text{Ch} \in \text{ADM} \). To show \( S \in \text{COMP} \), we need to show \( d(S) \subseteq S \).

This is done as follows:

- \( a \in d(S) \)
- \( \iff a^- \subseteq S^+ \)
- \( \iff a^- \subseteq \left( \bigcup_{C \in \text{Ch}} C \right)^+ \) by Corollary B.3 (page 87),
- \( \iff a^- \subseteq \bigcup_{C \in \text{Ch}} C^+ \) as the “\( \Rightarrow \)” direction follows as \( \text{Ch} \) is a chain\(^{[21]} \)
- \( \iff (\exists C \in \text{Ch}) a^- \subseteq C^+ \)
- \( \iff (\exists C \in \text{Ch}) a \in d(C) \)
- \( \iff (\exists C \in \text{Ch}) a \in C \)
- \( \iff a \in \bigcup \text{Ch} = S, \)

and therefore, \( d(S) \subseteq S \), so \( S \in \text{COMP} \). Therefore, \( \langle \text{COMP}, \subseteq \rangle \) is chain complete. \( \square \)

\(^{[21]}\)See Footnote 20 (page 53).
It immediately follows that \( \langle \text{COMP}, \subseteq \rangle \) is \( \omega \)-complete. We can generalise further:

**Theorem 6.32.** \( \langle \text{COMP}, \subseteq \rangle \) is directed complete.

**Proof.** Let \( \langle D, \subseteq \rangle \subseteq \langle \text{COMP}, \subseteq \rangle \) be any directed subset. By Theorem 6.27 \( \langle D, \subseteq \rangle \) is also a directed subset of \( \langle \text{ADM}, \subseteq \rangle \). By Corollary 5.26 (page 47), \( \bigcup D \in \text{ADM} \). Therefore, it is sufficient to show \( d(\bigcup D) \subseteq \bigcup D \). We can apply the same reasoning in Theorem 6.31 and by invoking that \( D \) is a directed subset. Therefore, \( \bigcup D \) is a fixed point of \( d \) and hence \( \bigcup D \in \text{COMP} \). As \( D \) is an arbitrary directed subset, the result follows.

**Theorem 6.33.** Let \( \emptyset \neq C \subseteq \text{COMP} \), then there exists \( S \in \text{COMP} \) such that \( S \) is the \( \subseteq \)-glb of \( C \).

**Proof.** If \( \bigcap C \in \text{COMP} \), then the result follows by choosing \( C = \bigcap C \).

Otherwise, similar to the proof of Corollary 5.27 (page 47), let \( LB := \{ T \in \text{ADM} \mid \forall C' \in C \ T \subseteq C' \} \). By Corollary 5.11 (page 44), \( \emptyset \in LB \) so \( LB \neq \emptyset \). Let \( T \in LB \) and \( C' \in C \) be arbitrary. As \( T \subseteq C' \), then \( d(T) \subseteq d(C') = C' \), hence \( \forall C' \in C \) \( d(T) \subseteq C' \) as well. Therefore, \( d(T) \in LB \) and hence \( d : LB \to LB \) is well-defined. Now let \( S := \bigcup LB \), which for the same reasons as Corollary 5.27 it follows that \( S \in \text{ADM} \). Furthermore, as \( S = \bigcup_{T \in LB} T \) and \( \forall C' \in C \ T \subseteq C' \) and hence \( S \in LB \). Therefore, \( d(S) \in LB \). As \( S \in \text{ADM} \) we have \( S \subseteq d(S) \), and as \( S \) is \( \subseteq \)-maximal in \( LB \), \( d(S) \subseteq S \). This establishes that \( S \in \text{COMP} \).

Therefore, we have found a complete extension \( S \) that is a lower bound of \( C \) as \( S \in LB \), and is a greatest such lower bound because for any \( S' \in LB \), \( S' \subseteq S \) by definition. The result follows.

**Theorem 6.34.** [14, Theorem 25(3)] \( \langle \text{COMP}, \subseteq \rangle \) is a complete semilattice.

**Proof.** This follows from Theorems 6.31 and 6.33.

The main difference between the proofs of Theorem 6.33 and [14, Theorem 25(3)] is that the latter applies transfinite induction (Corollary 6.18) to locate the complete extension that is the \( \subseteq \)-glb, while Theorem 6.33 avoids transfinite induction by showing \( d : LB \to LB \) and applies maximality to show that \( \bigcup LB \) is indeed the \( \subseteq \)-glb complete extension.

### 6.4 Summary

- \( S \subseteq A \) is complete iff it is both conflict-free and a fixed point of \( d \). The set of all complete extensions is \( \text{COMP}(AF) \) or just \( \text{COMP} \).
- It can be shown that \( \text{COMP} \neq \emptyset \), and the \( \subseteq \)-smallest complete extension is the least fixed point of \( d \).

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\(^{22}\) Note that from Example 6.3 (page 50), if we choose \( C = \{ \{a, e\}, \{b, e\} \} \), we have \( \bigcap C = \{e\} \notin \text{COMP} \).
• If \( U = \emptyset \) then \( \emptyset \in COMP \). For all \( C \in COMP, U \subseteq C \).

• Clearly, \( COMP \subseteq ADM \), and \( d : COMP \rightarrow COMP \). For \( S \in ADM \) there exists a \( \subseteq \)-least \( C \in COMP \) such that \( S \subseteq C \). Further, \( ADM = \{ \emptyset \} \) iff \( COMP = \{ \emptyset \} \).

• If the underlying AF is limited controversial then there is a non-empty complete extension.

• Let \( G \in COMP \) be the \( \subseteq \)-minimal complete extension. If the underlying AF is uncontroversial then for \( a \notin G \cup G^+ \), there are complete extensions \( C_1 \) and \( C_2 \) such that \( a \in C_1 \cap C_2^+ \).

• \( \langle COMP, \subseteq \rangle \) is a complete semilattice that is also directed complete.
7 Preferred, Stable and Grounded Extensions

We now discuss the most important types of complete extensions used to draw inferences from AFs.

7.1 Preferred Extensions

7.1.1 Definition

Definition 7.1. [14, Definition 7] A **preferred extension** is a \( \subseteq \)-maximal admissible set.

Preferred extensions are thus the (order-theoretic) largest admissible sets. We will prove that they are also complete extensions in what follows.

Corollary 7.2. We have that \( \text{PREF} = \max \subseteq \text{ADM} \).

*Proof.* Immediate by Definition 7.1. \( \square \)

Definition 7.3. Given an AF, let \( \text{PREF} \subseteq \mathcal{P}(A) \) denote the set of preferred extensions.

If we need to make the underlying AF \( A \) explicit, we may write \( \text{PREF}(A) \).

Example 7.4. (Example 2.2 continued, page 8, from [14, Example 9]) In the Nixon diamond, we have \( \text{PREF} = \{\{a\}, \{b\}\} \).

Example 7.5. (Example 2.5 continued, page 9, from Example [14, Example 8]) In this case, we have \( \text{PREF} = \{\{a\}, \{c\}\} \).

Example 7.6. (Example 5.4 continued, page 42) Here, \( A = \{a, b, c, f, e\} \) and \( R = \{(a, b), (c, b), (c, f), (f, c), (f, e), (c, c)\}, \) so \( \text{PREF} = \{\{a, c\}, \{a, f\}\} \).

Do not confuse the meaning of “\( \subseteq \)-maximal” with “containing the most arguments”, as the following example shows:

Example 7.7. Consider the AF with arguments \( A = \{a, b, c\} \) and attacks \( R = \{(a, b), (b, a), (b, c), (c, b)\} \). This AF is depicted in Figure 7.1.

![Figure 7.1: The AF from Example 7.7](image_url)

In this case, \( \text{PREF} = \{\{a, c\}, \{b\}\} \).

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Corollary 7.8. \[14, \text{Theorem 11(2)}\] For any \( S \in \text{ADM} \) there exists an \( E \in \text{PREF} \) such that \( S \subseteq E \).

Proof. Given \( S \in \text{ADM} \), consider its upper-set \( S := \{ T \in \text{ADM} | S \subseteq T \} \). We show \( S \) has a \( \subseteq \)-maximal element, which would be our \( E \).

Let \( D \subseteq S \) be a directed subset. As \( D \) consists of sets of the form \( T \in \text{ADM} \) and \( S \subseteq T \), then \( S \subseteq \bigcup D \in \text{ADM} \) by Corollary 5.25 (page 46). But as \( S \subseteq \bigcup D \), we have that \( \bigcup D \in S \). As \( D \) is any directed subset of \( S \), and every directed subset is a chain, and \( D \) has an upper bound \( \bigcup D \), we have shown that every chain in \( S \) has an upper bound. Therefore, by Zorn’s lemma, \( S \) has a maximal element, which we will call \( E \). As \( S \) is a \( \subseteq \)-up-set, we have that \( E \in \text{PREF} \). The result follows.

We offer another proof of Corollary 7.8 that makes use of the fundamental lemma (Lemma 5.29, page 47).

Proof. (Of Corollary 7.8) Let \( S \in \text{ADM} \), so \( S \subseteq d(S) \). Consider the set \( d(S) - S \) and invoke the well-ordering theorem such that we can write its elements as an ordinal-indexed list. Starting from the element indexed by the least ordinal, say \( a_0 \), by the fundamental lemma, \( S \cup \{ a_0 \} \in \text{ADM} \). We can append such elements one by one to \( S \) and the fundamental lemma guarantees that the result is still in \( \text{ADM} \). For the limit case, we apply the fact that \( \text{ADM} \) is a dcpo (Corollary 5.26, page 47) such that the union is still in \( \text{ADM} \). More precisely, this is done via transfinite induction on the ordinal-valued indices of the elements in \( d(S) - S \).

1. (Base) \( S \in \text{ADM} \)
2. (Successor) \( S \cup \{ a_0, a_1, \ldots, a_\beta \} \in \text{ADM} \) and \( a_{\beta+1} \in d(S) - S \) implies that \( S \cup \{ a_0, a_1, \ldots, a_\beta, a_\beta+1 \} \in \text{ADM} \) by the fundamental lemma.
3. (Limit) If \( S \cup \{ a_\beta \}_\beta<\gamma \in \text{ADM} \) where \( \gamma \) is a limit ordinal, then the sequence of sets \( T_\beta := S \cup \{ a_\beta \}_\beta<\gamma \) is an ascending chain of admissible sets, such that \( \bigcup_{\beta<\gamma} T_\beta \in \text{ADM} \) by Corollary 5.26.

Therefore, transfinite induction shows that \( d(S) \in \text{ADM} \). We then apply transfinite induction a second time to show that for a suitably large ordinal \( \gamma \), \( E := d^+(S) \in \text{ADM} \) (recall also Theorem 5.21, page 46), and is also a fixed point of \( d \) (Theorem 6.11, page 52). Let \( a \in A - E \), then \( a \notin d(E) \) so \( a^- \not\subseteq E^+ \). Therefore, \( E \cup \{ a \} \notin \text{ADM} \). Therefore, \( E \in \text{PREF} \) and \( S \subseteq E \).

The following result applies Corollaries 5.11 (page 44) and 7.8 to give a different proof of Corollary 5.24 (page 46).

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This is because we are not making any assumptions that the underlying AF is finite. Further, the well-ordering theorem is equivalent to the axiom of choice.
Corollary 7.9. [14, Corollary 12] Any AF has a preferred extension, i.e. \( \text{PREF} \neq \emptyset \).

Proof. By Corollary 5.11 we can choose \( S = \emptyset \in \text{ADM} \) and invoke Corollary 7.8. Alternatively, this follows from Corollary 5.24 (page 46).

Note that in proving Corollary 7.8, we have used either Zorn’s lemma or the well-ordering theorem, and hence the axiom of choice (AC), to prove that every AF has a preferred extension. Clearly, AC is sufficient to demonstrate this. It is reasonable to ask whether it is necessary. In Appendix D (page 96), we show that AC is also necessary, hence the assertion that all AFs have preferred extensions is equivalent to AC.

Theorem 7.10. [14, Theorem 25(1)] \( \text{PREF} \subseteq \text{COMP} \), and the converse is not true.

Proof. Let \( S \in \text{PREF} \), then \( S \in \max \subseteq \text{ADM} \). We need to show \( d(S) \subseteq S \). Let \( a \in d(S) \), then \( S \cup \{a\} \in \text{ADM} \) by Dung’s fundamental lemma (Lemma 5.29, page 47). Trivially, \( S \subseteq S \cup \{a\} \), but as \( S \in \max \subseteq \text{ADM} \), \( S \cup \{a\} \subseteq S \) and hence \( a \in S \). Therefore, as \( a \) is arbitrary, \( d(S) \subseteq S \). Therefore, \( S \in \text{COMP} \).

The converse is not true. From Example 5.4 (page 42), \( \{a\} \in \text{COMP} \) but \( \{a\} \notin \text{PREF} \).

Theorem 7.11. \( \text{PREF} = \max \subseteq \text{COMP} \).

Proof. \((\Rightarrow)\) Let \( P \in \text{PREF} \). Let \( C \in \text{COMP} \) such that \( P \subseteq C \). Clearly, \( C \in \text{ADM} \) so \( C \subseteq P \). Therefore, \( C = P \) and hence \( P \in \max \subseteq \text{COMP} \).

\((\Leftarrow, \text{contrapositive})\) Let \( P \notin \text{PREF} \), then either \( P \notin \text{ADM} \), or \( P \in \text{ADM} \) and is not \( \subseteq \)-maximal.

1. If \( P \notin \text{ADM} \), then \( P \notin \text{COMP} \), which means \( P \notin \max \subseteq \text{COMP} \).

2. If \( P \in \text{ADM} \) and is not \( \subseteq \)-maximal, then by Corollary 7.8, there is some \( E \in \text{PREF} \) such that \( P \subseteq E \). By assumption, \( P \subset E \). Clearly, \( E \in \text{COMP} \). Assume that \( P \in \text{COMP} \) (else \( P \notin \max \subseteq \text{COMP} \) and the result follows), we have that \( P \subset E \) and hence \( P \notin \max \subseteq \text{COMP} \).

The result follows.

This means that every complete extension is contained in a preferred extension.

Example 7.12. [7] Exercise 7(a)] From Figure 4.1 (page 29), \( \text{COMP} = \{\{a,c\}\} \) and hence \( \text{PREF} = \{\{a,c\}\} \).

Example 7.13. [7] Exercise 7(b)] From Figure 5.2 (page 43), \( \text{COMP} = \{\{a\}, \{a,c,f\}, \{a,e\}\} \) and hence \( \text{PREF} = \{\{a,c,f\}, \{a,e\}\} \).

Example 7.14. [7] Exercise 7(c)] From Figure 5.3 (page 44), \( \text{COMP} = \{\{b\}\} \) and hence \( \text{PREF} = \{\{b\}\} \).
Example 7.15.  [Exercise 7(d)] From Figure 4.3 (page 31), COMP = {∅, {a}, {b, e}} and hence PREF = {{a}, {b, e}}.

Corollary 7.16. COMP = {∅} if and only if PREF = {∅}.

Proof. (⇒) If COMP = {∅}, then max COMP = {∅} = PREF.
(⇐, contrapositive) Let COMP ≠ {∅}. As COMP ≠ ∅, either ∅ ∈ COMP or ∅ ∉ COMP. If ∅ ∈ COMP and COMP ≠ {∅}, then there is some non-empty C ∈ COMP. Therefore, C ∈ max COMP = PREF and hence PREF ≠ {∅}. If ∅ ∉ COMP, then as PREF ⊆ COMP, ∅ ∉ PREF either so PREF ≠ {∅}. In either case, PREF ≠ {∅}.

Theorem 7.17. If the AF is isomorphic to Cn where n is even, then |PREF| = 2.

Proof. WLOG let A = {a1, . . . , an} where n is even. Partition this into two sets S1 := {ai ∈ A | 1 ≤ i ≤ n, i odd} and S2 := {ai ∈ A | 1 ≤ i ≤ n, i even}. Clearly both are cf sets as R(ai, ai±1) for 1 ≤ i ≤ n − 1 and R(ai, ai1) are the only attacks. Further, they are self-defending: for any ak−1 for i = 1, 2, there is some ak−2 ∈ S1 that attacks ak−1. These are clearly ⊆-maximal. Therefore, PREF = {S1, S2} and hence |PREF| = 2.

As Cn for n even are bipartite graphs, any AF isomorphic to Cn will have each partition forming a preferred extension.

Theorem 7.18. If the AF is isomorphic to Cn where n is odd, then PREF = {∅}.

Proof. WLOG let Cn = {a1, . . . , an} for n odd. Let S ⊆ A be a non-empty cf set, which can have at most ⌊|S|/2⌋ arguments. For the ⊆-largest such set, there is at least a pair of arguments (say a and b) in S whose path length is 3 (say from a to b), which implies that there is some argument (in this case b) in S whose attacker is not attacked by S (as a cannot reach it). Therefore, any such S cannot be self-defending, and any subset of this S cannot be self-defending either. However, as U = ∅, the only admissible extension is ∅ and hence PREF = {∅}.

So AFs whose underlying digraph is an odd cycle will have PREF = {∅}, and hence COMP = ADM = {∅} (Corollary 7.16 above and Corollary 6.29, page 50).

Theorem 7.19. [Theorem 2.3.1] If the underlying AF is finite and has no even cycle, then the preferred extension is unique.

Proof. (From Theorem 2.6], contrapositive) Suppose there are at least two preferred extensions, and pick two of them P and Q, where P ≠ Q. Consider the sets P − Q and Q − P, neither of which can be empty else P ⊆ Q or Q ⊆ P, which contradicts the fact that P and Q are distinct preferred extensions. As the AF is finite, WLOG let P − Q = {p1, . . . , pm} and Q − P = {q1, . . . , qn} for
some \( n, m \in \mathbb{N}^+ \). Let \( p \in P - Q \) and \( q \in Q - P \). Either \( R(p, q) \) or \( R(q, p) \), for if neither, then (say) \( p \) can be added to \( Q \) making \( Q \cup \{ p \} \supset Q \), contradicting that \( Q \in \text{PREF} \).

We now use this to construct our even cycle. WLOG suppose \( R(p, q) \), then as \( q \in Q - P \subseteq Q \subseteq \text{PREF} \subseteq \text{ADM} \), there is some \( r_1 \in Q \) such that \( R(r_1, p) \). If \( r_1 = q \), then \{ \( p, q \) \} forms an even cycle. If \( r_1 \neq q \), then as \( p \in P - Q \subseteq P \in \text{PREF} \subseteq \text{ADM} \), there exists an \( r_2 \in P \) such that \( R(r_2, r_1) \). If \( r_2 = p \) then \{ \( p, r_1 \) \} forms an even cycle. If \( r_2 \neq p \), then by the same argument as above by invoking an appropriate counter-attack which is guaranteed to exist as preferred extensions are self-defending, we will yield an even cycle as we alternate between the preferred extensions \( P \) and \( Q \). This process must terminate as we have assumed that the AF is finite. Therefore, the AF has an even cycle. \( \square \)

The following result strengthens Theorem 7.19.

**Theorem 7.20.** [1, Corollary 2.3.1] Let \( \langle A, R \rangle \) be a finite AF such that \( U = \emptyset \) and has no even cycle, then \( \text{PREF} = \{ \emptyset \} \).

**Proof.** (Contrapositive) Assume that our AF is finite and \( U = \emptyset \). Assume that \( \text{PREF} \neq \{ \emptyset \} \), we prove that there must exist an even cycle.

As \( \text{PREF} \neq \emptyset \), we have some \( S \in \text{PREF} \) such that \( S \neq \emptyset \). Clearly, \( S \) is a finite set. Choose \( a_0 \in S \). As \( U = \emptyset \), there is some \( b_0 \in A - S \) such that \( R(b_0, a_0) \). As \( U = \emptyset \), and \( S \in \text{PREF} \), there is some \( a_1 \in S \) such that \( R(a_1, b_0) \). If \( a_1 = a_0 \), then we have an even cycle \( a_0 \) attacks \( b_0 \) attacks \( a_0 \) and the result follows.

Otherwise, \( a_1 \neq a_0 \), and as before, we have some \( b_1 \in A - S \) attacking \( a_1 \), and some \( a_2 \in S \) attacking \( b_1 \). If \( a_2 = a_0 \), then we have an even cycle \( a_0 \) attacks \( b_1 \) attacks \( a_1 \) attacks \( b_0 \) attacks \( a_0 \), and the result follows. Similarly, if \( a_2 = a_1 \), we have an even cycle again.

We cannot extend this path \( \langle a_0, b_0, a_1, b_1, a_2, b_2, \ldots \rangle \) such that \( a_i \notin \{ a_j \}_{j<i} \) indefinitely, because \( A \) is finite. Therefore, there exists some \( a_k \) that repeats twice in this path, and this guarantees the existence of an even cycle. \( \square \)

**Definition 7.21.** Let \( S \) be a set. We say a family \( \mathcal{F} \) of sets covers \( S \) iff \( S \subseteq \bigcup \mathcal{F} \).

**Corollary 7.22.** [12, Lemma 1] For any AF \( \langle A, R \rangle \), if \( \text{PREF} \) covers \( A \), then \( \bigcap \text{PREF} = U \). If the hypothesis is not true, then the consequent may or not be true.

**Proof.** (\( \Leftarrow \)) By Theorem 7.19 (page 61), \( \text{PREF} \subseteq \text{COMP} \). By Corollary 6.11 (page 52), for all \( C \subseteq \text{COMP} \), \( U \subseteq C \). Therefore, it follows that for all \( E \in \text{PREF} \), \( U \subseteq E \). Therefore, \( U \subseteq \bigcap_{E \in \text{PREF}} E = \bigcap \text{PREF} \).

(\( \Rightarrow \)) Let \( a \in \bigcap \text{PREF} \). Assume for contradiction that \( a \notin U \). Therefore, there is some \( b \in a^- \). But as this \( b \in A \), there is some \( E \in \text{PREF} \) such that \( b \in E \) by our hypothesis. As \( a \in \bigcap \text{PREF} \), clearly \( a \in E \) as well. Therefore, we have found \( a, b \in E \) in \( \text{PREF} \) such that \( b \in a^- \), which means \( E \notin \text{CF} \) – contradiction, as \( \text{PREF} \subseteq \text{CF} \). Therefore, \( a \in U \).
Now suppose that the preferred extensions do not cover $A$. We give two examples where $\bigcap PREF = U$ and $\bigcap PREF \neq U$.

1. In Example 5.4 (page 42), $PREF = \{\{a, c\}, \{a, f\}\}$, which clearly does not cover $A = \{a, b, c, f, e\}$, but $\bigcap PREF = \{a\} = U$.

2. In Example 2.3 (page 8), $PREF = \{\{a, c\}\}$ and $A = \{a, b, c\}$. Therefore, $PREF$ does not cover $A$, and $\bigcap PREF = \{a, c\} \neq U = \{c\}$.

This shows the result.

**Corollary 7.23.** [10, Proposition 6] For all symmetric AFs, if there are no self-attacking arguments, then $PREF$ covers $A$.

**Proof.** Let $(A, R)$ be such an AF. Let $a \in A$, then $\{a\} \in CF$ because there are no self-attacking arguments. By Theorem 5.31 (page 49), $\{a\} \in ADM$. By Theorem 7.8 (page 59), there is a preferred extension that contains $\{a\}$. As $a$ is arbitrary, then every argument in $A$ belongs to some preferred extension $P \subseteq A$, hence $PREF$ covers $A$.

The lattice-theoretic properties are trivial for $PREF$.

**Corollary 7.24.** $(PREF, \subseteq)$ is an antichain.

**Proof.** Immediate, as $PREF = \max_{\subseteq} ADM$, by Corollary 7.2 (page 59).

### 7.1.3 Relationship with Naive Extensions

Recall that naive extensions are $\subseteq$-maximal $CF$ sets. Of course, $PREF \subseteq CF$ and $PREF = \max_{\subseteq} COMP$. How are preferred and naive extensions related? It turns out that neither implies the other.

**Lemma 7.25.** Preferred extensions are not always naive extensions.

**Proof.** Consider the AF whose underlying digraph is $C_3$, i.e. $A = \{a, b, c\}$ and $R = \{(a, b), (b, c), (c, a)\}$. Clearly, $CF = \{\emptyset, \{a\}, \{b\}, \{c\}\}$ and hence $NAI = \{\{a\}, \{b\}, \{c\}\}$. However, by Theorem 7.18, $PREF = \emptyset$. Therefore, preferred extensions are not always naive extensions.

**Lemma 7.26.** Naive extensions are not always preferred extensions.

**Proof.** Consider the AF $A = \{a, b, c, d\}$ and $R = \{(a, b), (a, c), (b, d), (c, d)\}$. This AF is depicted in Figure 7.2.
Clearly, $CF = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,d\}, \{b,c\}\}$. By inspection, we have that $NAI = \{\{a,d\}, \{b,c\}\}$. However, $PREF = \{\{a,d\}\}$. Therefore, we have a naive extension $\{b,c\}$ that is not preferred.

**Corollary 7.27.** If an AF is symmetric, then $PREF = NAI$.

**Proof.** If an AF is symmetric, then by Theorem 5.31 (page 49), $CF = ADM$, hence $NAI = \max \subseteq CF = \max \subseteq ADM = PREF$.

### 7.2 Stable Extensions

#### 7.2.1 Definition

**Theorem 7.28.** [14, Lemma 14] For $S \subseteq A$, $S = n(S)$ iff $S$ is cf and $A - S \subseteq S^+$.

**Proof.** $(\Rightarrow)$ If $S = n(S)$, then $S = A - S^+$ by Definition 3.11 which means $S^+ = A - S$, hence $S$ attacks all arguments outside of it. Further, $S = n(S)$ implies $S \subseteq n(S)$, hence $S$ is cf by Definition 3.11.

$(\Leftarrow)$ Let $S$ be a cf set that attacks all arguments outside of it. The latter means $S^+ = A - S$ and hence $S = A - S^+ = n(S)$, hence $S = n(S)$ so $S$ must be a stable extension.

**Definition 7.29.** [14, Definition 13] We say $S \subseteq A$ is a **stable extension** iff $S$ satisfies any one of the two equivalent properties in Theorem 7.28.

#### 7.2.2 Existence

**Definition 7.30.** Given an underlying AF, let $STAB \subseteq \mathcal{P}(A)$ denote the set of stable extensions.

If the underlying AF $A = \langle A, R \rangle$ needs to be explicitly specified, we can write $STAB(A)$ or $STAB(\langle A, R \rangle)$.

**Example 7.31.** (Example 5.4 continued, page 42) For the AF $A = \{a, b, c, f, e\}$ and $R = \{(a, b), (c, b), (c, f), (f, c), (f, e), (e, e)\}$, we have $STAB = \{\{a, f\}\}$.

**Corollary 7.32.** [14, Theorem 2.2.3] $A = \emptyset$ iff $\emptyset \in STAB$.
Proof. ∅ is stable iff \( n(∅) = ∅ = A \) by Corollary 3.3 (page 22).

**Corollary 7.33.** [4, Example 2.2.5] It is possible for \( STAB = ∅ \).

Proof. Let \( (A, R) \) be the 1-cycle, i.e. \( A = \{a\} \) and \( R = \{(a, a)\} \). We have \( P(A) = \{∅, \{a\}\} \). Further, \( n(\{a\}) = ∅ \) and \( n(∅) = \{a\} \). Therefore, \( n \) has no fixed points, hence by Theorem 7.28, \( STAB = ∅ \).

**Theorem 7.34.** [14, Lemma 15] \( STAB \subseteq PREF \), and the converse is not true in general.

Proof. This is trivial if \( STAB = ∅ \), so assume that \( STAB \neq ∅ \). If \( S \in STAB \), then \( S^+ = A - S \) and hence any strict superset of \( S \) cannot be in \( CF \). Therefore, as \( S \) is cf and self-defending (the latter follows from Corollary 4.45 (page 40), \( S \in PREF \).

The converse is not true, e.g. in Example 5.4 (page 42), \( \{a, c\} \in PREF \) but \( \{a, c\} \notin STAB \).

It also follows by Corollary 7.24 that \( STAB \) is also a \( \subseteq \)-antichain thus its lattice-theoretic properties are trivial.

**Corollary 7.35.** For a non-empty AF, if \( PREF = \{∅\} \), then \( STAB = ∅ \).

Proof. By Theorem 7.34, \( STAB \subseteq PREF = \{∅\} \). As \( A \neq ∅ \), by Corollary 7.32, \( ∅ \notin STAB \), hence \( STAB = ∅ \).

**Corollary 7.36.** For all \( S \in STAB, U \subseteq S \), and it is not necessarily the case that \( U = S \).

Proof. From Corollary 6.11 (page 52) and Theorems 7.10 and 7.34, we have that \( STAB \subseteq PREF \subseteq COMP \), so for \( S \in STAB, S \in COMP \) and hence \( U \subseteq S \).

Alternatively, for any \( a \in U, a^- = ∅ \) by definition so for any \( S \subseteq A, a \in n(S) \). If \( S \in STAB, a \in n(S) = S \) and the result follows.

For the converse, Example 7.33 (page 5), we have \( \{a, c\} \in STAB \) but \( U = \{c\} \subseteq \{a, c\} \).

**Example 7.37.** [7, Exercise 8(a)] From Figure 4.4 (page 29), we have that \( PREF = \{\{a, c\}\} \) and because \( a \) attacks \( b \) and \( c \) attacks \( d \), we have \( STAB = \{\{a, c\}\} \).

**Example 7.38.** [7, Exercise 8(b)] From Figure 5.2 (page 43), we have that \( PREF = \{\{a, c, f\}, \{a, e\}\} \), and \( STAB = PREF \).

**Example 7.39.** [7, Exercise 8(c)] From Figure 5.3 (page 44), we have that \( PREF = \{\{b, e\}\} \), but \( STAB = ∅ \) because the self-attacking argument \( a \) is not attacked by anything else.

**Example 7.40.** [7, Exercise 8(d)] From Figure 4.3 (page 31), we have that \( PREF = \{\{a\}, \{b, e\}\} \), but \( \{a\} \) does not attack all arguments outside of it while \( \{b, e\} \) does, so \( STAB = \{\{b, e\}\} \).
7.2.3 Relationship with Naive and Preferred Extensions

Corollary 7.41. For all AFs, \( STAB \subseteq NAI \).

Proof. The result is trivial for \( STAB = \emptyset \), so suppose we have an AF \( \langle A, R \rangle \) with \( S \in STAB \), then \( S \in CF \). Let \( a \in A - S \), then \( a \in S^+ \) hence \( S \cup \{a\} \notin CF \). Therefore, \( S \) is a \( \subseteq \)-maximal \( CF \) set, i.e. \( S \in NAI \).

It is natural to ask when the converse of Theorem 7.34 is true. One sufficient condition is:

Lemma 7.42. Assume an AF with no self-attacking arguments, then if \( PREF \subseteq NAI \) or \( NAI \subseteq PREF \), then \( PREF = STAB \). The assumption that there are no self-attacking arguments is necessary.

Proof. Let \( \langle A, R \rangle \) be any AF without self-attacking arguments. It is sufficient to show that \( PREF \subseteq STAB \) by Theorem 7.34.

Assume \( PREF \subseteq NAI \), then let \( S \in PREF \) be naive. Therefore, for any \( a \notin S \), \( S \cup \{a\} \notin CF \). We have three possibilities:

1. \( a \) is self-attacking, which we have excluded.
2. \( a \in S^+ \).
3. \( a \in S^- \), but as \( S \in PREF \subseteq ADM \subseteq SD \), \( a \in S^- \subseteq S^+ \) by Theorem 4.23 (page 34), therefore, \( a \in S^+ \).

Therefore, in all cases, \( a \in S^+ \). This means for all \( a \notin S \), \( a \in S^+ \), hence \( A - S \subseteq S^+ \), so \( S = A - S^+ = n(S) \), therefore \( S \in STAB \). As \( S \in PREF \) is arbitrary, we have shown \( STAB = PREF \).

Assume \( NAI \subseteq PREF \), then let \( S \in NAI \) be preferred, i.e. \( S \in SD \). For \( a \notin S \), \( S \cup \{a\} \notin CF \) because \( S \in NAI \). By analogous reasoning to the above case, we have that \( a \in S^+ \) and hence \( S \in STAB \). Therefore, \( STAB = PREF \). \( \square \)

7.3 The Grounded Extension

We have already encountered the \( \subseteq \)-least complete extension \( G \) (Section 6.2.1, page 50). We now study its properties further.

7.3.1 Definition

Definition 7.43. [14, Definition 20] Given an AF, its grounded extension, \( G \), is the \( \subseteq \)-least fixed point of \( d \).

This just names Definition 6.8 (page 51). We now give some examples:

Example 7.44. (Example 2.2, page 8 continued) In the Nixon diamond, \( G = \emptyset \). This is because \( COMP = \{\emptyset, \{a\}, \{b\}\} \) and hence \( \emptyset \cap COMP = \emptyset \).
Example 7.45. (Example 2.3, page 8 continued) In simple reinstatement, \( G = \{a, c\} \). This is because \( \text{COMP} = \{\{a, c\}\} \).

Example 7.46. [7, Exercise 5(a)] For Figure 4.1 (page 29), \( G = \{a, c\} \). This is because \( \text{COMP} = \{\{a, c\}\} \).

Example 7.47. [7, Exercise 5(b)] For Figure 5.2 (page 43), \( G = \{a\} \). This is because \( \text{COMP} = \{\{a\}, \{a, c, f\}, \{a, e\}\} \) hence \( \bigcap \text{COMP} = \{a\} \).

Example 7.48. [7, Exercise 5(c)] For Figure 5.3 (page 44), \( G = \{b, e\} \). This is because \( \text{COMP} = \{\{b, e\}\} \).

Example 7.49. [7, Exercise 5(d)] For Figure 4.3 (page 31), \( G = \emptyset \), because \( U = \emptyset \). This is because \( \text{COMP} = \{\emptyset, \{a\}, \{b, e\}\} \), hence \( \bigcap \text{COMP} = \emptyset \).

7.3.2 Existence and Uniqueness

Theorem 7.50. Given an AF, its grounded extension exists and is unique.

Proof. This is immediate from the Knaster-Tarski theorem (Theorem C.32, page 92), as \( d \) is a \( \subseteq \)-monotonic function on the complete lattice \( \langle \mathcal{P}(A), \subseteq \rangle \).

Intuitively, the grounded extension captures skeptical reasoning. The agent believes in only those arguments that are either in \( U \) or ultimately reinstated by, i.e. grounded in, \( U \).

Theorem 7.51. Given an AF, its grounded extension is the \( \subseteq \)-smallest complete extension.

Proof. This follows from Theorem 6.6 (page 50).

Corollary 7.52. \( U \subseteq G \).

Proof. By Theorem 6.6 (page 50), \( G \in \text{COMP} \). By Corollary 6.11 (page 52), \( U \subseteq G \).

Recall also from Corollary 6.10 that \( U = \emptyset \) iff \( G = \emptyset \).

Example 7.53. [14, Example 22] In Example 2.2 (page 8), as \( U = \emptyset \), we have \( G = \emptyset \).

Corollary 7.54. If \( U \neq \emptyset \) and \( a \in A \) is indirectly defended by \( U \) (Definition 2.51, page 17), then \( a \in G \).

Proof. If \( U \neq \emptyset \) and \( a \in A \) is indirectly defended by \( U \), then there exists \( b \in U \) such that there is an even-length path from \( b \) to \( a \). Suppose the length is \( 2n \) for some \( n \in \mathbb{N} \). We prove by induction on \( n \).

1. (Base) If \( n = 0 \), then \( b = a \), i.e. the path length is 0, then \( a \in U \subseteq G \) by Corollary 7.52
2. (Inductive) Suppose the path length is $2n$ and that $a \in G$. Let $a'$ be of length 2 away from $a$, therefore for all $b \in a^-, b \in a^+$ so $b \in G^+$. Therefore, $a' \in d(G) = G$.

The result follows by mathematical induction.

The following result is important as it provides an efficient way of calculating the grounded extension for finite argumentation frameworks.

**Theorem 7.55.** If $d$ is $\omega$-continuous, then $\bigcup_{i \in \mathbb{N}} d^i(\emptyset)$ is the grounded extension.

**Proof.** We instantiate Kleene’s fixed point theorem (Theorem C.46, page 95) where $\langle D, \leq, \bot \rangle = \langle P(A), \subseteq, \emptyset \rangle$ and $f = d$. Then $F := \bigcup_{i \in \mathbb{N}} d^i(\emptyset)$ is the least fixed point of $d$. By Definition 7.43, $F = G$. □

**Example 7.56.** (Example 2.5 continued, page 8, from [14, Example 21]) We have that $d(\emptyset) = \{c\}$, $d^2(\emptyset) = \{a, c\} = d^k(\emptyset)$ for $k > 2$. Therefore, $G = \{a, c\}$.

**Corollary 7.57.** If $S \subseteq G$ then $S$ is not necessarily admissible.

**Proof.** From Example 2.3 (page 8), as $G = \{c, a\}$, let $S = \{a\}$, which is clearly not admissible because it does not defend itself against the attack from $b$. □

### 7.4 Summary

On preferred extensions:

- $S \subseteq A$ is preferred iff it is a $\subseteq$-maximal admissible extension. The set of all preferred extensions of a given AF $A$ is $\text{PREF}(A)$ or just $\text{PREF}$.
  
- For any AF, $\text{PREF} \neq \emptyset$, and for any $S \in \text{ADM}$ there is an $E \in \text{PREF}$ such that $S \subseteq E$.
  
- We have $\text{PREF} = \text{max}_{\subseteq} \text{ADM} = \text{max}_{\subseteq} \text{COMP}$, and that $\text{COMP} = \{\emptyset\}$ iff $\text{ADM} = \{\emptyset\}$.
  
- If $(A, R)$ is an even cycle, then $\text{PREF}$ has two sets. If $(A, R)$ is an odd cycle, then $\text{PREF} = \{\emptyset\}$.
  
- If the underlying AF has no even cycle, then $\text{PREF}$ is unique.
  
- If the underlying AF has no even cycle and $U = \emptyset$, then $\text{PREF} = \{\emptyset\}$.
  
- $\text{PREF}$ is a $\subseteq$-antichain.
  
- If $\text{PREF}$ covers $A$, then $\bigcap \text{PREF} = U$.
  
- In general, $\text{PREF} \not\subseteq \text{NAI}$ and $\text{NAI} \not\subseteq \text{PREF}$, but if the AF is symmetric, then $\text{PREF} = \text{NAI}$.
On stable extensions:

- $S \subseteq A$ is stable iff it is a fixed point of $n$. The set of all stable extensions is $STAB(AF)$ or just $STAB$.
- $STAB \subseteq PREF$, $STAB \subseteq NAI$, and if $PREF = \{\varnothing\}$, then $STAB = \varnothing$.
- $A = \varnothing$ iff $STAB = \{\varnothing\}$.
- If either $PREF \subseteq NAI$ or $NAI \subseteq PREF$, then $STAB = PREF$.

On the grounded extension

- The $\subseteq$-least complete extension is the grounded extension, $G$, which exists and is unique for all AF.
- $U \subseteq G$, and $U = \varnothing$ iff $G = \varnothing$.
- If $d$ is $\omega$-continuous, then the limit of the $\subseteq$-ascending chain $\{d^k(\varnothing)\}_{k \in \mathbb{N}}$ is $G$. 
8 Which Arguments are Justified?

8.1 Credulous and Sceptical Justification

The Dung semantics are the four main semantics that we have discussed, and all are variations of admissible sets.

**Definition 8.1.** For an AF, its Dung semantics are: complete, preferred, stable and grounded.

Given that there are multiple notions of what it means for a set of arguments to be justified, we now define what it means for an individual argument to be justified. Let AF be an argumentation framework.

**Definition 8.2.** We say \( a \in A \) is **skeptically justified w.r.t. preferred / stable semantics** iff

\[
a \in \bigcap \text{PREF} \quad \text{and} \quad a \in \bigcap \text{STAB}
\]

respectively. If \( \text{STAB} = \emptyset \) then no argument can be skeptically justified w.r.t. stable semantics.

**Example 8.3.** (Example 2.7, page 9 continued), we have \( \bigcap \text{PREF} = \{e\} \) hence the argument \( e \) is skeptically justified w.r.t. preferred semantics.

**Definition 8.4.** We say \( a \in A \) is **credulously justified w.r.t. complete / preferred / stable semantics** iff

\[
a \in \bigcup \text{COMP}, \quad a \in \bigcup \text{PREF} \quad \text{and} \quad a \in \bigcup \text{STAB}
\]

respectively. If \( \text{STAB} = \emptyset \) then no argument can be credulously justified w.r.t. stable semantics.

**Example 8.5.** (Example 2.7, page 9 continued), we have \( \bigcup \text{PREF} = \{a, b, e\} \) hence the argument \( a \) is credulously justified w.r.t. preferred semantics.

**Definition 8.6.** We say \( a \in A \) is **justified w.r.t. the grounded semantics** iff \( a \in G \).

Notice that as \( G \) is unique, skeptical inference and credulous inference coincide. Further, as \( G = \bigcap \text{COMP} \) (Corollary 6.9, page 51), we do not consider skeptical justification w.r.t. complete semantics.

**Example 8.7.** (Example 2.7, page 9 continued) As \( G = \emptyset \), no argument is justified w.r.t. the grounded semantics.

**Definition 8.8.** [1 Definition 2.13] An argument is **overruled w.r.t. a given semantics** iff it is not credulously justified w.r.t. to that semantics.

To summarise, the Dung semantics consists of the complete, preferred, stable and grounded extensions. An argument is skeptically justified w.r.t. a given semantics iff it is in all of the extensions of that given type. Notice this means that skeptical complete is the same as grounded. An argument is credulously justified w.r.t. a given semantics iff it is in at least one of the extensions of that given type.
8.2 Coincidence of Semantics

8.2.1 Equality of All Dung Semantics

So far we have established $\text{STAB} \subset \text{PREF} \subset \text{COMP} \subset \text{ADM} \subset \text{CF}$, because the reverse inclusions do not hold in general.\footnote{This result has been shown in Theorem 7.34 (page 66), Theorem 7.10 (page 61), Theorem 6.24 (page 55) and Corollary 6.14 (page 44).} Further, we have established that $G \in \text{COMP}$. We can illustrate this in the following Hasse diagram:

![Hasse diagram](image)

Figure 8.1: A Hasse diagram where the arrows represent the containment relations between each type of sets of arguments.

Under which circumstances for the AF can we have equality between the semantics? The strongest form of this equality is when all Dung semantics collapse and there is only one set of winning arguments. Figure 8.1 suggests that this will happen if $G \in \text{STAB}$. We prove the following result.

**Lemma 8.9.** If $G \in \text{STAB}$, then $\text{PREF} = \text{COMP} = \{G\} = \text{STAB}$. Therefore, all semantics coincide. The converse is false.

**Proof.** Let $C \in \text{COMP}$ be arbitrary, then $G \subseteq C$. But as $G \in \text{STAB} \subseteq \text{PREF} = \max_{\subseteq} \text{COMP}$, then $C \subseteq G$ and hence $C = G$. As $G \in \text{COMP}$ and
$C \in COMP$ is arbitrary, we have shown that any complete extension is equal to $G$ and hence $COMP = \{ G \}$. From $PREF = \max_{\subseteq} COMP$, we conclude $PREF = \{ G \}$. As $G \in STAB$, then $\emptyset \neq STAB \subseteq PREF = \{ G \}$ and hence $STAB = \{ G \}$ as well. Therefore, all semantics coincide because $G$ is grounded, complete, preferred and stable.

The converse is false. Suppose we have a non-empty AF where $COMP = \{ \emptyset \}$ hence $G = \emptyset$ and $STAB = \emptyset$. Therefore, $G \notin STAB$. □

**Example 8.10.** (Example 2.5 continued, page 9) As $G = \{ a, c \}$, we have $G \in STAB$ because the only argument outside is $b$, which is attacked by $c$. Therefore, $STAB = PREF = COMP = \{ \{ a, c \} \}$.

**Corollary 8.11.** If the AF is empty then all four semantics coincide.

*Proof.* If $A = \emptyset$ then $G = \emptyset$ because $U = \emptyset$, and further, $n(\emptyset) = \emptyset - \emptyset^+ = \emptyset$, hence the grounded extension $\emptyset$ is stable. By Lemma 5.1, all four semantics coincide. □

One further sufficient condition to achieve equality of all semantics is the following.

**Theorem 8.12.** [14, Theorem 30] If the AF is non-empty and well-founded (Definition 2.40, page 14), then all four semantics coincide. The converse is not true in general.

*Proof.* (Contrapositive) If all four semantics do not coincide, then the grounded extension $G$ is not stable by Lemma 5.9. Therefore, $(\exists a \notin G) a \notin G^+$. Define the set $S := \{ x \in A - G | x \notin G^+ \}$. As $a \in S$, then $S \neq \emptyset$. As $a \notin G = d(G)$, then $a \notin d(G)$, so $a^- \not\subseteq G^+$. There is some $b \in a^-$ such that $b \notin G^+$. Note as $G \in CF$, $b \notin G$ hence $b \in A - G$ and $b \notin G^+$. Therefore, we have found some $b \in S$ such that $R(b, a)$, so $a \in S^+$. As $a$ is arbitrary, $S \subseteq S^+$ and $S \neq \emptyset$. By Corollary 2.40 (page 16), the underlying $\langle A, R \rangle$ is not well-founded.

If $G = \emptyset$, which does not conflict with the fact that $G$ is not stable as $A \neq \emptyset$, then $S = A \neq \emptyset$ and by the same argument we show that $A \subseteq A^+$, hence $\langle A, R \rangle$ is also not well-founded.

For the converse, Example 2.5 (page 9) satisfies $STAB = PREF = COMP = \{ G \}$, but has a 2-cycle $(a, b), (b, a) \in R$, so this AF is not well-founded. □

**Corollary 8.13.** For finite acyclic AFs, there is only one extension of all four types.

*Proof.* This follows from Corollary 2.47 (page 16) and Theorem 8.12. □

It is easy to see that if all Dung semantics coincide, then $ADM$ has $G$ to be its $\subseteq$-greatest element, and thus becomes a bounded poset $ADM \subseteq \mathcal{P}(G)$.
8.2.2 Coherent Argumentation Frameworks

A weaker case is to investigate when $\text{PREF} = \text{STAB}$.

**Definition 8.14.** [14, Definition 31(1)] An AF is **coherent** iff $\text{PREF} = \text{STAB}$.

**Corollary 8.15.** If an AF is non-empty and well-founded, then it is coherent. The converse is not true.

**Proof.** If an AF is non-empty and well-founded, then by Theorem 8.12, all four semantics coincide. In particular $\text{PREF} = \text{STAB}$, hence by Definition 8.14, this AF is also coherent.

For the converse, consider the AF with $A = \{a, b, c, e\}$ and $R = \{(a, b), (b, c), (c, a), (e, a)\}$. This is depicted in Figure 8.2.

![Figure 8.2: The AF for the converse of Corollary 8.15](image)

We have that $\text{PREF} = \text{STAB} = \{\{b, c\}\}$. Therefore, this AF is coherent. However, the 3-cycle $\{a, b, c\}$ means this AF is not well-founded (although it is non-empty).

**Corollary 8.16.** If an AF is coherent, then $\text{STAB} \neq \emptyset$.

**Proof.** This follows from Definition 8.14 and Corollary 7.9 (page 61).

**Theorem 8.17.** [14, Theorem 33(1)] Every limited controversial AF is coherent. The converse is not true.

**Proof.** Let $(A, R)$ be a limited controversial AF. Assume for contradiction that it is not coherent, i.e. $\text{PREF} \neq \text{STAB}$. It follows that there is some $E \in \text{PREF} - \text{STAB}$. This means $E \cup E^+ \subset A$ and hence $A - (E \cup E^+) \neq \emptyset$. Let $A' := A - (E \cup E^+)$ and let $(A', R') \subseteq g (A, R)$. By Corollary 7.4 (page 63), $(A', R')$ is also limited controversial. By Theorem 6.25 (page 53), there is $C \subseteq A'$, $C \in \text{COMP} ((A', R'))$ such that $C \neq \emptyset$.

Now, as $(\forall a \in A') a \notin E^+$ and $C \subseteq A'$, we have that $E \cup C \in \text{CF}((A, R))$. Now let $b \in (E \cup C)^-$. Either $b \in E^-$ or $b \in C^-$ by Corollary 3.3 (page 87).
But as $E \in \text{PREF}(\langle A, R \rangle)$ and $C \in \text{COMP}(\langle A', R' \rangle)$, we must have $b \in E^+$ and $b \in C^+$. Therefore, $E \cup C \in \text{ADM}(\langle A, R \rangle)$. However, as $C \neq \emptyset$, we have constructed a strict superset of $E$ that is admissible, which contradicts the claim that $E$ is preferred.

For the converse, refer to the AF depicted in Figure 8.2 in Corollary 8.15 which is a coherent AF that is not limited controversial due to its 3-cycle.

**Corollary 8.18.** \[14\] Corollary 36] If the underlying AF is limited controversial then $\text{STAB} \neq \emptyset$.

**Proof.** Immediate from Corollary 8.16 and Theorem 8.17. \qed

**Theorem 8.19.** \[14\] Definition 33(2)] Every uncontroversial AF is coherent. The converse is not true.

**Proof.** Immediate from Corollary 2.64 (page 18) and Theorem 8.17. For the converse, the coherent AF depicted in Figure 8.2 has argument $a$ which indirectly attacks and defends itself and hence this AF is controversial. \qed

We have a useful result that gives an equivalent characterisation of coherent AFs that do not have self-attacking arguments.

**Lemma 8.20.** If an AF is coherent, then $\text{PREF} \subseteq \text{NAI}$.

**Proof.** By Corollary 7.41 (page 67), $\text{PREF} = \text{STAB} \subseteq \text{NAI}$. Therefore, $\text{PREF} \subseteq \text{NAI}$. \qed

**Lemma 8.21.** If an AF has no self-attacking arguments, and $\text{PREF} \subseteq \text{NAI}$, then the AF is coherent.

**Proof.** Let $S \in \text{PREF}$, then $S \in \text{NAI}$ and by Lemma 7.12, $S \in \text{STAB}$. It follows that $\text{PREF} \subseteq \text{STAB}$, but as $\text{STAB} \subseteq \text{PREF}$, this AF is coherent. \qed

**Theorem 8.22.** For AFs that do not have self-attacking arguments, $\text{PREF} \subseteq \text{NAI}$ iff the AF is coherent.

**Proof.** $(\Rightarrow)$ This is Lemma 8.21 $(\Leftarrow)$ This is Lemma 8.20. \qed

**Example 8.23.** We illustrate an example of Theorem 8.22 where we have an AF that has no self-attacking arguments, and is not coherent as $\text{PREF} \nsubseteq \text{NAI}$. Consider the AF where $A = \{a_0, a_1, a_2, a_3, a_4, a_5\}$ with the attacks illustrated in Figure 8.3 where we have abbreviated 2-cycles with two-headed arrows.
For the AF depicted in Figure 8.3, we have that

\[
\text{NAI} = \{\{a_2, a_0\}, \{a_2, a_3\}, \{a_2, a_5\}, \{a_3, a_1\}, \{a_4, a_5\}\}
\]
\[
\text{PREF} = \{\{a_2\}, \{a_4, a_5\}\}
\]
\[
\text{STAB} = \{\{a_4, a_5\}\}.
\]

This AF is not coherent despite not having any self-attacking arguments, because \(\text{PREF} \not\subseteq \text{NAI}\). Further, by the contrapositive of Theorem 8.17, this AF is not limited coherent, as shown by the 3-cycle \(a_1, a_3\) and \(a_5\). Similarly, by the contrapositive of Theorem 8.19, this AF is controversial, with (e.g.) the argument \(a_3\) being controversial with respect to \(a_5\).

Example 8.24. We again illustrate Theorem 8.22 with a second example. Consider the AF with \(A = \{a_0, a_1, a_2, a_3, a_4\}\) and attacks illustrated in Figure 8.4, where we have abbreviated 2-cycles with two-headed arrows.

For the AF depicted in Figure 8.4, we have

\[
\text{NAI} = \{\{a_1, a_2\}, \{a_0, a_3\}, \{a_1, a_3\}, \{a_1, a_4\}\}
\]
\[
\text{PREF} = \{\{a_1\}, \{a_0, a_3\}\}
\]
\[
\text{STAB} = \{\{a_0, a_3\}\}.
\]

Despite there not being any self-attacking arguments, we have that \(\text{PREF} \not\subseteq \text{NAI}\) and hence \(\text{PREF} \neq \text{STAB}\). Notice that the three-cycle \(a_2, a_4\) and \(a_3\) renders this AF not limited controversial.
Notice both Examples 8.23 and 8.24 each have a three-cycle \((a_5, a_1, a_0)\) in the former, and \(a_2, a_4, a_3\) in the latter. This is actually a general result for finite AFs:

**Corollary 8.25.** \([15, \text{Fact 19}]\) If a finite AF is not coherent, then it has an odd cycle. The converse is not true.

*Proof.* Let an AF be finite and not coherent, then by the contrapositive of Theorem 8.17 this AF cannot be limited controversial. By the contrapositive of Corollary 2.67 (page 19), this AF must have an odd cycle.

The converse is not true, e.g. Figure 8.2 depicts an AF with a three cycle and that it is coherent. \(\square\)

**Theorem 8.26.** \([17, \text{Proposition 5}]\) If an AF is symmetric with no self-attacking arguments, then it is coherent.

*Proof.* For a given AF with no self-attacking arguments, if it is symmetric, then \(\text{PREF} = \text{NAI}\) by Corollary 7.27 (page 65). It follows that \(\text{PREF} \subseteq \text{NAI}\) and, by Theorem 8.22, this AF is coherent. \(\square\)

### 8.2.3 Relatively Grounded Argumentation Frameworks

**Lemma 8.27.** \([14, \text{Remark 26}]\) The intersection of all preferred extensions may not be the grounded extension.

*Proof.* From Example 2.7 (page 9), we have \(\text{PREF} = \{\{a, e\}, \{b, e\}\}\) and hence \(\bigcap \text{PREF} = \{e\}\), but as \(U = \emptyset\), we have \(G = \emptyset\). Therefore, \(\bigcap \text{PREF} \neq G\). \(\square\)

Lemma 8.27 motivates us to classify AFs by whether they do satisfy the property that the intersection of all preferred extensions is the grounded extension.

**Definition 8.28.** \([14, \text{Definition 31(2)}]\) An AF is **relatively grounded** iff \(\bigcap \text{PREF} = G\).

We give two examples of relatively grounded AFs, one where \(G\) is empty and the other where \(G\) is not empty.

**Example 8.29.** (Example 8.24 continued) Notice for this AF, \(G = \emptyset\) because \(U = \emptyset\) (Corollary 6.10, page 57). Further, \(\text{PREF} = \{\{a_1\}, \{a_0, a_3\}\}\) and hence \(\bigcap \text{PREF} = G\). Therefore, this AF is relatively grounded. This is also an example of an AF that is not coherent and relatively grounded.

**Example 8.30.** Consider the following AF where \(A = \{a_0, a_1, a_2, a_3\}\) and the attacks are depicted in Figure 8.5, where we have used two-headed arrows to represent 2-cycles.
In this case, we have

\[ G = \{a_2\} \]
\[ \text{PREF} = \{\{a_2, a_0\}, \{a_2, a_1\}, \{a_2, a_3\}\} \]
\[ \text{STAB} = \{\{a_2, a_0\}, \{a_2, a_1\}, \{a_2, a_3\}\} \]

Clearly this is coherent and relatively grounded, because \( \bigcap \text{PREF} = G \).

**Corollary 8.31.** If an AF is well-founded and non-empty, then it is relatively grounded. The converse is not true.

**Proof.** If an AF is well-founded and non-empty, then by Theorem 8.12 its grounded extension \( G \) is its unique preferred extension. Hence \( \text{PREF} = \{G\} \) and \( \bigcap \text{PREF} = G \). Therefore, this AF is relatively grounded.

For the converse, Example 8.24 is a relatively grounded AF that has a 2-cycle \((a_0, a_1)\), hence it is not well-founded. \(\square\)

**Theorem 8.32.** [14, Theorem 33(2)] Every uncontroversial AF is relatively grounded. The converse is not true.

**Proof.** Assume for contradiction that \( G \neq \bigcap \text{PREF} \). By definition, \( G \subseteq \bigcap \text{PREF} \). Let \( a \in \bigcap \text{PREF} - G \neq \emptyset \). If \( a \in G^+ \), then as \( a \in \bigcap \text{PREF} \), for any \( P \in \text{PREF} \), \( a \in P \), and \( G \subseteq P \), so \( P \notin CF \) – contradiction, because \( \text{PREF} \subseteq CF \). Therefore, \( a \notin G^+ \). As the underlying AF is uncontroversial, by Theorem 6.26 (page 55), there exists complete extensions \( C_1 \) and \( C_2 \) such that \( a \in C_1 \cap C_2^+ \). But \( C_2 \in \text{ADM} \) and hence by Corollary 7.3 (page 60), there is some \( P \in \text{PREF} \) such that \( C_2 \subseteq P \), so \( a \in P^+ \), which contradicts that \( a \in \bigcap \text{PREF} \) as \( \text{PREF} \subseteq CF \). Therefore, \( G = \bigcap \text{PREF} \).

For the converse, Example 8.24 (page 76) is an AF that is relatively grounded (see Example 8.29) and by Figure S.31 has a controversial argument, i.e. \( a_0 \) is controversial with respect to \( a_4 \). \(\square\)

**Theorem 8.33.** [12] Lemma 1] Given an AF \( (A, R) \), if \( \text{PREF} \) covers \( A \) then this AF is relatively grounded.

**Proof.** By Corollary 7.22 (page 63), if \( \text{PREF} \) covers \( A \), then \( \bigcap \text{PREF} = U \). By Corollary 7.52 (page 68), \( \bigcap \text{PREF} \subseteq G \). However, as \( G \) is the \( \subseteq \)-least complete extension, and all preferred extensions are complete, we have \( G \subseteq \bigcap \text{PREF} \). The result follows. \(\square\)
8.3 Summary

- In any AF, if $G \in STAB$ then $STAB = PREF = COMP = \{G\}$.
- If the AF is non-empty and well-founded, then $STAB = PREF = COMP = \{G\}$.
- If the AF is limited controversial, then $PREF = STAB$.
- For any AF, $PREF \subseteq NAI$ iff $PREF = STAB$.
- If the AF is uncontroversial, then $\bigcap PREF = G$. 
9 Conclusion

In this note, we have reviewed [14, Section 2], with the aim of making all of the proofs explicit. We do not claim originality as many of the results in this note are likely folklore. We hope that this note will be useful for students and researchers approaching abstract argumentation theory, in particular [14], for the first time.

9.1 Summary of the Various Types of Sets of Arguments

Given an AF \((A, R)\) with neutrality function \(n\) and defence function \(d\):

| Type of Set of Args | Section | Definition |
|---------------------|---------|------------|
| \(CF\)              | 3.2     | \(S \in CF \iff S \subseteq n(S)\) |
| \(NAI\)             | 3.3     | \(NAI = \max_\subseteq CF\) |
| \(SD\)              | 4.2     | \(S \subseteq d(S)\) |
| \(ADM\)             | 5       | \(ADM = CF \cap SD\) |
| \(COMP\)            | 6       | \(S \in COMP \iff \{S \in CF, S = d(S)\}\) |
| \(PREF\)            | 7.1     | \(PREF = \max_\subseteq ADM\) |
| \(STAB\)            | 7.2     | \(S \in STAB \iff S = n(S)\) |
| \(G\)               | 7.3     | \(G = \bigcap COMP\) |

Table 9.1: A table summarising the types of sets of arguments and their definition.

All such sets of arguments, apart from the stable extensions, are non-empty for all AFs. Only the grounded extension is unique for all AFs. Lattice theoretically under \(\subseteq\), \(G\), \(STAB\), \(PREF\) and \(NAI\) are antichains. \(CF\), \(SD\) and \(ADM\) are bounded from below by \(\emptyset\). Further, \(CF\), \(ADM\) and \(COMP\) are all directed complete and hence chain complete. \(SD\) is closed under arbitrary unions and hence also directed complete, while \(CF\) is closed under arbitrary intersections. Both \(ADM\) and \(COMP\) are complete semilattices.

9.2 Acknowledgements

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A Directed Graphs

In abstract argumentation, arguments and attacks between arguments are respectively represented as nodes and edges of a directed graph. We therefore recap some elementary notions of graph theory.

**Definition A.1.** A directed graph (digraph) is a pair \( \langle A, R \rangle \) where \( A \) is a set of nodes and \( R \subseteq A^2 \) is a binary relation.

**Example A.2.** For \( n \in \mathbb{N} \), the digraph \( P_n \) is called the directed path graph on \( n \) nodes and has \( A = \{a_k\}_{k=1}^n \) and \( R = \{(a_k, a_{k+1})\}_{k=1}^{n-1} \). Notice \( P_0 = \langle \emptyset, \emptyset \rangle \). A typical non-empty graph \( P_3 \) is depicted in Figure A.1.

![Figure A.1: A depiction of \( P_n \), from Example A.2](image1)

**Example A.3.** For \( n \in \mathbb{N} \), the digraph \( C_n \) is called the directed cycle graph on \( n \) nodes and has \( A = \{a_k\}_{k=1}^n \) and \( R = \{(a_k, a_{k+1})\}_{k=1}^{n-1} \cup \{(a_n, a_1)\} \). Clearly, \( C_0 = \langle \emptyset, \emptyset \rangle \). A typical non-empty graph \( C_3 \) is depicted in Figure A.2.

![Figure A.2: A depiction of \( C_n \), from Example A.3](image2)

In what follows let \( \langle A, R \rangle \) be an arbitrary digraph.

**Definition A.4.** For \( a \in A \), we define two sets.

\[
\begin{align*}
    a^+ &:= \{b \in A \mid R(a, b)\} \quad \text{and} \\
    a^- &:= \{b \in A \mid R(b, a)\}.
\end{align*}
\]

We call \( a^+ \) the forward set of \( a \) and \( a^- \) the backward set of \( a \).

**Corollary A.5.** For all \( a, b \in A \), \( b \in a^- \) iff \( a \in b^+ \).

**Proof.** We have that \( b \in a^- \) iff \( R(b, a) \) iff \( a \in b^+ \) by Definition A.4 \( \square \)

**Definition A.6.** We say \( a \in A \) is a source node iff \( a^- = \emptyset \).
Definition A.7. For $S \subseteq A$, we define two sets:

\[
S^+ := \{ a \in A \mid \exists b \in S \ R(b, a) \} \tag{A.3}
\]

\[
S^- := \{ a \in A \mid \exists b \in S \ R(a, b) \} \tag{A.4}
\]

We call $S^+$ the \textit{forward set of} $S$ and $S^-$ the \textit{backward set of} $S$.

Corollary A.8. If $S = T$ then $S^\pm = T^\pm$. The converses are not in general true.

Proof. We have that $a \in S^+$ iff $(\exists b \in S) \ R(b, a)$ iff $(\exists b \in T) \ R(b, a)$ because $S = T$, iff $a \in T^+$. Similarly, $a \in S^-$ iff $(\exists b \in S) \ R(a, b)$ iff $(\exists b \in T) \ R(a, b)$ because $S = T$, iff $a \in T^-$. The result follows.

Now consider the digraph $A = \{a, b, c\}$ and $R = \{(a, c), (b, c)\}$. This is depicted in Figure A.3.

![Figure A.3: The first digraph mentioned in Corollary A.8](image)

Suppose $S = \{a\}$ and $T = \{b\}$. We have $S^+ = T^+ = \{c\}$ and $S \neq T$.

Now consider the dual digraph where $R^{\text{op}} = \{(c, a), (c, b)\}$, which is depicted in Figure A.4.

![Figure A.4: The second digraph mentioned in Corollary A.8](image)

Then $S^- = T^- = \{c\}$ while $S \neq T$. Therefore, the functions $S \mapsto S^\pm$ are not injective.

Corollary A.9. The functions $(\cdot)^\pm : \mathcal{P}(A) \to \mathcal{P}(A)$ where $S \mapsto S^\pm$ are well-defined.
Proof. Totality follows from Definition A.7. Single-valuedness follows from Corollary A.8.

**Corollary A.10.** We have:

\[ S^+ = \bigcup_{a \in S} a^+ \quad \text{and} \quad S^- = \bigcup_{a \in S} a^- . \] (A.5)

Proof. We have \( a \in S^+ \) iff \( \exists b \in S \) \( R(b, a) \) iff \( \exists b \in S \) \( a \in b^+ \) by Equation A.1 iff \( a \in \bigcup_{b \in S} b^+ \). Similarly, \( a \in S^- \) iff \( \exists b \in S \) \( R(a, b) \) iff \( \exists b \in S \) \( a \in b^- \) by Equation A.2 iff \( a \in \bigcup_{b \in S} b^- \). The result follows.

**Corollary A.11.** We have that \( \{a\}^\pm = a^\pm \).

Proof. Immediate from Corollary A.10 by setting \( S = \{a\} \).

**Corollary A.12.** We have that \( \emptyset \pm = \emptyset \).

Proof. Immediate from Corollary A.10 and the definition of the empty union.

**Corollary A.13.** If \( S \subseteq T \subseteq A \) then \( S^\pm \subseteq T^\pm \). The converse is not necessarily true.

Proof. The result follows from Corollary A.10 and the properties of set-theoretic union. From Figure A.3 it is the case that \( S^+ = T^+ = \{c\} \) but \( S = \{a\} \nsubseteq T = \{b\} \). Similarly, from Figure A.4 we have \( S^- \subseteq T^- \) and \( S \nsubseteq T \). Therefore, the converse is false.

Notice how Corollary A.8 follows trivially from Corollary A.13 using the definition of set equality.

**Definition A.14.** Let \( B \subseteq A \). The induced directed graph (digraph) w.r.t. \( B \) (a.k.a the full subgraph w.r.t. \( B \)) is the digraph \( (B, R_B) \) where \( R_B = B^2 \cap R \). We will write \( (B, R_B) \subseteq_g (A, R) \).

Clearly, \( \subseteq_g \) is a reflexive and transitive relation on the class of digraphs.
B  Unions and Intersections of Bounded Quantifiers

Definition B.1. Let $X$ be a set and $A, P \subseteq X$. We have

\[(\exists x \in A) P(x) \iff (\exists x \in X) (x \in A \land P(x)), \quad \text{and} \quad (B.1)\]
\[(\forall x \in A) P(x) \iff (\forall x \in X) (x \in A \rightarrow P(x)). \quad (B.2)\]

where we will occasionally write “,” instead of “\&&”, and “\implies” instead of “\implies”.

The following results are likely folklore; they are included as we make use of them.

Theorem B.2. Let $X$ be a set. Let $I$ be any index set such that $\{A_i\}_{i \in I} \subseteq \mathcal{P}(X)$. Let $P(x)$ be any unary predicate that may be true or false on the elements $x$ of $X$. We have the following results:

\[(\forall x \in \bigcup_{i \in I} A_i) P(x) \iff (\forall i \in I) (\forall x \in A_i) P(x), \quad (B.3)\]
\[(\exists x \in \bigcup_{i \in I} A_i) P(x) \iff (\exists i \in I) (\exists x \in A_i) P(x), \quad (B.4)\]
\[(\exists x \in \bigcap_{i \in I} A_i) P(x) \implies (\forall i \in I) (\exists x \in A_i) P(x) \quad \text{and} \quad (B.5)\]
\[(\forall x \in \bigcap_{i \in I} A_i) P(x) \iff (\exists i \in I) (\forall x \in A_i) P(x). \quad (B.6)\]

Further, the converses of Equations $[B.3]$ and $[B.6]$ are not true in general.

Proof. If $X = \emptyset$, then $\{A_i\}_{i \in I} = \{\emptyset\}$ for any index set $I$. Therefore, Equation $[B.3]$ reduces to true iff true, Equation $[B.4]$ reduces to false iff false, Equation $[B.5]$ reduces to false implies true if $I = \emptyset$ and false implies false if $I \neq \emptyset$, and Equation $[B.6]$ reduces to false implies true if $I = \emptyset$ and true implies true if $I \neq \emptyset$. In all cases, the four equations are true regardless of $I$.

If $X \neq \emptyset$ and $I = \emptyset$, then Equation $[B.3]$ reduces to true iff true, Equation $[B.4]$ reduces to false iff false, Equation $[B.5]$ reduces to true implies true, and Equation $[B.6]$ reduces to false implies either true or false, as $P(x)$ may not hold on all elements on $X$. In all cases, the four equations are true.

Now assume $X \neq \emptyset$ and $I \neq \emptyset$. For Equation $[B.3]$. 

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\[
\left( \forall x \in \bigcup_{i \in I} A_i \right) P(x) \iff (\forall x \in X) \left( x \in \bigcup_{i \in I} A_i \Rightarrow P(x) \right)
\]
\[
\iff (\forall x \in X) [(\exists i \in I) x \in A_i \Rightarrow P(x)]
\]
\[
\iff (\forall x \in X) (\forall i \in I) [x \in A_i \Rightarrow P(x)]
\]
\[
\iff (\forall i \in I) (\forall x \in X) [x \in A_i \Rightarrow P(x)]
\]
\[
\iff (\forall i \in I) (\forall x \in A_i) P(x),
\]
which proves Equation B.3.

For Equation B.4:
\[
\left( \exists x \in \bigcup_{i \in I} A_i \right) P(x) \iff (\exists x \in X) \left( x \in \bigcup_{i \in I} A_i \text{ and } P(x) \right)
\]
\[
\iff (\exists x \in X) [(\exists i \in I) x \in A_i \text{ and } P(x)]
\]
\[
\iff (\exists x \in X) (\exists i \in I) (x \in A_i \text{ and } P(x))
\]
\[
\iff (\exists i \in I) (\exists x \in X) (x \in A_i \text{ and } P(x))
\]
\[
\iff (\exists i \in I) (\exists x \in A_i) P(x),
\]
which proves Equation B.4.

For Equation B.5:
\[
\left( \exists x \in \bigcap_{i \in I} A_i \right) P(x) \iff (\exists x \in X) \left( x \in \bigcap_{i \in I} A_i \text{ and } P(x) \right)
\]
\[
\iff (\exists x \in X) [(\forall i \in I) x \in A_i \text{ and } P(x)]
\]
\[
\iff (\exists x \in X) (\forall i \in I) (x \in A_i \text{ and } P(x))
\]
\[
\Rightarrow (\forall i \in I) (\exists x \in X) (x \in A_i \text{ and } P(x))
\]
\[
\Rightarrow (\forall i \in I) (\exists x \in A_i) P(x),
\]
which proves Equation B.5. The converse may not be true because Equation B.7 is strictly stronger than Equation B.8. More concretely, let \( X = \mathbb{N} \) and \( P(x) \iff x \text{ is even} \). Let \( I = \{1, 2\} \) and let \( A_1 := A \) and \( A_2 := B \). Let \( A := \{2, 3, 4\} \) and \( B := \{5, 6, 7\} \). Clearly, \((\exists x \in A) P(x)\) is true with 2 as a witness. Further, \((\exists x \in B) P(x)\) is true with 6 as a witness. Therefore, the right hand side of Equation B.5 is true. However, \( A \cap B = \emptyset \), which has no even numbers. Therefore, \((\exists x \in A \cap B) P(x)\) is false.

Finally, for Equation B.6:
\[(\exists i \in I) (\forall x \in A_i) P(x) \iff (\exists i \in I) (\forall x \in X) (x \in A_i \Rightarrow P(x)) \quad (B.9)\]
\[\Rightarrow (\forall x \in X) (\exists i \in I) (x \in A_i \Rightarrow P(x)) \quad (B.10)\]
\[\iff (\forall x \in X) [(\forall i \in I) x \in A_i \Rightarrow P(x)] \]
\[\iff (\forall x \in X) \left( x \in \bigcap_{i \in I} A_i \Rightarrow P(x) \right) \]
\[\iff \left( \forall x \in \bigcap_{i \in I} A_i \right) P(x), \]

which proves half of Equation B.6. The converse may not be true because Equation B.9 is strictly stronger than Equation B.10. More concretely, let \( X = \mathbb{N} \) and \( P(x) \iff x \) is even. Let \( I = \{1, 2\} \) such that \( A_1 = A \) and \( A_2 = B \). Let \( A := \{1, 4\} \) and \( B := \{3, 4\} \). Clearly \( A \cap B = \{4\} \) and hence \( (\forall x \in A \cap B) P(x) \) is true. However, neither \((\forall x \in A) P(x)\) nor \((\forall x \in A \cap B) P(x)\) is true. In the first case, this is because \( 1 \in A \). In the second case, this is because \( 3 \in B \).

This proves all four equations and shows that the converses of the latter two are not true in general. \( \Box \)

We now apply Theorem B.2 (page 85) to digraphs.

**Corollary B.3.** We have that
\[
\left( \bigcup_{i \in I} S_i \right)^\pm = \bigcup_{i \in I} S_i^\pm \quad (B.11)
\]
\[
\left( \bigcap_{i \in I} S_i \right)^\pm \subseteq \bigcap_{i \in I} S_i^\pm, \quad (B.12)
\]

and the converse is not true in general for Equation B.12.

Notice that for \( I = \emptyset \), both results reduce to \( \emptyset = \emptyset \) and \( A^\pm \subseteq A \), which are trivially true.

**Proof.** For the + case in Equation B.11 we apply Equations A.3 and B.4
\[
a \in \left( \bigcup_{i \in I} S_i \right)^+ \iff (\exists b \in \bigcup_{i \in I} S_i) R(b, a)
\]
\[\iff (\exists i \in I) (\exists b \in S_i) R(b, a)
\]
\[\iff (\exists i \in I) a \in S_i^+
\]
\[\iff a \in \bigcup_{i \in I} S_i^+.\]

Notice that for the − case, the proof is the same but with + replaced by − and \( R(b, a) \) replaced by \( R(a, b) \). Therefore, Equation B.11 follows.
For the $+$ case in Equation B.12, we apply Equations A.3 and B.5:

$$a \in \left( \bigcap_{i \in I} S_i \right)^+ \iff \left( \exists b \in \bigcap_{i \in I} S_i \right) R(b, a)$$

$$\Rightarrow (\forall i \in I) (\exists b \in S_i) R(b, a)$$

$$\iff (\forall i \in I) a \in S_i^+$$

$$\iff a \in \bigcap_{i \in I} S_i^+.$$ 

Notice that for the $-$ case, the proof is the same but with $+$ replaced by $-$ and $R(b, a)$ replaced by $R(a, b)$. Therefore, Equation B.12 follows.

For the converse to the $+$ case in Equation B.12, consider the digraph

$$\langle \{a, b, c, x\}, \{(a, x), (c, x)\}\rangle.$$ 

This digraph is depicted in Figure B.1.

![Figure B.1](image_url)

Figure B.1: The digraph mentioned in Corollary B.3

Let $S_1 := \{a, b\}$ and $S_2 := \{b, c\}$. We have $S_1^+ = S_2^+ = \{x\}$ and $S_1^+ \cap S_2^+ = \{x\}$. However, $S_1 \cap S_2 = \{b\}$ and $(S_1 \cap S_2)^+ = \emptyset$. It is not the case that $\{x\} \subseteq \emptyset$. Therefore, the converse to Equation B.12 does not hold in general. 

For the converse in the $-$ case, take the dual of the digraph in Figure B.1 with the same definitions of $S_1$ and $S_2$ to conclude that $\{x\} \not\subseteq \emptyset$. Therefore, the converse to Equation B.12 does not hold in general. 

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C A Recap of Order and Lattices

Many of these results can be found in textbooks such as [13] or [21].

C.1 Partially Ordered Sets

In abstract argumentation, we will be concerned with partially ordered sets of the form \(\langle S, \subseteq \rangle\) and lattices of the form \(\langle S, \cap, \cup \rangle\), where \(S \subseteq P(A)\) for a set \(A\). We therefore recap the necessary elements of order and lattice theory.

Definition C.1. A binary relation \(R\) on a set \(A\) is a partial order iff it is reflexive, antisymmetric and transitive.

Definition C.2. A digraph \(\langle A, R \rangle\) is a partially ordered set (poset) iff \(R\) is a partial order on \(A\). We denote \(R\) with \(\leq\) and the underlying set will be denoted as \(P\).

Example C.3. For any set \(X\), the structure \(\langle P(X), \subseteq \rangle\) is a poset.

Definition C.4. A sub-poset of \(\langle P, \leq \rangle\) is a poset \(\langle Q, \leq' \rangle\) where \(Q \subseteq P\) and \(\leq' := Q^2 \cap \leq\).

Example C.5. For any set \(X\) and family of subsets \(S \subseteq P(X)\), the structure \(\langle S, \subseteq \rangle\) is a poset.

Definition C.6. Let \(\langle P, \leq \rangle\) be a poset. We say \(m \in U \subseteq P\) is a maximal element of \(U\) iff for all \(x \in U\), if \(m \leq x\) then \(m = x\). We let \(\text{max}_{\leq} U \subseteq U\) denote the set of all maximal elements of \(U\).

Definition C.7. Let \(\langle P, \leq \rangle\) be a poset. We say \(m \in U \subseteq P\) is the greatest element of \(U\) iff for all \(x \in U\), \(m \geq x\).

Note that greatest elements are necessarily unique and maximal, but maximal elements do not have to be unique, and a maximal element do not have to be the greatest.

Definition C.8. A poset \(\langle P, \leq \rangle\) is an antichain iff \(\leq\) is the diagonal relation on \(P\).

Clearly for any \(\langle P, \leq \rangle\) and \(U \subseteq P\), \(\leq\) restricted onto \(\text{max}_{\leq} U\) is an antichain.

Definition C.9. A poset \(\langle P, \leq \rangle\) is a totally ordered set (toset) a.k.a. chain iff for all \(x, y \in P\), either \(x \leq y\) or \(y \leq x\).

Definition C.10. Let \(\langle P, \leq \rangle\) be a poset. A chain in \(\langle P, \leq \rangle\) is a sub-poset that is also a toset.

Definition C.11. Let \(\langle P, \leq \rangle\) be a poset. An \(\omega\)-chain is a \(\geq\)-ascending sequence in \(\langle P, \leq \rangle\).\footnote{Recall for any set \(X\), an \(X\)-sequence is a function \(f: \mathbb{N} \to X\). If \(X\) has a partial order \(\leq\), then \(f\) is \(\geq\)-ascending iff \(f(i) \leq f(i + 1)\).}
Definition C.12. Let $\langle P, \leq \rangle$ be a poset and $Q \subseteq P$. We say $u \in P$ is an upper bound for $Q$ iff $(\forall x \in Q) \ x \leq u$. Similarly, $l \in P$ is a lower bound for $Q$ iff $(\forall x \in Q) \ l \leq x$.

Definition C.13. A poset $\langle P, \leq \rangle$ is directed iff every (not necessarily distinct) pair of elements in $P$ has an upper bound in $P$.

Definition C.14. We say $D$ is a directed subset of $\langle P, \leq \rangle$ iff $\langle D, \leq \cap D^2 \rangle$ is a poset and is directed$^{26}$.

Corollary C.15. Let $\langle D, \leq \rangle$ be an infinite directed poset and $x \in D$. Then there exists an $\omega$-chain in $\langle D, \leq \rangle$ starting from $x$.

Proof. We construct the chain as follows. Let $x_0 := x$ and $x_1 := x$. Trivially, $x_0 \leq x_1$. Now let $x_2 \in D$ be the upper bound of $x_0$ and $x_1$, which is well-defined because $D$ is directed and trivially $x_0 \leq x_1 \leq x_2$. Inductively, for $i \in \mathbb{N}$, $x_{i+2}$ is the upper bound of the set $\{x_{i+1}, x_i\}$, which always exists in $D$, and may not be equal to $x_i$ or $x_{i+1}$ because $D$ is infinite. Therefore, the $D$-sequence $\{x_i\}_{i \in \mathbb{N}}$ is well-defined, has cardinality $\aleph_0$, and satisfies $(\forall i \in \mathbb{N}) \ x_i \leq x_{i+1}$. This is our $\omega$-chain in $\langle D, \leq \rangle$, starting from $x_0 = x$. \qed

In what follows let $\langle P, \leq \rangle$ be an arbitrary poset.

Definition C.16. For $Q \subseteq P$, let $Q^{up} := \{u \in P \mid (\forall x \in Q) \ x \leq u\}$ denote the set of upper bounds for $Q$. The least upper bound of $Q$, sup $Q$, is $\min_{\leq} Q^{up}$.

Definition C.17. For $Q \subseteq P$, let $Q^{low} := \{l \in P \mid (\forall x \in Q) \ l \leq x\}$ denote the set of lower bounds for $Q$. The greatest lower bound of $Q$, inf $Q$, is $\max_{\leq} Q^{low}$.

Depending on $Q$, sup $Q$ and inf $Q$ may not exist, but if each does exist then each is unique.

Definition C.18. Let $\langle P, \leq \rangle$ and $\langle Q, \leq' \rangle$ be two posets. A function $f : P \rightarrow Q$ is monotone iff $(\forall x, y \in P) \ [x \leq y \Rightarrow f(x) \leq' f(y)]$.

Corollary C.19. Let $\langle P, \leq \rangle$ and $\langle Q, \leq' \rangle$ be two posets and $f : P \rightarrow Q$ be a monotone function. If $\{x_i\}_{i \in \mathbb{N}}$ is an $\leq$-chain in $P$, then $\{f(x_i)\}_{i \in \mathbb{N}}$ is an $\leq'$-chain in $Q$.

Proof. Immediate from the fact that $f$ is monotone. \qed

Definition C.20. Let $\langle P, \leq \rangle$ be a posets and let $f : P \rightarrow P$ be monotone. We say $x \in P$ is a:

1. prefixed point iff $f(x) \leq x$.
2. postfixed point iff $x \leq f(x)$.
3. fixed point iff $f(x) = x$.

$^{26}$N.B. The upper bound of every pair of elements now has to be in $D$. 

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C.2 Lattices

Definition C.21. A poset \( \langle P, \leq \rangle \) is a lattice iff for every pair \( x, y \in P \), \( \sup \{ x, y \} \) and \( \inf \{ x, y \} \) always exist. We formalise this as two binary operations:

\[
\begin{align*}
\wedge : P^2 & \to P \\
(x, y) & \mapsto x \wedge y := \inf \{ x, y \}. \\
\vee : P^2 & \to P \\
(x, y) & \mapsto x \vee y := \sup \{ x, y \}.
\end{align*}
\]

We call \( \wedge \) meet and \( \vee \) join.

Clearly \( x \wedge x = x \) and \( x \vee x = x \) for any lattice \( \langle L, \wedge, \vee \rangle \) and \( x \in L \).

Example C.22. For any set \( X \), \( \langle P(X), \cap, \cup \rangle \) is a lattice where \( \cap \) is meet and \( \cup \) is join. In abstract argumentation we will be mainly concerned with lattices of the form \( \langle S, \cap, \cup \rangle \) where \( S \subseteq P(X) \).

Every lattice is a poset where \( x \leq y \iff x \wedge y = x \iff x \vee y = y \). Not every poset is a lattice.

Example C.23. The antichain of length 2, i.e. \( P = \{0, 1\} \) and \( 0 = 1 \) is not a lattice because \( \sup \{0, 1\} \) and \( \inf \{0, 1\} \) do not exist. However, this is a well-defined poset.

Definition C.24. As a poset, the least element (if it exists) of a lattice is called the bottom element, and the greatest element (if it exists) of a lattice is called the top element. A lattice with both top and bottom elements is a bounded lattice.

Corollary C.25. Let \( \bot \) denote the least element of a lattice \( L \) if it exists, then for any \( x \in L \), \( x \vee \bot = x \).

Proof. Let \( x \) be as given, then by definition \( \bot \leq x \). We have that \( x \vee \bot = \sup \{ \bot, x \} = \max \{ \bot, x \} = x \).

Definition C.26. A poset is a complete lattice iff all subsets \( Q \subseteq P \) has a least upper bound \( \bigwedge Q \) and a greatest lower bound \( \bigvee Q \) in the poset.

Corollary C.27. A complete lattice is never empty.

Proof. Let \( \langle L, \leq \rangle \) be a complete lattice. \( \emptyset \subseteq L \) has a least upper bound and a greatest lower bound, both in \( L \). Therefore, \( L \neq \emptyset \).

Definition C.28. For a complete lattice, we define its greatest element to be \( \top := \bigwedge \emptyset \) and least element to be \( \bot := \bigvee \emptyset \).

Corollary C.29. Let \( \langle L, \leq \rangle \) be a complete lattice and let \( a, b \in L \). If \( a \leq b \), then \( \langle [a, b], \leq \rangle \) is also a complete lattice, where \( [a, b] := \{ x \in L \mid a \leq x \leq b \} \).
Proof. Let \( a \leq b \) so \([a, b] \neq \emptyset\) (else it cannot be a complete lattice by Corollary C.27). Let \( S \subseteq [a, b] \) be arbitrary. If \( S = \emptyset \) then \( \bigvee S = a \) and \( \bigwedge S = b \). If \( S \neq \emptyset \), then for all \( s \in S \), \( a \leq s \leq b \) so \( a \) is a lower bound of \( S \) and \( b \) is an upper bound of \( S \). As \( L \) is a complete lattice and \( S \subseteq [a, b] \subseteq L \), we have \( p := \bigwedge S, q := \bigvee S \in L \). By definition, \( a \leq p \) and \( q \leq b \). Now let \( x \in S \) be arbitrary, then \( a \leq p \leq x \leq q \leq b \). This means \( p, q \in [a, b] \) and hence \([a, b]\) contains the supremum and infimum of \( S \). As \( S \) is arbitrary, \([a, b]\) is also a complete lattice.

Example C.30. For any set \( X \), \( \langle \mathcal{P}(X), \subseteq \rangle \) is a complete lattice. For every family of subsets of \( X \), their collective union is the greatest lower bound and their collective intersection is the least upper bound.

Lemma C.31. (Knaster-Tarski lemma) Let \( \langle L, \leq \rangle \) be a complete lattice and \( f : L \to L \) be a monotone function. The set \( F := \{ x \in L | f(x) = x \} \) is a bounded lattice with respect to \( \leq \).

Proof. This is equivalent to showing that \( F \) has a greatest element (the greatest fixed point of \( f \)) and a least element (the least fixed point of \( f \)). Let \( D \) be the set of all postfixed points of \( L \), i.e. \( D = \{ x \in L | x \leq f(x) \} \) (Definition C.20). As \( L \) is a complete lattice, \( 0_L := \bigwedge \{ x \in L | x \leq f(x) \} = \bigwedge L \in L \) and for all \( x \in L \), \( f(x) \in L \) so \((\forall x \in L) 0_L \leq f(x) \). Thus, \( 0_L \in D \) and hence \( D \neq \emptyset \). Therefore, there is some \( x \in D \), iff \( x \leq f(x) \), implies that \( f(x) \leq f^2(x) \) because \( f \) is monotone, iff \( f(x) \in D \).

As \( D \subseteq L \) and \( L \) is a complete lattice, let \( u := \bigvee D \in L \). By definition, \( (\forall x \in D) x \leq u \) hence \( f(x) \leq f(u) \). But as \( x \in D \), \( x \leq f(x) \leq f(u) \) so \( f(u) \) is an upper bound of \( D \). As \( u \) is the supremum of \( D \), have \( u \leq f(u) \). Therefore, \( u \in D \). Further, \( u \leq f(u) \) implies \( f(u) \leq f^2(u) \) iff \( f(u) \in D \). But as \( f(u) \in D \) and \( u \) is the supremum of \( D \), \( f(u) \leq u \). Therefore, \( f(u) = u \). As \( P \subseteq D, D \) contains all fixed points, and \( u \in P \) is the greatest fixed point of \( f \).

Dually, as \( f \) also has a least fixed point of \( f \) by arguing as above on the dual lattice of \( L \). Therefore, \( P \) is a bounded lattice.

Theorem C.32. (Knaster-Tarski theorem) Let \( \langle L, \leq \rangle \) be a complete lattice and \( f : L \to L \) be a monotone function. The set \( F := \{ x \in L | f(x) = x \} \) is a complete lattice.

Proof. Let \( S \subseteq F \) be arbitrary. Let \( s := \bigvee S \in L \) as \( L \) is a complete lattice. We show that there is an element of \( F \) that is greater than all elements in \( S \), and this is the smallest such element of \( F \). It is sufficient to show the stronger result that this element of \( F \) is greater than \( s \).

Consider the interval \([s, 1_L] \subseteq L \) where \( 1_L := \bigvee L \in L \). We show that \( f : [s, 1_L] \to [s, 1_L] \). Clearly, for all \( a \in S \), \( a \leq s \). As \( S \subseteq F \), \( f(a) = a \). Therefore, \( a = f(a) \leq f(s) \) as \( f \) is monotone, which means \( a \leq f(s) \) and as \( a \in S \) is arbitrary, \( f(s) \) is an upper bound for \( S \). As \( s \) is the supremum for \( S \), we must have \( s \leq f(s) \). Let \( x \in [s, 1_L] \), then \( s \leq x \), and hence \( f(s) \leq f(x) \) and hence \( s \leq f(x) \), so \( f(x) \in [s, 1_L] \). Therefore, \( f : [s, 1_L] \to [s, 1_L] \) is well-defined.
As \([s, 1_L] \subseteq L\) is a complete lattice by Corollary [C.29] and \(f\) is a monotonic function, then by Lemma [C.31] \(f\) has a least fixed point which is the supremum of \(S\). By definition, this is in \(F\).

Dually, the infimum of \(S\) is also in \(F\) as the greatest fixed point of \(f\) as a function on the complete lattice \([0_L, \wedge S]\). Therefore, \(S \subseteq F\) has a supremum and infimum both in \(F\). As \(S \subseteq F\) is arbitrary, \(F\) is a complete lattice. 

C.3 Complete Partial Orders

Definition C.33. The limit of a chain \(C\) in a poset that is also a lattice is \(\sup C\).

Corollary C.34. Finite chains always have a limit in the chain.

Proof. Let \(\langle P, \leq \rangle\) be a poset and \(C \subseteq P\) be a chain. WLOG let \(C = \{c_1, \ldots, c_n\}\) as it is finite. Clearly \(\bigvee C = \max_{\leq} C =: c_{\text{max}}\), which exists and is in \(C\). Therefore, \(C\) contains its own limit. 

Clearly not every chain has a well-defined limit.

Example C.35. The poset \(\langle \mathbb{N}, \leq \rangle\) contains itself as a chain, and \(\sup \mathbb{N}\) is not contained in \(\mathbb{N}\). Therefore, \(\mathbb{N}\) itself does not have a limit.

Definition C.36. A poset is chain-complete iff the limit of every chain is in the poset.

Definition C.37. A poset is \(\omega\)-complete iff every \(\omega\)-chain has a least upper bound in the poset.

Clearly every chain-complete poset is \(\omega\)-complete.

Example C.38. \(\langle \mathcal{P}(X), \subseteq \rangle\) is chain complete, where for every chain \(\{S_i\}_{i \in I}\), its limit which is the union of all such elements of the chain is clearly in \(\mathcal{P}(X)\).

Definition C.39. [14, Footnote 5] A poset \(\langle P, \leq \rangle\) is a complete semilattice iff every non-empty subset of \(P\) has an infimum, and \(P\) is chain complete.

There is no universally accepted and consistent definition of a “complete semilattice” [2] Definition [C.39] comes from [14] Page 330, Footnote 5]. Further, the infimum is with respect to \(\subseteq\) and does not have to be set-theoretic intersection.

Corollary C.40. Let \(\langle P, \leq \rangle\) and \(\langle Q, \leq' \rangle\) be \(\omega\)-complete posets and \(f : P \to Q\) be a monotone function. Let \(\{x_i\}_{i \in \mathbb{N}}\) be a \(P\)-chain with limit \(x := \bigvee_{i \in \mathbb{N}} x_i \in P\) [28]. Then

\[
\bigvee_{i \in \mathbb{N}} f(x_i) \leq' f \left( \bigvee_{i \in \mathbb{N}} x_i \right) = f(x). \tag{C.3}
\]

[27] See https://en.wikipedia.org/wiki/Semilattice, last accessed 5/10/2017.
[28] This abuses notation as we use \(\bigvee\) to refer to the iterated meet of elements in both \(P\) and \(Q\).
Proof. By definition, for all \( i \in \mathbb{N} \), \( x_i \leq x \). By monotonicity, \( f(x_i) \leq f(x) \). Therefore, \( f(x) \) is an upper bound for \( \{ f(x_i) \}_{i \in \mathbb{N}} \), with limit \( \bigvee_{i \in \mathbb{N}} f(x_i) \). As the limit is the greatest lower bound, \( \bigvee_{i \in \mathbb{N}} f(x_i) \leq f(x) \), and the result follows. \( \square \)

**Definition C.41.** Let \( \langle P, \leq \rangle \) and \( \langle Q, \leq' \rangle \) be \( \omega \)-complete posets and \( f : P \to Q \) be a monotone function. We say \( f \) is \( \omega \)-**continuous** iff for every chain \( \{ x_i \}_{i \in \mathbb{N}} \) in \( P \) with limit \( x := \bigvee_{i \in \mathbb{N}} x_i \in P \),

\[
 f \left( \bigvee_{i \in \mathbb{N}} x_i \right) = f(x) \leq' \bigvee_{i \in \mathbb{N}} f(x_i). \quad (C.4)
\]

**Corollary C.42.** \( \omega \)-continuous functions preserve limits of chains, i.e.

\[
 \bigvee_{i \in \mathbb{N}} f(x_i) = f \left( \bigvee_{i \in \mathbb{N}} x_i \right). \quad (C.5)
\]

**Proof.** Immediate from Equations (C.3) and (C.4) and that \( \leq' \) is antisymmetric. \( \square \)

Notice that this definition is analogous to the continuity of real functions, where \( \land \) is replaced with \( \lim \).

**Example C.43.** Not all monotone functions between posets are \( \omega \)-continuous. Let \( \langle \mathbb{N} \cup \{ \infty \}, \leq \rangle \) be the poset of extended natural numbers with the usual order relation on \( \infty \). Let \( \langle \{ 0', \infty' \}, \leq' \rangle \) be another poset such that \( 0' <' \infty' \). Let \( f \) be the function

\[
 f : \mathbb{N} \cup \{ \infty \} \to \{ 0', \infty' \}
\]

\[
 n \mapsto 0'
\]

\[
 \infty \mapsto \infty'. \quad (C.6)
\]

Clearly, \( f \) is monotonic, because if \( n \leq m \) then \( f(n) = 0' \leq' f(m) = 0' \) and \( n \leq \infty \) means \( 0' \leq' \infty' \). However, \( f \) is not \( \omega \)-continuous. Consider the chain \( \{ n \}_{n \in \mathbb{N}} \) with limit \( \infty \). This chain is mapped to the constant sequence \( \{ 0' \}_{n \in \mathbb{N}} \) with limit \( 0' \). Therefore,

\[
 f \left( \bigvee_{n \in \mathbb{N}} \{ n \} \right) = f(\infty) = \infty' \leq' \bigvee_{n \in \mathbb{N}} f(n) = 0' \quad (C.7)
\]

is false.

**Definition C.44.** A poset is **directed-complete** (dcpo) iff every directed subset has its least upper bound in the poset.

**Definition C.45.** A poset is **pointed directed-complete** (cppo) iff it is a directed-complete poset with a least element.
We now recap a fixed point theorem in lattice theory: iterating an ω-continuous function \( f \) from a cppo to itself, starting from the bottom element, will eventually yield the least fixed point of \( f \).

**Theorem C.46. (Kleene’s fixed point theorem)** Let \( \langle D, \leq, \bot \rangle \) be a cppo and the function \( f : \langle D, \leq, \bot \rangle \to \langle D, \leq, \bot \rangle \) be ω-continuous. Let \( F = \bigvee_{n \in \mathbb{N}} f^n (\bot) \) be the supremum of the (\( \leq \)-increasing) chain \( \{f^n (\bot)\}_{n \in \mathbb{N}} \). Then \( F \) is the least fixed point of \( f \).

**Proof.** First we show that \( F \) is a fixed point of \( f \). Recall that \( f \) is ω-continuous.

\[
f(F) = f \left( \bigvee_{n \in \mathbb{N}} f^n (\bot) \right) = \bigvee_{n \in \mathbb{N}} f^{n+1} (\bot)
\]

\[
= \bigvee_{n \in \mathbb{N}^+} f^n (\bot) = \bot \vee \bigvee_{n \in \mathbb{N}^+} f^n (\bot) = F, \tag{C.8}
\]

as \( \bot \) is the identity of \( \vee \). Therefore, \( F \) is a fixed point of \( f \).

Then we show that \( F \) is the \( \leq \)-least fixed point. Let \( F' \) be any fixed point of \( f \), so \( f(F') = F' \). We show by induction that for all \( n \in \mathbb{N} \), \( f^n (\bot) \leq F' \).

1. (Base) For \( n = 0 \), \( f^0 (\bot) = \bot \leq F' \) by definition of it being a bottom element.

2. (Inductive) Assume \( f^k (\bot) \leq F' \). Apply \( f \) to both sides of the inequality to get \( f^{k+1} (\bot) \leq f(F') = F' \) as \( f \) is monotone and \( F' \) is a fixed point. Therefore, \( f^{k+1} (\bot) \leq F' \) as well.

By induction, this shows that \( F' \) is an upper bound of the chain \( \{f^n (\bot)\}_{n \in \mathbb{N}} \). As \( F \) is the supremum of this chain, by definition \( F \leq F' \). Therefore, \( F \) is the \( \leq \)-least fixed point of \( f \). \( \square \)
D  The Axiom of Choice in Abstract Argumentation Theory

In this appendix, we show that the axiom of choice is necessary to assert the existence of naive and preferred extensions for all AFs [20].

**Definition D.1.** Let $\mathcal{F}$ be any family of non-empty sets, i.e. $(\forall X \in \mathcal{F}) X \neq \emptyset$. A choice function is a function

$$f : \mathcal{F} \rightarrow \bigcup \mathcal{F}$$

(D.1)

that satisfies

$$\left( \forall X \in \mathcal{F} \right) f(X) \in X.$$  

(D.2)

**Definition D.2.** The axiom of choice (AC) states the following: any family $\mathcal{F}$ of non-empty sets has a choice function.

Intuitively, in a possibly infinite collection of non-empty sets, the axiom of choice permits one to choose exactly one element from each such set. We take as given that AC is equivalent to Zorn’s lemma (ZL) [16, 27]:

**Lemma D.3.** Zorn’s lemma (ZL): Let $\langle P, \leq \rangle$ be a poset. If every chain in $P$ has an upper bound in $P$, then $P$ has an $\leq$-maximal element.

Recall that AC (as ZL) is sufficient to show that naive extensions exist for all AFs (Theorem 3.32, page 26), and that preferred extensions exist for all AFs (Corollary 7.9, page 61). We now show that AC is also necessary for the existence of naive and preferred extensions. This was not stated explicitly in, e.g. [14, Corollary 12]

**Theorem D.4.** [20, Theorem 7] If all AFs $(A, R)$ have a naive extension, then AC is true.

**Proof.** Let $\mathcal{F}$ be any family of non-empty sets. We need to construct a choice function $f$. Recall that any function is a set of ordered pairs. The idea is to construct an AF from $\mathcal{F}$ such that its naive extensions correspond to choice functions on $\mathcal{F}$, hence verifying AC.

Let $A := \{(X, x) \mid X \in \mathcal{F}, x \in X\}$ be the arguments under consideration. Each argument picks a set $X \in \mathcal{F}$ and also an element $x \in X$. Let $R \subseteq A^2$ be defined as follows:

$$R((X_1, x_1), (X_2, x_2)) \iff [X_1 = X_2 \text{ and } x_1 \neq x_2].$$

(D.3)

Notice $R$ is a symmetric relation. The idea is that two such arguments attack each other if they disagree on which element of $X_1$ to choose. $(A, R)$ is a well-defined AF.

Note “any” means $\mathcal{F}$ can have a finite or any infinite number of these non-empty sets.
Let $N \subseteq A$ be a naive extension of $\langle A, R \rangle$, which exists our hypothesis. We show this corresponds to a choice function on $\mathcal{F}$. Given this $N$, suppose there is some $X \in \mathcal{F}$ such that $(\forall x \in X) (X, x) \notin N$. Then $N \cup \{(X, x)\} \supset N$ will be a conflict-free set - contradicting that $N$ is $\subseteq$-maximal conflict-free. Therefore, for every $X \in \mathcal{F}$, there is some $x \in X$ such that $(X, x) \in N$. Now suppose there is some $X \in \mathcal{F}$ such that for $x, y \in X$ distinct, $(X, x), (X, y) \in N$. But as $x \neq y$, the arguments $(X, x)$ and $(X, y)$ must be attacking, contradicting the conflict-freeness of $N$. It therefore follows that for each $X \in \mathcal{F}$, there is exactly one $x \in X$ such that $(X, x) \in N$. We can use this $N$, which exists, to define a choice function $f_N : \mathcal{F} \to \bigcup \mathcal{F}$

\[ X \mapsto f_N(X) = x \]

where $f_N(X) = x$ iff $(X, x) \in N$. Clearly, $(\forall X \in \mathcal{F}) f_N(X) \in X$, and $f_N$ is total on $\mathcal{F}$ and single-valued. As $\mathcal{F}$ is any family of non-empty sets, we have shown that this has a choice function. This verifies AC.

\textbf{Theorem D.5.} [20, Theorem 11] If all AFs $\langle A, R \rangle$ have a preferred extension, then AC is true.

\textbf{Proof.} Let $\mathcal{F}$ be any family of non-empty sets. We use the same construction as in Theorem D.4. Notice that the resulting AF is symmetric. By our hypothesis, this AF will have some preferred extension $P$, which is also naive by Corollary 7.27 (page 65). Therefore, the set of nodes in $P$ defines a choice function on $\mathcal{F}$ in the same manner as in the proof of Theorem D.4, thereby verifying AC.

We can summarise the preceding two theorems as follows:

\textbf{Corollary D.6.} TFAE:

1. The axiom of choice
2. Every AF has a naive extension.
3. Every AF has a preferred extension.

\textbf{Proof.} Theorem D.4 shows the equivalence of statements 1 and 2. Theorem D.5 shows the equivalence of statements 1 and 3. The result follows.

Note that (as usual) AC is not needed if we only consider finite AFs, e.g. in the context of modelling and implementing real argumentation.

\textbf{Theorem D.7.} All finite AFs $\langle A, R \rangle$ satisfy $\text{NAI} \neq \emptyset$ and $\text{PREF} \neq \emptyset$.

\textbf{Proof.} If $\langle A, R \rangle$ is finite, then $\langle \mathcal{P}(A), \subseteq \rangle$ is a finite poset. Hence $\text{CF}, \text{ADM} \subseteq \mathcal{P}(A)$ are also finite posets by inheriting $\subseteq$ on $\mathcal{P}(A)$. All finite posets have at least one maximal element (e.g. [13, page 16]). Therefore, $\text{max}_\subseteq \text{CF} =: \text{NAI} \neq \emptyset$ and $\text{max}_\subseteq \text{ADM} =: \text{PREF} \neq \emptyset$. 

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