Elliptic Wess-Zumino-Witten Model from Elliptic Chern-Simons Theory

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Abstract

This letter continues the program \cite{17,12,20,21} aimed at analysis of the scalar product of states in the Chern-Simons theory. It treats the elliptic case with group $SU_2$. The formal scalar product is expressed as a multiple finite dimensional integral which, if convergent for every state, provides the space of states with a Hilbert space structure. The convergence is checked for states with a single Wilson line where the integral expressions encode the Bethe-Ansatz solutions of the Lamé equation. In relation to the Wess-Zumino-Witten conformal field theory, the scalar product renders unitary the Knizhnik-Zamolodchikov-Bernard connection and gives a pairing between conformal blocks used to obtain the genus one correlation functions.

1 Introduction

A principal object of the conformal field theory are conformal blocks. For the group $SU_2$ Wess-Zumino-Witten (WZW) model of level $k$ and Virasoro central charge $c_k = \frac{3k}{k+2}$, on the elliptic curve $\mathbb{C}/2\pi(\mathbb{Z} + \tau \mathbb{Z})$, the conformal blocks are given by the expressions

$$\gamma_{\tau, z}(u) = \text{tr}_{\hat{V}_{k,l}} \left( \phi_{j_1}^{l_1} (z_N) \phi_{j_{N-1}}^{l_{N-1}} (z_{N-1}) \cdots \phi_{j_1}^{l_1} (z_1) \ q^{L_0-c_k/24} \ e^{4\pi i u J^3_0} \right).$$

(1)

Here $\phi_{j_1}^{l_1}(z) : \hat{V}_{k,l} \rightarrow \hat{V}_{k,l'} \otimes V_j$ are the primary fields of the model \cite{20} mapping the unitary highest weight irreducible modules $\hat{V}_{k,l}$ of the affine Kac-Moody algebra $\hat{sl}_2$ of level $k = 1, 2, \ldots$ and spin $l = 0, \frac{1}{2}, \ldots, \frac{k}{2}$. $V_j$ stands for the spin $j$ module of $sl_2$ and the conformal blocks take values in the vector space $\otimes_n V_{j_n}$. $L_0$ is the Virasoro generator and $J^3_0$ the Cartan subalgebra generator of $sl_2$ acting in $\hat{V}_{k,l}$. $q$ is shorthand for $e^{2\pi i u}$. 

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The elliptic conformal blocks $\gamma_{\tau,z}(u)$ given by Eq. (1) satisfy a second order differential equation \cite{4} generalizing the genus zero Knizhnik-Zamolodchikov (KZ) equations \cite{24}. In the absence of insertions, the genus one blocks coincide with the characters of the $\hat{\text{sl}}_2$ modules $\hat{V}_{k,l}$. For the case with insertions, free field realizations of modules $\hat{V}_{k,l}$ and of the primary fields \cite{29} allow to write integral representations for the genus one conformal blocks. An original approach was developed in \cite{9,10} where the elliptic conformal blocks were identified with vector-valued functions on the Kac-Moody group covariant w.r.t. the adjoint action of the loop group.

The genus one correlation functions of the WZW conformal field theory are sesqui-linear combinations of its conformal blocks. Geometrically, the conformal blocks may be viewed as holomorphic sections of (Friedan-Shenker \cite{14}) vector bundles over the Teichmüller space of punctured elliptic curves parametrized by $(\tau, z)$. It is a hermitian structure on the Friedan-Shenker bundle which determines the correlation functions.

There are various possible approaches to the construction of the Friedan-Shenker bundles for the WZW theory. In an algebraic approach, one considers the spaces of invariants (or coinvariants \cite{27}) arising by localization of irreducible representations of affine algebra $\hat{\text{sl}}_2$ at punctures of the complex curve. Alternatively, in an algebraic-geometric approach, one studies spaces of holomorphic sections of powers of the determinant bundle over the moduli space of $G$-bundles with parabolic structure at the punctures. Finally, in an approach inspired by the intrinsic relation between the two-dimensional conformal WZW field theory, and the three-dimensional topological Chern-Simons (CS) theory observed in \cite{30}, one considers spaces of quantum states of the group $G$ CS theory in the presence of Wilson lines. The above constructions are believed (and at least partially proven) to result in equivalent objects but different approaches lead to different insights into their nature and permit to use different technical tools in their analysis. For example, the Kac-Moody approach allows to apply the representation theory of the affine algebras and the determinant bundle approach the powerful tools of modular algebraic geometry. For genus one, the algebraic approach was recently extensively discussed (for general simple group) in \cite{15}.

One of the virtues of the approach via the CS theory is that the spaces of quantum states come with a formal scalar product which should provide the Friedan-Shenker bundles with a hermitian structure determining the correlation functions of the WZW theory. The scalar product of the CS states is given by a functional integral which, although not Gaussian, may be calculated by iterative Gaussian integration. This leads to finite-dimensional integral expressions obtained in \cite{17} (in the $SU_2$ case) and in \cite{12} (in the case of general simple group) for genus zero and in \cite{29} and \cite{21} for genus $>1$ (in the $SU_2$ case). For genus one, only the case with no insertions and general simple group was treated up to now in \cite{22} where it was shown that characters of the unitary representations of the Kac-Moody algebra are orthonormal with respect to the scalar product of the CS states. We partially fill this gap here by deriving the integral expressions for the scalar product of arbitrary elliptic CS states for the $SU_2$ group. For the case with a single Wilson line, we prove the convergence of the integrals and the compatibility of the scalar product with the Knizhnik-Zamolodchikov-Bernard (KZB) connection defining the differential equations satisfied by the conformal blocks. We also relate our integral expressions to those for conformal blocks derived by the free field techniques \cite{6}. For the genus zero case, the latter were used to
obtain the Bethe Ansatz solutions for simple spin chains \[3\] \[25\] and this relation has been recently extended to the elliptic case \[14\].

The paper is organized as follows. In Sect. 2, we recall the description of the elliptic CS states for the SU\(_2\) gauge group worked out in \[11\]. In Sect. 3, we perform the iterative Gaussian integral obtaining a finite-dimensional integral expression for the scalar product. In Sect. 3 we analyze the one-point case proving the unitarity of the KZB connection with respect to the scalar product of one-point states. Sect. 4 briefly illustrates the relation of our approach to the contour integral representations of conformal blocks and to the Bethe Ansatz on the example of elliptic one-point blocks. This discussion recaps the results of \[8\]. Finally, in Section 5, we explain in what sense our results provide an exact solution of the genus one WZW theory.

## 2 Elliptic CS states

In \[11\] we have described the space of states of the SU\(_2\) Chern-Simons theory on \(T^2 \times \mathbb{R}\) (\(T^2\) is the two dimensional torus and \(\mathbb{R}\) is the time axis) at level \(k\) and in the presence of Wilson lines in representations of spin \(j_n\), \(n = 1, \ldots, N\). We shall recall below the construction of this space.

A complex structure given by the coordinate \(z = \sigma_1 + \tau \sigma_2\) on \(T^2\) (\(\equiv \{(\sigma_1, \sigma_2) \in \mathbb{R}^2 / (2\pi \mathbb{Z}^2)\}\)), where \(\tau \equiv \tau_1 + i\tau_2\) with \(\tau_2 > 0\), induces a complex structure on the space of (smooth) SU\(_2\) connections on the torus and with the use of the latter one may carry out the Bargmann quantization of the theory. States \(\Psi\) are holomorphic functionals of the 0,1-component of the connection \(A^{01} \equiv A_{\bar{z}}\). They take values in the tensor product \(\otimes_n V_{j_n}\) of the representation spaces of SU\(_2\). Smooth maps \(g : T^2 \to SL_2\) act on \(A^{01}\) by complex gauge transformations

\[A^{01} \mapsto g A^{01} = g A^{01} g^{-1} + g \bar{\partial} g^{-1},\]

with \(\bar{\partial} = d\bar{z} \partial_z\). States corresponding to Wilson lines \(\{z_n\} \times \mathbb{R}\) should verify the chiral Ward identity

\[\Psi(g A^{01}) = e^{k S(g^{-1}, A^{01})} \otimes_n g(z_n)_{(n)} \Psi(A^{01})\]

(2)

where \(S(g, A^{01})\) is the action of the Wess-Zumino-Witten model coupled to \(A^{01}\), see \[11\], and the subsubscript \((n)\) indicates that the \(SL_2\) element acts on the space \(V_{j_n}\) of spin \(j_n\) representation.

Restricting \(\Psi\) to connections \(A^{01}_u = -u \sigma_3 d\bar{z} / (2\tau_2)\), \(u \in \mathbb{C}\), we can assign to every state a holomorphic map \(\gamma : \mathbb{C} \to \otimes_n V_{j_n}\) related to \(\Psi\) by the equation

\[\Psi(A^{01}_u) = e^{\pi k u^2 / \tau_2} \otimes_n (e^{\pi u^2 (z_n - z_n) / (2\tau_2)})_{(n)} \gamma(u)\]

(3)

It was shown in \[11\] that the correspondence between holomorphic maps \(\gamma\) and states \(\Psi\) is one to one provided that \(\gamma\) satisfies the following conditions

\[\gamma(u + 1) = \gamma(u)\]

(4,a)
\[
\gamma(u + \tau) = e^{-2\pi i k (\tau + 2u)} \otimes_n (e^{i u \sigma_3}) (u) \gamma(u), \tag{4.b}
\]
\[
0 = \oplus_n (\sigma_3)_n \gamma(u), \tag{4.c}
\]
\[
\gamma(-u) = \otimes_n (\nu)^n \gamma(u), \tag{4.d}
\]

and
\[
\partial_u^0 \partial_{v_1}^1 \ldots \partial_{v_N}^N \otimes_n \left( |j_n| \left( e^{u_\alpha \sigma_3} \right) (\alpha) \right) \gamma(u) \bigg|_{u=\nu=\exp[\alpha - \bar{\alpha} z_2]} = 0 \tag{5}
\]

for every \(N + 1\)-tuple of non negative integers with \(\sum_{n=0}^{N} l_n < \sum_{n=1}^{N} j_n\)

and \(\alpha = 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \).

\(w\) in \((4.d)\) is the generator of the Weyl group of \(SL_2\) and \(|j_n\rangle\) is the highest weight vector of the representation of spin \(j_n\) i.e. \((\sigma_3)_n |j_n\rangle = 2j_n |j_n\rangle\). We shall denote the space of maps \(\gamma(u)\) satisfying the above conditions by \(W_{\tau,z}\). The combined results of \([27][10]\) show that this space is isomorphic to the space of \(\text{Kac-Moody invariants}\) localized on the elliptic curve\(^1\).

### 3 Scalar product of CS states

We shall be interested in a study of the scalar product of Chern-Simons states. The latter will be expressed as a finite dimensional integral (the actual dimension will depend on the spin of the insertions) which, if finite, will provide \(W_{\tau,z}\) with a natural structure of a Hilbert space. The finiteness of the integral has been proven only in particular cases. Formally the scalar product in holomorphic quantization is given by the functional integral
\[
\| \Psi \|^2 = \int |\Psi(A^{01})|^2 \ e^{\frac{ik}{2}} \int \text{tr}(A^{10} A^{01}) \ DA \tag{6}
\]
where \(A^{10} = A_z dz\) with \(A_z = -A^*_z\). Our strategy for the calculation of the integral (6) will be as in \([22]\).

#### 3.1 Change of variables

The first observation is that an open dense set of connections \(A^{01}\) can be parametrized by \(g^{-1} A_u g\) with \(g\) a chiral gauge transformation and \(u\) in some fundamental domain of the action of translations \(u \rightarrow u + 1\) and \(u \rightarrow u + \tau\) and reflection \(u \rightarrow -u\) on \(\mathbb{C}\). With this restriction, the only ambiguity in the parametrization of \(A^{01}\) is left multiplication of \(g\) by a constant gauge transformation with values in the Cartan subgroup \(T^{\mathbb{C}}\). We shall see below how to treat this freedom.

Using Eq. (2) we obtain, see [22]
\[
|\Psi(A^{01})|^2 \ e^{\frac{ik}{2}} \int \text{tr}(A^{10} A^{01})
\]

\(^1\)we thank G. Felder for this remark
with $\zeta_u = e^{\sigma_3 u(z - \bar{z})/(2\tau_2)}$. Note that the right hand side of Eq. (7) depends only on the product $gg^*$. It is then natural to use the Iwasawa decomposition for $g \in SL_2$,

$$g = bU$$

with $U \in SU_2$ and

$$b = e^{\sigma_3 u^v \varphi/2},$$

for $v \in \mathbb{C}$ and $\varphi \in \mathbb{R}$. We may write Eq. (8) in the new variables as

$$|| \Psi ||^2 = \int \langle \Psi(A_u), \otimes_n ((gg^*)(z_n))^{-1}_{(u)} \Psi(A_u) \rangle j(u, g) \delta(\varphi(0)) \times e^{kS(\zeta_u gg^* \zeta_u^*) - \pi k(u^2 + \bar{u}^2)/\tau_2} j(u, g) \delta(\varphi(0)) Dg \, d^2u,$$

with $Dg = \Pi_z dg(z)$ the formal product of invariant measures on $SL_2$. The Jacobian from $A$ variables to $g$ and $u$ has been computed in [22]:

$$j(u, g) = C e^{4S(\zeta_u gg^* \zeta_u^*)} |\Pi(u, \tau)|^4$$

with $C$ a constant independent of $u$ and $\tau$ and

$$\Pi(u, \tau) = q^{1/8} \sin(2\pi u) \prod_{r=1}^{\infty} (1 - q^r)(1 - e^{4\pi i u q^r})(1 - e^{-4\pi i u q^r}),$$

The $\delta$-function in (10) fixes partially the ambiguity in the parametrization of $A^{01}$, the rest of the the freedom (left multiplication of $g$ by a constant gauge transformation with values in the Cartan subgroup $T$ of $SU_2$) is harmless: as the integrand is constant and $T$ is compact it only introduces an irrelevant normalization factor. The invariant measure of $SL_2$ in the parametrization of Eqs. (8) and (9) is

$$dg = d\varphi \, d^2(e^{\varphi} v) \, dU$$

with $dU$ the Haar measure of $SU_2$. The integrand of (11) does not depend on the $U$ variables. Their integration will give a factor independent of the other variables that we absorbe in the normalization. The action with this parametrization is given by

$$S(\zeta_u gg^* \zeta_u^*) = -\frac{i}{2\pi} \int (\partial \varphi)(\partial \bar{\varphi}) - \frac{i}{2\pi} \int e^{-2\varphi}(\partial \bar{\varphi}) \bar{v}(\partial \bar{\varphi}) + \frac{\pi}{\tau_2} (u - \bar{u})^2$$

where $\chi = (z - \bar{z})u/(2\tau_2)$, $v_u = v e^{(z - \bar{z})u/\tau_2}$ and $\varphi_\chi = \varphi + \chi + \bar{\chi}$. Notice that although $v$ is a function on the torus, $v_u$ is not but it has a monodromy in the $\tau$ direction depending on $u$

$$v_u(z + 2\pi) = v_u(z) \quad v_u(z + 2\pi \tau) = e^{4\pi i u}v_u(z).$$

Writing $(gg^*)^{-1}$ in terms of variables $v_u$ and $\varphi$,

$$(gg^*)^{-1} = e^{-\varphi} e^{-\varphi_3} e^{-\psi_3} = \zeta_u e^{-v_u - e^{-\varphi_3} e^{-\psi_3} \zeta_u}$$
and using the map $\gamma(u)$ introduced in relation (3), we obtain the following expression for the scalar product:

$$
\| \Psi \|^2 = C \int e^{-\frac{i(k+1)}{2\pi} \int (\partial \varphi)(\partial \varphi) + \int e^{-2\varphi}(\partial \varphi_{u})(\partial \varphi_{u}) + 2i\pi^2(u-\bar{u})^2/\tau_2} \\
\times \langle \gamma(u), \otimes_n \left( e^{-\delta_u \sigma} - e^{-\varphi_u \sigma_3} e^{-v_u \sigma_+} \right)(z_n)_{(n)} \gamma(u) \rangle D\varphi D(e^{\varphi}v) d^2u.
$$

(13)

### 3.2 $v$-integral

Now we must perform the functional integrals. First notice that the $v$-integral is Gaussian so it can be easily obtained using the two point function

$$
\int \nu_u(z) \bar{\nu}_u(z') e^{-i(k+4)(2\pi)^{-1} \int e^{-2\varphi}(\partial \varphi_{u})(\partial \varphi_{u})} D(e^{\varphi}v)
$$

(14)

$$
= \frac{\pi}{k+4} N(u,\tau) e^{i\pi^{-1} \int (\partial \varphi)(\partial \varphi) + 2i\pi^2(u-\bar{u})^2/\tau_2} \int e^{2\varphi}(y) P_u(z-y) D\varphi D(e^{\varphi}v) d^2y,
$$

where

$$
N(u,\tau) = (q\bar{q})^{1/24} |\Pi(u,\tau)|^{-2} \prod_{n=1}^{\infty} |1-q^n|^2
$$

and

$$
P_u(z) = -\frac{\vartheta_1'(0)}{\vartheta_1(\tau)} \vartheta_1(z) \vartheta_1(4\pi u)
$$

(15)

is the Green function of $\partial$ on functions with monodromies given by (12). Here $\vartheta_1$ is the theta-function s.t.

$$
\vartheta_1(z + 2\pi) = -\vartheta_1(z) \quad \vartheta_1(z + 2\pi \tau) = -e^{-i(z+\pi\tau)} \vartheta_1(z).
$$

(16)

It is odd, has simple zeros for $z \in 2\pi \mathbb{Z} + 2\pi \tau \mathbb{Z}$ and is non zero elsewhere. We shall take $\vartheta_1(z) = 2\Pi(\frac{z}{2\pi},\tau)$. $P_u(\cdot)$ is a meromorphic function for $u \notin (\mathbb{Z} + \tau \mathbb{Z})/2$ as one could expect from the fact that in those cases the line bundle defined by conditions (12) has no holomorphic sections different from zero.

Now let us insert into the matrix element $\langle \gamma(u), \cdots \gamma(u) \rangle$ a complete basis of $\otimes_n V_{j_n}$ composed of eigenvectors of every $(\sigma_3)_{(n)}$ with eigenvalues $2(p_n - j_n)$ and let us denote these vectors by $|p\rangle$ with $p \equiv (p_1, \ldots, p_N)$. Then

$$
\langle \gamma(u), \otimes_n \left( e^{-\delta_u \sigma} - e^{-\varphi_u \sigma_3} e^{-v_u \sigma_+} \right)(z_n)_{(n)} \gamma(u) \rangle
$$

$$
= \sum_{0 \leq p_n \leq 2j_n} \prod_n e^{2(j_n+p_n)\varphi_u(z_n)} |\langle p | \otimes_n \left( e^{-v_u \sigma_+} \right)(z_n)_{(n)} \gamma(u) \rangle|^2.
$$

(17)

We may perform the $v$-integral in Eq. (13) with the use of relations (14) and (17) obtaining

$$
\| \Psi \|^2 = C \sum_{p} \left( \frac{\pi}{k+4} \right)^K \frac{1}{K!} \int N(u,\tau) |\Pi(u,\tau)|^4
$$

6
\[ \times \left| \sum_{p' \in \mathbb{C}} F(z, y, p') \langle p | \otimes_{n} \tilde{p}_{n} \rangle \right|^2 \]

\[ \times \int \prod_{s=1}^{K} \frac{1}{2 \pi} \frac{1}{2 \pi} \prod_{n=1}^{N} e^{2(j_{n} - p_{n}) \varphi_{n}(z_{n})} e^{-i(k+2) \int (\partial \varphi)(\partial \varphi) + 2 \pi^{2}(u - \bar{u})^{2}/\tau_{2}} \delta(\varphi(0)) D\varphi d^{2K} y d^{2}u. \]

The condition

\[ \sum_{n=1}^{N} p'_{n} \equiv |p'| = K \equiv \sum_{n=1}^{N} (p_{n} - j_{n}) \]

in the second sum of (18) is a consequence of the invariance of \( \gamma(u) \) under the diagonal action of the Cartan subgroup.

\[ F(z, y, p') = \sum_{\rho \in S_{K}} \prod_{s=1}^{K} P_{\rho}(z_{n}^{p'} - y_{\rho(s)}) \] (19)

where \( S_{K} \) is the symmetric group, \( y \equiv (y_{1}, \ldots, y_{K}) \) and \( z_{n}^{p'} \equiv (z_{1}, \ldots, z_{1}, z_{2}, \ldots, z_{2}, \ldots, z_{N}) \) with every \( z_{n} \) repeated \( p'_{n} \) times.

### 3.3 \( \varphi \)-integral

We still have to evaluate the \( \varphi \)-integral,

\[ \int \prod_{s=1}^{K} \frac{1}{2 \pi} \frac{1}{2 \pi} \prod_{n=1}^{N} e^{2(j_{n} - p_{n}) \varphi_{n}(z_{n})} e^{-i(k+2) \int (\partial \varphi)(\partial \varphi) \delta(\varphi(0)) D\varphi} \] (20)

corresponding to a Coulomb gas system of zero total charge. As noted before, the neutrality is a consequence of the invariance of \( \gamma(u) \) under the diagonal Cartan subgroup. The Green function \( G(z - z') \) of the operator \( \partial_{z} \partial_{\bar{z}} \) restricted to functions on the torus orthogonal to constants is

\[ G(z) = \frac{\ln |\vartheta_{1}(z)|^{2}}{\pi} + \frac{(z - \bar{z})^{2}}{8\pi^{2}\tau_{2}} + \text{const.} \] (21)

The Gaussian integral (20) is equal to

\[ C(q\bar{q})^{-1/24} \tau_{2}^{-1/2} \prod_{n=1}^{\infty} (1 - q^{n})^{-2} \prod_{s} e^{2(\chi(y_{s}) + \bar{\chi}(y_{s}))} \prod_{n} e^{2(j_{n} - p_{n})(\chi(z_{n}) + \bar{\chi}(z_{n}))} \]

\[ \times e^{-\frac{k+2}{2\pi} \int f(z)G(z - z')f(z')} \] (22)

with

\[ f = \sum_{n} 2(j_{n} - p_{n}) \delta_{z_{n}} + \sum_{s} 2\delta_{y_{s}}. \]

The Green function behaves as \( \pi^{-1} \ln |z - z'|^{2} \) when \( z \to z' \) and, consequently, expression (22) diverges due to the terms diagonal in the \( \delta \)-functions of \( f \). These divergences may be
regularized by splitting the coinciding points to distance \( \epsilon \) and removed by multiplicative renormalization of the scalar product. The divergence is proportional to

\[
\epsilon^{-\frac{2}{k+2}} \left( \sum_n (p_n - j_n)^2 + \sum_n (p_n - j_n) \right).
\]

It is the most severe for \( p_n = 2j_n \) since \( 0 \leq p_n \leq 2j_n \). We shall then renormalize the point-split version of \((22)\) by multiplication by \( \epsilon^{\frac{2}{k+2}} \sum_n j_n (j_n + 1) \) and taking \( \epsilon \) to zero. The effect of the renormalization is to remove all terms in the sum over \( p \) in Eq. \((18)\) except those with the most severe divergence, i.e. \( p_n = 2j_n \).

Combining expressions \((18)\) and \((22)\), we obtain, after the renormalization,

\[
\| \Psi \|^2 = C \tau^{-1/2} \int e^{(\pi(k+2)\tau)^{-1} (\Theta - \overline{\Theta})^2} |\omega(z,y,u,\gamma)|^2
\]

with

\[
\Theta = \pi(k + 2)u + \frac{1}{2} \sum_s y_s - \frac{1}{2} \sum_n j_n z_n
\]

and \( \omega \) the multivalued form

\[
\omega(z,y,u,\gamma) = \partial_1'(0) - \frac{1}{\pi^2} \sum_n j_n (j_n + 1) \prod_{n < n'} \partial_1(z_n - z_{n'})^{-2j_n j_{n'}/(k+2)} \prod_{s,n} \partial_1(y_s - z_n)^{2j_n/(k+2)} \prod_{s < s'} \partial_1(y_s - y_{s'})^{-2/(k+2)} \Pi(u,\tau) dy_1 \wedge \cdots \wedge dy_J \wedge du \]

\[
\times \sum_p F(z,y,p) \otimes_n \left( \langle j_n | (p_n!)^{-1} (\sigma_+)^{p_n} \rangle \gamma(u) \right),
\]

where \( J = \sum_n j_n \). Note that \( |\omega(z,y,u,\gamma)|^2 \) is univalued in \( z_n, y_s, 2\pi u \in T^2 \). Eq. \((23)\) expresses the scalar product of current blocks as a finite dimensional integral in variables \( 2\pi u \) and \( y_s \) over \((T^2)^{1+J}\).

4 Convergence of integrals

The expression under the integral in \((23)\) has apparent non integrable divergencies. It is expected that singularities cancel or at least become integrable when \( \gamma \) satisfies conditions \((1)\) and \((3)\). For instance, we shall see below that, although \( \omega \) in \((23)\) seems to diverge at \( u = 0, 1/2, \tau/2, (\tau + 1)/2 \), it actually depends on \( u \) in a holomorphic way.

Before going to the general case let us study a simpler one with a unique insertion. This case may be easily controlled. Suppose that the insertion at \( z \) has spin \( j \) (necessarily \( j \) is an integer). We shall denote by \( \gamma_j \) the vector valued holomorphic map associated to the state \( \Psi_j \) by equation \((3)\). Function

\[
\theta(u) \equiv \langle j | (\sigma_+)^j \gamma_j(u) \rangle \Pi(u,\tau)
\]

\[(26)\]
is a theta-function of degree $2(k+2)$ satisfying
\[
\theta(u + 1) = \theta(u), \quad \theta(u + \tau) = e^{-2\pi i (k+2)(\tau + 2u)} \theta(u).
\] (27)

An explicit basis of such functions is formed by expressions
\[
\theta_n(u, \tau) = \sum_{r=-\infty}^{\infty} q^{(k+2)(\frac{n^2}{2(k+2)} + r)} e^{4\pi i u (k+2)(\frac{n^2}{2(k+2)} + r)}
\] (28)
with $n = 0, 1, \ldots, 2k+1$. $\theta_n$ satisfy the heat equation
\[
(\partial_\tau + \frac{i}{8\pi(k+2)} \partial_u^2) \theta_n = 0.
\] (29)

Besides, the functions $\theta$ coming from Eq. (26) have to have a definite parity
\[
\theta(-u) = (-1)^{j+1} \theta(u)
\]
and have to obey the selection rules [11]
\[
\partial_u^l \theta_j(u)|_{u=\alpha} = 0 \quad \text{for } l \leq j \text{ and } \alpha = 0, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}.
\] (30)

That leaves a $(k+1-2j)$-dimensional space of solutions. Labeling states $\gamma_j$ with one insertion by the theta-functions $\theta(u)$, we obtain from Eq. (23)
\[
\omega((y_s - z), u, \theta) = \frac{j!}{\pi^j} \theta(u) \prod_{s=1}^{j} P_u(z - y_s) \prod_{s < s'} \tilde{\vartheta}_1(y_s - y_{s'})^{-2j/(k+2)}
\]
\[
\times \prod_{s=1}^{j} \tilde{\vartheta}_1(z - y_s)^{2j/(k+2)} \prod_{s=1}^{j} \tilde{\vartheta}_1(z - y_s - 4\pi u)
\]
\[
\times \prod_{s=1}^{j} \tilde{\vartheta}_1(z - y_s)^{1+2j/(k+2)} dy_1 \wedge \cdots \wedge dy_j \wedge du,
\] (31)
where $\tilde{\vartheta}_1(y) \equiv \vartheta_1(y)/\vartheta_1'(0)$. The $y$ integral in Eq. (23) has a positive degree of divergence when $y_s \to z$ (recall that $\tilde{\vartheta}_1(y) \sim y$ when $y \to 0$) and it may be easily shown to converge. As for the $u$-dependence, the singular behavior $\sim (u - \alpha)^{-j}$ around $u = \alpha$ coming from $\tilde{\vartheta}_1(4\pi u)^{j}$ is regularized due to the conditions (30) for $\theta_j$ which guarantee that $\omega$ is holomorphic in $u$. We infer the convergence of the integral on the right hand side of Eq. (23):

the one-point states are normalizable.

Note that due to the presence of $\Pi(u, \tau)$ vanishing at $u = \alpha$ in the expression for $\omega$, the normalizability would still hold if we replaced the condition $l < j$ in (30) by $l < j - 1$ which, however, given the parity properties of the states, implies the stronger condition. Hence, for the one-point states, the normalizability is equivalent to the fusion conditions (30). We expect this to hold for general states.
Let us return to the general case and let us study the dependence of \( \omega \) on \( u \). We only have to examine the last line of Eq. (25) since the possible divergencies in \( u \) come from the zeros of \( \vartheta_1(u) \) in the denominator of \( P_u(z) \) in \( F \). Consider first the case \( u \to 0 \) and write

\[
P_u(z_n - y_s) = - \frac{1}{4\pi^2 u} + f_{n,s}(u)
\]

with \( f_{n,s} \) analytic around \( u = 0 \). Let us examine for fixed \( p \) the coefficient in \( F(z, y, p) \) of the term

\[
\left( \frac{-1}{4\pi^2 u} \right)^{J-L} f_{1,1}f_{1,2} \cdots f_{1,l_1}f_{2,l_1+1} \cdots f_{2,l_2} \cdots \cdots f_{N,L}
\]

where \( L = \sum_n l_n \). Denote this coefficient by \( c_{p,l} \) and call \( C_l \) similar coefficient arising after the sum over \( p \) in the last line of Eq. (25). With a little combinatorics one easily obtains

\[
c_{p,l} = (J-L)! \frac{p_1! \cdots p_N!}{(p_1 - l_1)! \cdots (p_N - l_N)!}
\]

and adding up all contributions from different \( p \)'s one has

\[
C_l = (J-L)! \sum_{\{p\} = j} \left( \langle j_n | (p_n - l_n) \rangle^{-1}(\sigma_+)^{p_n} n \right) \gamma(u)
\]

\[
= (J-L)! \partial_{v_1} \cdots \partial_{v_N} \otimes_n \left( \langle j_n | (e^{v_n\sigma_+})_{(n)} \rangle \gamma(u) \right)_{v_n=1}.
\]

From condition (5) we know that the last expression vanishes at least as \( u^{J-L} \) when \( u \to 0 \). Identical computation shows regularity of \( \omega \) at \( u = 1/2 \) and a slight modification of it allows to deal with cases \( u = \tau/2, (1 + \tau)/2 \). The final conclusion is that \( \omega \) depends holomorphically on \( u \) provided that conditions (5) are satisfied. That in the general case the singularities in \( y \)'s become integrable remains still to be proven.

## 5 Unitarity of KZB connection

The spaces of Chern-Simons states \( W_{\tau,z} \) form a holomorphic vector bundle \( W_{\tau,z} \) over the space \( \mathbb{C} \times (\mathbb{C}^N \setminus \Delta_N) \), where \( \Delta_N \) contains the \( N \)-tuples \((z_1, \ldots, z_N)\) with coincidences modulo \( 2\pi(\mathbb{Z} + \tau\mathbb{Z}) \). A (smooth) family of scalar products on \( W_{\tau,z} \) provides bundle \( W_{\tau,z} \) with a hermitian structure. On the other hand, \( W_{\tau,z} \) may be equipped with a holomorphic Knizhnik-Zamolodchikov-Bernard (KZB) connection first described in the elliptic case in [4], see [11] for its presentation using the same formalism as the one employed here. It is related to elliptic versions of integrable models [10] [15]. It has been conjectured in [17] [18] that the KZ connection and its higher genus generalizations (see [3] [23] [1]) are unitary i.e. that there exists a hermitian structure preserved by the parallel transport and that, moreover, such a hermitian structure is provided by the rigorous version of the formal scalar products [1] of Chern-Simons states. It is a simple fact that a hermitian structure determines a unique holomorphic unitary connection. This way, the hermitian structure preserved by the KZ connection, if existent, may be regarded as a more basic object that
the connection itself. The conjecture was proven for special insertions for genus zero. The main obstruction in proving it for general insertions was the lack of control of the convergence of the finite-dimensional integral expressions to which the functional integral in (3) was reduced for genus zero in [17] and for genera \( \geq 2 \) in [20][21]. For genus one and no insertions, the conjecture follows from the orthogonality of the Kac-Moody characters in the scalar product (24), established in [22], and from the well known fact that the characters satisfy a heat equation on the moduli space. Below, we shall provide a further support for the conjecture by proving it for genus one case with one insertion.

For a single insertion of spin \( j \) at point \( z \), the KZB connection is given by the formulae [11]

\[
\nabla_{\bar{z}} \theta(u) = \partial_{\bar{z}} \theta(u),
\]

\[
\nabla_{z} \theta(u) = \partial_{z} \theta(u),
\]

\[
\nabla_{\tau} \theta(u) = \partial_{\tau} \theta(u),
\]

\[
\nabla_{\bar{\tau}} \theta(u) = \left( \partial_{\bar{\tau}} + \frac{i}{8\pi(k+2)} \partial_{u}^{2} - \frac{\pi ij(j+1)}{2(k+2)} \partial_{u} P_{u}(y)|_{y=0} \right) \theta(u).
\]

The potential

\[
\partial_{u} P_{u}(y)|_{y=0} = 4 P(4\pi u) - 8 \sum_{n=1}^{\infty} \frac{q^{n}}{(1-q^{n})^{2}} + \frac{1}{3},
\]

where \( \mathcal{P}(y) \) is the Weierstrass function (meromorphic function on \( T^{2} \) with the only pole at 0 where \( \mathcal{P}(y) = y^{-2} + O(y^{2}) \)). Hence the \( \tau \)-component of the KZB connection contains the Lamé operator \( \partial_{u}^{2} + c \mathcal{P}(4\pi u) \) (which is replaced by the elliptic Calogero-Sutherland operator for the \( SU\_n \) group [10]).

The integral (23) giving the scalar product of the states has a natural geometric interpretation which we shall spell out for the one-point case. \( (j+1) \)-form \( \omega(y, u, \theta) \) may be viewed as a holomorphic form on \( T^{2} \times ((T^{2})^{j} \setminus \Delta_{j}) \) with values in a complex line bundle \( L \) (\( \Delta_{j} \) contains \( j \)-tuples \( y \) with \( y_{s} = y_{s}' \) for some \( s \neq s' \) or with \( y_{s} = 0 \) for some \( s \)). By definition, the sections \( \sigma \) of \( L \) have the same transformation properties under \( y_{s} \mapsto y_{s} + 2\pi, 2\pi \tau, u \mapsto u + 1, \tau \) and under the pure braiding of \( y_{s} \) as \( \omega \). The univalued expression

\[
e^{(\pi(k+2)r_{2})^{-1}(\Theta-\bar{\Theta})^{2}} |\sigma(y, u)|^{2}
\]

defines a hermitian structure on \( L \) which, in turn, determines a unique unitary connection

\[
\nabla_{y_{s}} \sigma = \partial_{y_{s}} \sigma,
\]

\[
\nabla_{y_{s}} \sigma = \left( \partial_{y_{s}} + \frac{1}{\pi(k+2)r_{2}} (\Theta - \bar{\Theta}) \right) \sigma,
\]

\[
\nabla_{u} \sigma = \partial_{u} \sigma,
\]

\[
\nabla_{u} \sigma = \left( \partial_{u} + \frac{2}{r_{2}} (\Theta - \bar{\Theta}) \right) \sigma.
\]
The connection acts naturally on differential forms with values in \( \mathcal{L} \) and its unitarity implies that, if \( \eta, \chi \) are two \( \mathcal{L} \)-valued forms of degree \( j+1 \) and \( j \), respectively, and \( \eta \) is holomorphic then

\[
d \left( e^{\pi(k+2)r_2} (\Theta - \overline{\Theta})^2 \eta \chi \right) = e^{\pi(k+2)r_2} (\Theta - \overline{\Theta})^2 \eta \nabla \chi.
\]  

Consequently,

\[
\int e^{\pi(k+2)r_2} (\Theta - \overline{\Theta})^2 \eta \nabla \chi = 0
\]

if the integral converges and there are no contributions from coinciding \( y_s \) or \( y_s = 0 \) to the integral of the left hand side of Eq. (35). This observation will be at the core of an argument showing that the KZB connection is unitary.

The unitarity of the KZB connection with respect to the hermitian structure defined by the one-point version of the formula \([\overline{\mathcal{L}}] \) is equivalent to the following relations:

\[
\frac{d}{d \tau} \| \Psi \|^2 = C \tau_2^{1/2} \int e^{\pi(k+2)r_2} (\Theta - \overline{\Theta})^2 \omega(y, u, \theta) \omega(y, u, \partial_\tau \theta),
\]

\[
\frac{d}{d \tau} \| \Psi \|^2 = C \tau_2^{1/2} \int e^{\pi(k+2)r_2} (\Theta - \overline{\Theta})^2 \omega(y, u, \theta) \omega(y, u, \nabla_\tau \theta),
\]

where we assumed that \( \theta \) depends holomorphically on \( z \) and \( \tau \). \( \Theta \) is given by Eq. (24) with \( z \) set to zero. The first of these equations is obvious since, after the shift \( y_s \mapsto y_s + z \), the \( z \)-dependence under the integral appears only in \( \theta \). The proof of the second relation is more involved because the \( \tau \)-dependence occurs both under the integral and in the integration domain parametrized by \( y_s, 2\pi u \in 2\pi(\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})) \). Reparametrizing \( y_s \) as \( y_s + \tau t_s \) and \( 2\pi u \) as \( \chi + \tau \phi \), we obtain by differentiation under the integral

\[
\frac{d}{d \tau} \| \Psi \|^2 = C \tau_2^{1/2} \int e^{\pi(k+2)r_2} (\Theta - \overline{\Theta})^2 \omega(y, u, \theta)
\]

\[
\times \left( \partial_\tau + \frac{u - \bar{u}}{2\tau_2} \partial_u + \sum_{s=1}^j \partial_{y_s} \frac{y_s - \bar{y}_s}{2\tau_2} - i \frac{1}{2\pi(k+2)r_2} (\Theta - \bar{\Theta})^2 - i \frac{1}{4\tau_2} \right) \omega(y, u, \theta).
\]

The differentiation under the integral is easily substantiated since it does not spoil the integrability. Writing

\[
\omega(y, u, \theta) \equiv \theta(u) \bar{\omega}(y, u),
\]

see Eq. (31), we obtain after a straightforward but somewhat tedious algebra:

\[
\left( \partial_\tau + \frac{u - \bar{u}}{2\tau_2} \partial_u + \sum_{s=1}^j \partial_{y_s} \frac{y_s - \bar{y}_s}{2\tau_2} - i \frac{1}{2\pi(k+2)r_2} (\Theta - \bar{\Theta})^2 - i \frac{1}{4\tau_2} \right) \omega(y, u, \theta)
\]

\[
= (\nabla_\tau \theta(u)) \bar{\omega}(y, u) + \nabla \chi(y, u) + \theta(u) \psi(y, u),
\]

where the \( j \)-form \( \chi \) and the \((j+1)\)-form \( \psi \) with values in the bundle \( \mathcal{L} \) are given by the following explicit expressions:

\[
\chi(y, u)
\]
factors, they act on forms with an improved y dependence, the form \( \varphi(y, u) \) in which

\[
\tilde{\omega}(y, u) \text{ stands for } \omega(y, u) \text{ without the } du \text{ differential and }
\]

\[
\psi(y, u) = \left( \partial_y - \frac{i}{\pi \sigma(k+2)} \partial_u^2 + \frac{\pi i(j+1)}{2(k+2)} \partial_y \varphi(y) \right) \tilde{\omega}(y, u)
\]

\[
+ \frac{j}{2} \sum_{s=1}^{j} \partial_{y_s} \left( (\partial_y \varphi(-y_s)) \prod_{s' \neq s} P_u(-y_{s'}) R_j(y)^{\frac{2}{k+2}} dy_1 \wedge \ldots \wedge dy_j \wedge du \right)
\]

(41)

in which \( \tilde{\omega} \) stands for \( \omega \) without the \( du \) differential and

\[
R_j(y) = \prod_{s < s'} (y_s - y_{s'})^{-1} \prod_{s=1}^{j} \tilde{\varphi}_1(y_s)^j.
\]

(42)

First note that \( \nabla \chi(y, u) \) does not contribute to the integral on the right hand side of Eq. (38) due to the relation (39). This, in fact, requires some analysis. We should prove that the corresponding terms are integrable and do not lead to boundary contributions. As for the \( u \)-dependence, the form \( \chi \) is still holomorphic in \( u \). Although the \( u \)-derivatives increase the degrees of the poles at \( u = \alpha \) by one, the fusion rules (31) still guarantee the regularity. As for the \( y_s \)-behavior, note that, although the \( y_s \)-derivatives produce additional \( \sim y_s^{-1} \) factors, they act on forms with an improved \( y_s \) behavior (\( \partial_y \varphi(-y_s) \) is regular at \( y_s = 0 \)).

The same argument is used to show that there are no boundary contributions from \( y_s = 0 \). The additional \( \sim (y_s - y_{s'})^{-1} \) factors which may be produced by the \( y_s \)-differentiation also do not spoil the integrability and there are no boundary contributions from \( y_s = y_{s'} \) (in showing this one uses the fact that such terms may be present only for \( k > 4 \) and that, due to the fusion rule, \( j < \frac{k}{2} \)). We leave the details to the reader.

In order to complete the proof of relation (37) and of the unitarity of the KZB connection for one-point insertions, we shall show that \( \psi(y, u) \), given by Eq. (41), vanishes. We have

\[
\psi(y, u) = \left( \sum_{s=1}^{j} ((\partial_y + \frac{i}{2} \partial_y \partial_u) P_u(-y_s)) \prod_{s' \neq s} P_u(-y_{s'}) \right.
\]

\[
+ \frac{2}{k+2} U_j(y, u) \left. R_j(y)^{\frac{2}{k+2}} dy_1 \wedge \ldots \wedge dy_j \wedge du \right,
\]

where

\[
U_j(y, u) \equiv \prod_{s=1}^{j} P_u(-y_s) \partial_y \ln R_j(y) - \frac{i}{16} \sum_{s=1}^{j} (\partial_y^2 P_u(-y_s)) \prod_{s' \neq s} P_u(-y_{s'})
\]

\[
- \frac{i}{8} \sum_{s < s'} (\partial_y P_u(-y_s)) (\partial_u P_u(-y_{s'})) \prod_{s'' \neq s, s'} P_u(-y_{s''})
\]

\[
+ \frac{i}{2} \sum_{s=1}^{j} (\partial_y^2 P_u(-y_s)) \prod_{s' \neq s} P_u(-y_{s'}) \partial_{y_s} \ln R_j(y)
\]

(40)
\[ F(y) \equiv (\partial_\tau + i\partial_y \partial_u)P_u(-y) \] is a holomorphic function of \( y \) such that
\[ F(y + 2\pi) = F(y), \quad F(y + 2\pi \tau) = e^{-4\pi i u} F(y). \]
Hence it vanishes. Similarly,
\[ U_j((y'_s + 2\pi \delta_{ss'}, y), u) = U_j(y, u), \]
\[ U_j((y'_s + 2\pi \tau \delta_{ss'}, y), u) = e^{-4\pi i u} U_j(y, u). \]

6 Integral representations of conformal blocks and Bethe Ansatz

It was remarked in [17][19] that the integral expressions for the scalar products of the genus zero Chern-Simons states are very closely related to the integral expressions [6][28] for the genus zero conformal blocks, i.e. for the sections horizontal w.r.t. the KZ connection.
The integral expressions for the conformal blocks are the consequence of the Wakimoto (free field) realization of the current algebra [29] and they are given by contour integrals of the genus zero counterpart of the forms \( \omega \) with \( k + 2 \) replaced by \( -(k + 2) \). Similarly, one may use the results obtained above to write down integral expressions for the genus one one-point conformal blocks, see also [6][8]. Namely, consider a vector of holomorphic \( j \)-forms with components
\[ \Omega_n(y, u) = \theta_n(u - \frac{\pi i (j + 1)}{4(k + 2)} \sum_s y_s) \hat{\omega}'(y, u) \]
where the theta-functions \( \theta_n \) are given by Eq. (28) and
\[ \hat{\omega}'(y, u) = \frac{j!}{\pi} \prod_{s=1}^j P_u(-y_s) R_j(y) \frac{\pi i}{k} dy_1 \wedge \ldots \wedge dy_j. \]
Note that \( \hat{\omega}' \) is same as \( \hat{\omega} \) used in Sect. 5 except for the flip of sign of \( k+2 \). In its dependence on \( y \), \( \Omega \) may be viewed as a holomorphic \( j \)-form on \((T^2)^j \setminus \Delta_j\) with values in a flat vector bundle. The one-point genus one conformal blocks may be expressed by the integrals
\[ \int_c \Omega_n(\cdot, u) \]
of the components of \( \Omega \) over cycles \( c \) of a homology with values in the dual bundle, see [6][15]. The horizontality of such integrals (in their dependence on the modular parameter \( \tau \) w.r.t. the KZ connection follows from the relation
\[ \left( \partial_\tau + \frac{i}{8\pi(k+2)} \partial_u^2 - \frac{\pi i j (j+1)}{2(k+2)} \partial_u P_u(y) \right) \bigg|_{y=0} \right) \Omega_n(y, u) \]
which is an immediate consequence of Eq. (29) and of the vanishing of the form \( \psi' \) given by Eq. (41) with the sign of \( k + 2 \) inverted.

Papers \[2\]\[3\]\[25\] have remarked that the integral expressions for the genus zero conformal blocks give rise, in the limit \( k \to -2 \) (\( k \to -d \)ual Coxeter number, for general simple groups) to Bethe Ansatz solutions for the Gaudin spin chains, see also \[13\]. This relation has its elliptic counterpart \[8\]\[14\]. In order to illustrate it on the example of the one-point functions, following \[8\], let us note that the identity

\[
\mathcal{U}_j(y, u) \equiv 0 \quad \text{with} \quad \mathcal{U}_j \text{ given by Eq. (43)},
\]

implies that

\[
\sum_{s=1}^{j} \partial_u \left( \theta_n(u) \partial_u P_u(-y_s) \right) \prod_{s' \neq s} P_u(-y_{s'}) R_j(y)^{\frac{1}{\sqrt{j}}} dy_1 \wedge \ldots \wedge dy_j
\]

if \( y \) is a stationary point of \( \ln R_j \), i.e. if

\[
\sum_{s=1}^{j} \frac{\theta_1'(y_s)}{\theta_1(y_s)} = \sum_{s' \neq s} \frac{\theta_1'(y_s-y_{s'})}{\theta_1(y_s-y_{s'})}
\]

for \( s = 1, \ldots, j \). Eq. (46) is the eigenvalue equation for the Lamé operator

\[
L \equiv \partial_u^2 - 4\pi^2 j(j + 1) \partial_u P_u(y) |_{y=0} \quad \text{or} \quad L' \equiv \partial_u^2 - 16\pi^2 j(j + 1) P(4\pi u).
\]

It states that the Bethe Ansatz meromorphic "multi-particle" wave function

\[
\psi_y(u) = \prod_{s=1}^{j} P_u(-y_s),
\]

built of one-particle functions \( u \mapsto P_u(-y_s) \), is an eigenfunction of \( L \) or \( L' \) with eigenvalue

\[
\lambda_y = -16\pi i \partial_u \ln R_j(y)
\]

or

\[
\lambda'_y = \lambda_y + 32\pi^2 j(j + 1) \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} - \frac{j}{12},
\]

respectively, if \( y_s \) satisfy Eq. (47). Note that

\[
\psi_y \simeq \left( \frac{-1}{4\pi^2 u} \right)^j \quad \text{as} \quad u \to 0
\]

and that

\[
\psi_y(u + \frac{1}{2}) = \psi_y(u), \quad \psi_y(u + \frac{\tau}{2}) = e^{-i \sum_s y_s} \psi_y(u)
\]

so that the eigenfunction \( \psi_y \) corresponds to the quasi-momenta \( 0 \) and \( \sum_s y_s \) along the two cycles of the torus. The above relations reduce the calculation of the eigenfunction of the Lamé operator \( L \) with properties (48) and (49) to solving the Bethe Ansatz equations (47).
7 Conclusions

We have reduced in Sect. 3 the formal functional-integral formula for the scalar product of elliptic CS states to finite-dimensional integral expressions. Under the assumption of convergence of the integrals, checked only for states with no or with a single Wilson line, our result determines a Hilbert space structure on the space of CS states. This structure allows, in turn, to define the genus one correlation functions of the ”diagonal” (A-series, in the terminology of [7]) SU\textsubscript{2} WZW model in the external field $A = -(A^0_1)^* + A^0_1$. They are given by the expressions

$$\sum_\alpha e^{\pi k(u-\bar{u})^2/\tau} \left( \otimes_n e^{i\sigma_3 u(z_n-\bar{z}_n)(2\tau_2)} \gamma_\alpha(u) \right) \otimes \left( \otimes_n e^{i\sigma_3 u(z_n-\bar{z}_n)(2\tau_2)} \gamma_\alpha(u) \right), \quad (50)$$

where $(\gamma_\alpha)$ corresponds through Eq. (3) to an orthonormal basis of the elliptic CS states. The correlation functions for the generic unitary gauge field $A = -(A^0_1)^* + A^0_1$ with $A^0_1 = g^{-1} A^0_1 u$ are obtained by acting on (50) by

$$e^{kS(g^*, -(A^0_1)^* + A^0_1)} \left( \otimes_n g(z_n)^{-1} \right) \otimes \left( \otimes_n g(z_n)^{-1} \right).$$

The above expressions are independent of the choice of the orthonormal basis $(\gamma_\alpha)$ of states and may be computed using an arbitrary basis by replacing

$$\sum_\alpha \gamma_\alpha \otimes \bar{\gamma}_\alpha \quad \text{by} \quad \sum_{\alpha,\beta} H^{\beta\alpha} \gamma_\alpha \otimes \bar{\gamma}_\beta,$$

where the matrix $(H^{\alpha\beta})$ is the inverse of the matrix of scalar products $H_{\alpha\beta} = (\gamma_\alpha, \gamma_\beta)$. Hence the calculation of the correlation functions reduces to the computation of the scalar products of elliptic CS states. The latter are given by (the polarized version of) Eq. (23) and require calculation of the finite-dimensional integrals. This way, our result provides an exact solution for the elliptic correlation functions of the diagonal SU\textsubscript{2} WZW model. For the non-diagonal D-series version of the model which corresponds to the SO\textsubscript{3} WZW theory, the correlation functions are given by similar formulae but with non-overlined $\gamma_\alpha$ replaced by

$$\frac{1}{2} \sum_\rho \rho \gamma_\alpha$$

where $\rho$ runs through the group $\text{Hom}(\pi_1(T^2 \setminus \{z_1, \ldots, z_N\}), \mathbb{Z}_2)$ which acts projectively and unitarily on the space of elliptic CS states, see [16] for the discussion of the case without insertions. In both diagonal and non-diagonal WZW theories, the scalar product of CS states is the essential structure which allows to determine the correlation functions. It is ultimately related to the KZ connection and its higher genus generalizations as well as to the Bethe Ansatz. It certainly deserves further study.

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