On Dynamic Coloring of Graphs

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Abstract

A dynamic coloring of a graph \(G\) is a proper coloring such that for every vertex \(v \in V(G)\) of degree at least 2, the neighbors of \(v\) receive at least 2 colors. In this paper we present some upper bounds for the dynamic chromatic number of graphs. In this regard, we shall show that there is a constant \(c\) such that for every \(k\)-regular graph \(G\), \(\chi_d(G) \leq \chi(G) + c \ln k + 1\). Also, we introduce an upper bound for the dynamic list chromatic number of regular graphs.

Key words: Dynamic chromatic number, Dynamic list chromatic number.

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1 Introduction

Let \(H\) be a hypergraph. The vertex set and the hyperedge set of \(H\) are mentioned as \(V(H)\) and \(E(H)\), respectively. The maximum degree and the minimum degree of \(H\) are denoted by \(\Delta(H)\) and \(\delta(H)\), respectively. For an integer \(l \geq 1\) we denote by \(\{1, 2, \ldots, l\}\) the set of all \(l\)-subsets of \([m]\). A proper \(l\)-coloring of a hypergraph \(H\) is a function \(c : V(H) \longrightarrow \{1, 2, \ldots, l\}\) in which there is no monochromatic hyperedge in \(H\). We say a hypergraph \(H\) is \(t\)-colorable if there is a proper \(t\)-coloring of it. For a hypergraph \(H\), the smallest integer \(l\) that \(H\) is \(l\)-colorable is called the chromatic number of \(H\) and denoted by \(\chi(H)\). Note that a graph \(G\) is a hypergraph such that the cardinality of each \(e \in E(G)\) is 2. We say a graph \(G\) is \(t\)-critical if \(\chi(G) = t\) and any proper induced subgraph of \(G\) has the chromatic number strictly less than \(t\). For a vertex \(v \in V(G)\), \(N(v)\) is the set of all adjacent vertices to \(v\) and \(\deg_T(v)\) is the number of neighbors of \(v\) that are lied in \(T \subseteq V(G)\), i.e., the cardinality of \(N(v) \cap T\).

We denote by \(\binom{[m]}{n}\) the collection of all \(n\)-subsets of \([m]\). The Kneser graph \(KG(m, n)\) is the graph with vertex set \(\binom{[m]}{n}\), in which \(A\) is connected to \(B\) if and only if \(A \cap B = \emptyset\). It was conjectured by Kneser [3] in 1955, and proved by Lovász [6] in 1978, that \(\chi(KG(m, n)) = m - 2n + 2\).

A proper vertex \(l\)-coloring of a graph \(G\) is called a dynamic \(l\)-coloring [8] if for every vertex \(u\) of degree at least 2, there are at least two different colors appeared in the neighborhood of \(u\). The smallest integer \(l\) that there is a dynamic \(l\)-coloring of \(G\) is called the dynamic chromatic number of \(G\) and denoted by \(\chi_d(G)\). Obviously, \(\chi(G) \leq \chi_d(G)\). Some properties of dynamic coloring was studied in [1] [4] [5] [8] [7]. It was proved in [5] that for a connected graph \(G\) if \(\Delta \leq 3\), then \(\chi_d(G) \leq 4\) unless \(G = C_5\), in which case \(\chi_d(C_5) = 5\) and if \(\Delta \geq 4\), then \(\chi_d(G) \leq \Delta + 1\). It was shown in [8] that the difference between chromatic number and dynamic chromatic number can be arbitrarily large. However, it was conjectured that for regular graphs the difference is at most 2.
Conjecture 1. \[8\] For any regular graph \( G \), \( \chi_d(G) - \chi(G) \leq 2 \)

Also, it was proved in \[8\] that if \( G \) is a bipartite \( k \)-regular graph, \( k \geq 3 \) and \( n < 2^k \) then \( \chi_d(G) \leq 4 \).

For a graph \( G \), \( L \) is called an \( l \)-list assignment for \( G \) if for each vertex \( v \in V(G) \), \( L(v) \) is an \( l \)-set of available colors at \( v \). An \( L \)-coloring, is a proper coloring \( c \) such that \( c(v) \in L(v) \), for each \( v \in V(G) \). The graph \( G \) is \( l \)-list colorable if for every \( l \)-list assignment \( L \) to the vertices of \( G \), \( G \) has a proper \( L \)-coloring. The list chromatic number, \( \chi_l(G) \) is the minimum number \( l \) such that \( G \) is \( l \)-list colorable. The list dynamic chromatic number of a graph \( G \), \( \chi_{dl}(G) \), is the minimum positive integer \( l \) such that for every \( l \)-list assignment, there is a dynamic coloring of \( G \) such that every vertex of \( G \) is colored with a color from its list. Clearly, the dynamic list chromatic number of graphs is a common generalization of both dynamic chromatic and list chromatic number of graphs and also \( \chi_{dl}(G) \leq \chi_l(G) \). It was proved in \[11\] that for a connected graph \( G \) if \( \Delta(G) \geq 3 \) then \( \chi_{dl}(G) \leq \Delta(G) + 1 \), and if \( \Delta(G) \leq 3 \) then \( \chi_{dl}(G) \leq 4 \) except \( G = C_5 \), in which case \( \chi_{dl}(C_5) = 5 \).

In a graph \( G \), a set \( T \subseteq V(G) \) is called a total dominating set in \( G \) if for every vertex \( v \in V(G) \), there is at least one vertex \( u \in T \) adjacent to \( v \). The set \( T \) is called double total dominating set if both of \( T \) and its complement \( V(G) \setminus T \) are total dominating sets.

2 Results

Suppose every vertex of a graph \( G \) appears in some triangles. It is clearly that \( \chi_{dl}(G) = \chi(G) \). We shall use this simple result to prove the next theorem.

Theorem 1. Let \( 0 < p < 1 \) be a constant. Almost all graphs in \( \mathcal{G}(n,p) \) have the same chromatic and dynamic chromatic number.

Proof. We show that for almost all graphs \( G \in \mathcal{G}(n,p) \), every vertex \( v \in V(G) \) appears in some triangles. Consider complete graph \( K_n \) with \( V(K_n) = V(\mathcal{G}(n,p)) = [n] \). Let \( v \) be an arbitrary vertex in \( V(G) \). Consider \( \lceil \frac{n-1}{2} \rceil \) edge disjoint triangles in \( K_n \) such that they all have the vertex \( v \). Note that no vertex, except \( v \), is in more than one triangle. Define \( \mathcal{A}_v \) to be the event that none of these triangles is happened in \( G \). Clearly, \( \Pr(\mathcal{A}_v) \leq (1 - p^3)^{\frac{n-1}{2}} \). As \( n(1 - p^3)^{\frac{n-1}{2}} \to 0 \) the proof is completed.

In the next theorem, we present an upper bound for the dynamic chromatic number of a \( k \)-regular graph \( G \) in terms of \( k \) and \( \chi(G) \).

Theorem 2. There exists a constant \( c \) such that for any \( k \)-regular graph \( G \), \( \chi_d(G) \leq \chi(G) + c \ln k + 1 \).

Proof. Define \( p = c' \frac{\ln k}{k} \) where \( c' \) will be specified later. Consider a random set \( T \) such that each vertex \( u \) lies in \( T \) with the probability \( p \). Assume that for each vertex \( u \), the random variable \( X_u \) is the number of neighbors of \( u \) that are in \( T \). Clearly, \( X_u \) is a binomial random variable and according to the Chernoff inequality we have

\[
\Pr(|X_u - E(X_u)| > \lambda) \leq 2e^{-\frac{\lambda^2}{3E(X_u)}}, \quad 0 < \lambda < E(X_u)
\]
Obviously, \( E(X_u) = c' \ln k \). For each vertex \( u \) define \( A_u \) to be the event that \( |\deg_T(u) - c' \ln k| > \lambda \). By Chernoff inequality we have \( \Pr(A_u) \leq 2e^{-\frac{\lambda^2}{3kX_u}} \). For each \( u \), \( A_u \) is mutually independent of all \( A_v \) events but at most \( k^2 \) number of them. If we set \( c' > 6 \) then there exist a threshold \( n(c') \) such that when \( k \geq n(c') \) we have

\[
3c' \ln k(1 + \ln 2 + \ln(k^2 + 1)) < c'^2(\ln k)^2.
\]

Let \( k \geq n(c') \) and consider a \( \lambda \) such that

\[
3c' \ln k(1 + \ln 2 + \ln(k^2 + 1)) \leq \lambda^2 < c'^2(\ln k)^2.
\]

In view of previous inequality, we have

\[
2e(k^2 + 1)e^{-\frac{\lambda^2}{3kX_u}} \leq 1.
\]

Therefore, By applying Lovasz Local Lemma there exists a set \( T \subseteq V(G) \) such that for every \( v \in V(G) \), \( A_v \) does not happen. Equivalently, there is a set \( T \subseteq V(G) \) such that for every vertex \( v \in V(G) \), \( |\deg_T(v) - c' \ln k| \leq \lambda \). Consequently, since \( 0 < \lambda < c' \ln k \), for each vertex \( v \in V(G) \) we have \( 0 < \deg_T(v) < 2c' \ln k \). It implies that for every vertex \( u \in V(G) \), \( \deg_T(u) > 0 \) and \( \deg_{V(G) \setminus T}(u) > 0 \). Note that \( \Delta(G[T]) \leq 2c' \ln k \) and therefore \( \chi(G[T]) = l \leq c \ln k + 1 \). Color the vertices in \( T \) with colors come from \( \{1, 2, \ldots, l\} \) and also color the vertices in \( V(G) \setminus T \) with colors come from \( \{l + 1, l + 2, \ldots, l + \chi(G)\} \). One can see that this coloring is a dynamic coloring of \( G \) and uses at most \( \chi(G) + c \ln k + 1 \) colors. Since the number of \( k \) is finite, there is a constant \( c \) such that for every \( k \)-regular graph \( G \), \( \chi_d(G) \leq \chi(G) + c \ln k + 1 \) and the proof is completed.

It was conjectured in [1] that for any graph \( G \), \( \chi_d(G) = \max\{\chi_l(G), \chi_d(G)\} \). This conjecture was disproved in [2]. It was shown in [2] that for any integer \( k \geq 5 \), there is a bipartite graph \( G_k \) such that \( \chi(G_k) = \chi_d(G_k) = 3 \) and \( \chi_d(G) \geq k \).

The next theorem provides an upper bound for list dynamic chromatic number of regular graphs in terms of their list chromatic number.

**Theorem 3.** Let \( \epsilon \) be a positive constant. If \( G \) is a \( k \)-regular graph then for large enough \( k \), \( \chi_d(G) \leq \chi_d(G) \leq \lceil (1 + \epsilon)\chi_l(G) \rceil \)

**Proof.** For convenience let \( \chi_l(G) = l \). Without loss of generality, we can assume that \( G \) is not a complete graph. Let \( m \geq l \) be a positive integer and consider an \( m \)-list assignment \( L \) for \( G \). For each vertex \( v \in V(G) \), choose a random \( l \)-set \( L'(v) \subset L(v) \) uniformly and independently. For \( v \in V(G) \), suppose that \( B_v \) denotes the event that \( \cap_{u \in N(v)} L(u) \neq \emptyset \). It is readily seen that \( \Pr(B_v) \leq m(l \choose m)^k \). Also, \( B_v \) is mutually independent of all the other events \( B_u \) but those for which \( N(v) \cap N(u) \neq \emptyset \). Therefore, at most \( k(k - 1) \) events are not mutually independent of \( B_v \). In view of Lovasz Local Lemma, if \( m \) is sufficiently large such that

\[
eq k^2 m(\frac{l}{m})^k \leq 1
\]
then there is a list assignment \( L' \) such that for each vertex \( v \in V(G) \), \( L'(v) \subseteq L(v) \), \( |L'(v)| = l \) and \( \cap_{u \in N(v)} L'(u) = \emptyset \). Obviously, \( ek^2m(\frac{k}{m}) \leq 1 \) if and only if \( l(ek^2) \rightarrow 1 \). Note that \( l \leq k + 1 \) and therefore \( (ek^2) \rightarrow 1 \). Hence, there is threshold \( M(\varepsilon) \) such that if \( k \geq M(\varepsilon) \) then \( (ek^2) \rightarrow 1 + \varepsilon \). Hence, if \( k \geq M(\varepsilon) \) then \( m = \left\lfloor (1 + \varepsilon)l \right\rfloor \) satisfies Equation 2. Since, for every \( v \in V(G) \), \( |L'(v)| = l = \chi_l(G) \), \( G \) has an \( L \)-coloring \( c \) such that \( c(u) \in L'(u) \) for each \( u \in V(G) \). Note that for every vertex \( v \in V(G) \), \( \cap_{u \in N(v)} L'(u) = \emptyset \) and it obviously implies that \( c \) is a dynamic coloring of \( G \).

Let \( H \) be a hypergraph. 2-colorability of hypergraphs has been studied in the literature and has lots of applications in some other concepts of combinatorics. Hereafter, we want to make a connection between 2-colorability of hypergraphs and the dynamic chromatic number of graphs.

**Lemma 1.** Let \( H \) be a hypergraph. Assume that there are two positive integers \( 2 \leq r_1 \leq r_2 \) such that for any \( f \in E(H) \), \( r_1 \leq |f| \leq r_2 \) and also \( er_2 \Delta(H)(\frac{1}{2})^{r_1-1} \leq 1 \). Then \( H \) is 2-colorable.

**Proof.** Consider a random 2-coloring of \( V(H) \) in which each vertex \( v \in V(H) \) is colored red or blue with the same probabilities. For any \( f \in E(H) \), define \( A_f \) to be the event that all the vertices in \( f \) have the same color. Obviously, \( \Pr(A_f) \leq \left( \frac{1}{2} \right)^{r_1-1} \) and two events \( A_f \) and \( A'_f \) are mutually independent when \( f \cap f' = \emptyset \). So, there are at most \( r_2(\Delta(H) - 1) + 1 \) events that are not mutually independent of \( A_f \). Lovasz Local Lemma implies that if \( er_2 \Delta(H)(\frac{1}{2})^{r_1-1} \leq 1 \), then \( H \) is 2-colorable.

It was shown in [8] that for any \( l \geq 2 \) there is a family of \( l \)-colorable graphs such that the difference between chromatic and dynamic chromatic number, in this family, is unbounded. The next theorem states that if \( \Delta(G) \) and \( \delta(G) \) are not so far from each other then its chromatic and dynamic chromatic number are not far from each other.

**Lemma 2.** Let \( G \) be a graph such that \( e\Delta^2 \leq 2\delta(G)-1 \). Then \( \chi_d(G) \leq 2\chi(G) \).

**Proof.** Define a hypergraph \( H \) whose vertex set is the same as the vertex set of \( G \) and its hyperedge set is defined as follows,

\[
E(H) \equiv \{ N(v) | v \in V(G) \}.
\]

Clearly, for any \( f \in E(H) \), \( \delta(G) \leq |f| \leq \Delta(G) \) and \( \Delta(G) = \Delta(H) \). By considering Lemma[1] \( H \) is 2-colorable. Let \( f \) be a 2-coloring of \( H \) and \( c \) be a \( \chi(G) \)-coloring of \( G \). It is readily to seen \( h = (f, c) \) is a \( 2\chi(G) \) dynamic coloring of \( G \).

Note that when \( G \) is a \( k \)-regular graph and \( k \geq 9 \), Lemma[2] implies that \( \chi_d(G) \leq 2\chi(G) \). It was shown by Thomassen [9] that for any \( k \)-uniform and \( k \)-regular hypergraph \( H \), if \( k \geq 4 \) then \( H \) is 2-colorable.

Assume that \( H \) is a \( k \)-uniform hypergraph and \( \Delta(H) \leq k \). One can construct a hypergraph \( H' \supseteq H \) such that \( H' \) is \( k \)-uniform and \( k \)-regular as follows. If \( \delta(H) = k \) then \( H' = H \). Assume that \( H' \) is constructed and \( k - \delta(H') = t > 0 \). Consider
Lemma 2, for any $k \chi$ in which case Corollary 1.

In the rest of this paper, we are focused on relationship between the total dominating set (res. double total dominating set) and the dynamic chromatic number of graphs.

Note that $T_n$, is a 2-coloring of $f$ coloring $h$ that the coloring $h$ in which the restriction of $h$ to $T$ is the same as $h'$ and the vertices in $G[V \setminus T]$ are colored with the colors that are not used in $h'$. One can easily check that the coloring $h$ is a dynamic coloring of $G$.

2. Let $h'$ be a coloring of $G[T]$ and $h$ be a coloring that is the same on $T$ as $h'$ and the vertices in $V \setminus T$ are colored by $h$ with colors that are distinct from the colors used in $T$. It is easy to see that $h$ is a dynamic coloring of $G$ and therefore $\chi_d(G) \leq \chi(G[V \setminus T]) + \chi(G[T])$.

As a consequence of the previous lemma, when a graph $G$ has a double total dominating set $T$, $G$ does not have a dynamic chromatic number far from its chromatic number. This result is restated in the next corollary.

Corollary 2. Let $G$ be a graph and assume that there is a double total dominating set $T \subset V(G)$. Then $\chi_d(G) \leq 2\chi(G)$.

Corollary 1. There exists a constant $c$ such that for any $k$-regular graph $G$, $\chi_d(G) \leq \min\{2\chi(G), \chi(G) + c\ln k\}$.

In the rest of this paper, we are focused on relationship between the total dominating set (res. double total dominating set) and the dynamic chromatic number of graphs.

Lemma 3. Let $G$ be a graph.

1. If $e\Delta^2 \leq 2^{\Delta(G)-1}$ and there is a total dominating set $T \subset V(G)$ then $\chi_d(G) \leq \chi(G[V \setminus T]) + 2\chi(G[T])$.

2. If $G$ has a double total dominating set $T \subset V(G)$ then $\chi_d(G) \leq \chi(G[V \setminus T]) + \chi(G[T])$.

Proof. For convenience let $\chi(G[T]) = s$.

(1) Assume that $H$ is a hypergraph with the vertex set $T$ and the hyperedge set defined as follows.

$$E(H) \overset{\text{def}}{=} \{N(u) \mid N(u) \subseteq T, \; u \in V(G)\}.$$ 

By proof of Lemma 2, $H$ is 2-colorable. Assume that $c$ is an $s$-coloring of $G[T]$ and $f$ is a 2-coloring of $H$. Obviously, $h' \overset{\text{def}}{=} (c, f)$ is a $2s$ coloring of $G[T]$. Consider the coloring $h$ of $G$ in which the restriction of $h$ to $T$ is the same as $h'$ and the vertices in $G[V \setminus T]$ are colored with the colors that are not used in $h'$. One can easily check that the coloring $h$ is a dynamic coloring of $G$.

(2) Let $h'$ be a coloring of $G[T]$ and $h$ be a coloring that is the same on $T$ as $h'$ and the vertices in $V \setminus T$ are colored by $h$ with colors that are distinct from the colors used in $T$. It is easy to see that $h$ is a dynamic coloring of $G$ and therefore $\chi_d(G) \leq \chi(G[V \setminus T]) + \chi(G[T])$. 

As a consequence of the previous lemma, when a graph $G$ has a double total dominating set $T$, $G$ does not have a dynamic chromatic number far from its chromatic number. This result is restated in the next corollary.

Corollary 2. Let $G$ be a graph and assume that there is a double total dominating set $T \subset V(G)$. Then $\chi_d(G) \leq 2\chi(G)$. 

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For a Kneser graph $KG(m,n)$, if $m \geq 3n$ then every vertex of $KG(m,n)$ is in some triangles and therefore $\chi_d(KG(m,n)) = \chi(KG(m,n))$. But when $m = 2n + t < 3n$ the graph $KG(m,n)$ is triangle free and so the dynamic chromatic number of $KG(m,n)$ is still interesting. Let $T = \{A \in V(KG(m,n))|A \subseteq \{t + 1, t + 2, \ldots, m\}\}$. Obviously, the induced subgraph $G[T]$ is a bipartite graph and $T$ is a total dominating set. Note that $c(B) \overset{\text{def}}{=} \min B$ is a $t$-coloring of the induced subgraph on $V(KG(m,n)) \setminus T$. By Lemma 3 we have $\chi_d(KG(m,n)) \leq t + 4 = \chi(KG(m,n)) + 2$. Although, the exact value of $\chi_d(KG(m,n))$ is not determined, but it is shown that the Conjecture [1] is true for Kneser graphs.

**Corollary 3.** Let $G$ be a $k$-critical graph. If $e\Delta(G)^2 \leq 2^{\delta(G) - 1}$ then $\chi_d(G) \leq 2k - 2$

**Proof.** Let $T$ be a random subset of $V(G)$ such that each vertex $v$ lies in $T$ with probability $\frac{1}{2}$, randomly and independently. Let $A_x$ be the event that all neighbors of $x$ are in $T$ or none of them are in $T$. For each vertex $v \in V(G)$, $A_v$ is mutually independent of all but at most $\Delta(G)(\Delta(G) - 1)$ events. Lovasz Local Lemma guarantees that with positive probability $\bigcap \bar{A}_x$ happens. Equivalently, there is a $T \subseteq V(G)$ such that for every vertex $u \in V(G)$, $\deg_{G}(u) > 0$ and $\deg_{G[V(G) \setminus T]}(u) > 0$. Since $G$ is a $k$-critical graph, $\chi(G[T]) < \chi(G)$ and also $\chi(G[V(G) \setminus T]) < \chi(G)$. By second part of Lemma 3 the proof is completed.

In the proof of the previous corollary it is shown that if for a graph $G$, $e\Delta(G)^2 \leq 2^{\delta(G) - 1}$ then $G$ has a total dominating set. Therefore, by Lemma 2, $\chi_d(G) \leq 2\chi(G)$. This provides another proof of Lemma 2.

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