CLASSIFICATION OF ISING VECTORS
IN THE VERTEX OPERATOR ALGEBRA $V_L^+$

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Abstract. Let $L$ be an even lattice without roots. In this article, we classify all Ising vectors in the vertex operator algebra $V_L^+$ associated with $L$.

Introduction

In vertex operator algebra (VOA) theory, the simple Virasoro VOA $L(1/2, 0)$ of central charge $1/2$ plays important roles. In fact, for each embedding, an automorphism, called a $\tau$-involution, is defined using the representation theory of $L(1/2, 0)$ ([Mi96]). This is useful for the study of the automorphism group of a VOA. For example, this construction gives a one-to-one correspondence between the set of subVOAs of the moonshine VOA isomorphic to $L(1/2, 0)$ and that of elements in certain conjugacy class of the Monster ([Mi96, Höl10]).

Many properties of $\tau$-involutions are studied using Ising vectors, weight 2 elements generating $L(1/2, 0)$. For example, the 6-transposition property of $\tau$-involutions was proved in [Sa07] by classifying the subalgebra generated by two Ising vectors. Hence it is natural to classify Ising vectors in a VOA. For example, this was done in [La99, LSY07] for code VOAs. However, in general, it is hard to even find an Ising vector.

Let $L$ be an even lattice and $V_L$ the lattice VOA associated with $L$. Then the subspace $V_L^+$ fixed by a lift of the $-1$-isometry of $L$ is a subVOA of $V_L$. There are two constructions of Ising vectors in $V_L^+$ related to sublattices of $L$ isomorphic to $\sqrt{2}A_1$ ([DMZ94]) and $\sqrt{2}E_8$ ([DLMN98, Gr98]).

The main theorem of this article is the following:

**Theorem 2.5.** Let $L$ be an even lattice without roots and $e$ an Ising vector in $V_L^+$. Then there is a sublattice $U$ of $L$ isomorphic to $\sqrt{2}A_1$ or $\sqrt{2}E_8$ such that $e \in V_U^+$.

We note that this theorem was conjectured in [LSY07] and that if $L/\sqrt{2}$ is even and if $L$ is the Leech lattice, then this theorem was proved in [LSY07] and in [LS07], respectively.

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We also note that if $L$ has roots then the automorphism group of $V_L^+$ is infinite, and $V_L^+$ may have infinitely many Ising vectors.

In this article, we prove Theorem 2.5, and hence we classify all Ising vectors in $V_L^+$. Our result shows that the study of $\tau$-involutions of $V_L^+$ is essentially equivalent to that of sublattices of $L$ isomorphic to $\sqrt{2}E_8$ (cf. [GL11] [GL12]).

The key is to describe the action of the $\tau$-involution on the Griess algebra $B$ of $V_L^+$. Let $e$ be an Ising vector in $V_L^+$ and $L(4; e)$ the norm 4 vectors in $L$ which appear in the description of $e$ with respect to the standard basis of $(V_L^+)_2$ (see Section 2 for the definition of $L(4; e)$). By [LS07], the $\tau$-involution $\tau_e$ associated to $e$ is a lift of an automorphism $g$ of $L$. We show in Lemma 2.1 that $g$ is trivial on $\{\{\pm v\} \mid v \in L(4; e)\}$. This lemma follows from the decomposition of $B$ with respect to the adjoint action of $e$ ([HLY12]), the action of $\tau_e$ on it ([M96]) and the explicit calculations on the Griess algebra ([FLM88]). By this lemma, we can obtain a VOA $V$ containing $e$ on which $\tau_e$ acts trivially. By [LSY07] $e$ is fixed by the group $A$ generated by $\tau$-involutions associated to elements in $L(4; e)$. Hence $e$ belongs to the subVOA $V^A$ of $V$ fixed by $A$. Using the explicit action of $A$, we can find a lattice $N$ satisfying $e \in V_N^+$ and $N/\sqrt{2}$ is even. This case was done in [LSY07].

1. Preliminaries

1.1. VOAs associated with even lattices. In this subsection, we review the VOAs $V_L$ and $V_L^+$ associated with even lattice $L$ of rank $n$ and their automorphisms. Our notation for lattice VOAs here is standard (cf. [FLM88]).

Let $L$ be a (positive-definite) even lattice with inner product $\langle \cdot, \cdot \rangle$. Let $H = \mathbb{C} \otimes \mathbb{Z} L$ be an abelian Lie algebra and $\hat{H} = H \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ its affine Lie algebra. Let $\hat{H}^- = H \otimes t^{-1}\mathbb{C}[t^{-1}]$ and let $S(\hat{H}^-)$ be the symmetric algebra of $\hat{H}^-$. Then $M_H(1) = S(\hat{H}^-) \cong \mathbb{C}[h(m) \mid h \in H, m < 0] \cdot 1$ is the unique irreducible $\hat{H}$-module such that $h(m) \cdot 1 = 0$ for $h \in H$, $m \geq 0$ and $c = 1$, where $h(m) = h \otimes t^m$. Note that $M_H(1)$ has a VOA structure.

The twisted group algebra $\mathbb{C}\{L\}$ can be described as follows. Let $\langle \kappa \rangle$ be a cyclic group of order 2 and $1 \rightarrow \langle \kappa \rangle \rightarrow \hat{L} \rightarrow L \rightarrow 1$ a central extension of $L$ by $\langle \kappa \rangle$ satisfying the commutator relation $[e^\alpha, e^\beta] = \kappa^{(\alpha, \beta)}$ for $\alpha, \beta \in L$. Let $L \rightarrow \hat{L}, \alpha \mapsto e^\alpha$ be a section and $\varepsilon(,) : L \times L \rightarrow \langle \kappa \rangle$ the associated 2-cocycle, that is, $e^\alpha e^\beta = \varepsilon(\alpha, \beta)e^{\alpha+\beta}$. We may assume that $\varepsilon(\alpha, \alpha) = \kappa^{(\alpha, \alpha)/2}$ and $\varepsilon(,)$ is bilinear by [FLM88 Proposition 5.3.1]. The twisted group algebra is defined by

$$\mathbb{C}\{L\} \cong \mathbb{C}[\hat{L}]/(\kappa + 1) = \text{Span}_{\mathbb{C}}\{e^\alpha \mid \alpha \in L\},$$

where $\mathbb{C}[\hat{L}]$ is the usual group algebra of the group $\hat{L}$. The lattice VOA $V_L$ associated with $L$ is defined to be $M_H(1) \otimes \mathbb{C}\{L\}$ ([Bo86] [FLM88]).
For any sublattice $E$ of $L$, let $\mathbb{C}\{E\} = \text{Span}_\mathbb{C}\{e^\alpha \mid \alpha \in E\}$ be a subalgebra of $\mathbb{C}\{L\}$ and let $H_E = \mathbb{C} \otimes_{\mathbb{Z}} E$ be a subspace of $H = \mathbb{C} \otimes_{\mathbb{Z}} L$. Then the subspace $S(\hat{H}_E) \otimes \mathbb{C}\{E\}$ forms a subVOA of $V_L$ and it is isomorphic to the lattice VOA $V_E$.

Let $O(\hat{L})$ be the subgroup of $\text{Aut}(\hat{L})$ induced from $\text{Aut}(L)$. By [FLM88, Proposition 5.4.1] there is an exact sequence of groups

$$1 \to \text{Hom}(L, \mathbb{Z}/2\mathbb{Z}) \to O(\hat{L}) \to \text{Aut}(L) \to 1.$$  

Note that for $f \in O(\hat{L})$

$$f(e^\alpha) \in \{\pm e^{\tilde{f}(\alpha)}\}. \tag{1.1}$$

By [FLM88, Corollary 10.4.8], $f \in O(\hat{L})$ acts on $V_L$ as an automorphism as follows:

$$f(h_{i_1}(n_1)h_{i_2}(n_2)\ldots h_{i_k}(n_k) \otimes e^\alpha) = \tilde{f}(h_{i_1})(n_1)\tilde{f}(h_{i_2})(n_2)\ldots \tilde{f}(h_{i_k})(n_k) \otimes f(e^\alpha), \tag{1.2}$$

where $n_i \in \mathbb{Z}_{\geq 0}$ and $\alpha \in L$. Hence $O(\hat{L})$ is a subgroup of $\text{Aut}(V_L)$.

Let $\theta$ be the automorphism of $\hat{L}$ defined by $\theta(e^\alpha) = e^{-\alpha}$, $\alpha \in L$. Then $\tilde{\theta} = -1 \in \text{Aut}(L)$. Using (1.2) we view $\theta$ as an automorphism of $V_L$. Let $V_L^+ = \{v \in V_L \mid \theta(v) = v\}$ be the subspace of $V_L$ fixed by $\theta$. Then $V_L^+$ is a subVOA of $V_L$. Since $\theta$ is a central element of $O(\hat{L})$, the quotient group $O(\hat{L})/\langle \theta \rangle$ is a subgroup of $\text{Aut}(V_L^+)$. Note that $V_L^+$ is a simple VOA of CFT type.

Later, we will consider the subVOA of $V_L^+$ generated by the weight 2 subspace.

**Lemma 1.1.** (cf. [FLM88, Proposition 12.2.6]) Let $L$ be an even lattice without roots. Let $N$ be the sublattice of $L$ generated by $L(4)$. Then the subVOA of $V_L^+$ generated by $(V_L^+)_{2}$ is $(V_N \otimes M_{H'}(1))^+$, where $H' = (\langle N \rangle_C)^\perp$ in $\langle L \rangle_C$.

### 1.2. Ising vectors and $\tau$-involutions.

In this subsection, we review Ising vectors and corresponding $\tau$-involutions.

**Definition 1.2.** A weight 2 element $e$ of a VOA is called an Ising vector if the vertex subalgebra generated by $e$ is isomorphic to the simple Virasoro VOA of central charge $1/2$ and $e$ is its conformal vector.

For an Ising vector $e$, the automorphism $\tau_e$, called the $\tau$-involution or Miyamoto involution, was defined in ([Mi96, Theorem 4.2]) based on the representation theory of the simple Virasoro VOA of central charge $1/2$ ([DMZ94]).

Let $V$ be a VOA of CFT type with $V_1 = 0$. Then the first product $(a, b) \mapsto a \cdot b = a_{(1)}b$ provides a (nonassociative) commutative algebra structure on $V_2$. This algebra $V_2$ is called the Griess algebra of $V$. The action of $\tau_e$ on the Griess algebra was described in [HLY12] as follows:
Lemma 1.3. [HLY12, Lemma 2.6] Let $V$ be a simple VOA of CFT type with $V_1 = 0$ and $e$ an Ising vector in $V$. Then $B = V_2$ has the following decomposition with respect to the adjoint action of $e$:

$$B = C e \oplus B^e(0) \oplus B^e(1/2) \oplus B^e(1/16),$$

where $B^e(k) = \{ v \in B \mid e \cdot v = kv \}$. Moreover, the automorphism $\tau_e$ acts on $B$ as follows:

- $1$ on $Ce \oplus B^e(0) \oplus B^e(1/2)$,
- $-1$ on $B^e(1/16)$.

In the proof of the main theorem, we need the following lemma:

Lemma 1.4. [LSY07, Lemma 3.7] Let $V$ be a VOA of CFT type with $V_1 = 0$. Suppose that $V$ has two Ising vectors $e, f$ and that $\tau_e = id$ on $V$. Then $e$ is fixed by $\tau_f$, namely $e \in V^{\tau_f}$.

Let $L$ be an even lattice of rank $n$ without roots, that is, $L(2) = \{ v \in L \mid \langle v, v \rangle = 2 \} = \emptyset$. Then $(V_L^+)_1 = 0$, and we can consider the Griess algebra $B = (V_L^+)_2$ of $V_L^+$. Let $\{ h_i \mid 1 \leq i \leq n \}$ be an orthonormal basis of $H = \mathbb{C} \otimes \mathbb{Z} L = \langle L \rangle_\mathbb{C}$. Set $L(4) = \{ v \in L \mid \langle v, v \rangle = 4 \}$. For $1 \leq i \leq j \leq n$ and $\alpha \in L(4)$, set $h_{ij} = h_i(-1)h_j(-1)1$ and $x_\alpha = e^\alpha + e^{-\alpha} = e^\alpha + \theta(e^\alpha)$. Note that $x_\alpha = x_{-\alpha}$.

Lemma 1.5. [FLM88 Section 8.9]

1. The set

$$\{ h_{ij}, x_\alpha \mid 1 \leq i \leq j \leq n, \{ \pm \alpha \} \subset L(4) \}$$

is a basis of $B$.

2. The products of the basis of $B$ given in (1) are the following:

$$h_{ij} \cdot h_{kl} = \delta_{ik}h_{jl} + \delta_{il}h_{jk} + \delta_{jk}h_{il} + \delta_{jl}h_{ik},$$

$$h_{ij} \cdot x_\alpha = \langle h_i, \alpha \rangle \langle h_j, \alpha \rangle x_\alpha,$$

$$x_\alpha \cdot x_\beta = \begin{cases} 
\varepsilon(\alpha, \beta)x_{\alpha\pm\beta} & \text{if } \langle \alpha, \beta \rangle = \mp2, \\
\alpha(-1)^21 & \text{if } \alpha = \pm\beta, \\
0 & \text{otherwise}.
\end{cases}$$

Let $\alpha \in L(4)$. Then the elements $\omega^+(\alpha)$ and $\omega^-(\alpha)$ of $V_L^+$ defined by

$$\omega^\pm(\alpha) = \frac{1}{16}(-1)^2 \cdot 1 \pm \frac{1}{4} x_\alpha$$

are Ising vectors ([DMZ94 Theorem 6.3]). The following lemma is easy:

Lemma 1.6. The automorphisms $\tau_{\omega^\pm(\alpha)}$ of $V_L^+$ act by

$$u \otimes x_\beta \mapsto (-1)^{\langle \alpha, \beta \rangle} u \otimes x_\beta$$

for $u \in M_H(1)$ and $\beta \in L$. 

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In general, the following holds:

**Proposition 1.7.** [LS07, Lemma 5.5] Let $L$ be an even lattice without roots and $e$ an Ising vector in $V_L^+$. Then $\tau_e \in O(\hat{L})/\langle \theta \rangle$.

We note that the main theorem was proved if $L/\sqrt{2}$ is even as follows:

**Proposition 1.8.** [LSY07, Theorem 4.6] Let $L$ be an even lattice and $e$ an Ising vector in $V_L^+$. Assume that the lattice $L/\sqrt{2}$ is even. Then there is a sublattice $U$ of $L$ isomorphic to $\sqrt{2}A_1$ or $\sqrt{2}E_8$ such that $e \in V_U^+$.

## 2. Classification of Ising vectors in $V_L^+$

Let $L$ be an even lattice of rank $n$ without roots and $e$ an Ising vector in $V_L^+$. Then by Lemma 1.5 (1)

(2.1) \[ e = \sum_{i \leq j} c_{ij}^e h_{ij} + \sum_{\{\pm \alpha\} \subset L(4)} d_{\{\pm \alpha\}}^e x_\alpha, \]

where $c_{ij}^e, d_{\{\pm \alpha\}}^e \in \mathbb{C}$. Set $L(4; e) = \{\alpha \in L(4) \mid d^e_{\{\pm \alpha\}} \neq 0\}$, $H_1 = \langle L(4; e) \rangle_{\mathbb{C}}$ and $H_2 = H_1^\perp$ in $H$. Note that if $\alpha \in L(4; e)$ then $-\alpha \in L(4; e)$. Without loss of generality, we may assume that $h_i \in H_1$ if $1 \leq i \leq \dim H_1$. Then $H_2 = \text{Span}_{\mathbb{C}}\{h_j \mid \dim H_1 + 1 \leq j \leq n\}$.

By Proposition 1.7, $\tau_e \in O(\hat{L})/\langle \theta \rangle$. Since $e \in V_L$, we regard $\tau_e$ as an automorphism of $V_L$. Then $\tau_e \in O(\hat{L})$, and set $g = \overline{\tau_e} \in \text{Aut}(L)$. Since $\tau_e$ is of order 1 or 2, so is $g$. The following is the key lemma in this article:

**Lemma 2.1.** Let $\beta \in L(4; e)$. Then $g(\beta) \in \{\pm \beta\}$.

*Proof.* By (1.1) and (1.2),

(2.2) \[ \tau_e(x_\beta) \in \{\pm x_{g(\beta)}\}. \]

On the other hand, $\tau_e(x) = e$, (1.2) and (2.1) show

(2.3) \[ \tau_e(d_{\{\pm \beta\}}^e x_\beta) = d_{\{\pm g(\beta)\}}^e x_{g(\beta)}. \]

By (2.2) and (2.3),

(2.4) \[ \frac{d_{\{\pm g(\beta)\}}^e}{d_{\{\pm \beta\}}^e} \in \{\pm 1\}. \]

Suppose $g(\beta) \notin \{\pm \beta\}$. Then $x_\beta - \tau_e(x_\beta)$ is non-zero, and it is an eigenvector of $\tau_e$ with eigenvalue $-1$. By Lemma 1.3, we have

(2.5) \[ e \cdot (x_\beta - \tau_e(x_\beta)) = \frac{1}{16}(x_\beta - \tau_e(x_\beta)). \]
Let us calculate the image of both sides of (2.5) under the canonical projection \( \mu : (V_L^+)_2 \to \text{Span}_\mathbb{C}\{h_{ij} \mid 1 \leq i \leq j \leq n\} \) with respect to the basis given in Lemma 1.5 (1). By (2.2) the image of the right hand side of (2.5) under \( \mu \) is 0:

\[
\mu \left( \frac{1}{16}(x_{\beta} - \tau_e(x_{\beta})) \right) = 0.
\]

Let us discuss the left hand side of (2.5). By Lemma 1.5 (2) and (2.4), we have

\[
e \cdot (x_{\beta} - \tau_e(x_{\beta})) = \left( \sum_{i \leq j} c_{ij}^e h_{ij} + \sum_{\{\pm\alpha\} \subset L(4)} d_{\{\pm\alpha\}}^e x_\alpha \right) \cdot (x_{\beta} - \tau_e(x_{\beta}))
\]

\[
\in d_{\{\pm\}}^e \langle \beta(-1)^2 \mathbf{1} - g(\beta)(-1)^2 \mathbf{1} \rangle + \text{Span}_\mathbb{C}\{x_\gamma \mid \{\pm\} \subset L(4)\}.
\]

Thus

\[
\mu(e \cdot (x_{\beta} - \tau_e(x_{\beta}))) = d_{\{\pm\}}^e \langle \beta(-1)^2 \mathbf{1} - g(\beta)(-1)^2 \mathbf{1} \rangle
\]

\[
= d_{\{\pm\}}^e \langle \beta - g(\beta) \rangle (-1)(\beta + g(\beta))(1) \mathbf{1}.
\]

This is not zero by \( g(\beta) \notin \{\pm\} \), which contradicts (2.5) and (2.6). Therefore \( g(\beta) \in \{\pm\}. \)

For \( \varepsilon \in \{\pm\}, \) set \( L(4; e, \varepsilon) = \{v \in L(4; e) \mid g(v) = \varepsilon v\}, \) \( L^{e,\varepsilon} = \langle L(4; e, \varepsilon) \rangle_\mathbb{Z}, \) and \( H_1^\varepsilon = \langle L^{e,\varepsilon} \rangle_\mathbb{C}. \) Since \( g \) preserves the inner product, \( H_1 = H_1^+ \perp H_1^- \) and \( g \) acts on \( H_2 = H_1^ \downarrow \). Let \( H_2^\varepsilon \) be \( \pm 1 \)-eigenspaces of \( g \) in \( H_2. \) For \( \varepsilon \in \{\pm\}, \) let \( W^\varepsilon \) be a lattice of full rank in \( H_2^\varepsilon \) isomorphic to an orthogonal direct sum of copies of \( 2A_1. \) Then

\[
M_{H_2^\varepsilon}(1) \subset V_{W^\varepsilon}.
\]

**Lemma 2.2.** The Ising vector \( e \) belongs to the VOA \( V_{L^+, \oplus W^+}^+ \otimes V_{L^-, \oplus W^-}^+ \) and \( \tau_e = \text{id on this VOA}. \)

**Proof.** By Lemma 2.1, \( L(4; e) = L(4; e, +) \cup L(4; e, -). \) Hence, by (2.1) and (2.7),

\[
e \in (V_{L^+, \oplus M_{H_2^+}(1)} \otimes V_{L^-, \oplus M_{H_2^-}(1)})^+ \subset V_{L^+, \oplus W^+ + L^-, \oplus W^-}^+
\]

Since \( g \) acts by \( \pm 1 \) on \( L^{e, \pm} \oplus W^\pm, \) the subspace of (2.8) fixed by \( \tau_e \) is

\[
V_{L^+, \oplus W^+}^+ \otimes V_{L^-, \oplus W^-}^+.
\]

Since \( e \) is fixed by \( \tau_e, \) we have the desired result. \( \square \)

We now prove the main theorem.

**Theorem 2.3.** Let \( L \) be an even lattice without roots. Let \( e \) be an Ising vector in \( V_L^+. \) Then there is a sublattice \( U \) of \( L \) isomorphic to \( \sqrt{2} A_1 \) or \( \sqrt{2} E_8 \) such that \( e \in V_U^+. \)
Proof. Set $V = V_{L^+}^{+;+} \otimes V_{L^-}^{+;+}$. By Lemma 2.2, $e$ belongs to $V$ and $\tau_e = id$ on $V$. Let $A = \{\tau_{\omega^\pm(\beta)} \mid \beta \in L(4; e)\}$. By Lemma 1.3, $e$ belongs to the subVOA $V^A$ of $V$ fixed by $A$. Since $e$ is a weight 2 element, it is contained in the subVOA generated by $(V^A)_2$. By Lemmas 1.1 and 1.6 and (2.7) (cf. (2.8)),

$$e \in V_{N^+@K^+}^{+} \otimes V_{N^-@K^-}^{+} \subset V^+_N,$$

where for $\epsilon \in \{\pm\}$, $N^\epsilon = \text{Span}_Z\{v \in L(4; e, \epsilon) \mid \langle v, L(4; e) \rangle \in 2\mathbb{Z}\}$, $K^\epsilon$ is a lattice of full rank in $(\langle N^\epsilon \rangle_C)^+ \cap (H^+_1 \oplus H^+_2)$ isomorphic to an orthogonal direct sum of copies of $2A_1$, and $N = N^+ \oplus K^+ \oplus N^- \oplus K^-$. Since $N$ is generated by norm 4 and 8 vectors, and the inner products of the generator belong to $2\mathbb{Z}$, the lattice $N/\sqrt{2}$ is even. By Proposition 1.8 there is a sublattice $U$ of $N$ isomorphic to $\sqrt{2}A_1$ or $\sqrt{2}E_8$ such that $e \in V^+_U$. It follows from $K^+(4) = K^-(4) = \emptyset$ that $N(4) = N^+(4) \cup N^-(4) \subset L$. Since $\sqrt{2}A_1$ and $\sqrt{2}E_8$ are spanned by norm 4 vectors as lattices, we have $U \subset L$. Hence $V^+_U$ is a subVOA of $V^+_L$.

As an application of the main theorem, we count the total number of Ising vectors in $V^+_L$ for even lattice $L$ without roots.

Let us describe Ising vectors in $V^+_L$. The Ising vector $\omega^\pm(\alpha)$ associated to $\alpha \in L(4)$ was described in (1.3) as follows:

$$\omega^\pm(\alpha) = \frac{1}{16} \alpha(-1)^2 \cdot 1 \pm \frac{1}{4} x_\alpha.$$

Let $E$ be an even lattice isomorphic to $\sqrt{2}E_8$ and $\{u_i \mid 1 \leq i \leq 8\}$ an orthonormal basis of $\mathbb{C} \otimes_\mathbb{Z} E$. We consider the trivial 2-cocycle of $\mathbb{C}\{E\}$ for $V_E$. Then for $\varphi \in \text{Hom}(E, \mathbb{Z}/2\mathbb{Z})(\cong (\mathbb{Z}/2\mathbb{Z})^8)$

$$\omega(E, \varphi) = \frac{1}{32} \sum_{i=1}^{8} u_i(-1)^2 \cdot 1 + \frac{1}{32} \sum_{\{\pm \alpha\} \subseteq E(4)} (-1)^{\varphi(\alpha)} x_\alpha$$

is an Ising vector in $V^+_E$ ([DLMN98, G198]). Since $E(4)$ spans $E$ as a lattice, $\omega(E, \varphi) = \omega(E, \varphi')$ if and only if $\varphi = \varphi'$. Hence $V^+_E$ has 256 Ising vectors of form $\omega(E, \varphi)$. Thus $V^+_{\sqrt{2}A_1}$ and $V^+_{\sqrt{2}E_8}$ has exactly 2 and 496 Ising vectors, respectively ([LSY07, Proposition 4.2 and 4.3]).

Corollary 2.4. Let $L$ be an even lattice without roots. Then the number of Ising vectors in $V^+_L$ is given by

$$|L(4)| + 256 \times |\{U \subset L \mid U \cong \sqrt{2}E_8\}|.$$

Proof. Set $m = |L(4)| + 256 \times |\{E \subset L \mid E \cong \sqrt{2}E_8\}|$. Theorem 2.3 shows that the number of Ising vectors in $V^+_L$ is less than or equal to $m$. Let us show that there are exactly $m$
Ising vectors in $V_L^+$, that is, the Ising vectors $\omega^\pm(\alpha)$ and $\omega(E, \varphi)$ are distinct. By Lemma 1.5 (1), $\omega^\varepsilon(\alpha) = \omega^\delta(\beta)$ if and only if $\alpha = \beta$ and $\varepsilon = \delta$. Moreover, $\omega^\varepsilon(\alpha) \neq \omega(E, \varphi)$ for all $\alpha \in L(4), L \supset E \cong \sqrt{2}E_8$ and $\varphi \in \text{Hom}(E, \mathbb{Z}/2\mathbb{Z})$.

Let $E_1, E_2$ be sublattices of $L$ such that $E_1 \cong E_2 \cong \sqrt{2}E_8$. Let $\varphi_i \in \text{Hom}(E_i, \mathbb{Z}/2\mathbb{Z}), i = 1, 2$. Then it follows from Lemma 1.5 (1) and $\langle E_i(4) \rangle_{\mathbb{Z}} = E_i$ that $\omega(E_1, \varphi_1) = \omega(E_2, \varphi_2)$ if and only if $E_1 = E_2$ and $\varphi_1 = \varphi_2$. Therefore, there are exactly $m$ Ising vectors in $V_L^+$. □

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References

[Bo86] R.E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, Proc. Nat’l. Acad. Sci. U.S.A. 83 (1986), 3068–3071.
[DLMN98] C. Dong, H. Li, G. Mason, and S.P. Norton, Associative subalgebras of the Griess algebra and related topics. The Monster and Lie algebras (Columbus, OH, 1996), 27–42, Ohio State Univ. Math. Res. Inst. Publ., 7, de Gruyter, Berlin, 1998.
[DMZ94] C. Dong, G. Mason and Y. Zhu, Discrete series of the Virasoro algebra and the moonshine module, Proc. Sympos. Pure Math. 56 (1994), 295–316.
[FLM88] I. Frenkel, J. Lepowsky and A. Meurman, Vertex operator algebras and the Monster, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.
[Gr98] R.L. Griess, A vertex operator algebra related to $E_8$ with automorphism group $O^+(10, 2)$, Ohio State Univ. Math. Res. Inst. Publ. 7 (1998), 43–58.
[GL11] R. L. Griess and C. H. Lam, Dihedral groups and $EE_8$ lattices, Pure and Applied Math Quarterly (special issue for Jacques Tits) 7 (2011), 621–743.
[GL12] R. L. Griess and C. H. Lam, Diagonal lattices and rootless $EE_8$ pairs, J. Pure Appl. Algebra 216 (2012), 154-169.
[Hö10] G. Höhn, The group of symmetries of the shorter Moonshine module, Abh. Math. Semin. Univ. Hambg. 80 (2010), 275–283
[HLY12] G. Höhn, C.H. Lam, H. Yamauchi, McKay’s E7 observation on the Babymonster, Int. Math. Res. Not. IMRN 2012 (2012) 166-212.
[La99] C.H. Lam, Code vertex operator algebras under coordinates change, Comm. Algebra 27 (1999), 4587-4605.
[LSY07] C.H. Lam, S. Sakuma and H. Yamauchi, Ising vectors and automorphism groups of commutant subalgebras related to root systems, Math. Z. 255 (2007) 597–626.
[LS07] C.H. Lam and H. Shimakura, Ising vectors in the vertex operator algebra $V_L^+$ associated with the Leech lattice $\Lambda$, Int. Math. Res. Not. IMRN (2007) Art. ID rnm 132, 21 pp.
[Mi96] M. Miyamoto, Griess algebras and conformal vectors in vertex operator algebras, J. Algebra 179 (1996), 523–548.
[Sa07] S. Sakuma, 6-transposition property of $\tau$-involutions of vertex operator algebras, Int. Math. Res. Not. IMRN (2007), Art. ID rnm 030, 19 pp.
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