Superconformal field theories in analytic superspace

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Abstract
We summarise recent work on superconformal field theories using analytic superspace. All operators of $N = 4$ SYM can be given as unconstrained superfields on analytic superspace. We show how to write down operators as superfields on analytic superspace and how to completely solve the Ward identities for their correlation functions. We discuss the non-renormalisation of certain operators, and of some of their correlation functions. We discuss the relationship between harmonic and analytic superspace. Finally we discuss applications of these techniques to superconformal field theory in 6 dimensions.

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1 Introduction

Superconformal field theories have acquired considerable interest in recent years due to the discovery of the AdS/CFT correspondence which relates them to supergravity or superstring theories on anti-de-Sitter (AdS) spaces [1]. In particular $N = 4$ super Yang-Mills (SYM) provides the clearest and most concrete example of the AdS/CFT correspondence, via which it is related to IIB supergravity or string theory on an $AdS_5 \times S^5$ background. $N = 4$ SYM is also of fundamental interest in its own right due to a number of remarkable properties: it has the largest possible amount of flat space supersymmetry, it is uniquely determined by the coupling constant and the gauge group and it is superconformally invariant even as a quantum theory [2]. It is thus the most symmetric known four-dimensional gauge theory, thus providing an important testing ground for more physically interesting gauge theories. In particular, one would like to know to what extent the theory is determined by its symmetries.

The $(2,0)$ tensor multiplet in six dimensions is also of great interest as it is dual to $M$-theory on $AdS_7 \times S^4$ via the AdS/CFT correspondence. It is much more mysterious than $N = 4$ SYM, however, as the classical theory is not known and there is no dimensional coupling constant. Nevertheless one can still assume a conformally invariant quantum theory and investigate the consequences of conformal invariance.

Harmonic/analytic superspaces provide the clearest way to answering these questions since the full superconformal symmetry is manifest, and it acts on analytic superfields. In particular it is very easy to write down conformally invariant correlation functions.

In this talk we give an introduction to harmonic/analytic superspace and its applications in superconformal quantum field theory. We will begin in section 2 with the standard harmonic superspace techniques which allow one to write certain supermultiplets as analytic superfields (without superindices). We then discuss some technical details of supercoset spaces and super Dynkin diagrams. We show how one can obtain all superconformal fields as unconstrained analytic superfields. This is achieved with the aid of superindices (ie non-trivial linear representations of supergroups). The techniques are very general: any unitary irreducible superconformal representation can be given as an analytic superfield on any (non-twistor) superspace. In section 3 we consider some applications of this technique in $N = 4$ SYM. Section 4 deals with the relationship between harmonic superspace and analytic superspace, showing how to lift unconstrained fields on analytic superspace to constrained fields on harmonic superspace. Finally, in section 5, we look very briefly at some applications to six-dimensional superconformal field theory.

2 Harmonic/Analytic superspace

Harmonic superspace $\mathcal{M}_H$ is obtained by appending an internal manifold $K$ to extended Minkowski superspace $\mathcal{M}$

$$\mathcal{M}_H = \mathcal{M} \times K.$$  

(1)

The internal manifold is usually a coset manifold of the internal group (for example $SU(N)$ for $N$-extended supersymmetry in 4 dimensions) and we write the coordinates of $K$ accordingly as $u_I^j$ where $u_I^j$ are matrix elements of the internal group. So the coordinates of harmonic superspace are $(x, \theta^i, \bar{\theta}_i, u_I^j)$. Harmonic superspace was first
introduced by GIKOS in 1984 [3] for the case of $N = 2$ supersymmetry. In this case the internal group is $SU(2)$ carried by the indices $i = 1, 2$. The internal manifold is given by the coset $U(1) \backslash SU(2)$ which is just the Riemann sphere and can also be described by the coset $P \backslash SL(2; \mathbb{C})$ where $P$ is the set of 2x2 complex lower triangular matrices with unit determinant. A coset representative for this coset is given by

$$s(y) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

(2)

where $y$ is a complex number which covers all but one point of the sphere $K$. In order to cover the whole space one has to consider the new coordinate $y' = 1/y$ in the standard way for the Riemann sphere.

There are two alternative but equivalent ways of dealing with the internal coordinates $u_{ij}$. One can consider them to have any values in the entire group manifold $SU(2)$ (or $SL(2; \mathbb{C})$). By restricting oneself to superfields with special covariance properties this is equivalent to considering fields on the coset. Alternatively one can consider fields on harmonic superspace directly in which case one considers the internal coordinates $u_{ij} = s(y)_{ij}$. One must then also consider the other coordinate patch with coordinates $y'$. We adopt the latter point of view since it allows easier generalisation to the case of analytic superfields which transform non-trivially under supergroups (see section 2.6).

2.1 Why is harmonic superspace useful?

We will illustrate the usefulness of harmonic superspace through an example, the hypermultiplet in $N = 2$ supersymmetry. The hypermultiplet consists of 4 real scalars $f_i(x)$, and two spinors $\psi_\alpha(x), \kappa^{\alpha'}(x)$. These fields can be packaged into a single superfield $q_i(x)$ on $N = 2$ Minkowski superspace satisfying the constraint

$$D_{\alpha}(q_i) = \bar{D}_{\alpha'}(q_i) = 0 \implies q_i(x, \theta, \bar{\theta}) = f_i(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}^{\alpha'} \bar{\kappa}_{\alpha'}(x) + \ldots$$

(3)

where the dots indicate further terms which involve no further fields, and all component fields satisfy there equations of motion. On $N = 2$ harmonic superspace, the superfield $q_i$ becomes a harmonic superfield, $q(x, \theta, \bar{\theta}, u) := u_1^i q_i = q_i + y q_2$ and the constraints (3) become simple Grassmann analyticity conditions $D_{\alpha} q = \bar{D}_{\alpha'} q = 0$ where $D_{\alpha} := u_1^i D_{\alpha i}$, $\bar{D}_{\alpha'} := \bar{D}_{\alpha'} (u^{-1}) i^2$. Note that we have imposed analyticity on the internal manifold, H-analyticity, which leads to the short expansion in the internal coordinates. The Grassmann analyticity conditions are the generalisation of chirality constraints in $N = 1$ and as in that case one can easily solve the constraints. The solution here is $q = q(x_A, \theta^2, \bar{\theta}_1, y)$ with a suitably redefined $x$ coordinate $x_A$ and with $\theta^2 = \theta^i (u^{-1})^i_2$, $\bar{\theta}_1 = u_1^i \bar{\theta}_i$. The hypermultiplet can therefore be thought of as an analytic superfield on analytic superspace which has coordinates $(x_A, \theta^2, \bar{\theta}_1, y)$.

Strictly speaking analytic superspace is only defined when one complexifies the coordinates, so that $x$ is now complex and $\theta$ is unrelated to $\bar{\theta}$ ($y$ is already complex). To indicate this we will denote $\bar{\theta}$ by $\varphi$ from now on. We can always go back to the real case at the end of any calculation simply by considering $x$ real and $\varphi = \bar{\theta}$.
2.2 Supercosets and super Dynkin diagrams

A further advantage of considering the complex case is that all superspaces are then supercosets of the (complex) superconformal group $SL(4|N;\mathbb{C})$ allowing us to use many techniques from representation theory and parabolic induction. We use a non-standard representation for a matrix in the Lie algebra $sl(4|N)$, corresponding to changing the basis on which the matrix acts from $(4|N)$ to $(2|N|2)$. In this basis all superspaces will have the form $P\backslash SL(4|N)$ with $P$ a block lower triangular matrix [4].

The super Dynkin diagram of the superconformal group corresponding to this choice of basis is

\[
\begin{array}{c}
\bullet \quad \bullet \quad \cdots \quad \bullet \\
\end{array}
\]

\[N-1\]

Here the $N-1$ central black nodes represent the internal $\mathfrak{sl}(N)$ subalgebra ($su(N)$ in the real case), the two black nodes on the ends represent the space-time $\mathfrak{sl}(2)$ Lorentz group representations. The two white nodes represent odd roots in the Lie superalgebra.

We now consider coset spaces of the superconformal group $P\backslash SL(4|N)$. In the case where $P$ is a parabolic subgroup (corresponding to a block lower triangular matrix) these can be represent by putting crosses on the Dynkin diagram [5].

In fact all superspaces associated with four-dimesnional Minkowski space can be represented in this way [6].

Irreducible representations transform under the Levi subgroup $L \subset P$ which corresponds to the ‘block diagonal part’ of the block lower triangular matrices of $P$.

2.3 Examples of $N = 2$ superspaces $P\backslash SL(4|2)$

We illustrate the above points using a table containing some examples in the $N = 2$ case.

| superspace       | $P, L$ | coordinates                                                                 | Dynkin diagram |
|------------------|--------|------------------------------------------------------------------------------|----------------|
| Minkowski $(x, \theta^i, \varphi_i)$ | $\begin{pmatrix}
\bullet & \bullet \\
\circ & \circ \\
\circ & \circ \\
\circ & \circ \\
\end{pmatrix}$ | $\begin{pmatrix}
1^\alpha_\beta & \theta^\alpha_\beta & x^{\alpha\beta'} \\
1 & \varphi_i^{\beta'} & 1 \\
1 & \varphi_i^{\beta'} & 1 \\
1 & \varphi_i^{\beta'} & 1 \\
\end{pmatrix}$ | $l = sl(2)^3 \oplus \mathbb{C}^2$ |
| Harmonic $(x, \theta^i, \varphi_i, y)$ | $\begin{pmatrix}
\bullet & \bullet \\
\circ & \circ \\
\circ & \circ \\
\circ & \circ \\
\end{pmatrix}$ | $\begin{pmatrix}
1^\alpha_\beta & \theta^{\alpha1} & \theta^{\alpha2} & x^{\alpha\beta'} \\
1 & y & 1 & \varphi_1^{\beta'} \\
1 & y & 1 & \varphi_2^{\beta'} \\
1 & y & 1 & \varphi_3^{\beta'} \\
\end{pmatrix}$ | $l = sl(2)^2 \oplus \mathbb{C}^3$ |
| Analytic $(x, \theta^2, \varphi_1, y)$ | $\begin{pmatrix}
\bullet & \bullet \\
\circ & \circ \\
\circ & \circ \\
\circ & \circ \\
\end{pmatrix}$ | $\begin{pmatrix}
1^\alpha_\beta & \theta^{\alpha2} & x^{\alpha\beta'} \\
1 & y & 1 & \varphi_1^{\beta'} \\
1 & y & 1 & \varphi_2^{\beta'} \\
1 & y & 1 & \varphi_3^{\beta'} \\
\end{pmatrix}$ | $l = sl(2|1)^2 \oplus \mathbb{C}$ |

This table shows various aspects of the supercoset representation of the three $N = 2$ superspaces, Minkowski, harmonic and analytic superspace. In particular each space has the form $P\backslash SL(4|2)$ and the second column of this table gives the subgroup $P$: $P$ consists of matrices which have non-zero entries where there is a black or white circle and all other entries are zero. $P$ is always block lower triangular. Fields on this space will transform under the Levi subgroup $L$ which is the set of matrices which have non-zero entries only
where there are black nodes. This is the block diagonal part of $P$. The coordinates of the space can be thought of as lying in the zero components of $P$ as indicated in the third column. Finally one can give a corresponding Dynkin diagram with crosses through. The Dynkin diagram gives another simple way to read off the Levi subalgebra $\mathfrak{l}$ corresponding to $L$: crossed though nodes give $\mathbb{C}$ charges and the remaining Dynkin diagram after these nodes are taken away gives the rest of $\mathfrak{l}$.

In particular note that for analytic superspace $\mathfrak{l}$ is itself a superspace \cite{6,7}. This is indicated by the Dynkin diagram: on removing the crossed through node (corresponding to the $\mathbb{C}$ charge) one is left with two disconnected Dynkin diagrams each representing the superalgebra $\mathfrak{sl}(2|1)$.

### 2.4 Superfields on harmonic / analytic superspace

Superfields on the above spaces carry representations of the superconformal group $SL(4|N)$. These representations can be specified by putting Dynkin labels - specifying the highest weight state - above the Dynkin nodes.

Representations of the N-extended superconformal group are usually specified by the following labels

$$(\Delta, R, J_1, J_2; a_1 \ldots a_{N-1}) \quad (5)$$

where $\Delta$ is the dilation weight, $J_1, J_2$ the Lorentz spin, $R$ the $R$ charge and $a_i$ are Dynkin labels specifying the representation of the internal group $SL(N)$. These numbers are related in a straightforward manner to the $N + 3$ super Dynkin labels $n_i$ which one puts above the nodes of the Dynkin diagram: the two extremal nodes are related to the Lorentz spin $n_1 = 2J_1$, $n_{N+3} = 2J_2$ the two odd nodes are given by the linear combination $n_2 = \frac{1}{2}(\Delta - R) + J_1 + \frac{m}{N} - m_1$, $n_{N+2} = \frac{1}{2}(\Delta + R) + J_2 - \frac{m}{N}$ and the central nodes are given by the Dynkin labels of the internal group $SL(N)$. Here $m := \sum_{k=1}^{N-1} k a_k$ and $m_1 := \sum_{k=1}^{N-1} a_k$. We can now describe any unitary irreducible representation as a superfield on any superspace and also its transformation properties under the superconformal group.

Given a representation (specified for example by giving its quantum numbers as in (5)) and a superspace, one must work out the Dynkin labels (as shown above) and write them above the Dynkin diagram and put crosses on the Dynkin diagram as dictated by the superspace in question. The resulting Dynkin diagram then tells you how the field transforms on this superspace. Technically, one has to convert the Dynkin diagrams to Young tableaux using a simple formula and use (super)indices which are symmetrised as dictated by the Young tableau.

However, since our spaces are non-compact, the superfields one produces in this way are often not irreducible: superfields will in general have to satisfy differential constraints in order to be made irreducible.

### 2.5 Examples

We will now illustrate this procedure with some examples. Firstly we reconsider the hypermultiplet. This has dilation weight $\Delta = 1$, has no spin and no $R$-charge but transforms under the fundamental of $SU(2)$ ie it has $a_1 = 1$. From these quantum numbers one can
calculate the Dynkin labels and put them above the Dynkin diagram giving:
\[
\begin{array}{c}
0 & 0 & 1 & 0 & 0 \\
\end{array}
\]
\[
\begin{array}{c}
\circ & & \bullet & & \circ \\
\end{array}
\]
\[
\begin{array}{c}
\bullet & & \circ & & \circ \\
\end{array}
\]

(6)

This representation given as a superfield on various superspaces is represented by the relevant Dynkin diagrams:
\[
\begin{array}{c}
0 & 0 & 1 & 0 & 0 \\
\end{array}
\]
\[
\begin{array}{c}
\circ & & \bullet & & \circ \\
\end{array}
\]
\[
\begin{array}{c}
\bullet & & \circ & & \circ \\
\end{array}
\]
\[
\begin{array}{c}
\bullet & & \circ & & \circ \\
\end{array}
\]
\[
\begin{array}{c}
\circ & & \bullet & & \circ \\
\end{array}
\]
\[
\begin{array}{c}
\bullet & & \circ & & \circ \\
\end{array}
\]

Minkowski Harmonic Analytic

(7)

We can read off from these Dynkin diagrams the transformation properties of the hypermultiplet as a superfield on Minkowski, harmonic and analytic superspace.

In particular the ‘1’ above the central node indicates that the hypermultiplet transforms under the fundamental representation of the internal group $SL(2)$ on Minkowski superspace. On harmonic superspace and analytic superspace on the other hand since there is a cross through the central node the hypermultiplet carries no indices but has a non-trivial internal $\mathbb{C}$-charge.

We know from section 2.1 that the hypermultiplet satisfies constraints as a superfield on Minkowski and harmonic superspace, whereas it is unconstrained on analytic superspace due to analyticity in the internal coordinates. This turns out to be generally true: all unitary irreducible representations of the superconformal group are given as superfields on analytic superspace which are analytic but otherwise unconstrained. We illustrate this with another example.

### 2.6 Superfields with superindices

As already hinted at, for more general representations the superfield may transform linearly under a supergroup. One simply reads off the representation of the Levi subalgebra (under which the fields transform linearly) from the super Dynkin diagram. One can then express this representation as a tensor superfield by finding the corresponding (super) Young tableau (according to a straightforward formula.) The number of boxes of the Young tableau then indicates the number of superindices one needs and also the symmetrisation of these indices.

We will illustrate this with another simple example. The $N = 2$ super Maxwell multiplet can be described as a chiral superfield on $N = 2$ Minkowski superspace which also satisfies a second-order constraint. However, it can also be given on analytic superspace as follows. On calculating the Dynkin labels from the quantum numbers, one obtains the following Dynkin diagram which gives the Levi subalgebra indicated below it:
\[
\begin{array}{c}
0 & 1 & 0 & 0 & 0 \\
\end{array}
\]
\[
\begin{array}{c}
\circ & & \times & & \circ \\
\end{array}
\]
\[
\begin{array}{c}
\bullet & & \circ & & \circ \\
\end{array}
\]
\[
\begin{array}{c}
\bullet & & \circ & & \circ \\
\end{array}
\]
\[
\begin{array}{c}
\circ & & \bullet & & \circ \\
\end{array}
\]
\[
\begin{array}{c}
\bullet & & \circ & & \circ \\
\end{array}
\]

\[
\mathfrak{sl}(2|1)
\]

(8)

The representations of the two $SL(2|1)$ supergroups are carried by the superindices $A, A'$ respectively. We see from the lower Dynkin diagram that although the analytic superfield
carries a trivial representation of the right $\mathfrak{sl}(2|1)$ (as indicated by the zero Dynkin label) it carries a non-trivial representation of the left $\mathfrak{sl}(2|1)$ (as indicated by the non-zero Dynkin label.) In fact the diagram indicates that the superfield carries the anti-fundamental representation of the left $\mathfrak{sl}(2|1)$. This corresponds to a superfield $W_A = (W_\alpha, W)$ with one superindex.

The prescription of how to move from one patch of the Riemann sphere to another patch is completely determined. For $y \to y' = 1/y$, $W_A(x, \theta^2, \varphi_1, y) \to W'_A(x', \theta'^2, \varphi'_1, y')$ where

$$
\begin{pmatrix}
W'_\alpha \\
W'
\end{pmatrix} = 
\begin{pmatrix}
W_\alpha/y \\
-W_\beta(\theta^2)^\beta/y + W
\end{pmatrix},
\quad
x' = x - \frac{\theta^2 \varphi_1}{y}, \quad \theta'^2 = -\frac{\theta^2}{y}, \quad \varphi'_1 = \frac{\varphi_1}{y}.
$$

(9)

Demanding holomorphicity in $y$ on both patches of $S^2$ then gives the correct on-shell components $(\rho_{1\alpha}, \rho_{2\alpha}, F_{\alpha\beta}, C)$ all satisfying their equations of motion:

$$
W_\alpha = \rho_{1\alpha} + y \rho_{2\alpha} + (\theta^2)^\beta (\varphi_1)^{\alpha^{'}} \partial_\alpha C - (\theta^2)^{\alpha} (\varphi_1)^{\beta^{'}} \partial_\beta \rho_{2\alpha}
$$

$$
W = C - (\theta^2)^{\alpha} \rho_{2\alpha}.
$$

(10)

Note that as in the case of the hypermultiplet, no differential constraints are needed in order for the analytic superfield to carry an irreducible representation.

So to summarise, in this section we have seen how one can give any superconformal representation as a superfield on any superspace (except super twistor spaces for which the construction would lead to an infinite dimensional representation of the Levi subgroup). In general the superfields require additional differential constraints in order to carry irreducible representations. However, all unitary irreducible representations (UIRs) can be given as analytic superfields on analytic superspace, which are otherwise unconstrained [4, 8].

3 **$N = 4$ SYM on $N = 4$ analytic superspace**

From now on we specialise to $N = 4$ analytic superspace which has Dynkin diagram:

$$
\begin{array}{cccccccc}
\text{o} & \text{●} & \times & \text{●} & \text{o} \\
\end{array}
$$

(11)

It is a coset space of the $N = 4$ superconformal group, $P\backslash SL(4|4)$, with coset representative $s(X)$ where

$$
P = \begin{pmatrix}
\begin{array}{cccc}
\text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} \\
\end{array}
\end{pmatrix}
\end{pmatrix}
\quad
s(X) = \begin{pmatrix}
1_{2|2} & X \\
0_{2|2} & 1_{2|2}
\end{pmatrix}
\quad
X^{AA'} = \begin{pmatrix}
x^{\alpha\alpha'} \\
\pi^{\alpha\alpha'}
\end{pmatrix}
\lambda^{\alpha\alpha'}
\frac{\lambda^{\alpha\alpha'}}{y^{aa'}}.
$$

(12)

As previously, here the nodes correspond to non-zero elements of $P$ and black nodes correspond to non-zero elements of $L$ under which superfields transform.

Superfields on analytic superspace transform linearly under the Levi subalgebra $I = \mathfrak{sl}(2|2) \oplus \mathfrak{sl}(2|2) \oplus \mathbb{C}$. 

3.1 $N = 4$ SYM on analytic superspace

The component fields of $N = 4$ SYM, lie in a single analytic superfield:

$$\theta_{ij}, \psi^i, F_{\mu\nu} \rightarrow W(x, y, \lambda, \pi)$$  \hspace{1cm} (13)

All operators in the free theory can be obtained on analytic superspace by multiplying $W$’s, applying (super)derivatives and taking the trace over the gauge group $SU(N_c)$. The simplest examples are the ‘chiral primary operators’ or ‘CPOs’, given as

$$A_p = Tr(W^p)$$  \hspace{1cm} (14)

which are also known as ‘half BPS’ and are dual to Kaluza Klein states on $S^5$ via the AdS/CFT correspondence.

$$0 \quad 0 \quad 0 \quad p \quad 0 \quad 0 \quad 0$$  \hspace{1cm} (15)

The simplest CPO, $T := A_2$ contains the entire energy momentum multiplet.

3.2 Examples with superindices/ protected operators

A more complicated example of an operator on $N = 4$ analytic superspace is the Konishi operator. This is written in the free theory as

$$K_{AB, A'B'} = Tr(\partial_{(AA'}W\partial_{B)}W + ...)$$  \hspace{1cm} (16)

(the dots denote further total derivative terms needed to ensure the operator transforms correctly). This operator is known to develop an anomalous dimension in the interacting theory.

A seemingly similar operator is

$$O_{AB, A'B'} = \partial_{(AA'}T\partial_{B)}B'T + ...$$  \hspace{1cm} (17)

This operator is protected however from renormalisation, unlike the Konishi operator. The question arises as to why these two operators have such different properties. Analytic superspace provides a simple way to answer this question and to find all such protected operators [9]. Both $O$ and $K$ are short supermultiplets in the free theory (by short we simply mean that the operator does not have a full theta expansion when written as a superfield on $N = 4$ Minkowski superspace). However, $K$ cannot be extended to the interacting theory on analytic superspace (since there is no covariant superderivative there$^2$) whereas $O$ can be extended to the interacting case (since $T$, being gauge invariant, only requires the normal derivative $\partial$). Therefore $O$ must remain short in the interacting theory (since all tensor superfields on analytic superspace are irreducible and we know from the free theory that it must be short.) Superconformal representation theory tells us that operators with anomalous dimensions must be long and so we conclude that $O$ cannot develop an anomalous dimension and hence must be protected.

This then allows a very straightforward generalisation: all operators written in terms of CPOs which are short in the free theory are protected. So we see that analytic superspace gives a straightforward way of classifying protected operators.

$^2$It can however be written abstractly on analytic superspace in terms of “quasi-tensors” [12]
In fact, one can also prove the non-renormalisation of protected operators using correlation functions [10, 12] and this latter method is the only way to prove non-renormalisation of operators in the six dimensional (2,0) superconformal field theory since there is no known classical theory one can use in the latter case (see section 5).

3.3 Correlation functions

Using analytic superspace one can also completely solve the superconformal Ward identities for all correlation functions of gauge invariant operators. This is done by adapting the Minkowski space techniques of [11].

The correlation functions are written in terms of the analytic coordinates at points 1 to \( n \), \( X_1, X_2, \ldots X_n \). The propagators are given as \( g_{ij} = \text{sdet}(X_{ij}^{-1}) \).

The general solution of the Ward identities is then given in the following schematic form

\[
< 12 \ldots n > = \prod_{j=2}^{n} R_j (X_{1j}^{-1}) R'_j (X_{1j}^{-1}) \sum_t t^{R_2 \ldots R_n ; R'_2 \ldots R'_n} P_t F_t
\]

where \( < 12 \ldots n > := < \mathcal{O}_{R_1} X_{1} \ldots >, R_i \) are \( SL(2|2) \) representations (specified using superindices), \( t \) is a tensor which is a monomial in \( X_{12k} = X_{12} X_{2k}^{-1} X_{k1} \) and their inverses, \( F_t \) is an arbitrary invariant which depends on the coupling constant and \( P_t \) are monomials of the propagators \( g_{ij} \). Note that there are no non-trivial invariants for \( n \leq 3 \). The problem of finding the invariants has also been completely solved [13].

One has to check that the resulting correlator is analytic in the internal coordinates: if it is not then that correlator must be ruled out.

We illustrate the formula (18) with a couple of examples. The simplest cases are the correlators of CPOs, given by a monomial of the propagators, \( P \), times an invariant: \( < A_{p_1} A_{p_2} \ldots A_{p_n} > = PF \) [14, 15].

The simplest example with superindices is \( < \mathcal{O}_{AA'} TT > = P \ t_{AA'} \) where \( t \) is uniquely given by the monomial \( t^{AA'} = c(X_{123}^{-1})_{AA'} \) where \( c \) is a constant.

Using formula (18) together with the reduction formula and properties of correlators under an additional \( U(1)_Y \) symmetry [16], it can be proven that all two- and three-point functions of all protected operators are independent of the coupling constant [12].

4 From analytic superspace to harmonic superspace

We here show how one can lift an analytic superfield (with superindices) to a harmonic (in general, constrained) superfield.

The harmonic superspace is a fibration over analytic superspace. This fibration splits into two parts corresponding to the two supergroups \( SL(2|2) \) under which the analytic superfields transform. One thus obtains left and right coset spaces with representatives

\[
s(\rho)^A_B = \begin{pmatrix} \delta^\alpha_\beta & \rho^\alpha_b \\ 0 & \delta^a_b \end{pmatrix}, \quad s'(\eta)^{A'}_{A'B'} = \begin{pmatrix} \delta_{\alpha'^{'}a'^{'}b'} & 0 \\ \eta_{\alpha'^{'}b'} & \delta_{\alpha'^{'}a'} \end{pmatrix}.
\]

Here \( \rho \) and \( \eta \) will become the extra odd coordinates that harmonic superspace has above analytic superspace. To obtain a superfield on harmonic superspace from one on analytic
superspace, simply multiply the analytic superfield on the right by \( R(s^{-1}(\rho)) \) and on the left by \( R'(s'(\eta)) \) (where \( R \) and \( R' \) are the representations of the two \( GL(2|2) \) supergroups under which the analytic superfields transform) and choose the component with the maximum number of internal indices. This is best shown with some examples.

A one-half BPS state \( A \) has no indices, so it lifts trivially to harmonic superspace. It does not depend on the extra \((\rho, \eta)\) coordinates as we would expect. A one-quarter BPS operator with Dynkin labels of the form \([001d100]\) is given on analytic superspace by a superfield, \( V_{\alpha'\alpha}(x, \lambda, \pi, y) \) with two superindices. It lifts to a superfield \( v_{\alpha'\alpha}(x, \lambda, \pi, y, \rho, \eta) \) in harmonic superspace where

\[
v_{\alpha'\alpha} = s'_{\alpha'}^{B'}V_{B'B}(s^{-1})B_a = V_{\alpha'\alpha} - V_{\alpha'\beta}\rho_{\beta a} + \eta_{\alpha'\beta'}V_{\beta'\alpha} - \eta_{\alpha'\beta'}\eta_{\beta'\rho_{\beta a}}. \tag{20}
\]

The constraints on \( v_{\alpha'\alpha} \) then follow straightforwardly from the dependence on the ‘extra’ coordinates \( \pi, \eta \). In this way one can see that from this point of view the origin of the constraints of infinite dimensional superconformal representations on harmonic superspace (and hence also on Minkowski superspace) comes simply from the constraints of finite dimensional representations of \( GL(2|2) \).

5 Analytic superspace in six dimensions

One can also consider superconformal field theories in six dimensions from an analytic superspace point of view.

The (complexified) \( d = 6 \ (N, 0) \) superconformal group is \( Osp(8|2N) \) which has maximal bosonic subgroup \( SO(8) \times Sp(2N) \). The corresponding Lie algebra \( osp(8|2N) \) is simply the set of \((8|2N) \times (8|2N)\) supermatrices \( M \) s.t.

\[
osp(8|2N) = \{M|MJ + JM^{ST} = 0\} \quad J = \begin{pmatrix}
0_4 & 0_N & 0 & 0 \\
0_N & 0_4 & 0 & 0 \\
0 & 0 & 0_{N}^{-1} & 1_N \\
0 & 0 & 1_N & 0_{N^{-1}}
\end{pmatrix} \tag{21}
\]

where \( M^{ST} \) denotes the supertranspose. The corresponding Dynkin diagram is

\[
\begin{array}{c}
\cdots \\
\cdot \\
\cdot \\
\cdot
\end{array}
\]

As in the 4d case crosses can be placed arbitrarily on the Dynkin diagram to represent superspaces.

5.1 \( d = 6 \ (2, 0) \) superconformal symmetry

We would like to apply these techniques in particular to the \((2, 0)\) superconformal field theory first considered in [17] and reformulated into harmonic superspace in [18]. This is a somewhat mysterious theory which is dual to M theory on \( AdS_7 \times S^4 \) via the AdS/CFT correspondence. Superconformal symmetry provides a possible method to study properties of this theory.

We will use the simplest analytic superspace which has Dynkin diagram.
We can read off the parabolic subalgebra $\mathfrak{p}$ and a coset representative $s$

$$\mathfrak{p} = \left\{ \begin{pmatrix} -A^A_B & 0 \\ -C_{AB} & D_A B \end{pmatrix} \right\} \quad s(X) = \begin{pmatrix} 1 & X \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} x^{\alpha \beta} & \lambda^{\alpha \beta} \\ -\lambda^{\alpha \beta} & g^{\alpha \beta} \end{pmatrix} \quad (24)$$

where $D_A B = (-1)^{A(A+B)} A^B_A$ and $A = (\alpha, a)$ is a superindex, $\alpha = (1, 2, 3, 4)$ $a = (1, 2)$.

One can then proceed largely by analogy with the four dimensional case. The simplest superfield on analytic superspace is a scalar with charge 1 and is denoted $W$. This contains all the fundamental fields of the theory. Given that there is no known classical theory one can only form composite operators explicitly in the free theory. However, one can still abstractly consider superfields as representations of the superconformal group.

It is possible to prove that certain operators which lie on the threshold of the unitary bounds and which lie in the OPE of two short operators can not have anomalous dimensions, by considering restrictions on the three-point functions of these operators and the two short operators.

One can form correlation functions and find their Ward identities which can then be solved analogagously to the four dimensional case shown above. It is particularly interesting to study the 4-point function $<TTTT>$ and compare with the analogous correlator in $N = 4$ SYM. Arutyunov and Sokatchev have shown that this can be written in terms of a single function of two variables [19] in contrast to the four dimensional case where one also needs a (non-renormalised) function of 1 variable [20]. Using analytic superspace one can write down the complete four-point function of all operators in the energy momentum multiplet in a compact formula, one finds that the solution of the superconformal Ward identities can be solved in terms of a single function of two-variables even before crossing symmetry is taken into account. However, the relation of this function to operators appearing in the OPE of two Ts via the conformal wave expansion is more complicated than in the four dimensional case [21].

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