Robustness of tests for directional mean

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In this paper we study the robustness of the likelihood ratio, circular mean and circular trimmed mean test functionals in the context of tests of hypotheses regarding the mean direction of circular normal and wrapped normal distributions. We compute the level and power breakdown properties of the three test functionals and compare them. We find that the circular trimmed mean test functional has the best robustness properties for both the above-mentioned distributions. The level and power properties of the test statistics corresponding to these functionals are also studied. Two examples with real data are given for illustration. We also consider the problem of testing the mean direction of the von-Mises–Fisher distribution on the unit sphere and explore the robustness properties of the spherical mean direction and likelihood ratio test functionals.

Keywords: circular normal distribution; circular trimmed mean; level breakdown function; power breakdown function; spherical trimmed mean

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1. Introduction

All statistical methods rely on a number of assumptions either explicit or implicit. In reality, it often happens that one or more of these assumptions fail to hold. One common phenomenon encountered while analysing real-life data sets is the presence of one or a few ‘outliers’, i.e. observations which are very different from the rest. It is expected that a good statistical procedure would not be adversely affected by these outliers. Such statistical procedures are termed as robust procedures. One of the most well-studied problems in statistical inference is the problem of testing a statistical hypothesis. While there is substantial literature on robustness of estimates, not much work seems to have been done on the problem of studying the robustness of tests.

Directional data in two or three dimensions arise quite frequently in many natural, physical and social sciences like Biology, Medicine, Ecology, Geology, Meteorology, Image Analysis, Political Science, Finance, Demography, etc. A biologist may be interested in measuring the direction of flight of a bird\cite{1,2} or the orientation of an animal. In medical applications, circadian rhythms are often analysed as they control characteristics such as sleep–wake cycles, hormonal pulsatility, body temperature, mental alertness, reproductive cycles, etc. Because of the periodic nature of biological rhythm data, it can be put into the frame work of circular data analysis\cite{3}. Medical scientists have shown keen interest in topics such as chronobiology, chronotherapy, and the study

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of the biological clock [4,5]. Jammalamadaka et al. [6] discuss an interesting medical application, where the angle of knee flexion was measured to assess the recovery of orthopaedic patients. Gavin et al. [7] deal with circular data when comparing between performance of cervical collars and cervical thoracic orthoses in flexion and extension. In geology significant interest is shown in the study of paleocurrents to infer the direction of flow of rivers of the past [8]. Ginsberg [9] and Wallin [10] discuss the application of angular data in ecological and behavioural studies of animal orientation and habitat selection and also in ecological field studies [11]. Recently, Shimatani et al. [12] discuss a conceptual framework for movement ecology using circular data analysis. Apart from wind direction, other types of circular data arising in meteorology include the time of day at which thunderstorms occur and the times of year at which heavy rains occur [13]. Gill and Hangartner [14] discuss the application of circular data in political science in which they develop a circular regression model for domestic terrorism in which political nature of entities such as attacking groups, target groups, etc. are important factors. In demography, circular data arise in the studies like geographic marital patterns, [15] occupational relocation in the same city[16] and settlement trends. [17] More examples of applications of circular and spherical data analysis can be found in Fisher, [18] Fisher et al., [19] Jammalamadaka and SenGupta, [20] and Mardia and Jupp. [13] A short summary of the properties of circular normal, wrapped normal and von-Mises–Fisher distributions are given in the online supplementary material.

Robustness of estimates with circular data has been studied by several authors [21–28] and that with spherical data have also been reported in the literature [23,29–33], but no work seems to have been done on the problem of studying the robustness of tests with circular and spherical data, to the best of our knowledge. The growth in interest in testing statistical hypothesis with directional data can be seen from several recent papers that have studied this problem from different approaches. Tsai [34] proposed a general class of optimal and distribution-free rank tests for two sample modal directions on the sphere. Ehler and Galanis [35] and Ehler and Okoudjou [36] apply frame theory to obtain results which provide interesting explanation of non-robust behaviour of some well-known tests (like Bingham test) with spherical data.

In this paper we use the breakdown function approach introduced by He et al. [37] for studying robustness of tests. They introduced the concept of power and level breakdown functions (LBF) of a test statistic. We consider the problem of testing $H_0: \mu = 0$ against $H_1: \mu \neq 0$ in a location model $\{F_\mu(x) = F_0(x - \mu), \mu \in [0, 2\pi]\}$. Given a distribution $F$, let $G_\varepsilon(F) = (1 - \varepsilon)F + \varepsilon\delta_x$ where $\delta_x$ denotes a distribution which puts its entire mass at $x$. LBF $\varepsilon^{**}$ and the level breakdown point (LBP) $\varepsilon^{**}$ of a test functional $T$ is, respectively, defined as

$$\varepsilon^{**}(T) = \inf\{\varepsilon > 0: T(G_\varepsilon(F_0)) = T(F_\mu) \text{ for some } x\}$$

and

$$\varepsilon^{**} = \sup_{\mu}(\varepsilon^{**}(T)).$$

The power breakdown function (PBF) $\varepsilon^{*}_\mu$ and the LBP $\varepsilon^*$ of a test functional $T$ is, respectively, defined as

$$\varepsilon^{*}_\mu(T) = \inf\{\varepsilon > 0: T(G_\varepsilon(F_\mu)) = T(F_0) \text{ for some } x\}, \quad \varepsilon^* = \sup_{\mu}(\varepsilon^{*}_\mu).$$

We study the robustness of three single sample tests (based on different test functionals) for the mean direction of von-Mises and wrapped normal distributions on the circle, namely (a) the likelihood ratio test functional, (b) the circular mean (CM) test functional and (c) the $\gamma$-circular trimmed mean test functional (see online supplementary material for definition). We compare the performance of the three test functionals by studying their LBF and PBF and also their LBP and PBP. For the von-Mises–Fisher distribution $M(\mu, \kappa)$, we consider the testing problem for the mean direction and study the robustness of three single sample tests based on the following test functionals: (a) the likelihood ratio test functional, (b) the spherical mean test functional and (c) the $\gamma$-spherical trimmed mean test functional (see Section 7.2 for definition).
The organisation of this paper is as follows. In Section 2 we discuss the robustness of likelihood ratio test functional. In Section 3 we consider directional mean as a test functional and discuss its robustness. In Section 4 we consider $\gamma$-circular trimmed mean as a test functional and its robustness properties. In Section 5 we compare the robustness properties of the three test functionals. In Section 6 we extend the results of Section 5 for the wrapped normal distribution. In Section 7 we discuss the robustness properties of the three test functionals for the von-Mises–Fisher distribution on the unit sphere. In Section 8 we study the performance of the test statistics corresponding to the functionals with respect to their level and power through simulation. In Section 9 we discuss some examples with real data.

2. Robustness of the likelihood ratio test functional

Let $\Theta_1, \Theta_2, \ldots, \Theta_n$ be a random sample from $CN(\mu, \kappa)$ with $\kappa > 0$ and known. Consider the problem of testing $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$. A test statistic for testing $H_0$ against $H_1$ is $W = \bar{R} - \bar{C}$ where $\bar{C} = n^{-1} \sum_{i=1}^{n} \cos \Theta_i$, $\bar{S} = n^{-1} \sum_{i=1}^{n} \sin \Theta_i$ and $\bar{R} = \sqrt{\bar{C}^2 + \bar{S}^2}$ [13]. This corresponds to the functional $W(F) = \rho F - E_F(\cos \Theta)$. Theorem 2.1(a) gives the LBF and Theorem 2.1(b) gives the PBF and PBP corresponding to the test functional $W$. The LBP is obtained numerically.

Theorem 2.1

(a) Let $y = \rho(1 - \varepsilon) + \varepsilon \cos x$ and $c = 2\rho \sin^2(\mu/2)$. Then the LBF of $W$ is

$$\varepsilon^{**}_{\mu}(W) = \inf\{\varepsilon > 0 : \sqrt{y^2 + \varepsilon^2 \sin^2 x} = y + c \text{ for some } 0 \leq x < 2\pi\}.$$  

(b) The PBF of $W$ is $\varepsilon^*_{\mu}(W) = \rho|\sin \mu|/(1 + \rho|\sin \mu|)$ and the PBP of $W$ is $\varepsilon^* = \rho/(1 + \rho)$.

Proof (a) Using the definition of LBF we have $W(G_\varepsilon) = \sqrt{y^2 + \varepsilon^2 \sin^2 x - y}$ and $W(F_\mu) = c$ where $G_\varepsilon = (1 - \varepsilon)F_0 + \varepsilon\delta_x$. This yield:

$$\varepsilon^{**}_{\mu}(W) = \inf\{\varepsilon > 0 : \sqrt{y^2 + \varepsilon^2 \sin^2 x} = y + c \text{ for some } 0 \leq x < 2\pi\}.$$ 

In Figure 1 we show the variation $\varepsilon^{**}_{\mu}(W)$ with $\mu$ and for different values of $\kappa$.

We see from Figure 1 that the LBF of $W$ for a higher value of $\kappa$ dominates that of a lower value of $\kappa$. Also the LBP values of $W$ increases with $\kappa$ as seen from Table 1.

(b) A straightforward calculation using the definitions of PBF and PBP yields:

$$\varepsilon^*_{\mu}(W) = \frac{\rho|\sin \mu|}{1 + \rho|\sin \mu|} \quad \text{and} \quad \varepsilon^* = \frac{\rho}{1 + \rho}.$$ 

A detailed proof of part (a) and part (b) is given in the online supplementary material.
3. Robustness of the directional mean test functional

We now consider the directional mean \( W_1(F) = \arctan^* E_F(\sin \Theta)/E_F(\cos \Theta) \) and study its robustness based on its breakdown properties. Theorems 3.1(a) and (b), respectively, give the LBF, LBP, PBF and PBP corresponding to the test functional \( W_1 \).

**Theorem 3.1**

(a) The \( \text{LBF of } W_1 \) is \( \epsilon^{**}(W_1) = \rho |\sin \mu|/(1 + \rho |\sin \mu|) \) and the \( \text{LBP of } W_1 \) is \( \epsilon^{**} = \rho/(1 + \rho) \).

(b) The \( \text{PBF of } W_1 \) is \( \epsilon^*(W_1) = \rho |\sin \mu|/(1 + \rho |\sin \mu|) \) and the \( \text{PBP of } W_1 \) is \( \epsilon^* = \rho/(1 + \rho) \).

**Proof**  
(a) A straightforward calculation using the definitions of LBF and LBP yields 
\( \epsilon^{**}(W_1) = \rho |\sin \mu|/(1 + \rho |\sin \mu|) \) and \( \epsilon^{**} = \rho/(1 + \rho) \).

(b) A straightforward calculation using the definitions of PBF and PBP yields 
\( \epsilon^*(W_1) = \rho |\sin \mu|/(1 + \rho |\sin \mu|) \) and \( \epsilon^* = \rho/(1 + \rho) \).  

A detailed proof of part (a) and part (b) are given in the online supplementary material.

**Remark**  
In this case the LBF and PBF coincide and as a consequence LBP is equal to PBP.
4. Robustness of the circular trimmed mean test functional

We now consider $\gamma$-circular trimmed mean test functional for the hypothesis testing problem $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$ and study its robustness based on its breakdown properties. The definition of the $\gamma$-circular trimmed mean is given in Laha and Mahesh.[26] We reproduce the same in the online supplementary material for ease of reference.

Let $F_0$ be the cumulative distribution function (c.d.f.) of $CN(0, \kappa)$. Define $\rho_{\gamma, 0} = E_{\gamma, F_0}(\cos \Theta) = 1/\left(1 - 2\gamma\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta \ dF_0$ where $\tau = F_0^{-1}(\gamma)$ and $\eta = F_0^{-1}(1 - \gamma)$. Also let $\theta_1 = F_0^{-1}(\gamma/(1 - \epsilon))$, $\theta_2 = F_0^{-1}(1 - \gamma/(1 - \epsilon))$ and $\vartheta_{\gamma, 0} = \rho_{\gamma, 0}(1 - 2\gamma)$. Theorem 4.1 gives the lower bounds for the LBF and the LBP of $W_\gamma$.

**Theorem 4.1**

(a) The LBF of $W_\gamma$ satisfies $\varepsilon_{\mu, \gamma}(W_\gamma) \geq \vartheta_{\gamma, 0} |\sin \mu|/(1 + \vartheta_{\gamma, 0} |\sin \mu|)$.

(b) Let $k_1(\mu, \epsilon) = \sup \{|\sin(x - \mu) : x \in (\theta_1, \theta_2)|\}$. Then $\varepsilon_{\mu, \gamma}(W_\gamma) \leq (\theta_1 - \theta_2) |\sin \mu| + \epsilon k_1(\mu, \epsilon) = 0$.

(c) The LBP of $W_\gamma$ satisfies $\varepsilon_{\gamma, W}(W_\gamma) \geq \vartheta_{\gamma, 0} |\sin \mu|/(1 + \vartheta_{\gamma, 0} |\sin \mu|)$.

The proof of the theorem uses the two lemmas stated below, the proofs of which are given in the online supplementary material.

**Lemma 1** Let $\Theta$ be a circular random variable. Then we have $\mu_{\gamma, \mu} = \mu \forall \mu \in [-\pi, \pi]$.

**Lemma 2** $W_\gamma(G_x) \neq \mu$ for $x \leq \theta_1$ or $x \geq \theta_2$ where $\epsilon < \min(\gamma, 1 - \gamma)$.

We give an outline of the proof of the Theorem 4.1. A detailed proof can be found in the online supplementary material.

**Proof** (a) Let $F_\mu$ and $\delta_x$ be as in the proof of Theorem 2.1. Let $G_x = (1 - \epsilon)F_0 + \epsilon \delta_x$, $x \in [-\pi, \pi]$ and $0 \leq \gamma < 0.5$. Then

$$G_x(\Theta) = \begin{cases} (1 - \epsilon)F_0(\Theta) & \text{if } -\pi \leq \Theta < x, \\ (1 - \epsilon)F_0(\Theta) + \epsilon & \text{if } \Theta \leq x < \pi, \end{cases}$$

where $F_0(\Theta) = (1/2\pi I_0(\kappa)) \int_{-\pi}^{\pi} e^{x \cos \Theta} d\phi$. Suppose $\theta_1 = F_0^{-1}(\gamma/(1 - \epsilon))$, $\theta_2 = F_0^{-1}(1 - \gamma/(1 - \epsilon))$ and also note that since $F_0$ is symmetric about zero we have $\theta_1 = -\theta_2$. We have evaluated $W_\gamma(G_x)$ when $\theta_1 < x < \theta_2$, $x \leq \theta_1$ and $x \geq \theta_2$ and, respectively, obtained

$$W_\gamma(G_x) = \begin{cases} \arctan^* \left[ \frac{\vartheta_{\gamma, 0}(1 - \epsilon) \tan \mu_{\gamma, 0} + \epsilon x}{\vartheta_{\gamma, 0}(1 - \epsilon) + \epsilon x} \right], & \theta_1 < x < \theta_2, \\ \arctan^* \left[ \frac{\bar{E}_{\gamma, F_0}(\sin \Theta)}{\bar{E}_{\gamma, F_0}(\cos \Theta)} \right], & x \leq \theta_1, \\ \arctan^* \left[ \frac{\bar{E}_{\gamma, F_0}(\sin \Theta)}{\bar{E}_{\gamma, F_0}(\cos \Theta)} \right], & x \geq \theta_2. \end{cases}$$

Using Lemma 1 we have $W_\gamma(F_\mu) = \mu$ and using Lemma 2 we have $W_\gamma(G_x) \neq \mu$ for $x \leq \theta_1$ and $x \geq \theta_2$. Therefore, we obtain

$$\varepsilon_{\mu, \gamma}(W_\gamma) = \inf \left\{ \epsilon > 0 : \epsilon = \frac{\vartheta_{\gamma, 0}(1 - \epsilon) \sin \mu}{\sin(x - \mu) + \vartheta_{\gamma, 0} \sin \mu}, \text{ for some } x \in (\theta_1, \theta_2) \right\}. \quad (1)$$

Since $\sin(x - \mu) \leq 1$, we have $\varepsilon_{\mu, \gamma}(W_\gamma) \geq \vartheta_{\gamma, 0} |\sin \mu|/(1 + \vartheta_{\gamma, 0} |\sin \mu|)$ for $0 \leq \mu \leq \pi$. 
(b) Now, from (1) we get $x = \mu + \sin^{-1}(\Lambda)$ where $\Lambda = \epsilon^{-1}\vartheta_{q,0}(1 - \epsilon) \sin \mu$ has a solution in $x \in (\theta_1, \theta_2)$ if and only if $|\Lambda| < 1$. This implies $\vartheta_{q,0}|\sin \mu|/\tau_1 < \epsilon < \vartheta_{q,0}|\sin \mu|/\tau_2$ when $\tau_1, \tau_2 > 0$, $\vartheta_{q,0}|\sin \mu|/\tau_1 < \epsilon$ and $\epsilon < -\vartheta_{q,0}|\sin \mu|/\tau_2$ when $\tau_1 > 0, \tau_2 < 0$ and $\vartheta_{q,0}|\sin \mu|/\tau_2 < \epsilon < -\vartheta_{q,0}|\sin \mu|/\tau_1$ when $\tau_1, \tau_2 < 0$. Therefore, $\epsilon^{**}_{\mu, q}(W_q) \geq \vartheta_{q,0}|\sin \mu|/(k_1(\mu) + \vartheta_{q,0}|\sin \mu|)$.

(c) Further, the LBP of $W_q$ satisfies $\epsilon^{**} \geq \vartheta_{q,0}/(1 + \vartheta_{q,0})$.

Hence the theorem is established.

Theorem 4.2 gives the PBF of $W_q$ and the corresponding PBP is obtained numerically. Let $c_1(\mu) = \mu + F_0^{-1}(\gamma)$ and $c_2(\mu) = \mu + F_0^{-1}(1 - \gamma)$. Define $\varphi_{\mu} = \sup\{|\sin x : x \text{ lies on the arc } c_1 \text{ to } c_2 \text{ of the unit circle traversed anticlockwise}|$ and $\psi_{\mu} = \inf\{|\sin x : x \text{ lies on the arc } c_1 \text{ to } c_2 \text{ of the unit circle traversed clockwise}|$. Let $\lambda_{\mu} = 2C\gamma \sin \mu, C\gamma = \int_{0}^{\theta_1} \cos v_0(v) \, dv$, $\nu = F_0^{-1}(1 - \gamma)$, $\xi_{\mu} = \lambda_{\mu}/(\lambda_{\mu} - \psi_{\mu})$ and $\bar{\xi}_{\mu} = \lambda_{\mu}/(\lambda_{\mu} - \varphi_{\mu})$ where $F_0$ is the c.d.f of $CN(0, \kappa)$.

**Theorem 4.2**  The PBF of $W_q$ is given by

$$
\epsilon^{**}_{\mu, q}(W_q) = \begin{cases} 
\max(0, \min(\xi_{\mu}, 1)) & \text{if } 2C\gamma \geq \lambda_{\mu} \geq \varphi_{\mu}, \\
\min(1, \max(\xi_{\mu}, \xi_{\mu}, 0)) & \text{if } \psi_{\mu} \leq \lambda_{\mu} \leq \varphi_{\mu}, \\
\max(0, \min(\xi_{\mu}, 1)) & \text{if } -2C\gamma \leq \lambda_{\mu} \leq \psi_{\mu}.
\end{cases}
$$

The proof of the theorem uses the lemma stated below, the proof of which is given in the online supplementary material.

**Lemma 3**  Suppose that $\Theta \sim CN(0, \kappa)$. Then $C\gamma = \int_{0}^{\theta_1} \cos \theta f_0(\theta) \, d\theta > 0$ where $f_0$ is the p.d.f. of the circular normal distribution, $\theta \in (-\pi, \pi)$, $\theta^* = F_0^{-1}(1 - \gamma)$, and $0 \leq \gamma < 0.5$.

An outline of the proof of Theorem 4.1 is given. A detailed proof can be found in the online supplementary material.

**Proof**  Let $F_\mu$ and $\delta_\varepsilon$ be as in the proof of Theorem 2.1. Further, for $0 \leq \gamma < 0.5$ let $G_\varepsilon = (1 - \varepsilon)F_\mu + \varepsilon \delta_\varepsilon, x \in [\mu - \pi, \mu + \pi]$. Then

$$
G_\varepsilon(\theta) = \begin{cases} 
(1 - \varepsilon)F_\mu(\theta) & \text{if } \mu - \pi \leq \theta < x, \\
(1 - \varepsilon)F_\mu(\theta) + \varepsilon & \text{if } \theta \leq x < \mu + \pi,
\end{cases}
$$

where $F_\mu(\theta) = (1/2\pi I_0(\kappa)) \int_{\mu-\pi}^{\theta} e^{\cos \phi} \, d\phi = F_0(\theta - \mu)$. When $c_1(\mu) < x < c_2(\mu)$, we have $W_q(G_\varepsilon) = \arctan^{**}[E_{Y,G_\varepsilon}(\sin \theta)/E_{Y,G_\varepsilon}(\cos \theta)]$ and $W_q(F_\mu) = \mu$. Since $W_q(F_0) = 0$ under $H_0$ we have $W_q(G_\varepsilon) = 0 \Rightarrow \int_{c_1(\mu)}^{c_2(\mu)} \sin \theta \, dG_\varepsilon(\theta) = 0 \Rightarrow (1 - \varepsilon)S_\varepsilon + \varepsilon \sin x = 0$ where $S_\varepsilon = \lambda_{\mu}$. Using Lemma 3 we have $C\gamma > 0$. Therefore,

$$
\epsilon^{**}_{\mu, q}(W_q) = \inf\{|\varepsilon > 0 : (1 - \varepsilon)\lambda_{\mu} + \varepsilon \sin x = 0 \text{ for some } x \in (c_1(\mu), c_2(\mu))\}.
$$

Now, the equation $(1 - \varepsilon)\lambda_{\mu} + \varepsilon \sin x = 0$ has a solution in $x$ if and only if $\psi_{\mu} \leq \lambda_{\mu} \leq \varphi_{\mu}$. This yield:

$$
\epsilon^{**}_{\mu, q}(W_q) = \begin{cases} 
\max(0, \min(\xi_{\mu}, 1)) & \text{if } 2C\gamma \geq \lambda_{\mu} \geq \varphi_{\mu}, \\
\min(1, \max(\xi_{\mu}, \xi_{\mu}, 0)) & \text{if } \psi_{\mu} \leq \lambda_{\mu} \leq \varphi_{\mu}, \\
\max(0, \min(\xi_{\mu}, 1)) & \text{if } -2C\gamma \leq \lambda_{\mu} \leq \psi_{\mu}.
\end{cases}
$$

Hence the theorem is established.
Table 2. PBP of $W_\gamma$ for different values of $\kappa$. 

| Test functional $W_\gamma$ | $\kappa = 1$ | $\kappa = 2$ | $\kappa = 4$ | $\kappa = 10$ |
|---------------------------|--------------|--------------|--------------|---------------|
| PBP                       | 0.93         | 1            | 1            | 1             |

Remark The PBP of $W_\gamma$ is given by $\varepsilon^* = \sup_{\mu} (\varepsilon^*_{\mu,\gamma}(W_\gamma))$. Table 2 gives the PBP of $W_\gamma$ for different values of $\kappa$ computed numerically. In Figure A of the online supplement we show the variation of the PBF of $W_\gamma$ with $\mu$ and for different values of $\kappa$.

It can be seen that the PBF is periodic about $\pi$ and as $\kappa$ increases the PBF for values of $\mu$ outside a neighbourhood of 0 and a neighbourhood of $\pi$ are very close to one which is also reflected in Table 2.

5. Comparison of robustness of different types of test functionals

Table 3 and the table A given in the online supplement show the comparisons between the LBP and PBP of the test functionals $W$ and $W_1$ for different values of $\kappa$.

From these tables it can be seen that the likelihood ratio test functional has higher LBP and similar PBP compared with the directional mean functional for different values of $\kappa$. As pointed out in He et al. [37] a comparison of LBP and PBP may not be enough for deciding on the robustness of the test statistics. A more detailed comparison can be done by comparing the LBF’s and PBF’s. A graphical comparison of LBFs of the two test functionals for $\kappa = 1$ is given in Figure 2. It can be seen from the figure that the LBF of $W$ dominates the LBF of $W_1$ for all values of $\mu$.

Some graphical comparisons of PBFs of the three test functionals for different values of $\kappa$ are given in Figure 3(a)–(d).

From the figures we can see that the PBF of $W_\gamma$ clearly dominates that of $W_1$ and $W$ for $\kappa = 1, 2, 4$ and 10. Therefore, we can say with reasonable confidence that $W_\gamma$ has superior power breakdown property in comparisons to $W_1$ and $W$.

6. Robustness of tests with wrapped normal distribution

In this section we consider the test functionals $W$, $W_1$ and $W_\gamma$ and examine their robustness when the observations are coming from an underlying wrapped normal distribution. Theorems 6.1 and 6.2 give, respectively, the LBF, PBF and PBP of the likelihood ratio test functional ($W$) and LBF, LBP, PBF and PBP of the directional mean test functional ($W_1$) with respect to the wrapped normal distribution. These results are parallel to those that have been proved earlier for the circular normal distribution.

Table 3. LBP for different values of $\kappa$.

| Test functional | $\kappa = 1$ | $\kappa = 2$ | $\kappa = 4$ | $\kappa = 10$ |
|-----------------|--------------|--------------|--------------|---------------|
| $W$             | 0.62         | 0.82         | 0.93         | 0.97          |
| $W_1$           | 0.31         | 0.41         | 0.46         | 0.49          |
Figure 2. LBF of LRT and directional mean functionals for $\kappa = 1$. A, LRT functional and B, directional mean functional.

Figure 3. PBFs of the three test functionals for (a) $\kappa = 1$, (b) $\kappa = 2$, (c) $\kappa = 4$ and (d) $\kappa = 10$. A, LRT functional; B, Directional mean functional and C, $\gamma$-trimmed mean functional.
Theorem 6.1
(a) The LBF of $W$ is $\varepsilon_{\mu}^*(W) = \inf\{\varepsilon > 0 : \sqrt{y^2 + \varepsilon^2 \sin^2 x} = y + c \text{ for some } x \in [0, 2\pi]\}$ where $y = \rho(1 - \varepsilon) + \varepsilon \cos x, c = 2\rho \sin^2(\mu/2)$.
(b) The PBF of $W$ is $\varepsilon_{\mu}^*(W) = \rho|\sin \mu|/(1 + \rho|\sin \mu|)$ and the PBP of $W$ is $\varepsilon^* = \rho/(1 + \rho)$.

Proof The theorem is proved using arguments similar to that used in the proofs of Theorems 2.1 and 2.2. A detailed proof is given in the online supplementary material.

Remark The LBP can be computed numerically. The LBP values for $\rho$ close to 1 would be same as that given in Table 1 with $\kappa = A^{-1}(\rho)$ since in this case $WN(\mu, \rho)$ can be well approximated by $CN(\mu, A^{-1}(\rho))$ [38–40].

Theorem 6.2
(a) The LBF of $W_1$ is $\varepsilon_{\mu}^{**}(W_1) = \rho|\sin \mu|/(1 + \rho|\sin \mu|)$ and the LBP of $W_1$ is $\varepsilon^{**} = \rho/(1 + \rho)$.
(b) The PBF of $W_1$ is $\varepsilon_{\mu}^{**}(W_1) = \rho|\sin \mu|/(1 + \rho|\sin \mu|)$ and the PBP of $W_1$ is $\varepsilon^* = \rho/(1 + \rho)$.

Proof The theorem is proved using arguments similar to that used in the proofs of Theorems 3.1 and 3.2. A detailed proof is given in the online supplementary material.

Remark In this case the LBF and PBF coincide and as a consequence LBP is equal to PBP.

Let $F_0$ be the c.d.f. of $WN(0, \rho)$. Theorem 6.3 gives lower bounds for the LBF and LBP and also the PBF of the trimmed mean test functional ($W_\gamma$) when the underlying distribution is wrapped normal.

Theorem 6.3
(a) The LBF of $W_\gamma$ satisfies $\varepsilon_{\mu, \gamma}^{**}(W_\gamma) \geq \vartheta_{\gamma, 0}|\sin \mu|/(1 + \vartheta_{\gamma, 0}|\sin \mu|)$.
(b) Let $k_1(\mu, \varepsilon) = \sup \{\sin(x - \mu) : x \in (\theta_1, \theta_2)\}$. Then $\varepsilon_{\mu, \gamma}^{**}(W_\gamma)$ is the smallest $\varepsilon$ ($0 \leq \varepsilon < 1$) which satisfies the equation $(\varepsilon - 1)(\vartheta_{\gamma, 0}|\sin \mu|) + \varepsilon k_1(\mu, \varepsilon) = 0$.
(c) The LBP of $W_\gamma$ satisfies $\varepsilon^{**} \geq \vartheta_{\gamma, 0}/(1 + \vartheta_{\gamma, 0})$.
(d) The PBF of $W_\gamma$ is

$$
\varepsilon_{\mu, \gamma}^*(W_\gamma) = \begin{cases} 
\max(0, \min(\xi_\mu, 1)) & \text{if } 2C_\gamma \geq \lambda_\mu \geq \psi_\mu, \\
\min(1, \max(\xi_\mu, \xi_\mu, 0)) & \text{if } \psi_\mu \leq \lambda_\mu \leq \psi_\mu, \\
\max(0, \min(\xi_\mu, 1)) & \text{if } -2C_\gamma \leq \lambda_\mu \leq \psi_\mu,
\end{cases}
$$

where $\lambda_\mu = 2C_\gamma \sin \mu, C_\gamma = \int_0^{\nu_2} \cos \theta f_0(\theta) \, d\theta, \nu_2 = F_0^{-1}(1 - \gamma), F_0$ is the c.d.f. of $CN(0, \kappa), \xi_\mu = \lambda_\mu/\lambda_\mu - \phi_\mu$ and $\xi_\mu = \lambda_\mu/\lambda_\mu - \phi_\mu$.

The proof of the above theorem uses the Lemma 4 given along with Lemma 1 and Lemma 2 stated earlier in Section 4.

Lemma 4 Suppose that $\Theta \sim WN(0, \rho)$. Then $C_\gamma = \int_0^{\theta} \cos \theta f_0(\theta) \, d\theta > 0$ where $f_0$ is the p.d.f. of the wrapped normal distribution, $\theta \in (-\pi, \pi), \theta^* = F_0^{-1}(1 - \gamma)$ and $0 \leq \gamma < 0.5$.

Proof The result is proved using Lemma 4 and arguments similar to that used in the proofs of Theorems 4.1 and 4.2. A detailed proof is given in the online supplementary material.

Remark The LBP and PBP of $W_\gamma$ can be obtained numerically.
7. Robustness of tests with spherical distribution

In this section we consider robustness of the estimating functional of the mean direction of the von-Mises–Fisher distribution defined on the unit sphere.

7.1. Robustness of likelihood ratio test functional

We consider the testing problem \( H_0 : \tilde{\mu} = \tilde{\mu}_0 = (0, 0, 1) \) against \( H_1 : \tilde{\mu} \not= \tilde{\mu}_0 \). The LRT functional for this test of hypothesis is \( W^*(F) = \| E_F(\tilde{x}) \| - \tilde{\mu}_0^T E_F(\tilde{X}) \), which corresponds to the LRT statistic \( W^* = 2n\kappa(\bar{R} - \bar{C}) \) [13] where \( \bar{C} = \bar{\tilde{X}}^T \tilde{\mu}_0 \) denotes the component along \( \tilde{\mu}_0 \) of the vector mean \( \bar{X} \), \( \bar{R} = \| \bar{\tilde{X}} \| \) and \( \bar{\tilde{X}} = (\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta)^T \). Theorem 7.1 gives the LBF and PBF corresponding to the test functional \( W^* \).

Theorem 7.1

(a) Let \( y = \rho(1 - \varepsilon) + \varepsilon \cos \theta \) and \( c = 2\rho \sin^2(\alpha/2) \). Then the LBF of \( W^* \) is

\[
\varepsilon^*_\mu(W^*) = \inf\{\varepsilon > 0 : \sqrt{y^2 + \varepsilon^2 \sin^2 \theta} = y + c \text{ for some } \theta \in [0, \pi)\}.
\]

(b) The PBF of \( W^* \) is \( \varepsilon^*_\mu(W^*) = \min(\varepsilon_1, \varepsilon_2) \) where \( \varepsilon_1 \) and \( \varepsilon_2 \) are the roots of the quadratic equation \( (c^2 - 2cuv + u^2)\varepsilon^2 - 2c(uv - c)\varepsilon + c^2 = 0 \).

Proof (a) Using the definition of LBF we have

\[
W^*(G_\varepsilon) = \| (1 - \varepsilon)E_{F_0}(\bar{X}) + \varepsilon \bar{X} \| - \tilde{\mu}_0^T((1 - \varepsilon)E_{F_0}(\bar{X}) + \varepsilon \bar{X}) \text{ and } W^*(F_\mu) = 2\rho \sin^2\left(\frac{\alpha}{2}\right).
\]

This yields

\[
\varepsilon^*_\mu(W^*) = \inf\{\varepsilon > 0 : \sqrt{y^2 + \varepsilon^2 \sin^2 \theta} = y + c \text{ for some } \theta \in [0, \pi)\}.
\]

Remark We know that \( \sin^2 \theta \) is periodic with period \( \pi \). Therefore, if there is a solution to \( \sqrt{y^2 + \varepsilon^2 \sin^2 \theta} = y + c \) for \( \theta \) between \( \pi \) and \( 2\pi \), then there must also be a solution between \( 0 \) to \( \pi \). Also it can be seen that the LBF for this functional coincides with that of the LBF of the functional \( W \) (see Theorem 2.1). The LBP of \( W^* \) can be computed numerically.

(b) Using the definition of PBF we obtain \( W^*(G_\varepsilon) = \| (1 - \varepsilon)E_{F_0}(\bar{X}) + \varepsilon \bar{X} \| - \tilde{\mu}_0^T((1 - \varepsilon)E_{F_0}(\bar{X}) + \varepsilon \bar{X}) \) and \( W^*(F_0) = 0 \). This yield:

\[
\varepsilon^*_\mu(W^*) = \inf\left\{\varepsilon > 0 : (c^2 - 2cuv + u^2)\varepsilon^2 - 2c(uv - c)\varepsilon + c^2 = 0 \right\}.
\]

A detailed proof is given in the online supplementary material.

Remark The PBP of \( W^* \) can be computed numerically.
7.2. Robustness of spherical mean test functional

In this section we consider both spherical mean direction and spherical trimmed mean direction as test functionals. The spherical mean direction functional is defined by \( W^*_1(F) = E_F(\hat{X}) \) \[13\] where \( F \sim M(\mu, \kappa) \) and \( \hat{X} = (\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta)^T \) and the \( \gamma \)-spherical trimmed mean direction (\( \gamma \)-STM) is given in Definition 1. The hypothesis of interest is \( H_0 : \mu = \mu_0 = (0,0,1) \) against \( H_1 : \mu \neq \mu_0 \). Theorem 7.2 gives the LBF and PBF of \( W^*_1 \).

**Theorem 7.2** The LBF and PBF of \( W^*_1 \) coincides, i.e. \( \varepsilon^{*\ast}(W^*_1) = \varepsilon^*_\mu(W^*_1) = \min(\varepsilon_1, \varepsilon_2) \) for all \( \| \mu \| = 1, \mu \in \Re^3 \) where \( \varepsilon_1 \) and \( \varepsilon_2 \) are the roots of the quadratic equation

\[
(1 - \rho^2)\mu^2 - 2(1 - \rho^2 \cos \alpha)t + (1 - \rho^2) = 0.
\]

Here, we give an outline of the proof of the above theorem. A detailed proof is given in the online supplementary material.

**Proof** Let \( G_\varepsilon = (1 - \varepsilon)F_0 + \varepsilon \delta_x \) where \( F_0 \sim M(\mu_0, \kappa) \) and \( x \) is a point on the unit sphere. Then, we can write

\[
W^*_1(G_\varepsilon) = \begin{bmatrix}
\varepsilon \sin \theta \cos \varphi \\
\varepsilon \sin \theta \sin \varphi \\
(1 - \varepsilon)\rho + \varepsilon \cos \theta
\end{bmatrix}
\]

and \( W^*_1(F_\mu) = E_{F_\mu}(\hat{X}) = \begin{bmatrix}
\rho \sin \alpha \cos \beta \\
\rho \sin \alpha \sin \beta \\
\rho \cos \alpha
\end{bmatrix} \).

Now, using the definition of LBF and simplifying we obtain the quadratic equation

\[(1 - \rho^2)t^2 - 2(1 - \rho^2 \cos \alpha)t + (1 - \rho^2) = 0 \]

with solution \( t = (\varepsilon_1^2 \pm \sqrt{\varepsilon_1^2 - d_1^2})/d_1 \). Thus, \( \varepsilon^{*\ast}(W^*_1) = \min(\varepsilon_1, \varepsilon_2) \).

In the case of PBF, let \( G_\varepsilon = (1 - \varepsilon)F_\mu + \varepsilon \delta_x \) where \( F_\mu \sim M(\mu_0, \kappa) \) and \( x \) is a point on the unit sphere. Then, we can write

\[
W^*_1(G_\varepsilon) = \begin{bmatrix}
\rho(1 - \varepsilon) \sin \alpha \cos \beta + \varepsilon \sin \theta \cos \varphi \\
\rho(1 - \varepsilon) \sin \alpha \sin \beta + \varepsilon \sin \theta \sin \varphi \\
\rho(1 - \varepsilon) \cos \alpha + \varepsilon \cos \theta
\end{bmatrix}
\]

and \( W^*_1(F_0) = E_{F_0}(\hat{X}) = \begin{bmatrix}
0 \\
0 \\
\rho
\end{bmatrix} \).

Now, using the definition of PBF and arguing in a similar manner as in the case of LBF, we obtain the same quadratic equation. Now, when \( \phi = \beta + \pi \) then we have \( \cos \phi = -\cos \beta \), which also yields the same quadratic equation. \( \blacksquare \)

Hence the theorem is established.

**Remark** The LBP and PBP of \( W^*_1 \) can be obtained through numerical methods.

**Definition 1** Let \( C_\gamma \) be the geodesic disk of radius \( \gamma \) centred at \( \mu \) where \( \mu = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha)^T \) and \( \tau \) is the geodesic distance between \( \hat{X} \) and \( \mu \) where \( \hat{X} = (\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta)^T \). For \( 0 \leq \gamma < 0.5 \), define \( P_\gamma = \int_{C_\gamma} dF = 1 - 2\gamma \).

Then we define a \( \gamma \)-STM functional as \( W^{*\ast}_\gamma(F) = E_{F,\gamma}(\hat{X})/ \| E_{F,\gamma}(\hat{X}) \| \) where

\[
E_{F,\gamma}(\hat{X}) = \int \int_{C_\gamma} \hat{X} dF = \left[ \int \int_{C_\gamma} \sin \theta \cos \phi dF, \int \int_{C_\gamma} \sin \theta \cos \phi dF, \int \int_{C_\gamma} \cos \theta dF \right]^T.
\]

We introduce the following notations. Let \( P_\gamma = \int_{C_\gamma} dF \) and \( P^{*\ast}_\gamma = \int_{C_\gamma} dG_\varepsilon \) where \( C_\gamma \) is the geodesic disk of radius \( \gamma \) centred at \( \mu \) and \( G_\varepsilon = (1 - \varepsilon)F + \varepsilon \delta_x \) where \( F \sim M(\mu, \kappa) \). Also let \( \tau \)
be the geodesic distance between $\tilde{x}$ and $\tilde{\mu}$. Then we can write

$$P_y = \begin{cases} 
(1 - \varepsilon)P_y & \text{for } y < \tau, \\
(1 - \varepsilon)P_{\tau + \varepsilon} & \text{for } y \geq \tau.
\end{cases}$$

Note that if $\tau > y^*$ where $y^*$ is such that $P_{\gamma^*} = (1 - 2\gamma)(1 - \varepsilon)^{-1}$, then the trimmed mean does not change and hence $W_{\gamma^*}(G_{\varepsilon}) = W_{\gamma^*}(F)$. If $\tau < y^*$ then $P_{\gamma} = (1 - 2\gamma - \varepsilon)(1 - \varepsilon)^{-1}$. Hence we can write $W_{\gamma^*}(G_{\varepsilon})$ as

$$W_{\gamma^*}(G_{\varepsilon}) = \begin{cases} 
W_{\gamma^*}(F) & \text{if } \tau > y^*, \\
(1 - \varepsilon)\int_{C_{\varepsilon}} \tilde{X} dF + \varepsilon\tilde{X} & \text{if } \tau \leq y^*,
\end{cases}$$

where $\int_{C_{\varepsilon}} \tilde{X} dF = \left[\int_{C_{\varepsilon}} \sin \theta \cos \phi dF, \int_{C_{\varepsilon}} \sin \theta \cos \phi dF, \int_{C_{\varepsilon}} \cos \theta dF\right]^T$.

Therefore, $\varepsilon^*_\mu(W_{\gamma^*}) = \inf\{\varepsilon > 0 : W_{\gamma^*}(G_{\varepsilon}) = W_{\gamma^*}(F_{\tilde{\mu}}) \text{ for some } \theta \in [0, \pi) \text{ and } \phi \in [0, 2\pi]\}$ and $\varepsilon^*_\mu(W_{\gamma^*}) = \inf\{\varepsilon > 0 : W_{\gamma^*}(G_{\varepsilon}) = W_{\gamma^*}(F_{\tilde{\mu}}) \text{ for some } \theta \in [0, \pi) \text{ and } \phi \in [0, 2\pi]\}$.

8. Level and power of $\gamma$-CTM through simulations

In this section we examine the performance of the circular trimmed mean ($\gamma$-CTM) as a test statistic for testing $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$ when the observations come from circular normal distribution with contamination and compare that with the performance of the CM as a test statistic for the same problem. We study the power and level of both the test statistics for different values of $\kappa$, different levels of contamination ($\varepsilon$) and different sample sizes ($n$) through simulations. We run 10,000 simulations to estimate the level and power values for each of these cases. It is observed from the simulation results that for this testing problem the performance of $\gamma$-CTM with $\gamma = 0.05$ is better than that of the CM in terms of both level and power for different values of $\kappa$, $\varepsilon$ and $n$.

From Table B (online supplement) we see that both CM and $\gamma$-CTM perform similarly. They are both able to retain the level when the amount of contamination is small, but becomes anti-conservative as the amount of contamination increases. From Table C (online supplement) it can be seen that both CM and $\gamma$-CTM are able to retain their level despite contamination for different values of $n$. From Table 4, it can be observed that for small values of $\kappa$ both CM and $\gamma$-CTM retain their nominal level while as $\kappa$ increases $\gamma$-CTM becomes anti-conservative. From Table 5, it can be observed that for small values of $\kappa$, the $\gamma$-CTM has slightly better power performance. However, for larger values of $\kappa$, the CM has better power performance. From table D (online supplement), it is evident that the power performance of both CM and $\gamma$-CTM is similar for

| $\mu$ | $\gamma$-CTM ($\kappa = 1$) | $\gamma$-CTM ($\kappa = 2$) | $\gamma$-CTM ($\kappa = 10$) | $\gamma$-CTM ($\kappa = 10$) |
|-------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| $-\pi/4$ | 0.05                        | 0.05                        | 0.05                        | 0.05                        |
| $-\pi/6$ | 0.05                        | 0.05                        | 0.05                        | 0.05                        |
| $-\pi/8$ | 0.05                        | 0.04                        | 0.05                        | 0.05                        |
| $-\pi/50$ | 0.04                        | 0.05                        | 0.05                        | 0.05                        |
| $\pi/8$ | 0.05                        | 0.05                        | 0.05                        | 0.05                        |
| $\pi/6$ | 0.05                        | 0.05                        | 0.07                        | 0.05                        |
| $\pi/4$ | 0.05                        | 0.05                        | 0.05                        | 0.06                        |

Note: ($n = 100$, $\varepsilon = 0.01$ and $\gamma = 0.05$).
Table 5. Power values of CM and $\gamma$-CTM tests when observations come from \((1 - \varepsilon)CN(0, \kappa) + \varepsilon \delta_x\) for different values of $\kappa$.

| $\mu_a$ | CM ($\kappa = 1$) | $\gamma$-CTM ($\kappa = 1$) | CM ($\kappa = 2$) | $\gamma$-CTM ($\kappa = 2$) | CM ($\kappa = 10$) | $\gamma$-CTM ($\kappa = 10$) |
|--------|------------------|-------------------------------|------------------|-------------------------------|------------------|-------------------------------|
| $\pi/4$ | 0.99             | 0.99                          | 1.00             | 1.00                          | 1.00             | 1.00                          |
| $\pi/8$ | 0.68             | 0.70                          | 0.99             | 0.98                          | 1.00             | 1.00                          |
| $\pi/12$ | 0.36            | 0.40                          | 0.83             | 0.77                          | 1.00             | 1.00                          |
| $\pi/25$ | 0.11            | 0.14                          | 0.25             | 0.25                          | 0.93             | 0.83                          |
| $\pi/50$ | 0.06            | 0.06                          | 0.09             | 0.09                          | 0.38             | 0.32                          |
| $\pi/100$ | 0.05           | 0.05                          | 0.06             | 0.04                          | 0.12             | 0.12                          |

Note: ($n = 100$, $\varepsilon = 0.01$, $\mu_a = -\pi/2$ and $\gamma = 0.05$).

Table 6. Power values of CM and $\gamma$-CTM tests when observations come from \((1 - \varepsilon)CN(0, \kappa) + \varepsilon \delta_x\) for different values of $\varepsilon$.

| $\varepsilon$ | CM ($x = \pi/4$) | $\gamma$-CTM ($x = \pi/4$) | CM ($x = \pi/8$) | $\gamma$-CTM ($x = \pi/8$) | CM ($x = \pi/12$) | $\gamma$-CTM ($x = \pi/12$) |
|--------------|------------------|-------------------------------|------------------|-------------------------------|------------------|-------------------------------|
| 0.01         | 0.99             | 0.99                          | 0.68             | 0.68                          | 0.37             | 0.36                          |
| 0.05         | 0.99             | 0.99                          | 0.42             | 0.71                          | 0.19             | 0.39                          |
| 0.10         | 0.90             | 0.99                          | 0.22             | 0.70                          | 0.11             | 0.40                          |
| 0.15         | 0.69             | 0.99                          | 0.13             | 0.68                          | 0.23             | 0.40                          |
| 0.20         | 0.43             | 0.99                          | 0.34             | 0.70                          | 0.50             | 0.42                          |

Note: ($\kappa = 1$, $n = 100$, $\mu_a = -\pi/2$ and $\gamma = 0.05$).

different values of $n$. It can be seen from Table 6 that the $\gamma$-CTM is far better in terms of power than CM particularly when the contaminating observations are far away from zero.

9. Examples

In this section we consider two real-life data sets one given in Fisher and Lewis [41] and the other in Mardia.[42] Each of these data sets may be assumed to come from a circular normal distribution with possible contamination. This is justified since the analysis presented in Sengupta and Laha [43] shows that outliers are present in both of these data sets. We consider the hypothesis testing problem $H_0: \mu = 0$ against $H_1: \mu \neq 0$. We apply the three statistics LRT, CM and $\gamma$-CTM with $\gamma = 0.05$ and see their performance.

Example 1  Fisher and Lewis [41] give data from three samples of paleocurrent orientations from three bedded sandstone layers, measured on the Belford Anticline, New South Wales. Here, we consider the first sample. The dataset is given in SenGupta and Laha.[43] The LRT rejects the null hypothesis for large values of the LRT statistic. The LRT rejects the null hypothesis at 5% level of significance (obtained $p$-value < .01). In application of the CM and $\gamma$-CTM test statistics, we need to know the true value $\kappa$. We take an adaptive approach and treat the estimated value of $\kappa$ as the true value. The $\kappa$ value used for the test based on CM statistic is the estimated value of $\kappa$ from the entire sample which is 0.89. The CM-test rejects the null hypothesis if the observed value of the CM-test statistic falls outside the arc between the two cut-off values traversed in the anticlockwise direction. The cut-off values of the CM-test statistic are obtained through simulation under the null distribution. We find that the CM statistic based test rejects $H_0$ at 5% level of significance.

The value of $\kappa$ used for the test based on $\gamma$-CTM ($\gamma = 0.05$) statistic is 1.64, which is the estimated value of $\kappa$ from the trimmed sample. The $\gamma$-CTM test rejects the null hypothesis if the
observed value of the $\gamma$-CTM test statistic falls outside the arc between the two cut-off values traversed in the anticlockwise direction. The cut-off values of $\gamma$-CTM test statistic are obtained through simulation under the null distribution. We find that the $\gamma$-CTM statistic based test rejects $H_0$ at 5% level of significance.

**Example 2** Our next example is the famous roulette wheel data obtained from Mardia.[42] The LRT rejects the null hypothesis at 5% level of significance (obtained p-value < .01). Here also as in Example 1 we take an adaptive approach. For the test based on the CM statistic, the estimated value of $\kappa$ from the entire sample is obtained as 2.08 and the cut-off values of the test are obtained through simulation. We find that the test based on CM statistic rejects $H_0$ at 5% level of significance.

In the case of test based on $\gamma$-CTM ($\gamma = 0.05$) statistic, the estimated value of $\kappa$ from the trimmed sample is obtained as 4.78 and the cut-off values of $\gamma$-CTM test statistic are obtained through simulation. We find that the test based on $\gamma$-CTM statistic also rejects $H_0$ at 5% level of significance.

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**Supplemental materials**

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