Geometrical approach to Feynman integrals and the $\varepsilon$-expansion

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Abstract

Application of the geometrically-inspired representations to the $\varepsilon$-expansion of the two-point function with different masses is considered. Explicit result for an arbitrary term of the expansion is obtained in terms of log-sine integrals. Construction of the $\varepsilon$-expansion in the three-point case is also discussed.

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When studying Feynman diagrams, especially those with three or more external particles, one of the most important issues is how to choose appropriate variables. In particular, these variables should be chosen in such a way that the solutions of Landau equations [1], i.e. the positions of possible singularities, could be easily identified and understood. Studying the singularities of diagrams was the first reason of introducing geometrical ideas for the three-point [2] and four-point [3] functions at one loop. With a one-loop \(N\)-point diagram one can associate an \(N\)-dimensional Euclidean basic simplex (for details, see e.g. in [4]) whose sides are directly related to the masses and the external momenta invariants. Then, the situations when the volume (content) of this simplex vanishes correspond to the positions of possible singularities. For instance, for a two-point function with an external momentum \(k_{12}\) this basic simplex is just a triangle whose two sides are equal to \(m_1\) and \(m_2\) (the masses of the internal particles) and the third side is \(K_{12} = \sqrt{k_{12}^2}\). Its area vanishes when \(k_{12}^2 = (m_1 \pm m_2)^2\), i.e. at the threshold and pseudothreshold.

Another important step was to understand that the evaluation of Feynman integrals themselves, not only looking for the positions of the singularities, could be also reduced to a purely geometrical problem. In particular, as it is shown in [4] (see also in [5] for a brief review), the Feynman parametric representation of a one-loop \(N\)-point integral can be directly transformed into an integral over one of the \(N\)-dimensional solid angles of this basic simplex. The latter can be understood as the integral over an \((N-1)\)-dimensional simplex in non-Euclidean space of constant curvature. In particular, the four-point function in four dimensions is proportional to the volume of a non-Euclidean tetrahedron in spherical or hyperbolic space. The three-point function in three dimensions is just the area of a spherical (or hyperbolic) triangle; in this way, a very nice result of [6] was reproduced in a purely geometrical way. The three-point function in four dimensions is also an integral over the same triangle, but with an extra weight factor \(1/\cos \theta\), where \(\theta\) is the angle between the running vector of integration and the direction of the height of the basic simplex (which is a tetrahedron in this case). One can even get rid of this weight factor, reducing the three-point function to a special (asymptotic) case of the four-point function [7]. For some other links between Feynman diagrams and non-Euclidean geometry, see in [8].

Instead of repeating all conclusions given in [4], we have decided to present here an extended discussion of using the geometrically-inspired representations for the evaluation of integrals in the framework of dimensional regularization [9]. Sometimes it is possible to present results valid for an arbitrary \(\varepsilon\), usually in terms of hypergeometric functions. However, for practical purposes the coefficients of the expansion in \(\varepsilon\) are important. In particular, in multiloop calculations higher terms of the \(\varepsilon\)-expansion of the one-loop functions are needed, since one can get contributions where these functions are multiplied by singular factors containing poles in \(\varepsilon\). In this paper, we derive an explicit result for an arbitrary term of the \(\varepsilon\)-expansion of two-point function with arbitrary masses. We also discuss the construction of the \(\varepsilon\)-expansion of the three-point function.
2. According to section IV of [4], for the two-point function with an external momentum $k_{12}$ and internal masses $m_1$ and $m_2$, with unit powers of propagators ($\nu_1 = \nu_2 = 1$), in $n = 4 - 2\varepsilon$ dimensions, we get

$$J^{(2)}(4 - 2\varepsilon; 1, 1) = i \pi^{2-\varepsilon} \Gamma(\varepsilon) \frac{m_0^{1-2\varepsilon}}{\sqrt{k_{12}^2}} \left\{ \Omega_1^{(2;4-2\varepsilon)} + \Omega_2^{(2;4-2\varepsilon)} \right\},$$

(1)

where

$$\Omega_i^{(2;4-2\varepsilon)} = \int_{0}^{\tau_{0i}} \frac{d\theta}{\cos^{2-2\varepsilon}\theta}. \quad (2)$$

Here it is assumed that $(m_1 - m_2)^2 \leq k_{12}^2 \leq (m_1 + m_2)^2$. In other regions one needs to use analytic continuation. The following notation [4] is used:

$$\cos \tau_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1m_2}, \quad m_0 = \frac{m_1m_2 \sin \tau_{12}}{\sqrt{k_{12}^2}}, \quad \cos \tau_{0i} = \frac{m_0}{m_i}. \quad (3)$$

Note that $\tau_{01} + \tau_{02} = \tau_{12}$ and

$$\tan \tau_{01} = \frac{m_1^2 - m_2^2 + k_{12}^2}{2m_0\sqrt{k_{12}^2}}, \quad \tan \tau_{02} = \frac{m_2^2 - m_1^2 + k_{12}^2}{2m_0\sqrt{k_{12}^2}}. \quad (4)$$

The integral (2) can be presented in terms of the hypergeometric function $\, _2F_1$ given in eq. (4.12) of [4] (see also in [10]). Expanding (2) in $\varepsilon$, we get

$$\int_{0}^{\tau} \frac{d\theta}{\cos^{2-2\varepsilon}\theta} = \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{j!} \int_{0}^{\tau} \frac{d\theta}{\cos^2\theta} \ln^{j}(\cos \theta) = \sum_{j=0}^{\infty} \frac{(-2\varepsilon)^j}{j!} f_j(\tau), \quad (5)$$

$$f_j(\tau) \equiv (-1)^j \int_{0}^{\tau} \frac{d\theta}{\cos^2\theta} \ln^j(\cos \theta). \quad (6)$$

The lowest terms of the expansion are (cf. eqs. (4.9) and (4.11) of [4])

$$f_0(\tau) = \tan \tau, \quad (7)$$
$$f_1(\tau) = - \tan \tau \ln(\cos \tau) - \tan \tau + \tau, \quad (8)$$
$$f_2(\tau) = \tan \tau \left[ \ln^2(\cos \tau) + 2 \ln(\cos \tau) + 2 \right] - 2\tau(1 - \ln 2) - \text{Cl}_2(\pi - 2\tau). \quad (9)$$

Integrating by parts, we see that the $f_j$ function (6) can be reduced to the generalized Lobachevsky function

$$L_j(\tau) \equiv (-1)^{j-1} \int_{0}^{\tau} d\theta \ln^{j-1}(\cos \theta) \quad (10)$$

via

$$f_j(\tau) = (-1)^j \tan \tau \ln^j(\cos \tau) + jL_j(\tau) - jf_{j-1}(\tau). \quad (11)$$
Repeating this procedure, we arrive at

$$f_j(\tau) = j! \sum_{l=1}^{j} \frac{(-1)^{j-l}}{(l-1)!} L_l(\tau) + (-1)^j j! \tan \tau \sum_{l=0}^{j} \frac{\ln^l(\cos \tau)}{l!}. \quad (12)$$

When we consider the infinite sum over $j$, eq. (5), the logarithmic tail in eq. (12) can be summed up into a closed form, since

$$\sum_{j=0}^{\infty} (2\varepsilon)^j \sum_{l=0}^{j} \frac{\ln^l(\cos \tau)}{l!} \frac{1}{1 - 2\varepsilon} = \frac{\cos^{2\varepsilon} \tau}{1 - 2\varepsilon}. \quad (13)$$

Therefore,

$$\int_0^{\tau} \frac{d\theta}{\cos^{2-2\varepsilon} \theta} = \frac{\cos^{2\varepsilon} \tau}{1 - 2\varepsilon} \tan \tau + \sum_{j=1}^{\infty} (2\varepsilon)^j \sum_{l=1}^{j} \frac{(-1)^l}{(l-1)!} L_l(\tau). \quad (14)$$

It is obvious that $L_1(\tau) = \tau$. Then, note that $L_2(\tau) \equiv L(\tau)$ is what is usually called the Lobachevsky function (see e.g. in Appendix A of [4]). It is related to the Clausen function $\text{Cl}_2$ via

$$L_2(\tau) = -\frac{1}{2} \text{Cl}_2(\pi - 2\tau) + \tau \ln 2.$$

By a simple transformation of the integration variable, the $L_j$ function (10) can be presented as a finite sum of log-sine integrals (see in [11], chapter 7.9)

$$L_{s,j}(\theta) = -\int_0^{\theta} d\theta' \ln^{j-1} \left| 2 \sin \frac{\theta'}{2} \right|, \quad (15)$$

namely:

$$L_j(\tau) = \frac{(j-1)!}{2} \sum_{i=1}^{j} \frac{(-1)^i}{(j-i)! (i-1)!} \ln^{j-i} \frac{1}{2} \left[ L_{s,i}(\pi) - L_{s,i}(\pi - 2\tau) \right]. \quad (16)$$

Note that $L_{s,1}(\theta) = -\theta$, $L_{s,2}(\theta) = \text{Cl}_2(\theta)$, $L_{s,3}(\pi) = 0$. Some other values of $L_{s,i}(\pi)$ are given in eq. (7.113) of [11] in terms of $\zeta(j)$. The result for $L_{s,i}(\pi)$ with an arbitrary $i$ is given in eq. (45) of Appendix A.2.7 of [11] in terms of a differential operator, which can be represented in a more compact form as

$$L_{s,j+1}(\pi) = -\frac{\pi}{2j} \left. \left( \frac{d}{dz} \right)^j \frac{\Gamma(1+2z)}{\Gamma^2(1+z)} \right|_{z=0}. \quad (17)$$

For $j = 3$, eq. (16) yields

$$L_3(\tau) = \frac{1}{2} L_{s,3}(\pi - 2\tau) - \ln 2 \text{Cl}_2(\pi - 2\tau) + \tau \ln^2 2 + \frac{\pi^3}{24},$$

$$f_3(\tau) = -\tan \tau \left( \ln^3(\cos \tau) + 3 \ln^2(\cos \tau) + 6 \ln(\cos \tau) + 6 \right)$$

$$+ 3\tau \left( 2 - 2 \ln 2 + \ln^2 2 \right)$$

$$+ \frac{\pi^3}{8} + 3(1 - \ln 2) \text{Cl}_2(\pi - 2\tau) + \frac{3}{2} L_{s,3}(\pi - 2\tau). \quad (18)$$
Note that the function $L_{3}$ has also occurred in the $\varepsilon$-part of a triangle diagram with massless internal particles \[12\].

Substituting (16) into eq. (14), the resulting three-fold sum can be simplified,

$$\frac{1}{2} \sum_{j=1}^{\infty} (2\varepsilon)^j \sum_{l=1}^{j} \sum_{i=1}^{l} \frac{(-1)^{l-i} \ln^{l-i} 2}{(l-i)!(i-1)!} [L_{3}(\pi) - L_{3}(\pi - 2\tau)]$$

$$= \frac{2^{-2\varepsilon}}{1 - 2\varepsilon} \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{j!} [L_{3+j}(\pi) - L_{3+j}(\pi - 2\tau)].$$

(19)

Therefore,

$$\int_{0}^{\tau} \frac{d\theta}{\cos^{2-2\varepsilon} \theta} = \frac{\cos^{2\varepsilon} \pi}{1 - 2\varepsilon} \tan \tau + \frac{2^{-2\varepsilon}}{1 - 2\varepsilon} \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{j!} [L_{3+j}(\pi) - L_{3+j}(\pi - 2\tau)].$$

(20)

Finally, taking into account eqs. (3)–(4), we obtain for the integral (1)

$$J^{(2)}(4 - 2\varepsilon; 1, 1) = i\pi^{2-\varepsilon} \frac{\Gamma(1+\varepsilon)}{1 - 2\varepsilon} \left\{ \frac{m_{1}^{-2\varepsilon} + m_{2}^{-2\varepsilon}}{2\varepsilon} + \frac{m_{1}^{2} - m_{2}^{2}}{2\varepsilon k_{12}^{2}} (m_{1}^{-2\varepsilon} - m_{2}^{-2\varepsilon}) \right\}$$

$$+ \frac{2^{-2\varepsilon} (m_{1}m_{2} \sin \tau_{12})^{1-2\varepsilon}}{(k_{12}^{2})^{1-\varepsilon}} \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{j!}$$

$$\times [2L_{3+j}(\pi) - L_{3+j}(\pi - 2\tau_{01}) - L_{3+j}(\pi - 2\tau_{02})].$$

(21)

The coefficient at $\varepsilon^j$ has a closed form in terms of the $L_{3+j}$ functions, whose arguments have a simple geometrical interpretation. Note that, according to eq. (17), the infinite sum with $L_{3+j}(\pi)$ in eqs. (20)–(21) can be converted into

$$\sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{j!} L_{3+j}(\pi) = -\pi \frac{\Gamma(1 + 2\varepsilon)}{\Gamma^2(1 + \varepsilon)}.$$ 

(22)

As it was mentioned before, the obtained result (21) can be directly applied in the region $(m_{1} - m_{2})^{2} \leq k_{12}^{2} \leq (m_{1} + m_{2})^{2}$. In other regions, the analytic continuation of (21) gives expressions in terms of (generalized) polylogarithms. The result for the $\varepsilon$-term was obtained in \[13\]. For the case $m_{1} = 0$ ($m_{2} \equiv m$), the first terms of the expansion (up to $\varepsilon^3$) are presented in eq. (A.3) of \[14\].

An important special case of eq. (21) is $m_{1} = m_{2} \equiv m$, $k_{12}^{2} = m^{2}$. In this case, $\tau_{12} = \frac{1}{3} \pi$ and $\tau_{01} = \tau_{02} = \frac{1}{6} \pi$. Therefore, the $j$-th term of the expansion contains $L_{3+j}(2\pi/3)$. In particular, this is the reason why $L_{3}(2\pi/3)$ appears in the results for some two-loop diagrams considered in \[15\].

3. The geometrical approach to the three-point function is discussed in section V of \[4\]. The integration extends over a spherical (or hyperbolic) triangle, as shown in
Fig. 6 of [4], with a weight factor $1/\cos^{1-2\varepsilon}\theta$ (see eqs. (3.38)–(3.39) of [4]). This triangle 123 is split into three triangles 012, 023 and 031. Then, each of them is split into two rectangular triangles, according to Fig. 9 of [4]. We consider the contribution of one of the six resulting triangles, namely the left rectangular triangle in Fig. 9. Its angle at the vertex 0 is denoted as $\frac{1}{2}\varphi_{12}^+$, whereas the height dropped from the vertex 0 is denoted $\eta_{12}$.

As compared to the four-dimensional ($\varepsilon = 0$) case described in section VB of [4], eq. (5.14) should be transformed (for non-zero $\varepsilon$) into

$$-rac{1}{2\varepsilon} \frac{\partial}{\partial \xi} \cos^{2\varepsilon}\theta(\xi, \varphi),$$

where $\xi$ is an auxiliary variable ($0 \leq \xi \leq 1$). Integrating (23) over $\xi$ yields

$$\frac{1}{2\varepsilon} \left[ \cos^{2\varepsilon}\theta(0, \varphi) - \cos^{2\varepsilon}\theta(1, \varphi) \right] = \frac{1}{2\varepsilon} \left[ 1 - \left( 1 + \frac{\tan^2 \eta_{12}}{\cos^2 \varphi} \right)^{-\varepsilon} \right].$$

Therefore, the remaining $\varphi$ integration is (cf. eq. (5.16) of [4])

$$\frac{1}{2\varepsilon} \int_{0}^{\varphi_{12}/2} d\varphi \left[ 1 - \left( 1 + \frac{\tan^2 \eta_{12}}{\cos^2 \varphi} \right)^{-\varepsilon} \right]$$

$$= \frac{1}{2} \sum_{j=0}^{\infty} \frac{(-\varepsilon)^j}{(j + 1)!} \int_{0}^{\varphi_{12}/2} d\varphi \ln^{j+1} \left( 1 + \frac{\tan^2 \eta_{12}}{\cos^2 \varphi} \right).$$

In the limit $\varepsilon \to 0$ we get a combination of Cl$_2$ functions, eq. (5.17) of [4].

Collecting the results for all six triangles, we get the result for the three-point function with arbitrary masses and external momenta, corresponding (at $\varepsilon = 0$) to the analytic continuation of the well-known formula presented in [16]. The $\varepsilon$-term of the three-point function has been calculated in [13] in terms of Li$_3$. The case of massless internal particles has been considered in [17]. Its analytic continuation in terms of Li$_3$ functions has been constructed in [12]. Note that the same functions occur in the results for the two-loop vacuum diagrams with different masses. This connection was observed in [18] and then explained in [12]. We believe that the higher terms of the expansion (23) can also be presented in a compact form in terms of the Li$_j$ functions.

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