Article

Coincidences of the concave integral and the pan-integral

Yao Ouyang 1*, Jun Li 2 and Radko Mesiar 3,4

1 Faculty of Science, Huzhou teacher’s college, Huzhou, Zhejiang 313000, China; oyy@zjhu.edu.cn
2 School of Sciences, Communication University of China, Beijing 100024, China; lijun@cuc.edu.cn
3 Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 810 05 Bratislava, Slovakia; mesiar@math.sk
4 UTIA CAS, Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic

* Correspondence: oyy@zjhu.edu.cn
Academic Editor: name
Version May 28, 2017 submitted to Symmetry

Abstract: In this note, we discuss when the concave integral coincides with the pan-integral with respect to the standard arithmetic operations + and ·. The subadditivity of the underlying monotone measure is one sufficient condition for this equality. We show also another sufficient condition, which, in the case of finite spaces, is necessary, too. Some convergence results concerning pan-integrals are also included.

Keywords: Monotone measure; subadditivity; Concave integral; pan-integral.

1. Introduction

Integrals play a prominent role in almost any area dealing with quantitative information, varying from physics to sociology, including economy or engineering, but also many intelligent systems. The standard calculus is based on the Riemann integral [27]. Note that Riemann has generalized the earlier approaches known from antic Greece, and he has completed the ideas originated by Newton, Leibniz, Cauchy and others. Lebesgue [11] has further generalized this integral, working with sigma-additive measures, and thus he has enabled the development of many other theories, first of all the Kolmogorovian probability theory [10]. Even in Kolmogorov era, there were ideas of integrating some particular non-additive measures, especially outer and inner measures, see [32]. These efforts were completed by the introduction of the Choquet integral [3], which for sigma-additive measures coincides with the Lebesgue integral. Further development of integrals based on monotone but non necessarily additive measures was initiated first of all by needs of economy, multicriteria decision support, psychology, sociology, etc., i.e., by needs of branches where the phenomenon of interaction is crucial. Among these new types of integrals (based on monotone measures) recall Sugeno integral [29], Shilkret integral [28], pan-integral [35], and the concave integral introduced by Lehrer [12]. Note that there are successful efforts how to axiomatize some types of integrals, see, e.g. the concept of universal integrals from [8], or how to construct integrals, recall the decomposition integrals introduced in [5]. As already mentioned, the Choquet integral generalizes the Lebesgue integral, i.e., for any sigma-additive measure µ these integrals coincide. Similarly, when considering a sigma-additive measure µ, the Lebesgue integral coincides with the pan-integral, as well as with the concave integral. Note that this is not the case of the Shilkret integral neither of the Sugeno integral. Recall also that all three earlier mentioned integrals (Choquet, pan and concave integrals) are decomposition integrals. namely, the Choquet integral is based on finite chains, the pan-integral is based on finite partition while the concave integral is related to arbitrary finite set systems, for more details see [5]. The aim of this paper
is a further discussion of the coincidence of integrals, whose starting point is the above mentioned fact that, if a sigma-additive measure $\mu_j$ is considered, the all four Lebesgue, Choquet, pan and concave integrals coincide. Obviously, for $\mu_j$ which is not sigma-additive, the Lebesgue integral is not defined, and the remaining three integrals are different, in general. Nevertheless, for some particular monotone measure $\mu_j$, some of these integrals may coincide.

Lehrer [12,13] discussed the relationship between the concave integral and the Choquet integral, and showed that these two integrals coincide if and only if the underlying capacity $\nu$ is convex (also known as supermodular). In [34] the order relationship between the pan-integral (with respect to the usual addition $+$ and usual multiplication $\cdot$) and the Choquet integral was shown by using the subadditivity and superadditivity of monotone measures.

We have recently discussed the relationship between the concave integral and the pan-integral on finite spaces [25]. We have introduced the concept of minimal atom of a monotone measure. By means of two important structure characteristics related to minimal atoms: minimal atoms disjoint property and subadditivity for minimal atoms, we have shown a necessary and sufficient condition ensuring that the concave integral coincides with the pan-integral on finite spaces. A research on coincidences of the Choquet integral and the pan-integral on finite space was made by using the minimal atom of monotone measure (see [24]).

We pointed out that in the above-mentioned study we have only considered the case that the underlying space is finite. But our approach based on minimal atoms does not apply to infinite spaces, see [25].

This paper will focus on the relationship between the concave integrals and pan-integrals on general spaces (not necessarily finite). We shall show that if the underlying monotone measure $\mu$ is subadditive, then the concave integral coincides with the pan-integral w.r.t. the usual addition $+$ and usual multiplication $\cdot$.

2. Preliminaries

Let $X$ be a nonempty set and $\mathcal{A}$ a $\sigma$-algebra of subsets of $X$. $\mathcal{F}_+$ denotes the class of all finite nonnegative real-valued measurable functions on the measurable space $(X, \mathcal{A})$. Unless stated otherwise all the subsets mentioned are supposed to belong to $\mathcal{A}$, and all the functions mentioned are supposed to belong to $\mathcal{F}_+$.

**Definition 1.** ([34]) A monotone measure on $\mathcal{A}$ is an extended real valued set function $\mu : \mathcal{A} \to [0, +\infty]$ satisfying the following conditions:

1. $\mu(\emptyset) = 0$; (vanishing at $\emptyset$)
2. $\mu(A) \leq \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{F}$. (monotonicity)

When $\mu$ is a monotone measure, the triple $(X, \mathcal{A}, \mu)$ is called a monotone measure space ([15,26,34]). In some literature, such a monotone measure $\mu$ constrained by the boundary condition $\mu(X) = 1$ is also called a capacity or a fuzzy measure or a nonadditive probability, etc..

Let $\mu$ be a monotone measure on $(X, \mathcal{A})$. $\mu$ is said to be

(i) subadditive if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$;
(ii) superadditive if $\mu(A \cup B) \geq \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$ such that $A \cap B = \emptyset$ [4];
(iii) supermodular if $\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$ [4];
(iv) continuous from below (resp. from above), if $\lim_{n \to \infty} \mu(E_n) = \mu(E)$ whenever $E_n \nearrow E$ (resp. whenever $E_n \searrow E$ and $\mu(E_1) < \infty$) ([6]).

In our discussions we concern three types of nonlinear integrals, the Choquet integral, the concave integral and the pan-integral. We recall their definitions.

We consider a given monotone measure space $(X, \mathcal{A}, \mu)$, and let $f \in \mathcal{F}_+, \chi_A$ denote the indicator function of measurable set $A$. 
The Choquet integral [3] (see also [4,26]) of $f$ on $X$ with respect to $\mu$, is defined by

$$\int f \, d\mu = \int_0^\infty \mu(\{x : f(x) \geq t\}) \, dt,$$

where the right side integral is the Riemann integral.

Lehrer [13] introduced a new integral known as concave integral (see also [12,31]), as follows:

The concave integral of $f$ on $X$ is defined by

$$\int f \, d\mu = \sup \left\{ \sum_{i=1}^n \lambda_i \mu(A_i) : \sum_{i=1}^n \lambda_i \chi_{A_i} \leq f, \right.$$

$$\left. \{A_i\}_{i=1}^n \subset \mathcal{A}, \lambda_i \geq 0, n \in \mathbb{N} \right\}.$$

The concept of a pan-integral [34,35] involves two binary operations, the pan-addition $\oplus$ and pan-multiplication $\otimes$ of real numbers (see also [2,17,22,26,30,33,34]). In this paper we only consider the pan-integrals with respect to the usual addition $+$ and usual multiplication $\cdot$. Note that the general case of pan-integrals is discussed in Concluding Remarks.

The pan-integral of $f$ on $X$ w.r.t. the usual addition $+$ and usual multiplication $\cdot$ (in short, pan-integral), is given by

$$\int f \, d\mu = \sup \left\{ \sum_{i=1}^n \lambda_i \mu(A_i) : \sum_{i=1}^n \lambda_i \chi_{A_i} \leq f, \right.$$

$$\left. \{A_i\}_{i=1}^n \subset \mathcal{A} \text{ is a partition of } X, \lambda_i \geq 0, n \in \mathbb{N} \right\}.$$

All these integrals are covered by a resent concept of decomposition integrals by Even and Lehrer [5].

Note that the pan-integral is related to finite partitions of $X$, the concave integral to any finite set systems of measurable subsets of $X$. The Choquet integral is based on chains of sets, it can be expressed in the following form:

$$\int f \, d\mu = \sup \left\{ \sum_{i=1}^n \lambda_i \mu(A_i) : \sum_{i=1}^n \lambda_i \chi_{A_i} \leq f, \right.$$

$$\left. \{A_i\}_{i=1}^n \subset \mathcal{A} \text{ is a chain}, \lambda_i \geq 0, n \in \mathbb{N} \right\}.$$

Comparing above three definitions, it is obvious that for each $f \in \mathcal{F}_+$,

$$\int f \, d\mu \geq \int f \, d\mu,$$

and

$$\int f \, d\mu \geq \int f \, d\mu.$$

In general, $\int f \, d\mu \neq \int f \, d\mu$, $\int f \, d\mu \neq \int f \, d\mu$.

**Example 2.** Let $X = \mathbb{N}$ (the set of all positive integers). The monotone measure $\mu : 2^\mathbb{N} \to [0,1]$ is defined by

$$\mu(E) = \begin{cases} 
1 & \text{if } |E| = \infty \text{ and } 1 \in E, \\
0 & \text{otherwise}. 
\end{cases}$$

We take

$$f(x) = \begin{cases} 
2, & \text{if } x = 1; \\
1, & \text{if } x = 2, 3, \ldots.
\end{cases}$$
Then $\int_{\text{cav}} f \, d\mu = 2$, and $\int_{\text{pan}} f \, d\mu = \int_{\text{Cho}} f \, d\mu = 1$. Thus, $\int_{\text{cav}} f \, d\mu \neq \int_{\text{pan}} f \, d\mu$, $\int_{\text{cav}} f \, d\mu \neq \int_{\text{Cho}} f \, d\mu$.

Observe that the Choquet integral and the pan-integral are not comparable.

**Example 3.** Let $X = \{1, 2\}$, $A = 2^X$, and the monotone measure $\mu$ be defined as $\mu(X) = 3$, $\mu(\{1\}) = \mu(\{2\}) = 1$, $\mu(\emptyset) = 0$. Let $f(x) = x$. Then

$$\int_{\text{Cho}} f \, d\mu = \mu(X) + \mu(\{2\}) = 4$$

and

$$\int_{\text{pan}} f \, d\mu = \max \left( \mu(X), \mu(\{1\}) + 2\mu(\{2\}) \right) = 3.$$

Thus, we have $\int_{\text{Cho}} f \, d\mu > \int_{\text{pan}} f \, d\mu$.

**Example 4.** Let $X = \{1, 2\}$, $A = 2^X$, and the monotone measure $\mu$ be defined as $\mu(A) = 1$ if $A \neq \emptyset$ and $\mu(\emptyset) = 0$. Let $f(x) = x$. Then

$$\int_{\text{Cho}} f \, d\mu = \mu(X) + \mu(\{2\}) = 2$$

and

$$\int_{\text{pan}} f \, d\mu = \max \left( \mu(X), \mu(\{1\}) + 2\mu(\{2\}) \right) = 3.$$

Thus, $\int_{\text{Cho}} f \, d\mu < \int_{\text{pan}} f \, d\mu$.

The above examples indicate that any two of the three integrals do not coincide, in general. They are significantly different from each other.

### 3. The main results

We consider a given measurable space $(X, A)$, and let $\mathcal{M}$ be the class of all monotone measures defined on $(X, A)$.

For the convenience of our discussion, we denote $\text{Ch}_\mu(f) = \int_{\text{Cho}} f \, d\mu$, $\text{Cav}_\mu(f) = \int_{\text{cav}} f \, d\mu$ and $\text{Pan}_\mu(f) = \int_{\text{pan}} f \, d\mu$.

In [13] (see also [1,12,14]) the relationship between the the concave integral and the Choquet integral was discussed, as follows:

**Theorem 5.** Given $\mu \in \mathcal{M}$. Then $\text{Cav}_\mu \equiv \text{Ch}_\mu$, i.e., for each $f \in \mathcal{F}_+$,

$$\int_{\text{cav}} f \, d\mu = \int_{\text{Cho}} f \, d\mu$$

if and only if $\mu$ is supermodular, i.e., for any $A, B \in A$

$$\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B).$$

The following results were shown in [34] (Theorem 10.7 and 10.8 in [34]).

**Theorem 6.** Given $\mu \in \mathcal{M}$. Then

(i) if $\mu$ is superadditive, then $\text{Pan}_\mu \leq \text{Ch}_\mu$, i.e., for each $f \in \mathcal{F}_+$, $\text{Pan}_\mu(f) \leq \text{Ch}_\mu(f)$;

(ii) if $\mu$ is subadditive, then $\text{Pan}_\mu \geq \text{Ch}_\mu$.

Moreover, we have the following result (see also Mesiar et al. [?]):
**Theorem 7.** Given $\mu \in \mathcal{M}$. If $\text{Pan}_\mu \equiv \text{Ch}_\mu$, i.e., for each $f \in \mathcal{F}_+$,

$$\int \text{pan} f \, d\mu = \int \text{Cho} f \, d\mu,$$

then $\mu$ is superadditive.

**Proof.** Observe that $\text{Ch}_\mu(\chi_E) = \mu(E)$ for any $E \subseteq X$ and, thus for any $A, B \subseteq X$, $A \cap B = \emptyset$, we have

$$\mu(A \cup B) = \text{Ch}_\mu(\chi_{A \cup B}) = \text{Pan}_\mu(\chi_{A \cup B})$$

$$= \sup \left\{ \sum_{i=1}^{k} \lambda_i \cdot \mu(D_i) \mid (D_i)_{i=1}^{k} \text{ is a disjoint system, } \lambda_1, \lambda_2, \ldots, \lambda_k \geq 0 \text{ and } \sum_{i=1}^{k} \lambda_i \chi_{A_i} \leq \chi_{A \cup B} \right\}$$

$$\geq \mu(A) + \mu(B),$$

i.e., $\mu$ is superadditive. \(\Box\)

**Remark 8.** The converse of Theorem 7 may not be true. Observe that in Example 2, the monotone measure $\mu$ is superadditive, but $\int \text{Cho} f \, d\mu > \int \text{pan} f \, d\mu$.

Now we present our main result.

**Theorem 9.** Given $\mu \in \mathcal{M}$. If $\mu$ is subadditive, then $\text{Cav}_\mu \equiv \text{Pan}_\mu$, i.e., for each $f \in \mathcal{F}_+$,

$$\int \text{pan} f \, d\mu = \int \text{cav} f \, d\mu.$$

**Proof.** It suffices to prove that $\int \text{pan} f \, d\mu \geq \int \text{cav} f \, d\mu$ holds for any $f \in \mathcal{F}_+$. To prove this fact, it suffices to prove that for any $\{A_i\}_{i=1}^{N} \subseteq \mathcal{A}$ and $\lambda_i \geq 0, i = 1, 2, \ldots, N$, there is a sequence of pairwise disjoint subsets $\{B_j\}_{j=1}^{M} \subseteq \mathcal{A}$ and a sequence of nonnegative numbers $l_j, j = 1, 2, \ldots, M$ such that

$$\sum_{i=1}^{N} \lambda_i \chi_{A_i} = \sum_{j=1}^{M} l_j \chi_{B_j} \quad (3.3)$$

and

$$\sum_{i=1}^{N} \lambda_i \mu(A_i) \leq \sum_{j=1}^{M} l_j \mu(B_j). \quad (3.4)$$

For $N = 2$, observe that

$$\lambda_1 \chi_{A_1} + \lambda_2 \chi_{A_2} = \lambda_1 \chi_{A_1 - (A_1 \cap A_2)} + \lambda_2 \chi_{A_2 - (A_1 \cap A_2)} + (\lambda_1 + \lambda_2) \chi_{A_1 \cap A_2}.$$

If we let

$$l_1 = \lambda_1, \quad l_2 = \lambda_2, \quad l_3 = \lambda_1 + \lambda_2$$

and

$$B_1 = A_1 - (A_1 \cap A_2), \quad B_2 = A_2 - (A_1 \cap A_2), \quad B_3 = A_1 \cap A_2,$$

then

$$\sum_{i=1}^{N} \lambda_i \chi_{A_i} = \sum_{j=1}^{M} l_j \chi_{B_j}$$

and

$$\sum_{i=1}^{N} \lambda_i \mu(A_i) \leq \sum_{j=1}^{M} l_j \mu(B_j).$$
then
\[ \sum_{i=1}^{2} \lambda_i \chi_{A_i} = \sum_{j=1}^{3} l_j \chi_{B_j}. \]

Moreover, thanks to the subadditivity of \( \mu \), we have
\[
\lambda_1 \mu(A_1) + \lambda_2 \mu(A_2) \\
\leq \lambda_1 (\mu(B_1) + \mu(B_3)) + \lambda_2 (\mu(B_2) + \mu(B_3)) \\
= l_1 \mu(B_1) + l_2 \mu(B_2) + l_3 \mu(B_3).
\]

Now suppose that (3.3) and (3.4) hold for \( N = k \), we need to verify that they are also true for \( N = k + 1 \).

For \( \sum_{i=1}^{k+1} \lambda_i \chi_{A_i} \), we have
\[
\sum_{i=1}^{k+1} \lambda_i \chi_{A_i} = \sum_{i=1}^{k} \lambda_i \chi_{A_i} + \lambda_{k+1} \chi_{A_{k+1}} \\
= \sum_{j=1}^{N'} \alpha_j \chi_{C_j} + \lambda_{k+1} \chi_{A_{k+1}},
\]

where \( C_j, j = 1, 2, \ldots, N' \) are pairwise disjoint subsets of \( X \), \( \alpha_j \geq 0 \) with \( \sum_{i=1}^{k} \lambda_i \mu(A_i) \leq \sum_{j=1}^{N'} \alpha_j \mu(C_j) \).

Observe the facts that
\[
C_j = (C_j - (C_j \cap A_{k+1})) \cup (C_j \cap A_{k+1})
\]
and
\[
A_{k+1} = \left( A_{k+1} - \bigcup_{j=1}^{N'} (A_{k+1} \cap C_j) \right) \cup \left( \bigcup_{j=1}^{N'} (A_{k+1} \cap C_j) \right).
\]

If we let
\[
B_j = C_j - (C_j \cap A_{k+1}), \quad j = 1, 2, \ldots, N'
\]
\[
B_{N'+j} = C_j \cap A_{k+1}, \quad j = 1, 2, \ldots, N',
\]
\[
B_{2N'+1} = A_{k+1} - \bigcup_{j=1}^{N'} (A_{k+1} \cap C_j)
\]
and let
\[
l_j = \alpha_j, \quad l_{N'+j} = \alpha_j + \lambda_{k+1}, j = 1, 2, \ldots, N', \quad l_{2N'+1} = \lambda_{k+1},
\]
then
\[
\sum_{i=1}^{k+1} \lambda_i \chi_{A_i} = \sum_{j=1}^{2N'+1} l_j \chi_{B_j}.
\]
and
\[
\sum_{i=1}^{k+1} \lambda_i \mu(A_i)
\leq \sum_{j=1}^{N'} \alpha_j \mu(C_j) + \lambda_{k+1} \mu(A_{k+1})
\leq \sum_{j=1}^{N'} \alpha_j \left( \mu(B_j) + \mu(B_{N'+j}) \right)
\]
\[
+ \lambda_{k+1} \left( \mu(B_{2N'+1}) + \sum_{j=1}^{N'} \mu(B_{N'+j}) \right)
\]
\[
= \sum_{j=1}^{N'} \alpha_j \mu(B_j) + \sum_{j=1}^{N'} (\alpha_j + \lambda_{k+1}) \mu(B_{N'+j}) + \lambda_{k+1} \mu(B_{2N'+1})
\]
\[
= \sum_{j=1}^{2N'+1} l_j \mu(B_j). \quad \Box
\]

The following example shows that the subadditivity in Theorem 9 is not a necessary condition.

Example 10. Let \(X = [0, 1]\) and \(\mathcal{A} = \mathcal{B}(X)\) (the Borel σ-algebra over \(X\)). Let a monotone measure \(\mu\) be defined as
\[
\mu(E) = \begin{cases} 
1 & \text{if } E = X, \\
0 & \text{if } E \neq X.
\end{cases}
\]

Then, for all \(f \in F_+\)
\[
\int f \, d\mu = \int \mu f \, d\mu = \int \mu f \, d\mu = \inf \{ f(x) \mid x \in X \}.
\]

But \(\mu\) is not subadditive. Indeed, for any Borel measurable proper subset \(E\) of \(\mathcal{A}\), we have \(\mu(E \cup E^c) = \mu(X) = 1 > 0 = \mu(E) + \mu(E^c)\).

The next theorem gives another sufficient condition ensuring the coincidence of the pan-integral and concave integral, now covering Example 10, too.

Theorem 11. Let \(\mu\) be a monotone measure on \((X, \mathcal{A})\). If there is a countable partition \(\{E_t \mid t \in T\} \subset \mathcal{A}\) of \(X\), so that \(e_t = \mu(E_t), t \in T, and \)
\[
\mu(E) \leq \sum_{t \in T, E_t \subset E} e_t, \forall E \in \mathcal{A},
\]
then the concave integral coincides with the pan-integral with respect to the usual arithmetic operation “+” and “·”.

Proof. It is not difficult to check that under the above constraints on \(\mu\), for any \(f \in F_+\) it holds
\[
\int f \, d\mu = \int \mu f \, d\mu = \sum_{t \in T} e_t \cdot \inf \{ f(x) \mid x \in E_t \}. \quad \Box
\]
Yao Ouyang provided a draft of this paper; Jun Li and Radko Mesiar gave many constructive suggestions and Radko Mesiar improved the language quality.

Acknowledgments: This work was supported by the grant APVV-14-0013, the National Natural Science Foundation of China (Grants No. 11371332 and No. 11571106) and the NSF of Zhejiang Province (No. LY15A010013).

Author Contributions: Yao Ouyang provided a draft of this paper; Jun Li and Radko Mesiar gave many constructive suggestions and Radko Mesiar improved the language quality.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Azrieli, Y., Lehrer, E.: Extendable cooperative games, J. Public Econ. Theory 2007, 9, 1069-1078.
2. Benvenuti, P., Mesiar, R., Vivona, D.: Monotone set functions-based integrals. In: E. Pap, editor, Handbook of Measure Theory, Vol II, Elsevier 2002.
3. Choquet, G.: Theory of capacities, Ann. Inst. Fourier 1953, 5, 131-295.
4. Denneberg, D.: Non-additive Measure and Integral, Kluwer Academic Publishers, Dordrecht 1994.
5. Even, Y., Lehrer, E.: Decomposition Integral: Unifying Choquet and the Concave Integrals, Economic Theory 2014, 56, 33-58.
6. Kawabe, J.: Continuity and compactness of the indirect product of two non-additive measures, Fuzzy Sets and Systems 2009, 160, 1327-1333.
7. Grabisch, M.: k-order additive discrete fuzzy measures and their representation, Fuzzy Sets and Systems 1997, 92, 167-189.
8. Klement, E. P., Mesiar, R., Pap, E.: A universal integral as common frame for Choquet and Sugeno integral, IEEE Trans. Fuzzy Syst. 2010, 18, 178-187.
9. Kolesárová, A., Li, J., Mesiar, R.: Pseudo-concave Benvenuti integral, in: Commun. Comput. and Inf. Sci., vol. 300, Springer-Verlag Berlin Heidelberg, 2012, pp.565-570.
10. Kolmogorov, A. N., Foundations of the Theory of Probability, Chelsea, New York, 1950; first published in German in 1933.
11. Lebesgue, H.: Intégrale, longueur, aire, Ann. Mat. 1902, 3(7), 231-359.
12. Lehrer, E.: A new integral for capacities, Economic Theory 2009, 39, 157-176.
13. Lehrer, E., Teper, R.: The concave integral over large spaces, Fuzzy Sets and Systems 2008, 159, 2130-2144.
14. Lovász, L.: Submodular functions and convexity, in: A. Bachem, et al. (eds.) Mathematical Programming: The state of the Art, Springer 1983, pp. 235-257.
15. Li, J., Mesiar, R., Pap, E.: Atoms of weakly null-additive monotone measures and integrals, Information Sciences 2014, 257, 183-192.
16. Li, J., Mesiar, R., Pap, E., Klement, E. P.: Convergence theorems for monotone measures, Fuzzy Sets and Systems 2015, 281, 103-127.
17. Li, J., Mesiar, R., Struk, P.: Pseudo-optimal measures, Information Sciences 2010, 180, 4015-4021.
18. Mesiar, R.: Choquet-like integrals, J. Math. Anal. Appl. 1995, 194, 477-488.
19. Mesiar, R.: A note on de Finetti’s lower probabilities and belief measures, Rend. Matem. Appl. 2008, 28(7), 229-235.
20. Mesiar, R., Li, J., Pap, E.: Pseudo-concave integrals, in: NLMUA’2011, in: Adv. Intell. Syst. Comput., vol.100, Springer-Verlag, Berlin Heidelberg, 2011, pp. 43-49.
21. Mesiar, R., Li, J., Pap, E.: Discrete pseudo-integrals, Int. J. of Approx. Reasoning 2013, 54, 357-364.
22. Mesiar, R., Rybárik, J.: Pan-operations structure, Fuzzy Sets and Systems 1995, 74, 365-369.
23. R. Mesiar, R., Stupna ˇnová, A.: Decomposition Integrals, Int. J. of Approx. Reasoning 2013, 54, 1252-1259.
24. Mesiar, R., Li, J., Ouyang, Y.: On the equality of integrals, Information Sciences 2017, 393, 82-90.
25. Ouyang, Y., Li, J., Mesiar, R.: Relationship between the concave integrals and the pan-integrals on finite spaces, J. Math. Anal. Appl. 2015, 424, 975-987.
26. Pap, E.: Null-Additive Set Functions, Kluwer, Dordrecht, 1995.
27. Riemann, B.: Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe, Habilitation thesis, Universität Göttingen, 1854; published in Götting. Abh. 13, 1968.
28. Shilkret, N.: Maxitive measure and integration, Indag. Math. 1971, 33, 109-116.
29. Sugeno, M.: Theory of fuzzy integrals and its applications, PhD thesis, Tokyo Institute of Technology, 1974.
30. Sugeno, M., Murofushi, T.: Pseudo-additive measures and integrals, J. Math. Anal. Appl. 1987, 122, 197-222.
31. Teper, R.: On the continuity of the concave integral, Fuzzy Sets and Systems 2009, 160, 1318–1326.
32. Vitali, G.: Sulla definizione di integrale delle funzioni di una variabile, Ann. Mat. Pura Appl. 1925, 2, 111-121.
33. Wang, Z., Wang, W., Klir, G. J.: Pan-integrals with respect to imprecise probabilities, Int. J. General Systems 1996, 25(3), 229-243.
34. Wang, Z., Klir, G. J.: Generalized Measure Theory, Springer, New York, 2009.
35. Yang, Q.: The pan-integral on fuzzy measure space, Fuzzy Mathematics 1985, 3, 107-114 (in Chinese).
36. Zhang, Q., Mesiar, R., Li, J., Struk, P.: Generalized Lebesgue integral, Int. J. Approx. Reasoning 2011, 52, 427-443.
© 2017 by the authors. Submitted to *Symmetry* for possible open access publication under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).