Gauss-Bonnet Term on Vacuum Decay

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Abstract

We study the effect of the Gauss-Bonnet term on vacuum decay process in the Coleman-De Luccia formalism. The Gauss-Bonnet term has an exponential coupling with the real scalar field, which appears in the low energy effective action of string theories. We calculate numerically the instanton solution, which describes the process of vacuum decay, and obtain the critical size of bubble. We find that the Gauss-Bonnet term has a nontrivial effect on the false vacuum decay, depending on the Gauss-Bonnet coefficient.
I. INTRODUCTION

Vacuum decay is an old subject in field theory [1]. Coleman and Callan [2, 3] have shown that a quantum tunneling process from a false vacuum to a true vacuum can be realized via the nucleation of a true vacuum bubble in the surrounding of the false vacuum. Furthermore, Coleman and De Luccia [4] have found that gravity has a significant effect on the vacuum decay process.

In semiclassical approximation, the decay rate per unit time per unit volume is given by $\Gamma/V = Ae^{-B/\hbar}[1 + O(\hbar)]$, where the factor $A$ has been discussed in Ref.[4, 5] and the exponent $B$ is the difference of Euclidean actions between the instanton solution $\phi_b$ (bounce solution) and false vacuum solution $\phi_F$. Recently, some authors have discussed the vacuum decay in the different situations such as nonminimal coupling between the scalar field and curvature scalar [6], DBI action [7], and non-thin-wall limit [8], etc. Especially, a new kind of bounce solutions in de Sitter spacetime, which is called by oscillating bounce, has been found by [9, 10].

The finite temperature effect on the false vacuum decay process has also been discussed by Linde et. al [11], where one should look for the $O(3)$-symmetric solution due to periodicity in the time direction with the period of inverse temperature $T^{-1}$, instead of the $O(4)$-symmetric solution at zero temperature. The cosmological applications of false vacuum decay process to various inflation cosmological models have been extensively discussed in [12].

Recently, the so-called stringy landscape scenario [13] predicts that there is a big number of vacua in the effective theory of string theories. On the other hand, a lot of astronomical observations indicate a tiny positive cosmological constant exists in our universe. These motivate many new investigations on the vacuum decay. It is well-known that the higher derivative terms of gravity naturally appear in the low energy effective action of string theories. By field redefinition, $R^2$ terms corrections can be recast to a Gauss-Bonnet form [14, 15]. In particular, the low energy theory with the Gauss-Bonnet is free of ghosts, evading any problem with the unitarity [15]. The possible role of such a Gauss-Bonnet term in the inflation and dark energy models has been investigated recently [16, 17, 18, 19, 20, 21, 22, 23]. The instability of vacua for the Gauss-Bonnet branch in the Gauss-Bonnet gravity is also
investigated by a very recent paper [24]. Therefore, it is of great interest to see whether the Gauss-Bonnet term has any effect on the vacuum decay. This is just the aim of the present paper. We find that the Gauss-Bonnet term indeed has a significant effect on the vacuum decay process.

Before proceeding, let us first stress the issue of stability of vacua. In the absence of gravity, any local minimum of potential for a scalar field can be viewed as a vacuum of the scalar field. The vacuum with the lowest energy density called true one, while others false vacua. Energy density of the field at the vacuum can be positive, zero or negative, the vacuum is always classically stable even for the case with a negative energy density if the potential has a lower bound. When gravity appears, three cases correspond to de Sitter, Minkowski and anti-de Sitter spacetimes, respectively. Due to the absence of ghost in the Gauss-Bonnet gravity, these vacua are classically stable. However, if the true vacuum has a negative energy density, the spacetime inside bubble is anti-de Sitter universe by quantum tunneling from a false vacuum. The anti-de Sitter universe is unstable and will collapse. On the other hand, Minkowski and de Sitter universe are stable. Therefore in this paper we will not consider the case with a negative energy density in true vacuum.

This paper is organized as follows. In Sec. II we present the Euclidean action, equation of motion (EoM) of the scalar field \( \phi \), as well as the Einstein equations. In Sec. III we numerically calculate the nucleation of a Minkowski true vacuum from a de Sitter false vacuum. And we consider three different Gauss-Bonnet coefficient \( \alpha \) values to investigate the Gauss-Bonnet term effect. In Sec. IV we compute the exponent \( B \) and the critical size of bubble radius analytically in thin-wall approximation. We also mention the classical growth of the bubble. Sec. V includes our conclusion and discussion.

### II. ACTION AND EQUATIONS OF MOTION

We consider the following low energy effective action with an exponential coupling between the Gauss-Bonnet term and the scalar field

\[
S = \int d^4x dt \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - U(\phi) + \alpha e^{\beta \phi} R_{GB}^2 \right\},
\]  

\( (1) \)
where $\kappa^2 = 8\pi G = 8\pi/M_p^2 = 8\pi l_p^2$, the signature of the metric is $(-, +, +, +)$, $R^2_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, $U(\phi)$ is the potential of the scalar field, $\beta$ is the coupling constant of the scalar field to the Gauss-Bonnet term and $\alpha$ is called Gauss-Bonnet coefficient. Here we have neglected the boundary term $S_b$ associated with the scalar curvature and the Gauss-Bonnet term, because it will be canceled in our computation.

Changing to the Euclidean signature by virtue of $t = i\eta$, we obtain the Euclidean action

$$S_E = -\int d^3x d\eta \sqrt{g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - U(\phi) + \alpha \varepsilon^{\beta\phi} R^2_{GB} \right\}. \tag{2}$$

Following \[4\], we consider the metric of $SO(4)$-symmetry

$$ds^2_E = d\eta^2 + \rho^2(\eta)d\Omega^2_{(3)} = d\eta^2 + \rho^2(\eta)(d\theta^2 + \sin^2 \theta d\chi^2 + \sin^2 \theta \sin^2 \chi d\psi^2), \tag{3}$$

which has the curvature scalar and Gauss-Bonnet term $R = -\frac{6(1+\rho^2+\rho\dot{\rho})}{\rho^2}$, and $R^2_{GB} = -24\varepsilon^{\beta\phi}(1 - \dot{\rho}^2)$, respectively. Here an overdot stands for the derivative with respect to $\eta$.

Plugging into Eq.\((2)\), the Euclidean action reduces to

$$S_E = 2\pi^2 \int d\eta \rho^3 \left\{ \frac{1}{2} \dot{\phi}^2 + U(\phi) - \frac{3}{\kappa^2} \frac{(1 - \dot{\rho}^2 - \rho\ddot{\rho})}{\rho^2} + 24\alpha \varepsilon^{\beta\phi} \frac{\ddot{\rho}}{\rho^3} (1 - \dot{\rho}^2) \right\}. \tag{4}$$

Within the action \([\ref{16, 17}]\), the Einstein equations read \([\ref{16, 17}]\)

$$0 = \frac{1}{\kappa^2} R_{\mu\nu} - \partial_{\mu} \phi \partial_{\nu} \phi + g_{\mu\nu} \left[ -\frac{1}{2\kappa^2} R + 4\nabla^2 f - 8R^{\sigma\tau} \nabla_{\sigma} \nabla_{\tau} f + \frac{1}{2} \partial_{\mu} \phi \partial^{\rho} \phi + U(\phi) \right]$$

$$-4R\nabla_{\mu} \nabla_{\nu} f - 8R_{\mu\nu} \nabla^2 f - 8R_{(\mu}^{\sigma\tau \nu)} \nabla_{\sigma} \nabla_{\tau} f + 16R_{\sigma(\mu} \nabla^{\sigma} \nabla_{\nu)} f, \tag{5}$$

where $f = \alpha \varepsilon^{\beta\phi}$. The $\eta\eta$ component and $\theta\theta$ component of Einstein equations are

$$0 = \frac{1}{2} \dot{\phi}^2 - U(\phi) + \frac{3}{\kappa^2} \frac{1}{\rho^2} (1 - \dot{\rho}^2) - 24\alpha \beta e^{\beta\phi} \frac{\dot{\rho} \ddot{\rho}}{\rho^3} (1 - \dot{\rho}^2), \tag{6}$$

$$0 = \rho^2 \left[ \frac{1}{2} \dot{\phi}^2 + U(\phi) \right] - \frac{1}{\kappa^2} (1 - \dot{\rho}^2 - 2\rho\ddot{\rho}) - 24\alpha \beta e^{\beta\phi} \frac{\dot{\rho} \ddot{\rho}}{\rho} (1 - \dot{\rho}^2) - 24\alpha \beta e^{\beta\phi} \dot{\phi} \ddot{\phi} + 8\alpha \beta e^{\beta\phi} (\beta\dot{\phi}^2 + \ddot{\phi})(1 - \dot{\rho}^2). \tag{7}$$

Varying the Euclidean action \([\ref{16, 17}]\) with respect to $\phi$ yields the EoM of $\phi$

$$\ddot{\phi} + 3\frac{\dot{\phi}}{\rho} \frac{dU}{d\phi} - 24\alpha \beta e^{\beta\phi} \frac{\ddot{\rho}}{\rho^3} (1 - \dot{\rho}^2) = 0. \tag{8}$$
III. NUMERICAL CALCULATION

As usual, we consider the one-loop effective potential as follows [4].

\[ U(\phi) = U_0 + 1 \text{ loop term} = \frac{\lambda}{8}(\phi^2 - \mu^2/\lambda)^2 + \frac{\epsilon}{2\sqrt{\mu^2/\lambda}}(\phi - \sqrt{\mu^2/\lambda}), \]  

(9)

where \( \mu \) and \( \lambda \) are Higgs mass and coupling constant, respectively, and the second term corresponds to one-loop correction. When the correction is absent, the potential \( U_0 \) has two degenerate vacua at \( \phi_\pm = \pm \sqrt{\mu^2/\lambda} \), with vanishing potential. The correction term eliminates the degeneracy. The constant \( \epsilon \) stands for the potential energy difference between two vacua. The correction does not change the value of the potential at \( \phi_+ \), but a positive \( \epsilon \) shifts the potential at \( \phi_- \) down and a negative \( \epsilon \) shifts the potential up. In Fig. 1 we plot the potential profile with a negative \( \epsilon \). In that case, the vacuum at \( \phi_T = \sqrt{\mu^2/\lambda} \) corresponds to a Minkowski one with vanishing potential, while the vacuum at \( \phi_F = -\sqrt{\mu^2/\lambda} \) is a false vacuum, corresponding to a de Sitter vacuum in gravity theory. If we take \( \epsilon \) to be a positive one, then the vacuum at \( \phi_+ \) turns to be a false vacuum with vanishing potential, while the vacuum at \( \phi_- \) is a true one with a negative potential, corresponding to an anti-de Sitter vacuum, once gravity is taken into account.

We see from (8) that when the GB term is absent, the vacuum structure of the potential is completely determined by the potential itself. However, when the GB term appears, the vacuum becomes to be determined by the third and fourth terms in (8). We can define an
effective potential

\[ U_{\text{eff}} = U + 24\alpha e^{\beta \phi} \hat{\rho}(1 - \rho^2)/\rho^3. \]  

(10)

Then new vacuum is determined by \( dU_{\text{eff}}/d\phi|_{\phi_{\text{new}}} = 0 \). To be more clear, for a Minkowski vacuum, one has \( \rho = \eta, \) \( U_{\text{eff}} = U \), i.e., GB term takes no effect on Minkowski vacuum; for a de Sitter vacuum, \( \rho = \Lambda_1 \sin \frac{\eta}{\Lambda_1}, \) (6) and (10) give \( \Lambda_1 = \left( \frac{3}{\kappa U(\phi_{\text{new}})} \right)^{1/2} \) and \( U_{\text{eff}} = U - 24\frac{\alpha}{\Lambda_1^2} e^{\beta \phi_{\text{new}}}, \) respectively, where \( \phi_{\text{new}} \) represents new false vacuum value \( \phi_{\text{F}} \) or new true vacuum value \( \phi_{\text{true}} \) which is determined by \( dU_{\text{eff}}/d\phi|_{\phi_{\text{old}}} = 0 \); similarly, for an anti-de Sitter vacuum, \( \rho = \Lambda_2 \sinh \frac{\eta}{\Lambda_2}, \) \( \Lambda_2 = \left( -\frac{3}{\kappa U(\phi_{\text{new}})} \right)^{1/2} \) and \( U_{\text{eff}} = U - 24\frac{\alpha}{\Lambda_2^2} e^{\beta \phi_{\text{new}}}, \) in which \( \phi_{\text{new}} \) is also given by \( dU_{\text{eff}}/d\phi|_{\phi_{\text{old}}} = 0 \). As a summary, in Minkowski case, GB term keeps potential \( U(\phi) \) unchanged; in both de Sitter and anti-de Sitter case, GB term shifts the vacuum value of scalar field from an old one to a new one \( (\phi_{\text{old}} \to \phi_{\text{new}}) \), meanwhile, the original potential energy is replaced by an effective potential \( (U \to U_{\text{eff}}) \), in which \( \phi_{\text{old}} (\phi_{\text{new}}) \) is computed by \( dU/d\phi|_{\phi_{\text{old}}} = 0 \) \( (dU_{\text{eff}}/d\phi|_{\phi_{\text{new}}} = 0) \).

We conclude, from above analysis, that, at a de Sitter vacuum, \( U_{\text{eff}} = U - 24\frac{\alpha}{\Lambda_1^2} e^{\beta \phi_{\text{new}}}, \) a positive \( \alpha \) is analogous to the case to decrease \( |\epsilon| \) in the Einstein gravity, while a negative \( \alpha \) leads to an opposite effect (See Fig. 2); at anti-de Sitter vacuum, the cases are reverse, \( U_{\text{eff}} = -|U| - 24\frac{\alpha}{\Lambda_2^2} e^{\beta \phi_{\text{new}}}. \) That is, a positive \( \alpha \) corresponds to increasing \( |\epsilon| \), while a negative \( \alpha \) to decreasing \( |\epsilon| \); at Minkowski vacuum, the GB term takes no effect.

In addition, we must point out that the shifts of vacuum potential energy due to the appearance of GB term can not change the topology of the original spacetime manifold. For example, if the original potential energy is definitely positive \( (U(\phi) > 0) \), we can only get new de Sitter solution. Neither anti-de Sitter solution nor Minkowski solution are permitted in the new effective potential. That is to say, a de Sitter vacuum could not become a Minkowski vacuum or an anti-de Sitter vacuum due to the Gauss-Bonnet term. This can be seen from (6). The same holds for the Minkowski vacua case and anti-de Sitter vacua case. Note that in the non-minimal coupling case (6), such a change is possible. That is, in that case, a true vacuum could turn to be a false vacuum due to the non-minimal coupling.

In the low energy effective action of string theories, the GB term can be parameterized
FIG. 2: Potential profile with different $\tilde{\epsilon}$. The solid line corresponds to $\tilde{\epsilon} = -0.1$, the dashed line to $\tilde{\epsilon} = -0.15$ and the dotted line to $\tilde{\epsilon} = -0.05$. as [18]

$$\alpha e^{\beta \phi} R^2_{GB} = -\frac{1}{2} \alpha' \gamma e^{-\phi/M_{pl}} R^2_{GB},$$

(11)

where $M_{pl}$ is the Planck mass, and $\alpha' = l_s^2$ is the string slope. That is to say, we take $\alpha = -\frac{1}{16\pi^2} \gamma (\frac{h}{l_s^4})^2$, and $\beta = -1/M_{pl}$. Note that $\gamma = -\frac{1}{4}, -\frac{1}{8}$, and 0, correspond to the cases in the low energy effective theory of bosonic, heterotic, and type II superstring theory, respectively. In order not to make confusion, we should stress here that in fact, there is no quadratic correction in type II superstring theory.

For numerical calculations, we make the following rescalings so that these quantities become dimensionless.

$$\tilde{\phi} = \sqrt{\frac{\lambda}{\mu^2}} \phi, \quad \tilde{\eta} = \mu \eta, \quad \tilde{\epsilon} = \frac{\lambda}{\mu^4} \epsilon, \quad \tilde{\rho} = \mu \rho, \quad \tilde{\beta} = \sqrt{\frac{\mu^2}{\lambda}} \beta, \quad \tilde{\alpha} = \lambda \alpha, \quad \tilde{U}(\tilde{\phi}) = \frac{\lambda}{\mu^2} U(\phi), \quad \tilde{\kappa}^2 = \frac{\mu^2}{\lambda} \kappa^2.$$

In principle we can discuss the quantum tunneling among various vacua as was done in Ref.[6]. Note that if the true vacuum is an anti-de Sitter one, the resulting spacetime is dynamically unstable as stressed in Introduction. To demonstrate the role of the GB term in the vacuum decay, here we focus on the case in which a de Sitter vacuum decays into a Minkowski vacuum only by using the potential [19]. In the following numerical calculation we fix $\tilde{\epsilon} = -0.1$ as well as $\tilde{\kappa}^2 = 0.1$ and keep $\tilde{\alpha}$ free, which mainly depends on the string length scale. In convenience, we will drop the tilde symbol in the following when we use dimensionless quantities.
FIG. 3: Reversed potential profile with $\bar{\epsilon} = -0.1$.

FIG. 4: $\phi$ profile v.s. $\eta$. The solid, dashed and dotted curves correspond to $\alpha = 0, -10^3, \text{and } 300$, respectively.

As Coleman [2] has demonstrated that, at semiclassical level, the quantum tunneling from a false vacuum to a true vacuum in Lorentzian spacetime is analogous to the problem that a classical particle rolls down from a higher peak to a lower peak in the minus potential in Euclidean spacetime. That is to say, if the particle is released at rest at a proper position near $\phi_T$ it will come to rest at $\eta = \eta_{\text{max}}$ at $\phi_F$. (In de Sitter spacetime there exists a maximum $\eta_{\text{max}}$, so we let the particle come to rest at $\eta_{\text{max}}$, while in Minkowski or anti-de Sitter spacetime there does not exist such $\eta_{\text{max}}$, so in those cases, we will let particle reach the lower peak when $\eta$ goes to infinity.) Thus, we can solve the EoM of $\phi$ and Einstein
Among Eqs. (6-8) only two equations are independent, because the three equations are related by Bianchi identity. We combine the EoM of $\phi$ and $\theta$ component equation to solve the problem and choose the $\eta \eta$ component equation as a constraint which, on one hand, constrains our solution, on the other hand, determines the initial value of $\dot{\rho}$, i.e., the value of $\dot{\rho}_0$.

Although in string theories $\alpha$ is always positive definite, we still consider a negative $\alpha$ case because that case corresponds to shift up the false vacuum (de Sitter vacuum). In this paper, we take three different $\alpha$ values to see the effect of GB term. Case 1: $\alpha = 0$, which corresponds to the case without the GB term; case 2: $\alpha = -10^3$, which corresponds to de Sitter vacuum shifting up; case 3: $\alpha = 300$, corresponding to de Sitter vacuum shifting down but still a de Sitter vacuum. The numerical results are shown in Fig. 4 and Fig. 5. We can see from Fig. 4 that when $\alpha$ is negative, quantum tunneling can happen at smaller $\eta$, compared to the case with $\alpha = 0$, while it can occur at larger $\eta$ for a positive $\alpha$. We can also find that GB term makes the false vacua values of scalar field $\phi_{\text{new}}^F$ (dashed and
dotted curves) have tiny shifts compared with $\phi^{old}_{T}$ (solid curve), however, GB term leaves $\phi_{T}$ (Minkowski vacuum) unchanged (three curves coincide with each other at true vacuum). As a qualitative approximation, if neglect the thickness of the wall of the bubble, we can write down the metric of the whole spacetime

$$
\begin{align*}
\rho &= \eta, \quad (0 \leq \eta \leq \eta_{w}), \\
\rho &= \Lambda_{eff} \sin(\eta/\Lambda_{eff}), \quad (\eta_{w} \leq \eta \leq \eta_{\text{max}}),
\end{align*}
$$

(13)

where $\eta_{w}$ represents for the position of the wall, and $\eta_{\text{max}} = \pi \Lambda_{eff}$. Crossing the the wall at $\eta_{w}$, spacetime metric and matter field should be matched by the conjunction conditions \[25, 26\]. Of course, this is just a rough approximation. In fact, due to the existence of the wall supported by the scalar field, the spacetime metric and matter field should be smoothly continuous across the wall. In that case, the conjunction condition is not necessary.

We can see from figures that as we analyzed above, indeed, in the case we considered, the effect of the GB term is qualitatively equivalent to changing the potential difference $\epsilon$ between two vacua in Einstein gravity. A negative $\alpha$ term corresponds to increasing the potential difference, and a positive $\alpha$ term results in an opposite effect.

**IV. THIN WALL APPROXIMATION**

In general it is impossible to find an analytic solution to describe the process of vacuum decay, even in the case of absence of gravity. But as shown in \[4\], it is possible to find an approximate solution in the thin wall approximation. The so-called thin wall approximation means that the thickness of the wall is quite small compared to the size of the bubble. This happens $|\bar{\epsilon}| \ll 1$ for the potential \[9\], which implies that the energy difference between the false vacuum and true vacuum is small compared to the height of the barrier between these two vacua. As shown in \[4\], in the thin wall approximation, the thickness of the wall is of $O(\mu^{-1})$, while the size of the bubble is proportional to the inverse of the energy difference of two vacua.

In this section we will calculate the critical size of the bubble in the thin wall approximation. The critical size of the bubble is determined by requiring that the Euclidean action difference be stationary, between the bounce (instanton) solution and the false vacuum so-
olution. If the radius of the bubble after nucleation is smaller than the critical radius, the bubble cannot grow up, because the decrement of volume energy is less than the increment of surface energy. That is to say, only when \( \bar{\rho} \geq \bar{\rho}_c \) the nucleated bubble can grow up. Here \( \bar{\rho} \) denotes the size of the bubble. Now we calculate the critical radius \( \bar{\rho}_c \) of the bubble. Note that \( B \) is the action difference between the bounce solution \( \phi_b \) and the constant false vacuum solution \( \phi_F \). Then the critical radius of the bubble is determined by \( dB/d\bar{\rho}|_{\bar{\rho}=\bar{\rho}_c} = 0 \).

Following [4], we calculate the Euclidean action \( S_E \) by dividing the solution into three parts: inside the wall \( S^i_E \), the wall \( S^w_E \), and outside the wall \( S^o_E \). Outside the wall, \( S^o_E(\phi_b) \) and \( S^o_E(\phi_F) \) cancels each other, so we have \( B \) as

\[
B = S^{i+w}_E(\phi_b) - S^{i+w}_E(\phi_F). \tag{14}
\]

The Euclidean action reads

\[
S_E = 2\pi^2 \int d\eta \, \bar{\rho}^3 \left\{ \frac{1}{2} \dot{\phi}^2 + U(\phi) - \frac{3}{\kappa^2} \frac{1 - \dot{\rho}^2}{\rho^2} + \frac{3\bar{\rho}}{\kappa^2 \rho} + 24\alpha e^{\beta \phi} \frac{\ddot{\rho}}{\rho^4} (1 - \dot{\rho}^2) \right\}. \tag{15}
\]

On the wall, in the thin wall approximation, the second and fourth terms in the equation of motion for \( \phi \) in (8) can be neglected (we will discuss this below). This implies that the last term in (15) can be neglected as well on the wall. By virtue of integration by parts and Einstein equation (9), the Euclidean action is changed to

\[
S_E(\phi) = 4\pi^2 \int_{\eta_1}^{\eta_3} d\eta \, \left( \rho^3 U - \frac{3}{\kappa^2 \rho} \right), \tag{16}
\]

where we drop the surface terms because those terms are always cancelled when we calculate the difference of the Euclidean action (\( B \)). The action (16) can be further approximated as

\[
S^w_E(\phi_b) = 4\pi^2 \int_{\eta_1}^{\eta_3} d\eta \, \left( \bar{\rho}^3 U_0(\phi_b) - \frac{3}{\kappa^2 \bar{\rho}} \right), \tag{17}
\]

where we have used \( \rho \simeq \bar{\rho} \), and \( U(\phi) \simeq U_0(\phi) \) (thin wall approximation) [4].

For the false vacuum solution, on the wall, we can also replace \( \rho \) by \( \bar{\rho} \), and \( U(\phi) \) by \( U_0(\phi) \), then the Euclidean action reads

\[
S^w_E(\phi_F) = 4\pi^2 \int_{\eta_1}^{\eta_3} d\eta \, \left( \bar{\rho}^3 U_0(\phi_F) - \frac{3}{\kappa^2 \bar{\rho}} \right). \tag{18}
\]

Thus we have
FIG. 6: The instanton solution profile $\phi$ v.s. $\eta$.

\[ B^w \equiv S_{E}^w(\phi_b) - S_{E}^w(\phi_F) \approx 2\pi^2 \bar{\rho}^3 S_1 \]  

(19)

where $S_1 \equiv \int_{\eta_1}^{\eta_2} d\eta \ 2[U_0(\phi) - U_0(\phi_F)]$. This result indicates that the Gauss-Bonnet term has no contribution to the wall part of the Euclidean action difference. This is an expected result since one can see from (19) that even the Hilbert-Einstein term has no contribution by noting that the form (19) is completely the same as the case without gravity [4].

Inside the wall, the Euclidean action reads

\[ S_{E}^i(\phi) = 4\pi^2 \int_0^{\eta_2} d\eta \left( \rho^3 U(\phi) - \frac{3}{\kappa^2} \rho \right) + 96\pi^2 \alpha \int_0^{\eta_2} d\eta \ e^{\beta \rho} \dot{\rho}^2 \]  

(20)

and the Einstein equations are

\[ \dot{\rho}^2 = 1 - \frac{1}{3} \kappa^2 U(\phi) \rho^2, \]  

(21)

\[ \ddot{\rho} = \frac{1}{2\rho} [1 - \rho^2 - \kappa^2 U(\phi) \rho^2] = -\frac{1}{3} \kappa^2 U(\phi) \rho. \]  

(22)

By virtue of Eqs. (21) and (22), the Euclidean action becomes

\[ S_{E}^i(\phi_b) = -\frac{12\pi^2}{\kappa^2} \int_0^{\bar{\rho}} d\rho \ \rho \sqrt{1 - \frac{\kappa^2}{3} U(\phi_T) \rho^2} \]

\[ = -\frac{6\pi^2 \bar{\rho}^2}{\kappa^2}, \]  

(23)

where we have used the fact $U(\phi_T) = 0$, $\phi_b \simeq \phi_T$. It is so because inside the wall, the true vacuum is a Minkowski spacetime, where the Gauss-Bonnet term has no contribution. On
the other hand, the Euclidean action from the false vacuum is

\[ S^i_{E}(\phi_F) = -\frac{12\pi^2}{\kappa^2} \int_0^{\tilde{\rho}} d\rho \rho \sqrt{1 - \frac{1}{3}D_2\rho^2 - 32\pi^2\alpha \int_0^{\tilde{\rho}} d\rho e^{\beta\phi_F} D_2\rho \sqrt{1 - \frac{1}{3}D_2\rho^2}} \]

\[ = \left( \frac{12\pi^2}{\kappa^2D_2} + 32\pi^2\alpha e^{\beta\phi_F} \right) \left[ (1 - \frac{1}{3}D_2\bar{\rho}^2)^{3/2} - 1 \right]. \] (24)

We have defined the position of the wall as \( \bar{\rho} = \rho(\eta_2) \approx \rho(\eta_1) \) because of the thin wall approximation, and \( D_2 \equiv \kappa^2U(\phi_F) \). As a result, we get the contribution \( B^i \) inside the wall

\[ B^i \equiv S^i_{E}(\phi_b) - S^i_{E}(\phi_F) \]

\[ = -\frac{6\pi^2\bar{\rho}^2}{\kappa^2} - \left( \frac{12\pi^2}{\kappa^2D_2} + 32\pi^2\alpha e^{\beta\phi_F} \right) \left[ (1 - \frac{1}{3}D_2\bar{\rho}^2)^{3/2} - 1 \right]. \] (25)

Finally we obtain the total Euclidean action difference \( B \) as

\[ B = 2\pi^2\bar{\rho}^3S_1 - \frac{6\pi^2\bar{\rho}^2}{\kappa^2} - \left( \frac{12\pi^2}{\kappa^2D_2} + 32\pi^2\alpha e^{\beta\phi_F} \right) \left[ (1 - \frac{1}{3}D_2\bar{\rho}^2)^{3/2} - 1 \right]. \] (26)

The critical size of bubble is determined through

\[ \frac{dB}{d\bar{\rho}} \bigg|_{\bar{\rho}=\bar{\rho}_c} = 0. \] (27)

Substituting Eq. (26) into the above equation yields

\[ 3S_1\bar{\rho}_c - \frac{6}{\kappa^2} + D_3(1 - \frac{1}{3}D_2\bar{\rho}_c^{1/2}) = 0, \] (28)

where \( D_3 = 6/\kappa^2 + 16\alpha D_2e^{\beta\phi_F} \). As \( \alpha = 0 \), Eq. (28) leads to the Coleman-de Luccia’s result

\[ \bar{\rho}_c = \frac{12S_1}{4U(\phi_F) + 3\kappa^2S_1^2}. \] (29)
If $\alpha \neq 0$, one has

$$\bar{\rho}_c = \frac{36S_I + \sqrt{(36S_I)^2 - 4(36 - \kappa^4 D_3^2)(\frac{1}{3} D_2 D_3^2 + 95^2)}}{2\kappa^2(\frac{1}{3} D_2 D_3^2 + 95^2)}.$$  \hspace{1cm} (30)

In Fig. 7 we plot the critical size of the bubble versus the GB coefficient $\alpha$. As expected, when $\alpha < 0$, the critical size of bubble becomes smaller, which implies that bubble nucleation becomes easier, and vice versa. These analytical results are consistent with our previous numerical calculation.

Before we turn to the issue on the growth of the bubble, we discuss the validity of the thin-wall approximation. In order to get an instant solution over the wall, the second and the fourth terms in Eq. (8) have been neglected, i.e., one has $\dot{\bar{\rho}}/\bar{\rho} \ll 1$ and $(1 - \dot{\bar{\rho}}^2)/\bar{\rho}^2 \ll 1$. Now we justify this. By Eq. (6),

$$\frac{\dot{\bar{\rho}}^2}{\bar{\rho}^2} + 8\alpha \beta \kappa^2 \phi e^{2\phi} \frac{\phi (1 - \dot{\rho}^2)}{\rho^2} = \frac{1}{\bar{\rho}^2} + \frac{\kappa^2}{3} \left( \frac{\phi^2}{2} - U \right).$$  \hspace{1cm} (31)

The left hand side of this equation is certainly small if both terms on the right are small. $1/\bar{\rho}^2 \ll 1$ is required as in the absence of gravity \cite{2}. In fact, this is also a natural consequence of large bubble. The quantity in parentheses on the righthand side can be viewed as total energy of the particle in inverse potential. The total energy vanishes inside the wall and has absolute value $|\epsilon|$ outside the wall. So, the absolute value of the energy density over the wall must be smaller than $|\epsilon|$, i.e. the absolute value of the second term on the righthand side is smaller than $1/\Lambda^2 \equiv \kappa^2 |\epsilon|/3$. Besides that, the continuity condition of the metric on the wall gives $\bar{\rho}/\Lambda \ll 1$. In conclusion, the thin-wall approximation is valid when $\Lambda \gg \bar{\rho} \gg \mu^{-1}$.

The classical growth of the bubble after its quantum nucleation is similar to the result in Ref.\cite{6}. For completeness, here we just briefly mention main results. We obtain the Lorentzian solution from the Euclidean solution by employing the analytic continuation in \cite{3}

$$\theta \rightarrow i\theta + \frac{\pi}{2}.$$  \hspace{1cm} (32)

And then transform the coordinate into the static spherically symmetric coordinate by
applying following coordinate transformation

\[
\begin{align*}
  r &= \eta \cosh \theta, \quad t = \eta \sinh \theta, \\
  r &= \Lambda_1 \sin \frac{\eta}{\Lambda_1} \cosh \theta, \quad t = \frac{\Lambda_1}{2} \ln \frac{\cos \frac{\eta}{\Lambda_1} + \sin \frac{\eta}{\Lambda_1} \sinh \theta}{\cos \frac{\eta}{\Lambda_1} - \sin \frac{\eta}{\Lambda_1} \sinh \theta}.
\end{align*}
\]

(33) \hspace{1cm} (34)

Here the first line corresponds to the Minkowski true vacuum, while the second line corresponds to the de Sitter false vacuum. After these transformations, we obtain a Minkowski spacetime inside the bubble

\[
d s^2 = -dt^2 + dr^2 + r^2(d\chi^2 + \sin^2 \chi d\psi^2),
\]

while outside the bubble, we have a de Sitter space

\[
d s^2 = -(1 - \frac{r^2}{\Lambda_1^2})dt^2 + \frac{dr^2}{1 - \frac{r^2}{\Lambda_1^2}} + r^2(d\chi^2 + \sin^2 \chi d\psi^2).
\]

(35) \hspace{1cm} (36)

The proper velocity of the bubble wall observed by an observer outside the wall (spacelike observer) is

\[
\frac{dr}{d\tau} = \sqrt{\frac{r^2}{\eta_c^2} - 1},
\]

(37)

where the measure of proper time equals to \(d\tau = \sqrt{dt^2 - dr^2} = \eta_c d\theta\), and \(\eta_c\) is a constant value very closed to \(\bar{\eta}\).

V. CONCLUSIONS

In this paper we have investigated the effect of a Gauss-Bonnet term on vacuum decay process of a scalar field. The Gauss-Bonnet term has an exponential coupling with the scalar field. Such a coupling appears in the low energy effective action of some string theories. We found that the Gauss-Bonnet term could change the vacuum structure of the scalar field but could not change the topology of the original spacetime manifold, i.e., a de Sitter vacuum could become a new de Sitter vacuum but could not become a Minkowski vacuum or an anti-de Sitter vacuum if the potential \(U\) is positive definite. Similar case happens in an original anti-de Sitter vacuum. As to an original Minkowski vacuum, Gauss-Bonnet term takes no effect, so the Minkowski vacuum remains unchanged.
Concretely, in this paper, we considered the effect of the Gauss-Bonnet term on a de Sitter vacuum decaying into a Minkowski vacuum. In this case, the Gauss-Bonnet term shifts the de Sitter vacuum up or down depending on a negative or positive Gauss-Bonnet coefficient $\alpha$, and keeps the Minkowski vacuum unchanged. We calculated numerically the instanton solution with different Gauss-Bonnet coefficient, and found that the effect of the Gauss-Bonnet term is qualitatively equivalent to increasing or decreasing the potential energy difference between the false vacuum and the true vacuum, which makes the bubble nucleation easier or harder.

We also computed the exponent coefficient $B$ in the decay rate and the critical radius of the bubble in the thin-wall approximation. If the radius of nucleated bubble is smaller than its critical size, the bubble will shrink, while if it is larger than the critical size, it can grow up after quantum nucleation. We found that a negative Gauss-Bonnet coefficient $\alpha$ leads to a smaller critical radius, and a positive $\alpha$ to a larger critical radius. That is to say, a negative $\alpha$ makes bubble nucleation easier, while a positive $\alpha$ makes bubble nucleation harder, which is consistent with our numerical calculations. In this paper, we investigated the vacuum decay from a de Sitter vacuum to a Minkowski vacuum. We expect the Gauss-Bonnet term has a similar effect to other decay processes.

Acknowledgments

BH thanks H. Dong for lots of good suggestions and L. M. Cao for many useful discussions. SK thanks W. Lee for many useful discussions. This work was supported in part by a grant from Chinese Academy of Sciences, grants from NSFC with No. 10325525 and No. 90403029.

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