Classifying the Arithmetical Complexity of Teaching Models

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Abstract. This paper classifies the complexity of various teaching models by their position in the arithmetical hierarchy. In particular, we determine the arithmetical complexity of the index sets of the following classes: (1) the class of uniformly r.e. families with finite teaching dimension, and (2) the class of uniformly r.e. families with finite positive recursive teaching dimension witnessed by a uniformly r.e. teaching sequence. We also derive the arithmetical complexity of several other decision problems in teaching, such as the problem of deciding, given an effective coding $\{L_0, L_1, L_2, \ldots\}$ of all uniformly r.e. families, any $e$ such that $L_e = \{L_{e0}, L_{e1}, \ldots\}$, any $i$ and $d$, whether or not the teaching dimension of $L_{ei}$ with respect to $L_e$ is upper bounded by $d$.

1 Introduction

A fundamental problem in computational learning theory is that of characterising identifiable classes in a given learning model. Consider, for example, Gold’s \cite{gold1967} model of learning from positive data, in which a learner is fed piecewise with all the positive examples of an unknown target language – often coded as a set of natural numbers – in an arbitrary order; as the learner processes the data, it outputs a sequence of hypotheses that must converge syntactically to a correct conjecture. Of particular interest to inductive inference theorists is the learnability of classes of recursively enumerable (r.e.) languages. Angluin \cite{angluin1980} demonstrated that a uniformly recursive family is learnable in Gold’s model if and only if it satisfies a certain “tell-tale” condition. As a consequence, the family of nonerasing pattern languages over any fixed alphabet and the family of regular expressions over $\{0, 1\}$ that contain no operators other than concatenation and Kleene plus are both learnable in the limit. On the other hand, even a relatively simple class such as one consisting of an infinite set $L$ and all the finite subsets of $L$ cannot be learnt in the limit \cite{gold1967} Theorem I.8. Analogous characterisations of learnability have since been discovered for uniformly r.e. families as well as for other learning models such as behaviourally correct learning \cite{pitt1988}.
Intuitively, the structural properties of learnable families seem to be related to the “descriptive complexity” of such families. By fixing a system of describing families of sets, one may wish to compare the relative descriptive complexities of families identifiable under different criteria. One idea, suggested by computability theory and the fact that many learnability criteria may be expressed as first-order formulae, is to analyse the quantifier complexity of the formula defining the class of learnable families. In other words, one may measure the descriptive complexity of identifiable classes that are first-order definable by determining the position of their corresponding index sets in the arithmetical hierarchy. This approach to measuring the complexity of learnable classes was taken by Brandt [4], Klette [9], and later Beros [3]. More specifically, Brandt [4, Corollary 1] showed that every identifiable class of partial-recursive functions is contained in another identifiable class with an index set that is in $\Sigma_3 \cap \Pi_3$, while Beros [3] established the arithmetical complexity of index sets of uniformly r.e. families learnable under different criteria.

The purpose of the present work is to determine the arithmetical complexity of various decision problems in algorithmic teaching. Teaching may be viewed as a natural counterpart of learning, where the goal is to find a sample efficient learning and teaching protocol that guarantees learning success. Thus, in contrast to a learning scenario where the learner has to guess a target concept based on labelled examples from a truthful but arbitrary (possibly even adversarial) source, the learner in a cooperative teaching-learning model is presented with a sample of labelled examples carefully chosen by the teacher, and it decodes the sample according to some pre-agreed protocol. We say that a family is “teachable” in a model if and only if the associated teaching parameter – such as the teaching dimension [6], the extended teaching dimension [7] or the recursive teaching dimension [14] – of the family is finite. Due to the ubiquity of numberable families of r.e. sets in theoretical computer science and the naturalness of such families, our work will focus on the class of uniformly r.e. families. Our main results classify the arithmetical complexity of index sets of uniformly r.e. families that are teachable under the teaching dimension model and a few variants of the recursive teaching dimension model.

From the viewpoint of computability theory, our work provides a host of natural examples of complete sets, thus supporting Rogers’ view that many “arithmetical sets with intuitively simple definitions … have proved to be $\Sigma^0_n$-complete or $\Pi^0_n$-complete (for some $n$)” [12, p. 330]. From the viewpoint of computational learning theory, our results shed light on the recursion-theoretic structural properties of the classes of uniformly r.e. families that are teachable in some well-studied models.

2 Preliminaries

The notation and terminology from computability theory adopted in this paper follow in general the book of Rogers [12].

$\forall^\infty x$ denotes “for almost every $x$”, $\exists^\infty x$ denotes “for infinitely many $x$” and $\exists! x$ denotes “for exactly one $x$”. $\mathbb{N}$ denotes the set of natural numbers,
Definition 1. \[2\] A set \( A \subseteq \mathbb{N} \) is in \( \Sigma_0 = \Pi_0 = \Delta_0 \) iff \( A \) is recursive. \( A \) is in \( \Sigma_n \) iff there is a recursive relation \( R \) such that

\[
x \in B \iff (\exists y_1)(\forall y_2) \ldots (Q_n y_n) R(x, y_1, y_2, \ldots, y_n)
\]  

(1)

where \( Q_n = \forall \) if \( n \) is even and \( Q_n = \exists \) if \( n \) is odd. A set \( A \subseteq \mathbb{N} \) is in \( \Pi_0 \) iff its complement \( \overline{A} \) is in \( \Sigma_0 \) (\( \exists y_1)(\forall y_2) \ldots (Q_n y_n) \) is known as a \( \Sigma_n \) prefix; \( (\forall y_1)(\exists y_2) \ldots (Q_n, y_n) \), where \( Q_n = \exists \) if \( n \) is even and \( Q_n = \forall \) if \( n \) is odd, is known as a \( \Pi_n \) prefix. The formula on the right-hand side of \( \mathbf{1} \) is called a \( \Sigma_n \) formula and its negation is called a \( \Pi_n \) formula. A set \( A \) is in \( \Delta_n \) iff \( A \) is in \( \Sigma_n \) and \( A \) is in \( \Pi_n \). Sets in \( \Sigma_n \) (\( \Pi_n \), \( \Delta_n \)) are known as \( \Sigma_n \) sets (\( \Pi_n \) sets, \( \Delta_n \) sets). For any \( n \geq 1 \), a set \( A \) is \( \Sigma_n \)-hard \( (\Pi_n \)-hard) iff every \( \Sigma_n \) \( (\Pi_n) \) set \( B \) is many-one reducible to it, that is, there exists a recursive function \( f \) such that \( x \in B \iff f(x) \in A \). \( A \) is \( \Sigma_n \)-complete \( (\Pi_n \)-complete) iff \( A \) is definable with a \( \Sigma_n \) (\( \Pi_n \) formula and \( A \) is \( \Sigma_n \)-hard \( (\Pi_n \)-hard).

The following proposition collects several useful equivalent forms of \( \Sigma_n \) or \( \Pi_n \) formulas (for any \( n \)).

Proposition 2. (i) For every \( \Sigma^0_{n+1} \) set \( A \), there is a \( \Pi^0_n \) predicate \( P \) such that for all \( x \),

\[
x \in A \iff (\forall^\infty a) P(a, x) \iff (\exists a) P(a, x).
\]
closely related to the query complexity of learning $C$ and is denoted by $\text{ord}_C$ for $C$ expresses the complexity of unique specification with respect to a concept class. This parameter may be viewed as a generalisation of the teaching dimension; it

\begin{itemize}
    \item[(i)] Hegedüs [7] showed, the extended teaching dimension of a concept class $C$ is defined as $\text{inf}_{X(T) = T \cup T'}$ for all $X(T)$ consistent with $T$ and $T' = T \cup T'$, if $T'$ is consistent with $C$ and $T$ is not consistent with $C$. If $T'$ is a teaching set for $C$ with respect to $X(T)$, then $X(T)$ is known as a distinguishing set for $C$ with respect to $X(T)$. Every element of $\mathbb{N} \times \{+,-\}$ is known as a labelled example.

**Definition 3.** [6][13] Let $L$ be any family of subsets of $\mathbb{N}$. Let $L \in L$ be given. The size of a smallest teaching set for $L$ with respect to $L$ is called the teaching dimension of $L$ with respect to $L$, denoted by $\text{TD}(L,L)$. The teaching dimension of $L$ is defined as $\sup\{\text{TD}(L,L) : L \in L\}$ and is denoted by $\text{TD}(L)$. If there is a teaching set for $L$ with respect to $L$ that consists of only positive examples, then the positive teaching dimension of $L$ with respect to $L$ is defined to be the smallest possible size of such a set, and is denoted by $\text{TD}^+(L,L)$. If there is no teaching set for $L$ w.r.t. $L$ that consists of only positive examples, then $\text{TD}^+(L,L)$ is defined to be $\infty$. A teaching set for $L$ with respect to $L$ that consists of only positive examples is known as a positive teaching set for $L$ with respect to $L$. The positive teaching dimension of $L$ is defined as $\sup\{\text{TD}^+(L,L) : L \in L\}$.

Another complexity parameter recently studied in computational learning theory is the recursive teaching dimension. It refers to the maximum size of teaching sets in a series of nested subfamilies of the family.

**Definition 4.** (Based on [13][10]) Let $L$ be any family of subsets of $\mathbb{N}$. A teaching sequence for $L$ is any sequence $\text{TS} = ((F_0, d_0), (F_1, d_1), \ldots )$ where (i) the families $F_i$ form a partition of $L$ with each $F_i$ nonempty, and (ii) $d_i = \sup\{\text{TD}(L', L) : L' \in F_i\}$ for all $i$. $\text{sup}\{d_i : i \in \mathbb{N}\}$ is called the order of $\text{TS}$, and is denoted by $\text{ord}(\text{TS})$. The recursive teaching dimension of $L$ is defined as $\inf\{\text{ord}(\text{TS}) : \text{TS} \text{ is a teaching sequence for } L\}$ and is denoted by $\text{RTD}(L)$.

We shall also briefly consider the extended teaching dimension (XTD) of a class. This parameter may be viewed as a generalisation of the teaching dimension; it expresses the complexity of unique specification with respect to a concept class $C$ for every concept (not just members of $C$) over a given instance space $X$. As Hegedüs [7] showed, the extended teaching dimension of a concept class $C$ is closely related to the query complexity of learning $C$. 

\begin{itemize}
    \item[(ii)] For every $\Sigma^0_{n+1}$ set $B$, there is a $\Pi^0_n$ predicate $Q$ such that for all $x$,
    \[ x \in B \iff (\exists a)Q(a,x) \iff (\exists a)Q(a,x). \] 
\end{itemize}
Definition 5. Let \( \mathcal{L} \) be a family of subsets of \( \mathbb{N} \), and let \( L \) be a subset of \( \mathbb{N} \). A set \( S \subseteq \mathbb{N} \) is a specifying set for \( L \) with respect to \( \mathcal{L} \) iff there is at most one concept \( L' \) in \( \mathcal{L} \) such that \( L \cap S = L' \cap S \). Define the extended teaching dimension (XTD) of \( \mathcal{L} \) as \( \inf \{d : \text{for every set } L \subseteq \mathbb{N} \text{ there exists an at most } d\text{-element specifying set with respect to } \mathcal{L} \} \).

A set \( S \subseteq L \) is a positive specifying set for \( L \) with respect to \( \mathcal{L} \) iff there is at most one concept in \( \mathcal{L} \) that contains \( S \). Define the positive extended teaching dimension (XTD*) of \( \mathcal{L} \) as \( \inf \{d : \text{for every set } \emptyset \neq L \subseteq \mathbb{N} \text{ there exists an at most } d\text{-element positive specifying set with respect to } \mathcal{L} \} \). If there is a nonempty set \( L \) that does not have a positive specifying set w.r.t \( \mathcal{L} \), define XTD*(\( \mathcal{L} \)) = \( \infty \).

The next series of definitions will introduce various subsets of \( \mathbb{N} \), each of which is a set of codes for u.r.e. families that satisfy some notion of teachability.

\[
\begin{align*}
(1) & \quad I_{TD}^\infty = \{ e : (\forall L \in \mathcal{L}_e)[\text{TD}(L, \mathcal{L}_e) < \infty] \} \\
(2) & \quad I_{TD} = \{ e : \text{TD}(\mathcal{L}_e) < \infty \} \\
(3) & \quad I_{T_{D*}}^\infty = \{ e : (\forall L \in \mathcal{L}_e)[\text{TD}(L, \mathcal{L}_e) < \infty] \} \\
(4) & \quad I_{T_{D*}} = \{ e : (\forall L \in \mathcal{L}_e)[\text{TD}^*(L, \mathcal{L}_e) < \infty] \} \\
(5) & \quad I_{TD}^- = \{ e : \text{TD}(\mathcal{L}_e) < \infty \} \\
(6) & \quad I_{T_{D*}}^- = \{ e : (\forall L \in \mathcal{L}_e)[\text{TD}^*(L, \mathcal{L}_e) < \infty] \} \\
(7) & \quad I_{XTD} = \{ e : \text{XTD}(\mathcal{L}_e) < \infty \} \\
(8) & \quad I_{XTD}^- = \{ e : \text{XTD}^+(\mathcal{L}_e) < \infty \} \\
\end{align*}
\]

Owing to space constraints, many proofs will be omitted or sketched.

4 Teaching Dimension

In this section we study the arithmetical complexity of the class of u.r.e. families with finite teaching dimension; several related decision problems will also be considered.

Before proceeding with the main theorems on the arithmetical complexity of the teaching dimension model and its variants, a series of preparatory results will be presented. Theorem 7 addresses the question: how hard (arithmetically) is it to determine whether or not, given \( e \in \mathbb{N} \) and a finite set \( D \), \( D \) can distinguish an r.e. set \( L_j^e \subseteq \mathcal{L}_e \) from the other members of \( \mathcal{L}_e \)?

Definition 6. \( DS := \{ (e, x, u) : (\forall y)[L_j^e \neq L_y^e \rightarrow L_j^e \cap D_u \neq L_y^e \cap D_u] \} \)

Theorem 7. \( DS \) is \( \Pi^0_2 \)-complete.

Proof. By the definition of DS, \( (e, x, u) \in DS \iff (\forall y)(\forall t)(\exists v > t)[[L_j^e \cap D_u \neq L_y^e \cap D_u] \vee (\forall p)(\forall a)[\exists b > a][L_j^e(p) = \bot] \]. Thus DS has a \( \Pi^0_2 \) description. Now, since \( \text{INF} \) is \( \Pi^0_2 \)-complete \[12\], it suffices to show that \( \text{INF} \) is many-one reducible to DS. Let \( g \) be a one-one recursive function with

\[
W_{g(e,i)} = \begin{cases} 
\mathbb{N} & \text{if } i = 0 \lor (\exists j > i)[j \in W_e]; \\
\{0,1,\ldots,i\} & \text{otherwise.}
\end{cases}
\]

\({}^3\) DS stands for “distinguishing set.”
Let \( f \) be a recursive function such that \( \mathcal{L}_{f(e)} = \{ W_{g(e,i)} : i \in \mathbb{N} \} \) and \( \mathcal{L}_{f(0)} = W_{g(e,0)} \). Recall that \( D_0 = \emptyset \). It is readily verified that \( e \in \text{INF} \iff \langle f(e), 0, 0 \rangle \in \text{DS} \).

The expectation that the arithmetical complexity of determining if a finite \( D \) is a smallest possible distinguishing set for some \( W_x \) belonging to \( \mathcal{L}_x \) is at most one level above that of DS is confirmed by Theorem 9.

**Definition 8.** \( \text{MDS} := \{ \langle e, x, u \rangle \in \text{DS} : (\forall u')[(D_{u'} | > | D_u|) \lor (e, x, u') \notin \text{DS}] \} \)

**Theorem 9.** \( \text{MDS} \) is \( \Pi^0_3 \)-complete.

**Proof.** (Sketch.) By the definition of MDS,

\[
\langle e, x, u \rangle \in \text{MDS} \iff \langle e, x, u \rangle \in \text{DS} \land (\forall u')[|D_{u'}| \geq |D_u|] \lor \langle e, x, u' \rangle \notin \text{DS}.
\]

By Theorem 7, \( \text{DS} \) has a \( \Pi^0_3 \) description and \( \overline{\text{DS}} \) has a \( \Sigma^0_2 \) description. Thus MDS has a \( \Pi^0_3 \) description. We omit the proof that MDS is \( \Pi^0_3 \)-hard.

Another problem of interest is the complexity of determining whether or not the teaching dimension of some \( W_x \) w.r.t. a class \( \mathcal{L}_x \) is upper-bounded by a given number \( d \). For \( d = 0 \), this problem is just as hard as DS (see Proposition 11); for \( d > 0 \), however, the complexity of the problem is exactly one level above that of DS (see Theorem 12). We omit the proofs.

**Definition 10.** \( \text{TDDP} := \{ \langle e, x, d \rangle : d \geq 1 \land (\exists u)[|D_u| \leq d \land \langle e, x, u \rangle \in \text{DS}] \} \)

**Proposition 11.** \( \{ \langle e, x \rangle : \mathcal{L}_x = \{ L^e_x \} \} \) is \( \Pi^0_2 \)-complete.

**Theorem 12.** \( \text{TDDP} \) is \( \Sigma^0_3 \)-complete.

Our first main result states that the class of all u.r.e. families \( \mathcal{L} \) such that any finite subclass \( \mathcal{L}' \subseteq \mathcal{L} \) has finite teaching dimension with respect to \( \mathcal{L} \) is \( \Pi^0_4 \)-complete.

**Theorem 13.** \( \Pi^0_{TD} \) is \( \Pi^0_4 \)-complete.

**Proof.** First, note the following equivalent conditions:

\[
e \in \Pi^0_{TD} \iff (\exists i)[\text{TD}(L^e_x, \mathcal{L}_e) < \infty]] \iff (\forall i)(\exists u)[\langle e, i, u \rangle \in \text{DS}].
\]

By Theorem 7, \( \langle e, i, u \rangle \in \text{DS} \) may be expressed as a \( \Pi^0_3 \) predicate, so \( \Pi^0_{TD} \) is \( \Pi^0_3 \).

Now consider any \( \Pi^0_3 \) unary predicate \( P(e) \); \( P(e) \) is of the form \( (\forall x)[Q(e, x)] \), where \( Q \) is a \( \Sigma^0_3 \) predicate. Since COF is \( \Sigma^0_3 \)-complete \( [12] \), there is a recursive function \( g(e, x) \) such that \( P(e) \iff (\forall x)[Q(e, x)] \iff (\forall x)[g(e, x) \in \text{COF}] \) holds. For each triple \( \langle e, x, i \rangle \), define

\[
H(e, x, i) = \begin{cases} 
\langle \langle e, x \rangle \rangle \cup (W_{g(e,x)} \cup \{i\}) & \text{if } i > 0; \\
\langle \langle e, x \rangle \rangle \cup W_{g(e, x)} & \text{if } i = 0 
\end{cases}
\]

Let \( h \) be a recursive function such that for all \( e \), \( \mathcal{L}_{h(e)} = \{ H(e, x, i) : x, i \in \mathbb{N} \} \).

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4 MDS stands for “minimal distinguishing set.”

5 TDDP stands for “Teaching dimension decision problem.”
By Theorem 13, the predicate \(\Sigma_{\text{cof}}\) is co-finite. Thus for all \(x\) and each \(i > 0\) such that \(i \notin W_{g(e,x)}\), \(H(e,x,i)\) has the teaching set \(\{2(e,x),+\}, \{2i+1,1\}\) with respect to \(L_{h(e)}\). Furthermore, for all \(x\) and each \(i\) such that either \(i \neq 0\) or \(i = 0\), \(H(e,x,i)\) has the teaching set \(\{2(e,x),+\}\cup\{2i+1,-1\}\) with respect to \(L_{h(e)}\). Therefore TD\((H(e,x,i))\), \(L_{h(e)}\) is co-finite for every pair \((x,i)\), so that \(h(e) \in I^y_{TD}\).

**Case (ii):** \(^{-}P(e)\) holds. Then \(W_{g(e,x)}\) is co-finite for some \(x\). Fix such an \(x\).

Then \(L_{h(e)}\) contains \(L' = \{H(e,x,i) : i \in \mathbb{N}\}\). Furthermore, for each positive \(i \notin W_{g(e,x)}\), since \(\{\langle e,x \rangle\} \cup (W_{g(e,x)} \cup \{i\}) \in \mathcal{L}'\), any teaching set for \(H(e,x,0)\) w.r.t. \(L_{h(e)}\) must contain \(\{2i+1,-1\}\). Hence TD\((H(e,x,0),L_{h(e)}) = \infty\), so that \(h(e) \notin I^y_{TD}\).

Thus \(I^y_{TD}\) is \(\Pi^0_4\)-complete.

Extending \(I^y_{TD}\) to include u.r.e. families \(L\) for which there is a co-finite subclass \(\mathcal{L}' \subseteq \mathcal{L}\) belonging to \(I^y_{TD}\) increases the arithmetical complexity of \(I^y_{TD}\) to \(\Sigma^0_5\).

**Theorem 14.** \(I^\infty_{TD}\) is \(\Sigma^0_5\)-complete.

**Proof.** (Sketch.) For any \(e, s\), let \(g\) be a recursive function such that \(L_{g(e,s)} = \{L^g_i : i > s\}\). Note that the expression for \(I^\infty_{TD}\) can be rewritten as \(e \in I^\infty_{TD} \iff (\exists g)(\forall s > t)g(e,s) = e\). Since \(L_e \subseteq L_{g(e,t)}\) is finite, it follows that \(e \in I^\infty_{TD}\).

By Theorem 13, the predicate \(g(e,s) \in I^y_{TD}\) has a \(\Pi^0_4\) description, so that \(I^\infty_{TD}\) is definable with a \(\Sigma^0_5\) predicate.

Now let \(P\) be any \(\Sigma^0_5\) predicate. By Proposition 2 and the \(\Sigma^0_3\)-completeness of \(\text{COF}\) [12], there is a recursive function \(h\) such that \(P(e) \iff (\forall^\infty a)(\forall b)[h(e,a,b) \in \text{COF}] \iff (\exists h)(\forall b)[h(e,a,b) \in \text{COF}] \iff (\exists h)(h(e,a,b) \in \text{COINF}]\). Now let \(g\) be a one-one recursive function such that

\[
W_{g(e,a,b,i)} = \begin{cases} \{\langle e, a, b \rangle\} \cup (W_{h(e,a,b)} \cup \{i\}) & \text{if } i > 0; \\ \{\langle e, a, b \rangle\} \cup W_{h(e,a,b)} & \text{if } i = 0. 
\end{cases}
\]

Let \(f\) be a one-one recursive function such that \(L_{f(e)} = \{W_{g(e,a,b,i)} : a, b, i \in \mathbb{N}\}\). Note that for all \(a, b, i \in \mathbb{N}\), TD\((W_{g(e,a,b,i)}) < \infty \iff h(e,a,b) \in \text{COF}\). One can show as in the proof of Theorem 13 that TD\((L, L_{f(e)}) < \infty\) holds for almost all \(L \in L_{f(e)}\) iff \(P(e)\) is true.

The next theorem shows that the index set of the class consisting of all u.r.e. families with finite teaching dimension is \(\Sigma^0_5\)-complete.

**Theorem 15.** \(I_{TD}\) is \(\Sigma^0_5\)-complete.

**Proof.** (Sketch.) From TD\((L_e) < \infty \iff (\exists a)(\forall b)[\langle e, b, a \rangle \in \text{TDPP}]\) and the fact that TDPP is \(\Sigma^0_3\)-complete by Theorem 12 we have that \(I_{TD}\) is \(\Sigma^0_5\).

To prove that \(I_{TD}\) is \(\Sigma^0_5\)-hard, consider any \(\Sigma^0_5\) predicate \(P(e)\). There is a binary recursive function \(g\) such that \(P(e) \iff (\exists a)(\forall b)[g(e,a,b) \in \text{COF}]\) and \(^{-}P(e) \iff (\forall^\infty a)(\forall b)[g(e,a,b) \in \text{COINF}]\). Now fix \(e, b \in \mathbb{N}\). For each \(a\), let \(\{H_{(a,0)}, H_{(a,1)}, H_{(a,2)}, \ldots\}\) be a numbering of the union of two u.r.e. families \(\{C_{(a,0)}, C_{(a,1)}, C_{(a,2)}, \ldots\}\) and \(\{L_{(a,0)}, L_{(a,1)}, L_{(a,2)}, \ldots\}\), which are defined as follows. (For simplicity, the notation for dependence on \(e\) and \(b\) is dropped.)
1. \( \{C(a,0), C(a,1), C(a,2), \ldots \} \) is a numbering of \( \{X \subseteq \mathbb{N} : |\mathbb{N} \setminus X| = a\} \).

2. Let \( E(a,0), E(a,1), E(a,2), \ldots \) be a one-one enumeration of \( \{X \subseteq \mathbb{N} : |\mathbb{N} \setminus X| < \infty \land |\mathbb{N} \setminus X| \neq a\} \). Let \( f \) be a recursive function such that for all \( n, s \in \mathbb{N} \), \( f(n, s) \) is the \((n + 1)\)st element of \( \mathbb{N} \setminus W_{g(e,a,b)} \). For all \( n, s \in \mathbb{N} \), define

\[
L_{(a,(n,s))} = \begin{cases} 
E_{(a,n)} & \text{if } (\forall t \geq s)[f(n, s) = f(n, t)]; \\
\mathbb{N} & \text{if } (\exists t > s)[f(n, s) \neq f(n, t)].
\end{cases}
\]

Note that \( \{H_{(a,0)}, H_{(a,1)}, H_{(a,2)}, \ldots \} \neq \emptyset \). Now construct a u.r.e. family \( \{G_{(e,b,0)}, G_{(e,b,1)}, \ldots \} \) with the following properties:

(i) For all \( s \), \( G_{(e,b,s)} \) is of the form \( \{b\} \oplus \{s\} \oplus \bigoplus_{j \in \mathbb{N}} H_{(j,i,j)} \).

(ii) For every nonempty finite set \( \{H_{(c_{0},d_{0})}, \ldots ,H_{(c_{k},d_{k})}\} \) with \( c_{0} < \ldots < c_{k} \), there is at least one \( t \) for which \( G_{(e,b,t)} = \{b\} \oplus \{t\} \oplus \bigoplus_{i \in \mathbb{N}} A_{i} \), where \( A_{i} = H_{(c_{i},d_{i})} \) for all \( i \in \{0, \ldots , k\} \) and \( A_{i} \in \{H_{(i,j)} : j \in \mathbb{N}\} \) for all \( i \notin \{c_{0}, \ldots , c_{k}\} \).

(iii) For every \( t \) such that \( G_{(e,b,t)} = \{b\} \oplus \{t\} \oplus \bigoplus_{j \in \mathbb{N}} H_{(j,i,j)} \), there are infinitely many \( t' \neq t \) such that \( G_{(e,b,t')} = \{b\} \oplus \{t'\} \oplus \bigoplus_{j \in \mathbb{N}} H_{(j,i,j)} \).

The family \( \{G_{(e,i,j)} : i,j \in \mathbb{N}\} \) may be thought of as an infinite matrix \( M \) in which each row represents a set parametrised by \( g(e,a,b) \) for a fixed \( b \) and \( a \) ranging over \( \mathbb{N} \). Furthermore, if there exists an \( a \) such that \( W_{g(e,a,b)} \) is cofinite for all \( b \), then the \( \alpha \)th column of \( M \) contains all cofinite sets with complement of size \( a \) plus a finite number of other cofinite sets; if no such \( a \) exists then every column of \( M \) contains all cofinite sets. Let \( h \) be a recursive function with \( L_{h(e)} = \{b\} \oplus \bigoplus_{i \in \mathbb{N}} \emptyset : b \in \mathbb{N}\} \cup \{G_{(e,b,s)} : b, s \in \mathbb{N}\} \). Showing that \( P(e) \) iff \( h(e) \in I_{TD} \) proves \( I_{TD} \) to be \( \Sigma_{3}^{0} \)-complete.

To conclude our discussion on the general teaching dimension, we demonstrate that the criterion for a u.r.e. family to have finite extended teaching dimension is so stringent that only finite families have a finite XTD.

**Theorem 16.** \( I_{XTD} \) is \( \Sigma_{3}^{0} \)-complete.

**Proof.** We show \( XTD(L_{e}) < \infty \iff |L_{e}| < \infty \). First, suppose \( L_{e} = \{L_{1}, \ldots ,L_{k}\} \).

As \( L_{e} \) is finite, \( TD(L_{1}, L_{e}) \leq k - 1 \) for all \( i \in \{1, \ldots , k\} \). Consider any \( L \notin L_{e} \).

For each \( i \in \{1, \ldots , k\} \), fix \( y_{i} \) with \( \mathbb{I}_{L}(y_{i}) \neq \mathbb{I}_{L_{i}}(y_{i}) \). Then \( \{y_{i,+} : 1 \leq i \leq k \land y_{i} \notin L\} \cup \{(y_{j},-) : 1 \leq j \leq k \land y_{j} \notin L\} \) is a specifying set for \( L \) with respect to \( L_{e} \) of size \( k \).

Second, suppose \( |L_{e}| = \infty \). Let \( T = \{\sigma \in \{0,1\}^{*} : (\exists \infty L \in \mathbb{L}_{e})(\forall x < |\sigma|)[\sigma(x) = \mathbb{I}_{L}(x)]\} \). Note that \( |L_{e}| = \infty \) implies \( T \) is an infinite binary tree. Thus by König’s lemma, \( T \) has at least one infinite branch, say \( B \). Then for all \( n \in \mathbb{N} \), there exist infinitely many \( L \in L_{e} \) such that \( \mathbb{I}_{B}(x) = \mathbb{I}_{L}(x) \) for all \( x < n \). Therefore \( B \) has no finite specifying set with respect to \( L_{e} \) and so \( XTD(L_{e}) = \infty \). Consequently, \( XTD(L_{e}) < \infty \iff |L_{e}| < \infty \iff (\exists a)(\forall b)(\exists c \leq a)\ldots \)
\[ a(\forall x)(\exists t > s)[1_{L_x}(x) = 1_{L_{x,t}}(x)]; \text{as any two quantifiers, at least one of which is bounded, may be permuted, it follows that the last expression is equivalent to a } \Sigma^0_3 \text{ formula. To show that } I_{XTD} \text{ is } \Sigma^0_3 \text{-hard, consider any } \Sigma^0_3 \text{ predicate } P, \text{ and let } g \text{ be a recursive function such that}
\]

\[ P(e) \leftrightarrow g(e) \in \text{COF.} \]

Let \( f \) be a recursive function with \( L_f(e) \) equal to \( \{W_{g(e)} \cup D : N \supseteq |D| < \infty\} \), the class of all sets consisting of the union of \( W_{g(e)} \) and a finite set of natural numbers. Then

\[ P(e) \leftrightarrow g(e) \in \text{COF} \leftrightarrow |L_f(e)| < \infty \leftrightarrow \text{XTD}(L_{f(e)}) < \infty, \]

and so \( I_{XTD} \) is indeed \( \Sigma^0_3 \)-complete. 

### 5 Positive Teaching Dimension

We now consider the analogues of the results in the preceding section for the positive teaching dimension model. In studying the process of child language acquisition, Pinker [11, p. 226] points to evidence in prior research that children are seldom “corrected when they speak ungrammatically”, and “when they are corrected they take little notice”. It seems likely, therefore, that children learn languages mainly from positive examples. Thus, as a model for child language acquisition, the positive teaching dimension model is probably closer to reality than the general teaching dimension model in which negative examples are allowed. The next two results are the analogues of Theorems 13 and 15 for the positive teaching dimension model. It is noteworthy that \( I^\forall_{TD} \) and \( I^\forall_{TD^+} \) have equal arithmetical complexity; that is to say, restricting the teaching sets of each \( L \in \mathcal{L} \) with \( e \in I^\forall_{TD} \) to positive teaching sets has no overall effect on the arithmetical complexity of \( I^\forall_{TD} \).

**Theorem 17.** \( I^\forall_{TD^+} \) is \( \Pi^0_4 \)-complete.

**Proof.** (Sketch.) Observe that

\[ (\forall L \in \mathcal{L}_e)[TD^+(L, \mathcal{L}_e) < \infty] \leftrightarrow (\forall i)(\exists u)[(\exists s)[D_u \subseteq L_{i,s}] \wedge (\forall j)[(\forall x)(\exists t > s)[1_{L_{j,t}}(x) = 1_{L_{i,s,t}}(x)] \vee \forall s'[D_u \not\subseteq L_{j,s'}]]], \]

Since the right-hand side simplifies to a \( \Pi^0_4 \) predicate, we know that \( I^\forall_{TD^+} \) is \( \Pi^0_4 \).

For the proof that \( I^\forall_{TD^+} \) is \( \Pi^0_4 \)-hard, take any \( \Pi^0_4 \) predicate \( P \), and let \( g \) be a recursive function such that \( P(e) \leftrightarrow (\forall a)[g(e,a) \in \text{COF}] \). Define a u.r.e. family \( \mathcal{L} = \{L_i\}_{i \in \mathbb{N}} \) as follows. (For notational simplicity, the notation for dependence on \( e \) is dropped.) For all \( a, i \in \mathbb{N} \),

\[ L_{i,a} = \{a\} \oplus W_{g(e,a)}, \]
\[ L_{(a,i+1)} = \begin{cases} \{a\} \oplus W_{g(e,a)} & \text{if } i \in W_{g(e,a)}; \\ \{a\} \oplus \{i\} \cup \{x : x < i \land x \in W_{g(e,a)}\} & \text{if } i \notin W_{g(e,a)}. \end{cases} \]

Let \( f \) be a recursive function for which \( L_{f(e)} = \mathcal{L} \). One can show that \( \text{TD}^+(L,f(e)) < \infty \) holds for all \( L \in \mathcal{L}_{f(e)} \) iff \( P(e) \) is true. 

**Theorem 18.** \( I_{TD^+} \) is \( \Sigma^0_5 \)-complete.

**Proof.** (Sketch.) For any \( e \), one has \( \text{TD}^+(\mathcal{L}_e) < \infty \leftrightarrow (\exists a)(\forall b)[\text{TD}^+(L^e_b, \mathcal{L}_e) < a] \) and
\[
\text{TD}^+(L^e_b, \mathcal{L}_e) < a \leftrightarrow (\exists a)(\forall c)[(\exists s')|D_a| \subseteq L^e_b(a) \land |D_a| < a \\
\land[(\forall x)(\exists s)(\forall t > s)[1_{L^e_b}(x) = 1_{L^e_b}(x)] \land (\forall t')|D_a| \not\subseteq L^e_{c,t}]].
\]

Simplifying the last equivalence yields a \( \Sigma^0_5 \)-predicate for \( \text{TD}^+(\mathcal{L}_e) < \infty \).

The proof that \( I_{TD^+} \) is \( \Sigma^0_5 \)-hard is similar to the earlier proof that \( I_{TD} \) is \( \Sigma^0_5 \)-hard (but requires some additional ideas). Given any \( \Sigma^0_5 \) formula \( P \), let \( R \) be a recursive predicate such that \( P(e) \rightarrow (\exists a)(\forall b)(\forall c)(\forall d)(\exists \bar{l})[R(e,a,b,c,d,\bar{l})] \) and \( \neg P(e) \rightarrow (\forall x)(\forall b)(\forall c)(\exists \bar{l})[\neg R(e,a,b,c,d,\bar{l})] \). Now fix any \( e,b \in \mathbb{N} \). Let \( ((a_0^e, b_0^e), \ldots, (a_{k_e}^e, b_{k_e}^e), (a_0^b, b_0^b, c_0^b), \ldots, (a_{k_b}^b, b_{k_b}^b, c_{k_b}^b)) \) be a one-one enumeration of all non-empty finite sequences of triples such that \( a_0^e < \ldots < a_{k_e}^e \) for all \( j \in \mathbb{N} \). Define the set (dropping the notation for dependence on \( e \))
\[
S_b := \{i : (\exists j \in \{0, \ldots, k_e\})[|D_{b_j}| \neq a_i^e + 1 \land (\exists \bar{l})[R(e,a_i^e, b_j, b_{j+1}^e, \bar{l})]]\},
\]
using our fixed numbering \( D_0, D_1, D_2, \ldots \) of all finite sets. For each \( b \) (with \( e \) fixed), construct a u.r.e. family \( \{G_{(b,-1)}\} \cup \bigcup_{s \in \mathbb{N}} \{G_{(b,s)}\} \) as follows:
\[
G_{(b,s)} = \{b\} \oplus S_b \oplus \bigoplus_{i \in \mathbb{N}} \mathbb{N} \text{ if } s = -1 \text{ or } s \in S_b.
\]
If \( s \notin S_b \), set \( G_{(b,s)} = \{b\} \oplus (S_b \cup \{s\}) \oplus \bigoplus_{i \in \mathbb{N}} H_i \), where
\[
H_i = \begin{cases} \emptyset & \text{if } i \notin \{a_0^e, \ldots, a_{k_e}^e\} \\
D_{b_i} & \text{if } i \in \{a_0^e, \ldots, a_{k_e}^e\}.
\end{cases}
\]
Let \( f \) be a recursive function such that \( L_{f(e)} = \bigcup_{e \in \mathbb{N}}(\{G_{(b,-1)}\} \bigcup \bigcup_{s \in \mathbb{N}} \{G_{(b,s)}\}) \) (note again that the notation for dependence on \( e \) in the definition of \( G_{(b,s)} \) has been dropped). We omit the proof that \( P(e) \) holds iff \( \text{TD}^+(L_{f(e)}) < \infty \).

**Theorem 19.** \( I_{TD^+}^{\infty} \) is \( \Sigma^0_5 \)-complete.

**Proof.** (Sketch.) The condition
\[
(\forall i)[\text{TD}^+(L^i_{f(e)}, \mathcal{L}_e) < \infty] \leftrightarrow (\exists i)(\forall j \geq i)(\exists a)[\text{TD}^+(L^e_j, \mathcal{L}_e) < a],
\]

together with the fact that \( \text{TD}^+(L^e_j, \mathcal{L}_e) < a \) is a \( \Sigma^0_3 \) predicate (as shown in the proof of Theorem 18), shows that \( I_{TD^+}^{\infty} \) is \( \Sigma^0_5 \). The proof that \( I_{TD^+}^{\infty} \) is \( \Sigma^0_5 \)-hard is very similar to that of Theorem 14. 

\[\]
Finally, consider the positive extended teaching dimension. Like the u.r.e. families with finite extended teaching dimension, those with finite positive extended teaching dimension have a particularly simple structure.

**Theorem 20.** $I_{XTD^+}$ is $\Pi_2^0$-complete.

**Proof.** (Sketch.) One may verify directly that

$$XTD^+(L_e) < \infty \iff (\forall i,j) [L_i^e = L_j^e \lor L_i^e \cap L_j^e = \emptyset].$$

Note that $(\forall i,j) [L_i^e = L_j^e \lor L_i^e \cap L_j^e = \emptyset]$ if and only if

$$(\forall i,j)(\forall x,s)(\exists t > s)[\exists L_{i,t}^e(x) = \top_{L_j^e(x)} \lor (\forall s')[\exists L_{i,s'}^e \cap L_{j,s'}^e = \emptyset]],$$

and that the latter expression reduces to a $\Pi_2^0$ predicate. Hence $I_{XTD^+}$ is $\Pi_2^0$-hard. To establish that $I_{XTD^+}$ is $\Pi_2^0$-complete, take any $\Pi_2^0$ predicate $P$, and let $g$ be a recursive function such that $P(e) \leftrightarrow g(e) \in \text{INF}$. Let $f$ be a recursive function such that $L_{f(e)} = \{N, G\}$, where

$$G = \begin{cases} 
\mathbb{N} & \text{if } |W_{g(e)}| = \infty; \\
\{0\} \cup \{x : x < m\} & \text{if } m \text{ is the least number such that } (\forall s \geq m)[W_{g(e),s} = W_{g(e),s+1}].
\end{cases}$$

Then $P(e) \leftrightarrow g(e) \in \text{INF} \leftrightarrow XTD^+(L_{f(e)}) < \infty$. 

## 6 Recursive Teaching Dimension

Although the classical teaching dimension model is quite succinct and intuitive, it is rather restrictive. For example, let $L$ be the concept class consisting of the empty set $L_0 = \emptyset$ and all singleton sets $L_i = \{i\}$ for positive $i \in \mathbb{N}$. Then $TD(L_i, L) = 1$ for all $i \in \mathbb{N} \setminus \{0\}$ but $TD(L_0, L) = \infty$. Thus $TD(L) = \infty$ even though the class $L$ is relatively simple. One deficiency of the teaching dimension model is that it fails to exploit any property of the learner other than the learner being consistent. The recursive teaching model [13][10], on the other hand, uses inherent structural properties of concept classes to define a teaching-learning protocol in which the learner is not just consistent, but also “cooperates” with the teacher by learning from a sequence of samples that is defined relative to the given concept class. In this section, we shall consider the arithmetical complexity of the index set of the class of all u.r.e. families that are teachable in some variants of the recursive teaching model. The complexities of interesting problems relating to the original recursive teaching model remain open.

**Definition 21.** A positive teaching plan for $L$ is any sequence $TP = ((L_0, d_0), (L_1, d_1), \ldots)$ where (i) the families $\{L_i\}$ form a partition of $L$, and (ii) $d_i = TD^+(L_i, L \setminus \bigcup_{0 \leq j < i}(L_j))$ for all $i$. $RTD^+_1(L)$ is defined to be $\text{inf}\{\text{ord}(TP) : TS \text{ is a positive teaching plan for } L\}$. Since this paper only considers positive teaching plans, positive teaching plans will simply be called “teaching plans”. Note that a positive teaching plan for $L$ is essentially a teaching sequence for $L$ that employs only positive examples and partitions $L$ into singletons.
We begin with a lemma; the proof is omitted.

**Lemma 22.** Let \( \{ \mathcal{F}_i \}_{i \in \mathbb{N}} \) be any sequence of families. If \( \sup \{ \text{RTD}^+_i(\mathcal{F}_i) : i \in \mathbb{N} \} < \infty \), then \( \sup \{ \text{RTD}^+_i(\mathcal{F}_i) : i \in \mathbb{N} \} \leq \text{RTD}^+_i(\bigcup_{i \in \mathbb{N}} \mathcal{F}_i) \leq \sup \{ \text{RTD}^+_i(\mathcal{F}_i) : i \in \mathbb{N} \} + 1 \); otherwise, \( \text{RTD}^+_i(\bigcup_{i \in \mathbb{N}} \mathcal{F}_i) = \infty \).

**Definition 23.** We denote by \( R^+_1 \) the set of codes for u.r.e. families, \( \mathcal{L} \), such that \( \text{RTD}^+_1(\mathcal{L}) \) is finite.

**Theorem 24.** \( R^+_1 \) is \( \Sigma^0_4 \)-complete.

**Proof.** To show that \( R^+_1 \) is \( \Sigma^0_4 \), fix \( e \), a code for a u.r.e. family, \( \mathcal{F} \). Given \( n \in \mathbb{N} \), whether or not \( \text{RTD}^+_1(\mathcal{F}) \leq n \) can be decided by executing the following algorithm. Find the \( F \in \mathcal{F} \) of least index such that \( \text{TD}^+(F, \mathcal{F} \setminus \{ F \}) \leq n \) and call this set \( F_0 \). Having defined \( F_0, \ldots, F_i \), let \( F_{i+1} \) be the set of least index in \( \mathcal{F} \setminus \{ F_0, \ldots, F_i \} \) such that \( \text{TD}^+(F_{i+1}, \mathcal{F} \setminus \{ F_0, \ldots, F_{i+1} \}) \leq n \). If there is a teaching plan for \( \mathcal{F} \) with order at most \( n \), then the above algorithm will produce such a teaching plan because \( \text{RTD}^+_1(\mathcal{F} \setminus \{ F_0, \ldots, F_i \}) \leq n \) for all \( i \in \mathbb{N} \). Conversely, if there is no such teaching plan, then clearly the algorithm must initiate a non-terminating search at some stage.

Observe that the statement \( \text{TD}^+(F_{i+1}, \mathcal{F} \setminus \{ F_0, \ldots, F_{i+1} \}) \leq n \) is \( \Pi^0_3 \), therefore \( \text{RTD}^+_1(\mathcal{F}) \leq n \) is \( \Pi^0_0 \). Finally, \( \text{RTD}^+_1(\mathcal{F}) < \infty \) is equivalent to \( (\exists n)(\text{RTD}^+_1(\mathcal{F}) \leq n) \); hence, it is \( \Sigma^0_4 \). It remains to show that \( R^+_1 \) is \( \Sigma^0_4 \)-hard.

Every \( \Sigma^0_3 \) predicate is of the form \( (\forall \in \mathbb{N})(g(e, n) \in \text{CONF}) \), where \( g \) is a computable function. Fix such a predicate, \( P \), and computable function \( g \).

For \( k \in \mathbb{N} \), let \( f_k : \mathbb{N} \to \mathbb{N} \) be a uniformly computable sequence of functions such that (1) \( D_{f_k(n)} \subseteq [0, k] \), (2) \( (\forall S \subseteq [0, k])(\exists n)(D_{f_k(n)} = S) \) and (3) \( (\forall n \in \text{ran}(f_k))(|f_k^{-1}(n)| = \infty) \). Define \( L_k = \{(0, k) \oplus \emptyset \} \cup \{ D_{f_k(n)} \oplus \{ n \} : n \in \mathbb{N} \} \). We will denote the members of \( L_k \) by \( L^k_0, L^k_1 \), where \( L^k_0 = [0, k] \oplus \emptyset \) and \( L^k_{n+1} = D_{f_k(n)} \oplus \{ n \} \). Observe that \( \text{RTD}^+_1(L_k) = k + 1 \). However, for any finite \( L' \subset L_k \), \( \text{RTD}^+_1(L') = 1 \).

Let \( m \) be a computable function such that \( m(a, n, s) \) is the number of \( t < s \) such that the \( n \)-th element of the complement of \( W_{a,t} \) differs from the \( n \)-th element of the complement of \( W_{a,t+1} \). Define \( G_{a,i} = \bigcup_{n \in \mathbb{N}} \{ L^i_j : (\exists s)(j < m(a, n, s)) \} \) and let \( \mathcal{F}_e = \bigcup_{i \in \mathbb{N}} G_{g(e,i),i} \). By the construction, there is a computable function, \( h \), such that \( W_{h(e)} = \mathcal{F}_e \). We omit the proof that \( \text{RTD}^+_1(\mathcal{F}_e) < \infty \) iff \( P(e) \). 

**Definition 25.** Given \( \sigma \in \mathbb{N}^* \) and \( S = \{ s_0, \ldots, s_n \} \subset \mathbb{N} \) with \( s_0 < \cdots < s_n < |\sigma| \), define \( \sigma[S] \in \mathbb{N}^* \) by \( \sigma[S](i) = \sigma(s_i) \) for \( i \leq n \).

We now consider a “semi-effective” version of the recursive teaching model in which the teacher presents only positive teaching sets to the learner.

**Definition 26.** Let \( \mathcal{L} \) be any family of subsets of \( \mathbb{N} \). A **positive teaching sequence** for \( \mathcal{L} \) is any sequence \( TS = ((\mathcal{F}_0, d_0), (\mathcal{F}_1, d_1), \ldots) \) such that (i) the families \( \mathcal{F}_i \) form a partition of \( \mathcal{L} \) with each \( \mathcal{F}_i \) nonempty, and (ii) for all \( i \) and all \( L \in \mathcal{F}_i \), there is a subset \( S_L \subseteq L \) with \( |S_L| = d_i < \infty \) such that for all \( L' \in \bigcup_{j \geq i} \mathcal{F}_j \), it
holds that $S_L \subseteq L' \rightarrow L = L'$. This is called the order of $TS$, and is denoted by $\text{ord}(TS)$. The positive recursive teaching dimension of $L$ is defined as $\inf\{\text{ord}(TS) : TS$ is a positive teaching sequence for $L\}$ and is denoted by $\text{RTD}^+(L)$.

We denote by $R_{\text{are}}^+$ the set of codes for u.r.e. families, $L$, such that $\text{RTD}^+(L)$ is finite and witnessed by a u.r.e. teaching sequence. In this section, a “teaching sequence” will always mean a u.r.e. teaching sequence.

Our last major result is that $R_{\text{are}}^+$ is $\Sigma^0_5$-complete, which we establish in the following three theorems.

**Theorem 27.** $R_{\text{are}}^+$ is $\Sigma^0_5$-hard.

**Proof.** Fix a $\Sigma^0_5$-predicate $P$. As in the proof of Theorem 4.4 from [3], let $g$ be computable such that $P(e) \rightarrow (\exists x)(\forall x' > x)(\forall y)(g(e, x', y) \in \text{COINF}) \land (\forall x' \leq x)(\exists y)(g(e, x', y) \in \text{COINF})$ and $\neg P(e) \rightarrow (\forall x)(\exists y)(g(e, x, y) \in \text{COINF})$.

As in the proof of Theorem 24, let $\{f_k\}_{k \in \mathbb{N}}$ be a uniformly computable sequence of functions such that for all $k, n \in \mathbb{N}$, (1) $D_{f_k(n)} \subseteq [0, k]$, (2) $\forall S \subseteq [0, k] (\exists m) (D_{f_k(m)} = S)$ and (3) $\forall m \in \text{ran}(f_k) (|f_k^{-1}(m)| = \infty)$.

Fix $a, n \in \mathbb{N}$ and $\sigma \in \mathbb{N}^*$. Define $H_a(x, \sigma) = \{\langle \sigma, x \rangle\}$ if $x \in W_a$ and $H_a(x, \sigma) = \emptyset$ otherwise. Using this notation, we define the set $A^n = \bigoplus_{i \in \mathbb{N}} ([0, n] \oplus \emptyset)$ and the sets $A^n_{i, \sigma} = \left(D_{f_a(i, \sigma)} \cup H_a(i, \sigma)\right) \oplus \left(\bigoplus_{j < i}([0, n] \oplus \emptyset)\right) \oplus \left(\bigoplus_{1 \leq j < |\sigma|} (D_{f_a(i, j)} \cup \{\sigma[j] \cup [i, |\sigma|]\})\right)$. Using the above sets, we define the following families: $G^n_{i, \sigma} = \{A^n_{i, \sigma} : \sigma \in \mathbb{N}^* \land |\sigma| \geq 2\}; G^n = \{A^n\} \cup \bigcup_{i \in \mathbb{N}} G^n_{i, \sigma}$.

Suppose that $a \in \text{COINF}$ and let $x_0, \ldots, x_k$ be an increasing enumeration of $W_a$. The following is a teaching sequence for $G^n$: $\left((\bigcup_{i \in \mathbb{N}} W_a G^n_{i, \sigma}, 1), (G^n_{0, \sigma}, 1), \ldots, (G^n_{i, \sigma}, 1), \{A^n\}, 1\right)$. Thus, $\text{RTD}^+(G^n) = 1$. Now suppose that $a \in \text{COFINF}$ and let $x_0, x_1, \ldots$ be an increasing enumeration of $W_a$. Suppose $TS = ((L_0, d_0), (L_1, d_1), \ldots)$ is a teaching sequence for $G^n$ and $\text{ord}(TS) \leq n$. Consider an arbitrary $A^n_{i, \sigma} \in G^n_{i, \sigma}$ for $i \geq 1$. $A^n_{i, \sigma} \notin \mathcal{L}_0$, because $n + 1$ points are needed to distinguish $A^n_{i, \sigma}$ from every member of $G^n_{i, \sigma}$. Since $G^n_{i, \sigma} \cap \mathcal{L}_0 = \emptyset$, we know that $A^n \notin \mathcal{L}_0$. Now suppose that $G^n_{i, \sigma} \subseteq \bigcup_{j \geq k} \mathcal{L}_i$ and $A^n \notin \bigcup_{i < k} \mathcal{L}_i$, then $\mathcal{L}_k$ cannot contain any member of $G^n_{i, \sigma}$ for $i \geq k + 1$ because $n + 1$ points are needed to distinguish the members of $G^n_{i, \sigma}$ from the members of $G^n_{i, \sigma}$. As before, this also implies $A^n \notin \mathcal{L}_{k+1}$. By induction, we conclude that $A^n \notin \mathcal{L}_i$ for any $i \in \mathbb{N}$. This is a contradiction, so $\text{RTD}^+(G^n) \geq n + 1$. Since $TS = \left((\{A^n\}, n + 1), (G^n_{0, \sigma}, 1), (G^n_{1, \sigma}, 1), \ldots\right)$ is a teaching sequence for $G^n$ and $\text{ord}(TS) = n + 1$, we conclude that $\text{RTD}^+(G^n) = n + 1$. Finally, define $F_{e, x} = \bigcup_{y \in \mathbb{N}} G^n_{(e, x, y), x}$ and $F_e = \bigcup_{x \in \mathbb{N}} F_{e, x}$. 
We wish to prove that RTD\(^+\)(\(F_e\)) \(\neq \infty\) if and only if \(P(e)\). First, suppose \(P(e)\). For all but finitely many \(x\), \(g(e, x, y) \in \text{COF}\) for all \(y\). This means that RTD\(^+\)(\(F_{e,x}\)) = 1 for all but finitely many \(x\). For each \(x\) for which RTD\(^+\)(\(F_{e,x}\)) \(\neq 1\) the dimension is still finite, hence, there is a uniform bound, \(n\), on the recursive teaching dimension of all the \(F_{e,x}\). We conclude that RTD\(^+\)(\(F_e\)) \(\neq \infty\).

On the other hand, if \(\neg P(e)\), then for every \(x\) there is exactly one \(y\) such that \(g(e, x, y) \in \text{COINF}\). Hence, \(F_e\) is the disjoint union of families whose RTD is unbounded. We have thus reduced an arbitrary \(\Sigma^0_5\)-predicate to \(R^+_{\text{ure}}\).

**Theorem 28.** \(R^+_{\text{ure}}\) is \(\Sigma^0_5\).

**Proof.** Let \(\{S_n\}_{n \in \mathbb{N}}\) enumerate the u.r.e teaching sequences, with \(S_n = ((\mathcal{L}_0^n, d_0^n), (\mathcal{L}_1^n, d_1^n), \ldots)\). Consider a u.r.e. family, \(F = \{F_0, F_1, \ldots\}\) coded by \(e\).

\[
e \in R^+_{\text{ure}} \iff (\exists a, n) (\text{ord}(TS_a) \leq n \land \text{TS}_a \text{ is a teaching sequence for } F)
\]

\[
\text{ord}(TS_a) \leq n \iff (\forall i) (d_i \leq n)
\]

To say that \(S_a\) is a teaching sequence for \(F\) is equivalent to (1) \(\{\mathcal{L}_0, \mathcal{L}_1, \ldots\}\) is a partition of \(F\) and (2) \(\forall i \in \mathbb{N}, L \in \mathcal{L}_i\) (TD\(^+\)(\(L, \bigcup_{j \geq i} \mathcal{L}_j\)) \(\leq d_i\)).

Since the statement TD\(^+\)(\(L, \bigcup_{j \geq i} \mathcal{L}_j\)) \(\leq d_i\) is \(\Sigma^0_3\), we know that the statement \(\forall i \in \mathbb{N}, L \in \mathcal{L}_i\) (TD\(^+\)(\(L, \bigcup_{j \geq i} \mathcal{L}_j\)) \(\leq d_i\)) is \(\Pi^0_3\). The statement that \(\{\mathcal{L}_0, \mathcal{L}_1, \ldots\}\) is a partition of \(F\) is equivalent to

\[
(\forall i \in \mathbb{N}, F \in \mathcal{L}_i)(\exists F' \in \mathcal{F}) (F = F') \land (\forall F \in \mathcal{F})(\exists i \in \mathbb{N}, F' \in \mathcal{L}_i) (F = F')
\]

\[
\land (\forall i, j \in \mathbb{N}, F \in \mathcal{L}_i, F' \in \mathcal{L}_j) (i \neq j \rightarrow F \neq F')
\]

which is \(\Pi^0_3\). Thus, \(e \in R^+_{\text{ure}}\) is \(\Sigma^0_5\).

**Theorem 29.** \(R^+_{\text{ure}}\) is \(\Sigma^0_5\)-complete.

7 Conclusion

This paper studied the arithmetical complexity of index sets of classes of u.r.e. families that are teachable under various teaching models. Our main results are summarised in Table 1. While u.r.e. families constitute a very special case of families of sets, many of our results may be extended to the class of families of countably many sets; more precisely, if we define \(C^X_j = \{x : \langle j, x \rangle \in X\}\) and \(C_X = \{C^X_j : j \in \mathbb{N}\}\) for any \(X \subseteq \mathbb{N}\), it is not difficult to apply our results to determine the position of \(\{X : I(C_X) < \infty\}\) for different teaching parameters \(I\) in the hierarchy of sets of sets [12, §15.1]. We also determined first-order formulas with the least possible quantifier complexity defining some fundamental decision problems in algorithmic teaching. Our work may be extended in several directions. For example, it might be interesting to investigate the arithmetical.
complexity of index sets of classes of general – even non-u.r.e. – families that are teachable under the teaching models considered in the present paper. In particular, it may be asked whether the arithmetical complexity of the class of teachable families with one-one numberings is less than that of the class of teachable families that do not have one-one numberings.

| Index Set | Arithmetical Complexity |
|-----------|--------------------------|
| \{e : (\forall L \in \mathcal{L}_e)[\text{TD}(+)(L, \mathcal{L}_e) < \infty]\} | \Pi_0^0\text{-complete (Thms }13, 17) |
| \{e : \text{TD}(+)(\mathcal{L}_e) < \infty\} | \Sigma_0^0\text{-complete (Thms }15) |
| \{e : \text{XTD}(\mathcal{L}_e) < \infty; \{e : \text{XTD}^+(\mathcal{L}_e) < \infty\}\} | \Sigma_0^0\text{-complete (Thm }10); \Pi_0^0\text{-complete (Thm }20) |
| \{e : \text{RTD}_1^+(\mathcal{L}_e) < \infty\} | \Sigma_0^0\text{-complete (Thm }20) |
| \{e : \text{RTD}_1^{+\text{u.r.e.}}(\mathcal{L}_e) < \infty\} | \Sigma_0^0\text{-complete (Thm }29) |

Table 1. Summary of main results on u.r.e. families. The notation TD(+) indicates that the result holds for both TD and TD^+.

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