Efficient Streaming Algorithms for Submodular Maximization with Multi-Knapsack Constraints

Yanhao Wang*, Yuchen Li† and Kian-Lee Tan‡

School of Computing, National University of Singapore

June 16, 2017

Abstract

Submodular maximization (SM) has become a silver bullet for a broad class of applications such as influence maximization, data summarization, top-k representative queries, and recommendations. In this paper, we study the SM problem in data streams. Most existing algorithms for streaming SM only support the append-only model with cardinality constraints, which cannot meet the requirements of real-world problems considering either the data recency issues or more general d-knapsack constraints. Therefore, we first propose an append-only streaming algorithm KnapStream for SM subject to a d-knapsack constraint (SMDK). Furthermore, we devise the KnapWindow algorithm for SMDK over sliding windows to capture the recency constraints. Theoretically, the proposed algorithms have constant approximation ratios for a fixed number of knapsacks and sublinear complexities. We finally evaluate the efficiency and effectiveness of our algorithms in two real-world datasets. The results show that the proposed algorithms achieve two orders of magnitude speedups over the greedy baseline in the batch setting while preserving high quality solutions.

1 Introduction

Extracting representative elements from massively generated data is becoming the de facto standard in the big data era. In many scenarios, the utility value (or representativeness) of the selected subset is defined by a submodular function and the problem is thus formulated as maximizing a submodular function subject to a certain constraint. In fact, submodular functions are pervasive in a broad class of applications, e.g., influence maximization [8,20,28,30], data summarization [2,15,17,18,31,33], top-k representative queries [4,9] and recommendations [1,22,23,32]. This is because submodular functions are not only adequate and general to model different subset selection problems but also have deep theoretical consequences for designing efficient approximation algorithms.

Most existing literature on the submodular maximization (SM) problem focuses on maximizing a monotone submodular function with a cardinality constraint [4,8,9,17,22,28,31]. Although the classical greedy algorithm proposed by Nemhauser et al. in [19] provides a \((1 - 1/e)\)-approximate solution, it has several drawbacks that limit its use in a broader spectrum of applications.

First, the greedy algorithm can only work with a single cardinality constraint. There is a large body of real-world problems \([1,13,14,16,18,20,24,25,32]\) that are modeled as SM with (1) more than one constraint and (2) more general constraints beyond cardinality constraints. The d-knapsack constraints are a class of general constraints that have been widely adopted in SM problems \([14,16,18,21,24,32]\). A knapsack constraint assigns a cost to each element and restricts the total cost of selected elements to a given budget, and the cardinality constraint is a special case of a knapsack constraint where each element is assigned to a uniform cost of 1 and the budget is constrained to \(k\). Here, we illustrate the d-knapsack constraints with two concrete
We devise a novel algorithm utilities. It maintains a fixed-size buffer in each candidate solution and performs a post-processing with buffered elements before returning the results. It spends extra time but produces solutions with significantly higher quality. (Section 4)

We further propose \textsc{KnapWindow} for SMDK over sliding windows. \textsc{KnapWindow} always maintains a logarithmic number of checkpoints w.r.t. the range of the utility values while returning solutions with provable approximation ratios. To the best of our knowledge, it is the first algorithm for SMDK over sliding windows. In addition, we devise a buffer and post-processing optimization for \textsc{KnapWindow}. It maintains a fixed-size buffer in each candidate solution and performs a post-processing with buffered elements before returning the results. It spends extra time but produces solutions with significantly higher utilities.

Here, we summarize our main contributions as follows:

- We devise a novel algorithm \textsc{KnapStream} for SMDK in append-only streams. It improves the approximation ratio from $\frac{1}{1+\epsilon} - O(1)$ to $\frac{1}{1+\epsilon} - O(1))$. (Section 5)
- We propose the first algorithm \textsc{KnapWindow} for SMDK over sliding windows. It provides $(\frac{1}{2(1+\epsilon)} - O(1))$ approximate solutions for SMDK. We also devise the buffer and post-processing optimization for \textsc{KnapWindow}, which achieves better utilities with extra overhead. (Section 5)
- We experimentally evaluate the efficiency and effectiveness of our proposed algorithm with two applications in real-world datasets. The results show that our algorithms are much more efficient than existing algorithms on SMDK in both batch and append-only streaming settings while producing solutions of competitive quality. (Section 6)

2 Related Work

In this section, we survey the literature that is the most relevant to our work: (1) SM in the batch setting and (2) SM in data streams.

SM in the batch setting. Due to its theoretical consequences, SM has been widely applied to various problems in databases and data mining, including influence maximization [8,20,25,30], data summariza-
tion, top-k representative queries, recommendations, to just name a few. This has triggered a large body of research on SM in recent years. We focus on reviewing existing techniques for SM in the batch setting as they are the most related to our work. Sviridenko first proposes a \((1-1/e)\) approximation algorithm for SM subject to 1-knapsack constraints with \(O(n^5)\) complexity. Kulik et al. propose a \((1-1/e-\varepsilon)\) approximation algorithm for SM subject to \(d\)-knapsack constraints with \(O(n^{d-1}e)\) complexity. Both algorithms have high-order polynomial complexities and are not practical for real-world applications in large datasets. Efficient algorithms for SM subject to 1-knapsack constraints with approximation factors of \(\frac{1}{2}(1-1/e)\) and \(1-1/\sqrt{e}\) are proposed by Leskovec et al. and Lin et al. respectively. Both algorithms cannot be directly applied to SM with \(d\)-knapsack constraints. In this paper, we adapt the algorithm in Lin for SM and use the adapted algorithm as the batch baseline. We further propose algorithms for SMDK in data stream models with much higher efficiency than the baseline.

**SM in data streams.** Next, we review existing streaming SM algorithms. Badanidiyuru et al. propose a \((1/2-\varepsilon)\) approximation algorithm for SM with cardinality constraints in append-only streams with sublinear time and space complexities. Append-only stream algorithms for SMDK are proposed in and . Both algorithms achieve a \((1/2+O(1))\) approximation for SM. To the best of our knowledge, there is no existing algorithm with a better approximation ratio. Our proposed KnapStream algorithm in Section improves the approximation ratio to \((1-1/e)\).

The sliding window model is widely adopted in many streaming applications to capture the data recency constraint. Exponential histograms and smooth histograms are common methods to estimate the values of functions over sliding windows. Both methods achieve sublinear time and space complexities w.r.t. the window size with bounded error ratios. Having said that, they are only applicable to the functions with special properties. Specifically, exponential histograms can only approximate “weakly additive” functions, i.e., \(f(A) + f(B) \leq f(A \cup B) \leq C_f \cdot (f(A) + f(B))\) for any disjoint sub-streams \(A, B\) and a small constant \(C_f\). Submodular functions are not “weakly additive” as they have \(f(A) + f(B) \geq f(A \cup B)\). Smooth histograms give non-trivial results only when the target function can be approximated with a factor of at least 0.8 in append-only streams. However, there is no polynomial time algorithms for SMDK achieving an approximation factor of over \((1 - e^{-1}) \approx 0.632\) unless \(P = NP\). Therefore, both methods cannot be directly applied to SMDK over sliding windows. To the best of our knowledge, there are only a few research efforts on SM over sliding windows, but they only focus on cardinality constraints. Epasto et al. propose \((1/3 - \varepsilon)\) and \((1/2 - \varepsilon)\) approximation algorithms for SM with cardinality constraints over sliding windows. In , influence maximization in social streams is defined by SM with cardinality constraints over sliding windows. They propose \((1/2 - \varepsilon)\) and \((1/4 - \varepsilon)\) approximation algorithms for the problem. In this paper, we address the problem of SM over sliding windows with more general \(d\)-knapsack constraints.

### 3 Preliminaries

In this section, we first introduce the definitions of monotone submodular functions and \(d\)-knapsack constraints in Section 3.1. Subsequently, we discuss the classical SMK problem in the batch setting in Section 3.2. In Section 3.3, we present the problems of SMDK in the append-only stream model as well as the sliding window model respectively.

#### 3.1 Monotone Submodular Functions and \(d\)-knapsack Constraints

**Monotone Submodular Functions.** Given a ground set of elements \(V\), we consider a set function \(f : 2^V \rightarrow \mathbb{R}_{\geq 0}\), which maps any subset of elements in \(V\) to a utility value. We first introduce the concept of marginal gain on \(f\).

**Definition 1 (Marginal Gain).** For a set of elements \(S \subseteq V\) and an element \(v \in V \setminus S\), the marginal gain of \(f\) is defined by \(\Delta_f(v|S) := f(S \cup \{v\}) - f(S)\).

Then, we formally define the monotonicity and submodularity\(^1\) of \(f\) based on the marginal gains of \(f\).

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\(^1\)We also consider \(f\) is nonnegative as \(f\) is aligned to \(f(\emptyset) = 0\) and \(f\) is monotone.
Definition 2. SMDK in the append-only stream model returns a set with an additional constraint: is observed from the stream and SMDK requires to maintain an optimal solution at each time and any element must be processed once it arrives. At each time 

\[ \exists v \in V \setminus S \text{ such that } S \cup \{v\} \in \xi \]

3.3 SMDK in Data Stream Models

Algorithm 1 CEGreedy

Input: A ground set of elements \( V \)
Output: A result set \( S^G \)

1: \( v_{max} \leftarrow \arg\max_{v \in V} f(\{v\}) \)
2: \( S \leftarrow \emptyset \)
3: while \( \exists v \in V \setminus S \text{ such that } S \cup \{v\} \in \xi \) do
4: \( v^* \leftarrow \arg\max_{v \in V \setminus S \cup \{v\} \in \xi} \Delta_f(v|S) \)
5: \( S \leftarrow S \cup \{v^*\} \)
6: if \( f(S) \geq f(\{v_{max}\}) \) then
7: \( \text{return } S^G \leftarrow S \)
8: else
9: \( \text{return } S^G \leftarrow \{v_{max}\} \)

We say \( f \) is monotone iff \( \Delta_f(v|S) \geq 0 \) for any \( S \subseteq V \) and \( v \in V \setminus S \). Furthermore, \( f \) is submodular iff \( \Delta_f(v|S) \geq \Delta_f(v|T) \) for any \( S \subseteq T \subseteq V \) and \( v \in V \setminus T \). Intuitively, monotonicity means adding more elements to a set does not decrease its utility value, and submodularity captures the “diminishing returns” property that the marginal gain of adding any new element decreases as the set grows larger.

The \( d \)-knapsack Constraints. A knapsack is defined by a cost function \( c : V \rightarrow \mathbb{R}_+ \) that assigns a cost to each element in \( V \). Let \( c(v) \) denote the cost of \( v \in V \). The cost \( c(S) \) for a set of elements \( S \subseteq V \) is the sum of the costs of all its members, i.e., \( c(S) = \sum_{v \in S} c(v) \). Given a budget \( b \), we say \( S \) satisfies the knapsack constraint if \( c(S) \leq b \). W.l.o.g., we set the budget to 1 and normalize the costs of all elements to \((0,1)\). A \( d \)-knapsack constraint \( \xi \) is defined by \( d \) cost functions \( c_1(\cdot), \ldots, c_d(\cdot) \). Formally, we define \( \xi = \{S \subseteq V : c_j(S) \leq 1, \forall j \in [1,d]\} \). We say \( S \) satisfies the \( d \)-knapsack constraint iff \( S \in \xi \).

3.2 SMDK in the Batch Setting

We define the optimization problem of maximizing a monotone submodular function \( f \) subject to a \( d \)-knapsack constraint \( \xi \) (SMDK) as follows:

\[
\max_{S \subseteq V} f(S) \quad \text{s.t. } S \in \xi
\]

we use \( S^* \) to denote the optimal solution of SMDK, i.e., \( S^* = \arg\max_{S \in \xi} f(S) \). According to the definition of the \( d \)-knapsack constraint, a cardinality constraint with a budget \( k \) is a special case of a \( 1 \)-knapsack constraint with \( c(v) = 1/k, \forall v \in V \). As SM with a cardinality constraint has been proved to be NP-hard \[19\], SMDK is also NP-hard.

The cost-effective greedy algorithm CEGreedy proposed in \[15\] for SM with \( 1 \)-knapsack constraints can be adopted for SMDK\(^2\). As shown in Algorithm 1 CEGreedy starts from an empty set \( S = \emptyset \) and iteratively adds an element \( v^* \) into \( S \) until \( S \cup \{v\} \notin \xi \) for any remaining element \( v \). The criterion for picking \( v^* \) is based on its cost-effectiveness, i.e., \( v^* \) maximizes \( \frac{\Delta_f(v|S)}{\max_{j \in [1,d]} c_j(v)} \) at the \( i \)-th iteration among all elements \( v \in V \setminus S \). Finally, CEGreedy compares \( S \) with the singleton element \( v_{max} \) with the largest utility value and returns the better one as the final solution.

3.3 SMDK in Data Stream Models

SMDK in Append-only Streams. The append-only stream model considers all elements in \( V \) arrive one at a time and any element must be processed once it arrives. At each time \( t \in [1,n] \), an element \( v_t \in V \) is observed from the stream and SMDK requires to maintain an optimal solution \( S^*_t \) regarding Equation 1 with an additional constraint: \( S^*_t \subseteq \{v_1, \ldots, v_t\} \). Formally,

\[
\text{Definition 2. SMDK in the append-only stream model returns a set } S^*_t = \arg\max_{S \subseteq \xi \land S \subseteq \{v_1, \ldots, v_t\}} f(S).
\]

\(^2\)We do not use the \((1 - 1/e - \epsilon)\) approximation algorithm in \[11\] as the baseline for its \( O(n^{de^{-4}}) \) complexity.
For ease of presentation, we use \( c_{tj} \) to denote the cost of \( v_t \) in the \( j \)-th knapsack.

**SMDK over Sliding Windows.** We further consider SMDK in the *sliding window* model\(^3\) to capture the recency constraint. The sliding window model considers the *active window* \( A_t \) always contains the \( W \) most recent elements (or active elements) at time \( t \), i.e., \( A_t = \{ v_s, \ldots, v_t \} \) where \( s = \max(1, t - W + 1) \). At each time \( t \), SMDK requires to maintain a solution \( S^*[A_t] \) regarding Equation 1 with an additional constraint: \( S^*[A_t] \subseteq A_t \). Formally,

**Definition 3.** *SMDK in the sliding window model returns a set* \( S^*[A_t] = \arg\max_{S \subseteq \xi \wedge S \subseteq A_t} f(S) \).

A naïve approach to SMDK in the aforementioned streaming scenarios is to store all observed/active elements in a data structure and rerun CEGreedy from scratch for each observed element. This naïve approach is obviously undesirable to satisfy the requirements of stream processing since it iteratively scans all elements and invokes a large number of function calls. In the subsequent sections, we will present our algorithms for SMDK both in append-only streams and over sliding windows. They are more efficient than the naïve approach and are adequate to process data streams with high arrival rates.

Before introducing the proposed techniques, we summarize the frequently used notations in Table 1.

### Table 1: Frequently Used Notations

| Notation | Description |
|----------|-------------|
| \( V, v_t \) | \( V = \{ v_1, \ldots, v_n \} \) is a ground set of \( n \) elements, \( v_t \in V \) is the element arriving at time \( t \) in the stream. |
| \( d, \xi \) | \( d \) is the number of knapsacks, \( \xi \) is a family of sets defined by a \( d \)-knapsack constraint. |
| \( c_j(v), c_{tj} \) | \( c_j(v) \) is the cost of \( v \in V \) in the \( j \)-th knapsack, \( c_{tj} \) is the cost of \( v_t \) in the \( j \)-th knapsack. |
| \( \gamma, \delta \) | \( \gamma = \min_{v \in V} c_j(v) \) and \( \delta = \max_{v \in V} c_j(v) \) are the lower and upper bounds of the costs of elements. |
| \( f(\cdot), \Delta f(\cdot) \) | \( f \) is a monotone submodular set function defined on \( 2^X \). \( \Delta f(\cdot) \) is the marginal gain function of \( f \). |
| \( W_d \) | the size of the sliding window. |
| \( A_t \) | \( A_t = \{ v_s, \ldots, v_t \} \) is the active window at time \( t \) where \( s = \max(1, t - W + 1) \). |
| \( S^*, S^*, S^*[A_t] \) | the optimal solution for SMDK in the ground set \( V \), a sub-stream \( \{ v_1, \ldots, v_t \} \) and the active window \( A_t \) respectively. We use \( \text{OPT} = f(S^*) \), \( \text{OPT}_1 = f(S^*_1) \), \( \text{OPT}[A_t] = f(S^*[A_t]) \) for the utility values. |
| \( S_n, S(A_t) \) | the approximate solution for SMDK in a sub-stream \( \{ v_1, \ldots, v_t \} \) and \( A_t \) respectively. |
| \( X_t, x_t \) | \( X_t \) is the set of checkpoints in SubKnapChk at time \( t \), \( x_t \) is the \( t \)-th checkpoint in \( X_t \). |
| \( S^*[x, y], S_{x,y} \) | \( S^*[x, y] \) and \( S_{x,y} \) are the optimal solution and the approximate solution for SMDK in a sub-stream \( \{ v_1, \ldots, v_t \} \). We use \( f[x, y] = f(S[x, y]) \) and \( \text{OPT}_x = f(S^*[x, y]) \) for the utility values. |

### 4 The Append-only Streaming Algorithm

In this section, we propose an algorithm KnapStream for SMDK in append-only streams. We first give an algorithmic description of KnapStream in Section 4.1. Then, we theoretically analyze the approximation ratio and complexity of KnapStream in Section 4.2.

#### 4.1 The Algorithmic Description

The **KnapStream** algorithm uses a threshold-based framework proposed in [2] and [12] for streaming SM. The basic idea is to estimate the optimal utility value \( \text{OPT} \) from the observed elements and to maintain the candidate solutions based on the estimations. As \( \text{OPT} \) cannot be determined unless \( P = NP \), KnapStream only tracks the lower and upper bounds of \( \text{OPT} \) and maintains a set of candidate solutions with different estimated values for \( \text{OPT} \) in the range. Each candidate solution derives a unique threshold for marginal gains according to its estimated value of \( \text{OPT} \). Whenever a new element arrives, a candidate solution includes the element if the marginal gain of adding the element exceeds the threshold. After processing all elements, the candidate solution with the largest utility value is returned as the final result. Although having similar basic ideas, KnapStream is different from the algorithms in [2] and [12] in two aspects: (1) the criterion for adding an element considers not only the marginal gain of adding the element but also its costs, i.e., it checks the cost-effectiveness of adding the element in each knapsack and includes the element only when its cost-effectivenesses reach the threshold in \( d \) knapsacks; (2) the singleton element with the largest utility value is also considered as a candidate solution, which guarantees the approximation factor of the algorithm against undesirable cost distributions. Next, we will present the procedure of KnapStream in detail.

\(^3\) We only discuss the sequence-based sliding window in this paper but the proposed algorithms naturally support the time-based sliding window.
Algorithm 2 KnapStream

Input: A ground set of elements $V$, a parameter $\varepsilon$

Output: A result set $S_t$ at time $t$

1: $O = \{(1 + \varepsilon)^i | i \in \mathbb{Z}\}$
2: For each $\rho \in O$, $S_\rho \leftarrow \emptyset$
3: Initialize $m \leftarrow 0$, $M \leftarrow 0$ and $v_{\text{max}} \leftarrow \text{nil}$
4: for $t \leftarrow 1, \ldots, n$ do
5:   if $f(\{v_t\}) > f(\{v_{\text{max}}\})$ then
6:     $v_{\text{max}} \leftarrow v_t$
7:   if $M < f(\{v_t\})$ then
8:     $M \leftarrow f(\{v_t\})$, $m \leftarrow f(\{v_t\})$
9: $O_t = \{(1 + \varepsilon)^i | i \in \mathbb{Z}, m \leq (1 + \varepsilon)^i \leq M(1 + d)\}$
10: Delete all $S_\rho$ such that $\rho \notin O_t$
11: for all $\rho \in O_t$ do
12:   if $\Delta f(\{v_t\}S_\rho) \geq \frac{\rho}{1 + d} \forall j \wedge S_\rho \cup \{v_t\} \in \xi$ then
13:     $S_\rho \leftarrow S_\rho \cup \{v_t\}$
14: For returning the result set $S_t$ at time $t$:
15: $\bar{S} \leftarrow \text{argmax}_{\rho \in O_t} f(S_\rho)$
16: if $f(S) \geq f(\{v_{\text{max}}\})$ then
17:     return $S_t \leftarrow S$
18: else
19:     return $S_t \leftarrow \{v_{\text{max}}\}$

The pseudo-code of KnapStream is given in Algorithm 2. It maintains three auxiliary variables: $v_{\text{max}}$ maintains the maximum singleton element among observed elements at time $t$; $m$ and $M$ track the lower and upper bounds of the optimal value $\text{OPT}_t$ at time $t$ respectively (Lines 3–8). When $m$ and $M$ are updated, KnapStream will set up new candidate solutions with the updated estimations of optimal values in $O_t$ and delete candidates whose estimated optimal values are outside of $O_t$ (Lines 9–10). Then, each maintained candidate will process $v_t$ at time $t$. For each $\rho \in O_t$, if the marginal gain $\Delta f(\{v_t\}S_\rho)$ is at least $\frac{\rho}{1 + d}$ for all $1 \leq j \leq d$ and the $d$-knapsack constraint is still satisfied after adding $v_t$ into $S_\rho$, $v_t$ will be included into $S_\rho$ (Lines 11–13). Finally, to return the solution $S_t$ at time $t$, it first finds the set $S$ with the largest utility value among all candidates, then compares $S$ with the maximum singleton element $v_{\text{max}}$, and returns the better one as $S_t$ (Lines 15–19).

4.2 Theoretical Analysis

We analyze the approximation ratio and complexity of KnapStream in this subsection. For the theoretical analysis, we consider the cost of any element is bounded by $\gamma$ and $\delta$, i.e., $0 < \gamma \leq c_{ij} \leq \delta < 1, \forall t \in [1, n], \forall j \in [1, d]$. It is noted that KnapStream does not need to know $\gamma$ and $\delta$ in advance. In summary, we show that KnapStream requires only one pass over the ground set, provides a $(\frac{1}{1 + d} - O(1))$ approximate solution for SMDK in an append-only stream, stores at most $O(\frac{\log(\varepsilon^{-1})}{\varepsilon})$ elements, and needs $O(\frac{\log(\delta^{-1})}{\varepsilon})$ function calls per element.

The roadmap of our analysis is as follows. First of all, we show that if we knew the optimal utility $\text{OPT}^\gamma$ in advance, the candidate solution whose estimated optimal value is the closest to $\text{OPT}$ would be a $(1 - \varepsilon)(1 - \delta)$ approximate solution (Lemma 1). However, this approximation ratio depends on $\delta$ and may degrade arbitrarily when $\delta$ increases. So we further prove that by considering the maximum singleton element among all candidates, there is a lower bound for the approximation ratio regardless of $\delta$ (Lemma 2). Then, as $\text{OPT}$ is unknown, we analyze how KnapStream keeps track of the lower and upper bounds for $\text{OPT}$ and how many different estimations are needed to guarantee that at least one of them approximates $\text{OPT}$ within a bounded error ratio (Theorem 1). Finally, as KnapStream maintains one candidate solution for each estimation, we can get its time and space complexities accordingly (Theorem 2).
Lemma 1. Assuming there exists $q \in O_t$ such that $(1 - \varepsilon)\text{OPT}_t \leq q \leq \text{OPT}_t$, $S_q$ satisfies that $f(S_q) \geq \frac{(1 - \varepsilon)(1 - \delta)}{1 + \delta} \text{OPT}_t$.

Proof. Let $s_i$ be the $i$-th element added to $S_q$, $S_q^i$ be $\{s_1, \ldots, s_i\}$ for $i \in [0, |S_q|]$ ($S_q^0 = \emptyset$ accordingly), $b_j = c_j(S_q)$ for $j \in [1, d]$ be the cost of $S_q$ in the $j$-th knapsack, and $b = \max_{j \in [1, d]} b_j$ be the maximum cost of $S_q$ among $d$ knapsacks. According to Line 12 in Algorithm 2, we have $\frac{\Delta f(s_i | S_q^{i-1})}{c_j(s_i)} \geq \frac{q}{1 + \delta}$ for $j \in [1, d]$. Then, it holds that

$$f(S_q) = \sum_{i=1}^{|S_q|} \Delta f(s_i | S_q^{i-1}) \geq \frac{\theta}{1 + \delta} c_j(S_q) = \frac{\theta}{1 + \delta} b_j$$

Thus, $f(S_q) \geq \frac{\theta}{1 + \delta} b$.

Next, we discuss two cases separately as follows:

Case 1: When $b \geq (1 - \delta)$, we have $f(S_q) \geq \frac{\theta}{1 + \delta} b \geq \frac{\theta (1 - \delta)}{1 + \delta} \geq \frac{(1 - \varepsilon)(1 - \delta)}{1 + \delta} \text{OPT}_t$.

Case 2: When $b < (1 - \delta)$, we have $\forall v \in V \setminus S_q, S_q \cup \{v\} \in \xi$. Let $S_q^i$ be the optimal solution and $a$ be an element in $S_q^i \setminus S_q$. Since $a$ is not added to $S_q$, there must exist $\mu(a) \in [1, d]$ such that $\frac{\Delta f(a | S_q^i)}{c_{\mu(a)}(a)} < \frac{q}{1 + \delta}$, where $S_q^i \subseteq S_q$ is the partial solution when $a$ is processed. We consider $S_q^j = \{a | a \in S_q^i \setminus S_q \wedge \mu(a) = j\}$ for $j \in [1, d]$. Due to the submodularity of $f(\cdot)$, we acquire:

$$f(S_q \cup S_q^j) - f(S_q) \leq \sum_{a \in S_q^j} \Delta f(a | S_q) < \frac{\theta}{1 + \delta} c_j(S_q^j) \leq \frac{\theta}{1 + \delta}$$

Then, due to $S_q^i \setminus S_q = \bigcup_{j=1}^d S_q^j$, we have:

$$f(S_q^i \cup S_q) - f(S_q) \leq \sum_{j=1}^d f(S_q \cup S_q^j) - f(S_q) < \frac{d\theta}{1 + \delta}$$

Finally, we get $f(S_q) > \text{OPT}_t - \frac{d\theta}{1 + \delta} \text{OPT}_t \geq \frac{1}{1 + \delta} \text{OPT}_t$.

Considering both cases, we conclude the proof.

Lemma 1 has proved that KnapStream achieves a good approximation ratio when $\delta$ is small. Next, we further analyze the case where $\delta > 0.5$ and prove that the approximation ratio has a lower bound regardless of $\delta$.

Lemma 2. When $\delta > 0.5$, it satisfies that at least one of $f(S_q)$ and $f(\{v_{\text{max}}\})$ is greater than $\frac{0.5(1 - \varepsilon)}{1 + \delta}$.

Proof. Lemma 2 naturally follows when $b \geq 0.5$ (Case 1 of Lemma 1) or for all $a \in S_q^i \setminus S_q$, $a$ is excluded from $S_q$ because its marginal gain does not reach the threshold in some knapsack (Case 2 of Lemma 1).

Thus, we only need to consider the following case: there exists some elements whose marginal gains reach the threshold in all knapsacks but are excluded from $S_q$ because including them into $S_q$ violates the $d$-knapsack constraint. Assuming $a$ is such an element for $S_q$, we have $\frac{\Delta f(a | S_q^i)}{c_j(a)} \geq \frac{\theta}{1 + \delta}$ and $c_j(S_q^i) + c_j(a) > 1$ for some $j \in [1, d]$. In this condition, we have $f(S_q^i \cup \{a\}) \geq \frac{\theta}{1 + \delta} (c_j(S_q^i) + c_j(a)) > \frac{\theta}{1 + \delta}$. From the monotonicity and submodularity of $f(\cdot)$, we get:

$$\frac{\theta}{1 + \delta} \leq f(S_q^i \cup \{a\}) \leq f(S_q) + f(\{a\}) \leq f(S_q) + f(\{v_{\text{max}}\})$$

Therefore, at least one of $f(S_q)$ and $f(\{a\})$ is greater than $\frac{0.5(1 - \varepsilon)}{1 + \delta}$. As $f(\{v_{\text{max}}\}) \geq f(\{a\})$, we prove the lemma.

Given Lemmas 1 and 2, we prove that KnapStream achieves an approximation factor of $\frac{(1 - \varepsilon)(1 - \delta)}{1 + \delta}$ (when $\delta \leq 0.5$) or $\frac{0.5(1 - \varepsilon)}{1 + \delta}$ (when $\delta > 0.5$).

Theorem 1. KnapStream satisfies that: at any time $t$,

$$f(S_t) \geq \begin{cases} \frac{(1 - \varepsilon)(1 - \delta)}{1 + \delta} f(S_q^i), & \delta \leq 0.5 \\ \frac{0.5(1 - \varepsilon)}{1 + \delta} f(S_q^i), & \delta > 0.5 \end{cases} \quad (2)$$
Proof. By Lemmas 1 and 2, we can say Theorem 1 naturally holds if there exists at least one $g \in \mathcal{O}_t$ such that $(1 - \varepsilon)OPT_t \leq g \leq OPT_t$. To prove this, we first show that $m$ and $M$ are the lower and upper bounds for $OPT_t$. It is easy to see $m \leq OPT_t$ as $m \leq f\{v_{\text{max}}\}$ and $\{v\} \in \xi$ for any $v \in V$. $M$ maintains the maximum cost-effectiveness of all observed elements. We have $M \geq \frac{f\{v_j\}}{c_i}$ for $i \in [1, \ell]$ and $j \in [1, d]$. Let $S^*_t = \{a_1, \ldots, a|S^*_t|\}$ be the optimal solution at time $t$. As $f(\cdot)$ is monotone submodular, $OPT_t \leq \sum_{i=1}^{\epsilon(S^*_t)} f\{a_i\} \leq c_i(S^*_t)M$ for $j \in [1, d]$. As $c_j(S^*_t) \leq 1$, we are safe to say $M \geq OPT_t$. KnapStream estimates $OPT_t$ by a sequence of values $\{(1 + \varepsilon)^l | l \in \mathbb{Z}, m \leq (1 + \varepsilon)^l \leq M(1 + d)\}$. Then, we can find an estimation $g$ such that $g \leq OPT_t \leq (1 + \varepsilon)g$. Equivalently, $(1 - \varepsilon)OPT_t \leq g \leq OPT_t$. Finally, we conclude the proof by combining this result with Lemmas 1 and 2. \hfill \Box

Next, we analyze the complexity of KnapStream. KnapStream adopts the same candidate solution maintenance strategy as used in [2]. As only one pass over the stream is permitted, to avoid missing elements with marginal gains of more than $\frac{M}{1 + \gamma}$, KnapStream maintains candidates in an increased range $[m, (1 + d)M]$ instead of $[m, M]$. The following theorem shows the complexity of KnapStream.

**Theorem 2.** KnapStream requires one pass over the ground set $V$, stores at most $O\left(\frac{\log(d\gamma^{-1})}{\varepsilon}\right)$ elements, and has $O\left(\frac{\log(d\gamma^{-1})}{\varepsilon}\right)$ update complexity per element.

**Proof.** We first analyze the number of candidate solutions maintained at the same time. As $\frac{M}{A} \leq \gamma$, the number of candidates is bounded by $\lceil \log_{1 + \varepsilon} \gamma^{-1} (1 + d) \rceil$. Thus, we have KnapStream maintains $O\left(\frac{\log(d\gamma^{-1})}{\varepsilon}\right)$ candidates. For each candidate, one function call is required to evaluate whether to add a new element. Thus, the update complexity per element is also $O\left(\frac{\log(d\gamma^{-1})}{\varepsilon}\right)$. Finally, for each candidate, at most $\gamma^{-1}$ elements can be maintained. Otherwise, the $d$-knapsack constraint must not be satisfied. Therefore, the total number of elements stored is $O\left(\frac{\log(d\gamma^{-1})}{\varepsilon}\right)$. \hfill \Box

A Comparison with [32]. The algorithm in [32] also uses the threshold-based framework and works for SMDK in append-only streams. It maintains $O\left(\frac{\log d\gamma^{-1}}{\varepsilon}\right)$ candidates and is $\frac{1}{1 + \varepsilon}$ approximate for SMDK. The approximation ratio of KnapStream is better than this algorithm when the maximum cost of any element $\delta$ is less than $\frac{d}{1 + \varepsilon}$. It is noted that practical applications often use small values of $\delta$ (e.g., 0.1 in [6] and 0.25 in [32]), where KnapStream has obviously better approximation ratios. In addition, KnapStream is more robust than the algorithm in [32] when $d$ increases as $O\left(\frac{\log d}{\varepsilon}\right)$ more candidates are maintained.

## 5 The Sliding Window Algorithm

In this section, we propose an efficient algorithm KnapWindow for SMDK over sliding windows. We first give an algorithmic description of KnapWindow in Section 5.1. Then, we analyze KnapWindow theoretically in Section 5.2. Finally, we propose a buffer-based optimization technique to further improve the solution quality of KnapWindow and discuss how to handle the scenario where the sliding window shifts for more than one element in Section 5.3.

### 5.1 The Algorithmic Description

The high level idea of KnapWindow is to maintain several instances of KnapStream (also called checkpoints) with different starting points. Each checkpoint maintains a solution for SMDK over all elements from the element corresponding to its starting point to the up-to-date element. At time $t$, it returns the solution of the first non-expired checkpoint as the result for $A_t$. The idea of maintaining a sequence of checkpoints over sliding windows is inspired by smooth histograms [3]. However, as has been shown in Section 2 smooth histograms cannot be applied to SMDK because no algorithm for SMDK in append-only streams meets the requirement of approximation ratios. Therefore, we devise a novel data structure Submodular Knapsack Checkpoints (SubKnapChk) for our problem.

**Submodular Knapsack Checkpoints.** SubKnapChk at time $t$ comprises a sequence of $s$ checkpoints $X_t = \{x_1, \ldots, x_s\}$ where $x_1 < \ldots < x_s = t$. Each checkpoint $x_i$ maintains an instance of KnapStream denoted by $\mathcal{H}(x_i)$. The instance $\mathcal{H}(x_i)$ processes all elements from $v_{x_i}$ to $v_t$ at time $t$ and will be terminated
Algorithm 3 \textsc{KnapWindow}

\textbf{Input:} A ground set $V$, the window size $W$, a parameter $\beta$

\textbf{Output:} A result set $S[A_t]$ for $A_t$ at time $t$

1: Initialize $s \leftarrow 0$, the set of checkpoints $X_0 \leftarrow \emptyset$

2: \textbf{for} $t \leftarrow 1, \ldots, n$ \textbf{do}

3: \hspace{1em} $s \leftarrow s + 1$, $x_s \leftarrow t$, and $X_t \leftarrow X_{t-1} \cup \{x_s\}$

4: \hspace{1em} Initiate a \texttt{KnapStream} instance $H(x_s)$

5: \hspace{1em} \textbf{while} $t > W \land x_2 < t - W + 1$ \textbf{do}

6: \hspace{2em} $X_t \leftarrow X_t \setminus \{x_1\}$ and terminate $H(x_1)$

7: \hspace{2em} Shift the remaining checkpoints accordingly

8: \hspace{1em} $s \leftarrow s - 1$

9: \hspace{1em} \textbf{for} $i \leftarrow 1, \ldots, s$ \textbf{do}

10: \hspace{3em} $H(x_i)$ processes $v_i$ according to \texttt{KnapStream}

11: \hspace{3em} \textbf{while} $\exists i \in [1, s - 2] : f[x_{i+2}, t] \geq (1 - \beta)f[x_i, t]$ \textbf{do}

12: \hspace{4em} $X_t \leftarrow X_t \setminus \{x_{i+1}\}$ and terminate $H(x_{i+1})$

13: \hspace{4em} Shift the remaining checkpoints accordingly

14: \hspace{3em} $s \leftarrow s - 1$

15: \hspace{1em} For returning the result set $S[A_t]$ at time $t$:

16: \hspace{2em} \textbf{if} $x_1 \geq \max(1, t - W + 1)$ \textbf{then}

17: \hspace{3em} \textbf{return} the result of $H(x_1)$ at time $t$

18: \hspace{2em} \textbf{else}

19: \hspace{3em} \textbf{return} the result of $H(x_2)$ at time $t$

when $x_i$ is deleted from \texttt{SubKnapChk}. When $t > W$, the first checkpoint $x_1$ is expired (i.e., $x_1 < t - W + 1$) but is not deleted from \texttt{SubKnapChk} immediately. It is still maintained to track the upper bound of $\texttt{OPT}[A_t]$. However, we do not use the result of $H(x_1)$ as the solution because it may contain expired elements. It is noted that as there is at most one expired checkpoint in \texttt{SubKnapChk} at the same time and $x_2$ must not expire, the result of $H(x_2)$ is returned as the solution in this case.

The most critical problem is how to find an adequate sequence of checkpoints to maintain. The maintenance strategy should achieve two objectives: (1) the number of maintained checkpoints should be small for high efficiency; (2) the utility values of returned solutions are theoretically bounded. Towards both objectives, we propose the following strategy to maintain the checkpoints: (1) create a new checkpoint and a new \texttt{KnapStream} instance for each arrival element; (2) delete a checkpoint and terminate its \texttt{KnapStream} instance once it can be approximated by its successors. Let $f[x_i, t]$ denote the utility value of the result returned by $H(x_i)$ at time $t$ and $\texttt{OPT}_t^x$ represent the optimal utility value for SMDK over elements from $x_s$ to $v_t$ at time $t$ respectively. Given three checkpoints $x_{i-1}, x_i, x_{i+1}$ and a parameter $\beta > 0$, if $f[x_{i+1}, t] \geq (1 - \beta)f[x_{i-1}, t]$, we consider that the second checkpoint $x_i$ can be approximated by the third one $x_{i+1}$ and can be deleted from \texttt{SubKnapChk}. Obviously, $f[x_{i+1}, t]$ is $(1 - \beta)$ approximate to $f[x_{i-1}, t]$ at time $t$. According to Theorem 1, $f[x_{i+1}, t]$ is at least $(1 - \beta)(1 - \delta)(1 - \epsilon)$ approximate to $\texttt{OPT}_t^{x_{i-1}}$. It is essential to ensure that this ratio will not degrade too much over time. Utilizing the submodularity of $f(\cdot)$ and the properties of \texttt{KnapStream}, we have proved that $f[x_{i+1}, t']$ is at least $(1 - \beta)(1 - \delta)(1 - \epsilon)\frac{1 + \delta}{2(1 + \delta)}$ approximate to $\texttt{OPT}_t^{x_{i-1}}$ for any $t' > t$. We will formally analyze this property in Section 5.2. For the remaining part of this subsection, we focus on presenting the procedures of maintaining \texttt{SubKnapChk} and providing solutions for SMDK over sliding windows.

The pseudo-code of \texttt{KnapWindow} is presented in Algorithm 3. The maintenance of \texttt{SubKnapChk} is as illustrated in Lines 3-14. At time $t$, a new checkpoint $x_s = t$ and a \texttt{KnapStream} instance $H(x_s)$ are created for $v_t$. Then, if there is more than one expired checkpoint in \texttt{SubKnapChk}, all except the last one will be deleted (Lines 8-10). This guarantees that there is only one expired checkpoint at any time. Subsequently, all remaining checkpoints process $v_t$ and update their results accordingly. This procedure follows Lines 5-13 of Algorithm 2. Next, \texttt{KnapWindow} performs the maintenance of \texttt{SubKnapChk}. It identifies all checkpoints that can be approximated by its successors and deletes them from \texttt{SubKnapChk} (Lines 11-14). After the maintenance of \texttt{SubKnapChk}, for any $x_i \in X_t$, there is at most one checkpoint $x \in X_t$ such that $x > x_i$ and...
Lemma 3. Case 1: If \( SMDK \) over \( f \), it holds that
\[
SMDK_{\text{opt}}(w) = \sum_{t=1}^{\text{end}} f[x_t, t].
\]
Finally, the solution \( S[A_t] \) for \( A_t \) is the result of \( H(x_1) \) if \( x_1 \) does not expire yet; otherwise, \( S[A_t] \) is the result of \( H(x_2) \).

5.2 Theoretical Analysis

We will analyze the approximation ratio and complexity of \text{KnapWindow} theoretically in this subsection. Generally, \text{KnapWindow} requires one pass over the stream and provides an \( \left( \frac{1}{2(1+d)} - O(1) \right) \) approximate solution for \( SMDK \) over sliding windows. It maintains \( O(\log \frac{\theta}{\delta}) \) checkpoints and \text{KnapStream} instances, requires \( O(\log \theta \log \frac{\theta}{\delta} - 1) \) function calls per element, and stores \( O(\log \theta \log \frac{\theta}{\delta} - 1) \) elements, where \( \theta \) is the ratio between the maximum and minimum utility values of \( SMDK \) and \( \beta \in (0, 1) \) is a predefined parameter.

First of all, we analyze the properties of \text{SubKnapChk} maintained by Algorithm 3.

Lemma 3. At time \( t \), \text{SubKnapChk} contains a sequence of \( n \) checkpoints \( X_i = \{x_1, \ldots, x_n\} \). For any \( 1 \leq i < n \) and \( \beta \in (0, 1) \), \( x_i \) satisfies one of the following properties:

1. if \( f[x_i+1, t] \geq (1-\beta) f[x_i, t] \), then \( f[x_i+2, t] < (1-\beta) f[x_i, t] \) or \( x_{i+1} = x_i \).
2. \( x_{i+1} \neq x_i + 1 \) and \( f[x_i+1, t] \geq (1-\beta) f[x_i, t] \), then there exists some \( t' < t \) such that \( f[x_i+1, t'] \geq (1-\beta) f[x_i, t] \).
3. \( x_{i+1} = x_i + 1 \) and \( f[x_i+1, t] < (1-\beta) f[x_i, t] \).

Proof. We prove the lemma by induction on \( t \). As the base case, we first consider the condition when \( t = 2 \).

We have \( x_2 = \{x_1 = 1, x_2 = 2\} \). Then property (1) holds if \( f[x_2, 2] \geq (1-\beta) f[x_1, 2] \) and otherwise property (3) holds.

Next, we assume Lemma 3 holds at time \( t \) and show that it still holds after performing Lines 3–14 of Algorithm 3 at time \( t+1 \). Let \( x_i \) be a checkpoint created before \( t + 1 \) and not deleted at time \( t + 1 \). Then, \( x_{i+1} \) is the next checkpoint of \( x_i \) at time \( t \). We discuss all possible cases during the update procedure at time \( t + 1 \).

Case 1: \( x_{i+1} \neq x_i + 1 \) and \( x_{i+1} \) is deleted from \text{SubKnapChk} at time \( t + 1 \). In this case, we have \( f[x_{i+1} + 2, t+1] \geq (1-\beta) f[x_i, t+1] \) (Line 14 of Algorithm 3). As \( x_{i+2} \) becomes the successor of \( x_i \) at time \( t + 1 \), property (1) holds at \( t + 1 \).

Case 2: \( x_{i+1} \neq x_i + 1 \) and \( x_{i+1} \) is not deleted from histograms at time \( t + 1 \). In this case, we consider \( x_{i+1} \) becomes the successor of \( x_i \) at some time \( t' < t \). Then, it must hold that \( f[x_{i+1}, t'] \geq (1-\beta) f[x_i, t'] \). Since \( x_{i+1} \) is not deleted at time \( t + 1 \), we have either property (1) (when \( f[x_{i+1}, t'] \geq (1-\beta) f[x_i, t'] \)) or property (2) (when \( f[x_{i+1}, t+1] < (1-\beta) f[x_i, t+1] \)) at time \( t + 1 \).

Case 3: \( x_{i+1} = x_i + 1 \). No matter whether \( x_{i+1} \) is deleted at time \( t + 1 \), property (1) holds as long as \( f[x_{i+1}, t+1] \geq (1-\beta) f[x_i, t+1] \); otherwise, property (3) holds.

We show that the properties of \text{SubKnapChk} still hold at time \( t + 1 \) in all possible cases and conclude the proof.

Given the properties of \text{SubKnapChk} in Lemma 3, we can analyze the approximation ratio of \( S[A_t] \) returned by \text{KnapWindow} for \( SMDK \) w.r.t. \( A_t \).

Theorem 3. It holds that \( f(S[A_t]) \geq \frac{(1-\beta)(1-\delta)(1-\epsilon)}{2(1+d)} \cdot \text{OPT}[A_t] \) at any time \( t \).

Proof. We consider the first two checkpoints \( x_1 \) and \( x_2 \) of \text{SubKnapChk} maintained by \text{KnapWindow} at time \( t \). If \( t \leq W \), \( x_1 = 1 \) does not expire and \( H(x_1) \) is maintained over \( A_t = \{v_1, \ldots, v_1\} \). Thus, \( f(S[A_t]) = f[x_1, t] \geq \frac{(1-\delta)(1-\epsilon)}{1+d} \cdot \text{OPT}[A_t] \) for \( t \leq W \) by Theorem 1. Next, we consider the case that \( t > W \) and \( x_2 = x_1 + 1 \). In this case, \( x_1 \) expires and \( x_2 \) corresponds to the starting point of \( A_t \). Similarly, \( f(S[A_t]) = f[x_2, t] \geq \frac{(1-\delta)(1-\epsilon)}{1+d} \cdot \text{OPT}[A_t] \).

Subsequently, we consider other cases for \( t > W \). We use \( \text{OPT}^t_y \) to denote the optimal utility value of \( SMDK \) over \( \{v_2, \ldots, v_y\} \).

Case 1: If \( f[x_2, t] \geq (1-\beta) f[x_1, t] \), \( f(S[A_t]) = f[x_2, t] \geq (1-\beta) f[x_1, t] \). By Theorem 1, \( f[x_1, t] \geq \frac{(1-\delta)(1-\epsilon)}{1+d} \cdot \text{OPT}^t \). As \( x_1 < t - W + 1 \), we have \( A_t \subset \{v_3, \ldots, v_y\} \) and \( \text{OPT}[A_t] \leq \text{OPT}^t \). Finally, we have \( f(S[A_t]) \geq \frac{(1-\beta)(1-\delta)(1-\epsilon)}{2(1+d)} \cdot \text{OPT}[A_t] \).
Case 2: If \( f[x_2, t] < (1 - \beta)f[x_1, t] \), we have \( f[x_2, t] \geq (1 - \beta)f[x_1, t] \) for some \( t' < t \). Let \( S^*[x_1, t] \) denote the optimal solution for \( \{v_2, \ldots, v_t\} \). We can split \( S^*[x_1, t] \) into two subsets \( S_1 \) and \( S_2 \), where 
\[
S_1 = \{v_i | v_i \in S^*[x_1, t] \land i \in [x_1, t']\} \quad \text{and} \quad S_2 = \{v_i | v_i \in S^*[x_1, t] \land i \in [x_2, t]\}.
\]
Let \( \text{OPT}_1 = f(S_1) \) and \( \text{OPT}_2 = f(S_2) \). For \( S^*[x_1, t] = S_1 \cup S_2 \) and the submodularity of \( f(\cdot) \), \( \text{OPT}_{t_1} \leq \text{OPT}_1 + \text{OPT}_2 \). Then, as \( S_1 \in \xi \) and \( S_2 \in \xi \), it holds that \( \text{OPT}_1 \leq \text{OPT}_{t_1}^{i_1} \) and \( \text{OPT}_2 \leq \text{OPT}_{t_2}^{i_2} \). In addition, for any \( t_1 < t_2 \), the solution returned by \( \text{KnapStream} \) satisfies that \( f[x, t_1] \leq f[x, t_2] \). As \( t > t' \), we have:
\[
f[x_2, t] \geq \frac{(1 - \beta)(1 - \delta)(1 - \varepsilon)}{1 + d}\text{OPT}_{t_1} \geq \frac{(1 - \beta)(1 - \delta)(1 - \varepsilon)}{1 + d}\text{OPT}_1 \tag{3}
\]
We also have:
\[
f[x_2, t] \geq \frac{(1 - \delta)(1 - \varepsilon)}{1 + d}\text{OPT}_{t_2} \geq \frac{(1 - \delta)(1 - \varepsilon)}{1 + d}\text{OPT}_2 \tag{4}
\]
Combining Equations (3) and (4), we prove that:
\[
f[x_2, t] \geq \frac{(1 - \beta)(1 - \delta)(1 - \varepsilon)}{1 + d} \cdot \frac{\text{OPT}_1 + \text{OPT}_2}{2}
\]
\[
\geq \frac{(1 - \beta)(1 - \delta)(1 - \varepsilon)}{2(1 + d)}\text{OPT}_{t_1}
\]
\[
\geq \frac{(1 - \beta)(1 - \delta)(1 - \varepsilon)}{2(1 + d)}\text{OPT}[A_t]
\]
Combining both cases, we conclude the proof.

Finally, the following theorems analyze the complexity of \( \text{KnapWindow} \).

**Theorem 5.** \( \text{KnapWindow} \) performs \( O\left(\frac{\log \frac{\theta}{\beta}}{\beta \gamma} \log(\frac{dv}{\theta})^{-1} \right) \)
function calls per element and stores at most \( O\left(\frac{\log \frac{\theta}{\beta} \log(\frac{dv}{\theta})^{-1}}{\beta \gamma} \right) \) elements.

It is easy to prove Theorem 5 by combining the results of Theorem 2 with Theorem 4.

### 5.3 Optimization and Discussion

#### 5.3.1 Optimization

Although the utilities of the results returned by \( \text{KnapWindow} \) are much better than the theoretical lower bound in practice, they are often inferior to the solutions returned by \( \text{CEGreedy} \) according to our investigation. We identify the major reasons for the discrepancy as follows: First, the solutions of \( \mathcal{H}(x_2) \) are always used as the final results for \( A_t \). However, \( x_2 \) is often much later than the starting point of \( A_t \) (i.e., \( x_2 \gg t - W + 1 \)). This implies that all elements between \( v_{t-W+1} \) and \( v_{x_2-1} \) are missing from the solutions. Second, the candidate solutions maintained by \( \text{KnapStream} \) with high thresholds are hard to be filled, even if more elements could still be added without violating the \( d \)-knapsack constraints. Based on these observations, we propose the buffer and post-processing optimization to improve the solution quality of \( \text{KnapWindow} \).

**The Buffer Stage.** When processing a new element \( v_t \), \( \text{KnapStream} \) will discard \( v_t \) from a candidate solution \( S_e \) directly if the marginal gain of adding \( v_t \) is less than \( \frac{v_{i,j}}{i+j} \) for any \( j \in [1, d] \), regardless of how much the marginal gain is. For further improvements, instead of dropping these elements directly, we maintain a buffer for each candidate solution \( S_e \). Let \( B_o \) be the buffer maintained for \( S_e \). If adding \( v_t \) to \( S_e \) achieves a
by processing elements from with trivial adaptations. It just creates one checkpoint at each time receives from solutions. Then, we run than one element shifted at a time. Thus, the adaptations does not affect the approximation ratios and adaptations do not affect the ⌈⌉ is

In practical scenarios, we often consider a batch update strategy. Specifically, at time ⋈, we consider a batch update strategy. At time t, a sliding window receives L new elements while the earliest L elements become expired. KnapsWindow can handle this case with trivial adaptations. It just creates one checkpoint at each time t and updates all maintained checkpoints by processing elements from t\((\neg t+1)\)l+1 to t\ell collectively. In this way, the total number of checkpoints created is \(\left\lceil \frac{t}{\ell} \right\rceil \) but the number of maintained checkpoints is still bounded by the utility values. In addition, the adaptations do not affect the buffer and post-processing optimization.

It is also easy to see the properties of SubKnappChk are still preserved for sliding windows with more than one element shifted at a time. Thus, the adaptations does not affect the approximation ratios and complexities of KnapsWindow and KnapsWindowOpt as well.

Algorithm 4 KnapsWindowOpt
Input: A ground set V, the window size w, the buffer size σ, a parameter α
Output: A result set S[A_t] for A_t at time t
1: Initialize all variables and the set of checkpoints
2: for t ← 1,...,n do
3: Perform Lines 3–10 of KnapsWindow
4: Maintain the buffer of each candidate solution in each checkpoint w.r.t. v_t according to Strategy (1) to (3)
5: Perform Lines 11–14 of KnapsWindow
6: The post-processing for returning S[A_t] at time t:
7: if \(x_1 \geq \max(1,t−W+1)\) then
8: For each \(S_\varepsilon \in \mathcal{H}(x_1)\), perform Lines 3–10 of CEGreedy to add elements in \(B_\varepsilon\) to \(S_\varepsilon\)
9: return the result of \(\mathcal{H}(x_1)\)
10: else
11: For each \(S_\varepsilon \in \mathcal{H}(x_2)\), (1) add any element \(v_i\) in \(S_\varepsilon\) and \(B_\varepsilon\) of \(\mathcal{H}(x_1)\) to \(B_\varepsilon'\) if \(i \geq t−W+1 \wedge S_\varepsilon \cup \{v_i\} \in \xi\); (2) perform Lines 3–10 of CEGreedy to add elements in \(B_\varepsilon\) and \(B_\varepsilon'\) to \(S_\varepsilon\)
12: return the result of \(\mathcal{H}(x_2)\)

The post-processing stage. Before returning the final solutions, we perform the post-processing using the buffered elements in the candidate solutions of \(\mathcal{H}(x_1)\) and \(\mathcal{H}(x_2)\) (if \(t < W\), we only use \(\mathcal{H}(x_1)\)). Specifically, for each \(S_\varepsilon\) in \(\mathcal{H}(x_2)\), we consider not only the elements in \(B_\varepsilon\) but also the non-expired elements in \(S_\varepsilon\) and \(B_\varepsilon\) of \(\mathcal{H}(x_1)\) (as \(B_\varepsilon'\)) if \(S_\varepsilon\) exists in \(\mathcal{H}(x_1)\). This avoids missing the elements between \(v_{t−W+1}\) and \(v_{t\ell−1}\) from solutions. Then, we run CEGreedy starting from \(S_\varepsilon\) and iteratively add the elements in \(B_\varepsilon\) and \(B_\varepsilon'\) to \(S_\varepsilon\). After post-processing each \(S_\varepsilon\) in \(\mathcal{H}(x_2)\), we also return the solution with the largest utility value. The pseudo-code of KnapsWindowOpt with the buffer and post-processing optimization is shown in Algorithm 4.

Obviously, the utility values of the results do not decrease after post-processing. In addition, the buffer maintenance procedure does not change the set of checkpoints. Therefore, KnapsWindowOpt achieves at least the same approximation ratio as KnapsWindow. However, it brings extra overhead for maintaining the buffer and post-processing. In our implementation, the buffer is maintained in a min-heap. The complexity of adding an element is \(O(\log \eta)\) and dropping elements is \(O(\eta)\). Thus, the extra computational cost brought by maintaining the buffer is \(O(\eta \log \theta \log(d\gamma^{-1}))\) and the total number of buffered elements is \(O(\frac{\eta \log \theta \log(d\gamma^{-1})}{\beta \epsilon})\) as well. The post-processing procedure for any candidate solution handles at most \((2\eta + \gamma^{-1})\) elements and runs at most \(\gamma^{-1}\) iterations. Therefore, the post-processing requires \(O((2\eta + \gamma^{-1}) \log(d\gamma^{-1}))\) function calls.

5.3.2 Discussion
In practical scenarios, we often consider a batch update strategy. Specifically, at time t, a sliding window receives L new elements while the earliest L elements become expired. KnapsWindow can handle this case with trivial adaptations. It just creates one checkpoint at each time t and updates all maintained checkpoints by processing elements from \(v_{t\ell−1}\)L+1 to \(v_t\ell\) collectively. In this way, the total number of checkpoints created is \(\left\lceil \frac{t}{\ell} \right\rceil \) but the number of maintained checkpoints is still bounded by the utility values. In addition, the adaptations do not affect the buffer and post-processing optimization.
6 Experiments

We discuss the experiments in this section. First, we introduce the applications for experimental evaluation as well as the datasets used in Section 6.1. Second, we present the compared approaches and parameter settings in Section 6.2. Finally, we evaluate the effectiveness and efficiency of our proposed algorithms and analyze the results in Section 6.3.

6.1 Applications and Datasets

In this subsection, we show how to formulate two real-world applications, namely representative subset selection and maximum coverage, as SMDK and describe the datasets used for both applications in our experiments.

Representative Subset Selection is a popular way for data summarization. It selects a small subset of elements out of the full dataset subject to some budget constraints as the representatives. A key issue is how to measure the representativeness of the selected subset. In this paper, we adopt the widely used Informative Vector Machine (IVM) as the metric for the representativeness of any set of elements \( S \), which is defined as follows:

\[
f(S) = \frac{1}{2} \log \det(\mathbf{I} + \sigma^{-2} \mathbf{K}_{S,S})
\]

where \( \mathbf{K}_{S,S} \) is an \(|S| \times |S|\) kernel matrix indexed by \( S \) and \( \sigma > 0 \) is a regularization parameter. For each element \( v_i, v_j \in S \), the \((i,j)\)-th entry \( K_{i,j} \) of \( \mathbf{K} \) represents the similarity between \( v_i \) and \( v_j \) measured via a symmetric positive definite kernel function. Here, we adopt the squared exponential kernel embedded in the Euclidean space \( K_{i,j} = \exp(-\frac{\|v_i - v_j\|^2}{2}) \). It has been proved that \( f(\cdot) \) defined by Equation 5 is a monotone submodular function [7].

Representative subset selection is formulated as maximizing the IVM function with a cardinality constraint in previous work [2,6]. In the experiments, we consider the cardinality constraint as a special case of the 1-knapsack constraint. We use the Yahoo! Webscope dataset consisting of 45,811,883 user visits from the Featured Tab of the Today module on the Yahoo! front page. Each user visit is a 5-dimensional feature vector with a timestamp. In data preprocessing, we normalize all feature vectors to unit vectors. We set parameters \( h = 0.75 \) and \( \sigma = 1 \) as used in [2]. The ground set \( V \) consists of all user visits in the dataset. We assign a uniform cost \( c(v) = c \) to each user visit \( v \), which is equivalent to a cardinality constraint of \( k = [1/c] \). The objective is to select a representative subset \( S \) such that \( f(S) \) is maximized under the 1-knapsack constraint \( \xi = \{ S \subseteq V : c(S) \leq 1 \} \). In the experiments, we stream all user visits to each approach one by one in ascending order of their timestamps.

Maximum Coverage with \( d \)-knapsack Constraints. The maximum coverage problem is a classic NP-hard problem. Given a domain of items \( \mathcal{E} = \{e_1, \ldots, e_m\} \) and a ground set of sets \( \mathcal{S} = \{s_1, \ldots, s_n\} \) with each set \( s_i \subseteq \mathcal{E} \), it aims to select a set of sets \( S' \subseteq \mathcal{S} \) subject to a given constraint such that \( S' \) covers as many items in \( \mathcal{E} \) as possible. Many real-world problems such as document summarization [15], Blog watching [26], and resource allocation [24] are modeled as maximum coverage with 1 or \( d \) knapsack constraints [10]. As the coverage function is monotone submodular, algorithms for SM can naturally be used for maximum coverage.

In our experiments, we consider a simple application of maximum coverage. We define a problem of selecting influential tweets as maximum coverage with a 2-knapsack constraint. We use a Twitter dataset collected by the streaming API containing 6,860,349 users and 8,890,118 tweets. The objective is to monitor influential tweets in social streams by tracking a set of tweets covering as many users as possible. We consider a user is covered by a tweet if he retweets the tweet. Then, the set of all users is treated as the domain of items and the coverage of each tweet forms the ground set of sets. We adopt the constraints as used in [14] for our experiments: the number as well as the total length of selected tweets are bounded. To fulfill both constraints, we assign two costs to each tweet \( v \). The first one is a uniform cost \( c_1(v) = c \) and the second one, i.e., \( c_2(v) \), is derived from the length of \( v \). To normalize the cost, we first compute the average number of words \( \hat{w} \) of all tweets and then assign \( c_2(v) \) as follows: if a tweet \( v \) contains \( w \) words and \( c_1(v) = c \), the cost \( c_2(v) \) of this tweet will be \( c_2(v) = \frac{c}{\hat{w}} \). E.g., if \( \hat{w} = 20 \) and \( c = 0.1 \), a tweet \( v \) containing 40 words will be assigned

\[\text{http://webscope.sandbox.yahoo.com/}
\]

\[\text{https://dev.twitter.com/streaming/overview}\]
a cost of $c_2(v) = 0.2$. In the experiments, we stream each tweet along with all users covered by it to each approach one by one in ascending order of their timestamps.

6.2 Experimental Settings

**Approaches.** The following approaches are compared in the experiments:

- **CEGreedy (CEG):** We implement CEGreedy [15] as the batch baseline. As stated in Section 3, we do not use the algorithm in [11] for its efficiency. CEGreedy is $(1 - 1/e)$ approximate for representative subset selection and $\frac{1}{2}(1 - 1/e)$ approximate for influential tweets monitoring in the experiments. To work for sliding windows, it stores all active elements in the current window and recomputes the result from scratch for each window slide.
- **Random (RAND):** We use a set of randomly selected elements as a baseline. For experiments in append-only streams, it picks elements from the ground set one by one uniformly at random until the $d$-knapsack constraint is not satisfied. For experiments over sliding windows, it selects elements from $A_t$ one by one uniformly at random until the $d$-knapsack constraint is not satisfied for each window slide.
- **Streaming (STR):** We implement the algorithm proposed in [22] as the append-only streaming baseline. For experiments over sliding windows, it requires to recompute the result from scratch for each window slide.
- **KnapStream (KS):** We implement KnapStream proposed in Section 4 for SMDK in append-only streams.
- **KnapWindow (KW):** We implement KnapWindow in Section 5.1 for SMDK over sliding windows.
- **KnapWindowOpt (KW\textsuperscript{+}):** We implement KnapWindowOpt with the buffer and post-processing optimization in Section 5.3 for SMDK over sliding windows. We use $\alpha = 0.8$ and $\eta = 20$ across all experiments.

For experiments in append-only streams, we compare our proposed KS with CEG, RAND, and STR. For experiments over sliding windows, we compare KW and KW\textsuperscript{+} with the same baselines. All algorithms are implemented in Java 8 and all experiments are conducted on a server running CentOS 7.2 with a 3.6 GHz Intel i7-3820 processor and 64 GB memory.

**Parameter settings.** The parameters used in our experiments are listed as follows: (1) the parameter $\varepsilon$ for STR, KS, KW, and KW\textsuperscript{+} ranges from 0.05 to 0.25, 0.1 by default. (2) the cost parameter $c$ ranges from 0.01 to 0.05, 0.02 by default. (3) the size of the sliding window $W$ ranges from 100$K$ to 500$K$ for representative subset selection and 1$M$ to 5$M$ for maximum coverage, 200$K$ and 2$M$ by default respectively. We adopt the sliding window model with multiple elements shifted at a time and set the number of elements for each window slide to 0.1% of $W$. And (4) the parameter $\beta$ for KW and KW\textsuperscript{+} ranges from 0.05 to 0.25, 0.1 by default.

**Metrics for performance and quality.** (1) We use the CPU time to measure the efficiency of compared approaches. For experiments in append-only streams, we use the total time for each approach to run over the full datasets. For experiments over sliding windows, we compute the average time for each approach to process one window slide. (2) We use the utility values of the solutions returned by compared approaches as the measure for quality. For experiments in append-only streams, we use the utility values of the solutions returned by each approach after processing the full datasets. For experiments over sliding windows, we compute the average utility values of the solutions returned by each approach among all window slides. (3) We also use the number of checkpoints and the number of elements maintained to measure the space usage of KW and KW\textsuperscript{+}. As other approaches require to store the full active window to work in the sliding window model, the number of elements stored is always $W$ and thus is omitted.

6.3 Experimental Results

6.3.1 An Overview of Experimental Results

**Overview for Append-only Streams.** We first compare KS with CEG, RAND, and STR in append-only streams. Figure 1 illustrates the CPU time and utility values of compared approaches in the default setting. Generally speaking, CEG spends the most time while achieving the best utility values as well. KS runs much faster than CEG (9x speedups on Yahoo! Webscope and 23x speedups on Twitter) while the utility values are only slightly worse (< 6%). RAND spends little time for stream processing but returns undesirable
solutions. Compared with STR, KS is 70% slower but achieves 5% and 12% increases in solution quality in the two datasets respectively. This is because KS maintains \( O\left(\frac{\log d}{\varepsilon}\right) \) more candidate solutions than STR, which needs more function calls to process each element. Since all parameters used in append-only streams are also used similarly in the experiments for sliding windows, we only present the experimental results with different parameter settings for sliding windows in the remaining part of this section.

**Overview for Sliding Windows.** We compare KW and KW\(^+\) with CEG, RAND, and STR in the sliding window model. In Figure 2, we present the CPU time and utility values of compared approaches in the default setting. KW and KW\(^+\) achieve speedups of more than two orders of magnitude over CEG in both datasets. At the same time, KW\(^+\) provides solutions with utility values of at least 89% of CEG and the solution quality of KW also reaches at least 87% of CEG. We can see that KW\(^+\) improves the quality over KW, with 2% on Yahoo! Webscope and 7% on Twitter. Meanwhile, it spends 8% and 40% more time respectively. In addition, KW and KW\(^+\) run much faster than STR while providing solutions with equivalent or better quality. Not surprisingly, RAND uses little time but cannot provide any useful results for both applications. The experimental results verify the superiority of KW and KW\(^+\) for SMDK over sliding windows.

**Changes in Utility Values over Time.** In Figure 3, we show the changes in the utility values of compared
The CPU time decreases on size of result sets becomes smaller. For \( U \) distribution generate \( 5 \) d effectiveness of the proposed algorithms for monitoring hot spots from social streams in real time. Since all approaches except RAND include them into the results, the utility values show a drastic increase when they are observed at time \( 7.7 \times 10^6 \) and \( 8.3 \times 10^6 \) respectively. This results verify the effectiveness of the proposed algorithms for monitoring hot spots from social streams in real time.

### 6.3.2 Scalability

#### Varying \( d \). To study the scalability of compared approaches w.r.t. the number of knapsacks, i.e., \( d \), we generate additional constraints by assigning random costs to the elements in the ground set. Specifically, we generate 5 costs for each element in both datasets. The costs are generated independently from a uniform distribution \( U(0.01, 0.05) \). We range \( d \) from 1 to 5 in the experiments and set the \( j \)-th cost of each element to construct the \( j \)-th knapsack constraint.

The CPU time and utility values of compared approaches with varying \( d \) are shown in Figures 4a-4d. The time usage of \( STR \) and \( CEG \) decreases as \( d \) increases. With the number of knapsacks increasing, the expected size of result sets becomes smaller. For \( KW \) and \( KW^+ \), different trends are observed in the two datasets. The CPU time decreases on Yahoo! Webscope but increases on Twitter with larger \( d \). This is because \( KS \)
maintains the candidate solutions with estimated optimal values from \( m \) to \( M(1 + d) \) (see Algorithm 2). As the range increases with \( d \), more candidate solutions are maintained by \( KS \). Since \( KW \) and \( KW^+ \) maintain \( KS \) in checkpoints, they also have more candidate solutions when \( d \) increases. On the Twitter dataset, extra costs brought by more candidate solutions overwhelm the benefits of smaller result sets and the overall time increases. But it is a different case for Yahoo! Webscope dataset. The time complexity of the IVM function evaluation for \( S \) is \( O(|S|^3) \). Therefore, as the result set becomes smaller, the time for function evaluations decreases largely. Although more candidate solutions are maintained, the overall time decreases.

For solution quality, the first thing to note is that although all compared approaches only achieve approximation ratios of \( O(d^{-1}) \), they show good robustness when \( d \) increases. The solution returned by \( CEG \) for \( d = 1 \) is guaranteed to be \((1 - \delta)(1 - 1/e)\) approximate (\( \delta = 0.05 \)) and thus can be seen as an upper bound estimation of the optimal value in practice (see the black lines in Figures 4c–4d). In addition, the optimal value can only decrease when \( d \) increases as the solution space becomes smaller. When \( d \) increases, the utility values of the compared approaches decrease. However, we can see that their solutions are much better than the theoretical lower bounds as they are close to the optimal utility upper bound of the solution for \( d = 1 \). \( KW \) and \( KW^+ \) remain stable w.r.t. \( d \) and can always guarantee solutions with utility values of at least 80% compared to \( CEG \). Nevertheless, the solution quality of \( STR \) becomes highly unstable as \( d \) increases. When \( d = 5 \), its solution quality degrades to only 70% of \( CEG \).

Varying \( W \). In Figures 5a–5d, the CPU time and utility values of compared approaches with varying \( W \) are presented. The processing time for each window slide increases proportionally with \( W \). This is because we set the number of elements for each window slide to 0.1% of \( W \) and the number of elements to process increases linearly with \( W \). The utility values increase with \( W \) and the solution quality ratio between different approaches remain stable. As shown in Table 2, the number of checkpoints and the number of elements show a slightly decreasing trend as \( W \) increases. In fact, they are not directly related to \( W \) and are bounded by the ratio between the utility values of the first and the last checkpoints, where the ratio decreases with a
larger $W$.

**Varying** $c$. The CPU time and utility values of compared approaches with varying the cost $c$ are illustrated in Figures 6a–6d. When $c$ decreases, it takes a longer processing for all compared approaches. This is because the size of the result set is inversely proportional to $c$. We can see the CPU time for the Yahoo! Webscope dataset increases rapidly when $c$ decreases. This is because the time complexity for each IVM function evaluation is $O(|S|^3)$. Thus, all approaches spend much more CPU time for each function call of $f(S)$ when the size of result set grows. The utility values of each approach drops with a smaller cost due to the smaller size of result sets. We also note that the relative ratios between different approaches keep steady.

6.3.3 Parameters $\varepsilon$ and $\beta$

**Varying** $\varepsilon$. We compare the performance of $KW$, $KW^+$, and $STR$ with varying $\varepsilon$ in Figures 7a–7d. The results show that all approaches have lower CPU time for a larger $\varepsilon$. This can be easily explained by their time complexities, all of which are inversely proportional to $\varepsilon$. For solution quality, all approaches keep steady w.r.t. $\varepsilon$ in the Yahoo! Webscope dataset. On the Twitter dataset, the quality of $STR$ degrades seriously when $\varepsilon$ grows. The phenomenon has demonstrated that, in practical scenarios, the approximation algorithms can provide solutions with much better quality than theoretical lower bounds. We note that $KW$ and $KW^+$ are more robust in terms of solution quality for different $\varepsilon$. Finally, $KW$ and $KW^+$ are much more efficient than $STR$ in all experiments.

**Varying** $\beta$. The results for $KW$ and $KW^+$ with varying $\beta$ are shown in Figures 8a–8d. Both the running time and utility values decrease as $\beta$ increases, which is consistent with the analysis in Section 5.2. As the intervals among checkpoints increase with $\beta$, the error of approximating a checkpoint by its successors increases. At the expense of post-processing, $KW^+$ returns better solutions than $KW$. On the Twitter dataset, when $\beta$ is 0.2 or 0.25, the time for post-processing exceeds the time for stream processing. As a result, the solution quality hardly degrades when $\beta$ increases. As is shown in Table 2, the number of checkpoints and the number of elements stored by $KW$ and $KW^+$ decrease as $\beta$ grows. In addition, the difference between the number of elements kept by $KW$ and $KW^+$ is the number of buffered elements in $KW^+$. The ratios of buffered elements remain stable in different parameter settings (about 20% on Yahoo! Webscope and about 40% on Twitter).

6.3.4 Summary

Finally, we summarize all experimental results. In the append-only stream model, $KS$ runs at least one order of magnitude faster than $CEG$ while achieving close solution quality ($> 90\%$). It improves the solution quality of $STR$ at the expense of speed. In the sliding window model, $KW$ and $KW^+$ achieve speedups of more than two orders of magnitude over $CEG$ in all parameter settings, while returning solutions with competitive quality ($> 80\%$ for $KW$ and $> 85\%$ for $KW^+$). Compared with $STR$, they not only run much faster but also return solutions with equivalent or better utility values. Furthermore, $KW^+$ shows better solution quality than $KW$ with the overhead for maintaining the buffer and post-processing.

7 Conclusion

In this paper, we studied the problem of maximizing a monotone submodular function with a $d$-knapsack constraint (SMDK) in data stream models. We first proposed an algorithm $\text{KnapStream}$ for SMDK in append-only streams with an approximation ratio of $(\frac{1}{1+d} - O(1))$, which improved the state-of-the-art $(\frac{1}{1+2d} - O(1))$ approximation algorithm. Furthermore, we proposed $\text{KnapWindow}$ and its optimized version $\text{KnapWindowOpt}$ for SMDK over sliding windows with an approximation factor of $(\frac{1}{2(1+d)} - O(1))$. The experimental results showed the efficiency and effectiveness of our proposed algorithms compared with baselines. A broad class of real-world applications can benefit from the results in this paper due to the prevalence of submodular functions and the generality of $d$-knapsack constraints. For future work, we would like to explore the solution for SMDK in a completely dynamic setting, where random insertions and deletions in the ground set are allowed.
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