PRIMITIVE WONDERFUL VARIETIES

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Abstract. We complete the classification of wonderful varieties initiated by D. Luna. We review the results that reduce the problem to the family of primitive varieties, and report the references where some of them have already been studied. Finally, we analyze the rest case-by-case.

Introduction

In the article [11] Luna started a research program to classify wonderful $G$-varieties, for $G$ a reductive connected linear algebraic group over the field of complex numbers.

In this program, wonderful varieties are to be classified by means of certain invariants called their spherical systems, which can be represented as combinatorial objects attached to the Dynkin diagram of $G$. A strategy to prove the classification, also known as the Luna conjecture, consists in reducing the problem to a distinguished class of wonderful varieties called primitive. This approach was already used in [11], where groups $G$ of semisimple type $A$ were considered, and in other works: [5], [2], [4].

In [6], we have extended the existing reduction techniques to groups $G$ of any semisimple type, and introduced new ones. The resulting notion of primitive wonderful varieties is given here in Definition 2.5.1. Their combinatorial counterparts, the primitive spherical systems, are discussed by the first-named author in [3], where a complete list of them is also obtained.

It is known that these invariants distinguish between different $G$-isomorphism classes of wonderful varieties (see [10]), therefore the classification is achieved if one proves that each primitive system is geometrically realizable, i.e. comes from a wonderful variety.

In this paper we complete this program. We briefly review the results leading to the definition of primitive spherical systems, and discuss each case of the lists of [6]. Some of them are already well-known, for example those corresponding to reductive wonderful subgroups of $G$ (a reference with their spherical systems will be contained in a further work). We refer for brevity a few other known cases to existing publications, and we analyze in detail the remaining ones.

A relevant byproduct of this proof of the Luna conjecture is an explicit description, albeit laborious, of a generic stabilizer of a wonderful variety using only its spherical system.

Indeed, if a wonderful $G$-variety $X$ is non-primitive, or admits a so-called quotient of higher defect, then our reduction techniques provide a generic stabilizer $H \subset G$ of $X$. The description of $H$ is concise, and relates $H$ to the generic stabilizers of those varieties that can be considered the “primitive components” of $X$. If $X$ is
1.1. Definitions and statement. We start with the statement of the main theorem, known as Luna’s conjecture, on the classification of wonderful varieties.

Let $G$ be a connected reductive algebraic group over the complex numbers $\mathbb{C}$.

Definition 1.1.1. A wonderful $G$-variety (of rank $r$) is a complete non-singular $G$-variety with an open $G$-orbit whose complement is the union of $r$ non-singular prime $G$-divisors $D_1, \ldots, D_r$ with non-empty transversal intersection such that the $G$-orbit closures are the intersections $\cap_{i \in I} D_i$ for any $I \subseteq \{1, \ldots, r\}$.

A wonderful $G$-variety is a simple toroidal spherical variety. It is actually projective, and the radical of $G$ is known to act trivially on it. Therefore, the group $G$ will be assumed to be semi-simple.

Let us fix a maximal torus $T$ and a Borel subgroup $B \supset T$. The corresponding set of simple roots of the root system of $(G, T)$ will be denoted by $S$. The opposite Borel subgroup w.r.t. $T$ will be denoted by $B_-$.

We now define the spherical system of a wonderful $G$-variety $X$. Let $P_X$ be the stabilizer of the open $B$-orbit of $X$ and denote by $S_X^P$ the subset of simple roots corresponding to $P_X$, a parabolic subgroup of $G$ containing $B$. Let $\Sigma_X$ be the set of spherical roots of $X$, a basis of the lattice of $B$-weights that are spherical roots of wonderful $G$-varieties of rank 1 (recall that wonderful varieties of rank 1 are classified) and $B$ is the set of all colors that are not stable under a minimal parabolic $B$-stable and not $G$-stable prime divisors (also called colors) of $X$; by definition $A_X$ is the set of $B$-stable and not $G$-stable prime divisors (also called colors) of $X$; by definition $A_X$ is the set of all colors that are not stable under a minimal parabolic containing $B$ and corresponding to a simple root belonging to $\Sigma_X$. Recall that there is a $B$-bilinear pairing, called Cartan pairing, between colors and spherical roots induced by the valuations of $B$-stable divisors on functions in $C(X)^{(B)}$. The triple $\mathcal{S}_X = (S_X^P, \Sigma_X, A_X)$ is a spherical $G$-system in the sense of the following definition, and it is called the spherical system of $X$.

Definition 1.1.2. Let $(S^p, \Sigma, A)$ be a triple such that $S^p \subset S$, $\Sigma$ is a linearly independent set of $B$-weights that are spherical roots of wonderful $G$-varieties of rank 1 (recall that wonderful varieties of rank 1 are classified) and $A$ a finite set endowed with a $B$-bilinear pairing $c: \mathbb{Z}A \times \mathbb{Z}\Sigma \to \mathbb{Z}$. For every $\alpha \in \Sigma \cap S$, let $A(\alpha)$ denote the set $\{D \in A : c(D, \alpha) = 1\}$. Such a triple is called a spherical $G$-system if:

1. For every $D \in A$ we have $c(D, -) \leq 1$, and if $c(D, \sigma) = 1$ for some $\sigma \in \Sigma$ then $\sigma \in S \cap \Sigma$;
2. For every $\alpha \in \Sigma \cap S$, $A(\alpha)$ contains two elements and denoting with $D_\alpha^+$ and $D_\alpha^-$ these elements, it holds $c(D_\alpha^+, -) + c(D_\alpha^-, -) = \langle \alpha^\vee, - \rangle$;
3. The set $A$ is the union of $A(\alpha)$ for all $\alpha \in \Sigma \cap S$;
4. If $2\alpha \in \Sigma \cap 2S$ then $\frac{1}{2}\langle \alpha^\vee, \sigma \rangle$ is a non-positive integer for every $\sigma \in \Sigma \setminus \{2\alpha\}$;
5. If $\alpha, \beta \in S$ are orthogonal and $\alpha + \beta$ belongs to $\Sigma$ or $2\Sigma$ then $\langle \alpha^\vee, \sigma \rangle = \langle \beta^\vee, \sigma \rangle$ for every $\sigma \in \Sigma$;
6. For every $\sigma \in \Sigma$, there exists a wonderful $G$-variety $X$ of rank 1 with $S_X^P = S^p$ and $\Sigma_X = \{\sigma\}$. 

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1.2. Colors and quotients. Let \( \mathcal{X} = (S^p, \Sigma, A) \) be a spherical \( G \)-system.

Definition 1.2.1. The set of colors of \( \mathcal{X} \) is the finite set \( \Delta \) obtained as disjoint union \( \Delta = \Delta^a \cup \Delta^2 \cup \Delta^b \) where:

- \( \Delta^a = A \).
- \( \Delta^2 = \{ D_\alpha : \alpha \in S \cap \frac{1}{2} \Sigma \} \).
- \( \Delta^b = \{ D_\alpha : \alpha \in S \setminus (S^p \cup \Sigma \cup \frac{1}{2} \Sigma) \}/ \sim, \) where \( D_\alpha \sim D_\beta \) if \( \alpha + \beta \) are orthogonal and \( \alpha + \beta \in \Sigma \).

For all \( \alpha \in S \) set:

\[
\Delta(\alpha) = \begin{cases} 
\emptyset & \text{if } \alpha \in S^p \\
A(\alpha) & \text{if } \alpha \in \Sigma \\
\{ D_\alpha \} & \text{otherwise}
\end{cases}
\]

The full Cartan pairing of \( \mathcal{X} \) is the \( \mathbb{Z} \)-bilinear map \( c : \mathbb{Z} \Delta \times \mathbb{Z} \Sigma \to \mathbb{Z} \) defined as:

\[
c(\Delta, \sigma) = \begin{cases} 
c(D, \sigma) & \text{if } D \in \Delta^a \\
\frac{1}{2}(\alpha^\vee, \sigma) & \text{if } D = D_\alpha \in \Delta^2 \\
(\alpha^\vee, \sigma) & \text{if } D = D_\alpha \in \Delta^b
\end{cases}
\]

If \( X \) is a wonderful \( G \)-variety, the set of colors \( \Delta_X \) of \( \mathcal{X} \) identifies with the set of colors of \( X \), then \( \Delta_X(\alpha) \) corresponds to the colors that are not stable under the minimal parabolic containing \( B \) corresponding to \( \alpha \), and the Cartan pairing equals the Cartan pairing of \( X \).

This allows to define a quotient of a spherical system. Let \( \mathcal{X} = (S^p, \Sigma, A) \) be a spherical \( G \)-system with set of colors \( \Delta \) and Cartan pairing \( c \).

Definition 1.2.2. A subset of colors \( \Delta' \subset \Delta \) is distinguished if there exist \( a_D > 0 \) for all \( D \in \Delta' \) such that \( \sum_{D \in \Delta'} a_D c(D, \sigma) \geq 0 \) for all \( \sigma \in \Sigma \).

Proposition 1.2.3 ([3 Theorem 3.1]). If \( \Delta' \subset \Delta \) is distinguished then:

- the monoid \( \{ \sigma \in \mathbb{N} \Sigma : c(D, \sigma) = 0 \text{ for all } D \in \Delta' \} \) is free;
- setting \( S^p/\Delta' = \{ \alpha : \Delta(\alpha) \subset \Delta' \}, \Sigma/\Delta' \) the basis of the above monoid and \( A/\Delta' = \bigcup_{\alpha \in S \cap \Sigma/\Delta'} A(\alpha) \), the triple \( (S^p/\Delta', \Sigma/\Delta', A/\Delta') \) is a spherical \( G \)-system.

In this case, the spherical \( G \)-system \( \mathcal{X}/\Delta' = (S^p/\Delta', \Sigma/\Delta', A/\Delta') \) is called quotient of \( \mathcal{X} \) by \( \Delta' \). We also use the notation \( \mathcal{X} \to \mathcal{X}/\Delta' \). The set of colors of \( \mathcal{X}/\Delta' \) can be identified with \( \Delta \setminus \Delta' \).

At the level of wonderful varieties this corresponds to certain morphisms. Namely, let \( f : X \to Y \) be a surjective \( G \)-morphism with connected fibers between wonderful \( G \)-varieties. Then the subset \( \Delta_f = \{ D \in \Delta_X : f(D) = Y \} \) is distinguished and \( \mathcal{X}_Y = \mathcal{X}_X/\Delta_f \).
Proposition 1.2.4 ([11] Section 3.3]). Let \( X \) be a wonderful \( G \)-variety. The assignment \( f \mapsto \Delta_f \) induces a bijective correspondence between \( G \)-isomorphism classes of surjective \( G \)-morphisms with connected fibers onto wonderful \( G \)-varieties and distinguished subsets of \( \Delta_X \).

2. Reduction to the primitive cases

The proof of Theorem [11, Section 3.3] can be reduced to a certain subclass of wonderful varieties and spherical systems called primitive. Let us recall the necessary definitions and results on localizations, decompositions into fiber product, positive combs and tails.

2.1. Localizations. For all \( \sigma = \sum n_\alpha \alpha \in \mathbb{N}S \), set \( \text{supp } \sigma = \{ \alpha : n_\alpha \neq 0 \} \). For all \( \Sigma \subset \mathbb{N}S \), set \( \text{supp } \Sigma = \bigcup_{\sigma \in \Sigma} \text{supp } \sigma \).

Definition 2.1.1. Let \( \mathcal{S} = (S^p, \Sigma, A) \) be a spherical \( G \)-system. For all subsets of simple roots \( S' \subseteq S \), consider a semi-simple group \( G_{S'} \) with set of simple roots \( S' \); we define the localization \( \mathcal{S}_{S'} \) of \( \mathcal{S} \) as the spherical \( G_{S'} \)-system \( ((S')^p, \Sigma', A') \) as follows:

- \((S')^p = S^p \cap S'\),
- \( \Sigma' = \{ \sigma \in \Sigma : \text{supp } \sigma \subseteq S' \} \),
- \( A' = \bigcup_{\alpha \in S \setminus \Sigma} A(\alpha) \).

Let \( X \) be a wonderful \( G \)-variety. For all subsets of simple roots \( S' \subseteq S \) define the localization \( X_{S'} \) of \( X \) to be the subvariety \( X_{P''} \) of points fixed by the radical \( P'' \) of \( P \), where \( P \) is the parabolic subgroup containing \( B_+ \) and corresponding to \( S' \). Under the action of \( G_{S'} = P/P'' \) the variety \( X_{S'} \) is wonderful, and

\[ \mathcal{S}(X_{S'}) = (\mathcal{S}_X)_S. \]

Proposition 2.1.2 ([11] Section 3.4]). Let \( \mathcal{S} = (S^p, \Sigma, A) \) be a spherical \( G \)-system and let \( S' \) be a subset of simple roots containing \( S^p \cup \text{supp } \Sigma \). If there exists a wonderful \( G_{S'} \)-variety \( Y \) with spherical system \( \mathcal{S}_{S'} \), then there exists a wonderful \( G \)-variety \( X \) with spherical system \( \mathcal{S} \). Precisely, \( X \) is a parabolic induction of \( Y \), that is,

\[ X \cong G \times_p Y \]

where \( P \) is the parabolic subgroup containing \( B_+ \) corresponding to \( S' \) and \( Y \) is a wonderful \( P/P'' \)-variety (a \( P \)-variety with trivial action of \( P'' \)) with spherical system \( \mathcal{S}_{S'} \).

Let \( \mathcal{S} = (S^p, \Sigma, A) \) be a spherical \( G \)-system with \( S^p \cup \text{supp } \Sigma = S \), then \( \text{supp } \Sigma \) and \( S^p \setminus \text{supp } \Sigma \) are orthogonal, thus \( G \cong G_{\text{supp } \Sigma} \times G_{S^p \setminus \text{supp } \Sigma} \) and the second factor acts trivially on any wonderful \( G \)-variety \( X \) with spherical system \( \mathcal{S} \). Therefore, the previous proposition can be rewritten as follows.

Proposition 2.1.3. Let \( \mathcal{S} = (S^p, \Sigma, A) \) be a spherical \( G \)-system and let \( S' \) be a subset of simple roots containing \( \text{supp } \Sigma \). If there exists a wonderful \( G_{S'} \)-variety with spherical system \( \mathcal{S}_{S'} \), then there exists a wonderful \( G \)-variety with spherical system \( \mathcal{S} \).

Definition 2.1.4. A spherical \( G \)-system \( \mathcal{S} = (S^p, \Sigma, A) \) is called cuspidal if \( \text{supp } \Sigma = S \).
2.2. Decompositions into fiber product.

**Definition 2.2.1.** Let $\mathcal{S} = (S^p, \Sigma, A)$ be a spherical $G$-system with set of colors $\Delta$. Two non-empty distinguished subsets of colors $\Delta'$ and $\Delta''$ decompose $\mathcal{S}$ if

- $(S^p/\Delta') \setminus S^p$ and $(S^p/\Delta'') \setminus S^p$ are orthogonal,
- $\Sigma$ is included in $\Sigma/\Delta' \cup \Sigma/\Delta''$.

In this case the spherical $G$-system $\mathcal{S}$ is called decomposable.

One has that $\Delta''$ (resp. $\Delta'$) is a distinguished subset of colors of $\mathcal{S}/\Delta'$ (resp. $\mathcal{S}/\Delta''$), hence $\Delta' \cup \Delta''$ is a distinguished subset of colors of $\mathcal{S}$ and $(\mathcal{S}/\Delta')/\Delta'' = (\mathcal{S}/\Delta'')/\Delta' = \mathcal{S}/(\Delta' \cup \Delta'')$.

**Proposition 2.2.2 (6, Section 4).** Let $\mathcal{S} = (S^p, \Sigma, A)$ be a spherical $G$-system with set of colors $\Delta$ and let $\Delta'$ and $\Delta''$ be two distinguished subsets that decompose $\mathcal{S}$. If there exist wonderful varieties $X'$ and $X''$ with spherical systems $\mathcal{S}/\Delta'$ and $\mathcal{S}/\Delta''$, respectively, then there exists a wonderful $G$-variety $X$ with spherical system $\mathcal{S}$. One has that $X$ is a fiber product, that is,

$$X \cong X' \times_{X''} X''$$

where $X''$ is a wonderful $G$-variety with surjective morphisms with connected fibers $f'' : X' \to X''$ and $f' : X'' \to X'''$ such that $\Delta f'' = \Delta''$ and $\Delta f' = \Delta'$.

2.3. Positive combs.

**Definition 2.3.1.** Let $\mathcal{S} = (S^p, \Sigma, A)$ be a spherical $G$-system. A positive comb is an element $D$ of $A$ such that $c(D, \sigma) \geq 0$ for all $\sigma \in \Sigma$. It is also called positive $n$-comb if $n = \text{card}\{\alpha \in S \cap \Sigma : c(D, \alpha) = 1\}$.

Let $\mathcal{S} = (S^p, \Sigma, A)$ be a spherical $G$-system with a positive comb $D$. Set $S_D = \{\alpha \in S \cap \Sigma : c(D, \alpha) = 1\}$. For all $\alpha \in S_D$, define $\mathcal{S}_\alpha = (S^p, \Sigma_\alpha, A_\alpha)$ where $\Sigma_\alpha = \Sigma \setminus (S_D \setminus \{\alpha\})$ and $A_\alpha = \cup_{\beta \in S \setminus \Sigma_\alpha} A(\beta)$. The spherical $G$-system $\mathcal{S}_\alpha$ has a positive 1-comb in $A_\alpha(\alpha)$.

**Proposition 2.3.2 (6).** Let $\mathcal{S} = (S^p, \Sigma, A)$ be a spherical $G$-system with a positive $n$-comb $D$, with $n > 1$. If for all $\alpha \in S_D$ there exists a wonderful $G$-variety with spherical system $\mathcal{S}_\alpha$, then there exists a wonderful $G$-variety with spherical system $\mathcal{S}$.

In this case the principal isotropy group of the wonderful $G$-variety with spherical system $\mathcal{S}$ can be explicitly constructed starting from the principal isotropy groups of the wonderful $G$-varieties with spherical systems $\mathcal{S}_\alpha$, for $\alpha \in S_D$ (see 6 Section 5.4 for details).

2.4. Tails.

**Definition 2.4.1.** Let $\mathcal{S} = (S^p, \Sigma, A)$ be a spherical $G$-system. A tail is a subset of spherical roots $\Sigma \subset \Sigma$ with $\text{supp} \Sigma$ included in a connected component $S_0 = \{\alpha_1, \ldots, \alpha_n\}$ of $S$ such that there exists a distinguished subset of colors $\Delta'$ with $\Sigma/\Delta' = \bar{\Sigma}$, under one of the following cases:

- (type $b(m)$) $S_0$ is of type $B_n$, $1 \leq m \leq n$, $\bar{\Sigma} = \{\alpha_{n-m+1} + \ldots + \alpha_n\}$ and $\alpha_n \in S^p$ if $m > 1$ (or $c(D^+_\alpha, \sigma') = c(D^-_{\alpha_n}, \sigma')$ for all $\sigma' \in \Sigma$ if $m = 1$);
- (type $2b(m)$) $S_0$ is of type $B_n$, $1 \leq m \leq n$, and $\bar{\Sigma} = \{2\alpha_{n-m+1} + \ldots + 2\alpha_n\}$;
Proposition 2.4.2 \[ \text{(6)} \]. Let \( \mathcal{S} = (S^p, \Sigma, A) \) be a spherical G-system with a tail \( \Sigma \). Set \( S' = \text{supp}(\Sigma \setminus \Sigma) \). If there exists a wonderful \( G_{S'} \)-variety with spherical system \( \mathcal{S}_S \), then there exists a wonderful \( G \)-variety with spherical system \( \mathcal{S} \).

In this case the principal isotropy group of the wonderful \( G \)-variety with spherical system \( \mathcal{S} \) can be explicitly constructed starting from the principal isotropy group of the wonderful \( G_{S'} \)-variety with spherical system \( \mathcal{S}_S \) (see \[ \text{[6]} \text{ Section 6} \] for details).

2.5. Primitive cases. The above results lead to the following.

**Definition 2.5.1.**
- A spherical \( G \)-system is called primitive if it is cuspidal, not decomposable, without positive combs and without tails.
- A positive 1-comb of a spherical \( G \)-system \( \mathcal{S} \) is called primitive if \( \mathcal{S} \) is cuspidal, not decomposable and without tails.

Theorem 1.1.3 holds provided that all primitive spherical systems and all spherical systems with a primitive positive 1-comb are geometrically realizable.

Primitive spherical systems and spherical systems with a primitive positive 1-comb are classified in \[ \text{[3]} \].

2.6. Known cases. The geometrical realizability of spherical systems is known in many particular cases.

Wonderful varieties with rank \( \leq 2 \) are well known after \[ \text{[1]} \text{[8]} \text{[14]} \] and in that case Theorem 1.1.3 holds.

Affine spherical homogeneous spaces are well known, see \[ \text{[9]} \text{[12]} \text{[7]} \]. On the other hand, the wonderful \( G \)-varieties \( X \) whose open \( G \)-orbit is affine are characterized by the existence of \( n_\sigma \geq 0 \) for all \( \sigma \in \Sigma_X \) such that \( c_X(D, \sum_{\sigma \in \Sigma_X} n_\sigma \sigma) > 0 \) for all \( D \in \Delta_X \). It has been shown that all spherical systems with the above property are geometrically realizable; they are also called reductive spherical systems. In the notations of \[ \text{[3]} \], they are: the entire clan \( R \), S-1, S-2, S-3, S-5, S-68, T-9, T-12, T-15, T-15', T-25.

Wonderful \( G \)-varieties for \( G \) with a simply-laced Dynkin diagram have been considered in \[ \text{[11]} \text{[5]} \text{[2]} \] and under this hypothesis Theorem 1.1.3 has been proved. The following cases have this property and cannot be treated with other reduction techniques: S-50, S-62, S-67, S-75, S-76, S-105, T-2, T-3 of rank 6, T-4 of rank 5 and 7, T-8, T-10, T-11.

Theorem 1.1.3 has also been proved for strict wonderful varieties, \[ \text{[4]} \]. A wonderful variety \( X \) is strict if all its isotropy groups are self-normalizing and this is equivalent to a combinatorial condition on \( \mathcal{S}_X \): for every \( \sigma \in \Sigma_X \), there exists no wonderful \( G \)-variety \( X' \) with \( S^p_{X'} = S^p_X \) and \( \Sigma_{X'} = \{2\sigma\} \). We may apply this result to the cases: T-1, T-22.
3. Quotients of type $\mathcal{L}$

Let $\mathcal{S} = (S^p, \Sigma, A)$ be a primitive spherical $G$-system with set of colors $\Delta$. Typically, it admits a quotient spherical system $\mathcal{S}/\Delta'$ of type $\mathcal{L}$ which is somewhat simpler and known to be geometrically realizable. Therefore, to construct the principal isotropy group $H$ of a wonderful $G$-variety with spherical system $\mathcal{S}$ we use the explicit knowledge of the principal isotropy group $K$ of a wonderful $G$-variety of $\mathcal{S}/\Delta'$.

Let us here skip the combinatorial notion of quotient $\mathcal{S} \to \mathcal{S}/\Delta'$ of type $\mathcal{L}$ at the level of spherical $G$-systems (see [6, Section 5]) but recall that, if $\mathcal{S}$ is geometrically realizable, such quotient corresponds to a minimal co-connected inclusion $H \subset K$ of spherical subgroups of $G$ such that $H^u$ is strictly contained in $K^u$, $\text{Lie}(K^u)/\text{Lie}(H^u)$ is a simple $H$-module and Levi subgroups of $H$ and $K$ (i.e. $L$ and $L_K$, respectively) differ only by their connected centers (actually, $L = N_{L_K}(H^u)$).

Let us describe some special classes of such quotients more in detail. This will allow us to reduce the final case-by-case analysis to a smaller set of primitive spherical systems.

3.1. Minimal quotients of higher defect. Let us recall that the defect of a spherical system $\mathcal{S} = (S^p, \Sigma, A)$ with set of colors $\Delta$ is defined as

$$d(\mathcal{S}) = \text{card } \Delta - \text{card } \Sigma.$$ 

If $H$ is the principal isotropy group of a wonderful variety with spherical system $\mathcal{S}$, then $d(\mathcal{S})$ equals the rank of the character group of $H$, i.e. the dimension of the connected center of a Levi subgroup $L$ of $H$.

Let $\mathcal{S} \to \mathcal{S}/\Delta'$ be a quotient of type $\mathcal{L}$, notice that, in the above notation, $L = L_K$ if and only if the quotient has constant defect, i.e. $d(\mathcal{S}) = d(\mathcal{S}/\Delta')$.

In [6, Section 5.3] it is studied the case of a minimal quotient $\mathcal{S} \to \mathcal{S}/\Delta'$ of higher defect, i.e. $d(\mathcal{S}/\Delta') > d(\mathcal{S})$. Every such quotient is of type $\mathcal{L}$.

Under the assumption that $\mathcal{S}/\Delta'$ is spherically closed (for every $\sigma \in \Sigma \setminus S$ there exists no wonderful $G$-variety $X$ with $S^p_X = S^p$ and $\Sigma_X = \{2\sigma\}$) and fulfills further technical combinatorial conditions (see [6, Conjecture 5.3.1]), the principal isotropy group of $\mathcal{S}$ can be explicitly constructed starting from the principal isotropy groups of certain spherical systems $\mathcal{S}$ that have the same quotient spherical system $\mathcal{S}/\Delta'$ (of type $\mathcal{L}$) but with constant defect.

On the list of [6] one can check that the above combinatorial conditions are satisfied by all the minimal quotients with higher defect of all primitive spherical systems (this is a long but easy verification).

Therefore, we have the following.

**Proposition 3.1.1.** *Theorem 1.1.3 follows from the geometric realizability of the primitive spherical systems without minimal quotients of higher defect.*

We apply this result to the systems: $S-4, S-6, S-8, \ldots, S-13, S-15, S-16, S-17, S-19, \ldots, S-49, S-51, S-52, S-54, \ldots, S-60, S-66, S-82, S-83, S-84, S-89, \ldots, S-94, S-97, \ldots, S-104, S-106, \ldots, S-122, T-3$ of rank 5 and 7, $T-4$ of rank 6, $T-5, T-6, T-7, T-13, T-14, T-16, T-17$.

3.2. Minimal quotients of rank 0. Many primitive spherical systems $\mathcal{S}$ of defect 1 admit a rank 0 (i.e. with $\Sigma = \emptyset$) spherical system $\mathcal{S}/\Delta'$ as quotient of type $\mathcal{L}$ of constant defect.
Rank 0 spherical systems correspond to partial flag varieties, namely the corresponding principal isotropy groups are parabolic subgroups, which are maximal if the defect is equal to 1. More precisely, such a spherical system has only one color, say $D_\alpha$ where $S \setminus S^\alpha = \{\alpha\}$: then up to conjugation we have as principal isotropy group the parabolic subgroup $Q$ containing $B_-$ corresponding to $S \setminus \{\alpha\}$. The Lie algebra of the unipotent radical of $Q$ decomposes under the action of the standard Levi subgroup $L_Q \supset T$ as

$$\text{Lie } Q^u \cong V(-\alpha) \oplus [\text{Lie } Q^u, \text{Lie } Q^u],$$

where $V(-\alpha)$ is the simple $L_Q$-module of highest $T$-weight $-\alpha$. This leads to a unique possible candidate $H$ for the principal isotropy group of a wonderful variety with spherical system $\mathcal{S}$. Namely, the unipotent radical $H^u$ must be $(Q^u, Q^u)$ and a Levi subgroup of $H$ must be $L_Q$.

With this choice, $H$ is a subgroup of $G$ and it is equal to its normalizer, hence it is the principal isotropy group of a wonderful variety $X$. Its spherical system $\mathcal{S}_X$ is primitive and admits $\mathcal{S}/\Delta'$ as a quotient of type $\mathcal{L}$ of constant defect. Finally, it is easy to check on the list in [3] that these properties identify $\mathcal{S}$ uniquely, so $\mathcal{S} = \mathcal{S}_X$.

Therefore, we have the following.

**Proposition 3.2.1.** If $\mathcal{S}$ admits a quotient spherical system $\mathcal{S}/\Delta'$ of type $\mathcal{L}$ of constant defect of rank 0 then it is geometrically realizable and, with the above notation, we have $H = H^u L$ where $L = L_Q$ and $H^u = (Q^u, Q^u)$.

We may apply this result to the following spherical systems: S-53, S-73, T-23, T-26.

3.3. **Localizations.** Finally, we want to recall the following obvious fact.

**Proposition 3.3.1.** Let $\mathcal{S}$ be a spherical $G$-system. Let $S'$ be a subset of $S$. Then the geometric realizability of its localization $\mathcal{S}'$ follows from the geometric realizability of $\mathcal{S}$.

This may be applied to those primitive spherical systems that are localizations of other primitive systems, but some care is needed since this reduction technique works “backwards” with respect to the rank. We will thus apply Proposition 3.3.1 only if the geometric realizability of $\mathcal{S}$ does not depend (through other reduction techniques) on systems of lower rank.

This can be done in the cases: S-64 which is a localization of S-70 (found in [4]), S-65 which is a localization of S-72 (found in [4]), S-74 and S-87 which are localizations of S-73 (where Proposition 3.2.1 applies), S-95 which is a localization of S-96 (found in [4]), T-24 which is a localization of T-25 (a reductive case), T-29 which is a localization of T-26 (where Proposition 3.2.1 applies).

4. **Explicit Computations**

In this section we study all the remaining primitive spherical systems. To be precise we are left with the primitive spherical $G$-systems $\mathcal{S}$ such that:

- the rank (i.e. card $\Sigma$) is $> 2$,
- there does not exist $n_\sigma \geq 0$ for all $\sigma \in \Sigma$ such that $c(D, \sum_{\sigma \in \Sigma} n_\sigma \sigma) > 0$ for all $D \in \Delta$,
- the Dynkin diagram of $G$ is not simply-laced,
there exists $\sigma \in \Sigma$ such that there exists a wonderful $G$-variety $X$ with $S^p_X = S^p$ and $\Sigma_X = \{\sigma\}$.

- there does not exist a minimal quotient of higher defect,
- there does not exist a quotient $\mathcal{I} \to \mathcal{I}/\Delta'$ of type $\mathcal{L}$ with $\text{card}(\Sigma/\Delta') = \emptyset$ and constant defect,
- $\mathcal{I}$ is not one of the localizations listed in §3.3.

They consist of 24 cases, which in the notation of [3] are: $ab^\gamma(p - 1, p)$, $ag^\gamma(1, 2)$, $b^\gamma(4)$, $b^\alpha(3)$, S-63, S-69, ..., S-72, S-77, ..., S-81, S-85, S-86, S-88, S-96, T-18, ..., T-21, T-27 and T-28.

4.1. Non-essential quotients of type $\mathcal{L}$. Let $\mathcal{I} = (S^p, \Sigma, A)$ be a spherical $G$-system.

**Definition 4.1.1.** A minimal quotient $\mathcal{I} \to \mathcal{I}/\Delta'$ is called essential if

$$\left(\Sigma/\Delta'\right) \cap \Sigma = \emptyset.$$  

Among the 24 cases above, the following admit a non-essential quotient of type $\mathcal{L}$ of constant defect. Roughly speaking, the subset of spherical roots $(\Sigma/\Delta') \cap \Sigma$ plays no role in the co-connected inclusion $H \subset K$ corresponding to the quotient $\mathcal{I} \to \mathcal{I}/\Delta'$.

For all $D \in \Delta'$ and $\sigma \in (\Sigma/\Delta') \cap \Sigma$, one clearly has $c(D, \sigma) = 0$. Therefore, the subset $\Delta'$ can be identified with a distinguished subset of the spherical $G$-system $\tilde{\mathcal{I}} = (S^p, \tilde{\Sigma}, \tilde{A})$ with $\tilde{\Sigma} = \Sigma \setminus (\Sigma/\Delta')$ and $\tilde{A} = \cup_{\alpha \in S \cap A} \tilde{A}(\alpha)$. The quotient $\tilde{\mathcal{I}} \to \tilde{\mathcal{I}}/\Delta'$ is still of type $\mathcal{L}$ of constant defect, but clearly essential.

Let $\tilde{H} \subset \tilde{K}$ be the co-connected inclusion corresponding to $\tilde{\mathcal{I}} \to \tilde{\mathcal{I}}/\Delta'$, recall that $\tilde{H}^u \subset \tilde{K}^u$ and $W = \text{Lie } \tilde{K}^u/\text{Lie } \tilde{H}^u$ is a simple $\tilde{H}$-module. More explicitly, we can fix the same Levi subgroup $\tilde{L}$ for $\tilde{H}$ and $\tilde{K}$: $W$ is a simple spherical $\tilde{L}$-module. There exists a direct factor $M$ of $\tilde{L}$ acting non-trivially on $W$: this conjecturally holds in general, and can easily be checked on the 12 cases above.

Let $K = K^u L$ be the principal isotropy group corresponding to $\mathcal{I}/\Delta'$, then we have a natural choice of a co-connected subgroup $H$ of $K$ such that $\text{Lie } K^u/\text{Lie } H^u$ is a simple $H$-module. Indeed, we can choose $H^u$ in $K^u$ such that the direct factor of $L$ acting non-trivially on $\text{Lie } K^u/\text{Lie } H^u$ is isomorphic to $M$ and $\text{Lie } K^u/\text{Lie } H^u \cong W$ as $M$-modules.

4.2. Remaining cases. We are left with 12 cases, which we subdivide as follows:

1. $ab^\gamma(p - 1, p)$, $ag^\gamma(1, 2)$, $b^\gamma(4)$, $b^\alpha(3)$, S-70, S-72, S-96 and T-28;
2. $b^\gamma(4)$;
3. S-63, S-81 and S-85;
4. T-21.

In the following we will make use of Luna diagrams, see [3] for their definition.

4.2.1. Let $G$ be of type $A_{p-1} \times B_p$, with $p \geq 2$: $S = \{a_1, \ldots, a_{p-1}, a'_1, \ldots, a'_p\}$. Let us consider the quotient $\mathcal{I} \to \mathcal{I}/\Delta'$ of type $\mathcal{L}$ of constant defect described
by the following diagrams.

The spherical system $\mathcal{S}/\Delta'$ is geometrically realizable, it is parabolic induction of the spherical system of a wonderful $Q/Q'$-variety with affine open $Q/Q'$-orbit, where $Q$ is the maximal parabolic subgroup of $G$ containing $B_-$ corresponding to $S \setminus \{\alpha'_p\}$. Indeed, the group $Q/Q'$ is semi-simple of type $A_{p-1} \times A_{p-1}$ and we can choose the principal isotropy group $K$ corresponding to $\mathcal{S}/\Delta'$ as the subgroup containing $Q_\mu$ and such that $K/Q_\mu$ is the semi-simple subgroup of type $A_{p-1}$ diagonally embedded in $Q/Q'$, a very reductive subgroup. Set $Q = Q_u L_Q$ with $L_Q \supset T$ and $K = K_u L_K$ with $L_K \subset L_Q$. As $L_Q$-modules we have

$$\text{Lie } Q_u \cong V(-\alpha'_p) \oplus [\text{Lie } Q_u, \text{Lie } Q_u],$$

where $V(-\alpha'_p)$ is the simple $L_Q$-module of highest $T$-weight $-\alpha'_p$. This module remains simple under the action of $L_K$. Let us now choose the principal isotropy group $H \subset K$ corresponding to $\mathcal{S}$: we can take $H = H_u L$ with $H_u = (Q_u, Q_u)$ and $L = L_K$.

The subgroup $H$ is spherical and self-normalizing. To prove that it is the principal isotropy group of the wonderful variety with spherical system $\mathcal{S}$ it is enough to notice that there is no other spherical system admitting $\mathcal{S}/\Delta'$ as quotient.

The other cases of this block are very similar (with one slight exception). We will put them one after the other keeping the same notation.

In this case $G$ is of type $A_1 \times G_2$, $S = \{\alpha_1, \alpha'_1, \alpha'_2\}$. The parabolic subgroup $Q \supset B_-$ corresponds to $S \setminus \{\alpha'_1\}$ and has semi-simple type $A_1 \times A_1$. The group $K/Q'$ is the semi-simple subgroup of type $A_1$ diagonally embedded in $Q/Q'$. As above we can take $H = H_u L$ with $H_u = (Q_u, Q_u)$ and $L = L_K$.

In this case $G$ is of type $B_3$, $S = \{\alpha_1, \alpha_2, \alpha_3\}$. The parabolic subgroup $Q \supset B_-$ corresponds to $S \setminus \{\alpha_2\}$ and has semi-simple type $A_1 \times A_1$. The group $K/Q'$ is the semi-simple subgroup of type $A_1$ diagonally embedded in $Q/Q'$. Here the simple $L_Q$-module $V_{L_Q}(-\alpha_2)$ does not remain simple under the action of $L_K$: as $L_K$-modules

$$V_{L_Q}(-\alpha_2) \cong V_{L_K}(-\alpha_2) \oplus W,$$
where $W$ is simple of dimension 2. We take $H = H^u L$ with $L = L_K$ and $\text{Lie } H^u$ equal to the $L$-complementary of $W$ in $\text{Lie } Q^u$.

In this case $G$ is of type $A_1 \times C_{p+2}$ with $p \geq 1$, $S = \{\alpha_1, \alpha'_1, \ldots, \alpha'_{p+2}\}$. The parabolic subgroup $Q \supset B_-$ corresponds to $S \setminus \{\alpha'_{p+1}\}$ and has semi-simple type $A_1 \times A_p \times A_1$. The group $K/Q^r$ is the semi-simple subgroup of type $A_1 \times A_p$ with the first factor diagonally embedded in the first and second factor of $Q/Q^r$. As in the first and second case of this block we can take $H = H^u L$ with $H^u = (Q^u, Q^u)$ and $L = L_K$.

In this case $G$ is of type $A_1 \times C_4$, $S = \{\alpha_1, \alpha'_1, \alpha'_2, \alpha'_4\}$. The parabolic subgroup $Q \supset B_-$ corresponds to $S \setminus \{\alpha'_3\}$ and has semi-simple type $A_1 \times A_2 \times A_3$. The group $K/Q^r$ is the semi-simple subgroup of type $A_1 \times A_2$ with the first factor diagonally embedded in the first and third factor of $Q/Q^r$. As in the above case we can take $H = H^u L$ with $H^u = (Q^u, Q^u)$ and $L = L_K$.

In this case $G$ is of type $A_1 \times F_4$, $S = \{\alpha_1, \alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4\}$. The parabolic subgroup $Q \supset B_-$ corresponds to $S \setminus \{\alpha'_5\}$ and has semi-simple type $A_1 \times A_2 \times A_3 \times A_1$. The group $K/Q^r$ is the semi-simple subgroup of type $A_1 \times A_2 \times A_1 \times A_2$ with the first factor diagonally embedded in the first and second factor of $Q/Q^r$. As in the above case we can take $H = H^u L$ with $H^u = (Q^u, Q^u)$ and $L = L_K$.

This can be seen as a generalization of the fifth case, $G$ is of type $A_1 \times C_{p+3}$ with $p \geq 1$, $S = \{\alpha_1, \alpha'_1, \ldots, \alpha'_{p+3}\}$. The parabolic subgroup $Q \supset B_-$ corresponds to $S \setminus \{\alpha'_2\}$ and has semi-simple type $A_1 \times A_1 \times A_{p+1}$. The group $K/Q^r$ is the semi-simple subgroup of type $A_1 \times A_{p+1}$ with the first factor diagonally embedded in the first and second factor of $Q/Q^r$. As in the above case we can take $H = H^u L$ with $H^u = (Q^u, Q^u)$ and $L = L_K$.

4.2.2. We keep the same notation as above.
The group $G$ is of type $B_4$, $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. The parabolic subgroup $Q \supset B_-$ corresponds to $S \setminus \{\alpha_2\}$ and has semi-simple type $A_1 \times B_2$. The group $K/Q^r$ is the semi-simple subgroup of type $A_1 \times A_1$ with the first factor diagonally embedded in the first and third factor of a semi-simple subgroup $K_2/Q^r$ of type $A_1 \times A_1 \times A_1$. The simple $L_Q$-module $V_{L_Q}(-\alpha_2)$ does not remain simple under the action of $L_K$: as $L_{K_2}$-modules

$$V_{L_Q}(-\alpha_2) \cong V_{L_{K_2}}(-\alpha_2) \oplus W_2,$$

where $W_2$ is simple of dimension 2, and as $L_K$-modules

$$V_{L_{K_2}}(-\alpha_2) \cong V_{L_K}(-\alpha_2) \oplus W,$$

where $W$ is simple of dimension 2. We take $H = H^u L$ with $L = L_K$ and Lie $H^u$ equal to the $L$-complementary of $W$ in Lie $Q^u$.

We now prove that the subgroup $H$ corresponds to $\mathcal{S}$. Notice that we could have taken as Lie $H^u$ the $L$-complementary of $W_2$ in Lie $Q^u$ obtaining a self-normalizing spherical subgroup, too. Indeed, let us also consider the following quotient of type $\mathcal{S}$ of constant defect (with fiber of dimension 2).

This is the (only) other possible choice of a spherical system corresponding to $H$. To show that $H$ actually corresponds to the spherical system represented by the first diagram above, it is enough to notice that the second one admits also the following (non-minimal) quotient, which would correspond to the inclusion of $H$ into a semi-simple subgroup of type $D_4$.

4.2.3.

The group $G$ is of type $B_{q+2}$ with $q \geq 1$, $S = \{\alpha_1, \ldots, \alpha_{q+2}\}$. The parabolic subgroup $Q \supset B_-$ corresponds to $S \setminus \{\alpha_{q+2}\}$ and has semi-simple type $A_{q+1}$. The subgroup $K/Q^r$ of $Q/Q^r$ is reductive of semi-simple type $A_q$. The simple $L_Q$-module $V_{L_Q}(-\alpha_{q+2})$ does not remain simple under the action of $L_K$: as $L_K$-modules

$$V_{L_Q}(-\alpha_{q+2}) \cong \mathbb{C} \oplus W,$$

where $W$ is simple of dimension $q+1$. We take $H = H^u L$ with $L = L_K$ and Lie $H^u$ equal to the $L$-complementary of $W$ in Lie $Q^u$. 
This generalizes to the following.

In this case the group $G$ is of type $B_{2p+q+2}$ with $p, q \geq 1$, $S = \{\alpha_1, \ldots, \alpha_{2p+q+2}\}$. The parabolic subgroup $Q \supset B_-$ corresponds to $S \setminus \{\alpha_{2p+q+2}\}$ and has semi-simple type $A_{2p+q+1}$. The subgroup $K/Q'$ of $Q/Q'$ is reductive of semi-simple type $A_p \times A_{p+q}$. The simple $L_Q$-module $V_{L_Q}(-\alpha_{2p+q+2})$ does not remain simple under the action of $L_K$: as $L_K$-modules

$$V_{L_Q}(-\alpha_{2p+q+2}) \cong W_1 \oplus W_2,$$

where $W_1$ and $W_2$ are simple of dimension $p + 1$ and $p + q + 1$, respectively. We take $H = H^u L$ with $L = L_K$ and Lie $H^u$ equal to the $L$-complementary of $W_2$ in Lie $Q^u$.

Let us now consider the following.

In this case the parabolic subgroup $Q \supset B_-$ corresponds to $S \setminus \{\alpha_2\}$ and has semi-simple type $A_1 \times A_2$. The subgroup $K/Q'$ of $Q/Q'$ is reductive of semi-simple type $A_1 \times A_1$. The simple $L_Q$-module $V_{L_Q}(-\alpha_2)$ does not remain simple under the action of $L_K$: as $L_K$-modules

$$V_{L_Q}(-\alpha_{2p+q+2}) \cong W_1 \oplus W_2 \oplus W_3,$$

where $W_1, W_2$ and $W_3$ are simple of dimension 6, 4 and 2, respectively. We take $H = H^u L$ with $L = L_K$ and Lie $H^u$ equal to the $L$-complementary of $W_2$ in Lie $Q^u$.

4.2.4. To describe the last case we do not use its quotient of type $\mathcal{L}$, let us consider the following quotient, which is not minimal and is not the composition of quotients of type $\mathcal{L}$.

Here the subgroup $H$ is the parabolic subgroup of semi-simple type $B_2$ of the symmetric subgroup of type $B_4$ of $G$, which is of type $F_4$.

5. Primitive positive 1-combs

Let $\mathcal{S} = (S^p, \Sigma, A)$ be a spherical system with a primitive positive 1-comb $D \in A$. The quotient $\mathcal{S} \to \mathcal{S}/\{D\}$ is of type $\mathcal{L}$ and non-essential unless $\mathcal{S}$ is the rank 1 spherical $G$-system (with $G$ of type $A_1$) with the following diagram.
Nevertheless, in this case the explicit construction of the principal isotropy group \( H \) of a wonderful variety with spherical system \( \mathcal{S} \) found in Section 4.1 is not very convenient.

In general, morphisms between wonderful varieties corresponding to quotients by subsets of colors of the form \( \{D\} \) where \( D \) is a positive comb consist of projective fibrations: smooth (surjective) morphisms with fibers isomorphic to projective spaces. In particular the following is known.

**Proposition 5.0.1** ([11] Section 3.6). Let \( \mathcal{S} = (S^p, \Sigma, A) \) be a spherical system with a positive comb \( D \in A \) such that \( S_D \cap \text{supp}(\Sigma \setminus S) = \emptyset \). Then the geometric realizability of \( \mathcal{S} \) follows from the geometric realizability of \( \mathcal{S}/\{D\} \).

Therefore, we can restrict here to rank > 2 spherical systems with a primitive positive 1-comb \( D \) such that \( S_D \cap \text{supp}(\Sigma \setminus S) \neq \emptyset \). There are only 4 such spherical systems.

**5.1.** Here \( G \) is of type \( B_n \). We describe the subgroup \( H \) in case of even \( n \) (the odd case is slightly more complicated but follows by localization). Let us consider the following quotients \( \mathcal{S} \to \mathcal{S}_1 \to \mathcal{S}_2 \) of type \( \mathcal{L} \).

\[
\text{Let } Q \text{ be the parabolic subgroup containing } B_- \text{ corresponding to } S \setminus \{\alpha_n\}. \text{ The subgroup } K_2 \text{ corresponding to } \mathcal{S}_2 \text{ contains } Q^r \text{ and } K_2/Q^r \text{ is very reductive of type } C_{n/2} \text{ in } Q/Q^r \text{ which is of type } A_n. \text{ As } L_Q \text{-modules }
\]

\[
\text{Lie } Q^n \cong V(-\alpha_n) \oplus [\text{Lie } Q^n, \text{Lie } Q^n] \cong V(-\alpha_n) \oplus V(-\alpha_{n-1} - 2\alpha_n),
\]

and \( V(-\alpha_n) \) remains simple under the action of \( L_{K_2} \). The subgroup \( K_1 \) corresponding to \( \mathcal{S}_1 \) has \( L_{K_1} = L_{K_2} \) and \( \text{Lie } K_1 \setminus = [\text{Lie } Q^n, \text{Lie } Q^n] \). As \( L_{K_2} \)-modules

\[
[\text{Lie } Q^n, \text{Lie } Q^n] \cong V(-\alpha_{n-1} - 2\alpha_n) \oplus V(0).
\]

The subgroup \( H \) corresponding to \( \mathcal{S} \) has \( L = L_{K_2} \) and \( \text{Lie } H^n \) of codimension 1 in \( [\text{Lie } Q^n, \text{Lie } Q^n] \).

**5.2.** Let us consider the following minimal quotients \( \mathcal{S} \to \mathcal{S}_1 \to \mathcal{S}_2 \): the second one, \( \mathcal{S}_1 \to \mathcal{S}_2 \), is of type \( \mathcal{L} \) while the first one, \( \mathcal{S} \to \mathcal{S}_1 \), is not.

Here \( G \) is of type \( F_4 \). Let \( Q \) be the parabolic subgroup containing \( B_- \) corresponding to \( S \setminus \{\alpha_4\} \). The subgroup \( K_2 \) corresponding to \( \mathcal{S}_2 \) contains \( Q^r \) and \( K_2/Q^r \) is very reductive of type \( A_3 \) (or equivalently \( D_3 \)) in \( Q/Q^r \) which is of type \( B_3 \). To be
more precise, the root subsystem of $K_2/Q^\ast$ is generated by $\{\alpha_1, \alpha_2, \alpha_2 + 2\alpha_3\}$. As $L_\Delta$-modules

$$\text{Lie } Q^u \cong V(-\alpha_4) \oplus [\text{Lie } Q^u, \text{Lie } Q^u],$$

and the 8-dimensional $L_\Delta$-module $V(-\alpha_4)$ decomposes into two 4-dimensional $L_{K_2}$-submodules. The subgroup $K_1$ corresponding to $\mathcal{J}_1$ has $L_{K_1} = L_{K_2}$ and $\text{Lie } K_1^u$ as the $L_{K_2}$-complementary in $\text{Lie } Q^u$ of the $L_{K_2}$-simple submodule $V(-\alpha_4)$. The subgroup $H$ corresponding to $\mathcal{J}$ is the parabolic subgroup of $K_2$ containing $B_- \cap K_2$ corresponding to $\{\alpha_1, \alpha_2\}$.

5.3. Let us consider the following minimal quotient $\mathcal{J} \to \mathcal{J}/\Delta'$ which is not of type $\mathcal{Z}'$.

The subgroup $K$ corresponding to $\mathcal{J}/\Delta'$ is the symmetric subgroup of type $B_4$ of $G$, which is of type $F_4$. The subgroup $H$ corresponding to $\mathcal{J}$ is the parabolic subgroup of $K$ of semi-simple type $A_3$.

5.4. Let us consider the following quotients of type $\mathcal{Z}$, $\mathcal{J} \to \mathcal{J}_1 \to \mathcal{J}_2$.

Here $G$ is still of type $F_4$. Let $P$ be the parabolic subgroup containing $B_-$ corresponding to $\{\alpha_2\}$. The subgroup corresponding to $\mathcal{J}_2$ is $K_2 = K_2^u L$ with $L$ that differs from $L_P$ only by its connected center and $\text{Lie } K_2^u$ that consists of an $L$-complementary in $\text{Lie } P^u$ of an $L$-submodule $W_2$ diagonally embedded in $V(-\alpha_1) \oplus V(-\alpha_3)$. The subgroup corresponding to $\mathcal{J}_1$ is $K_1 = K_1^u L$ where $\text{Lie } K_1^u$ is the $L$-complementary in $\text{Lie } K_2^u$ of $V(-\alpha_4)$. The $L$-submodule $W_1 = [W_2, V(-\alpha_3 - \alpha_4)]$ of $\text{Lie } K_2^u$ is diagonally embedded in $V(-\alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4) \oplus V(-\alpha_2 - 2\alpha_3 - \alpha_4)$. The subgroup corresponding to $\mathcal{J}$ is $H = H^u L$ with $\text{Lie } H^u$ the $L$-complementary in $\text{Lie } K_1^u$ of $W_1$, containing $[\text{Lie } K_1^u, \text{Lie } K_1^u]$.

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