An approximate equivalence for the GNS representation of the Haar state of $SU_q(2)$

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July 11, 2024

**Abstract**

We use the crystallised $C^*$-algebra $C(SU_q(2))$ at $q = 0$ to obtain a unitary that gives an approximate equivalence involving the GNS representation on the $L^2$ space of the Haar state of the quantum $SU(2)$ group and the direct integral of all the infinite dimensional irreducible representations of the $C^*$-algebra $C(SU_q(2))$ for nonzero values of the parameter $q$. This approximate equivalence gives a $KK$ class via the Cuntz picture in terms of quasihomomorphisms as well as a Fredholm representation of the dual quantum group $\tilde{SU}_q(2)$ with coefficients in a $C^*$-algebra in the sense of Mishchenko.

**2020 AMS Subject Classification No.:** 20G42, 58B32, 46L67, 19K35

**Keywords.** Quantum groups, representations, approximate equivalence.

1 **Introduction**

The quantum group $SU_q(2)$ was studied in the context of Connes’ set up of spectral triples independently by Chakraborty & Pal \(^1\) and Dabrowski et al \(^7\). Subsequently Connes \(^5\) studied the equivariant spectral triple in \(^1\) for $SU_q(2)$ in great detail, in particular determined its dimension spectrum, established regularity and showed that the local index theorem proved by him and Moscovici applies to this spectral triple. His idea was then carried forward by a host of authors for studying equivariant spectral triples for certain quantum groups and their homogeneous spaces that were known at that time \((15, 8, 3, 14)\). A few years later,
Neshveyev & Tuset [13] constructed an equivariant Dirac operator for a large class of quantum groups, namely the $q$-deformations of all simple simply connected compact Lie groups.

However, despite the work of Neshveyev & Tuset setting the stage, further work on local index formulas did not take off. In particular regularity, discreteness of dimension spectrum and computation of local cyclic cocycles still remain open. This calls for a closer look at the papers by Connes [5] and Neshveyev & Tuset [13]. The most important idea used in Connes’ paper is the observation that if one substitutes $q = 0$ in the expressions for the actions of the generating elements $\alpha$ and $\beta$ on $L^2(SU_q(2))$, one gets a representation of the $C^*$-algebra $C(SU_q(2))$ at $q = 0$ (which is just the $C^*$-algebra if one replaces $q = 0$ in the defining relations of $C(SU_q(2))$), and a lot of simplifications occur. Connes was then able to connect the GNS representation $\lambda_q$ on the $L^2$ space of the Haar state with another faithful representation $\pi_q$ of $C(SU_q(2))$ that one gets by combining all the irreducible representations for these $C^*$-algebras, which have been characterized and listed by results of Soibelman [10]. Following [11], henceforth we will refer to this representation $\pi_q$ as the Soibelman representation. It is computationally much more tractable and acts on the Hilbert space $\ell^2(\mathbb{Z} \times \mathbb{N})$. The Dirac operator of Neshveyev & Tuset, on the other hand, comes from the classical Dirac through a twisting procedure, and while it has a sound conceptual origin, as Hilbert space operators, both this Dirac $D$ as well as the $C(G_q)$ elements are rather difficult to work with. Therefore, following Connes’ idea, it is reasonable to try to make use of the Soibelman representation.

With this in mind, we tried to see what is going on behind the computations in Connes’ paper at the $C^*$-algebra level. In the present paper, we make our observation precise and explicit. In particular, we prove a certain approximate equivalence between the GNS representation for the quantum $SU(2)$ group and an ampliation of the Soibelman representation. The key step in proving this approximate equivalence is obtaining the unitary that gives the equivalence. We obtain this unitary by using the crystallised quantised function algebra $C(SU_q(2))$ at $q = 0$. We obtain the unitary in Section 2 and prove the approximate equivalence in Section 3. In Section 4, we list a few consequences of this equivalence. The approximate equivalence also gives an element of the $KK$ group via the Cuntz picture. Further, we extend the notion of a Fredholm representation of a discrete group introduced by Mishchenko [12] and show that the approximate equivalence presents us with an example of a Fredholm representation for the dual quantum group $SU_q(2)$. Finally we describe how the approximate equivalence sets up a relation between the equivariant spectral triple constructed in [1] and another spectral triple studied by the authors in [2], which is precisely what lies behind Connes’ computations in [4].

**Notation:** $\mathcal{H}$ will denote a complex separable Hilbert space. $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(X)$ will denote the space of bounded linear maps on a Hilbert space $\mathcal{H}$ and the space of adjointable operators on a Hilbert $C^*$-module $X$ respectively. Similarly, $\mathcal{K}(\mathcal{H})$ and $\mathcal{K}(X)$ will denote the spaces of compact operators on them. We will denote the Toeplitz algebra by $\mathcal{T}$. For a real number $t$, $t_+$ and $t_-$ will denote its positive and negative parts respectively. Throughout the paper, $q$ will
denote a real number in the interval $(-1, 1)$ that will be assumed to be nonzero unless explicitly stated otherwise. For a positive integer $n$, we will denote by $g(n)$ the number $(1 - q^{2n})^{1/2}$.

2 Unitary equivalence at $q = 0$

2.1 Quantum $SU(2)$ group

To fix notation, let us give here a very brief description of the quantum $SU(2)$ group. The $C^*$-algebra associated with $SU_q(2)$, usually denoted by $C(SU_q(2))$, is the $C^*$-algebra generated by two elements $\alpha$ and $\beta$ satisfying the following relations:

\begin{align*}
\alpha^* \alpha + \beta^* \beta &= I, \\
\alpha \beta &= q \beta \alpha, \\
\alpha^* = q^2 \beta^* = I, \\
\alpha \beta^* &= q \beta^* \alpha, \\
\beta^* \beta &= \beta \beta^*.
\end{align*}

We will also use the symbol $A_q$ to denote this $C^*$-algebra and use $\alpha_q$ and $\beta_q$ for the generating elements of $A_q$ instead of just $\alpha$ and $\beta$. The quantum group structure is given by the coproduct $\Delta$ which is a unital $*$-homomorphism from $A_q$ to $A_q \otimes A_q$ given by

\begin{align*}
\Delta(\alpha_q) &= \alpha_q \otimes \alpha_q - q \beta_q^* \otimes \beta_q, \\
\Delta(\beta_q) &= \beta_q \otimes \alpha_q + \alpha_q^* \otimes \beta_q.
\end{align*}

For two continuous linear functionals $\rho_1$ and $\rho_2$ on $A_q$, one defines their convolution product by: $\rho_1 * \rho_2(a) = (\rho_1 \otimes \rho_2)\Delta(a)$. It is known [16] that $A_q$ admits a faithful state $h$, called the Haar state, that satisfies

\[ h * \rho(a) = h(a) \rho(I) = \rho * h(a) \]

for all continuous linear functionals $\rho$ and all $a \in A_q$. We will be concerned with the GNS space associated with this state.

2.2 Representation on $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z})$

Let $\{e_n\}_{n \in \mathbb{N}}$ be the canonical orthonormal basis for $\ell^2(\mathbb{N})$, where $\mathbb{N}$ denotes the set of nonnegative integers. Let us denote by $P_0$ the projection onto $\mathbb{C}e_0$, by $N$ the number operator on $\ell^2(\mathbb{N})$, and by $S$ the left shift:

\[ Ne_k = ke_k, \quad Se_k = \begin{cases} 
  e_{k-1} & \text{if } k \geq 1, \\
  0 & \text{if } k = 0.
\end{cases} \]

It is well-known and easy to show that as $z$ varies over the unit circle $S^1$, the following constitute all inequivalent infinite dimensional irreducible representations of the $C^*$-algebra $A_q$:

\[ \alpha_q \mapsto S \sqrt{1 - q^{2N}}, \quad \beta_q \mapsto z q^N. \]
The direct integral of these representations gives a faithful representation \( \pi_q \) of \( A_q \) on the Hilbert space \( \mathcal{H}_\pi := \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z}) \) given by

\[
\pi_q(\alpha_q) = S\sqrt{I - q^{2N}} \otimes I, \quad \pi_q(\beta_q) = q^N \otimes S, \tag{2.4}
\]

where we have used the same symbol \( S \) to denote the left shift \( e_k \mapsto e_{k-1} \) on \( \ell^2(\mathbb{Z}) \).

### 2.3 GNS representation on the \( L^2 \) space

Let \( \mathcal{H} \equiv L^2(SU_q(2)) \) denote the GNS space for the Haar state on \( A_q \). Let \( \lambda_q \) denote the GNS representation (i.e. the representation by left multiplication) of \( A_q \) on \( \mathcal{H} \). By the Peter-Weyl theorem for compact quantum groups, and the representation theory for the quantum group \( SU_q(2) \), it follows that \( \mathcal{H} \) has a natural orthonormal basis \( \{ e_{ij}^n : n \in \frac{1}{2}\mathbb{N}, i, j = -n, -n+1, \ldots, n-1, n \} \) where \( e_{ij}^n \)'s are the normalized matrix entries of the irreducible unitary (co-)representations of \( SU_q(2) \). Thus \( \mathcal{H} \) can be identified with \( \ell^2(\Gamma) \) where

\[
\Gamma = \left\{ (n, i, j) : n \in \frac{1}{2}\mathbb{N}, i, j \in \{-n, -n+1, \ldots, n-1, n\} \right\}.
\]

The representation \( \lambda_q \) of \( A_q \) can be written down explicitly using Clebsch-Gordan coefficients (see Equations (2.1–2.2), [1]) as follows:

\[
\lambda_q(\alpha_q) : e_{ij}^n \mapsto a_+(n, i, j)e_{i, j}^{n+\frac{1}{2}, \frac{1}{2}} + a_-(n, i, j)e_{i, j}^{n-\frac{1}{2}, \frac{1}{2}}, \tag{2.5}
\]

\[
\lambda_q(\beta_q) : e_{ij}^n \mapsto b_+(n, i, j)e_{i, j}^{n+\frac{1}{2}, -\frac{1}{2}} + b_-(n, i, j)e_{i, j}^{n-\frac{1}{2}, -\frac{1}{2}}, \tag{2.6}
\]

where

\[
a_+(n, i, j) = g^{2n+i+j+1}g(n-j+1)g(n-i+1)g(2n+2), \tag{2.7}
\]

\[
a_-(n, i, j) = g(n+j)g(n+i), \tag{2.8}
\]

\[
b_+(n, i, j) = -q^{n+j}g(n-j+1)g(n+i+1)g(2n+1), \tag{2.9}
\]

\[
b_-(n, i, j) = q^{n+i}g(n+j)g(n-i)g(2n+1). \tag{2.10}
\]

One can view the representations \( \lambda_q \) as acting on the single Hilbert space \( \ell^2(\Gamma) \) with an orthonormal basis \( \{ e_{ij}^n : n \in \frac{1}{2}\mathbb{N}, i, j = -n, -n+1, \ldots, n-1, n \} \). Thus for each \( q \neq 0 \), we have a faithful representation \( \pi_q \) of \( A_q \) acting on \( \mathcal{H}_\pi \) and another faithful representation \( \lambda_q \) acting on \( \ell^2(\Gamma) \). The actions of \( \alpha_q \) and \( \beta_q \) are shown in the following diagram, where the black double headed arrows represent the action of \( \alpha_q \) and the red arrows represent the action of \( \beta_q \). For both, the solid colored arrow stands for the second terms in (2.5) and (2.6) and the dashed arrow stands for the first terms in (2.5) and (2.6).
2.4 The crystallised $C^*$-algebra $A_0$

The relations (2.1) at $q = 0$, namely,

$$
\alpha_0^*\alpha_0 + \beta_0^*\beta_0 = I, \quad \alpha_0\beta_0 = 0,
\alpha_0^*\alpha_0 = I, \quad \alpha_0\beta_0^* = 0,
\beta_0^*\beta_0 = \beta_0\beta_0^*,
$$

(2.11)

generate a universal $C^*$-algebra $A_0$ which is isomorphic to the $C^*$-algebra $A_q \equiv C(SU_q(2))$ for $q \neq 0$. Analogous to the representations $\pi_q$ and $\lambda_q$ for $A_q$, the $C^*$-algebra $A_0$ has the following faithful representations on $\mathcal{H}_\pi$ and $\mathcal{H}$ respectively:

$$
\pi_0(\alpha_0) = S \otimes I, \quad \pi_0(\beta_0) = P_0 \otimes S.
$$

(2.12)

$$
\lambda_0(\alpha_0)e_{ij}^n = \begin{cases} 
e^{-\frac{i}{2}j-j^2} & \text{if } i, j > -n, \\
0 & \text{otherwise.}
\end{cases}
\lambda_0(\beta_0)e_{ij}^n = \begin{cases} e^{\frac{n+1}{2}i-j-j^2} & \text{if } i = -n, \\
e^{-\frac{n+1}{2}i-j-j^2} & \text{if } j = -n, \\
0 & \text{otherwise.}
\end{cases}
$$

(2.13)

Remark 2.1 As mentioned in the introduction, Pal & Giri [9] have introduced the notion of crystallisation of quantised function algebras for $q$-deformations of classical compact Lie groups in the type $A_n$ case and Matassa & Yuncken [11] for the general case. The $C^*$-algebra described above is the crystallised $C^*$-algebra according to their notions for the $SU_q(2)$ case. However, in this case the $C^*$-algebra is very easy to obtain and was described by Woronowicz in [16].

2.5 The unitary

In this section, we obtain the unitary that will be used in the main theorem for the equivalence. This is done by passing to $q = 0$ and studying the behaviour of the operators $\lambda_0(\alpha_0)$ and $\lambda_0(\beta_0)$.
As observed by Connes in [5], significant simplifications happen at \( q = 0 \): one term from (2.5) and (2.6) disappear and the actions of \( \alpha_0 \) and \( \beta_0 \) becomes simpler, as shown in the following diagram:

\[
\begin{array}{c}
\lambda_0(\alpha_0) \\
\lambda_0(\beta_0)
\end{array}
\]

Let us denote the sheet consisting of the right and rear face of the pyramid by \( \Gamma_0 \), i.e.

\[
\Gamma_0 = \left\{ (n, i, j) : n \in \frac{1}{2}\mathbb{N}, \ i, j \in \{-n, -n+1, \ldots, n-1, n\}, \ \max\{i, j\} = n \right\}.
\]

Then one can naturally identify \( \Gamma_0 \) with \( \mathbb{N} \times \mathbb{Z} \) as the next diagram illustrates, which means there is a unitary \( V_0 \) between \( \ell^2(\Gamma_0) \) and \( \ell^2(\mathbb{N} \times \mathbb{Z}) \). As the actions of \( \lambda_0(\alpha_0) \) and \( \lambda_0(\beta_0) \) keep the sheet \( \Gamma_0 \) invariant, through this unitary, they are equivalent to operators on \( \ell^2(\mathbb{N} \times \mathbb{Z}) \). In fact they turn out to be precisely \( \pi_0(\alpha_0) \) and \( \pi_0(\beta_0) \). Next, taking the sheet \( \Gamma_0 \) out, \( \Gamma \) is left with an identical replica of itself. Denote by \( \Gamma_1 \) by the union of the right and rear face of this remaining part. Continue in this fashion and define, for \( k \in \mathbb{N} \),

\[
\Gamma_k = \left\{ (n, i, j) : n \in \frac{1}{2}\mathbb{N}, \ n \geq \frac{k}{2}, \ i, j \in \{-n, -n+1, \ldots, n-1, n\}, \ \max\{i, j\} = n \right\}.
\]

Then just like \( \Gamma_0 \), each \( \Gamma_k \) is kept invariant by \( \lambda_0(\alpha_0) \) and \( \lambda_0(\beta_0) \), can naturally be identified with \( \mathbb{N} \times \mathbb{Z} \) through a unitary \( V_k \) and \( V_k\lambda_0(\alpha_0)V_k^* \) and \( V_k\lambda_0(\beta_0)V_k^* \) are precisely the operators \( \pi_0(\alpha_0) \) and \( \pi_0(\beta_0) \) on \( \ell^2(\mathbb{N} \times \mathbb{Z}) \).
Taking the direct sum of all these unitaries $V_k$, one gets a unitary $U$ from $\ell^2(\Gamma)$ to the space $\ell^2(N \times Z)$:

$$U \equiv \bigoplus V_k$$

In the remaining part of this section, we carry out the proof outlined above.

We will denote by $\mathcal{H}_{mult}$ the multiplicity space $\ell^2(N)$ in what follows. The unitary $U : \ell^2(\Gamma) \cong L_2(SU_q(2)) \rightarrow \mathcal{H}_{mult} \otimes \mathcal{H}_\pi$ is given by

$$U e_{ij}^n = \begin{cases} 
  e(n - j, n + i, j - i) & \text{if } i < j, \\
  (-1)^{j-i} e(n - i, n + j, j - i) & \text{if } i \geq j
\end{cases}$$

$$= (-1)^{(i \lor j) - i} e(n - (i \lor j), n + (i \land j), j - i).$$

(2.14)

We then have

$$U^* e(r, s, t) = \begin{cases} 
  e^{-\frac{s + |t|}{2} + r r + |t|} & \text{if } t > 0, \\
  (-1)^{|t|} e^{\frac{s + |t|}{2} + r r + |t|} & \text{if } t \leq 0
\end{cases}$$

$$= (-1)^t e^{-\frac{s + |t|}{2} + r r + |t|}.$$  

(2.15)
Theorem 2.2 For any $a \in A_0$, one has $U\lambda_0(a)U^* = I \otimes \pi_0(a)$.

Proof: We will show that the equality holds for the generating elements $\alpha_0$ and $\beta_0$. From (2.15), one has

$$U\lambda_0(\alpha_0)U^*e(r,s,t) = (-1)^{t_+}U\lambda_0(\alpha_0)e^{\frac{r}{2} + \frac{s}{2} + \frac{|t|}{2} - \frac{\delta}{2} - \frac{\delta + \delta + \delta}{2}}.$$ 

Note that

1. $-\frac{r}{2} - \frac{s}{2} - \frac{|t|}{2} = -\frac{r}{2} + \frac{s}{2} - \frac{t}{2}$ if and only if $s + t_\bot = 0$,
2. $-\frac{r}{2} - \frac{s}{2} - \frac{|t|}{2} = -\frac{r}{2} + \frac{s}{2} + \frac{t}{2}$ if and only if $s + t_\top = 0$,
3. $(-\frac{r}{2} + \frac{s}{2} - \frac{t}{2} - \frac{1}{2}) \lor (-\frac{r}{2} + \frac{s}{2} + \frac{t}{2} - \frac{1}{2}) = -\frac{r}{2} + \frac{s}{2} + \frac{|t|}{2} - \frac{1}{2}$,
4. $(-\frac{r}{2} + \frac{s}{2} - \frac{t}{2} - \frac{1}{2}) \land (-\frac{r}{2} + \frac{s}{2} + \frac{t}{2} - \frac{1}{2}) = -\frac{r}{2} + \frac{s}{2} - \frac{|t|}{2} - \frac{1}{2}$.

Therefore it follows that

$$U\lambda_0(\alpha_0)U^*e(r,s,t) = \begin{cases} (-1)^{t_-}Ue^{\frac{r}{2} + \frac{s}{2} + \frac{|t|}{2} - \frac{1}{2}} & \text{if } s \neq -t_- \text{ and } s \neq -t_+, \\ 0 & \text{otherwise} \end{cases} \quad \text{if } s \neq -t_- \text{ and } s \neq -t_+,$$

$$= \begin{cases} (-1)^{t_-} + (-\frac{r}{2} + \frac{s}{2} + \frac{|t|}{2} - \frac{1}{2}) - (-\frac{r}{2} + \frac{s}{2} + \frac{|t|}{2} - \frac{1}{2}) \left( (-\frac{r}{2} + \frac{s}{2} + \frac{|t|}{2} - \frac{1}{2}) + (-\frac{r}{2} + \frac{s}{2} - \frac{|t|}{2} - \frac{1}{2}), t \right) & \text{if } s \neq -t_- \text{ and } s \neq -t_+, \\ 0 & \text{otherwise} \end{cases}$$

$$= e(r,s-1,t) \quad \text{if } t \geq 0 \text{ and } s \neq 0,$$

$$= e(r,s-1,t) \quad \text{if } t < 0 \text{ and } s \neq 0,$$

$$= (I \otimes \pi_0(\alpha_0))e(r,s,t).$$

Similarly one has

$$U\lambda_0(\beta_0)U^*e(r,s,t) = (-1)^{t_+}U\lambda_0(\beta_0)e^{\frac{r}{2} + \frac{s}{2} + \frac{|t|}{2} - \frac{\delta}{2} - \frac{\delta + \delta + \delta}{2}}.$$ 

Note that

1. $-\frac{r}{2} - \frac{s}{2} - \frac{|t|}{2} = -\frac{r}{2} + \frac{s}{2} - \frac{t}{2}$ if and only if $s + t_- = 0$,
2. $-\frac{r}{2} - \frac{s}{2} - \frac{|t|}{2} = -\frac{r}{2} + \frac{s}{2} + \frac{t}{2}$ if and only if $s + t_+ = 0$. 

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Therefore

\[
U\lambda_0(\beta_0)U^*e(r,s,t) = \begin{cases}
0 & \text{if } s \neq 0, \\
(-1)^t_+\left(-\frac{a}{2} + \frac{s}{2} + \frac{|q|}{2} + \frac{1}{2}\right) - \left(-\frac{a}{2} + \frac{s}{2} + \frac{1}{2}\right) & \text{if } s = -t_+,
\end{cases}
\]

\[
\begin{cases}
(-1)^t_+\left(-\frac{a}{2} + \frac{s}{2} + \frac{|q|}{2} - \frac{1}{2}\right) - \left(-\frac{a}{2} + \frac{s}{2} + \frac{1}{2}\right) & \text{if } s = -t_+ \text{ and } s = -t_-,
\end{cases}
\]

\[
0 & \text{if } s = -t_+ \text{ and } s \neq -t_-.
\]

Observe that

1. one has \( s \neq -t_- \) and \( s \neq -t_+ \) if and only if \( s \neq 0 \),
2. one has \( s \neq -t_+ \) and \( s = -t_- \) if and only if \( s = 0 \) and \( t > 0 \),
3. \( s = -t_+ \) if and only if \( s = 0 \) and \( t \leq 0 \),
4. \( (-\frac{a}{2} + \frac{s}{2} + \frac{|q|}{2} + \frac{1}{2}) \lor (-\frac{a}{2} + \frac{s}{2} + \frac{1}{2}) = -\frac{a}{2} + \frac{s}{2} + (\frac{|q|}{2})_+ \),
5. \( (-\frac{a}{2} + \frac{s}{2} + \frac{|q|}{2} - \frac{1}{2}) \land (-\frac{a}{2} + \frac{s}{2} + \frac{1}{2}) = -\frac{a}{2} + \frac{s}{2} - (\frac{|q|}{2})_- \).

Therefore

\[
U\lambda_0(\beta_0)U^*e(r,s,t) = \begin{cases}
0 & \text{if } s \neq 0, \\
(-1)^t_+\left(-\frac{a}{2} + \frac{s}{2} + \frac{|q|}{2} + \frac{1}{2}\right) - \left(-\frac{a}{2} + \frac{s}{2} + \frac{1}{2}\right) & \text{if } s = -t_+ \text{ and } s = -t_-,
\end{cases}
\]

\[
\begin{cases}
(-1)^t_+\left(-\frac{a}{2} + \frac{s}{2} + \frac{|q|}{2} - \frac{1}{2}\right) - \left(-\frac{a}{2} + \frac{s}{2} + \frac{1}{2}\right) & \text{if } s = -t_+ \text{ and } s \neq -t_-.
\end{cases}
\]

Since we have \( U\lambda(a)U^* = (I \otimes \pi_0(a)) \) for \( a = \alpha_0 \) and \( a = \beta_0 \), the result follows. \( \square \)

3 Approximate equivalence for nonzero \( q \)

We now come to the main result of the paper, which says that the unitary that gave us equivalence in the \( q = 0 \) case gives an approximate unitary equivalence between \( \lambda_q \) and \( I \otimes \pi_q \) for \( q \neq 0 \).
Theorem 3.1 For any $a \in C(SU_q(2))$, one has

$$U\lambda_q(a)U^* - I \otimes \pi_q(a) \in \mathcal{T} \otimes \mathcal{K}(\mathcal{H}_\pi).$$ \hspace{1cm} (3.1)

Proof: Observe that it is enough to show that (3.1) holds for the generating elements $\alpha_q$ and $\beta_q$ of $A_q$. Let us first prove that

$$U\lambda_q(\alpha_q)U^* - I \otimes \pi_q(\alpha_q) \in \mathcal{T} \otimes \mathcal{K}(\mathcal{H}_\pi).$$ \hspace{1cm} (3.2)

Note that

$$U\lambda_q(\alpha_q)U^*e(r, s, t)$$

$$= (-1)^{t_r}U\lambda_q(\alpha_q)e^{rac{r+s+t}{2}}$$

$$= (-1)^{t_r}U\left(q^{2s+|t|+1}g(r + t_+ + 1)g(r + t_+ + 1) + \frac{r+s+t}{2} \frac{r+s+t+1}{2} \frac{r+s+t+1}{2} \right)$$

$$+ g(s + t_+)g(s + t_-) \frac{r+s+t}{2} \frac{r+s+t+1}{2} e(r, s - 1, t).$$ \hspace{1cm} (3.3)

Since $\max\{\frac{-r+s-t-1}{2}, \frac{-r+s-t+1}{2}\} = \frac{-r+s-t-1}{2}$ and $\min\{\frac{-r+s-t-1}{2}, \frac{-r+s-t+1}{2}\} = \frac{-r+s-t+1}{2}$, we have

$$U\lambda_q(\alpha_q)U^*e(r, s, t)$$

$$= q^{2s+|t|+1}g(r + t_+ + 1)g(r + t_+ + 1) + \frac{r+s+t}{2} \frac{r+s+t+1}{2} e(r, s - 1, t).$$ \hspace{1cm} (3.4)

For the representation $\pi$, one has

$$(I \otimes \pi_q(\alpha_q))e(r, s, t) = g(s)e(r, s - 1, t).$$ \hspace{1cm} (3.5)

From (3.3) and (3.4), we have

$$\left( U\lambda_q(\alpha_q)U^* - I \otimes \pi_q(\alpha_q) \right)e(r, s, t)$$

$$= q^{2s+|t|+1}g(r + t_+ + 1)g(r + t_+ + 1) + \frac{r+s+t}{2} \frac{r+s+t+1}{2} e(r, s - 1, t)$$

$$+ \left( g(s + t_+)g(s + t_-) - g(s) \right) e(r, s - 1, t).$$ \hspace{1cm} (3.5)

where $S$ is the left shift on $\ell^2(\mathbb{N})$ and

$$R_1e_{r,s,t} = q^{2s+|t|+1}g(r + t_+ + 1)g(r + t_+ + 1) + \frac{r+s+t}{2} \frac{r+s+t+1}{2} e(r, s, t),$$ \hspace{1cm} (3.6)

$$R_2e_{r,s,t} = \left( g(s + t_+ + 1)g(s + t_- + 1) - g(s + 1) \right) e(r, s, t).$$ \hspace{1cm} (3.7)
Define operators $R_3$ and $R_4$ on $H_\pi$ as follows:

\begin{align*}
R_3 e_{s,t} &= q^{2s+|t|+1} e(s,t), \\
R_4 e_{s,t} &= g(s + 1) (g(s + |t| + 1) - 1) e(s,t).
\end{align*}

It is straightforward to check that $R_3$ and $R_4$ are compact operators. We will show that $R_1 - (I \otimes R_3)$ and $R_2 - (I \otimes R_4)$ are both compact operators on $H_{\text{mult}} \otimes H_\pi$. It will then follow from (3.5) that $U \lambda_q(\alpha_q)U^* - I \otimes \pi_q(\alpha_q)$ is in $\mathcal{F} \otimes \mathcal{K}(H_\pi)$.

Observe that

\[ (R_1 - (I \otimes R_3)) e(r,s,t) = q^{2s+|t|+1} \left( \frac{g(r + t_- + 1)g(r + t_+ + 1)}{g(r + s + |t| + 1)g(r + s + |t| + 2)} - 1 \right) e(r,s,t). \]

We will now use the following estimates for the proof of this:

\[ |1 - g(k)| < q^{2k} \quad \text{for all } k \geq 1. \quad (3.10) \]

\[ \text{There exists } c > 0 \text{ such that } |1 - g(k)| < cq^{2k} \quad \text{for all } k \geq 1. \quad (3.11) \]

From the above estimates, it follows that

\[ \left| q^{2s+|t|+1} \left( \frac{g(r + t_- + 1)g(r + t_+ + 1)}{g(r + s + |t| + 1)g(r + s + |t| + 2)} - 1 \right) \right| = O(q^{2r+2s+|t|+1}), \]

so that

\[ R_1 - (I \otimes R_3) \in \mathcal{K}(H_{\text{mult}} \otimes H_\pi). \quad (3.12) \]

Similarly, we have

\[ (R_2 - (I \otimes R_4)) e(r,s,t) = \left( \frac{g(s + t_+ + 1)g(s + t_- + 1)}{g(r + s + |t| + 1)g(r + s + |t| + 2)} - g(s + 1) + g(s + 1)(1 - g(s + |t| + 1)) \right) e(r,s,t) \]
\[ = \left( \frac{g(s + 1)g(s + |t| + 1)}{g(r + s + |t| + 1)g(r + s + |t| + 2)} - g(s + 1)g(s + |t| + 1) \right) e(r,s,t). \]

Using (3.10) and (3.11), we obtain

\[ \left| \frac{g(s + 1)g(s + |t| + 1)}{g(r + s + |t| + 1)g(r + s + |t| + 2)} - g(s + 1)g(s + |t| + 1) \right| \]
\[ \leq g(s + 1)(1 - g(s + |t| + 1)) \left| 1 - \frac{1}{g(r + s + |t| + 1)g(r + s + |t| + 2)} \right| \]
\[ = O(q^{2r+2s+2|t|}). \]

Hence

\[ R_2 - (I \otimes R_4) \in \mathcal{K}(H_{\text{mult}} \otimes H_\pi). \quad (3.13) \]
Combining (3.5), (3.12) and (3.13), we have (3.2).

Next let us show that

\[ U \lambda_q(\beta_q)U^* - I \otimes \pi_q(\beta_q) \in \mathcal{T} \otimes \mathcal{K}(H_\pi). \] (3.14)

As before, note that

\[ U \lambda_q(\beta_q)U^* e(r, s, t) \]
\[ = (-1)^{t-r} U \lambda_q(\beta_q) e_{\frac{r+s-|t|}{2}, \frac{r+s+|t|}{2}} \]
\[ = (-1)^{t-r} U \left( -q^{s+t} \frac{g(r + t_+ + 1)g(s + t_- + 1)}{g(r + s + |t| + 1)g(r + s + |t| + 2)} e_{\frac{r+s-|t|+1}{2}, \frac{r+s+|t|+1}{2}} \right. \]
\[ + q^{s+t} \frac{g(s + t_+ + 1)g(r + t_+)}{g(r + s + |t|)g(r + s + |t| + 1)} e_{\frac{r+s+|t|-1}{2}, \frac{r+s+|t|-1}{2}} \).

Since \( \max\left\{ \frac{r+s-|t|+1}{2}, \frac{r+s+|t|-1}{2} \right\} = \frac{r+s+|t|-1}{2} \), we have

\[ U \lambda_q(\beta_q)U^* e(r, s, t) \]
\[ = \begin{cases} 
-q^{s+t} \frac{g(r+1)g(s+1)}{g(r+s+t+1)g(r+s+t+2)} e(r + 1, s + 1, t - 1) \\
+ q^{s} \frac{g(s+t)g(r+t)}{g(r+s+t)g(r+s+t+1)} e(r, s, t - 1) & \text{if } t \geq 1,
\end{cases} \] (3.15)
\[ q^{s} \frac{g(r+|t|+1)g(s+|t|+1)}{g(r+s+|t|+1)g(r+s+|t|+2)} e(r, s, t - 1) - q^{s+|t|} \frac{g(s)g(r)}{g(r+s+|t|)g(r+s+|t|+1)} e(r - 1, s - 1, t - 1) & \text{if } t < 1.
\]

For the representation \( \pi \),

\[ (I \otimes \pi_q(\beta_q))e(r, s, t) = q^s e(r, s, t - 1). \] (3.16)

From (3.15) and (3.16), we have

\[ \left( U \lambda_q(\beta_q)U^* - (I \otimes \pi_q(\beta_q)) \right) e(r, s, t) \]
\[ = \begin{cases} 
-q^{s+t} \frac{g(r+1)g(s+1)}{g(r+s+t+1)g(r+s+t+2)} e(r + 1, s + 1, t - 1) \\
+ q^{s} \left( \frac{g(s+t)g(r+t)}{g(r+s+t)g(r+s+t+1)} - 1 \right) e(r, s, t - 1) & \text{if } t \geq 1,
\end{cases} \]
\[ q^{s} \left( \frac{g(r+|t|+1)g(s+|t|+1)}{g(r+s+|t|+1)g(r+s+|t|+2)} - 1 \right) e(r, s, t - 1) - q^{s+|t|} \frac{g(s)g(r)}{g(r+s+|t|)g(r+s+|t|+1)} e(r - 1, s - 1, t - 1) & \text{if } t < 1.
\]
\[ = (T_1(S^* \otimes S^* \otimes S + S \otimes S \otimes S) + T_2(I \otimes I \otimes S))e(r, s, t), \] (3.17)
where $S$ denotes the left shift and

\[
T_1 e_{r,s,t} = \begin{cases} 
q^{s+|t|}\frac{g(r)g(s)}{g(r+s+|t|)g(r+s+|t|+1)}e(r,s,t) & \text{if } (r, s) \neq (0, 0), t \geq 0, \\
q^{s+|t|}\frac{g(r+1)g(s+1)}{g(r+s+|t|+1)g(r+s+|t|+2)}e(r,s,t) & \text{if } (r, s) \neq (0, 0), t < 0, \\
0 & \text{if } (r, s) = (0, 0)
\end{cases}
\]

where

\[
T_2 e_{r,s,t} = \begin{cases} 
q^{s}(\frac{g(r+|t|+1)g(s+|t|+1)}{g(r+s+|t|+1)g(r+s+|t|+2)} - 1)e(r,s,t) & \text{if } t \geq 0, \\
q^{s}(\frac{g(r+|t|)g(s+|t|)}{g(r+s+|t|)g(r+s+|t|+1)} - 1)e(r,s,t) & \text{if } t < 0.
\end{cases}
\]

Define operators $T_3$ and $T_4$ on $\mathcal{H}_\pi$ as follows:

\[
T_3 e_{s,t} = \begin{cases} 
-q^{s+|t|}g(s)e(r,s,t) & \text{if } (r, s) \neq (0, 0), t \geq 0, \\
-q^{s+|t|}g(s+1)e(r,s,t) & \text{if } (r, s) \neq (0, 0), t < 0, \\
0 & \text{if } (r, s) = (0, 0),
\end{cases}
\]

and

\[
T_4 e_{s,t} = \begin{cases} 
q^{s}(g(s+|t|+1) - 1)e(r,s,t) & \text{if } t \geq 0, \\
q^{s}(g(s+|t|) - 1)e(r,s,t) & \text{if } t < 0.
\end{cases}
\]

It is straightforward to check that $T_3$ and $T_4$ are compact operators on $\mathcal{H}_\pi$. We will show that $T_1 - (I \otimes T_3)$ and $T_2 - (I \otimes T_4)$ are both compact operators on $\mathcal{H}_{\text{mult}} \otimes \mathcal{H}_\pi$. It will then follow from (3.17) that $U\lambda_q(\beta_q)U^* - I \otimes \pi_q(\beta_q)$ is in $\mathcal{T} \otimes \mathcal{K}(\mathcal{H}_\pi)$.

Observe that

\[
\left(T_1 - (I \otimes T_3)\right)e(r,s,t) = \begin{cases} 
-q^{s+|t|}g(s)\left(\frac{g(r)}{g(r+s+|t|)g(r+s+|t|+1)} - 1\right)e(r,s,t) & \text{if } (r, s) \neq (0, 0), t \geq 0, \\
-q^{s+|t|}g(s+1)\left(\frac{g(r+1)}{g(r+s+|t|+1)g(r+s+|t|+2)} - 1\right)e(r,s,t) & \text{if } (r, s) \neq (0, 0), t < 0, \\
0 & \text{if } (r, s) = (0, 0).
\end{cases}
\]

From the estimates in (3.10) and (3.11), it follows that the right hand side above is $O(q^{r+s+|t|})$. Hence

\[
T_1 - (I \otimes T_3) \in \mathcal{K}(\mathcal{H}_{\text{mult}} \otimes \mathcal{H}_\pi).
\]

Similarly, one has

\[
\left(T_2 - (I \otimes T_4)\right)e(r,s,t) = \begin{cases} 
q^{s}g(s+|t|)\left(\frac{g(r+|t|+1)}{g(r+s+|t|+2)} - 1\right)e(r,s,t) & \text{if } t \geq 0, \\
q^{s}g(s+|t|)\left(\frac{g(r+|t|)}{g(r+s+|t|+1)} - 1\right)e(r,s,t) & \text{if } t < 0.
\end{cases}
\]

Using (3.10) and (3.11), we conclude that the right hand side is $O(q^{r+s+|t|})$. Therefore

\[
T_2 - (I \otimes T_4) \in \mathcal{K}(\mathcal{H}_{\text{mult}} \otimes \mathcal{H}_\pi).
\]

Thus combining (3.17), (3.22) and (3.23), we get (3.14).
4 Applications

We now describe three different contexts in noncommutative topology and geometry where the main result of the last section can be used.

4.1 $KK$-groups

The first is in the context of $KK$ theory. The approximate equivalence tells us that

$$U_{\lambda}(a)U^* - I \otimes \pi_q(a) \in \mathcal{T} \otimes K(\mathcal{H}_\pi) \text{ for all } a \in A_q.$$  \hspace{1cm} (4.1)

Since $I \otimes \pi_q(a) \in \mathcal{T} \otimes \mathcal{L}(\mathcal{H}_\pi)$ and $\mathcal{T} \otimes \mathcal{L}(\mathcal{H}_\pi) \subseteq M(\mathcal{T} \otimes K(\mathcal{H}_\pi))$, one gets

$$U_{\lambda}(a)U^* \in M(\mathcal{T} \otimes K(\mathcal{H}_\pi)) \text{ for all } a \in A_q.$$  \hspace{1cm} (4.2)

It now follows that the pair $(U_{\lambda}(\cdot)U^*, I \otimes \pi_q(\cdot))$ gives a quasihomomorphism in the Cuntz description (see [6]) of the $KK$ group $KK(A_q, \mathcal{T})$. In other words, we have the following theorem.

**Theorem 4.1** There is a unitary $U : L^2(SU_q(2)) \rightarrow H_{mult} \otimes H_{\pi}$ such that $(U_{\lambda}(\cdot)U^*, \pi_q)$ is a quasihomomorphism and gives a $KK$-class in the group $KK(A_q, \mathcal{T})$.

4.2 Fredholm representation of $\widehat{SU_q(2)}$

Next we will show that the above approximate equivalence also gives an example of a Fredholm representation of the dual quantum group $SU_q(2)$. Let us first recall the notion of a Fredholm representation of a discrete group.

**Definition 4.2** (Mishchenko [12]) Let $\Gamma$ be a discrete group. A pair of unitary representations $\pi_1$ and $\pi_2$ of $\Gamma$ acting on a Hilbert space $\mathcal{H}$ together with a Fredholm operator $F \in \mathcal{L}(\mathcal{H})$ is called a Fredholm representation of $\Gamma$ if $F\pi_1(g) - \pi_2(g)F \in K(\mathcal{H})$ for all $g \in \Gamma$.

This notion admits an immediate extension to discrete quantum groups where the representations $\pi_1$ and $\pi_2$ will be elements of $M(\hat{A} \otimes K(\mathcal{H}))$, where $\hat{A}$ stands for the $C^*$-algebra associated with the quantum group and the condition $F\pi_1(g) - \pi_2(g)F \in K(\mathcal{H})$ gets replaced by $(I \otimes F)\pi_1 - \pi_2(I \otimes F) \in \hat{A} \otimes K(\mathcal{H})$.

We now extend this notion further and incorporate coefficients from a $C^*$-algebra $B$.

**Definition 4.3** Assume that the Hilbert space $\mathcal{H}$ on which the two representations act is of the form $\mathcal{H}_1 \otimes \mathcal{H}_2$, and that $B$ is a $C^*$-subalgebra of $\mathcal{L}(\mathcal{H}_1)$ such that one has

$$\pi_1, \pi_2 \in M(\hat{A} \otimes B \otimes K(\mathcal{H}_2)).$$
We call \((\pi_1, \pi_2, F)\) a **Fredholm representation with coefficients in** \(B\) if

\[
(I \otimes F)\pi_1 - \pi_2(I \otimes F) \in \hat{A} \otimes \mathcal{K}(B \otimes \mathcal{H}),
\]

where \(B \otimes \mathcal{H}\) denotes the standard Hilbert \(B\)-module \(\ell^2(B)\).

Given a representation \(\pi\) of the \(C^*\)-algebra \(A = C(G)\) for a compact quantum group \(G\), one can associate a co-representation of the dual quantum group \(\hat{G}\) with its opposite coproduct as follows. Let \(\Gamma\) be the set of equivalence classes of unitary representations of the compact quantum group \(G\). Denote by \(u^{(\gamma)}\) the irreducible unitary representation indexed by \(\gamma \in \Gamma\) acting on a finite dimensional Hilbert space \(\mathcal{H}_\gamma\). Thus \(u^{(\gamma)} \in \mathcal{L}(\mathcal{H}_\gamma) \otimes A\). Let \(\hat{A}\) denote the \(c_0\)-direct sum \(\bigoplus_{\gamma \in \Gamma} \mathcal{L}(\mathcal{H}_\gamma)\). Then \(u := \bigoplus_{\gamma} u^{(\gamma)}\) is a unitary element of \(M(\hat{A} \otimes A)\) and satisfies the following two identities:

\[
(id \otimes \Delta_G)u = u_{1,2}u_{1,3}, \quad (\Delta_{\hat{G}} \otimes id)u = u_{2,3}u_{1,3}.
\]

Now define \(w_\pi := (id \otimes \pi)u\). Then \(w_\pi\) is a unitary element of \(M(\hat{A} \otimes \mathcal{H})\) and satisfies \((\Delta_{\hat{G}} \otimes id)w_\pi = (w_\pi)_{2,3}(w_\pi)_{1,3}\). Thus the element \(w_\pi\) gives a unitary (co-)representation of the dual quantum group \(\hat{G}\), which has the same set of intertwiners as \(\pi\).

Next, note that

\[
\mathcal{F} \otimes \mathcal{K}(\mathcal{H}_\pi) \cong \mathcal{K}(\mathcal{F} \otimes \mathcal{H}_\pi), \quad M(\mathcal{F} \otimes \mathcal{K}(\mathcal{H}_\pi)) \cong \mathcal{L}(\mathcal{F} \otimes \mathcal{H}_\pi),
\]

where \(\mathcal{K}(\mathcal{F} \otimes \mathcal{H}_\pi)\) and \(\mathcal{L}(\mathcal{F} \otimes \mathcal{H}_\pi)\) denote the space of compact operators and the space of adjointable operators respectively on the Hilbert \(\mathcal{F}\)-module \(\mathcal{F} \otimes \mathcal{H}_\pi\). Thus we have for all \(a \in A_q\),

\[
U_{\lambda_q}(a)U^*, \quad I \otimes \pi_q(a) \in \mathcal{L}(\mathcal{F} \otimes \mathcal{H}_\pi), \quad U_{\lambda_q}(a)U^* - I \otimes \pi_q(a) \in \mathcal{K}(\mathcal{F} \otimes \mathcal{H}_\pi).
\]

If we denote by \(w_\lambda\) and \(w_\pi\) the co-representations of the dual quantum group \(\widehat{SU_q(2)}\) corresponding to the representations \(U_{\lambda_q}(\cdot)U^*\) and \(I \otimes \pi_q(\cdot)\) respectively of the \(C^*\)-algebra \(A_q\), and by \(\widehat{A_q}\) the \(C^*\)-algebra associated with the dual \(SU_q(2)\), then \(w_\lambda\) and \(w_\pi\) are unitary elements of \(M(\widehat{A_q} \otimes \mathcal{K}(\ell^2(\mathbb{N}) \otimes \mathcal{H}_\pi))\). It follows from (4.1) that

\[
w_\lambda - w_\pi \in \widehat{A_q} \otimes \mathcal{K}(\mathcal{F} \otimes \mathcal{H}_\pi).
\]

Thus \((w_\lambda, w_\pi, I)\) gives a Fredholm representation of the dual \(\widehat{SU_q(2)}\) with coefficients in the Toeplitz algebra.
4.3 Spectral triples

The spectral triples corresponding to the two representations $\lambda_q$ and $\pi_q$ given in [2] and [1] are $(\ell^2(\mathbb{N} \times \mathbb{Z}), \pi_q, D_\pi)$ and $(L^2(SU_q(2)), \lambda_q, D_\lambda)$, and one has

$$U\lambda_q(a)U^* = I \otimes \pi_q(a) \text{ modulo } \mathcal{F} \otimes \mathcal{K}(\ell^2(\mathbb{N} \times \mathbb{Z})), \quad U|D_\lambda|U^* = I \otimes |D_\pi| + N \otimes I,$$

$$U(\text{sign } D_\lambda)U^* = P_0 \otimes (\text{sign } D_\pi) - (I - P_0) \otimes I.$$

One can now establish that the differences $U\lambda_q(a)U^* - I \otimes \pi_q(a)$ belong to certain finer ideals of the compacts for specific collections of elements $a \in A_q$ and this enables one to use the knowledge of regularity and dimension spectrum of the spectral triple in [2] to draw similar conclusions on the spectral triple in [1].

**Remark 4.4** For odd dimensional quantum spheres $S^{2n+1}_q$, which are the homogeneous spaces $SU_q(n+1)/SU_q(n)$, using the $C^*$-algebra $C(S^{2n+1}_q)$ at $q = 0$ one can prove an approximate equivalence between the representation of $C(S^{2n+1}_q)$ on the $L^2$-space of the invariant state and a faithful representation that is easier to work with computationally. In this case, if one works with the basis for the $L^2$-space that comes from the GT basis for the $L^2$-space of the Haar State on $SU_q(n+1)$, then the resulting $\Gamma$, though more complicated than the $SU_q(2)$ case, is very similar and can be handled in essentially the same way to arrive at the unitary. The approximate equivalence can then be derived in a very similar manner as in the present case.

**Remark 4.5** It is worthwhile to make a remark at this point about crystallisations of the $C^*$-algebras $C(G_q)$ where $G_q$ stands for the $q$-deformation of a connected simply connected compact Lie group. In two recent papers, Pal & Giri [9] and Matassa & Yuncken [11] introduced the notion of crystallisation of these $C^*$-algebras. The paper [9] then focuses on studying the irreducible representations of the crystallised $C^*$-algebra in the type $A$ case, while in [11] the authors exploit the link with Kashiwara’s crystal basis theory to study certain properties of the crystallised $C^*$-algebras. Our present paper is an illustration of how these crystallised $C^*$-algebras can be useful in the type $A_1$ case. One can hope to obtain similar decomposition results for higher rank cases and for deformations of Lie groups of other types ($B$, $C$, $D$ etc.) using the crystallised $C^*$-algebras introduced in [9] and [11], which in turn should pave the way for a detailed study of the Neshveyev-Tuset Dirac operators for $q$-deformations of Lie groups of higher ranks.

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