NON-UNITAL ALGEBRAIC $K$-THEORY AND ALMOST MATHEMATICS

YUKI KATO

ABSTRACT. The Gersten conjecture is still an open problem of algebraic $K$-theory for mixed characteristic discrete valuation rings. In this paper, we establish non-unital algebraic $K$-theory which is modified to become an exact functor from the category of non-unital algebras to the stable $\infty$-category of spectra. We prove that for any almost unital algebra, the non-unital $K$-theory homotopically decomposes into the non-unital $K$-theory the corresponding ideal and the residue algebra, implying the Gersten property of non-unital $K$-theory of the the corresponding ideal.

1. Introduction

Let $V$ be a Noetherian regular local ring and $F$ denote the fractional field. Then the inclusion $j : V \to F$ induces the pullback

$$j^* : K_n(V) \to K_n(F)$$

of algebraic $K$-groups for any integer $n$. Gersten [Ger73] conjectured the induced homomorphism $j^*$ is injective for any integer $n$. This conjecture is called the Gersten conjecture for algebraic $K$-theory. In the case of positive characteristic, Panin [Pan03] proved Gersten’s conjecture is true by using Quillen’s result [Qui73] of the case $V$ is essentially smooth over a field and Popescu’s result [Pop86]. In the mixed characteristic case, the Gersten conjecture is still open.

This paper considers the Gersten conjecture for valuation rings $V$ with idempotent maximal ideals and non-unital algebras for non-unital algebraic $K$-theory $\text{K}^\text{nu}$ which is modified to become an exact functor from the category of non-unital algebras to the stable $\infty$-category of spectra. Let $m$ be an idempotent maximal ideal of $V$ and $A$ an almost $V$-algebra. We assume that $m$ is a flat $V$-module. We prove that the non-unital $K$-theory $\text{K}^\text{nu}(A)$ homotopically decompose into the non-unital $K$-theory of the ideal $mA$ and of the residue algebra $A/mA$ (Theorem 3.4):

$$\text{K}^\text{nu}(A) \cong \text{K}^\text{nu}(mA) \oplus \text{K}^\text{nu}(A/mA),$$

implying that one has a decomposition

$$\text{K}^\text{nu}(A \otimes_V F) \cong \text{K}^\text{nu}(mA) \oplus \text{K}^\text{nu}((A \otimes_V F)/mA),$$

where $\text{K}^\text{nu}((A \otimes_V F)/mA)$ denotes the homotopy cofiber of the map $\text{K}^\text{nu}(mA) \to \text{K}^\text{nu}(A \otimes_V F)$ of spectra. In particular, the induced map $\text{K}^\text{nu}(m) \to \text{K}^\text{nu}(F)$ is homotopically split injective.

Date: February 28, 2023.

Key words and phrases. Algebraic $K$-theory, non-unital algebra, perfectoid algebra.
2. NON-UNITAL ALGEBRAIC K-THOERY

2.1. A review of non-unital rings. Let $A$ be a commutative ring. Then the category of $A$-modules is equivalent to the category of $\mathbb{Z} \oplus A$-modules.

**Proposition 2.1 ([Qui96]).** If $A$ has a multiplicative unit $1_A$, then one has a categorical equivalence $\text{Mod}_{\mathbb{Z} \oplus A} \simeq \text{Mod}_\mathbb{Z} \times \text{Mod}_A$ which induces an equivalence $\text{PMod}_{\mathbb{Z} \oplus A} \simeq \text{PMod}_\mathbb{Z} \times \text{PMod}_A$.

**proof.** The distinguish element $e = (0, 1_A) \in \mathbb{Z} \oplus A$ is clearly a projector. For any $\mathbb{Z} \oplus A$-module $M$, $M$ decomposes into $M \simeq eM \oplus (1 - e)M$. If $M$ is a projective $\mathbb{Z} \oplus A$-module, $eM$ and $(1 - e)M$ are also projective $\mathbb{Z} \oplus A$-modules. The module $eM$ is killed by $\mathbb{Z}$, having an $A$-module structure. Hence $eM$ is also a projective $A$-module. qed

**Definition 2.2.** Let $A$ be a commutative ring. The $K(A)$ is defined to be the homotopy fiber of the map $K(\mathbb{Z} \oplus A) \to K(\mathbb{Z})$ induced by the projection.

If $A$ has a multiplicative unit $1_A$, then $K(A)$ is an ordinal $K$-theory. Indeed, the functor $A \otimes_{\mathbb{Z} \oplus A} - : \text{Mod}_{\mathbb{Z} \oplus A} \to \text{Mod}_A$ induces a projector $K(\mathbb{Z} \oplus A) \to K(A)$ of spectra.

**Theorem 2.3 ([Wei89] Theorem 2.1 (Excision for ideals)).** Let $I$ be an ideal of a ring $A$. Then $KH(I) \to KH(A) \to KH(A/I)$ is a fiber sequence. □

In order to use the above property of homotopy invariant $K$-theory for any exact sequence of non-unital algebra, we adjust the $K$-theory spectrum as the following: Let $\text{Sp}$ denote the stable $\infty$-category of spectra and $\text{Fun}^{\text{rex}}(\text{CAlg}^{\text{nu}}, \text{Sp})$ the full subcategory of $\text{Fun}(\text{CAlg}^{\text{nu}}, \text{Sp})$ spanned by right exact functors. Then the inclusion $\text{Fun}^{\text{rex}}(\text{CAlg}^{\text{nu}}, \text{Sp}) \to \text{Fun}(\text{CAlg}^{\text{nu}}, \text{Sp})$ admits a left adjoint

$$P_1 : \text{Fun}(\text{CAlg}^{\text{nu}}, \text{Sp}) \to \text{Fun}^{\text{rex}}(\text{CAlg}^{\text{nu}}, \text{Sp})$$

by [Lur17, p.759, Theorem 6.1.1.10].

**Lemma 2.4.** Let $\mathcal{C}$ be a stable $\infty$-category. Given two fiber sequences $X \to Y \to Z$ and $X \to Y' \to Z'$ of objects of $\mathcal{C}$ and a homotopically commutative square:

![Diagram]

the square is both homotopy Cartesian and coCartesian, and the following properties are equivalent:

1. The object $Y$ homotopically decomposes into the direct sum of $X$ and $Z$.
2. The object $Y'$ homotopically decomposes into the direct sum of $X$ and $Z'$. 2
proof. Note that, in a stable \( \infty \)-category, a homotopy coCartesian product is also a homotopy Cartesian product. By a formal argument, one has \( Z \simeq Y' \amalg_{Y} Z \): Indeed, one has a chain of weak equivalences: \( Y' \amalg_{Y} Z \simeq Y' \amalg_{Y} (Y \amalg_{Y} 0) \simeq Y' \amalg_{Y} 0 \simeq Z' \).

If condition (1) holds, \( Y \) is weakly equivalent to the coproduct of \( X \) and \( Z \). Hence, we have a chain of weak equivalences: \( X \amalg Z' \simeq X \amalg (Z \amalg Y') \simeq (X \amalg Z) \amalg Y' \simeq Y \amalg Y' \simeq Y' \)

Conversely, condition (2) implies \( Y' \simeq X \times Z' \), giving a chain of equivalences: \( Y \simeq Y' \times_{Z'} Z \simeq (X \times Z') \times_{Z'} Z \simeq X \times X \).

By Lemma 2.4, we obtain the following proposition:

**Proposition 2.5.** Let \( m \subset V \subset F \) be a sequence of commutative rings. For any \( V \)-algebra \( A \), write \( A_{F} = A \otimes_{V} F \). Let \( \mathcal{F} : \text{CAlg}^{nu} \to \text{Sp} \) be a functor. Then the spectrum \( P_{1}(\mathcal{F})(A) \) decomposes into a direct sum of \( P_{1}(\mathcal{F})(mA) \) and \( P_{1}(\mathcal{F})(A/mA) \) if and only if \( P_{1}(\mathcal{F})(A_{F}) \) has also a decomposition into a direct sum of \( P_{1}(\mathcal{F})(mA) \) and \( P_{1}(\mathcal{F})(A_{F}/mA) \), where \( P_{1}(\mathcal{F})(A_{F}/mA) \) denotes the homotopy cofiber of \( P_{1}(\mathcal{F})(mA) \to P_{1}(\mathcal{F})(A_{F}) \).

**Corollary 2.6.** Let \( m \subset V \subset F \) be a sequence of commutative rings. Assume that \( P_{1}(\mathcal{F})(A) \) decomposes into a direct sum of \( P_{1}(\mathcal{F})(mA) \) and \( P_{1}(\mathcal{F})(A/mA) \). Then the pullback \( j^{*} : P_{1}(\mathcal{F})(A) \to P_{1}(\mathcal{F})(A_{F}) \) induces an injection \( \pi_{n}(P_{1}(\mathcal{F})(A)) \to \pi_{n}(P_{1}(\mathcal{F})(A_{F})) \) for each \( n \geq 0 \) if and only if the induced map \( \pi_{n}(P_{1}(\mathcal{F})(A/mA)) \to \pi_{n}(P_{1}(\mathcal{F})(A_{F}/mA)) \) is injective for each \( n \geq 0 \).

### 3. Non-unital \( K \)-theory of almost mathematics

#### 3.1. The almost Gersten property of non-unital \( K \)-theory

Let \( V \) be a commutative unital ring with an idempotent ideal \( m \). A \( V \)-module \( M \) is said to be *almost zero* if \( mM = 0 \). A \( V \)-homomorphism \( f : M \to N \) of \( V \)-modules is called an *almost isomorphism* if both the kernel and the cokernel of \( f \) are almost zero. An *almost \( V \)-module* is an object of the (bi)localization of the category of \( V \)-modules by the Serre subcategory spanned by almost zero modules, and an *almost \( V \)-algebra* is a commutative algebra object of the category of almost \( V \)-modules. Write \( \tilde{m} = m \otimes_{V} m \). In this section, we always assume that \( m \) is a flat \( V \)-module. Then the multiplication \( \tilde{m} \to m \) is an isomorphism.

We define (recall) of the definition of almost \( K \)-theory [Kat22]. Let \( A \) be an almost \( V \)-algebra and \( \text{APerf}(A) \) denote the full subtriangulated category generated by \( \tilde{m} \otimes_{V} A \) of the derived category of \( A \)-modules. Further, let \( \text{Perf}^{+}(A) \) denote the full subtriangulated category generated by \( A \) and \( \tilde{m} \otimes_{V} A \) of the derived category of \( A \)-modules.

**Definition 3.1.** Let \( A \) be an almost \( V \)-algebra and \( K^{nu}(A)^{al} \) denote the non-unital \( K \)-theory spectrum of the triangulated category of \( \text{APerf}(A) \). We call \( K^{nu}(A)^{al} \) the *non-unital almost \( K \)-theory spectrum* of \( A \). Further, let \( K^{nu}(A)^{+} \) denote the non-unital \( K \)-theory spectrum of the triangulated category \( \text{Perf}^{+}(A) \).
Consider a diagram $B \leftarrow A \rightarrow C$ of non-unital $V$-algebras. Let $K$ denote the kernel the argumentation $(V \oplus B) \otimes_{V \oplus A} (V \oplus C) \rightarrow V$. Then the unitalization $V \oplus K$ is isomorphic to $(V \oplus B) \otimes_{V \oplus A} (V \oplus A)$ by the equivalence $V \oplus (-)$ from the category of non-unital $V$-algebras to the category of unital augmented $V$-algebras. Therefore $K$ represents the colimit of the diagram $B \leftarrow A \rightarrow C$. We write $B \square_A C = K$ for the colimit of $B \leftarrow A \rightarrow C$. By definition of non-unital $K$-theory, the induced square

$$
\begin{array}{ccc}
K^\text{nu}(A) & \rightarrow & K^\text{nu}(B) \\
\downarrow & & \downarrow \\
K^\text{nu}(C) & \rightarrow & K^\text{nu}(B \square_A C)
\end{array}
$$

is homotopy coCartesian.

**Lemma 3.2.** Let $A$ be an almost $V$-algebra and $K^\text{nu}(A/\mathfrak{m}A)$ (resp. $K^\text{nu}(A/\mathfrak{m}A)^+$) denote the homotopy cofiber of $K^\text{nu}(\mathfrak{m}A) \rightarrow K^\text{nu}(A)$ (resp. $K^\text{nu}(\mathfrak{m}A)^+ \rightarrow K^\text{nu}(A)^+$). Then the augmentation $V \otimes_{V \oplus A} (V \oplus A) \rightarrow V$ is an almost isomorphism. Furthermore, the induced map $K^\text{nu}(A/\mathfrak{m}A) \rightarrow K^\text{nu}(A/\mathfrak{m}A)^+$ is a weak equivalence of spectra.

**proof.** The proof is a formal argument: One has a chain of isomorphisms

$$\tilde{\mathfrak{m}} \otimes_V (V \otimes_{V \oplus A} (V \oplus A) \simeq \tilde{\mathfrak{m}} \otimes_{V \oplus \mathfrak{m}A} (V \oplus A) \oplus (\tilde{\mathfrak{m}} \otimes_{V \oplus \mathfrak{m}A} A)$$

$$\simeq (\tilde{\mathfrak{m}} \otimes_{V \oplus \mathfrak{m}A} V) \oplus (\tilde{\mathfrak{m}} \otimes_{V \oplus \mathfrak{m}A} \mathfrak{m}A) \simeq \tilde{\mathfrak{m}} \otimes_{V \oplus \mathfrak{m}A} (V \oplus \mathfrak{m}A) \simeq \tilde{\mathfrak{m}}.$$

The kernel $A \square_{\mathfrak{m}A} 0$ of the augmentation $V \otimes_{V \oplus \mathfrak{m}A} (V \oplus A) \rightarrow V$ represents the homotopy cofiber $K^\text{nu}(A/\mathfrak{m}A)$ (resp. $K^\text{nu}(A/\mathfrak{m}A)^+$). Then the projection $A \square_{\mathfrak{m}A} 0 \rightarrow (A \square_{\mathfrak{m}A} 0) \otimes_V V/\mathfrak{m}$ is an isomorphism. Therefore $K(A \square_{\mathfrak{m}A} 0) \rightarrow K(V/\mathfrak{m}) \otimes_{V \oplus \mathfrak{m}A} (V \oplus A)) \rightarrow K(V/\mathfrak{m})$ is a homotopy fiber sequence. Since the inclusion functors $\text{Perf}((V/\mathfrak{m}) \otimes_{V \oplus \mathfrak{m}A} (V \oplus A)) \rightarrow \text{Perf}^+(V/\mathfrak{m}) \otimes_{V \oplus \mathfrak{m}A} (V \oplus A))$ and $\text{Perf}(V/\mathfrak{m}) \rightarrow \text{Perf}^+(V/\mathfrak{m})$ are categorical equivalence, the induced morphism $K^\text{nu}(A \square_{\mathfrak{m}A} 0) \rightarrow K^\text{nu}(A \square_{\mathfrak{m}A} 0)^+$ is a weak equivalence. \hfill $\square$

**Theorem 3.3** (c.f. [Kat22] Theorem 3.15 and Theorem 3.21). Let $A$ be an almost $V$-algebra. Then the non-unital $K$-theory $K^\text{nu}(A)^+$ homotopically decomposes into the product of $K^\text{nu}(A)^\text{al}$ and $K^\text{nu}(A/\mathfrak{m}A)$.

**proof.** The proof is a similar argument of the proof of [Kat22] Theorem 3.11]. The functor $\tilde{\mathfrak{m}} \otimes_V (-) : \text{Perf}^+(A) \rightarrow \text{Perf}^+(A)$ is categorical idempotent, and the essential image is equivalent to $A\text{Perf}(A)$. Therefore the functor $\tilde{\mathfrak{m}} \otimes_V (-) : \text{Perf}^+(A) \rightarrow \text{Perf}^+(A)$ induces a homotopically splitting $K(\tilde{\mathfrak{m}} \otimes_V (-)) : K^\text{nu}(A)^\text{al} \rightarrow K^\text{nu}(A)^+$ and a decomposition $K^\text{nu}(A)^+ \simeq K^\text{nu}(A)^\text{al} \oplus K^\text{nu}(A)^m$, where $K^\text{nu}(A)^m$ denotes the homotopy fiber of $K^\text{nu}(A)^+ \rightarrow K^\text{nu}(A)^\text{al}$.

Let $K^\text{nu}(A/\mathfrak{m}A)^+$ (resp. $K^\text{nu}(A/\mathfrak{m}A)^\text{al}$) denote the homotopy cofiber of $K^\text{nu}(\mathfrak{m}A)^+ \rightarrow K^\text{nu}(A)^+$ (resp. $K^\text{nu}(\mathfrak{m}A)^\text{al} \rightarrow K^\text{nu}(A)^\text{al}$). By Lemma 3.2, $K^\text{nu}(A/\mathfrak{m}A)^\text{al}$ is contractible, entailing that $K^\text{nu}(\mathfrak{m}A)^\text{al} \rightarrow K^\text{nu}(A)^\text{al}$ is a weak equivalence.
Next, we show that the induced map $K^\nu(mA)^+ \to K^\nu(mA)^{al}$ is a weak equivalence. Consider the unitalization $V \oplus mA$ and the application $\varepsilon : V \oplus mA \to V$. Then $K^\nu(V \oplus mA)^+$ is decomposed into the direct sum $K^\nu(mA)^+ \oplus K^\nu(V)^+$. Note that $E \simeq E \otimes_{V \oplus mA} (V \oplus mA) \simeq (E \otimes_{V \oplus mA} V) \oplus (E \otimes_{V \oplus mA} mA)$ for any $V \oplus mA$-complex $E$. Therefore those projections $K^\nu(V \oplus mA)^+ \to K^\nu(V)^+$ and $K^\nu(V \oplus mA)^+ \to K^\nu(mA)^+$ is induced by those functors $(-) \otimes_{V \oplus mA} V$ and $(-) \otimes_{V \oplus mA} mA$, respectively. Furthermore, for any $V \oplus mA$-complex $E$, the canonical morphism $\hat{m} \otimes_{V} E \otimes_{V \oplus mA} mA \to E \otimes_{V \oplus mA} mA$ is already an isomorphism by $\hat{m} \otimes_{V} mA \simeq m^3 A = mA$. Hence, we have a weak equivalence $K^\nu(mA)^+ \simeq K^\nu(mA)^{al}$.

Finally, one has weak equivalences: $K^\nu(mA)^+ \simeq K^\nu(mA)^{al} \simeq K^\nu(A)^{al}$ and $K^\nu(A)^+ \simeq K^\nu(A)^{al} \oplus K^\nu(A)^m \simeq K^\nu(mA)^+ \oplus K^\nu(A)^m \simeq K^\nu(mA)^+ \oplus K^\nu(A/mA)^+$, giving us the conclusion by the second part of Lemma 3.2.

\begin{proof}
By the argument of the proof of Theorem 3.3, all of the canonical maps $K^\nu(mA)^{al} \to K^\nu(mA)^+ \to K^\nu(A)^{al}$ are weak equivalences, implying that one has a homotopy Cartesian square

$$
\begin{array}{ccc}
K^\nu(mA) & \longrightarrow & K^\nu(A) \\
\downarrow & & \downarrow \\
K^\nu(A)^{al} & \longrightarrow & K^\nu(A)^+,
\end{array}
$$

where both of the cofibers of horizontal maps are the same $K^\nu(A/mA)$ up to weak equivalence. Since the lower horizontal map is homotopically split, the upper one is also homotopically split by Lemma 2.4.

\end{proof}

\begin{corollary}
Let $V$ be a valuation ring with an idempotent maximal ideal $m$ and $F$ denote the fractional field of $V$. Assume that $m$ is flat. Let $A$ be an almost $V$-algebra. Then the canonical morphism $K^\nu(mA) \to K^\nu(A \otimes_{V} F)$ is homotopically split injective. Furthermore, for any $F$-algebra $B$, we have an canonical weak equivalence

$$K^\nu(B) \simeq K^\nu(mA) \oplus K^\nu(B/mA),$$

where $K^\nu(B/mA)$ denotes the homotopy cofiber of $K^\nu(mA) \to K^\nu(B)$.

\end{corollary}

\begin{proof}
This corollary is immediately obtained by Theorem 3.4. Since all of those functors $\text{Perf}(B) \to \text{A Perf}(B) \to \text{Perf}^+(B)$ are canonically categorical equivalences by the isomorphism: $m \otimes_{V} B \simeq B$, one has weak equivalences: $K^\nu(B) \simeq K^\nu(B)^{al} \simeq K^\nu(B)^+$. Therefore one has a homotopy coCartesian square

$$
\begin{array}{ccc}
K^\nu(A) & \longrightarrow & K^\nu(A/mA) \\
\downarrow & & \downarrow \\
K^\nu(B) & \longrightarrow & K^\nu(B/mA).
\end{array}
$$

\end{proof}
In particular, one has the following splittings: \( K_{\nu}(V) \simeq K_{\nu}(m) \oplus K_{\nu}(V/m) \) and \( K_{\nu}(F) \simeq K_{\nu}(m) \oplus K_{\nu}(F/m) \).

**Corollary 3.6.** Let \( V \) be a valuation ring with an idempotent maximal ideal \( m \) and \( F \) denote the fractional field of \( V \). Then the pullback \( K_{\nu}(V) \to K_{\nu}(F) \) induces injections between their all homotopy groups if and only if \( K_{\nu}(V/m) \to K_{\nu}(F/m) \) has the same property. \( \square \)

### 3.2. A remark on the case an integral perfectoid valuation ring.

We will apply Corollary 3.6 to the case \( V \) a perfectoid valuation ring. Recall the definition of perfectoid algebra:

**Definition 3.7.** Let \( F \) be a complete non-Archimedean non-discrete valuation field of rank 1, and \( V \) denote the subring of powerbounded elements. We say that \( F \) is a perfectoid field if the Frobenius \( \Phi : V/pV \to V/pV \) is surjective, where \( p \) is the characteristic of the residue field of \( V \).

In this case, it is known that the maximal ideal \( m \) of \( V \) is flat and idempotent (See [Bha17, Example 4.1.3].) For any \( V \)-algebra \( A \), let \( A^\flat \) denote the tilting algebra \( \varprojlim_{n \to x \flat} A/pA \) of \( A \). The tilting ideal \( m^\flat \subset V^\flat \) is a flat \( V^\flat \)-module as \( m \) is. Note that \( A^\flat/m^\flat A^\flat \to A/mA \) is an isomorphism of commutative unital rings of positive characteristic.

Under the assumption that we are given an weak equivalence: \( K_{\nu}(F^\flat/V^\flat) \simeq K_{\nu}(F/V) \), which is weaker than \( K_{\nu}(F/V) \simeq K_{\nu}(k)[1] \simeq K_{\nu}(F^\flat/V^\flat) \), one has the following:

**Proposition 3.8.** Let \( V \) be a mixed characteristic integral perfectoid valuation ring with an idempotent maximal ideal \( m \) and \( F \) denote the fractional field of \( V \). Assume that the non-unital \( K \)-theories \( K_{\nu}(V) \) and \( K_{\nu}(V^\flat) \) hold the condition: We are given weak equivalences \( K_{\nu}(F^\flat/V^\flat) \simeq K_{\nu}(F/V) \). Then the pullback \( K_{\nu}(V) \to K_{\nu}(F) \) induces injections \( K_{n}(V) \to K_{n}(F) \) for any integers \( n \), where we write \( K_{n}(V) = \pi_{n}(K_{\nu}(V)) \) if and only if \( K_{\nu}(V^\flat) \to K_{\nu}(F^\flat) \) has the same property, where \(( \cdot )^\flat \) denote the tilting functor of perfectoid algebras.

**proof.** By the assumption \( K_{\nu}(F/V) \simeq K_{\nu}(F^\flat/V^\flat) \) and the isomorphism \( V^\flat/m^\flat \simeq V/m \), one has a weak equivalence \( K_{\nu}(F^\flat/m^\flat) \simeq K_{\nu}(F/m) \). The result follows from corollary 3.6. \( \square \)

**Remark 3.9.** By the result [KM21, Theorem 3.1], in the case the (ordinal) \( K \)-theories, the induced map \( K_{n}(V^\flat) \to K_{n}(F^\flat) \) is injective for any integer \( n \).

### References

[Ba10] **Barwick**, Clark: On left and right model categories and left and right Bousfield localizations. In: *Homology Homotopy Appl.* 12 (2010), Nr. 2, S. 245–320. – ISSN 1532–0073

[Bha17] **Bhatt**, Bhargav: Lecture notes for a class on perfectoid spaces. Available at: [http://www-personal.umich.edu/~bhattb/teaching/mat679w17/lectures.pdf](http://www-personal.umich.edu/~bhattb/teaching/mat679w17/lectures.pdf) 2017
[CMM21] Clausen, Dustin ; Mathew, Akhil ; Morrow, Matthew: $K$-theory and topological cyclic homology of Henselian pairs. In: *J. Am. Math. Soc.* 34 (2021), Nr. 2, S. 411–473. [http://dx.doi.org/10.1090/jams/961] – DOI 10.1090/jams/961. – ISSN 0894–0347

[DS75] Dennis, R. K. ; Stein, Michael R.: $K_2$ of discrete valuation rings. In: *Advances in Math.* 18 (1975), Nr. 2, S. 182–238. – ISSN 0001–8708

[Fal88] Faltings, Gerd: $p$-adic Hodge theory. In: *J. Amer. Math. Soc.* 1 (1988), Nr. 1, S. 255–299. – ISSN 0894–0347

[Ger73] Gersten, S. M.: *Some exact sequences in the higher $K$-theory of rings*. Algebr. $K$-Theory I, Proc. Conf. Battelle Inst. 1972, Lect. Notes Math. 341, 211–243 (1973). [link.springer.com/chapter/10.1007/BFb0067059] Version: 1973

[GL87] Gillet, Henri ; Levine, Marc: The relative form of Gersten’s conjecture over a discrete valuation ring: The smooth case. In: *J. Pure Appl. Algebra* 46 (1987), S. 59–71. [http://dx.doi.org/10.1016/0022-4049(87)90043-0] – DOI 10.1016/0022–4049(87)90043–0. – ISSN 0022–4049

[GR03] Gabber, Ofer ; Ramero, Lorenzo: Lecture Notes in Mathematics. Bd. 1800: *Almost ring theory*. Springer-Verlag, Berlin, 2003. – vi+307 S. – ISBN 3–540–40594–1

[Hovey14] Hovey, Mark: *Smith ideals of structured ring spectra*. Available at [https://arxiv.org/abs/1401.2856](https://arxiv.org/abs/1401.2856) 2014

[Kal76] Kalassen, Wilberd van d.: The $K_2$’s of a 2-dimensional regular local ring and its quotient field. In: *Comm. Algebra* 4 (1976), Nr. 7, S. 677–679. – ISSN 0092–7872

[Kat22] Kato, Yuki: *Algebraic $K$-theory and algebraic cobordism of almost mathematics*. Available at [https://arxiv.org/abs/2203.08081](https://arxiv.org/abs/2203.08081)

[KM21] Kelly, Shane ; Morrow, Matthew: $K$-theory of valuation rings. In: *Compos. Math.* 157 (2021), Nr. 6, S. 1121–1142. [http://dx.doi.org/10.1112/S0010437X21007119] – DOI 10.1112/S0010437X21007119. – ISSN 0010–437X

[Lur09] Lurie, Jacob: *Annals of Mathematics Studies*. Bd. 170: *Higher topos theory*. Princeton, NJ : Princeton University Press, 2009. – xviii+925 S. – ISBN 978–0–691–14049–0; 0–691–14049–9

[Lur17] Lurie, Jacob: *Higher Algebra*. available at [https://www.math.ias.edu/~lurie/papers/HA.pdf](https://www.math.ias.edu/~lurie/papers/HA.pdf) 2017

[Pan03] Panin, I. A.: The equicharacteristic case of the Gersten conjecture. In: *Tr. Mat. Inst. Steklova* 241 (2003), Nr. Teor. Chisel, Algebra i Algebr. Geom., S. 169–178. – ISSN 0371–9685

[Pop86] Popescu, Dorin: General Néron desingularization and approximation. In: *Nagoya Math. J.* 104 (1986), S. 85–115. – ISSN 0027–7630

[Qui73] Quillen, Daniel: *Higher algebraic $K$-theory. I*. Algebr. $K$-Theory I, Proc. Conf. Battelle Inst. 1972, Lect. Notes Math. 341, 85–147 (1973).

[Qui96] Quillen, Daniel: *Module theory over nonunital rings*. available at [https://ncatlab.org/nlab/files/QuillenModulesOverRngs.pdf](https://ncatlab.org/nlab/files/QuillenModulesOverRngs.pdf) 1996

[Sch12] Scholze, Peter: Perfectoid spaces. In: *Publ. Math. Inst. Hautes Études Sci.* 116 (2012), S. 245–313. – ISSN 0073–8301
THOMASON, R. W.; TROROUGH, Thomas: Higher algebraic K-theory of schemes and of derived categories. Appendix A: Exact categories and the Gabriel-Quillen embedding. Appendix B: Modules versus quasi-coherent modules. Appendix C: Absolute noetherian approximation. Appendix D: Hypercohomology with supports. Appendix E: The Nisnevich topology. Appendix F: Invariance under change of universe. The Grothendieck Festschrift, Collect. Artic. in Honor of the 60th Birthday of A. Grothendieck. Vol. III, Prog. Math. 88, 247-435. Appendix A: 398-408; appendix B: 409-417; appendix C: 418-423; appendix D: 424-426; appendix E: 427-430; appendix F: p. 431 (1990).

WALDHUSEN, Friedhelm: Algebraic K-theory of spaces, localization, and the chromatic filtration of stable homotopy. Algebraic topology, Proc. Conf., Aarhus 1982, Lect. Notes Math. 1051, 173-195 (1984).

WALDHUSEN, Friedhelm: Algebraic K-theory of spaces. In: Algebraic and geometric topology (New Brunswick, N.J., 1983) Bd. 1126. Berlin : Springer, 1985, S. 318–419

WEIBEL, Charles A.: Homotopy algebraic K-theory. Algebraic K-theory and algebraic number theory, Proc. Semin., Honolulu/Hawaii 1987, Contemp. Math. 83, 461-488 (1989).

Email address: ykato@ube-k.ac.jp