UNIQUENESS OF THE SCATTERER FOR ELECTROMAGNETIC FIELD
WITH ONE INCIDENT PLANE WAVE

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Abstract. In this paper, we solve a longstanding open problem for determining the shape of an obstacle from the knowledge of the electric (or magnetic) far field pattern for the scattering of time-harmonic electromagnetic field. We show that the electric (or magnetic) far field pattern $E^\infty(\beta, \alpha_0, k_0)$ (or $H^\infty(\beta, \alpha_0, k_0)$), known for all $\beta \in S^2$, where $S^2$ is the unit sphere in $\mathbb{R}^3$, $\alpha_0 \in S^2$ is fixed, $k_0 > 0$ is fixed, determines the obstacle $D$ and the boundary condition on $\partial D$ uniquely. The boundary condition on $\partial D$ is either the perfect conductor or the impedance one.

1. Introduction

Throughout this paper, $D$ is assumed to be a bounded domain with boundary $\partial D$ of class $C^2$ and with the connected complement $\mathbb{R}^3 \setminus \bar{D}$. The time-harmonic electromagnetic waves in the homogeneous isotropic medium $\mathbb{R}^3 \setminus \bar{D}$ must satisfy the reduced Maxwell equations

$$\begin{cases}
\nabla \times E - ikH = 0 & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\
\nabla \times H + ikE = 0 & \text{in } \mathbb{R}^3 \setminus \bar{D}.
\end{cases}$$

Here $E$ and $H$ denote the space dependent parts of the electric field $\frac{1}{\sqrt{\varepsilon}}E(x)e^{-i\omega t}$ and the magnetic field $\frac{1}{\sqrt{\mu}}H(x)e^{-i\omega t}$ respectively, $k$ is the positive wave number given by $k = \sqrt{\varepsilon \mu \omega}$ in terms of the frequency $\omega$, the electric permittivity $\varepsilon$ and the magnetic permeability $\mu$. The scattering of time-harmonic electromagnetic waves by an impenetrable bounded obstacle $D$ in $\mathbb{R}^3$ yields the exterior boundary value inverse scattering problems for the Maxwell equations. Therefore, the total electromagnetic wave $E, H$ is decomposed $E = E^i + E^s, H = H^i + H^s$ into the given incident wave $E^i, H^i$ and the unknown scattered wave $E^s, H^s$ which is required to satisfy the Silver-Müller radiation condition

$$\lim_{|x| \to \infty} (H^s \times x - |x|E^s) = 0 \quad (\text{or} \quad \lim_{|x| \to \infty} (E^s \times x + |x|H^s) = 0)$$

1991 Mathematics Subject Classification. 35P25, 35R30, 78A25, 78A46.
Key words and phrases. The Maxwell equations; Electric (or magnetic) far field pattern; Electric (or magnetic) scatterer; Uniqueness theorem.
uniformly with respect to all directions. On the boundary \( \partial D \), the total field has to satisfy a boundary condition of the form

\[
T(\mathbf{E}, \mathbf{H}) = 0 \quad \text{on} \quad \partial D
\]

with the operator \( T \) depending on the nature of the scatterer \( D \). For a perfect conductor we have \( T(\mathbf{E}, \mathbf{H}) = \mathbf{\nu} \times \mathbf{E} \), where \( \mathbf{\nu} \) denotes the unit normal to the boundary \( \partial D \) pointing out of \( D \), i.e., the total electric field has a vanishing tangential component:

\[
\mathbf{\nu} \times \mathbf{E} = 0 \quad \text{on} \quad \partial D.
\]

The scattering by an obstacle that is not perfectly conducting but that does not allow the electromagnetic wave to penetrate deeply into the obstacle is modeled by an impedance boundary condition:

\[
\mathbf{\nu} \times (\nabla \times \mathbf{E}) - i\psi(\mathbf{\nu} \times \mathbf{E}) \times \mathbf{\nu} = 0 \quad \text{on} \quad \partial D
\]

with a positive function \( \psi \), that is, \( T(\mathbf{E}, \mathbf{H}) = \mathbf{\nu} \times (\nabla \times \mathbf{E}) - i\psi(\mathbf{\nu} \times \mathbf{E}) \times \mathbf{\nu} \). It is well-known that the existence and well-posedness of the Silver-Müller radiating solution for the above exterior boundary value problems of the Maxwell equations have been established by boundary integral equations (see, e.g. [20], [5] or [2]), and the scattering field \( \mathbf{E}^s, \mathbf{H}^s \) has the asymptotic form

\[
\mathbf{E}^s(x) = e^{ik|x|}\left\{ \mathbf{E}^\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty,
\]

\[
\mathbf{H}^s(x) = e^{ik|x|}\left\{ \mathbf{H}^\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty,
\]

uniformly in all directions \( \hat{x} = \frac{x}{|x|} \) where the vector fields \( \mathbf{E}^\infty \) and \( \mathbf{H}^\infty \) defined on the unit sphere \( \mathbb{S}^2 \) are known as the electric far field pattern and magnetic far field pattern, respectively. They satisfy

\[
\mathbf{H}^\infty = \mathbf{\nu} \times \mathbf{E}^\infty \quad \text{and} \quad \mathbf{\nu} \cdot \mathbf{E}^\infty = \mathbf{\nu} \cdot \mathbf{H}^\infty = 0
\]

with the unit outward normal \( \mathbf{\nu} \) on \( \mathbb{S}^2 \). An important cases of incident fields are plane waves

\[
\mathbf{E}^i(x, \alpha, k, p) = e^{ik\alpha \cdot x} p, \quad \mathbf{H}^i(x, \alpha, k, p) = e^{ik\alpha \cdot x}(\alpha \times p)
\]

with propagation direction \( \alpha \in \mathbb{S}^2 \), wave number \( k \) and polarization vector \( p \). The corresponding scattered waves and far field patterns (or scattering amplitudes) are denoted by \( \mathbf{E}^s(x, \alpha, k, p) \), \( \mathbf{H}^s(x, \alpha, k, p) \) and \( \mathbf{E}^\infty(\frac{x}{|x|}, \alpha, k, p) \), \( \mathbf{H}^\infty(\frac{x}{|x|}, \alpha, k, p) \), respectively. Because of the linearity of the scattering problem with respect to the incident waves, we see that the scattered waves and the far field patterns are both linear respect to the polarization vector \( p \). Therefore we can write \( \mathbf{E}^s(x, \alpha, k, p) \) as \( \mathbf{E}^s(x, \alpha, k)p \), and so forth. The scattering amplitudes \( \mathbf{E}^\infty(\frac{x}{|x|}, \alpha, k) \) and \( \mathbf{H}^\infty(\frac{x}{|x|}, \alpha, k) \) are 3 by 3 matrices, which are physics quantities and can be measured experimentally. It follows from [3] [5] that for smooth bounded obstacles the far field patterns \( \mathbf{E}^\infty(\beta, \alpha, k) \) and \( \mathbf{H}^\infty(\beta, \alpha, k) \) are analytic matrices of \( \beta \) and \( \alpha \) on the unit sphere \( \mathbb{S}^2 \). For a fixed \( \alpha \in \mathbb{S}^2 \), if \( \mathbf{E}^\infty(\beta, \alpha, k) \) as a matrix of \( \beta \) is known on an open subset of \( \mathbb{S}^2 \), it is uniquely extended to all of \( \mathbb{S}^2 \) by analyticity. The same is true for \( \mathbf{H}^\infty(\beta, \alpha, k) \).

The basic inverse problem in scattering theory is to determine the shape of the scatterer \( D \) from a knowledge of the electric far field pattern \( \mathbf{E}_\infty(\frac{x}{|x|}, \alpha, k)p \) (or the magnetic far field pattern \( \mathbf{H}_\infty(\frac{x}{|x|}, \alpha, k)p \)) for one or several incident plane waves with incident directions \( \alpha \) and polarizations \( p \). The study of inverse scattering problem for electromagnetic wave is of fundamental important to many areas of science and technology, such as radar, sonar, geophysical exploration, medical imaging and nondestructive testing.
Until the 1980’s, very little was known concerning the mathematical properties of far field patterns (cf. [5]). However, in the past three decades results have been obtained for the inverse electromagnetic problems. In [5], based on the ideas of Kirsch and Kress [7], D. Colton and R. Kress proved that for perfect conductor, one fixed incident direction $\alpha$ and polarization $p$, and all wave number contained in some interval $0 < k_1 < k < k_2 < \infty$ can determine $D$.

It has been shown by Liu, Yamamoto and Zou [15] that a perfectly conducting polyhedron is uniquely determined by the far field pattern for plane wave incidence with one direction $\alpha$ and two polarizations $p_1$ and $p_2$. D. Colton and R. Kress proved (see [5]) that if $D_1$ and $D_2$ are two scatterers with boundary conditions $T_1$ and $T_2$ such that for a fixed wave number the far field patterns coincide for all incident directions $\alpha$, all polarizations $p$, and all observation directions $\frac{\alpha}{|\alpha|}$, then $D_1 = D_2$ and $T_1 = T_2$. In [5], D. Colton and R. Kress also showed that a ball and its boundary condition (for constant impedance $\psi$) is uniquely determined by the far field pattern for plane wave incidence with one direction $\alpha$ and $p$. We refer to [12], [3], [9], [14], [22] for a review of this topic. In the inverse acoustic obstacle scattering (i.e., the Helmholtz equation), by using a completely new technique the author [10] showed that the scattering amplitude for one single incident direction and one wave number uniquely determines the acoustic obstacle.

However, it has been a challenging open problem (see p. 6 of [1] or p. 4 of [12]) that for a fixed wave number $k$, a fixed incident direction $\alpha$, and a fixed polarization $p$, whether the electric (or magnetic) far field pattern can uniquely determine the general scatterer $D$ and its boundary condition?

In this paper, using a novel idea and an elementary means by discussing all possible positions of two scatterers and applying the electric (or magnetic) eigenvalue theory, we solve the above inverse scattering problem for the electromagnetic field. Our main result is the following:

**Theorem 1.1.** Assume that $D_1$ and $D_2$ are two scatterers with boundary condition $T_1$ and $T_2$ such that for a fixed wave number $k_0$, a fixed incident direction $\alpha_0$, and a fixed polarization $p_0$, the electric (or magnetic) far field pattern of both scatterers coincide (i.e., $E_\infty^e(\beta, \alpha_0, k_0)p_0 = E_\infty^e(\beta, \alpha_0, k_0)p_0$ (or $H_\infty^m(\beta, \alpha_0, k_0)p_0 = H_\infty^m(\beta, \alpha_0, k_0)p_0$) for all $\beta$ in an open subset of $S^2$). Then $D_1 = D_2$ and $T_1 = T_2$.

Let us point out that our method is completely new. In particular, we subtly apply three basic tools: the property of the eigenfunction in a bounded domain, the interior analyticity of the solutions for the time-harmonic Maxwell equations, and the asymptotic property of the scattered waves as $|x| \to \infty$.

**Remark 1.2.** For the Maxwell equations, we only need to be concerned with the study of three-dimensional inverse scattering problems since the two-dimensional case can be reduced to the two-dimensional Helmholtz equation that has been solved by the author in [10].

This paper is organized as follows. In Section 2, we present some known results. In Section 3, we prove a key lemma (Lemma 3.1) which shows that the electric (or magnetic) far field pattern determines the total electromagnetic scattering wave in the unbounded connected component of $\mathbb{R}^3 \setminus (D_1 \cup D_2)$. Section 4 is devoted to the proof of the main result.
2. Preliminaries

Let $g(x)$ be a real-valued function defined in an open set $\Omega$ in $\mathbb{R}^n$. For $y \in \Omega$ we call $g$ real analytic at $y$ if there exist $a_\gamma \in \mathbb{R}$ and a neighborhood $U$ of $y$ (all depending on $y$) such that

$$g(x) = \sum_\gamma a_\gamma (x - y)^\gamma$$

for all $x \in U$, where $\gamma = (\gamma_1, \cdots, \gamma_n)$ is a multi-index (a set of non-negative integers), $|\gamma| = \sum_{j=1}^n \gamma_j$, and $(x - y)^\gamma = (x_1 - y_1)^{\gamma_1} \cdots (x_n - y_n)^{\gamma_n}$. We say $g$ is real analytic in $\Omega$, if $g$ is real analytic at each $y \in \Omega$.

Lemma 2.1 (Unique continuation of real analytic function, see, for example, p. 65 of [8]). Let $\Omega$ be a connected open set in $\mathbb{R}^n$, and let $g$ be real analytic in $\Omega$. Then $g$ is determined uniquely in $\Omega$ by its values in any nonempty open subset of $\Omega$.

Lemma 2.2 (The interior real analyticity of the solutions for real analytic elliptic equations, see [16], [17], [18] or [19]). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $L$ be a strongly elliptic linear differential operator of order $2m$

$$Lu = \sum_{|\gamma|\leq 2m} a_\gamma(x) D^\gamma u(x).$$

If the coefficients $a_\gamma(x)$, $|\gamma| \leq 2m$, and the right-hand side $f(x)$ of the equation $Lu = f$ are real analytic with respect to $x = (x_1, \cdots, x_n)$ in the domain $\Omega$, then any solution $u$ of this equation is also real analytic in $\Omega$.

Lemma 2.3 (see Theorem 6.4 of [5]). Let $E$, $H$ be a solution to the Maxwell equations

$$\nabla \times E - ikH = 0, \quad \nabla \times H + ikE = 0.$$ 

Then $E$ and $H$ are divergence free (i.e., $\nabla \cdot E = 0$ and $\nabla \cdot H = 0$) and satisfy the vector Helmholtz equation

$$\Delta E + k^2 E = 0 \quad \text{and} \quad \Delta H + k^2 H = 0.$$ 

Conversely, let $E$ (or $H$) be a solution to the vector Helmholtz equation satisfying $\nabla \cdot E = 0$ (or $\nabla \cdot H = 0$). Then $E$ and $H := \frac{1}{ik} \nabla \times E$ (or $H$ and $E := -\frac{1}{ik} \nabla \times H$) satisfy the Maxwell equations.

Lemma 2.4 (see Theorem 6.3 of [5]). Any continuously differentiable solution to the Maxwell equations has analytic cartesian components. In particular, the cartesian components of solutions to the Maxwell equations are automatically two times continuously differentiable.

Lemma 2.5 (see Theorem 6.7 of [5]). Assume the bounded domain $D$ is the open complement of an unbounded domain of class $C^2$. Let $E$, $H \in C^1(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$ be a solution to the Maxwell equations

$$\nabla \times E - ikH = 0, \quad \nabla \times H + ikE = 0 \quad \text{in} \ \mathbb{R}^3 \setminus \overline{D}$$

satisfying the Silver-Müller radiation conditions (1.2). Then the radiating solutions $E$, $H$ to the Maxwell equations automatically satisfy

$$E(x) = O\left(\frac{1}{|x|}\right), \quad H(x) = O\left(\frac{1}{|x|}\right), \quad |x| \to \infty,$$

uniformly for all directions $\frac{x}{|x|}$. 

Lemma 2.6 (see p. 198 of [5]). Let $E, H$ be a solution to the Maxwell equations in $\mathbb{R}^3$ satisfying the Silver-Müller radiation conditions. Then $E, H$ must vanish identically in $\mathbb{R}^3$.

Lemma 2.7 (Holmgren’s uniqueness theorem for the scattering total solutions of the Maxwell equations, see Theorem 6.5 of [5]). Let $D$ be a bounded domain with $C^2$-smooth boundary $\partial D$ and let $\Gamma \subset \partial D$ be an open subset with $\Gamma \cap (\mathbb{R}^3 \setminus D) \neq \emptyset$. Assume that $E, H$ is a solution of the scattering problem for the Maxwell equations

$$\begin{cases}
\nabla \times E - ikH = 0 & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\
\nabla \times H + ikE = 0 & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\
E = E^i + E^s, \quad H = H^i + H^s & \text{in } \mathbb{R}^3 \setminus \bar{D},
\end{cases}$$

where $E^i$ and $H^i$ are defined in (1.8), and $E^s$ and $H^s$ satisfy the Silver-Müller radiation condition, such that $\nu \times E = \nu \times H = 0$ on $\Gamma$.

Then $E \equiv 0$ and $H \equiv 0$ in $\mathbb{R}^3 \setminus \bar{D}$.

The following Lemma will be needed in the proof of Lemma 3.1.

Lemma 2.9 (Rellich’s lemma, see p. 33 of [5] or p. 178 of [23]). Assume the bounded domain $D$ is the open complement of an unbounded domain and let $v \in C^2(\mathbb{R}^3 \setminus D)$ be a solution to
the Helmholtz equation \((\Delta + k^2)v = 0\) satisfying \(\int_{\partial B_r(0)} |v|^2 ds \to 0\) as \(r \to \infty\), where \(\partial B_r(0)\) is the sphere \(\{x \in \mathbb{R}^3 | |x| = r\}\). Then \(v(x) = 0\) for \(x \in \mathbb{R}^3 \setminus \bar{D}\).

3. Uniqueness of scattering solutions in the exterior of two scatterers

We consider the scattering of electromagnetic plane waves with incident direction \(\alpha \in \mathbb{S}^2\) and polarization vector \(p\) as described by the matrices \(E^i(x, \alpha, k)\) and \(H^i(x, \alpha, k)\) defined by

\[
E^i(x, \alpha, k)p := e^{ikx \cdot p}, \quad H^i(x, \alpha, k)p := e^{ix \cdot p}. \quad (3.2)
\]

Let \(D_j\) be a bounded domain in \(\mathbb{R}^3\) with a connected boundary \(\partial D_j\) of class \(C^2\) (\(j = 1, 2\)). Let \(E_j(x, \alpha, k)p, \quad H_j(x, \alpha, k)p\) be the solution of the scattering problem in \(\mathbb{R}^3 \setminus \bar{D}_j\), i.e., \(E_j(x, \alpha, k)p := E^i(x, \alpha, k)p + E^i_0(x, \alpha, k)p, \quad H_j(x, \alpha, k)p := H^i(x, \alpha, k)p + H^i_0(x, \alpha, k)p\), \(j = 1, 2\) satisfy the Maxwell equations

\[
\begin{cases}
\nabla \times E_j - ikH_j = 0, & \text{in } \mathbb{R}^3 \setminus \bar{D}_j,

\nabla \times H_j + ikE_j = 0, & \text{in } \mathbb{R}^3 \setminus \bar{D}_j,

\end{cases}
\]

\[
E_j = e^{ikx \cdot p} + O\left(\frac{1}{|x|}\right), \quad H_j = e^{x \cdot p} + O\left(\frac{1}{|x|}\right), \quad \text{as } |x| \to \infty
\]

uniformly for all direction \(\frac{x}{|x|}\). As pointed out in Section 1, we can write

\[
E_j(x, \alpha, k)p = e^{ikx \cdot p} + e^{\frac{i|p|}{|x|}}E_j^\infty(\beta, \alpha, k)p + O\left(\frac{1}{|x|^2}\right),
\]

\[
\text{as } |x| \to \infty, \quad \beta = \frac{x}{|x|}.
\]

\[
H_j(x, \alpha, k)p = e^{ikx \cdot p} + e^{\frac{i|p|}{|x|}}H_j^\infty(\beta, \alpha, k)p + O\left(\frac{1}{|x|^2}\right),
\]

\[
\text{as } |x| \to \infty, \quad \beta = \frac{x}{|x|},
\]

where \(E_j^\infty(\beta, \alpha, k)p\) and \(H_j^\infty(\beta, \alpha, k)p\) are the electric and magnetic far field patterns for the exterior domains \(\mathbb{R}^3 \setminus \bar{D}_j\), \(j = 1, 2\) with polarization \(p\), respectively.

Now, we have the following basic lemma:

**Lemma 3.1.** Let \(E_j(x, \alpha_0, k_0)p_0, \quad H_j(x, \alpha_0, k_0)p_0\) be the solution of the scattering problem for Maxwell equations in \(\mathbb{R}^3 \setminus \bar{D}_j\) (\(j = 1, 2\)). If \(E_2^\infty(\beta, \alpha_0, k_0)p_0 = E_2^\infty(\beta, \alpha_0, k_0)p_0\) (or \(H_2^\infty(\beta, \alpha_0, k_0)p_0 = H_2^\infty(\beta, \alpha_0, k_0)p_0\)) for all \(\beta = \frac{x}{|x|} \in \mathbb{S}^2\), a fixed \(\alpha_0 \in \mathbb{S}^2\), a fixed \(k_0 \in \mathbb{R}^1\) and a fixed \(p_0 \in \mathbb{R}^3\), then

\[
E_2(x, \alpha_0, k_0)p_0 = E_1(x, \alpha_0, k_0)p_0 \quad \text{for } x \in D_{12},
\]

and

\[
H_2(x, \alpha_0, k_0)p_0 = H_1(x, \alpha_0, k_0)p_0 \quad \text{for } x \in D_{12},
\]

where \(D_{12}\) is the unbounded connected component of \(\mathbb{R}^3 \setminus (\bar{D}_1 \cup \bar{D}_2)\).
Proof. For each $j$ and any boundary condition $T(E_j, H_j)$, by (3.2) and (3.3) we have

\begin{equation}
E_2(x, \alpha_0, k_0)p_0 - E_1(x, \alpha_0, k_0)p_0 = \frac{e^{ik|x|}}{|x|^2} [E_2^\infty(\beta, \alpha_0, k_0)p_0 - E_1^\infty(\beta, \alpha_0, k_0)p_0] + O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty, \quad \beta = \frac{x}{|x|},
\end{equation}

(3.6)

\begin{equation}
H_2(x, \alpha_0, k_0)p_0 - H_1(x, \alpha_0, k_0)p_0 = \frac{e^{ik|x|}}{|x|^2} [H_2^\infty(\beta, \alpha_0, k_0)p_0 - H_1^\infty(\beta, \alpha_0, k_0)p_0] + O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty, \quad \beta = \frac{x}{|x|},
\end{equation}

(3.7)

In view of

\begin{equation*}
E_1^\infty(\beta, \alpha_0, k_0)p_0 = E_2^\infty(\beta, \alpha_0, k_0)p_0 \quad \text{for all } \beta \in S^2,
\end{equation*}

(or \(H_1^\infty(\beta, \alpha_0, k_0)p_0 = H_2^\infty(\beta, \alpha_0, k_0)p_0 \quad \text{for all } \beta \in S^2),\)

we obtain

\begin{equation}
E_1(x, \alpha_0, k_0)p_0 - E_2(x, \alpha_0, k_0)p_0 = O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty, \quad \beta = \frac{x}{|x|},
\end{equation}

(3.8)

\begin{equation}
\text{(or } H_1(x, \alpha_0, k_0)p_0 - H_2(x, \alpha_0, k_0)p_0 = O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty, \quad \beta = \frac{x}{|x|}).
\end{equation}

(3.9)

With the aid of Lemma 2.3, we get that \(E_1 - E_2\) (or \(H_1 - H_2\)) satisfies the vector Helmholtz equations, i.e,

\begin{equation*}
\Delta(E_1(x, \alpha_0, k_0)p_0 - E_2(x, \alpha_0, k_0)p_0) + k_0^2 (E_1(x, \alpha_0, k_0)p_0 - E_2(x, \alpha_0, k_0)p_0) = 0 \quad \text{in } D_{12},
\end{equation*}

(\text{or } \Delta(H_1(x, \alpha_0, k_0)p_0 - H_2(x, \alpha_0, k_0)p_0) + k_0^2 (H_1(x, \alpha_0, k_0)p_0 - H_2(x, \alpha_0, k_0)p_0) = 0 \quad \text{in } D_{12}).

It follows from (3.8), (3.9) and Lemma 2.9 (Rellich’s lemma) that

\begin{equation*}
E_1(x, \alpha_0, k_0)p_0 - E_2(x, \alpha_0, k_0)p_0 = 0 \quad \text{for } x \in D_{12},
\end{equation*}

(\text{or } H_1(x, \alpha_0, k_0)p_0 - H_2(x, \alpha_0, k_0)p_0 = 0 \quad \text{for } x \in D_{12}).

Furthermore, by applying any one of the above two relations to the Maxwell equations

\begin{equation*}
\nabla \times E_j - ikH_j = 0, \quad \nabla \times H_j + ikE_j = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D}_j,
\end{equation*}

we see that (5.4) and (5.5) hold simultaneously. \(\square\)

4. Proof of main theorem

Proof of theorem 1.1. For convenience, we assume below the obstacle has the perfect conductor boundary condition, but our proof is valid for the impedance boundary condition as well. Also, we only discuss unique determination of the scatterer by the electric far field pattern because the magnetic case can be similarly dealt with. It is an obvious fact that if two bounded domains \(D_1\) and \(D_2\) of class \(C^2\) satisfying \(D_1 \neq D_2\), then either \(D_1 \neq D_2\) and \(D_1 \cap D_2 = \emptyset\), or \(D_1 = D_2\) and \(D_1 \cap D_2 \neq \emptyset\). We will show that the above two cases can never occur.

Case 1. Suppose by contradiction that \(D_1 \neq D_2\) and \(D_2 \cap D_1 = \emptyset\). Since \(E_1^\infty(\beta, \alpha_0, k_0)p_0 = E_2^\infty(\beta, \alpha_0, k_0)p_0\) for all \(\beta \in S^2\) in an open subset of \(S^2\), we immediately get that the above relation is still true for all \(\beta \in S^2\) by analyticity. From Lemma 3.1 we get that

\begin{equation*}
E_1(x, \alpha_0, k_0)p_0 = E_2(x, \alpha_0, k_0)p_0 \quad \text{and } H_1(x, \alpha_0, k_0)p_0 = H_2(x, \alpha_0, k_0)p_0 \quad \text{for all } x \in D_{12},
\end{equation*}

(3.10)

Furthermore, we see that (3.4) and (3.5) hold simultaneously. \(\square\)
where \( E_j(x, \alpha_0, k_0)p_0 \), \( H_j(x, \alpha_0, k_0)p_0 \) is the solution of scattering problem for the Maxwell equations in \( \mathbb{R}^3 \setminus D_j \) (\( j = 1, 2 \)), and \( D_{12} \) is the unbounded connected component of \( \mathbb{R}^3 \setminus (D_1 \cup D_2) \). Note that the real part and imaginary part of cartesian components of \( E_j \), \( H_j \) are both real analytic in \( \mathbb{R}^3 \setminus D_j \) (\( j = 1, 2 \)) by Lemma 2.4. Since \( E_2(x, \alpha_0, k_0)p_0 \), \( H_1(x, \alpha_0, k_0)p_0 \) is defined in \( D_2 \) and satisfies there the Maxwell equations, the unique continuation property implies that \( E_2(x, \alpha_0, k_0)p_0 \), \( H_2(x, \alpha_0, k_0)p_0 \) can be defined in \( D_2 \) and satisfies there the Maxwell equations. Consequently, \( E_2(x, \alpha_0, k_0)p_0 \), \( H_2(x, \alpha_0, k_0)p_0 \) is defined in \( \mathbb{R}^3 \), it is a smooth function that satisfies the Maxwell equations in \( \mathbb{R}^3 \), and the same is true for \( E_1(x, \alpha_0, k_0)p_0 \), \( H_1(x, \alpha_0, k_0)p_0 \). Therefore the scattered parts \( E_1^s(x, \alpha_0, k_0)p_0 \), \( H_1^s(x, \alpha_0, k_0)p_0 \) and \( E_2^s(x, \alpha_0, k_0)p_0 \), \( H_2^s(x, \alpha_0, k_0)p_0 \) of the scattering solutions \( E_1(x, \alpha_0, k_0)p_0 \), \( H_1(x, \alpha_0, k_0)p_0 \) and \( E_2(x, \alpha_0, k_0)p_0 \), \( H_2(x, \alpha_0, k_0)p_0 \) satisfy the Maxwell equations \( \nabla \times E - i\kappa H = 0 \), \( \nabla \times \bar{H} + i\kappa E = 0 \) in \( \mathbb{R}^3 \) and have the Silver-Müller radiation conditions. It follows from Lemma 2.6 that \( E_1^s(x, \alpha_0, k_0)p_0 = E_2^s(x, \alpha_0, k_0)p_0 = 0 \), \( H_1^s(x, \alpha_0, k_0)p_0 = H_2^s(x, \alpha_0, k_0)p_0 = 0 \) in \( \mathbb{R}^3 \) and hence \( E_1(x, \alpha_0, k_0)p_0 = E_2(x, \alpha_0, k_0)p_0 = e^{i\omega_0 x}p_0 \), \( H_1(x, \alpha_0, k_0)p_0 = H_2(x, \alpha_0, k_0)p_0 = e^{i\omega_0 x}p_0 \) in \( \mathbb{R}^3 \). This is impossible since \( \nu \times E_j(x, \alpha_0, k_0)p_0 = 0 \) on \( \partial D_j \), \( j = 1, 2 \), while \( e^{i\omega_0 x}p_0 \) can not vanish identically for all \( x \in \partial D_j \). Thus, we must have \( D_1 = D_2 \).

Case 2. Suppose by contradiction that \( D_2 \neq D_2 \) and \( D_1 \cap D_2 \neq \emptyset \). Then either \( \mathbb{R}^3 \setminus (D_1 \cup D_2) \) or \( \mathbb{R}^3 \setminus (D_1 \cup D_2) \) has only finitely many connected components, and each of them adjoins the unbounded domain \( D_{12} \) by sharing a common \( C^2 \)-smooth surface, where \( D_{12} \) is the unbounded connected component of \( \mathbb{R}^3 \setminus (D_1 \cup D_2) \). Let us assume that \( \Omega \) be any one of the above connected components. Clearly, \( \Omega \) is a bounded domain with piecewise \( C^2 \)-smooth boundary. Without loss of generality, we let \( \Omega \subset \mathbb{R}^3 \setminus D_1 \). Since \( E_1^s(\beta, \alpha_0, k_0)p_0 = E_2^s(\beta, \alpha_0, k_0)p_0 \) for all \( \beta \in \mathbb{S}^2 \) by analyticity, applying Lemma 3.1 once more we find that

\[
E_1(x, \alpha_0, k_0)p_0 = E_2(x, \alpha_0, k_0)p_0, \quad H_1(x, \alpha_0, k_0)p_0 = H_2(x, \alpha_0, k_0)p_0 \quad \text{for all} \quad x \in D_{12},
\]

where \( E_j(x, \alpha_0, k_0)p_0 \), \( H_j(x, \alpha_0, k_0)p_0 \) is the solution of scattering problem for the Maxwell equations in \( \mathbb{R}^3 \setminus D_j \) (\( j = 1, 2 \)). Note that \( (\nu \times E_j)|_{\partial D_j} = 0 \), \( j = 1, 2 \), and \( (\nu \times E_1)|_{\partial D_{12}} = (\nu \times E_2)|_{\partial D_{12}} = 0 \). It is easy to see from this and the definition of \( \Omega \) that the restriction of \( E_1(x, \alpha_0, k_0)p_0 \), \( H_1(x, \alpha_0, k_0)p_0 \) to \( \Omega \) satisfies

\[
\begin{align*}
\nabla \times E - i\kappa \bar{H} &= 0 & \text{in} \ \Omega, \\
\nabla \times \bar{H} + i\kappa E &= 0 & \text{in} \ \Omega, \\
\nu \times E &= 0 & \text{on} \ \partial \Omega,
\end{align*}
\]

i.e., the restriction of \( E_1(x, \alpha_0, k_0)p_0 \), \( H_1(x, \alpha_0, k_0)p_0 \) to \( \Omega \) is a Maxwell eigen-field corresponding to the Maxwell eigenvalue \( k \). We find by Lemma 2.4 that \( \text{Re} E_1(x, \alpha_0, k_0)p_0 \) and \( \text{Im} E_1(x, \alpha_0, k_0)p_0 \) respectively, \( \text{Re} H_1(x, \alpha_0, k_0)p_0 \) and \( \text{Im} H_1(x, \alpha_0, k_0)p_0 \) are both real analytic vector-valued function in \( \mathbb{R}^3 \setminus D_1 \), where \( \text{Re} E_1(x, \alpha_0, k_0)p_0 \) and \( \text{Im} E_1(x, \alpha_0, k_0)p_0 \) respectively, \( \text{Re} H_1(x, \alpha_0, k_0)p_0 \) and \( \text{Im} H_1(x, \alpha_0, k_0)p_0 \) are the real part and imaginary part of the electric field \( E_1(x, \alpha_0, k_0)p_0 \) (respectively, the magnetic field \( H_1(x, \alpha_0, k_0)p_0 \)), i.e., \( E_1(x, \alpha_0, k_0)p_0 = \text{Re} E_1(x, \alpha_0, k_0)p_0 + i \text{Im} E_1(x, \alpha_0, k_0)p_0 \) (respectively, \( H_1(x, \alpha_0, k_0)p_0 = \text{Re} H_1(x, \alpha_0, k_0)p_0 + i \text{Im} H_1(x, \alpha_0, k_0)p_0 \)). By the definition of the electric field \( E_1(x, \alpha_0, k_0)p_0 \),
we have that for all $x \in \mathbb{R}^3 \setminus \bar{D}_1$,

\begin{equation}
E_1(x, \alpha_0, k_0)p_0 = e^{ik_0\alpha_0 \cdot x}p_0 + E_1^s(x, \alpha_0, k_0)p_0
\end{equation}

\begin{align*}
&= (\cos(k_0\alpha_0 \cdot x) + i \sin(k_0\alpha_0 \cdot x))p_0 + (\text{Re} E_1^s(x, \alpha_0, k_0) + i \text{Im} E_1^s(x, \alpha_0, k_0))p_0 \\
&= (\cos(k_0\alpha_0 \cdot x) + \text{Re} E_1^s(x, \alpha_0, k_0))p_0 + i(\sin(k_0\alpha_0 \cdot x) + \text{Im} E_1^s(x, \alpha_0, k_0))p_0.
\end{align*}

Combining Lemma 2.3, (4.1) and (2.4), we see that the electric field $E_1(x, \alpha_0, k_0)p_0$ of the Maxwell eigen-field in $\Omega$ must be a real analytic vector-valued function in $\Omega$. From this and (4.2), we get that

\begin{equation}
\text{Im} E_1^s(x, \alpha_0, k_0)p_0 = -\sin(k_0\alpha_0 \cdot x)p_0 \quad \text{for all } x \in \Omega.
\end{equation}

With the aid of Lemma 2.1, we know that the real analytic vector-valued function $\text{Im} E_1^s(x, \alpha_0, k_0)p_0$ is uniquely determined in $(\Omega \cup D_{12} \cap ((\partial \Omega) \cap \partial D_{12}))^\circ$ by its values in the subset domain $\Omega$, where $(\Omega \cup D_{12} \cap ((\partial \Omega) \cap \partial D_{12}))^\circ$ is the interior of $\Omega \cup D_{12} \cap ((\partial \Omega) \cap \partial D_{12})$. Let us remark that $(\Omega \cup D_{12} \cap ((\partial \Omega) \cap \partial D_{12}))^\circ$ is still an unbounded connected component (i.e., a unbounded domain in $\mathbb{R}^3$). Note also that the real analytic vector-valued function $-\sin(k_0\alpha_0 \cdot x)p_0$ defined for $x \in \Omega$ has just a unique real analytic extension to $(\Omega \cup D_{12} \cap ((\partial \Omega) \cap \partial D_{12}))^\circ$, that is,

\begin{equation}
-\sin(k_0\alpha_0 \cdot x)p_0 \quad \text{for } x \in (\Omega \cup D_{12} \cup ((\partial \Omega) \cap \partial D_{12}))^\circ.
\end{equation}

Thus, we have that for all $x \in (\Omega \cup D_{12} \cup ((\partial \Omega) \cap \partial D_{12}))^\circ$,

\begin{equation}
\text{Im} E_1^s(x, \alpha_0, k_0)p_0 = -\sin(k_0\alpha_0 \cdot x)p_0.
\end{equation}

Since $E_1^s(x, \alpha_0, k_0)p_0$ is the electric scattering solution of the Maxwell equations in $\mathbb{R}^3 \setminus \bar{D}_1$ satisfying the Sommerfeld radiation condition, by (2.1) of Lemma 2.5 we get $\lim_{|x| \to \infty} |E_1^s(x, \alpha_0, k_0)p_0| = 0$. On the other hand, from (4.6) we see that

\begin{align*}
|E_1^s(x, \alpha_0, k_0)p_0| &= \left[|\text{Re} E_1^s(x, \alpha_0, k_0)p_0|^2 + |\text{Im} E_1^s(x, \alpha_0, k_0)p_0|^2\right]^{1/2} \\
&= \left[|\text{Re} E_1^s(x, \alpha_0, k_0)p_0|^2 + |\sin(k_0\alpha_0 \cdot x)p_0|^2\right]^{1/2} \\
&\geq \left[|\sin(k_0\alpha_0 \cdot x)p_0|\right] \quad \text{for all } x \in (\Omega \cup D_{12} \cup ((\partial \Omega) \cap \partial D_{12}))^\circ,
\end{align*}

and so $|E_1^s(x, \alpha_0, k_0)p_0|$ can’t tend to zero as $|x| \to \infty$. Here $|b|$ denotes the Euclidean norm of a vector $b$ in $\mathbb{R}^3$. This is a contradiction, which implies that any domain $\Omega$ mentioned above can never appear. Therefore we must have $D_1 = D_2$.

Finally, denoting $D = D_1 = D_2$, $E = E_1 = E_2$, and $H = H_1 = H_2$, we assume that we have different boundary condition $T_1(E, H) \neq T_2(E, H)$. For the sake of generality, consider the case where we have impedance boundary conditions with two different continuous impedance functions $\psi_1 \neq \psi_2$. Then, from $\nu \times H - i\psi_j(\nu \times E) \times \nu = 0$ on $\partial D$ for $j = 1, 2$ we observe that $i(\psi_1 - \psi_2)(\nu \times E) \times \nu = 0$ on $\partial D$. Therefore for the open set $\Gamma := \{x \in \partial D | \psi_1(x) \neq \psi_2(x)\}$ we have that $(\nu \times E) \times \nu = 0$ on $\Gamma$ so that $(\nu \times E) = 0$ on $\Gamma$. Consequently, we obtain $\nu \times H = 0$ on $\Gamma$ by the boundary condition. Hence, by Holmgren’s uniqueness theorem for the Maxwell equations (see Lemma 2.6), $E = H = 0$ in $\mathbb{R}^3 \setminus D$, which implies that the scattered wave $E^s$, $H^s$ is an entire solution of the Maxwell equations, and $E^s$ and $H^s$ satisfy the Silver-Müller radiation condition. But the incident field $E^i, H^i$ doesn’t satisfy the Silver-Müller radiation condition. This is a contradiction. Hence $\psi_1 = \psi_2$. The case where one of the boundary conditions is the perfect boundary condition can be treated analogously. \qed
Acknowledgments

This research was supported by NNSF of China (11171023/A010801) and NNSF of China (11671033/A010802).

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