LOCAL SAMPLING AND APPROXIMATION OF OPERATORS WITH BANDLIMITED KOHN-NIRENBERG SYMBOLS

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Abstract. Recent sampling theorems allow for the recovery of operators with bandlimited Kohn-Nirenberg symbols from their response to a single discretely supported identifier signal. The available results are inherently non-local. For example, we show that in order to recover a bandlimited operator precisely, the identifier cannot decay in time nor in frequency. Moreover, a concept of local and discrete representation is missing from the theory. In this paper, we develop tools that address these shortcomings.

We show that to obtain a local approximation of an operator, it is sufficient to test the operator on a truncated and mollified delta train, that is, on a compactly supported Schwarz class function. To compute the operator numerically, discrete measurements can be obtained from the response function which are localized in the sense that a local selection of the values yields a local approximation of the operator. Further, we exhibit that the derived measurements allow for signal processing methods based on local features, such as coarse quantization.

Central to our analysis is to conceptualize the meaning of localization for operators with bandlimited Kohn-Nirenberg symbol.

Keywords. Operator identification, channel measurement, local approximation, pseudodifferential operators, Sigma-Delta quantization.

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1. Introduction

In communications engineering, the effect of a slowly time-varying communication channel is commonly modeled as superposition of translations (time shifts due to multipath propagation) and modulations (frequency shifts caused by Doppler effects). In order to recover transmitted signals from their channel outputs, precise knowledge of the nature of the channel is required. A common procedure for channel identification in this sense is to periodically send short duration test signals.
The resulting outputs are then used to estimate channel parameters which allow for an inversion of the operator \[14, 2, 15, 25, 1, 13\].

Kailath \[14\] and Bello \[2\] analyzed the identifiability of such channels. In mathematical terms, these channels are characterized by bandlimited Kohn-Nirenberg symbols and the channel identification problem becomes an operator identification problem: can an operator with bandlimited Kohn-Nirenberg symbol be identified from the output corresponding to a given test input signal?

Kozek and Pfander \[15\], and Pfander and Walnut \[25\] gave mathematical proof of the assertions by Kailath and Bello that a suitable test signal exists as long as the band support of the symbol of the operator has outer Jordan content less than one. The suggested test signals are periodically weighted regularly spaced Dirac-delta distributions. In \[22\], Pfander coined the term operator sampling as the resulting theory has many direct parallels to the sampling theory for bandlimited functions. For example, an operator sampling reconstruction formula was established which generalizes the reconstruction formula in the classical sampling theorem for bandlimited functions (see \[22\] and Theorem 2.2 below).

The operator sampling results in \[15, 25, 22, 24\] rely on using test functions as those described above. These decay neither in time nor in frequency and cannot be realized in practice. In this paper, we show that indeed, for stable identification of operator classes defined by a bandlimitation of the Kohn-Nirenberg symbol, test signals that lack decay in time and frequency are necessary. When seeking to recover only the operator’s action on a time-frequency localized subspace, then this ideal but impractical signal can be replaced with a mollified and truncated copy; the test signal can thereby be chosen to be a compactly supported Schwartz function as shown below.

Furthermore, an important difference to the sampling theory for bandlimited functions is that the response to a test signal in operator sampling is a square-integrable function rather than a discrete set of sample values. Of course, one can obtain a discrete representation using any basis or frame of the range space of square integrable functions, but the question remains which of the multitude of commonly considered representations allow to recover the operator most efficiently. In the case of a bandlimited function, one feature that distinguishes the representation by samples is locality: a sample is the function value at a given location; due to the smoothness of bandlimited functions it represents the function in the neighborhood of the sampling point. This feature has many fundamental approximation theoretic consequences. First, it allows to approximate the function in a given region using only samples taken in a fixed-size neighborhood of it. Second, locality is a key ingredient for many signal processing methods. Examples are coarse quantization schemes such as Sigma-Delta modulation, where the quantization accuracy depends on the good control over finite differences between neighboring samples.

In this paper we develop discrete representations of operators with bandlimited Kohn-Nirenberg symbols that, on the one hand, can be computed in a direct and simple way from the output corresponding to a test signal and, on the other hand, have locality properties analogous to those we appreciate in the classical sampling theory. We work with the same concept of locality as in the localized sampling results mentioned above, namely, locality will be defined through the action of the operator on time-frequency localized functions.

Combining the results of our analysis, we show that a suitable local discrete representation arises from time-frequency measurements of the output corresponding to a truncated and mollified weighted sum of Dirac delta distributions. Moreover, these discrete measurements allow for arbitrary oversampling rates, which allows to design coarse quantization schemes based on the resulting representations.

The paper is organized as follows. We summarize our main results and put them in the context of previous work in Section 2. In Section 3 we recall operator sampling terminology in some detail. Section 4 provides results on local approximations of operators; in Section 5 we discuss identification using smooth and finite duration test signals. Section 6 uses Gabor frames to derive our novel discretization scheme for operators with bandlimited Kohn-Nirenberg symbols, and in Section 7 we use the resulting representations to devise a coarse quantization scheme for such operators.
2. Main Results

Every bounded linear operator on $L^2(\mathbb{R})$ has a formal Kohn-Nirenberg symbol representation

$$Hf(x) = \int \sigma_H(x, \xi) \widehat{f}(\xi) e^{2\pi i x\xi} \, d\xi$$

where $\sigma_H$ is a tempered distribution on $\mathbb{R}^2$, the integral is understood to converge weakly, and here and in the following $\widehat{f}$ denotes integration over $\mathbb{R}$. The Fourier transform $\widehat{f}$ is normalized by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int f(t) e^{-2\pi i t\xi} \, dt$$

for integrable $f$.

The space of bounded operators whose Kohn-Nirenberg symbols are bandlimited to a given set $M$ — we will also use the shorthand terminology bandlimited operators — is called operator Paley-Wiener space$^1$; it is denoted by

$$\text{OPW}(M) = \{ H \in \mathcal{L}(L^2(\mathbb{R})) : \text{supp} \mathcal{F}_s \sigma_H \subseteq M \},$$

where the symplectic Fourier transform $\mathcal{F}_s$ is defined densely by

$$\mathcal{F}_s \sigma(t, \gamma) = \int \int \sigma(x, \xi) e^{-2\pi i (x\gamma - t\xi)} \, dx \, d\xi.$$

The Kohn-Nirenberg symbol of an $L^2$-bounded operator with $\text{supp} \mathcal{F}_s \sigma_H$ compact is bounded. In fact, for some $A, B > 0$ we have,

$$A \| \sigma_H \|_{L^\infty(\mathbb{R})} \leq \| H \|_{\mathcal{L}(L^2(\mathbb{R}))} \leq B \| \sigma_H \|_{L^\infty(\mathbb{R})},$$

for all $H \in \text{OPW}(M)$, where $\| H \|_{\mathcal{L}(L^2(\mathbb{R}))}$ is the operator norm of $H$ (Proposition 4.1 below).

Certainly, if we have direct access to $\sigma_H$, then some of our approximation theoretic goals can be accomplished using classical two-dimensional sampling results applied to $\sigma_H$. In the model considered here, however, we do not have access to any of the values of the symbol $\sigma_H$ of the operator $H$ directly, but we must rely on the operator output $Hw$ resulting from applying $H$ to a single test input $w$. Due to stability consideration, we say that the linear space $\text{OPW}(M)$ is identifiable by $w$ if for $A, B > 0$ we have

$$A \| H \|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \| Hw \|_{L^2(\mathbb{R})} \leq B \| H \|_{\mathcal{L}(L^2(\mathbb{R}))},$$

for all $H \in \text{OPW}(M)$ [15]. “Sampling” the operator means that the identifier $w$ in (2.4) is a weighted sequence of Dirac delta distributions, that is,

$$w = \sum_{k \in \mathbb{Z}} c_k \delta_{kT},$$

where $c_k$ is an appropriately chosen periodic sequence [18, 25, 22].

A guiding paradigm in the sampling theory of operators is the direct analogy to sampling of bandlimited functions. To illustrate this analogy, we compare the classical sampling theorem (often credited to Cauchy, Kotelnikov, Shannon, and Whittaker, among others), Theorem 2.1, with the corresponding result for operators, Theorem 2.2 [22]. Note that Theorem 2.1 formally follows from Theorem 2.2 by choosing the operator $H$ in Theorem 2.2 to be the pointwise multiplication operator $f \mapsto \sigma \cdot f$ [22].

The engineering intuition underlying sampling theorems is that reducing a function to periodic samples at a rate of $1/T$ samples per unit interval corresponds to a periodization with shift $1/T$ in

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$^1$In general terms, operator Paley-Wiener spaces are defined by requiring its members to have bandlimited Kohn-Nirenberg symbol which are in a prescribed weighted and mixed $L^p$ space [22]. For example, to restrict the attention to bandlimited Hilbert-Schmidt operators, we would consider only operators with square integrable symbols. These form a subset of the operators considered in this paper.
frequency space [20]. Thus, as long as $T\Omega \leq 1$, a function bandlimited to $[-\frac{\Omega}{2}, \frac{\Omega}{2}]$ can be recovered via a convolution with a low-pass kernel, that is, a function $\phi$ that satisfies

$$\hat{\phi}(\xi) = \begin{cases} 1/\Omega, & \text{if } |\xi| \leq \frac{\Omega}{2}, \\ 0, & \text{if } |\xi| \geq \frac{1}{2T}. \end{cases}$$

If $T\Omega = 1$, the only such function is $\phi(t) = \text{sinc}(\pi t/T) = \frac{\sin(\pi t/T)}{\pi t/T}$. For $T\Omega < 1$, there are many such functions; in particular $\phi$ in the Schwartz class is possible. With this notion, the classical sampling theorem reads as follows.

**Theorem 2.1.** For $g \in L^2(\mathbb{R})$ with $\text{supp} \mathcal{F}g \subseteq [-\frac{\Omega}{2}, \frac{\Omega}{2}]$ and $T\Omega \leq 1$, we have

$$g(x) = \sum_{n \in \mathbb{Z}} g(nT) \phi(x - nT)$$

with uniform convergence and convergence in $L^2(\mathbb{R})$. Here, $\phi$ is any low-pass kernel satisfying (2.5).

Recall that every operator $H$ on $L^2(\mathbb{R})$ is in one-to-one correspondence with its kernel $\kappa_H$, that is, for a unique tempered distribution $\kappa_H$, we have $Hf(x) = \int \kappa_H(x,y) f(y) \, dy$ weakly. In the following, $\chi_A$ denotes the characteristic function of a set $A$.

**Theorem 2.2.** [22] For $H : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ with $\sigma_H \in L^2(\mathbb{R}^2)$, $\text{supp} \mathcal{F}_x \sigma_H \subseteq [0, T] \times [-\frac{\Omega}{2}, \frac{\Omega}{2}]$, and $T\Omega \leq 1$, we have

$$\kappa_H(x + t, x) = \chi_{[0,T]}(t) \sum_{n \in \mathbb{Z}} (H \sum_{k \in \mathbb{Z}} \delta_{kT})(t + nT) \phi(x - nT),$$

with convergence in $L^2(\mathbb{R}^2)$ and uniform convergence in $x$ for each $t$. Again, $\phi$ is any low-pass kernel satisfying (2.5).

We point to an important difference between the applicability of Theorems 2.1 and 2.2: in Theorem 2.1, a bandlimitation to a large set $[-\frac{\Omega}{2}, \frac{\Omega}{2}]$ can be resolved by choosing a small $T$; on the other side, Theorem 2.2 is not applicable if the bandlimiting set $[0, T] \times [-\frac{\Omega}{2}, \frac{\Omega}{2}]$ has area greater than one. Indeed, in [25, 23] the following is shown.

**Theorem 2.3.** $\text{OPW}(M)$ is identifiable in the sense of (2.4) with appropriate $w = \sum_{n \in \mathbb{Z}} c_n \delta_{nT}$ if $M$ is compact with measure less than 1. If $M$ is open and has area greater than 1, then exists no tempered distribution $w$ identifying $\text{OPW}(M)$.

Hence, it is necessary to restrict ourselves to operator Paley-Wiener spaces defined by compact sets $M$ with Lebesgue measure one. For such spaces, one can extend Theorem 2.2 to the following.

**Theorem 2.4.** [24] Let $M$ be compact with Lebesgue measure less than one. Then exists $T, \Omega > 0$ with $T\Omega = \frac{1}{T'}$, $L$ prime, $\delta > 0$, and an $L$-periodic sequence $(c_n)$ so that with $w = \sum_n c_n \delta_{nT}$, we have for $H \in \text{OPW}(M)$

$$\kappa_H(x + t, x) = LT \sum_{j=0}^{L-1} r(t - k_j T)(\sum_{q \in \mathbb{Z}} b_{jq} Hw(t - (k_j - q)T)\phi(x + (k_j - q)T) e^{2\pi i n, \Omega x},$$

where $r, \phi$ are Schwartz class functions that satisfy

$$r(t) \hat{\phi}(\gamma) = 0 \text{ if } (t, \gamma) \notin (-\delta, T + \delta) \times (-\delta, \Omega + \delta),$$

and

$$\sum_{k \in \mathbb{Z}} r(t - kT) \equiv 1 \equiv \sum_{n \in \mathbb{Z}} \hat{\phi}(\gamma - n\Omega).$$

Moreover, (2.8) converges in $L^2(\mathbb{R})$ and uniformly in $t$. 

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In both sampling scenarios, working with Schwartz class kernels $r,\phi$ is of advantage. Indeed, in the classical sampling theorem, the slow decay of the sinc function $\sin x / x$ in (2.6) implies that a small perturbation of just a few coefficients $g(nT)$ can lead to significant deviations of all values $g(t)$ outside of the sampling grid $TZ$; this includes values achieved at locations far from the sampling points $nT$. Hence to approximately recover the function values locally, that is, on an interval $[a,b]$, it does not suffice to know the function samples in a constant size neighborhood of that interval. When working with Schwartz class kernels, in contrast, such a local approximate reconstruction is possible; one has

$$g(x) \approx \sum_{n \in [a-d,b+d]} g(nT) \phi(x-nT),$$

where the neighborhood size $d$ does not depend on the interval $[a,b]$.

A corresponding possibility of using local information for local reconstruction is not given in Theorem 2.2. Moreover, the identifier $w = \sum_{n \in \mathbb{Z}} \delta_{nT}$ neither decays in time or in frequency, clearly showing that in practice, this input signal is not usable. However, in the framework of Theorem 2.2, this is unavoidable, as the following theorem shows.

**Theorem 2.5.** If the tempered distribution $w$ identifies $\text{OPW}([0,T] \times [-\Omega/2, \Omega/2])$, $T\Omega > 0$, then $w$ decays weakly neither in time nor in frequency, that is, we have neither

$$\langle w, \varphi(-x) \rangle \xrightarrow{x \to \pm \infty} 0 \quad \text{nor} \quad \langle \hat{w}, \varphi(-\xi) \rangle \xrightarrow{\xi \to \pm \infty} 0$$

for all Schwartz class functions $\varphi$.

We address this problem by developing a concept of “local recovery” of an operator, in analogy to the local recovery of a function in (2.10). Indeed, the key to most results presented in this paper is the local recovery of a function in (2.10). Hence, a natural way to define time-frequency localized functions is that all but certain expansion coefficients are small.

The definition of functions localized in time and frequency is based on Gabor frames. Their definition involves translation and modulation operators,

$$T_t f : f \mapsto f(\cdot - t) \quad \text{and} \quad M_\nu : f \mapsto e^{2\pi i \nu \cdot} f.$$  

These operators are unitary on $L^2(\mathbb{R})$ and isomorphisms on all function and distribution spaces considered in this paper.

For any $g \in L^2(\mathbb{R})$ and $a, b > 0$, we say that the Gabor system

$$(g, a\mathbb{Z} \times b\mathbb{Z}) = \{T_{ka} M_{\ell b} g \}_{k, \ell \in \mathbb{Z}}$$

is a tight frame for $L^2(\mathbb{R}^d)$ if for some $A > 0$, we have

$$f = A \sum_{k, \ell \in \mathbb{Z}} \langle f, T_{ka} M_{\ell b} g \rangle \ T_{ka} M_{\ell b} g$$

for all $f \in L^2(\mathbb{R}^d)$. Each coefficient in this expansion can be interpreted to reflect the local behavior of the function near the indexing point in time-frequency space. Hence, a natural way to define time-frequency localized functions is that all but certain expansion coefficients are small.

**Definition 2.6.** Let $(g, a\mathbb{Z} \times b\mathbb{Z})$, $g \in \mathcal{S}(\mathbb{R})$, be a tight frame for $L^2(\mathbb{R})$ with frame bound 1. We say that $f \in L^2(\mathbb{R})$ is $\epsilon$-time-frequency localized on the set $S$ if

$$\sum_{(ka, \ell b) \in S} |\langle f, M_{\ell b} T_{ka} g \rangle|^2 \geq (1 - \epsilon^2) \sum_{(ka, \ell b) \in \mathbb{R}^2} |\langle f, M_{\ell b} T_{ka} g \rangle|^2.$$
Our first result states that a sufficient condition for two operators to approximately agree on functions $\epsilon$-time-frequency localized on a set $S$ is that their Kohn-Nirenberg symbols almost agree on a neighborhood of $S$. Below, $B(r)$ denotes the Euclidean unit ball with radius $r$ and center 0; the dimension will always be clear from the context. For brevity of notation, we set $S - B(r) = (S^c + B(r))^c$ for $S \subseteq \mathbb{R}^2$.

**Theorem 2.7.** Fix $M$ compact with $\mu(M) < 1$ and let $(g, a\mathbb{Z} \times b\mathbb{Z}), g \in S(\mathbb{R})$, be a tight frame for $L^2(\mathbb{R})$ with frame bound 1. Then exists $C > 0$ and a strictly monotone function $\epsilon : (0, 1) \rightarrow \mathbb{R}^+$, $\lim_{\epsilon \rightarrow 0} \epsilon = +\infty$, with the property that if $H, \tilde{H} \in \operatorname{OPW}(M)$ satisfy on a set $S \subseteq \mathbb{R}^2$ the bounds

$$\|\sigma_H\|_{L^\infty(\mathbb{R}^2)}, \|\sigma_{\tilde{H}}\|_{L^\infty(\mathbb{R}^2)} \leq \mu \quad \text{and} \quad \|\sigma_H - \sigma_{\tilde{H}}\|_{L^\infty(S)} \leq \epsilon \mu,$$

then

$$\|Hf - \tilde{H}f\|_{L^2(\mathbb{R})} \leq C \epsilon \mu \|f\|_{L^2(\mathbb{R})}$$

for all $f \in L^2(\mathbb{R})$ that are $\epsilon$-time-frequency localized on $S - B(\epsilon)$ in the sense of Definition 2.6.

This observation is a key ingredient in the proof of our next main result. It concerns truncated and mollified versions of the identifier $\sum_n c_n \delta_{qT}$ and provides localized versions of Theorems 2.2 and 2.4. For $S = \mathbb{R}^2$, it reduces to Theorems 2.2 and 2.4.

**Theorem 2.8.** Fix $M$ compact with $\mu(M) < 1$ and let $(g, a\mathbb{Z} \times b\mathbb{Z}), g \in S(\mathbb{R})$, be a tight frame for $L^2(\mathbb{R})$ with frame bound 1. Choose $\delta > 0$ such that $\mu(M + [-3\delta, 3\delta]^2) < 1$. Then exists $C > 0$ and a strictly monotone function $\epsilon : (0, 1) \rightarrow \mathbb{R}^+$, $\lim_{\epsilon \rightarrow 0} \epsilon = +\infty$, with the following property:

Let $H \in \operatorname{OPW}(M)$ satisfy the bound $\|\sigma_H\|_{L^\infty(\mathbb{R}^2)} \leq \mu$ and let $S \subseteq I_1 \times I_2 \subseteq \mathbb{R}^2$, where $I_1$ and $I_2$ may coincide with $\mathbb{R}$. Furthermore, let

$$\tilde{w} = \sum_{nT \in I_1} c_n \varphi(\cdot - nT),$$

where the tempered distribution $\varphi$ is chosen such that $\varphi \geq 0$ and $\hat{\varphi} \equiv 1$ on $I_2$ and define for $H \in \operatorname{OPW}(M)$ the operator $\tilde{H}$ via

$$\kappa_{\tilde{H}}(x + t, x) = LT \sum_{j=0}^{L-1} r(t - k_j T) \left( \sum_{q \in \mathbb{Z}} b_{jq} H \tilde{w}(t - (k_j - q) T) \phi(x + (k_j - q) T) \right) e^{2\pi i n T x},$$

where $r, \phi$ are Schwartz class functions defined as in Theorem 2.4, but for the above choice of $\delta$.

For rectangular bandlimitation domains $M = [0, T] \times [-\frac{\Omega}{2}, \frac{\Omega}{2}]$ one can choose the identifier $\sum_{nT \in I_1} \varphi(\cdot - nT)$ and define $\tilde{H}$ via the formula

$$\kappa_{\tilde{H}}(x + t, x) = T \sum_{n \in \mathbb{Z}} \left( H \sum_{nT \in I_1} \varphi(\cdot - nT) \right) (t + n T) \phi(x - n T).$$

In both cases one has

$$\|Hf - \tilde{H}f\|_{L^2(\mathbb{R})} \leq C \epsilon \mu \|f\|_{L^2(\mathbb{R})}$$

for all $f \in L^2(\mathbb{R})$ that are $\epsilon$-time-frequency localized on $S - B(\epsilon)$ in the sense of Definition 2.6.

Note that this theorem is completely analogous to the condition (2.10) for localized function sampling. Due to the two-dimensional nature of the operator, however, localization is an issue in both time (restricting to a finite number of deltas) and frequency (replacing the deltas by approximate identities). If one is interested in localization only in time or only in frequency, one can choose one of the $I_i$ to be $\mathbb{R}$ and thus consider

$$\tilde{w} = \sum_{nT \in I_1} c_n \delta_{nT} \quad \text{or} \quad \tilde{w} = \sum_n c_n \varphi(\cdot - nT),$$

and with $(c_n) \equiv 1$ in case of rectangular domains $M$.  

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where

with \( \phi \) functions, not scalars. Among the many possibilities to discretely represent the operator’s response to the identifier \( w \), we consider Gabor representations of this sample function. A time-frequency localized subset of the coefficients will then yield a corresponding local approximation of the operator. Theorem 2.9 below establishes a reconstruction formula based on Gabor coefficients that allows for the exact recovery of the operator; Theorem 2.10 shows that a local subset of the coefficients yields a local approximation of the operator. Again, one obtains considerably simpler formulas for rectangular domains, but for reasons of brevity, we focus on the comprehensive setup of arbitrary domains.

For a Schwartz class function \( \phi \) and a tempered distribution \( f \) on \( \mathbb{R} \) we call

\[
V_\phi f(x, \xi) = \langle f, M_\xi T_x \phi \rangle, \quad x, \xi \in \mathbb{R},
\]

the short-time Fourier transform of \( f \) with respect to the window function \( \phi \). Throughout this paper, all pairings \( \langle \cdot, \cdot \rangle \) are taken to be linear in the first component and antilinear in the second.

**Theorem 2.9.** For \( M \) compact with \( \mu(M) < 1 \) exists \( L \) prime, \( \delta > 0, T, \Omega > 0 \) with \( T \Omega = 1/L \), an \( L \)-periodic sequence \( (c_n) \), and a sequence \( \{B_{jq}\}_{j=0,...,L-1,q \in \mathbb{Z}} \) (which is \( L \)-periodic in \( q \) and depends only on the sequence \( (c_n) \)), such that with \( w = \sum_n c_n \delta_{nT} \) we have for \( H \in \text{OPW}(M) \)

\[
\sigma_H(x, \xi) = \frac{LT}{\beta_1 \beta_2} \sum_{j=0}^{L-1} e^{-2\pi i(xn, \Omega + \xi k, T)} e^{2\pi i n, \Omega k, T} \sum_{m, \ell \in \mathbb{Z}} \sigma_{m, \ell}^{(j)} \gamma(x - (mL/\beta_1 + k)T, \xi - (\ell \beta_2 + n)\Omega),
\]

where

\[
\sigma_{m, \ell}^{(j)} = \sum_{q \in \mathbb{Z}} B_{jq} \phi((-q - k - mL/\beta_1)T) \langle Hw, T_q T_\xi M_{\ell \Omega/L} \gamma \rangle,
\]

and \( r, \phi \) are Schwartz class functions such that \( r \) and \( \widehat{\phi} \) are real valued and satisfy

\[
r(t) = 0 \quad \text{if} \quad t \notin (-\delta, \delta + T), \quad \widehat{\phi}(\gamma) = 0 \quad \text{if} \quad \gamma \notin (-\delta - \Omega/2, \delta + \Omega/2),
\]

and

\[
\sum_{k \in \mathbb{Z}} |r(t + kT)|^2 \equiv 1 \equiv \sum_{n \in \mathbb{Z}} |\widehat{\phi}(\gamma + n\Omega)|^2,
\]

with oversampling rates \( \beta_2 \geq 1 + 2\delta/T \) and \( \beta_1 \geq 1 + 2\delta/\Omega).^3

Observe that the reconstruction formulas given in Theorems 2.4 and 2.8 require \( r \) and \( \widehat{\phi} \) to generate partitions of unity (2.15), while (2.9) above requires that their modulus squared form partitions of unity.

**Theorem 2.10.** Fix \( M \) compact with \( \mu(M) < 1 \), let \( T, \Omega, L \) and \( w, r, \phi \) be defined in Theorem 2.4, and let \( (g, a\mathbb{Z} \times b\mathbb{Z}), g \in \mathcal{S}(\mathbb{R}), \) be a tight frame for \( L^2(\mathbb{R}) \) with frame bound 1. Then exists \( C > 0 \) and a strictly monotone function \( d : (0, 1) \to \mathbb{R}^+ \), \( \lim_{\epsilon \to 0} d(\epsilon) = +\infty \), with the following property:

Let \( H \in \text{OPW}(M) \) satisfy the bound \( \|\sigma_H\|_{L^\infty(\mathbb{R}^2)} \leq \mu \), fix a not necessarily bounded set \( S \subseteq I_1 \times I_2 \) in \( \mathbb{R}^2 \), choose \( \varphi \) and \( \tilde{\varphi} \) as in Theorem 2.8, and define the operator \( \tilde{H} \) via its symbol

\[
\tilde{\sigma}(x, \xi) = \frac{LT}{\beta_1 \beta_2} \sum_{j=0}^{L-1} e^{-2\pi i(xn, \Omega + \xi k, T)} e^{2\pi i n, \Omega k, T} \sum_{(mL/\beta_1 + k)T, (\ell \beta_2 + n)\Omega) \in S} \sigma_{m, \ell}^{(j)} \gamma(x - (mL/\beta_1 + k)T, \xi - (\ell \beta_2 + n)\Omega).
\]

^2For example, we can choose \( r = \chi_{(0, T)} \ast \varphi_\delta \), where \( \varphi_\delta \) is an approximate identity, that is, a non-negative function with \( \varphi_\delta \in \mathcal{S}(\mathbb{R}) \), \( \text{supp} \varphi_\delta \subseteq [-\delta, \delta/2] \), and \( \int \varphi_\delta = 1 \).

^3Then the Gabor systems \( \{T_{k, \ell}, M_{\ell \Omega/L} \gamma \}_{k, \ell \in \mathbb{Z}} \), \( \{T_{m, n} \gamma, M_{\ell \Omega/L} \gamma \}_{m, n \in \mathbb{Z}} \), and \( \{\varphi_{m, n, l, k} = T_{mL/\beta_1, \ell \Omega/L} M_{\ell \Omega/L} \gamma \}_{m, n, k, \ell \in \mathbb{Z}} \) are tight Gabor frames with \( A = \beta_2/T, \beta_1/\Omega, \) and \( A = \beta_1 \beta_2/\Omega \).

\[
\begin{align*}
\text{Theorem 2.4} & \quad \text{Theorem 2.8} & \quad \text{Theorem 2.9} & \quad \text{Theorem 2.10}
\end{align*}
\]
where
\[ \tilde{\sigma}_{m,\ell}^{(j)} = \sum_{q \in \mathbb{Z}} B_{jq} \phi((-q - k_j - mL/\beta_1)T) \langle H\tilde{w}, T_q T M_{Hw/\beta_2} r \rangle. \]

Then \( \tilde{H} \) satisfies
\[ \| Hf - \tilde{H}f \|_{L^2(\mathbb{R})} \leq C \varepsilon \mu \| f \|_{L^2(\mathbb{R})} \]
for all \( f \in L^2(\mathbb{R}) \) which are \( \varepsilon \)-time-frequency localized on \( S - B(D(\varepsilon)) \) with respect to \((g, a\mathbb{Z} \times b\mathbb{Z})\) in the sense of Definition 2.6.

The discrete representations introduced in Theorems 2.9 and 2.10 resolve a fundamental conceptual difference between classical sampling and operator sampling. In contrast to classical sampling, which yields a set of separate function values, the contributions of the different Dirac-deltas in the operator sampling formula are combined in a single function and cannot easily be separated. Hence, while choosing a higher sampling rate in the function case yields more information, in the operator case, this additional information is mixed in an inseparable way. These aliasing effects [15] make it impossible to obtain redundant representations merely by oversampling in Theorem 2.2 or Theorem 2.4. In reconstruction formula (2.13), however, the oversampling parameters \( \beta_i \) can be chosen arbitrarily, allowing for representations of arbitrarily large redundancy.

The interplay of large redundancy and good local representation properties of the discrete coefficients allows for coarse quantization methods to be applied to operators. In the mathematical literature, the most common scenarios for such methods deal with frames in \( \mathbb{R}^n \) [3, 4, 17] or the space of bounded bandlimited functions on \( \mathbb{R}^n \) [6, 12, 7]. The underlying idea is to sample at a high rate and use the resulting redundancy to lower the number of bits needed to represent each sample. The following establishes a corresponding result for bandlimited operators.

**Theorem 2.11.** Fix a compact set \( M \) with \( \mu(M) < 1 \) and the associated parameters \( L, \Omega, \) and \( T \) as in Theorem 2.9. Then there are constants \( c, C \) such that for each \( H \in \text{OPW}(M) \) satisfying \( \| \sigma_H \|_{L^\infty(\mathbb{R}^2)} \leq c \) and all oversampling rates \( \beta_1, \beta_2 > 1 \), one can compute quantized values \( \tilde{q}_{m,\ell}^{(j)} \in \{\pm 1, \pm 3\} \) directly from \( Hw \) with \( w = \sum_k c_k \delta_{kT} \) via a recursive procedure. These quantized values give rise to an approximate reconstruction of \( H \) through the operator \( \tilde{H} \) with symbol
\[ \tilde{\sigma}(x, \xi) = \frac{LT}{\beta_1 \beta_2} \sum_{j=0}^{L-1} e^{-2\pi i (x + k_j \xi)} e^{2\pi i n_j \xi} T_{q_{m,\ell}^{(j)} V_{\phi r}(x - (mL/\beta_1 + k_j)T, \xi - (\ell L/\beta_2 + n_j)\Omega), \]
which satisfies
\[ \| Hf - \tilde{H}f \|_{L^2(\mathbb{R})} \leq \frac{C}{\beta_1 \beta_2} \| f \|_{L^2(\mathbb{R})} \]
for all \( f \in L^2(\mathbb{R}) \).

Combining all these results, we obtain a result about local reconstruction from local quantized values resulting from a localized identifier.

**Corollary 2.12.** Fix a compact set \( M \) with \( \mu(M) < 1 \) and the associated parameters \( L, \Omega, \) and \( T \) as in Theorem 2.9. Then there are constants \( c, C \), and a strictly monotone function \( d : (0, 1) \to \mathbb{R}^+ \), \( \lim_{\varepsilon \to 0} d(\varepsilon) = +\infty \) such that for each \( H \in \text{OPW}(M) \) satisfying \( \| \sigma_H \|_{L^\infty(\mathbb{R}^2)} \leq c \) and all oversampling rates \( \beta_1, \beta_2 > 1 \), one can compute quantized values \( \tilde{q}_{m,\ell}^{(j)} \in \{\pm 1, \pm 3\} \) from \( H\tilde{w} \), with \( \tilde{w} \) as in Theorem 2.8, via a recursive procedure. Then using the finitely many quantized values with indices \( (m, \ell) \) satisfying \( (mLT/\beta_1, \ell L\Omega/\beta_2) \in S \), one obtains a local approximate reconstruction of \( H \) on \( S - d(\varepsilon) \) through the operator \( \tilde{H} \) with symbol
\[ \tilde{\sigma}(x, \xi) = \frac{LT}{\beta_1 \beta_2} \sum_{j=0}^{L-1} e^{-2\pi i (x + k_j \xi)} e^{2\pi i n_j \xi} T_{q_{m,\ell}^{(j)} V_{\phi r}(x - (mL/\beta_1 + k_j)T, \xi - (\ell L/\beta_2 + n_j)\Omega), \]
where
This operator satisfies
\[ \| Hf - \tilde{H}f \|_{L^2(\mathbb{R})} \leq \frac{C}{\beta_1 \beta_2} \| f \|_{L^2(\mathbb{R})} + C \varepsilon \mu \| f \|_{L^2(\mathbb{R})} \]
for all \( f \in L^2(\mathbb{R}) \) which are \( \varepsilon \)-time-frequency localized on \( S - B(D(\varepsilon)) \) with respect to \((g, a\mathbb{Z} \times b\mathbb{Z})\) in the sense of Definition 2.6.

3. Bandlimited pseudodifferential operators and operator Paley-Wiener spaces

It is well known that every bounded linear operator \( H : S(\mathbb{R}^d) \to S'(\mathbb{R}^d) \) is of the form
\[ Hf(x) = \int \kappa(x, t) f(t) dt \]
for some \( \kappa \in S'(\mathbb{R}^{2d}) \), where \( S(\mathbb{R}^d) \) is the Schwartz space, and \( S'(\mathbb{R}^d) \) is its dual, the space of tempered distributions [10]. This integral representation is understood in the weak sense, that is,
\[ \langle Hf, g \rangle = \langle \kappa, f \otimes g \rangle \]
for all \( f, g \in S(\mathbb{R}^d) \), where \( f \otimes g(x, y) = f(x)g(y) \) and \( \langle \cdot, \cdot \rangle \) is the sesquilinear pairing between \( S \) and \( S' \) functions. Each such operator has a spreading function representation
\[ Hf = \int \int \eta(t, \gamma) M_{\gamma} T_t f \ dt \ d\gamma, \tag{3.1} \]
a time-varying impulse response representation
\[ Hf(x) = \int h(x, t)f(x - t) dt, \]
and a Kohn-Nirenberg symbol representation
\[ Hf(x) = \int \sigma(x, \xi) \hat{f}(\xi) e^{2\pi i x \xi} \ d\xi. \]

We write \( H_\sigma \) and \( \sigma_H, \eta_H, \kappa_H \) when it is necessary to emphasize the correspondence between \( H \) and \( \sigma, \eta, \kappa \). The symbols \( \sigma \) and \( \eta \) are related via the symplectic Fourier transform \( \mathcal{F}_s \) defined in (2.2), that is, \( \sigma = \mathcal{F}_s \eta \). For convenience, we use the symbol \( \eta_H(t, \gamma) = e^{2\pi i \gamma t} \eta_H(t, \gamma) \), and denote its symplectic Fourier transform by
\[ \sigma_H(x, \xi) = \int \int \eta_H(t, \gamma) e^{2\pi i x \gamma - t \xi} \ dt \ d\gamma. \]
The relationship between \( \sigma_H \) and \( \sigma_{H^*} \) is given by
\[ \sigma_{H^*} = \sigma_{\bar{H}}, \tag{3.2} \]
where \( H^* \) denotes the adjoint of \( H \). Indeed, we have for Schwartz functions \( f, g \)
\[
\langle H^* f, g \rangle = \langle f, Hg \rangle = \langle f(x), \int \eta_H(t, \nu) M_{\nu} T_t g(x) \ dt \ d\nu \rangle \\
= \int \int \overline{\eta_H(t, \nu)}(f, M_{\nu} T_t g) \ dt \ d\nu \\
= \int \int \overline{\eta_H(t, \nu)}(T_{-t} M_{-\nu} f, g) \ dt \ d\nu \\
= \langle \int \int \overline{\eta_H(-t, -\nu)} T_{-t} M_{\nu} f \ dt \ d\nu, g \rangle \\
= \langle \int \int \overline{\eta_H(-t, -\nu)} e^{-2\pi i t \nu} M_{\nu} T_t f \ dt \ d\nu, g \rangle 
\]
and conclude using a density argument that \( \eta_{H^*}(t, \nu) = \eta_H(-t, -\nu) e^{-2\pi it\nu} \). Hence,

\[
\sigma_{H^*}(x, \xi) = \mathcal{F}_s \eta_{H^*}(x, \xi) = \int \int \eta_H(-t, -\nu) e^{-2\pi it\nu} e^{-2\pi i\xi\nu} dt \, d\nu
\]

\[
= \int \int \eta_H(t, \nu) e^{2\pi it\nu} e^{-2\pi i\xi\nu} dt \, d\nu
\]

\[
= \int \mathcal{F}_s \eta_H(x, \xi) = \sigma_H(x, \xi).
\]

To prove our results, we shall frequently transition from \( \sigma \) to \( \sigma \). This does not cause a problem in our analysis since (2.3) combined with \( ||H||_{\mathcal{L}(L^2(\mathbb{R}))} = ||H^*||_{\mathcal{L}(L^2(\mathbb{R}))} \) shows that for \( M \subseteq \mathbb{R}^2 \) compact exist \( A, B > 0 \) with

\[
A ||\sigma||_{L^\infty(\mathbb{R}^2)} \leq ||\sigma||_{L^\infty(\mathbb{R}^2)} \leq B ||\sigma||_{L^\infty(\mathbb{R}^2)}
\]

for all \( H \in \text{OPW}(M) \).

4. Local approximation of bandlimited operators

In this section we show that a local approximation of an operator’s symbol always yield a local approximation of the operator in the sense of Definition 2.6. The given results are of general interest and will be stated in more general terms than other results in this paper. This does not increase the difficulty of proof, but necessitates to recall additional terminology from time-frequency analysis.

For that, recall that for any full rank lattice \( \Lambda = A \mathbb{Z}^2 \subseteq \mathbb{R}^2 \), \( A \neq 0 \), \( \ell^p(\Lambda) \) denotes the set of sequences \( (c_\lambda)_{\lambda \in \Lambda} \) for which

\[
||c||_{\ell^p(\Lambda)} = (\sum_{\lambda \in \Lambda} |c_\lambda|^p)^{1/p} < \infty.
\]

A time-frequency shift by \( \lambda = (t, \nu) \in \Lambda \) is denoted by \( \pi(\lambda) = M_\nu T_t \) and in the following we will consider Gabor systems of the form \( (g, \Lambda) = \{\pi(\lambda)g\}_{\lambda \in \Lambda} \).

Among the many equivalent definitions of modulation spaces, we choose the following. Let \( g_0(x) = e^{-\|x\|} \) and \( 1 \leq p \leq \infty \). Then

\[
M^p(\mathbb{R}^d) = \{f \in S'(\mathbb{R}^d) : ||f||_{M^p(\mathbb{R}^d)} = ||(f, \pi(\lambda)g_0)_\lambda||_{\ell^p(\frac{1}{2} \mathbb{Z}^d)} < \infty \}
\]

(see, for example, [11, 8]). In the following we shall use the fact that whenever \( (g, \Lambda) \) is a tight \( L^2 \)-Gabor frame (see below for a precise definition) with \( g \in M^1(\mathbb{R}^d) \) then replacing the \( L^2 \)-Gabor frame \( (g_0, \frac{1}{2} \mathbb{Z}^d) \) in (4.1) with \( (g, \Lambda) \) leads to an equivalent norm on \( M^p(\mathbb{R}^d) \) [11]. That is, there exist positive constants \( A \) and \( B \) with

\[
A ||f||_{M^p(\mathbb{R}^d)} \leq \sum_{\lambda \in \Lambda} \|f, \pi(\lambda)g\|_p \leq B ||f||_{M^p(\mathbb{R}^d)}, \quad f \in M^p(\mathbb{R}^d)
\]

if \( 1 \leq p < \infty \) and

\[
A ||f||_{M^\infty(\mathbb{R}^d)} \leq \sup_{\lambda \in \Lambda} \|f, \pi(\lambda)g\| \leq B ||f||_{M^\infty(\mathbb{R}^d)}, \quad f \in M^\infty(\mathbb{R}^d)
\]

if \( p = \infty \). In either case, we call \( (g, \Lambda) \) an \( \ell^p \)-frame with lower frame bound \( A \) and upper frame bound \( B \). If we can choose \( A = B \) in case of \( p = 2 \) then we call \( (g, \Lambda) \) a tight Gabor frame.

The norm equivalence (2.3) follows from the following result since \( M^2(\mathbb{R}) = L^2(\mathbb{R}) \).

Theorem 4.1. Let \( 1 \leq p \leq \infty \) and \( M \) compact. Then exist positiv constants \( A = A(M, p) \) and \( B = B(M, p) \) with

\[
A \|\sigma_H\|_{L^\infty(\mathbb{R}^2)} \leq ||H||_{\mathcal{L}(M^p(\mathbb{R}))} \leq B \|\sigma_H\|_{L^\infty(\mathbb{R}^2)}, \quad H \in \text{OPW}(M).
\]
Proof. Theorem 2.7 in [22] (see for example the proof of Theorem 3.3 in [22]) provides \( C = C(M, p) \) with
\[
\|Hf\|_{M^p(\mathbb{R})} \leq C \|\sigma_H\|_{L^\infty(\mathbb{R}^2)} \|f\|_{M^p(\mathbb{R})}
\]
for all \( H \in \text{OPW}(M) \). This establishes the existence of \( B = B(M, p) \) above.

In addition, we shall use the following facts. In [9, 11] it is shown that the operator norm of an operator mapping the modulation space \( M^1(\mathbb{R}) \) into its dual \( M^\infty(\mathbb{R}) \) is equivalent to the \( M^\infty(\mathbb{R}^2) \) norm of its kernel \( \kappa \), which can easily shown to be equivalent to the \( M^\infty(\mathbb{R}^2) \) norm of the time-varying impulse response \( h \). Moreover, we use the fact that \( M^\infty(\mathbb{R}^2) \) is invariant under Fourier transforms (in some or all variables) and that the \( M^\infty(\mathbb{R}^2) \) norm can be replaced by the \( L^\infty(\mathbb{R}^2) \) norm if we restrict ourselves to functions bandlimited to a fixed set \([19, 22]\). Last but not least, we use that the identity map embedding \( M^p(\mathbb{R}) \) into \( M^q(\mathbb{R}) \), \( p \leq q \), is bounded.

Writing \( \lesssim \) to express that \( A \leq CB \) for some constant \( C \) depending only on the support \( M \) and \( A \approx B \) to denote equivalence in norms, i.e., \( A \lesssim B \) and \( B \lesssim A \), we obtain for all \( H \in \text{OPW}(M) \)
\[
\|\sigma_H\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\sigma_H\|_{M^\infty(\mathbb{R}^2)} \approx \|h\|_{M^\infty(\mathbb{R}^2)} \preceq \|\kappa\|_{M^\infty(\mathbb{R}^2)} \preceq \|H\|_{L(\mathbb{M}^1(\mathbb{R}), M^\infty(\mathbb{R}))} \lesssim \|H\|_{L(M^p(\mathbb{R}))}
\]
and the result follows. \( \square \)

We proceed to prove the following generalization of Theorem 2.7. Indeed, the earlier stated result follows again from the fact that \( L^2(\mathbb{R}) = M^2(\mathbb{R}) \). We focus on the case of arbitrary domains; a simpler proof for rectangular domains can be obtained using Theorem 2.2 instead of Theorem 2.4.

**Theorem 4.2.** Fix \( M \) compact and \( p \in [1, \infty) \). Let \((g, \Lambda), g \in M^1(\mathbb{R}), \) be a tight frame for \( L^2(\mathbb{R}) \) with frame constant 1. Then exists a constant \( C \) and a strictly monotone function \( d : (0, 1) \rightarrow \mathbb{R}^+ \), \( \lim_{\epsilon \rightarrow 0} d(\epsilon) = +\infty \), with the property that if \( H \in \text{OPW}(M) \) satisfies
\[
\|\sigma_H\|_{L^\infty(\mathbb{R}^2)} \leq \mu \quad \text{and} \quad \|\sigma_H\|_{L^\infty(S)} \leq \epsilon \mu,
\]
then
\[
\|Hf\|_{M^p(\mathbb{R})} \leq C \epsilon \mu \|f\|_{M^p(\mathbb{R})}
\]
for all \( f \in M^p(\mathbb{R}) \) time-frequency localized on \( S - B(d(\epsilon)) = \left(S^c + B(d(\epsilon))\right)^c \) in the sense that, for \( p < \infty \),
\[
\sum_{\lambda \in \Lambda^c(S - B(d(\epsilon)))} |\langle f, \pi(\lambda)g \rangle|^p \geq (1 - \epsilon^p) \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^p,
\]
or, for \( p = \infty \),
\[
\sup \left\{ \left| \langle f, \pi(\lambda)g \rangle \right|, \lambda \in \Lambda \cap (S - B(d(\epsilon))) \right\} \geq (1 - \epsilon) \sup \left\{ \left| \langle f, \pi(\lambda)g \rangle \right|, \lambda \in \Lambda \right\}.
\]

**Proof.** Step 1. Preliminary observations and choice of auxiliary objects. Choose a nonnegative \( \phi \in \mathcal{S}(\mathbb{R}^2) \) with \( \int \phi(x) \, dx = 1 \) and \( \text{supp} \phi \subseteq [-\frac{1}{2}, \frac{1}{2}]^2 \). Recall that \( \Lambda^+ = \{ \mu \in \mathbb{R}^2 : e^{2\pi i(\mu, \lambda)} = 1 \text{ for all } \lambda \in \Lambda \} \) is called dual lattice of the lattice \( \Lambda \) in \( \mathbb{R}^2 \). Let \( \widetilde{\Lambda} \) be a lattice containing \( \Lambda \) with the property that there exists a compact and convex fundamental domain \( D \) of \( \widetilde{\Lambda}^+ \) which contains \( M + [-\frac{1}{2}, \frac{1}{2}]^2 \). Set \( \sigma_P = \|\chi_D \ast \phi\|_{L^1(\mathbb{R}^2)}^{-1} F(\chi_D \ast \phi) \) and, using the sampling theorem for lattices in \( \mathbb{R}^n \) [21, 11], we obtain for all \( H \in \text{OPW}(M) \)
\[
\sigma_H = \sum_{\lambda \in \Lambda} \sigma_H(\lambda) \mathcal{T}_\lambda \sigma_P
\]
and hence
\[
(4.3) \quad H = \sum_{\lambda \in \Lambda} \sigma_H(\lambda) \pi(\lambda) P \pi(\lambda)^*.
\]
As explained above, the fact that \((g, \Lambda)\) is a Gabor frame in \(L^2(\mathbb{R})\) with \(g \in M^1(\mathbb{R})\), implies that it is also an \(\ell^p\)-frame for \(M^p(\mathbb{R})\) and there exists \(C_1, C_2 > 0\) with
\[
\|f\|_{M^p(\mathbb{R})} \leq C_1 \|\{(f, \pi(\lambda)g)\}_{\lambda \in \Lambda}\|_{\ell^p(\Lambda)} \leq C_1 C_2 \|f\|_{M^p(\mathbb{R})}, \quad f \in M^p(\mathbb{R}).
\]
As the synthesis map is the adjoint of the analysis map, we also have
\[
\left\| \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g \right\|_{M^p(\mathbb{R})} \leq C_2 \|\{c_\lambda\}_{\lambda \in \Lambda}\|_{\ell^p(\Lambda)}.
\]

For some \(\mu_1, \mu_2, \ldots, \mu_n\), we have \(\Lambda = \bigcup_{\ell=1}^n (\Lambda + \mu_\ell)\), where \(n\) depends only on \(M\) and \((g, \Lambda)\). It is easily seen that \((g, \Lambda + \mu_\ell), \ell = 1, \ldots, n\), also satisfies (4.4) and (4.5). Setting \(\tilde{g} = n^{-1/2} g \in M^1(\mathbb{R})\), we conclude that the Gabor system \((\tilde{g}, \Lambda)\) is a tight frame for \(L^2(\mathbb{R})\) with frame bounds equal 1 and an \(\ell^p\)-frame with for \(M^p(\mathbb{R})\) with
\[
\|f\|_{M^p(\mathbb{R})} \leq C_1 n^{1-\frac{1}{p}} \|\{(f, \pi(\lambda)\tilde{g})\}_{\lambda \in \Lambda}\|_{\ell^p(\Lambda)} \leq C_1 n^{1-\frac{1}{p}} C_2 n^{\frac{1}{p}-\frac{1}{2}} \|f\|_{M^p(\mathbb{R})} = C_1 C_2 \|f\|_{M^p(\mathbb{R})}, \quad f \in M^p(\mathbb{R}).
\]

We claim that
\[
\left\{ \langle P\pi(\lambda)\tilde{g}, \pi(\lambda)\tilde{g} \rangle \right\} \in \ell^1(\tilde{\Lambda} \times \tilde{\Lambda}).
\]
To see this, recall that \(\sigma_P \in S(\mathbb{R}^2) \subseteq M^1(\mathbb{R}^2)\), and, hence, \(\tilde{\sigma}_P\) given by \(\sigma_P(\lambda, \xi) e^{2\pi i \xi \cdot \lambda} \) is in \(M^1(\mathbb{R}^2)\) as \(e^{2\pi i \xi \cdot \lambda} \) is a Fourier multiplier and hence also a time multiplier for \(M^1(\mathbb{R}^2)\) (Theorem 11, [5]). A direct computation implies that for \(\lambda = (t, \nu)\) and \(\tilde{\lambda} = (\tilde{t}, \tilde{\nu})\) we have
\[
|\langle P\pi(\lambda)\tilde{g}, \pi(\lambda)\tilde{g} \rangle| = \left| \int \int \sigma_P(x, \xi) e^{2\pi i \xi \cdot \lambda} \mathcal{M}_\nu \mathcal{T}_\tilde{\nu}(\tilde{g})(\xi)\mathcal{M}_\nu \mathcal{T}_\tilde{\nu}(\tilde{g})(x) \, dx \, d\xi \right|
= |\int \int \sigma_P(x, \xi) e^{2\pi i \xi \cdot \lambda} \mathcal{M}_{-t} \mathcal{T}_{\tilde{t}} \mathcal{M}_\nu \mathcal{T}_\tilde{\nu}(\tilde{g})(x) \, dx \, d\xi |
= |\langle \tilde{\sigma}_P, \mathcal{M}_{(\tilde{\nu}, \tilde{t})} \mathcal{T}_{(\tilde{\nu}, \tilde{t})}(\tilde{g} \otimes \tilde{g}) \rangle|.
\]
Equation (4.2) implies that the right hand side is in \(\ell^1(\tilde{\Lambda} \times \tilde{\Lambda})\) since \(\tilde{\sigma}_P \in M^1(\mathbb{R}^2)\) and \((g \otimes \tilde{g}, \tilde{\Lambda} \times \tilde{\Lambda})\) is a Gabor frame with window \(\tilde{g} \otimes \tilde{g}\) in \(M^1(\mathbb{R}^2)\).

Fix \(\epsilon > 0\) and choose \(d(\epsilon) > 0\) so that
\[
\sum_{\lambda \in \Lambda} \sum_{\lambda \in \Lambda \cap B(d(\epsilon))} |\langle P\pi(\lambda)\tilde{g}, \pi(\lambda)\tilde{g} \rangle| < \epsilon.
\]
Now, set \(A(\lambda, \tilde{\lambda}) = \langle P\pi(\lambda)\tilde{g}, \pi(\lambda)\tilde{g} \rangle\) if \(\lambda \in \Lambda \cap B(d(\epsilon))\) and \(0\) else.

**Step 2. Decomposing \(Hf\) as \(Hf = H_{\text{in}}f_{\text{in}} + H_{\text{out}}f_{\text{in}} + Hf_{\text{out}}\).**

Set
\[
\Lambda_{\text{in}} = \Lambda \cap (S - B(d(\epsilon))), \quad \tilde{\Lambda}_{\text{in}} = \tilde{\Lambda} \cap (S - B(d(\epsilon))), \quad \Lambda_{\text{out}} = \Lambda \setminus \Lambda_{\text{in}}, \quad \tilde{\Lambda}_{\text{out}} = \tilde{\Lambda} \setminus \tilde{\Lambda}_{\text{in}}.
\]
Let
\[
f_{\text{in}} = \sum_{\lambda \in \Lambda_{\text{in}}} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g = \sum_{\lambda \in \Lambda_{\text{in}}} c_\lambda \pi(\lambda)\tilde{g}, \quad f_{\text{out}} = f - f_{\text{in}},
\]
where \(c_\lambda = \sqrt{\pi} \langle f, \pi(\lambda)g \rangle\) if \(\lambda \in \Lambda\) and \(0\) else. Similarly, inspired by (4.3), we set for \(H \in OPW(M)\)
\[
H_{\text{in}} = \sum_{\lambda \in \Lambda \cap S} \sigma_H(\lambda) \pi(\lambda)P\pi(\lambda)^*, \quad H_{\text{out}} = H - H_{\text{in}}
\]
and note that \(H_{\text{in}}, H_{\text{out}} \in OPW(D + [-\frac{1}{2}, \frac{1}{2}]^2)\).
Step 3. Bounding $\|H_{\text{out}} f_{\text{in}}\|_{L^p(\mathbb{R})}$. We use the separation of $\tilde{\Lambda}_{\text{in}}$ and $\tilde{\Lambda} \cap S^c$ by $d(\epsilon)$ to compute

$$
\langle \{H_{\text{out}} f_{\text{in}}, \pi(\tilde{\lambda}) \tilde{g}\} \rangle_{\tilde{\Lambda} \in \tilde{\Lambda}} \{d_\lambda\} \in \ell^p(\tilde{\Lambda}), \frac{1}{p + 1/q} = 1,
$$

we conclude

$$
\begin{align*}
\langle \{H_{\text{out}} f_{\text{in}}, \pi(\tilde{\lambda}) \tilde{g}\} \rangle_{\tilde{\Lambda} \in \tilde{\Lambda}} \{d_\lambda\} & \leq \|H_{\text{out}} f_{\text{in}}\|_{L^p(\mathbb{R})} \|\sigma_H\|_{L^\infty(\mathbb{R}^2)} \{\{d_\lambda\}\} \|\{\nu, \tilde{\lambda}\} \|_{\ell^p(\tilde{\Lambda})} \sum_{\lambda \in \Lambda} \sum_{\nu \in \Lambda} A(\lambda, \tilde{\lambda})
\end{align*}
$$

and

$$
\begin{align*}
\|H_{\text{out}} f_{\text{in}}\|_{L^p(\mathbb{R})} & \leq n^{\frac{1}{2} - \frac{1}{p}} C_1 \{\{H_{\text{out}} f_{\text{in}}, \pi(\tilde{\lambda}) \tilde{g}\}\} \|\{\nu, \tilde{\lambda}\} \|_{\ell^p(\tilde{\Lambda})} \sum_{\lambda \in \Lambda} \sum_{\nu \in \Lambda} A(\lambda, \tilde{\lambda})
\end{align*}
$$

Step 4. Bounding $\|H f_{\text{out}}\|_{L^p(\mathbb{R})}$. By Proposition 4.1 we have

$$
\|H f_{\text{out}}\|_{L^p(\mathbb{R})} \leq B(M, p) \|\sigma_H\|_{L^\infty(\mathbb{R}^2)} \|f_{\text{out}}\|_{L^p(\mathbb{R})}.
$$

By hypothesis, for $p < \infty$ we have

$$
\|f_{\text{out}}\|_{L^p(\mathbb{R})} = \left\| \sum_{\lambda \in \Lambda_{\text{out}}} \langle f, \pi(\lambda) g \rangle \pi(\lambda) g \right\|_{L^p(\mathbb{R})}^p
$$

$$
\begin{align*}
& \leq C_2^p \sum_{\lambda \in \Lambda_{\text{out}}} |\langle f, \pi(\lambda) g \rangle|^p
\end{align*}
$$

$$
\begin{align*}
& \leq C_2^p \epsilon^p \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda) g \rangle|^p
\end{align*}
$$

$$
\begin{align*}
& \leq C_2^p \epsilon^p \|f\|_{L^p(\mathbb{R})}^p.
\end{align*}
$$
and for \(p = \infty\) we have
\[
\|f_{\text{out}}\|_{M^\infty(\mathbb{R})} = \| \sum_{\lambda \in \Lambda_{\text{out}}} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g \|_{M^\infty(\mathbb{R})}
\]
\[
\leq C_2 \| \langle f, \pi(\lambda)g \rangle \|_{\ell^\infty(\Lambda_{\text{out}})}
\]
\[
\leq C_2 \epsilon \| \langle f, \pi(\lambda)g \rangle \|_{\ell^\infty(\Lambda)}
\]
\[
\leq C_2^2 \epsilon \| f \|_{M^\infty(\mathbb{R})}.
\]

We conclude
\[
\|Hf_{\text{out}}\|_{M^p(\mathbb{R})} \leq B(M, p) C_2 \epsilon \| \sigma_H \|_{L^\infty} \| f \|_{M^p(\mathbb{R})} \leq B(M, p) C_2 \epsilon \mu \| f \|_{M^p(\mathbb{R})}.
\]

Step 5. Bounding \(\|H_{\text{in}}f_{\text{in}}\|_{M^p(\mathbb{R})}\). Since \(\sigma_P \in \mathcal{S}(\mathbb{R}^2)\), the operator
\[
\ell^\infty(\Lambda) \to L^\infty(\mathbb{R}^2), \quad \{c_\lambda\} \to \sum_{\lambda \in \Lambda} c_\lambda T_\mu \sigma_P
\]
is bounded, say with operator norm bound \(C_3\). Then, Proposition 4.1 implies
\[
\|H_{\text{in}}f_{\text{in}}\|_{M^p(\mathbb{R})} \leq B(D+[-\frac{1}{2}, \frac{1}{2}]^2, p) \| \sigma_{H_{\text{in}}} \|_{L^\infty(\mathbb{R}^2)} \| f_{\text{in}} \|_{M^p(\mathbb{R})}
\]
\[
\leq B(D+[-\frac{1}{2}, \frac{1}{2}]^2, p) C_3 \| \sigma_{H}(\Lambda) \|_{\ell^\infty(\Lambda \cap \mathcal{S})} (1 + \epsilon) \| f \|_{M^p(\mathbb{R})}
\]
\[
\leq 2 B(D+[-\frac{1}{2}, \frac{1}{2}]^2, p) C_3 \epsilon \mu \| f \|_{M^p(\mathbb{R})}.
\]

Since all constants are independent of \(\epsilon, \mu, H, \) and \(f\), we summarize
\[
\|Hf\|_{M^p(\mathbb{R})} = \|H_{\text{in}}f_{\text{in}} + H_{\text{out}}f_{\text{in}} + H_{\text{out}}f_{\text{out}}\|_{M^p(\mathbb{R})} \leq C \epsilon \mu \| f \|_{M^p(\mathbb{R})}.
\]

5. OPERATOR IDENTIFICATION USING LOCALIZED IDENTIFIERS

This section analyzes identifiers that are localized in time and frequency. Theorem 2.5 shows that such functions cannot serve as an identifier for the complete operator Paley-Wiener space as a whole.

Proof of Theorem 2.5. Let \(r \neq 0\) be a Schwartz function with \(\text{supp } r \subseteq [0, T] \) and \(\phi \neq 0\) be a Schwartz function with \(\text{supp } \hat{\phi} \subseteq [-\Omega/2, \Omega/2]\). Let \(H_n\) be defined via its kernel \(\kappa_n(x, y) = \phi(x - n)r(x - y)\), so \(h_n(x, t) = \phi(x - n)r(t)\) and \(\eta_n(t, \nu) = \int h_n(x, t)e^{-2\pi i x \nu} dx = r(t)e^{2\pi i \nu} \hat{\phi}(\nu)\), so \(H_n \in OPW([0, T] \times [-\Omega/2, \Omega/2])\) with \(\|\sigma_{H_n}\|_{L^\infty(\mathbb{R}^2)} = \|\hat{\tau}\|_{L^\infty(\mathbb{R})} \|\hat{\phi}\|_{L^\infty(\mathbb{R})}\).

If \(w\) identifies \(OPW([0, T] \times [-\Omega/2, \Omega/2])\), then by definition \(H_n w \in L^2(\mathbb{R})\). Then
\[
\int |H_n w(x)|^2 dx = \int |(\kappa_n(x, y), w(y))_y|^2 dx
\]
\[
= \int |\phi(x - n)|^2 |\langle r(x - y), w(y) \rangle_y|^2 dx.
\]
Clearly, \(\langle r(x - y), w(y) \rangle_y \xrightarrow{y \to \pm \infty} 0\) would imply \(\|H_n w\|_{L^2(\mathbb{R})} \xrightarrow{n \to \pm \infty} 0\) and contradict identifiability (2.4) since by (2.3) we have \(\|H_n\|_{\mathcal{B}(L^2(\mathbb{R}))} \geq A\|\sigma_{H_n}\|_{L^\infty(\mathbb{R}^2)} = A\|\hat{\tau}\|_{L^\infty(\mathbb{R})} \|\hat{\phi}\|_{L^\infty(\mathbb{R})}\) for all \(n \in \mathbb{Z}\).

To show that an identifier \(w\) cannot decay in frequency, we choose \(H_n \in OPW([0, T] \times [-\frac{1}{2}, \frac{1}{2}])\) to have spreading functions \(\eta_n(t, \nu) = r(t) e^{2\pi i nt} \hat{\phi}(\nu)e^{-2\pi i \nu}\). Let \(g\) be a Schwartz function and compute using Fubini’s Theorem and, for notational simplicity, using bilinear pairings in place of
sesquilinear ones,
\[ \langle H_n w(x), g(x) \rangle_x = \langle \eta_n(t, \nu), (e^{2\pi i x \nu} w(x - t), g(x))_x \rangle_{t, \nu} \]
\[ = \langle r(t) e^{2\pi i (x - \nu) t} w(x - t), (g(x))_x \rangle_{t, \nu} \]
\[ = \langle r(t) e^{2\pi i \nu} w(x - t) g(x), (\bar{\phi}(\nu), e^{2\pi i (x - \nu) t} \nu)_{t, x} \rangle_{t, x} \]
\[ = \langle (r(t) e^{2\pi i \nu}, w(x - t) \phi(x - t))_t, g(x) \rangle_x \]
\[ = \langle (\tilde{\nu}(\xi - n), e^{-2\pi i \xi \nu} \tilde{w} \ast \tilde{\phi}(\nu), g(x) \rangle_x \]
\[ = \langle \tilde{\nu}(\xi - n) \tilde{w} \ast \tilde{\phi}(\xi), \tilde{g}(\xi) \rangle_{\xi}. \]

Hence,
\[ \| H_n w \|_{L^2(\mathbb{R})}^2 = \| \tilde{H_n w} \|_{L^2(\mathbb{R})}^2 = \int |\tilde{\nu}(\xi - n)|^2 |\langle \tilde{w}(\xi - \nu), \tilde{\phi}(\nu) \rangle_{\nu}|^2 d\xi, \]
and we can conclude as above. \(\square\)

We proceed by showing that local identification of operators is possible with identifiers localized both in time and frequency, Theorem 2.8.

**Proof of Theorem 2.8.** The proof proceeds in two steps. First we show that replacing each Dirac-delta by a suitable smoothed out version locally introduces only a small error and identification using the resulting smooth identifier can be interpreted as sampling a modified bandlimited operator. Second we show that reducing to a finite number of samples also locally yields only a small error. Applying this to the modified operator arising in the first part proves that both reductions together also yield only a small error.

For the first part, choose \( \varphi \in \mathcal{S} \) with \( \text{supp} \varphi \subseteq [-\delta, \delta] \), \( \| \tilde{\varphi} \|_{L^\infty(\mathbb{R})} = 1 \), and \( |\tilde{\varphi}(\xi) - 1| \leq \epsilon \) for \( \xi \in I_2 \). Define \( C_\varphi : f \mapsto f \ast \varphi \) and set \( H_C = H \circ C_\varphi \). Observe that
\[ H_C f(x) = \int \int \eta_H(t, \nu) e^{2\pi i x \nu} f \ast \varphi(x - t) dt d\nu \]
\[ = \int \int \int \eta_H(t, \nu) e^{2\pi i x \nu} f(x - t - y) \varphi(y) dy dt d\nu \]
\[ = \int \int \int \eta_H(t - y, \nu) e^{2\pi i x \nu} f(x - t) \varphi(y) dy dt d\nu \]
\[ = \int \left( \int \eta_H(t - y, \nu) \varphi(y) dy \right) e^{2\pi i x \nu} f(x - t) dt d\nu, \]
that is, \( \eta_{H_C}(t, \nu) = \eta_H(\cdot, \nu) \ast \varphi(t) \) and
\[ \text{supp} \eta_{H_C} \subseteq \text{supp} \eta_H + [-\delta, \delta] \times \{0\}. \]

We can apply Theorem 2.4 for the operator \( H_C \) with \( M_1 := M + [-\delta, \delta] \times \{0\} \) in place of \( M \). As by assumption \( M_1 + [-\delta, \delta]^2 \) still has measure less than one, this can be done with \( \delta, r \) and \( \phi \) as given in the theorem. Defining \( w_1 := \varphi \ast w \), we obtain
\[ \kappa_{H_C}(x + t, x) = LT \sum_{j=0}^{\Omega x} r(t - k_j T) \left( \sum_{q \in \mathbb{Z}} b_{j} w_1(t - (k_j - q) T) \phi(x + (k_j - q) T) \right) e^{2\pi i n_1 (x + t)} \]
(5.1) \( \sigma_{H_C}(x, \xi) = \mathcal{F}_x \eta_{H_C}(x, \xi) = \sigma_H(x, \xi) \tilde{\phi}(\xi), \)
and, by hypothesis, we have \( \| \sigma_{H_C} \|_{L^\infty(\mathbb{R})} \leq \| \sigma_H \|_{L^\infty(\mathbb{R})} \leq \mu \) and \( \| \sigma_H - \sigma_{H_C} \|_{L^\infty(S)} \leq \epsilon \mu \).

Note that for \( I_1 = \mathbb{R} \), (5.1) agrees with (2.11) and we have \( H_C = \tilde{H} \), so this establishes the result.

For the second part, let us assume \( S \subseteq I_1 \times \mathbb{R} \) and \( M_1 \subseteq [c, d] \times \mathbb{R} \). Let \( \psi \in \mathcal{S}(\mathbb{R}) \) be nonnegative and satisfy \( \sum_n \psi(x - nT) = 1 \) and \( \text{supp} \tilde{\psi} \subseteq [-1/T, 1/T] \). Such a function can be
obtained by choosing an arbitrary bandlimited, nonnegative $\psi_0 \in S$ with $\|\psi_0\|_{L^1} = 1$ and defining $\psi = \chi_{[0,T^*]} \ast \psi_0$.

Set $P_A(x) = \sum_{n \in A} \psi(x-nT)$, so $P_{[-N,N]} \to 1$ and $P_{[-N,N]^c} \to 0$ uniformly on compact subsets as $N \to \infty$. Moreover, $|P_A(x)| \leq 1$ for all $A$. Choose $N(\epsilon)$ so that $|P_{I_1+[-N(\epsilon),N(\epsilon)]}(x)| - 1| \leq \epsilon$ for $x \in I_1 + [c,d]$ and choose $R(\epsilon)$ with

$$
(5.2) \quad \sum_{qT \notin I_1 + [R(\epsilon), R(\epsilon)]} \|P_{I_1+[-N(\epsilon),N(\epsilon)]}(x) V_{\theta^*} r(x-q, \xi)\|_{L^1(\mathbb{R})} < \epsilon(1-\epsilon) D.
$$

where the nature of $D$ is derived by the computations below. The existence of such $R(\epsilon)$ follows from the fact that $P_{I_1+[-N(\epsilon),N(\epsilon)]}(x)$ and $V_{\theta^*} r$ decay faster than any polynomial.

Let $w_2 = \sum_{kT \in I_1+[-R(\epsilon), R(\epsilon)]+[\delta,T+\delta]} c_k \delta_k T$ and observe that $\tilde{H}$ as defined in the theorem satisfies

$$
h_{\tilde{H}}(x+t,t) = \kappa_{\tilde{H}}(x+t,x) = LT \sum_{j=0}^{L-1} r(t-k_j T) \left( \sum_{q \in \mathbb{Z}} b_{jq} H_{C} w_2(t-(k_j - q)T) \phi(x+(k_j - q)T) \right) e^{2\pi i n \Omega x}.
$$

Since $M_1 \subset [c,d] \times \mathbb{R}$, we have supp $H_{C} \delta_y \subseteq [c+y, d+y]$, and therefore,

$$
H_{C} w(x) = H_{C} \sum_{k \in \mathbb{Z}} c_k \delta_k T(x) = H_{C} \sum_{kT \in I_1+[-R(\epsilon), R(\epsilon)]+[\delta,T+\delta]+[c,d]} c_k \delta_k T(x) = H_{C} w_2(x),
$$

$$
x \in K \equiv I_1 + [-R(\epsilon), R(\epsilon)] + [-T, T-\delta, \delta].
$$

Note that $\tilde{H} \in OPW(M_2)$, where $M_2 = M_1 + [-\delta, \delta]^2$ (for details, see, for example, [24]). As $M_2 + [-\delta, \delta]^2$ still has measure less than one, this implies that we can apply Theorem 2.4 again with the same $\delta$. We obtain

$$
h_{H_{C}}(x+t,t) - h_{\tilde{H}}(x+t,t)
$$

$$
= LT \sum_{j=0}^{L-1} r(t-k_j T) \left( \sum_{q \in \mathbb{Z}} b_{jq} H_{C} (w-w_2)(t-(k_j - q)T) \phi(x+(k_j - q)T) \right) e^{2\pi i n \Omega x}
$$

$$
= LT \sum_{j=0}^{L-1} r(t-k_j T) \left( \sum_{q \notin K -(t-k_j T)} b_{jq} H_{C} (w-w_2)(t-(k_j - q)T) \phi(x+(k_j - q)T) \right) e^{2\pi i n \Omega x}
$$

$$
= LT \sum_{j=0}^{L-1} r(t-k_j T) \left( \sum_{q \notin I_1 + [-R(\epsilon), R(\epsilon)]} b_{jq} H_{C} (w-w_2)(t-(k_j - q)T) \phi(x+(k_j - q)T) \right) e^{2\pi i n \Omega x}.
$$
Setting $\tilde{K} = K^c + [-\delta, T + \delta]$ and using that $(\sigma_{H_C}(x, \xi) - \sigma_{\tilde{H}}(x, \xi)) P_{t_1 + [-N(e), N(e)]}(x)$ is bandlimited to $M + \{0\} \times [-1/T, 1/T)$, we compute

$$\|\sigma_{H_C} - \sigma_{\tilde{H}}\|_{L^\infty(S)} \leq 1/(1 - \epsilon) \|\|\sigma_{H_C}(x, \xi) - \sigma_{\tilde{H}}(x, \xi)) P_{t_1 + [-N(e), N(e)]}(x)\|_{L^\infty(\mathbb{R}^2)}$$

$$\leq 1/(1 - \epsilon) \|\|\sigma_{H_C}(x, \xi) - \sigma_{\tilde{H}}(x, \xi)) P_{t_1 + [-N(e), N(e)]}(x)\|_{M^\infty(\mathbb{R}^2)}$$

$$\leq 1/(1 - \epsilon) \left\|\left(\sum_{j=0}^{L-1} b_{j,q} H_C(w - w_2)(t - (k_j - q)T) \phi(x - t + (k_j - q)T)\right)_{M^\infty(\mathbb{R}^2)} \right\|_{L^1(\mathbb{R})}$$

$$\leq LT/(1 - \epsilon) \left\|\left(\sum_{j=0}^{L-1} b_{j,q} H_C(w - w_2)(t - (k_j - q)T) \phi(x - t + (k_j - q)T)\right)_{M^\infty(\mathbb{R}^2)} \right\|_{L^1(\mathbb{R})}$$

where we used the invariance of the $M^\infty$ and $M^1$ norm under translation and modulation and, for the last inequality, Theorem 4.1 - noting that, for functions constant in one of the coordinate directions, the $M^\infty(\mathbb{R})$ and $M^\infty(\mathbb{R}^2)$ norms agree. The second to last inequality is based on $M^1(\mathbb{R}^2)$ being a Banach algebra, namely on $\|g_1 g_2\|_{M^1(\mathbb{R}^2)} \leq \|g_1\|_{M^1(\mathbb{R}^2)} \|g_2\|_{M^1(\mathbb{R}^2)}$ for $g_1, g_2 \in M^1(\mathbb{R}^2)$. Indeed, for $f \in M^\infty(\mathbb{R}^2)$ and $g \in M^1(\mathbb{R}^2)$, we have

$$\|fg\|_{M^\infty(\mathbb{R}^2)} = \sup_{|f|_{M^1(\mathbb{R}^2)} = 1} \|\langle f, g \rangle\| \leq \sup_{|\tilde{f}|_{M^1(\mathbb{R}^2)} = 1} \|\tilde{f}\|_{M^1(\mathbb{R}^2)} \|\bar{g}\|_{M^1(\mathbb{R}^2)}$$

Note that with $\phi^*(t) = \bar{\phi}(-t)$, we have

$$\int r(t) \phi(x - t) e^{-2\pi i tx} dt = V_{\phi^*} r(x, \xi),$$

which is a bandlimited function since

$$\int \int V_{\phi^*} r(x, \xi) e^{2\pi i \xi \cdot x} dx d\xi = \int \int r(t) \phi(x - t) e^{-2\pi i tx} dx = r(t) \bar{\phi}(\nu) e^{2\pi i \nu}. $$

Using that the $M^1$-norm is invariant under partial Fourier transforms and the equivalence between the $M^1$ and $L^1$ norms which is implied by the bandlimitation of $P_{t_1 + [-N(e), N(e)]}(x + q) V_{\phi^*} r(x, \xi)$ to $(-1/T, 1/T) \times \{0\} + (-\delta, \Omega + \delta) \times (-\delta, T + \delta)$, we obtain

$$\left\|P_{t_1 + [-N(e), N(e)]}(x + q) r(t) \phi(x - t)\right\|_{M^1(\mathbb{R}^2)} \leq \left\|P_{t_1 + [-N(e), N(e)]}(x + q) V_{\phi^*} r(x, \xi)\right\|_{M^1(\mathbb{R}^2)}$$

$$\leq \left\|P_{t_1 + [-N(e), N(e)]}(x + q) V_{\phi^*} r(x, \xi)\right\|_{L^1(\mathbb{R}^2)}.$$
Fix \( g \in \mathcal{S}(\mathbb{R}) \) and observe that \( \|V_g f\|_{L^p(\mathbb{R}^2)} \) defines a norm on \( M^p(\mathbb{R}) \) equivalent to the \( M^p(\mathbb{R}) \) norm given in (4.1) [11]. For any \( A \subset \mathbb{R} \) we obtain the uniform bound
\[
\| \sum_{nT \in A} c_n \delta_{nT} \|_{M^\infty(\mathbb{R})} \approx \| V_g \sum_{nT \in A} c_n \delta_{nT} \|_{L^\infty(\mathbb{R})} = \| \sum_{nT \in A} c_n g(nT - t) e^{2\pi i nT} \|_{L^\infty(\mathbb{R})} \\
\leq \| \sum_{nT \in A} |c_n| |g(nT - t)| \|_{L^\infty(\mathbb{R})} \leq \| \sum_{n \in \mathbb{Z}} |c_n| |g(nT - t)| \|_{L^\infty(\mathbb{R})} < \infty.
\]
The first norm inequality stems from the fact that for all \( g \in M^1(\mathbb{R}) \), \( \|V_g f\|_{L^p(\mathbb{R}^2)} \) defines a norm on \( M^p(\mathbb{R}) \) equivalent to the \( M^p(\mathbb{R}) \) norm given in (4.1).

Combining this upper bound on \( \|w - w_2\|_{M^\infty(\mathbb{R})} \) with the above estimate for \( \|\sigma_{H_C} - \sigma_{H_c}\|_{L^\infty(S)} \) and (5.2), we conclude
\[
\|\sigma_{H_C} - \sigma_{H}\|_{L^\infty(S)} \leq \tilde{D} \|H_C\|_{L^\infty(M^\infty(\mathbb{R}))} L^2 T \left( 1 - \epsilon \right) \|g_{jq}\|_{L^\infty} \sum_{q \neq 1} \|P_{H + [-N(\epsilon), N(\epsilon)]}(x + q)V_{\phi,r}(x, \xi)\|_{L^1(\mathbb{R}^2)} \\
\leq \tilde{D} \|H_C\|_{L^\infty(M^\infty(\mathbb{R}))} \|\sigma_{H_C}\|_{L^\infty(\mathbb{R}^2)} \leq \tilde{D} \|\sigma_{H}\|_{L^\infty(\mathbb{R}^2)} \leq \tilde{D} \epsilon \mu.
\]
Choosing \( R(\epsilon) \) above large to yield \( \tilde{D} \) small enough to compensate all the multiplicative constants, we obtain
\[
\|\sigma_{H_C} - \sigma_{H}\|_{L^\infty(S)} \leq \epsilon \mu.
\]
As a meaningful statement is only obtained for \( \epsilon < 1 \), this bound directly implies that
\[
\|\sigma_{H}\|_{L^\infty(\mathbb{R}^2)} \leq 2 \mu.
\]
Combining this with the bound
\[
\|\sigma_{H} - \sigma_{H}\|_{L^\infty(\mathbb{R}^2)} \leq \|\sigma_{H} - \sigma_{H_C}\|_{L^\infty(\mathbb{R}^2)} + \|\sigma_{H_C} - \sigma_{H}\|_{L^\infty(\mathbb{R}^2)} \leq 2 \epsilon \mu,
\]
Theorem 2.7 directly yields the result with a constant of twice the size as in Theorem 2.7. \( \square \)

6. RECONSTRUCTION OF BANDLIMITED OPERATORS FROM DISCRETE MEASUREMENTS

This section concerns the discrete representation given in Theorem 2.9. First, we prove this theorem, hence establishing that indeed this representation is globally exact.

Proof of Theorem 2.9: The proof is similar to the proof of Theorem 2.4 given in [24]. The main idea is to use a Jordan domain argument to cover a fixed compact set \( M \) of size less than one by shifts of a rectangle that still have combined area less than one and then to combine identifiability results for each of them to obtain identifiability for the whole set. Indeed, there exist \( L \) prime and \( T, \Omega > 0 \) with \( T \Omega = \frac{1}{L} \) such that
\[
\text{supp}(\eta) \subseteq \bigcup_{j=0}^{L-1} [R + (k_j T, n_j \Omega)] \subseteq [-((L - 1)T/2, (L + 1)T/2) \times [-L\Omega/2, L\Omega/2] \setminus \left( \frac{-1}{2\Omega} + T/2, \frac{1}{2\Omega} + T/2 \right) \times [-1/2T, 1/2T] \\
= [-1/(2\Omega) + T/2, 1/(2\Omega) + T/2] \times [-1/(2T), 1/(2T)]
\]
where \( R = [0, T) \times [-\Omega/2, \Omega/2) \), and the sequence \( (k_j, n_j) \in \mathbb{Z}^2 \) consists of distinct pairs. For \( \delta > 0 \) small enough (and possibly slightly smaller \( T, \Omega \), and a larger prime \( L \)), one can even achieve
\[
M_{\delta} \subseteq \bigcup_{j=0}^{L-1} [R + (k_j T, n_j \Omega)] \subseteq [-((L - 1)T/2, (L + 1)T/2) \times [-L\Omega/2, L\Omega/2] \\
\]
where \( M_{\delta} \) is the \( \delta \)-neighborhood of \( M \).

Fix such \( \delta \) and let \( r, \phi \in \mathcal{S}(\mathbb{R}) \) satisfy (2.14) and (2.15) for this \( \delta \). Clearly,
\[
(6.1) \ (k, n) \neq (k_j, n_j) \text{ for all } j \text{ implies } S_{\delta} \cap \left( R + (kT, n\Omega) \right) = \emptyset \text{ and } \eta(t, \gamma)r(t-kT)\hat{\phi}(\gamma-n\Omega) = 0,
\]
a fact that we shall use below.
Define the identifier \( w = \sum_{n \in \mathbb{Z}} c_n \delta_n T \), where \( \{c_n\} \) is \( L \)-periodic and observe that

\[
H w(x) = \int \int \eta(t, \gamma) e^{2\pi i x \gamma} w(x - t) \, dt \, d\gamma
= \int \int \eta(t, \gamma) e^{2\pi i x \gamma(x-t)} \sum_{k \in \mathbb{Z}} c_k \delta_{kT}(x - t) \, dt \, d\gamma
= \sum_{k \in \mathbb{Z}} c_k \int \eta(x - kT, \gamma) e^{2\pi i x kT} \, d\gamma
= \sum_{m \in \mathbb{Z}} \sum_{k=0}^{L-1} c_{k+m} \int \eta(x - (mL + k + p)T, \gamma) e^{2\pi i x (mL + k + p)T} \, d\gamma
\]

for any \( p \in \mathbb{Z} \). We shall use the non-normalized Zak transform \( Z_{LT} : L^2(\mathbb{R}) \rightarrow L^2([0,LT) \times [-\Omega/2,\Omega/2]) \) defined by

\[
Z_{LT} f(t, \gamma) = \sum_{n \in \mathbb{Z}} f(t - nLT) e^{2\pi i nLT \gamma}.
\]

We compute using the Poisson summation formula and the fact that \( \Omega = 1/LT \)

\[
(Z_{LT} \circ H) w(t, \nu)
= \sum_{n \in \mathbb{Z}} H w(t - nLT) e^{2\pi i nLT \nu}
= \sum_{m, n \in \mathbb{Z}} e^{2\pi i nLT} \sum_{k=0}^{L-1} c_{k+m} \int \eta(t - (nL + mL + k + p)T, \gamma) e^{2\pi i x (mL + k + p)T} \, d\gamma
= \sum_{m, n \in \mathbb{Z}} \sum_{k=0}^{L-1} c_{k+m} \int \eta(t - (mL + k + p)T, \gamma) e^{2\pi i x (mL + k + p)T} \sum_{n \in \mathbb{Z}} e^{2\pi i nLT \nu} \, d\gamma
= \sum_{m, n \in \mathbb{Z}} \sum_{k=0}^{L-1} \int \eta(t - (mL + k + p)T, \gamma) e^{2\pi i x (mL + k + p)T} \frac{1}{LT} \sum_{n \in \mathbb{Z}} \delta_{n/LT} \delta_{\nu - \gamma} \, d\gamma
= \Omega \sum_{k=0}^{L-1} \sum_{m, n \in \mathbb{Z}} \eta(t - (mL + k + p)T, \nu + n\Omega) e^{2\pi i (\nu + n\Omega)(mL + k + p)T}
\]

By (6.1) we get for \( p = 0, \ldots, L - 1 \),

\[
(6.2) \quad r(t) \hat{\phi}(\nu)(Z_{LT} \circ H) w(t + pT, \nu)
= \Omega \sum_{k=0}^{L-1} \sum_{m, n \in \mathbb{Z}} r(t) \hat{\phi}(\nu) \eta(t - (mL + k)T, \nu + n\Omega) e^{2\pi i T (\nu + n\Omega)(mL + k + p)}
= \Omega \sum_{j=0}^{L-1} \sum_{k=0}^{L-1} r(t) \hat{\phi}(\nu) \eta(t + kT, \nu + n\Omega) e^{2\pi i (\nu + n\Omega)(p + k_j)}
= \Omega e^{2\pi i \nu pT} \sum_{j=0}^{L-1} (T^{k_j} M^{n_j} c)_p e^{2\pi i k_j T r(t) \hat{\phi}(\nu) \eta(t + kT, \nu + n\Omega)}
\]

where here and in the following, \( T : (c_0, c_1, \ldots, c_{L-2}, c_{L-1}) \mapsto (c_{L-1}, c_0, \ldots, c_{L-3}, c_{L-2}) \) and \( M : (c_0, c_1, \ldots, c_{L-2}, c_{L-1}) \mapsto (c^{2\pi i 0/L} c_0, c^{2\pi i 1/L} c_1, \ldots, c^{2\pi i (L-2)/L} c_{L-2}, c^{2\pi i (L-1)/L} c_{L-1}) \), that
is, \((T^{k_j}M^{n_j})_p = e^{2\pi i \frac{n_j(p+k_j)}{p}}c_{p+k_j}\). Equivalently, we obtain the matrix equation

\[
(6.3) \quad [e^{-2\pi i p T}r(t)\hat{\phi}(\nu)(Z_{LT} \circ H)w(t+pT, \nu)]_{p=0}^{L-1}
= \Omega A[e^{2\pi i v k}T^r(t)\hat{\phi}(\nu)\eta(t+k_j T, \nu + n_j \Omega)]_{j=0}^{L-1}
\]

where \(A\) is a \(L \times L\) matrix, whose \(j\)th column is \(T^{k_j}M^{n_j}c \in \mathbb{C}^L\). \(A\) is a submatrix of the \(L \times L^2\) matrix \(G\), whose columns are \(\{T^{k_j}M^{n_j}\}_{k,l=0}^{L-1}\). It was shown in [16] that if \(L\) is prime, then we can choose \(c \in \mathbb{C}^L\) such that every \(L \times L\) submatrix of \(G\) is invertible. In fact, the set of such \(c \in \mathbb{C}^L\) is a dense open subset of \(\mathbb{C}^L\) [16]. Hence we can apply the matrix \(A^{-1} = [b_{jp}]_{j,p=1}^{L} \) on both sides of Equation (6.3) to obtain

\[
(6.4) \quad e^{2\pi i v k}T^r(t)\hat{\phi}(\nu)\eta(t+k_j T, \nu + n_j \Omega)
= LT \sum_{p=0}^{L-1} b_{jp} e^{-2\pi i p T} r(t)\hat{\phi}(\nu)(Z_{LT} \circ H)w(t+pT, \nu)
\]

for every \(j = 0, 1, \ldots, L - 1\).

In fact, until this point the proof agrees with the proof of (2.8) in Theorem 2.4. Indeed, if we extend \(\{b_{jp}\}_p\) to a \(L\)-periodic sequence by setting \(b_{j,p+mL} = b_{jp}\), replace the so far unused property (2.15) by (2.9) then further computations [22] give

\[
h(x, t) = LT \sum_{j=0}^{L-1} r(t-k_j T) \left( \sum_{q \in \mathbb{Z}} b_{jq} hw(t-(k_j + q)T)\phi(x-t+(k_j + q)T) \right) e^{2\pi in_j \Omega(x-t)}.
\]

Observe that (2.15) implies that \((r, TZ \times \frac{\Omega L}{\beta_2}) = \{T_{LT}M_{\ell L \Omega_1/\beta_2}\}_{k, \ell \in \mathbb{Z}}\) is a tight Gabor frame whenever \(\beta_2 \geq 1+2\delta/T\) as, in this case, \((r, \frac{\beta_2}{\beta_1}Z \times \frac{T}{\beta_2}) = (r, \frac{\beta_2}{\beta_1}T \times \Omega LZ)\) is an orthogonal sequence and the Ron-Shen criterion applies [11, 26]. The same arguments imply that \((\hat{\phi}, \Omega X \times \frac{LT}{\beta_1}Z)\) is a tight Gabor frame. Using a simple tensor argument, we obtain that \(\{\Psi_{m,n,l,k}\}_{m,n,l,k} \in \mathbb{C}^L\) forms a tight Gabor frame where

\[
\Psi_{m,n,l,k}(t, \nu) = \mathcal{T}_{kT, n\Omega_1} M_{\ell \Omega_1/\beta_1} T_{\nu/\beta_1}(\nu) \cong \hat{\phi}(t, \nu)
= e^{2\pi i L \left( \frac{m\nu-n\Omega_1}{\beta_1} + \frac{\nu-t-kT}{\beta_2} \right)} r(t-kT) \hat{\phi}(\nu-n\Omega).
\]

The frame bound is \(T \Omega L^2 \Omega_1/(\beta_1 \beta_2) = 1/(\beta_1 \beta_2)\). We set \(\Phi_{m,n,l,k} = F_x \Psi_{m,n,l,k}\). Clearly, as \(F_x\) is unitary, we have that \(\{\Phi_{m,n,l,k}\}_{m,n,l,k} \in \mathbb{C}^L\) forms a tight frame with frame bound \(1/(\beta_1 \beta_2)\), in fact, a tight Gabor frame as

\[
\Phi_{m,n,l,k}(x, \xi) = F_x \Psi_{m,n,l,k}(x, \xi)
= (F T^{-kT} M_{\ell \Omega_1/\beta_2}) (x) \left( F^{-1} T_{-n\Omega_1} M_{m TL/\beta_1} \hat{\phi} \right) (x)
= (M_{kt} T_{\ell \Omega_1/\beta_2}) (x) \left( M_{n\Omega_1} T_{m TL/\beta_1} \hat{\phi} \right) (x)
= e^{2\pi i (nm+k) \Omega_1/\beta_1} T_{\ell \Omega_1/\beta_2} M_{kT} \hat{\phi} (x).\]

Note that (6.1) together with the fact that the symplectic Fourier transform is unitary implies that the coefficients in the Gabor frame expansion of \(\sigma\) satisfy

\[
\langle \sigma, \Phi_{m,n,l,k} \rangle = \langle \eta, \Psi_{m,n,l,k} \rangle = 0 \quad \text{unless} \quad (n, k) = (n_j, k_j) \quad \text{for some} \quad j.
\]

Hence we need to estimate \(\sigma_{m,l}^{(j)} = \langle \sigma, \Phi_{m,n,l,k} \rangle \) for \(j = 0, 1, \ldots, L - 1\). We obtain by (6.4)
where

\[ \mathbf{B}_{\sigma} = \mathbf{B}_{\sigma} \]

for \( q = mL + q' \) with \( q' = 0, 1, \ldots, L - 1 \).

Set

\[ C_{q, l}(H w) = \langle H w, T_{q} M_{l L / \beta_{2}} r \rangle. \]

In sum,

\[
\sigma(x, \xi) = \frac{1}{\beta_{1} \beta_{2}} \sum_{j=0}^{L-1} \sum_{m, \ell \in \mathbb{Z}} \langle \sigma, \Phi_{m, -n_{j}, l, -k_{j}} \rangle \Phi_{m, -n_{j}, l, -k_{j}}(x, \xi)
\]

(6.5)

\[
= \frac{LT}{\beta_{1} \beta_{2}} \sum_{j=0}^{L-1} e^{-2\pi i (x \Omega + \xi k_{j} T)} \sum_{m, \ell \in \mathbb{Z}} \sigma_{m, \ell}^{(j)} \hat{r}(\xi - \frac{\ell T}{\beta_{2}}) \phi(x - \frac{m T L}{\beta_{1}}),
\]

where

\[ \sigma_{m, \ell}^{(j)} = \sum_{q \in \mathbb{Z}} B_{jq} \phi(a(-q - k_{j} - mL / \beta_{1})) C_{q, l}(H w). \]
Applying the symplectic Fourier transform to (6.5) yields
\[ \eta(t, \nu) = e^{-2\pi i t \nu} \eta(t, \nu) \]
\[ = e^{-2\pi i t} \frac{LT}{\beta_1 \beta_2} \sum_{j=0}^{L-1} \sum_{m, \ell \in \mathbb{Z}} \sigma^{(j)}_{m, \ell} \mathcal{F}_s \left( \mathcal{M}(-n_j \Omega, -k_T) \mathcal{T}_{\frac{mTL}{\beta_1}, \frac{LL\Omega}{\beta_2}} \right) \left( t, \nu \right) \]
\[ = e^{-2\pi i t} \frac{LT}{\beta_1 \beta_2} \sum_{j=0}^{L-1} \sum_{m, \ell \in \mathbb{Z}} \sigma^{(j)}_{m, \ell} \mathcal{T}(k_T, -n_j \Omega) \mathcal{M}_{\frac{2LL}{\beta_2}, \frac{LL\Omega}{\beta_2}} \left( r \otimes \hat{\phi} \right) \left( t, \nu \right) \]
\[ = \frac{LT}{\beta_1 \beta_2} \sum_{j=0}^{L-1} \sum_{m, \ell \in \mathbb{Z}} \sigma^{(j)}_{m, \ell} e^{2\pi i \nu j \Omega k_T} \mathcal{T}(k_T, -n_j \Omega) \mathcal{M}_{\frac{LL\Omega}{\beta_2}, \frac{LL\Omega}{\beta_2}, \frac{LL\Omega}{\beta_2}, \frac{LL\Omega}{\beta_2}} \left( r \otimes \hat{\phi} \right) \left( t, \nu \right) e^{-2\pi i t \nu} \]

For \( U(t, \nu) = r \otimes \hat{\phi}(t, \nu) e^{-2\pi i t} \), we have
\[ \mathcal{F}_s U(x, \xi) = \int \int r(t) \hat{\phi}(\nu) e^{-2\pi i t \nu} e^{-2\pi i t (\xi - \nu)} d\nu dt \]
\[ = \int r(t) \phi(x - t) e^{-2\pi i t \xi} dt = \int r(t) \overline{\phi} e^{-2\pi i t \xi} dt = V \phi r(x, \xi), \]
where we used that \( \hat{\phi} \) real valued implies \( \phi(y) = \overline{\phi}(-y) \). Now, we compute
\[ \sigma(x, \xi) = \mathcal{F}_s \eta(x, \xi) \]
\[ = \frac{LT}{\beta_1 \beta_2} \sum_{j=0}^{L-1} \sum_{m, \ell \in \mathbb{Z}} \sigma^{(j)}_{m, \ell} e^{2\pi i \nu j \Omega k_T} \mathcal{F}_s \left( \mathcal{T}(k_T, -n_j \Omega) \mathcal{M}_{\frac{LL\Omega}{\beta_2}, \frac{LL\Omega}{\beta_2}, \frac{LL\Omega}{\beta_2}, \frac{LL\Omega}{\beta_2}} \right) \left( x, \xi \right), \]
\[ = \frac{LT}{\beta_1 \beta_2} \sum_{j=0}^{L-1} \sum_{m, \ell \in \mathbb{Z}} \sigma^{(j)}_{m, \ell} e^{2\pi i \nu j \Omega k_T} \mathcal{M}(-n_j \Omega, k_T) \mathcal{T}_{\frac{mT\Omega}{\beta_1}, \frac{LL\Omega}{\beta_2} - n_j \Omega} V \phi r \left( x, \xi \right), \]
\[ (6.6) \hspace{1cm} = \frac{LT}{\beta_1 \beta_2} \sum_{j=0}^{L-1} e^{-2\pi i (\xi n_j + \xi k_T) T} e^{2\pi i \nu j \Omega k_T} \sum_{m, \ell \in \mathbb{Z}} \sigma^{(j)}_{m, \ell} V \phi r \left( x - \frac{mT\Omega}{\beta_1} + k_T, \xi - \frac{LL\Omega + n_j \Omega}{\beta_2} \right). \]

The convergence in (6.5) and (6.6) is defined in the weak sense, but can be shown to converge absolutely and uniformly on compact subsets.

Next we prove Theorem 2.10, that is, the direct local correspondence between the discretization values and the operator action.

**Proof of Theorem 2.10.** We intend to apply Theorems 2.7 and 2.9. We assume that the set \( M \) as well as its enclosing rectangular grid are fixed, hence also the parameters \( T, \Omega, \) and \( L \). The dependence of the constants, auxiliary functions, etc., in the following derivations on these parameters will be suppressed for notational convenience; this should be seen as analogue to the one-dimensional scenario where the arising constants also depend on the shape and not just the size of the frequency support. Furthermore, set \( Q = \max(LT, L\Omega) \).

We can bound using (3.3)
\[ |\sigma^{(j)}_{m, \ell}| = |\langle \sigma, \Phi_{m, -n_j, \ell, -k_T} \rangle| \]
\[ \leq \| \sigma \|_\infty \| \Phi_{m, -n_j, \ell, -k_T} \|_1 \]
\[ \leq B \| \sigma \|_\infty \| \hat{r} \otimes \hat{\phi} \|_1 \]
\[ \leq \hat{B}_\mu \]

For the second inequality, we used that the \( L_1 \)-norm is invariant under translations and modulations.
Furthermore, note that \( V_\sigma r \in \mathcal{S}(\mathbb{R}^2) \), so there is a decreasing positive function \( \rho \in \mathcal{S}([0, \infty)) \) such that for \( \tilde{\rho}(x, \xi) = \rho(|x|)\rho(|\xi|) \) one has \( |V_\sigma r| \leq \frac{1}{8CT} \tilde{\rho} \) pointwise.

Now observe that, as \( \rho \) is decreasing,

\[
\sum_{j=0}^{\infty} \alpha \rho(\alpha j) \leq \rho(0) + \sum_{j=1}^{\infty} \int_{\alpha(j-1)}^{\alpha j} \rho(t) dt = \|\rho\|_1 + \|\rho\|_\infty.
\]

We use this estimate to bound for arbitrary \((x, \xi)\)

\[
|\tilde{\sigma}(x, \xi)| = \left| \frac{LT}{\beta_1 \beta_2} \sum_{j=0}^{L-1} e^{-2\pi i(xn+\xi k_j)T} e^{2\pi i j n \Omega} \sum_{(mLT/\beta_1, \ell L\Omega/\beta_2) \in S} \sigma_{m,\ell}^{(j)} V_\sigma r \left( x - \left( \frac{mL}{\beta_1} + k_j \right) T, \xi - \left( \frac{\ell L}{\beta_2} + n_j \right) \Omega \right) \right|
\]

\[
\leq \frac{LT}{\beta_1 \beta_2} \sum_{(mLT/\beta_1, \ell L\Omega/\beta_2) \in S} \sum_{j=0}^{L-1} \tilde{C}_\mu \frac{\rho}{23} \left( x - \left( \frac{mL}{\beta_1} + k_j \right) T \right) \rho \left( \xi - \left( \frac{\ell L}{\beta_2} + n_j \right) \Omega \right)
\]

\[
\leq \frac{\mu}{8CLT} \sum_{m,\ell=0}^{L-1} \sum_{j=0}^{\infty} \frac{LT}{\beta_1} \rho \left( \frac{mL}{\beta_1} T \right) \frac{L\Omega}{\beta_2} \rho \left( \frac{\ell L}{\beta_2} \Omega \right)
\]

\[
\leq \frac{\mu}{2} \left( \|\rho\|_1 + \|\rho\|_\infty \right)^2
\]

and hence

\[
\|\sigma - \tilde{\sigma}\|_\infty \leq \|\sigma\|_\infty + \|\tilde{\sigma}\|_\infty \leq \mu + \frac{\mu}{2} \left( \|\rho\|_1 + \|\rho\|_\infty \right)^2 =: C_1 \mu.
\]

By the definition of \( S \), for every \( \delta > 0 \), there is a constant \( C(\delta) \) such that for any fixed \( 0 \leq j < L \),

\[
\delta \geq 8\|\rho\|_{L^1(\mathbb{R}^2)} \|\rho\|_{L^1([C(\delta) - 2Q, \infty))} \geq 8\|\tilde{\rho}\|_{L^1([-C(\delta) + Q, C(\delta) - Q])},
\]

and hence, for \((x, \xi) \in S - B(C(\delta))\),

\[
\delta \geq \frac{L^2}{\beta_1 \beta_2} \sum_{\ell, m \in \mathbb{Z}} \rho \left( \left| x - \left( \frac{mL}{\beta_1} + k_j \right) T \right| \rho \left( \left| \xi - \left( \frac{\ell L}{\beta_2} + n_j \right) \Omega \right| \right)
\]

\[
\left( x - \frac{mL}{\beta_1} T, \xi - \frac{\ell L}{\beta_2} \Omega \right) \notin [-C(\delta), C(\delta)]^2
\]

To obtain (6.9) from (6.8), the boundary term in the discretization of the integral and the shifts by \( k_j \) and \( n_j \), respectively, are each compensated by increasing the dimensions of the integration/summation domain by \( LT \) and \( L\Omega \) in time and frequency, respectively, both of which are bounded by \( Q \).

Note furthermore that, as \((x, \xi) \in S - B(C(\delta))\), a necessary condition for

\[
\left( x - \frac{mL}{\beta_1} T, \xi - \frac{\ell L}{\beta_2} \Omega \right) \notin [-C(\delta), C(\delta)]^2
\]

is that

\[
(mLT/\beta_1, \ell L\Omega/\beta_2) \notin S.
\]

Thus, using (6.7) and the triangle inequality, we can bound (6.9) from below obtaining

\[
\delta \mu \geq \left| \frac{L^2T}{\beta_1 \beta_2} \sum_{(mLT/\beta_1, \ell L\Omega/\beta_2) \notin S} \sigma_{m,\ell}^{(j)} V_\sigma r \left( x - \left( \frac{mL}{\beta_1} + k_j \right) T, \xi - \left( \frac{\ell L}{\beta_2} + n_j \right) \Omega \right) \right|
\]
Hence forming a weighted average (with complex weighting factors of modulus one) of Equation (6.10) over the \( L \) choices of \( j \), we obtain
\[
\delta \mu \geq \left| \frac{LT}{\beta_1 \beta_2} \sum_{j=0}^{L-1} e^{-2\pi i(x_n \Omega + \xi k_j T)} e^{2\pi i m \xi k_j T} \left[ \phi_{m,\xi}^{(j)}(x - \left( \frac{mL}{\beta_1} + k_j \right) T, \xi) - \left( \frac{L}{\beta_2} + n_j \right) \Omega \right] \right|
\]
\[
= |\sigma(x, \xi) - \tilde{\sigma}(x, \xi)|.
\]
This yields \( \| \sigma - \tilde{\sigma} \|_{L^\infty(S - B(C(\delta)))} \leq \delta \mu \). Hence by Theorem 2.7, we conclude that
\[
\| Hf - \tilde{H}f \|_2 \leq C_2 \frac{\delta}{C_1} \mu
\]
for all functions \( f \) which are \( \frac{\delta}{C_1} \)-time-frequency-localized to \( S - B(C(\delta)) - B(d(\epsilon)) \). The result follows by choosing \( \delta = \min \left( \frac{C_2}{C_1} \epsilon, C_1 \epsilon \right) \) and \( D(\epsilon) = C(\delta) + d(\epsilon) \).

\[\square\]

### 7. Quantization of Bandlimited Operators

The underlying idea of the quantization schemes in Theorem 2.11 and Corollary 2.12 is based on \( \Sigma \Delta \) modulation. At the core of this arguably most influential coarse quantization paradigm is the observation that the reconstruction formula in Theorem 2.1 directly corresponds to the application of a low-pass filter. The key idea is then that, while, due to the coarseness of the alphabet, the sequence \( y_n - q_n \) cannot be made uniformly small, it can be chosen to be approximately high-pass, that is, close to the kernel of the low-pass operator; hence it almost vanishes in the reconstruction procedure. In other words, \( \tilde{f}(t) = T \sum_{n \in \mathbb{Z}} q_n \phi(t - nT) \) is a good approximation for \( f \).

The main goal of this section is to show, in general terms, the possibility of combining the discretization procedure presented above with coarse quantization schemes. Hence, we will restrict ourselves to the simplest possible \( \Sigma \Delta \) modulator, a so-called first-order \( \Sigma \Delta \) modulator. Usually, with so-called higher order modulators \([6, 12, 7]\) considerably better error decay rates can be achieved. There are no specific obstacles that would prevent the direct application of such higher order modulators in the operator context. However, the estimates would be considerably more complicated without providing much additional insight, which is why we refrain from presenting them here.

To define a first-order one-bit \( \Sigma \Delta \) modulator, we first fix a kernel \( \phi \) such that \( \phi \in S \) and \( \text{supp } \phi \subset \left[ -\frac{\lambda_0}{4}, \frac{\lambda_0}{2} \right] \) for some \( \lambda_0 > \Omega \). Then the minimal sampling step is \( T_0 = \frac{\lambda_0}{2\Omega} \). So sampling at step size \( T \) corresponds to an over sampling ratio of \( \lambda = \frac{T}{2\Omega} \). As we are interested in the approximation behavior when the redundancy of the dictionary increases, we do not want to consider multiple quantization schemes. Hence we fix an underlying quantization alphabet with an associated quantization rule. Again, we will focus on the simplest possible scenario, namely one-bit quantization, where the quantization alphabet just has two elements.

Using this quantization rule, a first order \( \Sigma \Delta \) modulator computes a sequence of quantized values \( \{q_n\} \) by means of the iterative scheme

\[
\begin{align*}
    u_n &= u_{n-1} + f(nT) - q_n \\
    q_n &= \text{sign}(f(nT) + u_{n-1}),
\end{align*}
\]

with an initial condition \( u_0 \).

One can show (see for example [6]) that if \( |f(t)| \leq 1 \) and \( |u_0| \leq 1 \), then the state variable \( u \) in (7.1) satisfies
\[
\forall n \in \mathbb{Z}, \quad |u_n| \leq 1.
\]

This entails a bound for the quantization error of
\[
\| f - \tilde{f} \|_{L^\infty(\mathbb{R})} \leq \frac{1}{2\Omega \lambda} \| \phi' \|_{L^1(\mathbb{R})}.
\]
Proof of Theorem 2.11. By Theorem 2.9, \( \sigma = LT \sum_{j=1}^{L} \omega_j \sigma^{(j)} \), where \( \omega_j \) are phase factors and

\[
\sigma^{(j)}(x, \xi) = \frac{1}{\beta_1 \beta_2} \sum_{m, \ell \in \mathbb{Z}} \sigma_{m, \ell}^{(j)} V_{\phi} r(x - \left( \frac{mL}{\beta_1} + k_j \right) T, \xi - \left( \frac{\ell L}{\beta_2} + n_j \right) \Omega).
\]

We will quantize the doubly-indexed coefficient sequence \( \{ \sigma_{m, \ell}^{(j)} \} \) of the expansion in (7.2), separately for each \( j = 0, 1, \ldots, L - 1 \). For simplicity, we will drop the superscript \( (j) \) for now, since most of the proof proceeds independently for each \( j \). We also drop the \( n_j \) and \( k_j \), as for fixed \( j \), they can be absorbed in \( x \) and \( \xi \). Hence we seek to quantize an expansion of the form

\[
\sigma(x, \xi) = \frac{1}{\beta_1 \beta_2} \sum_{m, \ell \in \mathbb{Z}} \sigma_{m, \ell} V_{\phi} r(x - \frac{mL}{\beta_1} T, \xi - \frac{\ell L}{\beta_2} \Omega).
\]

We proceed by applying first order Sigma-Delta modulators subsequently in both time and frequency. Such an approach has been successfully applied by Yılmaz [27] to devise a coarse quantization scheme for Gabor expansions.

More specifically, we recursively define doubly indexed sequences \( u, p, v, r \) as follows.

\[
\begin{align*}
&u_{n_1, n_2} = u_{n_1-1, n_2} + \sigma_{n_1, n_2} - p_{n_1, n_2}, \\
&p_{n_1, n_2} = \text{sign}(u_{n_1-1, n_2} + \sigma_{n_1, n_2}), \\
&v_{n_1, n_2} = u_{n_1, n_2-1} + u_{n_1, n_2} - r_{n_1, n_2}, \\
&r_{n_1, n_2} = \text{sign}(v_{n_1, n_2-1} + u_{n_1, n_2}).
\end{align*}
\]

(7.2)

For a bivariate sequence \( a \), we denote \( (\Delta_1 a)_{n_1, n_2} = a_{n_1, n_2} - a_{n_1-1, n_2} \) and \( (\Delta_2 a)_{n_1, n_2} = a_{n_1, n_2} - a_{n_1, n_2-1} \). Then we have

\[
\begin{align*}
(\Delta_1 \Delta_2 v)_{n_1, n_2} &= \sigma_{n_1, n_2} - q_{n_1, n_2}, \\
q_{n_1, n_2} &= p_{n_1, n_2} + (\Delta_1 r)_{n_1, n_2}.
\end{align*}
\]

(7.3)

The resulting sequence \( q_{m, \ell} \in \{ \pm 1, \pm 3 \} \) of quantized values will be used for reconstructing \( \sigma \), these equation hence define the bivariate Sigma-Delta modulator. Note that to properly define the Sigma-Delta modulator, one also needs to set an initial condition for some finite index. However, the above recurrence relations are reversible in the sense that they allow to apply the quantization procedure backwards in time and frequency, hence allowing for the acquisition of quantized values for all \( (m, \ell) \in \mathbb{Z}^2 \). A more realistic scenario with only finitely many quantized values used for the approximation is obtained by combining this result with the localisation result given in Theorem 2.10.

As the Sigma-Delta scheme given by (7.2) and (7.3) is a combination of two first order Sigma-Delta schemes, its stability is a direct consequence of the stability of (7.1), see also [27]. We summarize this observation in the following proposition.

Proposition 7.1. Suppose that \( |\sigma_{n_1, n_2}| \leq 1 \). If \( |u_{0, n_2}| \leq 1 \) for each \( n_2 \in \mathbb{Z} \), and \( |v_{n_1, 0}| \leq 1 \) for each \( n_1 \in \mathbb{Z} \), then \( |u_{n_1, n_2}| \leq 1 \), \( |v_{n_1, n_2}| \leq 1 \) for every \( n_1, n_2 \in \mathbb{Z} \).

Stability is a crucial ingredient for the error analysis of the bivariate Sigma-Delta scheme.

Proposition 7.2. Let \( \sigma : \mathbb{R}^2 \to \mathbb{C} \) be as in (7.2) and \( q_{m, \ell} \), \( m, \ell \in \mathbb{Z} \) be the quantized values resulting from the bivariate Sigma-Delta quantization given by (7.2) and (7.3) with \( \sigma_{m, \ell} \leq 1 \) for all \( m, \ell \in \mathbb{Z}^2 \). Then the symbol reconstructed according to the formula

\[
\tilde{\sigma}(x, \xi) = \frac{1}{\beta_1 \beta_2} \sum_{m, \ell \in \mathbb{Z}} q_{m, \ell} V_{\phi} r(x - \frac{mL}{\beta_1} T, \xi - \frac{\ell L}{\beta_2} \Omega),
\]

satisfies

\[
\|\sigma - \tilde{\sigma}\|_{L^\infty} \leq \frac{1}{\beta_1 \beta_2} \left\| \frac{\partial^2}{\partial y \partial x} V_{\phi} r \right\|_{L^1}.
\]
Proof. We compute
\[
\beta_1 \beta_2 |\sigma(x, \xi) - \bar{\sigma}(x, \xi)|
\]
\[
= \left| \sum_{m, \ell \in \mathbb{Z}} (\sigma_{m, \ell} - q_{m, \ell}) \phi \left( x - \frac{m}{\beta_1} LT, \xi - \frac{\ell}{\beta_2} L \Omega \right) \right|
\]
\[
= \left| \sum_{m, \ell \in \mathbb{Z}} (\Delta_1 \Delta_2 v)_{m, \ell} \phi \left( x - \frac{m}{\beta_1} LT, \xi - \frac{\ell}{\beta_2} L \Omega \right) \right|
\]
\[
= \left| \sum_{m, \ell \in \mathbb{Z}} \sum_{q} v_{m, \ell} \int_{\mathbb{R}} \frac{\partial}{\partial x} \phi r(x - u, \xi - \frac{\ell}{\beta_2} L \Omega) - \frac{\partial}{\partial x} \phi r(x - u, \xi - \frac{\ell + 1}{\beta_2} L \Omega) \, du \right|
\]
\[
\leq \left| \sum_{m, \ell \in \mathbb{Z}} \sum_{q} v_{m, \ell} \int_{\mathbb{R}} \frac{\partial^2}{\partial x^2} \phi r(x - u, \xi - v) \, dv \right|
\]
\[
\leq \iint \left| \frac{\partial^2}{\partial y \partial x} \phi r(u, v) \right| \, dv \, du.
\]
In the last step we used the stability of the scheme, as established by Proposition 7.1. \(\square\)

To complete the proof of Theorem 2.11, we recall from (6.7) that there exists a constant \(\tilde{C}\) such that, for all \(m, \ell \in \mathbb{Z}\), \(|\sigma_{m, \ell}^{(j)}| \leq \tilde{C}\|\sigma\|_{\infty} \leq c\tilde{C}\) by assumption of the Theorem. Hence choosing \(c = \frac{1}{\tilde{C}}\), we ensure that \(|\sigma_{m, \ell}^{(j)}| \leq 1\) and we can apply Proposition 7.2 to conclude that
\[
\|\sigma^{(j)} - \bar{\sigma}^{(j)}\|_{L^\infty} \leq \frac{1}{\beta_1 \beta_2} \left\| \frac{\partial^2}{\partial x \partial y} \phi r \right\|_{L^1}.
\]
The right hand side is bounded by an absolute constant, so by Proposition 4.1 for \(p = 2\), we conclude that \(\|H^{(j)} f - \hat{H}^{(j)} f\|_{L^2} \leq \frac{C}{\beta_1 \beta_2} \|f\|_{L^2}\), where \(H^{(j)}\) and \(\hat{H}^{(j)}\) are the operators with symbol \(\sigma^{(j)}\) and \(\bar{\sigma}^{(j)}\), respectively. Choosing \(C = L^2 T \tilde{C}\), the theorem follows via the triangle inequality. \(\square\)

Sketch of proof of Corollary 2.12. We quantize \(H\) as described above. Due to the recursive nature, we need only finitely many of the \(q_{m, \ell}\) to compute any given finite set of \(q_{m, \ell}\)’s. This gives rise to the first summand in the error bound. The step from the quantized representation to the quantized representation with only finitely summands proceeds completely analogously to the proof of Theorem 2.10; in particular, the error arising in this step can be bounded by the second summand of the error bound. Due to the completely analogous nature of the proof, we will not repeat the details here.

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