Diurnal variation of VLF signals

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1 Introduction

Like a number of amateurs, I have been recording VLF signal strength using a home-
made loop antenna, amplifier, and a computer sound-card. Having obtained a number
of days data, it seemed to be an interesting exercise to fit the theoretically predicted
logsec variation with zenith angle of the reflection layer height in the ionosphere D-
layer. This short paper attempts just that.

I am no expert in radio astronomy, or radio engineering, or even of the physics
involved, though I do have some background knowledge and expertise in mathematics
and programming. So much of this short paper explains some of the theory I have
learnt in the process of doing this work. I don’t claim anything particularly new here,
but some of the techniques I use may be of interest and my explanations of the back-
ground theory and description of the investigation here may be of interest to other VLF
amateurs. There is mathematics here, at the upper end of the current A-level standard,
including some simple differential equations, but hopefully the presentation will be
straightforward enough for readers at this level. It is quite reassuring that some quite
significant results on the ionosphere and VLF needs nothing more complicated than
this.

2 The theory

The logsec variation of the height of the reflection layer is due to Chapman [1]. In
the form needed for this work, the theory is very straightforward and accessible to
anyone with a knowledge of simple differential equations. I have learnt this theory from
reading Ratcliffe [6] though no doubt many other texts are available. The following is
a slightly simplified account that gives the results needed.

The first stage (prior to applying Chapman’s theory of the production layer) is to
understand the height variation of concentration of particles (atoms, molecules, ions)
in the atmosphere.

Let h denote height (in m) above some reference level (for convenience, the Earth’s
surface) and \( n = n(h) \) the concentration (in \( \text{Molm}^{-3} \)) of some species of molecule
relevant to a particular ionisation process, such as NO. If each molecule has mass \( m \)
and \( g = g(h) \) is the acceleration due to gravity then the force downwards due to gravity

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on the molecules in the unit volume is \( nmg \). This is balanced by the difference in pressure \( p \) between the top and bottom of the volume so that

\[
\frac{dp}{dh} = -nmg.
\]  

(1)

Pressure is given by \( p = nkT \) where \( k \) is Boltzmann’s constant and \( T = T(h) \) is temperature in K, so

\[
\frac{d}{dh}(nkT) = -nmg.
\]  

(2)

Now for the range of heights to be taken here that are relevant to the lower ionosphere, \( h \) ranging from around 60 to 95 km, and compared to the radius of the Earth of 6371 km, both \( g \) and \( T \) may be assumed to be more or less constant. Thus

\[
\frac{1}{n} \frac{dn}{dh} = -\frac{mg}{kT}.
\]  

(3)

We set \( H = kT/mg \) (ideally with values of \( g \) and \( T \) at or near those at the D-layer) and call \( H \) the scale height or distribution height of the species represented by \( n \). The solution of the differential equation above is

\[
n = n_0e^{-h/H}
\]  

(4)

where \( n_0 \) is a constant representing the value of \( n \) at the reference height \( h = 0 \). In other words, concentration \( n \) is theoretically an inverse exponential distribution with constant \( H \).

For readers unfamiliar with this and unclear on the significance of \( H \), this distribution is somewhat similar to the familiar distribution in time of the number \( N_0e^{-t/t_0} \) of atoms of a radioactive element undergoing decay. The constant \( t_0 \) (a length of time) is the half-life of the the element, and waiting one half-life results in halving the number of atoms. Similarly, the scale height \( H \) is the height one must travel upwards to decrease \( n \) by a factor of \( e = 2.71828 \ldots \). For the D-layer it is typically about 5 km, as we shall see.

The next stage is to imagine ionising radiation being applied from above, i.e. from the sun. The sun, we shall assume, is at an angle \( \chi \) from the zenith, i.e. \( \chi = 0 \) corresponds to the sun being directly overhead and \( \chi = 90^\circ \) it being on the horizon. Just as it was the case that not all air molecules are relevant for ionisation of the D-layer, so it is that not all frequencies are relevant here either. We will assume that a band of frequencies are responsible for ionisation, and the power flux from the sun in this band is \( I_\infty \), measured in W m\(^{-2}\), so if an area of one square metre were mapped out in space on a plane perpendicular to solar rays, in one second \( I_\infty \) Joules of energy in the relevant band would pass through this area. As should be clear, if the plane were not perpendicular to the solar rays the effective area available is less and less energy would pass through it. In fact if the solar radiation were at an angle \( \chi \) to the plane’s perpendicular then \( I_\infty \sec \chi \) Joules of energy would pass through the plane, where \( \sec \chi = 1/\cos \chi \), and again \( \chi = 0 \) refers to the rays being exactly perpendicular i.e. directly above.

The sun’s energy is absorbed by the atmosphere, and the amount it is absorbed by is proportional to \( n \)—the constant of proportionality (called the ‘absorption cross-section’) will be denoted \( \sigma \). So the ionisation radiation \( I \) varies with height \( h \) and as it passes through each unit of volume is decreased by \( \sigma nI \sec \chi \). Thus the differential equation for \( I \) is

\[
\frac{dI}{dh} = \sigma nI \sec \chi
\]  

(5)
Some care is needed to get the sign right here, but the above is correct since the ionisation energy is coming from above and is absorbed in the atmosphere, so $I$ is decreasing as $h$ decreases.) Our previous equation (4) can be substituted in here and the equation rearranged to give,

$$\frac{1}{I} \frac{dI}{dh} = \sigma n_0 \sec \chi e^{-h/H}$$

which when solved gives

$$\log \left( \frac{I}{I_\infty} \right) = -H \sigma n_0 \sec \chi e^{-h/H},$$

log being natural logarithm to base e, or

$$I = I_\infty \exp(-H \sigma n_0 \sec \chi e^{-h/H}).$$

The energy absorbed by the atmosphere doesn’t disappear but goes somewhere: it is either converted to heat or used to ionise the atmosphere. Thus the production rate $q$ of electrons (or other charged particles that can reflect radio waves) is proportional to the amount of energy $\sigma n I \sec \chi$ absorbed. Letting $C$ denote the constant of proportionality, we have

$$q = C \sigma n I \sec \chi$$

or

$$q = C \sigma n_0 e^{-h/H} \sec \chi I_\infty \exp(-H \sigma n_0 \sec \chi e^{-h/H}).$$

To complete the story, the electrons produced in this way either diffuse to a different height or recombine with other molecules in the air according to one of a number of possible reactions. More details on this are not needed here. What we need to observe here is that the height $h_m$ of the reflecting layer corresponds to the position of greatest rate of electron production. (The reason why this is the right condition is slightly complicated, but my understanding is that it is because the rate of electron recombination is proportional to the concentration of electrons and to the concentration of particles with which they can combine with. It is these simple proportionalities that ensure that the place of greatest change of electron concentration is the same as the place of greatest electron production.) Thus to find the height $h_m$ of the reflecting layer we find the height where $q$ is maximum, and the simple technique of differentiating $q$ and setting the derivative equal to zero is used. The derivative of $q$ is obtained by the chain and product rules (noting the double exponential in $h$) and $dq/dh = 0$ simplifies to

$$-\frac{1}{H} + \sigma n_0 \sec \chi e^{-h_m/H} = 0$$

or

$$h_m = H \log(H \sigma n_0 \sec \chi) = H \log(H \sigma n_0) + H \log \sec \chi$$

which is the equation alluded to in the introduction. Notice that this is of the form

$$h_m = H + H \log \sec \chi$$

where $H$ is the scale height, a value of some physical importance. The constant $A = H \log(H \sigma n_0)$ represents the height the reflecting layer would have been at, given steady conditions with the same radiation energy but with the sun exactly overhead.

Of course, a VLF receiver does not measure the height of the reflecting layer directly, but this height can sometimes be inferred from measurements. The varying
strength of a signal is an indication of an interference between different paths of propagation. Normally, there are many different paths, but for signals from nearby transmitters we can reasonably model the process as being the interference effects between a ground-wave and a bounced sky-wave. This part of the modelling process is essentially just one of geometry without any calculus. The details have no doubt been worked out many times, I summarise the results here, and Mark Edwards [2] gives more detail and additional explanations should they be required.

Given that the ground-wave travels a distance $D$ along the curved surface of the earth and the sky-wave travels a distance $L$, the phase difference between them (in radians) is

$$
\phi = 2\pi (L - D)f/c + \pi 
$$

(13)

where $f$ is the frequency of the transmission (in Hz), $c$ is the speed of light and the additional $\pi$ is due to a phase change on reflection. By geometry and the cosine rule, the distance $L$ is related to $D$, $h_m$ and $R$ (the radius of the Earth) by

$$
L = 2\sqrt{R^2 + (h_m + R)^2} - 2R(h_m + R)\cos(D/2R)
$$

(14)

and the power $P$ of the received wave is proportional to

$$
P = G^2 + S^2 + 2GS \cos \phi
$$

(15)

where $G$ is the amplitude of the ground wave and $S$ the amplitude of the sky-wave.

3 The practice

The proposal is to look at the variation of the received power over the course of a quiet day, for a nearby transmitter and see how well the observed data for the theoretical pattern described here. Note that there are four unknown variables in the theory: the scale height $H$, the quantity $H \log(H \sigma_0)$ representing the height of the reflecting layer at $\chi = 0$, and the amplitudes of the ground and sky-waves.

Thus we want to fit

$$
\text{power} = Q + 2P \cos \phi
$$

(16)

$$
\phi = 2\pi (L - D)f/c + \pi
$$

(17)

$$
L = 2\sqrt{R^2 + (h_m + R)^2} - 2R(h_m + R)\cos(D/2R)
$$

(18)

$$
h_m = A + H \log \sec \chi
$$

(19)

$$
\chi = \text{sun’s zenith angle at midpoint}
$$

(20)

to our data, where $Q = G^2 + S^2$ and $P = GS$ in (15). I used a downloadable algorithm for the sun’s zenith angle\(^1\) and readily available data on the position of the transmitter—and hence derived the longitude and latitude of the midpoint. Thus the unknowns are $Q, P, A, H$ only.

In any curve fitting algorithm, having initial estimates for the unknown values being sought is very useful indeed. In this case the constants can be given rough estimates quite quickly: $H$ is known to be about 8 km at ground level, independent of the species involved [6, page 5]; the height of the reflecting layer is nominally around 90 km; and

\(^1\)From http://www.psa.es/sdg/sunpos.htm
the quantities \( Q = G^2 + S^2 \) and \( 2P = 4GS \) can be estimated quickly from the VLF measurements, as follows. In the daytime, excluding some complicated sunrise/sunset effects which are due to more complicated geometry of a spherical Earth and different propagation paths, the measured signal varies from a minimum at \( \phi = 2N\pi - \pi/2 \) to a maximum at \( \phi = 2N\pi + \pi/2 \) (for some integer \( N \) which cannot be directly estimated) and thus from (15) the difference between this maximum and minimum is about \( 4GS \). The quantity \( G^2 + S^2 \) is then the value exactly halfway between this maximum and minimum.

Even with these initial guesses for the parameters involved, I do not have an ideal curve-fitting algorithm. The main problem is that rather different height estimates sometimes record a ‘good fit’ simply because the \( \cos \phi \) function in (16) is periodic and differing values of \( \phi \) do indeed give reasonably good fits. To say this in another way, it is not really possible to obtain the height \( h_m \) from the measured phase information as the mapping from \( h_m \) to \( \phi \) is many-to-one. A second problem is the possible occurrence of SIDs—periods when the data do not fit the usual quiet diurnal pattern.

As a compromise, my experimental algorithm discounts a certain percentage of the data (say 10%, though this parameter can be varied). The measure of ‘fit’ is the sum of the \( (v_{\text{measured}} - v_{\text{predicted}})^2 \) for all but the 10% greatest values of this quantity. (These squared differences are stored in a heap so that the best 90% can be extracted quickly.) Rather than risking a ‘clever’ algorithm rapidly settling on a ‘bad’ value of \( h_m \), I test many values of \( A, H \) differing by only a small amount in succession before selecting the ‘best’ and then refining this value in a similar way. But as it turns out, the curve fitting is relatively stable in the other two parameters \( S, P \) so that it is possible to find reasonably good values for \( A, H \) using the initial estimates for \( S, P \), using these values to refine the estimates for \( S, P \), and then using these values to refine the values for \( A, H \). Exactly how often this process should continue and in what order and with what step size is still very much open for experiment, but as can be seen, reasonably good fits can indeed be obtained.

In the month of October 2013, the 21st was a comparatively ‘quiet’ day and will be used to illustrate these methods. The signal from Skelton, UK, on 22.1kHz is the strongest nearby signal at my location, being about 263km distant. I entered the co-ordinates of the midpoint and the distance to the transmitter and started to fit the data. This was the result.
The blue line shows the actual measured values. The red line shows the model’s value for the height of the reflecting layer—which is only defined for $0 \leq \chi < 90^\circ$ since $\sec \chi$ approaches infinity near sunset. (Outside this region I arbitrarily set it to 100km.) The green line shows the fitted curve.

The fit seems reasonably good, though not by any means perfect. The values the fit took for the height of the reflecting layer at various times and for $A, H$ in (19) above were $A = 76.84$ and $H = 5.06$. These values seem encouraging, especially as they were chosen from the fitting algorithm over a range of plus or minus 20% and we read

For VLF waves incident on the ionosphere at steep incidence, the reflection height, $h$, appears to vary as $h_0 + H \ln \sec \chi$ where $\chi$ is the solar zenith angle. $h_0$ is about 72 km, and $H$ is about 5 km, which happens to be the scale height of the neutral gas in the mesosphere.

(from Hunsucker and J. K. Hargreaves [4, page 35]).

Unfortunately, one worry is that (as already mentioned) quite different values for the height parameter also fit quite well through using a different period in the $\cos \phi$ function. For example, the following fit was found with parameters $A = 64.06$ and $H = 5.12$ showing that value obtained by the fit for $A$ is not particularly robust. Similarly values for $H$ from reasonable looking fits were found ranging from 5 to 6.

One possible approach is to look at a number of different signals and compare them. For example, this is the nearby Anthorn signal on 19.6kHz on the same day.
The fit here had $A = 78.23$ and $H = 5.06$. This suggests that these parameters are in the right sort of ‘ballpark’, but the evidence isn’t particularly convincing.

Mark Edwards has pointed out (especially in his presentation to the BAA Radio group in 2011) that the *combination* of these two signals from two transmitters very close together can *together* give an accurate fix on the height of the reflecting D-layer, because they are operating at different frequencies and it so happens that at his location the result is that the daytime signals from these two locations appear almost a mirror image of each other. What’s more, the reflection points for the sky-waves for these two transmissions are very close to each other so it is reasonable to assume that the D-layer height is the same in both cases. In this context the simultaneous fit of the log sec model to these data may be only feasible for $A$ around 77 or 78 and $H$ about 5. This is a very sensible suggestion and well worth undertaking where feasible, but in general this will depend on specific local circumstances (such as the availability of suitable nearby transmissions and the distance to the transmitters and frequency of these transmissions). In general, the hope is that an intelligent examination of all the various possible heights in the case of two or more separate transmissions will rule out all but the correct D-layer height, especially if the reflection points in question are very close together. There is obviously more work to be done here.

In both cases, the fit is noticeably not so good near sunrise and sunset, particularly near sunset. Of course one cannot expect a perfect fit near these limits, because for example the model predicts an infinite height at sunrise/sunset, whereas in fact the curvature of the Earth has effects that are not taken into account by the model (such as the possibility that, at 90km above the ground, the ionosphere is radiated by solar radiation even when $\chi$ is greater than 90°). Also, other propagation paths come into effect at such extremes, and other mechanisms ionising mechanisms (such as cosmic rays) will become more significant at such times. Some indications that different mechanisms are at play are already evident in the multiple peak structure in the sunrise/sunset pattern for 19.6kHz, which (since the peaks are not at the maximum) cannot be predicted by the simple Chapman model with a single ionising source, and perhaps suggests evidence for more than one source of ionisation. This could be investigated further. Possible improvements to the model include: (a) reworking it for a spherical Earth; (b) incorporating any tilt of the D-layer into the calculations, since there is no particular reason why the D-layer will always be horizontal, especially at sunset and sunrise; and (c) investigating other ray paths. For (c), Edwards [3] reports improvements when an additional double bounce model is added.
Irrespective of the situation at sunrise, the shape of the measured and modelled curves are rather different at sunset, though a casual look at the data prior to making these attempts at fitting the model to them did not suggest there might be a problem. A little investigation explains why.

The next graphic shows the same raw data alongside GOES satellite measurements of X-ray solar flux.

One sees there was enhanced solar X-ray activity from 15:00UT onwards, and particularly from 15:40UT. At its maximum (at 16:12UT) this was at the C2.7 level, which is often small enough to be neglected, and in this case not sharp enough to be an obvious ‘flare’ creating a peak in the VLF trace. It seems highly likely that the lack of ‘fit’ at this time and the enhanced solar X-ray activity are related. Indeed this seems to be the main value for this technique: that comparing measured data with the model, the places where the measured data does not fit are more obvious and these often will reflect some interesting phenomena going on—in this case a minor X-ray induced ionospheric disturbance—that might have been easy to miss otherwise. Or to put it another way, such analyses have the potential to dramatically enhance the sensitivity of the measurements without changing the hardware in any way.

4 Conclusions

Fitting the Chapman model of diurnal variation can be done, and often seems successful except very close to the points of sunrise and sunset where the model (at least in the form given here) is not meaningful. However drawing conclusions from the model fitting has its difficulties, mainly because the mathematical mapping of reflection layer height (as predicted by the model) to phase difference (as measured) is many-to-one, hence different heights can result in fits that are or appear to be just as good. Any further experiments that exploit this model to obtain measurements of (for example) the scale height will have to resolve this problem and make very clear why the values for heights chosen are indeed the correct ones.

Nevertheless, even if the actual numerical values obtained from the process are not believed, the technique can provide a source of evidence for ionospheric disturbances.
measured from VLF data near sunrise or sunset when no obvious traditional ‘SID pattern’ is present in the data. In other words, these techniques can in principle be used to dramatically increase the sensitivity of a SID detector especially near sunrise/sunset.

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