BEM ANALYSIS OF LAMB WAVE SCATTERING
BY LAYERED PLATE DEBONDING

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This paper presents an analysis of reflection and transmission coefficients as boundary element solutions of guided Lamb wave scattering in a two-dimensional, isotropic, and linear elastic layered plate. A scatterer, defined as a debonding, is located on the interface between layers. The numerical model is formulated based on the boundary element method with elastodynamic traction and displacement fundamental solutions in frequency domain, where the boundary integral equations, including artificial boundaries, are discretized with constant elements. Lamb wave modal functions in each layer can be calculated by using the partial wave technique. Wave fields on the artificial boundaries, which are set long enough from the debonding to neglect near-field evanescent scattering modes, are treated as the superposition of incident wave and possible scattered waves. Then, the traction boundary condition on the artificial boundary can be expressed by unknown total displacement fields. In addition to the artificial boundary condition, the continuity condition of layer interface and the traction-free boundary conditions yield a system of linear equations that can be solved by the efficient indirect solver. The reflection and transmission coefficients are obtained from the total displacement fields after solving the system of linear equations. Various configurations of problems, including materials and length of the debonding are considered. Furthermore, the mode conversion and resonance phenomena of scattering coefficients can be seen directly related to the length of the debonding.

Key Words: BEM, Lamb wave scattering, debonding, scattering coefficients, mode conversion

1. INTRODUCTION

Nowadays, in many fields of engineering, multi-layered or laminated structures are being used due to their superior mechanical properties compared to traditional structures. For example, in aerospace engineering, laminated plates or shells are often used in aircraft wings. In the civil engineering field as well, carbon fiber sheets or plates, which are light and strong, are often used in infrastructure maintenance to reinforce or repair damaged structures.1) For any structures, when their lifetime has passed, the demand for inspection increases to improve the safety of a structure in service and to reduce the cost in life-cycle management. Non-destructive testing (NDT) based on guided waves has been well known for its application to practical inspection of large waveguide structures, and it allows detection of some defects or cracks inside the structures by investigating the scattering behavior of guided waves.2) The key advantages for guided wave NDT can be summarized as follows3): (1) guided waves have a potential to inspect a long and large structure, such as plate or pipe with a single inspection.
(2) Guided waves can detect, classify, and characterize flaws inside a structure in the hidden zone without causing any damage. (3) A guided wave NDT is one of the most cost-effective inspection methods due to its simple and rapid processes.

Figure 1 illustrates the overview concept of a guided wave inspection in a laminated plate structure. Guided waves generated by the transducer propagate as “incident wave” or “excited wave” in a layered plate to the receiver. During the propagation, if these waves encounter some flaws or cracks, scattering behavior will occur. In Fig.1, the transducer for transmitter also works as a receiver. The scattered waves that propagate back to the transmitter are known as “reflected waves.” The measurement of reflected waves makes it possible for the inspector to detect, classify, and characterize flaws in the structure being tested.

The guided wave propagating in a plate is called Lamb wave, which was named after the investigator Horace Lamb in 1917. The traditional method to formulate the governing equation for guided Lamb waves can be done by using the method of displacement potentials, which are limited only to isotropic plates. Solie and Auld proposed the technique called the partial waves method to obtain Lamb wave solutions, which can be applied for nonhomogeneous and anisotropic plates. Several numerical methods have been developed to solve the guided Lamb wave problems. A finite element method (FEM) has been implemented for scattering problems in analyzing the scattering of Lamb wave for many applications, such as in elastic plate waveguide and a normal rectangular strip weldment. Hayashi and Rose used a semi-analytical FEM to analyze the propagation of guided wave in plate and pipe. Cho and Rose developed a numerical method based on the boundary element method (BEM) together with a normal-mode expansion technique called Hybrid BEM to calculate scattering coefficients for edge reflection and surface breaking in a single plate scattering problem. Gunawan and Hirose proposed a method called the mode-exciting method for solving the scattering problem of Lamb waves in an infinite plate. They obtained the reflection and transmission coefficients for scattering problems with a horizontal crack in a single plate. The advantage of this method is the direct application of FEM or BEM without any coupling with other methods. From these paper reviews, it is found that most of numerical methods have been developed for the scattering of guided waves in a single plate, whereas very few studies have been done for scattering problems in a multiple-layered plate due to the technical difficulties, such as mathematical formulations, computational costs, and complexity of scattering behavior.

When dealing with a multiple-layered structure, it is important to detect inner defects, because a debonding often occurs on the interface of multiple layers as well as surface defects in mother materials. For the analysis of multiple-layered plates, a more sophisticated technique than one for a single plate is required to ensure the accuracy and precision of the calculation.

In this research, the computation technique based on the boundary element method for a single plate by Cho and Rose is extended to calculate scattering coefficients of guided Lamb waves generated by a debonding of a double-layered plate. The convergence of calculation results is discussed and the accuracy is verified for various configurations by comparison with previous studies to find new scattering phenomena seen in a multi-layered plate. The results from this study will be useful as reference information on quantitative NDT to detect inner defects of a layered structure.

This paper is organized as follows. In section 2, we present the problem description and the formulation of basic equations involved in the BEM analysis. In section 3, the mode expansion technique for multi-mode propagation with the treatment of artificial boundary is explained in detail and the procedure for calculating reflection and transmission coefficients is shown. In section 4, numerical results are demonstrated with different configurations and the calculation is verified. Lastly, this paper ends with conclusions and future plans.

2. PROBLEM STATEMENT AND FORMULATION

(1) Problem description

Consider a two-dimensional, linearly isotropic, elastic, double-layered plate as illustrated in Fig.2. The horizontal length of the plate is assumed to be infinite for the propagation of Lamb waves. For each layer, material properties of Lamé constants $\lambda^{(n)}$, $\mu^{(n)}$, $\rho^{(n)}$, and $\nu^{(n)}$.
Fig.3 A domain for two-dimensional elastodynamic problem.

\( \mu^{(n)} \), density \( \rho^{(n)} \) and height \( h^{(n)} \) are given; where \( n \) is the number of layers, i.e., \( n=1 \) for steel and \( n=2 \) for steel, aluminum, or titanium. The interface is perfectly bonded between layers except for a finite debonding zone with traction-free boundary condition, and the top and bottom surfaces of the double-layered plate are also traction-free.

In practice, the incident wave may be a Lamb wave excited by a transmitter located on the top or bottom surface of a plate as shown in Fig.1. However, since the objective of our research is focused on wave scattering in a multi-layered plate, we do not mind how the incident wave is generated by a transducer. We assume that the incident wave is a Lamb wave with a given amplitude, propagating from the far field on the left side of the plate, and scattered waves, which are scattered from the debonding zone, are calculated in terms of the amplitudes of Lamb wave functions, which have relations with reflection and transmission coefficients.

For convenience in further reference, the Cartesian reference coordinate system \((x_1, x_2)\) with the origin located at the bottom left of the boundary domain is used as indicated in Fig.2. It is noted that the region enclosed by red lines in Fig.2 is a finite domain used in the BEM analysis of guided waves in an infinite layered plate.

### Fundamental equations

Before formulating the boundary integral equations for the double-layered plate, the equation of motion for a general two-dimensional domain \( \Omega \) with the surface \( S = S_1 + S_2 \) in Fig.3 is shown in terms of the stress tensor \( \sigma_{y} \), body force \( b_i \), and displacement \( u_i \) as

\[
\sigma_{y,j} + b_j = \rho \ddot{u}_i \quad (i, j = 1, 2),
\]

where \( \rho \) is the mass density, the dot indicates \( \partial / \partial t \), and \( j \) denotes the derivative of \( \partial / \partial x_j \). The boundary conditions are given by

\[
u_i = \tilde{u}_i \quad \text{on} \quad S_1, \quad (2)
\]
\[
t_i = \tilde{t}_i \quad \text{on} \quad S_2, \quad (3)
\]

and the Cauchy formula between stress and surface traction is expressed as

\[
t_i = \sigma_{y,j} n_j, \quad (4)
\]

where \( n_j \) is the \( j \) component of the vector normal to the surface.

For time-harmonic motions with the factor \( e^{i\omega t} \) in time, eq.(1) without body force \( b_i \) is reduced to

\[
\sigma_{y,j} + \rho \omega^2 u_i = 0. \quad (5)
\]

The constitutive equation for isotropic material is

\[
\sigma_{y} = \lambda \epsilon_{y} \delta_{ij} + 2 \mu \epsilon_{yij} \quad (i, j = 1, 2),
\]

where \( \lambda \) and \( \mu \) are Lamé constants, and \( \delta_{ij} \) is the Kronecker delta. In eq.(6), \( \epsilon_{yij} \) is the strain tensor defined by the deformation equation

\[
\epsilon_{yij} = \frac{1}{2} (u_{i,j} + u_{j,i}).
\]

### Boundary element formulation

A standard weighted residuals technique is utilized to establish the weak-form equation\(^{14} \). By multiplying eqs.(3) and (5) by general weight function \( u'_i \) and eq.(2) by the traction field \( t'_i \), corresponding to \( u'_i \), and then integrating them over the domain and the surface, respectively, it follows that

\[
\int_{\Omega} (\sigma_{y,j} + \rho \omega^2 u_i) u'_i d\Omega = -\int_{S_1} (u_i - \tilde{u}_i) t'_i dS + \int_{S_1} (t_i - \tilde{t}_i) u'_i dS. \quad (8)
\]

By carrying out the integration by parts of the first integral on the left-hand side of (8), it leads to

\[
\int_{\Omega} n_j \sigma_{y,i} u'_i dS - \int_{\Omega} \sigma_{y,i} u'_i d\Omega + \int_{\Omega} \rho \omega^2 u_i u'_i d\Omega = -\int_{S} (u_i - \tilde{u}_i) t'_i dS + \int_{S} (t_i - \tilde{t}_i) u'_i dS. \quad (9)
\]

Utilizing the reciprocity theorem, we have

\[
\int_{\Omega} u'_i \sigma_{y,j} d\Omega = \int_{\Omega} n_j u_i \sigma_{y,j} d\Omega \quad (10)
\]

Substituting (10) and (4) into (9) yields

\[
\int_{\Omega} t'_i u'_i dS - \int_{\Omega} \sigma_{y,i} u'_i d\Omega + \int_{\Omega} \rho \omega^2 u_i u'_i d\Omega = -\int_{S} (u_i - \tilde{u}_i) t'_i dS + \int_{S} (t_i - \tilde{t}_i) u'_i dS. \quad (11)
\]

Then, performing the integration by parts again, it leads to

\[
\int_{\Omega} t'_i u'_i dS - \int_{\Omega} u_i t'_i d\Omega + \int_{\Omega} (\sigma_{y,i} + \rho \omega^2 u'_i) u_i d\Omega = -\int_{S} (u_i - \tilde{u}_i) t'_i dS + \int_{S} (t_i - \tilde{t}_i) u'_i dS. \quad (12)
\]

For the weight functions, \( u'_i \) and \( t'_i \), we employ the
elastodynamic displacement and traction fundamental solutions in the frequency domain\textsuperscript{(5), (6)}, respectively. The displacement fundamental solution \( u_{ik}^* \) is given by

\[
u_{ik}^*(\xi, \chi) = \frac{i}{4\mu} \left[ \delta_{ik} AU1 - r_i r_j AU2 \right], \tag{13}
\]

where

\[
AU1 = H^{(i)}_0(k_r r) - \frac{1}{k_r r} H^{(i)}_1(k_r r), \tag{14}
\]

\[
AU2 = -H^{(i)}_1(k_r r). \tag{15}
\]

\( H^{(i)}_m \) is the \( m \)-th kind Hankel function of \( l \) order and \( i \) means the imaginary unit. The elastodynamic fundamental solutions for traction\textsuperscript{(5), (6)} is

\[
t_{ik}^*(\xi, \chi) = \frac{i}{4} \left[ \left( \frac{\lambda}{\mu} + 2 \right) n_i r_i + \frac{\delta}{\partial n} r_i + n_i r_j \frac{\delta}{\partial n} r_j - \frac{4}{\mu} \frac{\partial}{\partial n} r_i r_j \right] + \left( \frac{\lambda}{\mu} + 2 \right) n_i r_i + \frac{\delta}{\partial n} r_i + n_i r_j \frac{\delta}{\partial n} r_j \right] \right] . \tag{16}
\]

In eqs.(13) and (16), the subscript \( i \) is the direction of displacement or traction at the field point \( \chi \), the subscript \( k \) is the direction of a point force at the source point \( \xi \), \( n_i \) is the \( i \)-component of the normal vector at the point \( \chi \), \( r \) is the distance between the source point and field point defined as

\[
r = |\xi - \chi| = \sqrt{(\xi_x - \chi_x)^2 + (\xi_y - \chi_y)^2}, \tag{17}
\]

\( k_l \) and \( k_r \) are the longitudinal and transverse wavenumber, respectively, and \( r_j = \partial / \partial \chi_j \).

Since the fundamental solution \( u_{ik}^* \) satisfies eq.(1) with the body force \( b = \delta_{ik} \delta(\xi - \chi) \), where \( \delta(\xi - \chi) \) is Dirac delta function, representing a unit point load at \( \xi = \chi \), substituting the displacement and traction fundamental solution in eqs.(13) and (16) as general weight functions in eq.(12) yields

\[
u_k(\xi) = -\int_{S} \nu_i(\chi) H^{(i)}_0(\xi, \chi) dS_{\chi} + \int_{S} (\nu_i(\chi) - \bar{u}_i(\chi)) \nu_{ik}^*(\xi, \chi) dS_{\chi} \]

\[
-\int_{S} (\bar{t}_i(\chi) - \bar{u}_i(\chi)) u_{ik}^*(\xi, \chi) dS_{\chi} \chi \in \Omega. \tag{18}
\]

Rearranging eq.(18) with the consideration that the closed boundary surface \( S \) consists of sub-boundaries \( S_1 \) and \( S_2 \), it then follows that

\[
\nu_k(\xi) + \int_{S_1} \nu_{ik}^*(\xi, \chi) u_i(\chi) dS_{\chi} - \int_{S_1} \nu_{ik}^*(\xi, \chi) \bar{u}_i(\chi) dS_{\chi} = \]

\[
-\int_{S_2} \nu_{ik}^*(\xi, \chi) u_i(\chi) dS_{\chi} + \int_{S_2} \nu_{ik}^*(\xi, \chi) \bar{u}_i(\chi) dS_{\chi} \chi \in \Omega. \tag{19}
\]

Finally, taking the limit \( \xi \in \Omega \rightarrow \xi_0 \) on \( S \), the boundary integral becomes

\[
c_i(\xi_0) u_i(\xi_0) \]

\[
+ PV \int_{S_1} \nu_{ik}^*(\xi_0, \chi) u_i(\chi) dS_{\chi} - \int_{S_1} \nu_{ik}^*(\xi_0, \chi) \bar{u}_i(\chi) dS_{\chi} = \]

\[
- PV \int_{S_2} \nu_{ik}^*(\xi_0, \chi) u_i(\chi) dS_{\chi} + \int_{S_2} \nu_{ik}^*(\xi_0, \chi) \bar{u}_i(\chi) dS_{\chi}, \tag{20}
\]

where \( c_i \) is the free term depending on the boundary smoothness, and \( PV \int \) indicates the principal value of the integral. In the case of the smooth boundary, \( c_i = 1 / 2 \).

(4) Domain decomposition and discretization

To solve the scattering problem in an interface-layered plate as shown in Fig.2, the boundary integral equation is applied to the finite domain enclosed by the red lines in Fig.2, which is decomposed into two sub-domains, as shown in Fig.4. The boundary surface is then separated into twelve sub-surfaces from \( \Gamma 1 \) to \( \Gamma 12 \) depending on the boundary conditions of the problem including a debonding of the interface. For the next step, the surface boundary of the \( n \)-th layer is discretized into \( N^{(n)} \) constant elements where the node is located at the center of each element.

Consequently, the boundary integral equation (20) for the \( p \)-th element becomes

\[
\begin{align*}
&\begin{bmatrix}
0 & 0 \\
0 & 1/2
\end{bmatrix}
\begin{bmatrix}
u_{11}^{(n)} \\
u_{21}^{(n)}
\end{bmatrix} + \sum_{q=1}^{N^{(n)}} \int_{S_q} \left[ \begin{bmatrix}
u_{12}^{(n)} & \nu_{22}^{(n)} & \nu_{32}^{(n)} & \nu_{42}^{(n)} & \nu_{52}^{(n)} & \nu_{62}^{(n)} & \nu_{72}^{(n)} & \nu_{82}^{(n)} & \nu_{92}^{(n)} & \nu_{102}^{(n)} & \nu_{112}^{(n)} & \nu_{122}^{(n)}
\end{bmatrix} \begin{bmatrix}
u_{12}^{(n)} \\
u_{22}^{(n)} \\
u_{32}^{(n)} \\
u_{42}^{(n)} \\
u_{52}^{(n)} \\
u_{62}^{(n)} \\
u_{72}^{(n)} \\
u_{82}^{(n)} \\
u_{92}^{(n)} \\
u_{102}^{(n)} \\
u_{112}^{(n)} \\
u_{122}^{(n)}
\end{bmatrix}
\right] dS_q,
\end{align*}
\]

\[
\begin{align*}
&= \sum_{q=1}^{N^{(n)}} \int_{S_q} \left[ \begin{bmatrix}
u_{12}^{(n)} & \nu_{22}^{(n)} & \nu_{32}^{(n)} & \nu_{42}^{(n)} & \nu_{52}^{(n)} & \nu_{62}^{(n)} & \nu_{72}^{(n)} & \nu_{82}^{(n)} & \nu_{92}^{(n)} & \nu_{102}^{(n)} & \nu_{112}^{(n)} & \nu_{122}^{(n)}
\end{bmatrix} \begin{bmatrix}
u_{12}^{(n)} \\
u_{22}^{(n)} \\
u_{32}^{(n)} \\
u_{42}^{(n)} \\
u_{52}^{(n)} \\
u_{62}^{(n)} \\
u_{72}^{(n)} \\
u_{82}^{(n)} \\
u_{92}^{(n)} \\
u_{102}^{(n)} \\
u_{112}^{(n)} \\
u_{122}^{(n)}
\end{bmatrix}
\right] dS_q,
\end{align*}
\]

(21)
where the indices of $2p-1$ and $2p$ mean the $x_i$ and $x_j$ components, respectively, on the element $p$. In eq.(21), $\Delta \Gamma^{(n)}$ is the $q$-th element of the surface boundary $\Gamma^{(n)}$ of the $n$-th layer, where $\Gamma^{(1)}$ and $\Gamma^{(2)}$ consist of $\Gamma_1$–$\Gamma_6$ and $\Gamma_7$–$\Gamma_{12}$, respectively. Since $p=1,\ldots,N^{(n)}$ in eq.(21), the equation yields the $2 \times N^{(n)}$ set of equations for each layer, resulting in the total $2 \times (N^{(1)} + N^{(2)})$ set of equations, which can be solved by substituting the boundary conditions and the continuity conditions given as follows.

The traction-free boundary condition is applied to the bottom surface of the first layer $\Gamma_2$ and the top surface of the second layer $\Gamma_{12}$ in eq.(21) as well as the debonding zone $\Gamma_5$ and $\Gamma_9$,

$$t_i = 0 \quad (i = 1,2).$$

Next, the continuity conditions are imposed on traction and displacement on the interfaces between $\Gamma_6$ and $\Gamma_8$ as well as $\Gamma_4$ and $\Gamma_{10}$ on the assumption that each interface is perfectly bonded, the relationship for displacement and traction is

$$u_i^{(1)} = u_i^{(2)} \quad (i = 1,2),$$

$$t_i^{(1)} = -t_i^{(2)} \quad (i = 1,2),$$

where the negative sign of tractions in eq.(24) denotes that the unit outward normal vectors on the surfaces of two layers have the opposite direction to each other. After substituting eqs.(22), (23), and (24) into eq.(21), the unknown wave field on each element becomes either displacement $u_i$ or traction $t_i$, except for the elements on the artificial boundaries of $\Gamma_1, \Gamma_7, \Gamma_3$, and $\Gamma_{11}$, where both displacement and traction from scattered waves are still unknown.

3. TREATMENT OF ARTIFICIAL BOUNDARY AND SCATTERING COEFFICIENT

(1) Mode expansion technique

A single propagation mode of Lamb wave is incident from the left side and propagates to the right side. When the incident wave encounters the debonding, this wave will be scattered into all directions with many possible modes, including both propagating modes and non-propagating modes, as shown in Fig.5. The scattered waves propagating in the left direction are so-called reflected waves and the scattered waves traveling in the right direction are so-called transmitted waves.

On the artificial boundaries, $\Gamma_1 + \Gamma_7$ and $\Gamma_3 + \Gamma_{11}$, the total displacement and traction fields can be expressed as the superposition of incident wave and scattered waves. In the mode expansion method, the scattered waves on the artificial boundaries are approximated by the superposition of only propagating modes on the assumption that non-propagating modes decay immediately away from the debonding. It is, therefore, worth noting that the artificial boundary on each side must be placed far enough to neglect the near-field evanescent scattering modes. Now, the total displacement and traction fields on the artificial boundaries are obtained as

$$u^{\text{total}} = u^{\text{inc}} + u^R, \quad t^{\text{total}} = t^{\text{inc}} + t^R \quad \text{on} \quad \Gamma_1 + \Gamma_7,$$

$$u^{\text{total}} = u^{\text{inc}} + u^T, \quad t^{\text{total}} = t^{\text{inc}} + t^T \quad \text{on} \quad \Gamma_3 + \Gamma_{11},$$

where the superscripts, $\text{inc}$, $R$, and $T$, represent the fields of incident, reflected, and transmitted waves, respectively.

The displacement and traction of the incident wave of a single $p$-th mode propagation in $+x_i$-direction are easily obtained by calculating wave functions using the partial wave technique (see APPENDIX for calculation) and can be expressed as

$$u^{\text{inc}} = \rho \alpha \left\{ \alpha \Gamma_1, \alpha \Gamma_2 \right\}^T, \quad t^{\text{inc}} = \rho \alpha \left\{ \alpha \Gamma_1, \alpha \Gamma_2 \right\}^T,$$

where $T$ indicates the transposed vector, $\rho \alpha$ is the amplitude of the incident wave, and $\alpha \Gamma_1$ and $\alpha \Gamma_2$ are the $i$-component of normalized displacement and traction fields, respectively, of the $p$-th mode propagating in $+x_i$-direction.

On the other hand, the scattering displacement fields can be expressed in terms of the sum of possible propagating modes $M$, which can be found from the dispersion curves at a certain frequency, e.g., $M = 2$ at $f = 1$ MHz (see Fig.13) as

$$u^R = \sum_{m=1}^M u^R_m \left\{ \alpha \Gamma_1, \alpha \Gamma_2 \right\}^T,$$  

$$u^T = \sum_{m=1}^M u^T_m \left\{ \alpha \Gamma_1, \alpha \Gamma_2 \right\}^T,$$

where $u^R_m$ and $u^T_m$ are the unknown amplitudes of reflected waves propagating in $-x_i$-direction and transmitted waves propagating in $+x_i$-direction of $m$-th mode, respectively. By substituting eqs.(28) and (29) into eqs.(7), (6) and (4) in sequel, the traction of the scattered waves can be similarly expressed as

$$t^R = \sum_{m=1}^M t^R_m \left\{ \alpha \Gamma_1, \alpha \Gamma_2 \right\}^T,$$  

where

![Fig.5](image-url) Incident wave and scattered waves.
\[ t^{T} = \sum_{n=1}^{M} m^\beta \left\{ n \, \hat{\mathbf{T}} \right\} M_{n} \right\}^{T}. \]  

(31)

The artificial boundaries \( \Gamma_{7}(\Gamma_{1} \cap \Gamma_{3}) \) and \( \Gamma_{1}(\Gamma_{3}) \) for first and second layers \((n = 1 \text{ and } 2)\) are discretized into \( k \) and \( l \) elements, respectively. Equations (25) and (26) can then be expressed as linear combinations in matrix forms of

\[
\begin{align*}
\mathbf{u}_{\text{total}, \Gamma_{7}(\Gamma_{1} \cap \Gamma_{3})}^{(2k+2) \times 1} & = [m \, \hat{\mathbf{u}}]^{\Gamma_{7}(\Gamma_{1} \cap \Gamma_{3})} \{ \rho \, \alpha \delta_{\text{pm}} \} M_{x=1}^{(2k+2) \times 1} + [m \, \hat{\mathbf{u}}]^{\Gamma_{7}(\Gamma_{1} \cap \Gamma_{3})} \{ \rho \, \beta \} M_{x=1}^{(2k+2) \times 1}, \\
\mathbf{t}_{\text{total}, \Gamma_{7}(\Gamma_{1} \cap \Gamma_{3})}^{(2k+2) \times 1} & = [m \, \hat{\mathbf{t}}]^{\Gamma_{7}(\Gamma_{1} \cap \Gamma_{3})} \{ \rho \, \alpha \delta_{\text{pm}} \} M_{x=1}^{(2k+2) \times 1} + [m \, \hat{\mathbf{t}}]^{\Gamma_{7}(\Gamma_{1} \cap \Gamma_{3})} \{ \rho \, \beta \} M_{x=1}^{(2k+2) \times 1}, \\
\mathbf{u}_{\text{total}, \Gamma_{3}(\Gamma_{1} \cap \Gamma_{3})}^{(2k+2) \times 1} & = [m \, \hat{\mathbf{u}}]^{\Gamma_{3}(\Gamma_{1} \cap \Gamma_{3})} \{ \rho \, \alpha \delta_{\text{pm}} \} M_{x=1}^{(2k+2) \times 1} + [m \, \hat{\mathbf{u}}]^{\Gamma_{3}(\Gamma_{1} \cap \Gamma_{3})} \{ \rho \, \beta \} M_{x=1}^{(2k+2) \times 1}, \\
\mathbf{t}_{\text{total}, \Gamma_{3}(\Gamma_{1} \cap \Gamma_{3})}^{(2k+2) \times 1} & = [m \, \hat{\mathbf{t}}]^{\Gamma_{3}(\Gamma_{1} \cap \Gamma_{3})} \{ \rho \, \alpha \delta_{\text{pm}} \} M_{x=1}^{(2k+2) \times 1} + [m \, \hat{\mathbf{t}}]^{\Gamma_{3}(\Gamma_{1} \cap \Gamma_{3})} \{ \rho \, \beta \} M_{x=1}^{(2k+2) \times 1}. 
\end{align*}
\]

(32) \( \sim \) (35)

In the matrices \([m \, \hat{\mathbf{u}}]\) and \([m \, \hat{\mathbf{t}}]\), the column means the possible modes of propagating wave and the row is the location of a nodal point centered at the elements.

Since the amplitudes \( \rho \beta \) and \( \rho \beta \) of the scattered waves are common for both displacement and traction fields, the unknown total traction fields can be expressed in terms of the unknown total displacement fields. Rearranging (32) and (34) with respect to the unknown scattered amplitudes \( \rho \beta \) and \( \rho \beta \), it follows that

\[
\begin{align*}
\begin{bmatrix} \rho \beta \end{bmatrix}_{M_{x=1}^{(2k+2) \times 1}} & = [m \, \hat{\mathbf{u}}]^{-1} \Gamma_{7}(\Gamma_{1} \cap \Gamma_{3}) \mathbf{u}_{\text{total}, \Gamma_{7}(\Gamma_{1} \cap \Gamma_{3})}^{(2k+2) \times 1} \{ \rho \, \alpha \delta_{\text{pm}} \} M_{x=1}^{(2k+2) \times 1}, \\
\begin{bmatrix} \beta \end{bmatrix}_{M_{x=1}^{(2k+2) \times 1}} & = [m \, \hat{\mathbf{u}}]^{-1} \Gamma_{3}(\Gamma_{1} \cap \Gamma_{3}) \mathbf{u}_{\text{total}, \Gamma_{3}(\Gamma_{1} \cap \Gamma_{3})}^{(2k+2) \times 1} \{ \rho \, \alpha \delta_{\text{pm}} \} M_{x=1}^{(2k+2) \times 1}, \\
\end{align*}
\]

(36) \( \sim \) (37)

where \([\hat{\mathbf{u}}]^{-1}\) is the generalized complex inverse matrix of \([\hat{\mathbf{u}}]^{-1}\) defined as

\[
\begin{align*}
[\hat{\mathbf{u}}]^{-1} = \left( [\hat{\mathbf{u}}]^{T} \right) \left( [\hat{\mathbf{u}}] \right)^{-1} [\hat{\mathbf{u}}]^{T},
\end{align*}
\]

(38)

and \([\hat{\mathbf{u}}]^{T}\) is the transpose of the complex conjugate of \([\hat{\mathbf{u}}]^{-1}\). Next, substituting eqs.(36) and (37) into traction fields in eqs.(33) and (35), respectively, yields

\[
\begin{align*}
\mathbf{t}^{(2k+2) \times 1} & \equiv \left\{ n \, \hat{\mathbf{u}} \right\} M_{n} \right\}^{T},
\end{align*}
\]

(39)

(40)

Now, the unknown total traction fields on the artificial boundaries are expressed in terms of the unknown total displacement fields. Then, eqs.(39) and (40) are substituted into eq.(21) to delete the number of unknown total traction fields. Then, the unknown variables included in eq.(21) are the total displacement fields only.

(2) Scattering coefficients calculation

After solving the total displacement fields, the scattered amplitudes \( \rho \beta \) and \( \rho \beta \) can be calculated by substituting the obtained total displacement fields into (36) and (37). Finally, the scattering coefficients are obtained as the following absolute values of the ratios of scattered amplitudes and the amplitude of the incident wave,

\[
R_{pm} = \left| \frac{\alpha \beta}{\beta \alpha} \right|,
\]

(41)

\[
T_{pm} = \left| \frac{\beta \alpha}{\beta \alpha} \right|,
\]

(42)

where \( R_{pm} \) is the reflection coefficient and \( T_{pm} \) is the transmission coefficient of \( m \)-th mode for the incident wave of \( p \)-th mode.

4. NUMERICAL RESULTS AND DISCUSSION

(1) Verification

If the same material constants are given to both domains in a double-layered plate model, the numerical results are expected to be the same as the solutions for a single plate. In this configuration, the numerical results of reflection and transmission coefficients for a double-layered plate with debonding domains in a double-layered plate model, the numerical results obtained from

| Material   | \( \lambda \) (GPa) | \( \mu \) (GPa) | \( \rho \) (g/cm³) |
|------------|-------------------|----------------|-----------------|
| Steel      | 115.5             | 79.9           | 7.8             |
| Aluminum   | 58.2              | 26.1           | 2.7             |
| Titanium   | 66.9              | 44.6           | 4.5             |

Table 1 Material properties.
Gunawan and Hirose\textsuperscript{12}, which are the reflection and transmission coefficients for scattering problems with a horizontal crack in a single plate. The material used here is steel, the material constants of which are listed in Table 1. The thickness of each layer is 0.5 mm and the debonding is located at the middle of the plate. The length of the debonding zone is equal to 1 mm. As mentioned before, for the Lamb wave scattering problem, the debonding causes mode conversion from an incident propagating mode to an infinite number of scattering modes including non-propagating modes or evanescent modes.

However, the non-propagating modes decay exponentially at far field. Therefore, the length of the analysis domain $L$ must be long enough to meet the far-field approximation to obtain converged results and keep the accuracy in calculation processes. The optimum model length $L$ and the number of elements are selected depending on wavelength $\lambda$ of the incident wave and the length of the debonding $l$, so that $L$ is larger than at least $2\lambda + l$, and the element number is at least more than 10 per one wavelength. $L$ about 8 mm – 12 mm long and 800 – 1000 elements are used in this study to ensure the numerical accuracy and efficient performance in the computation time depending on the incident frequency.

In Fig.6, scattering coefficients are plotted with frequency range from 0.1 MHz to 2.5 MHz for A0 incidence with 1 mm horizontal debonding. “R-An” indicates the reflection coefficient of the antisymmetric $n$-th mode and “T-An” represents the transmission coefficient of the antisymmetric $n$-th mode. “Ref” with white symbols represents the benchmark solution.\textsuperscript{12} For the present method, the specific mode of incident waves must be chosen and the scattering coefficients are directly determined by using BEM coupling with normal mode expansion technique. For the reference method\textsuperscript{12}, all Lamb wave modes are simultaneously excited by appropriate boundary condition given on both ends of a finite plate and lead to a system of equations to determine the scattering coefficients for all Lamb wave modes. Since both methods in the present paper and the reference are numerical ones, a small error tolerance should be acceptable. In fact, after investigating the errors in Fig.6, it can be said that the results obtained by the present method show less than 5% of relative error, compared with the benchmark solution. It is therefore, verified that the proposed computation technique is capable for the homogenous double-layered plate problem with reliable results and readily extends to further study with different configurations.

(2) Scattering coefficients for layered plate

Next, we calculate the scattering coefficients for a double-layered plate consisting of two different materials with debonding on the interface. The material difference between two layers makes the wave field complex where the multi-mode scattered wave cannot be classified into the symmetric and antisymmetric modes, since they are not uncoupled from each other. Thus, all possible modes at a certain frequency might occur from scattering phenomena. In this paper, the mode numbers of the guided waves are sequentially ordered as first mode, second mode,

Fig.6 Reflection and transmission coefficients for the steel plate of 1 mm thickness with 1 mm horizontal debonding on the middle plane at $x_2 = 0.5$ mm subjected to A0 incidence.

Fig.7 Reflection and transmission coefficients of the first and second modes for the double-layered plate of 1 mm thickness with 1 mm horizontal debonding on the middle plane at $h^{(1)} = 0.5$ mm subjected to first mode incidence.
third mode, ..., depending on the value of the phase velocity at a certain frequency, in which the Lamb wave with the smallest velocity is labeled as the first mode. In the sequel, the scattering coefficients are calculated for various frequencies, materials, and lengths of the debonding zone to discuss the scattering behavior. Figure 7 shows the reflection and transmission coefficients of the first and second modes as a function of frequency from 0.1 MHz to 1.4 MHz, scattered by 1 mm debonding on the middle interface of 1 mm double-layered plate composed by two different materials listed in Table 1, i.e., steel-aluminum, steel-titanium, and steel-steel. “R_n-m and T_n-m” indicate the reflection and transmission coefficients of n-th mode with material m. The results demonstrate different scattering characteristics depending on material properties. It is also pointed out that reflection and transmission coefficients show reverse trend to each other, satisfying the energy conservation. For example, the reflection coefficient increases when the transmission coefficient decreases, and vice versa. The resonance phenomena can be seen in the frequency range between 1.2 MHz –1.4 MHz, where very little power passes through debonding and the reflection coefficient becomes nearly unity18), 19). Furthermore, there are some points where the scattering coefficients suddenly change around the frequency of 0.8 MHz – 1 MHz for steel-aluminum and steel-titanium plates. This behavior is the mode conversion phenomenon where a part of energy is converse from one mode to another mode as seen in Fig.7 that occurs in the bi-material layered plate, explained as follows. For a single homogeneous plate with symmetric geometry in defect configuration, it is well known that symmetric and antisymmetric modes are uncoupled in the scattered wave field3), 5).

For a bi-material plate, however, multi-modes always interact with each other because of non-symmetric material property. This phenomenon will be discussed in detail later. Figures 8 and 9 show the reflection and transmission coefficients of the first mode with frequency range from 0.1 MHz to 1.4 MHz, scattered by 1 mm debonding at various locations of the interface in steel-aluminum and steel-titanium plates, respectively, where the bottom layer is steel and the upper layer is aluminum or titanium. The total thickness of the bi-material layered plate is 1 mm and the location of the interface is given by the thickness of steel \( h(1) = 0.3, 0.5, \text{ and } 0.7 \text{ mm} \). It can be seen that the location of the interface or debonding zone, as well as the material difference, gives a large impact on the scattering coefficients behavior.

As mentioned before, normally for a single plate with symmetric scatterer, the symmetric and anti-
symmetric modes are uncoupled. For example, only antisymmetric modes can occur for A0 incidence. But for layered plates with different materials, all possible modes can occur due to the mode conversion as seen in Fig. 13 in the Appendix. There exist for both the first and second modes in frequency range below 1.5 MHz. Figures 10 and 11 show scattering coefficients of all possible modes with frequency range from 0.5 MHz to 1 MHz for the first mode incidence in steel-aluminum and steel-titanium plates of \( h^{(1)} = h^{(2)} = 0.5 \) mm. Various debonding lengths \( l \) at the interface are chosen as \( l = 1.0, 1.2, \) and \( 1.4 \) mm. “\( R_{n-l} \) and \( T_{n-l} \)” indicate the reflection and transmission coefficients of \( n \)-th mode with length of debonding \( l \). It is clearly shown that the mode conversion plays an important role in affecting the scattering coefficient. As the length of debonding increases, the frequency range for mode conversion shifts to the lower frequency zone, \( i.e., 0.85 - 0.95 \) MHz, \( 0.7 \) MHz - \( 0.8 \) MHz and \( 0.55 \) MHz - \( 0.65 \) MHz for 1.0 mm, 1.2 mm, and 1.4 mm debonding, respectively, for both steel-aluminum and steel-titanium plates. The shift of the frequency zone is correlated with the phase correction due to radiation of scattered field from layered debonding.\(^{19}\)

5. CONCLUSION

The conclusions obtained in this paper are summarized below:

1. The boundary element method in conjunction with the normal mode expansion technique was proposed to calculate the scattering coefficients of guided Lamb wave scattering for layered plate debonding problems.
2. The accuracy of numerical results was verified by comparing the proposed method and the mode-exciting method with good agreement.
3. The results showed that different scattering behavior can be seen with the change in material properties, location, and length of debonding zone. In the bi-material plate, especially, it was found that the fluctuations in reflection and transmission coefficients were caused by mode conversion, and frequency shifts had close relation with the length of debonding, which will provide useful information for quantitative guided wave NDT to detect inner defects of a layered structure with prescribed materials and layer configuration.

In this paper, the method was applied to 2D problems with simple horizontal line debonding only. For future work, it will be an interesting topic to inves-
toigate 3D problems with more complex configurations, such as anisotropic-layered plate and complex geometry of the scatterer.

APPENDIX: LAMB WAVE FUNCTION AND DISPERSION CURVE

The partial wave technique is used to understand the fundamental properties of dispersive guided Lamb waves in a double-layered plate. First, we assume the waves propagating upward and downward in each layer, as shown in Fig. 12.

Then, Lamb wave modal functions for displacement and stress propagating in the positive $x_1$-direction of the $n$-th layer can be expressed as

$$u_n^{(a)} = \sum_{a=1}^{4} B_n^{(a)} \exp \left[ ik(x_1 + \alpha_n^{(a)} x_2 - c_p t) \right], \quad (A.1)$$

$$u_2^{(a)} = \sum_{a=1}^{4} B_n^{(a)} U_2^{(a)} \exp \left[ ik(x_1 + \alpha_n^{(a)} x_2 - c_p t) \right], \quad (A.2)$$

$$\sigma_{11}^{(a)} = \sum_{a=1}^{4} B_n^{(a)} (\mu^{(a)}(1 + U_2^{(a)} \alpha_n^{(a)}) + 2\mu^{(a)})(ik) \exp \left[ ik(x_1 + \alpha_n^{(a)} x_2 - c_p t) \right], \quad (A.3)$$

$$\sigma_{12}^{(a)} = \sum_{a=1}^{4} B_n^{(a)} (U_2^{(a)} + \alpha_n^{(a)}) \mu^{(a)}(ik) \exp \left[ ik(x_1 + \alpha_n^{(a)} x_2 - c_p t) \right], \quad (A.4)$$

$$\sigma_{22}^{(a)} = \sum_{a=1}^{4} B_n^{(a)} (\mu^{(a)}(1 + 2\mu^{(a)}) U_2^{(a)} \alpha_n^{(a)}) (ik) \exp \left[ ik(x_1 + \alpha_n^{(a)} x_2 - c_p t) \right], \quad (A.5)$$

where $\alpha_n^{(a)}$ is the ratio of the wavenumbers in the $x_2$-direction to the wavenumber $k$ in the $x_1$-direction for the $n$-th layer, expressed as

$$\alpha_n^{(a)} = \sqrt{-1 + \frac{\rho_n^{(a)}}{\mu_n^{(a)} c_p^2}}, \quad (A.6)$$

$$\alpha_2^{(a)} = -\sqrt{-1 + \frac{\rho_n^{(a)}}{\mu_n^{(a)} c_p^2}}, \quad (A.7)$$

$$\alpha_3^{(a)} = -\sqrt{-1 + \frac{\rho_n^{(a)}}{\mu_n^{(a)} c_p^2}}, \quad (A.8)$$

$$\alpha_4^{(a)} = -\sqrt{-1 + \frac{\rho_n^{(a)}}{\mu_n^{(a)} c_p^2}}. \quad (A.9)$$

In eqs. (A.1)–(A.5), the polarization vectors $U_n^{(a)}$ of the partial waves can be expressed as

$$U_{21}^{(a)} = \frac{1}{\alpha_1}, \quad (A.10)$$

$$U_{22}^{(a)} = \frac{1}{\alpha_2}, \quad (A.11)$$

$$U_{23}^{(a)} = \alpha_3, \quad (A.12)$$

$$U_{24}^{(a)} = \alpha_4, \quad (A.13)$$

and the unknowns are $c_p$ and $B_n^{(a)}$, which can be determined by the following boundary conditions.

$$\sigma_{21}^{(1)} = 0 \quad \text{at} \quad x_1 = 0, \quad (A.14)$$

$$\sigma_{22}^{(1)} = 0 \quad \text{at} \quad x_1 = 0, \quad (A.15)$$

$$u_1^{(1)} = u_2^{(2)} \quad \text{at} \quad x_2 = h^{(1)}, \quad (A.16)$$

$$u_2^{(1)} = u_2^{(2)} \quad \text{at} \quad x_2 = h^{(1)}, \quad (A.17)$$

$$\sigma_{21}^{(1)} = \sigma_{21}^{(2)} \quad \text{at} \quad x_2 = h^{(1)}, \quad (A.18)$$

$$\sigma_{22}^{(1)} = \sigma_{22}^{(2)} \quad \text{at} \quad x_2 = h^{(1)}, \quad (A.19)$$

$$\sigma_{21}^{(2)} = 0 \quad \text{at} \quad x_1 = h^{(1)} + h^{(2)}, \quad (A.20)$$

$$\sigma_{22}^{(2)} = 0 \quad \text{at} \quad x_1 = h^{(1)} + h^{(2)}. \quad (A.21)$$

All boundary conditions in eqs. (A.14)–(A.21) are assembled into a system of linear equations, which is called the characteristic equation,

$$DB = 0, \quad (A.22)$$

where $D$ is the coefficient matrix containing the phase velocity $c_p$ and the wavenumber $k$, which leads to the construction of dispersion curves; and $B$ is the unknown vector consisting of the amplitude $B_n^{(a)}$, which leads to the wave structure of modal functions.

To get the non-trivial solutions of eq. (A.22), the determinant of the coefficient matrix must be zero,

$$|D| = 0, \quad (A.23)$$

Fig. 12 Component of partial waves in double-layered plate problem.
which gives the dispersive relation between the phase velocity $c_p$ and the wavenumber $k$ or the frequency $f$. The example of dispersion curves for a double-layered plate is shown in Fig.13. Three combinations of materials shown in Table 1, i.e., steel-aluminum, steel-titanium, and steel-steel, are considered. The distinction among the dispersion curves for three combinations of materials can clearly be seen in higher-order modes. Also, the mode shapes for $u_i$ of the first mode with the frequency of 1 MHz are illustrated along the depth of the plate in Fig.14. The results show that the displacement distribution is symmetric for the steel–steel plate, but neither symmetric nor anti-symmetric for the steel–aluminum and steel–titanium plates.

As shown in Fig.13, there are finite numbers of dispersion curves of propagating modes at a certain frequency. For the particular mode $m$, the coefficients of $nB^{(a)}_i$ ($n = 1$ or 2, $a = 1$ to 4) can be determined as relative amplitudes by means of eigenvectors of eq.(A.22). To construct the base functions of the $m$-th Lamb wave mode, $u_i^{(m)}$ is normalized as

$$u_i^{(m)} = N \cdot u_i^{(n)},$$  \hspace{1cm} \text{(A.24)}

where $n = 1$ or 2 is selected depending on the depth position in layer $n = 1$ or 2, and $N$ is the normalized factor defined as

$$N = \sqrt{\frac{P_i}{\text{inc} \cdot P_i}},$$  \hspace{1cm} \text{(A.25)}

where the asterisk means the complex conjugate of displacement.\textsuperscript{10,11,20}

Furthermore, we put the sign of $\pm$ to the normalized displacement like $u_i^{(m)}$ to demonstrate the base functions of the $m$-th Lamb wave mode propagating in the $\pm x_i$-directions, which can be obtained by putting the sign $\pm$ to $x_i$, such as $\pm x_i$ in eqs.(A.1)–(A.5). Similarly, the traction component is normalized as $t_i^{(m)}$.

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