Canonical Tensor Product Subfactors

K.-H. Rehren

Institut für Theoretische Physik, Universität Göttingen
Bunsenstraße 9, D-37073 Göttingen
E-mail: rehren@theorie.physik.uni-goettingen.de

Abstract

Canonical tensor product subfactors (CTPS’s) describe, among other things, the embedding of chiral observables in two-dimensional conformal quantum field theories. A new class of CTPS’s is constructed some of which are associated with certain modular invariants, thereby establishing the expected existence of the corresponding two-dimensional theories.

1 Introduction and results

There is a common mathematical note which recurs again and again in the areas of conformal quantum field theory and modular invariants on the one hand, and asymptotic subfactors and quantum doubles on the other hand. In all these areas, there arise inclusions of von Neumann factors of the form $A \otimes B \subset C$ sharing a “canonical” property (see Def. 1.1 below) for which we call them “canonical tensor product subfactors” (CTPS, cf. [17]). E.g., in chiral quantum field theories on $S^1$, CTPS’s describe the violation of Haag duality for disjoint intervals (Jones-Wassermann subfactors, cf. [10, 21]), or the embedding of a coset model into a given ambient model. In two-dimensional conformal quantum field theories they describe the embedding of chiral subtheories [17] which is (incompletely) reflected also by modular invariant coupling matrices [13]. Ocneanu’s asymptotic subfactors [10] which are sometimes regarded as generalized quantum doubles [8] are also CTPS’s.

The main result in this article is the presentation (Thm. 1.4) of a class of new CTPS’s associated with extensions of closed systems of endomorphisms (Def. 1.2, 1.3). Among them there is a subclass of considerable importance for the understanding of modular invariants. Namely, with every modular invariant constructed by a method due to Böckenhauer, Evans and Kawahigashi [3] one can associate one of the new CTPS’s which, if interpreted as a local inclusion of an algebra of
chiral observables into an algebra of two-dimensional observables, allows to prove the existence of a complete two-dimensional local conformal quantum field theory associated with the given modular invariant (Cor. 1.6).

The mathematical abstraction of this physical problem as a problem on von Neumann algebras and subfactors is most efficient. It is based on the seminal realization that positive-energy representations (“superselection charges”) and their fusion are conveniently expressed in terms of endomorphisms, promoting particle statistics to a unitary operator representation (braiding) on the physical Hilbert space, and identifying the statistical dimension as (the square root of) a Jones index.

Extensions or embeddings of quantum field theories can also be coded into single subfactors [12]. The characterization of a subfactor, in turn, in terms of a “Q-system” is particularly useful in this context since these data directly describe the charged field content of the extended theory in terms of superselection charges of the embedded theory [12, 19]. The new subfactors presented in Thm. 1.4 are also defined by specification of their Q-systems (in terms of certain matrix elements for the transition between two extensions), thus making as close contact with the structure of modular invariants as possible.

CTPS’s are very special cases of “symmetric joint inclusions”, i.e., triples of von Neumann algebras \((A, B, C)\) such that \(A\) and \(B\) are commuting subalgebras of \(C\). After a survey of some general properties of symmetric joint inclusions in Sect. 4, we give a characterization of “normality” for CTPS’s in Proposition 4.3. This is a maximality property which, in the case of the embedding of chiral observables into a two-dimensional conformal quantum field theory, corresponds to the maximally extended chiral algebras and diagonal or permutation invariants [17].

The canonical property mentioned before is a natural feature of the embedding of chiral quantum field theories into a two-dimensional conformal quantum field theory, reflecting the independence of left and right moving degrees of freedom [17]. It is defined as follows.

**Definition 1.1.** A tensor product subfactor of the form \(A \otimes B \subset C\) is called a canonical tensor product subfactor (CTPS) if either \(A, B, C\) are type II factors and \(C\) considered as an \(A \otimes B\)-\(A \otimes B\) bimodule decomposes into irreducibles which are all tensor products of \(A\)-\(A\) bimodules with \(B\)-\(B\) bimodules, or if \(A, B, C\) are type III factors and the dual canonical endomorphism \(\theta \equiv \bar{\iota} \circ \iota \in \text{End}(A \otimes B)\) decomposes into irreducibles which are all tensor products of endomorphisms of \(A\) with endomorphisms of \(B\). Let, in the type III case,

\[
\theta \simeq \bigoplus_{\alpha,\beta} Z_{\alpha,\beta} \alpha \otimes \beta
\]

where the sum extends over two sets of mutually inequivalent irreducible endomorphisms of \(A\) and of \(B\), respectively. Then we call the matrix of multiplicities \(Z\) with
non-negative integer entries the coupling matrix of the CTPS. The coupling matrix in the type II case is defined analogously in terms of mutually inequivalent irreducible \( A-A \) and \( B-B \) bimodules.

Here, as always throughout this paper, \( \iota : A \otimes B \to C \) denotes the inclusion homomorphism of the subfactor under consideration, and \( \bar{\iota} : C \to A \otimes B \) a conjugate homomorphism \[13\].

In order to state our main result, we have to introduce some further notions. We consider type III von Neumann factors \( N \), and denote by \( \operatorname{End}_{\text{fin}}(N) \) the set of unital endomorphisms \( \lambda \) of \( N \) with finite dimension \( d(\lambda) \).

**Definition 1.2.** A closed \( N \)-system is a set \( \Delta \subset \operatorname{End}_{\text{fin}}(N) \) of mutually inequivalent irreducible endomorphisms such that (i) \( \text{id}_N \in \Delta \), (ii) if \( \lambda \in \Delta \) then there is a conjugate endomorphism \( \bar{\lambda} \in \Delta \), and (iii) if \( \lambda, \mu \in \Delta \) then \( \lambda \mu \) belongs to \( \Sigma(\Delta) \), the set of endomorphisms which are equivalent to finite direct sums of elements from \( \Delta \).

**Definition 1.3.** Let \( N \subset M \) be a subfactor with inclusion homomorphism \( \iota : N \to M \). An extension of the closed \( N \)-system \( \Delta \) is a pair \((\iota, \alpha)\), where \( \iota \) is as above, and \( \alpha \) is a map \( \Delta \to \operatorname{End}_{\text{fin}}(M) \), \( \lambda \mapsto \alpha_\lambda \), such that

\[
\begin{align*}
\text{(E1)} & \quad \iota \circ \lambda = \alpha_\lambda \circ \iota, \\
\text{(E2)} & \quad \iota(\operatorname{Hom}(\nu, \lambda \mu)) \subset \operatorname{Hom}(\alpha_\nu, \alpha_\lambda \alpha_\mu).
\end{align*}
\]

After these preliminaries, we can state our main result.

**Theorem 1.4.** Let \( N_1 \subset M \) and \( N_2 \subset M \) be two subfactors of \( M \), and \((\iota_1, \alpha^1)\) and \((\iota_2, \alpha^2)\) a pair of extensions of a finite closed \( N_1 \)-system \( \Delta_1 \) and a finite closed \( N_2 \)-system \( \Delta_2 \), respectively. Then there exists an irreducible CTPS

\[ A \equiv N_1 \otimes N_2^{\text{opp}} \subset B \]

with dual canonical endomorphism

\[ \theta \equiv \bar{\iota} \circ \iota \simeq \bigoplus_{\lambda_1 \in \Delta_1, \lambda_2 \in \Delta_2} Z_{\lambda_1, \lambda_2} \lambda_1 \otimes \lambda_2^{\text{opp}}, \]

whose coupling matrix \( Z \) of multiplicities is given by

\[ Z_{\lambda_1, \lambda_2} = \dim \operatorname{Hom}(\alpha^1_{\lambda_1}, \alpha^2_{\lambda_2}). \]

The special case when \( \Delta_i \) are braided systems is of particular interest for the problem of modular invariants in conformal quantum field theory:

**Proposition 1.5.** Assume in addition that the closed systems \( \Delta_1 \) and \( \Delta_2 \) are braided with unitary braidings \( \varepsilon_1 \) and \( \varepsilon_2 \), respectively, turning \( \Pi(\Delta_1) \) and \( \Pi(\Delta_2) \) into braided monoidal categories. If for all \( \lambda_i, \mu_i \in \Delta_i \) and all \( \phi \in \operatorname{Hom}(\alpha^1_{\lambda_i}, \alpha^2_{\lambda_i}) \),
ψ ∈ Hom(α_{µ_1}^1, α_{µ_2}^2), one has
(E3) \((ψ \times φ) \circ ε_1(λ_1, µ_1) = ε_2(λ_2, µ_2) \circ (φ \times ψ)\),
then the canonical isometry \(w_1 \in Hom(θ, θ)\) (defined below in the proof of the Theorem) and the braiding operator \(ε(θ, θ)\) naturally induced by the braidings \(ε_1\) and \(ε_2^{\text{opp}}\) satisfy
\(ε(θ, θ)w_1 = w_1.\)

This result answers an open question in quantum field theory, where possible matrices \(Z\) are classified which are supposed to describe the restriction of a given two-dimensional modular invariant conformal quantum field theory to its chiral subtheories, while it is actually not clear whether any given solution \(Z\) does come from a two-dimensional quantum field theory. This turns out to be true for a large class of solutions, obtained in [3]:

**Corollary 1.6.** Let \(A : I \mapsto A(I)\) be a chiral net of local observables associated with the open intervals \(I \subset \mathbb{R}\), such that each \(A(I)\) is a type III factor. Let \(Δ_{DHR}\) be a closed system of mutually inequivalent irreducible DHR-endomorphisms [7] of \(A\) with finite dimension, localized in some interval \(I_0\), and put \(N := A(I_0)\). We assume that \(A\) is conformally covariant, implying [4] that the system of restrictions \(Δ := \{λ = λ_{DHR}|_N : λ_{DHR} \in Δ_{DHR}\}\) is a closed \(N\)-system. Let \(N_1 \subset N\) be a subfactor with canonical endomorphism \(θ \in Σ(Δ)\), and \(N \subset M\) its Jones extension. Put
\[Z_{λ, µ} := \dim Hom(α_{λ}^+, α_{µ}^-)\]
where \(α^\pm\) are the pair of \(α\)-inductions [12, 4, 20] of endomorphisms of \(N\) to endomorphisms of \(M\), associated with the braidings given by the DHR statistics and its opposite. Then there is a two-dimensional local conformal quantum field theory described by a net \(B : O \mapsto B(O)\) of observables associated with the doublecones \(O = I × J\) in \(\mathbb{R}^2\), containing subnets of left and right chiral observables \(A_L\) and \(A_R\) both isomorphic with \(A\), such that the local inclusions of chiral observables \(A_L(I) × A_R(J) \subset B(O)\) are CTPS’s with coupling matrix \(ZC\). (Here \(C\) is the matrix describing sector conjugation in \(Δ\).) Equivalently, the restriction of the vacuum representation of the two-dimensional quantum field theory \(B\) to its chiral subtheories is given by
\[π^0|_{A_L ⊗ A_R} = \sum_{λ, µ ∈ Δ} Z_{λ, µ} π_λ ⊗ π_µ.\]

The corollary combines and adapts results from [3, 12]. The point is that if the dual canonical endomorphism \(θ\) associated with \(N \subset M\) belongs to \(Σ(Δ)\), then \(α\)-induction [12, 4, 20] provides a pair of extensions \((ι, α^+)\) and \((ι, α^-)\) which satisfies (E1), (E2) as well as (E3) (e.g., [4] I; Def. 3.3, Lemma 3.5 and 3.25)). The associated
coupling matrix $Z_{\lambda,\mu}$ is automatically a modular invariant \cite{3}. By the characterization of extensions of local quantum field theories given in \cite{12, Prop. 4.9], the local subfactor $A_L(I_0) \otimes A_R(I_0) \subset B(O_0)$ given by the Thm. 1.4 induces an entire net of subfactors, indexed by the double-cones $O$ of two-dimensional Minkowski space.

(The charge conjugation $C$ arises due to an anti-isomorphism between $N_{opp}$ and $N$, cf. \cite{12, Prop. 4.10 ff}.) The statement of Proposition 1.5 is precisely the criterium given in \cite{12} for the resulting two-dimensional quantum field theory to be local.

Thus, every modular invariant found by the $\alpha$-induction method given in \cite{3} indeed corresponds to a two-dimensional local conformal quantum field theory extending the given chiral nets of observables.

## 2 Extensions of systems of endomorphisms

We collect some immediate consequences of the definition of an extension, Def. 1.3, using terminology and notations as in \cite{4, 13}.

**Proposition 2.1.** An extension $(\iota, \alpha)$ of a closed $N$-system gives rise to a monoidal functor from the full monoidal C* subcategory of $\text{End}_{\text{fin}}(N)$ with objects $\Pi(\Delta)$ (the set of finite products of elements from $\Delta$) into the monoidal C* category $\text{End}_{\text{fin}}(M)$. In particular, $\lambda \mu \simeq \bigoplus_{\nu} N_{\lambda \mu}^{\nu}$ implies $\alpha_{\lambda \nu} \simeq \bigoplus_{\nu} N_{\lambda \nu}^{\nu} \alpha_{\nu}$ (notwithstanding $\alpha_{\lambda}$ will be reducible in general), and $\alpha_{\lambda}$ is conjugate to $\alpha_{\lambda}$.

**Proof.** The functor maps objects $\lambda_1 \circ \ldots \circ \lambda_n$ to $\alpha_{\lambda_1} \circ \ldots \circ \alpha_{\lambda_n}$, and intertwiners $T$ to $\iota(T)$ which are again intertwiners by iteration of (E2). It follows from (E2) that $\alpha$ preserves the fusion rules as stated. In particular, $\alpha_{id_N}$ is an idempotent within $\text{End}_{\text{fin}}(M)$, implying that its dimension is 1, hence it is invertible and must coincide with $id_M$. Thus the functor preserves the monoidal unit object. It preserves the right monoidal product of intertwiners trivially, and the left monoidal product by (E1). Conjugacy between $\alpha_{\lambda}$ and $\alpha_{\lambda}$ is a consequence of the following lemma. \hfill $\Box$

**Lemma 2.2.** Let $(\iota, \alpha)$ be an extension of a closed $N$-system $\Delta$. Let $R \in \text{Hom}(id_N, \lambda \lambda)$ and $\tilde{R} \in \text{Hom}(id_N, \lambda \lambda)$ be a pair of standard isometries solving the conjugate equations

$$(1_{\lambda} \times R_{\lambda}^*) \circ (\tilde{R}_{\lambda} \times 1_{\lambda}) = d(\lambda)^{-1} 1_{\lambda} = (1_{\lambda} \times \tilde{R}_{\lambda}^*) \circ (R_{\lambda} \times 1_{\lambda}),$$

and thus implementing the unique left- and right-inverses \cite{13} $\Phi_{\lambda}$ and $\Psi_{\lambda}$ for $\lambda \in \Delta$. Then $\iota(R_{\lambda})$ and $\iota(\tilde{R}_{\lambda})$ induce left- and right-inverses $\Phi_{\alpha_{\lambda}}$ and $\Psi_{\alpha_{\lambda}}$ for $\alpha_{\lambda}$. If either $N \subset M$ has finite index, or $\Delta$ is a finite system, then $d(\alpha_{\lambda}) = d(\lambda)$, and $\Phi_{\alpha_{\lambda}}$ and $\Psi_{\alpha_{\lambda}}$ are the unique standard left- and right-inverses.

**Proof.** The first statement is an obvious consequence of (E2). If the index $d(\iota)^2$ is finite, then (E1) implies $d(\alpha_{\lambda}) = d(\lambda)$. If $\Delta$ is finite, then the minimal dimensions
$d(\alpha_\lambda)$ are uniquely determined by the fusion rules of $\{\alpha_\lambda, \lambda \in \Delta\}$, and the latter coincide with those of $\{\lambda \in \Delta\}$. Hence again $d(\alpha_\lambda) = d(\lambda)$. Since $d(\lambda)$ are also the dimensions associated with the pair of isometries $\iota(R_\lambda), \iota(R_\lambda)$, the last claim follows by [13, Thm. 3.11].

Thus, general properties of standard left- and right-inverses [13] are applicable. We shall in the sequel repeatedly exploit the trace property

$$d(\rho)\Phi_\rho(S^*T) = d(\tau)\Phi_\tau(TS^*) \quad \text{if} \quad S, T \in \text{Hom}(\rho, \tau)$$

for standard left-inverses of $\rho, \tau \in \text{End}_{\text{fin}}(M)$, their multiplicativity $\Phi_{\tau \rho} = \Phi_\tau \Phi_\rho$, as well as the equality of standard left- and right-inverses $\Phi_\rho = \Psi_\rho$ on $\text{Hom}(\rho, \rho)$.

### 3 Construction of the new CTPS’s

We shall prove Theorem 1.4 by the specification of “Q-systems” (or “canonical triples”) $(\theta, w, w_1)$, which uniquely determine subfactors [11].

Longo’s characterization states that $\theta \in \text{End}_{\text{fin}}(A)$ is the dual canonical endomorphism associated with $A \subset B$ if (and only if) there is a pair of isometries $w \in \text{Hom}(\text{id}_A, \theta)$ and $w_1 \in \text{Hom}(\theta, \theta^2)$ satisfying

(Q1) $w^*w_1 = \theta(w^*)w_1 = d(\theta)^{-1/2}1_A$,
(Q2) $w_1w_1 = \theta(w_1)w_1$, and
(Q3) $w_1w_1^* = \theta(w_1^*)w_1$.

Namely, then the map $w^*\theta(\cdot)w_1$ is the minimal conditional expectation onto its image $A_1 = w_1^*\theta(A)w_1 \subset A$. For $\iota_1 : A_1 \rightarrow A$ the inclusion map and $\bar{\iota}_1 : A \rightarrow A_1$ defined by $\theta = \iota_1\bar{\iota}_1$, the pair of isometries $w \in \text{Hom}(\text{id}_A, \iota_1\bar{\iota}_1)$ and $\bar{\iota}_1^{-1}(w_1) \in \text{Hom}(\text{id}_{A_1}, \bar{\iota}_1\iota_1)$ achieves the conjugacy between $\iota_1$ and $\bar{\iota}_1$. By the Jones construction [9], then, the subfactor $A_1 \subset A$ determines its dual subfactor (the Jones extension) $A \subset B$ such that $\theta = \bar{\iota}_1$. 

**Proof of Theorem 1.4.** First notice that the multiplicity of $\text{id}_A$ in $\theta$ is $Z_{\text{id}_{N_1}, \text{id}_{N_2}} = \dim \text{Hom}(\text{id}_M, \iota_1\bar{\iota}_1) = 1$, so the asserted subfactor is automatically irreducible.

In order to show that $\theta$ given in the Theorem is the dual canonical endomorphism associated with a subfactor $A \subset B$, we construct the Q-system $(\theta, w, w_1)$ as follows. We first choose a complete system of mutually inequivalent isometries $W_{(\lambda_1, \lambda_2, l)} \equiv W_l \in A \equiv N \otimes N^{\text{opp}}$, where $l$ is considered as a multi-index ($\lambda_1 \in \Delta_1, \lambda_2 \in \Delta_2, l = 1, \ldots, Z_{\lambda_1, \lambda_2}$), and put

$$\theta = \sum_l W_l^* (\lambda_1 \otimes \lambda_2^{\text{opp}})(\cdot) W_l^*.$$ 

The choice of these isometries is immaterial and affects the subfactor to be constructed only by inner conjugation.
Since $Hom(id_A, \theta)$ is one-dimensional, the isometry $w$ is already fixed up to an irrelevant complex phase: $w = W_0$, where 0 refers to the multi-index $l = 0 \equiv (id_{N_1}, id_{N_2}, 1)$. The second isometry, $w_1$, must be of the form

$$w_1 = \sum_{l,m,n} (W_l \times W_m) \circ T_{lm} \circ W_n^*$$

where $T_{lm}^n \in Hom(\nu_1 \otimes \nu_2^{opp}, (\lambda_1 \otimes \lambda_2^{opp}) \circ (\mu_1 \otimes \mu_2^{opp}))$, since these operators span $Hom(\theta, \theta^2)$.

In turn, $T_{lm}^n$ must be of the form

$$T_{lm}^n = \sum_{e_1,e_2} \zeta_{lm,e_1e_2} T_{e_1} \otimes (T_{e_2}^{opp}) \quad (\zeta_{lm,e_1e_2} \in \mathbb{C})$$

where $T_{ei}$ constitute orthonormal bases of the intertwiner spaces $Hom(\nu_i, \lambda_i \mu_i)$, since these operators span the spaces $Hom(\nu_1 \otimes \nu_2^{opp}, (\lambda_1 \otimes \lambda_2^{opp}) \circ (\mu_1 \otimes \mu_2^{opp})) \equiv Hom(\nu_1, \lambda_1 \mu_1) \otimes Hom(\nu_2^{opp}, \lambda_2^{opp} \mu_2^{opp})$. Note that if $T \in Hom(\alpha, \beta)$ is isometric in $N$, then $(T^*)^{opp} \in Hom(\beta, \alpha)^{opp} \equiv Hom(\alpha^{opp}, \beta^{opp})$ is isometric in $N^{opp}$. The labels $e_i$ are again multi-indices ($\lambda, \mu, \nu, e = 1, \ldots \text{dim} Hom(\nu, \lambda \mu)$).

It remains therefore to determine the complex coefficients $\zeta_{lm,e_1e_2}^n$, such that $w_1$ is an isometry satisfying Longo’s relations (Q1–3) above. To specify these coefficients, we equip the spaces $Hom(\alpha_{\lambda_1}, \alpha_{\lambda_2}^2)$ with the non-degenerate scalar products $(\phi, \phi') := \Phi_{\lambda_1}^1(\phi \phi')$ (where $\Phi_{\lambda_i}^1$ stand for the induced left-inverses for $\alpha_{\lambda_i}^1, i = 1, 2$, cf. Lemma 2.2). With respect to these scalar products, we choose orthonormal bases \{\phi_l, l = 1, \ldots Z_{\lambda_1, \lambda_2}\} for all $\lambda_1 \in \Delta_1, \lambda_2 \in \Delta_2$, and put

$$\zeta_{lm,e_1e_2}^n = \sqrt{\frac{d(\lambda_2)d(\mu_2)}{d(\delta)d(\nu_2)}} \Phi_{\nu_1}^1 [\nu_1(T_{ei}^*) (\phi_1^* \times \phi_m^*) \nu_2(T_{e_2}^*) \phi_n].$$

This formula is only apparently asymmetric under exchange 1 $\leftrightarrow$ 2: by the trace property $d(\lambda_2)\Phi_{\lambda_2}^2(\phi \phi^*) = d(\lambda_1)\Phi_{\lambda_1}^2(\phi^* \phi)$, an orthonormal basis $\psi_l$ of $Hom(\alpha_{\lambda_1}^2, \alpha_{\lambda_2}^1)$ differs from $\phi_l^*$ by a factor $\sqrt{\frac{d(\lambda_1)}{d(\lambda_2)}}$, so that in fact

$$\overline{\zeta_{lm,e_1e_2}^n} = \sqrt{\frac{d(\lambda_1)d(\mu_1)}{d(\delta)d(\nu_1)}} \Phi_{\nu_2}^2 [\nu_2(T_{e_2}^*) (\psi_1^* \times \psi_m^*) \nu_1(T_{ei}) \psi_n].$$

With these coefficients, condition (Q1) is trivially satisfied, since left multiplication of $w_1$ by $w^*$ singles out the term $l = 0$ due to $W_0^* W_l = \delta_{0l}$. This leaves only terms with $\lambda_i = id_{N_i}$, hence $\mu_i = \nu_i$, for which $T_{ei}$ are trivial and $\sqrt{d(\delta)} \overline{\zeta_{lm,e_1e_2}^n} = \delta_{mn}$ (up to cancelling complex phases), so $\sqrt{d(\delta)}w^* w_1 = \sum_n W_n W_n^* \equiv \mathbb{1}_A$. For $\theta(w^*) w_1$ the argument is essentially the same.
We turn to the conditions (Q2) and (Q3). Whenever we compute either of the four products occurring, we obtain a Kronecker delta $W_s^* W_t = \delta_{st}$ for one pair of the labels $l, m, n, \ldots$ involved, while the remaining operator parts are of the form

$$(W_l \times W_m \times W_k) \left[ (T_{e_1} \times 1_{\kappa_1}) T_{f_1} \otimes (((1_{\lambda_2} \times 1_{\kappa_2}) T_{g_2})^\text{opp}) W_n^* \right]$$

for the left- and right-hand side of (Q2), $w_1 w_1^* = \theta(w_1) w_1$, and in turn,

$$(W_l \times W_m) \left[ T_{e_1} T_{f_1}^* \otimes (((T_{e_2} T_{f_2}^*)^\text{opp}) (W_n \times W_k)^* \right]$$

for the left- and right-hand side of (Q3), $w_1 w_1^* = \theta(w_1^*) w_1$. (In these expressions, we do not specify the respective intertwiner spaces to which the various operators $T$ belong, since these are determined by the context.)

The numerical coefficients of these operators are sums over products of two $\zeta$’s or one $\zeta$ and one $\overline{\zeta}$, respectively, with a summation over one common label $s = 1, \ldots Z_{\sigma_1, \sigma_2}$ due to the above Kronecker $\delta_{st}$. These sums can be carried out.

Namely, the coefficients of the above operators on both sides of (Q2) involve one factor $\zeta^* s$, which is a scalar product of the form $\Phi^1_{\sigma_1} (X \phi_s) = (X^*, \phi_s)$ in $\text{Hom}(\alpha^1_{\sigma_1}, \alpha^2_{\sigma_2})$, so summation with the operator $\phi^* s$ contributing to the other factor $\zeta$ yields $\sum_s (X^*, \phi_s) \phi^* s = X$. Then the coefficients of the above operators on both sides of (Q2) are easily cast into the respective form

$$\sum_s \zeta^*_{lm,e_1 e_2} \zeta^s_{nk,f_1 f_2} = \sqrt{\frac{d(\lambda_2) d(\mu_2) d(\kappa_2)}{d(\theta)^2 d(\nu_2)}} \times \Phi^1_{\nu_1} [\iota_1 (T_{e_1}^* (1_{\kappa_1}) (\phi^* \times \phi^*_m \times \phi^*_h) \iota_2 ((1_{\lambda_2} \times 1_{\kappa_2}) T_{f_2}) \phi_n],$$

$$\sum_s \zeta^s_{mk,g_1 g_2} \zeta^*_{ln,h_1 h_2} = \sqrt{\frac{d(\lambda_2) d(\mu_2) d(\kappa_2)}{d(\theta)^2 d(\nu_2)}} \times \Phi^1_{\nu_1} [\iota_1 (T_{h_1}^* (1_{\lambda_1} \times T_{g_1}^*) (\phi^*_l \times \phi^*_m \times \phi^*_k) \iota_2 ((1_{\lambda_2} \times T_{g_2}) T_{h_2}) \phi_n].$$

Now, since the passage from bases of the form $(T_e \times 1_\kappa) T_f$ to bases $(1_\lambda \times T_g) T_h$ of $\text{Hom}(\nu, \lambda \mu \kappa)$ for any fixed $\nu, \lambda, \mu, \kappa$ is described by unitary matrices, equality of both sides of (Q2) follows.

The case of (Q3) is in the same vein, but slightly more involved. In the coefficients on the left-hand $\sum_s \zeta^*_{lm,e_1 e_2} \zeta^s_{nk,f_1 f_2}$, we read again the first factor as a scalar product $(X^*, \phi_s)$ within $\text{Hom}(\alpha^1_{\sigma_1}, \alpha^2_{\sigma_2})$ and perform the summation $\sum_s (X^*, \phi_s) \phi^* s = X$ with the operator $\phi^* s$ contributing to the second factor. This yields, after application of
the trace property for standard left-inverses, the coefficients on the left-hand side of (Q3)

\[ \sum_s \zeta^s_{lm,e_1e_2} \overline{\zeta^s_{nk,f_1f_2}} = \sqrt{\frac{d(\lambda_2)d(\mu_2)d(\kappa_2)d(\nu_2)}{d(\theta)^2d(\sigma_2)^2}} \frac{d(\lambda_1)d(\mu_1)}{d(\sigma_1)} \times \Phi^1_{\mu_1\lambda_1}[(\phi^*_l \times \phi^*_m)_{l_2}(T_{e_2}T^*_f)(\phi_n \times \phi_k)_{l_1}(T_{f_1}T^*_e)]. \]

To compute the coefficients \( \sum_s \zeta^s_{sk,g_1} \zeta^n_{ls,h_1h_2} \) on the right-hand side of (Q3), we first rewrite the second factor as a scalar product \( (\phi_s, X) \) within \( \text{Hom}(\alpha^1_{\sigma_1}, \alpha^2_{\sigma_2}) \). This is achieved by applying the trace property:

\[ \zeta^n_{ls,h_1h_2} = \sqrt{\frac{d(\lambda_2)d(\sigma_2)d(\lambda_1)d(\sigma_1)}{d(\theta)d(\nu_2)d(\mu_2)}} \Phi^1_{\sigma_1}[\phi^*_s \Phi^1_{\lambda_1}((\phi^*_l \times 1_{\alpha^2_{\sigma_2}})_{l_2}(T_{h_2})(\phi_n \times \phi_k)_{l_1}(T^*_h))]. \]

Now the sum over \( s \) with \( \phi_s \) contributing to \( \bar{\zeta}^m_{sk,g_1} \zeta^n_{ls,h_1h_2} \) can be performed as before, yielding the coefficients on the right-hand side of (Q3) in the form

\[ \sum_s \zeta^m_{sk,g_1} \zeta^n_{ls,h_1h_2} = \zeta^m_{sk,g_1} \zeta^n_{ls,h_1h_2} = \sqrt{\frac{d(\lambda_2)d(\kappa_2)d(\sigma_2)^2}{d(\theta)^2d(\nu_2)d(\mu_2)}} \frac{d(\lambda_1)d(\mu_1)}{d(\nu_1)} \times \Phi^1_{\mu_1\lambda_1}[(\phi^*_l \times \phi^*_m)_{l_2}(1_{\lambda_2} \times T^*_{g_2})(T_{h_2} \times 1_{\kappa_2})((\phi_n \times \phi_k)_{l_1}(1_{\lambda_1} \times T^*_h))]. \]

Noting that the passage from bases \( \sqrt{\frac{d(\mu)}{d(\sigma)}}T_e T^*_f \) to bases \( \sqrt{\frac{d(\sigma)}{d(\nu)}}(1_{\lambda} \times T^*_g)(T_h \times 1_{\kappa}) \) of \( \text{Hom}(\nu, \kappa, \lambda, \mu) \) is again described by unitary matrices for any fixed \( \nu, \kappa, \lambda, \mu \), we obtain equality of both sides of (Q3).

It remains to show that \( w_1 \) is an isometry, \( w^*_1w_1 = 1 \).

Performing the multiplication \( w^*_1w_1 \) yields two Kronecker delta’s from the factors \( W_1 \times W_m \), and two more Kronecker delta’s from the factors \( T_{e_1} \otimes (T^*_e)^{\text{opp}} \). Thus

\[ w^*_1w_1 = \sum_{ns} \left( \sum_{lm,e_1e_2} \zeta^s_{lm,e_1e_2} \zeta^n_{lm,e_1e_2} \right) W_1^* W_n, \]

and we have to perform the sums over \( l, m, e_1, e_2 \) (involving, as sums over multi-indices, the summation over sectors \( \lambda_i, \mu_i \in \Delta_i \) for fixed \( \nu_i \in \Delta_i, i = 1, 2 \).

It turns out convenient to express \( \zeta^n_{lm,e_1e_2} \) as a scalar product \( (\phi_m, X) \) within \( \text{Hom}(\alpha^1_{\mu_1}, \alpha^2_{\mu_2}) \) as before (with indices relabelled), and to perform the sum over \( m \) first. This yields

\[ \sum_{lm,e_1e_2} \zeta^s_{lm,e_1e_2} \zeta^n_{lm,e_1e_2} = \sum_{l,e_1e_2} \frac{d(\lambda_2)d(\mu_2)\lambda_1)d(\mu_1)}{d(\theta)d(\nu_2)d(\nu_1)} \times \Phi^1_{\nu_1}[\phi^*_{l_2}(T^*_{e_2})((\phi_l \times \Phi^1_{\lambda_1}((\phi^*_l \times 1_{\alpha^2_{\mu_1}})_{l_2}(T_{e_2})(\phi_n \times \phi_k)_{l_1}(T^*_e)))](T_{e_1}). \]
In this expression, we can perform the sums over \((e_1, \mu_1)\) and over \((e_2, \mu_2)\) after a unitary passage from the bases of orthonormal isometries \(T_e\) of \(\text{Hom}(\nu, \lambda \mu)\) to the bases \(\sqrt{\frac{d(\lambda_1)}{d(\mu)}}(1_\lambda \times T_e^*)(R_\lambda \times 1_\nu)\), and obtain after use of the conjugate equations for \(R_\lambda, \bar{R}_\lambda\)

\[
\sum_{l_{m,e_1 e_2}} \zeta_{lm, e_1 e_2}^n \zeta_{lm, e_1 e_2} = \sum_{l, \lambda_1 \lambda_2} \frac{d(\lambda_2)^2}{d(\theta)} \Phi_{n_1}^1 [\Psi_{\lambda_2}^{\lambda_1} (\phi_l \phi_l^*) \times (\phi_s \phi_n)].
\]

Here \(\Psi_{\lambda_2}^{\lambda_1}\) is the standard right-inverse implemented by \(\nu_2(\bar{R}_{\lambda_2})\) which coincides with \(\Phi_{\lambda_2}^{\lambda_1}\) on \(\text{Hom}(\alpha_{\lambda_2}, \alpha_{\lambda_2})\), and can be evaluated by the trace property: \(\Psi_{\lambda_2}^{\lambda_1} (\phi_l \phi_l^*) = \Phi_{\lambda_2}^{\lambda_1} (\phi_l \phi_l^*) = \Phi_{\lambda_2}^{\lambda_1} (\phi_l \phi_l^*) = \Phi_{\lambda_2}^{\lambda_1} (\phi_l \phi_l^*)\), while the sum over \(l\) yields the multiplicity factor \(Z_{\lambda_1, \lambda_2}\). Hence

\[
\sum_{l_{m,e_1 e_2}} \zeta_{lm, e_1 e_2}^n \zeta_{lm, e_1 e_2} = \left( \sum_{l_{m,e_1 e_2}} \frac{d(\lambda_1) d(\lambda_2) Z_{\lambda_1, \lambda_2}}{d(\theta)} \right) \Phi_{n_1}^1 (\phi_s \phi_n) = \delta_{sn},
\]

and hence \(w_1^* w_1 = \sum_n W_n W_n^* = 1\). This completes the proof of the Theorem. \(\Box\)

**Proof of Proposition 1.5.** Left multiplication of \(w_1\) with the induced braiding operator

\[
\varepsilon(\theta, \theta) = \sum_{m \mu m' \mu'} (W_{n' \times W_{m'}} \circ (\varepsilon_1(\lambda_1, \mu_1) \otimes (\varepsilon_2(\lambda_2, \mu_2)^{\mu = \mu' = (1)} \circ (W_l \times W_n)^*)
\]

amounts to a unitary passage from bases \(T_e \in \text{Hom} (\nu, \lambda \mu)\) to bases \(\varepsilon(\lambda, \mu) T_e \in \text{Hom}(\nu, \mu \lambda)\). But by (E3), the coefficients \(\zeta_{lm, e_1 e_2}^n\) are invariant under these changes of bases. Hence \(\varepsilon(\theta, \theta) w_1 = w_1\). \(\Box\)

4 Joint inclusions and normality

The main purpose of this section is to introduce and discuss the notions of “normality” and “essential normality”. These properties are of interest since the embedding of chiral subtheories into two-dimensional conformal quantum field theories should always give rise to essentially normal CTPS’s [17]. We start by introducing these and related notions in the broader context of “joint inclusions” of von Neumann algebras, i.e., triples \((A, B, C)\) such that \(A \lor B \subset C\). We first record some more or less elementary properties of joint inclusions, before we give a simple characterization of normality in the case of CTPS’s in terms of the coupling matrix.

**Definition 4.1.** Let \(\Lambda = (A, B, C)\) be a joint inclusion of von Neumann algebras. We denote by \(\Lambda^c := (B^c, A^c, C)\) the joint inclusion of the relative commutants in \(C\).
We write $\Lambda_1 \subset \Lambda_2$ if $C_1 = C_2$ and $A_1 \subset A_2$, $B_1 \subset B_2$, and call $\Lambda_2$ intermediate w.r.t. $\Lambda_1$. We call $\Lambda$ symmetric if $\Lambda \subset \Lambda^c$ (i.e., $A$ and $B$ commute with each other). We call $\Lambda$ normal if $\Lambda = \Lambda^c$ (i.e., $A$ and $B$ are each other’s relative commutants). We call $\Lambda$ essentially normal if $\Lambda^c = \Lambda^{cc}$.

One has the following elementary facts.

**Proposition 4.2.**

1. $\Lambda^c = \Lambda^{ccc}$.
2. If $\Lambda_1 \subset \Lambda_2$ then $\Lambda_2^c \subset \Lambda_1^c$.
3. If $\Lambda$ is symmetric then $\Lambda \subset \Lambda^{cc} \subset \Lambda^c$.
4. $\Lambda$ is essentially normal if and only if $\Lambda$ and $\Lambda^c$ are both symmetric.
5. Every symmetric $\Lambda$ has a normal intermediate joint inclusion.
6. If $\Lambda$ is normal then one has $Z(A) = (A \vee B)^c = Z(B) \supset Z(C)$, so $A$ and likewise $B$ are factors if and only if $A \vee B \subset C$ is irreducible, and in this case $C$ necessarily is also a factor.

**Proof.** Assertions 1.–4. are obvious. A normal intermediate joint inclusion is given by, e.g., $(B^c, B^{cc}, C)$. Assertion 6 holds since $(A \vee B)^c = A^c \cap B^c$. □

While these statements are in quite some parallelism to the theory of self-adjoint extensions of symmetric unbounded operators, assertion 5 is a departure from this parallelism, since self-adjoint extensions do not always exist for symmetric operators. The parallelism seems to become closer if one restricts to the subclass of tensor product subfactors (canonical or not) which are obviously symmetric joint inclusions. But neither $((1 \otimes B)^c, (A \otimes 1)^c, C)$ nor the joint inclusion $((1 \otimes B)^c, (1 \otimes B)^{cc}, C)$ in 4.2(5) will again be a tensor product subfactor in general.

While we have no general criterium for the existence of normal intermediate tensor product subfactors in general, the following proposition gives a simple characterization of normality in the case of CTPS’s, which entails certain constraints on the structure of $A_1 \otimes B_1 \subset C$ for which a normal CTPS $A \otimes B \subset C$ can possibly be intermediate. These constraints will apply to the embeddings of left and right chiral subtheories into two-dimensional conformal quantum field theories, which by [17] give rise to CTPS’s whose relative commutants are again tensor product subfactors, hence symmetric. Thus these local subfactors are essentially normal CTPS’s by Prop. 4.2(4), and the normal intermediate subfactor corresponds to the maximally extended chiral algebras (going along with permutation modular invariants). We do not evaluate these constraints here, but it is clear that the total dual canonical endomorphism must be of the form $(i_A \otimes i_B) \circ \theta \circ (i_A \otimes i_B) \simeq \bigoplus_\alpha (i_B \sigma(\alpha) i_A) \otimes (i_B \sigma(\alpha) i_B)$ where $\theta$ corresponding to the normal intermediate inclusion is of the special “permutational” form (N3) as described in the following proposition.
**Proposition 4.3.** Let $A \otimes B \subset C$ be a CTPS of type III with coupling matrix $Z$, i.e., the dual canonical endomorphism is of the form

$$
\theta \simeq \bigoplus_{\alpha \in \Delta_A, \beta \in \Delta_B} Z_{\alpha, \beta} \alpha \otimes \beta,
$$

where $\Delta_A \ni \text{id}_A$ and $\Delta_B \ni \text{id}_B$ are two sets of mutually inequivalent irreducible endomorphisms in $\text{End}_{\text{fin}}(A)$ and $\text{End}_{\text{fin}}(B)$. Then the following conditions are equivalent.

1. **(N1)** The joint inclusion $(A \otimes \text{id}_B, \text{id}_A \otimes B, C)$ is normal, i.e., $A \otimes \text{id}_B$ and $\text{id}_A \otimes B$ are each other’s relative commutants in $C$.

2. **(N2)** The coupling matrix couples no non-trivial sector of $A$ to the trivial sector of $B$, and vice versa, i.e.,

$$
Z_{\alpha, \text{id}_B} = \delta_{\alpha, \text{id}_A} \quad \text{and} \quad Z_{\text{id}_A, \beta} = \delta_{\beta, \text{id}_B}.
$$

3. **(N3)** The sets $\Delta_A$ and $\Delta_B$ are closed $A$- and $B$-systems, respectively, i.e., they are both closed under conjugation and fusion. There is a bijection $\sigma : \Delta_A \rightarrow \Delta_B$ which preserves the fusion rules, i.e.,

$$
\dim \text{Hom}(\alpha_1, \alpha_2 \alpha_3) = \dim \text{Hom}(\sigma(\alpha_1), \sigma(\alpha_2)\sigma(\alpha_3)).
$$

The matrix $Z$ is the permutation matrix for this bijection, i.e.,

$$
Z_{\alpha, \beta} = \delta_{\sigma(\alpha), \beta}.
$$

The proof is published in [17, Lemma 3.4 and Thm. 3.6].

Making contact with the new CTPS’s in Thm. 1.4, we first point out that in general they will not be normal, since among the coupling matrices constructed in [3] there are those which are not of the form (N2,N3).

The most simple case $N_1 = N_2 = M$ hence $Z = \mathbb{1}$ is known for a while [12], and clearly is normal by Prop. 4.3. Specifically, it describes the “diagonal” extension of the chiral observables by local two-dimensional observables carrying opposite chiral charges. We conclude from Prop. 4.3 that the left and right chiral observables are each other’s relative commutants within this two-dimensional theory, and the same holds whenever the coupling matrix in Cor. 1.6 satisfies condition (N2,N3), i.e., describes a permutation modular invariant (cf. [17]).

In the abstract mathematical setting, the subfactors with $Z = \mathbb{1}$ constructed in [12] were recognized [14] (up to some trivial tensoring with a type III factor) as the type II asymptotic subfactor [10] associated with $\sigma(M) \subset M$ where $\sigma \equiv \bigoplus_{\lambda \in \Delta} \lambda$. As the asymptotic subfactor $M \vee M^c \subset M_\infty$ associated with a fixed point inclusion $M^G \subset M$ for an outer action of a group $G$, provides the same category of $M_\infty$-$M_\infty$ subfactors.
bimodules as a fixed point inclusion for an outer action of the quantum double $D(G)$ on $M_\infty$, general asymptotic subfactors in turn are considered \[16, 3\] as generalized quantum doubles.

General asymptotic subfactors are CTPS’s, i.e., $M \vee M^c \simeq M \otimes M^c$ are in a tensor product position within $M_\infty$, and every irreducible $M \vee M^c$-$M \vee M^c$ bimodule associated with the asymptotic subfactor respects the tensor product \[16\]. They are normal, i.e., $M$ and $M^c$ are each other’s relative commutant in $M_\infty$. Moreover, the system of $M_\infty$-$M_\infty$ bimodules associated with an asymptotic subfactor has a non-degenerate braiding \[16, 8\]. We do not know at present whether the new CTPS’s always share this braiding property, which ought to be tested with methods as in \[8\].

5 Conclusion

We have shown the existence of a class of new subfactors associated with extensions of closed systems of sectors. The proof proceeds by establishing the corresponding Q-systems in terms of certain matrix elements for the transition between two extensions. The new subfactors are canonical tensor product subfactors and include the asymptotic subfactors.

Among the new subfactors, there are the local subfactors of two-dimensional conformal quantum field theory associated with certain modular invariants, thereby establishing the expected existence of these theories. We also gave a characterization of normality of CTPS’s which corresponds to the maximal subtheories of chiral observables in these models.

Acknowledgement

I am deeply indebted to Y. Kawahigashi, M. Izumi, T. Matsui, and I. Ojima who made possible my visit to Japan during summer 1999 where the present work was completed. I want to thank all of them as well as H. Kosaki, Y. Watatani, and T. Masuda for discussions, and for their hospitality extended to me at the Department of Mathematical Sciences, University of Tokyo, the Graduate School of Mathematics, Kyushu University, and the Research Institute for Mathematical Sciences, Kyoto University. I also thank H. Kurose for giving me the opportunity to present these results at the workshop “Advances in Operator Algebras” \[18\] held at RIMS, Kyoto. Financial support by a Grant-in-Aid for Scientific Research from the Ministry of Education (Japan) is gratefully acknowledged. Finally, I thank J. Böckenhauer for sending me a preliminary manuscript on related issues from a complementary perspective \[9\].
References

[1] Böckenhauer, J., Evans, D.E.: Modular invariants, graphs and α-induction for nets of subfactors. I, Commun. Math. Phys. 197, 361-386 (1998), II, ibid. 200, 57-103 (1999), and III, ibid. 205, 183-229 (1999)

[2] Böckenhauer, J., Evans, D.E.: Modular invariants from subfactors: Type I coupling matrices and intermediate subfactors. Preprint math.OA/9911239 (1999)

[3] Böckenhauer, J., Evans, D.E., Kawahigashi, Y.: On α-induction, chiral generators and modular invariants for subfactors. Preprint math.OA/9904109 (1999), to appear in Commun. Math. Phys.

[4] Doplicher, S., Roberts, J.E.: A new duality theory for compact groups. Invent. Math. 98, 157-218 (1989)

[5] Evans, D.E., Kawahigashi, Y.: Quantum Symmetries on Operator Algebras, Oxford University Press, 1998

[6] Guido, D., Longo, R.: The conformal spin and statistics theorem. Commun. Math. Phys. 181, 11-36 (1996)

[7] Haag, R.: Local Quantum Physics, Springer, 1996

[8] Izumi, M.: The structure of sectors associated with the Longo-Rehren inclusions. I, Kyoto preprint (1999)

[9] Jones, V.F.R.: Index for subfactors. Invent. Math. 72, 1-25 (1983)

[10] Kawahigashi, Y., Longo, R., Müger, M.: Multi-interval subfactors and modularity of representations in conformal field theory. Preprint math.OA/9903104 (1999)

[11] Longo, R.: A duality for Hopf algebras and for subfactors. I, Commun. Math. Phys. 159, 133-150 (1994)

[12] Longo, R., Rehren, K.-H.: Nets of subfactors. Rev. Math. Phys. 7, 567-597 (1995)

[13] Longo, R., Roberts, J.E.: A theory of dimension. K-Theory 11, 103-159 (1997)

[14] Masuda, T.: An analogue of Longo’s canonical endomorphism for bimodule theory and its application to asymptotic inclusions. Int. J. Math. 8, 249-265 (1997)

[15] Moore, G., Seiberg, N.: Naturality in conformal field theory. Nucl. Phys. B313, 16-40 (1989)

[16] Ocneanu, A.: Quantum symmetry, differential geometry, and classification of subfactors. Univ. Tokyo Seminary Notes 45 (1991) (notes recorded by Y. Kawahigashi)

[17] Rehren, K.-H.: Chiral observables and modular invariants. Preprint hep-th/9903262, to appear in Commun. Math. Phys.

[18] Rehren, K.-H.: New subfactors associated with closed systems of sectors. Preprint math.OA/9911148 (1999), contribution to the proceedings of the workshop “Advances in Operator Algebras”, RIMS, Kyoto, Sept. 1999

[19] Rehren, K.-H., Stanev, Ya.S., Todorov, I.T.: Characterizing invariants for local extensions of current algebras. Commun. Math. Phys. 174, 605-633 (1996)

[20] Xu Feng: New braided endomorphisms from conformal inclusions. Commun. Math. Phys. 192, 349-403 (1998)

[21] Xu Feng: Jones-Wassermann subfactors for disconnected intervals. Preprint hep-th/9704003 (1997)