COUNTEREXAMPLE TO A THEOREM ON QUASI-BIRTH-AND-DEATH PROCESSES

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Abstract. The paper disproves the basic theorem on quasi-birth-and-death processes given in [M. F. Neuts (1995). Matrix Geometric Solutions in Stochastic Models: An Algorithmic Approach. Dover, New York].

1. Introduction

My original aim was to develop some recent results from the theory of birth-and-death processes [1] to the more general objects such as quasi-birth-and-death (QBD) processes and level dependent QBD processes. Previously I was not familiar with matrix geometric methods in queueing problems. So, I started to read a well-known book by Neuts [8] and together with this book a number of more recent sources, a few most representative of which are [2], [5], [6], [7] and [9].

With critical reading the aforementioned book by Neuts [8], I arrived at the conclusion that the main theorem has a serious gap. Since 1981 a huge number of papers that are based on that theorem have been published, and all of them made the same error taking the statement of the theorem on belief. Therefore, I decided that it is very important to publish an example showing that an algebraic result that is commonly used by mathematicians in the areas of quasi-birth-and-death processes and queueing theory is incorrect.

Quasi-birth-and-death process is a continuous time Markov process, the transition rate matrix of which has a trigonal block structure

\[
Q = \begin{pmatrix}
B_{0,0} & B_{0,1} & & \\
B_{1,0} & A_1 & A_0 & \\
& A_2 & A_1 & A_0 \\
& & \ddots & \ddots & \ddots
\end{pmatrix},
\]

where each of sub-matrices of the matrix \( Q \) is of order \( d \times d \). To simplify our observation, assume \( B_{0,1} = A_0 \) and \( B_{1,0} = A_2 \). The matrices \( A_0 \) and \( A_2 \) are nonnegative matrices, and \( A_1 \) and \( B_{0,0} \) have nonnegative off-diagonal

1991 Mathematics Subject Classification. 60J80, 15A24.

Key words and phrases. Quasi-birth-and-death-processes; nonlinear matrix equation; matrix geometric solution.
elements, while the diagonal elements are strictly negative. The row sums in the matrices \( A_0 + A_1 + A_2 \) and \( B_{0,0} + A_0 \) must be equal to zero.

Assume that the QBD process is irreducible and ergodic. The conditions for ergodicity of the QBD process are known. All vectors considered below are row vectors. Transpose of a vector is specified with the superscript \( t \).

For ergodic process, let \( x \) denote a vector of stationary probabilities of the process, satisfying

\[
xQ = 0,
\]

where \( 0 \) denotes the vector of zeros, and

\[
x1^t = 1,
\]

where \( 1 \) denotes the vector of units. The vectors \( x, 0 \) and \( 1 \) are infinite-dimensional vectors, and the vector \( x \) is partitioned by level into subvectors \( x_0, x_1, \ldots \) of dimension \( d \) each. So,

\[
x = (x_0, x_1, x_2, \ldots).
\]

The following theorem was established in [8] for \( d < \infty \) and then extended in [6] for \( d = \infty \).

**Theorem 1.1.** Under the assumption that the QBD process is irreducible and ergodic, the stationary distribution of \( x \) is defined by

\[
x_n = x_0 R^n, \quad R^0 = I,
\]

where \( I \) is the \( d \times d \) unit matrix, \( R \) is a least nonnegative solution of the matrix equation

\[
R^2 A_2 + RA_1 + A_0 = 0,
\]

where \( O \) denotes the \( d \times d \) zero matrix. The stationary probability vector \( x_0 \) satisfies the equation

\[
x_0 (B_{0,0} + RA_2) = \theta,
\]

where \( \theta \) is \( d \)-dimensional vector of zeros, with the normalization condition

\[
\sum_{i=0}^{\infty} x_i e^t = x_0 (I - R)^{-1} e^t = 1,
\]

where \( e \) is the \( d \)-dimensional vector of units.

There is no special definition in [8] explaining what the meaning of a least nonnegative root of matrix equation is. (We use article a rather than the since the situation of equal roots is not excluded.) The proof of Lemma 1.2.3 on page 9 and an example on page 20 enable us to understand that it is said about element-wise comparison and the word least implies that all elements of one matrix (that is a solution of (1)) are not greater than the corresponding elements of any other matrix (that is also a solution of (1)). The main result of [8] is given on page 19 of the book in Theorem 1.3.2. The matrix equation more general than (1) is given in the book by relation (1.3.15). The proof of that theorem is provided in Section 1.3, where the
main reference is given to Lemma 1.2.3 on page 9. The second order matrix polynomial equation (1) appeared at the first time in the dissertation of Wallace [10], which is mentioned in [8] together with another pioneering work by Evans [3]. Having no access to the aforementioned dissertation, I analysed the proofs of Theorem 1.3.2 and Lemma 1.2.3 in [8] and did not find any justification of the existence of a least nonnegative solution to polynomial matrix equations that appeared in the proofs. Furthermore, overwhelming majority of the papers in this area provide numerical methods and algorithms for solution of (1) and derive the stationary probabilities, and there is none paper to study (1) analytically in order to understand how the roots of a nonlinear matrix equations look like. It will be shown in the next section that a least nonnegative root of equation (1) does not need to exist.

2. Counterexample

Let $I$ denote $2 \times 2$ unit matrix and $E$ the matrix of units of the same dimension. Set $A_2 = I$, $A_0 = \ell I$, $0 < \ell < 1$, and $A_1 = \epsilon E - (\ell + 1 + 2\epsilon)I$, where $\epsilon$ is a small positive value.

Since $A_0 < A_2$, then the condition of ergodicity is satisfied. (The inequality between the matrices is understood as the inequalities between their corresponding elements.) The matrices $A_0$, $A_1$ and $A_2$ all are of the required structures: the matrices $A_0$ and $A_2$ are positive, $A_1$ has negative diagonal elements and positive off-diagonal elements and the sums of the elements in each row is zero. So, the matrix $Q$ describes an irreducible QBD process.

For further simplification of the expressions denote $m = \ell + 1 + 2\epsilon$. Then matrix equation (1) takes the form

$$R^2 - R[mI - \epsilon E] + \ell I = O.$$  

The elements of the unknown matrix $R$ will be denoted by

$$R = \begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix}.$$

**Theorem 2.1.** There is no least nonnegative solution of matrix equation (2) in the element-wise sense.

**Proof.** Our goal is to demonstrate the following two properties.

**P1.** There is no nonnegative solution of the matrix equation in the form

$$R = \begin{pmatrix} r_{1,1} & 0 \\ 0 & r_{2,2} \end{pmatrix}.$$  

**P2.** There exists a continuously infinite set of solutions of the matrix equation in the form

$$R(x) = \begin{pmatrix} a & -g\epsilon + bx \\ -g\epsilon + bx^{-1} & a \end{pmatrix}, \quad a > 0, \quad b > 0, \quad g \geq 0,$$  

$$a > 0, \quad b > 0, \quad g \geq 0,$$  

$$\epsilon > 0.$$
for all real \( x > 0 \) and small \( \epsilon > 0 \).

These two properties mean that (2) has no least nonnegative solution.

It is easily seen that Property \( P1 \) is satisfied. Direct substitution of (3) into (2) shows that there is no matrix (3) that can be a solution of matrix equation (2), because of the presence of the nonzero off-diagonal elements in the matrix \( A_1 \). Our goal is to prove the correctness of Property \( P2 \). This proof is given in the sections below.

2.1. Matrix equation and its representations. To find a solution of matrix equation (2), we are to seek a matrix \( C = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix} \) depending on that solution \( R \) of the matrix equation such that

\[
R(mI - \epsilon C) = (mI - \epsilon C)R = R(mI - \epsilon E).
\]

From (4) we have the linear system of equations

\[
RC = CR = RE,
\]

the solution of which yields the required values \( c_{1,1}, c_{1,2}, c_{2,1} \) and \( c_{2,2} \).

Note that the system of the equations \( RC = CR \) has infinitely many solutions. Then the additional system of the equations \( RC = RE \) serves as a boundary condition and yields the exact values of \( c_{1,1}, c_{1,2}, c_{2,1} \) and \( c_{2,2} \).

Thus, original matrix equation (2) can be rewritten as a series of equations

\[
R^2 - \frac{1}{2} R(mI - \epsilon C) - \frac{1}{2} (mI - \epsilon C)R + \ell I = O,
\]

each of which is written for a specified root only, and the elements \( c_{1,1}, c_{1,2}, c_{2,1} \) and \( c_{2,2} \) of the matrix \( C \) depend on that root \( R \).

In the particular case when \( r_{1,1} = r_{2,2} \) and \( r_{1,2} = r_{2,1} \) from the system of equations (5) we obtain \( C = E \). In this case (6) can be rewritten as follows:

\[
R^2 - \frac{1}{2} R(mI - \epsilon C) - \frac{1}{2} (mI - \epsilon C)R + \ell I = O.
\]

2.2. Solutions of equation (2). In this section, the only solutions of (2) that are relevant to the proof of Theorem 2.1 are considered. Specifically, below we consider the solutions \( R \) satisfying the property \( r_{1,1} = r_{2,2} \) and \( r_{1,2} = r_{2,1} \). There are at least two roots of equation (2) satisfying this property, the simple explanation of which is as follows.

The complex matrices \( R \) in which \( r_{1,1} = r_{2,2} \) and \( r_{1,2} = r_{2,1} \) fall into the category of circulant matrices [4]. The class of circulant matrices forms Abelian ring, and the field of such matrices is an extension of the field of complex numbers. Since the matrices \( A_0, A_1 \) and \( A_2 \) are circulant matrices, then matrix equation (2) has at least two solutions of this class.
Assuming $\epsilon$ small, consider the expression $\sqrt{(mI - \epsilon E)^2 - 4\ell I}$. For this expression write

(8) $\sqrt{(mI - \epsilon E)^2 - 4\ell I} = \sqrt{m^2 - 4\ell I - \epsilon D}$

with some matrix $D = \begin{pmatrix} d_{1,1} & d_{1,2} \\ d_{2,1} & d_{2,2} \end{pmatrix}$. It follows from (8) that

(9) $D - \frac{\epsilon D^2}{2\sqrt{m^2 - 4\ell}} = \frac{mE}{\sqrt{m^2 - 4\ell}} - \frac{\epsilon E^2}{2\sqrt{m^2 - 4\ell}}$.

Keeping in mind that $E^2 = 2E$, we obtain the two solutions for the elements of the matrix $D$, both of which satisfy the property $d_{1,1} = d_{1,2} = d_{2,1} = d_{2,2} = d$. Then $d$ can be found as the least root of the equation

(10) $x - \frac{\epsilon x^2}{\sqrt{m^2 - 4\ell}} = \frac{m - \epsilon}{\sqrt{m^2 - 4\ell}}$.

This root is greater than 1. (The other root of equation (10) is proportional to $1/\epsilon$ and tends to infinity as $\epsilon$ vanishes.)

Then, based on (8) and the obtained presentation for the matrix $D$ we have the properties:

$$R(\sqrt{(mI - \epsilon E)^2 - 4\ell I}) = (\sqrt{(mI - \epsilon E)^2 - 4\ell I})R,$$

$$(mI - \epsilon E)(\sqrt{(mI - \epsilon E)^2 - 4\ell I}) = (\sqrt{(mI - \epsilon E)^2 - 4\ell I})(mI - \epsilon E),$$

and hence, (7) can be written in the product form

$$R^2 - \frac{1}{2}R(mI - \epsilon E) - \frac{1}{2}(mI - \epsilon E)R + \ell I$$

(11) $$= \left\{ R - \frac{1}{2}(mI - \epsilon E) - \frac{\sqrt{(mI - \epsilon E)^2 - 4\ell I}}{2} \right\}$$

$$\times \left\{ R - \frac{1}{2}(mI - \epsilon E) + \frac{\sqrt{(mI - \epsilon E)^2 - 4\ell I}}{2} \right\}$$

$$= \left\{ R - \frac{1}{2}(mI - \epsilon E) - \frac{\sqrt{m^2 - 4\ell I}}{2} I + \frac{\epsilon D}{2} \right\}$$

$$\times \left\{ R - \frac{1}{2}(mI - \epsilon E) + \frac{\sqrt{m^2 - 4\ell I}}{2} I - \frac{\epsilon D}{2} \right\}$$

$$= (R - R_1)(R - R_2)$$

$= O$.

Then the roots $R_1$ and $R_2$ obtained directly from (11) are

$$R_1 = \begin{pmatrix} \frac{m + \sqrt{m^2 - 4\ell}}{2} & -\frac{\epsilon(1+d)}{2} & -\frac{\epsilon(1+d)}{2} \\ -\frac{\epsilon(1+d)}{2} & \frac{m + \sqrt{m^2 - 4\ell}}{2} & -\frac{\epsilon(1+d)}{2} \\ -\frac{\epsilon(1+d)}{2} & -\frac{\epsilon(1+d)}{2} & \frac{m + \sqrt{m^2 - 4\ell}}{2} - \frac{\epsilon(1+d)}{2} \end{pmatrix},$$
\[ R_2 = \begin{pmatrix} \frac{m - \sqrt{m^2 - 4\ell}}{2} + \frac{\epsilon(d-1)}{2} & \frac{\epsilon(d-1)}{2} \\ \frac{\epsilon(d-1)}{2} & \frac{m - \sqrt{m^2 - 4\ell}}{2} + \frac{\epsilon(d-1)}{2} \end{pmatrix}, \]

and the root \( R_2 \) is positive.

Let us now find the set of roots \( R_3(z) \) from the following equation

\[
(R_3(z) - R_1)(R_3(z) - R_2) = \begin{pmatrix} a_1 & -a_1z \\ -a_2z^{-1} & a_2 \end{pmatrix} \begin{pmatrix} b_1 & b_2z \\ b_1z^{-1} & b_2 \end{pmatrix} = O
\]

for any complex \( z \neq 0 \), in which \( a_1, a_2, b_1, b_2 \) are unknowns. That is, the system of the equations we are going to solve is as follows:

\[
R_3(z) - R_1 = \begin{pmatrix} a_1 & -a_1z \\ -a_2z^{-1} & a_2 \end{pmatrix},
\]

\[
R_3(z) - R_2 = \begin{pmatrix} b_1 & b_2z \\ b_1z^{-1} & b_2 \end{pmatrix}.
\]

For \( z = 1 \) this is the system of eight equations with eight unknowns. We obtain:

\[
a_1 = a_2 = \frac{\epsilon d}{2} - \frac{\sqrt{m^2 - 4\ell}}{2},
\]

\[
b_1 = b_2 = \frac{\sqrt{m^2 - 4\ell}}{2} - \frac{\epsilon d}{2},
\]

and

\[
R_3(z) = \begin{pmatrix} \frac{m}{2} - \frac{\epsilon}{2} & \frac{\sqrt{m^2 - 4\ell} - \epsilon d}{2} \frac{z - \frac{\epsilon(1+d)}{2}} {z \frac{m}{2} - \frac{\epsilon}{2}} \\ \frac{\sqrt{m^2 - 4\ell} - \epsilon d}{2} \frac{z - \frac{\epsilon(1+d)}{2}} {z \frac{m}{2} - \frac{\epsilon}{2}} & \frac{m}{2} - \frac{\epsilon}{2} \end{pmatrix}.
\]

Property \( P2 \) is established, and the counterexample is built. The proof of Theorem 2.1 is completed.

\[\square\]

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