Solvable Subgroups of Locally Compact Groups

Karl Heinrich Hofmann and Karl-Hermann Neeb

Abstract. It is shown that a closed solvable subgroup of a connected Lie group is compactly generated. In particular, every discrete solvable subgroup of a connected Lie group is finitely generated. Generalizations to locally compact groups are discussed as far as they carry.

Mathematics Subject Classification 2000: 22A05, 22D05, 22E15;
Key Words and Phrases: Connected Lie group, almost connected locally compact group, solvable subgroup, compactly generated, finitely generated.

A topological group $G$ with identity component $G_0$ is said to be almost connected if $G/G_0$ is compact. We shall prove the following result.

Main Theorem. A closed solvable subgroup of a locally compact almost connected group is compactly generated.

This result belongs to a class of “descent” type results that are on record for compactly generated groups. The essay [8] provides a good background of their history. It follows, in particular, that a discrete solvable subgroup of an almost connected locally compact group is finitely generated.

Example S. The connected simple Lie group $\text{PSL}(2, \mathbb{R})$ contains a discrete free group of infinite rank; such a closed subgroup is not compactly generated.

We remark that a nonabelian free group is countably nilpotent (see e.g. [4], Definition 10.5); that is, the descending central series terminates at the singleton subgroup after $\omega$ steps. The Main Theorem therefore fails for transfinitely solvable subgroups in place of solvable ones.

The following example shows that subgroups of finitely generated solvable groups need not be finitely generated:

Example SOL. Let $\Gamma \subseteq \mathbb{Q} \times \mathbb{Q}^\times$ be the subgroup generated by the two elements $a := (0, 2)$ and $b := (1, 0)$. Then

$$\Gamma \cong \left( \frac{1}{2^\omega} \mathbb{Z} \right) \times \mathbb{Z},$$

is a 2-generator metabelian group, while the abelian subgroup $\frac{1}{2^\omega} \mathbb{Z} \times \{0\}$ is not finitely generated.

Thus, in the Main Theorem, the hypothesis “$G/G_0$ compact” cannot be relaxed to “$G/G_0$ compactly generated”.

1
For abelian subgroups the Main Theorem will allow us to derive a characterisation theorem for compactly generated locally compact abelian groups as follows.

**Theorem.** For a locally compact abelian group \( A \) the following conditions are equivalent:

1. \( A \) is compactly generated.
2. \( A \cong \mathbb{R}^k \oplus C \oplus \mathbb{Z}^n \) for a unique largest compact subgroup \( C \) and natural numbers \( k, n \).
3. The character group \( \hat{A} \) is a Lie group.
4. There is an almost connected locally compact group \( G \) and a closed subgroup \( H \) such that \( A \cong H \).

**Proof.**

1. \( \Rightarrow \) 2: See e.g. [3], Theorem 7.57(ii).

2. \( \Rightarrow \) 3: If \( \hat{A} \) is a Lie group, then \( (\hat{A})_0 \) is open and isomorphic to \( \mathbb{R}^k \oplus \mathbb{T}^n \) for some \( k \) and \( n \); it is divisible, whence \( \hat{A} \cong (\mathbb{R}^k \oplus \mathbb{T}^n) \oplus D \) for a discrete subgroup \( D \). Hence \( A = \mathbb{R}^k \oplus \hat{D} \oplus \mathbb{T}^n \cong \mathbb{R}^k \oplus C \oplus \mathbb{Z}^n \) for the unique largest compact subgroup \( C \) of \( A \).

2. \( \Rightarrow \) 4: \( A \subseteq \mathbb{R}^k \times C \times \mathbb{R}^n \cong \mathbb{R}^{k+n} \oplus C \), an almost connected locally compact group.

4. \( \Rightarrow \) 1: Let \( G \) be an almost connected locally compact group and \( A \) a closed abelian subgroup. Then \( A \) is, in particular, solvable. Hence the Main Theorem provides the required implication.

By comparison with Example SOL, the situation for abelian groups is distinctly simpler than it is for metabelian groups:

**Corollary.** (Morris’ Theorem [5], [8]) A closed subgroup of a compactly generated locally compact abelian group is compactly generated.

**Proof.** We proved \( 2 \Leftrightarrow 3 \) in the Theorem independently of the Main Theorem. Thus if \( G \) is a locally compact compactly generated abelian group, then \( \hat{G} \) is an abelian Lie group. The character group \( \hat{A} \) of a closed subgroup \( A \) of \( G \), by duality, is a quotient of the Lie group \( \hat{A} \) and thus is a Lie group. Hence \( A \) is compactly generated.

As we now begin a proof of the main theorem we first reduce it to one on connected Lie groups and its closed subgroups:

**Reduction.** The Main Theorem holds if every closed solvable subgroup \( H \) of a connected Lie group \( G \) is compactly generated.

**Proof.** Indeed let \( G \) be an almost connected locally compact group and \( N \) a compact normal subgroup such that \( G/N \) is a Lie group. The existence of \( N \) is a consequence of Yamabe’s Theorem saying that each almost connected locally
compact group is a pro-Lie group ([9,10]). Then $HN$ is a closed subgroup and $HN/N$ is a closed solvable subgroup $A$ of the Lie group $L = G/N$ with finitely many components. If our claim is true for connected Lie groups $G$, then $A \cap L_0$ is compactly generated. We may assume $L = L_0A$. Then $A \cap L_0$ has finite index in $A$. Therefore $A = HN/N$ is compactly generated. Then $HN$ is compactly generated. So $H$ is compactly generated. (See [1], Chap. VII, §3, Lemma 3. Also see [8].) □

This reduction allows us to concentrate on connected Lie groups $G$ and closed solvable subgroups $H$. Since any locally compact connected group, and so in particular every connected Lie group, is compactly generated we shall have to prove that $\pi_0(H) \overset{\text{def}}{=} H/H_0$ is finitely generated.

**Lemma 1.** For a closed subgroup $H$ of a connected solvable connected Lie group $G$ any subgroup of $\pi_0(H)$ is finitely generated.

**Proof.** This is proved in [7], Proposition 3.8. □

This shows that the two generator metabelian group $\Gamma$ of Example SOL cannot be realized as $\pi_0(H)$ for a closed subgroup $H$ of a connected solvable Lie group $G$—let alone be discretely embedded into $G$.

**Lemma 2.** Let

$$1 \to A \to B \overset{q}{\to} C \to 1$$

be a short exact sequence of groups. If $A$ and $C$ have the property that each subgroup is finitely generated, then $B$ has this property as well.

**Proof.** Each subgroup $\Gamma \subseteq B$ is an extension of the finitely generated group $q(\Gamma)$ by the finitely generated group $A \cap \Gamma$, hence is finitely generated itself. □

**Lemma 3.** Assume that the solvable Lie group $G$ has the property that each subgroup of $\pi_0(G)$ is finitely generated. Let $H$ be a closed subgroup of $G$. Then each subgroup of $\pi_0(H)$ is finitely generated.

**Proof.** Let $q: G \to \pi_0(G)$ denote the quotient map. Then we have a short exact sequence

$$1 \to \pi_0(H \cap G_0) \to \pi_0(H) \to q(H) \to 1.$$ 

As a subgroup of $\pi_0(G)$, the group $q(H)$ has the property that all its subgroups are finitely generated, and the group $\pi_0(H \cap G_0)$ has this property by Lemma 1. Now Lemma 2 implies that each subgroup of $\pi_0(H)$ is finitely generated. □

**Lemma 4.** If $H$ is a closed solvable subgroup of $GL_n(\mathbb{C})$, then each subgroup of $\pi_0(H)$ is finitely generated.

**Proof.** Let $S$ denote the Zariski closure of $H$. Then $S$ is a solvable linear algebraic group, so that $\pi_0(S)$ is finite (see e.g. [6], Theorems 3.1.1 and 3.3.1). Since $H$ is a closed subgroup of the Lie group $S$, the assertion follows from Lemma 3. □
In order to proceed we need a further line of lemmas. We shall call a Lie group *linear* if it has a faithful linear representation. The following statement is of independent interest.

**Proposition 5.** A connected linear Lie group has a faithful linear representation with a closed image.

**Proof.** By [2], Theorem IV.3 a connected Lie group $G$ is linear if and only if it is isomorphic to a semidirect product $B \rtimes H$ where $B$ is a simply connected solvable Lie group and $H$ is a linear reductive Lie group with compact center. We set $G = B \rtimes H$ and deduce that the commutator subgroup $G'$ equals $(G, B) \rtimes (H, H)$. From [2], Theorem IV.5 it follows that $G'$ is closed in $G$. The quotient group $G/G'$ is a direct product $B/(G, B) \times H/(H, H) \cong B/(G, B) \times Z(H)_0/(Z(H)_0 \cap (H, H))$, where $B/(G, B)$ is a vector group and $Z(H)_0/(Z(H)_0 \cap (H, H))$ is a torus. This group has a representation mapping the vector group $B/(G, B)$ homeomorphically on a unipotent subgroup. That is, we have a representation $\rho: G \to \text{GL}(W)$ such that

$$\ker \rho = (G, B)H \quad \text{and} \quad \overline{\text{im} \rho} = \text{im} \rho,$$

the image being unipotent.

Now let $\pi: G \to \text{GL}(V)$ be a faithful linear representation and define $\zeta = \pi \oplus \rho$. We shall show that $\zeta$ has a closed image. Suppose this is not the case. Then there is an $X \in \mathfrak{g}$ such that $T \overset{\text{def}}{=} \overline{\zeta(\exp \mathbb{R}X)}$ is a torus not contained in $\zeta(G)$ (see [2], Proposition XVI.2.3 and Theorem XVI.2.4). In the Appendix we shall show that, under any representation of a connected Lie group $G$, the commutator subgroup $G'$ has a closed image. Thus $\zeta(G')$ is closed and $\zeta(Z(H))$ is compact since $H$ has a compact center. Thus $\zeta(G'Z(H)) = \zeta(G')\zeta(Z(H))$ is closed and contained in $\zeta(G)$. Accordingly, $X$ cannot be contained in $\mathfrak{g}' + \mathfrak{z}(\mathfrak{h}) = [\mathfrak{g}, \mathfrak{b}] + \mathfrak{h}$. Thus by (1), $\exp \mathbb{R}X$ fails to be in $\ker \rho$. It follows that $\rho \circ \exp$ maps $\mathbb{R}X$ homeomorphically onto a unipotent one-parameter group. Then $\zeta \circ \exp$ maps $\mathbb{R}X$ homeomorphically as well, and that contradicts the fact that $T$ is a torus. This contradiction proves the proposition. \(\square\)

We now complete the proof of the Main Theorem by proving the last lemma:

**Lemma 6.** Let $G$ be a connected Lie group and $H$ a closed solvable subgroup. Then $H$ is compactly generated.

**Proof.** Let $Z = Z(G)$ be the center of $G$. Then $A \overset{\text{def}}{=} \overline{ZH}$ is a closed solvable subgroup of $G$ containing $H$. By Lemma 3 for $H$ to be compactly generated it will suffice to show that all subgroups of $\pi_0(A) = A/A_0$ are finitely generated. Let $A_1$ be a subgroup of $A$ containing $A_0$. Then $A_1$ is open in $A$, and so $A_1Z$ is open and thus closed in $A$. Therefore

$$A_1/(A_1 \cap (A_0Z)) \cong A_1Z/A_0Z.$$
By the modular law,

\[ A_1 \cap (A_0 Z) = A_0 (A_1 \cap Z). \]

We have the following isomorphism of discrete groups

\[ A_0 (A_1 \cap Z)/A_0 \cong (A_1 \cap Z)/(A_0 \cap (A_1 \cap Z)) = (A_1 \cap Z)/(A_0 \cap Z). \]

Taking (1), (2) and (3) together we recognize the following exact sequence

\[ 1 \to \frac{A_1 \cap Z}{A_0 \cap Z} \to \frac{A_1}{A_0} \to \frac{A_1 Z}{A_0 Z} \to 1. \]

In order to show that \( A_1/A_0 \) is finitely generated it therefore suffices that

(a) \( (A_1 \cap Z)/(A_0 \cap Z) \) is finitely generated,

(b) \( (A_1 Z)/(A_0 Z) \) is finitely generated.

Ad (a): The center \( Z \) of the connected Lie group \( G \) is compactly generated. (Indeed the fundamental group \( \pi_1(G/Z) \) is finitely generated abelian and \( \pi_0(Z) = Z/Z_0 \) is the kernel of the covering morphism \( G/Z_0 \to G/Z \) and is therefore finitely generated as a quotient of \( \pi_1(G/Z) \). Thus \( Z \) is compactly generated.) Since \( A_1 \) is open in \( A \), the group \( A_1 \cap Z \) is open in \( Z \) and thus compactly generated, and so (a) follows.

Ad (b): The adjoint representation \( \text{Ad}: G \to \text{Aut} g \subseteq \text{GL}(g) \) induces a faithful linear representation of \( G/Z \). Then by Lemma 4 and Proposition 5, \( A_1 Z/Z \), a closed solvable subgroup of \( G/Z \), is compactly generated. Then the discrete factor group \( A_1 Z/A_0 Z \cong (A_1 Z/Z)/(A_0 Z/Z) \) is finitely generated. Thus (b) is proved as well and this completes the proof of Lemma 6 and thereby the proof of the Main Theorem. \( \square \)

**Appendix**

In the proof of Proposition 5 we used the following

**Theorem A.** For any finite dimensional representation of a connected Lie group \( G \), the image of the commutator subgroup is closed.

**Proof.** It is no loss of generality to assume that \( G \) is simply connected. Then we have Levi decomposition \( G = R \times_\alpha S \) and \( G' = (G, R) \times S \). Let \( \pi: G \to \text{GL}(V) \) be a finite dimensional representation and let

\[ V_0 = \{0\} \subseteq V_1 \subseteq \cdots \subseteq V_n = V \]

be a maximal flag of \( G \)-submodules of \( V \) such that all quotient modules \( V_{j+1}/V_j \) are simple. Since \( \pi|S \) is a semisimple representation, we may choose \( S \)-invariant decompositions \( V_j = V_{j-1} \oplus W_j \). Then

\[ \pi(G) \subseteq G_F \overset{\text{def}}{=} \{ g \in \text{GL}(V) : (\forall j) gV_j = V_j \}, \]
and we have a semidirect decomposition \( G_F = U_F \rtimes L_F \), where

\[
U_F = \{ g \in \text{GL}(V) : (\forall j)(g - 1)(V_j) = V_{j-1} \}
\]

and \( L_F = \prod_j \text{GL}(W_j) \). Note also that \( \pi(S) \subseteq L_F \). Furthermore, Theorem I.5.3.1 of [1] implies that the ideal \([g, r]\) acts trivially on each simple \( g \)-module and so \( \pi((G, R)) \subseteq U_F \). Hence \( \pi((G, R)) \) is a unipotent analytic group and is therefore closed. Moreover, \( \pi(S) \) is closed (see [2], Chapter XVI) and this shows that \( \pi(G') \cong \pi((G, R)) \rtimes \pi(S) \) is closed.

The proof of Theorem A can be derived from the theory of algebraic groups, since the commutator algebra of a linear Lie algebra is the Lie algebra of an algebraic group [6]. We gave a more direct proof inspired by the discussion of linear Lie groups in [2].

**References**

[1] Bourbaki, N., Groupes et algèbres de Lie, Chap. I-III, reprinted by Springer-Verlag, Berlin etc., 1989.

[2] Hochschild, G., The Structure of Lie Groups, Holden Day, San Francisco, 1965.

[3] Hofmann, K. H. and S. A. Morris, The Structure of Compact Groups, W. DeGruyter, Berlin 1998 and 2006.

[4] —, The Lie Theory of Connected Pro-Lie Groups, European Mathematical Society Publishing House, Zürich, 2007.

[5] Morris, S. A., Locally compact abelian groups and the variety of topological groups generated by the reals, Proc. Amer. Math. Soc. 34 (1972), 290–292.

[6] Onishchik, A. L., and E. B. Vinberg, Lie Groups and Algebraic Groups, Springer-Verlag, Berlin etc., 1990.

[7] Raghunathan, M. S., “Discrete Subgroups of Lie Groups,” Ergebnisse der Math. 68, Springer, Berlin etc., 1972.

[8] Ross, K., Closed subgroups of compactly generated LCA group are compactly generated, [http://www.uoregon.edu/~ross1/subgroupsofCGLCA6.pdf](http://www.uoregon.edu/~ross1/subgroupsofCGLCA6.pdf).

[9] Yamabe, H., *On the Conjecture of Iwasawa and Gleason*, Ann. of Math. 58 (1953), 48–54.

[10] —, *Generalization of a theorem of Gleason*, Ann. of Math. 58 (1953), 351–365.