Persistence of locality in systems with power-law interactions

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Motivated by recent experiments with ultra-cold matter, we derive a new bound on the propagation of information in D-dimensional lattice models exhibiting $1/r^\alpha$ interactions with $\alpha > D$. The bound contains two terms: One accounts for the short-ranged part of the interactions, giving rise to a bounded velocity and reflecting the persistence of locality out to intermediate distances, while the other contributes a power-law decay at longer distances. We demonstrate that these two contributions not only bound but qualitatively reproduce the short- and long-distance dynamical behavior following a local quench in an XY chain and a transverse-field Ising chain. In addition to describing dynamics in numerous intractable long-range interacting lattice models, our results can be experimentally verified in a variety of ultracold-atomic and solid-state systems.

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Lieb-Robinson bounds [1, 2] on the propagation of information in many-body quantum systems underly our understanding of numerous equilibrium and non-equilibrium phenomena, including entanglement growth after quenches [3–5], stability to perturbations of the area law for entanglement entropy [6, 7], exponential decay of correlations in gapped ground states [8], and the Lieb-Schultz-Mattis theorem [9]. While such bounds are well established for systems with short-range interactions, their generalization to systems with long-range interactions is far from complete [8, 10, 11]. Meanwhile, numerous currently available atomic, molecular, and optical systems exhibiting long-range interactions are emerging as versatile platforms for studying quantum many-body physics both in and out of equilibrium. These long-range interactions include dipolar ($1/r^3$) interactions between electric [12, 13] or magnetic [14–19] dipoles, strong van-der-Waals ($1/r^6$) interactions between Rydberg atoms [12, 20] or polaritons [21], along with $1/r^\alpha$ and even more general forms of interactions between trapped ions [22–25] or atoms in multimode cavities [26].

An important consequence of Lieb-Robinson bounds for short-range interacting systems is an emergent velocity for the propagation of information, which gives rise to a linear light cone and the associated notion of locality. While these bounds have been generalized to long-range interacting systems by Hastings and Koma [8], it is not yet clear to what extent such locality persists. For example, while the Hastings-Koma bound allows for a causal region that grows exponentially in time, and thus a divergent velocity (see Fig. 1), to the best of our knowledge there are no models that explicitly demonstrate such behavior. Conversely, linear light cones have been observed in certain systems with long-ranged interactions [10, 11], yet are manifestly absent in the existing Hastings-Koma bound. In this Letter, we derive a new Lieb-Robinson-type bound for systems with $1/r^\alpha$ interactions, which is consistent with the Hastings-Koma bound at long distances, but explicitly captures the possibility for a linear light cone at intermediate distances. Our results are relevant to recent experiments in trapped ions [24, 25], and build on recent theoretical work studying post-quench dynamics in several long-range interacting systems, including Ising models both with [10, 27, 28] and without [11, 29] a transverse field, the XXZ chain [11, 30], and spin models with boson-mediated interactions [31].

We consider general spin Hamiltonians of the form $H = \frac{1}{2} \sum_{i>j} h_{ij}$, defined on an infinite D-dimensional regular cubic lattice ($D = 1, 2, 3$), where $h_{ij}$ is an operator supported on sites $i$ and $j$ and $h_{ij} \equiv h_{ji}$ [39]. We note that all of the results presented below apply to Hamiltonians with time dependence and/or arbitrary onsite interactions [32]. We further assume that all interactions are bounded by a power-law decay: $||h_{ij}|| \leq J_{ij} \equiv 1/r_{ij}^{\alpha}$, where $r_{ij}$ is the Euclidean distance (in lattice units) between sites $i$ and $j$, and for convenience we define $J_{ii} = 1$. The notation $||O||$ represents the norm of the operator $O$ (the largest magnitude of an eigenvalue of $O$). In what follows, we prove that arbitrary operators $A$ and $B$, supported on single sites a distance $r$ apart, obey the bound

$$\frac{||A(t), B||}{2 ||A|| ||B||} \leq \left( c_1 e^{v_1 t} - c_2 e^{v_2 t} \right) \frac{1}{(1 - \mu r)^{\alpha}}$$

where $v_1 = v_2 = v$. The boundary switches from linear to logarithmic at a critical $r_c$ satisfying $r_c \sim \alpha \log r_c$. (b) The decay of the signal outside the causal region changes from exponential to algebraic at $r_c$. 

![FIG. 1: (Color online). (a) Illustration of the causal region (shaded) resulting from Eq. (1) for the case $v_1 = v_2 = v$. The boundary switches from linear to logarithmic at a critical $r_c$ satisfying $r_c \sim \alpha \log r_c$. (b) The decay of the signal outside the causal region changes from exponential to algebraic at $r_c$.](image-url)
where $A(t) = e^{iHt} A e^{-iHt}$. The constants $c_1, c_2, v_1, v_2$ are finite for all $\alpha > D$ and independent of $t$ and $r$, while $\mu \in (0, 1)$ is an adjustable parameter. As shown in Fig. 1(a), we can define a causal region as the part of the $r$-$t$ plane where the right-hand-side of Eq. (1) is larger than a given value. The first term on the right-hand-side of Eq. (1) is the familiar short-range bound of Ref. [1]; alone it would lead to a causal region bounded by a linear light cone ($v_1 t \geq r$), and thus to a finite velocity for the propagation of information. The second term is similar to the long-range bound of Ref. [8]; alone it would lead to a causal region with a logarithmic boundary $v_2 t \geq \alpha \log r$, and an actual velocity that grows exponentially in time. The two terms together give a hybrid boundary, which switches from linear to logarithmic behavior at a critical $r_c$ satisfying $r_c \sim \alpha \log r_c$. As shown in Fig. 1(b), the decay of the signal outside the causal region changes from exponential to polynomial at $r_c$. Note that, as $\alpha \to \infty$, and the Hamiltonian approaches a nearest-neighbor model, $r_c \to \infty$ and the bound reduces to the short-range case.

We emphasize that this hybrid bound cannot be obtained by simply adding a short-range contribution to the long-range bound of Ref. [8]. The long-range bound alone, in order to remain valid in the large-$\alpha$ limit, has the property that $v_2$ diverges as $\alpha \to \infty$. Therefore, as shown in Fig. 2(a), it leads to a causal region that grows larger for shorter-range interactions. To the contrary, in the long-range piece of Eq. (1), $v_2$ remains finite in the large-$\alpha$ limit. Thus we obtain a much more physical scenario, in which the causal region shrinks for progressively shorter-range interactions, eventually coinciding with the linear light cone familiar from short-range Lieb-Robinson bounds [Fig. 2(b)].

Equation (1) has many important applications; for example it can be used to predict dynamics following a local quench of a lattice system [25]. Specifically, we envision applying a unitary operator $U$ (acting only on site $j$) to an arbitrary state $|\psi\rangle$. The effect of this quench on the expectation value of an arbitrary operator $A$, acting only on site $i$ a distance $r$ from site $j$, is captured by the experimentally measurable quantity $Q(t) = |\langle \psi | U^\dagger A(t) U | \psi \rangle - \langle \psi | A(t) | \psi \rangle|$. Because $Q(t) = |\langle \psi | U^\dagger A(t), U | \psi \rangle| \leq \| A(t), U \|$, we can bound $Q(t)$ using Eq. (1).

Long-range Lieb-Robinson bounds. — We first briefly summarize the origin of the Hastings-Koma bound derived in Ref. [8]. For operators $A$ and $B$ supported on sites $i$ and $j$, respectively, $\| A(t), B \|$ can be bounded by an infinite series in time [8],

$$\frac{\| A(t), B \|}{2 \| A \| \| B \|} \leq \sum_{n=1}^{\infty} \frac{(2\lambda)^n}{n!} \mathcal{J}_n(i,j), \quad (2)$$

$$\mathcal{J}_n(i,j) \equiv \sum_{k_1, \ldots, k_{n-1}} J_{k_1} J_{k_1 k_2} \ldots J_{k_{n-1} j}. \quad (3)$$

Here $\mathcal{J}_n(i,j)$ can be thought of as the total contribution from all $n$th order “hopping” processes connecting sites $i$ and $j$, and the factor $\lambda = \sum_k J_{k k}$ is finite for all $\alpha > D$.

Following Ref. [8], one can show that, for any $i$, $j$, and $\alpha > D$, the following reproducibility condition holds

$$\sum_k J_{i k} J_{k j} \leq 2^n \lambda J_{i j}. \quad (4)$$

Repeated application of this bound to Eq. (3) yields

$$\mathcal{J}_n(i,j) \leq (2^n \lambda)^{n-1}/r^n, \quad (5)$$

where $v = 2\lambda^{2\alpha}$ and $c = (\lambda^{2\alpha})^{-1}$. Equation (5) holds for all $\alpha > D$, so naively one would expect to be able to recover a short-ranged Lieb-Robinson bound [e.g. the first term in Eq. (1)] by taking the limit $\alpha \to \infty$. However, because the velocity $v$ in Eq. (5) diverges exponentially with $\alpha$, the causal region encompasses all $r$ and $t$ for short-range ($\alpha \to \infty$) interactions [Fig. 2(a)]. Below, we derive the new bound in Eq. (1), which recovers the correct short-range physics in the large $\alpha$ limit, and manifestly preserves the effects of short-range interactions at intermediate distance scales.

Recovering locality. — Our strategy for obtaining the bound on $\| A(t), B \|$ given in Eq. (1) begins by separating the infinite series in Eq. (2) into two parts:

$$\frac{\| A(t), B \|}{2 \| A \| \| B \|} \leq \sum_{n=1}^{\lfloor \mu r \rfloor - 1} \frac{(2\lambda)^n}{n!} \mathcal{J}_n(i,j) + \sum_{n=\mu r}^{\infty} \frac{(2\lambda)^n}{n!} \mathcal{J}_n(i,j), \quad (6)$$

where $\mu$ is an adjustable parameter satisfying $0 < \mu < 1$, and $\lfloor \mu r \rfloor$ represents the smallest integer satisfying $\lfloor \mu r \rfloor \geq \mu r$. The intuition for this separation is that the first part contains a relatively small number of hops, where the long-range part of the interactions gives important contributions, while the second part contains a relatively
large number of hops, where the short-range part of the interactions is dominant.

By using the inequality $J_{k_{n-1}j} \leq 1$, we immediately obtain $J_n(i, j) \leq \lambda^{n-1}$. Therefore the second term in Eq. (6) is bounded by

$$\sum_{n=\lfloor \alpha \rfloor}^{\infty} \frac{(2\lambda t)^n}{n!} J_n(i, j) \leq \sum_{n=\lfloor \alpha \rfloor}^{\infty} \frac{(2\lambda^2 t)^n}{n! \lambda^{\alpha r - n}} < c_1 \frac{e^{vt} - 1}{e^{\mu r}},$$

(7)

where $v_1 = 2\lambda^2 e$ and $c_1 = \lambda^{-1}$. For nearest-neighbor interactions ($\alpha \to \infty$), we can simply take the limit $\mu \to 1$ so that $\lfloor \alpha \rfloor = [r]$, and the first sum in Eq. (6) vanishes as one needs at least $[r]$ hops to get from site $i$ to site $j$. Hence the bound is directly given by Eq. (7), which is the usual Lieb-Robinson bound for short-range interactions.

The first term in Eq. (6) could be bounded using Eq. (4); however, this would once again lead to a velocity that diverges with $\alpha$. The origin of this divergence is the attempt to bound repeated nearest-neighbor hops (which have unity amplitude for all $\alpha$) by a single long-range hop (whose amplitude decreases with $\alpha$). To resolve this issue, we include the contribution from nearest-neighbor hops explicitly, thereby arriving at the following modified reproducibility condition (valid for all $i, j$) [33]

$$\sum_k J_{ik} J_{kj} \leq 3^D \lambda \sum_{r_{ik} < 2} J_{ik} J_{kj}.$$

(8)

Here the notation $\sum_{r_{ik} < 2}$ implies a sum over all sites $k$ for which $r_{ik} < 2$. This bound is a major improvement over Eq. (4) used for deriving the Hastings-Koma bound, as it does not contain a coefficient that grows exponentially with $\alpha$. Applying this result iteratively in Eq. (3), we find

$$J_n(i, j) \leq (3^D \lambda)^{n-1} \sum_{r_{ik} < 2} \sum_{r_{n-2}k_{n-1} \leq 2} J_{i, k_1} \ldots J_{k_{n-1}, j}.$$

(9)

The maximum possible value for each summand is given by $(r - n + 1)^{-\alpha}$, which is achieved by combining $n - 1$ nearest-neighbor hops with one remaining hop of distance $r - n + 1$ (see Fig. 3) [40]. The number of sites with $r_{ik} < 2$ for a D-dimensional cubic lattice is bounded by $3^D$, therefore $J_n(i, j) \leq (9^D \lambda)^{n-1} (r - n + 1)^{-\alpha}$ and

$$\sum_{n=1}^{\lfloor \alpha \rfloor} \frac{(2\lambda t)^n}{n!} J_n(i, j) \leq c_2 \frac{e^{vt} - 1}{(1 - \mu r)^\alpha},$$

(10)

with $v_2 = 2\lambda^2 g D$ and $c_2 = (\lambda g D)^{-1}$. Combining Eqs. (6, 7, 10) we arrive at our bound in Eq. (1).

A key feature of our combined bound is that both velocities $c_1$ and $v_2$ actually decrease (through the implicit $\alpha$-dependence of $\lambda$) with shorter interaction range (larger $\alpha$), consistent with the expected physical picture. Note that the parameter $\mu$ can be optimized to give the best possible bound for a particular range of interactions. For small $\alpha$, $\mu \ll 1$ recovers the Hastings-Koma bound at large $r$ (since the long-range part of the bound dominates at sufficiently large $r$ for any $\mu \neq 0$), whereas for large $\alpha$, one can choose $\mu$ closer to unity in order to improve the short-range part of the bound.

Applications to experimentally realizable models— We now show that the coexistence of behavior consistent with both terms in Eq. (1) can be seen in experimentally realizable lattice spin models. We consider a 1D 1/r$^\alpha$ interacting spin-1/2 chain, with (a) XY interactions: $H_{XY} = \frac{1}{4} \sum_{i \neq j} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y)/r_{ij}^\alpha$, and (b) Ising interactions with a transverse field (TFIM): $H_{TFIM} = \frac{1}{2} \sum_{i \neq j} \sigma_i^z \sigma_j^z/r_{ij}^\alpha + B \sum_i \sigma_i^z$. Ions in a linear rf-Paul trap have already been used to simulate $H_{TFIM}$ with variable $B_2$ and $\alpha \in (0, 3)$ [22, 24, 25], and in the limit $B_2 \gg 1$ (where the total spin excitation fraction is conserved) can also simulate $H_{XY}$ [24, 25, 34]. Alternately, for $\alpha = 3$, both $H_{XY}$ and $H_{TFIM}$ can be simulated with polar molecules [13, 30, 35, 36]. In both models, we take a spin-polarized initial state $|\psi\rangle = \bigotimes_i |\sigma_i^z = -1\rangle$ and apply a local quench operator $U_0 = \frac{1}{\sqrt{2}} (1 + \sigma_0^y)$ on the 0th site. We then numerically calculate the measurable quantity $Q_r(t) \equiv \langle \psi | U_0^\dagger \sigma_r^z(t) U_0 | \psi \rangle - \langle \psi | \sigma_r^z(t) | \psi \rangle$ at a specific time $t$. For a chain with $N$ spins, we choose the time $1 \leq t \ll N$ small enough to avoid boundary effects and large enough to prohibit a perturbative treatment of the time evolution.

For the long-range XY model subjected to the stated local quench, we can restrict our attention to the single spin-excitation subspace during the entire time evolution. As a result, we can map the spin model to a solitary free particle, making numerical calculation trivial for hundreds of spins. It is also easy to show that $Q_r(t) = \| [\sigma_r^z(t), U_0] \|$, and thus the commutator norm in Eq. (1) is measurable. For $N = 501$ spins and $\alpha = 2, 3, 6, \infty$, Fig. 4(a) demonstrates that at a specific time, the distance dependence of $Q_r(t)$ can be divided into several regions: (I) $1 \leq r \leq r_{LC} \equiv v_{\text{max}} t$, where $v_{\text{max}}$ denotes the maximum group velocity of the free particle, $Q_r(t)$ increases to its maximum value at $r \approx r_{LC}$. (II) $r_{LC} \leq r < r_c$, where $Q_r(t)$ decays faster than a power law. Note that for $\alpha = 3$ and $\alpha = 6$, $Q_r(t)$ is almost unchanged by the addition of long-ranged interactions for $r < r_c$. Thus the behavior of $Q_r(t)$ in this region is
a direct consequence of nearest-neighbor interactions in the system, and is captured by the first term in Eq. (1).

(III) $r > r_c$, where $Q_r(t)$ decays algebraically as $1/r^\alpha$ due to the second term in Eq. (1). Note, however, that $Q_r(t) \approx t/r^\alpha$ (which is asymptotically exact in the limit of $t/r \to 0$ [37]) does not saturate the time dependence $\exp(v_2t) - 1$ in Eq. (1). This exponential time dependence in our bound (as well as in Ref. [8]) results from the $J_n$ in Eq. (2) adding in phase. For the $XY$ chain (periodic boundary conditions, $N = 23$, $B_z = 0.5$, $t = 1$).

In the TFIM, the long-ranged interactions prevent a mapping onto a free model, and therefore our numerical calculation is limited to a relatively small chain size ($N = 23$). Setting $B_z = 0.5$, which accentuates the role of quantum fluctuations, we calculate $Q_r(t)$ numerically for $\alpha = 2, 3, 6, \infty$ using a Krylov-subspace projection method. Figure 4(b) shows that a local quench of the TFIM yields qualitatively similar behaviors to the $XY$ model. For large $r$, we see a clear power-law decay $\sim 1/r^\alpha$. For intermediate $r$, we see hints of faster than power-law decay similar to the nearest-neighbor case.

**Bound on the propagation of correlations.**—We now use Eq. (1) to bound the spread of correlations following a global quench [24, 27]. Specifically, we consider connected correlation functions $C(t) = \langle A(t)B(t) \rangle - \langle A(t) \rangle \langle B(t) \rangle$, evaluated for initial product states, with $A$ and $B$ supported on sites $i$ and $j$, respectively.

We first define an operator $\hat{A}(t) = \exp(iHt) A \exp(-iHt)$ evolving under $H_i = \frac{1}{2} \sum_{r_{ik}, r_{il} < r} h_{kl}$, with $r = r_{ij}/2$ (similarly for $\hat{B}$). Because the supports of $\hat{A}(t)$ and $\hat{B}(t)$ are nonoverlapping, the operators $A(t) - \hat{A}(t)$ and $B(t) - \hat{B}(t)$ determine the correlation function $C(t)$, leading to

$$|C(t)| \leq 2|\|A(t) - \hat{A}(t)\|B\| + |\|B(t) - \hat{B}(t)\|A\||.$$

Because $H_i$ agrees with $H$ in the neighborhood of $i$, we expect $\hat{A}(t) - A(t)$ to be small; indeed, it is bounded as

$$\|A(t) - \hat{A}(t)\| \leq \int_0^t d\tau \sum_{r_{ik}, r_{il} \geq r} \|\hat{A}(\tau), h_{kl}\|,$$

where $h_{kl}$ are the terms in $H$ that expand the support of $\hat{A}(t)$. Applying Eq. (1) and carrying out the summations, we arrive at the following bound for correlation functions

$$|C(t)| \leq 2|A|\|B\| \left( c_3 \frac{e^{v_1 t} - 1}{e^{\mu r/2}} + c_4 \frac{e^{v_2 t} - 1}{(1 - \mu)r/2} \right)^\alpha,$$

where the constants $c_3$ and $c_4$ are finite for all $\alpha > D$ [33]. We see that, interestingly, while local-quench and global-quench dynamics obey the same form of the short-range part of the bound, the long-range part has different $r$ dependence for the two types of quenches ($1/r^\alpha$ vs. $1/r^{\alpha-D}$). The intuition behind the appearance of $1/r^{\alpha-D}$ is in the summation over $l$ in Eq. (12), which can be thought of as an effective integration of $1/r^{\alpha}$ over the $D$-dimensional cubic lattice outside the support of $\hat{A}(t)$.

**Outlook.**—In addition to being relevant to a variety of equilibrium [2–5] and short-time non-equilibrium [3–5] phenomena, we also expect the derived bound to shed light on long-time relaxation processes in quantum many-body systems [4]. It would be very interesting to try either saturate or tighten the time dependence in the long-range part of the bound, thereby proving or ruling out the possibility of quantum state transfer [38] in time $t \propto \log r$. Similarly, whether it is possible to saturate or tighten the $1/r^{\alpha-D}$ dependence for the spread of correlations after a global quench is an interesting open question deserving of further study.

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Consider first the case when $i \neq j$, then for any $\alpha > D \geq 1$,

$$\sum_{k} J_{ik}J_{kj} \leq \sum_{r_{ik} < 2} J_{ik}J_{kj} + \sum_{r_{ij} < 2} J_{ik}J_{kj} + \sum_{r_{ik} \geq 2} r_{ik}^{-\alpha}r_{kj}^{-\alpha}$$

(S1)

$$\leq 2 \sum_{r_{ik} < 2} J_{ik}J_{kj} + 2^{\alpha} r_{ij}^{-\alpha} \sum_{r_{ik} \geq 2} r_{ik}^{-\alpha}$$

(S2)
where the second inequality uses \( r_{ik}^\alpha \leq (r_{ik} + r_{kj})^\alpha \leq [(2r_{ik})^\alpha + (2r_{kj})^\alpha]/2 \). The notation \( \sum_{r_{ik} < r} \) indicates a summation over all \( b \) for which \( r_{ab} < c \). Now let \( x_\beta = 2y_\beta + w_\beta \) be the Cartesian coordinates of the vector \( \mathbf{r}_{ik} \), where \( w_\beta = 0, 1 \) for \( y_\beta \geq 1, w_\beta = 0, \pm 1 \) for \( y_\beta = 0 \), and \( w_\beta = 0, -1 \) for \( y_\beta \leq -1 \). The constraint \( r_{ik} \geq 2 \) is then equivalent to \( \sum_{\beta=1}^{D} y_\beta^2 \neq 0 \) for \( D = 1, 2, 3 \). We can now write

\[
\sum_{r_{ik} \geq 2} r_{ik}^{-\alpha} = \sum_{\beta=1}^{D} \left( \sum_{y_\beta \neq 0} \left( \sum_{\beta=1}^{D} (2y_\beta + w_\beta)^2 \right)^{-\alpha/2} \right) 
\leq 3D \left[ \sum_{\beta=1}^{D} (2y_\beta)^2 \right]^{-\alpha/2} 
= 3D 2^{-\alpha}(\lambda - 1).
\]

As a result,

\[
\sum_{k} J_{ik}J_{kj} \leq 2 \sum_{r_{ik} < 2} J_{ik}J_{kj} + J_{ij}3D(\lambda - 1) 
\leq 3D \sum_{r_{ik} < 2} J_{ik}J_{kj} + 3D(\lambda - 1) \sum_{r_{ik} < 2} J_{ik}J_{kj} 
= 3D \lambda \sum_{r_{ik} < 2} J_{ik}J_{kj}.
\]

Consider now the situation when \( i = j \), in which case we have \( \sum_{k} J_{ik}^2 \leq \sum_{k} J_{ik} < 3D \lambda \sum_{r_{ik} < 2} J_{ik}^2 \). Therefore, Eq. (9) of the main text holds for any \( i \) and \( j \).

## S2. PROOF OF EQS. (12-14) IN THE MAIN TEXT

For an initial product state, we have \( \langle \tilde{A}(t)\tilde{B}(t) \rangle = \langle \tilde{A}(t) \| \tilde{B}(t) \rangle \). Eq. (12) in the main text then follows from

\[
|C(t)| = |\langle [A(t) - \tilde{A}(t)]B(t) + \langle \tilde{A}(t)B(t) - \tilde{B}(t) \rangle - \langle A(t) - \tilde{A}(t) \rangle \| \tilde{B}(t) \rangle - \langle A(t) \| B(t) - \tilde{B}(t) \rangle |,
\]

and Eq. (13) follows immediately from the inequality

\[
\| A(t) - \tilde{A}(t) \| = \| \int_{0}^{t} d\tau \frac{d}{d\tau} (e^{-iH_\tau} e^{iH_\tau} A e^{-iH_\tau} e^{iH_\tau}) \| \leq \int_{0}^{t} d\tau \| [A(\tau), H - H_\tau] \|.
\]

To prove Eq. (14), we first note the bound given by Eq. (1) in the main text can be generalized to operators supported on more than one site by using the shortest distance between the supports of the two operators, and by multiplying the bound by sizes (the number of sites) of the two supports [S1]. Thus, using our bound Eq. (1) in the main text, for \( i \neq k \) we obtain

\[
\| [\tilde{A}(\tau), h_{kl}] \| \leq 4 \| A \| \| h_{kl} \| \left( \frac{c_1 e^{\mu r_{ik}} + c_2 e^{\nu_{2\tau}}}{e^{\mu r_{ik}} + \sum_{\mu < r_{ik} < \tau} r_{ik}^{-\alpha} (1 - \mu) r_{ik}^{-\alpha}} \right).
\]

The case of \( i = k \) can be bounded by the first term of Eq. (S11) if we replace \( c_1 = 1/\mu \) with 1. Therefore,

\[
\sum_{r_{ik} < \tau} \left( \frac{[\tilde{A}(\tau), h_{kl}]}{4 \| A \|} \right) \leq \sum_{r_{ik} < \tau} r_{ik}^{-\alpha} c_1 e^{\mu r_{ik}} + \sum_{\mu < r_{ik} < \tau} r_{ik}^{-\alpha} \left( \frac{c_2 e^{\nu_{2\tau}}}{(1 - \mu) r_{ik}^{-\alpha}} \right).
\]

The second sum above can be easily bounded using Eq. (4) in the main text, giving

\[
\sum_{0 < r_{ik} < \tau} r_{ik}^{-\alpha} \left( \frac{c_2 e^{\nu_{2\tau}}}{(1 - \mu) r_{ik}^{-\alpha}} \right) \leq \frac{c_2 e^{\nu_{2\tau}}}{(1 - \mu)^{2\alpha}} \sum_{r_{il} \leq \tau} r_{il}^{-\alpha} \leq \frac{2}{9(1 - \mu)^{1/D} \lambda^{1/2}} e^{\nu_{2\tau}},
\]

\[ \leq \frac{2}{9(1 - \mu)^{1/D} \lambda^{1/2}} e^{\nu_{2\tau}}, \]
where we have used the inequality $\sum_{r_{ik} \geq r} r_{il}^{-\alpha} \left( r_{il} - \frac{1+\mu}{2} r \right)^{-\alpha} \leq \lambda_1/r^{\alpha-D}$ for some $D$-dependent constant $\lambda_1$. To bound the first sum in Eq. (S12), we note that for $r_{ik} \leq \frac{1+\mu}{2} r$,

$$\sum_{r_{ik} \leq \frac{1+\mu}{2} r} r_{kl}^{-\alpha} e^{-\mu r_{ik}} \leq \sum_{r_{il} \geq r} (r_{il} - \frac{1+\mu}{2} r)^{-\alpha} \sum_k e^{-\mu r_{ik}}$$  \hspace{1cm} (S15)

\[ \leq \frac{\lambda_2}{[(1-\mu)r/2]^{\alpha-D}} \zeta_1, \]  \hspace{1cm} (S16)

where we have used the fact that $\sum_{r_{il} \geq r} (r_{il} - \frac{1+\mu}{2} r)^{-\alpha} \leq \lambda_2/[(1-\mu)r/2]^{\alpha-D}$ for some constant $\lambda_2$, and have defined $\zeta_1 \equiv \sum_k e^{-\mu r_{ik}}$. For $r_{ik} > \frac{1+\mu}{2} r$, the inequality $\sum_{r_{ik} > \frac{1+\mu}{2} r} e^{-\mu r_{ik}} \leq \zeta_2 e^{-\mu r/2}$ holds for some constant $\zeta_2$, and thus

$$\sum_{\frac{1+\mu}{2} r < r_{ik} < r} r_{kl}^{-\alpha} e^{-\mu r_{ik}} \leq \sum_{r_{ik} > \frac{1+\mu}{2} r} e^{-\mu r_{ik}} \sum_{r_{il} > 0} r_{kl}^{-\alpha}$$  \hspace{1cm} (S17)

\[ \leq \lambda_2 \zeta_2 e^{-\mu r/2}. \]  \hspace{1cm} (S18)

Combining Eqs. (S12,S14,S16,S18) and noting that $v_2 > v_1$, we obtain

$$\sum_{r_{ik} < r \atop r_{il} \geq r} \frac{\|A(\tau),h_{kl}\|}{4 \|A\|} \leq \lambda_2 e^{v_2 r-\mu r/2} + (\lambda_2 \zeta_1 + \frac{2}{9(1-\mu)} D \lambda_1) \frac{e^{v_2 r}}{[(1-\mu)r/2]^{\alpha-D}}.$$  \hspace{1cm} (S19)

As the above inequality has exactly the same form when replacing the operator $A$ by the operator $B$, one can readily obtain Eq. (13) in the main text by defining

$$c_3 \equiv 8\lambda_2 / v_1 \quad \text{and} \quad c_4 \equiv 8(\lambda_2 \zeta_1 + \lambda_1 \frac{2}{9(1-\mu)} D) / v_2.$$  \hspace{1cm} (S20)

Note that $\lambda_1, \lambda_2$ are $D$ dependent constants and $\zeta_1, \zeta_2$ both depend on $\mu$ and $D$. For $D = 1$, they have the simple explicit forms

$$\lambda_1 = \lambda_2 = \lambda \quad \text{and} \quad \zeta_1 = \zeta_2 = \coth(\mu/2).$$  \hspace{1cm} (S21)

[S1] M. Hastings and T. Koma, Comm. Math. Phys. 265, 781 (2006).