Discrete Convex Functions on Graphs and Their Algorithmic Applications

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Abstract
The present article is an exposition of a theory of discrete convex functions on certain graph structures, developed by the author in recent years. This theory is a spin-off of discrete convex analysis by Murota, and is motivated by combinatorial dualities in multiflow problems and the complexity classification of facility location problems on graphs. We outline the theory and algorithmic applications in combinatorial optimization problems.

1 Introduction
The present article is an exposition of a theory of discrete convex functions on certain graph structures, developed by the author in recent years. This theory is viewed as a spin-off of Discrete Convex Analysis (DCA), which is a theory of convex functions on the integer lattice and has been developed by Murota and his collaborators in the last 20 years; see [43, 44, 45] and [12, Chapter VII]. Whereas the main targets of DCA are matroid-related optimization (or submodular optimization), our new theory is motivated by combinatorial dualities arising from multiflow problems [23] and the complexity classification of certain facility location problems on graphs [27].

The heart of our theory is analogues of $L^\natural$-convex functions [10, 16] for certain graph structures, where $L^\natural$-convex functions are one of the fundamental classes of discrete convex functions on $\mathbb{Z}^n$, and play primary roles in DCA. These analogues are inspired by the following intriguing properties of $L^\natural$-convex functions:

- An $L^\natural$-convex function is (equivalently) defined as a function $g$ on $\mathbb{Z}^n$ satisfying a discrete version of the convexity inequality, called the discrete midpoint convexity:

$$g(x) + g(y) \geq g([x+y]/2) + g([(x+y)/2]) \quad (x, y \in \mathbb{Z}^n),$$

(1)

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are operators rounding down and up, respectively, the fractional part of each component.
L♮-convex function \( g \) is locally submodular in the following sense: For each \( x \in \mathbb{Z}^n \), the function on \( \{0, 1\}^n \) defined by \( u \mapsto g(x + u) \) is submodular.

Analogous to ordinary convex functions, L♮-convex function \( g \) enjoys an optimality criterion of a local-to-global type: If \( x \) is not a minimizer of \( g \), then there exists \( y \in (x + \{0, 1\}^n) \cup (x - \{0, 1\}^n) \) with \( g(y) < g(x) \).

This leads us to a conceptually-simple minimization algorithm, called the steeped descent algorithm (SDA): For \( x \in \mathbb{Z}^n \), find a (local) minimizer \( y \) of \( g \) over \( (x + \{0, 1\}^n) \cup (x - \{0, 1\}^n) \) via submodular function minimization (SFM). If \( g(y) = g(x) \), then \( x \) is a (global) minimizer of \( g \). If \( g(y) < g(x) \), then let \( x := y \), and repeat.

The number of iterations of SDA is sharply bounded by a certain \( l_\infty \)-distance between the initial point and minimizers [46].

L♮-convex function \( g \) is extended to a convex function \( \overline{g} \) on \( \mathbb{R}^n \) via the Lovász extension, and this convexity property characterizes the L♮-convexity.

Details are given in [44], and a recent survey [49] is also a good source of L♮-convex functions.

We consider certain classes of graphs \( \Gamma \) that canonically define functions (on the vertex set of \( \Gamma \)) having analogous properties, which we call L-convex functions on \( \Gamma \) (with \( \natural \) omitted). The aim of this paper is to explain these L-convex functions and their roles in combinatorial optimization problems, and to demonstrate the SDA-based algorithm design.

Our theory is parallel with recent developments in generalized submodularity and valued constraint satisfaction problem (VCSP) [39, 51, 54]. Indeed, localizations of these L-convex functions give rise to a rich subclass of generalized submodular functions that include \( k \)-submodular functions [29] and submodular functions on diamonds [15, 41], and are polynomially minimizable in the VCSP setting.

The starting point is the observation that if \( \mathbb{Z}^n \) is viewed as a grid graph (with order information), some of the above properties of L♮-convex functions are still well-formulated. Indeed, extending the discrete midpoint convexity to trees, Kolmogorov [38] introduced a class of discrete convex functions, called tree-submodular functions, on the product of rooted trees, and showed several analogous results. In Section 2, we discuss L-convex functions on such grid-like structures. We start by considering a variation of tree-submodular functions, where the underlying graph is the product of zigzagly-oriented trees. Then we explain that a theory of L-convex functions is naturally developed on a structure, known as Euclidean building [1], which is a kind of an amalgamation of \( \mathbb{Z}^n \) and is a far-reaching generalization of a tree. Applications of these L-convex functions are given in Section 3. We outline SDA-based efficient combinatorial algorithms for two important multiflow problems. In Section 4, we explain L-convex functions on a more general class of graphs, called oriented modular graphs. This graph class emerged from the complexity classification of the minimum 0-extension problem [34]. We outline that the theory of L-convex functions leads to a solution of this classification problem. This was the original motivation of our theory.

The contents of this paper are based on the results in papers [25, 26, 27, 28], in which further details and omitted proofs are found.

Let \( \mathbb{R} \), \( \mathbb{R}^+ \), \( \mathbb{Z} \), and \( \mathbb{Z}^+ \) denote the sets of reals, nonnegative reals, integers, and nonnegative integers, respectively. In this paper, \( \mathbb{Z} \) is often regarded as an infinite path.
obtained by adding an edge to each consecutive pair of integers. Let \( \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\} \), where \( \infty \) is the infinity element treated as \( a < \infty, a + \infty = \infty \) for \( a \in \mathbb{R} \), and \( \infty + \infty = \infty \). For a function \( g : X \rightarrow \overline{\mathbb{R}} \) on a set \( X \), let \( \text{dom} \ g := \{x \in X \mid g(x) < \infty\} \).

## 2 L-convex function on grid-like structure

In this section, we discuss L-convex functions on grid-like structures. In the first two subsections (Sections 2.1 and 2.2), we consider specific underlying graphs (tree-product and twisted tree-product) that admit analogues of discrete midpoint operators and the corresponding L-convex functions. In both cases, the local submodularity and the local-to-global optimality criterion are formulated in a straightforward way (Lemmas 2.1, 2.2, 2.3, and 2.4). In Section 2.3, we introduce the steepest descent algorithm (SDA) in a generic way, and present the iteration bound (Theorem 2.5). In Section 2.4, we explain that the L-convexity is naturally generalized to that in Euclidean buildings of type C, and that the Lovász extension theorem can be generalized via the geometric realization of Euclidean buildings and CAT(0)-metrics (Theorem 2.6).

We use a standard terminology on posets (partially ordered sets) and lattices; see e.g., [21]. Let \( P = (P, \preceq) \) be a poset. For an element \( p \in P \), the principal ideal \( I_p \) is the set of elements \( q \in P \) with \( q \preceq p \), and the principal filter \( F_p \) is the set of elements \( q \in P \) with \( p \preceq q \). For \( p, q \in P \) with \( p \preceq q \), let \( \max\{p, q\} := q \) and \( \min\{p, q\} := p \). A chain is a subset \( X \subseteq P \) such that for every \( p, q \in X \) it holds that \( p \preceq q \) or \( q \preceq p \). For \( p \preceq q \), the interval \([p, q]\) of \( p, q \) is defined as \([p, q] := \{u \in P \mid p \preceq u \preceq q\} \). For two posets \( P, P' \), the direct product \( P \times P' \) becomes a poset by the direct product order \( \preceq \) defined by \((p, p') \preceq (q, q')\) if and only if \( p \preceq q \) in \( P \) and \( p' \preceq q' \) in \( P' \).

For an undirected graph \( G \), an edge joining vertices \( x \) and \( y \) is denoted by \( xy \). When \( G \) plays a role of the domain of discrete convex functions, the vertex set \( V(G) \) of \( G \) is also denoted by \( G \). Let \( d = d_G \) denote the shortest path metric on \( G \). We often endow \( G \) with an edge-orientation, where \( x \rightarrow y \) means that edge \( xy \) is oriented from \( x \) to \( y \). For two graphs \( G \) and \( H \), let \( G \times H \) denote the Cartesian product of \( G \) and \( H \), i.e., the vertices are all pairs of vertices of \( G \) and \( H \) and two vertices \((x, y)\) and \((x', y')\) are adjacent if and only if \( d_G(x, x') + d_H(y, y') = 1 \).

### 2.1 L-convex function on tree-grid

Let \( G \) be a tree. Let \( B \) and \( W \) denote the color classes of \( G \) viewed as a bipartite graph. Endow \( G \) with a zigzag orientation so that \( u \rightarrow v \) if and only if \( u \in W \) and \( v \in B \). This orientation is acyclic. The induced partial order on \( G \) is denoted by \( \preceq \), where \( v \leftarrow u \) is interpreted as \( v \preceq u \).

Discrete midpoint operators \( \bullet \) and \( \circ \) on \( G \) are defined as follows. For vertices \( u, v \in G \), there uniquely exists a pair \((a, b)\) of vertices such that \( d(u, v) = d(u, a) + d(a, b) + d(b, v) \), \( d(a, a) = d(b, b) \), and \( d(a, b) \leq 1 \). In particular, \( a \) and \( b \) are equal or adjacent, and hence comparable. Let \( u \bullet v := \min\{a, b\} \) and \( u \circ v := \max\{a, b\} \).

Consider the product \( G^n \) of \( G \); see Figure 1 for \( G^2 \). The operators \( \bullet \) and \( \circ \) are extended on \( G^n \) component-wise: For \( x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in G^n \), let \( x \bullet y \) and \( x \circ y \) be defined by

\[
x \bullet y := (x_1 \bullet y_1, x_2 \bullet y_2, \ldots, x_n \bullet y_n), \quad x \circ y := (x_1 \circ y_1, x_2 \circ y_2, \ldots, x_n \circ y_n).
\]
Mimicking the discrete midpoint convexity [1], a function $g : G^n \rightarrow \mathbb{R}$ is called $L$-convex (or alternating $L$-convex [25]) if it satisfies
\begin{equation}
g(x) + g(y) \geq g(x \cdot y) + g(x \circ y) \quad (x, y \in G^n).
\end{equation}
By this definition, a local-to-global optimality criterion is obtained in a straightforward way. Recall the notation that $I_x$ and $F_x$ are the principal ideal and filter of vertex $x \in G^n$.

**Lemma 2.1** ([25]). Let $g : G^n \rightarrow \mathbb{R}$ be an $L$-convex function. If $x \in \text{dom} g$ is not a minimizer of $g$, there exists $y \in I_x \cup F_x$ with $g(y) < g(x)$.

**Proof.** Suppose that $x \in \text{dom} g$ is not a minimizer of $g$. There is $z \in \text{dom} g$ such that $g(z) < g(x)$. Choose such $z$ with minimum $\max_i d(x_i, z_i)$. By $g(x) + g(z) \geq g(x \cdot z) + g(x \circ z)$, it necessarily holds that $\max_i d(x_i, z_i) \leq 1$, and one of $x \cdot z \in I_x$ and $x \circ z \in F_x$ has a smaller value of $g$ than $x$.

This lemma says that $I_x$ and $F_x$ are “neighborhoods” of $x$ for which the local optimality is defined. This motivates us to consider the class of functions appearing as the restrictions of $g$ to $I_x$ and $F_x$ for $x \in G^n$, which we call the localizations of $g$. The localizations of $L$-convex functions give rise to a class of submodular-type discrete convex functions known as $k$-submodular functions [29]. To explain this fact, we introduce a class of semilattices isomorphic to $I_x$ or $F_x$.

For a nonnegative integer $k$, let $S_k$ denote a $(k+1)$-element set with a special element 0. Define a partial order $\preceq$ on $S_k$ by $0 \preceq u$ for $u \in S_k \setminus \{0\}$ with no other relations. Let $\sqcup$ and $\sqcap$ be binary operations on $S_k$ defined by
\begin{align*}
u \sqcap v &:= \begin{cases} 
\min\{u, v\} & \text{if } u \preceq v \text{ or } v \preceq u,
0 & \text{otherwise},
\end{cases} \\
u \sqcup v &:= \begin{cases} 
\max\{u, v\} & \text{if } u \preceq v \text{ or } v \preceq u,
0 & \text{otherwise}.
\end{cases}
\end{align*}
For an \( n \)-tuple \( k = (k_1, k_2, \ldots, k_n) \) of nonnegative integers, let \( S_k := S_{k_1} \times S_{k_2} \times \cdots \times S_{k_n} \). A function \( f : S_k \rightarrow \overline{\mathbb{R}} \) is \( k \)-submodular if it satisfies

\[
f(x) + f(y) \geq f(x \cap y) + f(x \cup y) \quad (x, y \in S_k),
\]

where operators \( \cap \) and \( \cup \) are extended to \( S_k \) component-wise. Then \( k \)-submodular functions are identical to submodular functions if \( k = (1, 1, \ldots, 1) \) and to bisubmodular functions if \( k = (2, 2, \ldots, 2) \).

Let us return to tree-product \( G^n \). For every point \( x \in G^n \), the principal filter \( F_x \) is isomorphic to \( S_k \) for \( k = (k_1, k_2, \ldots, k_n) \), where \( k_i = 0 \) if \( x_i \in W \) and \( k_i \) is equal to the degree of \( x_i \) in \( G \) if \( x_i \in B \). Similarly for the principal ideal \( I_x \) (with partial order reversed).

Observe that operations \( \bullet \) and \( \circ \) coincides with \( \cap \) and \( \cup \) (resp. \( \sqcap \) and \( \sqcup \)) in any principle filter (resp. ideal). Then an L-convex function on \( G^n \) is locally \( k \)-submodular in the following sense.

**Lemma 2.2** \([25]\). An L-convex function \( g : G^n \rightarrow \overline{\mathbb{R}} \) is \( k \)-submodular on \( F_x \) and on \( I_x \) for every \( x \in \text{dom} \, g \).

In particular, an L-convex function on a tree-grid can be minimized via successive \( k \)-submodular function minimizations; see Section 2.3.

### 2.2 L-convex function on twisted tree-grid

Next we consider a variant of a tree-grid, which is obtained by **twisting** the product of two trees, and by taking the product. Let \( G \) and \( H \) be infinite trees without vertices of degree one. Consider the product \( G \times H \), which is also a bipartite graph. Let \( B \) and \( W \) denote the color classes of \( G \times H \). For each 4-cycle \( C \) of \( G \times H \), add a new vertex \( w_C \) and new four edges joining \( w_C \) and vertices in \( C \). Delete all original edges of \( G \times H \), i.e., all edges non-incident to new vertices. The resulting graph is denoted by \( G \boxtimes H \). Endow \( G \boxtimes H \) with an edge-orientation such that \( x \rightarrow y \) if and only if \( x \in W \) or \( y \in B \). Let \( \preceq \) denote the induced partial order on \( G \boxtimes H \). See Figure 2 for \( G \boxtimes H \).

Discrete midpoint operators \( \bullet \) and \( \circ \) are defined as follows. Consider two vertices \( x, y \) in \( G \boxtimes H \). Then \( x \in C \) or \( x = w_C \) for some 4-cycle \( C \) in \( G \times H \). Similarly, \( y \in D \) or \( y = w_D \) for some 4-cycle \( D \) in \( G \times H \). There are infinite paths \( P \) in \( G \) and \( Q \) in \( H \) such that \( P \times Q \) contains both \( C \) and \( D \). Indeed, consider the projections of \( C \) and \( D \) to \( G \), which are edges. Since \( G \) is a tree, we can choose \( P \) as an infinite path in \( G \) containing both of them. Similarly for \( Q \). Now the subgraph \( P \boxtimes Q \) of \( G \boxtimes H \) coincides with (45-degree rotation of) the product of two paths with the zigzag orientation as in Section 2.1. Thus, in \( P \boxtimes Q \), the discrete midpoint points \( x \bullet y \) and \( x \circ y \) are defined. They are determined independently of the choice of \( P \) and \( Q \).

Consider the \( n \)-product \( (G \boxtimes H)^n \) of \( G \boxtimes H \), which is called a **twisted tree-grid**. Similarly to the previous case, an L-convex function on \( (G \boxtimes H)^n \) is defined as a function \( g : (G \boxtimes H)^n \rightarrow \overline{\mathbb{R}} \) satisfying

\[
g(x) + g(y) \geq g(x \bullet y) + g(x \circ y) \quad (x, y \in (G \boxtimes H)^n),
\]

where operations \( \bullet \) and \( \circ \) are extended on \( (G \boxtimes H)^n \) component-wise as before. Again the following holds, where the proof is exactly the same as Lemma 2.1.
Figure 2: Twisted tree-grid $G \boxtimes H$. Dotted lines represent edges of $G \times H$. Black and white (round) points represent vertices in $B$ and $W$, respectively, while square points correspond to 4-cycles in $G \times H$. The principal filter of a black vertex is picked out to the right, which is a semilattice isomorphic to $S_{2,3}$.

Lemma 2.3. Let $g : (G \boxtimes H)^n \to \overline{\mathbb{R}}$ be an L-convex function. If $x \in \text{dom } g$ is not a minimizer of $g$, there is $y \in I_x \cup F_x$ with $g(y) < g(x)$.

As before, we study localizations of an L-convex function on $(G \boxtimes H)^n$. This naturally leads to further generalizations of $k$-submodular functions. For positive integers $k, l$, consider product $S_k \times S_l$ of $S_k$ and $S_l$. Define a partial order $\leq'$ on $S_k \times S_l$ by $(0, 0) \leq' (a, b)$ and $(a, b) \leq' (0, b)$ for $(a, b) \in S_k \times S_l$ with $a \neq 0$ and $b \neq 0$. The resulting poset is denoted by $S_{k,l}$; note that $\leq'$ is different from the direct product order.

The operations $\sqcup$ and $\sqcap$ on $S_{k,l}$ are defined as follows. For $p, q \in S_{k,l}$, there are $a, a' \in S_k \setminus \{0\}$ and $b, b' \in S_l \setminus \{0\}$ such that $p, q \in \{0, a, a'\} \times \{0, b, b'\}$. The restriction of $S_{k,l}$ to $\{0, a, a'\} \times \{0, b, b'\}$ is isomorphic to $S_2 \times S_2 = \{-1, 0, 1\}^2$ (with order $\preceq$), where an isomorphism $\varphi$ is given by

$$(a, 0) \mapsto (0, 0),$$
$$(a, b) \mapsto (1, 0), (a', b) \mapsto (0, -1), (a, b') \mapsto (0, 1), (a', b') \mapsto (-1, 0),$$
$$(a, 0) \mapsto (1, 1), (a', 0) \mapsto (-1, -1), (0, b) \mapsto (1, -1), (0, b') \mapsto (-1, 1).$$

Then $p \sqcup q := \varphi^{-1}(\varphi(p) \sqcup \varphi(q))$ and $p \sqcap q := \varphi^{-1}(\varphi(p) \sqcap \varphi(q))$. Notice that they are determined independently of the choice of $a, a', b, b'$ and $\varphi$.

For a pair of $n$-tuples $k = (k_1, k_2, \ldots, k_n)$ and $l = (l_1, l_2, \ldots, l_n)$, let $S_{k,l} := S_{k_1,l_1} \times S_{k_2,l_2} \times \cdots \times S_{k_n,l_n}$. A function $f : S_{k,l} \to \overline{\mathbb{R}}$ is $(k, l)$-submodular if it satisfies

$$f(x) + f(y) \geq f(x \sqcap y) + f(x \sqcup y) \quad (x, y \in S_{k,l}). \quad (6)$$

Poset $S_{k,l}$ contains $S_{0,l} \simeq S_l$ and diamonds (i.e., a modular lattice of rank 2) as $(\sqcap, \sqcup)$-closed sets. Thus $(k, l)$-submodular functions can be viewed as a common generalization of $k$-submodular functions and submodular functions on diamonds.
Both the principal ideal and filter of each vertex in \((G \boxtimes H)^n\) are isomorphic to \(S_{k,l}\) for some \(k, l\), in which \(\{\bullet, \circ\}\) are equal to \(\{\cap, \cup\}\). Thus we have:

**Lemma 2.4.** An L-convex function \(g : (G \boxtimes H)^n \rightarrow \overline{\mathbb{R}}\) is \((k, l)\)-submodular on \(I_x\) and on \(F_x\) for every \(x \in \text{dom } g\).

### 2.3 Steepest descent algorithm

The two classes of L-convex functions in the previous subsection can be minimized by the same principle, analogous to the steepest descent algorithm for \(L^\flat\)-convex functions. Let \(\Gamma\) be a tree-product \(G^n\) or a twisted tree-product \((G \boxtimes H)^n\). We here consider L-convex functions \(g\) on \(\Gamma\) such that a minimizer of \(g\) exists.

**Steepest Descent Algorithm (SDA)**

**Input:** An L-convex function \(g : \Gamma \rightarrow \overline{\mathbb{R}}\) and an initial point \(x \in \text{dom } g\).

**Output:** A minimizer \(x\) of \(g\).

**Step 1:** Find a minimizer \(y\) of \(g\) over \(I_x \cup F_x\).

**Step 2:** If \(g(x) = g(y)\), then output \(x\) and stop; \(x\) is a minimizer.

**Step 3:** Let \(x := y\), and go to step 1.

The correctness of this algorithm follows immediately from Lemmas 2.1 and 2.3. A minimizer \(y\) in Step 1 is particularly called a **steepest direction** at \(x\). Notice that a steepest direction is obtained by minimizing \(g\) over \(I_x\) and over \(F_x\), which are \(k\)- or \((k, l)\)-submodular function minimization by Lemmas 2.2 and 2.4.

Besides its conceptual simplicity, there remain two issues in applying SDA to specific combinatorial optimization problems:

- **How to minimize** \(g\) **over** \(I_x\) **and over** \(F_x\).
- **How to estimate the number of iterations.**

We first discuss the second issue. In fact, there is a surprisingly simple and sharp iteration bound, analogous to the case of \(L^\flat\)-convex functions \([16]\). If \(\Gamma\) is a tree-grid \(G^n\), then define \(\Gamma^\Delta\) as the graph obtained from \(\Gamma\) by adding an edge to each pair of (distinct) vertices \(x, y\) with \(d(x_i, y_i) \leq 1\) for \(i = 1, 2, \ldots, n\). If \(\Gamma\) is a twisted tree-grid \((G \boxtimes H)^n\), then define \(\Gamma^\Delta\) as the graph obtained from \(\Gamma\) by adding an edge to each pair of (distinct) vertices \(x, y\) such that \(x_i\) and \(y_i\) belong to a common 4-cycle in \(G \boxtimes H\) for each \(i = 1, 2, \ldots, n\). Let \(d_\Delta := d_{\Gamma^\Delta}\).

Observe that \(x\) and \(y \in I_x \cup F_x\) are adjacent in \(\Gamma^\Delta\). Hence the number of iterations of SDA is at least the minimum distance \(d_\Delta(x, \text{opt}(g)) := \min\{d_\Delta(x, y) \mid y \in \text{opt}(g)\}\) from the initial point \(x\) and the minimizer set \(\text{opt}(g)\) of \(g\). This lower bound is almost tight.

**Theorem 2.5** \(([25, 26])\). The number of the iterations of SDA applied to L-convex function \(g\) and initial point \(x \in \text{dom } g\) is at most \(d_\Delta(x, \text{opt}(g)) + 2\).

For specific problems (considered in Section 3), the upper bound of \(d_\Delta(x, \text{opt}(g))\) is relatively easier to be estimated.

Thus we concentrate only on the first issue. Minimizing \(g\) over \(I_x\) and \(F_x\) is a \(k\)-submodular function minimization if \(\Gamma\) is a tree-grid and is a \((k, l)\)-submodular function.
minimization if $\Gamma$ is a twisted tree-grid. Currently no polynomial time algorithm is known for $k$-submodular function minimization under the oracle model. However, under several important special cases (including VCSP model), the above submodular functions can be minimizable in polynomial time, and SDA is implementable; see Section 4. Moreover, for further special classes of $k$- or $(k,l)$-submodular functions arising from our target problems in Section 3, a fast and efficient minimization via network or submodular flows is possible, and hence the SDA framework brings efficient combinatorial polynomial time algorithms.

2.4 L-convex function on Euclidean building

The arguments in the previous sections are naturally generalized to the structures known as spherical and Euclidean buildings of type $C$; see [4, 52] for the theory of buildings. We here explain a building-theoretic approach to L-convexity and submodularity. We start with a local theory; we introduce a polar space and submodular functions on it. Polar spaces are equivalent to spherical buildings of type $C$ [52], and generalize domains $S_k$ and $S_{k,l}$ for $k$-submodular and $(k,l)$-submodular functions.

A polar space $L$ of rank $n$ is defined as a poset endowed with a system of subposets, called polar frames, satisfying the following axioms:

P0: Each polar frame is isomorphic to $S_2^n$.

P1: For two chains $C, D$ in $L$, there is a polar frame $F$ containing them.

P2: If polar frames $F, F'$ both contain two chains $C, D$, then there is an isomorphism $F \to F'$ being identity on $C$ and $D$.

Namely a polar space is viewed as an amalgamation of several $S_2^n$. Observe that $S_k$ and $S_{k,l}$ (with $k, l \geq 2$) are polar spaces, and so are their products.

Operators $\sqcap$ and $\sqcup$ on polar space $L$ are defined as follows. For $x, y \in L$, consider a polar frame $F \simeq S_2^n$ containing $x, y$ (via P1), and $x \sqcap y$ and $x \sqcup y$ can be defined in $F$. One can derive from the axioms that $x \sqcap y$ and $x \sqcup y$ are determined independently of the choice of a polar frame $F$. Thus operators $\sqcap$ and $\sqcup$ are well-defined.

A submodular function on a polar space $L$ is a function $f : L \to \mathbb{R}$ satisfying

$$f(x) + f(y) \geq f(x \sqcap y) + f(x \sqcup y) \quad (x, y \in L). \quad (7)$$

Equivalently, a submodular function on a polar space is a function being bisubmodular on each polar frame.

Next we introduce a Euclidean building (of type $C$) and L-convex functions on it. A Euclidean building is simply defined from the above axioms by replacing $S_2^n$ by $\mathbb{Z}^n$, where $\mathbb{Z}$ is zigzagly ordered as

$$\cdots \succ -2 \prec -1 \succ 0 \prec 1 \succ 2 \prec \cdots.$$ 

Namely a Euclidean building of rank $n$ is a poset $\Gamma$ endowed with a system of subposets, called apartments, satisfying:

B0: Each apartment is isomorphic to $\mathbb{Z}^n$.

B1: For any two chains $A, B$ in $\Gamma$, there is an apartment $\Sigma$ containing them.
B2: If $\Sigma$ and $\Sigma'$ are apartments containing two chains $A, B$, then there is an isomorphism $\Sigma \rightarrow \Sigma'$ being identity on $A$ and $B$.

A tree-product $G^n$ and twisted tree-product $(G \boxtimes H)^n$ are Euclidean buildings (of rank $n$ and $2n$, respectively). In the latter case, apartments are given by $(P_1 \boxtimes Q_1) \times (P_2 \boxtimes Q_2) \times \cdots \times (P_n \boxtimes Q_n) \simeq \mathbb{Z}^{2n}$ for infinite paths $P_i$ in $G$ and $Q_i$ in $H$ for $i = 1, 2, \ldots, n$.

The discrete midpoint operators are defined as follows. For two vertices $x, y \in \Gamma$, choose an apartment $\Sigma$ containing $x, y$. Via isomorphism $\Sigma \simeq \mathbb{Z}^n$, discrete midpoints $x \bullet y$ and $x \circ y$ are defined as in the previous section. Again, $x \bullet y$ and $x \circ y$ are independent of the choice of apartments.

An L-convex function on $\Gamma$ is a function $g : \Gamma \rightarrow \mathbb{R}$ satisfying

$$g(x) + g(y) \geq g(x \bullet y) + g(x \circ y) \quad (x, y \in \Gamma).$$

Observe that each principal ideal and filter of $\Gamma$ are polar spaces. Then the previous Lemmas 2.1, 2.2, 2.3, 2.4, and Theorem 2.5 are generalized as follows:

- An L-convex function $g : \Gamma \rightarrow \mathbb{R}$ is submodular on polar spaces $I_x$ and $F_x$ for every $x \in \Gamma$.
- If $x$ is not a minimizer of $g$, then there is $y \in I_x \cup F_x$ with $g(y) < g(x)$. In particular, the steepest descent algorithm (SDA) is well-defined, and correctly obtains a minimizer of $g$ (if it exists).
- The number of iterations of SDA for $g$ and initial point $x \in \text{dom } g$ is bounded by $d_\Delta(x, \text{opt}(g)) + 2$, where $d_\Delta(y, z)$ is the $l_\infty$-distance between $y$ and $z$ in the apartment $\Sigma \simeq \mathbb{Z}^n$ containing $y, z$.

Next we discuss a convexity aspect of L-convex functions. Analogously to $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$, there is a continuous metric space $K(\Gamma)$ into which $\Gamma$ is embedded. Accordingly, any function $g$ on $\Gamma$ is extended to a function $\tilde{g}$ on $K(\Gamma)$, which is an analogue of the Lovász extension. It turns out that the L-convexity of $g$ is equivalent to the convexity of $\tilde{g}$ with respect to the metric on $K(\Gamma)$.

We explain this fact more precisely. Let $K(\Gamma)$ be the set of formal sums

$$\sum_{p \in \Gamma} \lambda(p)p$$

of vertices in $\Gamma$ for $\lambda : \Gamma \rightarrow \mathbb{R}_+$ satisfying that $\{p \in \Gamma \mid \lambda(p) \neq 0\}$ is a chain and $\sum_p \lambda(p) = 1$. For a chain $C$, the subset of form $\sum_{p \in C} \lambda(p)p$ is called a simplex. For a function $g : \Gamma \rightarrow \mathbb{R}$, the Lovász extension $\tilde{g} : K(\Gamma) \rightarrow \mathbb{R}$ is defined by

$$\tilde{g}(x) := \sum_{p \in \Gamma} \lambda(p)g(p) \quad \left( x = \sum_{p \in \Gamma} \lambda(p)p \in K(\Gamma) \right).$$

The space $K(\Gamma)$ is endowed with a natural metric. For an apartment $\Sigma$, the subcomplex $K(\Sigma)$ is isomorphic to a simplicial subdivision of $\mathbb{R}^n$ into simplices of vertices $z, z + s_{i_1}, z + s_{i_1} + s_{i_2}, \ldots, z + s_{i_1} + s_{i_2} + \cdots + s_{i_n}$ for all even integer vectors $z$, permutations $(i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$ and $s_i \in \{e_i, -e_i\}$ for $i = 1, 2, \ldots, n$, where $e_i$ denotes the $i$-th unit vector. Therefore one can metrize $K(\Gamma)$.
Figure 3: The space $K(G \boxtimes H)$, which is constructed from $G \boxtimes H$ by filling Euclidean right triangle to each chain of length two. The Lovász extension is a piecewise interpolation with respect to this triangulation. Each apartment is isometric to the Euclidean plane, in which geodesics are line segments. The midpoint $(x + y)/2$ of vertices $x, y$ lies on an edge (1-dimensional simplex). Discrete midpoints $x \circ y$ and $x \bullet y$ are obtained by rounding $(x + y)/2$ to the ends of the edge with respect to the partial order.

so that, for each apartment $\Sigma$, the subcomplex $K(\Sigma)$ is an isometric subspace of $K(\Gamma)$ and is isometric to the Euclidean space $(\mathbb{R}^n, l_2)$. Figure 3 illustrates the space $K(\Gamma)$ for twisted tree-grid $\Gamma = G \boxtimes H$. This metric space $K(\Gamma)$ is known as the standard geometric realization of $\Gamma$; see [1, Chapter 11]. It is known that $K(\Gamma)$ is a CAT(0) space (see [7]), and hence uniquely geodesic, i.e., every pair of points can be joined by a unique geodesic (shortest path). The unique geodesic for two points $x, y$ is given as follows. Consider an apartment $\Sigma$ with $x, y \in K(\Sigma)$, and identify $K(\Sigma)$ with $\mathbb{R}^n$ and $x, y$ with points in $\mathbb{R}^n$. Then the line segment $\{\lambda x + (1 - \lambda)y \mid 0 \leq \lambda \leq 1\}$ in $\mathbb{R}^n \simeq K(\Sigma) \subseteq K(\Gamma)$ is the geodesic between $x$ and $y$. In particular, a convex function on $K(\Gamma)$ is defined as a function $g : K(\Gamma) \to \mathbb{R}$ satisfying

$$\lambda g(x) + (1 - \lambda)g(y) \geq g(\lambda x + (1 - \lambda)y) \quad (x, y \in K(\Gamma), 0 \leq \lambda \leq 1),$$

where point $\lambda x + (1 - \lambda)y$ is considered in the apartment $\mathbb{R}^n \simeq K(\Sigma)$. Notice that the above $(x \bullet y, x \circ y)$ is equal to the unique pair $(z, z')$ with the property that $z \preceq z'$ and $(x + y)/2 = (z + z')/2$ (in $\mathbb{R}^n \simeq K(\Sigma)$). In this setting, the L-convexity is characterized as follows.

**Theorem 2.6** ([28]). Let $\Gamma$ be a Euclidean building of type C. For $g : \Gamma \to \mathbb{R}$, the following conditions are equivalent:

1. $g$ is L-convex.

2. The Lovász extension $\bar{g}$ of $g$ is convex.

3. $g$ is submodular on $I_x$ and on $F_x$ for every $x \in \text{dom } g$, and $\text{dom } g$ is chain-connected.

Here a subset $X$ of vertices in $\Gamma$ is chain-connected if every $p, q \in X$ there is a sequence $p = p_0, p_1, \ldots, p_k = q$ such that $p_i \preceq p_{i+1}$ or $p_i \succeq p_{i+1}$.
Remark 2.7. A Euclidean building $\Gamma$ consisting of a single apartment is identified with $\mathbb{Z}^n$, and $K(\Gamma)$ is a simplicial subdivision of $\mathbb{R}^n$. Then L-convex functions on $\Gamma$ as defined here coincide with UJ-convex functions by Fujishige [13], where he defined them via the convexity property (2) in Theorem 2.6.

3 Application

In this section, we demonstrate SDA-based algorithm design for two important multiflow problems from which L-convex functions above actually arise. The first one is the minimum-cost multiflow problem, and the second is the node-capacitated multiflow problem. Both problems also arise as (the dual of) LP-relaxations of other important network optimization problems, and are connected to good approximation algorithms. The SDA framework brings fast combinatorial polynomial time algorithms to both problems for which such algorithms had not been known before.

We will use standard notation for directed and undirected networks. For a graph $(V,E)$ and a node subset $X \subseteq V$, let $\delta(X)$ denote the set of edges $ij$ with $\{|i,j\} \cap X = 1$. In the case where $G$ is directed, let $\delta^+(X)$ and $\delta^-(X)$ denote the sets of edges leaving $X$ and entering $X$, respectively. Here $\delta(\{i\})$, $\delta^+(\{i\})$ and $\delta^-(\{i\})$ are simply denoted by $\delta(i)$, $\delta^+(i)$ and $\delta^-(i)$, respectively. For a function $h$ on a set $V$ and a subset $X \subseteq V$, let $h(X)$ denote $\sum_{i \in X} h(i)$.

3.1 Minimum-cost multiflow and terminal backup

3.1.1 Problem formulation

An undirected network $N = (V,E,c,S)$ consists of an undirected graph $(V,E)$, an edge-capacity $c : E \to \mathbb{Z}_+$, and a specified set $S \subseteq V$ of nodes, called terminals. Let $n := |V|$, $m := |E|$, and $k := |S|$. An $S$-path is a path connecting distinct terminals in $S$. A multiflow is a pair $(\mathcal{P}, f)$ of a set $\mathcal{P}$ of $S$-paths and a flow-value function $f : \mathcal{P} \to \mathbb{R}_+$ satisfying the capacity constraint:

$$f(e) := \sum \{f(P) \mid P \in \mathcal{P} : P \text{ contains } e\} \leq c(e) \quad (e \in E). \quad (11)$$

A multiflow $(\mathcal{P}, f)$ is simply written as $f$. The support of a multiflow $f$ is the edge-weight defined by $e \mapsto f(e)$. Suppose further that the network $N$ is given a nonnegative edge-cost $a : E \to \mathbb{Z}_+$ and a node-demand $r : S \to \mathbb{Z}_+$ on the terminal set $S$. The cost $a(f)$ of a multiflow $f$ is defined as $\sum_{e \in E} a(e) f(e)$. A multiflow $f$ is said to be $r$-feasible if it satisfies

$$\sum \{f(P) \mid P \in \mathcal{P} : P \text{ connects } s\} \geq r(s) \quad (s \in S). \quad (12)$$

Namely each terminal $s$ is connected to other terminals in at least $r(s)$ flows. The minimum-cost node-demand multiflow problem (MCMF) asks to find an $r$-feasible multiflow of minimum cost.

This problem was recently introduced by Fukumaga [18] as an LP-relaxation of the following network design problem. An edge-weight $u : E \to \mathbb{R}_+$ is said to be $r$-feasible if $0 \leq u \leq c$ and the network $(V,E,u,S)$ has an $(s,S \setminus \{s\})$-flow of value at least $r(s)$ for each $s \in S$. The latter condition is represented as the following cut-covering constraint:

$$u(\delta(X)) \geq r(s) \quad (s \in S, X \subseteq V : X \cap S = \{s\}). \quad (13)$$
The (capacitated) terminal backup problem (TB) asks to find an integer-valued \( r \)-feasible edge-weight \( u : E \to \mathbb{Z}_+ \) of minimum-cost \( \sum_{e \in E} a(e)u(e) \). This problem, introduced by Anshelevich and Karagiozova [2], was shown to be polynomially solvable [2, 6] if there is no capacity bound. The complexity of TB for the general capacitated case is not known.

The natural fractional relaxation, called the fractional terminal backup problem (FTB), is obtained by relaxing \( u : E \to \mathbb{Z}_+ \) to \( u : E \to \mathbb{R}_+ \). In fact, MCMF and FTB are equivalent in the following sense:

**Lemma 3.1** ([18]).
(1) For an optimal solution \( f \) of MCMF, the support of \( f \) is an optimal solution of FTB.

(2) For an optimal solution \( u \) of FTB, an \( r \)-feasible multiflow \( f \) in \( (V, E, u, S) \) exists, and is optimal to MCMF.

Moreover, half-integrality property holds:

**Theorem 3.2** ([18]). There exist half-integral optimal solutions in FTB, MCMF, and their LP-dual.

By utilizing this half-integrality, Fukunaga [18] developed a 4/3-approximation algorithm. His algorithm, however, uses the ellipsoid method to obtain a half-integral (extreme) optimal solution in FTB.

Based on the SDA framework of an L-convex function on a tree-grid, the paper [25] developed a combinatorial weakly polynomial time algorithm for MCMF together with a combinatorial implementation of the 4/3-approximation algorithm for TB.

**Theorem 3.3** ([25]). A half-integral optimal solution in MCMF and a 4/3-approximate solution in TB can be obtained in \( O(n \log(nAC) MF(kn, km)) \) time.

Here \( MF(n', m') \) denote the time complexity of solving the maximum flow problem on a network of \( n' \) nodes and \( m' \) edges, and \( A := \max\{a(e) \mid e \in E\} \) and \( C := \max\{c(e) \mid e \in E\} \).

It should be noted that MCMF generalizes the minimum-cost maximum free multiflow problem considered by Karzanov [32, 33]. To this problem, Goldberg and Karzanov [20] developed a combinatorial weakly polynomial time algorithm. However the analysis of their algorithm is not easy, and the explicit polynomial running time is not given. The algorithm in Theorem 3.3 is the first combinatorial weakly polynomial time algorithm having an explicit running time.

### 3.1.2 Algorithm

Here we outline the algorithm in Theorem 3.3. Let \( N = (V, E, c, S) \) be a network, \( a \) an edge cost, and \( r \) a demand. For technical simplicity, the cost \( a \) is assumed to be positive integer-valued \( \geq 1 \). First we show that the dual of MCMF can be formulated as an optimization over the product of subdivided stars. For \( s \in S \), let \( G_s \) be a path with infinite length and an end vertex \( v_s \) of degree one. Consider the disjoint union \( \bigcup_{s \in S} G_s \) and identify all \( v_s \) to one vertex \( O \). The resulting graph, called a subdivided star, is denoted by \( G \), and the edge-length is defined as 1/2 uniformly. Let \( d = d_G \) denote the shortest path metric of \( G \) with respect to this edge-length.

Suppose that \( V = \{1, 2, \ldots, n\} \). A potential is a vertex \( p = (p_1, p_2, \ldots, p_n) \) in \( G^n \) such that \( p_s \in G_s \) for each \( s \in S \).
Proposition 3.4 ([25]). The minimum cost of an \( r \)-feasible multiflow is equal to the maximum of
\[
\sum_{s \in S} r(s)d(p_s, O) - \sum_{ij \in E} c(ij) \max\{d(p_i, p_j) - a(ij), 0\}
\] (14)
over all potentials \( p = (p_1, p_2, \ldots, p_n) \in G^n \).

Sketch. The LP-dual of FTB is given by:
\[
\begin{align*}
\text{Max.} & \quad \sum_{s \in S} r(s) \sum_{X : X \cap S = \{s\}} \pi_X - \sum_{e \in E} c(e) \max\{0, \sum_{X : e \in \delta(X)} \pi_X - a(e)\} \\
\text{s.t.} & \quad \pi_X \geq 0 (X \subseteq V : |X \cap S| = 1).
\end{align*}
\]

By the standard uncrossing argument, one can show that there always exists an optimal solution \( \pi_X \) such that \( \{X \mid \pi_X > 0\} \) is laminar, i.e., if \( \pi_X, \pi_Y > 0 \) it holds that \( X \subseteq Y \), \( Y \subseteq X \), or \( X \cap Y = \emptyset \). Consider the tree-representation of the laminar family \( \{X \mid \pi_X > 0\} \). Since each \( X \) contains exactly one terminal, the corresponding tree is necessarily a subdivided star \( \hat{G} \) with center \( \hat{O} \) and non-uniform edge-length. In this representation, each \( X \) with \( \pi_X > 0 \) corresponds to an edge \( e_X \) of \( \hat{G} \), and each node \( i \) is associated with a vertex \( p_i \) in \( \hat{G} \). The length of each edge \( e_X \) is defined as \( \pi_X \), and the resulting shortest path metric is denoted by \( D \). Then it holds that \( \sum_{X : X \cap S = \{s\}} \pi_X = D(p_s, O) \) and \( \sum_{X : ij \in \delta(X)} \pi_X = D(p_i, p_j) \). By the half-integrality (Theorem 3.2), we can assume that each \( \pi_X \) is a half-integer. Thus we can subdivide \( \hat{G} \) to \( G \) so that each edge-length of \( G \) is 1/2, and obtain the formulation in (14).

Motivated by this fact, define \( \omega : G^n \to \overline{\mathbb{R}} \) by
\[
p \mapsto -\sum_{s \in S} r(s)d(p_s, O) + \sum_{ij \in E} c(ij) \max\{d(p_i, p_j) - a(ij), 0\} + I(p),
\]
where \( I \) is the indicator function of the set of all potentials, i.e., \( I(p) := 0 \) if \( p \) is a potential and \( \infty \) otherwise. The color classes of \( G \) are denoted by \( B \) and \( W \) with \( O \in B \), and \( G \) is oriented zigzagly. In this setting, the objective function \( \omega \) of the dual of MCMF is an \( L \)-convex function on tree-grid \( G^n \):

Proposition 3.5 ([25]). The function \( \omega \) is \( L \)-convex on \( G^n \).

In the following, we show that the steepest descent algorithm for \( \omega \) is efficiently implementable with a maximum flow algorithm. An outline is as follows:

- The optimality of a potential \( p \) is equivalent to the feasibility of a circulation problem on directed network \( D_p \) associated with \( p \) (Lemma 3.6), where a half-integral optimal multiflow is recovered from an integral circulation (Lemma 3.7).
- If \( p \) is not optimal, then a certificate (= violating cut) of the infeasibility yields a steepest direction \( p' \) at \( p \) (Lemma 3.8).

This algorithm may be viewed as a multiflow version of the dual algorithm [22] on minimum-cost flow problem; see also [49] for a DCA interpretation of the dual algorithm.
Figure 4: Construction of extended networks. Each node $i$ is split to a clique of size $|S|$ ($p_i = O$) or size two ($p_i \neq O$), where the edges in these cliques are bidirected edges of sign $(-,-)$. In the resulting bidirected network, drawn in the upper right, the path-decomposition of a bidirected flow gives rise to a multiflow flowing in $G$ geodesically. The network $D_p$, drawn in the lower right, is a directed network equivalent to the bidirected network.

Let $p \in G^n$ be a potential and $f : \mathcal{P} \to \mathbb{R}_+$ an $r$-feasible multiflow, where we can assume that $f$ is positive-valued. Considering $\sum_{e \in E} a(e) f(e) - (\omega(p)) \geq 0$, we obtain the complementary slackness condition: Both $p$ and $f$ are optimal if and only if

$$f(e) = 0 \quad (e = ij \in E : d(p_i, p_j) < a(ij)), \quad (15)$$

$$f(e) = c(e) \quad (e = ij \in E : d(p_i, p_j) > a(ij)), \quad (16)$$

$$\sum_{P \in \mathcal{P}} \{f(P) \mid P \text{ connects } s\} = r(s) \quad (s \in S : p_s \neq O), \quad (17)$$

$$\sum_{k=1, \ldots, \ell} d(p_{i_{k-1}}, p_{i_k}) = d(p_s, p_t) \quad (P = (s = i_0, i_1, \ldots, i_\ell = t) \in \mathcal{P}). \quad (18)$$

The first three conditions are essentially the same as the kilter condition in the (standard) minimum cost flow problem. The fourth one is characteristic of multiflow problems, which says that an optimal multiflow $f$ induces a collection of geodesics in $G$ via embedding $i \mapsto p_i$ for an optimal potential $p$.

Observe that these conditions, except the fourth one, are imposed on the support of $f$ rather than multiflow $f$ itself. The fourth condition can also be represented by a support condition on an extended (bidirected) network which we construct below; see Figure 4. An optimal multiflow will be recovered from a fractional bidirected flow on this network.

Let $E_=$ and $E_>$ denote the sets of edges $ij$ with $d(p_i, p_j) = a(ij)$ and with $d(p_i, p_j) > a(ij)$, respectively. Remove all other edges (by (15)). For each nonterminal node $i$ with
Let \( p_i = O \), replace \( i \) by \( |S| \) nodes \( i^s (s \in S) \) and add new edges \( i^s i^t \) for distinct \( s, t \in S \). Other node \( i \) is a terminal \( s \) or a nonterminal node with \( p_i \in G_s \setminus \{O\} \). Replace each such node \( i \) by two nodes \( i^s \) and \( i^O \), and add new edge \( i^s i^O \). The node \( i^s \) for terminal \( i = s \) is also denoted by \( s \). The set of added edges is denoted by \( E_- \). For each edge \( e = ij \in E_\leq \cup E_> \), replace \( ij \) by \( i^O j^s \) if \( p_i, p_j \in G_s \) and \( d(p_i, O) > d(p_j, O) \), and replace \( ij \) by \( i^O j^O \) if \( p_i \in G_s \setminus \{O\} \) and \( p_j \in G_t \setminus \{O\} \) for distinct \( s, t \in S \).

An edge-weight \( \psi : E_\leq \cup E_> \cup E_- \to \mathbb{R} \) is called a \( p \)-feasible support if

\[
0 \leq \psi(e) \leq c(e) \quad (e \in E_\leq),
\]
\[
\psi(e) = c(e) \quad (e \in E_>),
\]
\[
\psi(e) \leq 0 \quad (e \in E_-),
\]
\[
-\psi(\delta(s)) \geq r(s) \quad (s \in S, p_s = O),
\]
\[
-\psi(\delta(s)) = r(s) \quad (s \in S, p_s \neq O),
\]
\[
\psi(\delta(u)) = 0 \quad (\text{each nonterminal node } u).
\]

One can see from an alternating-path argument that any \( p \)-feasible support \( \psi \) is represented as a weighted sum \( \sum_{P \in \mathcal{P}} f(P) \chi^P \) for a set \( \mathcal{P} \) of \( S \)-paths with nonnegative coefficients \( f : \mathcal{P} \to \mathbb{R}_+ \), where each \( P \) is a path alternately using edges in \( E_- \) and edges in \( E_\leq \cup E_> \), and \( \chi^P \) is defined by \( \chi^P(e) := -1 \) for edge \( e \in P \) with \( e \in E_- \), \( \chi^P(e) := 1 \) for other edge \( e \) in \( P \), and zero for edges not in \( P \). Contracting all edges in \( E_- \) for all paths \( P \in \mathcal{P} \), we obtain an \( r \)-feasible multifold \( f^\psi \) in \( N \), where \( f^\psi(e) \leq c(e) \) and (16) are guaranteed by \((19)\) and \((20)\), and the \( r \)-feasibility and \((17)\) are guaranteed by \((22)\) and \((23)\). Also, by construction, each path in \( \mathcal{P} \) induces a local geodesic in \( G \) by \( i \mapsto p_i \), which must be a global geodesic since \( G \) is a tree. This implies \((18)\).

**Lemma 3.6** \((25)\). (1) A potential \( p \) is optimal if and only if a \( p \)-feasible support \( \psi \) exists.

(2) For any \( p \)-feasible support \( \psi \), the multifold \( f^\psi \) is optimal to MCMF.

Thus, by solving inequalities \((19)-(24)\), we obtain an optimal multifold or know the nonoptimality of \( p \). Observe that this problem is a fractional bidirected flow problem, and reduces to the following circulation problem. Replace each node \( u \) by two nodes \( u^+ \) and \( u^- \). Replace each edge \( e = uv \in E_- \) by two directed edges \( e^+ = u^+ v^- \) and \( e^- = u^- v^+ \) with lower capacity \( \underline{c}(e^+ - e^-) := 0 \) and upper capacity \( \overline{c}(e^+) = \overline{c}(e^-) := \infty \). Replace each edge \( e = uv \in E_\leq \cup E_> \) by two directed edges \( e^+ = u^+ v^+ \) and \( e^- = v^- u^- \), where \( \overline{c}(e^+) = \overline{c}(e^-) := c(e) \), and \( \underline{c}(e^+) = \underline{c}(e^-) := 0 \) if \( e \in E_\leq \) and \( \underline{c}(e^+) = \underline{c}(e^-) := c(e) \) if \( e \in E_> \). For each terminal \( s \in S \), add edge \( s^- s^+ \), where \( \underline{c}(s^- s^+) := r(s) \) and \( \overline{c}(s^- s^+) := \infty \) if \( p_s = O \) and \( \underline{c}(s^- s^+) = \overline{c}(s^- s^+) := r(s) \) if \( p_s \neq O \). Let \( D_p \) denote the resulting network, which is a variant of the double covering network in the minimum cost multifold problem \((32)-(33)\).

A circulation is an edge-weight \( \varphi \) on this network \( D_p \) satisfying \( \underline{c}(e) \leq \varphi(e) \leq \overline{c}(e) \) for each edge \( e \), and \( \varphi(\delta^+(u)) = \varphi(\delta^-(u)) \) for each node \( u \). From a circulation \( \varphi \) in \( D_p \), a \( p \)-feasible support \( \psi \) is obtained by

\[
\psi(e) := (\varphi(e^+) + \varphi(e^-))/2 \quad (e \in E_\leq \cup E_> \cup E_-).
\]

It is well-known that a circulation, if it exists, is obtained by solving one maximum flow problem. Thus we have:
Figure 5: The correspondence between movable cuts in $D_p$ and neighbors of $p$. There is a one-to-one correspondence between $I_p \cup F_p$ and $\{X \cap V_i \mid X: \text{movable cut}\}$, where $X \cap V_i$ is surrounded by a closed curve, and nodes with $+$ label and $-$ label are represented by white and black points, respectively. By a movable cut $X$, a potential $p$ can be moved to another potential $p^X$ for which $(p^X)_i \in I_p \cup F_p$ ($i \in V$).

Lemma 3.7 ([25]). From an optimal potential, a half-integral optimal multiflow is obtained in $O(MF(kn, m + k^2n))$ time.

Next we analyze the case where a circulation does not exist. By Hoffman’s circulation theorem, a circulation exists in $D_p$ if and only if

$$
\kappa(X) := \zeta(\delta^-(X)) - \zeta(\delta^+(X))
$$

is nonpositive for every node subset $X$. A node subset $X$ is said to be a violating cut if $\kappa(X)$ is positive, and is said to be maximum if $\kappa(X)$ is maximum among all node-subsets.

From a maximum violating cut, a steepest direction of $\omega$ at $p$ is obtained as follows. For (original) node $i \in V$, let $V_i$ denote the set of nodes in $D_p$ replacing $i$ in this reduction process; see Figure 4. Let $V_i^+$ and $V_i^-$ denote the sets of nodes in $V_i$ having $+$ label and $-$ label, respectively. A node subset $X$ is said to be movable if $X \cap V_i = \emptyset$ or $\{u^+\} \cup V_i^- \setminus \{u^-\}$ for some $u^+ \in V_i^+$. For a movable cut $X$, the potential $p^X$ is defined by

$$
(p^X)_i = \begin{cases} 
\text{the neighbor of } p_i \text{ closer to } O & \text{if } X \cap V_i^+ = \{i^{O+}\}, \\
\text{the neighbor of } p_i \text{ in } G_s \text{ away from } O & \text{if } X \cap V_i^+ = \{i^{s+}\}, \\
 p_i & \text{if } X \cap V_i = \emptyset.
\end{cases}
$$

(26)

See Figure 5 for an intuition of $p^X$. Let $V_I := \bigcup_{i \in W} V_i$ and $V_F := \bigcup_{i \in B} V_i$. Since $a$ is integer-valued, edges between $V_I$ and $V_F$ have the same upper and lower capacity. This implies $\kappa(X) = \kappa(X \cap V_I) + \kappa(X \cap V_F)$. Thus, if $X$ is violating, then $X \cap V_I$ or $X \cap V_F$ is violating, and actually gives a steepest direction as follows.

Lemma 3.8 ([25]). Let $p$ be a nonoptimal potential. For a minimal maximum violating cut $X$, both $X \cap V_I$ and $X \cap V_F$ are movable. Moreover, $p^{X \cap V_I}$ is a minimizer of $\omega$ over $I_p$ and $p^{X \cap V_F}$ is a minimizer of $\omega$ over $F_p$.

Now SDA is specialized to MCMF as follows.

Steepest Descent Algorithm for MCMF
Step 0: Let $p := (O, O, \ldots, O)$.

Step 1: Construct network $D_p$.

Step 2: If a circulation $\varphi$ exists in $D_p$, then obtain a $p$-feasible support $\psi$ by (25), and an optimal multiflow $f^\psi$ via the path decomposition; stop.

Step 3: For a minimal maximum violating cut $X$, choose $p' \in \{p^X \cap V_i, p^X \cap V_F\}$ with $\omega(p') = \min\{\omega(p^X \cap V_i), \omega(p^X \cap V_F)\}$, let $p := p'$, and go to step 1.

Proposition 3.9 (25). The above algorithm runs in $O(nA \cdot \text{MF}(kn, m + k^2n))$ time.

**Sketch.** By Theorem 2.5, it suffices to show $\max_i d(O, p_i) = O(nA)$ for some optimal potential $p$. Let $p$ be an arbitrary optimal potential. Suppose that there is a node $i^*$ such that $p_{i^*} \in G_s$ and $d(p_{i^*}, O) > nA$. Then there is a subpath $P$ of $G_s$ that has 2A edges and no node $i$ with $p_i \in P$. Let $U(\ni^*)$ be the set of nodes $i$ such that $p_i$ is beyond $P$ (on the opposite side to $O$). For each $i \in U$, replace $p_i$ by its neighbor closer to $O$. Then one can see that $\omega$ does not increase. By repeating this process, we obtain an optimal potential as required.

This algorithm can be improved to a polynomial time algorithm by a domain scaling technique. For a scaling parameter $\ell = -1, 0, 1, 2, \ldots, \lceil \log nA \rceil$, let $G_\ell$ denote the graph on the subset of vertices $x$ of $G$ with $d(x, O) \in 2^\ell \mathbb{Z}$, where an edge exists between $x, y \in G_\ell$ if and only if $d(x, y) = 2^\ell$. By modifying the restriction of $\omega$ to $G_\ell^n$, we can define $\omega_\ell : G_\ell^n \to \mathbb{R}$ with the following properties:

- $\omega_\ell$ is L-convex on $G_\ell^n$.
- $\omega_-1 = \omega$.
- If $x^*_\ell$ is a minimizer of $\omega_\ell$ over $G_\ell \subseteq G_{\ell-1}$, then $d_\Delta(x^*_\ell, \text{opt}(\omega_{\ell-1})) = O(n)$, where $d_\Delta$ is defined for $G_{\ell-1}^n$ (with unit edge-length).

The key is the third property, which comes from a proximity theorem of L-convex functions [25]. By these properties, a minimizer of $\omega$ can be found by calling SDA $\lceil \log nA \rceil$ times, in which $x^*_\ell$ is obtained in $O(n)$ iterations in each scaling phase. To solve a local $k$-submodular minimization problem for $\omega_\ell$, we use a network construction in [31], different from $D_p$. Then we obtain the algorithm in Theorem 3.3.

### 3.2 Node-capacitated multiflow and node-multiway cut

#### 3.2.1 Problem formulation

Suppose that the network $N = (V, E, b, S)$ has a node-capacity $b : V \setminus S \to \mathbb{R}_+$ instead of edge-capacity $c$, where a multiflow $f : \mathcal{P} \to \mathbb{R}_+$ should satisfy the node-capacity constraint:

$$\sum\{f(P) \mid P \in \mathcal{P} : P \text{ contains node } i\} \leq b(i) \quad (i \in V \setminus S). \quad (27)$$

Let $n := |V|$, $m := |E|$, and $k := |S|$ as before. The node-capacitated maximum multiflow problem (NMF) asks to find a multiflow $f : \mathcal{P} \to \mathbb{R}_+$ of the maximum total flow-value $\sum_{P \in \mathcal{P}} f(P)$. This problem first appeared in the work of Garg, Vazirani, and
Yannakakis [19] on the node-multiway cut problem. A node-multiway cut is a node subset \( X \subseteq V \setminus S \) such that every \( S \)-path meets \( X \). The minimum node-multiway cut problem (NMC) is the problem of finding a node-multiway cut \( X \) of minimum capacity \( \sum_{i \in X} b(i) \).

The two problems NMF and NMC are closely related. Indeed, consider the LP-dual of NMF, which is given by

\[
\begin{align*}
\text{Min.} & \quad \sum_{i \in V \setminus S} b(i)w(i) \\
\text{s.t.} & \quad \sum \{w(i) \mid i \in V \setminus S: P \text{ contains node } i\} \geq 1 \quad (P: S\text{-path}), \\
& \quad w(i) \geq 0 \quad (i \in V \setminus S).
\end{align*}
\]

Restricting \( w \) to be 0-1 valued, we obtain an IP formulation of NMC. Garg, Vazirani, and Yannakakis [19] proved the half-integrality of this LP, and showed a 2-approximation algorithm for NMC by rounding a half-integral solution; see also [53]. The half-integrality of the primal problem NMF was shown by Pap [47, 48]; this result is used to solve the integer version of NMF in strongly polynomial time. These half-integral optimal solutions are obtained by the ellipsoid method in strongly polynomial time.

It is a natural challenge to develop an ellipsoid-free algorithm. Babenko [3] developed a combinatorial polynomial time algorithm for NMF in the case of unit capacity. For the general case of capacity, Babenko and Karzanov [4] developed a combinatorial weakly polynomial time algorithm for NMF. As an application of \( \ell \)-convex functions on a twisted tree-grid, the paper [26] developed the first strongly polynomial time combinatorial algorithm:

**Theorem 3.10** ([26]). A half-integral optimal multiflow for NMF, a half-integral optimal solution for its LP-dual, and a 2-approximate solution for NMC can be obtained in \( O(m(\log k)\text{MSF}(n,m,1)) \) time.

The algorithm uses a submodular flow algorithm as a subroutine. Let \( \text{MSF}(n,m,\gamma) \) denote the time complexity of solving the maximum submodular flow problem on a network of \( n \) nodes and \( m \) edges, where \( \gamma \) is the time complexity of computing the exchange capacity of the defining submodular set function. We briefly summarize the submodular flow problem; see [11, 12] for detail. A submodular set function on a set \( V \) is a function \( h: 2^V \to \mathbb{R} \) satisfying

\[
h(X) + h(Y) \geq h(X \cap Y) + h(X \cup Y) \quad (X,Y \subseteq V).
\]

Let \( N = (V,A,c,l,\bar{c}) \) be a directed network with lower and upper capacities \( c, l, \bar{c} : A \to \mathbb{R} \), and let \( h : 2^V \to \mathbb{R} \) be a submodular set function on \( V \) with \( h(\emptyset) = h(V) = 0 \). A feasible flow with respect to \( h \) is a function \( \varphi: A \to \mathbb{R} \) satisfying

\[
\begin{align*}
c(e) & \leq \varphi(e) \leq \bar{c}(e) \quad (e \in A), \\
\varphi(\delta^-(X)) - \varphi(\delta^+(X)) & \leq h(X) \quad (X \subseteq V).
\end{align*}
\]

For a feasible flow \( \varphi \) and a pair of nodes \( i,j \), the exchange capacity is defined as the minimum of

\[
h(X) - \varphi(\delta^-(X)) + \varphi(\delta^+(X)) \quad (\geq 0)
\]

over all \( X \subseteq V \) with \( i \in X \not\ni j \).
The maximum submodular flow problem (MSF) asks to find a feasible flow \( \varphi \) having maximum \( \varphi(e) \) for a fixed edge \( e \). This problem obviously generalizes the maximum flow problem. Generalizing existing maximum flow algorithms, several combinatorial algorithms for MSF have been proposed; see [14] for survey. These algorithms assume an oracle of computing the exchange capacity (to construct the residual network). The current fastest algorithm for MSF is the pre-flow push algorithm by Fujishige-Zhang [17], where the time complexity is \( O(n^3 \gamma) \). Thus, by using their algorithm, the algorithm in Theorem 3.10 runs in \( O(mn^3 \log k) \) time.

### 3.2.2 Algorithm

Let \( N = (V, E, b, S) \) be a network. For several technical reasons, instead of NMF, we deal with a perturbed problem. Consider a small uniform edge-cost on \( E \). It is clear that the objective function of NMF may be replaced by \( \sum_{p \in P} Mf(P) - \sum_{e \in E} 2f(e) \) for large \( M > 0 \). We further perturb \( M \) according to terminals which \( P \) connects. Let \( \Sigma \) be a tree such that each non-leaf vertex has degree 3, leaves are \( u_s \) (\( s \in S \)), and the path-metric on \( G \) is denoted by \( d \). Let \( v_s \) denote the vertex in \( P_s \) with \( d(u_s, v_s) = (2|E| + 1)[\log k] \). The perturbed problem PNMM is to maximize

\[
\sum_{p \in P} d(v_{sp}, v_{tp})f(P) - \sum_{e \in E} 2f(e)
\]

over all multiflows \( f : P \to \mathbb{R}_+ \), where \( s_p \) and \( t_p \) denote the ends of an \( S \)-path \( P \).

**Lemma 3.11** ([26]). Any optimal multiflow for PNMF is optimal to NMF.

Next we explain a combinatorial duality of PNMM, which was earlier obtained by [24] for more general setting. Consider the edge-subdivision \( G^* \) of \( G \), which is obtained from \( G \) by replacing each edge \( pq \) by a series of two edges \( pv_pq \) and \( v_pq \) with a new vertex \( v_{pq} \). The edge-length of \( G^* \) is defined as \( 1/2 \) uniformly, where \( G \) is naturally regarded as an isometric subspace of \( G^* \) (as a metric space). Let \( Z^* := \{z/2 \mid z \in \mathbb{Z}\} \) denote the set of half-integers. Suppose that \( V = \{1, 2, \ldots, n\} \). Consider the product \( (G^* \times \mathbb{Z}^*)^n \). An element \( (p, r) = ((p_1, r_1), (p_2, r_2), \ldots, (p_n, r_n)) \in (G^* \times \mathbb{Z}^*)^n = (G^*)^n \times (\mathbb{Z}^*)^n \) is called a potential if

\[
\begin{align*}
    r_i &\geq 0 & (i \in V), \\
    d(p_i, p_j) - r_i - r_j &\leq 2 & (ij \in E), \\
    (p_s, r_s) &\neq (v_s, 0) & (s \in S).
\end{align*}
\]

and each \((p_i, r_i)\) belongs to \( G \times \mathbb{Z} \) or \((G^* \setminus G) \times (\mathbb{Z}^* \setminus \mathbb{Z}) \). Corresponding to Proposition 3.4, the following holds:

**Proposition 3.12** ([24]). The optimal value of PNMF is equal to the minimum of \( \sum_{i \in V \setminus S} 2b(i)r_i \) over all potentials \((p, r)\).

A vertex \((v_{pq}, z + 1/2)\) in \((G^* \setminus G) \times (\mathbb{Z}^* \setminus \mathbb{Z}) \) corresponds to a 4-cycle \((p, z), (p, z + 1), (q, z + 1), (q, z)\) in \( G \times \mathbb{Z} \). Thus any potential is viewed as a vertex of a twisted tree-grid.
\((G \otimes Z)^n\). Define \(\varpi : (G \otimes Z)^n \to \mathbb{R}\) by

\[
\varpi(p, r) := \sum_{i \in V \setminus S} 2b(i)r_i + I(p, r) \quad ((p, r) \in (G \otimes Z)^n),
\]

where \(I\) is the indicator function of the set of all potentials.

**Theorem 3.13** ([20]). \(\varpi\) is \(L\)-convex on \((G \otimes Z)^n\).

As in the previous subsection, we are going to apply the SDA framework to \(\varpi\), and show that a steepest direction at \((p, r) \in (G \otimes Z)^n\) can be obtained by solving a submodular flow problem on network \(D_{p, r}\) associated with \((p, r)\). The argument is parallel to the previous subsection but is technically more complicated.

Each vertex \(u \in G\) has two or three neighbors in \(G\), which are denoted by \(u^\alpha\) for \(\alpha = 1, 2, 3\). Accordingly, the neighbors in \(G^*\) are denoted by \(u^{*\alpha}\), where \(u^{*\alpha}\) is the vertex replacing edge \(uu^\alpha\). Let \(G_3 \subseteq G\) denote the set of vertices having three neighbors.

Let \((p, r)\) be a potential. Let \(E_\pm\) denote the set of edges \(ij\) with \(d(p_i, p_j) - r_i - r_j = 2\). Remove other edges. For each nonterminal node \(i\), replace \(i\) by two nodes \(i^1, i^2\) if \(p_i \notin G_3\) and by three nodes \(i^1, i^2, i^3\) if \(p_i \in G_3\). Add new edges \(i^{*\alpha}\) for distinct \(\alpha, \beta\). The set of added edges is denoted by \(E_-\). For \(ij \in E_\pm\), replace each edge \(ij \in E_-\) by \(i^\alpha j^\beta\) for \(\alpha, \beta \in \{1, 2, 3\}\) with \(d(p_i, p_j) = d(p_i, p_j)^{*\alpha} + d((p_i)^{*\alpha}, (p_j)^{*\beta}) + d((p_j)^{*\beta}, p_j)\). Since \(d(p_i, p_j) \geq 1\) and \(G^*\) is a tree, such neighbors \((p_i)^{*\alpha}\) and \((p_j)^{*\beta}\) are uniquely determined. If \(i = s \in S\), the incidence of \(s\) is unchanged, i.e., let \(i^{*\alpha} = s\) in the replacement. An edge-weight \(\psi : E_\pm \cup E_- \to \mathbb{R}\) is called a \((p, r)\)-feasible support if it satisfies

\[
\begin{align*}
\psi(e) & \geq 0 \quad (e \in E_\pm), \\
\psi(e) & \leq 0 \quad (e \in E_-), \\
\psi(\delta(i^{*\alpha})) &= 0 \quad (\alpha \in \{1, 2, 3\}), \\
-\psi(i^{1,2}) & \leq b(i) \quad (p_i \notin G_3, r_i = 0), \\
-\psi(i^{1,2}) &= b(i) \quad (p_i \notin G_3, r_i > 0), \\
-\psi(i^{1,2}) & - \psi(i^{2,3}) - \psi(i^{3,1}) \leq b(i) \quad (p_i \in G_3, r_i = 0), \\
-\psi(i^{1,2}) &- \psi(i^{2,3}) - \psi(i^{3,1}) = b(i) \quad (p_i \in G_3, r_i > 0)
\end{align*}
\]

for each edge \(e\) and nonterminal node \(i\). By precisely the same argument, a \((p, r)\)-feasible support \(\psi\) is decomposed as \(\psi = \sum_{P \in \mathcal{P}} f^\psi(P)\chi_P\) for a multflow \(f^\psi : \mathcal{P} \to \mathbb{R}_+\), where the node-capacity constraint (27) follows from (31)-(34). Corresponding to Lemma 3.6, we obtain the following, where the proof goes along the same argument.

**Lemma 3.14** ([20]).

1. A potential \((p, r)\) is optimal if and only if a \((p, r)\)-feasible support exists.

2. For any \((p, r)\)-feasible support \(\psi\), the multflow \(f^\psi\) is optimal to PNMF.

The system of inequalities (28)-(34) is similar to the previous bidirected flow problem ([19]-[24]). However, (33) and (34) are not bidirected flow constraints. In fact, the second constraint (34) reduces to a bidirected flow constraint as follows. Add new vertex \(i^0\), replace edges \(i^1, i^2, i^3, i^0\) by \(i^{0,1}, i^{0,2}, i^{0,3}\), and replace (34) by

\[
-\psi(\delta(i^0)) = 2b(i), \quad -\psi(i^{0,\alpha}) \leq b(i) \quad (\alpha = 1, 2, 3).
\]
Then \((\psi(i^1i^2), \psi(i^2i^3), \psi(i^1i^3))\) satisfying (34) is represented as
\[
\psi(i^\alpha i^\beta) = (\psi(i^\alpha i^\gamma) + \psi(i^\gamma i^\beta) - \psi(i^\omega i^\tau))/2 \quad (\{\alpha, \beta, \gamma\} = \{1, 2, 3\})
\]
for \((\psi(i^\alpha i^1), \psi(i^\alpha i^2), \psi(i^\alpha i^3))\) satisfying (35).

We do not know whether (33) admits such a reduction. This makes the problem difficult. Node \(i \in V \setminus S\) with \(p_i \in G_3, r_i = 0\) is said to be special. For a special node \(i\), remove edges \(i^1i^2, i^2i^3, i^1i^3\). Then the conditions (30) and (33) for \(u = i^1, i^2, i^3\) can be equivalently written as the following condition on degree vector \((\psi(\delta(i^1)), \psi(\delta(i^2)), \psi(\delta(i^3)))\):
\[
\psi(\delta(X)) - \psi(\delta(Y)) \leq g_i(X, Y) \quad (X, Y \subseteq \{i^1, i^2, i^3\} : X \cap Y = \emptyset), \quad (36)
\]
where \(g_i\) is a function the set of pairs \(X, Y \subseteq \{i^1, i^2, i^3\}\) with \(X \cap Y \neq \emptyset\). Such \(g_i\) can be chosen as a bisubmodular (set) function, and inequality system (36) says that the degree vector must belong to the bisubmodular polyhedron associated with \(g_i\); see [12] for bisubmodular polyhedra. Thus our problem of solving (28)-(32), (35), and (36) is a fractional bidirected flow problem with degrees constrained by a bisubmodular function, which may be called a bisubmodular flow problem. However this natural class of the problems has not been well-studied so far. We need a further reduction. As in the previous subsection, for the bidirected network associated with (28)-(32), and (35), we construct the equivalent directed network \(D_{p,r}\) with upper capacity \(\bar{c}\) and lower capacity \(c\), where node \(\bar{r}^a\) and edge \(e\) are doubled as \(i^{a+}, i^{a-}\) and \(e^+, e^-\), respectively. Let \(V_i\) denote the set of nodes replacing original \(i \in V\), as before. We construct a submodular-flow constraint on \(V_i = \{i^{1+}, i^{2+}, i^{3+}, i^{1-}, i^{2-}, i^{3-}\}\) for each special node \(i\) so that a \((p, r)\)-feasible support \(\psi\) is recovered from any feasible flow \(\varphi\) on \(D_{p,r}\) by the following relation:
\[
\psi(e) = (\varphi(e^+) + \varphi(e^-))/2. \quad (37)
\]
Such a submodular-flow constraint actually exists, and is represented by some submodular set function \(h_i\) on \(V_i\); see [26] for the detailed construction of \(h_i\).

Now our problem is to find a flow \(\varphi\) on network \(D_{p,r}\) having the following properties:

- For each edge \(e\) in \(D_{p,r}\), it holds that \(c(e) \leq \varphi(e) \leq \bar{c}(e)\).
- For each special node \(i\), it holds that
  \[
  \varphi(\delta^-(U)) - \varphi(\delta^+(U)) \leq h_i(U) \quad (U \subseteq V_i). \quad (38)
  \]
- For other node \(i \in V \setminus S\), it holds that
  \[
  \varphi(\delta^-(i^{a\sigma})) - \varphi(\delta^+(i^{a\sigma})) = 0 \quad (\sigma \in \{+, -\}).
  \]

This is a submodular flow problem, where the defining submodular set function is given by \(X \mapsto \sum_{i \text{special}} h_i(X \cap V_i)\), and its exchange capacity is computed in constant time. Consequently we have the following, where the half-integrality follows from the integrality theorem of submodular flow.

**Lemma 3.15** ([26]). From an optimal potential \((p, r)\), a half-integral optimal multiflow is obtained in \(O(MSF(n, m, 1))\) time.
By Frank’s theorem on the feasibility of submodular flow (see [11, 12], a feasible flow
\( \varphi \) exists if and only if
\[
\kappa(X) := \zeta(\delta^-(X)) - \zeta(\delta^+(X)) - \sum_{i: \text{special}} h_i(X \cap V_i).
\]
(39)
is nonpositive for every vertex subset \( X \). A violating cut is a vertex subset \( X \) having
positive \( \kappa(X) \). By a standard reduction technique, finding a feasible flow or violating
cut is reduced to a maximum submodular flow problem, where a minimal violating cut
is naturally obtained from the residual graph of a maximum feasible flow.

Again, we can obtain a steepest direction from a minimal violating cut, according to
its intersection pattern with each \( V_i \). A vertex subset \( X \) is called movable if for \( i \) with
\( p_i \in G^* \setminus G \) it holds that \(|X \cap V_i| \leq 1\) and for node \( i \) with \( p_i \in G \) it holds that \( X \cap V_i = \emptyset\)
\( V_i^+, V_i^-, \{i^{2+}\}, V_i^+ \setminus \{i^{2-}\} \), or \( \{i^{2+}\} \cup V_i^- \setminus \{i^{2-}\} \) for some \( \alpha \in \{1, 2, 3\} \). For a movable

cut \( X \), define \((p, r)^X\) by

\[
(p, r)_{i, r}^X := \begin{cases} 
(p_i, r_i) & \text{if } X \cap V_i = \emptyset, \\
(p_i^{\alpha}, r_i + 1/2) & \text{if } X \cap V_i = \{i^{\alpha+}\}, \\
(p_i^{\alpha}, r_i - 1/2) & \text{if } X \cap V_i = V_i^- \setminus \{i^{\alpha-}\}, \\
(p_i^{\alpha}, r_i) & \text{if } X \cap V_i = \{i^{\alpha+}\} \cup V_i^- \setminus \{i^{\alpha-}\}, \\
(p_i, r_i + 1) & \text{if } X \cap V_i = V_i^+, \\
(p_i, r_i - 1) & \text{if } X \cap V_i = V_i^-.
\end{cases}
\]

(40)

See Figure 3 for an intuition of \((p, r)^X\). Let \( V_F \) be the union of \( V_i \) over \( i \in V \) with \((p_i, r_i) \in B\) and \( \{i^{2+}, i^{2-}\} \) over \( i \in V \) and \( \alpha \in \{1, 2\} \) with \( p_i \in G^* \setminus G \) and \((p_i^{\alpha}, r_i - 1/2) \in B\). Let \( V_i \) be defined by replacing the role of \( B \) and \( W \) in \( V_F \). Edges between \( V_i \) and \( V_F \) have the same (finite) lower and upper capacity. If \( X \) is violating, then \( X \cap V_i \) or \( X \cap V_F \) is
violating.

Lemma 3.16 ([26]). Let \( X \) be the unique minimal maximum violating cut, and let \( \bar{X} \) be
obtained from \( X \) by adding \( V_i^+ \) for each node \( i \in V \setminus S \) with \( p_i \in G_3 \) and \(|X \cap V_i^+| = 2\).
Then \( \bar{X} \cap V_i \) and \( \bar{X} \cap V_F \) are movable, one of them is violating, and \((p, r)^{\bar{X} \cap V_F}\) is a
minimizer of \( \varpi \) over \( I_{p, r} \) and \((p, r)^{\bar{X} \cap V_F}\) is a minimizer of \( \varpi \) over \( F_{p, r} \).

Now the algorithm to find an optimal multiflow is given as follows.

Steepest Descent Algorithm for PNMF

Step 0: For each terminal \( s \in S \), let \((p_s, r_s) := (v_s, 0)\). Choose any vertex \( v \in \Sigma \). For
each \( i \in V \setminus S \), let \( p_i := v \) and \( r_i := 2(m + 1)[\log k] \).

Step 1: Construct \( D_{p, r} \) with submodular set function \( X \mapsto \sum_{i: \text{special}} h_i(X \cap V_i) \).

Step 2: If a feasible flow \( \varphi \) exists in \( D_{p, r} \), then obtain a \((p, r)\)-feasible support \( \psi \) from \( \varphi \)
and an optimal multiflow \( f^\psi \) via the path decomposition; stop.

Step 3: For a minimal maximum violating cut \( X \), choose \((p'_i, r'_i) \in \{(p, r)^{\bar{X} \cap V_i}, (p, r)^{\bar{X} \cap V_F}\}
with \( \varpi(p'_i, r'_i) = \min \{\varpi((p, r)^{\bar{X} \cap V_i}), \varpi((p, r)^{\bar{X} \cap V_F})\} \), let \((p, r) := (p'_i, r'_i)\), and go to
step 1.

One can see that the initial point is actually a potential. By the argument similar to the
proof of Proposition 3.9, one can show that there is an optimal potential \((p^*, r^*)\) such
that \( r^*_i = O(m \log k) \) and \( d(v, p^*_i) = O(m \log k) \). Consequently, the number of iterations
is bounded by \( O(m \log k) \), and we obtain Theorem 3.10.
Figure 6: The correspondence between movable cuts in $D_{p,r}$ and neighbors of $(p,r)$. For $(p_i, r_i) \in B$, there is a one-to-one correspondence between $F_{p_i}$ and $\{X \cap V_i \mid X : \text{movable cut}\}$, where the meaning of this figure is the same as in Figure 5.

**Remark 3.17.** As seen above, $k$- and $(k,l)$-submodular functions arising from localizations of $\omega$ and $\varpi$ can be minimized via maximum (submodular) flow. A common feature of both cases is that the domain $S_k$ or $S_{k,l}$ of a $k$- or $(k,l)$-submodular function is associated with special intersection patterns between nodes and cuts on which the function-value is equal to the cut-capacity (up to constant). A general framework for such network representations is discussed by Iwamasa [30].

4 L-convex function on oriented modular graph

In this section, we explain L-convex functions on oriented modular graphs, introduced in [27, 28]. This class of discrete convex functions is a further generalization of L-convex functions in Section 2. The original motivation of our theory comes from the complexity classification of the minimum 0-extension problem. We start by mentioning the motivation and highlight of our theory (Section 4.1), and then go into the details (Sections 4.2 and 4.3).

4.1 Motivation: Minimum 0-extension problem

Let us introduce the minimum 0-extension problem (0-EXT), where our formulation is different from but equivalent to the original formulation by Karzanov [31]. An input $I$ consists of number $n$ of variables, undirected graph $G$, nonnegative weights $b_{iv}$ ($1 \leq i \leq n, v \in G$) and $c_{ij}$ ($1 \leq i < j \leq n$). The goal of 0-EXT is to find $x = (x_1, x_2, \ldots, x_n) \in G^n$...
that minimizes
\[ \sum_{i=1}^{n} \sum_{v \in G} b_{iv}d(x_i, v) + \sum_{1 \leq i < j \leq n} c_{ij}d(x_i, x_j), \] (41)
where \( d = d_G \) is the shortest path metric on \( G \). This problem is interpreted as a facility location on graph \( G \). Namely we are going to locate new facilities \( 1, 2, \ldots, n \) on graph \( G \) of cities, where these facilities communicate each other and communicate with all cities, and communication costs are propositional to their distances. The problem is to find a location of minimum communication cost. In facility location theory [50], 0-EXT is known as the multifacility location problem. Also 0-EXT is an important special case of the metric labeling problem [50], which is a unified label assignment problem arising from computer vision and machine learning. Notice that fundamental combinatorial optimization problems can be formulated as 0-EXT for special underlying graphs. The minimum cut problem is the case of \( G = K_2 \), and the multiway cut problem is the case of \( G = K_m \) \((k \geq 3)\).

In [34], Karzanov addressed the computational complexity of 0-EXT with fixed underlying graph \( G \). This restricted problem class is denoted by 0-EXT[\( G \)]. He raised a question: What are graphs \( G \) for which 0-EXT[\( G \)] is polynomially solvable? An easy observation is that 0-EXT[\( K_m \)] is in P if \( m \leq 2 \) and NP-hard otherwise. A classical result [37] in facility location theory is that 0-EXT[\( G \)] is in P for a tree \( G \). Consequently, 0-EXT[\( G \)] is in P for a tree-product \( G \). It turned out that the tractability of 0-EXT is strongly linked to median and modularity concept of graphs. A median of three vertices \( x_1, x_2, x_3 \) is a vertex \( y \) satisfying
\[ d(x_i, x_j) = d(x_i, y) + d(y, x_j) \quad (1 \leq i < j \leq 3). \]
A median is a common point in shortest paths among the three points, may or may not exist, and is not necessarily unique even if it exists. A median graph is a connected graph such that every triple of vertices has a unique median. Observe that trees and their products are median graphs. Chepoi [9] and Karzanov [34] independently showed that 0-EXT[\( G \)] is in P for a median graph \( G \).

A modular graph is a further generalization of a median graph, and is defined as a connected graph such that every triple of vertices admits (not necessarily unique) a median. The following hardness result shows that graphs tractable for 0-EXT are necessarily modular.

**Theorem 4.1 (34).** If \( G \) is not orientable modular, then 0-EXT[\( G \)] is NP-hard.

Here a (modular) graph is said to be orientable if it has an edge-orientation, called an admissible orientation, such that every 4-cycle \((x_1, x_2, x_3, x_4)\) is oriented as: \( x_1 \rightarrow x_2 \) if and only if \( x_4 \rightarrow x_3 \). Karzanov [34, 35] showed that 0-EXT[\( G \)] is polynomially solvable on special classes of orientable modular graphs.

In [27], we proved the tractability for general orientable modular graphs.

**Theorem 4.2 (27).** If \( G \) is orientable modular, then 0-EXT[\( G \)] is solvable in polynomial time.

For proving this result, [27] introduced L-convex functions on oriented modular graphs and submodular functions on modular semilattices, and applied the SDA framework to 0-EXT. An oriented modular graph is an orientable modular graph endowed with an
admissible orientation. A modular semilattice is a semilattice generalization of a modular lattice, introduced by Bandelt, Van De Vel, and Verheul [5]. Recall that a modular lattice $L$ is a lattice such that for every $x, y, z \in L$ with $x \geq z$ it holds $x \wedge (y \vee z) = (x \wedge y) \vee z$. A modular semilattice is a meet-semilattice $L$ such that every principal ideal is a modular lattice, and for every $x, y, z \in L$ the join $x \vee y \vee z$ exists provided $x \vee y$, $y \vee z$, and $z \vee x$ exist. These two structures generalize Euclidean buildings of type $C$ and polar spaces, respectively, and are related in the following way.

**Proposition 4.3.**

1. A semilattice is modular if and only if its Hasse diagram is oriented modular [5].

2. Every principal ideal and filter of an oriented modular graph are modular semilattices [27]. In particular, every interval is a modular lattice.

3. A polar space is a modular semilattice [8].

4. The Hasse diagram of a Euclidean building of type $C$ is oriented modular [8].

An admissible orientation is acyclic [27], and an oriented modular graph is viewed as (the Hasse diagram of) a poset.

As is expected from these properties and arguments in Section 2, an L-convex function on an oriented modular graph is defined so that it behaves submodular on the local structure (principal ideal and filter) of each vertex, which is a modular semilattice. Accordingly, the steepest descent algorithm is well-defined, and correctly obtain a minimizer.

We start with the local theory in the next subsection (Section 4.2), where we introduce submodular functions on modular semilattices. Then, in Section 4.3, we introduce L-convex functions on oriented modular graphs, and outline the proof of Theorem 4.2.

**Remark 4.4.** The minimum 0-extension problem $0\text{-EXT}[\Gamma]$ on a fixed $\Gamma$ is a particular instance of finite-valued CSP with a fixed language. Thapper and Živný [51] established a dichotomy theorem for finite-valued CSPs. The complexity dichotomy in Theorems 4.1 and 4.2 is a special case of their dichotomy theorem, though a characterization of the tractable class of graphs (i.e., orientable modular graphs) seems not to follow directly from their result.

### 4.2 Submodular function on modular semilattice

A modular semilattice, though not necessarily a lattice, admits an analogue of the join, called the fractional join, which is motivated by fractional polymorphisms in VCSP [39] and enables us to introduce a submodularity concept.

Let $L$ be a modular semilattice, and let $r : L \to \mathbb{Z}_+$ be the rank function, i.e., $r(p)$ is the length of a maximal chain from the minimum element to $p$. The fractional join of elements $p, q \in L$ is defined as a formal sum

$$\sum_{u \in E(p, q)} [C(u; p, q)] u$$

of elements $u \in E(p, q) \subseteq L$ with nonnegative coefficients $[C(u; p, q)]$, to be defined soon. Then a function $f : L \to \overline{\mathbb{R}}$ is called submodular if it satisfies

$$f(p) + f(q) \geq f(p \wedge q) + \sum_{u \in E(p, q)} [C(u; p, q)] f(u) \quad (p, q \in L).$$
The fractional join of \( p, q \in L \) is defined according to the following steps; see Figure 7 for intuition.

- Let \( I(p, q) \) denote the set of all elements \( u \in L \) represented as \( u = a \lor b \) for some \((a, b)\) with \( p \preceq a \preceq p \land q \preceq b \preceq q\). This representation is unique, and \((a, b)\) equals \((u \land p, u \land q)\) \[27\].

- For \( u \in I(p, q) \), let \( r(u; p, q) \) be the vector in \( \mathbb{R}_2^+ \) defined by
  \[
  r(u; p, q) = (r(u \land p) - r(p \land q), r(u \land q) - r(p \land q)).
  \]

- Let \( \text{Conv} \ I(p, q) \subseteq \mathbb{R}_2^+ \) denote the convex hull of vectors \( r(u; p, q) \) over all \( u \in I(p, q) \).

- Let \( E(p, q) \) be the set of elements \( u \) in \( I(p, q) \) such that \( r(u; p, q) \) is a maximal extreme point of \( \text{Conv} \ I(p, q) \). Then \( u \mapsto r(u; p, q) \) is injective on \( E(p, q) \) \[27\].

- For \( u \in E(p, q) \), let \( C(u; p, q) \) denote the nonnegative normal cone at \( r(u; p, q) \):
  \[
  C(u; p, q) := \{ c \in \mathbb{R}_2^+ \mid \langle c, r(u; p, q) \rangle = \max_{x \in \text{Conv} I(p,q)} \langle c, x \rangle \},
  \]
  where \( \langle \cdot, \cdot \rangle \) is the standard inner product.

- For a convex cone \( C \subseteq \mathbb{R}_2^+ \) represented as
  \[
  C = \{(x, y) \in \mathbb{R}_2^+ \mid y \cos \alpha \leq x \sin \alpha, y \cos \beta \geq x \sin \beta \}
  \]
  for \( 0 \leq \alpha \leq \beta \leq \pi/2 \), define nonnegative value \([C]\) by
  \[
  [C] := \frac{\sin \alpha}{\sin \alpha + \cos \alpha} - \frac{\sin \beta}{\sin \beta + \cos \beta}.
  \]

Figure 7: The construction of the fractional join. By \( u \mapsto r(u; p, q) \), the set \( I(p, q) \) is mapped to points in \( \mathbb{R}_2^+ \), where \( r(p \land q; p, q) \) is the origin, \( r(p; p, q) \) and \( r(q; p, q) \) are on the coordinate axes. Then \( \text{Conv} \ I(p, q) \) is the convex hull of \( r(u; p, q) \) over \( u \in I(p, q) \). The fractional join is defined as the formal sum of elements mapped to maximal extreme points of \( \text{Conv} \ I(p, q) \).
The fractional join of \( p, q \) is defined as
\[
\sum_{u \in E(p,q)} [C(u; p, q)]u.
\]
This weird definition of the submodularity turns out to be appropriate. If \( L \) is a modular lattice, then the fractional join is equal to the join \( 1 \cdot \lor = \lor \), and our definition of submodularity coincides with the usual one. In the case where \( L \) is a polar space, it is shown in [28] that the fractional join of \( p, q \) is equal to
\[
\frac{1}{2}(p \sqcup q) \sqcup q + \frac{1}{2}(p \sqcup q) \sqcup p,
\]
and hence a submodular function on \( L \) is a function satisfying
\[
f(p) + f(q) \geq f(p \land q) + \frac{1}{2}f((p \sqcup q) \sqcup q) + \frac{1}{2}f((p \sqcup q) \sqcup p) \quad (p, q \in L).
\]
(42)

It is not difficult to see that systems of inequalities (42) and (7) define the same class of functions. Thus the submodularity concept in this section is consistent with that in Section 2.4.

An important property relevant to 0-EXT is its relation to the distance on \( L \). Let \( d : L \times L \to \mathbb{R} \) denote the shortest path metric on the Hasse diagram of \( L \). Then \( d \) is also written as
\[
d(p, q) = r(p) + r(q) - 2r(p \land q) \quad (p, q \in L).
\]

**Theorem 4.5** ([27]). Let \( L \) be a modular semilattice. Then the distance function \( d \) is submodular on \( L \times L \).

Next we consider the minimization of submodular functions on a modular semilattice. The tractability under general setting (i.e., oracle model) is unknown. We consider a restricted situation of valued constraint satisfaction problem (VCSP); see [39, 54] for VCSP. Roughly speaking, VCSP is the minimization problem of a sum of functions with small number of variables. We here consider the following VCSP (submodular-VCSP on modular semilattice). An input consists of (finite) modular semilattices \( L_1, L_2, \ldots, L_n \) and submodular functions \( f_i : L_{i_1} \times L_{i_2} \times \cdots \times L_{i_k} \to \mathbb{R} \) with \( i = 1, 2, \ldots, m \) and \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \), where \( k \) is a fixed constant. The goal is to find \( p = (p_1, p_2, \ldots, p_n) \in L_1 \times L_2 \times \cdots \times L_n \) to minimize
\[
\sum_{i=1}^{m} f_i(p_{i_1}, p_{i_2}, \ldots, p_{i_k}).
\]

Each submodular function \( f_i \) is given as the table of all function values. Hence the size of the input is \( O(nN + mN^k) \) for \( N := \max_i |L_i| \).

Kolmogorov, Thapper, and Živný [39] proved a powerful tractability criterion for general VCSP such that an LP-relaxation (Basic LP) exactly solves the VCSP instance. Their criterion involves the existence of a certain submodular-type inequality (fractional polymorphism) for the objective functions, and is applicable to our submodular VCSP (thanks to the above weird definition).

**Theorem 4.6** ([27]). Submodular-VCSP on modular semilattice is solvable in polynomial time.

**Remark 4.7.** Kuivinen [40, 41] proved a good characterization for general SFM on product \( L_1 \times L_2 \times \cdots \times L_n \) of modular lattices \( L_i \) with \( |L_i| \) fixed. Fujishige, Király, Makino, Takazawa, and Tanigawa [15] proved the oracle-tractability for the case where each \( L_i \) is a diamond, i.e., a modular lattice of rank 2.
4.3 L-convex function on oriented modular graph

Here we introduce L-convex functions for a slightly restricted subclass of oriented modular graphs; see Remark 4.11 for general case. Recall Proposition 4.3 (2) that every interval of an oriented modular graph \( \Gamma \) is a modular lattice. If every interval of \( \Gamma \) is a complemented modular lattice, i.e., every element is a join of rank-1 elements, then \( \Gamma \) is said to be well-oriented. Suppose that \( \Gamma \) is a well-oriented modular graph. The L-convexity on \( \Gamma \) is defined along the property (3) in Theorem 2.6, not by discrete midpoint convexity, since we do not know how to define discrete midpoint operations on \( \Gamma \). Namely an L-convex function on \( \Gamma \) is a function \( g : \Gamma \rightarrow \mathbb{R} \) such that \( g \) is submodular on every principal ideal and filter, and \( \text{dom} \, g \) is chain-connected, where the chain-connectivity is similarly defined as in Theorem 2.6. By this definition, the desirable properties hold:

**Theorem 4.8** ([27, 28]). Let \( g \) be an L-convex function on \( \Gamma \). If \( x \in \text{dom} \, g \) is not a minimizer of \( g \), then there is \( y \in I_x \cup F_x \) with \( g(y) < g(x) \).

Thus the steepest descent algorithm (SDA) is well-defined, and correctly obtains a minimizer of \( g \) (if it exists). Moreover the \( l_\infty \)-iteration bound is also generalized. Let \( \Gamma^\Delta \) denote the graph obtained from \( \Gamma \) by adding an edge \( pq \) if both \( p \land q \) and \( p \lor q \) exist in \( \Gamma \), and let \( d^\Delta := d_{\Gamma^\Delta} \). Then Theorem 4.9 is generalized as follows.

**Theorem 4.9** ([28]). The number of iterations of SDA applied to L-convex function \( g \) and initial point \( x \in \text{dom} \, g \) is at most \( d^\Delta(x, \text{opt}(g)) + 2 \).

Corresponding to Theorem 4.5 the following holds:

**Theorem 4.10** ([27]). Let \( G \) be an orientable modular graph. The distance function \( d \) on \( G \) is L-convex on \( G \times G \).

We are ready to prove Theorem 4.2. Let \( G \) be an orientable modular graph. Endow \( G \) with an arbitrary admissible orientation. Then the product \( G^n \) of \( G \) is oriented modular. It was shown in [27, 28] that the class of L-convex functions is closed under suitable operations such as variable fixing, nonnegative sum, and direct sum. By this fact and Theorem 4.10, the objective function of 0-EXT[\( \Gamma \)] is viewed as an L-convex function on \( G^n \). Thus we can apply the SDA framework to 0-EXT[\( \Gamma \)], where each local problem is submodular-VCSP on modular semilattice. By Theorem 4.6 a steepest direction at \( x \) can be found in polynomial time. By Theorem 4.9 the number of iterations is bounded by the diameter of \((G^n)^\Delta\). Notice that for \( x, x' \in G^n \), if \( \max_i d(x_i, x'_i) \leq 1 \), then \( x \) and \( x' \) are adjacent in \((G^n)^\Delta\). From this, we see that the diameter of \((G^n)^\Delta\) is not greater than the diameter of \( G \). Thus the minimum 0-extension problem on \( G \) is solved in polynomial time.

**Remark 4.11.** Let us sketch the definition of L-convex function on general oriented modular graph \( \Gamma \). Consider the poset of all intervals \([p, q]\) such that \([p, q]\) is a complemented modular lattice, where the partial order is the inclusion order. Then the Hasse diagram \( \Gamma^* \) is well-oriented modular [8, 27]. For a function \( g : \Gamma \rightarrow \mathbb{R} \), let \( g^* : \Gamma^* \rightarrow \mathbb{R} \) be defined by \( g^*([p, q]) = (g(p) + q(q))/2 \). Then an L-convex function on \( \Gamma \) is defined as a function \( g : \Gamma \rightarrow \mathbb{R} \) such that \( g^* \) is L-convex on \( \Gamma^* \). With this definition, desirable properties hold. In particular, the original L^\sharp-convex functions coincide with L-convex functions on the product of directed paths, where \( \mathbb{Z} \) is identified with an infinite directed path.
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