Unconventional cycles and multiple adiabatic points

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Abstract
The so-called unconventional cycles provide a useful didactic resource to
discuss the second law of thermodynamics applied to heat engines and their
efficiency. In the simplest, previous and thoroughly studied case involving a
negatively sloped linear process, interesting physics follow from the presence
of an adiabatic point. At such point, the process is tangent to an adiabatic
curve and $\delta Q = 0$, signalling where the heat flow is reversed. In order to deal
with the follow up question on the possibility of having more than one
adiabatic point, we introduce a parabolic process whose behavior is richer but
still amenable to analytical exploration. In addition, we define pseudoadiabatic
processes that are reversible and non-isoentropic, but whose total heat
exchanged is zero. These processes are useful when emphasizing the necessary
and sufficient conditions for an actual adiabatic process. The linear-parabolic
cycle is then introduced, and a few particular cases are discussed.

Keywords: heat engines, unconventional cycles, second law, circular cycle

(Some figures may appear in colour only in the online journal)

1. Introduction

Along some of the idealized, conventional processes most often considered in thermo-
dynamics, the exchanged heat $\delta Q$ is either constant (e.g. isochoric or isobaric for ideal gases),
null (adiabatic) or monotonously varying with the volume (e.g. isothermal or positively
sloped linear processes). Importantly, in all these cases, $\delta Q$ is either positive or negative along
the whole process. Nonetheless, in the so-called unconventional (albeit idealized) cycles, $\delta Q$
has a smooth, continuous reversal of direction (signal) [1–8], changing from endo- to exo-
thermic. Where it happens, the so-called adiabatic point, as the name indicates, the process
described by the behavior of the pressure as a function of the volume, $P(V)$, is tangent to an
adiabatic curve [2] (that, for example, for a monoatomic ideal gas, is given by the Poisson equation $PV^{2/3} = $ constant) and $\delta Q = 0$ (the notation $\delta Q$ denotes an inexact differential). Obtaining such point(s) is, in some cases, straightforward (the simplest, most discussed example is the linear, negative slope process [1–6, 8]) while being difficult and error-prone in more complex cases (e.g. the circular [2, 9, 10], elliptical [11] and multilobed [12] cycles).

Although for the linear process, only a single [2] adiabatic point is allowed (because adiabatic curves cannot cross), numerical results show that more than two adiabatic points are possible on the circular cycle [10] (two such points were obviously necessary, albeit not always correctly located; see discussion in [2, 9]). The possibility of several adiabatic points existing has attracted little attention, despite being a frequent question following the discussion of the linear process, and because the circular cycle usually resorts to numerical solutions, it is important to have an intermediate and treatable case. Our main objective here is to present a simple example where multiple adiabatic points may appear, the parabolic process, and to explore the interesting new phenomenology associated with their presence. For example, in the parabolic process it is possible to have a single adiabatic point without reversal of the heat flux, at variance with the linear case. Moreover, combined with a linear process, it results in a linear-parabolic cycle that is visually similar to the well-known ‘Sadly Cannot’ cycle [1] (with an adiabatic process instead of the parabolic one). Indeed, by a proper choice of the parameters, either the parabolic or the linear process may have zero net heat exchanged along the whole path, what we then call pseudoadiabatic, in opposition to an actual adiabatic process that has $\delta Q = 0$ everywhere. This provides a good starting point to discuss the related subtleties of these processes and the cycles they may belong to.

2. The linear process

Before introducing the parabolic process, we revisit the linear expansion and explore the pseudoadiabatic case, a reversible, non-isentropic process with zero net exchanged heat ($Q = 0$). Consider one mol of a monoatomic ideal gas and the linear process passing through $(V_0, P_0)$ and $(2V_0, \alpha P_0)$, where $\alpha < 1$ (otherwise the slope is positive and no adiabatic point is possible) is a control parameter. Introducing the adimensional variables $p \equiv P/P_0$ and $v \equiv V/V_0$, we get

$$p = (\alpha - 1)(v - 1) + 1,$$

(1)

where $0 \leq v \leq v_{\text{MAX}} = (\alpha - 2)/(\alpha - 1)$. The adimensional, rescaled temperature, $t \equiv RT/P_0V_0 = pv$ ($R$ is the gas constant and $T$, the absolute temperature), has a maximum at the point T, where $v_T = v_{\text{MAX}}/2$. Following Dickerson and Mottmann [2], the heat exchanged during an infinitesimal step along the process is obtained from the first law of thermodynamics, $\delta Q = c_VdT + Pdv$ (where the specific heat at constant volume for a monoatomic ideal gas is used, $c_V = 3R/2$), and the equation of state $PV = RT$,

$$\delta Q = P_0V_0\left[4(\alpha - 1)v + \frac{5}{2}(2 - \alpha)\right]dv. \quad (2)$$

The point A where the heat flow is reversed, $\delta Q = 0$, is

$$v_A = \frac{5}{8} v_{\text{MAX}}, \quad (3)$$

thus occurring at a larger volume than T, i.e. $v_A > v_T$. Eliminating $\alpha$ we write $p_A = 3v_A/(8v_A - 5)$, locating all adiabatic points for the chosen parametrization (equivalently, $p_T = v_T/(2v_T - 1)$ for the point T). By changing the declivity of the linear process,
\( \alpha \) goes from \( 5/8 \) \((\alpha \rightarrow -\infty)\) to \( \infty \) \((\alpha \rightarrow 1)\). Moreover, the specific heat along the linear process \([4]\) (albeit we write \( \partial Q \) in the equation below, remember that heat is not a state variable),

\[
c = \left( \frac{\partial Q}{\partial T} \right)_{\text{lin}} = 2R \frac{v - \nu_A}{v - \nu_T},
\]

is negative for \( \nu_T < v < \nu_A \) (positive when \( v < \nu_T \) or \( v > \nu_A \)), zero at \( A \) and divergent at \( T \). That the specific heat may be both positive and negative along a process is an often misunderstood point.

Particularly interesting is the case in which the total heat exchanged is zero. This occurs when the points are symmetrically chosen around \( \nu_A \) \((\text{e.g. } a \text{ and } b \text{ in figure 1})\). Indeed, integrating \( \delta Q \) from \( v_a \) to \( v_b \) and imposing \( Q_{ab} = 0 \), we get that \( \nu_A \) is the average between \( v_a \) and \( v_b \):

\[
v_a + v_b = \frac{5}{4} v_{\text{MAX}} = 2 \nu_A.
\]

This result is expected from equation \( (2) \), whose dependence on the volume is linear. Such a process, which we call pseudoadiabatic, offers a myriad of interesting problems to be explored in itself and as part of thermodynamical cycles, not only to emphasize the differences with an actually adiabatic process, but also as a simple way to introduce those unconventional cycles and discuss the subtleties on the evaluation of thermodynamical efficiencies.

### 3. The parabolic process

Consider again one mol of a monoatomic ideal gas and the parabolic process \( P = a_1 V^2 + a_2 V + a_3 \). The (dimensional) coefficients \([a_i]\) are such that the parabola goes through the point \((V_0, P_0)\) and has a minimum at \((2V_0, \alpha P_0)\) where it is, by construction, parallel to an isobaric process. Using the adimensional variables \( p \) and \( v \) defined before, we write
where $\alpha < 1$ remains as a free parameter (for $\alpha > 1$, a single adiabatic point is possible, similar to the first quadrant of the circular cycle). An example with $\alpha = 1/3$ is shown in figure 2. If $\alpha \equiv \alpha_T < 4/7$, the rescaled temperature $t$ has extremes at two points, $T_+$ and $T_-$, a minimum and a maximum, respectively:

$$v_T^\pm = \frac{4}{3} \pm \frac{1}{3} \sqrt{\frac{4 - 7\alpha}{1 - \alpha}}. \quad (7)$$

At these points the curve $p(v)$ is tangent to different isothermal processes. Between these two points, and generalizing the linear case, there may be two adiabatic points, $A_-$ and $A_+$. By analogy with equation (2), the heat exchanged along the parabolic process is

$$\delta Q = q(v)P_0V_0dv$$

with

$$q(v) = \frac{1}{2}(1 - \alpha)(11v^2 - 32v + 15) + \frac{5}{2}. \quad (8)$$

The two possible adiabatic points are located at

$$v_\pm^A = \frac{16}{11} \pm \frac{1}{11} \sqrt{\frac{36 - 9\alpha}{1 - \alpha}}, \quad (9)$$

shown in figure 3 along with $v_T^\pm$. For an expansion, in the interval $v^- < v < v^+, \delta Q < 0$ and heat flows out of the system. Thus, $A_-$ plays a role similar to $A$ in the linear process, separating regions, as $v$ increases, with $\delta Q > 0$ and $\delta Q < 0$, respectively. Crossing $A_+$, the order is reversed, from negative to positive $\delta Q$. For $\alpha < 0$, the point $A_+$ is not present anymore, an unphysical region (for an ideal gas) with $p < 0$ develops, instead, above $v = 2 - \sqrt{\alpha}/(\alpha - 1)$, shaded in figure 3. The $A_-$ and $A_+$ points merge at $\alpha_T = 36/91$. For this value of $\alpha$ there is a single adiabatic point at $v^-_A = v^+_A = 16/11$ but, differently from the linear case, before and after this point the sign of $\delta Q$ is the same and there is no change in the heat flow direction. For $\alpha_T < \alpha < \alpha_T$, albeit $\delta Q > 0$ along the whole process, there are still extremes of temperature at $T_+$ and $T_-$. At $\alpha = \alpha_T$ these points merge and the temperature has
an inflection point associated with a divergent specific heat (see below). The region where both adiabatic points may be present is thus 0 ≤ α < αₐ. Of course, the presence or not of such points on a given process also depends on the initial and final states chosen along the parabolic curve.

In the detail of figure 2, the parabolic process is plotted on a log-log scale. On such a logarithm scale, both the isothermal (not shown) and adiabatic processes are seen as straight lines with declivity 1 and γ, respectively, and provide a graphical way to locate the adiabatic points, a clear example where using a logarithmic scale facilitates the understanding. Indeed, the two adiabatics that are tangent to the parabola at A⁺ and A⁻ are clearly seen as straight, parallel lines. Between A⁻ and A⁺ the slope is smaller than the slope of an adiabatic, dp/dvₚ = 0, and integrating q(v) up to the point where the overall exchanged heat would be zero, one gets two non-trivial solutions, v₁ < v₂ < v₃ and v₄ > v₅. Figure 2 shows an example (thick, black line).

The specific heat along the process, similar to equation (4), for the linear process, is

\[
c = \frac{11}{6} R \frac{(v - v₅)(v - v₆)}{(v - v₇)(v - v₈)}.
\]

The specific heat diverges at v₅⁺ where the parabola is tangent to isothermal processes and is zero at v₆ where it is tangent to the two adiabatic ones. It is negative for v⁻ < v < v₅ and v₆ < v < v₇ or positive otherwise. Notice also that v = 2 corresponds, by construction, to the minimum of the parabolic process. Thus, because the infinitesimal process is flat and parallel to an isobaric curve at that point, the specific heat converges to the specific heat at constant pressure, c = γ = 5R/2. The behavior of c in the region containing the points T⁺ and A⁻ is reminiscent of the linear case, equation (4).
4. The linear-parabolic cycle

It is possible to combine the two processes discussed above in a single cycle. Starting at (1, 1), the system expands linearly, equation (1), up to (2, α), where the final pressure is \( p(2) = \alpha \), with \( 0 < \alpha < 1 \) being a parameter. It is then parabolically compressed, equation (6), returning to the initial state. An example is shown in the inset of figure 4 for \( \alpha = 1/3 \). The internal area of the cycle, the delivered work, is

\[
W = PV_1 - PV_2 \quad \text{and decreases with } \alpha.
\]

Heat may enter the system both in the linear and parabolic branches (L and P indices, respectively), and its total, \( Q_m = Q_L + Q_P \), is related to the presence and location of the adiabatic points that, obviously, depend on \( \alpha \) as well. Along the linear process, heat enters the system in the interval \( v \leq v_1 \leq v_2 < A \), i.e.

\[
\frac{Q_L}{P_0V_0} = \begin{cases} 
\frac{(3\alpha + 2)^2}{32(1 - \alpha)}, & 0 < \alpha \leq 6/11 \\
\frac{7\alpha - 2}{2}, & 6/11 < \alpha < 1
\end{cases}
\]

where the interval has been divided because for \( 6/11 < \alpha < 1 \), \( v_2 > 2 \). In the parabolic process, instead, heat enters only for \( \alpha < \alpha_A \). For \( 1/6 \leq \alpha < \alpha_A \), the contributing region is \( v_\lambda < v < v_\lambda^+ \),

\[
\frac{Q_P}{P_0V_0} = \frac{11}{12} (1 - \alpha)(v_\lambda^+ - v_\lambda)^3,
\]

while for \( 0 < \alpha < 1/6 \), the region with positive heat is \( 1 < v < v_\lambda^+ \) and the result is slightly different (not shown because, as can be seen in figure 4, the interesting behavior occurs close to \( \alpha_A \)). Indeed, figure 4 shows that the efficiency \( \eta(\alpha) \) is a non-monotonic function. To the right of the dashed vertical line at \( \alpha = \alpha_A \), there is only the adiabatic point on the linear process and \( Q_m = Q_L \). As \( \alpha \) decreases, from above \( \alpha_A \), \( \eta \) increases. Once entering the region \( \alpha < \alpha_A \), there is a second interval where heat enters the system, now on the parabolic process, \( v_\lambda^+ < v < v_\lambda^- \), and the efficiency, after attaining its maximum value at \( \alpha = (16982 - 880\sqrt{122})/22207 \approx 0.327 \), starts decreasing.

Figure 4. Efficiency \( \eta \) as a function of \( \alpha \) for the linear-parabolic cycle (shown in the inset for \( \alpha = 1/3 \)). The peak close to \( \alpha = \alpha_A \) (dashed vertical line) is due to the appearance of a region with positive heat in the parabolic path that decreases the efficiency.
Besides the above value of $\alpha$ where the efficiency has a maximum, several other values give interesting situations, all of them close to $\alpha_\text{A}$. For example, for $\alpha = 2/7$ and $7/22$, the linear and the parabolic processes, respectively, are pseudoadiabatic. On the other hand, at $\alpha = 2^{-5/3}$, the points at $\nu = 1$ and 2 (but not the intermediate ones) belong to the same adiabatic. In all cases, the cycles contain a process with some characteristic of an actual adiabatic (either $Q = 0$ or $p_1 v_1^\gamma = p_2 v_2^\gamma$, with $\gamma = c_p/c_V$), emphasizing that these are necessary but not sufficient conditions for being an adiabatic process.

5. Discussion and conclusions

Despite the name, unconventional cycles offer an excellent conceptual tool to discuss some subtleties related to thermodynamic cycles \cite{14,15} and shed light on some issues that students have while evaluating their efficiencies \cite{2}. For example, in the ‘Sadly Cannot’ cycle \cite{1}, two points on an adiabatic curve are connected through a linear process similar to equation (1). When considering only the net exchanged heat in each process (zero and positive, respectively) to evaluate the efficiency, one may erroneously obtain a value larger than the reversible case, in a clear violation of the second law of thermodynamics. By identifying the adiabatic point where $\delta Q = 0$, it is possible to dismiss the apparent paradox, find the missing region with negative $\delta Q$ and correctly evaluate the efficiency. Another instance where recognizing those points is important is in less idealized, more realistic cycles as they lack a clear, sharp transition between well-known processes, and the round, broad crossovers (see, for example, \cite{16}) make them more prone to exhibit adiabatic points, probably several of them. Although the linear, negative slopped process may only present a single one, the existence of processes with multiple adiabatic points had been numerically observed \cite{10}, en passant, for the circular cycle in a region of constant curvature (several points are to be expected when the curvature keeps changing, as in the multilobed process \cite{12}). Thus, within this context, our objectives were twofold. First, to introduce a process with more than one adiabatic point, yet simple enough to be tackled analytically. For that we considered a parabolic process, whose curvature is constant, and that may present up to two such points. This leads to the second objective, the discussion of pseudoadiabatic processes that, due to the presence of at least one adiabatic point, have a zero total exchanged heat because what enters on one side of an adiabatic point is compensated on the other side, at variance with an adiabatic process that has $\delta Q = 0$ everywhere. A pseudoadiabatic parabolic process, differently from the linear one, can be easily exchanged for an actual adiabatic one and be a source of error and confusion when only the net heat exchanged is evaluated. Addressing these differences and discussing the necessary and sufficient conditions for adiabaticity can prove very useful for students.

More than two adiabatic points are also possible \cite{10,12}. If one considers only the third quadrant of the circular cycle, it is possible to have up to three such points. By finding a curve that conveniently changes its curvature, it might be possible to have even more points. The parabolic process is interesting not only because we can obtain simple, closed expressions for it, but also because it provides a myriad of different, interesting physical situations for well-chosen values of $\alpha$. It is thus a very useful didactic resource while discussing the subtleties of thermodynamical cycles and the respective efficiencies.
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