Moonshine for Rudvalis’s sporadic group II *

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Abstract

In Part I we introduced the notion of enhanced vertex operator superalgebra, and constructed an example which is self-dual, has rank 28, and whose full symmetry group is a seven-fold cover of the sporadic simple group of Rudvalis. In this article we construct a second enhanced vertex operator superalgebra whose full automorphism group is a cyclic cover of the Rudvalis group. This new example is self-dual and has rank −28. As in Part I, we can compute all the McKay–Thompson series associated to the action of the Rudvalis group explicitly. We observe that these series, when considered together with those of Part I, satisfy a genus zero property.

Contents

0 Introduction 2
0.1 Outline . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
0.2 Notation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
1 Vertex algebras 7
1.1 Vertex algebra structure . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
1.2 Vertex Lie algebra structure . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8
2 Enhanced vertex algebras 9
3 Weyl algebras 10
3.1 Weyl algebra structure . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 10
3.2 Metaplectic groups . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 11
3.3 Weyl module VOAs . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 12
3.4 Hermitian structure . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 15
4 Linear groups 16

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0 Introduction

Monstrous Moonshine is, in a word, the unexpected connection between modular functions\(^1\) and the largest sporadic simple group, the Fischer–Griess Monster. Since the initial observations of McKay and Thompson

\[
\begin{align*}
196884 & = 1 + 196883 \\
21493760 & = 1 + 196883 + 21296876 \\
864299970 & = (2)1 + (2)196883 + 21296876 + 842609326
\end{align*}
\]

(0.0.1)

relating coefficients of the Fourier expansion of Klein’s modular invariant (on the left) with degrees of irreducible representations of the Monster group (on the right), much fascinating and beautiful mathematics has been developed in order to explicate these coincidences.

It is very striking that the largest sporadic simple group has a connection with modular functions. Even more surprising are the particular special properties that these “Monstrous” modular functions \(F_g(\tau)\) for \(g \in M\) appear to satisfy. Roughly speaking, each one is a generator for the field of functions on the curve \((\mathbb{H}/\Gamma_g)^*\) defined by its invariance group \(\Gamma_g \subset PSL_2(\mathbb{R})\), and in particular, the curve in each case has genus zero. Armed only with the character table of the (at that time conjectural) Monster group, and with the observations (0.0.1) of McKay and Thompson, Conway and Norton were able to collect a wealth of information about what the connection should entail [CN79], and in particular, formulated their Moonshine conjectures:

There exists a graded vector space \(V = \bigoplus_n V_n\) with an action by \(M\) such that for each \(g \in M\) the graded trace function \(F_g(\tau) = \sum_n \text{tr}|_{V_n} g q^n\) is a generator for the function field determined by a discrete group \(\Gamma_g < PSL_2(\mathbb{R})\) of genus zero.

---

\(^1\)Here, by modular function we understand a holomorphic function on the upper half plane that is invariant for the action of some discrete subgroup of \(PSL_2(\mathbb{R})\) that is commensurable with the modular group \(PSL_2(\mathbb{Z})\).
The proof of these conjectures is due to Borcherds [Bor92], and is a fine example of the far reaching developments in mathematics that have occurred since the inception of Moonshine. Important roles in the proof of the Moonshine conjectures are played by vertex algebras introduced by Borcherds [Bor86], and by a particular vertex operator algebra (VOA) constructed by Frenkel–Lepowsky–Meurman [FLM88] called the Moonshine VOA denoted \( V^2 \). Indeed, the VOA \( V^2 \) furnishes an algebraic structure whose full automorphism group is the Monster group \( \mathbb{M} \) and whose underlying vector space is a natural candidate for the \( V \) in the statement of the Moonshine conjectures.

In brief, a vertex operator algebra is a vertex algebra equipped with extra structure (and conditions). A simple theme in this article and its companion [Dun06] (see also [Dun07]) is to explore what happens when one specializes further the structure of vertex algebra; we arrive at the notion of enhanced vertex algebra.

Having observed aspects of Monstrous Moonshine, it is natural to ask if there are analogous phenomena with other finite simple groups in place of the Monster. Perhaps of greatest importance are the sporadic simple groups, since at present they resist a uniform description, and must be studied via somewhat ad. hoc. methods. Any approach that might furnish a uniform treatment of the sporadic groups would be of great interest. Curiously, many (indeed most) of the 26 sporadic simple groups are involved in the Monster and for these sporadic groups there is substantial evidence of Moonshine type phenomena (c.f. [Que81]). In fact, these observations for sporadic groups that are involved in the Monster (monstrous sporadics) may be regarded as the content of the generalized Moonshine conjectures due to Norton:

For each \( g \in \mathbb{M} \) there is graded vector space \( V_g = \bigoplus_n (V_g)_n \) with a (projective) action by \( C_g(g) \) such that for each \( h \in C_g(g) \) the graded trace function \( Z(g,h,\tau) = \sum_n \text{tr}_{(V_g)_n} h q^n \) is either a generator for the function field determined by a discrete group \( \Gamma_{g,h} < \text{PSL}_2(\mathbb{R}) \) of genus zero, or is constant.

In the preceding article [Dun06] we focused attention on a particular sporadic group that is not involved in the Monster: the sporadic simple group of Rudvalis. We introduced the notion of enhanced vertex operator superalgebra which is, in brief, a vertex operator superalgebra equipped with extra structure. We constructed a particular example \( A_{Ru} \) whose full automorphism group is a cyclic cover of the Rudvalis group. Even more than this, the object \( A_{Ru} \) is somewhat distinguished among vertex algebras in that it is self-dual (that is, has a unique simple module) and satisfies a strong vanishing condition on certain homogeneous subspaces of low degree. This suggests that \( A_{Ru} \) may admit a characterisation analogous to those which hold for classical objects such as a the Golay code, and the Leech lattice (c.f. [Con69]), and which have been established to some extent for other distinguished (enhanced) vertex operator algebras such as the Moonshine VOA \( V^2 \) (c.f. [FLM88], [DGL05], [LY06], and see also [Dun07]). The action of the Rudvalis group \( Ru \) on \( A_{Ru} \) allows one to attach modular functions (and modular forms) to elements \( g \in Ru \), and in fact, the extra structure that constitutes the particular enhanced vertex operator superalgebra structure on \( A_{Ru} \) gives rise to Jacobi forms in a natural way, and we obtain, for example, the following analogues.

\(^2\)A group is involved in the Monster if it can be identified with a quotient of some subgroup of the Monster.
of McKay and Thompson’s observations (0.0.1)

\[378 = 378\]
\[784 = 1 + 783\]
\[20475 = 20475\]
\[92512 = (2)378 + 406 + 91350\]
&c.

relating coefficients of a particular Jacobi form (on the left) with degrees of irreducible representations of the Rudvalis group (on the right).

In the present article we construct a second enhanced vertex operator algebra \(\tilde{\mathcal{V}}_{Ru}\) whose full automorphism group is a cyclic cover of the sporadic group of Rudvalis. The object \(\tilde{\mathcal{V}}_{Ru}\) is in some sense a mirror to \(A_{Ru}\); it shares many properties with \(A_{Ru}\) in a kind of “reciprocal” way. For example, \(\tilde{\mathcal{V}}_{Ru}\) is self-dual, as is \(A_{Ru}\), and \(\tilde{\mathcal{V}}_{Ru}\) has rank –28 — just the opposite of the rank of \(A_{Ru}\). Even at a practical level, \(\tilde{\mathcal{V}}_{Ru}\) admits a construction which runs directly parallel to that given for \(A_{Ru}\) in the preceding article [Dun06], and the format of the present article has been organized so as to emphasize this.

The structure on \(\tilde{\mathcal{V}}_{Ru}\) gives rise to Jacobi forms just as in the case of \(A_{Ru}\), and we obtain a second collection of coincidences in analogy with (0.0.1), (0.0.2).

\[406 = 406\]
\[784 = 1 + 783\]
\[31465 = 1 + 783 + 3276 + 27405\]
\[114464 = (2)378 + (3)406 + 3654 + 45500 + 63336\]
&c.

This may reassure the reader who noticed an absence of some small representations of \(Ru\) in (0.0.2); relations (0.0.2) and (0.0.3) incorporate all of the irreducible characters of \(Ru\) of degree less than \(10^5\).

Each of the enhanced vertex operator superalgebras \(A_{Ru}\) and \(\tilde{\mathcal{V}}_{Ru}\) admit canonical automorphisms of order two. (In the former case the automorphism is that associated to the super structure — a natural \(\mathbb{Z}/2\)-grading on the underlying vector space. In the latter case there is again a natural \(\mathbb{Z}/2\)-grading, but the chosen vertex algebra structure is nonetheless purely even.) Since each vertex algebra is self-dual, there are unique canonically twisted modules \((A_{Ru})_g\) and \((\tilde{\mathcal{V}}_{Ru})_g\) over \(A_{Ru}\) and \(\tilde{\mathcal{V}}_{Ru}\), respectively, and the group \(Ru\) acts projectively on each module. This allows one to attach further functions with modular properties to elements \(g \in Ru\), and in the last part of the article we use these four spaces: \(A_{Ru}\), \(\tilde{\mathcal{V}}_{Ru}\), and their canonically twisted modules, to construct certain distinguished functions \(\tilde{F}_g^A(\tau)\) and \(\tilde{F}_g^{\tilde{V}}(\tau)\) for each \(g \in Ru\).

Our main motivation for introducing this second realization of the Rudvalis group by vertex operators on \(\tilde{\mathcal{V}}_{Ru}\) is the following observation.

For each \(g \in Ru\) the functions \(\tilde{F}_g^A(\tau)\) and \(\tilde{F}_g^{\tilde{V}}(\tau)\) span a (two dimensional) representation of a discrete subgroup of \(PSL_2(\mathbb{R})\) that is commensurable with \(PSL_2(\mathbb{Z})\) and has genus zero.
Thus we obtain a genus zero property for the sporadic group of Rudvalis.

As in the case of the Monster, the reason for this property remains mysterious. We do have the advantage here that the actions of $Ru$ are sufficiently transparent that it is not hard to compute the functions $\tilde{F}_g^A$ and $\tilde{F}_g^\nu$ explicitly, and check the invariance groups directly. The genus zero property for $Ru$ is certainly weaker than that which holds for the Monster group. For example, the functions $\tilde{F}_g^A$ and $\tilde{F}_g^\nu$ are constant for some choices $g \in Ru$ (giving rise to the trivial two dimensional representation of $PSL_2(\mathbb{Z})$), but this should be compared with the statement of the generalized Moonshine conjectures above.

Evidently the genus zero phenomena, which is such a surprising aspect of Monstrous Moonshine, extends in some sense to at least one sporadic group beyond the Monster. It is now of great importance to determine if this behavior is not shared by the other non-monstrous sporadics.

The main result of this article is the following theorem.

**Theorem (5.6).** The quadruple $(\tilde{\omega}_{Ru}, Y, 1, \tilde{\omega}_{Ru})$ is a self-dual enhanced VOA of rank $-28$. The full automorphism group of $(\tilde{\omega}_{Ru}, \tilde{\omega}_{Ru})$ is a four-fold cover of the sporadic simple group of Rudvalis.

### 0.1 Outline

The format of this article runs parallel to that of [Dun06], with Weyl algebras playing the role formerly taken by Clifford algebra. For this reason, we recall only briefly the background material that is discussed more fully in [Dun06].

In §1 we review some facts about VOAs. We recall the definition in §1.1, and we recall the notion of vertex Lie algebra in §1.2. The notion of enhanced VOA is reviewed in §2.

In §3.1 we make our conventions regarding Weyl algebras, and in §3.2 the metaplectic groups. In §3.3 we review a previously known construction of VOA structure on certain Weyl algebra modules constructed in turn from a finite dimensional vector space with antisymmetric bilinear form, and in §3.4 we make some conventions regarding the Hermitian structures arising in the case that the initial vector space comes equipped also with a suitable Hermitian form.

In §4 we present a family of enhanced VOAs whose full automorphism groups are the general linear groups $GL_N(\mathbb{C})$. This section prepares the way for our main example, which is considered in §5. The construction is parallel to that of the main example of Part I [Dun06], although it is less involved, since the enhanced conformal structure admits an extremely simple description in terms of the Conway–Wales lattice. We describe this construction in §5.1 and then the construction of $\tilde{\omega}_{Ru}$ is given in §5.3. We determine the symmetry group of $\tilde{\omega}_{Ru}$ in §5.4.

In §6 we furnish explicit expressions for the McKay–Thompson series associated to the Rudvalis group via its action on $\tilde{\omega}_{Ru}$. The results of this section, along with those of the corresponding section of Part I allow us to formulate a genus zero property for certain modular functions associated to the Rudvalis group. This property is described in §6.2. We also mention in §6.2 an extension of this to the third group of Janko.

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3Here, and from here on, we will usually suppress the “super” in super-objects, so that unless extra clarification is necessary, superspaces and superalgebras will be referred to as spaces and algebras, respectively, and the term VOA for example, will be used even when the underlying vector space comes equipped with a $\mathbb{Z}/2$-grading.
0.2 Notation

Our notation follows that of [Dun06]. For example, $\mathfrak{i}$ denotes a square root of $-1$ in $\mathbb{C}$. We use $\mathbb{F}^*$ to denote the non-zero elements of a field $\mathbb{F}$. More generally, $A^\times$ shall denote the set of invertible elements in an algebra $A$. In this article we shall use $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

A superspace is a vector space with a grading by $\mathbb{Z}/2 = \{0, \bar{1}\}$. When $M$ is a superspace, we write $M = M_0 \oplus M_\bar{1}$ for the superspace decomposition, and for $u \in M$ we set $|u| = \gamma \in \{0, \bar{1}\}$ when $u$ is $\mathbb{Z}/2$-homogeneous and $u \in M_\gamma$. The dual space $M^*$ has a natural superspace structure such that $(M^*)_\gamma = (M_\bar{1})^*$ for $\gamma \in \{0, \bar{1}\}$. The space $\text{End}(M)$ admits a structure of Lie superalgebra when equipped with the Lie superbracket $[\cdot, \cdot]$ which is defined so that $[a, b] = ab - (-1)^{|a||b|}ba$ for $\mathbb{Z}/2$-homogeneous $a, b$ in $\text{End}(M)$. All formal variables will be regarded as even, so that $M((z))_{\bar{1}} = M_1((z))$ for example, for $\gamma \in \mathbb{Z}/2$.

We write $\wedge(M)$ for the full exterior algebra of a vector space $M$. We write $\wedge(M) = \wedge(M)^0 \oplus \wedge(M)^1$ for the parity decomposition of $\wedge(M)$, and we write $\wedge(M) = \bigoplus_{k \geq 0} \wedge^k(M)$ for the natural $\mathbb{Z}$-grading on $\wedge(M)$.

We denote by $D_z$ the operator on formal power series which is formal differentiation in the variable $z$, so that if $f(z) = \sum f_n z^{-n-1} \in V[[z^{\pm 1}]]$ is a formal power series with coefficients in some space $V$, we have $D_z f(z) = \sum_n (-n) f_{n-1} z^{-n-1}$. For $m$ a non-negative integer, we set $D_z^{(m)} = \frac{1}{m!} D_z^m$.

We use $\eta(\tau)$ to denote the Dedekind eta function.

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n) \quad (0.2.1)$$

Here $q = e^{2\pi i \tau}$ and $\tau$ is a variable in the upper half plane, which we denote by $\mathfrak{h}$. Recall also the Jacobi theta function $\vartheta(z|\tau)$ which we normalize so that

$$\vartheta(z|\tau) = \sum_{m \in \mathbb{Z}} e^{2\pi i zm + \pi i \tau m^2} \quad (0.2.2)$$

for $\tau \in \mathfrak{h}$ and $z \in \mathbb{C}$. According to the Jacobi Triple Product Identity we have

$$\vartheta(z|\tau) = \prod_{m \geq 0} (1 - q^{m+1})(1 + e^{2\pi i z} q^{m+1/2})(1 + e^{-2\pi i z} q^{m+1/2}) \quad (0.2.3)$$

with $q = e^{2\pi i \tau}$ as before.

The most specialized notations arise in §3. We include here a list of them, with the relevant subsections indicated in brackets. They are grouped roughly according to similarity of appearance, rather than by order of appearance, so that the list may be easier to search through, whenever the need might arise.

- a, A complex vector space with non-degenerate Hermitian form (§3.4).
- $a^*$ The dual space to $a$ (§3.4).
- $u$ A real or complex vector space of even dimension with non-degenerate bilinear form. In the case that $u = a \oplus a^*$, the bilinear form is assumed to be $1/2$ times the symmetric linear extension of the natural pairing between $a$ and $a^*$ (§3.4).
\{a_i\}_{i \in \Delta} \quad \text{A basis for } a, \text{ orthonormal in the sense that } (a_i, a_j) = \delta_{ij} \text{ for } i, j \in \Delta \ (\S 3.4).
\{a_i^\ast\}_{i \in \Delta} \quad \text{The dual basis to } \{a_i\}_{i \in \Delta} \ (\S 3.4).

g(\cdot) \quad \text{We write } g \mapsto g(\cdot) \text{ for the natural homomorphism } Mp(u) \to Sp(u). \text{ Regarding } g \in Sp(u) \text{ as an element of } Weyl(u) \times \text{ we have } g(u) = gug^{-1} \text{ in } Weyl(u) \text{ for } u \in u. \text{ More generally, we write } g(x) \text{ for } gxg^{-1} \text{ when } x \text{ is any element of } Weyl(u).

\theta \quad \text{The map which is } -1 \text{ Id on } u, \text{ or the parity involution on } Weyl(u) \ (\S 3.1), \text{ or the parity involution on } \forall(u)_\Theta \ (\S 3.3).

\rho^{1/2} \quad \text{The map which is } i \text{ Id on } a \text{ and } -1 \text{ Id on } a^*, \text{ or the lift of this map to } Weyl(u) \text{ or } \forall(u)_\Theta \ (\S 3.1).

\Delta \quad \text{A finite ordered set indexing an orthonormal basis for } u \ (\S 3.1).

\Delta' \quad \text{A second copy of } \Delta \text{ with the natural identification } \Delta \leftrightarrow \Delta' \text{ denoted } i \leftrightarrow i' \text{ for } i \in \Delta \ (\S 3.4).

\Sigma \quad \text{A finite ordered set indexing an orthonormal basis for } u \ (\S 3.1). \text{ In the case that } u = a \oplus a^*, \text{ we set } \Sigma = \Delta \cup \Delta' \text{ and insist that } \Sigma \text{ be ordered according to the ordering on } \Delta \text{ and the rule } i < j' \text{ for } i, j \in \Delta \ (\S 3.4).

\mathcal{E} \quad \text{A label for the basis } e_i \in \Sigma \ (\S 3.1).

\forall(u) \quad \text{The Weyl module VOA associated to the vector space } u \ (\S 3.3).

\forall(u)_\Theta \quad \text{The canonically twisted module over } A(u) \ (\S 3.3).

\forall(u)_\Theta \quad \text{The direct sum of } A(u)\text{-modules } A(u) \oplus \forall(u)_\Theta \ (\S 3.3).

Weyl(u) \quad \text{The Weyl algebra associated to the vector space } u \ (\S 3.1).

Mp(u) \quad \text{The metaplectic group associated to the vector space } u \ (\S 3.2).

\langle x, \cdot \rangle \quad \text{A non-degenerate antisyymmetric bilinear form on } u \text{ or on } Weyl(u) \ (\S 3.1).

\langle \cdot, \cdot \rangle \quad \text{A non-degenerate Hermitian form on } a. \text{ The Hermitian forms arising will always be antilinear in the right hand slot.}

1 Vertex algebras

We briefly review the definition of vertex algebra, and of vertex operator algebra (VOA) in \S 1.1. We recall our notational conventions regarding vertex Lie algebras in \S 1.2. For more background, we refer reader to [Dun06 \S 1] and references therein.

1.1 Vertex algebra structure

For a vertex algebra structure on a (super)space \( U = U_0 \oplus U_1 \) over a field \( \mathbb{F} \) we require the following data.

- **Vertex operators:** an even morphism \( Y : U \otimes U \to U(\{z\}) \) such that when we write \( Y(u, z)v = \sum_{n \in \mathbb{Z}} u(\alpha)vz^{-n-1} \), we have \( Y(u, z) = 0 \) only when \( u = 0 \).
- **Vacuum:** a distinguished vector \( 1 \in U_0 \) such that \( Y(1, z)u = u \) for \( u \in U \), and \( Y(u, z)1|_{z=0} = u \).

This data furnishes a vertex algebra structure on \( U \) just when the following identity is satisfied.

- **Jacobi identity:** for \( \mathbb{Z}/2 \)-homogeneous \( u, v \in U \), and for any \( m, n, l \in \mathbb{Z} \) we have

\[
\text{Res}_z Y(u, z)Y(v, w)\iota_{z, w} F(z, w) - \text{Res}_z (-1)^{|u||v|}Y(v, w)Y(u, z)\iota_{w, z} F(z, w) = \text{Res}_z (-1)^{|u||v|} Y(Y(u, z - w)v, w)\iota_{w, z - w} F(z, w) \tag{1.1.1}
\]

where \( F(z, w) = z^m w^n (z - w)^l \).
We denote such an object by the triple \((U, Y, 1)\).

A vertex operator algebra (VOA) is vertex algebra \((U, Y, 1)\) equipped with a distinguished element \(\omega \in U_0\) called the Virasoro element, such that \(L(-1) := \omega_0\) satisfies

\[
[L(-1), Y(u, z)] = D_z Y(u, z)
\]

for all \(u \in U\), and such that the operators \(L(n) := \omega(n+1)\) furnish a representation of the Virasoro algebra on the vector space underlying \(U\), so that we have

\[
[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12}\delta_{m+n,0}\text{Id}
\]

for some \(c \in \mathbb{F}\). We also require the following grading condition.

- \(L(0)\)-grading: the action of \(L(0)\) on \(U\) is diagonalizable with rational eigenvalues bounded from below, and is such that the \(L(0)\)-homogeneous subspaces \(U_n := \{u \in U \mid L(0)u = nu\}\) are finite dimensional.

When these conditions are satisfied we write \((U, Y, 1, \omega)\) in order to indicate the particular data that constitutes the VOA structure on \(U\). The value \(c\) in (1.1.3) is called the rank of \(U\), and we denote it by \(\text{rank}(U)\).

We refer the reader to [FHL93] for a discussion of VOA modules, twisted modules, adjoint operators, and intertwining operators. A VOA is said to be self-dual in the case that it has no non-trivial irreducible modules other than itself.

1.2 Vertex Lie algebra structure

The notion of vertex Lie algebra is an axiomatic formulation of the kind of object one obtains by replacing the formal series \(Y(u, z)v\) with its singular terms \(\text{Sing} Y(u, z)v = \sum_{n \geq 0} u(n) vz^{-n-1}\) in the definition of vertex algebra. More precisely, for a structure of vertex Lie algebra on a superspace \(R = R_0 \oplus R_1\) we require morphisms \(Y_\cdot : R \otimes R \to z^{-1}R[z^{-1}]\) and \(T : R \to R\) such that the following axioms are satisfied.

- **Translation:** \(Y_\cdot(Tu, z) = D_z Y(u, z)\).

- **Skew–symmetry:** \(Y_\cdot(u, z)v = \text{Sing} e^{T}Y_\cdot(v, -z)u\).

- **Jacobi identity:** for \(\mathbb{Z}/2\)-homogeneous \(u, v \in R\), and for any \(m, n, l \in \mathbb{Z}\) we have

\[
\text{Sing} \text{Res}_z Y_\cdot(u, z)Y_\cdot(v, w) \mu_{z,w} F(z, w) = \text{Sing} \text{Res}_z(-1)^{|u||v|} Y_\cdot(v, w)Y_\cdot(u, z) \mu_{w,z} F(z, w)
\]

\[
= \text{Sing} \text{Res}_{z-w} Y_\cdot(Y_\cdot(u, z-w)v, w) \mu_{w,z-w} F(z, w)
\]

where \(F(z, w) = zm w^n(z - w)^l\).
We denote such an object by $R = (R, Y, T)$. We write the image of $u \otimes v$ under $Y$ as $Y_-(u, z)v = \sum_{n \geq 0} u_n(v)u^z_n$. For any vertex algebra $U = (U, Y, 1)$ we obtain a vertex Lie algebra $(U, Y, T)$ by setting $Y_-=\text{Sing} Y$, and by setting $T$ to be the translation operator: the morphism on $U$ defined so that $Tu = u_{-2}1$ for $u \in U$. (If $(U, Y, 1, \omega)$ is a VOA, then $T$ so defined satisfies $T = L(-1)$. We abuse notation somewhat to write $\text{Sing} U$ for this object $(U, \text{Sing} Y, T)$. Conversely, to any vertex Lie algebra one may canonically associate a vertex algebra called the enveloping vertex algebra, and this construction plays an analogous role for vertex Lie algebras as universal enveloping algebras do for ordinary Lie algebras.

On the other hand, to each vertex Lie algebra $R$ is canonically associated a Lie algebra $\text{Lie}(R)$ called the local Lie algebra of $R$. As a vector space, we have $\text{Lie}(R) = R[t, t^{-1}]/\text{Im} \partial$ where $\partial$ is the operator $T \otimes 1 + \text{Id}_R \otimes D_t$ on $R[t, t^{-1}] = R \otimes \mathbb{F}[t, t^{-1}]$. Writing $u[m]$ for the image of $u \otimes t^m$ in $\text{Lie}(R)$ we have

Proposition 1.1. $\text{Lie}(R)$ is a Lie algebra under the Lie bracket

$$[u[m], v[n]] = \sum_{k \geq 0} \binom{m}{k} (u(k)v)[m+n-k]$$

and $(Tu)[m] = -mu[m-1]$ for all $u$ in $R$.

Given a vertex Lie algebra $R$ and a subset $\Omega \subset R$ we may consider the vertex Lie subalgebra generated by $\Omega$. This is by definition just the intersection of all vertex Lie subalgebras of $R$ that contain $\Omega$. When $\Omega \subset U$ for some vertex algebra $(U, Y, 1)$, we will write $[\Omega]$ for the vertex Lie subalgebra of $\text{Sing} U$ generated by $\Omega$.

2 Enhanced vertex algebras

We recall the definitions of enhanced vertex algebra and enhanced vertex operator algebra in this section. We refer the reader to [Dun06, §2] for a fuller discussion.

Definition. An enhanced vertex algebra is a quadruple $(U, Y, 1, R)$ such that $(U, Y, 1)$ is a vertex algebra, and $R$ is a vertex Lie subalgebra of $\text{Sing}(U, Y, 1)$. We say that $(U, Y, 1, R)$ is an enhanced vertex operator algebra (enhanced VOA) if there is a unique $\omega \in R$ such that

1. $(U, Y, 1, \omega)$ is a VOA, and
2. $\omega_n u = 0$ for all $n \geq 2$ whenever $u \in R$ and $\omega_1 u = u$.

When $(U, Y, 1, R)$ is an enhanced VOA, we call $(U, Y, 1, \omega)$ the underlying VOA. The element $\omega$ is called the Virasoro element of $(U, Y, 1, R)$.

In many instances, the vertex Lie algebra $R$ will be of the form $R = [\Omega]$ for some subset $\Omega \subset U$, and in such a case we will write $(U, Y, 1, \Omega)$ in place of $(U, Y, 1, [\Omega])$ since there is no loss of information. We will write $(U, \Omega)$ or even $U$ in place of $(U, Y, 1, \Omega)$ when no confusion will arise.
When \( U = (U,Y,1,R) \) is an enhanced vertex algebra we say that \( R \) determines the \textit{conformal structure} on \( U \). The automorphism group of an enhanced vertex algebra \((U,\Omega)\) is the subgroup of the group of vertex algebra automorphisms of \( U \) that fixes each element of \( \Omega \). In practice, we may write \( \text{Aut}(U,\Omega) \) in order to emphasize this.

For \( U = (U,Y,1,\Omega) \) an enhanced vertex algebra, the set \( \Omega \) is called a \textit{conformal generating set} for \( U \), and the elements of \( \Omega \) are called \textit{conformal generators}.

\textbf{Definition.} Let \((U,\Omega) = (U,Y,1,\Omega)\) be an enhanced vertex algebra and set \( A(\Omega) = \text{Lie}(\Omega) \), so that \( A(\Omega) \) is the local Lie algebra of the vertex Lie subalgebra of \( \text{Sing}(U,Y,1) \) generated by \( \Omega \). We call \( A(\Omega) \) the \textit{local algebra} associated to \((U,\Omega)\).

The \( U(1) \) \textit{Virasoro algebra} is the (purely even) Lie algebra spanned by symbols \( J_m, L_m, \text{ and } c \), for \( m \in \mathbb{Z} \), and subject to the following relations.

\[
\begin{align*}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c, \quad [L_m, c] = 0, \\
[L_m, J_n] &= -nJ_{m+n}, \quad [J_m, J_n] = -m\delta_{m+n,0}c, \quad [J_m, c] = 0.
\end{align*}
\]

\textbf{Definition.} We say that an enhanced VOA \((U,Y,1,\Omega)\) is an \textit{enhanced} \( U(1) \)-VOA if there is a unique (up to sign) \( \omega \in \Omega \) such that the Fourier components of the operators \( Y(\omega,z) = \sum L(m)z^{-m-2} \) and \( Y(\omega,z) = \sum J(m)z^{-m-1} \) furnish a representation of the \( U(1) \) Virasoro algebra \((2.0.3)\) under the assignment \( L_m \mapsto L(m) \), \( J_m \mapsto J(m) \). We require also that \( J(0) \) acts semi-simply on \( U \).

Note that for an enhanced \( U(1) \)-VOA the element \( \omega_\alpha = \omega + \alpha T_j \) may render \((U,Y,1,\omega_\alpha)\) a VOA for many choices of \( \alpha \in \mathbb{F} \). On the other hand, there is only one choice \((\alpha = 0)\) for which \( \langle \omega_\alpha | (2)J \rangle = 0 \). Recall from [Dun06, §6] that enhanced \( U(1) \)-VOAs admit two variable analogues of the usual McKay–Thompson series that are defined for ordinary VOAs.

\section{Weyl algebras}

The construction of VOAs that we will use arises from certain Weyl algebra modules. In this section we recall some basic properties of Weyl algebras. We discuss briefly the metaplectic groups \( \text{Mp}_N \) in §3.2 and in §3.3 we recall the construction of VOA module structure on modules over certain (infinite dimensional) Weyl algebras. In §3.4 we review the Hermitian structure that arises naturally on these objects given the existence of a suitable Hermitian form.

\subsection{Weyl algebra structure}

Recall that \( \mathbb{F} \) denotes either \( \mathbb{R} \) or \( \mathbb{C} \). We suppose that \( u \) is an \( \mathbb{F} \)-vector space of even dimension with non-degenerate antisymmetric bilinear form \( \langle \cdot , \cdot \rangle \).

We write \( \text{Weyl}(u) \) for the Weyl algebra over \( \mathbb{F} \) generated by \( u \). More precisely, we set \( \text{Weyl}(u) = T(u)/I(u) \) where \( T(u) \) is the tensor algebra of \( u \) over \( \mathbb{F} \) with unit denoted \( 1 \), and \( I(u) \) is the ideal

\[\text{Then there is a convenient coincidence with the terminology of [Kac98], where the objects we refer to as vertex Lie algebras are called \textit{conformal algebras}.}\]
of $T(u)$ generated by all expressions of the form $u \otimes v - v \otimes u + 2 \langle \langle u, v \rangle \rangle 1$ for $u, v \in u$. The natural algebra structure on $T(u)$ induces an associative algebra structure on $\text{Weyl}(u)$. The vector space $u$ embeds naturally in $\text{Weyl}(u)$, and when it is convenient we identify $u$ with its image in $\text{Weyl}(u)$. We also write $a$ in place of $\alpha 1 + I(u) \in \text{Weyl}(u)$ for $\alpha \in F$ when no confusion will arise. For $u, v \in u$ we have the relation $uv - vu = -2 \langle \langle u, v \rangle \rangle$ in $\text{Weyl}(u)$, and more generally

$$u^m v^n = \sum_{k \geq 0} \binom{m}{k} \binom{n}{k} 2^k \langle \langle v, u \rangle \rangle^{k} u^{m-k}$$

(3.1.1)

for nonnegative integers $m$ and $n$.

The linear transformation on $u$ which is $-1$ times the identity map lifts naturally to $T(u)$ and preserves $I(u)$, and hence induces an involution on $\text{Weyl}(u)$ which we denote by $\theta$. The map $\theta$ is known as the parity involution. We have $\theta(u_1 \cdots u_k) = (-1)^k u_1 \cdots u_k$ for $u_1 \cdots u_k \in \text{Weyl}(u)$ with $u_i \in u$, and we write $\text{Weyl}(u) = \text{Weyl}(u)^{0} \oplus \text{Weyl}(u)^{1}$ for the decomposition into eigenspaces for $\theta$.

A linear isomorphism $\lambda : \sqrt{u} \to \text{Weyl}(u)$ is defined by setting

$$\lambda(u_1 \vee \cdots \vee u_k) = \frac{1}{k!} \sum_{\sigma \in S_k} u_{\sigma 1} \cdots u_{\sigma k} 1$$

(3.1.2)

for $u = u_1 \vee \cdots \vee u_k$ with $u_i \in u$. The antisymmetric bilinear form on $u$ extends naturally to the symmetric algebra $\sqrt{u}$ by decreeing that $\langle \langle \langle \lambda^{-1}(a), \lambda^{-1}(b) \rangle \rangle = 0$ in $\text{Weyl}(u)$. Evidently, this form is antisymmetric on odd symmetric powers of $u$, and symmetric on even powers. Now we may use the map $\lambda$ to imbue $\text{Weyl}(u)$ with a bilinear form by setting

$$\langle \langle a, b \rangle \rangle = \langle \langle \lambda^{-1}(a), \lambda^{-1}(b) \rangle \rangle$$

(3.1.4)

for $a, b \in \text{Weyl}(u)$. With this definition we have $\langle \langle 1, 1 \rangle \rangle = 1$ and $\langle \langle ua, b \rangle \rangle = -\langle \langle a, ub \rangle \rangle$ for $a, b \in \text{Weyl}(u)$ and $u \in u$, and the restriction to $u \to \text{Weyl}(u)$ agrees with the original form on that space.

**Example.** We can compute $\langle \langle 1, uv \rangle \rangle$ for $u, v \in u$, by observing that

$$\lambda(u \vee v) = \frac{1}{2}(uv + vu) = uv + \langle \langle u, v \rangle \rangle$$

(3.1.5)

in Weyl($u$), so that $\langle \langle 1, uv \rangle \rangle = 0$ in $\sqrt{u}$ implies $\langle \langle 1, uv \rangle \rangle = -\langle \langle u, v \rangle \rangle$.

### 3.2 Metaplectic groups

We would like to define an enlargement of Weyl($u$) that will contain expressions like $\exp(tu^2)$ for $t \in F$ and $u \in u$. Our construction follows [Cru90].

Let $\{e_i, f_i\}_{i=1}^{N}$ be a basis for $u$ satisfying $\langle \langle e_i, f_j \rangle \rangle = \delta_{ij}$. For $H = (h_1, \ldots, h_N)$ in $\mathbb{Z}_{\geq 0}^N$ we write $e^H$ for $e_{h_1} \cdots e_{h_N}$, and similarly for $f^H$. Then Weyl($u$) is spanned by the expressions $e^H f^K$ for
\( H, K \in \mathbb{Z}_{\geq 0} \). We set \( H! = \prod h_i! \) and \( |H| = \sum h_i \). We define \( \tilde{\text{Weyl}}(u) \) to be the set of expressions of the form

\[
\tilde{u} = \sum_{H,K} \frac{M_{H,K} e^{HfK}}{(H!K!)^{1/2}}
\]

(3.2.1)

where the \( M_{H,K} \in \mathbb{F} \) should satisfy \( |M_{H,K}| \leq C/2^{|H|+|K|} \) for some constant \( C \), for all \( H \) and \( K \) such that \( |H| \) and \( |K| \) are sufficiently large. Then \( \text{Weyl}(u) \) embeds naturally in \( \tilde{\text{Weyl}}(u) \), and the latter space is closed under the algebra structure it naturally inherits from the former.

Let \( \tilde{\text{Weyl}}(u)^\times \) denote the set of invertible elements in \( \tilde{\text{Weyl}}(u) \). For \( x \in \tilde{\text{Weyl}}(u)^\times \) and \( a \in \text{Weyl}(u) \) we set \( x(a) = xax^{-1} \). We define the \textit{metaplectic group associated to } \( u \) to be the subgroup \( Mp(u) \) of \( \text{Weyl}(u) \) generated by expressions of the form \( \alpha \exp(tu^2) \) for \( \alpha, t \in \mathbb{F} \) and \( u \in u \), with \( |\alpha| = 1 \). The identity

\[
\exp(tu^2)v\exp(-tu^2) = v - 4\langle u, v \rangle tu
\]

(3.2.2)

for \( v \in u \) shows that \( x(\cdot) \) preserves \( u \) for \( x \inMp(u) \), and further, that the map \( x \mapsto x(\cdot) \) defines a surjection \( Mp(u) \to Sp(u) \). We have a short exact sequence

\[
1 \to T \to Mp(u) \to Sp(u) \to 1
\]

(3.2.3)

where \( T \) denotes the elements of unit norm in \( T \).

### 3.3 Weyl module VOAs

In this section we review the construction of VOA structure on modules over a certain infinite dimensional Weyl algebra associated to \( u \). The construction appears also in [Wei94], and we refer the reader there for further details. Another important precursor is the article [FFS5] in which the construction below, and generalizations thereof, are applied to the theory of affine Lie (super)algebras.

Suppose that \( u \) has dimension \( 2N \). Let \( \Delta \) be a set of cardinality \( N \), and let \( \Delta' \) be a second copy of this set, with the natural correspondence denoted \( i \leftrightarrow i' \). (The inverse of the map \( i \mapsto i' \) will also be denoted by \( ' \), so that \( i'' = i \).) We assume chosen a basis \( E = \{ e_i, e_{i'} \}_{i \in \Delta} \) for \( u \) such that

\[
\langle e_i, e_{i'} \rangle = -\langle e_{i'}, e_j \rangle = \delta_{ij}
\]

(3.3.1)

Let \( \hat{u} \) and \( \hat{u}_\theta \) denote the infinite dimensional vector spaces with antisymmetric bilinear form described as follows.

\[
\hat{u} = \prod_{m \in \mathbb{Z}} u \otimes t^{m+1/2}, \quad \hat{u}_\theta = \prod_{m \in \mathbb{Z}} u \otimes t^m,
\]

(3.3.2)

\[
\langle u \otimes t^r, v \otimes t^s \rangle = \langle u, v \rangle \delta_{r+s,0}, \quad \text{for } u, v \in u \text{ and } r, s \in \frac{1}{2} \mathbb{Z}.
\]

(3.3.3)

We write \( u(r) \) for \( u \otimes t^r \) when \( u \in u \) and \( r \in \frac{1}{2} \mathbb{Z} \). We consider the Weyl algebras \( \text{Weyl}(\hat{u}) \) and \( \text{Weyl}(\hat{u}_\theta) \). By the conventions of 3.1 we have the following \textit{fundamental relation}

\[
[u(r), u'(s)] = -2\langle u, u' \rangle \delta_{r+s,0}
\]

(3.3.4)

for any \( u, u' \in u \), holding in \( \text{Weyl}(\hat{u}) \) for \( r, s \in \mathbb{Z} + \frac{1}{2} \), and holding in \( \text{Weyl}(\hat{u}_\theta) \) for \( r, s \in \mathbb{Z} \).
Remark. Our generators $e_i(r)$ and $e_i′(r)$ correspond to $\sqrt{2}a_{i}^{-}(r)$ and $\sqrt{2}a_{i}^{+}(r)$, respectively, in the notation of [Wei94].

The inclusion of $u$ in $\hat{u}_\theta$ given by $u \mapsto u(0)$ induces an embedding of algebras $\text{Weyl}(u) \hookrightarrow \text{Weyl}(\hat{u}_\theta)$.

We write $\mathcal{B}(\hat{u})$ for the subalgebra of $\text{Weyl}(\hat{u})$ generated by the $u(m + \frac{1}{2})$ for $u \in \mathfrak{u}$ and $m \in \mathbb{Z}_{\geq 0}$. We write $\mathcal{B}(\hat{u}_\theta)$ for the subalgebra of $\text{Weyl}(\hat{u}_\theta)$ generated by the $u(m)$ for $u \in \mathfrak{u}$ and $m \in \mathbb{Z}_{> 0}$, and by the $u(0)$ for $u \in \mathfrak{a}^*$. Let $\mathbb{F}_0$ denote a one-dimensional module for either $\mathcal{B}(\hat{u})$ or $\mathcal{B}(\hat{u}_\theta)$, spanned by a vector $1_0$, such that $u(r)1_0 = 0$ whenever $r \in \frac{1}{2}\mathbb{Z}_{> 0}$, and such that $u(0)1_0 = 0$ for $u \in \mathfrak{a}^*$. We write $\forall(u)$ (respectively $\forall(u)_\theta$) for the $\text{Weyl}(\hat{u})$-module (respectively $\text{Weyl}(\hat{u}_\theta)$-module) induced from the $\mathcal{B}(\hat{u})$-module structure (respectively $\mathcal{B}(\hat{u}_\theta)$-module structure) on $\mathbb{F}_0$.

$$\forall(u) = \text{Weyl}(\hat{u}) \otimes_{\mathcal{B}(\hat{u})} \mathbb{F}_0, \quad \forall(u)_\theta = \text{Weyl}(\hat{u}_\theta) \otimes_{\mathcal{B}(\hat{u}_\theta)} \mathbb{F}_0.$$ (3.3.5)

We write $1$ for the vector $1 \otimes 1_0$ in $\forall(u)$, and we write $1_\theta$ for the vector $1 \otimes 1_0$ in $\forall(u)_\theta$.

The space $\forall(u)$ supports a structure of VOA. In order to define the vertex operators we review the notion of bosonic normal ordering for elements in $\text{Weyl}(\hat{u})$ and $\text{Weyl}(\hat{u}_\theta)$. The bosonic normal ordering on $\text{Weyl}(\hat{u})$ is the multi-linear operator defined so that for $u_i \in \mathfrak{u}$ and $r_i \in \mathbb{Z} + \frac{1}{2}$ we have

$$:u_1(r_1) \cdots u_k(r_k): = u_{\sigma 1}(r_{\sigma 1}) \cdots u_{\sigma k}(r_{\sigma k})$$ (3.3.6)

where $\sigma$ is any permutation of the index set $\{1, \ldots, k\}$ such that $r_{\sigma 1} \leq \cdots \leq r_{\sigma k}$. For elements in $\text{Weyl}(\hat{u}_\theta)$ the bosonic normal ordering is defined in steps by first setting

$$:u_1(0) \cdots u_k(0): = \frac{1}{k!} \sum_{\sigma \in S_k} u_{\sigma 1}(0) \cdots u_{\sigma k}(0)$$ (3.3.7)

for $u_i \in \mathfrak{u}$. Then in the situation that $n_i \in \mathbb{Z}$ are such that $n_i \leq n_{i+1}$ for all $i$, and there are some $s$ and $t$ (with $1 \leq s \leq t \leq k$) such that $n_j = 0$ for $s \leq j \leq t$, we set

$$:u_1(n_1) \cdots u_k(n_k): = u_1(n_1) \cdots u_{s-1}(n_{s-1}) :u_s(0) \cdots u_t(0) :u_{t+1}(n_{t+1}) \cdots u_k(n_k)$$ (3.3.8)

Finally, for arbitrary $n_i \in \mathbb{Z}$ we set

$$:u_1(n_1) \cdots u_k(n_k): = :u_{\sigma 1}(n_{\sigma 1}) \cdots u_{\sigma k}(n_{\sigma k}):$$ (3.3.9)

where $\sigma$ is again any permutation of the index set $\{1, \ldots, k\}$ such that $n_{\sigma 1} \leq \cdots \leq n_{\sigma k}$, and we extend the definition multi-linearly to $\text{Weyl}(\hat{u}_\theta)$.

For $u \in \mathfrak{u}$ we now define the generating function, denoted $u(z)$, of operators on $\forall(u)_\theta = \forall(u) \oplus \forall(u)_\theta$ by setting

$$u(z) = \sum_{r \in \frac{1}{2}\mathbb{Z}} u(r)z^{-r-1/2}$$ (3.3.10)
Note that \( u(r) \) acts as 0 on \( \forall(u) \) if \( r \in \mathbb{Z} \), and acts as 0 on \( \forall(u)_\theta \) if \( r \in \mathbb{Z} + \frac{1}{2} \). To an element \( a \in \forall(u) \) of the form \( a = u_1(-m_1 - \frac{1}{2}) \cdots u_k(-m_k - \frac{1}{2}) \) for \( u_i \in u \) and \( m_i \in \mathbb{Z}_{\geq 0} \), we associate the operator valued power series \( \overline{Y}(a,z) \), given by

\[
\overline{Y}(a,z) = D_z^{(m_1)} u_{i_1}(z) \cdots D_z^{(m_k)} u_{i_k}(z) : \quad (3.3.11)
\]

We define the vertex operator correspondence

\[
Y(\cdot, z) : \forall(u) \otimes \forall(u)_\Theta \to \forall(u)_\Theta((z^{1/2})) \quad (3.3.12)
\]

by setting \( Y(a, z)b = \overline{Y}(a, z)b \) when \( b \in \forall(u) \), and by setting \( Y(a, z)b = \overline{Y}(e^{\Delta z}a, z)b \) when \( b \in \forall(u)_\theta \), where \( \Delta_z \) is the expression defined by

\[
\Delta_z = -\frac{1}{2} \sum_i \sum_{m,n \in \mathbb{Z}_{\geq 0}} C_{mn} e_i(m + \frac{1}{2}) e_i(n + \frac{1}{2}) z^{-m-n-1} \quad (3.3.13)
\]

\[
C_{mn} = \frac{1}{2} m + n + 1 \begin{pmatrix} \frac{1}{2} m - n \\ n \end{pmatrix} \quad (3.3.14)
\]

Define \( \omega \) by setting

\[
\omega = \frac{1}{4} \sum_i (e_i(-\frac{1}{2}) e_i(-\frac{3}{2}) 1 - e_i(-\frac{1}{2}) e_i'(\frac{3}{2}) 1) \in \forall(u)_2 \quad (3.3.15)
\]

Then we have the following

**Theorem 3.1** ([Wei94]). The map \( Y \) defines a structure of VOA of rank \(-N\) on \( \forall(u) \) when restricted to \( \forall(u) \otimes \forall(u) \), and the Virasoro element is given by \( \omega \). The map \( Y \) defines a structure of \( \theta \)-twisted \( \forall(u) \)-module on \( \forall(u)_\theta \) when restricted to \( \forall(u) \otimes \forall(u)_\theta \).

Given \( I = (i_1, \ldots, i_N) \in \Sigma^N \) with \( \Sigma = \Delta \cup \Delta' \), let us write \( e_I(-\frac{1}{2}) \) for \( e_{i_1}(-\frac{1}{2}) \cdots e_{i_N}(-\frac{1}{2}) \). Observe then that \( \forall(u)_2 \) is spanned by vectors of the form \( e_I(-\frac{1}{2}) 1 \) for \( I \in \Sigma^4 \), and by the \( e_i(-\frac{3}{2}) e_j(-\frac{1}{2}) 1 \) with \( i, j \in \Sigma \).

All we need to know about the expressions \( Y(a, z)b \) for \( b \in \forall(u)_\theta \) is contained in the following

**Proposition 3.2.** Let \( b \in \forall(u)_\theta \).

1. If \( a = 1 \in \forall(u)_0 \) then \( Y(a, z)b = b \).
2. If \( a \in \forall(u)_1 \) then \( \Delta_z a = 0 \) so that \( Y(a, z)b = \overline{Y}(a, z)b \).
3. If \( a \in \forall(u)_2 \) and \( a \in \text{Span} \{ e_I(-\frac{1}{2}) 1, e_i(-\frac{3}{2}) e_j(-\frac{1}{2}) 1 \mid i' \neq j \} \) then \( \Delta_z a = 0 \) and \( Y(a, z)b = \overline{Y}(a, z)b \).
4. For \( a = e_i(-\frac{1}{2}) e_i(-\frac{3}{2}) 1 \) and \( i \in \Delta \) we have \( \Delta_z a = -\frac{1}{4} z^{-2} \) and \( \Delta_z^2 a = 0 \) so that \( Y(a, z)b = \overline{Y}(a, z)b - \frac{1}{4} b z^{-2} \) in this case.
5. For \( a = e_i(-\frac{1}{2}) e_i'(-\frac{3}{2}) 1 \) and \( i \in \Delta \) we have \( \Delta_z a = \frac{1}{4} z^{-2} \) and \( \Delta_z^2 a = 0 \) so that \( Y(a, z)b = \overline{Y}(a, z)b + \frac{1}{4} b z^{-2} \) in this case.
As a corollary of Proposition 3.2 we have that

\[ Y(\omega, z)1_\theta = -\frac{N}{8}1_\theta z^{-2} \]  

(3.3.16)

and consequently the \(L(0)\)-grading on \(\forall(u)_\theta\) is given as follows.

\[ \forall(u) = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \forall(u)_n, \quad \forall(u)_\theta = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_{\geq 0}} (\forall(u)_\theta)_{n-N/8}. \]  

(3.3.17)

Note that \(\forall(u)\) is strongly generated (c.f. [Kac98]) by it’s degree 1/2 subspace in the sense that we have

\[ \forall(u) = \text{Span} \left\{ u^i(-m_1 - \frac{1}{2})\cdots u^k(-m_k - \frac{1}{2})1 \mid u^i \in u, \ m_i \in \mathbb{Z}_{\geq 0} \right\} \]  

(3.3.18)

Armed with this observation it is not difficult to see that \(\forall(u)\) is a self-dual VOA.

**Proposition 3.3.** Let \(M\) be an irreducible \(\forall(u)\)-module with grading bounded from below. Then \(M\) is isomorphic to \(\forall(u)\) as a \(\forall(u)\)-module. In particular, \(\forall(u)\) is a self-dual VOA.

**Proof.** Let \(v_0 \in M\) be a homogeneous non-zero element of minimal degree. Then we have \(u(m + \frac{1}{2})v_0 = 0\) for any \(u \in u\) whenever \(m \geq 0\). We claim that \(u(-m - \frac{1}{2})v_0 \neq 0\) for any non-zero \(u \in u\) and for all \(m \geq 0\) since otherwise we have

\[ 0 = u'(m + \frac{1}{2})u(-m - \frac{1}{2})v_0 = [u'(m + \frac{1}{2}), u(-m - \frac{1}{2})]v_0 = \langle u, u' \rangle v_0 \]  

(3.3.19)

with the second equality holding because \(u'(m + \frac{1}{2})v_0 = 0\). Evidently then, the map \(\forall(u) \to M\) given by

\[ u^1(-m_1 - \frac{1}{2})\cdots u^k(-m_k - \frac{1}{2})1 \mapsto u^1(-m_1 - \frac{1}{2})\cdots u^k(-m_k - \frac{1}{2})v_0 \]  

(3.3.20)

for \(u^i \in u\) and \(m_i \in \mathbb{Z}_{\geq 0}\) is an embedding of \(\forall(u)\)-modules. (The map is defined on all of \(\forall(u)\) by (3.3.18).) Since \(M\) is irreducible by hypothesis, the image of this map is all of \(M\), and the proposition follows.

The metaplectic group \(Mp(u)\) acts naturally on \(\forall(u)_\theta\). This action is generated by exponentials of the operators \(x_0\) for \(x \in \forall(u)_1\). In particular, any \(a \in Mp(u) \subset \text{Weyl}(u)\) may be regarded as a VOA automorphism of \(\forall(u)\), and as an equivariant linear isomorphism of the \(\forall(u)\)-module \(\forall(u)_\theta\).

### 3.4 Hermitian structure

In this section we take special interest in the case that \(\mathbb{F} = \mathbb{C}\) and \(u\) is of the form \(u = a \oplus a^*\) for \(a\) a complex vector space with non-degenerate Hermitian form, and \(a^*\) the dual space to \(a\). The Hermitian form will be denoted \(\langle \cdot, \cdot \rangle\), and we assume it to be antilinear in the second slot. We extend \(\langle \cdot, \cdot \rangle\) in the natural way to a Hermitian form on \(u\). Suppose that \(a\) has dimension \(N\). We
take $\Delta$ to be some set of cardinality $N$, and we use it to index some orthonormal basis $\{a_i\}_{i \in \Delta}$ for $\mathfrak{a}$. We then let $\{a_i^*\}_{i \in \Delta}$ denote the dual basis, so that the expressions
\[(\alpha a_i, \beta a_j) = (\alpha a_i^*, \beta a_j^*) = \alpha \beta \delta_{ij}, \quad (a_i, a_j^*) = (a_i^*, a_j) = 0, \quad (3.4.1)\]
completely describe the Hermitian form on $\mathfrak{u}$.

There is a natural symmetric bilinear form on $\mathfrak{u}$ induced from the pairing between $\mathfrak{a}$ and $\mathfrak{a}^*$. We take $\langle \cdot, \cdot \rangle$ to be $1/2$ times this form. More precisely, we set $\langle a, f \rangle = \langle f, a \rangle = \frac{1}{2} f(a)$ for $a \in \mathfrak{a}$ and $f \in \mathfrak{a}^*$. We make the convention that $\mathcal{E}$ shall denote the basis for $\mathfrak{u}$ obtained in the following way. We let $\Delta'$ be another copy of the set $\Delta$ with the natural identification $\Delta \leftrightarrow \Delta'$ denoted $i \leftrightarrow i'$, and we set $\mathcal{E} = \{e_i, e_i'\}_{i \in \Delta}$ where
\[e_i = a_i + a_i^*, \quad e_i' = i(a_i - a_i^*), \quad (3.4.2)\]
for $i \in \Delta$. One can check that the basis $\mathcal{E}$ is orthonormal with respect to the symmetric form $\langle \cdot, \cdot \rangle$.

Let $\theta^{1/2}$ be the map which is multiplication by $i$ on $\mathfrak{a}$, and multiplication by $-i$ on $\mathfrak{a}^*$. Then $\theta^{1/2}$ preserves both the Hermitian form $\langle \cdot, \cdot \rangle$ and the symmetric form $\langle \cdot, \cdot \rangle$, and we may define an antisymmetric form $\langle \langle \cdot, \cdot \rangle$ on $\mathfrak{u}$ by setting
\[\langle\langle u, v \rangle\rangle = \langle \theta^{1/2} u, v \rangle \quad (3.4.3)\]
for $u, v \in \mathfrak{u}$. (Mnemonic: “$\langle\rangle$” is a shorthand for “$\langle \theta^{1/2}\rangle$.”) We then have $\langle\langle e_i, e_j' \rangle\rangle = -\langle\langle e_i', e_j \rangle\rangle = \delta_{ij}$ for $i, j \in \Delta$. (Compare with the convention of $\langle\langle \cdot, \cdot \rangle\rangle$.) We also have $\langle\langle a_i, a_j^* \rangle\rangle = -\langle\langle a_j^*, a_i \rangle\rangle = \frac{1}{2} i \delta_{ij}$ so that
\[a_i^* a_j = i \delta_{ij} + a_j a_i^* \quad (3.4.4)\]
in Weyl($\mathfrak{u}$), by the relation $\langle\langle \cdot, \cdot \rangle\rangle$.

### 4 Linear groups

In this section we define a family of enhanced VOAs with symmetry groups of the form $GL_N(\mathbb{C})$ for $N$ a positive integer. In defining these objects we are laying some of the groundwork for the construction of an enhanced VOA for the Rudvalis group in §5.

Let $N$ be a positive integer and let $\mathfrak{a}$ be a complex vector space of dimension $N$ equipped with a positive definite Hermitian form denoted $\langle \cdot, \cdot \rangle$. As in §3.4 we set $\mathfrak{u} = \mathfrak{a} \oplus \mathfrak{a}^*$, we extend the Hermitian form to $\mathfrak{u}$, and equip this space also with the antisymmetric bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$, naturally determined as in §3.4.

Let us denote $\mathcal{O}_U = \{\omega, j\}$ where $\omega$ and $j$ are given by
\[\omega = \frac{1}{4} \sum_{i \in \Delta} (e_i(-\frac{1}{2})e_i(-\frac{3}{2})1 - e_i(-\frac{1}{2})e_i(-\frac{3}{2})1) = \frac{1}{2} \sum_{i \in \Delta} (a_i(-\frac{1}{2})a_i^*(-\frac{3}{2})1 - a_i^*(-\frac{1}{2})a_i(-\frac{3}{2})1) \in \forall(\mathfrak{u})_2 \quad (4.0.5)\]
\[j = \sum_{i \in \Delta} a_i(-\frac{1}{2})a_i^*(-\frac{3}{2})1 \in \forall(\mathfrak{u})_1, \quad (4.0.6)\]
so that $\omega$ is just the usual Virasoro element for $\forall(u)$, and $j$ is an element dependent upon the Hermitian structure on $a$. We define operators $L(n)$ and $J(n)$ for $n \in \mathbb{Z}$ by setting

$$Y(\omega, z) = L(z) = \sum L(n)z^{-n-2},$$  \hspace{1cm} (4.0.7)

$$Y(j, z) = J(z) = \sum J(n)z^{-n-1}.$$

We then have

**Proposition 4.1.** The operators $L(z)$ and $J(z)$ satisfy the following OPEs.

$$L(z)L(w) = \frac{-N/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{D_wL(w)}{(z-w)} + \text{reg.}$$  \hspace{1cm} (4.0.9)

$$L(z)J(w) = \frac{J(w)}{(z-w)^2} + \frac{D_wJ(w)}{(z-w)} + \text{reg.}$$  \hspace{1cm} (4.0.10)

$$J(z)J(w) = \frac{N}{(z-w)^2} + \text{reg.}$$  \hspace{1cm} (4.0.11)

**Corollary 4.2.** The operators $L(m)$ and $J(m)$ satisfy the following commutation relations for $m, n \in \mathbb{Z}$.

$$[L(m), L(n)] = (m-n)L(m+n) - \frac{m^3-m}{12}N\delta_{m+n,0} \text{Id}$$  \hspace{1cm} (4.0.12)

$$[L(m), J(n)] = -nJ(m+n)$$  \hspace{1cm} (4.0.13)

$$[J(m), J(n)] = mN\delta_{m+n,0} \text{Id}$$  \hspace{1cm} (4.0.14)

**Proposition 4.3.** For $N$ a positive integer, the quadruple $(\forall(u), Y, 1, U_{\text{U}})$ is a self-dual $U(1)$–VOA of rank $-N$.

**Proof.** We will show that $(\forall(u), U_{\text{U}})$ admits a $U(1)$–VOA structure (c.f. §2). The fact that the 4-tuple $(\forall(u), Y, 1, \omega)$ is a VOA of rank $-N$ is the content of Theorem 4.1 and self-duality was shown in Proposition 3.3.

From Proposition 4.1 we see that $U_{\text{U}}$ has defect 0, so that the vertex Lie algebra generated by $U_{\text{U}}$ is just $[U_{\text{U}}] = \text{Span}\{1, T^kj, T^k\omega \mid k \in \mathbb{Z}\}$ (c.f. [Dun06, §2]). From this explicit spanning set we see that the only Virasoro elements for $\forall(u)$ in $[U_{\text{U}}]$ are of the form $\omega_\alpha = \omega + \alpha Tj$ for some $\alpha \in \mathbb{C}$. (The operator $T$ maps $u = u_{(-1)}$ to $u_{(-2)}$.) Using Corollary 4.2 we check that the Fourier coefficients of $Y(\omega, z) = L_\alpha(m)z^{-m-2}$ furnish a representation of the Virasoro algebra of rank $-N(1 + 12\alpha^2)$, and we have $L_\alpha(n) = L(n) + \alpha(-n-1)J(n)$ where $Y(\omega, z) = \sum L(m)z^{-m-2}$ and $Y(j, z) = \sum J(m)z^{-m-1}$. In particular, $L_\alpha(-1) = L(-1) = T$ for all $\alpha$, and $L_\alpha(0) = L(0) = \alpha J(0)$. 
Thus if \( u \in \mathfrak{U} \) and \( L_\alpha(0)u = u \) then \( u \in \text{Span}\{j\} \). Now we compute \( L_\alpha(1)j = \alpha(-2)J(1)j = 2\alpha N \), and it follows that \( (\forall(u), Y, 1, \mathfrak{U}) \) is an enhanced VOA with the Virasoro element given by \( \omega = \omega_0 \). From Corollary 4.2 we see that \( j \) is the unique (up to sign) choice of element in \( [\Omega_\mathfrak{U}] \) for which the component operators of \( Y(j, z) \) satisfy the required relations \( (2.0.3) \), and thus \( (\forall(u), \mathfrak{U}) \) is a \( U(1) \)-VOA.

We would like to compute the automorphism group of \( (\forall(u), \mathfrak{U}) \). First we will compute the automorphism group of the underlying VOA \( (\forall(u), \omega) \).

**Proposition 4.4.** The automorphism group of \( (\forall(u), \omega) \) is \( \text{Sp}(u) \).

**Proof.** Observe that there is a natural correspondence \( u \leftrightarrow \forall(u)_{1/2} \) given by \( u \leftrightarrow u(-\frac{1}{2})1 \), and that we recover the corresponding antisymmetric form on \( \forall(u)_{1/2} \) by defining \( \langle a\mid b \rangle \) for \( a, b \in \forall(u)_{1/2} \) in such a way that \( a_0b = \langle a\mid b \rangle 1 \). (The skew-symmetry of the form follows from the skew-symmetry property of a VOA: \( Y(u, z)v = e^{zL(-1)}Y(v, -z)u \).) Any automorphism of the VOA \( \forall(u) \) must preserve the homogeneous subspace \( \forall(u)_{1/2} \), and must preserve the bracket \( [u, v] = u_0v \), and thus we obtain a map \( \phi : \text{Aut}(\forall(u), \omega) \rightarrow \text{Sp}(u) \). On the other hand, \( \forall(u) \) is strongly generated by its subspace of degree 1/2 (c.f. \([33.18]\)), in the sense that we have

\[
\forall(u) = \text{Span}\left\{u_{-n_1}^1 \cdots u_{-n_k}^k 1 \mid u^i \in \forall(u)_{1/2}, n_i > 0, k \geq 0\right\} \tag{4.0.15}
\]

and we obtain a map \( \psi : \text{Sp}(u) \rightarrow \text{Aut}(\forall(u), \omega) \), by letting \( g \in \text{Sp}(u) \) act on \( \forall(u) \) in the following way.

\[
g : u_{-n_1}^1 \cdots u_{-n_k}^k 1 \mapsto (gu^1)_{-n_1} \cdots (gu^k)_{-n_k} 1 \tag{4.0.16}
\]

Evidently, \( \psi \) is an inverse to \( \phi \). \qed

**Proposition 4.5.** The automorphism group of \( (\forall(u), \mathfrak{U}) \) is \( \text{GL}(a) \).

**Proof.** Recall that \( j = \sum a_i(-\frac{1}{2})a_i(-\frac{1}{2})1 \). From the previous proposition we have \( \text{Aut}(\forall(u), \omega) = \text{Sp}(u) \). The group \( \text{Aut}(\forall(u), \mathfrak{U}) \) is just the subgroup of \( \text{Aut}(\forall(u)) \) that fixes \( j \). Consider the automorphism of \( \forall(u) \) obtained by setting \( \theta^{1/2} = \exp(\pi J(0)/2) \). It is the automorphism of \( \forall(u) \) induced by the orthogonal transformation of \( u \) which is multiplication by \( i \) on \( a \), and multiplication by \( -1 \) on \( a^* \). Clearly, any element of \( \text{Aut}(\forall(u), \mathfrak{U}) \) commutes with \( \theta^{1/2} \). On the other hand, if \( g \in \text{Sp}(u) \) commutes with \( \theta^{1/2} \) then \( g \) preserves the decomposition \( u = a \oplus a^* \), and with respect to the basis \( \{a_1, \ldots, a_1^*, \ldots\} \) is represented by a block matrix

\[
g \sim \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{4.0.17}
\]

and we must be able to write \( A = T_g^d \) and \( B = T_g^{-1} \) for some invertible \( N \times N \) matrix \( T_g \) if it is the case that \( g \) represents an element of \( \text{Sp}(u) \). (Here \( T_g^d \) denotes the transpose of \( T_g \).) Evidently \( \text{Aut}(\forall(u), \mathfrak{U}) \) is the centralizer of \( \theta^{1/2} \) in \( \text{Sp}(u) \), and this group is all matrices of the form \( (4.0.17) \), with \( A \) invertible and \( B \) the inverse transpose of \( A \). This is a copy of the group \( \text{GL}(a) \). \qed

We record the results of this section in

**Theorem 4.6.** For \( N \) a positive integer, the quadruple \( (\forall(u), Y, 1, \mathfrak{U}) \) is a self-dual \( U(1) \)-VOA of rank \(-N\). The full automorphism group of \( (\forall(u), \mathfrak{U}) \) is \( \text{GL}(a) \).
5 The Rudvalis group

In this section we realize the sporadic simple group of Rudvalis as symmetries of an enhanced VOA $(\tilde{\mathcal{V}}_{\text{Ru}}, \tilde{\mathcal{V}}_{\text{Ru}})$. Our plan for realizing the Rudvalis group as symmetry of an enhanced VOA is to consider first the enhanced VOA for $GL_N(\mathbb{C})$ for suitable $N$, constructed in §4, and then to find a single extra conformal generator with which to refine the conformal structure, just to the point that the Rudvalis group becomes visible.

We give two constructions of this extra conformal generator. The first construction arises directly from the geometry of the Conway–Wales lattice [Con 77], and is given in §5.1. We include also in this section an explicit description of this lattice. The second construction, which is perhaps more convenient for computations, is a description in terms of a particular maximal subgroup of a double cover of the Rudvalis group, and is given in §5.2. Just as in the companion article [Dun06], §5.1 and §5.2 are independent, and the reader may safely skip one in favor of the other. The approach of §5.1 is extremely simple.

The enhanced VOA structure for $(\tilde{\mathcal{V}}_{\text{Ru}}, \tilde{\mathcal{V}}_{\text{Ru}})$ is described in §5.3. We show in §5.4 that the full automorphism group of $(\tilde{\mathcal{V}}_{\text{Ru}}, \tilde{\mathcal{V}}_{\text{Ru}})$ is a certain four fold cover of the Rudvalis group.

5.1 Geometric description

The Conway–Wales lattice is a certain Hermitian lattice $\Lambda_{\text{Ru}}$ of rank 28 over $\mathbb{Z}[i]$. Regarded as a lattice of rank 56 over $\mathbb{Z}$, it is even and self-dual, and has no roots (elements of square length 2). The full automorphism group of $\Lambda_{\text{Ru}}$ is a four fold cover of the sporadic simple group $\text{Ru}$.

We now recall Conway’s description of $\Lambda_{\text{Ru}}$ from [Con77]. We use a complex quaternionic notation which allows to regard a typical vector in $\Lambda_{\text{Ru}}$ as a seven tuple $(q_0, \ldots, q_6)$ where each $q_i$ is an element of the complex quaternions which is the $\mathbb{C}$-algebra with generators $j, k, l$ satisfying the usual quaternion relations

$$jl = -1, \quad jk = -kj, \quad kl = -lk, \quad &c.$$  \hspace{1cm} (5.1.1)

and with $i$ as usual denoting a square root of $-1$ in $\mathbb{C}$ (so that $i$ commutes with $j, k$, and $l$).

We take $\{i, j, k, l\}$ as a basis for the complex quaternions as a vector space over $\mathbb{C}$. We will write a vector $\lambda \in \Lambda_{\text{Ru}}$ as a $4 \times 7$ array of complex numbers, with the rows displaying our component quaternions, and the columns denoting the $\mathbb{C}$-coefficients of a given quaternion with respect to our chosen basis elements $i, j, k, l$. Actually, all the entries will lie in $\frac{1}{2}\mathbb{Z}[i]$.

Regarding elements of $\mathbb{C}^{28}$ as seven-tuples of complex quaternions allows to describe easily the action of a certain group of the shape $Q_8 \times 2^3 : 7$ on $\mathbb{C}^{28}$. The factor $Q_8$ is the quaternion group of order 8 consisting of componentwise right multiplications by the quaternions $\pm 1, \pm j, \pm k, \pm l$. The $2^3$ of the right hand factor consists of sign changes on coordinates $q_n, q_{n+3}, q_{n+5}, q_{n+6}$, for $n \in \mathbb{Z}/7$ (with indices read modulo 7), and the 7 of the right hand factor is the group of cyclic permutations of quaternion components $(q_0, q_1, \ldots, q_6) \mapsto (q_n, q_{1+n}, \ldots, q_{6+n})$.

We now define $\Lambda_{\text{Ru}}$ to be the $\mathbb{Z}[i]$-module generated by the vectors of Tables 1 and 2 and their images under the group $Q_8 \times 2^3 : 7$ just described. (In the tables we write $\bar{a}$ as a shorthand for $-a$. The content of Tables 1 and 2 is just Table 1 of [Con77].)
Table 1: Some vectors in $\Lambda_{Ru}$

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
|   | 1 | 1 | 1 |   |   |   |   |
|   |   |   |   |   |   |   |   |
|   |   |   |   |   |   |   |   |
|   |   |   |   |   |   |   |   |
|   | 1 | 1 | 1 |   |   |   |   |
| 1/2 |   |   |   |   |   |   |   |

Now set $r = \mathbb{C} \otimes_{\mathbb{Z}[i]} \Lambda_{Ru}$ and $s = r \oplus r^*$, and consider the element $\delta \in \bigvee^4(r) \oplus \bigvee^4(r^*)$ described as

$$\delta = \sum_{\lambda \in (\Lambda_{Ru})_2} \lambda^4 + (\lambda^*)^4$$  \hspace{1cm} (5.1.2)

where we write $(\Lambda_{Ru})_2$ for the type 2 (i.e. square norm 4) vectors in $\Lambda_{Ru}$.

**Proposition 5.1.** The element $\delta$ is non-zero.

**Proof.** The vector $\sum_{(\Lambda_{Ru})_2} (\lambda^*)^4$ may be viewed as a polynomial function on $r$. Consider the value of this function at the first vector of Table 1, say, so that we are just computing the sum of the fourth powers of the top left entries of each minimal length vector in $\Lambda_{Ru}$. Since all entries lie in $\mathbb{Z} \cup \mathbb{Z}i$, each term in the sum is non-negative. At least one term is positive. We conclude that $\delta$ is non-zero. \hfill $\square$

Note that the space $\bigvee^n(r)$ embeds naturally in $\forall(s)$ under the map

$$a_{i_1} \lor \cdots \lor a_{i_n} \mapsto a_{i_1}(\frac{-1}{2}) \cdots a_{i_n}(\frac{-1}{2})1$$  \hspace{1cm} (5.1.3)

and similarly for $\bigvee^n(r^*)$. In §5.3 we will identify $\delta$ with its image in $\forall(s)_2$ under this embedding.

**5.2 Monomial description**

In this section we consider the invariants of the monomial group $M$ in the spaces $\bigvee^4(r)$ and $\bigvee^4(r^*)$, which both embed naturally in $\forall(s)_2$. We will assume that the reader is acquainted with the notations and terminology of §5.2 of the preceding article [Dun06].
Table 2: Some more vectors in $A_{Ru}$

|    | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 |
|----|---------|---------|---------|---------|---------|
| 1/2|         |         |         |         |         |
| 1 0 0 ɪ | 1 0 0 ɪ | 0 ɪ ɪ 0 | 0 ɪ ɪ 0 | 0 ɪ ɪ 0 | 0 ɪ ɪ 0 |
| 1 ɪ 0 0 | 1 ɪ 0 0 | 0 0 0 ɪ | 0 0 0 ɪ | 0 0 0 ɪ | 0 0 0 ɪ |
| 0 0 ɪ 1 | 0 0 ɪ 1 | ɪ ɪ 0 0 | ɪ ɪ 0 0 | ɪ ɪ 0 0 | ɪ ɪ 0 0 |
| 1 0 1 0 | 1 0 1 0 | 0 1 0 ɪ | 0 1 0 ɪ | 0 1 0 ɪ | 0 1 0 ɪ |
| 0 1 1 0 | 0 1 1 0 | 0 ɪ 0 ɪ | 0 ɪ 0 ɪ | 0 ɪ 0 ɪ | 0 ɪ 0 ɪ |
| 0 1 0 1 | 0 1 0 1 | ɪ 0 1 0 | ɪ 0 1 0 | ɪ 0 1 0 | ɪ 0 1 0 |

|    | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 |
|----|---------|---------|---------|---------|---------|
| 1/2|         |         |         |         |         |
| 1 1 ɪ ɪ | 1 1 ɪ ɪ | 1 1 ɪ ɪ | 1 1 ɪ ɪ | 1 1 ɪ ɪ | 1 1 ɪ ɪ |
| 0 1 ɪ 0 | 0 1 ɪ 0 | 1 0 1 0 | 1 0 1 0 | 1 0 1 0 | 1 0 1 0 |
| 0 0 ɪ 1 | 0 0 ɪ 1 | 0 1 0 ɪ | 0 1 0 ɪ | 0 1 0 ɪ | 0 1 0 ɪ |
| 0 1 0 ɪ | 0 1 0 ɪ | 1 0 1 0 | 1 0 1 0 | 1 0 1 0 | 1 0 1 0 |
| 1 0 1 0 | 1 0 1 0 | 0 1 0 ɪ | 0 1 0 ɪ | 0 1 0 ɪ | 0 1 0 ɪ |

|    | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 |
|----|---------|---------|---------|---------|---------|
| 1/2|         |         |         |         |         |
| 1 ɪ 1 1 | 1 ɪ 1 1 | 1 ɪ 1 1 | 1 ɪ 1 1 | 1 ɪ 1 1 | 1 ɪ 1 1 |
| 1 0 0 ɪ | 1 0 0 ɪ | 0 ɪ ɪ 0 | 0 ɪ ɪ 0 | 0 ɪ ɪ 0 | 0 ɪ ɪ 0 |
| 0 0 1 ɪ | 0 0 1 ɪ | 0 1 0 ɪ | 0 1 0 ɪ | 0 1 0 ɪ | 0 1 0 ɪ |
| 0 1 0 ɪ | 0 1 0 ɪ | 1 0 1 0 | 1 0 1 0 | 1 0 1 0 | 1 0 1 0 |
| 1 0 1 0 | 1 0 1 0 | 0 1 0 ɪ | 0 1 0 ɪ | 0 1 0 ɪ | 0 1 0 ɪ |

|    | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 |
|----|---------|---------|---------|---------|---------|
| 1/2|         |         |         |         |         |
| 0 1 ɪ 0 | 0 1 ɪ 0 | ɪ 0 1 0 | ɪ 0 1 0 | ɪ 0 1 0 | ɪ 0 1 0 |
| 1 ɪ 0 0 | 1 ɪ 0 0 | 0 ɪ ɪ 0 | 0 ɪ ɪ 0 | 0 ɪ ɪ 0 | 0 ɪ ɪ 0 |
| 1 ɪ 0 0 | 1 ɪ 0 0 | 0 ɪ ɪ 0 | 0 ɪ ɪ 0 | 0 ɪ ɪ 0 | 0 ɪ ɪ 0 |
| 0 1 0 1 | 0 1 0 1 | ɪ 0 1 0 | ɪ 0 1 0 | ɪ 0 1 0 | ɪ 0 1 0 |
| 1 0 0 ɪ | 1 0 0 ɪ | 1 0 0 ɪ | 1 0 0 ɪ | 1 0 0 ɪ | 1 0 0 ɪ |

|    | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 | 1 1 1 1 |
|----|---------|---------|---------|---------|---------|
| 1/2|         |         |         |         |         |
| 0 1 0 1 | 0 1 0 1 | ɪ 0 1 0 | ɪ 0 1 0 | ɪ 0 1 0 | ɪ 0 1 0 |
| ɪ 0 0 ɪ | ɪ 0 0 ɪ | 1 0 0 ɪ | 1 0 0 ɪ | 1 0 0 ɪ | 1 0 0 ɪ |
| 0 1 0 0 | 0 1 0 0 | ɪ 0 1 0 | ɪ 0 1 0 | ɪ 0 1 0 | ɪ 0 1 0 |
| 0 0 0 1 | 0 0 0 1 | 1 0 0 1 | 1 0 0 1 | 1 0 0 1 | 1 0 0 1 |
Consider first the space $\mathcal{V}^4(\tau)$. A basis for this space is given by the homogeneous monomials of degree 4 in the symbols $\{a_i\}_{i \in \Delta}$, and the embedding in $\mathbb{V}(\mathfrak{s})_2$ is given by $f(a_i) \mapsto f(a_i(-\frac{1}{2}))\mathbf{1}$, for $f$ a monomial (homogeneous of degree 4).

Recall the code $\mathcal{D}$ generated by the dozens of $\Delta$. Suppose that $t \in \mathcal{V}^4(\tau)$ is invariant for the action of $M$, and let us consider the expansion of $t$ with respect to the basis of monomials of degree 4. Then it is easy to see that the coefficient of a monomial of the form $a_i^3a_j$ must be zero for all $\{i,j\} \subset \Delta$, since for each such pair in $\Delta$ there is an element of $A \subset M$ that fixes $a_i$ say, and negates $a_j$; in other words, the dual code to $\mathcal{D}$ has no words of weight 2. Similarly, the coefficient of any monomial of the form $a_i^2a_{jk}$ must also vanish. The coefficient of a monomial of the form $a_i$ with $I \subset \Delta$ of cardinality 4 must be zero unless $I$ lies in the dual to $\mathcal{D}$, and the words of weight four in $\mathcal{D}^*$ are precisely the quartets of $\Delta$. The ringed quartets $\mathcal{R}$ and the non-ringed quartets $\mathcal{N}$ constitute the two orbits of $\overline{M}$ on weight four words in $\mathcal{D}^*$.

We see that there are four kinds of monomials that can appear with non-trivial coefficient in an invariant for the action of $M$ on $\mathcal{V}^4(\tau)$. Namely, those of the form $a_i^4$ for $i \in \Delta$, or $a_i^2$. for $C \subset C$ a couple in $\Delta$, or $a_N$ for $N \in \mathcal{N}$, or $a_R$ for $R \in \mathcal{R}$. Since $\overline{M}$ is transitive on each of the sets $\Delta$, $\mathcal{C}$, $\mathcal{R}$, and $\mathcal{N}$, it follows that there is at most a four-space of invariants for the action of $M$ on $\mathcal{V}^4(\tau)$. Candidates for a spanning set for this space are the vectors

$$t_1 = \sum_{i \in \Delta} \varepsilon_i a_i^4, \quad t_2 = \sum_{C \subset C} \varepsilon_C a_C^2, \quad t_3 = \sum_{R \in \mathcal{R}} \varepsilon_R a_R, \quad t_4 = \sum_{N \in \mathcal{N}} \varepsilon_N a_N$$

(5.2.1)

where the functions $X \mapsto \varepsilon_X$ are uniquely determined up to scalar multiplication, so long as they exist; that is, give rise to a non-zero invariant. Recall that each element of $\overline{M}$ may be written as a coordinate permutation followed by a diagonal matrix with entries in $\{\pm 1, \pm i\}$. It follows that we may set $\varepsilon_i = 1$ for all $i \in \Delta$, and that $t_1$ is then invariant for the action of $M$. More generally, we have

**Proposition 5.2.** Set $\varepsilon_i = 1$ for each $i \in \Delta$, and $\varepsilon_R = 1$ for each $R \in \mathcal{R}$. Set $\varepsilon_C = 1$ for each couple $C$ associated to a dozen containing $\infty$, and $\varepsilon_C = -1$ otherwise. Set $\varepsilon_N = -1$ for each non-ringed quartet $N$ that does not contain $\infty$ and has intersection at most one with each block, and $\varepsilon_N = 1$ otherwise. Then the vectors $t_1$, $t_2$, $t_3$, and $t_4$ of (5.2.1), are each invariant for the action of $M$ on $\mathcal{V}^4(\tau)$.

Note that the functions $\varepsilon$ may all be chosen to take values in $\{\pm 1\}$. We thus obtain four linearly independent invariants $t_i^*$ for the action of $M$ on $\mathcal{V}^4(\mathfrak{t}^*)$ by replacing $a_i$ with $a_i^*$ in the expressions of (5.2.1).

We identify the $t_i$ and $t_i^*$ with their images in $\mathbb{V}(\mathfrak{s})_2$, so that $t_1 = \sum a_i(-\frac{1}{2})^4\mathbf{1}$, and $t_1^* = \sum a_i^*(-\frac{1}{2})^4\mathbf{1}$, &c., and we define a vector $\delta \in \mathbb{V}(\mathfrak{s})_2$ by setting

$$c_0 \delta = 13t_1 + (18 + 12i)t_2 + (-192 - 24i)t_3 + (-48 + 72i)t_4$$

$$+ 13t_1^* + (18 - 12i)t_2^* + (-192 + 24i)t_3^* + (-48 - 72i)t_4^*$$

(5.2.2)

where $c_0 = 2^7.3.5.13.29$. 

Moonshine for Rudvalis’s sporadic group II
5.3 Construction

Recall the VOA $\mathcal{V}(u)$ of §3.3, and the vector space $s$ of §5.1. We now define $\tilde{\mathcal{V}}_{\text{Ru}}$ to be the enhanced VOA with underlying VOA structure given by $\mathcal{V}_{\text{Ru}} = \mathcal{V}(s)$, and with enhanced conformal generators given by $\tilde{\mathcal{O}}_{\text{Ru}} = \{\omega, \eta, \delta\}$, where $\delta$ is as in §5.1 or §5.2 (the descriptions are equivalent), and $\omega$ and $\eta$ are as in §4. We have

**Theorem 5.3.** The quadruple $(\tilde{\mathcal{V}}_{\text{Ru}}, Y, 1, \tilde{\mathcal{O}}_{\text{Ru}})$ is a self-dual enhanced $U(1)$–VOA of rank $-28$.

5.4 Symmetries

We now consider the automorphism group of the enhanced VOA $\tilde{\mathcal{V}}_{\text{Ru}}$ with enhanced conformal structure given by $\tilde{\mathcal{O}}_{\text{Ru}}$. We opt to use the construction of §5.1 for the purposes of describing $\text{Aut}(\tilde{\mathcal{V}}_{\text{Ru}})$.

Let $F$ be the subgroup of $GL(\tau)$ that fixes $\delta$. Then $F$ is the full automorphism group of the enhanced VOA structure on $\tilde{\mathcal{V}}_{\text{Ru}}$. The element $\delta$ is evidently invariant under $\text{Aut}(\Lambda_{\text{Ru}})$, and thus we have an embedding $4.Ru \hookrightarrow F$.

**Proposition 5.4.** The group $F$ contains a group isomorphic to $4.Ru$.

We write $R$ for the copy of $4.Ru$ in $GL(\tau)$. The proof of the following proposition is directly parallel to the proof of Proposition 5.10 of Part I [Dun06]. The main ingredient is the fact that the only elements of the Lie algebra of $GL(\tau)$ that annihilate $\delta$ are those that exponentiate to scalar matrices. The nature of $\delta$ then forces these scalars to be fourth roots of unity.

**Proposition 5.5.** The group $F$ is finite, and is contained in $SL(\tau)$.

We now apply Proposition 5.12 of Part I [Dun06] to conclude that the group $F$ is precisely the group $4.Ru$. Actually, the Schur multiplier of the Rudvalis group has order 2 [CCN+85]; the center of $F$ is evidently generated by scalar multiplications by $i$, and the simple group $Ru$ has no non-trivial representation of degree 28, so the only possibility is that the group $4.Ru$ is in fact a central product of this cyclic group $\langle 1\text{Id} \rangle$ with the perfect double cover of $Ru$.

We record this and the other results of this section in the following

**Theorem 5.6.** The quadruple $(\tilde{\mathcal{V}}_{\text{Ru}}, Y, 1, \tilde{\mathcal{O}}_{\text{Ru}})$ is a self-dual enhanced $U(1)$–VOA of rank $-28$. The full automorphism group of $(\tilde{\mathcal{V}}_{\text{Ru}}, \tilde{\mathcal{O}}_{\text{Ru}})$ is a central product of a cyclic group of order four with the perfect double cover of the sporadic simple group of Rudvalis.

6 McKay–Thompson series

In this section we consider the McKay–Thompson series arising from the enhanced VOA constructed in §5. These series furnish an analogue of Monstrous Moonshine for the sporadic group of Rudvalis, and we will derive explicit expressions for them here.

The main tool for expressing the series explicitly is the notion of weak Frame shape, discussed in the companion article [Dun06 §6.3]. We refer to there for the definition of weak Frame shape.
We recall briefly the notion of two variable McKay–Thompson series associated to the action of an element of the automorphism group of an enhanced $U(1)$–VOA in §6.1 and we then present explicit expressions for all the McKay–Thompson series arising from the action of $2.Ru$ on $\forall_{Ru}$.

As mentioned in §6, we obtain a genus zero property for the Rudvalis group by considering together the series arising here and those of the preceding article [Dun06]. We define the functions $\tilde{F}_A(g)$ and $\tilde{F}_\forall(g)$ for $g \in Ru$, and make the main observation regarding these functions in §6.2.

### 6.1 Two variable McKay–Thompson series

Let $(U,Y,1_{\mathbb{A}},\{\omega,\eta,\ldots\})$ be an enhanced $U(1)$–VOA of rank $c$ (see §2). Then for $g$ an automorphism of $(U,\{\omega,\eta,\ldots\})$, the two variable McKay–Thompson series associated to the action of $g$ on $U$ is the series in variables $p$ and $q$ given by

$$\text{tr}|_U gp^{J(0)}q^{L(0)} - c/24 = \sum_{m,n}(\text{tr}|_{U^m} g)p^m q^n - c/24$$

(6.1.1)

where $U^m$ denotes the subspace of $U$ of degree $n$ consisting of vectors of charge $m$ (see §2). In the limit as $p \to 1$ we recover the (ordinary) McKay–Thompson series associated to the action of $g$ on $U$. In the case that $g$ is the identity we obtain what we call the two variable character of $U$.

We would like to compute the two variable McKay–Thompson series arising from our main example $\forall_{Ru}$. In this case we are in the situation of §4, so that the relevant automorphisms lie in $GL(a)$ where $a$ is a Hermitian vector space, and the VOA underlying the relevant enhanced VOA is of the form $\forall(u)$ for $u = a \oplus a^*$. We suppose that $g$ lies in $GL(a)$, and has a weak Frame shape $\prod_j (k_j a_j)$ say, for its action on $a$. Recall the Jacobi theta function $\vartheta(z|\tau)$ from §0.2. Then we set

$$\phi_g(z|\tau) = \prod_j \frac{\vartheta(k_j z + 1/2 + a_j k_j \tau)^{m_j}}{\eta(k_j \tau)^{m_j}}$$

(6.1.2)

$$\psi_g(z|\tau) = \prod_j p^{k_j m_j/2} q^{k_j m_j/8} \frac{\vartheta(k_j z + 1/2 + a_j + k_j \tau/2 k_j \tau)^{m_j}}{\eta(k_j \tau)^{m_j}}$$

(6.1.3)

( just as in [Dun06 §6]) and we then have

**Theorem 6.1.** Let $g \in GL(a)$. Then the McKay–Thompson series associated to the actions of $g$ and $\theta g$ on $\forall(u)$ and $\forall(u)_g$ admit the following expressions.

$$\text{tr}|_{\forall(u)} gp^{J(0)}q^{L(0)} - c/24 = \frac{1}{\phi_g(z|\tau)}$$

(6.1.4)

$$\text{tr}|_{\forall(u)} \theta g p^{J(0)}q^{L(0)} - c/24 = \frac{1}{\phi_{-g}(z|\tau)}$$

(6.1.5)

$$\text{tr}|_{\forall(u)} g p^{J(0)}q^{L(0)} - c/24 = \frac{1}{\psi_g(z|\tau)}$$

(6.1.6)

$$\text{tr}|_{\forall(u)} \theta g p^{J(0)}q^{L(0)} - c/24 = \frac{1}{\psi_{-g}(z|\tau)}$$

(6.1.7)
The verification of Theorem 6.1 is very similar to that of the corresponding Theorem 6.1 in [Dun06].

Recall that the vector space underlying the enhanced vertex algebra $A_{Ru}$ constructed in Part I [Dun06] may be expressed as a direct sum $A_{Ru} = A(s)^0 \oplus A(s)_0$. (Where the $s$ there and in this article coincide as Hermitian vector spaces.) The action of $Ru$ preserves this decomposition, and more than this, $Ru$ acts projectively on the companion spaces $A(s)_1$ and $A(s)_1^\theta$. In fact, we may regard the object $A(s) = A(s)^0 \oplus A(s)_1$ as an enhanced $U(1)$–VOA with a projective action by $Ru$, and $A(s)_\theta$ is then its canonically twisted module, again with a projective action by $Ru$. We now set $\tilde{A}_{Ru} = A(s)$ and $(\tilde{A}_{Ru})_\theta = A(s)_\theta$. We regard $\tilde{A}_{Ru}$ as a twisting of $A_{Ru}$, and $(\tilde{A}_{Ru})_\theta$ is then its canonically twisted module. This twisting breaks much of the enhanced conformal structure on $A_{Ru}$, but the $U(1)$–VOA structure at least survives.

For convenience later we now record the analogue of Theorem 6.1 with $A(u)$ and $A(u)_\theta$ in place of $\forall(u)$ and $\forall(u)_\theta$.

**Theorem 6.2.** Let $g \in GL(a)$. Then the McKay–Thompson series associated to the actions of $g$ and $\theta g$ on $A(u)$ and $A(u)_\theta$ admit the following expressions.

$$
\text{tr}_{A(u)} | g^p J^{(0)} q^{L(0) - c/24} = \phi_g(z|\tau) \quad (6.1.8)
$$

$$
\text{tr}_{A(u)} | g^p J^{(0)} q^{L(0) - c/24} = \phi_g(z|\tau) \quad (6.1.9)
$$

$$
\text{tr}_{A(u)_\theta} | g^p J^{(0)} q^{L(0) - c/24} = \psi_g(z|\tau) \quad (6.1.10)
$$

$$
\text{tr}_{A(u)_\theta} | \theta g^p J^{(0)} q^{L(0) - c/24} = \psi_g(z|\tau) \quad (6.1.11)
$$

### 6.2 Moonshine beyond the Monster

Using Theorem 6.1, we may now compute the two variable McKay–Thompson series associated to the action of any element $g \in 4.Ru$ on $\tilde{\forall}_{Ru}$ as soon as the corresponding weak Frame shape is known. We reproduce in Table 3 the Frame shapes and weak Frame shapes for the Rudvalis group, given also in [Dun06, §6].

In Table 4 we tabulate the terms of lowest charge and degree in the character of $\tilde{\forall}_{Ru}$. The column headed $m$ is the coefficient of $p^m$ (as a series in $q$), and the row headed $n$ is the coefficient of $q^n$ (as a series in $p$). The coefficients of $p^{-m}$ and $p^m$ coincide, and the coefficient of $p^n q^n$ vanishes unless $2m$ and $n$ are of the same parity. The entries of Table 4 extend the coincidences (0.0.3) noted in [Dun06] and in addition, we now see that small degree characters of the covering group $2.Ru$ participate in similar identities also.

$$
28 = 28
$$
$$
4060 = 28 + 4032 \quad (6.2.1)
$$
$$
11396 = (2)28 + 4032 + 7308
$$
$$
201376 = 28 + (2)4032 + 7308 + 87696 + 98280
$$

We now focus attention on the series arising from the projective action of the Rudvalis group on $(\widetilde{A}_{Ru})_\theta$ and $(\tilde{\forall}_{Ru})_\theta$. For an arbitrary rational vertex operator superalgebra (satisfying some
Table 3: Frame Shapes for 2.Ru

| Class | $SO_{56}$ | $SU_{28}$ | Class | $SO_{56}$ | $SU_{28}$ |
|-------|-----------|-----------|-------|-----------|-----------|
| 1A    | $^{1}_5$  | $^{1}_2$  | 12B   | $^{4}_12^{1/2}$ | $^{1}_6^{1/2}$ |
| 2A    | $^{1}_82^{24}$ | $^{1}_42^{12}$ | 13A   | $^{1}_213^{4}$  | $^{1}_213^{2}$  |
| 2B    | $^{4}_28/2^{28}$ | $^{4}_2^{14}/2^{14}$ | 14A   | $^{2}_8^{4}/14^{1}$ | $^{2}_8^{2}/14^{2}$ |
| 3A    | $^{1}_23^{18}$  | $^{1}_3^{9}$  | 14B   | $^{2}_8^{4}/14^{1}$ | $^{2}_8^{2}/14^{2}$ |
| 4A    | $^{1}_8^{4}/2^{12}$ | $^{1}_4^{4^{6}}$ | 14C   | $^{2}_8^{4}/14^{1}$ | $^{2}_8^{2}/14^{2}$ |
| 4B    | $^{1}_4^{2}/2^{4}$  | $^{1}_4^{4^{6}}$ | 15A   | $^{1}_215^{4}/3^{2}$ | $^{1}_215^{2}/3^{1}$ |
| 4C    | $^{1}_4^{2}/2^{4}$  | $^{1}_4^{4^{6}}$ | 16A   | $^{1}_216^{4}/2^{18^{1}}$ | $^{1}_211^{1}/14^{2}\times16^{2}/8^{1}$ |
| 4D    | $^{2}_4^{12}$  | $^{2}_4^{4^{6}}$ | 16B   | $^{1}_216^{4}/2^{18^{1}}$ | $^{1}_3/4^{1}/4^{1}16^{2}/8^{1}$ |
| 5A    | $^{1}_6^{5}/10^{5}$  | $^{1}_3^{5^{5}}$ | 20A   | $^{2}_4^{2}10^{2}/2^{25^{2}}$ | $^{1}_4^{1}10^{1}/2^{15^{1}}$ |
| 5B    | $^{5}/1^{2}$  | $^{1}_2^{6}/2^{4}$ | 20B   | $^{2}_4^{1}2^{10^{2}/10^{1}}$ | $^{1}_21^{5}/3^{1}/4^{20^{1}}$ |
| 6A    | $^{1}_2^{3}6^{8}$  | $^{1}_3^{3}/4^{1}$ | 20C   | $^{2}_4^{1}2^{10^{3}/10^{1}}$ | $^{1}_21^{5}/3^{1}/4^{20^{1}}$ |
| 7A    | $^{7^{8}}$  | $^{7^{1}}$ | 24A   | $^{2}_4^{1}2^{12}/2^{4}/2^{16^{1}}$ | $^{1}_3/4^{1}/4^{24^{1}}$ |
| 8A    | $^{4}_48^{6}/2^{4}$ | $^{1}_2^{2}4^{1}8^{2}/2^{1}$ | 24B   | $^{2}_4^{1}2^{12}/2^{4}/2^{16^{1}}$ | $^{1}_3/4^{1}/4^{24^{1}}$ |
| 8B    | $^{8^{2}/4^{2}}$  | $^{1}_2^{2}/4^{1}8^{2}/4^{2}$ | 26A   | $^{4}_2^{5}2^{6}/2^{26^{2}}$ | $^{4}_2^{1}/2^{26^{1}}$ |
| 8C    | $^{4^{8}}$  | $^{4^{8}}$ | 26B   | $^{4}_2^{5}2^{6}/2^{26^{2}}$ | $^{4}_2^{1}/2^{26^{1}}$ |
| 10A   | $^{2}_4^{5}2^{10^{2}/1^{2}}$  | $^{2}_4^{2}5^{1}/10^{2}/1^{1}$ | 26C   | $^{4}_2^{5}2^{6}/2^{26^{2}}$ | $^{4}_2^{1}/2^{26^{1}}$ |
| 10B   | $^{2}_22^{6}/4^{2}10^{6}$  | $^{2}_4^{2}2^{3}/4^{1}10^{3}$ | 29A   | $^{2}_2^{9}/1^{2}$ | $^{2}_9^{1}/1^{1}$ |
| 12A   | $^{1}_2^{3}2^{12^{4}}$  | $^{1}_3^{3}/12^{2}$ | 29B   | $^{2}_2^{9}/1^{2}$ | $^{2}_9^{1}/1^{1}$ |
Table 4: Character of $\tilde{\nu}_{Ru}$

|   | 0    | 2    | 4    | 6    | 8    |
|---|------|------|------|------|------|
| 0 | 1    |      |      |      |      |
| 1 | 784  | 406  |      |      |      |
| 2 | 166404 | 114464 | 31465 |      |      |
| 3 | 17122560 | 13207964 | 5752208 | 1107568 |      |
| 4 | 1083938457 | 889479360 | 480258212 | 156267776 | 23535820 |
| 5 | 48023166576 | 40957573860 | 25100492032 | 10592918798 | 2786641088 |
| 6 | 1612815529556 | 1412141567872 | 941094824285 | 466052603296 | 163110431792 |
| 7 | 43324776509184 | 38661856383304 | 27348457208240 | 15108948493100 | 6331332324352 |

Table 5: Character for the action of $2\cdot Ru$ on $\Lambda(\tau)$

| $Ru$ | 1A | 2B | 3A | 5A | 5B | 10B | 13A |
|------|----|----|----|----|----|-----|-----|
| $2\cdot Ru$ | 1A | 2A | 4A | 6A | 10A | 12B | 13A | 26A |
| $\chi$ | $2^8$ | 0 | $2^{14}$ | 0 | $2^8$ | 0 | 16 | 0 | 4 | 16 | 0 |
| $Ru$ | 14A | 14B | 14C | 15A | 26A | 26B | 26C | 29A | 29B |
| $2\cdot Ru$ | 28A | 28B | 28C | 15A | 30A | 52A | 52B | 52C | 29A | 58A | 29B | 58B |
| $\chi$ | 4 | 4 | 4 | 4 | 0 | 4 | 4 | 1 | 29 | 1 | 29 |

technical conditions) the characters of ordinary modules are not usually invariant for the action of the modular group $PSL_2(\mathbb{Z})$, but rather for the index 3 subgroup generated by $\tau \mapsto \tau + 2$ and $\tau \mapsto -1/\tau$. However, considering the super characters of canonically twisted modules one does expect to recover a representation of $PSL_2(\mathbb{Z})$; indeed, this has been shown in [DZ05]. By a similar token, we expect the best modular properties to be enjoyed by the series associated to the Rudvalis group via its action on the twisted modules $(\tilde{A}_{Ru})_{\theta}$ and $(\tilde{\nu}_{Ru})_{\theta}$.

The subspace of $(\tilde{A}_{Ru})_{\theta}$ of lowest degree is a copy of the exterior algebra $\Lambda(\tau)$. We record the traces of elements of $2\cdot Ru$ on this $2\cdot Ru$-module in Table 5. A class determined by an element $\hat{g}$ such that $\hat{g}$ and $-\hat{g}$ both have vanishing trace on $\Lambda(\tau)$ is omitted from the table. We also record in Table 5 the fusion of conjugacy classes under the natural map $2\cdot Ru \to Ru$, but only for the classes in $Ru$ that have a lift to $2\cdot Ru$ with non-trivial trace on $\Lambda(\tau)$. Our naming of conjugacy classes follows the computer system [GAP05].
We now define functions $\tilde{F}^A_g(\tau)$ and $\tilde{F}^\forall_g(\tau)$ for $g \in Ru$ by setting

$$
\tilde{F}^A_g(\tau) = \lim_{p \to 1} \text{tr}|(\hat{\lambda}_{Ru})_g - \hat{g}p^{J(0)}q^{L(0) - c/24}
$$

(6.2.2)

$$
\tilde{F}^\forall_g(\tau) = \lim_{p \to 1} \text{tr}|(\tilde{\lambda}_{Ru})_g - \hat{g}p^{J(0)}q^{L(0) - c/24}
$$

(6.2.3)

where $\hat{g}$ is a preimage of $g$ in $2.Ru$ of minimal order. One may check by consulting the character table of $2.Ru$ that any two preimages of minimal order have the same Frame shape. We define a vector space $\mathfrak{F}_g$, for each $g \in Ru$, by setting

$$
\mathfrak{F}_g = \text{Span}_C \left\{ \tilde{F}^A_g(\tau), \tilde{F}^\forall_g(\tau) \right\}.
$$

(6.2.4)

In the case that $-\hat{g}$ has trace 0 on $\wedge(\tau)$, this amounts to setting $\tilde{F}^A_g(\tau) = 0$ and $\tilde{F}^\forall_g(\tau) = \infty$, and we invite the reader to regard the corresponding space $\mathfrak{F}_g$ as a copy of the trivial two dimensional representation of $PSL_2(\mathbb{Z})$.

Care must be applied when taking the limit in (6.2.3). In the case of (6.2.2) each coefficient in $q$ is a polynomial in $p$, and there is no problem regarding the a priori formal variable $p$ as a variable on $C^\infty$. In the case of (6.2.3) there is a factor of the form

$$
\prod_i \frac{1}{(1 - \xi_i p)}
$$

(6.2.5)

where $\xi_i$ ranges over the eigenvalues of $\hat{g}$ (c.f. Theorem 6.1), which is formally a power series in $p$, and this series converges only for $|p| < 1$. In order to make sense of the limit in (6.2.3) we should analytically continue each coefficient in $q$ as a function in $p$, so that the series represented by (6.2.5) may be replaced with the product expression there — in other words, the factor (6.2.5) should be regarded as the reciprocal of the trace of $-\hat{g}$ on $\wedge(\tau)$.

From Table 5 we see that $\mathfrak{F}_g$ is comprised of non-constant functions just in the case that $g$ belongs to one of the following 10 conjugacy classes.

$$
2B, 10B, 14A, 14B, 14C, 26A, 26B, 26C, 29A, 29B
$$

(6.2.6)

We will examine these cases in some detail in the following sections.

We refer the reader to [CN79], [CMS04] and [Cum04] for more information about discrete subgroups of $PSL_2(\mathbb{R})$ of genus zero. We adopt the notation of [CN79] for discrete subgroups of $PSL_2(\mathbb{R})$.

6.2.1 $\tilde{\mathfrak{F}}_{2B}$

An element $g$ of class $2B$ in $Ru$ lifts to a unique class of order 4 in the covering group $2.Ru$, and an element of this class has trace $2^{14}$ on $\wedge(\tau)$. We see from Table 5 that the Frame shape of $\hat{g}$ is $4^{28}/2^{28}$, so we have

$$
\tilde{F}^A_{2B}(\tau) = \frac{2^{14}}{1} \frac{\eta(2\tau)^{28}}{\eta(4\tau)^{28}}, \quad \tilde{F}^\forall_{2B}(\tau) = \frac{1}{2^{14}} \frac{\eta(2\tau)^{28}}{\eta(4\tau)^{28}}.
$$

(6.2.7)
After rescaling the variable τ (i.e. after conjugating by the element of $PSL_2(\mathbb{R})$ represented by the matrix in (6.2.8))

$$
\begin{pmatrix}
1/2 & 0 \\
0 & 1
\end{pmatrix}
$$

we obtain a space spanned by $(\eta(2\tau)/\eta(\tau))^{28}$ and its reciprocal. This space is evidently preserved by the action of $\Gamma_0(2) + 2$. The Fricke involution $\tau \mapsto -1/2\tau$ interchanges (the rescalings of) $\tilde{F}_2B$ and $\tilde{F}_2B^\vee$.

### 6.2.2 $\tilde{\mathfrak{F}}_{10B}$

Elements of class 10B also lift to a unique class in the covering group, and any such lift has trace 4 on $\bigwedge(\tau)$. Consulting Table 3 we see that $\pm \hat{g} \sim 2^2 20^6/4^2 10^6$.

$$
\tilde{F}^A_{10B}(\tau) = \frac{4}{1} \frac{\eta(2\tau)^2 \eta(20\tau)^6}{\eta(4\tau)^2 \eta(10\tau)^6}, \quad \tilde{F}^\vee_{10B}(\tau) = \frac{1}{4} \frac{\eta(4\tau)^2 \eta(10\tau)^6}{\eta(2\tau)^2 \eta(20\tau)^6}
$$

(6.2.9)

After rescaling we obtain the space spanned by $\eta(\tau)^2\eta(10\tau)^6/\eta(2\tau)^2\eta(5\tau)^6$ and its reciprocal. This space is preserved by the action of the group $\Gamma_0(10)$.

### 6.2.3 $\tilde{\mathfrak{F}}_{14A}$, $\tilde{\mathfrak{F}}_{14B}$, $\tilde{\mathfrak{F}}_{14C}$

Similar to the previous cases, elements of class 14A, 14B, or 14C, all lift to unique classes of order 28 in the cover. Any such lift has trace 4 on $\bigwedge(\tau)$, and we have $\pm \hat{g} \sim 28^4/14^4$.

$$
\tilde{F}^A_{14ABC}(\tau) = \frac{4}{1} \frac{\eta(28\tau)^4}{\eta(14\tau)^4}, \quad \tilde{F}^\vee_{14ABC}(\tau) = \frac{1}{4} \frac{\eta(14\tau)^4}{\eta(28\tau)^4}
$$

(6.2.10)

Then similar to the 2B case we see that a conjugate of $\Gamma_0(2) + 2$ preserves the spaces $\tilde{\mathfrak{F}}_{14ABC}$.

### 6.2.4 $\tilde{\mathfrak{F}}_{26A}$, $\tilde{\mathfrak{F}}_{26B}$, $\tilde{\mathfrak{F}}_{26C}$

Again, all elements of class 26A, 26B, or 26C, lift to unique classes in the covering group, and have trace 4 on $\bigwedge(\tau)$. From Table 3 we have $\pm \hat{g} \sim 4^2 52^2/2^2 26^2$, and this shows

$$
\tilde{F}^A_{26ABC}(\tau) = \frac{4}{1} \frac{\eta(52\tau)^2}{\eta(2\tau)^2 \eta(26\tau)^2}, \quad \tilde{F}^\vee_{26ABC}(\tau) = \frac{1}{4} \frac{\eta(2\tau)^2 \eta(26\tau)^2}{\eta(4\tau)^2 \eta(52\tau)^2}
$$

(6.2.11)

Here we obtain a representation of (the conjugate by (6.2.8) of) the genus zero group $\Gamma_0(26) + 26$. The Fricke involution interchanges the two one dimensional representations of the subgroup (conjugate to) $\Gamma_0(26)$. (Incidentally, this subgroup is not a group of genus zero.)
6.2.5  $\mathfrak{F}_{29A}, \mathfrak{F}_{29B}$

Elements of class $29A$ or $29B$ have two lifts to the covering group $2.\text{Ru}$. A lift $\hat{g}$ of minimal order has order 29, and $-\hat{g}$ then has trace 29 on $\wedge(\tau)$. The corresponding Frame shape is $\hat{g} \sim 29^2/1^2$, and thus we have

$$\tilde{F}_{29AB}^A(\tau) = \frac{29}{2} \frac{\eta(29\tau)^2}{\eta(\tau)^2}, \quad \tilde{F}_{29AB}^\forall(\tau) = \frac{1}{29} \frac{\eta(\tau)^2}{\eta(29\tau)^2}.$$  \hspace{1cm} (6.2.12)

The spaces $\mathfrak{F}_{29AB}$ are each preserved by the action of $\Gamma_0(29)+29$. The Fricke involution $\tau \mapsto -1/29\tau$ interchanges the one dimensional representations of $\Gamma_0(29)$ spanned by $\tilde{F}_{29A}^A$ and $\tilde{F}_{29A}^\forall$, respectively (and similarly for the class $29B$).

We conclude with the following observation.

For each $g \in \text{Ru}$ the functions $\tilde{F}_{g}^A(\tau)$ and $\tilde{F}_{g}^\forall(\tau)$ span a (two dimensional) representation of a discrete subgroup of $PSL_2(\mathbb{R})$ that is commensurable with $PSL_2(\mathbb{Z})$ and has genus zero.

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