Gromov-Hausdorff-like distance function
defined in the aspect of
Riemannian submanifold theory

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November 20, 2018

Abstract

In this paper, we discuss how a Gromov-Hausdorff-like distance function over the space of all isometric classes of compact $C^k$-Riemannian manifolds should be defined in the aspect of the Riemannian submanifold theory, where $k \geq 1$. The most important fact in this discussion is as follows. The Hausdorff distance function between two spheres of mutually distinct radii isometrically embedded into the hyperbolic space of curvature $c$ converges to zero as $c \to -\infty$. The key in the construction of the Gromov-Hausdorff-like distance function given in this paper is to define the distance of two $C^{k + 1}$-isometric embeddings of distinct compact $C^k$-Riemannian manifolds into a higher dimensional Riemannian manifold by using the Hausdorff distance function in the tangent bundle of order $k + 1$ equipped with the Sasaki metric.

Keywords: Hausdorff distance function, Gromov-Hausdorff distance function

1 Introduction

First we shall recall the Gromov-Hausdorff distance function introduced by M. Gromov ([G1], [G2]). Denote by $\tilde{M}$ the set of all metric spaces and $M_c$ the space of all isometric classes of compact metric spaces and $[(X, d)]$ the isometric class of a compact metric space $(X, d)$. For metric spaces $(X, d)$ and $(\tilde{X}, \tilde{d})$, denote by $\text{Emb}_{d.p.}((X, d), (\tilde{X}, \tilde{d}))$ the space of all distance-preserving embeddings of $(X, d)$ into $(\tilde{X}, \tilde{d})$. Let $\tilde{M}_c(\tilde{X}, \tilde{d})$ be the set of all compact subsets of a metric space $(\tilde{X}, \tilde{d})$. The Hausdorff distance function $d_{H, (\tilde{X}, \tilde{d})}$ over $\tilde{M}_c(\tilde{X}, \tilde{d})$ is defined by

$$d_{H, (\tilde{X}, \tilde{d})}(K_1, K_2) := \inf \{\varepsilon > 0 | K_2 \subset B(K_1, \varepsilon) & K_1 \subset B(K_2, \varepsilon) \} \quad (K_1, K_2 \in \tilde{M}_c(\tilde{X}, \tilde{d})),$$

where $B(K_i, \varepsilon)$ denotes the $\varepsilon$-neighborhood of $K_i$. By using this distance function, the Gromov-Hausdorff distance function $d_{GH}$ over $M_c$ is defined by

$$d_{GH}([(X_1, d_1)], [(X_2, d_2)]) := \inf_{(\tilde{X}, \tilde{d}) \in \tilde{M}} \inf \{d_{H, (\tilde{X}, \tilde{d})}(f_1(X_1), f_2(X_2)) | f_i \in \text{Emb}_{d.p.}((X_i, d_i), (\tilde{X}, \tilde{d})) \quad (i = 1, 2)\},$$

where $\inf \{d_{H, (\tilde{X}, \tilde{d})}(f_1(X_1), f_2(X_2)) | f_i \in \text{Emb}_{d.p.}((X_i, d_i), (\tilde{X}, \tilde{d})) \quad (i = 1, 2)\}$ implies $\infty$ in the case where $\text{Emb}_{d.p.}((X_1, d_1), (\tilde{X}, \tilde{d})) = \emptyset$ or $\text{Emb}_{d.p.}((X_2, d_2), (\tilde{X}, \tilde{d})) = \emptyset$. It is well-known that this function $d_{GH}$ gives a distance function over $M_c$ and furthermore $(M_c, d_{GH})$ is a complete metric space.

In this paper, we introduce a Gromov-Hausdorff-like distance function over the space of all isometric classes of compact $C^k$-Riemannian manifolds in the aspect of the Riemannian submanifold theory, where $k \geq 1$. 

2 Some important examples

Let $k \in \mathbb{N}$ or $k = \infty$. Denote by $\mathcal{R}M_k^c$ the set of all $C^k$-Riemannian manifolds and $\mathcal{R}M_c^k$ the space of all isometric classes of compact $C^k$-Riemannian manifolds and $[(M, g)]$ the isometric class of a compact Riemannian manifold $(M, g)$. Here we note that “$C^k$-Riemannian manifold” means a $C^{k+1}$-manifold equipped with a $C^k$-Riemannian metric. For $C^k$-Riemannian manifolds $(M, g)$ and $(\tilde{M}, \tilde{g})$, denote by $\text{Emb}^+_k((M, g), (\tilde{M}, \tilde{g}))$ (resp. $\text{Emb}^+_g((M, g), (\tilde{M}, \tilde{g}))$) the space of all $C^{k+1}$-isometric embeddings (resp. all totally geodesic $C^{k+1}$-isometric embeddings) of $(M, g)$ into $(\tilde{M}, \tilde{g})$. Here we note that the following facts hold:

**Fact.** Let $f$ be a $C^{k+1}$-isometric embedding of $(M, g)$ into $(\tilde{M}, \tilde{g})$. If $f$ is not totally geodesic, then it is not a distance-preserving embedding of $(M, d_g)$ into $(\tilde{M}, d_{\tilde{g}})$, where $d_g$ (resp. $d_{\tilde{g}}$) denotes the Riemannian distance function of $g$ (resp. $\tilde{g}$). Even if $f$ is a totally geodesic $C^{k+1}$-isometric embedding, it is not necessarily a distance-preserving embedding of $(M, d_g)$ into $(\tilde{M}, d_{\tilde{g}})$ (see Figure 1). On the other hand, if $f$ is a totally geodesic $C^{k+1}$-isometric embedding and if $(\tilde{M}, \tilde{g})$ is a Hadamard manifold, then it is a distance-preserving embedding of $(M, d_g)$ into $(\tilde{M}, d_{\tilde{g}})$.

![Figure 1: Gap between distance-preserving embeddings and isometric embeddings](image)

By using the Hausdorff distance functions, we define a function $d_{GH}^k$ over $\mathcal{R}M_c^k \times \mathcal{R}M_c^k$ by

$$d_{GH}^k([(M, g_1)], [(M, g_2)]) := \inf_{(\tilde{M}, \tilde{g}) \in \mathcal{R}M_c^k} \inf \{d_{H, (\tilde{M}, d_{\tilde{g}})}(f_1(M_1), f_2(M_2)) \mid f_i \in \text{Emb}^+_k((M_i, g_i), (\tilde{M}, \tilde{g})) (i = 1, 2)\}.$$

This definition seems to be natural. However, we can show that $d_{GH}^k$ is not a distance function over $\mathcal{R}M_c^k$. In fact, we can give some counter-examples as follows. Let $S^n(r-2)$ be the sphere of radius $r$ centered at the origin $o = (0, \ldots, 0)$ in the $(n + 1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ (i.e., $S^n(r-2) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{E}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = r^2\}$) and denote by $g_E$ the Euclidean metric of $\mathbb{E}^{n+1}$. Let $i_S^S$ be the inclusion map of $S^n(r-2)$ into $\mathbb{E}^{n+1}$ and denote by $g_r^S$ the induced metric $(i_S^S)^* g_E$. In the sequel, we abbreviate $(S^n(r-2), g_r^S)$ as $S^n(r-2)$. In the case of $n \geq 2$, $S^n(r-2)$ is the $n$-dimensional sphere of constant curvature $r^{-2}$. We consider the case of $n = 1$. The length of $(S^1(r^{-2}), g_r^S)$ is equal to $2\pi r$. Fix positive numbers $r_1 < r_2$. It is clear that $S^1(r_1^{-2})$ is not isometric to $S^1(r_2^{-2})$. However, we can show

$$\inf \{d_{H, \mathbb{E}^2}(f_1(S^1(r_1^{-2})), f_2(S^1(r_2^{-2}))) \mid f_i \in \text{Emb}^+_k(S^1(r_i^{-2}), \mathbb{E}^2) (i = 1, 2)\} = 0$$

as follows. Take any positive number $\varepsilon$. Let $N_\varepsilon(S^1(r_1^{-2}))$ be the $\varepsilon$-tubular neighborhood of $S^1(r_1^{-2})$,
that is,

\[ N_\varepsilon(S^1(r_1^{-2})) := \{(x_1, x_2) \in \mathbb{E}^2 \mid (r_1 - \varepsilon)^2 < x_1^2 + x_2^2 < (r_1 + \varepsilon)^2 \}. \]

For any sufficiently small positive number \( \varepsilon \), \( S^1(r_2^{-2}) \) can be isometrically embedded into \( N_\varepsilon(S^1(r_1^{-2})) \) (see Figure 2). This fact implies that

\[
\inf \{ d_{H, \mathbb{E}^2}(f_1(S^1(r_1^{-2})), f_2(S^1(r_2^{-2}))) \mid f_i \in \text{Emb}^{k+1}(S^1(r_i^{-2}), \mathbb{E}^2) \ (i = 1, 2) \} = 0.
\]

Hence we obtain

\[ d_{GH}^k([S^1(r_1^{-2})], [S^1(r_2^{-2})]) = 0. \]

Thus \( d_{GH}^k \) is not a distance function over \( \mathcal{RM}_k \). For any Riemannian manifold \( (M, g) \), we consider the product Riemannian manifolds \( M \times S^1(r_i^{-2}) \) \((i = 1, 2)\). In more general, it is shown that \([M \times S^1(r_1^{-2})] \neq [M \times S^1(r_2^{-2})]\) but \( d_{GH}^k([M \times S^1(r_1^{-2})],[M \times S^1(r_2^{-2})]) = 0\). Isometric embeddings \( f_1 \) and \( f_2 \) in Figure 2 are sufficiently close as \( C^0 \) embeddings but they are not close as \( C^1 \)-embeddings and they are very far as \( C^2 \)-embeddings. On the other hand, isometric embeddings \( \hat{f}_1 \) and \( \hat{f}_2 \) in Figure 3 are sufficiently close as \( C^\infty \)-embeddings, where \( 2r_1 < r_2 < 2r_1 + \varepsilon \) \((\varepsilon: a \text{ sufficiently small positive number})\).

![Figure 2: The first example showing \( d_{GH}^k([S^1(r_1^{-2})], [S^1(r_2^{-2})]) = 0 \)](image)

![Figure 3: The second example showing \( d_{GH}^k([S^1(r_1^{-2})], [S^1(r_2^{-2})]) = 0 \)](image)

We shall give third example showing that \( d_{GH}^k \) is not a distance function. Let \( \mathbb{R}^{n+2}_1 \) be the \((n + 2)\)-dimensional Lorentzian space and \( g_L \) the Lorentzian metric of \( \mathbb{R}^{n+2}_1 \), that is, \( g_L = -dx_1^2 + dx_2^2 + \cdots + dx_{n+2}^2 \). Put

\[ \mathbb{H}^{n+1}(-\tilde{r}^{-2}) := \{(x_1, \cdots, x_{n+2}) \in \mathbb{R}^{n+2}_1 \mid -x_1^2 + x_2^2 + \cdots + x_{n+2}^2 = -\tilde{r}^2 \} \quad (\tilde{r} > 0). \]
Denote by \( i^H \) the inclusion map of \( \mathbb{H}^{n+1}(\overline{t}^{-2}) \) into \( \mathbb{R}^{n+2} \) and \( g^H_\overline{t} \) the induced metric \( (i^H)^*g_L \). The space \( (\mathbb{H}^{n+1}(\overline{t}^{-2}), g^H_\overline{t}) \) is the \((n+1)\)-dimensional hyperbolic space of constant curvature \(-\overline{t}^{-2} \). The sphere \((\mathbb{S}^n(r^{-2}), g^S_r)\) is isometrically embedded into \( (\mathbb{H}^{n+1}(\overline{t}^{-2}), g^H_\overline{t}) \) by the following \( C^\infty \)-embedding:

\[
(f_{r,\overline{t}}(x_1, \cdots, x_{n+1}) = (\sqrt{\overline{t}^2 + r^2}, x_1, \cdots, x_{n+1}) \quad ((x_1, \cdots, x_{n+1}) \in \mathbb{S}^n(r^{-2})).
\]

Take distinct positive constants \( r_1 \) and \( r_2 \) \((r_1 < r_2)\). We shall calculate

\[
d_{H,\mathbb{H}^{n+1}(\overline{t}^{-2})}(f_{r_1,\overline{t}}(\mathbb{S}^n(r_1^{-2})), f_{r_2,\overline{t}}(\mathbb{S}^n(r_2^{-2}))),
\]

which is equal to

\[
d_{g^H_\overline{t}}(f_{r_1,\overline{t}}(r_1, 0, \cdots, 0), f_{r_2,\overline{t}}(r_2, 0, \cdots, 0)) = d_{g^H_\overline{t}}((\sqrt{\overline{t}^2 + r_1^2}, r_1, 0, \cdots, 0), (\sqrt{\overline{t}^2 + r_2^2}, r_2, 0, \cdots, 0)).
\]

The shortest geodesic \( \gamma_{\overline{t}} \) (in \( \mathbb{H}^{n+1}(\overline{t}^{-2}) \)) connecting \((\sqrt{\overline{t}^2 + r_1^2}, r_1, 0, \cdots, 0)\) and \((\sqrt{\overline{t}^2 + r_2^2}, r_2, 0, \cdots, 0)\) is given by

\[
\gamma_{\overline{t}}(t) := (\overline{t} \cosh t, \overline{t} \sinh t, 0, \cdots, 0) \quad (\sinh^{-1}(\frac{r_1}{\overline{t}}) \leq t \leq \sinh^{-1}(\frac{r_2}{\overline{t}})),
\]

where \( \sinh^{-1} \) denotes the inverse function of \( \sinh \) \([0, \infty)\). For the simplicity, put \( a(r_i) := \sinh^{-1}(\frac{r_i}{\overline{t}}) \) \((i = 1, 2)\). The length \( L(\gamma_{\overline{t}}) \) of \( \gamma \) is given by

\[
L(\gamma_{\overline{t}}) = \int_{a(r_1)}^{a(r_2)} \|\gamma'_{\overline{t}}(t)\| dt = \int_{a(r_1)}^{a(r_2)} \sqrt{|-\overline{t}^2|} dt = \sqrt{r_2^2 - r_1^2} \cdot \sinh^{-1}(\frac{r_2}{\overline{t}}) - \sinh^{-1}(\frac{r_1}{\overline{t}})).
\]

For the simplicity, denote by \( F_{r_1, r_2}(\overline{t}) \) the right-hand side of this relation. Then we have

\[
d_{g^H_\overline{t}}(f_{r_1,\overline{t}}(r_1, 0, \cdots, 0), f_{r_1,\overline{t}}(r_2, 0, \cdots, 0)) = F_{r_1, r_2}(\overline{t}),
\]

that is,

\[
d_{H,\mathbb{H}^{n+1}(\overline{t}^{-2})}(f_{r_1,\overline{t}}(\mathbb{S}^n(r_1^{-2})), f_{r_2,\overline{t}}(\mathbb{S}^n(r_2^{-2}))) = F_{r_1, r_2}(\overline{t}).
\]

By using L'Hôpital's theorem, we have

\[
\lim_{\overline{t} \to +0} \frac{r_2 \sqrt{r_2^2 + r_1^2} - r_1 \sqrt{r_2^2 + r_2^2}}{\sqrt{r_2^2 + r_1^2} \cdot \sqrt{r_2^2 + r_2^2}} = 0.
\]

Hence we obtain

\[
\lim_{\overline{t} \to +0} d_{H,\mathbb{H}^{n+1}(\overline{t}^{-2})}(f_{r_1,\overline{t}}(\mathbb{S}^n(r_1^{-2})), f_{r_2,\overline{t}}(\mathbb{S}^n(r_2^{-2}))) = 0,
\]

that is,

\[
\tilde{d}_{GH}^{\infty}([\mathbb{S}^n(r_1^{-2})], [\mathbb{S}^n(r_2^{-2})]) = 0.
\]

In more general, we can give the following counter-examples. Let \( \mathbb{M} \) be a compact submanifold in \( \mathbb{S}^n(1) \) embedded by a \( C^{k+1} \)-embedding \( f \). Then, since \((\mathbb{M}, f^*g_1^S)\) and \((\mathbb{M}, r \cdot f^*g_1^S)\) \((r > 0, r \neq 1)\) are compact \( C^k \)-Riemannian submanifolds in \( \mathbb{S}^n(1) \) and \( \mathbb{S}^n(r^{-2}) \), respectively, it is shown that

\[
d_{H,\mathbb{H}^{n+1}(\overline{t}^{-2})}((\mathbb{M}, f^*g_1^S), (\mathbb{M}, r \cdot f^*g_1^S)) = F_{r_1, r_2}(\overline{t})
\]

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and hence
\[ d_{GH}^k([M, f^*g^1]), ([M, r \cdot f^*g^1]) = 0. \]
On the other hand, it is clear that \([M, f^*g^1] \neq [M, r \cdot f^*g^1]\). Denote by \(D_{r_1, r_2}\) the domain of \(H^{n+1}(-\tilde{r}^{-2})\) surrounded by \(f_{r_1} \tilde{f}(S^n(r_1^{-2}))\) and \(f_{r_2} \tilde{f}(S^n(r_2^{-2}))\). The Riemannian manifold \((D_{r_1, r_2}, g^1_{\tilde{r}}|_{D_{r_1, r_2}})\) cannot be isometrically embedded into \(\mathbb{E}^m\) for any \(m \in \mathbb{N}\) (see Figure 5). We consider that this fact arises

\[
\lim_{\tilde{r} \to +0} d_{H, H^{n+1}(-\tilde{r}^{-2})}(f_{r_1} \tilde{f}(S^n(r_1^{-2})), f_{r_2} \tilde{f}(S^n(r_2^{-2}))) = 0.
\]

**Figure 4**: The third example showing \(d_{GH}^\infty([S^n(r_1^{-2})], [S^n(r_2^{-2})]) = 0\)

**Figure 5**: A domain in \(H^{n+1}(-\tilde{r}^{-2})\) isometrically embedded into \(\mathbb{E}^{n+2}\)

### 3 Gromov-Hausdorff-like distance function

By referring three examples in the previous section, we shall define a Gromov-Hausdorff-like distance function over \(\mathcal{RM}^k\). We use the notations in the previous section. The first example in the previous section indicates that \(d_{H, (M, d_{k})}\) in the definition of \(d_{GH}^k\) should be replaced by a distance function including informations of the \(i\)-th derivatives \((1 \leq i \leq k + 1)\) of the isometric embeddings because \(\text{Emb}^{k+1}((M_i, g_i), (\tilde{M}, \tilde{g}))\) is a very wider class than \(\text{Emb}_{d.p.}((M_i, d_{g_i}), (\tilde{M}, d_{\tilde{g}}))\). Also, the third example indicates that, in the definition of \(d_{GH}^k\), the range which \((\tilde{M}, \tilde{g})\) moves should be restricted to the class of \(C^k\)-Riemannian manifolds isometrically embedded into a Euclidean space. On the basis of these reasons, we shall define another function over \(\mathcal{RM}_k^0 \times \mathcal{RM}_k^0\). We shall prepare some notions to state the definition. Let \(\pi_1 : TM \to M\) be the tangent bundle of \(M\), \(\pi_2 : T(TM) \to TM\) be the tangent bundle of the manifold \(TM\). Denote by \(T^2M\) the manifold \(T(TM)\). Let \(\pi_3 : T(T^2M) \to T^2M\) be the tangent bundle of the manifold \(T^2M\) and denote by \(T^3M\) the manifold \(T(T^2M)\). In the sequel,
we define $T^iM$ and $\pi_l$ ($l = 4, 5, \ldots$) inductively. Let $f$ be a $C^{k+1}$-map from a Riemannian manifold $(M, g)$ to another Riemannian manifold $(\tilde{M}, \tilde{g})$. The differential $df : TM \to T\tilde{M}$ of $f$ is defined by $df|_{T_vM} = df_x (x \in M)$ and the differential $d^2f := df(df)$ : $T^2M \to T^2\tilde{M}$ is defined similarly. In the sequel, $d^f : T^iM \to T^i\tilde{M}$ ($l = 3, \ldots, k$) are defined inductively. The Sasaki metric $g_\tilde{S}$ of $T\tilde{M}$ with respect to $g$ is defined by

$$
(g_\tilde{S})_{v_1}(v_2, v_3) := g_{\pi_1(v_1)}((d\pi_1)v_1(v_2), (d\pi_1)v_1(v_3)) + g_{\pi_1(v_1)}((v_2)_{\pi_1}, (v_3)_{\pi_1})
$$

where $(v_i)_{\pi_1}$ denotes the vertical component of $v_i$ with respect to $T_{v_i}(TM) = V_{v_i} \oplus H_{v_i}$ ($V$ : the vertical distribution, $H$ : the horizontal distribution associated to the Riemannian connection of $g$). Here we note that $V_{v_i} := T_{v_i}(\pi^{-1}_1(\pi_1(v_1)))$ is identified with $T_{v_i}(\pi_1(v_1))M = T_{\pi_1(v_1)}M$. Similarly, the Sasaki metric $g^{\tilde{S}}$ of $T^2\tilde{M}$ with respect to $g^{\tilde{S}}$ is defined. In the sequel, the Sasaki metric $g^{\tilde{S}}$ of $T^iM$ with respect to $g^{\tilde{S}}$ ($l = 3, 4, \ldots, k$) are defined inductively. Similarly, $T^i\tilde{M}$ and $g^{\tilde{S}}$ are defined for $(\tilde{M}, \tilde{g})$.

Let $S^iM$ be the unit tangent bundle of the Riemannian manifold $(T^{l-1}M, g^{l-1})$ (i.e., $S^iM : = \{v \in T^iM | g^{l-1}(v, v) = 1\}$). Denote by $m(M, g_i)$ the minimum of natural numbers $l$'s such that $(M, g_i)$ is isometrically embedded into $E^l$. For simplicity, set $m_i := m(M, g_i)$. Here we note that, since $M_i$ is compact, the existence of such a minimum number is assured by the Nash’s isometric embedding theorem $(\text{NI, N2})$. On the basis of the above reasons and the consideration of the second example in the previous section, we define a function $d^k_{G^H} : \mathcal{RM}_c^k \times \mathcal{RM}_c^k \to \mathbb{R}$ by

$$
d^k_{G^H}([(M_1, g_1)], [(M_2, g_2)]) := \inf \{d_{H, (T^{i+1}E, \cdot)}(E_{g_1}, E_{g_2}) \mid f_i \in \text{Emb}^{k+1}_t((M_1, g_1), E^{m_i}) \text{ and } i_i \in \text{Emb}^{k+1}_t(E^{m_1}, E^{m_2}) \ (i = 1, 2)\},
$$

where $m$ is a any natural number with $m \geq \max\{m_1, m_2\}$. It is easy to show that this definition is independent of the choice of the natural number $m$ with $m \geq \max\{m_1, m_2\}$.

The following fact holds for $d^k_{G^H}$.

**Theorem 3.1.** $d^k_{G^H}$ is a distance function over $\mathcal{RM}_c^k$.

First we prepare the following lemma.

**Lemma 3.2.** Let $[(M_1, g_1)] \in \mathcal{RM}_c^k$ ($i = 1, 2$) and $m_1, m_2, m$ be as above. Also, let $\{f^i_{M_1, g_1}\}_{i=1}^\infty$ be a sequence in $\text{Emb}^{k+1}_t((M_1, g_1), E^{m_1})$ satisfying

$$
\|f^i\|_{C^{k+1}} \to \inf \{\|f\|_{C^{k+1}} \mid f \in \text{Emb}^{k+1}_t((M_1, g_1), E^{m_1}) \} \ (l \to \infty),
$$

where $\|\cdot\|_{C^{k+1}}$ denotes the $C^{k+1}$-norm of the vector space $C^{k+1}(M_1, E^{m_1})$ of all $C^{k+1}$-maps from $M_1$ to $E^{m_1}$ (which is regarded as a vector space) with respect to $g_1$ and $g_2$.

Then there exists a sequence $\{f^i_{M_2, g_2}\}_{i=1}^\infty$ in $\text{Emb}^{k+1}_t((M_2, g_2), E^{m_2})$ satisfying the following two conditions:

1. $\|f^i_2\|_{C^{k+1}} \to \inf \{\|f\|_{C^{k+1}} \mid f \in \text{Emb}^{k+1}_t((M_2, g_2), E^{m_2})\}$, where $\|\cdot\|_{C^{k+1}}$ denotes the $C^{k+1}$-norm of the vector space $C^{k+1}(M_2, E^{m_2})$ with respect to $g_2$ and $g_2$.

2. $d_{H, (T^{i+1}E, \cdot)}(E_{g_1}, E_{g_2}) \to d^k_{G^H}([(M_1, g_1)], [(M_2, g_2)]) \ (l \to \infty)$ holds for suitable totally geodesic embeddings $i_i$'s of $E^{m_1}$ into $E^{m_2}$.

It is clear that the distance $d^k_{G^H}([(M_1, g_1)], [(M_2, g_2)])$ is attained when the $C^{k+1}$-norms of isometric embeddings of $(M_1, g_1)$'s ($i = 1, 2$) into $E^{m_1}$ are as small as possible and $E^{m_1}, E^{m_2}$ are isometrically embedded into $E^{m_1}$ by suitable totally geodesic isometric embeddings. Hence it is clear that the statement of this lemma holds. For example, in the case of $(M_1, g_1) = \mathbb{S}^0(r_i^{-2})$ ($i = 1, 2$), we can confirm easily that the statement of Lemma 3.2 holds as follows. Since $\mathbb{S}^0(r_i^{-2})$'s are isometrically embedded into $E^{n+1}$, $m_i$ ($i = 1, 2$) in Lemma 3.2 are equal to $n + 1$ and hence $m$ in Lemma 3.2 also is equal to $n + 1$. First we consider the case of $n \geq 2$. Then, since isometric embeddings of $\mathbb{S}^0(r_i^{-2})$ into $E^{n+1}$ are rigid, that is, they are congruent to one another. In more
detail, they are congruent to the totally umbilic isometric embedding of $S^n(r_1^{-2})$ into $E^{n+1}$ the barycenter of whose origin is equal to the origin $o$ of $E^{n+1}$. Denote by $f_t^n$ this totally umbilic isometric embedding. Hence, in this case, the sequence $\{f_t^n\}_{t=1}^\infty$ in Lemma 3.2 are given by $\{\phi_t \circ f_t^n\}$ in terms of some sequences $\{\phi_t\}_{t=1}^\infty$ of isometries of $E^{n+1}$. Set $f_t^n := \phi_t \circ f_t^n$, then we have

$$d_{H,T^{k+1}E^{n+1}}(d_k^{k+1}(t_1 \circ f_t^n)(S^{k+1}(S^n(r_1^{-2}))), d_k^{k+1}(t_2 \circ f_t^n)(S^{k+1}(S^n(r_2^{-2})))) = |r_1 - r_2| = d_{GH}^k(S^n(r_1^{-2}), S^n((r_2^{-2})))$$

and hence

$$d_{H,T^{k+1}E^{n+1}}(d_k^{k+1}(t_1 \circ f_t^n)(S^{k+1}(S^n(r_1^{-2}))), d_k^{k+1}(t_2 \circ f_t^n)(S^{k+1}(S^n(r_2^{-2})))) \to d_{GH}^k(S^n(r_1^{-2}), S^n((r_2^{-2}))) \quad (l \to \infty).$$

Next we consider the case of $n = 1$. Let $f_t^n$ be the isometric embedding of $S^1(r_1^{-2})$ into $E^2$ whose origin is equal to the circle of radius $r_1$ centered at $o$. Let $\{f_t^n\}_{t=1}^\infty$ be a sequence in Lemma 3.2. Then, it follows from $\|f_t^n\|_{C^{k+1}} \to \inf\{\|f_t^n\|_{C^{k+1}} \mid f \in \text{Emb}^{k+1}_l(S^1(r_1^{-2}), E^2)\}$ that $\|f_t^n - \phi \circ f_t^n\|_{C^{k+1}} \to 0 \quad (l \to \infty)$ holds for some isometry $\phi$ of $E^2$. Let $o_t$ be the barycenter of $f_t^n(S^1(r_1^{-2}))$ and $\tau_{l,o_t} : E^2 \to E^2$ be the parallel translation by $\pm o_t$ (i.e., $\tau_{l,o_t}(p) := p \pm o_t \quad (p \in E^2)$). Set $f_t^n := \tau_{l,o_t} \circ f_t^n$. Take a sequence $\{f_t^{k+1}\}_{t=1}^\infty$ such that $\|f_t^{k+1}\|_{C^{k+1}} \to \inf\{\|f_t^{k+1}\|_{C^{k+1}} \mid f \in \text{Emb}^{k+1}_l((M_1, g_1), E^{m_1})\} \quad (l \to \infty)$. Then, according to Lemma 3.2, there exist sequences $\{f_t^{k+1}\}_{t=1}^\infty$ in $\text{Emb}^{k+1}_l((M_1, g_1), E^{m_1})$ $(i = 1, 3)$ such that

$$\lim_{l \to \infty} d_{H,T^{k+1}E^{m_1},(g_1),E^{m_2}}(d_k^{k+1}(t_2 \circ f_t^n)(S^{k+1}(M_2)), d_k^{k+1}(t_1 \circ f_t^n)(S^{k+1}(M_1))) = a_{21} \quad (i = 1, 3)$$

holds for suitable totally geodesic embeddings $t_{23} \quad (i = 1, 3)$ of $E^{m_2}$ into $E^{m_{23}}$ and suitable totally geodesic embeddings $t_i, t_{23} \quad (i = 1, 3)$ of $E^{m_1}$ into $E^{m_{23}}$. Then we have

$$d_{H,E^{m_2}}(d_k^{k+1}(t_2 \circ f_t^n)(S^{k+1}(M_2)), d_k^{k+1}(t_1 \circ f_t^n)(S^{k+1}(M_1))) \leq d_{H,E^{m_2}}(d_k^{k+1}(f_t^n)(S^{k+1}(M_2)), d_k^{k+1}(f_t^n)(S^{k+1}(M_1)))$$

and hence

$$\lim_{l \to \infty} d_{H,E^{m_2}}(d_k^{k+1}(f_t^n)(S^{k+1}(M_1)), d_k^{k+1}(f_t^n)(S^{k+1}(M_2))) \leq a_{12} + a_{23}.$$ 

Therefore we obtain $a_{13} \leq a_{12} + a_{23}$. Thus $d_{GH}^k$ satisfies the triangle inequality and hence it is a pseudo-distance function.

Furthermore we show that $d_{GH}^k$ is a distance function. Assume that $d_{GH}^k([M_1, g_1],[M_2, g_2]) = 0$. Set $m_{12} := \max\{m_1, m_2\}$. Take a sequence $\{f_t^{k+1}\}_{t=1}^\infty$ in $\text{Emb}^{k+1}_l((M_1, g_1), E^{m_1})$ satisfying $\|f_t^{k+1}\|_{C^{k+1}} \to \inf\{\|f_t^{k+1}\|_{C^{k+1}} \mid f \in \text{Emb}^{k+1}_l((M_1, g_1), E^{m_1})\} \quad (l \to \infty)$. Then, according to Lemma 3.2, there exist a sequence $\{f_t^{k+1}\}_{t=1}^\infty$ in $\text{Emb}^{k+1}_l((M_1, g_1), E^{m_{23}})$ such that

$$\lim_{l \to \infty} d_{H,T^{k+1}E^{m_1},(g_1),E^{m_2}}(d_k^{k+1}(t_1 \circ f_t^n)(S^{k+1}(M_1)), d_k^{k+1}(t_2 \circ f_t^n)(S^{k+1}(M_2))) = d_{GH}^k([M_1, g_1],[M_2, g_2])(= 0)$$

holds for suitable totally geodesic embeddings $t_i, t_{23} \quad (i = 1, 2)$ of $E^{m_1}$ into $E^{m_{23}}$. This implies that $M_1$ and $M_2$ are $C^{k+1}$-diffeomorphic. Take a $C^{k+1}$-diffeomorphism $\psi$ of $M_1$ onto $M_2$. Since $[M_1, \psi \circ g_1] = ([M_2, g_2])$, we
have \( d_{GH}^k([M_1, g_1]), ([M_1, \psi^* g_2]) = 0 \) by the assumption. According to Lemma 3.2, there exists a sequence \( \{\tilde{f}_j\}_{j=1}^\infty \) in \( \text{Emb}_{k+1}((M_1, \psi^* g_2), E^{m_2}) \) such that

\[
\lim_{j \to \infty} d_{H, (T^{k+1}E^{m_1}, g_1^{k+1})} (d^{k+1}(\iota_1 \circ \tilde{f}_j)(S^{k+1} M_1), d^{k+1}(\tilde{\iota}_1 \circ \tilde{f}_j)(S^{k+1} M_1)) = d_{GH}^k([M_1, g_1]), ([M_1, \psi^* g_2]) = 0
\]

holds for a suitable totally geodesic isometric embedding \( \tilde{\iota}_1 \) of \( (M_1, \psi^* g_2) \) into \( E^{m_2} \). Clearly we have \( \lim_{j \to \infty} \| (\iota_1 \circ f_j) - (\tilde{\iota}_1 \circ \tilde{f}_j) \|_{C^{k+1}} = 0 \) and hence \( \lim_{j \to \infty} \| (\iota_1 \circ f_j)^* g_E - (\tilde{\iota}_1 \circ \tilde{f}_j)^* g_E \|_{C^k} = 0 \), where \( \| \cdot \|_{C^k} \) denotes the \( C^k \)-norm of the space of all \( C^k \)-sections of the tensor bundle \( T^* M_1 \otimes T^* M_1 \). Therefore, by noticing \( (\iota_1 \circ f_j)^* g_E = g_1 \) and \( (\tilde{\iota}_1 \circ \tilde{f}_j)^* g_E = \psi^* g_2 \), we obtain \([M_1, g_1]) = ([M_1, \psi^* g_2]) \), that is, \([M_1, g_1]) = ([M_2, g_2]) \). Therefore \( d_{GH}^k \) is a distance function over \( \mathcal{R}M_\xi^k \).

**Problem.** Does \( d_{GH}^k \) coincide with \( d_{GH} \mid_{\mathcal{R}M}_\xi^k \times \mathcal{R}M_\xi^k \)?

If this problem were solved affirmatively, then we can define a completion of \( (\mathcal{R}M_\xi^k, d_{GH}^k) \) as \( (\overline{\mathcal{R}M}_\xi^k, d_{GH}) \mid_{\overline{\mathcal{R}M}_\xi^k \times \overline{\mathcal{R}M}_\xi^k} \), where \( \overline{\mathcal{R}M}_\xi^k \) denotes the closure of \( \mathcal{R}M_\xi^k \) in \( (\mathcal{M}_\xi, d_{GH}) \).

**References**

[G1] M. Gromov, Structures métriques pour les variétés riemanniannes, edited by Lafontaine and Pierre Pansu, 1981.

[G2] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, Progr. Math., 152, Birkhauser Boston, Boston, MA, 1999.

[N1] J. Nash, \( C^1 \)-isometric imbeddings, Ann of Math. 60 (1954), 383–396.

[N2] J. Nash, The imbedding problem for Riemannian manifolds, Ann of Math. 63 (1956), 20–63.