ON VARIATIONS OF THE NEUMANN EIGENVALUES OF 
p-LAPLACIAN GENERATED BY MEASURE PRESERVING 
QUASICONFORMAL MAPPINGS

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We study variations of the first nontrivial eigenvalue of the two-dimensional p-Laplace operator, p > 2, generated by measure preserving quasiconformal mappings. The study is based on the geometric theory of composition operators in Sobolev spaces and sharp embedding theorems. Using a sharp version of the reverse Hölder inequality, we obtain a lower estimate for the first nontrivial eigenvalue in the case of Ahlfors type domains.

Bibliography: 24 titles.

1 Introduction

In the present paper, we consider the Neumann eigenvalue problem for the two-dimensional degenerate p-Laplace operator (p > 2)

$$\Delta_p u = \text{div} (|\nabla u|^{p-2} \nabla u).$$

This operator arises in the study of vibrations of nonelastic membranes [1, 2]. The weak statement of the frequency problem for vibrations of a nonelastic membrane is formulated as follows: a function u solves the problem if $u \in W^1_p(\Omega)$ and

$$\int_{\Omega} (|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x)) \, dx = \mu_p \int_{\Omega} |u(x)|^{p-2} u(x) v(x) \, dx \quad \forall \ v \in W^1_p(\Omega).$$

As known, to calculate exact Neumann eigenvalues is possible only in a limited number of cases. Thus, estimates for the Neumann eigenvalues are important in the spectral theory of elliptic operators.

The lower estimate for the first nontrivial Neumann eigenvalue of the p-Laplace operator, p > 2, is known for convex domains $\Omega \subset \mathbb{R}^n$ [3]:

$$\mu_p(\Omega) \geq \left( \frac{\pi_p}{d(\Omega)} \right)^p,$$

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where \(d(\Omega)\) is the diameter of \(\Omega\) and \(\pi_p = 2\pi(p - 1)^{\frac{1}{p}}/(p\sin(\pi/p))\). In nonconvex domains, \(\mu_p(\Omega)\)
cannot be characterized in terms of the Euclidean diameters, which can be seen by considering
a domain consisting of two identical squares connected by a thin corridor \([4]\). A method for
estimating the Neumann eigenvalues in nonconvex domains was proposed in \([5]\). The method
was based on the geometric theory of composition operators in Sobolev spaces \([6]–[9]\). In a
series of works \([10]–[14]\), this method was used to obtain a lower estimate for the first nontrivial
Neumann eigenvalue of the degenerate \(-Laplace \) operator \((p > 1)\) in a large class of nonconvex domains
in terms of (quasi)conformal geometry. In \([15]\), a lower estimate was obtained for the first
nontrivial Neumann eigenvalue of the degenerate \(p\)-Laplace operator, \(p > 2\), in a large class of
planar domains in terms of the conformal radii of domains.

The goal of this paper is to extend the results of \([10]\) to the case of measure preserving
quasiconformal mappings. Namely, we derive a lower estimate for the first nontrivial Neumann
eigenvalue of the degenerate \(p\)-Laplace operator in a quasiconformal regular domain generated
by a measure preserving quasiconformal mapping. To this purpose we apply the method based
on the geometric theory of composition operators in Sobolev spaces and theory of quasiconformal
mappings (cf., for example, \([6]–[9]\)).

We recall that a homeomorphism \(\varphi : \Omega \to \Omega'\) is a \(K\)-\textit{quasiconformal mapping} if \(\varphi \in W^1_{n,loc}(\Omega)\)
and there exists a constant \(1 \leq K < \infty\) such that \(|D\varphi(x)|^n \leq K|J(x, \varphi)|\) for almost all \(x \in \Omega\).

A connected domain \(\Omega \subset \mathbb{R}^2\) is called a \(K\)-\textit{quasiconformal} \(\beta\)-\textit{regular domain} if there exists a
\(K\)-\textit{quasiconformal mapping} \(\varphi : \mathbb{D} \to \Omega\) such that \(\|J(\cdot, \varphi)\|_{L^\beta(\Omega)} \| < \infty\) for some \(\beta > 1\), where
\(J(z, \varphi)\) is the Jacobian of \(\varphi : \mathbb{D} \to \Omega\) (cf. \([13]\)). A domain \(\Omega \subset \mathbb{R}^2\) is called a \textit{quasiconformal
regular domain} if it is a \(K\)-quasiconformal \(\beta\)-regular domain for some \(\beta > 1\).

We note that the class of quasiconformal regular domains includes the class of Gehring
domains \([16]\) and can be described in terms of quasihyperbolic geometry \([17]–[19]\).

Let \(\varphi : \mathbb{D} \to \Omega\) be a \(K\)-quasiconformal mapping. We note that there exists a measure
preserving mapping, i.e., \(|J(z, \varphi)| = 1, z \in \mathbb{D}\). Some examples of such mappings are described in
Section 3. We also note that, in the class of conformal mappings, there exists a unique mapping
with the Jacobian equal to 1; namely, the identity mapping \(\varphi(z) = z, z \in \mathbb{D}\).

In this paper, we prove that for a \(K\)-quasiconformal \(\beta\)-regular domain \(\Omega\) generated by a
measure preserving \(K\)-quasiconformal mapping \(\varphi : \mathbb{D} \to \Omega\) and \(r = p\beta/(\beta - 1), p > 2\) the following estimate holds:

\[
\frac{1}{\mu_p(\Omega)} \leq \inf_{q \in (q^*, 2]} \left\{ B_{r,q}(\mathbb{D})^{\frac{p}{\beta}} K^p \frac{1}{\pi} \frac{1}{\pi - 1} \right\},
\]

where \(q^* = 2\beta p/(\beta p + 2(\beta - 1))\), \(B_{r,q}(\mathbb{D})\) is the best constant in the (non-weighted) \((r, q)\)-
Poincaré–Sobolev inequality in \(\mathbb{D} \subset \mathbb{R}^2\) estimated from above by (cf., for example, \([14]\))

\[
B_{r,q}(\mathbb{D}) \leq 2^{-\delta} \left( \frac{1 - \delta}{1 - 2\delta} \right)^{1-\delta} \pi^{\delta}, \quad \delta = \frac{1}{q} - \frac{1}{r}.
\]

We also obtain a lower estimate for the first nontrivial Neumann eigenvalue of the degenerate
\(p\)-Laplace operator for Ahlfors type domains (quasidisics) by using the sharp estimate for the constant
in the reverse Hölder inequality for the Jacobians of quasiconformal mappings \([12, 15]\). We
recall that \(K\)-\textit{quasidisics} are the images of unit discs under \(K\)-quasiconformal homeomorphisms
of the plane \(\mathbb{R}^2\). The class of Ahlfors type domains includes all Lipschitz connected domains and
some class of fractal domains (for example, the von Koch snowflake). The Hausdorff dimension
of the quasidisc boundary can be any number in \([1, 2]\).
Let \( \Omega \) be a \( K \)-quasidisc. Then
\[
\mu_p(\Omega) \geq \frac{M_p(K)}{|\Omega|^{\frac{p}{2}}} = \frac{M_p^*(K)}{R_*^p},
\]
where \( R_* \) is the radius of a disc \( \Omega^* \) having the same area as \( \Omega \) and \( M_p^*(K) = M_p(K)\pi^{-p/2} \)
deeps only on \( p \) and the quasiconformality coefficient \( K \) of the domain \( \Omega \). The exact value of \( M_p(K) \) is given in Theorem 4.2.

## 2 Sobolev Spaces and Quasiconformal Mappings

In this section, we recall definitions of Lebesgue and Sobolev spaces, basic facts on composition operators in the Lebesgue and Sobolev spaces, and some facts of the theory of quasiconformal mappings. Let \( \Omega \subset \mathbb{R}^n, n \geq 2, \) be an \( n \)-dimensional domain in the Euclidean space. The Lebesgue space \( L^p(\Omega), 1 \leq p < \infty, \) is the space of all locally integrable functions with the finite norm
\[
\|f|_{L^p(\Omega)}\| = \left( \int_{\Omega} |f(x)|^p \, dx \right)^{\frac{1}{p}}.<\infty.
\]

The following assertion concerning composition operators in the Lebesgue spaces is well known (cf., for example, [9]).

**Theorem 2.1.** Let \( \varphi : \Omega \to \Omega' \) be a weakly differentiable homeomorphism between two domains \( \Omega \) and \( \Omega' \). Then the composition operator \( \varphi^* : L_r(\Omega') \to L_s(\Omega), 1 \leq s \leq r < \infty, \) is bounded if and only if \( \varphi^{-1} \) possesses the Luzin N-property and
\[
\left( \int_{\Omega'} |J(y, \varphi^{-1})|^{\frac{r-s}{r}} \, dy \right)^{\frac{r-s}{rs}} = K < \infty, \quad 1 \leq s < r < \infty,
\]
\[
\text{ess sup}_{y \in \Omega'} |J(y, \varphi^{-1})|^{\frac{1}{s}} = K < \infty, \quad 1 \leq s = r < \infty.
\]

The composition operator has the norm \( \|\varphi^*\| = K. \)

The Sobolev space \( W^1_p(\Omega), 1 \leq p \leq \infty, \) is the Banach space of all locally integrable weakly differentiable functions \( f : \Omega \to \mathbb{R} \) with the finite norm
\[
\|f|_{W^1_p(\Omega)}\| = \left( \int_{\Omega} |f(x)|^p \, dx \right)^{\frac{1}{p}} + \left( \int_{\Omega} |\nabla f(x)|^p \, dx \right)^{\frac{1}{p}},
\]
where \( \nabla f \) is the weak gradient of a function \( f \). Recall that the Sobolev space \( W^1_p(\Omega) \) coincides with the closure of the space of smooth functions \( C^\infty(\Omega) \) in the \( W^1_p(\Omega) \)-norm.

The homogeneous seminormed Sobolev space \( L^1_p(\Omega), 1 \leq p < \infty, \) is the space of all locally integrable weakly differentiable functions equipped with the seminorm
\[
\|f|_{L^1_p(\Omega)}\| = \left( \int_{\Omega} |\nabla f(x)|^p \, dx \right)^{\frac{1}{p}}.
\]
We regard the Sobolev spaces as the Banach spaces of equivalence classes of functions up to a set of $p$-capacity zero [20].

Let $\Omega \subset \mathbb{R}^n$ be an open set. A mapping $\varphi : \Omega \to \mathbb{R}^n$ belongs to the space $L^{1,p}_{\text{loc}}(\Omega)$, $1 \leq p \leq \infty$, if its coordinate functions $\varphi_j$ belong to the spaces $L^{1,p}_{\text{loc}}(\Omega)$, $j = 1, \ldots, n$. In this case, the formal Jacobi matrix

$$D\varphi(x) = \left( \frac{\partial \varphi_i}{\partial x_j}(x) \right), \quad i, j = 1, \ldots, n,$$

and its Jacobian $J(x, \varphi) = \det D\varphi(x)$ are well defined at almost all points $x \in \Omega$. The norm $|D\varphi(x)|$ of the matrix $D\varphi(x)$ is the norm of the corresponding linear operator $D\varphi(x) : \mathbb{R}^n \to \mathbb{R}^n$ defined by the matrix $D\varphi(x)$.

A weakly differentiable mapping $\varphi : \Omega \to \tilde{\Omega}$ has finite distortion if $|D\varphi(z)| = 0$ for almost all $x \in Z = \{ z \in \Omega : J(z, \varphi) = 0 \}$.

We say that a mapping $\varphi : \Omega \to \mathbb{R}^n$ possesses the Luzin $N$-property if the image of any zero measure set has measure zero. We note that any Lipschitz mapping possesses the Luzin $N$-property.

The following theorem yields an analytic description of composition operators in Sobolev spaces.

**Theorem 2.2** (cf. [8, 9]). A homeomorphism $\varphi : \Omega \to \Omega'$ between two domains $\Omega$ and $\Omega'$ induces a bounded composition operator $\varphi^* : L^{1,p}_{\text{loc}}(\Omega') \to L^{1,q}(\Omega)$, $1 \leq q < p < \infty$, if and only if $\varphi$ belongs to $W^{1,1}_{1,\text{loc}}(\Omega)$, has finite distortion, and

$$K_{p,q}(\Omega) = \left( \int_\Omega \left( \frac{|D\varphi(x)|^p}{|J(x, \varphi)|^{p-q}} \right)^{\frac{p-q}{pq}} \, dx \right)^{\frac{1}{p-q}} < \infty.$$

Quasiconformal mappings have finite distortion, i.e., $D\varphi(x) = 0$ for almost all $x \in Z = \{ x \in \Omega : J(x, \varphi) = 0 \}$, and any quasiconformal mapping possesses the Luzin $N$-property. The inverse of a quasiconformal mapping is a quasiconformal mapping.

If $\varphi : \Omega \to \Omega'$ is a $K$-quasiconformal mapping, then $\varphi$ is differentiable almost everywhere in $\Omega$ and

$$|J(x, \varphi)| = J_{\varphi}(x) := \lim_{r \to 0} \frac{|\varphi(B(x, r))|}{|B(x, r)|^r}$$

for almost all $x \in \Omega$.

If $K \equiv 1$, then 1-quasiconformal homeomorphisms are conformal mappings and are exhausted by Möbius transformations in $\mathbb{R}^n$, $n \geq 3$.

3 Eigenvalue Problem for Neumann $p$-Laplacian

A lower estimate for the first nontrivial eigenvalue of the degenerate $p$-Laplace operator ($p > 2$) with the Neumann boundary condition in quasiconformal regular domains was obtained in [10].

**Theorem 3.1.** Assume that $\Omega$ is a $K$-quasiconformal $\beta$-regular domain, $r = p\beta/(\beta - 1)$, $p > 2$. Then

$$\frac{1}{\mu_p(\Omega)} \leq \inf_{q \in (q^*, 2]} \left\{ 2p \left( \frac{1}{2} - \frac{1}{q} + \frac{1}{p} \right)^{p-\frac{q}{p}+\frac{q}{q}} \frac{p-\frac{q}{p}+\frac{q}{q}}{\pi^{\frac{p}{2}}} \cdot K_{p,q}^{\frac{p-q}{pq}} \cdot ||J_{\varphi}||_{L^p(\mathbb{D})} \right\},$$
where $q^\ast = 2\beta p/(\beta p + 2(\beta - 1))$.

In the case of $K$-quasiconformal $\infty$-regular domains, the following assertion holds [10].

**Theorem 3.2.** Let $\Omega$ be a $K$-quasiconformal $\infty$-regular domain. Then for any $p > 2$

$$
\frac{1}{\mu_p(\Omega)} \leq \inf_{q \in (q^\ast, 2]} \left\{ 2^p \left( \frac{1 - \frac{1}{q} + \frac{1}{p}}{\frac{1}{2} - \frac{1}{q} + \frac{1}{p}} \right)^{p+1-\frac{2}{q}} \pi^{1-\frac{2}{q}} \right\} K^\frac{p}{2} |\Omega|^{\frac{p-2}{p}} \cdot ||J_\varphi| L_\infty(\mathbb{D})||,
$$

where $q^\ast = 2p/(p + 2)$.

For an example we consider the domain bounded by an epicycloid. Since such domains are $K$-quasiconformal $\infty$-regular, we can apply Theorem 3.2.

**Example 3.1.** For $n \in \mathbb{N}$ the homeomorphism

$$
\varphi(z) = A \left( z + \frac{z^n}{n} \right) + B \left( \overline{z} + \overline{z^n} \right), \quad z = x + iy, \quad A > B \geq 0,
$$

is quasiconformal with $K = (A + B)/(A - B)$ and maps the unit disc $\mathbb{D}$ onto the domain $\Omega_n$ bounded by the epicycloid of $(n - 1)$ cusps inscribed in the ellipse with the $(A + B)(n + 1)/n$- and $(A - B)(n + 1)/n$-semi-axes. Now, we calculate the area of domain $\Omega_n$ and estimate $||J_\varphi| L_\infty(\mathbb{D})||$. A direct calculation shows that

$$
|\Omega_n| = \int \int_{\mathbb{D}} |J(z, \varphi)| \, dx \, dy = \int \int_{\mathbb{D}} (A^2 - B^2)|1 + z^{n-1}|^2 \, dx \, dy = (A^2 - B^2)^{n+1} n \pi,
$$

$$
||J_\varphi| L_\infty(\mathbb{D})|| = \text{ess sup}_{|z| < 1} [(A^2 - B^2)|1 + z^{n-1}|^2] \leq 4(A^2 - B^2).
$$

By Theorem 3.2,

$$
\frac{1}{\mu_p(\Omega)} \leq \inf_{q \in (q^\ast, 2]} \left\{ \left( \frac{1 - \frac{1}{q} + \frac{1}{p}}{\frac{1}{2} - \frac{1}{q} + \frac{1}{p}} \right)^{p+1-\frac{2}{q}} \pi^{1-\frac{2}{q}} \right\} 2^{p+2}(A + B)^p \left( \frac{n+1}{n} \right)^{\frac{p}{2} - 1},
$$

where $q^\ast = 2p/(p + 2)$.

We construct a $K$-quasiconformal measure preserving mapping $\varphi : \mathbb{D} \to \Omega$, i.e., $|J(z, \varphi)| = 1$, $z \in \mathbb{D}$.

**Example 3.2.** The homeomorphism

$$
\varphi(z) = \sqrt{a^2 + 1} z + a \overline{z}, \quad z = x + iy, \quad a \geq 0,
$$

is a $K$-quasiconformal with $K = \frac{\sqrt{a^2 + 1} + a}{\sqrt{a^2 + 1} - a}$ and maps the unit disc $\mathbb{D}$ onto the interior of the ellipse

$$
\Omega_e = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{(\sqrt{a^2 + 1} + a)^2} + \frac{y^2}{(\sqrt{a^2 + 1} - a)^2} = 1 \right\}.
$$

It is easy to verify that $J(z, \varphi) = |\varphi_z|^2 - |\varphi_{\overline{z}}|^2 = 1$. 

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Example 3.3. The homeomorphism
\[ \varphi(z) = \sqrt{2}(1 + z)\frac{3}{2}(1 + \overline{z})\frac{1}{2}, \quad z = x + iy, \]
is a $K$-quasiconformal mapping with $K = 2$ sending the unit disc $\mathbb{D}$ onto the interior of the “rose petal”
\[ \Omega_p := \left\{ (\rho, \theta) \in \mathbb{R}^2 : \rho = 2\sqrt{\cos(2\theta)}, \quad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \right\}. \]
It is easy to verify that $J(z, \varphi) = |\varphi_x|^2 - |\varphi_y|^2 = 1$.

Example 3.4. Let $f \in L^\infty(\mathbb{R})$. Then $\varphi(x, y) = (x + f(y), y)$ is a quasiconformal mapping with $K = \lambda/J_\varphi(x, y)$. Here, $\lambda$ is the largest eigenvalue of the matrix $Q = DD^T$, where $D = D\varphi(x, y)$ is the Jacobi matrix of the mapping $\varphi = \varphi(x, y)$ and $J_\varphi(x, y) = \det D\varphi(x, y)$ is its Jacobian. It is easy to see that the Jacobi matrix corresponding to the mapping $\varphi = \varphi(x, y)$ has the form
\[ D = \begin{pmatrix} 1 & f'(y) \\ 0 & 1 \end{pmatrix}. \]
A calculation shows that $J_\varphi(x, y) = 1$ and
\[ \lambda = \left(1 + \frac{(f'(y))^2}{2}\right) \left(1 + \sqrt{1 - \frac{4}{(2 + (f'(y))^2)^2}} \right). \]
Therefore, any mapping $\varphi = \varphi(x, y)$ is quasiconformal from $\mathbb{R}^2$ to $\mathbb{R}^2$ with $J_\varphi(x, y) = 1$ and an arbitrary large quasi-conformality coefficient. We can use the restriction $\varphi|_\mathcal{D}$ to the unit disc $\mathcal{D}$. The images can be very exotic quasidisics. If $a > 0$, then the mapping $\varphi(x, y) = (ax + f(y), \frac{1}{a}y)$ have similar properties.

Thus, the class of measure preserving $K$-quasiconformal mappings is not empty. If $\Omega$ is a $K$-quasiconformal $\beta$-regular domain generated by a measure preserving $K$-quasiconformal mapping $\varphi : \mathbb{D} \to \Omega$, then we obtain a lower estimate for the first nontrivial Neumann eigenvalue of the degenerate $p$-Laplace operator ($p > 2$) in terms of the Sobolev–Poincaré constant for the unit disc $\mathbb{D}$ and the quasiconformality coefficient $K$ of the domain $\Omega$.

Theorem 3.3. Assume that $\Omega$ is a $K$-quasiconformal $\beta$-regular domain generated by a measure preserving $K$-quasiconformal mapping $\varphi : \mathbb{D} \to \Omega$ and $r = p\beta/2(\beta - 1)$, $p > 2$. Then
\[ \frac{1}{\mu_p(\Omega)} \leq \inf_{q \in [q^*, 2]} \left\{ 2^p \left( \frac{1}{r} - \frac{1}{q} + \frac{1}{p} \right)^{\frac{p-2}{2}} + \|L_\varphi|_{L_\beta(\mathbb{D})}\|^p \right\} K_\Omega^\beta, \]
where $q^* = 2\beta p/(\beta p + 2(\beta - 1))$.

Proof. According to Theorem 3.1,
\[ \frac{1}{\mu_p(\Omega)} \leq \inf_{q \in [q^*, 2]} \left\{ 2^p \left( \frac{1}{r} - \frac{1}{q} + \frac{1}{p} \right)^{\frac{p-2}{2}} + \|L_\varphi|_{L_\beta(\mathbb{D})}\|^p \right\} K_\Omega^\beta, \]
where $q^* = 2\beta p/(\beta p + 2(\beta - 1))$. Since $\Omega$ is a $K$-quasiconformal $\beta$-regular domain generated by a measure preserving $K$-quasiconformal mapping $\varphi : \mathbb{D} \to \Omega$, we have $||J_\varphi|_{L_\beta(\mathbb{D})}|| = \pi^\beta$ and $|\Omega| = |\mathcal{D}| = \pi$. After some computations we obtain the required estimate. \[ \square \]
In the case of a $K$-quasiconformal $\infty$-regular domain generated by a measure preserving $K$-quasiconformal mapping $\varphi : \mathbb{D} \rightarrow \Omega$, the following assertion holds.

**Theorem 3.4.** Assume that $\Omega$ is a $K$-quasiconformal $\infty$-regular domain generated by a measure preserving $K$-quasiconformal mapping $\varphi : \mathbb{D} \rightarrow \Omega$. Then for any $p > 2$

$$\frac{1}{\mu_p(\Omega)} \leq \inf_{q \in (q^*, 2]} \left\{ 2^p \left( \frac{1 - \frac{1}{q} + \frac{1}{p}}{1 - \frac{1}{q} + \frac{1}{p}} \right)^{p+1 - \frac{p}{q}} \right\} K^{\frac{p}{q}},$$

where $q^* = 2p/(p + 2)$.

From Theorem 3.4 we obtain a lower estimate for the first nontrivial eigenvalue of the Neumann problem for the degenerate $p$-Laplace operator in the domains from Examples 3.2 and 3.3, i.e., $\Omega = \Omega_e$ and $\Omega = \Omega_p$. In these cases,

$$\frac{1}{\mu_p(\Omega_e)} \leq \inf_{q \in (q^*, 2]} \left\{ 2^p \left( \frac{1 - \frac{1}{q} + \frac{1}{p}}{1 - \frac{1}{q} + \frac{1}{p}} \right)^{p+1 - \frac{p}{q}} \left( \frac{\sqrt{a^2 + 1 + a}}{\sqrt{a^2 + 1 - a}} \right)^{\frac{p}{q}} \right\}$$

and

$$\frac{1}{\mu_p(\Omega_p)} \leq \inf_{q \in (q^*, 2]} \left\{ \left( \frac{1 - \frac{1}{q} + \frac{1}{p}}{1 - \frac{1}{q} + \frac{1}{p}} \right)^{p+1 - \frac{p}{q}} \right\} \left( 2\sqrt{2} \right)^p,$$

where $q^* = 2p/(p + 2)$.

### 4 Spectral Estimates in Quasidiscs

In this section, we extend Theorem 3.1 to the class of Ahlfors type domains (i.e., quasidiscs) by using the weak reverse Hölder inequality and the sharp estimates for the constants in doubling conditions for measures generated by the Jacobians of quasiconformal mappings [12].

According to [21], the boundary of any $K$-quasidisc $\Omega$ admits a $K^2$-quasiconformal reflection. Hence any quasiconformal homeomorphism $\varphi : \mathbb{D} \rightarrow \Omega$ can be extended to a $K^2$-quasiconformal homeomorphism from the whole plane to itself.

We recall the known sharp result: $J(z, \varphi) \in L^p_{\text{loc}}(\Omega)$ for any planar $K$-quasiconformal homeomorphism $\varphi : \Omega \rightarrow \Omega'$ and any $1 \leq p < K/(K - 1)$ (cf. [22, 23]).

In fact, the following result concerning estimates for the constant in the reverse Hölder inequality for the Jacobians of quasiconformal mappings was proved in [12], although it was not formulated in an explicit form.

**Theorem 4.1.** Let $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a $K$-quasiconformal mapping. Then for every disc $\mathbb{D} \subset \mathbb{R}^2$ and $1 < \kappa < K/(K - 1)$ the following reverse Hölder inequality holds:

$$\left( \int_{\mathbb{D}} |J(z, \varphi)|^\kappa \, dx \, dy \right)^{\frac{1}{\kappa}} \leq C_\kappa K \pi^{\frac{1}{\kappa} - 1} \exp \left\{ \frac{K\pi^2(2 + \pi^2)^2}{2\log 3} \right\} \int_{\mathbb{D}} |J(z, \varphi)| \, dx \, dy,$$

where

$$C_\kappa = \frac{10^6}{\left[ (2\kappa - 1)(1 - \nu) \right]^{1/2\kappa}}, \quad \nu = 10^{6\kappa} \frac{2\kappa - 2}{2\kappa - 1} (24\pi^2 K)^{2\kappa} < 1.$$
If $\Omega$ is a $K$-quasidisc, then the above theorem and the fact that the quasiconformal mapping $\varphi : \mathbb{D} \to \Omega$ admits a $K^2$-quasiconformal reflection [24, 21] imply the following assertion.

**Corollary 4.1.** Let $\Omega \subset \mathbb{R}^2$ be a $K$-quasidisc, and let $\varphi : \mathbb{D} \to \Omega$ be a $K$-quasiconformal mapping. Assume that $1 < \kappa < K/(K - 1)$. Then

$$
\left( \iint_{\mathbb{D}} |J(z, \varphi)|^{\kappa} \, dx \, dy \right)^{\frac{1}{\kappa}} \leq \frac{C_{\kappa}}{\kappa} \left( \frac{K^2 \pi^{\frac{1}{2} - 1}}{4} \exp \left\{ \frac{K^2 \pi^2 (2 + \pi^2)^2}{2 \log 3} \right\} \cdot |\Omega|, \right.
$$

where

$$
C_{\kappa} = \frac{10^6}{[2(2\kappa - 1)(1 - \nu)]^{1/2}\nu}, \quad \nu = 10^{8\kappa} \frac{2\kappa^2}{2\beta - 1} \frac{(24\pi^2 K^2)^{2\kappa}}{1 - \kappa} < 1.
$$

Combining Theorem 3.1 and Corollary 4.1, we obtain spectral estimates in the case of the degenerate $p$-Laplace operator $(p > 2)$ with the Neumann boundary conditions in Ahlfors type domains. We denote $\beta^* = \min (K/(K - 1), \beta)$, where $\beta$ is the unique solution of the equation

$$
\nu(\beta) := 10^{8\beta} \frac{2\beta^2 - 2}{2\beta - 1} (24\pi^2 K^2)^{2\beta} = 1.
$$

The function $\nu(\beta)$ is monotone and increasing. Hence $(1 - \nu(\beta)) > 0$ and $C_\beta > 0$ for any $\beta < \beta^*$.

**Theorem 4.2.** Let $\Omega$ be a $K$-quasidisc. Then

$$
\mu_p(\Omega) \geq \frac{M_p(\Omega)}{|\Omega|^\frac{1}{p}} = \frac{M_p^*(\Omega)}{R_p^*},
$$

where $R_*$ is a radius of a disc $\Omega^*$ of the same area as $\Omega$ and $M_p^*(\Omega) = M_p(\Omega)\pi^{-p/2}$. The quantity $M_p^*(\Omega)$ depends only on $p$ and the quasiconformality coefficient $K$ of the domain $\Omega$:

$$
M(K) := \frac{\pi^\frac{1}{2}}{2^{p-2} K^{\frac{p}{2} + 2}} \exp \left\{ - \frac{K^2 \pi^2 (2 + \pi^2)^2}{2 \log 3} \right\} \inf_{\beta \in (1, \beta^*)} \inf_{q \in (q^*, 2]} \left\{ \frac{1 - \frac{1}{q} + \frac{1}{\beta}}{2 - \frac{1}{q} + \frac{1}{\beta}} \right\} \left( \frac{\nu(\beta) - p + \frac{p-1}{2} - \frac{p-1}{\nu(\beta)}}{\nu(\beta) - \frac{p-1}{\nu(\beta)}} \right) C_{\beta},
$$

$$
C_{\beta} = \frac{10^6}{[(2\beta - 1)(1 - \nu(\beta))]^{1/2}\beta}.
$$

**Proof.** We note that $K$-quasidiscs, $K \geq 1$, are $K$-quasiconformal $\beta$-regular domains if $1 < \beta < K/(K - 1)$. By Theorem 3.1, for $1 < \beta < K/(K - 1)$, $r = p\beta/(\beta - 1)$, $p > 2$,

$$
\frac{1}{\mu_p(\Omega)} \leq \inf_{q \in (q^*, 2]} \left\{ 2^p \left( \frac{1 - \frac{1}{q} + \frac{1}{\beta}}{2 - \frac{1}{q} + \frac{1}{\beta}} \right)^{p - \frac{p+1}{q} + \frac{p}{\pi^2 - \frac{p-1}{\nu(\beta)}}} K_{\beta} \cdot |\Omega|^\frac{1}{p} \cdot ||J_\varphi| L_\beta(\mathbb{D})||, \right.
$$

where $q^* = 2\beta p/(\beta p + 2(\beta - 1))$. Using Corollary 4.1, we estimate $||J_\varphi| L_\beta(\mathbb{D})||$. A direct calculation shows that

$$
||J_\varphi| L_\beta(\mathbb{D})|| = \left( \iint_{\mathbb{D}} |J(z, \varphi)|^{\beta} \, dx \, dy \right)^{\frac{1}{\beta}} \leq \frac{C_{\beta} K^2 \pi^{\frac{1}{2} - \beta}}{4} \exp \left\{ \frac{K^2 \pi^2 (2 + \pi^2)^2}{2 \log 3} \right\} \cdot |\Omega|. \quad (4.2)
$$

Finally, combining the inequalities (4.1) and (4.2) and making some computations, we obtain the required inequality.

**Remark 4.1.** In the case of conformal mappings, the lower estimate for the first nontrivial Neumann eigenvalue of the degenerate $p$-Laplace operator in a quasidisc was obtained in [15].
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References

1. G. Astarita and G. Marrucci, *Principles of Non-Newtonian Fluids Mechanics*, McGraw-Hill, New York (1974).

2. G. Poliquin, “Principal frequency of the $p$-Laplacian and the inradius of Euclidean domains,” *J. Topol. Anal.* 7, No. 3, 505–511 (2015).

3. L. Esposito, C. Nitsch, and C. Trombetti, “Best constants in Poincaré inequalities for convex domains,” *J. Convex Anal.* 20, No. 1, 253–264 (2013).

4. B. Brandolini, F. Chiacchio, E. B. Dryden, and J. J. Langford, “Sharp Poincaré inequalities in a class of non-convex sets,” *J. Spectr. Theory* 8, No. 4, 1583–1615 (2018).

5. V. Gol’dshtein and A. Ukhlov, “On the first eigenvalues of free vibrating membranes in conformal regular domains,” *Arch. Ration. Mech. Anal.* 221, No. 2, 893–915 (2016).

6. V. Gol’dshtein and L. Gurov, “Applications of change of variables operators for exact embedding theorems,” *Integral Equations Oper. Theory* 19, No. 1, 1–24 (1994).

7. V. Gol’dshtein and A. Ukhlov, “About homeomorphisms that induce composition operators on Sobolev spaces,” *Complex Var. Elliptic Equ.* 55, No. 8–10, 833–845 (2010).

8. A. Ukhlov, “On mappings, which induce embeddings of Sobolev spaces,” *Sib. Math. J.* 34, No. 1, 165–171 (1993).

9. S. K. Vodop’yanov, A. D. Ukhlov, “Superposition operators in Sobolev spaces,” *Russ. Math.* 46, No. 10, 9–31 (2002).

10. V. Gol’dshtein, R. Hurri-Syrjänen, V. Pchelintsev, and A. Ukhlov, “Space quasiconformal composition operators with applications to Neumann eigenvalues,” *Anal. Math. Phys.* 10, Article No. 78 (2020).

11. V. Gol’dshtein, V. Pchelintsev, and A. Ukhlov, “Spectral estimates of the $p$-Laplace Neumann operator and Brennan’s conjecture,” *Boll. Unione Mat. Ital.* 11, No. 2, 245–264 (2018).

12. V. Gol’dshtein, V. Pchelintsev, and A. Ukhlov, “Integral estimates of conformal derivatives and spectral properties of the Neumann-Laplacian,” *J. Math. Anal. Appl.* 463, No. 1, 19–39 (2018).

13. V. Gol’dshtein, V. Pchelintsev, and A. Ukhlov, “Spectral properties of the neumann-laplace operator in quasiconformal regular domains,” *Contemp. Math.* 734, 129–144 (2019).

14. V. Gol’dshtein and A. Ukhlov, “Spectral estimates of the $p$-Laplace Neumann operator in conformal regular domains,” *Trans. A. Razmadze Math. Inst.* 170 No. 1, 137–148 (2016).

15. V. Gol’dshtein, V. Pchelintsev, and A. Ukhlov, “On the first eigenvalue of the degenerate $p$-Laplace operator in non-convex domains,” *Integral Equations Oper. Theory* 90, No. 4, Paper No. 43 (2018).
16. K. Astala and P. Koskela, “Quasiconformal mappings and global integrability of the derivative,” *J. Anal. Math.* **57**, 203–220 (1991).

17. F. W. Gehring and O. Martio, “Lipschitz classes and quasiconformal mappings,” *Ann. Acad. Sci. Fenn., Ser. A I, Math.* **10**, 203–219 (1985).

18. R. Hurri, “Poincaré domains in $\mathbb{R}^n$,” *Ann. Acad. Sci. Fenn., Ser. A I, Math.* **71**, 1–42 (1988).

19. P. Koskela, J. Onninen, and J. T. Tyson, “Quasihyperbolic boundary conditions and Poincaré domains,” *Math. Ann.* **323**, No. 4, 811–830 (2002).

20. V. Maz’ya, *Sobolev Spaces. With Applications to Elliptic Partial Differential Equations*, Springer, Berlin (2011).

21. F. W. Gehring and K. Hag, “Reflections on reflections in quasidicks,” *Report. Univ. Jyväskylä* **83**, 81–90 (2001).

22. K. Astala, “Area distortion of quasiconformal mappings,” *Acta Math.* **173**, No. 1, 37–60 (1994).

23. V. M. Gol’dshtein, “The degree of summability of generalized derivatives of quasiconformal homeomorphisms,” *Sib. Math. J.* **22**, No. 6, 821–836 (1981).

24. L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, Am. Math. Soc., Providence, RI (2006).

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