Abstract. The purpose of this paper is to present a generalized hole argument for gauge field theories and their geometrical setting in terms of fiber bundles. The generalized hole argument is motivated and extended from the spacetime hole arguments which appear in spacetime theories based on differentiable manifolds such as general relativity. Analogously, the generalized hole argument rules out fiber bundle substantivalism and, thus, a relationalistic interpretation of the geometry of fiber bundle spaces is favoured. Along the way, the concept of gauge field theories will be analyzed via considering the gauge principle and thereby hopefully clarifying certain terminological ambiguities.

1 Introduction

There can be no doubt: gauge field theories (for short, ‘gauge theories’), nowadays, provide a most powerful tool in modern physics with regard to a unification of the four known interaction forces. In this connection, the so-called gauge principle lays the foundation of these theories in terms of an elegant derivation of the interaction coupling. The principle works by satisfying a gauge postulate, the heartpiece of any gauge theory, which demands the theory’s invariance under local gauge transformations of the matter fields. Unfortunately, we are far away from a proper understanding of the gauge principle’s conceptual meaning – it actually works just heuristically. But since the theoretical and, most of all, experimental success of the gauge approach is hardly understandable as pure coincidence, we are challenged with a deep physical and philosophical problem.

It is well known that gauge field theories allow a natural mathematical description in the framework of fiber bundles, which may therefore be considered as an enlarged geometrical arena of physics. Thus, from the philosopher’s point of view a first step into a better understanding could be made by analyzing the status of this geometry and its internal spaces. This paper will deal with these questions in terms of a confrontation of relationalism vs. substantivalism.
with regard to bundle spaces. First, I will consider the concepts of the gauge principle, gauge transformations, and gauge freedom. After introducing fiber bundles I propose a definition of the notion of gauge field theory. I will, finally, turn to the spacetime hole argument, and will propose a generalized bundle-space hole argument, which rules out fiber bundle substantivalism.

2 The gauge principle

We start from the empirical fact that there exist certain conserved quantities in nature. Actually, Noether’s theorem tells us that, given any global symmetry, there is a corresponding conserved quantity.

**Noether’s theorem.** Let \( \phi_i(x) \) be some field variable (with general index \( i \) of the field components). Then the invariance of the action functional \( S[\phi] = \int \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \, d^4x \) under some \( k \)-dimensional Lie group leads to the existence of \( k \) conserved currents.

As a paradigm case I shall consider the free Dirac field \( \psi(x) \) with the Lagrangian density

\[
\mathcal{L}_D = \bar{\psi}(x) (i\gamma^\mu \partial_\mu - m) \psi(x).
\]

Clearly, the free Dirac Lagrangian is form invariant under global gauge transformations of the spinor wavefunctions,

\[
\psi(x) \to \psi'(x) = e^{iq\alpha} \psi(x), \quad \bar{\psi}(x) \to \bar{\psi}'(x) = e^{-iq\alpha} \bar{\psi}(x),
\]

with some arbitrary constant phase parameter \( \alpha \) and charge \( q \). The Noether current corresponding to the transformations (2) is given by

\[
j^\mu = -q \bar{\psi}(x) \gamma^\mu \psi(x).
\]

It satisfies the continuity equation,

\[
\partial_\mu j^\mu = 0,
\]

which expresses the conservation of charge. In order to identify \( q \) in (4) empirically with the elementary charge \( e \), we have to couple the Dirac particle – perhaps an electron – to the electromagnetic field. Thus, the free Lagrangian, which is an idealization anyway, must be replaced by some Lagrangian describing interaction. Miraculously, it turns out that this coupling can in fact be derived just by postulating the invariance of (4) under local gauge transformations instead of the corresponding global ones (2).

**Gauge postulate.** The Lagrangian of a free matter field \( \phi_i(x) \) should remain invariant under local gauge transformations \( \phi_i(x) \to \phi'_i(x) = \phi'_i(\phi_i(x), \alpha_s(x)) \).

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1 I refer to Emmy Noether’s first theorem simply as Noether’s theorem, whereas her second theorem, which is related to infinite symmetry transformations, i.e. transformations with arbitrary functions instead of parameters, for these purposes will be better described in terms of local gauge transformations, which in fact play the central role in gauge theories.
To see ‘how the miracle occurs’ in the example, we consider the free Dirac equation:

\[(i\gamma^\mu \partial_\mu - m) \psi(x) = 0.\]  

(5)

Due to the gauge postulate we have to replace (2) by

\[\psi(x) \rightarrow \psi'(x) = e^{iq\alpha(x)} \psi(x)\]  

(6)

with a local, i.e. spacetime dependent, phase function \(\alpha(x)\). Obviously (5) is not invariant under this local gauge transformation. If we, however, identify

\[A_\mu(x) = -\partial_\mu \alpha(x)\]  

(7)

and thereby introduce a coupling field, which itself satisfies the local gauge transformations

\[A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x),\]  

(8)

we may get a new interaction equation

\[(i\gamma^\mu \partial_\mu - m) \psi(x) = q \gamma^\mu A_\mu(x) \psi(x).\]  

(9)

This equation is indeed invariant under the combined transformations (6) and (8).

Formally it seems reasonable to identify \(A_\mu\) with the electromagnetic potential, since the construction of a field strength tensor (as the derivative of the potential) which is invariant under (8) gives

\[F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x).\]  

(10)

This tensor satisfies the vacuum Maxwell equations

\[\partial^\mu F_{\mu\nu}(x) = 0\]  

(11)

and, as a Bianchi identity,

\[\partial_{[\mu} F_{\nu\rho]}(x) = 0.\]  

(12)

Maxwell’s equations follow from the Lagrangian of the free electromagnetic field

\[\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x).\]  

(13)

The coupling can explicitly be seen by introducing a covariant derivative

\[\partial_\mu \rightarrow D_\mu = \partial_\mu - iq A_\mu(x)\]  

(14)

and therefore

\[\mathcal{L}_{int} = -j_\mu(x) A^\mu(x).\]  

(15)

Since

\[(i\gamma^\mu D_\mu - m) \psi(x) = 0\]  

(16)

is equivalent to (8), the coupling of matter and interaction fields can be derived in just one step via (14), thereby satisfying the gauge postulate by means of

\[\mathcal{L}_D \rightarrow \mathcal{L}'_D = \mathcal{L}_D + \mathcal{L}_{int}.\]  

(17)

\(^2\)This is the Euler–Lagrange equation belonging to (1).

\(^3\)The experimental evidence for this maneuver is usually seen in the existence of the Aharonov–Bohm effect, which should justify the crucial identification (8).
Gauge principle. The coupling of the Noether current corresponding to the global gauge transformations of the Lagrangian of free matter fields can be introduced via replacing the usual derivative by the covariant derivative $\partial_{\mu} \rightarrow D_{\mu}$ corresponding to local gauge transformations.

Now, the merit of the concept of gauge field theories in modern physics becomes evident since the gauge principle provides a most successful and elegant ‘recipe’ for introducing interaction, e.g. in our example deriving from the free theory (1) via (6) the coupling structure of quantum electrodynamics.

$$L_{QED} = L_D + L_{EM} + L_{int}. \quad (18)$$

3 Gauge transformations and gauge freedom

As we have seen, quantum electrodynamics can be understood as a gauge field theory proper with gauge group $U(1)$, since the gauge transformations occurring are global and local $U(1)$ transformations. Unfortunately, the usage of the terms ‘gauge theory’ and ‘gauge transformations’ is by no means uniform throughout the literature, which sometimes leads to conceptual confusions. In order to clarify the terminology I shall make some necessary distinctions:

1. Global gauge transformations, also called ‘gauge transformations of the first kind’ ($GT_1$), with corresponding gauge group $G_1$.

2. Local gauge transformations, also called ‘gauge transformations of the second kind’ ($GT_2$), with corresponding gauge group $G_2$.

On closer inspection of $GT_2$ one should distinguish two kinds:

2a. Matter field transformations, hereby called ‘type a’ gauge transformations of the second kind ($GT_{2a}$).

2b. Gauge field transformations, hereby called ‘type b’ gauge transformations of the second kind ($GT_{2b}$).

Regarding the example from the preceding section the $GT_{2a}$ are given by (1), whereas the $GT_{2b}$ are given by (8). Usually the $GT_{1}$-$GT_{2}$ distinction is made, seldom however $GT_{2a}$-$GT_{2b}$. One exception is for instance Wolfgang Pauli, who in an early influential article concerning the gauge approach in relativistic field theories indicates the $GT_{2a}$-$GT_{2b}$ distinction, however calling it “... gauge transformations of the first ... and ... of the second type” (Pauli 1941, p.207). I very much agree with Pauli in regarding this as an important distinction, which is, as

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4 In order to obtain full quantum electrodynamics the fields $\psi(x), \bar{\psi}(x)$, and $A_\mu(x)$ have to be quantized.

5 This is the usual terminology, although there may exist fundamental particle fields with mass zero such as, perhaps, neutrinos – ‘energy-matter field’ is certainly the more precise term.

6 This is again the usual terminology, although ‘gauge potential’ is more precise. The derivative gives the gauge field strength.
he points out, "... manifested through the fact that only expressions which are bilinear in $U$ and $U^*$ [comparable with the matter fields $\psi$ and $\bar{\psi}$ in (1)] are associated with physically measurable quantities ..." and "... that, in principle, only gauge invariant quantities can be obtained by direct measurement". Moreover, the structure of the GT2b – although they already appear in the free Maxwell theory (13) – is forced by the structure of the GT2a, thus, expressing the 'miracle' of the gauge principle in other terms: the gauge field appears as an appendix of the matter field!

In addition to the 3-fold distinction (GT1, GT2a, GT2b) concerning the usage of the notion of 'gauge transformation', the term 'gauge theory' should be clarified. It is already a common practice to call classical electrodynamics, i.e. the free Maxwell theory, a gauge theory. This is due to the fact that (13) is form invariant under the GT2b transformations (8). But this is certainly a misleading terminology, since we better should refer to this invariance as a gauge freedom of the theory, whereas only the combined Dirac-Maxwell theory, i.e. quantum electrodynamics (8), is to be considered as a true gauge theory. Note that the same argument also holds for the diffeomorphism invariance of our known spacetime theories. Whether general relativity, for instance, may nonetheless be considered as a gauge theory is another question and should not be confused with the obvious gauge freedom concerning the group of diffeomorphisms of the spacetime manifold. Thus, a theory comprising some gauge freedom is not yet a gauge theory, but only a theory incorporating the gauge principle.

4 The fiber bundle structure of gauge theories

I shall recall the definition of a fiber bundle $\langle E, M, \pi, F, G \rangle$ with bundle space $E$, base manifold $M$, projection map $\pi : E \to M$, fiber space $F$, and structure group $G$ – compare e.g. (NAKAHARA 1990). Fiber bundles can be considered as generalizations of direct product spaces, locally looking like $M \times F$. Thus, a local trivialisation is given by a diffeomorphic map $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ within some open set $U_i \subset M$. In order to obtain the global bundle structure the local charts $\phi_i$ must be glued together via transition functions $t_{ij}(p) = \phi^{-1}_{i,p} \circ \phi_{j,p}$ with $\phi_{i,p}(f) \equiv \phi_i(p,f)$, $p \in M$, $f \in F$. A bundle section is a mapping $s : M \to E$ and can be considered as a generalization of a tangent vector field. With $\pi(s(p)) = p$ the section $s(p) \in F_p$ is local. A bundle is called trivial, if it admits a global section. In physics two classes of bundles play a central role. If the fiber is given by some $n$ dimensional linear vector space $V^n$ the bundle is called a vector bundle $E(M, V^n, GL(n, \mathbb{V}))$. For $\mathbb{V}^n = \mathbb{R}^n$ the general structure group is $G = GL(n, \mathbb{R})$. For principal bundles $P(M,G)$ the fiber is itself a Lie group $F \equiv G$ with a natural action on the bundle from the right, $F \times G \to P$. To any principal bundle there naturally exists an associated vector bundle with the same structure group and transition functions.

\(^{7}\text{Pauli was always concerned by questions relating to the measurability of quantized fields. Already in [\text{Pauli} 1933, p. 579] he noticed "... daß für das Photonfeld ... der Begriff der raum-zeitlich-lokalen Teilchendichte W(\vec{x}, t) nicht sinnvoll existiert" [... that for the photon field ... the notion of a particle density $W(\vec{x}, t)$ located in space-time has no meaningful existence]. This is due to the fact that the four current for the photon field identically vanishes, $j^\mu_E = \partial_\mu F^{\mu\nu} = 0$, which has considerable consequences for the interpretation of quantum fields. Since there is no local conservation law for the number of photons, the concept of a well-defined particle density is not in the same sense meaningful for the photon gauge field as it is for the Dirac matter field [\text{Lyre} 1996].}\)
It is crucial for our considerations to understand that the geometrical framework of fiber bundles provides indeed a natural mathematical setting for the representation of physical gauge field theories. It turns out that any of the four fundamental interactions can be represented within the framework of a principal bundle. For instance, quantum electrodynamics in section \[ \text{section 2} \] is to be considered as a $U(1)$ gauge field theory with a trivial bundle $\mathbb{P}(\mathbb{R}^{(1,3)}, S^1)$ – the triviality of this bundle being due to the fact that Minkowskian spacetime $\mathbb{R}^{(1,3)}$ is contractible. Within the fiber bundle language, gauge physical notions obey the following dictionary: A matter field is given as a section $s$ in the associated vector bundle $(\mathbb{E}(\mathbb{R}^{(1,3)}, \mathbb{C}, U(1))$ in our example). In order to describe the gauge potential the concept of a bundle connection $\omega$ is needed: $\omega$ is a 1-form with values in the Lie algebra $\mathfrak{g}$ of the structure group $G$ which defines a unique decomposition $T\mathbb{P} = V\mathbb{P} \oplus H\mathbb{P}$ of the tangent space $T\mathbb{P}$ of the principal bundle $\mathbb{P}$ into a ‘vertical’ and a ‘horizontal’ part. Note that the vertical subspace $V\mathbb{P}$ is isomorphic to $\mathfrak{g}$. Hence, the Dirac-Maxwell gauge potential can be represented as $A = s^*\omega = A_\mu dx^\mu$, where $s^*$ is the pull back. The local gauge transformations \[ (8) \], which stem from the transition functions $t_{ij}$ above, may in general be written as $\omega' = g^{-1}\omega g + g^{-1}dg$ with $g \in G$. Thus, the structure group serves as the gauge group.

It is important to note that, again, this terminology is not uniform. Due to the local gauge postulate the gauge group consists of spacetime-dependent group elements and is, thus, infinite dimensional. Therefore sometimes the gauge group is to be considered as a subgroup $G \subset Aut(\mathbb{P})$ of the automorphism group of $\mathbb{P}$. The ‘pure gauge transformations’, which preserve the connection, are then given by the group $G_o$ of just the vertical automorphisms. This is of particular interest when considering general relativity as a gauge theory. Here, the bundle structure is given by the orthonormal frame bundle $L_\omega \mathcal{M}$ (sometimes called tetrad bundle since the frames are orthonormalized tetrads) with the homogeneous Lorentz group as structure group. Trickily, as authors like Andrzej Trautman have pointed out \[ \text{TRAUTMAN 1980} \], in general relativity the ‘gauge group’ $G$ is simply isomorphic to the diffeomorphism group $Diff(\mathcal{M})$ of the spacetime manifold $\mathcal{M}$, whereas the group of ‘pure gauge transformations’ shrinks to the identity $G_o = id$. Although this is certainly an important characteristic of general relativity, I prefer calling the structure group the gauge group – in contrast to $G$ as the group of local gauge transformations. In gravitational gauge theories this refers to the central conceptual importance of tetradial reference frames, which are locally free to rotate and translate due to the structure group. Therefore sometimes general relativity can alternatively be considered as a translational gauge theory, since the tetrads represent the gauge potentials of the translation group isomorphic to $\mathbb{R}^{(1,3)}$. In this case the corresponding Noether currents are given by the energy-momentum density current, a procedure which exactly mimics the way the gauge principle works.\[ \text{footnote} \]

I shall now claim – as an important feature of any gauge approach – that the local gauge transformations $GT2$ only allow for an active interpretation. This can directly be seen from the gauge postulate. More precisely, this point of view holds inherently for the local gauge transformations of the matter fields $GT2a$, whereas the gauge fields turn out to be a consequence

\footnote{These questions are closely related to the fact that in general relativity the fiber is soldered to the base manifold, which indicates the most important conceptual difference from quantum gauge field theories. Moreover, it seems quite natural to extend the structure group, which leads to generalized theories of gravitation – most of them including torsion. Thus, orthodox general relativity fits into the gauge theoretic framework although this framework forces one to think about more general structures – compare \[ \text{HEHL et al. 1995} \] for an overview.}
of this postulate (which is the very idea of the gauge principle). In general relativity, ‘matter fields’ are represented in terms of reference frames. We therefore are forced to think about reference frames as the building blocks of gravitational gauge theories \cite{Hehl1994} – even more so in encountering quantum gravity \cite{Rovelli1991}.

Concluding this, the following definition of gauge field theories can be given:

\begin{definition}
We call a \textbf{gauge field theory} any theory being derived from the gauge principle and representing the geometry of a principal fiber bundle. The gauge group is given by the structure group of the bundle.
\end{definition}

5 Hole arguments till now

In 1913-1914, during his crucial years of developing general relativity, \textsc{Einstein} claimed mistakenly that the new theory might not be generally covariant, i.e. form invariant under general coordinate transformations. He tried to convince himself by the so-called “\textit{Lochbetrachtung}”, i.e. by considering a hole in the matter energy distribution where $T_{\mu\nu} = 0$, which would lead to the possibility of describing the metric $g_{\mu\nu}$ via, in modern terminology, different diffeomorphic tensors – a possibility which for \textsc{Einstein} seemed to contradict the law of causality. This is the first famous “hole argument” and its historical and philosophical background has been scrutinized in detail – see e.g. \cite{Norton1984}, \cite{Stachel1989}. For the purpose of this paper only the structure of the argument will be of interest. It can be repeated in two steps:

1. \textbf{Leibniz equivalence of diffeomorphic models of usual spacetime theories (general covariance):} In the relationalist’s view, diffeomorphic models of any spacetime theory making use of the concept of smooth manifolds to represent spacetime are equivalent with regard to any observation, i.e. they represent one and the same physical situation.

2. \textbf{Failure of determinism:} The substantivalist’s assumption of an existence of spacetime points independent from the matter content of spacetime leads to an indeterminism by considering different diffeomorphic models of a theory whose predictions cannot be used to make out any empirical distinction between the ‘different’ models. Thus, indeterminism arises due to the substantivalist’s denying of \textit{Leibniz} equivalence.

In 1987, \textsc{John Earman} and \textsc{John Norton} presented a new hole argument following this 2-step structure, which is valid for the whole class of spacetime theories containing diffeomorphic models \cite{EarmanNorton1987}. In general, such a model is given by a tupel $\langle M, O_1, ... O_n \rangle$ with a spacetime manifold $M$ and quantities $O_1, ... O_n$ denoting certain geometric objects. Thus, a model of general relativity is given by $M = \langle M, g_{\mu\nu}, T_{\mu\nu} \rangle$ with metric $g_{\mu\nu}$ and energy-momentum tensor $T_{\mu\nu}$. The authors claim a \textit{gauge theorem} which reads as follows:

\textit{“Gauge Theorem (General covariance): If $\langle M, O_1, ... O_n \rangle$ is a model of a local spacetime theory and $h$ is a diffeomorphism from $M$ to $M$, then the carried along tuple $\langle M, h^*O_1, ... h^*O_n \rangle$ is also a model of the theory.”} \cite[Earman and Norton 1987, p. 520]
Recall that any diffeomorphism \( f : M \to M \) induces a map (carry along) \( f^* : \mathcal{V}_p \to \mathcal{V}_{f(p)} \) at points \( p \in M \), which means that any tensor \( T \) given in the coordinates \( \{x^i\} \) of \( \mathcal{V}_p \) is also given as \( f^*T \) in new coordinates \( \{y^i\} \) of \( \mathcal{V}_{f(p)} \). Hence, the model \( f^*M = \langle M, f^*g_{\mu\nu}, f^*T_{\mu\nu} \rangle \) is LEIBNIZ equivalent to \( M \), i.e. \( M \) and \( f^*M \) are empirically indistinguishable. This is the first part of the argument. For the second part EARMAN und NORTON chose a special “hole diffeomorphism \( h \)” with

\[
h = id, \quad t \leq t_o, \quad \text{and} \quad h \neq id, \quad t > t_o,
\]

which, of course, obeys usual smoothness and differentiability conditions at \( t_o \). Hence, we have \( M = h^*M \) for times \( t \leq t_o \), whereas \( M \neq h^*M \) for \( t > t_o \). Since the spacetime substantivalist must claim that at \( t_o \) the world splits into two physically distinct models \( M \) and \( h^*M \) – although the theory cannot predict any empirical difference –, for him a radical inherent indeterminism arises.

It is important to note that EARMAN and NORTON for the first step of the argument consider active point transformations instead of mere passive coordinate transformations. They indicate this by the term “gauge theorem”, which they explain by quoting ROBERT WALD: “... the diffeomorphisms comprise the gauge freedom of any theory formulated in terms of tensor fields on a spacetime manifold” (WALD 1984, p. 438). Thus, by converting the notion of general covariance into an active language of point transformations, the hole argument becomes vivid.

After all, we are confronted with the following alternatives in order to escape the hole argument: either to give up spacetime substantivalism, or to accept indeterminism as a consequence of any generally covariant spacetime theory. But the latter way out, EARMAN und NORTON close, seems to be “... far too heavy a price to pay for saving a doctrine that adds nothing empirically to spacetime theories” (EARMAN and NORTON 1987, p. 524).

6 The generalized hole argument

The above manifold hole argument rules out spacetime substantivalism for orthodox spacetime theories including general relativity. As shown in section 4, in gauge field theories one naturally has to take into account the enlarged geometrical arena of the underlying fiber bundles to represent matter and gauge fields. The question arises, which kind of status is appropriate for the geometry of fiber bundles. In particular: Does a relationalistic or a substantivalistic point of view hold? In order to rule out the latter one, I argue in the following that there exists a straightforward extension of the spacetime manifold hole argument to a generalized bundle space hole argument.

One first of all should ask whether there exist reasons at all for believing in fiber bundle substantivalism. Since the spacetime hole argument already rules out manifold substantivalism, the fiber bundle substantivalist will claim the independent existence of fiber spaces as internal geometrical spaces in which matter and gauge fields ‘live’. First, this is the decisive argument for the general need of fiber bundles: Fields do not live in spacetime itself; rather, they live in state spaces defined on spacetime.\(^9\) As indicated in section 4, at each spacetime point we need two

\(^9\)This statement surely holds for quantum gauge field theories in our standard model. Whether this is even
additional spaces constituting the geometrical arena of our world: A group space constituting
the state space of gauge fields, which is provided by some principal bundle $\mathbb{P}$, and a vector
space for matter fields, provided by the associated vector bundle $\mathbb{E}$. Now, the fiber bundle
substantivalist will consider bundle, or at least fiber-space points (if he has already accepted
the spacetime hole argument) as individuated substances. Analogous to the usual spacetime
substantivalist’s emphasis on the existence of vacuum solutions in general relativity, the fiber
space substantivalist could point out the activity and effectiveness of the vacuum in quantum
gauge field theories. Indeed, Yuval Ne’eman claims, that “... the vacuum is ... the arena for
the nonlinear interaction of the gauge fields. As a result, spacetime is a physical entity – as a set
of fields – at the classical level already.” Therefore, “... physics selects the realist or substantivist
view, and contradict[s] the tenets of relationalism or conventionalism, with respect to spacetime ...
” (Ne’eman 1995). Curiously enough, Ne’eman applies his argument to spacetime alone
– obviously ignoring the spacetime hole argument –, although, if at all, his argument should a
fortiori hold for fiber spaces, too.

I shall now confront these points of view with the relationalist’s arguments. This will be
done in the same two steps as for the spacetime hole argument. However, I like to argue that
one does not necessarily need a second step. At least for the case of principal bundles the first
step of the argument is already sufficient. Thus, interestingly, one does not need a proper hole
argument in this case.

6.1 Generalized hole argument: First step

Consider the usual gauge freedom arising in any gauge theory $T$. Thus, $T$ admits gauge trans-
formations $GT_1 : T \to T'$ and $GT_2^b : T(x) \to T'(x)$ such that $T$ and $T'$ resp. $T(x)$ and $T'(x)$
are Leibniz equivalent. In other words, only gauge invariant quantities are observable. In order
to make empirical use of $T$, it is necessary to fix a gauge. This is evidently just a convent-
onal operation, such as introducing coordinates. The gauge by its own nature has no significant
physical meaning. Surely, the substantivalist will not deny this, but he will nevertheless insist
on the ontological individuality of points in spaces in which the gauge is applied.

As demonstrated, the Earman-Norton hole argument makes a decisive use of an active
interpretation of the considered transformations as point transformations instead of passive
coordinate transformations. In order to take over the argument I shall refer to $GT_2^b$ as point
transformations in the group manifold of $G$. This reflects the very nature of the fiber spaces in
the framework of a principal bundle $\mathbb{P}$: The right action of $G$ on $\mathbb{P}$ leads to the fibration of the
bundle, i.e. $G$-orbits (fibers) are equivalence classes of physically indistinguishable states. Here,
a crucial point arises: Since the fiber space is the group itself, its points have per definition
no significant physical meaning as entities per se. One can see this by recalling the idea of

true for the connection forms, i.e. the gauge fields, of general relativity, is of course a matter of a more detailed
analysis and interpretation of the theory’s underlying gauge structure. In the light of the few remarks at the end
of section 3 I like to assume this to be the case: Gauge fields of gravity – as well as their derived properties
such as curvature – live primarily in the fibers. They are constituted by actual transformations of local reference
frames. Speaking about the curvature of the manifold, though, appears from the gauge theoretic point of view as
a conventional maneuver due to the fact that fiber and base space are soldered. Surely, a detailed discussion of
this topic remains to be a further task.
thinking about Lie groups in terms of their parameter manifolds. This leads to a most natural representation: the Lie group as its own homogeneous space, thus, the group acting transitively on itself. From this point of view we are forced to primarily interpret the abstract group in terms of its algebraic rather than its analytic structure, which seems to be the natural way of an application of Lie groups within the framework of principal bundles. Therefore – since gauge fields just take values in the group – merely a relationalist’s interpretation of the group’s natural homogeneous representation space holds: no points are distinguished, and, moreover, no group space point has any physical significance whatsoever. It is indeed the key idea of gauge theories that only relations of gauge transformations within different fibers, given by the connection forms, have any empirical meaning.

Beside the structure of principal bundles \( \mathbb{P} \), gauge theories make use of their associated vector bundles \( \mathbb{E} \) as well. How to proceed with their fiber spaces? Clearly, it is one and the same abstract group \( G \) which constitutes the fibers of \( \mathbb{P} \) and acts on the vector space fibers of \( \mathbb{E} \). The matter field ‘lives’ in the latter ones and is thereby just a representation of \( G \) in some vector space. Matter fields transform according to local gauge transformations \( G^T2a \). In section I gave an interpretation of \( G^T2a \) as inherently active transformations, namely, a local change of a matter field at some point \( p \) compared to a different point \( p' \) in spacetime changes the physical situation, i.e. constitutes an interaction represented as a gauge field. Changing the physical situation can be understood as the general meaning of transformations considered to be active. Note however, that this does not necessarily refer to active point transformations (of spaces whatsoever). Actually, in stressing local gauge transformations being actively interpreted, we are by no means forced to consider them as point transformations – they rather represent active changes of general state space reference frames. The vector bundle substantivalist, however, will consider local gauge transformations \( G^T2a \) as active point transformations in the vector space fibers. The relationalist’s arguments must then prove this point of view to be untenable.

However, as we will see now, the mere representation of \( G \) in the vector space fibers of \( \mathbb{E} \) does not allow for the same argument to rule out vector bundle substantivalism as for the group manifold fibers of \( \mathbb{P} \). The gauge fixing mentioned above acts in the vector space as distinguishing a certain basis. Again, this is a pure conventional maneuver: only local changes of state space reference frames have a physical meaning. But, the substantivalist may still claim that vector space points are entities per se, since gauge fixing is related to a mere passive operation such as choosing coordinates. Here, the relationalist has to accept that representing the bundle’s structure group in a vector space is not reason enough to derive a pure relational status of such a space, simply because the group theoretic argument as in the principal bundle case above does not apply.

Thus, the first step of the generalized argument is only sufficient to rule out principal fiber bundle substantivalism, since these fiber spaces are, unlike base manifolds or any kind of vector spaces, group manifolds, i.e. spaces in which the structure group is homogeneously represented on itself. Regardless of the question about an appropriate active interpretation of local gauge transformations, I see no way to individuate the points of such kinds of spaces. With regard to vector bundle fiber spaces, our mere mathematical tools, however, do still allow a substantivalist’s viewpoint - even if this position turns out to be absurdly extreme. We can make this clear

\(^{10}\)E.g., each component \( \psi^i \) of the Dirac bispinor \( \psi \) gives a fundamental representation of \( U(1) \) in \( \mathbb{C} \).
by emphasizing an inherent fundamental conventionalism in any gauge physics: we must fix a
gauge in order to make empirical use of a gauge theory. This clearly indicates (but does not
prove) that fiber space points are physically indistinguishable and non-individuated because of
our freedom of choosing a particular gauge.

6.2 Generalized hole argument: Second step

I shall now confront ‘hard core substantivalists’, who are still not convinced, with the proper
version of the generalized hole argument. For this purpose, I will consider bundle isomorphisms
instead of base manifold diffeomorphisms. Recall the following commutative diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{\phi} & E' \\
\downarrow \pi & & \downarrow \pi' \\
M & \xrightarrow{f} & M'
\end{array}
\] (20)

If \(\phi\) is a diffeomorphism, we may call it a bundle isomorphism. Per definition, bundle iso-
morphisms preserve the fiber structure of the bundle. In particular, \(\phi : E \to E\) is a bundle
automorphism. It can be read from the diagram that any bundle isomorphism uniquely induces
a manifold diffeomorphism \(f : M \to M'\).

I shall now choose an appropriate “hole isomorphism \(\tau\)\(^{11}\). To begin with, one simply might
use a bundle isomorphism which induces the hole manifold diffeomorphism (19) – this already
would be sufficient. But one may even think about a most general hole isomorphism
\(\tau = \text{id}, \, t \leq t_o, \, \text{and} \, \tau \neq \text{id}, \, t > t_o\). (21)

In this way we are able to perfectly take over the second step of the hole argument: Since for
the fiber space substantivalist the action of \(\tau\) changes the ‘real’ arrangement of bundle space
points, i.e. the physical situation, the world splits again into different models, thus, leading to
indeterminism.

Note that this kind of indeterminism has nothing to do with the type of indeterminism arising
in quantum theories (and, thus, in quantum gauge field theories). The DIRAC-MAXWELL or,
in general, YANG-MILLS field equations, which govern the temporal development of the fields
are strict deterministic field equations. Therefore, the existence of symmetry properties of the
fields such as bundle morphisms are clearly not related to indeterminism arising in the quantum
measurement process.

Hence, it should have become clear from the above arguments that there is no possibility left
for the substantivalist to hold his position, since the proper use of bundle isomorphisms in the
generalized hole argument rules out fiber bundle substantivalism in the same manner as base
manifold diffeomorphisms rule out manifold substantivalism. Moreover, since it can be argued
that the second part of the argument is not necessary at least in the principal bundle case, the
substantivalist’s possible escape into indeterminism is even more eroded. Thus, one ends up
with a clear result: Fiber bundles refer to a relationalistic interpretation.

\(^{11}\)There is no Greek counterpart of the letter ‘h’ – the reader may guess why ‘\(\tau\)’ is chosen instead ...
7 The meaning of gauging?

Once again, the idea of an active interpretation of local gauge transformations refers to the active relational change of reference frames of the matter fields. This considerably changes the physical situation, whereas the idea of actively shifting fiber space points remains without any empirical meaning because of these points being genuinely a representation of a group. Due to the gauge principle gauge fields appear to be a consequence and, thus, a mere appendix of the matter fields. This is another way of arguing that the notion of a matter free spacetime is without any empirical meaning. Since we must regard the gauge principle as a tremendous successful heuristic principle in modern theoretical physics, the more so we should be puzzled with the unsolved philosophical question concerning the meaning of this principle. Until today it still remains a pure miracle why the postulate of local gauge transformations, i.e. replacing the transformation parameters $\alpha_s \rightarrow \alpha_s(x)$, leads to the coupling of matter and interaction fields.

It seems quite clear that hand waving arguments such as “field physics has to be local, therefore the transformations must be local”, which one finds throughout the textbook literature, are philosophically by no means satisfying. Maybe, the curious interplay between global and local considerations in the gauge approach gives us a hint for considering new ideas of spacetime – not referring to it as being primarily a differentiable manifold \(^{[\text{Lyre 1998}]}\). But these questions touch the deep conceptual roots of physics in general. At this stage, the real puzzle begins and therefore a lot of work needs to be done by physicists as well as philosophers of physics to find the true meaning of gauging.

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