The Smectic A-C Phase Transition in Biaxial Disordered Environments

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We study the smectic A-C phase transition in biaxial disordered environments, e.g. fully anisotropic aerogel. We find that both the A and C phases belong to the universality class of the "XY Bragg glass", and therefore have quasi-long-ranged translational smectic order. The phase transition itself belongs to a new universality class, which we study using an $\epsilon = 7/2 - d$ expansion. We find a stable fixed point, which implies a continuous transition, the critical exponents of which we calculate.

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I. INTRODUCTION

The effect of quenched disorder on condensed matter systems has been a widely studied topic for many years[1,2], both for practical reasons (since disorder is always present in real systems) and fundamental ones. One of the fascinating results to emerge from the study of this problem is that even arbitrarily weak disorder can drastically affect the fundamental properties of many systems. Among other effects, such weak disorder can destroy many types of long ranged order (e.g., ferromagnetic order in systems with quenched random fields[3]), and it can radically change the critical behavior of many phase transitions[3].

Such effects have been found in, e.g., superconductors[2], charge density waves[3-5], Josephson junction arrays[3], superfluid helium in aerogel[6], and ferromagnetic superconductors[7].

Some of the most novel and dramatic effects of quenched disorder are found in liquid crystals confined in random porous media[11,12]. These intriguing systems exhibit a variety of exotic "Bragg Glass" phases. They also undergo unique types of phase transitions[13], one of which, the Smectic A to Smectic C (hereafter, AC) transition[14,15], is the subject of this paper.

In the high temperature phase (A phase), the nematic director $\hat{n}$, which points along the axis of alignment of the constituent long molecules that make up the smectic material, and the normal to the smectic layers $\hat{N}$, are parallel. In the low temperature phase (C phase), $\hat{n}$ and $\hat{N}$ tilt away from each other.

Here, we wish to study how this transition changes when the smectic is confined in a quenched, random, anisotropic environment. This can be achieved experimentally by confining the smectic within aerogel. By changing the preparation of the aerogel, it should be possible to study the smectic A-C transition in different anisotropic environments.

In previous work[13], the nature of this transition was studied for a liquid crystal confined in a uniaxially stretched aerogel. In this paper we consider the transition in a biaxial disordered environment, in which the aerogel is first stretched along one axis, then compressed in an arbitrary direction perpendicular to the stretch[16].

As illustrated in Fig. 1, we refer the stretched direction as the $\hat{z}$ axis, the compressed direction as the $\hat{h}$ axis, and the direction perpendicular to both $\hat{z}$ and $\hat{h}$ as the $\hat{s}$ axis. We denote the entire plane perpendicular to $\hat{z}$ (i.e., the $s-h$ plane) as $\perp$.

We find that, in this geometry, the AC transition is a continuous transition belonging to a new universality class. Unusually, the upper critical dimension, below which the transition exhibits non-mean-field critical behavior, is $d_{UC} = 7/2$. We have studied this phase transition using an $\epsilon = \frac{7}{2} - d$ expansion, and find that there exists a stable fixed point, which implies a second-order phase transition with universal critical exponents. Furthermore, we find anisotropic scaling at the transition.

One of these universal exponents is the usual order parameter exponent $\beta$, defined via the temperature dependence of the tilt angle $\theta$ between the local molecular axis (i.e., the nematic director $\hat{n}$), and the normal to the smectic layers $\hat{N}$. As $T \rightarrow T_{AC}$, this angle, which is the magnitude of the order parameter for this transition, obeys the universal scaling law:

$$\theta \propto (T_{AC} - T)^{\beta}$$  \hspace{1cm} (1.1)

with

$$\beta = \frac{1}{2} - \frac{2}{9} \epsilon + O(\epsilon^2).$$  \hspace{1cm} (1.2)

In the high temperature phase (A phase), both the nematic directors $\hat{n}$ and the layer normals $\hat{N}$ lie, on average along $\hat{z}$. In the low temperature phase (C phase), $\hat{n}$ and $\hat{N}$ tilt simultaneously away from $\hat{z}$ in opposite directions in the $s-z$ plane. Near $T_{AC}$, the angles $\theta_N(T)$ between $\hat{N}$ and $\hat{z}$ and $\theta_n(T)$ between $\hat{n}$ and $\hat{z}$ are not independent, but rather have the same power-law temperature dependence as the total angle $\theta(T) \equiv \theta_N(T) - \theta_n(T)$ between $\hat{N}$ and $\hat{n}$. The ratio of $\theta_N(T)$ to $\theta_n(T)$ is a non-universal (negative) constant.
This geometrical constraint forbids azimuthal rotations of $\hat{n}$ and $N$ in the $C$ phase. As a result, there are no additional “Goldstone modes” associated with the breaking of azimuthal symmetry in this anisotropic situation, in contrast to the isotropic case [17, 18]. Indeed, we find that both the $A$ and $C$ phases belong to the universality class of the random field $XY$ model (RFXY) [16, 17, 18].

The correlation length of the transition is defined as the length beyond which the two point real space correlations of the order parameter $\hat{c}$ become small. Here $\hat{c}$ is the projection of $N$ along $\hat{s}$; its magnitude is therefore proportional to the tilt angle $\theta_N$. As $T \rightarrow T_{AC}$, the correlation lengths $\xi_{s,h,z}$ along $\hat{s}$, $\hat{h}$, and $\hat{z}$ become very different: they diverge as power laws with different exponents.

$$\xi_{s,h,z} \propto |T - T_{AC}|^{\nu_{s,h,z}}. \quad (I.3)$$

To first order in $\epsilon$, we find

$$\nu_s = \frac{1}{2} + \frac{7}{27} \epsilon + O(\epsilon^2), \quad (I.4)$$
$$\nu_h = \frac{1}{2} - \frac{2}{27} \epsilon + O(\epsilon^2), \quad (I.5)$$
$$\nu_z = \frac{1}{2} + \frac{10}{27} \epsilon + O(\epsilon^2). \quad (I.6)$$

The critical exponents most closely analogous to the exponent $\eta$ in conventional phase transitions [19, 20] are those associated with “anomalous elasticity” right at the critical point. Anomalous elasticity means that the elastic moduli characterizing the smectic are not, as in conventional materials, constants, but rather become dependent on the length scale $\lambda$ or wavevector $\vec{q}$ on which they are measured. More specifically, the bend modulus $K$ that determines the free energy cost of bending the smectic layers diverges as $\vec{q} \rightarrow 0$ according to scaling laws

$$K(\vec{q}) = q_s^{-\eta_s} f_K(q_h/q_s^{\zeta_h}, q_z/q_s^{\zeta_z}); \quad (I.7)$$

while the compression modulus $B$ characterizing the resistance of the smectic layers to being squeezed closer together vanishes according to

$$B(\vec{q}) = q_s^{-\eta_h} f_B(q_h/q_s^{\zeta_h}, q_z/q_s^{\zeta_z}). \quad (I.8)$$

The anisotropy exponents $\zeta_h$ and $\zeta_z$ are defined as $\zeta_h = 2 - \eta_K/2$ and $\zeta_z = 2 - (\eta_B + \eta_K)/2$. Our $\epsilon$-expansion results for $\zeta_{s,z}$ quoted below show that neither of these exponents equals one. This implies that the system exhibits anisotropic scaling. In fact, since $\zeta_h \neq \zeta_z$, there is completely anisotropic scaling between all three directions: soft ($s$), hard ($h$), and along the stretch axis ($z$).

The strength of the quenched disorder - specifically, of the random torques and random compression exerted by the aerogel on the smectic layers - is characterized by the disorder invariance $\Delta_{t,c}$ where the subscripts $t,c$ respectively denote “torque” and “compression”. This also becomes wavevector dependent right at the AC transition, with an anisotropic scaling form:

$$\Delta_{t,c}(\vec{q}) = q_s^{-\nu_{t,c}} f_{t,c}(q_h/q_s^{\zeta_{h,t}}, q_z/q_s^{\zeta_{z,t}}). \quad (I.9)$$

To leading order in $\epsilon = \frac{7}{27} - d$, the anomalous exponents $\eta_K, \eta_B$, and $\eta_t$ are given by

$$\eta_K = \frac{16}{27} \epsilon + O(\epsilon^2) \approx 0.3, \quad (I.10)$$
$$\eta_B = \frac{4}{9} \epsilon + O(\epsilon^2) \approx 0.22, \quad (I.11)$$
$$\eta_t = \frac{4}{27} \epsilon + O(\epsilon^2) \approx 0.07, \quad (I.12)$$
$$\eta_c = 2 - \frac{8}{9} \epsilon + O(\epsilon^2) \approx 1.55. \quad (I.13)$$

The anisotropy exponents are:

$$\zeta_z = 2 - (\eta_B + \eta_K)/2 = 2 - \frac{14}{27} \epsilon + O(\epsilon^2) \approx 1.74, \quad (I.14)$$
$$\zeta_h = 2 - \eta_K/2 = 2 - \frac{8}{27} \epsilon + O(\epsilon^2) \approx 1.85. \quad (I.15)$$

Finally, we find that the specific heat exponent $\alpha$ is given by

$$\alpha = \frac{2}{9} \epsilon + O(\epsilon^2). \quad (I.16)$$

Since $\epsilon = 1/2$ is quite small in the physical dimension $d = 3$, we expect that our predictions for $\alpha$, $\beta$, and $\eta_{B,K,t}$ are quantitatively accurate.

In addition, there are several exact scaling relations between these critical exponents:

$$\beta = \nu_s/4 [4d - 10 + (5 - d)\eta_K - \eta_H - 2\eta_t] \quad (I.17)$$
$$\alpha = 2 - \nu_s/2 [4d - 6 + (3 - d)\eta_K - 2\eta_t - \eta_B], \quad (I.18)$$
$$\eta_c = 2 + \eta_t - \eta_K - \eta_B, \quad (I.19)$$
$$\nu_z = \zeta_z \nu_s, \quad (I.20)$$
$$\nu_h = \zeta_h \nu_s. \quad (I.21)$$

As in the AC transition in uniaxially disordered environments [14], $\alpha$ does not obey the usual anisotropic hyperscaling relation $\alpha = 2 - \nu_s - (d - 2)\nu_h - \nu_z$: this is due to the strongly relevant disorder.

The behavior of the correlation functions near the AC transition is more complicated than in conventional phase transitions; in particular, it cannot be summarized by a single correlation length exponent $\nu$. This is in part due to the fact, mentioned above, that the transition exhibits anisotropic scaling. It is also due to the fact that the correlations within the $A$ and $C$ phases themselves are quite non-trivial.

Since both the $A$ and $C$ phases belong to the RFXY universality class, we can use the results of reference [17] to obtain the x-ray scattering intensity (which is a probe of the smectic translational order) far away from the critical point. The most surprising feature of this is that
neither phase shows true long-ranged translational order; rather, they exhibit quasi-long-ranged translational order, but with a universal exponent. The experimental signature of this quasi-long-ranged order is a power-law, rather than delta function, singularity of scattering as the Bragg peak is approached:

\[ I_n (\delta \vec{q}) \propto [(\delta q_h)^2 + \alpha_h (\delta q_h)^2 + \alpha_z (\delta q_z)^2]^{-\lambda + \frac{\nu z^2}{2}} \]

where \( n \) is the index of the Bragg peak, and \( \delta \vec{q} = \vec{q} - \vec{G}_n \) with \( \vec{G}_n \) the reciprocal lattice vector. In the A phase \( \vec{G}_n = \frac{2\pi}{a} \sigma \hat{z} \) with \( a \) the layer spacing constant. In the C phase \( \vec{G}_n = \frac{2\pi}{a} (\cos \theta_N \hat{\sigma} \pm \sin \theta_N \hat{s}) \), where “\( \pm \)” reflects that the tilting of the layer normals breaks two-fold symmetry, and “\( + \)” (“\( - \)”) is taken if the tilting is along \( \hat{s} \) (\( -\hat{s} \)). \( \alpha_{h,z} \) are non-universal constants of order 1. The exponent \( -\lambda + \frac{\nu z^2}{2} \) implies that \( I_n (\delta \vec{q}) \) diverges as a power law only for the first two Bragg peaks.

In contrast, right at the AC critical point, the system only has short-ranged translational order; as a result, the Bragg peaks are broad.

The X-ray scattering pattern near, but not at, the critical point interpolates between these two limits, as illustrated in Fig. 2. Sufficiently far away from the center of the Bragg peak, the line shape of \( I (\delta \vec{q}) \) versus \( \delta \vec{q} \) is qualitatively Lorenzian squared. Close to the center of the Bragg peak, the line shape diverges as the power law given by Eq. (I.22). The crossover between these two distinct patterns - or, equivalently, the width of the spike on the top of the line shape, is isotropic in scaling and strongly temperature dependent,

\[ \delta \vec{q}^c \propto \exp (-A|t|^{-\Omega}) \]

where \( A \) is non-universal and of order 1, and \( t \) is the reduced temperature \( t = \frac{T - T_{AC}}{T_{AC}} \). \( \Omega \) is universal, and we can bound it above and below:

\[ .25 < \Omega < .65 \]

The width of the quasi sharp spike decreases extremely rapidly as \( T \to T_{AC} \), and vanishes completely right at \( T = T_{AC} \). Defining a correlation length \( \xi \) for the AC transition via \( \xi \equiv q_c^{-1} \), we see that \( \xi \) diverges like a stretched exponential as the AC transition is approached; hence, the usual correlation length exponent \( \nu = \infty \), in this sense.

Another detailed experimental probe of our predictions is light scattering. Consider first an experiment in which the magnitude \( q \) of the wavenumber \( \vec{q} \) is varied for fixed, generic direction \( \hat{q} \) of the scattering wavevector \( \vec{q} \). By “generic direction \( \hat{q} \)”, we mean a direction of \( \hat{q} \) for which no component of \( \hat{q} \) is \( \ll 1 \). We find that in such an experiment, in both the A and C phases, the light scattering intensity \( I(q) \) plotted as a function of \( q \) shows three distinct scaling regimes:

\[ I(q) \propto \begin{cases} q^{-\frac{\nu c}{2}}, & q \gg \xi_c^{-1}, \\ \text{constant}, & q^F (T) \ll q \ll \xi_c^{-1} \\ q^{-1}, & q \ll q^F (T) \end{cases} \]

where \( q^F (T) \) vanishes as \( T \to T_{AC} \) according

\[ q^F (\hat{q}) \propto |T - T_{AC}|^{\phi} \]

where the universal exponent \( \phi \) is given by

\[ \phi = \nu_s \left( 1 + \eta_c + \frac{\eta_B - \eta_K}{2} \right) = \frac{3}{2} + \frac{8}{27} \epsilon + O(\epsilon^2) \]

The first equality in (I.27) is exact, while the second follows from the known \( \epsilon \)-expansions (I.10), (I.11), and (I.13), for \( \eta_K, \eta_B, \) and \( \eta_c \).

This prediction for \( I(q) \) as a function of \( q \) for generic directions \( \hat{q} \) of \( \vec{q} \) is plotted in Fig. 3.

More information can be obtained by restricting the scattering to the subspace of \( \vec{q} \) with \( q_h = 0 \). In this case, we find that the light scattering exhibits different behaviors in the six different regions of the \( q_h - q_z \), \( q_h = 0 \) plane illustrated in Fig. 4.

In this figure, the various boundaries between regions...
are given by:

\[ FG : \quad q_z \xi_s^N = (q_{\perp} \xi_s^N)^{\xi_z}, \quad (I.28) \]
\[ EF : \quad q_z = \xi_s^{-1}, \quad (I.29) \]
\[ DF : \quad q_s = \xi_s^{-1}, \quad (I.30) \]
\[ OF : \quad q_z = \frac{\xi_s}{\xi_z} q_s, \quad (I.31) \]
\[ BC : \quad q_z = (\xi_s^N)^{-1}, \quad (I.32) \]
\[ AC : \quad q_s = (\xi_s^N)^{-1}. \quad (I.33) \]

and the light scattering in each region obeys:

\[
I(\vec{q}) \propto \begin{cases}
 q_z^2 \left( q_z \xi_s^N \right)^{2\eta_K-\eta_t-6} \xi_z, & \text{region 1} \\
 q_s^{2\eta_K-\eta_t-4} \left( q_{\perp} \xi_s^N \right)^{2\eta_K-\eta_t-6}, & \text{region 2} \\
 q_s^2 \left( \frac{q_z}{\xi_z^N} \right)^{2\eta_K-\eta_t-6} (q_z \xi_z)^{-2}, & \text{region 3} \\
 \left( q_z \right)^{2\eta_K-\eta_t-6} \left( \xi_s^{-3} \right), & \text{region 4} \\
 q_s^2 \left( \frac{\xi_s^N}{\xi_z} \right)^{1-\frac{2\eta_B}{\eta_t}} (q_z \xi_z)^{-3}, & \text{region 5} \\
 \left( q_z \right)^{2\eta_K-\eta_t-6} \left( \xi_s^{-3} \right) q_s^{-1}, & \text{region 6}
\end{cases}
\quad (I.34)
\]

The two new characteristic lengths \( \xi_z^r \) and \( \xi_s^r \) are given by:

\[
\xi_z^r = \xi_s \left( \frac{\xi_s}{\xi_s^N} \right)^{\Gamma_s}, \quad (I.35)
\]
\[
\xi_s^r = \xi_s \left( \frac{\xi_s}{\xi_s^N} \right)^{\Gamma_s}, \quad (I.36)
\]
with
\[ \Gamma_s = 2 - \eta_K + \eta_h + \frac{1}{2} \eta_B, \]  
\[ \Gamma_z = 3 - \frac{3}{2} \eta_K + \eta_h. \]

We also find that the transition temperature \( T_{AC} \) depends on the anisotropy of the disorder. Compared to the bulk transition temperature it is lowered by the stretch of the aerogel but raised by the compression. Hence, there is no a priori way to predict whether the net effect of the anisotropy is to raise or lower the AC transition temperature.

In this problem there are two ways continuing the model from 3 dimension to higher dimensions. The \( \epsilon = 7/2 - d \) expansion uses the hard analytical continuation, in which we take the \( h \)-space to be \( d - 2 \) dimensional, while both the \( z \)- and \( s \)-spaces are kept one dimensional. It is complemented by the \( \hat{\epsilon} = 4 - d \) expansion, which uses the soft analytical continuation, where both \( h \)- and \( z \)-space are one dimensional, and \( s \)-space is \( d - 2 \) dimensional. The \( \hat{\epsilon} \) expansion also leads to a stable non-Gaussian fixed point, which again implies a second-order phase transition. However, the numerical estimate of the exponents are not in good agreement with the \( \epsilon \) expansion in \( d = 3 \). The technical details are given in Appendix A.

The remainder of this paper is organized as follows. In Sec. II we derive our model for the AC transition in biaxial disordered environments. In Sec. III we derive the renormalization-group recursion relations for the transition. In Secs. IV and V we calculate the thermodynamic critical exponents, and wavevector dependences of the elastic constants and disorder variances, respectively. In Sec. VIII we determine the experimentally observable correlation functions and make predictions for X-ray and visible light scattering. In Sec. VIII we show that the transition is stable against dislocations in the smectic layers.

II. MODEL

In keeping with earlier work on the AC transition in clean and uniaxially disordered systems, we assume that the local layer displacement \( u(r) \) and the local nematic director \( \hat{n}(r) \) are the only important fluctuating quantities in this problem. The complete Hamiltonian, like Gauß, can be divided into three parts. The first part is obtained by a slight modification of the Hamiltonian for the smectic A phase in isotropic disordered environments. It reads
\[
H_1 = \int d^3 r \left[ \frac{K}{2} (\nabla \cdot u)^2 + \frac{B}{2} (\nabla \times u)^2 - \frac{g}{2} (\nabla \times u) |\nabla \cdot u|^2 \right]
+ \frac{w}{8} \left| \nabla \cdot u \right|^4 + \tilde{h}(r) \cdot \nabla \cdot u + V_u(u - \phi(r)).
\]  

Since in our problem the liquid crystal is in an anisotropic environment, the Hamiltonian Eq. (II.1) need not be invariant under rigid rotations of the liquid crystal. Therefore, the values of the coefficients \( B \), \( g \) and \( w \) are not required to satisfy the constraint \( B = g = w \), which they do in the isotropic case. The \( \perp \) and \( z \) components of the random field \( \tilde{h}(r) \) in Eq. (II.1) respectively incorporate the random torques and random compressions exerted on the smectic by the aerogel. We take \( \tilde{h}(r) \) to be Gaussian, zero mean, and characterized by anisotropic short-ranged correlations:
\[ \tilde{h}_i(r) \tilde{h}_j(r') = \left( \Delta_i \delta_{ij} + \Delta_h \delta_{ij}^h + \Delta_e \delta_{ij}^e \right) \delta^d(r - r'), \]  
where \( \delta_{ij}^h = 1 \) for \( i = j \), and both \( i \) and \( j \) taking values in one of the \( d - 2 \) hard directions, and zero otherwise; \( \delta_{ij}^e = 1 \) for \( i = j = s \), and zero otherwise; and \( \delta_{ij}^h = 1 \) for \( i = j = z \), and zero otherwise. The function \( V(u - \phi(r)) \) is a periodic function with period \( a \), the smectic layer spacing. The field \( \phi(r) \) is a random length uniformly distributed between 0 and \( a \) with short-ranged correlations. Thus, \( V(u - \phi(r)) \) represents the random pinning of the smectic layers.

The second part of our Hamiltonian describes the coupling between layer normals \( \hat{N} \) and directors \( \hat{n} \). The simplest such coupling is given by
\[ H_2 = \int d^3 r \left| \hat{N} - \hat{n} \right|^2. \]  

This part of the Hamiltonian is responsible for the AC transition. To see this, simply note that when \( P > 0 \), the lowest energy state has \( \hat{N} \parallel \hat{n} \); this corresponds to the A phase. In contrast, when \( P < 0 \), the energy can be lowered by making \( \hat{N} \) and \( \hat{n} \) point in different directions; this corresponds to the C phase. Thus, the point \( P = 0 \) corresponds, in mean field theory, and in the absence of other couplings between \( \hat{N} \) and \( \hat{n} \), to the critical point for the pure smectic AC transition.

Since the layer normal \( \hat{N} \) is related to the smectic layer displacement \( u \) by the geometric relation
\[ \hat{N} = \frac{\hat{z} - \nabla \cdot u}{|\hat{z} - \nabla \cdot u|}, \]  
we can rewrite Eq. (II.3) as a coupling between \( u \) and \( \hat{n} \); to lowest non-trivial order in \( u \), this reads:
\[ H_2 = \int d^3 r \left| \nabla \cdot u + \delta \hat{n} \right|^2. \]  

Since the aerogel is stretched and compressed, the disorder is no longer isotropic. Instead, on average it picks
out a preferred orientation for both the layer normals and the directors. Therefore, in general the Hamiltonian should also include
\[
H_3 = \int d^3r \left[ M|\vec{\nabla}_\perp u|^2 + Q|\delta n_\perp|^2 + (M' - M)(\partial_u u)^2 + (Q' - Q)(\delta n)^2 \right],
\]
(II.6)
which is not rotation invariant and hence forbidden in the isotropic case. It is expected that the stretching of the aerogel tends to hold both \(N\) and \(\hat{n}\) along \(\hat{z}\), while the compression tends to hold both \(\hat{N}\) and \(\hat{n}\) within the \(zs\) plane. Therefore, the coefficients in Eq. (II.6) obey the bounds
\[
M' > M > 0 \quad , \quad Q' > Q > 0 \quad .
\]
(II.7)
The values of \(M\) and \(Q\) reflect how much the aerogel is stretched; those of \(M' - M\) and \(Q' - Q\) likewise reflect the degree of compression.

After a linear change of variables
\[
\delta n'_s = \delta n_s + R\partial_u u \quad , \\
\delta n'_h = \delta n_h + R'\partial_u u \quad ,
\]
(II.8)
the sum of \(H_2\) and \(H_3\) becomes
\[
H_s = \int d^3r \left[ \frac{D(T)}{2}(\partial_u u)^2 + (P + Q)(\delta n'_s)^2 \right.
\]
\[+ \frac{\gamma}{2}(\partial_u u)^2 + (P + Q')(\delta n'_h)^2 \right],
\]
(II.9)
where \(R \equiv P/(Q + P), \quad R' \equiv P/(Q' + P), \quad D(T) \equiv 2M + 2QP/(Q + P), \quad \gamma \equiv 2M' + 2QP/(Q' + P). \) It is easy to verify that under the constraints (II.7) \(D(T) < \gamma < P + Q < P + Q'\). Therefore, if we start in the high temperature (A) phase and lower the temperature, \(D(T)\) is the first among the coefficients in Eq. (II.9) to become negative. As we will see in a moment, the critical point (ignoring fluctuations) is given by \(D(T_{AC}) = 0\), which leads to \(P(T_{AC}) = -MQ/(Q + M)\). Therefore, stretching the aerogel lowers the critical temperature. Stretching aerogel suppresses the transition. Since \(P_{AC}\) has no dependence on \(M'\) and \(Q'\), \(T_{AC}\) is not affected by the compression, in this mean field approximation in which fluctuations are neglected.

Now we look for the ground state of the system, averaged over many realizations of the disorder. Since the disorder pieces in Eq. (II.1) are completely random and isotropic, they have no averaged contribution to the ground state. Therefore, they can be neglected. Minimizing the sum of the Hamiltonians Eq. (II.1) and Eq. (II.9) we find two distinct states separated by \(D(T) = 0\). For \(D(T) > 0\) (i.e. \(T > T_{AC}\)), the ground state is given by \(|\vec{\nabla}_\perp u| = |\delta n_\perp| = 0\), which implies that both \(N\) and \(\hat{n}\) point along \(\hat{z}\). Thus the system is in the \(A\) phase.

For \(D(T) < 0\) (i.e. \(T < T_{AC}\)), the ground state shifts to \(\partial_u u = \pm\sqrt{-2D(T)/(w - g^2/B)}, \quad \delta n_s = -R\partial_u u, \quad \delta n_h = 0\). This implies that both \(N\) and \(\hat{n}\) tilt away from \(\hat{z}\). However, the tilting is constrained since \(\hat{N}\) and \(\hat{n}\) are restricted within \(zs\) plane with \(\hat{z}\) in between the two (this constraint arises from the fact that the last two terms in (II.9) will raise the energy of any state in which \(\hat{N}\) and \(\hat{n}\) do not lie in this plane above that of an otherwise identical state in which they do). In this case the system is in the \(C\) phase. Near the critical point, the magnitudes of the two tilting angles \(\theta_n^s\) and \(\theta_n^h\) are very small such that approximately, \(\theta_n^s(T) = |\partial_u u|\) and \(\theta_n^h(T) = |\delta n_s|\). Therefore, they obey the relation
\[
\theta_n^s(T) = \frac{P}{P + Q}.
\]
(II.10)

Although Eq. (II.10) is derived through mean field theory, since we are simply minimizing the free energy while ignoring fluctuation effects, it still holds near the critical point. This is because the fluctuations of \(\delta n'_\perp\) are always massive. For the same reason we can set \(\delta n'_\perp = 0\) in (II.9) and use the sum of (II.1) and (II.9) as our simplified model:
\[
H = \int d^3r \left[ \frac{K}{2}(\nabla^2 u)^2 + \frac{B}{2}(\partial_u u)^2 + \frac{D}{2}(\partial_u u)^2 \right.
\]
\[- \frac{g}{2}(\partial_u u)(\partial_u u)^2 + \frac{\gamma}{2}(\partial_u u)^2 + \frac{w}{8}(\partial_u u)^4
\]
\[+ \left( \vec{h} (\vec{r}) \cdot \vec{\nabla} u + V_p(u - \phi(\vec{r})) \right) \right],
\]
(II.11)
where the anharmonic terms involving \(\partial_u u\) are not included since they are irrelevant compared to \(\gamma(\partial_u u)^2/2\).

To cope with the quenched disorder we employ the replica trick\(^2\). We assume that the free energy of the system for a specific realization of the disorder is the same as the one averaged over many realizations. To calculate the averaged free energy \(\mathcal{F} = \ln \mathcal{Z}\), where \(\mathcal{Z}\) is the partition function, we use the mathematical identity \(\ln Z = \lim_{n \to 0} \mathcal{Z}^n\). When calculating \(\mathcal{Z}^n\), one can first compute the average over the two random fields \(\vec{h}(\vec{r})\) and \(\phi(\vec{r})\), whose statistics have been given earlier. Implementing this procedure gives an replicated Hamiltonian with the effect of the random fields transformed into couplings between replicated fields:
\[
H[\alpha_\sigma] = \frac{1}{2} \int d^3r \left( \sum_{\alpha_\sigma=1}^{n} \left[ \frac{K}{2}(\partial^2_{\alpha_\sigma} u_{\alpha_\sigma})^2 + B(\partial_u u_{\alpha_\sigma})^2 \right.ight.
\]
\[- g(\partial_u u_{\alpha_\sigma})(\partial_u u_{\beta_\sigma})^2 + \frac{w}{4}(\partial_u u_{\alpha_\sigma})^4 + D(\partial_u u_{\alpha_\sigma})^2
\]
\[+ \gamma(\partial_u u_{\alpha_\sigma})^2 \right] - \sum_{\alpha_\sigma, \beta_\sigma=1}^{n} \frac{\Delta p_\perp}{T}(\vec{\nabla}_\perp u_{\alpha_\sigma} \cdot \vec{\nabla}_\perp u_{\beta_\sigma} + \vec{n}_{\perp}(\partial_u u_{\alpha_\sigma})(\partial_u u_{\beta_\sigma}) + \frac{1}{T}\Delta p(u_{\alpha_\sigma} - u_{\beta_\sigma}) \right). \]
(II.12)

We will now analyze this Hamiltonian using the renormalization group.
III. RENORMALIZATION GROUP

Right at the critical point, since the tilt term \((\partial_z u_\alpha)^2\) vanishes, the Hamiltonian Eq. (II.12) becomes very similar to that for a smectic A liquid crystal confined in compressed aerogel; such a system has been described\cite{17} as an “m = 1 Bragg Glass”. For the latter Ref.\cite{17} showed that under the renormalization group (RG), the random pinning \(\Delta_p\) is irrelevant, in the RG sense, in spatial dimensions \(d < \frac{4}{3}\). Repeating the (virtually identical) calculation we find that within the critical region, \(\Delta_p\) is also irrelevant in our problem. Furthermore, it is easy to show that the random compression \(\Delta_c\) is also subdominant to \(\Delta_t\). Therefore, to construct the RG for the transition we can use the following truncated Hamiltonian:

\[
H[u_\alpha] = \frac{1}{2} \int d^d r \sum_{\alpha=1}^{n} \left[ K (\partial_x u_\alpha)^2 + B (\partial_z u_\alpha)^2 \right. \\
- g(\partial_z u_\alpha)(\partial_z u_\alpha)^2 + w \left. (\partial_x u_\alpha)^4 \right] + D(\partial_x u_\alpha)^2 \\
+ \gamma |\nabla_h u_\alpha|^2 - \sum_{\alpha,\beta=1}^{n} \frac{\Delta_t}{T} \nabla_{\perp} u_\alpha \cdot \nabla_{\perp} u_\beta \right),
\]

(III.1)

where we have adopted the hard analytical continuation with a single stretch (z) axis, a single soft (s) axis, and \(d - 2\) hard (h) axes.

First we calculate the harmonic propagator. In fourier space the quadratic part of the Hamiltonian can be written as

\[
H = \frac{1}{2} \sum_q A_{\alpha\beta}(q) u_\alpha(q) u_\beta(-q),
\]

(III.2)

where

\[
A_{\alpha\beta} = (Bq_s^2 + \gamma q_h^2 + Dq_s^2 + Kq_s^4) \delta_{\alpha\beta} + \frac{\Delta_t}{T} q_s^2,
\]

(III.3)

and

\[
u_\alpha(q) = \frac{1}{\sqrt{V}} \int d^d r \ u_\alpha(\vec{r}) e^{-\vec{q} \cdot \vec{r}}.
\]

(III.4)

The correlation function of this model can be shown to be

\[
\langle u_\alpha(q) u_\beta(q') \rangle = T A^{-1}_{\alpha\beta}(q) \delta_{q,-q'}
\]

(III.5)

where \(A^{-1}_{\alpha\beta}\) is the matrix inverse of \(A_{\alpha\beta}\). For any \(n \times n\) matrix \(M_{\alpha\beta}\) of the form

\[
M_{\alpha\beta} = a \delta_{\alpha\beta} + b,
\]

(III.6)

it is straightforward to verify by explicit matrix multiplication that

\[
M^{-1}_{\alpha\beta} = \frac{1}{a} \delta_{\alpha\beta} + \frac{b}{a(a + bn)}.
\]

(III.7)

Since \(A_{\alpha\beta}\) is if the form \(\text{[III.1]}\), with \(a = Bq_s^2 + \gamma q_h^2 + Dq_s^2 + Kq_s^4\) and \(b = \frac{\Delta_t}{T} q_s^2\), we obtain, from \(\text{[III.5]}\), upon taking the limit \(n \to 0\):

\[
\langle u_\alpha(q) u_\beta(-q) \rangle = TG(q) \delta_{\alpha\beta} + \Delta q_s^2 G(q)^2
\]

(III.8)

with

\[
G(q) = \frac{1}{Bq_s^2 + \gamma q_h^2 + Dq_s^2 + Kq_s^4}.
\]

(III.9)

The momentum shell RG procedure consists of tracing over the short wavelength Fourier modes of \(u_\alpha(\vec{r})\) followed by a rescaling of the length. As usual in the momentum shell RG procedure, we initially restrict wavevectors to lie in a bounded Brillouin zone; here, we will take our Brillouin zone to be the infinite slab \(-\Lambda < q_s < \Lambda, -\infty < |q_h| < \infty, -\infty < q_z < \infty\), where \(\Lambda \sim 1/a\) is an ultra-violet cutoff, and \(a\) is the smectic layer spacing. The displacement field \(u_\alpha(\vec{r})\) is separated into high and low wave vector parts \(u_\alpha(\vec{r}) = u_\alpha^h(\vec{r}) + u_\alpha^l(\vec{r})\), where \(u_\alpha^l(\vec{r})\) has support in the wave vector space \(\Delta \sim dt < q_s < \Lambda, -\infty < q_h < \infty, -\infty < q_z < \infty\). We first integrate out \(u_\alpha^l(\vec{r})\). This integration is done perturbatively in the anharmonic couplings in \(\text{[III.1]}\) as usual, this perturbation theory can be represented by Feynman graphs, with the order of perturbation theory reflected by the number of loops in the graphs we consider. After this perturbative step, we anisotropically rescale lengths, with \(r_s = r_s' e^\epsilon, r_h = r_h' e^{\omega_\epsilon}, r_z = r_z' e^{\omega_\epsilon}\), so as to restore the UV cutoff back to \(\Lambda\). This is then followed by rescaling the long wave length part of the field with \(u_\alpha^h(\vec{r}) = u_\alpha(\vec{r})' e^{\epsilon}\). After this procedure we obtain the following RG flow equations to one-loop order, keeping only lowest order terms in \(D\), since we are interested in the critical point where \(D\), to leading order in \(\epsilon\), vanishes:

\[
\frac{d\gamma}{dt} = [(d - 4)\omega_h + \omega_z + 2\chi + 1] \gamma,
\]

(III.10)

\[
\frac{dB}{dt} = [(d - 2)\omega_h - \omega_z + 2\chi + 1 - \frac{3}{32 \sqrt{2}} g_1] B,
\]

(III.11)

\[
\frac{dK}{dt} = [(d - 2)\omega_h + \omega_z + 2\chi - 3 + \frac{1}{8 \sqrt{2}} g_1] K,
\]

(III.12)

\[
\frac{dg_1}{dt} = [(d - 2)\omega_h + 3\chi - 1 + \frac{3}{16 \sqrt{2}} g_1 - \frac{9}{32 \sqrt{2}} g_2] g_1,
\]

(III.13)

\[
\frac{dw}{dt} = [(d - 2)\omega_h + \omega_z + 4\chi - 3 - \frac{3}{32 \sqrt{2}} g_1^2] w
\]

\[
+ \left[ \frac{9}{8 \sqrt{2}} g_1 - \frac{27}{32 \sqrt{2}} g_2 \right],
\]

(III.14)

\[
\frac{d\Delta_t}{dt} = [(d - 2)\omega_h + \omega_z + 2\chi - 1 + \frac{g_1}{32 \sqrt{2}}] \Delta_t,
\]

(III.15)

\[
\frac{dD}{dt} = [(d - 2)\omega_h + \omega_z + 2\chi - 1 + \frac{9}{32 \sqrt{2}} (g_1 - g_2)] D
\]

\[
+ \frac{3}{8 \sqrt{2}} K(g_2 - g_1).
\]

(III.16)
\( g_1 \) and \( g_2 \) are two dimensionless couplings defined by
\[
g_1 = C_{d-1}(g/B)^2\Delta_t(B\gamma^{2-d}K^{d-7})^{1/2}\Lambda^{2d-7}, \quad (\text{III.17})
\]
\[
g_2 = C_{d-1}(w/B)\Delta_t(B\gamma^{2-d}K^{d-7})^{1/2}\Lambda^{2d-7}, \quad (\text{III.18})
\]
where \( C_d = S_d/(2\pi)^d \) with \( S_d \) the surface area of a \( d \)-dimensional sphere of radius one. The RG recursion relations \( (\text{III.11, III.12, III.13, III.15, III.16}) \) imply that \( g_1 \) and \( g_2 \) themselves flow according to the recursion relations
\[
\frac{dg_1}{dl} = [2\epsilon + \frac{21}{64\sqrt{2}}g_1 - \frac{9}{16\sqrt{2}}g_2] g_1, \quad (\text{III.19})
\]
\[
\frac{dg_2}{dl} = [2\epsilon + \frac{63}{64\sqrt{2}}g_1 - \frac{27}{32\sqrt{2}}g_2 - \frac{3}{8\sqrt{2}}g_2^2] g_2, \quad (\text{III.20})
\]
where \( \epsilon = 7/2 - d \). These two flow equations are independent of the rescaling factors \( \omega_{h,z} \) and \( \chi \) since \( g_{1,2} \) are dimensionless.

To discuss the critical behavior it is convenient, but not necessary, to make a special choice of \( \omega_{h,z} \) such that \( \gamma, B, \) and \( K \) are fixed at their bare values. This choice is given by
\[
\omega_h = 2 - \frac{1}{16\sqrt{2}}g_1, \quad (\text{III.21})
\]
\[
\omega_z = 2 - \frac{7}{64\sqrt{2}}g_1, \quad (\text{III.22})
\]
\[
\chi = \frac{5 - 2d}{2} + \frac{4d - 9}{128\sqrt{2}}g_1. \quad (\text{III.23})
\]

After this choice Eq. (III.10) becomes
\[
\frac{dD}{dl} = \left[ 2 + \frac{5g_1}{32\sqrt{2}} - \frac{9g_2}{32\sqrt{2}} \right] D + \frac{3}{8\sqrt{2}}K(g_2 - g_1). \quad (\text{III.24})
\]

For \( d < 7/2 \), there are three fixed points to Eqs. (III.19 \( \text{III.20, III.24}) \). All three lie on the “critical surface” \( (\text{III.19, III.20, III.24}) \), which separates flows that asymptotically run to large, positive \( D \) from those that run to large, negative \( D \). More physically, this surface is the phase boundary between the \( A \) and the \( C \) phase. The RG flow on this critical surface, projected onto the \( g_{1,2} \) parameter space, is illustrated in Fig. [7].

This flow is topologically identical to that for the AC transition in a clean system in an external field in spatial dimensions \( d = 3 - \epsilon \), for \( \epsilon \ll 1 \) \( (\text{III.14}) \). This is scarcely surprising, since our problem is simply a disordered version of that problem.

The Gaussian fixed point \( G \), given by \( g^*_1 = 0, g^*_2 = 0, D^* = 0 \), is trivial, and unstable. Hence, it does not control the AC transition.

The uniaxial fixed point \( U \), given by \( g^*_1 = g^*_2 = \frac{128\sqrt{2}}{15} \epsilon + O(\epsilon^2), D^* = 0 \), is stable only within the parameter subspace \( g_1 = g_2, D = 0 \). This fixed point was first found in the study of smectics in uniaxially compressed aerogel \( (\text{III.17}) \), and controls the phase reference \( (\text{III.17}) \) calls the “m = 1 Bragg Glass”. In that system, the constraint \( g_1 = g_2, D = 0 \) is enforced by the symmetry that the Hamiltonian is invariant under any rigid rotation of the liquid crystal about the compressed direction. This fixed point controls the power law anomalous elasticity of the \( m = 1 \) BG.

In our problem, since the rotation invariance is destroyed by the stretching of the aerogel, the conditions \( g_1 = g_2 \) and \( D = 0 \) are not generally obeyed. Thus, we expect that the critical behavior is controlled by a different fixed point with \( g^*_1 \neq g^*_2 \), which we call the biaxial fixed point, and denote as \( B \). In fact, the fixed point \( B \) must satisfy \( g^*_1 < g^*_2 \), as required by the stability of the model Eq. (III.11). To see this we rewrite the model as
\[
H = \int d^3r \left[ \frac{K}{2}(\nabla^2 u)^2 + \frac{B}{2}(\partial_\tau u - \left( \frac{g}{2B} \right)(\partial_r u)^2 \right] + \frac{D}{2}(\partial_\tau u)^2 + \frac{\gamma}{2}(\partial_\tau h)^2 + \frac{1}{8}(w - g^2/B)(\partial_r u)^4
\]
\[
+ \hat{h}(\vec{r}) \cdot \nabla u + V_p(u - \phi(\vec{r})). \quad (\text{III.25})
\]

Clearly, for the Hamiltonian to have a well defined minimum at \( u = 0 \) the coefficient of \( (\partial_r u)^4 \) must be positive, which implies \( g^*_1 < g^*_2 \). This stability condition should be preserved under the RG and, indeed, satisfied by fixed point \( B \). To the first order in \( \epsilon \), the fixed point \( B \) is given by
\[
g^*_1 = \frac{128\sqrt{2}}{27} \epsilon + O(\epsilon^2), \quad (\text{III.26})
\]
\[
g^*_2 = \frac{512\sqrt{2}}{81} \epsilon + O(\epsilon^2), \quad (\text{III.27})
\]
\[
D^* = -\frac{8K}{27} \epsilon + O(\epsilon^2) \quad (\text{III.28})
\]

It is easy to show that this fixed point is stable in two directions (to leading order in \( \epsilon \), the \( g_{1,2} \) plane), and unstable in a single direction (to leading order in \( \epsilon \), the \( D \) direction). This stability/instability structure is precisely that associated in the standard theory of critical phenomenon \( (\text{III.19}) \) with a fixed point controlling a phase transition. Therefore, the fixed point \( B \) controls the critical behavior of the AC transition.

IV. ANOMALOUS ELASTICITY WITHIN THE CRITICAL REGION

It has long been known that both pure \( (\text{III.14}) \) and disordered \( (\text{III.12}) \) smectics exhibit anomalous elasticity, which is a novel phenomenon found, thus far, only in liquid crystals, fluctuating membranes, and at phase transitions \( (\text{III.19, III.20}) \). In smectics, the anomalous elasticity is characterized by wavevector dependent compression modulus \( B \) and bend modulus \( K \) at large length scales.
The origin of this exotic behavior is the non-linear coupling of fluctuations. For a smectic in an isotropic environment (either clean or disordered), rotation invariance symmetry, which is spontaneously broken, prohibits certain terms (e.g., $\nabla \cdot \mathbf{u}(\mathbf{r})$) in the elastic model, and, hence, the system is soft in the $\perp$ directions. This softness leads to big fluctuations. As a result, the anharmonic terms in the model become nontrivial and the conventional harmonic theory breaks down. In the presence of quenched disorder, since the fluctuations are bigger, the anharmonic effect becomes even more important. Thus, in contrast to the marginally weak anomalous elasticity in pure smectics with logarithmic wavevector dependence, disordered smectics show a strong one with power law wavevector dependence.

In our problem, since rotation invariance is broken, in general all $\perp$ directions are hard. However, near the critical point (i.e., at length scales shorter than the correlation lengths), when $(\partial_\perp u_\perp)^2$ is subdominant to $(\partial^2 u_\perp)^2$ and hence negligible, the softness in the $s$ direction is recovered. Indeed, in this case the elastic theory of our system becomes very similar to that for the $m = 1$ BG phase of a smectic in uniaxially compressed aerogel[17]. Therefore, we expect similar anomalous behavior. However, the anomalous exponents will be different, since the system flows under the RG to a different stable fixed point, as illustrated in Fig. 5.

To get an estimate of how important the anharmonic effect is, we calculate the graphical corrections to the compression modulus $B$ using standard perturbation theory. The one-loop graphical corrections to $B$ are given by

$$
\delta B = -\frac{g^2}{2} \int d^d p \left[T G^2(\mathbf{p}) + 2\Delta_t p_s^2 G^3(\mathbf{p})\right] p_s^4
\approx \frac{C_{d-1} \Theta_{d-1}}{(2d-7)} g^2 \Delta_t \left(\frac{\gamma^{2-d} K_{d-7}^2}{B}\right)^{\frac{1}{2}} L^{7-2d}, \quad (IV.1)
$$

where $L^{-1}$ is the infrared cutoff of $q_s$. $L$ will typically be of order the system’s length in the soft direction, for systems that are much shorter in the soft than the hard or $z$ directions. For $d < 7/2$, $\delta B$ grows with $L$ and becomes comparable to $B$ at $L = \xi_s^N$, the “nonlinear crossover length”. This length $\xi_s^N$ can be calculated by equating $|\delta B| = B$, which gives

$$
\xi_s^N = \left[\frac{(7 - 2d)B\gamma_d^2 K_{d-7}^2}{C_{d-1} \Theta_{d-1} g^2 \Delta_t}\right]^{\frac{1}{2d}}, \quad (IV.2)
$$

where $\Theta \equiv \Gamma(d/2)\Gamma(3 - d/2)/2$. The corresponding lengths along the $h$ and $z$ axes can be obtained by looking for the wavenumbers $q_h^N \equiv 1/\xi_h^N$ and $q_z^N \equiv 1/\xi_z^N$ at which the $B$ and $\gamma$ terms in the propagator $G(\mathbf{p})$ become comparable to the $K$ term evaluated at $\xi_s^N$. This gives: $\xi_h^N = (\xi_s^N)^2/\lambda_h$ and $\xi_z^N = (\xi_s^N)^2/\lambda_z$, where $\lambda_z \equiv \sqrt{K/B}$ and $\lambda_h \equiv \sqrt{K/\gamma}$. The conventional harmonic theory only applies when at least one of the length scales being probed in the soft, hard, or $z$ directions is smaller than the corresponding non-linear crossover lengths $\xi_{s,h}^N$. To go beyond these crossover lengths, we employ the RG which has been derived in section III.

To do this we use the trajectory integral matching formalism[23], as used in[12] and many other papers. In this approach, the RG is used established a connection between a correlation function at small wave vector in the unrenormalized system, and the same correlation function at a larger wavevector in the renormalized system. While the former cannot be evaluated using the harmonic approximation theory, the latter can, if we rescale to the point where the rescaled wavevector lies within the harmonic regime just described. Here, we chose the connected disorder averaged correlation function is $C(q) = \langle [u(q)]^2 \rangle - \langle u(-q)u(q)\rangle$.

For this correlation function, the connection just described is given by:

$$
C(\mathbf{q}, B, K, \gamma, \Delta_t, g, w)
= e^{[2x + (d-2)\omega_{\mathbf{q}} + \omega_{\mathbf{q}} + 1]L} C(q_s e^\epsilon, q_s e^{\omega_{\mathbf{q}} \epsilon}, q_h e^{\omega_{\mathbf{q}} \epsilon}, B(\ell), K(\ell), \gamma(\ell), \Delta_t(\ell), g(\ell), w(\ell)), \quad (IV.3)
$$

where on the right-hand side the prefactor comes from the dimensional and field rescalings.

First we consider special $\mathbf{q}$’s with $q_s = 0, q_h = 0$. In this case we will choose the rescaling parameter is taken to be $\ell = \ell^* \equiv \ln(\Lambda/q_s)$, so that the rescaled $\mathbf{q}$’s lies on the boundary of the Brillouin zone. Thus, we can accurately evaluate the correlation function on the right-hand
the solution of this equation reduces to

\[ C(q, B, K, \gamma, \Delta_t, g, w) = e^{[2\chi + (d-2)\omega_h + \omega_\perp + 1]t^*} \frac{1}{K(t^*)\Lambda^2} \]

\[ = \frac{1}{K(q_s, q_z, h = 0)q_s^2} \quad \text{(IV.4)} \]

where

\[ K(q_s, q_z, h = 0) = K \left( \xi^N q_s \right)^{-\eta_K} \quad \text{(IV.5)} \]

with

\[ \eta_K = \frac{g^4}{6\sqrt{2}} = \frac{16}{27} \epsilon + O(\epsilon^2). \quad \text{(IV.6)} \]

In the second equality of Eq. (IV.4) we have eliminated \( t^* \) in favor of \( q_s \), and used

\[ K(t^*) = K \left( \xi^N \Lambda \right)^{(d-2)\omega_h + \omega_\perp + 2\chi - 3} \times \left( \xi^N q_s \right)^{(2-d)\omega_h - \omega_\perp - 2\chi + 3 - \frac{g^4}{6\sqrt{2}}} \]

\[ = \left( \xi^N q_s \right)^{-\eta_K} \left( \frac{\Lambda}{q_s} \right)^{(d-2)\omega_h + \omega_\perp + 2\chi - 3} \quad \text{(IV.7)} \]

which is obtained by integrating the flow Eq. (III.12).

The above calculations can be generalized for arbitrary direction of \( \vec{q} \) by using a more sophisticated choice of \( t^* \), in which we rescale until the propagator \( G(\vec{p}) \) takes on the same value as its smallest on the Brillouin zone boundary. This leads to the condition:

\[ K(t^*)\Lambda^4 = K(t^*)q_s^4 e^{t^*} + B(t^*)(q_s e^{t^*})^2 + \gamma(t^*) (q_s e^{t^*})^2. \quad \text{(IV.8)} \]

This choice again ensures that the correlation function on the right-hand side of Eq. (IV.3) can be accurately evaluated using the harmonic approximation. Clearly, the solution of this equation reduces to \( t^* = \ln (\Lambda/q_s) \) for the special case \( q_h = 0, q_z = 0 \). In general, the solution is

\[ e^{t^*} = \left( \frac{\Lambda}{q_s} \right) f(X, Y) \quad \text{(IV.9)} \]

where we have defined the scaling variables

\[ X = \frac{q_s \xi^N}{(q_s \xi^N)^{\xi^*_z}}, \quad \text{(IV.10)} \]

and

\[ Y = \frac{q_h \xi^N}{(q_s \xi^N)^{\xi^*_h}}. \quad \text{(IV.11)} \]

The universal anisotropy exponents \( \xi^*_z \) and \( \xi^*_h \) are given by

\[ \xi^*_z = 2 - \frac{\eta_B + \eta_K}{2}, \quad \text{(IV.12)} \]

\[ \xi^*_h = 2 - \frac{\eta_K}{2}. \quad \text{(IV.13)} \]

In equation (IV.9), \( f(X, Y) \) is a scaling function. Its asymptotic behaviors can be deduced by noting that, for \( X \ll 1, Y \ll 1 \), it should recover the result for \( q_s = q_h = 0 \), while for \( X \gg 1, X^\xi \gg Y^\xi \), \( t^* \) should only depend on \( q_s \), and for \( Y \gg 1, Y^\xi \gg X^\xi \), \( t^* \) should only depend on \( q_h \). These conditions force \( f(X, Y) \) to have the asymptotic forms:

\[ f(X, Y) = \begin{cases} 1, & X \ll 1, Y \ll 1, \\ X^{-\frac{1}{\xi^*_h}}, & X \gg 1, X^\xi \gg Y^\xi, \\ Y^{-\frac{1}{\xi^*_z}}, & Y \gg 1, Y^\xi \gg X^\xi. \end{cases} \quad \text{(IV.14)} \]

Using \( t^* \) as given by Eq. (IV.9) and repeating our earlier calculation for the \( q_h = q_z = 0 \) case, we obtain the anomalous elasticity for \( B, K \) and \( \Delta_t \) for an arbitrary direction of \( \vec{q} \):

\[ B(\vec{q}) = B \left( \xi^N q_s \right)^{\eta_B} f_B(X, Y) \]

\[ = \left( \xi^N q_s \right)^{\eta_B} f_B(X, Y) \quad \text{(IV.15)} \]

\[ K(\vec{q}) = K \left( \xi^N q_s \right)^{-\eta_K} f_K(X, Y) \]

\[ = \left( \xi^N q_s \right)^{-\eta_K} f_K(X, Y) \quad \text{(IV.16)} \]

\[ \Delta_t(\vec{q}) = \Delta_t \left( \xi^N q_s \right)^{-\eta_t} f(X, Y)^{\eta_t} \]

\[ = \left( \xi^N q_s \right)^{-\eta_t} f_t(X, Y) \quad \text{(IV.17)} \]

where we have defined the \( B, K, \) and \( \Delta_t \) scaling functions:

\[ f_B(X, Y) \equiv B(f(X, Y))^{-\eta_B}, \quad \text{(IV.18)} \]

\[ f_K(X, Y) \equiv (K, \Delta_t)(f(X, Y))^{\eta_K}, \quad \text{(IV.19)} \]

Here, the exponents \( \eta_B \) and \( \eta_t \) are given by:

\[ \eta_B = \frac{3}{32\sqrt{2}} g^4 = \frac{4}{9} \epsilon + O(\epsilon^2), \quad \text{(IV.20)} \]

\[ \eta_t = \frac{1}{32\sqrt{2}} g^4 = \frac{4}{27} \epsilon + O(\epsilon^2). \quad \text{(IV.21)} \]

In deriving Eq. (IV.17), we have used the correlation function \( \langle u(-\vec{q})\rangle \langle u(\vec{q}) \rangle = \Delta_t q^4(\vec{q}) \) instead of \( C(\vec{q}) \), since the latter has no dependence on \( \Delta_t \).

In the physical dimension \( d = 3 \), to leading order in \( \epsilon, \eta_B = 0.2222, \eta_K = 0.2963, \) and \( \eta_t = 0.07407 \). These results are quantitatively reliable, since \( \epsilon = \frac{1}{2} \), which is small. Since both \( \eta_B \) and \( \eta_t \) are positive, the compression modulus \( B \) and the bend modulus \( K \) vanish and diverge, respectively, as \( \vec{q} \to \vec{0} \). Physically, this can be understood as follows:

In the presence of layer fluctuations, a compression can be relieved by smoothing out these fluctuations, thus leading to a vanishing effective compression modulus. Similarly, bending a fluctuating smectic leads to compression of the layers, thus leading to a diverging effective bend modulus.
V. CRITICAL BEHAVIOR

In this section we calculate the critical exponents for the transition following standard RG procedures \[24\]. Linearizing the flow Eqs. \[11,13\] around the fixed point \( g_1^* = \frac{128}{27} \epsilon + O(\epsilon^2), g_2^* = \frac{812}{81} \epsilon + O(\epsilon^2), D^* = \frac{8}{27} \epsilon + O(\epsilon^2) \), we obtain

\[
\frac{d\delta g_1}{dt} = \frac{14 \epsilon}{9} \delta g_1 - \frac{8 \epsilon}{3} \delta g_2, \quad (V.1)
\]

\[
\frac{d\delta g_2}{dt} = \frac{8 \epsilon}{3} \delta g_1 - 4 \epsilon \delta g_2, \quad (V.2)
\]

\[
\frac{d\delta D}{dt} = \lambda_D \delta D + \frac{3}{8 \sqrt{2}} \delta g_1 - \frac{3}{8 \sqrt{2}} \delta g_2, \quad (V.3)
\]

where

\[
\lambda_D = 2 - \frac{28}{27} \epsilon + O(\epsilon^2). \quad (V.4)
\]

These recursion relations have the solution:

\[
\delta g_1 = C_1 e^{-2 \epsilon t} + C_2 e^{-4 \epsilon t}, \quad (V.5)
\]

\[
\delta g_2 = \frac{4}{3} C_1 e^{-2 \epsilon t} + \frac{3}{4} C_2 e^{-4 \epsilon t}, \quad (V.6)
\]

\[
\delta D = C_3 e^{(\lambda_D - 1) \epsilon t} + \frac{K C_1}{16 \sqrt{2}} e^{-2 \epsilon t} - \frac{3 K C_2}{64 \sqrt{2}} e^{-4 \epsilon t}, \quad (V.7)
\]

where the \( C \)'s are determined by the bare values of the coefficients in the Hamiltonian Eq. \[11,13\].

Note that \( \delta D \) flows to \( \infty \) for positive \( C_3 > 0 \), to \( -\infty \) for \( C_3 < 0 \), and to zero for \( C_3 = 0 \). Thus positive \( C_3 \) corresponds to the A phase, negative \( C_3 \) to the C phase, and \( C_3 = 0 \) to the AC transition. In terms of the bare values of \( g_{1,2} \) and \( D \), the AC transition occurs when \( C_3 = 0 \); a little algebra shows that this happens when

\[
D(T_{AC}) = \frac{3K}{16 \sqrt{2}} (g_2 - g_1), \quad (V.8)
\]

which describes a plane in the \( g_1, g_2, D \) parameter space. Unlike in mean field theory, the critical value of \( D \), and hence the critical temperature \( T_{AC} \), depend on both \( g_1 \) and \( g_2 \).

\( C_3 \) measures how far the system is away from the critical point. Assuming that \( C_3 \) is a smooth, analytic function of temperature near \( T_{AC} \) (and we have no reason to assume otherwise), then near the critical point, \( C_3 \propto t \). This is the sole input we will need in our analysis to get the critical dependence of all quantities on \( t \).

For \( t \to 0^+ \), the correlation lengths of the original system and those of the rescaled system are connected by

\[
\xi_s = e^{\ell} \xi_s(\ell), \quad (V.9)
\]

\[
\xi_h = e^{\nu_h} \xi_h(\ell), \quad (V.10)
\]

\[
\xi_z = e^{\nu_z} \xi_z(\ell). \quad (V.11)
\]

This allows us to calculate the singular \( t \)-dependence of the correlation lengths. Under the RG, according to flow Eqs. \[V.5, V.6, V.7\] \( \delta g_1 \to 0, \delta g_2 \to 0, \) and \( \delta D \) grows as \( \delta D(\ell) \propto t e^{\lambda_D \ell} \). At

\[
\ell = \ell^* \equiv - \frac{1}{\lambda_D} \ln t, \quad (V.12)
\]

\( \delta D(\ell) \) is of order one, and the rescaled system becomes non-critical with finite \( \xi_{s,h,z}(\ell) \). Plugging \( \ell = \ell^* \) into the above equations we obtain

\[
\xi_s \propto t^{-\nu_s}, \quad (V.13)
\]

\[
\xi_h \propto t^{-\nu_h}, \quad (V.14)
\]

\[
\xi_z \propto t^{-\nu_z}, \quad (V.15)
\]

The temperature dependence of the order parameter \( \theta_N \), the angle between \( \hat{N} \) and \( \hat{z} \), can be calculated in a similar way. For \( t \to 0^- \), \( \theta_N \) is small and approximately \( \theta_N = |\partial_s u|/|\partial_s u| \) in the original system and that in the rescaled system are connected by

\[
|\partial_s u| = |\partial_s u(\ell^*)| e^{(\nu_s - 1)\ell^*}. \quad (V.16)
\]

\[|\partial_s u(\ell^*)| \] can be calculated using mean field theory since the rescaled system is deep into the \( C \) phase, and given by

\[
|\partial_s u(\ell^*)| = \sqrt{-2D(\ell^*)} \left| \frac{w(\ell^*) - g^2(\ell^*)}{B(\ell^*)} \right|. \quad (V.17)
\]

Since the dimensionless coupling \( g_{1,2} \) are finite constants near the RG fixed point, and we have chosen our RG rescaling factors so that, \( K(\ell), B(\ell), \) and \( \gamma(\ell) \) are kept fixed at their bare values, we have, from the definitions Eqs. \[11,13\], \( w(\ell^*) \sim 1/\xi_s(\ell^*), \) and \( g^2 \sim 1/\xi_z(\ell^*) \). Furthermore, \( D(\ell^*) \) is of order one. Therefore, Eq. \[V.20\] leads to \( |\partial_s u(\ell^*)| \sim \sqrt{\Delta_{t}(\ell^*)} \). Substituting this result into Eq. \[V.19\], using the recursion relation equation \[11,13\] for \( \Delta_t \), and using equation \[V.12\] for \( \ell^* \), we get

\[
|\partial_s u| \sim |t|^{2(d-2)\nu_s - \nu_h} \sim t^{2(d-2)\nu_s - \nu_h}. \quad (V.21)
\]

This implies

\[
\beta = \frac{\nu_s}{4} [4d - 10 + (5 - d)\eta_K - \eta_B - 2\eta] \quad (V.22)
\]

The first and second equalities are respectively the exact scaling relation \[1,17\] and the \( \epsilon \)-expansion result \[1,12\] quoted in the introduction.
Thus, the disorder averaged free energy of the system is invariant under the RG, that is,

$$\mathcal{F} = V_r \mathcal{F}_r \left[ K(\ell^*), B(\ell^*), \gamma(\ell^*), D(\ell^*), \Delta_t(\ell^*) \right]$$

$$+ \Delta F(\ell^*), \quad (V.23)$$

where $V_r$ and $\mathcal{F}_r$ are the volume and disorder averaged free energy density of the rescaled system, respectively, and $\Delta F(\ell^*)$ represents the contribution to the free energy from the degrees of freedom already integrated out in the RG process. We will focus in what follows on the first term on the right hand side of equation (V.23), since it scales the same way as the full free energy. If renormalize for the same renormalization group time $\ell^*$ that we used above to calculate the correlation lengths, we will again have renormalized out of the critical region, which will permit us to calculate the free energy in mean field theory. With this choice, $V_r$ is related to the volume of the original system by $V_r = V_0 e^{-[(d-2)\omega_h + \omega_z + 1] \ell^*}$. Furthermore, because we have renormalized out of the critical regime, $f_r$ only depends on $K(\ell^*), B(\ell^*), \gamma(\ell^*), D(\ell^*)$, and $\Delta_t(\ell^*)$ since it can be evaluated by ignoring the anharmonic terms. The only singular temperature dependence of $\mathcal{F}_r(\ell^*)$ comes from $\Delta_t(\ell^*)$, since $K(\ell^*), B(\ell^*)$, and $\gamma(\ell^*)$ are fixed at their bare values and $D(\ell^*)$ is of order one. This singular temperature dependence can be calculated by noting that $\mathcal{F}_r$ is linear in $\Delta_t(\ell^*)$; that is

$$\frac{\partial \mathcal{F}_r(\ell^*)}{\partial \Delta_t(\ell^*)} = \lim_{n \to 0} \frac{1}{(2\pi)^d d^r} \int dq \ q^2 \sum_{\alpha\beta} \{u_\alpha(\hat{q})u_\beta(-\hat{q})\}$$

$$= \text{constant. \quad (V.24)}$$

Therefore,

$$\mathcal{F} \propto V_0 e^{-[(d-2)\omega_h + \omega_z + 1] \ell^*} \left[ \Delta_t(\ell^*) + \text{cons.} \right]$$

$$\propto [t]^{\nu_s(d-2)\omega_h + \omega_z - 1 + \eta_K - \eta_B} \quad (V.25)$$

where in the second line we have used the solution of the recursion relation (III.15) and the expression (V.12) relating $\ell^*$ to the reduced temperature $t$.

Because the specific heat is the second temperature derivative of the free energy, the power of $t$ on the right hand side of the bottom line of equation (V.23) is just $2 - \alpha$, where $\alpha$ is the usual specific heat critical exponent. Thus,

$$\alpha = 2 - \frac{\nu_s}{2}(4d - 6 + (3 - d)\eta_K - 2\eta_t - \eta_B)$$

$$= \frac{2}{9} + O(\epsilon^2). \quad (V.26)$$

The first and second equalities are respectively the exact scaling relation (I.18) and the $\epsilon$-expansion result (I.16) quoted in the introduction.

Once we probe the system at length scales longer than the correlation lengths $\xi_{s,h,z}$ in the corresponding directions, the elastic constants and disorder variances become wavevector independent. This can be seen either by repeating the RG analysis just done above for small $D$ for large $D$; alternatively, one could repeat the perturbation calculation equation (IV.11) with a nonzero $D$, and find that all perturbative corrections now converge in the infra-red.

Instead of wavevector dependent, however, all of the elastic constants and disorder variances now become the temperature dependent. We can obtain this temperature dependence by matching.

For example, the temperature dependence of $B$ can be obtained by matching, right at $q_s = \xi_s^{-1}$, the wavevector dependence of $B(q_s, q_z = 0, q_h = 0)$ for $q_s \gg \xi_s^{-1}$ given in equation (IV.13) onto the wavevector independent behavior we expect for $q_s \ll \xi_s^{-1}$. Doing so, we obtain

$$B(T) = B(\xi_s^N/\xi_s(T))^{\eta_B}, \quad (V.27)$$

$$K(T) = K(\xi_s^N/\xi_s)^{-\eta_K} \propto (T - T_{AC})^{-\eta_K \nu_s}, \quad (V.28)$$

$$D(T) = D(\xi_s^N/\xi_s)^{2 - \eta_K - \eta_B} \propto (T - T_{AC})^{(2 - \eta_K)\nu_s}, \quad (V.29)$$

$$\Delta_t(T) = \Delta_t(\xi_s^N/\xi_s) \propto (T - T_{AC})^{-\eta_t \nu_s}, \quad (V.30)$$

$$\Delta_c(T) = \Delta_c(\xi_s^N/\xi_s) \propto (T - T_{AC})^{-\eta_c \nu_s}. \quad (V.31)$$

These temperature dependences are needed to predict the singular dependence of the light scattering in the A and C phases near the critical point, as will be shown in section VII.

VI. PROPERTIES OF THE A AND C PHASES

The analysis of the preceding two sections has focused on the critical region near the AC transition. What about the properties of the A and C phases themselves in this random biaxial environment? These prove to be quite interesting and novel as well: we find that both the A and C phases behave, at long distances, like an XY ferromagnet in a random field [39]. This should be contrasted to the behavior of this system in uniaxial disordered environments, in which case the A
phase also looks like an XY model in a random field, but the C phase belongs to the far more disordered “m=1 Bragg Glass” universality class.

We begin with the A phase. At long length scales \(r_s \ll \xi_s, r_z \ll \xi_z\), and \(r_h \ll \xi_h\) (i.e., \(D > 0\)), the effective Hamiltonian for this phase is:

\[
H = \int d^d r \left[ \frac{B(T)}{2} (\partial_z u)^2 + \frac{D(T)}{2} (\partial_u u)^2 + \gamma (\partial_h u)^2 + V_p(u - \phi(\vec{r})) \right].
\]  

(VI.1)

Note that we have dropped the term \((\partial_z u)^2\), because it is less important than the \((\partial_u u)^2\); near the transition, we could not do this, since \(D(T)\) vanished there. But in the A phase, since \(D > 0\), we have this term, and it dominates the long length scales as defined above.

An additional effect of this extra term is that it suppresses fluctuations of \(u\), and the coefficient \(D(T)\) in the Hamiltonian Eq. (VI.1) are given by their bare values. Instead, both are renormalized by the critical fluctuations and become non-trivially dependent. They are given by Eqs. (V.27) and (V.29).

Like the model for the smectic A phase in \textit{uniaxially\cite{17}} stretched aerogel, this model is effectively a random field XY model (RFXY), with an anisotropic spin wave stiffness.

To show this, we rescale lengths as follows:

\[
r_s = r_s' \sqrt{D/B}, \quad r_h = r_h' \sqrt{\gamma/B},
\]

(VI.2)

and replace \(u\) by a rescaled variable

\[
\theta(\vec{r}') = q_0 u(\vec{r}),
\]

(VI.3)

where \(q_0 = 2\pi/a\) is the reciprocal lattice constant. This latter transformation has the property that the physical state of the system is now invariant under the transformation:

\[
\theta \rightarrow \theta + 2\pi n,
\]

(VI.4)

where \(n\) is any integer, since the smectic state is invariant under translations by any integral number \(n\) of layer spacings \(a\). This is precisely the symmetry obeyed by the angle variable \(\theta_{XY}\) giving the angle between the local spins and some reference direction in a ferromagnetic XY model. And, indeed, in the rescaled variables, the model becomes:

\[
H = \int d^d r' \left[ \frac{K_{XY}}{2} \nabla' \theta)^2 + V_p^{XY} (\theta - \phi_{XY}(\vec{r}')) \right],
\]

(VI.5)

where \(V_p^{XY}(\theta) \equiv V_p(\frac{\pi}{a})\) is a periodic function with period \(2\pi, \phi_{XY}(\vec{r}) \equiv \frac{2\pi \phi(\vec{r})}{a}\) is a short-range correlated random variable uniformly distributed between 0 and \(2\pi\), and, in spatial dimension \(d = 3\), the spin wave stiffness

\[
K_{XY} = \frac{\sqrt{\pi}}{4\alpha}. \quad \text{(VI.6)}
\]

This Hamiltonian (VI.5) is easily recognized\cite{26} as just the long-wavelength Hamiltonian for a ferromagnetic XY model in a random field, which has been extensively studied\cite{18}. Fisher\cite{18} showed that, in this model, the spin wave stiffness \(K_{XY}\) is an constant of order one and \(C(3) \approx 1.10\pi^2\).

Using the rescaling (VI.3) with this result then implies:\

\[
\langle u(\vec{q}') u(\vec{-q}')\rangle = \frac{1.10\pi^2 \sqrt{BD\gamma}}{(Bq_z^2 + Dq_z^2 + \gamma^2 q_0^2)^{1.2} q_0^2}.
\]

(VI.6)

The real space correlation function

\[
C(\vec{r}) \equiv \langle |u(\vec{r}) - u(\vec{0})|^2\rangle
\]

(VI.7)

can be calculated using a similar strategy. The real space correlation function governed by Hamiltonian (VI.5) has also been derived by Fisher\cite{18}, and is, in \(d = 3\),

\[
\langle \theta(\vec{r'}) - \theta(\vec{0})\|^2 = 1.10\ln (r'/b'),
\]

where \(b'\) is the cutoff length in \(\vec{r}'\) space. Then rescaling \(\theta\) and \(\vec{r}'\) back to \(u\) and \(\vec{r}\), respectively, using the rescalings described above, we obtain

\[
C(\vec{r}) \approx \frac{1.10}{q_0} \ln \sqrt{\left(\frac{r_s}{b_s}\right)^2 + \left(\frac{r_h}{b_h}\right)^2 + \left(\frac{r_z}{b_z}\right)^2}
\]

(VI.8)

where \(b_{s,h,z}\) are respectively cutoffs along \(\hat{s}, \hat{h}, \text{ and } \hat{z}\) in \(\vec{r}\) space, and related to \(b'\) by the rescalings (VI.2), which give \(b_s = b' \sqrt{D/B}\), \(b_h = b' \sqrt{\gamma/B}\), and \(b_z = b'\).

The physically more important correlation is \(F_n(\vec{r}) = \langle e^{i n q_0 u(\vec{r})} - u(\vec{0})\rangle\), which can be experimentally measured by X-ray scattering, as will be discussed in next section. In the approximation that the fluctuations of \(u\) are Gaussian, which Fisher\cite{18} has shown to give the correct power law decay of the XY analog of \(F_n(\vec{r})\) to leading order in \(\epsilon \equiv 4 - d\),

\[
F_n(\vec{r}) \approx \exp \left[ -\frac{n^2 q_0^2}{2} C(\vec{r}) \right].
\]

(VI.9)

Plugging Eq. (VI.8) into Eq. (VI.9) we get

\[
F_n(\vec{r}) \approx \left(\sqrt{\left(\frac{r_s}{b_s}\right)^2 + \left(\frac{r_h}{b_h}\right)^2 + \left(\frac{r_z}{b_z}\right)^2}\right)^{-0.55 n^2}
\]

(VI.10)
i.e., quasi-long-ranged translational order. However, unlike in bulk smectics, the power law exponent has no dependence on either the temperature or the quantitative elastic properties (e.g., the bare values of the elastic coefficients), and is therefore universal. Furthermore, \( F_n(\vec{r}) \) is isotropic in scaling, unlike the clean case.

Now we discuss \( \langle u(q)u(-q) \rangle \) at small \( q \)'s in the C phase. Since \( D < 0 \), \( u = 0 \) is not the disorder-averaged ground state. Therefore, we need to first expand the Hamiltonian around the state \( \partial_z u = -\frac{2\nu}{Bw-\eta^2} \), \( \partial_h u = 0 \), which is one of the disordered averaged ground states in the C phase, as obtained by minimizing the Hamiltonian Eq. (VI.11) with the disordered terms excluded. We will denote the displacement of the layer from the ground state position as \( u'(\vec{r}) \). Expanding in small \( u' \), and ignoring all irrelevant terms, we obtain a Hamiltonian which is essentially the same as Eq. (VI.1) but with \( u(\vec{r}) \) replaced by \( u'(\vec{r}) \). Thus, we conclude that the universality class of the C phase is also that of the random field XY model, and \( \langle u'(\vec{q})u'(\vec{-q}) \rangle \) is also given by equation (VI.6). Likewise, real space correlations are also given by equation (VI.10).

VII. PREDICTIONS FOR SCATTERING EXPERIMENTS

In this section we work out the implications of our theory for both visible light and X-ray scattering experiments.

A. X-ray Scattering

X-ray scattering experiments provide a direct experimental measure of the density-density correlation function \( \langle \rho(\vec{r})\rho(\vec{r}') \rangle \), where \( \rho(\vec{r}) \) is the molecular number density. This correlation function is an important quantity in smectics, since it reflects the translational order of the system. The X-ray scattering intensity is related to its Fourier transform by

\[
I(\vec{q}) \propto \int d^dr' d^dr'' \langle \rho(\vec{r})\rho(\vec{r}') \rangle e^{-i\vec{q} \cdot (\vec{r}' - \vec{r}'')}.
\]  

(VII.1)

In a smectic, \( \rho(\vec{r}) \) can be expanded in a Fourier series with period \( a \), the layer spacing between nearest layers, via

\[
\rho(\vec{r}) = \frac{1}{\sqrt{V}} \sum_n \rho_n e^{i\vec{q}_n \cdot [\hat{z} + u(\vec{r})]}.
\]  

(VII.2)

where \( \rho_n \) is the complex amplitude of the \( n_{th} \) harmonic of the density wave. In writing this equation, we have included the effect of fluctuations \( u(\vec{r}) \) in the positions of the layers, but not those of the magnitudes. The latter fluctuations can be shown to be irrelevant, in the RG sense, close to the transition.

Inserting Eq. (VII.2) into Eq. (VII.1) we obtain

\[
I(\vec{q}) \propto \sum_n |\rho_n|^2 \int d^dr e^{i(nq_0z-\vec{q} \cdot \vec{r})} F_n(\vec{r}),
\]  

(VII.3)

where

\[
F_n(\vec{r}) = \langle e^{i\nu_0 q_0 u(\vec{r}) - u(\vec{0})} \rangle.
\]  

(VII.4)

In deriving Eq. (VII.3), we have changed variables of integration from \( \vec{r}' \) to \( \vec{r}' = \vec{r}'' - \vec{r}'' \) and used the fact that, for homogeneous systems,

\[
\langle e^{i\nu_0 q_0 u(\vec{r}'')} \rangle = \langle e^{i\nu_0 q_0 u(\vec{r}) - u(\vec{0})} \rangle.
\]  

(VII.5)

Because all of the \( F_n(\vec{r}) \) are slowly varying functions of position (because \( u(\vec{r}) \) itself is), the sum in Eq. (VII.3) is dominated by the term with \( nq_0z \) the nearest Bragg peak to \( \vec{q} \). Therefore, the X-ray scattering intensity near the \( n_{th} \) Bragg peak can be simply written as

\[
I_n(\vec{q}) \propto |\rho_n|^2 \int d^dr e^{i\delta q \cdot \vec{r}} F_n(\vec{r}),
\]  

(VII.6)

where \( \delta \vec{q} \) is defined as the deviation from the center of the \( n_{th} \) Bragg peak.

We have already obtained the scaling of the limiting, large \( r \) behavior of \( F_n(\vec{r}) \) in both the A and the C phases in section (VI).

To get an idea of the behavior of \( F_n(\vec{r}) \) for smaller \( r \)'s, which proves to be extremely important for getting the full scattering, we will begin by approximating \( F_n(\vec{r}) \) by

\[
F_n(\vec{r}) \approx \exp \left[ -\frac{n^2q_0^2}{2} C(\vec{r}) \right],
\]  

(VII.7)

where

\[
C(\vec{r}) = \frac{\langle |u(\vec{r}) - u(\vec{0})|^2 \rangle}{2} = \frac{2}{(2\pi)^d} \int d^d\vec{q} [1 - \cos (\vec{q} \cdot \vec{r})] \langle u(-\vec{q})u(\vec{q}) \rangle.
\]  

(VII.8)

This expression would be exact if the fluctuations of the displacement field \( u \) were Gaussian. As discussed above, this approximation can be shown to give the correct scaling of \( F_n(\vec{r}) \) at the longest length scales in both the A and the C phase. Its validity for length scales shorter than the correlation lengths (i.e., for \( r_s \ll \xi_s \), \( r_z \ll \xi_z \), and \( r_h \ll \xi_h \)) is more questionable; here we will begin by assuming equation (VII.8) and exploring the consequences of that assumption. Then we will discuss possible perils of the approximation (VII.8), and argue that the qualitative form of the scattering is given correctly by it, although certain quantitative results (in particular, the precise numerical value of the universal exponent \( \Omega \) in equation (IV.23) for the spike width in the X-ray scattering), may be incorrectly given by it.
We can calculate $C(\vec{r})$ in the Gaussian approximation using our knowledge of the disorder averaged fluctuations in momentum space, $\langle u(-\vec{q})u(\vec{q}) \rangle$.

At large $\vec{q}$s (i.e., $q_s \gg \xi_s^{-1}$, or $q_z \gg \xi_z^{-1}$, or $q_h \gg \xi_h^{-1}$), the term $(\partial_u)^2$ in the Hamiltonian Eq. (VI.11) is subdominant to $(\partial^2_{\vec{q}})^2$ and hence negligible. In this case the random pinning disorder is irrelevant, and we treat the tilt-only model using the RG derived in Sec. III.

$\langle u(-\vec{q})u(\vec{q}) \rangle$ can be calculated by following the method described in Sec. IV and given by Eq. (VII.9),

$$\langle u(-\vec{q})u(\vec{q}) \rangle = \frac{\Delta_0(q)q_s^2}{[B(q)q_s^2 + \gamma q_h^2 + K(q)q_z^2]} \tag{VII.9}$$

where the wavevector dependencies of $B(q)$, $K(q)$, and $\Delta_0(q)$ are given by equations (IV.15,IV.16,IV.17), respectively.

Plugging $\langle u(-\vec{q})u(\vec{q}) \rangle$ into Eq. (VII.10) we calculate $C(\vec{r})$ asymptotically. For $\vec{r}$ larger than the correlation lengths (i.e., $r_s, r_h, r_z \ll \xi_s, \xi_h, \xi_z$), the dominant contribution to $C(\vec{r})$ in Eq. (VII.10) comes from the integral over large $\vec{q}$'s: $n\sim q_\nu$, or $q_z \gg \xi_z^{-1}$, or $q_h \gg \xi_h^{-1}$), and $C(\vec{r})$ is thus given by

$$C(\vec{r}) \approx \lambda^2 \left| \begin{array}{c} \frac{r_s}{\xi_s} \frac{r_z}{\xi_z} \frac{r_h}{\xi_h} \\ \frac{r_s}{\xi_s} \frac{r_z}{\xi_z} \frac{r_h}{\xi_h} \\ \frac{r_s}{\xi_s} \frac{r_z}{\xi_z} \frac{r_h}{\xi_h} \end{array} \right| \tag{VII.10}$$

where $\Gamma = 1 + \nu_h - \eta_K + \nu_h$. Plugging this result into Eq. (VII.11) gives an exponentially decaying translational correlation function; i.e., short-ranged smectic translational order.

For $\vec{r}$ larger than the correlation lengths (i.e., $r_s \gg \xi_s$, or $r_z \gg \xi_z$, or $r_h \gg \xi_h$), $C(\vec{r})$ crossovers to

$$C(\vec{r}) \approx \frac{1.10}{q^2_0} \ln \left( \frac{r_s^2}{\xi_s^2} + \frac{r_z^2}{\xi_z^2} + \frac{r_h^2}{\xi_h^2} \right) \tag{VII.11}$$

The first piece comes from the integral over small $\vec{q}$'s (where we have used “small” in the sense described earlier). It follows from Eq. (VI.18) with the cutoff lengths $b_{s,h,z}$ replaced by correlation lengths $\xi_{s,h,z}$, respectively. The second piece comes from the integral over large $\vec{q}$'s and connects with $C(\vec{r})$ Eq. (VII.11) at the crossover point.

The x-ray scattering intensity near the $n_{th}$ Bragg peak $I_n(\delta q)$ can be calculated using Eq. (VII.12). Plugging the above calculated $C(\vec{r})$ into Eq. (VII.11), we find $F_n(\vec{r})$ has the following behavior. For $\vec{r}$ larger than the correlation lengths $F_n(\vec{r})$ decays exponentially fast, while for $\vec{r}$ larger than the correlation lengths $F_n(\vec{r})$ decays much slower and as a power law. Thus, in the large $\delta q$ limit the main contribution to $I_n(\delta q)$ Eq. (VII.12) from the integral over small $\vec{r}$. This implies that far away from the center of the Bragg peak, the line shape of the scattering intensity is qualitatively Lorentzian-squared. The line widths of this squared-Lorentzian are anisotropic and given respectively by

$$\delta q^{w}_{s} = (\xi^{w}_{s})^{-1} \left( \frac{a}{\lambda_{s}} \right)^{-2/\Gamma}, \tag{VII.12}$$

$$\delta q^{w}_{z,h} = (\xi^{w}_{z,h})^{-1} \left( \frac{a}{\lambda_{z}} \right)^{-2\xi_{z,h}/\Gamma}, \tag{VII.13}$$

where $\xi^{w}_{s,h,z}$ are the line widths of the correlation function $F_n(\vec{r})$ and obtained via $C(\vec{r}) = a^2$. In the limit $\delta q \rightarrow 0$, $I_n(\delta q)$ converges for $n > 2$ and diverges for $n \leq 2$. Therefore, only the first two Bragg peaks diverge, following the power law:

$$I_n(\delta q) \sim \propto_{n} \left[ (\delta q_{s}\xi_{s})^{2} + (\delta q_{h}\xi_{h})^{2} + (\delta q_{z}\xi_{z})^{2} \right]^{-\frac{1}{2\xi_{s,h,z}}} \tag{VII.14}$$

where $\propto_{n} = \xi_{s,z,h} \xi_{\nu} \exp \left[ -\frac{1}{2} n^{2} \lambda^{2}_{s,h,z} (\xi_{s}/\xi_{s}^{N})^{\nu} \right]$ has an exponentially strong dependence on the temperature, which is responsible for the fast vanishing of the divergent Bragg peak approaching the critical point. The divergence shown by Eq. (VII.14) comes from the integral in Eq. (VII.1) over large $\vec{r}$.

The crossover between these two distinct scattering patterns can be estimated as the follows. Since the divergent (power law) Bragg peak vanishes extremely fast approaching the critical point, we expect that near the critical point, the crossover occurs at very small $\delta q$ which are much less than the widths of the Lorentzian square. Thus, at this crossover the Lorentzian part of $I_n(\delta q)$ is approximately proportional to $\xi^{w}_{s,h,z} \xi_{s,z,h}$. Equating this to the power law part given by Eq. (VII.14) we get, for the $n_{th}$ Bragg peak,

$$\delta q^{nc}_{s,h,z} = \xi_{s,h,z}^{-1} \left( \frac{\xi_{s,h,z}}{\xi_{s}^{N}} \right)^{-1/2} \propto_{n} \propto_{n} \exp \left( -A_n \Omega^{G} \right), \tag{VII.15}$$

where $A_n$ is non-universal, $\Omega^{G}$ (where the subscript “G” stands for “Gaussian”) and $\propto_{s,h,z}$ are universal and given, respectively, by

$$\Omega^{G} = \nu_{s} \Gamma = \frac{1}{2} + \frac{4}{27} \epsilon = O(\epsilon^{2}), \tag{VII.16}$$

$$\propto_{s,h,z} = \propto_{s,h,z} + \frac{\nu_{s} \nu_{h} \nu_{z}}{-3 + 55n^{2}}. \tag{VII.17}$$

These results were derived in the Gaussian approximation. Since we have shown that the smectic elasticity and fluctuations are controlled by a non-Gaussian fixed
point, one might well question the validity of the Gaussian approximation.

How could we go beyond the Gaussian approximation? In principle, this could be done by integrating all of the degrees of freedom out of the Hamiltonian (completeH) except for the single variable that enters the X-ray scattering near the $n$'th Bragg peak, namely:

$$x_n(\vec{r}) \equiv nq_0[u(\vec{r}) - u(0)]. \quad (VII.18)$$

The hypothetical integration out of all other variables would leave us with a known probability distribution $P(x_n(\vec{r}))$ of this single variable $x_n(\vec{r})$.

In reality, of course, it is far too difficult to perform this hypothetical integration. We do, however, know something about $P(x_n(\vec{r}))$: its variance, which is just $C(\vec{r})$, equation (VII.8). This sets the scale of $x_n(\vec{r})$ fluctuations; hence, we can write $P(x_n(\vec{r}))$ in a scaling form:

$$P(x_n(\vec{r})) = \sqrt{k} e^{-U(\sqrt{k(\vec{r})}x_n(\vec{r}))}, \quad (VII.19)$$

where

$$k(\vec{r}) \equiv \frac{O(1)}{(nq_0)^2 C(\vec{r})}. \quad (VII.20)$$

If we take $U(y) = y^2$, then we recover the Gaussian theory just presented. More generally, we can write:

$$F_n(\vec{r}) = \sqrt{k} \int_{-\infty}^{\infty} dx \ e^{ix-U(\sqrt{k(\vec{r})}x)}. \quad (VII.21)$$

For $|\vec{r}|$ large, $k(\vec{r})$ is small, and one can accurately evaluate this integral, for a given $U(y)$, using the method of saddle points. That is, one finds the place where the argument of the exponential in (VII.21) is extremized, expands about that point, and performs a Gaussian integral using this expansion. In general, locating the saddle point requires analytically continuing the integral in (VII.21) into the complex plane.

The first step is, of course, locating the saddle point. Defining

$$f(x) \equiv ix - U(\sqrt{k(\vec{r})}x), \quad (VII.22)$$

this point is where

$$f'(x) = i - \sqrt{k(\vec{r})}U'(\sqrt{k(\vec{r})}x) = 0. \quad (VII.23)$$

For example, consider $U(y) = y^4$ [28]. The condition (VII.23) becomes

$$4k^2x^3 = i, \quad (VII.24)$$

whose solution is [29]:

$$x_{\pm} = \left(\frac{i}{4k^2}\right)^{1/3} = \frac{\pm \sqrt{3} + i}{(32k^2)^{1/3}}. \quad (VII.25)$$

To exploit these saddle points, we deform our contour of integration from the real axis to a line parallel to the real axis, but displaced up (i.e., in the imaginary direction) from it by the imaginary part of $x_{\pm}$, as illustrated in Fig. 6. (The contour is connected to the original contour by vertical segments at $\pm \infty$ that contribute nothing to the integral.) This passes the contour of integration through both saddle points $x_{\pm}$.

Since the integral is now dominated by the region near these two saddle points (as we will verify a posteriori in a moment), we can expand the integrand around each of the saddle points. Noting that

$$f''(x_{\pm}) = -12k^2x_{\pm}^3 = -6 \left(\frac{k}{4}\right)^2 (1 \pm \sqrt{3}i) \quad (VII.26)$$

and

$$f(x_{\pm}) = ix_{\pm} - k^2x_{\pm}^4 = \frac{3(\pm \sqrt{3}i - 1)}{8(2k)^4}, \quad (VII.27)$$

we can use the expansion

$$f(x) \approx f(x_{\pm}) + \frac{1}{2} f''(x_{\pm}) u^2 \quad (VII.28)$$

with $u$ real, near each of the saddle points. Making this approximation, and extending the region of integration over $u$ near each saddle point to run from $-\infty$ to $\infty$ (an approximation that we will justify for small $k$ in a moment), we are then left with two Gaussian integrals, whose sum is $F_n(\vec{r})$. This gives

$$F_n(\vec{r}) \approx \sqrt{\frac{2\pi}{3}} (4\sqrt{k})^{1/4} \exp\left(-\frac{3}{8(2k)^4}\right) \times \cos\left(\frac{3\sqrt{3}}{8(2k)^{1/4}} - \frac{\pi}{6}\right). \quad (VII.29)$$

Now, to justify the saddle point approximation:

This approximation will be valid provided that the expansion (VII.28) is valid over the range of $u$ that dominates the integral near each of the saddle points. This
implies that it must work for

\[ u \sim \frac{1}{\sqrt{f'(x_\pm)}}. \tag{VII.30} \]

The neglected next order term in the expansion (VII.28) is \( \sim f'''(x_\pm) u^3 \sim \frac{f'''(x_\pm)}{f'(x_\pm)^{3/2}} \), where we have used the estimate (VII.30) for the range of \( u \) that dominates the integral near each saddle point. As long as this is \( \ll 1 \), the saddle point approximation will be good. Since, from (VII.26), \( f'''(x_\pm) \sim k^4 \) while \( f'''(x_{\pm}) \sim k^2 x_{\pm} \sim k^2 k^{-4} \sim k^4 \) (where we have used (VII.25) for \( x_\pm \)), we have \( \frac{f'''(x_\pm)}{f'(x_\pm)^{3/2}} \sim \frac{k^4}{k^2} \ll 1 \), provided \( k \ll 1 \), as it is for large \( \vert \vec{r} \vert \), as noted earlier.

Thus, for large \( \vert \vec{r} \vert \), the saddle point approximation is valid.

More generally, if we take \( U(y) = y^n \), then we get, by exactly the same sort of steepest descent analysis,

\[ F_n(\vec{r}) \propto \exp \left( -O(1) \times k^{-\Theta(n)} \right) \tag{VII.31} \]

with

\[ \Theta(n) \equiv \frac{n}{2(n-1)}. \tag{VII.32} \]

This is readily seen to reproduce the results for the Gaussian case \((n = 2)\), and the quartic case we just examined in detail \((n = 4)\). Note that for all values of \( n \geq 2 \), the exponent \( \Theta \) is bounded above and below:

\[ \frac{1}{2} \leq \Theta(n) \leq 1, \tag{VII.33} \]

with the upper bound satisfied by the Gaussian case, while the lower bound is approached in the limit \( n \rightarrow \infty \).

Since we expect \( n \geq 2 \), since \( \mathcal{H} \) contains higher powers of \( u \) and the value of \( \Theta \) is determined by the highest power, we expect \( \Theta \) to satisfy (VII.33).

As a final example, let us consider \( U(y) = e^{-y} \). The steepest descent analysis now gives

\[ F_n(\vec{r}) \propto \exp \left( -O(\ln n) \times k^{-\frac{1}{2}} \right), \tag{VII.34} \]

which is effectively (up to slowly varying logarithmic corrections) of the form (VII.31) with \( \Theta = \frac{1}{2} \), i.e., just the limiting value of \( \Theta(n) \) as \( n \rightarrow \infty \) (as is hardly surprising, since the Taylor series for \( e^{-y} \) involves arbitrarily large powers of \( y \)).

While hardly exhaustive, these examples give us confidence that the leading behavior of \( F_n(\vec{r}) \) is always given by (VII.31), with \( \Theta \) satisfying (VII.33). Using this, and the definition (VII.20), and repeating the analysis given above of the Gaussian case, we find that \( \delta q^{nc}_{s,h,z} \) is given by:

\[ \delta q^{nc}_{s,h,z} = \xi_{s,h,z}^{-1} \left( \frac{\xi_{s,h,z}}{\xi_{s,h,z}} \right)^{-1} \propto t^{\Theta} \exp \left( -A_n t^\Theta \right), \tag{VII.35} \]

with

\[ \Omega = \Omega_G \Theta, \tag{VII.36} \]

where \( \Omega_G \) is given by (VII.10), \( Y \) by (VII.17), and \( \Theta \) obeys the bound (VII.33). Combining this bound on \( \Theta \) with an estimate of the uncertainties in the \( e \)-expansion result (VII.16) for \( \Omega_G \) leads to the bounds on \( \Omega \) quoted in the introduction.

Near the critical point the dominant t-dependence of \( \delta q^{nc}_{s,h,z} \) is exponential. Thus, a plot of \( \ln \ln \delta q^{nc}_{s,h,z} \) versus \( \ln t \) should give a straight line with slope \( \Omega \).

### B. Visible Light Scattering

The fluctuations of the nematic director scatter the light strongly. The scattering intensity is proportional to the fluctuations [24]:

\[ I(\vec{q}) \propto (\hat{e}_L \cdot \vec{n}_0) \delta(\hat{\delta}_U \cdot \vec{q} + \hat{e}_T \cdot \vec{n}_0) \delta(\hat{\delta}_U \cdot \vec{q} + \hat{e}_T \cdot \vec{n}_0) \delta(\hat{\delta}_U \cdot \vec{q} + \hat{e}_T \cdot \vec{n}_0) \delta(\hat{\delta}_U \cdot \vec{q} + \hat{e}_T \cdot \vec{n}_0), \tag{VII.37} \]

where \( \hat{e}_L \) and \( \hat{e}_T \) are respectively the polarization directions of the incident and the transmitted light, and \( \delta \hat{n} \) is the deviation of the nematic director from its averaged value \( \vec{n}_0 \): \( \delta \hat{n} \equiv \hat{n} - \vec{n}_0 \).

In the \( A \) phase \( \vec{n}_0 = \hat{z} \), so the largest components of \( \delta \hat{n} \) are those \( \delta n_{1\perp} \) perpendicular to \( \hat{z} \). Thus, the scattering intensity is proportional to a linear sum of \( \langle \delta n_{1\perp}^2 \delta n_{-1\perp}^2 \rangle \); namely,

\[ I(\vec{q}) \sim \sum_{ij} D_{ij} \langle \delta n_{1\perp}(\vec{q}) \delta n_{-1\perp}(\vec{q}) \rangle, \tag{VII.38} \]

where the coefficients \( D_{ij} \) are of order one and solely determined by the polarization directions of the incident and transmitted light, and \( \delta n_{1\perp}^2 \) denotes the \( i \)’th component of \( \delta \hat{n}_{1\perp} \).

Since, in Sec. [III] we have shown that the fluctuations of the linear combination \( \delta \hat{n}_{1\perp} \) (equation (II.8) of \( \delta n_{1\perp}^2 \)) and \( \partial^2 u \) are massive, \( \langle \delta n_{1\perp}^2(\vec{q}) \delta n_{-1\perp}^2(\vec{q}) \rangle \approx \langle \partial^2 u(\vec{q}) \rangle \), where \( \partial^2 u \) denotes the partial derivative along the \( i \)’th component in the \( \perp \) plane. Therefore, equation (VII.38) can be rewritten as

\[ I(\vec{q}) \sim \sum_{ij} E_{ij} C_{ij}(\vec{q}), \tag{VII.39} \]

where

\[ C_{ij}(\vec{q}) \equiv L_{ij}(\vec{q}) q_{\perp}^2 \langle |\vec{u}(\vec{q})|^2 \rangle, \tag{VII.40} \]

with \( L_{ij}(\vec{q}) \equiv q_{\perp}^2 q_{\perp}^2 / q_{\parallel}^2 \), the projection operator along \( \perp \), and the coefficients \( E_{ij} \) are independent of \( \vec{q} \) and of order one.

An expression for \( \langle |\vec{u}(\vec{q})|^2 \rangle \) has been obtained in Sec. [VII.A]. However, those results have to be improved to predict the light scattering. Specifically, in addition to the
fluctuations induced by the random tilt in the $\hat{s}$ direction and the random field, we need to include in $\langle |u(\vec{q})|^2 \rangle$ the fluctuations induced by the random tilt in $\hat{h}$ direction and the random compression, as embodied in the $\Delta_h$ and $\Delta_c$ terms of equation (VII.12). These last two types of disorder, though "irrelevant" in the RG sense, have experimentally detectable effects on light scattering.

If any component of $\vec{q}$ is large compared to the corresponding inverse correlation length, (i.e., if $q_s \gg \xi^{-1}_s$, or $q_h \gg \xi^{-1}_h$, or $q_z \gg \xi^{-1}_z$), but all components are small compared to the ultraviolet cutoff $\Lambda$, we find that the $\langle |u(\vec{q})|^2 \rangle$ is given to a good approximation by

$$\langle |u(\vec{q})|^2 \rangle = \frac{\Delta_t(q)}{B(q)q^2 + \gamma q^2 + D(q)q^4},$$

(VII.41)

$$\langle |u(\vec{q})|^2 \rangle = \frac{\Delta_t(q)}{B(q)q^2 + \gamma q^2 + D(q)q^4} + \frac{1.10 \pi^2 \sqrt{B(T)D(T)}}{(B(q)q^2 + \gamma q^2 + D(q)q^4)^2} q^2,$$

(VII.42)

where the $\vec{q}$-dependences of $B(q)$, $K(q)$, and $\Delta_t(q)$ were derived in section [IV] and are given by Eqs. [IV.13] [IV.16] and [IV.17] respectively. The wavevector dependence $\Delta_c(q)$ of the $z$ component of the random field $\hat{h}$ is derived in Appendix B, and given by equation (IV.17). The correlations $\Delta_h(q)$ of the $\hat{h}$ component of the random field $\hat{h}$ do not acquire any anomalous wavevector dependence since there is no graphical correction to $\Delta_h(q)$. The fluctuations caused by the random field disorder are not included in Eq. (VII.41), since their contributions to $\langle |u(\vec{q})|^2 \rangle$ are always dominated by the others for large $\vec{q}$.

Likewise, if all of the components of $\vec{q}$ are small compared to the corresponding inverse correlation length, (i.e., if $q_s \ll \xi^{-1}_s$, and $q_h \ll \xi^{-1}_h$, and $q_z \ll \xi^{-1}_z$), we obtain

$$\langle |u(\vec{q})|^2 \rangle = \frac{\Delta_t(q)}{B(q)q^2 + \gamma q^2 + D(q)q^4},$$

(VII.43)

For $q$'s satisfying the condition (VII.43), with a generic direction of $\vec{q}$, the dominant cutoff in all of the scaling functions for all of the anomalous elastic coefficients in equation (VII.41) is $q_h$.

To see this, begin by noting that the scaling arguments

$$X \equiv \frac{q_{\xi z} N}{(q_s \xi s)^{\xi_s}},$$

(VII.44)

and

$$Y \equiv \frac{q_{\xi h} N}{(q_s \xi s)^{\xi s}},$$

(VII.45)

that appear in all of the scaling functions (e.g., equation [IV.8] for $\xi^s$) are both $\gg 1$, since $X \propto q^{\xi z} \rightarrow \infty$ and $Y \propto q^{\xi h} \rightarrow \infty$ as $q \rightarrow \infty$, since both $\xi_z$ and $\xi_h$ are $> 1$. This implies that either $q_h$ or $q_z$ dominates the scaling functions. To determine which, we form the only possible $q_s$-independent ratio of $X$ and a power of $Y$:

$$\frac{X}{Y \frac{1}{h}} = \frac{q_{\xi s} N}{(q_s \xi s)^{\xi s}} \propto q^{\xi s \frac{\xi z}{\xi h}} \propto q^{\xi z} \rightarrow 0 \text{ as } q \rightarrow \infty.$$

The vanishing of this ratio implies that $q_h$ is the dominant cutoff in all of the scaling functions.

Using this fact implies that, for the generic directions of $\vec{q}$ that we are considering, $\Delta_t(q) \propto q^{-\xi s}$, and $\Delta_c(q) \propto q^{-\xi s}$. This in turn implies that the $\Delta_t(q)$ term in the numerator of eqn. (VII.11) dominates the $\Delta_t(q)$ and $\Delta_h$ terms, since the ratio of the $\Delta_t(q)$ term to the $\Delta_h(q)$ term vanishes like $q^{-\xi h}$, which vanishes as $q \rightarrow 0$, since $\eta_c - \eta_t = 2 - \eta_B - \eta_K > 0$; likewise, the ratio of the $\Delta_h$
term to the $\Delta_c(\hat{q})$ term vanishes like $q_s^{2n}$, which vanishes as $q \rightarrow 0$, since $\eta_h > 0$.

Similar arguments show that the $\gamma$ term, which scales like $q^2$ for the generic directions of $\hat{q}$ that we are considering, dominates the denominator of eqn. (VII.31).

Taking these two terms as dominant in the numerator and denominator of eqn. (VII.41), respectively, we find:

$$<|u(\hat{q})|^2> \sim q^{-2} \frac{q_s^2}{4 + \frac{2}{q_{sh}}} \xi_h^{-1}, \quad q \gg \xi_h^{-1},$$

for generic directions of $\hat{q}$. Using this result in equation (VII.40), we obtain for the light scattering intensity, in this regime of wavevector:

$$I(\hat{q}) \propto q^{-\frac{2}{q_{sh}}} \xi_h^{-1}, \quad q \gg \xi_h^{-1}. \quad \text{(VII.47)}$$

Using the epsilon-expansion results (I.13) and (I.15) for $\eta_c$ and $\xi_h$, we obtain for the expansion for the power law in this expression:

$$\frac{\eta_c}{\xi_h} = 1 - \frac{8\epsilon}{27} + O(\epsilon^2), \quad \text{(VII.48)}$$

As discussed above, once $q_h$ decreases to $\xi_h^{-1}$, the correlation function $<|u(\hat{q})|^2>$ crosses over to the form equation (VII.42). In this regime, as mentioned earlier, all of the elastic moduli and disorder variances cease to be wavevector dependent, but become temperature dependent, as given by equations (VII.27-VII.31). As $q$ first decreases below $\xi_h^{-1}$, the $\Delta_c$ term continues to dominate $<|u(\hat{q})|^2>$; in this regime, this implies $<|u(\hat{q})|^2> \propto \frac{1}{q}$, and, hence, that the light scattering intensity $I(\hat{q})$ is constant. Eventually, however, as $q$ continues to decrease, the “random field” term (i.e., the $\sqrt{D B T}$ term, must start to dominate, since it diverges like $\frac{1}{q}$ as $q \rightarrow 0$.

For smaller $q$’s, $<|u(\hat{q})|^2> \propto \frac{1}{q}$, and, hence, that the light scattering intensity $I(\hat{q})$ obeys

$$I(\hat{q}) \propto q^{-\frac{1}{q_{sh}}}. \quad \text{(VII.49)}$$

We can estimate the critical value $q_c^F$ of $q$ at which this crossover between constant $I(\hat{q})$ and $I(\hat{q}) \propto q^{-3}$ takes place by equating the $\Delta_c$ term and the random field term; i.e., setting:

$$\frac{\Delta_c(T)(q_c^F)^2}{2(q_c^F)^4} = \frac{\sqrt{B(T)D(T)}}{(q_c^F)^2} \xi_h^{-1}, \quad \text{(VII.50)}$$

where we have used the fact that the $q_h$ term continues to dominate the denominators of all terms for this generic direction of $\hat{q}$, since its coefficient $\gamma$ is the only one in the denominator that is not renormalized to small values by the critical fluctuations. Solving equation (VII.50) for $q_c$ gives:

$$q_c^F(\hat{q}) = \frac{\sqrt{B(T)D(T)} \xi_h^{-1}}{\Delta_c(T)(q_c^F)^2} \propto |T - T_{AC}|^\phi, \quad \text{(VII.51)}$$

where the universal exponent $\phi$ is given by

$$\phi = \nu_s \left( 1 + \eta_c + \frac{\eta_B - \eta_K}{2} \right) = \frac{3}{2} + \frac{8}{27} \epsilon + O(\epsilon^2), \quad \text{(VII.52)}$$

where the first equality is obtained using the known temperature dependences equations (V.27), (V.29), and (V.31) for $B(T)$, $D(T)$, and $\Delta_c(T)$, while the second follows from the known $\epsilon$-expansions (I.10), (I.11), and (I.13), for $\eta_K$, $\eta_B$, and $\eta_c$. Equation (VII.52) is, of course, precisely equation (I.27) of the Introduction.

To summarize, for generic directions $\hat{q}$ of $\hat{q}$, the light scattering is given by equation (I.25) of the Introduction, and plotted there in Fig. 3.

As mentioned in the introduction, more information can be obtained by restricting the scattering to the subspace of $\hat{q}$ with $q_h = 0$. In this case the only nonzero $C_{ij}^s(\hat{q})$ is $C_{ss}(\hat{q})$. Therefore, equation (VII.39) implies that

$$I(\hat{q}) \propto C_{ss} = q_s^{2n} I(\hat{q})^2, \quad \text{(VII.53)}$$

As we just did for the case of a generic direction of $\hat{q}$, here, too, we can distinguish between the cases of $q_s$’s large compared to the correlation lengths, (which now means $q_s \gg \xi_s^{-1}$, or $q_s \gg \xi_s^{-1}$), but small compared to the ultraviolet cutoff $\Lambda$, and the reverse. In the large $q_s$ case, the correlation function $<|u(\hat{q})|^2>$ takes a simple scaling form:

$$<|u(\hat{q})|^2> = f_s(X) q_s^{2n} - \eta_t \eta_s \quad \text{(VII.54)}$$

where $X$ is the scaling ratio defined in equation (IV.10). This can be seen by noting that, when $q_h = 0$, both the numerator and denominator of (VII.41) take a scaling form.

The numerator is

$$\Delta_t(\hat{q}) q_s^2 + \Delta_c(\hat{q}) q_z^2 = q_s^{2-\eta_t} f_t(X) + q_z^{2-\eta_t} q_s^{2-\eta_t} f_c(X) = q_s^{2-\eta_t} f_t(X) + q_z^{2-\eta_t} q_s^{2-\eta_t} f_c(X),$$

where we have used the scaling forms (IV.17) and (B.7) for $\Delta_t$ and $\Delta_c$.

Using the exact scaling relation (I.19) for $\eta_c$, we find that $\eta_t - 2 - \eta_c = -4 + \eta_B + \eta_K = -2 (2 - \frac{2\eta_s}{2 - 2\eta_B + \eta_K}) = -2(2 - 2\eta_s/2 - 2\eta_B + \eta_K) = -2 \xi_z$, where we have used the exact scaling law $\xi_z = 2 - 2\eta_s/2 - 2\eta_B + \eta_K$.

Using this in (VII.55) implies that

$$\Delta_t(\hat{q}) q_s^2 + \Delta_c(\hat{q}) q_z^2 = q_s^{2-\eta_t} f_t(X) + q_z^{2-\eta_t} q_s^{2-\eta_t} f_c(X)$$

$$= q_s^{2-\eta_t} f_{num}(X), \quad \text{(VII.56)}$$

where we have defined the scaling function for the numerator

$$f_{num}(X) \equiv f_t(X) + \left( \frac{q_z^{\xi_z}}{q_s^{\xi_z}} \right)^2 X^{2} f_c(X). \quad \text{(VII.57)}$$
and used the definition \([IV.10]\) of the scaling variable \(X\).

Similarly, the square root of the denominator of \([VII.41]\) can be written, when \(q_b = 0\), as:

\[
B(\mathbf{q}) q_s^2 + K(\mathbf{q}) q_s^4 = q_s^{-\eta_K}\left(q_s^{-2\eta_B+\eta_K} f_B(X) + f_K(X)\right),
\]

\[
\equiv q_s^{-\eta_K} f_{\text{den}}(X) \quad (VII.58)
\]

where we have once again used both the definition \([IV.10]\) of the scaling variable \(X\), and the exact scaling relation \(2 - 2\eta_B + \eta_K = \zeta_c\).

Using the final equalities of \([VII.55]\) and \([VII.58]\) in \([VII.41]\) implies the scaling form equation \([VII.54]\), with

\[
f_u(X) \equiv \frac{f_{\text{num}}(X)}{f_{\text{den}}(X)} \quad (VII.60)
\]

and used the definition \([IV.10]\) of the scaling variable \(X\).

The scaling function \(f_u(X)\) must have limits such that the full \(\langle |u(\mathbf{q})|^2 \rangle\) correlation function becomes a function only of \(q_s\) for \(X \ll 1\), and only of \(q_s\) for \(X \gg 1\). This implies, by arguments virtually identical to those used in, e.g., section \([IV]\),

\[
f_u(X) \propto \begin{cases} 
\text{constant}, & X \ll 1, \\
X^{2\eta_B-\eta_K}, & X \gg 1,
\end{cases} \quad (VII.61)
\]

which in turn implies

\[
\langle |u(\mathbf{q})|^2 \rangle \propto \begin{cases} 
q_s^{2\eta_B-\eta_K}, & q_s \zeta_c^N \ll (q_s \xi_s^N)\zeta_c, \\
q_s^{2\eta_B-\eta_K}, & q_s \zeta_c^N \gg (q_s \xi_s^N)\zeta_c.
\end{cases} \quad (VII.62)
\]

Using equation \([VII.53]\), we can obtain the light scattering intensity from this:

\[
I(\mathbf{q}) \propto \begin{cases} 
q_s^{2\eta_B-4-\eta_K}, & q_s \zeta_c^N \ll (q_s \xi_s^N)\zeta_c, \\
q_s^{2\eta_B-4-\eta_K}, & q_s \zeta_c^N \gg (q_s \xi_s^N)\zeta_c.
\end{cases} \quad (VII.63)
\]

The above result only applies for both \(q_{s,z}\) each much larger than its associated inverse correlation length \(\xi_{s,z}^{-1}\). Once this ceases to be true, there will, as in the case of a generic direction of \(\mathbf{q}\) just discussed, be two subsequent crossovers as we decrease \(\mathbf{q}\): first, at \(q_{s,z} = \xi_{s,z}^{-1}\), at which point all of the parameters \(\Delta_{s,z}, B,\) and \(K\) cease to be wavevector dependent; (i.e., when the crossover between Eqs. \([VII.41]\) and \([VII.42]\) occurs). Then, as we continue lowering \(q\), at a smaller \(q\), a crossover from the \(u - u\) correlations being dominated by random compressions and tilts to being dominated by the random field term (i.e., crossover from the first to the second term dominating in equation \([VII.42]\)). Taking all this into account leads to different behaviors of \(\langle |u(\mathbf{q})|^2 \rangle\) in the six different regions of the \(q_h-q_z\) plane illustrated in Fig. 4.

The locus \(FG\), given by

\[
FG : q_s \xi_c^N = (q_s \xi_s^N)\zeta_c, \quad (VII.64)
\]

is where the crossover between the two limits of equation \([VII.63]\) occurs.

The locus \(EF\), given by

\[
EF : q_z = \xi_z^{-1}, \quad (VII.65)
\]

is where we crossover between expressions Eqs. \([VII.41]\) and \([VII.42]\) on the \(q_s \xi_c^N \gg (q_s \xi_s^N)\zeta_c\) side of \(FG\) occurs, while \(DF\), given by

\[
DF : q_s = \xi_s^{-1}, \quad (VII.66)
\]

plays the same role on the \(q_s \xi_c^N \ll (q_s \xi_s^N)\zeta_c\) side.

The locus \(OF\), given by

\[
OF : q_z = \xi_z q_s, \quad (VII.67)
\]

determines which of \(B(T)q_s^2\) and \(D(T)q_s^2\) dominates the denominators in Eq. \([VII.42]\).

The locus \(BC\), given by

\[
BC : q_z = (\xi_z^{-1})_s, \quad (VII.69)
\]

is where we crossover between the random compression and the random field in Eq. \([VII.42]\), while \(AC\), given by

\[
AC : q_s = (\xi_s^{-1})_z, \quad (VII.71)
\]

plays the same role between the random tilt and the random field.

In these six regions, the \(\mathbf{q}\)-dependence of \(\langle |u(\mathbf{q})|^2 \rangle\) is given by
with the random field piece \( (\xi_s \xi_z^N \eta)^{1/2} \) to the random tilt and random compression pieces, in the appropriate regime of (VII.42) to the random tilt and random compression.

The two characteristic lengths \( \xi_s \) and \( \xi_z \) are obtained by equating the random field piece \( (\sqrt{D(T)B(T)}\gamma(T)) \) of (VII.42) to the random tilt and random compression pieces, in the appropriate regime of \( \vec{q} \), and are given by

\[
\xi_s = \xi_s^N \left( \frac{\xi_s}{\xi_s^N} \right)^{\Gamma_s}, \quad \xi_z = \xi_z^N \left( \frac{\xi_z}{\xi_z^N} \right)^{\Gamma_z},
\]

with

\[
\Gamma_s = 2 - \eta \kappa + \eta \zeta + \frac{1}{2} \eta \beta, \quad \Gamma_z = 3 - \frac{3}{2} \eta \zeta + \eta \zeta
\]

Along locus (1) we fix \( q_s, h \) at 0 and vary \( q_s \). We predict this in turn implies for the light scattering:

\[
I(\vec{q}) \propto \left\{ \begin{array}{ll}
q_s^2 (q_z \xi_z^N)^{2\eta \kappa - \eta \eta - 6} & \text{region 1} \\
q_s^{2\eta \kappa - \eta \eta - 4} (\xi_s^N)^{2\eta \kappa - \eta \eta - 6} & \text{region 2} \\
q_s^2 (\xi_z^N)^{2\eta \kappa - \eta \eta - 6} (q_z \xi_z)^{2} & \text{region 3} \\
(\xi_s^N)^{2\eta \kappa - \eta \eta - 6} (\xi_z)^{2} & \text{region 4} \\
q_s^2 (\xi_z^N)^{1- 1/2 \eta \kappa - \eta \beta} (q_z \xi_z)^{3} & \text{region 5} \\
(\xi_s^N)^{\eta \kappa + 1/2 \eta \beta - 2} (\xi_z)^{3} & \text{region 6}
\end{array} \right.
\]

Now imagine experiments in which one varies the light scattering wavevector \( \vec{q} \) along the two different experimental loci we have indicated in Fig. 4.
Thus, a log-log plot of $I(q_s)$ verses $q_s$ will be a straight line with slope $2\eta_K - \eta_t - 4 \approx -3.5$ for $q_s \gg \xi_s^{-1}$, where in the approximate equality we have used our $\epsilon$-expansion estimates [110] and [112] for $\eta_K$ and $\eta_t$. This straight line is then replaced by a horizontal section for $(\xi'_s)^{-1} \ll q_s \ll \xi_s^{-1}$, followed by another straight section with slope $-1$ for smaller $q_s$'s. We can read off $\xi_s$ and $\xi'_s$ from the place where the slope changes. Fitting $\xi^{-1}_s$ to $(T - T_{AC})^{-\nu_s}$ determines $\nu_s$. Knowing $\xi_s$ and $\xi'_s$ allows us to calculate $\Gamma_s$ from Eq. [VII.80].

Along locus 2 we vary $q_z$ and keep $q_s$ fixed at a value in the range $0 \ll q_s \ll (\xi'_s)^{-1}$. We find

$$I(q_z) \propto \begin{cases} \left(\frac{q_z \xi'_N}{\xi'_c}\right)^{2\eta_K - \eta_t - 4}, & q_z \gg \xi'_z^{-1}, \\ \left(\frac{\xi'_z}{\xi'_c}\right)^{2\eta_K - \eta_t - 4}, & (\xi'_z)^{-1} \ll q_z \ll \xi'_z^{-1}, \\ \left(\frac{\xi'_z}{\xi'_c}\right)^{-1}, & q_z \ll (\xi'_z)^{-1}, \end{cases}$$

(VII.84)

Using an approach similar to that just described for locus 1, we can obtain $(2\eta_K - \eta_t - 4)/\xi_z$, $\xi'_z$, and hence $\nu_z$, and $\Gamma_z$. These results combined with those from locus 1 allow us to find $\eta_B, K, t$. Then plugging $\nu_s$ and $\eta_B, K, t$ into Eqs. [118] [120] can be examined.

Finally, since $\eta_B, K, t$, $\nu_{s, z}$, $\beta$, and $\alpha$ can all be measured experimentally, the exact scaling relations Eqs. [117] [118] [119] can be examined.

In the $C$ phase the relation between the light scattering intensity and the fluctuations is given by a formula similar to [VII.39]:

$$I(\vec{q}) \sim \sum_{ij} E'_{ij} C'_{ij}(\vec{q})$$

(VII.86)

where

$$C'_{ij}(\vec{q}) \equiv L_{ij}(\vec{q}) q_{\perp}^2 \langle |u'(\vec{q})|^2 \rangle.$$ (VII.87)

The coefficients $E'_{ij}$ are independent of $\vec{q}$. However, their value is affected by the tilt angle of the layer normals, and therefore has a weak dependence on the temperature. Since we have argued in Sec. VI that $\langle |u'(\vec{q})|^2 \rangle$ has the same expression as $\langle |u(\vec{q})|^2 \rangle$, we conclude that the $C$ phase has qualitatively the same light scattering behavior as the $A$ phase.

VIII. STABILITY OF THE SMECTIC PHASE AT THE CRITICAL POINT

All of our discussion so far has assumed that the smectic state remains stable right up to the AC critical point. There are two conceivable mechanisms by which this assumption might fail: first, the orientational order of the smectic layers could be destroyed; and second, the integrity of the smectic layers themselves could fail via the proliferation of dislocations in the smectic layers.

In this section, we argue that neither of these happens, and that the smectic state is stable right up to, and through, the AC transition.

Consider first the orientational order of the system. In deriving our model we have implicitly assumed that the both the real space fluctuations $\langle |\delta \hat{n}(\vec{r})|^2 \rangle$ of the nematic directors $\hat{n}(\vec{r})$, and those $\langle |\delta N(\vec{r})|^2 \rangle$ of the normals $N(\vec{r})$ to the smectic layers, are small, so that the expansions $\hat{n} = \hat{z} + \delta \hat{n}$ and $N = \hat{z} + \delta N$ are justified. Thus, for our theory to be valid, long-ranged orientational order must exist. To show that it does, even at the AC critical point, at least for sufficiently small disorder, we need to calculate $\langle |\delta \hat{n}(\vec{r})|^2 \rangle$ and $\langle |\delta N(\vec{r})|^2 \rangle$.

We can obtain both of these quantities by calculating $\langle |\nabla \cdot u(\vec{r})|^2 \rangle$, since, as shown in section [11], fluctuations of both $\delta N(\vec{r})$ and $\delta \hat{n}(\vec{r})$ away from $\nabla \cdot u$ are massive.
Hence, up to small corrections,  
\[ \langle (\delta \tilde{n}(\vec{r})^2) \rangle = \langle (\delta \tilde{N}(\vec{r})^2) \rangle = \langle (\nabla \cdot \vec{u})^2 \rangle . \]

The most problematical point is right at the AC transition, since there the fluctuations are biggest. Hence, if \( \langle (\delta \tilde{n}(\vec{r})^2) \rangle \) and \( \langle (\delta \tilde{N}(\vec{r})^2) \rangle \) are finite at this point, they’ll be finite everywhere.

We will now show that they are, indeed, finite right at the AC transition.

Because of the anisotropic scaling, \( \langle (\partial_s u(\vec{r})^2) \rangle \) is the dominant part of \( \langle (\nabla \cdot \vec{u})^2 \rangle \). Using our earlier equation \( \text{(VIII.9)} \) for \( \langle (u(\vec{q})^2) \rangle \), we obtain:

\[ \langle (\partial_s u(\vec{r})^2) \rangle = \frac{1}{(2\pi)^d} \int d^3q \frac{\Delta_s(\vec{q})q_s^2}{B(\vec{q})q_s^2 + \gamma q_s^4 + \kappa(\vec{q})q_s^2} \]

\[ \times L^{\Gamma'} \]  \( \text{(VIII.1)} \)

with

\[ \Gamma' = -1 + \eta_l + \eta_K + \frac{\eta u}{2}, \]  \( \text{(VIII.2)} \)

where \( L \) is the infrared cutoff in the \( s \) directions. Thus, \( \Gamma' < 0 \) is required for the convergence of \( \langle (\nabla \cdot \vec{u})^2 \rangle \) and hence the convergence of \( \langle (\delta \tilde{n}(\vec{r})^2) \rangle \). Clearly, in \( d = 3 \) our numerical estimate to first order in \( \epsilon \) does satisfy this condition. This shows that at the AC transition point, the smectic is stable against orientational fluctuations.

In previous sections we have shown that right at the critical point the long-ranged translational order is destroyed in the presence of the quenched disorder. Thus one might reasonably question what is the difference between the smectic and the isotropic (nematic) phases, and even worry whether the smectic phase survives at the AC critical point at all. However, there are examples in which the low-temperature and the high-temperature phases are not distinguished by the order of the system, but rather, by whether the topological defects are bound or unbound. The most famous one is the Kosterlitz-Thouless transition in the \( d = 2 \) XY model \( [30] \). Another famous example is the “Bragg glass” in pinned superconducting flux lines \( [18, 21, 32] \). We will now show that the same thing happens in our problem. Since similar calculations have been given in Refs \( [12, 17] \) in great detail, and ours are virtually the same, our description will be very brief.

The starting point of the theory is the tilt only harmonic Hamiltonian at the critical point:

\[ H = \int d^d\vec{r} \left[ \frac{K}{2} (\partial_s^2 u)^2 + \frac{B}{2} (\partial_s^2 u)^2 + \frac{\gamma}{2} (\partial_s u)^2 + \hbar \cdot \nabla \cdot \vec{u} \right]. \]  \( \text{(VIII.3)} \)

where the random field disorder is not included. It can be justified that the random field disorder is irrelevant when dislocations are included.

When the smectics contains dislocations, the displacement field \( u \) is no longer single valued. Mathematically this can be represented as

\[ \nabla \times \nabla u = \vec{m}, \]  \( \text{(VIII.4)} \)

with

\[ \vec{m}(\vec{r}) = \sum_i a_i M_i t(s_i) \delta^3 (\vec{r} - \vec{r}_i(s_i)) \, ds_i, \]  \( \text{(VIII.5)} \)

where \( s_i \) parameterizes the path of the \( i \)’th dislocation loop, \( M_i \) is an integer giving the charge of that loop, \( t(s_i) \) is the local tangent of the loop, and \( \vec{r}_i \) is a position of the point on the loop parameterized by \( s_i \) the loop. Furthermore, equation \( \text{(VIII.4)} \) implies

\[ \nabla \cdot \vec{M} = 0, \]  \( \text{(VIII.6)} \)

which means that dislocation lines can not end in the bulk of the sample.

To obtain a dislocation Hamiltonian, we need to trace over field \( u \) which is constrained by equation \( \text{(VIII.3)} \). This can be done in the following standard way. We separate the field \( \vec{u} = \vec{u}_d + \delta \vec{u}, \) \( \text{(VIII.7)} \)

where \( \vec{u}_d \) minimizes Hamiltonian \( \text{(VIII.3)} \) for a given dislocation configuration \( \vec{m}(\vec{r}) \). \( \delta \vec{u} \) can be viewed as the fluctuation from the ground state. Inserting \( \text{(VIII.7)} \) into Hamiltonian \( \text{(VIII.3)} \), we find that \( \vec{u}_d \) and \( \delta \vec{u} \) are decoupled due to the construction that \( \vec{u}_d \) minimizes Hamiltonian \( \text{(VIII.3)} \). Thus we obtain the effective model for \( \vec{m}(\vec{r}) \):

\[ H_d = \sum_{\vec{q}} \left[ \left( \frac{1}{2B^2} \frac{K}{\gamma} q_s^2 |m_s(\vec{q})|^2 + B K q_s^2 |m_h(\vec{q})|^2 \right) + B \gamma |m_s(\vec{q})|^2 \right] + \vec{m}(\vec{q}) \cdot \vec{a}(\vec{q}), \]  \( \text{(VIII.8)} \)

where \( \Gamma_q = q_s^2 + (\gamma/B) q_h^2, \) and

\[ \vec{a} = \int \left[ \frac{\vec{q} \times \vec{h}}{q^2} + \left( B q_z \delta + \gamma q_h \hat{h} + K q_s^2 s \right) \right] \frac{\vec{q} \cdot \vec{h}}{B T q^2} \]  \( \text{(VIII.9)} \)

By putting the model on a a simple cubic lattice, the partition function can be written as

\[ Z = \sum_{\vec{m}} e^{-S[\vec{m}]}, \]  \( \text{(VIII.10)} \)

where

\[ S[\vec{m}] = \frac{1}{T} \left[ H_d[\vec{m}] + \frac{E_q d^4}{a^2} \sum_{\vec{r}} |\vec{m}(\vec{r})|^2 \right], \]  \( \text{(VIII.11)} \)
where the $n_i$'s are integers, $d$ is the cubic lattice constants used in the discretization, and $\frac{2d}{a} \sum |\vec{m}(\vec{r})|^2$ is the core energy term coming from the core of the defect line that is not accurately treated by the continuum elastic theory.

To cope with the constraint $\nabla \cdot \vec{m} = 0$, we introduce an auxiliary field $\theta(\vec{r})$, rewriting the partition function equation \textbf{(VIII.10)} as

$$Z = \prod_{\vec{r}} \int d\theta(\vec{r}) \sum_{\vec{m}(\vec{r})} e^{-S[\vec{m},\theta,\vec{A}]} \delta(\nabla \cdot \vec{A}) \delta(A_z) \quad \text{(VIII.13)}$$

where the sum over $\vec{m}(\vec{r})$ is now unconstrained.

Then we introduce a dummy vector field $\vec{A}$ to mediate the long-range interaction between defect loops in the Hamiltonian \textbf{(VIII.8)}. This is accomplished by rewriting the partition function as

$$Z = \prod_{\vec{r}} \int d\theta(\vec{r}) d\vec{A}(\vec{r}) \sum_{\vec{m}(\vec{r})} e^{-S[\vec{m},\theta,\vec{A}]} \delta(\nabla \cdot \vec{A}) \delta(A_z) \quad \text{(VIII.14)}$$

with

$$S = \frac{1}{T} \sum_{\vec{r}} \left[ \vec{m} \cdot \left( -T \frac{\partial^2}{\partial \theta^2} \theta(\vec{r}) + d^2 [i\vec{A}(\vec{r}) + \vec{a}(\vec{r})] \right) ight.$$

$$\left. + E_c d^4 |\vec{m}|^2 \right] + \frac{1}{2T} \sum_{\vec{r}} \left( \frac{BT}{Kq^2_s} |A_z(\vec{q})|^2 + \frac{\Gamma_q}{Kq^2_s} |A_h(\vec{q})|^2 + \frac{\Gamma_\gamma}{\gamma} |A_s(\vec{q})|^2 \right). \quad \text{(VIII.15)}$$

The sum over $\vec{m}(\vec{r})$ is just the “periodic Gaussian” introduced by Villain \cite{34}. Then the partition function equation \textbf{(VIII.14)} can be rewritten as

$$Z = \prod_{\vec{r}} \int d\theta(\vec{r}) d\vec{A}(\vec{r}) \delta(\nabla \cdot \vec{A}) \delta(A_z)$$

$$\times \exp \left[ -\sum_{\vec{r}} V_p \left[ \theta(\vec{r} + \hat{\vec{r}}) - \theta(\vec{r}) - \frac{ad}{T} |A_1(\vec{r})| \right. 
$$

$$\left. - i\alpha(\vec{r}) \right] - \frac{1}{2T} \sum_{\vec{q}} \left( \frac{BT_q}{Kq^2_s} |A_z(\vec{q})|^2 + \frac{\Gamma_q}{Kq^2_s} |A_h(\vec{q})|^2 + \frac{\Gamma_\gamma}{\gamma} |A_s(\vec{q})|^2 \right] \right], \quad \text{(VIII.16)}$$

where the 2\pi-period Villain potential $V_p(x)$ is defined as

$$e^{-V_p(x)} = \sum_{n=-\infty}^{\infty} e^{-n^2 E_c/T + i\pi n}. \quad \text{(VIII.17)}$$

Since $V_p(x)$ has sharper minima for smaller $E_c/T$, which corresponds to higher temperature, raising the temperature in the original model is equivalent to lowering temperature in the dual model equation \textbf{(VIII.16)}.

Standard universality class arguments imply that the model equation \textbf{(VIII.16)} has the same universality class as the “soft spin”, or Landau-Ginsburg-Wilson, with the complex “action”

$$S_r = \sum_{r,\alpha} \left[ \frac{c}{2} \left( \vec{\nabla}_r + \frac{ad}{T} (i\vec{A}_r + \vec{a}) \right) \psi_\alpha \cdot \left( \vec{\nabla}_r - \frac{ad}{T} (i\vec{A}_r + \vec{a}) \right) \psi_\alpha + t_d |\psi_\alpha|^2 + u_d |\psi_\alpha|^4 \right] + \sum_{\vec{q}} \left( \frac{1}{\Gamma_q} |A_z(\vec{q})|^2 \right.$$

$$\left. + \frac{1}{\gamma} |A_s(\vec{q})|^2 \right), \quad \text{(VIII.18)}$$

where $\psi$ is a complex order parameter whose phase is $\theta(\vec{r})$. Because of the duality transformation’s inversion of the temperature axis, the reduced temperature $t_d$ is a monotonically decreasing function of the temperature $T$ (of the original dislocation loop model), which vanishes at the mean-field transition temperature of the dislocation model \textbf{(VIII.16)}. Disorder is included in the model \textbf{(VIII.18)} through $\vec{a}(\vec{r})$, which is related to the random tilt field $\hat{h}(\vec{r})$ by equation \textbf{(VIII.9)}.

Because of the duality inversion of the temperature axis, the ordered phase of the dual model \textbf{(VIII.18)} corresponds to the disordered phase (dislocation loops unbound) of the original dislocation model.

An complete analysis of the dislocation loop unbinding transition described by the model \textbf{(VIII.18)} is beyond the scope of this paper. The goal here is to know if a dislocation bound state ever exists near the critical point of the $AC$ transition. Let us check the one-loop graphical correction to the dual temperature:

$$t_R = t_0 - \frac{ca^2d^2}{T^2} \int \frac{d^3 \vec{q}}{(2\pi)^3} \left( \frac{\Delta_t (B^2 q_s^2 + \gamma_0^2 q_0^2)}{B^2 \Gamma_q} \right). \quad \text{(VIII.19)}$$

If $K$ and $\Delta_t$ are treated as constant it is easy to show that the integral diverges. This result, if true, would imply that the smectic phase becomes unstable against dislocation unbinding sufficiently close to the putative AC transition.

However, this conclusion only holds within the harmonic approximation. In sections \textbf{III} and \textbf{IV} we have shown that anharmonic effects are important. A crude way to include their effects is to replace the constant $K$ and $\Delta$ in the harmonic calculation above with the anomalous, wavevector-dependent $K(\vec{q})$ and $\Delta(\vec{q})$ found earlier in section \textbf{IV}. Implementing this in the integral and requiring it to converge, we obtain a restriction on $\eta_K$ and $\eta_t$:

$$1 - \eta_t - \frac{\eta_B}{2} > 0. \quad \text{(VIII.20)}$$

This condition is satisfied by the values of $\eta_K$ and $\eta_t$ obtained by the $\epsilon$-expansion to $O(\epsilon)$ in the physical dimension $d = 3$. This implies that the second-order smectic $A$-$C$ phase transition is stable against dislocation unbinding.
IX. CONCLUSION

We have developed a theory of the smectic-A to smectic-C phase transition in a biaxial disordered environment. Using an $\epsilon = \frac{1}{2} - d$ expansion, we have shown that the phase transition can be second order. The critical exponents were calculated to first order in $\epsilon$. We also calculated the smectic correlation functions near the critical point, which lead to exotic X-ray and light scattering patterns. All these predictions should be testable provided that the biaxial disordered environment can be realized in experiments.

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Appendix A:

There are two ways of continuing our model Eq. (1.11) from three dimensions to higher spatial dimensions. In the main text we have focused on the “hard” continuation, in which we keep the h-space $d-2$ dimensional and the $s$-space one dimensional. In this appendix we briefly describe the calculation which uses the soft continuation, in which we keep the h-space one dimensional and the $s$-space $d-2$ dimensional.

The RG procedure is essentially the same as that used in the hard continuation. After integrating out the freedom of the fast varying part of $u(\vec{r})$ and rescaling, we obtain the following RG flow equations:

\begin{align}
\frac{d\gamma}{d\ell} &= [d - \omega_h + \omega_z + 2\chi - 2] \gamma, \quad (A.1) \\
\frac{dB}{d\ell} &= \left[d + \omega_h - \omega_z + 2\chi - 2 - \frac{1}{4}\tilde{g}_1\right] B, \quad (A.2) \\
\frac{dK}{d\ell} &= \left[d + \omega_h + \omega_z + 2\chi - 6 + \frac{1}{8}\tilde{g}_1\right] K, \quad (A.3) \\
\frac{dg}{d\ell} &= \left[d + \omega_h + 3\chi - 4 + \frac{1}{4}\tilde{g}_1 - \frac{1}{2}\tilde{g}_2\right] g, \quad (A.4) \\
\frac{dw}{d\ell} &= \left[d + \omega_h + \omega_z + 4\chi - 6 - \frac{3}{8}\tilde{g}_1\right] w \\
&\quad + \left[\frac{5}{4}\tilde{g}_1 - \frac{9}{8}\tilde{g}_2\right] w, \quad (A.5) \\
\frac{d\Delta_t}{d\ell} &= \left[d + \omega_h + \omega_z + 2\chi - 4 + \frac{1}{24}\tilde{g}_1\right] \Delta_t, \quad (A.6) \\
\frac{dD}{d\ell} &= \left[d + \omega_h + \omega_z + 2\chi - 4 + \frac{3}{8}(\tilde{g}_1 - \tilde{g}_2)\right] D \\
&\quad + \frac{3}{8}K(\tilde{g}_2 - \tilde{g}_1), \quad (A.7)
\end{align}

where $\tilde{g}_1$ and $\tilde{g}_2$ are two dimensionless couplings defined by:

\begin{align}
\tilde{g}_1 &= C_d \tilde{g}(B/\gamma)^{1/2}K^{-2}\Lambda^{d-4}, \quad (A.8) \\
\tilde{g}_2 &= C_d \tilde{g}(w/B)\Delta_t(B/\gamma)^{1/2}K^{-2}\Lambda^{d-4}. \quad (A.9)
\end{align}

These couplings $\tilde{g}_1$ and $\tilde{g}_2$ flow according to

\begin{align}
\frac{d\tilde{g}_1}{d\ell} &= \tilde{\epsilon} + \frac{2}{3}\tilde{g}_1 - \tilde{g}_2, \quad (A.10) \\
\frac{d\tilde{g}_2}{d\ell} &= \tilde{\epsilon} - \frac{3}{8}\tilde{g}_1^2 - \frac{9}{8}\tilde{g}_2 + \frac{7}{6}\tilde{g}_1, \quad (A.11)
\end{align}

where $\tilde{\epsilon} = 4 - d$.

Once again, it is convenient but not necessary to make a special choice of $\chi$, $\omega_h$, and $\omega_z$ such that $B$, $K$, and $\gamma$ are fixed at their bare values. This requirement can only be fulfilled by

\begin{align}
\omega_h &= 2 - \frac{1}{16}\tilde{g}_1, \quad (A.12) \\
\omega_z &= 2 - \frac{3}{16}\tilde{g}_1, \quad (A.13) \\
\chi &= 2 - \frac{d}{2} + \frac{1}{16}\tilde{g}_1. \quad (A.14)
\end{align}

This choice turns flow Eq. (A.7) into

\begin{align}
\frac{dD}{d\ell} &= \left[2 - \frac{1}{8}\tilde{g}_1 + \frac{3}{8}(\tilde{g}_1 - \tilde{g}_2)\right] D + \frac{3}{8}K(\tilde{g}_2 - \tilde{g}_1). \quad (A.15)
\end{align}

Flow Eqs. (A.7, A.10, A.11) have three fixed points in total. One of them is the trivial Gaussian fixed point.
Another one was also found previously for the $m = 1$ Bragg Glass problem\textsuperscript{17}, which preserves the symmetry that the Hamiltonian is invariant under any rigid rotation of the liquid crystal about the $h$ axis. This fixed point is:

$$
\tilde{g}_1^* = \tilde{g}_2^* = 3\tilde{\epsilon} + O(\tilde{\epsilon}^2), \quad (A.16)
$$

$$
D^* = 0, \quad (A.17)
$$

Both of these fixed points are unstable in two or more directions.

The third fixed point is:

$$
\tilde{g}_1^* = \frac{3}{7} \tilde{\epsilon} + O(\tilde{\epsilon}^2), \quad (A.18)
$$

$$
\tilde{g}_2^* = \frac{9}{7} \tilde{\epsilon} + O(\tilde{\epsilon}^2), \quad (A.19)
$$

$$
D^* = -\frac{9K}{56} \tilde{\epsilon} + O(\tilde{\epsilon}^2), \quad (A.20)
$$

which is stable in two directions, and unstable in a third. Therefore, this fixed point controls the critical behavior of the phase transition.

With this fixed point in hand, the anomalous elasticity and the critical exponents can be calculated by following the steps described in Secs. \[\text{IV} \text{ and V}\]. Here we only give the results:

$$
\tilde{\eta}_B = \frac{3}{28} \tilde{\epsilon} + O(\tilde{\epsilon}^2), \quad (A.21)
$$

$$
\tilde{\eta}_K = \frac{3}{56} \tilde{\epsilon} + O(\tilde{\epsilon}^2), \quad (A.22)
$$

$$
\tilde{\eta}_t = \frac{1}{56} \tilde{\epsilon} + O(\tilde{\epsilon}^2), \quad (A.23)
$$

$$
\tilde{\nu}_s = \frac{1}{2} + \frac{3}{32} \tilde{\epsilon} + O(\tilde{\epsilon}^2), \quad (A.24)
$$

$$
\tilde{\nu}_h = 1 + \frac{39}{224} \tilde{\epsilon} + O(\tilde{\epsilon}^2), \quad (A.25)
$$

$$
\tilde{\nu}_z = 1 + \frac{33}{224} \tilde{\epsilon} + O(\tilde{\epsilon}^2), \quad (A.26)
$$

$$
\tilde{\beta} = \frac{1}{2} - \frac{9}{32} \tilde{\epsilon} + O(\tilde{\epsilon}^2), \quad (A.27)
$$

$$
\tilde{\alpha} = \frac{9}{56} \tilde{\epsilon} + O(\tilde{\epsilon}^2). \quad (A.28)
$$

Now we compare the numerical results obtained from the two continuations. In $d = 3$, to first order in $\epsilon$ and $\tilde{\epsilon}$ we find, for the hard continuation discussed earlier,

$$
\eta_B = 0.2222, \quad (A.29)
$$

$$
\eta_K = 0.2963, \quad (A.30)
$$

$$
\eta_t = 0.07407, \quad (A.31)
$$

$$
\nu_s = 0.6296, \quad (A.32)
$$

$$
\nu_h = 1.1852, \quad (A.33)
$$

$$
\nu_z = 1.1296, \quad (A.34)
$$

$$
\beta = 0.3889, \quad (A.35)
$$

$$
\alpha = 0.1111. \quad (A.36)
$$

while for the soft continuation treated in this appendix,

$$
\tilde{\eta}_B = 0.1071, \quad (A.37)
$$

$$
\tilde{\eta}_K = 0.05357, \quad (A.38)
$$

$$
\tilde{\eta}_t = 0.01786, \quad (A.39)
$$

$$
\tilde{\nu}_s = 0.5938, \quad (A.40)
$$

$$
\tilde{\nu}_h = 1.1741, \quad (A.41)
$$

$$
\tilde{\nu}_z = 1.1473, \quad (A.42)
$$

$$
\tilde{\beta} = 0.2188, \quad (A.43)
$$

$$
\tilde{\alpha} = 0.1607, \quad (A.44)
$$

Unfortunately, the two sets of results are not in good agreement. We trust the hard continuation more, since $\epsilon$ is smaller ($\epsilon = \frac{1}{2}$) for that configuration than the expansion parameter $\tilde{\epsilon} = 1$ for the soft continuation.

**Appendix B: Wavevector-dependences of Irrelevant Disorder Variances**

In our RG calculations in Section \[\text{III} \text{ A} \] we did not include in the Hamiltonian \[\text{III} \text{ B} \] the random compression disorder

$$
-\frac{\Delta_c}{2T} \int d^d r \sum_{\alpha,\beta=1}^n \partial_z u_\alpha \cdot \partial_z u_\beta, \quad (B.1)
$$

because simple power counting shows that this term is irrelevant. However, while it has no effect on the critical exponents, the disorder invariance $\Delta_c$ itself develops strong power-law dependence on wavevector at long wavelengths. We need to know this power-law exponent to calculate the fluctuations caused by the random compression, which have a significant effect on the light scattering intensity.

The most important one-loop graphical correction to $\Delta_c$ comes from the Feynman diagram in Fig. 7 from...
which we obtain
\[
\Delta_c = \frac{g^2}{2} \int_{p}^{\infty} \frac{\Delta_{c}^{2} p^{\delta} G(p^2)}{2} \int_{-2}^{\infty} dp_z \int_{-\infty}^{\infty} \frac{d^{d-2} p_h}{(2\pi)^{d-2}} \int_{-\infty}^{\infty} \frac{dp_x}{(2\pi)^{d-2}} \frac{g_{c}}{\lambda^{2}} G(p^2) \Delta_{c},
\]
with the anomalous exponent
\[
\eta_c = 2 - \frac{8}{9} + O(\epsilon^2). \tag{B.8}
\]
Here \(f_c(X, Y)\) is another crossover function of the scaling variables \(X\) and \(Y\) given by \([\text{IV.10}]\) and \([\text{IV.11}]\), respectively. Note that \(\eta_c\) is nonzero even to zeroth order in \(\epsilon\), which is quite common for irrelevant variables.

We can also obtain Eq. \([\text{B.3}]\) through an exact scaling relation between \(\eta_c\) and other anomalous exponents. This scaling relation is implied by the fact that \(g_3\) flows to a non-zero stable fixed point. For large enough \(\ell\), \(g_3\) reaches a fixed point and thus
\[
\frac{d\ln g_3}{d\ell} = 0. \tag{B.9}
\]
This equation, after rewriting \(g_3\) in terms of other parameters using its definition \([\text{B.4}]\), leads to
\[
\frac{d\ln B}{d\ell} + \frac{d\ln \Delta_c}{d\ell} + \frac{d\ln g_1}{d\ell} - \frac{d\ln K}{d\ell} - \frac{d\ln \Delta_c}{d\ell} = 0. \tag{B.10}
\]
Then we replace the graphical part in flow Eqs. \([\text{III.11}]\) \([\text{III.12}]\) \([\text{III.15}]\) \([\text{III.19}]\) by \(-\eta_B, \eta_K, t, c\), respectively, and plug them and Eq. \([\text{III.19}]\) into the above equation. The rescaling factors \(\chi\) and \(\omega\) vanish, and we are left with
\[
\eta_c = 2 + \eta_B - \eta_B - \eta_K. \tag{B.11}
\]
Using the already known values of \(\eta_B\) and \(\eta_K\) to \(O(\epsilon)\), using \([\text{B.11}]\) also leads to the epsilon expansion for \(\eta_c\); reassuringly, this expansion is exactly \([\text{B.3}]\).

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Since aerogel tends to be rather brittle, it may be more practical to prepare aerogel with the desired type of anisotropy by growing it anisotropically, rather than by stretching and compressing an already formed isotropic aerogel. All that matters is that the final aerogel state be the same as one constructed by a gedanken experiment in which an originally isotropic aerogel is stretched and compressed in the manner we described here. This has already been achieved experimentally for a liquid crystal compressed in the manner we described here. This has its applications, due in that case to thermal fluctuations, but with a non-universal (and, indeed, temperature-dependent) exponent. Furthermore, the clean smectic peaks exhibit anisotropic scaling, while the RFXY peaks we describe here do not. See references [22].

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[26] One easy way to see this is to imagine that the random field couples to the magnetization via a conventional external field-magnetization coupling Hamiltonian $H_{\text{field}} = -\frac{1}{2} \int d^d r \left[ \hat{h}(\vec{r}) \cdot \hat{M}(\vec{r}) \right]$, with the $\hat{h}(\vec{r})$'s all of the same magnitude $|\vec{h}(\vec{r})| = h$, but oriented completely randomly. This then leads to a model precisely of the form (VI.5), with $V^{XY}_{\phi} (\theta - \phi_{XY}(\vec{r})) = -h \cos(\theta - \phi_{XY}(\vec{r}))$, and $\phi_{XY}(\vec{r})$ just the angle between the random field at point $\vec{r}$ and some reference direction.

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[28] One could more generally take $U(y) = Cy^4$, where $C = O(1)$; we can, without loss of generality, absorb this $O(1)$ constant into that in our definition (VI.20) of $k(\vec{r})$, and we choose to do so.

[29] Equation (VI.24) also has a solution $x = -\left(\frac{C}{4}\right)^{\frac{1}{3}} i$; we ignore this because it proves to not be a saddle point.

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