10-VERTEX GRAPHS WITH CYCLIC AUTOMORPHISM GROUP
OF ORDER 4

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Abstract. We describe computational results about undirected graphs having 10 vertices and
automorphism group isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

1. Introduction.

This paper deals with a special case of the problem of finding undirected graphs
having a given automorphism group and minimal number of vertices. All graphs in
this paper are undirected.

Studies of automorphism groups of graphs started in 1930s with the classical
results of Frucht [5] who proved in the late 1930s that for any finite group $G$
there is a graph $\Gamma = (V, E)$ such that $\text{Aut}(\Gamma) \cong G$. In the 1970s it was proved by Babai
[1] in a constructive way that for any finite group $G$ there is a graph $\Gamma$
such that $\text{Aut}(\Gamma) \cong G$ and $|V(\Gamma)| \leq 2|G|$ if $G$ is not cyclic of order 3, 4 or 5. An estimate
$|V(\Gamma)| \leq 3|G|$ and a construction in the three exceptional cases was obtained by
Sabidussi [7]. Examples of graphs with $3n$ vertices and cyclic automorphism group
$\mathbb{Z}/n\mathbb{Z}$ are easy to construct and widely known since 1960s, see [6]. We can mention
that there are 4 isomorphism types of graphs with 9 vertices which form 2 isomorphism
types up to complementarity. See Babai [2] for a comprehensive exposition of this
area.

Although it has been mentioned in the literature that 10-vertex graphs with
cyclic automorphism group of order 4 do exist, see [4], we could not find details about
such graphs in research surveys or textbooks. Therefore we present a short note
summarizing computational results related to this problem.

We use standard notations of graph theory, see Diestel [4]. For a graph $\Gamma = (V, E)$
the subgraph induced by $X \subseteq V$ is denoted by $\Gamma[X]$.

2. Main computational results.

Denote by $F$ the set of isomorphism classes of graphs $\Gamma = (V, E)$ such that
$|V| = 10$ and $\text{Aut}(\Gamma) \cong \mathbb{Z}/4\mathbb{Z}$. By default we mean that isomorphism types of graphs

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considered in this paper belong to $F$.

**Proposition 2.1.** Let $\Gamma \in F$.

1. $|F| = 12$. Elements of $F$ form 6 isomorphism classes up to complementarity.
2. $18 \leq |E(\Gamma)| \leq 27$.
3. $3 \leq \delta(\Gamma) \leq 5$, $4 \leq \Delta(\Gamma) \leq 6$ (minimal and maximal degree).
4. $\text{girth}(\Gamma) = 3$.
5. $3 \leq \omega(\Gamma) \leq 4$ (clique number).
6. $3 \leq \kappa(\Gamma) \leq 5$, $\kappa(\Gamma) = \lambda(\Gamma)$ (vertex and edge connectivity).
7. $2 \leq \text{diam}(\Gamma) \leq 3$.
8. $3 \leq \chi(\Gamma) \leq 4$ (chromatic number).
9. $F$ contains one planar graph.
10. $F$ contains one Eulerian graph.
11. All graphs in $F$ are Hamiltonian.
12. None of graphs in $F$ is vertex, edge or distance transitive.
13. There are no graphs having less than 10 vertices and automorphism group isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

**Proof.** All statement are proved by direct computation. \[\square\]

**Cases.**

We describe two elements of $F$.

**The planar graph.**

The only planar graph $\Gamma_1 \in F$ is shown in Fig.1. It can be thought as embedded in the 3D space, a plane embedding is not given. $\text{Aut}(\Gamma_1)$ is generated by the vertex permutation $g = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10)$.

Subgraphs $\Gamma_1[1, 2, 3, 4, 5, 7, 9]$ and $\Gamma_1[1, 2, 3, 4, 6, 8, 10]$ which can be thought as
being drawn above and below the orbit $\Gamma_1[1, 2, 3, 4]$ are interchanged by $g$.

The graph with minimal number of edges.

The graph $\Gamma_2 \in F$ with minimal number of edges (18 edges) is shown in Fig. 2. $\text{Aut}(\Gamma_2)$ is generated by the vertex permutation $g = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10)$.

Other graphs.

All other graphs in $F$ can be obtained starting from $\Gamma_1$ or $\Gamma_2$ and adding or removing edges in $\Gamma_2[1, 2, 3, 4]$, the edge $(9, 10)$ and edges in $\Gamma_2[5, 6, 7, 8]$. 

Fig. 1. - $\Gamma_1$ the planar graph in $F$.

Fig. 2. - $\Gamma_2$ - the graph in $F$ with minimal number of edges.
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