Engineering frequency-dependent superfluidity in Bose-Fermi mixtures

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Abstract

Unconventional superconductivity or superfluidity are among the most exciting and fascinating quantum states in condensed matter physics. Usually these states are characterized by non-trivial spatial symmetry of the pairing order parameter, such as in $^3\text{He}$ and high-$T_c$ cuprates. Besides spatial dependence the order parameter could have unconventional frequency dependence, which is also allowed by Fermi-Dirac statistics. For instance, odd-frequency pairing is an exciting paradigm when discussing exotic superfluidity or superconductivity and is yet to be realized in the experiments. In this paper we propose a symmetry-based method of controlling frequency dependence of the pairing order parameter via manipulating the inversion symmetry of the system. First, a toy model is introduced to illustrate that frequency dependence of the order parameter can be adjusted by controlling the inversion symmetry of the system. Second, taking advantage of the recent rapid developments of shaken optical lattices in ultracold gases, we propose a Bose-Fermi mixture to realize such frequency dependent superfluids. The key idea is introducing the frequency-dependent attraction between Fermions mediated by Bogoliubov phonons with asymmetric dispersion. Our proposal should pave an alternative way for exploring frequency-dependent superconductors or superfluids with cold atoms.

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I. INTRODUCTION

Symmetry plays an important role in physics of superconductors or superfluids and influences their properties in a profound way. For example, research on unconventional superconductors where the pairing order parameter has non-trivial spatial symmetry, such as spin-singlet $d$-wave in cuprates and spin-triplet $p$-wave in $Sr_2RuO_4$ has generated tremendous interest in strongly correlated electron systems. In addition to spatial dependence the order parameter may also depend on frequency. Indeed, in models describing the electron-electron attraction induced by interaction with lattice phonons (Frohlich Hamiltonian) the effective electron-electron attraction, and therefore the gap, are frequency-dependent \[1\]. The gap happens to be even in frequency but this does not have to be the case in general. In fact the idea of pairing order parameter with odd frequency dependence was proposed by V.L. Berezinskii in 1974 \[2\]. This Berezinskii conjecture has attracted considerable attention, in particular with respect to some exotic phenomena like the anomalous proximity effect in superconductor-ferromagnet junctions \[3\].

However the question of controlling the frequency dependence of the pairing order parameter remains unresolved. The idea of symmetry breaking may offer a clue: indeed, it has been skillfully used recently to obtain exotic superconducting and superfluid ordering in the ultracold atomic setting \[4–6\]. In this paper we report on a method of engineering frequency-dependent superconductors or superfluids based on manipulating the inversion symmetry. The main idea can be understood through the following symmetry argument. The superconducting order parameter is defined by a non-vanishing value of the anomalous correlator

$$F(\alpha, \alpha') = \langle \psi(\alpha)\psi(\alpha') \rangle,$$  \hspace{1cm} (1)

where $\alpha, \alpha'$ stand for the internal degrees of freedom, such as coordinate, time, spin and band indices of the Fermi field $\psi$ and $\langle ... \rangle$ stands for the ground state or thermal state average. Let us assume that fermions pair up in a spin-singlet state (as is the case for the BCS model \[7\]). According to Fermi-Dirac statistics $F(\alpha, \alpha')$ should be antisymmetric with respect to exchange of $\alpha$ and $\alpha'$. Spin-singlet state is already antisymmetric therefore the remaining part of the anomalous correlator should satisfy

$$F(q, \omega_n) = F(-q, -\omega_n),$$  \hspace{1cm} (2)
where \( q \) and \( \omega_n \) are the momentum and Matsubara frequency. In order to control the frequency dependence of the pairing we suggest manipulating the inversion symmetry of the system. Indeed, if the system is inversion-symmetric then

\[
F(q, \omega_n) = F(-q, \omega_n),
\]

which according to Eq. (2) makes it also even in frequency,

\[
F(q, \omega_n) = F(q, -\omega_n).
\]

By introducing the inversion symmetry breaking the odd-frequency superconductivity

\[
F(q, \omega_n) = -F(-q, \omega_n),
\]

\[
F(q, \omega_n) = -F(q, -\omega_n)
\]

may be feasible. In the following we will first use a toy model to show how this idea works. Then a cold atom based experiment is proposed.

II. TOY MODEL

Let us first consider a 1D toy model of a Bose-Fermi mixture:

\[
\hat{H} = \hat{H}_F + \hat{H}_B + \hat{H}_{BF}.
\]

Here for simplicity we assume that \( \hat{H}_F \) describes non-interacting spin-1/2 fermions,

\[
\hat{H}_F = \sum_{k,\sigma} (\epsilon_F(k) - \mu_F) \hat{f}^\dagger_{k,\sigma} \hat{f}_{k,\sigma},
\]

and \( \hat{H}_B \) describes spinless bosons,

\[
\hat{H}_B = \sum_q (\epsilon_B(q) - \mu_B) \hat{b}^\dagger_q \hat{b}_q.
\]

The key idea for controlling the frequency dependence of superconducting order parameter relies on introducing the inversion symmetry breaking via asymmetric energy dispersion of bosons, \( \epsilon_B(-q) \neq \epsilon_B(q) \), that will be illustrated in detail below. Finally, \( \hat{H}_{BF} \) describes the fermion-boson interaction,

\[
\hat{H}_{BF} = \frac{U_{BF}}{\sqrt{N}} \sum_q \hat{\rho}_q \left( \hat{b}_{-q} + \hat{b}^\dagger_q \right)
\]
where \( \hat{\rho}_q = N^{-1} \sum_{k,\sigma} \hat{f}_{k,\sigma}^\dagger \hat{f}_{q+k,\sigma} \) stands for the Fourier transform of fermion density and \( N \) is the total number of lattice sites. In this toy model we can integrate out the bosons exactly using the path integral formalism, see Appendix A for details, and the resulting induced density-density type interaction between fermions is given by

\[
V_{\text{Ind}}(q, \omega_n) = -\frac{U_{\text{BF}}^2}{2} \frac{i \epsilon_B(q) + \epsilon_B(-q) - 2 \mu_B}{\omega_n^2 + \epsilon_B(q) (\epsilon_B(-q) - \mu_B) + i \omega_n (\epsilon_B(q) - \epsilon_B(-q))},
\]

where \( \omega_n = (2n+1)\pi T \) are Matsubara frequencies and \( \hbar = 1, k_B = 1 \) throughout the paper.

To demonstrate how the inversion symmetry breaking can be used to control frequency dependence of the order parameter we assume the following dispersion for bosons

\[
\epsilon_B(q) = \begin{cases} 
c_R |q|, & q \geq 0 
c_L |q|, & q < 0 
\end{cases}
\]

as a concrete example. We employ Eliashberg equations \cite{8} to find the superconducting gap \( \Delta(k, \omega_n) \) as a function of momentum and frequency:

\[
\Delta(k, \omega_n) = -\frac{T}{N} \sum_{k',n'} \frac{V_{\text{Ind}}(k - k', \omega_n - \omega_{n'}) \Delta(k', \omega_{n'})}{\omega_{n'}^2 Z^2(k', \omega_{n'}) + \xi^2(k', \omega_{n'}) + |\Delta(k', \omega_{n'})|^2},
\]

\[
(1 - Z(k, \omega_n)) i \omega_n = \frac{T}{N} \sum_{k',n'} \frac{V_{\text{Ind}}(k - k', \omega_n - \omega_{n'}) i \omega_{n'} Z(k', \omega_{n'})}{\omega_{n'}^2 Z^2(k', \omega_{n'}) + \xi^2(k', \omega_{n'}) + |\Delta(k', \omega_{n'})|^2},
\]

\[
\chi(k, \omega_n) = \frac{T}{N} \sum_{k',n'} \frac{V_{\text{Ind}}(k - k', \omega_n - \omega_{n'}) \xi(k', \omega_{n'})}{\omega_{n'}^2 Z^2(k', \omega_{n'}) + \xi^2(k', \omega_{n'}) + |\Delta(k', \omega_{n'})|^2},
\]

where \( \xi(k, \omega_n) = \epsilon_F(k) + \chi(k, \omega_n) \), \( Z(k, \omega_n) \) is the fermion mass renormalization and \( \chi(k, \omega_n) \) the renormalization of the chemical potential. To simplify the problem, here we consider 1D case and assume the fermions to have s-band-like dispersion, \( \epsilon_F(k) = -t_F \cos(ka) \), \( k \) being lattice momentum and \( a \) the lattice constant.

As shown in Fig. 1, when bosons have asymmetric dispersion (i.e. \( c_R/c_L \neq 1 \)) and the inversion symmetry is broken, the odd-frequency component of the superconducting gap emerges. Furthermore, as shown on Fig. 2, the frequency-dependence of the gap can be controlled by manipulating the degree of asymmetry, i.e. the ratio \( c_R/c_L \). It shows that the stronger the boson dispersion asymmetry the more favorable is the odd-frequency component of the superconducting gap. We note that the superconducting pairing considered here is in the spin-singlet channel, since the spin-singlet pairing always has lower free energy compared to the spin-triplet counterpart, as confirmed numerically.
Figure 1. Odd-frequency component of the superconducting gap, in the units of $t_F$, for the case of non-isotropic dispersion with $c_R/c_L = 3$ and $c_L/a = t_F$, $\mu_B = -0.005t_F$, $t_F = 1$, $\mu_F = 0$, $U = t_F$, $T = 0.001t_F$. Lattice momentum is plotted in units of $\pi/a$, $a$ being the lattice constant. Throughout the paper $k_B = 1$, $\hbar = 1$.

III. SUPERCONDUCTING ORDERING VIA PAIRING WITH SHAKEN BOSONS

In the following we explore the possibility of realizing frequency-dependent superfluidity via a cold-atom based system. Let us consider a Bose-Fermi mixture in a 1D optical superlattice as shown in Fig. 3 and further consider the bosons to be in a shaken lattice. Lattice shaking will change the energy dispersion of bosons to a double-dip shape, see Appendix B. By further assuming that the bosons are in the plane-wave state (i.e. that condensation in one of the minima of the double-dip dispersion has occurred), the induced attraction between fermions can be obtained as follows.
Figure 2. $\frac{\max |\Delta_{odd}|}{\max |\Delta_{even}|}$ plotted as a function of $r$, where $c_R/c_L = \frac{1+r}{1-r}$.

Figure 3. Boson (blue, deeper lattice) and fermion (orange, shallower lattice) mixture in a 1D superlattice. The scheme could use alkali-earth atoms in different orbital states, $^1S_0$ and $^3P_0$. The Bose-Fermi interactions can be tuned using orbital Feshbach resonance techniques, e.g. see [9], [10] and [11].
We start with the Hamiltonian which is a generalization of the Bose-Hubbard model where the momentum-space dispersion of free particles $\epsilon_B(k)$ has a double-dip (see details in Appendix B):

$$\hat{H}_B = \sum_k (\epsilon_B(k) - \mu_B) \hat{b}_k^\dagger \hat{b}_k + \frac{g}{2N} \sum_{k,k',q} \hat{b}_{k+q} \hat{b}_{k'-q} \hat{b}_{k'} \hat{b}_k,$$

(13)

$k$ is lattice momentum in 1D and $\hat{b}_k^\dagger, \hat{b}_k$-s are the bosonic creation/annihilation operators. The fermions are described by

$$\hat{H}_F = \sum_{k,\sigma} (\epsilon_F(k) - \mu_F) \hat{f}_{k,\sigma}^\dagger \hat{f}_{k,\sigma},$$

(14)

where possible existing repulsion between the fermions is treated at the mean-field level and is absorbed into the chemical potential. The interaction between bosons and fermions has the form

$$\hat{H}_{BF} = \frac{U_{BF}}{N} \sum_q \hat{n}_{-q} \hat{\rho}_q,$$

(15)

where $\hat{\rho}_q$ has the same definition as given below Eq. (9) and, similarly, $\hat{n}_q$ is the bosonic density operator.

We assume that bosons condense in the state with momentum $k_0$, corresponding to one of the minima of the double-dip dispersion $\epsilon_B(k)$. The effective attraction between fermions mediated by Bogoliubov phonons is derived with the help of the path integral formalism. Details are shown in Appendix C and here we only state the result. The induced density-density interaction between fermions is given by

$$V_{Ind}(k, \omega_n) = -\frac{U_{BF}^2 n_0}{2} \left( \frac{-\omega_n + \epsilon_B(k_0 + k) - \epsilon_B(k_0 - k) + 2\epsilon_B(k_0)}{\epsilon_B(k_0 + k) + \epsilon_B(k_0 - k) - 2\epsilon_B(k_0) + gn_0 - g^2 n_0^2},

(16)

where $n_0 = N_0/N$ is the dimensionless “density” of the condensate ($N_0$ is the number of bosons in the condensate, $N$ is the total number of lattice sites, as before).

We now proceed to solve the Eliashberg equations for the superconducting order parameter and accompanying quantities but this time with the attraction given by Eq. (16). Below we show sample order parameters calculated based on the inter-fermion attraction plotted on Fig. 4. As shown on Fig. 5 the frequency-dependent superfluid is obtained. We find that the real and imaginary parts of the order parameter have even- and odd-frequency dependence, respectively. This is in line with the structure of induced attraction Eq. (16).
Figure 4. Strength of induced fermion-fermion attraction potential (in the units of $E_R$), as shown in Eq. (16), $\Re V_{Ind}(k, \omega_n)$ (right) and $\Im V_{Ind}(k, \omega_n)$ (left). The shaken bosons have lattice depth $V = 3.5E_R$ and shaking frequency $\Omega = 5E_R$, where $E_R$ is the recoil energy of boson lattice. The boson interaction strength is such that $g_{n0} = 0.05E_R$ and $T = 0.001E_R$.

Namely, $\Re V_{Ind}(k, \omega_n)$ is even in frequency and $\Im V_{Ind}(k, \omega_n)$ is odd, see Fig. 4. By tuning the shaking frequency it is possible to go from the case of symmetric ($\epsilon_B(k_0 + k) = \epsilon_B(k_0 - k)$) to asymmetric ($\epsilon_B(k_0 + k) \neq \epsilon_B(k_0 - k)$) Bogoliubov dispersion, and hence to introduce an odd-frequency component in the superconducting order parameter.

IV. CONCLUSIONS

We have demonstrated the theoretical feasibility of fermions forming mixed-frequency superconducting order parameter by means of interacting with bosons with non-isotropic dispersion. To achieve such non-isotropy we considered bosons condensing in one of the minima of the double dip dispersion formed by (quasi) 1D shaking. We have shown that by tuning the aforementioned parameters it is possible for the superconducting order parameter to be 10% odd-frequency or greater. As a final note we want to mention that shaking the lattice is not the only possibility for creating non-isotropic and/or the double-dip dispersion, see for example recent work on spin-orbit coupled BEC \cite{12}, and our scheme can be perfected and/or modified to achieve realization of a mixed-frequency superconductor in ultracold
Figure 5. Real (left) and imaginary (right) parts of the order parameter, in the units of $t_F$, for a non-symmetric interaction potential Eq. (16). The fermion dispersion $t_F = 1$, $\mu_F = 0$, $U_{BF} = t_F$ and $T = 0.01t_F$. The boson parameters are the same as in Fig. 4.

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Appendix A: Integrating out the toy model bosons

Toy model Hamiltonian Eq. (6) is quadratic in bosonic operators. This makes integrating out bosons a straightforward exercise in Gaussian integration. Using the path integral formalism we go from Hamiltonian

$$\hat{H} = \sum (\epsilon_B(q) - \mu_B) \hat{b}^\dagger_q \hat{b}_q + \frac{U_{BF}}{\sqrt{N}} \sum \hat{\rho}_q \left( \hat{b}_{-q} + \hat{b}^\dagger_q \right)$$

(A1)

to partition function

$$Z = \int DbDb \exp \left( -\sum_0^\beta \int d\tau \; \hat{b}_q(\tau) \left( \partial_\tau + \epsilon_B(q) - \mu_B \right) \hat{b}_q(\tau) - \frac{U_{BF}}{\sqrt{N}} \sum_0^\beta \int d\tau \; \hat{\rho}_q(\tau) \left( \hat{b}_{-q}(\tau) + \hat{b}_q(\tau) \right) \right)$$

(A2)
where

\[ D\bar{b}Db = \prod d\bar{b}_q db_q \]  

(A3)

up to an arbitrary normalization factor and \( \beta = 1/T \) is the inverse temperature (throughout this paper \( k_B = 1, \hbar = 1 \)). Instead of operators, \( \bar{b}_q(\tau), b_q(\tau) \)-s now stand for complex-valued bosonic fields and \( \rho_q(\tau) \)-s are fields corresponding to Fourier components of fermion density. In Eq. (A2) \( \rho_q(\tau) \)-s are treated as external fields.

We will now integrate out the bosons and obtain the induced interaction between the fermions. To this end we employ the Matsubara frequency representation, \( b_q(\tau) = \sqrt{T} \sum_n b_q(\omega_n)e^{i\omega_n \tau} \), \( \bar{b}_q(\tau) = \sqrt{T} \sum_n \bar{b}_q(\omega_n)e^{-i\omega_n \tau} \) and \( \rho_q(\tau) = T \sum_n \rho_q(\omega_n)e^{-i\omega_n \tau} \), where \( \omega_n = 2n\pi T, n \) being integers. We re-express the imaginary time integrals in Eq. (A2) as sums over Matsubara frequencies \( \omega_n \),

\[
Z = \int D\bar{b}Db \exp \left( -\sum_{q,n} \bar{b}_q(\omega_n) (-i\omega_n + \epsilon_B(q) - \mu_B) b_q(\omega_n) - \frac{U_{BF} \sqrt{T}}{\sqrt{N}} \sum_{q,n} \rho_q(\omega_n) \left( b_{-q}(-\omega_n) + \bar{b}_q(\omega_n) \right) \right),
\]

(A4)

then integrate out \( \bar{b}_q(\omega_n), b_q(\omega_n) \) with the help of the following identity:

\[
\int d\bar{z}dz e^{-zG^{-1}z + Jz + \bar{z}J} = \frac{e^{JGJ}}{\det G^{-1}}.
\]

(A5)

This results in the following induced fermion-fermion interaction action

\[
S_{Ind} = -\frac{U_{BF}^2 T}{2N} \sum_{q,n} \rho_{-q}(-\omega_n) \frac{\epsilon_B(q) + \epsilon_B(-q) - 2\mu_B}{\omega_n^2 + (\epsilon_B(q) - \mu_B)(\epsilon_B(-q) - \mu_B) + i\omega_n (\epsilon_B(q) - \epsilon_B(-q))} \rho_q(\omega_n),
\]

(A6)

from which \( V_{Ind}(k, \omega_n) \) Eq. (10) is obtained. The minus sign in front of the sum in Eq. (A6) ensures that induced interaction is attractive.

**Appendix B: Shaken lattice dispersion**

Here we outline the physics of shaken lattices [13]. Consider two lowest bands (\( s- \) and \( p- \)) of a 1D optical lattice and their hybridization when the shaking frequency matches the band gap. The Hamiltonian is

\[
H(t) = \frac{k_x^2}{2m} + V \cos^2 \left( k_0 x + \frac{\theta(t)}{2} \right),
\]

(B1)
where $\theta(t) = f \cos(\Omega t)$ describes the periodic shaking of the lattice. Because of its special form the time-dependent Hamiltonian Eq. (B1) can be expanded using the properties of Bessel functions. Write the potential term

$$V \cos^2 \left( k_0 x + \frac{f}{2} \cos(\Omega t) \right) = \frac{V}{2} \left( 1 + \cos (2k_0 x + f \cos(\Omega t)) \right)$$

as

$$\cos (2k_0 x + f \cos(\Omega t)) = \frac{1}{2} \left( e^{i2k_0 x} e^{if \cos(\Omega t)} + e^{-i2k_0 x} e^{-if \cos(\Omega t)} \right) ,$$

then use the Jacobi-Anger expansion

$$e^{iz \cos(\phi)} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\phi},$$

where $J_n(z)$ are the $n$–th order Bessel functions of the first kind. Applying the expansion and using the symmetry of Bessel functions,

$$J_n(-z) = (-1)^n J_n(z),$$

$$e^{if \cos(\Omega t)} = \sum_{n=-\infty}^{\infty} i^n J_n(f) e^{in\Omega t},$$

$$e^{-if \cos(\Omega t)} = \sum_{n=-\infty}^{\infty} (-i)^n J_n(f) e^{in\Omega t}$$

end up with the following expansion for the driving term:

$$\cos (2k_0 x + f \cos(\Omega t)) = J_0(f) \cos(2k_0 x) - 2J_1(f) \sin(2k_0 x) \sin(\Omega t) + ...$$

where terms with frequencies $2\Omega t$ and higher have been neglected. Therefore, up to a constant energy shift, the shaken Hamiltonian takes the form

$$H(t) = \frac{k_x^2}{2m} + V J_0(f) \cos^2(k_0 x) - V J_1(f) \sin(2k_0 x) \sin(\Omega t) + ...$$

The above exercise in algebra allows us to connect physics of shaken lattice and band mixing. The $s$– and $p$– bands of the time-averaged Hamiltonian

$$H_{av} = \frac{1}{T} \int_0^T H(t) dt = \frac{k_x^2}{2m} + V J_0(f) \cos^2(k_0 x)$$

mix under the influence of periodic perturbation

$$V(t) = -V J_1(f) \sin(2k_0 x) \sin(\Omega t) + ...$$
when the shaking frequency matches the inter-band spacing, forming two Floquet bands. It was shown in [13] that for a certain parameter window of $V, f, \Omega$ one of the Floquet bands possess two degenerate minima at $k_{\min}, -k_{\min} \neq 0, \pi$ (e.g. see Fig. 1b in [13]).

To find Floquet bands of the shaken lattice we can follow [13] and project the periodic driving term Eq. (B11) on the lowest two bands ($s-$ and $p-$) of the time-averaged Hamiltonian Eq. (B10) which results in

$$V(t) = -V J_1(f) \begin{pmatrix} 0 & C(k, V, f) \\ C^*(k, V, f) & 0 \end{pmatrix} \sin(\Omega t)$$  \hspace{1cm} (B12)

where $C(k) = \langle \Psi_{sk} | \sin(2k_0 x) | \Psi_{pk} \rangle$ and the diagonal terms vanish due to symmetry considerations. $|\Psi_{sk}\rangle$ and $|\Psi_{pk}\rangle$ are the Bloch waves corresponding to the $s-$ and $p-$ bands of the cosine lattice and can be either computed numerically using the plane wave expansion or expressed exactly in terms of Mathieu functions [14]. When shaking frequency $\Omega$ is close to the value of the band gap (the resonance condition) it is justified to use the rotating wave approximation [15] which results in the following approximate Floquet Hamiltonian:

$$H(k) = \begin{pmatrix} E_p(k) - \frac{\Omega}{2} - \frac{i V J_1(f) C(k)}{2} & -\frac{i V J_1(f) C^*(k)}{2} \\ \frac{i V J_1(f) C^*(k)}{2} & E_s(k) + \frac{\Omega}{2} \end{pmatrix}$$  \hspace{1cm} (B13)

where $E_p(k) = \langle \Psi_{sk} | H_{av} | \Psi_{sk} \rangle$ and similarly for $E_s(k)$. All the terms in Eq. (B13) can be obtained numerically exactly in terms of Mathieu functions and by diagonalizing Eq. (B13) the emergent shaken (Floquet) bands can be found. Including the inter-boson interactions will result in Bose condensation around either $k_{\min}$ or $-k_{\min}$ and the spectrum of excitations above that state (the Bogoliubov phonons) will end up not being inversion-symmetric [16].

**Appendix C: Integrating out the shaken bosons**

The bosonic part of the partition function takes the form

$$Z = \int D\bar{b} D\bar{b} \exp \left[ -S_B - S_{BF} \right],$$  \hspace{1cm} (C1)

where

$$S_B + S_{BF} = \int_0^\beta d\tau \sum_k \bar{b}_k(\tau) \left( \partial_\tau + \epsilon_B(k) - \mu_B \right) b_k(\tau) + \frac{g}{2N} \int_0^\beta d\tau \sum_{k,k',q} \bar{b}_{k+q} \bar{b}_{k'-q} b_{k'} b_k +$$
\[ S_{Bog} = \beta \int_0^\beta d\tau \sum_i \tilde{b}_i b_i \rho_i \] (C2)

and

\[ D\tilde{b}D\tilde{b} = \prod_k d\tilde{b}_k d\tilde{b}_k. \] (C3)

In the above \( \tilde{b}_i, b_i \) are the Bose fields at lattice site \( i \) and \( \tilde{b}_k, b_k \) are their momentum-space counterparts. Assuming condensation in the state with momentum \( +k_0 \), \( b_{k_0} \) and \( \tilde{b}_{k_0} \) acquire non-zero mean-field values which can be accounted for by a shift of variables,

\[
\begin{align*}
    b_i &\rightarrow \sqrt{\frac{N_0}{N}} e^{ik_0 r_i} + b_i, \\
    \tilde{b}_i &\rightarrow \sqrt{\frac{N_0}{N}} e^{-ik_0 r_i} + \tilde{b}_i,
\end{align*}
\] (C4)

where \( r_i \) stands for \( i \)-th lattice site position. We also assume that the number of atoms above the condensate, \( N_B - N_0 \), is small, where

\[ N_0 = \tilde{b}_{k_0} b_{k_0} = N_B - \sum_{k \neq k_0} \tilde{b}_k b_k. \] (C5)

Then substituting the mean-field value Eq. (C4) into the action Eq. (C2) and expanding in creation/annihilation operators up to the second order the following effective quadratic action is obtained:

\[
S_{Bog} = \beta \int_0^\beta d\tau \sum_k \tilde{b}_k(\tau) (\partial_\tau + \epsilon_B(k) - \mu_B + 2gn_0) b_k(\tau)
\]

\[
+ \frac{gn_0}{2} \int_0^\beta d\tau \sum_k \tilde{b}_{k_0+k}(\tau) \tilde{b}_{k_0-k}(\tau) + \frac{gn_0}{2} \int_0^\beta d\tau \sum_k \tilde{b}_{k_0+k}(\tau) b_{k_0-k}(\tau)
\] (C6)

where in all the sums \( k \) runs over the first BZ momenta and in the second and third sums it is understood that if \( k_0 + k \) becomes greater than \( \pi \) its value “wraps around” the 1BZ edge and becomes \( k_0 + k - 2\pi \). \( n_0 \) is the condensate density, \( n_0 = N_0/N \), and can be determined from the equation of state, namely the equation that demands that the linear terms in the action vanish, \( gn_0 = \mu_B - \epsilon_B(k_0) \).

The final step is integrating out the bosons which is made easier by introducing the particle-hole notation:

\[
S_{Bog} = \int_0^\beta d\tau \frac{1}{2} \sum_k \begin{pmatrix} b_{k_0+k}(\tau) & b_{k_0-k}(\tau) \end{pmatrix} \begin{pmatrix} \partial_\tau + \epsilon_B(k_0 + k) - \mu_B + 2gn_0 & gn_0 \\
-gn_0 & -\partial_\tau + \epsilon_B(k_0 - k) - \mu_B + 2gn_0 \end{pmatrix} \begin{pmatrix} b_{k_0+k}(\tau) \\
\tilde{b}_{k_0-k}(\tau) \end{pmatrix}.
\] (C7)
or, in momentum-frequency space,

\[
S_{Bog} = \frac{1}{2} \sum_{k,\omega} \left( \bar{b}_{k_0+k}(\omega_n) b_{k_0-k}(-\omega_n) \right) G^{-1}(k,\omega_n) \left( \bar{b}_{k_0+k}(-\omega_n) b_{k_0-k}(\omega_n) \right)
\]

(C8)

where

\[
G^{-1}(k,\omega_n) = \begin{pmatrix}
i \omega_n + \epsilon_B(k_0 + k) - \mu_B + 2gn_0 & gn_0 \\
gn_0 & -i \omega_n + \epsilon_B(k_0 - k) - \mu_B + 2gn_0
\end{pmatrix}.
\]

(C9)

We then expand the boson-fermion interaction term Eq. (15) to the first order in \( \bar{b}_k, b_k \):

\[
S_{BF} = U_{BF} \int_0^\beta d\tau \sum_i \rho_i n_0 + \frac{U_{BF} \sqrt{n_0}}{\sqrt{N}} \int_0^\beta d\tau \sum_{i,k \neq k_0} \rho_i \left( \bar{b}_k e^{i(k_0-k)\tau} + b_k e^{-i(k_0-k)\tau} \right).
\]

(C10)

The first term renormalizes the fermion chemical potential,

\[
\mu_F \rightarrow \mu_F + U_{BF} n_0,
\]

(C11)

whereas the second-order terms in \( \bar{b}_k, b_k \) (not shown in Eq. (C10)) will produce higher-order contributions to inter-fermion interactions (three-body interactions and higher). It is the first-order term in Eq. (C10) that produces the attractive density-density interaction between fermions. To see this make the change of variables in the sum, \( k_0 - k \rightarrow -k \). Then the second term of Eq. (C10) can be represented as

\[
\frac{1}{2} \int_0^\beta d\tau \sum_k J^\dagger(k,\tau) \left( \begin{pmatrix}
\bar{b}_{k_0+k}(\tau) \\
\bar{b}_{k_0-k}(\tau)
\end{pmatrix}
\right) + \left( \begin{pmatrix}
b_{k_0+k}(\tau) \\
b_{k_0-k}(\tau)
\end{pmatrix}
\right) J(k,\tau)
\]

(C12)

with

\[
J(k,\tau) = \begin{pmatrix}
\rho_k(\tau) \\
\rho_{-k}(\tau)
\end{pmatrix}, \quad J^\dagger(k,\tau) = \begin{pmatrix}
\rho_{-k}(\tau) \\
\rho_k(\tau)
\end{pmatrix}.
\]

(C13)

Comparing with Eq. (C8) and employing the rules of Gaussian integration Eq. (A5) we obtain the induced fermion density-density interaction

\[
V_{Ind}(k,\omega_n) = -\frac{U_{BF}^2 n_0}{2} \frac{\epsilon_B(k_0 + k) + \epsilon_B(k_0 - k) - 2\epsilon_B(k_0)}{(i \omega_n + \epsilon_B(k_0 + k) - \epsilon_B(k_0) + gn_0)(-i \omega_n + \epsilon_B(k_0 - k) - \epsilon_B(k_0) + gn_0) - g^2 n_0^2}
\]

(C14)

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