GLOBAL EXISTENCE AND REGULARITY RESULTS FOR STRONGLY COUPLED NONREGULAR PARABOLIC SYSTEMS VIA ITERATIVE METHODS

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Dedicated to Professor Stephen Cantrell on the occasion of his 60th birthday

Abstract. The global existence of classical solutions to strongly coupled parabolic systems is shown to be equivalent to the availability of an iterative scheme producing a sequence of solutions with uniform continuity in the BMO norms. Amann’s results on global existence of classical solutions still hold under much weaker condition that their BMO norms do not blow up in finite time. The proof makes use of some new global and local weighted Gagliardo-Nirenberg inequalities involving BMO norms.

1. Introduction. Among the long standing questions in the theory of strongly coupled parabolic systems and its applications are the global existence and regularity properties of their solutions. We consider in this paper the following system

\[
\begin{cases}
  u_t = \text{div}(A(u)Du) + f(u) & (x,t) \in Q = \Omega \times (0,T), \\
  u(x,0) = U_0(x) & x \in \Omega \\
  \text{Boundary conditions for } u \text{ on } \partial \Omega \times (0,T). 
\end{cases}
\]

(1.1)

Here, \( \Omega \) is a bounded domain with smooth boundary \( \partial \Omega \) in \( \mathbb{R}^n \), \( n > 1 \), and \( u : \Omega \to \mathbb{R}^m, f : \mathbb{R}^m \to \mathbb{R}^m \) are vector valued functions. \( A(u) \) is a full \( m \times m \) matrix. Thus, the above is a system of \( m \) equations. The vector valued solution \( u \) satisfies either Dirichlet or Neumann boundary condition on \( \partial \Omega \times (0,T) \).

The system (1.1) arises in many mathematical biology and ecology applications as well as in differential geometry theory. In the last few decades, papers concerning such strongly coupled parabolic systems usually assumed that the solutions under consideration were bounded, a very hard property to check as maximum principles had been unavailable for systems in general. In addition, past results usually relied on the following local existence result of Amann.

Theorem 1.1. (2, 3) Suppose \( \Omega \subset \mathbb{R}^n, n \geq 2 \), with \( \partial \Omega \) being smooth. Assume that (1.1) is normally elliptic. Let \( p_0 \in (n, \infty) \) and \( U_0 \) be in \( W^{1,p_0}(\Omega) \). Then there exists a maximal time \( T_0 \in (0, \infty] \) such that the system (1.1) has a unique classical solution in \( (0, T_0) \) with

\[ u \in C([0, T_0), W^{1,p_0}(\Omega)) \cap C^{1,2}((0, T_0) \times \Omega). \]

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Moreover, if \( T_0 < \infty \) then
\[
\lim_{t \to T_0^-} \|u(\cdot, t)\|_{W^{1,p_0}(\Omega)} = \infty.
\] (1.2)

The proof of the above result worked directly with the system and based on semigroup and interpolation of functional spaces theories. We refer the readers to [2] for the definition of normal ellipticity. The checking of (1.2) is the most difficult one as known techniques for the regularity of solutions to scalar equations could not be extended to systems and counterexamples were available.

In this paper, we propose a different approach using iterative techniques and depart from the boundedness assumptions. Namely, we consider the following schemes
\[
(u_k)_t = \text{div}(A(u_{k-1})Du_k) + f(u_{k-1}, Du) \quad k \geq 1.
\]

Under very weak assumptions on the uniform boundedness and continuity of the BMO norms of the solutions to the above systems, we will show the global existence of a classical solution to (1.1) if the above scheme converges in weak norms. Thus, global existence and regularity of solutions are established at once. Furthermore, without the boundedness assumptions the systems are no longer regular elliptic, we will only assume that the matrix \( A(u) \) in (1.1) is uniformly elliptic.

We also improve Theorem 1.1 by replacing the condition (1.2) with a weaker one using the BMO or \( W^{1,p_0} \) norms of \( u \) with \( p_0 = n \). In a forthcoming work, we will show that the results in this paper can apply to a class of generalized Shigesada-Kawasaki-Teramoto models ([14]) consisting of more than 2 equations. Namely, we will establish the global existence of classical solutions to the following system
\[
u_t = \Delta(P(u)) + f(u),
\] (1.3)
where \( P(u), f(u) \) are vector valued functions whose components have quadratic (or even polynomial) growth in \( u \).

In the proof we make use of some new local Gagliardo-Nirenberg inequalities involving BMO norms and weights in \( A_p \) classes. These inequalities are established in [12] and are generalizations of those by Strzelecki and Rivière in [16].

2. Preliminaries and main results. Throughout this paper \( \Omega \) is a bounded domain with smooth boundary in \( \mathbb{R}^n \), \( n > 1 \). To describe our assumptions we recall the definitions of BMO spaces and \( A_p \) classes.

For any locally integrable vector valued function \( u \in L^1_{loc}(\Omega, \mathbb{R}^m) \) and measurable set \( B \subset \Omega \) with its Lebesgue measure \( |B| \neq 0 \), we denote
\[
\int_B u \, dx = \frac{1}{|B|} \int_B u \, dx.
\]

For a smooth function \( u \) defined on \( \Omega \times (0, T), \ T > 0 \), its temporal and spatial derivatives are denoted by \( u_t, Du \) respectively. If \( A \) is a function in \( u \) then we also abbreviate \( \frac{\partial A}{\partial u} \) by \( A_u \).

We will frequently work with a ball \( B(z, R) \) centered at \( z \in \mathbb{R}^n \) with radius \( R \). If \( z \) is understood we will write \( B_R \) for \( B(z, R) \). For any \( x_0 \in \Omega \) and \( R > 0 \), we also denote \( \Omega(x_0, R) = \Omega \cap B(x_0, R) \). For any locally measurable function \( u \) on \( \Omega \) and \( x_0 \in \Omega \) and \( R > 0 \) we write \( u_{x_0,R} \) for the average of \( u \) over \( \Omega(x_0, R) \). If \( x_0 \) is understood, we simply write \( u_R \) for \( u_{x_0,R} \).
Lemma 2.1. If elements are vector valued functions \( u : \Omega \to \mathbb{R}^m \) is BMO (Bounded Mean Oscillation) if the seminorm
\[
[u]_{BMO(\Omega)} = \sup_{x_0 \in \Omega, R > 0} \frac{1}{|\Omega(x_0, R)|} \int_{\Omega(x_0, R)} |u - u(x_0, r)| \, dx < \infty,
\]
where the supremum is taken over all balls \( B \subset \Omega \). The space \( BMO(\Omega) \) is the Banach space of BMO functions on \( \Omega \) with norm
\[
\|u\|_{BMO(\Omega)} = \|u\|_{L^1(\Omega)} + [u]_{BMO(\Omega)}.
\]

We recall the following well known fact (e.g., see [8]).

**Lemma 2.1.** If \( \lambda : \mathbb{R}^m \to \mathbb{R} \) is Hölder function then \( \lambda(u) \) is BMO if \( u \) is BMO.

There is a connection between BMO functions and the so called \( A_\gamma \) weights, which are defined as follows. Let \( \Psi \) be a measurable nonnegative function on \( \Omega \) and \( \gamma > 1 \). We say that \( \Psi \) belongs to the class \( A_\gamma \) if the seminorm
\[
\int_{B(x,R)} \Psi \, dx \int_{B(x,R)} \Psi^{1-\gamma'} \, dx < \infty.
\]

Here, \( \gamma' = \gamma/(\gamma - 1) \). The \( A_{\infty} \) class is defined by \( A_{\infty} = \bigcup_{\gamma > 1} A_\gamma \). For more details on these classes we refer the readers to [13] [15].

We also recall the following result from [8] on the connection between BMO functions and weights.

**Lemma 2.2.** Let \( \Psi \) be a positive function and \( \mu \) is a nonnegative Radon measure on \( \Omega \) such that \( \Psi, \Psi^{-1} \) are in \( BMO(\Omega) \) then \( \Psi \) belongs to \( \cap_{\gamma > 1} A_\gamma \) and \( [\Psi]_\gamma \) is bounded by a constant depending on \( [\Psi]_{BMO(\Omega)} \) and \( [\Psi^{-1}]_{BMO(\Omega)} \).

We also recall the definition of the Campanato spaces \( \mathcal{L}^{p,\gamma}(\Omega, \mathbb{R}^m) \). For any \( p \geq 1 \) and \( \gamma > 0 \) and \( u \in L^p(\Omega, \mathbb{R}^m) \), we define
\[
[u]^p_{p,\gamma} = \sup_{x_0 \in \Omega, \rho > 0} \rho^{-\gamma} \int_{B(x_0, \rho)} |u - u(x_0, \rho)|^p \, dx.
\]

Then \( \mathcal{L}^{p,\gamma}(\Omega, \mathbb{R}^m) \) is the Banach space of such functions with finite norm
\[
\|u\|_{p,\gamma} = \|u\|_p + [u]_{p,\gamma}.
\]

Clearly, \( \mathcal{L}^{p,\alpha}(\Omega, \mathbb{R}^m) = BMO(\Omega, \mathbb{R}^m) \). Moreover, it is well known that ([7 Theorem 2.9, p.52]) \( \mathcal{L}^{p,\gamma}(\Omega, \mathbb{R}^m) \) is isomorphic to \( C^{0,\alpha}(\Omega) \) if \( \alpha = \frac{2-n}{p} > 0 \).

As usual, \( W^{1,p}(\Omega, \mathbb{R}^m) \), \( p \geq 1 \), will denote the standard Sobolev spaces whose elements are vector valued functions \( u : \Omega \to \mathbb{R}^m \) with finite norm
\[
\|u\|_{W^{1,p}(\Omega, \mathbb{R}^m)} = \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)},
\]
where \( Du \) is the the derivative of \( u \).

We now state our structural conditions on the system (1.1).

**A.1:** (Uniform ellipticity) There are positive constants \( C, \lambda_0 \) and smooth functions \( \lambda(u), \Lambda(u) \) such that \( \lambda(u) \geq \lambda_0 \) and \( \Lambda(u) \leq C\lambda(u) \) for all \( u \in \mathbb{R}^m \) and
\[
\lambda(u)\xi \leq \langle A(u)\xi, \xi \rangle \leq \Lambda(u)\xi \quad \forall u \in \mathbb{R}^m, \xi \in \mathbb{R}^m.
\]
A.2): Assume that $A \in C^1(\mathbb{R}^m)$. Let $\Phi_0, \Phi$ be defined as

$$\Phi_0(u) = \lambda^{\frac{1}{2}}(u) \quad \text{and} \quad \Phi(u) = \frac{|A_u(u)|}{\lambda^{\frac{1}{2}}(u)} \quad u \in \mathbb{R}^m.$$ 

Assume that the quantities

$$k_1 := \sup_{u \in \mathbb{R}^m} \frac{|\Phi_u|}{\Phi}, \quad k_2 := \sup_{u \in \mathbb{R}^m} \frac{\Phi}{\Phi_0} \tag{2.2}$$

are finite.

A.3): (Weights) If $u \in BMO(\Omega)$ then $\Phi^{\frac{3}{2}}(u)$ belongs to the $A_\frac{4}{3}$ class and the quantity $[\Phi(u)^{\frac{3}{2}}]_\frac{4}{3}$ can be controlled by the norm $\|u\|_{BMO(\Omega)}$.

In applications, if $A(u)$ has a polynomial growth in $u$ then we can assume that $|A(u)| \sim \lambda(u)$ and $|A_u| \sim |\lambda_u|$. In this case

$$\Phi \sim \frac{|\lambda_u|}{\lambda^\frac{2}{3}}, \quad |\Phi_u| \sim \frac{|\lambda_{uu}|}{\lambda^\frac{2}{3}} + \frac{|\lambda|^2}{\lambda^\frac{2}{3}}.$$ 

Therefore, if $\sup_u |\lambda_u|$ and $\sup_u |\lambda_{uu}|$ are bounded then (2.2) of A.2) is verified. It is clear that this is the case if $\lambda(u)$ is a polynomial in $|u|$, say $\lambda(u) \sim (1 + |u|)^{k}$ for some $k \geq 0$. Similarly, concerning A.3), we see that $\Phi^\alpha(u) \sim (1 + |u|)^{\alpha(k/2 - 1)}$. Thus, if $2 \leq k < 5$ then $\alpha(k/2 - 1) \leq 1$ for some $\alpha > 2/3$. In this case, $\Phi^\alpha(u)$ is Hölder in $u$ so that it is BMO if $u$ is. Of course, $\Phi^{-1}(u)$ is bounded so that it is also BMO. Lemma 2.2 shows that $\Phi^\alpha(u)$ is then an $A_\gamma$ weight for all $\gamma > 1$ and A.3) is satisfied.

On the other hand, it is clear from the definition of weights that $[\Phi^\alpha]_{\beta+1} = [\Phi^{-\frac{\beta}{2}}]_{\frac{\beta}{2}+1}$ so that if $1 < k < 2$ then $\Phi^\frac{k}{2}$ is bounded from above and $\Phi^\frac{k}{2} \sim (1 + |u|)^{\frac{k}{2}}(1 - k/2)$. Thus, we can find $\alpha > 2/3$ and $\beta < 1/3$ such that $\Phi^{-\frac{\beta}{2}}$ is Hölder in $u$ and therefore BMO if $u$ is. Again, this gives that $\Phi^{-\frac{\beta}{2}}$ is a weight and belongs to $\cap_{\gamma > 1} A_\gamma$.

Concerning global existence of classical solutions, we also assume that the ellipticity constants $\lambda, \Lambda$ in A.1) are not too far apart.

R): (The ratio condition) There is $\delta \in (0, 1)$ such that

$$\frac{n - 2}{n} = \delta \sup_{u \in \mathbb{R}^m} \frac{\lambda(u)}{\Lambda(u)}. \tag{2.3}$$

One should note that there are examples in $\Pi$ of blow up solutions to (1.1) if the condition R) is violated. We assume the following growth conditions on the nonlinearity $f$.

F): There is a positive constant $C$ and a $C^1$ function $g : \mathbb{R}^m \to \mathbb{R}$ such that for any vector valued functions $u \in C^1(\Omega, \mathbb{R}^m)$ and $p \in C^1(\Omega, \mathbb{R}^{mn})$

$$|f(u, p)| \leq C \lambda^{\frac{1}{2}}(u) |p| + g(u). \tag{2.4}$$

$$|Df(u, p)| \leq C |D(\lambda^{\frac{1}{2}}(u)|p|)| + |g_u(u)||Du|. \tag{2.5}$$

We also assume that $|g_u| \leq C\lambda(u)$ and $\lambda(u)$ has polynomial growth in $u$.

To solve (1.1), we can make use of the following iterative scheme. We start with any smooth vector valued function $u_0$ on $Q$ and define a sequence $\{u_k\}$ of solutions to the following linear systems

$$(u_k)_t = \text{div}(A(u_{k-1})Du_k) + f(u_{k-1}, Du_k) \quad k \geq 1. \tag{2.6}$$
The initial and boundary conditions for the above systems are those of $u$ in (1.1). Note that the global existence of the strong solutions to the above systems is not generally available by standard theories (see [1]) because of the presence of $Du_k$ in $f$. However, this is the case if we assume R) and the linear growth of $f$ in $Du$ of F) and make use of the results in [1].

Concerning the approximation sequence $\{u_k\}$, we assume the following uniform bound and continuity of their BMO norms.

V): Let $\{u_k\}$ and $\Phi$ be defined by (2.6) and A.2). There exists a continuous function $C$ on $(0, \infty)$ such that for any $T > 0$

$$\|u_0(\cdot, t)\|_{C^1(\Omega)}, \|u_k(\cdot, t)\|_{BMO(\Omega)} \leq C(T) \quad \forall t \in (0, T), \ k = 1, 2, \ldots$$

Moreover, for any $\varepsilon > 0$ and $(x, t) \in Q$, there exists $R = R(\varepsilon, T) > 0$ such that

$$\|u_k(\cdot, t)\|_{BMO(B_R(x))} < \varepsilon \quad \forall t \in (0, T), \ k = 0, 1, \ldots$$

In addition, for all $(x, t) \in Q$ and integer $k \geq 1$ we assume that

either $\Phi(u_k(x, t)) \leq C(T)$ or $\lambda(u_k(x, t)) \leq C(T).$ (2.7)

The uniform boundedness assumption on the BMO norm of $u_k$ is of course much weaker than the $L^\infty$ boundedness assumptions in literature. Moreover, the uniform continuity assumption (2.7) on the BMO norms is somehow necessary for the regularity of the limit solution $u$.

The assumption (2.8) seems to be technical at first glance but it is clearly necessary if we would like to produce a sequence $\{u_k\}$ that converges in $L^\infty(Q)$ to a solution of (1.1). In particular, if $\lambda(u)$ behaves like $\lambda(u) \sim (1 + |u|)^k$ for some $k \geq 0$ then, as discussed earlier, we see that $\Phi(u) \sim (1 + |u|)^{k/2}$. Thus,

$$\Phi(u_k(x, t)) \sim \left(1 + |u_k(x, t)|^k\right)^{k/2}, \quad \lambda(u_k(x, t)) \sim \left(1 + |u_k(x, t)|^k\right)^{k/2}.$$

Hence, if $k \in [0, 2]$ then (2.8) is clearly verified. The generalized Shigesada-Kawasaki-Teramoto model (1.3) clearly is a typical example.

We then have our main result of this paper as follows.

**Theorem 2.3.** Assume A.1)-A.3), R), F) and V). If the sequence $\{u_k\}$ converges weakly to $u$ in $L^2(\Omega \times (0, \infty))$ then $u$ a classical solution of (1.1).

A simple consequence of the above theorem is the following.

**Corollary 2.4.** Assume A.1)-A.3), R), F) and V). Then the system (1.1) has a classical solution $u$ that exists globally on $\Omega \times (0, \infty)$ if and only if there is $u_0$ and a sequence $\{u_k\}$ satisfying V) such that $\{u_k\}$ converges weakly to $u$ in $L^2(\Omega \times (0, T))$ for each $T > 0$.

The necessary part is trivial as we can take $u_k = u$ for all $k \geq 0$, the solution sequence is then a constant one. The sufficient part comes from Theorem 2.3 and the uniqueness of weak limits in $L^2(\Omega \times (0, T))$.

In the next theorem we discuss the global existence of classical solutions when their local existence can be achieved by other methods (e.g., Theorem 1.1).

**Theorem 2.5.** Assume A.1)-A.3), R) and F). Let $p_0 \in (n, \infty)$ and $U_0$ be in $W^{1,p_0}(\Omega)$. Suppose that $T_0 \in (0, \infty]$ is the maximal existence time for a classical solution

$$u \in C([0, T_0), W^{1,p_0}(\Omega)) \cap C^{1,2}((0, T_0) \times \Omega)$$

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for the system \( \text{(1.1)} \). Suppose that there is a function \( C \) in \( C^0((0,T_0]) \) such that
\[
\|u(\cdot,t)\|_{BMO(\Omega)} \leq C(t) \quad \forall t \in (0,T_0).
\]
Moreover, for any \( \varepsilon > 0 \) and \( (x,t) \in Q \), there exists \( R = R(\varepsilon) > 0 \) such that
\[
\|u(\cdot,t)\|_{BMO(B_R(x))} < \varepsilon \quad \forall t \in (0,T_0).
\]
Then \( T_0 = \infty \).

By Poincaré’s inequality,
\[
\int_{B_R} |u - u_R|^n \, dx \leq C(n) \int_{B_R} |Du|^n \, dx,
\]
it is easy to see that if \( Du(\cdot,t) \in L^n(\Omega) \) then \( u(\cdot,t) \) is BMO and \( \|u(\cdot,t)\|_{BMO(B_R)} \)
is small if \( R \) is small. Therefore, as a simple consequence of the above theorem, we have the following improvement of Theorem \( \text{[1.1]} \). We will see that the assumption \( p_0 > n \) in \( \text{[1.2]} \) can be replaced by \( p_0 = n \).

**Corollary 2.6.** In addition to the assumptions of Theorem \( \text{[1.1]} \) on the structural conditions of \( \text{[1.1]} \), we assume the ratio condition \( R \). Then there exists a maximal time \( T_0 \in (0,\infty) \) such that the system \( \text{[1.1]} \) has a unique classical solution in \((0,T_0)\) with
\[
u \in C([0,T_0), W^{1,p_0}(\Omega)) \cap C^{1,2}((0,T_0) \times \Omega).
\]
Moreover, if \( T_0 < \infty \) then
\[
\lim_{t \to T_0^-} \|u(\cdot,t)\|_{W^{1,n}(\Omega)} = \infty.
\]

Again, we remark that if the condition \( R \) is violated then there are counterexamples for finite time blow up solutions to \( \text{[1.1]} \).

3. **Weighted Gagliardo-Nirenberg inequalities.** In this section we will establish our main tool, Lemma \( \text{[3.3]} \) which allows us to control the \( L^p \) norm of \( Du_k \) in the proof of our main theorems.

First of all, we recall the following global weighted Gagliardo-Nirenberg inequality which is proved in \( \text{[12]} \).

**Lemma 3.1.** [12] Lemma 2.1] Let \( u, U : \Omega \to \mathbb{R}^m \) be vector valued functions with \( u \in C^4(\Omega), U \in C^2(\Omega) \) and \( \Phi : \mathbb{R}^m \to \mathbb{R} \) be a \( C^3 \) function. Suppose that either \( U \) or \( \Phi^2(u) \frac{\partial \Phi}{\partial u} \)
vanish on the boundary \( \partial \Omega \) of \( \Omega \). We set
\[
I_1 := \int_{\Omega} \Phi^2(u)|DU|^{2p+2} \, dx, \quad \bar{I}_1 := \int_{\Omega} \Phi^2(u)|Du|^{2p+2} \, dx, \quad (3.1)
\]
\[
\bar{I}_1 := \int_{\Omega} |\Phi_u(u)|^2(|DU|^{2p+2} + |Du|^{2p+2}) \, dx, \quad (3.2)
\]
and
\[
I_2 := \int_{\Omega} \Phi^2(u)|DU|^{2p-2}|D^2U|^2 \, dx. \quad (3.3)
\]

Suppose that
\[
\text{GN): } \Phi(u) \mathcal{F} \text{ belongs to the } A_{\frac{p}{p-2}+1} \text{ class.}
\]

Then for any \( \varepsilon > 0 \) there is a constant \( C_{\varepsilon,\Phi} \) depending on \( \varepsilon \) and \( [\Phi^2(u)]_{\frac{p}{p-2}+1} \)
for which
\[
I_1 \leq \varepsilon \bar{I}_1 + C_{\varepsilon,\Phi} \|U\|^2_{BMO(\Omega)} \left[ \bar{I}_1 + I_2 \right]. \quad (3.4)
\]
We also have the following local version of the above lemma.

**Lemma 3.2.** [12] Lemma 2.2] Let \( u, U : \Omega \to \mathbb{R}^m \) be vector valued functions with \( u \in C^1(\Omega), U \in C^2(\Omega) \) and \( \Phi : \mathbb{R}^m \to \mathbb{R} \) be a \( C^1 \) function such that the condition \( GN \) in Lemma 3.1 holds. For any ball \( B_t \) in \( \Omega \) we set

\[
I_1(t) := \int_{B_t} \Phi^2(u)|DU|^{2p+2} \, dx, \quad \tilde{I}_1(t) := \int_{B_t} \Phi^2(u)|Du|^{2p+2} \, dx, \tag{3.5}
\]

and

\[
I_2(t) := \int_{B_t} \Phi^2(u)|DU|^{2p-2}|D^2U|^2 \, dx. \tag{3.7}
\]

Consider any ball \( B_s \), concentric with \( B_t \), \( 0 < s < t \), and any nonnegative \( C^1 \) function \( \psi \) such that \( \psi = 1 \) in \( B_s \) and \( \psi = 0 \) outside \( B_t \). Then, for any \( \varepsilon > 0 \) there are constant \( C_{\varepsilon, \Psi, \Phi} \) depending on \( \varepsilon \) and \([\Phi^{\frac{2}{p+2}}(u)]_{p+2,1} \), such that

\[
I_1(s) \leq \varepsilon(I_1(t) + \tilde{I}_1(t)) + C_{\varepsilon, \Psi, \Phi} \|U\|_{BMO(B_t)}^2 \left[ I_1(t) + I_2(t) \right] \tag{3.8}
\]

\[
+ C_{\varepsilon, \Psi} \|U\|_{BMO(B_t)} \sup_{x \in B_t} |D\psi(x)|^2 \int_{B_t} |\Phi(u)|^{2p} \, dx.
\]

**Remark 3.3.** By approximation, see [19], the lemma also holds for \( u \in W^{1,2}(\Omega) \) and \( U \in W^{2,2}(\Omega) \) provided that the quantities \( I_1, I_2 \) and \( \tilde{I}_1 \) defined in [3.1] and [3.3] are finite.

We now state the main assumptions of our key lemma in this section.

**GN.1:** Let \( \Phi_0, \Phi \) be positive functions on \( \mathbb{R}^m \) with \( \Phi \in C^1(\mathbb{R}^m) \). Assume that the following quantities are finite

\[
k_1 := \sup_{u \in \mathbb{R}^m} \frac{|\Phi_u|}{\Phi}, \quad k_2 := \sup_{u \in \mathbb{R}^m} \frac{\Phi}{\Phi_0}. \tag{3.9}
\]

**GN.2:** For some \( p \geq 1 \) suppose that \( \Phi(u)^{\frac{2}{p+2}} \) belongs to the \( A_{\frac{p}{p+2}+1} \) class if \( u \) is \( BMO \) and the quantity \([\Phi^{\frac{2}{p+2}}(u)]_{p+2,1}\) can be controlled by the norm \( \|u\|_{BMO(\Omega)} \).

The following lemma will be crucial in obtaining uniform estimates for the approximation sequence \( \{u_k\} \) using the condition V.

**Lemma 3.4.** Assume GN.1 and GN.2. Let \( u, U : \Omega \to \mathbb{R}^m \) be vector valued functions with \( U \in C^2(\Omega), u \in C^1(\Omega) \). Suppose further that either \( U \) or \( \Phi^2(u) \frac{\partial U}{\partial \nu} \) vanish on the boundary \( \partial \Omega \) of \( \Omega \). For a ball \( B_t \) in \( \Omega \) we set

\[
I_1(t) := \int_{B_t} \Phi^2(u)|DU|^{2p+2} \, dx, \quad \tilde{I}_1(t) := \int_{B_t} \Phi^2(u)|Du|^{2p+2} \, dx, \tag{3.10}
\]

and

\[
I_2(t) := \int_{B_t} \Phi_0^2(u)|DU|^{2p-2}|D^2U|^2 \, dx. \tag{3.11}
\]

Then there is \( \varepsilon_0 \) depending on \([\Phi^{\frac{2}{p+2}}(u)]_{p+2,1}, k_1, k_2 \) such that if \( \|U\|_{BMO(B_t)} < \varepsilon_0 \) then there is a constant \( C_0 \) depending on \([\Phi^{\frac{2}{p+2}}(u)]_{p+2,1}, k_1, k_2 \) such that

\[
I_1(s) \leq C_0\|U\|_{BMO(B_t)} \left[ \frac{1}{(t-s)^2} \int_{B_t} \Phi_0^2|DU|^{2p} \, dx + \tilde{I}_1(t) + I_2(t) \right] \tag{3.12}
\]
for any \( s, t \) such that \( 0 < s < t < R \).

For the proof of this lemma and later use, let us recall the following elementary iteration result (e.g., see [7] Lemma 6.1, p.192).

**Lemma 3.5.** Let \( f, g, h \) be bounded nonnegative functions in the interval \([\rho, R]\) with \( g, h \) being increasing. Assume that for \( \rho \leq s < t \leq R \) we have

\[
f(s) \leq [(t - s)^{-\alpha} g(t) + h(t)] + \varepsilon f(t)
\]

with \( C \geq 0, \alpha > 0 \) and \( 0 < \varepsilon < 1 \). Then

\[
f(\rho) \leq c(\alpha, \varepsilon)[(R - \rho)^{-\alpha} g(R) + h(R)].
\]

The constant \( c(\alpha, \varepsilon) \) can be taken to be \((1 - \nu)^{-\alpha}(1 - \nu^{-\alpha} \nu_0)^{-1}\) for any \( \nu \) satisfying \( \nu^{-\alpha} \nu_0 < 1 \).

We are now ready to give the proof of Lemma 3.4.

**Proof.** From the definition of the quantities in Lemma 3.2, it is clear that \( I_1(t) = I_1(t) \) for any \( \rho, R \). By GN.1, we have \( I_1(t) \leq k_1[I_1(t) + I_1(t)] \) and \( I_2(t) \leq k_2I_2(t) \).

Let \( B_s, B_t \) be two concentric balls in \( \Omega \) with radii \( t > s > 0 \) and \( \psi \) be a \( C^1 \) function such that \( \psi = 1 \) in \( B_s \) and \( \psi = 0 \) outside \( B_t \). For any \( \varepsilon = \frac{1}{2} \) we now see that the following comes from (3.8) for some positive constants \( C_1 \) and \( C_2 \) depending on \( C_{1, \Phi}, k_1, k_2 \)

\[
I_1(s) \leq \frac{1}{2} I_1(t) + C_1 \|U\|_{BMO(B_s)} \left[I_1(t) + I_1(t) + I_2(t)\right] + C_2 \|U\|_{BMO(B_s)} \sup_{x \in B_t} |D\psi(x)|^2 \int_{B_t} |\Phi|^2(u)|DU|^{2p} \, dx.
\]

(3.13)

For any \( s, t, \rho, R \) such that \( 0 < \rho < s < t < R \), let \( \psi \) be a cutoff function for \( B_s, B_t \) with \( |D\psi| \leq 1/(t-s) \). If \( \varepsilon_0 \) is sufficiently small such that

\[
\nu_0 := \frac{1}{2} + C_1 \|U\|_{BMO(B_R)} < 1,
\]

then, in noting that \( \|U\|_{BMO(B_r)} \leq \|U\|_{BMO(B_R)} \)

\[
I_1(s) \leq \nu_0 I_1(t) + C_4 \|U\|_{BMO(B_R)} \left[\frac{1}{(t-s)^2} \int_{B_t} \Phi^2 |DU|^{2p} \, dx + \hat{I}_1(t) + I_2(t)\right].
\]

Since \( \Phi(u) \leq k_2 \Phi_0(u) \), the above yields a constant \( C_4 \) such that

\[
I_1(s) \leq \nu_0 I_1(t) + C_4 \|U\|_{BMO(B_R)} \left[\frac{1}{(t-s)^2} \int_{B_t} \Phi_0^2 |DU|^{2p} \, dx + \hat{I}_1(t) + I_2(t)\right].
\]

We now apply Lemma 3.5 to \( f = I_1 \) and get

\[
I_1(\rho) \leq C(\nu_0)C_4 \|U\|_{BMO(B_R)} \left[\frac{1}{(R-\rho)^2} \int_{B_R} \Phi_0^2 |DU|^{2p} \, dx + \hat{I}_1(R) + I_2(R)\right].
\]

Obviously, the above also holds for \( \rho, R \) being replaced by \( s, t \). For \( C_0 = C(\nu_0)C_4 \) the lemma then follows. \( \Box \)
4. Proofs of the main theorems. We now go back to the iterative scheme (2.6) and prove our main theorems in this section. In the rest of this paper we will slightly abuse our notations and write the dot product \( \langle u, v \rangle \) as \( uv \) for any two vectors \( u, v \) because its meaning should be clear in the context. Similarly, when there is no ambiguity \( C \) will denote a universal constant that can change from line to line in our argument. Furthermore, \( C(\cdots) \) is used to denote quantities which are bounded in terms of their parameters.

The following lemma is the main vehicle of the proof of Theorem 2.3.

Lemma 4.1. Assume A.1)-A.3), R)and V). Let \( \beta, \alpha > 0 \) and \( \alpha > 1 \). Let \( Q \) be a number such that

\[
\frac{2p - 2}{2p} < \sup_{u \in \mathbb{R}^m} \frac{\lambda(u)}{\Lambda(u)}.
\]

If \( R \) is sufficiently small then for any two concentric balls \( B_R \subset B_R \) with center in \( \Omega \) there is a constant \( C(T) \) such that the following holds for all integers \( k \geq 1 \)

\[
\sup_{t \in (0,T)} \int_{B_R \cap \Omega} |Du_k|^{2p} \, dx + \int_{Q_R} \lambda(u_{k-1})|Du_k|^{2p-2}|D^2 u_k|^2 \, dz \leq C_1(T) \int_{Q_R} \left[ \lambda(u_{1})|Du_{2}|^{2p-2}|D^2 u_{2}|^2 + \frac{|A_{u_1}(u_{1})|^2}{\lambda(u_{1})} |Du_{2}|^{2p+2} \right] \, dz
\]

\[
+ C_1(T) \max_{1 \leq k \leq R} \int_{Q_R} \lambda((u_{k-1})|Du_k|^{2p} \, dz.
\]

Here, \( Q_R = (B_R \cap \Omega) \times (0, T) \).

Before going to the proof, we recall the following elementary fact in [1], Lemma 2.1.

Lemma 4.2. Assume the ellipticity condition A). Let \( \alpha \) be a number such that there is \( \delta_\alpha \in (0, 1) \) such that \( \frac{2p - 2}{2p} = \frac{1}{\delta_\alpha} \). We then have

\[
AD\zeta D(\zeta|\zeta|^\alpha) \geq \hat{\lambda}|\zeta|^\alpha|D\zeta|^2, \quad \hat{\lambda} = (1 - \delta_\alpha^2)\lambda.
\]

Furthermore, since \( u_{k-1}, u_k \) are \( C^2 \) in \( x \), we can differentiate (2.6) with respect to \( x \) to get

\[
(Du_k)_t = div((A(u_{k-1})D^2 u_k + A_{u}(u_{k-1})Du_{k-1} Du_k) + Df(u_{k-1}, Du_k) \quad k \geq 1.
\]

Proof. (Proof of Lemma 4.1) We consider the interior case \( B_R \subset B_R \subset \Omega \) and leave the boundary case, when the center of \( B_R \) is on the boundary \( \partial \Omega \), to Remark 4.4 following the proof. For any \( s, t \) such that \( 0 \leq s < t \leq R \) let \( \psi \) be a cutoff function for \( B_s \) to \( B_t \). That is, \( \psi \equiv 1 \) in \( B_s \) and \( \psi \equiv 0 \) outside \( B_t \) with \( |D\psi| \leq 1/(t - s) \). Testing (4.4) with \( |Du_k|^{2p-2}Du_k \psi^2 \). The assumption (4.4) shows that \( \alpha = 2p - 2 \) satisfies the condition of Lemma 4.2 so that we can find a positive constant \( C(p) \) such that

\[
\sup_{\tau \in (0,T)} \int_{\Omega} |Du_k|^{2p} \psi^2 \, dx + C(p) \int_{Q} \lambda(u_{k-1})|Du_k|^{2p-2}|D^2 u_k|^2 \psi^2 \, dz \leq
\]

\[
\int_{Q} |A(u_{k-1})| |D^2 u_k| |Du_k|^{2p-1} \psi |D\psi| \, dz
\]

\[
- \int_{Q} A_{u}(u_{k-1})Du_{k-1} Du_k |Du_k|^{2p-2} \psi \, dz
\]

\[
+ \int_{Q} Df(u_{k-1}, Du_k) |Du_k|^{2p-2} \psi \, dz.
\]
For simplicity, we will assume in the sequel that \( f \equiv 0 \). The presence of \( f \) will be discussed in Remark 4.3 after the proof. We also note that
\[
|A_{u}(u_k-1)Du_k - Du_k D([Du_k]^{2p-2} D u_k \psi^2)| \leq \\
|A_{u}(u_k-1)||Du_k|^{2p-1}|D^2 u_k \psi^2| + |A_{u}(u_k-1)||Du_k|^{2p}|D \psi|.
\]

By Young’s inequality, we can find a constant \( C(\varepsilon) \) such that the followings estimates hold for any given \( \varepsilon > 0 \)
\[
|A_{u}(u_k-1)||D^2 u_k|^{2p-1} D u_k \psi^2| \leq \\
\varepsilon A_{u}(u_k-1)||D^2 u_k|^{2p-2} D u_k \psi^2 + C(\varepsilon) A_{u}(u_k-1)||D^2 u_k\|^{2p} D \psi^2,
\]
\[
|A_{u}(u_k-1)||Du_k|^{2p-1} D u_k \psi^2 \leq \\
\varepsilon A_{u}(u_k-1)||Du_k|^{2p-2} D u_k \psi^2 + \Phi^2(u_k-1)||Du_k|^{2p} D \psi^2,
\]
where \( \Phi^2(u) = \frac{A_u(u)\|^2}{\lambda(u)} \) as in A.2. Also, with \( \Phi^2_0(u) = \lambda(u) \), we have
\[
|A_{u}(u_k-1)||Du_k|^{2p} D \psi^2 = \Phi(u_k-1)||Du_k|^{p} D \psi^2 \leq \\
\Phi^2(u_k-1)||Du_k|^{2p} D \psi^2 + |D \psi^2| \Phi^2_0(u_k-1)||Du_k|^{2p}.
\]

Therefore, for small and fixed \( \varepsilon \) we deduce
\[
\sup_{\tau \in (0,T)} \int_{Q_t} |Du_k|^{2p} D \psi^2 \, dx + \int_{\Omega} \int_{Q} \Phi^2_0(u_k-1)||Du_k|^{2p-2} D u_k \psi^2 \, dx \, dz \leq \\
C_1 \int_{Q_t} \Phi^2(u_k-1)||Du_k|^{2p} D \psi^2 \, dz + \sup_{Q_t} |D \psi|^2 \int_{Q_t} \Phi^2_0(u_k-1)||Du_k|^{2p} \, dz.
\]

Here, we denoted \( Q_t = B_t \times (0, T) \). Again, a use of Young’s inequality to the first integral on the right yields
\[
\sup_{\tau \in (0,T)} \int_{\Omega} |Du_k|^{2p} D \psi^2 \, dx + \int_{Q} \Phi^2_0(u_k-1)||Du_k|^{2p-2} D u_k \psi^2 \, dx \, dz \leq \\
C_2 \int_{Q_t} \Phi^2(u_k-1)||Du_k|^{2p+2} + |Du_k|^{2p+2} D \psi^2 \, dz + C \sup_{Q_t} |D \psi|^2 \int_{Q_t} \Phi^2_0(u_k-1)||Du_k|^{2p} \, dz.
\]

By the choice of \( \psi \), we obtain from the above the following
\[
\sup_{\tau \in (0,T)} \int_{Q_t} |Du_k|^{2p} D \psi^2 \, dx + H(s) \leq C_2(B_0(t) + B_1(t)) + C \frac{1}{(t-s)\zeta^2} G(t).
\] (4.5)

Here, for any fixed integer \( k \geq 1 \), we set
\[
H(t) = \int_{Q_t} \Phi^2_0(u_k-1)||Du_k|^{2p+2} D u_k \psi^2 \, dz, B_1(t) = \int_{Q_t} \Phi^2(u_k-1)||Du_k|^{2p} \, dz,
\]
and
\[
B_0(t) = \int_{Q_t} \Phi^2(u_k-1)||Du_k|^{2p+2} D \psi^2 \, dz, G(t) = \int_{Q_t} \Phi^2_0(u_k-1)||Du_k|^{2p} \, dz.
\]

We now apply Lemma 3.4 for \( u = u_k-1 \) and \( U = u_k \). We will see that our assumptions A.2) and A.3) imply the assumptions GN.1) and GN.2) of Lemma 3.4 for any \( p \geq 1 \). Indeed, by our assumption \( \Phi_0, \Phi \in A.2) \) the constants in GN.1) are finite. Furthermore, since \( u_k-1 \) is BMO with uniform bounded norm and the assumption A.3), \( \Phi^2(u_k-1) \) belongs to the \( A_{\frac{3}{2}} \) class. As \( \frac{4}{3} \geq \frac{2}{p+2} \) and \( \frac{4}{3} \leq \frac{2}{p+2} + 1 \), \( \Phi^2(u_k-1) \) belongs to the \( A_{\frac{3}{2}+1} \) class. Thus, the quantity \( C_0 \) in (4.6) is finite. Also, our continuity assumption \( \Phi_0, \Phi \in A.2) \) on the BMO norm of \( u_k \)
implies the smallness of \( C_0\|u_k\|_{BMO(B_R)} \) if \( R \) is small. Hence, for any given \( \mu_1 > 0 \) if \( R = R(\mu_1) > 0 \) is sufficiently small then we have from (3.13) of Lemma 3.4 the following estimate.

\[
\int_{B_s} \Phi^2(u_{k-1})|Du_k|^{2p+2} \, dx \leq \mu_1 \int_{B_s} \Phi^2(u_{k-1})|Du_k|^{2p-2}|D^2u_k|^2 \, dx + \\
\mu_1 \left[ \int_{B_t} \Phi^2(u_{k-1})|Du_k|^{2p+2} \, dx + \frac{1}{(t-s)^2} \int_{B_t} \Phi^2(u_{k-1})|Du_k|^{2p} \, dx \right].
\] 

Then (4.6) and (4.5) give a positive constant \( C_2 \), which is redefined and can depend on \( k_1, k_2 \), such that

\[
B_1(s) \leq \mu_1[H(t) + B_0(t) + \frac{1}{(t-s)^2}G(t)] \quad \rho < s < t < R, \tag{4.7}
\]

\[
H(s) \leq C_2B_1(t) + C_2B_0(t) + \frac{1}{(t-s)^2}G(t) \quad \rho < s < t < R. \tag{4.8}
\]

Let \( t' = s + (t-s)/2 \). Using (4.7) with \( s \) being \( t' \) in the inequality (4.8) with \( t' \) being \( t' \) and the fact that \( H, B_0, G \) are increasing, we get

\[
H(s) \leq C_2\mu_1H(t) + C_2(\mu_1 + 1)B_0(t) + \frac{4(\mu_1 + 1)}{(t-s)^2}G(t) \quad \rho < s < t < R. \tag{4.9}
\]

We can assume that \( \mu_2 = C_2\mu_1 < 1 \). By Lemma 3.5, (4.9) yields

\[
H(\rho) \leq C_3(C_2(\mu_1 + 1)B_0(R) + \frac{4(\mu_1 + 1)}{(R-\rho)^2}G(R)).
\]

Here, \( C_3 = (1-\nu)^{-2}(1-\nu^{-2}\mu_2)^{-1} \) for any \( \nu \) satisfying \( \nu^{-2}\mu_2 < 1 \). Obviously, the above also hold with \( \rho, R \) replaced by \( s, t \) with \( \rho \leq s < t \leq R \). So,

\[
H(s) \leq C_3[C_2(\mu_1 + 1)B_0(t) + \frac{4(\mu_1 + 1)}{(t-s)^2}G(t)]. \tag{4.10}
\]

We now let \( t' = (s+t)/2 \) and use (4.7) with \( t \) being \( t' \) and then (4.10) with \( s \) being \( t' \) to see that

\[
B_1(s) \leq \mu_1[H(t') + B_0(t') + \frac{1}{(t'-s)^2}G(t')] \\
\leq \mu_1[C_3(C_2(\mu_1 + 1)B_0(t) + \frac{4(\mu_1 + 1)}{(t-s)^2}G(t))] + B_0(t') + \frac{1}{(t'-s)^2}G(t')].
\]

Since \( B_1, G \) are increasing functions, the above yields

\[
B_1(s) \leq \mu_3B_0(t) + C_4 \frac{1}{(t-s)^2}G(t), \tag{4.11}
\]

where \( \mu_3 = \mu_1(C_3C_2(\mu_1 + 1) + 1) \) and \( C_4 = 4(\mu_1(\mu_1 + 1) + \mu_1) \).

We now consider \( B_0 \). Applying Lemma 3.4 with \( u = U = u_{k-1} \), so that \( I_1 = \hat{I}_1 \), and \( \Phi(u) = \Phi_0(u) \), we see that if \( \|u_{k-1}\|_{BMO(B_R)} \), or \( R \), is sufficiently small then there is a constant \( C_0(\Phi, \Phi_0) \) such that for any \( s, t \) satisfying \( 0 < s < t < R \)

\[
I_1(s) \leq C_0(\Phi)\|u_{k-1}\|_{BMO(B_R)} \left[ \frac{1}{(t-s)^2} \iint_{Q_t} |\Phi(u_{k-1})|^2 |Du_{k-1}|^{2p} \, dz + I_2(t) \right]
\]

with

\[
I_1(t) = \iint_{Q_t} \Phi^2(u_{k-1})|Du_{k-1}|^{2p+2} \, dz = B_0(t),
\]

\[
I_2(t) = \iint_{Q_t} \Phi^2(u_{k-1})|Du_{k-1}|^{2p-2}|D^2u_{k-1}|^2 \, dz.
\]
Going back to the notation $\Phi_0(u) = \lambda^2(u)$, by $[2.8]$, we can split $Q_t$ into two disjoint sets

$$Q_{(1)} := \left\{(x, t) \in Q_t : \Phi(u_{k-1})(x, t) \leq C(T)\Phi(u_{k-2})(x, t)\right\},$$

$$Q_{(2)} := \left\{(x, t) \in Q_t : \Phi_0(u_{k-1})(x, t) \leq C(T)\Phi_0(u_{k-2})(x, t)\right\}.$$

Recall that $\Phi(u) \leq k_2\Phi_0(u)$ by $A.2$. On $Q_{(1)}$, we have $\Phi(u_{k-1}) \leq C(T)\Phi(u_{k-2}) \leq C(T)k_2\Phi_0(u_{k-2})$. Meanwhile, on $Q_{(2)}$, $\Phi(u_{k-1}) \leq k_2\Phi_0(u_{k-1}) \leq C(T)k_2\Phi_0(u_{k-2})$. Thus,

$$I_2(t) = \int_{Q_t} \Phi^2(u_{k-1})|Du_{k-1}|^{2p-2}|D^2u_{k-1}|^2 \, dz \leq C(T)k_2 \int_{Q_t} \Phi^2_0(u_{k-2})|Du_{k-1}|^{2p-2}|D^2u_{k-1}|^2 \, dz.$$

Similarly,

$$\int_{Q_t} |\Phi(u_{k-1})|^2|Du_{k-1}|^{2p} \, dz \leq C(T)k_2 \int_{Q_t} |\Phi_0(u_{k-2})|^2|Du_{k-1}|^{2p} \, dz. \quad (4.13)$$

Using these estimates in $(4.12)$, we obtain

$$B_0(s) \leq C_1(\Phi, \Phi_0, T)||u_{k-1}||_{BMO(B_0)}\left[\frac{1}{(t-s)^2}G_0(t) + H_0(t)\right],$$

where

$$G_0(t) = \int_{Q_t} |\Phi_0(u_{k-2})|^2|Du_{k-1}|^{2p} \, dz,$$

$$H_0(t) = \int_{Q_t} \Phi^2_0(u_{k-2})|Du_{k-1}|^{2p-2}|D^2u_{k-1}|^2 \, dz.$$

Using the above estimate for $B_0$ in $[4.10]$ and $[4.11]$ and adding the results, we can easily see that if $||u_{k-1}||_{BMO(B_0)}$ is sufficiently small then

$$H(s) + B_1(s) \leq \mu_4H_0(t) + \frac{C_5}{(t-s)^2}[G(t) + G_0(t)], \quad (4.14)$$

for some $C_5$ depending on $\Phi, \Phi_0, k_1, k_2, T$ and

$$\mu_4 = C_1(\Phi, \Phi_0, T)||u_{k-1}||_{BMO(B_0)}[C_2(\mu_1 + 1) + \mu_3].$$

We now define

$$B_k(t) = \int_{Q_t} [\lambda(u_{k-1})|Du_k|^{2p-2}|D^2u_k|^2 + \Phi^2(u_{k-1})||Du_k||^{2p+2}] \, dz,$$

$$G_k(t) = \int_{Q_t} [\lambda(u_{k-1})|Du_k|^{2p} + \lambda(u_{k-2})|Du_{k-1}|^{2p}] \, dz.$$

We then have from $(4.14)$ that

$$B_k(s) \leq \mu_4B_{k-1}(t) + \frac{C_5}{(t-s)^2}G_k(t). \quad (4.15)$$

As before, we can assume that $R$ is sufficiently small such that $\mu_4 < 1$. For any $a \in (0, 1)$ such that $\mu_4a^{-2} < 1$ we define the sequences $t_0 = \rho$ and $t_{i+1} = t_i + (1 - a)a^i(R - \rho)$. Iterate the above $k - 2$ times to get

$$B_k(\rho) \leq \mu_4^kB_2(t_{k-2}) + \sum_{i=0}^{k-2} \mu_4^i a^{-2i} \frac{C_5}{(t-a)^2(R-\rho)^2}G_{k-i}(t_{i+1}) \leq B_2(R) + \frac{C_6(a, \mu_4)}{(R-\rho)^2} \max_{2 \leq i \leq k} G_i(R).$$
This shows that the quantity
\[ \int_{Q_t} [\lambda(u_{k-1})|Du_k|^{2p-2}|D^2u_k|^2 + \Phi^2(u_{k-1})||Du_k|^{2p+2}] \, dz, \quad t \geq \rho, \ k \geq 1 \]
can be bounded by
\[ C_1(T) \int_{Q_R} \left[ \lambda(u_1)|Du_2|^{2p-2}|D^2u_2|^2 + \frac{|A_{u_1}|^2}{\lambda(u_1)}|Du_2|^{2p+2} \right] \, dz \]
\[ + \frac{1}{(R-\rho)^n} \max_{1 \leq i < k} \int_{Q_R} \lambda((u_{k-1}))|Du_k|^{2p} \, dz. \]

Using this and (4.13) in (4.15), and the estimate for \( B_0(t) \), we obtain (4.2) of the lemma.

\[ \square \]

**Remark 4.3.** The presence of \( f(u, Du) \neq 0 \) causes few more extra terms which can be handled as follows. Assume that \( f(u, Du) \) satisfying F such that
\[ |f(u, p)| \leq C \lambda^{\frac{1}{2}}(u)|p| + g(u) \]
and
\[ |Df(u, p)| \leq C |D(\lambda^{\frac{1}{2}}(u))| + |g_u(u)||Du|. \]
Then we have
\[ |Df(u, p)| \leq C \lambda^{\frac{1}{2}} |Dp| + C \Phi^{\frac{1}{2}}(u)|Du||p| + |g_u(u)||Du|. \]
Therefore, the extra term \( |Df(u_{k-1}, Du_k)||Du_k|^{2p-1} \) in the proof can be treated by using the following estimates, which are the results of a simple use of Young's inequality.
\[ |Df(u_{k-1}, Du_k)||Du_k|^{2p-1} \leq C|\lambda^{\frac{1}{2}}(u_{k-1})|D^2u_k| + C\Phi^{\frac{1}{2}}(u_{k-1})|Du_k| + g_u(u_{k-1})||Du_k|^{-1} \]
\[ \leq \lambda(u_{k-1})|Du_k|^{2p-2}|D^2u_k|^2 + C(\varepsilon)|\lambda(u_{k-1})|Du_k|^{2p-1} + C\Phi^{\frac{1}{2}}(u_{k-1})||Du_k|-1||Du_k|^{-1}. \]

By Young's inequality, we easily see that
\[ \Phi^{\frac{1}{2}}(u_{k-1})||Du_k|-1||Du_k|^{2p} \leq C(\Phi(u_{k-1}))|Du_k|^{2p+1} + C(\Phi(u_{k-1})) \frac{1}{2p} |Du_k|-1|^{p+1}. \]

The case \( p = 1 \) is easy so we consider \( p > 1 \). We have
\[ \Phi(u_{k-1}) \frac{1}{2p} |Du_k|-1|^{p+1} = \Phi(u_{k-1}) \frac{1}{2p} \lambda(u_{k-1})^{-\frac{p+1}{p}} \lambda(u_{k-1})^{\frac{p+1}{p}} |Du_k|-1|^{p+1} \]
\[ \leq C\lambda(u_{k-1})||Du_k|-1|^{2p} + C\Phi(u_{k-1})^{\frac{1}{2p}} \lambda(u_{k-1})^{1/p}. \]

As we assume that \( \lambda(u) \) has polynomial growth in \( u \) then so does \( \Phi(u_{k-1})^{\frac{1}{2p}} \lambda(u_{k-1})^{\frac{1}{p}} \) and thus the integral of the latter is finite because \( u_{k-1} \) is BMO. On the other hand, since \( |g_u(u)| \leq C\lambda(u) \) for some constant \( C \)
\[ |g_u(u_{k-1})||Du_k|-1||Du_k|^{2p-1} \leq C\lambda(u_{k-1})||Du_k|-1|^{2p} + C\lambda(u_{k-1})||Du_k|^{2p}. \]

We then see that the presence of \( f \) will give rise to terms that can be handled as in the proof.

**Remark 4.4.** We discuss the case when the centers of \( B_r, B_R \) are on the boundary \( \partial\Omega \). Let us first consider the case \( u \) satisfies the Dirichlet condition \( u = 0 \) on \( \partial\Omega \). By flattening the boundary we can assume that \( B_R \cap \Omega \) is the set
\[ B^+ = \{ x : x = (x_1, \ldots, x_n) \text{ with } x_n \geq 0 \text{ and } |x| < R \}. \]

For any point \( x = (x_1, \ldots, x_n) \) we denote by \( \tilde{x} \) its reflection across the plane \( x_n = 0 \), i.e., \( \tilde{x} = (x_1, \ldots, -x_n) \). Accordingly, we denote by \( B^- \) the reflection of
For $u = u_k$ we define the odd reflection of $u$ by $\bar{u}$, i.e. $\bar{u}(x,t) = -u(x,t)$ for $x \in B^-$. We then consider the odd extension $U$ in $B = B^+ \cup B^-$ of $u$

$$U(x,t) = \begin{cases} u(x,t) & \text{if } x \in B^+, \\ \bar{u}(x,t) & \text{if } x \in B^-. \end{cases}$$

It is easy to see that $\bar{u}$ satisfies in $B^-$ a system similar to (4.4) for $u$ in $B^+$. As in the proof of the lemma, we test the system for $u_k$ with $|Du_k|^{p-2}Du_k\psi^2$ and the system for $\bar{u}$ with $|D\bar{u}_k|^{2p-2}D\bar{u}_k\psi^2$ and then sum the results. The integration parts result in the extra boundary terms along the flat boundary parts $\partial B^+$ and $\partial B^-$. Using the facts that either $D_{x_i}u = D_{x_i}\bar{u} = 0$ for $i \neq n$ or $D_{x_n}u = D_{x_n}\bar{u}$ and the outward normal vectors of $B^+$ and $B^-$ are opposite we can easily see that those boundary terms are either zero or cancel each others in the summation. Thus, we can obtain (4.5) again with $u_{k-1}, u_k$ being replaced by $U_{k-1}, U_k$. Since $U_k$ belong to $W^{2,\infty}(B)$ the argument can continue and the lemma holds for $U_k$ and then $u_k$.

The same argument applies for the Neumann boundary condition if we use the even extension for $u_k$.

We now give the proof of Theorem 2.3

Proof. We test the systems (2.6) with $u_k$ and use Young’s inequality to have

$$\sup_{t \in [0,T]} \int_{\Omega} |u|^2 \, dx + \int_{Q} \lambda(u_{k-1})|Du_k|^2 \, dz \leq C \int_{Q} [u_{k-1}]^2 + |u_{k-1}|^{b+1} + 1 \, dz.$$

The uniform bound assumption on the BMO norms of $u_{k-1}$ yields that the right hand side is bounded uniformly for all $k$. Thus, there is a constant $C$ such that

$$\int_{Q} \lambda(u_{k-1})|Du_k|^2 \, dz \leq C \quad \forall k.$$

Now, for any $0 < \rho < R$ and concentric balls $B_{\rho}, B_{R}$ with centers in $\bar{\Omega}$ let us assume that there is some $\rho \geq 1$ such that there is a constant $C_0(\rho, R, u_0)$ depending on $\rho, R$ and $sup_{t \in (0,T)} \|u_0(\cdot,t)\|_{C^1(\Omega)}$ on such that $(Q_R = B_R \times (0,T))$

$$\int_{Q_R} \lambda(u_{k-1})|Du_k|^{2p} \, dz \leq C_0(\rho, R, u_0) \quad \forall k. \quad (4.16)$$

It is well known that the $C^1$ norms of $u_1$ and $u_2$ can be bounded by that of $u_0$. Now, if $p$ satisfies (4.11) then Lemma 4.1 and (4.16) establish the existence of a constant $C_1(\rho, R)$ such that the following holds for all integers $k$

$$\sup_{t \in (0,T)} \int_{B_{\rho}} |Du_k|^{2p} \, dx + \int_{Q_{\rho}} \Phi_1(u_{k-1})|Du_k|^{2p-2}|Du_k|^2 \, dz \leq C_1(\rho, R, u_0) \quad (4.17)$$

if $0 < \rho < R$ and $R$ is sufficiently small.

Let $\chi_0$ be any number such that $1 < \chi_0 < 1 + \frac{2}{n}$. Denote $V = |Du_k|^p$ and use Hölder’s inequality to get

$$\int_{Q} \lambda V^{2\chi_0} \, dz \leq \left( \int_{Q} \lambda^r \, dz \right)^{\frac{1}{r}} \left( \int_{Q} V^{2(1+\frac{2}{\chi_0})} \, dz \right)^{\frac{1}{\chi_0}}$$

where $\lambda = \lambda(u_{k-1})$ and $r$ is a number such that $r'\chi_0 = 1 + \frac{2}{n}$.

Recall the Sobolev imbedding inequality

$$\|V\|_{L^{\frac{2(n+2)}{n}}(Q)} \leq C \sup_t \|V(\cdot,t)\|_{L^2(\Omega)} + C \left( \int_{Q} |DV|^2 \, dz \right)^{\frac{1}{2}}$$
and the fact that $u_k$ is BMO so that $\lambda(u_{k-1})$ belongs to $L^r(\Omega)$ for any $r > 1$ (see \[1\]). The above estimates for $Q = Q_0$ show that there is a constant $C(\rho)$ such that

$$
\int_{Q_0} \lambda V^{2\chi_0} \, dz \leq C(\rho) \left[ \left\| \nabla V(z, t) \right\|_{L^2(B_\rho)} + \left( \int_{Q_0} |D\lambda|^2 \, dz \right)^{\frac{1}{2}} \right]. \tag{4.18}
$$

From the ellipticity condition $A)$ and (4.17) we see that the right hand side is bounded. Hence

$$
\int_{Q} \lambda(u_{k-1})|Du_k|^{2p\chi_0} \, dz \leq C_2(\rho, R, u_0) \quad \forall k.
$$

Therefore, (1.16) holds again with $p$ now is $p\chi_0$. We already showed that (1.16) is valid for $p = 1$. Thus, we can repeat the argument $k$ times until $2\chi_0^n > n$ as long as the ratio condition (1.1) of Lemma 1.1 is verified for $p = \chi_0^n$. The assumption R) shows that we can choose $\chi_0^n, k$ such that $1 < \chi_0^n < 1 + \frac{2}{n}, 2\chi_0^n > n$ and the ratio condition (1.1) holds for $p = \chi_0^n$. Therefore, (4.17) holds for $2p = 2\chi_0^n$. We now cover $\Omega$ with finitely many balls of radius $R/2$ to obtain

$$
\sup_{t \in (0, T)} \int_{\Omega} |Du_k|^2 \, dx + \int_{Q} \Phi_0^2(u_k) |Du_k|^{2p-2} |D^2u_k|^2 \, dz \leq C(\Omega, T, u_0). \tag{4.19}
$$

For each $t \in (0, T)$, (4.19) shows that the norms $\|u_k(\cdot, t)\|_{W^{1, 2p}(\Omega)}$ for some $2p > n$ are bounded uniformly in $t$ by a constant depending only on the size of $\Omega, T$ and $u_0$. By Sobolev’s imbedding theorem $\{u_k(\cdot, t)\}$ is a bounded sequence in $C^\alpha(\Omega)$ for some $\alpha > 0$. From the system for $u_k$, (4.19) with $p = 1$ also shows that $\|u_k(\cdot, t)\|_{L^2(\Omega)}$ is uniformly bounded. Together, these facts show that the solutions $u_k$ are uniformly Hölder continuous in $(x, t)$ and that $\{u_k\}$ is bounded in $C^\beta(Q)$ for some $\beta > 0$. We then see that there is a subsequence of $\{u_k\}$ converging in $C^\beta(\Omega)$ to $u$, the weak limit of $\{u_k\}$ in $L^2(Q)$ from the assumption of the theorem. Using difference quotient in $t$ we see that $u_t \in L^2(Q)$. We can see from the estimate (4.19) that $Du_{k+1}(\cdot, t)$ converges weakly to $Du(\cdot, t)$ in $L^2(\Omega)$ for each $t \in (0, T)$. By the continuity of $A$ in its variable $u$, we see that $u$ weakly solves (1.1).

By the semicontinuity of norms, (4.19) implies

$$
\sup_{t \in (0, T)} \int_{\Omega} |Du|^2 \, dx + \int_{Q} \lambda(u)|Du|^{2\beta-2} |D^2u|^2 \, dz \leq C(\Omega, u_0). \tag{4.20}
$$

Since $2p > n$, the above implies that $u$ is Hölder continuous and its regularity in $x$. Since $u_t$ is in $L^2(Q)$, it is easy to derive from these facts that $u$ is Hölder in $(x, t)$.

We now turn to the proof of our second theorem.

**Proof.** (Proof of Theorem 2.3) Let $u$ be the classical solution of the system (1.1) in $\Omega \times (0, T_0)$. We can differentiate (1.1) to have

$$
(Du)_t = \text{div}(A(u)D^2u + A_u(u)DuD\nu) + Df(u, Du). \tag{4.21}
$$

For any $s, t$ such that $0 < s < t < R$ let $\psi$ be a cutoff function for two concentric balls $B_s, B_t$ with centers in $\Omega$. That is, $\psi \equiv 1$ in $B_s$ and $\psi \equiv 0$ outside $B_t$ with $|D\psi| \leq 1/(t-s)$. As in the proof of the previous theorem, we test (4.21) with
Thus, as in the proof of Lemma 4.1, we can use Lemma 3.4 to obtain
\[
\int_{B_t} \Phi^2 |Du|^{2p+2} \, dx \leq \mu_1 \int_{B_t} \Phi_0^2 |Du|^{2p-2} |D^2u|^2 \, dx + \\
\mu_1 \int_{B_t} \Phi^2 |Du|^{2p+2} \, dx + \frac{1}{(t-s)^2} \int_{B_t} \Phi^2 |Du|^{2p} \, dx.
\] (4.22)

Here, \( \Phi_0^2 = \lambda(u) \) and \( \Phi^2 = \frac{A_2(u)}{\lambda(u)} \). We now set
\[
H(t) = \int_{Q_t} \Phi_0^2(u)|Du|^{2p-2} |D^2u|^2 \, dz, \quad B(t) = \int_{Q_t} \Phi^2(u)|Du|^{2p+2} \, dz,
\]
\[
G(t) = \int_{Q_t} \Phi^2(u)|Du|^{2p} \, dz.
\]

Because A.4) obviously holds for \( u_k = u_{k-1} = u \), it is clear that the proof of Lemma 4.1 with \( H, H, B_0, B_1, B_k, \) and \( G \) being replaced by the new definitions, and \( B_0 = B_1 = B_k = B \), now leads to
\[
B(s) \leq \mu_4 B(t) + C_4 \frac{1}{(t-s)^2} G(t), \quad 0 < \rho < s < t < R,
\] (4.23)

where \( \mu_4 < 1 \). For any \( a \in (0,1) \) such that \( \mu_4 a^{-2} < 1 \) we define the sequences \( t_0 = \rho \) and \( t_{i+1} = t_i + (1-a) \alpha(R-\rho) \). Iterate the above to get
\[
B(\rho) \leq \mu_4^k B(t_k) + \sum_{i=0}^{k-1} \mu_4^i a^{-2i} \frac{C_4}{(1-a)^2(R-\rho)^2} G(t_k).
\]

Let \( k \) tend to infinity and use the fact that \( \mu_4 \in (0,1) \) and \( B(R) \) is finite to get
\[
B(\rho) \leq \frac{C_5(a, \mu_4)}{(R-\rho)^2} G(R).
\]

We now see that a similar argument as in the proof of Theorem 2.3 with \( u_k \) being \( u \) now gives
\[
\sup_{t \in (0,T)} \int_{B_\rho} |Du|^{2p} \, dx + \int_{Q_\rho} \Phi_0^2 |Du|^{2p-2} |D^2u|^2 \, dz \leq C_1(\rho, R)
\] (4.24)

if \( 0 < \rho < R \) and \( R \) is sufficiently small and some \( p \) such that \( 2p > n \). Finite covering \( \Omega \) with balls \( B_{R/2} \) yields
\[
\sup_{t \in (0,T)} \int_{\Omega} |Du|^{2p} \, dx + \int_{Q} \lambda(u)|Du|^{2p-2} |D^2u|^2 \, dz \leq C(\Omega, R).
\] (4.25)

Hence \( u \) is Hölder continuous in \( x \). From the system for \( u \) and the above, with \( p = 1 \), we see that \( u_t \) is in \( L^2(Q) \). It is now standard to show that \( u \) is Hölder in \( (x,t) \) and \( Du \) is Hölder continuous. We now can refer to Amann’s results to see that \( u \) exists globally. \( \Box \)

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