Querying RDF Databases with Sub-CONSTRUCTs

Dominique Duval  
LJK - Univ. Grenoble Alpes  
dominique.duval@univ-grenoble-alpes.fr

Rachid Echahed  
LIG - Univ. Grenoble Alpes  
rachid.echahed@imag.fr

Frédéric Prost  
LIG - Univ. Grenoble Alpes  
frederic.prost@univ-grenoble-alpes.fr

Graph query languages feature mainly two kinds of queries when applied to a graph database: those inspired by relational databases which return tables such as SELECT queries and those which return graphs such as CONSTRUCT queries in SPARQL. The latter are object of study in the present paper. For this purpose, a core graph query language GrAL is defined with focus on CONSTRUCT queries. Queries in GrAL form the final step of a recursive process involving so-called GrAL patterns. By evaluating a query over a graph one gets a graph, while by evaluating a pattern over a graph one gets a set of matches which involves both a graph and a table. CONSTRUCT queries are based on CONSTRUCT patterns, and sub-CONSTRUCT patterns come for free from the recursive definition of patterns. The semantics of GrAL is based on RDF graphs with a slight modification which consists in accepting isolated nodes. Such an extension of RDF graphs eases the definition of the evaluation semantics, which is mainly captured by a unique operation called Merge. Besides, we define aggregations as part of GrAL expressions, which leads to an original local processing of aggregations.

1 Introduction

Graph database query languages are becoming ubiquitous. In contrast to classical relational databases where SQL language is a standard, different languages [1] have been proposed for querying graph databases, like SPARQL [9] or Cypher [5]. Among the most popular models for representing graph databases, one may quote for instance the sets of triples (or RDF graphs [10]) used by SPARQL or the property graphs used by Cypher. In addition to the lack of a standard model to represent graph databases, there are different kinds of queries in the context of graph query languages. One may essentially distinguish two classes of queries: those inspired by relational databases which return tables such as SELECT queries and those which return graphs such as CONSTRUCT queries in SPARQL. Such CONSTRUCT queries are graph-to-graph queries specific to graph databases.

The graph-to-graph queries received less attention than the graph-to-table queries. For instance, for the SPARQL language, a semantics of SELECT queries and subqueries is proposed in [6], a semantics of CONSTRUCT queries in [7] and a semantics of CONSTRUCT queries with nested CONSTRUCT queries in FROM clauses in [2, 8], where the outcome of a subCONSTRUCT is a graph. All these works consider graphs as sets of triples. Unfortunately, such a definition of graphs prevents having a uniform semantics of all patterns and in particular for BIND patterns.

In this paper, we focus on graph-to-graph queries and subqueries for RDF graphs and we propose a new semantics for CONSTRUCT subqueries which departs from the one in [2, 8]. First, we propose to change slightly the definition of graphs by allowing isolated nodes. Indeed, this new definition of graphs allows us to have a uniform semantics of all patterns. In fact, we define CONSTRUCT subpatterns rather...
than CONSTRUCT subqueries. For this purpose, we introduce a core query language GrAL based on RDF graphs. The syntactic categories of GrAL include both queries and patterns. When evaluating a CONSTRUCT query over a graph one gets a graph, whereas when evaluating a CONSTRUCT pattern over a graph one gets a set of matches which involves both variable assignments and a graph. In fact, a CONSTRUCT query first acts as a CONSTRUCT pattern and then returns only the constructed graph. As the definition of patterns is recursive, CONSTRUCT subpatterns are obtained for free.

In order to define the semantics of GrAL, we introduce an algebra of operations over sets of matches, where a match is a morphism between graphs. We propose to base the semantics of GrAL upon an algebra on sets of matches, like the semantics of SQL is based upon relational algebra. All operations in our algebra essentially derive from a unique operation called \textit{Merge}, which generalizes the well-known \textit{Join} operation. As stated earlier, we consider graphs consisting of classical RDF triples possibly augmented with some additional isolated nodes. This slight extension helps formulating the semantics of patterns and queries without using some cumbersome notations to handle, for instance, environments defined by variable bindings. The proposed algebra is used to define an evaluation semantics for GrAL. As for aggregations, they are handled locally inside expressions. The semantics of the various patterns and queries is uniform, as it is based on instances of the \textit{Merge} operation.

The paper is organized as follows. Section 2 introduces the algebra designed to express the semantics of the query language GrAL. In Section 3, the language GrAL is defined by its syntax and semantics. Concluding remarks are given in Section 4.

2 The Graph Query Algebra

The Graph Query Algebra is a family of operations which are used in Section 3 for defining the evaluation of queries in the Graph Algebraic Query Language GrAL. Graphs and matches are introduced in Section 2.1, then operations on sets of matches are defined in Section 2.2.

2.1 Graphs and matches

In this paper, graphs are kinds of generalised RDF graphs that may contain isolated nodes. Let $\mathcal{L}$ be a set, called the set of \textit{labels}, union of two disjoint sets $\mathcal{C}$ and $\mathcal{V}$, called respectively the set of \textit{constants} and the set of \textit{variables}.

\textbf{Definition 2.1} (graph). Every element $t = (s, p, o)$ of $\mathcal{L}^3$ is called a \textit{triple} and its members $s$, $p$ and $o$ are called respectively the \textit{subject}, \textit{predicate} and \textit{object} of $t$. A graph $X$ is made of a subset $X_N$ of $\mathcal{L}$ called the set of \textit{nodes} of $X$ and a subset $X_T$ of $\mathcal{L}^3$ called the set of \textit{triples} of $X$, such that the subject and the object of each triple of $X$ are nodes of $X$. The nodes of $X$ which are neither a subject nor an object are called the \textit{isolated nodes} of $X$. The set of \textit{labels} of a graph $X$ is the subset $\mathcal{L}(X)$ of $\mathcal{L}$ made of the nodes and predicates of $X$, then $\mathcal{C}(X) = \mathcal{C} \cap \mathcal{L}(X)$ and $\mathcal{V}(X) = \mathcal{V} \cap \mathcal{L}(X)$. Given two graphs $X_1$ and $X_2$, the graph $X_1$ is a \textit{subgraph} of $X_2$, written $X_1 \subseteq X_2$, if $(X_1)_N \subseteq (X_2)_N$ and $(X_1)_T \subseteq (X_2)_T$, then obviously $\mathcal{L}(X_1) \subseteq \mathcal{L}(X_2)$. The \textit{union} $X_1 \cup X_2$ is the graph defined by $(X_1 \cup X_2)_N = (X_1)_N \cup (X_2)_N$ and $(X_1 \cup X_2)_T = (X_1)_T \cup (X_2)_T$, then $\mathcal{L}(X_1 \cup X_2) = \mathcal{L}(X_1) \cup \mathcal{L}(X_2)$.

We will not use the \textit{intersection} $X_1 \cap X_2$, which could be defined by $(X_1 \cap X_2)_N = (X_1)_N \cap (X_2)_N$ and $(X_1 \cap X_2)_T = (X_1)_T \cap (X_2)_T$, then the intersection of two graphs without isolated nodes might have isolated nodes and $\mathcal{L}(X_1 \cap X_2)$ might be strictly smaller than $\mathcal{L}(X_1) \cap \mathcal{L}(X_2)$, as for instance when $X_1 = \{(x, y, z)\}$ and $X_2 = \{(y, z, x)\}$ so that $X_1 \cap X_2 = \{x\}$.
Example 2.2. We introduce here a toy database representing a simplified view of a social media network. We will use it as a running example to illustrate various definitions. The network consists in authors publishing messages. Each message is timestamped at a certain date (a day). A message can refer to other messages and an author may like a message. An instance of such a network is described by the following graph $G_0$ (written “à la” RDF):

\[
G_0 = \begin{align*}
\text{auth1 publishes mes1} & . \quad \text{auth1 publishes mes2} . \\
\text{auth2 publishes mes3} & . \quad \text{auth3 publishes mes4} . \quad \text{auth3 publishes mes5} . \\
\text{mes1 stampedAt date1} & . \quad \text{mes2 stampedAt date2} . \\
\text{mes3 stampedAt date1} & . \quad \text{mes4 stampedAt date4} . \quad \text{mes5 stampedAt date4} . \\
\text{mes3 refersTo mes1} & . \quad \text{mes4 refersTo mes1} . \quad \text{mes4 refersTo mes2} . \\
\text{auth1 likes mes3} & . \quad \text{auth1 likes mes4} . \quad \text{auth1 likes mes5} . \\
\text{auth2 likes mes1} & . \quad \text{auth2 likes mes3} . \quad \text{auth2 likes mes4} 
\end{align*}
\]

The meaning of $G_0$ is that author auth1 has published messages mes1 and mes2, which have been stamped respectively at dates date1 and date2, etc.

Definition 2.3 (match). A match $m$ from a graph $X$ to a graph $G$, denoted $m : X \rightarrow G$, is a function from $\mathcal{L}(X)$ to $\mathcal{L}(G)$ which preserves nodes and preserves triples and which fixes $\mathcal{C}$, in the sense that $m(X_N) \subseteq G_N$, $m^3(X_T) \subseteq G_T$ and $m(x) = x$ for each $x \in \mathcal{C}(X)$. The set of all matches from $X$ to $G$ is denoted $\text{Match}(X,G)$. An isomorphism of graphs is an invertible match.

When $n$ is an isolated node of $X$ then the node $m(n)$ does not have to be isolated in $G$. A match $m : X \rightarrow G$ determines two functions $m_N : X_N \rightarrow G_N$ and $m_T : X_T \rightarrow G_T$, restrictions of $m$ and $m^3$ respectively. A match $m : X \rightarrow G$ is an isomorphism if and only if both functions $m_N : X_N \rightarrow G_N$ and $m_T : X_T \rightarrow G_T$ are bijections. This means that a function $m$ from $\mathcal{L}(X)$ to $\mathcal{L}(G)$ is an isomorphism of graphs if and only if $\mathcal{C}(X) = \mathcal{C}(G)$ with $m(x) = x$ for each $x \in \mathcal{C}(X)$ and $m$ is a bijection from $\mathcal{V}(X)$ to $\mathcal{V}(G)$: thus, $X$ is the same as $G$ up to variable renaming. It follows that the symbol used for naming a variable does not matter as long as graphs are considered only up to isomorphism.

Definition 2.4 (image of a graph by a function). Let $X$ be a graph. Every function $f : \mathcal{V}(X) \rightarrow \mathcal{L}$ can be extended in a unique way as a function $f' : \mathcal{L}(X) \rightarrow \mathcal{L}$ that fixes $\mathcal{C}$. The image $f(X)$ of $X$ by $f$ is the graph made of the nodes $f'(n)$ for $n \in X_N$ and the triples $(f')^3(t)$ for $t \in X_T$. The function $f$ can be extended in a unique way as a match $f^2 : X \rightarrow f(X)$.

Definition 2.5 (compatible matches). Two matches $m_1 : X_1 \rightarrow G_1$ and $m_2 : X_2 \rightarrow G_2$ are compatible, written as $m_1 \sim m_2$, if $m_1(x) = m_2(x)$ for each $x \in \mathcal{V}(X_1) \cap \mathcal{V}(X_2)$. Given two compatible matches $m_1 : X_1 \rightarrow G_1$ and $m_2 : X_2 \rightarrow G_2$, let $m_1 \bowtie m_2 : X_1 \cup X_2 \rightarrow G_1 \cup G_2$ denote the unique match such that $m_1 \bowtie m_2 \sim m_1$ and $m_1 \bowtie m_2 \sim m_2$ (which means that $m_1 \bowtie m_2$ coincides with $m_1$ on $X_1$ and with $m_2$ on $X_2$).

We will see in Section 3 that the execution of a query in GrAL is a graph-to-graph transformation, which main part is a graph-to-set-of-matches transformation.

Definition 2.6 (set of matches, assignment table). Let $X$ and $G$ be graphs. A set $\mathfrak{m}$ of matches, all of them from $X$ to $G$, is denoted $\mathfrak{m} : X \Rightarrow G$. The assignment table $\text{Tab}(\mathfrak{m})$ of $\mathfrak{m}$ is the two-dimensional table with the elements of $\mathcal{V}(X)$ in its first row, then one row for each $m$ in $\mathfrak{m}$, and the entry in row $m$ and column $x$ equals to $m(x)$.

Thus, the assignment table $\text{Tab}(\mathfrak{m})$ describes the set of functions $\mathfrak{m}_\mathcal{V} : \mathcal{V}(X) \Rightarrow \mathcal{L}$, made of the functions $m_\mathcal{V} : \mathcal{V}(X) \rightarrow \mathcal{L}$ for all $m \in \mathfrak{m}$. The set of matches $\mathfrak{m} : X \Rightarrow G$ is determined by the graphs $X$ and $G$ and the assignment table $\text{Tab}(\mathfrak{m})$. This property is used hereafter to describe some examples.
Example 2.7. Here are some examples of matches in the graph \( G_0 \) (defined in Example 2.2).

- Let \( P_{ps} \) be the following graph (written “à la” SPARQL: variable names begin with “?”):
  \[
P_{ps} = \text{?a publishes ?m . ?m stampedAt ?d}
  \]

  The set \( m_{ps} \) of all matches from \( P_{ps} \) to \( G_0 \) is:
  \[
m_{ps} : P_{ps} \Rightarrow G_0 \text{ with } \text{Tab}(m_{ps}) = \\
  \begin{array}{|c|c|c|}
  \hline
  ?a & ?m & ?d \\
  \hline
  \text{auth1} & \text{mes1} & \text{date1} \\
  \text{auth1} & \text{mes2} & \text{date2} \\
  \text{auth2} & \text{mes3} & \text{date1} \\
  \text{auth3} & \text{mes4} & \text{date4} \\
  \text{auth3} & \text{mes5} & \text{date4} \\
  \hline
  \end{array}
  \]

- Let \( P_{pl} \) be the graph:
  \[
P_{pl} = \text{?a1 publishes ?m . ?a2 likes ?m}
  \]

  The set \( m_{pl} \) of all matches from \( P_{pl} \) to \( G_0 \) is:
  \[
m_{pl} : P_{pl} \Rightarrow G_0 \text{ with } \text{Tab}(m_{pl}) = \\
  \begin{array}{|c|c|c|}
  \hline
  ?a1 & ?m & ?a2 \\
  \hline
  \text{auth1} & \text{mes1} & \text{auth2} \\
  \text{auth2} & \text{mes3} & \text{auth1} \\
  \text{auth3} & \text{mes4} & \text{auth1} \\
  \text{auth3} & \text{mes4} & \text{auth2} \\
  \text{auth3} & \text{mes5} & \text{auth1} \\
  \hline
  \end{array}
  \]

- Let \( P'_{pl} \) be the following subgraph of \( P_{pl} \), made of two isolated nodes:
  \[
P'_{pl} = \text{?a1 . ?a2}
  \]

  The subset \( m'_{pl} \) of \( m_{pl} \) made of the restrictions to \( P'_{pl} \) of the matches in \( m_{pl} \) is:
  \[
m'_{pl} : P'_{pl} \Rightarrow G_0 \text{ with } \text{Tab}(m'_{pl}) = \\
  \begin{array}{|c|c|}
  \hline
  ?a1 & ?a2 \\
  \hline
  \text{auth1} & \text{auth2} \\
  \text{auth2} & \text{auth1} \\
  \text{auth3} & \text{auth1} \\
  \text{auth3} & \text{auth2} \\
  \hline
  \end{array}
  \]

- Let \( P_{prp} \) be the graph:
  \[
P_{prp} = \text{?a1 publishes ?m1 . ?m1 refersTo ?m2 . ?a2 publishes ?m2}
  \]

  The set \( m_{prp} \) of all matches from \( P_{prp} \) to \( G_0 \) is:
  \[
m_{prp} : P_{prp} \Rightarrow G_0 \text{ with } \text{Tab}(m_{prp}) = \\
  \begin{array}{|c|c|c|c|}
  \hline
  ?a1 & ?m1 & ?m2 & ?a2 \\
  \hline
  \text{auth2} & \text{mes3} & \text{mes1} & \text{auth1} \\
  \text{auth3} & \text{mes4} & \text{mes1} & \text{auth1} \\
  \text{auth3} & \text{mes4} & \text{mes2} & \text{auth1} \\
  \hline
  \end{array}
  \]
**Definition 2.8** (image of a graph by a set of functions). The *image* of a graph \( X \) by a set of functions \( f \) from \( \mathcal{Y}(X) \) to \( \mathcal{L} \), denoted \( f(X) \), is the graph union of the graphs \( f(x) \) for every \( f \) in \( f \). Every set of functions \( f : \mathcal{Y}(X) \Rightarrow \mathcal{L} \) can be extended in a unique way as a set of matches \( f^2 : X \Rightarrow \mathcal{L}(X) \).

**Remark 2.9** (about RDF graphs). RDF graphs \([10]\) are graphs (as in Definition 2.1) without isolated nodes, where constants are either IRIs (Internationalized Resource Identifiers) or literals and where all predicates are IRIs and only objects can be literals. Blank nodes in RDF graphs are the same as variable nodes in our graphs. Thus an isomorphism of RDF graphs, as defined in \([10]\), is an isomorphism of graphs as in Definition 2.3.

### 2.2 Operations on sets of matches

In this Section we introduce some operations on sets of matches which are used in Section 3 for defining the semantics of GrAL. The prominent one is the **merging** operation (Definition 2.10), which is a kind of generalized **joining** operation (see Definition 2.14). Other basic operations are the simple **restriction** and **extension** operations (Definitions 2.12 and 2.13). Then, these basic operations are combined in order to get some derived operations (Definition 2.14).

**Definition 2.10** (Merge). Let \( m : X \Rightarrow G \) be a set of matches and \( p_m : Y \Rightarrow H_m \) a family of sets of matches indexed by \( m \in m \), and let \( H = \bigcup_{m \in m} H_m \). The **merging** of \( m \) along the family \( (p_m)_{m \in m} \) is the set of matches \( m \bowtie p \) for every \( m \in m \) compatible with \( m \):

\[
\text{Merge}(m, (p_m)_{m \in m}) = \{ m \bowtie p \mid m \in m \land p \in p_m \bowtie m \bowtie p \} : X \cup Y \Rightarrow G \cup H.
\]

Let \( q = \text{Merge}(m, (p_m)_{m \in m}) \), then \( q \) is made of a match \( m \bowtie p \) for each pair \( (m, p) \) with \( m \in m \) and \( p \in p_m \bowtie m \), compatible with \( m \) (so that for each \( m \in m \) the number of \( m \bowtie p \) in \( q \) is between 0 and \( \text{Card}(p_m) \)). The match \( m \bowtie p : X \cup Y \Rightarrow G \cup H \) is such that \( m \bowtie p(x) = m(x) \) when \( x \in X \) and \( m \bowtie p(y) = p(y) \) when \( y \in Y \), which is unambiguous because of the compatibility condition.

**Example 2.11.** Here are some examples of merging, based on the sets of matches in Example 2.7.

- Let \( p_{pl}' \) be similar to \( m_{pl}' \), up to renaming the variables \(?a_1\) and \(?a_2\) as \(?a_2\) and \(?a_1\), respectively, so that \( p_{pl}' \) is:

\[
p_{pl}' : P_{pl} \Rightarrow G_0 \text{ with } \text{Tab}(p_{pl}') = \begin{array}{|c|c|} \hline ?a_1 & ?a_2 \\ \hline \text{auth2} & \text{auth1} \\ \text{auth1} & \text{auth2} \\ \text{auth1} & \text{auth3} \\ \text{auth2} & \text{auth3} \\ \hline \end{array}
\]

For each match \( m \) in \( m_{pl}' \) let \( p_m = p_{pl}' \), which does not depend on \( m \). Then \( \text{Merge}(m_{pl}', (p_m)_{m \in m_{pl}'}) \) is denoted simply \( \text{Join}(m_{pl}', p_{pl}') \) (as in Definition 2.14 and Remark 2.15) and:

\[
q_{pl}' = \text{Join}(m_{pl}', p_{pl}') : P_{pl} \Rightarrow G_0 \text{ with } \text{Tab}(q_{pl}') = \begin{array}{|c|c|} \hline ?a_1 & ?a_2 \\ \hline \text{auth1} & \text{auth2} \\ \text{auth2} & \text{auth1} \\ \hline \end{array}
\]

- Assume that there is some operation **concat** that builds a string from any given date and string. For each match \( m \) in \( m_{ps} \) let \( p_m = \{ p_m \} : \{ ?dm \} \Rightarrow H_{ps} \) where \( p_m(?dm) = \text{concat}(m(?d), m(?m)) \) and...
\( H_{ps} \) is any graph that contains all strings \( \text{concat}(d, m) \) as nodes. Then:

\[
q_{ps} = \text{Merge}(m_{ps}, (p_{m})_{m \in m_{ps}}) : P_{ps} \cup \{?dm\} \Rightarrow G_{0} \cup H_{ps}
\]

with \( \text{Tab}(q_{ps}) = \)

\[
\begin{array}{|c|c|c|c|}
\hline
?a & ?m & ?d & ?dm \\
\hline
\text{auth1} & \text{mes1} & \text{date1} & \text{date1mes1} \\
\text{auth1} & \text{mes2} & \text{date2} & \text{date2mes2} \\
\text{auth2} & \text{mes3} & \text{date1} & \text{date1mes3} \\
\text{auth3} & \text{mes4} & \text{date4} & \text{date4mes4} \\
\text{auth3} & \text{mes5} & \text{date4} & \text{date4mes5} \\
\hline
\end{array}
\]

- For each match \( m \) in \( m_{ps} \) let \( p_{m} = \{p_{m}\} : \{?r\} \Rightarrow H_{var} \) where \( p_{m}(?r) \) is some fresh variable \( ?r_{m} \) and \( H_{var} \) is any graph that contains all variables \( ?r_{m} \) as nodes. Then:

\[
q_{var} = \text{Merge}(m_{ps}, (p_{m})_{m \in m_{ps}}) : P_{ps} \cup \{?r\} \Rightarrow G_{0} \cup H_{var}
\]

with \( \text{Tab}(q_{var}) = \)

\[
\begin{array}{|c|c|c|c|}
\hline
?a & ?m & ?d & ?r \\
\hline
\text{auth1} & \text{mes1} & \text{date1} & ?r1 \\
\text{auth1} & \text{mes2} & \text{date2} & ?r2 \\
\text{auth2} & \text{mes3} & \text{date1} & ?r3 \\
\text{auth3} & \text{mes4} & \text{date4} & ?r4 \\
\text{auth3} & \text{mes5} & \text{date4} & ?r5 \\
\hline
\end{array}
\]

**Definition 2.12** (Restrict). Let \( m : X \Rightarrow G \) be a set of matches. For every graph \( Y \) contained in \( X \) and every graph \( H \) contained in \( G \) such that \( m(Y) \subseteq H \), the restriction \( \text{Restrict}(m, Y, H) : Y \Rightarrow H \) is made of the restrictions of the matches in \( m \) as matches from \( Y \) to \( H \). When \( H = G \) the notation may be simplified: \( \text{Restrict}(m, Y) = \text{Restrict}(m, Y, G) : Y \Rightarrow G \).

**Definition 2.13** (Extend). Let \( m : X \Rightarrow G \) be a set of matches. For every graph \( H \) containing \( G \), the extension \( \text{Extend}(m, H) : X \Rightarrow H \) is made of the extensions of the matches in \( m \) as matches from \( X \) to \( H \). New operations are obtained by combining the previous ones (assuming that \( \text{true} \) is a constant). Comments on Definition 2.14 are given in Remark 2.15. We will see in Section 3.2 that these derived operations provide the semantics of the operators of the language GrAL.

**Definition 2.14** (derived operations).

- For every sets of matches \( m : X \Rightarrow G \) and \( p : Y \Rightarrow H \), let \( p_{m} = p \) for each \( m \in m \), then:

\[
\text{Join}(m, p) = \text{Merge}(m_{ps}, (p_{m})_{m \in m}) : X \cup \overline{Y} \Rightarrow G \cup H.
\]

- For every set of matches \( m : X \Rightarrow G \), every family of constants \( c = (c_{m})_{m \in m} \) and every variable \( x \), let \( p_{m} = \{p_{m}\} \) and \( p_{m}(x) = c_{m} \) for each \( m \in m \), then:

\[
\text{Bind}(m, c, x) = \text{Merge}(m, (p_{m})_{m \in m}) : X \cup \{x\} \Rightarrow G \cup c.
\]

- For every set of matches \( m : X \Rightarrow G \) and every family of constants \( c = (c_{m})_{m \in m} \), for some fresh variable \( x \), let \( \text{true} = (\text{true})_{m \in m} \):

\[
\text{Filter}(m, c) = \text{Restrict}(\text{Bind}(m, c, x), \text{true}, x, G) : X \Rightarrow G.
\]

- For every set of matches \( m : X \Rightarrow G \) and every graph \( R \), for every \( m \in m \) let \( p_{m} : R \Rightarrow p_{m}(R) \) be the match such that:

\[
p_{m}(x) = m(x) \text{ if } x \in \mathcal{V}(R) \cap \mathcal{V}(X)
\]

\[
\text{and } p_{m}(x) = \text{var}(x, m) \text{ is a fresh variable if } x \in \mathcal{V}(R) \setminus \mathcal{V}(X)
\]

and let \( p_{m} = \{p_{m}\} \) and \( p(R) = \cup_{m \in m}(p_{m}(R)) \), then:

\[
\text{Construct}(m, R) = \text{Restrict}(\text{Merge}(m, (p_{m})_{m \in m}),R) : R \Rightarrow G \cup p(R).
\]
• For every sets of matches \( m : X \Rightarrow G \) and \( p : X \Rightarrow H \):
\[
\text{Union}(m, p) = \text{Extend}(m, G \cup H) \cup \text{Extend}(p, G \cup H) : X \Rightarrow G \cup H.
\]

**Remark 2.15.** Let us analyse these definitions. Note that the definition of Bind and Filter rely on the fact that isolated nodes are allowed in graphs.

• Operation Join is Merge when the set of matches \( p_m \) does not depend on \( m \), so that:
\[
\text{Join}(m, p) = \{ m \uplus p \mid m \in m \land p \in p \land m \sim p \} : X \cup Y \Rightarrow G \cup H.
\]
It follows that Join is commutative.

• Operation Bind is Merge when \( p_m \) has exactly one element \( p_m \) for each \( m \), which is such that \( p_m(x) = c_m \). There are two cases:

  - If \( x \in \mathcal{V}(X) \) then this operation selects the matches \( m \) in \( m \) such that \( m(x) = c_m \):
    \[
    \text{Bind}(m,c,x) = \{ m \mid m \in m \land m(x) = c_m \} : X \Rightarrow G.
    \]
  
  - If \( x \notin \mathcal{V}(X) \) then this operation extends each match \( m \) in \( m \) by assigning the value \( c_m \) to the variable \( x \). Let us denote the resulting match as \( m \uplus (x \mapsto c_m) \), so that:
    \[
    \text{Bind}(m,c,x) = \{ m \uplus (x \mapsto c_m) \mid m \in m \} : X \cup \{ x \} \Rightarrow G \cup \{ c \}.
    \]

• Operation Filter applies Bind twice, first when \( x \notin \mathcal{V}(X) \) for extending each \( m \in m \) by assigning \( c_m \) to \( x \), then since \( x \in \mathcal{V}(X \cup \{ x \}) \) for selecting the matches \( m \) in \( m \) such that \( c_m = \text{true} \). Now the value of the auxiliary variable \( x \) is always \text{true} \), so that \( x \) can be dropped: this is the role of the last step which restricts the domain of the matches from \( X \cup \{ x \} \) to \( X \) and its range from \( G \cup \{ c \} \) to \( G \).

• The first step in operation Construct is Merge when \( p_m \) has exactly one element \( p_m \) for each \( m \) (as for Bind), which is determined by \( p_m(x) = \text{var}(x,m) \) for each variable \( x \) in \( R \) that does not occur in \( X \). Each \( \text{var}(x,m) \) is a fresh variable, which means that it is distinct from the variables in \( G \), \( X \), \( R \), and the variables \( \text{var}(x,m) \) are pairwise distinct. Note that the precise symbol used for denoting \( \text{var}(x,m) \) does not matter. The second step in operation Construct restricts the domain of the matches from \( X \cup R \) to \( R \). Thus:

  \[
  \text{Construct}(m,R) \text{ is the set of matches from } R \text{ to } G \cup p(R) \text{ determined by the functions } f_m : \mathcal{V}(R) \rightarrow \mathcal{L} \text{ (for each } m \in m \text{) such that } f_m(x) = m(x) \text{ if } x \in \mathcal{V}(R) \cap \mathcal{V}(X) \text{ and } f_m(x) = \text{var}(x,m) \text{ if } x \in \mathcal{V}(R) \setminus \mathcal{V}(X).
  \]
Thus, the graph \( G \cup p(R) \) is obtained by “gluing” one copy of \( G \) with \( \text{Card}(m) \) copies of \( R \) in the right way. Note that the functions \( f_m \) are pairwise distinct when \( \mathcal{V}(R) \) is not included in \( \mathcal{V}(X) \), but it needs not be the case in general. Also, note that the domain \( R \) of \( \text{Construct}(m,R) \) may be quite different from the domain \( X \) of \( m \), whereas every other operation in Definition 2.14 either keeps or extends the domain of \( m \).

• Operation Union is simply the set-theoretic union of sets of matches which share the same domain (by assumption) and the same range (by extending the range if necessary). This operation differs from the previous ones in the sense that it is not defined by examining the matches in its arguments. Note that Union is commutative.

**Proposition 2.16.** The sets of matches obtained by the operations previously defined in this Section have bounded cardinals, as follows.
The proof of Proposition 2.16 follows easily from the definitions.

3 The Graph Algebraic Query Language

In this Section we introduce the syntax and semantics of the Graph Algebraic Query Language GrAL. There are three syntactic categories in GrAL: expressions, patterns and queries. Expressions are considered in Section 3.1. Patterns are defined in Section 3.2, their semantics is presented as an evaluation function which maps every pattern \( P \) and graph \( G \) to a set of matches \( \llbracket P \rrbracket_G \). Queries are defined in Section 3.3, they are essentially specific kinds of patterns and their semantics is easily derived from the semantics of patterns, the main difference is that the execution of a query on a graph returns simply a graph instead of a set of matches.

To each expression \( e \) or pattern \( P \) is associated a set of variables called its in-scope variables and denoted \( \mathcal{V}(e) \) or \( \mathcal{V}(P) \), respectively. An expression \( e \) is over a pattern \( P \) if \( \mathcal{V}(e) \subseteq \mathcal{V}(P) \). In this Section, as in Section 2, the set of labels \( \mathcal{L} \) is the union of the disjoint sets \( \mathcal{C} \) and \( \mathcal{V} \), of constants and variables respectively. We assume that the set \( \mathcal{C} \) of constants contains the numbers and strings and the boolean values \( true \) and \( false \), as well as a symbol \( \perp \) for errors.

3.1 Expressions

The expressions of GrAL are built from the labels using operators, which are classified as either basic operators (unary or binary) and aggregation operators (always unary). Remember that typing constraints are not considered in this paper. Typically, and not exclusively, the sets \( Op_1, Op_2 \) and \( Agg \) of basic unary operators, basic binary operators and aggregation operators can be:

\[
\begin{align*}
Op_1 &= \{-, \text{NOT}\}, \\
Op_2 &= \{+, -, \times, /, =, >, <, \text{AND}, \text{OR}\}, \\
Agg &= Agg_{\text{elem}} \cup \{ \text{agg} \text{ DISTINCT} \mid \text{agg} \in Agg_{\text{elem}} \}. \\
\text{where } Agg_{\text{elem}} &= \{ \text{MAX, MIN, SUM, AVG, COUNT} \}
\end{align*}
\]

A group of expressions is a non-empty finite list of expressions.

**Definition 3.1** (syntax of expressions). The expressions \( e \) of GrAL and their set of in-scope variables \( \mathcal{V}(e) \) are defined recursively as follows:

- A constant \( c \in \mathcal{C} \) is an expression with \( \mathcal{V}(c) = \emptyset \).
- A variable \( x \in \mathcal{V} \) is an expression with \( \mathcal{V}(x) = \{x\} \).
- If \( e_1 \) is an expression and \( op \in Op_1 \) then \( op \ e_1 \) is an expression with \( \mathcal{V}(op \ e_1) = \mathcal{V}(e_1) \).
- If \( e_1 \) and \( e_2 \) are expressions and \( op \in Op_2 \) then \( e_1 \ op \ e_2 \) is an expression with \( \mathcal{V}(e_1 \ op \ e_2) = \mathcal{V}(e_1) \cup \mathcal{V}(e_2) \).
• If \(e_1\) is an expression and \(agg \in Agg\) then \(agg(e_1)\) is an expression with \(\forall'(agg(e_1)) = \forall'(e_1)\).

• If \(e_1\) is an expression, \(agg \in Agg\) and \(gp\) a group of expressions with all its variables distinct from the variables in \(e_1\), then \(agg(e_1 \text{ BY } gp)\) is an expression with \(\forall'(agg(e_1 \text{ BY } gp)) = \forall'(e_1)\).

The value of an expression with respect to a set of matches \(m\) (Definition 3.2) is a family of constants \(ev(m, e) = (ev(m, e)_m)_{m \in m}\) indexed by the set \(m\). Each constant \(ev(m, e)_m\) depends on \(e\) and \(m\) and it may also depend on other matches in \(m\) when \(e\) involves aggregation operators. The value of a group of expressions \(gp = (e_1, ..., e_k)\) with respect to \(m\) is the list \(ev(m, gp)_m = (ev(m, e_1), ..., ev(m, e_k))\). To each basic operator \(op\) is associated a function \([[op]]\) (or simply \(op\)) from constants to constants if \(op\) is unary and from pairs of constants to constants if \(op\) is binary. To each aggregation operator \(agg\) in \(Agg\) is associated a function \([[agg]]\) (or simply \(agg\)) from multisets of constants to constants. Note that each family of constants determines a multiset of constants: for instance a family \(c = (c_m)_{m \in m}\) of constants indexed by the elements of a set of matches \(m\) determines the multiset of constants \(\{c_m \mid m \in m\}\), which is also denoted \(c\) when there is no ambiguity. Some aggregation operators \(agg\) in \(Agg_{\text{elem}}\) are such that \([[agg]](c)\) depends only on the set underlying the multiset \(c\), which means that \([[agg]](c)\) does not depend on the multiplicities in the multiset \(c\); for instance this is the case for MAX and MIN but not for SUM, AVG, COUNT. When \(agg = agg_{\text{elem}}\) DISTINCT with \(agg_{\text{elem}}\) in \(Agg_{\text{elem}}\) then \([[agg]](c)\) is \([[agg_{\text{elem}}]]\) applied to the underlying set of \(c\). For instance, COUNT \((c)\) counts the number of elements of the multiset \(c\) with their multiplicities, while COUNT DISTINCT \((c)\) counts the number of distinct elements in \(c\).

Definition 3.2 (evaluation of expressions). Let \(X\) be a graph, \(e\) an expression over \(X\) and \(m: X \Rightarrow Y\) a set of matches. The value of \(e\) with respect to \(m\) is the family of constants \(ev(m, e) = (ev(m, e)_m)_{m \in m}\) defined recursively as follows (with notations as in Definition 3.1):

- \(ev(m, c)_m = c\).
- \(ev(m, x)_m = m(x)\).
- \(ev(m, op \ e_1)_m = [[op]] ev(m, e_1)_m\).
- \(ev(m, e_1 \ op \ e_2)_m = ev(m, e_1)_m [[op]] ev(m, e_2)_m\).
- \(ev(m, agg(e_1))_m = [[agg]](ev(m, e_1))\) (which is the same for every \(m\) in \(m\)).
- \(ev(m, agg(e_1) \text{ BY } gp)_m = [[agg]](ev(m, gp, m, e_1))\) where \(m_{gp, m}\) is the subset of \(m\) made of the matches \(m'\) in \(m\) such that \(ev(m, gp)_{m'} = ev(m, gp)_m\) (which is the same for every \(m\) and \(m'\) in \(m\) such that \(ev(m, gp)_m = ev(m, gp)_{m'}\).

Example 3.3. Let \(m_{pl}\) be as in Example 2.7. For every \(m\) in \(m_{pl}\) we have:

\[ev(m_{pl}, \text{COUNT} \ (\text{likes}))_m = 5\]

whereas \(ev(m_{pl}, \text{COUNT} \ (\text{likes \ BY \ ?a1}))_m\) depends on the match \(m\) in \(m_{pl}\):

\[ev(m_{pl}, \text{COUNT} \ (\text{likes \ BY \ ?a1}))_m = \begin{cases} 
1 & \text{when } m(?a1) = \text{auth1} \\
1 & \text{when } m(?a1) = \text{auth2} \\
3 & \text{when } m(?a1) = \text{auth3}
\end{cases}\]

Definition 3.4 (equivalence of expressions). Two expressions over a graph \(X\) are equivalent if they have the same value with respect to every set of matches \(m: X \Rightarrow Y\).
3.2 Patterns

In Definition 3.5 the patterns of GrAL are built from graphs by using five operators: JOIN, BIND, FILTER, CONSTRUCT and UNION. In Definition 3.6 the semantics of patterns is given by an evaluation function. Some patterns have an associated graph called a template, such as pattern \( P \) may give rise to a query \( Q \) as explained in Section 3.3 then the result of query \( Q \) is built from the template of \( P \).

**Definition 3.5** (syntax of patterns). The patterns \( P \) of GrAL, their set of in-scope variables \( \mathcal{V}(P) \) and their template graph \( \mathcal{T}(P) \) when it exists are defined recursively as follows.

- A graph is a pattern, called a basic pattern, and \( \mathcal{V}(P) \) is the set of variables of the graph \( P \).
- If \( P_1 \) and \( P_2 \) are patterns then \( P_1 \) JOIN \( P_2 \) is a pattern and \( \mathcal{V}(P_1 \text{ JOIN } P_2) = \mathcal{V}(P_1) \cup \mathcal{V}(P_2) \).
- If \( P_1 \) is a pattern, \( e \) an expression over \( P_1 \) and \( x \) a variable then \( P_1 \) BIND \( e \) AS \( x \) is a pattern and \( \mathcal{V}(P_1 \text{ BIND } e \text{ AS } x) = \mathcal{V}(P_1) \cup \{x\} \).
- If \( P_1 \) is a pattern and \( e \) an expression over \( P_1 \) then \( P_1 \) FILTER \( e \) is a pattern and \( \mathcal{V}(P_1 \text{ FILTER } e) = \mathcal{V}(P_1) \).
- If \( P_1 \) is a pattern and \( R \) a graph then \( P_1 \) CONSTRUCT \( R \), also written CONSTRUCT \( R \) WHERE \( P_1 \), is a pattern and \( \mathcal{V}(P_1 \text{ CONSTRUCT } R) = \mathcal{V}(R) \).

In addition this pattern has a template \( \mathcal{T}(P_1 \text{ CONSTRUCT } R) = R \).

- If \( P_1 \) and \( P_2 \) are patterns with template and if \( \mathcal{T}(P_1) = \mathcal{T}(P_2) = R \) then \( P_1 \) UNION \( P_2 \) is a pattern and \( \mathcal{V}(P_1 \text{ UNION } P_2) = \mathcal{V}(R) \).

In addition this pattern has a template \( \mathcal{T}(P_1 \text{ UNION } P_2) = \mathcal{T}(P_1) = \mathcal{T}(P_2) \).

The value of a pattern over a graph is a set of matches, as defined now.

**Definition 3.6** (evaluation of patterns). The value of a pattern \( P \) of GrAL over a graph \( G \) is a set of matches \( [P]_G : [P] \Rightarrow G^{(P)} \) from a graph \( [P] \) that depends only on \( P \) to a graph \( G^{(P)} \) that contains \( G \). This value \( [P]_G : [P] \Rightarrow G^{(P)} \) is defined inductively as follows (with notations as in Definition 3.1):

- If \( P \) is a basic pattern then \( [P]_G = \text{Match}(P,G) : P \Rightarrow G \).
- \( [P_1 \text{ JOIN } P_2]_G = \text{Join}([P_1]_G, [P_2]_G) : [P_1] \cup [P_2] \Rightarrow G^{(P_1)} \cup G^{(P_2)} \).
- \( [P_1 \text{ BIND } e \text{ AS } x]_G = \text{Bind}([P_1]_G, \mathcal{V}([P_1]_G,e), x) : [P_1] \cup \{x\} \Rightarrow G^{(P_1)} \cup [P_1]_G(e) \).
- \( [P_1 \text{ FILTER } e]_G = \text{Filter}([P_1]_G, \mathcal{V}([P_1]_G,e)) : [P_1] \Rightarrow G^{(P_1)} \).
- \( [P_1 \text{ CONSTRUCT } R]_G = \text{Construct}([P_1]_G,R) : R \Rightarrow G^{(P_1)} \cup [P_1]_G(R) \).
- \( [P_1 \text{ UNION } P_2]_G = \text{Union}([P_1]_G, [P_2]_G) : R \Rightarrow G^{(P_1)} \cup G^{(P_2)} \) where \( R = \mathcal{T}(P_1) = \mathcal{T}(P_2) \).

**Remark 3.7.** Note that, syntactically, each operator \( OP \) builds a pattern \( P \) from a pattern \( P_1 \) and a parameter \( param \), which is either a pattern \( P_2 \) (for JOIN and UNION), a pair \( (e,x) \) made of an expression and a variable (for BIND), an expression \( e \) (for FILTER) or a graph \( R \) (for CONSTRUCT). Semantically, for every non-basic pattern \( P = P \text{ OP } param \), we denote \( m_1 : X_1 \Rightarrow G_1 \) for \( [P_1]_G : [P_1] \Rightarrow G^{(P_1)} \) and \( m : X \Rightarrow G' \) for \( [P]_G : [P] \Rightarrow G^{(P)} \). In every case it is necessary to evaluate \( m_1 \) before evaluating \( m \); for JOIN and UNION this is because pattern \( P_2 \) is evaluated on \( G_1 \), for BIND and FILTER because expression \( e \) is evaluated with respect to \( m_1 \), and for CONSTRUCT because of the definition of Construct. According to Definition 3.6 given a pattern \( P \) and a graph \( G \), the value \( m : X \Rightarrow G' \) of \( P \) is determined as follows:

- When \( P \) is a basic pattern then \( X = P \), \( G' = G \) and \( m \) is made of all matches from \( P \) to \( G \).
• $P = P_1 \text{ OP param}$ then the semantics of $P$ is easily derived from Definition 2.14 (see also Remark 2.15). However, note that the semantics of $P_1 \text{ JOIN } P_2$ and $P_1 \text{ UNION } P_2$ is not symmetric in $P_1$ and $P_2$ in general, unless $G^{(P_1)} = G$ and $G^{(P_2)} = G$, which occurs when $P_1$ and $P_2$ are basic patterns.

• The graph $G^{(P)}$ is built by adding to $G$ “whatever is required” for the evaluation, in examples we often avoid its precise description.

Given a non-basic pattern $P = P_1 \text{ OP param}$, the pattern $P_1$ is a subpattern of $P$, as well as $P_2$ when $P = P_1 \text{ JOIN } P_2$ or $P = P_1 \text{ UNION } P_2$. The semantics of patterns is defined in terms of the semantics of its subpatterns (and the semantics of its other arguments, if any). Thus, for instance, CONSTRUCT patterns can be nested at any depth.

**Proposition 3.8.** For every pattern $P$, the set $\mathcal{V}(P)$ of in-scope variables of $P$ is the same as the set $\mathcal{V}([P])$ of variables of the graph $[P]$.

**Example 3.9.** In each item below we consider first some pattern $P_i$ and some template $R_i$, then the pattern $C_i = P_i \text{ CONSTRUCT } R_i$ also written as $C_i = \text{ CONSTRUCT } R_i \text{ WHERE } P_i$.

We refer to Examples 2.7 and 3.3.

• $C_1 = \text{ CONSTRUCT } \{ \text{ ?a1 cites ?a2 } \}$
  WHERE $\{ \text{ ?a1 publishes ?m1 . ?m1 refersTo ?m2 . ?a2 publishes ?m2 } \}$

Here, $C_1 = P_1 \text{ CONSTRUCT } R_1$ where $P_1 = \text{ prp}$, so that the value of $P_1$ over $G_0$ is $m_{\text{prp}} : P_1 \Rightarrow G$. Note that $\mathcal{V}(R_1) \subseteq \mathcal{V}(P_1)$. Let $G_1 = G_0 \cup \{ \text{ auth2 cites auth1 . auth3 cites auth1 } \}$, the value of $C_1$ over $G_0$ is:

$[[C_1]]_{G_0} : R_1 \Rightarrow G_1$ with $\text{ Tab}([[C_1]]_{G_0}) = \begin{bmatrix} \text{ ?a1 } & \text{ ?a2 } \\ \text{ auth2 } & \text{ auth1 } \\ \text{ auth3 } & \text{ auth1 } \end{bmatrix}$

• $C_2 = \text{ CONSTRUCT } \{ \text{ ?n } \}$
  WHERE $\{ \text{ ?a likes ?m }$
  BIND $\text{ COUNT}(\text{likes}) \text{ AS } \text{ ?n } \}$

Here, $C_2 = P_2 \text{ CONSTRUCT } R_2$ where the template $R_2$ is the graph made of only one isolated node which is the variable ?n. The graph $[P_2]$ is $\{ \text{ ?a likes ?m . ?n } \}$. Let $G_2 = G_0 \cup \{ 5 \}$, the value of $C_2$ over $G_0$ is:

$[[P_2]]_{G_0} : [P_2] \Rightarrow G_2$ with $\text{ Tab}([[P_2]]_{G_0}) = \begin{bmatrix} \text{ ?a } & \text{ ?m } & \text{ ?n } \\ \text{ auth1 } & \text{ mes3 } & 5 \\ \text{ auth1 } & \text{ mes4 } & 5 \\ \text{ auth1 } & \text{ mes5 } & 5 \\ \text{ auth2 } & \text{ mes1 } & 5 \\ \text{ auth2 } & \text{ mes4 } & 5 \end{bmatrix}$

Then the graph $[[C_2]]$ is $\{ \text{ ?n } \}$ and the value of $C_2$ over $G_0$ is:

$[[C_2]]_{G_0} : [C_2] \Rightarrow G_2$ with $\text{ Tab}([[C_2]]_{G_0}) = \begin{bmatrix} \text{ ?n } \\ 5 \end{bmatrix}$
• \( C_3 = \text{CONSTRUCT} \{ \text{?a1 nbOfLikes ?n} \} \)
  
  \[ \text{WHERE} \{ \text{?a1 publishes ?m . ?a2 likes ?m} \} \]
  
  \[ \text{FILTER (NOT(?a1=?a2))} \]
  
  \[ \text{BIND COUNT(likes BY ?a1) AS ?n} \] 

Here, \( C_3 = P_3 \text{CONSTRUCT} R_3 \) where \( R_3 = \{ \text{?a1 nbOfLikes ?n} \} \) is made of one triple and \( P_3 = \{ \text{?a1 publishes ?m . ?a2 likes ?m . ?n} \} \). The evaluation of \( C_3 \) over \( G_0 \) starts from \( m_{pl} \). Let \( G_3 = G_0 \cup \{ \text{auth1 nbOfLikes 1 . auth2 nbOfLikes 1 . auth3 nbOfLikes 3} \} \), the value of \( C_3 \) over \( G_0 \) is:

\[
[[P_3]]_{G_0} \Rightarrow G_3 \text{ with } Tab([[P_3]]_{G_0}) =
\begin{array}{|c|c|c|}
\hline
\text{?a1} & \text{?m} & \text{?a2} \\
\text{auth1} & \text{mes1} & \text{auth2} \\
\text{auth2} & \text{mes3} & \text{auth1} \\
\text{auth3} & \text{mes4} & \text{auth1} \\
\text{auth3} & \text{mes4} & \text{auth2} \\
\text{auth3} & \text{mes5} & \text{auth1} \\
\hline
\end{array}
\]

Then the graph \( [C_3] \) is \( R_3 = \{ \text{?a1 nbOfLikes ?n} \} \) and the value of \( C_3 \) over \( G_0 \) is:

\[
[[C_3]]_{G_0} \Rightarrow G_3 \text{ with } Tab([[C_3]]_{G_0}) =
\begin{array}{|c|c|c|}
\hline
\text{?a1} & \text{?n} \\
\text{auth1} & 1 \\
\text{auth2} & 1 \\
\text{auth3} & 3 \\
\hline
\end{array}
\]

• \( C_4 = \text{CONSTRUCT} \{ \text{?a1 nb0fFriends ?n} \} \)

\[ \text{WHERE} \{
\text{CONSTRUCT} \{ \text{?a1 friend ?a2} \}
\] 

\[ \text{WHERE} \{
\text{?a1 publishes ?m1 . ?a2 likes ?m1 .}
\text{?a2 publishes ?m2 . ?a1 likes ?m2}
\}
\] 

\[ \text{BIND COUNT (friend BY ?a1) AS ?n} \] 

Here \( C_4 = P_4 \text{CONSTRUCT} R_4 \) where \( P_4 \) itself contains a subpattern \( C_4' = P_4' \text{CONSTRUCT} R_4' \).

The evaluation of the basic pattern \( P_4' \) over \( G_0 \) gives

\[
[[P_4']]_{G_0} : P_4' \Rightarrow G_0 \text{ with } Tab([[P_4']]_{G_0}) =
\begin{array}{|c|c|c|c|}
\hline
\text{?a1} & \text{?m1} & \text{?a2} & \text{?m2} \\
\text{auth1} & \text{mes1} & \text{auth2} & \text{mes3} \\
\text{auth2} & \text{mes3} & \text{auth1} & \text{mes1} \\
\hline
\end{array}
\]

Then we get \( [C_4'] = \{ \text{?a1 friend ?a2} \} \) and:

\[
[[C_4']]_{G_0} : [C_4'] \Rightarrow G_0^{(G_4)} \text{ with } Tab([[C_4']]_{G_0}) =
\begin{array}{|c|c|}
\hline
\text{?a1} & \text{?a2} \\
\text{auth1} & \text{auth2} \\
\text{auth2} & \text{auth1} \\
\hline
\end{array}
\]

Finally \( [C_4] = \{ \text{?a1 Nb0fFriends ?n} \} \) and:

\[
[[C_4]]_{G_0} : [C_4] \Rightarrow G_0^{(G_4)} \text{ with } Tab([[C_4]]_{G_0}) =
\begin{array}{|c|c|}
\hline
\text{?a1} & \text{?n} \\
\text{auth1} & 1 \\
\text{auth2} & 1 \\
\hline
\end{array}
\]
Definition 3.10 (equivalence of patterns). Two patterns are equivalent if they have the same value over \( G \) for every graph \( G \).

Proposition 3.11. For every basic patterns \( P_1 \) and \( P_2 \), the basic pattern \( P_1 \cup P_2 \) is equivalent to \( P_1 \) JOIN \( P_2 \) and to \( P_2 \) JOIN \( P_1 \).

3.3 Queries

A query in GrAL is essentially a pattern which has a template. The main difference between patterns and queries is that, while a pattern is interpreted as a function from graphs to sets of matches, a query is interpreted as a function from graphs to graphs. The operator for building queries from patterns is denoted \( \text{GRAPH} \). According to Definition 3.6, the value of a pattern \( P \) with template \( R \) over a graph \( G \) is a set of matches \([ [P] ]_G : R \Rightarrow G^{(P)}\), and the semantics of patterns is defined recursively in terms of their values. Thus, patterns have a graph-to-set-of-matches semantics, while queries have a graph-to-graph semantics, as defined below, based on Definition 2.8 of the image of a graph by a set of functions.

Definition 3.12 (syntax of queries). A query \( Q \) of GrAL is written \( \text{GRAPH} (P) \) where \( P \) is either a \text{CONSTRUCT} or a \text{UNION} pattern. Then the pattern of \( Q \) is \( P \) and the template \( \mathcal{T}(Q) \) of \( Q \) is the template of \( P \).

Definition 3.13 (result of queries). The result of a query \( Q \) with pattern \( P \) and template \( R \) over a graph \( G \) is the subgraph of \( G^{(P)} \) image of \( R \) by \([ [P] ]_G \), it is denoted \( \text{Result}(Q,G) \).

Thus, when \( Q = \text{GRAPH} (P \text{ CONSTRUCT} R) \), the result of \( Q \) over \( G \) is the graph \( \text{Result}(Q,G) = [ [P] ]_G (R) \) built by “gluing” the graphs \( m(R) \) for \( m \in [ [P] ]_G \), where \( m(R) \) is a copy of \( R \) with each variable \( x \in \mathcal{V}(R) \setminus \mathcal{V}(X) \) replaced by a fresh variable \( \text{var}(x,m) \). And when \( Q = \text{GRAPH} (P \text{ UNION} P_2) \), the result of \( Q \) over \( G \) is the graph \( \text{Result}(Q,G) = H_1 \cup H_2 \) where \( H_i = \text{Result}(\text{GRAPH}(P_i),G) \) and the fresh variables occurring in \( H_1 \) are distinct from the ones in \( H_2 \).
Example 3.14. It is now easy to compute the result of the GrAL queries:

\[ Q_i = \text{GRAPH}(G_i) \]

over \( G_0 \) when \( C_i \) is a pattern from Example 3.9. We know that the result of \( Q_i \) applied to \( G_0 \) is an instance of \( R_i \) when \( \mathcal{Y}(R_i) \subseteq \mathcal{Y}(C_i) \), and that in general it is built by “gluing” together several instances of \( R_i \) (as for query \( Q_5 \) below).

- **Author citations.** Let us say that an author \( a1 \) cites an author \( a2 \) when \( a1 \) has published a message that refers to a message published by \( a2 \). In order to build the graph of author citations we use the query:

\[ Q_1 = \text{GRAPH}(C_1). \]

From \([C_1]_{G_0}\) we get the graph:

\[ \text{Result}(Q_1, G_0) = \{ \text{auth2 cites auth1, auth3 cites auth1} \}. \]

- **Number of likes.** Let us count the number of *likes* in the database. We can get this result by counting the number of triples with predicate *likes*, or equivalently the number of predicates *likes*. We use the query:

\[ Q_2 = \text{GRAPH}(C_2). \]

From \([C_2]_{G_0}\) we get the graph:

\[ \text{Result}(Q_2, G_0) = \{ 5 \}. \]

This result is the number 5, which is considered in GrAL as a graph made of only one isolated node. Note that we would get a query equivalent to \( Q_2 \) by counting either the number of authors \(?a\) who like a message (with multiplicity the number of messages liked by \(?a\)), or by counting the number of messages \(?m\) which are liked by someone (with multiplicity the number of authors who like \(?m\)). This means that the line \( \text{BIND COUNT(likes) AS ?n} \) could be replaced either by \( \text{BIND COUNT(?a) AS ?n} \) or by \( \text{BIND COUNT(?m) AS ?n} \).

- **Number of likes per author.** Let us now count the number of *likes per author* in the database, which means, for each author count the number of likes of messages published by this author, except for self-likes. We display the result as the graph made of the triples \(?a1 nb0fLikes ?n\) where \(?n\) is the number of likes of author \(?a1\), by using the query:

\[ Q_3 = \text{GRAPH}(C_3). \]

From \([C_3]_{G_0}\) we get the graph:

\[ \text{Result}(Q_3, G_0) = \{ \text{auth1 nb0fLikes 1, auth2 nb0fLikes 1, auth3 nb0fLikes 3} \}. \]

- **Number of friends per author.** Let us count the number of friends of each author, where friendship is the symmetric relation between authors defined as follows: two authors are *friends* when each one likes a publication by the other (here self-friends are allowed). We use the query:

\[ Q_4 = \text{GRAPH}(C_4). \]

From \([C_4]_{G_0}\) we get the graph:

\[ \text{Result}(Q_4, G_0) = \{ \text{auth1 nb0fFriends 1, auth2 nb0fFriends 1} \}. \]

- **Generation of fresh variables.** Now let us build, for each author \(?a\) and each message \(?m\) published by \(?a\) and stamped at date \(?d\), a tree with a fresh variable as root and with two branches, one
named author towards ?a and the other one named date towards ?d. We use the query:

\[ Q_5 = \text{GRAPH}(C_5) \]

From \([[[C_5]]]_{G_0}\) we get the graph:

\[ \text{Result}(Q_5, G_0) = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 \]

where each \(T_i\) is the copy of \(R_5\) corresponding to the \(i\)-th row in \(\text{Tab}([[C_5]]_{G_0})\), so that:

\[
\begin{align*}
T_1 &= \{ ?r1 \text{ author auth1 . } ?r1 \text{ date date1} \} \\
T_2 &= \{ ?r2 \text{ author auth1 . } ?r2 \text{ date date2} \} \\
T_3 &= \{ ?r3 \text{ author auth2 . } ?r3 \text{ date date1} \} \\
T_4 &= \{ ?r4 \text{ author auth3 . } ?r4 \text{ date date4} \} \\
T_5 &= \{ ?r5 \text{ author auth3 . } ?r5 \text{ date date4} \}
\end{align*}
\]

In fact, query \(Q_5\) “mimicks” the following SELECT query \(Q'_5\):

\[
\text{SELECT } ?a \text{ ?d } \\
\text{WHERE } \{ ?a \text{ publishes ?m . } ?m \text{ stampedAt ?d } \}
\]

As explained in [4], the various copies of the variable ?\(r\) in the result of \(Q_5\) act as identifiers for the rows in the table result of \(Q'_5\) over \(G_0\) (as for instance in SPARQL), which is obtained by dropping the column ?\(r\) from \(\text{Tab}([[C_5]]_{G_0})\). Note that the table \(\text{Tab}([[P_5]]_{G_0})\) has all its rows distinct by definition, whereas this becomes false when the column ?\(r\) is dropped.

**Definition 3.15** (equivalence of queries). Two queries are *equivalent* if they have the same template and the same result over every graph.

It follows that queries with equivalent patterns are equivalent, but this condition is not necessary.

**Remark 3.16** (about SPARQL queries). CONSTRUCT queries in SPARQL are similar to CONSTRUCT queries in GrAL: the variables in \(\mathcal{V}(R) \setminus \mathcal{V}(X)\) in GrAL play the same role as the blank nodes in SPARQL. However the subCONSTRUCT patterns are specific to GrAL. There is no SELECT query in this core version of GrAL, however following [4] we may consider SELECT queries as kinds of CONSTRUCT queries.

### 4 Conclusion

We considered the problem of the evaluation of graph-to-graph queries, namely CONSTRUCT queries, possibly involving nested sub-queries. We proposed a new uniform evaluation semantics of such queries which rests on a recursive definition of the notion of patterns and a new definition of the considered graphs which are allowed to have isolated nodes. Hence, the evaluation of a pattern always yields a pair consisting of a graph and a set of matches (variable assignments). Notice that we did not tackle explicitly graph-to-table queries such as the well-known SELECT queries. We have shown recently in [4] that SELECT queries are particular case of CONSTRUCT queries. This stems from an easy encoding of tables as graphs. Thus, the proposed semantics can be extended immediately to SELECT queries involving Sub-SELECT queries.

The present work opens several perspectives including a generalization of the proposed semantics to other models of graphs such as *property graphs*. Such an extension needs to ensure the existence of the main operations of the proposed algebra such as the *Merge* operation. An operational semantics, based
on rewriting systems, which is faithful with the evaluation semantics proposed in this paper is under progress. Its underlying rewrite rules are inspired by the algebraic approach in [3]. Furthermore, the core language GrAL contains only simple patterns needed to illustrate our uniform semantics. Comparison with other expressions such as EXISTS(pattern) or patterns such as FROM(query) [2][8] remains to be investigated.

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