ASYMPTOTIC STABILITY OF VISCOUS CONTACT WAVE FOR THE INFLOW PROBLEM OF THE ONE-DIMENSIONAL RADIATIVE EULER EQUATIONS

LILI FAN
School of Mathematics and Computer Science
Wuhan Polytechnic University, Wuhan 430023, China

LIZHI RUAN*
Hubei Key Laboratory of Mathematical Physics, School of Mathematics and Statistics
Central China Normal University, Wuhan 430079, China

WEI XIANG
Department of Mathematics
City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong, China

(Communicated by Irena Lasiecka)

Abstract. This paper is devoted to the study of the inflow problem governed by the radiative Euler equations in the one-dimensional half space. We establish the unique global-in-time existence and the asymptotic stability of the viscous contact discontinuity solution. It is different from the case involved with the rarefaction wave for the inflow problem in our previous work [6], since the rarefaction wave is a nonlinear expansive wave, while the contact discontinuity wave is a linearly degenerate diffusive wave. So we need to take good advantage of properties of the viscous contact discontinuity wave instead. Moreover, series of tricky argument on the boundary is done carefully based on the construction and the properties of the viscous contact discontinuity wave for the radiative Euler equations. Our result shows that radiation contributes to the stabilization effect for the supersonic inflow problem.

1. Introduction. The radiative Euler equations describe the fundamental motion of the compressible gas with the radiative heat transfer phenomena, which has many concrete applications in astrophysics and nuclear explosions. Mathematically, the radiative Euler equations in the Eulerian coordinates are a hyperbolic-elliptic coupled system with the following form:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla p &= 0, \\
\left\{ \rho \left( e + \frac{|u|^2}{2} \right) \right\}_t + \text{div} \left\{ \rho u \left( e + \frac{|u|^2}{2} \right) + pu \right\} + \text{div} q &= 0, \\
-\nabla \text{div} q + a q + b \nabla \theta^4 &= 0,
\end{align*}
\]

(1)

2020 Mathematics Subject Classification. Primary: 35B35, 35B40, 35M30, 35Q35, 76N10; Secondary: 76N15.

Key words and phrases. Radiative Euler equations, inflow problem, viscous contact wave, asymptotic stability, supersonic.

* Corresponding author: Lizhi Ruan, rlz@mail.ccnu.edu.cn.
where \( \rho, u, p, e \) and \( \theta \) are respectively the density, velocity, pressure, internal energy and absolute temperature of the gas, and \( q \) is the radiative heat flux. Positive constants \( a \) and \( b \) depend only on the gas itself. Like the classic compressible Euler equations, the first three equations in (1) stand for the conservation of the mass, momentum and energy respectively. The fourth equation in (1) is related to the radiative heat transfer phenomenon, and one can refer \([2, 8, 24, 37]\) for more details. System (1) can be also derived by the non-relativistic limit (speed of light tending to \( +\infty \)) from a hyperbolic-kinetic system, where the rigorous mathematical derivation can be found in \([14]\). Such a radiative hydrodynamics model was also considered in the other works such as \([4]\).

1.1. **One dimensional inflow problem.** This is the second one of our papers on the initial-boundary value problem for the following one-dimensional radiative Euler equations (2), where the inflow problems with rarefaction wave are well understood based on our previous work in \([6]\). In this paper, we will continue to study the inflow problem with viscous contact discontinuity. The inflow problem of the one-dimensional radiative Euler equations in the Eulerian coordinates is the following system

\[
\begin{aligned}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + p)_x &= 0, \\
\left\{ \rho \left( e + \frac{u^2}{2} \right) \right\}_t + \left\{ \rho u \left( e + \frac{u^2}{2} \right) + pu \right\}_x + q_x &= 0, \\
-q_{xx} + a q + b \left( \theta^4 \right)_x &= 0
\end{aligned}
\]

(2)

on \( 0 \leq x < \infty \) and \( 0 \leq t < \infty \) with the initial data

\[
(\rho, u, \theta)(x, 0) = (\rho_0, u_0, \theta_0)(x), \quad \text{and} \quad \inf_{x \in \mathbb{R}^+} (\rho_0, \theta_0)(x) > 0, \quad \text{for} \quad x \geq 0,
\]

(3)

the asymptotic boundary condition at the far field \( x = +\infty \)

\[
(\rho, u, \theta, q)(+\infty, t) = (\rho_+, u_+, \theta_+, 0), \quad t \geq 0
\]

(4)

with \( \rho_+ > 0, u_+, \theta_+ > 0 \) being given constants, and some appropriate boundary conditions on boundary \( x = 0 \). In this paper, we will consider the polytropic gas, \( i.e. \), the compressible flow satisfies the following thermal relation that

\[
p = R\rho\theta, \quad e = \frac{R}{\gamma - 1} \theta,
\]

(5)

where \( \gamma > 1 \) is the adiabatic exponent and \( R > 0 \) is the specific gas constant.

Assume solution \((\rho, u, \theta)\) is in a sufficiently small neighborhood of \( z_+ := (\rho_+, u_+, \theta_+) \). Based on the characteristics on the boundary, there are two types of boundary condition on \( x = 0 \) for the inflow problem:

**Case (1).** If \( 0 < u_+ < \sqrt{\gamma R\theta_+} \), the boundary condition of (2) is

\[
u(0, t) = u_-, \quad \theta(0, t) = \theta_-, \quad q(0, t) = 0.
\]

(6)

**Case (2).** If \( u_+ > \sqrt{\gamma R\theta_+} \), the boundary condition of (2) is

\[
\rho(0, t) = \rho_-, \quad u(0, t) = u_-, \quad \theta(0, t) = \theta_-, \quad q(0, t) = 0.
\]

(7)

In this paper, we will consider the second case with a viscous contact wave, that is, \( u_+ > \sqrt{\gamma R\theta_+} \) with \( \gamma > 1 \). We will establish the unique global-in-time existence and the asymptotic stability of the solution with a viscous contact discontinuity under some smallness conditions, which will be given in Theorem 2.2.
First let us introduce the Lagrangian coordinates transformation: by \((2)_1\), there exists a function \(y\) with \(y(0,0) = 0\) such that
\[
y_t = -\rho u \quad \text{and} \quad y_x = \rho.
\]
Then the coordinates transformation is defined as \((x,t) \rightarrow (y,t)\). Let \(v = \frac{1}{\rho}\) be the specific volume, then the one-dimensional radiative Euler equations (2) in the Lagrangian coordinates is
\[
\begin{align*}
v_t - u_y &= 0, \quad y > s_t, \quad t > 0, \\
u_t + py &= 0, \quad y > s_t, \quad t > 0, \\
\left(e + \frac{u^2}{2}\right)_t + (pu)_y + q_y &= 0, \quad y > s_t, \quad t > 0, \\
- \left(\frac{qv}{v}\right)_y + avq + b (\theta^4)_y &= 0, \quad y > s_t, \quad t > 0
\end{align*}
\]
with the initial condition, the moving boundary condition and the asymptotic boundary condition that
\[
\begin{align*}
(v, u, \theta)(y, 0) &= (v_0, u_0, \theta_0)(y) \rightarrow (v_+, u_+, \theta_+), \quad \text{as} \quad y \rightarrow +\infty, \\
(v, u, \theta, q)(s_- t, t) &= (v_-, u_-, \theta_-, 0), \\
(v, u, \theta, q)(y, t) &= (v_+, u_+, \theta_+, 0), \quad \text{as} \quad y \rightarrow +\infty,
\end{align*}
\]
where \(v_0(y) = \frac{1}{\rho_0(y)}\) and \(v_\pm = \frac{1}{\rho_\pm}\), and the boundary moves with the constant speed
\[
s_- = -\frac{u_-}{v_-} < 0.
\]
In order to fix the moving boundary \(y = s_- t\), we introduce a new variable \(\xi = y - s_- t\). Then we have
\[
\begin{align*}
v_t - s_- v_\xi - u_\xi &= 0, \quad \xi > 0, \quad t > 0, \\
u_t - s_- u_\xi + p_\xi &= 0, \quad \xi > 0, \quad t > 0, \\
\left(e + \frac{u^2}{2}\right)_t - s_- \left(e + \frac{u^2}{2}\right)_\xi + (pu)_\xi + q_\xi &= 0, \quad \xi > 0, \quad t > 0, \\
- \left(\frac{qv}{v}\right)_\xi + avq + b (\theta^4)_\xi &= 0, \quad \xi > 0, \quad t > 0, \\
(v, u, \theta)(\xi, 0) &= (v_0, u_0, \theta_0)(\xi) \rightarrow (v_+, u_+, \theta_+), \quad \xi \geq 0, \quad \text{as} \quad \xi \rightarrow +\infty, \\
(v, u, \theta, q)(0, t) &= (v_-, u_-, \theta_-, 0), \quad t \geq 0, \\
(v, u, \theta, q)(\xi, t) \rightarrow (v_+, u_+, \theta_+, 0), \quad t \geq 0, \quad \text{as} \quad \xi \rightarrow +\infty.
\end{align*}
\]
The argument for the case with a viscous contact wave is different from the one with a rarefaction wave for the inflow problem, which was studied in our previous work [6]. The main reason is that the rarefaction wave is expansive, while the contact wave is diffusive. The process in the boundary is very different from the case of rarefaction wave too. Series of tricky operations on the boundary need to be carefully taken by taking good advantages of the construction on the viscous contact wave and the structures of the radiative Euler to derive the energy estimates, such as (59), (81) and (97).

Moreover, we remark that the dissipative mechanism of perturbation system (31) around viscous contact discontinuity solution comes from the terms related to the
radiation, which is mathematically demonstrated from the term $\int_0^t \| w(\tau) \|^2_2 d\tau$ (resp. $\int_0^t \| w_t(\tau) \|^2_2 d\tau$) in the energy estimate (34), (52), (93) (resp. (71), (76)). Then the dissipation of the radiation is transferred to the temperature as indicated in (70) and (104). This mechanism leads to close the energy estimates.

There are also some results on the study of viscous contact discontinuity wave the inflow or outflow problem governed by other systems such as the Navier-Stokes system (see [11, 12, 13, 9, 30, 31]). However, the dissipation of the radiative Euler system is much weaker than the Navier-Stokes system.

1.2. Related literature. As far as we know, so far most of the existing results deal with the one-dimensional case. For the one-dimensional Cauchy problem, the global-in-time existence of the solutions around a constant state was shown in [15]. If the initial data is a small perturbation of a given rarefaction wave with small wave strength, it was proved in [20] that the solutions converge to the rarefaction wave as $t \to +\infty$. Then in [10], the authors showed that when the absorption coefficient $\alpha$ tends to $+\infty$, the solutions converge to the rarefaction wave with the convergence rate $\alpha^{-\frac{1}{4}} |\ln \alpha|^2$, where the absorption coefficient $\alpha$ is defined by the relationship $a = 3\alpha^2$ and $b = 4\alpha \sigma$ for positive constants $a, b$ and the Stefan-Boltzmann constant $\sigma$. Meanwhile, the asymptotic stability of a single viscous contact wave was proved in [38, 39]. The existence and stability for zero mass perturbation of the small amplitude shock profile were respectively studied in [21] and [22]. The authors in [26] showed the nonlinear orbital asymptotic stability of small amplitude shock profiles for general hyperbolic-elliptic coupled systems of the type modeling the radiative gas. Analysis of large amplitude shock profiles was given in [3, 25]. Finally, for the case of composite waves, the stability of the composite wave of rarefaction waves and a viscous contact wave was investigated in [32, 40]. Recently the authors in [5] studied the unique global-in-time existence and the asymptotic stability of the composite wave of two viscous shock waves by employing the anti-derivative method.

Due to the complexity and difficulties resulted from the boundary effect, up to now, only one rigorous mathematical result on the well-posedness of the initial-boundary value problem governed by the one-dimensional radiative Euler equations was established by the authors in [6]. Precisely speaking, we in [6] investigated the inflow problem where the velocity of the inward flow on the boundary is given as a positive constant. We gave a rigorous proof of the asymptotic stability of the rarefaction wave, provided that the data on the boundary is supersonic. It is the first rigorous result on the initial-boundary value problem for the radiative Euler equations. The result in the present paper is the second result on the initial-boundary value (inflow) problem for the radiative Euler equations.

We need to mention that the investigations on the simplified radiative Euler equations (Hamer model) provide good understanding on the radiative effect although the full radiative Euler equations (8) is comparatively mathematically underdeveloped. The exhaustive literature list is beyond the scope of the paper, and thus only few closely related results on the wave patterns of Riemann problem are mentioned, c.f. [1, 7, 16, 19, 18, 27, 28, 29, 33, 34, 35, 17].

The present paper is organized as follows. In Section 2, the viscous contact discontinuity wave is constructed and the main result is stated. Then series of $a$ priori estimates are established and main Theorem 2.2 is proved in Sections 3-5.
2. **Viscous contact wave and main result.** In this section, we construct viscous contact wave and give some properties, which will play an important role in the energy estimates done in the next three sections. And then the main result in this paper is also stated.

First, the viscous contact wave is constructed and some basic properties are collected in the following. When radiation is ignored, the one-dimensional radiative Euler equations (8) becomes the well-known compressible non-isentropic Euler system. Based on the classic theories of hyperbolic conservation law (see [23, 36]), assume \((v_+, p_+, \theta_+)\) is on the contact discontinuity curve passing through \((v_-, p_-, \theta_-)\) in the phase plane, that is,

\[
\begin{align*}
(u_+, p_+, \theta_+) & = (u_-, p_-, \theta_-), \\
& \quad y < 0,
\end{align*}
\]

then the Riemann problem of the Euler system in the Lagrangian coordinates \((y, t)\)

\[
\begin{cases}
v_t - u_y = 0, \\
u_t + p_y = 0, \\
(e + \frac{u^2}{2})_t + (pu)_y = 0, \\
(v, u, \theta)(y, 0) = \begin{cases} (v_-, u_-, \theta_-), & y < 0, \\ (v_+, u_+, \theta_+), & y > 0 \end{cases}
\end{cases}
\]

(13)

with \(p = \frac{R\theta}{v}\) admits a single contact discontinuity solution as

\[
\begin{cases}
v_+ = R\theta_+, \\
\frac{p_+}{v_+} = \frac{p_+}{v_+}, \\
q_+ = -\frac{b}{av^2} \left\{ (\theta^2 \theta_y) \right\}_y = -\frac{4bp_+}{aR} \left( \frac{\theta^2 \theta_y}{\gamma - 1} \right) y.
\end{cases}
\]

(15)

Substituting \((v_+, u_+, q_+)\) defined by (15) with \(p_+ = p_+\) into (8), we have

\[
C_v \frac{\partial \theta_+^d}{\partial y} + p_+ u_y^d = -q_y^d = \frac{4bp_+}{aR} \left( \frac{\theta^2 \theta_y}{\gamma - 1} \right) y.
\]

(16)

Let us further simplify equation (16), such that \(\theta^d(y, t) = \bar{\theta}(\eta)(\eta = \frac{y}{\sqrt{1+t}})\) is the unique self-similar solution of the nonlinear diffusion equation

\[
\begin{cases}
\bar{\theta}_t = \frac{\gamma - 1}{R\gamma} \frac{4bp_+}{aR} \left( \frac{\theta^2 \theta_y}{\gamma - 1} \right) y, \\
\bar{\theta}(\pm \infty, t) = \theta_{\pm}.
\end{cases}
\]

(17)

That is, \(\bar{\theta}(\eta)\) is a solution of the following ordinary equation

\[
\begin{cases}
-\eta \bar{\theta}' = \frac{\gamma - 1}{R\gamma} \frac{4bp_+}{aR} \left( \frac{\theta^2 \theta_y}{\gamma - 1} \right)', \\
\bar{\theta}(\pm \infty) = \theta_{\pm}.
\end{cases}
\]

(18)
Once $\bar{\theta}$ is solved, then the viscous contact discontinuity wave $z^{cd} = (v^{cd}, u^{cd}, \theta^{cd}, q^{cd})$ $(y, t)$ is expressed by $\bar{\theta}$ as follows:

\[
\begin{aligned}
   v^{cd}(y, t) &= \frac{R\theta^{cd}}{p_+}, \\
v^{cd}(y, t) &= u_+ + \frac{\gamma - 1}{\alpha} \frac{\theta^{cd}}{\theta^{cd}} \frac{1}{y}, \\
q^{cd}(y, t) &= \frac{4bp_+}{aR} \left( \theta^{cd} \right)^2 \frac{1}{y}.
\end{aligned}
\]

(19)

After the coordinates transformation $(y, t) \to (\xi, t)$ with $\xi = y - s - t$, the viscous contact discontinuity wave $(v^{cd}, u^{cd}, \theta^{cd}, q^{cd})(\xi + s - t, t)$ satisfies the system

\[
\begin{aligned}
v^{cd}_t - s_v v^{cd}_\xi - u^{cd}_\xi &= 0, \\
v^{cd}_t - s_u u^{cd}_\xi &= G_1, \\
C_v v^{cd} - s_c v^{cd}_\xi + p^{cd} u^{cd}_\xi + q^{cd}_\xi &= G_2, \\
a v^{cd} q^{cd} + b \left\{ (\theta^{cd})_\xi \right\} &= 0, \\
\left\{ v^{cd}, u^{cd}, \theta^{cd}, q^{cd} \right\} (\pm \infty, t) &= (v_\pm, u_\pm, \theta_\pm, 0),
\end{aligned}
\]

(20)

where $p^{cd} = \frac{R\theta^{cd}}{\rho^{cd}} = p_+ = p_-$, $G_1 := u^{cd}_t - s_\nu u^{cd}_\xi$ and $G_2 := u^{cd}(u^{cd}_t - s_\nu u^{cd}_\xi)$.

By the direct calculations based on solving problem (18) and employing (19), one has the following decay properties on viscous contact discontinuity wave and source term $G_1$ and $G_2$, which will be frequently used later.

**Lemma 2.1.** When $|\xi + s - t| \to + \infty$, $\theta^{cd}, G_1$ and $G_2$ satisfy the following decay estimates:

\[
\left| \theta^{cd} - \theta_\pm \right| + (1 + t)^\frac{1}{2} |\theta^{cd}| + (1 + t)^\frac{1}{2} |\theta^{cd}_\xi| = O(1) e^{- \frac{(\xi + s - t)^2}{\delta}}.
\]

(21)

and

\[
G_1 = O(1) \delta (1 + t)^{-1} e^{- \frac{(\xi + s - t)^2}{\delta}}, \quad G_2 = O(1) \delta (1 + t)^{-1} e^{- \frac{(\xi + s - t)^2}{\delta}}.
\]

(22)

with the radiation estimates

\[
q^{cd} = O(1) \theta^{cd}_\xi, \quad q^{cd}_\xi = O(1) (|\theta^{cd}_\xi|^2 + |\theta^{cd}_\xi|),
\]

(23)

\[
q^{cd}_\xi = O(1) (|\theta^{cd}_\xi|^3 + |\theta^{cd}_\xi \theta^{cd} + |\theta^{cd}_\xi|).
\]

In particular, it holds on the boundary $\xi = 0$

\[
\left| (v_\nu - v^{cd}, u_\nu - u^{cd}, \theta_\nu - \theta^{cd}) \right| (s - t, t) = O(1) e^{- \frac{(s - t)^2}{\delta}}.
\]

(24)

and

\[
q^{cd} = O(1) \delta (1 + t)^{-1} e^{- \frac{(\xi + s - t)^2}{\delta}}.
\]

(25)

Here $\delta := |\theta_+ - \theta_-|$. 

**Proof.** The proof of (21) can be found in [39, p.1032], so we omit it for the shortness. Then (23) and (24) can be proved by straightforward calculation. (24) comes from (21) and the fact that $s_- < 0$ for the inflow problem. (25) comes from (21) with (23).
In this paper, for the notational simplicity, we restrict the domain of functions \((v^{cd}, u^{cd}, \theta^{cd}, q^{cd})(\xi, t)\) such that they are defined only on the half line \(\mathbb{R}^+\), i.e., we denote functions \((v^{cd}, u^{cd}, \theta^{cd}, q^{cd})(\xi, t)\) as \((v^{cd}, u^{cd}, \theta^{cd}, q^{cd})(\xi, t)\).

Before introducing our main result, first let us introduce several notations which are frequently used throughout this paper.

Space \(L^p(\mathbb{R}^+)(1 \leq p \leq \infty)\) represents the standard Sobolev space on \(\mathbb{R}^+\) with norm \(\|f\|_{L^p(\mathbb{R}^+)} = \left(\int_{\mathbb{R}^+} |f(x)|^p \, dx\right)^{\frac{1}{p}}\). In particular, we denote the norm \(\|f\| = \|f\|_{L^2(\mathbb{R}^+)}\). Space \(\mathcal{H}^l(\mathbb{R}^+)\) for \(l \in \mathbb{Z}^+\) denotes the standard \(l\)-th order Sobolev space with norm \(\|f\|_{\mathcal{H}^l(\mathbb{R}^+)} = \|f\|_l = \left(\sum_{i=0}^{l} \|\partial_x^i f\|_2^2\right)^{\frac{1}{2}}\). For the notational simplicity, for any fixed time \(t\), \(\|f(\cdot, t)\|_{L^2}\) and \(\|f(\cdot, t)\|_{H^1}\) are denoted by \(\|f(t)\|\) and \(\|f(t)\|_t\) respectively. In addition, we denote by \(C^k(\mathbb{R}^+; \mathcal{H}^l)\) the \(k\)-times continuously differentiable functions in the time interval \(\mathbb{R}^+\) with the range in \(\mathcal{H}^l(\mathbb{R}^+)\); and denote by \(L^2(\mathbb{R}^+; \mathcal{H}^l)\) the space of \(L^2\) functions in the time interval \(\mathbb{R}^+\) with the range in \(\mathcal{H}^l(\mathbb{R}^+)\). Finally, the solution space is defined as follows:

\[
X_{\frac{1}{2}v_-, \frac{1}{2}\theta_- \cdot M}(0, t) := \left\{ (\phi, \psi, \zeta) \in C([0, t]; H^2(\mathbb{R}^+)), w \in C([0, t]; H^3(\mathbb{R}^+)), \right. \\
(\phi_{t\xi}, \psi_{t\xi}, \zeta_{t\xi}, w_{t\xi}) \in C([0, t]; L^2(\mathbb{R}^+)), \\
(\phi_{\xi}, \psi_{\xi}, \zeta_{\xi}) \in L^2(0, t; \mathcal{H}^1(\mathbb{R}^+)), 
\left. w \in L^2(0, t; H^3(\mathbb{R}^+)), 
\inf_{[0, t] \times \mathbb{R}^+} v(\xi, t) \geq \frac{1}{4} v_-, \quad \inf_{[0, t] \times \mathbb{R}^+} \theta(\xi, t) \geq \frac{1}{4} \theta_-, 
\sup_{\tau \in [0, t]} \{ ||(\phi, \psi, \zeta)(\tau)||_2 + ||w(\tau)||_3 + ||(\phi_{t\xi}, \psi_{t\xi}, \zeta_{t\xi}, w_{t\xi})(\tau)||_3 \} \leq M \right\},
\]

where \(M\) is a positive constant.

Then our main result is stated as follows:

**Theorem 2.2.** Assume that \(u_+ > \sqrt{\gamma R \theta_+}\) and \(u_- > \sqrt{\gamma R \theta_-}\), and assume that (12) holds, then there exist some small positive constants \(\epsilon_0\) and \(\eta_0\) such that if \(\delta = |\theta_+ - \theta_-| \lesssim \epsilon_0\) and

\[
\| (v^0 - v^{cd}, u^0 - u^{cd}, \theta_0 - \theta^{cd})(\xi, 0) \|_2 \lesssim \eta_0,
\]

where \((v^{cd}, u^{cd}, \theta^{cd}, q^{cd})\) is given by (18) and (19), then the inflow problem (11) admits a unique solution \((v, u, \theta, w)(\xi, t)\) satisfying

\[
(v - v^{cd}, u - u^{cd}, \theta - \theta^{cd}, q - q^{cd})(\xi, t) \in X_{\frac{1}{2}v_-, \frac{1}{2}\theta_- \cdot C(\eta_0 + \epsilon_0)^{\frac{1}{2}}}(0, +\infty).
\]

Furthermore, it holds that

\[
\lim_{t \to +\infty} \sup_{\xi \in \mathbb{R}^+} |(v, u, \theta, q)(\xi, t) - (v^{cd}, u^{cd}, \theta^{cd}, q^{cd})(\xi + s_- t)| = 0.
\]

3. **Mathematical reformulation and propositions.** In this section, we will reformulate system (11) based on the viscous contact wave defined by (20). Then two propositions on the reformulated system, which concludes the proof of Theorem 2.2, are given.

Define the functions related to the perturbation by

\[
(\phi, \psi, \zeta, w)(\xi, t) = (v, u, \theta, q)(\xi, t) - (v^{cd}, u^{cd}, \theta^{cd}, q^{cd})(\xi, t),
\]

...
then direct calculations yield that the functions satisfy the following reformulated system:

\[
\begin{align*}
\phi_t - s_- \phi_\xi - \psi_\xi &= 0, \quad \xi > 0, \quad t > 0, \\
\psi_t - s_- \psi_\xi + \left( \frac{R \zeta}{v} \right)_\xi - \left( \frac{p_- \phi}{v} \right)_\xi &= -G_1, \\
C_a \zeta_t - s_- C_a \zeta_\xi + p \psi_\xi + w_\xi &= -(p - p_+) u_\xi^{cd} - G_2, \\
- \left( \frac{w_\xi}{v} \right)_\xi + av \theta^2 + 4b \vartheta_3 \zeta_\xi + 4b \vartheta_3 \zeta_\xi \left\{ \vartheta^2 + \theta \vartheta^{cd} + (\vartheta^{cd})^2 \right\} &= \left( \frac{\vartheta^{cd}}{v} \right)_\xi - a \vartheta \vartheta^{cd}, \\
(\phi, \psi, \zeta)(\xi, 0) &= (\phi_0, \psi_0, \zeta_0)(\xi) \rightarrow (0, 0, 0), \quad \text{as} \quad \xi \rightarrow +\infty, \\
(\phi, \psi, \zeta)(0, t) &= (v_-, v_\xi, u_\xi^{cd}, \theta_-, \vartheta^{cd})(s_-, t), \\
w(0, t) &= -q^{cd}(s_-, t) = \frac{4b}{a} \left( \frac{\vartheta^{cd}}{v} \right)^3 \theta^{cd}(s_-, t) = \frac{4b}{aR} \left( \theta^{cd} \right)^2 \theta^{cd}(s_-, t).
\end{align*}
\]  

(31)

First of all, the local unique existence of the solution to system (31) is as follows:

**Proposition 1** (Local existence). There exist positive constants \( \epsilon_2, \eta_2 \) and \( C, (C \eta_2 \leq \eta_0) \) such that if \( \eta \leq \eta_2 \) and \( \epsilon \leq \epsilon_2 \), for any constant \( M \in (0, \eta_2) \), there exists a positive constant \( t_0 = t_0(M) \), which does not depend on \( \tau \), such that if \( \| (\phi, \psi, \zeta)(\tau)\|_2 + \| w(\tau)\|_3 \leq M \) and \( \inf_{[0, \tau] \times \mathbb{R}_+} v(\xi, t) \geq \frac{1}{4} v_- \), \( \inf_{[0, \tau] \times \mathbb{R}_+} \theta(\xi, t) \geq \frac{1}{4} \theta_- \), problem (31) has a unique solution \( (\phi, \psi, \zeta, w)(\xi, t) \in \mathcal{X}_{v_-, v_\xi, \theta_-}^{\mathcal{C}M}(\tau, \tau + t_0) \).

Proposition 1 can be proved similarly as the proof of Proposition 3.1 in [6]. Thus the proof of Proposition 1 is omitted. Suppose that \((\phi, \psi, \zeta, w)(\xi, t)\) has been extended to the time \( t \geq t_0 \), by applying Proposition 1, in order to show the global existence of solutions of problem (31), we only need to show the following a priori estimate.

**Proposition 2** (A priori estimate). Under the assumptions stated in Theorem 2.2, there exist positive constants \( \eta_2 \leq \eta_1, \epsilon_2 \leq \min\{\epsilon_1, 1\} \) and \( C \), such that for any \( t > 0 \), if \( (\phi, \psi, \zeta, w) \in X([0, t]) \) with \( \epsilon \leq \epsilon_2 \) and

\[
N(t) := \sup_{0 \leq \tau \leq t} \left\{ \| (\phi, \psi, \zeta)(\tau)\|_2 + \| w(\tau)\|_3 \right\} \leq \eta_2,
\]

then it follows the estimate that

\[
\begin{align*}
&\sup_{0 \leq \tau \leq t} \left\{ \| (\phi, \psi, \zeta)(\tau)\|_2^2 + \| w(\tau)\|_3^2 + \| (\phi_t, \psi_t, \zeta_t, w_t)(\tau)\|_1^2 \right\} \\
&+ \sum_{m_1 + m_2 \leq 2} \int_0^t \left\| \partial_t^{m_1} \partial_\xi^{m_2} \phi_t \partial_\xi^{m_2} \psi_t \partial_\xi^{m_2} \zeta_t \partial_\xi^{m_2} w_t \right\|^2 (0, \tau) d\tau \\
&+ \int_0^t \left( \| (\phi_\xi, \psi_\xi, \zeta_\xi)(\tau)\|_1^2 + \| w(\tau)\|_3^2 + \| w_t(\tau)\|_2^2 \right) d\tau
\end{align*}
\]
\[ \lesssim \| (\phi_0, \psi_0, \zeta_0) \|^2_2 + \| (\phi_t, \psi_t, \zeta_t)(0) \|^2_1 + \delta. \] (33)

Based on Proposition 2, the asymptotic behaviors stated in (29) can easily be obtained by combining estimate (33) with the Sobolev inequality. This concludes the proof of Theorem 2.2. In the next two sections, our main task is to show the \textit{a priori} estimate (33). First, in Section 4, the basic energy estimate is obtained. Then in Section 5, the higher order energy estimates are established.

4. Basic energy estimate. In this section, we will show the following basic energy estimate.

\textbf{Lemma 4.1 (Basic energy estimate).} Under the same assumptions listed in Proposition 2, if \( \epsilon \) and \( N(t) \) are suitable small, then it holds

\[ \| (\phi, \psi, \zeta)(t) \|^2 + \int_0^t \| w(\tau) \|^2_2 d\tau + \int_0^t (\phi, \psi, \zeta, w)(\tau) \|_2 d\tau \lesssim \| (\phi_0, \psi_0, \zeta_0) \|^2_2 + \delta + (\delta + N(t)) \int_0^t \left( \| \zeta(\tau) \|^2 + \| w(\tau) \|^2 \right) d\tau \] (34)

\[ + \delta \int_0^t \| (\phi, \psi)(\tau) \|^2 d\tau. \]

\textit{Proof.} Let \( \Phi(s) = s - 1 - \ln s \). By the direct calculations, one has the following equalities

\[ -R \theta^{cd} \left( \frac{1}{v} - \frac{1}{\nu^{cd}} \right) \phi_t = \left\{ R \theta^{cd} \Phi \left( \frac{v}{\nu^{cd}} \right) \right\}_t + \frac{p + u^{cd}}{v \nu^{cd}} \phi^2 - R \theta^{cd} \Phi \left( \frac{v}{\nu^{cd}} \right), \]

\[ C_v \zeta_t \frac{\zeta}{\theta} = \left\{ C_v \theta^{cd} \Phi \left( \frac{\theta}{\theta^{cd}} \right) \right\}_t + C_v \theta^{cd} \Phi \left( \frac{\theta^{cd}}{\theta} \right). \] (35)

In addition, from the definition of \( \theta^{cd} \) in (17) and (19), direct calculations also yield the following equation

\[ \theta^{cd}_t - s_\nu \theta^{cd} = \frac{\gamma - 1}{R \gamma} \frac{a \theta^{cd}}{a R} \left( (\theta^{cd})^2 \theta^{cd} \right)_\xi. \] (36)

Multiplying (31)_1 by \( -R \theta^{cd} \left( \frac{1}{v} - \frac{1}{\nu^{cd}} \right) \), it follows from (35) and (36)

\[ \left\{ R \theta^{cd} \Phi \left( \frac{v}{\nu^{cd}} \right) \right\}_t - \left\{ s_\nu R \theta^{cd} \Phi \left( \frac{v}{\nu^{cd}} \right) \right\}_\xi + \frac{p + u^{cd}}{v \nu^{cd}} \phi^2 - \frac{p + \phi}{v} \psi \xi \]

\[ = R \left( \theta^{cd}_t - s_\nu \theta^{cd} \right) \Phi \left( \frac{v}{\nu^{cd}} \right) \] (37)

\[ = \frac{\gamma - 1}{\gamma} \frac{a \theta^{cd}}{a R} \left( (\theta^{cd})^2 \theta^{cd} \right)_\xi \Phi \left( \frac{v}{\nu^{cd}} \right). \]

Multiplying (31)_2 by \( \psi \), it holds that

\[ \left( \frac{\nu^2}{2} \right)_t - \left\{ \frac{s_\nu}{2} \psi^2 - (p + p_\nu) \psi \right\}_\xi - \frac{R \kappa}{v} \psi \xi + \frac{p + \phi}{v} \psi \xi = -G_\psi. \] (38)
Multiplying (31) by $\frac{f}{b}$, it follows from (20) and (35)

$$\left\{ C_c \theta^{\alpha} \Phi \left( \frac{\theta}{b} \right) \right\}_t - \left\{ s C_c \theta^{\alpha} \Phi \left( \frac{\theta}{b} \right) \right\} \xi + \frac{C_c \zeta w \xi + R \zeta \psi}{v}$$

$$= (p+u_c^c + q_c^c) - G_2 \Phi \left( \frac{\theta}{b} \right) - \frac{\zeta}{\theta} (p-p+) u_c^c - G_2 \frac{\zeta}{\theta}$$

(39)

Multiplying (31) by $\frac{w}{\beta \theta^4}$, we get

$$- \left( \frac{w \zeta}{\beta \theta^4} \right)_\xi + \frac{w \zeta}{v} \left( \frac{w}{\beta \theta^2} \right)_\xi + \frac{avw^2}{\beta \theta^4} + \frac{\zeta}{w}$$

$$= - \frac{\theta^2 + \theta^{\alpha} \theta + \theta^{\alpha} \theta w \zeta + \left\{ \left( \frac{q_c^c}{v} \right)_\xi - a \phi q^{\alpha} \right\}}{4 \beta \theta^4}$$

(40)

Combination of (37)-(40) yields

$$\left\{ R \theta^{\alpha} \Phi \left( \frac{v}{\beta \eta} \right) + \frac{\psi^2}{2} + C_c \theta^{\alpha} \Phi \left( \frac{\theta}{b} \right) \right\}_t + \frac{w \zeta}{v} \left( \frac{w}{\beta \theta^2} \right)_\xi + \frac{avw^2}{\beta \theta^4}$$

$$= \tilde{G} + \left( \frac{1}{\theta} \right)_\xi \zeta w - \frac{\theta^2 + \theta^{\alpha} \theta + (q^{\alpha})^2 \theta w \zeta + \left\{ \left( \frac{q_c^c}{v} \right)_\xi - a \phi q^{\alpha} \right\}}{4 \beta \theta^4} \frac{w}{\beta \theta^4}$$

(41)

$$+ s \theta^{\alpha} \Phi \left( \frac{v}{\beta \eta} \right) + s \frac{\psi^2}{\beta} + s C_c \theta^{\alpha} \Phi \left( \frac{\theta}{b} \right) - (p-p+) \zeta w - \frac{w \zeta}{\beta \theta^4} + \frac{\zeta}{\theta} w$$

where

$$\tilde{G} = - \frac{p+u_c^c}{\beta \theta^4} \phi^2 + \gamma - \frac{1}{\gamma} \frac{4 \beta \theta}{aR} \left\{ \left( \frac{\theta^{\alpha}}{\beta} \right)^2 \theta \xi \right\} \Phi \left( \frac{v}{\beta \eta} \right) - G_1 \psi$$

(42)

Integrating (41) on $[0, t] \times \mathbb{R}^+$, we have

$$\| (\phi, \psi, \zeta) \|^2 + \int_0^t \| w(\tau) \|^2 d\tau$$

$$\leq \| (\phi_0, \psi_0, \zeta_0) \|^2 + \int_0^t \int_{\mathbb{R}^+} \left\{ c \xi \xi + \left( \left( \frac{q^{\alpha}}{\xi} \right)^2, \frac{q^{\alpha}}{\xi} \right) \right\} d\zeta d\tau + \int_0^t \int_{\mathbb{R}^+} |\tilde{G}| d\zeta d\tau$$

(43)

$$+ \int_0^t \left( \| (\phi, \psi, \zeta) \|^2 + \| w\xi | + | w\zeta | \right) (0, \tau) d\tau + \int_0^t \int_{\mathbb{R}^+} \frac{\beta^2 + \theta^{\alpha} \theta + \frac{(q^{\alpha})^2}{\beta} \theta w \zeta}{\beta^4} d\zeta d\tau$$

$$= \| (\phi_0, \psi_0, \zeta_0) \|^2 + \sum_{i=1}^4 J_i$$

Thus, it suffices to control $J_1, J_2, J_3$ and $J_4$ on the right-hand side of (43) as follows.

First, it is obvious to get

$$\int_0^t \int_{\mathbb{R}^+} | \xi \xi |^2 d\zeta d\tau \lesssim N(t) \int_0^t \| \xi \|^2 d\tau$$

(44)
In addition, it follows from (21) and (23) that
\[
\int_0^t \int_{\mathbb{R}^+} \left| \begin{pmatrix} \vartheta^{cd} \xi \omega \xi \xi \xi \\ \vartheta^{cd} \xi \omega \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \x
\begin{align}
&\lesssim \delta^2 \int_0^t e^{-\frac{2s^2-\tau^2}{4s^2+\tau^2}} d\tau \\
&\lesssim \delta^2 \int_0^t e^{-\tau} d\tau \lesssim \delta^2 \tag{49}
\end{align}

and

\begin{align}
\int_0^t |ww_\xi|(0,\tau)d\tau &\lesssim \delta \int_0^t (1+\tau)^{-\frac{3}{2}} e^{-\tau^2} \|w_\xi(\tau)\|_\infty d\tau \\
&\lesssim \delta \int_0^t \|w_\xi(\tau)\| \|w_\xi(\tau)\|_2^\delta d\tau \\
&\lesssim \delta \int_0^t \|w_\xi(\tau)\|^2 d\tau + \delta \int_0^t (1+\tau)^{-1} e^{-2\tau} d\tau \\
&\lesssim \delta \int_0^t \|w_\xi(\tau)\|^2 d\tau + \delta. \tag{50}
\end{align}

If let \( \delta \ll 1 \), then the combination of (49) and (50) together yields to the estimate on the boundary integral \( J_3 \) that:

\[ J_3 \lesssim \delta \int_0^t \|w_\xi(\tau)\|^2 d\tau + \delta. \tag{51} \]

Putting (46), (47), (48) and (51) into (43), we get (34). This completes the proof of Lemma 4.1.

5. Higher order energy estimates. In this section, based on the basic energy estimate developed in the previous section, we are going to derive the higher order estimates. First, the first order energy estimates is derived as follows:

**Lemma 5.1.** Under the same assumptions as listed in Proposition 2, if \( \epsilon \) and \( N(t) \) are suitable small, then it holds

\[
\| (\phi_\xi, \psi_\xi, \zeta_\xi) (t) \|^2 + \int_0^t \left( \| \zeta_\xi(\tau) \|^2 + \| w_\xi(\tau) \|^2 \right) d\tau \\
+ \int_0^t |(\phi_\xi, \psi_\xi, \zeta_\xi, w_\xi, w_\xi)|^2 (0,\tau)d\tau \\
\lesssim \| (\phi_0, \psi_0, \zeta_0) \|^2_1 + \delta + (\delta + N(t)) \int_0^t \| (\phi_\xi, \psi_\xi, \zeta_\xi)(\tau) \|^2_1 d\tau. \tag{52}
\]

**Proof.** First, we will consider the boundary integral of the radiative terms \( \int_0^t |w_\xi|^2 (0,\tau) d\tau \) and \( \int_0^t |w_\xi|^2 (0,\tau) d\tau \) on the left-hand side of (52), which can be controlled by the fluid quantities as follows.

In fact, from (31)_3 and (31)_4 respectively, one has

\[
\int_0^t w_\xi^2(0,\tau)d\tau \lesssim \int_0^t \left\{ \zeta_\xi^2 + \psi_\xi^2 + (p-p_+)^2 (u_\xi^d)^2 + G_\xi^2 \right\} (0,\tau)d\tau \\
\lesssim \int_0^t (\zeta_\xi^2 + \psi_\xi^2) (0,\tau)d\tau + \delta \tag{53}
\]
and
\[
\int_{0}^{t} w^2_{\xi}(0, \tau) \, d\tau
\]
\[
\lesssim \int_{0}^{t} \left\{ w^2_{\xi} + w^2 + \xi^2 + \left( (\frac{q_{\xi}^{cd}}{v})^2 \right) - a \phi q^{cd} \right\} (0, \tau) \, d\tau \tag{54}
\]
\[
\lesssim \int_{0}^{t} (w^2_{\xi} + \xi^2) (0, \tau) \, d\tau + \delta.
\]
Thus, in order to show (52), it suffices to show the following estimate:
\[
\|(\phi_{\xi}, \psi_{\xi}, \xi_{\xi})(t)\|^2 + \int_{0}^{t} \left( \|(\xi_{\xi}(\tau))\|^2 + \|w_{\xi}(\tau)\|_{1}^{2} \right) \, d\tau + \int_{0}^{t} \|(\xi_{\xi}, \psi_{\xi}, \xi_{\xi})(\tau)\|^{2} (0, \tau) \, d\tau
\]
\[
\lesssim \|(\phi_{0}, \psi_{0}, \xi_{0})\|_{1}^{2} + \delta + (\delta + N(t)) \int_{0}^{t} \|(\phi_{\xi}, \psi_{\xi}, \xi_{\xi})(\tau)\|^{2} \, d\tau.
\] (55)

By the straightforward calculation from (31)_{1} \times (- \frac{p_{+}}{v} \phi_{\xi \xi}) + (31)_{2} \times (- \psi_{\xi \xi}) + (31)_{3} \times (- \frac{\xi_{\xi \xi}}{v}) + (31)_{4} \times \frac{w_{\xi}}{4b\theta t^4}, we get that
\[
\left\{ \left( \frac{p_{+}}{2v} \phi_{\xi}^2 + \frac{\xi_{\xi} \theta_{\xi}}{2} \right) \right\}_{t} + \frac{w_{\xi} \xi}{4b\theta t^4} \frac{w_{\xi}}{v} + \left\{ \left( \frac{c_{\xi}}{2 \theta} \right) \right\}_{t} \xi_{\xi} + \left\{ \left( \frac{s - c_{\xi}^2}{2 \theta} \right) \right\}_{t} \xi_{\xi} + (p_{+})_{\xi} \xi_{\xi} + G_{1} \psi_{\xi \xi} + G_{2} \xi_{\xi \xi} - \frac{\xi_{\xi \xi \xi}}{\theta} \tag{56}
\]
\[
- \left\{ \left( \frac{p_{+}}{v} \xi \phi \right) \right\}_{t} - \left( \frac{R}{v} \right) \xi \psi_{\xi \xi} - \left( \frac{R}{v} \right) \xi \phi_{\xi \xi} + (p_{+})_{\xi} \xi_{\xi} \psi_{\xi} + (p_{+})_{\xi} \xi_{\xi} \psi_{\xi} - \frac{C_{\xi}}{\theta} \xi_{\xi \xi} \psi_{\xi} + (p_{+})_{\xi} \xi_{\xi} \psi_{\xi}
\]
\[
+ \frac{w_{\xi} \xi w_{\xi} w_{\xi}}{4b\theta t^4} - \left( \frac{1}{4b\theta t^4} \right) w_{\xi} \psi_{\xi} - \frac{\xi}{4b\theta t^4} \xi \psi_{\xi} - \frac{\xi}{4b\theta t^4} \xi \phi_{\xi \xi} + \left( \frac{C_{\xi}}{\theta} \right) \xi \xi \psi_{\xi}
\]
\[
+ \frac{w_{\xi}}{4b\theta t^4} \left( \frac{\phi_{\xi}^{cd}}{v} \right) \xi - a \phi q^{cd} \phi - 4b\theta \xi \phi \left( \phi^2 + \theta \phi^{cd} + (\theta^{cd})^2 \right) \right\}_{t}. \]

Integrating (56) on $[0, t] \times \mathbb{R}^+$, we obtain
\[
\|(\phi_{\xi}, \psi_{\xi}, \xi_{\xi})(t)\|^2 + \int_{0}^{t} \|w_{\xi}(\tau)\|_{1}^{2} \, d\tau + \int_{0}^{t} I_{1} (\phi_{\xi}, \psi_{\xi}, \xi_{\xi})(0, \tau) \, d\tau
\]
\[
\leq \|(\phi_{0}, \psi_{0}, \xi_{0})\|_{1}^{2} + \delta + (\delta + N(t)) \int_{0}^{t} \|(\phi_{\xi}, \psi_{\xi}, \xi_{\xi})(\tau)\|^{2} \, d\tau
\]
\[
+ \int_{0}^{t} \int_{\mathbb{R}^+} \left( |q_{\xi}^{cd} \xi_{\xi \xi}| + |q_{\xi}^{cd}|^2 + |q_{\xi}^{cd}|^2 \right) \, d\xi \, d\tau.
\]
The combination of (57)-(60) together yields

$$- \int_0^t \left\{ \frac{p_+}{v_-} \phi_t \phi_\xi + \psi_t \psi_\xi + \frac{C_v}{\theta_-} \zeta_\xi \zeta_t + \frac{w_\xi}{4\delta \theta_-} (\frac{w_\xi}{v_\xi}) \right\} (0, \tau) d\tau, \quad (57)$$

where

$$I_1 (\phi_\xi, \psi_\xi, \zeta_\xi) (\xi, t) := \left( -\frac{s_- p_+}{2v_-} \phi_\xi^2 - \frac{s_-}{2} \psi_\xi^2 - \frac{s_- C_v}{2\theta_-} \zeta_\xi^2 + \frac{R}{v_-} \zeta_\xi \psi_\xi - \frac{p_+}{v_-} \phi_t \phi_\xi \right) (\xi, t). \quad (58)$$

Taking the good advantages of the region, where the asymptotic states on the boundary \( z_- := (v_-, u_-, \theta_-) \) lie in, that is \( u_-^2 > \gamma R \theta_-, -s_- = \frac{v_-}{v_+} > 0 \) and \( v_- > 0 \), direct calculations show that the quadratic form

$$I_1 (\phi_\xi, \psi_\xi, \zeta_\xi) = (\phi_\xi, \psi_\xi, \zeta_\xi) \left( \begin{array}{ccc} -\frac{s_- p_+}{2v_-} & -\frac{p_+}{v_-} & 0 \\ -\frac{p_+}{v_-} & -\frac{s_-}{2} & \frac{R}{2v_-} \\ 0 & \frac{R}{2v_-} & -\frac{s_- C_v}{2\theta_-} \end{array} \right) (\phi_\xi, \psi_\xi, \zeta_\xi)$$

is definitely positive at \( \xi = 0 \), i.e.,

$$I_1 (\phi_\xi, \psi_\xi, \zeta_\xi) (0, t) \geq C \left| (\phi_\xi, \psi_\xi, \zeta_\xi) \right|^2 (0, t). \quad (59)$$

In addition, it follows from the Cauchy inequality that

$$- \int_0^t \left\{ \frac{p_+}{v_-} \phi_t \phi_\xi + \psi_t \psi_\xi + \frac{C_v}{\theta_-} \zeta_\xi \zeta_t \right\} (0, \tau) d\tau \leq \frac{C}{2} \int_0^t \left| (\phi_\xi, \psi_\xi, \zeta_\xi) \right|^2 (0, \tau) d\tau + \int_0^t \left| (\phi_t, \psi_t, \zeta_t) \right|^2 (0, \tau) d\tau. \quad (60)$$

The combination of (57)-(60) together yields

$$\left\| (\phi_\xi, \psi_\xi, \zeta_\xi) (t) \right\|^2 + \int_0^t \left\| w_\xi (\tau) \right\|^2 d\tau + \int_0^t \left| (\phi_\xi, \psi_\xi, \zeta_\xi) \right|^2 (0, \tau) d\tau$$

$$\lesssim \left\| (\phi_0, \psi_0, \zeta_0) \right\|_1^2 + \delta + (\delta + N(t)) \int_0^t \left\| (\phi_\xi, \psi_\xi, \zeta_\xi) (\tau) \right\|^2 d\tau$$

$$+ \int_0^t \int_{\mathbb{R}^+} \left( |q_\xi^d \zeta_\xi| + |q_\xi^e \zeta_\xi | + |q_\xi^d |^2 + |q_\xi^e |^2 \right) d\xi d\tau$$

$$+ \int_0^t \left| (\phi_t, \psi_t, \zeta_t) \right|^2 (0, \tau) d\tau - \int_0^t \frac{w_\xi}{4\delta \theta_-} \left( \frac{w_\xi}{v_\xi} \right) (0, \tau) d\tau$$

$$= \left\| (\phi_0, \psi_0, \zeta_0) \right\|_1^2 + \delta + (\delta + N(t)) \int_0^t \left\| (\phi_\xi, \psi_\xi, \zeta_\xi) (\tau) \right\|^2 d\tau + \sum_{i=1}^3 K_i. \quad (61)$$

Thus, it suffices to control \( K_1, K_2 \) and \( K_3 \) on the right-hand side of (61) as follows.

First, one has from (23) that

$$\int_0^t \int_{\mathbb{R}^+} |q_\xi^d \zeta_\xi| \, d\xi d\tau \lesssim \int_0^t \left( |q_\xi^e |^2 + |q_\xi^d | \right) |\zeta_\xi| \, d\xi d\tau$$

$$\lesssim \delta \int_0^t \int_0^\infty (1 + \tau)^{-1} e^{-\frac{(\xi + s \tau)^2}{4\sigma^2}} |\zeta_\xi| \, d\xi d\tau$$
\[
\lesssim \delta \int_0^t \|\zeta_{\xi}(\tau)\|^2 d\tau + \delta \int_0^t \int_0^\infty (1 + \tau)^{-2} e^{-\frac{2(\xi + \tau)^2}{1 + \tau^2}} d\xi d\tau
\]

\[
\lesssim \delta \int_0^t \|\zeta_{\xi}(\tau)\|^2 d\tau + \delta.
\] (62)

Thus, by using (23) again, we have

\[
K_1 \lesssim \delta \int_0^t \|\zeta_{\xi}(\tau)\|^2 d\tau + \delta.
\] (63)

Based on the direct calculations, it follows from (11)_6, (30) and Lemma 2.1 that

\[
K_2 \lesssim \int_0^t \left\{ \|v_{cd}\|^2 + \|u_{cd}\|^2 + \|\theta_{cd}\|^2 \right\} (0, \tau) d\tau
\]

\[
\lesssim \delta^2 \int_0^t (1 + \tau)^{-2} e^{-\frac{2(\xi + \tau)^2}{1 + \tau^2}} d\tau \lesssim \delta^2.
\] (64)

Finally, for the estimate of \(K_3\), it is a little more complicated to obtain. By (31)_4, we notice that

\[
\left( \frac{w_{\xi}}{v} \right)_{\xi} (0, t) = av_{-} w(0, t) + 4b\theta^3 \zeta_{\xi}(0, t)
\]

\[
+ 4b\theta^3 \zeta \left\{ \theta^2 + \theta \theta_{cd} + (\theta_{cd})^2 \right\} (0, t) - \left\{ \left( \frac{q_{cd}}{v} \right)_{\xi} - a\phi q_{cd} \right\} (0, t),
\] (65)

which means

\[
\frac{w_{\xi}}{4b\theta^2} \left( \frac{w_{\xi}}{v} \right)_{\xi} (0, t) = \frac{av_{-} w_{\xi}}{4b\theta^2} (0, t) + \frac{\zeta_{\xi} w_{\xi}}{\theta_{-}} (0, t)
\]

\[
+ 4b\theta^3 \zeta \left\{ \theta^2 + \theta \theta_{cd} + (\theta_{cd})^2 \right\} \frac{w_{\xi}}{4b\theta^2} (0, t)
\]

\[
- \left\{ \left( \frac{q_{cd}}{v} \right)_{\xi} - a\phi q_{cd} \right\} \frac{w_{\xi}}{4b\theta^2} (0, t)
\]

\[
= O(1) \left\{ w_{\xi} + \zeta_{\xi} w_{\xi} + \theta_{cd} \zeta_{\xi} w_{\xi} + \left[ \left( \frac{q_{cd}}{v} \right)_{\xi} - a\phi q_{cd} \right] w_{\xi} \right\} (0, t).
\] (66)

Thus, \(K_3\) can be estimated as follows

\[
K_3 \lesssim \int_0^t \left\{ \|w_{\xi}\| + \|\theta_{cd} w_{\xi}\| + \| (q_{cd}, q_{cd} w_{\xi}, q_{cd} \phi) \| w_{\xi} \right\} (0, \tau) d\tau + \int_0^t \|\phi_{\xi} w_{\xi}\| (0, \tau) d\tau
\]

\[
\lesssim \delta \int_0^t \|w_{\xi}(\tau)\|^2 d\tau + \delta \int_0^t \|\phi_{\xi}(\tau)\|^2 (0, \tau) d\tau + \delta + \int_0^t \|\zeta_{\xi} w_{\xi}\| (0, \tau) d\tau.
\] (67)
where we have used (49), (50) and (53). Moreover, for the last term on the right-hand side of estimate (67), we see that
\[
\int_0^t |\xi w| (0, \tau) d\tau \lesssim \int_0^t \|w(\tau)\|_\infty \xi (0, \tau) d\tau
\]
\[
\lesssim \frac{1}{4} \int_0^t \xi (0, \tau) d\tau + \int_0^t \|w(\tau)\|_\infty d\tau
\]
\[
\lesssim \frac{1}{4} \int_0^t \xi (0, \tau) d\tau + \int_0^t \|w_x(\tau)\|_\infty d\tau
\]
(68)
\[
\lesssim \frac{1}{4} \int_0^t \xi (0, \tau) d\tau + \frac{1}{4} \int_0^t \xi (0, \tau) \|w_{xx}(\tau)\|_\infty d\tau + \int_0^t \|w(\tau)\|_\infty d\tau.
\]
Putting (63), (64), (67) and (68) into (61), we get
\[
\|(\phi, \psi, \zeta) (t)\|^2 + \int_0^t \|w(\tau)\|^2_1 d\tau + \int_0^t |(\phi, \psi, \zeta)|^2 (0, \tau) d\tau
\]
\[
\lesssim \|(\phi_0, \psi_0, \zeta_0)\|^2 + \delta + (\delta + N(t)) \int_0^t \|w(\phi, \psi, \zeta) (\tau)\|^2_1 d\tau.
\]
(69)
Finally, it follows from (31) that
\[
\int_0^t \|\xi(\tau)\|^2_2 d\tau
\]
\[
\lesssim \int_0^t \int_{R_+} \left\{ \left( \frac{w}{v} \right)^2 + w^2 + \theta_x^d \xi^2 + \left[ \left( \frac{q_{\xi x}}{v} \right)^2 - a \phi q_{\xi} \right] \right\} d\xi d\tau
\]
\[
\lesssim \int_0^t \|w(\tau)\|^2_2 d\tau + \int_0^t \int_{R_+} \left( |\theta_x|^2 + |\theta_{\xi x}| \right) |(\phi, \zeta)|^2 d\xi d\tau
\]
\[
+ \int_0^t \int_{R_+} \left( |q_{x}^d|^2 + |q_{\xi x}^d| \right) |(\phi, \psi)|^2_1 d\xi d\tau
\]
\[
\lesssim \|(\phi_0, \psi_0, \zeta_0)\|^2 + \delta + (\delta + N(t)) \int_0^t \|w(\phi, \psi, \zeta) (\tau)\|^2_1 d\tau.
\]
(70)
The combination of estimates (69) and (70) together yields estimate (52). Hence, this completes the proof of Lemma 5.1.

In order to obtain the high order energy estimates, now let us estimate the tangential derivatives, i.e., the time derivatives first.

**Lemma 5.2 (Estimate on the tangential derivatives).** Under the same assumptions listed in Proposition 2, if \( \epsilon \) and \( N(t) \) are suitable small, it holds that
\[
\|(\phi_t, \psi_t, \zeta_t) (t)\|^2 + \int_0^t \|w_t(\tau)\|_2^2 d\tau
\]
\[
\lesssim \|(\phi_0, \psi_0, \zeta_0)\|^2 + \|(\phi_t, \psi_t, \zeta_t) (0)\|^2 + \delta
\]
Proof. By paying attention on the terms with\( G_1 \) and \( G_2 \), it follows from the computation \((31)_t \times \frac{\psi}{v}\phi_t + (31)_t \times \psi_t + (31)_t \times \frac{\theta}{\theta} + (31)_t \times \frac{w}{\theta}\) that,

\[
\begin{align*}
&\left(\frac{p_t + \phi_t^2}{2v} + \frac{\psi_t^2}{2} + \frac{C_v \theta}{\theta^2} \right)_t - s\left(\frac{p_t + \phi_t^2}{2v} + \frac{\psi_t^2}{2} + \frac{C_v \theta}{\theta^2} \right)\xi \\
&+ \left\{\frac{R}{v} \xi \psi_t - \frac{p_t}{v} \phi_t \psi_t + \frac{w_t}{\theta} \xi \right\} - \frac{w_t}{\theta^2} \left(\frac{\psi}{v}\right)_t \xi + \frac{w_t \theta}{\theta^2} \xi_t + p_t \psi_t \xi_t \\
&+ \frac{w_t^2}{4v \theta^4} - v_t w_t w_t \xi + \left(1 - \frac{s - p_t}{2v}\right) \xi - \frac{w_t}{\theta^2} \left(\frac{C_v}{\theta}\right)_t - \frac{w_t}{\theta^2} \left(\frac{s - C_v}{\theta}\right)_t \xi \\
&+ \frac{w_t}{\theta^2} \left\{\left(\frac{\psi}{v}\right)_t - \left(\frac{\psi}{v}\right) \xi \right\} - a q \frac{\phi - 4b \theta \xi \xi}{v} - 4b \theta \xi \xi \left[\theta^2 + \theta \theta + (\theta \theta)^2\right] \xi \right\}.
\end{align*}
\]

Noticing that on the boundary \( v_t(0, t) = 0 \), we have

\[
\int_0^t w_t \left(\frac{w}{v}\right) (0, \tau) d\tau \lesssim \int_0^t |w_t w_{t\xi}| (0, \tau) d\tau \\
\lesssim \delta \int_0^t (1 + \tau)^{-\frac{\alpha}{2}} e^{-\frac{\tau^2}{2 + \tau}} |w_{t\xi}(0, \tau)| d\tau \\
\lesssim \delta + \delta \int_0^t w_{t\xi}^2 (0, \tau) d\tau.
\]

Now, integrating \((72)\) on \([0, t] \times \mathbb{R}^+\), it holds

\[
\|(\phi_t, \psi_t, \xi_t) (t)\|^2 + \int_0^t \|(w_{t\xi}, w_{t\xi}) (\tau)\|^2 d\tau \\
\lesssim \|(\phi_0, \psi_0, \xi_0)\|^2 + \|(\phi_t, \psi_t, \xi_t) (0)\|^2 + \delta + \int_0^t \|(\phi_t, \psi_t, \xi_t, w_{t\xi})^2 (0, \tau) d\tau \\
+ \int_0^t \|(\phi_t, \psi_t, \xi_t, w_{t\xi})^2 (0, \tau) d\tau + \delta \int_0^t w_{t\xi}^2 (0, \tau) d\tau \\
+ (\delta + N(t)) \int_0^t \|(\phi_t, \psi_t, \xi_t) (\tau)\|^2 d\tau.
\]

Moreover, it is easy to get

\[
\int_0^t w_{t\xi}^2 (0, \tau) d\tau \lesssim \int_0^t |w_{t\xi}(\tau)|^2 d\tau \lesssim \int_0^t \|w_{t\xi}(\tau)\|^2_1 d\tau.
\]
Now, by the boundary estimates (75) and Lemma 5.1, we obtain estimate (71). This completes the proof of Lemma 5.2.

Based on the estimate on the tangential derivatives of solutions, now we can consider the higher order energy estimates one by one. First, it is the estimate on the derivative \(\partial_\xi\) of solutions.

**Lemma 5.3.** Under the same assumptions listed in Proposition 2, if \(\epsilon\) and \(N(t)\) are suitable small, then it holds

\[
\| (\phi_\xi, \psi_\xi, \zeta_\xi) (t) \|^2 + \int_0^t \| (\phi_{t\xi}, \psi_{t\xi}, \zeta_{t\xi}, w_{t\xi}) \|^2 (0, \tau) + \| w_{t\xi}(\tau) \|^2_1 \, d\tau \\
\lesssim \| (\phi_0, \psi_0, \zeta_0) \|^2 + \| (\phi_\xi, \psi_\xi, \zeta_\xi)(0) \|^2_1 + \delta + (\delta + N(t)) \int_0^t \| (\phi_\xi, \psi_\xi, \zeta_\xi)(\tau) \|^2_1 \, d\tau.
\]  

(76)

**Proof.** By direct calculation, we have

\[
(p - p_+)_{t\xi} = \left( \frac{R}{v} \zeta_{t\xi} - \frac{p_+}{v} \phi_{t\xi} \right) + \left( \frac{R}{v} \right)_{t} \zeta - \left( \frac{p_+}{v} \right)_{t} \phi
\]

\[
+ \left( \frac{R}{v} \right)_{t\xi} \zeta - \left( \frac{p_+}{v} \right)_{t\xi} \phi + \left( \frac{R}{v} \right)_{\xi} \zeta - \left( \frac{p_+}{v} \right)_{\xi} \phi
\]

and

\[
(p - p_+)_{t\xi\xi} = \frac{R}{v} \zeta_{t\xi\xi} - \left( \frac{p_+}{v} \phi_{t\xi\xi} \right) + \left( \frac{R}{v} \right)_{t} \zeta_{t\xi} + \left\{ \left( \frac{R}{v} \right)_{t} \zeta - \left( \frac{p_+}{v} \right)_{t} \phi \right\}_{\xi} + \left( \frac{R}{v} \right)_{t\xi} \zeta
\]

\[
- \left( \frac{p_+}{v} \right)_{t\xi} \phi + \left( \frac{R}{v} \right)_{t\xi} \zeta - \left( \frac{p_+}{v} \right)_{t\xi} \phi + \left\{ \left( \frac{R}{v} \right)_{\xi} \zeta - \left( \frac{p_+}{v} \right)_{\xi} \phi \right\}_{\xi} \]

\[
= \frac{R}{v} \zeta_{t\xi\xi} - \left( \frac{p_+}{v} \phi_{t\xi\xi} \right) + \left( \frac{R}{v} \right)_{t\xi} \zeta - \left( \frac{p_+}{v} \right)_{t\xi} \phi
\]

\[
+ O(1)(\delta + N(t)) \| (\phi_\xi, \psi_\xi, \zeta_\xi, \phi_\xi, \psi_\xi, \zeta_\xi) \|^2.
\]

Therefore, it follows from the straightforward calculation by considering \((31)_{1t} \times (-\frac{p_+}{v} \phi_{t\xi\xi}) + (31)_{2t} \times (-\psi_{t\xi\xi}) + (31)_{3t} \times (-\frac{\zeta_{t\xi\xi}}{\theta}) + (31)_{4t} \times \frac{w_{t\xi}}{4b\theta^3}\) that

\[
\left( \frac{p_+}{2v} \phi_{t\xi\xi} + \frac{\psi_{t\xi\xi}^2}{2} + \frac{C_0 \zeta_{t\xi\xi}^2}{\theta^3} \right)_{t} + I_{2t} + \frac{w_{t\xi\xi}}{4b\theta^3} + \frac{aw_{t\xi}}{4b\theta^3}
\]

\[
+ \frac{w_{t\xi\xi}}{4b\theta^3} \left\{ w_{t\xi} \left( \frac{1}{v} \right)_{\xi} + w_{\xi} \left( 1_{t\xi} \right) + w_{\xi} \left( \frac{1}{v} \right)_{t\xi} \right\} - \frac{w_{t\xi} \theta_{\xi}}{b\theta^2} \left( \frac{w_{t\xi}}{b\theta^3} \right)_{\xi}
\]

\[
+ \frac{aw_{t\xi}}{4b\theta^3} (v_{t\xi} w_{t\xi} + w_{t\xi} + w_{t\xi}) + \left\{ (4b\theta^3)_{t\xi} \zeta_{\xi} + (4b\theta^3)_{t\xi} \zeta_{t\xi} + (4b\theta^3)_{t\xi} \zeta_{t\xi} \right\} \frac{w_{t\xi}}{4b\theta^3}
\]

\[
= \left( \frac{R}{v} \right)_{t\xi\xi} \zeta_{\psi_{t\xi}} - \left( \frac{p_+}{v} \right)_{t\xi\xi} \phi \psi_{t\xi} + O(1)(\delta + N(t)) \| (\phi_\xi, \psi_\xi, \zeta_\xi, \phi_\xi, \psi_\xi, \zeta_\xi) \|^2
\]
\begin{align*}
- G_{tt} \psi_t - \left\{ p_t \psi_t + (p - p_+) u_t^{ed} + (p - p_+) u_t^{cd} + G_{1t} \right\} \frac{\zeta}{\theta} \\
+ \left\{ \left( \frac{\xi_t^{ed}}{v} \right) - a q^{ed} \phi - 4 b_0 \zeta \left( \theta^2 + \theta \zeta^{cd} + \theta^{cd} \right) \right\} \frac{w_t}{4 \theta^2} \tag{79}
\end{align*}

where

\begin{align*}
I_2 := & \frac{s - p_+}{2v} \phi_t^2 - \frac{s}{2} \psi_t^2 - \frac{R}{v} \zeta_t \psi_t + \frac{p_+}{v} \phi_t \psi_t + \frac{s - C_v \zeta_t}{2} \\
- \frac{p_+}{2v} \phi_t \psi_t - \frac{\zeta_t}{\theta} - \left( \frac{R}{v} \right) \phi_t \psi_t - \left( - \frac{p_+}{v} \right) x \psi_t \\
- \left\{ \left( \frac{R}{v} \right) \zeta - \left( \frac{p_+}{v} \right) \phi + \left( \frac{R}{v} \right) \psi_t - \left( \frac{p_+}{v} \right) \phi_t \right\} \psi_t - G_{1t} \psi_t \\
- \left[ p_t \psi_t + (p - p_+) u_t^{ed} + (p - p_+) u_t^{cd} + G_{2t} \right] \frac{\zeta}{\theta} - \frac{w_t}{4 \theta^2} \tag{80}
\end{align*}

In particular, based on the boundary condition \( \psi_t = 0 \) on the boundary \( \xi = 0 \) and the argument (58)-(59) for \( I_1(0,t), I_2(0,t) \) satisfies

\begin{align*}
- I_2(0,t) \\
\geq \left( \frac{w_t}{v} \right) \frac{w_t}{4 \theta^2} (0,t) + O(1) |(\phi_t, \psi_t, \zeta_t)| \|(\psi_{tt}, \psi_{tt}, \zeta_{tt}, \phi, \psi, \zeta, G_{1t}, G_{2t})|(0,t) \\
+ O(1) |(\phi_t, \psi_t, \zeta_t)| \|(\phi_t, \psi_t, \zeta_t, \phi, \psi, G_{1t}, G_{2t})|(0,t). \tag{81}
\end{align*}

Now, integrating (79) on \( [0,t] \times \mathbb{R}^+ \) and by applying Lemmas 5.1-5.2, we have

\begin{align*}
\|(\phi_t, \psi_t, \zeta_t)(t)\|^2 + \int_0^t \|(\phi_t, \psi_t, \zeta_t)(\tau)\|^2 d\tau \\
\leq \|(\phi_0, \psi_0, \zeta_0)\|^2 + \|(\phi_t, \psi_t, \zeta_t)(0)\|^2 + \delta \\
+ (\delta + N(t)) \int_0^t \|(\phi_t, \psi_t, \zeta_t)(\tau)\|^2 d\tau + \sum_{i=4}^7 K_i, \tag{82}
\end{align*}

where

\begin{align*}
\sum_{i=4}^7 K_i \\
= \int_0^t \int_{\mathbb{R}^+} \left( \frac{R}{v} \right) \zeta - \left( \frac{p_+}{v} \right) \phi \psi_t d\xi d\tau + \int_0^t \int_{\mathbb{R}^+} \left( \frac{\xi_t^{ed}}{v} \right) \frac{w_t}{4 \theta^2} d\xi d\tau \tag{83}
\end{align*}

The argument for the estimate on \( K_4 \) is similar to the one for (4.50) in [6], thus we omit the details, and are now going to estimate the remaining terms \( K_5, K_6 \) and
Lemma 2.1 that

and

Therefore, one has

\[
K_5 = \int_0^t \int_{\mathbb{R}^+} \left\{ \left( \frac{q_{\xi}^{cd}}{v} \right) \frac{w_{\xi} \xi}{4b \theta^4} \right\} d\xi d\tau - \int_0^t \int_{\mathbb{R}^+} \left( \frac{q_{\xi}^{cd}}{v} \right) \frac{w_{\xi} \xi}{4b \theta^4} d\xi d\tau
\]

\[
\lesssim \int_0^t \int_{\mathbb{R}^+} \left| (q_{\xi \xi \xi}^{cd}, q_{\xi \xi \xi}^{cd} v_{\xi}, q_{\xi \xi \xi}^{cd} v_{\xi}, q_{\xi \xi \xi}^{cd} v_{\xi}) \right| |(w_{\xi}, w_{\xi \xi \xi})| d\xi d\tau
\]

\[
+ \int_0^t \int_{\mathbb{R}^+} \left| (q_{\xi \xi \xi}^{cd}, q_{\xi \xi \xi}^{cd} v_{\xi}, q_{\xi \xi \xi}^{cd} v_{\xi}, q_{\xi \xi \xi}^{cd} v_{\xi}) \right| |w_{\xi}| (0, \tau) d\tau
\]

\[
\lesssim (\delta + N(t)) \int_0^t \| (w_{\xi}, w_{\xi \xi \xi}, \phi, \phi_{\xi \xi \xi}, \psi, \psi_{\xi \xi \xi}) (\tau) \|^2 d\tau
\]

\[
+ \delta \int_0^t \| (\phi_{\xi}, \phi_{\xi \xi \xi}, \phi_{\xi \xi \xi}) \|^2 (0, \tau) d\tau + \delta.
\]

Next, for \( K_6 \), note that \( G_1 = u_{t}^{cd} - s_{-}u_{\xi}^{cd} \) and \( G_2 = u_{\xi}^{cd}G_1 \), so it follows from Lemma 2.1 that

\[
|(G_{1 \xi}, G_{2 \xi})| \lesssim \frac{\delta}{(1 + \tau)} e^{-\frac{(\xi + \tau - \tau)^2}{1 + \tau}},
\]

\[
|(G_{11 \xi}, G_{22 \xi})| \lesssim \frac{\delta}{(1 + \tau)} e^{-\frac{(\xi + \tau - \tau)^2}{1 + \tau}}
\]

and

\[
|(\psi_{\xi}, \zeta_{\xi})| \lesssim O(1) |(\phi_{\xi}, \psi_{\xi}, \zeta_{\xi}, w_{\xi}, \psi_{\xi}, \zeta_{\xi}, G_{1 \xi}, G_{2 \xi})|
\]

\[
+ O(1) |\theta_{\xi}^{cd}|| (\phi_{\xi}, \psi_{\xi}, \zeta_{\xi})| + O(1) |u_{\xi \xi \xi}^{cd}|| (\phi, \zeta)|.
\]

Therefore, one has

\[
K_6 \lesssim \int_0^t \int_{\mathbb{R}^+} \frac{\delta}{(1 + \tau)^{\frac{3}{2}}} e^{-\frac{(\xi + \tau - \tau)^2}{1 + \tau}} |(\phi_{\xi}, \psi_{\xi}, \zeta_{\xi}, w_{\xi}, \phi_{\xi}, \psi_{\xi}, \zeta_{\xi}, G_{1 \xi}, G_{2 \xi})| d\xi d\tau
\]

\[
+ \int_0^t \int_{\mathbb{R}^+} \frac{\delta}{(1 + \tau)^{\frac{3}{2}}} e^{-\frac{(\xi + \tau - \tau)^2}{1 + \tau}} |\theta_{\xi}^{cd}|| (\phi_{\xi}, \psi_{\xi}, \zeta_{\xi})| d\xi d\tau
\]

\[
+ \int_0^t \int_{\mathbb{R}^+} \frac{\delta}{(1 + \tau)^{\frac{3}{2}}} e^{-\frac{(\xi + \tau - \tau)^2}{1 + \tau}} |u_{\xi \xi \xi}^{cd}|| (\phi, \zeta)| d\xi d\tau
\]

\[
\lesssim \delta \int_0^t \| (\phi_{\xi}, \psi_{\xi}, \zeta_{\xi}, w_{\xi}, \phi_{\xi}, \psi_{\xi}, \zeta_{\xi}) (\tau) \|^2 d\tau
\]

\[
+ \int_0^t \int_{\mathbb{R}^+} |u_{\xi \xi \xi}^{cd}|| (\phi, \zeta) ||^2 d\xi d\tau + \int_0^t \int_{\mathbb{R}^+} \delta^2 \frac{2(\xi + \tau - \tau)^2}{1 + \tau} e^{-\frac{2(\xi + \tau - \tau)^2}{1 + \tau}} d\xi d\tau
\]

\[
\lesssim \delta \int_0^t \| (\phi_{\xi}, \psi_{\xi}, \zeta_{\xi}, w_{\xi}, \phi_{\xi}, \psi_{\xi}, \zeta_{\xi}) (\tau) \|^2 d\tau + \delta.
\]
Finally, in order to estimate the boundary integral $K_7$, first we see the facts that
\[
\left( \frac{w_\xi}{v} \right)_t \xi = a \left( vw + 4b \theta^3 \zeta \right)_t \\
+ \left\{ 4b \theta^d \xi \zeta \left[ \theta^2 + \theta \theta^d + (\theta^d)^2 \right] - \left( \frac{q^d}{v} \right) \xi + a \phi q^d \right\} t
\]
and
\[
\left( \frac{w_\xi}{v} \right)_t \xi \frac{w_\xi}{4b \theta^4} (0, t) \\
= \frac{aw_\xi}{4b \theta^4} \left( vw + 4b \theta^3 \zeta \right)_t (0, t) \\
+ \frac{aw_\xi}{4b \theta^4} \left\{ 4b \theta^d \xi \zeta \left[ \theta^2 + \theta \theta^d + (\theta^d)^2 \right] - \left( \frac{q^d}{v} \right) \xi + a \phi q^d \right\} (0, t)
\]
\[
\lesssim \frac{1}{8} \zeta \xi (0, t) + w_\xi^2 (0, t) + w_\xi^2 (0, t) \\
+ \left\{ 4b \theta^d \xi \zeta \left[ \theta^2 + \theta \theta^d + (\theta^d)^2 \right] - \left( \frac{q^d}{v} \right) \xi + a \phi q^d \right\} (0, t).
\]

So
\[
K_7 \lesssim \frac{1}{8} \int_0^t \zeta \xi (0, \tau) d\tau + \int_0^t (w^2 + w_\xi^2) (0, \tau) d\tau \\
+ \int_0^t \left\{ \theta^d \xi \zeta \left[ \theta^2 + \theta \theta^d + (\theta^d)^2 \right] - \left( \frac{q^d}{v} \right) \xi + a \phi q^d \right\} (0, \tau) d\tau.
\]

Inserting (84), (87) and (90) into (83), we obtain
\[
\| (\phi_\xi, \psi_\xi, \zeta_\xi) (t) \|^2 + \int_0^t \left( \| (\phi_\xi, \psi_\xi, \zeta_\xi) \|^2 (0, \tau) + \| w_\xi (\tau) \|^2 \right) d\tau \\
\lesssim \| (\phi_0, \psi_0, \zeta_0) \|^2 + \| (\phi_\xi, \psi_\xi, \zeta_\xi) (0) \|^2 + \delta \\
+ (\delta + N(t)) \int_0^t \| (\phi_\xi, \psi_\xi, \zeta_\xi) (\tau) \|^2 d\tau \\
+ \int_0^t \| (\phi_\xi, \psi_\xi, \zeta_\xi) \|^2 (0, \tau) d\tau + \int_0^t w_\xi^2 (0, \tau) d\tau.
\]

Finally, the boundary integral $\int_0^t w_\xi^2 (0, \tau) d\tau$ is estimated as follows:
\[
\int_0^t w_\xi^2 (0, \tau) d\tau \lesssim \int_0^t \| w_\xi (\tau) \| \| w_\xi (\tau) \| d\tau \\
\lesssim \frac{1}{8} \int_0^t \| w_\xi (\tau) \|^2 d\tau + \int_0^t \| w_\xi (\tau) \|^2 d\tau.
\]
The combination of (91) and (92) yields to estimate (76). This completes the proof of Lemma 5.3. □

Next, let us consider the estimate of the second-order derivative \( \partial_{\xi\xi} \) on the solutions.

**Lemma 5.4.** Under the same assumptions listed in Proposition 2, if \( \epsilon \) and \( N(t) \) are suitably small, then it holds

\[
\| (\phi_{\xi\xi}, \psi_{\xi\xi}, \xi_{\xi\xi}) (t) \|^2 + \int_0^t \left( \left| (\phi_{\xi\xi}, \psi_{\xi\xi}, \xi_{\xi\xi}) \right|^2 (0, \tau) + \| w_{\xi\xi}(\tau) \|^2 \right) d\tau
\leq \left\| (\phi_0, \psi_0, \zeta_0) \right\|^2 + \left( (\phi, \psi, \zeta) (0) \right)^2 + \delta + (\delta + N(t)) \int_0^t \| (\phi_{\xi\xi}, \psi_{\xi\xi}, \zeta_{\xi\xi}) (\tau) \|^2 d\tau.
\]

**Proof.** By the direct calculation, it follows from (31) that

\[
\left( p_+ \frac{\phi_{\xi\xi}^2}{v} + \frac{\psi_{\xi\xi}^2}{v} + C_v \frac{\zeta_{\xi\xi}^2}{\theta^2} \right) I_{3\xi} + \frac{w_{\xi\xi\xi\xi}}{4v\theta^4} \left\{ w_{\xi\xi} \left( \frac{2}{v} \right) + w_{\xi} \left( \frac{1}{v} \right) \right\}
+ \left\{ a \left( v_{\xi\xi} w + 2v_{\xi} w_{\xi} \right) + 4b \left[ (\theta^4)_{\xi\xi} \xi_{\xi} + 2 (\theta^3)_{\xi\xi} \xi_{\xi} \right] \right\} \frac{w_{\xi\xi}}{4v\theta^4}
= \frac{p_+ \phi_{\xi\xi} - R}{v^2 \xi} \phi_{\xi\xi} \psi_{\xi\xi\xi\xi} + O(1) \left| (\theta_{\xi\xi}^2 \phi_{\xi\xi}) + (\theta_{\xi\xi\xi\xi}^2) \right| |(\phi, \psi, \xi)|^2
+ O(1)(\delta + N(t)) |(\phi_{\xi\xi}, \psi_{\xi\xi}, \xi_{\xi\xi}, w_{\xi\xi}, \phi_{\xi\xi}, \psi_{\xi\xi}, \zeta_{\xi\xi}, w_{\xi\xi})|^2
+ \left( \frac{\zeta_{\xi\xi}}{v} \right) \frac{w_{\xi\xi}}{4v\theta^4} + O(1) |G_{1\xi\xi} \psi_{\xi\xi} + G_{2\xi\xi} \xi_{\xi\xi}|,
\]

where

\[
I_{3\xi}(\xi, t) = s \frac{\psi_{\xi\xi}^2}{2} - \psi_{\xi\xi} \psi_{\xi\xi} - \frac{R}{v} \xi_{\xi\xi} \psi_{\xi\xi} + \frac{p_+}{v} \phi_{\xi\xi} \psi_{\xi\xi} + \frac{s - p_+}{v} \phi_{\xi\xi}^2
- \frac{p_+}{v} \phi_{\xi\xi} \phi_{\xi\xi} - \frac{s - C_v}{\theta} \frac{\zeta_{\xi\xi}}{2} - \frac{C_v}{\theta} \phi_{\xi\xi} \xi_{\xi\xi} - \frac{p_+}{v^2} \phi - \frac{R}{v^2 \xi} \psi_{\xi\xi} \phi_{\xi\xi}
- \left[ \frac{2 R}{v^2} \right] \xi_{\xi} - \left( \frac{2p_+}{v} \right) \xi_{\xi} \psi_{\xi\xi} + \frac{2 R v_{\xi\xi}^2}{v^3} \xi_{\xi} - \frac{2 p_+}{v^2} \psi_{\xi\xi} \phi_{\xi\xi} + G_{1\xi\xi} \right] \psi_{\xi\xi}
- \left[ p_+ \psi_{\xi\xi} + (p - p_+) \xi v_{\xi\xi} + (p - p_+) \psi_{\xi\xi} \xi_{\xi\xi} + G_{2\xi\xi} \right] \frac{\zeta_{\xi\xi}}{\theta} - \frac{w_{\xi\xi}}{4v\theta^4} \left( \frac{w_{\xi\xi}}{v} \right) \xi_{\xi\xi}.
\]
In particular, on the boundary \( \xi = 0 \), \( I_3(\xi, t) \) satisfies that

\[
-I_3(0, t) = \left\{ \begin{aligned}
&-s_p \frac{\psi^2}{\nu} - s_p \frac{\psi^2}{2} + R \frac{\psi^2}{\xi} \psi \xi - \frac{p_p}{\nu} \phi \xi \psi \xi - s_C \frac{\psi^2}{\theta} \\
&+ \left\{ \frac{p_p}{\nu} \phi \xi \psi \xi + \psi \xi \psi \xi + \frac{C_v}{\psi} \psi \xi \psi \xi + \left( \frac{p_p}{\nu^2} \phi - \frac{R}{\nu^2} \phi \right) \psi \xi \psi \xi \psi \xi \xi \right\} (0, t) \\
&+ \left[ \left( \frac{2R}{\xi} \right) \phi \xi - \left( \frac{2p_p}{\nu} \right) \phi \xi + \frac{2R \nu^2}{\psi^2} \xi - \frac{2p_p}{\psi^3} \xi + G_{1 \xi} \right] \xi \psi \xi \psi \xi (0, t) \\
&+ \left[ p_p \phi \psi + (p - p_p) u^{\xi \xi} + (p - p_p) u^{\xi \xi} + G_{2 \xi} \right] \xi \psi \xi \psi \xi (0, t) \\
&+ \frac{w_{\xi \xi}}{4 \theta^2} \left( \frac{w_{\xi \xi}}{v} \right) \xi \xi (0, t) \\
\geq &c_1 \left| (\phi \xi \psi, \psi \xi \psi, \xi \psi \psi \xi) \right|^2 (0, t) - c_2 \left| (\phi \xi \psi, \psi \xi \psi, \xi \psi \psi \xi) \right|^2 (0, t) \\
&- c_3 \left| (u^{\xi \xi} c_{11}^{\xi}, u^{\xi \xi} c_{11}^{\xi}) (\phi, \xi) \right|^2 (0, t) - c_4 \left| (\phi \xi \psi, \psi \xi \psi, \xi \psi \psi \xi, \phi, \psi, \xi) \right|^2 (0, t) \\
&- (G_{2 \xi}^2 + G_{2 \xi}^2)(0, t) - \frac{w_{\xi \xi}}{4 \theta^2} \left( \frac{w_{\xi \xi}}{v} \right) \xi \xi (0, t).
\end{aligned} \right.
\]

Integrating (95) on \([0, t] \times \mathbb{R}^+\), and using Lemmas 5.1-5.3 and (97), we have

\[
\| (\phi \xi \psi, \psi \xi \psi, \xi \psi \psi \xi) \|_2^2 + \int_0^t \left| (\phi \xi \psi, \psi \xi \psi, \xi \psi \psi \xi) \right|^2 (0, \tau) + \| w_{\xi \xi} \xi \psi \xi \psi \xi \|_2^2 d\tau \\
\leq \| (\phi_0, \psi_0, \xi_0) \|_2^2 + \delta + (\delta + N(t)) \int_0^t \| (\phi \xi \psi, \psi \xi \psi, \xi \psi \psi \xi) \|_2^2 d\tau + \sum_{i=8}^{11} K_i,
\]

where

\[
\sum_{i=8}^{11} K_i = \int_0^t \int_{\mathbb{R}^+} \left( \frac{p_p}{\nu^2} \phi - \frac{R}{\nu^2} \phi \right) \phi \xi \psi \xi \psi \xi d\xi d\tau + \int_0^t \int_{\mathbb{R}^+} \left( \frac{q_{\xi \xi}^{\xi \xi}}{\psi} \right) \psi \xi \psi \xi \psi \xi d\xi d\tau \\
+ \int_0^t \int_{\mathbb{R}^+} (G_{1 \xi \xi} \xi \xi \xi) \left| (\psi \xi \psi, \xi \psi \psi \xi) \right| d\xi d\tau + \int_0^t \frac{w_{\xi \xi}}{4 \theta^2} \left( \frac{w_{\xi \xi}}{v} \right) \xi \xi (0, \tau) d\tau.
\]

The estimate on \( K_8 \) is similar to (4.68) in [6]. Next, for \( K_9 \)

\[
K_9 = \int_0^t \int_{\mathbb{R}^+} \left\{ \left( \frac{q_{\xi \xi}^{\xi \xi}}{\psi} \right) \frac{w_{\xi \xi}}{4 \theta^2} \psi \xi \psi \xi \psi \xi d\xi d\tau - \int_0^t \int_{\mathbb{R}^+} \left( \frac{q_{\xi \xi}^{\xi \xi}}{\psi} \right) \psi \xi \psi \xi \psi \xi d\xi d\tau \\
\leq \int_0^t \int_{\mathbb{R}^+} \left| (\xi_{\xi \xi}, \xi_{\xi \xi} \psi \xi \psi \xi, \xi_{\xi \xi} \psi \xi \psi \xi) \right| \left| (w_{\xi \xi}, w_{\xi \xi}) \right| d\xi d\tau \\
+ \int_0^t \left| (q_{\xi \xi}^{\xi \xi}, q_{\xi \xi}^{\xi \xi} \psi \xi \psi \xi, q_{\xi \xi}^{\xi \xi} \psi \xi \psi \xi) \right| \left| w_{\xi \xi} \xi \psi \xi \psi \xi \right| (0, \tau) d\tau
\]
Thus, it follows from Lemma 5.3 that
\[ \leq (\delta + N(t)) \int_0^t \| (w_{\xi \xi}, \phi_\xi) (\tau) \|_1 \, d\tau + \delta \int_0^t \| (\phi_\xi, \psi_{\xi \xi}, w_{\xi \xi}) \|^2 (0, \tau) \, d\tau + \delta. \] (100)

For \( K_{10} \), we have
\[ K_{10} \leq \int_0^t \int_{\mathbb{R}^+} \delta (1 + \tau)^{-3} e^{-\frac{[\xi + e^\tau]^{\frac{2}{3}}}{\tau^{\frac{2}{3}}} } |(\psi_{\xi \xi}, \zeta_{\xi \xi})| \, d\xi d\tau \]
\[ \leq \delta \| \psi_{\xi \xi} \|^2 d\xi d\tau + \delta \int_0^t \int_{\mathbb{R}^+} (1 + \tau)^{-3} e^{-\frac{[\xi + e^\tau]^{\frac{2}{3}}}{\tau^{\frac{2}{3}}} } d\xi d\tau \] (101)
\[ \leq \delta \| \psi_{\xi \xi} \|^2 d\xi d\tau + \delta. \]

Finally, in order to estimate the boundary integral \( K_{11} \), we note that
\[ \frac{w_{\xi \xi}}{4b_0^4} \left( \frac{w_\xi}{v} \right)_{\xi} \xi (0, t) \]
\[ = \frac{w_{\xi \xi}}{4b_0^4} (av_\xi w + av \xi + 4b_0^3 \zeta_{\xi \xi} + 4b_0^2 \theta \xi \zeta) (0, t) \]
\[ + \frac{w_{\xi \xi}}{4b_0^4} \left\{ 4b_0^3 \xi \left[ \theta^2 + \theta \theta^{cd} + (\theta^{cd})^2 \right] - \left( \frac{q_{\xi}^{cd}}{v} \right)_\xi + a \phi q^{cd} \right\} (0, t) \] (102)
\[ \leq \frac{1}{8} (\phi_{\xi \xi}^2 + \zeta_{\xi \xi}^2) (0, t) + \| (w_{\xi \xi}, w_\xi, \phi_\xi, \zeta_\xi) \|^2 (0, t) + \| (q_{\xi \xi}^c, q_{\xi \xi}^{cd}, q_{\xi \xi}^{cd}) \|^2 (0, t). \]

Thus, it follows from Lemma 5.3 that
\[ K_{11} \leq \frac{1}{8} (\phi_{\xi \xi}^2 + \zeta_{\xi \xi}^2) (0, t) + \| (\phi_\xi, \psi_{\xi \xi}, \psi_\xi) \|^2_1 + \delta \]
\[ + (\delta + N(t)) \int_0^t \| (\phi_\xi, \psi_\xi, \xi) (\tau) \|_1^2 \, d\tau. \] (103)

Plugging (100), (101) and (103) into (98), we obtain (93). This completes the proof of Lemma 5.4. \( \Box \)

Finally, let us consider the term \( \int_0^t \| (\phi_\xi, \psi_\xi) (\tau) \|_1^2 \, d\tau \), which is estimated by the following lemma.

**Lemma 5.5.** Under the same assumptions listed in Proposition 2, if \( \epsilon \) and \( N(t) \) are suitably small, then it holds
\[ \int_0^t \| (\phi_\xi, \psi_\xi) (\tau) \|_1^2 \, d\tau \leq \| (\phi_\xi, \psi_\xi_0, \zeta_0) \|^2_2 + \| (\phi_\xi, \psi_\xi_0, \zeta_0) (0) \|^2_1 + \delta. \] (104)

**Proof.** Based on the straightforward calculation and equation (31), it follows from (31) \( 2 \times (\xi \phi_\xi) \) + (31) \( 3 \times \psi_\xi \) that
\[ \begin{align*}
&\left( C_p \psi_{\xi \xi} - \frac{p}{v} \phi_\xi \psi \right)_t - \left( C_p \psi_{\xi \xi} - \frac{p}{v} \phi_\xi \psi \right)_\xi + \frac{p}{2} \psi_\xi^2 + \frac{pp_\psi}{2v} \phi_\xi^2 \\
&= \frac{p_\xi}{v} \phi_\xi \xi - \frac{p}{v} \phi_\xi^2 - \frac{p}{2} \phi_\xi \left( \frac{p}{v} \right)_\xi \phi - \left( \frac{p}{v} \right)_t \phi_\xi \psi
\end{align*} \]
Thus, collecting all the estimates on $K_{16}$, we obtain

$$
\int_0^t \int_\mathbb{R}^+ (\phi^2 + \psi^2) \, d\xi \, d\tau \lesssim \| (\phi, \psi, \psi, \zeta) (t) \|^2 + \| (\phi_0, \psi_0, \psi_0, \zeta_0) (t) \|^2 + \sum_{i=12}^{16} K_i,
$$

(106)

where

\[
\sum_{i=12}^{16} K_i = \left( \frac{1}{4} + \delta + N(t) \right) \int_0^t \int_\mathbb{R}^+ (\phi^2 + \psi^2) \, d\xi \, d\tau
\]

\[
+ \int_0^t \| (\zeta, w) (\tau) \|^2 \, d\tau + \int_0^t \int_\mathbb{R}^+ |u^d|^2 \| (\phi, \xi) \|^2 d\xi \, dt
\]

\[
+ \int_0^t |\psi_t + \phi_t| (0, \tau) \, d\tau + \int_0^t \int_\mathbb{R}^+ |G_1, G_2| \| (\phi, \psi, \zeta, \xi) \| \, d\xi \, d\tau.
\]

(107)

$K_{12}$ can be absorbed by the left-hand side of the estimate (106) due to the smallness of $\delta$ and $N(t)$. $K_{16}$ can be estimated as the estimate for $K_6$ in (87). $K_{13}$ can be controlled by applying Lemma 5.1. Then we will estimate $K_{14}$-$K_{15}$ one by one as follows.

First, for $K_{14}$, by the definition in (19), we see that

$$
u^c = \gamma - 1 \frac{4b}{R^\gamma} \left( \frac{\theta^c}{\theta} \right)_\xi \lesssim |\theta^c|^2 + |\theta^c|_{\xi}^2.
$$

(108)

So similarly as the one in (47), it holds

$$
K_{14} \lesssim \int_0^t \int_\mathbb{R}^+ \left( |\theta^c|^2 + |\theta^c_{\xi} | \right) \| (\phi, \psi, \xi) \|^2 d\xi \, d\tau
\]

\[
\lesssim \delta \int_0^t \frac{1}{1 + \tau} \left\{ \int_\mathbb{R}^+ |(\phi, \psi, \xi)|^2 e^{-\frac{1}{\tau} (\xi + \tau)^2} d\xi \right\} \, d\tau
\]

\[
\lesssim \delta + \delta \int_0^t \| (\phi, \psi, \xi, \zeta) (\tau) \|^2 \, d\tau.
\]

(109)

Finally, it is easy to see that the boundary integral $K_{15}$ is bounded by

$$
\delta^2 \int_0^t (1 + \tau)^{-1} e^{-\frac{\tau^2}{\tau + 1}} \, d\tau \lesssim \delta^2.
$$

(110)

Thus, collecting all the estimates on $K_{12}$-$K_{16}$, we obtain

$$
\int_0^t \| (\phi, \psi, \xi) (\tau) \|^2 \, d\tau \lesssim \| (\phi_0, \psi_0, \zeta_0) \|_2^2 + \| (\phi_0, \psi_t, \zeta_0) (0) \|_1^2 + \delta
\]

\[
+ (\delta + N(t)) \int_0^t \| (\phi, \psi, \xi) (\tau) \|_1^2 \, d\tau.
\]

(111)
Similarly, multiplying \((31)_{2\xi}\) by \(-\frac{2}{3}\phi_{\xi\xi}\) and multiplying \((31)_{3\xi}\) by \(\psi_{\xi\xi}\), we can also get (we omit the details for the shortness since the argument is similar but long)

\[
\int_0^t \| (\phi_{\xi\xi}, \psi_{\xi\xi}) (\tau) \|^2 d\tau \lesssim \| (\phi_0, \psi_0, \zeta_0) \|^2 + \| (\phi_\tau, \psi\tau, \zeta) (0) \|^2 + \delta \]

\[
+ (\delta + N(t)) \int_0^t \| (\phi\xi, \psi\xi) (\tau) \|^2 d\tau.
\]

(112)

Then (104) follows from (111) and (112). This completes the proof of Lemma 5.5.

Now we are ready to finish the proof of Proposition 2.

Proof of Proposition 2. By applying Lemma 4.1 with Lemmas 5.1-5.5 together, we have

\[
\| (\phi, \psi, \zeta)(t) \|^2 \lesssim \| (\phi_0, \psi_0, \zeta_0) \|^2 + \| (\phi_\tau, \psi_\tau, \zeta_\tau)(0) \|^2 + \delta.
\]

(113)

Therefore, in order to complete the proof of Proposition 2, it is sufficient to prove

\[
\| w(t) \|^2 \lesssim \| (\phi_0, \psi_0, \zeta_0) \|^2 + \| (\phi_\tau, \psi_\tau, \zeta_\tau)(0) \|^2 + \delta.
\]

(114)

In fact, by integrating the resulted equations obtained from \((31)_4 \times w\) and \((31)_{4\xi} \times (-w_{\xi\xi\xi})\) with respect to \(\xi\) on \(\mathbb{R}^+\) respectively, one has

\[
\int_{\mathbb{R}^+} \left( \frac{w_{\xi}^2}{v} + avw^2 \right) (\xi, t) d\xi
\]

\[
\lesssim \int_{\mathbb{R}^+} \left\{ 4b\theta^2\zeta_\xi + 4b\theta\xi\zeta \left( \theta^2 + \theta\theta_{\xi\xi} + \theta_{\xi\xi} \right) - \left( \frac{q_{\xi\xi} cd}{v} \right) + aq_{\xi\xi} \phi \right\}^2 d\xi
\]

\[
+ |w_{\xi\xi\xi}| (0, t)
\]

\[
\lesssim \int_{\mathbb{R}^+} \left\{ \zeta_\xi^2 + \theta_{\xi\xi}^2 (\zeta^2 + \phi^2) + \left( \frac{q_{\xi\xi} cd}{v} \right)^2 \right\} d\xi + \delta (1 + t) - \frac{1}{2} e^{-\frac{(s-t)^2}{4}} \| w_\xi(t) \| \infty
\]

\[
\lesssim \| (\phi_0, \psi_0, \zeta_0) \|^2 + \| (\phi_\tau, \psi_\tau, \zeta_\tau)(0) \|^2 + \delta + \delta \| w_\xi(t) \|^2
\]

and

\[
\int_{\mathbb{R}^+} \left( avw_{\xi\xi\xi} + \frac{w_{\xi\xi\xi}}{v} \right) (\xi, t) d\xi
\]
Here we used the estimate on the following boundary term in (116)

\[
\begin{aligned}
&\lesssim \int_{\mathbb{R}^+} \left\{ 4\delta^3 \zeta_{\xi} + 4\delta^2 \zeta \left( \theta^2 + \theta \theta_{\xi}^{cd} + \theta_{\xi}^{cd2} \right) - \left( \frac{q_{\xi}^{cd}}{v} \right)_t + aq_{\xi}^{cd} \phi \right\}^2 d\xi \\
&\quad + \int_{\mathbb{R}^+} (\phi_{\xi}^2 + \phi_{\xi\xi}^2) d\xi + |a(vw)\xi w_{\xi\xi}| (0, t) \\
&\lesssim \|(\phi_0, \psi_0, \zeta_0)\|_2^2 + \|(\phi_t, \psi_t, \zeta_t) (0)\|_1^2 + \delta.
\end{aligned}
\]  

(116)

Here we used the estimate on the following boundary term in (116)

\[
|a(vw)\xi w_{\xi\xi}| (0, t)
\]

\[
= |u_{\xi\xi} w_{\xi}| (0, t) + |w_\phi w_{\xi\xi}| (0, t) + |w q_{\xi}^{cd} w_{\xi\xi}| (0, t)
\]

\[
\lesssim \|w_\xi(t)\|_\infty \|w_{\xi\xi}(t)\|_\infty + \delta \|\phi_\xi(t)\|_\infty \|w_{\xi\xi}(t)\|_\infty
\]

\[
+ \delta^2 (1 + t)^{-1} e^{-\frac{2(\xi - t)}{\xi + t}} \|w_{\xi\xi}(t)\|_\infty
\]

\[
\lesssim \|w_\xi(t)\|^{\frac{1}{2}} \|w_{\xi\xi}(t)\|^{\frac{1}{2}} + \delta \|\phi_\xi(t)\|^{\frac{1}{2}} \|w_{\xi\xi}(t)\|^{\frac{1}{2}} + \frac{1}{2} \|w_{\xi\xi\xi}(t)\|^{\frac{1}{2}} + \delta^2
\]

\[
\lesssim \frac{1}{8} \|w_{\xi\xi}(t)\|^{2} + \|w_\xi(t)\| \|w_{\xi\xi\xi}(t)\| + \delta \|\phi_\xi(t)\| \|\phi_\xi(t)\|
\]

\[
+ \delta \|w_{\xi\xi}(t)\| \|w_{\xi\xi\xi}(t)\| + \delta^2 \|w_{\xi\xi}(t)\|_1^2 + \delta^2
\]

\[
\lesssim \left( \frac{1}{8} + \delta \right) \|w_{\xi\xi\xi}(t)\|^{2} + \|w_\xi(t)\|^{2} + \delta \|\phi_\xi(t)\|^{2} + \delta^2.
\]

Combining the estimates (113) and (114) together, we obtain (33). This completes the proof of Proposition 2. \qed

Acknowledgments. The authors will express heartfelt appreciation to the anonymous referees for valuable suggestions and comments. The research of Lili Fan was supported by the Natural Science Foundation of China #11871388 and in part by the Natural Science Foundation of China #11701439. The research of Lizhi Ruan was supported in part by the Natural Science Foundation of China #1171169, #11331005, #1187236, Program for Changjiang Scholars and Innovative Research Team in University #IRT17R46, and the Special Fund for Basic Scientific Research of Central Colleges #CCNU19TS031. The research of Wei Xiang was supported in part by the Research Grants Council of the HKSAR, China (Project No. CityU 21305215, Project No. CityU 11332916, Project No. CityU 11304817 and Project No. CityU 11303518).

REFERENCES

[1] A. M. Blokhin and Yu. L. Trakhinin, Shock-wave stability for one model of radiation hydrodynamics, J. Appl. Mech. Tech. Phys., 37 (1996), 775–784.

[2] C. Buet and B. Despres, Asymptotic analysis of fluid models for the coupling of radiation and hydrodynamics, J. Quant. Spectrosc. Radiat. Transfer, 85 (2004), 385–418.
[3] J.-F. Coulombel, T. Goudon, P. Lafitte and C. Lin, Analysis of large amplitude shock profiles for non-equilibrium radiative hydrodynamics: Formation of Zeldovich spikes, *Shock Waves*, 22 (2012), 181–197.

[4] B. Ducomet, E. Feireisl and S. Nečasova, On a model in radiation hydrodynamics, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28 (2011), 797–812.

[5] L. Fan, L. Ruan and W. Xiang, Asymptotic stability of a composite wave of two viscous shock waves for the one-dimensional radiative Euler equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36 (2019), 1–25.

[6] L. Fan, L. Ruan and W. Xiang, Asymptotic stability of rarefaction wave for the inflow problem governed by the one-dimensional radiative Euler equations, *SIAM J. Math. Anal.*, 51 (2019), 595–625.

[7] W. Gao, L. Ruan and C. Zhu, Decay rates to the planar rarefaction waves for a model of the radiating gas in $n$-dimensions, *J. Differential Equations*, 244 (2008), 2614–2640.

[8] P. Godillon-Lafitte and T. Goudon, A coupled model for radiative transfer: Doppler effects, equilibrium and non equilibrium diffusion asymptotics, *Multiscale Model. Simul.*, 4 (2005), 1245–1279.

[9] H. Hong, Global stability of viscous contact wave for 1-D compressible Navier-Stokes equations, *J. Differential Equations*, 252 (2012), 3482–3505.

[10] F. Huang and X. Li, Convergence to the rarefaction wave for a model of radiating gas in one-dimension, *Acta Math. Appl. Sin. Engl. Ser.*, 32 (2016), 239–256.

[11] F. Huang, J. Li and A. Matsumura, Asymptotic stability of combination of viscous contact wave with rarefaction waves for one-dimensional compressible Navier-Stokes system, *Arch. Ration. Mech. Anal.*, 197 (2010), 89–116.

[12] F. Huang, A. Matsumura and Z. Xin, Stability of contact discontinuities for the 1-D compressible Navier-Stokes equations, *Arch. Ration. Mech. Anal.*, 179 (2007), 55–77.

[13] F. Huang, Z. Xin and T. Yang, Contact discontinuity with general perturbations for gas motions, *Adv. Math.*, 219 (2008), 1246–1297.

[14] S. Jiang, F. Li and F. Xie, Nonrelativistic limit of the compressible Navier-Stokes-Fourier-P1 approximation model arising in radiation hydrodynamics, *SIAM J. Math. Anal.*, 47 (2015), 3726–3746.

[15] S. Kawashima, Y. Nikkuni and S. Nishibata, The initial value problem for hyperbolic-elliptic coupled systems and applications to radiation hydrodynamics, *Analysis of Systems of Conservation Laws (Aachen, 1997)*, Chapman Hall/CRC Monogr. Surv. Pure. Appl. Math., Chapman Hall/CRC, Boca Raton, FL, 99 (1999), 87–127.

[16] S. Kawashima and S. Nishibata, Shock waves for a model system of a radiating gas, *SIAM J. Math. Anal.*, 30 (1999), 95–117.

[17] S. Kawashima and Y. Tanaka, Stability of rarefaction waves for a model system of a radiating gas, *Kyushu J. Math.*, 58 (2004), 211–250.

[18] C. Lattanzio, C. Mascia, T. Nguyen, R. Plaza and K. Zumbrun, Stability of scalar radiative shock profiles, *SIAM J. Math. Anal.*, 41 (2009/10), 2165–2206.

[19] C. Lattanzio, C. Mascia and D. Serre, Shock waves for radiative hyperbolic-elliptic systems, *Indiana Univ. Math. J.*, 56 (2007), 2601–2640.

[20] C. Lin, Asymptotic stability of rarefaction waves in radiative hydrodynamics, *Commun. Math. Sci.*, 9 (2011), 207–223.

[21] C. Lin, J.-F. Coulombel and T. Goudon, Shock profiles for non-equilibrium radiating gas, *Phys. D*, 218 (2006), 83–94.

[22] C. Lin, J.-F. Coulombel and T. Goudon, Asymptotic stability of shock profiles in radiative hydrodynamics, *C. R. Math. Acad. Sci. Paris*, 345 (2007), 625–628.

[23] T.-P. Liu, Linear and nonlinear large-time behavior of solutions of general systems of hyperbolic conservation laws, *Comm. Pure Appl. Math.*, 30 (1977), 767–796.

[24] R. B. Lowrie, J. E. Morel and J. A. Hittinger, The coupling of radiation and hydrodynamics, *Astrophys. J.*, 521 (1999), 432–450.

[25] C. Mascia, Small, medium and large shock waves for radiative Euler equations, *Phys. D*, 245 (2013), 46–56.

[26] T. Nguyen, R. G. Plaza and K. Zumbrun, Stability of radiative shock profiles for hyperbolic-elliptic coupled systems, *Phys. D*, 239 (2010), 428–453.

[27] M. Nishikawa and S. Nishibata, Convergence rates toward the travelling waves for a model system of the radiating gas, *Math. Methods Appl. Sci.*, 30 (2007), 649–663.
[28] M. Ohnawa, Convergence rates towards the traveling waves for a model system of radiating gas with discontinuities, *Kinet. Relat. Models*, 5 (2012), 857–872.
[29] M. Ohnawa, $L^\infty$-stability of continuous shock waves in a radiating gas model, *SIAM J. Math. Anal.*, 46 (2014), 2136–2159.
[30] X. Qin and Y. Wang, Stability of wave patterns to the inflow problem of full compressible Navier-Stokes equations, *SIAM J. Math. Anal.*, 41 (2009), 2057–2087.
[31] X. Qin and Y. Wang, Large-time behavior of solutions to the inflow problem of full compressible Navier-Stokes equations, *SIAM J. Math. Anal.*, 43 (2011), 341–366.
[32] C. Rohde, W. Wang and F. Xie, Hyperbolic-hyperbolic relaxation limit for a 1D compressible radiation hydrodynamics model: Superposition of rarefaction and contact waves, *Commun. Pure Appl. Anal.*, 12 (2013), 2145–2171.
[33] L. Ruan and J. Zhang, Asymptotic stability of rarefaction wave for hyperbolic-elliptic coupled system in radiating gas, *Acta Math. Sci. Ser. B Engl. Ed.*, 27 (2007), 347–360.
[34] L. Ruan and C. Zhu, Asymptotic decay toward rarefaction wave for a hyperbolic-elliptic coupled system on half space, *J. Partial Differential Equations*, 21 (2008), 173–192.
[35] L. Ruan and C. Zhu, Asymptotic behavior of solutions to a hyperbolic-elliptic coupled system in multi-dimensional radiating gas, *J. Differential Equations*, 249 (2010), 2076–2110.
[36] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, 2nd edition, Springer-Verlag, New York, 1994.
[37] W. G. Vincenti and C. H. Kruger Jr, *Introduction to Physical Gas Dynamics*, Wiley, New York, 1965.
[38] J. Wang and F. Xie, Singular limit to strong contact discontinuity for a 1D compressible radiation hydrodynamics model, *SIAM J. Math. Anal.*, 43 (2011), 1189–1204.
[39] J. Wang and F. Xie, Asymptotic stability of viscous contact wave for the 1D radiation hydrodynamics system, *J. Differential Equations*, 251 (2011), 1030–1055.
[40] F. Xie, Nonlinear stability of combination of viscous contact wave with rarefaction waves for a 1D radiation hydrodynamics model, *Discrete Contin. Dyn. Syst. Ser. B*, 17 (2012), 1075–1100.

Received March 2020; revised August 2020.

E-mail address: f118104live.cn
E-mail address: rlz@mail.ccnu.edu.cn
E-mail address: weixiang@cityu.edu.hk