Understanding the Capability of PD Control for Uncertain Stochastic Systems

Cheng Zhao ©, Member, IEEE, and Yanbin Zhang ©

Abstract—In this article, we focus on the global stabilizability problem for a class of uncertain stochastic control systems, where both the drift term and the diffusion term are nonlinear functions of the state variables and the control variables. We will show that the widely applied proportional–derivative (PD) control in engineering practice has the ability to globally stabilize such systems in the mean square sense, provided that the upper bounds of partial derivatives of the nonlinear functions satisfy a certain algebraic inequality. It will also be proved that the stabilizing PD parameters can only be selected from a two-dimensional bounded convex set, which differs significantly from the existing results for PD controlled stochastic systems where the diffusion term does not depend on the control variables. Moreover, a particular polynomial on these bounds is introduced, which can be used to determine under what conditions the system is not stabilizable by the PD control, and thus demonstrating the fundamental limitations of PD control.

Index Terms—Global stabilizability, nonlinear dynamics, proportional–derivative (PD) control, stochastic systems, uncertain structure.

I. INTRODUCTION

Feedback is a basic concept in automatic control, which has had a revolutionary influence in practically all areas. Its primary objective is to reduce the effects of the plant uncertainty on the desired control performance (e.g., stability, optimality of tracking, etc.). Plenty of control methods have been developed for dealing with uncertainties over the past 60 years, such as adaptive control [20], robust control, active disturbance rejection control [5], [9], and sliding-mode control. However, the classical proportional–integral–derivative (PID) control, perhaps the most basic form of feedback, has been at the heart of control engineering practice for several decades [3]. In fact, the PID control is used in more than 90% of industrial processes [2]. One may naturally believe that such basic controller has been deeply understood in both theory and practice. However, as mentioned in [14], many practical PID loops are poorly tuned, and there is strong evidence that its rationale remains to be unclear.

Recently, the PID control has attracted more and more attention from the research community. For example, the stabilization problems of PD (or PID) controlled linear systems with time-delay are investigated, see, e.g., [11], [12], [13], [17]. There are also abundant works on PD controlled mechanical systems [4], [8], [18], among which [18] is probably the most notable, where a PD controller was constructed to globally stabilize fully actuated robot manipulators. For more general class of nonlinear uncertain systems without special structures, some rigorous mathematical investigations have been made on the theory and design of PID in recent years (see, e.g., [7], [21], [22], [23]). For instance, it has been shown that for a class of second-order single-input–single-output affine nonlinear system, one can select the three PID parameters to globally stabilize the system and at the same time to make the system output converge to any desired setpoint, provided that the partial derivatives of the system nonlinear functions are bounded [22]. Extensions to multi-input–multi-output (MIMO) nonlinear systems without stochastic disturbances are discussed in [23]. It is worth mentioning that stochastic systems are being widely used as realistic models of physical phenomena, since random disturbances and measurement noises are inevitable in practice. Of course, the stability property is a basic requirement for stochastic control systems, which has been studied extensively in the literature [1], [15], [19]. In view of this, the performance and design of PID controlled nonaffine stochastic systems are discussed in [21], where the diffusion term does not depend on the control input.

As a special case of PID, the PD controller has also attracted many scientists and scholars, see [8], [12], [18], [21], [22]. In order to understand the mechanism of the linear PD control, it is of vital importance to take nonlinearity, uncertainty, and randomness into consideration. Moreover, efforts must be taken to investigate the limitations of PD control in a general framework. However, to the best of authors’ knowledge, these issues have not been fully explored. In this article, we are devoted to this fundamental problem by considering a basic class of MIMO stochastic nonlinear uncertain systems, where both the drift term and the diffusion term are functions of the state variables and the control variables. The main contributions are summarized as follows.

1) We have shown that the PD control has the ability to globally stabilize such systems in mean square, if the upper bounds of the partial derivatives of the nonlinear functions satisfy a certain algebraic inequality. Moreover, a particular polynomial is introduced, which can be used to determine under what conditions the system is not stabilizable by PD control, and thus demonstrating the fundamental limitations of PD control.

2) Open and bounded parameter sets for the controller gains are also constructed, which are based on some knowledge of both the drift and diffusion functions. Besides, it will be shown that the PD parameters cannot be chosen arbitrarily large, which is also a significant difference from the existing literature on PD controlled nonlinear uncertain systems, see, e.g., [10], [22], [7], [21].

The rest of this article is organized as follows. In Section II, we will introduce the mathematical formulation. The main results are presented in Section III. Section IV contains the proofs of the main theorems. Finally, Section V concludes this article.
II. MATHEMATICAL FORMULATION

A. Notations and Definitions

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space, $\mathbb{R}^{m \times n}$ be the space of $m \times n$ real matrices, $I_n$ be the $n \times n$ identity matrix. Denote $\|x\|$ as the Euclidean norm of a vector $x$, and $x^T$ as the transpose of a vector or matrix $x$. The norm of a matrix $A \in \mathbb{R}^{m \times n}$ is defined by $\|A\| = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{|Ax|}{\|x\|}$. For a square matrix $A$, denote $A^{\text{sym}} := (A + A^T)/2$ and $(A)$ as the trace of $A$. For a symmetric matrix $S \in \mathbb{R}^{n \times n}$, let $\lambda_{\text{max}}(S)$ and $\lambda_{\text{min}}(S)$ be the smallest and the largest eigenvalues of $S$, respectively. For two symmetric matrices $S_1$ and $S_2$ in $\mathbb{R}^{n \times n}$, the notation $S_1 \geq S_2$ implies that $S_1 - S_2$ is a positive definite matrix; $S_1 \geq S_2$ implies that $S_1 - S_2$ is a positive semidefinite matrix.

Consider a basic class of second-order nonlinear uncertain stochastic control system

$$
\begin{align*}
\dot{x}_1 &= x_2 dt \\
\dot{x}_2 &= f(x_1, x_2, u)dt + g(x_1, x_2, u)dB_t
\end{align*}
$$

(1)

where $x_1, x_2 \in \mathbb{R}^n$ are the system state vector, $u \in \mathbb{R}^n$ is the control input, $B_t \in \mathbb{R}^n$ is an $n$-dimensional standard Brownian motion, and the nonlinear functions $f$ and $g$ belong to $C^1(\mathbb{R}^{m \times n} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n)$, which may contain unknown dynamics.

In this article, we aim to study the capability of the classical PD control (also abbreviated as “the PD control”):

$$
\begin{align*}
\dot{e}_1 &= k_p e(t) + k_d \dot{e}(t), \\
\dot{e}_2 &= g(y^*, 0, 0) = 0
\end{align*}
$$

(2)

where $y^* \in \mathbb{R}^n$ is the setpoint, $e(t)$ is the regulation error, $k_p$ and $k_d$ are the PD parameters. The objective is to design suitable PD parameters to globally stabilize and regulate system (1) in mean square, i.e.,

$$
\lim_{t \to \infty} \mathbb{E} \left[ \left( \|e(t)\|^2 + \|\dot{e}(t)\|^2 \right) \right] = 0 \quad \forall (x_1(0), x_2(0)) \in \mathbb{R}^{2n}
$$

(3)

where $\mathbb{E}$ denotes the expectation of a random variable.

We first introduce a basic assumption that will be used throughout this article.

Assumption 1: The setpoint $y^* \in \mathbb{R}^n$ is an equilibrium of the uncontrollable stochastic system (1).

To be precise

$$
\begin{align*}
f(y^*, 0, 0) &= 0, \\
g(y^*, 0, 0) &= 0
\end{align*}
$$

(4)

It is worth noting that Assumption 1 is necessary for the existence of $(k_p, k_d)$ to achieve the control objective (3). Specifically, we have the following proposition.

Proposition 1: Consider the PD controlled system (1), (2), where the functions $f$ and $g$ are Lipschitz continuous. Suppose that there exist some PD parameters $k_p$ and $k_d$ and some $(x_1(0), x_2(0)) \in \mathbb{R}^{2n}$, such that the solution of the closed-loop system satisfies $\lim_{t \to \infty} \mathbb{E}[\|e(t)\|^2 + \|\dot{e}(t)\|^2] = 0$, then $f(y^*, 0, 0) = 0$ and $g(y^*, 0, 0) = 0$.

The proof of Proposition 1 is given in Appendix B. Note that both $f(\cdot)$ and $g(\cdot)$ are uncertain functions, we need to find a suitable measure to quantitatively describe the size of uncertainty. The upper bounds of the partial derivatives of the uncertain functions are a natural choice for such measurement, see, e.g., [20], [22]. In addition, in order to enable the input signal to affect the state of the controlled system, the control gain matrix $\frac{\partial f}{\partial u}$ should not vanish. These natural intuitions inspired us to introduce the following assumption.

Assumption 2: The drift function $f(\cdot)$ is in $\mathcal{F}_{L_1, L_2}$, where

$$
\mathcal{F}_{L_1, L_2} := \left\{ f : \left\| \frac{\partial f}{\partial x_1} \right\| \leq L_1, \left\| \frac{\partial f}{\partial u} \right\| \geq I_n \quad \forall x_1, x_2, u \right\}
$$

(5)

where $L_1$ and $L_2$ are positive constants. Besides, the diffusion term $g(\cdot)$ belongs to

$$
\mathcal{G}^{M}_{N_1, N_2} := \left\{ g : \left\| \frac{\partial g}{\partial x_1} \right\| \leq N_1, \left\| \frac{\partial g}{\partial u} \right\| \leq M \quad \forall x_1, x_2, u \right\}
$$

(6)

where $N_1$ and $N_2$ are positive constants and the constant $M \geq 0$.

Next, we introduce the following definition.

Definition 1: We say that the uncertain stochastic system (1) is (globally) stabilizable by the PD control (2), if there exist some PD parameters $k_p$ and $k_d$, such that the control performance (3) is satisfied for all functions $f$ and $g$ that satisfy Assumptions 1 and 2. Otherwise, we say that system (1) is not stabilizable by the PD control (2).

Remark 1: It is known that, if system (1) has the following special form (the diffusion term does not depend on $u$):

$$
\begin{align*}
\dot{x}_1 &= x_2 dt \\
\dot{x}_2 &= f(x_1, x_2, u)dt + g(x_1, x_2)dB_t
\end{align*}
$$

(7)

then for any positive quadruple $(L_1, L_2, N_1, N_2)$, the uncertain stochastic system (7) is globally stabilizable by the PD control (2), see [21, Th. 3.9]. Moreover, the solution of the PD parameters $k_p$ and $k_d$ has wide flexibility, since they can be arbitrarily chosen from an open and unbounded set in $\mathbb{R}^2$. Thus, one might naturally conjecture that this result can be extended to the more general system (1) considered in this article, where $g(\cdot)$ is a function of both the state variables and the control variables. To be precisely, for any given positive constants $L_1, L_2, N_1, N_2$, and $M$, an open and unbounded PD parameter set could be constructed, from which the PD control (2) has the ability to globally stabilize the system (1), for all functions $f$ and $g$ satisfying Assumptions 1 and 2. Surprisingly, the answer to the abovementioned conjecture is no! In fact, these five constants have to meet suitable constraints before such stabilizing PD parameters can be found.

III. MAIN RESULTS

A. Uncertain Nonlinear Stochastic System

For given positive constants $L_1, L_2, N_1$, and $N_2$, we first define a family of parameter set $\Omega_0(M)$, $M \geq 0$ as follows:

$$
\Omega_0 := \left\{ (k_p, k_d) \in \mathbb{R}_+^2 \left| k_d^2 > \bar{k} + k_d T_2^2 \right\} \right\}
$$

(8)

where $\bar{k} := (L_1 + L_2)(k_p + k_d)$, and $T_1, T_2$ are defined by

$$
T_1 := N_1 + Mk_p, \quad T_2 := N_2 + Mk_d.
$$

(9)

Next, we list some geometric properties of the set $\Omega_0$.

1) If $M = 0$, $\Omega_0$ is an open and unbounded set in $\mathbb{R}^2$.
2) $\Omega_0(M_2) \subset \Omega_0(M_1)$, if $0 \leq M_2 < M_1$.
3) $\Omega_0 = \emptyset$ if $M \geq M_0$, where $M_0$ is the unique positive solution of

$$
16L_1 s^4 + 16N_1 s^4 + 4L_2 s^4 + 4N_2 s = 1.
$$
Let $M_1'$ be the supremum of the set consisting of $M$ that makes $\Omega_0$ nonempty. More precisely
\[
M_1' := \sup \{ M > 0 : \Omega_0(M) \neq \emptyset \}.
\]

Theorem 1: Consider the nonlinear stochastic system (1) and (2), where Assumptions 1 and 2 are satisfied.
(i) If $0 \leq M < M_1'$, system (1) is stabilizable by the PD control (2).
(ii) If $M \geq M_2'$, where $M_2'$ is the unique positive root of the following quartic polynomial:
\[
4L_1 s^4 + 4N_1 s^3 + 2L_2 s^2 + 2N_2 s - 1 = 0
\]
system (1) is not stabilizable by the PD control (2).

Remark 2: Note that $M \geq M_2'$ is equivalent to
\[
U := 4L_1 M^4 + 4N_1 M^3 + 2L_2 M^2 + 2N_2 M \geq 1.
\]

Hence, it can be seen from Theorem 1(ii) that, if we regard the quantity $U$ as a measure of system uncertainty, the PD control (2) will have fundamental limitations in dealing with the uncertain nonlinear stochastic system (1), once the uncertainty of the system is too large, namely, $U \geq 1$.

Remark 3: The constant $M_1'$ is always positive and satisfies
\[
16L_1 M_1'^4 + 16N_1 M_1'^3 + 4L_2 M_1'^2 + 4N_2 M_1' \leq 1.
\]
Therefore, $M_1' < M_2'$. So, one naturally ask, whether system (1) is stabilizable if $M_1' \leq M < M_2'$? Furthermore, does it exist a positive constant $M'$, such that system (1) is stabilizable by the PD control (2) if and only if $M < M'$? In general, these problems can be very challenging due to the inherent nonlinearity, uncertainties, and the strong coupling of high-dimensional state variables and nonaffine control input, and remains open. However, the stabilizability problems have been solved in this article, when system (1) has a specific linear structure. In fact, the necessary and sufficient condition for a class of uncertain linear stochastic systems to be stabilized by the PD control (2) is $U < 1$, see Theorem 2 for details.

Remark 4: First, we remark that both the symmetry assumption of $\frac{\partial^2}{\partial u^2}$ and the lower bound condition $\frac{\partial^2}{\partial u^2} \geq I_n$ can be weakened in the definition of $F_{k_1, k_2}$. To be precisely, if $\frac{\partial^2}{\partial u^2} \geq I_n$ is replaced by $\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial u^2} \geq \frac{\partial^2}{\partial u^2} b(L_2)$ for some constant $b > 0$, we can deal with this situation similarly. Next, for bounded time-varying reference signal $y^*(t)$ with bounded first and second order derivatives, system (1) can still be stabilizable by PD control (2) in the sense
\[
\sup_{t \geq 0} \mathbb{E} \left( \|x_1(t)\|^2 + \|x_2(t)\|^2 \right) < \infty \quad \forall(x_1(0), x_2(0)) \in \mathbb{R}^{2n}
\]
provided that $\frac{\partial^2}{\partial u^2}$ satisfies an additional boundedness condition. However, the tracking error $\mathbb{E} \|x_1(t) - y^*(t)\|^2$ may not converge to zero in general.

IV. PROOFS OF THE MAIN RESULTS

A. Proof of Theorem 1

To prove Theorem 1(i), we apply the Lyapunov method for stochastic systems, where the construction of the Lyapunov function plays a key role. To prove Theorem 1(ii), we first find a necessary condition that $k_p$ and $k_d$ should satisfy, that is, $(k_p, k_d)$ belongs to the set $\Omega$ defined in (17). Next, we prove $\Omega \neq \emptyset$ is equivalent to $M < M_2'$.

Firstly, we prove the first half of Theorem 1 in three steps.

Step 1: Some properties of the PD parameter set $\Omega_0$

Firstly, for given positive constants $L_1, L_2, N_1, N_2,$ and $M \geq 0$, we define a set $\Omega'$ as follows:
\[
\Omega' := \{(k_p, k_d) \mid k_p > L_1, \quad \bar{k}_1 \bar{k}_2 > T_1^2 + \bar{k}_1 T_2^2 \}
\]
where $T_1, T_2, \bar{k}_1,$ and $\bar{k}_2$ are defined in (9) and (18).

Property 1: The sets $\Omega_0, \Omega'$, and $\Omega$ satisfy
\[
\Omega_0 \subset \Omega' \subset \Omega
\]
where $\Omega_0$ and $\Omega$ are defined in (8) and (17), respectively. The inclusion $\Omega' \subset \Omega$ is obvious by definitions of $\Omega'$ and $\Omega$. We only need to show $\Omega_0 \subset \Omega'$. Indeed, if $(k_p, k_d) \in \Omega_0$, then by definition (8), we know
\[
k_p^2 > \bar{k} := (L_1 + L_2)(k_p + k_d) > k_p(L_1 + L_2)
\]
which yields $k_p > L_1$. Moreover, combine $k_p^2 > \tilde{k} + k_d T_f^2$ with $\tilde{k} > k_p L_1$, it can be obtained that $k_p^2 - k_p L_1 > \tilde{k} T_f^2$, hence, $k_p - L_1 > k_d T_f^2 / k_p$. Recall $\tilde{k} := k_p - L_1$, we have

$$\tilde{k}_1 > k_d T_f^2 / k_p.$$  \hfill (22)

On the other hand, since $k_p^2 - k_p > \tilde{k} + k_d T_f^2 > k_d L_2 + k_d T_f^2$, we have $k_d (k_d - L_2) > k_p + k_d T_f^2$. Therefore, we have

$$k_2 := k_d - L_2 > k_p / k_d + T_f^2.$$  \hfill (23)

Combine (22) with (23), it is easy to obtain

$$\tilde{k}_1 (T_f^2 - T_2^2) > T_2^2.$$  \hfill (24)

From (24) and recall $k_p > L_1$, we conclude that $(k_p, k_d) \in \Omega$. 

**Property 2:** $\Omega_0$ will become smaller when $M$ increase, i.e.,

$$\Omega_0 (M_1) \subset \Omega_0 (M_2), \quad \text{if } 0 \leq M_2 < M_1.$$  \hfill (25)

In fact, let $0 \leq M_2 < M_1$, and suppose $(k_p, k_d) \in \Omega_0 (M_1)$, then by (8) and note that $k_p > 0, k_d > 0$, it is easy to obtain

$$k_p^2 > \tilde{k} + k_d (N_1 + M_2 k_p)^2 > \tilde{k} + k_d (N_1 + M_2 k_p)^2,$$

$$k_p^2 > \tilde{k} + k_d (N_2 + M_2 k_d)^2 > \tilde{k} + k_d (N_2 + M_2 k_d)^2,$$

hence, $(k_p, k_d) \in \Omega_0 (M_2)$, which yields the relationship (25).

**Property 3:** If $M = 0$, $\Omega_0$ is an open and unbounded set. To this end, let $k_p = k_d = k > 0$, then

$$k_p^2 - \tilde{k} - k_d T_f^2 = k^2 - 2k (L_1 + L_2) - k N_f^2 > 0, \text{ as } k \to \infty.$$  \hfill (26)

Similarly, $k_p^2 - k^2 - k_d T_f^2 > 0$ for $k$ large enough. Consequently, $\Omega_0$ is open and unbounded when $M = 0$. 

**Property 4:** For given positive constants $L_1, L_2, N_1, N_2$, let $M_0^* be the unique positive solution of the polynomial

$$16 L_1 \ s^4 + 16 N_1 \ s^3 + 4 L_2 \ s^2 + 4 N_2 \ s = 1$$

then $\Omega_0 = \emptyset$ if $M > M_0^*$. 

First, by the definition of $M_0^*$, we know that $M \geq M_0^*$ is equivalent to $16 L_1 \ M^4 + 16 N_1 \ M^3 + 4 L_2 \ M^2 + 4 N_2 \ M \geq 1$. From Lemma 2 in Appendix A, we know $M^*$ is empty if and only if $M \geq M_0^*$. Besides, by the inclusion relationship (20), we conclude that $\Omega_0$ is also empty if $M \geq M_0^*$.

Let $M_0^*$ be the supremum that makes $\Omega_0$ nonempty, i.e.,

$$M_0^* := \sup \{ M > 0 | \Omega_0 (M) \neq \emptyset \}.$$  \hfill (27)

Then, it is easy to obtain the following facts.

1) The set $\Omega_0$ is not empty, if $0 \leq M < M_0^*$.

2) The constant $M_0^*$ depends on $L_1, L_2, N_1, N_2$ only.

3) $M_0^* \leq M_0^*$.

**Step 2:** (Transform the closed-loop system into a linearity-like form)

Let us denote

$$z_1 (t) := x_1 (t) - y, \quad z_2 (t) := x_2 (t)$$

then the PD controlled system (1), (2) turns into

$$\dot{z}_1 = z_2 dt,$$

$$\dot{z}_2 = f (z_1 + y^*, z_2, u) dt + g (z_1 + y^*, z_2, u) dB_t,$$

$$u = -k_p z_1 - k_d z_2.$$  \hfill (29)

By Assumption 1, we know $(0, 0) \in \mathbb{R}^{2n}$ is an equilibrium of (29). Recall $f (y^*, 0, 0) = 0$ and $f \in \mathcal{F}_1, L_2, f (z_1 + y^*, z_2, u)$ can be expressed by (details can be found in [23]):

$$f (z_1 + y^*, z_2, u) = a (z_1) z_1 + b (z_1, z_2) z_2 + \theta (z_1, z_2, u)$$  \hfill (30)

where $a, b$, and $\theta$ are $n \times n$ matrices satisfying

$$\|a (z_1)\| \leq L_1, \|b (z_1, z_2)\| \leq L_2, \theta (z_1, z_2, u) \geq \eta u.$$  \hfill (31)

for all $z_1, z_2, u$. Similarly, one can obtain

$$g (z_1 + y^*, z_2, u) = c (z_1) z_1 + d (z_1, z_2) z_2 + e (z_1, z_2, u) u,$$

where $c, d, e$ are $n \times n$ matrices satisfying

$$\|c (z_1)\| \leq N_1, \|d (z_1, z_2)\| \leq N_2, \|e (z_1, z_2, u)\| \leq M.$$  \hfill (32)

By (30) and (32), the nonlinear system (29) turns into the linearity-like form

$$\dot{z}_1 = z_2 dt,$$

$$\dot{z}_2 = [a z_1 + b z_2] dt + [\hat{e} z_1 + \hat{d} z_2] dB_t,$$

where $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ are nonlinear (matrix-valued) functions of $z = (z_1, z_2)$ defined by

$$\hat{a} = a (z_1) - k_p \theta (z, u), \quad \hat{b} = b (z_1, z_2) - k_d \theta (z, u)$$

$$\hat{c} = c (z_1) - k_p \theta (z, u), \quad \hat{d} = d (z_1, z_2) - k_d \theta (z, u)$$

with $u = k_p z_1 + k_d z_2$. Step 3: (Construction of Lyapunov function)

We adopt a similar Lyapunov function $V (z)$ as that used for deterministic system (see [23])

$$V (z) = k_p k_d z_1^2 + k_p z_1 z_2 + k_d z_2^2 / 2 = \frac{1}{2} z^T P z$$  \hfill (37)

where

$$z = (z_1, z_2)^T, \quad P = \begin{bmatrix} k_p & k_n \\ k_p & k_d \end{bmatrix}.$$  \hfill (38)

Note that $(k_p, k_d) \in \Omega_0$, we obtain $P > 0$.

**Step 4:** (Stability analysis based on Lyapunov methods) By some manipulations, the operator $L$ acting on the function $V (z)$ along the trajectories of (34) is given by

$$LV (z) = \frac{\partial V}{\partial z} \begin{bmatrix} z_1 \\ \hat{e} z_1 + \hat{d} z_2 \end{bmatrix} + \frac{k_d}{2} \| \hat{e} z_1 + \hat{d} z_2 \|^2.$$  \hfill (39)

Denote $A = \begin{bmatrix} 0_n \\ a_n \\ b \\ \bar{b} \end{bmatrix}$, where $\bar{a}$ and $\bar{b}$ are defined in (35). Then, the first term in (38) can be estimated as follows (see [23, proof of Proposition 4.3]):

$$I = z^T P A z = \frac{1}{2} z^T (PA + A^T P) z$$

$$\leq -\left( k_p^2 - \tilde{k} \right) \| z \|^2 - \left( k_d^2 - k_p - \tilde{k} \right) \| z \|^2.$$  \hfill (39)

By definitions of $\hat{a}, \hat{d}$ in (36) and the properties (33), we have $\| \hat{e} \| \leq N_1 + k_p M = T_1$ and $\| d \| \leq N_2 + k_d M = T_2$. Therefore, the second term has the following upper bound:

$$\bar{I} = \frac{k_d}{2} \| \hat{e} z_1 + \hat{d} z_2 \|^2 \leq k_d (T_1^2 \| z \|^2 + T_2^2 \| z \|^2).$$  \hfill (40)

Combining (38) and (40), we obtain the upper bounds of $LV (z)$

$$LV (z) \leq \left( k_p^2 - \tilde{k} - k_d T_f^2 \right) \| z \|^2 - \left( k_d^2 - k_p - \tilde{k} - k_d T_f^2 \right) \| z \|^2.$$  \hfill (39)

Since $(k_p, k_d) \in \Omega_0$, there is some constant $\eta > 0$ such that $LV (z) \leq -\eta \| z \|^2$ for all $z \in \mathbb{R}^{2n}$. Recall $z_1 (t) = -\hat{e} (t), z_2 (t) = -\hat{d} (t)$, we conclude that the PD control system (1) and (2) will satisfy the control objective (3).
Next, we prove the second half of Theorem 1. For this, it suffices to show the following statement.

If there exist some \((k_p, k_d) \in \mathbb{R}^2\), such that the closed-loop system \((1)\) and \((2)\) satisfies the performance \((3)\) for all \(f(\cdot)\) and \(g(\cdot)\) satisfying Assumptions 1 and 2, then \(M < M_2^*\).

By the definition of \(M_2^*\) in \((10)\), one can see that
\[
M < M_2^* \iff 4L_1 M^4 + 4N_1 M^3 + 2L_2 M^2 + 2N_2 M < 1.
\]

Now, suppose that the functions \(f(\cdot)\) and \(g(\cdot)\) are given by
\[
f(x_1, x_2, u) = a(x_1 - y^1) + bx_2 + u, x_1, x_2, u \in \mathbb{R}^n \quad (41)
g(x_1, x_2, u) = c(x_1 - y^2) + dx_2 - eu, x_1, x_2, u \in \mathbb{R}^n \quad (42)
\]
where \(a, b, c, d, e\) are five real numbers satisfying \(|a| \leq L_1, |b| \leq L_2, |c| \leq N_1, |d| \leq N_2, |e| \leq M\). Then, it is easy to see that Assumptions 1 and 2 hold.

Denote \(z_1(t) = x_1(t) - y^*\) and \(z_2(t) = x_2(t)\)
\[
z(t) := \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \quad A := \begin{bmatrix} a_0 & I_n \\ a_0 I_n & b_0 I_n \end{bmatrix}, \quad B := \begin{bmatrix} 0_n \\ c_0 I_n \\ d_0 I_n \end{bmatrix}
\]
where
\[
a_0 = a - k_p, \quad b_0 = b - k_d \\
c_0 = c + k_p e, \quad d_0 = d + k_d e
\]
then closed-loop equation \((1), (2)\) with \(f\) and \(g\) defined by \((41)\) and \((42)\) turns into
\[
dz = A z dt + B z dB_t.
\]

Define a \(2n \times 2n\) time-varying matrix \(\tilde{P}(t) := \begin{bmatrix} p(t) & r(t) \\ r^T(t) & q(t) \end{bmatrix}\) where
\[
p(t), r(t), \text{ and } q(t) \text{ are } n \times n \text{ matrix defined by}
\[
p(t) := \mathbb{E} \left[ z_1(t) z_1^T(t) \right], \quad r(t) := \mathbb{E} \left[ z_1(t) z_2^T(t) \right], \quad q(t) := \mathbb{E} \left[ z_2(t) z_2^T(t) \right]
\]
then it can be seen that \(\tilde{P}(t) = \mathbb{E}[z(t) z^T(t)]\). From [1, Th. 8.5.5], we know that \(\tilde{P}(t)\) is the unique nonnegative-definite symmetric solution of the equation
\[
\frac{d\tilde{P}}{dt} = A \tilde{P}(t) + \tilde{P}(t) A^T + B \tilde{P}(t) B^T.
\]
From \((47)\), it can be obtained that
\[
\dot{p} = r + r^T \\
\dot{r} = a_0 p + b_0 r + q \\
\dot{q} = c_0 p + (a_0 + c_0 d_0)(r + r^T) + (2b_0 + d_0^2) q
\]
Let \(r_0(t) = (r(t) + r^T(t))/2\) and define
\[
Q = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 2 & c_0 & 2(b_0 + d_0^2) \end{bmatrix}
\]
then it follows from \((48)\) that
\[
\frac{d}{dt} \begin{bmatrix} p(t) \\ r_0(t) \\ q(t) \end{bmatrix} = Q \otimes I_n \begin{bmatrix} p(t) \\ r_0(t) \\ q(t) \end{bmatrix}
\]
where \(\otimes\) denotes the Kronecker product.

Since for any initial state \((z_1(0), z_2(0)) \in \mathbb{R}^{2n}\), the solution of \((44)\) satisfies \(\lim_{t \to \infty} \mathbb{E}[\|z_1(t)\|^2 + \|z_2(t)\|^2] = 0\), which implies that \(\lim_{t \to \infty} \|\tilde{P}(t)\| = 0\) for all initial state \((z_1(0), z_2(0))\). We conclude that \(Q \otimes I_n\) is a Hurwitz matrix. Note that the matrix \(Q \otimes I_n\) shares the same spectrum with \(Q\). Hence, \(Q\) is also Hurwitz.

From the expression of \(Q\) in \((49)\), the characteristic polynomial of \(Q\) can be calculated as follows:
\[
det(\lambda I_3 - Q) = \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0
\]
where \(\alpha_0, \alpha_1, \text{ and } \alpha_2\) are given by
\[
\alpha_1 = b_0 d_0^2 + 2b_0^2 - 4a_0 - 2c_0 d_0 \\
\alpha_2 = -3b_0 + d_0^2
\]
From the Routh–Hurwitz stability criterion for third order polynomials, the matrix \(Q\) is Hurwitz if and only if the following inequalities holds:
\[
\alpha_2 > 0, \quad \alpha_0 > 0, \quad \alpha_1 \alpha_2 > \alpha_0
\]

We next proceed to show the following statement:

Suppose that the matrix \(Q\) defined in \((49)\) is Hurwitz for all \(|a| \leq L_1, |b| \leq L_2, |c| \leq N_1, |d| \leq N_2, |e| \leq M\), then the parameters \(k_p\) and \(k_d\) belong to the set \(\Omega\) defined in \((17)\).

Proof: From the definitions of \(\alpha_2\) and \(b_0\) in \((53)\) and \((48)\), we have
\[
\alpha_2 = -(3b_0 + d_0^2) = 3(k_d - b) - d_0^2
\]
In addition, since \(\alpha_2 > 0\), it follows that \(3(k_d - b) \geq \alpha_2 > 0\). Choose \(b = L_2\), we conclude that \(k_d > L_2\).

Next, suppose for all \(|a| \leq L_1, |b| \leq L_2, |c| \leq N_1, |d| \leq N_2, |e| \leq M\), there is
\[
\alpha_0 = 2(a - k_p)(b - k_d) + (a - k_p)(d - ek_p)^2 - (c - ek_p)^2 > 0.
\]
Choose \(c = d = e = 0\), it follows from \((55)\) that \(k_p > L_1\). Moreover, if we choose \(a = L_1, b = L_2, c = -N_1, d = -N_2,\) and \(e = M\), then we have
\[
2k_1k_2 - k_1(N_2 + M k_d)^2 - (N_1 + M k_d)^2 > 0
\]
Combine \((55), (56)\), we conclude that \((k_p, k_d)\) belongs to \(\Omega\).

Finally, by Lemma 1 in Appendix A, we know that the necessary and sufficient condition for the set \(\Omega\) to be nonempty is \(4L_1 M^4 + 4N_1 M^3 + 2L_2 M^2 + 2N_2 M < 1\). Hence, if \(M \geq M_2^*\), where \(M_2^*\) is the unique positive solution of \((10)\), then \(\Omega\) is empty, and thus, there does not exist \(k_p\) and \(k_d\) such that \(Q\) is Hurwitz for all \(|a| \leq L_1, |b| \leq L_2, |c| \leq N_1, |d| \leq N_2, |e| \leq M\). Therefore, system \((1)\) is not stabilizable by the PD control \((2)\). \(\Box\)

B. Proof of Theorem 2

Sufficiency: Suppose that \((16)\) is satisfied. From Lemma 1 in Appendix A, we know that the set \(\Omega\) defined by \((17)\) is not empty. Now, suppose \((k_p, k_d) \in \Omega\) and the matrices \(a, b, c, d, \) and \(e\) satisfy \((14)\), we proceed to show that the closed-loop system \((13)\) and \((15)\) will satisfy
\[
\lim_{t \to \infty} \mathbb{E}[\|z_1(t)\|^2 + \|z_2(t)\|^2] = 0
\]
for all initial states. Substituting \((15)\) into \((13)\), we have
\[
\frac{dz_1}{dt} = x_2 dt \\
\frac{dz_2}{dt} = [a_0 x_1 + b_0 x_2] dt + [c_0 x_1 + d_0 x_2] dB_t
\]
where \(a_0, b_0, c_0, d_0\) are \(n \times n\) constant matrices defined by
\[
a_0 = a - k_p I_n, \quad b_0 = b - k_d I_n, \quad c_0 = c - k_p e, \quad d_0 = d - k_d e
\]
Define
\[ A := \begin{bmatrix} a_0 & I_n \\ a_0 & b_0 \end{bmatrix}, \quad X := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad B := \begin{bmatrix} 0_n & 0_n \\ 0_n & d_0 \end{bmatrix} \]
then (57) can be rewritten in a more compact form
\[ dX = AX dt + BX dB_t. \]
Let us define a 2n \times 2n matrix \( P \) as follows:
\[ P := \begin{bmatrix} m & rI_n \\ r^T I_n & m \end{bmatrix} \]
where \( m \) is an \( n \times n \) matrix defined by
\[ m := -r b_0 - a_0^2 - c_0^T d_0 \]
and \( r > 0 \) is a constant defined by
\[ r := \left( 2\tilde{k}_2 T_2^2 - \tilde{k}_1 T_2^2 \right) / \left( 4\tilde{k}_1 \right). \]
Now, let \( V(X) = X^T PX \), where \( P \) is defined in (60), then the differential operator \( L \) acting on \( V \) is
\[ LV(X) = X^T \left( PA + AT^P + B^T PB \right) X \]
where
\[ a_0^s = (a_0 + \tilde{a}_0)/2, \quad b_0^s = (b_0 + \tilde{b}_0)/2 \]
are the symmetrization matrices of \( a_0 \) and \( b_0 \). Denote \( Q := PA + AT^P + B^T PB \), then from (61), it is easy to obtain
\[ Q = \begin{bmatrix} 2r a_0 + c_0^T c_0 & m + r b_0 + a_0^2 + c_0^T d_0 \\ m + r b_0^T + a_0 + d_0^T c_0 & 2r I_n + 2b_0 + d_0^T d_0 \end{bmatrix}. \]
Note that \( ||a|| \leq L_1 \) and \( ||b|| \leq L_2 \), it follows that
\[ \lambda_{\max}(a_0^s) \leq L_1 - k_p = -\tilde{k}_2, \quad \lambda_{\max}(b_0^s) \leq -\tilde{k}_1. \]
Moreover, since \( ||c|| \leq N_1, ||d|| \leq N_2, ||e|| \leq M \), we have
\[ \|c_0\| = ||c - k_p e|| \leq T_1, \quad \|d_0\| = ||d - k_d e|| \leq T_2 \]
where \( T_1, T_2 \) are defined in (9). From (64), and recall \( (k_p, k_d) \in \Omega \), it can be seen that
\[ \lambda_{\max}\left[ 2r a_0 + c_0^T c_0 \right] \leq -2\tilde{k}_1 r + T_1^2 \leq \left( \tilde{k}_1 T_2^2 + T_1^2 - 2\tilde{k}_1 k_2 \right) / 2 < 0 \]
\[ \lambda_{\max}\left[ 2r + 2b_0 + d_0^T d_0 \right] \leq 2(2\tilde{k}_1 + T_2^2 - 2\tilde{k}_1 k_2) / (2\tilde{k}_1) < 0 \]
which implies \( LV(X) \) is a negative definite function.

Finally, we prove \( V(X) = X^T PX \) is positive definite. By the definition of \( P \) in (60), it suffices to show \( m^y \geq -r^2 I_n \). Note that \( m^y = -r b_0 - a_0^2 - c_0^T d_0 \), we have
\[ \lambda_{\min}(m^y) \geq -r \lambda_{\min}(b_0^s) - \lambda_{\max}(a_0^s) - \|c_0^T d_0\| \]
Therefore, it follows from (64) that
\[ \lambda_{\min}(m^y - r^2 I_n) \geq \left( r \tilde{k}_2 - r^2 + \tilde{k}_1 - T_1 T_2 \right) \]
\[ = - \left( T_1^2 / (4\tilde{k}_1) - T_2^2 / 4 \right)^2 + 2\tilde{k}_2 / 4 + \tilde{k}_1 - T_1 T_2 \]
Consequently
\[ \lambda_{\max}\left[ 16\tilde{k}_1^2 \left( m^y - r^2 I_n \right) \right] \geq - \left( T_1^2 - T_2^2 \tilde{k}_1 \right)^2 + 4\tilde{k}_1^2 \tilde{k}_2^2 + 16\tilde{k}_1^3 - 16\tilde{k}_1^2 T_1 T_2 \]
\[ = 4\tilde{k}_1^2 \tilde{k}_2^2 - T_1^4 + T_2^4 \tilde{k}_1^2 + 2T_1^2 T_2^2 \tilde{k}_1 + 16\tilde{k}_1^3 - 16\tilde{k}_1^2 T_1 T_2 \]
\[ > 4T_1^2 T_2^2 \tilde{k}_1 + 16\tilde{k}_1^3 - 16\tilde{k}_1^2 T_1 T_2 = 4\tilde{k}_1 T_1 T_2 - 2\tilde{k}_1^2 \geq 0 \]
which implies that \( V(X) \) is positive definite. As a consequence, the PD control system (13) and (15) will satisfy (3) exponentially, for all initial values \( x_1(0), x_2(0) \in \mathbb{R}^n \).

**Necessity:** The necessity of Theorem 2 is similar to the proof of Theorem 1(ii), we omit it here due to page limitation. \( \square \)

**V. CONCLUSION**

This article investigates the capabilities and limitations of the classical PD control for a class of uncertain nonlinear stochastic systems. We have shown that the uncertain stochastic system can be globally stabilized by PD control in mean square, if the upper bounds of the partial derivatives of the system nonlinear functions satisfy a certain algebraic inequality. Moreover, we have shown that the PD control will have fundamental limitations in stabilizing the considered stochastic systems, once the size of the system uncertainty exceeds a critical value. Furthermore, necessary and sufficient conditions on the selection of the PD parameters are also provided for a class of uncertain linear stochastic systems. For further investigation, it would be interesting to consider whether the PD controller’s ability to handle uncertainty can be enhanced by using more general feedback controllers, such as the PID controller, the ConVex optimization-based stochastic steady-state tracking error minimization control [19], the state-dependent Riccati equation control [6], etc. It would also be meaningful to optimize the PD parameters to get better transient performance, and to consider more practical situations including time-delay and saturation.

**APPENDIX**

**A. Auxiliary Results**

We provide two lemmas that are used in the proof of the main results.

**Lemma 1:** A necessary and sufficient condition for the set \( \Omega \) defined by (17) to be nonempty is
\[ 4L_1 M^4 + 4N_1 M^3 + 2L_2 M^2 + 2N_2 M - 1 < 0. \]

**Sufficiency:** First, suppose that (68) holds, we will show \( \Omega \neq \emptyset \) by verifying \( (k_p^*, k_d^*) \in \Omega \), where
\[ k_p^* := \frac{1 - 2N_2 M - 2L_2 M^2 - 2N_1 M^3}{2 M^4}, \quad k_d^* := \frac{1 - N_2 M}{M^2}. \]
First, note that
\[ \tilde{k}_1 = k_p^* - L_1 \]
\[ = (1 - 2N_2 M - 2L_2 M^2 - 2N_1 M^3 - 2L_1 M^4) / (2 M^4) \]
then it follows from (68) that \( \tilde{k}_1 > 0 \). Moreover, it can be obtained that
\[ \tilde{k}_2 = k_d^* - L_2 = (1 - N_2 M - L_2 M^2) / M^2 \]
\[ T_1 = N_1 + M k_p^* = (1 - 2N_2 M - 2L_2 M^2) / (2 M^2) \]
\[ T_2 = N_2 + M k_d^* = 1 / M. \]
It follows from (71) to (73) that
\[ 2\tilde{k}_2 - T_2^2 = (1 - 2N_2 M - 2L_2 M^2) / M^2 = 2MT_1. \]
From (70) and (72), we obtain the following identity:

\[
2M^3 (2MK_1 - T_1) = 4M^4 k_1 - 2M^3 T_1
\]

\[
= 2 - 4N_2 M - 4L_2 M^2 - 4N_1 M^3 - 4L_1 M^4
\]

\[
- (1 - 2N_2 M - 2L_2 M^2)
\]

\[
= 1 - 2N_2 M - 2L_2 M^2 - 4N_1 M^3 - 4L_1 M^4.
\]  

(75)

Thus, we have \(2MK_1 - T_1 > 0\). Consequently, it follows from (74) and (75) that

\[
2k_1 \bar{k}_2 - T_2^2 - \bar{k}_1 T_2^2 = k_1(2k_2 - T_2^2) - T_1^2
\]

\[
= 2MT_1 k_1 - T_1^2 = (2M k_1 - T_1) T_1 > 0.
\]  

(76)

From (70) and (76), we know that \((k^*_p, k^*_d) \in \Omega\), which implies the nonempty property of \(\Omega\).

**Necessary:** Suppose that \(\tilde{\Omega}\) is nonempty, we proceed to show that (68) holds. It suffices to consider the case \(M > 0\), since (68) is automatically satisfied when \(M = 0\).

Let \(\tilde{\Omega}\) be the closure of \(\Omega\), i.e.,

\[
\tilde{\Omega} = \{ (k_p, k_d) \mid k_p \geq L_1, \ 2k_1 \bar{k}_2 - T_2^2 - \bar{k}_1 T_2^2 \geq 0 \}.
\]  

(77)

First, we show that \(\tilde{\Omega}\) is bounded (hence, it is compact).

Suppose that \((k_p, k_d) \in \tilde{\Omega}\), then \(2k_1 \bar{k}_2 - T_2^2 \geq 0\), which yields \(k_1 \geq 2/M^2\). Hence, \(k_1 < 2/M^2\). Also, from \(2k_1 \bar{k}_2 > 0\), we know that \(k_2 > 0\), i.e., \(k_d > L_2\).

Next, we estimate the bounds of \(k_d\). It is easy to obtain

\[
4k_1/M^2 > 4k_1/M^2 > 2k_1 k_d > 2k_1 \bar{k}_2 \geq T_2^2 > M^2 k_d^2
\]

therefore, \(L_1 < k_p < 4/M^4\). Combine this with the bounds of \(k_d\), we find that \(\tilde{\Omega}\) is bounded.

Define a function \(H(\cdot)\) as follows:

\[
H(k_p, k_d) = 2k_1 \bar{k}_2 - T_2^2 - \bar{k}_1 T_2^2, \quad (k_p, k_d) \in \tilde{\Omega}.
\]

By the definition (77) of \(\tilde{\Omega}\), we know that \(H(k_p, k_d) \geq 0\), for \((k_p, k_d) \in \tilde{\Omega}\) and \(H(k_p, k_d) > 0\), for \((k_p, k_d) \notin \tilde{\Omega}\).

Since \(\tilde{\Omega}\) is compact, we know that \(H(\cdot)\) can attain its maximum value. Note that \(H(k_p, k_d) = 0\) on the boundary of \(\tilde{\Omega}\), thus, the maximum point \((k^*_p, k^*_d) \in \tilde{\Omega}\), and therefore \(\partial H/\partial k_p |_{(k^*_p, k^*_d)} \neq \partial H/\partial k_d |_{(k^*_p, k^*_d)} = 0\). By simple manipulations, we have

\[
\frac{\partial H}{\partial k_p} |_{(k^*_p, k^*_d)} = 2 \bar{k}_2 - 2T_1 M - T_2^2 = 0
\]  

(79)

\[
\frac{\partial H}{\partial k_d} |_{(k^*_p, k^*_d)} = 2 \bar{k}_1 (1 - T_2 M) = 0.
\]  

(80)

It follows from (79) and (80) that

\[
M(N_2 + MK^*_d) = 1, \ 2(N_1 + MK^*_p)M + (N_2 + MK^*_d)^2 = 2 \bar{k}_2.
\]

Hence, it can be obtained that \(k^*_p = (1 - MN_2)/M^2\) and

\[
k^*_d = (1 - 2MN_2 - 2M^2 L_2 - 2M^2 N_1)/(2M^4)
\]

which is exactly the formula given in (69).

Note that \(H(k^*_p, k^*_d) > 0\), and from (75) to (76), we know

\[
1 - 2N_2 M - 2L_2 M^2 - 4N_1 M^3 = 2M^3 (2M k_1 - T_1) = 2M^3 H(k^*_p, k^*_d)/T_1 > 0.
\]  

(81)

Hence, Lemma 1 is proved.

Similar to the proof of Lemma 1, the following can be obtained.

**Lemma 2:** A necessary and sufficient condition for the set \(\Omega'\) defined by (19) to be nonempty is

\[
16L_1 M^4 + 16N_1 M^3 + 4L_2 M^2 + 4N_2 M - 1 < 0.
\]  

(82)

**B. Proof of Proposition 1**

Without loss of generality, we assume that \(y' = 0\). Suppose that for some \(k_p\) and \(k_d\) and for some initial state \((x_1(0), x_2(0)) \in \mathbb{R}^n\), the closed-loop equation (1) and (2) satisfies

\[
\lim_{t \to \infty} \mathbb{E}[x_1(t)]^2 = 0 \quad \text{and} \quad \lim_{t \to \infty} \mathbb{E}[x_2(t)]^2 = 0
\]  

(83)

we proceed to show \(f(0, 0, 0) = g(0, 0, 0) = 0\).

Note that \(u(t) = -k_p x_1(t) - k_d x_2(t)\), it follows from (83) that \(\lim_{t \to \infty} \mathbb{E}[u(t)]^2 = 0\). Recall \(dx_2 = f(x_1, x_2, u)dt + g(x_1, x_2, u) dB_t\), it follows that

\[
x_2(t + 1) - x_2(t) = X_t + Y_t
\]  

(84)

where

\[
X_t = \int_t^{t+1} f(x_1(s), x_2(s), u(s))ds
\]  

(85)

\[
Y_t = \int_t^{t+1} g(x_1(s), x_2(s), u(s))dB_s.
\]  

(86)

Next, we proceed to show that

\[
\mathbb{E}[X_t^2] \to 0, \quad \text{as} \quad t \to \infty.
\]  

(87)

To this end, we first need to prove the following two facts:

\[
\lim_{t \to \infty} \mathbb{E}[X_t - f(0, 0, 0)]^2 = 0
\]  

(88)

\[
\lim_{t \to \infty} \mathbb{E}[Y_t - g(0, 0, 0)(B_{t+1} - B_t)]^2 = 0.
\]  

(89)

From the Cauchy–Schwarz inequality, we can obtain

\[
\lim_{t \to \infty} \mathbb{E}[X_t - f(0, 0, 0)]^2
\]

\[
\leq \lim_{t \to \infty} \mathbb{E}\left[ \int_t^{t+1} \left| f(x_1(s), x_2(s), u(s)) - f(0, 0, 0) \right| ds \right]^2
\]

\[
\leq \lim_{t \to \infty} \int_t^{t+1} \mathbb{E}\left[ \left| f(x_1(s), x_2(s), u(s)) \right|^2 \right] ds.
\]  

(90)

Moreover, from (90) and the Lipschitz property of \(f\), we have

\[
\lim_{t \to \infty} \mathbb{E}[X_t - f(0, 0, 0)]^2
\]

\[
\leq \lim_{t \to \infty} \mathbb{E}\left[ \int_t^{t+1} C \left[ \|x_1(s)\|^2 + \|x_2(s)\|^2 + \|u(s)\|^2 \right] ds \right]
\]

\[
= \lim_{t \to \infty} C \int_t^{t+1} \mathbb{E}[\|x_1(s)\|^2] ds = 0.
\]  

(91)

for some constant \(C > 0\). Hence, (88) is proved. Similarly, by the Ito’s isometry and the Lipschitz property of \(g\), one can prove (89) in a similar way. From (89), we know that \(\mathbb{E}[\|Y(t)\|^2] = 0\) is a bounded function of \(t \in [0, \infty)\).

By applying (88)–(89) again, it can be obtained that

\[
\lim_{t \to \infty} \mathbb{E}[X_t^2] = f^T(0, 0, 0)g(0, 0, 0)(B_{t+1} - B_t) = 0.
\]  

(92)

On the other hand, note that

\[
\mathbb{E}[f^T(0, 0, 0)g(0, 0, 0)(B_{t+1} - B_t)] = 0 \quad \forall t \geq 0.
\]  

(93)
Consequently, (87) follows from (92) and (93).

From (84), we know that

\[ \mathbb{E} \left[ \| X_1 \|^2 + \| Y_1 \|^2 + 2 X_1^T Y_1 \right] = \mathbb{E} \left[ \| x_2(t+1) - x_2(t) \|^2 \right]. \]

Recall \( \lim_{t \to \infty} \mathbb{E} \| x_2(t) \|^2 = 0 \), we conclude that

\[ \lim_{t \to \infty} \mathbb{E} \left[ \| X_1 \|^2 + \| Y_1 \|^2 + 2 X_1^T Y_1 \right] = 0. \quad (94) \]

From (87), we have \( \lim_{t \to \infty} \mathbb{E} \| X_1 \|^2 + \| Y_1 \|^2 = 0 \). Combine this with (88) and (89), we can obtain \( f(0, 0, 0) = 0 \) and \( g(0, 0, 0) = 0 \). □

**References**

[1] L. Arnold, *Stochastic Differential Equations*. Hoboken, NJ, USA: Wiley, 1974.

[2] K. J. Åström and T. Hägglund, *PID Controllers: Theory, Design and Tuning*. Research Triangle Park, NC, USA: Instrument Society of America, 1995.

[3] K. J. Åström and T. Hägglund, “The future of PID control,” *Control Eng. Pract.*, vol. 9, no. 11, pp. 1163–1175, 2000.

[4] P. Borja, R. Ortega, and J. M. A. Scherpen, “New results on stabilization of port-Hamiltonian systems via PID passivity-based control,” *IEEE Trans. Autom. Control*, vol. 66, no. 2, pp. 625–636, Feb. 2021.

[5] S. Chen, W. Xue, and Y. Huang, “On active disturbance rejection control for nonlinear systems with multiple uncertainties and nonlinear measurement,” *Int. J. Robust Nonlinear Control*, vol. 30, no. 8, pp. 3411–3435, 2020.

[6] T. Çimen, “State-dependent Riccati equation (SDRE) control: A survey,” *IFAC Proc. Volume*, vol. 41, no. 2, pp. 3761–3775, 2008.

[7] X. Cong and L. Guo, “PID control for a class of nonlinear uncertain stochastic systems,” in *Proc. IEEE 56th Annul. Conf. Decis. Control*, 2017, pp. 612–617.

[8] R. Kelly, “Global positioning of robot manipulators via PD control plus a class of nonlinear integral actions,” *IEEE Trans. Autom. Control*, vol. 43, no. 7, pp. 934–938, Jul. 1998.

[9] J. Han, “PID to active disturbance rejection control,” *IEEE Trans. Ind. Electron.*, vol. 56, no. 3, pp. 900–906, Mar. 2009.

[10] H. K. Khalil, “Universal integral controllers for minimum-phase nonlinear systems,” *IEEE Trans. Autom. Control*, vol. 45, no. 3, pp. 490–494, Mar. 2000.

[11] X. G. Li, S. I. Niculescu, J. X. Chen, and T. Chai, “Characterizing PID controllers for linear time-delay systems: A parameter-space approach,” *IEEE Trans. Autom. Control*, vol. 66, no. 10, pp. 4499–4513, Oct. 2021.

[12] D. Ma and J. Chen, “Delay margin of low-order systems achievable by PID controllers,” *IEEE Trans. Autom. Control*, vol. 64, no. 5, pp. 1958–1973, May 2019.

[13] J. Chen, D. Ma, Y. Xu, and J. Chen, “Delay robustness of PID control of second-order systems: Pseudoconcavity, exact delay margin, and performance tradeoff,” *IEEE Trans. Autom. Control*, vol. 67, no. 3, pp. 1194–1209, Mar. 2022.

[14] A. O’Dwyer, “PI and PID controller tuning rules: An overview and personal perspective,” in *Proc. IET In. Signals Syst. Conf.*, Dublin, Ireland, 2006, pp. 161–166.

[15] Q. C. Pham, N. Tabareau, and J. J. Slotine, “A contraction theory approach to stochastic incremental stability,” *IEEE Trans. Autom. Control*, vol. 54 no. 4, pp. 816–820, Apr. 2009.

[16] T. Samad, “A survey on industry impact and challenges thereof,” *IEEE Control Syst. Mag.*, vol. 37, no. 1, pp. 17–18, Feb. 2017.

[17] G. J. Silva, A. Datta, and S. P. Bhattacharyya, *PID Controllers for Time-Delay Systems*. Boston, MA, USA: Birkhäuser, 2005.

[18] M. Takegaki and S. Arimoto, “A new feedback method for dynamic control of manipulators,” *J. Dyn. Syst., Meas. Control, Trans. ASME*, vol. 103 no. 2, 119–125, 1981.

[19] H. Tsukamoto and S. J. Chung, “Robust controller design for stochastic nonlinear systems via convex optimization,” *IEEE Trans. Autom. Control*, vol. 66, no. 10, pp. 4731–4746, Oct. 2021.

[20] L. L. Xie and L. Guo, “How much uncertainty can be dealt with by feedback,” *IEEE Trans. Autom. Control*, vol. 45, no. 12 pp. 2203–2217, Dec. 2000.

[21] J. K. Zhang, C. Zhao, and L. Guo, “On PID control theory for nonaffine uncertain stochastic systems,” *J. Syst. Sci. Complexity*, vol. 36, no. 1, pp. 165–186, 2023.

[22] C. Zhao and L. Guo, “PID controller design for second order nonlinear uncertain systems,” *Sci. China Inf. Sci.*, vol. 60, no. 2, pp. 1–13, 2017.

[23] C. Zhao and L. Guo, “Towards a theoretical foundation of PID control for uncertain nonlinear systems,” *Automatica*, vol. 142, 2022, Art. no. 110360.