Abstract. Let \((X, d)\) be a metric space and \(A\) be a commutative Banach algebra such that \(\Delta(A)\) be nonempty. In this paper, we provide necessary and sufficient conditions, under which \(C_{BSE}(\Delta(A))\) is a Dual Banach algebra. Moreover, we provide some conditions for that \(\text{Lip}_\alpha(X, A)\) is a Dual Banach algebra.

0. Introduction and Preliminaries

Let \((A, \| \cdot \|)\) be a commutative Banach algebra and \(A'\) and \(A''\) be the dual and second dual Banach spaces, respectively. Let \(a \in A, f \in A'\) and \(\Phi, \Psi \in A''\). Then \(f \cdot a\) and \(a \cdot f\) are defined as \(f \cdot a(x) = f(ax)\) and \(a \cdot f(x) = f(xa)\), for all \(x \in A\), making \(A'\) an \(A\)–bimodule. Moreover for all \(f \in A'\) and \(\Phi \in A''\), we define \(\Phi \cdot f\) and \(f \cdot \Phi\) as the elements of \(A''\) by

\[
\langle \Phi \cdot f, a \rangle = \langle \Phi, f \cdot a \rangle \quad \text{and} \quad \langle f \cdot \Phi, a \rangle = \langle f, \Phi \cdot a \rangle \quad (a \in A).
\]

This defines two Arens products \(\square\) and \(\diamond\) on \(A''\) as

\[
\langle \Phi \square \Psi, f \rangle = \langle \Phi, \Psi \cdot f \rangle \quad \text{and} \quad \langle \Phi \diamond \Psi, f \rangle = \langle \Psi, f \cdot \Phi \rangle,
\]

making \(A''\) a Banach algebra with each. The products \(\square\) and \(\diamond\) are called respectively, the first and second Arens products on \(A''\). Note that \(A\) is embedded in its second dual via the identification

\[
\langle a, f \rangle = \langle f, a \rangle \quad (f \in A');
\]

see [5], for more information. The space, consisting of all nonzero multiplicative linear functionals on \(A\) will be denote by \(\Delta(A)\) and is called the Gelfand space of \(A\). A bounded continuous function \(\sigma\) on \(\Delta(A)\) is called a BSE-function if there exists a constant \(M > 0\) such that for every finite number of \(\varphi_1, \cdots, \varphi_n\) in \(\Delta(A)\) and the same number of \(c_1, \cdots, c_n\) in \(\mathbb{C}\), the inequality

\[
\sum_{j=1}^{n} c_j \sigma(\varphi_j) \leq M \left\| \sum_{j=1}^{n} c_j \varphi_j \right\|_{A^*}
\]

holds. The BSE-norm of \(\sigma\) is the infimum of all such \(M\) and is denoted by \(\| \sigma \|_{BSE}\). The set of all BSE-functions will be denoted by \(C_{BSE}(\Delta(A))\). It has been shown that under the norm \(\| \cdot \|_{BSE}\), \(C_{BSE}(\Delta(A))\) is a commutative and semisimple Banach algebra, embedded in \(C_b(\Delta(A))\) as a subalgebra. We refer to [11], for a full information about the history of BSE functions and BSE algebras.

Now we provide some information about Lipschitz algebras; see [6], [7] and [8], for more information. Let \((X, d)\) be a metric space and \(A\) be a Banach algebra.

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For \( f : X \to A \) let
\[
\|f\|_{\infty,E} = \sup_{x \in X} \|f(x)\|.
\]
Then \( f \) is called bounded if \( \|f\|_{\infty} < \infty \). Now let
\[
\rho_{d,E}(f) = \sup_{x \neq y} \frac{\|f(x) - f(y)\|_E}{d(x,y)}.
\]
Following [6] and [7], the vector-valued Lipschitz space \( Lip_d(X,A) \), is the space consisting of all bounded maps \( f : X \to A \) with \( \rho_{d,E}(f) < \infty \). Furthermore, \( Lip_d(X,A) \), equipped with the norm
\[
\|f\| = \max\{\rho_{d,E}(f), \|f\|_{\infty,E}\}
\]
and pointwise product, is a Banach algebra. Moreover, we denote by \( lip_d(X,A) \), the subspace of \( Lip_d(X,A) \), consisting of all functions \( f \) such that
\[
\lim_{d(x,y) \to 0} \frac{\|f(x) - f(y)\|}{d(x,y)} = 0.
\]
Then \( lip_d(X,A) \) is a Banach algebra with pointwise product. In the case where \( A \) is the field of complex numbers \( \mathbb{C} \), to simplify the notation, we will write \( Lip_dX \) and \( lip_dX \), rather than \( Lip_d(K,\mathbb{C}) \) and \( lip_d(X,\mathbb{C}) \), respectively.

Recently, some important algebraic properties of Lipschitz algebras have been investigated; see [1], [2] and [3]. Here, we study another feature of this algebras. We first recall the notion of Dual Banach algebras.

Following [10], a dual Banach algebra is a pair \((A,A_*)\) such that:

- \( A = (A_*)' \).
- \( A \) is a Banach algebra, and multiplication in \( A \) is separately \( \sigma(A,A_*) \) continuous; or equivalently \( A \cdot A_* \subseteq A_* \) and \( A_* \cdot A \subseteq A_* \).

In [8, Theorem 4.1], it has been proved that for any metric space \((X,d)\), \( Lip_\alpha(X,E) \) is a dual space, whenever \( E \) is. In the present paper, we first provide necessary and sufficient conditions for that \( C_{BSE}(\Delta(A)) \) is a Dual Banach algebra. Then we give some conditions, under which vector-valued Lipschitz algebra \( Lip_\alpha(X,A) \) is a Dual Banach algebra.

1. Main Results

We commence our results with the with following propositions.

**Proposition 1.1.** Let \( A \) be a commutative Banach algebra such that \( \Delta(A) \) be nonempty. Then the following assertions are equivalent.

(i) The weak topology and weak-* topology are the same on \( \Delta(A) \),

(ii) \( A^{**}|_{\langle \Delta(A) \rangle} = \overline{A|_{\langle \Delta(A) \rangle}}^{\|\cdot\|_{A^{**}}} \), as isometric Banach algebras.

**Proof.**

(i) \( \Rightarrow \) (ii). Since \( \hat{A} \subseteq A^{**} \), it follows that \( \hat{A}|_{\langle \Delta(A) \rangle} \subseteq A^{**}|_{\langle \Delta(A) \rangle} \) and so \( \hat{A}|_{\langle \Delta(A) \rangle} \subseteq (\Delta(A))^* \). Consequently, \( \hat{A}|_{\langle \Delta(A) \rangle}^{\|\cdot\|_{A^{**}}} \subseteq (\Delta(A))^* \). In other words,
\[
\hat{A}|_{\langle \Delta(A) \rangle}^{\|\cdot\|_{A^{**}}} \subseteq A^{**}|_{\langle \Delta(A) \rangle}.
\]

Now let \( W := \hat{A}|_{\langle \Delta(A) \rangle}^{\|\cdot\|_{A^{**}}} \) and suppose on the contrary that \( W \) is a proper subset of \( A^{**}|_{\langle \Delta(A) \rangle} \). Thus there exist \( F \in A^{**} \) such that \( F|_{\langle \Delta(A) \rangle} \not\in W \). By Hahn-Banach theorem, there exists \( G \in (\Delta(A))^{**} \) such that \( G(W) = 0 \) and
\[ G(F|_{\Delta(A)}) = 1. \] Let \( (x_\alpha)_{\alpha \in I} \) be a net in \( A \) such that \( ||x_\alpha|| \leq ||F|| \) \( (\alpha \in I) \) and \( \hat{x}_\alpha \to_\alpha F \), in the weak* topology of \( A^{**} \). By the assumption, we have \( \hat{x}_\alpha|_{\Delta(A)} \to_\alpha F|_{\Delta(A)} \), in the weak topology of \( A^{**} \). Consequently,
\[
0 = G(\hat{x}_\alpha|_{\Delta(A)}) \to G(F|_{\Delta(A)}) = 1,
\]
which is a contradiction.

(ii) \( \Rightarrow \) (i). If \( \phi_\alpha \to_\alpha \phi \), in the weak* topology of \( \Delta(A) \) and \( F \in A^{**} \), then 
\[
F|_{\Delta(A)} \in A|_{\Delta(A)}^{||.||^{A^{**}}}.
\]
Thus there exists a sequence \( (x_n)_{n \in \mathbb{N}} \) in \( A \) such that,
\[
x_n \to \Delta(A) \to_\alpha F|_{\Delta(A)}^{*},
\]
in the norm topology of \( A^{**} \). Consequently for every \( \varepsilon > 0 \) exists \( N > 0 \) such that for all \( n \geq N \) we have
\[
||x_n|_{\Delta(A)} - F|_{\Delta(A)}|| < \varepsilon.
\]
Thus by taking a subnet if necessary,
\[
\lim_{n} \lim_{\alpha} \phi_\alpha(x_n) - \lim_{\alpha} F(\phi_\alpha) \leq \varepsilon
\]
and
\[
\lim_{n} \lim_{\alpha} \phi_\alpha(x_n) - \lim_{\alpha} F(\phi_\alpha) \leq \varepsilon.
\]
It follows that
\[
\lim_{n} \lim_{\alpha} \phi_\alpha(x_n) = \lim_{\alpha} \phi_\alpha(x_n)
\]
and so \( \lim_{n} F(\phi_\alpha) = F(\phi) \). Note that since the nets are bounded, the limits exist. Consequently, \( \phi_\alpha \to_\alpha \phi \), in the weak topology of \( A^{**} \).

**Proposition 1.2.** Let \( A \) be a commutative Banach algebra such that \( \Delta(A) \) be nonempty. Then the following assertions are equivalent.

\( \text{(i)} \) \( A^{**}|_{\Delta(A)} = \overline{A|_{\Delta(A)}}^{||.||^{A^{**}}} \), as isometric Banach algebras,

\( \text{(ii)} \) \( A^{**}|_{\Delta(A)}^{||.||^{A^{**}}} = \overline{A|_{\Delta(A)}}^{||.||^{A^{**}}} \), as isometric Banach algebras.

**Proof.** (i) \( \Rightarrow \) (ii). It is clear that \( \overline{A|_{\Delta(A)}}^{||.||^{A^{**}}} \subseteq \overline{A^{**}|_{\Delta(A)}}^{||.||^{A^{**}}} \). For the proof of the reverse of the inclusion, take \( F \in A^{**}|_{\Delta(A)} \) and define \( \overline{F} \) on \( \overline{\Delta(A)} \) as
\[
\overline{F}(\psi) := \lim_{n} F(\psi_n),
\]
where, \( \{\psi_n\} \subseteq \langle \Delta(A) \rangle \) and \( \psi_n \to_\psi \psi \), in the norm topology. Note that \( \overline{F} \) is well-defined. In fact, for all \( m, n \in \mathbb{N} \) we have
\[
||F(\psi_n) - F(\psi_m)|| \leq ||F|| ||\psi_n - \psi_m||.
\]
Thus sequence \( (F(\psi_n))_{n \in \mathbb{N}} \) is Cauchy and so convergent. Moreover,
\[
||\overline{F}(\psi)|| = \lim_{n} F(\psi_n) \leq ||F|| \lim_{n} ||\psi_n|| = ||F|| ||\psi||,
\]
which implies that \( \overline{F} \in \langle \Delta(A) \rangle^{*} \). By the assumption, we have
\[
A^{**}|_{\Delta(A)} = \overline{A|_{\Delta(A)}}^{||.||^{A^{**}}}.
\]
It follows that \( \overline{F} \in \overline{A|_{\Delta(A)}}^{||.||^{A^{**}}} \). Thus there exists a sequence \( (x_n)_{n \in \mathbb{N}} \) in \( A \) such that
\[
x_n|_{\Delta(A)} \to_n \overline{F},
\]
in the norm topology. It follows that
\[
x_n|_{\Delta(A)} \to_n F|_{\Delta(A)}.
\]
in the norm topology.

(ii)⇒(i). Let $A^{**}(\Delta(A)) = \overline{A\langle \Delta(A) \rangle}$. It is clear that

$$\overline{A\langle \Delta(A) \rangle}^{||.||A^{**}} \subseteq A^{**}(\Delta(A)).$$

Now take $H \in A^{**}(\Delta(A))$ and set

$$F := H|\Delta(A)) \in A^{**}(\Delta(A)).$$

Thus $F \in \overline{A\langle \Delta(A) \rangle}^{||.||A^{**}}$ and so there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $A$ such that

$$x_n|\Delta(A)) \rightarrow_n F,$$

in the norm topology. It follows that $x_n(\psi) \rightarrow_n F(\psi)$, for all $\psi \in \Delta(A)$ with $||\psi|| \leq 1$. Now take $k \in \Delta(A)$ such that $||k|| \leq 1$. There exists the sequence $(k_n)_{n \in \mathbb{N}}$ in $\Delta(A)$ such that, $k_n \rightarrow_n k$, in the norm topology. Thus there exists $N \in \mathbb{N}$ such that for all $n, \ell \geq N$, $||k_\ell|| \leq 1$ and

$$|x_n(k_\ell) - F(k_\ell)| < \varepsilon.$$

Consequently,

$$|x_n(k) - F(k)| = |\lim_\ell (x_n(k_\ell) - F(k_\ell))| = \lim_\ell |x_n(k_\ell) - F(k_\ell)| \leq \varepsilon.$$

It follows that for all $n \geq N,$

$$||x_n|\Delta(A)) - F|| \leq \varepsilon,$$

which implies that $F \in \overline{A\langle \Delta(A) \rangle}^{||.||A^{**}}$. Therefore,

$$A^{**}(\Delta(A)) \subseteq \overline{A\langle \Delta(A) \rangle}^{||.||A^{**}},$$

and the proof is completed. \hfill \Box

We state here the main result of the present paper.

**Theorem 1.3.** Let $A$ be a commutative Banach algebra such that $\Delta(A)$ be nonempty. Then the following assertions are equivalent.

(i) $C_{BSE}(\Delta(A)) = \langle \Delta(A) \rangle^{*}$, as isometric Banach algebras,

(ii) The weak topology and weak-* topology are the same on $\Delta(A)$.

**Proof.** (i)⇒(ii). Suppose that $\{\phi_n\}$ is a net in $\Delta(A)$, converging to $\phi \in \Delta(A)$, in the weak* topology of $\Delta(A)$. If $F \in A^{**}$, then $F|\Delta(A)) \in \Delta(A)^{*}$ and so $F|\Delta(A)) \in C_{BSE}(\Delta(A))$. It follows that $F|\Delta(A)) \in C_{BSE}(\Delta(A))$ and we have

$$||F|\Delta(A))||_{BSE} = ||F|\Delta(A))||_{A^{**}} \leq ||F|| < \infty.$$

Since $F|\Delta(A) \in C_b(\Delta(A))$, thus

$$F|\Delta(A)(\phi_n) \rightarrow_\alpha F|\Delta(A)(\phi).$$

Consequently $F(\phi_n) \rightarrow_\alpha F(\phi)$, which implies that $\{\phi_n\}$ converges to $\phi$, in the weak topology of $\Delta(A)$. (ii)⇒(i). Define

$$\Phi : C_{BSE}(\Delta(A)) \rightarrow \langle \Delta(A) \rangle^{*},$$

by

$$\Phi(H)\left(\sum_{k=1}^n \lambda_k \phi_k\right) := \sum_{k=1}^n \lambda_k H(\phi_k) \quad (H \in C_{BSE}(\Delta(A))).$$
It is obvious that $\Phi$ is linear and
\[ |\Phi(H)\sum_{k=1}^{n} \lambda_k \phi_k| \leq ||H||_{BSE} ||\sum_{k=1}^{n} \lambda_k \phi_k||_{A^*} .\]

Thus $||\Phi(H)|| \leq ||H||_{BSE}$ and so $\Phi$ is continuous. For any $G \in (\Delta(A))^*$ there exists $\sigma := G|_{\Delta(A)}$ such that $\Phi(\sigma) = G$. In fact,
\[
\tag{1.1} \Phi(\sigma)\sum_{k=1}^{n} \lambda_k \phi_k = \sum_{k=1}^{n} \lambda_k G(\phi_k) = G(\sum_{k=1}^{n} \lambda_k \phi_k),
\]
which implies that $\Phi(\sigma) = G$. Now we show that $\sigma$ is. Let $\phi_\alpha \rightarrow_\alpha \phi$, in the weak* topology of $\Delta(A)$. By the assumption, $\phi_\alpha \rightarrow_\alpha \phi$, in the weak topology of $\Delta(A)$ and so $G(\phi_\alpha) \rightarrow G(\phi)$. Thus $\sigma(\phi_\alpha) \rightarrow \sigma(\phi)$. Consequently $\sigma \in C_b(\Delta(A))$ and
\[ |\sum_{k=1}^{n} c_k \sigma(\phi_i)| \leq ||G|| ||\sum_{k=1}^{n} c_k \phi_i||_{A^*},\]
which implies that $\sigma \in C_{BSE}(\Delta(A))$ and $||\sigma||_{BSE} \leq ||G||$. Thus $\Phi$ is surjective. Moreover, by $[.,.]$, for each $H \in C_{BSE}(\Delta(A))$, $H = F|_{\Delta(A)}$ for some $F \in A^{**}$. Then
\[
||H||_{BSE} = \sup \left\{ |\sum_{k=1}^{n} \lambda_k H(\phi_k)| : ||\sum_{k=1}^{n} \lambda_k \phi_k||_{A^*} \leq 1 \right\} \leq \sup \left\{ ||F||_{BSE} : ||G|| \leq 1 \right\} = \sup \left\{ ||F||_{\Delta(A)} : ||G|| \leq 1 \right\} = ||H||_{A^{**}} = ||\Phi(H)||_{A^{**}}.
\]
Therefore $\Phi$ is isometric. Now we show that $\Phi$ is isomorphism. For $\sigma_1, \sigma_2 \in C_{BSE}(\Delta(A))$, there exist $F_1, F_2 \in A^{**}$ such that $\sigma_1 = F_1|_{\Delta(A)}, \sigma_2 = F_2|_{\Delta(A)}$. Then for each $\phi \in \Delta(A)$, we have
\[
(\sigma_1, \sigma_2)(\phi) := \sigma_1(\phi), \sigma_2(\phi) = (F_1|_{\Delta(A)})(\phi), (F_2|_{\Delta(A)})(\phi) = F_1(\phi)F_2(\phi).
\]
Furthermore,
\[
[\Phi(\sigma_1) \square \Phi(\sigma_2)](\phi) = \Phi(\sigma_1)[\Phi(\sigma_2)](\phi)
\]
and so for each $x \in A$,
\[
\Phi(\sigma_2)\phi(x) = \Phi(\sigma_2)(\phi, x),
\]
where,
\[
(\phi, x)(y) = \phi(xy) = \phi(x)\phi(y) \quad (y \in A).
\]
Thus for all $x \in A$ and $\phi \in \Delta(A)$, $\phi, x = \phi(x)\phi$. Consequently,
\[
\Phi(\sigma_2)\phi(x) = \Phi(\sigma_2)(\phi, x) = \phi(x)\Phi(\sigma_2)(\phi)
\]
and so for any $\sigma_2 \in C_{BSE}(\Delta(A))$,
\[
\Phi(\sigma_2)\phi = \Phi(\sigma_2)(\phi)\phi.
\]
It follows that
\[
[\Phi(\sigma_1) \square \Phi(\sigma_2)](\phi) = \Phi(\sigma_1)[\Phi(\sigma_2)](\phi) = \Phi(\sigma_2)(\phi)\Phi(\sigma_1)(\phi).
\]
It follows that
\[
[\Phi(\sigma_1) \square \Phi(\sigma_2)](\phi) = F_1(\phi)F_2(\phi) = \sigma_1(\phi)\sigma_2(\phi) = \Phi(\sigma_1\sigma_2)(\phi).
\]
Therefore
\[ \Phi(\sigma_1 \sigma_2) = \Phi(\sigma_1) \Box \Phi(\sigma_2). \]
and so \( \Phi \) is isomorphism.

\[ \square \]

**Corollary 1.4.** Let \( A \) be a reflexive commutative Banach algebra such that \( \Delta(A) \) be nonempty. Then \( \mathcal{C}_{BSE}(\Delta(A)) = \langle \Delta(A) \rangle^* \).

**Example 1.5.** Let \((X, d)\) be an infinite compact metric space and \( A = \text{Lip}_dX \). Then
\[ \Delta(A) = \{ \delta_x : x \in X \}. \]
Moreover, we have
\[ A = \langle \delta_x : x \in X \rangle^*. \]
Since \( A \) is a unital BSE-algebra [9, Example 6.1], we obtain
\[ C_{BSE}(\Delta(A)) = \overline{\text{M}(A)} = \hat{A} = \langle \delta_x : x \in X \rangle^* = \langle \Delta(A) \rangle^*. \]
Note that \( A \) is not reflexive. In fact, by [5],
\[ \text{lip}_d(X)^{**} = \text{Lip}_dX \quad \text{and} \quad \Delta(\text{lip}_dX) = \Delta(\text{Lip}_dX). \]
Thus \( A \) is reflexive if and only if \( \text{lip}_d(X)^{**} \) is reflexive. It follows that \( \text{lip}_d(X)^* \)
and so \( \text{lip}_dX \) is reflexive. Consequently, \( \text{Lip}_dX = \text{lip}_dX \), which implies that \( X \)
is discrete uniform. Since \( X \) is compact it follows that \( X \) is finite, which is a contradiction.

**Proposition 1.6.** Let \( A \) be a commutative and semisimple Banach algebra such that weak topology and weak* topology be the same on \( \Delta(A) \). Then \( \mathcal{C}_{BSE}(\Delta(A)) \)
is a dual Banach algebra by predual \( B = \langle \Delta(A) \rangle^{\langle 1+\|\cdot\|^* \rangle} \).

**Proof.** According to Theorem 1.3,
\[ C_{BSE}(\Delta(A)) \cong \langle \Delta(A) \rangle^* = \overline{\langle \Delta(A) \rangle}^* = B^*, \]
as two isometric Banach algebras. It is sufficient to show that
\[ C_{BSE}(\Delta(A)) \cdot \Delta(A) \subseteq B. \]
Let \( \sigma \in C_{BSE}(\Delta(A)) \) and \( \phi \in \Delta(A) \). Then there exists \( F \in A^{**} \) such that
\[ \sigma = F|\Delta(A). \]
Thus
\[ (\sigma \cdot \phi)(x) = \sigma(\phi \cdot x) = F(\phi \cdot x) \quad (x \in A). \]

Thus
\[ \phi \cdot x = \phi(x)\phi, \]
and so
\[ (\sigma \cdot \phi)(x) = F(\phi(x)\phi) = \phi(x)F(\phi). \]
It follows that
\[ \sigma \cdot \phi = F(\phi)\phi \in \langle \Delta(A) \rangle \subseteq B. \]
In general, if \( \psi = \sum_{i=1}^n \lambda_i \phi_i \in \langle \Delta(A) \rangle \), where \( \phi_i \in \Delta(A), (i = 1, \cdots, n) \), we have
\[ (\sigma \cdot \psi) = \sum_{i=1}^n \lambda_i F(\phi_i)\phi_i \in B. \]
Now, take \( \psi \in B. \) There exists a sequence \( \{ \psi_n \} \in \langle \Delta(A) \rangle \) such that \( \psi_n \to_n \psi \),
in the norm topology. Thus
\[ ||\sigma \cdot \psi - \sigma \cdot \psi_n|| \leq ||\sigma|| ||\psi - \psi_n|| \to_n 0, \]
and so \( \sigma \cdot \psi \in B. \) Therefore \( (C_{BSE}(\Delta(A)), B) \) is a dual Banach algebra. \( \square \)
Theorem 1.7. Let \((X,d)\) be a metric space and \(E\) be a Banach algebra. Suppose that
\[
W = \langle \delta_s \otimes f : f \in E^*, s \in X \rangle \subseteq \text{Lip}_d(X,E)^*,
\]
where,
\[
(\delta_s \otimes f)(h) = f(h(s)) \quad (s \in X, f \in E^*, h \in \text{Lip}_d(X,E)).
\]
Then \(\text{Lip}_d(X,E)\) can be imbedded isomerically in the dual Banach algebra \(W^*\).

Proof. Define
\[
\Phi : \text{Lip}_d(X,E) \to W^*
\]
h \mapsto \Phi_h,
where
\[
\Phi_h(\sum_{k=1}^n \lambda_k(\delta_{s_k} \otimes f_k)) := \sum_{k=1}^n \lambda_k h(s_k),
\]
for each \(h \in \text{Lip}_d(X,E)\). It is clear that \(\Phi\) is well-defined and linear. We show that \(\Phi\) is an isometric mapping. It is easily verified that \(\|\Phi_h\| \leq \|h\|\). For the reverse inequality, note that for each \(h \in \text{Lip}_d(X,E)\) and \(x \in X\) we have
\[
\|h(x)\|_E = \|h(x)\|_E = \sup \{ |\sigma(h(x))| : \sigma \in E^*, \|\sigma\| \leq 1 \}
\]
\[
= \sup \{ |(\delta_x \otimes \sigma)(h)| : \sigma \in E^*, \|\sigma\| \leq 1 \}.
\]
Moreover, for any \(x \in X\) and \(\sigma \in E^*\) with \(\|\sigma\| \leq 1\) we have \(\|\delta_x \otimes \sigma\| \leq 1\) and
\[
\|\Phi_h\| = \sup \{ |\Phi_h(u)| : u \in W, \|u\| \leq 1 \}
\]
\[
= \sup \left\{ \|\Phi_h(\sum_{k=1}^n \delta_{x_k} \otimes f_k)\| : \left\| \sum_{k=1}^n \delta_{x_k} \otimes f_k \right\| \leq 1 \right\}
\]
\[
\geq |\Phi_h(\delta_x \otimes \sigma)|.
\]
Thus we obtain
\[
\|h(x)\|_E = \sup \{ |\sigma(h(x))| : \sigma \in E^*, \|\sigma\| \leq 1 \}
\]
\[
= \sup \{ |\delta_x \otimes \sigma(h)| : \sigma \in E^*, \|\sigma\| \leq 1 \}
\]
\[
= \sup \{ |\Phi_h(x \otimes \sigma)| : \sigma \in E^*, \|\sigma\| \leq 1 \}
\]
\[
\leq \sup \left\{ \|\Phi_h(\sum_{k=1}^n \delta_{x_k} \otimes f_k)\| : \left\| \sum_{k=1}^n \delta_{x_k} \otimes f_k \right\| \leq 1 \right\}
\]
\[
= \|\Phi_h\|.
\]
Consequently,
\[
\|h\|_{\infty,E} \leq \|\Phi_h\|.
\]
Moreover, for all \(x, y \in X\) with \(x \neq y\) we have
\[
\frac{\|h(x) - h(y)\|_E}{d(x,y)} \leq \frac{\|h(x) - h(y)\|_E}{\|\delta_x - \delta_y\|}
\]
\[
= \sup \left\{ \left| \frac{\sigma(h(x)) - \sigma(h(y))}{\|\delta_x - \delta_y\|} \right| : \sigma \in E^*, \|\sigma\| \leq 1 \right\}
\]
\[
= \sup \left\{ \left| \frac{\|\delta_x \otimes \sigma(h) - \delta_y \otimes \sigma(h)\|}{\|\delta_x - \delta_y\|} \right| : \sigma \in E^*, \|\sigma\| \leq 1 \right\}.
\]
On the other hand, for any $\sigma \in E^*$ with $\|\sigma\| \leq 1$
\[\|\delta_x \otimes \sigma - \delta_y \otimes \sigma\| = \sup \{|(\delta_x \otimes \sigma - \delta_y \otimes \sigma)(h)| : h \in \text{Lip}_d(X,E), \|h\| \leq 1\}\]
\[= \sup \{|\sigma(h(x)) - \sigma(h(y))| : h \in \text{Lip}_d(X,E), \|h\| \leq 1\}\]
\[\leq \|h(x) - h(y)\|\]
\[= \|(\delta_x - \delta_y)(h)\|\]
\[\leq \|\delta_x - \delta_y\| \|h\|\]
\[\leq \|\delta_x - \delta_y\|.
\]
Consequently,
\[\frac{1}{\|\delta_x - \delta_y\|} \leq \frac{1}{\|\delta_x \otimes \sigma - \delta_y \otimes \sigma\|}.
\]
Thus
\[\frac{|h(x) - h(y)|_E}{d(x,y)} \leq \sup \left\{ \frac{|\delta_x \otimes \sigma(h) - \delta_y \otimes \sigma(h)|}{\|\delta_x - \delta_y\|} : \|\sigma\| \leq 1 \right\}
\leq \sup \left\{ \frac{|(x \otimes \sigma - y \otimes \sigma)(h)|}{\|\delta_x \otimes \sigma - y \otimes \sigma\|} : \|\sigma\| \leq 1 \right\}
\leq \sup \left\{ \frac{\|\Phi(h(u))\|}{\|u\|} : u \neq 0 \right\}
= \|\Phi\|.
\]
Then for all $h \in \text{Lip}_d(X,E)$, we obtain
\[\rho_{d,E}(h) = \sup \left\{ \frac{|h(x) - h(y)|_E}{d(x,y)} : x \neq y \right\} \leq \|\Phi_h\|.
\]
Then
\[\|h\| = \max \{\rho_{d,E}(h), \|h\|_{\infty,E} \} \leq \|\Phi_h\|.
\]
Therefore $\Phi$ is isometry, as claimed. 

**Proposition 1.8.** Let $(X,d)$ be a metric space and $E$ be a natural Banach algebra. Then for each $f \in \text{Lip}_d(X,E)$ we have
\[\|f\| = \|\hat{f}\| = \sup \{|h(f)| : h \in E^*, \|h\| \leq 1\} = \sup \{|\phi(f)| : \phi \in \Delta(E)\}.
\]

**Proof.** Since $E$ is natural, $\Delta(E) = \{\delta_t : t \in X\}$. It is clear that,
\[\sup \{|\phi(f)| : \phi \in \Delta(E)\} = \sup \{|\delta_t(f)| : t \in X\} = \|f\|_{\infty} \leq \|f\|.
\]
Moreover,
\[|\Phi(f)| = |\Sigma_k \phi_k(f(x_k))| \leq \|u\| \|f\|_{\text{Lip}_d(X,E)}.
\]
Thus
\[\|\Phi_f\| \leq \|f\|_{\text{Lip}_d(X,E)}.
\]
Since $u = \Sigma_k \delta_x \otimes \phi_k \in W \subseteq (\text{Lip}_d(X,E))^*$, thus
\[\|u\| = \sup \{|\Sigma_k \delta_x \otimes \phi_k(f)| : f \in \text{Lip}_d(X,E), \|f\| \leq 1\} = \sup \{|\Sigma_k \phi_k(f(x_k))| : f \in \text{Lip}_d(X,E), \|f\| \leq 1\}.
\]
Now we show that $P_{d,E}(|f|) \leq \|\Phi_f\|$ and $\|f\|_{\infty,E} \leq \|\Phi_f\|$. Note that
\[\|f\|_{\infty,E} = \sup \{|h(f(s))| : s \in X\},
\]
where
\[\|f(s)|_E = \|f(s)|_{E^*} = \sup \{|h(f(s))| : h \in E^*, \|h\| \leq 1\}.
\]
\[ ||\Phi_f|| = \sup \left\{ \sum_{k=1}^{n} \phi_k(f_{x_k}) : \sum_{k=1}^{n} \delta_{x_k} \otimes \phi_k \leq 1 \right\}. \]

By the hypothesis, for any \( z \in E \),

\[ ||z|| = \sup \{ |\phi(z)| : \phi \in \Delta(E) \}. \]

Thus

\[ ||f(s)|| = \sup \{ |\phi(f(s))| : \phi \in \Delta(E) \} = \sup \{ |(\delta_s \otimes \phi)(f)| : \phi \in \Delta(E) \}. \]

Since

\[ ||\delta_s \otimes \phi|| = ||\delta_s|| ||\phi|| = 1 \]

and for every \( s \in X \)

\[ ||f(s)|| \leq ||\phi(f)||, \]

it follows that

\[ ||f||_{\infty,E} \leq ||\Phi_f||. \]

Furthermore, for all \( s,t \in X \) with \( s \neq t \) we have

\[ \frac{||f(s) - f(t)||_E}{d(s,t)} \leq \frac{||f(s) - f(t)||_E}{||\delta_s - \delta_t||} = \sup \left\{ \frac{|\phi(f(s)) - \phi(f(t))|}{||\delta_s - \delta_t||} : s \neq t \right\} \]

\[ = \sup \left\{ \frac{|(\delta_s \otimes \phi - \delta_t \otimes \phi)(f)|}{||\delta_s - \delta_t||} : s \neq t \right\} \]

\[ \leq \sup \left\{ \frac{||\Phi_f(u)||}{||u||} : u \neq 0 \right\} = ||\Phi_f||. \]

Therefore \( \rho_{d,E}(f) \leq ||\Phi_f|| \) and so \( \Phi \) is isometry. \( \square \)

**Proposition 1.9.** Let \( Y \) be a Hausdorff locally compact space, \((X,d)\) be a metric space and \( E = (C_0(Y), ||.||_{\infty}) \). Suppose that

\[ W = \langle \delta_x \otimes \delta_y : x \in X, y \in Y \rangle. \]

Then

\[ Lip_d(X,E) = \langle \delta_x \otimes \delta_y : x \in X, y \in Y \rangle^* := W^* \]

as a dual Banach algebra.

**Proof.** Define

\[ \Phi : Lip_d(X,E) \to W^* \]

\[ f \mapsto \Phi_f, \]

where

\[ \Phi_f(\Sigma_{k=1}^{n} \lambda_k(\delta_{x_k} \otimes \delta_{y_k})) = \sum_{k=1}^{n} \lambda_k f(x_k)(y_k). \]

It is easily verified that \( \Phi \) is linear, and

\[ ||\Phi_f|| \leq ||f||. \]
We prove the reverse of the inequality. For all \(s, t \in X\) with \(s \neq t\) we have
\[
\frac{||f(s) - f(t)||_E}{d(s, t)} \leq \frac{||f(s) - f(t)||_{\infty}}{||\delta_s - \delta_t||_{Lip_d(X, E)^*}}
\]
\[
= \sup \left\{ \frac{|f(s)(y) - f(t)(y)|}{||\delta_s - \delta_t||_{Lip_d(X, E)^*}} : y \in Y \right\}
\]
\[
\leq \sup \left\{ \frac{||\Phi_f(u)||}{||u||} : u \in W, u \neq 0 \right\}.
\]
Since
\[
|f(s)(y) - f(t)(y)| = |\Phi_f(\delta_s \otimes \delta_y) - \Phi_f(\delta_t \otimes \delta_y)|
\]
\[
= |\Phi_f(\delta_s - \delta_t) \otimes \delta_y)|
\]
\[
\leq |\Phi_f|| ||\delta_s - \delta_t|| |\delta_y|
\]
\[
= |\Phi_f|| ||\delta_s - \delta_t||,
\]
it follows that
\[
(1.2) \quad \frac{|f(s)(y) - f(t)(y)|}{||\delta_s - \delta_t||} \leq ||\Phi_f||.
\]
For all \(s, t \in X\) with \(s \neq t\) we have
\[
||\delta_s - \delta_t|| = \sup \{||\delta_s(f) - \delta_t(f)|| : f \in Lip_d(X, E), ||f||_{d, E} \leq 1\}
\]
\[
= \sup \{||f(s) - f(t)|| : f \in Lip_d(X, E), ||f||_{d, E} \leq 1\}
\]
\[
\leq \sup \{P_{d, E}(f) d(s, t) : f \in Lip_d(X, E), ||f||_{d, E} \leq 1\}
\]
\[
\leq d(s, t).
\]
This inequality together with (1.2) imply that
\[
\frac{||f(s) - f(t)||_E}{d(s, t)} \leq ||\Phi_f||.
\]
Consequently,
\[
\rho_{d, E}(f) \leq ||\Phi_f|| = ||\Phi(f)||.
\]
Moreover, it is easily verified that
\[
||f||_{\infty, E} \leq ||\Phi_f||.
\]
It follows that
\[
\max \{||f||_{\infty, E, \rho_{d, E}(f)} = ||f|| \leq ||\Phi(f)||,
\]
and so \(\Phi\) is isometry. Now we show that \(\Phi\) is surjective. Take \(F \in W^*\) and for every \(x \in X\) and \(y \in Y\), define
\[
f(x)(y) := F(\delta_x \otimes \delta_y).
\]
Then for all \(s, t \in X\) with \(s \neq t\) we have
\[
\frac{||f(s) - f(t)||_E}{d(s, t)} \leq ||F||
\]
and so
\[
\rho_{d, E}(f) \leq ||F||.
\]
Also
\[
||f||_{\infty, E} \leq ||F||,
\]
obviously. Consequently, \(\|f\| \leq \|F\|\) and so \(f \in \text{Lip}_d(X,E)\). It is not hard to see that \(\Phi(f) = F\). Thus \(\Phi\) is surjective. Finally, we show that

\[
\text{Lip}_d(X,E) \cdot W \subseteq W \quad \text{and} \quad W \cdot \text{Lip}_d(X,E) \subseteq W.
\]

Let \(f \in \text{Lip}_d(X,E)\), \(y, z \in Y\) and \(s, t \in X\). For every \(g \in \text{Lip}_d(X,E)\) we have

\[
((\delta_s \otimes \delta_y) \cdot \Phi_f)(g) = \Phi_f(g \cdot (\delta_s \otimes \delta_y)).
\]

But

\[
g \cdot (\delta_s \otimes \delta_y)(f) = (\delta_s \otimes \delta_y)(fg) = (fg(s))(y) = (f(s)g(s))(y) = f(s)(y) \cdot g(s)(y).
\]

Thus

\[
g \cdot (\delta_s \otimes \delta_y)(f) = (\delta_s \otimes \delta_y)(f) \cdot (\delta_s \otimes \delta_y)(g),
\]

and so

\[
g \cdot (\delta_s \otimes \delta_y) = (\delta_s \otimes \delta_y)(g) \cdot (\delta_s \otimes \delta_y).
\]

Thus

\[
[(\delta_s \otimes \delta_y) \cdot \Phi_f](g) = \Phi_f[(\delta_s \otimes \delta_y)(g) \cdot (\delta_s \otimes \delta_y)]
\]

\[
= (\delta_s \otimes \delta_y)(g) \cdot \Phi_f(\delta_s \otimes \delta_y).
\]

Consequently,

\[
(\delta_s \otimes \delta_y) \cdot \Phi_f = \Phi_f(\delta_s \otimes \delta_y) \cdot (\delta_s \otimes \delta_y)
\]

and so

\[
(\delta_s \otimes \delta_y) \cdot \Phi_f = f(s)(y) \cdot (\delta_s \otimes \delta_y) \in W.
\]

Now let \(u = \sum_{k=1}^{n} \lambda_k \delta_{s_k} \otimes \delta_{y_k} \in W\). Then

\[
u \cdot \Phi_f = \sum_{k=1}^{n} \lambda_k f(s_k)(y_k) (\delta_{s_k} \otimes \delta_{y_k}) \in W.
\]

In general, if \(u_n \to u\) in the norm topology, where \(u_n \in (\delta_s \otimes \delta_y)\), then

\[
\|u_n \cdot \Phi_f - u \cdot \Phi_f\| \leq \|u_n - u\| \|\Phi_f\| \to 0.
\]

Therefore

\[
W \cdot \text{Lip}_d(X,E) \subseteq W.
\]

Similarly, one can show

\[
\text{Lip}_d(X,E) \cdot W \subseteq W,
\]

and so the proof is completed. \(\square\)

**Proposition 1.10.** Let \((X,d)\) be a metric space and \(E\) be a dual Banach algebra such that \(E = F^*\). Then

\[
\text{Lip}_d(X,E) \cong (X \otimes F)^*.
\]

**Proof.** Define

\[
\Phi : \text{Lip}_d(X,E) \to V^*
\]

by \(f \mapsto \Phi_f\), where

\[
V = \langle x \otimes z : x \in X, z \in F \rangle,
\]

and for each \(f \in \text{Lip}_d(X,E)\),

\[
\Phi_f(x \otimes z) = f(x)(z).
\]

C_{BSE}(\Delta(A)) AS A DUAL BANACH ALGEBRA 11
Johnson [8] showed that the mapping of $\Phi$ is linear and isometric and 

$$\operatorname{Lip}_d(X, F^*) = V^*,$$

as two Banach spaces. Moreover, for all $f, g \in \operatorname{Lip}_d(X, E)$ we have

$$\Phi_{fg}(x \otimes z) = (fg)(x)(z) = f(x)(z)g(x)(z) = \Phi_f(x \otimes z)\Phi_g(x \otimes z).$$

Thus $\Phi$ is a homomorphism. Finally, we show that

$$\operatorname{Lip}_d(X, E) \cdot V \subseteq V \quad \text{and} \quad V \cdot \operatorname{Lip}_d(X, E) \subseteq V.$$

Let $k \in \operatorname{Lip}_d(X, E)$ and $x \otimes z \in V$. Then for each $h \in \operatorname{Lip}_d(X, E)$ we have

$$((x \otimes z), k)(h) = (x \otimes z)(kh) = (kh(x))(z) = (k(h))(z) = k(x)(z)h(x)(z) = (k(x))(z(h)).$$

It follows that

$$(x \otimes z) \cdot k = (k(x))(z)(x \otimes z).$$

Thus

$$\Phi_{f}((x \otimes z) \cdot k) = \Phi_{f}((k(x))(z)(x \otimes z)) = k(x)(z)\Phi_{f}(x \otimes z) = (x \otimes z)(k)\Phi_{f}(x \otimes z) = (\Phi_{f} \cdot (x \otimes z))(k),$$

which implies that

$$\Phi_{f} \cdot (x \otimes z) = (\Phi_{f}(x \otimes z)) \cdot (x \otimes z) \in V.$$

Similarly, one can prove that

$$(x \otimes z) \cdot \Phi_{f} = (x \otimes z) \cdot (\Phi_{f}(x \otimes z)) \in V,$$

and so the proof is completed.

\[\square\]

Remark 1.11. Proposition 1.10 asserts that if $E$ is a dual Banach algebra, then $\operatorname{Lip}_d(X, E)$ is a dual Banach algebra, as well. But the converse of this statement is not true, by Proposition 1.9.

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