The first Pontryagin class of a quadratic Lie 2-algebroid

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Abstract

In this paper, first we give a detailed study on the structure of a transitive Lie 2-algebroid and describe a transitive Lie 2-algebroid using a morphism from the tangent Lie algebroid $TM$ to a strict Lie 3-algebroid constructed from derivations. Then we introduce the notion of a quadratic Lie 2-algebroid and define its first Pontryagin class, which is a cohomology class in $H^5(M)$.

To a CLWX 2-algebroid, there is a quadratic Lie 2-algebroid naturally. Conversely, we show that the first Pontryagin class of a quadratic Lie 2-algebroid is the obstruction class of the existence of a CLWX-extension. Finally we construct a quadratic Lie 2-algebroid from a trivial principle 2-bundle with a $\Gamma$-connection and show that its first Pontryagin class is trivial.

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Key words: transitive Lie 2-algebroid, quadratic Lie 2-algebroid, the first Pontryagin class, CLWX 2-algebroid, principle 2-bundle, $\Gamma$-connection

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1 Introduction

The notion of a Lie algebroid was introduced by Pradines in 1967, which is a generalization of Lie algebras and tangent bundles. Just as Lie algebras are the infinitesimal objects of Lie groups, Lie algebroids are the infinitesimal objects of Lie groupoids. See [16] for general theory about Lie algebroids. A transitive Lie algebroid (i.e. $\rho$ is surjective) $(A, [\cdot, \cdot], \rho)$ is called a quadratic Lie algebroid if there is a nondegenerate symmetric invariant bilinear form $S$ on $\ker(\rho)$. A connection of a quadratic Lie algebroid is a splitting $\sigma : TM \to A$ of the short exact sequence

$$0 \to \ker(\rho) \to A \xrightarrow{\rho} TM \to 0.$$

The curvature $R \in \text{Hom}(\wedge^2 TM, \ker(\rho))$ is defined by

$$R(X, Y) = \sigma[X, Y] - [\sigma(X), \sigma(Y)]_A, \quad \forall X, Y \in \Gamma(TM).$$

Then the first Pontryagin class of a quadratic Lie algebroid is defined to be the cohomology class $[S(R, R)]$ in $H^4(M)$.

The notion of a Courant algebroid was introduced in [14] in the study of the double of a Lie bialgebroid. An alternative definition was given in [20]. See the review article [12] for more information. Roughly speaking, a Courant algebroid is a vector bundle $E$, whose section space is a Leibniz algebra, together with an anchor map $\rho : E \to TM$ and a nondegenerate symmetric bilinear form $S$, such that some compatibility conditions are satisfied. For a transitive Courant algebroid (i.e. $\rho$ is surjective), $E/\rho^*(T^*M)$ is a quadratic Lie algebroid. Conversely, it is proved in [8, 9] that a quadratic Lie algebroid admits a Courant-extension if and only if its first Pontraygin class is trivial.

Recently, people have paid more attention to higher categorical structures by reasons in both mathematics and physics. A Lie 2-algebra is the categorification of a Lie algebra [3]. The Jacobi identity is replaced by a natural transformation, called the Jacobiator, which also satisfies some coherence laws of its own. If a skew-symmetric bracket is used in the definition of a Courant algebroid, the authors showed that the underlying algebraic structure of a Courant algebroid is a Lie 2-algebra [21]. Along this approach, the author defined split Lie $n$-algebroids in [26] using the language of graded vector bundles. Usually an NQ-manifold of degree $n$ is considered as a Lie $n$-algebroid [27]. The equivalence between the category of split Lie $n$-algebroids and the category of NQ-manifolds of degree $n$ was given in [6].

CLWX 2-algebroids (named after Courant-Liu-Weinstein-Xu) were introduced in [13], which can be viewed as the categorification of Courant algebroids, and one-to-one correspond to symplectic NQ-manifolds of degree 3. See [14] for more applications in 4D topological field theory. Similar as the case of Courant algebroids, we can obtain a quadratic Lie 2-algebroid (called the ample Lie 2-algebroid) from an exact CLWX 2-algebroid by modulo $\rho^*(T^*M)$. Then it is natural to ask the following question:

- Whether every quadratic Lie 2-algebroid admits a CLWX-extension? If not, what is the obstruction?

Motivated by the above question, first we study the structure of a transitive Lie 2-algebroid $(A_{-1} \oplus A_0; \rho, l_1, l_2, l_3)$. It turns out that a transitive Lie 2-algebroid is a nonabelian extension of the tangent Lie algebroid $TM$ by a graded bundle of Lie 2-algebra $A_{-1} \oplus \ker(\rho)$. By choosing a splitting (a connection), we give precise formulas of a transitive Lie 2-algebroid. One important property that we need to stress here is that the corresponding curvatures include a $\ker(\rho)$-valued 2-form.
and an $A_{-1}$-valued 3-form. Furthermore, we construct a strict Lie 3-algebroid using derivations of $A_{-1} \oplus \ker(\rho)$ and characterize a transitive Lie 2-algebroid by a morphism from the tangent Lie algebroid $TM$ to this strict Lie 3-algebroid. Then we go on to give the notion of a quadratic Lie 2-algebroid. For a quadratic Lie 2-algebroid, we construct a 5-form using the aforementioned curvatures. We show that this 5-form is closed and does not depend on the choices of connections. Thus, it gives rise to a characteristic class, which we call the first Pontryagin class of a quadratic Lie 2-algebroid. The first Pontryagin class of a quadratic Lie 2-algebroid plays the same role as the one for a quadratic Lie algebroid, namely, a quadratic Lie 2-algebroid admits a CLWX-extension if and only if its first Pontryagin class is trivial. This is the reason why we call this characteristic class the first Pontryagin class of a quadratic Lie 2-algebroid.

In the past ten years, principle 2-bundles and higher gauge theory are deeply studied in [2, 5, 7, 10, 11, 13, 22, 23, 28, 29, 30]. For a trivial principle Γ-2-bundle with a connection, where Γ is a strict Lie 2-group, we construct a transitive Lie 2-algebroid which can be viewed as its Atiyah algebroid. We further show that if the Lie 2-algebra corresponding to the Lie 2-group Γ is quadratic, then this Lie 2-algebroid is quadratic and its first Pontryagin class is trivial. We will go on to study the infinitesimal of principle 2-bundles in a separate paper.

The paper is organized as follows. In Section 2, we recall Leibniz 2-algebras, Lie n-algebroids and CLWX 2-algebroids. In Section 3, we study transitive Lie 2-algebroids. In Subsection 3.1, we give precise conditions that the structure maps of a transitive Lie 2-algebroid satisfy. In Subsection 3.2, we construct a strict Lie 3-algebroid using derivations and characterize a transitive Lie 2-algebroid by a morphism from the tangent Lie algebroid $TM$ to this strict Lie 3-algebroid. In Section 4, we give the notion of a quadratic Lie 2-algebroid and define its first Pontryagin class, which is a cohomology class in $H^4(M)$. In Section 5, we study exact CLWX 2-algebroids. In Subsection 5.1, we show that skeletal exact CLWX 2-algebroids are classified by the higher analogue of the Ševera class, which is a cohomology class in $H^3(M)$. In Subsection 5.2, we show that a CLWX 2-algebroid gives rise to a quadratic Lie 2-algebroid and conversely, a quadratic Lie 2-algebroid admits a CLWX-extension if and only if its first Pontryagin class is trivial. In Section 6, for a trivial principle 2-bundle with a Γ-connection, we construct a transitive Lie 2-algebroid and show that its first Pontryagin class is trivial.

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2 Preliminaries

2.1 Leibniz 2-algebras

As a model for “Leibniz algebras that satisfy Jacobi identity up to all higher homotopies”, the notion of a strongly homotopy Leibniz algebra, or a $Lod_{\infty}$-algebra was given in [15] by Livernet, which was further studied by Ammar and Poncin in [11]. In [25], the authors introduced the notion of a Leibniz 2-algebra, which is the categorification of a Leibniz algebra, and proved that the category of Leibniz 2-algebras and the category of 2-term $Lod_{\infty}$-algebras are equivalent.

**Definition 2.1.** A Leibniz 2-algebra $V$ consists of the following data:

- a complex of vector spaces $V : V_{-1} \xrightarrow{d} V_0$,
• bilinear maps $l_2 : V_{-i} \times V_{-j} \rightarrow V_{-i-j}$, where $0 \leq i + j \leq 1$,
• a trilinear map $l_3 : V_0 \times V_0 \times V_0 \rightarrow V_{-1}$,
such that for all $w, x, y, z \in V_0$ and $m, n \in V_{-1}$, the following equalities are satisfied:

(a) $dl_2(x, m) = l_2(x, dm)$,
(b) $dl_2(m, x) = l_2(dm, x)$,
(c) $l_2(dm, n) = l_2(m, dn)$,
(d) $dl_3(x, y, z) = l_2(x, l_2(y, z)) - l_2(l_2(x, y), z) - l_2(y, l_2(x, z))$,
(e1) $l_3(x, y, dm) = l_2(x, l_2(y, m)) - l_2(l_2(x, y), m) - l_2(y, l_2(x, m))$,
(e2) $l_3(x, dm, y) = l_2(x, l_2(m, y)) - l_2(l_2(x, m), y) - l_2(m, l_2(x, y))$,
(e3) $l_3(dm, x, y) = l_2(m, l_2(x, y)) - l_2(l_2(m, x), y) - l_2(x, l_2(m, y))$,
(f) the Jacobiator identity:

\[
\begin{align*}
& l_2(w, l_3(x, y, z)) - l_2(x, l_3(w, y, z)) + l_2(y, l_3(w, x, z)) + l_2(l_3(w, x, y), z) \\
& -l_3(l_2(w, x), y, z) - l_3(x, l_2(w, y), z) - l_3(y, l_2(x, w)) \\
& +l_3(w, l_2(x, y), z) + l_3(w, y, l_2(x, z)) - l_3(w, x, l_2(y, z)) = 0.
\end{align*}
\]

We usually denote a Leibniz 2-algebra by $(V_{-1} \oplus V_0; d, l_2, l_3)$, or simply by $V$. In particular, if both $l_2$ and $l_3$ are skew-symmetric, we obtain the notion of a Lie 2-algebra [3].

**Definition 2.2.** Let $V$ and $V'$ be Leibniz 2-algebras, a morphism $\hat{\dagger}$ from $V$ to $V'$ consists of

• linear maps $f_0 : V_0 \rightarrow V'_0$ and $f_1 : V_1 \rightarrow V'_1$ commuting with the differential, i.e.

\[
f_0 \circ d = d' \circ f_1;
\]

• a bilinear map $f_2 : V_0 \times V_0 \rightarrow V'_1$,
such that for all $x, y, z \in L_0$, $m \in L_1$, we have

\[
\begin{align*}
& f_0 l_2(x, y) - l'_0(f_0(x), f_0(y)) = d' f_2(x, y), \\
& f_1 l_2(x, m) - l'_1(f_1(x), f_1(m)) = f_2(dm, x), \\
& f_1 l_2(m, x) - l'_1(f_1(m), f_0(x)) = f_2(dm, x),
\end{align*}
\]

and

\[
\begin{align*}
& -f_1(l_2(x, y, z)) + l'_2(f_0(x), f_2(y, z)) - l'_2(f_0(y), f_2(x, z)) - l'_2(f_2(x, y), f_0(z)) \\
& -f_2(l_2(x, y, z)) + f_2(x, l_2(y, z)) - f_2(y, l_2(x, z)) + l'_2(f_0(x), f_0(y), f_0(z)) = 0.
\end{align*}
\]
2.2 Lie n-algebroids and CLWX 2-algebroids

The notion of a split Lie n-algebroid was introduced in [26].

**Definition 2.3.** A split Lie n-algebroid is a graded vector bundle \( A = A_{-n+1} \oplus \cdots \oplus A_{-1} \oplus A_0 \) over a manifold \( M \) equipped with a bundle map (the anchor) \( \rho : A_0 \to TM \), and brackets \( l_i : \Gamma(A) \to \Gamma(A) \) with degree \( 2 - i \) for \( i = 1, 2, \cdots, l_{n+1} \), such that

1. \((\Gamma(A); l_1, l_2, \cdots, l_{n+1})\) is a Lie n-algebra (n-term \( L_\infty \)-algebra);
2. \( l_2 \) satisfies the Leibniz rule with respect to the anchor \( \rho \):
   \[
   l_2(X^0, fY) = fl_2(X^0, Y) + \rho(X^0)(f)Y, \quad \forall X^0 \in \Gamma(A_0), f \in C^\infty(M), Y \in \Gamma(A);
   \]
3. for \( i \neq 2, l_i \) are \( C^\infty(M) \)-linear.

Denote a Lie n-algebroid by \((A; \rho, l_1, l_2, \cdots, l_{n+1})\).

A split Lie n-algebroid is said to be **strict** if \( l_i = 0 \) for all \( i > 2 \). In this paper, we will only use split Lie 2-algebroids and strict split Lie 3-algebroids. See [6] for more details about the category of Lie n-algebroids.

**Lemma 2.4.** Let \((A; \rho, l_1, l_2, \cdots, l_{n+1})\) be a Lie n-algebroid. Then we have

\[
\rho \circ l_1 = 0, \quad \rho l_2(X^0, Y^0) = [\rho(X^0), \rho(Y^0)], \quad \forall X^0, Y^0 \in \Gamma(A_0).
\]

When \( n = 1 \), a Lie 1-algebroid is exactly a Lie algebroid. Associated to any vector bundle \( E \), the covariant differential operator bundle \( \mathcal{D}(E) \) is a Lie algebroid naturally. Let \((A; \alpha, l_2)\) be a Lie algebroid and \( \nabla : A \to \mathcal{D}(E) \) a bundle map. Then there is a differential operator \( d_\nabla : \Gamma(\text{Hom}(\wedge^k A, E)) \to \Gamma(\text{Hom}(\wedge^{k+1} A, E)) \) defined by

\[
d_\nabla \theta(X_1, \cdots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \nabla X_i \theta(X_1, \cdots, \widehat{X}_i, \cdots, X_{k+1})
+ \sum_{i<j} (-1)^{i+j} \theta(l_2(X_i, X_j), X_1, \cdots, \widehat{X}_i, \cdots, \widehat{X}_j, \cdots, X_{k+1}),
\]

for all \( X_1, \cdots, X_{k+1} \in \Gamma(A) \). \( d_\nabla = 0 \) if and only if \( \nabla \) is a Lie algebroid morphism, i.e. a representation of \( A \) on \( E \). See [16] for more details.

The notion of a CLWX 2-algebroid was introduced in [13] as the categorification of a Courant algebroid [13, 20].

**Definition 2.5.** A CLWX 2-algebroid is a graded vector bundle \( E = E_{-1} \oplus E_0 \) over \( M \) equipped with a nondegenerate symmetric bilinear form \( S \) on \( E \), a bilinear operation \( \circ : \Gamma(E_{-1}) \times \Gamma(E_{-1}) \to \Gamma(E_{-1+i+j}) \), \( 0 \leq i + j \leq 1 \), which is skewsymmetric restricted on \( \Gamma(E_0) \times \Gamma(E_0) \), an \( E_{-1} \)-valued 3-form \( \Omega \) on \( E_0 \), two bundle maps \( \vartheta : E_{-1} \to E_0 \) and \( \rho : E_0 \to TM \), such that \( E_{-1} \) and \( E_0 \) are isotropic and the following conditions are satisfied:

1. \((\Gamma(E_{-1}), \Gamma(E_0), \vartheta, \rho, \Omega)\) is a Leibniz 2-algebra,
2. for all \( e \in \Gamma(E) \), \( e \circ e = \frac{1}{2} DS(e, e) \),
consists of a section $\sigma$ and a bundle map $\gamma$ where $A$ is a Lie 2-algebroid ($A$-3). On the double of a Lie 2-bialgebroid, there is a CLWX 2-algebroid structure naturally. See \ref{11\ref{12}} for more details.

3 Transitive Lie 2-algebroids

In this section, we study the structure of a transitive Lie 2-algebroid. In Subsection 3.1, by choosing a splitting, we obtain some structure maps and we give the conditions that these structure maps satisfy. Then in Subsection 3.2, we characterize these conditions by a morphism from the tangent Lie algebroid $TM$ to a strict Lie 3-algebroid constructed from derivations of a graded bundle of Lie 2-algebras.

3.1 General description of a transitive Lie 2-algebroid

A Lie 2-algebroid $(A_{-1} \oplus A_0; \rho, l_1, l_2, l_3)$ is said to be transitive if the anchor $\rho : A_0 \to TM$ is surjective. Denote by $\mathcal{G} = \ker(\rho)$. By Lemma 2.4 we deduce that $l_2(u, v) \in \Gamma(\mathcal{G})$ for all $u, v \in \Gamma(\mathcal{G})$. Then it is obvious that $(A_{-1} \oplus \mathcal{G}; l_1, l_2, l_3)$ is a graded bundle of Lie 2-algebras, where $l_1 = l_1$, $l_2$ and $l_3$ are restrictions of $l_2$ and $l_3$ respectively.

Definition 3.1. A splitting (connection) of a transitive Lie 2-algebroid $(A_{-1} \oplus A_0; \rho, l_1, l_2, l_3)$ consists of a section $\sigma : TM \to A_0$ of the following short exact sequence of vector bundles:

$$0 \to \mathcal{G} \xrightarrow{i} A_0 \xrightarrow{\rho} TM \to 0,$$

and a bundle map $\gamma : \wedge^2 TM \to A_{-1}$. Here $i$ denotes the inclusion map.

After choosing a splitting, we have $A_0 \cong TM \oplus \mathcal{G}$ and $\rho$ is simply the projection $\text{pr}_{TM}$. Define $\nabla^0 : TM \to \mathcal{D}(\mathcal{G})$ and $\nabla^1 : TM \to \mathcal{D}(A_{-1})$ by

$$\nabla^0_X u = l_2(\sigma(X), u), \quad \nabla^1_X m = l_2(\sigma(X), m), \quad \forall X \in \mathfrak{X}(M), u \in \Gamma(\mathcal{G}), m \in \Gamma(A_{-1}).$$

Define the bundle map $R : \wedge^2 TM \to \mathcal{G}$ by

$$R(X, Y) = \sigma[X, Y] - l_2(\sigma(X), \sigma(Y)) - l_1(\sigma(X), Y), \quad \forall X, Y \in \mathfrak{X}(M).$$

Define the bundle map $I : \wedge^3 TM \to A_{-1}$ by

$$I(X, Y, Z) = -d_{\varphi^* \gamma}(X, Y, Z) - l_3(\sigma(X), \sigma(Y), \sigma(Z)), \quad \forall X, Y, Z \in \mathfrak{X}(M).$$
Define totally skewsymmetric bundle map $J : \wedge^2 TM \otimes \mathcal{G} \rightarrow A_{-1}$ by
\[ J(X, Y, u) = -l_3(\sigma(X), \sigma(Y), u), \quad \forall X, Y \in \mathfrak{X}(M), \quad u \in \Gamma(\mathcal{G}). \] (10)

Define totally skewsymmetric bundle map $K : TM \otimes \wedge^2 \mathcal{G} \rightarrow A_{-1}$ by
\[ K(X, u, v) = l_3(\sigma(X), u, v), \quad \forall X \in \mathfrak{X}(M), \quad u, v \in \Gamma(\mathcal{G}). \] (11)

Transfer the transitive Lie 2-algebroid structure from $A_{-1} \oplus A_0$ to $A_{-1} \oplus (\mathcal{G} \oplus TM)$, for which we use the same notation $(\rho, l_1, l_2, l_3)$, we have
\[
\begin{align*}
\rho &= \text{pr}_{TM}, \\
l_1(m) &= l_1(m), \\
l_2(X + u, Y + v) &= [X, Y] - R_\gamma(X, Y) + \nabla^0_X v - \nabla^0_Y u + l_2(u, v), \\
l_2(X + u, m) &= -l_2(m, X + u) = \nabla^1_X m + l_2(u, m), \\
l_3(X + u, Y + v, Z + w) &= -l_3(X, Y, Z) - J(X, Y, w) - J(X, v, Z) - J(u, Y, Z) + K(X, v, w) + K(u, v, w) + l_3(u, v, w),
\end{align*}
\] (12)

where
\[ R_\gamma = R + l_1 \circ \gamma, \quad I_\gamma = I + d_{\nabla_\gamma}. \]

**Remark 3.2.** $R_\gamma$ and $I_\gamma$ can be viewed as curvatures.

**Theorem 3.3.** Let $(A_{-1} \oplus \mathcal{G}; l_1, l_2, l_3)$ be a graded bundle of Lie 2-algebras. Then $(A_{-1} \oplus (\mathcal{G} \oplus TM); \rho, l_1, l_2, l_3)$ is a transitive Lie 2-algebroid, where $\rho, l_1, l_2, l_3$ are given by (12) for totally skewsymmetric $I, J, K$, if and only if for all $X, Y, Z \in \mathfrak{X}(M)$, $u, v, w \in \Gamma(\mathcal{G})$ and $m \in \Gamma(A_{-1})$, the following equalities hold:

\[
\begin{align*}
I_1 \circ \nabla^1_X &= \nabla^0_X \circ I_1, \quad (13) \\
\nabla^0_X l_2(v, w) - l_2(\nabla^0_X v, w) - l_2(v, \nabla^0_X w) &= I_1 K(X, v, w), \quad (14) \\
\nabla^1_X l_2(v, m) - l_2(\nabla^0_X v, m) - l_2(v, \nabla^1_X m) &= K(X, v, l_1 m), \quad (15) \\
l_2(u, K(v, w, X)) - l_2(v, K(u, w, X)) + l_2(w, K(u, v, X)) &= -K(l_2(u, v, w), X) - K(v, l_2(u, w), X) + K(u, l_2(v, w), X), \\
-\nabla^0_X l_3(u, v, w) + l_3(\nabla^0_X u, v, w) + l_3(v, \nabla^0_X u, w) + l_3(u, v, \nabla^0_X w) &= 0, \quad (16) \\
\nabla^0_X \nabla^0_X w - \nabla^0_X \nabla^0_X w - \nabla^0_{[X,Y]} w + l_2(\tau_1(X, Y), w) + l_1(\tau_1(X, Y), w) &= 0, \quad (17) \\
\n\nabla^1_X \nabla^1_X m - \nabla^1_Y \nabla^1_X m - \nabla^1_{[X,Y]} m + l_2(\tau_1(X, Y), m) + J(X, Y, l_1 m) &= 0, \quad (18) \\
d_{\nabla_\gamma} R_\gamma(X, Y, Z) &= l_1 I_\gamma(X, Y, Z), \quad (19) \\
l_2(u, I_\gamma(X, Y, Z)) - \nabla^1_X J(u, Y, Z) + \nabla^1_Y J(u, X, Z) &= 0, \quad (20) \\
\nabla^1_Z J(u, X, Y) + J(\nabla^0_X u, Y, Z) + J(X, \nabla^0_X u, Z) + J(X, Y, \nabla^0_X u) + J(u, [X, Y], Z) + J(u, [Y, X], Y) + J(u, [Y, Z], X) &= 0, \quad (21) \\
- K(u, Z, R_\gamma(X, Y)) - K(u, Y, R_\gamma(Z, X)) - K(u, X, R_\gamma(Y, Z)) &= 0, \\
d_{\nabla_\gamma} I_\gamma + J \circ R_\gamma &= 0,
\end{align*}
\]

where $J \circ R_\gamma : \wedge^3 \mathfrak{X}(M) \rightarrow \Gamma(A_{-1})$ is given by
\[
J \circ R_\gamma(X_1, \cdots, X_4) = \frac{1}{4} \sum_{\tau \in S_4} \text{sgn}(\tau) J(R_\gamma(X_{\tau(1)}), X_{\tau(2)}), X_{\tau(3)}, X_{\tau(4)}).
\] (22)
Proof. Assume that \((A_1 \oplus (G \oplus TM); \rho, l_1, l_2, l_4)\) is a transitive Lie 2-algebroid. By the equality \(l_1l_2(X, m) = l_2(X, l_1(m))\), we deduce that (13) holds. By the equality
\[
l_2(X, l_2(v, w)) + l_2(v, l_2(w, X)) + l_2(w, l_2(X, v)) = l_1l_3(X, v, w),
\]
we deduce that (14) holds. Similarly, we deduce that (15) holds. By the equality
\[
l_2(u, l_3(v, w, X)) = l_2(v, l_3(u, w, X)) + l_3(l_2(u, v, w)) - l_3(l_2(u, v, w, X)),
\]
we deduce that (16) holds. By the equality
\[
l_2(X, l_2(Y, w)) + l_2(Y, l_2(w, X)) + l_2(w, l_2(X, Y)) = l_1l_3(X, Y, w),
\]
we deduce that (17) holds. Similarly, we deduce that (18) holds. By the equality
\[
l_2(X, l_2(Y, Z)) + l_2(Y, l_2(Z, X)) + l_2(Z, l_2(X, Y)) = l_1l_3(X, Y, Z),
\]
we deduce that (19) holds. Then by the equality
\[
l_2(u, l_3(X, Y, Z)) = l_2(X, l_3(u, Y, Z)) + l_2(Y, l_3(u, X, Z)) + l_2(Z, l_3(u, X, Y)) - l_3(l_2(u, X), Y, Z)
+ l_3(l_2(u, X), Y, Z) - l_3(l_2(u, Z), Y, X) - l_3(l_2(u, X, Y), u, Z) + l_3(l_2(X, Z), u, Y) - l_3(l_2(Y, Z), u, X) = 0,
\]
we deduce that (20) holds. Finally, by the equality
\[
\sum_{i=1}^{4} (-1)^{i+1}l_2(X_i, l_2(X_1, \ldots, \widehat{X_i}, \ldots, X_4)) + \sum_{i<j} (-1)^{i+j}l_3(l_2(X_i, X_j), X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_4) = 0,
\]
we deduce that (21) holds. Conversely, by (16) and (17), we can deduce that
\[
-l_2(u, J(v, Y, Z)) + l_2(v, J(u, Y, Z)) + J(l_2(u, v), Y, Z)
+ \nabla^Y_K(u, v, Z) - \nabla^Z_K(u, v, Y) - K(u, v, [Y, Z]) + l_3(u, v, R_{u}(Y, Z))
+ K(v, \nabla^u_K u, Z) + K(v, Y, \nabla^u_Z u) - K(u, \nabla^y_Z v, Z) - K(u, Y, \nabla^y_Z v) = 0. \tag{23}
\]
Then by (16) - (21) and (23), we can deduce that \((A_1 \oplus (G \oplus TM); \rho, l_1, l_2, l_4)\) is a transitive Lie 2-algebroid. ■

3.2 Transitive Lie 2-algebroids and Lie 3-algebroid morphisms

In this subsection, we give a conceptual explanation of equations listed in Theorem 3.2. We show that they give rise to a morphism from the Lie algebroid \(TM\) to a strict Lie 3-algebroid constructed from derivations of the graded bundle of Lie 2-algebras \((A_1 \oplus G; l_1, l_2, l_3)\).

First we give the definition of a morphism from a Lie algebroid to a strict Lie 3-algebroid. See Section 4.1 for details for general formulas of a morphism between Lie \(n\)-algebroids.

**Definition 3.4.** Let \(A = (A; \rho, l_2)\) be a Lie algebroid and \(A' = (A'_1 \oplus A'_1 \oplus A'_0; \rho', l'_1, l'_2)\) a strict Lie 3-algebroid. A morphism \(F\) from \(A\) to \(A'\) consists of:

- a bundle map \(F^1 : A \to A'_0\),
- a bundle map \(F^2 : \wedge^2 A_0 \to A'_1\),
- a bundle map \(F^3 : \wedge^3 A_0 \to A'_2\).
such that for all $X,Y,Z,X_i \in \Gamma(A)$, $i = 1,2,3,4$, we have

$$\rho' \circ F^1 = \rho,$$

$$F^1 l_2(X,Y) - l_2(F^1(X),F^1(Y)) = l_2' F^2(X,Y),$$

$$l_2'(F^1(X),F^2(Y,Z)) - F^2(l_2(X,Y),Z) + c.p. = l_1' F^3(X,Y,Z),$$

and

$$
\sum_{i=1}^{4} (-1)^{i+1} l_2'(F^1(X_i),F^3(X_1,\ldots,\hat{X_i},\ldots,X_4)) \\
+ \sum_{i<j} (-1)^{i+j} \left( F^3(l_2(X_i, X_j),X_k,X_l) - \frac{1}{2} l_2'(F^2(X_i, X_j),F^2(X_k,X_l)) \right) = 0,
$$

where $k < l$ and $\{k,l\} \cap \{i,j\} = \emptyset$.

Then we construct a strict Lie 3-algebroid using derivations of a graded bundle of Lie 2-algebras $\mathfrak{g} = (\mathcal{G}_{-1} \oplus \mathfrak{g}_0; l_1, l_2, l_3)$. Denote by $\mathcal{D}(\mathfrak{g}_0)$ and $\mathcal{D}(\mathcal{G}_{-1})$ the covariant differential operator bundle associated to $\mathfrak{g}_0$ and $\mathcal{G}_{-1}$ respectively. We have the following exact sequences:

$$
0 \xrightarrow{\mathfrak{gl}(\mathfrak{g}_0)} \mathcal{D}(\mathfrak{g}_0) \xrightarrow{j_0} TM \xrightarrow{} 0,
$$

$$
0 \xrightarrow{\mathfrak{gl}(\mathcal{G}_{-1})} \mathcal{D}(\mathcal{G}_{-1}) \xrightarrow{j_1} TM \xrightarrow{} 0.
$$

**Definition 3.5.** A degree 0 derivation of a graded bundle of Lie 2-algebras $(\mathcal{G}_{-1} \oplus \mathfrak{g}_0; l_1, l_2, l_3)$ is a triple $\mathfrak{d} = (\mathfrak{d}_0, \mathfrak{d}_1, l_0)$, where $\mathfrak{d}_0 \in \Gamma(\mathcal{D}(\mathfrak{g}_0))$, $\mathfrak{d}_1 \in \Gamma(\mathcal{D}(\mathcal{G}_{-1}))$ and $l_0 \in \Gamma(\text{Hom}(\bigwedge^2 \mathfrak{g}_0, \mathcal{G}_{-1}))$, such that we have

$$
\begin{align*}
\mathfrak{d}_0 \circ l_1 &= l_1 \circ \mathfrak{d}_1, \\
\mathfrak{d}_0 l_2(u,v) &= l_2(\mathfrak{d}_0(u),v) + l_2(u,\mathfrak{d}_0(v)) + l_1 l_2(u,v), \\
\mathfrak{d}_1 l_2(u,m) &= l_2(\mathfrak{d}_0(u),m) + l_2(u,\mathfrak{d}_1(m)) + l_0(u,l_1(m)),
\end{align*}
$$

and

$$
l_0(u,l_2(v,w)) + l_2(l_0(u),l_2(v,w)) + l_3(\mathfrak{d}_0(u),v,w) + l_3(u,\mathfrak{d}_0(v),w) + l_3(u,v,\mathfrak{d}_0(w))
= \mathfrak{d}_1 l_3(u,v,w) + l_3(l_2(u,v),w) + l_3(l_2(u),l_2(v),w) + l_3(l_2(u),v,l_2(w)),
$$

for all $u,v,w \in \Gamma(\mathfrak{g}_0)$ and $m \in \Gamma(\mathcal{G}_{-1})$.

It is obvious that the set of degree 0 derivations is the section space of a vector bundle, which we denote by $\text{Der}^0(\mathfrak{g})$. Furthermore, we define $\text{Der}^{-1}(\mathfrak{g}) = \text{Hom}(\mathfrak{g}_0, \mathcal{G}_{-1})$.

**Example 3.6.** For any $u \in \Gamma(\mathfrak{g}_0)$, define $\text{ad}_u^0 : \Gamma(\mathfrak{g}_0) \rightarrow \Gamma(\mathfrak{g}_0)$ and $\text{ad}_u^1 : \Gamma(\mathcal{G}_{-1}) \rightarrow \Gamma(\mathcal{G}_{-1})$ by

$$
\text{ad}_u^0 v = l_2(u,v), \quad \text{ad}_u^1 m = l_2(u,m), \quad \forall v \in \Gamma(\mathfrak{g}_0), m \in \Gamma(\mathcal{G}_{-1}).
$$

Define $l_{\text{ad}_u} : \bigwedge^2 \mathfrak{g}_0 \rightarrow \Gamma(\mathcal{G}_{-1})$ by

$$
l_{\text{ad}_u}(v,w) = l_3(u,v,w), \quad \forall v,w \in \Gamma(\mathfrak{g}_0).
$$

Then it is straightforward to see that $(\text{ad}_u^0, \text{ad}_u^1, l_{\text{ad}_u}) \in \Gamma(\text{Der}^0(\mathfrak{g}))$. 

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Define \( j : \text{Der}^0(\mathcal{G}) \rightarrow TM \) by
\[
j(\partial_0, \partial_1, t_0) := j_0(\partial_0).
\]
On the graded bundle \( \text{Der}(\mathcal{G}) = \text{Der}^0(\mathcal{G}) \oplus \text{Der}^{-1}(\mathcal{G}) \), we define \( \delta : \text{Der}^{-1}(\mathcal{G}) \rightarrow \text{Der}^0(\mathcal{G}) \) and skew-symmetric bracket operation \( [\cdot, \cdot]_{\text{Der}} : \Gamma(\text{Der}^{-1}(\mathcal{G})) \times \Gamma(\text{Der}^{-1}(\mathcal{G})) \rightarrow \Gamma(\text{Der}^{-(i+j)}(\mathcal{G})), 0 \leq i + j \leq 1 \), by
\[
\delta(\phi) = (l_1 \circ \phi \circ l_1, l_{\delta(\phi)}),
\]
\[
[[\partial_0, \partial_1, t_0], (t_0, t_1, t_1)]_{\text{Der}} = ([\partial_0, t_0], [-\partial_1, t_1]),
\]
\[
[[\partial_0, \partial_1, t_0], \phi + u]_{\text{Der}} = [\partial_0, \partial_1, t_0]_{\text{Der}} + l_{\phi} (u, \cdot) + \phi (u),
\]
\[
\phi + u, \psi + v]_{\text{Der}} = \phi (v) + \psi (u),
\]
where \( l_{\delta(\phi)}, l_{[\partial_0, \partial_1]} \in \Gamma(\text{Hom}(\wedge^2 \mathcal{G}_0, \mathcal{G}_1)) \) are defined by
\[
l_{\delta(\phi)}(u, v) = \phi (l_2 (u, v)) - l_2 (\phi (u), v) - l_2 (u, \phi (v)),
\]
\[
l_{[\partial_0, \partial_1]}(u, v) = \partial_1 l_2 (u, v) - l_1 (\partial_0 (u), v) - l_1 (u, \partial_0 (v)) + l_2 (\partial_0 (u), v) + l_2 (u, \partial_0 (v)) - t_1 l_0 (u, v).
\]
It is straightforward to deduce that

**Proposition 3.7.** Let \( \mathcal{G} = (\mathcal{G}_1 \oplus \mathcal{G}_0; t_1, t_2, t_1) \) be a graded bundle of Lie 2-algebras. Then \( (\text{Der}^{-1}(\mathcal{G}) \oplus \text{Der}^{-1}(\mathcal{G}); j, \delta, [\cdot, \cdot]_{\text{Der}}) \) is a strict Lie 2-algebroid.

We go on constructing a strict Lie 3-algebroid. On the 3-term graded vector bundles
\[
\text{DER}(\mathcal{G}) = \mathcal{G}_{-1} \oplus (\text{Der}^{-1}(\mathcal{G}) \oplus \mathcal{G}_0) \oplus \text{Der}^0(\mathcal{G}),
\]
where the degree 0 part \( \text{DER}^0(\mathcal{G}) \) is \( \text{Der}^0(\mathcal{G}) \), the degree \(-1\) part \( \text{DER}^{-1}(\mathcal{G}) \) is \( \text{Der}^{-1}(\mathcal{G}) \oplus \mathcal{G}_0 \) and the degree \(-2\) part \( \text{DER}^{-2}(\mathcal{G}) \) is \( \mathcal{G}_{-1} \), we define \( d : \text{DER}^{-i}(\mathcal{G}) \rightarrow \text{DER}^{-i+1}(\mathcal{G}), i = 1, 2 \) and \( [\cdot, \cdot]_{\text{DER}} : \Gamma(\text{DER}^{-i}(\mathcal{G})) \times \Gamma(\text{DER}^{-j}(\mathcal{G})) \rightarrow \Gamma(\text{DER}^{-(i+j)}(\mathcal{G})), 0 \leq i + j \leq 2 \), by
\[
d(m) = -l_2 (m, \cdot) + l_1 (m),
\]
\[
d(\phi + u) = \delta(\phi) + (\text{ad}^0 \circ \text{ad}^1 \circ l_{\delta(\phi)}),
\]
\[
[[\partial_0, \partial_1, t_0], (t_0, t_1, t_1)_{\text{DER}} = (\partial_0, \partial_1), (t_0, t_1, t_1)],
\]
\[
[[\partial_0, \partial_1, t_0], \phi + u]_{\text{DER}} = [\partial_0, \partial_1, t_0]_{\text{DER}} + l_{\phi} (u, \cdot) + \phi (u),
\]
\[
\phi + u, \psi + v]_{\text{DER}} = \phi (v) + \psi (u),
\]
for all \( (\partial_0, \partial_1, t_0), (t_0, t_1, t_1) \in \Gamma(\text{DER}^0(\mathcal{G})), \phi, \psi \in \Gamma(\text{Der}^{-1}(\mathcal{G})), u, v \in \Gamma(\mathcal{G}_0) \) and \( m \in \Gamma(\mathcal{G}_{-1}) \).

**Theorem 3.8.** Let \( \mathcal{G} = (\mathcal{G}_1 \oplus \mathcal{G}_0; t_1, t_2, t_1) \) be a graded bundle of Lie 2-algebras. Then
\[
(\text{DER}^{-2}(\mathcal{G}) \oplus \text{DER}^{-1}(\mathcal{G}) \oplus \text{DER}^0(\mathcal{G}); j, d, [\cdot, \cdot]_{\text{DER}})
\]
is a strict Lie 3-algebroid.

**Proof.** The proof is straightforward verification. We leave it to readers. \( \blacksquare \)

With above preparations, we go back to Subsection 3.1. For a transitive Lie 2-algebroid \( (\mathcal{A}_1 \oplus \mathcal{A}_0; \rho, t_1, t_2, t_3) \), let \( \mathcal{G} = (\mathcal{A}_1 \oplus \mathcal{G}; t_1, t_2, t_3) \) be the corresponding graded bundle of Lie 2-algebras, where \( \mathcal{G} = \ker(\rho) \), and \( \text{DER}(\mathcal{G}) \) the strict Lie 3-algebroid given above.
Define \( F^1 : TM \to \text{DER}^0(\mathfrak{g}) \) by
\[
F^1(X) = (\nabla^0_X, \nabla^1_X, K(X, \cdot, \cdot)), \quad \forall X \in \mathfrak{x}(M).
\] (27)

By [13], [14], [15], [16], \( F^1 \) is well-defined.

Define \( F^2 : \wedge^2 TM \to \text{DER}^{-1}(\mathfrak{g}) \) by
\[
F^2(X, Y) = J(X, Y, \cdot) + R_\gamma(X, Y), \quad \forall X, Y \in \mathfrak{x}(M).
\] (28)

Define \( F^3 : \wedge^3 TM \to \text{DER}^{-2}(\mathfrak{g}) \) by
\[
F^3(X, Y, Z) = I_\gamma(X, Y, Z), \quad \forall X, Y, Z \in \mathfrak{x}(M).
\] (29)

**Theorem 3.9.** Let \((A_{-1} \oplus A_0; \rho, l_1, l_2, l_3)\) be a transitive Lie 2-algebroid. Then \((F^1, F^2, F^3)\) is a morphism from the Lie algebroid \( TM \) to the strict Lie 3-algebroid \( \text{DER}(\mathfrak{g}) \).

**Proof.** We only give a sketch of the proof and leave details to readers. By [17], [18] and [23], we deduce that
\[
F^1([X, Y]) - [F^1(X), F^1(Y)]_{\text{DER}} = dF^2(X, Y), \quad \forall X, Y \in \mathfrak{x}(M).
\]

By [19] and [20], we deduce that
\[
[F^1(X), F^2(Y, Z)]_{\text{DER}} - F^2([X, Y], Z) + c.p. = dF^3(X, Y, Z).
\]

By [21], we deduce that
\[
\sum_{i=1}^{4} (-1)^{i+1}[F^1(X_i), F^3(X_1, \cdots, \widehat{X}_i, \cdots, X_4)]_{\text{DER}}
\]
\[\quad + \sum_{i<j} (-1)^{i+j} \left( F^3([X_i, X_j], X_k, X_l) - \frac{1}{2}[F^2(X_i, X_j), F^2(X_k, X_l)]_{\text{DER}} \right)
\]
\[\quad = (d_{\nabla^1} I_\gamma + J \circ R_\gamma)(X_1, \cdots, X_4)
\]
\[\quad = 0.
\]

Thus, \((F^1, F^2, F^3)\) is a morphism form \( TM \) to \( \text{DER}(\mathfrak{g}) \).

\[\square\]

4 The first Pontryagin class of a quadratic Lie 2-algebroid

In this section, we give the notion of a quadratic Lie 2-algebroid and define its first Pontryagin class, which is a cohomology class in \( H^3(M) \).

**Definition 4.1.** A transitive Lie 2-algebroid \((A_{-1} \oplus A_0; \rho, l_1, l_2, l_3)\) is said to be a **quadratic Lie 2-algebroid** if there is a degree 1 nondegenerate graded symmetric bilinear form \( S \) on the graded vector bundle \( A_{-1} \oplus \ker(\rho) \) satisfying
\[
S(l_1(m), p) = S(m, l_1(p)), \quad \forall m, p \in \Gamma(A_0) \tag{30}
\]
\[
\rho(e^0) S(u, m) = S(l_2(e^0, u), m) + S(u, l_2(e^0, m)), \quad \forall u, v \in \Gamma(A_0), e^0, e^0_1, e^0_2 \in \Gamma(A_0) \tag{31}
\]
\[
S(l_3(e^0_1, e^0_2, u), v) = -S(l_3(e^0_1, e^0_2, v), u), \quad \forall u, v \in \Gamma(A_{-1}) \tag{32}
\]
Let \((A_1 \oplus A_0; \rho, l_1, l_2, l_3)\) be a quadratic Lie 2-algebroid. As before, denote by \(\mathcal{G} = \ker(\rho)\). By the nondegeneracy of \(\mathcal{S}\), we deduce that \(A_{-1} \cong \mathcal{G}^*\) and we will write \(A_{-1} = \mathcal{G}^*\) directly. Then the pairing \(\mathcal{S}\) is simply given by
\[
\mathcal{S}(u + m, v + p) = (u, p) + (v, m), \quad \forall u, v \in \Gamma(\mathcal{G}), m, p \in \Gamma(\mathcal{G}^*). \tag{33}
\]
By \((30) - (32)\), we deduce that for the graded bundle of Lie 2-algebras \((\mathcal{G}^* \oplus \mathcal{G}; l_1, l_2, l_3)\), there holds:
\[
\langle l_1(m, p) \rangle = \langle m, l_1(p) \rangle, \quad \forall m, p \in \Gamma(\mathcal{G}^*), \tag{34}
\]
\[
\langle l_2(u, v, m) \rangle = -\langle v, l_2(u, m) \rangle, \quad \forall u, v \in \Gamma(\mathcal{G}), m \in \Gamma(\mathcal{G}^*), \tag{35}
\]
\[
\langle l_3(u, v, w, x) \rangle = -\langle w, l_3(u, v, x) \rangle, \quad \forall u, v, w, x \in \Gamma(\mathcal{G}). \tag{36}
\]

**Proposition 4.2.** With the same assumption and conditions in Theorem 3.3, the transitive Lie 2-algebroid \((\mathcal{G}^* \oplus \mathcal{G}; l_1, l_2, l_3)\) is a quadratic Lie 2-algebroid, where \(\rho, l_1, l_2, l_3\) are given by \((12)\) for totally skew-symmetric \(I, J, K\), if and only if \((34) - (36)\) and the following equalities hold:
\[
\langle \nabla_X^0 u, m \rangle + \langle \nabla_X^1 m \rangle = X(u, m), \tag{37}
\]
\[
\langle J(X, Y, u, v) \rangle + \langle J(X, Y, v, u) \rangle = 0, \tag{38}
\]
\[
\langle K(X, u, v), w \rangle + \langle K(X, u, w), v \rangle = 0. \tag{39}
\]

**Proof.** It follows from that the invariant conditions in the definition of a quadratic Lie 2-algebroid is equivalent to \((37) - (39)\). \(\blacksquare\)

Let \((\mathcal{G}^* \oplus \mathcal{G}; l_1, l_2, l_3)\) be a quadratic Lie 2-algebroid given by \((12)\) with \(\mathcal{S}\) the graded symmetric bilinear form given by \((33)\). Define \(S(R_\gamma, I_\gamma) \in \Omega^5(M)\) by
\[
S(R_\gamma, I_\gamma)(X_1, \cdots, X_5) = \frac{1}{12} \sum_{\tau \in S_5} \text{sgn}(\tau)S(R_\gamma(X_{\tau(1)}, X_{\tau(2)}), I_\gamma(X_{\tau(3)}, X_{\tau(4)}, X_{\tau(5)})). \tag{40}
\]

**Theorem 4.3.** The 5-form \(S(R_\gamma, I_\gamma)\) is closed, i.e. \(dS(R_\gamma, I_\gamma) = 0\), and its cohomology class \([S(R_\gamma, I_\gamma)]\) in \(H^5(M)\) does not depend on the choices of \(\sigma\) and \(\gamma\).

The cohomology class \([S(R_\gamma, I_\gamma)]\) in \(H^5(M)\) is called the first Pontryagin class of the quadratic Lie 2-algebroid \((A_{-1} \oplus A_0; \rho, l_1, l_2, l_3)\). To prove the above theorem, we need the following technical lemma.

**Lemma 4.4.** For all \(\Xi \in \Gamma(\text{Hom}(\wedge^k TM, \mathcal{G}))\) and \(\Pi \in \Gamma(\text{Hom}(\wedge^k TM, \mathcal{G}^*))\), we have
\[
dS(\Xi, \Pi) = S(d\varphi_0 \Xi, \Pi) + (-1)^k S(\Xi, d\varphi_1 \Pi), \tag{41}
\]
where \(S(\Xi, \Pi) \in \Omega^{k+1}(M)\) is defined by
\[
S(\Xi, \Pi)(X_1, \cdots, X_{k+1}) = \frac{1}{k!} \sum_{\tau \in S_{k+1}} \text{sgn}(\tau)S(\Xi(X_{\tau(1)}, \cdots, X_{\tau(k)}), \Pi(X_{\tau(k+1)}, \cdots, X_{\tau(k+1)})). \tag{42}
\]

for all \(X_1, \cdots, X_{k+1} \in \mathfrak{X}(M)\).
We define
\[ dS(\Xi, I_\gamma)(X_1, \cdots, X_{k+l+1}) \]
\[ = \sum_{i=1}^{k+l+1} (-1)^{i+1} X_i S(\Xi, I_\gamma)(X_1, \cdots, \hat{X}_i, \cdots, X_{k+l+1}) \]
\[ + \sum_{i<j} (-1)^{i+j} S(\Xi, I_\gamma)([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{k+l+1}). \]

Then by \[ (37) \], we can deduce that \[ (41) \] holds after a careful computation. We omit details. ■

**Proof.** First we have
\[ dS(\Xi, I_\gamma)(X_1, \cdots, X_{k+l+1}) = S(d_{\nabla^0} R_{\gamma}, I_\gamma) + S(R_{\gamma}, d_{\nabla^1} I_\gamma) \]
\[ = S(I_1 I_\gamma, I_\gamma) - S(R_{\gamma}, J \circ R_{\gamma}) = 0. \]

Thus, \[ S(R_{\gamma}, I_\gamma) \] is a closed 5-form.

Furthermore, by Lemma \[ 4.4 \] and \[ 19 \], we have
\[ S(R_{\gamma}, I_\gamma) = S(R, I) + S(R, d_{\nabla^1} I_\gamma) + S(I_1 \circ I_\gamma, I) + S(I_1 \circ I_\gamma, d_{\nabla^1} I_\gamma) \]
\[ = S(R, I) + dS(R, I) + \frac{1}{2} dS(I_1 \circ I_\gamma, I_\gamma), \]

which implies that the cohomology class does not depend on the choices of \( \gamma \).

If we choose another section \( \sigma': TM \rightarrow A_0 \), then we define \( \theta: TM \rightarrow G \) by
\[ \theta(X) = \sigma(X) - \sigma'(X), \quad \forall X \in \mathfrak{X}(M). \]

We have
\[ R'(X, Y) = \sigma'[X, Y] - l_2(\sigma'(X), \sigma'(Y)) - l_1 \circ \gamma(X, Y) \]
\[ = R(X, Y) + d_{\nabla^0} \theta(X, Y) - l_2(\theta(X), \theta(Y)) \]
and
\[ I'(X, Y, Z) = -l_3(\sigma'(X), \sigma'(Y), \sigma'(Z)) - d_{\nabla^1} \gamma(X, Y, Z) \]
\[ = I(X, Y, Z) + l_2(\theta(X), \theta(Y), \theta(Z)) - \left( J(X, Y, \theta(Z)) + K(X, \theta(Y), \theta(Z)) + c.p. \right). \]

We define \( I_2 \circ \theta : \wedge^2 TM \rightarrow G, I_3 \circ \theta : \wedge^3 TM \rightarrow G^*, J \circ \theta : \wedge^3 TM \rightarrow G^* \) and \( K \circ \theta : \wedge^3 TM \rightarrow G^* \) respectively by

\[ I_2 \circ \theta(X, Y) = I_2(\theta(X), \theta(Y)), \]
\[ I_3 \circ \theta(X, Y, Z) = I_3(\theta(X), \theta(Y), \theta(Z)), \]
\[ J \circ \theta(X, Y, Z) = J(\theta(X), Y, Z) + J(X, \theta(Y), Z) + J(X, Y, \theta(Z)), \]
\[ K \circ \theta(X, Y, Z) = K(X, \theta(Y), \theta(Z)) + K(\theta(X), Y, \theta(Z)) + K(\theta(X), \theta(Y), Z). \]
Therefore, by the following Lemma 4.5-Lemma 4.9, we have
\[ S(R', I') = S(R, dC_\gamma - l_2 \circ \theta, I_\gamma - J \circ \theta - K \circ \theta + l_3 \circ \theta) \]
\[ = S(R, I_\gamma) + S(dC_\gamma, I_\gamma) - S(R, J \circ \theta) - S(R, K \circ \theta) + S(R, l_3 \circ \theta) \]
which implies that \( S(R', I') \) and \( S(R, I_\gamma) \) are in the same cohomology class. 

For any \( \mathcal{G} \)-valued 1-form \( \theta \in \Gamma(\text{Hom}(TM, \mathcal{G})) \), we define a 4-form \( C_4^I \in \Omega^4(M) \) by
\[ C_4^I = \frac{1}{2} S(\theta, J \circ \theta). \]

**Lemma 4.5.** With the above notations, we have
\[ dC_4^I = S(dC_\gamma, I_\gamma) - S(R, J \circ \theta). \]

**Proof.** By Lemma 4.4, 4.7 and 4.8, we have
\[ dS(\theta, I_\gamma) = S(dC_\gamma, I_\gamma) - S(\theta, dC_\gamma I_\gamma) \]
\[ = S(dC_\gamma, I_\gamma) + S(\theta, J \circ R, I_\gamma) \]
\[ = S(dC_\gamma, I_\gamma) - S(R, J \circ \theta). \]
The proof is finished. 

For any \( \mathcal{G} \)-valued 1-form \( \theta \in \Gamma(\text{Hom}(TM, \mathcal{G})) \), we define a 4-form \( C_4^K \in \Omega^4(M) \) by
\[ C_4^K = \frac{1}{3} S(\theta, K \circ \theta). \]

**Lemma 4.6.** With the above notations, we have
\[ dC_4^K = S(dC_\gamma, J \circ \theta) + S(R, K \circ \theta) + S(l_2 \circ \theta, I_\gamma). \]

**Proof.** By Lemma 4.4, 4.9, 4.8 and 4.10, we obtain
\[ dC_4^K = \frac{1}{2} dS(\theta, J \circ \theta) \]
\[ = \frac{1}{2} S(dC_\gamma, J \circ \theta) - S(\theta, dC_\gamma J \circ \theta) \]
\[ = \frac{1}{2} S(dC_\gamma, J \circ \theta) + S(dC_\gamma, J \circ \theta) + 2S(R, K \circ \theta) + 2S(l_2 \circ \theta, I_\gamma) \]
The proof is finished. 

For any \( \mathcal{G} \)-valued 1-form \( \theta \in \Gamma(\text{Hom}(TM, \mathcal{G})) \), we define a 4-form \( C_4^K \in \Omega^4(M) \) by
\[ C_4^K = \frac{1}{3} S(\theta, K \circ \theta). \]
Lemma 4.7. With the above notations, we have
\[ dC_4^K = S(\delta\varphi \circ \theta, K \circ \theta) - S(R_{\gamma}, l_3 \circ \theta) - S(l_2 \circ \theta, J \circ \theta). \tag{47} \]

Proof. By Lemma 4.4, (23), (38), (39), (35) and (36), we have
\[
dC_4^K = \frac{1}{3} dS(\theta, K \circ \theta) \\
= \frac{1}{3} \left( S(\delta\varphi \circ \theta, K \circ \theta) - S(\theta, d\varphi_1(K \circ \theta)) \right) \\
= \frac{1}{3} \left( S(\delta\varphi \circ \theta, K \circ \theta) + 2S(d\varphi \circ \theta, K \circ \theta) - 3S(R_{\gamma}, l_3 \circ \theta) - 3S(l_2 \circ \theta, J \circ \theta) \right) \\
= S(\delta\varphi \circ \theta, K \circ \theta) - S(R_{\gamma}, l_3 \circ \theta) - S(l_2 \circ \theta, J \circ \theta).
\]
The proof is finished. ■

For any \( G \)-valued 1-form \( \theta \in \Gamma(\text{Hom}(TM, G)) \), we define a 4-form \( C_{4}^{l_3} \in \Omega^4(M) \) by
\[
C_{4}^{l_3} = \frac{1}{4} S(\theta, l_3 \circ \theta). \tag{48}
\]
Equivalently,
\[ C_{4}^{l_3}(X_1, X_2, X_3, X_4) = S(\theta(X_1), l_3(\theta(X_2), \theta(X_3), \theta(X_4))). \]

Lemma 4.8. With the above notations, we have
\[ dC_4^{l_3} = S(\delta\varphi \circ \theta, l_3 \circ \theta) + S(l_2 \circ \theta, K \circ \theta). \tag{49} \]

Proof. By Lemma 4.4, (16), (36) and (39), we have
\[
dC_4^{l_3} = \frac{1}{4} dS(\theta, l_3 \circ \theta) \\
= \frac{1}{4} \left( S(d\varphi \circ \theta, l_3 \circ \theta) - S(\theta, d\varphi_1 l_3 \circ \theta) \right) \\
= \frac{1}{4} \left( S(d\varphi \circ \theta, l_3 \circ \theta) + 3S(d\varphi \circ \theta, l_3 \circ \theta) + 4S(l_2 \circ \theta, K \circ \theta) \right) \\
= S(d\varphi \circ \theta, l_3 \circ \theta) + S(l_2 \circ \theta, K \circ \theta).
\]
The proof is finished. ■

Lemma 4.9. With the above notations, we have
\[ S(l_2 \circ \theta, l_3 \circ \theta) = 0. \tag{50} \]

Proof. It follows from (35), (36) and the Jacobiator identity for \( l_3 \). ■

5 Exact CLWX 2-algebroids

A CLWX 2-algebroid \((E_{-1}, E_0, \partial, \rho, S, \circ, \Omega)\) is called exact if we have the following exact sequence of vector bundles:
\[
0 \longrightarrow T^*M \xrightarrow{\rho^*} E_{-1} \xrightarrow{\partial} E_0 \xrightarrow{\rho} TM \longrightarrow 0,
\]
where \( \rho^*: T^*M \longrightarrow E_{-1} \) is defined by
\[
S(\rho^*(\alpha), e^0) = \langle \alpha, \rho(e^0) \rangle, \quad \forall \alpha \in \Omega^1(M), e^0 \in \Gamma(E_0).
\]
5.1 Skeletal exact CLWX 2-algebroids

In this subsection, we associate a skeletal exact CLWX 2-algebroid a cohomology class in $H^4(M)$, can call it the higher analogue of the Ševera class.

**Definition 5.1.** An exact CLWX 2-algebroid $(E_-, E_0, \partial, \rho, S, \circ, \Omega)$ is called skeletal if $\partial = 0$.

For a skeletal exact CLWX 2-algebroid $(E_-, E_0, \partial, \rho, S, \circ, \Omega)$, by the exactness, we have

$$E_- \cong T^* M, \quad E_0 \cong TM.$$  

In the sequel, we will write $E_- = T^* M, E_0 = TM$ directly. By definition, $\partial = 0$. $S$ is given by

$$S(X + \alpha, Y + \beta) = \langle \alpha, Y \rangle + \langle \beta, X \rangle, \quad \forall X, Y \in \mathfrak{X}(M), \alpha, \beta \in \Omega^1(M).$$  

(51)

Obviously, we have

$$X \circ Y = [X, Y].$$  

(52)

By Axiom (iv) in Definition 2.5, we have

$$X \langle \alpha, Y \rangle = XS(\alpha, Y) = S(X \circ \alpha, Y) + S(\alpha, X \circ Y) = \langle X \circ \alpha, Y \rangle + \langle \alpha, [X, Y] \rangle,$$

which implies that

$$X \circ \alpha = L_X \alpha.$$  

(53)

Then by Axiom (ii), we deduce that

$$\alpha \circ X = -\iota_X d\alpha.$$  

(54)

Finally, by Axiom (v), $\Omega : \wedge^3 \mathfrak{X}(M) \rightarrow \Omega^3(M)$ gives rise to a 4-form $H \in \Omega^4(M)$ by

$$H(X, Y, Z, W) = S(\Omega(X, Y, Z), W).$$  

(55)

By Axiom (i), $H$ is closed. Summarize the above discussion, we have

**Proposition 5.2.** Any skeletal exact CLWX 2-algebroid must be of the form $(T^*[1] M, TM, \partial = 0, \rho = \text{id}, S, \circ, \Omega)$, where $S, \circ$ are given by (51), (52), (53), (54), and $\Omega$ is equivalent to a closed 4-form $H$.

We will denote a skeletal exact CLWX 2-algebroid simply by $(T^*[1] M, TM, S, \circ, H)$.

**Definition 5.3.** An isomorphism from a CLWX 2-algebroid $(E_-, E_0, \partial, \rho, S, \circ, \Omega)$ to a CLWX 2-algebroid $(E'_-, E'_0, \partial', \rho', S', \circ', \Omega')$ consists of two bundle isomorphisms $\Phi_0 : E_0 \rightarrow E'_0, \Phi_1 : E_- \rightarrow E'_-$ and a bundle morphism $\Phi_2 : \wedge^2 E_0 \rightarrow E'_-$ such that

(i) $\rho' \circ \Phi_0 = \rho$,

(ii) $S(\Phi_0(e^0), \Phi_1(e^1)) = S(e^0, e^1)$, for all $e^0 \in \Gamma(E_0)$ and $e^1 \in \Gamma(E_-)$,

(iii) $\Phi_0, \Phi_1, \Phi_2$ is an isomorphism from the Leibniz 2-algebra $(\Gamma(E_-) \oplus \Gamma(E_0), \partial, \circ, \Omega)$ to the Leibniz 2-algebra $(\Gamma(E'_-) \oplus \Gamma(E'_0), \partial', \circ', \Omega')$. 

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We have seen that different skeletal exact CLWX 2-algebroids are differed by closed 4-forms. Now we show that if two closed 4-forms $H$ and $H'$ are in the same cohomology class, the corresponding skeletal exact CLWX 2-algebroids are isomorphic. Thus, skeletal exact CLWX 2-algebroids are classified by $H^4(M)$. This is a higher analogous result of that exact Courant algebroids are classified by $H^3(M)$.

**Theorem 5.4.** Let $(T^*[1]M, TM, S, \circ, H)$ and $(T^*[1]M, TM, S, \circ, H')$ be two skeletal exact CLWX 2-algebroids. They are isomorphic if and only if $H$ and $H'$ are in the same cohomology class.

**Proof.** If $H$ and $H'$ are in the same cohomology class, we assume that $H = H' + dh$. Then $(\Phi_0 = \text{id}, \Phi_1 = \text{id}, \Phi_2 = h)$ gives the required isomorphism from $(T^*[1]M, TM, S, \circ, H)$ to $(T^*[1]M, TM, S, \circ, H')$. In fact, what we need to show is that $(\Phi_0 = \text{id}, \Phi_1 = \text{id}, \Phi_2 = h)$ is a Leibniz 2-algebra morphism from the Leibniz 2-algebra $(\Omega^1(M), X(M), \partial = 0, \circ, H)$ to $(\Omega^1(M), X(M), \partial = 0, \circ, H')$. The only nontrivial thing is to verify that

$$X \circ h(Y, Z) - Y \circ h(X, Z) + h(X, Y) \circ Z + H'(X, Y, Z) = h([X, Y], Z) + c.p. + H(X, Y, Z),$$

which is equivalent to $dh + H' = H$. ■

It is known that exact Courant algebroids are classified by the Ševera class [24]. Thus, for a skeletal exact CLWX 2-algebroid $(T^*[1]M, TM, S, \circ, H)$, we call the cohomology class $[H] \in H^4(M)$ the higher analogue of the Ševera class.

### 5.2 CLWX-extension of a quadratic Lie 2-algebroid

In this subsection, we show that every exact CLWX 2-algebroid gives rise to a quadratic Lie 2-algebroid and a quadratic Lie 2-algebroid admits a CLWX-extension if and only if its first Pontryagin class vanishes.

Let $(E_{-1}, E_0, \partial, \rho, S, \circ, \Omega)$ be an exact CLWX 2-algebroid. Denote by $F_{-1} = E_{-1}/\rho^*(T^*[1]M)$ and we have the following short exact sequence:

$$0 \rightarrow T^*[1]M \xrightarrow{\rho^*} E_{-1} \xrightarrow{\text{pr}} F_{-1} \rightarrow 0.$$

On the graded vector bundle $F_{-1} \oplus E_0$, define $l_1 : F_{-1} \rightarrow E_0$ by

$$l_1(m) = \partial(\tilde{m}), \quad \forall m \in \Gamma(F_{-1}), \quad \tilde{m} \in \Gamma(E_{-1}) \text{ such that } \text{pr}(\tilde{m}) = m; \quad (56)$$

define $l_2$ by

$$l_2(e^0_1, e^0_2) = e^0_1 \circ e^0_2, \quad l_2(e^0_m, m) = \text{pr}(e^0 \circ \tilde{m}), \quad l_2(m, e^0) = \text{pr}(\tilde{m} \circ e^0); \quad (57)$$

and define $l_3 : \wedge^3 \Gamma(E_0) \rightarrow \Gamma(F_{-1})$ by

$$l_3(e^0_1, e^0_2, e^0_3) = \text{pr}(e^0_1, e^0_2, e^0_3). \quad (58)$$

**Proposition 5.5.** Let $(E_{-1}, E_0, \partial, \rho, S, \circ, \Omega)$ be an exact CLWX 2-algebroid. Then $(F_{-1} \oplus E_0; \rho, l_1, l_2, l_3)$ is a quadratic Lie 2-algebroid, which is called the **ample Lie 2-algebroid**. Consequently, the exact CLWX 2-algebroid $(E_{-1}, E_0, \partial, \rho, S, \circ, \Omega)$ is an extension, called CLWX-extension, of the quadratic Lie 2-algebroid $(F_{-1} \oplus E_0; \rho, l_1, l_2, l_3)$ by $T^*[1]M$, i.e.

$$0 \rightarrow T^*[1]M \xrightarrow{\rho^*} E_{-1} \xrightarrow{\text{pr}} F_{-1} \rightarrow 0 \quad (59)$$

$$0 \rightarrow 0 \rightarrow E_0 \rightarrow E_0 \rightarrow 0.$$
Proof. By the exactness, we can deduce that $l_1$ is well-defined. By [13] Lemma 3.7, we can deduce that $l_2$ is also well-defined. Since the obstruction of the operation $\circ$ being skew-symmetric is given in $\rho^*(T^*[1]M)$, it follows that $l_2$ is skew-symmetric. Since $\partial \circ \rho^* = 0$, we obtain a Lie 2-algebra $(\Gamma(F_{-1}) \oplus \Gamma(E_0); l_1, l_2, l_3)$. Therefore, $(F_{-1} \oplus E_0; \rho, l_1, l_2, l_3)$ is a transitive Lie 2-algebroid. It is obvious that the pairing $S$ vanishes restricting on $\rho^*(T^*[1]M) \oplus \ker(\rho)$. Thus, it induces a nondegenerate graded symmetric pairing $S$ on $F_{-1} \oplus \ker(\rho)$ by

$$S(u, m) = S(u, \tilde{m}), \quad \forall m \in \Gamma(F_{-1}), \quad \tilde{m} \in \Gamma(E_{-1}) \quad \text{such that} \quad \text{pr}(\tilde{m}) = m. \quad (60)$$

It is straightforward to see that invariant conditions in Definition [14] are satisfied. Thus, the transitive Lie 2-algebroid $(F_{-1} \oplus E_0; \rho, l_1, l_2, l_3)$ is a quadratic Lie 2-algebroid. ■

Let $(A_{-1} \oplus A_0; \rho, l_1, l_2, l_3)$ be a quadratic Lie 2-algebroid, in which $l_1$ is injective. Now we use the notations as in Subsection [15]. Denote by $\mathcal{G} = \ker(\rho)$. By the nondegeneracy of $\mathcal{S}$, we have $A_{-1} \cong \mathcal{G}^*$ and we will write $A_{-1} = \mathcal{G}^*[1]$. The transitive Lie 2-algebroid $(\mathcal{G}^*[1] \oplus A_0; \rho, l_1, l_2, l_3)$ fits the following exact sequence:

$$0 \longrightarrow \mathcal{G}^*[1] \longrightarrow \mathcal{G}^*[1] \longrightarrow 0 \longrightarrow 0$$

$$0 \longrightarrow \mathcal{G} \longrightarrow \ker(\rho) \longrightarrow A_0 \longrightarrow TM \longrightarrow 0. \quad (61)$$

We use $(l_1, l_2, l_3)$ to denote the graded bundle of quadratic Lie 2-algebra structures on $\mathcal{G}^*[1] \oplus \mathcal{G}$. Since $l_1$ is injective, $l_1$ is an isomorphism. Choose a splitting $(\sigma, \gamma)$ of the quadratic Lie 2-algebroid $(\mathcal{G}^*[1] \oplus A_0; \rho, l_1, l_2, l_3)$, then $A_0 \cong TM \oplus \mathcal{G}$ and $\rho$ is exactly the projection $\text{pr}_M$. Define $\nabla^0 : \Gamma(TM) \times \Gamma(\mathcal{G}) \longrightarrow \Gamma(\mathcal{G})$, $\nabla^1 : \Gamma(TM) \times \Gamma(\mathcal{G}^*) \longrightarrow \Gamma(\mathcal{G}^*)$ and $R \in \Omega^2(TM, \mathcal{G})$ by [16] and [3] respectively. Define totally skew-symmetric bundle maps $I : \wedge^3 TM \longrightarrow \mathcal{G}^*$, $J : \wedge^2 TM \otimes \mathcal{G} \longrightarrow \mathcal{G}^*$ and $K : TM \otimes \wedge^2 \mathcal{G} \longrightarrow \mathcal{G}^*$ by [9], [11] respectively. Then $I_\gamma, J, K$ induce totally skew-symmetric bundle maps $I_\gamma : \wedge^3 TM \otimes \mathcal{G} \longrightarrow T^*M$, $J^\gamma : TM \otimes \wedge^3 \mathcal{G} \longrightarrow T^*M$ and $K^\gamma : \wedge^3 \mathcal{G} \longrightarrow T^*M$ by

$$\langle I_\gamma(X, Y, u), Z \rangle = -\langle u, I_\gamma(X, Y, Z) \rangle,$$

$$\langle J^\gamma(X, u, v), Y \rangle = -\langle v, J(X, u, Y) \rangle,$$

$$\langle K^\gamma(u, v, w), X \rangle = -\langle w, K(X, u, v) \rangle,$$

for all $X, Y, Z \in \mathfrak{X}(M)$ and $u, v, w \in \Gamma(\mathcal{G})$.

Theorem 5.6. Let $(A_{-1} \oplus A_0; \rho, l_1, l_2, l_3)$ be a quadratic Lie 2-algebroid, in which $l_1$ is injective. Then it admits a CLWX-extension if and only if its first Pontryagin class $[S(R_\gamma, I_{\gamma})] \in H^5(M)$ is trivial.

Proof. Assume that the quadratic Lie 2-algebroid $(A_{-1} \oplus A_0; \rho, l_1, l_2, l_3)$ admits a CLWX-extension. Using the same notations as in Section 4, let $(\mathcal{G}^* \oplus \mathcal{G}; l_1, l_2, l_3)$ be the corresponding graded bundle of Lie 2-algebras. Let $(E_{-1}, E_0, \partial, \rho, S, \circ, \Omega)$ be an exact CLWX 2-algebroid whose ample Lie 2-algebroid is $(A_{-1} \oplus A_0; \rho, l_1, l_2, l_3)$. Then we have $E_0 = A_0$. We go on choosing a section $\lambda$ on $\mathcal{G}^*[1]$ that is orthogonal to $\sigma$, i.e. $S(\lambda(m), \sigma(X)) = 0$ for all $m \in \Gamma(\mathcal{G}^*[1])$ and $X \in \mathfrak{X}(M)$. It turns out that $E_{-1} \cong \rho^*(T^*[1]M) \oplus \lambda(\mathcal{G}^*[1])$. Since $\rho$ is surjective, $\rho^*$ is injective, we deduce that $E_{-1} \cong T^*[1]M \oplus \mathcal{G}^*[1]$ and the nondegenerate bilinear form is exactly given by

$$S(X + u, \alpha + m) = \langle \alpha, X \rangle + \langle m, u \rangle, \quad \forall X \in \mathfrak{X}(M), \alpha \in \Omega^1(M), u \in \Gamma(\mathcal{G}), m \in \Gamma(\mathcal{G}^*). \quad (62)$$

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Define $H \in \Omega^1(M)$ by

$$H(X, Y, Z, W) = S(\sigma(X), \sigma(Y), \sigma(Z), \sigma(W)).$$  \hfill (63)

Transfer the \textit{CLWX} 2-algebroid structure to $(T^*[1]M \oplus \mathcal{G}^*[1]) \oplus (TM \oplus \mathcal{G})$, we have

\begin{align*}
\partial(\alpha + m) &= I_4(m), \\
(X + u) \circ (Y + v) &= [X, Y] + \nabla^0_X v - \nabla^0_Y u - R_\gamma(X, Y) + I_2(u, v), \\
(X + u) \circ (\alpha + m) &= L_X \alpha + \nabla^1_X m + Q(X, m) + I_2(u, m) + P_1(u, m), \\
(\alpha + m) \circ (X + u) &= -\iota_X d\alpha - \nabla^1_X m - Q(X, m) + I_2(m, u) + P_2(m, u), \\
\Omega(X + u, Y + v, Z + w) &= H(X, Y, Z) - P_1^X(X, Y, w) - P_1^Z(u, Z, Z) - I_2^X(u, Y, Z) \\
&\quad - J^X(u, v, w) - J^Y(u, Y, w) - J^Z(u, v, Z) + K^3(u, v, w) \\
&\quad - R_\gamma(X, Y, Z) - J(X, Y, w) - J(Y, Z, Z) - J(u, Y, Z) \\
&\quad + K(X, v, w) + K(u, Y, w) + K(u, v, Z) + I_3(u, v, w), \hfill (64)
\end{align*}

where $Q : \mathfrak{X}(M) \times \Gamma(\mathcal{G}^*) \rightarrow \Omega^1(M)$ is defined by

$$\langle Q(X, m), Y \rangle = \langle m, R_\gamma(X, Y) \rangle,$$

and $P_1 : \Gamma(\mathcal{G}) \times \Gamma(\mathcal{G}^*) \rightarrow \Omega^1(M)$ and $P_2 : \Gamma(\mathcal{G}^*) \times \Gamma(\mathcal{G}) \rightarrow \Omega^1(M)$ are defined by

$$\langle P_1(u, m), X \rangle = \langle m, \nabla^0_X u \rangle, \quad \langle P_2(m, u), X \rangle = \langle u, \nabla^1_X m \rangle.$$

Then by the Jacobiator identity that $\Omega$ should satisfy, for all $W, X, Y, Z \in \mathfrak{X}(M)$, we have

\begin{align*}
W \circ \Omega(X, Y, Z) - X \circ \Omega(W, Y, Z) + Y \circ \Omega(W, X, Z) + \Omega(W, X, Z) &
\quad - \Omega(X, Y, W) - \Omega(W, Z, Y) - \Omega(Y, Z, X) - \Omega(Y, W, Z) + \Omega(Z, X, W) - \Omega(Z, W, X) \\
&= W \circ (H(X, Y, Z) - I(X, Y, Z)) - X \circ (H(W, Y, Z) - I(W, Y, Z)) \\
&\quad + Y \circ (H(W, X, Z) - I(W, X, Z)) + (H(W, X, Y) - I(W, X, Y)) \circ Z \\
&\quad - \Omega([W, X] - R_\gamma(W, X, Y)) + \Omega([W, Y] - R_\gamma(W, Y, X, Z) - \Omega([W, Z] - R_\gamma(W, Z, X, Y) \\
&\quad - \Omega([X, Y] - R_\gamma(X, Y, W, Z)) + \Omega([X, Z] - R_\gamma(X, Z, W, Y) - \Omega(Y, Z) - R_\gamma(Y, Z, W, X) \\
&= L_W H(X, Y, Z) - L_X H(W, Y, Z) + L_Y H(W, X, Z) - \iota_Z d(H(W, X, Y)) \\
&\quad - H([W, X, Y, Z]) + H([W, Y, X, Z]) - H([W, Z, X, Y]) \\
&\quad - H([X, W, Y, Z]) + H([X, Z, W, Y]) - H([Y, Z, W, X]) \\
&\quad - \nabla^1 W I_3(X, Y, Z) - Q(W, I_3(X, Y, Z)) + \nabla^1 X I_3(W, Y, Z) + Q(X, I_3(W, Y, Z)) \\
&\quad - \nabla^1 Y I_3(W, X, Z) - Q(Y, I_3(W, X, Z)) + \nabla^1 Z I_3(W, X, Y) + Q(Z, I_3(W, X, Y)) \\
&\quad + I_3([W, X], Y, Z) - I_3([W, Y], X, Z) + I_3([W, Z], X, Y) \\
&\quad + I_3([X, W], Y, Z) - I_3([X, Z], W, Y) + I_3([Y, Z], W, X) \\
&\quad - I_3^X(R_\gamma(W, X, Y, Z) + I_3^Y(R_\gamma(W, Y, X, Z) - I_3^Z(R_\gamma(W, Z, X, Y) \\
&\quad - I_3^X(R_\gamma(X, Y, W, Z)) + I_3^Y(R_\gamma(X, Z, W, Y) - I_3^Z(R_\gamma(X, Z, W, Y) \\
&\quad - J(R_\gamma(W, X, Y, Z) + J(R_\gamma(W, X, Y, Z) - J(R_\gamma(W, Z, X, Y) \\
&\quad - J(R_\gamma(X, Y, W, Z) + J(R_\gamma(X, Y, W, Z) - J(R_\gamma(X, Y, W, Z) \\
&= dH(W, X, Y, Z, \cdot) + S(R_\gamma, I_3)(W, X, Y, Z, \cdot) - (d\varphi_1 I_3 + J \circ R_\gamma)(W, X, Y, Z) \\
&= 0.
\end{align*}
In particular, we have \( S(R_\gamma, I_\gamma) + dH = 0 \), which implies that the first Pontryagin class \([ S(R_\gamma, I_\gamma) ] \in H^5(M)\) is trivial.

Conversely, if the first Pontryagin class is trivial, then there exists a 4-form \( H \in \Omega^4(M) \) such that

\[
S(R_\gamma, I_\gamma) + dH = 0.
\]

On the graded bundle \((T^*[1]M \oplus G^*[1]) \oplus (TM \oplus G)\), define \( \partial, \phi, \Omega \) by \((\ref{46})\). We are going to show that \(((T^*[1]M \oplus G^*[1]) \oplus (TM \oplus G), \partial, pr_{TM}, S, \phi, \Omega)\) is a CLWX 2-algebroid. First we show that under conditions in Theorem \((\ref{6})\) and by Proposition \((\ref{7})\) and \((\ref{67})\), \((\Omega^4(M) \oplus \Gamma(G^*), \mathcal{X}(M) \oplus \Gamma(G), \partial, \phi, \Omega)\) is a Leibniz 2-algebra. This is the most intrinsic part in the proof. By \((\ref{8})\) and \((\ref{44})\), we can deduce that Axioms (a), (b), (c) in Definition \((\ref{2})\) hold by straightforward computations. Note that the restriction of \( \phi \) on \( \wedge^2 \Gamma(TM) \oplus G\) is the same as the one for the transitive Lie 2-algebroid given in \((\ref{12})\). We leave the proof to readers. This finishes the proof of \((\ref{12})\), we deduce that Axiom (d) in Definition \((\ref{2})\) holds by the fact \( \partial|_{\Omega^4(M)} = 0 \). By straightforward computation, we have

\[
(X + u) \circ ((Y + v) \circ (\alpha + m)) - ((X + u) \circ (Y + v)) \circ (\alpha + m) = 0.
\]

Thus, Axiom \((e_1)\) in Definition \((\ref{2})\) holds if and only if

\[
L_X(Q(Y, m) + P_1(v, m)) + Q(X, \nabla^1_m v + l_2(v, m)) + P_1(u, \nabla^1_m v + l_2(v, m)) \quad \text{for (e2)}
\]

\[
- Q(Y, m) - P_1(\nabla^1_m v - \nabla^1_m v - R_\gamma(X, Y) + l_2(u, m)) \quad \text{for (e3)}
\]

\[
- l_2(Y, u, l_2(Y, v, m)) + l_2(Y, u, l_2(Y, v, m)) - l_2(Y, u, l_2(Y, v, m)) \quad \text{for (e4)}
\]

\[
= - J(X, Y, l_1(m)) + K(v, l_1(m)) + J(u, Y, l_1(m)) + l_1(v, u, l_1(m)).
\]

Similarly, we can show that Axiom \((e_2)\) and \((e_3)\) in Definition \((\ref{2})\) hold.

The last step to show that \(((\Omega^4(M) \oplus \Gamma(G^*)) \oplus (\mathcal{X}(M) \oplus \Gamma(G)); \partial, \phi, \Omega)\) is a Leibniz 2-algebra is to show the Jacobiator identity for \( \Omega \). Roughly speaking, the Jacobiator identity for \( \Omega \) is due to \((\ref{15})\), \((\ref{20})\), \((\ref{21})\), \((\ref{23})\) and \((\ref{67})\). We leave the proof to readers. This finishes the proof of \(((\Omega^4(M) \oplus \Gamma(G^*)) \oplus (\mathcal{X}(M) \oplus \Gamma(G)); \partial, \phi, \Omega)\) being a Leibniz 2-algebra.

By \((\ref{37})\), we have

\[
(X + u) \circ (\alpha + m) + (\alpha + m) \circ (X + u) = L_X \alpha - \iota_X \alpha + P_1(u, m) + P_2(m, u)
\]

\[
d(\langle X, \alpha \rangle + \langle u, m \rangle)
\]

\[
dS(X + u, \alpha + m),
\]
which implies that Axiom (ii) in Definition 2.5 holds.

Finally, by Proposition 4.2, we can deduce that Axioms (iii)-(iv) in Definition 2.5 hold directly. The proof is finished. 

6 The first Pontryagin class of a trivial principle 2-bundle with a Γ-connection

6.1 Strict Lie 2-groups and strict Lie 2-algebras

A group is a monoid where every element has an inverse. A 2-group is a monoidal category where every object has a weak inverse and every morphism has an inverse. Denote the category of smooth manifolds and smooth maps by Diff, a (semistrict) Lie 2-group is a 2-group in DiffCat, where DiffCat is the 2-category consisting of categories, functors, and natural transformations in Diff. For more details, see [3, 4]. Here we only give the definition of a strict Lie 2-group.

Definition 6.1. A strict Lie 2-group is a Lie groupoid $C$ such that

(a) The space of morphisms $C_1$ and the space of objects $C_0$ are Lie groups.

(b) The source and the target $s, t : C_1 \rightarrow C_0$, the identity assigning function $i : C_0 \rightarrow C_1$ and the composition $\circ : C_1 \times_{C_0} C_1 \rightarrow C_1$ are all Lie group morphisms.

In the following we will denote the composition $\circ$ in the Lie groupoid structure by $\cdot^v : C_1 \times_{C_0} C_1 \rightarrow C_1$ and call it the vertical multiplication. Denote the Lie 2-group multiplication by $\cdot^h : C_1 \times C_1 \rightarrow C_1$ and call it the horizontal multiplication.

It is well known that strict Lie 2-groups can be described by crossed modules of Lie groups.

Definition 6.2. A crossed module of Lie groups is a quadruple $(H_1, H_0, \Psi, \Phi)$, which we denote simply by $H$, where $H_1$ and $H_0$ are Lie groups, $\Psi : H_1 \rightarrow H_0$ is a Lie group morphism, and $\Phi : H_0 \times H_1 \rightarrow H_1$ is an action of $H_0$ on $H_1$ preserving the Lie group structure of $H_1$ such that the Lie group morphism $\Psi$ is $H_0$-equivariant:

$$\Phi(\Psi(h)) = g\Psi(h)g^{-1}, \quad \forall \, g \in H_0, \, h \in H_1,$$

(68)

and $\Psi$ satisfies the so called Perffer identity:

$$\Phi_{\Psi(h)}(h') = hh'h^{-1}, \quad \forall \, h, h' \in H_1.$$

(69)

Theorem 6.3. There is a one-to-one correspondence between crossed modules of Lie groups and strict Lie 2-groups.

Roughly speaking, given a crossed module $(H_1, H_0, \Psi, \Phi)$ of Lie groups, there is a strict Lie 2-group for which $C_0 = H_0$ and $C_1 = H_0 \rtimes H_1$, the semidirect product of $H_0$ and $H_1$. In this strict Lie 2-group, the source and target maps $s, t : C_1 \rightarrow C_0$ are given by

$$s(g, h) = g, \quad t(g, h) = t(h) \cdot g,$$

the vertical multiplication $\cdot^v$ is given by:

$$(g', h') \cdot^v (g, h) = (g, h' \cdot h), \quad \text{where} \quad g' = t(h) \cdot g,$$

(70)

the horizontal multiplication $\cdot^h$ is given by

$$(g, h) \cdot^h (g', h') = (g \cdot g', h \cdot \Phi_{\Psi g}h').$$

(71)
Definition 6.4. A crossed module of Lie algebras is a quadruple $(h_1, h_0, \psi, \phi)$, which we denote by $M$, where $h_1$ and $h_0$ are Lie algebras, $\psi : h_1 \to h_0$ is a Lie algebra morphism and $\phi : h_0 \to \text{Der}(h_1)$ is an action of Lie algebra $h_0$ on Lie algebra $h_1$ as a derivation, such that

$$\psi(\phi_u(m)) = [u, \psi(m)]_{h_0}, \quad \phi_{\psi(m)}(p) = [m, p]_{h_1}, \quad \forall u \in h_0, m, p \in h_1.$$ 

Example 6.5. For any Lie algebra $\mathfrak{t}$, the adjoint action $\text{ad}$ is a Lie algebra morphism from $\mathfrak{t}$ to $\text{Der}(\mathfrak{t})$. Then $(\mathfrak{t}, \text{Der}(\mathfrak{t}), \text{ad}, \text{id})$ is a crossed module of Lie algebras.

Theorem 6.6. There is a one-to-one correspondence between 2-term DGLAs (strict Lie 2-algebras) and crossed modules of Lie algebras.

In short, the formula for the correspondence can be given as follows: A 2-term DGLA $(V_1 \oplus V_0; l_1, l_2)$ gives rise to a Lie algebra crossed module with $h_1 = V_1$ and $h_0 = V_0$, where the Lie brackets are given by:

$$[m, p]_{h_1} = l_2(l_1(m), p), \quad [u, v]_{h_0} = l_2(u, v), \quad \forall m, p \in V_1, u, v \in V_0,$$

and $\psi = l_1$, $\phi : h_0 \to \text{Der}(h_1)$ is given by $\phi_u(m) = l_2(u, m)$. The DGLA structure gives the Jacobi identity for $[\cdot, \cdot]_{h_1}$ and $[\cdot, \cdot]_{h_0}$, and various other conditions for crossed modules.

Conversely, a crossed module $(h_0, h_1, \psi, \phi)$ gives rise to a 2-term DGLA with $V_1 = h_1$, $V_0 = h_0$, $l_1 = \psi$, and $l_2$ given by:

$$l_2(u, v) = [u, v]_{h_0}, \quad l_2(u, m) = \phi_u(m), \quad \forall u, v \in h_0, m \in h_1.$$

6.2 The transitive Lie 2-algebroid associated to a trivial principle 2-bundle with a $\Gamma$-connection

First we review the notion of a principle 2-bundle for a strict Lie 2-group $\Gamma$ on the basis of [28, 30]. Let $\Gamma$ be a strict Lie 2-group corresponding to the Lie group crossed module $(H_1, H_0, \Psi, \Phi)$. Let $(h_1, h_0, \psi, \phi)$ be the crossed module of Lie algebras corresponding to $(H_1, H_0, \Psi, \Phi)$, and $(h_1 \oplus h_0; l_1, l_2)$ the associated strict Lie 2-algebra.

Definition 6.7. A principle $\Gamma$-2-bundle over a differential manifold $M$ is a Lie groupoid $P$, a surjective submersion functor $\pi : P \to M_{\text{dis}}$, and a smooth right action $R$ of $\Gamma$ on $P$ that preserves $\pi$, such that the smooth functor

$$(\text{pr}_1, R) : P \times \Gamma \to P \times_M P$$

is a weak equivalence, where $M_{\text{dis}}$ is the Lie groupoid with objects $M$ and only identity morphisms.

It is obvious that $M_{\text{dis}} \times \Gamma$ is a principle $\Gamma$-2-bundle, which is called the trivial principle $\Gamma$-2-bundle.

A $\Gamma$-connection on $M$ is a pair $(A, B)$ consisting of an $h_0$-valued 1-form $A \in \Omega^1(M, h_0)$ and an $h_1$-valued 2-form $B \in \Omega^2(M, h_1)$. The curvature $\text{curv}$ and the fake-curvature $f\text{curv}$ of a $\Gamma$-connection $(A, B)$ are defined by

$$\text{curv}(A, B) \triangleq dB + l_2(A, B) \in \Omega^3(M, h_1),$$

$$f\text{curv}(A, B) \triangleq dA + \frac{1}{2}l_2(A, A) - l_1(B) \in \Omega^2(M, h_0).$$

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Every $\Gamma$-connection can give rise to a connection on the trivial principle $\Gamma$-2-bundle $\widetilde{M}_{dis} \times \Gamma$. See Lemma 5.4.1 for more details.

Given a $\Gamma$-connection $(A, B)$, we can construct a transitive Lie 2-algebroid, which can be viewed as the infinitesimal of the trivial principle $\Gamma$-2-bundle with a connection, as follows: let $A_{-1} = M \times h_1$, $A_0 = TM \oplus (M \times h_0)$, and define $l_1, l_2, l_3$ by

\[
\begin{align*}
    l_1(m) &= I_1(m), \\
    l_2(X + u, Y + v) &= [X, Y] + L_X v + l_2(A(X), v) - L_Y u - l_2(A(Y), u) + \text{curv}(A, B)(X, Y) + l_2(u, v), \\
    l_3(X + u, Y + v, Z + w) &= -\text{curv}(A, B)(X, Y, Z) + l_2(B(X, Y), w) + l_2(B(Y, Z), u) + l_2(B(Z, X), v),
\end{align*}
\]

for all $X, Y, Z \in \mathfrak{X}(M), u, v, w \in \Gamma(M \times h_0)$ and $m \in \Gamma(M \times h_1)$.

**Theorem 6.8.** Let $(A, B)$ be a $\Gamma$-connection on $M$. Then $((M \times h_1) \oplus (TM \oplus (M \times h_0)); \rho = \text{pr}_M, l_1, l_2, l_3)$ is a transitive Lie 2-algebroid, where $l_1$ are given by (72).

**Proof.** Compare to (72), we write

\[
\begin{align*}
    \nabla_X^0 u &= L_X u + l_2(A(X), u), \\
    \nabla_X^1 m &= L_X m + l_2(A(X), m), \\
    R &= -dA - \frac{1}{2} l_2(A, A), \\
    \gamma &= B, \\
    J(X, Y, w) &= -l_2(B(X, Y), w),
\end{align*}
\]

and $I = 0$, $K = 0$, $I_3 = 0$. Now we have

\[
I_\gamma = I + d\varphi_1 \gamma = d\varphi_1 B = dB + l_2(A, B) = \text{curv}(A, B).
\]

Then $((M \times h_1) \oplus (TM \oplus (M \times h_0)); \rho = \text{pr}_M, l_1, l_2, l_3)$ is a transitive Lie 2-algebroid if and only if (13)–(21) hold. Since $l_1$ is $C^\infty(M)$-linear, (13) holds naturally. Since $K = 0$ and $I_3 = 0$, we can deduce that (14)–(16) hold. It is also straightforward to deduce that (17) and (18) hold. By $d l_2(A, A) = 2 l_2(dA, A)$ and the Jacobi identity for $l_2$, we have

\[
\begin{align*}
    \nabla_X^0 R(Y, Z) - R([X, Y], Z) + c.p. \\
    = L_X R(Y, Z) - R([X, Y], Z) + c.p. + l_2(A(X), R(Y, Z)) + c.p. \\
    = dR(X, Y, Z) + l_2(A, R)(X, Y, Z) \\
    = d(-dA - \frac{1}{2} l_2(A, A))(X, Y, Z) + l_2(A, -dA - \frac{1}{2} l_2(A, A))(X, Y, Z) \\
    = 0,
\end{align*}
\]

which implies that (19) holds. It is straightforward to deduce that (20) holds. Finally, by $dl_2(A, \gamma) = l_2(dA, \gamma) - l_2(A, d\gamma)$ and $l_2(I_1(m), p) = l_2(m, l_1(p))$, we have

\[
\begin{align*}
    d\varphi_1 (I + d\varphi_1 \gamma) &= d\varphi_1 d\varphi_1 \gamma \\
    &= d(d\varphi_1 \gamma) + l_2(A, d\varphi_1 \gamma) \\
    &= d(d\gamma + l_2(A, \gamma)) + l_2(A, d\gamma + l_2(A, \gamma)) \\
    &= l_2(dA, \gamma) + l_2(A, l_2(A, \gamma)) \\
    &= I_2(dA, \gamma),
\end{align*}
\]

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and
\[
J \circ R = -i_2(\gamma, R) = -i_2(\gamma, R + i_1 \circ \gamma) = i_2(R, \gamma),
\]
which implies that (21) holds. Thus, \((M \times h_1) \oplus (TM \oplus (M \times h_0)); \rho = pr_T M, i_1, i_2, i_3)\) is a transitive Lie 2-algebroid.

**Remark 6.9.** As proposed by the referee, it is natural to consider the case of nontrivial principle 2-bundles. It is known that 2-bundles are classified by the nonabelian differential cohomology. See [19, 29, 30] for more details. A nonabelian cocycle is represented by an open cover \(\{U_i\}\) of \(M\), together with a collection of smooth maps \(g_{ij} : U_{ij} \rightarrow H_0\) and \(a_{ijk} : U_{ijk} \rightarrow H_1\) such that \(g_{ki} = \Psi(a_{ijk})g_{ij}g_{jk}\) and \(a_{klm} = a_{ijl}a_{jki}\). One can see that \(g_{ij}\) does not satisfy the cocycle condition. Thus, the naive idea of gluing the local standard model given in Theorem 6.8 to obtain a global object does not work. We will study this interesting question in the future.

### 6.3 The first Pontryagin class of the quadratic Lie 2-algebroid associated to a \(\Gamma\)-connection

In this subsection, we show that the first Pontryagin class of the quadratic Lie 2-algebroid associated to an \(\Gamma\)-connection is trivial. First we give an interesting example of quadratic strict Lie 2-algebras.

**Example 6.10.** Let \((g, [\cdot, \cdot]_g, K)\) be a quadratic Lie algebra. That is, \(K\) is a symmetric nondegenerate bilinear form on \(g\). Denote by \(K^2\) the induced map from \(g\) to \(g^*\), i.e.
\[
K^2(u)(v) = K(u, v), \quad \forall u, v \in g.
\]
Then \(K^2\) is an isomorphism. On the graded vector space \(g^*[1] \oplus g\), define \(I_1\) and \(I_2\) by
\[
I_1 = (K^2)^{-1}, \quad I_2(u + \xi, v + \eta) = [u, v]_g + ad^*_\eta \xi - ad^*_\xi \eta, \quad \forall u, v \in g, \xi, \eta \in g^*.\]
Then \((g^*[1] \oplus g; I_1, I_2, S)\) is a quadratic strict Lie 2-algebra, where \(S\) is given by
\[
S(u + \xi, v + \eta) = \langle \xi, v \rangle + \langle \eta, u \rangle.
\]
In fact, the only nontrivial part of proving \((g^*[1] \oplus g; I_1, I_2)\) to be a strict Lie 2-algebra is to show the equality
\[
I_2((K^2)^{-1}(\xi), \eta) = I_2(\xi, (K^2)^{-1}(\eta)), \quad (K^2)^{-1}I_2(u, \eta) = I_2(u, (K^2)^{-1}(\eta)).
\]
Since \(K^2\) is an isomorphism, we can assume that \(\xi = K^2(u)\) and \(\eta = K^2(v)\). Since \(K\) is invariant, we have
\[
\langle I_2((K^2)^{-1}(\xi), \eta) - I_2(\xi, (K^2)^{-1}(\eta)), w \rangle = \langle I_2(u, K^2(v)) - I_2(K^2(u), v), w \rangle = \langle ad^*_\eta K^2(v) + ad^*_\xi K^2(u), w \rangle = -K(v, [u, w]_g) - K(u, [v, w]_g) = 0.
\]
Similarly, for any \(\gamma = K^2(w)\), we have
\[
\langle (K^2)^{-1}I_2(u, \eta) - I_2(u, (K^2)^{-1}(\eta)), \gamma \rangle = \langle I_2(u, \eta), w \rangle - \langle I_2(u, v), K^2(w) \rangle = -K(v, [u, w]_g) - K(u, [v, w]_g) = 0.
\]
Thus, \((g^*|1 \oplus g; l_1, l_2, S)\) is a strict Lie 2-algebra. It is obvious that \(S\) is invariant. Therefore, \((g^*|1 \oplus g; l_1, l_2, S)\) is a quadratic strict Lie 2-algebra.

Now let \(\Gamma\) be a strict Lie 2-group such that the corresponding strict Lie 2-algebra is a quadratic strict Lie 2-algebra \((h^*|1 \oplus h; l_1, l_2, S)\). Let \((A, B)\) be a \(\Gamma\)-connection on \(M\). Then the transitive Lie 2-algebroid given in Theorem 6.8 is a quadratic Lie 2-algebroid naturally. Consider its first Pontryagin class, we have

\textbf{Theorem 6.11.} Let \(\Gamma\) be a strict Lie 2-group such that the corresponding strict Lie 2-algebra is a quadratic strict Lie 2-algebra, and \((A, B)\) a \(\Gamma\)-connection on \(M\). Then the first Pontryagin class associated to the quadratic Lie 2-algebroid given in Theorem 6.8, which is represented by the 5-cocycle \(-S(f_{\text{curv}}(A, B), \text{curv}(A, B))\), is trivial.

\textbf{Proof.} First we have

\begin{align*}
S(f_{\text{curv}}(A, B), \text{curv}(A, B)) &= S(dA + \frac{1}{2}l_2(A, A) - l_1(B), dB + l_2(A, B)) \\
&= S(dA, dB) + S(dA, l_2(A, B)) + S(l_2(A, A), dB) \\
&+ S(l_2(A, A), l_2(A, B)) - S(l_1(B), dB) - S(l_1(B), l_2(A, B)).
\end{align*}

By the Jacobi identity that \(l_2\) satisfies and (35), we have

\begin{align*}
S(l_2(A, A), l_2(A, B)) &= -S(\frac{1}{6}l_2(A, l_2(A, A)), B) = 0.
\end{align*}

By (35) and the property that \(l_2\) being skew-symmetric, we have

\begin{align*}
S(l_1(B), l_2(A, B)) &= -S(A, l_2(l_1(B), B)) = 0.
\end{align*}

It is not hard to see that \(d l_1(B) = l_1 dB\). Thus, by (34), we have

\begin{align*}
S(l_1(B), dB) &= \frac{1}{2} dS(l_1(B), B).
\end{align*}

Finally by (35), we have

\begin{align*}
dS\left(\frac{1}{2}l_2(A, A), B\right) &= S\left(\frac{1}{2}d l_2(A, A), B\right) + S\left(\frac{1}{2}l_2(A, A), dB\right) \\
&= S(l_2(dA, A), B) + S\left(\frac{1}{2}l_2(A, A), dB\right) \\
&= S\left(dA, l_2(A, B)\right) + S\left(\frac{1}{2}l_2(A, A), dB\right).
\end{align*}

Therefore, we have

\begin{align*}
S(f_{\text{curv}}(A, B), \text{curv}(A, B)) &= d\left(S(A, dB) + S\left(\frac{1}{2}l_2(A, A), B\right) - \frac{1}{2} S(l_1(B), B)\right),
\end{align*}

which implies that the first Pontryagin class is trivial.

\[\square\]
At the end of this subsection, we analyze how does the primitive form of the first Pontryagin class behave under the gauge transformation. Recall from [28, Section 5.4] that a gauge transformation between \( \Gamma \)-connections \((A, B)\) and \((A', B')\) on \(M\) is a pair \((g, \phi)\) consisting of a smooth map \(g : M \rightarrow H_0\) and a 1-form \(\phi \in \Omega^1(M, h_1)\) such that
\[
A' = \text{Ad}_g A - g^*\bar{\theta} - l_1(\phi), \\
B' = (\Phi_g)_* B - d\phi + \frac{1}{2} l_2(l_1(\phi), \phi) - l_2(\text{Ad}_g A, \phi) + l_2(g^*\bar{\theta}, \phi),
\]
where \(\bar{\theta}\) is the Maurer-Cartan 1-form on the Lie group \(H_0\).

**Proposition 6.12.** Let \((A, B)\) and \((A', B')\) be two gauge equivalent \(\Gamma\)-connections on \(M\). Then we have
\[
S(\text{fcurv}(A', B'), \text{curv}(A', B')) = S(\text{fcurv}(A, B), \text{curv}(A, B)).
\]  

**Proof.** By the Maurer-Cartan equation that \(\bar{\theta}\) satisfies, we have
\[
\text{fcurv}(A', B') = dA' + \frac{1}{2} l_2(A', A') - l_1(B') \\
= d(\text{Ad}_g A - g^*\bar{\theta} - l_1(\phi)) + \frac{1}{2} l_2(\text{Ad}_g A - g^*\bar{\theta} - l_1(\phi), \text{Ad}_g A - g^*\bar{\theta} - l_1(\phi)) \\\n- l_1((\Phi_g)_* B - d\phi + \frac{1}{2} l_2(l_1(\phi), \phi) - l_2(\text{Ad}_g A, \phi) + l_2(g^*\bar{\theta}, \phi)) \\\n= \text{Ad}_g dA + l_2(g^*\bar{\theta}, \text{Ad}_g A) - d\phi^*\bar{\theta} - l_1(d\phi) \\\n+ \frac{1}{2} \text{Ad}_g l_2(A, A) - l_2(\text{Ad}_g A, g^*\bar{\theta}) - l_2(\text{Ad}_g A, l_1(\phi)) \\\n+ \frac{1}{2} l_2(g^*\bar{\theta}, g^*\bar{\theta}) + \frac{1}{2} l_2(l_1(\phi), l_1(\phi)) + l_2(g^*\bar{\theta}, l_1(\phi)) \\\n- \text{Ad}_g l_1(\phi) + l_1(d\phi) + \frac{1}{2} l_2(l_1(\phi), l_1(\phi)) + l_2(\text{Ad}_g A, l_1(\phi) - l_2(g^*\bar{\theta}, l_1(\phi)) \\\n= \text{Ad}_g dA + \frac{1}{2} l_2(A, A) - l_1(B)) \\\n= \text{Ad}_g \text{fcurv}(A, B).
\]

Similarly, by a tedious computation, we obtain
\[
\text{curv}(A', B') = (\Phi_g)_* \text{curv}(A, B) - l_2(\text{Ad}_g \text{fcurv}(A, B), \phi).
\]

Therefore, by the invariance condition that \(S\) satisfies, we have
\[
S(\text{fcurv}(A', B'), \text{curv}(A', B')) = S(\text{Ad}_g \text{fcurv}(A, B), (\Phi_g)_* \text{curv}(A, B) - l_2(\text{Ad}_g \text{fcurv}(A, B), \phi)) \\\n= S(\text{fcurv}(A, B), \text{curv}(A, B)) + S(\text{Ad}_g l_2(\text{fcurv}(A, B), \text{fcurv}(A, B)), \phi) \\\n= S(\text{fcurv}(A, B), \text{curv}(A, B)).
\]

The proof is finished. \(\blacksquare\)

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