Deterministic Algorithms for the Hidden Subgroup Problem

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Abstract

We consider deterministic algorithms for the well-known hidden subgroup problem (HSP): for a finite group $G$ and a finite set $X$, given a function $f: G \rightarrow X$ and the promise that for any $g_1, g_2 \in G$, $f(g_1) = f(g_2)$ iff $g_1H = g_2H$ for a subgroup $H \leq G$, the goal of the decision version is to determine whether $H$ is trivial or not, and the goal of the identification version is to identify $H$. An algorithm for the problem should query $f(g)$ for $g \in G$ at least as possible.

Nayak \cite{25} asked whether there exist deterministic algorithms with $O\left(\sqrt{|G|/|H|}\right)$ query complexity for HSP. We answer this problem by proving the following results, which also extend the main results of Ref. \cite{30}, since here the algorithms do not rely on any prior knowledge of $H$.

(i) When $G$ is a general finite Abelian group, there exist an algorithm with $O\left(\sqrt{|G|/|H|}\right)$ queries to decide the triviality of $H$ and an algorithm to identify $H$ with $O\left(\sqrt{|G|/|H|} \log |H| + \log |H|\right)$ queries.

(ii) In general there is no deterministic algorithm for the identification version of HSP with query complexity of $O\left(\sqrt{|G|/|H|}\right)$, since there exists an instance of HSP that needs $\omega\left(\sqrt{|G|/|H|}\right)$ queries to identify $H$. On the other hand, there exist instances of HSP with query complexity far smaller than $O\left(\sqrt{|G|/|H|}\right)$, whose query complexity is $O\left(\log |G|/|H|\right)$ and even $O(1)$.

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\textsuperscript{1}$f(x)$ is said to be $\omega(g(x))$ if for every positive constant $C$, there exists a positive constant $N$ such that for $x > N$, $f(x) \geq C \cdot g(x)$, which means $g$ is a strict lower bound for $f$. 

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1. Introduction

1.1. Background

The hidden subgroup problem plays an important role in the history of quantum computing. Several important quantum algorithms such as Deutsch-Jozsa algorithm [10], Simon algorithm [29], and Shor algorithm [28] have a uniform description in the framework of the hidden subgroup problem [19]. Moreover, many quantum algorithms were proposed for the instances of the hidden subgroup problem, e.g., [1, 6, 12, 14, 17, 20, 22].

The hidden subgroup problem is formally defined as follows.

**Definition 1.** [The hidden subgroup problem]

**Given:** A finite group $G$; an (unknown) function $f : G \rightarrow X$, where $X$ is a finite set.

**Promise:** There exists a subgroup $H$ such that for any $g_1, g_2 \in G$, $f(g_1) = f(g_2)$ iff $g_1H = g_2H$.

**Problem:** Identify $H$.

The hidden subgroup problem consists of the Abelian hidden subgroup problem and the non-Abelian hidden subgroup problem according to whether $G$ is commutative or not. Many problems are special cases of the Abelian hidden subgroup problem, such as Simon’s problem [29], generalized Simon’s problem [30] and some important number-theoretic problems [16, 13, 27]. The non-Abelian hidden subgroup problem also received much attention [14, 18, 22, 13, 17, 23, 26]. While there exist efficient quantum algorithms to solve the Abelian hidden subgroup problem [29, 12, 21, 3, 24, 11], many instances of the non-Abelian hidden subgroup problem are not known to have efficient quantum algorithms, such as the dihedral hidden subgroup problem and the symmetric hidden subgroup problem [22, 1].

In this paper, we focus on deterministic algorithms for the hidden subgroup problem. Deterministic and randomized algorithms are called classical algorithms. A deterministic algorithm for the above problem usually computes $f(g)$
for enough amount of $g \in G$ to find $H$ (or check whether $H$ is trivial or not). The number of $g \in G$ computed for $f$ is called the query complexity of the algorithm. For example, a naive algorithm with query complexity $O(|G|)$ is to compute $f(g)$ for all $g \in G$. A nontrivial problem worthy of consideration is: how difficult is this problem in terms of query complexity and how to find an optimal query algorithm for it. There exist a lot of discussions about the classical query complexity for some instances of the hidden subgroup problem. For example, the classical query complexity of Simon’s problem was proven to be $\Theta(\sqrt{2^n})$ \cite{29, 4, 8}. Ye et al. \cite{30} proved that a nearly optimal bound for the classical query complexity of generalized Simon’s problem (GSP). For the order-finding problem over $\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n}$, Cleve \cite{7} proved that the deterministic query complexity is at least $\Omega\left(\sqrt{2^n m}\right)$, and the randomized query complexity is at least $\Omega\left(\frac{2^{n/3}}{\sqrt{m}}\right)$. Kuperberg \cite{22} proved the classical query complexity of the dihedral hidden subgroup problem over the dihedral group $D_n$ is $\Omega(\sqrt{N})$. Childs \cite{5} showed that a classical algorithm must make $\Omega(\sqrt{N})$ queries if there are $N$ candidate subgroups whose only common element is the identity element.

Recently, Nayak \cite{25} considered deterministic algorithms for the hidden subgroup problem. He gave a deterministic algorithm with $O(\sqrt{|G|})$ queries to identify $H$ for the Abelian hidden subgroup problem, and showed there exists a deterministic algorithm with query complexity $O(\sqrt{|G| \ln |G|})$ for the general hidden subgroup problem. Finally, he proposed an open problem: is there a deterministic algorithm to the hidden subgroup problem with query complexity $O(\sqrt{|G|})$?

1.2. Our contributions

In this paper, we will answer the above open problem. In the following, the hidden subgroup problem will be abbreviated to HSP. The one in Definition 1 will be called the identification version of HSP. We will also consider its decision version where the goal is to decide whether the hidden subgroup $H$ is trivial or not, instead of finding $H$.

Our main results are the following theorems and corollaries, which not only answer the open problem proposed by Nayak \cite{25}, but also extend the main results of Ref. \cite{30}. Theorem 1 tells that there exist instances of HSP with query complexity far smaller than $O(\sqrt{|G|})$. Theorem 2 shows that there exist deterministic algorithms with query complexity $O(\sqrt{|G|})$ for the decision version of the Abelian hidden subgroup problem and deterministic algorithms with query complexity not far from $O(\sqrt{|G|})$ for the identification version. Theorem 3 and
its corollary indicate that in general there is no deterministic algorithm with query complexity $O(\sqrt{|G|/|H|})$ for the identification version of HSP. In addition, Corollary 1 extends the main results of Ref. [30], since the algorithm in Ref. [30] requires to know in advance the rank of the hidden group $H$, whereas the algorithms in this paper do not rely on any prior knowledge of $H$.

**Theorem 1.** For $G = \mathbb{Z}_{p^k}$, where $p$ is a prime, there exist a deterministic algorithm using 2 queries to solve the decision version of HSP, and a deterministic algorithm using $O(\log |G|/|H|)$ queries to solve the identification version.

**Theorem 2.** There exist a deterministic algorithm using at most $O(\sqrt{|G|/|H|})$ queries to solve the decision version of HSP over a general finite Abelian group $G$, and a deterministic algorithm using at most $O(\sqrt{|G|/|H|} \log |H| + \log |H|)$ queries to solve the identification version.

By Theorem 2, we have the following corollary.

**Corollary 1.** For $G = \mathbb{Z}_p^n$, where $p$ is a prime, there exist a deterministic algorithm using at most $O(\sqrt{p^n - k})$ queries to decide whether $H$ is trivial or not, and a deterministic algorithm using at most $O(\max\{k, \sqrt{k \cdot p^n - k}\})$ queries to identify $H$, where $k = \log_p |H|$.

**Proof.** First, $k = \log_p |H|$ is equivalent to $|H| = p^k$. Then substituting $|G| = p^n$ and $|H| = p^k$ into Theorem 2 we have $O(\sqrt{|G|/|H|}) = O(\sqrt{p^n - k})$ and

$$O(\sqrt{|G|/|H|} \log |H| + \log |H|) = O(\sqrt{p^n - k} \cdot k + k) = O(\max\{k, \sqrt{k \cdot p^n - k}\}),$$

where the last equation follows from the fact that for $a, b \geq 0$, since $\frac{a+b}{2} \leq \frac{a^2 + b^2}{2} + \frac{|a-b|}{2} = \max\{a, b\} \leq a + b$, we have $\Theta(\max\{a, b\}) = \Theta(a + b)$. \qed

Note that Ref. [30] also gave an identification algorithm with the same query complexity as Corollary 1. The difference between them is that the algorithm in Ref. [30] requires that $k$ is known in advance, whereas our algorithms do not need to know $k$ initially. Thus, Corollary 1 is a stronger conclusion and extend Theorem 3 of Ref. [30].

**Theorem 3.** Any deterministic algorithm needs $\Omega(\frac{\log |H|}{\log |H'|})$ queries to solve the identification version of HSP, where $H = \{H' \leq G | H'| = |H|\}$. 
It is easy to get the following corollary from Theorem 3. It was also implied by Lemma 2 of [30].

**Corollary 2.** Any deterministic algorithm needs $\Omega(\log |H|)$ queries to solve the identification version of HSP over $G = \mathbb{Z}_p^n$.

**Proof.** For $G = \mathbb{Z}_p^n$, suppose $H$ is a subgroup of rank $k$ in $G$. Then $|H| = p^k$. By Ref. [30],

$$|H| = k - 1 \prod_{j=0}^{k-1} p^j - p^{(n-k)k} > p^{(n-k)k}.$$ 

Thus, by Theorem 2, a lower bound on the query complexity is

$$\Omega \left( \frac{\log |H|}{\log |H|/|G|} \right) = \Omega \left( \frac{\log p^{(n-k)k}}{\log p^{n-k}} \right) = \Omega (k) = \Omega (\log |H|).$$

Remark 1. If $G = \mathbb{Z}_p^n$ and $|H| = p^{n-1}$, then any deterministic algorithm needs $\Omega(n)$ queries to identify $H$ by Corollary 3. Note that in this case we have $\sqrt{|G|/|H|} = O(1)$. Therefore, any deterministic algorithm for the problem requires $\omega(\sqrt{|G|/|H|})$ queries to identify $H$.

1.3. Organization

The remainder of the paper is organized as follows. In Section 2, we review some notations used in this paper. In Section 3, two deterministic algorithms are presented for solving HSP over $G = \mathbb{Z}_p^n$. In Section 4, algorithms are designed to solve HSP over general finite Abelian groups. In Section 5, a general lower bound is given for the query complexity of the identification version of HSP. Finally, a conclusion is made in Section 6.

2. Preliminary

In this section, we present some notations used in this paper. Let $[m] = \{1, 2, \ldots, m\}$ and $\mathbb{Z}_{p^k}$ denote the additive group of elements $\{0, 1, \ldots, p^k - 1\}$ with addition modulo $p^k$. For two groups $G_1, G_2$, let $G_1 \times G_2$ denote the direct product of $G_1$ and $G_2$. For an arbitrary finite Abelian group $G$, we have $G \cong \{0\} \times \mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_l^{k_l}}$ for $a, b \in G$, suppose $a = (0, a_1, \ldots, a_l)$, $b = (0, b_1, \ldots, b_l)$.

\[\text{Generally, we say } G \cong \mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_l^{k_l}}. \text{ In this paper, we add } \{0\} \text{ for the simplicity of algorithms discription.}\]
Then we define the group operation "+" as follows:
\[ a + b = (0, a_1 + b_1 \mod p_1^{k_1}, \ldots, a_l + b_l \mod p_l^{k_l}). \]
Correspondingly, we define
\[ a - b = (0, a_1 - b_1 \mod p_1^{k_1}, \ldots, a_l - b_l \mod p_l^{k_l}). \]
For \( W_1, W_2 \subseteq G \), we have
\[ W_1 + W_2 = \{ a + b | a \in W_1, b \in W_2 \}, \]
\[ W_1 - W_2 = \{ a - b | a \in W_1, b \in W_2 \}. \]

\( W_1, W_2 \) is called a \textit{generating pair} of \( V \) if \( V = W_1 - W_2 \). \( W_1, W_2 \) is called to have a collision if there exist \( x \in W_1, y \in W_2 \) such that \( f(x) = f(y) \).

We denote that \( H \) is a subgroup of \( G \) by \( H \leq G \). For \( H \leq G \), a subset \( S \) is said to be a generating set for \( H \) if all elements in \( H \) can be expressed as the finite sum of elements in \( S \) and their inverses, i.e., \( H = \langle S \rangle = \{ l_1a_1 + l_2a_2 + \cdots + l_na_n | a_i \in S, l_i = \pm 1, n \in \mathbb{N} \} \). For \( g \in G \), we have \( \langle g \rangle = \{ g + \cdots + g | n \in \mathbb{Z} \} \) and \( gH = \{ g + h | h \in H \} \). If \( g \) is a nonzero element, we let \( I_g \) denote the maximum nonzero coordinate. For example, if \( g = (0, g_1, \ldots, g_i, 0, \ldots, 0) \) and \( g_i \neq 0 \), then \( I_g = i \). If \( g = 0 \), then we let \( I_g = 0 \). For \( g_1, g_2 \in G \), \( g_1H = g_2H \) is equivalent to \( g_1 - g_2 \in H \). The group with only one element, the identity element, is called the \textit{trivial group}.

Additionally, we use \( W_1 \times \cdots \times W_i \) to represent \( W_1 \times \cdots \times W_i \times \{0\} \times \cdots \times \{0\}^{l-1+1} \) for simplicity. In the description and analysis of Algorithm 3 and 4, let \( G_i = \{0\} \times \mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_i^{k_i}}, H_i = \{ h \in H | I_h \leq i \} \).

3. Algorithms for HSP over \( G = \mathbb{Z}_{p^k} \)

In this section, we propose algorithms to solve the hidden subgroup problem over \( G = \mathbb{Z}_{p^k} \) (\( p \) is a prime), which establishes Theorem 1.

3.1. Decision version

The algorithm of the decision version is as follows: Query \( f(0) \) and \( f(p^{k-1}) \). If \( f(0) = f(p^{k-1}) \), then \( H \) is non-trivial; otherwise, \( H \) is trivial. The correctness of the algorithm relies on the following fact.

\textbf{Fact 1.} The subgroups of \( \mathbb{Z}_{p^k} \) consist of \( \{0\}, (p^{k-1}), (p^{k-2}), \ldots, (1) \).

Since all non-trivial subgroups of \( G \) contain the element \( p^{k-1} \), we only need to check whether \( p^{k-1} \in H \) or not to decide the triviality of \( H \).
3.2. Identification version

In the following, we give an algorithm to identify $H$ (Algorithm 1). The idea of Algorithm 1 is to check whether $p^i \in H$ for $0 \leq i \leq k - 1$. If there exists some $i$ such that $p^i \in H$, then $H = \langle p^i \rangle$; otherwise, $H$ is trivial.

**Algorithm 1** Find $H$ in $\mathbb{Z}_{p^k}$

1: Query 0;
2: for $i = 0 \rightarrow k - 1$ do
3: Query $p^i$;
4: if $f(0) = f(p^i)$ then
5: return $\langle p^i \rangle$;
6: end if
7: end for
8: return $\{0\};$

Specifically, we analyze the correctness and query complexity of Algorithm 1 as follows. If $H = \langle p^j \rangle$ for some $0 \leq j \leq k - 1$, then $f(0) = f(x)$ iff $x \in H$, i.e., $p^j|x$. Thus, the algorithm will perform the loop repeatedly for $0 \leq i \leq j$. Finally, the algorithm returns $\langle p^j \rangle$. Since $|G| = p^k, |H| = p^{k-j}$, the query complexity of the algorithm is $1 + (j + 1) = O(\log \frac{|G|}{|H|})$.

If $H = \{0\}$, then the algorithm will perform the loop entirely since $f(0) \neq f(p^i)$ for any $0 \leq i \leq k - 1$. Finally, the algorithm returns $\{0\}$. Since $|H| = 1$, the query complexity of the algorithm is $1 + k = O(\log \frac{|G|}{|H|})$.

4. General algorithms for HSP over finite Abelian groups

In this section, we will discuss the general Abelian hidden subgroup problem. Since any finite Abelian group is isomorphic to $\{0\} \times \mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_l^{k_l}}$, where $p_i$’s are primes and $k_i \geq 1$ for any $i$, we only need to consider the case $G = \{0\} \times \mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_l^{k_l}}$. The decision and identification versions are discussed in Section 4.2 and 4.3 respectively. The query complexities of our algorithms are related to $H$, even if we do not know any information about $H$ initially.

4.1. Finding generating pairs

We first give Algorithm 2 as follows, which is a subroutine of the main algorithms solving HSP (i.e., Algorithm 3 and 4). The goal of Algorithm 2 is to find a generating pair $W_1, W_2$ such that $W_1 - W_2 = V, |W_1| \leq 2\lfloor \sqrt{|V|/r} \rfloor$, $|W_2| \leq \lceil \sqrt{|V|/r} \rceil$. This subroutine does not make queries.
Algorithm 2 findPair

Input: $V = \{0\} \times \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_l}$ and $r \in [1, l)$, where $p_i$’s are primes and $j_i \geq 0$ for any $i$;

Output: $W_1, W_2$ s.t. $W_1 - W_2 = V$, $|W_1| \leq 2\sqrt{|V| \cdot r}$, $|W_2| \leq \sqrt{|V| / r}$.

1: Select $i$ such that $\prod_{m=i+1}^{l} p_m^{j_m} < \lceil \sqrt{|V| / r} \rceil \leq \prod_{m=i}^{l} p_m^{j_m}$;

2: Let $a = \lfloor \sqrt{|V| / r} \rfloor$, $b = \lceil \sqrt{|V| / r} \rceil$;

3: $W_1 = \{0\} \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{i-1}} \times \{0, 1, \ldots, b-1\} \times \{0\} \times \cdots \times \{0\}$;

4: $W_2 = \{0\} \times \cdots \times \{0\} \times \{0, -b, \ldots, -(a-1)b\} \times \mathbb{Z}_{p_{i+1}} \times \cdots \times \mathbb{Z}_{p_l}$.

5: return $W_1, W_2$.

In the following, we show that the above returned values are satisfied. Since $a \geq \lceil \sqrt{|V| / r} \rceil$, we have $\prod_{m=i+1}^{l} p_m^{j_m} \geq \lceil \sqrt{|V| / r} \rceil$. Since $\sqrt{|V| / r} \leq \prod_{m=i}^{l} p_m^{j_m}$, we have

$$\prod_{m=1}^{i-1} p_m^{j_m} = \frac{|V|}{\prod_{m=i}^{l} p_m^{j_m}} \leq \frac{|V|}{\sqrt{|V| \cdot r}} \leq \sqrt{|V| \cdot r}.$$ 

Thus,

$$|W_1| = \prod_{m=1}^{i-1} p_m^{j_m} \cdot b$$

$$= \prod_{m=1}^{i-1} p_m^{j_m} \cdot \left\lfloor \frac{\sqrt{|V| / r}}{a} \right\rfloor$$

$$\leq \prod_{m=1}^{i-1} p_m^{j_m} + \prod_{m=1}^{i-1} p_m^{j_m} \cdot \frac{\sqrt{|V| / r}}{a}$$

$$\leq \left\lceil \sqrt{|V| / r} \right\rceil + \frac{|V|}{\sqrt{|V| \cdot r} \cdot \prod_{m=i+1}^{l} p_m^{j_m}} \cdot \frac{\sqrt{|V| / r}}{a}$$

$$\leq 2\sqrt{|V| \cdot r}.$$

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and
\[ W_2 = a \prod_{m=i+1}^{l} p_m^{l_m} \leq \left\lceil \sqrt{|V|/r} \right\rceil \prod_{m=i+1}^{l} p_m^{l_m} = \left\lceil \sqrt{|V|/r} \right\rceil. \]

Moreover, since \( ab \geq p_i^{k_i} \), we have \( \{0, 1, ..., b - 1\} - \{0, b, ..., -(a - 1)b\} = \{0, 1, ..., ab - 1\} \equiv Z_{p_i^{k_i}} (mod p_i^{k_i}) \). Thus, \( W_1 - W_2 = V \).

### 4.2. Decision version

In the following, we give Algorithm 3 to solve the decision version of Abelian hidden subgroup problem. The idea is to verify whether \( G_i \) has nonzero elements in \( H \) for \( 1 \leq i \leq l \). If for some \( i \) the answer is yes, then \( H \) is nontrivial; otherwise, \( H \) is trivial. We use the following observation: in order to verify whether a set has nonzero elements in \( H \), it suffices to check whether a generating pair of the set have collisions.

**Algorithm 3** The algorithm for the decision version

**Input:** \( G \cong \{0\} \times Z_{p_1^{k_1}} \times \cdots \times Z_{p_l^{k_l}} \), where \( p_i \)'s are primes and \( k_i \geq 1 \) for any \( i \);

**Output:** whether \( H \) is trivial or not;

1. \( W_1 = W_2 = \{0\} \);
2. for \( i = 1 \rightarrow l \) do
3. Query all not queried elements in \( W_1 \times \{0\}, W_2 \times \{p_i^{k_i-1}\} \);
4. if there exist \( x \in W_1 \times \{0\}, y \in W_2 \times \{p_i^{k_i-1}\} \) s.t. \( f(x) = f(y) \) then
5. return false;
6. end if
7. \( W_1, W_2 \leftarrow \text{findPair}(G_i, 1) \);
8. end for
9. return true;

Specifically, the process of Algorithm 3 is as follows. In Step 1, we initialize \( W_1 = W_2 = \{0\} \). Then we go into the loop. We query all the elements not queried in \( W_1 \times \{0\} \) and \( W_2 \times \{p_i^{k_i-1}\} \). If there exist \( x \in W_1 \times \{0\}, y \in W_2 \times \{p_i^{k_i-1}\} \) such that \( f(x) = f(y) \), then the algorithm returns false; otherwise we update the values of \( W_1, W_2 \) and go to the next iteration. After performing \( l \) iterations, if the algorithm does not return false, then it returns true.
We first analyze the correctness of Algorithm 3. Let $d = \min_{h \in H} I_h$. We discuss the following two cases: i) $H$ is trivial. In this case, $f(x) \neq f(y)$ for any $x \neq y$, and thus Algorithm 3 returns true. ii) $H$ is non-trivial. In this case, if the algorithm is over after no more than $d - 1$ iterations of the loop, then the algorithm must return false, which is a correct result; if the algorithm is not over after $d - 1$ iterations, then $W_1 - W_2 = G_{d-1}$. Since $d = \min_{h \in H} I_h$, there exists $h \in H$ such that $h = (0, h_1, \ldots, h_d, 0, \ldots, 0)$, where $h_d \neq 0$. Then there exists $b$ such that $bh_d = p_d^{k_d-1}$ by Lemma 1, i.e., $bh = (0, bh_1, \ldots, bh_{d-1}, p_d^{k_d-1}, 0, \ldots, 0)$. Thus,

$$\forall b \in G_{d-1} \times \{p_d^{k_d-1}\} = W_1 \times \{0\} - W_2 \times \{p_d^{k_d-1}\}.$$}

Since $bh \in H$, for some $x, y$ satisfying that $x - y = bh$, we will find $f(x) = f(y)$ in Step 4 of the $d$-th iteration. Hence, Algorithm 3 will return false. Totally, Algorithm 3 always outputs the correct result.

**Lemma 1.** For any nonzero element $h \in \mathbb{Z}_{p^k}$, there exists an element $b \in \mathbb{Z}_{p^k}$ such that $bh \equiv p^k-1 \pmod{p^k}$.

**Proof.** Let $h' = h \mod p$. Since $h' \in \{1, \ldots, p - 1\}$, there exists $a \in \mathbb{Z}_p$ such that $ah' \equiv 1 \pmod{p}$. Thus, $ah'p^{k-1} \equiv p^k-1 \pmod{p^k}$. Let $b \equiv ap^{k-1} \pmod{p^k}$. We have $bh \equiv ap^{k-1}h \equiv ap^{k-1}h' \equiv ah'p^{k-1} \equiv p^k-1 \pmod{p^k}$. \hfill \square

Now we analyze the query complexity of Algorithm 3. In the following, since Step 3 is the only step to makes queries, it suffices to compute the values of $|W_1|, |W_2|$ in Step 3. Similarly, we discuss the following two cases:

i) $H$ is nontrivial. Since $H_i = \{h \in H | I_h \leq i\}$, we have $H_i = H \cap G_i$. And because $H_i \subseteq H$ and for any $h_1, h_2 \in H_i$ we have $h_1 + h_2 \in H_i$, $H_i$ is a subgroup of $H$. Since $d = \min_{h \in H} I_h$, we have $|H_{d-1}| = 1$. By Lemma 2, we have $|H| \leq \prod_{i=d}^{|G_i|} k_i$. Additionally, Algorithm 3 will go over in the first $d$ iterations during the above correctness proof. Since $G_i = \{0\} \times \mathbb{Z}_{p_i} \times \cdots \times \mathbb{Z}_{p_i}$, we have $i \leq \log |G_i|$ and $|G_{i+1}|/|G_i| = p_i^{k_i+1} \geq 2$. Thus,

\[
\sum_{m=0}^{i} \sqrt{|G_i|} \leq \sqrt{|G_i|}(1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{2^{i-1}}}) \leq \frac{1}{1 - 1/\sqrt{2}} \sqrt{|G_i|}. \tag{1}
\]

$W_1, W_2$ are updated by calling Algorithm 2 in Step 4. Note that after each iteration, $W_1$ is non-decrease. Additionally, $|W_1| \leq 2(\sqrt{|G_i|} + 1), |W_2| \leq \sqrt{|G_i|} + 1$ after running the $i$-th iteration. Thus, the query complexity of Algorithm 3 is
we have

\[ \text{pairs. Meanwhile, in each iteration, the algorithm updates} \]

\[ 4.3. \text{Identification version} \]

\[ \text{to i), we can obtain the query complexity is at most} \]

\[ \text{Lemma 2.} \]

\[ \text{Proof.} \]

\[ \text{Algorithm 3, the idea of Algorithm 4 is to find} \]

\[ \text{Then there exists} \]

\[ \text{Since} \]

\[ \text{HSP} \]

\[ \text{we give Algorithm 4 to solve the identification version of} \]

\[ \text{is trivial. In this case, Algorithm 3 will perform all} \]

\[ \text{We give Algorithm 4 to solve the identification version of} \]

\[ \text{ii) $H$ is trivial. In this case, Algorithm 3 will perform all} \]

\[ \text{is the minimum integer such that} \]

\[ \text{Thus,} \]

\[ \text{which implies} \]

\[ \text{Lemma 2. If $H_i \neq H_{i-1}$, then there exist} \]

\[ \text{Additionally,} \]

\[ \text{Proof. Suppose} \]

\[ \text{Then there exists} \]

\[ \text{For any} \]

\[ \text{If} \]

\[ \text{On the other hand, since} \]

\[ \text{Additionally,} \]

\[ \text{Thus,} \]

\[ \text{Hence,} \]

\[ \text{which implies} \]

\[ 4.3. \text{Identification version} \]

\[ \text{We give Algorithm 4 to solve the identification version of} \]

\[ \text{Similar to} \]

\[ \text{Algorithm 3, the idea of Algorithm 4 is to find} \]

\[ \text{Meanwhile, in each iteration, the algorithm updates} \]

\[ \text{for} \]

\[ \text{Finally, the algorithm returns} \]

\[ \text{at most} \]

\[ 2(\sqrt{|G_{d-1}|} + 1) + \sum_{i=0}^{d-1}(\sqrt{|G_i|} + 1) \]

\[ \leq 2(\sqrt{|G_{d-1}|} + 1) + \frac{1}{1-1/\sqrt{2}} \sqrt{|G_{d-1}|} + d \]

\[ = O(\sqrt{|G_{d-1}|}). \]

\[ \text{Since} \]

\[ \text{Suppose} \]

\[ \text{w.l.o.g., let} \]

\[ \text{If} \]

\[ \text{Hence,} \]

\[ \text{Thus,} \]

\[ \text{Thus,} \]

\[ \text{Thus,} \]

\[ \text{Thus,} \]

\[ \text{Thus,} \]

\[ \text{which implies} \]

\[ 11 \]
Algorithm 4: The algorithm for the identification version

**Input:** $G = \{0\} \times \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_l^{k_l}}$, where $p_i$’s are primes and $k_i \geq 1$ for any $i$;

**Output:** The hidden subgroup $H$;

1. $V = W_1 = W_2 = H' = \{0\}, r = 0$;
2. for $i = 1 \to l$ do
3. Query all not queried elements in $W_1 \times \{0\}$;
4. $H' \leftarrow H' \times \{0\}, t_i = k_i$;
5. for $j = 0 \to k_i - 1$ do
6. Query the elements in $W_2 \times \{p_i^j\}$;
7. if there exist $x \in W_1 \times \{0\}, y \in W_2 \times \{p_i^j\}$ such that $f(x) = f(y)$ then
8. $H' \leftarrow H' + \langle x - y \rangle, t_i = j$;
9. if $j = 0$ then
10. $r = r + 1$;
11. end if
12. break;
13. end if
14. end for
15. $V \leftarrow V \times \mathbb{Z}_{p_i^{t_i}}$;
16. $(W_1, W_2) \leftarrow \text{findPair}(V, \max\{1, r\})$;
17. end for
18. return $H'$;
In the following, we first describe the process of Algorithm 4 and then analyze its correctness and query complexity. In Step 1, we first initialize $V = W_1 = W_2 = H' = \{0\}$ and $r = 0$. Then we jump into the outer loop. We first query all the elements not queried in $W_1 \times \{0\}$ in Step 3 and initialize $t_i = k_i$ in Step 4. Then we jump into the inner loop. In Step 6, we query the elements in $W_2 \times \{p'_i\}$. If there exist some collisions, then we do the following things: i) update $H$ and set $t_i = j$; ii) if $j = 0$, let $r = r + 1$. iii) jump out the inner loop. Then we update $V, W_1, W_2$ in Step 15 and Step 16.

The correctness analysis of Algorithm 4 is as follows. Note that we always have $V = W_1 - W_2$ in the algorithm procedure by calling Algorithm 2. In the following, for the sake of clarity, let $V_0, H'_0, r_0$ be the initial values of $V, H', r$ in Algorithm 4 and $V_i, H'_i, r_i$ represent the values of $V, H', r$ after running the $i$-th iteration of the outer loop ($1 \leq \ i \leq l$). Let $V, r$ be the final values, i.e., $V = V_l$ and $r = r_l$.

We first prove $V_i + H'_i = G_i$ by induction. i) $i = 0$, $H'_0 = V_0 = G_0 = \{0\}$, so $V_0 + H'_0 = G_0$. ii) suppose $V_m + H'_m = G_m$ for any $m < i$. If we do not find collisions in the inner loop, then $H'_i = H'_{i-1} \times \{0\}$, $V_i = V_{i-1} \times Z_{p'_i}$, and thus

$$H'_i + V_i = G_{i-1} \times Z_{p'_i} = G_i.$$  

If there exist collisions in the $j$-th iteration of the inner loop, since $x \in W_1 \times \{0\}, y \in W_1 \times \{p'_i\}$ and $W_1 - W_2 = V_{i-1}$, we have $x - y \in V_{i-1} \times \{p'_i\}$. Thus, $H'_i = H'_{i-1} \times \{0\} + \langle x - y \rangle$, $V_i = V_{i-1} \times Z_{p'_i}$. Since $\langle p'_i \rangle + Z_{p'_i} \equiv Z_{p'_i} \pmod{p'_i}$, we have $H'_i + V_i = G_{i-1} \times Z_{p'_i} = G_i$.

Then we will prove $H'_i = H_i$ by induction. First, $i = 0$, $H'_0 = \{0\} = H_0$. Second, suppose $H'_m = H_m$ for any $m < i$. Then we go into the $i$-th iteration of the outer loop. i) If we do not find collisions in the inner loop, then $H'_i = H'_{i-1} = H_{i-1}$. By Lemma 2 if $H_i \neq H_{i-1}$, then there exists $h \in H_i$ such that $h = p'_i$ for some $j$ and $H_i = H_{i-1} + \langle h \rangle$. Additionally, since $W_1 \times \{0\} - W_2 \times \{p'_i\} = V_{i-1} \times \{p'_i\}$ in the $j$-th iteration of the inner loop, we have $v \notin H$ for any $v \in V_{i-1} \times \{p'_i\}$. Due to

$$H_{i-1} + V_{i-1} = H'_{i-1} + V_{i-1} = G_{i-1},$$

for any $g \in G_{i-1} \times \{p'_i\}$, there exist $h \in H_{i-1}, v \in V_{i-1} \times \{p'_i\}$ such that $g = h + v$. Since $h \in H, v \notin H$, we have $g \notin H$. Thus, there exists no such $h \in H_i$ that $h_i = p'_i$, which implies $H_i = H_{i-1}$, i.e., $H'_i = H_i$. ii) If we find collisions in the $j$-th iteration of the inner loop, then let $h = x - y = (0, h_1, ..., h_{i-1}, p'_i, 0, ..., 0)$. Since $H'_i = H'_{i-1} + \langle h \rangle$ and $H'_{i-1} = H_{i-1}$, it suffices to prove $H_{i-1} + \langle h \rangle = H_i$. For $h' \in H_i/H_{i-1}$, suppose there exists no integer $c$ such that $h'_i = cp'_i$. Without
Hence, for any $h' \in H_i/H_{i-1}$, there exists an integer $c$ such that $h'_i = c p_i^{j_i}$. By the proof of Lemma 2, we have $H_i = H_{i-1} + \langle h \rangle$. Thus, $H'_i = H_i$. Totally, after $l$ iterations, Algorithm 4 returns $H'_i = H_i = H$, i.e., Algorithm 4 returns the correct result.

Next, we analyze the query complexity of Algorithm 4. Note that Step 3 and 6 are the only steps to make queries. For $1 \leq i \leq l$, the running time of Step 6 in the $i$-th iteration is at most $t_i + 1$; let $a_i = |W_1|$ and $b_i = |W_2|$ when running the $i$-th iteration of the outer loop. Let $a_0 = 0$. It is worth noting that $W_1$ is non-decrease after each iteration. Since $W_1$ and $W_1 \times \{0\}$ are actually the same set, we essentially query all the elements not queried in $W_1$ in Step 3 each time. Thus, in the $i$-th iteration, the number of queries in Step 3 is $a_i - a_{i-1}$; also, the number of queries in Step 6 is at most $(t_i + 1)b_i$. Thus, the total number of queries is at most $a_1 + \sum_{i=1}^{l} (t_i + 1)b_i$.

In the following, we first prove $|H_i| \leq p_i^{k_i-t_i}|H_{i-1}|$. i) If $H_i = H_{i-1}$, then $t_i = k_i$ and thus $|H_i| = p_i^{k_i-t_i}|H_{i-1}|$. ii) If $H_i \neq H_{i-1}$, then there exists $h$ such that $h_i = p_i^{k_i}$ and $H_i = H_{i-1} + \langle h \rangle$. By Lemma 2 we have $|H_i| \leq p_i^{k_i-t_i}|H_{i-1}|$.

Hence, $|H_i| \leq p_i^{k_i-t_i}|H_{i-1}| \leq \prod_{m=1}^{i} p_m^{k_m-t_m}$. Since $|V_i| = \prod_{m=1}^{i} p_m^{k_m}$, we have $|H_i| : |V_i| = \prod_{m=1}^{i} p_m^{k_m} = |G_i|$. Thus, $|V| = |V_i| = \frac{|G_i|}{|H_i|} = \frac{|G|}{|H_{i-1}|}$.

Let $S_1 = \{i|r_i = r_{i-1}, 1 \leq i \leq l\}$ and $S_2 = \{i|r_i = r_{i-1} + 1\}$. Then $|S_1| = l - r$ and $|S_2| = r$. Moreover, for $i \in S_1$, we have $t_i \geq 1$ and $b_i \leq \lceil \sqrt{|V_i|/\max\{1,r\}} \rceil \leq \sqrt{|V_i|} \leq \sqrt{|V|} + 1$; for $i \in S_2$, we have $t_i = 0$. Similar to Eq. (1), we have $\sum_{m=1}^{i} \sqrt{|V_m|} \leq \frac{1}{1-1/\sqrt{2}} \sqrt{|V|}$. Since $t_{i_1}^2 \leq 2 t_{i_1} + 2 \leq 4 \cdot p_i^{t_i}$ for $t_i \geq 1$, we have $t_i \leq 2 \sqrt{|V_{i-1}|}$. So $t_i \sqrt{|V_{i-1}|} \leq 2 \sqrt{|V_i|}$. Since $|V| = \prod_{i=1}^{l} p_m^{l_m}$, we have

$$\sum_{i=1}^{l} t_i \leq \sum_{i=1}^{l} t_m \log p_m = \log |V|,$$

which means $\sum_{i=1}^{l} t_i \leq \log p |V|$. Moreover, since

$$|V| = \prod_{i=1}^{l} p_m^{l_m} \geq \prod_{i \in S_1} p_m^{l_m} \geq \prod_{i \in S_1} p_m \geq 2^{l-r},$$

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we have \( l - r \leq \log |V| \). Thus, we have

\[
\sum_{i \in S_1} (t_i + 1)b_i \leq \sum_{i \in S_1} (t_i + 1)(\sqrt{|V_{i-1}|} + 1)
\]

\[
= \sum_{i=1}^{l} (t_i \sqrt{|V_{i-1}|} + \sqrt{|V_{i-1}|}) + \sum_{i=1}^{l} t_{i+1} + l - r
\]

\[
\leq \sum_{i=1}^{l} (2 \sqrt{|V_i|} + \sqrt{|V_{i-1}|}) + 2 \log |V|
\]

\[
\leq \sum_{i=1}^{l} 3 \sqrt{|V_i|} + 2 \log |V|
\]

\[
\leq \frac{3}{1 - 1/\sqrt{2}} \sqrt{|V|} + 2 \log |V|
\]

\[
= O(\sqrt{|V|}),
\]

\[
\sum_{i \in S_2} (t_i + 1)b_i = \sum_{i \in S_2} b_i
\]

\[
\leq \sum_{i \in S_2} (\sqrt{|V|} \max\{1, r_{i-1}\} + 1)
\]

\[
< \sqrt{|V|} (1 + \sum_{i=1}^{r-1} \frac{1}{\sqrt{i}}) + r
\]

\[
< \sqrt{|V|} (1 + \int_{0}^{r-1} \frac{1}{\sqrt{x}} dx) + r
\]

\[
< 2 \sqrt{|V| \cdot r} + r
\]

\[
= O(r \sqrt{|V| \cdot r}).
\]

Since \( a_l \leq 2 \left\lfloor \sqrt{|V| \cdot r} \right\rfloor \), the total number of queries is \( O(r + \sqrt{|V| \cdot r}) \). Since \( |H| \geq \prod_{i \in S_2} p_i^{k_i} \geq 2^r \), we have \( r = O(\log |H|) \). Since \( |V| = |G/\mathbb{H}| \), the total number of queries is \( O(\sqrt{|G/\mathbb{H}| \log |H| + \log |H|}) \). Finally, Theorem 2 is implied by the above analysis of Algorithm 3 and 4.

5. A general lower bound on the query complexity of HSP

In this section, we give a general lower bound for HSP by proving Theorem 3.

Proof of Theorem 3. We consider the identification version of the hidden subgroup problem with additional promise (HSP+). In HSP+, suppose we have
known the size of $H$ and $|X| = \frac{|G|}{|H|}$. Since any algorithm solving HSP also can solve HSP$^+$, it suffices to obtain the lower bound on the query complexity of HSP$^+$. Since HSP$^+$ has at least $|\mathcal{H}|$ possible outputs, where $|\mathcal{H}| = \{H' \leq G ||H'| = |H|\}$, any decision tree solving the problem must have at least $\Omega(|\mathcal{H}|)$ leaves. Since every query has at most $|G| |H|$ possible answers, the depth of the decision tree must be at least $\Omega\left(\frac{\log |\mathcal{H}|}{\log |G| \log |H|}\right)$.

6. Conclusion

In this paper, we considered deterministic algorithms to solve the hidden subgroup problem (HSP). First, for HSP over $G = \mathbb{Z}_{p^k}$ with $p$ being a prime, we presented a deterministic algorithm with 2 queries to decide whether $H$ is trivial, and an algorithm with $O(\log \frac{|G|}{|H|})$ queries to identify $H$. Second, for HSP over a general finite Abelian group $G$, we devised an algorithm with $O(\sqrt{\frac{|G|}{|H|}})$ queries to decide the triviality of $H$ and an algorithm to identify $H$ with $O(\sqrt{\frac{|G|}{|H|}} \log |H| + \log |H|)$ queries. Third, a general lower bound on the query complexity of HSP was given, which leads to the observation that there exists an instance of HSP that needs $\omega(\sqrt{\frac{|G|}{|H|}})$ queries to identify $H$.

These results not only answer the open problem proposed by Nayak [25], but also extend the main results of Ref. [30]. Furthermore, it is worth exploring whether there exist similar deterministic algorithms for the non-Abelian hidden subgroup problem.

References

[1] Dave Bacon, Andrew M. Childs, and Wim van Dam. From optimal measurement to efficient quantum algorithms for the hidden subgroup problem over semidirect product groups. In Proceedings of the 46th Annual Symposium on Foundations of Computer Science, pages 469–478, 2005. doi:10.1109/SFCS.2005.38

[2] Dan Boneh and Richard J. Lipton. Quantum cryptanalysis of hidden linear functions (extended abstract). In Advances in Cryptology - CRYPTO ’95, 15th Annual International Cryptology Conference, pages 424–437, 1995. doi:10.1007/3-540-44750-4_34

[3] Gilles Brassard and Peter Høyer. An exact quantum polynomial-time algorithm for Simon’s problem. In Proceedings of the 5th Israeli
Symposium on Theory of Computing and Systems, pages 12–23, 1997. doi:10.1109/ISTCS.1997.595153

[4] Guangya Cai and Daowen Qiu. Optimal separation in exact query complexities for Simon’s problem. Journal of Computer and System Sciences, 97:83–93, 2018. doi:10.1016/j.jcss.2018.05.001

[5] Andrew M. Childs. Lecture notes on quantum algorithms. https://www.cs.umd.edu/~amchilds/qa/qa.pdf, 2021.

[6] Andrew M. Childs and Wim van Dam. Quantum algorithm for a generalized hidden shift problem. In Proceedings of the 18th Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1225–1232, 2007. http://dl.acm.org/citation.cfm?id=1283383.1283515.

[7] Richard Cleve. The query complexity of order-finding. Information and Computation, 192(2):162–171, 2004. doi:10.1016/j.ic.2004.04.001

[8] Ronald de Wolf. Quantum Computing: Lecture Notes. arXiv preprint, 2019. arXiv:1907.09415.

[9] Thomas Decker, Gábor Ivanyos, Miklos Santha, and Pawel Wocjan. Hidden symmetry subgroup problems. SIAM Journal of Computing, 42(5):1987–2007, 2013. doi:10.1137/120864416

[10] David Deutsch and Richard Jozsa. Rapid solution of problems by quantum computation. Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences, 439(1907):553–558, 1992.

[11] Kirsten Eisenträger, Sean Hallgren, Alexei Y. Kitaev, and Fang Song. A quantum algorithm for computing the unit group of an arbitrary degree number field. In Proceedings of the 46th Annual ACM Symposium on Theory of Computing, pages 293–302, 2014. doi:10.1145/2591796.2591860

[12] Mark Ettinger, Peter Høyer, and Emanuel Knill. The quantum query complexity of the hidden subgroup problem is polynomial. Information Processing Letters, 91(1):43–48, 2004. doi:10.1016/j.ipl.2004.01.024

[13] Katalin Friedl, Gábor Ivanyos, Frédéric Magniez, Miklos Santha, and Pranab Sen. Hidden translation and orbit coset in quantum computing. In Proceedings of the 35th Annual ACM Symposium on Theory of Computing, pages 1–9, 2003. doi:10.1145/780542.780544
[14] Michelangelo Grigni, Leonard J. Schulman, Monica Vazirani, and Umesh V. Vazirani. Quantum mechanical algorithms for the non-abelian hidden subgroup problem. *Combinatorica*, 24(1):137–154, 2004. doi:10.1007/s00493-004-0009-8.

[15] Sean Hallgren. Fast quantum algorithms for computing the unit group and class group of a number field. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing*, pages 468–474, 2005. doi:10.1145/1060590.1060660.

[16] Sean Hallgren. Polynomial-time quantum algorithms for Pell’s equation and the principal ideal problem. *Journal of ACM*, 54(1):4:1–4:19, 2007. doi:10.1145/1206035.1206039.

[17] Sean Hallgren, Alexander Russell, and Amnon Tashma. The hidden subgroup problem and quantum computation using group representations. *SIAM Journal on Computing*, 32(4):916–934, 2003. doi:10.1137/S009753970139450X.

[18] Gábor Ivanyos, Frédéric Magniez, and Miklos Santha. Efficient quantum algorithms for some instances of the non-Abelian hidden subgroup problem. In *Proceedings of the 13th Annual ACM Symposium on Parallel Algorithms and Architectures*, pages 263–270, 2001. doi:10.1145/378580.378679.

[19] Richard Jozsa. Quantum algorithms and the Fourier transform. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 454(1969):323–337, 1998.

[20] Julia Kempe and Aner Shalev. The hidden subgroup problem and permutation group theory. In *Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1118–1125, 2005. http://dl.acm.org/citation.cfm?id=1070432.1070592.

[21] Alexei Y. Kitaev. Quantum measurements and the Abelian stabilizer problem. *arXiv preprint*, 1995. quant-ph/9511026.

[22] Greg Kuperberg. A subexponential-time quantum algorithm for the dihedral hidden subgroup problem. *SIAM Journal on Computing*, 35(1):170–188, 2005. doi:10.1137/s0097539703436345.

[23] Cristopher Moore, Daniel N. Rockmore, Alexander Russell, and Leonard J. Schulman. The power of strong Fourier sampling: Quantum algorithms for
affine groups and hidden shifts. *SIAM Journal of Computing*, 37(3):938–958, 2007. doi:10.1137/S0097539705447177

[24] Michele Mosca and Artur Ekert. The hidden subgroup problem and eigenvalue estimation on a quantum computer. In *Proceedings of the 1st NASA International Conference on Quantum Computing and Quantum Communications*, pages 174–188, 1998. doi:10.1007/3-540-49208-9_15

[25] Ashwin Nayak. Deterministic algorithms for the hidden subgroup problem. *arXiv preprint*, 2021. arXiv:2104.14436

[26] Oded Regev. Quantum computation and lattice problems. *SIAM Journal of Computing*, 33(3):738–760, 2004. doi:10.1137/S0097539703440678

[27] Arthur Schmidt and Ulrich Vollmer. Polynomial time quantum algorithm for the computation of the unit group of a number field. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing*, pages 475–480, 2005. doi:10.1145/1060590.1060661

[28] Peter W. Shor. Algorithms for quantum computation: Discrete logarithms and factoring. In *Proceedings of the 35th Annual Symposium on Foundations of Computer Science*, pages 124–134, 1994. doi:10.1109/SFCS.1994.365700

[29] Daniel R. Simon. On the power of quantum computation. In *Proceedings of the 35th Annual Symposium on Foundations of Computer Science*, pages 116–123, 1994. doi:10.1109/SFCS.1994.365701

[30] Zekun Ye, Yunqi Huang, Lvzhou Li, and Yuyi Wang. Query complexity of generalized Simon’s problem. *Information and Computation*, https://doi.org/10.1016/j.ic.2021.104790, 2021. Also see arXiv:1907.07367