On Hypothesis Testing Against Independence with Multiple Decision Centers

Sadaf Salehkalaibar, Michèle Wigger, and Roy Timo

Abstract

A distributed binary hypothesis testing problem is studied with one observer and two decision centers. The type-II error exponents region is derived for testing against independence when the observer communicates with the two decision centers over one common and two individual noise-free bit pipes. When there is only a common noise-free bit pipe, the type-II error exponents region is derived for testing against conditional independence. Finally, when the observer can communicate to the two decision centers over a discrete memoryless broadcast channel, an achievable type-II error exponents region is derived for testing against conditional independence. The last type-II error exponent is obtained by splitting the observations into subblocks, having the transmitter apply hybrid joint source-channel coding with side-information independently to each subblock, and having each receiver apply a Neyman-Pearson test jointly over the subblocks. This decision approach avoids introducing further error exponents due to the binning or the decoding procedures.

I. INTRODUCTION

Consider the distributed hypothesis testing problem in Fig. 1, where a first node (the transmitter) can communicate with two remote nodes (the receivers) and each of the latter two wishes to

S. Salehkalaibar and M. Wigger are with LTCI, Telecom ParisTech, Université Paris-Saclay, 75013 Paris, michele.wigger@telecom-paristech.fr,

S. Salehkalaibar is also with the Department of Electrical and Computer Engineering, College of Engineering, University of Tehran, Tehran, Iran, s.saleh@ut.ac.ir,

R. Timo is with Ericsson Research, Stockholm, Sweden, roy.timo@ericsson.com,

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Fig. 1. Multi-terminal hypothesis testing with side information.

decide on the joint probability distribution underlying the observations $X^n, Y^n_1, Y^n_2$ at the three terminals. In the scenario we consider, communication from the transmitter to the receivers either takes place over one or multiple noise-free bit pipes or over a discrete memoryless broadcast channel. For simplicity we will restrict attention to a binary hypothesis where either $\mathcal{H} = 0$ or $\mathcal{H} = 1$. The focus of this paper is on the asymptotic regime where the length of the observed sequences $n$ tends to infinity and where both the type-I error probability (i.e., the probability of deciding on hypothesis 1 when $\mathcal{H} = 0$) and the type-II error probability (i.e., the probability of deciding on hypothesis 0 when $\mathcal{H} = 1$) vanish. We follow the approach in [1], [2], and aim to quantify the fastest possible exponential decrease of the type-II error probabilities, while we allow the type-I probabilities of error to vanish arbitrarily slowly. Ahlswede and Csiszar [1], Han [2], and Shimokawa, Han, and Amari [3] studied the problem with only a single receiver. They presented general upper and lower bounds on the maximum type-II error exponents, and these bounds match when under $\mathcal{H} = 1$ the joint distribution of the observations equals the product of the marginal distributions under $\mathcal{H} = 0$. This problem formulation is widely known as testing against independence.

Rahman and Wagner [4] and Zhao and Lai [5] found similar results for scenarios with multiple observers. Interactive observers, interactive multi-round communications between observers and decision centers, successive refinement and privacy setups were considered in [6]–[9]. Recently, the optimal type-II error exponent was found for distributed hypothesis testing over a noisy communication channel [10]. In contrast to previous works, here we consider only a single observer but multiple decision centers. There are two practical motivations for this assumption.

- **Multiple Decision Centers Interested in the Same Information:** The information collected
at a given sensor is required at multiple decision centers to make a correct guess on their desired hypotheses. In this sense, the sensor will be interested in sending information about its observed sequence to multiple decision centers. This work treats the scenario where communication to the two decision centers takes place over a common network.

- **Single Decision Center with Uncertain Local Observation:** There is only a single decision center, and the probability distribution of the decision center’s observation under each of the two hypotheses is uncertain a priori. For example, there could be two possible options, depending on some other phenomenon that the decision center is not interested to detect. In this case, \( Y_1^n \) and \( Y_2^n \) model the observations at the decision center for each of the two options.

A main feature of the scenario that we consider, is that the observer is interested in extracting and transmitting information about its observation \( X^n \) that is useful to both decision centers. There can thus be an inherent tradeoff in the problem, in that some information might be more beneficial for Receiver 1 than for Receiver 2 or vice versa. The goal of this paper is to shed light on this tradeoff. In fact, we will present two examples, where in the first example there is no tradeoff between the maximum type-II error exponents that can simultaneously be achieved at both receivers, whereas in the second example such a tradeoff exists. As will be explained shortly, in this paper, we consider communications of positive rates. Interestingly, for zero-rate communication, such a tradeoff never exists. That means, there is a single strategy at the transmitter that is optimal for both decision centers. This optimal strategy is simply the strategy from \([2], [3]\) where the transmitter sends a single bit indicating whether its observation is typical with respect to the distribution under \( \mathcal{H} = 0 \), irrespective of the distribution of the receiver observation.

In this paper, we consider the special case of testing against independence or testing against conditional independence, and we propose coding schemes for three different one-to-two communication scenarios. In the first scenario, communication takes place over a Gray-Wyner source coding network \([11]\) with three noise-free bit pipes, the first connecting the transmitter with both receivers and the other two connecting the transmitter with only one of the two receivers. For
this scenario, the type-II error exponents region achieved by our scheme is optimal when testing against independence. In the second scenario, communication takes place over a Kaspi/Heegard-Berger source coding network \[12\], \[13\] with a single common noise-free bit pipe connecting the transmitter to both receivers. Moreover, here the first decision center has access to an additional side information \(Z_1^n\). For this scenario, the type-II error exponents region achieved by our scheme is optimal when testing against conditional independence given the additional observation \(Z_1^n\).

In the third scenario, Receiver 1 again observes side-information \(Z_1^n\), and communication takes place over a discrete memoryless broadcast channel (BC).

For the second and third scenarios, our coding scheme is based on random binning so as to exploit the side-information at Receiver 1. The idea of using binning for distributed hypothesis testing was introduced by Shimokawa, Han, and Amari [3] for a single-receiver setup. Interestingly, the standard approach to first analyze the probability that the receiver can recover the correct codeword within the indicated bin and then condition on the successful recovery of this codeword to analyze the type-II error probability, yields a result with two competing error exponents. A first exponent that results from the recovery of the correct codeword and a second one that results from hypothesis testing. Rahman and Wagner [4] showed that when testing against conditional independence the binning error exponent can be made to disappear by changing the way the receiver takes its decision and by applying Stein’s lemma to analyze the performance. A similar approach has been recently proposed also by Sreekuma and Gunduz [10] for hypothesis testing over noisy multi-access channels.

Inspired by [4] and [10], we propose the following scheme: Each terminal splits its observation into many subblocks and then applies a Heegard-Berger coding scheme [12], [13] to each subblock. The receivers collect all the i.i.d. blocks and apply a Neyman-Pearson test over all these subblocks to decide on the desired hypothesis. The analysis of the scheme is performed by analysing a related hypothesis testing problem where the code constructions of all blocks are revealed to all terminals in the form of an additional i.i.d. observation that can be used to test on the hypothesis. To analyse the error probabilities of this related scenario, similar steps to the ones indicated in [4] are followed.
The same coding and analysis approach is also used for the third scenario where the communication link is noisy. That means:

- Each terminal splits its sequence of observations into subblocks.
- The transmitter applies hybrid joint source-channel coding with side-information [14] to each of subblock.
- Each receiver groups the subblocks and applies a Neyman-Pearson test over these i.i.d. subblocks.

Following similar analysis steps as in [4] allows to condition on correct decoding of the quantized sequences without introducing additional error exponents. Notice that in our analysis, hybrid coding can be replaced by any joint source-channel coding technique that builds on a random code construction where the receivers’ reconstructions are codewords generated at the transmitter.

We conclude this section with an outline of the paper and some notation.

A. Paper Outline

In Section II, we present our scheme and result for the Gray-Wyner network and evaluate it for a Gaussian example. In Section III, we present our scheme and result for the Heegard-Berger network with side information, and evaluate it for a similar Gaussian example. In Section IV, we present our scheme and result for discrete memoryless BCs. Section V concludes the paper.

B. Notation

Random variables are denoted by capital letters, e.g., $X$, $Y$, and their realizations by lower case letters, e.g., $x$, $y$. Script symbols such as $\mathcal{X}$ and $\mathcal{Y}$ stand for alphabets of random variables and realizations, and $\mathcal{X}^n$ and $\mathcal{Y}^n$ for the corresponding $n$-fold Cartesian products. Sequences of random variables $(X_i, ..., X_j)$ and realizations $(x_i, ..., x_j)$ are abbreviated by $X^j_i$ and $x^j_i$. When $i = 1$, then we also use the notations $X^j$ and $x^j$ instead of $X^j_1$ and $x^j_1$.

The probability mass function (pmf) of a finite random variable $X$ is written as $P_X$; the conditional pmf of $X$ given $Y$ is written as $P_{X|Y}$. Entropy, conditional entropy, and mutual information of random variables $X$ and $Y$ are denoted by $H(X)$, $H(X|Y)$, and $I(X;Y)$. 

August 15, 2017 DRAFT
Differential entropy and conditional differential entropy of continuous random variables $X$ and $Y$ are indicated by $h(X)$ and $h(X|Y)$. All entropies and mutual informations in this paper are meant with respect to the distribution under hypothesis $\mathcal{H} = 0$. The term $D(P||Q)$ stands for the Kullback-Leibler divergence between two pmfs $P$ and $Q$ over the same alphabet.

For a given $P_X$ and a constant $\mu > 0$, let $T^n_\mu(P_X) = \{x^n : |\{i : x_i = x\}|/n - P_X(x)| \leq \mu P_X(x), \forall x \in \mathcal{X}\}$ be the set of $\mu$-typical sequences in $\mathcal{X}^n$ [15]. Similarly, $T^n_\mu(P_{X,Y})$ stands for the set of jointly $\mu$-typical sequences.

The expectation operator is written as $\mathbb{E}[\cdot]$. A Gaussian distribution with mean $a$ and variance $\sigma^2$ is written as $\mathcal{N}(a, \sigma^2)$. We abbreviate independent and identically distributed by i.i.d.. Finally, the $\log(.)$-function is taken with respect to base 2.

II. HYPOTHESIS TESTING OVER A GRAY-WYNER NETWORK

A. Setup

Consider the distributed hypothesis testing problem with one transmitter (the observer) and two receivers (the decision centers) in Fig. 2. The transmitter observes the sequence $X^n$, and Receivers 1 and 2 observe sequences $Y^n_1$ and $Y^n_2$, where under the null hypothesis

$$\mathcal{H} = 0: (X^n, Y^n_1, Y^n_2) \sim \text{i.i.d. } P_{XY_1Y_2},$$

and under the alternative hypothesis

$$\mathcal{H} = 1: (X^n, Y^n_1, Y^n_2) \sim \text{i.i.d. } P_X P_{Y_1Y_2}.$$ 

Here, $P_{XY_1Y_2}$ is a given joint distribution over the finite product alphabet $\mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2$, and $P_X$ and $P_{Y_1Y_2}$ denote its marginals, i.e.,

$$P_X(x) = \sum_{y_1 \in \mathcal{Y}_1, y_2 \in \mathcal{Y}_2} P_{XY_1Y_2}(x, y_1, y_2), \quad x \in \mathcal{X},$$
\[ P_{Y_1Y_2}(y_1, y_2) = \sum_{x \in \mathcal{X}} P_{XY_1Y_2}(x, y_1, y_2), \quad (y_1, y_2) \in \mathcal{Y}_1 \times \mathcal{Y}_2. \]

The transmitter communicates with the two receivers over 1 common and 2 individual noise-free bit pipes. Specifically, it computes messages \((M_0, M_1, M_2) = \phi(n)(X^n)\), using a possibly stochastic encoding function \(\phi(n) : \mathcal{X}^n \to \{0, ..., 2^{nR_0}\} \times \{0, ..., 2^{nR_1}\} \times \{0, ..., 2^{nR_2}\}\), and sends message \(M_0\) over the common pipe and messages \(M_1\) and \(M_2\) over the two individual pipes. Receiver 1 observes messages \(M_0\) and \(M_1\) and decides on the hypothesis \(\mathcal{H} \in \{0, 1\}\) by means of a decoding function \(g_1^{(n)} : \mathcal{Y}_1^n \times \{0, ..., 2^{nR_0}\} \times \{0, ..., 2^{nR_1}\} \to \{0, 1\}\). It produces \(\hat{\mathcal{H}}_1 = g_1^{(n)}(Y_1^n, M_0, M_1)\). Similarly, Receiver 2 observes messages \(M_0\) and \(M_2\) and decides on the hypothesis \(\mathcal{H}\) by means of a decoding function \(g_2^{(n)} : \mathcal{Y}_2^n \times \{0, ..., 2^{nR_0}\} \times \{0, ..., 2^{nR_2}\} \to \{0, 1\}\). It then produces \(\hat{\mathcal{H}}_2 = g_2^{(n)}(Y_2^n, M_0, M_2)\).

**Definition 1:** For each \(\epsilon \in (0, 1)\), an exponents-rates tuple \((\theta_1, \theta_2, R_0, R_1, R_2)\) is called \(\epsilon\)-achievable over the Gray-Wyner network if there exists a sequence of encoding and decoding functions \(\{(\phi(n), g_1^{(n)}, g_2^{(n)})\}_{n=1}^{\infty}\) such that for \(i \in \{1, 2\}\) and all positive integers \(n\), the corresponding sequences of type-I error probabilities
\[ \alpha_{i,n} \triangleq \Pr[\hat{\mathcal{H}}_i = 1 | \mathcal{H} = 0], \]
and type-II error probabilities
\[ \beta_{i,n} \triangleq \Pr[\hat{\mathcal{H}}_i = 0 | \mathcal{H} = 1], \]
satisfy the following inequalities,
\[ \alpha_{i,n} \leq \epsilon, \]
and
\[ -\lim_{n \to \infty} \frac{1}{n} \log \beta_{i,n} \geq \theta_i. \]

**Definition 2:** Given nonnegative rates \((R_0, R_1, R_2)\), define the exponents region \(\mathcal{E}_{GW}(R_0, R_1, R_2)\) as the closure of all non-negative exponent pairs \((\theta_1, \theta_2)\) for which \((\theta_1, \theta_2, R_0, R_1, R_2)\) is \(\epsilon\)-achievable over the Gray-Wyner network for every \(\epsilon \in (0, 1)\).

**Remark 1:** The exponents region \(\mathcal{E}_{GW}(R_0, R_1, R_2)\) only depends on the marginal distributions \(P_{XY_1}\) and \(P_{XY_2}\) under both hypotheses. In particular, \(\mathcal{E}_{GW}(R_0, R_1, R_2)\) remains unchanged if under hypothesis \(\mathcal{H} = 1\), we replace \(P_{Y_1Y_2}\) by \(P_{Y_1} \cdot P_{Y_2}\).
B. Coding Scheme

Fix $\mu > 0$, a sufficiently large blocklength $n$, and a joint conditional distribution $P_{U_0U_1U_2|X}$. Consider nonnegative rates $(R_0, R_1, R_2)$ satisfying the rate constraints

\begin{align*}
R_0 &> I(U_0; X), \quad (3) \\
R_1 &> I(U_1; X|U_0), \quad (4) \\
R_2 &> I(U_2; X|U_0). \quad (5)
\end{align*}

**Codebook Generation:** Randomly generate a codebook $C_0 := \{U_0^n(m_0) : m_0 \in \{1, \ldots, 2^{nR_0}\}\}$ by drawing each entry of the $n$-length codeword $U_0^n(m_0)$ in an i.i.d. manner according to the pmf $P_{U_0}$. For each index $m_0 \in \{1, \ldots, 2^{nR_0}\}$ and each $i \in \{1, 2\}$, randomly generate a codebook $C_i(m_0) := \{U_i^n(m_i|m_0) : m_i \in \{1, \ldots, 2^{nR_i}\}\}$ by drawing the $j$-th entry of each codeword $U_i^n(m_i|m_0)$ in a memoryless manner according to the conditional pmf $P_{U_i|U_0}(., U_{0,j}(m_0))$, where $U_{0,j}(m_0)$ denotes the $j$-th symbol of codeword $U_0^n(m_0)$. Reveal the realizations $C_0$, $\{C_1(\cdot)\}$, and $\{C_2(\cdot)\}$ of the randomly generate codebooks to all terminals.

**Transmitter:** Given that it observes the source sequence $x^n$, the transmitter looks for a tuple of indices $(m_0, m_1, m_2)$ such that $(x^n, u_0^n(m_0), u_1^n(m_1|m_0), u_2^n(m_2|m_0)) \in T_{\mu/2}^n(P_{XU_0U_1U_2})$, where $u_0^n(m_0), u_1^n(m_1|m_0), u_2^n(m_2|m_0)$ denote the corresponding codewords in codebooks $C_0, C_1(\cdot)$, and $C_2(\cdot)$. If this check is successful, the transmitter picks one of these tuples uniformly at random and sends the corresponding indices $m_0, m_1, m_2$ over the bit pipes: $m_0$ over the common bit pipe, $m_1$ over the individual bit pipe to Receiver 1, and $m_2$ over the individual bit pipe to Receiver 2. Otherwise, it sends the value 0 over each of the three bit pipes.

**Receiver $i \in \{1, 2\}$:** Assume that Receiver $i$ observes messages $M_0 = m_0$ and $M_1 = m_1$ and source sequence $Y_i^n = y_i^n$. If $m_0 = 0$, it declares $\hat{H}_i = 1$. If $m_0 \neq 0$, it checks whether $(y_i^n, u_0^n(m_0), u_i^n(m_i|m_0)) \in T_{\mu}^n(P_{Y_iU_0U_i})$. If this test is successful, Receiver $i$ declares $\hat{H}_i = 0$. Otherwise, it declares $\hat{H}_i = 1$. 

DRAFT August 15, 2017
C. Exponents Region

The main result of this section is a single-letter characterization of the exponents region \( E_{GW}(R_0, R_1, R_2) \). It is achieved by the scheme described in the preceding subsection.

**Theorem 1:** The exponents region \( E_{GW}(R_0, R_1, R_2) \) is given by the set of all nonnegative pairs \((\theta_1, \theta_2)\) that satisfy

\[
\theta_1 \leq I(U_0, U_1; Y_1), \\
\theta_2 \leq I(U_0, U_2; Y_2),
\]

for some auxiliary random variables \((U_0, U_1, U_2)\) satisfying the Markov chain \((U_0, U_1, U_2) \rightarrow X \rightarrow (Y_1, Y_2)\) and the rate constraints

\[
R_0 \geq I(U_0; X), \\
R_1 \geq I(U_1; X|U_0), \\
R_2 \geq I(U_2; X|U_0).
\]

**Proof:** See [16].

In the above theorem it suffices to consider auxiliary random variables \(U_0, U_1, U_2\) over alphabets \(\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2\) whose sizes satisfy: \(|\mathcal{U}_0| \leq |\mathcal{X}| + 4, \ |\mathcal{U}_1| \leq |\mathcal{X}| \cdot |\mathcal{U}_0| + 1\) and \(|\mathcal{U}_2| \leq |\mathcal{X}| \cdot |\mathcal{U}_0| + 1\). This follows by simple applications of Carathodory’s theorem.

D. Example

We propose an example to investigate the exponents region of Theorem 1. Theorem 1 was stated and proved for discrete memoryless sources. It can be shown that it remains valid also when sources are memoryless and jointly Gaussian [15, Chap. 3]. We focus on the regime \(R_1 = R_2 = 0\). Consider a setup where under both hypotheses \(X \sim \mathcal{N}(0, 1)\). Moreover, under hypothesis

\[
\mathcal{H} = 0: \quad Y_1 = X + N_1, \\
Y_2 = X + N_2,
\]
where $N_1 \sim \mathcal{N}(0, \sigma_1^2)$ and $N_2 \sim \mathcal{N}(0, \sigma_2^2)$ are independent of each other and of $X$. Under hypothesis
\begin{equation}
\mathcal{H} = 1: \quad Y_1 = X' + N_1, \quad (13)
\end{equation}
\begin{equation}
Y_2 = X' + N_2, \quad (14)
\end{equation}
where $X' \sim \mathcal{N}(0, 1)$ is independent of the triple $X, N_1, N_2$.

As we show in the following, for this example the exponents region $\mathcal{E}_{GW}(R_0, 0, 0)$ evaluates to the set of all nonnegative exponent pairs $(\theta_1, \theta_2)$ that satisfy
\begin{equation}
\theta_i \leq \frac{1}{2} \log \left( \frac{1 + \sigma_i^2}{2^{-2R_0} + \sigma_i^2} \right), \quad i \in \{1, 2\}.
\end{equation}
That these exponent pairs lie in $\mathcal{E}_{GW}(R_0, 0, 0)$ can be seen by evaluating (6)–(8) for an auxiliary $U_0$ that satisfies $X = U_0 + W_0$ with independent zero-mean Gaussians $W_0$ and $U_0$ of variances $2^{-2R_0}$ and $1 - 2^{-2R_0}$. To prove that $\mathcal{E}_{GW}(R_0, 0, 0)$ is no larger than the proposed region, notice that by the Entropy Power Inequality (EPI) [15]:
\begin{equation}
h(Y_i|U_0) \geq \frac{1}{2} \log \left( 2^{2h(X|U_0)} + 2^{2h(N_i)} \right), \quad i \in \{1, 2\}. \quad (15)
\end{equation}
Since (8) is equivalent to $R_0 \geq h(X) - h(X|U_0)$ and since $h(X|U_0) \leq h(X)$, the $\mathcal{E}_{GW}$ is included in the set of all pairs $(\theta_1, \theta_2)$ that satisfy
\begin{equation}
\theta_i \leq h(Y_i) - \frac{1}{2} \log \left( 2^{2\alpha} + 2^{2h(N_i)} \right), \quad i \in \{1, 2\}, \quad (16)
\end{equation}
for some $\alpha \in [h(X) - R_0, h(X)]$. The desired inclusion follows then by noting that the right-hand side of (16) is decreasing in $\alpha$ and there is thus no loss in optimality when replacing $\alpha$ by $h(X) - R_0$, or equivalently replacing $2^{2\alpha}$ by $2^{2h(X)} 2^{-2R_0} = 2\pi e \cdot 2^{-2R_0}$.

Fig. 3. Exponents region of Theorem 1 for $\sigma_1^2 = 0.2$ and $\sigma_2^2 = 0.3$. 
III. HYPOTHESIS TESTING OVER A HEEGARD-BERGER NETWORK WITH SIDE-INFORMATION

A. Setup

This section focuses on a related hypothesis testing problem with only a single, common bit pipe from the transmitter to the receivers, see Fig. 4. As before, the transmitter observes the sequence $X^n$, and Receivers 1 and 2 observe $Y_1^n$ and $Y_2^n$, respectively. In this new model, Receiver 1 additionally also observes a side information $Z_1^n$ whose pairwise distribution with $X^n$ and with $Y_1^n$ does not depend on the hypothesis $\mathcal{H}$. In fact, under the null hypothesis

$$\mathcal{H} = 0: (X^n, Y_1^n, Y_2^n, Z_1^n) \sim \text{i.i.d. } P_{XY_1Y_2Z1},$$

and under the alternative hypothesis,

$$\mathcal{H} = 1: (X^n, Y_1^n, Y_2^n, Z_1^n) \sim \text{i.i.d. } P_{XZ1}P_{Y1|Z1}P_{Y2}. \quad (18)$$

Here $P_{XY_1Y_2Z1}$ is a given joint distribution over a finite product alphabet $\mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Z}_1$, and $P_{XZ1}$, $P_{Y1|Z1}$ and $P_{Y2}$ denote its conditional marginals. The test here is “against conditional independence” because $Z_1$ has the same joint distribution with the source $X$ under both hypotheses and because under $\mathcal{H} = 1$, $Y_1$ is conditionally independent of $X$ given $Z_1$.

The transmitter communicates with the two receivers over a common noise-free bit pipe. Specifically, it computes the message $M = \phi^{(n)}(X^n)$ using a possibly stochastic encoding function $\phi^{(n)}$ of the form $\phi^{(n)} : \mathcal{X}^n \rightarrow \{0, ..., 2^{nR}\}$ and sends message $M$ over the common pipe. Receiver 1 observes message $M$ and decides on the hypothesis $\mathcal{H} \in \{0, 1\}$ by means of a decoding function $g_1^{(n)} : \mathcal{Y}_1^n \times \mathcal{Z}_1^n \times \{0, ..., 2^{nR}\} \rightarrow \{0, 1\}$. It produces $\hat{\mathcal{H}}_1 = g_1^{(n)}(Y_1^n, Z_1^n, M)$. 

*Figure 4.* Hypothesis testing over a Heegard-Berger network with additional side information at one receiver.

*Figure 3* illustrates this region $\mathcal{E}_{GW}(R_0, 0, 0)$ for different rates $R_0$. 

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**August 15, 2017 DRAFT**
Similarly, Receiver 2 observes message \( M \) and decides on the hypothesis \( \mathcal{H} \) by means of a decoding function \( g_2(n) : Y_2^n \times \{0, ..., 2^nR\} \rightarrow \{0, 1\} \). It then produces \( \hat{\mathcal{H}}_2 = g_2(n)(Y_2^n, M) \).

**Definition 3:** For each \( \epsilon \in (0, 1) \), an exponents-rate tuple \( (\theta_1, \theta_2, R) \) is called\( \epsilon \)-achievable over the Heegard-Berger network if there exists a sequence of encoding and decoding functions \( \{(\phi(n), g_1(n), g_2(n))\}_n \) such that for \( i \in \{1, 2\} \) and for all sufficiently large \( n \), we have \( \alpha_{i,n} \leq \epsilon \) and
\[
-\lim_{n \to \infty} \frac{1}{n} \log \beta_{i,n} \geq \theta_i.
\]

**Definition 4:** For a given rate \( R \), we define the exponents region \( \mathcal{E}_{HB}(R) \) as the closure of all non-negative exponent pairs \( (\theta_1, \theta_2) \) for which \( (\theta_1, \theta_2, R) \) is \( \epsilon \)-achievable over the Heegard-Berger network for every \( \epsilon \in (0, 1) \).

Notice that similar to Remark 1, according to the distribution under \( \mathcal{H} = 1 \), \( \mathcal{E}_{HB}(R) \) depends only on the marginal distributions \( P_{XZ_1}, P_{XY_1|Z_1} \) and \( P_{XY_2} \).

**B. Coding Scheme**

Fix \( \mu > 0 \), sufficiently large positive integers \( k \) and \( B \), and a joint conditional distribution \( P_{U_0U_1|X} \). Consider also nonnegative rates \( R_0 \) and \( R_1 \) that satisfy
\[
R_0 > I(U_0, X),
\]
\[
R_1 + R_1' > I(U_1; X|U_0),
\]
\[
R_1' < I(U_1; Z_1|U_0).
\]

Choose rate and blocklength as \( R = R_0 + R_1 \) and \( n = kB \).

**Codebook Generation:** Let \( P_{U_0} \) and \( P_{U_1|U_0} \) be the marginal and conditional marginal pmfs of \( P_X \cdot P_{U_0U_1|X} \).

For each block \( b \in \{1, \ldots, B\} \), randomly generate a codebook \( C_{0,b} := \{U_{0,b}^k(m_0) : m_0 \in \{1, \ldots, 2^{kR_0}\}\} \) by drawing each entry of the \( n \)-length codeword \( U_{0,b}^k(m_0) \) i.i.d. according to the marginal pmf \( P_{U_0} \). For each block \( b \in \{1, \ldots, B\} \) and each index \( m_0 \), randomly generate a codebook \( C_{1,b}(m_0) := \{U_{1,b}^k(s_1|m_0) : s_1 \in \{1, \ldots, 2^{k(R_1+R_1')}\}\} \) by drawing each entry of the \( n \)-length codeword \( U_{1,b}^k(s_1|m_0) \) i.i.d. according to the conditional marginal pmf \( P_{U_1|U_0}(s_1|U_{0,b,j}(m_0)) \).

Here, \( U_{0,b,j}(m_0) \) denotes the \( j \)-th symbol of codeword \( U_{0,b}^k(m_0) \).
Reveal the realizations \( \{C_{0,b}\} \) and \( \{C_{1,b}(\cdot)\} \) of the randomly generated codebooks to all terminals.

Randomly assign a bin index \( m_1 \in \{1, \ldots, 2^{kR_1}\} \) to each index \( s_1 \in \{1, \ldots, 2^{k(R_1 + R'_1)}\} \). Let

\[
\nu: \{1, \ldots, 2^{k(R_1 + R'_1)}\} \rightarrow \{1, \ldots, 2^{kR_1}\},
\]
denote the chosen bin assignment, which is revealed to all terminals.

**Transmitter:** The transmitter observes a source sequence \( x^n \) and splits it into \( B \) blocks, \( x^n = (x^k_1, \ldots, x^k_B) \). For each block \( b \in \{1, \ldots, B\} \), it considers codebooks \( C_{0,b} \) and \( \{C_{1,b}(\cdot)\} \) and looks for a pair of indices \((m_{0,b}, s_{1,b})\) such that

\[
(x^k_b, u^k_{0,b}(m_{0,b}), u^k_{1,b}(s_{1,b}|m_{0,b})) \in T^k_{\mu/2}(P_{XU0U1}).
\]

If there are multiple such pairs, it picks one of them uniformly at random.

If for some block \( b \), the transmitter cannot find a desired pair \((m_{0,b}, s_{1,b})\), it sends the message \( m = 0 \) over the common bit pipe. Otherwise, it sends the tuple

\[
m = (m_{0,1}, \ldots, m_{0,B}, m_{1,1}, \ldots, m_{1,B}),
\]

where \( m_{1,b} = \nu(s_{1,b}) \).

**Receiver 1:** Assume that Receiver 1 observes message \( M = m \) and source sequences \( Y^n_1 = y^n_1 \) and \( Z^n_1 = z^n_1 \). If \( m = 0 \), Receiver 1 declares \( \hat{H}_1 = 1 \). Otherwise, it decomposes its observations according to the \( B \) blocks, \( \{(m_{0,b}, m_{1,b}, y^k_{1,b}, z^k_{1,b}, C_{0,b}, \{C_{1,b}(\cdot)\})\}_{b=1}^B \).

It then applies a Neyman-Pearson test\(^1\) to decide on \( H \) based on these i.i.d. blocks, in a way that the type-I error probability does not exceed \( \epsilon/8 \).

**Receiver 2:** If \( m = 0 \), Receiver 2 declares \( \hat{H}_2 = 1 \). Otherwise, it decomposes its observations according to the \( B \) blocks, \( \{(m_{0,b}, y^k_{2,b}, C_{0,b})\}_{b=1}^B \). It then applies a Neyman-Pearson test to decide on \( H \) based on these i.i.d. blocks, so that the type-I error probability does not exceed \( \epsilon/8 \).

**C. Results on Exponents Region**

Let \( E_{\text{in}}^{\text{HB}}(R) \) be the set of all nonnegative pairs \((\theta_1, \theta_2)\) that satisfy

\[
\theta_1 \leq I(U_0, U_1; Y_1|Z_1)
\]

\(^1\)Notice that the Neyman Pearson test is designed for the random experiment where the codebooks \( C_{0,b} \) and \( \{C_{1,b}(\cdot)\} \) are part of the observations at the receivers.
\[ \theta_2 \leq I(U_0; Y_2) \]  
(24)

for some auxiliary random variables \( U_0, U_1 \) so that the Markov chain \( (U_0, U_1) \to X \to (Y_1, Y_2, Z_1) \), and the following rate constraint hold:

\[ R \geq I(U_0; X) + I(U_1; X|U_0, Z_1). \]  
(25)

Notice that to calculate \( \mathcal{E}_{\text{HB}}^\text{in}(R) \) it suffices to consider auxiliary random variables \( U_0, U_1, U_2 \) over alphabets \( \mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2 \) whose sizes satisfy: \( |\mathcal{U}_0| \leq |\mathcal{X}| + 2 \) and \( |\mathcal{U}_1| \leq |\mathcal{X}| \cdot |\mathcal{U}_0| + 1 \).

The exponents region \( \mathcal{E}_{\text{HB}}^\text{in}(R) \) can be achieved by the scheme presented in the preceding subsection.

**Theorem 2:** The set \( \mathcal{E}_{\text{HB}}^\text{in}(R) \) is achievable, i.e., \( \mathcal{E}_{\text{HB}}^\text{in}(R) \subseteq \mathcal{E}_{\text{HB}}(R) \).

**Proof:** See Appendix A.

The scheme in the preceding subsection that achieves \( \mathcal{E}_{\text{HB}}^\text{in} \) performs codeword binning. As the following Theorem 3 shows, it is optimal in some special cases. Binning has previously been considered for other distributed hypothesis testing problems [1], [4], [7], [8], [17–20]. In particular, [4], [8] proved that binning is necessary to attain an optimal error exponent.

**Definition 5:** We say that \( Z_1 \) is less noisy than \( Y_2 \), if for all auxiliary random variables \( U \) satisfying the Markov chain \( U \to X \to (Y_1, Y_2, Z_1) \) the following inequality holds:

\[ I(U; Z_1) \geq I(U; Y_2). \]  
(26)

**Theorem 3:** When \( Z_1 \) is less noisy than \( Y_2 \), \( \mathcal{E}_{\text{HB}}^\text{in} = \mathcal{E}_{\text{HB}} \).

**Proof:** Achievability follows from Theorem 2. The converse is proved in Appendix B.

**D. Example**

Theorem 3 was stated and proved for discrete memoryless sources. It can be shown that it remains valid also when sources are memoryless and jointly Gaussian [15, Chap. 3].

Consider the following scenario. Under both hypotheses, \( X \sim \mathcal{N}(0, 1) \) and \( Z_1 = X + N_z \), where \( N_z \sim \mathcal{N}(0, \sigma_z^2) \) is independent of \( X \). Moreover, under hypothesis

\[ \mathcal{H} = 0: \quad Y_1 = X + Z_1 + N_1 \]  
(27)

\[ Y_2 = Z_1 + N_2, \]  
(28)
where \( N_1 \sim \mathcal{N}(0, \sigma_1^2) \) and \( N_2 \sim \mathcal{N}(0, \sigma_2^2) \) are independent of each other and of \((X, Z_1)\), and under hypothesis

\[
\mathcal{H} = 1: \quad Y_1 = X' + \frac{2 + \sigma_x^2}{1 + \sigma_x^2} \cdot Z_1 + N_1, \quad (29)
\]

\[
Y_2 = Z_1' + N_2, \quad (30)
\]

where \( X' \sim \mathcal{N}(0, \frac{\sigma_x^2}{1 + \sigma_x^2}) \) and \( Z_1' \sim \mathcal{N}(0, 1 + \sigma_x^2) \) are independent of each other and of the tuple \((X, Z_1, N_1, N_2)\).

The described scenario satisfies the less noisy condition in (26). By Theorem 3, for this example region \( \mathcal{E}_{HB} \) is thus equal to \( \mathcal{E}_{HB}^{in} \). Moreover, the exponents region \( \mathcal{E}_{HB}(R) \) evaluates to the set of all nonnegative exponent pairs \((\theta_1, \theta_2)\) that satisfy

\[
\theta_1 \leq \frac{1}{2} \log \left( \frac{\sigma_x^2 + \sigma_z^2 (1 + \sigma_x^2)}{2^{\alpha_x} \sigma_x^2 + \sigma_z^2 (1 + \sigma_x^2)} \right), \quad (31a)
\]

\[
\theta_2 \leq \frac{1}{2} \log \left( \frac{1 + \sigma_x^2 + \sigma_z^2}{2^{-2(\alpha_x + R)} (1 + \sigma_x^2) + \sigma_z^2} \right), \quad (31b)
\]

for some \( \alpha_x \in [-R, 0] \).

That these exponent pairs lie in \( \mathcal{E}_{HB}(R) \) can be seen by evaluating (6)–(8) for auxiliaries \( U_0 \) and \( U_1 \) that are jointly Gaussian with \( X \) and so that \( X = U_1 + W_1 \) and \( U_1 = U_0 + W_0 \) for independent zero-mean Gaussians \( W_1, W_0 \) and \( U_0 \) that are of variances \( \frac{\sigma_x^2}{(\sigma_x^2 + 1)^2} \), \( (\sigma_x^2 + 1)^2 \), \( (\sigma_x^2 + 2^{-2(\alpha_x + R)} - \sigma_x^2) (1 + \frac{1}{(\sigma_x^2 + 1)^{2 - 2\alpha_x - 1}}) \) and \( (1 + \sigma_x^2) (1 - 2^{-2(\alpha_x + R)}) \), respectively.

The proof that \( \mathcal{E}_{HB}(R) \) is no larger than the proposed region is similar to the converse proof for the example in Section II-D. Indeed, by the EPI:

\[
h(Y_2|U_0) \geq \frac{1}{2} \log \left( 2^{2h(Z_1|U_0)} + 2^{2h(N_2)} \right),
\]

\[
h(Y_1|U_0, U_1, Z_1) \geq \frac{1}{2} \log \left( 2^{2h(X|U_0, U_1, Z_1)} + 2^{2h(N_1)} \right). \quad (32)
\]

Moreover, rate-constraint (25) is equivalent to

\[
I(U_0; X) + I(U_1; X|U_0, Z_1) = h(X) - h(X|U_0) + h(X|U_0, Z_1) - h(X|U_0, U_1, Z_1)
\]

\[
= h(X) - I(X; Z_1|U_0) - h(X|U_0, U_1, Z_1)
\]

\[
= h(X) - h(Z_1|U_0) + h(Z_1|X, U_0) - h(X|U_0, U_1, Z_1)
\]

\[
= h(X, Z_1) - h(Z_1|U_0) - h(X|U_0, U_1, Z_1), \quad (33)
\]
where the last equality follows from the Markov chain $U_0 \rightarrow X \rightarrow Z_1$.

Defining now
\[
\alpha := h(X|U_0, U_1, Z_1) \quad \text{and} \quad \beta := h(Z_1|U_0),
\]
above inequalities show that $\mathcal{E}_{HB}(R)$ is included in the set of all pairs $(\theta_1, \theta_2)$ that satisfy
\[
\theta_1 \leq h(Y_1|Z_1) - \frac{1}{2} \log \left( 2^{2\alpha} + 2^{2h(N_1)} \right) \tag{35}
\]
\[
\theta_2 \leq h(Y_2) - \frac{1}{2} \log \left( 2^{2\beta} + 2^{2h(N_2)} \right), \tag{36}
\]
for some choice of parameters $\alpha \leq h(X|Z_1)$ and $\beta \leq h(Z_1)$ so that
\[
(\alpha - h(X|Z_1)) + (\beta - h(Z_1)) \geq -R. \tag{37}
\]
Now, since the right-hand sides of (35) and (36) are decreasing in the parameters $\alpha$ and $\beta$, these parameters should be chosen so that the rate-constraint (37) is satisfied with equality. In other words, for fixed $\alpha$, the optimal $\beta$ is obtained by solving (37) under the equality constraint.

Defining $\tilde{\alpha} := (\alpha - h(X|Z_1)) \leq 0$ and expressing the optimal $\beta$ in terms of $\tilde{\alpha}$ then establishes the desired inclusion of $\mathcal{E}_{HB}(R)$ in the set of pairs $(\theta_1, \theta_2)$ given in (31).

The boundary of the exponents region $\mathcal{E}_{HB}(R)$ is illustrated in Fig. 5 for different values of the rate $R$. Generally, on this boundary $\theta_1 > \theta_2$, because Receiver 1 has the additional side-information $Z_1$. One further observes a trade-off between the two exponents $\theta_1$ and $\theta_2$. In other words, having a larger exponent $\theta_1$ comes at the expense of a smaller exponent $\theta_2$, and vice versa. In the example without side-information in Section II-D such a tradeoff did not arise.
E. Extension: Gray-Wyner Network with Side-Information

The exponents regions of Theorems 1 and 2 can be combined to derive an exponents region that is achievable in a scenario with three bit pipes and the additional side information $Z_1^n$ at Receiver 1.

For this extended scenario, we propose to apply the code construction, encoding and decision at Receiver 2 described in Section II-B and the decision at Receiver 1 described in Section III-B. This strategy achieves the exponents region $E_{\text{general}}$ presented in the following remark.

**Remark 2:** Let $E_{\text{general}}(R_0, R_1, R_2)$ be the set of all nonnegative pairs $(\theta_1, \theta_2)$ that satisfy

\begin{align}
\theta_1 & \leq I(U_0, U_1; Y_1|Z_1), \\
\theta_2 & \leq I(U_0, U_2; Y_2),
\end{align}

for some auxiliary random variables $(U_0, U_1, U_2)$ satisfying the Markov chain $(U_0, U_1, U_2) \to X \to (Y_1, Y_2, Z_1)$ and

\begin{align}
R_0 & \geq I(U_0; X), \\
R_1 & \geq I(U_1; X|U_0, Z_1), \\
R_2 & \geq I(U_2; X|U_0).
\end{align}

The exponents region $E_{\text{general}}(R_0, R_1, R_2)$ is achievable over the Heegard-Berger network with side-information when there is an additional noise-free bit pipe of rate $R_i$ from the transmitter to Receiver $i$, for $i \in \{1, 2\}$. Moreover, to evaluate $E_{\text{general}}(R_0, R_1, R_2)$ it suffices to consider auxiliary random variables $U_0, U_1, U_2$ over alphabets $U_0, U_1, U_2$ whose sizes satisfy the following three conditions: $|U_0| \leq |X| + 4$, $|U_1| \leq |X| \cdot |U_0| + 2$, and $|U_2| \leq |X| \cdot |U_0| + 2$.

IV. HYPOTHESIS TESTING OVER NOISY BCs

A. Setup

This section considers hypothesis testing over a discrete memoryless BC $(\mathcal{W}, \mathcal{V}_1, \mathcal{V}_2, P_{V_1V_2|W})$, where $\mathcal{W}$ denotes the finite channel input alphabet, $\mathcal{V}_1$ and $\mathcal{V}_2$ the finite channel output alphabets at Receivers 1 and 2, and $P_{V_1V_2|W}$ the BC transition pmf. We assume that the BC is degraded, so that for each $(w, v_1, v_2) \in \mathcal{W} \times \mathcal{V}_1 \times \mathcal{V}_2$ the BC transition pmf decomposes as $P_{V_1V_2|W}(v_1, v_2|w) =$
Trans.  \[ P_{V_1|W}(v_1|w) \cdot P_{V_2|V_1}(v_2|v_1) \]. The transmitter observes a sequence \( X^n \) and produces its channel inputs \( W^n := (W_1, \ldots, W_n) \) as \( W^n = \Phi(n)(X^n) \) by means of a possibly stochastic encoding function \( \Phi(n): \mathcal{X}^n \to \mathcal{W}^n \). Receivers 1 and 2 observe the corresponding channel outputs \( V^n_1 := (V_{1,1}, \ldots, V_{1,n}) \) and \( V^n_2 := (V_{2,1}, \ldots, V_{2,n}) \), as well as the source sequences \( (Y^n_1, Z^n_1) \) and \( Y^n_2 \) defined in the previous sections. Both receivers guess the hypothesis \( \mathcal{H} \) underlying the joint distribution of the source sequences \( X^n, Y^n_1, Y^n_2, Z^n_1 \). Assume that under hypothesis

\[
\mathcal{H} = 0: (X^n, Y^n_1, Y^n_2, Z^n_1) \sim \text{i.i.d. } P_{X,Y_1,Y_2,Z_1},
\]

and under hypothesis

\[
\mathcal{H} = 1: (X^n, Y^n_1, Y^n_2, Z^n_1) \sim \text{i.i.d. } P_{X,Z_1}P_{Y_1|Z_1}P_{Y_2}.
\]

Type-I and type-II errors as well as the exponents region are defined in analogy to the previous two sections. We denote the exponents region for this scenario by \( \mathcal{E}_{\text{noisy}} \).

**B. Coding Scheme**

Fix \( \mu > 0 \), sufficiently large positive integers \( k \) and \( B \), and a joint conditional distribution \( P_{U_0U_1|X} \) over finite auxiliary alphabets \( \mathcal{U}_0 \) and \( \mathcal{U}_1 \). Consider also nonnegative rates \( R_0, R_1, R'_1 \) that satisfy

\[
R_1 \leq I(U_1; V_1, Z_1|U_0) \tag{45}
\]

\[
R_0 \leq I(U_0; V_2) \tag{46}
\]

\[
R_0 > I(U_0; X), \tag{47}
\]

\[
R_1 > I(U_1; X|U_0). \tag{48}
\]
Finally, fix a function \( f : \mathcal{U}_0 \times \mathcal{U}_1 \times \mathcal{X} \to \mathcal{W} \).

**Code Construction:** For each block \( b \in \{1, ..., B\} \), randomly generate a codebook \( \mathcal{C}_{0,b} = \{ U_{0,b}^k(m_0) : m_0 \in \{1, ..., 2^{kR_0}\} \} \) by drawing each entry of the \( n \)-length codeword \( U_{0,b}^k(m_0) \) i.i.d. according to the pmf \( P_{U_0} \). Moreover, for each index \( m_0 \), randomly generate a codebook \( \mathcal{C}_{1,b}(m_0) := \{ U_{1,b}^k(m_1|m_0) : m_1 \in \{1, ..., 2^{kR_1}\} \} \) by drawing each entry of the \( k \)-length codeword \( U_{1,b}^k(m_1|m_0) \) i.i.d. according to the conditional pmf \( P_{U_1|U_0}(\cdot|U_{0,b,j}(m_0)) \), where \( U_{0,b,j}(m_0) \) denotes the \( j \)-th symbol of \( U_{0,b}^k(m_0) \). Reveal the realizations \( \{ \mathcal{C}_{0,b} \} \) and \( \{ \mathcal{C}_{1,b}(\cdot) \} \) of the randomly generated codebooks to all terminals.

**Transmitter:** It observes a source sequence \( x^n \) and splits it into \( B \) subblocks \( x^n = (x_1^k, ..., x_B^k) \).

For each block \( b \), it looks for a pair of indices \( (m_{0,b}, m_{1,b}) \in \{1, ..., 2^{nR_1}\} \times \{1, ..., 2^{nR_2}\} \) such that \( (x_b^k, u_{0,b}^k(m_{0,b}), u_{1,b}^k(m_{1,b}|m_{0,b})) \in T_{n/2}^k(P_{XU_0U_1}) \). where \( u_{0,b}^k(m_{0,b}) \) and \( u_{1,b}^k(m_{1,b}|m_{0,b}) \) are codewords from the chosen codebooks \( \mathcal{C}_{0,b} \) and \( \{ \mathcal{C}_{1,b}(\cdot) \} \). If the typicality test is successful, the transmitter picks one of the pairs satisfying the test at random. Otherwise, it picks a pair \( (m_{0,b}, m_{1,b}) \) uniformly at random over \( \{1, ..., 2^{nR_1}\} \times \{1, ..., 2^{nR_2}\} \). It finally sends the \( k \) inputs \( w_{(b-1)k+j} = f(u_{0,b,j}(m_{0,b}), u_{1,b,j}(m_{1,b}|m_{0,b}), x_{(b-1)k+j}), j \in \{1, ..., k\} \), over the channel.

**Receiver 1:** Assume that it observes the sequence of channel outputs \( v_1^n \) and the source sequences \( y_1^n \) and \( z_1^n \). It decomposes its observations into \( B \) blocks, \( \{(v_{1,b}^k, y_{1,b}^k, z_{1,b}^k, \mathcal{C}_{0,b}, \{ \mathcal{C}_{1,b}(\cdot) \})\}_{b=1}^B \). It then applies a Neyman-Pearson test to decide on \( \mathcal{H} \) based on these i.i.d. blocks, in a way that the type-I error probability does not exceed \( \epsilon/4 \).

**Receiver 2:** Assume that it observes the sequence of channel outputs \( v_2^n \) and the source sequences \( y_2^n \). It decomposes its observations according to the \( B \) blocks, \( \{(v_{2,b}^k, y_{2,b}^k, \mathcal{C}_{0,b})\}_{b=1}^B \). It then applies a Neyman-Pearson test to decide on \( \mathcal{H} \) based on these i.i.d. blocks, so that the type-I error probability does not exceed \( \epsilon/4 \).

**C. Exponents Region**

Let \( \mathcal{E}_{\text{hyb}} \) be the set of all nonnegative pairs \((\theta_1, \theta_2)\) satisfying

\[
\theta_1 \leq I(U_0; U_1; Y_1|Z_1), \tag{49}
\]

\[
\theta_2 \leq I(U_0; Y_2), \tag{50}
\]
for some auxiliary random variables \((U_0, U_1)\) over finite auxiliary alphabets \(U_0 \times U_1\) satisfying the Markov chains

\[
(U_0, U_1) \to X \to (Z_1, Y_1, Y_2),
\]

and some function \(f : U_0 \times U_1 \times \mathcal{X} \to \mathcal{W}\) where \(W = f(U_0, U_1, X)\) and

\[
I(U_1; X | U_0) \leq I(U_1; V_1, Z_1 | U_0),
\]

\[
I(U_0; X) \leq I(U_0; V_2).
\]

**Theorem 4:** The set of \(\mathcal{E}_{\text{hyb noisy}}\) is achievable, i.e., \(\mathcal{E}_{\text{hyb noisy}} \subseteq \mathcal{E}_{\text{noisy}}\).

*Proof:* See Appendix C.

To evaluate the region \(\mathcal{E}_{\text{noisy}}\), it suffices to consider auxiliaries whose alphabets satisfy the following two conditions: \(|U_0| \leq |\mathcal{X}| + 4\) and \(|U_1| \leq |\mathcal{X}| \cdot |U_0| + 2\). The exponents region \(\mathcal{E}_{\text{hyb noisy}}\) is achieved by means of hybrid joint source-channel coding with side-information. Constraints (52), (53), and (54) ensure that the receivers can decode their intended hybrid coding codewords; a \(U_0\)-codeword is decoded at both receivers and a \(U_1\)-codeword at Receiver 1 only. These codewords are then used at the receivers for testing against conditional independence, see the exponents expression in (49) and (50). Notice that hybrid joint source-channel coding also includes separate source-channel coding as a special case [14]. In fact, the separate scheme’s exponents region can be derived by considering \(U_0 = (W_0, \tilde{U}_0)\) and \(U_1 = (W, \tilde{U}_1)\) where \((\tilde{U}_0, \tilde{U}_1, W_0)\) are auxiliary random variables which satisfy the Markov chains \((\tilde{U}_0, \tilde{U}_1) \to X \to Z_1\) and \(W_0 \to W \to (V_1, V_2)\), and \((W_0, W)\) are independent of \((\tilde{U}_0, \tilde{U}_1, X, Y_1, Z_1, Y_2)\).

The analysis steps in Appendix C can be adapted to any joint source-channel coding scheme based on random coding other than hybrid coding. Combined with Neyman-Pearson testing at the receivers, such a scheme can be shown to achieve error exponents as in \(\mathcal{E}_{\text{hyb noisy}}\) except that (52)–(54) have to be replaced by the conditions of the new joint source-channel coding scheme and \(U_0\) and \(U_1\) by the random variables associated with the codewords decoded at each of the two receivers.
V. CONCLUSION AND DISCUSSION

This paper considers a distributed hypothesis testing problem in a one-observer, two-decision center setup. The type-II error exponents region is derived for testing against independence when communication from the observer to the decision centers takes place over a common and two individual noise-free bit pipes. The type-II error exponents region is also derived for testing against conditional independence when there is only a single noise-free bit pipe and the additional observation at the first decision center is less noisy than the observation at the other decision center. An achievable type-II error exponents region is finally derived for the problem of testing against conditional independence when communication takes place over a discrete memoryless broadcast channel.

The coding scheme achieving the described type-II error exponents in the last two scenarios applies the following approach:

- all terminals split their observations into many subblocks;
- the transmitter encodes using Heegard-Berger compression with binning or using hybrid joint source-channel coding;
- the receiver applies a Neyman-Pearson test over the multiple subblocks of channel and source observations.

In above approach, the “multi-letter” decision over subblocks avoids introducing a competing error exponent due to the binning or the channel decoding procedure. Closely related schemes were also proposed in [4] and [10] for noise-free or noisy multi-access channels. The analysis introduced here is easily adapted to any desired joint source-channel coding scheme that is based on random coding arguments.

VI. ACKNOWLEDGEMENT

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APPENDIX A

PROOF OF THEOREM 2

We analyze the probability of error of the scheme in Section III-B, but assuming that the code construction is random and independent across blocks. We treat the realization of this construction as an additional observation at all terminals. Notice that the codebook observation is independent of the source observations $X^n, Y^n_1, Y^n_2, Z^n_1$. The type-I and type-II error probabilities of this random experiment represent expected values of the corresponding error probabilities achieved with fixed choices of codebooks. We denote these expected values by $\mathbb{E}_{C_0^b, C_1^b}[\alpha_{i,n}]$ and $\mathbb{E}_{C_0^b, C_1^b}[\beta_{i,n}]$, for $i \in \{1, 2\}$.

The steps in the following analysis are inspired by the analysis in [4]. We first analyze the type-I error probabilities. To this end, define $\mathcal{E}^{(1)}_{\text{NP},i}$ to be the event that the Neyman-Pearson test at Receiver $i$ decides on the hypothesis $\mathcal{H} = 1$. We then have

$$\mathbb{E}_{C_0^b, C_1^b}[\alpha_{i,n}] = \Pr[\mathcal{E}_{\text{NP},i}^{(1)} \text{ or } M = 0 | \mathcal{H} = 0]$$

$$\leq \Pr [M = 0 | \mathcal{H} = 0] + \Pr[\mathcal{E}_{\text{NP},i}^{(1)} | \mathcal{H} = 0] \leq \epsilon/8 + \epsilon/8 = \epsilon/4,$$

where (a) holds by the covering lemma and by the rate constraints (19) and (20) which imply that $\Pr [M = 0 | \mathcal{H} = 0] \leq \epsilon/8$, and because the Neyman-Pearson test has been designed so that $\Pr[\mathcal{E}_{\text{NP},i}^{(1)} | \mathcal{H} = 0] \leq \epsilon/8$.

We now analyze the type-II error probabilities. Recall that each Receiver $i$ only declares $\hat{H}_i = 0$ if the applied Neyman-Pearson test produces $0$. Notice that the sequence of tuples

$$\{M_{0,b}, M_{1,b}, S_{1,b}, U^k_{0,b}(M_{0,b}), U^k_{1,b}(S_{1,b}|M_{0,b}), X^k_b, Y^k_{1,b}, Y^k_{2,b}, Z^k_{1,b}, C_0, C_1, \{C_i(\cdot)\}\}_{b=1}^B$$

is i.i.d. according to a pmf $P_{M_{0,b}M_{1,b}S_{1,b}U^k_{0,b}S_{1,b}|M_{0,b}}X^k_bY^k_{1,b}Y^k_{2,b}Z^k_{1,b}C_0C_1$. The Chernoff-Stein Lemma [21] can thus be applied to bound the probabilities of type-II error.

Consider first Receiver 1. By the Chernoff-Stein Lemma, for any $\mu' > 0$ and sufficiently large
because these tuples are independent of the codebooks.

\[ \frac{1}{n} \log \mathbb{E}_{c_0, c_1} [\beta_{1,n}] \geq \frac{1}{k} D( P_{M_0, M_1, Y_1^k Y_2^k Z_1^k c_0 c_1 | H=0} \parallel P_{M_0, M_1, Y_1^k Y_2^k Z_1^k c_0 c_1 | H=1}) - \mu' \]

(a) \[ \frac{1}{k} I(M_0, M_1; Y_1^k | Z_1^k, C_0, C_1) - \mu' \]

= \frac{1}{k} H(Y_1^k | Z_1^k, C_0, C_1) - \frac{1}{k} H(Y_1^k | M_0, M_1, Z_1^k, C_0, C_1) - \mu'

(b) \[ H(Y_1 | Z_1) - \frac{1}{k} H(Y_1^k | M_0, M_1, Z_1^k, C_0, C_1) - \mu' \]

(c) \[ H(Y_1 | Z_1) - \frac{1}{k} H(Y_1^k | M_0, S_1, Z_1^k, C_0, C_1) - \mu' \]

\[ - \frac{1}{k} I(Y_1^k; S_1 | M_0, M_1, Z_1^k, C_0, C_1). \] \tag{57}

Here, (a) holds by the assumptions on the tuples \( X^n, Y_1^n, Y_2^n, Z_1^n \) under the two hypotheses and because these tuples are independent of the codebooks \( C_0, \{ C_1 \} \). Step (b) holds by the same independence and because \( (Y_1^n, Z_1^n) \) are i.i.d. Step (c) holds by adding and subtracting the term \( \frac{1}{k} H(Y_1^k | S_1, M_0, M_1, Z_1^k, C_0, C_1) \) and by noting that it is equal to \( \frac{1}{k} H(Y_1^k | S_1, M_0, Z_1^k, C_0, C_1) \).

We show that the last term in (57) becomes arbitrarily small for \( k \to \infty \). In fact, by the covering lemma, the Markov lemma, and the packing lemma [15], if the rate constraints in (19)–(21) hold, there exists a function \( \zeta^k \) such that for any \( \mu_1 > 0 \) and all sufficiently large \( k \) such that \( p_e \overset{\Delta}{=} \Pr[\zeta^k(M_0, M_1, Z_1^k, C_0, C_1) \neq S_1] \leq \mu_1 \). Therefore,

\[ I(Y_1^k; S_1 | M_0, M_1, Z_1^k, C_0, C_1) \leq H(S_1 | M_0, M_1, Z_1^k, C_0, C_1) \]

\[ \leq H(S_1 | \zeta^k(M_0, M_1, Z_1^k, C_0, C_1)) \]

(a) \[ \leq h_b(p_e) + p_e \cdot k(R_1 + R_1') \leq 1 + \mu_1 \cdot k(R_1 + R_1'), \] \tag{58}

where (a) holds by Fano’s Inequality. Inserting (58) into (57) yields:

\[ \frac{1}{n} \log \mathbb{E}_{c_0, c_1} [\beta_{1,n}] \geq H(Y_1 | Z_1) - \frac{1}{k} H(Y_1^k | Z_1^k, M_0, S_1, C_0, C_1) - \frac{1}{k} - \mu_1(R_1 + R_1') - \mu' \]

\[ \geq H(Y_1 | Z_1) - \frac{1}{k} H(Y_1^k | Z_1^k, U_0^k(M_0), U_1^k(S_1 | M_0), C_0, C_1) - \frac{1}{k} - \mu_1(R_1 + R_1') \]

\[ - \mu' \]

\[ \geq H(Y_1 | Z_1) - \frac{1}{k} H(Y_1^k | Z_1^k, U_0^k(M_0), U_1^k(S_1 | M_0)) - \frac{1}{k} - \mu_1(R_1 + R_1') - \mu' \] \tag{59}
We continue by defining the event
\[ \mathcal{E}_V \triangleq \{(U_0^k(M_0), U_1^k(S_1|M_0), Y_1^k, Z_1^k) \in T^k_{\mu}(P_{U_0U_1Y_1Z_1})\}. \]
Let \( \mathbb{1}_V \) be the indicator function of event \( \mathcal{E}_V \). By the covering lemma and the Markov lemma, and the rate-constraints \((19)\) and \((20)\), for any \( \mu_2 > 0 \) and for all sufficiently large \( k \),
\[
\Pr[\mathbb{1}_V = 0] \leq \mu_2. \tag{60}
\]
For ease of notation, in the following we abbreviate \( U_0^k(M_0) \) by \( U_0^k \) and \( U_1^k(S_1|M_0) \) by \( U_1^k \).
The second term on the RHS of \((59)\) can be upper bounded as follows:
\[
H(Y_1^k | Z_1^k, U_0^k, U_1^k)
\]
\[\leq H(Y_1^k | Z_1^k, U_0^k, U_1^k, \mathbb{1}_V) + H(\mathbb{1}_V | Z_1^k U_0^k, U_1^k)\]
\[\leq \sum_{(u_0^k, u_1^k, z_1^k) \in T_{\mu}^k} \Pr[Z_1^k = z_1^k, U_0^k = u_0^k, U_1^k = u_1^k | \mathbb{1}_V = 1] \cdot H(Y_1^k | Z_1^k, U_0^k = u_0^k, U_1^k = u_1^k, \mathbb{1}_V = 1)
\]
\[+ k \log |\mathcal{Y}_1| \cdot \mu_2 + 1\]
\[\leq \sum_{(u_0^k, u_1^k, z_1^k) \in T_{\mu}^k} \Pr[Z_1^k = z_1^k, U_0^k = u_0^k, U_1^k = u_1^k | \mathbb{1}_V = 1] \cdot \log(|T_{\mu}^k(Y_1^k | u_0^k, u_1^k, z_1^k)|) + k \log |\mathcal{Y}_1| \cdot \mu_2 + 1\]
\[\leq kH(Y_1^k | Z_1^k, U_0^k, U_1^k) + k\delta'(\mu) + k \log |\mathcal{Y}_1| \cdot \mu_2 + 1. \tag{61}\]
The steps leading to \((61)\) are justified as follows:

- (a) follows from the fact \( H(\mathbb{1}_V | Z_1^k, M_0, S_1) \leq 1 \) since \( \mathbb{1}_V \) is a binary random variable,
- (b) follows from \((60)\), \( \Pr[\mathbb{1}_V = 1] \leq 1 \) and \( H(Y_1^k | Z_1^k, M_0, S_1, \mathbb{1}_V = 0) \leq k \log |\mathcal{Y}_1| \),
- (c) follows because entropy is maximized by a uniform distribution,
- (d) follows by bounding the size of the typical set \([15]\) where \( \delta'(\mu) \) is a function that goes to 0 as \( \mu \to 0 \).
We combine (59) with (61) to obtain that for all $\mu, \mu', \mu_1, \mu_2 > 0$ and sufficiently large $k, B$:
\[
-\frac{1}{n} \log \mathbb{E}_{C_0^B, C_1^P}[\beta_{1,n}] \geq I(U_0, U_1; Y_1|Z_1) - \log |Y_1| \cdot \mu_2 - \frac{2}{k} - \mu_1 \cdot (R_1 + R_1') - \mu' - \delta'(\mu),
\]
where $\delta'(\mu)$ is a function that tends to 0 as $\mu \to 0$. Next, we consider the type-II error probability at Receiver 2. By the Chernoff-Stein lemma, we have
\[
-\frac{1}{n} \log \mathbb{E}_{C_0^B, C_1^P}[\beta_{2,n}] \geq \frac{1}{k} D(P_{MY_2^kC_0C_1|H=0}||P_{MY_2^kC_0C_1|H=1}) - \mu'
\]
\[
= \frac{1}{k} I(M; Y_2^k|C_0, C_1) - \mu'
\]
\[
\geq \frac{1}{k} I(M_0; Y_2^k|C_0, C_1) - \mu'
\]
\[
= \frac{1}{k} I(M_0, U_0^k, Y_2^k|C_0, C_1) - \mu'
\]
\[
= \frac{1}{k} H(Y_2^k|C_0, C_1) - \frac{1}{k} H(Y_2^k|M_0, U_0^k, C_0, C_1) - \mu'
\]
\[
= H(Y_2) - \frac{1}{k} H(Y_2^k|M_0, U_0^k, C_0, C_1) - \mu'
\]
\[
\geq H(Y_2) - \frac{1}{k} H(Y_2^k|U_0^k) - \mu'
\]
\[
= I(U_0; Y_2) - \mu',
\]
(62)
where again $\mu'$ can be chosen arbitrarily small as $B \to \infty$. We have thus proved that for $i \in \{1, 2\}$ and all $\tilde{\mu} > 0$ and sufficiently large $k, B$, we get
\[
\mathbb{E}_{C_0^B, C_1^P}[\alpha_{i,n}] \leq \epsilon/4
\]
(63)
\[
\mathbb{E}_{C_0^B, C_1^P}[\beta_{1,n}] \leq 2^{-n(I(U_0U_1; Y_1|Z_1) - \tilde{\mu})}
\]
(64)
\[
\mathbb{E}_{C_0^B, C_1^P}[\beta_{2,n}] \leq 2^{-n(I(U_0; Y_2) - \tilde{\mu})}.
\]
(65)
It can be shown that these expectations imply for each $\mu' > \tilde{\mu}$ and sufficiently large blocklength $n$, there exists a deterministic choice of codebooks so that
\[
\alpha_{i,n} \leq \epsilon/4,
\]
(66)
\[
\beta_{1,n} \leq 2^{-n(I(U_0U_1; Y_1|Z_1) - \mu')},
\]
(67)
\[
\beta_{2,n} \leq 2^{-n(I(U_0; Y_2) - \mu')}.
\]
(68)
We first order the sets of codebooks according to $\alpha_{1,n}$, and then restrict to the subset of codebooks that have smallest $\alpha_{1,n}$ and total probability at least $1/2$. Each of the codebooks in this restricted
set induces \( \alpha_{1,n} \leq \frac{\epsilon}{2} \). We can repeat this step by ordering the codebooks according to their values of \( \alpha_{2,n} \) and then \( \beta_{1,n}, \beta_{2,n} \). This way we construct a nonempty subset of codebooks so that for each of them \( \alpha_{i,n} \leq \epsilon, \beta_{1,n} \leq 16 \cdot 2^{-n(I(U_0;Y_1|Z_1)-\mu')} \) and \( \beta_{2,n} \leq 16 \cdot 2^{-n(I(U_0;Y_2)-\mu')} \). Introducing \( \mu' = \bar{\mu} + \frac{4}{n} \) completes the proof.

**APPENDIX B**

**PROOF OF THEOREM 3**

Fix a sequence of encoding and decoding functions \( \{\phi^{(n)}, g_1^{(n)}, g_2^{(n)}\} \) so that the inequalities in Definition 3 hold for sufficiently large blocklengths \( n \). Fix also such a sufficiently large \( n \). Then, define \( U_{0,t} \overset{\Delta}{=} (M, Z_{1,t}^{t-1}) \) and \( U_{1,t} \overset{\Delta}{=} (M, X_{1,t}^{t-1}, Z_{1,t+1}^{n}, Z_{1}^{t-1}) \). Following similar steps as in [16], it can be shown that

\[
D(P_{M,Y_1^n,Z_1^n|H=0}||P_{M,Y_1^n,Z_1^n|H=1}) \geq -(1-\epsilon) \log \beta_{1,n}.
\]

Therefore, the type-II error probability at Receiver 1 can be upper bounded as

\[
-\frac{1}{n} \log \beta_{1,n} \leq \frac{1}{n(1-\epsilon)} D(P_{M,Y_1^n,Z_1^n|H=0}||P_{M,Y_1^n,Z_1^n|H=1})
\]

\( \overset{(a)}{=} \frac{1}{n(1-\epsilon)} I(M; Y_1^n | Z_1^n) \)

\( = \frac{1}{n(1-\epsilon)} \sum_{t=1}^{n} I(M; Y_{1,t} | Y_1^{t-1}, Z_1^n) \)

\( \overset{(b)}{=} \frac{1}{n(1-\epsilon)} \sum_{t=1}^{n} I(M, Y_{1,t}^{t-1}, Z_1^{t-1}, Z_{1,t+1}^{n}; Y_{1,t}|Z_{1,t}) \)

\( \overset{(c)}{= \phantom{=}} \frac{1}{n(1-\epsilon)} \sum_{t=1}^{n} I(M, X_{1,t}^{t-1}, Z_1^{t-1}, Z_{1,t+1}^{n}; Y_{1,t}|Z_{1,t}) \)

\( = \frac{1}{n(1-\epsilon)} \sum_{t=1}^{n} I(U_{0,t}, U_{1,t}; Y_{1,t}|Z_{1,t}) \)

where (a) follows because under hypothesis \( H = 1 \) and given \( Z_1^n \), the sequence \( Y_1^n \) and message \( M \) are independent; (b) follows from the memoryless property of the sources; (c) follows from the Markov chain \( (Y_{1,t}, Z_{1,t}) \rightarrow (M, X_{1,t}^{t-1}, Z_1^{t-1}, Z_{1,t+1}^{n}) \) \( \rightarrow Y_1^n \). For the type-II error probability at Receiver 2, one obtains:

\[
-\frac{1}{n} \log \beta_{2,n} \leq \frac{1}{n(1-\epsilon)} D(P_{MY_2^n|H=0}||P_{MY_2^n|H=1}) = \frac{1}{n(1-\epsilon)} I(M; Y_2^n)
\]
\[
= \frac{1}{n(1 - \epsilon)} \sum_{t=1}^{n} I(M; Y_{2,t} | Y_{2,t+1}^{n})
\]
\[
= \frac{1}{n(1 - \epsilon)} \sum_{t=1}^{n} \left[ I(M, Z_{1,t}^{t-1}; Y_{2,t} | Y_{2,t+1}^{n}) - I(Z_{1,t}^{t-1}; Y_{2,t} | M, Y_{2,t+1}^{n}) \right]
\]
\[
= \frac{1}{n(1 - \epsilon)} \sum_{t=1}^{n} \left[ I(M, Z_{1,t}^{t-1}; Y_{2,t+1}^{n}; Y_{2,t}) - I(Z_{1,t}^{t-1}; Y_{2,t} | M, Y_{2,t+1}^{n}) \right]
\]
\[
= \frac{1}{n(1 - \epsilon)} \sum_{t=1}^{n} \left[ I(M, Z_{1,t}^{t-1}, Y_{2,t+1}^{n}; Y_{2,t}) - I(Y_{2,t+1}^{n}; Z_{1,t} | M, Z_{1,t}^{t-1}) \right]
\]
\[
\leq \frac{1}{n(1 - \epsilon)} \sum_{t=1}^{n} \left[ I(M, Z_{1,t}^{t-1}, Y_{2,t+1}^{n}; Y_{2,t}) - I(Y_{2,t+1}^{n}; Y_{2,t} | M, Z_{1,t}^{t-1}) \right]
\]
\[
= \frac{1}{n(1 - \epsilon)} \sum_{t=1}^{n} I(M, Z_{1,t}^{t-1}; Y_{2,t})
\]
\[
= \frac{1}{n(1 - \epsilon)} \sum_{t=1}^{n} I(U_{0,t}; Y_{2,t})
\]

where (b) follows from the memoryless property of the sources; (c) follows from Csiszar and Körner’s sum identity \([15]\); and (d) follows from the less noisy assumption (see Definition \([5]\) and the Markov chain \((M, Y_{2,t+1}^{n}, Z_{1,t}^{t-1}) \rightarrow X_t \rightarrow (Y_{1,t}, Y_{2,t}, Z_{1,t})\) which holds by the memoryless property of the sources and because \(M\) is a function of \(X^n\). For the rate \(R\), one finds:

\[
nR \geq H(M) \geq I(M; X^n, Z_1^n)
\]
\[
= I(M; X^n | Z_1^n) + I(Z_1^n; M)
\]
\[
= \sum_{t=1}^{n} \left[ I(M; X_t | X^{t-1}, Z_1^n) + I(M; Z_{1,t} | Z_1^{t-1}) \right]
\]
\[
= \sum_{t=1}^{n} \left[ I(M, X^{t-1}, Z_1^n, Z_{1,t+1}^{t}; X_t | Z_{1,t}) + I(M, Z_1^{t-1}; Z_{1,t}) \right]
\]
\[
= \sum_{t=1}^{n} \left[ I(X^{t-1}, Z_1^n, Z_{1,t+1}^{t}; X_t | M, Z_{1,t}, Z_1^{t-1}) + I(M, Z_1^{t-1}; X_t | Z_{1,t}) + I(M, Z_1^{t-1}; Z_{1,t}) \right]
\]
\[
= \sum_{t=1}^{n} \left[ I(X^{t-1}, Z_1^n, Z_{1,t+1}^{t}; X_t | M, Z_{1,t}, Z_1^{t-1}) + I(M, Z_1^{t-1}; Z_{1,t} | X_t) \right]
\]
\[
\geq \sum_{t=1}^{n} \left[ I(X^{t-1}, Z_1^n, Z_{1,t+1}^{t}; X_t | M, Z_{1,t}, Z_1^{t-1}) + I(M, Z_1^{t-1}; X_t) \right]
\]
\[
= \sum_{t=1}^{n} \left[ I(U_{1,t}; X_t | Z_{1,t}, U_{0,t}) + I(U_{0,t}; X_t) \right].
\]

Notice that by the memoryless property of the sources and because \(M\) is a function of \(X^n\), the Markov chain \((M, Z_1^n, X^{t-1}) \rightarrow X_t \rightarrow (Y_{1,t}, Y_{2,t}, Z_t)\) holds, and thus \((U_{0,t}, U_{1,t}) \rightarrow \)
$X_t \to (Y_{1,t}, Y_{2,t}, Z_t)$. The proof is then concluded by combining these observations with standard time-sharing arguments which require introducing the auxiliary random variables $T \in \{1, \ldots, n\}$, $U_0 \triangleq (U_{0,T}, T)$, $U_1 \triangleq U_{1,T}$, $X \triangleq X_T$, $Y_1 \triangleq Y_{1,T}$, $Y_2 \triangleq Y_{2,T}$, and $Z_1 \triangleq Z_{1,T}$.

**APPENDIX C**

**PROOF OF THEOREM 4**

We analyze the probability of error of the scheme in Section [V-B]. Consider the random experiment where the code construction is random and i.i.d. over blocks, and revealed to the terminals as additional observations. The type-I and type-II error probabilities of this random experiment coincide with the expected type-I and type-II error probabilities $E_0, E_1$ terms as additional observations. The type-I and type-II error probabilities of this random experiment where the code construction is random and i.i.d. over blocks, and revealed to the experiment. By construction of the Neyman-Pearson tests employed at the two receivers, we get $E_0, E_1$ the sequence of tuples

$$\{ M_{0,b}, M_{1,b}, U_{0,b}^k (M_{0,b}), U_{1,b}^k (M_{1,b} | M_{0,b}), W_0^k, V_{1,b}^k, V_{2,b}^k, X_{1,b}^k, Y_{1,b}^k, Y_{2,b}^k, Z_{1,b}^k, C_0, b, \{ C_{1,b}(\cdot) \} \}^{B}_{b=1}$$

is i.i.d. according to a pmf $P_{M_0 M_1 U_0^k U_1^k W_1^k Z_1^k Y_1^k Y_2^k X_1^k X_2^k Z_0^k C_0 C_1}$. Thus, by the Chernoff-Stein Lemma, for any $\mu' > 0$ and sufficiently large $k$:

$$-\frac{1}{n} \log E_{c_0, c_1} [\beta_{1,n}] \geq \frac{1}{k} D \left( P_{V_1^k Y_1^k Z_1^k C_0 C_1} | H = 0 \right) - \mu'$$

$$= \frac{1}{k} I (V_1^k, Y_1^k | Z_1^k, C_0, C_1) - \mu'$$

$$= \frac{1}{k} H (Y_1^k | Z_1^k, C_0, C_1) - \frac{1}{k} H (Y_1^k | V_1^k, Z_1^k, C_0, C_1) - \mu'$$

$$= H (Y_1 | Z_1) - \frac{1}{k} H (Y_1 | V_1, Z_1^k, C_0, C_1) - \mu'$$

$$= H (Y_1 | Z_1) - \frac{1}{k} H (Y_1 | V_1^k, M_0, M_1, Z_1^k, C_0, C_1) - \mu'$$

$$- \frac{1}{k} I (Y_1^k; M_0, M_1 | V_1^k, Z_1^k, C_0, C_1)$$

(69)

We show that the last term in (69) becomes arbitrarily small for $k \to \infty$. In fact, from the rate constraints (45)–(48) and by the covering, the Markov, and the packing lemmas [15], there exists a function $\zeta_k$ such that for any $\mu_1 > 0$ and all sufficiently large $k$, we have $p_e \triangleq$
\[ \Pr[\zeta^k(V^k_1, Z^k_1, C_0, C_1) \neq (M_0, M_1)] \leq \mu_1. \]

Therefore, with similar steps in (58), we get

\[ I(Y^k_1; M_0, M_1|V^k_1, Z^k_1, C_0, C_1) \leq 1 + \mu_1 \cdot k(R_0 + R_1), \]

(70)

Inserting (70) into (69) yields:

\[
-\frac{1}{n} \log \mathbb{E}[c_{\beta_1,n}] \geq H(Y_1|Z_1) - \frac{1}{k} H(Y^k_1|Z^k_1, U^k_0(M_0), U^k_1(M_1|M_0), C_0, C_1) - \frac{1}{k} - \mu_1(R_0 + R_1)
- \mu' \\
\geq H(Y_1|Z_1) - \frac{1}{k} H(Y^k_1|Z^k_1, U^k_0(M_0), U^k_1(M_1|M_0)) - \frac{1}{k} - \mu_1(R_0 + R_1) - \mu'.
\]

(71)

Following similar steps as the ones leading to (59), and considering the rate-constraints (47) and (48) yields the following inequality for all \( \mu, \mu', \mu_1, \mu_2 > 0 \) and all sufficiently large values of \( B \) and \( k \):

\[
-\frac{1}{n} \log \mathbb{E}[c_{\beta_1,n}] \geq I(U_0, U_1; Y_1|Z_1) - \log |Y_1| \cdot \mu_2 - \frac{2}{k} - \mu_1 \cdot (R_0 + R_1) - \mu' - \delta'(\mu),
\]

(72)

where \( \delta'(\mu) \) is a function that tends to 0 as \( \mu \to 0 \).

The analysis of type-II error probability at Receiver 2 follows from the rate constraint (48) and similar steps to (70). We have thus proved that for \( i \in \{1, 2\} \) and all \( \bar{\mu} > 0 \) and sufficiently large \( k, B \), we have

\[
\mathbb{E}[c_{\beta_1,n}] \leq \epsilon/4
\]

(73)

\[
\mathbb{E}[c_{\beta_1,n}] \leq 2^{-n(I(U_0 U_1; Y_1|Z_1) - \bar{\mu})}
\]

(74)

\[
\mathbb{E}[c_{\beta_2,n}] \leq 2^{-n(I(U_0; Y_2) - \bar{\mu})}
\]

(75)

Following similar elimination steps as at the end of Appendix A, one can prove that there must exist at least one sequence of codebooks with the desired error probabilities. This concludes the proof of the theorem.

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