Lack of a genuine time crystal in a chiral soliton model

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In a recent publication [Phys. Rev. Lett. 124, 178902 (2020)], Öhberg and Wright claim that in a chiral soliton model it is possible to realize a genuine time crystal which corresponds to a periodic evolution of an inhomogeneous probability density in the lowest-energy state. We show that this result is incorrect and present a solution which possesses lower energy with the corresponding probability density that does not reveal any motion. It implies that the authors’ conclusion that a genuine time crystal can exist in the system they consider is not true.

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I. INTRODUCTION

The idea of a quantum time crystal was proposed by Wilczek in 2012 [1]. He considered attractively interacting bosons on a ring which formed a localized wave packet (more precisely, a bright soliton) and, in the presence of a magneticlike flux, were supposed to move periodically along a ring even if the energy of the system was the lowest possible. The existence of such a genuine time crystal would involve spontaneous breaking of the continuous time translation symmetry into a discrete time translation symmetry in the system’s ground state, in full analogy to the spontaneous breaking of the continuous time translation corresponding to a periodic evolution of an inhomogeneous probability density in the lowest-energy state. We show that this result is incorrect and present a solution which possesses lower energy with the corresponding probability density that does not reveal any motion. It implies that the authors’ conclusion that a genuine time crystal can exist in the system they consider is not true.

II. WILCZEK MODEL

A single particle on a ring (whose position is denoted by an angle $\theta$) in the presence of a constant magneticlike flux $\alpha$ is described by the Hamiltonian $H = (p - \alpha \dot{\theta})^2 / 2$. The periodic boundary conditions on a ring, i.e., $\Psi(\theta + 2\pi) = \Psi(\theta)$, imply the quantization of the particle momentum $p_\alpha = n$ where $n$ is an integer. If the flux $\alpha$ is not equal to an integer number,
then in the ground state, $\Psi_{n}(\theta) = e^{i\omega_{n}/\sqrt{2N}}$, the probability current is not zero,
\[
\frac{\partial H}{\partial P_{n}} = n - \alpha \neq 0,
\] (1)
where $n$ is the closest integer to $\alpha$. The corresponding probability density $|\Psi_{n}(\theta)|^{2}$ is spatially uniform and cannot be identified with a time crystal. Wilczek’s idea was to consider $N$ interacting bosons on a ring in the presence of the constant magneticlike flux $\alpha$ [1]. If the interactions between particles are attractive and sufficiently strong, it is known that the system forms a bright soliton in its lowest-energy state. That is, in the solitonic regime, spontaneous breaking of the space translation symmetry occurs and the system’s ground state collapses to a mean-field solution where all bosons occupy a bright soliton state [72]. Wilczek hoped that in the presence of the flux $\alpha$, not only could one observe spontaneous breaking of the space translation symmetry, but also the soliton would move periodically on a ring. However, it does not happen and the easiest way to see it is to analyze the center of mass of the system which is described by the Hamiltonian
\[
H_{CM} = (P - N\alpha)^{2}/(2N).
\] The center-of-mass momentum is quantized, $P_{n} = n \in \mathbb{Z}$, but in the ground state, the probability current related to the center-of-mass motion vanishes in the $N \to \infty$ limit regardless of a choice of $\alpha$,
\[
\frac{\partial H_{N}}{\partial P_{n}} = n/N - \alpha \approx 0.
\] (2)
Thus, if the $N$-particle system in the lowest-energy state forms a bright soliton, then the soliton does not move when $N \to \infty$ [5]. One might wonder whether the time crystal could be saved if we keep $N$ large but finite which, due to Eq. (2), would correspond to a slowly moving ground state soliton solution. It turns out that we do need the infinite $N$ limit, because otherwise the center-of-mass position is subjected to quantum fluctuations and the mean-field bright soliton description breaks down. In Wilczek’s model, the quantum fluctuations of the center-of-mass position require infinite time to appear, only when $N \to \infty$ but $N g = \text{const}$ (where $g$ is a contact interaction strength). Similarly, an ordinary space crystal is stable only in the thermodynamic limit ($N, V \to \infty$, $N/V = \text{const}$) where the energy difference between symmetry broken states and the true ground state is infinitesimally small. Otherwise, a space crystal would melt due to quantum fluctuations of the center-of-mass position [2].

It is worth analyzing the absence of a genuine time crystal also in the mean-field description. The mean-field approximation assumes that all $N$ bosons occupy the same single-particle wave function $\Psi(\theta,t)$ which fulfills the Gross-Pitaevskii equation (GPE) [73]. Assuming dimensionless variables as in Ref. [69], the GPE reads
\[
i\partial_{t}\Psi = [(-i\partial_{\theta} - \alpha)^{2} + g|\Psi|^{2}]\Psi,
\] (3)
with a contact interaction strength $g$, a constant $\alpha$, and $\langle \Psi|\Psi \rangle = 1$. As the system is confined in a ring geometry we assume that $\Psi$ fulfills periodic boundary conditions, $\Psi(\theta + 2\pi,t) = \Psi(\theta,t)$, and thus its phase can change only by $2\pi j$, where $j \in \mathbb{Z}$ is the phase winding number. The GPE, Eq. (3), is generated by the action associated with the energy functional,
\[
E_{\text{LAB}} = \int d\theta \Psi^{*} [(-i\partial_{\theta} - \alpha)^{2} + W + g|\Psi|^{2}]\Psi.
\] (4)
The energy $E_{\text{LAB}}$ is the energy of the system in the laboratory frame which we want to minimize if we are looking for a genuine time crystal. It turns out that for $g < -\pi$ and arbitrary $\alpha$, stable solitonic solutions of the GPE, Eq. (3), exist and they can move with any velocity $u$. These solutions are known analytically and can be expressed in terms of Jacobi elliptic functions and complete elliptic integrals [74–77]. Note that the fact that mean-field solitons on a ring can propagate with any velocity is consistent with the center-of-mass momentum quantization, i.e., in the limit $N \to \infty$ the momentum per particle $n/N$ becomes a continuous variable. Thus, in contrast to Wilczek’s initial claim, it does not matter if $\alpha$ is an integer or not, the lowest-energy state represented by a soliton solution reveals no periodic evolution.

### III. CHIRAL SOLITON MODEL

Let us consider the system of $N$ attractive bosons on a ring in the presence of a density-dependent gauge potential [69,71,78]. Within the mean-field description all bosons populate a Bose-Einstein condensate, where the condensate wave function $\Psi(\theta,t)$ fulfills periodic boundary conditions, i.e., $\Psi(\theta + 2\pi,t) = \Psi(\theta,t)$. In the dimensionless variables the laboratory frame energy per particle of the system reads
\[
E_{\text{LAB}} = \int d\theta \Psi^{*} [(-i\partial_{\theta} - A)^{2} + W + g|\Psi|^{2}]\Psi,
\] (5)
where $A = \frac{q}{2} + a|\Psi|^{2}$ is the density-dependent vector potential, $W = \frac{q^{2}}{4}$ is a scalar potential, $q$ is an integer, and $a$ determines the strength of the first-order density-dependent contribution to the vector potential. From now on we will refer to this model as a chiral soliton model. The chiral soliton model can be realized in ultracold atomic setups where the gauge fields $W,A$ arise as effective potentials due to light-matter interactions [78]. In particular, $q$ is related to the gradient of the laser’s phase and its quantization results from the winding number of the Laguerre-Gaussian laser beam [69]. The time evolution of the system is governed by the time-dependent GPE
\[
i\partial_{t}\Psi = [(-i\partial_{\theta} - A)^{2} - aj + W + g|\Psi|^{2}]\Psi,
\] (6)
generated by the action associated with Eq. (5), where
\[
f = -i\Psi^{*}(\partial_{\theta} - iA)\Psi + \text{c.c.}
\] (7)
is the nonlinear current. In order to answer the question whether a genuine time crystal exists in this system, we are going to look for the lowest-energy solution in the frame moving with a velocity $u$. After that, we shall return to the laboratory frame and evaluate its energy. If the soliton has a minimal energy for $u = 0$, then no genuine time crystal exists. It is crucial to be in the soliton regime where the formation of a localized wave packet is energetically favorable because only nonhomogeneous probability density that evolves periodically in time can represent a time crystal.
Switching to the frame moving with a velocity $u$, 
\[ \Psi'(\theta, t) = e^{iu\theta} \Psi(\theta, t) = \Psi(\theta + ut, t), \]
the GPE reads
\[ i\partial_t \Psi' = \left( (-i\partial_\theta - A')^2 - aJ^2 - (g - 2au)\Psi' \right) \Psi'. \tag{8} \]
where
\[ J^2 = -i\Psi'(\partial_\theta - iA')\Psi' + \text{c.c.}, \tag{9} \]
$A' = \frac{u}{2} + \frac{q}{2} + a|\Psi'|^2$, and constant contributions are accounted for in the chemical potential. Note that in the moving frame the wave function $\Psi'$ also fulfills periodic boundary conditions, i.e., $\Psi'(\theta + 2\pi, t) = \Psi'(\theta, t)$.

In order to find the lowest-energy stationary solution $\Psi'(\theta)$ in the moving frame we evolve the GPE, Eq. (8), in the imaginary time [79]. A uniform solution loses its stability for a sufficiently strong interparticle attraction $g$, where the formation of a localized lump—a soliton—becomes more energetically favorable. To identify a parameter regime of a solitonic phase in the moving frame, we perform a Bogoliubov stability analysis of the uniform solution with the phase winding number $J \in \mathbb{Z}$. $\Psi' = e^{i\theta} / \sqrt{2\pi}$. That is, we study the linear stability of the stationary solution $\Psi'$ of the GPE in Eq. (8) under a small perturbation $\delta \Psi'$, i.e., $\Psi' \rightarrow \Psi'_0 = \Psi' + \delta \Psi'$, where up to a trivial phase evolution,
\[ \delta \Psi'(\theta, t) = \sum_{k \in \mathbb{Z}} (u_k e^{ik\theta} e^{-i\omega_k t} + v_k e^{-ik\theta} e^{i\omega_k t}), \tag{10} \]
with $(u_k, v_k)$ being the eigenstates and eigenvalues of the Bogoliubov–de Gennes equations [72], respectively. It is worth emphasizing that for each real eigenvalue $\omega_k$ corresponding to the eigenvector with a positive norm $N_k = (u_k v_k) = +1$ the so-called “+ family” of the Bogoliubov modes [72], there exists also an eigenvalue $-\omega_k$ related to the eigenvector with a negative norm $N_k = -1$ (“− family”). By employing the Bogoliubov formalism one easily finds that for $g < g_{d}^{(J)}$,
\[ g_{d}^{(J)} = -\pi + 2a(2J - q) - 3a^2 / \pi, \tag{11} \]
the uniform solution $\Psi' = e^{i\theta} / \sqrt{2\pi}$ is dynamically unstable. However, for $g_{c}^{(J)} < g < g_{d}^{(J)}$, where
\[ g_{c}^{(J)} = -\pi + 2au + 4\pi (\Omega - J)^2, \tag{12} \]
with $\Omega = \frac{u}{2} + \frac{q}{2} + \frac{g c}{2}$, there is a negative eigenvalue of a “+ family” Bogoliubov mode [72], and consequently $\Psi' = e^{i\theta} / \sqrt{2\pi}$ describes a dynamically stable excited state of the system. In a result, for $g < g_{c}$,
\[ g_{c} = \min_{J \in \mathbb{Z}} g_{c}^{(J)}, \tag{13} \]
there exists a stationary soliton solution which represents the lowest-energy state of the system in the moving frame. Note that the critical values of the interaction strength $g$ are different than reported in Refs. [69,71]. In Fig. 1 we illustrate the influence of different values of the parameters $a$ and $q$ on the critical interaction strength $g_{c}$, Eq. (13), at which a chiral soliton moving with velocity $u$ appears [80].

Having the lowest-energy soliton solution $\Psi'(\theta)$ in the frame moving with a velocity $u$, we return to the laboratory frame. This yields a solution moving periodically on a ring $\Psi(\theta, t) = e^{-iu\theta} \Psi'(\theta) = \Psi'(\theta - ut)$, and in Fig. 2 we present the results obtained for parameters for which Öhberg and Wright claim an existence of the time crystal, i.e., $a = \pi / 2, g = -6$, and even $q$, but the final conclusion is the same for any choice of parameters $a$, $g$, and $q \in \mathbb{Z}$. While the soliton solutions that fulfill periodic boundary conditions exist for any $u \gtrsim -1.64$, FIG. 1. Critical interaction strength $g_{c}$, Eq. (13), for the chiral soliton model. (a) illustrates the situation where no density-dependent gauge potential is present, i.e., when $a = 0$, which corresponds to the Wilczek model with $a = \frac{1}{2}$. An influence of a density-dependent gauge potential is shown in (b) and (c), where $a = -1$ and $a = +1$, respectively. Note that $-\pi + 2au \leq g_{c} \leq 2au$, which is indicated by dotted lines. In every panel the case of even (odd) $q$ is represented by solid (dashed) lines.

FIG. 2. Results of numerical simulations for exemplary parameters $a = \pi / 2, g = -6$, and even $q$. (a) Color-coded plot of the chiral soliton density vs $u$. (b) Critical value $g_{c}$ of interaction strength vs $u$ (dashed line shows $g = -6$). (c) The energy $\mathcal{E}_{\text{LAB}}$ (5) of a chiral soliton moving with a velocity $u$. Circles indicate energies of solutions obtained within the ansatz used in Ref. [71], i.e., for $u = -0.5$ and $u = 1.5$. Clearly, the ansatz solutions do not represent the system ground state. The density $|\Psi(\theta)|^2$ and phase $\varphi(\theta) = \text{Arg}(\Psi)$ of the chiral soliton for $u = 0$ which minimizes $\mathcal{E}_{\text{LAB}}$ are depicted in the inset.
the laboratory frame energy $\mathcal{E}_{\text{LAB}}$, Eq. (5), is minimal when $u = 0$ and consequently in the lowest-energy state no motion of the soliton is allowed. The corresponding density $|\Psi|^2$ and phase $\varphi = \text{Arg}(\Psi)$ of the lowest-energy solution is depicted in the inset of Fig. 2(c). We stress that a nontrivial phase $\varphi$ of the stationary solution visible in the inset is due to a nonzero $A$. Indeed, according to the continuity equation, the probability current $j = 2|\Psi|^2(\partial_t \varphi - A)$ for stationary states can be nonzero but must be constant.

In Ref. [71] where the existence of a genuine time crystal is claimed, the authors introduce the following ansatz,

$$\Psi(\theta, t) = e^{i\Theta(\theta, t)} \Phi(\theta - ut, t),$$  

with

$$\Theta(\theta, t) = \frac{q\theta}{2} + \frac{ut\theta}{2} + \int_0^\theta d\theta'|\Phi(\theta', t)|^2.$$  

Substitution of the ansatz to the GPE, Eq. (6), significantly simplifies the equation but in general implies twisted boundary conditions $\Phi$,

$$\Phi(\theta - ut + 2\pi, t) = e^{-i(\Theta(\theta + 2\pi, t) - \Theta(\theta, t))} \Phi(\theta - ut, t).$$

However, if one insists (as it is done in Ref. [71]) that

$$\Theta(\theta + 2\pi, t) - \Theta(\theta, t) = 2\pi k, \quad k \in \mathbb{Z},$$  

then $\Phi$ fulfills periodic boundary conditions (similarly as $\Psi$) which enforces that the velocity $u$ is allowed to take quantized values only—e.g., for even $q$, the velocity $u = 2k - a/\pi$. Then, the ansatz represents a certain class of solutions only. If it was a general solution, then the velocity would be quantized even in the case of $a = 0$ where analytical soliton solutions propagating with any $u$ are known [74–77]. Importantly, the ansatz does not describe the ground state of the system [see Fig. 2(c)]. The latter corresponds to the stationary probability density and consequently the system does not represent a genuine time crystal.

\section*{IV. DISCUSSION AND CONCLUSIONS}

It turns out that it is not easy to realize a genuine time crystal. The initial proposition by Wilczek relied on attractively interacting bosons on a ring in the presence of a magnetictuslike flux (the so-called Aharonov-Bohm ring) [1]. In the single-particle case when the flux does not match the quantized values of the momentum of a particle on a ring, the probability current is nonzero even for the lowest-energy state, but the corresponding probability density is uniform and does not change over time. In a many-body case, the situation is quite the opposite: Although there exist spatially localized solutions which could travel nondispersively, the ground state probability current is zero in the thermodynamic limit [5].

Öhberg and Wright proposed an extension of the original Wilczek model, where they replaced a constant flux with a density-dependent gauge potential and claimed the existence of a time crystal behavior in the ground state of the system [69]. The idea was very attractive because such a genuine time crystal could be realized in ultracold atom laboratories. The publication triggered a debate in the literature whether the results are correct [70,71]. In this Rapid Communication we are taking the final step of the discussion.

We have reexamined the chiral soliton model and showed that in Ref. [71] a certain class of mean-field solutions is considered only and one can find other states which possess lower energy. It turns out that the ground state of the system is represented by a stationary probability density and consequently a genuine time crystal cannot be observed in the chiral soliton model.

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[80] Note that $\dot{q}$ does not change when we change $q$ within the same parity class. Such a feature manifests at the level of the GPE, Eq. (6), and energy functional, Eq. (5). Indeed, if $\Psi$ is a solution of Eq. (6) with $q = q_0$, then $\Psi = \Psi e^{i\theta}$ is a solution of the same problem but with $q = q_0 + 2s$ where $s \in Z$, i.e., the change $q \rightarrow q + 2s$ only modifies the phase winding numbers of the GPE solutions so that $J \rightarrow J + s$. Additionally, while the energies, Eq. (5), corresponding to $\Psi$ and $\Psi$, differ only by a constant associated with the change $W = q^2/4 \rightarrow (q + 2s)^2/4$, the probability current, Eq. (7), remains unchanged.