Mode coupling and conversion at anticrossings treated via stationary perturbation technique

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Abstract

Intermodal interactions displayed through the phenomena of mode coupling and conversion in optical systems are treated by means of the Lindstedt-Poincaré perturbation method of strained parameters more widely known in classical quantum mechanics and quantum chemistry as the stationary perturbation technique. The focus here is on the mode conversion at the points of virtual phase matching (otherwise called anticrossings or avoided crossings) associated with the maximum conversion efficiency. The method is shown to provide a convenient tool to deal with intermodal interactions at anticrossings — interactions induced by any kind of perturbation in dielectric index profile of the waveguide, embracing optical inhomogeneity, magnetization of arbitrary orientation, and nonlinearity. Closed-form analytic expressions are derived for the minimum value of mode mismatch and for the length of complete mode conversion (the coupling length, or the beat length) in generic waveguiding systems exhibiting anticrossings. Demonstrating the effectiveness of the method, these general expressions are further applied to the case of TE\textsubscript{n} \leftrightarrow TM\textsubscript{m} mode conversion in (i) a multilayer gyrotropic waveguide under piecewise-constant, arbitrarily oriented magnetization, and (ii) an optically-inhomogeneous planar dielectric waveguide — an example which the standard coupled-mode theory fails to describe.

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I. INTRODUCTION

Mode coupling and conversion are commonly known and primarily important phenomena in fiber and integrated optics, either hindering or fueling the operation of numerous devices and elements, e.g., electro-optical and acousto-optical switches and modulators, waveguide couples, power splitters, wavelength filters, and others [1]. When a mode of certain configuration transverses the structure and converts into some other mode, say, due to the externally induced perturbation in optical properties or geometrical configuration of the structure or due to the imperfectness of the materials used, or when the exchange of energy between the guided modes of adjacent waveguides occurs — anyway we face the phenomena of mode coupling and conversion. The crucial regime for the mode coupling is in the vicinity of the points of virtual phase matching (otherwise referred to as “anticrossings” or “avoided crossings” — a terminology brought about from quantum mechanics and solid-state physics). At anticrossings, the complete mode conversion can be achieved, which makes tailoring this regime particularly important for the applications. For a positive example, let me mention a promising method for controllable dispersion compensation in photonic bandgap fibers based on weak interactions at anticrossings between the core-guided mode and a mode localized in an intentionally introduced defect of the crystal [2]. An opposing example of technology strains due to the undesirable intermodal interactions is radiation losses caused by the coupling of the guided mode to radiation modes in a two-dimensional photonic crystal etched into a planar waveguide [3].

A dominating theoretical tool for the whole of mode coupling and conversion phenomena (except a few cases when rigorous analytical treatment is possible [4]) is the coupled-mode, or the coupled-wave, formalism [1, 5] that was first proposed more than fifty years ago and since then became commonplace in both optical engineering research and textbook literature. The mathematical foundation of all the modifications of the coupled-mode formalism is the method of variation of independent coefficients — one of the fruitful methods in perturbation theory [6]. Nevertheless, the formalism grounds upon the differing amount of approximations — e.g., that of slowly-varying dielectric function $\epsilon(r)$, so that the electric field $E(r)$ is assumed to satisfy $\nabla(\nabla \cdot E) = 0$; the so-called parabolic approximation; and others — which certainly limits the usefulness of the model. Moreover, the resultant “coupled modes” often fail to satisfy orthogonality relations and boundary conditions in the
actual (perturbed) structure. Finally, the coupled-mode formalism ignores all of the explicit mathematical parallels to the classical quantum-mechanics problems that also exploit perturbation techniques, in particular the widely known perturbation method for stationary levels — a modification of the Lindstedt-Poincaré method of strained parameters sometimes called the Rayleigh-Schrödinger method (see Refs. [6, 7] for the bibliographical details). Meanwhile, establishing and tracking the analogies between classical quantum mechanics and optics tends to be a really stimulating approach — both through adopting the formalism of electromagnetic-wave propagation in dielectric materials to collisionless propagation of effective-mass electron waves in semiconductor crystals, and on the other hand through applying the formalism of quantum-mechanical solid-state electronics to electromagnetic propagation in periodic dielectric structures which proved to be a very successful programme during the last decade [10, 11].

In this article, I translate the classical stationary perturbation technique — i.e., the method of strained parameters — to the case of mode coupling and conversion in optical waveguiding systems. The problem of source-free light propagation is formulated in Sec. II as an ordinary eigenvalue problem in the squared free-space wavenumber $\kappa \equiv k^2 = \omega^2/c^2$ (or, actually, in the angular frequency $\omega$) for the magnetic field $\mathbf{H}(\mathbf{r})$, and perturbations in dielectric index profile of the waveguide are considered, embracing the cases of optical inhomogeneity, magnetization of arbitrary orientation, and nonlinearity. Both the eigenvalues and the eigenvectors are expanded into the series in perturbation parameter, giving rise to an electromagnetic counterpart of the quantum-mechanics stationary perturbation method as an alternative to the coupled-mode formalism. Although physical interpretation of the former in the case of nondegenerate spectrum of eigenvalues looks rather questionable from the point of experiment, the method however provides promptly solvable and naturally interpretable treatment of the modal behavior at the anticrossings, which is an issue in the mode conversion analysis. To demonstrate the efficiency of the method (to distinguish from the coupled-mode theory, the qualifying term “stationary” is retained here with the primary meaning that the coefficients of an expansion into which an input wave is decomposed are not varying with coordinates), I refer in Sec. III to the two practice-targeted examples from integrated optics concerning $\text{TE} \leftrightarrow \text{TM}$ mode coupling: a multilayer gyrotropic waveguide subject to constant, arbitrarily oriented magnetic field, and an optically-inhomogeneous planar dielectric waveguide — an example which falls out of the scope of the standard
coulped-mode theory. The unperturbed basis for the both cases is a multilayer waveguide composed of linear isotropic dielectric materials — a simple structure known to exhibit under certain circumstances the perfect phase matching of guided TE and TM modes [12], thus it is natural to implement here the developed theory for the mode conversion at anticrossings. For both systems, I pursue the formulated way to derive closed-form first-order analytic expressions for the minimum value of mode mismatch and hence for the length of complete TE \( n \leftrightarrow \) TM \( m \) mode conversion (the coupling length, or the beat length). Finally, I conclude in Sec. IV with some claims concerning further possible applications and modifications of the stationary perturbation technique for the optical waveguiding theory.

II. H-EIGENPROBLEM AND ANTICROSSINGS

A. H-eigenproblem

Let me start with the wave equation for the magnetic field

\[
\nabla \times (\epsilon^{-1} \nabla \times H) = \kappa H, \tag{1}
\]

where \( \epsilon = \epsilon(r) \) is the dielectric permittivity distribution, \( \kappa \equiv k^2 = \omega^2/c^2 \) the squared free-space wave number, \( \omega \) the angular frequency, \( c \) the vacuum speed of light, and the field \( H \) satisfies additionally

\[
\nabla \cdot H = 0. \tag{2}
\]

Eq. (1) can be treated as an ordinary eigenvalue problem in \( \kappa \) (or, actually, in \( \omega \)) for the field \( H \), with Maxwellian operator \( \mathcal{M} \) defined by

\[
\mathcal{M}H \equiv \nabla \times (\epsilon^{-1} \nabla \times H). \tag{3}
\]

In fact, a similar equation can be written for, say, the \( E \) field:

\[
\epsilon^{-1} \nabla \times (\nabla \times E) = \kappa E, \tag{4}
\]

but there are two sound reasons to restrict oneself to the \( H \)-eigenproblem (1), but not to its \( E \)-counterpart (4): first, an accompanying to Eq. (1) divergence equation (2) is clearly simpler than \( \nabla \cdot (\epsilon E) = 0 \) — a satellite equation for the \( E \)-eigenproblem; second, given the dielectric index a real scalar function of coordinates — i.e., a real scalar field — or, more
generally, a Hermitian dyadic field, the Maxwellian (3) becomes Hermitian too — the fact that is though not crucial in macroscopic electrodynamics (non-Hermitian Maxwellian would generate a set of nonorthogonal eigenmodes and complex-valued eigenfrequencies — a situation normally faced in optics of lossy quasotropic media) nor pertinent to the present treatment, but aesthetically pleasant and legitimates many of the cross-references between Eq. (11) and the stationary Schrödinger eigenproblem with Hermitian Hamiltonian as well.

For a large class of waveguiding systems, namely, for those exhibiting continuous translation symmetry along the direction of light propagation (say, the z direction) we can assign harmonic dependence of the fields on that direction; in particular, the magnetic field reads

$$H(r; \beta) = e^{i\beta z}H(x, y),$$

(5)

where $\beta$ is the propagation constant, so that for a given $z$-independent dielectric index profile $\epsilon_0(x, y)$ Eq. (11) reduces to an eigenvalue problem for $H(x, y)$; the reduced eigenproblem operator depends parametrically on $\beta$ then and at a fixed $\beta$ spawns a set of eigenvalues $\kappa(\beta)$, thus yielding dispersion structure $D(\kappa, \beta) = 0$ of a perfect waveguide. If an actual dielectric permittivity distribution $\epsilon(r)$ of the waveguide happens to differ from $\epsilon_0(x, y)$ for which Eq. (5) holds, that is $\epsilon(r) = \epsilon_0(x, y) + \varepsilon \delta \epsilon(r)$ (we do not need to specify at this stage whether $\delta \epsilon(r)$ is a scalar or a tensor field) and

$$\epsilon^{-1}(r) = \xi_0(x, y) + \varepsilon \xi_1(r) + \varepsilon^2 \xi_2(r) + \ldots,$$

(6)

where $\xi_i(r) = [-\delta \epsilon(r)]^i/[\epsilon_0(x, y)]^{i+1}$, $i = 0, 1, \ldots$, then in the spirit of the Lindstedt-Poincaré perturbation method of strained parameters [6, ch. 3] one can rewrite Eq. (11) as

$$(\mathcal{M}^{(0)} + \varepsilon \mathcal{M}^{(1)} + \varepsilon^2 \mathcal{M}^{(2)} + \ldots)H = \kappa H,$$

(7)

$$\mathcal{M}^{(i)}H \equiv \nabla \times (\xi_i \nabla \times H),$$

(8)

in terms of the unperturbed Maxwellian $\mathcal{M}^{(0)}$ and the higher-order perturbation operators, and expand then the eigenvectors $H_n$ of the perturbed problem into a series in $\varepsilon$:

$$H_n = H_n^{(0)} + \varepsilon H_n^{(1)} + \varepsilon^2 H_n^{(2)} + O(\varepsilon^3),$$

(9)

and similarly for the eigenvalues:

$$\kappa_n = \kappa_n^{(0)} + \varepsilon \kappa_n^{(1)} + \varepsilon^2 \kappa_n^{(2)} + O(\varepsilon^3).$$

(10)
The procedure for further solving Eq. (7) using expansions (9) and (10), i.e., for finding the unknown functions \( H_n^{(i)} \) and \( \kappa_n^{(i)} \), is quite well-developed in perturbation theory. Substituting Eqs. (9), (10) into Eq. (7) yields for the zeroth-order and linear in \( \varepsilon \) terms:

\[
\mathcal{M}^{(0)} H_n^{(0)} = \kappa_n^{(0)} H_n^{(0)},
\]

(11)

\[
\mathcal{M}^{(0)} H_n^{(1)} + \mathcal{M}^{(1)} H_n^{(0)} = \kappa_n^{(0)} H_n^{(1)} + \kappa_n^{(1)} H_n^{(0)},
\]

(12)

with the homogeneous boundary conditions \( \nabla \cdot H_n^{(i)} = 0 \). The solution to the unperturbed problem (11) is assumed to be known; to say more — and this is an important point — I further assume all the \( H_n^{(0)} \) vectors to be of the form (5) which is likely to embrace all the cases of practical interest. The subsequent steps of the method depend essentially on whether the spectrum of \( \mathcal{M}^{(0)} \) is degenerate or not.

B. Nondegenerate spectrum

If there are no degenerate eigenvalues among those of \( \mathcal{M}^{(0)} \) spectrum, then we customarily obtain up to the first-order correcting terms [6]:

\[
H_n(r; \beta) \approx H_n^{(0)}(r; \beta) + \varepsilon \sum_{m \neq n} \frac{\mathcal{M}_{nm}^{(1)}(\beta)}{\kappa_n^{(0)} - \kappa_m^{(0)}} H_m^{(0)}(r; \beta)
\]

(13)

and

\[
\kappa_n(\beta) \approx \kappa_n^{(0)}(\beta) + \varepsilon \mathcal{M}_{nn}^{(1)}(\beta),
\]

(14)

where the first-order coupling matrix

\[
\mathcal{M}_{nm}^{(1)}(\beta) \equiv \mathcal{N}^{-1} \int_W H_m^{(0)}(r; \beta) \mathcal{M}^{(1)}(r) H_n^{(0)}(r; \beta) \, dv.
\]

(15)

The integration volume \( W \) embracing the waveguide is formally infinite, \( dv = dx \, dy \, dz \), the normalizing factor \( \mathcal{N} \) equals either \( Z \) — the distance of light propagation within a waveguide in the \( z \) direction (i.e., the length of the waveguide) — in the case of cylindrical (two-dimensional) geometry, or \( S = YZ \) — the \( yz \) square of a waveguiding cell — for planar (one-dimensional) systems. This normalization is due to specific orthogonality condition assigned to the \( H_n^{(0)} \) modes:

\[
\int_W H_n^{(0)} H_m^{(0)} \, dv = \delta_{nm} \int_W |H_m^{(0)}|^2 \, dv
\]

\[
= \begin{cases} 
\delta_{nm} Z \int |\mathcal{H}_m(x, y)|^2 \, dx \, dy = \delta_{nm} Z & \text{(cylindrical waveguide)}, \\
\delta_{nm} S \int |\mathcal{H}_m(x)|^2 \, dx = \delta_{nm} S & \text{(planar waveguide)}. 
\end{cases}
\]

(16)
FIG. 1: Modal dispersion curves (———) of the unperturbed hypothetical structure and \( \beta = \text{const.} \) lines (– – –) for the case of (a) nondegenerate spectrum of \( \mathcal{M}^{(0)} \) and (b) the two modes, \( \mathbf{H}_1^{(0)} \) and \( \mathbf{H}_2^{(0)} \), exhibiting degeneration. Here \( \Lambda \) is a characteristic dimension of the structure; the grey circles mark the modes related to the expansions (13) and (17).

I explicitly designated the (parametric) dependence of all the quantities in Eqs. (13), (14), and (15) on \( \beta \) entailed by the previous assumption that the unperturbed modes have the form as per Eq. (5). Now, the restriction \( \beta = \text{fixed} \) for the set of nondegenerate modes 

\[
\mathbf{H}_n^{(0)} e^{-i\omega_n t} \propto e^{i(\beta z - \omega_n t)} = e^{i\kappa_n (\tilde{\beta}_n z - ct)}
\]

into which the \( n \)th perturbed eigenmode \( \mathbf{H}_n \) is expanded via Eq. (13) — modes characterized thus by essentially different eigenwavenumbers \( k_n \) and differing normalized propagation constants \( \tilde{\beta}_n \equiv \beta/k_n \) — seems to correspond to an utterly odd situation from the point of experiment (see Fig. 1). To aid this shortcoming somehow, one might consider Eq. (1) as a generalized eigenproblem in, e.g., \( \beta \) or \( \tilde{\beta} \), parametrically dependent on \( \kappa \), which would lead to expansions like (13) but into the \( \{ \mathbf{H}_n^{(0)}(r; \kappa) \} \) set. Of course, this severely complicates the formalism — and probably here is an issue why the coupled-mode theory has been exclusively dominating for decades in the field. Fortunately, the case of quasi-degenerate eigenvalues allows promptly solvable and easily interpretable treatment in terms of the ordinary \( \mathbf{H} \)-eigenproblem in \( \kappa \).
C. Degenerate spectrum and anticrossings

Let us focus on the case of two-degenerate or nearly degenerate eigenmodes \( H_1^{(0)} \) and \( H_2^{(0)} \). Borrowing the known result from quantum mechanics [8], we see that in this case both the \( H_1^{(0)} \) and \( H_2^{(0)} \) modes dominate the expansion (9) of the related fundamental modes

\[
H_{\pm} = C_1^{\pm} H_1^{(0)} + C_2^{\pm} H_2^{(0)} + O(\varepsilon)
\]

with the constant (in particular, \( z \)-independent) coefficients

\[
C_1^{\pm} = \left[ \frac{M_{12}}{2\sqrt{M_{12} M_{21}}} \left( 1 \pm \frac{M_{11} - M_{22}}{\sqrt{(M_{11} - M_{22})^2 + 4 M_{12} M_{21}}} \right) \right]^{\frac{1}{2}},
\]

\[
C_2^{\pm} = \pm \left[ \frac{M_{12}}{2\sqrt{M_{12} M_{21}}} \left( 1 \mp \frac{M_{11} - M_{22}}{\sqrt{(M_{11} - M_{22})^2 + 4 M_{12} M_{21}}} \right) \right]^{\frac{1}{2}}.
\]

Here and below in the article I pursue only the first-order approximations and hence omit the overly superscript \( (1) \) over the coupling matrix elements \( M_{nm} \), now \( \beta \)-independent, defined by Eq. (15). The first-order correction to the corresponding eigenvalues leads to

\[
\kappa_{\pm} = \kappa_1^{(0)} + \kappa_2^{(0)} + \varepsilon \left( M_{11} + M_{22} \right) \pm \sqrt{\left[ \kappa_1^{(0)} - \kappa_2^{(0)} + \varepsilon \left( M_{11} - M_{22} \right) \right]^2 + \varepsilon^2 M_{12} M_{21}}.
\]

In quantum mechanics, this situation corresponds to quasi-degenerate energy levels of a quantum system and is associated with enormous variety of effects and treatments — take the Landau-Zener model for example [15], nourishing nearly 100 publications a year. In the realm of optics, Eq. (17) with coefficients given by Eqs. (18), (19) is fully consistent with the general statement that complete mode conversion can be achieved in any system exhibiting anticrossings. Indeed, instead of the \( H_1^{(0)} \) mode and the \( H_2^{(0)} \) mode crossing one another on the dispersion structure diagram (Fig. 1), at the anticrossings the \( H_1^{(0)} \) mode “transforms” continuously into the \( H_2^{(0)} \) one, and vice-versa, over a certain frequency range, so it is intuitive to expect that somewhere in the vicinity of the virtual crossing point the \( H_1^{(0)} \) and \( H_2^{(0)} \) modes would be equally unsuitable to play the role of the fundamental modes \( H_+ \) and \( H_- \), and if the original (input) wave is \( H_1^{(0)} \) or \( H_2^{(0)} \), it wouldn’t “feel” which of the fundamental modes, \( H_+ \) or \( H_- \), is more suited for it, and thus would oscillate between \( H_1^{(0)} \) and \( H_2^{(0)} \). Mathematically, these heuristic speculations are expressed via the requirement

\[
|C_1^{\pm}| = |C_1^{-}| = |C_2^{\pm}| = |C_2^{-}| = \frac{1}{\sqrt{2}}
\]
(under appropriate normalization) necessary for the complete mode conversion to occur, which is equivalent, as immediately follows from the comparison of Eqs. (18) and (19), to canceling out the $M_{11}$ and $M_{22}$ elements under the root in Eqs. (18), (19), and (20) due to

$$M_{11} = M_{22}. \quad (22)$$

This condition does not necessarily correspond to the case of stringently degenerate eigenvalues $\kappa_1^{(0)}$ and $\kappa_2^{(0)}$, but with a good deal of reason we may assume $\kappa_1^{(0)} - \kappa_2^{(0)} = O(\varepsilon^2)$ to estimate the minimum mode mismatch $\Delta k_{\text{min}}$ from Eq. (20). With this assumption, it is under condition (22) that the quantity

$$\Delta \kappa_{\text{min}} = 2k_0\Delta k_{\text{min}}, \quad (23)$$

where $k_0 = \omega_0/c$, $\omega_0$ is the frequency of a virtual mode crossing (phase matching), reaches its minimum value of $2\varepsilon\sqrt{M_{12}M_{21}}$, thus the minimum eigenmode mismatch $\Delta k_{\text{min}}$ along the $\beta = \text{const}$ curve corresponds to the complete mode conversion and reads

$$\Delta k_{\text{min}} = \varepsilon k_0^{-1}\sqrt{M_{12}M_{21}}. \quad (24)$$

In a nonabsorbing waveguide (in Sec. III we will deal with the two) $M_{nm}$ is a Hermitian matrix, hence $M_{12} = M_{21}^*$ and Eq. (24) is simplified to

$$\Delta k_{\text{min}} = \varepsilon k_0^{-1}|M_{12}|. \quad (25)$$

Complete mode conversion occurs when the $H_1^{(0)}$ and $H_2^{(0)}$ modes propagating in the $z$ direction accumulate a phase difference of $\pi$: $\Delta k_{\text{min}}\tilde{\beta}z_c = \pi$; substituting here Eq. (25), we obtain for the corresponding coupling length:

$$z_c = \frac{\pi k_0}{\varepsilon \beta |M_{12}|}, \quad (26)$$

where, as before [see Eq. (15)], the coupling matrix element

$$M_{12} = N^{-1}\int_W [\nabla \times (\xi_1 \nabla \times H_1^{(0)})] \cdot H_2^{(0)} dv. \quad (27)$$

In contrast to the coupled-mode theory, the $M_{12}$ element might and, in general, does depend on the length of a waveguide along the $z$ direction because of the possible $z$ dependence of the $\xi_1$ function. In what follows I calculate $M_{12}$ and hence $z_c$ explicitly for the two specific systems relevant to integrated optics: for a multilayer gyrotropic stack and for an optically-inhomogeneous waveguide.
III. MODE CONVERSION AT PLANAR GEOMETRY

A. Gyrotropic waveguide

Let the unperturbed waveguide be described by the scalar dielectric permittivity distribution \( \epsilon(0)(r) \). As argued by symmetry reasoning, relativity considerations, and energy conservation \[13, 14\], the most general form to allow for the electromagnetic disturbance induced by the \( \varepsilon h \) field applied to (initially) isotropic medium is

\[
\tilde{\epsilon} = \epsilon(0) \mathbf{I} + i \varepsilon h^x + \varepsilon^2 \eta (h \otimes h - h^2),
\]

where \( \mathbf{I} \) is a unit three-dimensional dyadic; \( \zeta \) and \( \eta \) are real functions of \( h^2 \); the superscript \( \times \) over the \( h \) vector denotes the antisymmetric dyadic dual to \( h \), so that \( h^x g = h \times g \) gives conventional vector product of \( h \) and \( g \); the circled cross \( \otimes \) denotes the outer product — the dyad: \( h \otimes g = h_i g_j, \ i, j = 1, 2, 3 \). If \( \varepsilon \) is small, we may assume \( \zeta \) and \( \eta \) to be constant (this assumption leads to the introduction of gyration vector \[13, 14\]) and write down for the inverse to the \( \tilde{\epsilon} \) dyadic \[14\]

\[
\tilde{\epsilon}^{-1} = \epsilon^{-1}(0) \mathbf{I} - i \varepsilon u^x + O(\varepsilon^2),
\]

in terms of \( u \equiv \varepsilon^{-2}(0) \varepsilon h \), so that the coupling matrix element (27) can be written as

\[
\mathcal{M}_{12}^{\text{gyr}} = -k S^{-1} \int_W \left[ \nabla \times (u \times D_1^{(0)}) \right] \cdot \mathbf{H}_2^{(0)} \, dv,
\]

where I used the frequency-domain Maxwell’s equation for \( \nabla \times \mathbf{H} \) in the source-free region, \( \nabla \times \mathbf{H} = -ik \mathbf{D} \). Equation (30) can be significantly simplified by applying the vector identity \[16\]

\[
\nabla \times (u \times D) = u(\nabla \cdot D) - D(\nabla \cdot u) + (D \cdot \nabla)u - (u \cdot \nabla)D
\]

valid for any three-vectors \( u \) and \( D \). The first term on the right vanishes immediately due to \( \nabla \cdot D = 0 \); then, in a specific but physically reasonable and normally considered case of \( u \) being a constant (or piecewise-constant) vector in a given volume \( V \) and equaling zero outside of it, the second and third terms of the sum in Eq. (31) being multiplied by \( \mathbf{H} \) and integrated as per Eq. (30) lead to the surface integrals instead of the volume ones. For the second term one obtains: \( \nabla \cdot u = -(\mathbf{n} \cdot u) \delta(\mathbf{r} - \mathbf{R}) \), where \( \mathbf{R} = \mathbf{R}(r) \) defines the boundary
For the third term we similarly have: \((D \cdot \nabla)u = -(D \cdot \hat{n})u\delta(r - R)\), hence for an integral
\[
\int_W [(D \cdot \nabla)u] \cdot H \, dv = - \int_{\Sigma} [(D \cdot \hat{n})u] \cdot H \, d\sigma. \tag{33}
\]

Thus for the case of constantly magnetized waveguide (up to this point, no assumptions have been made regarding its actual geometry) the integral in Eq. (30) is reduced via Eqs. (31), (32), and (33) to
\[
\int_W \left[ \nabla \times (u \times D) \right] \cdot H \, dv = \int_{\Sigma} \left[ D \left( \hat{n} \cdot u \right) \right] \cdot H \, d\sigma - \int_V \left[ (u \cdot \nabla)D \right] \cdot H \, dv. \tag{34}
\]

Now let us switch to the planar structures. In a planar unperturbed waveguide the eigenmodes are classified in terms of TE, or \(s\)-polarized, and TM, or \(p\)-polarized waves (for an elegant derivation of this common fact via symmetry considerations see Ref. [11]). In a coordinate system in which the \(x\) axes is normal to the bimedium interfaces and the light energy is guided in the \(z\) direction, we can write down, to evaluate the \(M_{sp}^{gr}\) element as per Eq. (30):
\[
D_1^{(0)} = D_s = e^{i\beta_z \phi_s(x)}\hat{y} \tag{35}
\]
for the \(s\)-polarized mode, and
\[
H_2^{(0)} = H_p = e^{i\beta_z \psi_p(x)}\hat{y} \tag{36}
\]
for the \(p\)-polarized one. Here \(\phi_s(x)\) and \(\psi_p(x)\) are the lateral distributions of \(D_s\) and \(H_p\) respectively, obtained from the unperturbed Maxwell’s equations for the given virtually-phase-matched modes. We see immediately from Eq. (35) that the second surface integral in Eq. (32) vanishes at planar geometry, since \(D_s \cdot \hat{n} = D_s \cdot \hat{x} = D_s \cdot (-\hat{x}) = 0\) holds identically in this case. Finally we arrive at
\[
M_{sp}^{gr} = -kS^{-1} \int_{\Sigma} \left[ D_s(\hat{n} \cdot u) \right] \cdot H_p^* \, d\sigma + kS^{-1} \int_V \left[ (u \cdot \nabla)D_s \right] \cdot H_p^* \, dv
\]
\[
= -kS^{-1} \int_V u_x \left( \frac{d\phi_s}{dx} \psi_p + \phi_s \frac{d\psi_p}{dx} \right) \, dv + kS^{-1} \int_V \left( u_x \frac{d\phi_s}{dx} \psi_p + i\beta u_x \phi_s \psi_p \right) \, dv
\]
\[
= -k \left[ u_x \frac{\alpha}{J^\alpha} \left( \frac{\phi_s}{\beta^{-1}} \right) \right] - iu_x \frac{\alpha}{J^\alpha} \left( \frac{\phi_s \psi_p}{\beta^{-1}} \right), \tag{37}
\]
where \( u^\alpha \) is the value of vector \( u \) in the \( \alpha \)th layer bounded between the \( x = x_{\alpha-1} \) and \( x = x_\alpha \) planes, and the integrals

\[
J_\alpha \left( \frac{\phi_s d\psi_p}{dx} \right) = \int_{x_{\alpha-1}}^{x_\alpha} \frac{\phi_s(x) d\psi_p(x)}{dx} \, dx, \tag{38}
\]

\[
J_\alpha \left( \frac{\phi_s \psi_p}{\beta^{-1}} \right) = \int_{x_{\alpha-1}}^{x_\alpha} \frac{\phi_s(x) \psi_p(x)}{\beta^{-1}} \, dx, \tag{39}
\]

\( \alpha = 1, \ldots, n \) (\( n \) is the number of layers).

Eq. (37) accounts for no mode conversion at transversal magnetization, in agreement with the long ago established result [17]. A few more implicit tips could also be deduced thereof:

1. In a symmetrically sandwiched stack, the polar magnetization (\( u = u_x \hat{x} \)) virtually doesn’t couple the modes exhibiting similar symmetry in the \( \phi_s(x) \) and \( \psi_p(x) \) functions (that is, when both \( \phi_s(x) \) and \( \psi_p(x) \) are either even or odd); on the contrary, the modes of opposite symmetry in \( \phi_s(x) \) and \( \psi_p(x) \) are virtually not sensitive to the longitudinal magnetization (\( u = u_z \hat{z} \)). Asymmetric sandwiching impairs this behavior.

2. The integral (39) contains scaling parameter \( \beta^{-1} \) that defines characteristic thickness of the waveguide corresponding to the comparable values of the both integrals, Eqs. (38) and (39), in Eq. (37), and hence to the comparable shares of polar and longitudinal polarizations in the mode conversion efficiency. For optical frequencies \( \beta^{-1} \approx 100 \, \text{nm} \); if the thickness of the waveguide considerably exceeds \( \beta^{-1} \), than \( \psi_p(x) \) would appear to be a too slowly varying function and therefore

\[
J_\alpha \left( \frac{\phi_s d\psi_p}{dx} \right) \ll J_\alpha \left( \frac{\phi_s \psi_p}{\beta^{-1}} \right). \tag{40}
\]

For the ultrathin layers the inverse inequality holds, but in this regime at most one mode is guided in the structure, which is apparently out of the scope here.

**B. Optically-inhomogeneous dielectric waveguide**

If we assume \( \xi = \xi(r) \) in Eq. (27) to be a scalar function within the waveguide volume \( V \), which conveys the cases of optical inhomogeneity and “isotropic” nonlinearity, then it is advantageous to simplify Eq. (27) using the identity

\[
\nabla \times (\xi \nabla \times H) = \nabla (\nabla \xi \cdot H) - (H \cdot \nabla) \nabla \xi - (\nabla \xi \cdot \nabla) H - \xi \nabla^2 H, \tag{41}
\]

where \( \nabla \cdot H = 0 \) is taken into account. In planar geometry, we are concerned as before with the TE and TM modes, but now expressed exclusively through the magnetic field vectors.
For the TM ($p$-polarized) mode we can readily apply Eq. (36); for the TE ($s$-polarized) mode we have for the electric field, in parallel with Eq. (35),

$$E_s = e^{i\beta z} \varphi_s(x) \hat{y},$$

(42)

hence

$$H_s = -ik^{-1} \nabla \times E_s = k^{-1} e^{i\beta z} \left( \beta \varphi_s(x) \hat{x} - i \frac{d\varphi_s(x)}{dx} \hat{z} \right),$$

(43)

where we once again encounter $\beta$ (in fact, $\beta^{-1}$) in the role of scaling parameter. If we put, say, $H = H_s$ in Eq. (41) and multiply the result by $H_p^\ast$, then only the first and second terms of the sum (41) will produce nonzero results, as it immediately follows from Eqs. (43) and (36). Equation (27) thus gives

$$M_{sp}^{inh} = \beta \left( kS \right)^{-1} \int_W \left[ \nabla \left( \nabla \cdot H_s \right) \right] \cdot H_p^\ast dv = \beta \left( kS \right)^{-1} \int_W \left[ \left( H_s \cdot \nabla \right) \nabla \xi \right] \cdot H_p^\ast dv. \quad (44)$$

Now care should be taken when dealing with $\nabla \xi$ function which is discontinuous in the vicinity of material boundaries [otherwise — if $\nabla \xi$ is assumed to be all-continuous — Eq. (44) gives $M_{sp}^{inh} = 0$]. The safest way is to introduce a new vector field $\varsigma = \varsigma(r)$ instead of $\nabla \xi$, with a requirement $\nabla \times \varsigma = 0$ everywhere except at the boundaries:

$$\nabla \times \varsigma = -(\hat{n} \times \varsigma) \delta(r - R),$$

(45)

and with this in mind to perform the brute-force evaluation of integrals in Eq. (44):

$$\int_W \left[ \nabla \left( \nabla \cdot H_s \right) \right] \cdot H_p^\ast dv = \beta k^{-1} \int_W \frac{\partial \varsigma_y}{\partial y} \varphi_s(x) \psi_p(x) \, dv - ik^{-1} \int_W \frac{\partial \varsigma_z}{\partial z} \frac{d\varphi_s(x)}{dx} \psi_p(x) \, dv, \quad (46)$$

$$\int_W \left[ \left( H_s \cdot \nabla \right) \nabla \xi \right] \cdot H_p^\ast dv = \beta k^{-1} \int_W \frac{\partial \varsigma_y}{\partial x} \varphi_s(x) \psi_p(x) \, dv - ik^{-1} \int_W \frac{\partial \varsigma_z}{\partial x} \frac{d\varphi_s(x)}{dx} \psi_p(x) \, dv, \quad (47)$$

hence

$$M_{sp}^{inh} = \beta \left( kS \right)^{-1} \int_W \left( -\nabla \times \varsigma \right)_z \varphi_s(x) \psi_p(x) \, dv + i \left( kS \right)^{-1} \int_W \left( -\nabla \times \varsigma \right)_x \frac{d\varphi_s(x)}{dx} \psi_p(x) \, dv$$

$$= \beta \left( kS \right)^{-1} \int_\Sigma \varsigma_y(r) \varphi_s(x) \psi_p(x) \, d\sigma$$

$$= \beta \left( kS \right)^{-1} \int_V \left( \xi_y \frac{d(\varphi_s \psi_p)}{dx} + \xi_{xy} \varphi_s \psi_p \right) \, dv. \quad (48)$$

We see that for the nonzero mode conversion, the $\xi(r)$ function should exhibit explicit dependence on the $y$ coordinate. For this reason $M_{sp}^{inh}$ would depend on the transversal dimension $Y$ of the waveguide and on the values of $\xi(r)$ at the $y = 0$ and $y = Y$ boundaries.
Another important consequence relates the effect of the $z$ modulation of $\xi(r)$ on the coupling length. If $\xi(r)$ does not depend on $z$, then $\mathcal{M}_{sp}^{inh}$ would in turn be independent on the waveguide length $Z$; but on the contrary, whenever $\xi(r)$ is a stochastically oscillating or periodic function of $z$, the coupling matrix $\mathcal{M}_{sp}^{inh}$ and hence the coupling length $z_c$ would become $Z$-dependent. Say, for the $\xi(z) \propto \cos(z/z_0)$ dependence we have $\mathcal{M}_{sp}^{inh}(Z) \propto Z^{-1} \sin(Z/z_0)$ which is of the order of $Z^{-1}$ and brings about an unexpected scaling rule $z_c \sim Z$ for the coupling length in a planar waveguide with periodic modulation of $\delta \epsilon(r)$ in the $z$ direction.

Finally, I would like to remind here that conventional expression for the off-diagonal coupling matrix element given by the standard coupled-mode theory \cite{1,5},

$$
\mathcal{M}_{12} \propto \int E_1 \delta \epsilon E_2^* \, dx \, dy, \quad (49)
$$
gives an identical zero for the TE $\leftrightarrow$ TM mode conversion, being the dielectric perturbation $\delta \epsilon(r)$ a scalar function.

**IV. CONCLUSION**

Mode conversion displays breaking the (initial) symmetry of the Hamiltonian of a system by some perturbation that distorts the mode spectrum, i.e., shifts the eigenvalues and alters polarization of the eigenmodes, hence a natural tool to treat the mode coupling and conversion phenomena is the perturbation technique. For the mode conversion at anticrossings, I presented in this article an electromagnetic counterpart of quantum-mechanical perturbation theory for quasi-degenerate levels based on the Lindstedt-Poincaré method of strained parameters as a specific alternative to the entrenched coupled-mode formalism grounded upon the method of variation of independent coefficients.

The general expressions derived for the minimum mode mismatch at anticrossings, Eq. (25), and for the coupling length, Eq. (26), are compact, transparent and premise on the calculation of just one element of the coupling matrix, Eq. (27). That matrix element was calculated explicitly for the two cases of interest in integrated optics: for a multilayer gyrotropic waveguide under piecewise-constant, arbitrarily oriented magnetization [Eq. (37)], and for an optically-inhomogeneous planar dielectric waveguide [Eq. (48)]. In a similar way a large variety of optical systems can be analyzed, including Bragg’s fibers, photonic crystals with broken periodicity, etc.
Finally, I should note that by means of an appropriate coordinate mapping, the problem of perturbation due to shifted material boundaries — so to say, geometrical perturbation known to spur difficulties when treated via conventional perturbation techniques — can be reduced to the problem of perturbation in permittivity and permeability profiles of a waveguide exhibiting perfect geometry in those (curvilinear) coordinates and governed by “Cartesian-looking” Maxwell’s equations. Such a trick that recently aided the FDTD modeling of high index-contrast photonic crystals would also significantly augment both the standard coupled-mode theory and the stationary perturbation technique described here.

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