Dynamics of vortex lattice formation in rotating two-component Bose–Einstein condensates

Toshiaki Kanai¹, Makoto Tsubota¹,²
¹Department of Physics, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-Ku, Osaka 558-8585, Japan
²The OCU Advanced Research Institute for Natural Science and Technology (OCARINA), Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-Ku, Osaka 558-8585, Japan

Abstract. We study the dynamics of rotating two-component Bose–Einstein condensates (BECs) toward vortex lattice formation by numerically solving the two-dimensional Gross–Pitaevskii equations. The dynamics are similar to that of single-component BEC, and the whole changes of physical quantities in two cases are qualitatively similar. However, there are some differences between two cases. First, the characteristic time toward vortex entering depends on the inter-component interaction. The time toward vortex entering becomes shorter with increasing the inter-component interaction. Second, the angular momenta exchange between two components with keeping the total angular momentum.

1. Introduction
Quantized vortices appear in diverse fields of quantum condensate systems. They have been studied in superfluid $^4$He for a long time [1]. The theoretical treatments of $^4$He are complicated because of high density and strong interaction. However, the theoretical treatments of cold atom BEC are easy because of low density and weak interaction [2]. Moreover, this system has several experimental advantages compared to the other system. The first advantage is the visualization of the density distribution. The vortices in cold atom BEC can be visualized. The second advantage is the high controllability by optical technology. It is possible to control the potential and the strength of interaction. The third advantage is the realization of multicomponent BECs. Therefore, it is important to study the physics of quantized vortices in cold atoms.

Vortex lattice formation has been studied experimentally, numerically, and theoretically in single-component BEC [3]. By rotating an asymmetric potential, the condensate is distorted to an elliptic shape and oscillates. Then, the surface of the condensate becomes unstable, leading to the excitation of the surface wave. From the surface wave, the quantized vortices enter the condensate and form a triangular lattice [4,5].

Multicomponent superfluids have been studied in superfluid $^3$He [6]. Recently, they are studied in cold atom BECs because of the experimental and theotitical advantages. The equilibrium states of two-component BECs are classified under miscible and phase-separated states depending on the strength of inter-component interaction. Rotating two-component BECs are known to have many kinds of equilibrium states of vortices [7]. Moreover, it is known that the inter-vortex forces of the inter-component BECs differ from that of the same component BEC [8]. For these reasons, it is expected that the inter-component interaction makes the dynamics of two-component BECs different from that of single-component BEC. Therefore, we...
study the dynamics toward the equilibrium states numerically. We confine ourselves to the case of miscible condensates.

2. Formulation and Equilibrium State

In this section, we describe the formulation of two-dimensional two-component BECs. Two-component BECs are described by two macroscopic wavefunctions $\psi_j(\mathbf{r}, t) = \sqrt{n_j(\mathbf{r}, t)} e^{i\phi_j(\mathbf{r}, t)}$ in the mean-field approximation at zero temperature, where the index $j$ refers to each component ($j = 1, 2$). Here $n_j$ and $\phi_j$ are the density and the phase of the $j$th component. Wavefunctions are normalized by the total particle number $N_j = \int dxdy |\psi_j|^{2}$. Wavefunctions in a rotating frame satisfy the coupled Gross–Pitaevskii equations (GPEs) [4]:

$$
(i - \gamma)\hbar \frac{\partial}{\partial t} \psi_j = \left( -\frac{\hbar^2}{2m_j}(\partial_x^2 + \partial_y^2) + V_j(\mathbf{r}) + \sum_{k=1,2} g_{jk}|\psi_k|^2 - \mu - \Omega \hat{L}_z \right) \psi_j.
$$

Here $\hat{L}_z = -i\hbar(x\partial_y - y\partial_x)$ is the angular momentum operator, $\Omega$ is the rotation frequency around z-axis, $m_j$ and $V_j(\mathbf{r})$ are the mass and the trapping potential of $j$th component, and the coefficients $g_{jk}$ represent the strength of interactions between the $j$th and $k$th components. In this paper, we assume the interactions are repulsive ($g_{jk} > 0$). The term with $\gamma$ introduces the dissipation [10]. In this paper, we use $\gamma = 0.03$ unless the value is mentioned explicitly, following the previous studies in single-component BEC [5, 10]. The potentials are the asymmetric harmonic potentials:

$$
V_j(x, y) = \frac{1}{2}m_j\omega^2\{(1 + \epsilon)x^2 + y^2\},
$$

with $\epsilon = 0.05$. In this paper, the number of parameters is reduced for simplicity by assuming that the components are symmetric namely $g_{11} = g_{22} \equiv g$, $m_1 = m_2 \equiv m$, and $N_1 = N_2 \equiv N$, and we use the parameters of miscible condensates ($g_{12} < g$). In a rotating frame, the energy $F$ of condensates is given in terms of the energy $E$ in the non-rotating frame by

$$
F = E - \Omega \sum_{j=1,2} <\hat{\psi}_j|\hat{L}_z|\hat{\psi}_j>.
$$

Under rotation, the equilibrium states of single component BEC are the triangular lattices [5], and those of miscible two-component BECs are triangular or square lattices depending on the strength of inter-component interaction and the rotation frequency [7]. The two vortex lattices are interlocked in such a manner that a peak in the density of one component is located at the density hole of the other component.

3. The dynamics toward vortex lattice formation

We perform numerical simulations of miscible two-component BECs. The time, the length, and the wavefunctions are normalized:

$$
\tilde{t} = \omega t, \\
\tilde{x} = \frac{x}{a}, \\
\tilde{\psi}_j = a\psi_j.
$$

Here $a = \sqrt{\hbar/m\omega}$ (the typical value of frequency is $\omega/2\pi = 219$Hz [9]). Figure 1 shows the time development of the density and the phase. We first prepare an equilibrium state of two-component BECs in a stationary potential (Fig.1 (a)).
Figure 1. Time development of the condensates density $|\tilde{\psi}_1|^2$ and $|\tilde{\psi}_2|^2$, and phase $\theta_1$ and $\theta_2$ after the trapping potential begins to rotate suddenly with $\Omega/\omega_0 = 0.7$, $g_{12}/g = 0.9$, and $g = 2000$. The time is $\tilde{t} = (a) 0$, (b) 40, (c) 74, (d) 100, (e) 230. The equilibrium state is a square lattice with these parameters.

By rotating an asymmetric potential, two-component BECs are distorted to an elliptic shape and oscillate together (Fig.1 (b)), and their angular momenta temporarily increase together (Fig.2). Then, the surface of the condensate becomes unstable, leading to the excitation of the surface wave. Surfaces of each component wave alternately (Fig.1 (c)). From the surface wave, the vortices enter the condensate, and the angular momenta increase rapidly (Fig.2 (a)). The vortices in the condensates move randomly (Fig.1 (d)). Then, they form a square lattice in both components with arranging themselves alternately in order to reduce the energy of inter-component interaction (Fig.1 (e)).

Two-component BECs exchange the angular momenta after the vortices enter the condensates (Fig.4 (b)). The vortices on the inside of condensates come to rest early on forming a lattice, but they near the surface continue to fluctuate. In order to reveal the mechanism of the exchange of angular momenta between two-component BECs, we take a boundary in the radial direction and separate the angular momenta of the inner and outer parts (Fig.3). Here, we take the Thomas–Fermi radii without rotation for the boundary. Figure 4 shows that the oscillation of the angular momentum comes from the whole condensates before $\tilde{t} \sim 140$ but from the outer part of the condensates after about $\tilde{t} \sim 140$ and come to rest gradually because of the dissipation. The vortices near the surface go in and out the condensates fluently with keeping total angular...
Figure 2. Time evolution of the energy and angular momenta per atom of component 1 and component 2. The Fig. (b) is the enlarged figure of the Fig. (a).

Figure 3. The boundary in the radial direction. We take the Thomas–Fermi radii without rotation for the boundary.

Figure 4. Time evolution of angular momentum per atom of component 1. The total angular momentum is separated into the contributions of the inner and outer parts.

momentum and cause the oscillation of each angular momenta. This oscillation of the angular momenta is a unique phenomenon in two-component BECs.

As the ratio $g_{12}/g$ increases with keeping $g = 2000$, the angular momentum $l_z$ of equilibrium state increases (Fig.5). Under the Thomas–Fermi approximation with rotation, the density profile of the sum of the condensates is

$$n_{TF} = \frac{2}{g + g_{12}} \left\{ \sqrt{\frac{N(g + g_{12})}{\pi}} m(\omega^2 - \Omega^2) - \frac{1}{2} m(\omega^2 - \Omega^2) r^2 \right\}. \quad (4)$$
Figure 5. Time evolution of the sum of angular momentums per atom of component 1 and component 2. We change $g_{12}/g$ with keeping $g = 2000$.

Figure 6. Time evolution of the sum of angular momentums per atom of component 1 and component 2 with $\gamma = 0.01$. We change $g_{12}/g$ with keeping $g + g_{12} = 3000$.

Thus the angular momentum of equilibrium state per atom is

$$ l_z = \frac{m\Omega}{2N} \int dr \, r^3 n_{TF} \propto \sqrt{g + g_{12}}. $$

When the sum $g + g_{12}$ increases, the density distribution spreads widely, so that $l_z$ becomes large.

We found that the characteristic time toward the surface wave excitation and the vortices entering the condensate depends on the inter-component interaction (Figs. 5 and 6). As the ratio $g_{12}/g$ increases with keeping $g = 2000$, the characteristic time becomes short (Fig. 5). As the sum $g + g_{12}$ increases, the radii of the condensates becomes large, and the velocity near the surface becomes fast. Thus the vortices enter the condensates easier. As the ratio $g_{12}/g$ increases with keeping $g + g_{12} = 3000$, the characteristic time becomes short not as much as the previous case with keeping $g = 2000$ (Fig. 5). The surface waves of each component are alternated and amplify that of the other component by the repulsive inter-component interaction. The larger the ratio $g_{12}/g$ is, the more remarkable this effect is. In these cases, the characteristic time depends on the sum $g + g_{12}$ strongly, and the ratio $g_{12}/g$ weakly.

4. Conclusion

The inter-component interaction makes the dynamics of two-component BECs different from that of single-component BEC. We found that the characteristic time and the behaviors are changed by the inter-component interaction. These behaviors reflect the differences of intervortex interaction between intra-component and inter-component. In this paper, we study the dynamics of two-component BECs under the symmetric parameters. Under the unsymmetric parameters, there are other equilibrium states [11]. Therefore, it is expected that there are unique behaviors under the unsymmetric parameters.

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