Quantum mechanics without spacetime IV
- A noncommutative Hamilton-Jacobi equation -

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Abstract

It has earlier been argued that there should exist a formulation of quantum mechanics which does not refer to a background spacetime. In this paper we propose that, for a relativistic particle, such a formulation is provided by a noncommutative generalisation of the Hamilton-Jacobi equation. If a certain form for the metric in the noncommuting coordinate system is assumed, along with a correspondence rule for the commutation relations, it can be argued that this noncommutative Hamilton-Jacobi equation is equivalent to standard quantum mechanics.

1 Introduction

The standard formulation of quantum mechanics assumes the existence of a classical background spacetime. However, such a classical spacetime becomes possible only because the present universe is dominated by classical matter fields. In the complete absence of such classical matter fields it becomes essential to formulate the rules of quantum mechanics without reference to a background classical spacetime [1], [2].

A possible reformulation is in the language of noncommutative geometry, as has been motivated in two preliminary papers [1, 3]. The essential idea is as follows. We assume that, in addition to the standard Minkowski and curvilinear coordinate systems, there exist in nature, noncommuting coordinate systems. That is, in such a coordinate system the coordinates do not commute with each other. We also assume that a description of quantum mechanics which is independent of a classical spacetime can be given using such noncommuting coordinates.

Consider first the case of a single relativistic particle, having a mass \( m \) much less than the Planck mass \( m_{Pl} \). As has been argued earlier [1], in this case one can legitimately neglect the gravitational field of the particle. The nature of the dynamics for \( m \ll m_{Pl} \), in the absence of a background classical spacetime, is the subject of this paper. It is assumed that there are many different possible noncommuting coordinate systems, which could all be equivalently used to describe dynamics (in the same spirit as Lorentz invariance, or general covariance). This equivalence is referred to as automorphism invariance - a natural generalisation when one makes a transition from Riemannian geometry to a noncommutative geometry [4].

When we examine the dynamics of this quantum mechanical particle from our vantage point, we are of course using standard commuting spacetime coordinates to describe its

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dynamics. It thus has to be shown that the fundamental description of the dynamics in terms of noncommuting coordinates reduces to quantum mechanics as we know it, when that dynamics is examined from a commuting coordinate system. The introduction of such commuting coordinate systems becomes possible because of the dominant presence of classical matter fields, in much the same way that Minkowski coordinate systems can be used very accurately in asymptotic regions, even though we well know that the real Universe is curved. In this spirit one is proposing that in the absence of a classical background, an appropriate description of quantum mechanics is via the use of noncommuting coordinate systems.

Here, without any pretense at achieving the rigor of noncommutative geometry, we construct a model for the noncommutative dynamics of a relativistic particle, and suggest how the model could be related to standard quantum mechanics.

2 A noncommutative Hamilton-Jacobi equation

Let us start by considering the relativistic Schrodinger equation [5] for a particle in 2-d spacetime

$$-\hbar^2 \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \psi = m^2 \psi \quad (1)$$

which, after the substitution $\psi = e^{iS/\hbar}$, becomes

$$\left( \frac{\partial S}{\partial t} \right)^2 - \left( \frac{\partial S}{\partial x} \right)^2 - i\hbar \left( \frac{\partial^2 S}{\partial t^2} - \frac{\partial^2 S}{\partial x^2} \right) = m^2 \quad (2)$$

This equation can further be written as

$$p^\mu p_\mu + i\hbar \frac{\partial p^\mu}{\partial x_\mu} = m^2 \quad (3)$$

where we have defined

$$p^t = -\frac{\partial S}{\partial t}, \quad p^x = \frac{\partial S}{\partial x} \quad (4)$$

and the index $\mu$ takes the values 1 and 2.

Equation (2) could be thought of as a generalisation of the classical Hamilton-Jacobi equation to the quantum mechanical case [also for reasons which will become apparent as we proceed], where the ‘action’ function $S(x,t)$ is now complex. Evidently, in (3) the $\hbar$ dependent terms appear as corrections to the classical term $p^\mu p_\mu$. We chose to consider the relativistic case, as opposed to the non-relativistic one, only because the available space-time symmetry makes the analysis more transparent.

Taking clue from the form of Eqn. (3) we now suggest a model for the dynamics, in the language of two noncommuting coordinates $\hat{x}$ and $\hat{t}$. We ascribe to the particle a ‘momentum’ $\hat{p}$, having two components $\hat{p}^t$ and $\hat{p}^x$, which do not commute with each other. The noncommutativity of these momentum components is assumed to be a consequence of the noncommutativity of the coordinates, as the momenta are defined to be the partial derivatives of the complex action $S(\hat{x}, \hat{t})$, with respect to the corresponding noncommuting coordinates.
We further assume that the coordinates $\hat{x}$ and $\hat{t}$ describe the non-commutative version of Minkowski space and that the noncommutative flat metric is

$$\hat{\eta}_{\mu\nu} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

The determinant is zero - it is not obvious that this is really an unsatisfactory feature in the present context (we remark further on this, below). This metric is obtained by adding an antisymmetric component to the Minkowski metric.

We now propose that the background independent description of the quantum dynamics is given by the equation

$$\hat{p}^\mu \hat{p}_\mu = m^2$$

Here, $\hat{p}_\mu = \hat{\eta}_{\mu\nu} \hat{p}^\nu$ is well-defined. Written explicitly, this equation becomes

$$(\hat{p}^t)^2 - (\hat{p}^x)^2 + \hat{p}^t \hat{p}^x - \hat{p}^x \hat{p}^t = m^2$$

Eqn. (6) appears an interesting and plausible proposal for the dynamics, because it generalizes the corresponding special relativistic equation to the noncommutative case. The noncommutative Hamilton-Jacobi equation is constructed from (7) by defining the momentum as gradient of the complex action function.

Of course we now need to ask if this dynamics looks the same as quantum mechanics, when seen from our classical spacetime. It should be apparent that a description of the noncommutative dynamics from the vantage point of a classical spacetime, is actually an approximate one. It is like trying to describe curved space dynamics using a flat spacetime metric - errors would be introduced, which have to be rectified by adding correction terms.

Thus, we propose the following rule for the transformation of the expression on the left of (7), when it is seen from our standard commuting coordinate system:

$$(\hat{p}^t)^2 - (\hat{p}^x)^2 + \hat{p}^t \hat{p}^x - \hat{p}^x \hat{p}^t = (p^t)^2 - (p^x)^2 + i\hbar \frac{\partial p^\mu}{\partial x^\mu}$$

Here, $p$ is the ‘momentum’ of the particle as seen from the commuting coordinate system, and is related to the complex action by Eqn. (4). This equality should be understood as an equality between the two equivalent equations of motion for the complex action function $S$ - one written in the noncommuting coordinate system, and the other written in the standard commuting coordinate system.

The idea here is that by using the Minkowski metric of ordinary spacetime one does not correctly measure the length of the ‘momentum’ vector, because the noncommuting off-diagonal part is missed out. The last, $\hbar$ dependent term in (8) provides the correction - the origin of this term’s relation to the commutator $\hat{p}^t \hat{p}^x - \hat{p}^x \hat{p}^t$ remains to be understood. If the relation (8) holds, there is equivalence between the background independent description (6) and standard quantum dynamics given by (3).

The proposal proceeds along analogous lines for four-dimensional spacetime. The metric $\hat{\eta}_{\mu\nu}$ is defined by adding an antisymmetric part (all entries of which are 1 and $-1$) to the Minkowski metric, and the off-diagonal contribution on the left-hand side of (8) is to be set equal to $i\hbar \partial p^\mu / \partial x^\mu$ on the right hand side.
3 Discussion

One could consider arriving at the noncommutative dynamics by a different route. Suppose we demand that the allowed class of noncommutative coordinate transformations are those which leave the line-element

\[ ds^2 = \hat{\eta}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu = d\hat{t}^2 - d\hat{x}^2 + d\hat{t}d\hat{x} - d\hat{x}d\hat{t} \]

invariant. The noncommutative flat metric is given by (5). This will be ‘flat noncommutative spacetime automorphism invariance’ - a generalisation of Lorentz invariance.

This line-element is left invariant by the Lorentz transformation

\[ \hat{x}' = \gamma(\hat{x} - \beta \hat{t}), \quad \hat{t}' = \gamma(\hat{t} - \beta \hat{x}) \]

where \( \gamma = (1 - \beta^2)^{-1/2} \). In the commutative limit, \( \beta \) has the interpretation of velocity: \( \beta = v/c \). In the noncommutative case, \( \beta \) should be thought of as defining a rotation in the non-commutative space by an angle \( \theta \) defined by \( \beta = \tanh \theta \).

It appears reasonable to expect that, since this class of transformations generalises Lorentz transformations to the noncommutative case, the dynamics should now be given by (6), which is a generalisation of the corresponding equation in special relativity. Hence the question posed here is: is quantum mechanics the same as the mechanics obtained by generalising special relativity to a noncommutative spacetime? The question as to exactly what is the form of the commutation relation for the noncommuting coordinates is still open.

A possible structure for the commutation relations might be as follows. We assume fundamental relations of the form

\[ [\hat{t}, \hat{x}] = iL_{Pl}^2, \quad [\hat{p}^t, \hat{p}^x] = iP_{Pl}^2 \]

where \( P_{Pl} \) is Planck-momentum (= \( m_{Pl}c \)). This would suggest uncertainty relations of the kind

\[ \Delta \hat{t} \Delta \hat{x} \sim L_{Pl}^2, \quad \Delta \hat{p}^t \Delta \hat{p}^x \sim P_{Pl}^2 \]

and hence

\[ \Delta \hat{t} \Delta \hat{x} \Delta \hat{p}^t \Delta \hat{p}^x \sim L_{Pl}^2 P_{Pl}^2 = \hbar^2 \]

which by symmetry suggests that

\[ \Delta \hat{t} \Delta \hat{p}^t \sim \hbar, \quad \Delta \hat{x} \Delta \hat{p}^x \sim \hbar \]

If this last uncertainty relation is invariant under transformation from the noncommuting coordinate system to the commuting one, we might be able to understand the standard quantum commutation relation \([q,p] = i\hbar\) as having its origin in the fundamental commutation relations (11). While its generally regarded as implausible that quantum mechanics could be a consequence of “Planck-scale” physics, we feel it may not be completely unreasonable to consider such a connection.

One could also try to derive the non-commutative Hamilton-Jacobi equation as a consequence of the variational principle

\[ \delta S = mc \delta \int ds = 0 \]
where $ds$ is defined by the non-commutative flat line-element (9). Work is in progress to address these unresolved issues, as well as the multi-particle case, the non-relativistic limit, and a detailed examination of the properties of a noncommutative special relativity.

In the 4-d case, the noncommutative line-element analogous to (9) is not invariant under a Lorentz transformation, thus suggesting that here the appropriate coordinate transformation is a generalization of Lorentz transformations.

When one assumes $m \ll m_{Pl}$, as was done here, one is neglecting gravity. Allowing for the mass to be comparable to Planck mass implies the introduction of curvature and a generalisation of the noncommutative flat metric to a noncommutative curved metric. In particular, the off-diagonal antisymmetric components of the metric will become mass-dependent, and the determinant will no longer be zero. Thus the vanishing of the determinant is strictly only a theoretical situation, and not an actual one. In reality, for $m \ll m_{Pl}$ the determinant is very close to zero, but not exactly zero.

The introduction of curvature and departure from the noncommutative flat metric brings into the picture general coordinate transformations of noncommuting coordinates (thus generalising general covariance to automorphism invariance). In such a case, one expects significant departures from the Schrodinger equation when $m \approx m_{Pl}$, an example of which was discussed in [6]. The experimental search for such a departure would go a long way in establishing contact between experiment and theoretical quantum gravity.

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