SUBRINGS OF INVARIANTS FOR ACTIONS OF FINITE-DIMENSIONAL HOPF ALGEBRAS

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Abstract. This paper is a survey of recent works on invariants of actions of Hopf algebras. Its highlights are results on integrality of \( H \)-module PI algebras over subrings of invariant elements obtained by P. Etingof and M. Eryashkin. Older results are also reviewed.

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Introduction

The classical invariant theory on the threshold of the 20th century considered finite generation of invariants as a key problem. Initially, only group actions on polynomial rings in several variables were looked at. In the case of a finite group, Emmy Noether's constructive approach via Galois resolutions yielded certain conclusions also in a more abstract situation. It showed that any commutative ring is integral over the subring of invariants with respect to a finite group of automorphisms. For a finitely generated commutative algebra over a field, the integrality over a subalgebra is equivalent to its finiteness as a module over this subalgebra, while the finite generation of the subalgebra is a consequence of these properties. This makes integrality and module-finiteness particularly important in the study of invariants.

Grothendieck and his school made a transition from group actions to actions of group schemes. As it turned out, there are general results on invariants incorporated in the construction of quotients by finite group schemes. These results can be interpreted in terms of coactions of commutative Hopf algebras or, dually, in terms of actions of cocommutative Hopf algebras.

In a different spirit, many people contributed to research on group actions and Lie algebra actions, and also group gradings, on noncommutative rings. The work on Hopf algebra actions started at
the 1980ies aimed to unify previously known results in those areas. In spite of the progress made in this study, considerable difficulties have been encountered on some questions. Even now the state of knowledge in the general Hopf algebra case has not reached the level recorded in the 1980 manuscript of Susan Montgomery on fixed rings of finite automorphism groups of associative rings (see [39]).

The present paper is intended primarily as a survey of recent works on invariants of Hopf algebra actions. Its highlights are results on integrality of $H$-module PI algebras over central invariants obtained independently by Etingof [27] and Eryashkin [5]. There is a different notion of integrality introduced by Schelter [48] which is suitable for extensions of noncommutative rings. Eryashkin has also proved that an arbitrary $H$-module PI algebra is Schelter integral over the subring of all invariants when the Hopf algebra $H$ is semisimple and cosemisimple (see [6]), thus answering a question of Montgomery (see [40]) in the PI case.

Other recent results on invariants are presented in the author’s own two papers. In [54], it has been proved that, given a semisimple Hopf algebra $H$, all nonzero $H$-stable one-sided ideals of any noetherian $H$-semiprime $H$-module algebra $A$ contain nonzero invariants, and the classical quotient ring of $A$ is obtained by localization at the Ore set of invariant regular elements. We will show that these conclusions are true even if $A$ is not noetherian provided that $A$ has an artinian classical quotient ring. Another article [56] answers the question of Bergen, Cohen, and Fischman (see [11]) on the equality of the left and right dimensions of a skew field over the subfield of invariants. We will also review older results.

In this paper, only finite-dimensional Hopf algebras over a field will be considered. However, it should be noted that many results discussed here can be formulated more generally when an arbitrary commutative ring is taken as a base ring and the Hopf algebras are finitely generated projective modules. In fact, this was the setting for several original papers.

1. Terminology and Notation

Throughout the whole paper, $H$ denotes a finite-dimensional Hopf algebra over a field $k$. We denote by $\Delta$, $\varepsilon$, and $S$ the comultiplication, counit, and antipode in either $H$ or the dual Hopf algebra $H^*$, depending on the context. For a general information on Hopf algebras and their actions on rings, we refer the reader to [1, 40] and other books.

Recall that the categories of $H$-modules and $H$-comodules are monoidal. If $V$ and $W$ are two (left) $H$-modules, then $V \otimes W$ is an $H \otimes H$-module, and $H$ acts on $V \otimes W$ via $\Delta : H \to H \otimes H$. If $V$ and $W$ are two (right) $H$-comodules, then $V \otimes W$ is an $H \otimes H$-comodule, and the coaction of $H$ is obtained by means of the map $H \otimes H \to H$, $a \otimes b \mapsto ab$. Here and in the sequel, $\otimes$ means $\otimes_k$ unless the base ring for the tensor product is indicated explicitly.

All algebras and rings are assumed to be associative and unital. An $H$-module algebra is a $k$-algebra $A$ equipped with a left $H$-module structure such that the multiplication map $A \otimes A \to A$ is $H$-linear, assuming that $H$ acts on $A \otimes A$ via $\Delta$. If this condition is satisfied, then $H$ acts trivially on the image of $k$ in $A$, so that $h1_A = \varepsilon(h)1_A$ for all $h \in H$, where $1_A$ is the unity of $A$ (see [17, Lemma 1.9]).

The $H$-invariant elements of an $H$-module algebra $A$ form a subalgebra

$$A^H = \{ a \in A \mid ha = \varepsilon(h)a \quad \text{for all } h \in H \}.$$ 

An $H$-comodule algebra is a $k$-algebra $A$ equipped with a right $H$-comodule structure such that the multiplication map $A \otimes A \to A$ is a homomorphism of comodules. This condition can be reformulated by saying that the comodule structure mapping $\rho : A \to A \otimes H$ is multiplicative, i.e.,

$$\rho(ab) = \rho(a)\rho(b) \quad \text{for all } a, b \in A.$$ 

Moreover, in this case, $\rho$ is a homomorphism of unital algebras. The fact that $\rho(1) = 1 \otimes 1$ and, therefore, $H$ coacts trivially on the image of $k$ in $A$, is easily seen as follows. Clearly, $\rho(1)x = x$ for
all $x$ in the right ideal $I$ of $A \otimes H$ generated by $\rho(A)$. Therefore, it suffices to verify the equality $I = A \otimes H$, but this does hold since the linear mapping

$$a \otimes h \mapsto \rho(a) \cdot (1 \otimes h)$$

is bijective. In fact, the assignment $a \otimes h \mapsto (id \otimes S)(\rho(a)) \cdot (1 \otimes h)$ defines the inverse mapping. This argument shows also that $\rho$ is an isomorphism of $A$ onto a subalgebra of $A \otimes H$, and $A \otimes H$ is a free $\rho(A)$-module with respect to the action by left multiplications. Similarly, $A \otimes H$ is free over $\rho(A)$ on the right.

With an $H$-comodule algebra, one associates its subalgebra consisting of coaction invariants

$$A^{\text{co}H} = \{a \in A \mid \rho(a) = a \otimes 1\}.$$ 

As is well known, the left $H$-module structures are in a bijective correspondence with the right $H^*$-comodule structures. This correspondence is compatible with the tensor products of modules and comodules. Therefore, each $H$-module algebra is an $H^*$-comodule algebra, and vice versa. Under the canonical identification of $A \otimes H^*$ with $\text{Hom}_k(H,A)$, the comodule structure on an $H$-module algebra $A$ is given by the mapping

$$\rho : A \to A \otimes H^* \cong \text{Hom}_k(H,A),$$

$$\rho(a)(h) = ha \quad \text{for } a \in A, h \in H.$$

In the rest of the paper, $A$ is assumed to be an $H$-module algebra. However, sometimes arguments are formulated more naturally in terms of comodule structures. Note, in particular, that $A^H = A^{\text{co}H^*}$.

By an ideal, we mean a two-sided ideal unless otherwise explicitly stated. $H$-Invariant ideals, i.e., ideals stable under the action of $H$, are of particular interest. Several properties of an $H$-module algebra $A$ are defined in terms of the collection of its $H$-stable ideals:

(i) $A$ is $H$-simple if $A \neq 0$ and $A$ has no $H$-stable ideals except for the zero ideal and the whole $A$;

(ii) $A$ is $H$-prime if $A \neq 0$ and $IJ \neq 0$ for all nonzero $H$-stable ideals $I$ and $J$ of $A$;

(iii) $A$ is $H$-semiprime if $A$ contains no nonzero nilpotent $H$-stable ideals.

An $H$-stable ideal $I$ of $A$ is called $H$-prime (respectively, $H$-semiprime) if the factor-algebra $A/I$ is $H$-prime (respectively, $H$-semiprime). For an arbitrary ideal $I$ of $A$, we denote by $I_H$ the largest $H$-stable ideal of $A$ contained in $I$. If $I$ is prime (respectively, semiprime), then $I_H$ is $H$-prime (respectively $H$-semiprime). Conversely, if $I$ is $H$-prime, then $I = P_H$ for some prime ideal $P$ of $A$ (see [15, Lemma 1.5]). An $H$-stable ideal is $H$-semiprime if and only if it is an intersection of $H$-prime ideals (see [41, Lemma 8.3]).

If a ring-theoretic notion is not prefixed by $H$, then it does not take into account the $H$-module structure. For example, an $H$-module algebra $A$ is $PI$ algebra if $A$ satisfies a polynomial identity as an ordinary algebra.

A left or right $A$-module $M$ is said to be $H$-equivariant if $M$ is equipped with a left $H$-module structure such that the action of $A$ on $M$ comes from an $H$-linear mapping $A \otimes M \to M$ or $M \otimes A \to M$, respectively, assuming that $H$ acts in the tensor products via $\Delta$. Denote by $H_A M$ and $H_M A$ the categories of $H$-equivariant left and right $A$-modules. Morphisms in these categories are mappings which are $A$-linear and $H$-linear simultaneously. Let $\mathcal{A}_M$ and $\mathcal{M}_A$ denote the categories of left and right $A$-modules.

Similarly, an $A$-bimodule $M$ is called $H$-equivariant if $M$ is equipped with a left $H$-module structure with respect to which $M$ is an object of both $H_A M$ and $H_M A$. We denote by $H_A M_A$ the category of $H$-equivariant $A$-bimodules. Note that each $H$-stable ideal of $A$ is an object of $H_A M_A$, and any homomorphism of $H$-module algebras $A \to B$ makes $B$ an object of $H_A M_A$.

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Recall that the smash product algebra $A \# H$ has $A \otimes H$ as its underlying vector space, the canonical mappings $A \to A \otimes 1$ and $H \to 1 \otimes H$ are isomorphisms of $A$ and $H$ onto subalgebras of $A \# H$, while 

$$(1 \# h)(a \# 1) = \sum h(1)a \# h(2) \quad \text{for } a \in A, \ h \in H.$$ 

The compatibility of the $A$-module and $H$-module structures required in the definition of the category $H_{-A}M$ means precisely that the two module structures come from a single $A \# H$-module structure. Thus, $H_{-A}M$ is identified with the category of left $A \# H$-modules.

Let $A^{\text{op}}$ be $A$ with the opposite multiplication, and let $H^{\text{cop}}$ be $H$ with the opposite comultiplication. Then $A^{\text{op}}$ is an $H^{\text{cop}}$-module algebra, and $H_{-A}M_A$ can be identified with the category of left $A^{\text{op}} \# H^{\text{cop}}$-modules.

The algebras $A \# H$ and $A^H$ are connected by a Morita context. Several articles (see [11, 12, 18, 20]) deduce various information about the invariant ring $A^H$ when $A \# H$ is known to be simple, or prime, or semiprime. These results are also discussed in [40]. However, for an arbitrary finite-dimensional Hopf algebra $H$, it is quite difficult to understand the ring structure of $A \# H$ in terms of the original algebra $A$. There is still a big gap between what is known in the general case and in the case of a finite group $G$ acting on a ring $R$, where the skew group ring $R \ast G$ is sufficiently well understood as a normalizing extension of $R$. In our paper, we rarely use ring-theoretic properties of $A \# H$ directly. However, equivariant modules are important for many considerations.

Recall that a left (respectively, right) integral in $H$ is an element $0 \neq t \in H$ such that $ht = \varepsilon(h)t$ (respectively, $th = \varepsilon(h)t$) for all $h \in H$. Let $t$ be a left integral. If $V$ is a left $H$-module, then the action of $t$ gives a mapping $\hat{t} : V \to V^H$, where $V^H$ is the subspace of $H$-invariant elements in $V$. In [20], this mapping was called a trace by analogy with the terminology used in the case of group actions.

By Maschke’s theorem, $H$ is semisimple if and only if $\varepsilon(t) \neq 0$. In this case, $t$ acts on $V^H$ as a nonzero scalar multiplication. In particular, we have $tV = V^H$, i.e., the trace $\hat{t}$ is always surjective.

### 2. Structural Properties of $H$-Module Algebras

In this section, we present several results concerned with the structure of $H$-module algebras, on which recent work on invariants of Hopf actions is based. Actually, most of these results can be formulated for a not necessarily finite-dimensional Hopf algebra $H$. Nevertheless, it will be assumed in all statements that $\text{dim} \ H < \infty$. With this assumption, we do not need to mention any additional restrictions, and also the proofs become considerably simpler.

A key argument used in deriving these results comes from the following theorem.

**Theorem 2.1** (see [51]). Assume that $A$ is a semilocal $H$-simple $H$-module algebra. Then each object $M \in H_{-A}M_A$ is projective in $M_A$. Moreover, a direct sum of several copies of $M$ is a free $A$-module. A similar conclusion holds in $H_{-A}M$.

There is one application of the freeness properties of $H$-equivariant modules, where the $H$-simplicity of the $H$-module algebra is not known in advance. To deal with this situation, one needs the next lemma stated under more technical assumptions about $A$ and $M$ than the previous theorem.

Denote by $\text{Max} \ A$ the set of maximal ideals of $A$. If $A$ is semilocal, then its factor-algebra by the Jacobson radical is semisimple artinian. This means that the set $\text{Max} \ A$ is finite and $A/P$ is simple artinian for each $P \in \text{Max} \ A$. An object $M$ of the category $H_{-A}M_A$ is said to be $A$-finite if $M$ is finitely generated as an $A$-module. If $M$ is $A$-finite, then $M/MP$ is an $A$-module of a finite length. The rank of $M$ at $P$ is defined as

$$r_P(M) = \frac{\text{length}_A M/MP}{\text{length}_A A/P} \in \mathbb{Q}.$$
Lemma 2.2. Let $A$ be a semilocal $H$-module algebra. Assume that $M \in H\cdot\mathcal{M}_A$ is $A$-finite and there exists $P \in \text{Max } A$ such that $P$ contains no nonzero $H$-stable ideals of $A$ and $r_P(M) \geq r_Q(M)$ for all $Q \in \text{Max } A$. Then $M^n$ is a free $A$-module for some integer $n > 0$.

For the proof, see [51, Lemma 7.5]. This lemma is valid even if $H$ is an infinite-dimensional Hopf algebra. However, the assumption $\dim H < \infty$ is needed to deduce the conclusion of Theorem 2.1 for objects $M$ which are not $A$-finite. Since each element of $M$ is contained in an $A$-finite subobject of $M$, a basis for $M^n$ over $A$ can be constructed using Zorn’s lemma.

It turns out that Lemma 2.2 and Theorem 2.1 lead to several fundamental facts concerning artinian $H$-module algebras very quickly. Sometimes, one has initially less information about an $H$-module algebra, but the left and right artinian conditions can be deduced. For this reason, we have to deal with semiprimary algebras. A semilocal ring is said to be \emph{semiprimary} if its Jacobson radical is nilpotent.

Lemma 2.3. Let $A$ be a semiprimary $H$-module algebra, and let $K \in \text{Max } A$. Then the largest $H$-stable ideal $K_H$ contained in $K$ is a maximal $H$-stable ideal of $A$.

Proof. Replacing $A$ with the factor-algebra $A/K_H$, we can assume that $K_H = 0$ and, therefore, we should prove that $A$ is $H$-simple. First, note that $A$ is $H$-prime. Indeed, if $I$ and $J$ are two nonzero $H$-stable ideals of $A$, then both $I \not\subset K$ and $J \not\subset K$, which implies $IJ \not\subset K$ and, therefore, $IJ \neq 0$.

Every semiprimary ring satisfies DCC on finitely generated one-sided ideals. Hence $A$ has a minimal nonzero $H$-stable finitely generated right ideal $M$. If $0 \neq x \in M$, then $M = (Hx)A$ since $(Hx)A$ is a nonzero $H$-stable finitely generated right ideal of $A$ contained in $M$. It follows that $M$ is minimal in the set of all nonzero $H$-stable ideals of $A$. If $I$ is any nonzero $H$-stable ideal of $A$, then $MI$ is an $H$-stable right ideal. Since $MI \subset M$ and $MI \neq 0$ by the $H$-primeness of $A$, we obtain $MI = M$.

We can consider $M$ as an $A$-finite object of $H\cdot\mathcal{M}_A$. We choose $P \in \text{Max } A$ for which $r_P(M)$ attains the maximum value. Since $M \neq 0$, we have $r_P(M) > 0$. This means that $M \neq MP$, but then $M \neq MP_H$ too, which is possible only for $P_H = 0$ by the previous argument. Thus, the assumptions of Lemma 2.2 are satisfied, and we deduce that $M^n$ is a free $A$-module for some $n > 0$. Hence $MI \neq M$ for each ideal $I \neq A$. If $I$ is $H$-stable and $I \neq A$, this implies $I = 0$. \qed

Theorem 2.4 (see [57]). Assume that $A$ is semiprimary and $H$-semiprime. There exists an isomorphism of $H$-module algebras

$$A \cong A_1 \times \ldots \times A_n,$$

where $A_1, \ldots, A_n$ are $H$-simple $H$-module algebras. If $A$ has a maximal ideal containing no nonzero $H$-stable ideals of $A$, then $A$ is $H$-simple.

Proof. By Lemma 2.3, the maximal $H$-stable ideals of $A$ are precisely the ideals $K_H$ with $K \in \text{Max } A$. In particular, there are finitely many of them. Let $I_1, \ldots, I_n$ be all maximal $H$-stable ideals. Then $I_1 \cap \ldots \cap I_n$ is contained in the Jacobson radical $J$ of $A$. Since $J$ is nilpotent and $A$ is $H$-semiprime, we obtain $I_1 \cap \ldots \cap I_n = 0$. But $I_k + I_l = A$ for each pair of indices $k \neq l$, which implies that the desired direct product decomposition of $A$ holds with $A_k = A/I_k$ by the Chinese remainder theorem. \qed

Corollary 2.5. Each right coideal subalgebra $B$ of $H^*$ is an $H$-simple $H$-module algebra and $H^*$ is a right and left $B$-free module.

Proof. Here $B$ is a subalgebra and a right coideal of $H^*$. The restriction of the comultiplication $\Delta$ in $H^*$ gives a mapping $B \to B \otimes H^*$ which makes $B$ an $H^*$-comodule algebra. Hence $B$ is also an $H$-module algebra. We set $B^+ = \text{Ker } \varepsilon|_B$ where $\varepsilon$ is the counit of $H^*$. Then $B^+ \in \text{Max } B$ with $B/B^+ \cong k$. Assume that $I$ is an $H$-stable ideal of $B$. Then $\Delta(I) \subset I \otimes H^*$. Recall that $(\varepsilon \otimes \text{id}) \circ \Delta$ is
the identity mapping by the definition of the counit. If $I \subseteq B^+$, we obtain $x = (\varepsilon \otimes \text{id})(\Delta x) = 0$ for each $x \in I$ since $\varepsilon(I) = 0$ and, therefore, $I = 0$. Thus, $B^+$ contains no nonzero $H$-stable ideals of $B$. By Theorem 2.4, $B$ is $H$-simple.

Now $H^*$ is also an $H$-module algebra and $B$ is its $H$-stable subalgebra. Hence we can consider $H^*$ as an object of both $H_{-B}M$ and $H_{-B}M$. The freeness of $H^*$ over $B$ follows from Theorem 2.1. □

Corollary 2.6. Assume that $A$ is semiprimary and $H$-semiprime. Then each object $M \in H_{-A}M$ is projective in $H_{-A}M$.

Proof. The direct product decomposition of $A$ given in Theorem 2.4 implies that $M \cong M_1 \times \ldots \times M_n$, where $M_i = M \otimes_A A_i \in H_{-A}M_i$. By Theorem 2.1, $M_i$ is projective in $M_{A_i}$ for each $i$, which implies the conclusion.

Corollary 2.7. Assume that $A$ is semiprimary and $H$-semiprime. If $I$ is an $H$-stable right ideal of $A$, then $I = eA$ for some idempotent $e \in A$.

Proof. We can consider $A/I$ as an object of $H_{-A}M$. By Corollary 2.6, $A/I$ is projective in $H_{-A}M$. Hence $I$ is a direct summand of $A$ as a right $A$-module. □

Theorem 2.8 (see [57]). Any semiprimary $H$-semiprime algebra $A$ is a quasi-Frobenius ring. In particular, $A$ is left and right artinian.

Proof. By Theorem 2.4, it suffices to consider the case where $A$ is $H$-simple. We are going to apply the general fact that a semiprimary ring is quasi-Frobenius whenever it is left and right self-injective (see [33, Theorem 10]). We show that $A$ is left self-injective. Applying this to the $H^\text{cop}$-module algebra $A^\text{cop}$, we deduce that $A$ is also right self-injective, and the conclusion follows.

Take any nonzero injective object $M \in H_{-A}M$. Then $M$ remains injective in $H_{-A}M$. To see this, recall that $H_{-A}M$ is identified with the category of left $B$-modules for $B = A#H$. The forgetful functor $H_{-A}M \to H_{-A}M$ is identified with the restriction functor $B_{-A}M \to H_{-A}M$ that arises from the canonical embedding of $A$ into $B$. The latter functor preserves injective objects since it has an exact left adjoint $B \otimes_A ?$.

But $M^n$ is a free $A$-module for some $n > 0$ by Theorem 2.1. Therefore, $A$ is an $H_{-A}M$-direct summand of $M^n$. Since $M^n$ is injective in $H_{-A}M$, so is $A$.

Thus, an $H$-semiprime algebra $A$ is semiprimary if and only if $A$ is left artinian, if and only if $A$ is right artinian, and we will say that $A$ is artinian in this case.

Theorem 2.9. Let $A$ be an $H$-stable subalgebra of an $H$-module algebra $B$. If $A$ is artinian and $H$-simple, then $B$ is free as an $A$-module with respect to the action by right (or left) multiplications.

Proof. We can consider $B$ as an object of $H_{-A}M$. Hence Theorem 2.1 can be applied to $B$. For each $P \in \text{Max } A$, denote by $F_P$ the projective cover in $H_{-A}M$ of a simple right $A/P$-module. These modules $F_P$ are indecomposable, and $A \cong \bigoplus_{P \in \text{Max } A} F_P^{m_P}$ for some multiplicities $m_P$. We set

$$E = \bigoplus_{P \in \text{Max } A} F_P^{m_P/d}, \quad \text{where } d = \gcd\{m_P \mid P \in \text{Max } A\}.$$ 

Then $A \cong E^d$. If $M$ is any right $A$-module such that $M^n$ is free for some $n > 0$, then, by the Krull–Schmidt theorem, $M$ is isomorphic to a direct sum of a family of copies of $E$. Moreover, $M$ is itself free if either $M$ is not finitely generated or $M \cong E^k$ with $d$ dividing $k$. If $M$ is, in fact, an $A$-bimodule, then $M \cong A \otimes_A M \cong N^d$, where $N = E \otimes_A M$. In this case, $N$ has to be isomorphic to a direct sum of a family of copies of $E$, which implies that $M$ is right $A$-free by the previous observation. It remains to apply this for $M = B$. □
Theorem 2.10 (see [57]). Assume that $H$ is semisimple, $A$ is artinian, and $H$-semiprime. Then $H \cdot A$ and $H \cdot A$ are semisimple categories. In other words, the smash product algebras $A^{op} \# H^{cop}$ and $A \# H$ are semisimple artinian.

Proof. A key ingredient in the proof is the fact established by Cohen and Fischman (see [18]) according to which a submodule $W$ of a left $A \# H$-module $V$ is a direct summand whenever $W$ is an $A$-module direct summand of $V$. For this, one needs only semisimplicity of $H$ but no assumptions about an $H$-module algebra $A$.

In the case of the category $H \cdot A$, a similar argument runs as follows. Let $M, N$ be two objects of $H \cdot A$. There exists a left $H$-module structure on $\text{Hom}_k(M, N)$ defined by the rule

$$(h \cdot f)(x) = \sum h(1) f(S(h(2))x)$$

for $h \in H$ with $\Delta h = \sum h(1) \otimes h(2)$, $f \in \text{Hom}_k(M, N)$, $x \in M$. It is straightforward to verify that $\text{Hom}_A(M, N)$ is stable under this action of $H$ and that a linear mapping $f \in \text{Hom}_k(M, N)$ is $H$-invariant if and only if $f$ is $H$-linear. Let $t \in H$ be an integral with $\varepsilon(t) = 1$. If $N$ is a submodule of $M$ which splits off as an $A$-module direct summand, then there exists an $A$-linear mapping $f : M \to N$ such that $f|_N = \text{id}$. Now the mapping $f' = t \cdot f$ is $A$-linear and $H$-linear simultaneously, and also $f'|_N = \text{id}$. Hence $M = N \oplus \text{Ker} f'$, a direct sum decomposition in $H \cdot A$.

Under the assumption that $A$ is artinian and $H$-semiprime, any submodule $N$ of $M$ is an $A$-module direct summand since the factor object $M/N \in H \cdot A$ is projective in $A$ by Corollary 2.6. Hence the previous conclusion holds. □

The result of Cohen and Fischman mentioned in the proof of Theorem 2.10 means that $A \# H$ is a semisimple extension of $A$ if $H$ is semisimple. In [18], it was used to show that $A \# H$ is semiprime artinian when so is $A$. If $A$ is not semiprime but only $H$-semiprime, the same conclusion requires Theorem 2.1, which was proved much later.

Let $R$ be a ring. A ring $Q$ is said to be a classical right quotient ring of $R$ if

(a) $Q$ contains $R$ as a subring,
(b) all regular elements, i.e., non-zero-divisors, of $R$ are invertible in $Q$, and
(c) each element $q \in Q$ can be written as $q = as^{-1}$, where $a, s \in R$, $s$ is regular.

Such a ring $Q$ exists if and only if the set of all regular elements of $R$ satisfies the right Ore condition. In this case, $Q$ is unique up to an isomorphism, and we will denote this ring by $Q(R)$.

If an $H$-module algebra $A$ is not artinian but $A$ has an artinian classical right quotient ring, the previous results can still be used to derive an information about $A$. There are two important cases where this happens.

Theorem 2.11 (see [57]). If $A$ is right noetherian and $H$-semiprime, then $A$ has a quasi-Frobenius classical right quotient ring.

Theorem 2.12 (see [5]). If $A$ is a PI algebra and it is $H$-semiprime with finitely many minimal $H$-prime ideals, then $A$ has a quasi-Frobenius classical right quotient ring. In particular, this holds if $A$ is a finitely generated and $H$-semiprime PI algebra.

In the proof of Theorem 2.11, one first constructs a generalized quotient ring $Q$ using the filter of $H$-stable essential right ideals of $A$. This ring turns out to be semiprimary. In the proof of Theorem 2.12 one starts from the $H$-equivariant Martindale quotient ring $Q$. As we will see in Sec. 6, it is a finite module over a central artinian subring. In both cases, $H$ acts on $Q$, and $H$-semiprimeness is preserved under the passage to $Q$. Hence $Q$ is quasi-Frobenius by Theorem 2.8, and the conclusion that $Q$ is a classical right quotient ring can be deduced from the following ring-theoretic fact.
**Proposition 2.13** (see [53]). Let $R$ be a subring of a quasi-Frobenius ring $Q$. Assume that $I$ is a topologizing filter of right ideals of $R$ with the following properties:

(a) each $I \in I$ has zero left and right annihilators in $Q$,
(b) for each $q \in Q$ there exists $I \in I$ such that $qI \subset R$.

Then each right ideal $I \in I$ contains a regular element of $R$, and $Q$ is a classical right quotient ring of $R$.

We say that a family $F$ of right ideals in a ring $R$ is a filter if, for any pair of right ideals $I, J \in F$, there exists $K \in F$ such that $K \subset I \cap J$. A filter $F$ is topologizing if for each $I \in F$ and each $a \in R$ there exists $I' \in F$ such that $aI' \subset I$. The condition that with each $I \in F$ all larger right ideals also belong to $F$ is often included in the definition of a filter, but omitting it will cause no harm.

With small improvements in the proof of [53, Proposition 1.4], the assumption that the filter $I$ is topologizing can be actually removed. For the case where $Q$ is semisimple artinian, see [54, Proposition 2.3].

**Theorem 2.14** (see [57]). Assume that $A$ has a right artinian classical right quotient ring $Q$. Then the $H$-module structure on $A$ has a unique extension to $Q$ with respect to which $Q$ becomes an $H$-module algebra.

**Proof.** We argue in terms of comodule structures. Since the mapping

$$A \otimes H^* \to A \otimes H^*, \quad a \otimes f \mapsto \rho(a) \cdot (1 \otimes f),$$

is invertible, $A \otimes H^*$ is a free $A$-module with respect to the action of $A$ given by left multiplications by the elements $\rho(a)$. For each regular element $s$ of $A$, it follows that $\rho(s)$ is right regular in $A \otimes H^*$, i.e., $\rho(s)x = 0$ for $x \in A \otimes H^*$ implies $x = 0$. Then $\rho(s)$ remains right regular in $Q \otimes H^*$ and, therefore, $\rho(s)$ has to be invertible in the right artinian ring $Q \otimes H^*$. This property shows that $\rho : A \to A \otimes H^*$ can be extended to an algebra homomorphism $\rho' : Q \to Q \otimes H^*$. Now $(\text{id} \otimes \Delta)\rho'$ and $(\rho' \otimes \text{id})\rho'$ are two algebra homomorphisms $Q \to Q \otimes H^* \otimes H^*$ which agree on $A$. Hence

$$(\text{id} \otimes \Delta)\rho' = (\rho' \otimes \text{id})\rho'.$$

Thus, $\rho'$ is a structure of an $H^*$-comodule algebra extending the given one on $A$. \hfill \Box

In the situation of Theorem 2.14, we have two subrings of invariants $A^H$ and $Q^H$. Clearly, $A^H = A \cap Q^H$. Theorem 2.14 is true even without assumption that $\dim H < \infty$.

The conclusions of Theorems 2.4, 2.8, and 2.11 hold also for some classes of infinite-dimensional Hopf algebras. Unfortunately, it is still unknown whether the assumption that $H$ has a bijective antipode is sufficient for their validity. However, if $A$ is right artinian and $H$-semiprime, then $A$ is a quasi-Frobenius ring, even if $H$ is an arbitrary infinite-dimensional Hopf algebra (see [52]).

### 3. Module-Finiteness over the Invariants

In this section, we examine those cases, where $A$ is known to be a finite $A^H$-module. Many results discussed here date back to as early as the 1980ies.

The comodule structure $\rho : A \to A \otimes H^*$ enables us to define a $k$-linear mapping

$$\gamma : A \otimes A^H A \to A \otimes H^*, \quad a \otimes b \mapsto (a \otimes 1) \cdot \rho(b)$$

(recall that $A^H = \{a \in A \mid \rho(a) = a \otimes 1\}$). Under the canonical identification of $A \otimes H^*$ with $\text{Hom}_k(H, A)$, we have

$$\gamma(a \otimes b)(h) = a(hb) \quad \text{for } a, b \in A \text{ and } h \in H.$$
The notion of Hopf Galois extensions of algebras is defined in terms of comodule structures (see [40, Chap. 8]). In the case where the Hopf algebra is finitely generated projective as a module over a commutative base ring, this notion was introduced by Kreimer and Takeuchi (see [34]).

Since $A$ is an $H$-module algebra by our convention, we say that $A$ is an $H^*$-Galois extension of the subalgebra $A^H$ if $\gamma$ is bijective. Since $\dim H^* < \infty$, it suffices to require surjectivity of $\gamma$ (see [34]).

**Theorem 3.1** (see [34]). Assume that $A$ is an $H^*$-Galois extension of $A^H$. Then $A$ is a finitely generated projective $A^H$-module on the left and on the right.

**Proof.** As explained in [40, 8.3.1], this conclusion can be proved by verifying the dual basis property which characterizes projective modules. To do this, let $t \in H$ be a left integral and $\lambda \in H^*$ be a right integral such that $\lambda(t) = 1$, and let $1 \otimes \lambda = \gamma(\sum a_i \otimes b_i)$ for some elements $a_i, b_i \in A$ ($i = 1, \ldots, n$). If $x \in A$, then

$$\gamma\left(\sum a_i \otimes b_i x\right) = (1 \otimes \lambda) \cdot \rho(x) = x \otimes \lambda,$$

and this means that

$$\sum a_i (h(b_i x)) = \lambda(h)x \quad \text{for all } h \in H.$$

For each $i$, we define a right $A^H$-linear mapping $f_i : A \to A^H$ by the relation $f_i(x) = t(b_i x)$. Taking $h = t$ above, we deduce that

$$\sum a_i f_i(x) = \sum a_i (t(b_i x)) = \lambda(t)x = x$$

for all $x \in A$. In the proof of the left-hand side version, one proceeds similarly replacing $\gamma$ with the mapping $a \otimes b \mapsto \rho(a) \cdot (b \otimes 1)$, which is also bijective by [34, Proposition 1.2]. □

Hopf Galois extensions form an important special class of comodule algebras for which more information is available than in general. However, the mapping $\gamma$ is quite useful, even if $\gamma$ is not bijective. We consider $A \otimes H^*$ as an $A$-bimodule with respect to the left and right actions defined by the rules

$$ax = (a \otimes 1) \cdot x, \quad xa = x \cdot \rho(a) \quad \text{for } a \in A \text{ and } x \in A \otimes H^*.$$

Then $\gamma$ is a homomorphism of $A$-bimodules. In particular, its image is a subbimodule of $A \otimes H^*$. Also, $\gamma$ corresponds to certain $H$-module structures. Recall the two natural left actions of $H$ on $H^*$ defined by the relations

$$(h \rightarrow \xi)(g) = \xi(gh), \quad (h \rightarrow \xi)(g) = \xi(S(h)g) \quad \text{for } g, h \in H \text{ and } \xi \in H^*.$$

**Lemma 3.2.** The mapping $\gamma$ is a morphism in $H$-$A$-$M$ with respect to the $H$-module structures on $A \otimes_A H$ and $A \otimes H^*$ defined by the relations

$$h(a \otimes b) = ha \otimes b, \quad h(a \otimes \xi) = h(1)_1 a \otimes (h(2) \rightarrow \xi), \quad \text{(\star)}$$

where $a, b \in A$, $h \in H$, and $\xi \in H^*$. On the other hand, $\gamma$ is a morphism in $H$-$M_A$ with respect to another pair of $H$-module structures

$$h(a \otimes b) = a \otimes hb, \quad h(a \otimes \xi) = a \otimes (h \rightarrow \xi). \quad \text{(\star\star)}$$

**Proof.** Clearly, the $H$-module structures (\star) are compatible with the left $A$-module structures, so that $A \otimes_A H$ and $A \otimes H^*$ become objects of $H$-$A$-$M$. To show that $\gamma$ is $H$-linear with respect to (\star), we need only to verify that $\rho(b)$ is $H$-invariant for each $b \in A$. In the monoidal category of left $H$-modules, the module $H^*$ with the action $\triangleright$ is the left dual of $H$ with the action by left multiplications. Therefore,

$$\text{Hom}_H(V, A \otimes H^*) \cong \text{Hom}_H(V \otimes H, A)$$

for each left $H$-module $V$. Taking $V = k$ with the trivial module structure, we obtain $(A \otimes H^*)^H \cong \text{Hom}_H(H, A)$. Under this bijection, the $H$-linear mapping $H \to A$, $h \mapsto hb$, corresponds to $\rho(b) \in A \otimes H^*$. In other words, $(A \otimes H^*)^H = \rho(A)$. 168
Now $H^*$ is an $H$-module algebra with respect to the action $\to$. Hence so is $A \otimes H^*$ with respect to the action given in (**). The mapping $\rho : A \to A \otimes H^*$ is $H$-linear with respect to this action and, therefore, $\rho$ is a homomorphism of $H$-module algebras. In particular, $A \otimes H^* \in H\mathcal{M}_A$. Clearly, $A \otimes_{AH} A \in H\mathcal{M}_A$ too. Finally, the mapping $\gamma$ is $H$-linear with respect to (**), since the elements of $A \otimes 1$ are $H$-invariant.

Lemma 3.3. We set $M = \text{Im } \gamma$. Assume that $M$ is left $A$-free and that there exist elements $a_1, \ldots, a_n \in A$ such that $\rho(a_1), \ldots, \rho(a_n)$ form a basis for $M$ over $A \otimes 1$. Then $a_1, \ldots, a_n$ form a basis for $A$ as an $A^H$-module with respect to the action by left multiplications. In particular, $A$ is left free of finite rank over $A^H$.

Proof. Given $a \in A$, there exist uniquely determined elements $c_1, \ldots, c_n \in A$ such that

$$\rho(a) = \sum (c_i \otimes 1)\rho(a_i) = \gamma\left(\sum c_i \otimes a_i\right).$$

Then $ha = \sum c_i(ha_i)$ for all $h \in H$. Taking $h = 1$, we obtain $a = \sum c_i a_i$. Now let $g \in H$. Since $\rho(a)$ is $H$-invariant and $\gamma$ is $H$-linear with respect to the $H$-module structures (*) considered in Lemma 3.2, we deduce that

$$\sum (gc_i \otimes 1)\rho(a_i) = \gamma\left(\sum gc_i \otimes a_i\right) = \varepsilon(g)\rho(a) = \varepsilon(g)\sum (c_i \otimes 1)\rho(a_i).$$

It follows that $gc_i = \varepsilon(g)c_i$ since $\rho(a_1), \ldots, \rho(a_n)$ are left linearly independent over $A \otimes 1$. Hence $c_i \in A^H$ for each $i$. On the other hand, if $a = \sum c'_i a_i$ for another collection of elements $c'_1, \ldots, c'_n \in A^H$, then $\rho(a) = \sum (c'_i \otimes 1)\rho(a_i)$, which entails $c'_i = c_i$ for each $i$.

If $A$ is artinian and $H$-simple, then the $A$-bimodule $M = \text{Im } \gamma$ is always left (and right) free. Indeed, $M$ can be considered as an object of $H\mathcal{M}_A$ by Lemma 3.2. Hence $M^n$ is left free for some $n > 0$ by Theorem 2.1. Moreover, this conclusion holds with $n = 1$, as explained in the proof of Theorem 2.9. As a left $A$-module, $M$ is generated by $\rho(A)$, but this does not mean that a basis can be chosen in $\rho(A)$, and, therefore, Lemma 3.3 does not apply in general. Here the source of possible misbehavior of the subring $A^H$ lies.

An example of Björk [14] produces a simple artinian ring $R$ of characteristic 2 which is not a finitely generated module over the subring $R^G$ of elements fixed by an automorphism group of order 4. In this example, $R^G$ is neither left nor right artinian.

The situation becomes much nicer when the $H$-module algebra $A$ has no nontrivial $H$-stable one-sided ideals.

Lemma 3.4. Assume that $A$ has no nontrivial $H$-stable left (or right) ideals. Then $A^H$ is a skew field and $\gamma$ is injective.

Proof. The condition imposed on $A$ means that $A$ is a simple object of $H\mathcal{M}_A$, i.e., a simple left $A\#H$-module. The fact that $A^H$ is a skew field is stated in [11, Lemma 2.1]. It follows from Schur’s lemma since $A^H \cong (\text{End}_{A\#H}A)^{\text{op}}$.

As explained in Lemma 3.2, $A \otimes_{AH} A$ can be considered as an object of $H\mathcal{M}_A$. It is a sum of its simple subobjects $A \otimes b$ with $0 \neq b \in A$, each isomorphic to $A$. Hence any subobject of $A \otimes_{AH} A$ in the category $H\mathcal{M}_A$ is equal to $A \otimes_{AH} V$ for some left vector subspace $V$ of $A$ over $A^H$. In particular, this applies to the kernel of $\gamma$. But $\gamma(1 \otimes b) = \rho(b)$ for each $b \in A$. This shows that the restriction of $\gamma$ to $1 \otimes A$ is injective, and therefore Ker $\gamma = 0$.

Under the hypothesis of Lemma 3.4, the $H$-module algebra $A$ can be considered as either a left or right vector space over the skew field $A^H$. Denote by $[A : A^H]_l$ and $[A : A^H]_r$ the dimensions of these two vector spaces.
Theorem 3.5 (see [11]). Assume that $A$ has no nontrivial $H$-stable left ideals and that $A$ has finite left Goldie rank (i.e., $A$ satisfies ACC on direct sums of left ideals). Then $[A : A^H]_l \leq n$, where $n$ is the dimension of the image of $H$ in $\text{End}_A H$.

This result of Bergen, Cohen, and Fischman [11, Theorem 2.2] was stated for a not necessarily finite-dimensional Hopf algebra $H$ with finite-dimensional image $\pi(H)$ in $\text{End}_A H$. In the proof given in [11], an application of Jacobson’s density theorem shows that, whenever $A$ contains $m$ right linearly independent over $A^H$ elements, there exists a free left $A$-submodule of rank $m$ in the image $\pi(A # H)$ of $A # H$ in $\text{End}_A H$. We choose $h_1, \ldots, h_n \in H$ whose images give a basis for $\pi(H)$. Then $\pi(A # H) = \pi(T)$, where $T \subset A # H$ is the left $A$-submodule generated by $1#h_1, \ldots, 1#h_n$. It follows that $T$ contains a free $A$-submodule of rank $m$. Since $T$ is a free $A$-module of rank $n$, this implies $m \leq n$ by finiteness of the Goldie rank.

Using the mapping $\gamma$, we can strengthen the previous theorem. We continue to work under the assumption that $\dim H < \infty$. However, we note that, replacing $A \otimes H^*$ by $\text{Hom}_k(H, A)$ in the preceding discussion and modifying all arguments appropriately, the next result can be proved for an infinite-dimensional Hopf algebra under the assumption $\dim \pi(H) < \infty$ used in [11]. In the semi-local case, one needs also bijectivity of the antipode.

Theorem 3.6. Assume that $A$ has no nontrivial $H$-stable left ideals and that either $A$ has a finite left Goldie rank or $A$ is semisimple.

In particular, $A$ is left and right artinian.

Proof. By Lemma 3.4, $\gamma$ is injective. Thus, the $A$-bimodule $M = \text{Im} \gamma$ is isomorphic to $A \otimes_{A#H} A$. Hence $M$ is left free of rank equal to $[A : A^H]_l$, and $M$ is right free of rank equal to $[A : A^H]_r$. Note that $\rho(A) \subset A \otimes C$, where $C$ is the sub-coalgebra of $H^*$ dual to the factor-algebra $\pi(H)$ of $H$. Hence $M \subset A \otimes C$ too. Since $\dim C = n$, this shows that $M$, regarded as an $A$-module with respect to the left action, embeds in a free $A$-module of rank $n$. If $A$ has a finite left Goldie rank, we obtain $[A : A^H]_l \leq n$, while the second inequality is the content of Theorem 3.5.

Further, assume that $A$ is semisimple. Since $C$ is stable under the action $\otimes$ of $H$ on $H^*$, the left $A$-module $A \otimes C$ is an object of $H#M$ with respect to the $H$-module structure described in (s) of Lemma 3.2. Hence $(A \otimes C)/M \subset H#M$ too. Since $A$ is $H$-simple, Theorem 2.1 shows that $(A \otimes C)/M$ is projective in $H#M$. Then $M$ is an $H#M$-direct summand of $A \otimes C$. Denoting by $J$ the Jacobson radical of $A$, we deduce that the free $A/J$-module $M/JM$ of rank equal to $[A : A^H]_l$ embeds into the free $A/J$-module $A/J \otimes C$ of rank $n$. Since $A/J$ is semisimple artinian, we must have $[A : A^H]_l \leq n$.

For the remaining part, we set $N = (1 \otimes S^{-1}(C)) \cdot \rho(A)$. Then $N$ is a subobject of $A \otimes H^*$ in the category $H#M_A$, and $N$ is $A$-free of rank $n$. Let $a \in A$. Writing symbolically $\rho(a) = \sum a_{(0)} \otimes a_{(1)} \in A \otimes C$, we have

$$a \otimes 1 = \sum a_{(0)} \otimes S^{-1}(a_{(2)})a_{(1)} = \sum (1 \otimes S^{-1}(a_{(1)})) \cdot \rho(a_{(0)}) \in N.$$ 

This shows that $A \otimes 1 \subset N$. Hence $M = (A \otimes 1) \rho(A) \subset N$. Applying Theorem 2.1 to the object $N/M \subset H#M_A$, we deduce that $M$ is an $M_A$-direct summand of $N$, and passing to quotients modulo $J$, we arrive at $[A : A^H]_r \leq n$. \hfill \Box

Another old result of Cohen, Fischman, and Montgomery determines when the extension $A/A^H$ is $H^*$-Galois for an $H$-module algebra satisfying the previous assumptions. We will show how the mapping $\gamma$ can be used in the proof.

Theorem 3.7 (see [20]). Assume that $A$ has no nontrivial $H$-stable left ideals and that $A$ has a finite left Goldie rank. Then the following conditions are equivalent:

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(1) \([A : A^H]_r = \dim H\);  
(1') \([A : A^H]_l = \dim H\);  
(2) \(A\) is a faithful left \(A\#H\)-module;  
(3) \(A\#H\) is a simple algebra;  
(4) \(A\) is an \(H^*-\)Galois extension of \(A^H\).

Proof. The mapping \(\gamma\) is an injective homomorphism of \(A\)-bimodules. Both \(A \otimes_{A^H} A\) and \(A \otimes H^*\) are free of finite rank as left \(A\)-modules and as right ones. Since \(A\) is left and right artinian by Theorem 3.6, all finitely generated \(A\)-modules have finite length. Therefore, \(\gamma\) is bijective if and only if the two bimodules have equal left ranks, and if and only if they have equal right ranks. This shows that \((4) \iff (1') \iff (1).\)

Denote by \(E\) the endomorphism ring of \(A\) as the subring of diagonal matrices. Therefore, one has to deal with two different images \(A\) of the natural embedding of \(A\) in \(E\). By one of general characterizations of Galois extensions, condition \((4)\) holds if and only if \(\gamma\) is an isomorphism \([11, \text{Question 2.4}]\). It was motivated by the classical result, due to Jacobson, that such an equality is indeed true in the case of group actions on skew fields.

Theorem 3.8 (see [56]). Assume that \(A\) is a semiprimary \(H\)-module algebra without nontrivial \(H\)-stable one-sided ideals. Then \([A A^H]_l = [A A^H]_r\).

Let us outline the proof of this theorem. The desired equality holds precisely when the \(A\)-bimodule \(M = A \otimes_{A^H} A\) has equal left and right ranks over \(A\). By Lemma 3.4, \(\gamma\) embeds \(M\) into \(N = A \otimes H^*\). The latter bimodule is also free on each side with the left and right ranks equal to \(\dim H\). The
conclusion will follow once it is shown that a direct sum of several copies of $M$ is isomorphic to a direct sum of several copies of $N$.

Each $A$-bimodule can be considered as a right module over the ring $A = A^{\text{op}} \otimes A$. We set $\mathcal{H} = H^{\text{cop}} \otimes H$ and consider $M$ and $N$ as left $\mathcal{H}$-modules using the two pairs of $H$-module structures described in Lemma 3.2. Note that $\mathcal{H}$ is a finite-dimensional Hopf algebra and $A$ is an $\mathcal{H}$-module algebra. One verifies the compatibility condition which makes $M$ and $N$ objects of the category $\mathcal{H} \cdot \mathcal{M}_A$.

We set $B = \text{End}_A M$, i.e., $B$ is the endomorphism ring of $M$ as an $A$-bimodule. Then $B$ is an $\mathcal{H}$-module algebra, and $B$ is semiprimary since $M$ is an $A$-bimodule of finite length. By the assumption on the $H$-stable one-sided ideals of $A$, there cannot exist nontrivial subbimodules of $M$ stable under the two $H$-module structures of Lemma 3.2. In other words, $M$ is a simple object of $\mathcal{H} \cdot \mathcal{M}_A$. This implies $H$-semiprimeness of $B$, and an application of Theorem 2.4 leads further to the conclusion that $B$ is $H$-simple.

As a left $A$-module, and, therefore, as a bimodule, $N$ is generated by $1 \otimes H^*$. For each $\xi \in H^*$, there exists a homomorphism of $A$-bimodules $\varphi_\xi : M \to N$ sending $a \otimes b \in A \otimes_A H$ to $(a \otimes \xi) \rho(b)$ (note that $(1 \otimes \xi) \rho(c) = c \otimes \xi$ for all $c \in A^H$), and $A \otimes \xi$ is contained in the image of $\varphi_\xi$. It follows that $N$, as an $A$-bimodule, is a homomorphic image of $M^d$, where $d = \dim H$. This means that the canonical mapping

$$\alpha : \text{Hom}_A(M, N) \otimes_B M \to N$$

is surjective. The trickiest part is to show that $\text{Ker} \alpha = 0$. We refer the reader to [56, Theorem 3.1] for details.

Thus, $N \cong F \otimes_B M$, where $F = \text{Hom}_A(M, N)$. Note that $F$ is an $\mathcal{H}$-equivariant right $B$-module. By Theorem 2.1, $F^n$ is a free $B$-module for some $n > 0$. Since $M$ and $N$ are $A$-finite, we deduce that $F$ is $B$-finite. Hence $F^n \cong B^r$ in $\mathcal{M}_B$ for some integer $r > 0$. This implies that $N^n \cong F^n \otimes_B M \cong B^r \otimes_B M \cong M^r$ in $\mathcal{M}_A$, completing the proof.

In some cases, the conclusion of Theorem 3.8 is almost obvious. Assume that $A$ is a skew field. If all $A$-bimodule composition factors of $N$ have equal left and right dimensions over $A$, the equality of the left and right dimensions over $A$ will be fulfilled for each subbimodule of $N$. This happens when $H$ is pointed and, more generally, when all simple $H$-comodules have dimension at most 2 over the ground field $k$. To see this, note that $A \otimes I$ is a sub-bimodule of $N$ for each right ideal $I$ of $H^*$. Taking a composition series of $H^*$ as a right module over itself, we obtain a series of sub-bimodules of $A \otimes H^*$ with factors isomorphic to $A \otimes V$ for various simple right $H^*$-modules $V$, where the right action of $A$ on $A \otimes V$ comes from $\rho$. We have $\dim V \leq 2$ for each $V$ by the previous assumption about $H$. The left and right dimensions of $A \otimes V$ over $A$ are both equal to $\dim V$. If $\dim V = 1$, then $A \otimes V$ cannot contain any nontrivial sub-bimodules. If $\dim V = 2$, then the $A$-bimodule $A \otimes V$ is either simple or has exactly two composition factors, each of left and right dimension over $A$ equal to 1.

A different approach to finiteness results exploits the trace $\hat{t} : A \to A^H$ given by the action of a left integral $t \in H$ on $A$. Note that $\hat{t}$ is left and right $A^H$-linear. In particular, $\hat{t}$ is surjective if and only if $ta = 1$ for some $a \in A$.

In terms of the comodule structure on $A$, the surjectivity of $\hat{t}$ is equivalent to the existence of a total integral $H^* \to A$, which is a homomorphism of right $H^*$-comodules sending $1 \in H^*$ to $1 \in A$ (see [19]). Total integrals were introduced by Doi [26] for comodule algebras over arbitrary Hopf algebras.

**Theorem 3.9** (see [40]). Assume that $A$ is right noetherian with a surjective trace $\hat{t}$. Then $A$ is a noetherian right $A^H$-module. In other words, $A^H$ is right noetherian and $A$ is right module-finite over $A^H$.

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This result is stated in [40, Theorem 4.4.2]. To prove this theorem, Montgomery shows that the lattice of $A^H$-submodules of any right $A$-module $V$ embeds into the lattice of submodules of the induced right $A\#H$-module. In particular, the lattice of right $A^H$-submodules of $A$ embeds into the lattice of right ideals of $A\#H$. Explicitly, this embedding is obtained by assigning to an $A^H$-submodule $U$ of $A$ the right ideal of $A\#H$ generated by $(U\#1)e$, where $e \in A\#H$ is an idempotent such that $A^H$ is isomorphic to $e(A\#H)e$.

The same argument shows that $A^H$ is right artinian if so is $A$.

The lemma below describes embeddings of lattices of submodules from our point of view in equivariant modules.

**Lemma 3.10.** Let $M \in H\cdot\mathcal{M}_A$. If the trace mapping $\bar{i} : A \to A^H$ is surjective, then:

(i) the lattice of $A^H$-submodules of $M^H$ embeds into that of $A$-submodules of $M$;
(ii) $M^H$ is an artinian $A^H$-module whenever $M$ is an artinian $A$-module;
(iii) $M^H$ is a noetherian $A^H$-module whenever $M$ is a noetherian $A$-module.

**Proof.** We have $t(vA) = vA^H$ for any $v \in M^H$ since the mapping $A \to M$ such that $a \mapsto va$ is $H$-linear. Hence $t(UA) = U$ for each $A^H$-submodule $U$ of $M^H$. Therefore, the assignment $U \mapsto UA$ gives the desired embedding of lattices. Assertions (ii) and (iii) follow immediately from (i). 

If $A$ is right noetherian, then each finitely generated right $A$-module is noetherian. Hence if $M$ is an $A$-finite object of $H\cdot\mathcal{M}_A$, then $M^H$ is a noetherian right $A^H$-module by Lemma 3.10. To deduce Theorem 3.9 from Lemma 3.10, one needs to find an $A$-finite object $M \in H\cdot\mathcal{M}_A$ such that $M^H \cong A$ as right $A^H$-modules. Recall that objects of $H\cdot\mathcal{M}_A$ are identified with left $A^H\#H^{\cop}$-modules. Let $M$ be a cyclic free $A^{\op}\#H^{\cop}$-module with a free generator $v$. Since the linear mapping $H \otimes A \to M$ such that $h \otimes a \mapsto h(va)$ is bijective, it follows that $M^H = tM = t(vA)$. Hence the assignment $a \mapsto t(va)$ defines a desired right $A^H$-linear isomorphism $A \to M^H$.

**Proposition 3.11.** Assume that $H$ is semisimple and that $A$ is semiprimary and $H$-semiprime. Then:

(i) each $H$-stable one-sided ideal of $A$ is generated by an $H$-invariant idempotent;
(ii) the subring of invariants $A^H$ is semisimple artinian;
(iii) $A$ is left and right module-finite over $A^H$.

**Proof.** By Theorem 2.8, $A$ is artinian, and by Theorem 2.10, all objects of $H\cdot\mathcal{M}_A$ are semisimple. In particular, the latter conclusion applies to $A$. This means that, whenever $I$ is an $H$-stable right ideal of $A$, there exists an $H$-stable right ideal $J$ such that $A = I \oplus J$. Then $I = eA$ for some idempotent $e \in A$. We have $e = p(1)$, where $p$ is the projection of $A$ onto $I$ with the kernel $J$. Since $1 \in A^H$ and $p$ is $H$-linear, it follows that $e \in A^H$. This proves assertion (i) for right ideals. Considering the $H^{\cop}$-module algebra $A^{\op}$, we obtain (i) for left ideals as well.

Let $U$ be any right ideal of $A^H$. Then $UA$ is an $H$-stable right ideal of $A$. By (i), $UA$ is generated by an idempotent $e \in A^H$. By Lemma 3.10 applied to $M = A$, the lattice of right ideals of $A^H$ embeds into that of right ideals of $A$. Since the right ideal $eA^H$ of $A^H$ has the same extension to $A$ as $U$, we deduce that $U = eA^H$. Thus, each right ideal of $A^H$ is generated by an idempotent. This yields (ii). Finally, (iii) follows from Theorem 3.9.

We have given a self-contained proof. It has been known for a long time that $A^H$ is semisimple artinian when so is $A\#H$ (see [20, Theorem 3.13]). The fact that $A\#H$ is semisimple artinian, under the hypothesis of Proposition 3.11, has been established in [57] (see Theorem 2.10).
4. Localization at Invariants and a Bergman–Isaacs Type Theorem

In [54], it was shown that, for a semisimple Hopf algebra \( H \), all right noetherian \( H \)-module algebras have, loosely speaking, “sufficiently many” \( H \)-invariant elements. The proofs of these results stem from a few statements from [57] that served as intermediate steps in the process of verifying the existence of artinian classical quotient rings. But, in fact, only the final conclusion from [57] is needed, and, therefore, the results of [54] are valid for a larger class of \( H \)-module algebras. This will be explained below.

Assume that an \( H \)-semiprime \( H \)-module algebra \( A \) has a right artinian classical right quotient ring \( Q \). By Theorem 2.14, the \( H \)-module structure extends to \( Q \). Then it is clear that \( Q \) has to be \( H \)-semiprime since \( J \cap A \neq 0 \) for each nonzero right ideal \( J \) of \( Q \). Therefore, all results concerning artinian \( H \)-semiprime algebras apply to \( Q \).

A right ideal \( I \) of a ring \( R \) is said to be essential if \( I \) has a nonzero intersection with each nonzero right ideal of \( R \).

**Lemma 4.1.** Assume that \( A \) is \( H \)-semiprime and that \( A \) has a right artinian classical right quotient ring \( Q \). For a right ideal \( I \) of \( A \), denote by \( I_H \) the largest \( H \)-stable right ideal of \( A \) contained in \( I \). The following conditions are equivalent:

(a) \( I_H \) is an essential right ideal of \( A \);
(b) \( I \) contains a regular element of \( A \).

**Proof.** Assume that \( I_H \) is an essential right ideal of \( A \). Then \( I_H \cap bA = 0 \) for some \( b \in A \), \( b \neq 0 \). Then \( (I \cap H^*) \cap \rho(bA) = 0 \). In particular,

\[
(sA \otimes H^*) \cap \rho(bA) = 0.
\]

Since \( s \otimes 1 \) is a regular element of the ring \( A \otimes H^* \), it follows that the sum

\[
\sum_{n=0}^{\infty} (s^n \otimes 1) \rho(bA)
\]

is direct, and each summand is a nonzero right \( \rho(A) \)-submodule of \( A \otimes H^* \). Now we consider \( F = A \otimes H^* \) as a right \( A \)-module with respect to the action of \( A \) given by right multiplications by the elements \( \rho(a) \), \( a \in A \). We know that this \( A \)-module is free of rank equal to the dimension of \( H \). Hence \( F \otimes_A Q \) is a finitely generated right \( Q \)-module containing an infinite direct sum of nonzero submodules. However, this is impossible since \( Q \) is right artinian. Thus, \( I_H \cap bA \neq 0 \) whenever \( b \neq 0 \), and, therefore, (b) \( \Rightarrow \) (a). \( \square \)

**Lemma 4.2.** Assume that \( H \) is semisimple, \( A \) is \( H \)-semiprime, and \( A \) has a right artinian classical right quotient ring \( Q \). Let \( I \) be an \( H \)-stable right ideal of \( A \), and let \( I^H = I \cap A^H \).

(i) If \( I^H = 0 \), then \( I = 0 \).
(ii) If \( I \) is an essential right ideal of \( A \), then \( Q^H I^H = Q^H \) and \( I^H Q^H = Q^H \).

**Proof.** By Proposition 3.11, \( Q^H \) is semisimple artinian and \( Q \) has a finite length as either left or right \( Q^H \)-module.

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We consider the \((Q^H, A)\)-sub-bimodule \(Q^H I\) of \(Q\). For each regular element \(s\) of \(A\), the \(Q^H\)-submodule \(Q^H I_s\) is isomorphic to \(Q^H I\). Hence these two \(Q^H\)-modules have the same length. Since \(Q^H I_s \subset Q^H I\), we obtain \(Q^H I_s = Q^H I\), and, therefore, \(Q^H I_s^{-1} = Q^H I\).

It follows that \(Q^H I\) is a right ideal of \(Q\). Since it is \(H\)-stable, Proposition 3.11 yields \(Q^H I = eQ\) for some \(e \in Q^H\). Let \(t \in H\) be an integral. The action of \(t\) on \(Q\) commutes with the left and right multiplications by \(H\)-invariant elements. Hence

\[
e \in eQ^H = t(eQ) = t(Q^H I) = Q^H I^H.
\]

If \(I^H = 0\), then the above inclusion implies \(e = 0\), i.e., \(I = 0\). This proves (i).

Assume that \(I\) is an essential right ideal of \(A\). By Lemma 4.1, \(I = I_H\) contains a regular element of \(A\), and, therefore, \(IQ = Q\). Since \(Q^H I\) is a right ideal of \(Q\) containing \(I\), we obtain \(Q^H I = Q\) as well. The previous argument with \(e = 1\) shows that \(1 \in Q^H I^H\). Thus, \(Q^H = Q^H I^H\).

Now we consider the \(H\)-stable right ideal \(I^H Q\) of \(Q\). By Theorem 2.10, there exists an \(H\)-stable right ideal \(J\) of \(Q\) such that \(Q = I^H Q \oplus J\). Then \(I \cap J\) is an \(H\)-stable right ideal of \(A\) such that

\[
(I \cap J)^H = I^H \cap J = 0.
\]

As we have already proved in part (i), this implies \(I \cap J = 0\). Since \(IQ = Q\), any element \(y \in J\) can be written as \(y = as^{-1}\), where \(a \in I\) and \(s\) is a regular element of \(A\); then \(a \in I \cap J\), so that \(a = 0\) and \(y = 0\). Therefore, \(J = 0\) and \(Q = I^H Q\). Hence

\[
Q^H = tQ = t(I^H Q) = I^H Q^H,
\]

and we are done.

\[\square\]

**Theorem 4.3.** Assume that \(H\) is semisimple, \(A\) is \(H\)-semiprime, and \(A\) has a right artinian classical right quotient ring \(Q\). Denote by \(\Sigma\) the set of regular elements of \(A^H\) and by \(E\) the set of right ideals of \(A\) which satisfy the equivalent conditions (a) and (b) of Lemma 4.1. Then

(i) the algebra \(A^H\) is semiprime right Goldie;

(ii) \(\Sigma\) is a right Ore subset of regular elements of \(A\);

(iii) \(Q\) is canonically isomorphic to the right localization of \(A\) with respect to \(\Sigma\);

(iv) the classical right quotient ring of \(A^H\) is isomorphic to \(Q^H\);

(v) \(I \cap \Sigma \neq \emptyset\) for each right ideal \(I \in E\).

**Proof.** We set \(I^H = I \cap A^H\) and \(F = \{I^H \mid I \in E\}\). If \(I, J \in E\), then \(I \cap J \in E\), which implies \(I^H \cap J^H = (I \cap J)^H \in F\). Therefore, \(F\) is a filter of right ideals of \(A^H\). If \(I \in E\) and \(a \in A^H\), then the right ideal \(I_a = \{x \in A \mid ax \in I_H\}\) is essential and \(H\)-stable; therefore, \(I_a \in E\) and \(I_a^H \in F\). Also, \(aI_a^H \subset I^H\) since \(aI_a \subset I_H \subset I\). This shows that \(F\) is a topologizing filter.

Moreover, \(F\) satisfies conditions (a) and (b) of Proposition 2.13 with \(A\) replaced with \(A^H\) and \(Q\) replaced with \(Q^H\). Indeed, if \(I \in E\), then \(I^H\) has zero left and right annihilators in \(Q\) since \(I^HQ\) and \(Q^H I\) contain the unity \(1\) by Lemma 4.2. If \(q \in Q^H\), then there exists a regular element \(s\) of \(A\) such that \(qs \in A\). We set \(K = sA\). Then \(K \in E\) by Lemma 4.1 and \(qK \subset A\). Hence \(K^H \in F\) and \(qK^H \subset A^H\).

It has already been observed that the ring \(Q^H\) is semisimple artinian. Now (iv) follows from Proposition 2.13, and (i) is its consequence since a ring \(R\) is semiprime right Goldie if and only if \(R\) has a semisimple artinian classical right quotient ring.

Given \(I \in E\), the equality \(I^H Q^H = Q^H\) of Lemma 4.2 means that \(I^H \cap \Sigma \neq \emptyset\), which amounts to (v). For any \(q \in Q\), the set

\[
I = \{x \in A \mid qx \in A\}
\]

is a right ideal of \(A\) containing a regular element of \(A\). By Lemma 4.1, \(I \in E\). Since \(qI \subset A\), assertion (v) shows that \(qs \in A\) for some \(s \in \Sigma\).
All elements of $\Sigma$ are invertible in $Q^H$ and, therefore, in $Q$. Hence all elements of $\Sigma$ are regular in $A$. Since each element of $Q$ can be written in the form $as^{-1}$ for some $a \in A$ and $s \in \Sigma$, (ii) and (iii) immediately follow (see [37, 2.2.4]).

In all corollaries below, we continue to assume that $H$ is semisimple.

**Corollary 4.4.** Let $A$ satisfy all conditions of Theorem 4.3. If $s$ is any regular element of $A$, then the right ideal $sA$ contains an $H$-invariant regular element of $A$.

**Proof.** Since $sA \in \mathcal{E}$ by Lemma 4.1, we have $sA \cap \Sigma \neq \emptyset$ by Theorem 4.3. □

**Corollary 4.5.** Let $A$ and $Q$ satisfy the conditions of Theorem 4.3, and let $I$ be any $H$-stable right ideal of $A$. There exists $x \in I^H$ such that $IQ = xQ$.

**Proof.** Since $IQ$ is an $H$-stable right ideal of $Q$, we have $IQ = eQ$ for some $e \in Q^H$ by Proposition 3.11. Now $J = \{a \in A \mid ea \in I\}$ is an $H$-stable right ideal of $A$ containing a regular element of $A$. Hence $J \cap \Sigma \neq \emptyset$ by Corollary 4.4. We choose any $s \in J \cap \Sigma$ and put $x = es$. Then $x \in I \cap Q^H = I^H$. Since $s$ is invertible in $Q$, we obtain $eQ = xQ$. □

**Corollary 4.6.** All conclusions of Theorem 4.3 as well as Corollaries 4.4 and 4.5 hold in each of the following three cases:

(a) $A$ is semiprime right Goldie;
(b) $A$ is right noetherian and $H$-semiprime;
(c) $A$ is PI and $H$-semiprime with finitely many minimal $H$-prime ideals.

**Proof.** A right artinian classical right quotient ring $Q$ exists in the case (a) by the Goldie theorem, and in the other cases by Theorems 2.11 and 2.12. □

Under the assumption that $A\#H$ is semiprime, a short argument given by Bergen and Montgomery (see [12, Proposition 2.4]) shows that $A^H$ is semiprime and that $\hat{t}(I) \neq 0$, where $\hat{t} : A \to A^H$ is the trace mapping (in particular, $I^H \neq 0$) for each nonzero $H$-stable one-sided ideal $I$ of $A$. From this, it was further deduced in [12, Lemma 3.4] that, among other things, regular elements of $A^H$ are regular in $A$, and that $A^H$ is Goldie when so is $A$. If $A, Q, H$ are as in Theorem 4.3, then $Q\#H$ is semisimple artinian by Theorem 2.10; since $Q\#H$ is a classical right quotient ring of $A\#H$, it follows that $A\#H$ is semiprime. This fact was not known at the time when [12] was written.

Several deeper results from [12] use the assumption that $A\#H$ is not only semiprime, but has the ideal intersection property (IIP for brevity) which means that each nonzero ideal of $A\#H$ has a nonzero intersection with $A$. In fact, in the presence of IIP, the ring $A\#H$ is semiprime if and only if $A$ is $H$-semiprime. The IIP is satisfied for $X$-outer group actions on semiprime rings and for $X$-outer actions of Lie algebras on prime rings. However, it seems that there are no approaches to analogs of such results for actions of arbitrary finite-dimensional or even semisimple Hopf algebras.

It was asked in [12], whether $Q(A)^H = Q(A^H)$ when $A\#H$ is semiprime with IIP. Part (iv) of Theorem 4.3 answers this question, imposing reasonable conditions on $A$ and $H$, but not assuming the IIP. In the case of a finite group $G$ acting on a semiprime ring $R$ without additive $|G|$-torsion, the fact that $R^G$ is right Goldie if and only if $R$ is right Goldie and the equality $Q(R)^G = Q(R^G)$ were proved by Kharchenko (see [7]); it was also observed by Montgomery [38] that $Q(R)$ is the localization of $R$ at the Ore set of regular $G$-invariant elements. Analogs of these results for group graded rings are due to Cohen and Rowen (see [21]).

There is a slightly weaker version of Lemma 4.2 for $H$-stable left ideals. The equality $Q^HI^H = Q^H$ in (ii) cannot be proved unless $Q$ is a two-sided quotient ring.
Lemma 4.7. Assume that $H$ is semisimple, $A$ is $H$-semiprime, and $A$ has a right artinian classical right quotient ring $Q$. Let $I$ be an $H$-stable left ideal of $A$, and let $I^H = I \cap A^H$.

(i) If $I^H = 0$, then $I = 0$.
(ii) If $QI = Q$, then $I^HQ^H = Q^H$.

Proof. We repeat the steps of the proof of Lemma 4.2 using the two-sided properties of $Q$. First, the $(A, Q^H)$-subbimodule $IQ^H$ is a left ideal of $Q$ since $Q$ has finite length as a right $Q^H$-module. Next, $IQ^H = Qe$ for some $e \in Q^H$ by Proposition 3.11. Applying the integral $t$, we deduce that $e \in I^HQ^H$.

If $I^H = 0$, then $e = 0$, which implies $I = 0$. If $QI = Q$, then $IQ^H = Q$, and, therefore, $1 \in I^HQ^H$.

\[ \square \]

Theorem 4.8. Denote by $N$ the prime radical of $A$ and by $N_H$ the largest $H$-stable ideal of $A$ contained in $N$. Assume that $H$ is semisimple, $N$ is nilpotent, and $A/N_H$ has a right artinian classical right quotient ring. If $I$ is any $H$-stable one-sided ideal of $A$ such that $I^H$ is nilpotent, then $I$ is nilpotent.

Proof. Denote by $\pi$ the canonical surjective homomorphism of $H$-module algebras $A \to A/N_H$. Then $\pi(I)$ is an $H$-stable one-sided ideal of $A/N_H$ and $\pi$ maps $I^H$ onto $\pi(I)^H$ since $H$ is semisimple. Hence $\pi(I)^H$ is nilpotent. The factor-algebra $A/N_H$ is $H$-semiprime since $N_H$ is an $H$-semiprime ideal of $A$. Hence its subring of invariants $(A/N_H)^H$ is semiprime by Theorem 4.3, which implies $\pi(I)^H = 0$. An application of Lemmas 4.2 and 4.7 yields $\pi(I) = 0$, i.e., $I \subset N$. Therefore, $I$ is nilpotent.

Corollary 4.9. Assume that $H$ is semisimple. The conclusion of Theorem 4.8 holds in each of the following three cases:

(a) $A$ is left noetherian;
(b) $A$ is right noetherian;
(c) $A$ is finitely generated and PI.

Proof. In all cases, $N$ is known to be nilpotent. In the cases (b) and (c), the $H$-semiprime factor-algebra $A/N_H$ has a right artinian classical right quotient ring by Theorems 2.11 and 2.12. In the case (a) we apply Theorem 4.8 to the right noetherian $H^{\text{cop}}$-module algebra $A^{\text{cop}}$.

Let $R$ be a nonunital ring and $G$ be a finite group of its automorphisms such that $R$ has no additive $|G|$-torsion. Consider the trace mapping $\hat{t} : R \to R^G$, $\hat{t}(a) = \sum_{g \in G} ga$.

A classical result of Bergman and Isaacs (see [13]) says that $R$ is nilpotent if $\hat{t}(R)$ is nilpotent. Moreover, if $\hat{t}(R) = 0$, then the nilpotency index of $R$ is bounded by a number which depends only on the order $|G|$ of the group $G$, but not on the ring $R$. An easy consequence of this result is that $R^G$ is semiprime when so is $R$.

A similar result for group graded rings proved by Cohen and Rowen (see [21]) is even simpler: if $R = \bigoplus_{g \in G} R_g$ is a nonunital ring graded by a finite group $G$, and if $R^d_i = 0$ for some integer $d > 0$, then $R^d = 0$. In fact, one needs only a grading with finite support, while the group $G$ may be infinite.

Theorem 4.8 is different not only in that the conclusion is stated for one-sided ideals of $H$-module algebras satisfying certain conditions, but also because it relies heavily on the $H$-semiprime case. The nilpotency index of $I$ can be bounded only by the nilpotency index of the ideal $N_H$, even if $I^H = 0$.

Bahturin and Linchenko (see [10]) investigated conditions under which one can conclude that $A$ is PI, knowing that $A^H$ is PI. They showed that, for a fixed finite-dimensional Hopf algebra $H$, in order
that each $H$-module algebra $A$ be PI whenever $A^H$ is PI, it is necessary and sufficient that there exists a natural number $n$ such that $A^n = 0$ for each nonunital $H$-module algebra $A$ with $A^H \cdot A^H = 0$, and this can happen only if $H$ is semisimple. Several other equivalent conditions are given in [10]. This work of Bahturin and Linchenko elucidates the need for a more precise analog of the Bergman–Isaacs result for Hopf algebra actions.

5. Hopf Actions on Commutative Algebras

Throughout this section, we assume that $A$ is a commutative $H$-module algebra. First, we are going to recall the algebraic interpretation of the classical result on quotients of affine schemes by actions of finite group schemes.

Given an associative algebra $U$ over a commutative ring $R$ such that $U$ is free of finite rank as an $R$-module, the norm $Nm_{U/R}(u) \in R$ of an element $u \in U$ is defined as the determinant of the operator $L_u \in \text{End}_R U$ of the left multiplication by $u$ in $U$. Considering the polynomial ring $U[t]$, where $t$ is a variable, as an algebra over $R[t]$, we also obtain the characteristic polynomial

$$P_{U/R}(u, t) = Nm_{U[t]/R[t]}(t - u) = \det(t \cdot \text{Id} - L_u) \in R[t].$$

In particular, $(-1)^r Nm_{U/R}(u)$, where $r = \text{rank}_R U$ is the coefficient of $t^r$ in this polynomial. Passing to localizations of the base ring $R$, these definitions extend to the case where $U$ is not free as an $R$-module, but only projective of finite constant rank.

By the Cayley–Hamilton theorem, $P_{U/R}(u, L_u) = 0$ in $\text{End}_R U$. Applying this operator to the identity element $1 \in U$, we obtain $P_{U/R}(u, u) = 0$ in $U$, which is a relation of integral dependence of the element $u$ over the ring $R$. The integral dependence of $U$ over $R$ is merely a consequence of module-finiteness. What is important for application to invariants is the fact that the characteristic polynomials satisfy several nice properties. In particular, they are functorial in the sense that, given a homomorphism of commutative rings $\zeta : R \rightarrow R'$, we have

$$P_{R' \otimes_R U/R'}(1 \otimes u, t) = \zeta^t P_{U/R}(u, t),$$

where $\zeta^t : R[t] \rightarrow R'[t]$ is the homomorphism extending $\zeta$ and sending $t$ to $t$.

Since $A$ is commutative, the mapping

$$\iota : A \rightarrow A \otimes H^*, \quad a \mapsto a \otimes 1,$$

is an isomorphism of $A$ onto a central subalgebra of $A \otimes H^*$. Therefore, we can consider $A \otimes H^*$ as an $A$-algebra via $\iota$. Clearly, this algebra is free of rank $d = \dim H$ as an $A$-module. Therefore, the polynomial $P_{A \otimes H^*/A}(u, t)$ is defined for each $u \in A \otimes H^*$. Making use of the comodule structure $\rho : A \rightarrow A \otimes H^*$, we obtain the polynomial

$$P_{A \otimes H^*/A}(\rho(a), t) \in A[t]$$

for $a \in A$. Assume that

$$P_{A \otimes H^*/A}(\rho(a), t) = t^d + \sum_{i=0}^{d-1} c_i t^i,$$

where $c_0, \ldots, c_{d-1} \in A$.

Then

$$\rho(a)^d + \sum_{i=0}^{d-1} (c_i \otimes 1) \rho(a)^i = 0 \quad \text{in } A \otimes H^*.$$

Applying the algebra homomorphism $\text{id} \otimes \varepsilon : A \otimes H^* \rightarrow A$ to the left-hand side of this equality, we obtain

$$a^d + \sum_{i=0}^{d-1} c_i a^i = 0.$$
If \( c_i \in A^H \) for all \( i \), then the relation above shows that \( a \) is integral over \( A^H \). The classical argument reproduced in the following theorem is based on this observation.

**Theorem 5.1.** Assume that \( H \) is cocommutative. Then, for each \( a \in A \), the characteristic polynomial \( P_{A \otimes H^*/A}(\rho(a), t) \) has all coefficients in \( A^H \). In particular, \( A \) is integral over \( A^H \).

**Proof.** By the condition imposed on \( H \), the dual Hopf algebra \( H^* \) is commutative, which implies that \( A \otimes H^* \) is also commutative. Since \( A^H \) is the equalizer of two algebra homomorphisms \( \iota, \rho : A \to A \otimes H^* \), we have to show that

\[
\iota^* P_{A \otimes H^*/A}(\rho(a), t) = \rho^* P_{A \otimes H^*/A}(\rho(a), t).
\]

Note that the commutative diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{\iota} & A \otimes H^* \\
\downarrow & & \downarrow \iota \otimes \text{id} \\
A \otimes H^* & \xrightarrow{\iota'} & A \otimes H^* \otimes H^*
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{\rho} & A \otimes H^* \\
\downarrow & & \downarrow \rho \otimes \text{id} \\
A \otimes H^* & \xrightarrow{\rho'} & A \otimes H^* \otimes H^*
\end{array}
\]

where \( \iota'(x) = x \otimes 1 \) for \( x \in A \otimes H^* \) are co-Cartesian in the category of commutative \( A \)-algebras in the sense that each diagram makes \( A \otimes H^* \otimes H^* \) the tensor product of two \( A \)-algebras given by the respective homomorphisms \( A \to A \otimes H^* \). Hence (\(*\)) can be rewritten as

\[
P_{A \otimes H^*/A \otimes H^*}(\iota \otimes \text{id}) = P_{A \otimes H^*/A \otimes H^*}((\rho \otimes \text{id}) \rho(a), t)
\]

by the functoriality of the characteristic polynomials. Here \( A \otimes H^* \otimes H^* \) is considered as an \( A \otimes H^* \)-algebra by means of \( \iota' \). Next, there exists an automorphism \( \varphi \) of the algebra \( A \otimes H^* \otimes H^* \) defined by the rule

\[
\varphi(x \otimes \xi) = (x \otimes 1)(1 \otimes \Delta \xi) \quad \text{for} \ x \in A \otimes H^* \text{ and } \xi \in H^*.
\]

Since \( \varphi \) acts as the identity on \( A \otimes H^* \otimes 1 \), we have

\[
P_{A \otimes H^*/A \otimes H^*}(y, t) = P_{A \otimes H^*/A \otimes H^*}(\varphi(y), t)
\]

for all \( y \in A \otimes H^* \otimes H^* \). If \( y = (\iota \otimes \text{id}) \rho(a) \), then \( \varphi(y) = (\text{id} \otimes \Delta) \rho(a) = (\rho \otimes \text{id}) \rho(a) \). This implies (\( **\)).

\( \square \)

By passage from \( A \) to \( A[t] \), the conclusion of Theorem 5.1 can be deduced from the fact that

\[
\text{Nm}_{A \otimes H^*/A}(\rho(a), t) \in A^H \quad \text{for all} \ a \in A.
\]

The arguments showing this inclusion in the proof of Theorem 5.1 above are tautological, up to different notation and the use of right comodule structures instead of left ones, to those in Mumford’s book on abelian varieties (see [43, Chap. III, Sec. 12]). Demazure and Gabriel describe quotients by actions of finite group schemes in [25, Chap. III, Sec. 2, Corollary 6.1] as a special case of a more general result on quotients of groupoid schemes [25, Chap. III, Sec. 2, Theorem 3.2].

Thus, Theorem 5.1 is a very old result. Somehow it had not been well known to Hopf algebra theorists for some time in the past. Integrality over invariants for commutative comodule algebras over commutative Hopf algebras was rediscovered by Ferrer Santos in [31]. In the language of module algebras, this approach was reformulated by Montgomery [40, Sec. 4.2]. It makes use of characteristic polynomials of endomorphisms of equivariant \( A \)-modules.

In [40], Montgomery raised the question on whether \( A \) is always integral over \( A^H \) in the case of an arbitrary finite-dimensional Hopf algebra \( H \). For pointed Hopf algebras, this question was answered shortly afterwards in the affirmative by Artamonov [2] if \( A \) is a domain and without any restrictions on \( A \) by Totok [8] and Zhu [58] if \( \text{char } k > 0 \). Both [8] and [58] provided counterexamples to integrality in characteristic 0. Zhu also proved that \( A \) is integral over \( A^H \) if \( H \) is involutory, i.e., \( S^2 = \text{id} \), and
char \, k \text{ does not divide the dimension of } H. \text{ At that time, it remained open what is actually needed for integrality to hold.}

The characteristic polynomials have reappeared in a later development.

**Theorem 5.2** (see [50]). If \( A \) is \( H \)-semiprime or, more generally, if there exists a homomorphism of commutative \( H \)-module algebras \( \varphi : A' \to A \) such that \( A = \varphi(A')A^H \) and \( A' \) is \( H \)-semiprime, then, for each \( a \in A \), the polynomial \( P_{A \otimes H^*/A}(\rho(a), t) \) has all coefficients in \( A^H \). In particular, \( A \) is integral over \( A^H \).

The case where \( A \) is \( H \)-semiprime, which is the main step here, has been subsumed in a recent work of Eryashkin [5] on invariants of \( H \)-module PI algebras. These results will be discussed in Sec. 7. The original proof of Theorem 5.2 had common elements with the proof of Theorem 7.5, but it did not use the Martindale quotient rings.

If \( A \) contains nonzero \( H \)-stable nilpotent ideals, then integrality over invariants may well be lost by the already mentioned examples of Totok and Zhu. There are still two important cases where the \( H \)-semiprimeness is not needed.

**Corollary 5.3.** The algebra \( A \) is integral over \( A^H \) in each of the following two cases:
(a) \text{ the trace mapping } \( A \to A^H \text{ is surjective,} \)
(b) \( \text{char } k = p > 0. \)

**Proof.** Let \( N \) be the largest \( H \)-stable ideal of \( A \) contained in the nil radical of \( A \). Since \( B = A/N \) is \( H \)-semiprime, Theorem 5.2 shows that \( B \) is integral over \( B^H \). Let \( \pi : A \to B \) be the canonical mapping.

In the case (a), \( \pi(A^H) = B^H \). Therefore, for each \( a \in A \) there exists a polynomial \( f \in A^H[t] \) with the leading coefficient 1 such that \( f(a) \in N \). Then \( f(a)^n = 0 \) for some integer \( n > 0 \) since \( N \) is nil. Hence \( a \) is integral over \( A^H \).

Assume that \( \text{char } k = p > 0 \). We set \( A' = \pi^{-1}(B^H) \). As in the case (a), we verify that \( A \) is integral over \( A' \). We claim that, for each \( c \in A' \), there exists \( n > 0 \) such that \( c^{p^n} \in A^H \). Indeed, \( \rho(c) - c \otimes 1 \in N \otimes H^* \) since \( \pi(c) \in B^H \). It follows that \( \rho(c) - c \otimes 1 \) is nilpotent. Hence
\[
\rho(c^{p^n}) - c^{p^n} \otimes 1 = (\rho(c) - c \otimes 1)^{p^n} = 0
\]
for sufficiently large \( n \), but this means that \( c^{p^n} \in A^H \). Thus, \( A' \) is integral over \( A^H \), and the final conclusion follows from the transitivity of the integrality.

In [58], Zhu conjectured that \( A \) is integral over \( A^H \) whenever \( H \) is involutory. If \( \text{char } k = 0 \), it is known that \( H \) is involutory if and only if \( H \) is semisimple. In this case, the trace \( A \to A^H \) is surjective. Thus, Zhu’s conjecture follows from Corollary 5.3. However, if \( \text{char } k > 0 \), the question on integrality does not depend on any condition on \( H \).

As was observed by Kalniuk and Tyc (see [32]), the fact that, in positive characteristic, each commutative \( H \)-module algebra is integral over the invariants implies a property of \( H \) similar to the geometric reductivity known in the theory of algebraic groups. This property was considered in [32] for a not necessarily finite-dimensional Hopf algebra \( H \), and its main consequence is that, whenever \( A \) is a finitely generated commutative \( H \)-module algebra, on which the action of \( H \) is locally finite, the algebra of invariants \( A^H \) is finitely generated. If \( \text{char } k > 0 \), each finite-dimensional Hopf algebra is geometrically reductive in this sense (see [32, Theorem 4]). This result can be reformulated as follows.

**Theorem 5.4** (see [32]). Assume that \( \text{char } k = p > 0 \). If \( \varphi : A \to B \) is a surjective homomorphism of commutative \( H \)-module algebras and \( b \in B^H \), then \( b^n \in \varphi(A^H) \) for some integer \( n > 0 \).

**Proof.** First, we consider the case where \( A \) and \( B \) are graded, \( \varphi \) corresponds to the grading, and \( b \) is homogeneous of degree 1. Let \( B' \) be the subalgebra of \( B \) generated by \( b \), and let \( A' = \varphi^{-1}(B') \).
Since $A'$ is integral over $A'^H$ by Corollary 5.3, $B' = \varphi(A')$ is integral over $\varphi(A'^H)$. But $\varphi(A'^H)$ is a graded subalgebra of $B'$. If $\varphi(A'^H) = \kappa$, then $b$ is integral over $\kappa$ and in this case $b$ has to be nilpotent. Otherwise, $\varphi(A'^H)$ contains a homogeneous element of a positive degree. But each homogeneous component of $B'$ is spanned by a power of $b$. Hence $b^n \in \varphi(A'^H)$ for some $n$ anyway.

In the general case, let $\varphi : A[t] \to B[t]$ be the extension of $\varphi$ to polynomial rings and extend the action of $H$ to $A[t]$ and $B[t]$ by making $t$ invariant. Then $bt \in B[t]$ is a homogeneous $H$-invariant element of degree 1 and, therefore, we are in the situation of the previous case.

The integrality of $A$ over $A^H$ implies several well-known nice properties of the ring extension $A^H \subset A$. In particular, the canonical mapping $\text{Spec} A \to \text{Spec} A^H$ between the prime spectra is surjective, closed, and satisfies the going-up. However, for deeper conclusions, the integrality alone is not sufficient, and the characteristic polynomials come into play in an essential way. One of the applications is as follows.

**Theorem 5.5 (see [50]).** Assume that, for each $a \in A$, the characteristic polynomial $P_{A \otimes H^* / A}(\rho(a),t)$ has all coefficients in $A^H$. Then the mapping $\text{Spec} A \to \text{Spec} A^H$ is open, has finite fibers, and satisfies the going-down property.

Theorem 5.5 and its proof generalize the classical results describing properties of the quotient morphism $X \to X/G$, where $X$ is an affine scheme and $X/G$ is its quotient by an action of a finite group scheme.

There are further applications of the technique used in the study of group scheme actions. For each $p \in \text{Spec} A$, denote by $k(p)$ the residue field of the local ring $A_p$. Let $\alpha_p : A \to k(p)$ be the canonical ring homomorphism. The composition

$$\delta_p : A \xrightarrow{\rho} A \otimes H^* \xrightarrow{\alpha_p \otimes \text{id}} k(p) \otimes H^*$$

is a homomorphism of $H$-module algebras, assuming that $H$ acts trivially on $k(p)$ and by the left hits $\to$ on $H^*$. Hence

$$O(p) = (k(p) \otimes 1) \cdot \delta_p(A)$$

is a commutative right coideal subalgebra of the Hopf algebra $k(p) \otimes H^*$ over the field $k(p)$. In [50], $O(p)$ was called the *orbital subalgebra* associated with $p$.

If $H$ is cocommutative and $G$ is the finite group scheme representable by the commutative Hopf algebra $H^*$, the algebra $O(p)$ represents the scheme-theoretic $G$-orbit of $p$ which is a closed subscheme in the affine scheme $\text{Spec}(k(p) \otimes A)$.

**Theorem 5.6 (see [50]).** Assume that $A$ is $H$-semiprime and that the function $p \mapsto \dim O(p)$ is locally constant on the whole $\text{Spec} A$. Then $A$ is a finitely generated projective $A^H$-module whose rank at a prime $q \in \text{Spec} A^H$ is equal to $\dim O(p)$, where $p$ is any prime ideal of $A$ lying above $q$.

Also, the assignment $I \mapsto I \cap A^H$ establishes a bijection between the $H$-stable ideals of $A$ and all ideals of $A^H$. The inverse correspondence is $J \mapsto JA$.

We will briefly explain the main ideas used in the proof. Given some elements $a_1, \ldots, a_n \in A$, the set $U$ of prime ideals $p$ of $A$ for which $\delta_p(a_1), \ldots, \delta_p(a_n)$ form a basis of $O(p)$ over $k(p)$ is open in $\text{Spec} A$. One can also verify that, whenever $p$ and $p'$ are two prime ideals of $A$ with $p \cap A^H = p' \cap A^H$, one has $p \in U$ if and only if $p' \in U$. Then, passing to the localizations $A[s^{-1}]$ of $A$ at suitable elements $s \in A^H$, one may assume that there exist $a_1, \ldots, a_n \in A$ such that $\delta_p(a_1), \ldots, \delta_p(a_n)$ form a basis of $O(p)$ over $k(p)$ for each $p \in \text{Spec} A$.

The technically complicated part of the proof is to show that the previous assumption implies that $\rho(a_1), \ldots, \rho(a_n)$ form a basis of $(A \otimes 1)\rho(A) \subset A \otimes H^*$ over $A$ with respect to the left module structure; once this has been done, the freeness of $A$ over $A^H$ follows from Lemma 3.3. However, we note that,
if $A$ is reduced (equivalently, semiprime), then there is a general ring-theoretic fact which states that a submodule $M$ of a finite rank free $A$-module $F$ is freely generated by elements $v_1, \ldots, v_n \in M$ provided that for each $p \in \text{Spec } A$, the image of $k(p) \otimes M$ in $k(p) \otimes F$ has a basis over $k(p)$ consisting of $1 \otimes v_1, \ldots, 1 \otimes v_n$; for

$$M = (A \otimes 1)\rho(A), \quad F = A \otimes H^*, \quad v_i = \rho(a_i),$$

the desired conclusion is immediate. Since $A$ is assumed to be only $H$-semiprime, one has to overcome several difficulties.

A special case of Theorem 5.6 was given in [49]. Although several fundamental facts of the classical theory can be generalized to commutative $H$-semiprime algebras, in the case where the Hopf algebra $H$ is not cocommutative, it may not admit sufficiently many actions on commutative algebras. The following result has been obtained by Etingof and Walton (see [28]) if either $\text{char } k = 0$ or $\text{char } k > 0$ and $H$ is also semisimple. Its extension to the case where $H$ is not necessarily semisimple has been given in [55].

**Theorem 5.7.** Assume that $k$ is algebraically closed. Then any action of a finite-dimensional cosemisimple Hopf algebra $H$ on a commutative domain $A$ factors through an action of a group algebra, i.e., there exists a Hopf ideal $I$ of $H$ such that $I$ annihilates $A$ and $H/I$ is spanned by group-like elements.

Etingof and Walton say that the action of $H$ on $A$ is inner faithful if $A$ is not annihilated by any nonzero Hopf ideal of $H$. In [29, 30] they investigated the question of the existence of inner faithful actions on commutative domains for pointed Hopf algebras. Some pointed Hopf algebras admit such actions, while others do not.

The fact that the annihilator of $A$ in $H$ is often nontrivial had been recognized much earlier. Cohen and Westreich pointed out in [22, Cor. 0.12] that $H$ can act faithfully (in the ordinary sense) on a field $A$ only if $H$ is involutory and all group-like elements of $H^*$ lie in the center of $H^*$. Here $A$ can be even a domain since, in this case, the action of $H$ extends to the quotient field.

All this shows that the class of commutative $H$-module algebras is too narrow when $H$ is not cocommutative, and there is a definite need to study the invariants in the larger class of algebras satisfying a polynomial identity. However, at the present time, not all results known for commutative $H$-module algebras have been extended to the PI case.

As an extension of the commutative theory in a different direction, Cohen and Westreich (see [23]) introduced quantum commutative $H$-module algebras. The commutativity law in these algebras comes from the braiding determined by a quasitriangular structure on $H$. Cohen, Westreich, and Zhu proved the following theorem.

**Theorem 5.8 (see [24]).** Let $A$ be a quantum commutative $H$-module algebra, where $H$ is triangular semisimple and either $\text{char } k = 0$ or $\text{char } k > \dim H$. Then $A$ is integral over $A^H$ and $A$ is PI.

One may wonder whether the conclusion of this theorem is valid under less restrictive conditions on $H$ and the characteristic of $k$ if $A$ is $H$-semiprime.

6. The $H$-Equivariant Martindale Quotient Ring

Here we present results of Eryashkin [5] on quotient rings of $H$-semiprime PI algebras. Generalized Martindale quotient rings can be defined with respect to any filter $\mathcal{F}$ of ideals of a ring $R$ subject to the conditions that each ideal $I \in \mathcal{F}$ has zero left and right annihilators in $R$ and that $IJ \in \mathcal{F}$ whenever $I, J \in \mathcal{F}$. Details of this construction are given, e.g., in [40, Sec. 6.4]. If $R$ is prime and $\mathcal{F}$ is the set of all nonzero ideals of $R$, this construction gives the left, right, and symmetric Martindale rings of quotients, as defined in [45, Chap. 3].

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Let $A$ be an $H$-module algebra. Denote by $\mathcal{F}_H(A)$ the set of all its $H$-stable ideals with zero left and right annihilators in $A$. If $A$ is $H$-prime, then $\mathcal{F}_H(A)$ consists of all nonzero $H$-stable ideals of $A$. The Martindale quotient rings with respect to this filter were introduced by Cohen (see [16]). The use of $H$-stable ideals in their definition accounts for the extension of the $H$-module structure to these quotient rings.

We will be concerned only with the $H$-symmetric ring of quotients $Q_H(A)$ (see [40, p. 98]). The larger left and right quotient rings are less useful. The ring $Q_H(A)$ is characterized by the following properties (cf. [45, Proposition 10.4]):

1. $Q_H(A)$ contains $A$ as a subring;
2. each $I \in \mathcal{F}_H(A)$ has zero left and right annihilators in $Q_H(A)$;
3. for each $q \in Q_H(A)$, there exists $I \in \mathcal{F}_H(A)$ such that $IQ \subset A$ and $qI \subset A$;
4. given $I \in \mathcal{F}_H(A)$, a left $A$-linear mapping $f_l : I \to A$, and a right $A$-linear mapping $f_r : I \to A$ such that $xf_r(y) = f_l(x)y$ for all $x, y \in I$, there exists $q \in Q_H(A)$ such that $f_l(x) = qx$ and $f_r(x) = qx$ for all $x \in I$.

We set $Q = Q_H(A)$. As was explained in [16], $Q$ is an $H$-module algebra, and $A$ embeds in $Q$ as an $H$-stable subalgebra. The centers $Z(A)$ and $Z(Q)$ of $A$ and $Q$ are not stable, in general, under the action of $H$. We set

$Z(A)^H = Z(A) \cap A^H$, $Z(Q)^H = Z(Q) \cap Q^H$.

It follows from (2) and (3) in the characterization of $Q_H(A)$ above that each nonzero left or right $A$-submodule of $Q$ has a nonzero intersection with $A$. Therefore, $Q$ has to be $H$-prime or $H$-semiprime whenever so is $A$.

There are several general properties of the $H$-equivariant Martindale quotient rings of $H$-prime algebras. In particular, the fact that $Z(Q)^H$ is a field was explicitly stated in Matczuk’s paper (see [36, Lemma 1.4]), which, however, used the right quotient rings. This field is called the $H$-extended centroid of $A$.

Lemma 6.1. Assume that $A$ is $H$-prime. Then $Z(Q)^H$ is a field. Furthermore, given any $H$-stable ideal $I$ of $Q$ and a morphism $f : I \to Q$ in $H$-$Q_MQ$, there exists $z \in Z(Q)^H$ such that $f(x) = zx$ for all $x \in I$.

Proof. Recall that $f$ is a homomorphism of $Q$-bimodules and $H$-modules. We set $I' = f^{-1}(A) \cap A$. This is an $H$-stable ideal of $A$. We may assume that $f \neq 0$. Then $I' \neq 0$, i.e., $I' \in \mathcal{F}_H(A)$. Note that $f(x)y = f(xy) = xf(y)$ for all $x, y \in I$. By (4), there exists $z \in Q$ such that $f(x) = zx = xz$ for all $x \in I'$.

If $u \in Q$ is any element, then $uJ \subset A$ for some $J \in \mathcal{F}_H(A)$. Replacing $J$ by $J^{'1}$, we will also have $J \subset I'$ and $uJ \subset I'$. Since $z$ centralizes all elements of $I'$, it follows that $(zu - uz)J = 0$. Hence $zu = uz$ by (2).

Since $f$ is $H$-linear, we deduce that

$(hz)x = \sum h_{(1)}f(S(h_{(2)})x) = \varepsilon(h)f(x) = \varepsilon(h)zx$ for $h \in H$ and $x \in I'$.

In other words, $(hz - \varepsilon(h)z)I' = 0$. Hence $hz - \varepsilon(h)z$ for all $h \in H$, and we conclude that $z \in Z(Q)^H$.

Define $g : I \to Q$ by the formula $g(x) = f(x) - zx$. Then $g|_{I'} = 0$. If $u \in I$ and $J \in \mathcal{F}_H(A)$ is such that $J \subset I'$ and $uJ \subset I'$, then $g(u)J = g(uJ) = 0$, yielding $g(u) = 0$. Hence $g = 0$, and, therefore, $f(x) = zx$ for all $x \in I$.

The annihilator of $z$ in $Q$ is an $H$-stable ideal. If this ideal is nonzero, then property (2) implies $z = 0$. Hence $f$ has to be injective whenever $z \neq 0$. In this case, $f(I)$ is a nonzero $H$-stable ideal of $Q$. Applying the already proved conclusion to the inverse mapping $f^{-1} : f(I) \to I$, we see that there exists $z' \in Z(Q)^H$ such that $f^{-1}(y) = z'y$ for all $y \in f(I)$. Then $(zz' - 1)I = 0$, and it follows that
By Posner’s theorem, the ring \( q \mapsto Q \) is a field. \( \square \)

**Lemma 6.2.** Assume that \( A \) is \( H \)-prime. Let \( S \) be any simple algebra whose center contains \( Z(Q)^H \). We consider \( S \otimes Z(Q)^H Q \) as an \( H \)-module algebra with respect to the action of \( H \) on the second tensorand. If \( I \) is a nonzero \( H \)-stable ideal of this algebra, then \( I \) has nonzero intersection with the image of \( Q \) in \( S \otimes Z(Q)^H Q \) under the mapping \( x \mapsto 1 \otimes x \).

**Proof.** Let \( n \) be minimal possible for which \( I \) contains an element \( u \neq 0 \) which can be written as \( u = a_1 \otimes b_1 + \ldots + a_n \otimes b_n \) with \( a_i \in S, b_i \in Q \). Fix such an element and its expression as a sum. We set

\[
M = \left\{ (x_1, \ldots, x_n) \in Q^n \mid \sum a_i \otimes x_i \in I \right\}.
\]

We consider \( Q^n \) as an object of \( H \otimes Q \) with respect to the natural actions of \( H \) and \( Q \) on each component. Then \( M \) is a subobject of \( Q^n \) in this category. Note that \( (b_1, \ldots, b_n) \in M \). Let \( p_i : Q^n \to Q, i = 1, \ldots, n \), be the projections. Then \( J = p_1(M) \) is an \( H \)-stable ideal of \( Q \). But \( \text{Ker} \ p_1|_M = 0 \) since otherwise \( I \) would contain a nonzero element written as \( \sum_{i=2}^n a_i \otimes x_i \) with less than \( n \) summands.

Thus, \( p_1|_M : M \to J \) is an isomorphism in the category \( H \otimes Q \). Setting \( f_i = p_i \circ (p_1|_M)^{-1} \), we obtain

\[
M = \left\{ (f_1(x), \ldots, f_n(x)) \mid x \in J \right\},
\]

and each mapping \( f_i : J \to Q \) is a morphism in \( H \otimes Q \). By Lemma 6.1, there exist \( z_1, \ldots, z_n \in Z(Q)^H \) such that \( f_i(x) = z_i x \) for all \( x \in J \). In particular, \( b_i = z_i b_1 \) for each \( i \), and, therefore, \( u = \sum a_i \otimes z_i b_1 = (\sum a_i z_i) \otimes b_1 \).

The minimality of \( n \) implies that \( n = 1 \), i.e., \( u = a_1 \otimes b_1 \). But then \( S a_1 S \otimes b_1 \subset I \). Since \( S \) is simple, we have \( S a_1 S = S \). Hence \( 1 \otimes b_1 \in I \). \( \square \)

**Lemma 6.3.** Assume that \( A \) is \( H \)-prime, and let \( P \) be any prime ideal of \( A \) such that \( P_H = 0 \). Denote by \( \overline{Q} \) the symmetric Martinumbe quotient ring of the prime ring \( \overline{A} = A/P \). The canonical mapping \( \pi : A \to \overline{A} \) extends to a ring homomorphism \( Q \to \overline{Q} \) which maps the center of \( Q \) into the center of \( \overline{Q} \).

**Proof.** Let \( q \in Q \). There exists \( I \in F_H(A) \) such that \( I q \subset A \) and \( q I \subset A \). Since \( P_H = 0 \), we have \( I \not\subset P \). Hence \( \pi(I) \) is a nonzero ideal of \( \overline{A} \). Note that \( I q(I \cap P) \) is contained in \( P \). Applying \( \pi \), we obtain \( \pi(I) \pi(q(I \cap P)) = 0 \), which implies \( \pi(q(I \cap P)) = 0 \) since \( \overline{A} \) is prime.

This shows that \( q(I \cap P) \subset P \). Similarly \( (I \cap P) q \subset P \). Therefore, the right and left multiplications by \( q \) induce, respectively, a left \( \overline{A} \)-linear mapping \( f_l : \pi(I) \to \overline{A} \) and a right \( \overline{A} \)-linear mapping \( f_r : \pi(I) \to \overline{A} \). The pair \((f_l, f_r)\) determines an element \( \overline{q} \in \overline{Q} \). It is easy to see that the assignment \( q \mapsto \overline{q} \) defines a ring homomorphism \( Q \to \overline{Q} \) whose restriction to \( A \) is \( \pi \).

Denote this extension of \( \pi \) by the same letter \( \pi \). If \( z \in Z(Q) \), then \( \pi(z) \) commutes with all elements of \( \pi(A) = \overline{A} \), but then \( \pi(z) \) commutes with all elements of \( \overline{Q} \). \( \square \)

**Theorem 6.4 (see [5]).** Assume that \( A \) is PI and \( H \)-prime. Then the \( H \)-symmetric quotient ring \( Q = Q_H(A) \) is an \( H \)-simple \( H \)-module algebra of finite dimension over \( Z(Q)^H \). Moreover, \( Q = Z(Q)^H A \).

**Proof.** Take any prime ideal \( P \) of \( A \) such that \( P_H = 0 \). Let \( \pi : Q \to \overline{Q} \) be the ring homomorphism of Lemma 6.3. Since \( \overline{A} \) is a prime PI algebra, \( \overline{Q} \) is the classical quotient ring of \( \overline{A} \) (see [45, Theorem 23.4]). By Posner’s theorem, the ring \( \overline{Q} \) is simple and finite dimensional over its center \( Z(\overline{Q}) \). The composite mapping

\[
\varphi : Q \xrightarrow{\rho} Q \otimes H^* \xrightarrow{\pi \otimes \text{id}} \overline{Q} \otimes H^*
\]
is a homomorphism of $H$-module algebras, assuming the trivial action of $H$ on $\mathcal{Q}$ and the hit action $\rightarrow$ on $H^*$. It extends to a homomorphism of $H$-module algebras

$$\psi : Z(Q) \otimes Z(Q)^H \rightarrow \mathcal{Q} \otimes H^*, \quad z \otimes q \mapsto (z \otimes 1) \cdot \varphi(q).$$

Since $(\text{id} \otimes \varepsilon) \circ \varphi = \pi$, we have $\ker \varphi \subseteq \ker \pi$. It follows that

$$A \cap \ker \varphi \subseteq A \cap \ker \pi = \ker \pi|_A = P.$$

But $\ker \varphi$ is an $H$-stable ideal of $Q$. Since $P_H = 0$, we obtain $A \cap \ker \varphi = 0$, which implies $\ker \varphi = 0$. Now $\ker \psi$ is an $H$-stable ideal of $Z(Q) \otimes Z(Q)^H$. It has zero intersection with the image of $Q$ by the previous conclusion, which implies $\ker \psi = 0$ by Lemma 6.2. The injectivity of $\psi$ implies an upper bound for the dimension

$$[Q : Z(Q)^H] = [Z(Q) \otimes Z(Q)^H : Q : Z(Q)] \leq [\mathcal{Q} \otimes H^* : Z(Q)] = [\mathcal{Q} : Z(Q)] \cdot \dim_k H < \infty.$$

Then the $H$-stable subalgebra $A' = Z(Q)^H A \subset Q$ is also finite dimensional over $Z(Q)^H$. Moreover, $A'$ is $H$-prime since each nonzero ideal of $A'$ has a nonzero intersection with $A$. By Theorem 2.4, $A'$ is $H$-simple. But for each $q \in Q$ there exists a nonzero $H$-stable ideal $I'$ of $A'$ such that $qI' \subset A'$. We must have $1 \in I'$ and, therefore, $q \in A'$. Thus, $Q = A'$. \hfill \Box \hfill

**Corollary 6.5.** If $A$ is PI and $H$-prime, then $A$ has finitely many minimal prime ideals, and $P_H = 0$ for each of them.

**Proof.** Since $Q$ is artinian, it has finitely many maximal ideals. Let $P_1, \ldots, P_n$ be their contractions to $A$. The intersection $\bigcap P_i$ is nilpotent since it is contained in the Jacobson radical of $Q$. Hence each prime ideal of $A$ contains $P_i$ for some $i$, i.e., all minimal primes are among $P_1, \ldots, P_n$. If $I$ is any $H$-stable ideal of $A$ contained in $P_i$, then $IQ$ is an $H$-stable ideal of $Q$ contained in a maximal ideal. It follows that $IQ = 0$ and, therefore, $I = 0$. \hfill \Box \hfill

**Corollary 6.6.** Assume that $A$ is PI and $H$-prime. If $P$ is any prime ideal of $A$ such that $P_H = 0$, then the ring homomorphism $\pi : Q \rightarrow \mathcal{Q}$ of Lemma 6.3 is surjective.

**Proof.** We have $\mathcal{A} = \pi(A) \subset \pi(Q) \subset \mathcal{Q}$. If $s$ is any regular element of $\mathcal{A}$, then $s$ is invertible in $\mathcal{Q}$ since $\mathcal{Q}$ is a classical quotient ring of $\mathcal{A}$. But $s \in \pi(Q)$, which implies that $s$ is a regular element of $\pi(Q)$. Since $\pi(Q)$ is a finite-dimensional algebra over a field, it follows that $s^{-1} \in \pi(Q)$. Then $\mathcal{Q} = \pi(Q)$. \hfill \Box \hfill

**Corollary 6.7.** If $A$ is PI and $H$-simple, then $A$ has a finite dimension over its central subfield $Z(A)^H$.

**Proof.** In this case $Q_H(A) = A$. \hfill \Box \hfill

**Lemma 6.8.** Assume that $K_1, \ldots, K_n$ are minimal $H$-prime ideals of $A$ such that $\bigcap K_i = 0$. Then $Q_H(A) \cong Q_H(A/K_1) \times \cdots \times Q_H(A/K_n)$.

**Proof.** We set $Q = Q_H(A)$, $A_i = A/K_i$, and $Q_i = Q_H(A_i)$ for each $i$. The canonical mapping $\pi_i : A \rightarrow A_i$ extends to a homomorphism of $H$-module algebras $Q \rightarrow Q_i$ as in the proof of Lemma 6.3. The main point here is that $A_i$ is $H$-prime and $I \not\subseteq K_i$ for each $I \in F_H(A)$. In fact, $K_i$ has a nonzero annihilator in $A$ since $K_i K'_i \subset K_i \cap K'_i = 0$, where $K'_i = \bigcap_{j \neq i} K_j$. Hence each element of $Q$ gives rise to a left $A_i$-linear mapping $f_I : \pi_i(I) \rightarrow A_i$ and a right $A_i$-linear mapping $f_r : \pi_i(I) \rightarrow A_i$, and the pair $(f_t, f_r)$ determines an element of $Q_i$.

Now the collection $\pi_1, \ldots, \pi_n$ gives a homomorphism of $H$-module algebras

$$\pi : Q \rightarrow Q_1 \times \cdots \times Q_n.$$
Since \( \text{Ker } \pi|_A = \bigcap K_i = 0 \), it follows that \( \text{Ker } \pi = 0 \). It remains to show that \( \pi \) is surjective. Assume that \( q_1 \in Q_1 \). There exists a nonzero \( H \)-stable ideal \( I_1 \) of \( A_1 \) such that \( I_1 q_1 \subset A_1 \) and \( q_1 I_1 \subset A_1 \). Take any \( H \)-stable ideal \( J \) of \( A \) with the property that \( 0 \neq \pi_1(J) \subset I_1 \). Replacing \( J \) with \( JK'_1 \), we will also have

\[
J \subset K'_1, \quad \pi_1(J) q_1 \subset \pi_1(K'_1), \quad q_1 \pi_1(J) \subset \pi_1(K'_1).
\]

Since \( K_1 \cap K'_1 = 0 \), there exists an isomorphism of \( A \)-bimodules \( K'_1 \cong \pi_1(K'_1) \). Define mappings \( f_l, f_r : J \to K'_1 \) by the rules

\[
\pi_1(f_l(x)) = \pi_1(x)q_1, \quad \pi_1(f_r(x)) = q_1\pi_1(x).
\]

We set \( I = J + \sum K'_1 \). Note that \( K'_1 \subset K_j \) for \( j \neq i \), while \( K'_j \cap K_j = 0 \). Hence the sum \( \sum K'_1 \) is direct, and there exist extensions of \( f_l, f_r \) to mappings \( I \to A \) vanishing on \( K'_1 \) for each \( i \neq 1 \). Note that \( f_l \) is left \( A \)-linear, while \( f_r \) is right \( A \)-linear.

Since \( \pi_i(I) \neq 0 \) for each \( i \), the left and right annihilators of \( I \) in \( A \) are contained in each \( K_i \). Then these annihilators are zero, i.e., \( I \in \mathcal{F}_H(A) \). Hence the pair \( (f_l, f_r) \) determines an element \( q \in Q \) such that \( \pi_1(q) = q_1 \) and \( \pi_i(q) = 0 \) for \( i \neq 1 \). By the symmetry, \( Q_1 \) in this argument can be replaced by \( Q_j \) for any \( j \).

**Corollary 6.9.** Assume that \( A \) is PI and \( H \)-semiprime with finitely many minimal \( H \)-prime ideals. Then \( Q_H(A) \cong Q_1 \times \cdots \times Q_n \), where \( Q_1, \ldots, Q_n \) are \( H \)-simple \( H \)-module algebras with \( [Q_i : Z(Q_i)]^H \) \(< \infty \) for each \( i \).

**Proof.** In any \( H \)-semiprime algebra, the intersection of all minimal \( H \)-prime ideals is zero. Therefore, Lemma 6.8 applies. It gives a direct product decomposition of \( Q_H(A) \), where each factor \( Q_i = Q_H(A/K_i) \) is \( H \)-simple and finite dimensional over \( Z(Q_i)^H \) by Theorem 6.4.

**Theorem 6.10.** Assume that \( A \) is PI and \( H \)-semiprime with finitely many minimal \( H \)-prime ideals. Then \( Q_H(A) \) is a classical right quotient ring of \( A \).

We have made some comments about the proof in the discussion following the statement of Theorem 2.12. Eryashkin proves the conclusion of Theorem 6.10 in the \( H \)-prime case. If \( A \) is PI and \( H \)-semiprime, and if \( K_1, \ldots, K_n \) are all its minimal \( H \)-prime ideals, then, knowing that \( Q_i = Q_H(A/K_i) \) is a classical quotient ring of \( A/K_i \) for each \( i \), he concludes that \( Q_1 \times \cdots \times Q_n \) is a classical quotient ring of \( A \) directly, not using Lemma 6.8.

For a special class of \( H \)-prime PI algebras, Theorems 6.4 and 6.10 were obtained in an earlier article [4].

**Lemma 6.11.** The set of minimal \( H \)-prime ideals of \( A \) is finite if \( A \) is finitely generated and PI. The same conclusion holds if \( A \) is either left or right noetherian.

**Proof.** Under each of these assumptions, \( A \) has finitely many minimal prime ideals, and the prime radical \( N \) of \( A \) is nilpotent (in the case of a finitely generated PI algebra, see [47, Corollary 6.3.36', Theorem 6.3.39]). Let \( P_1, \ldots, P_n \) be all minimal primes, and for each \( i \), let \( K_i \) be the largest \( H \)-stable ideal of \( A \) contained in \( P_i \). Since \( \prod K_i \subset N \) is nilpotent, any minimal \( H \)-prime ideal of \( A \) has to coincide with one of the \( H \)-prime ideals \( K_1, \ldots, K_n \).

**7. Integrality of PI Algebras over the Invariants**

If \( A \) is a noncommutative \( H \)-module algebra, it is meaningful to consider the integrality of \( A \) over invariants in two different senses. One question concerns the integrality over central invariants. For
this, it should be assumed at least that $A$ is integral over its center $Z(A)$. We denote by $Z(A)^H$ the subalgebra of $Z(A)$ consisting of $H$-invariant central elements.

If $H$ is cocommutative, then $Z(A)$ is stable under the action of $H$, and $Z(A)$ is integral over $Z(A)^H$ by the classical theory. If $H$ is not cocommutative, then the problem becomes highly nontrivial. If $H$ is pointed, or at least if the coradical of $H$ is cocommutative, the following result was obtained by Totok.

**Theorem 7.1** (see [9]). The center $Z(A)$ is integral over $Z(A)^H$ and, therefore, $A$ is integral over $Z(A)^H$ if $A$ is integral over $Z(A)$, in each of the following two cases:

(a) $\text{char } k > 0$ and $H$ has a cocommutative coradical;
(b) $\text{char } k = 0$, $H$ is pointed, $A$ is reduced, and $Z(A)$ is finitely generated.

Making use of the coradical filtration $H_0 \subset H_1 \subset \cdots \subset H_n = H$, Totok constructs a chain of subalgebras $Z_0 \supset Z_1 \supset \cdots \supset Z_n$ such that $Z_0 = Z(A)^{H_0}$, and for each $i > 0$ the ring $Z_{i-1}$ is integral over $Z_i$ and $H_i$ acts trivially on $Z_i$ in the sense that $h z = \epsilon(h) z$ for all $h \in H_i$ and $z \in Z_i$. Then the conclusion of Theorem 7.1 follows by the transitivity of the integrality since $Z(A)$ is integral over $Z_0$. This extends the technique applied by Artamonov (see [2]) in the case where $A$ is a commutative domain.

New results on integrality of $A$ over $Z(A)^H$ for an arbitrary finite-dimensional Hopf algebra have appeared very recently. In [27], Etingof observes that the problem admits a bimodule reformulation which can be studied independently of any Hopf algebra theory. In fact, $Z(A)^H$ consists precisely of those elements $a \in A$, for which the left multiplication by $a \otimes 1$ in the algebra $A \otimes H^*$ coincides with the right multiplication by $\rho(a)$ or, in other words, the left action of $a$ on $A \otimes H^*$ is the same as the right action with respect to the $A$-bimodule structure defined as in Sec. 3. This bimodule has a special property which has been used by Etingof to introduce the notion of *Galois bimodules*.

An $R$-bimodule $P$ for a ring $R$ is called Galois of rank $d \geq 1$ if $P$ is left and right free of rank $d$ and there exists an isomorphism of bimodules $P \otimes_R P \cong P^d$. Etingof derives a classification of Galois bimodules when $R$ is a semisimple artinian ring, which is module-finite over its center $Z(R)$. Let $R \cong R_1 \times \cdots \times R_n$, where $R_1, \ldots, R_n$ are simple rings. In the process of the classification, it is verified that, for each Galois bimodule $P$, the ring $R$ is a finite module over the *center* of $P$ defined as

$$Z(P) = \{ a \in R \mid ax = xa \text{ for all } x \in P \} \subset Z(R).$$

Let $\phi_i(a)$ be the $R_i$-linear endomorphism of $R_i \otimes_R P$ afforded by the right action of $a \in R$. Now $R_i \otimes_R P$ is a finite-dimensional vector space over the center $Z(R_i)$ of $R_i$, and $\phi_i(a)$ can be considered as a linear transformation of this vector space. Therefore, the characteristic polynomial $\chi_{\phi_i(a)} \in Z(R_i)[t]$ makes sense. Let

$$\chi_a \in Z(R)[t] \cong Z(R_i)[t] \times \cdots \times Z(R_n)[t]$$

be the polynomial whose $i$th component is $\chi_{\phi_i(a)}^{m_1^2, m_i^2}$, where $m_i^2 = [R_i : Z(R_i)]$ and $m$ is the least common multiple of $m_1, \ldots, m_n$. In [27], it is shown that, for each central element $a \in Z(R)$, all coefficients of $\chi_a$ belong to $Z(P)$. This is a key fact needed for applications to the integrality.

Assume that the $H$-module algebra $A$ has a semisimple artinian classical quotient ring $Q$ which is a finite module over its center $Z(Q)$. Then Etingof’s results discussed in the previous paragraph apply to the Galois $Q$-bimodule $Q \otimes H^*$, whose center is $Z(Q)^H$. In particular, $Q$ is module-finite over $Z(Q)^H$, and certain polynomials associated with central elements of $Q$ have coefficients in $Z(Q)^H$.

However, the ultimate goal is to find conditions ensuring the integrality of $A$. Etingof formulates results for comodule algebras, but we stick to the conventions set for the present paper.

**Theorem 7.2** (see [27]). Let $Z$ be a central subalgebra of $A$, whose total quotient ring $Q(Z)$ is a direct product of finitely many fields. Assume that
(1) $A$ is a finitely generated torsion-free $Z$-module,
(2) $Q(Z) \otimes_Z A$ is a semisimple ring with center $Q(Z),$
(3) either $Z$ is integrally closed in $Q(Z)$, or $A$ is a projective $Z$-module.

Then $A$ is integral over $Z \cap A^H$.

An $H$-module algebra $A$ is said to be indecomposable if it is not isomorphic to a direct product of two nonzero $H$-module algebras. If $A$ is artinian and $H$-semiprime, this is equivalent, by Theorem 2.4, to the $H$-simplicity of $A$.

The indecomposability of $A$ in the next proposition means that the corresponding Galois $A$-bimodule $P = A \otimes H^*$ is connected. In this case, $P^{m^2}$ is isomorphic to a multiple of $A \otimes_{Z(P)} A$ by the classification of Galois bimodules. Here $Z(P) = Z(A)^H$. Comparing the left ranks of the two bimodules, Etingof deduces a divisibility relation involving numeric characteristics of $A$.

**Proposition 7.3** (see [27]). Assume that $A$ is semisimple artinian, module-finite over $Z(A)$, and indecomposable as an $H$-module algebra. Let $A_1, \ldots, A_n$ be all simple factor-rings of $A$. We set

$$d_i = [Z(A_i) : Z(A)^H], \quad m_i = [A_i : Z(A_i)]^{1/2}, \quad m_* = \gcd(m_1, \ldots, m_n).$$

Then $\sum d_i(m_i/m_*)^2$ divides the dimension of $H$. In other words, $[A : Z(A)^H]$ divides $m^2_*(\dim H)$.

Now we describe a different approach, due to Eryashkin, which makes systematic use of structural properties of $H$-module algebras discussed earlier. The results have been obtained not only for semiprime $H$-module algebras but also for $H$-semiprime algebras.

The ring homomorphism $A \to A \otimes H^*$ given by the assignment $a \mapsto a \otimes 1$ maps the center of $A$ into a central subalgebra of $A \otimes H^*$. Therefore, $A \otimes H^*$ can be considered as a $Z$-algebra for any central subalgebra $Z$ of $A$. If $A$ is projective of finite constant rank as a $Z$-module, then so is $A \otimes H^*$. As was explained in Sec. 5, in this case there exist characteristic polynomials for the ring extension $A \otimes H^*/Z$. For each $a \in A$,

$$P_{A \otimes H^*/Z}(\rho(a), t) \in Z[t]$$

is the characteristic polynomial of the left multiplication operator by the element $\rho(a)$ in the $Z$-algebra $A \otimes H^*$. Alternatively, one could use the characteristic polynomials of the right multiplication operators.

At one point, we will need the ring-theoretic fact stated below. For the proof, see [5, Proposition 3.1].

**Proposition 7.4** (see [5, Proposition 3.1]). Let $R$ be a ring which has a right artinian classical right quotient ring $Q(R)$. Assume that $R$ is a finitely generated module over a central subring $Z$ such that $\operatorname{ann}_R(z) = \operatorname{ann}_Z(z)R$ for each $z \in Z$. Then $Q(R) \cong Q(Z) \otimes_Z R$, where $Q(Z)$ is the total quotient ring of $Z$.

**Theorem 7.5** (see [5]). Assume that $A$ is $H$-semiprime with finitely many minimal $H$-prime ideals. If $A$ is projective of finite constant rank as a module over its center $Z(A)$, then $A$ is integral over $Z(A)^H$. In fact, for each $a \in A$, the characteristic polynomial $P_{A \otimes H^*/Z(A)}(\rho(a), t)$ has all coefficients in $Z(A)^H$.

**Proof.** Before we treat the general case, let us verify the conclusion of this theorem under additional assumptions about $A$.

**Step 1.** Assume that $A$ is $H$-simple.

By Corollary 6.7, $Z(A)^H$ is a field and $[A : Z(A)^H] < \infty$. The integrality of $A$ over $Z(A)^H$ is immediate. We still have to prove the statement about the characteristic polynomials.
We consider the action of $H$ on $A \otimes H^*$ defined by the rule $h(a \otimes \xi) = a \otimes (h \mapsto \xi)$ for $h \in H$, $a \in A$, and $\xi \in H^*$. Then the mapping $\rho : A \rightarrow A \otimes H^*$ is a homomorphism of $H$-module algebras. Therefore, so is its extension
\[
\psi : Z(A) \otimes_{Z(A)^H} A \rightarrow A \otimes H^*, \quad z \otimes a \mapsto (z \otimes 1) \cdot \rho(a),
\]
with the action of $H$ on the first algebra defined by the formula $h(z \otimes a) = z \otimes ha$ for $h \in H$, $z \in Z(A)$, and $a \in A$. We set
\[
A = Z(A) \otimes_{Z(A)^H} A, \quad B = A \otimes H^*.
\]
We claim that $B$ is a free left $A$-module with respect to the action afforded by $\psi$. Since the ring $Z(A)$ is artinian and since $\psi$ is a homomorphism of $Z(A)$-algebras, both of which are free modules over $Z(A)$, it suffices to verify that, for each maximal ideal $m$ of $Z(A)$, the $A/mA$-module $B/mB$ is free of rank $r$, where $r$ does not depend on $m$. Now
\[
A/mA = K(m) \otimes_{Z(A)^H} A, \quad B/mB = A/mA \otimes H^*,
\]
where $K(m) = Z(A)/m$ is a field. Since $A$ is $H$-simple, it follows from Lemma 6.2 that $A/mA$ is also $H$-simple. Hence $B/mB$ is a free $A/mA$-module by Theorem 2.9. The rank $r(m)$ of this free module can be computed as
\[
r(m) = \frac{|B/mB : K(m)|}{|A/mA : K(m)|} = \frac{|A : Z(A)| \cdot (\dim H)}{|A : Z(A)^H|} = \frac{\dim H}{|Z(A) : Z(A)^H|},
\]
where $|A : Z(A)|$ is the rank of $A$ as a $Z(A)$-module. This shows that $r(m)$ has the same value for all $m$, as required.

Thus, $B \cong A^r$ as an $A$-module. Since $\rho(a) = \psi(1 \otimes a)$, we deduce that
\[
P_{B/Z(A)}(\rho(a), t) = P_{A/Z(A)}(1 \otimes a, t)^r = P_{A/Z(A)^H}(a, t)^r,
\]
which is a polynomial with coefficients in $Z(A)^H$.

**Step 2.** Assume that $A$ is artinian and $H$-semiprime.

By Theorem 2.4, $A = A_1 \times \ldots \times A_n$, where each $A_i$ is an $H$-simple $H$-module algebra. Clearly,
\[
Z(A) = Z(A_1) \times \ldots \times Z(A_n), \quad A^H = A_1^H \times \ldots \times A_n^H.
\]
Let $\pi_i : A \rightarrow A_i$ be the projection, $\zeta_i : Z(A) \rightarrow Z(A_i)$ be the restriction of $\pi_i$, and $\zeta_i^t : Z(A)[t] \rightarrow Z(A_i)[t]$ be the extension of $\zeta_i$ to polynomial rings. Note that
\[
A_i \cong Z(A_i) \otimes_{Z(A)} A \quad \text{and, therefore,} \quad A_i \otimes H^* \cong Z(A_i) \otimes_{Z(A)} (A \otimes H^*).
\]
Since $(\pi_i \otimes \text{id})\rho(a) = \rho(\pi_i a)$, we obtain
\[
\zeta_i^t P_{A_\otimes H^*/Z(A)}(\rho(a), t) = P_{A_i \otimes H^*/Z(A_i)}(\rho(\pi_i a), t) \in Z(A_i)^H [t]
\]
by Step 1. It follows that all coefficients of the polynomial $P_{A_\otimes H^*/Z(A)}(\rho(a), t)$ are $H$-invariant since they have $H$-invariant images in each $A_i$.

Now it is easy to complete the proof of Theorem 7.5 in full generality. By Theorem 2.12, $A$ has a right and left artinian classical right quotient ring $Q = Q(A)$, which is an $H$-semiprime $H$-module algebra since so is $A$. Note that $\text{ann}_A(z) = \text{ann}_{Z(A)}(z)A$ for each $z \in Z(A)$ since $A$ is a direct summand of a free $Z(A)$-module. By Proposition 7.4, $Q$ is a central localization of $A$. Then the total quotient ring of $Z(A)$ coincides with the center of $Q$ and, therefore, $Q \cong Z(Q) \otimes_{Z(A)} A$. The functorial properties of characteristic polynomials imply that
\[
P_{A_\otimes H^*/Z(A)}(\rho(a), t) = P_{Q_\otimes H^*/Z(Q)}(\rho(a), t).
\]
All coefficients of this polynomial lie in $Z(Q)^H$ by Step 2. Hence they actually lie in $Z(A) \cap Z(Q)^H = Z(A)^H$. 

\[\square\]
All ideas of this proof are taken from [5]. We have used Theorem 2.9 to make some arguments more transparent. Note that Step 1 in the proof yields also the following conclusion.

**Corollary 7.6.** Assume that $A$ is $H$-simple and $A$ is a free module of finite rank over its center $Z(A)$. Then the dimension $[Z(A) : Z(A)^H]$ of $Z(A)$ over $Z(A)^H$ divides the dimension of $H$.

It is not clear to what extent the conditions imposed on $A$ in Theorem 7.5 are optimal. One concern arising here is the finiteness of the set of minimal $H$-primes. An easy extension of Theorem 7.5 is stated below.

**Corollary 7.7.** Assume that $A$ is projective of finite constant rank as a module over its center $Z(A)$ and there exists a set $F$ of $H$-semiprime ideals of $A$ such that

1. each ideal in $F$ is an intersection of finitely many $H$-prime ideals;
2. each ideal $I \in F$ is generated by $I \cap Z(A)$;
3. for each $I \in F$, the image of $Z(A)$ in $A/I$ coincides with the center of $A/I$;
4. $\bigcap_{I \in F} I = 0$.

Then, for each $a \in A$, the characteristic polynomial $P_{A \otimes H^*/Z(A)}(\rho(a), t)$ has all coefficients in $Z(A)^H$.

**Proof.** For each $I \in F$, we have $A/I \cong Z(A/I) \otimes_{Z(A)} A$, and this algebra satisfies the hypothesis of Theorem 7.5. Hence all coefficients of $P_{A \otimes H^*/Z(A)}(\rho(a), t)$ have $H$-invariant images in $A/I$. But (4) ensures that any element $c \in A$ is $H$-invariant whenever $c + I$ is $H$-invariant in $A/I$ for each $I \in F$. □

The assumptions about $A$ in Corollary 7.7 are admittedly too restrictive. However, they are satisfied when $A$ is commutative and $H$-semiprime. Thus, Theorem 5.2 is a special case of Corollary 7.7.

Without projectivity of $A$ over $Z(A)$, the characteristic polynomials for the ring extension $A \otimes H^*/Z(A)$ are not defined. One can still exploit the finiteness of $Q = Q(A)$ over $Z(Q)^H$. Recall that, if $A$ is PI and $H$-prime, then $Q$ is $H$-simple and $Z(Q)^H$ is a field. The next result is based on [5, Proposition 3.3], although it was not stated this way.

**Proposition 7.8.** Assume that $A$ is $H$-prime and module-finite over its center $Z(A)$. Let $Q$ be the classical quotient ring of $A$. For each $a \in A$, all coefficients of the characteristic polynomial $P_{Q/Z(Q)^H}(a, t)$ are integral over $Z(A)$.

**Proof.** Since $[Q : Z(Q)^H] < \infty$, the set Max $Q$ of all maximal ideals of $Q$ is finite. For each $M \in$ Max $Q$, denote by $\pi_M$ the canonical mapping $Q \rightarrow Q/M$. Here $Q/M$ is a simple ring, which is finite dimensional over its center $Z(Q/M)$. Since $Z(A) \subset Z(Q)$, we have $\pi_M(Z(A)) \subset Z(Q/M)$.

An arbitrary element $q \in Q$ is integral over $Z(A)$ if and only if $\pi_M(q)$ is integral over $\pi_M(Z(A))$ for each $M \in$ Max $Q$. Indeed, if the latter property holds, then for each $M$ there exists a polynomial $f_M$ in one variable with all coefficients in $Z(A)$ and the leading coefficient 1 such that $f_M(q) \in M$. Putting

$$f = \prod_{M \in$ Max $Q} f_M,$$

we will have $f(q) \in J$, where $J$ stands for the Jacobson radical of $Q$. But $J$ is nilpotent, whence $f(q)^n = 0$ for some integer $n > 0$. Clearly, $f^n$ is a polynomial with all coefficients in $Z(A)$ and the leading coefficient 1.

We will verify that the necessary and sufficient condition of integrality from the previous paragraph is satisfied for all coefficients of $P_{Q/Z(Q)^H}(a, t)$. This will yield the final conclusion.

We consider the composite mapping

$$\rho_M : Q \overset{\rho}{\rightarrow} Q \otimes H^* \overset{\pi_M \otimes \text{id}}{\rightarrow} Q/M \otimes H^*$$

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and its extension
\[ \psi : Z(Q/M) \otimes_{Z(Q)^H} Q \to Q/M \otimes H^*, \quad z \otimes q \mapsto (z \otimes 1) \cdot \rho_M(q). \]

With the $H$-module structures as in the proof of Theorem 7.5, $\psi$ is a homomorphism of $H$-module algebras, and the first algebra is $H$-simple by Lemma 6.2. In particular, it follows that $\psi$ is injective. By Theorem 2.9, $\psi$ makes $Q/M \otimes H^*$ a free left module over $Z(Q/M) \otimes_{Z(Q)^H} Q$. Let $r$ be its rank (it depends on $M$).

Denote by $P_2$ and $P_1$ the characteristic polynomials of these two rings regarded as finite-dimensional algebras over the field $Z(Q/M)$. Since $\rho_M(a) = \psi(1 \otimes a)$, we obtain
\[ P_2(\rho_M(a), t) = P_1(1 \otimes a, t)^* = P_{Q/Z(Q)^H}(a, t)^*. \]

Here we identify the field $Z(Q)^H$ with its image in $Z(Q/M)$ under the mapping $\pi_M$.

Recall that $P_2(\rho_M(a), t)$ is the characteristic polynomial of the left multiplication operator associated with $\rho_M(a)$. But $\rho(a) \in A \otimes H^*$ and, therefore, $\rho_M(a)$ lies in the subring $\pi_M(A) \otimes H^*$ of $Q/M \otimes H^*$, which is a finitely generated module over $\pi_M(Z(A))$. It follows that $\rho_M(a)$ satisfies a polynomial relation of integral dependence over $\pi_M(Z(A))$, which implies that so does the corresponding multiplication operator. This means that all eigenvalues of this operator in any algebraic closure of $Z(Q/M)$ are integral over $\pi_M(Z(A))$. But these eigenvalues are precisely the roots of the characteristic polynomial, i.e., the roots of $P_{Q/Z(Q)^H}(a, t)$ in view of the equality above. The coefficients of $P_{Q/Z(Q)^H}(a, t)$ are evaluations of the elementary symmetric functions at the roots of this polynomial. Hence they are also integral over $\pi_M(Z(A))$.

\[ \text{Theorem 7.9 (see [5]).} \quad \text{Assume that} \ A \text{ is } H\text{-semiprime with finitely many minimal } H\text{-prime ideals.} \]
\[ \text{Let} \ Q \text{ be the classical quotient ring of} \ A. \ \text{If} \ Z(A) \text{ is integrally closed in} \ Z(Q) \text{ and} \ A \text{ is a finitely generated } Z(A)\text{-module, then} \ A \text{ is integral over} \ Z(A)^H. \]

**Proof.** By Theorem 2.4, $Q \cong Q_1 \times \ldots \times Q_n$, where each $Q_i$ is an $H$-simple $H$-module algebra. Let $e_i \in Q$ be the $H$-invariant central idempotent, whose projection to $Q_j$ is 1 for $j = i$ and 0 otherwise. Since $e_i \in Z(Q)$ is integral over $Z(A)$, we must have $e_i \in Z(A)$. Then
\[ A \cong A_1 \times \ldots \times A_n, \quad \text{where} \ A_i = A/(1 - e_i)A. \]

Clearly, $Q_i$ is the classical quotient ring of $A_i$. It follows that $A_i$ is $H$-prime, $A_i$ is module-finite over its center $Z(A_i)$, and $Z(A_i)$ is integrally closed in $Z(Q_i)$.

This reduces the proof to the case where $A$ is $H$-prime. But in this case, Proposition 7.8 applies. It shows that, for each $a \in A$, all coefficients of the characteristic polynomial $P_{Q/Z(Q)^H}(a, t)$ are in $Z(A) \cap Z(Q)^H = Z(A)^H$. Since $P_{Q/Z(Q)^H}(a, a) = 0$ by a general property of characteristic polynomials, $a$ is integral over $Z(A)^H$.

The statement of [5, Proposition 3.3] contains the additional assumption that $\text{ann}_A(z)$ is equal to $\text{ann}_{Z(A)}(z)A$ for each $z \in Z(A)$. The proofs given above show that this assumption is not needed.

In connection with Theorems 7.2, 7.5, and 7.9, we are prompted to ask the following question.

**Question 7.10.** Is there any example of an $H$-semiprime algebra $A$, which is module-finite over its center $Z(A)$ and such that $A$ is not integral over $Z(A)^H$?

**Proposition 7.11 (see [6]).** Assume that $H$ is semisimple. If $A$ is PI and $A^H \subset Z(A)$, then $A$ is integral over $A^H$.

**Proof.** It suffices to consider the case where $A$ is finitely generated. Then the integrality of $A$ over $A^H$ is equivalent to the module-finiteness. There exists a surjective homomorphism of $H$-module algebras $B \to A$, where $B$ has a grading $B = B_0 \oplus B_1 \oplus \ldots$ with finite-dimensional $H$-stable homogeneous...
components such that $B_0 = k$ and $B_1$ generates $B$. To this end, we can start from the tensor algebra of any finite-dimensional $H$-submodule $V \subset A$ which generates $A$ as an algebra. Taking the factor-algebra of $B$ by a suitable $H$-stable ideal, we may assume that $B$ is PI. Factoring out another ideal generated by all commutators $xy - yx$ with $x \in B$ and $y \in B^H$, we may also assume that $B^H \subset Z(B)$.

Since $H$ is semisimple, $B^H$ is mapped onto $A^H$. Therefore, it suffices to show that $B$ is a finite $B^H$-module. By the graded Nakayama lemma, this holds if and only if $\dim B/B^H_+ B < \infty$, where $B^H_+ = \sum_{i > 0} B^H_i$.

We set $D = B/B^H_+ B$. Note that $B^H_+ B$ is a homogeneous $H$-stable ideal of $B$. Hence $D$ inherits the structure of a graded $H$-module algebra and $D$ is PI. Since $B^H$ maps onto $D^H$, we have $D^H = k$. It follows that $D^H_+ = 0$, where $D_+ = \sum_{i > 0} D_i$.

Let $P$ be any maximal ideal of $D$, and let $P_H$ be the largest $H$-stable ideal contained in $P$. By Kaplansky’s theorem, the simple algebra $D/P$ is finite dimensional over its center, and, since $D$ is finitely generated, we have $\dim D/P < \infty$ (over $k$). Now $P_H$ is the kernel of the composite mapping

$$D \xrightarrow{\rho} D \otimes H^* \longrightarrow D/P \otimes H^*.$$ 

It follows that $\dim D/P_H < \infty$ too. By Theorem 2.4, $D/P_H$ is $H$-simple, which means that $P_H$ is a maximal $H$-stable ideal of $D$.

If $P_H \not\subset D_+$, then $P_H + D_+ = D$. Since all $H$-modules are completely reducible, we deduce that $P^H + D^H_+ = D^H$, where $P^H = P \cap D^H$. Hence there exists $d \in D^H_+$ such that $d \notin P$. This implies $D^H_+ \neq 0$, a contradiction.

Thus, $P_H \subset D_+$ is the only possibility. Since $D_+$ is an $H$-stable ideal of $D$, we obtain $P_H = D_+$ by the maximality of $P_H$, but then $P = D_+$ too. We conclude that $D$ has a single maximal ideal. Recall that the prime radical of any finitely generated PI algebra is nilpotent by the Braun theorem [47, Theorem 6.3.39] and coincides with the intersection of all maximal ideals by the Amitsur–Procesi theorem [47, Theorem 6.3.3]. This implies that $D_+$ is the prime radical of $D$ and that $D_+$ is nilpotent.

But then $D_i = 0$ for sufficiently large $i$. Hence $\dim D < \infty$, as required.

The condition $A^H \subset Z(A)$ may look artificial, but sometimes it arises very naturally. For example, this inclusion always holds when $A$ is quantum commutative. In [22], Cohen and Westreich investigated how the condition $A^H \subset Z(A)$ affects various properties of an $H$-module algebra, especially in the case where $A$ is an $H^*$-Galois extension of $A^H$.

In [3] and [4], Eryashkin considered a special class $A$ of $H$-module algebras. An $H$-module algebra $A$ belongs to $A$ if $A$ has an ideal $I$ such that the factor-algebra $A/I$ is commutative and $I$ contains no nonzero $H$-stable ideals of $A$. Such an algebra is PI since it is embedded into the algebra $A/I \otimes H^*$ which is a finite module over its center. If $z \in A^H$, then $\{za - az \mid a \in A\}$ is an $H$-submodule of $A$ contained in the ideal $I$, which implies $za = az$ for all $a \in A$ by the conditions imposed on $I$. This shows that $A^H \subset Z(A)$.

Starting from an arbitrary left $H$-module $V$, one obtains an $H$-prime algebra in $A$ taking $A = T(V)/I^*_H$, where $T(V)$ is the tensor algebra of $V$ and $I^*_H$ is its largest $H$-stable ideal contained in the ideal $I^*$ generated by all commutators. Here the ideal $I = I^*/I^*_H$ of $A$ has the property that $A/I$ is the symmetric algebra of $V$. By a careful examination, Eryashkin has verified that $A$ is not integral over $A^H$ in the case where $\text{char } k = 0$, $H$ is the 4-dimensional Hopf algebra described by Sweedler, and $V$ is one of its 2-dimensional indecomposable modules.

In particular, the semisimplicity of $H$ is necessary in Proposition 7.11, even if $A$ is assumed to be $H$-prime. In the case of a positive characteristic, the previous construction does not give such an example (see Corollary 8.4). This leaves open the following question.
Question 7.12. Assume that char $k > 0$. Is there any example of an $H$-prime PI algebra $A$ such that $A^H \subset Z(A)$, but $A$ is not integral over $A^H$?

If $A^H \not\subset Z(A)$, then the integrality of $A$ over $A^H$ should be understood as defined by Schelter [48]. An element $x \in A$ is called Schelter integral over $A^H$ if there exists an integer $m > 0$ such that $x^m$ can be written as a sum of several elements, each of which is a product of elements contained in $A^H \cup \{x\}$ with $x$ occurring as a factor in this product less than $m$ times. If all elements of $A$ are Schelter integral over $A^H$, then $A$ is said to be Schelter integral over $A^H$.

In the 1993 expository lectures, Montgomery asked whether Schelter integrality of $A$ over $A^H$ holds whenever $H$ is semisimple [40, Question 4.3.1]. At that time, the case of the group action had been settled in full generality by Quinn. If $G$ is a finite group of automorphisms of a ring $R$ such that $|G|R = R$, then $R$ is Schelter integral over the subring of invariants $R^G$. In fact, it was proved in [46] that $R$ is fully integral over $R^G$, which is a stronger property defined in terms of collections of elements rather than single elements. Quinn also obtained a particular result for Hopf actions.

Theorem 7.13 (see [46]). Assume that $H$ is semisimple and the action of $H$ on $A$ is inner. Then each ideal $I$ of $A$ is fully integral and, therefore, also Schelter integral over $I^H$ of degree bounded by a function of the dimension of $H$.

The condition that the action is inner means that there exists an invertible element $u \in A \otimes H^*$ such that

$$\rho(a) = u(a \otimes 1)u^{-1} \quad \text{for all } a \in A.$$  

In particular, two non-unital subalgebras $I \otimes 1$ and $\rho(I)$ of $A \otimes H^*$ are conjugate by an inner automorphism. It follows that $(I \otimes H^*)\#H \cong I \otimes \text{End}_k H$ is fully integral over $\rho(I)\#H$ since $I \otimes \text{End}_k H$ is known to be fully integral over $I \otimes k$ by the Paré–Schelter theorem [44]. Finally, Quinn deduces that $I$ is fully integral over $I^H$ using an idempotent $e \in (A \otimes H^*)\#H$ such that

$$e((I \otimes H^*)\#H)e \cong I, \quad e(\rho(I)\#H)e \cong I^H$$

as nonunital algebras. In the case of an inner action, all ideals of $A$ are $H$-stable.

At present, it is not known how to extend the previous theorem to arbitrary module algebras for a semisimple Hopf algebra. Special cases of the problem were considered in [9, 16]. Eryashkin has succeeded in answering Montgomery’s question in the case of PI algebras.

Theorem 7.14 (see [6]). Assume that $H$ is semisimple and cosemisimple. If $A$ is PI, then $A$ is Schelter integral over $A^H$.

Proof. The initial idea comes from the paper of Montgomery and Small [42], where a similar problem for group actions on PI rings was considered. By Zorn’s lemma, $A$ has an $H$-stable ideal $K$ maximal with respect to the property that all elements of $K$ are Schelter integral over $A^H$. It is easy to see that $K$ is $H$-semiprime.

Replacing $A$ by $A/K$, we may assume that $A$ is $H$-semiprime and $A$ has no nonzero $H$-stable ideals which are Schelter integral over $A^H$. We have to show that $A = 0$. This step requires more efforts as compared with the group action case.

Assume that $A \neq 0$. Since $H$ is cosemisimple, Linchenko and Montgomery tell us that the prime radical of $A$ is $H$-stable [35, Theorem 3.5]; hence $A$ is semiprime. Then, by the general PI theory, $A$ has a nonzero ideal $I'$ such that, for each $x \in I'$, the left ideal $Ax$ is contained in a finitely generated $Z(A)$-submodule of $A$. We set $I = H'H'$. This is a nonzero $H$-stable ideal of $A$. We will show that all elements of $I$ are Schelter integral over $A^H$, but this contradicts the assumptions about $A$.

Denote by $C$ the centralizer of $A^H$ in $A$. This is an $H$-stable subalgebra of $A$ with $C^H \subset Z(C)$ and $Z(A) \subset C$. Let $x \in I$. From the construction of $I$, it easily follows that $Ax$ is contained in a
Then, in the length of the coradical filtration of ring, the endomorphism \( \text{Proposition 8.1} \) (see [6]) conclusions, due to Eryashkin.

Without loss of generality, we may assume that \( C_0 \) is \( H \)-stable. Since \( C_0^H \subset Z(C_0) \), Proposition 7.11 shows that \( C_0 \) is integral and, therefore, module-finite, over \( C_0^H \). Hence \( N_0 \) is a finitely generated module over \( C_0^H \). Define \( r_x \in \text{End}_{C_0} N_0 \) by the rule \( r_x(y) = yx \) for \( y \in N_0 \). Since \( C_0^H \) is a commutative ring, the endomorphism \( r_x \) satisfies the relation

\[
r_x^m + c_1 r_x^{m-1} + \ldots + c_m \text{id} = 0
\]

for some integer \( m > 0 \) and elements \( c_1, \ldots, c_m \in C_0^H \). Applying this operator to \( x \in N_0 \), we deduce that \( x^{m+1} + c_1 x^m + \ldots + c_m x = 0 \). It follows that \( x \) is Schelter integral over \( C_0^H \subset A^H \).

\[ \square \]

8. Comparison with the Invariants of the Coradical

We continue to assume that \( A \) is an \( H \)-module algebra. If \( H \) is pointed with the group \( G \) of grouplike elements, it was observed by Artamonov (see [2]) that \( A^H = A^G \) when \( \text{char} \ k = 0 \) and \( A \) is a commutative domain. If \( \text{char} \ k = p > 0 \), the Hopf algebra is pointed, and \( A \) is commutative, then it follows from the results of Totok (see [8]) and Zhu (see [58]) that \( z^{p^r} \in A^H \) for all \( z \in A^G \), where \( s \) is the length of the coradical filtration of \( H \).

Etingof and Walton (see [29]) proved the equality \( Z(A)^H = Z(A)^{H_0} \), where \( H_0 \) is the coradical of \( H \) in the case where \( A \) is a prime Azumaya algebra and \( \text{char} \ k = 0 \). In this section, we present stronger conclusions, due to Eryashkin.

**Proposition 8.1** (see [6]). Let \( H_0 \subset H \) be a Hopf subalgebra containing the coradical of \( H \). Assume that \( A \) is PI and \( H \)-simple. Let \( K \) be a maximal \( H_0 \)-stable ideal of \( A \) and \( A_0 = A/K \). Denote by \( \nu \) the canonical mapping \( A \rightarrow A_0 \).

(i) If \( \text{char} \ k = 0 \), then \( Z(A_0)^{H_0} = \nu(Z(A)^H) \).

(ii) If \( \text{char} \ k = p > 0 \), then there exists an integer \( s \geq 0 \) such that \( z^{p^s} \in \nu(Z(A)^H) \) for all \( z \in Z(A_0)^{H_0} \).

**Proof.** By Corollary 6.7, \( Z(A)^H \) is a field and \( |A : Z(A)^H| < \infty \). Similarly, \( Z(A_0)^{H_0} \) is a field and \( |A_0 : Z(A_0)^{H_0}| < \infty \) since \( A_0 \) is PI and \( H_0 \)-simple. Note that \( \nu \) maps \( Z(A)^H \) into \( Z(A_0)^{H_0} \). Define \( H \)-module structures on

\[
A = Z(A_0)^{H_0} \otimes_{Z(A)^H} A, \quad B = A_0 \otimes H^*,
\]

as in the proof of Theorem 7.5. There exists a homomorphism of \( H \)-module algebras

\[
\psi : A \rightarrow B, \quad z \otimes a \mapsto (z \otimes 1) \cdot \varphi(a),
\]

where \( \varphi(a) = (\nu \otimes \text{id})(\rho(a)) \). By Lemma 6.2, \( A \) is \( H \)-simple. Hence \( B \) is a free left \( A \)-module by Theorem 2.9. Let \( r \) be its rank. Then

\[
P_{B/Z(A_0)^{H_0}}(\varphi(a), t) = P_{A/Z(A_0)^{H_0}}(1 \otimes a, t)' = \nu' P_{A/Z(A)^H}(a, t)'
\]

for all \( a \in A \). Here \( \nu' : Z(A)^H[t] \rightarrow Z(A_0)^{H_0}[t] \) is the homomorphism of polynomial rings induced by the restriction of \( \nu \) to \( Z(A)^H \). Thus, all coefficients of the polynomial above lie in \( \nu(Z(A)^H) \).

Now let \( z \in Z(A_0)^{H_0} \). Choose any \( a \in A \) such that \( \nu(a) = z \). We have \( H_0^* \cong H^*/J \), where \( J \) is a Hopf ideal of \( H^* \). The image of \( \rho(a) \in A \otimes H^* \) in \( A_0 \otimes H_0^* \) coincides with \( z \otimes 1 \) since \( z \) is \( H_0 \)-invariant. Then, in \( A_0 \otimes H^* \), we obtain \( \varphi(a) - z \otimes 1 \in A_0 \otimes J \). But \( J \) is nilpotent since \( H_0 \) contains the coradical of \( H \). Hence \( \varphi(a) - z \otimes 1 \) is nilpotent and, therefore,

\[
P_{B/Z(A_0)^{H_0}}(\varphi(a), t) = P_{B/Z(A_0)^{H_0}}(z \otimes 1, t) = (t - z)^m,
\]

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where \( m = [Z(A_0)^{H_0}] = [A_0 : Z(A_0)^{H_0}](\dim H) \). It follows that
\[
\left( \frac{m}{j} \right) z^j \in \nu(Z(A)^H) \quad \text{for all} \; j = 0, 1, \ldots, m.
\]
In particular, \( mz \in \nu(Z(A)^H) \). If \( \text{char} \; k = 0 \), this implies \( z \in \nu(Z(A)^H) \).

Assume that \( \text{char} \; k = p > 0 \). Let \( p^s \) be the largest power of \( p \) dividing \( m \). Taking \( j = p^s \), we have \( \left( \frac{m}{j} \right) \equiv 0 \pmod{p} \), which implies \( z^j \in \nu(Z(A)^H) \). \( \square \)

In [6, Proposition 3.1], it was assumed that \( H_0 \) coincides with the coradical of \( H \), but, clearly, the weaker assumption that \( H_0 \) contains the coradical of \( H \) is sufficient.

We have given a slightly different proof. The proof in [6] is based on the embedding of the simple \( H \)-module algebra \( Z(A/M) \otimes_{Z(A)^H} A \) into \( A/M \otimes H^s \), where \( M \) is any maximal ideal of \( A \) containing \( K \). Using this embedding, one can see that (ii) holds with \( p^s \) taken to be the largest power of \( p \) dividing the number
\[
[A/M : Z(A/M)] \cdot (\dim H).
\]
In other words, one obtains a possibly different value of \( s \).

**Corollary 8.2** (see [6]). Let \( H_0 \subset H \) be a Hopf subalgebra containing the coradical of \( H \). Assume that \( A \) is PI and prime (or at least \( H_0 \)-prime). If \( \text{char} \; k = 0 \), then
\[
Z(A)^{H_0} = Z(A)^H.
\]

**Proof.** The quotient ring \( Q = Q(A) \) is \( H_0 \)-simple and, therefore, Proposition 8.1 applies to the \( H \)-module algebra \( Q \) and its ideal \( K = 0 \). \( \square \)

When \( \text{char} \; k > 0 \), Eryashkin has investigated the relationship between the central invariants for \( H \) and for \( H_0 \) in \( H \)-prime PI algebras. The next result is a consequence of [6, Proposition 3.2].

**Theorem 8.3.** Assume that \( \text{char} \; k = p > 0 \). Let \( H_0 \subset H \) be a Hopf subalgebra containing the coradical of \( H \). Assume that \( A \) is PI and \( H \)-prime. Let \( A_0 = A/P_0 \), where \( P_0 \) is an \( H_0 \)-prime ideal of \( A \) containing no nonzero \( H \)-stable ideals of \( A \). If \( A_0 \) is integral over \( Z(A_0)^{H_0} \), then \( A \) is integral over \( Z(A)^H \).

In [6, Theorem 3.1], it was assumed additionally that \( H_0 \) is semisimple, which allows one to replace the condition that \( A_0 \) is integral over \( Z(A_0)^{H_0} \) by two weaker integrality assumptions for intermediate ring extensions.

**Corollary 8.4** (see [4]). Assume that \( \text{char} \; k = p > 0 \) and \( H \) is pointed. If \( A \) contains a prime ideal \( P \) such that the factor-algebra \( A/P \) is commutative and \( P \) contains no nonzero \( H \)-stable ideals of \( A \), then \( A \) is integral over \( Z(A)^H \).

**Proof.** Here the coradical \( H_0 \) of \( H \) is a group algebra \( kG \). For \( P_0 = \bigcap_{g \in G} gP \), the assumptions of Theorem 8.3 hold since \( A_0 = A/P_0 \) is commutative and, therefore, \( A_0 = Z(A_0) \) is integral over \( A_0^{H_0} = A_0^G \) by the classical theory. \( \square \)

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