HOLOMORPHIC SLICES, SYMPLECTIC REDUCTION AND MULTIPLICITIES OF REPRESENTATIONS

REYER SJAMAAR

ABSTRACT. I prove the existence of slices for an action of a reductive complex Lie group on a Kähler manifold at certain orbits, namely those orbits that intersect the zero level set of a momentum map for the action of a compact real form of the group. I give applications of this result to symplectic reduction and geometric quantization at singular levels of the momentum map. In particular, I obtain a formula for the multiplicities of the irreducible representations occurring in the quantization in terms of symplectic invariants of reduced spaces, generalizing a result of Guillemin and Sternberg.

INTRODUCTION

There has recently been much interest in formulas for multiplicities of Lie group representations arising in various different ways from group actions on manifolds. Typically, one can think of the manifold as being the phase space $M$ of a classical physical system acted upon by a group $G$ of symmetries, from which one obtains a unitary representation of $G$ through some kind of “quantization”. A prototype of such formulas is the multiplicity formula of Guillemin and Sternberg [11]. In their set-up the space $M$ is a compact Kähler manifold on which the compact group $G$ acts by holomorphic transformations, and the associated representation of $G$ is the space of holomorphic sections of a certain $G$-equivariant line bundle over $M$ (“geometric quantization”). The main result of [11] expresses the multiplicities of the irreducible components of this representation in terms of the Riemann-Roch numbers of the symplectic (or Marsden-Weinstein-Meyer) quotients $M_\lambda$ of $M$. The most important auxiliary result is that the symplectic quotient at the zero level, $M_0$, can be identified with a geometric quotient of $M$ by the complexified group $G^C$ as defined by Mumford [35].

The purpose of this paper is to generalize these results to the case where the symplectic quotient $M_\lambda$ is singular, a case which is of some interest in applications, but was excluded by Guillemin and Sternberg. This involves a closer study of the orbit structure of the action of the reductive group $G^C$ on $M$. The main technical result, expounded in Section 1, is that one can construct slices for the $G^C$-action at points that are in the zero level set of a momentum map. The proof of this holomorphic slice theorem utilizes Hörmander’s $L_2$-estimates for the Cauchy-Riemann operator. For affine algebraic manifolds it is a special case of Luna’s étale slice theorem [26].

Date: March 1993.

1991 Mathematics Subject Classification. Primary 58F06; Secondary 14L30, 19L10.

Key words and phrases. Symplectic reduction, geometric quantization, geometric invariant theory.

This work was partially supported by NSF grant DMS92-03398.
Kirwan [21] has introduced the notion of the Kähler quotient of \( M \) by \( G^\mathbb{C} \), which is the Kähler analogue of Mumford’s categorical quotient. It is roughly speaking defined as the space of closed \( G^\mathbb{C} \)-orbits in \( M \). Kirwan showed it is homeomorphic to the symplectic quotient \( M_0 \), generalizing the result of Guillemin and Sternberg referred to above. In [13] it was shown that \( M_0 \) is a so-called symplectic stratified space. In Section 2 I exploit the holomorphic slice theorem to study the analytic structure of Kirwan’s quotient and to compare it to the stratified symplectic structure of the symplectic quotient.

In the case of a projective manifold \( M \) endowed with the Fubini-Study symplectic form, Kirwan and, independently, Ness [37] showed that the symplectic quotient \( M_0 \) coincides with Mumford’s categorical quotient, the Proj of the invariant part of the homogeneous coordinate ring of \( M \). I show that the same conclusion holds when \( M \) has an arbitrary integral Kähler structure. (Under this hypothesis \( M \) has a unique algebraic structure by Kodaira’s Embedding Theorem, but the symplectic structure is not necessarily one coming from a Fubini-Study metric.) This result allows me to carry through the geometric quantization of the symplectic quotient. Now a theorem of Boutot [3] asserts that the singularities of a quotient such as \( M_0 \) are rational. This basically says that the Riemann-Roch numbers of \( M_0 \) are equal to those of any blowup, which, finally, leads to a generalization of Guillemin and Sternberg’s multiplicity formula.

Acknowledgement. I am grateful to Hans Duistermaat for suggesting to me the problem discussed in this paper and for showing me an unpublished manuscript of his, a succinct version of which appeared in [5]. Eugene Lerman has been a great help in carrying out this work. Part of it appears as a joint announcement in [25]. I would also like to thank Eugenio Calabi and Charlie Epstein for their generous help with the material in Section 1.2. I have furthermore benefited from helpful discussions with Victor Guillemin and Viktor Ginzburg.

1. Holomorphic Slices

Let \( X \) be a complex space and let \( G^\mathbb{C} \) be a reductive complex Lie group acting holomorphically on \( X \). I think of \( G^\mathbb{C} \) as being the complexification of a compact real Lie group \( G \).

**Definition 1.1.** A slice at \( x \) for the \( G^\mathbb{C} \)-action is a locally closed analytic subspace \( S \) of \( X \) with the following properties:

1. \( x \in S \);
2. \( G^\mathbb{C} \cdot S \) of \( S \) is open in \( X \);
3. \( S \) is invariant under the action of the stabilizer \( (G^\mathbb{C})_x \);
4. the natural \( G^\mathbb{C} \)-equivariant map from the associated bundle \( G^\mathbb{C} \times_{(G^\mathbb{C})_x} S \) into \( X \), which sends an equivalence class \([g, y]\) to the point \( gy \), is an analytic isomorphism onto \( G^\mathbb{C} \cdot S \).

It follows from (1) that for all \( y \in S \) the stabilizer \( (G^\mathbb{C})_y \) is contained in \( (G^\mathbb{C})_x \). Furthermore, if \( X \) is nonsingular at \( x \), a slice \( S \), if it exists, is nonsingular at \( x \) and transverse to the orbit \( G^\mathbb{C} \cdot x \).

The problem of constructing slices has been solved by Luna [26] for affine varieties and by Snow [44] for Stein spaces. One difficulty of the problem lies in the fact that an action of \( G^\mathbb{C} \) is typically not proper, unless it is locally free. One therefore faces the challenge of controlling the behaviour of the action “at infinity.
in the group”. Another snag is that there may be cohomological obstructions to analytically embedding the “normal bundle” $G^C \times (G^C)_x \times S$ of the orbit $G^C x$ into $X$. These obstructions vanish if the orbit is (analytically isomorphic to) an affine variety. A theorem of Matsushima’s [24] says that a $G^C$-orbit is affine if and only if the isotropy subgroup $(G^C)_x$ is reductive. But even if the isotropy of $x$ is reductive one cannot always construct a slice at $x$. (Cf. Richardson’s example [25, 26, 39] of the standard action of $SL(2)$ on homogeneous cubic polynomials, and also Trautman [40].) The additional condition that Luna, resp. Snow, impose in the context of an affine variety $X$, resp. a Stein space $X$, in order to deduce the existence of a slice is that the orbit should be closed in $X$.

The above notion of a slice is slightly weaker than that of Luna and Snow, who require the set $G^C S$ to be saturated with respect to a quotient mapping. In our context the definition of a quotient depends upon the choice of a momentum map. In the next section we shall see that for any choice of a momentum map there always exists slices $S$ such that $G^C S$ is saturated with respect to the corresponding quotient map (Proposition 1.4).

In this section I demonstrate the existence of slices at certain affine orbits of a $G^C$-action on a Kähler manifold. I was led to this result by the striking resemblance between Luna’s and Snow’s slice theorems and the normal forms in symplectic geometry due to Marle [28] and Guillemin and Sternberg [12]. Before formulating the theorem I have to state a number of definitions and auxiliary results. In Section 1.1 I discuss momentum maps on Kähler manifolds and the notion of orbital convexity. Section 1.2 contains a result concerning interpolation between Kähler metrics in the neighborhood of a totally real submanifold of a complex manifold, which relies on Hörmander’s $\bar{\partial}$-estimates, and which is the main ingredient in the proof of the slice theorem. In Section 1.3 I prove the slice theorem and discuss some of its immediate consequences.

1.1. Orbital convexity and isotropic orbits. Recall that the decomposition of the complexified Lie algebra $g^C = g \oplus \mathbb{C}$ into a direct sum $g^C = g \oplus \sqrt{-1} g$ gives rise to the polar (or Cartan) decomposition $G^C = G \cdot \exp \sqrt{-1} g$. The map $G \times \sqrt{-1} g \to g^C$ sending $(k, \sqrt{-1} \xi)$ to $k \exp \sqrt{-1} \xi$ is a diffeomorphism onto, and every element $g$ of $G^C$ can be uniquely decomposed into a product $g = k \exp \sqrt{-1} \xi$, with $k \in G$ and $\xi \in g$.

**Definition 1.2** (Heinzner [10]). A subset $A$ of a $G^C$-space $X$ is called orbitally convex with respect to the $G^C$-action if it is $G$-invariant and if for all $x \in U$ and all $\xi \in g$ the intersection of the curve $\{ \exp(\sqrt{-1} t \xi) x : t \in \mathbb{R} \}$ with $A$ is connected. Equivalently, $A$ is orbitally convex if and only if it is $G$-invariant and for all $x \in A$ and all $\xi \in g$ the fact that both $x$ and $\exp(\sqrt{-1} t \xi) x$ are in $A$ implies that $\exp(\sqrt{-1} t \xi) x \in A$ for all $t \in [0, 1]$.

**Remark 1.3.** If $f : X \to Y$ is a $G^C$-equivariant map between $G^C$-spaces $X$ and $Y$, and $C$ is an orbitally convex subset of $Y$, then it follows immediately from the definition that $f^{-1}(C)$ is orbitally convex in $X$.

A $G$-equivariant map defined on an orbitally convex open set can be analytically continued to a $G^C$-equivariant map.

**Proposition 1.4** (Heinzner [10], Koras [23]). Let $X$ and $Y$ be complex manifolds acted upon by $G^C$. If $A$ is an orbitally convex open subset of $X$ and $f : A \to Y$
is a $G$-equivariant holomorphic map, then $f$ can be uniquely extended to a $G^C$-equivariant holomorphic map $f^C: G^C A \to Y$.

Consequently, if the image $f(A)$ is open and orbitally convex in $Y$ and $f: A \to f(A)$ is biholomorphic, then the extension $f^C: G^C A \to Y$ is biholomorphic onto the open subset $G^C f(A)$.

Proof. The only way to extend $f$ equivariantly is by putting $f^C(g \exp(\sqrt{-\imath} \xi)x) = g\exp(\sqrt{-\imath} \xi)f(x)$ for all $x \in A$, $g \in G$ and $\xi$ in $\mathfrak{g}$. We have to check this is well-defined.

Let $x \in A$ and $\xi \in \mathfrak{g}$ be such that $\exp(\sqrt{-\imath} \xi)x \in A$. Then by assumption $\exp(\sqrt{-\imath} t \xi)x \in A$ for all $t$ between 0 and 1. So $f(\exp(\sqrt{-\imath} t \xi)x)$ is well-defined for $0 \leq t \leq 1$. Define the curves $\alpha(t)$ and $\beta(t)$ in $Y$ by $\alpha(t) = f(\exp(\sqrt{-\imath} t \xi)x)$ and $\beta(t) = \exp(\sqrt{-\imath} t \xi)f(x)$ for $0 \leq t \leq 1$. Then $\alpha(t)$ and $\beta(t)$ are integral curves of the vector fields $f^* (\sqrt{-\imath} \xi)_X$ and $\sqrt{-\imath} \xi_Y$ respectively, both with the same initial value $f(x)$. Now since $f$ is $G$-equivariant we have $f^*_\xi x = \xi_Y$, and, because $f$ is also holomorphic, $f^*_\xi (\sqrt{-\imath} \xi)_X = f^*_\xi J X = J f^*_\xi X = J \xi Y = \sqrt{-\imath} \xi_Y$. Hence $\alpha(t) = \beta(t)$, in other words $f(\exp(\sqrt{-\imath} t \xi)x) = \exp(\sqrt{-\imath} t \xi)f(x)$ for $0 \leq t \leq 1$.

It follows that for all $x \in A$ and all $\xi$ in $\mathfrak{g}$ such that $\exp(\sqrt{-\imath} \xi)x \in A$ we have $f(\exp(\sqrt{-\imath} \xi)x) = \exp(\sqrt{-\imath} \xi)f(x)$. It is easy to deduce from this that $f^C$ is well-defined.

Finally observe that if the image $f(A)$ is open and orbitally convex in $Y$ and $f: A \to f(A)$ is biholomorphic, then the inverse $f^{-1}$ also has a holomorphic extension $(f^{-1})^C: G^C f(A) \to G^C A$, and by uniqueness this must be the inverse of $f^C$. 



Remark 1.5. Suppose we drop the assumption that $A$ is orbitally convex from the statement of the proposition. Then it is not necessarily true that $f(\exp(\sqrt{-\imath} t \xi)x)$ is equal to $\exp(\sqrt{-\imath} t \xi)f(x)$ for all $t$ such that $\exp(\sqrt{-\imath} t \xi)x \in A$. But if we put $I = \{ t \in \mathbb{R} : \exp(\sqrt{-\imath} t \xi)x \in A \}$ and let $I^0$ be the connected component of $I$ containing 0, then the above proof shows that $f(\exp(\sqrt{-\imath} t \xi)x) = \exp(\sqrt{-\imath} t \xi)f(x)$ for all $t$ in $I^0$.

In the remainder of this section $M$ shall denote a Kähler manifold, not necessarily compact, with infinitely differentiable Kähler metric $ds^2$, Kähler form $\omega = -\text{Im} \, ds^2$, and complex structure $J$. Then $\text{Re} ds^2 = \omega(\cdot, J \cdot)$ is the corresponding Riemannian metric. We may assume without loss of generality that $ds^2$ is invariant under the compact group $G$. So the transformations on $M$ defined by $G$ are holomorphic and they are isometries with respect to the Kähler metric. The action of $G$ is called Hamiltonian if for all $\xi$ in the Lie algebra $\mathfrak{g}$ of $G$ the vector field $\xi_M$ on $M$ induced by $\xi$ is Hamiltonian. In this case we have a momentum map $\Phi$ from $M$ to the dual $\mathfrak{g}^*$ of the Lie algebra of $G$ with the property that $\d\Phi^\xi = \iota_{\xi_M} \omega$ for all $\xi$. Here $\Phi^\xi$ is the $\xi$-th component of $\Phi$, defined by $\Phi^\xi(m) = (\Phi(m))(\xi)$. After averaging with respect to the given action on $M$ and the coadjoint action on $\mathfrak{g}^*$ we may assume that the map $\Phi$ is $G$-equivariant. An equivariant momentum map is uniquely determined up to additive constants ranging over the $\text{Ad}^\mathfrak{g}^*$-fixed vectors in $\mathfrak{g}^*$. (So if $G$ is connected, the number of degrees of freedom is equal to the dimension of the centre of $G$.) It is easy to give sufficient conditions for the
existence of a momentum map, for instance, the first Betti number of $M$ is zero, or the \( \text{\textit{Kähler}} \) form $\omega$ is exact. (See e.g. [13, 17].) More surprisingly, by a theorem of Frankel [4] a momentum map always exists if the action has at least one fixed point and $M$ is \( \text{\textit{compact}} \). A necessary and sufficient condition for a holomorphic $G$-action on a compact \( \text{\textit{Kähler}} \) manifold to be Hamiltonian is that for every vector $\xi \in g$ the holomorphic vector field $\xi_M$ should be killed by every global holomorphic one-form $\alpha$ on $M$, $\alpha(\xi_M) = 0$. (This follows from Frankel’s theorem and a fixed point theorem of Sommese [45].) Note that this condition is independent of the \( \text{\textit{Kähler}} \) structure.

If $M$ is $\mathbb{C}^n$ with the standard Hermitian structure $dS^2 = \sum dz_i \otimes dz_i$ and the standard symplectic form $\Omega = \sqrt{-1}/2 \sum dz_i \wedge d\bar{z}_i$, then a momentum map $\Phi_{\mathbb{C}^n}$ is given by the formula

$$\Phi_{\mathbb{C}^n}(v) = 1/2 \Omega(\xi_{\mathbb{C}^n}(v), v), \tag{1.1}$$

where $\xi_{\mathbb{C}^n}$ denotes the image of $\xi \in g$ in the Lie algebra $\mathfrak{sp}(\mathbb{C}^n, \Omega)$, and $v \in \mathbb{C}^n$.

Because $G$ acts holomorphically on $M$, there is a natural way to define an action of the complexified Lie algebra $g^*$: For any $\xi$ in $g$ the vector field $(\sqrt{-1}\xi)_M$ induced by $\sqrt{-1}\xi$ is equal to $J\xi_M$. It follows easily from the definition of a momentum map that $J\xi_M$ is equal to the gradient vector field (with respect to the Riemannian metric $\text{Re} \; ds^2$) of the $\xi$-th component of the momentum map,

$$(\sqrt{-1}\xi)_M = J\xi_M = \text{grad} \Phi^\xi. \tag{1.2}$$

We will assume that these vector fields are \( \text{\textit{complete}} \) for all $\xi$ in $g$. This assumption implies that the action of $G$ extends uniquely to a holomorphic action of $G^C$. The assumption holds for instance if $M$ is compact, or if $M$ is the total space of a vector bundle over a compact manifold on which $G$ acts by vector bundle transformations. The identity (1.2) will enable us to gain control over the behaviour of the action “at infinity in the group”. For one thing, it implies that the trajectory $\gamma(t)$ of grad $\Phi^\xi$ through a point $x$ in $M$ is given by $\gamma(t) = \exp(\sqrt{-1}\xi)t)x$, which does not depend on the choice of the \( \text{\textit{Kähler}} \) metric or the momentum map.

Here is another application of (1.2). A submanifold $X$ of $M$ is called \( \text{\textit{totally real}} \) if $T_xX \cap J(T_xX) = \{0\}$ for all $x \in X$.

**Proposition 1.6.** Assume $G$ is connected. Consider the following conditions on a point $m \in M$:

1. $\Phi(m)$ is fixed under the coadjoint action of $G$ on $g^*$;
2. The orbit $Gm$ is isotropic with respect to the \( \text{\textit{Kähler}} \) form;
3. The complex stabilizer $(G^C)_m$ of $m$ is equal to the complexification $(G_m)^C$ of the compact stabilizer $G_m$, $(G_m)^C = (G^C)_m$;
4. The $G$-orbit through $m$ is totally real.

Conditions (1) and (2) are equivalent. Any one of these conditions implies (3); and (2) implies (4).

**Proof.** Put $\mu = \Phi(m)$. Let $G_\mu$ be the stabilizer of $\mu$ with respect to the coadjoint action; it is well-known that $G_\mu$ is a connected subgroup of $G$. Let $G_\mu^C$ be the coadjoint orbit through $\mu$. We regard $G_\mu$ as a symplectic manifold with the Kirillov-Kostant-Souriau symplectic form. Denote the tangent space $T_m(G_\mu m)$ to the compact orbit by $m$; then $m$ is isomorphic to $g/\mathfrak{g}_m$ as an $H$-module. Similarly, let $n$ denote the tangent space $T_m(G_\mu m)$ to the orbit $G_\mu m$; then $n \cong \mathfrak{g}_m/\mathfrak{g}_m$. We
have a fibration
\[ G_m \hookrightarrow Gm \xrightarrow{\Phi} G\mu, \]
which on the tangent level leads to an exact sequence of vector spaces
\[ 0 \to n \to m \overset{\partial}{\to} T\mu(\Phi\mu) \to 0. \]

The restriction of the symplectic form ω to m is an alternating bilinear (“presymplectic”) form. It follows from the fact that \( \Phi \) is a Poisson map that \( d\Phi \) preserves the presymplectic forms. Since \( T\mu(G\mu) \) is symplectic, \( n \) is exactly the nullspace of \( \omega|_m \). Therefore, \( m \) is isotropic if and only if \( T\mu(G\mu) = 0 \), that is, \( G\mu = G \), in other words, \( \mu \) is \( G \)-fixed. This shows that (1) is equivalent to (2).

We now prove (1) implies (3). It is easy to see that for any point \( m \) in \( M \) the complex stabilizer \((G^C)_m\) contains the complexification \((G_m)^C\) of the compact stabilizer \( G_m \). Now suppose \( \mu \) is fixed under the coadjoint action. Let \( g \exp \sqrt{-1}\xi \) be an arbitrary element of \( \text{dim}(G^C)_m \), where \( g \in G \) and \( \xi \in g \). We want to show that \( g \in G_m \) and \( \xi \in \mathfrak{g}_m \). (Cf. Kirwan \cite{kirwan} for this part of the argument.) By \( G \)-equivariance of the momentum map we have
\[ \Phi(\exp(\sqrt{-1}\xi)m) = g^{-1}\Phi(g \exp(\sqrt{-1}\xi)m) = g^{-1}\Phi(m) = g^{-1}\mu = \mu, \]
and therefore \( \Phi^\xi(\exp(\sqrt{-1}\xi)m) = \Phi^\xi(m) \). By (1.2) the curve \( \exp(\sqrt{-1}t\xi)m \) is the gradient trajectory of the vector field \((\sqrt{-1}\xi)_M \) through \( m \). So the function \( \Phi^\xi \) is increasing along this curve, and it is strictly increasing if and only if \( m \) is not a fixed point of \((\sqrt{-1}\xi)_M \). But it takes on the same values at \( t = 0 \) and \( t = 1 \), so \( m \) must be a fixed point of \((\sqrt{-1}\xi)_M \), that is, \( \xi \in \mathfrak{g}_m \). Hence \( gm = g \exp(\sqrt{-1}\xi)m = m \), so \( g \in G_m \).

Lastly we show (3) implies (4). If \((G_m)^C = (G^C)_m \), the (real) dimension of the complex orbit \( G^C_m \) equals twice the dimension of the compact orbit \( Gm \). Since the tangent space at \( m \) to \( G^C_m \) is equal to \( T_m(Gm) \oplus J(T_m(Gm)) \), the intersection \( T_m(Gm) \cap J(T_m(Gm)) \) has to be 0, that is, \( Gm \) is totally real.

\begin{remark}
1.7. The converse of the implications in the proposition are wrong. See \cite{kirwan} for a simple counterexample. If \( G \) is not connected, then the proof of the proposition shows the following implications hold: (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4). Moreover, if \( Gm \) is isotropic, then \( \Phi(m) \) is fixed under \( G^\mathfrak{a} \), where \( G^\mathfrak{a} \) denotes the component of the identity of \( G \).
\end{remark}

Let \( V \) be the orthogonal complement of \( T_m(G^C_m) \). Then using the notation of the proposition we have an \( H \)-invariant orthogonal direct sum decomposition of the tangent space:
\[ T_mM = n \oplus \mathfrak{J}n \oplus T_\mu(\Phi\mu) \oplus \mathfrak{U}. \tag{1.3} \]

This decomposition is symplectic in the sense that the summands \( n \oplus \mathfrak{J}n, T_\mu(G\mu) \) and \( V \) are symplectic subspaces, but it is not a complex-linear decomposition, since \( T_\mu(G\mu) \) need not be \( J \)-invariant.

\subsection{Interpolation of Kähler metrics near totally real submanifolds.}

Let \( M \) be a complex manifold and let \( \sigma \) be a real-valued \( C^\infty \) closed \((1, 1)\)-form on \( M \). On any sufficiently small open subset \( O \) of \( M \) we can find a potential for \( \sigma \), that is, a smooth real-valued function \( u \) defined on \( O \) such that \( \sigma = \sqrt{-1} \partial\bar{\partial}u \). Now let \( X \) be any real-analytic totally real submanifold of \( M \).
Theorem 1.8. If the form $\sigma$ is exact in a neighbourhood of $X$, then there exists a potential for $\sigma$ defined in a (possibly smaller) neighbourhood of $X$.

Assume $\sigma$ vanishes to $m$-th order on $X$, that is, in local coordinates its coefficients vanish to $m$-th order on $X$. Then $\sigma$ is exact near $X$, and there exists a potential for $\sigma$ defined near $X$ which vanishes to $(m+2)$-nd order on $X$.

Proof. Because $X$ is totally real, it has a basis of Stein tubular neighbourhoods in $M$. (See Grauert [8] and Reese and Wells [14, Theorem 2.2].) Without loss of generality we may replace $M$ by one of these tubular neighbourhoods. Let $\alpha$ be a solution to the equation

$$d\alpha = \sigma,$$

and let $\beta = \alpha^{01}$ be the $(0,1)$-part of $\alpha$. It is evident from the fact that $\sigma$ is of bidegree $(1,1)$ that $\bar{\partial}\beta = 0$. So we can solve the equation

$$\sqrt{-1}\bar{\partial}f = \beta,$$

since $M$ is a Stein manifold. It is easy to check that the function $u = f + \bar{f} = 2 \text{Re } f$ satisfies $\sigma = \sqrt{-1} \bar{\partial}\bar{\partial}u$. This proves the first statement.

Now suppose $\sigma$ vanishes to $m$-th order on $X$. Then, evidently, the restriction of $\sigma$ to $X$ is zero. Since $\sigma$ is also closed, it follows easily from De Rham’s Theorem that it is exact on the tubular neighbourhood $M$. Let $C: M \times [0,1] \to M$ be the homotopy defined by $C(p,t) = tp$, which contracts the bundle $M$ to the zero section along the fibres. There exists a very special solution to (1.4), namely the form $\alpha = \int_{[0,1]} C^*\sigma$. It is not hard to check that this form vanishes to $(m+1)$-st order on $X$. Therefore its $(0,1)$-part $\beta = \alpha^{01}$ also vanishes to $(m+1)$-st order on $X$. We now want to solve (1.3) augmenting the order of vanishing by one. We do this in three steps. First, we solve the problem locally and formally. That is, we assume $X$ is an open set in $\mathbb{R}^k$ and $M$ a strictly pseudoconvex open neighbourhood of $X$ in $\mathbb{C}^n$ (where $n \geq k$), and find a function $g$ vanishing to $(m+2)$-nd order on $X$ such that the $(0,1)$-form $\beta' = \beta - \sqrt{-1}\bar{\partial}g$ also vanishes to order $m+2$ on $X$. Secondly, and this is the crucial point, we use Hörmander’s $L^2$-estimates for the Cauchy-Riemann operator $\bar{\partial}$ to show that locally there exists a smooth solution $g'$ to the problem $\sqrt{-1}\bar{\partial}g' = \beta'$ which vanishes to $(m+2)$-nd order on $X$. Then the function $f = g + g'$ satisfies (1.4) and vanishes to $(m+2)$-nd order on $X$. Thirdly, we show that the local solutions to the problem can be glued together to obtain a global solution, which amounts to solving a Cousin type problem.

Step 1. A complexification $(X^C, i)$ of the real-analytic manifold $X$ is a complex manifold $X^C$ together with a real-analytic map $i: X \to X^C$ such that for every complex manifold $V$ and every real-analytic map $j: X \to V$ there exists, for a sufficiently small open neighbourhood $O$ of $i(X)$, a unique complex-analytic map $j^C: O \to V$ with $j^C \circ i = j$. The uniqueness of the complexification (more precisely, the uniqueness of the germ of $X^C$ at $i(X)$) is immediate from the definition; the existence was proven by Bruhat and Whitney [15]. The map $i$ is actually a closed embedding. If $j: X \to V$ is an embedding and the image $j(X)$ is totally real, the complexified map $j^C$ is an embedding (near $i(X)$).

So after shrinking the tube $M$ if necessary, we may assume that we have inclusions $X \subset X^C \subset M$. About every point of $X$ we can find an open neighbourhood that can be biholomorphically identified with a strictly pseudoconvex bounded open subset $U$ of $\mathbb{C}^k \times \mathbb{C}^l$, in such a manner that $U \cap X$ is given by the equations $w = y = 0$
and $U \cap X^C$ by $w = 0$. Here we write a point in $\mathbb{C}^k \times \mathbb{C}^l$ as a pair $(z, w)$ with $z = x + \sqrt{-1}y \in \mathbb{C}^k$ and $w \in \mathbb{C}^l$. We shall call a neighbourhood with such a coordinate system a distinguished neighbourhood.

Write $\beta = \sum_{\lambda=1}^k \zeta^\lambda d\bar{z}_\lambda + \sum_{\lambda=1}^l \eta^\lambda d\bar{w}_\lambda$ and consider the Taylor expansions at $X$ of the components $\zeta^\lambda$ and $\eta^\lambda$:

$$
\zeta^\lambda(x, y, w, \bar{w}) \sim \sum_{|I|+|J|+|K| \geq m+2} \zeta^\lambda_{I,J,K}(x)y^I w^J \bar{w}^K,
$$

$$
\eta^\lambda(x, y, w, \bar{w}) \sim \sum_{|I|+|J|+|K| \geq m+2} \eta^\lambda_{I,J,K}(x)y^I w^J \bar{w}^K,
$$

with coefficients $\zeta^\lambda_{I,J,K}$ and $\eta^\lambda_{I,J,K}$ in $C^\infty(X, \mathbb{C})$. Here $I$, $J$ and $K$ are multi-indices and $|I|$ denotes the norm $\sum i_\lambda$ of $I = (i_1, \ldots, i_k)$. The fact that $\bar{\partial}\beta = 0$ amounts to:

$$
\frac{\partial \zeta^\lambda}{\partial \bar{z}_\mu} = \frac{\partial \zeta^\mu}{\partial \bar{z}_\lambda}, \quad \frac{\partial \eta^\lambda}{\partial \bar{z}_\mu} = \frac{\partial \zeta^\mu}{\partial \bar{w}_\lambda}, \quad \frac{\partial \eta^\lambda}{\partial \bar{w}_\mu} = \frac{\partial \eta^\mu}{\partial \bar{w}_\lambda}.
$$

Plugging the Taylor expansions of $\zeta^\lambda$ and $\eta^\lambda$ into this system of equations and inspecting the lowest-order terms in the resulting equalities yields the following identities:

$$
(i_\mu + 1)\zeta^\lambda_{I+e_\mu,J,K} = (i_\lambda + 1)\zeta^\mu_{I,J+e_\lambda,K},
$$

$$
\sqrt{-1}(i_\mu + 1)\eta^\lambda_{I+e_\mu,J,K} = (k_\lambda + 1)\eta^\mu_{I,J,K+e_\lambda},
$$

$$
(k_\mu + 1)\eta^\lambda_{I,J,K+e_\mu} = (k_\lambda + 1)\eta^\mu_{I,J,K+e_\lambda},
$$

for all $I$, $J$ and $K$ such that $|I| + |J| + |K|$ equals $m + 1$. Here $e_\lambda$ denotes the multi-index whose entries are all $0$, except the $\lambda$-th, which is $1$. (For higher-order terms there are similar identities, but we will not need them.)

Our object is to find a smooth function $g$ such that $\sqrt{-1}\bar{\partial}g = \beta$ up to terms of total degree $\geq m + 3$ in $y$, $w$ and $\bar{w}$. Upon substitution of the Taylor expansion of $g$,

$$
g(x, y, w, \bar{w}) \sim \sum_{|I|+|J|+|K| \geq m+3} g_{I,J,K}(x)y^I w^J \bar{w}^K,
$$

we see this amounts to solving the equations

$$
g_{I+e_\mu,J,K} = -\frac{2\zeta^\lambda_{I,J,K}}{i_\lambda + 1}, \quad g_{I,J,K+e_\lambda} = \frac{2\eta^\lambda_{I,J,K}}{\sqrt{-1}(k_\lambda + 1)},
$$

for all $I$, $J$ and $K$ such that $|I| + |J| + |K| = m + 2$. There are no conditions on the terms of degree $\geq m + 3$ in the expansion of $g$. It is a straightforward exercise to check that the equations (1.7) admit solutions $g_{I,J,K}$, where $|I| + |J| + |K| = m + 3$. So if we put $g(x, y, w, \bar{w}) = \sum_{|I|+|J|+|K| = m+3} g_{I,J,K}(x)y^I w^J \bar{w}^K$, then $g$ is a smooth function defined on $U$ and vanishing to $(m + 2)$-nd order on $X$, and the $(0, 1)$-form $\beta' = \beta - \sqrt{-1}\bar{\partial}g$ also vanishes to order $m + 2$ on $X$.

Step 2. Obviously the form $\beta'$ is $\bar{\partial}$-closed. We now want to find a smooth solution $g'$ defined on $U$ to the problem $\sqrt{-1}\bar{\partial}g' = \beta'$ together with an order estimate. But first note that every locally square integrable solution to this equation is actually smooth. This follows from the fact that there exists a smooth solution (Corollary 4.2.6), and that the difference of any two solutions is a $\bar{\partial}$-closed function, therefore harmonic, and therefore smooth by the ellipticity of the Laplacian on $\mathbb{C}^n$. 

Let \( \rho \) be the distance squared to the submanifold \( X \), \( \rho(z, w) = ||y||^2 + ||w||^2 \).
Because \( \beta' \) vanishes to order \( m + 2 \) on \( X \), the integral \( \int_U |\beta'|^2 r^{-r} dzd\bar{z}dwd\bar{w} \) is finite for all \( r < k + 2l + m + 3 \). In Hörmander’s paralogue, \( \beta' \) is an element of the weighted \( L^2 \)-space \( L^2_{(0,1)}(U, \phi) \) with weight \( \phi = r \log \rho \). It is easy to check that for every positive \( r \) the weight function \( \phi \) is pluri-subharmonic. Let us take \( r = k + 2l + m + 2 \). By Theorem 4.4.2 of Hörmander [17], we can find a solution to the equation \( \sqrt{-1} \partial g' = \beta' \) such that
\[
\int_U |g'|^2 e^{-\phi}(1 + ||z||^2 + ||w||^2)^{-2} dzd\bar{z}dwd\bar{w} \leq \int_U |\beta'|^2 e^{-\phi} dzd\bar{z}dwd\bar{w} < \infty.
\]
But \( g' \) is smooth as noted before, so this is only possible if \( g' \) vanishes to order \( \geq r - k - 2l = m + 2 \) on \( X \). The function \( f = g + g' \) defined on \( U \) satisfies (1.5) and vanishes to order \( m + 2 \) on \( X \).

**Step 3.** Let \( \{U_i\} \) be a Stein cover of \( M \). By the previous result we can find smooth functions \( f_i \) defined on \( U_i \), which vanish to \((m + 2)\)-nd order on \( U_i \cap X \) and satisfy \( \sqrt{-1} \partial f_i = \beta U_i \). Put \( f_{ij} = f_j - f_i \); then \( \partial f_{ij} = 0 \), so \( f_{ij} \) is a holomorphic function on \( U_{ij} = U_i \cap U_j \). It also vanishes to \((m + 2)\)-nd order on \( U_{ij} \cap X \), so, by Lemma 1.9 below, it has to vanish to order \( m + 2 \) on \( U_{ij} \cap V \), where \( V \) denotes the complexification \( X^C \) of \( X \). In other words, the collection of \( f_{ij} \)'s defines a Čech 1-cocycle with coefficients in the coherent sheaf \( I^2_{V^C} \), where \( I_V \) denotes the ideal sheaf of the complex submanifold \( V \). Since \( M \) is Stein, Cartan’s Theorem B implies this cocycle is a coboundary (cf. [17, Theorem 7.4.3]), so there exist holomorphic functions \( g_i \in \Gamma(U_i, I^2_{V^C}) \) such that \( f_{ij} = g_j - g_i \). Consider the smooth functions \( f_i + g_i \) defined on \( U_i \). Clearly \( f_i + g_i = f_j + g_j \) on \( U_{ij} \), so \( f_i + g_i = f|_{U_i} \) for a global smooth function \( f \). By construction \( f \) vanishes to \((m + 2)\)-nd order on \( X \), and because the \( g_i \)'s are holomorphic, we have \( \sqrt{-1} \partial f = \beta \).

The proof of the theorem used the following little lemma.

**Lemma 1.9.** Suppose \( f \) is a holomorphic function on \( M \) vanishing to \( m \)-th order on the totally real submanifold \( X \). Then \( f \) vanishes to \( m \)-th order on the complexification \( X^C \subset M \) of \( X \).

**Proof.** We compute in a distinguished system of coordinates \( (z, w) \), writing \( z = x + \sqrt{-1} y \), as in the proof of Theorem 1.8. First we prove the statement for \( m = 0 \). So suppose \( f \) vanishes on \( X \); we have to show it vanishes on \( X^C \). The assumption implies that the partial derivatives of \( f \) along \( X \), \( \partial^{|I|} f/\partial x_I \), vanish identically on \( X \) for all multi-indices \( I \). Since \( \partial^{|I|}/\partial z_I = \partial^{|I|}/\partial x_I \) on holomorphic functions, we conclude that the power series of the restriction \( f|_{X^C} \) of \( f \) to \( X^C \) is trivial at any point of \( X \). By the identity principle \( f|_{X^C} = 0 \).

Now suppose \( f \) vanishes to order \( m > 0 \) on \( X \). This means the holomorphic functions \( \partial^{|I|+|J|} f/\partial z_I \partial w_J \) vanish identically on \( X \) for all \( I \) and \( J \) with \( |I| + |J| \leq m \). Then by the previous result \( \partial^{|I|+|J|} f/\partial z_I \partial w_J = 0 \) on \( X^C \) if \( |I| + |J| \leq m \), so \( f \) vanishes to order \( m \) on \( X^C \).
property: For all open \( U_1 \) with \( X \subset U_1 \subset U \) there exist an open subset \( U_2 \) with \( X \subset U_2 \subset U_1 \) and a smooth Kähler metric \( d\tilde{s}^2 \) on \( M \) such that \( d\tilde{s}^2 = dS^2 \) on \( U_2 \) and \( d\tilde{s}^2 = ds^2 \) on \( M \setminus U_1 \).

In the presence of a compact group \( G \) of holomorphic transformations on \( M \) leaving the submanifold \( X \) and the metrics \( dS^2 \) and \( ds^2 \) invariant, the metric \( d\tilde{s}^2 \) may be taken to be invariant. If the \( G \)-action is Hamiltonian with respect to the Kähler form \( -\text{Im} \, ds^2 \), it is Hamiltonian with respect to the Kähler form \( -\text{Im} \, d\tilde{s}^2 \).

**Proof.** For the first part of the theorem we may again assume that \( M \) is a Stein tubular neighbourhood of \( X \). Let \( \Omega = -\text{Im} \, dS^2 \) and \( \omega = -\text{Im} \, ds^2 \) be the Kähler forms corresponding to \( dS^2 \) and \( ds^2 \), and put \( \sigma = \omega - \Omega \). Then \( \sigma \) vanishes to order 0 on \( X \), so by Theorem 1.13 there exists a smooth function \( u \) vanishing to second order on \( X \) such that \( \sigma = \sqrt{-1} \partial \bar{\partial} u \). Let \( \rho \) be the square of some distance function on the tube \( M \). Then \( \rho \) vanishes to first order on \( X \), so \( u \) is of order \( O(\rho^{3/2}) \) as \( \rho \) tends to zero. Let \( \chi : \mathbb{R} \to [0,1] \) be a smooth function with \( \chi(t) = 0 \) for \( t \leq 1 \) and \( \chi(t) = 1 \) for \( t \geq 2 \). For \( \lambda > 0 \) define a smooth function \( \varepsilon \) on \( M \) by \( \varepsilon(x) = \chi\left(\rho(x)/\lambda^2\right) \). Put \( M_\rho = \{ x \in M : \rho(x) < \rho \} \) and define a smooth two-form \( \tilde{\omega} \) on \( M \) by

\[
\tilde{\omega} = \begin{cases}
\Omega + \sqrt{-1} \partial \bar{\partial} \varepsilon u & \text{on } M_{3\lambda^2}, \\
\omega & \text{on } M - M_{2\lambda^2}.
\end{cases}
\]

Then on \( M_{3\lambda^2} \) the form \( \tilde{\omega} \) is equal to \( \Omega \). On \( M_{3\lambda^2} \) we have \( \tilde{\omega} - \omega = \sqrt{-1} \partial \bar{\partial} (\varepsilon - 1) u \).

In a distinguished neighbourhood \( U \) of a point of \( X \) with coordinates \( v = (z, w) \) we can write \( \partial \bar{\partial} (\varepsilon - 1) u = \sum_{\alpha,\beta=1}^n f_{\alpha\beta} \, dv_\alpha \wedge d\bar{v}_\beta \) with

\[
f_{\alpha\beta}(v) = \frac{\partial^2}{\partial v_\alpha \partial \bar{v}_\beta} \left((\varepsilon(v) - 1) u(v)\right).
\]

By carrying out the differentiation one can check in a straightforward manner that for every compact subset \( K \) of \( U \) the supremum of \( |f_{\alpha\beta}(v)| \) over all \( v \in K \cap M_{3\lambda^2} \) is of order \( O(\lambda) \) as \( \lambda \) tends to zero. For instance, one of the terms involved in \( f_{\alpha\beta} \) is

\[
\frac{\chi''(\rho/\lambda^2)}{\lambda^4} \frac{\partial \rho}{\partial v_\alpha} \frac{\partial \rho}{\partial \bar{v}_\beta} u = \frac{\chi''(\rho/\lambda^2)}{\lambda^4} O(\rho^{5/2}),
\]

where we used \( u = O(\rho^{3/2}) \) and the fact that the first derivatives of \( \rho \) are of order \( O(\rho^{1/2}) \) as \( \rho \) tends to zero, since they vanish on \( X \). Since \( \chi''(\rho/\lambda^2) = 0 \) for \( \rho \geq 2\lambda^2 \), we have

\[
\sup_{\rho \leq 3\lambda^2} \left(\frac{\chi''(\rho/\lambda^2)}{\lambda^4} \frac{\partial \rho}{\partial v_\alpha} \frac{\partial \rho}{\partial \bar{v}_\beta} u\right) = \frac{1}{\lambda^4} O(\lambda^5) = O(\lambda).
\]

The other terms can be dealt with similarly. From the compactness of \( X \) it now follows that \( \tilde{\omega} \) becomes arbitrarily close to \( \omega \) uniformly on \( M \) as \( \lambda \) tends to zero. Hence, for \( \lambda \) small enough the symmetric bilinear form \( \tilde{\omega} \langle \cdot, J \cdot \rangle \) is positive-definite, and therefore \( \tilde{\omega} \) is the imaginary part of a Kähler metric \( d\tilde{s}^2 \). By construction \( d\tilde{s}^2 \) is equal to \( dS^2 \) on \( M_{\lambda^2} \) and equal to \( ds^2 \) on \( M - M_{2\lambda^2} \).

In the proof of the second part of the theorem we denote the Stein tube around \( X \) by \( N \) to distinguish it from the whole of \( M \). Suppose the compact Lie group \( G \) acts on \( M \) by holomorphic transformations leaving \( X \), \( dS^2 \) and \( ds^2 \) invariant. After averaging over \( G \) we may assume the potential \( u \) is invariant. It is not hard to verify by inspecting the proof in [14] that the tube \( N \) can be chosen to be invariant. If we choose an invariant distance function, the shrunken tubes \( N_\rho \) are also invariant. It is clear from the definition that the form \( \tilde{\omega} \) is then also \( G \)-invariant.
Now assume that there exists a momentum map $\Phi$ for the action with respect to the symplectic form $\omega$. By construction we have $\tilde{\omega}_x = \omega_x$ for all $x$ in $X$, so by the equivariant Darboux-Weinstein Theorem (see e.g. [13, §22]) for sufficiently small $\lambda$ there exists a $G$-equivariant diffeomorphism $\Gamma: N_{3\lambda^2} \rightarrow N_{3\lambda^2}$ fixing the manifold $X$ such that $\Gamma^* \tilde{\omega} = \omega$. Then the map $\tilde{\Phi}: N_{3\lambda^2} \rightarrow g^*$ defined by $\Gamma^* \Phi = \Phi$ is a momentum map with respect to the form $\tilde{\omega}$. On $N_{3\lambda^2} - N_{2\lambda^2}$ we have $\tilde{\omega} = \omega$, so there $\tilde{\Phi}$ differs by a locally constant function $c$ from the $\omega$-momentum map $\Phi$.

Let us assume, as we may, that the manifolds $M$ and $X$ are connected. If $X$ is of codimension greater than one, the subset $N_{G}$ is connected. In this case $c$ is a constant, so after shifting $\Phi$ by $c$ we can paste $\Phi$ and $\Phi$ together to obtain a global $\tilde{\omega}$-momentum map for the $G$-action.

The totally real submanifold $X$ can only be of codimension one if $\dim X = 1$ and $\dim M = 2$. The only Riemann surfaces $M$ that admit a continuous group of automorphisms are $\mathbb{P}^1$, $\mathbb{C}$, $\mathbb{C}^\times$, elliptic curves $\mathbb{C}/\Lambda$, the unit disc $\Delta$ and annuli $\Delta_r = \{ z \in \mathbb{C} : |z| < 1, r \leq |z| < 1 \}$, for $0 \leq r < 1$. (Cf. Farkas and Kra [3, Section V.4].) No subgroup of $\text{Aut}(\mathbb{C}/\Lambda) = \mathbb{C}/\Lambda$ acts on $\mathbb{C}/\Lambda$ in a Hamiltonian fashion, so elliptic curves are out. In the other examples the only compact connected group of automorphisms is the circle acting in the standard way. In each of these cases $X$ has to be a circle, the complement of $X$ in the tube $N$ consists of two components, and $M - X$ also consists of two components. We can therefore glue together the two momentum maps $\Phi$ and $\Phi$ by adding appropriate constants to $\Phi|_{M - N_{2\lambda^2}}$ on each of the two components of $M - N_{2\lambda^2}$.

**Remark 1.11.** If $X$ connected, then the $\tilde{\omega}$-momentum map $\tilde{\Phi}$ is equal to $\Phi$ on $X$, and if $\Phi$ is equivariant, then so is $\tilde{\Phi}$.

### 1.3. Holomorphic slices.

We now come to the main result of Section 1.

**Theorem 1.12** (Holomorphic Slice Theorem). Let $M$ be a Kähler manifold and let $G^C$ act holomorphically on $M$. Assume the action of the compact real form $G$ is Hamiltonian. Let $m$ be any point in $M$ such that the $G$-orbit through $m$ is isotropic. Then there exists a slice at $m$ for the $G^C$-action.

If $S$ is a slice at $m$, then $gS$ is a slice at $gm$. So the theorem implies the existence of a slice at any point $m$ such that the $G^C$-orbit through $m$ contains an isotropic $G$-orbit. Moreover, if $G'$ is another compact real form of $G^C$, then $G'$ is conjugate to $G$ by some element $g$ of $G^C$, $g = gg^{-1}$. Then $G'$ leaves invariant the symplectic form $g\omega$, and a $G'$-momentum map is given by $\Phi' = (\text{Ad}^* g) \circ \Phi \circ g^{-1}$, where $\Phi: M \rightarrow g^*$ is an $\text{Ad}^*$-equivariant momentum map for the $G$-action. So the choice of the compact real form is irrelevant.

**Proof of Theorem 1.12**. We divide the proof into several steps. Using the analytic continuation argument of Proposition 1.4 we shall first reduce the question of the existence of a slice to the existence of orbitally convex open neighbourhoods of the compact orbit $Gm$. Next we consider the special case where the compact orbit $Gm$ has a $G$-invariant neighbourhood that can be embedded in a holomorphic, $G$-equivariant and isometric fashion into a unitary representation space of $G$. The last step of the proof consists in showing that an arbitrary metric $ds^2$ can always be deformed to a metric which close to $Gm$ is the pullback of a flat metric via some embedding into a Euclidean space, and which is still compatible with all the relevant data. The details are as follows.
By Remark 1.3, the vector \( \Phi(m) \) is \( \text{Ad}^* G^0 \)-fixed. After shifting the momentum map we may assume that \( \Phi(m) = 0 \). (If \( \Phi(m) \) is not fixed under the whole of \( G \), then the shifted momentum map \( \Phi - \Phi(m) \) is merely \( G^0 \)-equivariant. This will however be sufficient in what follows.) The tangent space \( T_m M \) at \( m \) is a Hermitian vector space, which we shall identify with standard \( C^n \). Then the value of the Kähler form \( \omega \) at \( M \) is the standard symplectic form \( \Omega \) on \( C^n \). Let \( H \) be the stabilizer of \( m \) with respect to the \( G \)-action. Then by Remark 1.3, the stabilizer with respect to the \( G^C \)-action is the complexification \( H^C \) of \( H \). The tangent action of \( H^C \) defines a linear representation \( H^C \to \text{GL}(n, C) \), the restriction of which to \( H \) is a unitary representation \( H \to U(n) \). Let \( \phi: O \to M \) be a local holomorphic coordinate system on \( M \) with \( \phi(0) = m \) and \( d\phi_0 = \text{id}_{C^n} \), where \( O \) is a small \( H \)-invariant open ball about the origin in \( C^n \). Then the pullback of the form \( \omega \) is equal to \( \Omega \) at the origin. Let \( O' = \phi(O) \) and let \( \psi: O' \to O \) be the inverse of \( \phi \). After averaging over \( H \) and shrinking \( O \) if necessary we may assume that \( \psi \) and hence \( \phi \) are \( H \)-equivariant.

The tangent space to the complex orbit \( G^C m \) is a Hermitian subspace of \( T_m M \cong C^n \). Denote its orthogonal complement by \( V \); then \( V \) is an \( H^C \)-invariant subspace, which can be identified with \( C^l \) for some \( l \leq n \). Now let \( B \) be the intersection of the ball \( O \) with \( V \), and let \( B' \) be the image of \( B \) under \( \phi \), \( B' = \phi(B) \). We claim that if \( B' \) is sufficiently small the \( H^C \)-saturation \( S' = H^C B' \) of \( B' \) is a slice at \( m \). (In Snow’s terminology \( B' \) is a local slice.) To verify this claim we have to show that the natural map from the associated bundle \( G^C \times_{H^C} S' \) into \( M \) is biholomorphic onto an open subset of \( M \). We shall show this indirectly by proving that the map \( \phi: B \to B' \) can be uniquely extended to a \( G^C \)-equivariant map from \( G^C \times_{H^C} S \) into \( M \), which is biholomorphic onto an open subset. Here \( S \) is defined to be the open subset \( H^C B \) of \( V \).

Let us define \( E \) to be the associated bundle

\[
E = G^C \times_{H^C} V,
\]

and let \( e \) be the “base point” \( [1, 0] \in E \). Consider the \( G \)-equivariant map \( G \times m \to \mathfrak{g}^C / \mathfrak{g}^C \) sending a pair \(( g, \sqrt{-1} \xi ) \) to \( g \exp(\sqrt{-1} \xi) H^C \). This map descends to a \( G \)-equivariant map

\[
G \times_H \sqrt{-1} m \to \mathfrak{g}^C / \mathfrak{g}^C,
\]

(1.8)

which by a refinement of the Cartan decomposition due to Mostow \( 32, 33, 34 \) is a diffeomorphism. In other words, every element of \( G^C \) can be written as a product \( g \exp(\sqrt{-1} \xi) h \) with \( g \in G \), \( \xi \in m \) and \( h \in H^C \); and if \( g \exp(\sqrt{-1} \xi) h = g' \exp(\sqrt{-1} \xi') h' \), then \( g' = g k^{-1} \), \( \xi' = (\text{Ad} k) \xi \) and \( h' = k h \) for some \( k \in H \). It follows that the map

\[
G \times_H (\sqrt{-1} m \times \mathfrak{g}) \to \mathfrak{g}^C \times_{\mathfrak{g}^C} \mathfrak{g}
\]

sending the equivalence class \([ g, \sqrt{-1} \xi, v ] \) to the equivalence class \([ g \exp(\sqrt{-1} \xi) v ] \) is likewise a diffeomorphism. We conclude that the sets \( U = G \exp(\sqrt{-1} D) B \cong G \times_H (\sqrt{-1} D \times B) \), for \( D \), resp. \( B \), ranging over all balls about the origin in \( m \), resp. \( V \), form a basis of neighbourhoods of the compact orbit \( Ge \) inside the space \( E \). Furthermore, we can extend the \( H \)-equivariant holomorphic map \( \phi: B \to M \) to a \( G \)-equivariant holomorphic map \( U \to M \) by defining

\[
[g, \sqrt{-1} \xi, v] \mapsto g \exp(\sqrt{-1} \xi) \phi(v),
\]

(1.9)
for \( g \in G, \xi \in D \) and \( v \in B \). For simplicity we still call this map \( \phi \). From the decomposition \([3.3]\), where now \( \mu = 0 \) and \( n = m \), it is clear that \( G \times_H (\sqrt{-1}m \times \mathbb{B}) \) is nothing but the normal bundle to the compact orbit \( Gm \cong G/H \) in \( M \). Consequently, for \( D \) and \( B \) small enough \( \phi: U \to M \) is a biholomorphic map onto an open neighbourhood of \( Gm \) in \( M \). Clearly \( G^C U = G^C \times_H C^C B = G^C \times_H C^C S \).

We will prove:

**Claim 1.13.**  
1. The compact orbit \( Ge \subset E \) possesses a basis of orbitally convex open neighbourhoods; and  
2. The compact orbit \( Gm \subset M \) possesses a basis of orbitally convex open neighbourhoods.

In view of Proposition \([1.4]\) this will imply there is a \( G^C \)-equivariant biholomorphic map \( \phi^C: G^C \times_H C^C S \to G\tilde{C} S \) extending the map \( \phi \), which will conclude the proof of Theorem \([1.2]\). Heinzner \([10]\) gives a proof of \([1]\). We shall present an adapted version of his argument, which can be utilized to give a proof of \([3]\). The argument bears a certain similarity to a convexity argument of Kempf and Ness \([20]\). Let us start with the simple case where \( e \) is a fixed point. Then \( G = H \), the space \( E \) is just \( C^n \) and \( e \) is the origin. On \( C^n \) we have the constant Kähler metric denoted by \( dS^2 \), the standard symplectic form \( \Omega \) and the quadratic momentum map \( \Phi_{C^n}: C^n \to \mathfrak{g}^* \) given by \([3.3]\). As above, \( B \) is an open ball about the origin in \( C^n \). By \( r(v) \) we denote the Riemannian distance of \( v \in B \) to the origin, and by \( \langle \cdot, \cdot \rangle \) the positive-definite inner product \( \text{Re} \, dS^2 \).

**Lemma 1.14.** For all \( \xi \in \mathfrak{g} \) and \( v \in B \) the momentum function \( \Phi_{C^n}^\xi \) measures the inner product of the outward pointing normal \( r^2 \) to the metric sphere of radius \( r \) about the origin and the vector field \( J_{C^n} = \text{grad} \Phi_{C^n}^\xi \), as follows:

\[
\langle \text{grad} r^2, \text{grad} \Phi_{C^n}^\xi \rangle = 4\Phi_{C^n}^\xi . \tag{1.10}
\]

It follows that \( B \) is orbitally convex with respect to the \( G^C \)-action.

**Proof.** The path \( \delta(t) = \exp(\sqrt{-1}t\xi)v \) is the trajectory of the gradient of \( \Phi_{C^n}^\xi \) through \( v \). On one hand,

\[
\frac{d}{dt} r^2(\delta(t)) = \langle \text{grad} r^2(\delta(t)), \delta'(t) \rangle = \langle \text{grad} r^2(\delta(t)), \text{grad} \Phi_{C^n}^\xi(\delta(t)) \rangle.
\]

On the other hand,

\[
\frac{d}{dt} r^2(\delta(t)) = \frac{d}{dt} \|\delta(t)\|^2 = \frac{d}{dt} \langle \delta(t), \delta(t) \rangle = 2\langle \delta'(t), \delta(t) \rangle = 2\langle \delta'(t), \delta(t) \rangle = 2\Omega(\xi_{C^n}(\delta(t)), \delta(t)) = 4\Phi_{C^n}^\xi(\delta(t)),
\]

where we have used \([1.1]\) and \([1.2]\). Taking \( t = 0 \) yields \([1.10]\).

Now \([1.10]\) implies that the curve \( \delta(t) \) can only enter \( B \) at a point \( p \) in the boundary \( \partial B \) for which \( \Phi_{C^n}^\xi(p) \leq 0 \) and leave it at a point \( q \in \partial B \) where \( \Phi_{C^n}^\xi(q) \geq 0 \). But \( \delta(t) \) is also a gradient curve of the function \( \Phi_{C^n}^\xi \) and so \( \Phi_{C^n}^\xi \) is increasing along \( \delta(t) \). If \( \delta(t) \) is not constant, \( \Phi_{C^n}^\xi(\delta(t)) \) is strictly increasing. Therefore, if \( \delta(t) \) leaves the ball \( B \) at some point, it can never sneak back in. Consequently \( \{ \delta(t) : t \in \mathbb{R} \} \cap B \) is connected. If \( \delta(t) \) is constant it is trivially true that \( \{ \delta(t) : t \in \mathbb{R} \} \cap B \) is connected. \( \square \)

Observe that the proof does not use that the metric is flat on all of \( C^n \); it works for any Kähler metric that is flat in a neighbourhood of the origin.
We shall make repeated use of the following result of Kempf and Ness [20]. (Cf. also Procesi and Schwarz [38].)

**Proposition 1.15.** Suppose $G$ acts unitarily on $C^N$. Consider the complexified representation $G^C \rightarrow GL(N,C)$. An orbit $O$ of $G^C$ in $C^N$ is closed if and only if the restriction $r|_O$ of the length function $r$ has a stationary point. If $v \in O$ is a stationary point of $r|_O$, then

1. $r|_O$ takes on its minimum at $v$, and for all $w \in O$, $r(w) = r(v)$ implies $w \in Gv$;
2. $v$ is in the zero level set of the momentum map $\Phi_{CN}$;
3. $(G^C)_v = (G_v)^C$. \hfill $\square$

To jack up Lemma 1.14 we embed the homogeneous bundle $E$ equivariantly into a representation space.

**Lemma 1.16.** There exists a $G^C$-equivariant, holomorphic and proper embedding $\iota$ of $E$ into a finite-dimensional representation space $C^N$ of $G^C$.

Choose any $G$-invariant Hermitian inner product on $C^N$. Then the sets $\iota^{-1}(B)$, where $B$ ranges over the collection of open balls about the origin in $C^N$, form a basis of orbitally convex open neighbourhoods of the orbit $Ge$ in $E$.

**Proof.** It is not hard to find an orthogonal representation of $G$ on $R^{N_1}$ for some $N_1$ containing a vector $w$ whose stabilizer is exactly $G_w = H$. (See [15].) Then the map assigning to $gH$ the vector $gw$ is a real-analytic $G$-equivariant embedding of the homogeneous space $G/H$ into $R^{N_1}$. Complexifying the representation $G \rightarrow O(N_1)$ yields a unitary representation $G \rightarrow U(N_1)$, which extends to a complex-linear representation $G^C \rightarrow GL(N_1,C)$. Consider the inclusions $Gw \subset R^{N_1}$ and $G^Cw \subset C^{N_1}$. Since the $G$-representation on $R^{N_1}$ is orthogonal, the tangent space to the orbit $T_w(Gw)$ is a subspace of the tangent space to the $(N_1 - 1)$-dimensional sphere about the origin in $R^{N_1}$ containing $w$. It follows that the tangent space to the complexified orbit $T_w(G^Cw) = T_w(Gw) + JT_w(Gw)$ is a subspace of the tangent space to the $(2N_1 - 1)$-dimensional sphere about the origin in $C^{N_1}$ containing $w$.

In other words, $w$ is a critical point of the function $r^2|_{G^Cw}$, where $r^2$ is the distance to the origin in $C^{N_1}$. Proposition 1.15 now implies that $(G_w)^C = (G_v)^C$, and that the orbit $G^Cw$ is closed in $C^{N_1}$. We conclude that the map $\iota_1: G^C/H^C \rightarrow C^{N_1}$ sending $gH^C$ to $gw$ is an equivariant, holomorphic and proper embedding.

Next we show how to find an embedding of the $H^C$-module $V$ into a finite-dimensional $G^C$-module $C^{N_2}$, that is, an $H^C$-equivariant injective complex-linear map $\iota_2: V \rightarrow C^{N_2}$. Let $V_1, V_2, \ldots, V_k$ be the irreducible components of $V$. It is an easy consequence of the Peter-Weyl Theorem that every irreducible $H$-module can be embedded $H$-equivariantly into an irreducible $G$-module. (Consider the decomposition of the left-regular representation $L^2(G) = \bigoplus_i W_i$ into $G$-irreducibles. Decompose each of the $W_i$ into $H$-irreducibles, $W_i = \bigoplus_{ij} Z_{ij}$. Let $\chi: H \rightarrow C$ be the character of some irreducible $H$-representation; pushing $\chi$ forward as a measure to $G$ gives a measure on $G$, and the convolution product $f \mapsto \chi * f$ defines a nonzero $H$-equivariant projection operator $\pi$ in $L^2(G)$. Now $\pi|_{Z_{ij}} = \text{id}$ or $\pi|_{Z_{ij}} = 0$ depending on whether or not $Z_{ij}$ has character $\chi$. Since $\pi \neq 0$ at least one of the $W_i$ has to contain a $Z_{ij}$ with character $\chi$.) So we can find irreducible $G$-modules $C^{n_k}$ with $H$-equivariant injective complex-linear maps $j_k: V_k \rightarrow C^{n_k}$. Each of the $j_k$'s is necessarily $H^C$-equivariant. We can take $\iota_2$ to be the direct sum of the $j_k$'s.
It is now easy to check that the map $\iota: E = G^C \times_{H^C} V \to C^{N_1+N_2}$ mapping $[g,v]$ to $gw + g\iota_2(v)$ is a $G^C$-equivariant, holomorphic and proper embedding.

By Proposition 1.13 $Gw$ is precisely the subset of vectors in $G^Cw$ of minimal length. From the inequality $\|\iota(g,v)\|^2 = \|gw + g\iota_2(v)\|^2 = \|gw\|^2 + \|g\iota_2(v)\|^2 \geq \|gw\|^2$ it is clear that $Gw$ is also equal to the subset of vectors of minimal length in the submanifold $\iota(E)$. Because of this and the $G$-invariance of the metric on $C^{N_1+N_2}$, any open ball $B$ about 0 such that $B \cap \iota(E)$ is nonempty contains the orbit $Gw$, and the sets $B \cap \iota(E)$ are a basis of open neighbourhoods of the orbit $Gw = G \cdot \iota(e)$. The second assertion of the lemma now follows from Lemma 1.14 and Remark 1.3.

This proves part (i) of Claim 1.13.

Now consider the $G$-equivariant holomorphic embeddings

$$C^N \mapsto U \mapsto M,$$

where $\phi$ is the map defined in (1.4). Pulling back the metric $ds^2$ on $M$ via $\phi$ we obtain a metric on $E$ defined on the neighbourhood $U$ of the compact orbit $Ge$.

The proof of Lemma 1.16 allows us to deduce the following stronger assertion.

**Lemma 1.17.** Suppose the linear embedding $\iota$ is isometric, that is, $\iota^*dS^2 = \phi^*ds^2$, where $dS^2$ is the flat metric on $C^N$. Then for any orbitally convex open subset of the form $\iota^{-1}(B)$ contained in $U$ the image $\phi(\iota^{-1}(B))$ is orbitally convex in $M$.

**Proof.** Put $U' = \phi(U)$ and let $\psi: U' \to U$ be the inverse of $\phi$. We have two $G$-invariant Kähler metrics on $U'$, namely $ds^2$ and $\psi^*\iota^*dS^2$, with corresponding momentum maps $\Phi$ and $\Phi' = \psi^*\iota^*\Phi_{CN}$. By assumption $ds^2$ is equal to $\psi^*\iota^*dS^2$. Moreover, $\Phi(m) = 0$ and, by Proposition 1.12, $\Phi_{CN}(\iota(e)) = 0$. This implies $\Phi(m) = \Phi'(m)$, and $\Phi = \Phi'$. Put $O = \phi(\iota^{-1}(B))$ and pick any point $x$ in $O$. Let $\gamma(t) \subset M$ be the curve $\exp(\sqrt{-1}t\xi)x$; then $\gamma(t)$ is contained in $U'$ for small $t$. Put $v = \psi(x)$ and $\delta(v) = \exp(\sqrt{-1}t\xi)v$. Let $I = \{t \in R : \gamma(t) \in O\}$ and let $I^0$ be the connected component of $I$ containing 0. Because the map $\psi$ is $G$-equivariant and holomorphic, we have $\delta(t) = \exp(\sqrt{-1}t\xi)\psi(x) = \psi\left(\exp(\sqrt{-1}t\xi)x\right) = \psi(\gamma(t))$ for all $t \in I^0$. (See Remark 1.3.) It now follows from the proof of Lemma 1.14 that the curve $\gamma(t)$ can only enter the set $O$ at time $t_0$ if $\Phi(\gamma(t_0)) \leq 0$ and leave it at time $t_1$ if $\Phi(\gamma(t_1)) \geq 0$. Because the function $\Phi^2$ is increasing along $\gamma$, it follows that $I = I^0$, i.e. $\gamma$ intersects $O$ in a connected set.

Of course, in general the map $\iota$ will not be an isometry for the given metric $ds^2$ on $M$. We claim, however, that we can arrange for $\iota$ to be an isometry along the compact orbit $Ge$.

**Lemma 1.18.** The representation $G^C \to GL(N,C)$ and the $G$-invariant Hermitian inner product on $C^N$ in Lemma 1.16 can be chosen in such a way that the embedding $\iota$ is a Kähler isometry at all points of the orbit $Ge$, that is, $\iota^*dS^2 = \phi^*ds^2$ on $T_xE$ for all $x \in Ge$.

**Proof.** We use the notation of the proof of Lemma 1.16. Moore 30 has shown that the representation $G \to O(N_1)$ can be chosen in such a way as to make the embedding $G/H \to R^{N_1}$ an isometry of Riemannian manifolds. (Cf. also 31.) The associated embedding $\iota_1$ of the complexified homogeneous space $G^C/H^C$ into $C^{N_1}$ is holomorphic and the complex structure $J$ is an orthogonal map (at each point of
G^C/H^C and C^{N_1}). So the differential \(d\iota_1\) is a unitary map \(T_x(G^C/H^C) \to C^{N_1}\) for all \(x \in G/H\).

We can also arrange for the embedding \(\iota_2: V \to C^{N_2}\) to be a unitary map. Indeed, we obtained \(\iota_2\) by embedding each irreducible component \(V_k\) of \(V\) into an irreducible unitary representation \(C^{n_k}\) of \(G\). By Schur's Lemma the invariant Hermitian inner products on \(V_k\) and \(C^{n_k}\) are unique up to constant multiples. By suitably rescaling the metric on each \(C^{n_k}\) the embedding \(\iota_2: \bigoplus_k V_k \to \bigoplus_k C^{n_k}\) becomes unitary.

The embedding \(\iota: E \to C^N\) is now a K"ahler isometry along the orbit \(Ge\).

With a choice of embedding as in this lemma Theorem 1.10 tells us we can deform the metric \(ds^2\) in such a manner that \(\iota\) becomes an isometry. Theorem 1.11 plus Lemmas 1.17 and 1.18 therefore imply part (2) of Claim 1.13. This finishes the proof of Theorem 1.12.

Along the lines of [26] one can deduce from the slice theorem many results on the local structure of a \(G^C\)-action. Let us list a few for the record.

**Theorem 1.19.** Every point in \(M\) the \(G\)-orbit through which is isotropic possesses a \(G^C\)-invariant Stein open neighbourhood.

**Proof.** Let \(m \in M\) and suppose \(Gm\) is isotropic. Put \(H = G_m\). Let \(S\) be a slice at \(m\) as constructed in the proof of Theorem 1.12. Then \(S\) is biholomorphically equivalent to the set \(H^CB\) swept out by a ball \(B\) in the tangent space \(T_mS\). It is not hard to show that \(H^CB\) is Stein. We conclude \(m\) has an open \(G^C\)-invariant neighbourhood that is biholomorphically equivalent to a bundle with affine base \(G^Cm\), Stein fibre \(S\) and reductive structure group \((G_m)^C\). By a theorem of Matsushima the total space of this bundle is Stein.

**Theorem 1.20.** Let \(m\) be any point in \(M\) such that the \(G\)-orbit through \(m\) is isotropic. Then for every point \(x\) nearby \(m\) the stabilizer subgroup \((G^C)_x\) is conjugate to a subgroup of \((G^C)_m\).

**Theorem 1.21.** Let \(m \in M\) be any fixed point of the \(G^C\)-action. Then the action of \(G^C\) can be linearized in a neighbourhood of \(m\) in the sense that there exist a \(G^C\)-invariant open neighbourhood \(U\) of \(m\) in \(M\), a \(G^C\)-invariant open neighbourhood \(U'\) of the origin \(0\) in the tangent space \(T_mM\) and a biholomorphic \(G^C\)-equivariant map \(U \to U'\).

**Proof.** A fixed point is obviously isotropic. The result now follows immediately from the Holomorphic Slice Theorem.

**Remark 1.22.** This theorem was also stated by Koras [24], but my proof is different from Koras', which I have trouble understanding in places. In particular, I fail to see a justification for his application of the curve selection lemma.

Recall that the \(G^C\)-action is called proper at the point \(m\) if for all sequences \((m_i) \subset M\) and \((g_i) \subset G\) the following holds: If \((m_i)\) converges to \(m\) and \((g_i m_i)\) converges to some point in \(M\), then \((g_i)\) converges to some element of \(G\). If the action is proper at \(m\), the stabilizer \((G^C)_m\) is compact.

**Theorem 1.23.** Suppose the \(G\)-orbit through a point \(m \in M\) is isotropic. Then the following conditions are equivalent:
1. The action of $G^C$ is proper at $m$;
2. The stabilizer $(G^C)_m$ is finite;
3. $m$ is a regular point of the momentum map $\Phi$.

Proof. First we show (1) is equivalent to (2). If the $G^C$-action is proper at $m$, the stabilizer $(G^C)_m$ is a compact complex submanifold of $G^C$, which is a Stein manifold. Therefore $(G^C)_m$ is finite. Conversely, assume $(G^C)_m$ is finite. Then it is easy to see that the left action of $G^C$ on the homogeneous space $G^C/(G^C)_m$ is proper. It follows that the left $G^C$-action on the homogeneous vector bundle $G^C \times (G^C)_m V$ is also proper, $V$ being the tangent space at $m$ to a slice at $m$. By the Holomorphic Slice Theorem the point $m$ has an invariant neighbourhood which is equivariantly isomorphic to an invariant open subset of $G^C \times (G^C)_m V$, so the $G^C$-action on $M$ is proper at $m$.

Next we show (2) is equivalent to (3). If $(G^C)_m$ is finite, obviously the real stabilizer $G_m$ is also finite, so the stabilizer subalgebra $g_m$ is trivial. Now the annihilator of $g_m$ in $\mathfrak{g}$ is equal to the range of $d\Phi_m$ (see [13, §26]), so $d\Phi_m$ is surjective. Conversely, if $d\Phi_m$ is surjective, $g_m$ is trivial, so $G_m$ is finite, so by Proposition 1.6 $(G_m)^C = (G^C)_m$ is finite.

Theorem 1.24. Suppose $G$ is a torus. Then slices for the $G^C$-action exist at all points of $M$.

Proof. If $G$ is a torus, the coadjoint representation of $G$ is trivial, so by Proposition 1.6 all $G$-orbits in $M$ are isotropic. Now apply the Holomorphic Slice Theorem.

Our results can also be used to give a short proof of a theorem of Snow’s [44].

Theorem 1.25 (Snow). Let $X$ be a Stein space on which $G^C$ acts holomorphically. Let $x$ be any point in $X$ such that the orbit $G^C x$ is closed. Then there exists a slice at $x$ for the $G^C$-action.

Proof. The first part of the proof is the same as in [44]. Snow proves there exists a $G^C$-equivariant holomorphic map $h$ of $X$ into a $G^C$-representation space $C^n$ that is an immersion at $x$ (and hence at all points of the orbit $G^C x$) and whose restriction to $G^C x$ is a proper embedding ([44 Proposition 2.5]). It follows the orbit $G^C \cdot h(x)$ is closed in $C^n$, and therefore by Proposition 1.13 the compact orbit $G \cdot h(x)$ is contained in the zero level set of the quadratic momentum map $\Phi_{C^n}$. So by Lemma 1.16 the orbit $G \cdot h(x)$ possesses a basis $U$ of orbitally convex neighbourhoods in $C^n$. From the fact that $h|_{Gx}$ is injective, that $h$ is an immersion at all points of $Gx$ and that $Gx$ is compact, we conclude $h$ is a diffeomorphism from a neighbourhood of $Gx$ onto a neighbourhood of $G \cdot h(x)$. It follows that the sets $h^{-1}(U)$ for $U \in U$ form a basis of neighbourhoods of $Gx$. By Remark 1.3 they are also orbitally convex. The theorem now follows from Proposition 1.4 (or rather, the generalization of Proposition 1.4 to arbitrary complex spaces, which is just as easy to prove; see [16]).

Remark 1.26. One can talk of holomorphic actions and momentum maps in the setting of Kähler spaces (“Kähler manifolds with singularities”) in the sense of Grauert [3, §3]. It seems reasonable to expect that the Holomorphic Slice Theorem can be extended to this more general situation.
2. Kähler Quotients and Geometric Quantization

In this section I apply the Holomorphic Slice Theorem to the study of symplectic quotients of a Kähler manifold $M$. The upshot is that such a quotient has a natural structure of an analytic space, and that if $M$ is integral, the quotient is a complex-projective variety.

Of course, if $M$ is integral, it is a complex-projective manifold by Kodaira’s Embedding Theorem, but the embedding given by Kodaira’s theorem is usually not a symplectic embedding into projective space. (For a simple example where it is not, consider any non-singular $X \subset \mathbb{CP}^n$. Let $\Omega$ be the restriction of the Fubini-Study form to $X$. For any smooth function $f$ on $X$, put $\Omega_f = \Omega + \sqrt{-1} \partial \bar{\partial} f$. If $f$ is $C^2$-small, $\Omega_f$ is a Kähler form. But for most $f$, for instance, those $f$ that are not real-analytic, no holomorphic embedding of $(X, \Omega_f)$ into any projective space $\mathbb{CP}^N$ is an isometry.) Under the assumption that Kodaira’s embedding does preserve the symplectic form Kirwan [21] and Ness [37] proved that the symplectic quotient of $M$ agrees with a categorical quotient of a semistable subset of $M$ in the sense of geometric invariant theory. I show that this conclusion still holds even if Kodaira’s map is not a symplectic embedding. Thus the result says roughly that the class of symplectic quotients of an integral Kähler manifold is not bigger than the class of its algebraic quotients. Alternatively, it says that there are many non-equivalent symplectic structures on the algebraic quotients of $M$.

The abovementioned result of Kirwan and Ness is a generalization of earlier work of Guillemin and Sternberg [11], and Kempf and Ness [20]. Guillemin and Sternberg dealt with the case where the quotient of $M$ is non-singular. This case is technically simpler mainly owing to the fact that here the action of $G^C$ is proper at all points of the zero level set of the momentum map. (See Theorem 1.23.) Kempf and Ness handled the case of a linear action on a Hermitian vector space. In fact, I shall reduce the general case to that of a linear action by locally “flattening out” the Kähler metric.

Section 2.1 is a discussion of quotients of Kähler manifolds in the general setting of Section 1. Section 2.2 focuses on the case of integral Kähler manifolds, placing the results of Section 2.1 in the context of geometric invariant theory. In Section 2.3 I rephrase some of the results in the language of geometric quantization and show how they lead to formulæ for multiplicities of representations.

### 2.1. Reduction of Kähler manifolds

As in the previous section let us fix a connected Kähler manifold $(M, ds^2)$ on which $G^C$ acts holomorphically and assume there exists an equivariant momentum map $\Phi$ for the action of $G$. Let $\lambda \in g^*$. The symplectic quotient or reduced (phase) space of $M$ at the level $\lambda$ is by definition the topological space $M_\lambda = \Phi^{-1}(G\lambda)/G$, where $G\lambda$ is the coadjoint orbit through $\lambda$. By the results of [43] $M_\lambda$ has the structure of a symplectic stratified space. Roughly speaking, this means that $M_\lambda$ is a disjoint union of symplectic manifolds that fit together in a nice way, and that there is a unique open stratum, which is connected and dense in $M_\lambda$. We want to endow $M_\lambda$ with an analytic structure and show its stratification is analytic. We shall carry this out only for $\lambda = 0$: the general case follows from this by dint of the “shifting trick”. (See [1], [13].)

Define a point $m$ in $M$ to be (analytically) semistable if the closure of the $G^C$-orbit through $m$ intersects the zero level set $\Phi^{-1}(0)$, and denote the set of semistable points by $M^{ss}$. The point $m$ is called (analytically) stable if the closure of the $G^C$-orbit through $m$ intersects the zero level set $\Phi^{-1}(0)$ at a point where $d\Phi$ is surjective.
The set of stable points is denoted by $M^s$. The notions of analytic semistability and stability depend on the Kähler metric and on the momentum map. If $M$ is integral, they will turn out to be equivalent to semistability, resp. stability in the sense of Mumford [35] with respect to a suitable projective embedding (Theorem 2.15).

Introduce a $G$-invariant inner product on the Lie algebra of $G$. Let $\mu$ be the “Yang-Mills functional” $\|\Phi\|^2$ and let $F_t$ be the gradient flow of the function $-\mu$. Since $\mu$ is $G$-invariant, $F_t$ is $G$-equivariant. By Lemma 6.6 of Kirwan [21] the gradient of $\mu$ is given by

$$\text{grad}\,\mu(m) = 2J\Phi(m)_{M,m},$$

where we have identified $\Phi(m) \in g^*$ with a vector in $g$ using the inner product, and where $(\Phi(m))_{M,m}$ is the vector field on $M$ induced by $\Phi(m)$, evaluated at the point $m$. In particular, grad $\mu$ is tangent to the $G^C$-orbits, so these are preserved by the flow $F_t$. Let us call the momentum map admissible if for every $m \in M$ the path of steepest descent $F_t(m)$ through $m$ is contained in a compact set, as in Kirwan [21], §9. If $\Phi$ is admissible, the flow $F_t$ is defined for all $t \geq 0$. Kirwan has proved $M^{ss}$ is the set of points $m \in M$ with the property that the path $F_t(m)$ has a limit point in $\Phi^{-1}(0)$. Using the ideas of Neeman [30] one can show that for all $m \in M$ the limit $F_\infty(m) = \lim_{t \to \infty} F_t(m)$ actually exists and, moreover, that the restriction of the map $F_\infty$ to $M^{ss}$ is a continuous retraction of $M^{ss}$ onto $\Phi^{-1}(0)$. (See also Schwarz [22].)

All proper momentum maps are admissible. Here is another simple example, which will be important in what follows.

**Example 2.1.** Consider a linear action of $G^C$ on $\mathbb{C}^N$ with the standard momentum map $\Phi_{CN}$ given by (1.1). Let $\mu_{CN} = \|\Phi_{CN}\|^2$. An easy computation using (1.10) shows that $\langle \text{grad}\,r^2, \text{grad}\,\mu_{CN} \rangle = 8\mu_{CN}$, where $r$ denotes the distance to the origin. Consequently, at all points of the sphere bounding a ball $B$ about the origin the vector field $-\text{grad}\,\mu_{CN}$ points into $B$. Therefore $\Phi_{CN}$ is admissible. Let $(F_{CN})_t$ be the gradient flow of $-\mu_{CN}$. Then it is clear that the limit map $(F_{CN})_\infty$ retracts the set $G^CB$ onto $\Phi^{-1}_C(0) \cap B$.

Throughout this section we will assume $\Phi$ to be admissible. We now collect a number of basic results on the orbit structure of $M^{ss}$, which are either due to Kirwan [21], or which are refinements of her results.

**Proposition 2.2.** In the following “closed” means “closed in the relative topology of $M^{ss}$” and “closure” means “closure in $M^{ss}$”.

1. The semistable set $M^{ss}$ is the smallest $G^C$-invariant open subset of $M$ containing $\Phi^{-1}(0)$, and its complement is a complex-analytic subset;
2. A $G^C$-orbit in $M^{ss}$ is closed if and only if it intersects $\Phi^{-1}(0)$;
3. The intersection of a closed $G^C$-orbit with $\Phi^{-1}(0)$ consists of precisely one $G$-orbit;
4. For every semistable point $y$ the set $F_\infty(G^C y) \subset \Phi^{-1}(0)$ consists of precisely one $G$-orbit;
5. For any pair of points $x, y \in \Phi^{-1}(0)$ that do not lie on the same $G$-orbit there exist disjoint $G^C$-invariant open subsets $U$ and $V$ of $M$ with $x \in U$ and $y \in V$;
6. The closure of every $G^C$-orbit in $M^{ss}$ contains exactly one closed $G^C$-orbit.
Proof. See [24, §4] for a proof of (1).

We now prove (2). If \( x \) is semistable and \( G^C x \) is a closed subset of \( M^{ss} \), then \( F_\infty(x) \in G^C x \) because the flow \( F_t \) preserves the \( G^C \)-orbits, and also \( F_\infty(x) \in \Phi^{-1}(0) \), so \( G^C x \cap \Phi^{-1}(0) \) is non-empty. Conversely, suppose \( G^C x \cap \Phi^{-1}(0) \neq \emptyset \). Let \( \{y_i\} \) be a sequence in \( G^C x \) converging to \( y \in M^{ss} \). We have to show \( y \in G^C x \).

Clearly \( F_\infty(y) \in G^C y \) and by continuity \( F_\infty(y) = \lim_{i \to \infty} F_\infty(y_i) \in G^C x \). Therefore \( G^C y \) intersects every open neighbourhood of \( G^C x \). In particular, \( y \) is contained in every \( G^C \)-invariant open neighbourhood of \( x \). Since \( G^C x \) intersects \( \Phi^{-1}(0) \), the Holomorphic Slice Theorem tells us \( x \) has a \( G^C \)-invariant tubular neighbourhood \( U \). Evidently, \( G^C x \) is a closed subset of \( U \). Since \( y \in U \), this implies \( y = \lim_{i \to \infty} y_i \in G^C x \).

See [24, §6] for a proof of (3).

For the proof of (4), let \( x = F_\infty(y) \). Pick an arbitrary point \( z \in G^C y \). We need to show \( F_\infty(z) \in G x \). If \( G \) is connected, we can find a continuous path \( \gamma : [0, 1] \to M^{ss} \) with \( \gamma(0) = x \), \( \gamma(1) = z \) and \( \gamma(t) \in G^C y \) for \( t > 0 \). (If \( G \) is not connected, we can still do this, provided we replace \( x \) by a suitable translate \( gx \), where \( g \in G \).)

Consider the path \( F_\infty \circ \gamma \) contained in \( \Phi^{-1}(0) \), and let \( I \) be the set of all \( t \) in the unit interval such that the point \( x(t) \) defined by \( x(t) = F_\infty(\gamma(t)) \) is contained in \( C x \). We claim \( I \) is open in \([0, 1]\).

Indeed, suppose \( t \in I \). Recall that by Lemmas 2.1, 1.13 we have a slice \( S \) at \( x \) with the following special properties: There exists a \( G^C \)-equivariant embedding \( \iota \) of \( U = G^C S \) into a \( G^C \)-representation space \( C^N \), and \( U \) is equal to the set \( G^C O \), where \( O \) is the inverse image \( \iota^{-1}(B) \) of a Euclidean ball about the origin in \( C^N \). In order not to overburden the notation we shall identify \( U \) with its image in \( C^N \). Now let \( (F_{CN})_t \) be the gradient flow on \( C^N \) associated to the function \( -\mu_{CN} \) of Example 2.1. Then \( (F_{CN})_t = \lim_{i \to \infty} (F_{CN})_t \) retracts \( U \) onto \( \Phi_{CN}^{-1}(0) \cap O \).

By choosing \( B \) sufficiently small we can arrange that \( O \), and hence \( U \), are contained in \( M^{ss} \). Also, \( G^C y \subset U \), since \( x \) is in the closure of \( G^C y \). Since \( \gamma \in C^S \) we also have \( x(z) \in G^C y \). By (2) the orbit \( G^C x(z) \) is closed in \( M^{ss} \), and hence in \( U \). Furthermore, \( x(z) \in O \) for \( z \) sufficiently close to \( t \). Therefore, \( (F_{CN})_t(x(z)) \in \Phi_{CN}^{-1}(0) \cap G^C x(z) \). It now follows from (2) (applied to the momentum map \( \Phi_{CN} \) that for \( z \) sufficiently close to \( t \) the orbit \( G^C x(z) \) is closed in \( C^N \).

By construction each \( x(z) \) is contained in the closure of the orbit \( G^C y \subset U \). But in a \( G^C \)-representation space each orbit contains a unique closed orbit in its closure. (See e.g. Luna [24, §1].) We conclude that \( x(z) \in G^C x \) for all \( z \) close enough to \( t \). Since \( \Phi(x(z)) = 0 \), it follows from (3) that \( x(z) \in G x \), in other words \( z \in I \). Thus we have shown \( I \) is open.

Obviously, \( I \) is also closed and \( 0 \in I \). It follows \( I = [0, 1] \), and therefore \( F_\infty(z) = F_\infty(\gamma(1)) \in G x \). This finishes the proof of (4).

To prove (5), observe that (4) implies that for any \( G \)-invariant subset \( A \) of \( \Phi^{-1}(0) \) the preimage \( F_\infty^{-1}(A) \subset M^{ss} \) is \( G^C \)-invariant. Now suppose \( x, y \in \Phi^{-1}(0) \) and \( y \notin G x \). Because \( G \) is compact, there exist disjoint \( G \)-invariant open subsets \( A \) and \( B \) of \( \Phi^{-1}(0) \) with \( x \in A \) and \( y \in B \). Then \( F_\infty^{-1}(A) \) and \( F_\infty^{-1}(B) \) are disjoint \( G^C \)-invariant open sets containing \( x \), resp. \( y \).

Finally, (6) follows immediately from (2) and (5). \( \square \)

Call two semistable points \( x \) and \( y \) related if the closures of the orbits \( G^C x \) and \( G^C y \) intersect. (Again, “closure” means “closure in \( M^{ss} \).”) Assertion (1) of Proposition 2.2 implies this relation is an equivalence relation. Write \( M^{ss} // G^C \) for
the quotient space and Π for the quotient map $M^{ss} \to M^{ss} // G^{C}$. By (3) above, the space $M^{ss} // G^{C}$ is Hausdorff.

**Theorem 2.3.** The inclusion $\Phi^{-1}(0) \subset M^{ss}$ induces a homeomorphism $M_0 \to M^{ss} // G^{C}$.

**Proof.** By (2) and (3) of Proposition 2.2, the map $M_0 \to M^{ss} // G^{C}$ sending a $G$-orbit $Gm \subset \Phi^{-1}(0)$ to the equivalence class $\Pi(m)$ is a continuous injection. By (4) it is a bijection. Moreover, the inverse is induced by the retraction $F_{\infty} : M^{ss} \to \Phi^{-1}(0)$ and is therefore continuous.

**Remark 2.4.** Proposition 2.2 shows that $M_0$ can also be identified with the space of closed $G^{C}$-orbits in $M^{ss}$.

Let us say that a subset $A$ of $M^{ss}$ is saturated with respect to Π if $\Pi^{-1}(A) = A$. This means that for every $x$ in $A$ the closure of $G^{C}x$ is contained in $A$.

**Proposition 2.5.** At every point of $\Phi^{-1}(0)$ there exists a slice $S$ such that the set $G^{C}S$ is saturated with respect to the quotient mapping Π.

**Proof.** Let $x$ be any point in $\Phi^{-1}(0)$. We use the notation of the proof of part (2) of Proposition 2.2. We shall show that, after shrinking $O$ if necessary, the set $U$ becomes II-saturated. Choose a ball $B' \subset B$ so small that the $G$-invariant neighbourhood $O' = \iota^{-1}(B')$ of $Gx$ has the property that $F_{\infty}(O') \subset O$. This is possible because the sets $\iota^{-1}(B)$ form a basis of neighbourhoods of $Gx$ by Lemma 1.10 and because $F_{\infty}$ is the identity on $\Phi^{-1}(0)$.

Take any $y \in U'$. We claim that $G^{C}y$ is a subset of $U'$. Since $F_{\infty}(O') \subset O$, part (3) of Proposition 2.2 implies that $F_{\infty}(U') \subset O$, where $U' = G^{C}O'$. In particular $F_{\infty}(y) \in O$, and so $G^{C} \cdot F_{\infty}(y) \subset U$. Assertion (2) of Proposition 2.2 implies $G^{C} \cdot F_{\infty}(y)$ is closed in $M^{ss}$, and hence in $U$. Now $(F_{C^{N}})_{\infty}$ maps $U$ into $O$, so $(F_{C^{N}})_{\infty}(F_{\infty}(y)) \in O$. Moreover, since $G^{C} \cdot F_{\infty}(y)$ is closed in $M^{ss}$, $(F_{C^{N}})_{\infty}(F_{\infty}(y))$ sits in $G^{C} \cdot F_{\infty}(y)$. Therefore $G^{C} \cdot F_{\infty}(y)$ is closed in $C^{N}$ (by part (3) of Proposition 2.2 applied to the momentum map $\Phi_{C^{N}}$). But $(F_{C^{N}})_{\infty}(y)$ is contained in $O'$, and $G^{C} \cdot (F_{C^{N}})_{\infty}(y)$ is closed in $C^{N}$. Moreover, both orbits $G^{C} \cdot (F_{C^{N}})_{\infty}(y)$ and $G^{C} \cdot F_{\infty}(y)$ are contained in the closure of $G^{C}y$. It follows that $G^{C} \cdot F_{\infty}(y) = G^{C} \cdot (F_{C^{N}})_{\infty}(y) \subset U'$, and so $F_{\infty}(y) \in U'$. We now conclude from part (4) of Proposition 2.2 and the continuity of $F_{\infty}$ that $G^{C}y \subset U'$.

From now on we’ll identify the spaces $M^{ss} // G^{C}$ and $M_0$. We want to furnish $M_0$ with a complex-analytic structure in such a way that the quotient map Π becomes holomorphic. The richest possible such structure is obtained by declaring a function $f$ defined on an open subset $A$ of $M_0$ to be holomorphic if the pullback of $f$ to $\Pi^{-1}(A) \subset M^{ss}$ is holomorphic. Let $\mathcal{O}_{M}$ be the sheaf of holomorphic functions on $M_0$. We claim this indeed defines an analytic structure.

**Theorem 2.6.** The ringed space $(M_0, \mathcal{O}_{M})$ is an analytic space.

**Proof.** Let $p \in M_0$ and let $m$ be a point in $\Phi^{-1}(0)$ sitting over $p$. By the definition of $\mathcal{O}_{M}$, a neighbourhood of $p$ is isomorphic as a ringed space to a quotient $U // G^{C}$, where $U$ is a II-saturated open set containing $m$, equipped with the sheaf of $G^{C}$-invariant holomorphic functions. By Proposition 2.2 we may take $U$ to be of the form $G^{C}S$, where $S$ is a slice at $m$. Then $U$ can be identified with an invariant open subset of the bundle $E = G^{C} \times (G^{C})_{m} \cdot V$, where $V$ is the tangent space to a slice $S$. 
at $m$, and the quotient $U/G^C$ can be identified with an open subset of $E/G^C = V/(G^C)_m$. Now by a theorem of Luna [23] every $(G^C)_m$-invariant holomorphic function on the $(G^C)_m$-representation space $V$ is a holomorphic function of the invariant polynomials on $V$. Picking a finite number of generators $(\sigma_1, \ldots, \sigma_l)$ of the ring of invariant polynomials we get a Hilbert map $\sigma: V \to C^l$, sending $v$ to $(\sigma_1(v), \ldots, \sigma_l(v))$. The Hilbert map descends to a map $V/(G^C)_m \to C^l$, which by Luna’s theorem is a closed embedding of the ringed space $V/(G^C)_m$. It follows $U/G^C$ is isomorphic as a ringed space to an analytic subset of an open subset of $C^l$. Therefore $(M_0, O_{M_0})$ is an analytic space.

The proof of this theorem shows that in a neighbourhood of the point $p$ the quotient map $\Pi$ is equivalent to the quotient map $E \to E/G^C$ of the non-singular affine $G^C$-variety $E$.

**Corollary 2.7.** The quotient map $\Pi$ is locally biholomorphically equivalent to an affine map. In particular, the fibres of $\Pi$ are affine varieties.

The Holomorphic Slice Theorem implies that if the stabilizer of a point $m \in \Phi^{-1}(0)$ is finite, all $G^C$-orbits in an invariant neighbourhood of $m$ must have the same dimension. From this observation plus Theorem [23] and Proposition 2.2 one can easily deduce the following result.

**Theorem 2.8.** If $x \in M$ is stable, the orbit $G^C x$ is closed in $M^{ss}$ and the stabilizer $(G_x)^C$ is finite. Let $Z$ be the set of $m \in \Phi^{-1}(0)$ with the property that $d\Phi_m$ is surjective; then the stable set $M^s$ is equal to $F^{-1}_s(Z)$. Every fibre of $\Pi|_{M^s}$ consists of a single orbit.

In particular, if $0$ is a regular value of $\Phi$, $M^{ss}$ coincides with $M^s$ and $M_0 \cong M^s/G^C$, the space of stable orbits in $M$, is a Kähler orbifold.

Let $p$ be in $M_0$, let $x$ be a point in $\Phi^{-1}(0)$ mapping to $p$ and let $(H)$ be a conjugacy class of closed subgroups of $G$. Then $p$ is said to be of $G$-orbit type $(H)$ if the stabilizer $G_x$ is conjugate to $H$ in $G$. In [43] we showed that the set of all points of orbit type $(H)$ is a manifold carrying a natural symplectic structure and that the decomposition of $M_0$ into orbit type manifolds is a stratification.

Now let $(L)$ be a conjugacy class of reductive subgroups of $G^C$. We may assume $L = H^C$ for some closed subgroup $H$ of $G$. By Proposition 2.2 the fibre $\Pi^{-1}(p)$ contains a unique closed $G^C$-orbit, namely $G^C x$. Let us say $p$ is of $G^C$-orbit type $(H^C)$ if the stabilizer $(G^C)_x$ is conjugate to $H^C$ in $G^C$.

**Theorem 2.9.** The stratification of $M_0$ by $G$-orbit types is identical to the stratification by $G^C$-orbit types. Each stratum $S$ is a complex manifold and its closure is a complex-analytic subvariety of $M_0$. The reduced symplectic form on $S$ is a Kähler form.

**Proof.** The first assertion boils down to showing that if $H$ and $K$ are two closed subgroups of $G$, and $H^C$ and $K^C$ are conjugate in $G^C$, then $H$ and $K$ are conjugate in $G$. To say that $H^C$ and $K^C$ are conjugate in $G^C$ amounts to saying that there is a $G^C$-equivariant diffeomorphism of homogeneous spaces $f: G^C/H^C \to G^C/K^C$. By Mostow’s decomposition [18], for every closed subgroup $R$ of $G$ the complexified homogeneous space $G^C/R^C$ is a homogeneous vector bundle over $G/R$, so there exist a $G$-equivariant embedding $\iota_R: G/R \to G^C/R^C$ and a $G$-equivariant retraction $\rho_R: G^C/R^C \to G/R$. So the composite $\rho_R \circ f \circ \iota_R$ is a $G$-equivariant
map \( G/H \to G/K \). Therefore \( H \) is conjugate (in \( G \)) to a subgroup of \( K \). Switching the rôles of \( H \) and \( K \), we see that \( K \) is conjugate to a subgroup of \( H \). Therefore, since \( H \) and \( K \) have finitely many components, \( H \) is conjugate to \( K \) in \( G \).

For quotients of affine \( G^\mathbb{C} \)-varieties Luna proved in [26, §III.2] that each stratum is non-singular and that its closure is a variety. In view of the fact that \( \Pi \) is locally equivalent to a quotient map of an affine variety this implies the second statement of the theorem. (To be precise, Luna’s stratification is not the same as ours, but it is easy to see that they are the same up to connected components.)

Let \( S \) be the stratum of orbit type \((H)\). It is well-known that if \( G \) acts freely on the zero level set \( \Phi^{-1}(0) \) (which implies 0 is a regular value of \( \Phi \)) the reduced symplectic form is Kähler. (See e.g. [11].) Therefore, to prove that the reduced symplectic form on \( S \) is Kähler, it suffices to show that \( S \) can be obtained by carrying out reduction at a regular level on some Kähler manifold with respect to some group action.

Let \( N = N_G(H) \) be the normalizer of \( H \) in \( G \) and let \( M_H \) be the set of “symmetry type” \( H \), that is, the collection of all points whose stabilizer (with respect to the \( G \)-action) is exactly \( H \). Then \( M_H \) is a complex submanifold of \( M \), so it is Kähler. Moreover, \( M_H \) is \( N \)-invariant and \( H \) acts trivially on it. Let \( L = N/H \). By Theorem 3.5 of [43] the momentum map maps \( M_H \) into \( \mathfrak{l}^* \) and the restriction \( \Phi_H \) of \( \Phi \) to \( M_H \) is a momentum map for the \( N \)-action on \( M_H \). Moreover, 0 is a regular value of \( \Phi_H \) and the reduced space \( \Phi_H^{-1}(0) \) and the stratum \( S \) are symplectically diffeomorphic in a natural way.

Since \( M_H \) is complex, we have a well-defined action of \( L^\mathbb{C} = N^\mathbb{C}/H^\mathbb{C} \) on \( M_H \). Let \((M_H)^s = (M_H)^s \) denote the set of points in \( M_H \) stable with respect to the momentum map \( \Phi_H \). By Theorem 2.3 and Theorem 2.8 we have a map \((M_H)^s/L^\mathbb{C} \cong \Phi_H^{-1}(0)/L \to M_0\), which is a homeomorphism onto the image \( S \). To finish the proof, it suffices to show that this map is biholomorphic onto \( S \). Since \( \Phi(M_H) \) is a subset of \( \mathfrak{l}^* \), [24] implies that the flow of \( \|\Phi\|^2 \) leaves \( M_H \) invariant. Because also \( \Phi_H^{-1}(0) = \Phi^{-1}(0) \cap M_H \), we see that \((M_H)^s = M^s \cap M_H \). Moreover, we have a commutative diagram:

\[
\begin{array}{ccc}
(M_H)^s \overset{\Pi}{\longrightarrow} (M_H)^s/L^\mathbb{C} \\
i \downarrow & & \downarrow \tilde{i} \\
M^s \overset{\Pi_H}{\longrightarrow} M^s/G^\mathbb{C},
\end{array}
\]

where the inclusion \( \tilde{i} \) is biholomorphic onto its image. From the definition of the complex structures on \( M^s \// G^\mathbb{C} \) and \((M_H)^s/L^\mathbb{C} \) it now follows that \( \tilde{i} \) is biholomorphic onto its image. \( \blacksquare \)

Remark 2.10. The orbit type stratification is the minimal real-analytic Whitney stratification of \( M_0 \). However, it is not the minimal complex-analytic stratification. This is obvious from the following simple example. The \((1,-1)\)-resonance is the \( S^1 \) action on \( \mathbb{C}^2 \) defined by \( e^{\sqrt{-1}\theta}(z_1,z_2) = (e^{\sqrt{-1}\theta}z_1,e^{-\sqrt{-1}\theta}z_2) \). As a real-analytic space the reduced space is isomorphic to the cone in \( \mathbb{R}^3 \) given by \( x_1^2 = x_2^2 + x_3^2 \) and \( x_1 \geq 0 \). (See [4].) There are two strata: the vertex and the complement of the vertex. But from the complex-analytic point of view the singularity at the vertex is spurious: The ring of \( \mathbb{C}^\times \)-invariant polynomials is just \( \mathbb{C}[z_1z_2] \), so the quotient is simply \( \mathbb{C} \).
2.2. The integral case. The most important special case of the situation of
the previous section is that of a positive holomorphic line bundle over a complex
manifold $M$, that is, a holomorphic line bundle $\rho: L \to M$ with Hermitian fibre metric
$\langle \cdot, \cdot \rangle$ and curvature form $\Theta$ such that the real $(1,1)$-form $\omega = -(2\pi \sqrt{-1})^{-1} \Theta$
is Kähler. Recall that $\Theta$ is the unique two-form on $M$ satisfying $\rho^* \Theta = \partial \bar{\partial} \log r^2$,
where $r: L \to \mathbb{R}$ is the length function, $r(l) = \langle l, l \rangle^{1/2}$. The Kähler class $[\omega]$ is the
image of the Chern class $c_1(L)$ of $L$ under the natural map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$.

Now suppose that the compact group $G$ acts on $L$ by linear bundle transforma-
tions that leave the Hermitian metric invariant. Then the connection on $L$ is
invariant and $\omega$ is invariant under the induced action on the base $M$. This implies
that for each $\xi$ in $\mathfrak{g}$ there exists a unique real-valued function $\Phi^\xi$ on $M$ such that
the vector field $\xi_L$ is given by the following formula:

$$\xi_L = \xi_{M,\text{hor}} + 2\pi \Phi^\xi \nu_L. \quad (2.3)$$

Here $\xi_{M,\text{hor}}$ is the horizontal lift of $\xi_M$ to $TL$ with respect to the Hermitian connection,
and $\nu_L$ denotes the vector field on $L$ generating the circle action defined by
fibrewise multiplication by complex numbers of length one. It is not hard to check
that $\Phi^\xi$ is a Hamiltonian for the vector field $\xi_M$, and therefore the action on $M$ is
Hamiltonian. The momentum map $\Phi: M \to \mathfrak{g}^*$ is automatically equivariant. The
infinitesimal action of $G$ on smooth sections $s$ of $L$ is given by:

$$\xi \cdot s = -\nabla_{\xi_M} s + 2\pi \sqrt{-1} \Phi^\xi \cdot s.$$

As before, let us assume that the $G$-action on $M$ extends to a $G^C$-action and that
the momentum map $\Phi$ is admissible. Then the $G$-actions on $L$ and on its smooth
sections can both be uniquely extended to actions of $G^C$, and the projection $\rho$ is
$G^C$-equivariant. (See [4].)

Note that a holomorphic section of $L$ defined over a $G^C$-invariant open set is
$G$-invariant if and only if it is $G^C$-invariant. Let $\mathcal{L}$ be the sheaf of holomorphic
sections of $L$ and define a sheaf $\mathcal{L}_0$ on $M_0$, the sheaf of invariant sections, by putting
$\mathcal{L}_0(\mathcal{O}) = \mathcal{L}(\Pi^{-\infty}(\mathcal{O}))^G$. According to Roberts [4], $\mathcal{L}_0$ is a coherent $\mathcal{O}_{M,0}$-module.
The following result says that $\mathcal{L}_0$ is “almost” a holomorphic line bundle over $M_0$.

**Proposition 2.11.** The sheaf $\mathcal{L}_0$ is (the sheaf of sections of) a holomorphic $V$-line
bundle over $M_0 = M^G//G^C$.

**Proof.** Let $p \in M_0$. We have to show there exist a neighbourhood $O$ of $p$ that can
be written as a quotient of an analytic space $\tilde{O}$ by the action of a finite group $\Gamma$ and
a locally free sheaf $\tilde{\mathcal{L}}_0$ of rank one over $\tilde{O}$ such that $\tilde{\mathcal{L}}_0|_{\mathcal{O}}$ is isomorphic to the
sheaf of $\Gamma$-invariant sections of $\mathcal{L}_0$. Let $m$ be a point in $\Phi^{-1}(0)$ mapping to $p$ and
let $s$ be a slice at $m$ such that $U = G^C S$ is $\Pi$-saturated. Put $O = \Pi(U)$; then
$O \cong S//H^C$ as analytic spaces. Here $H$ denotes the stabilizer $G_m$ of $m$. The group
$H$ acts linearly on the fibre $L_m$. If $\eta \in \mathfrak{h}$ and $l \in L_m$, then (2.3) implies $\eta_L \cdot l = 0$,
since $\Phi(m) = 0$. In other words, the identity component $H^0$ acts trivially on $L_m$.

Now let $\mathcal{N}$ be the restriction of $\mathcal{L}$ to $S$ and let $s$ be a holomorphic section of
$\mathcal{N}$ that does not vanish at $m$. Then the section $\int_{H^0} h \cdot s \, dh$ is holomorphic, $H^C$-
invariant and does not vanish at $m$, because $H^0$ acts trivially on $L_m$. Hence we
may assume $s$ to be $H^0$-invariant. Define $\tilde{O} = S//(H^0)^C$ and let $\tilde{\mathcal{L}}_0$ be the sheaf
of $(H^0)^C$-invariant sections of $\mathcal{N}$. After shrinking $\tilde{O}$ if necessary, we may assume $\tilde{s}$
vanishes nowhere on $\tilde{O}$, so $\tilde{\mathcal{L}}_0$ is a free sheaf of rank one on $\tilde{O}$. 

By construction $O$ is the quotient of $\tilde{O}$ by the finite group $H^C/(H^0)^C = H/H^0$, and $L_0/O$ is isomorphic to the sheaf of $H/H^0$-invariant sections of $\tilde{L}_0$. It follows $L$ is the sheaf of sections of a holomorphic $V$-line bundle over $M_0$. □

In fact, the total space $L_0$ of this $V$-line bundle is simply the quotient of $L|_{\Phi^{-1}(0)}$ by $G$. Furthermore, as an analytic space $L_0$ can be identified with a quotient $L^{ss}//G^C$, where $L^{ss}$ is by definition the restriction of $L$ to $M^{ss}$. The proofs of these facts are sufficiently similar to the proofs in Section 2.1 that I can omit them.

To get a genuine line bundle on $M_0$, we have to replace $L$ by a suitable power. The proof of Proposition 2.11 shows that for every $m \in \Phi^{-1}(0)$ the image of $G_m \to \text{Aut}(L_m)$ is a finite cyclic group. Let $q(m)$ be the order of this group.

**Proposition 2.12.** Suppose $\Phi$ is proper. Let $q$ be the least common multiple of the $q(m)$ for $m$ ranging over $\Phi^{-1}(0)$. Then $(L^q)_0 = (L^q)^{ss}//G^C$ is a line bundle over $M_0$ satisfying $\Pi^*(L^q)_0 = L^q|_{M^{ss}}$.

**Proof.** First note that since $\Phi$ is proper, its zero level set is compact and so contains only finitely many orbit types. Furthermore, $q(m)$ is not greater than the order of the component group $G_m/(G_m)^0$. Therefore the integer $q$ is well-defined. It has the property that $G_m$ acts trivially on the fibre $L^q_m$ for all $m \in \Phi^{-1}(0)$. As in the proof of Proposition 2.11 we conclude that at every semistable point there exists a non-vanishing invariant local holomorphic section of $L^q$. By means of these sections we can define local trivializations of $(L^q)_0$, so $(L^q)_0$ is a holomorphic line bundle over $M_0$.

Moreover, the existence of these sections implies $G_x$ acts trivially on $L^q_x$ for all semistable $x$. Using this one can easily show that the commutative diagram

$$
\begin{array}{ccc}
L^q|_{M^{ss}} & \longrightarrow & (L^q)_0 \\
\rho \downarrow & & \downarrow \\
M^{ss} & \longrightarrow & M_0 \\
\end{array}
$$

is a pullback diagram. Therefore $\Pi^*(L^q)_0 = L^q|_{M^{ss}}$. □

Grauert [9] has defined a (holomorphic) line bundle $E$ over a complex space to be negative if the zero section in $E$ has a strictly pseudoconvex open neighbourhood. He called a bundle positive if its dual is negative. To show that $(L^q)_0$ is positive, we first need to discuss potentials for the reduced Kähler structure.

If $m \in \Phi^{-1}(0)$, a potential for the Kähler form $\omega$ on an open neighbourhood $U$ of $m$ is given by $u = -(2\pi)^{-1}\log(s, s)$, where $s$ is a local holomorphic section of $L$ that does not vanish at $m$. If $G$ acts freely on $\Phi^{-1}(0)$, the proof of Proposition 2.11 shows we can find an invariant such section. Then $u$ is a $G$-invariant smooth potential near $m$, and its restriction to $U \cap \Phi^{-1}(0)$ pushes down to a smooth function $u_0$ defined on $U_0 = (U \cap \Phi^{-1}(0))/G$. It is easy to see that $u_0$ is a potential for the reduced symplectic form $\omega_0$.

If $G$ does not act freely on $\Phi^{-1}(0)$, it may not be possible to find such a section, but we can certainly find an invariant local holomorphic section $s$ of the $q$-th power of $L$ that does not vanish at $m$. Then $u = -(2\pi q)^{-1}\log(s, s)$ is a $G$-invariant potential for $\omega$ near $m$, and as before its restriction to $U \cap \Phi^{-1}(0)$ pushes down to a function $u_0$ on the reduced space. Unfortunately, $u_0$ is not necessarily smooth or even $C^1$ on $M_0$. (By a smooth function on $M_0$ we mean a function that can be locally written as a differentiable function of the holomorphic functions on $M_0$.)

This notion of smooth functions differs from the one introduced in [43]. This is clear from the example in Remark 2.11, where \( u(z_1, z_2) = (|z_1|^2 + |z_2|^2)/2 \) and \( u_0(w) = |w|^2/2 \), with \( w = z_1z_2 \). Nonetheless, we claim \( u_0 \) is strictly plurisubharmonic in the sense of distributions. Recall that a continuous function \( f \) on \( M_0 \) is plurisubharmonic if for all discs \( D \subset \mathbb{C} \) and all analytic maps \( c: D \to M_0 \) the distribution \( \Delta (c^* f) \) is non-negative, where \( \Delta \) is the standard Laplacian on \( \mathbb{C} \).

It is strictly plurisubharmonic if for all smooth \( g \) with compact support the function \( f + \varepsilon g \) is plurisubharmonic for small \( \varepsilon \). (See Grauert and Remmert [10] and Lelong [24, p. 46].)

**Lemma 2.13.** The continuous function \( u_0 \) is strictly plurisubharmonic on the open subset \( U_0 \) of the analytic space \( M_0 \). It is a potential for the reduced Kähler structure in the following sense: Let \( S \) be any orbit type stratum in \( M_0 \) and let \( \omega_S \) be the reduced symplectic form on \( S \). Then \( u_0 \) is smooth on \( U_0 \cap S \) and there it satisfies

\[
\omega_S = \sqrt{-1} \partial \bar{\partial} u_0. 
\]

**Proof.** The second statement is an immediate consequence of the observation that the stratum \( S \) in \( M_0 \) of orbit type \((H)\) can be written as a quotient \( S = \Phi_H^{-\infty}(i)/\mathcal{L} = (\mathcal{M}_H)^*/\mathcal{L} \), where \( \mathcal{L} = N_G(H)/H \) and \( \mathcal{L} \) acts properly and freely on \((\mathcal{M}_H)^*\), as in diagram (2.2). The function \( u_{0|S} \) is equal to the pushforward of \( u_{|M_0} \) under the map \( \Phi_H^{-1}(0) \to S \). Therefore it is smooth and satisfies \( \omega_S = \sqrt{-1} \partial \bar{\partial} u_0 \mid_S \).

In particular, \( u_0 \) is strictly plurisubharmonic on every stratum of \( M_0 \). To see it is strictly plurisubharmonic as a function on \( M_0 \), we first consider the special case where \( M = \mathbb{C}^n \) is a \( G \)-representation with standard momentum map \( \Phi = \Phi_{\mathbb{C}^n} \) and standard flat metric with potential \( u = \|z\|^2/2 \). We embed the quotient \( M_0 = \mathbb{C}^n/G \) into \( \mathbb{C}^1 \) using homogeneous invariant complex polynomials \( \sigma_1, \ldots, \sigma_n \) as in the proof of Theorem 2.4. We shall identify \( M_0 \) with its image \( \sigma(M_0) \subset \mathbb{C}^1 \). Let \( w = (w_1, \ldots, w_l) \) be coordinates on \( \mathbb{C}^1 \). We claim \( u_0 \) is strictly plurisubharmonic at the “vertex” \( 0 \in M_0 \subset \mathbb{C}^1 \). It suffices to show that for sufficiently small \( \varepsilon \) the function \( u_0 - \varepsilon \|w\|^2 \) is plurisubharmonic close to the vertex. (For simplicity we have written \( \|w\|^2 \) for the restriction of \( \|w\|^2 \) to \( M_0 \).) Observe \( u_0 - \varepsilon \|w\|^2 \) is continuous on \( M_0 \), so by the extension theorem for plurisubharmonic functions of Grauert and Remmert [10] it suffices to show the restriction of \( u_0 - \varepsilon \|w\|^2 \) to the complement of a thin subset is plurisubharmonic close to the vertex. By Theorem 2.4 the complement of the top dimensional stratum \( T \) is a thin subset of \( M_0 \).

We now exploit the fact that the cone \( M_0 \) is quasi-homogeneous in \( \mathbb{C}^1 \). Consider the action \( A_\lambda \) of the positive real numbers on \( \mathbb{C}^n \), defined by scalar multiplication, \( A_\lambda z = \lambda z \). Let \( d_1, \ldots, d_l \) be the degrees of the homogeneous polynomials \( \sigma_1, \ldots, \sigma_l \), and define an action \( A_\lambda \) of \( \mathbb{R}_{>0} \) on \( \mathbb{C}^1 \) by putting \( A_\lambda(w_1, \ldots, w_l) = (\lambda^{d_1}w_1, \ldots, \lambda^{d_l}w_l) \). Then the Hilbert map \( \sigma: \mathbb{C}^n \to \mathbb{C}^1 \) is equivariant, \( A_\lambda \circ \sigma = \sigma \circ A_\lambda \), and the stratum \( T \) is \( A \)-invariant. The Hermitian bilinear forms on \( T \) corresponding to the real \((1, 1)\)-forms \( \sqrt{-1} \partial \bar{\partial} u_0 \mid_T \) and \( \sqrt{-1} \partial \bar{\partial} \|w\|^2 \mid_T \) are positive definite. Moreover, the flat metric on \( \mathbb{C}^n \) is conical, that is, \( A_\lambda^* (\partial \bar{\partial} u) = \lambda^2 \partial \bar{\partial} u \), and therefore the induced metric on the quotient is conical, \( A_\lambda^* (\partial \bar{\partial} u_0) = \lambda^2 \partial \bar{\partial} u_0 \). On the other hand,

\[
A_\lambda^* (\partial \bar{\partial} \|w\|^2) = A_\lambda^* \left( \sum_{ij} dw_i \wedge dw_j \right) = \sum_{ij} \lambda^{d_i + d_j} dw_i \wedge dw_j.
\]

Since \( d_i \geq 1 \) for all \( i \), we see that for sufficiently small \( \varepsilon \) the bilinear form corresponding to \( \sqrt{-1} \partial \bar{\partial} (u_0 - \varepsilon \|w\|^2) \mid_T \) is positive semidefinite on \( T \cap B \), where \( B \) is a
small ball about the origin in $\mathbb{C}^d$. Consequently $u_0 - \varepsilon \|w\|^2$ is plurisubharmonic on $M_0 \cap B$ for small $\varepsilon$, and so $u_0$ is strictly plurisubharmonic at the vertex $0 \in M_0$.

Now let $M$ be arbitrary and consider any point $m \in \Phi^{-1}(0)$. Let $H = G_m$ and let $V = (T_m(G^C_0)) \perp$ be the tangent space to the holomorphic slice at $m$. Then $V$ is a Hermitian vector space and by the Holomorphic Slice Theorem we have an analytic isomorphism $V//H^C \to M^s//G^C$ defined near the vertex $0$ of $V//H^C$ and mapping $0$ to $\Pi(m)$. We now have two Kähler metrics on the top stratum defined near $\Pi(m)$, namely the metric $ds^2_0$ with potential $u_0$ induced by the metric on $M$, and the metric $ds^2_0$ with potential $\tilde{u}_0$ induced by the flat metric on $V$. These metrics are not the same, but they are quasi-isometric near $\Pi(m)$ in the sense that there is an estimate of the type $C \Re ds^2_0 \leq \Re ds^2_0 \leq C^{-1} \Re ds^2_0$ in $O \cap T$, where $O$ is a neighbourhood of $\Pi(m)$ in $M_0$. From this and from the fact proved above that $\tilde{u}_0$ is strictly plurisubharmonic at $\Pi(m)$, it follows that $u_0$ is also strictly plurisubharmonic at $\Pi(m)$. $\square$

We conclude the analytic space $M_0$ is a Kähler space as defined by Grauert [3], if we extend Grauert’s definition to include local potentials that are not $C^2$.

**Theorem 2.14.** Assume $\Phi$ is proper. Let $L^*$ be the line bundle $(L^q)_0$, where $q$ is as in Proposition 2.13. Then $L$ is positive in the sense of Grauert. The reduced space $M_0$ is a complex-projective variety, a projective embedding being given by the Kodaira map $M_0 \to P\left(H^0(M_0, L^k)\right)$ for all sufficiently large $k$.

**Proof.** Let $L^*$ be the dual of $L$. We have to show the zero section of $L^*$ possesses a strictly pseudoconvex open neighbourhood. The fibre metric $\langle \cdot, \cdot \rangle_0$ on $L^q$ pushes down to a fibre metric $\langle \cdot, \cdot \rangle_0$ on $L$. Let $\langle \cdot, \cdot \rangle_0$ be the fibre metric on $L^*$ obtained by duality. On $L^q$ we have the distance function $r(l) = (l, l)^{1/2}$. Let $r_0$ and $r^*_0$ be the corresponding functions on $L$, resp. $L^*$. Let $\Delta \subset L^*$ be the tubular domain \{ $l : r^*_0(l) \leq 1$ \}.

In a local trivialization $(z, \zeta)$ of $L$ over an open subset $O$ of $M_0$ we can write $r_0(z, \zeta) = h_0(z) |\zeta|^2$ for a certain positive function $h_0$ on $O$. We can use the coordinates $(z, \zeta)$ to trivialize $L^*$ over $O$; then $r^*_0(z, \zeta) = h_0(z)^{-1} |\zeta|^2$. Also, $\Delta \cap (\rho_0^{-1})^{-1}(O)$ is given by $|\zeta|^2 \leq h_0(z)$, where $\rho_0^*: L^* \to M_0$ is the bundle projection. Up to a positive constant factor the function $u_0 = - \log h_0$ is a local potential for the reduced Kähler structure, so by Lemma 2.13 it is strictly plurisubharmonic. It follows immediately that $\Delta \cap (\rho_0^{-1})^{-1}(O)$ is strictly pseudoconvex in $(\rho_0^{-1})^{-1}(O)$. Thus we have shown $\Delta$ is a strictly pseudoconvex subset of $L^*$.

For the second part of the theorem, apply Grauert’s generalization of Kodaira’s Embedding Theorem, [3] §3, Satz 2. $\square$

Let us call a point $x \in M$ algebraically semistable if there exists an invariant global holomorphic section $s \in \Gamma(M, L^G)$ of some power $L^k$ of $L$ such that $s(x) \neq 0$. The point $x$ is called algebraically stable if in addition $G^C$ acts properly on the open set \{ $x \in M : s(x) \neq 0$ \}. If $M$ is algebraic, for instance, if $M$ is compact, these notions coincide with the ones introduced by Mumford [4] (except that Mumford uses the term “properly stable” where most authors nowadays use “stable”).

**Theorem 2.15.** If $\Phi$ is proper, the quotient map $\Pi: M^s \to M_0$ and the inclusion $M^s \subset M$ induce isomorphisms $\Gamma(M_0, L_0) \cong \Gamma(M^s, L)^G \cong \Gamma(M, L)^G$. It follows that a point in $M$ is analytically (semi)stable if and only if it is algebraically (semi)stable.
Proof. The first isomorphism follows from Proposition 2.11. The second isomorphism follows from the observation, essentially due to Guillemin and Sternberg [11], that the norm of an invariant holomorphic section \( s \) of \( L \) is increasing along the trajectories of \(-\operatorname{grad} \mu\). Indeed, for any invariant holomorphic section \( s \) defined on a \( G^C \)-invariant open subset and any \( \xi \in \mathfrak{g} \) we have \( J\xi_M \langle s, s \rangle = -4\pi \Phi^\xi \langle s, s \rangle \) (see [11]), so using (2.3) we get for any \( x \in M \):

\[
\frac{d}{dt} \langle s(F_t x), s(F_t x) \rangle = -\operatorname{grad} \langle s, s \rangle (F_t x) = -2J\Phi (F_t x) M (\langle s, s \rangle) (F_t x) = 8\pi \| \Phi (F_t x) \|^2 \langle s(F_t x), s(F_t x) \rangle = 8\pi \mu (F_t x) \langle s(F_t x), s(F_t x) \rangle \geq 0.
\]

It follows that for all \( x \in M^\text{ss} \) the restriction of the function \( \langle s, s \rangle \) to \( \overline{G^C x} \) takes on its maximum at \( F_\infty x \). Therefore, if \( s \) is defined on all of \( M^\text{ss} \), \( \langle s, s \rangle \) is bounded on \( M \), since \( \Phi \) is proper. An application of Riemann’s Extension Theorem now gives \( \Gamma (M^\infty, L^G) \cong \Gamma (M, L^G) \).

Now suppose \( x \in M \) is algebraically semistable. Then there exists \( s \in \Gamma (M, L^G) \) such that \( s(x) \neq 0 \). As \( t \) tends to infinity, \( F_t x \) approaches the critical point \( F_\infty x \), so \( \operatorname{grad} \mu (F_t x) \rightarrow 0 \). Letting \( t \rightarrow \infty \) in (2.4) we obtain

\[
\mu (F_\infty x) \langle s(F_\infty x), s(F_\infty x) \rangle = 0.
\]

But since \( F_\infty x \in \overline{G^C x} \), we have \( s(F_\infty x) = s(x) \neq 0 \). Therefore \( \mu (F_\infty x) = 0 \), in other words \( x \) is analytically semistable.

Conversely, suppose \( x \in M \) is analytically semistable, that is, \( \Phi (m) = 0 \), where \( m = F_\infty x \). By Proposition 2.13, \( L = (L^q)_0 \) is an ample bundle on \( M_0 \), so \( L^r \) is generated by global sections for big \( r \). Let \( s_0 \) be a global section of \( L^r \) with \( s_0 (\Pi (x)) = s_0 (\Pi (m)) \neq 0 \). By the first part of the theorem (applied to the bundle \( L^q \)) we can lift \( s_0 \) to a global invariant section \( s \in \Gamma (M, L^q)^G \). Evidently, \( s(x) \neq 0 \), so \( x \) is algebraically semistable.

One proves the equivalence of analytic and algebraic stability in a similar way, using Theorem 2.8.

To summarize, the set \( M - M^\text{ss} \) can be characterized as the collection of points where all invariant global holomorphic sections of all powers of \( L \) vanish. Also, the algebraic structure of \( M_0 \) depends only on the line bundle \( L \) and the lift of the \( G \)-action to \( L \), not on the symplectic form \( \omega \) or the momentum map \( \Phi \). (The symplectic structure of course does depend on \( \omega \) and \( \Phi \).) If \( M \) is compact, we conclude that as a projective variety \( M_0 \) is nothing but the quotient defined by Mumford [33].

2.3. Multiplicity formulae. In this section I have a stab at the “geometric multiplicity theory” of singular symplectic quotients. Let \( M \) be a Kähler manifold furnished with \( G \)-equivariant “prequantum data” \((L, \langle \cdot, \cdot \rangle)\) and momentum map \( \Phi \) as in the previous section. As before, let us assume that the \( G \)-action on \( M \) extends to a \( G^C \)-action. Let us also suppose for simplicity that the map \( \Phi \) is proper and that the group \( G \) is connected. Ideally, one would like to show that as a \( G \)-representation the space of sections \( \Gamma (L) \), sometimes called a “quantization” of \( M \), is a symplectic
invariant of $M$, in other words, that it is independent of the choice of the complex structure and the line bundle on $M$. One way of doing this would be to express the multiplicities of the unitary irreducible representations occurring in $\Gamma(L)$ in terms of symplectic data involving the reduced phase spaces $M_\lambda = \Phi^{-1}(G\lambda)/G$, where $\lambda$ ranges over the positive weights in the dual of a maximal torus of $G$. Guillemin and Sternberg [1] carried this out for those weights $\lambda$ for which $G$ acts freely on $\Phi^{-1}(G\lambda)$. (Then the reduced space $M_\lambda$ is non-singular.) Heckman [13] had earlier obtained related results in the important special case where $M$ is a coadjoint orbit of a big group containing $G$ as a subgroup. Using the results of the previous section, we can generalize their results.

If we regard the space of sections $\Gamma(M_0, L_0)$ of the $V$-line bundle $L_0$ as the quantization of the reduced space $M_0$, then Theorem 2.15 bears out the principle that quantization should commute with reduction.

**Remark 2.16.** The theory of geometric quantization is usually phrased in terms of polarizations, that is, involutive Lagrangian subbundles of the complexified tangent bundle of $M$. The quantization of $M$ with respect to a polarization $P$ is then the space of polarized sections of $L$, that is, sections $s$ such that $\nabla_X s = 0$ for all vectors $X$ tangent to the conjugate subbundle $\overline{P}$. It appears to be difficult to make sense of the notion of a polarization on a singular space, such as a symplectic quotient. In the case of a Kähler quotient $M_0$, however, the sheaf of holomorphic functions $\mathcal{O}_M$ seems to be a workable substitute for a polarization.

**Remark 2.17.** Properly speaking, the quantization of $M$ is not just the space of holomorphic sections of $L$, but the virtual representation $\bigoplus_i (-1)^i H^i(M, L)$, including all cohomology groups with coefficients in $L$. One might wonder whether it is true that reduction commutes with quantization in this broader sense, that is, $\bigoplus_i (-1)^i H^i(M, L)^G = \bigoplus_i (-1)^i H^i(M_0, L_0)$. It is not hard to prove that $H^i(M_0, L_0)$ is isomorphic to $H^i(M_{\text{ss}}, L)^G$ for all $i$, but I don’t know if $H^i(M_{\text{ss}}, L)^G$ is isomorphic to $H^i(M, L)^G$ for $i > 0$.

Theorem 2.15 obviously implies that the dimension of $\Gamma(M_0, L_0)$ is equal to the multiplicity of the one-dimensional trivial representation in $\Gamma(M, L)$. Let us briefly recall from [1] how this statement generalizes to arbitrary multiplicities by dint of the shifting trick. Choose a maximal torus $t$ in $\mathfrak{g}$ and a positive Weyl chamber $t^*_+\mathfrak{g}^*$ in $\mathfrak{g}^*$. For every positive weight $\lambda \in t^*_+$, the coadjoint orbit $G\lambda$ is a Kähler manifold carrying a naturally defined Hermitian line bundle $V_\lambda$, and the Borel-Weil Theorem asserts that $\Gamma(G\lambda, V_\lambda)$ is the unitary irreducible representation with highest weight $\lambda$. Let $G\lambda^-$ be the orbit $G\lambda$ with the opposite symplectic and complex structures and consider the Kähler manifold $M \times G\lambda^-$. Let $\pi_M$ and $\pi_{G\lambda}$ denote the projections of $M \times G\lambda^-$ on the respective factors and let $V_\lambda^*$ be the dual of $V_\lambda$. Then the Hermitian line bundle $\pi_M^*L \otimes \pi_{G\lambda}^*V_\lambda^*$ prequantizes $M \times G\lambda^-$. The reduced space at $0$ of $M \times G\lambda^-$ can be identified with $M_\lambda$, the reduced space of $M$ at the orbit $G\lambda$, and it comes equipped with a $V$-line bundle $L_\lambda = L|_{\Phi^{-1}(G\lambda)}/G$. By Theorem 2.15, $\Gamma(M_\lambda, L_\lambda)$ is isomorphic to the space of $G$-invariants in $\Gamma(M \times G\lambda^-, \pi_M^*L \otimes \pi_{G\lambda}^*V_\lambda^*)$. The Künneth theorem for coherent sheaves [1] now implies the following assertion.

**Theorem 2.18.** For every positive weight $\lambda$ of $G$ the space of sections $\Gamma(M_\lambda, L_\lambda)$ is naturally isomorphic to the space of intertwining operators

$$\text{Hom}(\Gamma(G\lambda, V_\lambda), \Gamma(M, L))^G.$$

\[\square\]
Corollary 2.19. If the orbit \( G\lambda \) does not lie in the image of the momentum map, the irreducible representation corresponding to \( \lambda \) does not occur in \( \Gamma(M, L) \). □

Let us write \( \mu(\lambda, L) \) for the multiplicity of the representation with highest weight \( \lambda \) occurring in \( \Gamma(M, L) \). By Theorem 2.18 \( \mu(\lambda, L) \) is equal to the dimension of \( \Gamma(M, \lambda L) \). By Theorem 2.14 for certain \( q \) (possibly depending on \( \lambda \)) the sheaf \( L = (L^q)_\lambda \) is an ample line bundle on the projective variety \( X = M_\lambda \), so for all sufficiently large \( r \) we have \( H^i(X, L^r) = 0 \) for \( i > 0 \). Then \( \mu(rq\lambda, L^r) \) is equal to the Euler characteristic \( \chi(X, L^r) \), so by the Hirzebruch-Riemann-Roch Theorem of Baum, Fulton and MacPherson \[1, 2\]

\[
\mu(rq\lambda, L^r) = \chi(X, L^r) = \varepsilon(ch L^r \cap \tau(X)). \tag{2.5}
\]

Here \( \tau(X) \) denotes the homological Todd class of \( X \), \( ch L^r \) denotes the Chern character of \( L^r \) and \( \varepsilon \) is the augmentation (the map \( H_\tau(X) \to C \) induced by mapping \( X \) to a point). If \( X \) is non-singular, \( \tag{2.5} \) comes down to the classical Hirzebruch-Riemann-Roch Theorem. As was pointed out by Guillemin and Sternberg, \( \mu(rq\lambda, L^r) \) is then a symplectic invariant, i.e., independent of the complex structure and the line bundle \( L \) on \( M \). It seems likely that this is also true if \( X \) is singular. I cannot quite prove this, but here follows some evidence.

For arbitrary singular spaces, the Todd class appears to be intractable, but for spaces with quotient singularities, such as \( X \), the situation is simpler. Namely, a theorem of Boutot \[3\] asserts that quotient singularities are rational, i.e.,

\[
f_* \mathcal{O}_Y = \mathcal{O}_X \quad \text{and} \quad \mathcal{R} \{ f_* \mathcal{O}_Y = t \quad \text{for} \} > t, \tag{2.6}
\]

where \( f: Y \to X \) is a resolution of singularities of \( X \). Therefore, by the functorial properties of the Todd class, \( \tau(X) = f_* (\tau(Y)) \), so by \( \tag{2.5} \) \( \mu(rq\lambda, L^r) = \chi(X, L^r) \) is equal to \( \chi(Y, f^* L^r) \).

In general, it is difficult to write down a desingularization of \( X \), but Kirwan \[22\] has explicitly constructed a “partial” resolution \( p: \tilde{X} \to X \). It has all the properties of a desingularization, except that it is not a smooth projective variety, but a complex-projective \( V \)-manifold (or orbifold). It is easy to see that the vanishing property \( \tag{2.7} \) also holds for the partial resolution \( p: \tilde{X} \to X \), and therefore \( \mu(rq\lambda, L^r) = \chi(X, p^* L^r) \). To construct a partial resolution, one performs a certain sequence of blowups on \( M \times GA^- \) at \( GC \)-invariant submanifolds, yielding a projective manifold \( \tilde{M} \) with a \( GC \)-action. The symplectic form on \( M \times GA^- \) pulls back to a degenerate \( (1, 1) \)-form \( \tilde{\omega} \) on \( \tilde{M} \), which descends to a degenerate \( (1, 1) \)-form \( \omega \) on \( \tilde{X} \). The class of \( q\tilde{\omega} \) is the Chern class of the pull-back \( p^* L \) of \( L \). To get a Kähler form on \( \tilde{M} \), one adds to the pullback of \( \omega \) at each stage in the sequence of blowups a small \( (1, 1) \)-form \( \sigma_\varepsilon \) supported on a neighbourhood of the exceptional divisor, such that the class of \( \sigma_\varepsilon \) is equal to \( \varepsilon \) times the dual class of the exceptional divisor. One then obtains \( \tilde{X} \) by taking the quotient of \( \tilde{M}; \tilde{X} = \tilde{M}^*/GC \simeq M_\theta \). See \[22\] for the details. Using Kawasaki’s formula \[11\], we can now write the multiplicity as a sum of integrals (still assuming that \( r \) is sufficiently large):

\[
\mu(rq\lambda, L^r) = \int_{\tilde{X}} e^{rq\tilde{\omega}} \wedge \det \frac{\sqrt{-1} R/2\pi}{1 - e^{-\sqrt{-1} R/2\pi}} + \Sigma, \tag{2.7}
\]

where \( R \) denotes the curvature two-form of \( \tilde{X} \) with respect to the Kähler metric, and where \( \Sigma \) denotes a sum of contributions from the singular strata in \( \tilde{X} \). In a local \( V \)-manifold chart, \( \Sigma \) can be written as a sum of integrals over fixed-point
manifolds, as in the holomorphic Lefschetz formula. Since the right-hand side is a function of the cohomology class of \(\tilde{\omega}_\lambda\) only, we conclude that it is a polynomial function of \(r\), and hence \(\mu(rq\lambda, L^r)\) is polynomial in \(r\) for \(r\) large. If the dimension of \(X\) is \(2n\), the form \(\tilde{\omega}_\lambda\) enters in the term \(\Sigma\) with exponents less than \(n\). The highest-order term in the multiplicity is therefore the term in \(r^n\), and the coefficient is \(q^n \int_X \tilde{\omega}_\lambda^n/n!\), which is equal to \(q^n \text{vol} X\), where \(\text{vol} X\) is the symplectic volume of the top-dimensional stratum of \(X = M_\lambda\). In particular, we see that the highest-order term is a symplectic invariant.

The following points seem to call for further clarification: (i) Is it really necessary in (2.5) and (2.7) to replace the \(V\)-bundle \(L_\lambda\) by the line bundle \(L = (L^9)_\lambda\)? In other words, does the Riemann-Roch formula of Baum, Fulton and MacPherson work for \(V\)-bundles on \(M_\lambda\)? It seems reasonable to guess that \(\chi(M_\lambda, L_\lambda)\) equals \(\chi(M_\lambda, f^*L_\lambda)\), which can then be computed by Kawasaki’s recipe. (ii) Under what conditions does vanishing of the cohomology of \(M\) with coefficients in \(L\) imply vanishing of the cohomology of the quotients? (iii) More importantly, the right-hand side of (2.7) makes sense even if the symplectic manifold \(M\) does not carry a complex structure: Kirwan’s partial resolution can be defined for any singular symplectic quotient, and all one needs to write down the form representing the Todd class is an almost-complex structure compatible with the symplectic form. It would be interesting to find out in how far (2.7) can be generalized to this more general situation.

References

[1] P. Baum, Riemann-Roch theorem for singular varieties, Differential Geometry (Stanford, 1973) (S. S. Chern and R. Osserman, eds.), Proceedings of Symposia in Pure Mathematics, vol. 27, Amer. Math. Soc., Providence, R. I., 1975, pp. 3–16.
[2] P. Baum, W. Fulton, and R. MacPherson, Riemann-Roch for singular varieties, Inst. Hautes Études Sci. Publ. Math. 45 (1976), 101–167.
[3] J.-F. Boutot, Singularités rationnelles et quotients par les groupes réductifs, Invent. Math. 88 (1987), 65–68.
[4] R. Cushman and R. Sjamaar, On singular reduction of Hamiltonian spaces, Symplectic Geometry and Mathematical Physics (Aix-en-Provence, 1990) (P. Donato et al., eds.), Progress in Mathematics, vol. 99, Birkhäuser, Boston, 1991.
[5] J. J. Duistermaat, Multiplicités de représentations de groupes et l’espace de phase réduit, Actions hamiltoniennes de groupes—Troisième théorème de Lie (Journées lyonnaises de la SMF, 1986) (P. Dazord et al., eds.), Travaux en cours, vol. 27, Séminaire sud-rhodanien de géométrie VIII, Hermann, Paris, 1988, pp. 71–81.
[6] H. M. Farkas and I. Kra, Riemann surfaces, second ed., Graduate Texts in Mathematics, vol. 71, Springer-Verlag, Berlin-Heidelberg-New York, 1991.
[7] T. Frankel, Fixed points on Kähler manifolds, Ann. of Math. (2) 70 (1959), 1–8.
[8] H. Grauert, On Levi’s problem and the imbedding of real-analytic manifolds, Ann. of Math. (2) 68 (1958), no. 2, 460–472.
[9] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann. 146 (1962), 331–368.
[10] H. Grauert and R. Remmert, Plurisubharmonische Funktionen in komplexen Räumen, Math. Z. 65 (1956), 175–194.
[11] V. Guillemin and S. Sternberg, Geometric quantization and multiplicities of group representations, Invent. Math. 67 (1982), 515–538.
[12] ______, A normal form for the moment map, Differential Geometric Methods in Mathematical Physics (Jerusalem, 1982) (S. Sternberg, ed.), Mathematical Physics Studies, vol. 6, D. Reidel Publishing Company, Dordrecht, 1984, pp. 161–175.
[13] ______, Symplectic techniques in physics, Cambridge Univ. Press, Cambridge, 1990, second reprint with corrections.
[14] F. R. Harvey and R. O. Wells, Jr., Holomorphic approximation and hyperfunction theory on a $C^1$ totally real submanifold of a complex manifold, Math. Ann. 197 (1972), 287–318.
[15] G. J. Heckman, Projections of orbits and asymptotic behaviour of multiplicities for compact Lie groups, Ph.D. thesis, Rijksuniversiteit te Leiden, 1980.
[16] P. Heinzner, Geometric invariant theory on Stein spaces, Math. Ann. 289 (1991), 611–662.
[17] L. Hörmander, An introduction to complex analysis in several variables, third revised ed., North-Holland Mathematical Library, vol. 7, North-Holland, Amsterdam, 1991.
[18] K. Jänich, Differenzierbare G-Mannigfaltigkeiten, Lecture Notes in Mathematics, vol. 59, Springer-Verlag, Berlin-Heidelberg-New York, 1968.
[19] T. Kawasaki, The Riemann-Roch theorem for complex V-manifolds, Osaka J. Math. 16 (1979), 151–159.
[20] G. Kempf and L. Ness, The length of vectors in representation space, Algebraic Geometry (Copenhagen, 1978) (K. Lønsted, ed.), Lecture Notes in Mathematics, vol. 732, Springer-Verlag, Berlin-Heidelberg-New York, 1979, pp. 233–244.
[21] F. C. Kirwan, Cohomology of quotients in symplectic and algebraic geometry, Mathematical Notes, vol. 31, Princeton Univ. Press, Princeton, 1984.
[22] P. Lelong, Plurisubharmonic functions and positive differential forms, Notes on Mathematics and its Applications, Gordon and Breach, New York, 1969.
[23] E. Lerman and R. Sjamaar, Reductive group actions on Kähler manifolds, preprint, Univ. of Pennsylvania, 1992.
[24] D. Luna, Slices étales, Sur les groupes algébriques, Mém. Soc. Math. France 33 (1973), 81–105.
[25] J. D. Moore, Equivariant embeddings of Riemannian homogeneous spaces, Indiana Univ. Math. J. 25 (1976), no. 3, 271–279.
[26] G. D. Mostow, On covariant fiberings of Klein spaces, Amer. J. Math. 77 (1955), 247–278.
[27] J. D. Moore and R. Schafly, On equivariant isometric embeddings, Math. Z. 173 (1980), 119–133.
[28] G. W. Schwarz, The topology of algebraic quotients, Topological Methods in Algebraic Transformation Groups (Rutgers Univ., 1988) (Hanspeter Kraft et al., eds.), Progress in Mathematics, vol. 80, Birkhäuser, Boston, 1989, pp. 135–151.
[29] R. W. Richardson, Jr., Deformations of Lie subgroups and the variation of isotropy subgroups, Acta Math. 129 (1972), 57–73.
[30] M. Roberts, A note on coherent G-sheaves, Math. Ann. 275 (1986), 573–582.
[31] J. H. Sampson and G. Washnitzer, A Küneth formula for coherent algebraic sheaves, Illinois J. Math. 3 (1959), 389–402.
[32] G. W. Schwarz, The topology of algebraic quotients, Topological Methods in Algebraic Transformation Groups (Rutgers Univ., 1988) (Hanspeter Kraft et al., eds.), Progress in Mathematics, vol. 80, Birkhäuser, Boston, 1989, pp. 135–151.
[33] R. Sjamaar and E. Lerman, Stratified symplectic spaces and reduction, Ann. of Math. (2) 134 (1991), 375–422.
[44] D. M. Snow, *Reductive group actions on Stein spaces*, Math. Ann. **259** (1982), 79–97.

[45] A. J. Sommese, *Extension theorems for reductive group actions on compact Kähler manifolds*, Math. Ann. **218** (1975), 107–116.

[46] K. Trautman, *Orbits that always have affine stable neighbourhoods*, Adv. in Math. **91** (1992), no. 1, 54–63.

[47] A. Weinstein, *Lectures on symplectic manifolds*, CBMS Regional Conf. Series in Math., vol. 29, Amer. Math. Soc., 1983, third printing.

[48] H. Whitney and F. Bruhat, *Quelques propriétés fondamentales des ensembles analytiques-réels*, Comment. Math. Helv. **33** (1959), no. 2, 132–160.

University of Pennsylvania, Department of Mathematics, Philadelphia, Pennsylvania 19104

Current address: Massachusetts Institute of Technology, Department of Mathematics, Cambridge, Massachusetts 02139-4307

E-mail address: sjamaar@math.mit.edu