SOME REMARKS ON INVARIANT POISSON QUASI-NIJENHUIS STRUCTURES ON LIE GROUPS

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Abstract. We study right-invariant (resp., left-invariant) Poisson quasi-Nijenhuis structures on a Lie group $G$ and introduce their infinitesimal counterpart, the so-called $r$-qn structures on the corresponding Lie algebra $g$. We investigate the procedure of the classification of such structures on the Lie algebras and then for clarity of our results we classify, up to a natural equivalence, all $r$-qn structures on two types of four-dimensional real Lie algebras. We mention some remarks on the relation between $r$-qn structures and the generalized complex structures on the Lie algebras $g$ and also the solutions of modified Yang-Baxter equation on the double of Lie bialgebra $g \oplus g^\ast$. The results are applied to some relevant examples.

1. Introduction

Poisson quasi-Nijenhuis ($P$-qN) structures on manifolds were introduced by Stiénon and Xu [12] as triples $(\Pi, \Phi, N)$ on a manifold $M$ for which $\Pi$ is a Poisson 2-vector, $N$ is a $(1,1)$-tensor field and $\Phi$ is a closed 3-form, such that $\Pi$ and $N$ are compatible in the sense of Poisson-Nijenhuis structures [7] and the Nijenhuis torsion of $N$ is

$$[N, N](X, Y) = \Pi^2(\iota_X \wedge \iota_Y \Phi), \quad \forall X, Y \in \mathfrak{X}(M).$$

In this work we study Poisson quasi-Nijenhuis structures on a Lie group $G$ which are appropriate right-invariant (or resp. left-invariant), so-called right-invariant $P$-qN structures (or resp. left-invariant $P$-qN structures) on $G$; we introduce their infinitesimal counterpart, the objects which we called $r$-qn structures on the Lie algebra $g$ of $G$.

In fact Poisson-Nijenhuis structures are trivial Poisson quasi-Nijenhuis structures since for them $\Phi \equiv 0$. The infinitesimal counterpart of right-invariant $P$-N structures, the structures which we called them $r$-n structures, can be used to construct compatible solutions of classical Yang-Baxter equations (for more details see [10]).

Since in the $r$-qn structures, $(1,1)$-tensor field $N$ is not Nijenhuis torsion free, in general the 2-vector $nr$ is not an $r$-matrix. We show that in a certain condition we can have compatible $r$-matrices by $r$-qn structures.

In the following, we show that how we can obtain all $r$-qn structures on the Lie algebra $g$ with finite dimension, or equivalently all right-invariant $P$-qN structures on the connected simply-connected Lie group $G$ corresponding to $g$. Many of this structures would be equivalent by an Lie algebra automorphism, so in order to classify such structures we need to define an equivalence relation. We study the procedure of classification of $r$-qn structures and classify, up to a natural equivalence, all $r$-qn structures on two chosen real Lie algebras in dimension four, symplectic Lie algebra $A_{4,1}$ and non-symplectic Lie algebra $A_{4,8}$.

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In [12], the authors studied generalized complex structures in terms of Poisson quasi-Nijenhuis manifold; they showed that a generalized complex manifold corresponds to a spacial class of Poisson quasi-Nijenhuis structures. So, it would be of interest to have r-qn structures since whose spacial class corresponds to the generalized complex structures on the Lie algebra.

In fact in the infinitesimal level, where we deal with a Lie bialgebra, the generalized tangent bundle \( g \oplus g^* \) of \( g \) which is called the double of Lie bialgebra is equipped with a Lie algebra structure, so in this case a generalized complex structure on \( g \) can be viewed as a spacial class of solutions of modified Yang-Baxter equation on the Lie algebra \( g \oplus g^* \).

We shall give some relevant examples of r-qn structures on some Lie algebras \( g \) which can be considered as a generalized complex structure on \( g \) or as a solution of modified Yang-Baxter equation on \( g \oplus g^* \).

The outline of the paper is as follows: In Section 2 we briefly recall the notion of P-qN structures on a manifold. We also take a review on cohomology of Lie algebra and then definition of Lie bialgebras, classical and modified Yang-Baxter equation. We end this section by the review of generalized complex structures on a manifold. In Section 3 we define right-invariant P-qN structures on the Lie group \( G \) as the main object of study and introduce their infinitesimal counterpart; the compatibility and equivalency of such structures will be also considered in this section. In Section 4 we describe the systematic way to get all r-qn structures on \( g \), equivalently all right-invariant P-qN structures on \( G \). The classification procedure of right-invariant P-qN structures is the subject of Section 5; we list the results of a classification of r-qn structures on two four dimensional Lie algebras, symplectic real Lie algebra \( A_{4,1} \) and non-symplectic Lie algebra \( A_{4,8} \); we explain all details in the procedure for Lie algebra \( A_{4,1} \). We shall consider some remarks on r-qn structures in Section 6, more precisely, we shall consider the conditions for which an r-qn structure on Lie algebra \( g \) defines a generalized complex structure on \( g \) or an \( R \)-matrix on double of Lie algebra \( g \oplus g^* \). We end the paper by some relevant examples of r-qn structures obtained in section 4.

2. Antecedents

In this section, we recall the definition of Poisson quasi-Nijenhuis structures [12]. We will briefly review the notion of Lie bialgebra, classical and modified Yang-Baxter equation. We also take a review on generalized complex structures on a manifold.

2.1. Poisson quasi-Nijenhuis structure. A Poisson quasi-Nijenhuis (P-qN) structure on a manifold \( M \) is a bivector field \( \Pi : T^*M \times T^*M \to \mathbb{R} \), a \((1,1)\)-tensor field \( N : TM \to TM \) together with a closed 3-form \( \Phi : TM \times TM \times TM \to \mathbb{R} \) on \( M \) satisfying the conditions:

\(\begin{align*}
(i) & \quad \Pi \text{ is a Poisson bivector, i.e. } [\Pi, \Pi] = 0, \\
(ii) & \quad [NX, NY] - N[NX, Y] - N[X, NY] + N^2[X, Y] = \Pi^2(\Phi^\sharp(X, Y)) , \quad \forall X, Y \in \mathfrak{X}(M) , \\
(iii) & \quad \Pi^2 = \Pi^\sharp N^t, \\
(iv) & \quad C(\Pi, N)(\alpha, \beta) = \{\alpha, \beta\}_\Pi N^t = \{\alpha, \beta\}_\Pi = 0, \quad \forall \alpha, \beta \in \Omega^1(M) , \\
(v) & \quad d[i_N^\sharp \Phi] = 0 ,
\end{align*}\)

where \( \Pi^2 : T^*M \to TM \) and \( \Phi^\sharp : TM \times TM \to T^*M \) are induced by \( \Pi \) and \( \Phi \), given by interior product,

\(\Pi^\sharp(\alpha) = i_\alpha \Pi , \quad \Phi^\sharp(X, Y) = i_{X \wedge Y} \Phi , \quad \alpha \in \Omega^1(M) \) and \( X, Y \in \mathfrak{X}(M) \).

\(N^t : T^*M \to T^*M \) is the dual \((1,1)\)-tensor field to \( N \) and \( C(\Pi, N) \) is a \((2,1)\)-tensor field on \( M \), a concomitant of \( \Pi \) and \( N \), where

\(\{\alpha, \beta\}_\Pi^N = \{\alpha^t, \beta\}_\Pi + \{\alpha, \beta^t\}_\Pi - \{\alpha^t, \beta^t\}_\Pi ,\)

\(1 \{\cdot, \cdot\} \) is the Schouten-Nijenhuis bracket.
and the bracket \{·, ·\}_\Pi is the bracket of 1-forms which is defined by the Poisson bivector \Pi as follows
\[
\{\alpha, \beta\}_\Pi = \mathcal{L}_{\Pi^\alpha} \beta - \mathcal{L}_{\Pi^\beta} \alpha + d(\Pi(\alpha, \beta)).
\] (2.1)

Similarly, the bracket \{\alpha, \beta\}_N is the bracket of 1-forms defined by the 2-contravariant tensor \N (for more details see, [7]).

Note that \i_N is the derivation of degree 0 defined by
\[
(i_N \alpha)(X_1, \ldots, X_p) = \sum_{i=1}^p \alpha(X_1, \ldots, N X_i, \ldots, X_p), \quad \forall \alpha \in \Omega^p(M).
\] (2.2)

\textbf{Example.} Poisson-Nijenhuis structures on \(M\) are trivial Poisson quasi-Nijenhuis, since for them the 3-form \(\Phi \equiv 0\).

2.2. Cohomology of Lie algebra. Let \(\mathfrak{g}\) be a finite dimensional Lie algebra and \(G\) be its corresponding simply connected Lie group. To each \(\mathfrak{g}\)-module \(M\) and arbitrary nonnegative integer \(k\) we can associate a \(k\)-cochain of \(\mathfrak{g}\) with values in \(M\) as a \(k\)-linear skew-symmetric map from \(\mathfrak{g}\) to \(M\). The 0-cochain is just an element of \(M\). Let us denote the space of \(k\)-cochains of \(\mathfrak{g}\) with value in \(M\) by \(C^k(\mathfrak{g}, M)\).

A linear map \(\partial : C^k(\mathfrak{g}, M) \to C^{k+1}(\mathfrak{g}, M)\), satisfying \(\partial^2 = 0\), is called a coboundary operator. We consider the definition of Chevalley-Eilenberg coboundary operator
\[
\partial \delta(X_0, \ldots, X_k) = \sum_{i=0}^k (-1)^i X_i(\delta(X_0, \ldots, \hat{X}_i, \ldots, X_k)) + \sum_{i<j} (-1)^{i+j} \delta([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k),
\]
for \(k\)-cochain \(\delta\) and \(X_0, \ldots, X_k \in \mathfrak{g}\). A \(k\)-cochain is called a \(k\)-cocycle if its coboundary is zero.

Now we set \(M := \mathbb{R}\) with the trivial action of \(\mathfrak{g}\) and use the abbreviate notation \(C^k(\mathfrak{g}, \mathbb{R})\) instead of \(C^k(\mathfrak{g}, M)\). Note that, in this case the cohomology group is just the cohomology group of right-invariant (resp. left-invariant) forms on \(G\) and the coboundary operator \(\partial\) is exactly the de Rham differential \(d\). The coboundary maps for 0, 1, 2 and 3-cochains \(\theta, \eta, \mu\) and \(\phi\) are:
\[
(\partial \theta)(X_i) = 0,
(\partial \eta)(X_i, X_j) = -\eta([X_i, X_j]),
(\partial \mu)(X_i, X_j, X_k) = -\mu([X_i, X_j], X_k) + \mu([X_i, X_k], X_j) - \mu([X_j, X_k], X_i),
(\partial \phi)(X_i, X_j, X_k, X_l) = -\phi([X_i, X_j], X_k, X_l) + \phi([X_i, X_k], X_j, X_l) - \phi([X_i, X_l], X_j, X_k)
-\phi([X_j, X_k], X_i, X_l) + \phi([X_j, X_l], X_i, X_k) - \phi([X_j, X_k], X_i, X_l),
\] (2.3)

We remark that the cocycle condition is equivalent to the corresponding \(k\)-cochain being closed. So, a \(k\)-form on \(\mathfrak{g}\) is closed if it is a \(k\)-cocycle in \(C^k(\mathfrak{g})\).

Now we can proceed to the definition of the Lie bialgebra.

A \textit{Lie bialgebra} \((\mathfrak{g}, \mathfrak{g}^*)\) is a Lie algebra with an additional structure, a linear map \(\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}\) such that:

\begin{enumerate}[(i)]
\item The linear map \(\delta : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}\) is a 1-cocycle, i.e.
\[
ad_X^2(\delta Y) - \ad_Y^2(\delta X) - \delta[X, Y] = 0, \quad \forall X, Y \in \mathfrak{g},
\]

where \(\ad_X^2\) is the adjoint representation of the Lie algebra \(\mathfrak{g}\) on the space \(\mathfrak{g} \otimes \mathfrak{g}\) defined by
\[
ad_X^2(Y_1 \otimes Y_2) = \ad_X Y_1 \otimes Y_2 + Y_1 \otimes \ad_X Y_2 \quad X, Y_1, Y_2 \in \mathfrak{g}.
\]

\item The dual map \(\delta^\ast : \mathfrak{g}^* \otimes \mathfrak{g}^* \longrightarrow \mathfrak{g}^*\) is a Lie bracket on \(\mathfrak{g}^*\).
\end{enumerate}
We denote the Lie bracket on $\mathfrak{g}^*$ by $[\alpha, \beta]_\ast := \delta^t(\alpha \otimes \beta)$ for $\alpha, \beta \in \mathfrak{g}^*$.

Coboundary Lie bialgebra is a Lie bialgebra defined by a 1-cocycle $\partial r$ which is the coboundary of an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ (for more details, see [6]).

We consider the following notation for the Sklaynin brackets on $\mathfrak{g}^*$ and $\mathfrak{g}$ defined by $r$-matrices $r$ and $r^{-1}$:

$[X^i, X^j] := \tilde{f}^{ij} X^k, \quad [X_i, X_j]^{r^{-1}} := [X_i, X_j] = f_{ij}^k X_k,$

where $\tilde{f}^{ij}$ and $f_{ij}$ are structure constants of $\mathfrak{g}^*$ and $\mathfrak{g}$, respectively.

### 2.2.1. Classical Yang-Baxter equation

Let $\mathfrak{g}$ be finite-dimensional Lie algebra and $\mathfrak{g}^*$ be its dual vector space with respect to a non-degenerate canonical pairing $\langle \cdot, \cdot \rangle$, so for basis $\{X_i\}$ and dual basis $\{X^i\}$ of $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively, we have:

$\langle X_i, X_j \rangle = \langle X^i, X^j \rangle = 0, \quad \langle X_i, X_j \rangle = \delta^j_i.$

To every element $r = r^{ij} X_i \otimes X_j$ of $\mathfrak{g} \otimes \mathfrak{g}$, we can associate the linear map $r^\sharp : \mathfrak{g}^* \to \mathfrak{g}$ defined by $r^\sharp(X^i)(X^j) := r(X^i, X^j)$. Let $\delta_r := \partial r$, then

$\delta_r(X) = ad^{(2)}_X r = r^{ij} (ad_X X_i \otimes X_j + X_i \otimes ad_X X_j), \quad X \in \mathfrak{g}.$

By definition, $\delta r$ is a 1-cocycle. We denote the bracket on $\mathfrak{g}^*$ in this case by $[X^i, X^j]^r$ instead of $[X^i, X^j]_\ast$.

To every element $r$ of $\mathfrak{g} \otimes \mathfrak{g}$ we can also associate a bilinear map $\langle r, r \rangle^\sharp : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}$ defined by

$\langle r, r \rangle^\sharp(X^i, X^j) = [r^\sharp X^i, r^\sharp X^j] - r^\sharp [X^i, X^j]^r,$

which can be identified with an element $\langle r, r \rangle \in \wedge^2 \mathfrak{g} \otimes \mathfrak{g}$, such that

$\langle r, r \rangle(X^i, X^j, X^k) := \langle X^k, \langle r, r \rangle^\sharp(X^i, X^j) \rangle.$

For a skew-symmetric element $r \in \Lambda^2 \mathfrak{g}$, we have

$[X^i, X^j]^r = ad^*_{r^{ij} X^i} X^j - ad^*_{r^{ij} X^j} X^i,$

where $ad^*_{X^i} = -(ad_{X^i})^t$ is the endomorphism of $\mathfrak{g}^*$ satisfying

$\langle X^i, ad_{X_k} X_j \rangle = -\langle ad^*_{X_k} X^i, X_j \rangle,$

which implies

$\langle ad^*_{X_k} X^i, X_j \rangle = -\langle ad^*_{X_j} X^i, X_k \rangle.$

On the other hand, in the case where $r$ is skew-symmetric, we have $\langle r, r \rangle = -\frac{1}{2}[r, r]$ where $[\cdot, \cdot]$ is Schouten-Nijenhuis bracket on the Lie algebra $\mathfrak{g}$ called the algebraic Schouten bracket.

For the skew-symmetric element $r \in \Lambda^2 \mathfrak{g}$, the condition $[r, r] = 0$ is called the classical Yang-Baxter equation (CYBE). A solution of the CYBE is called an $r$-matrix. For any $r$-matrix the bracket (2.5) is a Lie bracket on $\mathfrak{g}^*$, called the Sklyanin bracket. Therefore $r$-matrices can be identified with coboundary Lie bialgebras on the Lie algebra $\mathfrak{g}$ (for more details see, for instance, [6]).

**Remark 2.1.** $r \in \Lambda^2 \mathfrak{g}$ is a solution of classical Yang-Baxter equation if and only if

$r^\sharp [X^i, X^j]^r = [r^\sharp X^i, r^\sharp X^j], \quad \forall X^i, X^j \in \mathfrak{g}^*.$

}$\diamond$
2.2.2. Modified Yang-Baxter equation. Let $R$ be a linear map from finite-dimensional Lie algebra $g$ to itself. Consider the skew-symmetric bilinear form $\langle R, R \rangle_k$ on $g$ with values in $g$ defined by

$$\langle R, R \rangle_k (X, Y) = [RX, RY] - R[RX, Y] - R[X, RY] + k[X, Y], \quad \forall X, Y \in g, \quad k \in \mathbb{R}.$$ 

Condition $\langle R, R \rangle_k = 0$ is called a modified Yang-Baxter equation (MYBE) with coefficient $k$. A solution of MYBE is called a classical $R$-matrix or $R$-matrix.

We can define a bilinear skew-symmetric bracket $[\cdot, \cdot]_R$ on $g$ as

$$[X, Y]_R = [RX, Y] + [X, RY], \quad \forall X, Y \in g.$$ 

For $R$-matrix $R$, the above bracket defines a Lie algebra structure on $g$ which is called double Lie algebra (for more details see for example [6]).

2.3. Generalized complex structure. For a manifold $M$, the space $\mathcal{TM} := TM \oplus T^*M$ is called the generalized tangent bundle of $M$. The space of sections of $\mathcal{TM}$ is endowed with a bracket so called the Courant bracket given by

$$[(X + \alpha), (Y + \beta)] = [X, Y] + \mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2} d (\alpha(Y) - \beta(x)) \quad \forall X, Y \in \mathfrak{X}(M), \forall \alpha, \beta \in \Omega^1(M).$$ 

This bracket is not a Lie bracket since it does not satisfy the Jacobi identity. The non-degenerate symmetric bilinear form on the vector space $\mathcal{TM} := TM \oplus T^*M$ is defined by

$$\langle X + \alpha, Y + \beta \rangle = \alpha(X) + \beta(Y), \quad \forall X, Y \in \mathfrak{X}(M), \forall \alpha, \beta \in \Omega^1(M),$$ 

for more detail see [4] and [5].

Definition 2.2. A generalized complex structure on $M$ is a complex structures $J$ that is, a bundle map $J : \mathcal{TM} \to \mathcal{TM}$ which $J^2 = - Id$ and $(Jv, Jw) = \langle v, w \rangle$ satisfying the integrability condition

$$[Jv, Jw] - J[Jv, w] - J[v, Jw] - [v, w] = 0, \quad \forall v, w \in \mathcal{TM}.$$ 

Proposition 2.3. A generalized complex structure $J$ on $M$ is of the form

$$J = \begin{pmatrix} N & \Pi^\sharp \\ \Theta^\sharp & -N^t \end{pmatrix},$$

where $\Pi$ is a Poisson bivector on $M$ which is compatible with the vector bundle map $N : TM \to TM$ in the sense of Poisson-Nijenhuis structures, and $\Theta$ is a 2-form on $M$ for which we denote the map $\Theta^\sharp : TM \to T^*M$ such that $\Theta^\sharp(X) = \iota_X \Theta$; satisfying the following conditions

$$N^t \Theta^\sharp = \Theta^\sharp N, \quad N^2 + \Pi^\sharp \Theta^\sharp = - Id, \quad [N, N](X, Y) = \Pi^\sharp (\iota_{X \wedge Y} (d \Theta)), \quad d \Theta_N (X, Y, Z) = \iota_N (d \Theta)(X, Y, Z).$$

For more detail see for example [2] and [12].

Comparing the previous Proposition and the definition of Poisson-quasi Nijenhuis structures on a manifold we have the following Corollary.

Corollary 2.4. The Poisson quasi-Nijenhuis structure $(\Pi, \Phi, N)$ on $M$ defines a generalized complex structure of the form (2.9) on $M$ if 3-form $\Phi$ is exact and the two following conditions hold

$$N^t \Theta^\sharp = \Theta^\sharp N, \quad N^2 + \Pi^\sharp \Theta^\sharp = - Id,$$

where $\Theta$ is a 2-form on $M$ such that $\Phi = d \Theta$. 

3. Right-invariant Poisson quasi-Nijenhuis structures

In this section we define right-invariant Poisson quasi-Nijenhuis ($P$-$q$-$N$) structures on the Lie group $G$ and their infinitesimal counterpart on the Lie algebra $\mathfrak{g}$ of $G$. We also consider the concepts of compatibility and equivalence of those structures.

We are using the following notation. If $s$ is a $k$-vector on $\mathfrak{g}$ then $\overrightarrow{s}$ (resp. $\overleftarrow{s}$) is the right-invariant (resp. left-invariant) $k$-vector field on $G$ given by

$$\overrightarrow{s}(g) = (T_e R_g)(s), \quad \text{for } g \in G,$$

(resp. $\overleftarrow{s}(g) = (T_e L_g)(s)$, for $g \in G$) where $R_h : G \to G$ and $L_g : G \to G$ are the right and left translation by $h$ and $g$, respectively.

**Definition 3.1.** A $P$-$q$-$N$ structure $(\Pi, \Phi, N)$ on a Lie group $G$ is said to be right-invariant if:

(i) The Poisson structure $\Pi$ is right-invariant, that is, there exists $r \in \Lambda^2 \mathfrak{g}$ such that $\Pi = \overrightarrow{r}$.

(ii) The closed 3-form $\Phi$ is right-invariant, that is, there exist a real valued three linear, skew map $\phi \in C^3(\mathfrak{g})$ satisfying 3-cocycle condition, such that $\Phi = \overrightarrow{\phi}$.

(iii) The $(1,1)$-tensor field $N$ is right-invariant, that is, there exists a linear endomorphism $n : \mathfrak{g} \to \mathfrak{g}$ such that $N = \overrightarrow{n}$.

By the definition, we have

$$\Pi^r_{(g)} = T_e R_g \circ r^2 \circ T_e^* R_g, \quad N_{|T_e G} = T_e R_g \circ n \circ T_e R_g^{-1}, \quad \Phi^{2r}_{(g)} = \wedge^2(T^*_g R_g^{-1}) \circ \phi^{2r} \circ (T_g R_g^{-1}).$$

for $g \in G$. For right-invariant $P$-$q$-$N$ structures, we may prove the two following results which describe the infinitesimal version of such structures.

**Proposition 3.2.** Let $(\Pi, \Phi, N)$ be a right-invariant $P$-$q$-$N$ structure on a Lie group $G$ with Lie algebra $\mathfrak{g}$ and identity element $e \in G$. If $r \in \Lambda^2 \mathfrak{g}$ and $\phi \in \Lambda^3 \mathfrak{g}^*$ are the value of $\Pi$ and $\Phi$ at $e$, and $n$ is the restriction of $N$ to $\mathfrak{g}$, we have

(i) $r$ is a solution of the classical Yang-Baxter equation on $\mathfrak{g}$.

(ii) The Nijenhuis torsion $[n,n]$ of $n$ on $\mathfrak{g}$ equals

$$[n,n](X,Y) := [nX,nY] - n[nX,Y] - n[X,nY] + n^2[X,Y] = r^2(\phi^2(X,Y)), \quad \forall X,Y \in \mathfrak{g}. \quad (3.1)$$

(iii) $\phi$ and $i_n \phi$ are 3-cocycles with values in $\mathbb{R}$.

(iv) $n : \mathfrak{g} \to \mathfrak{g}$ satisfies the condition

$$n \circ r^2 = r^2 \circ n^t.$$

Conversely, let $\mathfrak{g}$ be a real Lie algebra of finite dimension, $r \in \Lambda^2 \mathfrak{g}$ be a 2-vector and $\phi \in \Lambda^3 \mathfrak{g}^*$ be a 3-form on $\mathfrak{g}$ and $n : \mathfrak{g} \to \mathfrak{g}$ be a linear endomorphism on $\mathfrak{g}$ which satisfy conditions (i), (ii), (iii), (iv) and (v); so-called $r$-$qn$ structure on the Lie algebra $\mathfrak{g}$. If $G$ is a Lie group with the Lie algebra $\mathfrak{g}$, then the triple $(\overrightarrow{r}, \overrightarrow{\phi}, \overrightarrow{n})$ is a right-invariant $P$-$q$-$N$ structure on $G$.

**Proof.** The proofs of (i), (ii), (v) and (iv) are straightforward by the definition of right invariant objects (see [10]). For (iii), it is the consequence of the fact that a $k$-form on $\mathfrak{g}$ is closed if it is a $k$-cocycle in $C^k(\mathfrak{g})$. 
We remark that, in case of \( r\)-qn structures the 2-vector \( nr \) is not an \( r \)-matrix since the operator \( n \) is not a Nijenhuis operator. In the following Proposition we will see that under a certain condition it would be an \( r \)-matrix.

If there is not risk of confusion, we will use the same notation \( r \) for the 2-vector \( r \) and the linear map \( r^* \).

**Proposition 3.3.** Let \( r \in \Lambda^2 g \) be an \( r \)-matrix and \( n \) be an linear operator on \( g \) which is compatible with \( r \), that is \( nr = rn^t \) and \( C(r,n) \equiv 0 \). Then, 2-vector \( nr \) is an \( r \)-matrix if and only if

\[
[n,n](rX^i,rX^j) = 0, \quad \forall X^i,X^j \in \mathfrak{g}^*. \tag{3.2}
\]

**Proof.** 2-vector \( nr \) defines a bracke \([X^i,X^j]^{nr} = ad^*_{nrX^i}X^j - ad^*_{nrX^j}X^i \) on \( g \). On the other hand \( C(r,n) \equiv 0 \) implies

\[
[X^i,X^j]^{nr} = [n^tX^i,X^j]^r + [X^i,n^tX^j] - n^t[X^i,X^j]^r.
\]

Equivalently, \( C(r,n) \equiv 0 \) garanties 2-vectors \( r \) and \( nr \) are compatible, that is \([r,nr] \equiv 0 \), so we have

\[
r[X^i,X^j]^{nr} + nr[X^i,X^j]^r = [rX^i,nrX^j] + [nrX^i,rX^j]. \tag{3.3}
\]

Suppose the condition (3.2) holds, that is

\[
[nrX^i,nrX^j] - n[nrX^i,rX^j] - n[rX^i,nrX^j] + n^2[rX^i,rX^j] = 0. \tag{3.4}
\]

Using (2.8) for \( r \)-matrix \( r \) and comparing two relations (3.3) and (3.4), we get

\[
r[X^i,X^j]^{nr} = [nrX^i,nrX^j]
\]

which from (2.8) means \( nr \) is an \( r \)-matrix. One proves the converse in a similar way.

**Corollary 3.4.** In the case that \( r \)-matrix \( r \) in the previous Proposition is non-degenerate, \( nr \) is an \( r \)-matrix if and only if \( n \) is a Nijenhuis operator.

### 3.1. Compatibility of right-invariant Poisson quasi-Nijenhuis structures.

Two \( P\)-\( qN \) structures \((\Pi,\Phi,N)\) and \((\Pi',\Phi',N')\) on a Lie group \( G \) are said to be compatible if the couple \((\Pi + \Pi',\Phi + \Phi',N + N')\) is a \( P\)-\( qN \) structure on \( G \).

In the case of the right-invariant \( P\)-\( qN \) structures, the compatibility of structures reduces to the compatibility of their infinitesimal version.

If \((r,\phi,n)\) and \((r',\phi',n')\) be the infinitesimal versions of the mentioned \( P\)-\( qN \) structures, then they are compatible if the couple \((r + r',\phi + \phi',n + n)\) is a \( r\)-\( qn \) structure on the Lie algebra \( \mathfrak{g} \) of \( G \), that is \( n + n' \) and \( r + r' \) are compatible and

\[
[r,r'] = 0, \quad [n,n'] = r\phi^\sharp + r\phi'^\sharp + r'\phi^\sharp + r'\phi'^\sharp.
\]

From the Proposition 3.2 \((r + r',\phi + \phi',n + n')\) would be the infinitesimal version of the right-invariant \( P\)-\( qN \) structure \((r + r',\phi + \phi',n + n')\) on the Lie group \( G \).

\[\text{[n,n]} \text{ is the Nijenhuis concomitant \([\Pi,\Pi']\) of two (1,1)-tensor fields n' and n on the Lie algebra g defined by}
\[
[n,n'](X_i,X_j) = [n,n](X_i,X_j) + [n',n'](X_i,X_j) + [n',n](X_i,X_j) + [n,n'](X_i,X_j) - n'[nX_i,X_j] - n[n'X_i,X_j] - n\circ n'(X_i,X_j) + n\circ n[X_i,X_j],
\]

for every two elements of basis \( \{X_i\} \) of \( \mathfrak{g} \).
3.2. Equivalence classes of right-invariant Poisson quasi-Nijenhuis structures. In \cite{10}, we defined the equivalence class of right-invariant Poisson-Nijenhuis structures. Now, we define the equivalence classes of right-invariant $P\text{-}qN$ structures. Two right-invariant $P\text{-}qN$ structures $(P, \Phi, N)$ and $(P', \Phi', N)$ on the Lie group $G$ are equivalent if two corresponding $r\text{-}qn$ structures $(r, \phi, n)$ and $(r', \phi', n')$ on the Lie algebra $\mathfrak{g}$ are equivalent.

**Definition 3.5.** Two $r\text{-}qn$ structures $(r, \phi, n)$ and $(r', \phi', n')$ are equivalent if there exist a Lie algebra automorphism $A$ such that the following diagrams commute,

\[ \begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\phi_i} & \mathfrak{g}^* \\
& \downarrow{\mathbb{A}} & \downarrow{\mathbb{A}^{-\ell}} \\
\mathfrak{g} & \xrightarrow{\phi'_i} & \mathfrak{g}^* \\
\end{array} \]

that is,

\[ r \sim_{\mathbb{A}} r', \quad n \sim_{\mathbb{A}} n', \quad \phi_i \sim_{\mathbb{A}} \phi'_i, \quad \text{for } i := 1, \ldots, \dim \mathfrak{g}, \]

where we define $\phi_i = \phi^Z(X_i) \in \Lambda^2 \mathfrak{g}^*$. In fact the map $\phi^Z : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}^*$ induces a map $\phi^\sharp : \mathfrak{g} \to \Lambda^2 \mathfrak{g}^*$ defined by

\[ \phi^\sharp(X_i)(X_j, X_k) = \phi^Z(X_i, X_j)(X_k) = \phi(X_i, X_j, X_k), \quad \{X_i\} \in \mathfrak{g} \]

which can be interpreted by the map $\phi_i : \mathfrak{g} \to \mathfrak{g}^*$ for every $i := 1, \ldots, \dim \mathfrak{g}$.

We will write $(r, \phi, n) \sim (r', \phi', n') ((r, \phi, n) \sim_{\mathbb{A}} (r', \phi', n')$ if we want to indicate $\mathbb{A})$.

4. $r\text{-}qn$ structures on Lie algebras

In this section we describe how we can get all $r\text{-}qn$ structures on $\mathfrak{g}$, equivalently right-invariant $P\text{-}qN$ structures on $G$. For this purpose we rewrite the five conditions of Proposition 3.2 in terms of coordinates. Throughout this section we denote the basis $\{X_i\}$ and the dual basis $\{X^j\}$ for Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively.

First, we write the structural constants $f^k_{ij}$ of the Lie algebra $\mathfrak{g}$, in terms of adjoint representation $X_i$, and antisymmetric matrices $y^i$, as

\[ f^k_{ij} = -(y^k)_{ij}, \quad f^k_{ij} = -(X_i)_{j}^{k}. \] (4.1)

**Condition (i)** Consider the tensor notation of the CYBE\footnote{$[r, r] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$, where $r_{12} = r^{ij}X_i \otimes X_j \otimes 1$, $r_{13} = r^{ij}X_i \otimes 1 \otimes X_j$ and $r_{23} = r^{ij}1 \otimes X_i \otimes X_j$.} (see \cite{6}). We can rewrite condition $[r, r] \equiv 0$ in the matrix form (see \cite{10})

\[ rY^i r - rX_i r^{id} - r^{id}X_i^d = 0, \quad i := 1, \ldots, \dim \mathfrak{g}. \] (4.2)

**Condition (ii)** We write the condition \cite{3.1} for two base elements $X_i$ and $X_j$ in $\mathfrak{g}$ and we get

\[ n^k_i n^l_j f^m_{kl} - n^k_i n^m_j f^l_{kj} - n^k_j f^l_{ik} n^m_i + f^k_{ij} n^l_k n^m_i = \phi_{ijk} r^{kl}. \]

By using (4.1) it can be rewritten in the matrix form

\[ -n^l_i n^k_i X_j - n^l_i X_i n^k - X_i n^l_i n^k + n^l_i X_i n^k = \phi_i r, \quad i := 1, \ldots, \dim \mathfrak{g}, \] (4.3)

where $\phi_i := \phi(X_i) = \phi_{ijk} X^j \wedge X^k$.

**Condition (iii)** The cocycle condition \cite{2.3} for $\phi$ in the base elements is

\[ \partial \phi(X_i, X_j, X_k, X_l) = -f^m_{ij} \phi_{mlk} + f^m_{ik} \phi_{mj} - f^m_{il} \phi_{mjk} - f^m_{jl} \phi_{mik} + f^m_{jl} \phi_{mik} = 0, \]

where $r_{12} = r^{ij}X_i \otimes X_j \otimes 1$, $r_{13} = r^{ij}X_i \otimes 1 \otimes X_j$ and $r_{23} = r^{ij}1 \otimes X_i \otimes X_j$.\]
which can be rewritten in the matrix form by using (4.1) as follows
\[ X_j \phi_k - X_k \phi_j + Y^m(\phi_m)_{jk} + (Y^m)_{jk} \phi_m + \phi_k X^t_j - \phi_j X^t_k = 0, \quad j, k := 1, \ldots, \dim \mathfrak{g}, \] (4.4)
where elements of the matrix \( \phi_i \) are \( \phi_i \) for all \( n \), for \( n = 1, \ldots, \dim \mathfrak{g} \).

Using the relation (2.2) for \( g \) with finite dimension, by applying matrices \( X^t \) on four-dimensional symplectic real Lie algebra \( n \), for \( n = 1, \ldots, \dim \mathfrak{g} \), we obtain \( X^t \) structures on \( g \), and the map \( n \) and \( r \) are the corresponding matrices to the linear operator \( n \) and the map \( r \) and \( n X_i = n^t X_j \) for \( n_i \in \mathbb{R} \).

**Condition (iv)** For every element \( X^i \) in \( \mathfrak{g} \), \( n \circ r^t(X^i) = r^t \circ n^t(X^i) \) implies
\[(nr)^i_j = (rn^t)^i_j, \quad i, j = 1, \ldots, \dim \mathfrak{g}, \] (4.6)

where \( n \) and \( r \) are the corresponding matrices to the linear operator \( n \) and the map \( r \) and \( n X_i = n^t X_j \) for \( n_i \in \mathbb{R} \).

**Condition (v)** By applying \( X^i \) and \( X_j \) in the concomitant and then, using (4.1), we get the matrix relation (see [10])
\[ rX_j n^i_j + X^t_j n^t_j r - rX_i n^t_j - nX^t_j r = 0, \quad i := 1, \ldots, \dim \mathfrak{g}. \] (4.7)

Given a Lie algebra \( \mathfrak{g} \) with finite dimension, by applying matrices \( X_i \) and \( Y^t \) using (4.1), in six relations (4.2), (4.3), (4.4), (4.5), (4.6), (4.7) and solving them by help of mathematical softwares, one can find all \( r-qn \) structures on \( \mathfrak{g} \) and so, all right-invariant \( P-qN \) structures on the Lie group \( G \).

5. CLASSIFICATION PROCEDURE

Many of \( r-qn \) structures obtained in section 4 would be equivalent by an Lie algebra automorphism. In this section we will proceed in five step to show how we can classify, up to an equivalence, all \( r-qn \) structures on a Lie algebra. For clarity of results, we exemplify the procedure by classifying all \( r-qn \) structures on two types of four dimensional real Lie algebras, symplectic real Lie algebra \( A_{4,1} \) and non-symplectic Lie algebra \( A_{4,8} \). We explain all details of classification procedure for Lie algebra \( A_{4,1} \). We did all computations using Maple.

The strategy is as follows.

**First step.** using (4.2) we find all \( r \)-matrices on four-dimensional symplectic real Lie algebra \( \mathfrak{g} \) and classify them up to equivalence
\[ r \sim r' \quad \Leftrightarrow \quad \exists \mathcal{A} \in Aut(\mathfrak{g}) \quad \mathcal{A} r \mathcal{A}^t = r'. \]
Second step. We take a representative of each class of $r$-matrices in first step, and find all endomorphisms $n$ on $\mathfrak{g}$ which are compatible with the chosen $r$-matrix by solving relations \((4.6)\) and \((4.7)\); they give five equations on four-dimensions. Then we classify all obtained pairs $(r,n)$ up to equivalence

$$(r, n') \sim_0 (r, n) \iff \exists A \in \text{Aut}(\mathfrak{g}) \quad ArA^t = r \& AnA^{-1} = n',$$  

where $\sim_0$ indicates the equivalence for the couple $(r, n)$ with the same $r$.

Third step. Now we find all 3-forms $\phi$ which satisfy in the relation \((4.3)\) for $(1,1)$-tensor fields $n$ we found in the second step. In order to, we write the skew symmetric maps $\phi_i$ in the matrix forms. In dimension four they are as follow

$\phi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_{123} & \phi_{124} \\ -\phi_{123} & 0 & \phi_{134} \\ 0 & -\phi_{124} & 0 & \phi_{134} \end{pmatrix} \quad \phi_2 = \begin{pmatrix} 0 & 0 & -\phi_{123} & -\phi_{124} \\ 0 & 0 & 0 & 0 \\ \phi_{123} & 0 & 0 & \phi_{234} \\ \phi_{124} & 0 & -\phi_{234} & 0 \end{pmatrix}$

$\phi_3 = \begin{pmatrix} 0 & \phi_{123} & 0 & -\phi_{134} \\ -\phi_{123} & 0 & 0 & -\phi_{234} \\ 0 & 0 & 0 & 0 \\ \phi_{134} & \phi_{234} & 0 & 0 \end{pmatrix} \quad \phi_4 = \begin{pmatrix} 0 & \phi_{123} & \phi_{134} & 0 \\ -\phi_{123} & 0 & 0 & \phi_{234} \\ -\phi_{134} & -\phi_{234} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

Fourth step: In this step we check if $(1,1)$-tensor fields $n$ from the second step and 3-forms $\phi$ from the third step satisfy in relations \((4.4)\) and \((4.5)\), or they may have some new conditions.

Fifth step. Finally, we classify all obtained pairs $(r, \phi, n)$ up to equivalence

$$(r, \phi, n) \sim_0 (r, \phi', n) \iff \exists A \in \text{Aut}(\mathfrak{g}) \quad ArA^t = r \& AnA^{-1} = n \& A^t\phi_iA = \phi'_i,$$  

for $i := 1,\ldots, dim \mathfrak{g}$. Here $\sim_0$ indicates the equivalence for $r$-$qn$ structures with the same $r$ and $n$.

Proposition 5.1. If \{$r_\alpha\}_\Lambda$ is a set of all representatives of the equivalence relation $r \sim r'$ and \{$(r_\alpha, n_\beta)\}_{(\alpha, \beta) \in \Psi}$ is a set of all representatives of the equivalence relations $(r_\alpha, n) \sim_0 (r_\alpha, n')$ and \{$(r_\alpha, (\phi_i)_\eta, n_\beta)\}_{(\alpha, \eta, \beta) \in \Gamma}$ is a set of all representatives of the equivalence relations $(r_\alpha, (\phi_i)_\eta, n_\beta) \sim_0 (r_\alpha, (\phi'_i)_\eta, n_\beta)$, then \{$(r_\alpha, (\phi_i)_\eta, n_\beta)\}_{(\alpha, \eta, \beta) \in \Gamma}$ is a set of representatives of the equivalence relation $(r, \phi_i, n) \sim (r', \phi'_i, n')$ (c.f. Definition 3.3).

Proof. Consider an $r$-$qn$ structure $(r, \phi, n)$. There exist $r_\alpha$ and $A$ such that $r_\alpha = ArA^t$. Take the representative $(r_\alpha, n_\beta)$ of $(r_\alpha, AnA^{-1})$. Then $(r_\alpha, n_\beta)$ represents the class of $(r, n)$ under $\sim$. Now take the representative $(r_\alpha, (\phi_i)_\eta, n_\beta)$ of $(r_\alpha, A^t\phi_iA, n_\beta)$. Then, it is easy to see that $(r_\alpha, (\phi_i)_\eta, n_\beta)$ represents the class of $(r, \phi_i, n)$ under $\sim$.

Moreover, different elements of \{$(r_\alpha, (\phi_i)_\eta, n_\beta)\}_{(\alpha, \eta, \beta) \in \Gamma}$ represent different elements of $\sim$. Indeed, if $(r_\alpha, (\phi_i)_\eta, n_\beta) \sim (r_\alpha', (\phi_i)_\eta', n_{\beta'})$, then there is $A$ such that

$$r_\alpha = Ar_\alpha A^t \& n_\beta = An_\beta A^{-1} \& (\phi_i)_\eta = A^t(\phi_i)_\eta'A.$$  

But then, by definition, $\alpha = \alpha'$ and hence $(r_\alpha, n_\beta) \sim_0 (r_\alpha, n_{\beta'})$, thus $\beta = \beta'$; which means different elements of \{$(r_\alpha, n_\beta)\}_{(\alpha, \beta) \in \Psi}$ represent different elements of $\sim$. Therefore for $(r_\alpha, (\phi_i)_\eta, n_\beta) \sim_0 (r_\alpha, (\phi_i)_\eta', n_{\beta'})$, implies $\eta = \eta'$.

In the following we clarify the above procedure by describing the details for the Lie algebra $A_{4,1}$. We list the results in any step for Lie algebras $A_{4,1}$ and $A_{4,8}$.
5.1. $r$-matrices. We consider the Lie algebra $A_{4,1}$ with non-zero commutators $[X_2, X_4] = X_1$ and $[X_3, X_4] = X_2$, from (5.1) we have the following matrices $X_i$ and $y^i$

\[
X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\
X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
y^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad y^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad y^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

We list the classification of $r$-matrices for Lie algebra $A_{4,1}$ and $A_{4,8}$ in the table 1. Note that we classified all such structures on four dimensional symplectic real Lie algebras in [10]. For self containing of the paper we bring the result of Lie algebra $A_{4,1}$ (see table 1 in [10]).

| $A_{4,1}$ | Equivalence classes of $r$-matrices |
|-----------|-----------------------------------|
| $f_{24}^1 = 1$ | $c^{12}X_1 \wedge X_4 + c^{13}X_1 \wedge X_3 + c^{14}X_1 \wedge X_4 + c^{23}X_2 \wedge X_3$ |
| $f_{34}^2 = 1$ | $c_{12}X_1 \wedge X_2 + c_{13}X_1 \wedge X_3 + c_{14}X_1 \wedge X_4 + c_{23}X_2 \wedge X_3$ |
| $f_{34}^3 = 1$ | $c_{12}X_1 \wedge X_2 + c_{13}X_1 \wedge X_3 + c_{14}X_1 \wedge X_4$ |

| $A_{4,8}$ | Equivalence classes of $r$-matrices |
|-----------|-----------------------------------|
| $f_{23}^1 = 1$ | $c^{12}X_1 \wedge X_2 + c^{23}X_2 \wedge X_3 + c^{24}X_2 \wedge X_4 + c^{34}X_3 \wedge X_4$ |
| $f_{23}^2 = 1$ | $c^{13}X_1 \wedge X_3 + c^{23}X_2 \wedge X_3 + c^{24}X_2 \wedge X_4 + c^{34}X_3 \wedge X_4$ |
| $f_{34}^3 = 1$ | $c^{12}X_1 \wedge X_2 + c^{13}X_1 \wedge X_3 + c^{14}X_1 \wedge X_4$ |

\[(5.4)\]
5.2. \textit{r-qn structures.} We take a representative on each class

\[ r_0 = X_1 \wedge X_4 - X_2 \wedge X_3, \quad r_1 = X_1 \wedge X_2 - X_2 \wedge X_3, \quad r_2 = X_1 \wedge X_2 - X_1 \wedge X_3 + X_1 \wedge X_4, \]
\[ r_3 = X_1 \wedge X_2 + X_1 \wedge X_3, \quad r_4 = X_1 \wedge X_2 - X_1 \wedge X_3, \quad r_5 = X_1 \wedge X_2. \]

Now, we find all \textit{r-qn} structures on Lie algebra \( A_{4,1} \) for each \textit{r-matrices} \( r_i \) \( (i := 0, ..., 5) \).

It is easy to see that every 3-form on this Lie algebra is close or equivalently is a 3-cocycle, in fact using (2.3), we have

\[ \partial \phi(X_1, X_2, X_3, X_4) = \phi(X_1, X_1, X_3) - \phi(X_2, X_1, X_2) = 0. \]

The same we see that \( i_n \phi \) is also a 3-cosycle, since

\[ \partial(i_n \phi)(X_1, X_2, X_3, X_4) = i_n \phi(X_1, X_1, X_3) - i_n \phi(X_2, X_1, X_2), \]

which vanishes apart from whatever \( i_n \phi \) is. Therefore, it only needs to be solved the equations (4.3), (4.6) and (4.7).

Take a generic (1,1)-tensor field \( n = \sum_{i,j=1}^{4} n^i_1 X_i \otimes X^j \). By inserting \( X_i \) and \( Y_i \) (5.4), the matrix forms \( n \) in the relations (4.6) and (4.7) we find all (1,1)-tensor fields \( n_i \) which are compatible with \( r_i \). For example for the two first \textit{r-matrices} we find the compatible couples \( (r_0,n_{r_0}) \) and \( (r_1,n_{r_1}) \) with the following (1,1)-tensor fields,

\[ n_{r_0} = \begin{pmatrix} n_1 & -n_2 & n_4 & 0 \\ 0 & n_3 & 0 & n_4 \\ 0 & 0 & n_3 & n_2 \\ 0 & 0 & 0 & n_1 \end{pmatrix}, \]

\[ n_{r_1} = \begin{pmatrix} n_1 + n_2 & 0 & n_3 - n_2 & n_4 \\ 0 & n_1 + n_3 & 0 & n_5 \\ n_1 & 0 & n_3 & n_6 \\ 0 & 0 & 0 & n_2 \end{pmatrix}. \]

We indicate by \( n_{r_i} \), the (1,1)-tensor fields compatible with \( r_i \); and for simplicity, we used the notation \( n_i \) for the element of matrices \( n_{r_i} \), instead of \( n_i^j \).

Finally, by inserting the matrices (5.4) of \( X_i \), \( Y_i \), matrix forms (5.2) of \( \phi_i \) and the matrix forms of \( r_i \) and \( n_{r_i} \) for \( (i := 1, ..., 4) \), in the relation (4.3) and solving equations we will find \( \phi_i \)'s and thus 3-forms \( \phi \) on \( g \).

We find \textit{r-qn} structures \( (r_0, \phi_{r_0}, n_{r_0}) \) and \( (r_1, \phi_{r_1}, n_{r_1}) \) where

\[ \phi_{r_0} \equiv 0, \quad \phi_{r_1} = (n_1^2 + n_1 n_3 - n_1 n_2) X_1 \wedge X_3 \wedge X_4. \]

So, there is no non-trivial \textit{r-qn} structure with \( r_0 \) on this Lie algebra. About \( r_1 \), we impose the following conditions in order to \( \phi_i \)'s are non-zero.

\[ n_1 \neq 0, \quad \text{and} \quad n_1 \neq n_2 - n_3. \quad (5.5) \]

All \textit{r-qn} structures on four-dimensional symplectic real Lie algebra \( A_{4,1} \) and non-symplectic real Lie algebra \( A_{4,8} \) are given in tables 2.a and 2.b, respectively. Note that, the first column gives the non-vanishing structural constants of the Lie algebra \( g^* \) defined by the corresponded \( r \)-matrix in column two.
### Table 2.a. \(r\)-\(q\)n structures on four-dimensional symplectic real Lie algebras \(A_{4,1}\).

| \(f^3_k\) | \(r\)-matrix \(r\) | (1,1)-tensor field \(n\) |
|-----------|-------------------|-------------------|
| \(f^3_1\) | \(r = X_1 \land X_4 - X_2 \land X_3\) | \(n(X_1) = n_1X_1\) |
| \(f^3_1\) | \(\phi \equiv 0\) | \(n(X_2) = -n_2X_1 + n_3X_2\), \(n(X_3) = n_4X_1 + n_3X_3\), \(n(X_4) = n_4X_2 + n_2X_3 + n_1X_4\) |
| \(f^2_2\) | \(r = X_1 \land X_2 + X_1 \land X_4 - X_3 \land X_3\) | \(n(X_1) = (n_1 + n_3)X_1\) |
| \(f^2_2\) | \(\phi = -(n_1^2 + n_1n_9)X^2 \land X^3 \land X^4\) | \(n(X_2) = (n_3 - n_4)X_1 + (n_2 + n_9)X_2\), \(n(X_3) = n_3X_1 + (n_1 + n_5 + n_6)X_2\), \(n(X_4) = n_4X_1 + n_5X_2 + n_6X_3 + n_2X_4\) |
| \(f^2_4\) | \(r = X_1 \land X_2 - X_1 \land X_3\) | \(n(X_1) = n_1X_1\) |
| \(f^2_4\) | \(\phi = (2n_1n_4 - n_1^2 - n_2^2)X^2 \land X^3 \land X^4\) | \(n(X_2) = n_2X_1 + n_4X_2\), \(n(X_3) = n_2X_1 + (n_4 - n_1)X_2 + n_4X_3\), \(n(X_4) = n_3X_1 + n_5X_2 + n_6X_3 + n_4X_4\) |
| \(f^1_3\) | \(r = X_1 \land X_2 - X_2 \land X_3\) | \(n(X_1) = (n_1 + n_2)X_1 + n_1X_3\) |
| \(f^1_3\) | \(\phi = (n_1^2 + n_1n_3 - n_1n_2)X^1 \land X^3 \land X^4\) | \(n(X_2) = (n_1 + n_3)X_2\), \(n(X_3) = (n_3 - n_2)X_1 + n_3X_3\), \(n(X_4) = n_4X_1 + n_5X_2 + n_6X_3 + n_2X_4\) |
| \(f^2_0\) | \(r = X_1 \land X_2\) | \(n(X_1) = n_1X_1\) |
| \(f^2_0\) | \(\phi = (n_1^2 - n_1n_6 - n_1n_9 - n_7n_8)X^1 \land X^3 \land X^4 + (n_1n_4 - n_4n_9)X^2 \land X^3 \land X^4 + n_5n_9X^2 \land X^3 \land X^4\) | \(n(X_2) = n_1X_2\), \(n(X_3) = n_2X_1 + n_4X_2 + n_6X_3 + n_8X_4\), \(n(X_4) = n_3X_1 + n_5X_2 + n_7X_3 + n_9X_4\) |

### Table 2.b. \(r\)-\(q\)n structures on four-dimensional non-symplectic real Lie algebras \(A_{4,8}\).

| \(f^3_k\) | \(r\)-matrix \(r\) | (1,1)-tensor field \(n\) |
|-----------|-------------------|-------------------|
| \(f^3_1\) | \(r = X_2 \land X_4\) | \(n(X_1) = n_1X_1\) |
| \(f^3_1\) | \(\phi \equiv 0\) | \(n(X_2) = n_2X_2\) |
| \(f^3_1\) | \(\phi \equiv 0\) | \(n(X_3) = n_3X_2 + n_1X_3 + n_4X_4\), \(n(X_4) = n_2X_4\) |
| \(f^2_2\) | \(r = X_3 \land X_4\) | \(n(X_1) = n_1X_1\) |
| \(f^2_2\) | \(\phi \equiv 0\) | \(n(X_2) = n_1X_2 + n_2X_3 + n_4X_4\), \(n(X_3) = n_3X_3\) |
| \(f^2_4\) | \(r = X_1 \land X_2 + X_1 \land X_3 + X_2 \land X_4\) | \(n(X_1) = (n_1 + n_2)X_1\) |
| \(f^2_4\) | \(\phi = -(n_2n_5)X^2 \land X^3 \land X^4\) | \(n(X_2) = -(n_3 + n_4)X_1 + (n_1 + n_2 - n_5)X_2\) |
| \(f^2_4\) | \(\phi \equiv 0\) | \(n(X_3) = n_3X_1 + n_1X_3\) |
| \(f^3_1\) | \(\phi \equiv 0\) | \(n(X_4) = n_4X_1 + n_5X_2 + n_2X_3 + (n_1 + n_2)X_4\) |
| \(f^2_2\) | \(r = X_1 \land X_2 + X_1 \land X_3\) | \(n(X_1) = (2n_1 - n_2)X_1\) |
| \(f^2_2\) | \(\phi = (2n_1^2 - 4n_1n_2 + 2n_2^2)X^1 \land X^2 \land X^4\) | \(n(X_2) = -n_3X_1 + n_1X_2 + (n_1 - n_2)X_3\) |
| \(f^2_2\) | \(\phi = (4n_1n_2 - 2n_1^2 - 2n_2^2)X^1 \land X^3 \land X^4\) | \(n(X_3) = n_3X_1 + (n_1 - n_2)X_2 + n_1X_3\) |
| \(f^2_2\) | \(\phi = (n_1 - n_1)(n_5 + n_6)X^2 \land X^3 \land X^4\) | \(n(X_4) = n_4X_1 + n_5X_2 + n_6X_3 + n_2X_4\) |

Table 2.b. \(r\)-\(q\)n structures on four-dimensional non-symplectic real Lie algebras \(A_{4,8}\).
5.3. Equivalence classes of \( r \)-\( qn \) structures. We use the following automorphism group element (classified in \([3]\), see also \([11]\)) of Lie algebra \( A_{4,1} \)

\[
A = \begin{pmatrix}
a_{11} a_{16}^2 & a_{7} a_{16} & a_{3} & a_{4} \\
0 & a_{11} a_{16} & a_{7} & a_{8} \\
0 & 0 & a_{11} & a_{12} \\
0 & 0 & 0 & a_{16}
\end{pmatrix}.
\]

Since for \( r_0 \) we have \( \phi \equiv 0 \), it means \((1,1)\)-tensor \( n \) is a Nijenhuis operator and in fact the couple \((r_0, n_{r_0})\) is an \( r \)-\( n \) structure and these structures classified in \([10]\), (see the table 3 of \([10]\)).

For the \( r \)-matrix \( r_1 = X_1 \wedge X_2 - X_2 \wedge X_3 \), we insert the above \( A \) in the relation \( A \circ r_1 - r_1 \circ A^{-t} = 0 \) and we get

\[
a_{16} = \frac{1}{(a_{11})^2}, \quad a_3 = \frac{(a_{11})^4 - 1}{(a_{11})^3}, \quad a_7 = 0,
\]

\[\text{(5.6)}\]

The automorphism group \( A \) in the new expression, given by \([5.6]\) is

\[
A = \begin{pmatrix}
\frac{1}{(a_{11})^2} & 0 & (a_{11})^4 - 1 & a_4 \\
0 & \frac{1}{a_{11}} & 0 & a_8 \\
0 & 0 & a_{11} & a_{12} \\
0 & 0 & 0 & \frac{1}{(a_{11})^2}
\end{pmatrix}.
\]

\[\text{(5.7)}\]

Since \( \det(A) = \frac{1}{(a_{11})^5} \), \( a_{11} \neq 0 \) and other parameters \( a_4, a_8, a_{12} \) can take any value.

Now, we find all equivalence classes of \((1,1)\)-tensor fields \( n_{r_1} \) such that \((r_1, n'_{r_1}) \sim_0 (r_1, n_{r_1})\), where

\[
n'_{r_1} = \begin{pmatrix}
n'_1 + n'_2 & 0 & n'_3 - n'_2 & n'_4 \\
0 & n'_1 + n'_3 & 0 & n'_5 \\
n'_1 & 0 & n'_3 & n'_6 \\
0 & 0 & 0 & n'_2
\end{pmatrix}.
\]

Therefore, we have all equivalence classes of the couples \((r_1, n'_{r_1})\) corresponding to the \( r \)-matrix \( r_1 \). In order to do, we insert the automorphism group \([5.7]\) in the relation \( n_{r_1} \circ A - A \circ n'_{r_1} = 0 \), we obtain the following equations

1. \( n_2 - n'_2 = 0, \quad n_3 - n'_3 = 0, \quad (n_1 + n_2) - (n'_1 + n'_2) = 0, \quad (n_1 + n_3) - (n'_1 + n'_3) = 0, \)
2. \( (a_{11})^4(n_3 - n'_3) + n'_1 + n'_2 - n_2 = 0, \)
3. \( a_8(a_{11})^2(n_2 - n'_2 - n'_3) + a_{11} n_5 - n'_5 = 0, \quad (a_{11})^2(a_{11} n_6 + a_{12} n_2 - a_{12} n'_3) - n'_6 = 0, \)
4. \( (a_{11})^3(n_6 a_{11} + a_{12} n_2 - a_{12} n'_1 - a_{12} n'_2 - a_{12} n'_3 + a_{12} n'_6) - a_{11} n'_4 + n_4 - n_6 = 0. \)
The set of equations (1) implies that \( n_1 := n_1', n_2 := n_2' \) and \( n_3 := n_3' \) which means three parameters \( n_1, n_2 \) and \( n_3 \) are free. Note that, we mean by free parameters \( n_i \), the parameters for which different values get non-equivalent Nijenhuis structures belonging to the different equivalence classes. The equation (2) implies that \( a_{11} = \pm 1 \). Applying \( a_{11} = \pm 1 \) in the equations (3) and (4) we get

\[
\begin{align*}
n_6 \pm n_6' &= \pm a_{12}(n_3 - n_2), \\
n_5 \pm n_5' &= \pm a_8(n_1 + n_3 - n_2), \\
n_4 \pm n_4' &= \pm (a_{12}(n_3 - n_2) + a_4 n_1).
\end{align*}
\]

From the first equation, if \( n_2 = n_3 \) then \( n_6 = \pm n_6' \); it means that for the structures whose \( n_2 = n_3 \) the parameter \( n_6 \) is free; note that in this case the structures with the opposite sign of \( n_6 \) are equivalent. For the structure whose \( n_2 \neq n_3 \) the parameter \( n_6 \) can be any constant because of arbitrary parameter \( a_{12} \). From the second and third equations, parameters \( n_4 \) and \( n_5 \) can be any arbitrary constant since parameters \( a_8 \) and \( a_{12} \) are arbitrary. We indicate arbitrary constants \( n_i \) by \( c_i \). We mean by arbitrary constants \( c_i \), the parameters such that for every different values of them, the corresponding Nijenhuis structures are equivalent belonging to the same class.

Finally, we get the following equivalence classes of \((1, 1)\)-tensor \( n \)

\[
n^{(1)}_{r_1} = \begin{pmatrix} n_1 + n_2 & 0 & n_3 - n_2 & c_4 \\ 0 & n_1 + n_3 & 0 & c_5 \\ n_1 & 0 & n_3 & c_6 \\ 0 & 0 & 0 & n_2 \end{pmatrix}, \quad n_2 \neq n_3 \neq 0,
\]

\[
n^{(2)}_{r_1} = \begin{pmatrix} n_1 + n_2 & 0 & n_3 - n_2 & c_4 \\ 0 & n_1 + n_3 & 0 & c_5 \\ n_1 & 0 & n_3 & n_6 \\ 0 & 0 & 0 & n_2 \end{pmatrix}, \quad n_6 \in \mathbb{R}^+, \ n_2 = n_3.
\]

Note that, all parameters are in \( \mathbb{R} \) unless we mention some conditions for them. Therefore, we get the equivalence classes of couples \((r_1, n^{(1)}_{r_1})\) and \((r_1, n^{(2)}_{r_1})\) with additional conditions \((5.3)\).

Now we consider the equivalence class of 3-form \( \phi \). We should solve the equations

\[
A^i \phi_i A - \phi_i' = 0, \quad \text{for} \quad i := 1, \ldots, 4, \tag{5.8}
\]

where \( \phi_i' \)'s are the same as \( \phi_i \) defined in \((5.2)\), the difference is that we denote the elements of \( \phi_i' \)'s by \( \phi_{ijk}' \) instead of \( \phi_{ijk} \), and the automorphism group \( A \) is \((5.7)\) with the condition \( a_{11} = \pm 1 \).

For \( \phi_1 \), the equation \((5.8)\) gets \( n_1^2 + n_1 n_3 - n_1 n_2 = 0 \) which can not be zero, so we have no more equivalence class of \( \phi \).

One may check the equation \((5.8)\) for \( \phi_2, \phi_3 \) and \( \phi_4 \) and get the same results, but actually it dose not need to be done because according to the Definition \((5.5)\) \( \phi \sim \phi' \) if and only if \( \phi_i \sim \phi_i' \) for all \( i = 1, \ldots, 4 \).

Eventually, we have the triples \((r_1, \phi_{r_1}, n^{(1)}_{r_1})\) and \((r_1, \phi_{r_1}, n^{(2)}_{r_1})\) as equivalence classes of \( r \)-form structures on Lie algebra \( A_{4,1} \) with \( r \)-matrix \( r_1 \).

All equivalence classes of \( r \)-form structures on four-dimensional symplectic real Lie algebra \( A_{4,1} \) and non-symplectic real Lie algebra \( A_{4,8} \) are listed in tables \( 3.a \) and \( 3.b \), respectively.

Note that we use the following Lie algebra automorphism for Lie algebra \( A_{4,8} \) in the procedure of classification

\[
A = \begin{pmatrix} a_{11} a_{16} & a_{12} a_6 & a_{11} a_8 & a_4 \\ 0 & a_6 & 0 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]
### Table 3.a. Equivalence classes of $r$-$qn$ structures on four-dimensional symplectic real Lie algebra $A_4$\(_1\)

| $r$-matrix $r$ | $(1,1)$-tensor field $n$ | 3-form $\phi$ | Comments |
|----------------|----------------------|---------------|----------|
| $r = X_1 \wedge X_2 - X_1 \wedge X_3$ | $(r_1) = n_1X_1$ | $(n_1) = n_1X_1$ | $n_1 \neq n_4$ |
| $\phi = (2n_1n_4 - n_1^2 - n_4^2)X^2 \wedge X^3 \wedge X^4$ | $(n_1) = n_1X_1$ | $(n_2) = n_2X_1 + n_4X_2$ | $n_1$ or $n_4 \neq 0$ |
| | $(n_3) = n_2X_1 - n_2X_2 + n_1X_3$ | $(n_4) = c_3X_1 + n_5X_2 + n_6X_3 + n_4X_4$ | $n_1 \neq n_4$ |
| | $(n_5) = c_3X_1 + n_5X_2 + n_6X_3 + n_4X_4$ | | $n_2 \neq n_1 - n_4$ |
| $r = X_1 \wedge X_2 - X_2 \wedge X_3$ | $(n_1) = n_1X_1$ | $(n_2) = n_2X_1 + n_4X_2$ | $n_2 \neq n_1 - n_4$ |
| $\phi = (n_1^2 + n_1n_3 - n_1n_2)X^1 \wedge X^3 \wedge X^4$ | $(n_2) = n_2X_1 + n_4X_2$ | $(n_3) = n_2X_1 + (n_4 - n_1)X_2 + n_1X_3$ | $n_1 \neq n_3$ |
| | $(n_4) = c_4X_1 + c_5X_2 + c_6X_3 + n_4X_4$ | | $n_1 \neq n_2 - n_3$ |
| $r = X_1 \wedge X_2 - X_2 \wedge X_3$ | $(n_1) = n_1X_1$ | $(n_2) = n_2X_1 + n_4X_2$ | $n_3 \neq 0$ |
| $\phi = (n_1n_3 - n_1n_2)X^1 \wedge X^3 \wedge X^4$ | $(n_2) = n_2X_1 + n_4X_2$ | $(n_3) = n_2X_3$ | $n_6 \in \mathbb{R}^+$ |
| | $(n_4) = c_4X_1 + c_5X_2 + n_6X_3 + n_4X_4$ | | |
| | | | |
| $r = X_1 \wedge X_2$ | $(n_1) = n_1X_1 + n_2X_3 + n_3X_4$ | $(n_2) = n_1X_1 + n_2X_3 + n_3X_4$ | $n_7n_8 \neq 0$ |
| $\phi = -(n_7n_8)X^1 \wedge X^3 \wedge X^4$ | $(n_2) = n_2X_3$ | $(n_3) = n_6X_3 + n_7X_4$ | $n_7n_8 \neq 0$ |
| | $(n_4) = n_8X_3 + n_1X_4$ | | |
| $r = X_1 \wedge X_2$ | $(n_1) = n_1X_1 + n_2X_3 + n_3X_4$ | $(n_2) = n_1X_1 + n_2X_3 + n_3X_4$ | $n_7n_8 \neq 0$ |
| $\phi = -(n_7n_8)X^1 \wedge X^3 \wedge X^4$ | $(n_2) = n_2X_2 + n_4X_3 + n_5X_4$ | $(n_3) = n_6X_3 + n_7X_4$ | $n_7n_8 \neq 0$ |
| | $(n_4) = n_8X_3 + n_1X_4$ | | |
| $r = X_1 \wedge X_2$ | $(n_1) = n_1X_1 + n_2X_3 + n_3X_4$ | $(n_2) = n_1X_1 + n_2X_3 + n_3X_4$ | $n_7n_8 \neq 0$ |
| $\phi = (n_5n_8)X^2 \wedge X^3 \wedge X^4$ | $(n_2) = n_2X_3$ | $(n_3) = n_6X_3 + n_7X_4$ | $n_7n_8 \neq 0$ |
| | $(n_4) = n_8X_3 + n_1X_4$ | | |
| $r = X_1 \wedge X_2$ | $(n_1) = n_1X_1 + n_2X_3 + n_3X_4$ | $(n_2) = n_1X_1 + n_2X_3 + n_3X_4$ | $n_7n_8 \neq 0$ |
| $\phi = (n_5n_8)X^2 \wedge X^3 \wedge X^4$ | $(n_2) = n_2X_3$ | $(n_3) = n_6X_3 + n_7X_4$ | $n_7n_8 \neq 0$ |
| | $(n_4) = n_8X_3 + n_1X_4$ | | |
| $r = X_1 \wedge X_2$ | $(n_1) = n_1X_1 + n_2X_3 + n_3X_4$ | $(n_2) = n_1X_1 + n_2X_3 + n_3X_4$ | $n_7n_8 \neq 0$ |
| $\phi = (n_1n_4 - n_4n_9)X^2 \wedge X^3 \wedge X^4$ | $(n_2) = n_1X_1 + n_2X_3 + n_3X_4$ | $(n_3) = n_1X_3 + n_7X_4$ | $n_4n_6 \neq n_4n_9$ |
| | $(n_4) = n_9X_4$ | | |
| $r = X_1 \wedge X_2$ | $(n_1) = n_1X_1 + n_2X_3 + n_3X_4$ | $(n_2) = n_1X_1 + n_2X_3 + n_3X_4$ | $n_7n_8 \neq 0$ |
| $\phi = -(n_7n_8)X^1 \wedge X^3 \wedge X^4$ | $(n_2) = n_2X_2 + n_4X_3 + n_5X_4$ | $(n_3) = n_1X_3 + n_7X_4$ | $n_7n_8 \neq 0$ |
| | $(n_4) = n_8X_3 + n_9X_4$ | | |
| $r = X_1 \wedge X_2$ | $(n_1) = n_1X_1 + n_2X_3 + n_3X_4$ | $(n_2) = n_1X_1 + n_2X_3 + n_3X_4$ | $n_7n_8 \neq 0$ |
| $\phi = (n_5n_8 - n_1n_4 - n_4n_9)X^2 \wedge X^3 \wedge X^4$ | $(n_2) = n_1X_2 + n_4X_3 + n_5X_4$ | $(n_3) = n_1X_3 + n_7X_4$ | $n_4n_6 \neq n_4n_9$ |
| | $(n_4) = n_8X_3 + n_9X_4$ | | |
| $r = X_1 \wedge X_2$ | $(n_1) = n_1X_1 + n_2X_3 + n_3X_4$ | $(n_2) = n_1X_1 + n_2X_3 + n_3X_4$ | $n_7n_8 \neq 0$ |
| $\phi = (n_5n_8 - n_1n_4 - n_4n_9)X^2 \wedge X^3 \wedge X^4$ | $(n_2) = n_1X_2 + n_4X_3 + n_5X_4$ | $(n_3) = n_1X_3 + n_7X_4$ | $n_7n_8 \neq 0$ |
| | $(n_4) = n_8X_3 + n_9X_4$ | | |
| $r = X_1 \wedge X_2 - X_1 \wedge X_3 + X_1 \wedge X_4$ | $(n_1) = (n_2 - n_1)X_1$ | $(n_2) = (c_3 - c_4)X_1 + (n_6 + n_2)X_2$ | $n_1 \neq 0$ |
| $\phi = -(n_5^2 + n_1n_6)X^2 \wedge X^3 \wedge X^4$ | $(n_2) = (c_3 - c_4)X_1 + (n_6 + n_2)X_2$ | $(n_3) = c_3X_1 + (n_1 + n_5 + n_6)X_2$ | $n_1 \neq 0$ |
| | $(n_4) = c_4X_1 + n_5X_2 + n_6X_3 + n_2X_4$ | | $n_1 \neq n_6$ |
Table 3.b. Equivalence classes of $r$-$qn$ structures on four-dimensional non-symplectic real Lie algebra $A_{4,8}$

| $r$-matrix $r$ | (1,1)-tensor field $n$ | 3-form $\phi$ | Comments |
|----------------|------------------------|---------------|----------|
| $r = X_2 \wedge X_4$ | $n(X_1) = n_1 X_1$ | $n(X_2) = n_2 X_2$ | $n(X_3) = n_3 X_3$ | $n(X_4) = n_2 X_4$ |
| $\phi \equiv 0$ | $n(X_1) = n_1 X_1$ | $n(X_2) = n_2 X_2$ | $n(X_3) = n_3 X_3$ | $n(X_4) = n_2 X_4$ |
| $r = X_2 \wedge X_4$ | $c_3 \in \mathbb{R} - \{0\}$ |
| $r = X_3 \wedge X_4$ | $n(X_1) = n_1 X_1$ | $n(X_2) = n_1 X_2$ | $n(X_3) = n_3 X_3$ | $n(X_4) = n_3 X_4$ |
| $r = X_3 \wedge X_4$ | $c_2 \in \mathbb{R} - \{0\}$ |
| $r = X_3 \wedge X_4$ | $n(X_1) = n_1 X_1$ | $n(X_2) = n_1 X_2$ | $n(X_3) = n_3 X_3$ | $n(X_4) = n_3 X_4$ |
| $r = X_3 \wedge X_4$ | $c_4 \in \mathbb{R} - \{0\}$ |
| $r = X_3 \wedge X_4$ | $n(X_1) = n_1 X_1$ | $n(X_2) = n_1 X_2$ | $n(X_3) = n_3 X_3$ | $n(X_4) = n_3 X_4$ |
| $r = X_3 \wedge X_4$ | $c_3 \in \mathbb{R}$ |
| $r = X_1 \wedge X_2 + X_1 \wedge X_3 + X_1 \wedge X_4$ | $n(X_1) = (n_1 + \ldots) X_1$ | $n(X_2) = -n_1 X_1 + n_1 X_2 + n_1 X_3 + n_1 X_4$ | $n(X_3) = n_3 X_1 + n_1 X_3$ | $n(X_4) = n_4 X_1 + n_5 X_2 + n_2 X_3 + n_1 X_4$ |
| $\phi = -n_2 n_5 X^2 \wedge X^3 \wedge X^4$ | $n_2 \in \mathbb{R} - \{0\}$ |
| $r = X_1 \wedge X_2 + X_1 \wedge X_3$ | $n(X_1) = (n_1 + \ldots) X_1$ |
| $\phi = (n_5 n_6 - n_1 n_5) X^2 \wedge X^3 \wedge X^4$ | $n_5 \in \mathbb{R} - \{0\}$ |
| $r = X_1 \wedge X_2$ | $n(X_1) = n_1 X_1$ |
| $\phi = (n_5 n_6 - n_1 n_5) X^2 \wedge X^3 \wedge X^4$ | $n_1 \not\in \mathbb{R}$ |
| $r = X_1 \wedge X_2$ | $n(X_1) = n_1 X_1$ |
| $\phi = (n_5 n_6) X^2 \wedge X^3 \wedge X^4$ | $n_4 \in \mathbb{R} - \{0\}$ |
| $r = X_1 \wedge X_3$ | $n(X_1) = n_1 X_1$ |
| $\phi = (n_5 n_6) X^2 \wedge X^3 \wedge X^4$ | $n_6 \in \mathbb{R} - \{0\}$ |
| $r = X_1 \wedge X_3$ | $n(X_1) = n_1 X_1$ |
| $\phi = (n_4 n_7 - n_1 n_7) X^2 \wedge X^3 \wedge X^4$ | $n_7 \in \mathbb{R} - \{0\}$ |
| $r = X_1 \wedge X_3$ | $n(X_1) = n_1 X_1$ |
| $\phi = (n_4 n_7 - n_1 n_7) X^2 \wedge X^3 \wedge X^4$ | $n_1 \not\in \mathbb{R}$ |
6. Some remarks on r-qn structures

In this section we shall consider the conditions for which an r-qn structure on Lie algebra \( g \) defines a generalized complex structure on \( g \) or an R-matrix on double of Lie algebra \( g \oplus g^* \). We bring some relevant examples of r-qn structures of previous section.

It is well-known that for a Lie bialgebra \((g, g^*)\), the vector space \( g \oplus g^* \), so-called the double of Lie bialgebra, is equipped with a Lie algebra structure defined by

\[
[X + \alpha, Y + \beta] = [X, Y] + [\alpha, \beta] + ad_\alpha^* \beta + ad_\beta^* Y - ad_Y^* \alpha - ad_\alpha^* X, \quad \forall X, Y \in g, \quad \alpha, \beta \in g^*. \tag{6.1}
\]

where \( ad_\alpha^* \beta \) and \( ad_\alpha^* X \) are the coadjoint representations of \( g \) on \( g^* \) and of \( g^* \) on \( g \) respectively. Let us use \( g \bowtie g^* \) to denote the vector space \( g \oplus g^* \) with the Lie algebra structure \((6.1)\). Recall that the non-degenerate symmetric bilinear form on the vector space \( g \oplus g^* \) defined by

\[
\langle X + \alpha, Y + \beta \rangle = \alpha(X) + \beta(Y), \quad \forall X, Y \in g, \forall \alpha, \beta \in g^*,
\]

for more details we refer to [6] and [8].

By definitions, the solutions of modified Yang-Baxter equation on double Lie bialgebra \( g \bowtie g^* \) with coefficient \( k = -1 \) are identified with the generalized complex structures on the Lie algebra \( g \).

**Corollary 6.1.** The r-qn structure \((r, \phi, n)\) on \( g \) defines an R-matrix \( J \) on the Lie algebra \( g \bowtie g^* \) of the form

\[
J = \begin{pmatrix} n & r^u \\ \theta^u \theta^r & -n \end{pmatrix} \tag{6.2}
\]

if \( \phi \) is a 3-coboundary and following two conditions hold

\[
\begin{align*}
n^t \theta^u &= \theta^2 N \\
n^2 + r^u \theta^r &= k \text{Id},
\end{align*} \tag{6.3}
\]

where \( \theta \) is a 2-cochain in \( C^2(g) \) such that \( \phi = \partial \theta \).

**Proof.** It is proved directly from the Corollary 2.4.

\[ \Box \]

**Example 1.** Consider the four-dimensional non-symplectic real Lie algebra \( g = A_{4,8} \) with non-zero commutators \([X_2, X_3] = X_1, [X_2, X_4] = X_2 \) and \([X_3, X_4] = -X_3\). We choose the third r-qn structure \((r, n)\) on this algebra of the table 3, as

\[
\begin{align*}
r^u(X^1) &= X_2 + X_3 + X_4, \\
r^u(X^2) &= -X_1, \\
r^u(X^3) &= -X_1, \\
r^u(X^4) &= -X_1,
\end{align*}
\]

\[
\phi = -(n_2 n_5) X^2 \wedge X^3 \wedge X^4,
\]

\[
\begin{align*}
n(X_1) &= (n_1 + n_2)X_1, \\
n(X_2) &= -(n_3 + n_4)X_1 + (n_1 + n_2 - n_5)X_2, \\
n(X_3) &= n_3X_1 + n_1X_3, \\
n(X_4) &= n_4X_1 + n_5X_2 + n_2X_3 + (n_1 + n_2)X_4.
\end{align*}
\]

The Lie algebra structure on the dual Lie algebra \( g^* \) of \( g \) induced by r-matrix \( r \) has the non-zero commutators \([X^1, X^2] = X^2 - X^4 \) and \([X^1, X^3] = X^4 - X^3\).

Integrating 3-cocycle \( \phi \) we find out it is a 3-coboundary, that is there exist 2-cochain \( \theta \) such that \( \partial \theta = \phi \), where

\[
\theta = (\frac{1}{2}n_2 n_5)X^1 \wedge X^4 - (\frac{1}{2}n_2 n_5)X^2 \wedge X^3.
\]

Applying \( n, r^u \) and \( \theta^r \) on two conditions \((6.3)\) and solving the equations, we get

\[
n_2 = n_3 = n_5 = -4n_1, \quad n_4 = 0, \quad k = n_1^2.
\]
Thus the linear map \( J : \mathfrak{g} \otimes \mathfrak{g}^* \to \mathfrak{g} \otimes \mathfrak{g}^* \) is a solution of MYBE with coefficient \( k = n_1^2 \), on the Lie algebra \( \mathfrak{g} \otimes \mathfrak{g}^* \) as

\[
J = \begin{pmatrix}
-3n_1 & 4n_1 & -4n_1 & 0 & 0 & -1 & -1 & 1 \\
0 & n_1 & 0 & -4n_1 & 1 & 0 & 0 & 0 \\
0 & 0 & n_1 & -4n_1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -3n_1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -8n_1^2 & -3n_1 & 0 & 0 & 0 \\
0 & 0 & 8n_1^2 & 0 & 4n_1 & n_1 & 0 & 0 \\
0 & -8n_1^2 & 0 & 0 & -4n_1 & 0 & n_1 & 0 \\
8n_1^2 & 0 & 0 & 0 & 0 & -4n_1 & -4n_1 & -3n_1
\end{pmatrix}.
\] (6.4)

**Remark 6.2.** Under the same assumption as in the Corollary 6.1 if \( k = -1 \), then \( J \) defines a generalized complex structure on \( \mathfrak{g} \).

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**Example 2.** Consider the four-dimensional symplectic real Lie algebra \( \mathfrak{g} := A_{1,1} \) with non-zero commutators \([X_2, X_4] = X_1\) and \([X_3, X_4] = X_2\). On this algebra there is an \( r\)-\( n \) structure (trivial \( r\)-\( qn \) structure) \((r, n)\) (see table 1), as

\[
r^2(X^1) = X_4, \quad r^2(X^2) = -X_3, \quad r^2(X^3) = X_2, \quad r^2(X^4) = -X_1,
\]

Note that, the Lie algebra structure on the dual Lie algebra \( \mathfrak{g}^* \) of \( \mathfrak{g} \) induced by \( r \)-matrix \( r \) has the non-zero commutators \([X^1, X^2] = X^3\) and \([X^1, X^3] = X^4\).

The Nijenhuis operator compatible with \( r \) is characterized by

\[
n(X_1) = n_1 X_1, \quad n(X_2) = -n_2 X_1 + n_3 X_2, \\
n(X_3) = n_4 X_1 + n_3 X_3, \quad n(X_4) = n_4 X_2 + n_2 X_3 + n_1 X_4.
\]

With straightforward computation we see there is no 3-coboundary on this Lie algebra. One can check all 2-cochains \( \theta \in C^2(\mathfrak{g}) \) which satisfy cocycle condition are the form

\[
\theta = c X^1 \wedge X^4 + d X^2 \wedge X^3 + e X^2 \wedge X^4 + f X^3 \wedge X^4, \quad a, b, d, e, f \in \mathbb{R},
\]

that is \( \phi := \partial \theta = 0 \). So we have

\[
\theta_1(X_1) = c X^4, \quad \theta_2(X_2) = d X^3 + e X^4, \\
\theta_3(X_3) = -d X^2 + f X^4, \quad \theta_4(X_4) = -c X^1 - e X^2 - f X^3.
\]

Applying \( n \), \( r^2 \) and \( \theta_2 \) on two conditions \((6.3)\), for \( k := -1 \) we get

\[
c = n_1^2 + 1, \quad d = -1 - n_1^2, \quad e = -2n_1 n_2, \quad f = 2n_1 n_4, \quad n_1 = n_3.
\]

Therefore, we have a generalized complex structure \( J : \mathfrak{g} \oplus \mathfrak{g}^* \to \mathfrak{g} \oplus \mathfrak{g}^* \) on the Lie algebra \( \mathfrak{g} := A_{1,1} \) as

\[
J = \begin{pmatrix}
n_1 & -n_2 & n_4 & 0 & 0 & 0 & -1 \\
0 & n_1 & 0 & n_4 & 0 & 0 & 1 \\
0 & 0 & n_1 & n_2 & 0 & -1 & 0 \\
0 & 0 & 0 & n_1 & 1 & 0 & 0 \\
0 & 0 & 0 & -(n_1^2 + 1) & -n_1 & 0 & 0 \\
0 & 0 & n_1^2 + 1 & 2n_1 n_2 & n_2 - n_1 & 0 & 0 \\
0 & -(n_1^2 + 1) & 0 & -2n_1 n_4 & -n_4 & 0 & n_1 \\
n_1^2 + 1 & -2n_1 n_2 & 2n_1 n_4 & 0 & 0 & -n_4 & -n_2 - n_1
\end{pmatrix}.
\] (6.5)

**Remark 6.3.** The trivial \( r\)-\( qn \) structures (\( r\)-\( n \) structures) on \( \mathfrak{g} \) for which \( \theta \equiv 0 \), the structure \( J \) defined in \((6.2)\) is a generalized complex structure on \( \mathfrak{g} \) if \( n^2 = -\text{Id} \).
Example 3. Consider the four-dimensional symplectic real Lie algebra \( g := II \oplus \mathbb{R} \) with non-zero commutator \([X_2, X_3] = X_1\). We have an \( r\)-n structure \((r, n)\) (trivial \( r\)-q\( n\) structure) on this algebra with following \( r\)-matrix \( r \) and Nijenhuis operator \( n \) (listed in the table 2 of \([10]\)).

The non-degenerate \( r\)-matrix \( r \) is as \( r = X_1 \wedge X_3 - X_2 \wedge X_4 \), so
\[
r^0(X^1) = X_3, \quad r^0(X^2) = -X_4, \quad r^0(X^3) = -X_1, \quad r^0(X^4) = X_2,
\]

Note that, the Lie algebra structure on the dual Lie algebra \( g^* \) of \( g \) induced by \( r\)-matrix \( r \) has the non-zero commutator \([X^1, X^4] = X^3\).

The Nijenhuis operator compatible with \( r \) is characterized by
\[
n(X_1) = n_1 X_1 + n_5 X_4, \quad n(X_2) = -n_2 X_1 + n_4 X_2 + n_5 X_3,
\]
\[
n(X_3) = n_3 X_2 + n_1 X_3 + n_2 X_4, \quad n(X_4) = n_3 X_1 + n_4 X_4.
\]

One can check in general, there is no 3-coboundary on this Lie algebra. Since \( \phi \equiv 0 \), we see that following 2-cochain \( \theta \in C^2(g) \)
\[
\theta = aX^1 \wedge X^2 + bX^1 \wedge X^3 + dX^2 \wedge X^3 + eX^2 \wedge X^4 + fX^3 \wedge X^4, \quad a, b, d, e, f \in \mathbb{R},
\]
are the possibilities for \( \phi := \partial \theta = 0 \). So we have
\[
\theta^k(X_1) = aX^2 + bX^3, \quad \theta^k(X_2) = -aX^1 + dX^3 + eX^4,
\]
\[
\theta^k(X_3) = -bX^1 - dX^2 + fX^4, \quad \theta^k(X_4) = -eX^2 - fX^3.
\]

Applying \( n \), \( r^2 \) and \( \theta^k \) on two conditions (6.3) for \( k = -1 \), we get
\[
a = b = d = e = f = 0, \quad n_1 = n_4 = 0, \quad n_3 = 1, \quad n_5 = -1.
\]

Therefore, we have a generalized complex structure \( J : g \oplus g^* \rightarrow g \oplus g^* \) on the Lie algebra \( g := II \oplus \mathbb{R} \) as

\[
J = \begin{pmatrix}
0 & -n_2 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & n_2 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & n_2 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\]

(6.6)

Note that, the statement of Remark 6.3 holds since \( n^2 = -Id \) and \( \theta \equiv 0 \).

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References

[1] J. Abedi-Fardad, A. Rezaei-Aghdam, Gh. Haghighatdoost, Classification of four dimensional real symplectic Lie bialgebras and their Poisson-Lie groups Theo. Math. Phys. 190 (2017) 1-17.
[2] Crainic, M. Generalized complex structures and Lie brackets, Bull. Braz. Math. Soc. (N.S.) 42(4), 559578 (2011).
[3] T. Christodoulakis, G.O. Papadopoulos, A. Dimakis: Automorphisms of Real 4 Dimensional Lie Algebras and the Invariant Characterization of Homogeneous 4-Spaces, J. Phys. A. 36 427-442, (2003).
[4] M.Gualtieri, Generalized complex geometry, D.Phil. thesis, Oxford University, 2003 , arXiv:math/0401221v4.
[5] N. Hitchin, Generalized Calabi-Yau manifold, Q.J. Math.,54(3):281-308, 2003, arXiv:math/0209099.
[6] Y. Kosmann-Schwarzbach, Lie bialgebras, Poisson Lie groups and dressing transformations, Integability of Nonlinear Systems, Second edition, Lecture Notes in Physics 638, Springer-Verlag, 2004, pp. 107-173.
[7] Y. Kosmann-Schwarzbach and F. Magri, Poisson-Nijenhuis structures, Ann. Inst. Henri Poincare, A Phys. Theor. 53 (1990) 35-81.
[8] Hu, J. H., Multiplicative and affine Poisson structures on Lie groups, PhD thesis, University of California, Berkeley, (1990).
[9] A. Nijenhuis, \(X_{n-1}\)-forming sets of eigenvectors, *Indag. Math* **13**, (1951), 200-212.

[10] Z. Ravanpak, A. Rezaei-Aghdam and Gh. Haghighatdoost, Invariant Poisson Nijenhuis structures on Lie groups and classification, *Int. J. Geom. Methods Mod. Phys.* (2017).

[11] A. Rezaei-Aghdam and M. Sefid, Complex and bi-Hermitian structures on four-dimensional real Lie algebras, *J. Phys. A: Math. Theor.* **43** (2010) 325210.

[12] Mathieu Stienon, Ping Xu Poisson Quasi-Nijenhuis Manifolds, Progress in Mathematics, *Comm. Math. Phys.* **270**, no. 3, 709-725, (2007).

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