Spin Squeezing via One-Axis Twisting with Coherent Light

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We propose a new method of spin squeezing of atomic spin, based on the interactions between atoms and off-resonant light which are known as paramagnetic Faraday rotation and fictitious magnetic field of light. Since the projection process, squeezed light, or special interactions among the atoms are not required in this method, it can be widely applied to many systems. The attainable range of the squeezing parameter is \( \zeta \gtrsim S^{-2/5} \), where \( S \) is the total spin, which is limited by additional fluctuations imposed by coherent light and the spherical nature of the spin distribution.

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Squeezed spin state (SSS) is one of the non-classical states in collective spin system. In SSS, the quantum uncertainty of the spins along an axis orthogonal to the mean spin vector \( \langle \Delta S^2_\perp \rangle \) is suppressed below the standard quantum limit (SCL) such as \( \langle \Delta S^2_\perp \rangle < |\langle S_\perp \rangle|/2 \), where \( \langle S \rangle \) is the mean spin vector, due to an entanglement formation among the individual spins. The degree of the squeezing is usually evaluated by the squeezing parameter \( \zeta \equiv 2\langle \Delta S^2_\perp \rangle/|\langle S_\perp \rangle| \), in terms of the variance to average ratio \( \sqrt{\frac{\langle \Delta S^2_\perp \rangle}{\langle S_\perp \rangle^2}} \).

For the last several years, SSS has been extensively interested in not only for precision measurement of a spin component \( \hat{S}_\perp \) [1, 2, 3, 4], but also for the application to the quantum infomation [5, 6, 7]. There have been many proposals and experiments to realize the spin squeezing of atoms. They can be put into three categories as follows: (i) Quantum non-demolition (QND) measurement of spin via paramagnetic Faraday rotation and spin squeezing by quantum projection [8, 9, 10, 11, 12, 13]. The QND measurement has been already performed by some groups for the electronic ground states of atom [8, 11, 12], and the squeezed parameter has reached about \( \zeta \sim 0.7 \) for \( S \approx 4 \times 10^7 \) [11], and \( \zeta \sim 0.1 \) for \( S \approx 10^{11} \) [13]. Since the projection causes the squeezing in this method, the degree of the squeezing will be finally determined by the performance of the detector. (ii) Quantum-state transfer from squeezed light to spin [9, 11, 13, 15, 17]. One type is based on the complete absorption of squeezed vacuum, and has been experimentally demonstrated for the electronic exited states of atom [9, 11, 13, 15]. Another type is based on the stimulated Raman adiabatic passage [9, 11, 13, 15]. Since the squeezed light is the source of spin squeezing in these methods, the degree of the squeezing will be finally determined by the quality of the squeezed light. (iii) Special systems to induce nonlinear interactions among the individual spins such as Bose-Einstein condensates [13, 14, 21], cold atoms in optical lattice [21], atoms in optical cavity [22, 23, 24]. They are not easy to prepare and difficult to operate after squeezed.

In this paper, we propose a new method to realize the spin squeezing, which can not be put into any of the three categories. Our method does not rely on the projection by the measurement, use of squeezed light, and the specialties of the systems. Instead, the new method only requires a coherent light pulse and a few linear optics, so it can be widely applied to many systems. It should be noted that a recent electronic archive by K.Hammerer et al. [25] includes another proposal of an unconditional spin squeezing with coherent light.

Our method is based on the interaction between atoms and off-resonant light, whose interaction Hamiltonian takes a form [3]

\[
H = \alpha J_z S_z, \tag{1}
\]

where \( \alpha \) is a real constant, and the \( z \)-axis is set parallel to the wave vector of the light. \( S \) is the summation over the individual spin, which obeys the usual commutation relation of angular momenta \( [S, S] = i\hbar S \). \( J \) is quantum-mechanical Stokes vector of light, which also obeys the usual commutation relation of angular momenta \( [J, J] = i\hbar J \). For a light pulse with the duration \( T \) propagating in free space, \( J \) can be written as

\[
J_z = \frac{T}{2} \int_0^T (a_+^\dagger a_- + a_-^\dagger a_+) dt, \quad J_y = \frac{1}{2T} \int_0^T (a_+^\dagger a_- - a_-^\dagger a_+) dt, \quad J_z = \frac{1}{2} \int_0^T (a_+^\dagger a_+ - a_-^\dagger a_-) dt
\]

where \( a_{\pm} \) is the annihilation operators of \( \sigma_{\pm} \), circular polarization mode, respectively [26]. The interaction of Eq. (1) represents the addition of the phase difference for \( \sigma_{\pm} \) light, which causes the rotation of the polarization plane for linear polarization at the angular frequency \( \alpha S_z / 2 \), known as paramagnetic Faraday rotation. It also represents the spin rotation around the \( z \)-axis at the angular frequency \( \alpha J_z \), known as fictitious magnetic field of light [27]. If we are able to apply a light pulse whose \( J_z \) is proportional to \( S_z \) as a fictitious magnetic field, the collective spin will nonlinarly rotate at angular frequencies proportional to \( S_z \), whose evolution will be similar to one-axis twisting [1]. This is the basic idea of our proposal.

To design such an interaction, we propose a system illustrated in Fig. 1. Initially a light pulse \( |\psi_i\rangle \) is lin-
early polarized along the $x$-axis and contains $2J(\geq 1)$ photons as an average. Atoms $|\psi_S\rangle$ are spin-polarized along the $x$-axis and contains total spin $S$. The light is weakly focused to match the atomic ensemble [26]. The averages of the Stokes components is then $\langle J_x \rangle = J_x$, $\langle J_y \rangle = \langle J_z \rangle = 0$, and the averages of the collective spin vector is $\langle S_z \rangle = S$, $\langle S_y \rangle = \langle S_z \rangle = 0$. Since the light pulse is a strong coherent state, we can approximate the commutation relation as $[J_y, J_z] = iJ_z$. Firstly, a light pulse passes through the atoms and the polarization plane is then rotated. We call it “the first interaction”, whose interaction time is labelled as $t_1$. The Stokes vector becomes $J^{(FI)} = e^{iS_z H} J e^{-iS_z H}$, whose $y$ component is approximately written as $J_y^{(FI)} \simeq J_y + \alpha t_1 S_z$, for $\alpha t_1 S_z \ll 1$. Since the average of the $J_y^{(FI)}$ becomes $\langle J_y^{(FI)} \rangle = \alpha t_1 S_z$, we can say that the information of $S_z$ is copied and held on $J_y^{(FI)}$ as a Faraday rotation angle. We note that $S_z$ is conserved because the interaction of Eq. (1) satisfies the back-action evasion (BAE) condition of $[S_z, H] = 0$. Secondly, the pulse passes through twice the $\lambda/8$ wave plate by the totally retroreflecting mirror. As a result, $\lambda/4$ phase difference is induced between the two orthogonal modes of linear polarization. We call it “the local operation” for the light. The Stokes vector becomes $J^{(LO)} = e^{i(\pi/2) J_z} J^{(FI)} e^{-i(\pi/2) J_z}$, whose $z$ component is $J_z^{(LO)} = J_y^{(FI)}$. We can say that the information of $S_z$ is shifted from $J_y^{(FI)}$ to $J_z^{(LO)}$, converting the angle of the polarization plane to the photon number difference of the $\sigma_\pm$ modes. Thus, the required light is achieved whose $J_z$ is approximately proportional to $S_z$. Finally, the pulse passes through the atomic ensemble again. We call it “the second interaction”, whose interaction time is labelled as $t_2$. The interaction Hamiltonian of the second interaction can be roughly written as $H^{(SI)} \sim \alpha J_z^{(LO)} S_z \propto S_z^2$, which takes a form similar to the one-axis twisting Hamiltonian $\chi S_z^2$ [11]. Thus, we can expect that the spin state becomes SSS after the second interaction.

Next, we derive the density operator of the spin after the second interaction to calculate the properties of the spin state obtained by this method. The initial density operator of the whole system can be written as $\rho_{SV} \equiv \rho_S \otimes |\psi_J\rangle \langle \psi_J|$, where $\rho_S = |\psi_S\rangle \langle \psi_S|$. After the second interaction, it becomes $\rho_{SVM} = U \rho_{SV} U^\dagger$ where $U = e^{-i\alpha t_2 H} e^{-i(\pi/2)J_z} e^{-i\alpha t_1 H}$. The reduced density operator representing the spin state after the second interaction $\rho_S$ can be written as $\rho_S = Tr_J(\rho_{SV})$, where $Tr_J$ is the partial trace for the light. For convenience, we consider the set of eigenstates for $S^2$ and $S_z$, say $|S, M\rangle$, where $S^2 |S, M\rangle = (S + 1) |S, M\rangle$ and $S_z |S, M\rangle = M |S, M\rangle$. The matrix elements take a form

$$\langle S, M | \rho_S | S', M' \rangle = \sigma_{MM'} \langle S, M | \rho_S | S', M' \rangle,$$

where we have set $\mu \equiv (\alpha t_1)J_z$ and $\mu' \equiv ((\alpha t_2)^2 + (\alpha t_2)^2)J_z/2$. If $t_1 = t_2$ then $\mu = \mu'$. Since the atoms $|\psi_S\rangle$ are polarized along the $x$-axis, the matrix elements of $\rho_S$ can be written as

$$\langle S, M | \rho_S | S', M' \rangle = \frac{1}{2^{2S}} \left( \frac{2S}{S + M} \right)^{1/2} \left( \frac{2S}{S + M} \right)^{1/2}.$$

In the following discussions, we use the expressions of Eq. (3) and Eq. (4). We note that the ideal one-axis twisted state [1] corresponds to the case of $\mu' = 0$.

To know how uncertainties evolve, we calculate the quasiprobability distributions (QPD), which is defined as $Q(\theta, \phi) = \langle \theta, \phi | \rho_S | \theta, \phi \rangle$, where $|\theta, \phi\rangle = e^{-i\theta S_x} e^{-i\phi S_y} |S, S\rangle$ is a spin state polarized along the direction whose polar and azimuth angles are $\theta$ and $\phi$, respectively [1]. The results of the calculations in the case of $S = 20$ are shown in Fig. 2(a) for the initial spin state (a), the spin state after the first interaction (b), and that after the second (c). The initial spin state is

![FIG. 2: State evolutions expressed as the quasiprobability distribution for $S = 20$. The value of QPD for $(\theta, \phi)$ direction is represented by the gray scale on the unit sphere, which is normalized by the maximum value. (a) The initial spin state. (b) The spin state after the first interaction, where we have set $(\alpha t_1)^2J_z/2 = 0.1$ and $t_2 = 0$, in other words, $\mu = 0$ and $\mu' = 0.1$. (c) The spin state after the second interaction, where we have set $(\alpha t_2)^2J_z/2 = (\alpha t_2)^2J_z/2 = 0.1$. In other words, $\mu = \mu' = 0.2$. The spin squeezing is realized along the $z'$-axis.](http://example.com/fig2.png)
is a little broadened along the $y$ direction, as Fig.3(b) indicates. This is explained by additional fluctuation imposed by coherent light. In fact, the $y$ components after the first interaction is approximately written as $S_{y}^{(F)} = e^{i\theta_{1}} S_{y} e^{-i\theta_{1} H} \approx S_{y} + \alpha_{1} J_{z} S_{x}$ for $\alpha_{1} J_{z} \ll 1$. Since $S_{z}$ is the BAE variable, the distribution along the $z$ direction does not change at all. After the second interaction, the distribution looks twisted around the $z$-axis, as Fig.3(c) indicates. Although not clear from the figure, the distribution is also broadened along the $y$-axis as in the case after the first interaction. In fact, the $y$ component after the second interaction is roughly written as $\tilde{S}_{y} \sim e^{i\theta_{2}} S_{y}^{(F)} e^{-i\theta_{2} H} \sim S_{y} + \mu S_{z} S_{x} + (\alpha_{1} J_{z} + \alpha_{2} J_{y}) S_{x}$.

By these additional fluctuations imposed by coherent light, the spin state after the second interaction is different from the ideal one-axis twisted state $\hat{1}$. The additional fluctuations would be reduced by use of a polarization squeezed light pulse whose squeezed component is $\hat{1} J_{z} + \hat{2} J_{y}$, approaching the ideal one-axis twisting interaction of $\mu' \rightarrow 0$. We mention that the additional fluctuations by light in the method of Ref.27 are imposed both on the $y$ and $z$ components, while the $z$ component is squeezed. Therefore, the squeezing parameter does not become small in that scheme.

From Eq.3 and Eq.4, we can derive the averages and variances of the spin components. The averages can be calculated as $\langle \tilde{S}_{z} \rangle = S e^{-\mu'/2} \cos^{2 S - 1} (\mu/2)$ and $\langle \tilde{S}_{y} \rangle = \langle \tilde{S}_{z} \rangle = 0$, where $S$ represents the spin operator after the second interaction. They indicate that the orientation of the mean spin vector remains $(\pi/2, 0)$ direction or the $x$-axis, as is shown in Fig.3(c). To characterize a elliptical distribution around the $z$-axis, we define the minor and major axis, say $z'$ and $y'$, respectively, as is shown in Fig.3(c), so that the variances of whose components $\langle \Delta \tilde{S}_{y'}^2 \rangle$ and $\langle \Delta \tilde{S}_{z'}^2 \rangle$ give the minimum and maximum on the $y$-$z$ plane, respectively. The variances can be calculated as $\langle \Delta \tilde{S}_{y'}^2 \rangle = S^2 - \langle \tilde{S}_{z} \rangle^2 - S(1/2) A/2$ and

$$\langle \Delta \tilde{S}_{z'}^2 \rangle = \frac{S}{2} + \frac{S - 1/2}{2} \left( A + \sqrt{A^2 + B^2} \right),$$

where we have set $A = 1 - e^{-2 \mu'/2} \cos^{2 S - 2} \mu$, and $B = 4 e^{-\mu'/2} \sin(\mu/2) \cos^{2 S - 2} (\mu/2)$. We can also calculate $\delta$, which is an angle between the directions of the $z'$- and $x$-axes, or the $y'$- and $y$-axes, as shown in Fig.3(c), and obtain $\delta = \arctan(B/A)/2$. For $S \gg 1$ and $S^{-1} \ll \mu \sim \mu' \ll S^{-1/2}$, we find the approximate value of the variance of the $z'$ component

$$\langle \Delta \tilde{S}_{z'}^2 \rangle \approx \frac{S}{2} \left( \gamma' + \gamma + 2 \beta^2 \right),$$

where we have set $\gamma = S \mu^2 / 2$, $\gamma' = S \mu'/2$ and $\beta = S \mu^2 / 4$. Also we find $\langle \tilde{S}_{z} \rangle \approx S(1 - \beta)$.

To examine the dependence on the interaction strength $\alpha_{1}, \alpha_{2}$ and the input photon number $2 J$, we plot the variances of the $y'$ components and the $z'$ components as a function of $\mu (= \mu')$ in Fig.3(a). We also plot the approximate value for the $z'$ components written as Eq.6. It is clearly known that the variance of the $z'$ component is reduced for small $\mu$, minimized at an optimal value of $\mu$, and becomes large for large $\mu$. It means that too strong interaction or too large photon number deteriorates the squeezing. This is explained by the spherical nature of the spin distribution, and in fact, the variance of the $y'$ component is almost saturated at the largest value of $S^2/2$ for large $\mu$, which was entirely ignored in the analysis in Ref.27. As a typical value of $\mu$, we introduce $\mu_{\text{half}}$ as the value of $\mu$ to attain $\langle \Delta \tilde{S}_{z'}^2 \rangle = S/4$, the half variance of the SQL. We also introduce $\mu_{\text{min}}$ as the value to attain the minimum of the $\langle \Delta \tilde{S}_{z'}^2 \rangle$. We plot the numerical solutions of $\mu_{\text{half}}$ and $\mu_{\text{min}}$ in Fig.3(b) for the case of $\mu = \mu'$. One can see that both $\mu_{\text{half}}$ and $\mu_{\text{min}}$ become small as $S$ increases but they obey different power laws. From Eq.1, we find $\mu_{\text{half}} \simeq 2 S^{-1}$ and $\mu_{\text{min}} \simeq 2(3/2)^{1/5} S^{-3/5}$. We show these approximate solutions in Fig.3(b), which are in good agreement with the numerical ones. We also find that the squeezing parameter at $\mu = \mu' = \mu_{\text{min}}$ becomes $\zeta_{\text{min}} \simeq (2/3)^{1/5} S^{-2/5}$. We note that it is slightly worse than $(1/3)^{1/3} S^{-2/3}$, which is the squeezing parameter for the ideal one-axis twisting, due to the additional fluctuations imposed by coherent light, as is mentioned above.

Finally, we discuss the feasibility of our method. In the following, we consider the case that the shape of the light pulse is a square wave with its peak power $P$ and pulse duration $T$. As in Ref.26, we assume $\Delta \gg \Gamma$, $\Omega \ll \Delta$, and $\Gamma T \ll (S \mu)^{-1} \ll 1$, where $\Delta$ represents the detuning from the resonance frequency, $\Gamma$ the full natural
linewidth at half maximum of the transition, $\Omega$ the Rabi frequency, and $r$ the photon scattering rate. After some calculations, we obtain $\mu = \mu' = rT\sigma_0/(2\pi w^2)$, where $w$ represents the beam waist and $\sigma_0$ the photon absorption cross section of an atom, which can be written as $\sigma_0 = 3\lambda_0^2/(2\pi)$ with the resonance wavelength $\lambda_0$. We note that $\Omega$, $r$ and $\mu$ are exactly the same as $2\sqrt{2N_p}/(cT)$, $4\varepsilon_\sigma/T$, and $\kappa^2/N_a$ in Ref. [24], respectively. The condition to obtain $w$ and has only nuclear spin 1/2 whose gyromagnetic ratio is the same as $\mu_0 = 2\pi \sigma_0/(\pi w^2)$ is the optical depth. We also note that this condition is $\kappa \gtrsim \sqrt{\gamma}$ similar to that of QND measurement [24]. Such a condition has been satisfied in several systems, such as atoms in a cell [3], laser cooled and trapped atom, and so on. The feasibility of our scheme also comes from the simple experimental setup depicted in Fig.1 which is also the great advantage over another scheme in Ref. [25]. This suggests efficient squeezing can be realized by the current technologies.

As one ideal example, we consider a ytterbium atoms ($^{171}$Yb) in optical trap [29, 30], which contains $S = 4 \times 10^6$. The atom collision and the precession due to the stray magnetic field, which causes the transverse relaxation, are well suppressed because it is ultracold fermion trapped atom, and so on. The feasibility of our scheme also comes from the simple experimental setup depicted in Fig.1 which is also the great advantage over another scheme in Ref. [25]. This suggests efficient squeezing can be realized by the current technologies.

To avoid the interference between the three steps of the first interaction, the local operation and the second interaction within one pulse, we can use a pulse train, each duration of which is so short that the three steps are separable and the repetition rate is so slow that the pulse number travelling in the path is at most one. Since the each matrix element $\langle S, M | \rho_S | S, M' \rangle$ would evolve like a geometric progression whose common ratio is $\sigma_{MM'}$ for the every pulse as Eq. [2] indicates, and $\sigma_{MM'}$ is the power of the mean photon number of the each pulse as Eq. [4] indicates, we can say that the same SSS would be obtained as long as the total mean photon number passed through the atomic ensemble are equal.

[1] M.Kitagawa and M.Ueda, Phys.Rev.A 47, 5138 (1993).