MAGNETIC NANOPARTICLES IN THE INTERSTELLAR MEDIUM: EMISSION SPECTRUM AND POLARIZATION

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1. INTRODUCTION

Iron is the fifth most abundant element by mass (after H, He, C, and O), assuming heavy element abundances approximately proportional to those in the Sun (Asplund et al. 2009). Interstellar iron is predominantly sequestered in solid grains (Jenkins 2009), although the composition of the Fe-bearing grains is largely unknown. The Fe could be in ferromagnesian silicates, but some authors have argued that much of the iron is in metallic Fe or Fe oxides (e.g., Schalen 1965; Wickramasinghe & Nandy 1971; Huffman 1977; Chlewicki & Laureijs 1988; Cox 1990; Jones 1990). If some of the iron is in ferro- or ferrimagnetic materials, then magnetic dipole radiation from these grains will contribute to the emission spectrum of interstellar dust.

Recent observations of low-metallicity dwarf galaxies (e.g., Galliano et al. 2003, 2005; Galametz et al. 2009, 2011; Grossi et al. 2010; O’Halloran et al. 2010), and the Small Magellanic Cloud (SMC) in particular (e.g., Israel et al. 2010; Bot et al. 2010), have revealed excess emission relative to predictions from standard dust models at millimeter and submillimeter wavelengths. The failure of conventional dust models (even those including electric dipole emission from spinning dust grains) to explain this excess is the impetus to consider alternative dust emission mechanisms (Bot et al. 2010; Planck Collaboration et al. 2011b).

Should magnetic dipole radiation furnish an explanation for the millimeter and submillimeter emission seen in low-metallicity dwarfs, it is likely that such grains also make a non-negligible contribution to Galactic dust emission at these wavelengths. Models for Galactic emission—both total intensity and polarization—require quantitative estimates for the contribution of magnetic dipole radiation. This is of considerable importance for experiments to measure the polarization of the cosmic microwave background.

The objective of the present study is to consider magnetic materials as possible constituents of interstellar dust, and to evaluate the absorption and extinction cross sections for interstellar grains composed of such materials. We will find that these materials have large absorption cross sections at millimeter-wave frequencies and below.

Models for the thermal emission of radiation from interstellar grains require calculation of the absorption cross section $C_{ab}($ for each grain type and size, as a function of frequency $\omega$. Calculations of $C_{ab}(\omega)$ for grains often assume the grain material to be nonmagnetic (magnetic permeability $\mu = 1$); the material is assumed to respond only to the local electric field. At optical and infrared frequencies, this is an excellent approximation; however, at submillimeter and microwave frequencies, the magnetic response of grain material to an oscillating magnetic field may not be negligible.

If the magnetic response enables a grain to absorb energy from an oscillating magnetic field, then there must also be a magnetic contribution to thermal emission. Magnetic dipole emission from magnetic grain materials was discussed by Draine & Lazarian (1999, hereafter DL99), who concluded that magnetic dipole radiation might be important at microwave frequencies if interstellar grains were composed in part of magnetic materials. At that time the frequencies of particular interest were $\sim 20–60\ GHz$, where so-called “anomalous microwave emission” (AME) had been detected from the interstellar medium (ISM; Kogut et al. 1996; de Oliveira-Costa et al. 1997; Leitch et al. 1997). Theoretical studies (Draine & Lazarian 1998a, 1998b) had shown that the observed AME could be electric dipole radiation from rapidly spinning ultrasmall grains, but DL99 also showed that magnetic dipole emission from magnetic grain materials could not be ruled out as a significant contributor to emission at these frequencies.

DL99 adopted a simple model for the frequency-dependent absorption in magnetic materials. The present study uses the...
Gilbert equation to model the magnetic response at microwave and submillimeter frequencies.

In Section 2 we introduce three candidate magnetic materials: metallic iron (bcc Fe), magnetite (Fe₃O₄), and maghemite (γ-Fe₂O₃). Section 3 presents a model for the magnetization dynamics of these materials. From the model, we derive the complex polarizability tensor $\alpha^{\text{mag}}(\omega)$ relating the oscillating magnetization response $m$ to an applied field $h$.

Because the objective is to calculate absorption cross sections $C_{\text{abs}}$, in Sections 4 and 5 we discuss calculation of $C_{\text{abs}}$ for small magnetic particles. The importance of eddy currents is discussed.

The purely magnetic contribution to the absorption cross section is calculated in Section 6. In Section 7 we calculate absorption cross sections for randomly oriented spheres of Fe, magnetite, and maghemite from optical to microwave frequencies. These calculations make use of dielectric functions for iron, magnetite, and maghemite that are derived in Appendices B–D. In Section 8 we estimate the temperatures for spherical grains of pure Fe, magnetite, or maghemite illuminated by interstellar starlight. In Section 9 we calculate the emission from small particles of metallic Fe, magnetite, and maghemite, and the polarization of this emission is discussed in Section 10. In Section 11 we find that a large fraction of interstellar Fe could be in magnetic material without violating constraints provided by observations of microwave and submillimeter emission, or by the observed wavelength-dependent extinction. The principal conclusions are summarized in Section 12.

In a separate paper (Draine & Hensley 2012) we show that Fe nanoparticles can account for the extremely strong submillimeter-microwave emission observed from the SMC.

2. MAGNETIC MATERIALS

All materials exhibit some magnetic response, but our attention here is on materials that have unpaired electron spins that are spontaneously ordered even in the absence of an applied magnetic field. These materials fall into three broad classes: ferromagnetic, ferrimagnetic, and antiferromagnetic (for an introduction to magnetic materials, see, e.g., Morrish 2001 or Coey 2010).

The magnetization of ferromagnetic and ferrimagnetic materials occurs because the Coulomb energy of the system is minimized for wavefunctions where the electrons have a nonzero net spin; this is often described as being due to the “exchange interaction.” We consider three candidate magnetic materials: metallic Fe, magnetite Fe₃O₄, and maghemite γ-Fe₂O₃. At low temperatures, metallic Fe is ferromagnetic (all unpaired Fe spins parallel), while magnetite and maghemite are ferrimagnetic (partial cancellation of Fe spins). As we will see below, the spontaneous magnetization of these materials leads to enhanced absorption at microwave and submillimeter frequencies.

Two other common iron oxides, ferrous oxide (wüstite) FeO and hematite α-Fe₂O₃ are antiferromagnetic, with zero net magnetization because the Fe spins alternate in direction from lattice site to lattice site, resulting in perfect cancellation. These also have interesting properties at microwave and submillimeter frequencies, but we do not discuss them here.

For spontaneously magnetized single-domain samples at low temperatures, the static magnetization $M_S = M_A$, where $M_A$ is the saturation magnetization. For ferrimagnetic systems (see Section 3.2 below) we also require the ratio $\beta = |M_A|/|M_B|$ of the magnetizations of the two opposed spin subsystems ($A$ and $B$), and a parameter $N_{AB}$ characterizing the coupling between the two spin subsystems.

The Coulomb energy of the crystal is minimized when the unpaired spins are aligned (i.e., it is magnetized). Because of the lattice structure, the Coulomb energy depends on the direction of the magnetization relative to the lattice, with the Coulomb energy being minimized when the magnetization is along certain preferred directions. The variations in Coulomb energy with direction of magnetization are referred to as the crystalline anisotropy energy. The so-called crystalline anisotropy field $H_K$ is a fictional magnetic field that conveniently characterizes the “energy cost” of small deviations of $M$ away from a direction that minimizes the energy of the material; the variations in Coulomb energy with small deviations $\theta$ of the direction of magnetization away from the preferred direction are as though there were an actual field $H_K = H_K \hat{e}_{\text{easy}}$ applied parallel to the direction $\hat{e}_{\text{easy}}$ where the crystal energy is minimized, corresponding to an effective potential $E_K = \text{const} - M \cdot H_K$. Thus $H_K$ is related to the second derivative $\partial^2 E_K/\partial \theta^2$ of the anisotropy energy $E_K(\theta)$, evaluated at $\partial E_K/\partial \theta = 0$. For a ferromagnetic material,

$$H_K = \frac{1}{M_s} \frac{\partial^2 E_K}{\partial \theta^2} \hat{e}_{\text{easy}}. \quad (1)$$

2.1. Metallic Iron

At $T < 1185$ K the thermodynamically favored structure for Fe is bcc. The low-temperature saturation magnetization of bcc Fe is $4\pi M_s = 2.202 \times 10^6$ G (Tebble & Craik 1969), corresponding to $2.2 \mu_B$ per Fe. Even very small clusters of Fe atoms are ferromagnetic. For clusters of $N = 20–50$ Fe atoms, the magnetic moment per atom is $\sim 3 \mu_B$, declining to the bulk value of $2.2 \mu_B$ for $N \gtrsim 400$ (Billas et al. 1993; Tiago et al. 2006). Fe spheres with diameters $\lesssim 1.7 \times 10^{-6}$ cm are expected to be single-domain; for prolate spheroids single-domain behavior persists to larger sizes (Butler & Banerjee 1975). Larger Fe particles are expected to contain more than one magnetic domain, as this lowers the free energy.

For a crystal with cubic symmetry (e.g., bcc Fe), the crystalline anisotropy energy can be written (Morrish 2001)

$$E_K = K_1 (\alpha_1^2 \alpha_2^2 + \alpha_2^2 \alpha_3^2 + \alpha_3^2 \alpha_1^2) + K_2 \alpha_1^2 \alpha_2^2 \alpha_3^2, \quad (2)$$

where the direction cosines $\alpha_i$ specify the direction of $M$ relative to the cubic axes of the crystal. For bcc Fe, $K_1 = 4.8 \times 10^5$ erg cm$^{-3}$ (Tebble & Craik 1969). For $K_1 > 0$, $E_K$ is minimized for spontaneous magnetization along one of the cubic axes (e.g., $\alpha_1 = 1, \alpha_2 = \alpha_3 = 0$), with $\partial^2 E_K/\partial \theta^2 = 2K_1$. The crystalline anisotropy field is

$$H_K = \frac{1}{M_s} \frac{\partial^2 E_K}{\partial \theta^2} = \frac{2K_1}{M_s} = 548 \text{ Oe}. \quad (3)$$

2.2. Magnetite Fe₃O₄

Magnetite Fe₃O₄ is spontaneously magnetized even at room temperature. Of the three Fe ions in the Fe₃O₄ unit cell, one (with magnetic moment $5 \mu_B$) is in the “A” sublattice, and two (with total magnetic moment $9 \mu_B$) are in the “B” sublattice. The spins in the two sublattices are antialigned, giving a net magnetic moment $4 \mu_B$ per Fe₃O₄. The ratio of the magnetic moments in the two sublattices $\beta = |M_A|/|M_B| = 5/9$.

Magnetite undergoes a phase transition at the Verwey transition temperature $T_V = 119$ K, with the crystal structure changing from cubic (at $T > T_V$) to monoclinic (at $T < T_V$). Interstellar grain temperatures will be below $T_V$. 


The low-temperature saturation magnetization $4\pi M_s = 6400$ Oe (Tebble & Craik 1969). The crystalline anisotropy energy for $T < T_C$ is of the form (Özdemir & Dunlop 1999)

$$E_K = K_a\alpha^2_a + K_b\alpha^2_b + K_{ab}\alpha_a\alpha_b + K_{ab}\alpha^2_a + K_{bb}\alpha^2_b - K_a\alpha^2_{111}$$  \(4\)

where $\alpha_a, \alpha_b$, and $\alpha_c$ are direction cosines relative to the [110], [110], and [001] directions of the $T > T_C$ cubic lattice, and $\alpha_{111}$ is the direction cosine relative to the [111] direction. The anisotropy constants $K_a, K_b, K_{ab}, K_{bb}, K_{ab}$, and $K_a$ have been measured at low temperatures (Abe et al. 1976).

If the $K_a$ term is neglected, then, in the absence of applied fields, spontaneous magnetization at $T < T_C$ will be along the [001] direction $\hat{e}_{easy}$. Let $\theta$ be the angle between $\mathbf{M}$ and $\hat{e}_{easy}$. If we write $\alpha_a = \sin \theta \cos \phi, \alpha_b = \sin \theta \sin \phi$, and average over $\phi$, we find

$$\frac{\partial^2 E_K}{\partial \theta^2} = 2(K_a \cos^2 \phi + K_b \sin^2 \phi) \to (K_a + K_b)$$

$$= 29.2 \times 10^4 \text{erg cm}^{-3},$$  \(5\)

and the crystalline anisotropy field

$$H_K = \frac{(1 + \beta)}{(1 - \beta)M_s} \frac{\partial^2 E_K}{\partial \theta^2} = 1640 \text{Oe}.$$  \(6\)

The interlattice coupling coefficient $N_{ab}$ will be discussed in Section 3.2.

### 2.3. Maghemite $\gamma$-Fe$_2$O$_3$

Fe$_2$O$_3$ exists in several different crystal forms, of which two are common in nature: hematite ($\alpha$-Fe$_2$O$_3$) and maghemite ($\gamma$-Fe$_2$O$_3$). Hematite is thermodynamically favored, but conversion of maghemite to hematite in nanoparticles requires very high temperatures, $T \gtrsim 600 \text{K}$ (Özdemir & Banerjee 1984; Kido et al. 2004). Maghemite is the low temperature oxidation product of magnetite.

Maghemite is widely used in magnetic recording media. It is ferrimagnetic below the Curie temperature $T_C \approx 870 \text{K}$, with a low temperature magnetization $4\pi M_s = 4890 \text{Oe}$ (Dutta et al. 2004). The $A$ lattice has a magnetic moment $5\mu_B$ per Fe$_2$O$_3$, and the $B$ lattice has $25\mu_B/3$ per Fe$_2$O$_3$ (Morrish 2001; Dionne 2009; Coey 2010), giving a net magnetic moment of $(10/3)\mu_B$ per formula unit, and $\beta \equiv |M_A|/|M_B| = 3/5$.\(^2\)

The crystalline anisotropy for maghemite is uncertain. For bulk maghemite at room temperature, $K_1 = -4.6 \times 10^9 \text{erg cm}^{-3}$ (Brabers 1995). Babkin et al. (1984) observed an increase in the anisotropy with decreasing temperature down to $T \approx 70 \text{K}$; extrapolation to $T \approx 20 \text{K}$ gives $K_1 \approx -1.8 \times 10^8 \text{erg cm}^{-3}$. For $K_1 < 0$, spontaneous magnetization occurs along the [111] axes, with

$$\frac{\partial^2 E_K}{\partial \theta^2} = -\frac{4}{5}K_1.$$  \(7\)

Much larger anisotropies have been reported for maghemite nanoparticles. Valstyn et al. (1962) found $K_1 = -2.5 \times 10^9 \text{erg cm}^{-3}$ at room temperature.

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1. At $4.2$ K, $K_a/K_a = 0.082$ (Abe et al. 1976), hence the $K_a$ term is a small correction in Equation (4).
2. $(10/3)\mu_B$ per Fe$_2$O$_3$ implies $4\pi M_s = 4\pi(\rho/160 \text{amu})10\mu_B/3 = 7100 \text{Oe}$. The actual measured magnetization $4\pi M_s = 4890 \text{Oe}$ (Dutta et al. 2004) is only ~70% of the expected value, indicating that real maghemite samples differ from the ideal model used here.

The crystalline anisotropy can also be studied by measurements of the “blocking temperature” for superparamagnetic behavior (Morrish 2001) to determine the energy barrier $K_{eff} V$ for magnetic reorientation in a particle of volume $V$. For maghemite, these studies lead to effective anisotropies that are often much larger, e.g., $K_{eff} = 2.3 \times 10^{10} \text{erg cm}^{-3}$ for ~7 nm (diameter) particles (Dutta et al. 2004) at $T \approx 85 \text{K}$, $K_{eff} \approx 1.2 \times 10^9 \text{erg cm}^{-3}$ at $T \approx 25 \text{K}$ (Shendruk et al. 2007), $K_{eff} = 7 \times 10^9 \text{erg cm}^{-3}$ for ~8 nm hollow particles at $T \approx 30 \text{K}$ (Cabot et al., 2009), and $K_{eff} = 6.0 \times 10^9 \text{erg cm}^{-3}$ for ~6 nm particles at $T \approx 60 \text{K}$ (Tsuzuki et al. 2011). It is not clear why these various studies reach such different conclusions. The blocking temperature studies probe the entire $E_K$ energy surface, particularly paths from the minima to saddle points. The present study, however, is concerned with small perturbations, and therefore we require only the local curvature of $E_K$ at the minima. For purposes of discussion, we will disregard the blocking temperature results and will take $K_1 = -1.8 \times 10^8 \text{erg cm}^{-3}$ based on the thin-film measurements by Babkin et al. (1984), giving

$$\frac{\partial^2 E_K}{\partial \theta^2} = 2.4 \times 10^8 \text{erg cm}^{-3}$$  \(8\)

and $H_K = 617 \text{Oe}$. In view of the much larger values of $K_{eff}$ obtained from blocking temperature studies, the adopted value of $H_K$ should be regarded as very uncertain. The interlattice coupling coefficient $N_{ab}$ will be discussed in Section 3.2.

### 3. MAGNETIC RESPONSE AT MICROWAVE AND SUBMILLIMETER FREQUENCIES

#### 3.1. Ferromagnetic Resonance

Consider a small, single-domain ferromagnetic sample subject to an applied field

$$\mathbf{H}(t) = H_0\hat{z} + \mathbf{h}(t)$$  \(9\)

$$\mathbf{h}(t) = \mathbf{m} e^{-i\omega t}.$$  \(10\)

We will later set $H_0 = 0$, but retain it here for generality. The oscillating field $\mathbf{h}$ is assumed to be small. The magnetization will include static and oscillating components:

$$\mathbf{M}(t) = \mathbf{M}_0 + \mathbf{m}(t)$$  \(11\)

$$\mathbf{m}(t) = \mathbf{m}_0 e^{-i\omega t}.$$  \(12\)

The dynamic magnetic response of ferromagnetic materials is a complex problem (see, e.g., Morrish 2001; Soohoo 1985). Phenomenological treatments of the damping include the Landau–Lifshitz equation (Landau & Lifshitz 1935), the Bloch–Bloembergen equation (Bloch 1946; Bloembergen 1950) and the Gilbert equation (Gilbert 1955, 2004).
it here. Iida (1963) argued that the Gilbert equation is more physically reasonable.

The Gilbert equation is

\[ \frac{dM}{dt} = γM \times H_T + αG \frac{M}{|M|} \times \frac{dM}{dt}, \]  

(13)

where

\[ γ = -\frac{e}{2m_{\text{eff}}} = -\frac{g}{2} 1.759 \times 10^7 \text{s}^{-1} \frac{\text{G}}{\text{Oe}} \]  

(14)

is the ratio of magnetic moment to angular momentum (g ≈ 2 is the usual gyromagnetic factor), and the total effective field \( H_T \) is discussed below. The term \( γM \times H_T \) in Equation (13) corresponds to precession of \( M \) around the effective field \( H_T \), while the second term describes relaxation toward a solution with \( M \parallel H_T \). The dimensionless parameter \( α_G \) is Gilbert’s phenomenological damping coefficient.

Morrish (2001) states that “typical” ferromagnetic absorption has \( α_G \approx 0.07 \). Wu et al. (2006) studied the response of micro- and submicron-sized Fe particles in a nonconducting, nonmagnetic matrix at frequencies between 0.5 and 16.5 GHz, and reported that their measurements were consistent with \( α_G \approx 0.4 \). Neo et al. (2010) used \( α_G = 0.1 \) to model the behavior of single-domain bcc Fe particles for frequencies up to 100 GHz (although experimental data were shown only for \( \leq 10 \) GHz). We will consider a range of possible values for \( α_G \).

The “total” field \( H_T \) in Equation (13) is an effective field. Consider an ellipsoidal grain. For a uniform applied field \( H_0 + h \), the magnetization \( M \) in the grain will be uniform. The effective field can be written

\[ H_T = H_0 + h - D \cdot M + H_K. \]  

(15)

Here \( D \) is the “demagnetization tensor,” giving the internal contribution to \( H_T \) arising from the “magnetic poles” at the surface of the sample. For an ellipsoid, \( D \) is diagonal, with diagonal elements

\[ D_{jj} = 4πL_j, \]  

(16)

where \( L_j \) is the same geometrical factor, or “shape factor,” that arises in relating the internal electric field to the applied electric field in a dielectric (see, e.g., Bohren & Huffman 1983, p. 146).

\[ \text{Table 1} \]

Candidate Magnetic Materials

| Material           | \( ρ \) (g cm\(^{-3}\)) | \( V_{Fe} \) (10\(^{-23}\) cm\(^3\)) | \( 4πM_s \) (Oe) | \( β \) | \( α^2E_K/α\theta^2 \) (erg cm\(^{-3}\)) | \( H_0 \) (Oe) | \( N_{AB} \) |
|-------------------|--------------------------|---------------------------------|------------------|------|---------------------------------|----------------|---------|
| Metallic Fe (bcc) | 7.87                     | 1.18                            | 22020            | -    | 9.6 \times 10^3                 | 548            | -       |
| Maghemite \( γ \)-Fe\(_2\)O\(_3\) | 5.18                     | 2.48                            | 6400\(^a\)       | 3/5  | 29.2 \times 10^3                | 1640           | 10800\(^d\) |
| Maghemite \( γ \)-Fe\(_2\)O\(_3\) | 4.86                     | 2.73                            | 4890\(^d\)       | 5/9  | 2.4 \times 10^4                 | 617            | 9280\(^b\) |

Notes.

\(^a\) Volume per Fe atom.

\(^b\) \( H_K = [(1-β)/(1+β)]M_s\times E_K/α\theta^2 \).

\(^c\) Tebble & Craik (1969).

\(^d\) Abbe et al. (1976) (see text).

\(^e\) Dionne (2009), Tables 4.3 and 4.4.

\(^f\) Özdemir & Dunlop (1999).

\(^g\) Abe et al. (1976) (see text).

\(^h\) Dutta et al. (2004).

\(^i\) Babkin et al. (1984) (see text).

\[ \text{Table 2} \]

Ferromagnetic Resonance Parameters for Metallic Fe

| Shape | \( L_x = L_y \) | \( L_z \) | \( ω_0/2\pi \) (GHz) | \( ω_M/2\pi \) (GHz) |
|-------|-----------------|---------|---------------------|---------------------|
| Sphere | 0.33333 | 0.33333 | 1.53 | 4.91 |
| 1:1 prolate spheroid | 0.35694 | 0.28613 | 5.90 | 4.91 |
| 2:1 prolate spheroid | 0.38351 | 0.23298 | 10.8 | 4.91 |
| 3:1 prolate spheroid | 0.41322 | 0.17356 | 16.3 | 4.91 |
| 4:1 prolate spheroid | 0.44565 | 0.10871 | 22.3 | 4.91 |
| 5:1 prolate spheroid | 0.46230 | 0.07541 | 25.4 | 4.91 |
| 6:1 prolate spheroid | 0.49986 | 0.03929 | 30.5 | 4.91 |
| 7:1 prolate spheroid | 0.50000 | 0.00000 | 32.4 | 4.91 |

For a sphere, \( L_x = 1/3 \). For ellipsoids, \( L_x + L_y + L_z = 1 \). In the discussion below we will consider prolate spheroids with \( L_x = L_y = 1/2 (1-L_z) \). Values of \( L_x \) and \( L_z \) are given in Table 2 for various axial ratios.

We will take \( ε_{\text{eff}} \), and the static (spontaneous) magnetization \( M_0 = M_0\hat{ε}_z \) (\( M_0 > 0 \)) to be in the \( z \) direction, which we take to be one of the principal axes of the ellipsoid. Then, to leading order in \( m \), Equation (13) becomes

\[ -iωM_{0x} = γM_0[H_0 + H_K - D_{xx}M_0] + γM_0[h_{oy} - D_{yy}m_{oy}] + iαGωm_{0y} \]  

(17)

\[ -iωM_{0y} = γM_0[H_0 + H_K - D_{yy}M_0] + γM_0[h_{ox} - D_{xx}m_{ox}] - iαGωm_{0x}. \]  

(18)

To simplify, assume the sample to be a spheroid, with \( D_{yy} = D_{xx} \). Define

\[ ω_0 \equiv γ[H_0 + H_K - (D_{xx} - D_{yy})M_0] \]  

(19)

\[ ω_M \equiv γM_0. \]  

(20)

We then obtain two coupled equations

\[ -iωM_{0x} = -ω_0^*m_{0y} + ω_Mh_{0y} \]  

(22)

\[ -iωM_{0y} = ω_Mm_{0x} - ω_Mh_{0x}, \]  

(23)

\[^3\] This is completely analogous to the depolarization electric field arising from the surface charge associated with the discontinuity in the polarization field \( P \) at the surface of a dielectric.
with solutions

\[ m_{0x} = \frac{\omega' \omega M}{(\omega'_0)^2 - \omega^2} h_{0x} - \frac{i \omega \omega M}{(\omega'_0)^2 - \omega^2} h_{0y}, \]  \hspace{1cm} (24)

\[ m_{0y} = \frac{i \omega \omega M}{(\omega'_0)^2 - \omega^2} h_{0x} + \frac{\omega' \omega M}{(\omega'_0)^2 - \omega^2} h_{0y}. \]  \hspace{1cm} (25)

If we define

\[ \chi_\pm \equiv \frac{(\omega'_0 \pm \omega) \omega M}{(\omega'_0)^2 - \omega^2} = \frac{\omega M}{\omega'_0 \pm \omega}, \]  \hspace{1cm} (26)

then the oscillating magnetization \( \mathbf{m} \) satisfies

\[ \begin{pmatrix} m_{0x} \\ m_{0y} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{1}{2} (\chi_+ - \chi_-) & -i \frac{1}{2} (\chi_+ - \chi_-) \\ i \frac{1}{2} (\chi_+ - \chi_-) & \frac{1}{2} (\chi_+ - \chi_-) \end{pmatrix} \begin{pmatrix} h_{0x} \\ h_{0y} \end{pmatrix}. \]  \hspace{1cm} (27)

If we define

\[ \mathbf{\hat{h}}_\pm \equiv \frac{1}{\sqrt{2}} (\mathbf{\hat{x}} \pm i \mathbf{\hat{y}}), \]  \hspace{1cm} (28)

then an arbitrary applied field \( h_0 e^{-i\omega t} \) will produce a magnetization

\[ \mathbf{m}_0 e^{-i\omega t} = \left[ \chi_+ (\mathbf{h}_0 \cdot \mathbf{\hat{h}}_+) \mathbf{\hat{h}}_+ + \chi_- (\mathbf{h}_0 \cdot \mathbf{\hat{h}}_-) \mathbf{\hat{h}}_- \right] e^{-i\omega t}. \]  \hspace{1cm} (29)

The eigenvectors \( \mathbf{\hat{h}}_\pm \) correspond to circular polarization modes, with \( \text{Re}(\mathbf{\hat{h}}_\pm e^{-i\omega t}) \) rotating either anticlockwise (\( \mathbf{\hat{h}}_+ \)) or clockwise (\( \mathbf{\hat{h}}_- \)) around \( \mathbf{h}_0 \). In this linearized treatment, the oscillating magnetization \( \mathbf{m} \) has no response to the component of the applied field \( \mathbf{h} \) parallel to \( \mathbf{h}_0 \)—this is because the magnetization along \( \mathbf{h}_0 \) is already saturated. The perpendicular component \( h_\perp = \sqrt{h_x^2 + h_y^2} \) can deflect the magnetization away from \( \mathbf{\hat{z}} \), leading to \( m_{0x} \propto h_\perp \) and \( m_{0y} \propto h_\perp \), but \( m_{0z} \) is second order in \( h_\perp \).

The real and imaginary parts of \( \chi_\pm \) are

\[ \text{Re}(\chi_\pm) = \frac{\omega M (\omega_0 + \omega)}{(\omega'_0)^2 + \omega^2} \]  \hspace{1cm} (30)

\[ \text{Im}(\chi_\pm) = \frac{\omega G \omega M (\omega_0 + \omega)}{(\omega'_0)^2 + \omega^2}. \]  \hspace{1cm} (31)

The dissipative part of the response is measured by \( \text{Im}(\chi_\pm) \). It is evident from Equation (31) that \( \chi_\pm \) has a resonance at \( \omega = \omega_0 \), corresponding to the applied field being in resonance with free precession of \( \mathbf{m} \) around the static magnetization \( \mathbf{M}_0 \).

The resonance frequency \( \omega_0 \) depends on the shape, because the effective field \( \mathbf{H}_F = \mathbf{H}_K - \mathbf{D} \cdot \mathbf{M} \) includes the demagnetization field \( -\mathbf{D} \cdot \mathbf{M} \), which is shape-dependent. Table 2 gives \( \omega_0 \) for spheres and selected prolate spheroids. For highly elongated spheroids, \( \omega_0 \to 32 \text{ GHz} \).

The \( \chi_\pm \) in Equation (26) give the ratio of the magnetization response \( \mathbf{m} \) to the external applied field \( \mathbf{h} \). If \( V \) is the volume, the magnetic dipole moment of the grain is

\[ \mathbf{p}_m = V \mathbf{m} = V \left[ \chi_+ (\mathbf{\hat{h}}_+ \cdot \mathbf{h}_0) + \chi_- (\mathbf{\hat{h}}_- \cdot \mathbf{h}_0) \right] e^{-i\omega t}. \]  \hspace{1cm} (32)

\[ = \mathbf{\alpha}_m \cdot \mathbf{h}, \]  \hspace{1cm} (33)

where the magnetic polarizability tensor \( \mathbf{\alpha}_m \) is

\[ \mathbf{\alpha}_m = V \left[ \chi_+ (\mathbf{\hat{h}}_+ \cdot \mathbf{h}_0) + \chi_- (\mathbf{\hat{h}}_- \cdot \mathbf{h}_0) \right]. \]  \hspace{1cm} (34)

At low frequencies, we have \( \chi_\perp(0) = \chi_\pm(\omega = 0) = \omega M / \omega_0 \).

For single-domain Fe spheres, we estimate \( \chi_\perp(0) \approx 3.2 \).

The polarizability tensor \( \mathbf{\alpha}_m \) describes the linear response to an applied magnetic field. For causal systems, linear response functions must satisfy the Kramers–Kronig relations (Landau et al. 1993). The phenomenological treatment of damping in the Gilbert equation might raise concern over whether the derived \( \chi_\perp \) satisfies the Kramers–Kronig relations. In Appendix G we show that they do.

3.2. Ferrimagnetic and Antiferromagnetic Resonance

Ferrimagnetic and antiferromagnetic materials have two oppositely aligned spin lattices, \( A \) and \( B \), with magnetizations \( \mathbf{M}_A \), \( \mathbf{M}_B \). Let \( |\mathbf{M}_B| > |\mathbf{M}_A| \). At low temperatures, the two spin lattices are each perfectly aligned. Recall that \( \beta \equiv |\mathbf{M}_A| / |\mathbf{M}_B| \). At low temperatures, the two spin lattices are each perfectly aligned. Recall that \( \beta \equiv |\mathbf{M}_A| / |\mathbf{M}_B| \). Ferrimagnetic materials have \( 0 < \beta < 1 \), and antiferromagnetic materials have \( \beta = 1 \).

Just as for ferromagnetism, the spin alignment results from minimization of the electronic energy. We will consider an idealization where all the spins in each sublattice (\( A \) and \( B \)) are mutually aligned, but \( A \) and \( B \) may not be perfectly aligned.

\footnote{\( \chi_\pm \) is not the same as the intrinsic magnetic susceptibility \( \chi_{\text{int}} \), because the latter is the ratio of the magnetization response to the local or internal field, which includes an internal depolarization field.}
with one another. There are then two alignments to consider: (1) antialignment of \( \mathbf{M}_A \) with \( \mathbf{M}_B \) and (2) alignment of the net magnetization \( \mathbf{M} = \mathbf{M}_A + \mathbf{M}_B \) with “easy” directions relative to the crystal axes.

Minimization of the energy when the two spin systems are antialigned can be described through fictitious fields \( -N_{AB} \mathbf{M}_B \) acting on \( A \) and \( -N_{AB} \mathbf{M}_A \) acting on \( B \). The dimensionless coupling coefficient \( N_{AB} \) has been estimated by Dionne (2006, 2009),\(^5\) with \( N_{AB} = 1.08 \times 10^4 \) for \( \text{Fe}_3\text{O}_4 \), and \( N_{AB} = 9.28 \times 10^3 \) for \( \gamma\)-\( \text{Fe}_2\text{O}_3 \). As we will see, the very strong coupling between the two sublattices results in a resonance in the infrared.

With the spin systems antialigned, the overall energy of the system is minimized when the magnetization is along an “easy” direction, which we take to be \( \mathbf{z} = \mathbf{M}_B, \mathbf{M}_A = -\beta \mathbf{M}_B. \mathbf{z} \).

The effective fields (or “crystalline fields”) \( \mathbf{H}_{KA} = \mathbf{H}_{KA} \mathbf{z} \) and \( \mathbf{H}_{KB} = \mathbf{H}_{KB} \mathbf{z} \) characterize the energy cost of departures of the magnetization \( \mathbf{M} \) away from the easy direction \( \mathbf{z} \). We take \( \mathbf{H}_{KB} = +H_K, \mathbf{H}_{KA} = -H_K \), where

\[
H_K = \frac{1}{(1 + \beta)} \frac{1}{|\mathbf{M}_B|} \frac{\partial^2 E_K}{\partial \theta^2} = \frac{(1 - \beta)}{(1 + \beta)} \frac{1}{|\mathbf{M}_A|} \frac{\partial^2 E_K}{\partial \theta^2} \quad (35)
\]

The Gilbert equations for the two spin systems are

\[
\frac{d}{dt} \mathbf{M}_A = \gamma_A \mathbf{M}_A \times (\mathbf{H}_0 + \mathbf{H}_A - N_{AB} \mathbf{M}_B) + \alpha_G \frac{\mathbf{M}_A}{|\mathbf{M}_A|} \times \frac{d \mathbf{M}_A}{dt}, \quad (36)
\]

\[
\frac{d}{dt} \mathbf{M}_B = \gamma_B \mathbf{M}_B \times (\mathbf{H}_0 + \mathbf{H}_B - N_{AB} \mathbf{M}_A) + \alpha_G \frac{\mathbf{M}_B}{|\mathbf{M}_B|} \times \frac{d \mathbf{M}_B}{dt} \quad (37)
\]

The contribution to the dynamics of the depolarization field \( -\mathbf{D} \cdot \mathbf{M} \) is neglected, as \( \mathbf{D} \cdot \mathbf{M} \sim \mathbf{M}_A \) is small compared with the crystalline fields \( \mathbf{H}_{KA}, \mathbf{H}_{KB} \), and the coupling fields \( N_{AB} \mathbf{M}_A, N_{AB} \mathbf{M}_B \).

As for the ferromagnetic case, we take \( \mathbf{H}_0 = 0 \),

\[
\mathbf{H}(t) = \mathbf{h}_0 e^{-i\omega t}, \quad (38)
\]

\[
\mathbf{M}_A(t) = \mathbf{M}_A \mathbf{z} + \mathbf{m}_A e^{-i\omega t}, \quad (39)
\]

\[
\mathbf{M}_B(t) = \mathbf{M}_B \mathbf{z} + \mathbf{m}_B e^{-i\omega t}, \quad (40)
\]

with \( M_B > 0, M_A < 0 \), and linearize around the equilibrium to obtain

\[
-i \omega m_{Ax} = -\gamma_A m_{Ay}(H_{KA} - N_{AB} M_B) + i \alpha_G \omega m_{Ay}, \quad (41)
\]

\[
-i \omega m_{Bx} = -\gamma_B m_{By}(H_{KB} - N_{AB} M_A) + i \alpha_G \omega m_{By}, \quad (42)
\]

Defining

\[
\omega_{0A} \equiv -\gamma_A (H_{KA} - N_{AB} M_B) \quad (43)
\]

\[
\omega_{0B} \equiv -\gamma_B (H_{KB} - N_{AB} M_A) \quad (44)
\]

\[
\omega_{MA} \equiv -\gamma_A M_A \quad (45)
\]

\[
\omega_{MB} \equiv -\gamma_B M_B \quad (46)
\]

\[
\omega_{0A} \equiv \omega_{0A} + i \alpha_G \omega \quad (47)
\]

\[
\omega_{0B} \equiv \omega_{0B} - i \alpha_G \omega \quad (48)
\]

the above equations become

\[
-i \omega m_{Ax} = -\omega_{0A} m_{Ay} - \omega_{MA} N_{AB} m_{By} + \omega_{MA} h_y \quad (51)
\]

\[
-i \omega m_{Bx} = -\omega_{0B} m_{By} - \omega_{MB} N_{AB} m_{Ay} + \omega_{MB} h_y \quad (52)
\]

\[
-i \omega m_{Ay} = \omega_{0A} m_{Ax} + \omega_{MA} N_{AB} m_{Bx} - \omega_{MA} h_x \quad (53)
\]

\[
-i \omega m_{By} = \omega_{0B} m_{Bx} + \omega_{MB} N_{AB} m_{Ax} - \omega_{MB} h_x \quad (54)
\]

Considering circularly polarized modes

\[
\mathbf{h} = \mathbf{h}_0 \mathbf{\hat{e}}_z e^{-i\omega t} \quad (55)
\]

\[
\mathbf{m}_\pm \equiv [(\mathbf{m}_A + \mathbf{m}_B) \cdot \mathbf{\hat{e}}_z] \mathbf{\hat{e}}_\pm \quad (56)
\]

we can solve to find

\[
m_{+x} = \frac{2N_{AB} \omega_{MA} \omega_{MB} - \omega_{MA} \omega_{0B} - \omega_{MB} \omega_{0A} \pm (\omega_{MA} + \omega_{MB}) \omega_h}{N_{AB}^2 \omega_{MA} \omega_{MB} - \omega_{MA} \omega_{0B} - \omega^2 \pm (\omega_{MA} + \omega_{MB}) \omega_h} \quad (57)
\]

with a similar equation for \( m_{-x} \).

\[
\chi_\pm = \frac{2N_{AB} \omega_{MA} \omega_{MB} - \omega_{MA} \omega_{0B} - \omega_{MB} \omega_{0A} \pm (\omega_{MA} + \omega_{MB}) \omega_h}{N_{AB}^2 \omega_{MA} \omega_{MB} - \omega_{MA} \omega_{0B} - \omega^2 \pm (\omega_{MA} + \omega_{MB}) \omega_h} \quad (58)
\]

The magnetization \( \mathbf{m}_0 e^{-i\omega t} \) in response to a general applied field \( \mathbf{h}_0 e^{-i\omega t} \) is given by Equation (29), but with \( \chi_\pm \) given by Equation (59). Values of \( \omega_{MA}, \omega_{MB}, \omega_{0A}, \) and \( \omega_{0B} \) for magnetite and maghemite are given in Table 3. Figure 1 shows \( \text{Im}(\chi_+) \) and \( \text{Im}(\chi_-) \) for single-domain particles of magnetite \( \text{Fe}_3\text{O}_4 \) and maghemite \( \gamma\)-\( \text{Fe}_2\text{O}_3 \).
Table 3
Ferrimagnetic Resonance Parameters

| Material       | $\omega_{\text{res}}/2\pi$ (GHz) | $\omega_{\text{MA}}/2\pi$ (GHz) | $\omega_{\text{MB}}/2\pi$ (GHz) | $\omega_{\text{A}}/2\pi$ (THz) | $\omega_{\text{B}}/2\pi$ (THz) | $\omega_{\text{res}}-\omega_{\text{A}}/2\pi$ (THz) |
|----------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|-----------------------------------------------|
| Magnetite Fe$_3$O$_4$ | 16.1                             | -4.01                            | 3.21                             | -34.7                            | 19.3                             | 15.4                                          |
| Maghemite $\gamma$-Fe$_2$O$_3$ | 7.83                             | -4.09                            | 2.71                             | -25.3                            | 15.2                             | 10.1                                          |

Figure 2. Predicted Im($\mu$) for Fe$_3$O$_4$ nanoparticles in nonmagnetic medium, for volume filling factor $f_{\text{fill}} = 0.007$, assuming $\alpha_G = 0.2$. Also shown are measurements of Kong et al. (2010) for Fe$_3$O$_4$ particles in a rubber matrix at room temperature. Agreement is poor. The laboratory sample shows much more absorption at low frequencies than expected for single-domain nanoparticles.

3.3. Comparisons to Experimental Data

As already noted, the Gilbert equation used here is phenomenological, and one would like to see it tested in the laboratory. Measurements of magnetic absorption at high frequencies are challenging. Laboratory measurements on Fe particles in insulating matrices are strongly affected by the conductivity of Fe unless very small particles are employed and great care is taken to avoid particle coagulation.

Low conductivity materials like magnetite and maghemite are much more suitable for laboratory work. Figure 2 shows 1–20 GHz measurements on $\sim$25 nm Fe$_3$O$_4$ particles with volume filling factor $f_{\text{fill}} = 0.007$ (4 wt%) in a nonmagnetic, nonconducting rubber matrix at room temperature (Kong et al. 2010). For $f_{\text{fill}} \ll 1$ the medium should have (see Appendix F, Equation (F10))

$$\mu \approx 1 + f_{\text{fill}} \frac{4\pi}{3} (\chi_+ + \chi_-).$$  (60)

if the particles are randomly oriented. In Figure 2 the laboratory sample exhibits much stronger absorption at low frequencies than predicted by the present model with $\alpha_G \approx 0.2$. The reason for this is unclear. Electron micrography indicates particle clumping, which complicates interpretation. More importantly, the particles may be large enough to be multidomain, in which case motion of domain walls will contribute to absorption. In addition, the room temperature electrical conductivity of Fe$_3$O$_4$ is large enough that, if clumping effects are severe, eddy currents could contribute strong absorption.

Figure 3 shows data from Valstyn et al. (1962) for three maghemite powders dispersed in paraffin wax with volume filling factor $f_{\text{fill}} = 0.15$. Typical particle sizes ranged from 120 nm (Powder 1) to $\sim$500 nm (Powders 2 and 3). For 1–10 GHz, the maghemite powders exhibit stronger absorption than predicted by the model for single-domain absorption. The particle sizes are large enough that multidomain structure is likely, and the extra absorption may be due to motion of domain walls. Of greater concern, however, is the very small absorption measured at the highest frequency, 12.3 GHz, well below what was expected from the model (if $\alpha_G \approx 0.2$).

To test the current model, what is needed are measurements of absorption by isolated, small, single-domain particles, at frequencies up to $\sim$300 GHz. We have been unable to find such data for Fe, Fe$_3$O$_4$, or $\gamma$-Fe$_2$O$_3$ at frequencies above 20 GHz, and even at lower frequencies the data are sparse.

4. POLARIZABILITIES OF SMALL PARTICLES

Magnetic effects appear to be of potential importance only for $\lambda \lesssim 500 \mu$m ($\nu \gtrsim 600$ GHz). We may therefore assume the particle to be small compared with the wavelength, in which case the electromagnetic response of the particle is predominantly through the electric and magnetic dipole moments $p_e(t)$ and $p_m(t)$ induced by the incident electromagnetic field. The electric
and magnetic dipole moments can be written

\[ p_e = \alpha_e \cdot E_{\text{inc}}, \quad (61) \]

\[ p_m = \alpha_m \cdot H_{\text{inc}}, \quad (62) \]

where \( E_{\text{inc}} \) and \( H_{\text{inc}} \) are the applied electric and magnetic fields, both assumed to be \( \propto e^{-\text{rot}} \), and \( \alpha_e(\omega) \) and \( \alpha_m(\omega) \) are the electric and magnetic polarizability tensors for the grain. We will assume the grain to be a spheroid, with symmetry axis and static magnetization parallel to \( \hat{z} \).

### 4.1. Electric Polarizability

We assume the dielectric function to be isotropic (i.e., a scalar). The electric polarizability tensor for the ellipsoidal particle has eigenvalues (van de Hulst 1957)

\[ \alpha_{e,j}(\omega) = \frac{V}{4\pi} (\epsilon - 1) L_j + \frac{1}{j}, \quad j = 1-3, \quad (63) \]

where \( \epsilon(\omega) = \epsilon_1 + i \epsilon_2 \) is the dielectric function, and \( L_j \) is the same geometrical factor that appears in Equation (16).

### 4.2. Magnetic Polarizability

#### 4.2.1. Magnetization

A small particle can acquire a net magnetic moment \( p_m \) from aligned electron spins, and also from electric currents circulating within the body of the grain. Therefore, we separate the magnetic polarizability tensor \( \alpha_m \) into two components:

\[ \alpha_m(\omega) = \alpha_m^{(\text{mag})}(\omega) + \alpha_m^{(\text{eddy})}(\omega); \quad (64) \]

\( \alpha_m^{(\text{mag})}H_{\text{inc}} \) is the magnetic moment contributed by magnetization of the grain material itself (i.e., alignment of electron spins), and \( \alpha_m^{(\text{eddy})}H_{\text{inc}} \) is the magnetic moment generated by the induced eddy currents (zero for nonconducting materials).

In principle, \( \alpha_m^{(\text{mag})} \) and \( \alpha_m^{(\text{eddy})} \) should be solved for self-consistently. Alignment of the electron spins will be in response to both \( H_{\text{inc}} \) and the field generated by the eddy currents; the eddy currents are induced by \( \partial B/\partial t \), which will include a contribution from the oscillating magnetization \( h \). Nevertheless, we will provisionally assume that \( \alpha_m^{(\text{eddy})} \) can be calculated neglecting the magnetic response of the grain; \( \alpha_m^{(\text{mag})} \) is then calculated with a simple estimate for the magnetic field generated by the eddy currents. The validity of this approximation will be verified below.

#### 4.2.2. Eddy Currents

The eddy current contribution is estimated from the solution for eddy currents in a nonmagnetic sphere of radius \( a \) (Landau et al. 1993): \( \alpha_m^{(\text{eddy})} \) is diagonal, with diagonal elements

\[ \alpha_m^{(\text{eddy})}(\omega, a) = \frac{3V}{8\pi} \left[ \frac{3}{y^2} - 3y\cot y - 1 \right], \quad (65) \]

where \( \omega = \sqrt{\epsilon_1 + \epsilon_2} \). The dielectric function \( \epsilon = 1 + \delta\epsilon(\text{bound}) + 4\pi\sigma/\omega \) where \( \delta\epsilon(\text{bound}) \) is the contribution of the bound electrons, and \( \sigma \) is the electrical conductivity of the grain material. While Equation (65) is for a spherical shape, we will use it as an estimate for spheroids, setting \( a = (3V/4\pi)^{1/3} \).

**Figure 4.** Dissipative part \( \Im(\alpha_m) \) of magnetic polarizability \( \alpha_m = \alpha_m^{(\text{mag})} + \alpha_m^{(\text{eddy})} \) in Fe grains. The Gilbert equation with \( \alpha_0 = 0.2 \) is assumed, and \( \alpha_m \) is shown for both circular polarizations (solid: \( \hat{h}_+ \); dashed: \( \hat{h}_- \)). Curves are labeled by grain radius \( a \). For \( a < 0.05 \mu m \), eddy currents are unimportant for \( \nu < 500 \text{ GHz} \).

(A color version of this figure is available in the online journal.)

The limiting case of a perfect conductor has \( \Im(\gamma) \to \infty \), and \( \alpha_m^{(\text{eddy})} \to -3V/8\pi \). In this limit, the magnetic field generated by eddy currents on the surface completely cancels the applied magnetic field \( h \) within the sphere. For finite conductivity, the eddy currents are not confined to the surface, and the magnetic field in the interior is not uniform. We take the typical field in the interior to be

\[ h_{\text{eff}} \approx h_0 (1 - \phi_{\text{eddy}}) e^{-\text{rot}} \quad (66) \]

\[ \phi_{\text{eddy}}(\omega, a) \equiv -\frac{8\pi \alpha_m^{(\text{eddy})}(\omega, a)}{3V}. \quad (67) \]

Our estimate (66) for effects of eddy currents is exact in the limits \( \alpha_m^{(\text{eddy})} = 0 \) (nonconducting: \( \phi_{\text{eddy}} = 0 \) and \( \alpha_m^{(\text{eddy})} \to -3V/8\pi \) (perfect conductor: \( \phi_{\text{eddy}} = 1 \)), and should be a reasonable approximation of the shielding for intermediate values of \( \alpha_m^{(\text{eddy})} \).

Shielding by eddy currents will act to lower the magnetization of the grain material. We will take the contribution of magnetization to the magnetic polarizability tensor to be

\[ \alpha_m^{(\text{mag})} \approx (1 - \phi_{\text{eddy}}) V \left( \chi_t \hat{h}_+ \hat{h}_+^* + \chi_b \hat{h}_- \hat{h}_-^* \right). \quad (68) \]

If \( \Re(\phi_{\text{eddy}}) \lesssim 0.2 \), the eddy currents do not reduce the magnetic field within the sphere by more than \( \sim 20\% \).

The electrical conductivity for interstellar metallic Fe particles with Ni impurities is estimated in Appendix B. Figure 4 shows \( 4\pi \left| \alpha_m^{(\text{mag})} \right| / V \) calculated for Fe spheres with radii \( a = 0.01, 0.05, 0.1, 0.2, \) and \( 0.5 \mu m \). From the figure, we see that for \( a < 0.05 \mu m \), eddy currents are unimportant for frequencies \( \nu < 500 \text{ GHz} \). Eddy currents do not appreciably affect the magnetization of submicron grains of magnetite and maghemite, because the electrical conductivity of these materials is much smaller than the conductivity of metallic Fe.
5. ABSORPTION BY SMALL PARTICLES: THE DIPOLE LIMIT

For a uniform sphere illuminated by a monochromatic plane wave, the electromagnetic scattering problem can be solved exactly (Mie 1908; Debye 1909) provided the material is isotropic, i.e., can be characterized by a scalar dielectric function $\epsilon(\omega)$ and a scalar magnetic permeability $\mu(\omega)$. This mathematical solution is commonly referred to as “Mie theory.” Many numerical implementations of Mie theory assume the material to be non-magnetic ($\mu = 1$), although a code for general scalar $\epsilon$ and $\mu$ has been implemented by Milham (1994). The magnetic response (Equation (29)) is, however, highly anisotropic, therefore the Mie theory solution is not directly applicable to the present problem. Lacking an exact solution, we seek an approximate treatment.

If the grain is small compared with $c/\omega$, then $E_{\text{inc}}$ and $H_{\text{inc}}$ can be approximated as uniform over the body of the grain. Let $p_e(t)$ and $p_m(t)$ be the electric and magnetic dipole moment of the grain. The time-averaged rate at which the incident wave does work on the grain is

$$\left\langle \frac{dW}{dt} \right\rangle = \left\langle \text{Re}(E_{\text{inc}}) \cdot \frac{d\text{Re}(p_e)}{dt} \right\rangle + \left\langle \text{Re}(H_{\text{inc}}) \cdot \frac{d\text{Re}(p_m)}{dt} \right\rangle$$

(69)

with a peak near

$$\omega_{\text{res},+} = \frac{\omega_0}{\sqrt{1 + \alpha_G^2}} \approx \omega_0.$$  

(76)

The resonance at $\omega_{\text{res},+}$ is associated with the $\hat{h}_r$ circular polarization mode. The height and width of the resonant absorption are determined by the damping coefficient $\alpha_G$. The resonance frequency $\omega_{\text{res},+}$ depends on the eigenvalues of the demagnetization tensor $D$ (see Equation (16)), and hence on the particle shape. Table 2 gives $\omega_0$ for Fe spheres and spheroids with selected shapes. Figures 5(a) and (b) show $C_{\text{abs}}/V$ for Fe spheres (absorption peak at $\sim 1.5$ GHz) and 2:1 prolate spheroids (absorption peak at $\sim 16$ GHz).

The asymptotic behavior for ferromagnetic material is

$$\left\langle C_{\text{abs}}^{(\text{mag})} \right\rangle \rightarrow \begin{cases} \frac{8\pi V}{3c} \frac{\alpha_G \omega M \omega^2}{\omega_0^2} & \text{for } \omega \ll \omega_0 \\ \frac{8\pi V}{3c} \frac{\alpha_G \omega M}{\omega_0^2} + \frac{4\pi \omega}{c} \text{Im}(\alpha_m^{(\text{eddy})}) & \text{for } \omega \gg \omega_0 \end{cases}$$

(77)

6.2. Ferrimagnetic Material

For ferrimagnetic materials, with $\chi$, given by Equation (59), $C_{\text{abs}}(\omega)$ has more complicated behavior. For $\alpha_G \ll 1$ there are two local extrema, near

$$\omega_{\text{res},\pm} = \left[ \frac{(\omega_0 A + \omega_0 B)^2}{4} - \frac{\omega_0 A \omega_0 B + N_{AB}^2 \omega_0 A \omega_0 B}{2} \right]^{1/2}.$$

(78)

The frequency $\omega_{\text{res},+}$ for resonance when $h \propto \hat{h}_r$ (i.e., applied field rotating anticlockwise around the static magnetization $\mathbf{M}$) is the frequency of free precession of the magnetizations $\mathbf{M}_A$ and $\mathbf{M}_B$ around the $\hat{z}$ axis, with $\mathbf{M}_A$ and $\mathbf{M}_B$ remaining antiparallel—it corresponds to ordinary ferromagnetic resonance, with $\mathbf{M}_A = -\beta \mathbf{m}_B$.

The frequency $\omega_{\text{res},-}$ associated with polarization $\hat{h}_r$ is much higher—at THz frequencies—and corresponds to precession with $\mathbf{M}_B$ and $\mathbf{M}_A$ no longer antiparallel, but instead with $\text{Re}(\mathbf{m}_A) \cdot \text{Re}(\mathbf{m}_B) > 0$. With $\mathbf{M}_A$ and $\mathbf{M}_B$ no longer antiparallel, the precession is driven primarily by the strong coupling $N_{AB}M_A \cdot M_B$.

The asymptotic behavior for nonconducting ferrimagnetic material is

$$\left\langle C_{\text{abs}}^{(\text{mag})} \right\rangle \rightarrow \begin{cases} \frac{8\pi V}{3c} \alpha_G A \omega^2 & \text{for } \omega \ll \omega_{\text{res},+} \\ \frac{8\pi V}{3c} \alpha_G B & \text{for } \omega \gg \omega_{\text{res},-} \end{cases},$$

(79)

where

$$A = \frac{(2N_{AB} \omega M_A \omega M_B - \omega M_A \omega_0 - \omega M_B \omega_0 A)(\omega_0 - \omega_0 A)}{(N_{AB}^2 \omega M_A \omega M_B - \omega_0 A \omega_0 A)^2}$$

(80)

$$- \frac{(\omega_0 M_B - \omega_0 A)}{(N_{AB}^2 \omega M_A \omega M_B - \omega_0 A \omega_0 B)}$$

(81)
Figures 5(c) and (d) show the magnetic contribution to the absorption cross section per unit volume calculated for small particles of magnetite Fe₃O₄, and maghemite γ-Fe₂O₃. The dip in absorption at the upper resonance frequency $\omega_{\text{res}}$ is notable in Figures 5(c) and (d), but we will see below that at these frequencies electric dipole absorption is dominant, and this feature in the magnetic dipole absorption is probably not observable.

7. OPACITIES

Above we have discussed the response of metallic Fe, magnetite, and maghemite to oscillating electric and magnetic fields in the dipole limit, $a \ll \lambda$. We now calculate the absorption cross section, as a function of wavelength, including both electric and magnetic effects, over a broad range of frequencies. We treat the grains as spherical and calculate $C_{\text{MT}}^{\text{abs}}$ using Mie theory with $\mu = 1$. To this we add the “magnetic” contribution to magnetic dipole absorption calculated in the magnetic dipole limit, including the eddy current correction. Thus, for randomly oriented grains, we take

$$C_{\text{abs}} = C_{\text{abs}}^{\text{MT}} + \frac{4\pi \omega V}{3c} \text{Im}[(1 - \phi_{\text{eddy}})(\chi_+ + \chi_-)].$$

The magnetization response for Fe is strongly dependent on the shape of the Fe particles. Interstellar Fe nanoparticles—which free-flying or inclusions within larger grains—are presumably nonspherical. To compute the magnetization contribution to the absorption, we will assume the Fe to be in 2:1 prolate spheroids (magnetized along the symmetry axis).

Figure 6(a) shows $\lambda C_{\text{abs}}^{\text{mag}}/V$ as a function of $\lambda$ for Fe spheres with selected radii ranging from $a = 100 \text{ nm}$ to $1 \mu$ m. For comparison, we also show $\lambda C_{\text{abs}}^{\text{mag}}/V$ for $a = 100 \text{ nm}$ amorphous silicate spheres. Because amorphous silicate is nonconducting, electric dipole absorption dominates in the FIR, and $C_{\text{abs}}^{\text{mag}}/V$ is independent of $a$ for $2\pi a/\lambda \ll 1$. By contrast, the eddy current absorption in the Fe particles causes $C_{\text{abs}}^{\text{mag}}/V$ at far-infrared and submillimeter frequencies to be sensitive to $a$ (see also Figure 4). For $a \gtrsim 100 \text{ nm}$, “magnetic dipole” absorption due to eddy current dissipation is important for $\lambda \gtrsim 30 \mu$ m. True magnetic absorption dominates only at very long wavelengths: for $a = 100 \text{ nm}$, magnetic absorption dominates for $\lambda > 1 \text{ mm}$ ($\nu < 300 \text{ GHz}$).

From Figure 6(a) we see that at $\lambda = 10^3 \mu$ m = 1 mm ($\nu = 300 \text{ GHz}$) an $a = 100 \text{ nm}$ Fe sphere has $C_{\text{abs}}^{\text{mag}}/V$ that is a factor ~2 times larger than for astrosilicate, whereas for $a = 300 \text{ nm}$, $C_{\text{abs}}^{\text{mag}}/V$ exceeds astrosilicate by a factor ~10, because of eddy currents in the conducting particle. Therefore,
Figure 6. $\lambda C_{\text{abs}}/V$ as a function of wavelength $\lambda$. (a) Fe spheres with radii $a = 1, 10, 100, 200, 300,$ and $10^3$ nm. The increase in opacity near $\lambda = 100$ $\mu$m as $a$ increases from 10 nm to 300 nm is due to the increased dissipation by eddy currents. The eddy current absorption peak shifts to longer wavelengths with increasing grain size. For $a < 100$ nm, absorption for $\lambda > 1000$ $\mu$m ($\nu < 300$ GHz) is dominated by magnetic absorption, with a ferromagnetic resonance peak at $\sim 15$ GHz (the ferromagnetic absorption has been calculated for 2:1 prolate spheroids—see text). (b) Magnetite spheres with radii $a = 10$ and 100 nm. The ferrimagnetic resonance peak is at 15 GHz. (c) Maghemite spheres with radii $a = 10$ and 100 nm. The ferrimagnetic resonance peak is at 8 GHz. Because of the low electrical conductivity, eddy current absorption is unimportant for magnetite and maghemite. All results have been calculated assuming $\alpha G = 0$.2. Also shown is $\lambda C_{\text{abs}}/V$ for $a = 100$ nm amorphous silicate spheres.

(A color version of this figure is available in the online journal.)

if a significant fraction of the interstellar Fe is in particles of such size, it could noticeably affect the overall submillimeter and millimeter-wave emission.

8. GRAIN TEMPERATURES

With absorption cross sections $C_{\text{abs}}(\omega)$ calculated as described above, we can evaluate the rate

$$P_{\text{heat}}(a) = \int dv C_{\text{abs}}(v) u_v c$$

(83)

at which a grain of radius $a$ will absorb energy from the interstellar radiation field (ISRF) with specific energy density $u_v$. We assume $u_v$ to have the spectrum estimated by Mathis et al. (1983, hereafter MMP83) for the local ISRF multiplied by a factor $U$ ($U = 1$ corresponds to the solar-neighborhood ISRF). We then solve for the “steady-state” temperature $T_s$ for which the time-averaged thermal emission is equal to $P_{\text{heat}}$:

$$\int dv C_{\text{abs}}(v) 4\pi B_\nu(T_s) = P_{\text{heat}}.$$  (84)

The actual grain temperature will fluctuate around $T_s$. For $a \gtrsim 0.01$ $\mu$m the temperature fluctuations are small enough that they can be neglected here.

Figure 7 shows the steady-state grain temperature $T_s$ as a function of radius $a$ for Fe, magnetite, and maghemite grains heated by radiation with the MMP83 spectrum. Free-flying nanoparticles of maghemite or magnetite with $a \lesssim 0.01$ $\mu$m would have $T_s \approx 22 U^{1/6}$ K, similar to the temperatures of the $a \approx 0.1$–0.3 $\mu$m grains that dominate the visual extinction. Fe nanoparticles, however, are considerably hotter, $T_s \approx 42 U^{0.2}$ K, because the high electrical conductivity suppresses the radiative cooling.

9. INFRARED EMISSION SPECTRUM

Figure 8(a) shows the emission spectrum, per unit grain volume, for $a = 0.1$ $\mu$m spheres of Fe, Fe$_3$O$_4$, and $\gamma$-Fe$_2$O$_3$ heated by the MMP83 starlight spectrum with the intensity ($U = 1$) estimated for the solar neighborhood. Because of their enhanced absorption cross sections in the optical and near-IR, the Fe, Fe$_3$O$_4$, and $\gamma$-Fe$_2$O$_3$ grains are somewhat warmer than amorphous silicate grains heated by the same $U = 1$ radiation fields, with grain temperatures of $\sim 20$ K versus the 15.8 K temperature for the amorphous silicate grain, but all have emission spectra peaking near $\sim 150$ $\mu$m. At longer wavelengths, $\lambda \gtrsim 1$ mm ($\nu < 300$ GHz), the magnetic grains...
radiate more strongly than the amorphous silicate, as expected from the cross sections shown in Figure 6.

The magnetic grains may be inclusions in larger grains (rather than free-fliers), in which case it is more appropriate to compare the emissivities of the different grain materials at a common temperature. Figure 8(b) shows the emission per unit grain volume at $T = 18\, K$ for the same four grain materials. At 300 GHz ($\lambda = 10^3\, \mu m$), the power radiated per grain volume by the magnetic grains ranges from 1.2 (for $\gamma$-Fe$_2$O$_3$) times larger than the emission from the amorphous silicate. At 100 GHz the difference between amorphous silicate and the magnetic materials is even more pronounced: factors of 5 (for $\gamma$-Fe$_2$O$_3$), 8 (for Fe$_3$O$_4$), and 9 (for Fe).

10. POLARIZATION

10.1. Davis–Greenstein Alignment

Let $\mathbf{h}_{\text{ism}}$ be the static interstellar magnetic field; $\mathbf{h}_{\text{ism}}$ is very weak compared with either $4\pi \mathbf{M}_0$ or the crystalline anisotropy field $\mathbf{H}_K$. However, as pointed out by Davis & Greenstein (1951), $\mathbf{h}_{\text{ism}}$ can exert systematic torques on a spinning grain.

Consider a magnetic particle spinning with angular velocity $\omega$. Grain rotation can be excited by many processes. At a minimum, elastic collisions with gas atoms will excite “Brownian” rotation with $\frac{\omega}{2\pi} \approx 1 \left( \frac{15 k T}{8\pi \rho a^3} \right)^{1/2} \approx 20\, kHz \left( \frac{T}{10^2\, K} \right)^{1/2} \left( \frac{5\, cm^{-3}}{\rho} \right)^{1/2} \left( \frac{0.1\, \mu m}{a} \right)^{5/2}$. (85)

However, grains with $a \gtrsim 0.1\, \mu m$ appear likely to be spinning suprathermally, as a result of systematic torques due to formation of $H_2$ on the grain surface and emission of photoelectrons (Purcell 1979) as well as radiative torques due to starlight (Draine & Weingartner 1996). Suprathermal rotation of small $a \lesssim 0.05\, \mu m$ grains is thought to be suppressed by “thermal flipping” (Lazarian & Draine 1999).

Let $\psi$ be the angle between $\omega$ and $\mathbf{h}_{\text{ism}}$. The grain’s spontaneous magnetization $\mathbf{M}_0$ need not be aligned with its angular velocity $\omega$; let $\Theta$ be the angle between $\mathbf{M}_0$ and $\omega$. Let $\hat{x}_m$, $\hat{y}_m$, $\hat{z}_m$ be unit vectors “frozen into” the magnetic material, with $\hat{z}_m \parallel \mathbf{M}_0$. The magnetic field can be separated into two components, one that appears stationary to the rotating grain, and another, denoted $\mathbf{h}_{\text{ism,\omega}}$, oscillating with frequency $\omega$. Choose $\hat{x}_m$ to be perpendicular to the $\omega - \mathbf{M}_0$ plane, and $\hat{y}_m = \hat{z}_m \times \hat{x}_m$. If $\text{Re}(\mathbf{h}_{\text{ism,\omega}})$ is in the $\hat{x}_m$ direction at $t = 0$ then the magnetic field in “grain coordinates” is

$$\mathbf{H}_{\text{ism}} = H_{\text{ism}} \cos \psi \cos \Theta \hat{z}_m + \text{Re}(\mathbf{h}_{\text{ism,\omega}})$$

$$\mathbf{H}_{\text{ism,\omega}} = H_{\text{ism}} \sin \psi (-\hat{x}_m + i \cos \Theta \hat{y}_m - i \sin \Theta \hat{z}_m)e^{-i\omega t}.$$ (87)

The grain is spinning in the static field $\mathbf{H}_{\text{ism}}$. Following Davis & Greenstein, assume that the magnetic dissipation in the grain material is the same as in a stationary sample in a rotating magnetic field corresponding to $\mathbf{H}_{\text{ism,\omega}}$. As before, the magnetic eigenmodes are $\hat{h}_\pm = (\hat{x}_m \pm i \hat{y}_m)/\sqrt{2}$, and the time-averaged energy dissipation rate is (see Equation (70))

$$\left\langle \frac{dW}{dt} \right\rangle = \frac{\omega}{2} V (\text{Im}(\chi_+)|\mathbf{H}_{\text{ism,\omega}} \cdot \hat{h}_+|^2 + \text{Im}(\chi_-)|\mathbf{H}_{\text{ism,\omega}} \cdot \hat{h}_-|^2)$$

$$= \frac{1}{4} \omega V H_{\text{ism}}^2 \sin^2 \psi (1 - \cos \Theta)^2 \text{Im}(\chi_+) + (1 + \cos \Theta)^2 \text{Im}(\chi_-)$$

$$\approx \frac{1}{2} \omega V H_{\text{ism}}^2 \sin^2 \psi (1 + \cos^2 \Theta) \text{Im} \left( \frac{\chi_+ + \chi_-}{2} \right)$$

$$= K V H_{\text{ism}}^2 \omega^2 \sin^2 \psi$$

$$K(\Theta) = \frac{\text{Im}(\chi_+ + \chi_-)}{2\omega} \left( \frac{1 + \cos^2 \Theta}{2} \right)$$

$$\approx \frac{\alpha_G \omega_M}{2 \omega_0} (1 + \cos^2 \Theta),$$ (93)

where the approximation of Equation (90) takes $\text{Im}(\chi_+ - \chi_-) \ll \text{Im}(\chi_+ + \chi_-)$ because the grain rotational frequency $\omega \ll \omega_0$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{(a) Emissivity per grain volume of $a = 0.10\, \mu m$ grains of astrosilicate, metallic Fe, magnetite Fe$_3$O$_4$, and maghemite $\gamma$-Fe$_2$O$_3$, heated by the MMP83 radiation field. (b) Emissivity per grain volume of small grains of astrosilicate, metallic Fe, magnetite, and maghemite at $T = 18\, K$. All results shown are for $\alpha_G = 0.2$. (A color version of this figure is available in the online journal.)}
\end{figure}
The grain, spinning in a static field, is converting rotational kinetic energy into heat. Davis & Greenstein (1951) showed that the associated torques would act to reduce the transverse component of the angular momentum, bringing the grain rotation axis into alignment with the local magnetic field. If \( K \) is constant, the dissipation causes tan \( \psi \) to decay exponentially:

\[
\tan \psi = \tan \psi(t = 0)e^{-t/\tau_{DG}}
\]

where \( I \) is the moment of inertia. For metallic Fe in a 2:1 prolate spheroid,

\[
K \approx 1.47 \times 10^{-12} \alpha_G(1 + \cos^2 \Theta) \text{ s}.
\]

If \( \alpha_G \approx 0.2 \), this gives a rate of magnetic dissipation that is only slightly larger than what is estimated for normal paramagnetic materials, \( K \approx 10^{-13}(18 \text{K}/T_m) \text{ s} \) (Jones & Spitzer 1967). However, as discussed in Appendix A, at low or intermediate frequencies, we might expect the effective value of \( \alpha_G \) to be \( \alpha_G \approx 1/\omega \tau_2 \), where \( \tau_2 \) is the spin–spin relaxation time. For a typical spin–spin relaxation time \( \approx 10^{-10} \text{ s} \), we would then have \( \alpha_G \approx 8 \times 10^4(20 \text{kHz}/(\omega/2\pi))(10^{-10} \text{ s}/\tau_2) \), suggesting that \( K \approx 10^{-7} \text{ s} \) for grains where metallic Fe is a significant fraction of the grain volume, giving a Davis–Greenstein alignment time \( \tau_{DG} \approx 1 \text{ yr} \). For extremely rapid alignment, the grain angular momentum with the local magnetic field.6

The discussion above applies to an isolated single-domain grain. Jones & Spitzer (1967) pointed out that a grain containing many magnetic inclusions could exhibit superparamagnetism, with enhanced dissipation at frequencies below the characteristic frequency \( \tau_{flip}^{-1} \) for thermal fluctuations to “flip” the magnetization of an individual inclusion. This could result in \( K \) as large as \( 10^{-6} \text{ s} \). Rapid alignment by ferromagnetic inclusions has been discussed by various authors, including Duley (1978), Mathis (1986), Goodman & Whittet (1995), and Martin (1995).

### 10.2. Polarization of the Magnetic Dipole Emission

#### 10.2.1. Single-domain Magnetic Grains

Single-domain magnetic grains, if their directions of static magnetization are aligned, will produce strong polarization, as previously noted by DL99. Let us assume that the grains are prolate spheroids, magnetized along the long axis. The grains are assumed to be spinning around an axis perpendicular to the symmetry axis. Without loss of generality, let \( \hat{x} \) point toward the observer, let \( \hat{z} \) be orthogonal to \( \hat{x} \) and \( \hat{H}_\text{ism} \), and let \( \hat{y} = \hat{z} \times \hat{x} \). Then \( \hat{H}_\text{ism} \) is in the \( \hat{x}\hat{y} \) plane.

If \( \hat{z}_m \) is a unit vector along the prolate symmetry axis (the direction of magnetization), then the magnetic absorption cross section for linearly polarized radiation with \( \hat{H}_\text{inc} = H_\text{inc}\hat{h}e^{-i\omega t} \) can be written

\[
C_{\text{abs}}(\text{mag}) = \frac{2\pi}{c} \omega V \text{Im}[(1 - \phi_{\text{eddy}})(\chi_+ + \chi_-)][1 - \langle \hat{h} \cdot \hat{z}_m \rangle^2].
\]

The brackets \( \langle \ldots \rangle \) denote averaging over the grain rotation and precession. Let \( C_{\text{abs},\perp} \) and \( C_{\text{abs},\parallel} \) be \( C_{\text{abs}} \) for \( \hat{E}_\text{inc} \parallel \hat{y} \) (i.e., \( \hat{H}_\text{inc} \parallel \hat{y} \)) and \( \hat{E}_\text{inc} \perp \hat{y} \) (i.e., \( \hat{H}_\text{inc} \parallel \hat{z} \)), respectively:

\[
C_{\text{abs},\perp}(\text{mag}) = \frac{2\pi}{c} \omega V \text{Im}[(1 - \phi_{\text{eddy}})(\chi_+ + \chi_-)][1 - \langle \hat{y} \cdot \hat{z}_m \rangle^2],
\]

\[
C_{\text{abs},\parallel}(\text{mag}) = \frac{2\pi}{c} \omega V \text{Im}[(1 - \phi_{\text{eddy}})(\chi_+ + \chi_-)][1 - \langle \hat{z} \cdot \hat{z}_m \rangle^2].
\]

The fractional polarization of the emitted radiation is just

\[
p = \frac{C_{\text{abs},\perp} - C_{\text{abs},\parallel}}{C_{\text{abs},\perp} + C_{\text{abs},\parallel}} = \frac{(\langle \hat{z} \cdot \hat{z}_m \rangle^2 - \langle \hat{y} \cdot \hat{z}_m \rangle^2)}{2 - (\langle \hat{z} \cdot \hat{z}_m \rangle^2 - \langle \hat{y} \cdot \hat{z}_m \rangle^2)}.\]

The numerator is proportional to the familiar “Rayleigh reduction factor” \( (3/2)(\cos^2 \psi - 1/3) \), which varies from 0 to 1 as \( \cos^2 \psi \) varies from 1/3 to 1. The polarization \( p \) is plotted as a function of \( \phi \) in Figure 9, for selected values of \( \langle \cos^2 \psi \rangle \), including 1 (perfect alignment), 1/3 (random alignment), and \( \langle \cos^2 \psi \rangle = 0 \) (perfect antialignment).

Single-domain grains aligned with the long axis (and magnetization) \( \hat{z}_m \parallel \hat{H}_\text{ism} \) give \( p \approx 0 \). The assumed perfect alignment of the grain’s magnetization with the principal axis of smallest moment of inertia results in very large polarizations, ranging from \(-0.2 \) to \(1 \) depending on the values of \( \langle \cos^2 \psi \rangle \) and \( \sin^2 \phi \).

#### 10.2.2. Randomly Oriented Magnetic Inclusions

Magnetic material might be primarily in the form of “inclusions” in larger grains. If the nonspherical inclusions are themselves perfectly oriented relative to the principal axes of the host grain, then Equation (103) will approximate the polarization of the magnetic dipole emission.

If, as seems more likely, the inclusions are randomly oriented relative to the principal axes of the host, and the inclusions have a sufficiently low volume filling factor \( f_{\text{fill}} \), then, using effective medium theory (see Appendix F) we may approximate the magnetic permeability of the grain as a sum over the magnetic response of the individual inclusions (assumed to be spherical—see Equation (F10)):

\[
\mu \approx 1 + f_{\text{fill}} \frac{2\pi}{3} (\chi_+ + \chi_-).\]

---

6 Jones & Spitzer (1967, p. 961–962) estimated

\[2 \times 10^{-13} < K < 4 \times 10^{-7} \text{ s} \] for single-domain Fe particles, with “the lower limit extremely unlikely.”
The host material is assumed to be nonmagnetic. For simplicity, we let both inclusions and host have the same complex dielectric function $\varepsilon(\omega)$. We assume the grain to be spinning around $\hat{a}_1$, the principal axis of largest moment of inertia; $\omega$ precesses around $H_{\text{tot}}$, with an angle $\psi$ between $\omega$ and $H_{\text{tot}}$. Let $\Phi$ be the angle between the static magnetic field and the line-of-sight. For a grain with shape factors $L_1$, $L_2$, $L_3$ (for principal axes $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$), the absorption cross section in the dipole limit is

$$C_{\text{abs},\perp} = V \frac{\omega}{c} \left[ \text{Im(}$\varepsilon$) \left( \frac{\langle \hat{z} \cdot \hat{a}_1 \rangle^2}{|1 + L_1(\varepsilon - 1)|^2} + \frac{\langle \hat{z} \cdot \hat{a}_2 \rangle^2}{|1 + L_2(\varepsilon - 1)|^2} + \frac{\langle \hat{z} \cdot \hat{a}_3 \rangle^2}{|1 + L_3(\varepsilon - 1)|^2} \right) \right] \left[ \text{Im(}$\varepsilon$) \left( \frac{\langle \hat{y} \cdot \hat{a}_1 \rangle^2}{|1 + L_1(\mu - 1)|^2} + \frac{\langle \hat{y} \cdot \hat{a}_2 \rangle^2}{|1 + L_2(\mu - 1)|^2} + \frac{\langle \hat{y} \cdot \hat{a}_3 \rangle^2}{|1 + L_3(\mu - 1)|^2} \right) \right] \right]$$

$$C_{\text{abs},\parallel} = V \frac{\omega}{c} \left[ \text{Im(}$\varepsilon$) \left( \frac{\langle \hat{y} \cdot \hat{a}_1 \rangle^2}{|1 + L_1(\varepsilon - 1)|^2} + \frac{\langle \hat{y} \cdot \hat{a}_2 \rangle^2}{|1 + L_2(\varepsilon - 1)|^2} + \frac{\langle \hat{y} \cdot \hat{a}_3 \rangle^2}{|1 + L_3(\varepsilon - 1)|^2} \right) \right] \left[ \text{Im(}$\varepsilon$) \left( \frac{\langle \hat{z} \cdot \hat{a}_1 \rangle^2}{|1 + L_1(\mu - 1)|^2} + \frac{\langle \hat{z} \cdot \hat{a}_2 \rangle^2}{|1 + L_2(\mu - 1)|^2} + \frac{\langle \hat{z} \cdot \hat{a}_3 \rangle^2}{|1 + L_3(\mu - 1)|^2} \right) \right] \right]$$

where the orientational averages are (see Appendix E)

$$\langle \hat{y} \cdot \hat{a}_1 \rangle^2 = \sin^2 \Phi \langle \cos^2 \psi \rangle + \frac{1}{2} \cos^2 \Phi \langle \sin^2 \psi \rangle$$

$$\langle \hat{y} \cdot \hat{a}_2 \rangle^2 = \langle \hat{y} \cdot \hat{a}_3 \rangle^2$$

$$\langle \hat{z} \cdot \hat{a}_1 \rangle^2 = \frac{1}{4} \cos^2 \Phi (1 + \langle \cos^2 \psi \rangle) + \frac{1}{2} \sin^2 \Phi \langle \sin^2 \psi \rangle$$

The fractional polarization of the emission is

$$p = \frac{C_{\text{abs},\perp} - C_{\text{abs},\parallel}}{C_{\text{abs},\perp} + C_{\text{abs},\parallel}} = \frac{2N}{D} [1 + 3 \cos(2\psi)] \sin^2 \Phi$$

$$N = B_1 B_2 B_3 [A_1(A_2 + A_3) - 2A_2 A_3] \text{Im}(\varepsilon)$$

$$- A_1 A_2 A_3 [B_1(B_2 + B_3) - 2B_2 B_3] \text{Im}(\mu)$$

$$D = \{10A_2 A_3 + 11A_2 (A_2 + A_3) + [A_1(A_2 + A_3) - 2A_2 A_3]$$

$$\times \langle \cos(2\psi) + \cos(2\Phi) [1 + 3 \cos(2\psi)] \rangle B_1 B_2 B_3 \text{Im}(\varepsilon)$$

$$+ [B_1 B_2 B_3 + 11B_1 (B_2 + B_3) + B_1 (B_2 + B_3) - 2B_2 B_3]$$

$$\times \langle \cos(2\psi) + \cos(2\Phi) [1 + 3 \cos(2\psi)] \rangle A_1 A_2 A_3 \text{Im}(\mu)$$

$$A_j \equiv |1 + L_j(\varepsilon - 1)|^2$$

$$B_j \equiv |1 + L_j(\mu - 1)|^2.$$
Interpretation is further complicated by the possibility that the FIR and submillimeter emission comes from more than one type of grain, with differing degrees of polarization, such as in the models discussed by Draine & Fraisse (2009). In models where the silicate grains are aligned, but the carbonaceous grains are not, Draine & Fraisse (2009) predicted that the polarization would increase with decreasing frequency—opposite to what is predicted for grains with magnetic inclinations. This effect would possibly overwhelm the polarization signature of magnetic dipole emission at frequencies $\nu > 100$ GHz.

11. DISCUSSION

11.1. Constraints from the Observed Extinction

If present as inclusions, magnetic grains could be accommodated within the grain population without significantly affecting the optical–UV extinction. If present as free-flying nanoparticles, their contribution to the extinction could potentially be significant. Figure 11 shows the contribution to the extinction if 100% of the Fe is in spheres with radii $a = 0.005$, 0.010, and 0.020 $\mu m$ composed of metallic Fe, Fe$_3$O$_4$, or $\gamma$-Fe$_2$O$_3$. The extinction contributed by metallic Fe or Fe$_3$O$_4$ never exceeds $\sim$40% of the observed extinction at optical–UV wavelengths, and the calculated extinction is quite smooth, lacking sharp features that might be recognizable. In the case of $\gamma$-Fe$_2$O$_3$ particles, 100% of the Fe in 0.02 $\mu m$ particles is probably incompatible with the observed extinction, but smaller particles appear to be permitted.

11.2. Constraints from the Observed IR–Microwave Emission

Figure 12 shows the predicted emissivity per H nucleon if 100% of the interstellar Fe is in particles of either metallic Fe, magnetite, or maghemite. For the magnetite and maghemite we assume $T \approx 18$ K, characteristic of the bulk of the dust in the diffuse ISM. For metallic Fe, we assume particle size $a \approx 0.01 \mu m$ (so that eddy currents are not important) and consider two temperatures: $T \approx 18$ K, appropriate if the metallic Fe is present as inclusions in larger grains, and $T \approx 40$ K, appropriate if the Fe is in free-flying nanoparticles heated by interstellar starlight (see Figure 7).

Figure 12 also shows the mean 110 $\mu m \leq \lambda \leq 1.3$ cm emission/H measured by COBE-FIRAS and Wilkinson Microwave Anisotropy Probe (WMAP) for regions with $|b| > 6^\circ$ and $I_v(240 \mu m) > 18$ MJy sr$^{-1}$ (Paradis et al. 2011), as well as the emissivity for low-velocity H I measured by IRAS (60 and 100 $\mu m$) and Planck (350, 550, and 850 $\mu m$) around the north ecliptic pole (Planck Collaboration et al. 2011a).

From Figure 12 we see that as much as 100% of the Fe could be in single-domain nanoparticles of metallic Fe, magnetite, or maghemite without exceeding the observed emission from the diffuse ISM of the Milky Way.

Figure 12 also shows the predicted emission if 100% of the Fe is in prolate spheroids of Fe metal, with axial ratios $a/b = 2, 5,$

---

To obtain the emission per H, we take this region to have $N(H_I) = 3 \times 10^{21}$ cm$^{-2}$, Paradis et al. (2011) state that $N(H_I) = 2 \times 10^{21}$ cm$^{-2}$, which we increase by a factor 1.5 to allow for both 21 cm self-absorption and H$_2$. 

---

Figure 10. Degree of linear polarization for emission from perfectly aligned 1.5:1 oblate spheroids composed of amorphous silicate only (curve labeled “nonmagnetic”) or amorphous silicate grains with randomly oriented single-domain magnetic inclusions. Magnetic inclusions considered are Fe spheroid and spheroids (1.5:1 and 2:1) with volume filling factor $f_{\text{fill}} = 0.15$. Fe$_3$O$_4$ magnetite with $f_{\text{fill}} = 0.3$, and $\gamma$-Fe$_2$O$_3$ maghemite with $f_{\text{fill}} = 0.3$. At high frequencies, electric dipole emission dominates, polarized with $E$ along the long axes ($p > 0$). For grains with magnetic inclinations, magnetic dipole emission dominates at low frequency, polarized with $\mathbf{H}$ along the long axes ($p < 0$). The detailed frequency dependence depends on the nature of the magnetic inclusions, but in all cases the degree of polarization drops by a factor $\sim 4$ as the frequency is reduced from 300 GHz to 50 GHz. $\alpha_{\text{FeO}} \approx 0.2$ is assumed.

(A color version of this figure is available in the online journal.)

Figure 11. Observed $R_V = 3.1$ extinction curve (Fitzpatrick 1999), and the extinction contribution if 100% of Fe were in metallic Fe, Fe$_3$O$_4$, or $\gamma$-Fe$_2$O$_3$ nanoparticles with radii $a = 0.005$, 0.010, or 0.020 $\mu m$. The observed extinction may be incompatible with 100% of the Fe being in $a = 0.02 \mu m$ $\gamma$-Fe$_2$O$_3$ particles, but the other cases shown here appear to be compatible with the observed extinction in the visible and ultraviolet.

(A color version of this figure is available in the online journal.)
and 10. For extreme axial ratios, $a/b = 5–10$, the ferromagnetic resonance in metallic Fe is shifted to 25–30 GHz (see Table 2). If much of the Fe is in metallic Fe nanoparticles with such extreme axial ratios, the thermal magnetic dipole emission could make a significant contribution to the observed 20–40 GHz AME. If, as seems more likely, the Fe particles have axial ratios $a/b < 2$, the contribution of magnetic dipole radiation to the AME will be minimal, in which case the observed AME would be primarily due to electric dipole radiation from spinning nanoparticles (Draine & Lazarian 1998a).

### 11.3. Comparison with DL99

Above we have concluded that the observed emission from the ISM does not preclude the bulk of the Fe being in single-domain Fe nanoparticles, whereas DL99 argued that no more than 5% of the Fe was allowed to be in metallic form. These differing conclusions arise from different models used for the magnetic response of Fe at microwave frequencies. DL99 modeled the magnetic response using a scalar susceptibility $\chi(\omega)$ with the frequency dependence of a damped harmonic oscillator. The resonance frequency $\omega_0$ was identified with the precession frequency of spins in the local magnetic field produced by the other spins in the system, and the damping time was set to $\tau_0 = 1/(2\omega_0)$. This choice of $\tau_0$ is probably reasonable for paramagnetic materials, because the local field fluctuates on a timescale related to the precession period of nearby spins. For ferromagnetic or ferrimagnetic materials, however, it is not clear that this damping time is appropriate.

Using this simple damped oscillator form for the magnetic susceptibility, DL99 concluded that nanoparticles of metallic Fe would have a strong absorption peak near 80 GHz, due to the magnetic analogue of a Fröhlich resonance when $\text{Re}(4\pi \chi) = -3$. Based on observations of dust-correlated emission by the COBE-DMR experiment (Kogut et al. 1996), DL99 argued that $\lesssim 5\%$ of interstellar Fe could be in metallic Fe particles.

The present study employs a more realistic dynamical model for the magnetic response, and reaches a different conclusion. The assumption of a scalar magnetic susceptibility used by DL99 provides a good phenomenological description of paramagnetism or the magnetization of bulk ferromagnetic materials at low frequencies (where the dynamic permeability $\mu(\omega)$ arises from motion of domain walls), but it does not describe the response of magnetic systems at frequencies $\nu \gtrsim 10$ GHz, where precession of the spins is important. The present study uses a tensor susceptibility derived from the Gilbert equation (13). For single-domain Fe particles, the magnetization response (26) has an absorption peak in the 1.5–20 GHz range, depending on shape, and the thermal emission is not as strong as estimated by DL99.

### 11.4. Validity of the Gilbert Equation

Our model for the magnetic response of Fe, maghemite, and magnetite is based on the Gilbert equation (13), with the dissipation characterized by an adjustable parameter $\alpha_G$. Empirical determinations of $\alpha_G$ for these materials have resulted in a range of values. While $\alpha_G \approx 0.2$ gives about the highest possible for the absorption at $\nu \gtrsim 100$ GHz (see Figure 5), if $\alpha_G$ were to be much smaller (e.g., $\alpha_G = 0.05$) the opacity would be reduced (by about a factor 3 in going from $\alpha_G = 0.2$ to 0.05), and the microwave–submillimeter emissivity would be reduced by the same factor.

Aside from the question of what value to use for the Gilbert damping parameter $\alpha_G$, we must recognize that the Gilbert equation uses a prescription for the dissipation that is simple and mathematically convenient, but not based on an underlying physical model. Empirical evidence for the accuracy of the Gilbert equation at high frequencies is scant. Laboratory measurements of electromagnetic absorption in Fe and Fe oxide nanoparticles at frequencies up to 500 GHz are required to validate use of the Gilbert equation at these frequencies.

The present treatment of dynamic magnetization assumed that the material was perfectly ordered and pure, but interstellar grain materials are likely to be non-crystalline and impure. We suspect that lattice defects and impurities will mainly lead to broadening of the resonance appearing near $\omega_0$ (for ferromagnetic materials) and $\omega_{\text{prez}}$ for ferrimagnetic materials, but this should be studied experimentally. Measurements of the frequency-dependent magnetic polarizability tensor $\alpha_{m}(\omega)$ for nanoparticles composed of amorphous Fe-rich oxides or silicates over the frequency range 1–500 GHz would be of great value to test the models that have been put forward here.

### 12. SUMMARY

A substantial fraction of the Fe in the ISM may be in magnetic materials such as metallic iron, magnetite $\text{Fe}_3\text{O}_4$, or maghemite $\gamma\text{-Fe}_2\text{O}_3$. We discuss the implications for the thermal emission from interstellar dust, and the polarization of this emission. The principal conclusions are as follows:

1. We obtain (in Sections 3 and 4) the frequency-dependent magnetic polarizability tensor $\alpha_{m}(\omega)$ of single-domain ferromagnetic and ferrimagnetic particles, and evaluate this for three candidate materials: metallic Fe, magnetite $\text{Fe}_3\text{O}_4$, and maghemite $\gamma\text{-Fe}_2\text{O}_3$.

2. Eddy currents are shown to have only a small effect on the magnetic absorption for $\nu \lesssim 500$ GHz.
3. We show (in Section 5) how to calculate the absorption cross section $C_{\text{abs}}(\omega)$ for magnetic particles that are small compared to the wavelength.

4. In order to calculate $C_{\text{abs}}(\omega)$ we obtain self-consistent dielectric functions for Fe (Appendix B), magnetite (Appendix C), and maghemite (Appendix D), based on a combination of theory and experiment.

5. We confirm the finding of Fischera (2004) for Fe grains that magnetic dipole absorption arising from eddy currents can result in very large absorption cross sections in the FIR and submillimeter for radii $a \gtrsim 0.05\mu m$.

6. The Gilbert equation implies that small metallic Fe, magnetite, and maghemite grains have opacities that scale approximately as $\kappa \approx v^3$ for $10\text{GHz} \lesssim v \lesssim 100\text{GHz}$ (see Figures 5 and 6).

7. We calculate the grain temperature $T$, as a function of radius, for particles of Fe, magnetite, and maghemite heated by starlight (Figure 7). Small ($a \lesssim 0.03\mu m$) Fe grains have $T \approx 40U^{0.2}\text{K}$.

8. If the Gilbert equation with $\alpha_G \approx 0.2$ applies at the $\sim 20\text{kHz}$ rotational frequencies of $\sim 0.1\mu m$ grains, then Davis–Greenstein alignment should take place on $\sim 10^3\text{yr}$ timescales, as estimated for ordinary paramagnetic dissipation (Jones & Spitzer 1967). However, it is possible that the Gilbert equation with $\alpha_G \approx 0.2$ may significantly underestimate magnetic dissipation at the rotation frequencies of grains, in which case grains with a significant fraction of the volume contributed by ferromagnetic or ferrimagnetic material may undergo rapid alignment by the Davis–Greenstein mechanism.

9. We calculate the expected polarization of the magnetic dipole emission from aligned free-flying magnetic nanoparticles. The polarization has the “normal” sense $(\mathbf{E} \perp \mathbf{H}_\text{om})$ and can be large (see Figure 9).

10. We calculate the expected polarization of the emission from a nonspherical silicate host with randomly oriented magnetic inclusions. The magnetic dipole emission becomes important at low frequencies, and can result in a reversal of the polarization direction (see Figure 10). The predicted decrease in polarization between 353 and 143 GHz may be measurable by Planck.

11. We show (Figure 12) that up to 100% of interstellar Fe could be in grains of metallic Fe, magnetite, and maghemite without exceeding the observed emission from interstellar dust at frequencies $v > 30\text{GHz}$.

12. Magnetic dipole radiation from magnetic grains will constitute an appreciable fraction of the observed 20–40 GHz anomalous microwave emission only if a large fraction of interstellar Fe is in metallic iron nanoparticles with extreme elongations (axial ratio $a/b \gtrsim 5$).

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APPENDIX A

BLOCH–BLOEMBERGEN EQUATION

Let $\mathbf{M}(t)$ be the magnetization. Equations for $d\mathbf{M}/dt$ all include a term $-\gamma \mathbf{M} \times \mathbf{H}_T$ for the torque arising from applied field and crystalline anisotropies. This torque is clearly nondissipative, and additional terms must be added to represent the effects of dissipation.

Many studies of paramagnetic resonance employ the Bloch–Bloembergen equation (Bloch 1946; Bloembergen 1950):

$$
\frac{d\mathbf{M}}{dt}_\text{BB} = \gamma \mathbf{M} \times \mathbf{H}_T - \frac{\mathbf{M}_\parallel - \mathbf{M}_\perp}{\tau_1} - \frac{\mathbf{M}_\perp}{\tau_2},
$$

(A1)

where $\mathbf{M}_\parallel \equiv (\mathbf{M} \cdot \mathbf{M}_0)/|\mathbf{M}_0|^2$, and $\mathbf{M}_\perp \equiv \mathbf{M} - \mathbf{M}_\parallel$ are the components of the magnetization parallel and perpendicular to the stationary magnetization $\mathbf{M}_0$ that would be present if the oscillating field $\mathbf{h}_0$ were set to zero and the magnetization were allowed to reach equilibrium. The time $\tau_1$ is interpreted as the spin–lattice relaxation time, the time required for the spin system to exchange heat with the lattice. The time $\tau_2$ is interpreted as the spin–spin relaxation time, the time for one spin to be perturbed by nearby spins. The Bloch–Bloembergen equation appears on the surface to be reasonable, and is frequently used. Here we show that this equation is unphysical.

Consider a weak periodic driving field of the form (10). Linearizing, we obtain

$$
-i\omega m_{0x} = \gamma m_{0y} [H_0 + H_{Kz} - D_{zz}M_0] - \gamma M_0 h_{yy} - D_{yy} m_{0y} - m_{0x} \tau_2^{-1} \quad (A2)
$$

and

$$
-i\omega m_{0y} = -\gamma m_{0x} [H_0 + H_{Kz} - D_{zz}M_0] + \gamma M_0 h_{xx} - D_{xx} m_{0x} - m_{0y} \tau_2^{-1}. \quad (A3)
$$

To simplify, assume the sample to be a spheroid, with $D_{yy} = D_{xx}$. With the definitions of $\omega_0$ and $\omega_M$ from Equations (19), (21) and

$$
q \equiv \omega \left(1 - \frac{1}{i\omega_0 \tau_2}\right) , \quad (A4)
$$

the oscillating magnetization $m_0$ obeys the tensor equation (27), but with

$$
\chi_{\pm}^\text{BB} \equiv \frac{(\omega_0 \pm q) \omega_M}{\omega_0 q^2 - \omega^2} = \frac{\omega_M}{\omega_0 \mp q}. \quad (A5)
$$

As before, the eigenvectors are the two circular polarizations (Equation (28)), with the magnetization given by Equation (29). For the two circular polarizations, the magnetization response is given by $\chi_{\pm}^\text{BB}$ and $\chi_{\mp}^\text{BB}$.

Dissipation is given by $\text{Im}(\chi)$: Straightforward algebra yields

$$
\text{Im}(\chi_{\pm}^\text{BB}) = \frac{\pm \omega_M \tau_2^{-1}}{(\omega_0 \mp \omega)^2 + \tau_2^{-2}}. \quad (A6)
$$

Thus, for anticlockwise circular polarization $\mathbf{h}_+$, the Bloch–Bloembergen equation has positive dissipation ($\text{Im}(\chi_+) > 0$). However, for the clockwise circular polarization $\mathbf{h}_-$, Equation (A6) has $\text{Im}(\chi_-) < 0$; the dissipation is negative, which is clearly unphysical. The Bloch–Bloembergen equation is frequently applied to studies of magnetic resonance. This unphysical aspect of the Bloch–Bloembergen equation has been previously noted (Lax & Button 1962; Jones & Spitzer 1967; Berger et al. 2000), but it does not appear to be widely recognized.

The Gilbert equation (13) was proposed by Gilbert (1955, 2004) as a simple way of including dissipation in the dynamics of ferromagnetic resonance. If we try to require these to have
similar response functions $\chi_{\pm}$ for some frequency $\omega$, we would have
\[ \chi_{\pm}^{G}(\omega) \approx \chi_{\pm}^{BB}(\omega). \] (A7)
Comparison of Equations (26) and (A5) implies that we would need to have
\[ \alpha_{G}(\omega) \approx \pm \tau^{-1}. \] (A8)
Clearly this cannot be satisfied simultaneously for both circular polarization modes. If we limit consideration to the $\hat{h}$ circular polarization mode, we would have $\alpha_{G}(\omega) \approx \tau^{-1}$: The assumption of constant $\alpha_{G}$ corresponds to a Bloch–Bloembergen spin–spin relaxation time $\tau_{2} \propto \omega^{-1}$. In other words, the Gilbert equation implies that for high driving frequencies $\omega$ the relaxation time is shorter than for low driving frequencies. This would presumably arise from a spectrum of relaxation processes with a broad range of characteristic timescales, with the “effective” timescale for a given driving frequency $\omega$ scaling as $1/\omega$.

As seen above, the Bloch–Bloembergen equation (A1) is unphysical, implying negative dissipation for the $\hat{h}$ circular polarization. Whether the phenomenological representation of dissipation in the Gilbert equation provides a good approximation to real materials is a question that can only be answered experimentally. We have not been able to find experimental data that provide a convincing answer.

APPENDIX B
DIELECTRIC FUNCTION AND CONDUCTIVITY OF METALLIC Fe
To calculate emission and absorption by Fe grains, we require the complex dielectric function $\epsilon(\omega)$ and complex magnetic permeability $\mu(\omega)$. Because we need to calculate heating of the grain by starlight as well as thermal emission, we require the dielectric function from the ultraviolet to microwave. We undertake to construct the complete dielectric function from microwave to X-rays.

Fe is body-centered cubic, and the dielectric function is a scalar $\epsilon = \epsilon_{1} + i\epsilon_{2}$, if magneto-optical effects are ignored. The real part $\epsilon_{1}$ will be obtained from the imaginary part $\epsilon_{2}$ using the Kramers–Kronig relation (Landau et al. 1993)
\[ \epsilon_{1}(\omega) = 1 + \frac{2}{\pi} P \int_{0}^{\infty} \frac{\epsilon_{2}(x)}{x^{2} - \omega^{2}} dx, \] (B1)
where $P$ indicates that the principal value is to be taken.
Figure 13 shows various determinations of $\epsilon_{2}$ for bulk Fe. When different determinations of $\epsilon_{2}$ in the optical and vacuum ultraviolet region disagree, we have tried to follow the data that appear to be most reliable. In the 1.24–4.75 eV range, we take the mean of the experimental results of Weaver et al. (1979) and

Figure 14. Real and imaginary components of the bound-electron contribution $\delta\epsilon_{b}^{(0)}(\omega)$ to the dielectric function of Fe.

\[ \delta\epsilon_{b}^{(0)}(\omega) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\delta\epsilon_{b}(x)}{x^{2} - \omega^{2}} dx, \]
the density functional theory (DFT) estimates of Werner et al. (2009). In the 5–26 eV range we adopt the values of Moravec et al. (1976). From 45–70.5 eV we take the reflection energy loss spectroscopy (REELS) results from Werner et al. (2009). Above 75 eV we use ε calculated by Lynch & Hunter (1991) using parameters from Henke et al. (1988).

The optical properties of bulk Fe have been measured over a wide range of frequencies (Lynch & Hunter 1991). The dielectric function can be separated into contributions from free electrons and bound electrons:

\[ \epsilon(\omega) = 1 + \delta \epsilon^{(f)}(\omega) + \delta \epsilon^{(b)}(\omega). \]  

\[ \delta \epsilon^{(f)}(\omega) = \frac{1}{\pi} \sum_{j=1}^{3} \frac{\omega_{p,j}^2 \tau_j}{\omega(1 - i \omega \tau_j)}. \]  

For free-electron components, the free-electron contribution can be approximated by a sum of three Drude free-electron components

\[ \delta \epsilon^{(f)}(\omega) = \sum_{j=1}^{3} \frac{\omega_{p,j}^2 \tau_j}{\omega(1 - i \omega \tau_j)}. \]

The free-electron contribution is taken to be large enough that \( \text{Im}(\delta \epsilon^{(b)}) \to 0 \) as \( \omega \to 0 \). The conductivity from the free-electron component is

\[ \sigma(\omega) = \frac{1}{4\pi} \omega \epsilon_2(\omega) \]

\[ = \frac{1}{4\pi} \sum_{j=1}^{3} \frac{\omega_{p,j}^2 \tau_{b,j}}{1 + (\omega \tau_j)^2}. \]

Taking \( \tau_j \) to be the bulk values \( \tau_{b,j} \) in Table 4, we obtain a zero-frequency conductivity \( \sigma(0) = 8.24 \times 10^{16} \text{ s}^{-1} \), in good agreement with the measured room temperature value \( \sigma(0) = 9.12 \times 10^{16} \text{ s}^{-1} \) (Ho et al. 1983).

The bound-electron contribution \( \delta \epsilon^{(b)}(\omega) \) (Figure 14) is taken to be the difference \( \epsilon_2(\omega) - \delta \epsilon^{(f)}(\omega) \). Because \( \epsilon_2 \) is dominated by \( \delta \epsilon^{(f)} \) at low frequencies, the low-frequency behavior of \( \delta \epsilon^{(b)} \) is unimportant. For mathematical convenience, we assume \( \delta \epsilon^{(b)} \propto \omega \) for \( h\nu \lesssim 0.3 \text{ eV} \).

\[ \delta \epsilon^{(b)} \propto \omega \] for \( h\nu \lesssim 0.3 \text{ eV} \).

### B.1. Temperature Dependence and Effects of Impurities

Fischera (2004) noted that the strong T dependence of the electrical conductivity \( \sigma(\omega) \) of bulk Fe implies strong T dependence of the absorption cross section in the FIR. The bulk electrical conductivity \( \sigma(0) \) of pure Fe increases by a factor 335 (from \( 9.2 \times 10^{16} \text{ s}^{-1} \) to \( 3.1 \times 10^{19} \text{ s}^{-1} \)) as \( T \) is reduced from 293 K to 20 K (Ho et al. 1983). Thermal contraction causes the \( \omega_{p,j}^2 \) to increase by only about 0.2%, and therefore the large increase in \( \sigma(0) \) is almost entirely due to a decrease in the scattering rates \( \tau_j^{-1} \). Pure Ni shows a similar increase in conductivity with decreasing \( T \).

The solar ratio of Fe:Ni is 95:5. If Fe grains exist in the ISM, it seems likely that they will contain appreciable amounts of Ni. Even a few % of Ni alloyed with Fe strongly limits the rise in electrical conductivity at low temperatures: 1% Ni by mass limits the \( T = 20 \text{ K} \) conductivity to \( 4.7 \times 10^{17} \text{ s}^{-1} \), and 5% Ni limits it to \( 1.1 \times 10^{17} \text{ s}^{-1} \) (Ho et al. 1983). The \( T = 20 \text{ K} \) conductivity of a Fe:Ne::95:5 alloy is only 2.4 times larger than the \( T = 293 \text{ K} \) value, and is similar to the \( T = 293 \text{ K} \) conductivity of pure Fe.

### B.2. Small Particle Effects

The Fermi velocity in Fe is \( v_F = 1.98 \times 10^8 \text{ cm} \text{s}^{-1} \) (Ashcroft & Mermin 1976). For Drude components 2 and 3 (see Table 4), the electron mean free path \( v_F \tau_j \) may be comparable to or larger than the grain size, and therefore the dielectric function must be modified to allow for electron scattering at the grain surface. To allow for this, we follow previous workers (Draine & Lee 1984; Fischera 2004) and set

\[ \tau_j^{-1} = \tau_{b,j}^{-1} + \frac{v_F}{a}. \]

This modification of the free-electron contribution to the dielectric function has no noticeable effect at optical frequencies or higher, but, because of the relatively long mean free path \( v_F \tau_{b,j} \) for free-electron components \( j = 2 \) and 3 (see Table 4), electron scattering from the grain surface can affect the far-infrared (FIR) absorption when the particle size drops below \( \sim 1 \mu \text{m} \).

### APPENDIX C

### DIELECTRIC FUNCTION AND CONDUCTIVITY OF MAGNETITE Fe₃O₄

Unfortunately, there does not appear to be a published dielectric function for magnetite extending from the far-infrared to the far-UV. In the infrared, there are often large disagreements between different studies. Here we attempt to synthesize a complete dielectric function for magnetite from microwave to X-ray frequencies.

At room temperature, magnetite is a conductor, with d.c. conductivity \( \sigma_{dc} \approx 250 \text{ ohm}^{-1} \text{ cm}^{-1} = 2.3 \times 10^{14} \text{ s}^{-1} \), but \( \sigma_{dc} \) decreases abruptly, by a factor \( \sim 10^2 \), upon cooling through the Verwey transition temperature \( T_V = 119 \text{ K} \). At lower temperatures magnetite behaves like a semiconductor. For high-quality samples, the d.c. conductivity is

\[ \sigma_{dc} \approx (3 \times 10^{16} e^{-1330 \text{K}/T} + 3 \times 10^{12} e^{-740 \text{K}/T}) \text{ s}^{-1} \] for \( 45 \text{ K} < T < T_V \)

\[ \approx 2.3 \times 10^{14} \text{ s}^{-1} \left( \frac{T - 100 \text{ K}}{160 \text{ K}} \right) \] for \( T_V < T \lesssim 260 \text{ K} \)

\[ \approx 2.3 \times 10^{14} \text{ s}^{-1} \] for \( 260 \text{ K} < T < 450 \text{ K} \)

(Miles et al. 1957; Degiorgi et al. 1987; Tsuda et al. 1991). Thus \( \sigma_{dc} \lesssim 3 \times 10^{14} \text{ s}^{-1} \) at the \( T \leq 40 \text{ K} \) temperatures of interstellar grains.

---

**Table 4**

| \( j \) | \( h\tau_j^{-1} \) | \( \hbar\omega_{p,j}^2 \tau_{b,j} \) | \( \hbar\omega_{p,j} \) | \( \tau_{b,j} \) | \( v_F \tau_{b,j} \) |
|---|---|---|---|---|---|
| 1 | 6.5 | 32 | 14.4 | 1.01 \times 10^{-16} | 2.00 \times 10^{-8} |
| 2 | 0.0165 | 600 | 3.146 | 3.99 \times 10^{-14} | 7.90 \times 10^{-6} |
| 3 | 0.010 | 50 | 0.707 | 6.59 \times 10^{-14} | 1.30 \times 10^{-5} |
The “free-electron” contribution to the dielectric function is (see Equation (B3))

$$\delta \epsilon^{(f)} = \frac{4\pi i \sigma_k}{\omega} \times \frac{1}{(1 - i\omega \tau)}$$.

(C4)

where $\tau$ is the mean-free-time between scatterings by phonons, impurities, or boundaries. Degiorgi et al. (1987) estimated $\tau_{\text{bulk}} = 3.1 \times 10^{-13}$ s at $T = 130$ K; we use this value for $T < 130$ K. For a Fermi speed $v_F \approx 10^5$ cm s$^{-1}$ this corresponds to a mean-free-path $v_F \tau_{\text{bulk}} \approx 3 \times 10^{-5}$ cm. For nanoparticles we take

$$\tau^{-1} = \tau_{\text{bulk}}^{-1} + \frac{v_F}{a}$$.

(C5)

$$\delta \epsilon^{(f)} = \frac{4\pi i \sigma_k \tau}{\omega \tau_{\text{bulk}}} \times \frac{1}{1 - i\omega \tau}$$.

(C6)

At room temperature $\sigma_k$ is large enough that $\delta \epsilon^{(f)}$ makes a significant contribution to $\epsilon(\omega)$ for $\omega < 10^{14}$ s$^{-1}$. For $T < 40$ K, $\sigma_k$ is small enough that $\delta \epsilon^{(f)}$ can be neglected for $\omega/2\pi \gtrsim 1$ GHz.

Guided by available data (shown in Figure 15), we adopt the function $\epsilon_2(\omega)$ shown in Figure 15. At optical and infrared frequencies, the dielectric function is approximated by

$$\epsilon(\omega) = 1 + \Delta \epsilon + \delta \epsilon^{(f)} + \sum_{j=1}^{13} \frac{S_j}{1 - (\omega/\omega_j)^2} - i\gamma_j/(\omega/\omega_j).$$

(C7)

where the $\omega_j$, $\gamma_j$, and $S_j$ are listed in Table 5 and $\Delta \epsilon = 0.375$ is the expected contribution to Re(\epsilon) from absorption at $h\nu \gtrsim 30$ eV. Resonance parameters for $j \leq 4$ are taken from the $T = 130$ K results of Degiorgi et al. (1987). Adding resonances $5 \leq j \leq 13$ generates a dielectric function that is consistent with a subset of the data in Figure 15.

For $h\nu > 30$ eV we estimate $\epsilon_2$ from the sum of the absorption cross sections for 3 neutral Fe atoms and 4 neutral O atoms. Our adopted $\epsilon_2(\omega)$, together with the various experimental determinations, are shown in Figure 15. We calculate the real part $\epsilon_1(\omega)$ using the Kramers–Kronig relation (Equation (B1)). The resulting $\epsilon_1$ is shown in Figure 15.

Tikhonov et al. (2010) measured the 12–145 GHz dielectric function of magnetite at $T \approx 300$ K. At 140 GHz they found $m \approx 20 + 0.1i$, or $\epsilon = m^2 \approx 400 + 4i$. With the room temperature conductivity $\sigma_k = 2.3 \times 10^{14}$ s$^{-1}$, we would expect $\text{Im}(\epsilon) = 4\pi \sigma_k / h\nu \approx 3300$ at 140 GHz—almost three orders of magnitude larger than the value found by Tikhonov et al. (2010). The results of Tikhonov et al. (2010) are inconsistent with the results of Degiorgi et al. (1987) and Pimenov et al. (2005) but the reason for the discrepancy is unclear.

### APPENDIX D

**DIELECTRIC FUNCTION OF MAGHEMITE \gamma-Fe_2O_3**

Maghemite ($\gamma$-$\text{Fe}_2\text{O}_3$) is considered to be a semiconductor. The band gap is variously estimated to be $E_g = 2.03$ eV (Cornell & Schwertmann 2003) and 2.43 eV (Chakrabarti et al. 2004). Maghemite has a spinel structure. Each unit cell contains vacancies that may or may not have long-range order. If the vacancies are located randomly, the material is said to have $Fd\bar{3}m$ symmetry. The infrared dielectric function of $Fd\bar{3}m$ $\gamma$-$\text{Fe}_2\text{O}_3$ has been determined by Pecharromán et al. (1995) from 0.015–0.15 eV reflectivity measurements. Pecharromán et al. (1995) fit the IR behavior with 5 damped oscillators.

Chakrabarti et al. (2004) have measured the absorption coefficient $\alpha(h\nu)$ in $\gamma$-$\text{Fe}_2\text{O}_3$ films over the range $1.95 < h\nu/eV < 4.95$. For $1.95 < (h\nu/eV) < 2.5$, the measured absorption coefficient greatly exceeded that expected for a semiconductor with a 2.43 eV bandgap.

We obtain $\epsilon_2$ from the measured $\alpha$:

$$\epsilon_2 = 2 \text{Re}(m) \text{Im}(m) = \frac{\text{Re}(m)\alpha\lambda_{vac}}{2\pi}.$$
where \( m \) is the complex refractive index, and \( \text{Re}(m) \approx 2.5 \) in the optical (Pecharromán et al. 1995). We assumed the electronic contribution to \( \epsilon_2 \propto h\nu \) for \( h\nu < 2 \text{ eV} \). At \( h\nu \approx 1 \text{ eV} \) this extrapolation approximately reproduces the measured absorption in amorphous Fe\(_2\)O\(_3\) films (Ozer & Tepehan 1999).

Tepper et al. (2004) have measured the optical constants of pulsed-laser-deposited iron oxide films between 0.6 and 4.1 eV. X-ray diffraction studies indicate the films consist of polycrystalline \( \gamma\)-Fe\(_2\)O\(_3\), but the measured absorption was 5–10 times stronger than in the “pure \( \gamma\)-Fe\(_2\)O\(_3\)” nanoparticles studied by Chakrabarti et al. (2004). Tepper et al. (2004) found the absorption to depend on the deposition conditions, weakening if oxygen is present, and they attribute the strong absorption to an excess of Fe\(^{2+}\) ions present in the films. Given the availability of oxygen in the ISM, we will adopt the Chakrabarti et al. (2004) results, but the Tepper et al. (2004) data indicate the large uncertainty concerning the optical absorption properties of \( \gamma\)-Fe\(_2\)O\(_3\) material.

Unfortunately, there do not appear to be published measurements of \( \gamma\)-Fe\(_2\)O\(_3\) above 5 eV. For \( E > 25 \text{ eV} \), we estimate the absorption as the sum of the photoelectric absorption by isolated Fe and O atoms, and obtain \( \epsilon_2 \) from Equation (D1) with \( \text{Re}(m) \approx 1 \). For \( h\nu > 5 \text{ eV} \), we adopt an entirely ad hoc \( \epsilon_2(E) \) that has sufficiently strong absorption in the 5–25 eV range to be consistent with the measured \( \epsilon_1 \) in the visible.

The resulting dielectric function should be reasonably accurate in the infrared and optical, but at vacuum-UV energies \( (h\nu > 5 \text{ eV}) \) should be regarded as merely illustrative of the expected strong absorption. Figure 16 shows the adopted dielectric function for \( \gamma\)-Fe\(_2\)O\(_3\).

**APPENDIX E**

**TIME-AVERAGED ORIENTATIONS FOR A SPINNING GRAIN**

Define a Cartesian coordinate system \( \hat{x}\hat{y}\hat{z} \) where \( \hat{x} \) is the direction to the observer. Let \( \Phi \) be the angle between the static magnetic field \( H_{\text{ism}} \) and the line of sight. Without loss of generality, assume that \( H_{\text{ism}} \) is in the \( \hat{x}\hat{y} \) plane:

\[
H_{\text{ism}} = H_{\text{ism}}(\hat{x}\cos\Phi + \hat{y}\sin\Phi).
\]

Consider a prolate spheroid, spontaneously magnetized along the long axis. The grain is assumed to be spinning with the long axis perpendicular to the angular momentum \( \mathbf{J} \). The angular momentum \( \mathbf{J} \) will precess around \( H_{\text{ism}} \). Let \( \psi \) be the angle between \( \mathbf{J} \) and \( H_{\text{ism}} \), and let \( \zeta \) be a precession angle:

\[
\mathbf{j} = (\hat{x}\cos\Phi + \hat{y}\sin\Phi)\cos\psi + \sin\psi[(\hat{x}\sin\Phi - \hat{y}\cos\Phi)\cos\zeta + \hat{z}\sin\zeta].
\]

Let \( \mathbf{j}_m \) be the direction of magnetization of the grain; the \( \mathbf{j}_m\hat{y}_m \) directions are perpendicular to the magnetization. Let \( \hat{x}_m \) be parallel to \( \mathbf{j} \); \( \hat{y}_m \) and \( \hat{z}_m \) will spin around \( \mathbf{j} \) with the grain’s angular velocity \( \Omega \):

\[
\hat{x}_m = (\cos\Phi\cos\psi + \sin\Phi\sin\psi\cos\zeta)\hat{x} + (\sin\Phi\cos\psi - \cos\Phi\sin\psi\cos\zeta)\hat{y} + (\sin\psi\sin\zeta)\hat{z}
\]

\[
\hat{y}_m = -\sin\Phi\sin\zeta\cos\Omega + (\sin\Phi\cos\psi\cos\zeta)
- \cos\Phi\sin\psi\sin\Omega\hat{x} + [\cos\Phi\sin\zeta\cos\Omega
- (\cos\Phi\cos\psi\cos\zeta + \sin\Phi\sin\psi)\sin\Omega]\hat{y}
+ [\cos\zeta\cos\Omega + \cos\psi\sin\zeta\sin\Omega]\hat{z}
\]

\[
\hat{z}_m = [\sin\Phi\sin\zeta\sin\Omega + (\sin\Phi\cos\psi\cos\zeta)
- \cos\Phi\sin\psi\cos\Omega]\hat{x} + [\cos\Phi\sin\zeta\sin\Omega
- (\cos\Phi\cos\psi\cos\zeta + \sin\Phi\sin\psi)\cos\Omega]\hat{y}
+ [-\cos\zeta\sin\Omega + \cos\psi\sin\zeta\cos\Omega]\hat{z}.
\]

Thus, noting that \( \langle \sin^2\zeta \rangle = \langle \cos^2\zeta \rangle = \langle \sin^2\Omega \rangle = \langle \cos^2\Omega \rangle = 1/2 \), we have

\[
\langle (\hat{y} \cdot \hat{x}_m)^2 \rangle = \sin^2\Phi \cos^2\psi + \frac{1}{2} \cos^2\Phi \sin^2\psi
\]

\[
\langle (\hat{y} \cdot \hat{y}_m)^2 \rangle = \langle (\hat{y} \cdot \hat{z}_m)^2 \rangle = \frac{1}{4} \cos^2\Phi(1 + \cos^2\psi) + \frac{1}{2} \sin^2\Phi \sin^2\psi
\]

\[
\langle (\hat{z} \cdot \hat{x}_m)^2 \rangle = \frac{1}{2} \sin^2\psi
\]

\[
\langle (\hat{z} \cdot \hat{y}_m)^2 \rangle = \langle (\hat{z} \cdot \hat{z}_m)^2 \rangle = \frac{1}{4}(1 + \cos^2\psi).
\]
APPENDIX F

EFFECTIVE MEDIUM THEORY

Laboratory measurements of absorption in magnetic nanoparticles generally study samples where the particles are dispersed in a nonmagnetic dielectric matrix. Let the matrix have scalar dielectric function \( \epsilon_{\text{mat}} \) and magnetic permeability \( \mu_{\text{mat}} = 1 \). Assume the magnetic nanoparticles to be spherical, with volume filling factor \( f_{\text{fill}} \), and to be characterized by scalar dielectric function \( \epsilon_m \) and magnetic polarizability tensor \( \alpha_m \) given by Equation (34).

We wish to approximate the composite medium by a uniform medium with an effective dielectric function \( \epsilon_{\text{eff}}(\omega) \) and effective permeability \( \mu_{\text{eff}}(\omega) \). Unfortunately, there is no exact solution to this problem, with various competing suggestions for how to estimate \( \epsilon_{\text{eff}} \) and \( \mu_{\text{eff}} \) (see the discussion in Bohren & Huffman 1983). The effective medium formulation due to Maxwell Garnett is often used, with

\[
\epsilon_{\text{eff}} = \frac{(1 - f_{\text{fill}}) \epsilon_{\text{mat}} (\epsilon_m + 2 \epsilon_{\text{mat}}) + 3 f_{\text{fill}} \epsilon_{\text{mat}} \epsilon_m}{(1 - f_{\text{fill}}) (\epsilon_m + 2 \epsilon_{\text{mat}}) + 3 f_{\text{fill}} \epsilon_{\text{mat}}}. \tag{F1}
\]

To obtain \( \mu_{\text{eff}} \) we must take into account the magnetic anisotropy of the magnetic nanoparticles. Suppose that the directions of spontaneous magnetization are randomly distributed. In the dipole limit, this is the same as if 1/6 of the particles have their static magnetization \( M_0 \) oriented in each of the \( \pm \hat{x}, \pm \hat{y}, \) and \( \pm \hat{z} \) directions. Thus we model the system as consisting of a matrix with six types of inclusions, \( j = 1 \)–6. As in Equation (29), each type of inclusion develops an internal magnetization in response to the field in the matrix \( \mathbf{h}_{\text{mat}} \):

\[
\mathbf{m}_j = \left[ \chi_{\pm} \mathbf{h}_j (\mathbf{h}_{\text{eff}} \cdot \mathbf{h}_{\text{mat}}) + \chi_{-} \mathbf{h}_j (\mathbf{h}_{\text{eff}} \cdot \mathbf{h}_{\text{mat}}) \right], \tag{F2}
\]

where

\[
\begin{align*}
\mathbf{h}_{1\pm} &= (\hat{x} \pm i \hat{y})/\sqrt{2} \quad \text{for } M_0 = M_0 \hat{z} \\
\mathbf{h}_{2\pm} &= (\hat{y} \pm i \hat{x})/\sqrt{2} \quad \text{for } M_0 = M_0 \hat{x} \\
\mathbf{h}_{3\pm} &= (\hat{z} \pm i \hat{x})/\sqrt{2} \quad \text{for } M_0 = M_0 \hat{y} \\
\mathbf{h}_{4\pm} &= (\hat{x} \mp i \hat{y})/\sqrt{2} \quad \text{for } M_0 = -M_0 \hat{z} \\
\mathbf{h}_{5\pm} &= (\hat{y} \mp i \hat{x})/\sqrt{2} \quad \text{for } M_0 = -M_0 \hat{x} \\
\mathbf{h}_{6\pm} &= (\hat{z} \mp i \hat{x})/\sqrt{2} \quad \text{for } M_0 = -M_0 \hat{y}.
\end{align*} \tag{F3}
\]

It is easily shown that

\[
\frac{1}{6} \sum_{j=1}^{6} \mathbf{m}_j = \left( \frac{\chi_+ + \chi_-}{3} \right) \mathbf{h}_{\text{mat}}. \tag{F4}
\]

The volume-averaged magnetization is

\[
\langle \mathbf{m} \rangle = \frac{f_{\text{fill}}}{6} \sum_{j=1}^{6} \mathbf{m}_j = f_{\text{fill}} \left( \frac{\chi_+ + \chi_-}{3} \right) \mathbf{h}_{\text{mat}}. \tag{F5}
\]

The oscillating field within an inclusion of type \( j \) is

\[
\mathbf{h}_j = \mathbf{h}_{\text{mat}} - D \mathbf{m}_j, \tag{F6}
\]

where \( D \) is the demagnetization tensor (see Equation (16)), with \( D = 4\pi/3 \) for spherical inclusions. The volume-averaged oscillating field is

\[
\langle \mathbf{h} \rangle = (1 - f_{\text{fill}}) \mathbf{h}_{\text{mat}} + \frac{f_{\text{fill}}}{6} \sum_{j=1}^{6} (\mathbf{h}_{\text{mat}} - D \mathbf{m}_j) \tag{F7}
\]

The effective permeability is

\[
\mu_{\text{eff}} \equiv 1 + 4\pi \left( \frac{m}{h} \right) = 1 + \frac{4\pi f_{\text{fill}} (\chi_+ + \chi_-)/3}{1 - f_{\text{fill}} D (\chi_+ + \chi_-)/3} \tag{F9}
\]

\[
= 1 + \frac{12\pi f_{\text{fill}} (\chi_+ + \chi_-)}{9 - 4\pi f_{\text{fill}} (\chi_+ + \chi_-)}, \tag{F10}
\]

where Equation (F10) is for spherical inclusions. The effective complex refractive index is \( m_{\text{eff}} = \sqrt{\epsilon_{\text{eff}} \mu_{\text{eff}}} \). A wave propagating through the material has an attenuation coefficient

\[
\alpha = \frac{2\omega}{c} \text{Im}(\sqrt{\epsilon_{\text{eff}} \mu_{\text{eff}}}). \tag{F11}
\]

APPENDIX G

KRAMERS–KRONIG RELATIONS

We wish to verify that the magnetic response \( \chi_{\pm} \) satisfies the Kramers–Kronig relation

\[
\text{Re}(\chi(\omega)) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\text{Im}(\chi(\omega'))}{\omega' - \omega} d\omega' \tag{G1}
\]

(Landau et al. 1993). For ferromagnetism, Equation (26) gives

\[
\text{Re}(\chi_\beta) = \frac{\omega_M (\omega_0 - \beta \omega)}{(\omega_0 - \beta \omega)^2 + \alpha_G^2 \omega^2}, \tag{G2}
\]

\[
\text{Im}(\chi_\beta) = -\frac{\omega_M \alpha_G \omega_0}{(\omega_0 - \beta \omega)^2 + \alpha_G^2 \omega^2}, \tag{G3}
\]

for \( \beta = \pm \). We form the complex function

\[
f(z) \equiv \frac{\alpha_G \omega_M z}{(z - \omega)(\omega_0 - \beta z)^2 + \alpha_G^2 z^2}, \tag{G4}
\]

with poles at \( z = \omega \) and \( z = \omega_0 (\beta \pm i \alpha_G)/(1 + \alpha_G^2) \). Consider a closed contour consisting of the real axis, indented at \( z = \omega_0 \), and a semicircle at infinity in the upper half plane. Because \( f \) can be expressed as the ratio of two polynomials, with the degree of the numerator two less than the degree of the denominator, Jordan’s Lemma states that the integral along the semicircle vanishes. Thus, Cauchy’s residue theorem gives the integral in terms of the residues at the pole on the real axis and the pole in the upper half-plane:

\[
\mathcal{P} \int_{-\infty}^{\infty} f(z)dz = \pi i \text{ Res}(\omega) + 2\pi i \text{ Res}\left( \frac{\omega_0 (\beta + i \alpha_G)}{1 + \alpha_G^2} \right). \tag{G5}
\]

Hence,

\[
\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\text{Im}[\chi_\beta(\omega')]}{\omega' - \omega} d\omega' \]

\[
= \frac{1}{\pi} \left( \frac{\pi i \alpha_G \omega_M \omega_0}{\omega_0 - (\beta - i \alpha_G) \omega} \right) \left[ \frac{\pi \omega_M}{\omega_0 - (\beta - i \alpha_G) \omega} \right]
\]

\[
= \frac{\omega_M (\omega_0 - \beta \omega)}{(\omega_0 - \beta \omega)^2 + \alpha_G^2 \omega^2}, \tag{G6}
\]

which verifies the relation (G1).
