Corrigendum to ”The $q$-analogue of bosons and Hall algebras” and some remarks

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1 Acknowledgements and backgrounds.

I thank Akira Masuoka very much for the following reasons. At first I am very grateful to him for his comments in [3] that the proof of [4, Proposition 3.2 (3)] is false: The argument that $0 = g(u_1) = Xg(u_2) = X$ in [4, line -3, p.4346] is wrong, since $X$ needs not to be in $B^+(\Lambda)$. In fact, the statement [4, Proposition 3.2 (3)] itself is wrong: There indeed exists an object in $O(B)$ which is not semisimple if the $q$-boson $B$ is determined by a Borcherds-Cartan (or generalized Kac-Moody) form on an infinite set (a counter example is given below), although there is only one isoclass of simple objects in $O(B)$.

Secondly, although I proved that the main statement, [4, Theorem 3.1], due to B. Sevenhant and M. Van den Bergh, can be deduced from [4, Proposition 3.2 (1)] and [4, Proposition 3.1], as Masuoka pointed out to me, [4, Theorem 3.1] can be proved much more directly using his result in [3].

Thirdly, as Masuoka pointed out to me, even the proof of [4, Proposition 3.2 (1)] can be simplified to large extent by using the natural skew-pairing on $U^- \otimes U^+$ described in [3]. Moreover, he also pointed out to me that the argument in [4, page 4345] may lead confusion between the left multiplication by $F_i \in U^+$ and the natural action by $F_i$ on $B^+(\Lambda) (= U^+)$. In deed, all formula in [4, page 4345] such that $F_i P_2 = 0 = F_i Q_2$, $F_i X_j = 0$, etc., should mean that $F_i P_2 = P'_2 F_i$, $F_i Q_2 = Q'_2 F_i$, $F_i X_j = X'_j F_i$ in $U$, etc., where $P'_2$, $Q'_2$ and $X'_j$ belong to $U^+$. For details please see Remark 1.2 below.

In the last month I’ve been trying to seek a ”correct” proof of [4, Proposition 3.2 (3)]. It is Masuoka who always finds mistakes in those arguments. I feel sorry for wasting so much time of him.

The counter example given below is motived by investigation of the semisimplicity of $O(B)$ for the case of $q$-boson $B$ determined by a Borcherds-Cartan
form on a finite set. Note that in this case the semisimplicity follows by using extremal projectors: The Kac-Moody case is due to Nakashima, while the more general case is due to Masuoka, see Remark 2.1 below. Moreover, Masuoka’s generalized extremal projectors deduces a nontrivial semisimple subcategory of $O(B)$ in the case of infinite indexed set, for details see [3, Theorem 4.4].

For self-contained purpose, we keep the the following notations. Let $\mathcal{I}$ be a countable set. A Borcherds-Cartan form on $\mathcal{I}$ is a non-degenerate $\mathbb{Q}$-valued bilinear form $(\cdot, \cdot)$ satisfying the following conditions (a)-(c):

(a) $(\cdot, \cdot)$ is symmetric;
(b) $(i, j) \leq 0$ for $i, j \in \mathcal{I}$ if $i \neq j$ and
(c) $2\frac{(i, j)}{(i, i)}$ is an integer if $(i, i)$ is positive.

The elements of $\mathcal{I}$ are called simple roots and we have a disjoint union $\mathcal{I} = \mathcal{I}^{\text{re}} \cup \mathcal{I}^{\text{im}}$ where $\mathcal{I}^{\text{re}}$ (resp. $\mathcal{I}^{\text{im}}$) contains the elements $i \in \mathcal{I}$ such that $(i, i) > 0$ (resp. $(i, i) \leq 0$). For a real root $i$, we set $a_{ij} = -2\frac{(i, j)}{(i, i)}$, and $d_i = \frac{(i, i)}{2}$, $q_i = q^{d_i}$, where $q$ is fixed to be an indeterminant.

By definition, the $q$-boson, also called Kashiwara algebra, $\mathbb{B}$ associated to the Borcherds-Cartan form on $\mathcal{I}$ is an associative algebra over $\mathbb{Q}(q)$ generated by symbols $E_i, F_i$ for $i \in \mathcal{I}$ subject to the following relations (1.1)-(1.4):

$$F_iE_j = q^{(i, j)}E_jF_i + \delta_{ij} \text{ for } i, j \in \mathcal{I};$$

$$\sum_{t=0}^{a_{ij}+1} (-1)^t \binom{a_{ij} + 1}{t} E_i^t E_j E_i^{a_{ij}+1-t} = 0 \text{ for real simple root } i,$$  

$$\sum_{t=0}^{a_{ij}+1} (-1)^t \binom{a_{ij} + 1}{t} F_i^t F_j F_i^{a_{ij}+1-t} = 0 \text{ for real simple root } i,$$  

$$E_iE_j - E_jE_i = 0, \quad F_iF_j - F_jF_i = 0 \text{ for any pair } i, j \text{ with } (i, j) = 0,$$  

where $[\cdot]_{d_i}$ is the standard notation of quantum binomials.
Remark 1.1 For a symmetrizable Kac-Moody algebra $\mathfrak{g}$, the $q$-boson $B_q(\mathfrak{g})$ is defined in [2, 3.3]. Here we adopt the “positive” version.

Following Kashiwara [3], we define $\mathcal{O}(B)$ to be the category containing left $B$-modules $M$ such that for any element $u$ of $M$ there exists an integer $l$ with $F_{i_1}F_{i_2} \cdots F_{i_l}u = 0$ for any $i_1, i_2, \ldots, i_l$ in $I$. Note that the category $\mathcal{O}(B)$ is closed under subs, quotients and extensions. Thus $\mathcal{O}(B)$ is an abelian subcategory of the category of left $B$-modules.

Let $B^+$ (resp. $B^-$) be the subalgebra of $B$ generated by $E_i$ (resp. $F_i$), $i \in I$. Since $B = B^+B^-$, due to (1.1), the Verma module $B/B^-$ is isomorphic to $B^+$ with module structure given by $F_i1 = 0$ for all $i \in I$. Then we have the following

Lemma 1.1 ([5, Proposition 3.2 (1)]). $B^+$ is a simple object of $\mathcal{O}(B)$. Moreover, $B^+$ represents the unique isoclass of simple objects in $\mathcal{O}(B)$. □

Remark 1.2 As mentioned above, this result can be obtained by using Masuoka’s result in [3]. My proof depends on [5, Lemma 3.4], and the formula

\[ F_i^a E_i^a Z = \frac{1 - q^{(a-1)(i,i)}}{1 - q^{(i,i)}} F_i^{a-1} E_i^{a-1} Z = \cdots = \prod_{t=1}^{a-1} \frac{1 - q^{t(i,i)}}{1 - q^{(i,i)}} Z \]

(Note also that $(i, i) = 0$ may appear) in [3, page 4345] should be replaced by the formula in $U$:

\[ F_i^a E_i^a Z \]

\[ = q^{(a-1)(i,i)} F_i^{a-1} E_i^{a-1} Z' F_i + (1 + q^{(i,i)} + \cdots + q^{(a-2)(i,i)}) F_i^{a-1} E_i^{a-1} Z, \quad (1.5) \]

where $F_i Z = Z' F_i$ for some $Z' \in U^+$. Since $F_i X_1 = X'_1 F_i$ for some $X'_1 \in U^+$ and $F_i u_\lambda = 0$, applying the action of $F_i^{l_j}$ to $E_i^{l_j} X_1 u_\lambda$ it follows that

\[ F_i^{l_j} E_i^{l_j} X_1 u_\lambda = (1 + q^{(i,i)} + \cdots + q^{(l_j-2)(i,i)}) F_i^{l_j-1} E_i^{l_j-1} X_1 u_\lambda, \]

and, if $b > a$ then $F_i^b E_i^a X u_\lambda = 0$, whenever $F_i X = X' F_i$. (For a more general expression see [1, (6.4)]). The remaining argument goes through and [5, Lemma 3.4] follows. □
We have the following

**Lemma 1.2** If a nonzero cyclic module $Bm \in \mathcal{O}(B)$ is simple then $Bm = B^+m$ as vector spaces. If a nonzero cyclic module $Bm$ satisfies that $F_i m = 0$ for all $i \in I$, then $Bm$ is simple and $Bm = B^+m$.

**Proof.** Assume that $Bm$ is simple. Since $Bm \in \mathcal{O}(B)$, there is a $Y \in B^-$ such that $Y m \neq 0$ but $F_i Y m = 0$ for all $j \in J$. By Lemma 1.1 it follows that $Bm = B^+Y m$, which is $\mathbb{Z}_+I$-graded. Thus there is a unique $X \in B^+$ such that $m = XY m$. If $Pm = Qm$ for some $P, Q \in B^+$, then $PXY m = QXY m$, which means that $PX = QX \in B^+$ by [5, Lemma 3.4]. Therefore $P = Q$ and hence there is an isomorphism of vector spaces $B^+m \cong B^+Y m$ induced by $m \mapsto Y m$. So $B^+m = Bm$ as required. \[\Box\]

Note that $B^+$ is $\mathbb{Z}_+I$-graded as a $Q(q)$-module:

$$B^+ = \oplus_{\alpha \in \mathbb{Z}_+I} B^+_\alpha,$$ (1.6)

where $B^+_\alpha$ is spanned by the monomials of the form $E_{i_1} \ldots E_{i_t}$ with $i_1 + \ldots + i_t = \alpha$.

## 2 Cases of finite indexed sets.

Let us recall the following

**Proposition 2.1** (Kashiwara-Masuoka-Nakashima). Assume that the indexed set $I$ is finite. Then the category $\mathcal{O}(B)$ is semisimple, that is, every nonzero object in $\mathcal{O}(B)$ is a sum of simple objects, and hence isomorphic to a sum of copies of $B^+$.

**Remark 2.1** For the $q$-boson $B_q(g)$ associated to a symmtrizable Kac-Moody algebra $g$, M. Kashiwara stated firstly that $\mathcal{O}(B_q(g))$ is semisimple in [2] without explicit proof. T. Nakashima [4] proved that there is a well defined element $\Gamma$ in some completion of $B_q(g)$, called the extremal projector, satisfying that

$$F_i \Gamma = \Gamma E_i = 0, \quad \Gamma^2 = \Gamma,$$

$$\sum_{k \geq 0} a_k \Gamma b_k = 1 \text{ for some } a_k \in B_q^+(g), \ b_k \in B_q^-(g).$$ (2.1)
Applying the action of $\Gamma$, Nakashima proved the semisimplicity of $\mathcal{O}(B_q(g))$. Masuoka generalized this construction to a more general situation, including the case of $q$-boson associated to symmetrizable Borcherds-Cartan form [3, Proposition 3.6]. These constructions generalize the rank 1 case due to Kashiwara [2] (see also [1]) in a remarkable and highly nontrivial way. □

Remark 2.2 Assume that $I$ is infinite. In [3] Masuoka considered a subcategory $\mathcal{O}'(B)$ of left $B$-modules $M$ such that

1. $M$ is an object of $\mathcal{O}(B)$.

2. For any $m \in M$, there is a finite set $F(m)$ such that $F_{i_1} \ldots F_{i_t} m = 0$ for any $i \not\in F(m)$.

(See [3, Definition 4.2]). Notations in [3] is adjusted here for brevity. Then the subcategory $\mathcal{O}'(B)$ is shown by Masuoka to be equivalent to $\text{Vec}$, which means that it is semisimple. Thus in this case Masuoka’s generalized extremal projector is crucial in my view.

3 A counter example to the case of infinite indexed sets.

Assume that $I = \{0,1,2,\ldots\}$ is infinite. For any sequence $\{a_j\}_{j \geq 1}$ with $0 \neq a_j \in \mathbb{Q}(q)$, set

$$N = B/J, \ J \text{ is the left ideal generated by } F_j - a_j F_0: j \geq 1, \ F_0^2. \quad (3.1)$$

Then $N$ becomes a left $B$-module in a natural way. Clearly $N$ is a nonzero object of $\mathcal{O}(B)$. Let $u \in N$ be the image of $1 \in B$. By definition in $N$ it holds that

$$F_j u = a_j F_0 u; \ F_r F_s u = 0, \ j \geq 1, r, s \in I. \quad (3.2)$$

Note that $N$ has a decomposition as vector spaces:

$$N = B^+ u \oplus B^+ F_0 u, \quad (3.3)$$
where $B^+F_0u$ is simple by Lemma 1.2, since $F_jF_0u = 0$ for all $j \in \mathcal{I}$.

We claim that $N$ is not semisimple. Assume contrarily that $N$ is semisimple. Then, by $\mathbb{Z}_+\mathcal{I}$-gradation there is a short exact sequence of the form

$$0 \rightarrow B^+F_0u \xrightarrow{f} N \xrightarrow{g} B^+ \rightarrow 0,$$

which must split. It follows that $N$ has a simple submodule of the form $B^+(u + QF_0u)$ for some $Q \in B^+$. By (1.1), for all $j \geq 1$ it holds that in $B$:

$$F_jQ = Q_jF_j + Q_j': \quad Q_j, Q_j' \in B^+. \tag{3.5}$$

Thus, for any $j \geq 1$, by (3.2) and (3.5) it follows that

$$0 = F_j(u + QF_0u) = F_ju + Q_jF_jF_0u + Q_j'F_0u
= F_ju + Q_j'F_0u = (a_j + Q_j')F_0u,$$

which means that $0 \neq Q_j' = -a_j \in Q(q)$ for all $j \geq 1$. But this is impossible in $B$, since $\mathcal{I}$ is infinite, there is always a $t \geq 1$ such that $E_t$ does not appear in $Q$, and hence $F_tQ = f_t(q)QF_t$ for some $f_t(q) \in Q(q)$ by (1.1), which implies that $a_t = 0$, a contradiction. Therefore $N$ is not semisimple as claimed.

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