Geometry of lightlike locus on mixed type surfaces in Lorentz-Minkowski 3-space from a contact viewpoint

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Abstract

A surface in the Lorentz-Minkowski 3-space is generally a mixed type surface, namely, it has the lightlike locus. We study local differential geometric properties of such a locus on a mixed type surface. We define a frame field along a lightlike locus, and using it, we define two lightlike ruled surfaces along a lightlike locus which can be regarded as lightlike approximations of the surface along the lightlike locus. We study a relationship of singularities of these lightlike surfaces and differential geometric properties of the lightlike locus. We also consider the intersection curve of two lightlike approximations, which gives a model curve of the lightlike locus.

1 Introduction

Let $f : U \rightarrow \mathbb{R}^3$ be a $C^\infty$ immersion or a “frontal” from an open set $U$ in $\mathbb{R}^2$ into the 3-dimensional Lorentz-Minkowski space $\mathbb{R}^3_1$. When $f$ is an immersion, spacelike, lightlike and timelike points are defined by the usual way, and it can be defined analogically for a frontal. The notion of lightlike points is an independent notion from singular points of $f$. In fact, it is a singular point of the induced metric. In this paper, we assume that the set of the lightlike points $L(f)$ of $f$ is a curve, and the lightlike locus $f|_{L(f)}$ is a spacelike regular curve. Under this assumption, we define a moving frame field along $L(f)$. The frame consists of the tangent vector of $f|_{L(f)}$ and the two lightlike vectors $L, N$. The
vector $L$ is tangent to the surface at $f|_{L(f)}$, and $N$ is normal to the surface in the Euclidean sense. Using it, we construct two special lightlike ruled surfaces whose ruling directions are $L$ and $N$. These surfaces can be regarded as lightlike approximations of $f$ along $L(f)$. We give conditions that singularities of these surfaces are cuspidal edges and swallowtails in terms of certain geometric properties of $f$. This kind of study is firstly given in [19] for an immersion in $\mathbb{R}^3$ with a given curve on it. In our case, different from the Euclidean case, rulings are lightlike lines. If singularities of a lightlike approximation is a constant point, then it is a lightcone. On the other hand, geometry and singularities of developable surface defined by a moving frame as a curve in $\mathbb{R}^3$ is studied in [16]. Our moving frame is associated to the immersion or frontal, geometric meanings are related to the properties of the curve as a curve on the original frontal $f$. Furthermore, we consider the pair of contacts of two lightlike approximations with the lightlike locus. Then the intersection curve of this pair can be considered as the model curve of the lightlike locus. We introduce a new notion of the contact orders of pairs. In our case, the contact orders of pairs of lightlike approximations characterize the singularities of the lightlike approximations and the pedals of the lightlike locus.

2 Preliminaries

2.1 Frontals in $\mathbb{R}^3_1$

Let $\mathbb{R}^3_1$ be the Lorentz-Minkowski 3-space equipped with the scalar product $\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2$, where $x = (x_0, x_1, x_2)$, $y = (y_0, y_1, y_2)$. Let $\text{Gr}(2, 3)$ be the Grassmanian of 2-planes in $\mathbb{R}^3_1$ and consider a subbundle $\mathbb{R}^3_1 \times \text{Gr}(2, 3) \subset \mathbb{R}^3_1 \times \mathbb{R}^3 = T\mathbb{R}^3_1$. It can be identified with the projective tangent bundle $PT\mathbb{R}^3_1$ via $PT_q \mathbb{R}^3_1 \ni v_q \mapsto (V_q)\perp$ by the scalar product $\langle \cdot, \cdot \rangle$, where $x^\perp = \{ y \in \mathbb{R}^3_1 | \langle x, y \rangle = 0 \}$. It is well known that $PT\mathbb{R}^3_1 = \mathbb{R}^3_1 \times P\mathbb{R}^3$ is a contact manifold. Let $U$ be a domain in $\mathbb{R}^2$. A map $f : U \rightarrow \mathbb{R}^3_1$ is a frontal if there exists a map $F : U \rightarrow \mathbb{R}^3_1 \times P\mathbb{R}^3$ of the form $F = (f, [\nu])$ such that $F$ is an isotropic map, namely, $\langle df(X), \nu \rangle = 0$ for any $X \in T_u U$ and $p \in U$. Such an $F$ is called an isotropic lift of $f$. Identifying $PT\mathbb{R}^3_1$ with $\mathbb{R}^3_1 \times \text{Gr}(2, 3)$, $F = (f, [\nu])$ can be identified with $(f, \nu^\perp)$, where $\nu^\perp : U \rightarrow \text{Gr}(2, 3)$ be a 2-dimensional subspace-valued map. We call $\nu^\perp(p)$ the limiting tangent plane at $p \in U$. A frontal is a front if the isotropic lift $F : U \rightarrow \mathbb{R}^3_1 \times P\mathbb{R}^3$ is an immersion. Let $f : U \rightarrow \mathbb{R}^3_1$ be an immersion, and let $(u, v)$ be a coordinate system on $U$. We set $[\nu] = [f_u \times f_v]$, where

$$
\begin{pmatrix}
-e_0 \\
e_1 \\
e_2
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \det
\begin{pmatrix}
-e_0 \\
e_1 \\
e_2
\end{pmatrix}
\cdot
\begin{pmatrix}
e_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{pmatrix}
$$

is the cross product. Then $f$ is a frontal. This $[\nu]$ is called a lightcone Gauss map of $f$ [4,29]. Thus, an immersion is a frontal. On the other hand, a frontal may have singular points. A singular point of a frontal germ $f$ at $p$ is a cuspidal edge if it is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^2, v^3)$ at the origin, where two map germs $f_i : (\mathbb{R}^2, p_i) \rightarrow (\mathbb{R}^2, q_i)$
(i = 1, 2) are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\phi : (\mathbb{R}^2, p_1) \to (\mathbb{R}^2, p_2)$ and $\Phi : (\mathbb{R}_1^3, q_1) \to (\mathbb{R}_1^3, q_2)$ such that $\Phi \circ f \circ \phi^{-1} = g$ holds. Cuspidal edges are the most fundamental singularities appear on fronts. The generic singularities of fronts are cuspidal edges and swallowtails. A singular point of a frontal germ $f$ at $p$ is a swallowtail if it is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, 4v^3+2uv, 4v^4+uv^2)$ at the origin. We denote by $S(f)$ the set of singular points. In this decade, differential geometric properties of frontals in $\mathbb{R}^3$ and Riemannian 3-manifold are investigated by many authors (see [20, 27, 32] for example).

On the other hand, points on frontals in $\mathbb{R}_1^3$ can be classified into the following three cases. A non-zero vector $\mathbf{x} \in \mathbb{R}_1^3$ is said to be spacelike (respectively, timelike, lightlike), if $\langle \mathbf{x}, \mathbf{x} \rangle$ is positive (respectively, negative, zero). We denote by $LC^*$ the set of lightlike vectors. A plane $P = \mathbf{x}^\perp \subset \mathbb{R}_1^3$ is said to be spacelike (respectively, timelike, lightlike), if $\mathbf{x}$ is timelike, (respectively, spacelike, lightlike). Let $f$ be a frontal and $(f, [v])$ its isotropic lift. A point $p \in U$ is said to be spacelike (respectively, timelike, lightlike) point, if $v^+(p)$ is spacelike (respectively, timelike, lightlike). We denote by $U_+$ (respectively, $U_-$, $L(f)$) the set of spacelike (respectively, timelike, lightlike) points. If $U_+, U_-$ and $L(f)$ are all non-empty, then $f$ is said to be mixed type.

There are many studies on differential geometric property of spacelike regular surfaces or spacelike frontals in $\mathbb{R}_1^3$ (see [8, 10, 13, 15, 35] for example), and surfaces whose lightlike point set $L(f)$ is non-empty (see [11, 14, 17, 21, 24, 26, 29, 31, 34, 36] for example). However, local behavior of a lightlike point are investigated only on frontals with special curvature properties.

### 2.2 Criteria for singularities of fronts in $\mathbb{R}_1^3$

To state criteria for singularities of fronts in $\mathbb{R}_1^3$, we firstly recall criteria for singularities of fronts in $\mathbb{R}^3$ given in [24]. Let $f : U \to \mathbb{R}^3$ be a frontal and $F = (f, [\nu_E]) : U \to PT\mathbb{R}^3 = \mathbb{R}^3 \times P\mathbb{R}^3$ its isotropic lift, namely, $\nu_E$ satisfies $df_p(X) \cdot \nu_E(p) = 0$ for any $p \in U$ and $X \in T_pU$, where the dot “.$$" stands for the Euclidean inner product. We set $\lambda_E = \det(f_u, f_v, \nu_E)$. Then $S(f) = \lambda_E^{-1}(0)$ holds. A function $\Lambda_E$ is called an identifier of singularities if it is a non-zero functional multiple of $\lambda_E$. Let $\Lambda_E$ be an identifier of singularities. Let $p \in U$ be a singular point satisfying rank $df_p = 1$. Then there exists a non-zero vector field $\eta$ on a neighborhood of $p$ such that the kernel of $df_\eta$ is generated by $\eta(q)$ for any $q \in S(f)$. We call it a null vector field. The following fact holds.

**Fact 2.1.** ([33 Corollary 2.5]) Let $f : U \to \mathbb{R}^3$ be a frontal, and let $p \in U$ be a singular point satisfying rank $df_p = 1$. Under the above notation,

1. $f$ at $p$ is a cuspidal edge if and only if $f$ is a front at $p$, and $\eta \Lambda_E(p) \neq 0$.
2. $f$ at $p$ is a swallowtail if and only if $f$ is a front at $p$, and $\eta \Lambda_E(p) = 0$, $\eta \eta \Lambda_E(p) \neq 0$, $d(\Lambda_E)_p \neq 0$.

A similar criteria hold for frontals in $\mathbb{R}_1^3$ by a slight modification. Let $f : U \to \mathbb{R}_1^3$ be a frontal and $F = (f, [\nu])$ its isotropic lift, where $[\nu] \in P\mathbb{R}_1^3$, namely, $\nu$ satisfies $\langle df_p(X), \nu(p) \rangle = 0$ for any $p \in U$ and $X \in T_pU$. Taking a vector field $\mathbf{T}$ along $f$ such
that $T$ is transverse to $\nu^\perp$, we set $\Lambda = \det(f_u, f_v, T)$. Then we see that $\Lambda$ is a non-zero functional multiple of $\lambda_E$. Thus $\Lambda$ is an identifier of singularities. By using $\Lambda$, we can recognize whether $f$ at $p$ is a cuspidal edge or a swallowtail.

2.3 Discriminant sets of functions

The lightlike approximations are envelopes of lightlike planes along the lightlike locus. To construct lightlike approximation, we use the theory of unfolding of a function and discriminant set which can describe the envelopes.

Let $a : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ be a function. For a manifold $X$ and $p \in X$, a function $A : (\mathbb{R} \times X, (0, p)) \rightarrow (\mathbb{R}, 0)$ is called an unfolding of $a$ if $A(u, p) = a(u)$ holds. In this setting, we regard $A$ as a parameter family of a function $a$. We assume that $a'(0) = 0$ (‘$= \partial/\partial u$’) and define the set $\Sigma_A$ and the discriminant set $D_A$ of $A$ as

$$
\Sigma_A = \{(u, q) \in \mathbb{R} \times X \mid A(u, q) = A_u(u, q) = 0\},
D_A = \{q \in X \mid \text{there exists } u \in \mathbb{R} \text{ such that } A(u, q) = A_u(u, q) = 0\}.
$$

If the map $(A, A_u)$ is a submersion at $(0, p)$, then $\Sigma_A$ is a manifold. By definition, the discriminant set is the envelope of the family $\{q \in X | A(u, q) = 0\}$. See [3, Section 7] or [18, Section 5] for the general theory of unfoldings and their discriminant sets.

3 Lightlike surfaces

3.1 Frame along lightlike locus

Let $f : U \rightarrow \mathbb{R}^3_1$ be a frontal whose lightlike point set $L(f)$ is non-empty, and let $F = (f, [\nu])$ be its isotropic lift. We take $p \in L(f)$, and assume that

$$
L(f) \text{ is a regular curve in } U \text{ which is parametrized by } \gamma : (-\varepsilon, \varepsilon) \rightarrow U \text{ near } p = \gamma(0).
$$

Under this assumption, $f(L(f))$ is called the lightlike locus. We set $\hat{\gamma} = f \circ \gamma$. We assume that

$$
\text{if } p \in S(f), \text{ then } \gamma'(0) \not\in \ker df_p,
$$

where $'=d/du$. This implies that $\hat{\gamma}$ is a regular curve in $\mathbb{R}^3_1$. Furthermore, we also assume that

$$
\hat{\gamma}'(u) \text{ is not lightlike}.
$$

By these assumptions, $\hat{\gamma}$ is a spacelike regular curve in $\mathbb{R}^3_1$. A frontal which satisfies the assumptions (3.1), (3.2) and (3.3) are said to be an admissible frontal.

A frame along a spacelike regular curve in $\mathbb{R}^3_1$ is obtained in [10]. Here we consider a frame along $L(f)$ as a curve on the surface $f$. Then we can take a parameter $u$ such that $|\hat{\gamma}'(u)| = 1$, where $|x| = \sqrt{|\langle x, x \rangle|}$. Setting $e(u) = \hat{\gamma}'(u)$, we have a frame $\{e, L, N\}$ along $\hat{\gamma}(u)$ satisfying

$$
\langle e, e \rangle = 1, \quad \langle e, L \rangle = 0, \quad \langle e, N \rangle = 0, \quad \langle L, L \rangle = 0, \quad \langle L, N \rangle = 1, \quad \langle N, N \rangle = 0,
$$

(3.4)
where
\[ l = \frac{\partial \alpha}{\partial u} = \frac{\partial \beta}{\partial u} \quad \text{and} \quad \alpha_{L}(u) = \langle \gamma''(u), L(u) \rangle, \quad \alpha_{N}(u) = \langle \gamma''(u), N(u) \rangle, \quad \alpha_{G}(u) = \langle L(u), N'(u) \rangle. \quad (3.5) \]

These three functions are determined by \( f \) and \( L \). We set \( \overline{L}(u) = \psi(u)L(u) \), where \( \psi \)
is a never vanishing function. Then setting \( \overline{N} = N/\psi \), the frame \( \{e, \overline{L}, \overline{N}\} \) satisfies the condition \( (3.4) \), and it holds that
\[ \overline{\alpha}_{L} = \psi \alpha_{L}, \quad \overline{\alpha}_{N} = \alpha_{N}/\psi, \quad \overline{\alpha}_{G} = \alpha_{G} + \psi (\psi^{-1})' \]
where \( \overline{\alpha}_{L}, \overline{\alpha}_{N}, \overline{\alpha}_{G} \) are defined by \( (3.5) \) with respect to the frame \( \{e, \overline{L}, \overline{N}\} \).

Let \( f : U \to \mathbb{R}^{3} \) be an immersion with non-empty lightlike point set \( L(f) \). Suppose that \( L(f) \) consists of lightlike points of the first kind (see \cite{12} Definition 2.2) for details). Then \( f \) is a mixed type surface, is admissible in the above sense, and \( \alpha_{L}, \alpha_{N}, \alpha_{G} \) can be regarded as invariants of \( f \) as follows: There exists a vector field \( l \) on \( U \) such that \( df(l)(q) = L(q) \) for any \( q \in L(f) \) and \( \beta = l \langle df(l), df(l) \rangle_{L(f)} \) does not vanish along \( L(f) \). Let \( \kappa_{L}, \kappa_{N}, \) and \( \kappa_{G} \) be the lightlike singular curvature, the lightlike normal curvature, and the lightlike geodesic torsion of \( f \) along \( L(f) \), respectively \( (\cite{12} Definition 3.2) \). By \( (12) \) Proposition 3.5, it holds that
\[ \alpha_{L} = \beta^{1/3} \kappa_{L}, \quad \alpha_{N} = \beta^{-1/3} \kappa_{N}, \quad \alpha_{G} = \kappa_{G} + \beta^{1/3}(\beta^{-1/3})'. \quad (3.7) \]

One can take \( l \) satisfying \( \beta = 1 \), by rechoosing \( \beta^{-1/3} l \) instead of \( l \). Then \( \alpha_{L} = \kappa_{L}, \alpha_{N} = \kappa_{N}, \alpha_{G} = \kappa_{G} \) hold.

### 3.2 Osculating and transversal lightlike surfaces

Let \( f : U \to \mathbb{R}^{3} \) be an admissible frontal. Under the notation in Section 3.1 we consider the discriminant sets of the following functions
\[ H_{L}(u, x) = \langle x - \gamma(u), L(u) \rangle : L(f) \times \mathbb{R}^{3} \to \mathbb{R}, \]
\[ H_{N}(u, x) = \langle x - \gamma(u), N(u) \rangle : L(f) \times \mathbb{R}^{3} \to \mathbb{R}, \]
\[ G(u, x) = \langle x - \gamma(u), x - \gamma(u) \rangle : L(f) \times \mathbb{R}^{3} \to \mathbb{R}, \quad (3.8) \]
\[ \tilde{H}(u, \tilde{v}, r) = \langle \gamma(u), \tilde{v} \rangle - r : L(f) \times S_{+}^{1} \times \mathbb{R}^{3} \to \mathbb{R}, \quad (3.9) \]
where \( S_{+}^{1} = \{(x_{1}, x_{2}) | (x_{1})^{2} + (x_{2})^{2} = 1 \} \) and \( \mathbb{R}^{3} = \mathbb{R} \setminus \{0\} \). Here we regard \( (X, p) = (\mathbb{R}^{3}, p) \) for \( p \in \mathbb{R}^{3} \) and \( A = H_{L}, H_{N}, G, H \) under the notation in Section 2.3. Since \( L \)

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*These three invariants \( \kappa_{L}, \kappa_{N}, \kappa_{G} \) are introduced in \cite{12} to investigate the behavior of the Gaussian curvature of mixed type surfaces at lightlike points, cf. \cite{12} Theorem 2.3.*
and $\mathcal{N}$ are lightlike, the discriminant set $\mathcal{D}_{\mathcal{H}_L}$ is the envelope of families of lightlike planes which are tangent to the frontal at $\hat{\gamma}(u)$, and $\mathcal{D}_{\mathcal{H}_N}$ is the envelope of families of transversal lightlike planes of $f$ at $\hat{\gamma}(u)$. We calculate the discriminant set of $H_L$ (respectively, $H_N$) under the assumption $\alpha_L \neq 0$ (respectively, $\alpha_N \neq 0$). Let us assume $\alpha_L \neq 0$ (respectively, $\alpha_N \neq 0$) for any $u \in I$. Since

$$H'_L = \langle x - \hat{\gamma}(u), -\alpha_L e - \alpha_G L \rangle$$  \quad \text{and} \quad $$H'_N = \langle x - \hat{\gamma}(u), -\alpha_N e + \alpha_G N \rangle,$$

the discriminant sets $\mathcal{D}_{\mathcal{H}_L}, \mathcal{D}_{\mathcal{H}_N}$ can be parametrized by

$$f_L(u, v) = \hat{\gamma}(u) + vL(u)$$  \quad \text{and} \quad $$f_N(u, v) = \hat{\gamma}(u) + vN(u),$$

respectively, where $' = \partial/\partial u$. Since $\langle f'_L, L \rangle = \langle (f'_L)_v, L \rangle = 0$ and $\langle f'_N, N \rangle = \langle (f'_N)_v, N \rangle = 0$ hold, $L$ and $N$ are lightcone Gauss map of $f_L$ and $f_N$ respectively even on the set of singular points. Since each lightcone Gauss map degenerates to a curve, $f_L$ and $f_N$ have both zero Gaussian curvature in the Euclidean sense ([14] Theorem 3.1). Moreover, the limiting tangent plane of $f_L$ coincides with that of $f$ along $L(f)$, and the limiting tangent plane of $f_N$ is the transversal lightlike plane of $f$ along $L(f)$. In this sense, we call $f_L(u, v)$ (respectively, $f_N(u, v)$) the osculating lightlike surface (respectively, transversal lightlike surface) of $f$ along $L(f)$. Now, let us investigate $f_L$ (respectively, $f_N$), without assuming $\alpha_L \neq 0$ (respectively, $\alpha_N \neq 0$), since such the assumptions are not necessary for the definition of $f_L$ and $f_N$. After obtaining these surfaces, it does not necessary the condition $\alpha_L \neq 0$ (respectively, $\alpha_N \neq 0$). We investigate these surfaces without the condition $\alpha_L \neq 0$ (respectively, $\alpha_N \neq 0$).

By a similar calculation, we see $\mathcal{D}_G = \mathcal{D}_{\mathcal{H}_L} \cup \mathcal{D}_{\mathcal{H}_N}$. On the other hand, since $\tilde{H}' = \langle \gamma, \tilde{v} \rangle$ holds, $\tilde{H}' = 0$ implies $\tilde{v} = aL + bN$. Since $\tilde{v}$ is lightlike, $ab = 0$. Moreover, $\tilde{v} \in S^1_L$, we have $\tilde{v} = \tilde{L}$ or $\tilde{v} = \tilde{N}$. Thus

$$\mathcal{D}_{\tilde{H}} = \left\{ \left( \hat{\gamma}(u), L(u), \tilde{L}(u) \right) \right\} \cup \left\{ \left( \tilde{N}(u), \tilde{\gamma}(u), \tilde{N}(u) \right) \right\} \left| u \in L(f) \right\}. $$

Let $\Phi : S_L^1 \times \mathbb{R} \times \rightarrow \mathcal{LC}^*$ be the diffeomorphism $\Phi(\tilde{v}, r) = r\tilde{v}$. We set

$$LP_L(u) = \langle \hat{\gamma}(u), \tilde{L}(u) \rangle \tilde{L}(u)$$  \quad \text{and} \quad $$LP_N(u) = \langle \hat{\gamma}(u), \tilde{N}(u) \rangle \tilde{N}(u).$$

The curve $LP_L$ (respectively, $LP_N$) is called the osculating lightcone pedal (respectively, transversal lightcone pedal) of $f$ (cf. [10]). The union of the images of oscillating lightcone pedal and transversal lightcone pedal coincides with $\Phi(\mathcal{D}_{\tilde{H}})$. We also consider discriminant sets $\mathcal{D}_G$ and $\mathcal{D}_{\tilde{H}}$ in the context of bi-contact of lightcones in Section 3.3

### 3.3 Singularities of osculating and transversal lightlike surfaces

We set two functions

$$\sigma_L(u) = \alpha'_L(u) + \alpha_L(u)\alpha_G(u)$$  \quad \text{and} \quad $$\sigma_N(u) = \alpha'_N(u) - \alpha_N(u)\alpha_G(u).$$

We have the following:
Theorem 3.1. Let \( f : U \rightarrow \mathbb{R}^3 \) be an admissible frontal. Then \( f_L \) and \( f_N \) are front. A point \((u, v)\) is a singular point of \( f_L \) (respectively, \( f_N \)) if and only if \( 1 - v\alpha_L = 0 \) (respectively, \( 1 - v\alpha_N = 0 \)). A singular point \((u, 1/\alpha_L)\) of \( f_L \) (respectively, \((u, 1/\alpha_N)\) of \( f_N \)) is

1. a cuspidal edge if and only if \( \sigma_L \neq 0 \) (respectively, \( \sigma_N \neq 0 \)) at \( u \).
2. a swallowtail if and only if \( \sigma_L = 0 \) and \( \sigma'_L \neq 0 \) (respectively, \( \sigma_N = 0 \) and \( \sigma'_N \neq 0 \)) at \( u \).

We remark that although \( \alpha_L, \alpha_N, \alpha_G \) do depend on the choice of \( L \), the positions of singular points of \( f_L \) and \( f_N \) do not depend on it, and the conditions (1) and (2) do not depend on it (cf. (3.6)). In fact, if \( L = \psi L \), then \( N = N/\psi \), and (3.6) holds. Moreover, by (3.6), we have

\[
\alpha_L + \alpha_L \alpha_G = \psi(\alpha'_L + \alpha_L \alpha_G), \quad \alpha_N - \alpha_N \alpha_G = \psi^{-1}(\alpha'_N - \alpha_N \alpha_G). \tag{3.10}
\]

Proof. We set \( g(u, v) = f_L(u, v) \) for simplicity. The isotropic lift of \( f_L \) is \( (f_L, [L]) \). Since \( g' = (1 - v\alpha_L)e - v\alpha_G L \), and \( g_v = L \), setting \( \Lambda = 1 - v\alpha_L \), \( \Lambda \) can be taken as an identifier of singularities. A null vector field \( \eta \) is \( \eta = \partial_u + v\alpha_G \partial_v \). Since \( \eta[L] \neq 0 \) if and only if \( L \) and \( \eta L \) are linearly independent, \( g \) at a singular point is a front if and only if \( \alpha_L \neq 0 \). If \( \alpha_L(u_0) = 0 \), then \((u_0, v)\) is not a singular point. Thus \( g \) is a front at any singular point. We have \( \eta \Lambda = -v\sigma_L \) and \( \eta \Lambda = -\sigma'_L \). Thus the assertion for \( g \) holds. The assertion for \( f_N \) can be shown by the same way taking an isotropic lift \( (f_N, [N]) \), an identifier of singularities \( \Lambda = 1 - v\alpha_N \) and a null vector field \( \eta = \partial_u - v\alpha_G \partial_v \).

If \( f \) is an immersion and \( L(f) \) consists of lightlike points of the first kind, then the conditions in Theorem 3.1 can be stated in terms of the curvatures \( \kappa_L, \kappa_N, \kappa_G \) as follows.

Corollary 3.2. Let \( f : U \rightarrow \mathbb{R}^3 \) be an immersion with non-empty lightlike point set \( L(f) \). Suppose that \( L(f) \) consists of lightlike points of the first kind. Then \( f_L \) and \( f_N \) are front. A point \((u, v)\) is a singular point of \( f_L \) (respectively, \( f_N \)) if and only if \( 1 - v\kappa_L = 0 \) (respectively, \( 1 - v\kappa_N = 0 \)). A singular point \((u, 1/\kappa_L)\) of \( f_L \) (respectively, \((u, 1/\kappa_N)\) of \( f_N \)) is

1. a cuspidal edge if and only if \( \bar{\sigma}_L \neq 0 \) (respectively, \( \bar{\sigma}_N \neq 0 \)) at \( u \).
2. a swallowtail if and only if \( \bar{\sigma}_L = 0 \) and \( \bar{\sigma}'_L \neq 0 \) (respectively, \( \bar{\sigma}_N = 0 \) and \( \bar{\sigma}'_N \neq 0 \)) at \( u \).

Here,

\[
\bar{\sigma}_L(u) = \kappa'_L(u) + \kappa_L(u)\kappa_G(u), \quad \text{and} \quad \bar{\sigma}_N(u) = \kappa'_N(u) - \kappa_N(u)\kappa_G(u).
\]

The functions \( \sigma_L \) and \( \sigma_N \) correspond to the invariants \( k' \neq k\tau \) in [16] which play an important role in their paper. In [16], they use the Frenet-Serret type frame, this is an invariant of the curve in \( \mathbb{R}_1^3 \).

We can state the condition of lightcone pedal curves in terms of \( \alpha_L \) and \( \alpha_N \).
Theorem 3.3. Under the same setting in Theorem 3.1, the point \( \gamma(0) \) is a singular point of the osculating lightcone pedal \( LP_L \) (respectively, the transversal lightcone pedal \( LP_N \)) if and only if \( \alpha_L(0) = 0 \) (respectively, \( \alpha_N(0) = 0 \)). Moreover, \( LP_L \) (respectively, \( LP_N \)) has a cusp at \( \gamma(0) \) if and only if \( \alpha_L(0) = 0, \alpha'_L(0) \neq 0 \) (respectively, \( \alpha_N(0) = 0, \alpha'_N(0) \neq 0 \)).

Like as we remarked just after Theorem 3.1 the conditions in this theorem do not depend on the choice of \( L \).

Proof. We see the function \( \tilde{h}(u) = \tilde{H}(u, \tilde{v}_0, r_0) \), where \( r_0 = \langle \gamma(0), \tilde{L}(0) \rangle \), \( \tilde{v}_0 = \tilde{L}(0) \) satisfies

\[
\tilde{h}'(0) = \langle e, \tilde{v}_0 \rangle = 0,
\tilde{h}''(0) = \langle \alpha_N L + \alpha_L N, \tilde{v}_0 \rangle = \alpha_L(0)/L_0(0) \quad (L = (L_0, L_1, L_2)),
\tilde{h}'''(0) = -2\alpha_L\alpha_N e - \sigma_N L + \sigma_L N, \tilde{v}_0 \rangle = \sigma_L(0)/L_0(0).
\]

Taking a parametrization \( \theta \) of \( S^1_+ \) by \( \tilde{v} = (1, \cos \theta, \sin \theta) \), then \( \tilde{H}_\theta(u, \tilde{v}, r) \) is the first component of \( e \times \tilde{v} \). Then we see \( \tilde{h}''(0) = 0 \) if and only if \( \alpha_L(0) = 0 \). If \( \tilde{h}''(0) = 0 \), then \( \tilde{h}'''(0) = 0 \) if and only if \( \alpha'_L(0) = 0 \). Furthermore, if \( \tilde{h}''(0) = 0 \) and \( \tilde{h}'''(u) \neq 0 \), then \( \tilde{H}(u, \tilde{v}, r) \) is a versal unfolding of \( \tilde{h} \). In fact, we assume the first component of \( e \times \tilde{v} \) is zero, then \( e \times \tilde{v} \) is lightlike, \( e_0^2 + e_1^2 + e_2^2 = 0 \) for \( e = (e_0, e_1, e_2) \).

On the other hand, since \( \langle e, \tilde{L} \rangle = 0 \), we have \( -e_0 + e_1 \cos \theta + e_2 \sin \theta = 0 \). This implies that \( -e_0^2 + e_1^2 + e_2^2 = 0 \), a contradiction. By the well-known fact of the versal unfolding of a function and its discriminant set (see \([3]\) for example), if \( \alpha_L \neq 0 \) (respectively, \( \alpha_L = 0, \alpha'_L \neq 0 \)) at 0 then \( D_{\tilde{H}} \) is locally diffeomorphic to a regular curve (respectively, a 3/2-cusp) at \((\tilde{N}(0), \langle \gamma(0), \tilde{N}(0) \rangle)\). By the diffeomorphism \( \Phi : S^1_+ \times \mathbb{R}^\times \to LC^* \), the discriminant set \( D_{\tilde{H}} \) is sent to the union of the images of \( LP_L(u) \) and \( LP_N(u) \). Since the diffeomorphism of the images implies the \( \mathcal{A} \)-equivalence and \( \tilde{L} \) and \( \tilde{N} \) are linearly independent, we have the assertion. \( \square \)

Like as Corollary 3.2 if \( f \) is an immersion and \( L(f) \) consists of lightlike points of the first kind, then the conditions in Theorem 3.3 can be stated in terms of \( \kappa_L, \kappa_N, \kappa_G \) as follows.

Corollary 3.4. Under the same setting in Corollary 3.2, the point \( \gamma(0) \) is a singular point of the osculating lightcone pedal \( LP_L \) (respectively, the transversal lightcone pedal \( LP_N \)) if and only if \( \kappa_L(0) = 0 \) (respectively, \( \kappa_N(0) = 0 \)). Moreover, \( LP_L \) (respectively, \( LP_N \)) has a cusp at \( \gamma(0) \) if and only if \( \kappa_L(0) = 0, \kappa'_L(0) \neq 0 \) (respectively, \( \kappa_N(0) = 0, \kappa'_N(0) \neq 0 \)).

3.4 Contact of \( L(f) \) with intersection curves

In this section, we study a lightlike locus by considering contact of intersection curves of two model surfaces defined by the lightlike locus with the frame defined in Section 3.1. Let \( f : U \to \mathbb{R}^3 \) be an admissible frontal. Regarding the discriminants of \( G, \tilde{H} \) in
under the notation in Section 3.1, we consider the following functions

\[
G_L(x) = \langle x - x_L, x - x_L \rangle, \quad G_N(x) = \langle x - x_N, x - x_N \rangle,
\]

\[
\tilde{H}_L(x) = \langle x - \tilde{\gamma}(0), \tilde{L}(0) \rangle, \quad \tilde{H}_N(x) = \langle x - \tilde{\gamma}(0), \tilde{N}(0) \rangle
\]

from \(\mathbb{R}^3\) into \(\mathbb{R}\), where \(x_L = \tilde{\gamma}(0) + \frac{L(0)}{\alpha_L(0)}, x_N = \tilde{\gamma}(0) + \frac{N(0)}{\alpha_N(0)}\) and \(\tilde{v} = v/v_0\) for \(v = (v_0, v_1, v_2)\). We also remark here that \(x_L\) and \(x_N\) do not depend on the choice of \(L\) (cf. (3.6)). Then \(G_L^{-1}(0)\) (respectively, \(G_N^{-1}(0)\)) is the lightcone with the vertex \(x_L\) (respectively, \(x_N\)), and \(H_L^{-1}(0)\) (respectively, \(H_N^{-1}(0)\)) is the lightlike plane with the lightlike normal vector \(L(0)\) (respectively, \(N(0)\)) passing through \(\tilde{\gamma}(0)\). We call \(G_L^{-1}(0)\), (respectively, \(G_N^{-1}(0)\), \(H_L^{-1}(0)\), \(H_N^{-1}(0)\)) the osculating contact lightcone (respectively, transversal contact lightcone, osculating contact lightlike plane transversal contact lightlike plane) at \(p\). We consider the intersections of two of them. The intersection of the osculating contact lightcone and the transversal contact lightcone, is a spacelike ellipse tangent to \(L(f)\) at \(p\), which is called an osculating ellipse. The intersection of the osculating contact lightlike plane and the transversal contact lightlike plane, is a spacelike line tangent to \(L(f)\) at \(p\), which is the tangent line. The intersection of the osculating contact lightcone and the transversal contact lightlike plane, (respectively, the osculating contact lightlike plane and the transversal contact lightcone) is a spacelike parabola tangent to \(L(f)\) at \(p\), which is called an \(N\)-osculating (respectively, an \(L\)-osculating) parabola. Since these curves are intersections of fundamental objects in the Lorentz-Minkowski 3-space, they can be regarded as model curves of a lightlike locus of a surface. In fact, we can interpret that the spacelike ellipse represents how \(L(f)\) looks round and the line represents how \(L(f)\) looks flat. Moreover, the spacelike parabola represents how \(L(f)\) looks semi-flat with respect to \(L\) or \(N\).

We set

\[
g_L(u) = G_L(\tilde{\gamma}(u)), \quad g_N(u) = G_N(\tilde{\gamma}(u)), \quad \tilde{h}_L(u) = \tilde{H}_L(\tilde{\gamma}(u)), \quad \tilde{h}_N(u) = \tilde{H}_N(\tilde{\gamma}(u)).
\]

**Definition 3.5.** Let \(F : (\mathbb{R}^3, \tilde{\gamma}(0)) \rightarrow (\mathbb{R}, 0)\) be a function. Then \(F^{-1}(0)\) and \(\tilde{\gamma}\) have a \(k\)-point contact at \(u = 0\) if \(f = F \circ \tilde{\gamma}\) satisfies \(f' = \cdots = f^{(k+1)} = 0, f^{(k+2)} \neq 0\) at \(u_0\). Let \(F_j : (\mathbb{R}^3, \tilde{\gamma}(0)) \rightarrow (\mathbb{R}, 0)\) \((j = 1, 2)\) be two functions. Then \(F_i^{-1}(0) \cap F_2^{-1}(0)\) and \(\tilde{\gamma}\) have a \((k_1, k_2)\)-point contact if \(F_i^{-1}(0)\) and \(\tilde{\gamma}\) have a \(k_i\)-point contact at \(u = 0\) for \(i = 1, 2\).

Applying this to \(f = g_L, g_N, \tilde{h}_L, \tilde{h}_N\), we see that the contact with the lightcone with the vertex \(x_L\) (respectively, \(x_N\)) can be measured by \(\sigma_L\) (respectively, \(\sigma_N\)), and the contact with the lightlike plane with the lightlike normal vector \(L(0)\) (respectively, \(N(0)\)), passing through \(\tilde{\gamma}(0)\) can be measured by the invariant \(\alpha_L\) (respectively, \(\alpha_N\)).

More precisely, the following theorem holds.

**Theorem 3.6.** (1) The function \(g_L\) satisfies \(g_L = g'_L = \cdots = g^{(k)}_L = 0\) at 0 if and only if \(\sigma_L = \sigma'_L = \cdots = \sigma^{(k-3)}_L = 0\) at 0 for any \(k \geq 3\). Similarly, the function \(g_N\) satisfies \(g_N = g'_N = \cdots = g^{(k)}_N = 0\) at 0 if and only if \(\sigma_N = \sigma'_N = \cdots = \sigma^{(k-3)}_N = 0\) at 0 for any \(k \geq 3\).
(2) The function \( \tilde{h}_L \) satisfies \( \tilde{h}_L = \tilde{h}'_L = \cdots = \tilde{h}^{(k)}_L = 0 \) at 0 if and only if \( \alpha_L = \alpha'_L = \cdots = \alpha^{(k-2)}_L = 0 \) for any \( k \geq 2 \). Similarly the function \( \tilde{h}_N \) satisfies \( \tilde{h}_N = \tilde{h}'_N = \cdots = \tilde{h}^{(k)}_N = 0 \) at 0 if and only if \( \alpha_N = \alpha'_N = \cdots = \alpha^{(k-2)}_N = 0 \) for any \( k \geq 2 \).

As we mentioned in (3.10), the condition \( \sigma_L = \sigma'_L = \cdots = \sigma^{(k-3)}_L = 0 \) (respectively, \( \sigma_N = \sigma'_N = \cdots = \sigma^{(k-3)}_N = 0 \)) at 0 for any \( k \geq 3 \) does not depend on the choice of \( L \). Proof of this theorem is not difficult, but complicated a little, we give a proof of this theorem in Appendix A. In [16], a similar consideration by using the Frenet frame along a curve in \( \mathbb{R}^3 \) is given. See [16] Propositions 2.1 and 2.2.

By Theorem 3.6 together with Theorems 3.1 and 3.3, we can conclude the contact of model curves can be measured by the singularities of lightlike surfaces and lightcone pedals. For the sake of simplified description, we set the following terminology. A frontal \( f : (\mathbb{R}^2, p) \rightarrow \mathbb{R}^3 \) has an \( A_2 \)-point (respectively, \( A_3 \)-point) at \( p \) if \( f \) at \( p \) is cuspidal edge (respectively, swallowtail). A frontal \( c : (\mathbb{R}, p) \rightarrow \mathbb{R}^2 \) has an \( A_1 \)-point (respectively, \( A_2 \)-point) at \( p \) if \( c \) at \( p \) is regular (respectively, a cusp).

**Corollary 3.7.** Let \( f : U \rightarrow \mathbb{R}^3 \) be an admissible frontal. Then the following hold for \( k_1, k_2 = 2, 3 \).

- The curve \( \hat{\gamma} \) and the osculating ellipse have \( (k_1, k_2) \)-point contact if and only if the osculating lightlike surface have an \( A_{k_1+1} \)-point, and the transversal lightlike surface have an \( A_{k_2+1} \)-point,

- the curve \( \hat{\gamma} \) and the \( N \)-osculating parabola have \( (k_1, k_2+1) \)-point contact if and only if the osculating lightlike surface have an \( A_{k_1+1} \)-point, and the transversal lightcone pedal have an \( A_{k_2+2} \)-point,

- the curve \( \hat{\gamma} \) and the \( L \)-osculating parabola have \( (k_1+1, k_2) \)-point contact if and only if the osculating lightcone pedal have an \( A_{k_1+2} \)-point, and the transversal lightlike surface have \( A_{k_2+1} \)-point,

- the curve \( \hat{\gamma} \) and the tangent line have \( (k_1+1, k_2+1) \)-point contact if and only if osculating lightcone pedal have an \( A_{k_1+2} \)-point, and transversal lightcone pedal have an \( A_{k_2+2} \)-point.

### 4 Special lightlike loci

We consider a frontal \( f \), where \( f_L \) or \( f_N \) has special properties. Since \( f_L |_{S(f_L)} = \gamma(u) + L(u)/\alpha_L \) (respectively, \( f_N |_{S(f_N)} = \gamma(u) + N(u)/\alpha_N \)), the singular value of \( f_L \) (respectively, \( f_N \)), is one point set if and only if \( \sigma_L \equiv 0 \) (respectively, \( \sigma_N \equiv 0 \)). As we mentioned above, these conditions do not depend on the choice of \( L \). We set two constant points \( V_L = \gamma(u) + L(u)/\alpha_L \) if \( \sigma_L \equiv 0 \), and \( V_N = \gamma(u) + N(u)/\alpha_N \) if \( \sigma_N \equiv 0 \), these points do not depend on the choice of \( L \) (cf. (3.9)). In this section, we consider
geometric meanings of $\sigma_L$ and $\sigma_N$. Since the function

$$d_L(u) = |\gamma(u) - V_C| = \left| \gamma(u) - \hat{\gamma}(u) - \frac{L(u)}{\alpha_L} \right|$$

(respectively, $d_N(u) = |\gamma(u) - V_N| = \left| \gamma(u) - \hat{\gamma}(u) - \frac{N(u)}{\alpha_N} \right|$)

vanishes identically, if $\sigma_L \equiv 0$ (respectively, $\sigma_N \equiv 0$), the curve $\hat{\gamma}$ lies on the lightcone $LC_L$ whose vertex is $V_L$ (respectively, the lightcone $LC_N$ whose vertex is $V_N$). If $\sigma_L \equiv 0$ and $\sigma_N \equiv 0$ holds simultaneously, then $\hat{\gamma}$ is an ellipse of an intersection $LC_L \cap LC_N$. Thus the pair $(\sigma_L, \sigma_N)$ measures how $L(f)$ is close to a ellipse which is obtained as an intersection of two lightcones. This ellipse is a (Euclidean) circle if and only if

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{L(u)}{\alpha_L(u)} + \frac{N(u)}{\alpha_N(u)} \right\rangle = 0.$$

**Example 4.1.** Let $lc(x, y) = (x, y, \sqrt{x^2 + y^2})$ be a lightcone, and let

$$\nu_l(x, u) = \frac{1}{\sqrt{2}} \left( -\frac{x}{\sqrt{x^2 + y^2}}, -\frac{y}{\sqrt{x^2 + y^2}}, 1 \right)$$

be a Euclidean unit normal of $lc(x, y)$. Let $\gamma(u)$ be a curve in the $xy$-plane, $\hat{\gamma}(u) = lc(\gamma(u))$, and let $\nu_l(\gamma(u)) = \nu_l(\gamma(u)) \times \hat{\gamma}(u)/|\nu_l(\gamma(u)) \times \hat{\gamma}(u)|$ be a Euclidean left-word unit normal vector of $\gamma$ as a curve on $lc$. We set

$$f_{\gamma}(u, v) = \hat{\gamma}(u) + r(\nu_l(\gamma(u)) + \cos v\nu_l(\gamma(u)) + \sin v\nu_l(\gamma(u)))$$

for $r > 0$. Then by the construction, $L(f_{\gamma})$ is the image of $\hat{\gamma}$, and the image of $(f_{\gamma})_L$ is the image of $lc$. We set $r = 1/10$.

1. Let us set $\gamma_1(u) = (\cos u, 2\sin u)/3$. Then the singular value of $(f_{\gamma_1})_L$ is a point, and $(f_{\gamma_1})_L$ is a lightcone. The images of $f_{\gamma_1}$ and $(f_{\gamma_1})_L$ are drawn in Figure 1 left.

2. Let us set $\gamma_2(u) = (2\sin u + 1, \sqrt{3}\cos u)/2$. Then both of the singular values of $(f_{\gamma_2})_L$ and $(f_{\gamma_2})_N$ are points, and $(f_{\gamma_2})_L$ and $(f_{\gamma_2})_N$ are lightcones. The axes of these lightcones do not coincide. The images of $f_{\gamma_2}$, $(f_{\gamma_2})_L$ and $(f_{\gamma_2})_N$ are drawn in Figure 1 center (i.e. $\gamma_2$ is the osculating ellipse).

3. Let us set $\gamma_3(u) = (\cos u, \sin u)$. Then both of the singular values of $(f_{\gamma_3})_L$ and $(f_{\gamma_3})_N$ are points, and $(f_{\gamma_3})_L$ and $(f_{\gamma_3})_N$ are lightcones. Furthermore, the axes of these lightcones coincide. The images of $f_{\gamma_3}$, $(f_{\gamma_3})_L$ and $(f_{\gamma_3})_N$ are drawn in Figure 1 right (i.e. $\gamma_3$ is the osculating ellipse (circle in the Euclidean sense)).

**Example 4.2.** We give an example $\sigma_L$ and $\sigma_N$ are constantly zero and $L(f)$ pass the cuspidal edge but it is not along cuspidal edge. Let $lc(x, y)$ is as in Example 4.1 and $lp(x, y) = (x, y, -(x - 1) - (y - 1))$ be a lightlike plane. We set $\hat{\gamma}(u) =$
Figure 1: Surfaces of Example 4.1

\[ 2(1, \cos u, \sin u)/(1 + \cos u + \sin u). \] Then \( \hat{\gamma}(u) \) is a parametrization of the intersection of the images of \( lc \) and \( lp \). Let us set \( c(u, v) = (c_1(u, v), c_2(u, v)) = (v(2u + v), v^2(3u + 2v)) \), and

\[ f(u, v) = \hat{\gamma}(u) + c_1(u, v) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2(u, v) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}. \]

Then \( f_v(u, u) = 0 \) and \( f \) at \( (u, u) \) is a cuspidal edge. On the other hand, \( f_v(u, 0) = 2u(1, 1, 0) \) holds, thus \( f \) is lightlike at \( f(u, 0) = \hat{\gamma}(u) \). We set \( L = (1, 1, 0) \) and \( N = (-1, 1, 0) \). Then \( \sigma_L = 0 \) and \( \alpha_N = 0 \) hold. In fact, by the above construction, \( \hat{\gamma} \) is contained in the osculating contact lightcone and the transversal contact lightlike plane (i.e. \( \hat{\gamma} \) is the \( N \)-osculating parabola).

A Proof of Theorem 3.6

One can easily see that \( g'_L(0) = g''_L(0) = 0 \). We set a function \( \beta_L(u) = \langle L(u), \hat{\gamma}(u) - x_L \rangle \). Since

\[
g^{(3)}_L = -2\alpha_N\alpha_L e + \sigma_N L + \sigma_L N, \hat{\gamma} - x_L \]
\[
= -2\alpha_N\alpha_L \langle e, \hat{\gamma} - x_L \rangle + \sigma_N \langle L, \hat{\gamma} - x_L \rangle + \sigma_L \langle N, \hat{\gamma} - x_L \rangle
\]
\[
= -2\alpha_N\alpha_L g'_L + \sigma_N\beta_L + \sigma_L \langle N, \hat{\gamma} - x_L \rangle,
\]
and \( \beta_L(0) = 0 \), we see \( g^{(3)}_L(0) = 0 \) if and only if \( \sigma_L(0) = 0 \). We have the following lemma.

Lemma A.1. If \( g^{(l)}_L(0) = 0 \), then \( \beta^{(l)}_L(0) = 0 \) \((l = 1, \ldots, k)\).

Proof. Since \( \beta_L(0) = 0 \), the case \( k = 1 \) follows from

\[
\beta'_L = \langle -\sigma_L e - \alpha_G L, \hat{\gamma} - x_L \rangle
\]
\[
= -\alpha_L \langle e, \hat{\gamma} - x_L \rangle - \alpha_G \langle L, \hat{\gamma} - x_L \rangle = -\alpha_L g'_L - \alpha_G \beta_L.
\] (A.2)

We assume that the assertion is true for \( k = 1, \ldots, K \), and we assume \( g^{(l)}_L(0) = 0 \) \( k = 1, \ldots, K + 1 \). Then by the assumption, \( \beta^{(l)}_L(0) = 0 \) \( k = 1, \ldots, K \) holds. Then \( \beta^{K+1}_L(0) = 0 \) follows from \( K \) times differentiation of (A.2). \qed
Proof of Theorem 3.6 (1). We have shown the case $k = 3$. We assume that the assertion is true for $k = 1, \ldots, K$. We assume that $g^{(l)}_L(0) = 0$ $(k = 3, \ldots, K)$, then by the assumption of induction, $\sigma^{(l-3)}_L(0) = 0$ $(l = 3, \ldots, K)$ holds. By Lemma (A.1) we have $\beta^{(l)}_L(0) = 0$ $(k = 3, \ldots, K)$. Then by $K - 2$ times differentiation of (A.1), we see

$$g^{(K+1)}_L(0) = \sigma^{(K-2)}_L(0) \langle \mathbf{N}(0), \hat{\gamma}(0) - x_L \rangle.$$

Thus the assertion is true for $k = K + 1$. The assertion for $g_N$ and $\sigma_N$ can be shown by just interchanging the subscripts $N$ and $L$.

One can easily see that $\tilde{h}_L(0) = \tilde{h}'_L(0) = 0$. We set a function $\delta_L(u) = \left\langle L(u), \widehat{L(0)} \right\rangle$.

Since

$$\tilde{h}''_L = \left\langle \alpha_N L + \alpha_L N, \widehat{L(0)} \right\rangle = \alpha_N \left\langle L, \widehat{L(0)} \right\rangle + \alpha_L \left\langle N, \widehat{L(0)} \right\rangle = \alpha_N \delta_L + \alpha_L \left\langle N, \widehat{L(0)} \right\rangle,$$

and $\delta_L(0) = 0$, we see $\tilde{h}''_L(0) = 0$ if and only if $\alpha_L(0) = 0$. We have the following lemma.

Lemma A.2. If $\alpha^{(l)}_L(0) = 0$, then $\delta^{(l+1)}_L(0) = 0$ $(l = 0, \ldots, k)$.

Proof. Since $\delta_L(0) = 0$, the case $k = 0$ follows from

$$\delta'_L = \left\langle -\alpha_L e - \alpha_G L, \widehat{L(0)} \right\rangle = -\alpha_L \left\langle e, \widehat{L(0)} \right\rangle - \alpha_G \left\langle L, \widehat{L(0)} \right\rangle = -\alpha_L \left\langle e, \widehat{L(0)} \right\rangle - \alpha_G \delta_L.$$

We assume that the assertion is true for $k = 0, \ldots, K$, and we assume $\alpha^{(l)}_L(0) = 0$ $k = 0, \ldots, K + 1$. Then by the assumption, $\delta^{(l+1)}_L(0) = 0$ $k = 1, \ldots, K$ holds. Thus $\delta^{K+2}_L(0) = 0$ follows from $K + 1$ times differentiation of (A.4).

Proof of Theorem 3.6 (2). We have shown the case $k = 2$. We assume that the assertion is true for $k = 2, \ldots, K$. We assume that $\tilde{h}^{(l)}_L(0) = 0$ $(k = 2, \ldots, K)$, then by the assumption of induction, $\alpha^{(l-2)}_L(0) = 0$ $(l = 2, \ldots, K)$ holds. By Lemma A.2, we have $\delta^{(l-1)}_L(0) = 0$ $(k = 2, \ldots, K)$. Then by $K - 1$ times differentiation of (A.3), we see

$$\tilde{h}^{(K+1)}_L(0) = \alpha^{(K-1)}_L(0) \left\langle \mathbf{N}(0), \widehat{L(0)} \right\rangle.$$

Thus the assertion is true for $k = K + 1$. The assertion for $\tilde{h}_N$ and $\sigma_N$ can be shown by just interchanging the subscripts $N$ and $L$. 

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