On a class of $n$-Leibniz deformations of the simple Filippov algebras

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Abstract

We study the problem of infinitesimal deformations of all real, simple, finite-dimensional Filippov (or $n$-Lie) algebras, considered as a class of $n$-Leibniz algebras characterized by having an $n$-bracket skewsymmetric in its $n-1$ first arguments. We prove that all $n > 3$ simple finite-dimensional Filippov algebras are rigid as $n$-Leibniz algebras of this class. This rigidity also holds for the Leibniz deformations of the semisimple $n = 2$ Filippov (i.e., Lie) algebras. The $n = 3$ simple FAs, however, admit a non-trivial one-parameter infinitesimal 3-Leibniz algebra deformation. We also show that the $n \geq 3$ simple Filippov algebras do not admit non-trivial central extensions as $n$-Leibniz algebras of the above class.
1 Introduction

Lie algebras can be generalized by relaxing the skewsymmetry of the Lie bracket. This leads to the Leibniz (or Loday’s) algebras \( L \) [1–4], defined as a vector space \( L \) endowed with a bilinear operation \( L \times L \rightarrow L \) that satisfies the Leibniz identity

\[
[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] \quad \forall X, Y, Z \in L ,
\]

which states that \( ad_X = [X, \ ] \) is a derivation of the Leibniz bracket. Lie algebras \( g \) are the special class of Leibniz algebras for which \([X, Y] = -[Y, X] \) \( \forall X, Y \). Since the Leibniz algebra bracket is not skewsymmetric, left and right derivations are not (anti)equivalent and correspondingly there are two possible versions of the Leibniz identity; eq. (1.1), which we shall adopt, is the left Leibniz identity and correspondingly defines a left Leibniz algebra.

Lie algebra deformations [5, 6] can be easily generalized to the Leibniz case. Infinitesimal Leibniz algebra deformations are defined by a deformed bracket \( [X_1, X_2]_t \)

\[
[X_1, X_2]_t = [X_1, X_2] + t\alpha(X_1, X_2) ,
\]

such that \( \alpha(X_1, X_2) \) is a bilinear \( L \)-valued map \( \alpha : L \otimes L \rightarrow L \), \( \alpha : (X_1, X_2) \mapsto \alpha(X_1, X_2) \) and \( [X_1, X_2]_t \) satisfies (1.1) (for deformations of Lie algebras \( g \), \( \alpha \) would be skewsymmetric in its two arguments). The non-trivial inequivalent infinitesimal deformations of Lie and Leibniz algebras are classified by the elements of the second cohomology groups \( H^2_{ad}(g, g) \) and \( H^2_{ad}(L, L) \) respectively. The Leibniz algebra cohomology has been discussed in [1–3, 7] (there for right \( L \)-s) and in [8]. The cohomology complex \( (C^\bullet(L, L), \delta) \) becomes the Lie algebra one \( (C^\bullet(g, g), \delta) \) when \( L = g \) and, as a result of the antisymmetry, the cochains are also required to be antisymmetric. But, since Lie algebras are also Leibniz, it is also possible to look for Leibniz deformations of Lie algebras when viewed as Leibniz ones. This may result in the appearance of more deformations, a fact recently discussed and observed in [9] for the nilpotent 3-dimensional Heisenberg algebra. In fact, and for a symmetric representation of \( L \) [12], there is a homomorphism [2] between the Leibniz and Lie algebra homologies as well as between the Lie algebra and Leibniz cohomologies for that representation.

Similarly, one may consider central extensions of Leibniz algebras. Given a real Leibniz algebra \( L \) with a basis \( \{X_a\} \), a central extension \( \tilde{L} \) is given by the vector space spanned by the vectors \( \tilde{X}_a \) plus an additional central generator \( \Xi \), endowed with the bracket

\[
[\tilde{X}_a, \tilde{X}_b] = C^c_{ab} \tilde{X}_c + \omega(X_a, X_b)\Xi , \quad [\tilde{X}_a, \Xi] = 0 = [\Xi, \tilde{X}_a] ;
\]

where the structure constants at the l.h.s. are those of the unextended \( L \) and \( \omega(X_a, X_b) \) is an \( \mathbb{R} \)-valued bilinear map which, in contrast with the Lie algebra case, does not have to be antisymmetric in its arguments. This also
follows from the fact that a central extension of a Leibniz algebra may be viewed as a deformation of the direct sum of \( \mathcal{L} \) and the one-dimensional algebra generated by \( \Xi \) that keeps the central character of \( \Xi \).

Other generalizations of the Lie algebra structure follow by considering brackets with \( n > 2 \) entries. A first \( n \)-ary generalization is obtained by extending the derivation property reflected by the ordinary (\( n = 2 \)) Jacobi identity (JI) to the \( n \)-bracket. This leads to the \( n \)-Lie or Filippov algebras \( \mathfrak{G} \) \[1, 10, 11, 12\] (other \( n \)-ary generalizations, based on the fact that the JI also follows from associativity, are possible \[13–15\]). Filippov or \( n \)-Lie algebras (FAs) are given by a vector space \( \mathfrak{G} \) endowed with a skew-symmetric, \( n \)-linear bracket,

\[
(X_1, \ldots, X_n) \in \mathfrak{G} \times \cdots \times \mathfrak{G} \mapsto [X_1, \ldots, X_n] \in \mathfrak{G}
\]  

(1.4)

that satisfies the Filippov identity (FI),

\[
[X_1, \ldots, X_{n-1}, [Y_1, \ldots, Y_n]] = \sum_{a=1}^{n} [Y_1, \ldots, Y_{a-1}, [X_1, \ldots, X_{n-1}, Y_a], Y_{a+1}, \ldots, Y_n]
\]  

(1.5)

which, for \( n = 2 \) is the JI; clearly, a 2-Lie algebra is an ordinary Lie one. The FI states that the left action of the linear operator \([X_1, \ldots, X_{n-1}, ] \equiv ad_X \in \text{End} \mathfrak{G}, \text{defined by}

\[
ad_X Z := [X_1, \ldots, X_{n-1}, Z] \equiv \mathcal{X} \cdot Z \quad \forall Z \in \mathfrak{G},
\]  

(1.6)

where \( \mathcal{X} = (X_1, \ldots, X_{n-1}) \in \wedge^{n-1} \mathfrak{G} \), is a derivation of the FA. It is convenient to have a name for the elements \( \mathcal{X} \in \wedge^{n-1} \mathfrak{G} \) that define inner derivations \( \mathcal{X} \cdot = ad_X \) of \( \mathfrak{G} \), since they will reappear when defining the different cohomology complexes; they are called \[17, 16\] ‘fundamental objects’ of the Filippov algebra. Detailed accounts of the properties of FAs are given in \[10, 12\] (see \[16\] for a review). FAs have been considered in physics in the context of Nambu mechanics \[18, 19\] and, recently (for \( n = 3 \)), in the search for the effective action of coincident M2-branes in M-theory initiated by the Bagger-Lambert-Gustavsson (BLG) model \[20, 21\] (further references on the physical applications of \( n \)-ary algebras are given in \[16\]).

The skewsymmetry of the FA \( n \)-bracket may also be relaxed. This gives rise to the \( n \)-Leibniz algebras \( \mathfrak{L} \) \[8, 7\] (for \( n = 2 \), Leibniz algebras \( \mathcal{L} \)). The left \( n \)-Leibniz algebras are defined by eqs. (1.4), (1.5) without requiring skewsymmetry for the \( n \)-bracket. As a result, the fundamental objects of \( \mathfrak{L} \) are no longer skewsymmetric and \( \mathcal{X} \in \otimes^{n-1} \mathfrak{L} \) in general. As for FAs, one may consider infinitesimal deformations of \( n \)-Leibniz algebras \( \mathfrak{L} \). Then one is led to the study of the \( n \)-Leibniz algebra cohomology for the adjoint action \( ad \). The expression that gives the action of the coboundary operator (Sec. 2.2) is the same as the one suitable to study FA deformations, the only difference being that antisymmetry is not required in the \( n \)-Leibniz case (in fact, since both FA and \( n \)-Leibniz cohomologies are based on the FI \[1, 3\], \( n \)-Leibniz cohomology underlies FA cohomology). We will see in Sec. 2.2 that the
non-equivalent infinitesimal deformations of a given $n$-Leibniz algebra are in one-to-one correspondence with the elements of the first cohomology group $H^1_{ad}(\mathcal{L}, \mathcal{L})$.

Besides deformations, one may consider central extensions of an $n$-Leibniz algebra by generalizing eq. (1.3) for the Leibniz $n$-bracket. The relevant cohomology here is formally the same as for the central extensions of FAs: central extensions of $n$-Leibniz algebras are classified by the first cohomology group $H^1(\mathcal{L}, \mathbb{R})$, as will be seen in Sec. 2.3. As before for deformations, the fact that it is the first cohomology group that matters (rather than the second, as it would be for extensions of Leibniz algebras $\mathcal{L}$ [1,3] with the standard counting for $n = 2$ algebras), is just a notational consequence of the natural labelling of $p$-cochains for general $n$-ary algebras, a point made clear in Sec. 2.2 where their deformation cohomology is presented.

Since all FAs are, in particular, $n$-Leibniz algebras with fully anticommuting $n$-brackets, one may again consider $n$-Leibniz infinitesimal deformations and central extensions of $n$-Lie algebras viewed as $n$-Leibniz ones. A FA may admit a non-trivial $n$-Leibniz deformation when looked at as an $n$-Leibniz algebra, even if the original FA is rigid under FA deformations. For instance, it has been proven [17] that a Whitehead Lemma holds for all semisimple $n$-Lie algebras. As a result, all simple (in fact semisimple) FAs are rigid under FA deformations [17]. However, the simple $n = 3$ Euclidean FA $A_4$ admits a non-trivial 3-Leibniz deformation [22]. A natural question to ask, which we shall address in this paper, is whether there exist non-trivial $n$-Leibniz deformations of the finite-dimensional simple FAs. Similarly, it is also natural to look for non-trivial $n$-Leibniz central extensions of simple FAs, in spite of the fact that the all semisimple FA central extensions are known to be trivial by the above extension to all $n$-Lie algebras [17] of the well known $n = 2$ Whitehead Lemma.

We shall study the above two problems, infinitesimal deformations and central extensions, for FAs considered as a particular case of $n$-Leibniz algebras, the class that keeps the antisymmetry of the first $n - 1$ entries in the $n$-bracket and thus has fundamental objects $\mathcal{X}$ that are skewsymmetric (see eq. (1.6)). For $n = 3$, examples of this type of real Leibniz algebras have appeared in the study of multiple M2-branes [23,24] (other examples weakening the skewsymmetry have been considered in physics, such as the ‘hermitean’ algebras [25] which will not be considered here; see also [26]). These non-fully commutative 3-algebras correspond to 3-Leibniz algebras with a 3-bracket that retains the skewsymmetry for the first two arguments or, equivalently, to 3-Leibniz algebras for which the fundamental objects $\mathcal{X} = (X_1, X_2)$ are still antisymmetric as in the FA case; the 3-Leibniz infinitesimal deformation of the FA $A_4$ in [22] is of this class. We shall show that this is, in fact, the only 3-Leibniz deformation and, further, that there are no non-trivial deformations of the above type when $n \neq 3$. The semisimple $n = 2$ FAs $\mathcal{G} = \mathfrak{g}$, for which the fundamental objects are single elements of the Lie algebra $\mathfrak{g}$ (hence with no restrictions), will also turn out to be Leibniz rigid. Finally,
we will also prove for simple $n \geq 3$ FAs that there are no non-trivial central $n$-Leibniz extensions of the mentioned class.

The plan of the paper is the following: in Sec. 2 we look at the deformation and the central extension theory of $n$-Leibniz algebras, and give the one-cocycle and one-coboundary conditions for the appropriate cohomologies both in their intrinsic forms and in coordinates. Sec. 3 proves that there are no $n$-Leibniz algebra deformations with $n$-brackets skewsymmetric in the first $n − 1$ entries (and thus with fully skewsymmetric fundamental objects) of simple $n$-Lie algebras when $n > 3$. This result will be obtained in Sec. 3.2 using the coordinate expressions given in Sec. 2.1 the proof for the $n = 2$ case is given in Sec. 3.1 Sec. 4 follows a similar procedure to prove that $n \geq 3$ simple FAs do not have non-trivial $n$-Leibniz central extensions of the mentioned class.

All algebras in this paper are real and finite-dimensional. Our results also hold in the complex case, but we prefer to consider real algebras since moving to $\mathbb{C}$ does not allow us to distinguish between the different physically interesting pseudoEuclidean simple FAs (see eq. (3.31)). Theorems 1 (Sec. 3.4) and 2 (Sec 4) are our results; Sec. 5 comments on possible extensions of the present work.

2 Deformations and extensions of $n$-Leibniz algebras and cohomology

2.1 Infinitesimal deformations of $n$-Leibniz algebras

Let $\mathfrak{L}$ be an $n$-Leibniz algebra. Its $n$-bracket obeys the $n$-Leibniz identity which, in fact, is formally identical to the FI (1.5), the only difference being that now the $n$-Leibniz bracket need not be fully skewsymmetric. A one-parameter infinitesimal deformation of $\mathfrak{L}$ is given by a new, deformed, bracket

$$\left[ X_1, \ldots, X_n \right]_t = \left[ X_1, \ldots, X_n \right] + t\alpha^1 (X_1, \ldots, X_n) ,$$

(2.7)

where $\alpha^1$ is a linear $\mathfrak{L}$-valued map $\alpha^1 : \mathfrak{L} \otimes \cdots \otimes \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L}$ (a $\mathfrak{L}$-valued one-cochain, as we shall see at the end of the section) and $t$ is the parameter of the infinitesimal deformation.

The requirement that the deformed bracket also obeys the $n$-Leibniz identity (the FI) leads to a condition on $\alpha^1$ that may be interpreted as the one-cocycle condition in the cohomology for the deformation of $n$-Leibniz
algebras. Namely, it is \( \delta \alpha^1 = 0 \) with
\[
\delta \alpha^1(X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}, Z)
= [X_1, \ldots, X_{n-1}, \alpha^1(Y_1, \ldots, Y_{n-1}, Z)] + \alpha^1(X_1, \ldots, X_{n-1}, [Y_1, \ldots, Y_{n-1}, Z])
- \sum_{r=1}^{n-1} [Y_1, \ldots, Y_{r-1}, \alpha^1(X_1, \ldots, X_{n-1}, Y_r, Y_{r+1}, \ldots, Y_{n-1}, Z)]
- [Y_1, \ldots, Y_{n-1}, \alpha^1(X_1, \ldots, X_{n-1}, Z)]
- \sum_{r=1}^{n-1} \alpha^1(Y_1, \ldots, Y_{r-1}, [X_1, \ldots, X_{n-1}, Y_r], Y_{r+1}, \ldots, Y_{n-1}, Z)
- \alpha^1(Y_1, \ldots, Y_{n-1}, [X_1, \ldots, X_{n-1}, Z]) .
\] (2.8)

The \( \delta \alpha^1 = 0 \) condition above is the same as for FA deformations, but now there are no antisymmetry conditions for the cochains. In terms of the fundamental objects of the \( n \)-Leibniz algebra \( \mathfrak{L} \), where now \( \mathfrak{L} = (X_1, \ldots, X_{n-1}) \in \otimes^{n-1} \mathfrak{L} \), eq. (2.8) may be rewritten as
\[
(\delta \alpha^1)(\mathfrak{L}, \mathfrak{Y}, Z) = ad_{\mathfrak{L}} \alpha^1(\mathfrak{Y}, Z) - ad_{\mathfrak{L}} \alpha^1(\mathfrak{L}, Z) - (\alpha^1(\mathfrak{L}, \mathfrak{Y}) \cdot Z)
- \alpha^1(\mathfrak{L} \cdot \mathfrak{Y}, Z) - \alpha^1(\mathfrak{Y}, \mathfrak{L} \cdot Z) + \alpha^1(\mathfrak{L}, \mathfrak{Y} \cdot Z) = 0 ,
\] (2.9)
where, for instance for \( n = 3 \), the term \( \alpha^1(\mathfrak{L}, \mathfrak{Y}) \cdot Z \) above is the fundamental object defined by
\[
\alpha^1(\mathfrak{L}, \mathfrak{Y}) := \alpha^1(\mathfrak{L}, \mathfrak{Y}) := (\alpha^1(\mathfrak{L}, Y_1) \cdot Y_2) + (Y_1, \alpha^1(\mathfrak{L}, Y_2))
= (\alpha^1(\mathfrak{L}, Y_1, Y_2) + (Y_1, \alpha^1(\mathfrak{L}, Y_2))) ,
\] (2.10)
\[
\mathfrak{L} \cdot Z = [\mathfrak{L}, Z] \equiv [X_1, \ldots, X_{n-1}, Z] \quad \text{and}
\]
\[
\mathfrak{L} \cdot \mathfrak{Y} := \sum_{a=1}^{n-1} [X_1, \ldots, X_{a-1}, [X_1, \ldots, X_{n-1}, Y_a], Y_{a+1}, \ldots, Y_{n-1}] \quad \text{above defines the composition of fundamental objects.}
\]

Let us choose a basis \( \{X_a\} \) of \( \mathfrak{L} \) for which
\[
[X_{a_1}, \ldots, X_{a_n}] = f_{a_1 \ldots a_n}^b X_b , \quad \text{ (2.12)}
\]
\[
\alpha^1(X_{a_1}, \ldots, X_{a_n}) = X_b (\alpha^1)^b_{a_1 \ldots a_n} . \quad \text{ (2.13)}
\]

Then, the one-cocycle condition (2.8) for the one-cochain \( \alpha^1 \) takes the form
\[
f_{a_1 \ldots a_n-1}^d (\alpha^1)^e_{b_1 \ldots b_{n-1}c} + (\alpha^1)^d_{a_1 \ldots a_{n-1}c} f_{b_1 \ldots b_{n-1}c}^e
- \sum_{r=1}^{n-1} f_{b_1 \ldots b_{r-1}c b_{r+1} \ldots b_{n-1}c}^d (\alpha^1)^e_{a_1 \ldots a_{n-1}b_r}
- f_{b_1 \ldots b_{n-1}c}^d (\alpha^1)^e_{a_1 \ldots a_{n-1}c}
- \sum_{r=1}^{n-1} (\alpha^1)^d_{b_1 \ldots b_{r-1}c b_{r+1} \ldots b_{n-1}c} f_{a_1 \ldots a_{n-1}b_r}^e
- (\alpha^1)^d_{b_1 \ldots b_{n-1}c} f_{a_1 \ldots a_{n-1}c}^e = 0 . \quad \text{ (2.14)}
\]
When \( n = 3 \), this reduces to
\[
\begin{align*}
 f_{a_1a_2e} d (\alpha^1)^e_{b_1b_2c} + (\alpha^1)^d_{a_1a_2c} f_{b_1b_2c}^e \\
 - f_{eb_2c} d (\alpha^1)^e_{a_1a_2b_1} - f_{b_1e} d (\alpha^1)^e_{a_1a_2b_2} \\
 - f_{b_1b_2e} d (\alpha^1)^e_{a_1a_2c} \\
 - (\alpha^1)^d_{eb_2c} f_{a_1a_2b_1}^e - (\alpha^1)^d_{b_1e} f_{a_1a_2b_2}^e \\
 - (\alpha^1)^d_{b_1b_2e} f_{a_1a_2c}^e = 0 .
\end{align*}
\]  

(2.15)

An \( n \)-Leibniz algebra deformation is trivial if there is a redefinition of the generators \( X'_i = X_i - t^0(X_i) \), defined by some \( \mathfrak{L} \)-valued zero-cochain \( \alpha^0 \), \( \alpha^0 : \mathfrak{L} \rightarrow \mathfrak{L} \), that removes the deforming term in eq. (2.7). This means that \( \alpha^1 \) is actually a one-coboundary i.e. \( \alpha^1 = \delta \alpha_0 \) since then
\[
\begin{align*}
\alpha^1(X_1, \ldots, X_{n-1}, Z) &= -\alpha^0([X_1, \ldots, X_{n-1}, Z]) \\
+ \sum_{r=1}^{n-1} [X_1, \ldots, X_r, \alpha^0(X_r), X_{r+1}, \ldots, X_{n-1}, Z] + [X_1, \ldots, X_{n-1}, \alpha^0(Z)] \\
&= \delta \alpha^0(X_1, \ldots, X_{n-1}, Z) .
\end{align*}
\]  

(2.16)

Again, this condition is the same (but for the skewsymmetry of both the bracket and the cochain in its arguments) as the one that establishes that an infinitesimal FA deformation is trivial. In terms of fundamental objects it is written as
\[
(\alpha^1)(\mathcal{X}, Z) \equiv (\delta \alpha^0)(\mathcal{X}, Z) = \mathcal{X} \cdot \alpha^0(Z) - \alpha^0(\mathcal{X} \cdot Z) + (\alpha^0(\mathcal{X}) \cdot \mathcal{X}) \cdot Z .
\]  

(2.17)

In the basis \( \{X_a\} \), the coordinates \( (\alpha^0)^b_a \) of \( \alpha^0 \) are defined through
\[
\alpha^0(X_a) = (\alpha^0)^b_a X_b ,
\]  

(2.18)

and the one-coboundary condition (2.17) that expresses \( \alpha^1 \) in terms of \( \alpha^0 \) is given by
\[
(\alpha^1)^b_{a_1 \ldots a_{n-1}c} = -(\alpha^0)^b_s f_{a_1 \ldots a_{n-1}c}^s \\
+ \sum_{r=1}^{n-1} f_{a_1 \ldots a_{r-1}s a_{r+1} \ldots a_{n-1}c}^b (\alpha^0)^s_{a_r} \\
+ f_{a_1 \ldots a_{n-1}s}^b (\alpha^0)^s_c .
\]  

(2.19)

For a 3-Leibniz algebra this gives
\[
(\alpha^1)^b_{a_1a_2c} = -(\alpha^0)^b_s f_{a_1a_2c}^s \\
+ f_{s a_2 c}^b (\alpha^0)^s_{a_1} + f_{a_1s c}^b (\alpha^0)^s_{a_2} \\
f_{a_1a_2s}^b (\alpha^0)^s_c .
\]  

(2.20)
2.2 \( n \)-Leibniz algebra deformations cohomology complex

The action of the coboundary operator \( \delta \) on zero- and one-cochains given above can be extended to arbitrary \( p \)-cochains \( \alpha^p \) so that, in Gerstenhaber’s sense, the deformation theory of \( n \)-Leibniz algebras generates the corresponding cohomology. In it, \( p \)-cochains are defined as elements \( \alpha^p \in \text{Hom}(\otimes^{p(n-1)+1}\mathcal{L}, \mathcal{L}) \); thus, a \( p \)-cochain takes \( p(n-1)+1 \) arguments in \( \mathcal{L} \), \( p(n-1) \) of which enter through \( p \) fundamental objects and \( \delta \alpha^p \) has order \( (p+1) \). The previous expressions for \( \delta \alpha^0 \), \( \delta \alpha^1 \) now generalize to provide the action of the \( n \)-Leibniz deformations cohomology coboundary operator \( \delta \) on an arbitrary \( p \)-cochain \( \alpha^p \). This action is best expressed in terms of fundamental objects, which explains why \( p \) and \( p+1 \) determine their order (see [17, 16]).

This leads to the \( n \)-Leibniz algebra deformation cohomology complex \( (\mathcal{C}^\bullet_{ad}(\mathcal{L}, \mathcal{L}), \delta) \), where

\[
(\delta \alpha^p)(\mathcal{X}_1, \ldots, \mathcal{X}_p, \mathcal{X}_{p+1}, Z) = \\
\sum_{1 \leq j < k}^{p+1} (-1)^j \alpha^p(\mathcal{X}_1, \ldots, \mathcal{X}_j, \ldots, \mathcal{X}_{k-1}, \mathcal{X}_j \cdot \mathcal{X}_k, \mathcal{X}_{k+1}, \ldots, \mathcal{X}_{p+1}, Z) \\
+ \sum_{j=1}^{p+1} (-1)^j \alpha^p(\mathcal{X}_1, \ldots, \mathcal{X}_j, \ldots, \mathcal{X}_{p+1}, \mathcal{X}_j \cdot Z) \\
+ \sum_{j=1}^{p+1} (-1)^j \mathcal{X}_j \cdot \alpha^p(\mathcal{X}_1, \ldots, \mathcal{X}_j, \ldots, \mathcal{X}_{p+1}) \\
+ (-1)^p(\alpha^p(\mathcal{X}_1, \ldots, \mathcal{X}_p, Z)) \cdot \mathcal{X}_{p+1} \cdot Z 
\]  

(2.21)

where, in the last term (see [17,16]),

\[
\alpha^p(\mathcal{X}_1, \ldots, \mathcal{X}_p, Z) \cdot \mathcal{Y} = \sum_{i=1}^{n-1} (Y_1, \ldots, \alpha^p(\mathcal{X}_1, \ldots, \mathcal{X}_p, Y_i), \ldots, Y_{n-1}) \; ; 
\]  

(2.22)

note that both the left and the right actions intervene. The nilpotency \( \delta^2 = 0 \) is guaranteed by the FI (1.5) for \( n \)-Leibniz algebras.

Consequently, we see that the infinitesimal deformations of \( n \)-Leibniz algebras are governed by the non-trivial elements of the first cohomology group \( H^1_{ad}(\mathcal{L}, \mathcal{L}) \).

The above cohomology complex is essentially equivalent to that previously given by Gautheron [27] in the context of Nambu algebras; see further [8,28,29].

\(^1\)When \( n = 2 \), \( \mathcal{L} = \mathcal{L} \) and a one-cochain contains two Leibniz algebra \( \mathcal{L} \) arguments. Thus, if its order were determined by this number (as it is usually the case when \( n = 2 \)), it would be a two- rather than a one-cochain.
2.3 Central extensions of \( n \)-Leibniz algebras

The central extensions of an \( n \)-Leibniz algebra \( L \) are obtained by adding to the \( L \) generators a central one. Thus, the \( n \)-brackets of the extended algebra are

\[
\begin{align*}
[X_1, \ldots, X_n] &= f_{a_1 \ldots a_n} b X_b + \alpha^1(X_1, \ldots, X_n) \Xi, \\
[X_1, \ldots, \tilde{X}_{i-1}, \Xi, \tilde{X}_{i+1}, \ldots, \tilde{X}_n] &= 0, \quad i = 1, \ldots n.
\end{align*}
\] (2.23)

In contrast with the FA case, the \( \mathbb{R} \)-valued \( n \)-Leibniz one-cochain \( \alpha^1 \) does not have to be skewsymmetric in its arguments.

As before, a one-cocycle condition arises for \( \alpha^1 \) when imposing that the centrally extended algebra is an \( n \)-Leibniz one, i.e., that it obeys the (left) Filippov identity. In terms of the fundamental objects \( \delta \alpha^1 = 0 \) reads

\[
(\delta \alpha^1)(\mathcal{X}, \mathcal{Y}, Z) = -\alpha^1(\mathcal{X} \cdot \mathcal{Y}, Z) - \alpha^1(\mathcal{Y}, \mathcal{X} \cdot Z) + \alpha^1(\mathcal{X}, \mathcal{Y} \cdot Z) = 0,
\] (2.24)

Using eq. (2.12), the coordinates expression for the one-cocycle condition is given by

\[
(\alpha^1)_{a_1 \ldots a_{n-1} c} f_{b_1 \ldots b_{n-1} e} - \sum_{r=1}^{n-1} (\alpha^1)_{b_1 \ldots b_{r-1} e b_{r+1} \ldots b_{n-1} c} f_{a_1 \ldots a_{n-1} b_r} e = 0.
\] (2.25)

An \( n \)-Leibniz algebra central extension is trivial if there is a redefinition of its generators \( \tilde{X}_i = \tilde{X}_i - t\alpha^0(X_i) \) that removes the central term in eq. (2.23). In this case \( \alpha^1 \) satisfies

\[
\alpha^1(X_1, \ldots, X_{n-1}, Z) = \delta \alpha^0(X_1, \ldots, X_{n-1}, Z) = -\alpha^0([X_1, \ldots, X_{n-1}, Z]),
\] (2.26)

where \( \alpha^0 \) is the zero-cochain that generates the one-coboundary \( \alpha^1 \). Again, this condition is formally the same that establishes that a FA central extension is trivial, although now \( \alpha^1 \in \text{Hom}(\otimes^{n-1} L \otimes L, \mathbb{R}) \) (rather than \( \alpha^1 \in \text{Hom}(\wedge^{n-1} G \otimes G, \mathbb{R}) \) as for a FA \( G \)). In terms of fundamental objects eq. (2.26) reads simply

\[
(\alpha^1)(\mathcal{X}, Z) = -\alpha^0(\mathcal{X} \cdot Z).
\] (2.27)

Using the basis (2.12) and

\[
\alpha^0(X_a) = \alpha^0_a,
\] (2.28)

the coboundary condition \( \alpha^1 = \delta \alpha^0 \) relates the coordinates of \( \alpha^1 \) to those of \( \alpha^0 \) by

\[
(\alpha^1)_{a_1 \ldots a_{n-1} c} = -(\alpha^0)_{a} f_{a_1 \ldots a_{n-1} c} s.
\] (2.29)

As before, the action of the coboundary operator \( \delta \) given above on the zero- and one-cochains of the central extension problem can be extended to
arbitrary cochains $\alpha^p \in \text{Hom}(\otimes^p (n-1)+1 \mathfrak{L}, \mathfrak{L})$; the nilpotency of $\delta$ is satisfied by virtue of the FI (1.5) for $\mathfrak{L}$. This leads to (see [17, 16])

$$(\delta \alpha^p)(\mathcal{X}_1, \ldots, \mathcal{X}_p, \mathcal{X}_{p+1}, Z) =$$

$$
\sum_{1 \leq j < k}^{p+1} (-1)^j \alpha^p(\mathcal{X}_1, \ldots, \hat{\mathcal{X}}_j, \ldots, \mathcal{X}_{k-1}, \mathcal{X}_j \cdot \mathcal{X}_k, \mathcal{X}_{k+1}, \ldots, \mathcal{X}_{p+1}, Z) +
\sum_{j=1}^{p+1} (-1)^j \alpha^p(\mathcal{X}_1, \ldots, \hat{\mathcal{X}}_j, \ldots, \mathcal{X}_p+1, \mathcal{X}_j \cdot Z),
$$

(2.30)

which determines the cohomology complex $(C^\bullet(\mathfrak{L}, \mathbb{R}), \delta)$. It follows that the central extensions of an $n$-Leibniz algebra are governed by the first cohomology group $H^1_0(\mathfrak{L}, \mathbb{R})$.

Note that all the expressions corresponding to the central extensions case can be obtained from those of the deformation cohomology by using $\mathbb{R}$-valued cochains in place of $\mathfrak{L}$-valued ones and by substituting the trivial action for the adjoint one. In coordinates this means removing the first, upper index of $(\alpha^1)^{a_1 \ldots a_{n-1} b}$ and ignoring all terms containing the ad action in the expression of $\delta$ in eqs. (2.9), (2.14).

### 3 A class of $n$-Leibniz algebra deformations of the real simple Filippov algebras

#### 3.1 The $n \geq 3$ simple FAs case

Consider now the simple real finite-dimensional Filippov algebras. These were given in [10] and found to be the only ones in [12]. They are constructed on $(n+1)$-dimensional vector spaces and are characterized by the structure constants [10][12]

$$f_{a_1 \ldots a_{n-1} c}^b = (-1)^n \varepsilon_{\delta \epsilon a_1 \ldots a_{n-1} c}^b, \quad a, b, c = 1, \ldots (n+1),$$

(3.31)

where $\epsilon$ is the Euclidean $(n+1)$-dimensional skewsymmetric tensor and, following Filippov’s notation, the $\varepsilon_b$ (no sum in $b$) are just signs that appear in the pseudoeuclidean algebras and that are absent when considering the Euclidean $(n+1)$-dimensional simple $n$-Lie algebra $A_{n+1}$.

Thus, the one-cocycle condition relevant for the deformation of a simple
An n-Lie algebra follows from eq. (2.14) and is given by

\[
\begin{align*}
\varepsilon_d & \varepsilon a_1 \ldots a_{n-1} e^d (\alpha^1 e)^a b_1 \ldots b_{n-1} c + \varepsilon_d (\alpha^1 e)^a_{a_1 \ldots a_{n-1}} e b_1 \ldots b_{n-1} c^e \\
- & \sum_{r=1}^{n-1} \varepsilon_d \varepsilon b_1 \ldots b_{r-1} e b_{r+1} \ldots b_{n-1} c^d (\alpha^1 e)^a_{a_1 \ldots a_{n-1}} b_r \\
- & \varepsilon_d b_1 \ldots b_{n-1} e^d (\alpha^1 e)^a_{a_1 \ldots a_{n-1}} c \\
- & \sum_{r=1}^{n-1} \varepsilon (\alpha^1)^d b_1 \ldots b_{r-1} e b_{r+1} \ldots b_{n-1} e a_1 \ldots a_{n-1} b_r e \\
- & \varepsilon (\alpha^1)^d b_1 \ldots b_{n-1} e a_1 \ldots a_{n-1} = 0 .
\end{align*}
\]

(3.32)

A one-cocycle for a simple n-Lie algebra is actually a one-coboundary (eq. (2.19)) when

\[
(\alpha^1)^b_{a_1 \ldots a_{n-1} c} = -\varepsilon_s (\alpha^0)^b_s e a_1 \ldots a_{n-1} c \\
+ \sum_{r=1}^{n-1} \varepsilon b e a_1 \ldots a_{r-1} e a_{r+1} \ldots a_{n-1} c^b (\alpha^0)^s_{a_r} \\
+ \varepsilon b e a_1 \ldots a_{n-1} e^b (\alpha^0)^s .
\]

(3.33)

To consider an n-Leibniz deformation of an n-Lie algebra one looks at the Filippov algebra $G$ as an n-Leibniz one $L$. If the deformed n-Leibniz algebra $L$ is required to have a fully antisymmetric n-bracket, then we are actually deforming FAs, and the answer is known: all semisimple FAs are rigid due to the Whitehead Lemma for n-Lie algebras [17], which holds for any $n \geq 2$. Rather than allowing for a general Leibniz bracket we will relax mildly the full skewsymmetry of the FA n-bracket by restricting it to its first $n-1$ arguments. As mentioned, this is a) natural, since it keeps the antisymmetry in the arguments of the fundamental objects (recall that $\mathcal{X} \cdot Z = [X_1, \ldots X_{n-1}, Z]$) and b) convenient, since 3-brackets that are antisymmetric in the first two arguments only have been used to define the ‘relaxed three algebras’ [23,24] that have appeared in the context of the BLG model and which correspond for $n = 3$ to the class of n-Leibniz algebras considered here (see further [25] for the ‘hermitean’ case). In coordinates, this means that the one-cocycles for the present cohomology problem have the form $(\alpha^1)^b_{a_1 \ldots a_{n-1} c}$: where only skewsymmetry in the $a_1 \ldots a_{n-1}$ indices is required. This is what will characterize the possible deformations of a FA $G$ considered as an n-Leibniz algebra of the above type. As stated, this corresponds to having $\mathcal{X} \in \wedge^{n-1} L$ in the resulting n-Leibniz algebra, as it is always the case for FAs.
3.2 Dualized cocycles

This above skewsymmetry restriction allows us to take the dual $\bar{\alpha}$ of $\alpha$ by defining

$$\begin{align*}
(\bar{\alpha})^b_{b_1 b_2 c} &= \frac{1}{(n-1)!} \epsilon^{a_1 \ldots a_n-1} b_1 b_2 (\alpha)^b_{a_1 \ldots a_n-1 c}, \\
(\alpha)^b_{a_1 \ldots a_n-1 c} &= \frac{1}{2} \epsilon^{a_1 \ldots a_n-1} b_1 b_2 (\bar{\alpha})^b_{b_1 b_2 c},
\end{align*}$$

which will be useful for calculational purposes. Note that the dual $\bar{\alpha}$ above always has four indices independently of $n$ and that it is $b_1, b_2$ skewsymmetric, $(\bar{\alpha})^b_{b_1 b_2 c} = -(\bar{\alpha})^b_{b_2 b_1 c}$. The invertibility is possible because, for fixed $b$ and $c$, both $\alpha$ and $\bar{\alpha}$ have the same \( \binom{n+1}{n-1} = \binom{n+1}{2} \) degrees of freedom.

In terms of $\bar{\alpha}$, the cocycle condition (3.32) now reads

$$\begin{align*}
\varepsilon_d \varepsilon_{a_1 \ldots a_{n-1} e} d \varepsilon_{b_1 \ldots b_{n-1} c_1 c_2} (\bar{\alpha})^e_{c_1 c_2} \\
+ \varepsilon_d \varepsilon_{a_1 \ldots a_{n-1} c_1 c_2} \varepsilon_{b_1 \ldots b_{n-1} e} (\bar{\alpha})^d_{c_1 c_2} \\
- \sum_{r=1}^{n-1} \varepsilon_d \varepsilon_{b_1 \ldots b_{n-1-1} e b_{r+1} \ldots b_{n-1}} d \varepsilon_{a_1 \ldots a_{n-1} c_1 c_2} (\bar{\alpha})^e_{c_1 c_2} \\
- \varepsilon_d \varepsilon_{b_1 \ldots b_{n-1-1} e} d \varepsilon_{a_1 \ldots a_{n-1}} c_1 c_2 (\bar{\alpha})^e_{c_1 c_2} \\
- \sum_{r=1}^{n-1} \varepsilon_d \varepsilon_{b_1 \ldots b_{n-1-1} e b_{r+1} \ldots b_{n-1}} c_1 c_2 \varepsilon_{a_1 \ldots a_{n-1} b_r} e (\bar{\alpha})^d_{c_1 c_2} \\
- \varepsilon_d \varepsilon_{b_1 \ldots b_{n-1-1} c_1 c_2} \varepsilon_{a_1 \ldots a_{n-1} c} e (\bar{\alpha})^d_{c_1 c_2} = 0.
\end{align*}$$

This expression is really $\delta \alpha^\epsilon (\mathcal{X}, \mathcal{Y}, Z) = 0$ for $\mathcal{X} = (X_{a_1}, \ldots, X_{a_{n-1}})$, $\mathcal{Y} = (X_{b_1}, \ldots, X_{b_{n-1}})$, so it must be skewsymmetric in $a_1, \ldots, a_{n-1}$ and in $b_1, \ldots, b_{n-1}$. Thus, without losing information, we can use equally the contraction of (3.35) with $c_1 c_2$. We obtain, after raising the free index $c$, the equivalent one-cocycle condition

$$\begin{align*}
\varepsilon_d \delta^{d}_{a_2} (\bar{\alpha})^d_{a_1 b_1 b_2} c - \varepsilon_d \delta^{d}_{a_1} (\bar{\alpha})^d_{a_2 b_1 b_2} c + \varepsilon_d \delta^{d}_{b_1} (\bar{\alpha})^d_{a_1 a_2 b_2} c - \varepsilon_d \delta^{d}_{b_2} (\bar{\alpha})^d_{a_1 a_2 b_1} c \\
- \varepsilon_d \delta^{d}_{b_1} (\bar{\alpha})^d_{a_1 a_2 b_2} c - \varepsilon_d \delta^{d}_{b_2} (\bar{\alpha})^d_{a_1 a_2 b_1} c \\
+ \varepsilon_d \delta^{d}_{a_1} (\bar{\alpha})^d_{a_2 b_1 b_2} c - \varepsilon_d \delta^{d}_{a_2} (\bar{\alpha})^d_{a_1 b_1 b_2} c - \varepsilon_d \delta^{d}_{b_1} (\bar{\alpha})^d_{a_1 a_2 b_2} c - \varepsilon_d \delta^{d}_{b_2} (\bar{\alpha})^d_{a_1 a_2 b_1} c \\
- \varepsilon_d \delta^{d}_{a_1} (\bar{\alpha})^d_{a_2 b_1 b_2} c + \varepsilon_d \delta^{d}_{a_2} (\bar{\alpha})^d_{a_1 b_1 b_2} c = 0.
\end{align*}$$

We may now extract consequences from this condition by taking different sums of indices. In particular one may, for instance, contract first $a_1'$ with $b_1'$ after multiplying by $\varepsilon_{a_1'}$. Then, contracting in the resulting equation (a) $c$ with $d$, (b) $a_2'$ with $b_2'$ (after multiplying by $\varepsilon_{a_2'}$) and (c) $d$ with $b_2'$ (after
Another possibility is to contract equations (3.41) in the first three indices. Lowering the superscript we obtain three expressions

\[ \begin{align*}
& a) \quad (\tilde{\alpha}^1)_{c a b}^e c = 0 ; \\
& b) \quad (\tilde{\alpha}^1)_{a b c}^e c = (\tilde{\alpha}^1)_{b a c}^e c ; \\
& c) \quad n\varepsilon_d (\tilde{\alpha}^1)_{a d}^e c + \varepsilon_c (\tilde{\alpha}^1)_{a d}^e c - \delta_{a d}^e \varepsilon_e (\tilde{\alpha}^1)_{c d}^e d = 0 .
\end{align*} \]  

(3.37)

Note that equations b) and c) above imply \( \varepsilon_c \varepsilon_e (\tilde{\alpha}^1)_{e d c}^e = \varepsilon_c \varepsilon_e (\tilde{\alpha}^1)_{e d c}^e \). Define a new quantity, \( \tilde{\alpha}^1 \), by

\[ (\tilde{\alpha}^1)_{a b c d} \equiv (\tilde{\alpha}^1)_{a b c d}^a + \frac{1}{n} \delta_{cd}(\tilde{\alpha}^1)_{b a c}^a + \frac{1}{n} \delta_{bd}(\tilde{\alpha}^1)_{a c b}^a . \]  

(3.38)

The \( \tilde{\alpha}^1 \) above has the following properties: (a) its traces, \( \varepsilon_a (\tilde{\alpha}^1)_{a c d}^c \), \( (\tilde{\alpha}^1)_{b c a}^c \) and \( (\tilde{\alpha}^1)_{b c a}^c \) vanish, and (b) \( \tilde{\alpha}^1 \) is a one cocycle (it satisfies (3.36)) cohomologous with \( \alpha^1 \). Condition (a) is true because of (3.37), and (b) follows because \( (\tilde{\alpha}^1)_{a b c d}^a - (\tilde{\alpha}^1)_{a b c d}^a \), as given by (3.38), is a one-coboundary. The quickest way to prove this is to show that any one-cocycle of the form \( (\beta^1)_{a b c d} = \delta_{b c} B_{a c} - \delta_{c d} B_{a b} \) with \( B_{a b} = B_{b a} \) (in our case \( B_{a b} = \frac{1}{n} (\tilde{\alpha}^1)_{a b c d}^a \) which, by the second equation in (3.37), is \( (a, b) \) symmetric) is trivial. To this aim, we write the corresponding dual one-cocycle \( (\beta^1)_{a_1 b_1 \ldots b_{n-1} d} \) by (3.34).

One finds that in this case

\[ (\beta^1)_{a b_1 \ldots b_{n-1} d} \propto \varepsilon_{b_1 \ldots b_{n-1}} \delta_{b a c} \bar{\beta}_{a b c}^d = \varepsilon_{b_1 \ldots b_{n-1}} \delta_{b c} B_{a c} - \delta_{c d} B_{a b} = 2 \varepsilon_{b_1 \ldots b_{n-1}} \delta_{b c} B_{a c} . \]  

(3.39)

This expression is a one-cocycle, as can be easily checked, and moreover one that is totally skew-symmetric in the \( n \) indices \( b_1, \ldots, b_{n-1}, d \), so that it corresponds to the Filippov algebra deformations of simple FA's. But since the Whitehead lemma holds for all \( n \geq 2 \) simple FAs [17], the \( \beta^1 \) part of the one-cocycle \( \tilde{\alpha}^1 \) is trivial and can be removed from the deformation.

Now we analyze the consequences of (3.36) for the traceless \( \bar{\alpha}^1 \) by taking suitable contractions. First, one may contract \( c \) with \( b_1 \) in (3.36) with \( \tilde{\alpha}^1 \) instead of \( \alpha^1 \). Then, using that the contractions of \( \bar{\alpha}^1 \) vanish, one obtains

\[ \varepsilon_{b_1} (\bar{\alpha}^1)_{a_1' b_1' b_2'}^d + \varepsilon_{d} (\bar{\alpha}^1)_{b_2' a_1' b_1'}^d = 0 . \]  

(3.40)

Another possibility is to contract \( d \) with \( a_2' \) after multiplying by \( \varepsilon_d \), which gives

\[ (n - 1) (\bar{\alpha}^1)_{a_1' b_1' b_2'}^c + (\bar{\alpha}^1)_{a_1' b_1' b_2'}^c - (\bar{\alpha}^1)_{b_2' a_1' b_1'}^c = 0 . \]  

(3.41)

Let us look at this expression closer by taking the cyclic permutations of (3.41) in in the first three indices. Lowering the superscript we obtain three equations,

\[ \begin{align*}
(n - 1) \bar{\alpha}_{abcd} &= -\bar{\alpha}_{b a c d} + \bar{\alpha}_{c a b d} \\
(n - 1) \bar{\alpha}_{cabd} &= -\bar{\alpha}_{c a b d} + \bar{\alpha}_{b c a d} \\
(n - 1) \bar{\alpha}_{b c d a} &= -\bar{\alpha}_{c b a d} + \bar{\alpha}_{a c b d} .
\end{align*} \]  

(3.42)
This yields a homogeneous linear system of three equations and three unknowns, $\tilde{\alpha}_{abcd}$, $\tilde{\alpha}_{bacd}$ and $\tilde{\alpha}_{cabd}$ (because $\tilde{\alpha}_{abcd} = -\tilde{\alpha}_{acbd}$, see (3.34)). By solving this system one finds

$$(\tilde{\alpha}^1)_{abcd} = -(\tilde{\alpha}^1)_{abcd}$$

$$(3 - n)(\tilde{\alpha}^1)_{abcd} = 0 .$$

We see that for $n \neq 3$ $\tilde{\alpha}^1$ vanishes. On the other hand, the first equation of (3.43), together with (3.40), implies that for $n = 3$ $\varepsilon_d(\tilde{\alpha}^1)_{abcd}$ is completely antisymmetric, hence proportional to $\epsilon_{abcd}$. It is then straightforward to check that $(\tilde{\alpha}^1)_{abc} = \varepsilon_d \epsilon^{abc}$ is a solution of eq. (3.36) for $n = 3$.

### 3.3 The $n = 3$ case

We have shown in Sec. 3.2 above that the most general one-cocycle in the cohomology for the infinitesimal $n$-Leibniz deformations (of the particular type specified above) of the $n > 2$ simple Filippov algebras is trivial except for $n = 3$, in which case

$$(\tilde{\alpha}^1)_{abcd} = t \epsilon_d \epsilon_{abcd} .$$

We now prove that in this $n = 3$ case $\tilde{\alpha}^1$ above is non-trivial since it cannot be generated by a zero-cochain. To see this, we rewrite the triviality condition of eq. (3.33) including the last term in the sum as follows:

$$(\alpha^1)^b_{a_1 \ldots a_n} = -\varepsilon_s (\alpha^0)^b_s \epsilon_{a_1 \ldots a_n}^s + \sum_{r=1}^n \varepsilon_b \epsilon_{a_1 \ldots ar-1sr+1 \ldots a_n}^b (\alpha^0)^s_{ar} .$$

The first term is fully skewsymmetric in $a_1 \ldots a_n$. The skewsymmetry of the second one follows from the identity $\epsilon_{a_1 \ldots a_n b} (\alpha^0)^s_s = 0$, which implies

$$\sum_{r=1}^n \varepsilon_{a_1 \ldots ar-1sr+1 \ldots a_n} b (\alpha^0)^s_{ar} = \epsilon_{a_1 \ldots a_n b} (\alpha^0)^s_s - \epsilon_{a_1 \ldots a_n s} (\alpha^0)^s_b .$$

This shows that the second term in (3.45) is also skewsymmetric in the $a_1, \ldots, a_n$ arguments. Thus, any one-coboundary of the simple FA is necessarily skewsymmetric. In in our case, however, $\tilde{\alpha}^1$ gives, using (3.34),

$$\alpha^1_{abcd} \propto \epsilon_{bc}^{b'} \epsilon_d \epsilon_{a'b'c'd} = 2(\delta_{ba} \delta_{cd} - \delta_{bd} \delta_{ca}) \epsilon_d ,$$

which is not skewsymmetric in $b, c, d$ and, therefore, is a non-trivial one-cocycle. For the Euclidean case, this recovers the $A_4$ deformation given in [22].

### 3.4 The simple 2-Lie (Lie) algebras case and general results

The proof above clearly applies to the Euclidean 2-Lie algebra $A_3$, i.e., to $so(3)$ (as well as to its pseudoEuclidean version $so(1,2)$), but it does not
extend to the other simple 2-Lie algebras of the Cartan classification. Nevertheless, it is easy to prove explicitly that the above result remains true for the Leibniz algebra deformations of all the \( n = 2 \) simple Filippov algebras i.e., that simple (in fact, semisimple) Lie algebras remain Leibniz rigid when viewed as Leibniz algebras.

To this end, we first write the one-cocycle condition (2.14) for \( n = 2 \),

\[
\begin{align*}
\alpha_1^{e} d^e (a_1^{bc}) - f_{e dc} a_1^{de} f_{e ac} & = 0. \\
\end{align*}
\]

Let us separate the symmetric and antisymmetric parts of the Leibniz algebra cocycle \( \alpha_1^{e} \),

\[
\alpha_1^{e} = \alpha_1^{S} + \alpha_1^{A}, \quad \alpha_1^{S} = \alpha_1^{e} d^e (a_1^{bc}) - f_{e dc} a_1^{de} f_{e ac}, \quad \alpha_1^{A} = -f_{e dc} a_1^{de} f_{e ac}.
\]

We show now that, if \( f_{a be} \) are the structure constants of a semisimple Lie algebra \( g \), then \( \alpha_1^{S} = 0 \). Indeed, taking the symmetric part in the indices \( a, b \) of eq. (3.48), we arrive at

\[
\begin{align*}
f_{e dc} a_1^{de} f_{e ac} & = 0. \\
\end{align*}
\]

Contracting this expression with \( f_{e' dc} \) we obtain \( k_{e' e} \alpha_1^{S} = 0 \), where \( k_{e' e} \) are the coordinates of the Cartan-Killing metric. Since \( g \) is semisimple, \( k \) is non-degenerate and necessarily \( \alpha_1^{S} = 0 \). Therefore, \( \alpha_1^{A} \) is antisymmetric and hence a one-cocycle for the deformation problem of semisimple Lie algebras. Since these deformations are all trivial by virtue of the Whitehead Lemma, it follows that there are no non-trivial Leibniz deformations of semisimple Lie algebras.

Collecting the above \( n \geq 2 \) results, we have thus proved the following

**Theorem 1.** The \( n \)-Leibniz algebra deformations of the \((n+1)\)-dimensional simple FA’s that preserve the skewsymmetry of the \((n-1)\) first elements in the \( n \)-Leibniz bracket (or that of the fundamental objects) are all trivial for \( n > 3 \). Further, all \( n = 2 \) semisimple Filippov (i.e., Lie) algebras are rigid as Leibniz algebras.

One may ask what makes the simple FA \( n = 3 \) deformation case special. In the present context this is simply answered by noticing that, for the class of \( n \)-Leibniz deformations we are considering, a one-cocycle with a dual given by the four index Levi-Civita fully antisymmetric symbol may exist only for a four dimensional simple Filippov algebra. Since all the simple real Filippov algebras are mixed signature versions of the Euclidean FA \( A_{n+1} \) and have dimension \( n + 1 \), this gives \( n = 3 \).

### 4 A class of \( n \)-Leibniz central extensions of simple FAs

Let us now move to the case of the \( n \)-Leibniz central extensions of the simple FAs when the resulting Leibniz algebra is of the type considered in Sec. 3.
that is, one with an \( n \)-bracket required to be skewsymmetric in its first \( n - 1 \) entries.

As stated in Sec. 3.2, all formulae relevant for the central extension cohomology may easily be derived from those in Secs. 3.1, 3.2 by taking \( \mathbb{R} \)-valued cochains \( \text{i.e.} \), by removing the first upper index and by eliminating from the expression of the action of the coboundary operator \( \delta \) the terms containing the action on the \( \mathbb{L} \)-valued cochains. In this way, the dualized coordinate expression for the one-cocycle and one-coboundary conditions become, respectively,

\[
\varepsilon \nu_2 \delta_{b_1}^c (\tilde{\alpha}^1)_{a_1' a_2'} - \varepsilon \nu_2 \delta_{b_2}^c (\tilde{\alpha}^1)_{a_1' a_2'} + \varepsilon \nu_2 \delta_{b_1 a_2'} (\tilde{\alpha}^1)_{a_1' a_2'} - \varepsilon \nu_2 \delta_{b_1 a_1'} (\tilde{\alpha}^1)_{a_2' b_2} - \varepsilon \nu_2 \delta_{b_2 a_1'} (\tilde{\alpha}^1)_{a_2' b_2} = 0 \tag{4.50}
\]

and

\[
(\delta \tilde{\alpha}^1)_{b_1 b_2} = -\varepsilon \nu_2 \delta_{b_2}^c (\alpha^0)_{b_2} + \varepsilon \nu_2 \delta_{b_2}^c (\alpha^0)_{b_1} , \tag{4.51}
\]

where, in a way analogous to eq. (3.34), \( \tilde{\alpha}^1 \) is given by

\[
\tilde{\alpha}^1_{b_1 b_2} = \frac{1}{(n - 1)!} \varepsilon^{b_1 b_2 a_1 ... a_{n-1}} \tilde{\alpha}^1_{a_1 ... a_{n-1} c} . \tag{4.52}
\]

Given an \( \tilde{\alpha}^1 \) that satisfies the one-cocycle condition (4.50), we now consider the equivalent cocycle \( \tilde{\alpha}^1 \) given by

\[
(\tilde{\alpha}^1)_{b_1 b_2} = (\tilde{\alpha}^1)_{b_1 b_2} - \frac{1}{n} \delta_{b_1}^c (\tilde{\alpha}^1)_{eb_2} + \frac{1}{n} \delta_{b_2}^c (\tilde{\alpha}^1)_{eb_1} e , \tag{4.53}
\]

where the last two terms define a one-coboundary generated by \( \alpha^0 = \frac{1}{2} \varepsilon_{b_2} (\tilde{\alpha}^1)_{eb_2} e \) (see (4.51)). This new cocycle \( \tilde{\alpha}^1 \) also obeys eq. (4.50), and has the property that the trace \( (\tilde{\alpha}^1)_{b_1} e \) vanishes. In this way, if one contracts \( b_1' \) with \( c \) in eq. (4.50) for \( \tilde{\alpha}^1 \) and takes into account the vanishing of the trace, one arrives at

\[
n(\tilde{\alpha}^1)_{b_1 b_2} = 0 , \tag{4.54}
\]

which shows that \( \tilde{\alpha}^1 = 0 \) and hence that all one-cocycles are trivial for any \( n \). We have thus proved the following

**Theorem 2.** The \( n \)-Leibniz algebra central extensions of simple FA’s that preserve the skewsymmetry of the \( (n - 1) \) first entries of the \( n \)-bracket (or of the fundamental objects) are all trivial for any \( n > 2 \).

The proof above also applies to the 2-Lie simple algebras \( A_3 \) (so(3)) and so(1, 2); for general simple Lie algebras, see [4] and [30] (Prop. 3.2 and Cor. 3.7).
5 Conclusions

We have shown that the simple Filippov algebras, when viewed as \( n \)-Leibniz algebras of the class that have \( n \)-brackets antisymmetric in the first \( n - 1 \) entries (and thus skewsymmetric fundamental objects), are rigid but for \( n = 3 \) (Theorem 1). Further, simple \( n \)-Lie algebras do not have \( n \)-Leibniz central extensions within the same class (Theorem 2).

Obviously, there is the question of whether a further relaxing of the full skewsymmetry condition, \textit{i.e.}, whether allowing for more general \( n \)-Leibniz brackets, results in more deformations or non-trivial central extensions. In the case of deformations, we expect that the situation will change, because the natural cochains dual to those with less than \( n - 1 \) antisymmetric entries have more than four indices, and Levi-Civita symbols with five or more indices may lead to additional non-trivial \( n \)-Leibniz deformations of the simple Filippov algebras.

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