Abstract. In this paper we prove that every quasi-projective base space $V$ of smooth family of minimal projective manifolds with maximal variation is pseudo Kobayashi hyperbolic, i.e. $V$ is Kobayashi hyperbolic modulo a proper subvariety $Z \subset V$. In particular, $V$ is algebraically degenerate, that is, every nonconstant entire curve $f : \mathbb{C} \to V$ has image $f(\mathbb{C})$ which lies in that proper subvariety $Z \subset V$. As a direct consequence, we prove the Brody hyperbolicity of moduli spaces of minimal projective manifolds, which answers a question by Viehweg-Zuo in 2003.

0. Introduction

0.1. Brody hyperbolicity of moduli spaces. In [VZ03] Viehweg-Zuo proved that moduli spaces of canonically polarized complex manifolds are Brody hyperbolic, which can be seen as the higher dimensional analytic Shafarevich hyperbolicity conjecture. Very recently, Popa-Taji-Wu [PTW18] extended Viehweg-Zuo’s theorem to moduli spaces of minimal projective manifolds of general type. In this paper, we prove a more general result, which answers a question by Viehweg-Zuo [VZ03, Question 0.2].

Theorem A (Brody hyperbolicity of moduli). Let $U \to V$ be a smooth family of polarized minimal projective manifolds over the quasi-projective manifold $V$ with the Hilbert polynomial $h$. Assume that the induced moduli map $V \to P_h$ is quasi-finite, where $P_h$ denotes to be the (quasi-projective) coarse moduli space associated to the moduli functor $\mathcal{P}_h$. Then $V$ is Brody hyperbolic, i.e. there are no nonconstant entire holomorphic curves on $V$.

0.2. Main tools: Viehweg-Zuo Higgs bundles. In the series of works [VZ01, VZ02, VZ03], Viehweg-Zuo studied families of minimal projective manifolds of maximal variation, by constructing the so-called Viehweg-Zuo Higgs bundles (see Definition 1.2) on the base spaces. The Viehweg-Zuo Higgs bundles ($VZ$ Higgs bundles for short) turned to be a powerful technique towards understanding the moduli spaces. For instance, combing the celebrated work by Campana-Pâun [CP15a,CP15b,CP16], the existence of VZ Higgs bundles on the aforementioned base spaces proved in [VZ02] implies the Viehweg hyperbolicity conjecture: the base spaces of smooth families of minimal projective manifolds with maximal variation are of log general type. These results were later extended in [PS17] to bases of families whose...
generic fibers admit a good minimal model. When the “generic Torelli property” (see Theorem C for the definition) holds for the VZ Higgs bundles, it was established in [VZ03, PTW18] that the base space is algebraically degenerate, i.e. all the non-constant entire curves lie on a proper subvariety of the base. In [VZ03, PTW18], such generic Torelli theorems are proved via vanishing theorems, which are unclear to us when the Kodaira dimension of fibers is not maximal. In Theorem C we prove that the generic Torelli property is indeed an intrinsic feature of all VZ Higgs bundles (not related to the Kodaira dimension of fibers!) using different approaches.

0.3. Pseudo Kobayashi hyperbolicity. A quasi-projective variety $V$ is called pseudo Kobayashi hyperbolic if $V$ is Kobayashi hyperbolic modulo a proper subvariety $Z \subsetneq V$, i.e. the Kobayashi-Royden infinitesimal pseudo-metric $\kappa_V$ of $V$ is positive definite outside $Z$. When $Z = \emptyset$, this definition reduces to the usual definition of Kobayashi hyperbolicity. Such a $V$ is in particular algebraically degenerate. In [Den18], we proved that quasi-projective base manifolds of effectively parametrized families of minimal projective manifolds of general type are Kobayashi hyperbolic, which generalized the previous work by To-yeung [TY15]. A crucial ingredient of our proof is the construction of (possibly degenerate) Finsler metrics on base spaces whose holomorphic sectional curvatures are bounded above by a negative constant via VZ Higgs bundles satisfying the generic Torelli property. Hence by [Den18, §3] and Theorem C, we conclude the following result.

**Theorem B** (Pseudo Kobayashi hyperbolicity of bases). Let $f : U \to V$ be a smooth family of minimal projective manifolds over the quasi-projective variety $V$, which is of maximal variation. Then $V$ is pseudo-Kobayashi hyperbolic. In particular, $V$ is algebraically degenerate.

As mentioned above, the Viehweg hyperbolicity conjecture proved by Campana-Păun shows that the base space $V$ in Theorem B is of log general type. Hence Theorem B is predicted by a famous conjecture by S. Lang, which stipulates that a quasi-projective variety is pseudo Kobayashi hyperbolic if it is of log general type.

**Remark 0.1.** Using the deep theories of Hodge modules, Popa-Taji-Wu constructed VZ Higgs bundles over quasi-projective base spaces of smooth families with maximal variation, whose generic fibers admit good minimal models (see [PTW18, Proposition 2.7]). Applying their result instead of Theorem 1.1 and using our generic Torelli theorem for all VZ Higgs bundles in Theorem C, Theorem B can be extended to the pseudo Kobayashi hyperbolicity of those base spaces.

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1. **Proofs of Theorems**

In this section, we first recall the Viehweg-Zuo Higgs bundles first constructed in [VZ01, VZ02, VZ03], and later developed in [PS17, PTW18] via Hodge modules. Then we collect some properties (see also [Den18, §2]), and prove a Torelli-type theorem for the VZ Higgs bundles, which is the main result of our paper.
Theorem 1.1 (Viehweg-Zuo, Popa-Taji-Wu). Let $U' \to V'$ be a smooth family of minimal projective manifolds over the quasi-projective variety $V'$, which is of maximal variation. Then there exists a birational morphism $\nu : V \to V'$ and a projective compactification $Y \supset V$, and two logarithmic Higgs bundles $(\bigoplus_{q=0}^{n} F_{n}^{q}, \bigoplus_{q=0}^{n} \tau_{n-q,q})$, $(\bigoplus_{q=0}^{n} E^{n-q,q}, \bigoplus_{q=0}^{n} \theta_{n-q,q})$ together with a big and nef line bundle $L$ over $Y$ satisfying the following properties:

(i) There is a diagram

$$
\begin{array}{ccc}
\mathcal{L}^{-1} \otimes E^{n-q,q} & \xrightarrow{1 \otimes \rho_{n-q,q}} & \mathcal{L}^{-1} \otimes E^{n-q-1,q+1} \otimes \Omega_{Y}(\log(D+S)) \\
F^{n-q,q} & \xrightarrow{\tau_{n-q,q}} & F^{n-q-1,q+1} \otimes \Omega_{Y}(\log(D))
\end{array}
$$

where both $D := Y \setminus V$ and $D + S$ are simple normal crossing divisors in $Y$, and $\iota : \Omega_{Y}(\log(D)) \to \Omega_{Y}(\log(D + S))$ is the natural inclusive map.

(ii) $(\bigoplus_{q=0}^{n} E^{n-q,q}, \bigoplus_{q=0}^{n} \theta_{n-q,q})$ is the logarithmic Higgs bundle underlying the Deligne extension with the real part of the eigenvalues of residues in $(0, 1)$ of a (polarized) variation of Hodge structures defined over $V_{0} := Y \setminus D \cup S$.

(iii) There is an injection $\mathcal{O}_{Y} \hookrightarrow F^{n,0}$ which is isomorphic over $V_{0}$.

(iv) $V_{0} \cap B_{+}(\mathcal{L}) = \emptyset$.

(v) For any $k = 1, \ldots, n$, (1.1) induces a map

$$
\tau_{k} : \text{Sym}^{k} \mathcal{F}_{Y}(\log D) \to \mathcal{L}^{-1} \otimes E^{n-k,k}.
$$

In [VZ02], the two Higgs bundles in Theorem 1.1 are only constructed on a big open set of $Y$. In order to extend them to the whole $Y$, we have to apply a technical lemma in [PTW18, Proposition 4.4], so that the discriminant of the zero divisor defined by a certain hypersurface used for the cyclic construction, is a priori normal crossing. Theorem 1.1.(v) is proved in [VZ03] for canonically polarized families, and in [PTW18] for smooth families admitting good minimal models. The readers can also refer to [Den18, §2] for further details.

Sketch of proof of Theorem 1.1. By [Den18, Remark 1.22.(ii)] (or [VZ02, Proposition 3.9] if we are allowed to take a birational model of $U' \to V'$), after passing to a $r$-folded fiber product $U^{nr} := U' \times_{V'} U' \times_{V'} \cdots \times_{V'} U'$, there exists a projective compactification

$$
\begin{array}{ccc}
U^{nr} & \xrightarrow{\nu} & X' \\
\downarrow & & \downarrow \\
V' & \xrightarrow{\iota} & Y'
\end{array}
$$

so that $mK_{X'/Y'} - m f^{*} \mathcal{A}$ is globally generated over $f^{-1}(\Omega)$ for $m \gg 0$, where $\Omega \subset V'$ is some Zariski open set, and $\mathcal{A}$ is a sufficiently ample line bundle. By [PTW18, Proposition 4.4] (see also [Den18, Theorem 1.23] if we need to control exceptional
locus of the birational morphism), there exists a birational morphism

$$
\begin{array}{c}
X' \leftarrow^\text{bir} \\
\downarrow \quad \downarrow f \\
Y' \leftarrow^\text{bir} Y
\end{array}
$$

so that there exists a hypersurface

$$
H \in \mid \ell \Omega^n_{X/Y}(\log \Delta) - \ell f^* \nu^* \mathcal{A} + E \mid
$$

satisfying

- There exists a reduced divisor $S$ so that $D + S$ is simple normal crossing, and $H \to Y$ is smooth over $V_0 := Y \setminus (D \cup S)$, where $D := \nu^{-1}(Y' \setminus V')$.
- $E$ is some $f$-exceptional divisor with $f(E) \subset \text{Supp}(D + S)$.

Here we denote by $\Delta := f^{-1}(D)$ so that $(X, \Delta) \to (Y, D)$ is a log morphism. Within this basic setup, let us define the two Higgs bundles in the theorem following [VZ02, §4]. Leaving out a codimension two subvariety of $Y$ supported on $D + S$, we assume that

- $f$ is flat, and $E$ in (1.3) disappears.
- The divisor $D + S$ is smooth. Moreover, both $\Delta$ and $\Sigma = f^{-1}S$ are relative normal crossing.

Set $\mathcal{L} := \Omega^*_{X/Y}(\log \Delta)$, and $\mathcal{L} := \nu^* \mathcal{A}$. Let $\delta : W \to X$ be a blow-up of $X$ with centers in $\Delta + \Sigma$ such that $\delta^*(H + \Delta + \Sigma)$ is a normal crossing divisor. One thus obtains a cyclic covering of $\delta^*H$, by taking the $\ell$-th root out of $\delta^*H$. Let $Z$ to be a strong desingularization of this covering. We denote the compositions by $h : W \to Y$ and $g : Z \to Y$. Write $\Pi := g^{-1}(S \cup D)$ which is assumed to be normal crossing. Leaving out more codimension two subvarieties supported $D + S$, we assume that $h$ and $g$ are also flat, and both $\delta^*(H + \Delta + \Sigma)$ and $\Pi$ are relative normal crossing. Then the restrictions of both $g$ and $h$ to $V_0$ are smooth. Set

$$
F^{n-q,q} := R^q h_* \left( \delta^* \left( \Omega^n_{X/Y}(\log \Delta) \right) \otimes \delta^* \mathcal{L}^{-1} \otimes \mathcal{O}_W \left( \frac{\delta^* H}{\ell} \right) \right) / \text{torsion}.
$$

It was shown in [VZ02, §4] that there exists a natural edge morphism

$$
\tau_{n-q,q} : F^{n-q,q} \to F^{n-q-1,q+1} \otimes \Omega_Y(\log D),
$$

which gives rise to the first Higgs bundle $\left( \bigoplus_{q=0}^n F^{n-q,q}, \bigoplus_{q=0}^n \tau_{n-q,q} \right)$ defined over a big open set of $Y$ containing $V_0$.

Write $Z_0 := Z \setminus \Pi$. Then the local system $R^n g_* \mathcal{C} \mid Z_0$ extends to a locally free sheaf $\mathcal{V}$ on $Y$ (here $Y$ is projective rather than the big open set!) equipped with the meromorphic connection

$$
\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega_Y \left( \log(D + S) \right),
$$

whose eigenvalues of the residues lie in $[0, 1]$ (the so-called lower canonical extension). By [Sch73, CKS86], the Hodge filtration of $R^n g_* \mathcal{C} \mid Z_0$ extends to a filtration $\mathcal{V} := \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^n$ of subbundles so that their graded sheaves $E^{p,n-p} := \mathcal{F}^p / \mathcal{F}^{p+1}$ are also locally free, and there exists

$$
\theta_{p,n-p} : E^{p,n-p} \to E^{p-1,n-p+1} \otimes \Omega_Y(\log D + S).
$$
This defines the second Higgs bundle \((\bigoplus_{q=0}^{n} E^{n-q,q}, \theta_{n-q,q})\). As observed in [VZ02, VZ03], \(E^{n-q,q} = R^q_g \Omega^{n-q}_{Z/Y}(\log \Pi)\) over a big open set of \(Y\) by the Steenbrink’s theorem, which in turn implies that (1.1) holds over a (smaller) big open set in which the Higgs bundle \((\bigoplus_{q=0}^{n} F^{n-q,q}, \tau_{n-q,q})\) is also defined, by [VZ03, Lemma 6.2] (cf. also [VZ02, Lemma 4.4]).

Note that all the objects are defined on a big open set of \(Y\) except for \((\bigoplus_{q=0}^{n} E^{n-q,q}, \theta_{n-q,q})\), which are defined on the whole \(Y\). Following [VZ03, §6], for every \(q = 0, \ldots, n\), we define \(F^{n-q,q}\) to be the reflexive hull, and the morphism \(\tau_{n-q,q}\) extends naturally. Since each \(E^{n-q,q}\) is locally free, and is defined on \(Y\), then the morphism \(\rho_{n-q,q}\) also extends. This leads to Theorems 1.1.(i) and 1.1.(ii). While Theorem 1.1.(iii) follows from [VZ02, Lemma 4.4.(ii)], Theorem 1.1.(iv) can be easily seen from the construction.

To prove Theorem 1.1.(v), we have to introduce a sub-Higgs bundle of \((\bigoplus_{q=0}^{n} \mathcal{L}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^{n} \mathbb{I} \otimes \theta_{n-q,q})\) following [VZ02, Corollary 4.5] (or [PTW18]). Write \(\tilde{\theta}_{n,q,q} := \mathbb{I} \otimes \theta_{n,q,q}\) for short. For each \(q = 1, \ldots, n\), we define a coherent torsion free sheaf \(\tilde{F}^{n,q} := \rho_{n,q,q}(\tilde{F}^{n,q,q}) \subset E^{n-q,q}\). By [VZ02, Lemma 4.4.(iv)], \(\rho_{n,q}\) is an injection, and thus \(\tilde{F}^{n,0} \simeq F^{n,0} \supset \mathcal{O}_Y\). By (1.1), one has

\[
\tilde{\theta}_{n,q,q} : \tilde{F}^{n,q,q} \to \tilde{F}^{n,q,q+1} \otimes \Omega_Y(\log D),
\]

and let us by \(\eta_{n,q,q}\) the restriction of \(\tilde{\theta}_{n,q,q}\) to \(\tilde{F}^{n,q,q}\). Then \((\bigoplus_{q=0}^{n} \tilde{F}^{n,q,q}, \bigoplus_{q=0}^{n} \eta_{n,q,q})\) is a sub-Higgs bundle of \((\bigoplus_{q=0}^{n} \mathcal{L}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^{n} \tilde{\theta}_{n,q,q})\). In particular, \(\eta_{n,q+1,q-1} \circ \cdots \circ \eta_{n,0}\) factors like

\[
F^{n,0} \to \tilde{F}^{n,q} \otimes \text{Sym}^q \Omega_Y(\log D) \subset \tilde{F}^{n,q} \otimes \bigotimes^q \Omega_Y(\log D).
\]

By Theorem 1.1.(iii), this induces a morphism

\[
\mathcal{O}_Y \to \tilde{F}^{n,0} \to \tilde{F}^{n,q} \otimes \text{Sym}^q \Omega_Y(\log D) \to \mathcal{L}^{-1} \otimes E^{n-q,q} \otimes \text{Sym}^q \Omega_Y(\log D),
\]

and equivalently

\[
\tau_q : \text{Sym}^q \mathcal{O}_Y(\log D) \to \mathcal{L}^{-1} \otimes E^{n-q,q},
\]

which is the desired morphism in (1.2).

\[
\square
\]

**Definition 1.2.** The negatively twisted Higgs bundle \((\bigoplus_{q=0}^{n} \mathcal{L}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^{n} \mathbb{I} \otimes \theta_{n-q,q})\) satisfying Properties (i)-(v) in Theorem 1.1 is called Viehweg-Zuo Higgs bundle, and \(\mathcal{L}\) is called the Viehweg-Zuo (big) sheaf.

**Theorem C** (Generic Torelli property). **Same assumption as Theorem 1.1.** Then \(\tau_1\) defined in (1.2) is generically injective.

Let us stress here that, differently from [Den18, Theorem 2.1.(vi)], we cannot give a precise description of the loci where \(\tau_1\) is injective since our method in proving Theorem C relies on the global aspects of the VZ Higgs bundles. Roughly speaking, the bigness of the Viehweg-Zuo sheaf \(\mathcal{L}\) forces \(\tau_1\) to be injective at least one point, which is analogous to Demailly’s (weak) holomorphic Morse inequality (see [Dem12, §8.2.(a)]).

Before we prove Theorem C, we will recall some technical preliminaries. It was initiated in [VZ03], and later developed in [PTW18] that, one has to take some
proper metric $g$ for $\mathcal{L}$ so that $g^{-1}$ can compensate the mild (blow-up) singularities of the Hodge metric $h_{\text{hod}}$ of $\bigoplus_{q=0}^{n} E^{n-q,q}$.

**Proposition 1.3** ([PTW18, Lemma 3.1, Corollary 3.4]). Same notation as Theorem 1.1. There exists a singular metric $g$ for $\mathcal{L}$ which is smooth over $V_0 := Y \setminus D \cup S$ such that

(i) over $V_0$, the curvature form $\sqrt{-1} \Theta_g(\mathcal{L})|_{V_0}$ is positive definite everywhere.

(ii) The singular hermitian metric $h := g^{-1} \otimes h_{\text{hod}}$ on $\mathcal{L}^{-1} \otimes \bigoplus_{q=0}^{n} E^{n-q,q}$ is locally bounded on $Y$, and smooth over $V_0$. Moreover, $h$ is degenerate on $D \cup S$.

Although the last statement of Proposition 1.3(ii) is not explicitly stated in [PTW18], it can be easily seen from the proof of [PTW18, Corollary 3.4].

**Proof of Theorem C.** By Theorem 1.1.(iii), $\rho_{n,0}$ induces a global section $s \in H^0(Y, \mathcal{L}^{-1} \otimes E^{n,0})$, which is generically non-vanishing over $V_0$. Set

$$V_1 := \{ y \in V_0 \mid s(y) \neq 0 \}$$

which is a non-empty Zariski open set of $V_0$. For the first stage of VZ Higgs bundle $(\mathcal{L}^{-1} \otimes E^{n,0}, h)|_{V_0}$ over $V_0$, let us denote by $\Theta_0$ its curvature form and set $D'$ to be the $(1,0)$-part of its Chern connection. Then by the Griffiths curvature formula of Hodge bundles (see e.g. [GT84]), over $V_0$ we have

$$\Theta_0 = -\Theta_{\mathcal{L},g} \otimes 1 + 1 \otimes \Theta_{h_{\text{hod}}}(E^{n,0})$$

$$= -\Theta_{\mathcal{L},g} \otimes 1 - 1 \otimes (\Theta^*_{n,0} \wedge \Theta_{n,0})$$

$$= -\Theta_{\mathcal{L},g} \otimes 1 - \tilde{\Theta}^*_{n,0} \wedge \tilde{\Theta}_{n,0},$$

where we set $\tilde{\Theta}_{n-k} := 1 \otimes \Theta_{n-k} : \mathcal{L}^{-1} \otimes E^{n-k,k} \to \mathcal{L}^{-1} \otimes E^{n-k-1,k+1} \otimes \Omega_Y(\log(D+S))$, and define $\tilde{\Theta}^*_{n,0}$ to be the adjoint of $\tilde{\Theta}_{n,0}$ with respect to the metric $h$. Hence over $V_1$ one has

$$-\sqrt{-1} \partial \bar{\partial} \log |s|^2_h = \left\{ \frac{\sqrt{-1} \Theta_0(s,s)}{|s|^2_h} \right\}_h + \frac{\sqrt{-1} \{ D's, s \}_h \wedge \{ s, D's \}_h}{|s|^4_h} - \frac{\sqrt{-1} \{ D's, D's \}_h}{|s|^2_h}$$

$$\leq \frac{\left\{ \sqrt{-1} \Theta_0(s,s) \right\}_h}{|s|^2_h}$$

(1.8)

thanks to the Lagrange’s inequality

$$\sqrt{-1} |s|^2_h \cdot \{ D's, D's \}_h \geq \sqrt{-1} \{ D's, s \}_h \wedge \{ s, D's \}_h.$$

Putting (1.7) to (1.8), over $V_1$ one has

$$\sqrt{-1} \Theta_{\mathcal{L},g} - \sqrt{-1} \partial \bar{\partial} \log |s|^2_h \leq - \frac{\left\{ \sqrt{-1} \tilde{\Theta}^*_{n,0} \wedge \tilde{\Theta}_{n,0}(s,s) \right\}_h}{|s|^2_h} = \frac{\sqrt{-1} \{ \tilde{\Theta}_{n,0}(s), \tilde{\Theta}_{n,0}(s) \}_h}{|s|^2_h}$$

(1.9)

where $\tilde{\Theta}_{n,0}(s) \in H^0(Y, \mathcal{L}^{-1} \otimes E^{n-1,1} \otimes \Omega_Y(\log(D+S)))$. By Proposition 1.3(ii), for any $y \in D \cup S$, one has

$$\lim_{y' \to y} |s|^2_h(y') = 0.$$
Therefore, it follows from the compactness of $Y$ that there exists $y_0 \in V_0$ so that $|s|^2_h(y_0) \gg |s|^2_h(y)$ for any $y \in V_0$. Hence $|s|^2_h(y_0) > 0$, and by (1.6), $y_0 \in V_1$. Since $|s|^2_h$ is smooth over $V_0$, then $\sqrt{-T\partial\bar{T}\log |s|^2_h(y_0)}$ is semi-negative. By Proposition 1.3.(i), $\sqrt{-T\partial\bar{T}g}$ is strictly positive at $y_0$. By (1.9) and $|s|^2_h(y_0) > 0$, we conclude that $\sqrt{-T\{\tilde{\theta}_{n,0}(s), \tilde{\theta}_{n,0}(s)\}}_h$ is strictly positive at $y_0$. In particular, for any non-zero $\xi \in \mathcal{R}_{Y,y_0}$, $\tilde{\theta}_{n,0}(s)(\xi) \neq 0$. For

$$\tau_1 : \mathcal{F}_Y(- \log D) \rightarrow \mathcal{L}^{-1} \otimes E^{n-1,1}$$

in (1.2), over $V_0$ it is defined by $\tau_1(\xi) := \tilde{\theta}_{n,0}(s)(\xi)$ by (1.1), which is thus injective at $y_0 \in V_1$. Hence $\tau_1$ is generically injective. The theorem is thus proved. \hfill \Box

Proof of Theorem B. By Theorem 1.1, there exists a VZ Higgs bundle $(\bigoplus_{q=0}^n \mathcal{L}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^n 1 \otimes \theta_{n-q,q})$ over a compactification $Y'$ of some birational model $\nu : V' \rightarrow V$. Write $D := Y' \setminus V'$. In [Den18, §3] we construct a (possibly degenerate) Finsler metric $\mathcal{F}$ over $\mathcal{F}_V$, defined by

\begin{equation}
\mathcal{F} := \left( \sum_{k=1}^n \alpha_k \mathcal{F}_k^{2} \right)^{1/2},
\end{equation}

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$, and $\mathcal{F}_k$ is the Finsler metric on $V'$ defined by

\begin{equation}
\mathcal{F}_k(\xi) := |\tau_k(\xi^{\otimes k})|_h^{\frac{1}{k}}, \quad \forall \xi \in \mathcal{F}_V,
\end{equation}

with $\tau_k : \text{Sym}^k \mathcal{F}_V(- \log D) \rightarrow E^{-1} \otimes E^{n-k,k}$ defined in (1.2), and $h$ the singular hermitian metric for $\bigoplus_{q=0}^n \mathcal{L}^{-1} \otimes E^{n-q,q}$ defined in Proposition 1.3.(ii). By Theorem C, $\tau_1$ is injective over a non-empty Zariski open set $V''$ of $V'$. Hence $\mathcal{F}_1$ is positive definite over $V_1 := V'' \cap V_0$ by (1.11), where $V_0 \subset V'$ is the Zariski open set defined in Theorem 1.1. By (1.10) $\mathcal{F}$ is also positive definite over the non-empty Zariski open set $V_1$. In [Den18, Proposition 3.14], we proved that when $\alpha_1, \ldots, \alpha_k \in \mathbb{R}^+$ are properly chosen, the holomorphic sectional curvature of $\mathcal{F}$ are bounded from above by a negative constant. It then follows from Demailly’s Ahlfors-Schwarz lemma [Dem97, Lemma 3.2] that the Kobayashi-Royden infinitesimal metric $\kappa_V$ of $V'$ is positive definite over $V_1$. By the bimeromorphic criteria of Kobayashi hyperbolicity in [Den18, Lemma 3.3], we conclude that $\kappa_V$ is positive definite over the non-empty Zariski open set $\nu(V_1) \subset V$. This proves the theorem. \hfill \Box

Remark 1.4. It is natural to ask whether the base space $V'$ of an effectively parametrized family of minimal projective manifolds is Kobayashi hyperbolic. When the fibers are of general type, the answer is positive by [Den18, Theorem A]. By the proof of Theorem B, in order to prove the Kobayashi hyperbolicity of $V'$, one has to assure that for any given point $p \in V$,

- (∗) there exists a birational morphism $\nu : V \rightarrow V'$ in Theorem 1.1 which is isomorphic at $p$, and $\nu^{-1}(p) \in V_0$.
- (♣) $\tau_1$ in (1.2) is injective at the point $\nu^{-1}(p)$.

When the fibers are of general type, (∗) relies on the result of positivity of direct images in [Den18, Theorem B.(iii)], and (♣) can be shown via the Bogomolov-Sommese vanishing theorem by [PTW18]. For general cases, both (∗) and (♣)
unclear to us. Indeed, as mentioned above, our proof of Theorem C cannot show the precise loci where \( \tau_1 \) is injective.

A standard inductive arguments in [VZ03] can easily show that Theorem B implies Theorem A.

**Proof of Theorem A.** We will proceed by contradiction. Suppose that there exists a non-constant holomorphic map \( f : \mathbb{C} \to V \). By Theorem B, \( f \) cannot be Zariski dense. Let \( Z := \overline{f(\mathbb{C})}^{\text{Zar}} \) be its Zariski closure. Take a desingularization \( \pi : Z' \to Z \).

Then there exists a lift \( f' : \mathbb{C} \to Z' \) so that \( \pi \circ f' = f \), which is also Zariski dense. Since the new family \( U \times_V Z' \to Z' \) is still of maximal variation by the quasi-finiteness of the moduli map, by Theorem B again, \( Z' \) must be algebraically degenerate. This is a contradiction. \( \square \)

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