Coulomb scattering of the Dirac fermions on de Sitter expanding universe

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Abstract
The lowest order contribution of the amplitude of the Dirac - Coulomb scattering in de Sitter spacetime is calculated assuming that the initial and final states of the Dirac field are described by exact solutions of the free Dirac equation on de Sitter spacetime with a given momentum and helicity. One studies the difficulties that arises when one passes from the amplitude to cross section.

1 Introduction

It is known that many difficult problems arises when one tries to combine the quantum theory of fields with general relativity. One of the main problems is related to the physical interpretation of the one-particle quantum modes that may indicate how to quantize the fields. The curved spacetime have specific symmetries which, in general, differ from that of the Minkowski spacetime. For this reason, here the symmetries generating conserved quantities have to be treated with specific methods. In other respects, it is known that the form of the fields equations and implicitly their solutions on curved spacetime are strongly dependent on the tetrad gauge and local chart in which one works. It is important to point out that in the latest years important steps in developing a quantum field theory on de Sitter spacetime was made by finding analytical solutions for the Dirac equation in moving or static local charts suitable for separation of variables [2], [8], [9], [10].

In the present the majority of investigations dedicated to Q.E.D on curved spacetime do not take in considerations scattering processes. Our aim in this paper is to calculate one scatter amplitude for Dirac field in curved spacetime and to analyze the physical consequences that emerge from this calculation. Differences with respect to the Minkowski case are that the modulus of the momentum is not conserved and a tendency for helicity conservation. Also in the limit of a vanishing expansion rate of the space the Minkowski amplitude
will not be recovered. Finally, we obtain that the cross section is a sum of two contributions, that corresponds to linear and nonlinear amplitude of scattering.

In section 2 we derive the formula for the scattering amplitude. In section 3 we discuss some of properties of scatter amplitude and we will examine the limit cases of a small/large expansion rate of the space compared with the mass of particle, in section 4 we calculate the cross section. Our conclusions are summarized in section 5 pointing out a series of aspects which remain to be clarified about this subject. The results are presented in natural units \(\hbar = c = 1\).

## 2 The scattering amplitude

We start with the exact solutions of the free Dirac equation in the de Sitter spacetime written in \[2\]. Let us write the de Sitter line element \[1\],

\[
    ds^2 = dt^2 - e^{2\omega t}d\vec{x}^2,
\]

(1)

where \(\omega\) is the expansion factor and \(\omega > 0\). Now we know that defining a spinor field on curved spacetime requires one to use the tetrad fields \(e^\mu_\nu(x)\) and \(\hat{e}^\mu_\nu(x)\), fixing the local frames and corresponding coframes which are labelled by the local indices \(\mu, \nu, \ldots = 0, 1, 2, 3\). The form of the line element allows one to chose the simple Cartesian gauge with the non-vanishing tetrad components:

\[
    e^0_\mu = e^{-\omega t}; \quad e^i_\mu = \delta^i_j e^{-\omega t},
\]

(2)

so that \(e_\mu = e^\mu_\nu e^\nu_\mu\) and have the orthonormalization properties \(e_\mu e_\nu = \eta_\mu_\nu\), \(\hat{e}_\mu e_\nu = \delta^\mu_\nu\) with respect to the Minkowski metric \(\eta = \text{diag}(1, -1, -1, -1)\).

Now let us introduce normalized helicity spinors for an arbitrary vector \(\vec{p}\) by notation: \(\xi_\lambda(\vec{p})\),

\[
    \hat{\sigma} \vec{p} \xi_\lambda(\vec{p}) = 2p_\lambda \xi_\lambda(\vec{p}),
\]

(3)

with \(\lambda = \pm 1/2\) and where \(\hat{\sigma}\) are the Pauli matrices and \(p = |\vec{p}|\). For writing the solutions of Dirac equation on de Sitter spacetime we set:

\[
    k = \frac{m}{\omega}, \quad \nu_\pm = \frac{1}{2} \pm ik.
\]

(4)

Then the positive frequency modes of momentum \(\vec{p}\) and helicity \(\lambda\) that were constructed in \[2\] using the gamma matrices in Dirac representation (with diagonal \(\gamma^0\)) are:

\[
    U_{\vec{p},\lambda}(t, \vec{x}) = \frac{\sqrt{p/\omega}}{(2\pi)^{3/2}} \left( \frac{1}{\lambda e^{\pi k/2} H_{\nu_+}^{(1)}(\omega e^{-\omega t})\xi_\lambda(\vec{p})}{\lambda e^{-\pi k/2} H_{\nu_-}^{(1)}(\omega e^{-\omega t})\xi_\lambda(\vec{p})} \right) e^{i\vec{p}\vec{x} - 2\omega t},
\]

(5)

where \(H_{\nu}^{(1)}(z)\) is the Hankel function of first kind.

Since the charge conjugation in a curved background is point independent \[7\], as in Minkowski case, the negative frequency modes can be obtained using the charge conjugation,

\[
    U_{\vec{p},\lambda}(x) \rightarrow V_{\vec{p},\lambda}(x) = i \gamma^2\gamma^0(\bar{U}_{\vec{p},\lambda}(x))^T.
\]

(6)
Thus we can restrict ourselves to analyze only the positive frequency modes. These spinors satisfy the orthonormalization relations [2]:

$$\int d^3x (-g)^{1/2} \bar{U}_{\vec{p}',\lambda}(x) \gamma^0 U_{\vec{p},\lambda}(x) = \delta_{\lambda\lambda'} \delta^3(\vec{p} - \vec{p}')$$

$$\int d^3x (-g)^{1/2} \bar{V}_{\vec{p}',\lambda}(x) \gamma^0 V_{\vec{p},\lambda}(x) = 0,$$

where the integration extends on an arbitrary hypersurface \( t = \text{const} \) and \((-g)^{1/2} = e^{3\omega t}\). They represent a complete system of solutions in the sense that

$$\int d^3p \sum_{\lambda} \left[ U_{\vec{p},\lambda}(t, \vec{x}) U_{\vec{p}',\lambda}(t, \vec{x}') + V_{\vec{p},\lambda}(t, \vec{x}) V_{\vec{p}',\lambda}(t, \vec{x}') \right] = e^{-3\omega t} \delta^3(\vec{x} - \vec{x}'),$$

In our calculation we need to find the form of the Coulomb potential on de Sitter spacetime which depends on metric. Here we can exploit the conformal invariance of the Maxwell equations since the de Sitter metric is conformal with the Minkowski one. We can write \( \tilde{A}_{\mu} \) for the Coulomb field in Minkowski spacetime. Then we find the corresponding de Sitter potential:

$$\tilde{A}^\mu(x) = \frac{Ze}{|\vec{x}|} e^{-\omega t}, \quad \tilde{A}^\lambda(x) = 0,$$

where the hated indices indicate label the components in the local Minkowski frames. We also observe that Eq. (9) is just the expression from flat space with distances dilated/contracted by the factor \( e^{-\omega t} \).

Having the above elements in our mind we can proceed to develop the theory for the scattering amplitude on de Sitter spacetime which can be reproduced from that in the Minkowski space. The necessary requirements for develop the scattering theory is the global hyperbolicity of the space and having a complete set of solutions of the free equation for incident field and scattered field (Born approximation) with the distinction between positive and negative frequencies. It is also important to specify that in our analysis both cases are fulfilled.

Let \( \psi_i(x) \) and \( \psi_f(x) \) be the waves freely propagating in the \textit{in} and \textit{out} sectors, and we assume that they are both of positive frequency. In direct analogy with the Minkowski [3], [4] theory we can define the lowest order contribution in the scattering amplitude as follows:

$$A_{i \rightarrow f} = -ie \int d^4x \left[ -g(x) \right]^{1/2} \bar{\psi}_f(x) \gamma_\mu A_\mu(x) \psi_i(x),$$

where \( e \) is the unit charge of the field. Our intention is to calculate the amplitude of Coulomb scattering for the external field (9) and for initial and final states of the form

$$\psi_i(x) = U_{\vec{p}_i,\lambda_i}(x), \psi_f(x) = U_{\vec{p}_f,\lambda_f}(x).$$
We introduce the following notation for simplify our formul\a: 

\[ \text{sgn} \]

that the interaction extends into the past and future.

Putting all the above notations together we can write for the scattering amplitude:

\[ \int d^3 \vec{x} \frac{e^{i(p_i - p_f) \cdot \vec{x}}}{|\vec{x}|} = \frac{4\pi}{|p_f - p_i|^2} \quad (12) \]

It is clear that the not so simple part is the temporal integral which contains the influence of the gravitational field via the expansion parameter,

\[ A_{i\rightarrow f} = -i\alpha Z \frac{\sqrt{p_ip_f}}{8\pi|p_f - p_i|^2} \xi^+_{\lambda_f}(p_f) \xi_{\lambda_i}(p_i) \left[ e^{i\pi k} \int_0^\infty dz z H^{(2)}_{\nu_z}(p_f z) H^{(1)}_{\nu_z}(p_i z) + \text{sgn}(\lambda_f, \lambda_i) e^{-i\pi k} \int_0^\infty dz z H^{(2)}_{\nu_z}(p_f z) H^{(1)}_{\nu_z}(p_i z) \right], \]

where we pass to a new variable of integration:

\[ z = \frac{e^{-\omega t}}{\omega}. \quad (14) \]

Note that the integration limits in (13) corresponds to \( t = \pm \infty \), we assume that the interaction extends into the past and future.

For simplification of our notation we introduce the following quantities:

\[ A_k(f,i) = A_k^+(p_fp_i) + \text{sgn}(\lambda_f, \lambda_i) A_k^-(p_fp_i), \]

with

\[ A_k^\pm(p_fp_i) = \frac{\sqrt{p_ip_f}}{2} e^{\pm i\pi k} \int_0^\infty dz z H^{(2)}_{\nu_z}(p_f z) H^{(1)}_{\nu_z}(p_i z). \]

Putting all the above notations together we can write for the scattering amplitude:

\[ A_{i\rightarrow f} = -\frac{i}{4\pi |p_f - p_i|^2} \alpha Z A_k(f,i) \xi^+_{\lambda_f}(p_f) \xi_{\lambda_i}(p_i), \]

with \( \alpha = e^2 \) and \( \xi_f, \xi_i \) stand for the initial and final helicity two spinors.

The evaluation of the integrals is discussed in Appendix, here we give the final result in terms of Dirac delta-function, hypergeometric functions, Euler Beta functions and unit step function

\[ A_k(f,i) = \delta(p_f - p_i) + \theta(p_i - p_f) \frac{1}{p_i} f_k \left( \frac{p_f}{p_i} \right) + \theta(p_f - p_i) \frac{1}{p_f} f^*_k \left( \frac{p_i}{p_f} \right) + \text{sgn}(\lambda_f, \lambda_i) \left[ \delta(p_f - p_i) + \theta(p_i - p_f) \frac{1}{p_i} f_{-k} \left( \frac{p_f}{p_i} \right) + \theta(p_f - p_i) \frac{1}{p_f} f^*_{-k} \left( \frac{p_i}{p_f} \right) \right]. \]

We introduce the following notation for simplify our formula:

\[ f_k(\chi) = i(\chi)^{-ik} \frac{e^{i\pi k}}{\cosh(\pi k)} 2F_1 \left( \frac{1}{2}, 1 - ik; \frac{1}{2} - ik; \chi^2 \right) \]

\[ -i(\chi)^{1+ik} \frac{e^{-i\pi k}}{\cosh(\pi k)} 2F_1 \left( \frac{3}{2}, 1 + ik; \frac{3}{2} + ik; \chi^2 \right), \]

\[ k = \frac{\sqrt{p_ip_f}}{2\pi |p_f - p_i|^2}, \]

\[ \text{B}(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)} \]

\[ \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad (19) \]
where \( \chi = \frac{p_f}{p_i} \) or \( \frac{p_i}{p_f} \) and \( f_{-k}(\chi) \) is obtain when \( k \to -k \) in (19).

The above formulas is our result and in remaining paper we will explore some of their physical consequences. Before that we must state that the argument \( \chi \) in (19) must be considered in interval \( 0 \leq \chi < 1 \) (note the argument in the \( \theta \) functions). This is the domain of convergence of the hypergeometric functions, because in the limit \( \chi \to 1 \) the \( _2F_1(a, b, c, \chi^2) \) functions diverge. The above observations are crucial when we compare our expression of scattering amplitude with the well known expression of scattering amplitude from Minkowski spacetime. In our case we see that in (19) appear terms that demand \( p_f \neq p_i \) and from this we conclude that there exist a nonvanishing probability for a scattering process where the law of conservation for total momentum is lost.

3 Limit cases

It is clear that the influence of the de Sitter geometry in the amplitude is contained in the \( A_k(f, i) \) factor which was obtained after integration with respect to time variable. This quantity encode via the dependence of the \( k \) parameter the effect of the expansion of the space on the scattering process.

We start our analysis having in mind that for \( \omega = 0 \) we obtain from de Sitter metric the Minkowski one and this corresponds to \( k = \infty \). We see that in our expression just the terms with Dirac delta-functions will not vanish. Also for opposite helicities \( A_\infty(f, i) = 0 \) and for equal helicities we will obtain:

\[
A_\infty(f, i) = 2\delta_{\lambda_f, \lambda_i} \delta(p_f - p_i). \tag{20}
\]

The correspondent factor from Minkowski case for equal helicities is:

\[
A_{\text{Minkowski}}(f, i) = 4\delta(E_f - E_i), \tag{21}
\]

where the notations are : \( E_f, E_i \) are the final and initial Minkowski energies and \( E^2 = p^2 + m^2 \). Now using the properties of distributions and the fact that helicity conservation increase in the ultra relativistic limit \( (p \simeq E \gg m) \) we obtain the amplitude ratio deSitter/Minkowski:

\[
\frac{2\delta(p_f - p_i)}{4\delta(E_f - E_i)} \simeq \frac{1}{2}. \tag{22}
\]

The fact that we don’t recover the Minkowski amplitude can be explained as follows. As we point out in the previous section the limits of integration in (13) for the time variable extends into the past and future \( t \to \pm \infty \). Observing that for \( t \to \infty \) the integrand vanishes as \( e^{-\omega t} \), while for \( t \to -\infty \) become divergent. In this case we can use the asymptotic formulas for Hankel functions

\[
H_{\nu}^{(1,2)}(z) = \left( \frac{2}{\pi z} \right)^{1/2} e^{\pm i(z - \nu \pi/2 - \pi/4)}, \quad z \to \infty. \tag{23}
\]
Passing to conformal time $t_c = -z$ and using (23) the contributions to the amplitude in (16) will be:

$$\frac{1}{\pi} \int dt_c e^{i(p_f - p_i)t_c}, \quad t_c \to -\infty. \tag{24}$$

The integral (24) produces the distributional term, integrating over an infinite interval. Observing that in (24) appears only one semi-infinite axis the explanation for the 1/2 factor is immediate. To summarize in de Sitter case the distributional terms will be produced by the evolution in the infinite past when the space was contracted. This contraction of the space in the infinite past is responsible for the 1/2 factor when we pass to Minkowski limit.

We will make a few comments about nonconservation of total momentum. The difference between (21) and the result in the previous section (18) is that the last one does not vanish for $p_f \neq p_i$. As we point out in the previous section the total momentum is not conserved in the scattering process on de Sitter spacetime. We know that in Minkowski case the conservation of the momentum modulus is a direct consequence of the conservation of total energy which in turns is a consequence of the translational invariance with respect to time, which is lost in de Sitter space.

Now let us take a closer analysis to helicities from which we deduce a interesting properties. First we observe that the delta terms $\delta(p_f - p_i)$ in (18), will add for equal helicities and vanish for opposite ones. Now if we admit that we deal with a process such that modulus of the final momentum is closer to the modulus of the initial momentum $p_f \sim p_i$ the probability of transition between different helicities states will be smaller that that between identical helicity states. From that we can conclude that on de Sitter space is manifesting a tendency of conservation for helicity.

Let us consider now the opposite case $k = 0$ which corresponds to a very large expansion rate of the space compared with mass of particle. The $f_{\pm k}(\chi)$ functions simplify in this case, observing that for null $k$ the second and the third argument in hypergeometric functions became the same and using $2F_1(a, b; b; z) = (1 - z)^{-a}$, we find

$$f_{(k=0)}(\chi) = \frac{1}{\pi} \frac{i}{1 - \chi}. \tag{25}$$

Introducing (25) in (18) we obtain a pair of terms which summed make the unit step functions unnecessary. Giving attention to the delta contributions, one obtain for identical initial and final helicities:

$$A_0(f, i) = 2\delta(p_f - p_i) + \frac{4}{\pi} \frac{i}{p_i - p_f}. \tag{26}$$

For opposite helicities the result is identically null, since for $k = 0$ it is easy to see that $A_k^\pm(p_f, p_i)$ exactly cancel.

An important observation is that (26) can be obtained if we put $k = 0$ in Hankel functions obtaining,

$$H_1^{(1,2)}(z) = \mp i \left(\frac{2}{\pi z}\right)^{1/2} e^{\pm iz}, \tag{27}$$
in which case the integrals in (16) can be easily evaluated.

Now we know that the Dirac theory is conformally invariant for a null fermion mass, and that our space is conformal with the Minkowski one. The case \( k = 0 \) is equivalent with a vanishing mass \( m = 0 \) and the explanation is immediate. Also note that the massless limit is consistent with the identically null amplitude in the case when helicity is not conserved.

In the above limit cases we observe that the amplitude will vanish for opposite helicities and the conclusion is immediate: in bought cases the total angular momentum is conserved. As we point out above this tendency is also manifested in the general case.

4 Calculation of cross section

We know from the scattering theory that the parameter measured by experiment in scattering processes is the cross section. In our case the situation is complicated and we must reconsider our vision about cross section. The first observation is that our amplitude of scattering contain two main contributions one linear part (terms with delta Dirac function) and one non-linear part. In this situation it is natural to define in general the differential cross section as a sum of a linear contribution and a non-linear one. Also it is important to interpret our non-linear amplitude at square as a probability of scatter when the modulus of momentum is not conserved. The definition of linear cross section for Coulomb scatter can be reproduced as in Minkowski spacetime

\[
\frac{d\sigma}{d\Omega} = \frac{1}{2} \sum_{\lambda_i\lambda_f} \frac{1}{J} \frac{dP_i}{dt},
\]

where the factor \( \frac{1}{2} \) transform one sum in mediation, \( P_i \) is the linear probability of transition and \( J \) is the incident flux in unit of time. When modulus of momentum is conserved just the linear probability of transition will give contribution to cross section.

For the non-linear differential cross section for Coulomb scatter we will use the following definition

\[
\frac{d\sigma}{d\Omega} = \frac{1}{2} \sum_{\lambda_i\lambda_f} \frac{1}{J} \frac{dP_{nl}}{dt},
\]

where \( P_{nl} \) is the non-linear probability of transition. In the case when modulus of momentum is not conserved just the non-linear probability of transition will give contribution to cross section.

For our further calculations we evaluate the incident flux in unit of time using the formula of Dirac current in local frames:

\[
J_\mu = e_\mu \bar{U}_{\vec{p}_i,\lambda_i}(x) \gamma^\mu U_{\vec{p}_i,\lambda_i}(x). \tag{30}
\]

The spatial component of Dirac current can be defined in our case as follows:

\[
J = \frac{e^{\omega t}}{\vec{p}_i} \bar{U}_{\vec{p}_i,\lambda_i}(x) (\vec{p}_i \gamma) U_{\vec{p}_i,\lambda_i}(x) = \frac{\pi P_i e^{-3\omega t}}{(2\pi)^3 4\omega} \left[ H^{(2)}_{\nu}(\frac{P_i}{\omega} e^{-\omega t}) H^{(1)}_{\nu}(\frac{P_i}{\omega} e^{-\omega t}) \right]
\]
\[ + H_{\nu_+}^{(2)}(\frac{p_\omega}{\omega} e^{-\omega t}) H_{\nu_-}^{(1)}(\frac{p_\omega}{\omega} e^{-\omega t}) \].

(31)

The spatial components of the four vector current of particles is the incident flux in unit of time. We see that this is a quantity that depends of time which is an unusual and complex problem. In addition another problem appears, we known that when we pass from scattering amplitude to cross section an essential role in the treatment when we deal with finitely extended wave is played by the energy conservation factor \( \delta(E_f - E_i) \). This factor assures that the cross section will not depend on the particular form of wave function of the incident particles. In our case the cross section calculate with our amplitude will depend on the specific form of the incident wave which is another unusual situation. All these aspects are consequences of the lost invariance with respect to time in de Sitter spacetime.

For calculating the cross section in this situation we will follow the next physical picture. According to the formalism of the scattering theory the amplitude obtained here has to be associated with the following picture: the in states is freely propagated from the moment \( t = 0 \) back to the infinite past, then evaluated in presence of the interaction up to infinite future, and again freely propagated back to \( t = 0 \) where is projected in out states. In this situation we can evaluate Eq. (31) at \( t = 0 \), thus obtaining one quantity that is not dependant of time,

\[ J = \frac{\pi p_i}{4 \omega (2\pi)^3} \left[ H_{\nu_+}^{(2)}(\frac{p_\omega}{\omega}) H_{\nu_-}^{(1)}(\frac{p_\omega}{\omega}) + H_{\nu_+}^{(2)}(\frac{p_\omega}{\omega}) H_{\nu_-}^{(1)}(\frac{p_\omega}{\omega}) \right]. \]

(32)

When we deal with particles with given helicity in processes of scattering we must average upon the helicities of incident particles and sum upon the helicities of the emergent particles. In our case the situation is immediate and we obtain:

\[ \frac{1}{2} \sum_{\lambda_i, \lambda_f} \left[ \xi_{\lambda_i}^\dagger (p_f) \xi_{\lambda_f} (p_i) \right]^2 = 2. \]

(33)

For calculation of linear cross section we need to evaluate \( \frac{dp_f}{dt} \) thus obtaining

\[ \frac{dP_f}{dt} = \frac{(\alpha Z)^2}{8 \pi^4 |p_f - p_i|^4} \delta(p_f - p_i) d^3 p_f. \]

(34)

Replacing (34) and (32) in (28) writing \( d^3 p_f = p_f^2 dp_f d\Omega p_f \) and integrating with respect to \( p_f \) the factor with Dirac delta-function will be eliminated and in addition using that \( p_f = p_i = p \) we will obtain for the linear part:

\[ \frac{d\sigma_l}{d\Omega} = \frac{(\alpha Z)^2 \omega}{2 \pi p^2 \sin^4 \left( \frac{\theta}{2} \right)} \left[ H_{\nu_+}^{(2)} (\frac{p_\omega}{\omega}) H_{\nu_-}^{(1)} (\frac{p_\omega}{\omega}) + H_{\nu_+}^{(2)} (\frac{p_\omega}{\omega}) H_{\nu_-}^{(1)} (\frac{p_\omega}{\omega}) \right]^{-1}. \]

(35)

We see that our cross section have the usual angular dependence as the cross section in Minkowski case. In the \( k = 0 \) limit we will obtain a much simple formula:

\[ \left( \frac{d\sigma_l}{d\Omega} \right)_{k=0} = \frac{(\alpha Z)^2}{8 \pi^2 \sin^4 \left( \frac{\theta}{2} \right)}. \]

(36)
Now for calculation of cross section in the Minkowski limit \((k = \infty)\) we must approximate the Hankel functions \(H^{(1,2)}_{\nu}(z)\) when bought arguments \(\nu\) and \(z\) have large values which is one of the most difficult problems. In our case one of the argument is imaginal and the other real and in the theory of cylindrical functions are given just the asymptotic formulas when bought are real or imaginal. Thus this problem of taking this limit is not solved, our calculations in this direction shows that this is a very difficult task. For this reason we can’t evolve the cross section in the Minkowski limit.

In the non-linear case the integration with respect to \(p_f\) give an expression which is to complicated to be given here. For this reason we restrict to give an expression for the differential probability of scatter in this case. Formulas necessary to solve integrals with respect to \(p_f\) for the calculation of cross section is given in Appendix B.

\[
dP_{nl} = \frac{(\alpha Z)^2}{16\pi^2|p_f - p_i|^4} \left[ \theta(p_i - p_f) \frac{1}{p_f^2} \right] M_k \left( \frac{p_f}{p_i} \right) + \text{sgn}(\lambda_i \lambda_f) B_k \left( \frac{p_f}{p_i} \right) \left[ \frac{p_i}{p_f} \right] d^3 p_f \tag{37}\]

In the above relation we introduce the following notations:

\[
B_k(\chi) = f_k(\chi) f^*_k(\chi) + f^*_k(\chi) f_{-k}(\chi), \quad M_k(\chi) = |f_k(\chi)|^2 + |f_{-k}(\chi)|^2. \tag{38}\]

In the \(k = 0\) limit we can evolve the non-linear contribution to cross section making the integral over final momentum and restrict the limit of integration to \(p_{f,\text{max}}\) because if we take the integrals between 0 and \(\infty\) our integral diverge. In fact this restriction to the limits of integration is suggested by the form of the scattering amplitude (26) observing that the final momentum can’t exceed one maxim value.

\[
\left( \frac{d\sigma_{nl}}{d\Omega} \right)_{k=0} = \frac{16(\alpha Z)^2}{\pi|p_f - p_i|^4} \frac{p_{f,\text{max}}(2p_i - p_{f,\text{max}}) - 2p_i(p_i - p_{f,\text{max}}) \ln\left( \frac{p_i}{p_i - p_{f,\text{max}}} \right)}{(p_i - p_{f,\text{max}})} \tag{39}\]

We see that the cross sections calculated with non-linear part of amplitude have much complicated expressions. In the general case as we can see from (37) the non-linear cross section will depend on the parameter of expansion and will have a complicated expression, which is a sum of expressions of the form (44)(see Appendix B). For very small values of \(\omega\) the spacetime appears as flat and contributions that contain negative powers of \(k\) can be neglected. In this approximation contributions to non-linear cross section will come from terms which are proportional to \(e^{2\pi k}\) and \(\chi^{2ik}\).

5 Conclusions

In this paper we investigated the Coulomb scattering amplitude for the Dirac field in the expanding de Sitter space. We have complete ignored complications.
due to the ambiguity of the particle concept in a curved spacetime. We considered the initial and final states of the field as exact solutions of the free Dirac equation in de Sitter space in the momentum and helicity basis. We also found that the amplitude depends in an essentially way on the parameter $k = m/\omega$. For a vanishing $k$ we recover the expected result due to the conformal invariance of the theory in the massless case.

Also we found that in the vanishing rate of expansion the Minkowski amplitude is not recovered. We explain this as a consequence of the contraction of space in the infinite past.

Now we want to draw attention to some aspects that remained untouched in the paper and which could be helpful for further investigations. A question of principle arises in our paper: how to define the cross section when this is a quantity which is dependent on the form of incident wave and in addition the incident flux is also a dependent of time quantity. Thus when we calculate the cross section we evolve the incident flux at $t = 0$. This is the only way to calculate the cross section in our situation and suggest a possible principle when we evolve the cross section in a spacetime where the translational symmetry with respect to time is lost. Also we found that the cross section can be defined generally as a sum of a linear and a non-linear part. From our point of view the next important step is to calculate one scatter amplitude when the projectile particle are at finite distances from target, because in this situation we may think to a better connection to experiment. It is also important to say that an experiment in our situation will depend of the orientation of the axes of local tetrad.

All these difficult problems that arise in our paper are consequences of the lost invariance with respect to time in de Sitter spacetime.

6 Appendix A: Integrals of Bessel functions

We will present here the main steps leading to (18). The formulas that we need are \cite{5,6}

\begin{align}
H^{(1)}_{\mu}(z) &= \frac{J_{-\mu}(z) - e^{-i\pi\mu}J_{\mu}(z)}{i \sin(\pi\mu)} \\
H^{(2)}_{\mu}(z) &= \frac{e^{i\pi\mu}J_{\mu}(z) - J_{-\mu}(z)}{i \sin(\pi\mu)}
\end{align}

and integrals of the type Weber-Schafheitlin \cite{5}

\[ \int_{0}^{\infty} dz z^{-s}J_{\mu}(az)J_{\nu}(\beta z). \]  

These integrals can be solved in the assumptions,

\begin{align}
\beta > \alpha > 0, \ Re(s) > -1, \ Re(\mu + \nu - s + 1) > 0 \\
\alpha > \beta > 0, \ Re(s) > -1, \ Re(\mu + \nu - s + 1) > 0,
\end{align}

10
bought cases are fulfilled in our analysis. For obtaining (40) in (16) and using Weber-Schafheitlin integrals we will arrive at the desired result. Now one sees that our integrals in (16) demand (40) in (16) and using Weber-Schafheitlin integrals we will arrive at the desired cases are fulfilled in our analysis. For obtaining (18) we must introduce lematic because the integral becomes oscillatory for \( z \to \infty \). Then a simple way to solve this problem is to consider an parameter \( s \) of the form:

\[
s = -1 + \epsilon,
\]

and let in the end \( \epsilon \to 0 \). No notably differences appear when evaluating the integrals in (16) directly with \( s = -1 \) and with \( \mu, \nu \) arguments following from (4).

Let us take a look to the distributional contribution \( \delta(p_f - p_i) \). This originates in the term \( \delta(\alpha - \beta) \) which is generally present in the integrals (41) for \( s = -1 \) (the case of real arguments \( \mu = \nu \) is a familiar one). In our case we must obtain this term for arbitrary \( \mu, \nu \) arguments, because in our case they depend of the parameter \( k \). In what follows we shall briefly indicate below a method to obtain it. We start with \( s = -1 + \epsilon \) parameter like in (43) and integrate in (41) with respect to \( \beta \) over the interval \( \alpha - \epsilon \leq \beta \leq \alpha + \epsilon \). Then the coefficient of \( \delta(\alpha - \beta) \) will be obtained then as the quantity which is left in the limit \( \epsilon \to 0 \).

7 Appendix B: Integrals with hypergeometric functions

For calculation of non-linear cross section we solve integrals of the form:

\[
\int_0^\infty \theta(p_i - p_f) F_1(b, c; d; \frac{p_f^2}{p_i^2})_2 F_1(e, f; g; \frac{p_f^2}{p_i^2}) p_i^n dp_i = \frac{1}{2} \left( -\frac{1}{p_i^2} \right)^{-n/2} p_i^2 \Gamma(d) \left[ \frac{\Gamma(-b + c) \Gamma(-1 + b + e - \frac{n}{2}) \Gamma(g) \Gamma(-1 + b + f - \frac{n}{2})}{\Gamma(c) \Gamma(-b + d) \Gamma(-1 + b + g - \frac{n}{2}) \Gamma(e) \Gamma(f)} \times \Gamma(1 - b + \frac{n}{2}) F_3(b, 1 + b - d, -1 + b + e - \frac{n}{2}; 1 - b - c \right. \\
\left. + \frac{b - \frac{n}{2}, 1 + b + g - \frac{n}{2}; 1}{\Gamma(\epsilon) \Gamma(\epsilon) \Gamma(\epsilon)} \times \frac{\Gamma(-b + c) \Gamma(g) \Gamma(-1 + b + e - \frac{n}{2}) \Gamma(b - c)}{\Gamma(c) \Gamma(-b + d) \Gamma(-1 + b + g - \frac{n}{2}) \Gamma(e) \Gamma(f)} \right. \\
\left. - \frac{1}{2} + b + f - \frac{n}{2}; 1 + b - c, \frac{1}{2} + b - \frac{n}{2}, -\frac{1}{2} + b + g - \frac{n}{2}; 1 + \Gamma(b - c) \right. \\
\left. \times \frac{\Gamma(-1 + c + e - \frac{n}{2}) \Gamma(g) \Gamma(-1 + c + f - \frac{n}{2}) \Gamma(1 - c + \frac{n}{2})}{\Gamma(c) \Gamma(-c + d) \Gamma(-1 + c + g - \frac{n}{2}) \Gamma(e) \Gamma(f)} \right. \\
\left. - 1 + c + e - \frac{n}{2}, -1 + c + f - \frac{n}{2}; 1 - b + c, c - \frac{n}{2}, -1 + c + g - \frac{n}{2}; 1 \right) \\
- \frac{1}{p_i^2} \Gamma(b - c) \Gamma(-\frac{1}{2} + c + e - \frac{n}{2}) \Gamma(g) \Gamma(-\frac{1}{2} + c + f - \frac{n}{2}) \Gamma(\frac{1}{2} (1 - 2c + n))}{\Gamma(b) \Gamma(-c + d) \Gamma(-\frac{1}{2} + c + g - \frac{n}{2}) \Gamma(e) \Gamma(f)}
\]
\begin{align}
&\times _4F_3(c,1+c-d,-\frac{1}{2}+c+e-n\frac{1}{2},-\frac{1}{2}+c+f-n\frac{1}{2};1-b+c,\frac{1}{2}+c-n\frac{1}{2})
&\times \frac{1}{p_t^2}p_t^i \frac{\Gamma(-\frac{b}{2}+\frac{1}{2}+d-\frac{n}{2})\Gamma(-\frac{b}{2}+c-\frac{n}{2})\Gamma(c)}{\Gamma(-\frac{b}{2}+c-n\frac{1}{2})\Gamma(c)\Gamma(b)}
&\times _4F_3(e,f,1+n\frac{3}{2}-d+n\frac{2}{2};g,3\frac{2}{2}-b+n\frac{3}{2}-c+n\frac{2}{2};1)
&\frac{\Gamma(-1+b-\frac{n}{2})\Gamma(-1+c-\frac{n}{2})\Gamma(1+\frac{n}{2})}{\Gamma(-1+d-\frac{n}{2})\Gamma(c)\Gamma(b)}
&\times _4F_3(e,f,1+n\frac{2}{2};2-d+n\frac{2}{2};1)
\end{align}

and integrals of the form \(\int_0^\infty \theta(p_i-p_f)[2F_1(b,c;d;p_t^2)^2p_t^i]^{2k+n}dp_f\) which is the same type as the above and have a closer result. The above integrals are valid in the assumption \(n \geq 0\).

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