Mixed order transition and condensation in an exactly soluble one dimensional spin model

Amir Bar and David Mukamel

Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 7610001, Israel
E-mail: amir.bracha@weizmann.ac.il and david.mukamel@weizmann.ac.il

Received 24 June 2014
Accepted for publication 7 September 2014
Published 4 November 2014

Online at stacks.iop.org/JSTAT/2014/P11001
doi:10.1088/1742-5468/2014/11/P11001

Abstract. Mixed order phase transitions (MOT), which display discontinuous order parameter and diverging correlation length, appear in several seemingly unrelated settings ranging from equilibrium models with long-range interactions to models far from thermal equilibrium. In a recent paper [1], an exactly soluble spin model with long-range interactions that exhibits MOT was introduced and analyzed both by a grand canonical calculation and a renormalization group analysis. The model was shown to form a bridge between two classes of 1D models exhibiting MOT, namely between spin models with inverse distance square interactions and surface depinning models. In this paper, we elaborate on the calculations performed in [1]. We also analyze the model in the canonical ensemble, which yields a better insight into the mechanism of MOT. In addition, we generalize the model to include Potts and general Ising spins and also consider a broader class of interactions that decay with distance using a power law different from 2.

Keywords: classical phase transitions (theory), correlation functions (theory), phase diagrams (theory), renormalisation group
1. Introduction

Classification of equilibrium phase transitions is a narrative in statistical physics and quantum field theory. A common starting point is the distinction between first order transitions and continuous—or critical—transitions. While in first order transitions the order parameter is discontinuous or, more generally, the free energy is non-differentiable, in continuous transitions, the order parameter changes continuously and the free energy is non-analytic but has a first derivative. Continuous transitions are known to possess universal features and can thus be categorized into universality classes, as opposed to first order transitions, which are non-universal. The universality of critical transitions is tightly related to the divergence of a correlation length. Similar features have also been discussed in the context of non-equilibrium phase transitions. Much progress has been made in classifying critical transitions based on methods such as renormalization group and conformal field theory, which rely on such divergence.

However, there are known examples that deviate from the above scheme, for instance, transitions for which, on the one hand, the free energy is non-differentiable but, on the other hand, display a diverging correlation length [2]. Early examples of such mixed order transitions (MOT) include (a) 1D discrete spin models with long-range interactions [3–8] and (b) models of depinning transitions such as DNA denaturation [9,10], wetting [11,12], and simplicial gravity [13]. More recently there has been renewed interest in MOT with studies of percolation models in the context of glass and jamming transitions [14–18], evolution of complex networks [19–22], and active biopolymer gels [23]. While all these examples exhibit MOT, they display some qualitative and quantitative differences. For instance the divergence of the correlation length is algebraic for some, while for others it is a stretched exponential. It would be interesting to understand the origin of the similarities and differences between the various models by studying them within a common framework.

In [1], a new model establishing a link between the classes of models (a) and (b) was introduced and analyzed, thus making a step towards a unified view. Specifically, this study focuses on two specific representative models of classes (a) and (b): the one dimensional Ising model with interactions decaying with distance $r$ between spins as $r^{-2}$ (dubbed IDSI, for inverse distance squared Ising model) [4,7,24] and the Poland Scheraga (PS) model for DNA denaturation [9,25]. The first class exhibits a Kosterlitz–Thouless
(KT) vortex unbinding transition [26], while the second class exhibits a condensation transition similar to the Bose Einstein condensation (BEC) transition of free bosons. The model presented in [1], which we refer to as the truncated IDSI (TIDSI) model thus provides an intriguing connection between KT and BEC transitions in one dimension. Another appealing property of the TIDSI model is that it is exactly soluble.

In this paper we extend the analysis of [1] in several directions:

• The calculations done in [1] are elaborated, and many features of the phase diagram which where only briefly mentioned in [1] are derived explicitly.

• The exact free energies of different ensembles are derived. These shed new light on the origin of criticality in the mixed-order transition appearing in the TIDSI model.

• We generalize the model to include spins other than Ising spins, so that these models could be compared with models of class (a) other than the IDSI [6]. We also consider long-range interactions decaying as \( r^{-\alpha} \), with \( \alpha \neq 2 \).

The paper is organized as follows: We start in section 2 by reviewing MOT in one dimension and describe the IDSI and PS models in detail. Then, in section 3, the TIDSI model is presented, and the relation to the IDSI and PS model is discussed. We also summarize the main results of the paper in this section, including the phase diagram and free energies. The following three sections are dedicated to deriving these results using grand canonical analysis (section 4), canonical analysis (section 5), and renormalization group (RG) analysis (section 6). While the TIDSI model is exactly soluble, the IDSI model is not. However, as we show in section 6, the RG analysis can be used as a common framework for studying both models. The TIDSI model is then extended to include more general interactions (section 7.1) and other spin variables (section 7.2).

2. Mixed order transitions (MOT) in one dimension

In this section, we briefly review two models exhibiting MOT in one dimension. As mentioned above, these models are prototypical representatives of wider classes of models.

2.1. The inverse distance squared Ising (IDSI) model

The IDSI is a 1D Ising model with long-range interactions that decay asymptotically as \( r^{-2} \) where \( r \) is the distance between the spins. Formally, it is defined on a 1D lattice of size \( L \), where in each lattice site there is a spin variable \( \sigma_i = \pm 1 \) and the Hamiltonian is

\[
H = -\sum_{i<j} J(j - i)\sigma_i\sigma_j; \quad J(r \gg 1) \sim r^{-2},
\]

with \( J(r) \geq 0 \) for all \( r \). More generally, interactions decaying as \( J(r \gg 1) \sim r^{-\alpha} \) can be considered. Ruelle [27] and Dyson [28] proved that \( \alpha = 2 \) is a borderline case, i.e. that models with \( \alpha > 2 \) exhibit no phase transition, while models with \( 1 < \alpha < 2 \) show a symmetry breaking transition at some finite critical temperature \( T_c > 0 \). Models with

doi:10.1088/1742-5468/2014/11/P11001
Mixed order transition and condensation in an exactly soluble 1D spin model

The phase diagram of the IDSI model in the temperature-magnetic field \((T, h)\) plane is given in figure 1(a). It resembles the \(d > 1\) dimensional Ising model phase diagram, with a low-temperature first order transition at zero magnetic field \(h\), which terminates at a critical point \(T_c > 0\). The only qualitative difference is that at \(T = T_c\), the transition of the IDSI is discontinuous, that is, the order parameter—the magnetization—jumps from 0 at \(T = T_c^+\) to \(\pm m_c \neq 0\) at \(T = T_c^-\). However, the transition, as mentioned above, is still critical as there is a diverging correlation length. This is different from more common first order symmetry breaking transitions found, for instance, in the Blume–Capel model

\[
H = -J \sum_{i \neq j} \sigma_i \sigma_j + \Delta \sum_i \sigma_i^2 - h \sum_i \sigma_i; \quad \sigma_i \in \{-1, 0, 1\}. \tag{2}
\]

in a certain range of the parameters \(J\) and \(\Delta\) and at \(h = 0\).

A schematic temperature-magnetic field phase diagram for the Blume–Capel model for fixed \(J\) and \(\Delta\) is presented in figure 1(b). Here, the \(T = T_c, h = 0\) point is actually a triple point in which three first order lines meet, i.e. in which three phases coexist. These are the + phase, the − phase, and the disordered phase. There are also finite \(h\) transition lines on which there is a coexistence of two magnetically ordered phases. These finite \(h\) lines terminate in a usual second order critical point [2].

Although the main features of the phase diagram of the IDSI model have been rigorously proven \([7, 29]\), there are many missing details; non-universal quantities such as the critical temperature are not known exactly. In addition, neither the partition function nor the thermodynamic potentials were calculated in any ensemble. The model analyzed
Mixed order transition and condensation in an exactly soluble 1D spin model

2.2. The Poland–Scheraga (PS) model

The PS model is a model of DNA denaturation [9]. Denaturation is the process in which the two strands of the double-stranded DNA molecule separate upon heating. The PS model idealizes the DNA molecule as an alternating chain of bound segments and denatured loops as depicted in figure 2. For homopolymers, while a bound segment of length \( l \) contributes an energy \(-\epsilon l\), \( \epsilon > 0 \) being the binding energy of DNA base-pairs, loops are assumed to contribute no energy but contribute entropy \( S \) of the form \( e^S \approx \omega l^s \), where \( \omega \) and \( s \) are non-universal constants and \( c \) is a universal exponent termed the loop exponent. Thus,

\[
S(l) = bl - c \log(l) + \Delta, \tag{3}
\]

with \( b \equiv \log(s) \) and \( \Delta \equiv \log(\omega) \). This form emerges from modeling the denatured loop as a closed path [10], which results in a universal exponent \( c \) that depends only on dimensionality and topological constraints, i.e. whether the path is considered as an ideal walk or self avoiding walk, among others [25]. The order parameter for this system is the fraction of bound base-pairs \( \theta \), with \( \theta = 0 \) above the melting temperature \( T_c \), and is positive below \( T_c \). The nature of the transition depends on the exponent \( c \), as depicted in the phase diagram in figure 3: For \( c \leq 1 \), the strands are bound at all temperatures and \( T_c = \infty \). For \( 1 < c \leq 2 \), there is a continuous transition at \( T = T_c \) with \( \theta \rightarrow 0 \) as \( T \rightarrow T_c \). For \( c > 2 \), the order parameter is discontinuous, i.e. \( \theta \rightarrow \theta_c > 0 \) as \( T \rightarrow T_c \). For all \( c > 1 \), the correlation length \( \xi \) diverges at the transition point as \( \xi \sim (T - T_c)^{-\nu} \) with \( \nu = \max \{ \frac{1}{c-1}, 1 \} \), and hence, for \( c > 2 \), the transition is of the MOT type. The mechanism of the transition is mathematically similar to the Bose–Einstein condensation (BEC) transition in non-interacting Bose gas, with the condensate being the macroscopic loop formed at the transition temperature \( T_c \). Other models belonging to the class of the PS model have been discussed in the past in a variety of contexts. These include wetting

\[\text{doi:10.1088/1742-5468/2014/11/P11001}\]
phenomena [11,12], mean field models of simplicial gravity [13], and non-conserving models of driven diffusive systems [32].

There are several clear differences between the MOT transition in the PS model and in the IDSI model. First, the correlation length in the two models diverges differently: algebraically in the PS model and as a stretched exponential in the IDSI model. Within a renormalization group analysis, the PS model displays a one-parameter family of fixed points with continuously varying exponent while the transition in the IDSI is controlled by a single fixed point. In addition, the PS transition is a condensation transition characterized by the formation of a single macroscopic loop. No such macroscopic object is associated with the IDSI transition. Finally, while the IDSI transition is a symmetry breaking transition, no symmetry is broken in the PS case. To understand the origin of the similarities and differences between these two models—and between the classes they represent—we introduced the TIDSI model, which serves as a bridge between these models, in [1]. We now define the model and explicitly show its similarity to those two models.

3. TIDSI model

3.1. Definition

Like the IDSI model, the TIDSI model of [1] is defined on a 1D Ising chain, where on each site $i$ there is a spin variable $\sigma_i = \pm 1$. The interaction between spins is composed of a nearest neighbor interaction term $-J_{NN}\sigma_i\sigma_{i+1}$ and a long-range interaction term $-J(i-j)\sigma_i\sigma_j I(i \sim j)$, where $I(i \sim j) = 1$ if sites $i$ and $j$ are in the same domain of either all up or all down spins and $I(i \sim j) = 0$ otherwise. Thus, the long-range interactions (beyond nearest neighbors) are confined within each of the domains. As in the IDSI model, we take the interaction strength $J(r)$ to decay as $r^{-2}$. The indicator function $I(i \sim j)$ can be written explicitly in terms of the spin variables

$$I(i \sim j) = \prod_{k=i}^{j-1} \delta_{\sigma_k\sigma_{k+1}} = \prod_{k=i}^{j-1} \frac{1 + \sigma_k\sigma_{k+1}}{2}.$$ 

Hence, the resulting Hamiltonian reads

$$H = -J_{NN} \sum_i \sigma_i\sigma_{i+1} - \sum_{i<j} J(j-i)\sigma_i\sigma_j \prod_{k=i}^{j-1} \frac{1 + \sigma_k\sigma_{k+1}}{2},$$

which explicitly contains multi-spin interactions of arbitrary order. For concreteness, we take $J(r) = C r^{-2}$ for $r \geq 1$, even though the results are qualitatively the same for any $J(r)$, which has the same asymptotic form.

An alternative representation of a configuration of the model is in terms of domains. Each microscopic configuration is defined in terms of the spin at the first site $\sigma_1$, the number of domains $1 \leq N \leq L$, and their lengths $\{l_a\}_{a=1}^N$, where $l_a \geq 1$ and $\sum_a l_a = L$. Due to the truncation of the long-range interactions to within domains, the energy (4)
can straightforwardly be expressed in terms of these variables. The nearest neighbors term sums up to

\[ H_{NN} = -J_{NN} [(L - 1) - 2(N - 1)] = 2J_{NN}N - J_{NN} (L + 1), \]  

while the energy contribution of the long-range term for each domain \( H_{LR} (l) \) is given by

\[ H_{LR} (l) = -C \sum_{k=1}^{l} \frac{l - k}{k^2} = -C \left[ \left( \zeta_2 - \frac{a}{l} + O \left( \frac{1}{l^2} \right) \right) - \log(l) + O \left( \frac{1}{l} \right) \right] \]

\[ = -bl + C \log(l) + \bar{\Delta} + O \left( \frac{1}{l} \right), \]  

where \( \zeta_2 \equiv \sum_{k=1}^{\infty} \frac{k^2}{k^2}, a > 0 \) is an expansion coefficient, \( b \equiv C \zeta_2 \), and \( \bar{\Delta} \equiv Ca \). Equation (6) is of the same form as \(-TS(l)\), where \( S \) is the loop entropy in the PS model given in equation (3) under the mapping \( c \rightarrow \beta C \) where \( \beta = T^{-1} \) (we use units in which the Boltzmann constant \( k_B \) is unity). In essence, the physics of both models stem from the logarithmic dependence of the energy (entropy) on the domain (loop) length. The mapping between energy and entropy implies an inversion of the temperature role and hence the mapping between the high temperature phase of the PS model and the low temperature phase of the TIDSI model. Other than this trivial difference, the phenomenology of the two models is almost identical. The only qualitative difference between the phase diagrams of the two models is due to the additional spin inversion symmetry, which exists in the TIDSI model and is lacking in the PS model, as discussed below. Another important difference is that while the parameter \( c \) in the PS model is a universal exponent, for the TIDSI model, the corresponding parameter \( \beta c C \), with \( \beta = T^{-1} \), depends on the details of the model such as the nearest-neighbors coupling \( J_{NN} \).

Neglecting terms of order \( O \left( \frac{1}{l} \right) \) and combining (5) and (6), the Hamiltonian expressed by the domain variables reads

\[ H \left( N; \{ l_n \} \right) = H_{NN} + \sum_n H_{LR} (l_n) = \Delta N + C \sum_n \log(l_n) + \text{Const}, \]  

where \( \Delta \equiv \bar{\Delta} + 2J_{NN} > 0 \). The constant term in (7), which also contains expressions such as \(-J_{NN} L\), can be set to 0 without loss of generality, and this is our convention henceforth. In terms of the mapping to the PS model, the Hamiltonian (7) can be viewed as the log-Boltzmann factor of a configuration made up of a sequence of loops without bound segments in between, as depicted in figure 4. Hence the TIDSI model can be viewed either as a truncated version of the IDS model or as a symmetrized version of the PS model. In addition, equation (7) reveals the equivalence of the TIDSI to the ‘balls in a box’ model of [13].

The transition in the TIDSI model can be described using two order parameters. First, the magnetization \( m = \frac{1}{L} \sum \sigma_i \), which in the domain variables reads

\[ m = -\frac{\sigma_1}{L} \sum_{a=1}^{N} (-1)^a l_a. \]  

The second order parameter is the density of domains \( n \equiv \frac{N}{L} \). A magnetic field \( h \) can be added to the Hamiltonian (7) by adding a term \(-hM\), where \( M \equiv Lm \). Similarly, a loop chemical potential \( \mu \) can be added as \(-\mu N\).
Finally, we should define the statistical ensemble on which we will focus. Both the IDSI and PS models are defined in the canonical ensemble in which the length of the chain $L$ is fixed. The partition function of the TIDSI model in this ensemble can be written in terms of the domain variables as

$$Z_C (L; \beta) = \sum_{N=1}^{\infty} \sum_{l_1=1}^{\infty} ... \sum_{l_N=1}^{\infty} e^{-\beta H} I \left( L = \sum_{a=1}^{N} l_a \right),$$

with $H$ given by (7) and the indicator function $I (\psi) = 1$ if $\psi$ is true and 0 otherwise.

However, we also consider more restricted ensembles in which either the magnetization $M$ or number of domains $N$ or both are fixed. The free energies in such ensembles play a similar role as a Landau free energy, enabling deeper understanding of the nature of the transition, as discussed below. The partition function in the ensemble in which both $M$ and $N$ are fixed reads

$$Z_0 (L, M, N; \beta) = \sum_{l_1} ... \sum_{l_N} e^{-\beta H} I \left( L = \sum_{a=1}^{N} l_a \right) I (M = L m (\{l_a\})).$$

From this partition function we can derive the marginal partition sums

$$Z_M (L, M; \beta) = \sum_{N} Z_0 (L, M, N; \beta),$$

$$Z_N (L, N; \beta) = \sum_{M} Z_0 (L, M, N; \beta),$$

$$Z_C (L; \beta) = \sum_{M, N} Z_0 (L, M, N; \beta).$$

3.2. Summary of results

In this section, we present a summary of the results derived for the TIDSI model in sections 4 and 5. As stated above, the transition in the TIDSI model is quite similar to the transition of the PS model (only the role of temperature is inverted): the high temperature phase of the TIDSI model is composed of a gas of microscopic (finite size) domains, while the low temperature phase consists essentially of a single macroscopic domain. The transition is thus a condensation transition: it is MOT for zero magnetic field but can be either continuous or MOT for nonzero field, as discussed in detail in the next subsection.
3.2.1. Phase diagram. The phase diagram of the TIDSI model is presented in figure 5. Figure 5(a) displays the transition temperature ($T_c$) as a function of the coupling $c$ at $h = 0$. Here,

$$c \equiv \beta_c C,$$

where $\beta_c = T_c^{-1}$ and $T_c$ is the critical temperature. The transition line is given by

$$\zeta(\beta_c C) = e^{\beta_c \Delta},$$

(14)

where $\zeta(z)$ is the Riemann Zeta function. The high temperature phase is disordered. Hence, the average magnetization vanishes, $\langle m \rangle = 0$, and the number of domains is macroscopic $\langle n \rangle > 0$. For $T < T_c$, the phase is ordered but, unlike spin models with two-body interactions, the magnetization is saturated, i.e. $\langle m \rangle = \pm 1$ throughout the low temperature phase. In addition, $\langle n \rangle = 0$ as there is essentially a single macroscopic domain. This jump between zero to saturated magnetization is an extreme example of the Thouless effect. As mentioned above, the Thouless effect—first conjectured to take place in the IDSI model—refers to a transition in which the magnetization is discontinuous while the correlation length diverges. In considering the other order parameter $\langle n \rangle$, the nature of the transition can be observed to change along the transition line: for $1 < c \equiv \beta_c C \leq 2$ (region I), the average density of the domains $\langle n \rangle$ drops continuously to 0; for $2 < c \leq 3$ (region II) there is a discontinuity in $\langle n \rangle$ from a finite density of domains $n_c$ to 0 at the transition, but the magnetic susceptibility $\chi_0 \equiv \left. \frac{\partial^2 \log Z_C}{\partial h^2} \right|_{h=0}$ diverges as $T \searrow T_c$; for $c > 3$ (region III) the density of domains $\langle n \rangle$ is discontinuous and $\chi_0$ is finite. The marginal distribution of domain size for $T > T_c$ can be written as $P(l) \sim l^{-\nu} \exp \left[-l/\xi\right]$. This form defines a natural length scale $\xi$ akin to a correlation length (it bounds from below the spin–spin correlation length). For any $c > 1$, this length scale diverges as

$$\xi \sim (T - T_c)^{-\nu}; \quad \nu = \max \left(1, \frac{1}{c - 1}\right).$$

(15)
The mechanism of the transition is similar to that of the Bose–Einstein condensation (BEC): while there is an extensive number of microscopic domains above the critical temperature, which can be referred to as a normal gas of domains, below $T_c$, there is a single macroscopic domain, which can be referred to as the condensate. Note that the first order nature of the transition may depend on the specific order parameter that is considered: while $\langle m \rangle$ changes discontinuously for any value of $c > 1$, $\langle n \rangle$ changes continuously for $1 < c \leq 2$ and discontinuously only for $c > 2$.

In figure 5(b), the phase diagram in the $(T, h)$ plane is plotted. The transition lines in the $(T, h)$ diagram are given by

$$\zeta(\beta C) \Phi_{\beta C}(e^{-2\beta_l|h|}) = e^{2\beta_l\Delta},$$

where $\Phi_{\gamma}(u)$ is the polylogarithm function, which satisfies $\Phi_{\gamma}(1) = \zeta(\gamma)$. Each of the two transition lines at finite $h$ separates two phases: the gas phase where $-1 < \langle m \rangle < 1$ and $\langle n \rangle > 0$, and the condensed phases where $\langle n \rangle = 0$ and $\langle m \rangle = 1$ or $\langle m \rangle = -1$. The nature of the transition changes along the transition line: for $c(h) \equiv \beta(h)C > 3$ (region III), there is a diverging length scale, finite susceptibility, and discontinuous $\langle m \rangle$ and $\langle n \rangle$. For $2 < c(h) \leq 3$ (region II), the susceptibility becomes divergent. For $1 < c(h) \leq 2$ (region I), the susceptibility diverges and the transition is continuous both in $\langle n \rangle$ and $\langle m \rangle$. This is unlike the $h = 0$ case, in which the magnetization is discontinuous. For $T < T_c(h)$, the transition line between the two condensed phases (region IV) is a normal first order transition as in short-range Ising models ($d \geq 2$).

### 3.2.2. Free energies

In addition to the phase diagram, the logarithm of the partition function was evaluated to leading and next to leading order in $L^{-1}$. The leading order term is just the free energy or, more generally, the large deviation function (LDF) [33]. The next to leading order (finite size correction) provides an insight into the mechanism of the transition, revealing a logarithmic (in $L$) barrier between coexisting phases. We argue in section 5.2.2.2 that this leads to a power-law decay of the spin–spin correlation function at large distance.

### Large deviation functions

The LDF for the magnetization in zero external magnetic field is given by

$$F_M(m; \beta) \equiv - \lim_{L \to \infty} \frac{1}{L} \log Z_M(L, mL; \beta)$$

$$= \left( \frac{1 + m}{2} \right) \log [z^+] + \left( \frac{1 - m}{2} \right) \log [z^-],$$

where, for $m \geq 0$, $z^\pm$ are given by

$$z^- = W_-(z^+, \beta),$$

$$z^+ = \begin{cases} W_+(m, \beta) & m < m_c \\ 1 & m \geq m_c \end{cases},$$

with $W_-$ and $W_+$ given implicitly by equations (44) and (48), respectively. Since $F_M$ is symmetric in $m$, this sets its value also for $m < 0$. Here, $W_\pm$ are analytic (and non-constant) functions of their arguments, and $W_+(m_c, \beta) = 1$, where $m_c = 1$ for $\beta C \leq 2$. doi:10.1088/1742-5468/2014/11/P11001
Figure 6. Free energies of the (a) constant magnetization ensemble and (b) constant number of domains ensemble. The X marks correspond to $\pm m_c$ in (a) and $n_c$ in (b), as defined in the text. Notice that in (a) for $T = 1.4$, there is no X mark as $m_c = 1$, and for $T = 1$ there is a single X mark as $m_c = 0$. The parameters used to produce these figures and the following ones are $c = 2.5$ and $\Delta = \log \zeta_c \approx 0.29$ so that $T_c = 1$.

while it is a decreasing function of $\beta$ which vanishes at $\beta_c$ given by equation (14). This implies that if $c \equiv \beta_c C > 2$, then for a given $\beta$ in the interval $[\frac{\beta_c}{C}, \beta_c]$, the LDF is linear for $|m| > m_c$, as can be seen in figure 6(a). This linearity is the usual Maxwell construction of a first order transition in which, under a magnetization constraint, the system phase separates and hence the free energy is essentially the weighted sum of the corresponding free energies of the phases. In this case, the phases are the normal gas of domains with $|m| = m_c$ and the condensate with $|m| = 1$. There are obviously two symmetric condensates, with $m = \pm 1$. Well below the critical temperature, when the gas phase becomes unstable, the coexistence is between those two condensates, and hence the free energy is completely flat. Note that detailed characterization of the nature of the phase transition requires knowledge of the $L^{-1}$ correction to the free energy, which is presented below.

The LDF of the domain density is given by

$$F_N (n; \beta) \equiv - \lim_{L \to \infty} \frac{1}{L} \log Z_N (L, nL; \beta) = \log z^* - n \log \left[ e^{-\beta \Delta} \Phi_{BC} (z^*) \right],$$

where $z^*$ is given by

$$z^* = \begin{cases} W_n (n, \beta) & n > n_c \\ 1 & n \leq n_c \end{cases},$$

with $W_n$ given implicitly by equations (56). Here, $W_n$ is an analytic function of its arguments such that $W_n (n, \beta) \leq 1$ for $n > n_c$, and is an increasing function of $n$ so that equality is achieved only for $n = n_c$. In addition, $n_c = 0$ for $\beta C \leq 2$ is an increasing function of $\beta$ and tends to 1 for $\beta \to \infty$. Thus, for $F_M$, we find that $F_N$ has a linear part for $0 < n < n_c$ and $c > 2$. It can also be seen that at the critical temperature, given by (14), the slope of the linear part of $F_N$ vanishes, while it is negative for $\beta < \beta_c$ and...
positive for $\beta > \beta_c$ (see figure 6(b)). This implies that the minimum of the free energy is at $n^* > n_c$ for $\beta < \beta_c$, at $n^* = 0$ for $\beta > \beta_c$, and is degenerate on the interval $[0, n_c]$ for $\beta = \beta_c$. Hence, for $c > 2$, we indeed find a discontinuous change of $\langle n \rangle$ at the transition while for $c \leq 2$, the change is continuous as $n_c = 0$ at the transition.

**Finite size corrections.** In section 5.2, we derive analytic expressions for $-\frac{1}{L} \log Z_N$ and $-\frac{1}{L} \log Z_M$ to leading orders in the large $L$ limit. Figure 7 presents an exact numerical calculation of these quantities (see appendix D for finite systems with $L = 1000$, compared with the large $L$ analytical expressions. The very good agreement between the results of the two approaches implies that the large $L$ analytical expressions can be used for finite $L$, down to around $L = 1000$.

Figure 7(a) shows that the transition is indeed first-order-like, with the usual picture of two competing wells and that the linearity of the LDF comes from a Maxwell construction. An immediate application of having an explicit approximation is the observation that the free energy barriers, which suppress fluctuations of the order parameters, are logarithmic in $L$. Hence, fluctuations are distributed following a power law, which implies divergence of the correlation length and criticality.

This analytical observation is verified by the exact numerical analysis presented in figure 8 in which the weight of fluctuations of the density of domains $n$ at $T_c$ are quantified. This is done by calculating the sum of the free energy of such fluctuations from $n = 0$ to the minimum of the free energy at $n^*$

$$R(L) \equiv \int_0^{n^*} - \log Z_N (L, Ln; \beta) \, dn.$$  

From equations (60) and (61), it is evident that there is a unique minimum of the free energy at $n^*(L) > n_c$, which represents a gas of domains. Figure 8 shows that $R(L)$ scales logarithmically with $L$, implying logarithmic free energy barriers.

doi:10.1088/1742-5468/2014/11/P11001
Mixed order transition and condensation in an exactly soluble 1D spin model

Figure 8. Scaling of $R(L)$ with $L$. The linearity of the graph on a semi-logarithmic scale shows the logarithmic scaling, i.e. $R(L) \sim \log L$.

3.3. Comparison with the PS and IDSI models

As shown above, the TIDSI model can be considered a symmetric version of the PS model or as a truncated version of the IDSI model. In this section, we discuss the similarities and differences in the phenomenology of the three models.

3.3.1. PS model. The TIDSI model has a very similar phase diagram to that of the PS model, as can be appreciated by comparing figures 3 and 5(a), recalling the inverted role of the temperature in the two models. Essentially, the mechanism of the phase transition is the same, i.e. a condensation transition in which a single domain—or a single loop—becomes macroscopic and encompasses the entire system. There are a few differences though, which we discuss here.

The most important difference is that at $h = 0$, the TIDSI model exhibits an extreme transition in magnetization, from $m = 0$ to $m = \pm 1$. This also happens for $1 < c \leq 2$, for which the PS model exhibits a second order transition. The reason for this difference is the symmetry between + and − spins in the TIDSI model, which is lacking in the PS model. Due to this symmetry, the magnetization vanishes above the transition (as long as there is no magnetic field). To break the symmetry, there must be either an external symmetry breaking field or a spontaneous symmetry breaking transition. Note that once a symmetry breaking field is applied, the transition becomes almost identical to the PS transition as, for $1 < c(h) \leq 2$, the transition is second order. A second related difference is the existence of two order parameters that behave differently for $1 < c \leq 2$: the density of domains, $n$, which is continuous and the magnetization, which is discontinuous. The discontinuity in the magnetization for $1 < c \leq 2$ is due to the symmetry of the model under magnetization reversal, which cannot be broken above the transition. There is no such symmetry for the $n$ order parameter. Hence, the behavior of $n$ in the TIDSI model is the same as that in the PS model.

Another difference involves the parameter $c$. In the PS model, this parameter is the universal exponent controlling the number of configurations of a self-avoiding random loop. The value of $c$ is hence independent of geometry and depends only on the spatial dimension.
and topological constraints. In the TIDSI model, on the other hand, the parameter $C$ itself is rather arbitrary. Moreover, the value of the critical temperature $\beta_c$ depends both on $C$ and $\Delta$, or the nearest neighbors interactions. Hence, $c \equiv \beta_c C$ depends on the model parameters rather than being universal.

3.3.2. IDSI model. The main similarity between the IDSI model and the TIDSI model is that both exhibit a mixed order symmetry breaking phase transition. There are, however, some distinctions between the two models. While the TIDSI has a finite magnetic field transition, the IDSI model exhibits no such transition, as expected from an Ising ferromagnet with two body interactions by the Lee-Yang theorem [2, 34, 35]. Another difference is the divergence of length-scale at the transition, which is algebraic for the TIDSI model, and essential singularity in the case of the IDSI model. The origin of this behavior is discussed in section 6 through the renormalization group analysis.

Perhaps a more fundamental difference between the models is the fact that, in the IDSI, there is no formation of a macroscopic domain (where a domain is defined as a consecutive set of spins all having the same sign). While the density of domains was not analyzed in the case of IDSI, it seems that it does not vanish for any finite temperature, though an interesting question is whether or not it is analytic at the transition. The lack of condensation in the IDSI is obviously the reason that the Thouless effect in this model is not extreme, i.e. that the magnetization jumps to a finite number $< 1$.

4. Grand canonical analysis

We now analytically establish the results stated above. The model (7) is defined in the canonical ensemble, where the chain length $L$ is fixed. In an ensemble where $L, N$ and $M$ are fixed, the partition function and the associated free energy (or large deviation function) are given by

$$Z_0(L, M, N; \beta) = e^{-\beta \Delta N} \sum_{l_1=1}^{\infty} \cdots \sum_{l_N=1}^{\infty} \prod_{a=1}^{N} l_a^{-\beta C} I(L = \sum l_a) \times I\left(M = -\sigma_1 \sum_{a=1}^{N} (-1)^a l_a\right), \tag{23}$$

$$F_0(M, N; \beta) = \lim_{L \to \infty} \frac{1}{L} \log Z_0(L, M, N; \beta). \tag{24}$$

As common in statistical mechanics, instead of evaluating the constrained partition sum (23) itself, we calculate the generating function

$$Q(p, h, \mu; \beta) = \sum_{L, M, N} Z_0(L, M, N; \beta) e^{Lp} e^{\beta h M} e^{\beta \mu N}. \tag{25}$$

For simplicity, we assume symmetric boundary conditions with

$$\sigma_1 = 1; \quad \sigma_L = -1,$$

doi:10.1088/1742-5468/2014/11/P11001 15
Mixed order transition and condensation in an exactly soluble 1D spin model

which implies an even number of domains. Equations (23)–(26) yield

\[ Q(p, h, \mu; \beta) = \sum_N e^{-\beta(\Delta - \mu)N} \prod_{a=1}^N \left[ \sum_{l=1}^\infty \frac{e^{\beta l}}{l^{\beta C}} \exp \left( \frac{(-1)^a \beta hl}{l^{\beta C}} \right) \right] \]

\[ = \frac{e^{2\beta(\mu - \Delta)} U(p + h) U(p - h)}{1 - e^{2\beta(\mu - \Delta)} U(p + h) U(p - h)}, \]  

(27)

with

\[ U(x) = \sum_l \frac{e^{\beta lx}}{l^{\beta C}} = \Phi_{\beta C}(e^{\beta x}). \]  

(28)

The function \( \Phi_{\gamma}(u) \) is the polylogarithm function, which has the following properties:

(a) \( \Phi_{\gamma}(u) \) is analytic in the complex plane except for a branch-cut along \([1, \infty)\)

(b) \( \Phi_{\gamma}(1) = \infty \) for \( \gamma \leq 1 \) and \( \Phi_{\gamma}(1) = \zeta(\gamma) \) for \( \gamma > 1 \), where \( \zeta(\gamma) \) is the Riemann Zeta function.

(c) \( \frac{d}{dx} \Phi_{\gamma}(e^x) = \Phi_{\gamma-1}(e^x) \)

(d) Expands around \( x = 0 \), \( \Phi_{\alpha}(e^x) = \Gamma(1 - \alpha)(-x)^{\alpha-1} + \sum_{k=0}^\infty \frac{\zeta(s-k)}{k!} x^k \), where \( \Gamma() \) is the Gamma function.

We proceed by using (27) to derive the results discussed in section 3, namely the discontinuity of the magnetization and the divergence of the correlation length. Then, we argue that the low temperature phase is characterized by a single macroscopic domain—a condensate. Finally, we justify the derivation for the low temperature phase using a regularization argument.

4.1. The normal phase

The thermodynamic behavior of the system is determined by the grand potential \([1,12]\)

\[ p^* = \min_{M, N} \left\{ F_0(M, N) - h\frac{M}{L} - \mu\frac{N}{L} \right\}. \]  

(29)

The average of the order parameters \( m \equiv M/L \) and \( n \equiv N/L \) are given by

\[ \langle m \rangle = -\frac{\partial p^*}{\partial h}; \quad \langle n \rangle = -\frac{\partial p^*}{\partial \mu}. \]  

(30)

According to equation (25), \( e^{\beta p^*} \) corresponds to the singularity of \( Q \) closest to the origin (most negative \( p^* \)), as it is the radius of convergence of its defining series. Inspecting (27), we see that the singularity can stem either from the denominator becoming 0, i.e.

\[ U(p^* + h) U(p^* - h) = e^{2\beta(\Delta - \mu)}, \]  

(31)

or from the branch point of \( U \), i.e.

\[ p^* + |h| = 0. \]  

(32)

doi:10.1088/1742-5468/2014/11/P11001
In the analysis below, we focus on the regime of small $\mu$. For high temperatures, $\beta \to 0$, $p^*$ is the solution of (31). By differentiating (31) with respect to $h$, we find

$$m(h) = \frac{\Psi_+(p^*, h) - \Psi_-(p^*, h)}{\Psi_+(p^*, h) + \Psi_-(p^*, h)}$$  

(33)

and hence $m \to 0$ for $h \to 0$ (as $\Psi_+(p, 0) = \Psi_-(p, 0)$). However, as $\beta$ increases, the RHS of (31) increases (for $\Delta > \mu$) while the LHS decreases (for a given $p^*$) and hence $p^*$ is an increasing function of $\beta$. Therefore, there is a critical value of $\beta$ such that equation (32) is also satisfied. This happens at $\beta_c$, which is the solution of

$$\Phi_{\beta_c C}(1) \Phi_{\beta_c C}(e^{-2\beta|h|}) = e^{2\beta_c(\Delta - \mu)}.$$  

(35)

From (35), it is clear that the parameter $c \equiv \beta_c C$ satisfies $c > 1$ since $\Phi_c(1)$ diverges for $c \leq 1$. As described in section 4.3, below the transition, namely for $\beta > \beta_c$, the singularity of $Q$ closest to the origin is given by equation (32), i.e. $p^* = -|h|$, and hence

$$m(h) = \text{sign}(h) \equiv \begin{cases} 1 & h > 0 \\ -1 & h < 0 \end{cases}. $$  

(36)

Equation (36) then proves the existence of a discontinuity of the magnetization, where $m(h \to 0)$ jumps from 0 to $\pm 1$ at some finite $\beta_c$. In addition, we see from (33)–(36) that there is also a phase transition for $h \neq 0$, since while for $T > T_c$ the magnetization is given by (33) so that $0 < |m(h)| < 1$, for $T < T_c$ the magnetization is $m(h) = \text{sign}(h)$. The order of this transition depends on $U''(0)$: When it diverges the transition is continuous as can be verified from equation (33), while when it is finite the transition involves a discontinuity of the magnetization. From the properties of the polylogarithm function this implies that the magnetization is continuous for $c(h) \equiv \beta_c(h) C \leq 2$ and discontinuous for $c(h) > 2$.

Calculating the average $N$ in the high temperature regime by differentiating (31) with respect to $\mu$ yields

$$n(h) = \frac{2\beta e^{2\beta(\Delta - \mu)}}{\Psi_+(p^*, h) + \Psi_-(p^*, h)}.$$  

In the low temperature phase, where $p^* = -|h|$, the result is just $n = 0$. Hence, $n$ is continuous through the transition if $U''(0)$ diverges (and therefore also $\Psi_+(-|h|, h)$ or $\Psi_-(-|h|, h)$ diverge), i.e. $c \leq 2$, and is discontinuous otherwise. Thus, the two order parameters $m$ and $n$ behave differently at $h = 0$. While $m$ is discontinuous at the transition for any $c$, $n$ is continuous for $1 < c \leq 2$ and discontinuous only for $c > 2$. On the other hand, on the $h \neq 0$ transition lines, both $m$ and $n$ change continuously for $c(h) \leq 2$ and jump for $c(h) > 2$.

The above results can also be used to calculate the magnetic susceptibility $\chi \equiv \frac{\partial m}{\partial h}$ and the distribution of domain sizes $P(l)$, which defines a typical length scale that diverges at the transition. Differentiating (33) with respect to $h$ yields

$$\chi = \frac{2(\Psi_-\partial_h \Psi_+ - \Psi_+\partial_h \Psi_-)(\Psi_+ + \Psi_-) + 2(\Psi_+ - \Psi_-)(\Psi_+\partial_p \Psi_- - \Psi_-\partial_p \Psi_+)}{(\Psi_+ + \Psi_-)^3}.$$  

(37)
As $\Psi_\pm$ involve $U'(p^* \pm h)$, $\chi$ involves $U''(p^* \pm h)$. It is easy to see that there is no cancellation of these terms, and hence if $U''(0)$ diverges then $\chi$ diverges. From the properties of the polylog, it is evident that $U''(0)$ diverges if $c \leq 3$.

Finally, the distribution of the size of $+$ and $-$ domains is given by

$$P_\pm (l) \simeq \frac{Z_C (L - l, h)}{Z_C (L, h)} \times \frac{e^{\pm \beta h l}}{l^{\beta C}} = \frac{e^{-l/\xi_\pm}}{l^{\beta C}},$$

where we used $Z_C \sim e^{-L \beta p^*}$ and defined $\xi_\pm \equiv \left[ \log (p^* \pm h) \right]^{-1}$. The length scales $\xi_\pm$ are not exactly the correlation length of the spin-spin correlation function but they are its lower bounds. For $h > 0$ ($h < 0$), the length scale $\xi_+$ ($\xi_-$) diverges as $T \to T_c$ for all $c$, which implies that the correlation length diverges as well. Expanding equation (31) near the transition, i.e. $t \equiv T - T_c \ll 1$, $\delta p \equiv -|h| - p^* \ll 1$, and using property four of the polylogarithm function, we get

$$t \sim (\delta p)^{\gamma}; \quad \gamma = \min (c - 1, 1).$$

The algebraic divergence of the correlation length equation (15) directly follows from this relation.

### 4.2. Appearance of condensate in the low temperature phase

In the low temperature phase, where $n = 0$, the number of domains in the system is sub-extensive. We argue that, in fact, this phase is composed of a single macroscopic domain by showing that this state is more favorable than having two condensates. The argument is similar to that given in [36] for condensation in the zero range process.

For a system with a single macroscopic domain and the boundary conditions (26), the partition sum scales as $Z_1 \sim L^{\beta C}$. For a system with two macroscopic domains, the partition function $Z_2$ is a sum of $O(L)$ terms, with each representing a different location of the domain wall. Each term scales as $(L^{-\beta C})^2$ and hence $Z_2 \sim L^{1 - 2\beta C}$. In the low temperature phase $\beta C > 1$ and hence $-\beta C > 1 - 2\beta C$ and the single condensate state is preferable. This argument can be easily extended to other configurations of condensates. From the fact that there is a single condensate, one can easily deduce that $m = \pm 1$ in the low temperature phase.

### 4.3. Condensate phase—regularization

The above argument for the relation $p^* = -|h|$ in the low temperature phase is not mathematically justified, as the sum defining the grand partition function (25) is not well-behaved in this regime.

One way to justify it is to invert the $z$-transform (25), thus finding $Z_0$ directly and validating (32) for $T < T_c$. This route is taken in section 5. Here, we follow a different procedure, whereby the grand canonical ensemble is regularized by introducing an upper cutoff to the domain length, thus making all quantities analytic, and then taking the upper cutoff to infinity. This procedure is well-suited for models with a condensation phenomena (similar to BEC) and has been used in such a context [36–38].
Let us consider the TIDSI model as defined in equation (7) and introduce an upper cutoff $\Lambda$ on the domain length. The partition function for this modified model is

$$Q_\Lambda(p, h, \mu; \beta) = e^{\beta \Delta} \sum_N e^{-\beta (\Delta - \mu) N} \prod_{a=1}^{N} \left[ \sum_{l=1}^{\Lambda} e^{\beta l} \exp \left( \frac{(-1)^n \beta hl}{\beta C} \right) \right]$$

$$= e^{\beta \Delta} \frac{e^{2\beta (\mu - \Delta)} U_\Lambda(p + h) U_\Lambda(p - h)}{1 - e^{2\beta (\mu - \Delta)} U_\Lambda(p + h) U_\Lambda(p - h)},$$

with

$$U_\Lambda(x) = \sum_{l=1}^{\Lambda} e^{\beta xl} = \Phi_{\beta C}^A(e^{\beta x}).$$

The function $\Phi_{\beta C}^A(u)$ is a truncated version of the polylogarithm, which is analytic for any $u$ and $\gamma$. For given $h$ and $\mu$, the thermodynamic limit $L \to \infty$ is again given by the most negative singularity of $Q_\Lambda$, i.e. by the solution of

$$U_\Lambda(p^* + h) U_\Lambda(p^* - h) = e^{2\beta (\Delta - \mu)}.$$

Since $U_\Lambda$ is an analytic function, this equation has a solution for all $\beta$. As before, $p^*$ is an increasing function of $\beta$ and thus there is a temperature $\beta_c(\Lambda)$ for which $p^*(h, \mu, \beta_c(\Lambda)) = -|h|$. For $\beta > \beta_c(\Lambda)$, there is $p^* > -|h|$. In the limit $\Lambda \to \infty$, $\beta_c(\Lambda) \to \beta_c$ as given by equation (35). For any $u > 1$, $\lim_{\Lambda \to \infty} \Phi_{\beta C}^A(u) = \infty$, and hence for $\beta > \beta_c$ and in the limit $\Lambda \to \infty$, the solution satisfies $p^* \to -|h|$ thus validating (32).

5. Canonical analysis

The above analysis proves the existence of the transition and its unique properties. To gain deeper understanding of the mechanism of the condensation transition, it is instructive to study the model within the canonical ensemble. Critical phenomena are commonly described by Landau theory of phase transitions, which provides more details on the nature of such transitions. While the Landau theory is generally based on phenomenological analysis, in our case, the model can be solved exactly and hence we can calculate the large deviation functions (LDF) $F_M$ and $F_N$ and their finite size corrections, which play the role of Landau free energy in this analysis. It turns out that, unlike the basic assumption of the Landau theory, namely that the free energy is an analytic function of the order parameter, the functions $F_M$ and $F_N$ are non-analytic in $n$ and $m$, respectively.

5.1. Large deviation functions

To find the LDF and their finite size correction at zero external field, we invert the $z$-transform (25) using the Cauchy integral formula, and use complex analysis techniques to evaluate the canonical free energy to relevant order.

doi:10.1088/1742-5468/2014/11/P11001
5.1.1. Magnetization large deviation function $F_M$. For ease of notation, we defined

$$L_{\pm} \equiv \frac{1}{2} (L \pm M); \quad z_{\pm} \equiv e^{\beta (p \pm k)}; \quad A \equiv e^{-\beta \Delta}. \quad (41)$$

We carried out the analysis for $M \geq 0$ (or $L_+ \geq L_-).$ The LDF for $M < 0$ is obtained from that of $M > 0$ by symmetry. The partition function $Z_M$ for the ensemble in which $L$ and $M$ (but not $N$) are fixed is given then by

$$Z_M (L, M; \beta) = \oint_{C_+} \frac{dz_+}{2\pi i} \oint_{C_-} \frac{dz_-}{2\pi i} \frac{Q(z_+, z_-; \beta)}{z_+^{L_+} + z_-^{L_-}}. \quad (42)$$

Carrying out the integration over $z_-$, we note that for sufficiently high temperatures, the singularity closest to the origin in the $z_-$ plane is a simple pole at $z_-^*(z_+)$, which satisfies

$$\Phi_{\beta C} (z_+) \Phi_{\beta C} (z_-^*) = A^{-2}. \quad (44)$$

This is the same equation as (31). Hence, the contour of integration $C_-$ can be deformed to a contour encircling this pole $C_p$, plus a contour with a larger radius $C_w$ (see figure 9(a)). Due to the $z_-^{L_-}$ factor, only the pole contour contributes, and hence

$$Z_M (L, M; \beta) \approx \frac{1}{2\pi i A^2} \oint_{C_+} \frac{dz_+}{z_+^{L_++1}} \Phi_{\beta C} (z_+) \frac{z_-^{L_-}}{\Phi_{\beta C} (z_-)}. \quad (45)$$

$$f_m (m, z_+; \beta) = \left( \frac{1 + m}{2} \right) \log [z_+] + \left( \frac{1 - m}{2} \right) \log [z_-^*] + O \left( \frac{1}{L} \right). \quad (46)$$

We proceed to carry out the integration over $z_+$. In the $z_+$ plane, there is a branch-cut for $z_+ \in [1, \infty)$ due to the polylogarithm function. If $f_m (m, z_+; \beta)$ has an extremum for $|z_+| < 1$, the saddle point method can be applied by deforming the contour $C_+$ to a contour $C_{sp}$, which passes through the saddle point as in figure 9(b), yielding

$$F_M (m; \beta) = - \lim_{L \to \infty} \frac{1}{L} \log Z_M (L, M; \beta) = f_m (m, z_+^* (m; \beta); \beta), \quad (47)$$

with the saddle point $z_+^*$ satisfying

$$0 = \frac{d}{dz_+} f_m (m, z_+; \beta) \bigg|_{z_+^*} = \frac{1 + m}{2z_+^*} + \left( \frac{1 - m}{2z_-^*} \right) \frac{dz_-^*}{dz_+^*}. \quad (48)$$

Equation (47) is the result for the LDF $F_M$ in the high temperature regime. We now show that, at a certain temperature, there is no longer a saddle point for $|z_+| < 1$ and hence a
Figure 9. Contours of integration used in the calculations of $Z_M$ and $Z_N$ (see text).

different approach should be followed. By differentiating equation (44) with respect to $z_+$ and using equation (48), the saddle point condition reads

$$ \frac{\Phi_{\beta C-1} (z_+^*)}{\Phi_{\beta C} (z_+^*)} \frac{\Phi_{\beta C-1} (z_-^*)}{\Phi_{\beta C} (z_-^*)} = \frac{1 + m}{1 - m}. $$

(49)

For fixed $z_+^*$, the LHS of (49) is a decreasing function of $\beta$ while for fixed $\beta$, it is an increasing function of $z_+$ as proven in appendix A. The RHS of (49) is an increasing function of both $\beta$ and $m$. Therefore, for a given $\beta$ such that $\beta C > 2$, there is a critical value of $m$, denoted $m_c$, such that for $m > m_c$, there is no saddle point for $|z_+| < 1$. The value of $m_c$ is given by

$$ \zeta (\beta C - 1) \frac{\Phi_{\beta C} (z_+^*)}{\Phi_{\beta C} (z_-^*)} = \frac{1 + m_c}{1 - m_c}. $$

(50)

If there is no saddle point, one can instead deform the contour $C_+$ to contour $C_{bc}$, which wraps the branch cut and close in a large circle with radius tending to infinity as in figure 9. The details of the calculations are involved and hence are deferred to appendix B. However, the result is simple, namely that $z_+$ is frozen at $z_+ = 1$ for all $m > m_c$, i.e.

$$ F_M (m; \beta) = f_m (m; 1; \beta) = \left(1 - \frac{m}{2}\right) \log \left[z_+^* (1)\right]. $$

(51)

Note that the analysis is valid for $m \geq 0$. For $m < 0$, the roles of $z_+$ and $z_-$ should be inverted. Hence, we find that for $m > m_c$, the LDF is linear in $m$ as depicted in figure 6(a).

From equation (44), it can be seen that $z_+^* (1)$ is an increasing function of $\beta$, which implies that the slope of $m$ decreases with $\beta$ (as $z_+^* < 1$). At some critical value $\beta_c$, one has

doi:10.1088/1742-5468/2014/11/P11001
$z^* = 1$ and thus $F_M$ becomes flat (with 0 slope) as this is also the point where $m_c = 0$. The critical temperature is given by

$$\zeta(\beta_c C) = e^{\Delta}, \quad (52)$$

which is the same as equation (35) for $h = 0$. For $\beta > \beta_c$, the singularity closest to the origin is no longer the pole (44) but the branch-point, and hence $F_M = 0$ for all $m$.

In summary, the free energy $F_M(m; \beta)$ is given by (47) for $|m| < m_c$ and by (51) for $|m| > m_c$, with $m_c$ given by (50) (see figure 6(a)). The free energy is linear in $m$ for $|m| > m_c$. Note that equation (44) implicitly defines the function $W^{-}(z_+, \beta)$, which appears in equation (18). Similarly, $W^{+}(m, \beta)$ is defined implicitly by equation (49) for $|m| < m_c$ and by $z_+ = 1$ for $|m| \geq m_c$.

The linearity of the LDF for $|m| > m_c$ is a manifestation of a phase coexistence, i.e. it is the Maxwell construction [39]. The coexistence is between a normal phase that consists of microscopic domains and a phase with a single macroscopic domain. That is, the most probable way to implement a high magnetization $m > m_c$ is by breaking the system to a ‘normal gas’ of domains with total density $m_c$ and a macroscopic condensate with total density $m - m_c$. At the transition, the slope of $F_M$ vanishes, which implies that the gas phase and the condensate phase have the same free energy (in the thermodynamic limit). This description may correspond just as well to a usual first order phase transition, and hence, from this analysis, it is not clear where the mixed nature of the transition comes from. As we see in section 5.2, this comes from the logarithmic barriers in $L$ between the phases.

5.1.2. Domain density large deviation function $F_N$. The analysis for $F_N$ is similar to the analysis of $F_M$, so some details will be spared. We start with

$$Z_N(L, N; \beta) = A \frac{1}{(2\pi i)^2} \oint dq \oint dz \Phi_{\beta C}(z) [z^L+1 q^{N+1}] \frac{q^2 \Phi_{\beta C}(z)^2}{1 - A^2 q^2 \Phi_{\beta C}(z)^2},$$

where $z = e^{3p}$ and $q = e^{3\mu}$. The pole equation that corresponds to the integration over $q$ is

$$q^*(z; \beta) = \frac{1}{A \Phi_{\beta C}(z)}. \quad (53)$$

Again, it is a special case of (31) for $h = 0$. Carrying out the pole integral yields

$$Z_N(L, N; \beta) = \int \frac{dz}{2\pi i} \frac{[A \Phi_{\beta C}(z)]^{Ln}}{A z^{L+1}} \approx \int \frac{dz}{2\pi i} e^{-L f_n(n, z; \beta)}, \quad (54)$$

$$f_n(n, z; \beta) \equiv \log z - n \log [A \Phi_{\beta C}(z)]. \quad (55)$$

A saddle point of $f_n$, when it exists, is given by

$$\frac{\Phi_{\beta C-1}(z^*)}{\Phi_{\beta C}(z^*)} = \frac{1}{n}, \quad (56)$$

doi:10.1088/1742-5468/2014/11/P11001
Deforming the contour to pass through it yields
\[ F_N (n; \beta) = - \lim_{L \to \infty} \frac{1}{L} \log Z_N (L, N; \beta) = f_n (n, \beta^*; \beta). \] (57)

The LHS of (56) is a decreasing function of \( \beta \) and an increasing function of \( z \), while the RHS is a decreasing function of \( n \), and thus \( z^* (n, \beta) \) is an increasing function of both \( \beta \) and \( n \). Hence, if \( \beta C > 2 \), there exists \( n_c \) such that for \( n < n_c \) there is no saddle point for \( |z| < 1 \), and \( n_c \) is given by
\[ n_c = \frac{\zeta(\beta C)}{\zeta(\beta C - 1)}. \] (58)

Then, by branch-cut integration (appendix C) we get
\[ F_N (n < n_c; \beta) = f_n (n, 1; \beta) = -n \log [A \zeta(\beta C)], \] (59)
which is a linear function of \( n \). This linearity is, again, a manifestation of the same coexistence between normal gas of microscopic domains (with domain density \( n = n_c \)) and a single macroscopic domain (with \( n = \frac{1}{L} \to 0 \)). The analysis is valid both above and below \( \beta_c \). For high temperatures (\( \beta < \beta_c \)), the slope is negative (as \( \zeta(\beta C) \) is a decreasing function of \( \beta \)) and hence \( n = 0 \) is disfavored. At the critical temperature, \( A(\beta_c) \zeta(\beta_c) = 1 \) by equation (52). Hence, \( F_N (n < n_c) = 0 \), implying coexistence between the gas \( (n = n_c) \) and the condensate \( (n = 0) \) in the unconstrained system. For \( \beta > \beta_c \), the slope is positive, which implies that the thermodynamically stable phase is one with \( n = 0 \).

For \( \beta C \leq 2 \), there is \( n_c = 0 \). From (56) and (57), it can be deduced that \( n^* \), the minimum of \( F_N \), is a decreasing function of temperature. If \( c \equiv \beta C \leq 2 \), this implies that \( n \to 0 \) in a continuous manner as \( \beta \to \beta_c \). On the other hand, if \( c > 2 \), then at high temperatures, there is no linearity (no coexistence with a condensate phase) but at temperatures \( \beta = 2/C < \beta_c \), a coexistence initiates for small \( n \) or for large \( L \).

In summary, the free energy \( F_N (n; \beta) \) is given by equations (55)–(57) for \( n > n_c \), and is a linear function given by equation (59) for \( n < n_c \), where \( n_c \) is given by equation (58). For \( c \leq 2 \), \( n_c = 0 \) and hence there is no linear part. The function \( W_n (n, \beta) \), appearing in equation (21), is defined by equation (56) for \( n > n_c \) and by \( z^* = 1 \) for \( n \leq n_c \).

### 5.2. Finite size corrections

The above analysis establishes the fact that \( m \) is discontinuous at the transition for all \( c \equiv \beta C > 1 \). Moreover, while \( n \) is discontinuous for \( c > 2 \), it is continuous for \( c \leq 2 \). The canonical analysis presents a scenario that seems to be a conventional first order transition. However, as discussed above, even though the order parameter is discontinuous, at the transition, there are features of a critical transition such as divergence of correlation length and diverging susceptibility (for \( c < 3 \)). How can this be explained in the framework of the canonical analysis, i.e. as an outcome of the form of the ‘Landau free energy’? This question is answered by looking into the finite size corrections to the large deviation functions, which turn out to be logarithmic in \( L \) as shown below. These logarithmic barriers can be understood as the usual surface energy between coexisting phases that appears in first order transitions. In \( d \)-dimensional models with short range interactions, the surface energy scales like \( L^{d-1} \), hence at \( d = 1 \) it is of order \( O(1) \) or \( O(\log(L)) \).
However, this is a somewhat simplistic argument since, in this case, the logarithmic scaling relies also on the effective $r^{-2}$ interaction between spins. Below, we derive the finite size correction for $Z_N$ and $Z_M$ and show that at and below the critical temperature there are logarithmic barriers.

5.2.1. Corrections to $Z_N$. To go beyond the LDF result (57), when a saddle point exists, i.e. for $n > n_c$, we note that the saddle point method yields a sub-leading term due to the second derivative of the integrand, i.e.

$$Z_N(L, Ln; \beta) \approx \sqrt{2\pi L |\partial_2^2 f_n(n, z^*; \beta)|^{Ln} A\Phi_{\beta C}(z^*)^{Ln}}.$$  

(60)

Explicitly calculating $\partial_2^2 f_n$ yields

$$Z_N(L, Ln; \beta) \approx \left[ \frac{2\pi L}{z^2 n} - \frac{\Phi_{\beta C}(z^*)}{\Phi_{\beta C}(z^*) n} \right]^{-\frac{1}{2}} \left[ A\Phi_{\beta C}(z^*) \right]^{Ln}.$$  

(61)

This implies that the correction to the extensive free energy is logarithmic as stated above. For $n < n_c$, the correction comes from the explicit result for the branch-cut integration (see equation (C.3)), i.e.

$$Z_N(L, Ln; \beta) \approx b_\phi(n) \frac{\Gamma(\beta C)}{\pi L^{\beta C - 1} (b_A(n))^{\beta C - 2}} [A\zeta(\beta C)]^{Ln},$$  

(62)

($\lambda$ here is not the same as the one defined in section 4.3) with

$$b_\phi(n) = \frac{n\zeta(\beta C - 1)}{\zeta(\beta C)},$$

$$b_A(n) = \frac{\pi}{\Gamma(\beta C) \zeta(\beta C)}.$$  

For $n = n_c = \zeta(\beta C) / \zeta(\beta C - 1)$ (equation (58)), $b_A(n_c) = 0$ and hence the approximation breaks down near $n = n_c$. The breakdown of the branch cut integration approximation is due to the proximity of the branch point and pole when $n \approx n_c$. Calculating the finite size corrections for $n \approx n_c$ requires a different approach, which is not discussed here.

In figure 7(a), we use the large $L$ expressions (60) and (61) to approximate the free energy for $L = 1000$ and compare it with exact numerical evaluation of the partition function as described in appendix D. The figure shows a very good fit between the two curves, except around $n = n_c$. At the critical point, i.e. when $z^* = 1$ and $A\zeta(\beta C) = 1$, $Z_N$ contain only powers of $L$ and no exponential (or stretched exponential) terms. This indicates that fluctuations of $n$ are distributed asymptotically as a power law and hence at the transition are scale invariant. Together with the discontinuity in $n$ (for $c > 2$), this implies that the transition is MOT.

To verify that the free energy barriers are logarithmic (corresponding to power law dependence of $Z_N$ in $L$), we define $R$ as in (22) to be the integral of $\tilde{F}_N = -1/L \log Z_N(L, N)$ from $n = 0$ to the minimum of $\tilde{F}_N$ at $n^*$. From equations (60) and (61), it can be seen that $\tilde{F}_N$ has a single minimum at $n^* > n_c$, and hence $R$ is well-defined. We then evaluate $R$ numerically and plot it in figure 8. From this figure, it is apparent that $R$ scales logarithmically. This confirms that any free energy barrier between 0 and $n^*$ is logarithmic in $L$. 

doi:10.1088/1742-5468/2014/11/P11001
5.2.2. Corrections to $Z_M$. Following similar steps as in the previous section, we can find the finite size corrections for $Z_M$ but, in this case, the approximation fails near the critical temperature and below it because equation (44) has no solution. Hence, instead we use a simple argument to obtain the leading order finite size behavior of $Z_M$ well below the critical temperature. Specifically, in the regime where the gas phase is unstable, we expect that any magnetization is realized (to leading order) by a phase separation of the chain to a $+$ condensate and a $-$ condensate, their lengths are set by the condition that the given magnetization is realized. This simply implies

$$Z_M(L, m; \beta > \beta_c) \sim \left( \frac{1 + m}{2} L \right)^{-\beta_c} \left( \frac{1 - m}{2} L \right)^{-\beta_c}.$$  \hspace{1cm} (62)

This approximation is plotted in figure 7(b) along with the exact numerical evaluation of the free energy (see appendix D) for $L = 1000$ and seems to fit very well. Here, again, we see that the barrier between the ordered states is logarithmic in $L$. This approach indicates that the logarithmic barriers can be translated to an algebraic decay of the spin-spin correlations function. This is evident from equation (62), according to which the probability of flipping a domain of length $l$ behaves as $l^{-\beta_c}$ for large $l$.

There are no exponential terms, and magnetization fluctuations are suppressed only in a power law manner and not exponentially.

6. RG analysis

As shown above, many features of the model can be obtained analytically in more than one way. However, the main tool that has been used to study the IDSI model and related models is renormalization group (RG) analysis [6, 24]. To study the relationship of our model to other models, we carried out an (approximate) RG analysis to our model and compared it to the RG of the IDSI model, stressing the similarities and differences.

6.1. Model definition

For the scaling—or renormalization group (RG)—analysis, it is useful to define an off-lattice version of the model (7). This version also makes the connection to Coulomb gas models and the Kosterlitz-Thouless scenario more explicit.

We consider a gas of $N$ particles on an interval $[0, L]$, or on the circle if periodic boundary conditions are considered. The particles represent the domain boundaries (kinks) of the lattice version, and following (7), each pair of nearest neighboring particles interact through an attractive logarithmic potential. To avoid divergences, there is an ultraviolet cutoff scale $a$, which is the hard-core of the particles, so that the Hamiltonian takes the form

$$H (\{r_i\}; N) = C \sum_{i=1}^{N} \log \left( \frac{r_{i+1} - r_i}{a} \right) + N \Delta \quad \text{with} \quad |r_{i+1} - r_i| \geq a.$$  

This constitutes the Coulomb gas picture of the model.
6.2. RG analysis

The grand canonical partition function takes the form

\[ Z_C(L; A, K) = \sum_{N=1}^{\infty} A^N \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \frac{dr_1}{a} \ldots \frac{dr_N}{a} \prod_{i=1}^{N} \left( \frac{r_{i+1} - r_i}{a} \right)^{-K} \Theta(r_{i+1} - r_i - a), \]

(63)

where \( A \equiv e^{-\beta \Delta} \) and \( K = \beta C \). This integral can be calculated exactly, as we have done above, but here we follow the RG protocol of [6]. First, we rescale the core-size \( A \) such that the typical distance between a pair is large and hence it is plausible to assume \( A \ll 1 \). Then, we express the partition sum in terms of rescaled parameters \( A_\kappa \) and \( K_\kappa \) such that, in terms of these parameters, it has the same form as the original partition sum. This implies

\[ Z_C(L; A, K) = \sum_{N=1}^{\infty} A_\kappa^N e^{N\kappa(K_\kappa-1)} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{dr_i}{a} \left( \frac{r_{i+1} - r_i}{a} \right)^{-K_\kappa} \Theta(r_{i+1} - r_i - a) \]

Applying the expansion \( \Theta(r_{i+1} - r_i - ae^\kappa) = \Theta(r_{i+1} - r_i - a) - a\kappa \delta(r_{i+1} - r_i - a) + O(\kappa^2) \), we get to first order in \( \kappa \)

\[ Z_C(L; A, K) = Z_C(L; A_\kappa e^{\kappa(K_\kappa-1)}, K_\kappa) - a\kappa \sum_{N=1}^{\infty} A_\kappa^N \sum_j \int_{-\infty}^{\infty} \prod_{i\neq j}^{N} \frac{dr_i}{a} \left( \frac{r_{i+1} - r_i}{a} \right)^{-K_\kappa} \Theta(r_{i+1} - r_i - a) \]

\[ \times \int_{r_{j+1}+a}^{r_{j+1}-a} \frac{(r_{j+1} - r_j)}{a}^{-K_\kappa} \left( \frac{r_j - r_{j-1}}{a} \right)^{-K_\kappa} dr_j \delta(r_{j+1} - r_j - a). \]

(64)

The integral in the last line of (64) results in

\[ \left( \frac{r_{j+1} - r_{j-1} - a}{a} \right)^{-K} \Theta(r_{j+1} - r_{j-1} - 2a). \]

Up to this point, the calculation is exact, but here we introduce a physical argument that enables writing closed RG equations. The physical picture is of kinks interacting in nearest-neighbor pairs. If the density of kinks is small (i.e. \( A \ll a^{-1} \)), it implies that the typical distance between a pair is large and hence it is plausible to assume \( r_{j+1} - r_{j-1} - a \approx r_{j+1} - r_{j-1} \). Therefore, the result (to first order in \( \kappa \)) is

\[ Z_C(L; A, K) = \sum_{N=1}^{\infty} A_\kappa^N (e^{N\kappa(K_\kappa-1)} - a\kappa NA_\kappa) \]

\[ \times \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{dr_i}{a} \left( \frac{r_{i+1} - r_i}{a} \right)^{-K_\kappa} \Theta(r_{i+1} - r_i - a). \]

To compensate for the additional factor \( (e^{N\kappa(K_\kappa-1)} - a\kappa NA_\kappa) \) such that the partition sum retains its original form, the renormalized parameters \( A_\kappa \) and \( K_\kappa \) are taken to be in leading order in \( \kappa \)

\[ A_\kappa = A(1 + \kappa (1 - K + aA)) + O(\kappa^2), \]

\[ K_\kappa = K + O(\kappa^2). \]
Defining \( x = 1 - K \) and \( y = aA \) the resulting flow equations are

\[
\begin{align*}
\frac{dy}{d\kappa} &= xy + y^2, \quad (65) \\
\frac{dx}{d\kappa} &= 0. \quad (66)
\end{align*}
\]

The physical intuition behind these flow equations is the following: A change of scale has two effects. First, due to the change of scale, the density—and hence the fugacity \( y \)—renormalizes. This is accounted for by the first term of (65). Second, rescaling of the core size might cause near-by kinks to merge. The likelihood of having two adjacent kinks scales is \( y^2 \), implying the second term. For far away (but nearest neighbors) kinks, the effect of two kinks that merged is the same as a single kink, as the interaction is between nearest neighbors. Since these are the only outcomes of the scale transformation (to leading order), the interaction \( x \) is kept constant.

The RG flow diagram has a line of fixed points at \( y = 0 \), which are stable for \( x < 0 \) and unstable for \( x > 0 \). It has another line of unstable fixed points at \( y = -x \) for \( x < 0 \), which is the line of phase transitions: flow lines which start above this line increase in \( y \) until the validity of the analysis breaks down (the condition \( y \ll 1 \) is no longer valid) and hence flow into the disordered phase, while flow lines that start below this line flow parallel to the \( y \) axis into the corresponding \( y = 0 \) fixed point, which is the ordered phase. The details of the flow diagram are presented in figure 10(a) together with the exact phase transition line.

From the RG equations, we can calculate the behavior of the correlation length near the transition. This is done by integrating the flow equations, starting just above the critical line \( y^* = -x \), i.e. at \( y = -x + \delta y \) with \( \delta y \sim T - T_c \), and ending at \( y = 1 \), where the analysis breaks down but the correlations are order 1. Then, equation (65) yields

\[
\kappa = \left[ \frac{1}{x} \log \left( \frac{y}{x + y} \right) \right]_{-x+\delta y}^1.
\]

Using \( \xi \sim e^\kappa \) and \( \delta y \sim T - T_c \), the leading order term is

\[
\xi \sim \left( \frac{T - T_c}{|x|} \right)^{\frac{1}{x}}.
\]

Using \( x = 1 - \beta_c C \), this is the same as equation (15). The exponent \( 1/x \) diverges for \( x \to 0_- \), suggesting that, in this limit, an essential singularity develops.

### 6.3. Comparison of the flow diagram of IDSI and related models

The IDSI model, like other models with \( r^{-2} \) interactions, has a very similar representation as an interacting gas of charges [6]. The only difference between the TIDSI model and the IDSI model is that in the IDSI, all the charges interact with each other logarithmically, while in the TIDSI model, the interactions are only between nearest neighbors. This implies that in the IDSI model, when two opposite charges are close by, they screen the effect of each other and thus to leading order they cancel each other out and contribute
only a dipole moment. This dipole moment turns out to renormalize the logarithmic interaction between other charges, exactly in the same manner that the screening of close by vortices renormalizes the interactions between vortices in the XY model. Hence, the $y^2$ term in (65), which appears to be due to the merging of two charges, is transferred to the flow equation for $x$ and the resulting flow equations for the IDSI model read

$$\frac{dy}{d\kappa} = xy,$$

(67)

$$\frac{dx}{d\kappa} = y^2,$$

(68)

where $x$ and $y$ are defined essentially the same as above. This flow has only one line of fixed points at $y = 0$ but also has a separatrix at $y = |x|$. The phase transition is along the line $y = -x$ for $x < 0$, where flow lines starting below it flow to the $y = 0$ fixed line (ordered phase) and flow lines that start above it flow to the disordered phase, but the transition is controlled by a single critical point $x = y = 0$. The flow in the IDSI and TIDSI cases can be compared in figure 10. A consequence of the different flow equations is the behavior of the correlation length, which diverges algebraically in the TIDSI model and exhibits an essential singularity for the IDSI model. The following is an outcome of the flow near the critical point (or critical line in the case of the TIDSI model): while linearizing equations (65) and (66) near $y = -x$ line, we find that the flow is linear in the TIDSI model and the flow is obviously quadratic in the IDSI case. This difference yields the different behavior of the correlation length.

Figure 10. RG flow for (a) TIDSI model, equations (65) and (66) and (b) IDSI model (or XY model), equations (67) and (68).

It is interesting to note that the lack of renormalization of the coupling constant $x$ also appears in the context of discrete Gaussian chain [8]

$$H = -\sum_{ij} J_{ij} (h_i - h_j)^2; \quad J_{ij} \sim |i - j|^{-2},$$

where the height variables $h_i$ are integers, and with the boundary condition $h_0 = 0$. This problem can also be mapped onto a dissipative quantum particle in a periodic potential [40]. In this case, there are an infinitely number of fugacities $y_k$ corresponding...
to kinks with $h_i - h_j = k$, all renormalized due to both the density scaling and merging of kinks, but the coupling coefficient does not rescale. The connection between the TIDSI model and the discrete Gaussian chain could be a subject of future investigations.

7. Generalizations

The TIDSI model can be generalized without losing its solubility. Below, we first generalize the decay of interactions between spins beyond the inverse square law of $J(r)$ in (4). We then consider the inverse squared law but with spin models other than the spin $\frac{1}{2}$ Ising model, namely the Potts model and the general Ising model.

7.1. General interaction decay law

7.1.1. Definition. In this section, we consider the Hamiltonian (4) with

$$J(r) = Cr^{-\alpha},$$

where $\alpha > 1$. The long-range self energy of a domain $H_{LR}$ can be estimated, as before, to be

$$H_{LR}(l) = -C \sum_{k=1}^{l} \frac{l-k}{k^\alpha} = -C \zeta(\alpha) l + C \left( \frac{1}{\alpha - 1} + \frac{1}{\alpha - 2} \right) l^{2-\alpha} + O(1).$$

As before, the linear term will contribute a constant in the total energy. For $\alpha > 2$, the subleading term is $O(1)$, and hence there is no transition. We thus restrict the discussion in this paper to $1 < \alpha < 2$. After readjusting the ground state energy, the Hamiltonian reads

$$H^{(\alpha)}(\{l_a\}; N) = C_\alpha \sum_{a=1}^{N} l_a^{2-\alpha} + \Delta N,$$

with $C_\alpha \equiv C \left( \frac{1}{\alpha - 1} + \frac{1}{\alpha - 2} \right)$ and $\Delta = 2J_{NN}$.

7.1.2. Analysis. The analysis of the above model in the grand canonical ensemble is very similar to the one outlined in section 4. Skipping some details, the generating function is now

$$Q(p, h, \mu; \beta) = \frac{e^{2\beta(\mu-\Delta)} W_\alpha(p+h) W_\alpha(p-h)}{1 - e^{2\beta(\mu-\Delta)} W_\alpha(p+h) W_\alpha(p-h)},$$

with

$$W_\alpha(x) = \sum_l \exp(\beta x l - \beta C_\alpha l^{2-\alpha}) \equiv \Psi_\alpha^{\beta C_\alpha}(e^{lx}).$$

Note that the functions $W_\alpha$ and $\Psi_\alpha^{\beta C}$ have nothing to do with the functions $W_\pm()$, $W_\gamma()$ and $\Psi_\pm()$ defined in previous sections. The relevant properties of $\Psi_\gamma^\alpha(u)$ for $1 < \alpha < 2$ are:

doi:10.1088/1742-5468/2014/11/P11001
7.2. General spins

We return now to the $J(r) \approx Cr^{-2}$ case but consider more general spin models. For concreteness, we focus on Potts spins and general Ising spins but other models can be analyzed following the same steps. Specifically, we show that such models can be solved exactly using the transfer matrix approach.
7.2.1. Potts spins. We now consider a chain of \( L \) spins \( \sigma_i \) that can take a value in \([1..K]\) with a Hamiltonian analogous to equation (4), i.e.

\[
H = -J_{NN} \sum \delta_{\sigma_i, \sigma_{i+1}} - \sum_{i<j} J(i-j) \delta_{\sigma_i, \sigma_j} I(i \sim j). \tag{75}
\]

Due to the truncation of the LR interactions, the term \( \delta_{\sigma_i, \sigma_j} \) within each domain is always unity. The model can be cast in the domain representation, in which a configuration is composed of \( N \) domains with sizes \( \{l_n\} \) and spins \( \{s_n\} \). A domain self energy then has a very similar form to equations (5) and (6), which yields the same Hamiltonian as (7), i.e.

\[
H (\{l_n\}, \{s_n\}; N) = C \sum_{n=1}^{N} \log (l_n) + \Delta N + \text{Const.} \tag{76}
\]

As there are \( K \) symmetric spin states in this case and not only two, a natural ensemble to consider is one where the set \( \{L_s\}_{s=1}^{K} \) is fixed and \( L_s \) is the total number of spins of type \( s \). The partition function of such an ensemble, where \( N \) is also fixed, is

\[
Z_0 (\{L_s\}, N; \beta) = \sum_{\{l_n\}} \sum_{\{s_n\}} \prod_{n=1}^{N} e^{-\beta \Delta} \prod_{s=1}^{K} \left( \sum l_n \delta_{s_n, s} = L_s \right). \]

A set of \( K \) order parameters can be constructed as

\[
m_s = \frac{KL_s - L}{L(K-1)}. \tag{77}
\]

To define the corresponding grand partition function, we thus introduce \( K + 1 \) fugacities (\( \mu \) and \( p_s, s = 1..K \)) corresponding to the \( K + 1 \) constraints (fixed \( N \) and \( \{L_s\} \)). Then, the grand partition function is

\[
Q (\{p_s\}, N; \mu; \beta) = \sum_{N} \prod_{n=1}^{N} e^{\beta (\mu - \Delta)} \sum_{s_n \neq s_{n-1}} U_{\beta C} (p_{s_n}), \tag{78}
\]

\[
U_{\rho} (x) \equiv \sum_{l} \frac{e^{\beta x l}}{l^\rho}. \tag{79}
\]

To proceed, we define the transfer matrix,

\[
\hat{T}_{\sigma \tau} = \begin{cases} 
  e^{\beta (\mu - \Delta)} U_{\beta C} (\beta p_\tau) & \sigma \neq \tau \\
  0 & \sigma = \tau 
\end{cases}. \tag{80}
\]

Assuming simplicity fixed boundary conditions \( s_1 \) and \( s_N \), and using conventional bra-ket notations, the grand-partition function can be expressed as

\[
Q (\{p_s\}, \mu; \beta) = \sum_{N=1}^{\infty} \langle s_1 | \hat{T}^N | s_N \rangle = \langle s_1 | \frac{\hat{T}}{1 - \hat{T}} | s_N \rangle. \tag{81}
\]
We can first set $\mu = 0$ and $p_s = p$ for all $s$, i.e. constraining only the total size $L$. As before, the most negative singularity of $Q$ in $p$ can stem either from the denominator, where the maximal eigenvalue of the matrix $\hat{T}(p)$ satisfies $\lambda_{\text{max}} = 1$, i.e.

$$p^* = \arg\min_p \{\lambda_{\text{max}}(p) = 1\} , \tag{82}$$

or from the branch point of $U_{\beta C}$, at

$$p^* = 0. \tag{83}$$

In this setting, the eigenvalues of $\hat{T}$ can be obtained exactly, and the maximal eigenvalue is

$$\lambda_{\text{max}} = (K - 1) e^{-\beta \Delta U_{\beta C}(p)} .$$

Hence, for sufficiently low temperature, $p^* = 0$, which, as in the TIDS1, implies a condensation transition.

To see that indeed there is condensation, we can set $p_1 = p + r$ and $p_s = p$ for all $s > 1$. Then,

$$\frac{L_1}{L} = - \frac{dp^*}{dr} \bigg|_{r=0} .$$

The largest eigenvalue of $\hat{T}$ is given by

$$\lambda_{\text{max}} = \frac{e^{-\beta \Delta}}{2} \left[ (K - 2) U_{\beta C}(p) + \sqrt{(K - 2)^2 U_{\beta C}(p)^2 + (4K - 4) U_{\beta C}(p) U_{\beta C}(p + r)} \right] .$$

Hence, in the high temperature phase, for which $p^*$ is set by the condition $\lambda_{\text{max}} = 1$, we find by straightforward calculation

$$L_1 = \frac{L}{K} \Rightarrow m_1 = 0,$$

while in the low temperature phase $p^* = -r$ (for $r > 0$) and, therefore,

$$L_1 = L \Rightarrow m_1 = 1.$$

This implies that the order parameter $m_1$ jumps from 0 to 1 at the transition just as in the original TIDS1. Finally, the domain size distribution (at $r = 0$) can be written, in analogy with equation (38), as

$$P(l) \simeq \frac{Z_C(L - l, h)}{Z_C(L, h)} \times \frac{1}{l^{\beta C}} = \frac{e^{-\beta p^* l}}{l^{\beta C}} = \frac{e^{-l/\xi}}{l^{\beta C}},$$

with $\xi = \log (p^*)^{-1}$ as the diverging length scale. Hence, the transition is MOT of the same kind as in the original TIDS1.

7.2.2. General Ising spins. Now we consider a spin $\frac{K}{2}$ Ising model, i.e. where each spin $\sigma_i$ can take one of the $K + 1$ different values $\{-K, -K + 2, ..., K\}$. The Hamiltonian is

$$H = -J_{NN} \sum \sigma_i \sigma_{i+1} - \sum_{i < j} J(j - i) \sigma_i \sigma_j I (i \sim j). \tag{84}$$

doi:10.1088/1742-5468/2014/11/P11001
Different domains have different energies according to their spin. The Hamiltonian in the domain variables reads

\[ H \left( \{ l_n \}, \{ s_n \}; N \right) = - \sum_{n=1}^{N} B_1 s_n^2 l_n + C \sum_{n=1}^{N} s_n^2 \log (l_n) + \sum_{n=1}^{N} \left( B_2 s_n^2 - J_{NN} s_n s_{n+1} \right) + \text{Const}, \]

where \( B_1 \) and \( B_2 \) are positive coefficients. The grand canonical partition function has the same form as in the Potts model, i.e. equation (81), with the transfer matrix

\[ \hat{T}_{\sigma \tau} = \exp \left[ \beta \left( \mu - B_2 \tau^2 + J_{NN} \sigma \tau \right) \right] U_{\beta C \tau^2} \left( p_\tau + B_1 \tau^2 \right), \]

for \( \sigma \neq \tau \) and \( \hat{T}_{\sigma \sigma} = 0 \). The function \( U_\rho(x) \) is defined by equation (79). Setting \( p_\tau = p \) and \( \mu = 0 \), the thermodynamic limit is obtained when either the maximal eigenvalue is unity, i.e.

\[ \lambda_{\text{max}} (p^*) = 1, \quad (85) \]

or at

\[ p^* = - \max_{\tau} \{ B_1 \tau^2 \} = -B_1 K^2. \quad (86) \]

Calculating \( \lambda_{\text{max}} \) is difficult and provides no new insights. However, as \( \hat{T} \) is a non-negative irreducible matrix, the Perron-Frobenius theorem implies that \( \lambda_{\text{max}} > 0 \). Moreover, for \( J_{NN} < B/K \), increasing \( \beta \) decreases (or does not change) all the elements of \( \hat{T} \) and thus, by Wielandt’s theorem [41], \( \lambda_{\text{max}} \) also decreases with \( \beta \). The reverse is true for increasing \( p \) and hence, in the high temperature phase, where \( p^* \) is set by equation (85), \( p^* \) is an increasing function of \( \beta \). Therefore, there is a critical \( \beta \) for which \( \lambda_{\text{max}} (B_1 K^2) = 1 \), which sets the transition temperature. Adding to the Hamiltonian, a magnetic field \( h \), which is coupled to the magnetization order parameter \( m = \frac{1}{L} \sum l_n s_n \), amounts to setting \( p_\tau = p + \tau h \). The maximal eigenvalue \( \lambda_{\text{max}} \) must be symmetric with respect to \( h \rightarrow -h \) and hence in the high temperature regime \( m = 0 \). In the low temperature regime, \( p^* = -B_1 K^2 - |h| K \), which implies that \( m = \pm K \) as expected. Hence, while there is also full magnetization in this model, the spin of the macroscopic domain is only two fold degenerate and not \( K \) (or \( K + 1 \)) fold degenerate as in the Potts model case.

8. Conclusions

In this paper, we present a detailed analysis of the TIDSI model, which was recently introduced in [1]. The study is motivated by the observation that this is an exactly soluble model that exhibits a mixed order transition and serves as a link between different classes of models exhibiting MOT. The steady state of the model and its phase diagram are first calculated in the grand canonical ensemble. In addition, a canonical analysis that sheds new light on the mechanism of a mixed order transition is presented. This analysis shows that for \( c > 2 \), where both order parameters, the magnetization \( m \), and the domain density \( n \) are discontinuous, criticality stems from logarithmic barriers in the effective Landau free energy. For \( c < 2 \), where \( n \) is continuous, the magnetization \( m \) remains discontinuous due to spin inversion symmetry in the high temperature phase. We also elaborate on...
the RG analysis presented in [1] and finally generalize the model by introducing general power-law decaying interactions \((1/r^\alpha)\) and several other types of spin variables. These generalizations elucidate the special features of the borderline case \(\alpha = 2\) and show that MOT can take place in a rather general class of discrete spin models.

The TIDSI model provides a bridge between 1D models with \(1/r^2\) interactions such as the IDS and 1D models exhibiting the depinning transition such as the PS model. This opens a window for a more general question regarding the connection between models exhibiting mixed order transitions. For instance, the spiral model of [15, 16] is a two-dimensional model that exhibits MOT. Can the mechanism that leads to the transition to the jammed state be related to that of the 1D models exhibiting MOT discussed in this paper? In the context of networks, there has been a recent debate [42,43] regarding the nature of transition of a process dubbed ‘explosive percolation’ in which irreversible network evolution models exhibit a rather abrupt appearance of a giant component. There is some evidence [21,22] that this process, or some version of it, leads to a mixed order transition, with a finite size behavior similar to TIDSI, i.e. logarithmic barriers. Can this process, and related percolation models such as \(k\)-core percolation [18] be connected with our model? A general framework for studying such mixed order transitions is still missing.

Another interesting and not thoroughly explored direction of research is the dynamics of (equilibrium) models exhibiting MOT. Phase separation kinetics is the dynamical behavior of systems quenched from a high temperature phase, usually infinite temperature, to a low temperature ordered phase. The phase ordering kinetics of systems exhibiting second order phase transitions, usually at zero temperature [44], has been the subject of a large body of work in recent years. However, there are no elaborate studies of the phase ordering kinetics in models exhibiting MOT. In [45], the phase ordering kinetics of the IDS and other models with long-range interactions were studied but only at zero temperature, while a more interesting case would be quenching to the critical temperature in which, unlike in second order transitions, real phase ordering is expected. Another intriguing question in this context is the connection of the dynamics of the TIDSI model with nonequilibrium models with absorbing states. This is left for future work.

Acknowledgments

We thank M Aizenman, O Cohen and O Hirschberg for helpful discussions. The support of the Israel Science Foundation (ISF) and of the Minerva Foundation with funding from the Federal German Ministry for Education and Research is gratefully acknowledged. We also thank the Galileo Galilei Institute for Theoretical Physics for their hospitality and the INFN for partial support during the completion of this work.

Appendix A. Analysis of the LHS of equation (49)

We wish to show that

\[
g(z_+) \equiv \frac{\Phi_{\beta C-1}(z_+) \Phi_{\beta C}(z^*_+ (z_+))}{\Phi_{\beta C}(z_+) \Phi_{\beta C-1}(z^*_+ (z_+))},
\]
is an increasing function of $z_+$ for fixed $\beta$. The function $z^*$ is given by the implicit relation (44), i.e.

$$\Phi_{\beta C}(z^*) = \left[A^2 \Phi_{\beta C}(z_+)\right]^{-1},$$

implying that $z^*$ is a decreasing function of $z_+$. The function $g$ can be written as

$$g(z_+) = \frac{h(z_+)}{h(z^*)},$$

where

$$h(u) = \Phi_{\beta C}(u) \Phi_{\beta C}(u) - \Phi_{\beta C}(u)^2,$$

implying that $h(u)$ is an increasing function thus proves that $g(z_+)$ is also an increasing function.

To show that $h$ is increasing, we inspect its derivative

$$\frac{dh(u)}{du} = \Phi_{\beta C}(u) \Phi_{\beta C}(u) - \Phi_{\beta C}(u)^2 = \frac{N}{D}_D,$$

The denominator $D$ is trivially positive and is left to show the same for the numerator $N$:

$$N = \sum_{k<l} u_l + k [l - k] > 0.$$

**Appendix B. Approximating (45) without saddle point**

Here, we calculate the integral (45) when $f_m(m, z_+)$ has no saddle point for $|z_+| < 1$, i.e. in the regime $\beta < \beta_c$ and $m > m_c$, which implies $\beta C > 2$. In this case, the contour of the integral can be deformed as presented in figure 9(b). The contour $C_{bc}$ can be expressed as the concatenation of four parts, which are

\begin{align*}
(I) & : [R - i\epsilon, 1 - i\epsilon], \\
(II) & : \left\{ 1 - \epsilon e^{i\theta} : \frac{\pi}{2} < \theta < \frac{3\pi}{2} \right\}, \\
(III) & : [1 + i\epsilon, R + i\epsilon], \\
(IV) & : \left\{ \text{Re}^{i\theta} : \delta < \theta < 2\pi - \delta \right\},
\end{align*}

where $R \gg 1$ and $\epsilon \ll 1$ are free parameters, and $tg(\delta) = \frac{\pi}{R}$. Along this contour, the function $f_m$ can have complex values, and hence we define

$$\Lambda \equiv \text{Re}[f_m]; \quad \phi \equiv \text{Im}[f_m].$$

doi:10.1088/1742-5468/2014/11/P11001
The integral can thus be written as

$$Z_M(L, M; \beta) = \frac{1}{2\pi L A^3} \oint_{C_{bc}} dz_+ e^{-L[A(m, z_+; \beta) + i\phi(m, z_+; \beta)]}. \quad (B.3)$$

The contribution of part (II) of $C_{bc}$ is $O(\epsilon)$ and hence can be neglected in the limit $\epsilon \to 0$. To show the same for part (IV), we note that for $|u| \gg 1$, $|\Phi_y(u)| \sim \log (u)^7$. Hence, from (44), we see that for $|z_+| = R \gg 1$, $z_+^* \sim (\log R)^{-7}$. Hence, along part (IV), the absolute value of the integrand scales is $(R^{-L_+} \log(R)^{7L_+})$, which vanishes faster than $R^{-2}$ for any extensive $L_+$ and thus the integral over (IV) is zero. Hence, only parts (I) and (III) contribute.

The polylogarithm function and hence also $f_m$, have a series expansion with real coefficients. Therefore, $f_m(m, z; \beta) = \overline{f_m(m, z; \beta)}$, where $\overline{z}$ is the complex conjugate of $z$.

Parts (I) and (III) traverse complex conjugated paths (in reverse order) and hence contribute.

As there is no saddle point for $z_+ < 1$, and $f_m(m, z \to 0; \beta) \to -\infty$, we see that $b_\Lambda > 0$. In addition, (equation (B.7)), together with the condition $L \phi (m, 1 + \delta z; \beta) = 1$, implies $\eta \sim L^{-\frac{1}{3\beta C}}$. In the thermodynamic limit, the upper limit tends to infinity as $\beta C > 2$. In addition, due to the $e^{-u}$ factor, only $u = O(1)$ contributes and, in this region, the argument of the sin () tends to 0 (as $\beta C > 2$). Hence, it can be expanded:

$$Z_M(L, M; \beta) \approx \frac{e^{-LA(m, 1; \beta)}}{\pi A^3 L b_\Lambda} \int_0^\eta d\delta z e^{-Lb_\Lambda \delta z} \sin \left(\frac{Lb_\phi \delta z}{(b_\Lambda L)^{\beta C - 1} u^{\beta C - 1}}\right).$$

As there is no saddle point for $z_+ < 1$, and $f_m(m, z \to 0; \beta) \to -\infty$, we see that $b_\Lambda > 0$. In addition, (equation (B.7)), together with the condition $L \phi (m, 1 + \delta z; \beta) = 1$, implies $\eta \sim L^{-\frac{1}{3\beta C}}$. In the thermodynamic limit, the upper limit tends to infinity as $\beta C > 2$. In addition, due to the $e^{-u}$ factor, only $u = O(1)$ contributes and, in this region, the argument of the sin () tends to 0 (as $\beta C > 2$). Hence, it can be expanded:

$$Z_M(L, M; \beta) \approx \frac{e^{-LA(m, 1; \beta)}}{\pi A^3 L b_\Lambda} \int_0^\infty du e^{-u} \sin \left(\frac{Lb_\phi \delta z}{(b_\Lambda L)^{\beta C - 1} u^{\beta C - 1}}\right).$$
B.1. Deriving equations (B.6) and (B.7)

We wish to prove (B.6) and (B.7). We define $z^∗(1 + \delta) = z^∗(1) + \chi$. Then, equation (44) implies

$$\Phi_{\beta c} (z^∗(1) + \chi) = \frac{1}{A^2 \Phi_{\beta c}(1 + \delta)}.$$ 

Expanding both sides in terms of $\chi$ and $\delta$ for $\beta C > 2$ yields

$$\Phi_{\beta c} (z^∗(1) + \chi) \approx \Phi_{\beta c} (z^∗(1)) + \frac{1}{z^∗} \Phi_{\beta c-1} (z^∗(1)) \chi,$$

$$\frac{1}{\Phi_{\beta c}(1 + \delta)} \approx \frac{1}{\zeta_{\beta c} + \zeta_{\beta c-1} \delta + i \pi \delta^{3c-1}/\Gamma(\beta c)} \approx \frac{1}{\zeta_{\beta c}} - \frac{\zeta_{\beta c-1}}{\zeta_{\beta c}^2} \delta - \frac{i \pi}{\zeta_{\beta c}^2 \Gamma(\beta c)} \delta^{3c-1}.$$ 

Hence,

$$\text{Re} [\chi] = - \frac{\zeta_{\beta c-1} z^*_c(1)}{A^2 \zeta_{\beta c}^2 \Phi_{\beta c-1}(z^*_c(1))} \delta + o(\delta),$$

$$\text{Im} [\chi] = - \frac{\pi z^*_c(1)}{\Phi_{\beta c-1}(z^*_c(1)) A^2 \zeta_{\beta c}^2 \Gamma(\beta c)} \delta^{3c-1} + o(\delta^{3c-1}).$$

Inserting these results into the definition of $f_m$, i.e. equation (45), yields

$$f_m(m, 1 + \delta; \beta) = \frac{1 + m}{2} \log (1 + \delta) + \frac{1 - m}{2} \log (z^*_c + \chi)$$

$$\approx \frac{1 - m}{2} \log (z^*_c) + b_A (m; \beta) \delta - i b_\phi (m; \beta) \delta^{3c-1},$$

$$b_A (m; \beta) = \frac{1 + m}{2} - \frac{(1 - m) \zeta_{\beta c-1} \Phi_{\beta c} (z^*_c)}{2 \zeta_{\beta c} \Phi_{\beta c-1} (z^*_c)},$$

$$b_\phi (m; \beta) = \frac{\pi (1 - m)}{\Phi_{\beta c-1} (z^*_c) A^2 \zeta_{\beta c}^2 \Gamma(\beta c)},$$

with $z^*_c = z^*_c(1)$. Hence,

$$b_A (m; \beta) \approx b_A (m; \beta) \delta,$$

$$b_\phi (m; \beta) \approx b_\phi (m; \beta) \delta^{3c-1}.$$ 

Appendix C. Approximating (54) without a saddle point

Here, we calculate the integral (54) when $f_n (n, z)$ has no saddle point for $|z| < 1$, i.e. in the regime $n < n_c$, which implies $\beta C > 2$. This case is very similar to the case considered in appendix B, and hence, we do not include all of the details. The contour of the integral
can be deformed to a contour \( C_{bc} \), defined by equation (B.1) and presented in figure 9(b). Along this contour, the function \( f_n \) can have complex values, and hence, we define

\[
\Lambda \equiv \text{Re} \left[ f_n \right]; \quad \phi \equiv \text{Im} \left[ f_n \right].
\]

The integral can then be written in a form equivalent to (B.3) and (B.4)

\[
Z_N(L, L n; \beta) \approx \frac{1}{\pi} \int_{C_{bc}} \text{d} z e^{-L \Lambda(n, z; \beta) + i \phi(n, z; \beta)}.
\]

where, as above, \( \eta \ll 1 \). Following similar steps to those outlined in section B.1, one can find

\[
\Lambda(n, 1 + \delta z; \beta) \approx \Lambda(n, 1; \beta) + b_\Lambda \delta z, \quad (C.1)
\]

\[
\phi(n, 1 + \delta z; \beta) \approx b_\phi \delta z^{\beta c - 1}, \quad (C.2)
\]

with

\[
b_\Lambda = 1 - \frac{n \zeta_{\beta c - 1}}{\zeta_{\beta c}} \frac{\zeta_{\beta c}}{\Gamma(\beta c)},
\]

\[
b_\phi = n \frac{\pi}{\Gamma(\beta c) \zeta_{\beta c}}.
\]

As \( n < n_c = \zeta_{\beta c}/\zeta_{\beta c - 1} \), this implies \( b_\Lambda > 0 \) as expected. Following the same steps as in appendix B, this implies

\[
Z_N(L, L n; \beta) \approx \frac{\exp(-L \Lambda(n, 1; \beta))}{\pi} \int_0^{\eta} \text{d} \delta z e^{-L b_\phi \delta z} \sin \left( L b_\phi \delta z^{\beta c - 1} \right)
\]

\[
\approx \frac{b_\phi \Gamma(\beta c)}{\pi L^{\beta c - 1} b_\Lambda^{\beta c}} e^{-L f_n(n, 1; \beta)}. \quad (C.3)
\]

**Appendix D. Numerical procedure for evaluating the partition function**

Here, we explain the numerical procedure used to evaluate \( Z_M(L, M; \beta) \) exactly. The partition function \( Z_M \), with the boundary conditions \( \sigma_1 = 1 \) and \( \sigma_L = -1 \), which we denote by \((+−)\), has the form

\[
Z^{(+−)}(L, M; \beta) = \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta(2\nu - 1)} \left( \prod_{a=1}^{2\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{2\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{2\nu} (-1)^a l_a \right)
\]

\[
= \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta \nu} \left( \prod_{a=1}^{\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{\nu} (-1)^a l_a \right)
\]

\[
\times \left( L \right) \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta \nu} \left( \prod_{a=1}^{\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{\nu} (-1)^a l_a \right)
\]

\[
= \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta \nu} \left( \prod_{a=1}^{\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{\nu} (-1)^a l_a \right)
\]

\[
+ \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta \nu} \left( \prod_{a=1}^{\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{\nu} (-1)^a l_a \right)
\]

\[
+ \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta \nu} \left( \prod_{a=1}^{\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{\nu} (-1)^a l_a \right)
\]

\[
\times \left( L \right) \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta \nu} \left( \prod_{a=1}^{\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{\nu} (-1)^a l_a \right)
\]

\[
\times \left( L \right) \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta \nu} \left( \prod_{a=1}^{\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{\nu} (-1)^a l_a \right)
\]

\[
\times \left( L \right) \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta \nu} \left( \prod_{a=1}^{\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{\nu} (-1)^a l_a \right)
\]

\[
\times \left( L \right) \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta \nu} \left( \prod_{a=1}^{\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{\nu} (-1)^a l_a \right)
\]

\[
\times \left( L \right) \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta \nu} \left( \prod_{a=1}^{\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{\nu} (-1)^a l_a \right)
\]

\[
\times \left( L \right) \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta \nu} \left( \prod_{a=1}^{\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{\nu} (-1)^a l_a \right)
\]

\[
\times \left( L \right) \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta \nu} \left( \prod_{a=1}^{\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{\nu} (-1)^a l_a \right)
\]

\[
\times \left( L \right) \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta \nu} \left( \prod_{a=1}^{\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{\nu} (-1)^a l_a \right)
\]

\[
\times \left( L \right) \sum_{\nu=1}^{\infty} \sum_{l_1=1}^{\infty} \ldots \sum_{l_N=1}^{\infty} e^{-\beta \Delta \nu} \left( \prod_{a=1}^{\nu} \frac{1}{l_a^{\beta c}} \right) \left( L = \sum_{a=1}^{\nu} l_a \right) \times I \left( M = -\sum_{a=1}^{\nu} (-1)^a l_a \right)
\]
Mixed order transition and condensation in an exactly soluble 1D spin model

\[
= \sum_{l_1=1}^{\infty} \frac{e^{-\beta \Delta}}{l_1^{\beta C}} \sum_{\nu=1}^{\infty} \sum_{l_2=1}^{\infty} \cdots \sum_{l_N=1}^{\infty} e^{-\beta \Delta (2\nu - 2)} \left( \prod_{a=2}^{2\nu-1} \frac{1}{l_a^{\beta C}} \right) \\
\times I \left( L = l_1 + \sum_{a=2}^{2\nu-1} l_a \right) \left( M = l_1 - \sum_{a=2}^{2\nu-1} (-1)^a l_a \right) \\
= \sum_{l_1=1}^{\infty} e^{-\beta \Delta} \frac{Z_{M}^{(-)} (L - l_1, M - l_1; \beta)}{l_1^{\beta C}},
\]

where \( Z_{M}^{(-)} \) is the partition function corresponding to boundary conditions \( s_1 = s_L = -1 \) (so that the number of domains is odd). A similar analysis for \( Z_{M}^{(+)} \) reads

\[
Z_{M}^{(+)} (L, M; \beta) = \sum_{l_1=1}^{\infty} \sum_{\nu=1}^{\infty} \cdots \sum_{l_N=1}^{\infty} e^{-\beta \Delta (2\nu - 2)} \left( \prod_{a=2}^{2\nu-3} \frac{1}{l_a^{\beta C}} \right) \\
\times I \left( L = l_1 + \sum_{a=2}^{2\nu-2} l_a \right) \left( M = -l_1 + \sum_{a=2}^{2\nu-2} (-1)^a l_a \right) \\
= \sum_{l_1=1}^{\infty} e^{-\beta \Delta} \frac{Z_{M}^{(+)} (L - l_1, M + l_1; \beta)}{l_1^{\beta C}}.
\]

Hence, we deduce the following coupled recursion relations

\[
Z_{M}^{(+)} (L, M; \beta) = \sum_{l_1=1}^{L+M} \frac{e^{-\beta \Delta}}{l_1^{\beta C}} \frac{Z_{M}^{(-)} (L - l_1, M - l_1; \beta)}, \tag{D.1}
\]

\[
Z_{M}^{(-)} (L, M; \beta) = \sum_{l_1=1}^{L+M} \frac{e^{-\beta \Delta}}{l_1^{\beta C}} \frac{Z_{M}^{(+)} (L - l_1, M + l_1; \beta)}. \tag{D.2}
\]

The total chain size \( L \) reduces in each step of applying these relations, and hence convergence is guaranteed. The base of the recursion is

\[
\forall L : \quad Z_{M}^{(-)} (L, \pm (L - 2); \beta) = \frac{e^{-\beta \Delta}}{(L - 1)^{\beta C}}, \tag{D.3}
\]

\[
\forall L : \quad Z_{M}^{(-)} (L, -L; \beta) = \frac{1}{L^{\beta C}}. \tag{D.4}
\]
Mixed order transition and condensation in an exactly soluble 1D spin model

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