Perturbation theory trispectrum in the time renormalization approach

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ABSTRACT

An accurate theoretical description of structure formation at least in the mildly non-linear regime is essential for comparison with data from next-generation galaxy surveys. In a recent approach, one follows the time evolution of correlators directly and finds a hierarchy of evolution equations with increasing order. So far, in this so-called time renormalization group method, the trispectrum was neglected in order to obtain a closed set of equations. In this work, we study the influence of the trispectrum on the evolution of the power spectrum. In order to keep the numerical cost at a manageable level, we use the tree-level trispectrum from Eulerian perturbation theory. In comparison to numerical simulations, we find improvement in the mildly non-linear regime up to $k \simeq 0.25 h \text{ Mpc}^{-1}$. Beyond $k \simeq 0.25 h \text{ Mpc}^{-1}$ the perturbative description of the trispectrum fails and the method performs worse than without the trispectrum included. Our results reinforce the conceptual advantage of the time renormalization group method with respect to the perturbation theory.

Key words: cosmology: large-scale structure of Universe – methods: analytical.

1 INTRODUCTION

In the contemporary picture of our Universe, structures evolve from nearly Gaussian-distributed small perturbations in the homogeneous density field. Sound waves formed in the coupled photon–baryon fluid before recombination left oscillatory features in the matter power spectrum, the so-called baryonic acoustic oscillations (BAO). Detections of this effect have become a valuable tool to constrain cosmological parameters since amplitude and position of the oscillations depend on the expansion history of the Universe (Eisenstein et al. 2005; Hütsi 2006; Blake et al. 2007; Eisenstein, Seo & White 2007; Padmanabhan et al. 2007). Current and upcoming galaxy surveys – such as Baryon Oscillation Spectroscopic Survey (BOSS),1 WFIRST,2 HETDEX (Hill et al. 2008) and WFMOS (Glazebrook et al. 2005) – will measure the power spectrum of the matter distribution to 1 per cent accuracy in the region of the baryonic oscillations $0.05 < k < 0.25 h \text{ Mpc}^{-1}$ (Eisenstein, Hu & Tegmark 1998; Seo & Eisenstein 2003).

Since mode-coupling effects can significantly influence the position of the first peak at low redshifts (Crocce & Scoccimarro 2008), it is necessary to find a robust theoretical description of structure formation in the linear and mildly non-linear regime. While standard perturbation theory (Bernardeau et al. 2002) is a powerful tool for comparison with observations from galaxy surveys on large scales (Jeong & Komatsu 2006, 2009), it breaks down at the scales of BAO (Jain & Bertschinger 1994).

The increase of computational power and efficiency of algorithms made N-body simulations the most established approach for structure formation (Springel 2005; Huff et al. 2007; Evrard et al. 2008; Heitmann et al. 2008, 2010; Takahashi et al. 2008). However, to extract statistical information from numerical simulations, either large sets of initial conditions or large volumes are needed and it is difficult to control measurement uncertainties in the mildly non-linear regime (Angulo et al. 2008). In a recent work, a simple physically motivated picture was used to reduce the sample variance to speed up the scanning for cosmological parameters with N-body simulations (Tassev & Zaldarriaga 2011), but it remains numerically expensive and we favour a fast semi-analytical tool to predict non-linear structure formation.

While the halo model approach (Peacock & Dodds 1996; Smith et al. 2003) was found to be incapable of reaching the required accuracy (Huff et al. 2007), different attempts to include corrections of specific types to all orders at the same time have been presented over the last few years. Field theoretical techniques motivated by the renormalization group (Matarrese & Pietroni 2007, 2008; Anselmi, Matarrese & Pietroni 2011) and resummation methods (Crocce & Scoccimarro 2006a,b) improved the results for power spectra in the mildly non-linear

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1 http://www.sdss3.org/surveys/boss.php
2 http://wfirst.gsfc.nasa.gov

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regime significantly down to \( z = 0 \) in comparison to \( N \)-body simulations. In the following, also higher order statistics were studied in these frameworks (Bernardeau, Crocce & Scoccimarro 2008; Valageas 2008; Guo & Jing 2009). However, these approaches are formulated for an Einstein–de Sitter cosmology and are later generalized to other cosmologies by substituting the respective growth function. The accuracy of this approximation is difficult to quantify at higher orders (Bernardeau et al. 2002).

Correlating the structure-formation equations with fields to different orders leads to an infinite hierarchy of evolution equations for correlators with increasing order. This hierarchy – similar to the well-known Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy (Peebles 1980) – was truncated at the level of the trispectrum to obtain a closed set of equations (Pietroni 2008). This so-called time renormalization group method (TRG) can easily be generalized to a large set of different cosmologies including models with scale-dependent growth functions. For example, this is important in the case of massive neutrinos (Lesgourgues & Pastor 2006) which was also studied within this framework (Lesgourgues et al. 2009). However, recent studies showed that results strongly depend on initial conditions for the bispectrum (Audren & Lesgourgues 2011) and show only little improvement with respect to the one-loop results from perturbation theory.

The objective of this work is to study the effect of the trispectrum on the TRG approach. Including the entire time evolution of the trispectrum would lead to an immoderate numerical effort. We therefore include the tree-level trispectrum from perturbation theory into the evolution equation of the bispectrum and study its effects on the power spectrum in the BAO regime. We work in a standard \( \Lambda \) cold dark matter (\( \Lambda \)CDM) cosmology close to the best-fitting cosmology (\( \Omega_m = 0.25, \Omega_b h^2 = 0.0224, h = 0.72, n = 0.97 \) and \( \sigma_8 = 0.8 \) and compare to power spectra obtained from \( N \)-body simulations of the same cosmology (Carlson, White & Padmanabhan 2009).

The paper is organized as follows. In Section 2, we review the time renormalization approach introduced by Pietroni (2008). How the perturbation theory trispectrum can be included into this method will be the subject of Section 3. In Section 4, we include the trispectrum into the formal analytic solution of the system and discuss the additional corrections in a diagrammatic representation and in Section 5, our numerical results and their comparisons to \( N \)-body simulations are presented. The results are summarized and discussed in Section 6.

## 2 TIME RENORMALIZATION

In this section, we write the structure-formation equations in a compact matrix form and review the time renormalization approach (Pietroni 2008) as a starting point for our further calculations. We will concentrate on spatially flat cosmologies with a dark matter component and a non-clustering dark energy fluid or \( \Lambda \)CDM. However, this method can be easily extended to more exotic cosmologies including cosmologies with a scale-dependent growth function \( D_*(k, a) \) (Pietroni 2008).

### 2.1 Non-linear structure formation

The time evolution of the dark matter density contrast \( \delta(x, \tau) \), the peculiar velocity \( v(x, \tau) \) and the fluctuation of the gravitational potential \( \Phi(x, \tau) \) is governed by the continuity, Euler and Poisson equations:

\[
\partial_t \delta(x) + \nabla \cdot \{ (1 + \delta(x)) v(x) \} = 0
\]

\[
\partial_t v(x) + \mathcal{H} v(x) + \left( \nabla \cdot v(x) \right) = -\nabla \Phi(x)
\]

\[
\nabla^2 \Phi(x) = \frac{3}{2} \mathcal{H}^2 \Omega_m \delta(x).
\]

Here, the Hubble function \( \mathcal{H} \) is the logarithmic derivative of the scale factor with respect to conformal time: \( \mathcal{H} = \frac{\dd \ln a}{dr} \). For notational clarity, the time dependences will be omitted for most of the quantities throughout the whole work. In the case of a one-component dark matter fluid we have \( \Omega_m = 1 \), while in the presence of an additional non-interacting dark energy fluid with a constant equation of state we only have to replace \( \Omega_m \) by

\[
\Omega_m = \left( 1 + \frac{\rho_{de}}{\rho_m} a^{-3w} \right)^{-1}.
\]

The quantities \( \rho_{de}^0 \) and \( \rho_m^0 \) represent the background densities of the dark energy and dark matter fluid components, respectively. \( w \) denotes the equation of state parameter of the dark energy fluid, \( p_{de} = w \rho_{de}^0 \). Models with a cosmological constant, as for example in \( \Lambda \)CDM, can then simply be described by \( w = -1 \). We assume an irrotational peculiar velocity field, \( \nabla \times v(x) = 0 \). Due to Helmholtz’s theorem, the velocity field can then be described by its divergence only, \( \theta(x) = \nabla \cdot v(x) \). For the two remaining scalar fields \( \theta(x) \) and \( \delta(x) \), the evolution in Fourier space reads (Bernardeau et al. 2002)

\[
\partial_t \delta(k) + \theta(k) = -\delta_0(k - q - p) \alpha(q, p) \theta(q) \delta(p)
\]

\[
\partial_t \theta(k) + \mathcal{H} \theta(k) + \frac{3}{2} \mathcal{H}^2 \Omega_m \delta(k) = -\delta_0(k - q - p) \beta(q, p) \theta(q) \theta(p).
\]

Here and in the following, repeated momentum arguments in products (for instance \( p, q \) in equation 5) imply an integration over the respective momenta \( \int d^3p \int d^3q \ldots \). The left-hand side of equation (5) represents the linear evolution of the fields while the right-hand side expresses...
the non-linear mode-coupling, which is determined by the two model-independent mode-coupling functions
\[
\alpha(q, p) = \frac{(p + q) \cdot q}{q^2},
\]
\[
\beta(q, p) = \frac{(p + q)^2 \cdot p \cdot q}{2q^2p^2}.
\] (6)

To write the equations in a more compact form, we introduce the logarithmic time variable \(\eta\), which explicitly contains the linear structure growth with respect to the scale factor:
\[
\eta = \ln \left( \frac{D_s(a)}{D_s(a_{in})} \right).
\] (7)

The scale factor \(a_{in}\) may be chosen at a time when the system could still be well approximated to be Gaussian and in the linear regime. Now a doublet field can be introduced for the density contrast \(\delta(k)\) and the divergence of the velocity dispersion \(\theta(k)\):
\[
\begin{pmatrix}
\varphi_1(k) \\
\varphi_2(k)
\end{pmatrix} = e^{-\eta} \begin{pmatrix}
\delta(k) \\
-\theta(k)/(H f_s)
\end{pmatrix}.
\] (8)

Here, \(f_s\) is the logarithmic derivative of the growth function \(D_s\) with respect to the scale factor \(a\), \(f_s = d\ln D_s/d\ln a\). The factor \(e^{-\eta}\) now compensates for the linear evolution of the fields \(\varphi_i(k)\). In other words, solving the linearized structure-formation equations for these fields would lead to no time dependence in the doublet field at all. Therefore, any evolution away from the initial field doublet is explicitly due to non-linear effects. The set of equations (5) can now be expressed in a very compact form:
\[
\partial_\eta \varphi_{a1}(k) = \Omega_{ab} \varphi_b(k) + e^\eta \tilde{\gamma}_{abc}(k, -q, -p) \varphi_b(q) \varphi_c(p).
\] (9)

Here, the linear evolution is governed by the matrix
\[
\Omega_{ab} = \begin{pmatrix}
1 & -1 \\
-3 \Omega_m/(2 f_s^2) & 3 \Omega_m/(2 f_s^2)
\end{pmatrix},
\] (10)

while the non-linear mode-coupling is moderated by the vertex functions \(\tilde{\gamma}_{abc}(k, q, p)\). The only non-vanishing vertex functions are
\[
\begin{align*}
\tilde{\gamma}_{121}(k, q, p) &= \delta_{12}(k + q + p) \frac{\alpha(q, p)}{2}, \\
\tilde{\gamma}_{222}(k, q, p) &= \delta_{12}(k + q + p) \beta(q, p).
\end{align*}
\] (11)

Since the vertex functions only appear in integrals, it is notationally convenient to introduce vertex functions for which the \(\delta_{12}\)-function is already integrated out:
\[
\gamma_{abc}^{k, q, k + q} = \int d^3p \tilde{\gamma}_{abc}(k, q, p).
\] (12)

These quantities turn out to be dependent only on the absolute values of the wave vectors \(k, q\) and \(|k + q|\).

### 2.2 Hierarchy of correlators

While in the standard perturbation theory one aims to solve the evolution equations of the fields themselves, in the time renormalization approach one formulates evolution equations directly for the final quantities of interest – the correlators of the fields. In order to do this, one can use equation (9) to write down a hierarchy of evolution equations for correlators of any order:
\[
\begin{align*}
\partial_\eta \langle \varphi_{a_1} \varphi_a \rangle &= -\Omega_{a_1a} \langle \varphi_{a_1} \varphi_a \rangle - \Omega_{aa_1} \langle \varphi_a \varphi_{a_1} \rangle + e^\eta [\tilde{\gamma}_{a_1bc} \langle \varphi_{a_1} \varphi_b \varphi_c \rangle + \tilde{\gamma}_{abc} \langle \varphi_a \varphi_b \varphi_c \rangle] \\
\partial_\eta \langle \varphi_{a_1} \varphi_{a_2} \varphi_a \rangle &= -\Omega_{a_1a} \langle \varphi_{a_1} \varphi_{a_2} \varphi_a \rangle + \text{c.p.}[a, b, c] + e^\eta [\tilde{\gamma}_{a_1bc} \langle \varphi_{a_1} \varphi_b \varphi_c \rangle + \text{c.p.}[a, b, c]] \\
\partial_\eta \langle \varphi_{a_1} \varphi_{a_2} \varphi_{a_3} \rangle &= \ldots.
\end{align*}
\] (13)

Here and in the following, we abbreviate the doublet field index \(a_1\) and the wave vector \(k_1\) to a single number index, i.e. \(\varphi_1 \equiv \varphi_{a_1}(k_1)\). As a natural property of this hierarchy, for the evolution of a correlator of order \(n\) the knowledge of correlators of the next higher order \(n + 1\) is needed. Therefore, one is obliged to truncate this hierarchy at a certain point in order to obtain a closed set of equations. Splitting up the four-point correlator into its connected and unconnected part yields by Wick’s theorem
\[
\begin{align*}
\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle &= \langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle_c + \langle \varphi_1 \varphi_2 \rangle \langle \varphi_3 \varphi_4 \rangle + \langle \varphi_1 \varphi_3 \rangle \langle \varphi_2 \varphi_4 \rangle + \langle \varphi_1 \varphi_4 \rangle \langle \varphi_2 \varphi_3 \rangle.
\end{align*}
\] (14)

For instance, one can close the system by neglecting the connected part of the four-point correlator, as was done by Pietroni (2008). Due to this approximation, one is left with the first two equations of the hierarchy and the simplified system is then fully described by its power spectra \(P_{ab}^{k_1}\) and its bispectra \(B_{abc}^{k_1, k_2, k_3}\):
\[
\begin{align*}
\langle \varphi_1 \varphi_2 \rangle &= \delta_{12}(k_1 + k_2) P_{ab}^{k_1}, \\
\langle \varphi_1 \varphi_3 \rangle &= \delta_{12}(k_1 + k_2 + k_3) B_{abc}^{k_1, k_2, k_3}.
\end{align*}
\] (15)
Due to isotropy, the bispectrum will only depend on the absolute values of the wave vectors. Integrating equation (13) over one wave vector and using equation (14) with \( \langle \varphi_1 \varphi_2 \varphi_3 \rangle_c = 0 \), one finds a closed system of equations in which the four-point function is represented in terms of power spectra \( P_{ab}^{k}: \)

\[
\begin{align*}
\partial_\eta P_{ab}^k &= -\Omega_{ab} P_{bc}^k - \Omega_{bc} P_{ab}^k + e^0 \int d^3 q \left[ \gamma_{acd}^k B_{bcf}^q + (a \leftrightarrow b) \right] \\
\partial_\eta B_{abc}^{k q} &= -\Omega_{abc} B_{bde}^{k q} - \Omega_{bde} B_{abc}^{k q} - \Omega_{abc} B_{bde}^{k q} + 2 e^0 \left[ \gamma_{ade}^k P_{db}^p + \gamma_{ade}^q P_{cb}^p + \gamma_{bde}^q P_{ac}^p + \gamma_{bde}^k P_{ac}^p \right].
\end{align*}
\]

(16)

Here and in the following, \( p \) will denote the vector \( -k + q \). How this system can be solved numerically will be the subject of the next subsection.

### 2.3 Solving the closed system

Formal and numerical solutions to the closed system in equation (16) have already been presented (Pietroni 2008). We follow the same numerical path to investigate the solution’s sensitivity to the perturbation theory trispectrum. Since we are mainly interested in the evolution of the power spectrum itself, one circumvents the necessity of tracking the total bispectrum by introducing auxiliary integrals

\[
I_{acd,bef}^{k} = \frac{k}{4\pi} \int d^3 q \left[ \frac{1}{2} \left[ \gamma_{acd}^{k q} B_{bcf}^{q} + (q \leftrightarrow p) \right] \right].
\]

(17)

The introduction of these integrals encapsulates the one-loop character of the power spectrum evolution equation (first equation in 16), which simplifies to the tree-level equation

\[
\partial_\eta P_{ab}^k = -\Omega_{ab} P_{bc}^k - \Omega_{bc} P_{ab}^k + e^0 \frac{1}{k} \left[ I_{acd,bed}^{k} + I_{bcd,aca}^{k} \right].
\]

(18)

Differentiating the integrals \( I_{acd,bef}^{k} \) with respect to time \( \eta \) and using the evolution equation of the bispectrum from equation (16) gives the following time evolution of these integrals:

\[
\partial_\eta I_{acd,bef}^{k} = -\Omega_{abc} I_{bde,acf}^{k} - \Omega_{bde} I_{acd,bef}^{k} - \Omega_{bde} I_{acd,bef}^{k} + 2 e^0 A_{acd,bef}^{k}.
\]

(19)

The \( k \)-space loop integral which was originally in the evolution of the power spectrum now appears in the mode-coupling integrals \( A_{acd,bef}^{k} \), which initially drive the system away from Gaussianity:

\[
A_{acd,bef}^{k} = \frac{k}{4\pi} \int d^3 q \left[ \frac{1}{2} \left[ \gamma_{acd}^{k q} \left( \gamma_{bde}^{k q} P_{ac}^{q} P_{bc}^{p} + \gamma_{bde}^{q k} P_{ac}^{q} P_{bc}^{p} + \gamma_{bde}^{q k} P_{ac}^{q} P_{bc}^{p} \right) + (q \leftrightarrow p) \right] \right].
\]

(20)

The calculation of the integrals \( A_{acd,bef}^{k} \) is so far the only time-consuming task in numerically solving the system.

### 2.4 Initial conditions and symmetries

We propagate the system of equations (18)–(19) forward in time starting from an initial time at which the dynamics of the fields could still be well approximated by the linearized evolution equations. As initial redshift, we choose \( \eta_{in} = 100 \) and start with linear growing mode initial conditions

\[
\begin{align*}
\varphi_{uL}(k, \eta = 0) &= u_a \varphi(k), \quad (u_1, u_2) = (1, 1) \\
\delta_3(k + k') P_{ab}^{k}(\eta = 0) &= \delta_3(k + k') u_a u_b P_{ab}^{k}(\eta = 0) = u_a u_b \langle \varphi(k) \varphi(k') \rangle.
\end{align*}
\]

(21)

Furthermore, we will assume Gaussian initial conditions. This implies a vanishing initial bispectrum \( B_{abc}^{k q p} \) and therefore also vanishing initial integrals \( I_{acd,bef}^{k} \):

\[
I_{acd,bef}^{k}(\eta = 0) = 0, \quad B_{abc}^{k q p}(\eta = 0) = 0.
\]

(22)

Following the next arguments, the full system of 64 integrals \( I_{acd,bef}^{k} \) and 3 power spectra \( P_{ab}^{k} \) can be reduced to 14 independent components. Since the only non-vanishing vertex contributions appear for the index triples \((acd) \in \{(112), (121), (222)\} \) only the integrand \( I_{acd,bef}^{k} \) with these triples will evolve away from zero, which can be seen from equations (19)–(20). The remaining 24 components can be further reduced by symmetry arguments. Using the following symmetries in the vertex functions and the bispectrum:

\[
\gamma_{acd}^{k q p} = \gamma_{cad}^{k q p}, \quad B_{bcf}^{k q p} = B_{bfc}^{k q p}.
\]

(23)

we can find the following symmetry for the integrals \( I_{acd,bef}^{k} \):

\[
I_{acd,bef}^{k} = I_{adc,bfe}^{k}.
\]

(24)

Due to this symmetry, only 14 independent integrals remain to be followed. The independent integrals are identified by the direct product of \((acd) = (112) \) and \((bef) = (b11), (b12), (b21) \) and \((b22) (b = 1, 2) \), and the direct product of \((acd) = (222) \) and \((bef) = (b11), (b12) \) and \((b22) (b = 1, 2) \). Including the three independent power spectra \( P_{11}^{k}, P_{12}^{k}, P_{22}^{k} \) implies a system of 17 components in total.
3 TRISPECTRUM

The connected part of the four-point correlator is called the trispectrum. To investigate the method’s sensitivity with respect to the trispectrum, we include the perturbation theory trispectrum to third order in the linear power spectrum \( P(k) \). Including the full time evolution of the non-perturbative trispectrum would increase the numerical effort disproportionately. This would be given by the third equation in the hierarchy of time evolution equations for correlators in equation (13). However, the time evolution of the tree-level perturbation theory trispectrum is exclusively given in terms of the growth function \( D_1(a)/D_0(a_0) = e^{\eta} \). Therefore, using this approximation for the bispectrum it is sufficient to compute trispectrum corrections at the initial time \( a_0 \). These corrections can then be included into the routine without increasing the computational cost significantly.

3.1 Perturbation theory

In the following, we adapt the standard perturbation theory (Bernardeau et al. 2002) to the compact matrix formulation of structure formation. We expand the fields \( \phi_a(k, \eta) \) in \( n \)-th-order perturbative contributions \( \psi^{(n)}(k, \eta) \), which can be written in terms of the perturbation theory kernels \( F_a(n)(k_1, \ldots, k_n) \) and the initial linear fields \( \phi_a^{(1)}(k) = \phi_a^{(1)}(k, \eta = 0) \):

\[
\phi_a^{(n)}(k) = \sum_{n=1}^{\infty} \epsilon(n-1) n \phi_a^{(n)}(k)
\]

\[
\psi^{(n)}(k) = \int d^3q_1 \cdot \int d^3q_n \delta(k - q_1, \ldots, q_n) F_a(n)(q_1, \ldots, q_n) \psi_a^{(1)}(q_1) \cdot \psi_a^{(1)}(q_n)
\]

with \( q_1, \ldots, q_n \). For this formulation, one has to drop the time dependence of \( \Omega_m \) from equation (10) to obtain separability of the structure-formation equations. However, the required relation \( f_1 = \Omega_m^{1/2} \) represents a good approximation in cosmologies close to \( \Lambda \) CDM (Bernardeau et al. 2002; Pietroni 2008). For notational simplicity we combined the two standard kernels of each order into a vector, since we will also need trispectrum correlations to the velocity field components:

\[
F_a(n)(k_1, \ldots, k_n) = \left( F_a^{(n)}(k_1, \ldots, k_n) \right)
\]

By inserting equations (25–26) into the evolution equations (9), one finds the following recursion relations by combinatorics (Goroff et al. 1986; Jain & Bertschinger 1994):

\[
F_a^{(n)}(k_1, \ldots, k_n) = \sum_{m=1}^{n-1} \frac{G_a^{(m)}(k_1, \ldots, k_m)}{(2n + 1)(1-n)} \left[ (2n + 1) \alpha(q_1, q_2) F_a^{(n-m)}(k_{m+1}, \ldots, k_n) + 2 \beta(q_1, q_2) G_a^{(n-m)}(k_{m+1}, \ldots, k_n) \right]
\]

\[
G_a^{(n)}(k_1, \ldots, k_n) = \sum_{m=1}^{n-1} \frac{G_a^{(m)}(k_1, \ldots, k_m)}{(2n + 1)(1-n)} \left[ 3 \alpha(q_1, q_2) F_a^{(n-m)}(k_{m+1}, \ldots, k_n) + 2 n G_a^{(n-m)}(k_{m+1}, \ldots, k_n) \right],
\]

where \( q_1 = k_1 + \cdots + k_n, q_2 = k_{m+1} + \cdots + k_n \) and \( F^{(1)} = G^{(1)} = 1 \). The explicit symmetrized expressions for the second-order perturbation theory kernels take a very simple and intuitive form:

\[
F^{(2)}(k_1, k_2) = \frac{5}{7} \left( \frac{k_1 \cdot k_2}{k_1 \cdot k_1} \right) + \frac{2}{7} \left( \frac{k_1 \cdot k_1}{k_1 \cdot k_1} \right)
\]

\[
G^{(2)}(k_1, k_2) = \frac{3}{7} \left( \frac{k_1 \cdot k_2}{k_1 \cdot k_1} \right) + \frac{4}{7} \left( \frac{k_1 \cdot k_1}{k_1 \cdot k_1} \right)
\]

One can see that mode-coupling to second order reaches its maximum when the contributing modes \( k_1 \) and \( k_2 \) are aligned, whereas the kernel vanishes for anti-parallel modes. When in equation (26) \( n \) different modes \( q_1, \ldots, q_n \) contribute to a mode \( k \), momentum conservation holds, enforced by the \( \delta_0 \)-distribution: \( k = q_1 + \cdots + q_n \). Similar to the renormalization vertex in equation (11) this also motivates a diagrammatic description for the mode coupling processes (see Fig. 1).

3.2 The four-point correlator in the perturbation theory

For an analytic expression of the perturbation theory four-point correlator, one has to expand the fields in the correlator. Due to the Gaussianity of the initial fields \( \psi^{(1)} \), the correlators with an even number of fields \( \psi^{(1)} \) will later simplify to products of initial power spectra \( P(k) \) while all uneven contributions vanish:

\[
\langle \psi_1 \psi_2 \psi_3 \psi_4 \rangle = \left( \psi_1^{(1)} + e^{\alpha} \psi_2^{(2)} + \cdots \right) \left( \psi_3^{(1)} + e^{\alpha} \psi_4^{(2)} + \cdots \right) \left( \psi_2^{(1)} + e^{\alpha} \psi_3^{(2)} + \cdots \right) \left( \psi_4^{(1)} + e^{\alpha} \psi_1^{(2)} + \cdots \right).
\]

Simple truncation of the expansion in equation (25) would lead to an inconsistent inclusion of powers of the linear power spectrum \( P(k) \). We take into account all terms up to third order in the linear power spectrum which is equivalent to including terms with initial fields up to sixth
order. Since the disconnected part will be represented by the full power spectra in equation (14), we are interested in the connected part of the correlator only. The connected part can be split into two contributions

\[ \langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle_c = \left\langle \left( \varphi_1^{(2)} \varphi_2^{(2)} \right) \left( \varphi_3^{(1)} \varphi_4^{(1)} \right) \right\rangle_c + \text{all pairs} \in \{1, 2, 3, 4\} + \left\langle \varphi_1^{(3)} \varphi_2^{(1)} \varphi_3^{(1)} \varphi_4^{(1)} \right\rangle_c + \text{c.p.} \{1, 2, 3, 4\}. \]  

(33)

The first contribution originates from second-order perturbation theory. In this case, two of the fields in the correlator have been expanded to second order. The expressions in terms of the initial power spectra and the second-order kernels are of the type

\[ r_{12}^{(2)}(k_1, k_2, (k_3, k_4)) = 4 e^{-\eta} p_{L}^{k_1} p_{L}^{k_2} \left( F^{(2)}_{ab}(k_{13}, -k_3) F^{(2)}_{ab}(k_{24}, -k_4) P_{L}^{k_1} + F^{(2)}_{ab}(k_{14}, -k_4) F^{(2)}_{ab}(k_{23}, -k_3) P_{L}^{k_1} \right). \]  

(34)

The second contribution is due to third-order perturbation theory. Here, one field is expanded to third order while the other three remain at linear order. For this reason, only one perturbation kernel appears in the expression for this type of contributions:

\[ r_{13}^{(3)}(k_1, k_2, (k_3, k_4)) = 6 e^{-\eta} F^{(3)}_{ab}(k_1, k_2, k_3) p_{L}^{k_1} p_{L}^{k_2} p_{L}^{k_3}. \]  

(35)

With these two functions the connected perturbation theory four-point correlator up to third order in the linear power spectrum \( P_{L}^{k} \) can be expressed by the following two tree-level contributions:

\[ \langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle_c = \delta_{0}(k_{14}) r_{12}^{(2)}((k_1, k_2), (k_3, k_4)) + \text{all pairs} \in \{1, 2, 3, 4\} + \delta_{0}(k_{14}) r_{13}^{(3)}(k_1, k_2, k_3, k_4) + \text{c.p.} \{1, 2, 3, 4\}. \]  

(36)

### 3.3 Trispectrum in TRG

Our main objective is to investigate the influence of the perturbation theory trispectrum on the evolution of the power spectrum. Writing the connected four-point correlator in terms of the trispectrum \( T_{abc}^{k_1,k_2,k_3} \),

\[ \langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle_c = \delta_{0}(k_{14}) T_{abc}^{k_1,k_2,k_3}, \]  

(37)

we can now include the corresponding corrections into our formalism. Taking the trispectrum in the hierarchy of equation (13) into account will change the closed system of equation (16) into

\[ \partial_{\eta} P_{ab}^{k} = -\Omega_{ab} p_{c}^{k} - \Omega_{bc} p_{a}^{k} + e^{\eta} \int d^{3}\varphi \left[ \gamma^{k,q,p} B_{bcd}^{k,q,p} \right] \]  

(38)

Since the trispectrum is taken from perturbation theory, the evolution of the next higher correlator is not needed for its description. One can stick to the same numerical solving procedure as was presented in Section 2.3. Modifications appear in the time evolution of the integrals \( I_{abcd}^{k} \) from equation (19) via changing the correction integrals \( A_{abcd}^{k} \) from equation (20):

\[ A_{abcd}^{k} \rightarrow A_{abcd}^{k} + \Delta A_{abcd}^{k}. \]  

(39)
\[ \Delta A_{\text{wdh}}^{k} = \frac{k}{8\pi} \int d^{3}q \int d^{3}r \left\{ \gamma_{\text{wdh}}^{k,q-p} \left[ \lambda_{r} T_{e,f}^{q,p,r} + \gamma_{e,g}^{q,r,p,r} T_{f,j}^{p,k,r} + \gamma_{f,g}^{r,q,p,r} T_{e,h}^{k,q,r} + (q \leftrightarrow p) \right] \right\}. \]  

While the former expression in equation (20) was a one-loop integral only, we now have to integrate twice over the full \( k \)-space. The reason for this is the additional \( \delta_{l} \)-functions in the disconnected parts of the four-point correlator in equation (14). The integration is performed numerically using Monte Carlo integration techniques from the multidimensional numerical integration library Cuba (Hahn 2005).

4 DIAGRAMMATIC DESCRIPTION

An analytic solution for the system in equation (16) can be formulated (Pietroni 2008). This is still the case for the system in equation (38) with additional trispectrum terms. Solving first the linearized evolution equations (9), one can write down the linear solutions for the fields \( \varphi_{\alpha \cdot, L}(k, \eta) \) with the help of the linear propagator \( g_{ab}(\eta, \eta') \) (Crocce & Scoccimarro 2006b; Matarrese & Pietroni 2007):

\[ \varphi_{\alpha \cdot, L}(k, \eta) = g_{ab}(\eta, \eta') \varphi_{b \cdot, L}(k, \eta'). \]  

Furthermore, the linear propagator has the following properties:

\[ \partial_{\eta} g_{ab}(\eta, \eta') = -\Omega_{ab}(\eta) g_{ab}(\eta, \eta') \]
\[ g_{ab}(\eta, \eta) = \delta_{ab} \]
\[ g_{ab}(\eta, \eta') g_{bc}(\eta', \eta'') = g_{ac}(\eta, \eta''). \]  

With the help of this linear propagator, a formal analytic solution can be given for the system (38):

\[ P_{ab}(\eta) = g_{ab}(\eta, 0) g_{cd}(\eta, 0) P_{cd}(\eta = 0) + \int_{0}^{\eta} \! d\eta' \int_{0}^{\eta} \! d\eta'' \int d^{3}q \! g_{ab}(\eta, \eta') g_{cd}(\eta, \eta'') \left[ \left( \beta_{k,q-p} T_{e,f}^{q,p,r} + \beta_{e,g}^{q,r,p,r} T_{f,j}^{p,k,r} + \beta_{f,g}^{r,q,p,r} T_{e,h}^{k,q,r} + (q \leftrightarrow p) \right) \right]. \]

For a better understanding of the trispectrum corrections to the power spectrum it is useful to analyse the equations in a diagrammatic representation. In the upper part of Fig. 1, symbols for the linear propagator \( g_{ab}(\eta_{s}, \eta_{h}) \), the renormalization vertex \( \gamma_{ab}^{k_{s},k_{h},k_{s}} \) and the linear power spectrum to different times \( P_{i}(\eta_{s}, \eta_{h}) \) are depicted. A diagrammatic representation for the perturbation theory kernels are also needed to describe the trispectrum. Since we used the perturbation theory trispectrum to third order in the power spectrum, the kernels of second and third order are sufficient.

For time-independent \( \Omega_{ab} \), i.e. \( f_{+} = \Omega_{m}^{1/2} \), one can invert the evolution matrix \( (\partial_{\eta} \delta_{ab} - \Omega_{ab}) \) of the structure formation equations (9) in Laplace space and finds the non-linear evolution as a time convolution of the fields \( \varphi \) and the linear propagator \( \tilde{g}_{ab} \) in the following form:

\[ \varphi_{a}(k, \eta) = \tilde{g}_{ab}(\eta, \eta') \varphi_{b}(k, \eta') + \int_{0}^{\eta} \! d\eta'' \tilde{g}_{ab}(\eta, \eta'') \left( \beta_{k,q-p}^{k_{s},k_{h},k_{s}} \varphi_{c}(q, \eta'') \varphi_{d}(p, \eta') \right). \]

Solving this equation iteratively, one can identify the perturbation theory kernels in terms of the vertex \( \gamma_{ab} \) and the linear propagator \( \tilde{g}_{ab} \), which is the linear propagator for time-independent \( \Omega_{ab} \),

\[ F_{a}^{(2)}(k_{1}, k - k_{1}) = e^{-\eta} \int_{0}^{\eta} \! d\eta'' \tilde{g}_{ab}(\eta, \eta'') \gamma_{b,c}^{k_{1},k_{h},k_{h}} u_{c} u_{d} \]
\[ F_{a}^{(3)}(k_{1}, k_{2}, k - k_{1} - k_{2}) = e^{-\eta} \int_{0}^{\eta} \! d\eta'' \int_{0}^{\eta''} \! d\eta''' \tilde{g}_{ab}(\eta, \eta'') \gamma_{b,c}^{k_{1},k_{h},k_{h}} \tilde{g}_{cd}(\eta''', \eta''') \gamma_{d,e}^{k_{1},k_{h},k_{h}} u_{e} u_{h} + (k_{1} \leftrightarrow k - k_{1}) \].

In the diagrammatic representation only, we omit the difference between the linear propagators and use \( g_{ab} = \tilde{g}_{ab} \). Then, the two trispectrum contributions from equations (34)–(35) can be expressed as in the last row of Fig. 1. The rules for the diagrams are as follows:

(i) Time flows monotonically from the linear power spectra towards the external legs.
(ii) All inner field indices are summed over.
(iii) All inner times at vertex intersections are integrated over in accordance with causality.
(iv) Momentum conservation holds at each vertex and all inner momenta are integrated over.

The perturbation theory trispectrum terms lead to corrections in the bispectrum in equation (44). The lowest-order corrections to the bispectrum are shown in Fig. 2. All the depicted terms are third order \( O(9) \) in the renormalization vertex since the trispectrum itself is second order in \( \gamma \) (Fig. 1) and equation (44) is first order in \( \gamma \). The diagrams (a)–(c) in Fig 2 originate from the second-order kernels \( F_{a}^{(2)} \) in equation (34), diagram (c) can be formed from both second-order kernels or third-order kernels \( F_{a}^{(3)} \) in equation (35), while diagrams (e)–(f) are only
due to the third-order perturbation theory in the trispectrum. Also here, one can see that all perturbative corrections, which we included, are of third order in the linear power spectrum. While the perturbation theory trispectrum is calculated at tree-level, both the evolution equations – for the bispectrum and for the power spectrum – introduce one momentum integration. Therefore, all the lowest-order corrections to the bispectrum are one-loop diagrams and the lowest-order corrections to the power spectrum are two-loop diagrams. The latter diagrams can easily be found by all possible contractions of the diagrams in Fig. 2 with a vertex $\gamma$, leading to the corresponding $\mathcal{O}(\gamma^4)$ corrections in the spectrum.

We want to emphasize at this point that the inclusion of the perturbation theory trispectrum does not lead to a simple perturbative correction only. At each time step, the perturbative trispectrum corrects the evolution of bispectrum and power spectrum. Therefore, from that moment on, these corrections will be involved in the non-perturbative method of time renormalization. In this work, we only discuss the trispectrum corrections to this method, since the quality and performance of the original time renormalization technique have been thoroughly discussed already (Pietroni 2008).

5 NUMERICAL RESULTS

We solved the system from equation (38) starting from redshift $z = 100$ well within the linear regime with linear initial power spectra (see equation 21) and vanishing bispectrum. We evolved the system to redshift $z = 1$ and $z = 0$ with and without trispectrum included and compared the power spectra with results from numerical simulations of the same $\Lambda$CDM cosmology (Carlson et al. 2009). Due to numerical complexity only power spectra up to $k = 1 \, h \, \text{Mpc}^{-1}$ were included in the trispectrum integrals $\Delta I_{acd,bef}^k$ from equation (40). However, in the integrals $I_{acd,bef}^k$ from equation (20), modes up to $k = 10 \, h \, \text{Mpc}^{-1}$ were taken into account, were the results saturate to per cent accuracy.

The results are shown in Fig. 3, in which also the linearly evolved power spectrum is depicted. All spectra were divided by a linear power spectrum $P_{L,0}$ without wiggles from BAO (Eisenstein & Hu 1998, equation 29). For $z = 1$, the results with the trispectrum included are in excellent agreement with numerical simulations up to $k \approx 0.17 \, h \, \text{Mpc}^{-1}$. For $0.17 \leq k \leq 0.25 \, h \, \text{Mpc}^{-1}$ the method performs still better than without trispectrum included. Beyond this regime, the perturbative description of the trispectrum breaks down and the results are in strong disagreement with simulations.

It is not surprising that below a certain scale the method performs better without the perturbative trispectrum included. For larger $k$ in the integrals $\Delta A_{acd,bef}^k$ in equation (40) also trispectra at smaller scales contribute to the evolution. Since in the perturbation theory loop corrections become more and more important at smaller scales, the tree-level trispectrum description breaks down above a specific wavenumber. For this reason, beyond $k \approx 0.25 \, h \, \text{Mpc}^{-1}$ time renormalization without trispectrum will lead to better results in comparison to numerical simulations.

At $z = 0$ trispectrum corrections overcompensate the too strong growth in the pure TRG approach on scales below $k \approx 0.15 \, h \, \text{Mpc}^{-1}$ and lead to too little growth. In this regime, our results agree with the numerical simulations within 2.5 per cent. The better agreement for $z = 1$ on these scales may simply be due to the breakdown of the tree-level perturbative description of the trispectrum at later times. Beyond $k \approx 0.15 \, h \, \text{Mpc}^{-1}$ the inclusion of the trispectrum leads to better agreement with simulations than pure TRG, while both methods show too strong growth further inside the non-linear regime.
Figure 3. Matter power spectra $P_k^L$ divided by a linear spectrum without baryonic wiggles $P_{L,\text{lin}}^k$ (Eisenstein & Hu 1998) in the mildly non-linear regime. The linear spectrum is depicted as a solid (black) line. A spectrum from $N$-body simulations (Carlson et al. 2009) serves as reference (yellow). The dashed (blue) line is TRG theory and the dashed (green) line is TRG theory with trispectrum included. Left-hand panel: $z = 1$, right-hand panel: $z = 0$.

The results for pure TRG may differ from the results of Pietroni (2008), since only 12 instead of 14 independent integrals $I_{abcd,def}^k$ were included in the original method. However, in later applications all 14 independent integrals were taken into account.

6 SUMMARY

In this work, the influence of the tree-level trispectrum on the time renormalization method (Pietroni 2008) has been studied. To keep the computational effort at a reasonable level we did not include the full trispectrum evolution from the hierarchy. Instead, we used tree-level perturbation theory for the trispectrum in the evolution of the bispectrum.

(1) The trispectrum was formulated in the standard perturbation theory to third order in the linear power spectrum $P_k^L$. This was included in the evolution equation of the bispectrum.

(2) The numerical method for solving the system was extended by the trispectrum corrections. Due to the linear time dependence of the perturbation theory trispectrum, it is sufficient to calculate the correction integrals at one fixed initial time. Once the corrections are derived the extended method operates at the same speed as the time renormalization method without trispectrum included. The diagrams for the lowest order corrections to the bispectrum were depicted in Fig. 2, from which all the lowest order terms for the power spectrum originating from the tree-level trispectrum can be induced. These two-loop corrections to the power spectrum are of order $O(y^4)$ in the renormalization vertex and third order in the linear power spectrum $O(P_k^L)$.

(3) Perturbative trispectrum corrections are fed into the systems at all times. Once a correction has been included, its evolution will be described by the non-perturbative formalism of time renormalization. Therefore, although the trispectrum was only taken into account perturbatively, its inclusion cannot be interpreted as a pure perturbative correction disentangled from renormalization.

(4) We solved the system numerically starting with Gaussian initial conditions and a linear power spectrum at an initial redshift of $z = 100$. In comparison to numerical simulations (Carlson et al. 2009), the inclusions of the trispectrum generally improves the results up to $k \approx 0.25 \, h \, \text{Mpc}^{-1}$. However, on larger scales the damping due to perturbative tree-level trispectrum overcompensates the deviation of pure TRG from $N$-body simulations. The results agree with the simulations within 1 per cent up to $k \approx 0.18 \, h \, \text{Mpc}^{-1}$ for $z = 1$ and within 2.5 per cent up to $k \approx 0.2 \, h \, \text{Mpc}^{-1}$ for $z = 0$.

(5) Beyond $k \approx 0.25 \, h \, \text{Mpc}^{-1}$ the perturbative description of the trispectrum breaks down and the method performs better without trispectrum included. This is due to the fact that loop corrections to the trispectrum are not included in our method and become more and more important on smaller scales. Adding the perturbation theory trispectrum therefore predominantly pays off on large scales and at the beginning of the mildly non-linear regime.

(6) Although the prediction of the amplitude and position of the first two peaks in the BAO was improved by the trispectrum, we are far from reaching per cent accuracy over the entire BAO regime. Also the speed of this method was reduced by including the trispectrum, since at a specific time the two-loop trispectrum corrections have to be derived. Finding a better analytical estimator for the trispectrum from other renormalization approaches, which includes higher order corrections, could improve the results of this method further inside the mildly non-linear regime. Besides this, studying the dependence of our results on the initial bispectrum would be an interesting objective for future work.
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