Subnormal and completely hyperexpansive completion problem for weighted shifts on directed trees

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Abstract

For a given directed tree and weights attached to a subtree, the completion problem is to determine if these weights may be completed in a way to obtain a bounded weighted shift on the whole tree, which further satisfies additional conditions. In this paper we consider subnormal and completely hyperexpansive completion problem for weighted shifts on directed trees with one branching point. We develop new results on backward extensions of truncated moment sequences and, exploiting these results, we obtain a characterization of existence of such a completion

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1. Introduction

Classical weighted shifts form an important class of operators, which is a rich source of examples in operator theory. In \textsuperscript{7} the author investigated the subnormality of weighted shifts and gave the construction of a minimal normal extension of a subnormal weighted shift. In \textsuperscript{7} there appeared another characterization of subnormal weighted shifts in terms of moment sequences (the characterization was mentioned also in \textsuperscript{8} as done formerly by Berger). In \textsuperscript{1} the author investigated complete hyperexpansivity of weighted shifts on the occasion of introducing such a class of operators; the obtained characterization of completely hyperexpansive weighted shifts involves Lévy-Khinchin representation of certain sequences.

In \textsuperscript{9} weighted shifts on directed trees have been introduced. It is a more general class of operators containing classical weighted shifts as well as their...
adjoints (cf. [9, Remark 3.4.2]). In the same paper there appeared criteria on subnormality and complete hyperexpansivity of weighted shifts on trees with one branching point (cf. [9, Theorems 6.2.1 and 7.2.1]). These criteria, as for classical weighted shifts, translate notions of subnormality and complete hyperexpansivity into certain moment sequences, but this time negative moments are also involved.

In [17] Stampfli initiated research on subnormal completions of weighted shifts. Thanks to his construction of minimal normal extension of a subnormal weighted shift, he obtained that for $0 < \lambda_1 < \lambda_2 < \lambda_3$ there exists a subnormal weighted shift with weights starting from $(\lambda_1, \lambda_2, \lambda_3)$. Moreover, he showed that such a condition does not suffice if we take four weights. Nevertheless, in his paper there appears a quite simple condition for a sequence $0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ to have a subnormal completion, namely

$$\lambda_3^2 \geq \lambda_3^2 + \frac{\lambda_3^2 (\lambda_3^2 - \lambda_2^2)^2}{\lambda_3^2 - \lambda_1^2}.$$ 

In [3] the authors recovered and generalized the results of [17] by studying truncated moment sequences and recursively generated completions. In [10] the completely hyperexpansive completion problem for weighted shifts was investigated – the authors used similar approach as in [3]. In [15] and [4] the authors studied subnormal completion problem for 2-variable weighted shifts. In [12] completions of $d$-variable weighted shifts were studied.

In [5] there was posed a subnormal completion problem for weighted shifts on directed trees with one branching point, and several general results were proved (concerning e.g. 1-generation completions and flat completions). In [6, Theorem 4.1] the solution to the 2-generation completion problem was obtained for directed trees with trunk of length 1.

The aim of the present paper is to give a full solution to $p$-generation subnormal and completely hyperexpansive completion problem for weighted shifts on directed trees with one branching point. The paper is organized as follows. In Section 3 we study truncated moment sequences on $(0, \infty)$ and $(0, 1]$ using methods from [13, Chapters III and IV]. The main result of this section is Theorem 3.7, which characterizes strict positivity in terms of index of a given sequence. In Section 4 we study backward extensions of truncated moment sequences on $(0, \infty)$ and $(0, 1]$. The key result is Theorem 4.4 which gives us the simple condition for strict positivity of backward extension. These all allow us to obtain crucial characterizations of truncated moment sequences in terms of extremal values of the integral of reciprocal function (see Theorems 4.7, 4.10). In Theorem 5.1 using the theory developed in Section 4 we give a solution (in the kind of [6, Theorem 4.1]) to the subnormal completion problem for directed trees with a finite trunk. The next part of Section 5 is devoted to study trees with infinite trunk – exploiting compactness of certain sets, we link these two cases. Theorem 6.1 is an analogue of Theorem 5.1 for the completely hyperexpansive completion problem.
2. Preliminaries

By \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) we denote the set of non-negative integers, integers, rational numbers, and real numbers, respectively. We also use the symbol \( \mathbb{N}^* \) for the set \( \mathbb{N} \cup \{\infty\} \) and \( \mathbb{R}^* \) for the set \( \mathbb{R} \cup \{-\infty, \infty\} \). For \( p \in \mathbb{R} \) and \( A \subset \mathbb{R} \) we denote \( A_p = \{ x \in A : x \geq p \} \). For \( x \in \mathbb{R} \) we denote \( \lceil x \rceil = \min\{ k \in \mathbb{Z} : x \leq k \} \).

For a subset \( A \subset \mathbb{R} \) we denote by \( \chi_A \) the characteristic function of \( A \). For a sequence \( (s_i)_{i \in I} \) of real numbers the notation \( (s_i)_{i \in I} \subset A \) means that all terms of the sequence \( (s_i)_{i \in I} \) are elements of the set \( A \subset \mathbb{R} \). If \( K \in \{\mathbb{R}, \mathbb{C}\} \) and \( n \in \mathbb{N} \), then \( K^n[x] \) stands for the space of polynomials of degree at most \( n \) with coefficients in \( K \). If \( X \) is a topological space, then \( \mathcal{B}(X) \) stands for the \( \sigma \)-algebra of Borel subsets of \( X \). If \( \mu \) is a positive regular measure on \( \mathcal{B}(X) \), then the closed support of \( \mu \) is denoted by \( \text{supp} \mu \).

A sequence \( (s_k)_{k=0}^{\infty} \subset [0, \infty) \) is called a Stjelties moment sequence, if there exists a measure \( \mu : \mathcal{B}([0, \infty)) \to [0, \infty) \) such that
\[
\int_{[0, \infty)} t^n \, d\mu(t) = s_k, \quad k \in \mathbb{N}.
\]  
(2.1)

Recall that a sequence \( (a_k)_{k=0}^{\infty} \subset \mathbb{R} \) is completely alternating if
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_{k+m} \leq 0, \quad m \in \mathbb{N}, \ n \in \mathbb{N}_1.
\]

Let \( H \) be a complex Hilbert space. By \( \mathcal{B}(H) \) we denote the \( C^* \)-algebra of linear and bounded operators on \( H \). An operator \( T \in \mathcal{B}(H) \) is called normal if \( T \) commutes with its adjoint. An operator \( T \in \mathcal{B}(H) \) is called subnormal if there exists a Hilbert space \( K \) containing \( H \) (in sense of isometric embeddings) and a normal operator \( S \in \mathcal{B}(K) \) such that \( S|_H = T \). We say that \( T \in \mathcal{B}(H) \) is completely hyperexpansive if
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} T^k T^* k \leq 0, \quad n \in \mathbb{N}_1.
\]  
(2.2)

In view of [1, Remark 2] we see that \( T \) is completely hyperexpansive if and only if for every \( f \in H \) the sequence \( (\|T^nf\|^2)_{n=0}^{\infty} \) is completely alternating.

Let us recall the definition of weighted shift on a directed tree (see [9]). For a directed tree \( T = (V, E) \) and a family \( \lambda = (\lambda_v)_{v \in V \setminus \{\text{root}\}} \subset \mathbb{C} \) satisfying the following condition:
\[
\sup_{v \in V} \sum_{(u,v) \in E} |\lambda_u|^2 < \infty,
\]  
(2.3)

we define an operator \( S_\lambda \in \mathcal{B}(l^2(V)) \), called weighted shift on \( T \) with weights \( \lambda \), by the formula
\[
S_\lambda e_v = \sum_{u \in V} \lambda_u e_u, \quad v \in V.
\]
Note that the classical weighted shifts are also weighted shifts in this more general setting: a suitable trees are simply \( \mathbb{Z} \) and \( \mathbb{N} \).

In this paper we are considering weighted shifts on directed trees with one branching point, that is, trees \( T_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa}) \), where \( \eta, \kappa \in \mathbb{R} \), \( \eta \geq 1 \), and

\[
\begin{align*}
V_{\eta,\kappa} &= \{-k: k \in \mathbb{N} \cap [0, \kappa]\} \cup \{(i, j): j \in \mathbb{N}_1, i \in \mathbb{N} \cap [1, \eta]\}, \\
E_{\kappa} &= \{(-k, -k + 1): k \in \mathbb{N} \cap [1, \kappa]\} \cup \{(0, (i, 1)): i \in \mathbb{N} \cap [1, \eta]\}, \\
E_{\eta,\kappa} &= E_{\kappa} \cup \{(i, j), (i, j + 1)): i \in \mathbb{N} \cap [1, \eta], j \in \mathbb{N}_1\}.
\end{align*}
\]

Given \( \eta, \kappa \in \mathbb{R} \), \( \eta \geq 2 \), and a weighted shift \( S_{\chi} \in \mathcal{B}(\ell^2(V_{\eta,\kappa})) \) we say that \( S_{\chi} \) is \( r \)-generation flat if \( \lambda_{i,j} = \lambda_{1,j} \) for \( j \in \mathbb{N}_r \) and \( i \in \mathbb{N} \cap [1, \eta] \).

If \( \eta, \kappa \in \mathbb{R} \), \( \eta \geq 2 \), \( p \in \mathbb{N}_1 \), and \( \lambda = \{\lambda_{-k}\}_{k=0}^{\kappa-1} \cup \{\lambda_i\}_{i=1}^{p} \subset \mathbb{C} \), we say that \( \lambda \) has a completion if there exists a weighted shift \( S_{\chi} \in \mathcal{B}(\ell^2(V_{\eta,\kappa})) \) such that

\[
\begin{align*}
\lambda_{-k} &= \lambda_{-1}, \\
\lambda'_{i,j} &= \lambda_{i,j},
\end{align*}
\]

for every such operator will be called a completion of \( \lambda \).

3. Moment problem on \((0, \infty)\) and \((0, 1]\)

In [3] the authors gave a solution to the truncated Stieltjes and Hausdorff moment problems. In this section we investigate the special cases of these problems, namely the moment problems on \((0, \infty)\) and \((0, 1]\). We use a different approach, which is similar to what Krein and Nudel’man presented in [12, Chapter III] for truncated moment problem on compact interval; actually, it turns out that the theory of moments on compact interval contains the theory of moments on \((0, \infty)\) and \((0, 1]\). This approach seems to be more suitable for studying backward extensions of truncated moment sequences in Section 4.

**Definition 3.1.** Let \( n \in \mathbb{N} \) and \( s = (s_0, \ldots, s_n) \subset \mathbb{R} \). Assume \( I \subset \mathbb{R} \) is an interval. We say that \( s \) is a moment sequence on \( I \) if there exists a measure \( \mu: \mathcal{B}(I) \rightarrow [0, \infty) \) satisfying the following conditions:

(i) there exists an interval \([a, b] \subset I \) \((a < b)\) such that \( \text{supp} \mu \subset [a, b] \),

(ii) \( \int_I t^k \, d\mu(t) = s_k, \ k \in \mathbb{N} \cap [0, n] \).

If such a measure is unique, we call \( s \) determinate on \( I \); otherwise, we call a moment sequence \( s \) indeterminate on \( I \).

If \( s = (s_0, \ldots, s_n) \subset \mathbb{R} \), \( n \in \mathbb{N} \), is a moment sequence on an interval \( I \subset \mathbb{R} \), then by \( \mathcal{M}_I(s) \) we denote the set of all measures satisfying Definition 3.1 for the further use we make the abbreviations: \( \mathcal{M}_\infty(s) := \mathcal{M}_{(0, \infty)}(s) \) and \( \mathcal{M}_1(s) := \mathcal{M}_{(0,1)}(s) \).

For \( s = (s_0, \ldots, s_n) \subset \mathbb{R} \), \( n \in \mathbb{N} \), define a linear functional \( \sigma: \mathbb{R}_n[x] \rightarrow \mathbb{R} \) by the formula

\[
\sigma(x^k) = s_k, \ k \in \mathbb{N} \cap [0, n].
\]
Definition 3.2. Let $n \in \mathbb{N}$ and $s = (s_0, \ldots, s_n) \subset \mathbb{R}$. Assume $I \subset \mathbb{R}$ is an interval. A sequence $s$ is positive on $I$ if there exists an interval $[a, b] \subset I$ $(a < b)$ such that for every polynomial $P \in \mathbb{R}[x]$ satisfying $P(x) \geq 0$, $x \in [a, b]$, we have $\sigma(P) \geq 0$. A positive sequence $s$ is strictly positive on $I$ if there exists an interval $[a, b] \subset I$ $(a < b)$ such that for every nonzero polynomial $P \in \mathbb{R}[x]$ satisfying $P(x) > 0$, $x \in [a, b]$, we have $\sigma(P) > 0$. If $s$ is positive on $I$, but not strictly positive, we call it singularly positive on $I$.

The first theorem reveals the connection between moment sequences and positive sequences (cf. [13, Theorem III.1.1]).

Theorem 3.3. Let $n \in \mathbb{N}$ and $s = (s_0, \ldots, s_n) \subset \mathbb{R}$. Assume $I \subset \mathbb{R}$ is an interval. Then $s$ is a moment sequence on $I$ if and only if $s$ is positive on $I$.

Proof. For the proof of the 'if' part suppose $\mu : \mathcal{B}(I) \to [0, \infty)$ satisfies Definition 3.1. Let $[a, b] \subset \mathbb{R}$ $(a < b)$ be such that $\text{supp} \mu \subset [a, b]$. Then for

$$
\nu : \mathcal{B}([a, b]) \ni A \mapsto \mu(A) \in [0, \infty)
$$

we have $\nu \in M_{[a, b]}(s)$. Hence, by [13, Theorem III.1.1], $s$ is positive on $[a, b]$, which implies that $s$ is positive on $I$.

Conversely, by Definition 3.2, $s$ is positive on some interval $[a, b] \subset I$ $(a < b)$. By [13, Theorem III.1.1], $s$ is a moment sequence on $[a, b]$. If $\mu \in M_{[a, b]}(s)$, then

$$
\nu : \mathcal{B}(I) \ni A \mapsto \mu(A \cap [a, b]) \in [0, \infty)
$$

satisfies Definition 3.1. \hfill \Box

Note that by [13, Theorem III.4.1] strictly positive moment sequences are always indeterminate; the converse will be proved later in Theorem 3.7.

Remark 3.4. Let $I \subset \mathbb{R}$ be an interval. If $s = (s_0, \ldots, s_n) \subset \mathbb{R}$, $n \in \mathbb{N}$, is positive on $I$, then

$$
\mathcal{M}_I(s) = \bigcup \{ M_{[a, b]}(s) : s \text{ is positive on } [a, b] \subset I \}
$$

with obvious identification

$$
M_{[a, b]}(s) \ni \mu \mapsto \mu(\cdot \cap [a, b]) \in \mathcal{M}_I(s), \quad a < b.
$$

Assume $I \subset \mathbb{R}$ is an interval and $s = (s_0, \ldots, s_n) \subset \mathbb{R}$ is a moment sequence on $I$. Following [11, III.3] we define the index of a measure $\mu \in \mathcal{M}_I(s)$ as follows: if $\text{supp} \mu$ is an infinite set, then we set $\text{ind}_I(s) = \infty$, otherwise we set

$$
\text{ind}_I(\mu) = \sum_{c \in I} \chi_{\text{supp} \mu}(c) \epsilon_I(c),
$$

where $\epsilon_I : I \to \{ \frac{1}{2}, 1 \}$ is defined by:

$$
\epsilon_I(c) = \begin{cases} 
1, & \text{if } c \in \text{int } I \\
\frac{1}{2}, & \text{otherwise}
\end{cases} \quad c \in I.
$$
Remark 3.5. In [11, Section III.§4.1] the authors used equivalent definition of index with the function \( \epsilon_I: I \to \{1, 2\} \) given by:

\[
\epsilon_I(c) = \begin{cases} 
2, & \text{if } c \in \text{int} I \\
1, & \text{otherwise} 
\end{cases} \quad c \in I.
\]

We define also the index of \( s \) as

\[
\text{ind}_I(s) = \min\{\text{ind}_I(\mu) : \mu \in M_I(s)\}.
\]

In view of [16, Theorem 1], \( \text{ind}_I(s) \) is always finite. As before, for the convenience we abbreviate: \( \text{ind}_\infty := \text{ind}_{(0, \infty)} \) and \( \text{ind}_1 := \text{ind}_{[0, 1]} \).

In [13, Chapter III] the authors was considering the case \( I = [a, b] \). In [13, Theorem III.4.1] there was proved that the sequence \( s \) is singularly positive on \( [a, b] \) if and only if \( \text{ind}_{[a, b]}(s) \leq \frac{n+1}{2} \). Moreover (cf. [13, Theorem III.5.1]), if \( s \) is strictly positive on \( [a, b] \), then there are exactly two measures (called principal measures) in \( M_I(s) \) of index \( \frac{n+1}{2} \); the principal measure with atom at \( b \) is called upper principal and the other is called lower principal. In the subsequent part of this section we present counterpart of this theory for the intervals \((0, \infty)\) and \((0, 1)\).

The next result gives the upper bound on the index of a positive sequence.

Lemma 3.6. Let \( n \in \mathbb{N} \) and \( s = (s_0, s_1, \ldots, s_n) \subset [0, \infty) \).

(i) If \( s \) is positive on \((0, \infty)\), then \( \text{ind}_\infty(s) \leq \lfloor \frac{n+1}{2} \rfloor \).

(ii) If \( s \) is positive on \((0, 1)\), then \( \text{ind}_1(s) \leq \frac{n+1}{2} \).

Proof. We prove (i). First, assume \( s \) is strictly positive on \((0, \infty)\). Let \([a, b] \subset (0, \infty)\) be an interval such that \( s \) is strictly positive on \([a, b]\). Assume \( n = 2m+1, m \in \mathbb{N} \), and let \( \mu \in M_{[a,b]}(s) \) be a lower principal measure. Then, in view of [13, Section III.§4.1],

\[
\text{ind}_{[a,b]}(\mu) = \text{ind}_\infty(\mu) = m + 1 = \frac{n+1}{2},
\]

so \( \text{ind}_\infty(s) \leq \frac{n+1}{2} = \lfloor \frac{n+1}{2} \rfloor \). Now, suppose \( n = 2m, m \in \mathbb{N} \), and let \( \mu \in M_{[a,b]}(s) \) be any principal measure. Then, again by [13, Section III.§4.1],

\[
\text{ind}_{[a,b]}(\mu) = \frac{n+1}{2} = m + \frac{1}{2}
\]

and

\[
\text{ind}_\infty(\mu) = m + 1 = \left\lfloor \frac{n+1}{2} \right\rfloor,
\]

so \( \text{ind}_\infty(s) \leq \left\lfloor \frac{n+1}{2} \right\rfloor \). Next, assume \( s \) is singularly positive on \((0, \infty)\), that is, for every interval \([a, b] \subset (0, \infty)\) if \( s \) is positive on \([a, b]\), then \( s \) is singularly positive.
on $[a, b]$. Suppose to the contrary that $\text{ind}_\infty(s) > \left\lceil \frac{n+1}{2} \right\rceil$. Then there exists an interval $[a, b] \subseteq (0, \infty)$ and a measure $\mu \in \mathcal{M}_{[a, b]}(s)$ satisfying

$$\infty > \text{ind}_\infty(\mu) > \left\lceil \frac{n+1}{2} \right\rceil \geq \frac{n+1}{2}.$$  

Enlarging the interval if necessary, we can assume that all atoms of $\mu$ are in the interior of $[a, b]$. Therefore, $\text{ind}_{[a, b]}(\mu) = \text{ind}_\infty(\mu) > \frac{n+1}{2}$. By [13, Theorem III.4.1], this implies that $s$ is strictly positive on $[a, b]$, contradictorily to our assumption. The proof of (ii) is a straightforward modification of the above reasoning (we leave details to the reader).

Below we present one of the main results in this section, which reveals the relationship between index and strict positivity.

**Theorem 3.7.** Let $n \in \mathbb{N}$ and let $I$ be either $(0, \infty)$ or $(0, 1]$. For a positive sequence $s = (s_0, s_1, \ldots, s_n) \subseteq [0, \infty)$ on $I$ the following conditions are equivalent:

(i) $s$ is strictly positive on $I$,

(ii) $\text{ind}_I(s) = \begin{cases} \left\lceil \frac{n+1}{2} \right\rceil, & \text{if } I = (0, \infty), \\ \frac{n+1}{2}, & \text{if } I = (0, 1] \end{cases}$,

(iii) $s$ is indeterminate on $I$.

**Proof.** We will show the equivalence in the case $I = (0, \infty)$. The proof in the case $I = (0, 1]$ (as a straightforward modification of the presented reasoning) is left to the reader.

(i)$\Rightarrow$(ii). From Lemma 3.6 it follows that $\text{ind}_\infty(s) \leq \left\lceil \frac{n+1}{2} \right\rceil$. Suppose to the contrary that $\text{ind}_\infty(s) < \left\lceil \frac{n+1}{2} \right\rceil$. Let $J_1 \subseteq (0, \infty)$ be a compact interval such that $s$ is strictly positive on $J_1$. Hence, by [13, Theorems III.4.1 and III.5.1], $\text{ind}_{J_1}(s) = \frac{n+1}{2}$. Take any $\mu \in \mathcal{M}_\infty(s)$ satisfying $\text{ind}_\infty(\mu) = \text{ind}_\infty(s)$. Then $\mu \in \mathcal{M}_{J_2}(s)$ for some compact interval $J_2 \subseteq (0, \infty)$; without loss of generality we can assume that all atoms of $\mu$ are in the interior of $J_2$. Take a compact interval $J \subseteq (0, \infty)$ such that $J_1 \cup J_2 \subseteq J$. Since $s$ is strictly positive on $J$, by [13, Theorem III.4.1] we have that every $\nu \in \mathcal{M}_J(s)$ satisfies $\text{ind}_J(\nu) \geq \frac{n+1}{2}$.

We also have $\mu \in \mathcal{M}_J(s)$ and $\text{supp} \mu \subseteq \text{int} J$. If $n = 2m + 1, m \in \mathbb{N}$, then

$$\text{ind}_J(\mu) = \text{ind}_\infty(\mu) < \frac{n+1}{2}.$$  

If $n = 2m, m \in \mathbb{N}$, then $\text{ind}_\infty(\mu) \leq \frac{n}{2}$, so

$$\text{ind}_J(\mu) = \text{ind}_\infty(\mu) \leq \frac{n}{2} < \frac{n+1}{2}.$$  

In both cases we get a contradiction with [13, Theorem III.4.1].

(ii)$\Rightarrow$(iii). Take $\mu \in \mathcal{M}_\infty(s)$ such that $\text{ind}_\infty(\mu) = \left\lceil \frac{n+1}{2} \right\rceil$. Then $\mu \in \mathcal{M}_J(s)$
for some compact interval \( J \subset (0, \infty) \); without loss of generality we can assume \( \text{supp}\mu \subset \text{int} \ J \). From this, it follows that

\[
\text{ind}_J(\mu) = \text{ind}_\infty(\mu) = \left\lceil \frac{n+1}{2} \right\rceil \geq \frac{n+1}{2},
\]

so, by [13, Theorem III.4.1], \( s \) is indeterminate on \( J \) and, consequently, indeterminate on \( (0, \infty) \).

(iii)\(\implies\)(i). Suppose \( \mu_1, \mu_2 \in \mathcal{M}_\infty(s) \) are distinct. Without loss of generality we can assume that \( \mu_1, \mu_2 \) are supported in the common compact interval \( J \subset (0, \infty) \). By [13, Theorem III.4.1], \( s \) is strictly positive on \( J \), hence strictly positive on \( (0, \infty) \).

**Corollary 3.8.** If \( n \in \mathbb{N} \) is odd and \( s = (s_0, \ldots, s_n) \subset [0, \infty) \) is strictly positive on \( (0, \infty) \), then there exists the unique measure \( \mu \in \mathcal{M}_\infty(s) \) with \( \text{ind}_\infty(\mu) = \left\lceil \frac{n+1}{2} \right\rceil \); its atoms are roots of the polynomial

\[
Q(t) = \det \begin{bmatrix}
s_0 & s_1 & \ldots & s_{m-1} & 1 \\
s_1 & s_2 & \ldots & s_m & t \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
s_m & s_{m+1} & \ldots & s_{2m-1} & t^m
\end{bmatrix},
\]

where \( n = 2m - 1 \), \( m \in \mathbb{N}_1 \).

**Proof.** Suppose \( \mu \in \mathcal{M}_\infty(s) \) is a measure satisfying \( \text{ind}_\infty(\mu) = \left\lceil \frac{n+1}{2} \right\rceil = m \). Then \( \mu \in \mathcal{M}_{[a,b]}(s) \) for some interval \( [a, b] \subset (0, \infty) \). Without loss of generality we can assume that \( \text{supp}\mu \subset (a, b) \), which implies that \( \text{ind}_{[a,b]}(\mu) = m \). Hence, \( \mu \) has to be a principal measure of \( s \) on \( [a, b] \). From the fact that all roots are in \( (a, b) \), by [13, Section III.\S 4.1], it follows that \( \mu \) is the lower principal measure of \( s \). Therefore, by [13, Section III.\S 5.3], atoms of \( \mu \) are roots of (3.1).

When \( n \) is even, the situation is not so simple as in the above corollary. Namely, if \( [a, b] \subset (0, \infty) \) is such that \( s \) is strictly positive on \( [a, b] \), then any principal measure \( \mu \in \mathcal{M}_{[a,b]}(s) \) has a support of cardinality \( \left\lceil \frac{n+1}{2} \right\rceil \) and its atoms depend on the interval \( [a, b] \) (in particular, they are different for different intervals \( [a, b] \); see [13, Section III.\S 5.3]). Hence, in this case we have infinitely many measures with minimal number of atoms, because \( s \) is strictly positive also on every interval \( (0, \infty) \supset [a', b'] \supset [a, b] \). Nevertheless, we can parametrize the set of all these measures, what will be seen in the next section.

In the case of interval \( (0, 1] \), the situation is much simpler, because the measure attaining the index of a strictly positive sequence is always unique.

**Corollary 3.9.** Let \( n \in \mathbb{N} \). If \( s = (s_0, \ldots, s_n) \subset [0, \infty) \) is strictly positive on \( (0, 1] \), then there exists the unique measure \( \mu \in \mathcal{M}_1(s) \) of index \( \frac{n+1}{2} \). Moreover,
(i) if \( n = 2m - 1, m \in \mathbb{N}_1 \), then atoms of \( \mu \) are roots of the polynomial

\[
Q(t) = \det \begin{bmatrix}
  s_0 & s_1 & \cdots & s_{m-1} & 1 \\
  s_1 & s_2 & \cdots & s_m & t \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  s_m & s_{m+1} & \cdots & s_{2m-1} & t^m
\end{bmatrix}, \tag{3.2}
\]

(ii) if \( n = 2m, m \in \mathbb{N} \), then atoms of \( \mu \) are roots of the polynomial

\[
Q(t) = (1 - t) \det \begin{bmatrix}
  s'_0 & s'_1 & \cdots & s'_{m-1} & 1 \\
  s'_1 & s'_2 & \cdots & s'_m & t \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  s'_m & s'_{m+1} & \cdots & s'_{2m-1} & t^m
\end{bmatrix}, \tag{3.3}
\]

where \( s'_k = s_k - s_{k+1}, k \in \mathbb{N} \cap [0, 2m-1] \).

**Proof.** Suppose \( \mu \in M_1(s) \) is a measure satisfying \( \text{ind}_1(\mu) = \frac{n+1}{2} \). Then \( \mu \in M_{[a,1]}(s) \) for some interval \([a,1] \subset (0,1)\). Without loss of generality we can assume that \( \text{supp} \mu \subset (a,1) \), so that \( \text{ind}_{[a,1]}(\mu) = \frac{n+1}{2} \). Hence, \( \mu \) has to be a principal measure of \( s \) on \([a,1]\). If \( n = 2m - 1, m \in \mathbb{N}_1 \), then, by [13, Section III.§4.1], \( \mu \) is the lower principal measure and, in view of [13, Section III.§5.3], its atoms are roots of (3.2). If \( n = 2m, m \in \mathbb{N} \), then, again by [13, Section III.§4.1], \( \mu \) is the upper principal measure of \( s \) and, in view of [13, Section III.§5.3], its atoms are roots of (3.3). \( \square \)

4. Backward extensions of moment sequences on \((0, \infty)\) and \((0,1]\

The question of backward extendibility of (infinite) moment sequence has a well-known answer. In [20] backward extensions were studied using methods of continued fractions. In [18] the methods of reproducing kernel Hilbert spaces are involved in the solution of the backward extendibility problem for (infinite) moment sequence. In this section, we present a new approach to characterization of backward extensions of (truncated) moment sequences using the theory developed in Section 3.

**Definition 4.1.** Let \( I \subset \mathbb{R} \) be either \((0, \infty)\) or \((0,1]\). Assume \( n \in \mathbb{N} \) and let \( s = (s_0, s_1, \ldots, s_n) \subset [0, \infty) \) be positive on \( I \). Suppose \( k \in \mathbb{N}_1 \) and \( s_{-k}, \ldots, s_{-1} \in [0, \infty) \). We say that a sequence \( s' = (s_{-k}, \ldots, s_{-1}, s_0, \ldots, s_n) \) is a backward extension of \( s \) on \( I \) if \( s' \) is positive on \( I \).

Since, by Theorem 3.7, singularly positive sequences are determinate on \( I \), backward extensions of such sequences are uniquely determined by \( \int_I \frac{1}{t} \, d\mu(t) \), where \( \mu \) is the unique representing measure. In what follows, we restrict our considerations on backward extensions to the case of strictly positive sequences.
If \( s = (s_0, \ldots, s_n) \subset [0, \infty) \), \( n \in \mathbb{N} \), is a strictly positive moment sequence on \((0, \infty)\), then we denote

\[
t_{a,b}(s) = \inf \left\{ \int_{[a,b]} \frac{1}{t} \text{d}\mu(t) : \mu \in \mathcal{M}_{[a,b]}(s) \right\}, \quad a, b \in (0, \infty), \ a < b,
\]

\[
t_\infty(s) = \inf_{0 < a < b} t_{a,b}(s),
\]

\[
T_{a,b}(s) = \sup \left\{ \int_{[a,b]} \frac{1}{t} \text{d}\mu(t) : \mu \in \mathcal{M}_{[a,b]}(s) \right\}, \quad a, b \in (0, \infty), \ a < b,
\]

\[
T_\infty(s) = \sup_{0 < a < b} T_{a,b}(s).
\]

If \( s \) is strictly positive on \((0,1]\), then we set

\[
t_1(s) = \inf_{0 < a < b \leq 1} t_{a,b}(s),
\]

\[
T_1(s) = \sup_{0 < a < b \leq 1} T_{a,b}(s).
\]

Observe that \( t_{a,b}(s) \) decreases, when \( b \) increases and \( T_{a,b}(s) \) increases, when \( a \) decreases. The next technical lemma states that, actually, \( T_{a,b}(s) \) increases to \( \infty \), which will be crucial in the subsequent results. Before the proof, we need a formulas for limiting values of integral of the reciprocal function on compact intervals. These formulas were given in [13, Section IV.§2.3]; unfortunately, they are wrong. The authors used the fact that if \( a, b, x \in \mathbb{R}, \ x < a < b, \) and \( n \in \mathbb{N} \), then the condition \( T_+(U) \) (see [13, p. 109]) holds for the function \((-1)^{n+1}\Omega(t) = \frac{(-1)^{n+1}}{x-t}\), which is not true. Using [13, P.1.1] it can be shown that the condition \( T_+(U) \) (see [13, p. 109]) holds for the function \((-1)^n\Omega(t) = \frac{(-1)^n}{x-t}\). Hence, if \( s = (s_0, s_1, \ldots, s_n) \subset [0, \infty) \) is a strictly positive moment sequence on the interval \([a, b]\), then the equality (2.10) in [13, p.116] should take the form

\[
(-1)^n \frac{P(x)}{Q(x)} \leq (-1)^n \int_a^b \frac{1}{x-t} \text{d}\mu(t) \leq (-1)^n \frac{P(x)}{Q(x)}, \quad \mu \in \mathcal{M}_{[a,b]}(s),
\]

where \( Q, \overline{Q} \) are polynomials with roots precisely at atoms of lower and upper principal measure, respectively (see [13, Section III.§5.3]), and

\[
P(x) = \sigma \left( \frac{Q(t) - Q(x)}{t-x} \right),
\]

\[
\overline{P}(x) = \sigma \left( \frac{Q(t) - \overline{Q}(x)}{t-x} \right).
\]

\(^1\text{We stick to the convention that } \inf \varnothing = \infty \text{ and } \sup \varnothing = -\infty.\)
Taking into account the rest of the reasoning in [13, Section IV.2.3], we obtain that for \( x = 0 \),

\[
\inf \left\{ (-1)^{n+1} \int_{[a,b]} \frac{1}{t} \, d\mu(t) : \mu \in \mathcal{M}_{[a,b]}(s) \right\} = (-1)^n \frac{P(0)}{Q(0)}, \tag{4.1}
\]

\[
\sup \left\{ (-1)^{n+1} \int_{[a,b]} \frac{1}{t} \, d\mu(t) : \mu \in \mathcal{M}_{[a,b]}(s) \right\} = (-1)^n \frac{\bar{P}(0)}{\bar{Q}(0)}, \tag{4.2}
\]

What is more, since \( \mathcal{M}_{[a,b]}(s) \) is strictly positive on \((0, \infty)\), the latter can be deduced from Banach-Alaoglu and Riesz representation theorems.

**Lemma 4.2.** Let \( n \in \mathbb{N} \). If \( s = (s_0, s_1, \ldots, s_n) \subset [0, \infty) \) is strictly positive on \((0, \infty)\) (resp. on \((0, 1]\)), then \( T_\infty(s) = \infty \) (resp. \( T_1(s) = \infty \)).

**Proof.** Assume that \( s \) is strictly positive on \((0, \infty)\). Fix \( b \in (0, \infty) \) such that \( s \) is strictly positive on \([a, b] \) for some \( a \in (0, b) \). Then, it holds \( T_{a,b}(s) > -\infty \). By the previous remark, it is enough to show that \( T_{a,b}(s) \xrightarrow{a \to 0^+} \infty \). For the further use, let us introduce the following notation: if \( N \in \mathbb{N} \) is odd and \( t = (t_0, \ldots, t_N) \subset \mathbb{R} \) and \( f = (f_0, \ldots, f_{N+1}) \subset \mathbb{R} \), then we denote

\[
\mathcal{D}(t, f) = \text{det} \begin{bmatrix}
t_0 & t_1 & \cdots & t_{\frac{N+1}{2}} & f_0 \\
t_1 & t_2 & \cdots & t_{\frac{N+1}{2}} & f_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
t_{\frac{N+1}{2}} & t_{\frac{N+1}{2}+1} & \cdots & t_N & f_{\frac{N+1}{2}}
\end{bmatrix}
\]

and

\[
\mathcal{E}(t, f) = \text{det} \begin{bmatrix}
f_0 & t_0 & t_1 & \cdots & t_{\frac{N+1}{2}} \\
f_1 & t_1 & t_2 & \cdots & t_{\frac{N+1}{2}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f_{\frac{N+1}{2}} & t_{\frac{N+1}{2}} & t_{\frac{N+1}{2}+1} & \cdots & t_N
\end{bmatrix}
\]

Case 1: \( n = 2m - 1, \ m \in \mathbb{N}_1 \). In this case, by (4.2), \( T_{a,b} = -\frac{\bar{P}(0)}{\bar{Q}(0)} \), where \( a \in (0, b) \) is such that \( s \) is strictly positive on \([a, b] \). Applying the formula for \( \bar{Q} \) from [13, Section III.5.3], by multilinearity of the determinant, we obtain the following:

\[
\bar{P}(0) = \sigma \left( \frac{\bar{Q}(t) - \bar{Q}(0)}{t} \right) = \mathcal{D} \left( (s_k')^{2m-3}_{k=0}, ((s_k')_{k=0}^{n-1}) \right),
\]

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where $s'_k = (a + b)s_{k+1} - abs_k - s_{k+2}$ for $k \in \mathbb{N} \cap [0, 2m - 2]$ and $s_{-1} = 0$. Set $s^{(1)}_k = s_{k+1} - bs_k$ for $k \in \mathbb{Z} \cap [-1, 2m - 2]$. Then, again by multilinearity of the determinant, we obtain

$$\frac{p(0)}{q(0)} = \frac{D \left( (s'_k)_{k=0}^{2m-3}, (s'_{k-1})_{k=0}^{m-1} \right)}{abD \left( (s'_k)_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)}$$

$$= \frac{D \left( (s'_k)_{k=0}^{2m-3}, (as^{(1)}_{k-1})_{k=0}^{m-1} \right)}{abD \left( (s'_k)_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)} + \frac{D \left( (s'_k)_{k=0}^{2m-3}, (-s^{(1)}_k)_{k=0}^{m-1} \right)}{abD \left( (s'_k)_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)}$$

Now, we calculate the limit of the first term of the right hand side of (4.3). Again, by basic properties of the determinant, we have

$$\frac{D \left( (s'_k)_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)}{abD \left( (s'_k)_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)} \xrightarrow{a \to 0+} \frac{D \left( (-s^{(1)}_{k+1})_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)}{bD \left( (-s^{(1)}_{k+1})_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)}$$

Observe that continuity of the determinant gives us

$$\frac{D \left( (s'_k)_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)}{abD \left( (s'_k)_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)} \xrightarrow{a \to 0+} \frac{D \left( (-s^{(1)}_{k+1})_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)}{bD \left( (-s^{(1)}_{k+1})_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)}$$

and that the above limit is non-zero by [13, Theorem 2.4] and [13, Remark 2.1]. Summarizing, we have

$$\frac{D \left( (s'_k)_{k=0}^{2m-3}, (as^{(1)}_{k-1})_{k=0}^{m-1} \right)}{abD \left( (s'_k)_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)} \xrightarrow{a \to 0+} \frac{D \left( (-s^{(1)}_{k+1})_{k=0}^{2m-3}, (s^{(1)}_{k-1})_{k=0}^{m-1} \right)}{bD \left( (-s^{(1)}_{k+1})_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)}$$

Next, we calculate the limit of the second term of the right hand side of (4.3). By the properties of the determinant, we see that

$$\frac{D \left( (s'_k)_{k=0}^{2m-3}, (-s^{(1)}_k)_{k=0}^{m-1} \right)}{abD \left( (s'_k)_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)} \xrightarrow{a \to 0+} \frac{D \left( (-s^{(1)}_{k+1})_{k=0}^{2m-3}, (-s^{(1)}_k)_{k=0}^{m-1} \right)}{bD \left( (-s^{(1)}_{k+1})_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)}$$

where, by [13, Theorem III.2.4] and [13, Remark III.2.1],

$$D \left( (-s^{(1)}_{k+1})_{k=0}^{2m-3}, (-s^{(1)}_k)_{k=0}^{m-1} \right) > 0.$$  

Similarly, we have

$$\frac{D \left( (s'_k)_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)}{bD \left( (s'_k)_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)} \xrightarrow{a \to 0+} \frac{D \left( (-s^{(1)}_{k+1})_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)}{bE \left( (-s^{(1)}_{k+1})_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)},$$

where, arguing as before, $E \left( (-s^{(1)}_{k+1})_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right) > 0$. Using (4.3) and (4.7), we obtain

$$\frac{D \left( (s'_k)_{k=0}^{2m-3}, (-s^{(1)}_k)_{k=0}^{m-1} \right)}{bD \left( (s'_k)_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)} \xrightarrow{a \to 0+} \frac{D \left( (-s^{(1)}_{k+1})_{k=0}^{2m-3}, (-s^{(1)}_k)_{k=0}^{m-1} \right)}{bE \left( (-s^{(1)}_{k+1})_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)} > 0.$$
Since $\frac{1}{a} \to \infty$ as $a \to 0^+$, it follows that

$$
\frac{\mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-3}, (-s_k^{(1)})_{k=0}^{m-1} \right)}{ab \mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-3}, (1, 0, \ldots, 0) \right)} \xrightarrow{a \to 0^+} \infty.
$$

(4.8)

Combining (4.3), (4.4), and (4.8), we get that $T_{a,b}(s) \xrightarrow{a \to 0^+} \infty$.

Case 2: $n = 2m$, $m \in \mathbb{N}$. In this case, by (4.1), $T_{a,b}(s) = -\frac{P(0)}{Q(0)}$. Applying the formula for $Q$ from [13, Section III.5.3], by multilinearity of the determinant, we obtain

$$
\frac{P(0)}{Q(0)} = \frac{\mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (s_k - as_{k-1})_{k=0}^{m} \right)}{a \mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right)}.
$$

(4.9)

Where $s_{-1} = 0$. Set $s_k^{(1)} = s_{k+1} - as_k$ for $k \in \mathbb{Z} \cap [-1, 2m - 2]$. Then, again using the formula from [13, Section III.5.3], by properties of the determinant, we see that

$$
\frac{P(0)}{Q(0)} \xrightarrow{a \to 0^+} \frac{\mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right)}{a \mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right)}.
$$

Now, we compute the limit of the first term of the right hand sider of (4.9).

Arguing similarly as to obtain (4.4), we have

$$
\frac{\mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right)}{a \mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right)} = \frac{\mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right)}{a \mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right)} \xrightarrow{a \to 0^+} \frac{\mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right)}{a \mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right)} > 0.
$$

Next, we compute the limit of the second term of the right hand sider of (4.9).

Using the same reasoning as to obtain (4.5) and (4.7), we get

$$
\mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (s_k^{(1)})_{k=0}^{2m-1}, (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right) \xrightarrow{a \to 0^+} \mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (s_k^{(1)})_{k=0}^{2m-1}, (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right) = (-1)^m \mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (s_k^{(1)})_{k=0}^{2m-1}, (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right),
$$

and

$$
\mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right) \xrightarrow{a \to 0^+} \mathcal{D} \left( (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right) = (-1)^m \mathcal{E} \left( (s_k^{(1)})_{k=0}^{2m-1}, (1, 0, \ldots, 0) \right).
$$
By [13, Theorem III.2.3] and [13, Remark III.2.1], we see that
\[ D \left( \left( s_k \right)^{2m-1}_{k=0}, \left( s_k \right)^{2m}_{k=m} \right) > 0, \quad E \left( \left( s_k+1 \right)^{2m-1}_{k=0}, (1, 0, \ldots, 0) \right) > 0. \]

Hence, arguing as to obtain (4.8), we have
\[ \frac{D \left( \left( s_k \right)^{(1)}_{k=0}, \left( s_k \right)^{m}_{k=0} \right)}{aD \left( \left( s_k \right)^{(1)}_{k=0}, (1, 0, \ldots, 0) \right)} \xrightarrow{a \to 0^+} \infty. \]

Summarizing, we get \( T_{a,b}(s) \xrightarrow{a \to 0^+} \infty \). If \( s \) is assumed to be strictly positive on \((0,1]\), then the same proof gives us the equality \( T_1(s) = \infty \). \( \Box \)

The following lemma gives us the necessary condition on the one-step backward extension of a strictly positive sequence.

**Lemma 4.3.** Let \( n \in \mathbb{N} \). If \( s' = (s_{-1}, s_0, \ldots, s_n) \subset [0, \infty) \) is a backward extension of a strictly positive sequence \( s = (s_0, s_1, \ldots, s_n) \) on \((0, \infty)\) (resp. on \((0,1]\)), then \( s_{-1} \in [t_\infty(s), \infty) \) (resp. \( s_{-1} \in [t_1(s), \infty) \)).

**Proof.** Assume that \( s' \) is strictly positive on \((0, \infty)\) and let \( \mu \in \mathcal{M}_\infty(s') \). Then \( \mu \in \mathcal{M}_{[a,b]}(s') \) for some interval \([a, b] \subset (0, \infty)\); without loss of generality we can assume that \( \text{supp} \mu \subset (a, b) \). For \( \nu = t \, d\mu(t) \) we have that \( \nu \in \mathcal{M}_{[a,b]}(s) \), so
\[ s_{-1} = \int_{[a,b]} t \, d\nu(t) \in [t_{a,b}(s), T_{a,b}(s)] \subset [t_\infty(s), \infty). \] \( \Box \)

With no substantial changes, the above proof works also when \( s \) is assumed to be strictly positive on \((0,1]\).

We are ready to prove one of the main results of this section, which characterizes strictly positive backward extensions.

**Theorem 4.4.** Let \( n \in \mathbb{N} \) and let \( s = (s_0, \ldots, s_n) \subset [0, \infty) \) be strictly positive on \((0, \infty)\) (resp. on \((0,1]\)). Assume \( s_{-1} \in [0, \infty) \). Then the following conditions are equivalent:

(i) \( s' = (s_{-1}, s_0, \ldots, s_n) \) is a strictly positive backward extension of \( s \) on \((0, \infty)\) (resp. on \((0,1]\)),

(ii) \( s_{-1} \in (t_\infty(s), \infty) \) (resp. \( s_{-1} \in (t_1(s), \infty) \)).

**Proof.** We concentrate on the case, when \( s \) is strictly positive on \((0, \infty)\).

(i)\(\Rightarrow\)(ii). By Theorem [13, Theorem IV.1.1], \( \text{ind}_\infty(s') = \left\lceil \frac{n+2}{2} \right\rceil \). Let \( \mu \in \mathcal{M}_\infty(s') \) be any measure satisfying \( \text{ind}_\infty(\mu) = \left\lceil \frac{n+2}{2} \right\rceil \). Then \( \mu \in \mathcal{M}_{[a,b]}(s') \) for some interval \([a, b] \subset (0, \infty)\); without loss of generality we can assume \( \text{supp} \mu \subset (a, b) \). This implies that \( \text{ind}_{[a,b]}(\mu) = \left\lceil \frac{n+2}{2} \right\rceil \). For \( \nu = t \, d\mu(t) \) we see that \( \nu \in \mathcal{M}_{[a,b]}(s) \) and \( \text{ind}_{[a,b]}(\mu) = \text{ind}_{[a,b]}(\nu) \geq \frac{n+2}{2} \). By [13, Theorem IV.1.1],
\[ s_{-1} = \int_{[a,b]} t \, d\nu(t) \in (t_{a,b}(s), T_{a,b}(s)) \subset (t_\infty(s), \infty). \]
(ii)⇒(i). From Lemma 4.2 it follows

$$(t_\infty(s), \infty) = \bigcup_{0 < a < b} (t_{a,b}(s), T_{a,b}(s)).$$  \hspace{1cm} (4.10)

By (4.10), there exist $a, b \in (0, \infty)$, $a < b$, such that

$$s_{-1} \in (t_{a,b}(s), T_{a,b}(s)).$$  \hspace{1cm} (4.11)

There exists a measure $\mu \in \mathcal{M}_{[a,b]}(s)$ satisfying $\int_{[a,b]} \frac{1}{t} \, d\mu(t) = s_{-1}$. In view of (4.11) and [13, Theorem IV.1.1], the measure $\mu$ is not a principal measure, so $\text{ind}_{[a,b]}(\mu) \geq \frac{n+2}{2}$. For $\nu = \frac{1}{t} \, d\mu(t)$ we have $\nu \in \mathcal{M}_{[a,b]}(s')$ and $\text{ind}_{[a,b]}(\nu) \geq \frac{n+3}{2} = \frac{n+1}{2} + 1$. By [13, Theorem III.4.1], this implies that $s'$ is strictly positive on $[a, b]$. Hence, $s'$ strictly positive on $(0, \infty)$.

It turns out that singularly positive backward extensions on $(0, \infty)$ can appear only when $n$ is odd.

**Corollary 4.5.** Let $n \in \mathbb{N}$ and let $s = (s_0, \ldots, s_n) \subset [0, \infty)$ be strictly positive on $(0, \infty)$. Assume $s_{-1} \in [0, \infty)$. Then the following conditions are equivalent:

(i) $s' = (s_{-1}, s_0, \ldots, s_n)$ is singularly positive backward extension of $s$ on $(0, \infty)$,

(ii) $n$ is odd and $s_{-1} = t_\infty(s)$.

Moreover, if (i) holds, then the atoms of the unique measure $\mu \in \mathcal{M}_\infty(s')$ are roots of the polynomial $Q$, where

$$Q(t) = \det \begin{bmatrix} s_0 & s_1 & \cdots & s_{m-1} & t \\ s_1 & s_2 & \cdots & s_m & t \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_m & s_{m+1} & \cdots & s_{2m-1} & t^m \end{bmatrix},$$  \hspace{1cm} (4.12)

and $n = 2m - 1$, $m \in \mathbb{N}_1$.

**Proof.** By Lemma 4.3 and Theorem 4.4 the sequence $s'$ is singularly positive if and only if $s_{-1} = t_\infty(s)$. It is enough to prove that this can happen only if $n$ is odd. From Theorem 3.7 we have $\text{ind}_{\infty}(s) = \lceil \frac{n+1}{2} \rceil$ and

$$\text{ind}_{\infty}(s) \leq \text{ind}_{\infty}(s') < \left\lceil \frac{n+2}{2} \right\rceil.$$  \hspace{1cm} (4.13)

Thus, $\text{ind}_{\infty}(s') = \lceil \frac{n+1}{2} \rceil$. Moreover, $s'$ is determinate on $(0, \infty)$. Let $\mu$ be the unique element of $\mathcal{M}_{\infty}(s')$; we can assume $\text{supp} \mu \subset (a, b)$ for some interval $[a, b] \subset (0, \infty)$. Let $\nu = \frac{1}{t} \, d\mu(t)$. Then $\nu \in \mathcal{M}_{[a,b]}(s)$ and

$$\text{ind}_{[a,b]}(\nu) = \text{ind}_{[a,b]}(\mu) = \text{ind}_{\infty}(\mu) = \left\lceil \frac{n+1}{2} \right\rceil.$$  \hspace{1cm} (4.14)
Suppose to the contrary that \( n \) is even. Then \( \text{ind}_{[a,b]}(\nu) = \frac{n+2}{2} > \frac{n+1}{2} \), so, by [13, Theorem IV.1.1] we have

\[
    s_{-1} = \int_{[a,b]} \frac{1}{t} \, d\nu(t) > t_{a,b}(s) \geq t_{\infty}(s),
\]

which gives us a contradiction. Hence, \( n \) is odd and, by Corollary 3.8 atoms of \( \nu \) (which are the same as atoms of \( \mu \)) are roots of (4.12). Conversely, if \( n \) is odd, then \( t_{\infty}(s) = t_{a,b}(s) \) for every \([a, b] \subset (0, \infty)\) such that \( s \) is strictly positive on \([a, b] \), because atoms of the lower principal measure of \( s \) do not depend on \( a \) and \( b \) (see [13, Section III.§5.3]).

In the case of interval \((0, 1]\) singularly positive backward extensions can appear in any case; we omit the proof, which is similar to the proof of Corollary 4.5.

**Corollary 4.6.** Let \( n \in \mathbb{N} \). Let \( s = (s_0, \ldots, s_n) \subset [0, \infty) \) be strictly positive on \((0, 1]\) and \( s_{-1} \in [0, \infty) \). Then the following are equivalent:

(i) \( s' = (s_{-1}, \ldots, s_n) \) is a singularly positive backward extension of \( s \),

(ii) \( s_{-1} = t_1(s) \).

Moreover, if (i) holds, then the atoms of unique measure \( \mu \in \mathcal{M}_1(s') \) are roots of the polynomial \( Q \), where

(i)

\[
    Q(t) = \det \begin{bmatrix} s_0 & s_1 & \cdots & s_{m-1} & 1 \\ s_1 & s_2 & \cdots & s_m & t \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_m & s_{m+1} & \cdots & s_{2m-1} & t^m \end{bmatrix}, \quad (4.13)
\]

if \( n = 2m - 1, m \in \mathbb{N}_1 \),

(ii)

\[
    Q(t) = (1 - t) \det \begin{bmatrix} s'_0 & s'_1 & \cdots & s'_{m-1} & 1 \\ s'_1 & s'_2 & \cdots & s'_m & t \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s'_m & s'_{m+1} & \cdots & s'_{2m-1} & t^m \end{bmatrix}, \quad (4.14)
\]

if \( n = 2m, m \in \mathbb{N} \), where \( s'_k = s_k - s_{k+1}, k \in \mathbb{N} \cap [0, 2m - 1] \).

Next result gives the characterization of moments of a positive sequence on \((0, \infty)\) in terms of numbers \( t_\infty(\cdot) \). This will be crucial in solving completion problems.

**Theorem 4.7.** Let \( n \in \mathbb{N} \) and let \( s = (s_0, \ldots, s_n) \subset [0, \infty) \) be positive on \((0, \infty)\). Set \( K = \text{ind}_\infty(s) \) and \( N = 2K - 1 \). Then

(i) \( s_k \in (t_\infty((s_{k+1}, \ldots, s_n)), \infty) \) for \( k \in \mathbb{N} \cap [n - N, n - 1] \),
Theorem 4.9. Let \( n \in \mathbb{N} \) and let \( s = (s_0, \ldots, s_n) \subset [0, \infty) \) be strictly positive on \((0, \infty)\). Suppose \( r \in \mathbb{N}_1 \) and \( K \in \mathbb{N} \cap \left[ \left\lceil \frac{n+1}{2} \right\rceil, \left\lceil \frac{n+3}{2} \right\rceil \right] \). Set \( N = 2K - 1 \) and let \( s-r, \ldots, s_1, \ldots, s_{-1} \in [0, \infty) \) be such that

(i) \( s_k \in \{t_\infty((s_{k+1}, \ldots, s_0), s_1, \ldots, s_n)\}, \infty\) for \( k \in \mathbb{Z} \cap [n-N, -1]\),
(ii) \( s_k = t_\infty((s_{k+1}, \ldots, s_{k+N+1})) \) for \( k \in \mathbb{Z} \cap [-r, n-N - 1] \).

Then \( s' = (s_{-r}, s_{-r+1}, \ldots, s_n) \) is a backward extension of \( s \) on \((0, \infty)\) satisfying \( \text{ind}_\infty(s') = K \). Moreover, if \( N \leq n + r \), then the atoms of the unique measure \( \mu \in \mathcal{M}_\infty(s') \) satisfying \( \text{ind}_\infty(\mu) = K \) are roots of the polynomial \( Q \), where

\[
Q(t) = \det \begin{bmatrix}
    s_{-r} & s_{-r+1} & \cdots & s_{-r+K-1} & 1 \\
    s_{-r+1} & s_{-r+2} & \cdots & s_{-r+K} & t \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    s_{-r+K} & s_{-r+K+1} & \cdots & s_{-r+2K-1} & t^K \\
\end{bmatrix} \tag{4.15}
\]

Proof. We show that for any \( k \in \mathbb{Z} \cap [n-N,0] \) the sequence \((s_k, s_{k+1}, \ldots, s_n)\) is strictly positive on \((0, \infty)\). The proof goes by induction. The case \( k = 0 \) follows from the assumption. If \( n-N \leq k-1 < 0 \), then by inductive hypothesis \( s_k = (s_k, \ldots, s_n) \) is strictly positive on \((0, \infty)\). Applying Theorem 3.7 to \( s_k \) and \( s_{k-1} \), we get that \((s_{k-1}, s_k, \ldots, s_n)\) is strictly positive on \((0, \infty)\). Next, from Corollary 4.5 applied to \( s_1 = (s_{n-N}, \ldots, s_n) \) and \( s_{n-N-1} \), we derive that \( s_1' = (s_{n-N-1}, \ldots, s_n) \) is singularly positive on \((0, \infty)\). Let \( \mu \) be the unique element of \( \mathcal{M}_\infty(s_1') \). Setting

\[
s'_k = \int_{(0,\infty)} t^{k+N-n+1} \, d\mu(t), \quad k \in \mathbb{Z} \cap [-r, n-N-2],
\]

we obtain from Theorem 3.7 that \((s'_1, \ldots, s'_{n-N-2}, s_{n-N-1}, \ldots, s_n)\) is singularly positive on \((0, \infty)\). The sequence \( s_1 \) is strictly positive on \((0, \infty)\), so \( \text{ind}_\infty(\mu) = \lceil \frac{N+1}{r} \rceil = K \). Therefore, using Corollary 4.5 we get

\[
s'_k = t_\infty((s_{k+1}, \ldots, s_{k+N+1})) = s_k, \quad k \in \mathbb{Z} \cap [-r, n-N-1].
\]

This shows that \( s' \) is a backward extension of \( s \) on \((0, \infty)\) of index \( K \). To prove the “moreover” part, consider \( s'' = (s_{-r}, \ldots, s_{-r+2K-1}) \). Then \( \text{ind}_\infty(s'') = \text{ind}_\infty(s') = K \). Hence, by Theorem 3.8 \( s'' \) is strictly positive and, by Corollary 3.8 atoms of the unique measure \( \mu \in \mathcal{M}_\infty(s) \) satisfying \( \text{ind}_\infty(\mu) = K \) are roots of \( \text{ind}_1(s') \). \( \square \)

Again, below we state (without the proof) the counterpart of the above theorem for moments on \((0,1)\).

**Theorem 4.10.** Let \( n \in \mathbb{N} \) and let \( s = (s_0, s_1, \ldots, s_n) \subset [0, \infty) \) be strictly positive on \([0,1]\). Suppose \( r \in \mathbb{N}_1 \) and \( K \in \mathbb{Q} \) is such that \( \frac{n+1}{2} \leq K \leq \frac{n+r+1}{2} \) and \( 2K \in \mathbb{N} \). Set \( N = 2K - 1 \) an let \( s_{-r}, \ldots, s_{-1} \in [0, \infty) \) be such that

(i) \( s_k \in (t_1((s_{k+1}, \ldots, s_n)), \infty) \) for \( k \in \mathbb{Z} \cap [n-N, n-1] \),

(ii) \( s_k = t_1((s_{k+1}, \ldots, s_{k+N+1})) \) for \( k \in \mathbb{Z} \cap [-r, n-N-1] \).

Then \( s' = (s_{-r}, \ldots, s_n) \) is a backward extension of \( s \) on \([0,1]\) and \( \text{ind}_1(s') = K \). Moreover, atoms of the unique measure \( \mu \in \mathcal{M}_1(s') \) satisfying \( \text{ind}_1(\mu) = K \) are roots of the polynomial
by Corollary 3.8, there exists the unique measure $\nu$ we see that Lemma 4.3, 4.5, we have

\[
\int r \quad \text{Applying Theorem 3.7 and Theorem 4.9 with } \int r
\]

First, observe that $\Phi$ is well-defined. Indeed, if

\[
\text{Proof.}
\]

\[
\text{At the end of the previous section we claimed that if } n \in \mathbb{N} \text{ is even and } s = (s_0, \ldots, s_n) \subset [0, \infty) \text{ is strictly positive on } (0, \infty), \text{ then we can parametrize the set of all measures in } \mathcal{M}_\infty(s) \text{ with support of minimal cardinality. We will do it now with help of our results on backward extensions.}
\]

**Corollary 4.11.** Let $s = (s_0, \ldots, s_n) \subset [0, \infty)$ be strictly positive on $(0, \infty)$, where $n \in \mathbb{N}$ is even. Denote by $\mathcal{M}^\text{min}_\infty(s)$ the set of all measures $\mu \in \mathcal{M}_\infty(s)$ satisfying $\text{ind}_\infty(\mu) = \left[\frac{n+1}{2}\right]$. Then the mapping

\[
\Phi: \mathcal{M}^\text{min}_\infty(s) \ni \mu \mapsto \int_{(0, \infty)} \frac{1}{t} \, d\mu(t) \in (t_\infty(s), \infty)
\]

is a bijection. Moreover, if $s_{-1} \in (t_\infty(s), \infty)$, then atoms of $\Phi^{-1}(s_{-1})$ are roots of the polynomial

\[
Q(t) = \det \begin{bmatrix}
    s_{-1} & s_{0} & \ldots & s_{K-2} & 1 \\
    s_{0} & s_{1} & \ldots & s_{K-1} & t \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    s_{K-1} & s_{K} & \ldots & s_{2K-2} & t^K
\end{bmatrix}, \quad (4.16)
\]

where $K = \left[\frac{n+1}{2}\right]$.

**Proof.** First, observe that $\Phi$ is well-defined. Indeed, if $\mu \in \mathcal{M}_\infty(s)$, then by Lemma 4.3, $\int_{(0, \infty)} \frac{1}{t} \, d\mu(t) \in (t_\infty(s), \infty)$. Since $n$ is even, in view of Corollary 4.5, we have $\int_{(0, \infty)} \frac{1}{t} \, d\mu(t) > t_\infty(s)$. Let $\mu \in \mathcal{M}^\text{min}_\infty(s)$ and $s_{-1} \in (t_\infty(s), \infty)$.

Applying Theorem 3.7 and Theorem 4.9 with $r = 1$ and $K = \left[\frac{n+2}{2}\right] = \left[\frac{n+1}{2}\right]$, we see that $s' = (s_{-1}, s_0, \ldots, s_n)$ is strictly positive on $(0, \infty)$. Moreover, by Corollary 3.8 there exists the unique measure $\nu_{s_{-1}} \in \mathcal{M}_\infty(s)$ satisfying $\text{ind}_\infty(\nu_{s_{-1}}) = \left[\frac{n+1}{2}\right]$; its atoms are roots of (4.15), which in this case becomes
Theorem 5.1. \( \mu \) \parallel d \mu \) directed trees with one branching point in full generality.

We obtain a solution of the subnormal completion problem for weighted shifts on \([6, \text{Theorem 4.1}]\) for the solution of the 2-generation subnormal completion. The same authors studied in details 2-generation subnormal completions (see \([9, \text{Theorem 6.2.1}]\)).

(4.16). Then \( (0 \infty (s, \infty) \ni s) \rightarrow t \nu s (t) \in M_{\infty}(s). \)

We show that \( \Psi \circ \Phi = \text{id}_{M_{\infty}(s)} \) and \( \Phi \circ \Psi = \text{id}_{(t_{\infty}(s), \infty)}. \) Let \( \mu \in M_{\infty}(s). \) Set \( s = 1. \) Because of the uniqueness of the measure \( \nu s \), we get that \( \frac{1}{t} d \mu (t) = \nu s. \) Hence, \( \mu = t \nu s (t) = \Psi (s). \) Conversely, let \( s = (t_{\infty}(s), \infty) \) and set \( \mu = \Psi (s) = t \nu s. \) Then \( \Phi (\mu) = \int_{(0, \infty)} \frac{1}{t} d \mu (t) = \int_{(0, \infty)} \frac{1}{t} t \nu s (t) = s. \) \( \square \)

5. Subnormal completions of weighted shifts on directed trees.

In \([6, \text{Chapter 6}]\) the authors studied subnormal weighted shifts on directed trees and characterized subnormality by certain moment sequences (the obtained characterization involves also negative moments, see \([6, \text{Theorem 6.2.1}]\)).

Using this approach, in \([3] \) the authors investigated the subnormal completion problem for weighted shifts on trees with one branching point in general; in \([6] \) the same authors studied in detail 2-generation subnormal completions (see \([6, \text{Theorem 4.1}]\) for the solution of the 2-generation subnormal completion).

In this section we will exploit the theory developed in the previous sections to obtain a solution of the subnormal completion problem for weighted shifts on directed trees with one branching point in full generality.

If \( T = (V, E) \) is a directed tree, then \([6, \text{Theorem 6.1.3}]\) states that a bounded weighted shift \( S \lambda \in B(t^{2}(V)) \) on \( T \) is subnormal if and only if the sequence \( (\| S \lambda^{n} u \|^{2})_{n=0}^{\infty} \) is a Stieltjes moment sequence for every \( u \in V. \) Moreover, the measure representing the moment sequence \( (\| S \lambda^{n} u \|^{2})_{n=0}^{\infty} \) is unique; we denote it by \( \mu \lambda. \)

The next result generalizes \([6, \text{Theorem 4.1}]\).

**Theorem 5.1.** Let \( \kappa \in \mathbb{N}, \eta \in \mathbb{N}, p \in \mathbb{N}_1. \) Let \( \lambda = \{ \lambda_{-k} \}_{k=0}^{\kappa+1} \cup \{ \lambda_{i,j} \}_{i,j=1}^{\eta,p} \subset (0, \infty) \) and let \( \{ K_{i} \}_{i=1}^{\eta} \subset \mathbb{N} \cap [1, \left[ \frac{p+\kappa+1}{2} \right] ]. \) Then the following conditions are equivalent:

(i) there exists a subnormal completion \( S \lambda^{'} \in B(t^{2}(V_{\eta,\kappa})) \) of \( \lambda^{'} \) such that for every \( i \in \mathbb{N} \cap [1, \eta] \) the measure \( \mu_{i,1}^{\lambda^{'}} \) is \( K_{i}, \text{-atomic}, \)

(ii) for every \( i \in \mathbb{N} \cap [1, \eta] \) there exists a sequence \( \{ s_{i,k} \}_{k=1}^{\kappa+1} \subset (0, \infty) \) such that for every \( i \in \mathbb{N} \cap [1, \eta] \)

\[
 s_{i,k} \in (t_{\infty}((s_{i,k+1}, \ldots, s_{p-1})), +\infty), \quad k \in \mathbb{Z} \cap [p - N_{i} - 1, p - 2], \tag{5.1}
\]
\[
 s_{i,k} = t_{\infty}((s_{i,k+1}, \ldots, s_{k+N_{i}+1})), \quad k \in \mathbb{Z} \cap [-\kappa - 1, p - N_{i} - 2] \tag{5.2}
\]

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and

\[
\sum_{i=1}^{\eta} \lambda_{i,1}^2 s_{i,-k-1} = \frac{1}{\prod_{j=0}^{k-1} \lambda_{-j}^2}, \quad k \in \mathbb{N} \cap [0, \kappa - 1], \tag{5.3}
\]

\[
\sum_{i=1}^{\eta} \lambda_{i,1}^2 s_{i,-k-\kappa} \leq \frac{1}{\prod_{j=0}^{k-1} \lambda_{-j}^2}, \tag{5.4}
\]

\[
\sup_{i \in \mathbb{N} \cap [1, \eta]} \frac{Q_{i}^{(K_{i}-1)}(0)}{Q_{i}^{(K_{i})}(0)} < \infty, \tag{5.5}
\]

where \( s_{i,k} = \prod_{j=2}^{k+1} \lambda_{i,j}^2, k \in \mathbb{N} \cap [0, p-1], N_{i} = 2K_{i} - 1 \) and \( Q_{i} \in \mathbb{R}_{K_{i}}[x] \) is a polynomial with simple zeros at atoms of \( \mu \in \mathcal{M}_{\infty}((s_{i,-\kappa-1}, \ldots, s_{i,p-1})) \) satisfying \( \#\text{supp} \mu = K_{i} \) for \( i \in \mathbb{N} \cap [1, \eta] \).

**Proof.** (i)\( \Rightarrow \) (ii). By (i) and \( [5, \text{Lemma 4.7}] \) we see that

\[
\int_{0}^{\infty} t^{k} d\mu_{i}(t) = \frac{1}{\prod_{j=0}^{k-1} \lambda_{-j}^2}, \quad k \in \mathbb{N} \cap [0, p-1], i \in \mathbb{N} \cap [1, \eta], \tag{5.6}
\]

\[
\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_{0}^{\infty} \frac{1}{t^{k+1}} d\mu_{i}(t) = \frac{1}{\prod_{j=0}^{k-1} \lambda_{-j}^2}, \quad k \in \mathbb{N} \cap [0, \kappa - 1], \tag{5.7}
\]

\[
\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_{0}^{\infty} \frac{1}{t^{k+1}} d\mu_{i}(t) \leq \frac{1}{\prod_{j=0}^{k-1} \lambda_{-j}^2}, \tag{5.8}
\]

\[
\sup_{i \in \mathbb{N} \cap [1, \eta]} \sup_{\text{supp} \mu_{i}} \mu_{i} < \infty. \tag{5.9}
\]

where \( \mu_{i} = \mu_{i,K}^{X_{i}} \) is \( K_{i} \)-atomic, \( i \in \mathbb{N} \cap [1, \eta] \). Set

\[
s_{i,-k} = \int_{0}^{\infty} \frac{1}{t^{k}} d\mu_{i}(t), \quad k \in \mathbb{N} \cap [1, \kappa + 1], i \in \mathbb{N} \cap [1, \eta].
\]

Then, for every \( i \in \mathbb{N} \cap [1, \eta] \) the sequence \( (s_{i,-\kappa-1}, \ldots, s_{i,p-1}) \) is a moment sequence on \( (0, \infty) \) of index \( K_{i} \). The application of Theorem 4.7 gives \( (5.1) \) and \( (5.2) \). The formulas \( (5.3) \) and \( (5.4) \) follow from the definition of \( s_{i,k} \). It is enough to show that \( (5.5) \) holds. Let \( \xi_{i,1}, \ldots, \xi_{i,K_{i}} \subset (0, \infty) \) be atoms of \( \mu_{i} \) written in the increasing order. Then \( \sup_{i \in \mathbb{N} \cap [1, \eta]} (\sup_{\text{supp} \mu_{i}}) = \sup_{i \in \mathbb{N} \cap [1, \eta]} \xi_{i,K_{i}} \). Next, observe that

\[
\sup_{i \in \mathbb{N} \cap [1, \eta]} \xi_{i,K_{i}} \leq \sup_{i \in \mathbb{N} \cap [1, \eta]} (\xi_{i} + \ldots + \xi_{i,K_{i}}) \leq \sup_{i \in \mathbb{N} \cap [1, \eta]} K_{i} \xi_{i,K_{i}} \leq \left\lfloor \frac{p + \kappa + 1}{2} \right\rfloor \sup_{i \in \mathbb{N} \cap [1, \eta]} \xi_{i,K_{i}}.
\]

\(^2\)Here and in the subsequent parts we follow the convention that \( \prod \emptyset = 1. \)
Hence,

\[
\sup_{i \in \mathbb{N} \cap [1, \eta]} \xi_{i,K_i} < \infty \iff \sup_{i \in \mathbb{N} \cap [1, \eta]} (\xi_{i,1} + \ldots + \xi_{i,K_i}) < \infty. \quad (5.10)
\]

Define \( Q_i(t) = \prod_{j=1}^{K_i} (t - \xi_{i,j}) \). Using Vieta’s formula (see [19]), we get

\[
\xi_{i,1} + \ldots + \xi_{i,K_i} = -\frac{1}{K_i} \frac{Q_i^{(K_i-1)}(0)}{Q_i^{(K_i)}(0)}, \quad i \in \mathbb{N} \cap [1, \eta]
\]

But

\[
-\frac{1}{\lfloor \frac{p+\kappa+1}{2} \rfloor} \frac{Q_i^{(K_i-1)}(0)}{Q_i^{(K_i)}(0)} \leq -\frac{1}{K_i} \frac{Q_i^{(K_i-1)}(0)}{Q_i^{(K_i)}(0)} \leq -\frac{Q_i^{(K_i-1)}(0)}{Q_i^{(K_i)}(0)}, \quad i \in \mathbb{N} \cap [1, \eta]
\]

so

\[
\sup_{i \in \mathbb{N} \cap [1, \eta]} (\xi_{i,1} + \ldots + \xi_{i,K_i}) < \infty \iff \sup_{i \in \mathbb{N} \cap [1, \eta]} Q_i^{(K_i-1)}(0) - Q_i^{(K_i)}(0) < \infty. \quad (5.11)
\]

(ii)\(\implies\)(i). It follows from (5.1), (5.2) and Theorem 4.9 that for every \( i \in \mathbb{N} \cap [1, \eta] \) the sequence \( s_i = (s_i, -\kappa_{-1}, \ldots, s_i, p-1) \) is a positive sequence on \((0, \infty)\) of index \( K_i \). Let \( \nu_i \in \mathcal{M}_\infty(s_i) \) be \( K_i \)-atomic. Set \( \mu_i = \nu_i^{p+1} \). It can be easily seen that, by (5.3) and (5.4), conditions (5.6), (5.7) and (5.8) are satisfied. Proceeding in the similar way as to obtain (5.10) and (5.11) we get (5.9). Using [8, Lemma 4.7] we obtain (i), which completes the proof. \(\square\)

**Remark 5.2.** Under the assumptions of Theorem 5.1, if \( \lambda \) has a subnormal completion \( S_{\lambda'} \), then for every \( i \in \mathbb{N} \cap [1, \eta] \) the sequence \( s_i = (s_i, -\kappa_{-1}, \ldots, s_i, p-1) \) is a moment sequence on \((0, \infty)\). By Lemma 5.7, this sequence has always a representing measure, which has at most \( \left\lfloor \frac{p+\kappa+1}{2} \right\rfloor \) atoms, so after changing weights \( \lambda'_{i,k} \) \( (i \in \mathbb{N} \cap [1, \eta], \ k \in \mathbb{N} \cap [1, \eta]) \) we can obtain another subnormal completion \( S_{\lambda''} \) such that all measures \( \mu_i^{\lambda''} \) \( (i \in \mathbb{N} \cap [1, \eta]) \) are finitely atomic with at most \( \left\lfloor \frac{p+\kappa+1}{2} \right\rfloor \) atoms. Therefore, Theorem 5.1 provides a full description of sequences \( \lambda \) having subnormal completion.

In the next example we show that [8, Theorem 4.1] can be derived from Theorem 5.1

**Example 5.3.** We investigate the case \( \kappa = 1, \eta \in \overline{\mathbb{N}}_2, p = 2 \) and \( K_i = 2 \) for \( i \in \mathbb{N} \cap [1, \eta] \). Since for any \( t \in (0, \infty) \) the sequence \((1, t)\) is a moment sequence on \((0, \infty)\), we have to check only (5.3) and (5.4). Note that, in our setting, the condition (5.2) does not hold for any \( i \in \mathbb{N} \cap [1, \eta] \). First, we compute \( t_{\infty}((1, \lambda_{i,2}^2)) \) from [13, Theorem IV.1.1], [13, Section III §5.3] and (4.1)–(4.2)
we know that $t_\infty((1, \lambda^2_{i,2})) = t_{a,b}((1, \lambda^2_{i,2})) = -\frac{P_i(0)}{Q_i(0)}$, where

$$Q_i(t) = \det \begin{bmatrix} 1 & 1 \\ \lambda^2_{i,2} & t \end{bmatrix}, \quad i \in \mathbb{N} \cap [1, \eta],$$

$$P_i(0) = \sigma \left( \frac{Q_i(t) - Q_i(0)}{t} \right), \quad i \in \mathbb{N} \cap [1, \eta].$$

By simple calculations we get $t_\infty((1, \lambda^2_{i,2})) = \frac{1}{\lambda^2_{i,2}}, \ i \in \mathbb{N} \cap [1, \eta]$. Next, assuming $s_{i,-1} \in (\frac{1}{\lambda^2_{i,2}}, \infty)$ is chosen, we compute $t_\infty((s_{i,-1}, 1, \lambda^2_{i,2}))$. Again, from [13, Theorem IV.1.1], [13, Section III.§5.3] and (4.1)–(4.2), it follows that $t_{a,b}((s_{i,-1}, 1, \lambda^2_{i,2}))$ depends on $b$ (but not on $a$). Hence, by the fact that $t_{a,b}(\cdot)$ decreases when $b$ increases, we have that for every $i \in \mathbb{N} \cap [1, \eta],$

$$t_\infty((s_{i,-1}, 1, \lambda^2_{i,2})) = \lim_{b \to \infty} t_{a,b}((s_{i,-1}, 1, \lambda^2_{i,2})) = \lim_{b \to \infty} \frac{P^{i,b}_i(0)}{Q^{i,b}_i(0)},$$

where

$$Q^{i,b}_i(t) = (b-t) \det \begin{bmatrix} b s_{i,-1} - 1 & 1 \\ b - \lambda^2_{i,2} & t \end{bmatrix}, \quad i \in \mathbb{N} \cap [1, \eta], \ a, b \in (0, \infty), \ a < b$$

$$P^{i,b}_i(0) = \sigma \left( \frac{Q^{i,b}_i(t) - Q^{i,b}_i(0)}{t} \right), \quad i \in \mathbb{N} \cap [1, \eta], \ a, b \in (0, \infty), \ a < b.$$

By simple calculations we get

$$t_\infty((s_{i,-1}, 1, \lambda^2_{i,2})) = \lim_{b \to \infty} \frac{b^2 s_{i,-1} - b s_{i,-1} + s_{i,-1} \lambda^2_{i,2}}{b^2 - b \lambda^2_{i,2}} = s^2_{i,-1} > 0, \quad i \in \mathbb{N} \cap [1, \eta].$$

Hence, $s_{i,-1} \in (\frac{1}{\lambda^2_{i,2}}, \infty)$ and $s_{i,-2} \in (s^2_{i,-1}, \infty)$. Set $r_1 = \lambda^2_{i,2} s_{i,-1} \in (1, \infty)$ and $\vartheta_1 = \frac{s_{i,-2}}{s^2_{i,-1}} \in (1, \infty), i \in \mathbb{N} \cap [1, \eta]$. Then (5.3) and (5.4) take the form

$$\sum_{i=1}^n \lambda_{i,1}^2 r_i = 1$$

$$\sum_{i=1}^n \lambda_{i,2}^2 r_i \vartheta_1^2 \leq \frac{1}{\lambda_0^2},$$

as in [1] Eq. (4.2) and (4.3). From Theorem 4.9 we get that the polynomials $Q_i$ in Theorem 5.1 are given by the formula:

$$Q_i(t) = \det \begin{bmatrix} s_{i,-2} & s_{i,-1} & 1 \\ s_{i,-1} & 1 & t \\ 1 & \lambda^2_{i,2} & t^2 \end{bmatrix}, \quad i \in \mathbb{N} \cap [1, \eta].$$
Then, we have

\[ Q_i'(0) = \det\begin{bmatrix} s_{i-2} & s_{i-1} & 0 \\ s_{i-1} & 1 & 1 \\ 1 & \lambda^2_{i,2} & 0 \end{bmatrix} = -\lambda^2_{i,2}s_{i-2} + s_{i-1}, \quad i \in \mathbb{N} \cap [1, \eta]. \]

and

\[ Q_i''(0) = \det\begin{bmatrix} s_{i-2} & s_{i-1} & 0 \\ s_{i-1} & 1 & 1 \\ 1 & \lambda^2_{i,2} & 1 \end{bmatrix} = s_{i-2} - s_{i-1}^2, \quad i \in \mathbb{N} \cap [1, \eta]. \]

Therefore,

\[-\frac{Q_i'(0)}{Q_i''(0)} = \frac{r_i^2\vartheta_i - r_{i-1}^2}{\lambda^2_{i,2}} = \lambda^2_{i,2} \vartheta_i - 1 - \frac{1}{\vartheta_i - 1}, \quad i \in \mathbb{N} \cap [1, \eta].\]

Hence, by the above equality, \([5.5]\) is equivalent to \([6, \text{Eq. (4.1)}]\). Consequently, \([6, \text{Theorem 4.1}]\) can be recovered from our result.

Let us make one more remark about \([5.5]\). Obviously, this condition matters only when \(\eta = \infty\). When we take \(K_i = \left\lceil \frac{p + \kappa + 1}{2} \right\rceil\), then polynomial \(Q_i\) is given as in Theorem \([4.9]\). However, when \(p + \kappa\) is even, then it can occur \(K_i = \left\lfloor \frac{p + \kappa + 2}{2} \right\rfloor\), so that Theorem \([4.9]\) gives no help in finding appropriate polynomial \(Q_i\). Of course, if there are only finitely many indices \(i \in \mathbb{N} \cap [1, \eta]\), for which \(K_i = \left\lfloor \frac{p + \kappa + 2}{2} \right\rfloor\), then we can simply skip these indices when checking \([5.5]\). The problem appears when there are infinitely many such indices. Observe that

\[
\sup_{i \in \mathbb{N} \cap [1, \eta]} \frac{Q_i^{(K_i-1)}(0)}{Q_i^{(K_i)}(0)} = \max \left\{ \sup_{i \in I_1} \frac{Q_i^{(K_i-1)}(0)}{Q_i^{(K_i)}(0)}, \sup_{i \in I_2} \frac{Q_i^{(K_i-1)}(0)}{Q_i^{(K_i)}(0)} \right\},
\]

where \(I_1 = \{ i \in \mathbb{N} \cap [1, \eta] : K_i = \left\lfloor \frac{p + \kappa + 1}{2} \right\rfloor \}\) and \(I_2 = \{ i \in \mathbb{N} \cap [1, \eta] : K_i = \left\lfloor \frac{p + \kappa + 2}{2} \right\rfloor \}\). For \(i \in I_2\) set \(s_i := (s_{i, \kappa-1}, \ldots, s_{i,p-1})\) and denote by \(\Phi_i\) the bijection given by Corollary \([4.11]\) for the sequence \(s_i\). If \(i \in I_2\), then by \(Q_{i,t}\) denote the polynomial with roots at atoms of the measure \(\Phi_{i,t}^{-1}(t)\), \(t \in (t_{\infty}(s_i), \infty)\). By Corollary \([4.11]\) the polynomials \(Q_i, i \in I_2\), are of the form \(Q_i = Q_{i, s_i, \ldots, s_{i,p-2}}\) for some \(s_{i, \kappa-2} \in (t_{\infty}(s_i), \infty)\). It is easy to see that \(\sup_{i \in \mathbb{N} \cap [1, \eta]} \frac{Q_i^{(K_i-1)}(0)}{Q_i^{(K_i)}(0)} < \infty\) if and only if \(\max\{S_1, S_2\} < \infty\), where

\[
S_1 = \sup_{i \in I_1} \frac{Q_i^{(K_i-1)}(0)}{Q_i^{(K_i)}(0)}
\]

\[
S_2 = \sup_{i \in I_2} \inf_{s_{i, \kappa-2} \in (t_{\infty}(s_i), \infty)} \left\{ \frac{(Q_{i, s_i, \ldots, s_{i,p-2}})^{(K_i-1)}(0)}{Q_i^{(K_i)}(0)} : s_{i, \kappa-2} \in (t_{\infty}(s_i), \infty) \right\},
\]

which gives us the convenient way of checking \([5.5]\) in any case.
Let $\kappa \in \mathbb{N}$, $\eta \in \mathbb{N}_2$, $p \in \mathbb{N}_1$. Assume $\lambda = \{\lambda_{-k}\}_{k=0}^{\kappa-1} \cup \{\lambda_{i,j}\}_{i,j=1}^{\eta,p} \subset (0, \infty)$ admits a subnormal completion. For $M \in (0, \infty)$ denote by $\mathcal{U}_M^\lambda$ the set of all sequences $(\mu_i)_{i=1}^{\eta}$ of Borel measures on $[0, M]$ satisfying

\[
\prod_{j=2}^{n+1} \lambda_{i,j}^2 \int_0^M t^n \mathrm{d}\mu_i(t), \quad i \in \mathbb{N} \cap [1, \eta], \ n \in \mathbb{N} \cap [0, p-1], \quad (5.12)
\]

\[
\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^M \frac{1}{t^{k+1}} \mathrm{d}\mu_i(t) = \frac{1}{\prod_{j=0}^{k-1} \lambda_{-j}^2}, \quad k \in \mathbb{N} \cap [0, \kappa-1], \quad (5.13)
\]

\[
\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^M \frac{1}{t^{k+1}} \mathrm{d}\mu_i(t) \leq \frac{1}{\prod_{j=0}^{\kappa-1} \lambda_{-j}^2}, \quad (5.14)
\]

Obviously, $\mathcal{U}_M^\lambda \subset \mathcal{P}([0, M])^\eta$, where $\mathcal{P}([0, M])$ stands for the set of all Borel probability measures on $[0, M]$. By Banach-Alaoglu and Riesz representation theorems we see that $\mathcal{P}([0, M])$ equipped with the weak topology is a compact space, hence, by Tychonoff’s theorem, so is $\mathcal{P}([0, M])^\eta$; moreover, it is metrizable (see [2]). We will prove that $\mathcal{U}_M^\lambda$ is a compact subset of $\mathcal{P}([0, M])$.

Before that, we need the following lemmas. For the sake of completeness we include the proof of Lemma 5.4; for the proof of Lemma 5.5 consult [2, Theorem I.5.3].

**Lemma 5.4.** Suppose $M \in (0, \infty)$ and $(\mu_n)_{n=0}^\infty \subset \mathcal{P}([0, M])$ is such that $\mu_n \rightharpoonup \mu \in \mathcal{P}([0, M])$ weakly. Let $k \in \mathbb{N}_1$ and assume that

\[
C := \sup_{n \in \mathbb{N}} \int_0^M \frac{1}{t^k} \mathrm{d}\mu_n(t) < \infty.
\]

Then $\frac{1}{t^k} \in L^1(\mu)$ and $\int_0^M \frac{1}{t^k} \mathrm{d}\mu(t) \leq C$.

**Proof.** For $m \in \mathbb{N}_1$ define a (continuous) function $f_m : [0, M] \to [0, \infty)$ by the formula

\[
f_m(t) = \min\{\frac{1}{t^k}, m^k\}, \quad t \in [0, M].
\]

Observe that $f_m \leq f_{m+1}$, $m \in \mathbb{N}_1$, and $f_m \xrightarrow{m \to \infty} \frac{1}{t^k}$ pointwise. Since

\[
\int_0^M f_m \mathrm{d}\mu_n \leq \int_0^M \frac{1}{t^k} \mathrm{d}\mu_n(t) \leq C, \quad m \in \mathbb{N}_1,
\]

and

\[
\int_0^M f_m \mathrm{d}\mu_n \xrightarrow{n \to \infty} \int_0^M f_m \mathrm{d}\mu,
\]

it follows that $\int_0^M f_m \mathrm{d}\mu \leq C$. By the Lebesgue’s monotone convergence theorem we obtain

\[
\int_0^M f_m \mathrm{d}\mu \xrightarrow{m \to \infty} \int_0^M \frac{1}{t^k} \mathrm{d}\mu(t).
\]
Hence, \( \int_0^M \frac{1}{t} \, d\mu(t) \leq C. \)

**Lemma 5.5.** Let \( M \in (0, \infty), k \in \mathbb{N}_1. \) Let \( (\mu_n)_{n=0}^{\infty} \subset \mathcal{P}([0, M]) \) be such that \( \mu_n \to \mu \in \mathcal{P}([0, M]) \) weakly. Assume that

\[
\sup_{n \in \mathbb{N}} \int_0^M \frac{1}{t^{k+1}} \, d\mu_n(t) < \infty.
\]

Then \( \frac{1}{t} \in L^1(\mu) \) and \( \int_0^M \frac{1}{t^2} \, d\mu_n(t) \to \int_0^M \frac{1}{t^2} \, d\mu(t). \)

Now we are in the position to prove that the sets \( \mathcal{U}_\lambda^M \) are compact in the weak topology.

**Theorem 5.6.** Let \( \kappa \in \mathbb{N}, \eta \in \mathbb{N}_2, p \in \mathbb{N}_1. \) Assume \( \lambda = \{\lambda_{-k}\}_{k=0}^{\kappa-1} \cup \{\lambda_{i,j}\}_{i,j=1}^p \subset (0, \infty) \) admits a subnormal completion. Then for every \( M \in (0, \infty) \) the set \( \mathcal{U}_\lambda^M \) is weakly compact.

**Proof.** Fix \( M \in (0, \infty). \) Since \( \mathcal{P}([0, M])^\eta \) is weakly compact, it is enough to prove that \( \mathcal{U}_\lambda^M \) is weakly closed in \( \mathcal{P}([0, M])^\eta. \) Let \( ((\mu_i^{(n)})_{n=0}^{\infty})_{i=1}^{\eta} \subset \mathcal{U}_\lambda^M \) be such that \( (\mu_i^{(n)})_{n=1}^{\infty} \to (\mu_i)_{n=1}^{\infty} \in \mathcal{P}([0, M])^\eta \) weakly. It is easy to see that (5.12) holds for \( (\mu_i)_{i=1}^\eta \). For \( n \in \mathbb{N} \) define a measure \( \tau_n \in \mathcal{P}([0, M]): \)

\[
\tau_n(A) = \frac{1}{\sum_{i=1}^\eta \lambda_{i,1}^2} \sum_{i=1}^\eta \lambda_{i,1}^2 \mu_i^{(n)}(A), \quad A \in \mathcal{B}([0, M]),
\]

and a measure \( \tau \in \mathcal{P}([0, M]): \)

\[
\tau(A) = \frac{1}{\sum_{i=1}^\eta \lambda_{i,1}^2} \sum_{i=1}^\eta \lambda_{i,1}^2 \mu_i(A), \quad A \in \mathcal{B}([0, M]).
\]

We will prove that \( \tau_n \to \tau \) weakly. Suppose \( f : [0, M] \to \mathbb{R} \) is continuous. Note that

\[
\left| \lambda_{i,1}^2 \int_0^M f \, d\mu_i^{(n)} \right| \leq \lambda_{i,1}^2 \sup_{[0, M]} |f|, \quad n \in \mathbb{N}, \ i \in \mathbb{N} \cap [1, \eta]
\]

and \( \sum_{i=1}^\eta \lambda_{i,1}^2 \sup_{[0, M]} |f| < \infty. \) Since

\[
\int_0^M f \, d\mu_i^{(n)} \xrightarrow{n \to \infty} \int_0^M f \, d\mu_i, \quad i \in \mathbb{N} \cap [1, \eta],
\]

by the Lebesgue’s dominated convergence theorem we have

\[
\int_0^M f \, d\tau_n = \frac{1}{\sum_{i=1}^\eta \lambda_{i,1}^2} \sum_{i=1}^\eta \lambda_{i,1}^2 \int_0^M f \, d\mu_i^{(n)} \xrightarrow{n \to \infty} \frac{1}{\sum_{i=1}^\eta \lambda_{i,1}^2} \sum_{i=1}^\eta \lambda_{i,1}^2 \int_0^M f \, d\mu_i = \int_0^M f \, d\tau.
\]
From (5.13) and (5.14), by Lemma 5.4 it follows that
\[
\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^M \frac{1}{t^{m+1}} \, dt(t) \leq \frac{1}{\prod_{j=0}^{m-1} \lambda_{j}^2}, \quad m \in \mathbb{N} \cap [0, \kappa].
\]
Now, we can apply Lemma 5.5 to obtain that for every \( m \in \mathbb{N} \cap [0, \kappa - 1] \) the following equality holds:
\[
\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^M \frac{1}{t^{m+1}} \, d\mu_i = \sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^M \frac{1}{t^{m+1}} \, dt(t) = \frac{1}{\prod_{j=0}^{m-1} \lambda_{j}^2}.
\]
This completes the proof.

As one can observe, Theorem 5.1 gives the solution for the subnormal completion problem only in case \( \kappa \in \mathbb{N} \). The case \( \kappa = \infty \) remains unsolved. In \cite{R} Problem 4.10 there was posed a question whether existence of subnormal completion for every \( \kappa \in \mathbb{N} \) implies existence of such a completion for \( \kappa = \infty \). The next theorem gives a partial affirmative answer to this question.

**Theorem 5.7.** Let \( \eta \in \mathbb{N}_2 \) and \( p \in \mathbb{N}_1 \). Let \( \lambda = \{ \lambda_{-j} \}_{j=1}^{\infty} \cup \{ \lambda_{i,j} \}_{i,j=1}^{p} \subset (0, \infty) \). Set \( \lambda_{\kappa} = \{ \lambda_{-j} \}_{j=0}^{\kappa-1} \cup \{ \lambda_{i,j} \}_{i,j=1}^{p} \), \( \kappa \in \mathbb{N}_1 \), and \( \lambda_0 = \{ \lambda_{i,j} \}_{i,j=1}^{p} \). The following conditions are equivalent:

(i) \( \lambda \) has a subnormal completion,

(ii) there exists a sequence \( (S_{\lambda_\kappa})_{\kappa=0}^{\infty} \) of weighted shifts such that \( S_{\lambda_\kappa} \) is a subnormal completion of \( \lambda_\kappa \) for every \( \kappa \in \mathbb{N} \) and \( \sup_{\kappa \in \mathbb{N}} \| S_{\lambda_\kappa} \| < \infty \).

**Proof.** (i)\( \Rightarrow \) (ii). If \( S_{\lambda_\kappa} \) is a subnormal completion of \( \lambda \), then \( S_{\lambda_\kappa} \mid \mathcal{L}(V_{\eta,\kappa}) \) is a subnormal completion of \( \lambda_\kappa \) for every \( \kappa \in \mathbb{N} \); moreover \( \sup_{\kappa \in \mathbb{N}} \| S_{\lambda_\kappa} \mid \mathcal{L}(V_{\eta,\kappa}) \| \leq \| S_{\lambda} \| \).

(ii)\( \Rightarrow \) (i). From (ii) and \cite{R} Lemma 4.7 we know that \( \mathcal{U}_{\lambda_\kappa}^{M} \neq \emptyset \) for every \( \kappa \in \mathbb{N} \), where \( M = \sup_{\kappa \in \mathbb{N}} \| S_{\lambda_\kappa} \| \). By Theorem 5.6, \( \mathcal{U}_{\lambda_\kappa}^{M} \) is compact. Since \( \mathcal{U}_{\lambda_{\kappa+1}}^{M} \subset \mathcal{U}_{\lambda_\kappa}^{M} \), by Cantor’s intersection theorem we obtain that
\[
\bigcap_{\kappa=0}^{\infty} \mathcal{U}_{\lambda_\kappa}^{M} \neq \emptyset.
\]

Let \( (\mu_i)_{i=1}^{\eta} \subset \bigcap_{\kappa=0}^{\infty} \mathcal{U}_{\lambda_\kappa}^{M} \). Then
\[
\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^M \frac{1}{t^{\kappa+1}} \, d\mu_i(t) = \frac{1}{\prod_{j=0}^{\kappa-1} \lambda_{j}^2}, \quad \kappa \in \mathbb{N}.
\]
Setting
\begin{align*}
\lambda'_{-j} &= \lambda_{-j}, \quad j \in \mathbb{N}, \\
\lambda'_{i,j} &= \lambda_{i,j}, \quad i \in \mathbb{N} \cap [1, \eta], \ j \in \mathbb{N} \cap [1, p], \\
\lambda'_{i,j} &= \sqrt{\int_0^M \frac{1}{t^{\kappa+1}} \, d\mu_i(t)} \int_0^M \frac{1}{t^{\kappa+2}} \, d\mu_i(t), \quad i \in \mathbb{N} \cap [1, \eta], \ j \in \mathbb{N}_{p+1},
\end{align*}

(5.15) (5.16) (5.17)
and using [9, Corollary 6.2.2] (as well as [9, Procedure 6.3.1]) we obtain a subnormal completion $S_\lambda$ of $\lambda$. This completes the proof.

In the hypothesis of [6, Problem 4.10] there is no assumption of uniform boundedness as in Theorem 5.7(ii). At this point, we do not know whether this assumption is superfluous or not, but it leads us to another natural problem.

**Problem 5.8.** Let $\kappa \in \mathbb{N}$, $\eta \in \mathbb{N}_2$, $p \in \mathbb{N}_1$. Assume $\lambda = \{\lambda_{-j}\}_{j=0}^{\kappa-1} \cup \{\lambda_{i,j}\}_{i,j=1}^{n,p} \subset (0, \infty)$ admits a subnormal completion. Compute

$$\inf\{||S_\lambda|| : S_\lambda \text{ is a subnormal completion of } \lambda\}.$$

### 6. Completely hyperexpansive completions of weighted shifts on directed trees.

It is the continuation of considerations from Section 5, but now we are interested in completely hyperexpansive completions. Several partial results in this area can be found in [14]. Recall from [1, Remark 1] that the sequence $(c_n)_{n=0}^\infty \subset \mathbb{R}$ is completely alternating if and only if there exists a Borel measure $\tau : B([0,1]) \to [0,\infty)$ such that

$$c_n = c_0 + \int_{[0,1]} (1 + t + \ldots + t^{n-1}) \, d\tau(t), \quad n \in \mathbb{N}_1. \quad (6.1)$$

Moreover, it can be easily seen that the measure $\tau$ satisfying (6.1) is unique; we call it a representing measure of $(c_n)_{n=0}^\infty$. If $T = (V,E)$ is a directed tree, then [3, Lemma 7.1.4] states that a bounded weighted shift $S_\lambda \in \ell^2(V)$ is completely hyperexpansive if and only if for every $v \in V$ the sequence $(||S_\lambda^v e_v||^2_{n=0})_{n=0}^\infty$ is completely alternating; the representing measure for the sequence $(||S_\lambda^v e_v||^2_{n=0})_{n=0}^\infty$ will be denoted by $\tau_\lambda^v$. Exploiting once again our results on backward extensions we obtain the following counterpart of Theorem 5.1.

**Theorem 6.1.** Let $\kappa \in \mathbb{N}$, $\eta \in \mathbb{N}_2$, $p \in \mathbb{N}_2$. Assume $\lambda = \{\lambda_{-k}\}_{k=0}^{\kappa-1} \cup \{\lambda_{i,j}\}_{i=1,j=1}^{n,p} \subset (0, \infty)$ is such that $\lambda_{i,j} > 1$ for $i \in \mathbb{N} \cap [1,\eta]$, $j \in \mathbb{N} \cap [2,p]$ and

$$\sum_{i=1}^\eta \lambda_{i,1}^2 < \infty.$$

Suppose $\{K_i\}_{1 \leq i \leq \eta} \subset \mathbb{Q} \cap \left[\frac{1}{2}, \frac{p+k}{2}\right]$ satisfies $2K_i \in \mathbb{N}$, $i \in \mathbb{N} \cap [1,\eta]$. Then the following conditions are equivalent:

(i) there exists a completely hyperexpansive completion $S_\lambda \in \mathcal{B}(\ell^2(V_{\kappa,\eta}))$ of $\lambda$ such that $\tau_\lambda^v$ is of index $K_i$ for $i \in \mathbb{N} \cap [1,\eta],$

(ii) for every $i \in \mathbb{N} \cap [1,\eta]$ there exists a sequence $\{s_{i-k}\}_{k=1}^{\kappa+1} \subset (0, \infty)$ such that for every $i \in \mathbb{N} \cap [1,\eta]$

$$s_{i,k} \in (t_1((s_{i,k+1}, \ldots, s_{i,p-1})), +\infty), \quad k \in \mathbb{Z} \cap [p-N_i-2, p-3]. \quad (6.2)$$

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\[ s_{i,k} = t_1((s_{i,k+1}, \ldots, s_{k+N_i+1})), \quad k \in \mathbb{Z} \cap [-\kappa - 1, p - N_i - 3], \quad \text{for} \quad i \in \mathbb{N} \sqcup [1, \eta]. \tag{6.3} \]

\[ \sup_{i \in \mathbb{N} \cup [1, \eta]} s_{i,0} < \infty. \tag{6.4} \]

and 4

(a) if \( \kappa = 0 \):

\[ 1 + \sum_{i=1}^{n} \lambda_{i,1}^2 s_{i,-1} \leq \sum_{i=1}^{n} \lambda_{i,1}^2, \tag{6.5} \]

(b) if \( \kappa > 0 \):

\[ 1 + \sum_{i=1}^{n} \lambda_{i,1}^2 s_{i,-1} = \sum_{i=1}^{n} \lambda_{i,1}^2, \tag{6.6} \]

\[ 1 + \prod_{j=0}^{k-1} \lambda_{i,j}^2 \sum_{i=1}^{n} \lambda_{i,1}^2 s_{i,-k-1} = \lambda_{i,k+1}^2, \quad k \in \mathbb{N} \cap [1, \kappa - 1], \tag{6.7} \]

\[ 1 + \prod_{j=0}^{\kappa-1} \lambda_{i,j}^2 \sum_{i=1}^{n} \lambda_{i,1}^2 s_{i,-\kappa-1} \leq \lambda_{i,-\kappa+1}^2, \tag{6.8} \]

where

\[ s_{i,k} = \prod_{j=2}^{k+2} \lambda_{i,j}^2 - \prod_{j=2}^{k+1} \lambda_{i,j}^2, \quad k \in \mathbb{N} \cap [0, p - 2]. \]

**Proof.** (i)⇒(ii). By (i) and [14, Proposition 2.8] there exist Borel measures \( \{\tau_i\}_{i=1}^{\eta} \) on \((0, 1]\) such that \( \tau_{i,1} = \tau_i \) is of index \( K_i \) for \( i \in \mathbb{N} \cap [1, \eta] \) and (6.4)–(6.8) hold with

\[ s_{i,-k} = \int_{(0,1]} \frac{1}{t^k} \, d\tau_i(t), \quad i \in \mathbb{N} \cap [1, \eta], \quad k \in \mathbb{N} \cap [0, \kappa + 1]. \]

Then for every \( i \in \mathbb{N} \cap [1, \eta] \) the sequence \( (s_{i,-\kappa-1}, \ldots, s_{i,p-2}) \) is a moment sequence on \((0, 1]\) of index \( K_i \), so Theorem 4.8 gives (6.2) and (6.3).

(ii)⇒(i). From (6.2) and (6.3), by Theorem 4.10, it follows that for every \( i \in \mathbb{N} \cap [1, \eta] \) the sequence \( s_i = (s_{i,-\kappa-1}, \ldots, s_{i,p-2}) \) is a positive sequence on \((0, 1]\) satisfying \( \text{ind}_1(s_i) = K_i \). Let \( \nu_i \in \mathcal{M}_1(s_i) \) be such that \( \text{ind}_1(\nu_i) = K_i \), \( i \in \mathbb{N} \cap [1, \eta] \). Set \( \tau_i = t^{\kappa+1} \, d\nu_i \), \( i \in \mathbb{N} \cap [1, \eta] \). Then for every \( i \in \mathbb{N} \cap [1, \eta] \) the measure \( \tau_i \) also satisfies \( \text{ind}_1(\tau_i) = K_i \). By [14, Proposition 2.8], \( \lambda \) has a completely hyperexpansive completion \( S_{\lambda} \) such that \( \tau_{i,1}^{\lambda} = \tau_i \) is of index \( K_i \) for every \( i \in \mathbb{N} \cap [1, \eta] \). \( \square \)

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4Recall the convention: \( \prod \emptyset = 1 \).
Again, when solving completely hyperexpansive completion problem we can always look for measures $\tau_{i,1}^k$ ($i \in \mathbb{N} \cap [1, \eta]$) of index at most $\frac{\eta + \kappa}{2}$ (cf. Remark 5.2).

Opposed to subnormal completions, the case $\kappa = \infty$ is trivial. Since by [9, Corollary 7.2.3] the only completely hyperexpansive weighted shifts on $T_{\eta,\infty}$ are isometries, the existence of completely hyperexpansive completion is guaranteed by the following three equalities

\[
\lambda - k = 1, \quad k \in \mathbb{N} \\
\lambda_{i,j} = 1, \quad i \in \mathbb{N} \cap [1, \eta], \quad j \in \mathbb{N}_2 \\
\sum_{i=1}^{\eta} \lambda_{i,1}^2 = 1.
\]

Declarations

Conflict of interests The author declares that he has no conflict of interests.

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