A new solution of thin elastic rod by dynamic analogy

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Abstract. Inspired by Kirchhoff dynamic analogy, we write the Kirchhoff equation of thin elastic rod in form of curvature expression. Compared it with nonlinear Schrödinger equation, we extend a Jacobi elliptic function analogy solution to elastic rod equation and give a new alternative way to solve the Kirchhoff equation.

1. Introduction
Kirchhoff’s kinetic analogy theory utilizes the similarity between the equilibrium differential equation of elastic rod and the fixed-point rotational differential equation of rigid body, infuses the concept and research method of dynamics into the statics of elastic rod, and lays the theoretical foundation of the statics of elastic rod. In recent years, elastic thin rod as a mechanical model of DNA molecule has attracted new attention [1-4].

Due to the extremely slender and soft material properties of the description object, the elastic thin rod mechanics is completely different from the classical elastic rod mechanics, showing more complicated geometric structure and nonlinearity, which brings difficulties to the solution. Most studies use numerical calculations to find numerical solution [5-6]. Shi et al [7-8] derived a class of one-dimensional fixed-state nonlinear Schrödinger equation by introducing complex vectors and complex bending moments, and gave a closed solution of the DNA molecular centerline. Xue et al. [9] further extended the results to the general case of non-circular cross sections. Wang et al. [10-12] used symmetry to derive some conserved quantities of elastic thin rods.

There has been a lot of outstanding work on the discussion of the exact solution of the nonlinear Schrödinger equation. Due to the particularity of the elastic thin rod, the Schrödinger equation of the elastic thin rod given by Shi [7-8] can not be solved by analogy method with the nonlinear Schrödinger equation directly. It is found that when the torsion of the rod is constant, the equilibrium differential equation of the elastic thin rod can be completely equivalent to the nonlinear Schrödinger equation in mathematical form, similar to the Kirchhoff kinetics, and a new class of exact solution of the elastic thin rod mechanics can be given. We call this analogy method the Schrödinger particle wave analogy.

In this paper, we use the Schrödinger particle wave analogy to give a new kind of exact solution of the elastic thin rod equation. This analogy method provides a new way to solve the elastic thin rod equation.
2. Nonlinear Kirchhoff equation of elastic thin rod

Study the elastic thin rods of length $L$ with Kirchhoff hypothesis. The inertial coordinate system $O-\xi\eta\zeta$ is established, and the unit base vector along the coordinate axis is $e_\xi$, $e_\eta$, $e_\zeta$. The radius vector of the centerline of the elastic thin rod is

$$ r = \xi(s)e_\xi + \eta(s)e_\eta + \zeta(s)e_\zeta $$

$s$ represents the arc length coordinate. The principal axis coordinate system $P-xyz$ fixed to the cross section is established, taking any point $P$ on the centerline as the origin. Where, $x,y$ axis is located on the cross section of the rod, and its base vectors $e_1$ and $e_2$ are respectively along the normal direction and the binormal direction of the centerline. They satisfying $e_1 \times e_2 = e_3$ and $e_3$ is tangential to the centerline. The centerline of rod is not stretchable, satisfying

$$ \frac{dr}{ds} = e_3 $$

The thin elastic rod can be regarded as the movement of the rod cross section along the centerline. The bending and torsional of the elastic rod can be expressed by curvature-twisting vector $\omega$ which is defined as [5]

$$ \frac{d e_i}{ds} = \omega \times e_i $$

where

$$ \omega = \kappa \sin \chi e_1 + \kappa \sin \chi e_2 + (\tau + \frac{d\chi}{ds}) e_3 $$

$\chi$ and $\tau$ are torsion and curvature of the rod elastica respectively, and $\chi$ is the twist angle of the cross section relative to the Frenet coordinate system. The equilibrium differential equation of elastic thin rod in the principal axis coordinate system is expressed as [5]

$$ \frac{d\hat{F}}{ds} + \omega \times \hat{F} + \hat{f} = 0 $$

$$ \frac{d\hat{M}}{ds} + \omega \times \hat{M} + e_3 \times \hat{F} + \hat{m} = 0 $$

Where $\hat{d}/ds$ represents the derivative with respect to the principal axis coordinate system. $\hat{F}$ and $\hat{M}$ are the principal vector and the principal moment of the internal force at the centroid of the rod section respectively. $\hat{f}$ and $\hat{m}$ are distributed force and distributed moment. To make the above equation closed, linear elastic constitutive equation is introduced

$$ M_1 = A(\omega_1 - \omega_1^0) $$

$$ M_2 = B(\omega_2 - \omega_2^0) $$

$$ M_3 = C(\omega_3 - \omega_3^0) $$

Where $A$ and $B$ are bending rigidity and $C$ is torsional rigidity. $\omega_1^0$ is intrinsic curvature-twisting vector.

3. Complex curvature differential equation expression

Assume that the distributed force is a contact force, perpendicular to the tangential direction, so tangential component of $\hat{f}$ is zero. Since there is no friction, the contact torque is $\hat{m} = 0$. The elastic rod equation (3) can be expressed as complex curvature equation [9].

$$ \frac{d^2\xi(s)}{ds^2} + id\frac{d\xi(s)}{ds} - b\xi(s) + \frac{1}{2}\left|\frac{1}{a}\right|\xi(s) - \frac{f_F}{A} \xi(s) = 0 $$

where
\[ \zeta(s) = \omega_1 = i\omega_2 \]

and
\[ a = \frac{(1+2\sigma)\omega_{30}}{1+\sigma} \]
\[ b = \frac{\omega_{10}^2 + \omega_{30}^2 + \sigma}{2} \frac{\omega_{10}^2}{1+\sigma} \]

are the integral constants, and \( \sigma \) is Poisson ratio.

\[ f_x = f_{\eta_1} + i f_{\eta_2} \]

\( \omega_{10}, \omega_{20} \) and \( \omega_{30} \) represent constant twist.

A class of nonlinear Schrödinger equation for elastic thin rod equation is given by Ref. [8, 9] but there was no further discussion, and the Euler-Lagrangian equation expressed by curvature is given. Because it contains the derivative of curvature cubed, we can not use Schrödinger equation analogy method. \( \zeta(s) \) is expressed as the following complex exponential equation
\[ \zeta(s) = \kappa(s) \exp(i\frac{\pi}{2} - \frac{as}{2}) \] (6)

Substitute it into equation (5), and we can get
\[ \frac{d^2\kappa(s)}{ds^2} - c\kappa(s) + \frac{1}{2} |\kappa'|^2 \kappa(s) - \frac{f_x}{A} \frac{\kappa(s)}{|\kappa'|} = 0 \] (7)

where
\[ c = b - \frac{1}{4} a^2 \]

It can be obtained from equation (6) as
\[ \chi(s) = -\frac{a}{2} s + \frac{\pi}{2} \]

Substitute it into the third term in equation (2), torsion can be obtained by compute
\[ \tau = \omega_{30} + \frac{a}{2} = \text{const} \] (8)

That is equation (7) representing an elastic rod with constant twist and torsion.

4. The analogy of curvature equation and Schrödinger equation

4.1. Jacobi elliptic function solution of one-dimensional nonlinear Schrödinger equation

The general form of the one-dimensional nonlinear Schrödinger equation is
\[ i \frac{\partial u}{\partial t} + \alpha \frac{\partial^2 u}{\partial x^2} + \beta |u|^2 u = 0, \quad i = \sqrt{-1} \] (9)

Where \( \alpha = \hbar/2m \) is the dispersion coefficient, and \( \beta = b/\hbar \) is the nonlinear interaction coefficient, also called Landau coefficient. Suppose the solution of the equation has the following form
\[ u(x,t) = \phi(x) \exp(iEt / \hbar) \] (10)

Where \( E \) is the energy of the wave function, and in the steady state the amplitude \( \phi(s) \) is still called the wave function. Substitute it into the NLS equation and get
\[ \frac{d^2 \phi(x)}{dx^2} = -\frac{\gamma}{\alpha} \phi(x) - \frac{\beta}{\alpha} \phi^3(x) \] (11)

where
\[ \gamma = \frac{E}{\hbar}, \quad \gamma > 0 \]
Equation (11) is called a one-dimensional stationary state nonlinear Schrödinger equation. If $\alpha > 0, \beta > 0$, the stationary state nonlinear Schrödinger equation has a Jacobi elliptic function solution.

$$\phi(x) = \pm \sqrt{\frac{2\gamma}{\beta(2-k^2)}} \text{dn} \left( \sqrt{\frac{\gamma}{\alpha(x-k^2)}}(x-x_0), k \right)$$

(12)

where $\text{dn}[\ ]$ is the third type of Jacobi elliptic function, and $k$ is the modulus of the Jacobi elliptic function.

4.2. Analogy of curvature equation and Schrödinger equation and exact solution

Ignoring the contact force, equation (7) becomes

$$\frac{d^2\kappa(s)}{ds^2} = c\kappa(s) - \frac{1}{2}\kappa^3(s)$$

(13)

In case of meeting the following conditions

$$\frac{\gamma}{\alpha} = c$$

$$\frac{\beta}{\alpha} = \frac{1}{2}$$

(14)

Equation (13) has the same mathematical form as the one-dimensional fixed-state nonlinear Schrödinger equation (11). We call equation (13) one-dimensional fixed-state nonlinear Schrödinger equation expressed by the curvature of the elastic thin rod. Similar to the Kirchhoff kinetics analogy, we give the analogy relation between the elastic thin rod equation of curvature and the one-dimensional fixed-state nonlinear Schrödinger equation.

By Schrödinger particle wave analogy, equation (13) has a Jacobi elliptic function solution:

$$\kappa(s) = \sqrt{\frac{4c}{2-k^2}} \text{dn} \left( \sqrt{\frac{c}{2-k^2}}(s-s_0), k \right)$$

(15)

According to the relationship between the Jacobi elliptic sine function $\text{sn}[\ ]$ and the elliptical Delta function $\text{dn}[\ ]$, we obtain the curvature of the elastic rod represented by the elliptical sine function:

$$\kappa(s) = \sqrt{\frac{4c}{2-k^2}} - \frac{4c}{2-k^2}k^2\text{sn}^3 \left( \sqrt{\frac{c}{2-k^2}}(x-x_0), k \right)$$

(16)

5. Conclusions

In this paper, by introducing a special representation of complex curvature, the elastic thin rod Kirchhoff equation is transformed into a curvature equation which is similar to the one-dimensional fixed-state nonlinear Schrödinger equation in mathematical form, and the analogy relation between Schrödinger equation and curvature equation of thin elastic rod.

According to the Schrödinger particle wave analogy, the Jacobi elliptic function form of the Kirchhoff elastic thin rod curvature changing with the arc coordinate is given, so that the nonlinear Schrödinger equation solution is transplanted into the elastic thin rod mechanics. Using the method of this paper, other solutions of the nonlinear Schrödinger equation can also be introduced into the elastic thin rod equation. At the same time, how to introduce the elastic thin rod equation solution into the Schrödinger equation is worthy of further study.

The significance of Schrödinger particle wave analogy is that: similar to the Kirchhoff kinetics analogy, for the different geometric images of elastic thin rod corresponding to the different quantum states of the stationary wave function by the Schrödinger particle wave analogy, and this paper provides a new way to solve the elastic thin rod equation.
References

[1] Benham C J and Mielke S P 2005 DNA mechanics[J]. Annu. Rev. Biomed. Eng. 7(1): 21-53
[2] Liu Y Z 2006 Nonlinear Mechanics of Thin Elastic Rod-Theoretical Basis of Mechanical Model of DNA[M]. Tsinghua University Press and Springer (in chinese)
[3] Liu Y Z and Sheng L W 2007 Stability and vibration of a helical rod with circular cross section in a viscous medium[J]. Chinese Physics 16(4): 891-896
[4] Xue Y and Liu Y Z 2009 Stability of a straight Kirchhoff elastic rod under the force screws[J]. Acta Physica Sinica 58(10): 6737-6742. (in chinese)
[5] Klapper I 1996 Biological Applications of the Dynamics of Twisted Elastic Rods[J]. Journal of Computational Physics 125: 325-337
[6] Huang L, Bao G W and Liu Y Z 2005 Solution of the Kirchhoff equation for thin elastic rod under bending by constraint violation correction method[J]. Acta Physica Sinica 54(6): 2457-2462. (in Chinese)
[7] Shi Y M and Hearst J E 1994 The Kirchhoff elastic rod, the nonlinear Schrödinger equation, and DNA supercoiling[J]. The Journal of Chemical Physics 101(6): 5186-5200
[8] Shi Y M, Borovik A E, Hearst J E. Elastic rod model incorporating shear and extension, generalized nonlinear Schrödinger equations, and novel closed-form solutions for supercoiled DNA[J]. The Journal of Chemical Physics 1998, 103(8):3166-3183
[9] Xue Y, Liu Y Z and Chen L Q 2004 The Schrödinger equation for a Kirchhoff elastic rod with noncircular cross section[J]. Chinese Physics 13(10): 794-797
[10] Wang P, Xue Y and Liu Y L 2012 Mei symmetry and conserved quantities in Kirchhoff thin elastic rod statics[J]. Chinese Physics B 21(7): 26-31
[11] Wang P and Xue Y 2016 Conformal invariance of Mei symmetry and conserved quantities of Lagrange equation of thin elastic rod[J]. Nonlinear Dynamics 83(4): 1815-1822
[12] Wang P, Feng H R and Lou Z M 2017 Conformal Invariance and Conserved Quantities for Lagrange Equation of Thin Elastic Rod[J]. Acta Physica Polonica A 131(2): 283-287