A BRUNN-MINKOWSKI TYPE INEQUALITY FOR FANO MANIFOLDS AND THE BANDO-MABUCHI UNIQUENESS THEOREM.

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ABSTRACT. For a metric on the anticanonical bundle, $-K_X$, of a Fano manifold $X$ we consider the volume of $X$

$$
\int_X e^{-\phi}.
$$

We prove that the logarithm of the volume is concave along bounded geodesics in the space of positively curved metrics on $-K_X$ and that the concavity is strict unless the geodesic comes from the flow of a holomorphic vector field on $X$. As a consequence we get a simplified proof of the Bando-Mabuchi uniqueness theorem for Kähler-Einstein metrics. We also prove a generalization of this theorem to 'twisted' Kähler-Einstein metrics and treat some classes of manifolds that satisfy weaker hypotheses than being Fano.

1. INTRODUCTION

Let $X$ be an $n$-dimensional projective manifold with seminegative canonical bundle and let $\Omega$ be a domain in the complex plane. We consider curves $t \to \phi_t$, with $t$ in $\Omega$, of metrics on $-K_X$ that have plurisubharmonic variation so that $i\partial\bar{\partial}_X \phi \geq 0$ (see section 2 for notational conventions). Then $\phi$ solves the homogenous Monge-Ampère equation if

$$(i\partial\bar{\partial}\phi)^{n+1} = 0.$$  

By a fundamental theorem of Chen, [10], we can for any given $\phi_0$ defined on the boundary of $\Omega$, smooth with nonnegative curvature on $X$ for $t$ fixed on $\partial\Omega$, find a solution of (1.1) with $\phi_0$ as boundary values. This solution does in general not need to be smooth (see [12]), but Chen’s theorem asserts that we can find a solution that has all mixed complex derivatives bounded, i.e.

$$
\partial\bar{\partial}_X \phi \text{ is bounded on } X \times \Omega.
$$

The solution equals the supremum (or maximum) of all subsolutions, i.e. all metrics with semipositive curvature that are dominated by $\phi_0$ on the boundary. Chen’s proof is based on some of the methods from Yau’s proof of the Calabi conjecture, so it is not so easy, but it is worth pointing out that the existence of a generalized solution that is only bounded is much easier, see section 2. On the other hand, if we assume that $\phi$ is smooth and $i\partial\bar{\partial}_X \phi > 0$ on $X$ for any $t$ fixed, then

$$(i\partial\bar{\partial}\phi)^{n+1} = nc(\phi)(i\partial\bar{\partial}\phi)^n \wedge idt \wedge d\bar{t}$$

with

$$
c(\phi) = \frac{\partial^2 \phi}{\partial t \partial \bar{t}} - |\partial_t \partial_{\bar{t}}(i\partial\bar{\partial}_X \phi)|^2.
$$

where the norm in the last term is the norm with respect to the Kähler metric $i\partial\bar{\partial}_X \phi$. Thus equation 1.1 is then equivalent to $c(\phi) = 0$. 

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The case when $\Omega = \{ t; 0 < \text{Re} t < 1 \}$ is a strip is of particular interest. If the boundary data are also independent of $\text{Im} t$ the solution to 1.1 has a similar invariance property. A famous observation of Semmes, [19] and Donaldson, [13] is that the equation $c(\phi) = 0$ then is the equation for a geodesic in the space of Kähler potentials. Chen’s theorem then almost implies that any two points in the space of Kähler potentials can be joined by a geodesic, the proviso being that we might not be able to keep smoothness or strict positivity along all of the curve. This problem causes some difficulties in applications, one of which we will address in this paper.

The next theorem is a direct consequence of the results in [7].

**Theorem 1.1.** Assume that $-K_X \geq 0$ and let let $\phi_t$ be a curve of metrics on $-K_X$ such that

$$i\partial \bar{\partial} \phi_t \geq 0$$

in the sense of currents. Then

$$F(t) := -\log \int_X e^{-\phi_t}$$

is subharmonic in $\Omega$. In particular, if $\phi_t$ does not depend on the imaginary part of $t$, $F$ is convex.

Here we interpret the integral over $X$ in the following way. For any choice of local coordinates $z^j$ in some covering of $X$ by coordinate neighbourhoods $U_j$, the metric $\phi_t$ is represented by a local function $\phi^j_t$. The volume form

$$c_n e^{-\phi^j_t} dz^j \wedge d\bar{z}^j,$$

where $c_n = i^{n^2}$ is a unimodular constant chosen to make the form positive, is independent of the choice of local coordinates. We denote this volume form by $e^{-\phi^j_t}$, see section 2.

The results in [7] deal with more general line bundles $L$ over $X$, and the trivial vector bundle $E$ over $\Omega$ with fiber $H^0(X, K_X + L)$ with the $L^2$-metric

$$\|u\|^2 = \int_X |u|^2 e^{-\phi_t},$$

see section 2. The main result is then a formula for the curvature of $E$ with the $L^2$-metric. In this paper we study the simplest special case, $L = -K_X$. Then $K_X + L$ is trivial so $E$ is a line bundle and Theorem 1.1 says that this line bundle has nonnegative curvature.

Theorem 1.1 is formally analogous to the Brunn-Minkowski inequality for the volumes of convex sets, and even more to its functional version, Prekopa’s theorem, [18]. Prekopa’s theorem states that if $\phi$ is a convex function on $\mathbb{R}^{n+1}$, then

$$f(t) := -\log \int_{\mathbb{R}^n} e^{-\phi_t}$$

is convex. The complex counterpart of this is that we consider a complex manifold $X$ with a family of volume forms $\mu_t$. In local coordinates $z^j$ the volume form can be written as above $c_n e^{-\phi^j_t} dz^j \wedge d\bar{z}^j$, and if $\mu_t$ is globally well defined $\phi^j_t$ are then the local representatives of a metric, $\phi_t$, on $-K_X$. Convexity in Prekopa’s theorem then corresponds to positive, or at least semipositive, curvature of $\phi_t$, so $X$ must be Fano, or its canonical bundle must have at least have
seminegative curvature (in some sense: \(-K_X\) pseudoeffective would be the minimal requirement). The assumption in Prekopa’s theorem that the weight is convex with respect to \(x\) and \(t\) together then correspond to the assumptions in Theorem 1.1.

If \(K\) is a compact convex set in \(\mathbb{R}^{n+1}\) we can take \(\phi\) to be equal to 0 in \(K\) and \(+\infty\) outside of \(K\). Prekopa’s theorem then implies the Brunn-Minkowski theorem, saying that the logarithm of the volumes of \(n\)-dimensional slices, \(K_t\) of convex sets are concave; concretely

\[
|K_{(t+s)/2}|^2 \leq |K_t||K_s|
\]

The Brunn-Minkowski theorem has an important addendum which describes the case of equality: If equality holds in (1.2) then all the slices \(K_t\) and \(K_s\) are translates of each other

\[
K_t = K_s + (t - s)v
\]

where \(v\) is some vector in \(\mathbb{R}^n\). A little bit artificially we can formulate this as saying that we move from one slice to another via the flow of a constant vector field.

**Remark 1.** It follows that from (1.2) and the natural homogeneity properties of Lebesgue measure that \(|K_t|^{1/n}\) is also concave. This (‘additive version’) is perhaps the most common formulation of the Brunn-Minkowski inequalities, but the logarithmic (or multiplicative) version above works better for weighted volumes and in the complex setting. For the additive version conditions for equality are more liberal; then \(K_t\) may change not only by translation but also by dilation (see [15]), but equality in the multiplicative case excludes dilation.

A natural question is then if one can draw a similar conclusion in the complex setting described above. In [7] we proved that this is indeed so if \(\phi\) is known to be smooth and strictly plurisubharmonic on \(X\) for \(t\) fixed. The main result of this paper is the extension of this to less regular situations. We keep the same assumptions as in Theorem 1.1.

**Theorem 1.2.** Assume that \(H^{0,1}(X) = 0\), and that the curve of metrics \(\phi_t\) is independent of the imaginary part of \(t\). Assume moreover that the metrics \(\phi_t\) are uniformly bounded in the sense that for some smooth metric on \(-K_X\), \(\psi\),

\[
|\phi_t - \psi| \leq C.
\]

Then, if the function \(\mathcal{F}\) in Theorem 1.1 is affine in a neighbourhood of 0 in \(Ω\), there is a (possibly time dependent) holomorphic vector field \(V\) on \(X\) with flow \(F_t\) such that

\[
F_t^\ast(\partial \bar{\partial} \phi_t) = \partial \bar{\partial} \phi_0.
\]

The same conclusion can also be drawn without the assumption that \(\phi_t\) be independent of the imaginary part of \(t\), and then assuming that \(\mathcal{F}\) be harmonic instead of affine, but the proof then seems to require more regularity assumptions. For simplicity we therefore treat only the case when \(\phi_t\) is independent of \(t\), which anyway seems to be the most useful in applications.

This theorem is useful in view of the discussion above on the possible lack of regularity of geodesics. As we shall see in section 2 the existence of a generalized geodesic satisfying the boundedness assumption in Theorem 1.2 is almost trivial. One motivation for the theorem is to give a new proof of the Bando-Mabuchi uniqueness theorem for Kähler-Einstein metrics on Fano
manifolds. Recall that a metric $\omega_\psi = i\partial\bar{\partial}\psi$, with $\psi$ a metric on $-K_X$ solves the Kähler-Einstein equation if

$$\text{Ric}(\omega_\psi) = \omega_\psi,$$

or equivalently if for some positive $a$

$$e^{-\psi} = a(i\partial\bar{\partial}\psi)^n,$$

where we use the convention above to interpret $e^{-\psi}$ as a volume form. By a celebrated theorem of Bando and Mabuchi any two Kähler-Einstein metrics $i\partial\bar{\partial}\phi_0$ and $i\partial\bar{\partial}\phi_1$ are related via the time-one flow of a holomorphic vector field. In section 4 we shall give a proof of this fact by joining $\phi_0$ and $\phi_1$ by a geodesic and applying Theorem 1.2.

It should be noted that a similar proof of the Bando-Mabuchi theorem has already been given by Berman, [2]. The difference between his proof and ours is that he uses the weaker version of Theorem 1.2 from [7]. He then needs to prove that the geodesic joining two Kähler-Einstein metrics is in fact smooth, which we do not need, and we also avoid the use of Chen’s theorem since we only need the existence of a bounded geodesic.

A minimal assumption in Theorem 1.2 would be that $e^{-\phi_t}$ be integrable, instead of bounded. I do not know if the theorem holds in this generality, but in section 6 we will consider an intermediate situation where $\phi_t = \chi_t + \psi$, with $\chi_t$ bounded and $\psi$ such that $e^{-\psi}$ is integrable, so that the singularities don’t change with $t$. Under various positivity assumptions we are then able to proof a version of Theorem 1.2.

Apart from making the problem technically simpler, this extra assumption that $\phi_t = \chi_t + \psi$ also introduces an additional structure, which seems interesting in itself. In section 6 we use it to give a generalization of the Bando-Mabuchi theorem to certain ’twisted’ Kähler-Einstein equations,

$$\text{Ric}(\omega) = \omega + \theta$$

considered in [20],[3] and [14]. Here $\theta$ is a fixed positive $(1,1)$-current, that may e g be the current of integration on a klt divisor. The solutions to these equations are then not necessarily smooth and it seems to be hard to prove uniqueness using the original methods of Bando and Mabuchi.

Another paper that is very much related to this one is [5], by Berman -Boucksom-Guedj-Zeriahi. There is introduced a variational approach to Monge-Ampere equations and Kähler-Einstein equations in a nonsmooth setting and a uniqueness theorem a la Bando-Mabuchi is proved, using continuous geodesics as we do here, but in a somewhat less general situation. I would like to thank all of these authors for helpful discussions, and Robert Berman in particular for proposing the generalized Bando-Mabuchi theorem in section 6.

2. Preliminaries

2.1. Notation. Let $L$ be a line bundle over a complex manifold $X$, and let $U_j$ be a covering of the manifold by open sets over which $L$ is locally trivial. A section of $L$ is then represented by a collection of complex valued functions $s_j$ on $U_j$ that are related by the transition functions of the bundle, $s_j = g_{jk}s_k$. A metric on $L$ is given by a collection of realvalued functions $\phi^j$ on $U_j$,
related so that
\[ |s_j|^2 e^{-\phi_j} = |s|^2 e^{-\phi} = \phi_s^2 \]
is globally well defined. We will write \( \phi \) for the collection \( \phi_j \), and refer to \( \phi \) as the metric on \( L \), although it might be more appropriate to call \( e^{-\phi} \) the metric. (Some authors call \( \phi \) the ‘weight’ of the metric.)

A metric \( \phi \) on \( L \) induces an \( L^2 \)-metric on the adjoint bundle \( K_X + L \). A section \( u \) of \( K_X + L \) can be written locally as
\[ u = dz \otimes s \]
where \( dz = dz_1 \wedge ...dz_n \) for some choice of local coordinates and \( s \) is a section of \( L \). We let
\[ |u|^2 e^{-\phi} := c_n dz \wedge d\bar{z}|s|^2; \]
it is a volume form on \( X \). The \( L^2 \)-norm of \( u \) is
\[ \|u\|^2 := \int_X |u|^2 e^{-\phi}. \]
Note that the \( L^2 \) norm depends only on the metric \( \phi \) on \( L \) and does not involve any choice of metric on the manifold \( X \).

In this paper we will be mainly interested in the case when \( L = -K_X \) is the anticanonical bundle. Then the adjoint bundle \( K_X + L \) is trivial and is canonically isomorphic to \( X \times \mathbb{C} \) if we have chosen an isomorphism between \( L \) and \( -K_X \). This bundle then has a canonical trivialising section, \( u \) identically equal to 1. With the notation above
\[ \|1\|^2 = \int_X |1|^2 e^{-\phi} = \int_X e^{-\phi}. \]
This means explicitly that we interpret the volume form \( e^{-\phi} \) as
\[ dz^j \wedge d\bar{z}^j e^{-\phi_j} \]
where \( e^{-\phi_j} = |(dz^j)^{-1}|^2 \) is the local representative of the metric for the frame determined by the local coordinates. Notice that this is consistent with the conventions indicated in the introduction.

2.2. Bounded geodesics. We now consider curves \( t \rightarrow \phi_t \) of metrics on the line bundle \( L \). Here \( t \) is a complex parameter but we shall (almost) only look at curves that do not depend on the imaginary part of \( t \). We say that \( \phi_t \) is a subgeodesic if \( \phi_t \) is upper semicontinuous and \( i\partial \bar{\partial}_{t,X} \phi_t \geq 0 \), so that local representatives are plurisubharmonic with respect to \( t \) and \( X \) jointly. We say that \( \phi_t \) is bounded if
\[ |\phi_t - \psi| \leq C \]
for some constant \( C \) and some (hence any) smooth metric on \( L \). For bounded geodesics the complex Monge-Ampere operator is well defined and we say that \( \phi_t \) is a (generalized) geodesic if
\[ (i\partial \bar{\partial}_{t,X} \phi_t)^{n+1} = 0. \]
Let $\phi_0$ and $\phi_1$ be two bounded metrics on $L$ over $X$ satisfying $i\partial\bar{\partial}\phi_{0,1} \geq 0$. We claim that there is a bounded geodesic $\phi_t$ defined for the real part of $t$ between 0 and 1, such that

$$\lim_{t \to 0,1} \phi_t = \phi_{0,1}$$

uniformly on $X$. The curve $\phi_t$ is defined by

$$\phi_t = \sup \{ \psi_t \}$$

where the supremum is taken over all plurisubharmonic $\psi_t$ with

$$\lim_{t \to 0,1} \psi_t \leq \phi_{0,1}.$$  

To prove that $\phi_t$ defined in this way has the desired properties we first construct a barrier

$$\chi_t = \max(\phi_0 - A Re t, \phi_1 + A(Re t - 1)).$$

Clearly $\chi$ is plurisubharmonic and has the right boundary values if $A$ is sufficiently large. Therefore the supremum in (2.1) is the same if we restrict it to $\psi$ that are larger than $\chi$. For such $\psi$ the onesided derivative at 0 is larger than $-A$ and the onesided derivative at 1 is smaller than $A$. Since we may moreover assume that $\psi$ is independent of the imaginary part of $t$, $\psi$ is convex in $t$ so the derivative with respect to $t$ increases, and must therefore lie between $-A$ and $A$. Hence $\phi_t$ satisfies

$$\phi_0 - A Re t \leq \phi_t \leq \phi_0 + A Re t$$

and a similar estimate at 1. Thus $\phi_t$ has the right boundary values uniformly. In addition, the upper semicontinuous regularization $\phi_t^*$ of $\phi_t$ must satisfy the same estimate. Since $\phi_t^*$ is plurisubharmonic it belongs to the class of competitors for $\phi_t$ and must therefore coincide with $\phi_t$, so $\phi_t$ is plurisubharmonic. That finally $\phi_t$ solves the homogenous Monge-Ampere equation follows from the fact that it is maximal with given boundary values, see e.g. [2].

Notice that as a byproduct of the proof we have seen that the geodesic joining two bounded metrics is uniformly Lipschitz in $t$. This fact will be very useful later on.

2.3. Approximation of metrics and subgeodesics. In the proofs we will need to approximate our metrics that are only bounded, and sometimes not even bounded, by smooth metrics. Since we do not want to lose too much of the positivity of curvature this causes some complications and we collect here some results on approximation of metrics that we will use. An extensive treatment of these matters can be found in [11]. Here we will need only the simplest part of this theory and we also refer to [9] for an elementary proof of the result we need.

In general a singular metric $\phi$ with $i\partial\bar{\partial}\phi \geq 0$ can not be approximated by a decreasing sequence of smooth metrics with nonnegative curvature. A basic fact is however (see [9], Theorem I.1) that this is possible if the line bundle in question is positive, so that it has some smooth metric of strictly positive curvature. This is all we need in the main case of a Fano manifold.

The approximation result for positive bundles also holds for $\mathbb{Q}$-line bundles; just multiply by some sufficiently divisible integer, and even for $\mathbb{R}$-bundles. In this paper we will also be interested in line bundles that are only semipositive. If $X$ is projective, as we assume, the basic
fact above implies that we then can approximate any singular metric with nonnegative curvature with a decreasing sequence of smooth $\phi^\nu$s, satisfying

$$i\bar{\partial}\partial\phi^\nu \geq -\epsilon_\nu \omega$$

where $\omega$ is some Kähler form and $\epsilon_\nu$ tends to zero. To see this we basically only need to apply the result above for the positive case to the $\mathbb{R}$-bundle $L + \epsilon F$ where $F$ is positive. If $\psi$ is a smooth metric with positive curvature on $F$, we approximate $\phi + \epsilon \psi$ by smooth metrics $\chi_\nu$ with positive curvature. Then $\phi^\nu = \chi_\nu - \epsilon \psi$ satisfies

$$i\bar{\partial}\partial\phi^\nu \geq -\epsilon \omega$$

for $\omega = i\bar{\partial}\partial\psi$. Then let $\epsilon$ go to zero and choose a diagonal sequence. This sequence may not be decreasing, but an easy argument using Dini’s lemma shows that we may get a decreasing sequence this way.

At one point we also wish to treat a bundle that is not even semipositive, but only effective. It then has a global holomorphic section, $s$, and the singular metric we are interested in is $\log |s|^2$, or some positive multiple of it. We then let $\psi$ be any smooth metric on the bundle and approximate by

$$\phi^\nu := \log(|s|^2 + \nu^{-1} e^\psi).$$

Explicit computation shows that $i\bar{\partial}\partial\phi^\nu \geq -C \omega$ where $C$ is some fixed constant. Moreover, outside any fixed neighbourhood of the zero divisor of $s$,

$$i\bar{\partial}\partial\phi^\nu \geq -\epsilon_\nu \omega$$

with $\epsilon_\nu$ tending to zero. This weak approximation will be enough for our purposes.

3. The smooth case

In this section we let $L$ be a holomorphic line bundle over $X$ and $\Omega$ be a smoothly bounded open set in $\mathbb{C}$. We consider the trivial vector bundle $E$ over $\Omega$ with fiber $H^0(X, K_X + L)$. Let now $\phi_t$ be a smooth curve of metrics on $L$ of semipositive curvature. For any fixed $t$, $\phi_t$ induces an $L^2$-norm on $H^0(X, K_X + L)$ as described in the previous section

$$\|u\|_t^2 = \int_X |u|^2 e^{-\phi_t},$$

and as $t$ varies we get an hermitian metric on the vector bundle $E$.

We now recall a formula for the curvature of $E$ with this metric from [6], [8]. Let for each $t$ in $\Omega$

$$\partial^{\phi_t} = e^{\phi_t} \partial e^{-\phi_t} = \partial - \partial \phi_t \land .$$

If $\alpha$ is an $(n, 1)$-form on $X$ with values in $L$, and we write $\alpha = v \land \omega$, where $\omega$ is our fixed Kähler form on $X$, then (modulo a sign)

$$\partial^{\phi_t} v = \partial^{\phi_t} \alpha,$$

the adjoint of the $\partial$-operator for the metric $\phi_t$. In particular this means that the operator $\partial^{\phi_t}$ is well defined on $L$-valued forms.
This also means that for any \( t \) we can solve the equation

\[
\partial^{\phi_t} v = \eta,
\]

if \( \eta \) is an \( L \)-valued \( (n, 0) \)-form that is orthogonal to the space of holomorphic \( L \)-valued forms (see remark 2 below). Moreover by choosing \( \alpha = v \wedge \omega \) orthogonal to the kernel of \( \bar{\partial}^{*}_{\phi_t} \), we can assume that \( \alpha \) is \( \bar{\partial} \)-closed, so that \( \bar{\partial} v \wedge \omega = 0 \). Hence, with this choice, \( \bar{\partial} v \) is a primitive form.

If, as we assume from now, the cohomology \( H^{n,1}(X, L) = 0 \), the \( \bar{\partial} \)-operator is surjective on \( \bar{\partial} \)-closed forms, so the adjoint is injective, and \( v \) is uniquely determined by \( \eta \).

**Remark 2.** The reason we can always solve this equation for \( t \) and \( \phi \) fixed is that the \( \bar{\partial} \)-operator from \( L \)-valued \( (n, 0) \)-forms to \( (n, 1) \)-forms on \( X \) has closed range. This implies that the adjoint operator \( \bar{\partial}^{*}_{\phi_t} \) also has closed range and that its range is equal to the orthogonal complement of the kernel of \( \bar{\partial} \). Moreover, that \( \bar{\partial} \) has closed range means precisely that for any \( (n, 1) \)-form in the range of \( \bar{\partial}^{*}_{\phi_t} \) we can solve the equation \( \bar{\partial} f = \alpha \) with an estimate

\[
\| f \| \leq C\| \alpha \|
\]

and it follows from functional analysis that we then can solve \( \partial^{\phi_t} v = \eta \) with the bound

\[
\| v \| \leq C\| \eta \|
\]

where \( C \) is the same constant. In case all metrics \( \phi_t \) are of equivalent size, so that \( |\phi_t - \phi_0| \leq A \) it follows that we can solve \( \partial^{\phi_t} v = \eta \) with an \( L^2 \)-estimate independent of \( t \). \( \square \)

Let \( u_t \) be a holomorphic section of the bundle \( E \) and let

\[
\dot{\phi}_t := \frac{\partial \phi}{\partial t}.
\]

For each \( t \) we now solve

\[
(3.1) \quad \partial^{\phi_t} v_t = \pi_{\perp}(\dot{\phi}_t u_t),
\]

where \( \pi_{\perp} \) is the orthogonal projection on the orthogonal complement of the space of holomorphic forms, with respect to the \( L^2 \)-norm \( \| \cdot \|^2 \). With this choice of \( v_t \) we obtain the following formula for the curvature of \( E \), see [6], [8]. In the formula, \( p \) stands for the natural projection map from \( X \times \Omega \) to \( \Omega \) and \( p_*(T) \) is the pushforward of a differential form or current. When \( T \) is a smooth form this is the fiberwise integral of \( T \).

**Theorem 3.1.** Let \( \Theta \) be the curvature form on \( E \) and let \( u_t \) be a holomorphic section of \( E \). For each \( t \) in \( \Omega \) let \( v_t \) solve (3.1) and be such that \( \bar{\partial}_X v_t \wedge \omega = 0 \). Put

\[
\hat{u} = u_t - dt \wedge v_t.
\]

Then

\[
(3.2) \quad \langle \Theta u_t, u_t \rangle_t = p_*(c_n i \bar{\partial} \bar{\partial} \dot{\phi} \wedge \hat{u} \wedge \bar{u} e^{-\phi}) + \int_X \| \bar{\partial} v_t \|^2 e^{-\phi_t} \sigma t \wedge d\bar{t}.
\]

**Remark 3.** This is not quite the same formula as the one used in [7] which can be seen as corresponding to a different choice of \( v_t \). \( \square \)
If the curvature acting on $u_t$ vanishes it follows that both terms in the right hand side of (3.2) vanish. In particular, $v_t$ must be a holomorphic form. To continue from there we first assume (like in [7]) that $i\partial\bar{\partial}\phi_t > 0$ on $X$. Taking $\bar{\partial}$ of formula 3.1 we get
\[ \partial\bar{\partial}^\phi v_t = \bar{\partial}\phi_t \wedge u_t. \]
Using
\[ \partial\bar{\partial}^\phi + \partial^\phi \bar{\partial} = \partial\bar{\partial}\phi_t \]
we get if $v_t$ is holomorphic that
\[ \partial\bar{\partial}\phi_t \wedge v_t = \bar{\partial}\phi_t \wedge u_t. \]
The complex gradient of the function $i\dot{\phi}_t$ with respect to the Kähler metric $i\partial\bar{\partial}\phi_t$ is the $(1,0)$-vector field defined by
\[ V_t = i\partial\bar{\partial}\phi_t. \]
Since $\partial\bar{\partial}\phi_t \wedge u_t = 0$ for bidegree reasons we get
\[ (3.3) \quad \partial\bar{\partial}\phi_t \wedge v_t = \bar{\partial}\phi_t \wedge u = (V_t \partial\bar{\partial}\phi_t) \wedge u = -\partial\bar{\partial}\phi_t \wedge (V_t \wedge u). \]
If $i\partial\bar{\partial}\phi_t > 0$ we find that
\[ -v_t = V_t \wedge u. \]
If $v_t$ is holomorphic it follows that $V_t$ is a holomorphic vector field - outside of the zero divisor of $u_t$ and therefore everywhere since the complex gradient is smooth under our hypotheses. If we assume that $X$ carries no nontrivial holomorphic vector fields, $V_t$ and hence $v_t$ must vanish so $\dot{\phi}_t$ is holomorphic, hence constant. Hence
\[ \partial\bar{\partial}\dot{\phi}_t = 0 \]
so $\partial\bar{\partial}\phi_t$ is independent of $t$. In general - if there are nontrivial holomorphic vector fields - we get that the Lie derivative of $\partial\bar{\partial}\phi_t$ equals
\[ L_{V_t} \partial\bar{\partial}\phi_t = \partial V_t \wedge \partial\bar{\partial}\phi_t = \partial\bar{\partial}\phi_t \wedge \frac{\partial}{\partial t} \partial\bar{\partial}\phi_t. \]
Together with an additional argument showing that $V_t$ must be holomorphic with respect to $t$ as well (see below) this gives that $\partial\bar{\partial}\phi_t$ moves with the flow of the holomorphic vector field which is what we want to prove.

For this it is essential that the metrics $\phi_t$ be strictly positive on $X$ for $t$ fixed, but we shall now see that there is a way to get around this difficulty, at least in some special cases.

The main case that we will consider is when the canonical bundle of $X$ is seminegative, so we can take $L = -K_X$. Then $K_X + L$ is the trivial bundle and we fix a nonvanishing trivializing section $u = 1$. Then the constant section $t \rightarrow u_t = u$ is a trivializing section of the (line) bundle $E$. We write
\[ F(t) = -\log \|u\|^2 = -\log \int_X |u|^2 e^{-\phi_t} = -\log \int_X e^{-\phi_t}. \]
Still assuming that $\phi$ is smooth, but perhaps not strictly positive on $X$, we can apply the curvature formula in Theorem 3.1 with $u_t = u$ and get
\[ \|u_t\|^2 i\partial\bar{\partial}_t F = \langle \Theta u_t, u_t \rangle_t = p_s(c_n i\partial\bar{\partial}\phi \wedge \dot{u} \wedge \bar{u} e^{-\phi_t}) + \int_X \|\partial v_t\|^2 e^{-\phi_t} dt \wedge d\bar{t}. \]
If $F$ is harmonic, the curvature vanishes and it follows that $v_t$ is holomorphic on $X$ for any $t$ fixed. Since $u$ never vanishes we can define a holomorphic vector field $V_t$ by

$$-v_t = V_t \lrcorner u.$$  

Almost as before we get

$$\bar{\partial}\dot{\phi}_t \wedge u = \partial\bar{\partial}\phi_t \wedge v_t = -\partial\bar{\partial}\phi_t \wedge (V_t \lrcorner u) = (V_t \lrcorner \partial\bar{\partial}\phi_t) \wedge u,$$

which implies that

$$V_t \lrcorner i\partial\bar{\partial}\phi_t = i\bar{\partial}\dot{\phi}_t.$$

if $u$ never vanishes. This is the important point; we have been able to trade the nonvanishing of $i\partial\bar{\partial}\phi_t$ for the nonvanishing of $u$. This is where we use that the line bundle we are dealing with is $L = -K_X$.

We also get the formula for the Lie derivative of $\partial\bar{\partial}\phi_t$ along $V_t$

$$L_{V_t} \partial\bar{\partial}\phi_t = \partial V_t \lrcorner \partial\bar{\partial}\phi_t = \partial\bar{\partial}\dot{\phi}_t - \frac{\partial}{\partial \bar{\partial}} (\partial\bar{\partial}\phi_t) \wedge u,$$

(3.4)

To be able to conclude from here we also need to prove that $V_t$ depends holomorphically on $t$. For this we will use the first term in the curvature formula, which also has to vanish. It follows that

$$i\partial\bar{\partial}\phi \wedge \hat{u} \wedge \bar{\hat{u}}$$

has to vanish identically. Since this is a semidefinite form in $\hat{u}$ it follows that

$$\partial\bar{\partial}\phi \wedge \hat{u} = 0.$$  

(3.5)

Considering the part of this expression that contains $dt \wedge d\bar{t}$ we see that

$$\mu := \frac{\partial^2 \phi}{\partial t \partial \bar{t}} - \partial \partial_X (\frac{\partial \phi}{\partial \bar{t}})(V_t) = 0.$$  

(3.6)

If $\partial\bar{\partial}_X \phi_t > 0$, $\mu$ is easily seen to be equal to the function $c(\phi)$ defined in the introduction, so the vanishing of $\mu$ is then equivalent to the homogenous Monge-Ampère equation. In [7] we showed that $\partial V_t / \partial \bar{t} = 0$ by realizing this vector field as the complex gradient of the function $c(\phi)$ which has to vanish if the curvature is zero. Here, where we no longer assume strict postivity of $\phi_t$ along $X$ we have the same problems as before to define the complex gradient. Therefore we follow the same route as before, and start by studying $\partial v_t / \partial \bar{t}$ instead.

Recall that

$$\partial^\phi_{\bar{t}} v_t = \dot{\phi}_t \wedge u + h_t$$

where $h_t$ is holomorphic on $X$ for each $t$ fixed. As we have seen in the beginning of this section, $v_t$ is uniquely determined, and it is not hard to see that it depends smoothly on $t$ if $\phi$ is smooth. Differentiating with respect to $\bar{t}$ we obtain

$$\partial^\phi_{\bar{t}} \frac{\partial v_t}{\partial t} = \left[ \frac{\partial^2 \phi}{\partial t \partial \bar{t}} - \partial \partial_X (\frac{\partial \phi}{\partial \bar{t}})(V_t) \right] \wedge u + \frac{\partial h_t}{\partial t}.$$
Since the left hand side is automatically orthogonal to holomorphic forms, we get that
\[ \partial \phi_t \partial v_t = \pi_\perp (\mu) = 0, \]
since \( \mu = 0 \) by (3.6). Again, this means that \( \partial v_t / \partial t = 0 \) since \( \partial v_t / \partial \bar{t} \land \omega \) is still \( \partial_X \)-closed, and the cohomological assumption implies that \( \partial \phi_t \) is injective on closed forms.

All in all, \( v_t \) is holomorphic in \( t \), so \( V_t \) is holomorphic on \( X \times \Omega \). We can now conclude the proof in the same way as in [7]. Define a holomorphic vector field \( V \) on \( X \times \Omega \) by
\[ V := V_t - \frac{\partial}{\partial t}. \]
Let \( \eta \) be the form \( \partial \partial_X \phi_t \) on \( X \). Then formula 2.4 says that the Lie derivative
\[ L_V \eta = 0 \]
on \( X \). It follows that \( \eta \) is invariant under the flow of \( V \) so \( \partial \partial \phi_t \) moves by the flow of a holomorphic family of automorphisms of \( X \).

4. THE NONSMOOTH CASE

In the general case we can write our metric \( \phi \) as the uniform limit of a sequence of smooth metrics, \( \phi^n \), with \( i \partial \partial \phi^n \geq -\epsilon_n \omega \), where \( \epsilon_n \) tends to zero, see section 2.3. Note also that in case we assume that \( -K_X > 0 \) we can even approximate with metrics of strictly positive curvature. The presence of the negative term \( -\epsilon_n \omega \) causes some minor notational problems in the estimates below. We will therefore carry out the proof under the assumptions that \( i \partial \partial \phi^n \geq 0 \) and leave the necessary modifications to the reader.

Let \( F^n \) be defined the same way as \( F \), but using the weights \( \phi^n \) instead. Then
\[ i \partial \partial F^n \]
goes to zero weakly on \( \Omega \). We get a sequence of \((n-1,0)\) forms \( v^n_t \), solving
\[ \partial \phi^n v^n_t = \pi_\perp (\dot{\phi}^n u) \]
for \( \phi = \phi^n \). By Remark 1, we have an \( L^2 \)-estimate for \( v^n_t \) in terms of the \( L^2 \) norm of \( \dot{\phi}^n \), with the constant in the estimate independent of \( t \) and \( n \). Since \( \dot{\phi}^n \) is uniformly bounded by section 2.2, it follows that we get a uniform bound for the \( L^2 \)-norms of \( v^n_t \) over all of \( X \times \Omega \). Therefore we can select a subsequence of \( v^n_t \) that converges weakly to a form \( v \) in \( L^2 \). Since \( i \partial \partial F^n \) tends to zero weakly, Theorem 2.1 shows that the \( L^2 \)-norm of \( \partial_X v^n \) over \( X \times K \) goes to zero for any compact \( K \) in \( \Omega \), so \( \partial_X v = 0 \). Moreover
\[ \partial_X \phi v = \pi_\perp (\dot{\phi} u) \]
in the (weak ) sense that
\[ \int_{X \times \Omega} dt \land d\bar{t} \land v \land \partial \overline{\partial} W e^{-\phi} = \int_{X \times \Omega} dt \land d\bar{t} \land \pi_\perp (\dot{\phi} u) \land \overline{\partial} W e^{-\phi} \]
for any smooth form \( W \) of the appropriate degree.
As before this ends the argument if there are no nontrivial holomorphic vector fields on $X$. Then $v$ must be zero, so $\dot{\phi}_t$ is holomorphic, hence constant. In the general case, we finish by showing that $v_t$ is holomorphic in $t$. The difficulty is that we don’t know any regularity of $v_t$ except that it lies in $L^2$, so we need to formulate holomorphicity weakly. We will use two elementary lemmas that we state without proof. The first one allows us get good convergence properties for geodesics, when the metrics only depend on the real part of $t$ and therefore are convex with respect to $t$.

**Lemma 4.1.** Let $f_\nu$ be a sequence of smooth convex functions on an interval in $\mathbb{R}$ that converge uniformly to the convex function $f$. Let $a$ be a point in the interval such that $f'(a)$ exists. Then $f'_\nu(a)$ converge to $f'(a)$. Since a convex function is differentiable almost everywhere it follows that $f'_\nu$ converges to $f'$ almost everywhere, with dominated convergence on any compact subinterval.

Another technical problem that arises is that we are dealing with certain orthogonal projections on the manifold $X$, where the weight depends on $t$. The next lemma gives us control of how these projections change.

**Lemma 4.2.** Let $\alpha_t$ be forms on $X$ with coefficients depending on $t$ in $\Omega$. Assume that $\alpha_t$ is Lipschitz with respect to $t$ as a map from $\Omega$ to $L^2(X)$. Let $\pi^t$ be the orthogonal projection on $\bar{\partial}$-closed forms with respect to the metric $\phi_t$ and the fixed Kähler metric $\omega$. Then $\pi^t(\alpha_t)$ is also Lipschitz, with a Lipschitz constant depending only on that of $\alpha$ and the Lipschitz constant of $\phi_t$ with respect to $t$.

Note that in our case, when $\phi$ is independent of the imaginary part of $t$, we have control of the Lipschitz constant with respect to $t$ of $\phi_t$, and also by the first lemma uniform control of the Lipschitz constant of $\dot{\phi}_t$, since the derivatives are increasing.

It follows from the curvature formula that

$$a_\nu := \int_{X \times \Omega'} i\partial\bar{\partial}\phi^\nu \wedge \hat{u} \wedge \hat{u}^e^{-\phi^\nu}$$

goes to zero if $\Omega'$ is a relatively compact subdomain of $\Omega$. Shrinking $\Omega$ slightly we assume that this actually holds with $\Omega' = \Omega$. By the Cauchy inequality

$$\int_{X \times \Omega} i\partial\bar{\partial}\phi^\nu \wedge \hat{u} \wedge \hat{W}e^{-\phi^\nu} \leq a_\nu \int_{X \times \Omega} i\partial\bar{\partial}\phi^\nu \wedge W \wedge \hat{W}e^{-\phi^\nu}$$

if $W$ is any $(n, 0)$-form. Choose $W$ to contain no differential $dt$, so that it is an $(n, 0)$-form on $X$ with coefficients depending on $t$. Then

$$\int_{X \times \Omega} i\partial\bar{\partial}\phi^\nu \wedge W \wedge \hat{W}e^{-\phi^\nu} = \int_{X \times \Omega} i\partial_t\phi^\nu \wedge W \wedge \hat{W}e^{-\phi^\nu}$$

We now assume that $W$ has compact support. The one variable Hörmander inequality with respect to $t$ then shows that the last integral is dominated by

$$\int_{X \times \Omega} |\partial_t^\nu W|^2 e^{-\phi^\nu}. \quad (4.1)$$
From now we assume that $W$ is Lipschitz with respect to $t$ as a map from $\Omega$ into $L^2(X)$. Then (4.1) is uniformly bounded, so
\[
\int_{X \times \Omega} idt \wedge d\bar{t} \wedge \mu^\nu \wedge \overline{W e^{-\phi^\nu}}
\]
goes to zero, where $\mu^\nu$ is defined as in (3.6) with $\phi$ replaced by $\phi^\nu$. By Lemma 4.2
\[
\int_{X \times \Omega} idt \wedge d\bar{t} \wedge \mu^\nu \wedge \overline{\pi_\perp W e^{-\phi^\nu}}
\]
also goes to zero. Therefore
\[
\int_{X \times \Omega} idt \wedge d\bar{t} \wedge \pi_\perp (\mu^\nu) \wedge \overline{W e^{-\phi^\nu}}
\]
goes to zero. Now recall that $\pi_\perp (\mu^\nu) = \partial^\phi_t (\partial v^\nu_t / \partial \bar{t})$ and integrate by parts. This gives that
\[
\int_{X \times \Omega} idt \wedge d\bar{t} \wedge \frac{\partial v^\nu_t}{\partial \bar{t}} \wedge \overline{\partial_X W e^{-\phi^\nu}}
\]
also vanishes as $\nu$ tends to infinity.

Next we let $\alpha$ be a form of bidegree $(n, 1)$ on $X \times \Omega$ that does not contain any differential $dt$. We assume it is Lipschitz with respect to $t$ and decompose it into one part, $\partial_X W$, which is $\partial_X$-exact and one which is orthogonal to $\partial_X$-exact forms. This amounts of course to making this orthogonal decomposition for each $t$ separately, and by Lemma 4.2 each term in the decomposition is still Lipschitz in $t$, uniformly in $\nu$. Since $v^\nu_t \wedge \omega$ is $\partial_X$-closed by construction, this holds also for $\partial v^\nu_t / \partial \bar{t}$. By our cohomological assumption, it is also $\partial$-exact, and we get that
\[
\int_{X \times \Omega} idt \wedge d\bar{t} \wedge \frac{\partial v^\nu_t}{\partial \bar{t}} \wedge \overline{\partial_X W e^{-\phi^\nu}} = \int_{X \times \Omega} idt \wedge d\bar{t} \wedge \frac{\partial v^\nu_t}{\partial \bar{t}} \wedge \overline{\partial_X W e^{-\phi^\nu}}.
\]
Hence
\[
\int_{X \times \Omega} dt \wedge v_t \wedge \overline{\partial^\phi_t \alpha e^{-\phi}}
\]
go to zero. By Lemma 4.1 we may pass to the limit here and finally get that
\[
(4.2) \int_{X \times \Omega} dt \wedge v_t \wedge \overline{\partial^\phi_t \alpha e^{-\phi}} = 0,
\]
under the sole assumption that $\alpha$ is of compact support, and Lipschitz in $t$. This is almost the distributional formulation of $\bar{\partial} v = 0$, except that $\phi$ is not smooth. But, replacing $\alpha$ by $e^{\phi - \psi} \alpha$, where $\psi$ is another metric on $L$, we see that if (4.2) holds for some $\phi$, Lipschitz in $t$, it holds for any such metric. Therefore we can replace $\phi$ in (4.2) by some other smooth metric. It follows that $v_t$ is holomorphic in $t$ and therefore, since we already know it is holomorphic on $X$, holomorphic on $X \times \Omega$. This completes the proof.
5. The Bando-Mabuchi theorem.

For $\phi_0$ and $\phi_1$, two metrics on a line bundle $L$ over $X$, we consider their relative energy

$$E(\phi_0, \phi_1).$$

This is well defined if $\phi_j$ are bounded with $i\partial\bar{\partial}\phi_j \geq 0$. It has the fundamental properties that if $\phi_t$ is smooth in $t$ for $t$ in $\Omega$, then

$$\frac{\partial}{\partial t} E(\phi_t, \phi_1) = \int_X \dot{\phi}_t (i\partial\bar{\partial}\phi_t)^n / \text{Vol}(L)$$

and

$$i\partial\bar{\partial} E(\phi_t, \phi_1) = p_* ((i\partial\bar{\partial} X, t\phi)^{n+1}) / \text{Vol}(L) = i dt \wedge d\bar{t} \int_X c(\phi_t) (i\partial\bar{\partial} X, \phi_t)^n / \text{Vol}(L),$$

where $p$ is the projection map from $X \times \Omega$ to $\Omega$. Here $\text{Vol}(L)$ is the normalizing factor

$$\text{Vol}(L) = \int_X (i\partial\bar{\partial} X, \phi)^n,$$

chosen so that the derivative of $E$ becomes 1 if $\phi_t = \phi + t$. If the family is only bounded, these formulas hold in the sense of distributions. In particular, if $\phi$ solves the homogeneous Monge-Ampère equation, so that $(i\partial\bar{\partial} X, t\phi)^{n+1} = 0$ or equivalently $c(\phi) = 0$, then $E(\phi_t, \phi_1)$ is harmonic in $t$. Hence this function is linear along geodesics.

Let now

$$G(t) = F(t) - E(\phi_t, \psi)$$

where $\psi$ is arbitrary. Then $\phi_0$ solves the Kähler-Einstein equation if and only if $G'(0) = 0$ for any smooth curve $\phi_t$. If $\phi_0$ and $\phi_1$ are two Kähler-Einstein metrics we connect them by a geodesic $\phi_t$ (a continuous geodesic will be enough). Now $\phi_t$ depends only on the real part of $t$ so $G$ is convex. We claim that since both end points are Kähler-Einstein metrics, 0 and 1 are stationary points for $G$. This would be immediate if the geodesic were smooth, but it is not hard to see that it also holds if the geodesic is only bounded, with boundary behaviour as described in section 2.2. The function $F$ is convex, hence has onesided derivatives at the endpoints, and using the convexity of $\phi$ with respect to $t$ one sees that they equal

$$\int \dot{\phi}_t e^{-\phi} / \int e^{-\phi}$$

(where $\dot{\phi}_t$ now stands for the onesided derivatives). The function $E(\phi_t, \psi)$ is linear so its distributional derivative

$$\int_X \dot{\phi}_t (i\partial\bar{\partial}\phi_t)^n / \text{Vol}(L)$$

is constant and simple convergence theorems for the Monge-Ampère operator show that it is equal to its values at the endpoints. Hence both end points are critical points for $G$ and the convexity implies that $G$ is constant so $F$ is linear.
By Theorem 1.2 \( \partial \bar{\partial} \phi_t \) are related via a holomorphic family of automorphisms. In particular \( \partial \bar{\partial} \phi_0 \) and \( \partial \bar{\partial} \phi_1 \) are related via an automorphism which is homotopic to the identity, which is the content of the Bando-Mabuchi theorem.

6. A GENERALIZED BANDO-MABUCHI THEOREM

6.1. A variant of Theorem 1.2 for unbounded metrics. One might ask if Theorem 1.2 is valid under even more general assumptions. A minimal requirement is of course that \( F \) be finite, or in other words that \( e^{-\phi_t} \) be integrable. For all we know Theorem 1.2 might be true in this generality, but here we will limit ourselves to the following situation:

Let \( t \to \tau_t \) be a curve of singular metrics on \( L = -K_X \) that can be written

\[ \tau_t = \phi_t + \psi \]

where \( \psi \) is a metric on an \( \mathbb{R} \)-line bundle \( S \) and \( \phi_t \) is a curve of metrics on \( -(K_X + S) \) such that:

(i) \( \phi_t \) is bounded and only depends on \( \text{Re} \, t \).

(ii) \( e^{-\psi} \) is integrable and \( \psi \) does not depend on \( t \)

and

(iii) \( i \partial \bar{\partial}_{\tau_t}(\pi_\tau) \geq 0 \).

**Theorem 6.1.** Assume that \( -K_X \geq 0 \) and that \( H^{0,1}(X) = 0 \). Let \( \tau_t = \phi_t + \psi \) be a curve of metrics on \( -K_X \) satisfying (i)-(iii). Assume that

\[ F(t) = -\log \int_X e^{-\tau_t} \]

is affine. Then there is a holomorphic vector field \( V \) on \( X \) with flow \( F \) such that

\[ F_t^*(\partial \bar{\partial} \tau_t) = \partial \bar{\partial} \tau_0. \]

The proof of this theorem is almost the same as the proof of Theorem 1.2. The main thing to be checked is that for \( \tau = \tau^\nu \) a sequence of smooth metrics decreasing to \( \tau \) we can still solve the equations

\[ \partial^\nu v_t = \pi_\nu(\tau_t u) \]

with an \( L^2 \) -estimate independent of \( t \) and \( \nu \).

**Lemma 6.2.** Let \( L \) be a holomorphic line bundle over \( X \) with a metric \( \xi \) satisfying \( i \partial \bar{\partial} \xi \geq 0 \). Let \( \xi_0 \) be a smooth metric on \( L \) with \( \xi \leq \xi_0 \), and assume

\[ I := \int_X e^{\xi_0 - \xi} < \infty. \]

Then there is a constant \( A \), only depending on \( I \) and \( \xi_0 \) (not on \( \xi_! \)) such that if \( f \) is a \( \bar{\partial} \)-exact \( L \) valued \( (n,1) \)-form with

\[ \int |f|^2 e^{-\xi} \leq 1 \]
there is a solution $u$ to $\bar{\partial}u = f$ with
\[ \int_X |u|^2 e^{-\xi} \leq A. \]
(The integrals are understood to be taken with respect to some arbitrary smooth volume form.)

Proof. The assumptions imply that
\[ \int |f|^2 e^{-\xi_0} \leq 1. \]
Since $\bar{\partial}$ has closed range for $L^2$-norms defined by smooth metrics, we can solve $\bar{\partial}u = f$ with
\[ \int |u|^2 e^{-\xi_0} \leq C \]
for some constant depending only on $X$ and $\xi_0$. Choose a collection of coordinate balls $B_j$ such that $B_j/2$ cover $X$. In each $B_j$ solve $\bar{\partial}u_j = f$ with
\[ \int_{B_j} |u_j|^2 e^{-\xi} \leq C_1 \int_{B_j} |f|^2 e^{-\xi} \leq C_1, \]
$C_1$ only depending on the size of the balls. Then $h_j := u - u_j$ is holomorphic on $B_j$ and
\[ \int_{B_j} |h_j|^2 e^{-\xi_0} \leq C_2, \]
so
\[ \sup_{B_j/2} |h_j|^2 e^{-\xi_0} \leq C_3. \]
Hence
\[ \int_{B_j/2} |h_j|^2 e^{-\xi} \leq C_3 I \]
and therefore
\[ \int_{B_j/2} |u|^2 e^{-\xi} \leq C_4 I. \]
Summing up we get the lemma.

By the discussion in section 2, the assumption that $-K_X \geq 0$ implies that we can write $\tau_t$ as a limit of a decreasing sequence of smooth metrics $\tau_t^\nu$ with
\[ i\partial\bar{\partial}\tau_t^\nu \geq -\epsilon_\nu\omega \]
where $\epsilon_\nu$ tends to zero. Applying the lemma to $\xi = \tau_t^\nu$ and $\xi_0$ some arbitrary smooth metric we see that we have uniform estimates for solutions of the $\bar{\partial}$-equation, independent of $\nu$ and $t$. By remark 2, section 3, the same holds for the adjoint operator, which means that we can construct $(n-1,0)$-forms $\nu_t^\nu$ just as in section 3, and the proof of Theorem 6.1 then continues as in section 3.
6.2. Yet another version. We also briefly describe yet another situation where the same conclusion as in Theorem 6.1 can be drawn even though we do not assume that $-K_X \geq 0$. The assumptions are very particular, and it is not at all clear that they are optimal, but they are chosen to fit with the properties of desingularisations of certain singular varieties. We then assume instead that $-K_X$ can be decomposed

$$-K_X = -(K_X + S) + S$$

where $S$ is the $\mathbb{R}$-line bundle corresponding to a klt-divisor $\Delta \geq 0$ and we assume $-(K_X + S) \geq 0$. We moreover assume that the underlying variety of $\Delta$ is a union of smooth hypersurfaces with simple normal crossings. We then look at curves

$$\tau_t = \phi_t + \psi$$

where $i\partial \bar{\partial}_t \phi_t \geq 0$ and $\psi$ is a fixed metric on $S$ satisfying $i\partial \bar{\partial} \psi = [\Delta]$. We claim that the conclusion of Theorem 6.1 holds in this situation as well. The difference as compared to our previous case is that we do not assume that $\tau_t$ can be approximated by a decreasing sequence of metrics with almost positive curvature. For the proof we approximate $\phi_t$ by a decreasing sequence of smooth metrics $\phi^\nu$ satisfying

$$i\partial \bar{\partial} \phi^\nu_t \geq -\epsilon^\nu \omega.$$ 

As for $\psi$ we approximate it following the scheme at the end of section 2 by a sequence satisfying

$$i\partial \bar{\partial} \psi^\nu \geq -C \omega$$

and

$$i\partial \bar{\partial} \psi^\nu \geq -\epsilon_0 \omega$$

outside of any neighbourhood of $\Delta$. Then let $\tau^\nu_t = \phi^\nu_t + \psi^\nu_t$. Now consider the curvature formula (3.2)

$$\langle \Theta^\nu u_t, u_t \rangle_t = p^\ast (c_n i\partial \bar{\partial} \tau^\nu_t \wedge \bar{\partial} e^{-\tau^\nu_t}) + \int_X \|\bar{\partial} \nu_t\|^2 e^{-\tau^\nu_t} idt \wedge d\bar{t}$$

We want to see that the second term in the right hand side tends to zero given that the curvature $\Theta^\nu$ tends to zero, and the problem is that the first term on the right hand side has a negative part. However,

$$p^\ast (c_n i\partial \bar{\partial} \tau^\nu_t \wedge \bar{\partial} e^{-\tau^\nu_t})$$

can for any $t$ be estimated from below by

$$-\epsilon^\nu \|\bar{\partial} u\|^2 - C \int_U |\nu_t|^2 e^{-\tau^\nu}$$

where $U$ is any small neighbourhood of $\Delta$ if we choose $\nu$ large. This means, first, that we still have at least a uniform upper estimate on $\bar{\partial} \nu_t$. This, in turn gives by the technical lemma below that the $L^2$-norm of $\nu_t$ over a small neighbourhood of $\Delta$ must be small if the neighbourhood is small. Shrinking the neighbourhood as $\nu$ grows we can then arrange things so that the negative part in the right hand side goes to zero. Therefore the $L^2$-norm of $\bar{\partial} \nu_t$ goes to zero after all, and after that the proof proceeds as before. We collect this in the next theorem.
**Theorem 6.3.** Assume that \(-(K_X + S) \geq 0\) and that \(H^{0,1}(X) = 0\). Let \(\tau_t = \phi_t + \psi\) be a curve of metrics on \(-K_X\) where

(i) \(\phi_t\) are metrics on \(-(K_X + S)\) with \(i\partial\bar{\partial}\phi_t \geq 0\),

and

(ii) \(\psi\) is a metric on \(S\) with \(i\partial\bar{\partial}\psi = [\Delta]\), where \(\Delta\) is a klt divisor with simple normal crossings. Assume that

\[ \mathcal{F}(t) = -\log \int_X e^{-\tau_t} \]

is affine. Then there is a holomorphic vector field \(V\) on \(X\) with flow \(F_t\) such that

\[ F_t^* (\partial\bar{\partial}\tau_t) = \partial\bar{\partial}\tau_0. \]

We end this section with the technical lemma used above.

**Lemma 6.4.** The term

\[ \int U |v^\nu|^2 e^{-\tau^\nu} \]

in (6.2) can be made arbitrarily small if \(U\) is a sufficiently small neighbourhood of \(\Delta\)

**Proof.** Covering \(\Delta\) with a finite number of polydisks, in which the divisor is a union of coordinate hyperplanes, it is enough to prove the following statement:

Let \(P\) be the unit polydisk in \(\mathbb{C}^n\) and let \(v\) be a compactly supported function in \(P\). Let

\[ \psi_\epsilon = \sum \alpha_j \log(|z_j|^2 + \epsilon) \]

where \(0 \leq \alpha_j < 1\). Assume

\[ \int_P (|v|^2 + |ar{\partial}v|^2)e^{-\psi} \leq 1. \]

Then for \(\delta >> \epsilon\)

\[ \int_{|z_j| \leq \delta} |v|^2 e^{-\psi_\epsilon} \leq c_\delta \]

where \(c_\delta\) tends to zero with \(\delta\).

To prove this we first estimate the integral over \(|z_1| \leq \delta\) using the one variable Cauchy formula in the first variable

\[ v(z_1, z') = \pi^{-1} \int v_{\zeta_1}(\zeta_1, z')/(\zeta_1 - z_1) \]

which gives

\[ |v(z_1, z')|^2 \leq C \int |v_{\zeta_1}(\zeta_1, z')|^2/|\zeta_1 - z_1|. \]

Then multiply by \((|z_1|^2 + \epsilon)^{-\alpha_1}\) and integrate with respect to \(z_1\) over \(|z_1| \leq \delta\). Use the estimate

\[ \int_{|z_1| \leq \delta} \frac{1}{(|z_1|^2 + \epsilon)^{\alpha_1}} |z_1 - \zeta_1| \leq c_\delta(|\zeta_1|^2 + \epsilon)^{-\alpha_1}, \]

multiply by \(\sum_2^n \alpha_j \log(|z_j|^2 + \epsilon)\) and integrate with respect to \(z'\). Repeating the same argument for \(z_2, \ldots, z_n\) and summing up we get the required estimate.

\(\square\)
6.3. A generalized Bando-Mabuchi theorem. As pointed out to me by Robert Berman, Theorems 6.1 and 6.3 lead to versions of the Bando-Mabuchi theorem for 'twisted Kähler-Einstein equations', [20], [3], and [14]. Let \( \theta \) be a positive \((1, 1)\)-current that can be written \( \theta = i\partial\bar{\partial}\psi \) with \( \psi \) a metric on a \( \mathbb{R} \)-line bundle \( S \). The twisted Kähler-Einstein equation is

\[
\text{Ric}(\omega) = \omega + \theta,
\]

for a Kähler metric \( \omega \) in the class \( c[-(K_X + S)] \). Writing \( \omega = i\partial\bar{\partial}\phi \), where \( \phi \) is a metric on the \( \mathbb{R} \)-line bundle \( F := -(K_X + S) \), this is equivalent to

\[
(i\partial\bar{\partial}\phi)^n = e^{-(\phi + \psi)}
\]

after adjusting constants.

To be able to apply Theorems 6.1 and 6.2 we need to assume that \( e^{-\psi} \) is integrable. By this we mean that representatives with respect to a local frame are integrable. When \( \theta = [\Delta] \) is the current defined by a divisor, it means that the divisor is klt.

Solutions \( \phi \) of (6.2) are now critical points of the function

\[
G_{\psi}(\phi) := -\log \int e^{-(\phi + \psi)} - \mathcal{E}(\phi, \chi)
\]

where \( \chi \) is an arbitrary metric on \( F \). Here \( \psi \) is fixed and we let the variable \( \phi \) range over bounded metrics with \( i\partial\bar{\partial}\phi \geq 0 \). If \( \phi_0 \) and \( \phi_1 \) are two critical points, it follows from the discussion in section 2 that we can connect them with a bounded geodesic \( \phi_t \). Since \( \mathcal{E} \) is affine along the geodesic it follows that

\[
t \rightarrow -\log \int e^{-(\phi_t + \psi)}
\]

is affine along the geodesic and we can apply Theorem 6.1.

**Theorem 6.5.** Assume that \(-K_X\) is semipositive and that \( H^{0,1}(X) = 0 \). Assume that \( i\partial\bar{\partial}\psi = \theta \), where \( e^{-\psi} \) is integrable. Let \( \phi_0 \) and \( \phi_1 \) be two bounded solutions of equation (6.3) with \( i\partial\bar{\partial}\phi_j \geq 0 \). Then there is a holomorphic automorphism, \( F \), of \( X \), homotopic to the identity, such that

\[
F^*(\partial\bar{\partial}\phi_1) = \partial\bar{\partial}\phi_0
\]

and

\[
F^*(\theta) = \theta.
\]

**Proof.** By Theorem 6.1 there is an \( F \) such that

\[
F^*(\partial\bar{\partial}\phi_1 + \partial\bar{\partial}\psi) = \partial\bar{\partial}\phi_0 + \partial\bar{\partial}\psi
\]

so we just need to see that \( F \) preserves \( \theta = i\partial\bar{\partial}\psi \). But this follows since \( \omega^j := i\partial\bar{\partial}\phi_j \) solves (6.1) and \( F^*(\text{Ric}(\omega^1)) = \text{Ric}(F^*(\omega^1)) \). Thus

\[
\omega^1 + \theta = \text{Ric}(\omega^1)
\]

implies

\[
\omega_0 + F^*(\theta) = \text{Ric}(\omega_0) = \omega_0 + \theta,
\]

and we are done.
Remark 4. Note that in case \( \theta \) is strictly positive we even get absolute uniqueness. This follows from the proof of Theorem 6.1 since both \( \phi_t \) and \( \phi_t + \psi \) must be geodesics, which forces \( \phi_t \) to be linear in \( t \) if \( i\partial\bar{\partial}\psi > 0 \). Certainly the assumption on strict positivity can be considerably relaxed here, see the end of the next section for a comment on this.

In the same way we get from Theorem 6.3

**Theorem 6.6.** Assume that \(-K_X = -(K_X + S) + S\) where \(-(K_X + S)\) is semipositive and \( S \) is the \( \mathbb{R} \)-line bundle corresponding to a klt divisor \( \Delta \geq 0 \) with simple normal crossings. Assume also that \( H^{0,1}(X) = 0 \). Let \( \phi_0 \) and \( \phi_1 \) be two bounded solutions of equation (6.2) with \( \theta = [\Delta] \) and with \( i\partial\bar{\partial}\phi_j \geq 0 \). Then there is a holomorphic automorphism, \( F \), of \( X \), homotopic to the identity, such that

\[
F^*(\partial\bar{\partial}\phi_1) = \partial\bar{\partial}\phi_0
\]

and

\[
F^*([\Delta]) = [\Delta].
\]

7. A CONCLUDING (WONKISH) REMARK ON COMPLEX GRADIENTS

The curvature formula in Theorem 3.1 is based on a particular choice of the auxiliary \((n-1,0)\) form \( v_t \) as the solution of an equation

\[
\partial^\phi v_t = \pi_\perp (\dot{\phi}_t u_t).
\]

In the case when \( \phi_t \) is smooth and \( i\partial\bar{\partial}_X \phi_t > 0 \) one could alternatively choose \( \tilde{v}_t \) as

\[
\tilde{v}_t = V_t \lceil u,
\]

where \( V_t \) is the complex gradient of \( \dot{\phi}_t \) defined by

\[
V_t \lceil \partial\bar{\partial}_X \phi_t = \bar{\partial} \dot{\phi}_t.
\]

This leads to a different formula for the curvature which is the one used in [7]:

\[
\langle \Theta^F u, u \rangle = \int_{X_t} c(\phi) |u|^2 e^{-\phi} + \langle (\Box + 1)^{-1} \bar{\partial} \tilde{v}_t, \tilde{v}_t \rangle,
\]

where \( \Box \) is the \( \partial\bar{\partial} \)-Laplacian for the metric \( i\partial\bar{\partial}_X \phi_t \). The relation between the two formulas is discussed in [8] in the more general setting of a nontrivial fibration. At any rate, the two choices \( v_t \) and \( \tilde{v}_t \) coincide in case the curvature vanishes, as we have seen in section 3.

Of course the definition of \( \tilde{v}_t \) makes no sense in our more general setting since we have no metric on \( X \) to help us define a complex gradient. Nevertheless, the methods of section 3 can perhaps be seen as giving a way to define a 'complex gradient' in a nonregular situation. We formulate the basic principle in the next proposition.

**Proposition 7.1.** Let \( L \) be a holomorphic line bundle over the compact Kähler manifold \( X \), and let \( \phi \) be a smooth metric on \( L \), not necessarily with positive curvature. Assume \( V \) is a holomorphic vector field on \( X \) such that

\[
V \lceil \partial\bar{\partial}\phi = 0.
\]
Then $V = 0$ provided that

$$H^{(0,1)}(X, K_X + L) = 0$$

and

$$H^0(X, K_X + L) \neq 0.$$

**Proof.** We follow the arguments in section 3. Let $u$ be a global holomorphic section of $K_X + L$, and put

$$v := V \rfloor u.$$ 

Then $v$ is a holomorphic $(n - 1, 0)$-form and

$$\partial \bar{\partial} \phi \wedge v = -(V \rfloor \partial \bar{\partial} \phi) \wedge u = 0.$$ 

Hence

$$\partial \bar{\partial} \phi \wedge v = -\partial \bar{\partial} \wedge v = 0.$$ 

Put $\alpha = v \wedge \omega$ where $\omega$ is the Kähler form. Then $\alpha$ is a smooth, $\bar{\partial}$-closed $(n, 1)$-form solving

$$\bar{\partial} \partial^* \phi \alpha = 0.$$ 

This means that $\partial^* \phi \alpha$ is a holomorphic, hence smooth $(n, 0)$-form. Integrating by parts we get

$$|\partial^* \phi \alpha|^2 = 0$$

Since we have assumed $H^{(n,1)} = 0$, $\alpha = \bar{\partial} g$ for some $g$. Then

$$\|\alpha\|^2 = \langle \alpha, \bar{\partial} g \rangle = 0$$

so $v$ and hence $V$ are 0.

This means that holomorphic solutions of

$$V \rfloor \partial \bar{\partial} \phi = \bar{\partial} \chi$$

are unique, if they exist.

Let us finally compare this to our first uniqueness result for twisted Kähler-Einstein equations, Theorem 6.5, and the remark immediately after it (Remark 4). There we noted that in case the twisting term $\theta$ is strictly positive, the automorphism $F$ must be the identity, so that we even get absolute uniqueness, and not just uniqueness up to a holomorphic automorphism. A considerably more general statement follows from Proposition 7.1: For absolute uniqueness it suffices to assume that some multiple of the $\mathbb{R}$ bundle $S$ satisfies the cohomological assumptions in Proposition 7.1, $H^0(X, K_X + mS) \neq 0$ and $H^1(X, K_X + mS) = 0$. This is certainly the case (by Kodaira vanishing) if $S > 0$, even if $\theta$ itself is not assumed positive. Of course it also holds in many other cases that are not covered by Kodaira’s theorem.

In this connection, notice also that some kind of regularity of $\phi$ in Proposition 7.1 is necessary, since for completely general metrics the meaning of the operators $\partial \phi$ and $\bar{\partial} \phi$ becomes unclear. This is not just a technical problem. The vector field $z \partial / \partial z$ vanishes at $z = 0$ and $z = \infty$ on the Riemann sphere. But, the divisor $\{0\} \cup \{\infty\}$ is certainly ample.
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