Spectrum, Effective Action and First Law of Wedge Holography

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Abstract

In this paper, we study the mass spectrum, the effective action and the first law of entanglement entropy for a novel doubly holographic model called wedge holography. Remarkably, there is always a massless mode of gravitons on the end-of-the-world branes in wedge holography. This happens because one imposes Neumann boundary condition on both branes. On the other hand, the massless mode disappears if one imposes Dirichlet boundary condition on one of the branes as in brane world theory and AdS/BCFT. We also study the effective action of quantum gravity on the branes. In the perturbative formulation, it is given by an infinite sum of Pauli-Fierz actions. In the non-perturbative formulation, the effective action is composed of a higher derivative gravity and a matter action. Usually, a higher derivative gravity can be renormalizable but suffers the ghost problem. For our case, since the effective theory on the brane is equivalent to Einstein gravity in the bulk, it must be ghost-free. We notice that the matter action plays an important role in eliminating the ghost. We also provide evidences that the higher derivative gravity on the brane is equivalent to a ghost-free multi-gravity. Besides, we prove that the non-perturbative effective action yields the correct Weyl anomaly. Remarkably, although the effective action on the brane is an infinite tower of higher derivative gravity, the holographic Weyl anomaly is exactly the same as that of Einstein gravity. Finally, we verify the first law of entanglement entropy for wedge holography. Interestingly, the massive fluctuations are irrelevant to the first order perturbation of the holographic entanglement entropy. Thus, in many aspects, the effective theory on the brane behaves like massless Einstein gravity.

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1 Introduction

Recently, a novel codimension two holography called wedge holography is proposed [1], which conjectures that the following dualities hold

\[
\text{Classical gravity on wedge } W_{d+1} \simeq (\text{Quantum) gravity on two branes } (Q_1 \cup Q_2) \simeq \text{CFT}_{d-1} \text{ on corner of wedge } \Sigma,
\]

where \( W_{d+1} \) is the \( d+1 \) dimensional wedge space, which is bounded by two end-of-the-world branes, i.e., \( \partial W = Q_1 \cup Q_2 \), and \( \Sigma \) is the corner of the wedge. See Fig.1 (left) for the geometry.

See also [2, 3, 4, 5] for some related works. Wedge holography is a generalization of the AdS/CFT correspondence [6, 7, 8], and is closely related to brane world holography [9, 10, 11], AdS/BCFT [14, 15, 16, 17, 18, 19, 20] and the doubly holographic model [21, 22, 23]. See [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45] for recent works on double holography and island. As shown in Fig.1 (right), wedge holography can be obtained from AdS/BCFT by taking the zero-volume limit \( M \to 0 \). In this limit, the bulk mode of CFTs on \( M \) disappears and only the edge mode on the boundary \( \Sigma \) survives. Thus wedge holography can be regarded as a holographic dual of the edge mode on the boundary (codim-1 defect) [4]. Generalizing wedge holography to codim-n defects, [4] proposes the so-called cone holography, which can be derived from suitable limit of holographic defect CFT [46, 47, 48].

The gravitational action of wedge holography is given by [1]

\[
I_W = \frac{1}{16\pi G_N} \int_W \sqrt{g} (R - 2\Lambda) + \frac{1}{8\pi G_N} \int_{Q_1 \cup Q_2} \sqrt{h} (K - T),
\]

where \( W \) denotes \( d+1 \) dimensional wedge space, \( Q_1 \) and \( Q_2 \) denote two end-of-the-world branes, \( K \) is the trace of the extrinsic curvature, and \( T \) is the tension of the brane. Following [14], [1] proposes to impose Neumann boundary condition (NBC) on the end-of-the-world branes

\[
\text{NBC} : \quad (K - T) h^{ij} \big|_Q = 0.
\]

Actually, the Dirichlet boundary condition (DBC) [17]

\[
\text{DBC} : \quad \delta h_{ij} \big|_Q = 0,
\]

and the conformal boundary condition (CBC) [20]

\[
\text{CBC} : \quad \begin{cases} 
K = \frac{d}{d-1} T, \\
\delta h_{ij} \big|_Q = 2\sigma(y) h_{ij} \big|_Q,
\end{cases}
\]

with \( \sigma(y) \) a conformal factor also work well. See also [49, 50, 51, 52, 53, 54] for discussions on various boundary conditions of gravity. For simplicity, we mainly focus on NBC in this
paper. It is found that wedge holography can yield the expected free energy, Weyl anomaly, entanglement/Rényi entropy, two point functions and so on [1, 3]. In particular, it obeys the holographic c-theorem [3]. For one novel class of solutions, it is proved that wedge holography is equivalent to AdS/CFT with Einstein gravity [3]. These are all strong supports for wedge holography.

The previous works [1, 3] mainly focus on a special class of solutions. In this paper, we discuss the general solutions and analyze the mass spectrum of wedge holography. Interestingly, we find that there is a massless mode on the end-of-the-world branes if we impose NBC on both of the two branes. On the other hand, the massless mode disappears if one imposes DBC/CBC on one of the branes as in brane world holography and AdS/BCFT. We also work out the effective action of quantum gravity on the end-of-the-world branes with NBC. In the perturbative formulation, the effective action is given by an infinite sum of Pauli-Fierz actions of massive gravity. In the non-perturbative formulation, it is given by a higher derivative gravity plus a matter action. Usually, a higher derivative gravity suffers the problem of ghost. However, since the effective action on the brane is equivalent to Einstein gravity in the bulk, it must be ghost-free. We discuss the mechanism to eliminate the ghost and find that the matter action plays an important role. We argue that the higher derivative gravity on the brane is equivalent to a ghost-free multi-gravity. Besides, we prove that the non-perturbative effective action can produce the correct Weyl anomaly. Interestingly, although the effective action on the brane is an infinite tower of higher derivative gravity, the holographic Weyl anomaly is exactly the same as that of Einstein gravity. Finally, we verify the first law of entanglement entropy and provide more supports for wedge holography.

The paper is organized as follows. In section 2, we study the mass spectrum and perturbative effective action of vectors on the end-of-the-world branes. In section 3, we generalize the discussions to the gravitons on the end-of-the-world branes. In section 4, we investigate the non-perturbative effective action of gravity and discuss the mechanism to eliminate the ghost. In section 5, we derive the holographic Weyl anomaly from the non-perturbative effective action of gravity on the brane. In section 6, we discuss the first law of entanglement entropy for wedge holography. Finally, we conclude with some open problems in section 6.
2 Vector on the brane

To warm up, we study the mass spectrum and effective action of vectors on the end-of-the-world brane in this section. The case of gravity is similar.

2.1 Mass spectrum

Let us first discuss the mass spectrum. For simplicity, we focus on the probe limit, where the bulk metric and the embedding function of \( Q \) are given by

\[
\begin{align*}
\text{ds}^2 &= dr^2 + \cosh^2(r)\bar{h}_{ij}^{(0)}(y)dy^i dy^j, \\
Q : r &= \pm \rho.
\end{align*}
\]

Here \( \bar{h}_{ij}^{(0)}(y) \) is the AdS metric with the unit radius \( L = 1 \). We take the following ansatz of Maxwell’s fields in the bulk

\[
A_r = 0, \quad A_i = S(r)A_i(y).
\]

Substituting (7) into Maxwell’s equations \( \nabla_{\mu}F^{\mu\nu} = 0 \) and separating variables, we obtain

\[
\begin{align*}
\bar{D}_i F^{ij} - m_v^2 A^j &= 0, \\
\cosh^2(r)S''(r) + (d - 2) \sinh(r) \cosh(r)S'(r) + m_v^2 S(r) &= 0,
\end{align*}
\]

where \( F = dA, \bar{D}_i \) are the covariant derivatives with respect to \( \bar{h}_{ij}^{(0)}(y) \), and \( m_v \) denotes the mass of vectors. Solving (9), we get

\[
S(r) = \text{sech} \frac{d-2}{2} r \left( c_1 P_{\lambda_v}^{\frac{d-2}{2}}(\tanh r) + c_2 Q_{\lambda_v}^{\frac{d-2}{2}}(\tanh r) \right),
\]

where \( P_{\lambda_v}^{\frac{d-2}{2}} \) and \( Q_{\lambda_v}^{\frac{d-2}{2}} \) are the Legendre polynomials, \( c_1 \) and \( c_2 \) are integral constants and \( \lambda_v \) is given by

\[
\lambda_v = \frac{1}{2} \left( \sqrt{(d-3)^2 + 4m_v^2} - 1 \right).
\]

From (11), we derive the expected Breitenlohner-Freedman (BF) bound of massive vectors in \( AdS_d \)

\[
m_v^2 \geq -\left( \frac{d-3}{2} \right)^2.
\]

One can impose either the absolute BC or the relative BC for vectors on the end-of-world branes

\[
\text{absolute BC} : F_{ni}\mid_{Q} = 0, \\
\text{relative BC} : \ast F_{ni}\mid_{Q} = 0,
\]

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where $n$ denotes the normal direction, $i$ denotes the tangent direction, $\mathcal{F} = dA$ is the bulk field strength and $^*\mathcal{F}$ is the Hodge dual of $\mathcal{F}$. For our ansatz (5,6,7), the absolute BC and relative BC become Neumann BC (NBC) and Dirichlet BC (DBC), respectively

\begin{align}
\text{NBC} : S'(\pm \rho) &= 0, \quad (14) \\
\text{DBC} : S(\pm \rho) &= 0. \quad (15)
\end{align}

The dynamical field on the brane is the induced vector and its conjugate momentum for NBC and DBC, respectively

\begin{align}
\delta A_i &= S'(\pm \rho) A_i, \quad \text{for absolute BC/NBC}, \\
\delta F_{ni} &= \pm S'(\pm \rho) A_i, \quad \text{for relative BC/DBC}. \quad (16)
\end{align}

Now we are ready to study the mass spectrum of vectors on the end-of-the-world branes. Let us first study the case of NBC. For the following purpose, we denote $S(r)$ by $S(r) = c_1 \bar{P}(r) + c_2 \bar{Q}(r)$ where $\bar{P}(r) = \text{sech}^{d-2}(r)P_{\lambda^2_v}(\tanh r)$ and $\bar{Q}(r) = \text{sech}^{d-2}(r)Q_{\lambda^2_v}(\tanh r)$. Imposing NBC (14), we get

\begin{align}
M \cdot c = \begin{pmatrix} \bar{P}'(\rho) & \bar{Q}'(\rho) \\ \bar{P}'(-\rho) & \bar{Q}'(-\rho) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0. \quad (17)
\end{align}

To have non-trivial solutions of $c_1$ and $c_2$, we must have

\begin{align}
|M| = \bar{P}'(\rho) \bar{Q}'(-\rho) - \bar{Q}'(\rho) \bar{P}'(-\rho) = 0, \quad (18)
\end{align}

which gives a constraint for the mass of the vector

\begin{align}
\text{NBC} : m_v^2 \text{sech}^d(\rho) \left( P_{\lambda_v}^{d-2}(x)Q_{\lambda_v}^{d-2}(-x) - P_{\lambda_v}^{d-2}(-x)Q_{\lambda_v}^{d-2}(x) \right) = 0, \quad (19)
\end{align}

where $x = \tanh(\rho)$ and $\lambda_v$ is a function of the mass $\lambda_v$.

It is clear that the massless mode with $m_v = 0$ is a solution to (19) for wedge holography with NBC. On the other hand, there is no massless mode for the usual brane world theory and AdS/BCFT defined in the region $-\rho \leq r \leq \infty$ with NBC on the brane $S'(\rho) = 0$ and DBC $S(\infty) = 0$ on the AdS boundary. Let us explain more on this point. Solving (9) with $m_v = 0$, we get

\begin{align}
S(r) = c_1 + c_2 \int_0^r \text{sech}^{d-2}(s)ds. \quad (20)
\end{align}

Imposing NBC $S'(\pm \rho) = 0$ for wedge holography, we derive

\begin{align}
S(r) = c_1, \quad (21)
\end{align}

\footnote{For simplicity, we focus on one side of the brane in brane world theory, where the junction condition is equivalent to NBC.}
which yields non-zero induced vector $A_i = S(\pm \rho)A_i = c_1 A_i$ on the brane. On the other hand, the BCs $S'(-\rho) = 0$ and $S(\infty) = 0$ for brane world theory and AdS/BCFT yield $S(r) = 0$. As a result, there is no massless mode in the usual brane world holography and AdS/BCFT. We take the existence of massless modes as an advantage of the wedge holography with NBC, since there are Maxwell’s fields on the end-of-the-world branes. Furthermore, the massless mode can be normalized since wedge holography is defined in a finite region $-\rho \leq r \leq \rho$.

For $\rho > 0$, the above solutions cannot satisfy NBC $S'(\pm \rho) = 0$, so we should rule out (22) from the spectrum. As a result, we have $m_v^2 \geq 0$ for the mass spectrum of vectors with NBC.

Let us draw some figures to illustrate the spectrum. See Fig.2, where the intersections of the curves and $m_v^2$-axis denote the mass squares. As shown in Fig.2, the massless mode is indeed a solution. Besides, the larger the tension $T = (d - 1) \tanh(\rho)$ is, the continuous the mass spectrum is.

To get more understandings of the spectrum, let us study two limits. For the large $\rho \to \infty$ ($x \to 1$), (19) can be approximately by

$$0 \approx \text{sech}^{d+2}(\rho) \begin{cases} \sin(\lambda_v \pi) & \text{even } d, \\ \cos(\lambda_v \pi) & \text{odd } d, \end{cases}$$

(24)

which has the roots

$$m_v^2 \approx k(k + d - 3), \quad \text{for large } \rho,$$

(25)

where $k \geq 0$ are integers. In the contrary limit of small $\rho$, $-\rho \leq r \leq \rho$ is also small. Then (9) becomes approximately by

$$S''(r) + m_v^2 S(r) = 0,$$

(26)
Figure 2: Mass spectrum for vectors with NBC. The left figure is for $d = 3$, and the right figure is for $d = 4$. $|M|$ is given by (19) and the roots of $|M| = 0$ denote the mass squares. The massless mode is always a solution to NBC. Besides, the larger the tension $T = (d-1) \tanh(\rho)$ is, the continuous the mass spectrum is.

which can be solved as

$$S(r) = c_1 \cos(|m_v|r) + c_2 \sin(|m_v|r).$$

(27)

Imposing NBC (14) and following the above approach, we get

$$m_v^2 \sin(2|m_v|\rho) = 0,$$

(28)

which yields the mass spectrum

$$m_v^2 \approx \frac{k^2 \pi^2}{4 \rho^2}, \quad \text{for small } \rho.$$

(29)

As a summary, the mass spectrum of vectors with NBC is non-negative, i.e., $m_v^2 \geq 0$. In particular, it contains a massless mode, which is quite different from the brane world holography and AdS/BCFT. For the large $\rho$, the mass spectrum is given by (25). While for small $\rho$, the mass spectrum is approximately by (29), which is independent of the dimensions of spacetime.

Let us go on to study the mass spectrum of DBC (15). Since the approach is quite similar to that of NBC (14), we do not repeat the calculations but just list main results below. The constraint of the mass spectrum with DBC is given by

$$\text{DBC} : \text{sech}^{d-2}(\rho) \left( P_{\lambda_v}^{4-1} (x) Q_{\lambda_v}^{4-1} (-x) - P_{\lambda_v}^{4-1} (-x) Q_{\lambda_v}^{4-1} (x) \right) = 0,$$

(30)

where $x = \tanh(\rho)$. The mass spectrum of vectors with DBC includes only positive $m_v^2 > 0$. Unlike NBC, the massless mode is no longer a solution to DBC. One can easily check that the solution (20) with $m_v = 0$ cannot satisfy the DBC $S(\pm \rho) = 0$ unless $S(r) = 0$. In the
Figure 3: Mass spectrum for vectors with DBC. The left figure is for $d = 3$, and the right figure is for $d = 4$, $|M|$ is given by (30) and the roots of $|M| = 0$ denote the mass squares. The massless mode should be removed in the above figures for DBC. The larger the tension $T = (d - 1) \tanh(\rho)$ is, the continuous the mass spectrum is.

large and small $\rho$ limit, except a massless mode, the mass spectrum of DBC is exactly the same as that of NBC

DBC : $m_v^2 \approx \begin{cases} l(l + d - 3), & \text{for large } \rho, \\ \frac{\rho^2 \pi^2}{4}, & \text{for small } \rho, \end{cases}$

where $l > 0$ is a positive integer. See Fig. 3 for the mass spectrum of vectors with DBC.

2.2 Perturbative action

Let us go on to study the perturbative action of vectors on the end-of-the-world branes. We expand the bulk vectors in infinite powers of Kaluza-Klein (KK) modes

$$A_r = 0, \quad A_i = \sum_{m_v} S^{(m_v)}(r) A^{(m_v)}_i(y),$$

where $\sum_{m_v}$ denotes the sum over the mass spectrum, $A^{(m_v)}_i(y)$ and $S^{(m_v)}(r)$ obey EOM (8,9) and the orthogonal condition

$$\int_{-\rho}^{\rho} \cosh(r) r^{-4} S^{(m_v)}(r) S^{(m'_v)}(r) dr = \delta^{m_v, m'_v}.$$  

Substituting (32) into the bulk action

$$I = -\frac{1}{4} \int d^{d+1}x \sqrt{|g|} F_{\mu\nu} F^{\mu\nu},$$

we get

$$I = -\frac{1}{4} \sum_{m_v, m'_v} \int_{-\rho}^{\rho} dr \int d^d y \sqrt{|\hat{h}(0)|} 2 \cosh(r)^{d-2} S^{(m_v)}(r) S^{(m'_v)}(r) A^{(m_v)}_i A^{(m'_v)}_i.$$
where $A^i$ and $F^{ij}$ are raised by $\bar{h}^{(0)ij}$. Integrating by parts and imposing either NBC $S'(\pm \rho) = 0$ or DBC $S(\pm \rho) = 0$, we derive

$$I = -\frac{1}{4} \sum_{m_v} \int_{-\rho}^{\rho} dr \int_Q d^d y \sqrt{|\bar{h}(0)|} \left[ - \frac{d}{dr} \left( 2 \cosh(r)^{d-2} S^{(m_v)}(r) \right) S^{(m_v)}(r) A_i^{(m_v)} A_i^{(m_v)}i 
$$
$$+ \cosh(r)^{d-4} S^{(m_v)} S^{(m_v)}(r) F_{ij}^{(m_v)} F_{ij}^{(m_v)} \right].$$

(36)

By using EOM (9) and orthogonal condition (33), the above action can be simplified as

$$I = -\frac{1}{4} \sum_{m_v} \int_{-\rho}^{\rho} dr \cosh(r)^{d-4} S^{(m_v)}(r)^2 \int_Q d^d y \sqrt{|\bar{h}(0)|} \left[ 2m_v^2 A_i^{(m_v)} A_i^{(m_v)}i + F_{ij}^{(m_v)} F_{ij}^{(m_v)} \right]$$

(37)

which is the correct action of massive vectors.

### 3 Gravity on the brane

In this section, we investigate the mass spectrum and effective action of gravitons on the end-of-the-world branes. Interestingly, the mass spectrum of gravitons in $d$ dimensions is the same as that of vectors in $(d + 2)$ dimensions. In particular, there is a massless mode when one imposes NBC on both branes.

#### 3.1 Mass spectrum

Let us first discuss the mass spectrum of gravitons on the end-of-the-world branes. We choose the following ansatz of the perturbation metric and the embedding function of $Q$

$$ds^2 = dr^2 + \cosh^2(r) \left( \bar{h}^{(0)}_{ij}(y) + \epsilon H(r) \bar{h}^{(1)}_{ij}(y) \right) dy^i dy^j + O(\epsilon^2),$$

(38)

$$Q : r = \pm \rho + O(\epsilon^2),$$

(39)

where $\bar{h}^{(0)}_{ij}(y)$ is the AdS metric with a unit radius and $\bar{h}^{(1)}_{ij}(y)$ denotes the perturbation. In terms of bulk metric perturbations, we have

$$\delta g_{r \mu} = 0, \quad \delta g_{ij} = \cosh^2(r) H(r) \bar{h}^{(1)}_{ij}(y).$$

(40)

Imposing the transverse traceless gauge

$$\nabla^\mu \delta g_{\mu \nu} = 0, \quad g^{\mu \nu} \delta g_{\mu \nu} = 0,$$

(41)

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we get
\[ \bar{D}^i \bar{h}^{(1)}_{ij} = 0, \quad \bar{h}^{(0)}_{ij} \bar{h}^{(1)}_{ij} = 0, \quad (42) \]
where \( \nabla_\mu \) and \( \bar{D}_i \) are the covariant derivatives with respect to \( g_{\mu\nu} \) and \( \bar{h}^{(0)}_{ij} \), respectively. One can impose either NBC [13], DBC [17] or CBC [20] on the end-of-the-world branes
\[ \text{NBC:} \quad H'(\pm \rho) = 0, \quad (43) \]
\[ \text{DBC/CBC:} \quad H(\pm \rho) = 0, \quad (44) \]
where CBC specifies the conformal geometry of the boundary and the trace of the extrinsic curvature. Note that DBC and CBC are the same at the linear order of perturbations, they are different at higher orders generally [20].

In the gauge (41), Einstein equations become
\[ (\nabla_\alpha \nabla^\alpha + 2) \delta g_{\mu\nu} = 0. \quad (45) \]
Substituting (40) together with (42) into (45) and separating variables, we obtain
\[ \left( \bar{D}_k \bar{D}^k + 2 - m_g^2 \right) \bar{h}^{(1)}_{ij} (y) = 0, \quad (46) \]
\[ \cosh^2(r) H''(r) + d \sinh(r) \cosh(r) H'(r) + m_g^2 H(r) = 0, \quad (47) \]
where \( m_g \) denotes the mass of gravitons. Solving (47), we get
\[ H(r) = \sech^2(r) \left( c_1 P^d_{\lambda_g} (\tanh r) + c_2 Q^d_{\lambda_g} (\tanh r) \right), \quad (48) \]
where \( P^d_{\lambda_g} \) and \( Q^d_{\lambda_g} \) are the Legendre polynomials, \( c_1 \) and \( c_2 \) are integral constants and \( \lambda_g \) is given by
\[ \lambda_g = \frac{1}{2} \left( \sqrt{(d-1)^2 + 4m_g^2} - 1 \right), \quad (49) \]
which yields the correct BH bound of massive gravity in \( AdS_d \)
\[ m_g^2 \geq - \frac{(d-1)^2}{2} \quad (50) \]
Recall that we have set the AdS radius \( L = 1 \).

Remarkably, the EOM of \( H(r) \) (47) in \( d \) dimensions is exactly the same as that of \( S(r) \) (9) in \( (d+2) \) dimensions. Furthermore, \( H(r) \) and \( S(r) \) obey the same boundary conditions [14, 15, 43, 44]. As a result, the mass spectrum of gravitons in \( d \) dimensions is the same as the spectrum of vectors in \( (d+2) \) dimensions. Thus there is no need to repeat the calculations again. For the convenience of readers, we list the key characteristics of the gravitational spectrum below.

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1. The gravitational spectrum obeys the following constraint

\[ \text{NBC} : m_g^2 \text{sech}^{d+2}(\rho) \left( P^\frac{d}{\lambda_g} (x) Q^\frac{d}{\lambda_g} (-x) - P^\frac{d}{\lambda_g} (-x) Q^\frac{d}{\lambda_g} (x) \right) = 0, \tag{51} \]

\[ \text{DBC/CBC} : \text{sech}^d(\rho) \left( P^\frac{d}{\lambda_g} (x) Q^\frac{d}{\lambda_g} (-x) - P^\frac{d}{\lambda_g} (-x) Q^\frac{d}{\lambda_g} (x) \right) = 0, \tag{52} \]

where \( x = \tanh(\rho) \) and \( \lambda_g = \lambda_v (d \to d + 2) \) is given by (49).

2. We have \( m_g^2 \geq 0 \) for NBC and \( m_g^2 > 0 \) for DBC/CBC. In particular, there is a massless mode when one imposes NBC on both branes. This is consistent with the result of [3], which finds that, for one class of solutions to NBC, the effective gravity on the branes is Einstein gravity. On the other hand, there is no massless mode in the brane-world holography [9, 10, 11] unless the brane tension approaches the critical value \( T = (d - 1) \).

3. In the large and small \( \rho \) limit, except a massless mode, the mass spectrum of NBC is exactly the same as that of DBC/CBC

\[ \text{NBC/DBC/CBC} : m_g^2 \approx \begin{cases} k(k + d - 1), & \text{for large } \rho, \\ \frac{k^2 \pi^2}{4 \rho^2}, & \text{for small } \rho, \end{cases} \tag{53} \]

where \( k \) is an integer, \( k \geq 0 \) for NBC and \( k > 0 \) for DBC/CBC. Recall that \( m_g^2 \) is defined in (47) with respect to \( \tilde{h}^{(1)}_{ij} \) instead of the induced metric \( \cosh^2(\rho) \tilde{h}^{(1)}_{ij} \). For the induced metric, the mass is defined by

\[ \left( \Box + \frac{2}{\cosh^2(\rho)} - M_g^2 \right) \tilde{h}^{(1)}_{ij} (y) = 0, \tag{54} \]

where \( \Box = \tilde{D}^i \tilde{D}_i / \cosh^2(\rho) \) is the D’Alembert operator with respect to \( \cosh^2(\rho) \tilde{h}^{(1)}_{ij} \). Comparing (47) with (54), we read off

\[ M_g^2 = \frac{m_g^2}{\cosh^2(\rho)}. \tag{55} \]

For the large \( \rho \), the mass spectrum of \( M_g^2 \) becomes almost continuous. While for the small \( \rho \), the massive modes are frozen at low energy due to the infinite masses and only the massless mode is excited. As a result, the effective theory of wedge holography at low energy is Einstein gravity.

4. Let us draw some figures to illustrate the spectrum. See Fig.4 and Fig.5 for NBC and DBC respectively, where the intersections of the curves and \( m_g^2 \)-axis denote the mass squares. We notice that the massless mode is indeed a solution to NBC. For both NBC and DNC, the larger the tension \( T = (d - 1) \tanh(\rho) \) is, the continuous the mass spectrum is.
Figure 4: Mass spectrum for gravitons with NBC. The left figure is for $d = 3$, and the right figure is for $d = 4$, $|M|$ is given by (51) and the roots of $|M| = 0$ denote the mass squares. The massless mode is always a solution to NBC. Besides, the larger the tension $T = (d - 1) \tanh(\rho)$ is, the continuous the mass spectrum is.

Figure 5: Mass spectrum for gravitons with DBC. The left figure is for $d = 3$, and the right figure is for $d = 4$, $|M|$ is given by (52) and the roots of $|M| = 0$ denote the mass squares. The massless mode should be removed in the above figures for DBC. The larger the tension $T = (d - 1) \tanh(\rho)$ is, the continuous the mass spectrum is.

3.2 Perturbative action

Let us go on to discuss the perturbative action of gravitons on the end-of-the-world branes. Similarly, we expand the metric perturbations (40) in powers of KK modes

$$\delta g_{r\mu} = 0, \quad \delta g_{ij} = \cosh(r)^2 \sum_{m_g} H^{(m_g)}(r) \tilde{h}^{(m_g)}_{ij}(y),$$

(56)

where $\sum_{m_g}$ denotes the sum over the gravitational mass spectrum, $\tilde{h}^{(m_g)}_{ij}(y)$ and $H^{(m_g)}(r)$ satisfy EOM (46,47) and the orthogonal condition

$$\int_{-\rho}^{\rho} \cosh(r)^{d-2} H^{(m_g)}(r) H^{(m'_g)}(r) dr = \delta^{m_g} m'_g.$$

(57)

The Einstein-Hilbert action expanded to second order of the metric perturbation is given
by
\[ I_{EH} = \int_{\partial N} d^{d+1}x \sqrt{|\mathcal{g}|} \left( -\frac{1}{4} \nabla_\alpha H_\mu \nabla^\alpha H^{\mu} + \frac{1}{2} \nabla_\alpha H_\mu \nabla^\nu H^{\mu \alpha} - \frac{d}{2} \bar{H}_\mu \bar{H}^{\mu} \right) \\
+ \int_{\partial N} d^{d}y \sqrt{|\mathcal{h}|} n^\mu \bar{H}^{\alpha \beta} (\nabla_\mu H_{\alpha \beta} - \nabla_\alpha \bar{H}_{\beta}^\mu), \] (62)
where we have used $\mathcal{g}$ by $\mathcal{h}$, which denote the metric perturbation and we have used the gauge (41) for simplicity. The second order of perturbations of the Gibbons-Hawking term plus the brane action are
\[ I_{GH} = 2 \int_{\partial N} d^{d}y \sqrt{|\mathcal{h}|} (K - T) = \int_{\partial N} d^{d}y \sqrt{|\mathcal{h}|} \tanh(\rho) \left( -\frac{1}{2} \bar{H}_{\alpha \beta} \bar{H}^{\alpha \beta} \right), \] (59)
where we have used $K - T = \tanh(\rho)$, which applies to NBC and CBC at all orders, to DBC at the linear order. For the metric ansatz (38) and the gauge (41), we have $\nabla_r \bar{H}_{ij}|_{\partial N} = 0$ and $\nabla_i \bar{H}_{jr} = -\tanh(\rho) \bar{H}_{ij}$, which yield
\[ n^\mu \bar{H}^{\alpha \beta} \nabla_\mu \bar{H}_{\alpha \beta}|_{\partial N} = 0, \quad n^\mu \bar{H}^{\alpha \beta} \nabla_\alpha \bar{H}_{\beta \mu}|_{\partial N} = -\tanh(\rho) \bar{H}_{\alpha \beta} \bar{H}^{\alpha \beta}. \] (60)
From (58-59), we obtain the total action
\[ I = \int_{N} d^{d+1}x \sqrt{|g|} \left( -\frac{1}{4} \nabla_\alpha H_\mu \nabla^\alpha H^{\mu} + \frac{1}{2} \nabla_\alpha H_\mu \nabla^\nu H^{\mu \alpha} - \frac{d}{2} \bar{H}_\mu \bar{H}^{\mu} \right) \\
+ \frac{1}{2} \int_{\partial N} d^{d}y \sqrt{|\mathcal{h}|} \tanh(\rho) \bar{H}_{\alpha \beta} \bar{H}^{\alpha \beta}, \] (61)
Substituting (60) into (61) and following the approach of vectors, we finally derive
\[ I = \sum_{m_g} \int_{Q} d^{d}y \sqrt{|\mathcal{h}(0)|} \left( -\frac{1}{4} \tilde{D}_k \bar{h}^{(m_g)}_{ij} \bar{D}^{k} \bar{h}^{(m_g)ij} + \frac{1}{2} \tilde{D}_k \bar{h}^{(m_g)}_{ij} \bar{D}^{k} \bar{h}^{(m_g)jk} \\
- \frac{d-1}{2} \bar{h}^{(m_g)ij} \bar{h}^{(m_g)ij} - \frac{1}{4} m_g^2 \bar{h}^{(m_g)}_{ij} \bar{h}^{(m_g)ij} \right), \] (62)
which is an infinite sum of the Fierz-Pauli action in $AdS_d$ with the gauge (42). To derive (62), we find the following formulas are useful
\[ \nabla_r \bar{H}_{ij} = \cosh^2(r) \sum_{m_g} \bar{h}^{(m_g)}_{ij}(y) \frac{d}{dr} H^{(m_g)}(r), \] (63)
\[ \nabla_i \bar{H}_{jr} = -\cosh(r) \sinh(r) \sum_{m_g} \bar{h}^{(m_g)}_{ij}(y) H^{(m_g)}(r), \] (64)
\[ \nabla_i \bar{H}_{jk} = \cosh^2(r) \sum_{m_g} \bar{D}_j \bar{h}^{(m_g)}_{ij}(y) H^{(m_g)}(r). \] (65)
Note that the induced metric on the brane is $\cosh^2(\rho) H(\pm \rho) \bar{h}^{(m_g)}_{ij}$ instead of $\bar{h}^{(m_g)}_{ij}$. Taking into account this fact and the normalization (57), from (62) we can recover the effective Newton’s constant of $\mathcal{K}$. 

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Note that action \(62\) has the correct sign of kinetic term, thus it is ghost-free. Of course, the effective action on the end-of-the-world branes should be ghost-free, since it is derived from Einstein gravity in the bulk. In the next section, we find that the non-perturbative effective action is an infinite power of higher derivative gravity. Usually, a higher derivative gravity suffers the ghost problem. However, as we have argued above, the effective action should be ghost-free. Thus, wedge holography provides an interesting model of ghost-free higher derivative gravity. Please see the following section for the mechanism to eliminate ghost.

4 Non-perturbative effective action

In this section, we investigate the non-perturbative effective action of wedge holography. For simplicity, we focus on NBC. We leave the study of DBC/CBC to future works. We consider only gravity in this section. The generalization to the vector and the scalar is straightforward, but a little complicated.

4.1 Higher derivative gravity on the brane

We apply the method of \([30]\), which is initially developed for the case of large brane tension \(T = (d - 1) \tanh(\rho) \to (d - 1)\), where the end-of-the-world brane approaches to the AdS boundary. In fact, the method of \([30]\) can be generalized to arbitrary brane tension. See \([55]\) for discussions of the case \(d = 3\).

To warm up, let us first consider the case of large brane tension, i.e., \(\rho \to \infty\) \((T \to d - 1)\), which means that the brane is set at infinity and the geometry of wedge holography becomes that of AdS/CFT. In this limit, the bulk gravitational action \(I_W\) is divergent. To get a finite action, one can perform the holographic renormalization by adding the following counterterms on the brane \([56, 57]\)

\[
I_c = \frac{-1}{16\pi G_N} \int_Q \sqrt{|h|} \left[ 2(d-1) - 2T + \frac{1}{d-2} \mathcal{R} + \frac{1}{(d-4)(d-2)^2} \left( \mathcal{R}^{ij} \mathcal{R}_{ij} - \frac{d}{4(d-1)} \mathcal{R}^2 \right) \right] + \mathcal{L}_{\mathcal{R}^3} + \ldots ,
\]

(66)

where \(Q\) denotes \(Q_1 \cup Q_2\), \(T = (d-1) \tanh(\rho)\) \(^2\), \(\mathcal{R}_{ij}\) denotes the intrinsic curvature on the brane and \(\mathcal{L}_{\mathcal{R}^3}\) is given by \([58]\)

\[
\mathcal{L}_{\mathcal{R}^3} = \frac{-2}{(d-2)^3(d-4)(d-6)} \left( \frac{3d+2}{4(d-1)} \mathcal{R} \mathcal{R}_{ij} \mathcal{R}^{ij} - \frac{d(d+2)}{16(d-1)^2} \mathcal{R}^3 - 2 \mathcal{R}^{ij} \mathcal{R}_{ikjl} \mathcal{R}^{kl} \right)
\]

\(^2\)We add \(T\) in the counterterms \([66]\) in order to compensate the brane action \(-2 \int_Q \sqrt{h} T\) in the bulk action \([1]\). In the standard holographic renormalization, we have \(T = 0\) due to the absence of brane action.
Now the renormalized action (the effective action of CFTs)
\[ I_{\text{CFT}} = I_W + I_c, \]  
(68)
becomes finite. From (68), we read off the action of wedge holography
\[ I_W = I_{\text{CFT}} - I_c, \]  
(69)
which is composed of a matter part \( I_{\text{CFT}} \) and a gravitational part \( (-I_c) \).

Now set the brane at a finite place \( r = \pm \rho \) instead of at infinity \( r = \pm \infty \). Then \( I_W \) becomes finite. One can always separate \( I_W \) into a matter part and a gravitational part. The natural conjecture is that the gravitational part is still given by \( (-I_c) \) (66) even if the brane is located at a finite position. One support for this proposal is that \( (-I_c) \) (66) yields the correct leading terms of entanglement entropy [30]. Let us explain more on this point. In double holography, one has two methods to calculate the entanglement entropy. One method is by using holographic entanglement entropy of higher derivative gravity \( (-I_c) \) (66) on the brane [59, 60, 61], the other way is by applying TR formula of Einstein gravity in the bulk [62]. It turns out that these two methods yield the same entanglement entropy for large but finite brane tension [30]. Another support for this proposal is that, as we will show in sect. 5, \( (-I_c) \) (66) yields the correct Weyl anomaly for wedge holography.

Note that, \( I_c \) (66) truncates at finite terms for the critical tension \( T \to (d-1) \), while \( I_c \) (66) includes infinite higher derivative terms for the finite tension \( T < (d-1) \). Note also that (66) works well only for odd \( d \). For even \( d \), one needs to correct \( I_c \) (66). See [30] for some examples. For simplicity, we focus on the case of odd \( d \) in this section.

To end this subsection, let us give a further evidence for the proposal (69). We verify that (69) yields the correct effective action for the following type of solutions
\[ ds^2 = dr^2 + \cosh^2(r)\tilde{h}_{ij}dy^i dy^j, \]  
(70)
where \(-\rho \leq r \leq \rho \), and \( \tilde{h}_{ij} \) obeys the vacuum Einstein equations on the brane
\[ R_{\tilde{h} \ ij} - R_{\tilde{h}} + \frac{(d-1)(d-2)}{2} \tilde{h}_{ij} = 0. \]  
(71)
Here \( R_{\tilde{h} \ ij} \) is the curvature with respect to \( \tilde{h}_{ij} \). The effective action of wedge holography for the solution (5) is given by [3]
\[ I_{W \text{Ein}} = \frac{1}{16\pi G_N} \int_0^\rho \cosh^{d-2}(r)dr \int_Q dy^d \sqrt{\tilde{h}} \left( R_{\tilde{h}} + (d-1)(d-2) \right), \]  
(72)
where we focus on half of the wedge space, i.e., \( 0 \leq r \leq \rho \), for simplicity. Rewriting the above action in form of the induced metric \( h_{ij} = \cosh^2(\rho)\tilde{h}_{ij} \) on the brane, we get
\[ I_{W \text{Ein}} = \frac{1}{16\pi G_N} \int_0^\rho \frac{\cosh^{d-2}(r)}{\cosh^{d-2}(\rho)} dr \int_Q dy^d \sqrt{h} \left( R + \frac{(d-1)(d-2)}{\cosh^2(\rho)} \right). \]  
(73)
For the class of solution \((70)\), we have
\[
\mathcal{R}_{ij} = -(d-1)\text{sech}^2(\rho)h_{ij}, \quad \mathcal{R} = -(d-1)\text{sech}^2(\rho). \tag{74}
\]

By applying the above formula, \((73)\) can be further simplified as
\[
I_{W \text{ Ein}} = \frac{1}{16\pi G_N} \int_\mathcal{Q} dy^d \sqrt{h}(d-1)\text{sech}^d(\rho)B_{\text{sech}^2(\rho)} \left(1 - \frac{d}{2} \frac{1}{2}\right) = \frac{1}{16\pi G_N} \int_\mathcal{Q} dy^d \sqrt{h}\left(-\frac{2(d-1)\epsilon^2}{d-2} + \frac{(1-d)\epsilon^4}{d-4} - \frac{3(d-1)\epsilon^6}{4(d-6)} + O(\epsilon^8)\right), \tag{75}
\]
for odd \(d\), where \(B_a(b, c)\) denotes the beta function and we have reparameterized \(\rho\) by 
\[
\epsilon = \text{sech}(\rho). \tag{76}
\]

Note that the \(\mathcal{R}^n\) terms in the counterterms \((66)\) are of orders \(O(\epsilon^{2n})\). Substituting \((74, 76)\) into the effective action \((69)\), we derive
\[
I_W = I_{\text{CFT}} + \frac{1}{16\pi G_N} \int_\mathcal{Q} dy^d \sqrt{h}\left(-\frac{2(d-1)\epsilon^2}{d-2} + \frac{(1-d)\epsilon^4}{d-4} - \frac{3(d-1)\epsilon^6}{4(d-6)} + O(\epsilon^8)\right), \tag{77}
\]
which agrees with \((75)\) provided that \(I_{\text{CFT}} = 0\). The vanishing of the CFT action has a natural explanation in odd dimensions. In the limit \(\rho \to \infty (\epsilon \to 0)\), the geometry of wedge holography becomes that of AdS/CFT. We can check by direct calculations that \(I_{\text{CFT}} \equiv 0\) indeed vanishes for the class of solutions \((5)\) when \(d\) is odd. It is natural that \(I_{\text{CFT}} = 0\) as we adiabatically pull the brane from infinity to a finite place. \(I_{\text{CFT}} = 0\) means that \((73)\) is the vacuum solution to AdS/CFT, and the complete expressions of the counterterms \((-I_c)\) \((66)\) is given by \((73)\) for the solutions \((5)\). Let us go on to discuss the case of finite tension. For simplicity, we assume that the geometry on the brane is an AdS. Recall that the vacuum energy of CFTs vanishes in odd AdS space, since there is no Weyl anomaly in odd dimensions \(^3\). As a result, it is natural that the effective action of CFTs also vanishes for odd \(d\).

Now we finish the verification that \(I_W\) \((69)\) gives the correct effective action of wedge holography for the class of solutions \((5)\). As a by-product, we notice that, \(I_{\text{CFT}} = 0\) and the complete expression of counterterms \((-I_c)\) \((66)\) is given by \((73)\) for the solutions \((5)\).

### 4.2 Mechanism to eliminate ghosts

As we have shown in the above subsection, the effective action on the end-of-the-world brane is an infinite tower of higher derivative gravity. Usually, a higher derivative gravity suffers

\(^3\)One the other hand, the vacuum energy is non-zero for CFTs in even AdS space, due to the non-trivial Weyl anomaly.
the ghost problem. However, since the effective action is obtained from Einstein gravity in
the bulk, it must be ghost-free. In this subsection, we discuss the mechanism to eliminate
the ghost and find that the matter action $I_{\text{CFT}}$ plays an important role. We also argue that
higher derivative gravity on the brane is equivalent to a ghost-free multi-gravity.

Let us first give a quick review of the ghost problem of higher derivative gravity. Take
curvature squared gravity as an example. In general, it includes a scalar mode, a massless
graviton and a massive graviton. Consider the linear perturbation equation around a flat-
space background

$$
\Box \left( \Box - m^2 \right) \delta h_{ij} = 0,
$$

(78)

where $\Box$ is the D'Alembert operator and the metric perturbation $\delta h_{ij}$ obeys $D_i \delta h^{ij} = 0$ and
$h^{ij} \delta h^{ij} = 0$. From (78), we read off the propagator

$$
D(p) \sim \left( \frac{1}{p^2} - \frac{1}{p^2 + m^2} \right),
$$

(79)

where we have ignored the indexes and the tensor structures. From (79), it is clear that
the massless graviton and the massive graviton cannot both have the correct sign. In other
words, one of them is a ghost. Our case is similar. The linear perturbation equation of the
higher derivative gravity on the brane takes the form

$$
\Pi_{n=0}^{\infty} \left( \Box + \frac{2}{L_{\text{eff}}^2} - m_n^2 \right) \delta h_{ij} = 0,
$$

(80)

where $m_0 = 0$ and $m_n$ denotes the mass of the nth mode. Without loss of generality, we set
$m_0^2 < m_1^2 < m_2^2 < ...$. For the large AdS radius $L_{\text{eff}} = \cosh(\rho) \to \infty$, (80) yields the following
propagator

$$
D(p) \sim \sum_{i=0}^{\infty} \left( \Pi_{j \neq i} \frac{1}{m_j^2 - m_i^2} \right) \frac{1}{p^2 + m_i^2}
\sim \left( \frac{a_0^2}{p^2} - \frac{a_1^2}{p^2 + m_1^2} + \frac{a_2^2}{p^2 + m_2^2} - \frac{a_3^2}{p^2 + m_3^2} + ... \right),
$$

(81)

where $a_i^2$ are some positive constants related to the mass of gravitons. The above propagator
implies that half of the massive modes are ghosts. However, this contradicts with the discus-
sions of sect.3.2, where the perturbation action [62] shows that all the massive modes have
the correct sign of kinetic energy.

Let us discuss the resolution to the above puzzle. To start, we want to mention an
enlightening example. It is found in [63] that, by eliminating one of the metric, the ghost-free
bimetric theory [64, 65]

$$
S[h, f] = m_h^{d-2} \int dy^d \left( \sqrt{h} R(h) + \alpha^{d-2} \sqrt{f} R(f) - 2m^2 \sqrt{h} \sum_{n=0}^{d} \beta_n e_n(S) \right),
$$

(82)
is equivalent to the higher derivative gravity

\[ S^{HD} = m_n^{d-2} \int dy^d \sqrt{h} \left( \Lambda + c_R \mathcal{R}(h) - \frac{c_{RR}}{m_N^2} \left( \mathcal{R}^{ij} \mathcal{R}_{ij} - \frac{d}{4(d-1)} \mathcal{R}^2 \right) + O\left(\frac{\mathcal{R}^3}{m_N^4}\right) \right), \quad (83) \]

where \( \alpha = m_I/m_h \) is the ratio of the Planck masses, \( \beta_n \) are dimensionless free parameters, \( m^2 \) sets the mass scale of the massive mode, \( \Lambda, c_R, c_{RR} \) are some unimportant constants depending on the parameters of bimetric gravity, \( S^I_{ij} = (\sqrt{h^{-1}} f)_j^i \) and \( e_n(S) \) is given by

\[ e_n(S) = -\frac{1}{n} \sum_{k=1}^{n} (-1)^k \text{Tr}(S^k) e_{n-k}(S). \quad (84) \]

Due to the equivalence to the ghost-free bimetric theory, the higher derivative gravity \( (83) \) must be ghost-free too. However, as we have reviewed above, the curvature squared gravity includes a ghost. \( [63] \) argues that the ghost problem is an artifact of the truncation at \( O(\mathcal{R}^2) \). If one considers the complete action with infinite higher derivative terms, the ghost can be removed. This is indeed the case in a toy model of higher derivative scalar theory with suitable couplings to the source \( [63] \). It is found that the source structure plays an important role in eliminating the ghost in the higher derivative theory \( [63] \).

Now turn to our case. Remarkably, the curvature squared term of \( (83) \) takes exactly the same form as that of the higher derivative gravity \( I_W \) \( [66,69] \) on the brane. Following the approach of \( [63] \), we find that, by choosing suitable parameters of the bimetric gravity, the \( O(\mathcal{R}^3) \) term \( [67] \) of \( I_W \) can also be recovered \( [66] \). This strongly implies that the higher derivative gravity on the brane is actually ghost-free. It should be mentioned that, although the \( O(\mathcal{R}^2), O(\mathcal{R}^3) \) terms are the same, the effective theory on the brane cannot be equivalent to the bimetric gravity \( [82] \). That is because they have different degrees of freedom. There are infinite massive modes on the brane, while there are only a massless mode and a massive mode in bimetric gravity. The inconsistency of degrees of freedom can be naturally resolved by considering multi-gravity with infinite metrics \( [67,68] \)

\[ S_N = \lim_{N \to \infty} \frac{M^{d-2}}{2} \int dy^d \sqrt{h_i} \left( \mathcal{R}_i + \frac{m_N^2}{2} \sum_{n=0}^{\infty} \alpha_n^{(i)} \mathcal{L}_n(\mathcal{K}(h_i, h_{i+1})) \right), \quad (85) \]

where \( h_i \) denotes the i-th metric, \( \mathcal{R}_i \) is the Ricci scalar of \( h_i \) and \( \mathcal{L}_n(\mathcal{K}(h_i, h_{i+1})) \) is the interaction between neighboring metrics. Naturally, \( h_1 \) and \( h_N \) are the metrics on the two branes. Following the approach of \( [63] \), we eliminate the other metrics and obtain two higher derivative gravity on the two branes

\[ \begin{align*}
S_N^{HD} &= \frac{M^{d-2}}{2} \int dy^d \sqrt{h_1} \left( \Lambda_1 + c_R \mathcal{R}_1 - \frac{c_{RR}}{m_N^2} \left( \mathcal{R}_1^{ij} \mathcal{R}_{1 ij} - \frac{d}{4(d-1)} \mathcal{R}_1^2 \right) + O\left(\frac{\mathcal{R}_1^3}{m_N^4}\right) \right) \\
&\quad + \frac{M^{d-2}}{2} \int dy^d \sqrt{h_N} \left( \Lambda_N + c_R \mathcal{R}_N - \frac{c_{RR}}{m_N^2} \left( \mathcal{R}_N^{ij} \mathcal{R}_{N ij} - \frac{d}{4(d-1)} \mathcal{R}_N^2 \right) + O\left(\frac{\mathcal{R}_N^3}{m_N^4}\right) \right) \\
&\quad + \frac{M^{d-2}}{2} \int dy^d \mathcal{L}_{int}(h_1, h_N), \quad (86)
\end{align*} \]
where $\mathcal{L}_{int}(h_1, h_N)$ denotes the interaction of $(h_1, h_N)$. See the appendix for the derivations. Remarkably, the $O(R^2)$ term takes exactly the same form as that of the effective action $I_W$ \cite{66,69}. This is a strong evidence that the effective theory on the brane is equivalent to a ghost-free multi-gravity. We leave a careful study of this equivalence to future work \cite{66}.

To end this section, we provide an alternative mechanism to remove the ghost in higher derivative gravity. To have a ghost-free unitary theory, we require that the Euclidean effective action satisfies the following condition

$$e^{-I_W} = e^{-I_{\text{CFT}}} + I_c \leq 1.$$ \hspace{1cm} (87)

Due to the ghost, the exponent of higher derivative action $e^{I_c}$ could be larger than one. Thus, the matter action $I_{\text{CFT}}$ plays an important role in defining a unitary theory on the brane. Indeed, $I_{\text{CFT}}$ is large enough to make $e^{-I_W}$ less than one. That is because the effective action $I_W$ on the brane is obtained from the Einstein gravity in the bulk. As a result, it must satisfy the above condition

$$e^{-I_W} = e^{-I_{\text{Ein bulk}}} \leq 1.$$ \hspace{1cm} (88)

This teaches us an interesting mechanism to eliminate the ghost. By adding suitable matter fields to the higher derivative gravity, it is possible to construct a ghost-free total theory. As we have discussed above, wedge holography provides such an example. However, the effective action of wedge holography includes infinite higher derivative terms, thus is too complicated. Is it possible to construct a ghost-free total theory with finite higher derivative terms? We hope this interesting problem could be addressed in futures.

5 Holographic Weyl anomaly

In this section, we derive the holographic Weyl anomaly from the non-perturbative effective action \cite{69} of wedge holography and show that it agrees with the result of \cite{3}. This can be regarded as a test of the non-perturbative effective action \cite{69}.

According to \cite{69}, holographic Weyl anomaly can be obtained from the UV logarithmic divergent term of the gravitational action. Since $I_{\text{CFT}}$ is finite, it is expected that it does not contribute to the UV logarithmic divergent term. Thus, we focus on the gravitational part $(-I_c)$ of the effective action \cite{69}. We assume that the spacetime on the brane is asymptotically AdS

$$ds_Q^2 = h_{ij} dy^i dy^j = \cosh^2(\rho) \frac{dz^2 + \sigma_{ab} dy^a dy^b}{z^2},$$ \hspace{1cm} (89)

where $y^i = (z, y^a)$, $\sigma_{ab} = \sigma_{ab}^{(0)} + z^2 \sigma_{ab}^{(1)} + ... + z^{d-1} (\sigma_{ab}^{(d-1)} + \lambda_{ab}^{(d-1)} \ln z)$. By analyzing the asymptotical symmetry of AdS, \cite{70} obtains a universal relation for general higher derivative
gravity

\[ \sigma^{(0)ab} \sigma^{(1)}_{ab} = -\frac{R\Sigma}{2}, \quad \text{for } d = 3 \]  
\[ \sigma^{(1)}_{ab} = -\frac{1}{d-3}(R\Sigma_{ab} - \frac{R\Sigma}{2(d-2)}\sigma^{(0)}_{ab}), \quad \text{for } d > 3 \]

where \( R\Sigma_{ab} \) is the curvature with respect to \( \sigma^{(0)}_{ab} \).

We use the background-field method developed in [71] to investigate the holographic Weyl anomaly. This method is quite useful for the study of Weyl anomaly [71], correlation functions [72], entanglement/Rényi entropy [73, 74, 75] of higher derivative gravity. Expanding \( I_W \) in terms of background curvature \( \bar{R} \) defined below

\[ \bar{R} = \bar{R} - d(d-1)\text{sech}^2(\rho), \]

\[ \bar{R}_{ij} = \bar{R}_{ij} - (d-1)\text{sech}^2(\rho)h_{ij}, \]

\[ \bar{R}_{ijkl} = \bar{R}_{ijkl} - \text{sech}^2(\rho)(h_{ik}h_{jl} - h_{il}h_{jk}), \]

we get

\[ I_W = I_{\text{CFT}} + I_{\text{Ein}} + \frac{1}{16\pi G_N} \int_Q \sqrt{|h|} \left[ \frac{\bar{R}^2}{d-4}L_{R^2} + O(R^{d-4}) \right], \]

where \( I_{\text{Ein}} \) is the action of Einstein gravity with an effective Newton’s constant and AdS radius

\[ \frac{1}{G_N^{\text{eff}}} = \frac{1}{G_N} \int_0^\rho \frac{cosh^{d-2}(r)}{cosh^{d-2}(\rho)} dr, \quad L_{\text{eff}} = \text{cosh}(\rho). \]

Recall that for the class of solutions [5], we have \( I_{\text{CFT}} = 0 \) and \( \bar{R}_{ij} = \bar{R} = L_{R^2} = O(R^4) = 0 \). As a result, we get \( I_W = I_{\text{Ein}} \) for solutions [5], which agrees with the discussions of sect.4.1. It should be stressed that, in general, \( I_W \) and \( I_{\text{Ein}} \) are different.

We are interested of only the Weyl anomaly for 2d, 4d and 6d CFTs \( (d = 3, 5, 7) \). According to [71], we have for general solutions

\[ \sqrt{|h|}(\bar{R}^2) \sim O(z^{d-2}), \]

\[ \sqrt{|h|}L_{R^2} \sim \sqrt{|h|}O(R^4) \sim O(z^{d-2}). \]

Thus, the \( O(R^2) \), \( O(R^3) \) and \( O(R^4) \) terms of \( (95) \) are irrelevant to the UV logarithmic divergent term of the effective action for \( d < 9 \). Recall that \( I_{\text{CFT}} \) is irrelevant to the UV logarithmic divergent term too. Thus only the Einstein action \( I_{\text{Ein}} \) of \( (95) \) contributes to the holographic Weyl anomaly. As a result, the holographic Weyl anomaly of wedge holography AdSW\(_{d+1}/\text{CFT}_{d-1}\) is exactly the same as that of AdS\(_d/\text{CFT}_{d-1}\) with Einstein gravity, provided that the Newton’s constant and AdS radius are given by the effective ones \( (96) \).
For the convenience of readers, we list the holographic Weyl anomaly for 2d, 4d and 6d CFTs below

\[ A_{2d} = \int \Sigma dx^2 \sqrt{\sigma} \frac{c}{2\pi} R_{\Sigma}, \]

\[ A_{4d} = \int \Sigma dx^4 \sqrt{\sigma} \left[ \frac{c}{16\pi^2} C_{ijkl} C_{ijkl} - \frac{a}{16\pi^4} (R_{ijkl} R_{ijkl} - 4R_{ij} R_{ij} + R_{ij}^2) \right], \]

\[ A_{6d} = \int \Sigma dx^6 \sqrt{\sigma} \left[ \sum_{n=1}^{3} B_n I_n + 2A E_6 \right], \]

where \( C_{ijkl} \) is the Weyl tensor on \( \Sigma \), \( I_n \) are the Weyl invariant terms constructed from curvatures and their covariant derivatives,

\[ I_1 = C_{ijkl} C_{imnl} C_{kl mn}, \quad I_2 = C_{ijkl} C_{kl mn} C_{mn ij}, \]

\[ I_3 = C_{ijkl} (\Box \delta^{ij} + 4R_{ij} - \frac{6}{5} R_{ij} \delta^{ij}) C_{kl mn}, \]

and \( E_{2p} \) is the Euler density defined by

\[ E_{2p}(R_{\Sigma}) = \frac{1}{(8\pi)^p\Gamma(p+1)} \delta_{j_1 j_2 ... j_{2p} - 1}^{i_1 i_2 ... i_{2p} - 1} R_{\Sigma}^{j_1 j_2 ... j_{2p}} R_{\Sigma}^{i_1 i_2 ... i_{2p} - 1}. \]

For wedge holography, the central charges of above Weyl anomaly are given by

\[ c_{2d} = \frac{3}{2G_N} \int_0^\rho \cosh(r) dr = \frac{3}{2G_N} \sinh(\rho), \]

\[ a = c = \frac{\pi}{8G_N} \int_0^\rho \cosh^3(r) dr = \frac{\pi}{96G_N} (9 \sinh(\rho) + \sinh(3\rho)), \]

\[ A = \frac{\pi^2}{16G_N} \int_0^\rho \cosh^4(r) dr, \quad B_1 = -\frac{1}{256\pi G_N} \int_0^\rho \cosh^4(r) dr, \]

\[ B_2 = -\frac{1}{1024\pi G_N} \int_0^\rho \cosh^4(r) dr, \quad B_3 = \frac{1}{3072\pi G_N} \int_0^\rho \cosh^4(r) dr. \]

Recall that we focus on half of the wedge space \( 0 \leq r \leq \rho \) in above discussions. For the whole wedge space, we should double the above central charges. Note also that although the metrics on the two branes \( Q_1 \) and \( Q_2 \) are different generally, the induced metrics \( \sigma_{ab} \) on the corner \( \Sigma \) are the same.

As a summary, we obtain the holographic Weyl anomaly for wedge holography with general solutions in this section. Remarkably, although the effective action on the brane are infinite towers of higher derivative gravity, the holographic Weyl anomaly is exactly the same as that of Einstein gravity. It should be mentioned that the holographic Weyl anomaly for 2d and 4d CFTs (99,100) has been derived in [3] for the special class of solution (5). Here we re-derive the same holographic Weyl anomaly for the most general solutions in wedge holography. Since Weyl anomaly is independent of the states of CFTs\(^4\) the holographic

\(^4CFTs\) in vacuum state and thermal state have the same central charges and thus the same Weyl anomaly.
Weyl anomaly should be irrelevant to the bulk solutions. As we have shown in this section, this is indeed the case. This self-consistency can be regarded as a test of wedge holography and the non-perturbative effective action \(^{(69)}\).

6 First law of entanglement entropy

In this section, we prove that the first law of entanglement entropy is satisfied for wedge holography. Since the holographic formula of entanglement entropy is still unknown for wedge holography with CBC/DBC \(^{(20)}\), we focus on NBC in this section. Interestingly, we find that the massive fluctuations do not change the holographic entanglement entropy. For simplicity, we focus on the first order perturbation including the massless and massive modes of gravity in this section.

To start, let us give a brief review of the relative entropy and the first law of entanglement entropy. The relative entropy measures the fundamental distance between two states in the same Hilbert space

\[
S(\rho_1|\rho_0) = \text{tr}(\rho_1 \ln \rho_1) - \text{tr}(\rho_1 \ln \rho_0).
\]

(109)

It is non-negative, i.e., \(S(\rho_1|\rho_0) \geq 0\), and vanishes if and only if the states are equal. The relative entropy can be re-expressed as

\[
S(\rho_1|\rho_0) = \Delta \langle H \rangle - \Delta S,
\]

(110)

with

\[
\Delta \langle H \rangle = \text{tr}(\rho_1 H) - \text{tr}(\rho_0 H), \quad \Delta S = S(\rho_1) - S(\rho_0),
\]

(111)

where \(H\) is the modular Hamiltonian defined by \(\rho_0 = e^{-H}/\text{tr}(e^{-H})\), and \(S(\rho) = -\text{tr}(\rho \ln \rho)\) is the entanglement entropy. In general, the positivity of the relative entropy requires

\[
\Delta \langle H \rangle \geq \Delta S,
\]

(112)

which is a quantum generalization of the Bekenstein bound. At the first order of perturbations, the above inequality is saturated \(^{(77)}\)

\[
\delta \langle H \rangle = \delta S.
\]

(113)

\(^{(5)}\)Recall that, for CBC/DBC, the dynamical gravity on the brane is the extrinsic curvature instead of the induced metric. So far we know little about this kind of gravity. What we know is that, the holographic entanglement entropy for CBC/DBC must be different from that of NBC, since they have different spectrum and effective action on the brane. Two natural questions are as follows. First, is the holographic entanglement entropy in the bulk still given by RT surface (extremal surface) with respect to the metric? Second, if one could impose CBC/DBC for gravity, can one impose CBC/DBC for the RT surface, that the endpoint of RT surface is fixed on the brane? See \(^{(40)}\) for some discussions on the RT surface with DBC. We leave a careful study of these problems to future works.
This is the so-called first law of entanglement entropy. Interestingly, by applying the first law of entanglement entropy \([113]\), one can derive the linearized Einstein Equations in the bulk \([78]\).

In the followings, we focus on the first order perturbations and verify that the first law of entanglement entropy is obeyed by wedge holography.

6.1 The massive fluctuations

Let us first discuss the massive fluctuations. We take the ansatz of the metric \([38]\)

\[
ds^2 = dr^2 + \cosh^2(r)\left(\frac{dz^2 + \eta_{ab} dy^a dy^b}{z^2} + \epsilon \sum_{m_g} H^{(m_g)}(r)\tilde{h}^{(m_g)}(y) dy^a dy^b\right),
\]

where \(y^i = (z, y^a), y^a = (t, x^A) = (t, x^1, ..., x^{d-2})\), the index \(a\) runs from 0 to \((d-2)\) and \(\sum_{m_g}\) denotes the sum over the spectrum.

Consider the entanglement entropy of CFT\(_{d-1}\) on a \((d-2)\) dimensional round disk \(\sum_{A=1}^{d-2}(x^A)^2 \leq R_0^2\), where \(R_0\) is the radius of the disk. According to \([62]\), it can be calculated by the area of the RT surface in the bulk

\[
S = \frac{\text{Area}(\gamma)}{4G_N},
\]

where \(\gamma\) is the minimal surface anchored at the entangling surface of the disk, i.e., \(\partial\gamma = \sum_{A=1}^{d-2}(x^A)^2 = R_0^2\). For the background metric \([114]\) with \(\epsilon = 0\), the RT surface in the bulk is given by \([1]\)

RT surface \(\gamma\) : \(z^2 + r_0^2 = R_0^2, t = \text{constant}\).

where \(r_0^2 = \sum_{A=1}^{d-2}(x^A)^2\). Since the RT surface is a minimal surface, the first order perturbation does not change its location. As a result, the RT surface for the perturbative metric \([114]\) is still given by \([116]\). From \([114]\) and \([116]\), we read off the induced metric on the RT surface

\[
ds_\gamma^2 = dr^2 + \frac{\cosh^2(r)}{z^2} \gamma_{AB} dx^A dx^B,
\]

where

\[
\gamma_{AB} = \delta_{AB} + \partial_A z \partial_B z + \epsilon \, z^2 \sum_{m_g} H^{(m_g)}(r)\tilde{h}^{(m_g)}_{AB}(y).
\]

By applying the RT formula \([115]\), we get the holographic entanglement entropy of a disk

\[
S = \frac{1}{4G_N} \int_{-\rho}^\rho dr \int_{r_0 \leq R_0} d^{d-2}x \frac{\cosh^{d-2}(r)}{z^{d-2}} \sqrt{\gamma_{AB}},
\]
which yields the first order variation of entanglement entropy

$$
\delta S = \sum_{m_g} \frac{\varepsilon}{8G_N} \int_{-\rho}^{\rho} dr \cosh^{d-2}(r) H^{(m_g)}(r) \int_{r_0 \leq R_0} d^{d-2}x f^{(m_g)}(x^A),
$$

with

$$
f^{(m_g)}(x^A) = \frac{1}{z^{d-4}} \left( \delta^{CD} h^{(m_g)}_{CD} \sqrt{1 + \delta^{AB} \partial_A z \partial_B z} - \frac{\tilde{h}^{(m_g)CD} \partial_A z \partial_B z}{\sqrt{1 + \delta^{AB} \partial_A z \partial_B z}} \right). \tag{121}\]

Here $h^{(m_g)CD}$ are raised by $\delta^{CD}$ and $z$ obeys (116). In the above derivations, we have used the formula $|g_{AB} + \partial_A z \partial_B z| = |g_{AB}|(1 + g^{AB} \partial_A z \partial_B z)$ [77].

Recall the orthogonal condition (57) and the fact that $H^{(0)}(r)$ is a constant for the massless mode. Immediately, we get

$$
\int_{-\rho}^{\rho} dr \cosh^{d-2}(r) H^{(m_g)}(r) \sim \int_{-\rho}^{\rho} dr \cosh^{d-2}(r) H^{(m_g)}(r) H^{(0)}(r) = 0, \tag{122}\]

for the massive modes $m_g > 0$. As a result, the first order variation of entanglement entropy (120) vanishes for the massive fluctuations.

The first order variation of the expectation value of the modular Hamiltonian $\delta \langle H \rangle$ for a spherical entangling surface is given by [77]

$$
\delta \langle H \rangle = \frac{\pi}{R_0} \int_{r_0 \leq R_0} d^{d-2}x \ z^2 \delta \langle T_{00} \rangle, \tag{123}\]

where $T_{00}$ is the stress tensor of CFT$_{d-1}$. Applying AdS$_d$/CFT$_{d-1}$ on the brane, we get the holographic stress tensor [57]

$$
\langle T_{ab} \rangle = \frac{d - 1}{16\pi G_N^{(d)}} h^{(d-1)}_{ab}, \tag{124}\]

where $G_N^{(d)}$ is the effective Newton’s constant, and $h^{(d-1)}_{ab}$ is defined in the Fefferman-Graham (FG) expansion on the brane

$$
d^2 s_Q = \frac{dz^2 + (\eta_{ab} + z^{d-1} h^{(d-1)}_{ab} + ...) dy^a dy^b}{z^2}. \tag{125}\]

Note that, for the massive modes, the FG expansion behaves as

$$
d^2 s_Q = \frac{dz^2 + (z^{d-1-\Delta} h^{(d-1-\Delta)}_{ab} + ...) dy^a dy^b}{z^2}, \tag{126}\]

where $\Delta = \frac{d-1}{2} + \sqrt{\frac{(d-1)^2}{4} + m_g^2}$ is the conformal dimension, which can be derived from EOM (46). Comparing (125) with (126), we get $\langle T_{ab} \rangle \sim h^{(d-1)}_{ab} = 0$ for the massive modes. As a result, the first order variation of the expectation value of the modular Hamiltonian (123) vanishes too. Now we have verified the first law of entanglement entropy for the massive fluctuations

$$
\delta S = \delta \langle H \rangle = 0, \quad \text{for } m_g^2 > 0. \tag{127}\]

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6.2 The massless fluctuations

Let us go on to discuss the massless fluctuations, where \( \bar{h}_{ab}^{(m_g=0)}(y) \) of (114) takes the form,

\[
\bar{h}_{ab}^{(m_g=0)}(y) = z^{d-3} \bar{h}_{ab}^{(d-1)}(y) = \frac{16\pi G_N^{(d)}}{d-1} z^{d-3} \langle T_{ab}(y) \rangle,
\]

(128)

and the effective Newton’s constant is given by

\[
\frac{1}{G_N^{(d)}} = \frac{1}{G_N} \int_{-\rho}^{\rho} dr \cosh^{d-2}(r).
\]

(129)

Note that we have set \( H^{(m_g=0)}(r) = 1 \) for simplicity. Combining equations (116,120,128), we get the first order perturbation of entanglement entropy

\[
\delta S = \frac{2\pi R_0 S_{d-4}}{d-1} \int_{0}^{R_0} dr_0 r_0^{d-3} \int_{0}^{\pi} d\theta \sin^{d-4}(\theta) \left( \langle \hat{T}^A_A(y) \rangle - \langle \hat{T}_{AB}(y) \rangle \frac{x^A p^B}{R_0^2} \right),
\]

(130)

where \( S_{d-4} \) is the volume of \( (d-4) \) dimensional unit sphere. On the other hand, the first order perturbation of the expectation value of the modular Hamiltonian is

\[
\delta \langle H \rangle = \frac{\pi}{R_0} \int_{r_0 \leq R_0} dr_0 r_0^{d-3} \int_{0}^{\pi} d\theta \sin^{d-4}(\theta) (R_0^2 - r_0^2) \langle \hat{T}_{00}(y) \rangle.
\]

(131)

It is convenient to express \( \delta S, \delta \langle H \rangle \) in the Fourier expansion

\[
T_{ab}(y) = \int d^{d-1} p e^{-ip \cdot y} \hat{T}_{ab}(p).
\]

(132)

Without loss of generality, we drop the integral \( \int d^{d-1} p e^{ip \cdot y} \) below. Following [77], we set the spatial direction of momentum in direction \( x^1 \). The conservation and tracelessness of \( \langle T_{ab} \rangle \) yield the following useful formulas [77]

\[
\langle \hat{T}^A_A \rangle = \langle \hat{T}_{00} \rangle, \quad \langle \hat{T}_{10} \rangle = -\frac{p^0}{p^1} \langle \hat{T}_{00} \rangle, \quad \langle \hat{T}_{11} \rangle = \left( \frac{p^0}{p^1} \right)^2 \langle \hat{T}_{00} \rangle,
\]

\[
\langle \hat{T}^A_A(p) \rangle - \langle \hat{T}_{AB}(p) \rangle \frac{x^A p^B}{R_0^2} = \langle \hat{T}_{00} \rangle \left( 1 - \left( \frac{p^0}{p^1} \right)^2 \frac{(r_0^2 - (x^1)^2)}{(d-3)R_0^2} \right)
\]

\[
= \langle \hat{T}_{00} \rangle \left( \frac{R_0^2 - r_0^2 \cos^2(\theta)}{R_0^2} + O(p^2) \right),
\]

(133)

where \( p^0 = \sqrt{(p^1)^2 - p^2} \).

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Substituting (132,133) into (130,131), we finally derive
\[ \delta S = \frac{2\pi R_0 S_{d-4}}{d-1} \langle \tilde{T}_{00} \rangle \int_0^{R_0} dr_0 r_0^{d-3} \int_0^\pi d\theta \sin^{d-4} \theta e^{-ip^1 r_0 \cos(\theta)} \left( \frac{R_0^2 - r_0^2 \cos^2(\theta)}{R_0^2} \right) \]
\[ = \frac{\pi^2 R_0^{d-1} S_{d-4} \Gamma \left( \frac{d-3}{2} \right)}{2\Gamma \left( \frac{d+2}{2} \right)} {}_0F_1 \left( \frac{d+2}{2}; -\frac{1}{4} (p^1 R_0)^2 \right) \langle \tilde{T}_{00} \rangle, \] (134)
and
\[ \delta \langle H \rangle = \frac{\pi S_{d-4}}{R_0} \langle \tilde{T}_{00} \rangle \int_0^{R_0} dr_0 r_0^{d-3} \int_0^\pi d\theta \sin^{d-4} \theta e^{-ip^1 r_0 \cos(\theta)} \left( \frac{R_0^2 - r_0^2}{R_0^2} \right) \]
\[ = \frac{\pi^2 R_0^{d-1} S_{d-4} \Gamma \left( \frac{d-3}{2} \right)}{2\Gamma \left( \frac{d+2}{2} \right)} {}_0F_1 \left( \frac{d+2}{2}; -\frac{1}{4} (p^1 R_0)^2 \right) \langle \tilde{T}_{00} \rangle, \] (135)
where \( {}_0F_1 (a; z) \) is the hypergeometric function. As expected, (134,135) yield the correct first law of entanglement entropy \( \delta S = \delta \langle H \rangle \). It is a strong support for wedge holography.

To end this section, let us make some comments. First, for the massive fluctuations, the first law of entanglement entropy (127) holds for more general background instead of only AdS. That is because the EOM (47), the spectrum and the orthogonal condition (57) of \( H(r) \) are independent of the choices of the background metric \( h_{ij}^{(0)} \), as long as it obeys the Einstein equations on the brane. As a result, \( \delta S \) (120) always includes a vanishing pre-factor \( \int d^dr \cosh^{d-2}(r) H^{(m)}(r) = 0 \) for the massive fluctuations. Similarly, we always have \( \delta \langle H \rangle = 0 \) for the massive fluctuations, since the massive modes are irrelevant to the holographic stress tensor \( \langle T_{ab} \rangle \) with the conformal dimension \( (d-1) \). Thus, we can prove the first law of entanglement entropy \( \delta S = \delta \langle H \rangle = 0 \) for the massive fluctuations around more general background. Second, it is straightforward to generalize the discussions of the massless modes to higher order perturbations. Recall that the effective theory of the massless modes is Einstein gravity (72) on the brane (3). According to (77), for Einstein gravity, we have \( \delta S \leq \delta \langle H \rangle \) at the second order perturbations. Third, inspired by (78), it is interesting to turn the logic around, and to derive the Einstein equations in the \( (d+1) \) dimensional wedge space from the first law of entanglement entropy of CFTs on the \( (d-1) \) dimensional corner of the wedge.

7 Conclusions and Discussions

In this paper, we have investigated the spectrum, the effective action and the first law of entanglement entropy for wedge holography. We find that the spectrum is discrete and non-negative for all kinds of boundary conditions. In particular, there is a massless mode if one imposes NBC on both branes. On the other hand, the massless mode disappears if one impose CBC/DBC on one or two of the branes. We work out the effective action on the brane. In the perturbative formulation, it is given by an infinite sum of Pauli-Fierz actions. In the
non-perturbative formulation, the effective action is composed of a higher derivative gravity and a matter action. Usually, a higher derivative gravity suffers the problem of ghost. Due to the equivalence to Einstein gravity in the bulk, the effective action on the brane must be ghost-free. We find that the matter action plays an important role in eliminating the ghost. Besides, we provide evidences that the higher derivative gravity on the brane is equivalent to a ghost-free multi-gravity [67, 68]. We show that the higher derivative gravity gives the correct Weyl anomaly of wedge holography, which is a support to both wedge holography and the non-perturbative effective action on the brane. Finally, we verify that the first law of entanglement entropy is satisfied for wedge holography. Interestingly, the massive fluctuations are irrelevant to the first order perturbations of entanglement entropy. This is also a strong support for wedge holography.

In this paper, we mainly focus on NBC. It is interesting to discuss the non-perturbative effective action and the first law of entanglement entropy for CBC/DBC [17, 20]. It is expected that, similar to the case of NBC, the non-perturbative effective action for CBC/DBC is also equivalent to a higher derivative gravity or a multi-gravity on the brane. The only difference are the parameters of the theory, since, as we have shown in sect.3, the perturbative effective action is given by an infinite sum of Pauli-Fierz massive gravity with different mass spectrums for different BCs. It is also interesting to study the holographic entanglement entropy for gravity with CBC/DBC. Recall that the dynamical gravity on the brane is the extrinsic curvature instead of the induced metric. Is the holographic entanglement entropy still given by the area of a minimal surface with respect to the metric in the bulk? Which boundary condition should one choose for the RT surface ending on the brane with CBC/DBC? Finally, inspired by [78], it is interesting to derive the Einstein equations in the \((d + 1)\) dimensional wedge space from the first law of entanglement entropy on the \((d - 1)\) dimensional corner of the wedge. We leave a carful study of these interesting problems to future works.

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A Multi-gravity on the brane

By applying the “deconstruction” method [68], one can obtain the so-called multi-gravity from Einstein gravity in the bulk. The main idea is to make discretization of Einstein gravity and replace the extra dimension \(r\) by a series of sites \(r_b\) (\(1 \leq b \leq N\)). See Fig. 6 for an
example. The Einstein gravity in the bulk can be written as

\[
I = \int_{-\rho}^{\rho} dr \int dy \sqrt{h} \left( \mathcal{R}[h] + K^2 - K_{ij}K^{ij} \right),
\]

(136)

where we have set \(16\pi G_N = 1\) for simplicity, \(\mathcal{R}[h]\) is the intrinsic Ricci scalar on the constant \(r\) surface, and \(K_{ij} = \frac{1}{2} \partial_r g_{ij}\) is the extrinsic curvature. By discretization, the extrinsic curvature becomes

\[
K^a_i = \epsilon^a_i K_{ij} \simeq m_N (e_{n+1}^a e_i^a - e_n^a e_i^a),
\]

\[
K^a_i[h_n, h_{n+1}] = -m_N \left( \delta^a_i - \left( \frac{1}{h_n} \right)_{ij} \right),
\]

(137)

where \(e_i^a\) is the vielbein and \(m_N = N/(2\rho)\) \(\simeq \partial_r\) denotes the discrete derivative. The integration over the whole of the extra dimension can be replaced by summing over all the sites after discretization. In this way, we get the multi-gravity (86) [67, 68]

\[
I = \sum_{n=1}^{N} \int dy \sqrt{h_n} \left( \mathcal{R}[h_n] + K^2[h_n, h_{n+1}] - K_{ij}[h_n, h_{n+1}]K^{ij}[h_n, h_{n+1}] \right)
\]

(138)

where \(\alpha_{m}^{(n)}\) are some coefficients, \(\mathcal{L}_m(Q) = \epsilon \epsilon^m Q^m\) denote the interaction between neighboring metrics and \(\epsilon\) is the Levi-Cevita symbol. For instance, we have \(\mathcal{L}_3(Q) = \epsilon^ijkl \epsilon_{ijkl} Q^i_k Q^j_l\) in four dimensions. By construction, the multi-gravity (138) is ghost-free [67, 68].

Taking the limit \(N \to \infty\), the multi-gravity (138) reduces to Einstein gravity (136) in the bulk. Since the multi-gravity (138) with \(N \to \infty\) and the effective higher derivative gravity (69) on the two branes are both obtained from the bulk Einstein gravity, they must be equivalent. Following the approach of [63], by eliminating the metrics \(h_2, h_3, ..., h_{N-1}\) of the multi-gravity with \(N \to \infty\), we should recover the higher derivative gravity (69) on the branes. We leave a careful study of this problem to future work [66]. In this appendix, we consider finite \(N\) for simplicity. We recover the correct form of \(O(R^2)\) terms in the higher derivative gravity (69). This is a strong evidence for the equivalence between the higher derivative gravity on the brane and the ghost-free multi-gravity. For simplicity, we focus on the symmetric deconstruction of the bulk gravity, so that we only need to consider half of the wedge spacetime \(-\rho \leq r \leq 0\) below.

### A.1 Multi-gravity with four metrics

Let us first discuss the multi-gravity with four metrics \(h_b\) (1 \(\leq b \leq N = 4\)). The action includes three parts

\[
I_4(h_b) = \int d^d y \left( \mathcal{L}_{12} + \mathcal{L}_{43} + \mathcal{L}_{int\ 23} \right),
\]

(139)
Figure 6: Schematic diagram of deconstruction. The main idea is to make discretization of Einstein gravity and replace the extra dimension $r$ by a series of sites $r_b$ ($1 \leq b \leq N$). In this way, one can derive ghost-free multi-gravity from the Einstein gravity in the bulk.

where $\mathcal{L}_{12}$ and $\mathcal{L}_{43}$ are the bi-metric Lagrangian density for $(h_1, h_2)$ and $(h_4, h_3)$ respectively, $\mathcal{L}_{int\,23}$ denotes the interaction of $(h_2, h_3)$

\begin{align}
\mathcal{L}_{12} &= m_{(1)}^{(d-2)} \left( \sqrt{h_1 R_1} + \gamma^{(2)}_h h_2 R_2 - 2m^2 \sqrt{h_1} \sum_{n=0}^{d-1} \beta_n^{(1)} e_n(S_{(1)}) \right), \\
\mathcal{L}_{43} &= m_{(4)}^{(d-2)} \left( \sqrt{h_4 R_4} + \gamma^{(3)}_h h_3 R_3 - 2m^2 \sqrt{h_4} \sum_{n=0}^{d-1} \beta_n^{(4)} e_n(S_{(4)}) \right), \\
\mathcal{L}_{int\,23} &= -m^2 \sum_{n=0}^{d-1} \left( m_{(1)}^{(d-2)} \gamma^{(2)}_h \sqrt{h_2} e_n(S_{(2)}) + m_{(4)}^{(d-2)} \gamma^{(3)}_h \sqrt{h_3} e_n(S_{(3)}) \right).
\end{align}

Let us recall some notations. $\gamma^{(2)}_h = m_{(2)}/m_{(1)}$ and $\gamma^{(3)}_h = m_{(3)}/m_{(4)}$ are the ratio of the Planck masses, $\beta_n^{(b)}$ are dimensionless free parameters, $R_b$ is the Ricci scalar of $h_b$, $S_{(b)} = \sqrt{h_{b-1}^\ast h_{b+1}}$ and $\bar{S}_{(b)} = \sqrt{h_{b-1}^\ast h_{b-1}}$. Note that $\mathcal{L}_{12}$ and $\mathcal{L}_{43}$ are symmetric under the interchanges \{h$_b$ ↔ h$_{N+1-b}$\} because of the symmetric deconstruction of the bulk gravity. The equations of motion (EOM) of $h_1$ and $h_2$ are given by

\begin{align}
P^{i}_{1\,j} - P^{i}_{1\,j} + m^2 \sum_{n=0}^{d-1} (-1)^n \beta_n^{(1)} \gamma^{\mu}_{(n)\nu}(S_{(1)}) &= 0, \\
P^{i}_{2\,j} - P^{i}_{2\,j} + m^2 \gamma^{(2)}_h \sum_{n=0}^{d-1} (-1)^n \left[ \beta_n^{(2)} \gamma^{\mu}_{(n)\nu}(S_{(2)}) + \beta_n^{(1)} \gamma^{\mu}_{(n)\nu}(S_{(1)}) \right] &= 0,
\end{align}

where $P_{b\,ij} = R_{b\,ij} - \frac{R_{b}}{2(d-1)} h_{b\,ij}$, $\gamma^{\mu}_{(n)\nu}(S_{(b)}) = \sum_{k=0}^{n} (-1)^k e_k(S_{(b)}) (S_{(b)}^{n-k})^{i\,j}$.
We aim to express $h_2$ in functions of $h_1$. To do so, we first solve $S_{(1)}^{i,j} = \left(\sqrt{h_{1}^{-1}}h_{2}\right)^{i,j}$ perturbatively. The general ansatz of $S_{(1)}^{i,j}$ is given by

$$S_{(1)}^{i,j} = a^{(1)}\delta_{i}^{j} + \frac{1}{m^2} \left( b_{1}^{(1)} P_{1}^{i} j + b_{2}^{(1)} P_{1} \delta_{i}^{j} \right) + \frac{1}{m^4} \left( c_{1}^{(1)} P_{1}^{i} P_{1}^{k} j + c_{2}^{(1)} P_{1} P_{1}^{i} j + c_{3}^{(1)} P_{1}^{i} P_{1}^{k} \delta_{i}^{j} + c_{4}^{(1)} P_{1}^{2} \delta_{i}^{j} \right) + \mathcal{O}(m^{-6}).$$  \quad (145)

where $a^{(1)}, b^{(1)}$ and $c^{(1)}$ are the coefficients to be determined below. Substituting (145) into (143), we get

$$\mathcal{P}_{1}^{i} j - \mathcal{P}_{1} \delta_{i}^{j} + m^2 \sum_{n=0}^{d-1} \beta_{n}^{(1)} a^{(1)n} \sum_{k=0}^{n-k} \left( \sum_{r=0}^{n} \sum_{m=0}^{k} (-1)^{n+k} C_{n-k}^{r} C_{d-m}^{k-m} (\mathcal{M}_{(1)}^{r})_{i}^{j} e_{m}(\mathcal{M}_{(1)}) \right) = 0,$$

where we have replaced $S_{(1)}$ by $M_{(1)}$ defined below

$$M_{(b)}^{i} j = \frac{1}{a^{(b)/a^{(1)}}} S_{(b)}^{i,j} - \delta_{i}^{j}, \quad \quad (147)$$

$$(S^{n-k})^{i} j = a^{(b)n-k} \sum_{k=0}^{d-1} (\mathcal{M}_{(b)}^{r})_{i}^{j}, \quad e_{k}(\mathcal{S}_{(b)}) = a^{(b)k} \sum_{m=0}^{k} C_{d-m}^{k} e_{m}(\mathcal{M}_{(b)}). \quad (148)$$

From (145) and (147), we have

$$M_{(1)}^{i} j = \frac{1}{m^2} a^{(1)} \left( b_{1}^{(1)} P_{1}^{i} j + b_{2}^{(1)} P_{1} \delta_{i}^{j} \right) + \frac{1}{m^4} \left( c_{1}^{(1)} P_{1}^{i} P_{1}^{k} j + c_{2}^{(1)} P_{1} P_{1}^{i} j + c_{3}^{(1)} P_{1}^{i} P_{1}^{k} \delta_{i}^{j} + c_{4}^{(1)} P_{1}^{2} \delta_{i}^{j} \right) + \mathcal{O}(m^{-6}). \quad (149)$$

It’s convenient to use the following notations and formulas

$$s_{k}^{(b)} = \sum_{n=k}^{d-1} C_{d-k-1}^{n-k} \beta_{n}^{(b)n}, \quad \quad (150)$$

$$\sum_{k=0}^{d} (-1)^{k+n} C_{n-k}^{r} C_{d-m}^{k-m} = (-1)^{r} C_{d-1}^{n-r+m}. \quad (151)$$

Combining (146, 150, 151), we obtain

$$\mathcal{P}_{1}^{i} j - \mathcal{P}_{1} \delta_{i}^{j} + s_{0}^{(1)} m^2 \delta_{i}^{j} + s_{1}^{(1)} m^2 \left[ -M_{(1)}^{i} j + \delta_{i}^{j} e_{1}(\mathcal{M}_{(1)}) \right] + s_{2}^{(1)} m^2 \left[ (M_{(1)}^{2})_{i}^{j} - M_{(1)}^{i} j e_{1}(\mathcal{M}_{(1)}) + \delta_{i}^{j} e_{2}(\mathcal{M}_{(1)}) \right] + \mathcal{O}(M_{(1)}^{3}) = 0. \quad (152)$$

Note that $S_{(1)}^{i,j}$ of orders $\mathcal{O}(m^{-2l})$ can be obtained from (152) of orders $\mathcal{O}(M_{(1)}^{l})$. Since $M$ starts with $m^{-2}$, we only need to calculate the terms which satisfy $r + m \leq l + 1$. For our
ansatz (163), we have \( l = -1, 0, 1 \). Substituting (149) into the above equation and solving it order by order in \( O(1/m^2) \), we get

\[
\mathcal{O}(m^2), \quad s_0^{(1)} = 0,
\]
\[
\mathcal{O}(m^0), \quad b_1^{(1)} = \frac{a_1^{(1)}}{s_1^{(1)}}, \quad b_2^{(1)} = 0,
\]
\[
\mathcal{O}(m^{-2}), \quad c_1^{(1)} = -c_2^{(1)} = \frac{a_1^{(1)} s_2^{(1)}}{s_1^{(1)}} \quad \text{and} \quad c_3^{(1)} = -c_4^{(1)} = -\frac{a_1^{(1)} s_2^{(1)}}{2(d-1)s_1^{(1)}}. \quad (153)
\]

Substituting (153) into (145), we finally obtain \( S(1) \)

\[
S_{(1)}^i j = a^{(1)} \delta^i_j + \frac{a_1^{(1)}}{s_1^{(1)} m^2} \mathcal{P}_1^i j + \frac{a_1^{(1)} s_2^{(1)}}{s_1^{(1)} m^4} \left( \mathcal{P}_1^i k \mathcal{P}_1^k j - \mathcal{P}_1^i j \mathcal{P}_1^k k + \frac{e_2(\mathcal{P}_1)}{d - 1} \delta^i_j \right) + \mathcal{O}(m^{-6}). \quad (154)
\]

and express \( h_2 \) in terms of \( h_1 \)

\[
h_2 ij = h_1 ik (S_{(1)}^2)_{jk}^k = a^{(1)} \left[ h_1 ij + \frac{2 \mathcal{P}_1^i j}{s_1^{(1)} m^2} + \frac{2 s_2^{(1)}}{s_1^{(1)} m^4} \left( 1 + \frac{s_1^{(1)}}{2 s_2^{(1)}} \right) \mathcal{P}_1^i k \mathcal{P}_1^k j - \mathcal{P}_1^i j \mathcal{P}_1^k k + \frac{e_2(\mathcal{P}_1)}{d - 1} h_1 ij \right] \quad + \mathcal{O}(m^{-6}). \quad (155)
\]

Following the same approach, we can express \( h_3 \) in terms of \( h_4 \). From \( h_2(h_1) \) (155) and \( h_3(h_4) \), the interaction potential \( \mathcal{L}_{\text{int},23}(h_2,h_3) \) can be converted to \( \mathcal{L}_{\text{int},14}(h_1,h_4) \). We find the following relations are useful in the calculations

\[
\sum_{n=0}^d \beta_n^{(b)} e_n(S_{(b)}) = \sum_{n=0}^d \alpha_n^{(b)} e_n(M_{(b)}), \quad \alpha_n^{(b)} = \sum_{k=n}^d C_{k-n}^{d-n} a^{(b)k} \beta_k^{(b)}. \quad (156)
\]

Finally, combining (139) \(140\) \(141\) \(154\) \(155\) \(156\), we obtain the higher derivative gravity with respect to two metrics \( h_1 \) and \( h_4 \)

\[
I_4^{\text{HD}}(h_1,h_4) = m^{d-2} \int_{Q_1} d^4 y \sqrt{h_1} \left[ \hat{\Lambda}^{(1)} + \hat{c}_{\mathcal{R}}^{(1)} \mathcal{R}_1 + \hat{c}_{\mathcal{R}^2}^{(1)} \mathcal{R}_1^2 + \frac{d}{4(d-1)} \mathcal{R}_1^2 \right] \quad + m^{d-2} \int_{Q_2} d^4 y \sqrt{h_4} \left[ \hat{\Lambda}^{(4)} + \hat{c}_{\mathcal{R}}^{(4)} \mathcal{R}_4 + \hat{c}_{\mathcal{R}^2}^{(4)} \mathcal{R}_4^2 + \frac{d}{4(d-1)} \mathcal{R}_4^2 \right] \quad + \mathcal{O}(\mathcal{R}_1^3) + \mathcal{O}(\mathcal{R}_4^3) + \mathcal{L}_{\text{int},14}(h_1,h_4), \quad (157)
\]

where the parameters are given by

\[
\hat{\Lambda}^{(1)} = -2 m^2 a_0^{(1)}, \quad \hat{c}_{\mathcal{R}}^{(1)} = 1 + (\gamma(2) a^{(1)})^{d-2} - \frac{a_1^{(1)} (d-2)}{s_1^{(1)} (d-1)},
\]

\[31\]
\[
\zeta_R^{(1)} = -\frac{1}{s_1^{(2)}} \left( 2s_1^{(1)} (\gamma^{(2)} a^{(1)})^{d-2} - \alpha_2^{(1)} + \frac{\alpha_1^{(1)} s_2^{(1)} (d-2)}{s_1^{(1)} (d-1)} \right),
\]

(158)

and \(\hat{\Lambda}^{(4)}, c_R^{(4)}, c_R^{(4)}\) can be obtained from (158) by replacing indexes \((b)\) by \((5 - b)\).

Remarkably, the higher derivative gravity (157) reduced from the multi-gravity takes similar forms as the effective action (69) on the brane, where the interaction \(\mathcal{L}_{\text{int} 14}(h_1, h_4)\) can be naturally explained as the CFT action \(I_{\text{CFT}}\).

### A.2 Multi-gravity with six metrics

Let us now discuss the multi-gravity with six metrics \(h_b (1 \leq b \leq N = 6)\)

\[
I_0(h_b) = \int d^d y (\mathcal{L}_{123} + \mathcal{L}_{654} + \mathcal{L}_{\text{int} 34}),
\]

(159)

where \(\mathcal{L}_{123}\) and \(\mathcal{L}_{654}\) are the Lagrangian density of multi-gravity with three metrics for \((h_1, h_2, h_3)\) and \((h_6, h_5, h_4)\) respectively, \(\mathcal{L}_{\text{int} 34}\) denotes the interaction of \((h_3, h_4)\)

\[
\mathcal{L}_{123} = \mathcal{L}_{12} + \mathcal{L}_3 = \mathcal{L}_{12} + m_2 d-2 \left( \gamma_{(3)}^{d-2} \sqrt{h_3 R_3} - 2m_2 \sqrt{h_2} \sum_{n=0}^{d} \beta_n^{(2)} e_n(S(2)) \right),
\]

(160)

\[
\mathcal{L}_{654} = \mathcal{L}_{65} + \mathcal{L}_4 = \mathcal{L}_{65} + m_2 d-2 \left( \gamma_{(4)}^{d-2} \sqrt{h_4 R_4} - 2m_2 \sqrt{h_5} \sum_{n=0}^{d} \beta_n^{(5)} e_n(S(5)) \right),
\]

(161)

\[
\mathcal{L}_{\text{int} 34} = -m_2^2 \sum_{n=0}^{d} \left( m_2^{d-2} \beta_n^{(3)} \sqrt{h_3 e_n(S(3))} + m_2^{d-2} \beta_n^{(4)} \sqrt{h_4 e_n(S(4))} \right),
\]

(162)

where \(\mathcal{L}_{12}\) is given by (140), \(\mathcal{L}_{65}\) can be obtained from (141) by replacing indexes \((4,3)\) with \((6,5)\). Since we have already rewritten \(\mathcal{L}_{12}\) in functions of \(h_1\) in (A.1), we focus on \(\mathcal{L}_3\) below.

In order to express \(h_3\) in functions of \(h_1\), we first solve \(S_{(2)}^i j = (\sqrt{h_2^{-1} h_3})^i j\) perturbatively. The general ansatz of \(S_{(2)}^i j\) is given by

\[
S_{(2)}^i j = a^{(2)} \delta^i j + \frac{1}{m^2} \left( b^{(2)} P_{1}^i j + b^{(2)} P_{2}^i j \right) + \frac{1}{m^4} \left( c_1^{(2)} P_{2}^i k P_{2}^j k + c_2^{(2)} P_{2}^i k P_{2}^j k + c_3^{(2)} P_{2}^{kl} P_{2}^{kli} + c_4^{(2)} P_{2}^{2li} \right) + \mathcal{O}(m^{-6}),
\]

(163)

where \(a^{(2)}, b^{(2)}\) and \(c^{(2)}\) are the coefficients to be determined below. Substituting (163) into (144), we get

\[
P_{2}^i j \mathcal{P}_{2}^i j + \frac{m^2}{s_{d-2}^{(2)}} \sum_{n=0}^{d-1} \beta_n^{(2)} a^{(2)n} \sum_{k=0}^{n-k} \left( \sum_{r=0}^{k} \sum_{m=0}^{k-r} (-1)^{n+k} C_{n-k}^{r} C_{d-m}^{k} (M_{(2)})^r j e_m M_{(2)} \right)
\]

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\[ + m^2 \sum_{n=0}^{d-1} \frac{\beta_{d-n}^{(1)}}{\gamma_2} \sum_{k=0}^{n} \left( \sum_{r=0}^{k} \sum_{m=0}^{r} (-1)^{n+k} C_{n-k}^r C_{d-m}^{k-m} (\hat{M}_{(1)}^r)_{j} e_m(\hat{M}_{(1)}) \right) = 0, \]  

where \( M_{(2)} \) is given by (147) [163] and we have replaced \( S_{-1}^{-1} \) by \( \hat{M}_{(1)} \) defined below

\[ \hat{M}_{(b)}^{i} \equiv a^{(b)} S^{i}_{(b)} - \delta^{i}_j, \]

(165)

\[ \left( (S^{i}_{(b)})^{n-k} \right)^{j} = a^{(b)k-n} \sum_{r=0}^{n-k} (\hat{M}_{(b)})^{i}_{j}, \quad e_k(S_{(b)})^{-1} = a^{(b)k} \sum_{m=0}^{k} C_{d-m}^{k-m} e_m(\hat{M}_{(b)}). \]

(166)

From (154) and (165), we have

\[ \hat{M}^{i}_{(1)} = - \frac{P^{i}_{1,j}}{s^{i}_{1}(m^2) + \frac{1}{s^{i}_{1}(m^2)} \left( s^{(1)}_{1} P^{i}_{1,j} + (s^{(1)}_{1} - s^{(2)}_{1}) P^{i}_{1,k} P^{k,j} - s^{(2)}_{1} \frac{\gamma}{d-1} \delta^{i}_{j} \right) + O(m^{-6}). \]

(167)

Combining (150) [151] [164], we obtain

\[ P^{i}_{2,j} = P^{i}_{2,j} + \frac{m^2}{\gamma_2^{d-2}} \left[ s^{(2)}_{0} \delta^{i}_{j} + s^{(2)}_{1} \left( -M_{(2)}^{i} + \delta^{i}_{j} e_1(M_{(2)}) \right) \right] 
\[ + \frac{m^2}{\gamma_2^{d-2}} \left[ s^{(2)}_{2} \left( (M_{(2)}^{i} - M_{(2)}^{i}) e_1(M_{(2)}) + \delta^{i}_{j} e_2(M_{(2)}) \right) \right] 
\[ + \frac{m^2}{\gamma_2^{d-2}} \left[ s^{(1)}_{2} \left( (\hat{M}_{(1)}^{i} - \hat{M}_{(1)}^{i}) e_1(\hat{M}_{(1)}) + \delta^{i}_{j} e_2(\hat{M}_{(1)}) \right) \right] 
\[ + \frac{m^2}{\gamma_2^{d-2}} \left[ s^{(1)}_{0} \delta^{i}_{j} + s^{(1)}_{1} \left( -\hat{M}_{(1)}^{i} + \delta^{i}_{j} e_1(\hat{M}_{(1)}) \right) \right] + O(M_{(1)}^{i}, M_{(2)}^{i}), \]

(168)

where \( q^{(b)}_{k} = \sum_{n=k}^{d-1} C_{d-k-1}^{n-k} \beta^{(b)}_{d-n} a^{(b) - n}. \) Note that \( P_{2} \) in (168) can be converted to \( P_{1} \) by (155)

\[ P^{i}_{2,j} - P^{i}_{2,j} = \frac{1}{a^{(1)2}} \left[ P^{i}_{1,j} - P_{1,j} + \frac{1}{s^{i}_{1}(m^2)} \left( (P^{1,k} P_{1,k} + \frac{P^{1}_{1}}{d-2}) \delta^{i}_{j} - 2(P^{i}_{1,k} P^{k,j} + \frac{P^{1}_{1,j}}{d-2}) \right) \right] 
\[ + O(m^{-4}), \]

(169)

where we have ignored the total derivative terms of \( P_{1}, \) which have no contribution to the gravitational action at order \( O(R^2) \). Substituting (167) [169] and \( M_{(2)} \) into (168) and solving it order by order in \( O(1/m^2) \), we obtain the coefficients of \( S_{(2)}^{i} \)

\[ O(m^2), \quad s_{0}^{(2)} = -s_{0}^{(1)}, \]

\[ O(m^0), \quad b_{1}^{(2)} = \frac{a^{(2)}}{s_{1}^{(2)}} \left( q_{1}^{(1)} + \frac{\gamma_{d-2}}{a^{(1)2}} \right), \quad b_{2}^{(2)} = 0, \]

\[ ^{6} \text{At order } O(R^2), \text{ the only possible terms including derivatives are } \square R \text{ and } \nabla_i \nabla_j R^{ij}, \text{ which are total derivative terms and can be dropped in the action.} \]
Finally, combining (156,159,160,170,172,173), we obtain the higher derivative gravity with
\[ c_1^{(2)} = -2(d-1)c - \frac{a^{(2)}q_1^{(1)}}{s_1^{(1)}s_1^{(2)}} - \frac{2a^{(2)}c_3^{(2)}}{a^{(1)}2s_1^{(1)}s_1^{(2)}}, \]
\[ c_2^{(2)} = 2(d-1)c - \frac{2a^{(2)}c_4^{(2)}}{(d-2)a^{(1)}2s_1^{(1)}s_1^{(2)}}, \]
where the parameter \( c \) is
\[ c = -\frac{a^{(2)}q_1^{(1)}}{2(d-1)s_1^{(1)}s_1^{(2)}} - \frac{a^{(2)}q_1^{(1)}}{2(d-1)s_1^{(1)}s_1^{(2)}} - \frac{a^{(2)}q_1^{(1)}}{2(d-1)s_1^{(1)}s_1^{(2)}} - \frac{a^{(2)}q_1^{(1)}}{(d-1)a^{(1)}2s_1^{(1)}s_1^{(2)}}, \]
\[ c_3^{(2)} = c + \frac{a^{(2)}c_3^{(2)}}{a^{(1)}2(d-1)s_1^{(1)}s_1^{(2)}}, \]
\[ c_4^{(2)} = -c + \frac{a^{(2)}c_4^{(2)}}{a^{(1)}2(d-1)s_1^{(1)}s_1^{(2)}}, \]
\[ \text{(170)} \]

where the parameters \( c_n^{(b)} \) are given by (170). Combining equations (155,172), we express the
metric \( h_3 \) in terms of the metric \( h_1 \)
\[ h_{3\, ij} = h_{2\, ik}(S^{(2)}_{(3)})_{jk} \]
\[ = a^{(1)2}a^{(2)} \left[ h_{1\, ij} + \frac{2(1 - s_1^{(1)}m^2 - 1)}{a^{(2)}s_1^{(1)}s_1^{(2)}} \right] \left[ \frac{a^{(1)2}}{a^{(2)}s_1^{(1)}s_1^{(2)}} \right] \left[ \frac{2a^{(2)}c_1^{(2)}}{a^{(1)2}m^2} \left( c_1^{(2)}s_1^{(1)}s_1^{(2)} \right) \right] \]
\[ + a^{(1)2}a^{(2)} \cdot \left( \frac{a^{(1)2}}{a^{(2)}s_1^{(1)}s_1^{(2)}} \right) \left[ \frac{a^{(1)2}}{a^{(2)}s_1^{(1)}s_1^{(2)}} \right] \left[ \frac{2a^{(2)}c_1^{(2)}}{a^{(1)2}m^2} \left( c_1^{(2)}s_1^{(1)}s_1^{(2)} \right) \right] \]
\[ + a^{(1)2}a^{(2)} \cdot \left( \frac{a^{(1)2}}{a^{(2)}s_1^{(1)}s_1^{(2)}} \right) \left[ \frac{a^{(1)2}}{a^{(2)}s_1^{(1)}s_1^{(2)}} \right] \left[ \frac{2a^{(2)}c_1^{(2)}}{a^{(1)2}m^2} \left( c_1^{(2)}s_1^{(1)}s_1^{(2)} \right) \right] \]
\[ \text{(173)} \]

Finally, combining (156,159,160,170,172,173), we obtain the higher derivative gravity with respect to two metrics \( h_1 \) and \( h_6 \)
\[ I_b^{HD}(h_1, h_6) = m^{d-2} \int_{Q_1} d^d y \sqrt{h_1} \left[ \Lambda^{(1)} + c^{(1)}R + \frac{c^{(1)}}{m^2} \left( \frac{R^{ij}R_{ij} - \frac{d}{2}}{d-1} \right) R^{ij} \right] \]
\[ + m^{d-2} \int_{Q_2} d^d y \sqrt{h_6} \left[ \Lambda^{(6)} + c^{(6)}R + \frac{c^{(6)}}{m^2} \left( \frac{R^{ij}R_{ij} - \frac{d}{2}}{d-1} \right) R^{ij} \right] \]
\[ + L_{int}(h_1, h_6) + O(R^{3}_1) + O(R^{3}_4), \]
\[ \text{(174)} \]

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where the parameters are given by

\[
\Lambda^{(1)} = -2m^2 (\alpha_0^{(1)} + a^{(1)} d \alpha_0^{(2)}), \quad c^{(1)}_{\mathcal{R} R} = \hat{c}_{\mathcal{R} R} + f_1 + f_2 + f_3,
\]

\[
c^{(1)}_R = \hat{c}_R + a^{(1)} d \left[ \frac{(a^{(2)} \gamma_{(3)})^{d-2}}{a^{(1/2)}} - \frac{(d-2)(a^{(2)} \alpha_0^{(2)} + b_1^{(2)} s_1^{(1)} \alpha_1^{(2)})}{(d-1)s_1^{(1)} a^{(2)}} \right],
\]

(175)

and

\[
f_1 = \frac{(a^{(1)} \gamma_{(2)})^{d-2} \left( q_1^{(1)} \alpha_2^{(2)} + \frac{a^{(1)} d}{a^{(2)} s_1^{(2)} a^{(1)}} \right)}{a^{(1/2)} s_1^{(2/2)}} - 2s_1^{(2)} (a^{(2)} \gamma_{(3)})^{d-2},
\]

\[
f_2 = \frac{2 \left( q_1^{(1)} \alpha_2^{(2)} + \frac{a^{(1)} d}{a^{(2)} s_1^{(2)} a^{(1)}} \right)}{s_1^{(1)} s_1^{(2/2)}} - 2s_1^{(2)} (a^{(1)} a^{(2)} \gamma_{(3)})^{d-2},
\]

\[
f_3 = \frac{a^{(1)d}}{a^{(2)} s_1^{(1/3)} s_1^{(2/2)}} \left[ a^{(2)} q_1^{(1)} s_1^{(1)} \alpha_2^{(2)} + a^{(2)} s_1^{(2)} \alpha_0^{(2)} s_1^{(1)} - \frac{d-2}{d-1} s_1^{(1)} \right]
\]

\[+ 2s_1^{(1)} s_1^{(2)} \alpha_1^{(2)} \left( a^{(2)} q_1^{(1)} + (d-2) c s_1^{(1)} s_1^{(2)} \right),
\]

(176)

Similarly, \( \Lambda^{(6)} \), \( c^{(6)}_R \) and \( c^{(6)}_{\mathcal{R} R} \) can be obtained from (175) by replacing indexes \( (b) \) with \( (N + 1 - b) \). Remarkably, the multi-gravity with eight metrics \( (N = 8) \) (174) yields the same curvature squared terms as the effective action (69) on the brane. This is a strong evidence for the equivalence between ghost-free multi-gravity and the higher derivative gravity (69) on the brane.

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