A computational proof of the linear Arithmetic Fundamental Lemma of GL$_4$

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ABSTRACT. Let $K/F$ be an unramified quadratic extension of non-Archimedian local fields with residue character not equals to $2$. We prove the linear Arithmetic Fundamental Lemma for GL$_4$ with the unit element in the spherical Hecke Algebra, using the formula in [Li18]. In this article, all measures are normalized by its hyperspecial subgroup.

CONTENTS

1. Introduction
2. The arithmetic-geometric side
3. A reduction formula
4. Some formula of inductive computation for intersection number
5. Computational Algorithm for Intersection Number
6. The analytic side
References

1. INTRODUCTION

The consideration of the linear Arithmetic Fundamental Lemma(linear AFL) originates from Zhang’s Relative trace Formula approach to the arithmetic Gan-Gross-Prasad (arithmetic GGP) conjecture( [Zha12]). The linear AFL is an arithmetic version of Jacquet-Guo’s Fundamental Lemma(Jacquet-Guo’s FL) [Guo96].

We introduce the linear AFL in details. Let $E/L$, $L/F$ be quadratic etale extensions of non-Archimedean local field, $\pi$ the uniformizer of $\mathcal{O}_F$ and $h$ a positive integer. Let $G_{2h} = \text{GL}_{2h}(F)$, and consider $H_h = \text{GL}_h(L)$ as a subgroup of $G_{2h}$ induced by choosing an $F$-isomorphism $F^{2h} \cong L^h$. Choose a double coset in $H_h \backslash G_{2h}/H_h$ and let $g \in G_{2h}$ be a representative. Let $f$ be a spherical Hecke function on $G_{2h}$. We define the relative orbital integral by

$$
\text{Orb}_L(f, g, s) := \int_{H_h \times H_h/I(g)} f(u_1^{-1}gu_2)\eta_{E/L}(u_2)\left|u_1u_2(u_1^{-1}u_2)^{-1}\right|^s_F du_1du_2.
$$

Here $\eta_{E/L}$ is the quadratic character for $E/L$ and by $\eta_{E/L}(u)$ and $|u|_F$ we mean $\eta_{E/L}(\det(u))$ and $\left|\det u\right|_F$. For $a \in L$, if $L$ is a field, we define $|a|_F = |a|_F^{\frac{1}{2}}$ and $\overline{a}$ is the conjugate of $a$. If $L = F \times F$, we define $|a|_F = |a_1|_F$ and $(a_1, a_2) = (a_2, a_1)$ for $a = (a_1, a_2)$. Subgroup $I(g)$ is defined by

$$
I(g) = \{(h_1, h_2) \in H_h \times H_h | h_1g = gh_2\}.
$$

For each double coset in $H_h \backslash G_{2h}/H_h$, we associate a degree $h$ polynomial over $F$ to it and called it the invariant polynomial. For some element $\gamma \in \text{GL}_{2h}(F)$, its invariant polynomial $P_\gamma$ could

For each double coset in $H_h \backslash G_{2h}/H_h$, we associate a degree $h$ polynomial over $F$ to it and called it the invariant polynomial. For some element $\gamma \in \text{GL}_{2h}(F)$, its invariant polynomial $P_\gamma$ could
match to some element in a division algebra, which give rise to intersection numbers $\text{Int}(f, \gamma)$ in the Lubin-Tate space. The linear AFL is the following statement. If $f$ is an element in Spherical Hecke algebra, $L = F \times F$, $K/F$ is an unramified quadratic extension and $E = K \times K$, then

$$\pm (2 \ln q)^{-1} \frac{d}{ds} \bigg|_{s=0} \text{Orb}_L(f, \gamma, s) = \text{Int}(f, \gamma).$$

The left-hand side of (1.2) is known as the analytic side (AS) and the right-hand side is known as the arithmetic-geometric side (GS). In this article, we prove the following result by computation.

**Theorem 1.1.** The equation (1.2) holds for $h = 2$ if $f$ is the characteristic function for $GL_4(O_F)$.

We prove the Theorem by straightforward computation. In Section 6, we compute the analytic side. In Section 5, we compute the arithmetic-geometric side by the formula in [Li18].

1.1. **Several Remarks.** Section 2.2 gives another way to state Guo’s work in [Guo96]. We define orbital integral for all quadratic etale extensions. Nevertheless, this article only uses the case of $L = K$ and $L = F \times F$. The Lemma 3.3 can be essentially deduced by applying Weyl Integration Formula and reduction formula of Guo’s Proposition 2.1 in [Guo96]. This paper proves linear AFL for unit Hecke function for $h = 2$, the method of computing intersection numbers, works for all prime $h$ with $v_F(\gamma)$ non-integer. Section 5 provides an algorithm for computing those cases.

2. The arithmetic-geometric side

In this section, we compute the arithmetic-geometric side by the following formula in [Li18].

$$\text{Int}(f, \gamma) = \frac{\epsilon_{F,2h}}{\epsilon_{K,h}^2} \int_{GL_{2h}(F)} f(g) \bigg|_{F} \text{Res}(P_\gamma, P_g)^{-1} \, dg.$$  

Here $K/F$ is unramified quadratic extension. The constant $\epsilon_{F,2h}$ and $\epsilon_{K,h}$ are relative volume of invertible matrices in $gl_{2h}(O_F)$ and $gl_h(O_K)$ respectively. By $\text{Res}(P_\gamma, P_g)$ we mean the resultant of $P_\gamma$ and $P_g$. The polynomial $P_g$ is the invariant polynomial which we defined as following.

Let $G_{2h} = GL_{2h}(F)$ and $H_h = GL_h(K)$. An $O_F-$isomorphism $\tau : O_K^h \cong O_F^{2h}$ give rise to an embedding $H_h \to G_{2h}$. Let $M = (\tau \cdot \tau) \in GL_{2h}(O_K)$, the element $M^{-1}gM$ is of the form

$$M^{-1}gM = \begin{pmatrix} g_+ & g_- \\ g_- & g_+ \end{pmatrix}$$

for some $g_+, g_- \in GL_h(K)$. For our convenience, we use subindexed symbols $g_+, g_-$ to denote the corresponding blocks in $M^{-1}gM$ as (2.2). If $g_+$ is invertible, use $g_+^{-1}$ to denote

$$g_+ = g_+^{-1}g_-.$$

The subgroup $H_h$ is cut out by $g_- = 0$. By an abuse of notation, we use $g$ for $M^{-1}gM$ and treat it as an element in $M^{-1}G_{2h}M \subset GL_{2h}(K)$. For any $g \in G_{2h}$ such that $g_+$ is invertible, we define its invariant polynomial $P_g$ by the characteristic polynomial of $(1 - g_+g_-^{-1})^{-1}$ as an element in $GL_h(K)$. This definition is well-defined on a Zaiski-dense subset and extends to all elements of $G_{2h}$(See Definition 1.2 of [Li18]). Therefore

$$\text{Res}(P_\gamma, P_g) = \det \left( P_\gamma (1 - g_+g_-^{-1}) \right).$$
For monic polynomials $P = P_\gamma$ or $P_g$, we define the unnormalized invariant polynomial $p(x) = p_\gamma(x)$ or $p_g(x)$ by the product of $(x-1+\lambda^{-1})$ for $\lambda$ run through roots of $P$. Furthermore, We have

$$P((1-x)^{-1}) = \frac{p(x)}{p(1)(1-x)^h}.$$ (2.4)

For any matrix $M$, by $|M|_F$ we mean $|\det(M)|_F$. From now on, we fix $p(x)$ to be the unnormalized invariant polynomial for $P_\gamma$, therefore we can write

$$\text{Int}(f, \gamma) = \frac{\epsilon_{F,2h}}{\epsilon_{K,h}^2} \int_{\GL_{2h}(F)} f(g)|p(1)|_F |1-g\#g^-|^h \frac{|p(g\#g^-)|_F}{|1-1-g\#g^-|^2h} dg.$$ (2.5)

**Lemma 2.1.** For any smooth compact supported function $f$ on $\GL_{2h}(F)$, define

$$\phi_f(g\#) = \int_{\GL_h(K)} f \left( \left( \begin{array}{c} g+ \\ g+ \\ g\# \\ 1 \end{array} \right) \right) dg_+dg_+.$$ (2.6)

Then

$$\frac{\epsilon_{F,2h}}{\epsilon_{K,h}^2} \int_{\GL_{2h}(F)} f(g)dg = \int_{\GL_h(K)} \phi_f(g\#) |g\#g^-|^h |1-1-g\#g^-|^2h dg_+dg_+.$$ (2.7)

**Proof.** Let $dg, dg_+$ and $dg_-$ be Haar measures of $\GL_{2h}(F)$ and $\GL_h(K)$ normalized by their hyperspecial subgroups. The normalized Haar measure for additive group $\GL_{2h}(F)$ is given by

$$\epsilon_{F,2h}|g|^2h dg.$$ 

The Haar measure of $\GL_h(K)$ normalized by $\GL_h(K)$ is given by

$$\epsilon_{K,h}|g_+g_+|^h_F dg_+dg_+; \quad \text{resp.} \quad \epsilon_{K,h}|g_-g_-|^h_F dg_-dg_-.$$ 

Since the Lie algebra naturally decomposes $\gl_{2h}(O_F) \cong \gl_h(O_K) \oplus \gl_h(O_K)$, we have

$$\epsilon_{F,2h}|g|^2h dg = \epsilon_{K,h}|g_+g_+|^h_F dg_+dg_+ + \epsilon_{K,h}|g_-g_-|^h_F dg_-dg_-.$$ 

Notice that $\det(g) = \det(g_+g_-) \det(1-1-g\#g^-)$ for $g \in \GL_{2h}(F)$, therefore

$$\frac{\epsilon_{F,2h}}{\epsilon_{K,h}^2} dg = \frac{|g_+g_+g_-g^-|^h_F}{|(g_+g_+)(1-g\#g^-)|^{2h}F}dg_+dg_-dg_+dg_- = \frac{|g\#g^-|^h_F}{|1-1-g\#g^-|^2h} dg_+dg_+dg_+dg_-. $$ 

we may write our integration as

$$\int_{\GL_h(K) \times \GL_h(K)} f \left( \left( \begin{array}{c} g+ \\ g+ \\ g\# \\ 1 \end{array} \right) \right) \frac{|g\#g^-|^h_F}{|1-1-g\#g^-|^2h} dg_+dg_+dg_+dg_-. $$

Therefore the Lemma follows. 

We calculate $\text{Int}(f, \gamma)$ when $f$ is the characteristic function of $\GL_{2h}(O_F)$. In this case $f(g) = 1$ is equivalent to $g_+, g_- \in \GL_h(O_K)$. So we can replace $f$ by the characteristic function of $\GL_{2h}(O_K)$. Apply Lemma 2.1 to (2.5) we can write

$$\text{Int}(f, \gamma) = |p_\gamma(1)|_F \int_{\GL_h(K)} \phi_f(g\#) |p_\gamma(g\#g^-)|_F^{-1} \frac{|g\#g^-|^h_F}{|1-1-g\#g^-|^2h} dg_+dg_+.$$ (2.8)

Here $\phi_f(g\#)$ is defined by (2.6) with $f$ the characteristic function of $\GL_{2h}(O_F)$. 

\[ \text{3} \]
2.1. **Properties of \( p_\gamma \).** In our application, \( P_\gamma \) is an invariant polynomial for an orbit of a division algebra \( D \). Let \( \gamma \# \) be a root for \( p_\gamma \) so the field \( T = F[\gamma \#] \) is the splitting field of \( P_\gamma \) and \( p_\gamma \). In this paper, we use the fact that the field \( T \) does not contains \( K \) as a subfield and the valuation \( v_T(\gamma \#) \) is an odd integer. Let \( \lambda_1, \ldots, \lambda_h \) be eigenvalues of \( g_\# \). We have

\[
|p_\gamma(g_\#)\rangle_F = \prod_{i=1}^{h} |\lambda_i - \gamma \#|_T
\]

In this paper, we fix

\[
r = v_F(\gamma \#)
\]

which is equivalent to \( |\gamma \#|_T = q^{-hr} \). In this paper, we only consider the case for \( h = 2 \), but our method works for all prime \( h \) with \( r \) not an integer, we have

**Lemma 2.2.** If \( h \) is a prime and \( r \) is not an integer, we have

\[
|\gamma \# - \lambda|_T = \max\{|\gamma \#|_T, |\lambda|_T\}
\]

**Proof.** By triangle inequality, this is true if \( |\lambda|_T \neq |\gamma \#|_T \). Otherwise if \( |\lambda|_T = q^{-hr} \), let \( n = F[\lambda] \), we have \( |\text{Nm}_{F[\lambda]/F}(\lambda)|_T = q^{-nhr} \) therefore \( nhr \) is divisible by \( h \) because \( \text{Nm}_{F[\lambda]/F}(\lambda) \in F \). Since \( \lambda \) is an eigenvalue of \( g_\# \), \( n \leq h \). But \( hr \) is coprime to \( h \), this implies \( n = h \). The Galois orbit \( \lambda \) is all eigenvalues of \( g_\# \). Therefore

\[
q^{-h^2r} = |\text{Nm}_{F[\lambda]/F}(\lambda)|_T = |g_\#|_T^{hr} = |g_\#|_F^{h}
\]

But since \( K/F \) is unramified, \( |\text{det}(g_\#)|_F = q^{a} \) for some integer \( a \), this implies if \( |g_\#|_F = q^{-hr} \) then \( hr \) must be an even number. This Contradiction implies \( |\lambda|_T \neq q^{-hr} \). \( \Box \)

Therefore, in this case, by equation (2.9) we have

\[
|p_\gamma(g_\#)\rangle_F = \prod_{i=1}^{h} \max\{q^{-hr}, |\lambda_i|_F^{h}\}
\]

This implies the following Lemma

**Lemma 2.3.** If \( h \) is a prime and \( r \) is not an integer, \( g_\# \in \text{GL}_a(K) \) then

- If all eigenvalues \( \lambda \) of \( g_\# \) has \( v_F(\lambda) > r \), then \( |p_\gamma(g_\#)|_F = q^{-rah} \)
- If all eigenvalues \( \lambda \) of \( g_\# \) has \( v_F(\lambda) \leq r \), then \( |p_\gamma(g_\#)|_F = |g_\#|_F^{h} = |g_\#|_T^{h} = |g_\#|_F^{h} \)

2.2. **Properties of the function \( \phi(g_\#) \).** To further simplify the equation (2.8), we discuss several properties of \( \phi \). Since the integrand in (2.6) does not vanish when \( g_\# \) is in the following subset

\[
S = \left\{ g_\# \in \text{GL}_h(K) : \left( \frac{k_+}{k_+} \right) \left( \frac{1}{g_#} \right) g_\# \in \text{GL}_{2h}(\mathcal{O}_K) \text{ for some } k \in H_1 \right\}
\]

For any logic expression \( p \), we define

\[
1[p] = \begin{cases} 
1 & \text{if } p \text{ is True.} \\
0 & \text{if } p \text{ is False.}
\end{cases}
\]

We found \( \phi(g_\#) = 1[g_\# \in S] \). For any two positive volume subset \( U, V \) of \( K^h \), denote the relative volume by

\[
[U : V] := \text{Vol}(U)/\text{Vol}(V).
\]
We denote $\mathcal{O} \doteq \mathcal{O}_K \subset K$. For any element $g_\# \in \GL_h(K)$, define
\[ ||g_\#|| = [g_\# \mathcal{O}^h + \mathcal{O}^h : \mathcal{O}^h] \]
In the following, we abbreviate the symbol $|k|$ as $|k|$. Note that $|k| = |kK|_F = [\mathcal{O}^h : k\mathcal{O}^h]$.

**Lemma 2.4.** There exists $k \in \gl_h(\mathcal{O}_K)$, such that $|k|^{-1} = ||g_\#||$ and $kg_\# \in \gl_h(\mathcal{O}_K)$

**Proof.** Let $k$ be an element such that
\[ k(\mathcal{O}^h + \mathcal{O}^h) = \mathcal{O}^h \]
Then $k(\mathcal{O}^h) \subset \mathcal{O}^h$ so $k \in \gl_h(\mathcal{O}_K)$ and
\[ (kg_\#)\mathcal{O}^h = kg_\#\mathcal{O}^h \subset \mathcal{O}^h + k\mathcal{O}^h = \mathcal{O}^h. \]
So $kg_\# \in \gl_h(\mathcal{O}_K)$. Furthermore,
\[ |k|^{-1} = [g_\#\mathcal{O}^h + \mathcal{O}^h : k(g_\#\mathcal{O}^h + \mathcal{O}^h)] = [g_\#\mathcal{O}^h + \mathcal{O}^h : \mathcal{O}^h] = ||g_\#||. \]
This proves the lemma.

**Proposition 2.5.** We have
\[ ||g_\#|| ||g_\#|| \geq |1 - g_\# g_\#|. \]
The equality holds if and only if $g_\# \in S$.

**Proof.** Let $k \in \gl_h(K)$ such that $kg_\# \in \gl_h(\mathcal{O}_K)$ and $|k|^{-1} = ||g_\#||$. Note that
\[
\left( \begin{array}{c}
1 \\
g_\#
\end{array} \right) \mathcal{O}^h + \mathcal{O}^h = \left( \begin{array}{c}
\frac{1}{g_\#} \\
1
\end{array} \right) \mathcal{O}^h + \mathcal{O}^h = \left( \begin{array}{c}
g_\# \\
1
\end{array} \right) \mathcal{O}^h + \mathcal{O}^h = (g_\# \mathcal{O} + \mathcal{O}) \oplus (\overline{g_\#} \mathcal{O} + \mathcal{O}).
\]
Therefore we have
\[ \left| \left( \begin{array}{c} 1 \\
g_\#
\end{array} \right) \right| = ||g_\#|| ||\overline{g_\#}|| = |kK|^{-1}. \]
Note that
\[
\left( \begin{array}{c} k \\
k
\end{array} \right) \left( \begin{array}{c} 1 \\
g_\#
\end{array} \right) = \left( \begin{array}{c} k \\
g_\#k\#
\end{array} \right) \in \gl_{2n}(\mathcal{O}_K).
\]
We have
\[ 1 \geq |kK(1 - g_\# \overline{g_\#})| = ||g_\#||^{-1} ||\overline{g_\#}||^{-1} |1 - g_\# \overline{g_\#}|. \]
The equality holds if and only if
\[
\left( \begin{array}{c} k \\
kg_\#
\end{array} \right) \in \GL_{2n}(\mathcal{O}_K),
\]
where this is equivalent to $g_\# \in S$. 

**Remark 2.6.** We can also write $g \in S$ if and only if $||g_\#|| = |1 - g_\# \overline{g_\#}|_F$.

**Proposition 2.7.** If $g_\# \in S$, then $g_\#^{-1} \in S$.

**Proof.** Note that
\[ ||g_\#^{-1}|| = [g_\#^{-1} \mathcal{O}^h + \mathcal{O}^h : \mathcal{O}^h] = |g_\#^{-1}|_K[\mathcal{O}^h + g_\# \mathcal{O}^h : \mathcal{O}^h] = |g_\#^{-1}|_K ||g_\#||. \]
Therefore,
\[ ||g_\#^{-1}|| ||\overline{g_\#^{-1}}|| = ||g_\#|| ||\overline{g_\#}|| ||g_\# \overline{g_\#}||^{-1}. \]
On the other hand,
\[ |1 - g_#^{-1}g_#^{-1}|_K = |g_# g_#^{-1}|_K^{-1} |g_# g_#^{-1} - 1|_K. \]
Therefore
\[ ||g_#^{-1}|| |g_#^{-1}| = |1 - g_#^{-1}g_#^{-1}|_K. \]
Then the proposition follows. \(\square\)

From now on, we use subindex \(\phi_h\) for the function \(\phi\) defined on \(\text{GL}_h(K)\). Define
\[ S_{< h, a} = \{ g \in \text{GL}_h(K) \mid a \text{ many eigenvalues of } g \text{ that has valuation less than } x \}. \]
Similarly we define the set \(S_{\leq h, a}, S_{> h, a}, S_{\geq 0, a}\). If \(a = h\), we omit \(a\). Let \(1_{< h, a}, 1_{\leq h, a}, 1_{> h, a}, 1_{\geq 0, a}\) be corresponding characteristic functions.

**Proposition 2.8.** We have
\[ 1_{< h}^0(x\overline{\pi})\phi(x) = 1 \quad [x \in \text{gl}_n(O_K) \text{ and } 1 - x\overline{\pi} \in \text{GL}_h(O_K)] . \]

**Proof.** \(1_{< h}^0(x\overline{\pi}) = 1\) implies all eigenvalues of \(x\overline{\pi}\) are integers, therefore \(|1 - x\overline{\pi}| \leq 1\). Then \(\phi(x) = 1\) if and only if
\[ \||x||_F ||\overline{\pi}|| = |1 - x\overline{\pi}|_F \leq 1. \]
Since \(\||x||_F ||\overline{\pi}|| \geq 1\). So \(\||x|| = |1 - x\overline{\pi}|_F = 1\). Therefore \(\||x|| = 1\) implies \(xO \subset O\) so \(x \in \text{gl}_h(O_K)\) and therefore \(1 - x\overline{\pi} \in \text{gl}_h(O_K)\). Since \(|1 - x\overline{\pi}| = 1\), we have \(1 - x\overline{\pi} \in \text{GL}_h(O_K)\). On the contrary, if \(1 - x\overline{\pi} \in \text{GL}_h(O_K)\) and \(x \in \text{gl}_h(O_K)\) we have both
\[ |1 - x\overline{\pi}| = 1 = \||x||_F ||\overline{\pi}||. \]
Therefore \(\phi(x) = 1\) and \(1_{< h}^0(x\overline{\pi}) = 1\). \(\square\)

**Corollary 2.9.** If \(1_{< h}^0(x\overline{\pi})\phi(x) = 1\), then
\[ |x\overline{\pi}|_F = |1 - x\overline{\pi}|_F \]

**Proof.** Since \(\phi(x) = \phi(x^{-1})\), we have \(1_{< h}^0(x^{-1}\overline{x^{-1}})\phi(x^{-1}) = 1\), this implies \(|1 - (x\overline{\pi})^{-1}|_F = 1\) therefore \(|x\overline{\pi}|_F = |1 - x\overline{\pi}|_F \). \(\square\)

**Theorem 2.10.** Let \(a \in \text{GL}_h(K)\) and \(c \in \text{GL}_m(K)\). We have
\[ \int \phi_{h+m} \left( \begin{array}{cc} a & b \\ c & 1 \end{array} \right) dB = |1 - a\overline{\pi}|_F |1 - c\overline{\pi}|_F \phi_h(a) \phi_m(c) \]

**Proof.** Let
\[ \mathcal{R} = a\mathcal{O}^h \oplus c\mathcal{O}^m. \]
Then
\[ \left\| \left( \begin{array}{ccc} a & bc \\ c & 1 \end{array} \right) \right\| = \left\| \left( \begin{array}{cc} 1 & b \\ 1 & 1 \end{array} \right) : \mathcal{O}^{h+m} \cap \mathcal{O}^{h+m} \right\| = \left\| \left( \begin{array}{cc} 1 & b \\ 1 & 1 \end{array} \right) : \mathcal{O}^{h+m} \cap \mathcal{O}^{h+m} \right\| = |a||c| I(b)^{-1}, \]
where
\[ I(b) := \text{Vol} \left( \left( \begin{array}{cc} 1 & b \\ 1 & 1 \end{array} \right) \mathcal{O}^{h+m} \cap \mathcal{R} \right) = \int_{\mathcal{O}^m} \int_{\mathcal{O}^h} 1 \left[ \left( \begin{array}{c} v_1 - bv_2 \\ v_2 \end{array} \right) \in \mathcal{R} \right] dv_1 dv_2. \]
The integrand is not vanishing if \( \mathbf{v}_1 - b\mathbf{v}_2 \in a\mathcal{O}^h \) and \( \mathbf{v}_2 \in c\mathcal{O}^m \) therefore we can write it into 

\[
I(b) = \int_{\mathcal{O}^m} \int_{\mathcal{O}^h} 1 [\mathbf{v}_1 - b\mathbf{v}_2 \in a\mathcal{O}^h] 1 [\mathbf{v}_2 \in c\mathcal{O}^m] d\mathbf{v}_1 d\mathbf{v}_2. 
\]

Note that 

\[
\int_{\mathcal{O}^h} 1 [\mathbf{v}_1 - b\mathbf{v}_2 \in a\mathcal{O}^h] d\mathbf{v}_1 = \text{Vol}(\mathcal{O}^h \cap a\mathcal{O}^h) 1 [b\mathbf{v}_2 \in a\mathcal{O}^h + \mathcal{O}^h],
\]

the equation (2.10) can be write into 

\[
I(b) = \text{Vol}(\mathcal{O}^h \cap a\mathcal{O}^h) \int_{\mathcal{O}^m} 1 [b\mathbf{v}_2 \in a\mathcal{O}^h + \mathcal{O}^h] 1 [\mathbf{v}_2 \in c\mathcal{O}^m] d\mathbf{v}_2.
\]

Since \( 1[b\mathbf{v}_2 \in a\mathcal{O}^h + \mathcal{O}^h] \leq 1 \), we have 

\[
I(b) \leq \text{Vol}(\mathcal{O}^h \cap a\mathcal{O}^h) \int_{\mathcal{O}^m} 1 [\mathbf{v}_2 \in c\mathcal{O}^m] d\mathbf{v}_2 = I(0).
\]

The equality holds if and only if \( b(c\mathcal{O}^m \cap \mathcal{O}^m) \subset a\mathcal{O}^h + \mathcal{O}^h \). Therefore, 

\[
\left\| \begin{pmatrix} a & bc \\ c \end{pmatrix} \right\| = \frac{\text{Vol}(\mathcal{R})}{I(b)} \geq \frac{\text{Vol}(\mathcal{R})}{I(0)} = \left\| \begin{pmatrix} a & bc \\ c \end{pmatrix} \right\| = ||a|| ||c||.
\]

The equality holds if and only if \( b(c\mathcal{O}^m \cap \mathcal{O}^m) \subset a\mathcal{O}^h + \mathcal{O}^h \). Therefore 

\[
\left\| \begin{pmatrix} a & bc \\ c \end{pmatrix} \right\| \left\| \begin{pmatrix} \bar{a} & bc \\ \bar{c} \end{pmatrix} \right\| \geq ||a|| ||a|| ||c|| ||c|| \geq |1 - a\bar{a}| |1 - c\bar{c}| = |1 - \begin{pmatrix} a & bc \\ c \end{pmatrix} \begin{pmatrix} \bar{a} & bc \\ \bar{c} \end{pmatrix}|.
\]

The equality holds if and only if \( b(c\mathcal{O}^m \cap \mathcal{O}^m) \subset a\mathcal{O}^h + \mathcal{O}^h \) and \( \phi(a) = \phi(c) = 1 \), therefore 

\[
J(a, c) := \int \phi_{h+m} \begin{pmatrix} a & bc \\ c \end{pmatrix} db = \phi(a) \phi(c) \int [b(c\mathcal{O}^m \cap \mathcal{O}^m) \subset a\mathcal{O}^h + \mathcal{O}^h] db.
\]

Let \( c', a'' \) be elements such that \( c'\mathcal{O}^m = c\mathcal{O}^m \cap \mathcal{O}^m \), and \( a''\mathcal{O}^h = a\mathcal{O}^h + \mathcal{O}^h \). The integral now can be written as 

\[
J(a, c) = \phi(a) \phi(c) \int 1[a''^{-1}b < M_{h+m}(\mathcal{O}^h}] = \phi(a) \phi(c) |a''| |c'|^{-h}
\]

Note that 

\[
|a''| = [a''\mathcal{O}^h : \mathcal{O}^h] = [a\mathcal{O}^h + \mathcal{O}^h : \mathcal{O}^h] = ||a||
\]

and 

\[
|c'|^{-1} = [\mathcal{O}^h : c'\mathcal{O}^h] = [\mathcal{O}^h : c\mathcal{O}^h \cap \mathcal{O}^h] = |c|^{-1}[c\mathcal{O}^h : c\mathcal{O}^h \cap \mathcal{O}^h] = ||c|| |c|^{-1}.
\]

We have 

\[
\int \phi_{h+m} \begin{pmatrix} a & b \\ c \end{pmatrix} db = |c|^h \int \phi_{h+m} \begin{pmatrix} a & bc \\ c \end{pmatrix} db = ||a||m ||c||^h \phi(a) \phi(c).
\]

Since \( \phi(a)|a||^m = \phi(a)|1 - a\bar{a}|^m_F \), the Theorem follows. \( \square \)
3. A REDUCTION FORMULA

To further simplify the integral in (2.8), we provide several reduction formula.

**Lemma 3.1.** For $B \in \text{GL}_n(K)$ such that
\[
B\bar{B} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},
\]
for $\alpha \in \text{GL}_m(F), \beta \in \text{GL}_{n-m}(F)$. Suppose furthermore the minimal polynomial $p_\alpha(x)$ of $\alpha$ and $p_\beta(x)$ of $\beta$ are coprime to each other, then
\[
B = \begin{pmatrix} u \\ v \end{pmatrix}
\]
for some $u \in \text{GL}_m(K)$ and $v \in \text{GL}_{n-m}(K)$.

**Proof.** Let $r(x)$ and $q(x)$ be polynomials such that $p_\beta(x)r(x) + p_\alpha(x)q(x) = 1$, we have
\[
p_\beta(B\bar{B})r(B\bar{B}) = \begin{pmatrix} I_m \\ 0 \end{pmatrix}.
\]
Since $B\bar{B} \in \text{GL}_n(F)$ we have $\bar{B}B = B\bar{B}$ so $p_\beta(B\bar{B})r(B\bar{B})$ commutes with $B$, its kernel and image are invariant subspaces for $B$, this implies $B$ must have the form as stated in the Lemma. □

**Theorem 3.2.** Let $f$ be a compact supported smooth function such that $f(g^{-1}xg) = f(x)$ for any $g \in \text{GL}_h(K)$, then we have
\[
\int_{\text{GL}_h(K)} f(x)1_{\geq m}(x)1_{\leq m}(x) |x|^h dx = \int_{G'} F(x_1, x_2)1_{a}(x_1)1_{-a}(x_2)|x_1^a x_2^h| dx_1 dx_2.
\]
where
\[
F(x_1, x_2) = \int_{M_{a}(h-a)(K)} f \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) dy; \quad G' = \text{GL}_a(K) \times \text{GL}_{h-a}(K).
\]

**Proof.** To prove this formula, note that the integrand is supported on the region
\[
R = S_{\geq m} \cap S_{\leq m}.
\]
For any $x \in R$, there exists a matrix $y$ such that
\[
y^{-1} x y = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}
\]
for some $\alpha \in \text{GL}_a(F), \beta \in \text{GL}_{h-a}(F)$.

Here the valuations of eigenvalues for $\alpha$ is bigger than $m$, and for $\beta$ is no bigger than $m$. Note that this means the product of $y^{-1} xy$ and $y^{-1} xy$ is a block diagonal matrix. By Lemma 3.1 we have
\[
y^{-1} xy = \begin{pmatrix} u \\ v \end{pmatrix}
\]
for some $u \in \text{GL}_a(K), v \in \text{GL}_{h-a}(K)$.

Apply the Iwasawa decomposition we can write
\[
y = y_1 y_2 = y_1 \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}
\]
with $y_1 \in \text{GL}_h(O_K)$, then we have the form
\[
(3.1) \quad \overline{y}_1^{-1} xy_1 = \overline{y}_2 \begin{pmatrix} u \\ v \end{pmatrix} \overline{y}_2^{-1} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} =: x'.
\]
Let $\Gamma$ be the subgroup of the form
\[
\Gamma(h, a) = \left\{ \begin{pmatrix} u_{11} & u_{12} \\ u_{22} \end{pmatrix} \middle| u_{11} \in \text{GL}_a(K), u_{22} \in \text{GL}_{h-a}(K) \right\}
\]

Then (3.1) suggests that every $x \in R$ has a decomposition
\[
(3.2) \quad x = \gamma x' y_1^{-1} =: \gamma^{-1} g
\]

For some $g \in \text{GL}_h(\mathcal{O}_K)$ and $\gamma \in \Gamma(h, a)$. But the choice of $g$ for this decomposition is not unique. All possible choice of $g$ is a left-coset of $\text{GL}_h(\mathcal{O}_K) \cap \Gamma(h, a)$. We denote
\[
\text{Gr}(h, a) := \text{GL}_h(\mathcal{O}_K) \cap \Gamma(h, a) \setminus \text{GL}_h(\mathcal{O}_K)
\]
and let $[g]$ be the image of $g$ in $\text{Gr}(h, a)$. We choose a smooth section $\beta: \text{Gr}(h, a) \longrightarrow \text{GL}_h(\mathcal{O}_K)$ and denote the image of it to be $S$. When we write $[g]$ we assume $g \in S$. There is an injective map
\[
(\gamma, [g]) \longmapsto \gamma^{-1} g
\]

Let $R$ be the image of $\alpha$. Let $dp$ and $dr$ be Haar measures on $\text{Gr}(h, a)$ and $\Gamma(h, a)$, normalized by $\text{Gr}(h, a)$ and $\Gamma(h, a) \cap \text{GL}_h(\mathcal{O}_K)$. We compute the Jacobian determinant of $\alpha$, denote
\[
i : \Gamma(h, a) \times \text{Gr}(h, a) \longrightarrow \text{GL}_h(K)
\]
\[
(\gamma, [g]) \longmapsto \gamma g
\]

let $i', \alpha'$ be maps so the following diagram commutes

\[
\begin{array}{ccc}
\Gamma(h, a) \times S & \xrightarrow{i'} & \text{Gr}(h, a) \\
\downarrow{id \times \beta} & & \downarrow{\alpha'} \\
\text{GL}_h(K) & \xrightarrow{i} & \Gamma(h, a) \times \text{Gr}(h, a) & \xrightarrow{\alpha} & \text{GL}_h(K)
\end{array}
\]

Maps $id \times \beta$, $i'$ preserves the Haar measure. Consider the induced tangent space map at $(g_0, \gamma_0)$
\[
(3.3) \quad T_{\gamma_0 g_0} \xrightarrow{d\alpha'} T_{\Gamma, \gamma_0} \oplus T_{S, g_0} \xrightarrow{d\alpha'} T_{\gamma_0 g_0}^{-1}
\]

\[
\begin{align*}
d(\gamma g) &= \gamma dg + (d\gamma)g \\
& \xlongleftarrow{d(\gamma X, dg)} \xlongleftarrow{(d\gamma, dg)} d(\gamma^{-1} g) = -\overline{g}^{-1}(d\overline{g}) \overline{g}^{-1} \gamma g + \overline{g}^{-1}(d\gamma)g + \overline{g}^{-1} \gamma dg
\end{align*}
\]

To give coordinates for these maps, take any $X \in T_{\Gamma, e}$ and $Y \in T_{S, e}$. To move it to $\gamma_0, g_0$ by left translation, we evaluate (3.3) by $g \mapsto g_0$, $\gamma \mapsto \gamma_0$, $d\gamma \mapsto \gamma_0 X$, $dg \mapsto g_0 Y$, the map is given by
\[
\begin{align*}
\gamma_0 g_0 (Y + g_0^{-1} X g_0) & \xrightarrow{\alpha'} (\gamma_0 X, g_0 Y) \\
& \xrightarrow{c_0} c_0(-c_0^{-1} Y c_0 + g_0^{-1} X g_0 + Y)
\end{align*}
\]
here $c_0 = g_0^{-1}h_0g_0$. Identify all tangent spaces to the tangent space at identity by left translation, we would like to calculate the determinant of the map

$$\phi : T_e \rightarrow T_e$$

$$Y + g_0^{-1}Xg_0 \mapsto -c_0^{-1}\nabla c_0 + g_0^{-1}Xg_0 + Y$$

The linear operator $\phi$ has an invariant subspace $R$ cut out by $Y = 0$.

$$R := g_0^{-1}T_{\Gamma,e}g_0 = T_{g_0^{-1}\Gamma g_0,e}$$

and $\phi|_R = \text{id}$. To calculate $\det(\phi)$, consider another linear transformation

$$\tilde{\phi} : T_e \rightarrow T_e$$

$$Z \mapsto -c_0^{-1}Zc_0 + Z$$

Then $R$ is also an invariant subspace of $\tilde{\phi}$. Linear transformations $\phi$ and $\tilde{\phi}$ agree on the quotient space $T_e/R$. Therefore

$$\det(\phi) = \frac{\det(\tilde{\phi})}{\det(\tilde{\phi}|_R)}$$

To calculate $\det(\tilde{\phi})$, note that $T_e \otimes_F K \cong T_e \oplus \overline{T_e}$. The base change map

$$\tilde{\phi} \otimes_F K : T_e \oplus \overline{T_e} \rightarrow T_e \oplus \overline{T_e}$$

$$(Z, \overline{Z}) \mapsto (-c_0^{-1}Zc_0 + Z, -c_0^{-1}Zc_0 + \overline{Z})$$

will have the same determinant over $K$. Let $C(c_0)$ denote the linear transformation of conjugation $Z \mapsto c_0^{-1}Zc_0$, the matrix of $\tilde{\phi} \otimes_F K$ in coordinates $(Z, \overline{Z})$ is given by

$$\begin{pmatrix}
1 & -C(c_0) \\
-C(\overline{c_0}) & 1
\end{pmatrix}$$

The determinant of the above matrix only depends on the conjugacy class of $c_0$ and invariant under base change, without loss of generality, we may assume $g_0 = 1$. Furthermore, since the set of diagonalizable matrices is Zariski dense, we may assume $c_0$ is a diagonal matrix with diagonal entries $s_1, s_2, \cdots, s_a, s_{a+1}, \cdots, s_h$. Let

$$\lambda_i = s_i\overline{s_i} \text{ for } 1 \leq i \leq h.$$ 

Let $E_{ij}$ denote the matrix with $1$ on $i$'th row and $j$'th column and $0$ elsewhere, then it is an eigenvector of $C(c_0\overline{c_0})$ with eigenvalue $\lambda_i^{-1}\lambda_j$. Note that

$$R = T_{\Gamma,e} = \text{span}\{E_{ij} : 1 \leq i \leq a \text{ or } a < i, j \leq h\},$$

$$T_e = \text{span}\{E_{ij} : 1 \leq i, j \leq h\}.$$ 

Therefore we have

$$\det(\tilde{\phi}) = \det(1 - C(c_0\overline{c_0})) = \prod_{1 \leq i, j \leq h} (1 - \lambda_i^{-1}\lambda_j)$$
\[ \det(\tilde{\phi}_R) = \det(1 - C(c_0c_0^t)|_R) = \prod_{1 \leq i \leq a \atop h < i, j \leq h} (1 - \lambda_i^{-1}\lambda_j) \]

Therefore we have
\[ \det(\phi) = \frac{\det(\tilde{\phi})}{\det(\tilde{\phi}_R)} = \prod_{1 \leq j \leq a < i \leq h} (1 - \lambda_i^{-1}\lambda_j) = \det(x_2)^{-a}\res(x_1, x_2). \]

Therefore, we have
\[ dx = |\res(x_1, x_2)|_F |x_2|^a d\gamma dg. \]

Furthermore, \( d\gamma \) is a left-invariant Haar measure for \( \Gamma(h, a) \),
\[ d\gamma = d\left(\begin{array}{cc} x_1 & y \\ x_2 & \end{array}\right) = d\left(\begin{array}{cc} x_1 & 1 \\ x_2 & 1 \end{array}\right) = |x_1|^{a-h} dx_1 dx_2 dy. \]

Plug this in, we have
\[ |x|^h dx = |\res(x_1, x_2)||x_2|^{h-a}|x_1|^a dy dg. \]

Let \( \lambda_1, \ldots, \lambda_a \) be eigenvalues of \( x_1x_1^t \) and \( \mu_1, \ldots, \mu_{h-a} \) be eigenvalues of \( x_2x_2^t \). We have
\[ |\res(x_1x_1^t, x_2x_2^t)|_F = \prod_{a=h-a}^{a} \prod_{k=1}^{a} \prod_{j=1}^{a} |\lambda_k - \mu_j|_F \]

In above factors, \( |\mu_j| > |\lambda_k| \) for any \( k, j \), so \( |\lambda_k - \mu_j|_F = |\mu_j|_F \), the above product equals to
\[ |\res(x_1x_1^t, x_2x_2^t)|_F = \prod_{j=1}^{h-a} |\mu_j|_F^a = |x_2x_2^t|_F^a \]

Therefore \( |x|^h dx = |x_2|^h |x_1|^a dy dg. \) Since \( \int_{GL_a(O_K)} dg = 1 \), this Theorem follows. \( \square \)

**Lemma 3.3.** Let \( P_a \) be complex-valued functions on \( GL_a(K) \) for every \( a \in \mathbb{Z}_{\geq 0} \) with

- \( P_h(g^{-1}tg) = P_h(t) \) for any \( g, t \in GL_h(K) \) and any integer \( h \).
- \( P_h\left(\begin{array}{cc} x_1 \\ x_2 \end{array}\right) = P_a(x_1)P_{a-h}(x_2) \) for any \( x_1 \in GL_a(K) \), \( x_2 \in GL_{h-a}(K) \)

Then for any \( r \in \mathbb{R} \), we can decompose
\[ \int_{GL_a(K)} \phi_h(x)P_h(x\overline{x})dx = \sum_{i=0}^{h} M_h(i, r)N_h(h - i, r). \]

Where
\[ M_h(i, r) = \int_{GL_a(K)} \phi_i(x)1_{i}^{\geq r}(x\overline{x})O(x\overline{x})dx \]
\[ N_h(h - i, r) = \int_{GL_{h-a}(K)} \phi_{h-i}(x)1_{h-i}^{\leq r}(x\overline{x})Q(x\overline{x})dx \]

with
\[ O(x\overline{x}) = P_i(x\overline{x}) \left| \frac{1 - x\overline{x}}{x\overline{x}} \right|_F^{-h-i} \]
\[ Q(x\overline{x}) = P_{h-i}(x\overline{x}) \left| 1 - x\overline{x} \right|_F. \]
Proof. We can decompose the integral into
\[ \sum_{i=0}^{h} \int_{\text{GL}_{h}(K)} \phi_{h}(x) \mathbf{1}_{i}^{X} \mathbf{1}_{h-1}(x) \mathbf{1}_{h} P_{h}(x) \, dx = \sum_{i=0}^{h} I(i) \]
Denote each summand by \( I(i) \). By Theorem \[3.2\] for \( G' = \text{GL}_{r} \times \text{GL}_{h-i}(K) \), \( I(i) \) equals to
\[ (3.4) \]
\[ \int_{G'} F(x_1, x_2) \mathbf{1}_{i}^{X} \mathbf{1}_{h-1}(x_2) \mathbf{1}_{h} P_{h}(x) \, dx_1 \, dx_2. \]

It remains to compute \( F(x_1, x_2) \). By Lemma \[2.10\] \( F(x_1, x_2) \) equals to
\[ \int \phi_{h} \left( \begin{array}{c} x_1 \\ b \\ x_2 \end{array} \right) P_{i}(x_1) P_{h-i}(x_2) \, db = |1 - x_1|^{h-i} |1 - x_2|^{i} \phi_{i}(x_1) \phi_{h-i}(x_2) P_{i}(x_1) P_{h-i}(x_2) \]
Therefore the equation \[3.4\] become
\[ \int_{G'} \phi_{i}(x_1) \phi_{h-i}(x_2) \mathbf{1}_{i}^{X} \mathbf{1}_{h-1}(x_2) |1 - x_1|^{h-i} |1 - x_2|^{i} |x_1|^{i} \phi_{i}(x_1) \phi_{h-i}(x_2) \, dx_1 \, dx_2. \]
Therefore we can decompose \( I(i) \) as a product
\[ M_{h}(i, r) = \int_{\text{GL}_{r}(K)} \phi_{i}(x) \mathbf{1}_{i}^{X} |x|^{h-i} \, dx \]
and
\[ N_{h}(h-i, r) = \int_{\text{GL}_{h-i}(K)} \phi_{h-i}(x) \mathbf{1}_{h-i}(x) |1 - x|^{i} \, dx. \]
The Lemma follows. \( \square \)

4. SOME FORMULA OF INDUCTIVE COMPUTATION FOR INTERSECTION NUMBER

Now we are on the way to calculate \[2.3\]. The following expressions would help calculation.
\[ (4.1) \]
\[ A(a, r, X) = \int_{\text{GL}_{a}(K)} \phi_{a}(x) \mathbf{1}_{a}^{X} |x|^{a} X^K(x) \, dx \]
\[ (4.2) \]
\[ B(a, r) = \int_{\text{GL}_{a}(K)} \phi_{a}(x) \mathbf{1}_{a}^{X} |1 - x|^{a} \, dx \]
\[ (4.3) \]
\[ C(a, r, X) = \int_{\text{GL}_{a}(K)} \phi_{a}(x) \mathbf{1}_{a}^{X} \mathbf{1}_{a}^{0} |1 - x|^{a} X^K(x) \, dx \]

Lemma 4.1. If \( f \) is the identity in the spherical Hecke Algebra, We have
\[ (4.4) \]
\[ \text{Int}(f, \gamma) = \sum_{a=0}^{h} q^{r a} A(a, r, 1) B(h - a, r). \]

Proof. Using \[2.3\], and note that \( |p_{\gamma}(1)|_{F} = 1 \), we can write \( \text{Int}(f, \gamma) \) as
\[ (4.5) \]
\[ \text{Int}(f, \gamma) = \int_{\text{GL}_{h}(K)} \phi_{h}(x) |p_{\gamma}(x)|_{F} | \frac{x}{1 - x} |^{h} \, dx. \]
Therefore we let

\[(4.6) \quad P_a(x^x) = |p_\gamma(x^x)|^{-1}_F \left| \frac{x^x}{1 - x^x} \right|^h_F \]

Apply Lemma 3.3 we can decompose (4.5) as

\[\sum_{i=0}^{h} M_h(i, r) N_h(h - i, r).\]

Use the same notation \(O(x^x), Q(x^x)\) as in Lemma 3.3 With \(P_a(x^x)\) in (4.6), We have

\[O(x^x) = \left| p_\gamma(x^x) \right|^{-1}_F \left| \frac{x^x}{1 - x^x} \right|^h \left| \frac{1 - x^x}{x^x} \right|^{h-a}_F = \left| p_\gamma(x^x) \right|^{-1}_F \left| \frac{1 - x^x}{x^x} \right|^{-a}_F\]

\[Q(x^x) = \left| p_\gamma(x^x) \right|^{-1}_F \left| \frac{x^x}{1 - x^x} \right|^h \left| 1 - x^x \right|^{2h}_F\]

By Lemma 2.3

\[\mathbb{1}^{>r}_a(x^x) O(x^x) = \mathbb{1}^{>r}_a(x^x) q^{r a h} \left| \frac{1 - x^x}{x^x} \right|^{-a}_F\]

\[\mathbb{1}^{<r}_h(a, x^x) Q(x^x) = \mathbb{1}^{<r}_h(a, x^x) \left| 1 - x^x \right|^{a-h}_F\]

Then note \(\mathbb{1}^{>r}_a(x^x) = 1\) implies det \((1 - x^x)\) is a unit, therefore

\[\mathbb{1}^{>r}_a(x^x) O(x^x) = \mathbb{1}^{>r}_a(x^x) q^{r a h} \left| x^x \right|^{a}_F\]

This implies \(M_h(i, r) = q^{r h} A(i, r, 1)\) and \(N_h(h - i, r) = B(h - i, r)\) This Lemma follows. \(\Box\)

Our next goal is to determine an algorithm for \(A(a, r, X)\) and \(B(a, r)\)

**Lemma 4.2.** We have

\[(4.7) \quad \sum_{i=0}^{a} A(i, 0, 1) B(a - i, 0) = \prod_{i=1}^{a} \frac{1 - q^{1-2i}}{1 - q^{-2i}}\]

**Proof.** In Lemma 2.1 let \(f\) be the characteristic function of \(GL_{2h}(O_F)\), we have

\[(4.8) \quad \prod_{i=1}^{a} \frac{1 - q^{1-2i}}{1 - q^{-2i}} = \frac{\epsilon_F}{\epsilon_K} = \int_{GL_{2h}(K)} \phi(x) \left| \frac{x^x}{|1 - x^x|^{2h}} \right| dx\]

Therefore we let

\[(4.9) \quad P_a(x^x) = \left| \frac{x^x}{|1 - x^x|^{2h}} \right| \]

Apply Lemma 3.3 we can decompose (4.8) as

\[\sum_{i=0}^{h} M_h(i, 0) N_h(h - i, 0).\]

Use the same notation \(O(x^x), Q(x^x)\) as in Lemma 3.3 With \(P_a(x^x)\) in (4.9), we have

\[O(x^x) = \left| \frac{x^x}{|1 - x^x|^{2h}} \right| \left| \frac{1 - x^x}{x^x} \right|^{h-i}_F = \left| \frac{x^x}{|1 - x^x|^{i+h}} \right|_F\]

\[Q(x^x) = \left| \frac{x^x}{|1 - x^x|^{2h}} \right| \left| \frac{1 - x^x}{x^x} \right|^{h-i}_F = \left| \frac{x^x}{|1 - x^x|^{i+h}} \right|_F\]
Then note $1_{i}^{>0}(x \overline{\mathcal{T}}) = 1$ implies $1 - x \overline{\mathcal{T}}$ is a unit, Therefore,

$$1_{i}^{>0}(x \overline{\mathcal{T}}) \phi_i(x) O(x \overline{\mathcal{T}}) = 1_{i}^{>0}(x \overline{\mathcal{T}}) \phi_i(x) |x \overline{\mathcal{T}}|^{i}_F$$

If $1_{h-i}^{<0}(x \overline{\mathcal{T}}) = 1$, $\phi_{h-i}(x) = 1$ by Lemma 2.9, $|x \overline{\mathcal{T}}|_F = |1 - x \overline{\mathcal{T}}|_F$. This implies

$$1_{h-i}^{<0}(x \overline{\mathcal{T}}) \phi_{h-i} (x) Q(x \overline{\mathcal{T}}) = 1_{h-i}^{<0}(x \overline{\mathcal{T}}) \phi_{h-i} (x) |1 - x \overline{\mathcal{T}}|^{-h}_F$$

This implies $M_h(i, 0) = A(i, 0, 1)$ and $N_h(h - i, 0) = B(h - i, 0)$ This Lemma follows. \(\Box\)

**Lemma 4.3.** We have

$$B(a, r) = \sum_{i=0}^{a} B(a - i, 0) C(i, r, q^{2(a-i)}) .$$

**Proof.** By definition

$$B(h, r) = \int_{GL_h(K)} \phi_h(x) 1_{h}^{<r}(x \overline{\mathcal{T}}) |1 - x \overline{\mathcal{T}}|^{-h}_F dx$$

Let

$$P_a(x \overline{\mathcal{T}}) = 1_{a}^{<r}(x \overline{\mathcal{T}})|1 - x \overline{\mathcal{T}}|^{-h}_F$$

Apply Lemma 2.3, we can decompose $B(h, r)$ as

$$\sum_{i=0}^{h} M_h(i, 0) N_h(h - i, 0).$$

Use the same notation $O(x \overline{\mathcal{T}}), Q(x \overline{\mathcal{T}})$ as in Lemma 2.3. With $P_a(x \overline{\mathcal{T}})$ in (4.11),

$$O(x \overline{\mathcal{T}}) = 1_{i}^{<r}(x \overline{\mathcal{T}})|1 - x \overline{\mathcal{T}}|^{-h}_F \left(1 - \frac{x \overline{\mathcal{T}}}{1}_F \right)^{h-i} = 1_{i}^{<r}(x \overline{\mathcal{T}})|1 - x \overline{\mathcal{T}}|^{-i}_F |x \overline{\mathcal{T}}|^{-h}_F$$

$$Q(x \overline{\mathcal{T}}) = 1_{h-i}^{<r}(x \overline{\mathcal{T}})|1 - x \overline{\mathcal{T}}|^{-h}_F|1 - x \overline{\mathcal{T}}|^{i}_F = 1_{h-i}^{<r}(x \overline{\mathcal{T}})|1 - x \overline{\mathcal{T}}|^{-i-h}_F$$

Note that $|x \overline{\mathcal{T}}|^{-i-h}_F = q^{(2h-2)a_{K}(x)}$. This implies $M_h(i, 0) = C(i, r, q^{2(h-i)})$ and $N_h(h - i, 0) = B(h - i, 0)$ This Lemma follows. \(\Box\)

**Lemma 4.4.** We have

$$A(a, 0, X) = \sum_{i=0}^{a} A(i, r, X) C(a - i, r, q^{-2a} X).$$

**Proof.** By definition

$$A(h, r, X) = \int_{GL_h(K)} \phi_h(x) 1_{h}^{<r}(x \overline{\mathcal{T}}) |x \overline{\mathcal{T}}|^{h}_F X^{v_{K}(x)} dx$$

Let

$$P_a(x \overline{\mathcal{T}}) = 1_{a}^{>0}(x \overline{\mathcal{T}})|x \overline{\mathcal{T}}|^{h}_F X^{v_{K}(x)}.$$
Apply Lemma 3.3, we can decompose (4.8) as

$$\sum_{i=0}^{h} M_h(i, r) N_h(h - i, r).$$

Use the same notation $O(x\overline{\tau}), Q(x\overline{\tau})$ as in Lemma 3.3. With $P_a(x\overline{\tau})$ in (4.13), Where

$$O(x\overline{\tau}) = 1_i^0(x\overline{\tau}) |x\overline{\tau}|^h F x^r \nu_{\overline{\tau}}(x) \left| 1 - \frac{x\overline{\tau}}{x} \right|^{-h-i} = 1_i^0(x\overline{\tau}) X^r \nu_{\overline{\tau}}(x) \left| 1 - \frac{x\overline{\tau}}{x} \right|^{-h}$$

and

$$Q(x\overline{\tau}) = 1_i^0(x\overline{\tau}) |x\overline{\tau}|^i X^r \nu_{\overline{\tau}}(x) \left| 1 - \frac{x\overline{\tau}}{x} \right|^{-h} = 1_i^0(x\overline{\tau}) (q^{-2h} X)^r \nu_{\overline{\tau}}(x) \left| 1 - \frac{x\overline{\tau}}{x} \right|^{-h}$$

Note that $1_i^0(x) = 1$ implies $|1 - x\overline{\tau}|_F = 1$. This implies $M_h(i, r) = A(i, r, X)$ and $N_h(h - i, r) = C(h - i, r, q^{-2h} X)$ This Lemma follows.

The previous calculation showed that it is enough to know the formula for $A(a, r, X)$. Let

$$F(a, X) = \int_{GL_a(K)} 1[x \in gl_a(O_K)] X^r \nu_{\overline{\tau}}(x) |x\overline{\tau}|_F a dx.$$

**Lemma 4.5.** We have

$$F(a, X) = \prod_{i=1}^{a} \frac{1}{1 - q^{-2i} X}.$$

**Proof.** By Iwasawa Decomposition, let $\Gamma$ be the set of upper triangular matrices, we can decompose $x = pt$ where $p \in \Gamma$ and $t \in GL_a(O_K)$. Let $dp$ be the Left Haar measure and $dt$ the right Haar measure, then $dx = dp dt$. Since $x \in gl_a(O_K)$ if and only if $p \in gl_a(O_K)$, $\nu_K(t) = 0$, We can write

$$F(a, X) = \int_{\Gamma} 1[p \in gl_a(O_K)] X^r \nu_{\overline{\tau}}(p) |p\overline{\tau}|_F^a dp.$$

Let $\Lambda$ be the group of diagonal matrix and $U$ the set of unipotent matrices. Decompose $p = \delta u$ where $\delta \in \Lambda$ and $u \in U$. Note that $\nu_K(u) = 0$,

$$F(a, X) = \int_{\Lambda} f(\delta) |\delta|^a X^r \nu_{\overline{\tau}}(\delta) d\delta.$$

Where

$$f(\delta) = \int_{U} 1[\delta u \in gl_a(O_K)] du$$

Let $\delta_{11}, \cdots, \delta_{aa} \in K$ be diagonal entries of $\delta$, $u_{ij}$ the entry of $u$ in $i$’th row and $j$’th column. $\delta u \in gl_a(O_K)$ is equivalent to that $u_{ij} \in \delta_{ij}^{-1} O_K$. Therefore

$$f(\delta) = \prod_{i=1}^{a} \prod_{j=i+1}^{a} \int_{K} 1[u_{ij} \in \delta_{ij}^{-1} O_K] du_{ij} = \prod_{i=1}^{a} \prod_{j=i+1}^{a} |\delta_{ij}|^{-1} = \prod_{i=1}^{a} |\delta_{ii}|^{-a}$$

Therefore,

$$F(a, X) = \int_{\Lambda \cap gl_a(O_K)} \prod_{i=1}^{a} |\delta_{ii}|^{-a} |\delta_{ii}|^a X^r \nu_{\overline{\tau}}(\delta_{ii}) d\delta$$
This equals to
\[ F(a, X) = \prod_{i=1}^{a} \int_{\mathcal{O}_K} (q^{-2i}X)^{y_K(h_{ii})}d\delta_{ii} = \prod_{i=1}^{a} \frac{1}{1 - q^{-2i}X}. \]

The Lemma follows.

**Lemma 4.6.** We have
\[ A(a, 0, X) = q^{-2a}X \prod_{i=1}^{a} \frac{1}{1 - q^{-2i}X}. \]

**Proof.** By Lemma 3.2, we write
\[ F(a, X) = \sum_{i=0}^{a} \int_{\text{GL}_{a-1}(K)} H_i(x_1, x_2) \mathbb{1}_{a}^{>0}(x_1 x_1) \mathbb{1}_{a-1}^{\leq0}(x_2 x_2)|x_1|^a|x_2|^bdx_1dx_2, \]
where
\[ H_i(x_1, x_2) = \int_{M_{i+1}(K)} \mathbb{1}_{\mathfrak{gl}_i(\mathcal{O}_K)}(x_1 \ y \ x_2) \ dy = \mathbb{1}_{\mathfrak{gl}_i(\mathcal{O}_K)}(x_1)\mathbb{1}_{\mathfrak{gl}_{a-i}(\mathcal{O}_K)}(x_2). \]

Therefore,
\[ F(a, X) = \sum_{i=0}^{a} \int_{\text{GL}_i(\mathcal{O}_K)} \mathbb{1}_{a}^{>0}(x_1 x_1)|x_1|^a dx_1 \int_{\mathfrak{gl}_i(\mathcal{O}_F)} \mathbb{1}_{a-1}^{\leq0}(x_2 x_2)|x_2|^bdx_2. \]

Note that the right factor in each summand is 1. By Proposition 2.8 we have \( \mathbb{1}_{a}^{>0}(x_1 x_1)\mathbb{1}_{[x_1 \in \mathfrak{gl}_n(\mathcal{O}_K)]} = \mathbb{1}_{a}^{>0}(x_1 x_1)\phi_a(x_1). \) This implies
\[ F(a, X) = \sum_{i=0}^{a} A(i, 0, X). \]

So \( A(a, 0, X) = F(a, X) - F(a-1, X). \) This Lemma follows by the formula in Lemma 4.5.

**Lemma 4.7.** For any \( n, r, \) the function \( A(n, r, X) \) is a meromorphic function with poles at \( X = q^2, q^4, \cdots, q^{2n}. \) Let \( a(n, r, q^{2m}) \) be the residual of \(-q^{-2m}A(n, r, X) \) at \( X = q^{2m}. \) Then we have
\begin{equation}
A(n, r, X) = \sum_{i=1}^{n} \frac{a(n, r, q^{2i})(q^{-2i}X)^{\lceil \frac{nr}{2} \rceil}}{1 - q^{-2i}X}.
\end{equation}

here \( \lceil r \rceil \) means the smallest integer larger than \( r. \)

**Proof.** We call the order of pole at infinity the degree. By Lemma 4.4, we have
\[ A(n, r, X) = A(n, 0, X) - \sum_{i=0}^{n-1} A(i, r, X)C(n - i, r, q^{-2n}X). \]

First we claim the degree of \( A(a, r, X) \) is at most \( \lceil \frac{nr}{2} \rceil - 1. \) We prove by induction, when \( n = 0, \) \( A(n, r, X) = 1 \) has degree 0 = \( \lceil 0 \rceil - 1. \) By Lemma 4.6, the degree of \( A(n, 0, X) \) is at most 0. For each summand \( A(i, r, X)C(n - i, r, q^{-2n}X), \) the degree of \( C(n - i, r, q^{-2n}X) \) is at most \( \lceil \frac{(n-i)q^r}{2} \rceil, \) the symbol \( \lceil r \rceil \) means the largest integer no larger than \( r. \) By induction, the degree of \( A(i, r, X) \) is at most \( \lceil \frac{nr}{2} \rceil - 1. \) Therefore, the degree of each summand is at most
\[ \lceil \frac{(n-i)r}{2} \rceil + \lceil \frac{ir}{2} \rceil - 1 \leq \lceil \frac{nr}{2} \rceil - 1. \]
We proved our claim for the degree of $A(a, r, X)$, therefore, it could be written as

$$A(n, r, X) = \sum_{i=1}^{n} a(n, r, q^{2i})(q^{-2i}X)^{\left\lfloor \frac{nr}{2} \right\rfloor} + P(X)$$

for some polynomial $P(X)$ of degree at most $\left\lfloor \frac{nr}{2} \right\rfloor - 1$. Since by definition of $A(n, r, X)$, the coefficient for $X^j$ must be 0 if $j < \frac{nr}{2}$. This proves $P(X) = 0$, and clear each coefficient $a(n, r, q^{2i})$ is the residual of $-q^{-2i}A(n, r, X)$ at $X = q^{2i}$. □

**Corollary 4.8.** We have

$$q^{2m-2n} \prod_{i=1 \atop i \neq m}^{n} \frac{1}{1-q^{2m-2i}} = \sum_{i=m}^{n} a(i, r, q^{2m})C(n-i, r, q^{2m-2n}).$$

**Proof.** Multiplying equation (4.12) by $1 - q^{-2m}X$ and evaluating at $X = q^{2m}$. By equation (4.14) and Lemma 4.6, the Corollary follows. □

## 5. Computational Algorithm for Intersection Number

This section gives an independent algorithm for intersection numbers for the case of $h$ prime and $r$ is not an integer, we neglect $r$ in the notation and use the following notation.

$$C[n, m] = C(n, r, q^{-2m}) \quad a[n, m] = a(n, q^m) \quad A[n, m] = A(n, r, q^m) \quad B[n] = B(n, r).$$

Let $\tilde{a}[n, m]$, $\tilde{A}[n, m]$ $\tilde{B}[n]$ be the numbers for $r = 0$. We can rewrite our formula into

\begin{align*}
(5.1) & \quad a[n, n-m] = \tilde{a}[n, n-m] - \sum_{i=1}^{m} C[i, m]a[n-i, n-m] \quad \text{for } 0 \leq m < n \\
(5.2) & \quad A[n, n-m] = \sum_{i=0}^{n-1} a[n, n-i]q^{2(i-m)\left\lfloor \frac{nr}{2} \right\rfloor} \quad \text{for } m \geq n \\
(5.3) & \quad C[n, m] = \tilde{A}[n, n-m] - \sum_{i=0}^{n-1} C[i, m]A[n-i, n-m] \quad \text{for } m \geq n \\
(5.4) & \quad \tilde{B}[a] = \prod_{i=1}^{a} \frac{1-q^{-2i}}{1-q^{2i}} - \sum_{i=1}^{a} \tilde{A}[i, 0]\tilde{B}[a-i] \\
(5.5) & \quad B[a] = \sum_{i=0}^{a} \tilde{B}[a-i]C[i, i-a].
\end{align*}

With all above equations, the intersection formula is given by

$$N(r) = \sum_{a=0}^{h} q^{-ra}A[a, 0]B[h-a].$$
5.1. **Computation for** \( h = 2 \). We use this machinery to compute the Intersection number for \( h = 2 \). Note that by Lemma 4.6, we have

\[
\tilde{a}[n, n - m] = q^{-2m} \prod_{i=0}^{n-1} \frac{1}{1 - q^{-2(m-i)}}
\]

Furthermore, by (5.1) we have \( \tilde{a}[n, n] = a[n, n] \) for any \( n \). Firstly, we have

\[
a[1, 1] = \tilde{a}[1, 1] = 1
\]

Therefore by Equation (5.2)

\[
A[1, 1 - m] = \frac{q^{-2m} \left[ \frac{r}{2} \right]}{1 - q^{-2}}
\]

Plug in \( m = -1 \) and \( m = 1 \), we have

\[
(5.8) \quad A[1, 2] = -q^{-2} - q^{-2} \left[ \frac{r}{2} \right] \quad A[1, 0] = q^{-2} \left[ \frac{r}{2} \right] \quad \tilde{A}[1, 0] = \frac{q^2 - 2 \left[ \frac{r}{2} \right]^2 - 1}{1 - q^{-2}}
\]

Use Equation (5.3) and note \( C[0, m] = 1 \) for any \( m \), we have

\[
C[1, m] = \tilde{A}[1, 1 - m] - A[1, 1 - m] = \frac{q^{-2m} - q^{-2} \left[ \frac{r}{2} \right]}{1 - q^{-2m}} - \frac{q^{-2m} \left[ \frac{r}{2} \right]}{1 - q^{-2m}}
\]

Evaluating this expression at \( m = 0 \) and \( m = 1 \), we have

\[
(5.9) \quad C[1, 1] = \frac{q^{-2} - q^{-2} \left[ \frac{r}{2} \right]}{1 - q^{-2}}; \quad C[1, 0] = \left[ \frac{r}{2} \right] - 1; \quad C[1, -1] = \frac{q^2 \left[ \frac{r}{2} \right]^2 - 1}{1 - q^{-2}}
\]

So \( C(1, r, 1) = \left[ \frac{r}{2} \right] - 1 \). Continue the same process. By (5.1) we have

\[
a[2, 2] = \tilde{a}[2, 2] = -\frac{q^{-2}}{1 - q^{-2}}
\]

Continue using (5.7) calculating \( \tilde{a}[2, 1] \) We see

\[
\tilde{a}[2, 1] = \frac{q^{-2}}{1 - q^{-2}}
\]

By (5.1), we have

\[
a[2, 1] = \tilde{a}[2, 1] - C[1, 1] a[1, 1] = \frac{q^{-2} \left[ \frac{r}{2} \right]}{1 - q^{-2}}
\]

Again apply (5.2) we have

\[
(5.10) \quad A[2, 2 - m] = \frac{-q^{-2-2m} [r]}{(1 - q^{-2})(1 - q^{-2m})} + \frac{q^{-2} \left[ \frac{r}{2} \right] + 2(1-m)[r]}{(1 - q^{-2})(1 - q^{-2(m-1)})}
\]

Let \( m = 2 \), we have

\[
(5.11) \quad A[2, 0] = \frac{-q^{-2-4} [r]}{(1 - q^{-2})(1 - q^{-4})} + \frac{q^{-2} \left[ \frac{r}{2} \right] - 2[r]}{(1 - q^{-2})^2}
\]

By Formula (5.3),

\[
(5.12) \quad C[2, m] = \tilde{A}[2, 2 - m] - A[2, 2 - m] - C[1, m] A[1, 2 - m]
\]
From (5.9) and (5.8), we already know \( C[1, 0]A[1, 2] = (1 - \left[ \frac{r}{2} \right]) \frac{q^{-2+2[r]} - q^{-2}}{1 - q^{-2}} \), therefore move this term to left and evaluating (5.12) at \( m = 0 \) we have

\[
(5.13) \quad C[2, 0] + \left(1 - \left[ \frac{r}{2} \right] \right) \frac{q^{-2+2[r]} - q^{-2}}{1 - q^{-2}} = \left(1 - \left[ \frac{r}{2} \right] \right) \frac{q^{-2} - q^{-2m[r]} - q^{-2m}}{(1 - q^{-2})(1 - q^{-2m})} + \frac{q^{-2}[r] - q^{-2}}{(1 - q^{-2})^2}
\]

Apply L’Hospital rule we have calculated out \( C(2, r, 1) \)

\[
C[2, 0] = \frac{q^{-2}[r] + 2[r]}{(1 - q^{-2})^2} + \frac{q^2 [r] - 2 [r]}{1 - q^{-2}} + \frac{q^{-2} [r] - q^{-2}}{1 - q^{-2}}
\]

Now we calculate \( \tilde{B}[1] \), by formula (5.4),

\[
\tilde{B}[1] = \frac{1 - q^{-1}}{1 - q^{-2}} - \tilde{A}[1, 0] = \frac{1 - q^{-1} - q^{-2}}{1 - q^{-2}}.
\]

Now we are ready to calculate \( B[1] \) by formula (5.5). Note \( C[1, 0] = \left[ \frac{r}{2} \right] - 1 \), so

\[
B[1] = B(1, 0) + C[1, 0] = \frac{-q^{-1}}{1 - q^{-2}} + \left[ \frac{r}{2} \right].
\]

To compute \( \tilde{B}[2] \), by formula (4.7),

\[
\tilde{B}[2] = \frac{(1 - q^{-1})(1 - q^{-3})}{(1 - q^{-2})(1 - q^{-4})} - \tilde{B}[1] A[1, 0] - \tilde{A}[2, 0].
\]

we have

\[
\tilde{B}[2] = \frac{q^{-3} - q^{-2}}{(1 - q^{-2})^2} + \frac{q^{-6} - q^{-3} + 1 - q^{-1} + q^{-4}}{(1 - q^{-2})(1 - q^{-4})}
\]

By formula (4.10),

\[
B[2] = \tilde{B}[2] + C[1, -1] \tilde{B}[1] + C[2, 0]
\]

By computation, this value equals to

\[
\frac{q^{-2+2[r]} - q^{-2}}{1 - q^{-2}} + \left(1 - \left[ \frac{r}{2} \right] \right) \frac{q^2 [r] - 2 [r]}{1 - q^{-2}} + \frac{q^{-2} [r] - q^{-2}}{(1 - q^{-2})^2} + \frac{q^2 [r] - q^{-2}}{(1 - q^{-2})^2}
\]

By our formula (4.4), the intersection number equals to

\[
N(r) = q^{4r} A[2, 0] + q^{2r} A[1, 0] B[1] + B[2].
\]

This equals to

\[
19
\]
\[
\frac{q^{-2-2\left\lfloor \frac{r}{2} \right\rfloor +2[r]}}{(1-q^{-2})^2} + \left( \left\lceil \frac{r}{2} \right\rceil - 1 \right) \frac{q^{2\left\lceil \frac{r}{2} \right\rceil} - 2 - \left\lfloor r \right\rfloor}{1-q^{-2}} - \frac{q^{-2}}{1-q^{-2}} - \frac{q^{-6}}{(1-q^{-2})(1-q^{-4})}
\]
\[+(q^{2\left\lceil \frac{r}{2} \right\rceil} - 2 - q^{-2})(1-q^{-1} - q^{-2}) + \frac{\left\lceil \frac{r}{2} \right\rceil q^{-2\left\lceil \frac{r}{2} \right\rceil +2r}}{1-q^{-2}} - \frac{q^{-1-2\left\lceil \frac{r}{2} \right\rceil +2r}}{(1-q^{-2})^2}.
\]

Since we have \(2r - \left\lfloor r \right\rfloor = -1\), so the intersection formula is simplified to

\[
N(r) = \frac{q^{-2-2\left\lfloor \frac{r}{2} \right\rfloor +2[r]}}{(1-q^{-2})^2} + \left( \left\lceil \frac{r}{2} \right\rceil - 1 \right) \frac{q^{2\left\lceil \frac{r}{2} \right\rceil} - 2 - \left\lfloor r \right\rfloor}{1-q^{-2}} - \frac{q^{-2}}{1-q^{-2}},
\]
\[+(q^{2\left\lceil \frac{r}{2} \right\rceil} - 2 - q^{-2})(1-q^{-1} - q^{-2}) + \frac{\left\lceil \frac{r}{2} \right\rceil q^{-2\left\lceil \frac{r}{2} \right\rceil +2r}}{1-q^{-2}}.
\]

(5.14)

We found \(N(\frac{1}{2}) = 1\) and \(N(\frac{3}{2}) = q + 2\). Furthermore, we compute

\[
N(r + 2) - N(r) = \frac{q^{-2-2\left\lfloor \frac{r}{2} \right\rceil +2[r]}}{(1-q^{-2})^2} + \left( \left\lceil \frac{r}{2} \right\rceil - 1 \right) \frac{q^{2\left\lceil \frac{r}{2} \right\rceil} - 2 - 2\frac{q^{-2}}{1-q^{-2}}}
\]
\[+q^{2\left\lceil \frac{r}{2} \right\rceil} (1-q^{-1} - q^{-2}) + \frac{\left\lceil \frac{r}{2} \right\rceil q^{-2\left\lceil \frac{r}{2} \right\rceil +2r+2}}{1-q^{-2}} + \frac{q^{-2\left\lceil \frac{r}{2} \right\rceil +2r}}{1-q^{-2}}.
\]

(5.15)

Note that \(2\left\lfloor r \right\rfloor = 2r + 1\). Simplify this equation, we may write \(N(r + 2) - N(r)\) as

\[
q^{-2\left\lceil \frac{r}{2} \right\rceil +2[r] +1} \left( \frac{1}{1-q^{-1}} + \left\lfloor \frac{r}{2} \right\rceil \right) + q^{2\left\lceil \frac{r}{2} \right\rceil} \left( \left\lceil \frac{r}{2} \right\rceil - \frac{1}{1-q^{-1}} \right) + \frac{2(1-q^{2\left\lceil \frac{r}{2} \right\rceil +2})}{1-q^{-2}}.
\]

(5.16)

6. The Analytic Side

In this section, we compute the analytic side of the formula we write \(\text{Orb}_L(f, g, s)\) in (1.1) as

\[\text{Orb}_L(f, g, s) = \int_{H_h \times H_h / I(g)} f(v^{-1}gu)\eta_{E/L}(u)[vu]^s d\nu d\mu.\]

Here we alleviate notation to use \([u]\) denote \([uu^{-1}]_F\). We use the same notation as in Section 2. We can assume

\[g = \left( \frac{1}{g^\#} \right) \quad u = \left( \frac{u^+}{u^+} \right) \quad v = \left( \frac{v^+}{v^+} \right).
\]

Therefore by change of variable \(v^+ \mapsto u^+ v^+\) we can write \(\text{Orb}_F(f, g, s)\) as

\[
\int_{H_h \times H_h / I(g)} f \left( \left( \frac{v^+}{v^+} \right)^{-1} \left( \frac{u^+}{u^+} \right)^{-1} \left( \frac{1}{g^\#} \right) \right) \left( \frac{u^+}{u^+} \right) \eta_{E/L}(u)[u]^s d\nu d\mu d\nu d\mu.
\]
Let
\begin{equation}
\phi_s(g\#) = \int_{\text{GL}_h(L)} f \left( \begin{pmatrix} g & \bar{g} \\ \bar{g} & g\# \end{pmatrix} \right) \left( \frac{1}{g\#} \right) \left[ |g|^s \right] dg
\end{equation}

be the function as defined similarly in (2.6), we can write \( \text{Orb}_F(f, g, s) \) as
\[
\int_{\text{GL}_h(L)/C(g\#)} \phi_s(g^{-1}g\#\bar{g})\eta_{E/L}(g)[g]^{2s}dg\bar{g}.
\]

Here \( C(g\#) \) is the set of elements \( g \) so that \( g\#\bar{g} = gg\# \). In our application we assume \( f \) is the characteristic function of \( \text{GL}_{2h}(\mathcal{O}_F) \). Since we assumed the characteristic polynomial of \( g\# \) is \( p_\gamma \), all eigenvalues of \( g\# \) has valuation large than 0 so \( \zeta_h^0(g\#g\#) = 1 \), apply Lemma [2.8] we have

\[ \phi_s(g\#) \neq 0 \iff g\# \in \mathfrak{gl}_h(\mathcal{O}_L). \]

So the integrand in (6.1) not vanish only when \( g \in \text{GL}_h(\mathcal{O}_L) \), this makes \( [g]^s = 1 \) and therefore \( \phi_s(g\#) = \phi_0(g\#) \), we denote them as \( \phi(g\#) \) since then its definition is the same as (2.6). In this case

\[ \phi(g\#) = \mathbb{1}_{\mathfrak{gl}_h(\mathcal{O}_L)}(g\#). \]

We can write \( \text{Orb}_L(f, g, s) \) as
\[
\text{Orb}_L(f, g, s) = \int_{\text{GL}_h(L)/C(g\#)} \mathbb{1}_{\mathfrak{gl}_h(\mathcal{O}_L)}(u^{-1}g\#\bar{u})\eta_{E/L}(u)[u]^{2s}dud\bar{u}.
\]

Note that the above formula is true for every quadratic etale extensions \( E/L, L/F \) when \( f \) is the characteristic function of \( \text{GL}_h(\mathcal{O}_F) \) and \( \zeta_h^0(g\#) = 1 \). To continue our computation, we have to specialize to the case \( L = F \times F \) and \( E = K \times K \). In this case, we write

\[ D = u^{-1}g\#\bar{u} \quad \text{dud\bar{u}} = Dd\bar{u}. \]

So

\[ D = u^{-1}g\#g\#u \quad \iff \quad D = D^{-1}g\#g\#. \]

For any elements \( \bullet \) over \( F \times F \), we use \( \bullet' \) to denote its first component therefore \( \bar{\bullet} \) is its second component. We have \( D \in \mathfrak{gl}_h(\mathcal{O}_L) \) if and only if

\[ D' \in \mathfrak{gl}_h(\mathcal{O}_F) \text{ and } \bar{D}' \in \mathfrak{gl}_h(\mathcal{O}_F). \]

Furthermore
\[
\eta_{E/L}(u) = \eta_{K/F}(u')\eta_{K/F}(\bar{u}) = \eta_{K/F}(D)\eta_{K/F}(g\#).
\]

\[ [u] = |u\bar{u}^{-1}| = |D^{-1}|g\#. \]

From now on, we will use \( \bullet \) and \( \bar{\bullet} \) to denote \( \bullet' \) and \( \bar{\bullet}' \), we could write \( \text{Orb}_L(f, g, s) \) as
\begin{equation}
|g\#|^s\eta_{K/F}(g\#) \int_{\text{GL}_h(F \times F)/C(g\#g\#)} \mathbb{1}_{\mathfrak{gl}_h(\mathcal{O}_F)}(D)\mathbb{1}_{\mathfrak{gl}_h(\mathcal{O}_F)}(D^{-1}g\#g\#u)\eta_{K/F}(D)[D]^{-2s}dDd\bar{u}.
\end{equation}

\( C(g\#g\#) \) is the centralizer of \( g\#g\# \). To simplify the function, we remind there is an injective \( \mathbb{C} \)-algebra homorphism

\[ \mathcal{H} \rightarrow \mathbb{C}(X_1, \ldots, X_h). \]
Proposition 6.1. Under Satake transform, we have

\[ C_{\mathbb{Q}^h} \subseteq \mathfrak{gl}_h(\mathbb{Q}_F) \text{ if } g \in \mathfrak{gl}_h(\mathbb{Q}_F) \text{ and } \chi(\det(g)). \]

Proposition 6.1. Under Satake transform, we have

\[ S_{\chi 1_{\mathfrak{gl}_h(\mathbb{Q}_F)}} = \prod_{j=1}^{h} \frac{1}{1 - \chi(\pi) q^\frac{h-j}{2} X_j} \]

Proof. For any Laurent series \( P \) of \( h \)-variables over \( \mathbb{C} \), if

\[ P(x_1, \cdots, x_h) = \sum_{I = (i_1, \cdots, i_h) \in \mathbb{Z}^h} b_I x_1^{i_1} \cdots x_h^{i_h} \]

let \( a = (a_1, a_2, \cdots, a_h) \) denote the diagonal matrix with entries \( a_1, \cdots, a_h \), we define

\[ P(a) := \sum_{I = (i_1, \cdots, i_h) \in \mathbb{Z}^h} b_I 1 \{|a_1| = q^{-i_1}\} \cdots 1 \{|a_h| = q^{-i_h}\}. \]

Then the Satake transformation \( S[f] \) is defined by a polynomial so that

\[ S[f](a) = \left| a_1^\frac{h+1}{2} a_2^\frac{h+3}{2} \cdots a_h^\frac{1+h}{2} \right| \int_U f(au) du. \]

If \( f(g) = 1 \in \mathfrak{gl}_h(\mathbb{Q}_F) \chi(\det(g)) \), we can compute the value of this integral

\[ S[f](a) = \prod_{i=1}^{h} \chi(a_i)|a_i|^{-\frac{h}{2}} 1[a_i \in \mathbb{Q}_F]. \]

Therefore we can write

\[ S[f](a) = \sum_{I = (i_1, \cdots, i_h) \in \mathbb{Z}_0^h} \prod_{j=1}^{h} \chi(\pi)^{i_j} q^{\frac{h+1}{2} i_j} 1 \{|a_j| = q^{-i_j}\}. \]

This implies

\[ S[f] = \sum_{I = (i_1, \cdots, i_h) \in \mathbb{Z}_0^h} \prod_{j=1}^{h} \chi(\pi)^{i_j} q^{i_j} X_j^{i_j} = \prod_{j=1}^{h} \frac{1}{1 - \chi(\pi) q^\frac{h-1}{2} X_j}. \]

The Proposition follows.

Let

\[ B(x) = \int_{\mathfrak{gl}_h(\mathbb{F})} 1_{\mathfrak{gl}_h(\mathbb{Q}_F)}(D) 1_{\mathfrak{gl}_h(\mathbb{Q}_F)}(D^{-1}x) \eta_{K/F}(D)|D|^{2s} dD. \]

Then \( B \) is the convolution of the function \( 1_{\mathfrak{gl}_h(\mathbb{Q}_F)} \) and \( \eta_{L/F} \cdot |F|^{-2s} 1_{\mathfrak{gl}_h(\mathbb{Q}_F)} \). By Proposition 6.1, and note that \( \eta_{K/F}(\pi) \), we know the satake transform of \( B \)

\[ S[B] = \prod_{j=1}^{h} \frac{1}{1 - q^{\frac{h-1}{2} X_j}} \left( 1 + q^{2s+\frac{h-1}{2} X_j} \right). \]
For our future use, we denote the first two terms of Taylor expansion of $\mathcal{B}$ by
$$
\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1 s + O(s^2)
$$
we have

(6.3)\[ S[\mathcal{B}_0] = S[\mathcal{B}]|_{s=0} = \prod_{j=1}^{h} \frac{1}{1 - q^{h-1}X_j^2}. \]

With $\mathcal{B}$ we can write (6.2) as

(6.4)\[ \text{Orb}_L(f, g, s) = |g_\#|^s \eta_{K/F}(g_\#) \int_{\text{GL}_h(F) \c/\eta_{g_\#}g_\#} \mathcal{B}(\overline{u}^{-1}g_\#g_\#\overline{u})\overline{d}u. \]

**Lemma 6.2.** If the invariant polynomial of $g$ is $P_\gamma$, then
$$
\text{Orb}_L(f, g, 0) = 0
$$

**Proof.** Invariant polynomial being $P_\gamma$ implies $\det(g_\#g_\#)$ has odd valuation, the power series $\mathcal{B}_0$ vanishes in odd degrees, therefore, we have
$$
1[| \det(g)|_F = q^k \text{ and } k \text{ is odd}] \times \mathcal{B}_0(g) = 0.
$$

Since all elements in the conjugacy class of $g_\#$ has same determinant with $g_\#$, the function $\mathcal{B}_0$ vanishes on whole orbits. \hfill \square

For derivative of this orbital integral, by Leibnitz Rule and equation (6.4),
$$
\frac{d}{ds} \bigg|_{s=0} \text{Orb}_L(f, g, s) = \ln |g_\#| \text{Orb}_L(f, g, 0) + \eta_{K/F}(g_\#) \int_{\text{GL}_h(F) \c/\eta_{g_\#}g_\#} \mathcal{B}_1(u^{-1}g_\#u)\overline{d}u.
$$

Now let $x = \overline{g_\#}g_\#$ so the characteristic polynomial of $x$ is $p_\gamma$ therefore $| \det(x)|_F = q^{-2r}$. So
$$
\tilde{N}(r) := (2\ln q)^{-1} \frac{d}{ds} \bigg|_{s=0} \text{Orb}_L(f, g, s) = -\int_{\text{GL}_h(F) \c/\eta_{g_\#}g_\#} \mathcal{B}_1(u^{-1}xu)\overline{d}u.
$$

Our goal is $\tilde{N}(r) = N(r)$. We specialize to $h = 2$, then $2r = v_F(\det(x))$ is an odd integer. Let
$$
\mathcal{B}_1^{\text{deg}=2r}(g) := 1[| \det(g)|_F = q^{-2r}] \times \mathcal{B}_1(g)
$$

The degree $2r$ term of $S[\mathcal{B}]$ is the degree $2r$ term of
$$
(1 - q^{\frac{1}{2}X_1})^{-1} \left(1 + q^{2s+\frac{1}{2}X_1}\right)^{-1} \left(1 - q^{\frac{1}{2}X_2}\right)^{-1} \left(1 + q^{2s+\frac{1}{2}X_2}\right)^{-1}
$$

The degree $2r$ term is sum of all possible degree $2r$ product of choosing an elements from each
$$
\{(q^{\frac{1}{2}X_1})^n\}_{n=0}^\infty, \quad \{(-q^{\frac{1}{2}+2s}X_1)^n\}_{n=0}^\infty, \quad \{((q^{\frac{1}{2}X_2})^n\}_{n=0}^\infty, \quad \{(-q^{\frac{1}{2}+2s}X_2)^n\}_{n=0}^\infty.
$$

Therefore
$$
S[\mathcal{B}]^{\text{deg}=r} = q^r \sum_{n=0}^{2r} \left(\sum_{j=0}^{m} (-1)^i q^{j2s}\right) X_1^n \left(\sum_{j=0}^{m} (-1)^i q^{j2s}\right) X_2^{n-m}.
$$

Taking derivative and evaluate at $s = 0$, we have

(6.5)\[ S[\mathcal{B}_1^{\text{deg}=2r}] = -2q^r \ln q \sum_{i=0}^{2r} \text{int} \left(\frac{i+1}{2}, r + \frac{1-i}{2}\right) X_1^i X_2^{2r-i}. \]
Here \( \text{int}(\frac{i+1}{2}, r + \frac{1-i}{2}) \) takes out the unique integer out of the two. We can write it into
\[
S \left[ B_1^{\text{deg}=2r} \right] = -2q^r \ln q \sum_{i=0}^{2r-1} \text{int} \left( \frac{i+1}{2}, r + \frac{1-i}{2} \right) \left( X_i^1 X_2^{2r-i} + X_1^{2r-i} X_i^2 \right).
\]

Let \( C_i^{\text{deg}=2r} \) be an element in Spherical Hecke Algebra such that
\[
S \left[ C_i^{\text{deg}=2r} \right] = q^r \left( X_i^1 X_2^{2r-i} + X_1^{2r-i} X_i^2 \right).
\]

Note that for any \( n \in \mathbb{Z}_{\geq 0} \)
\[
C_n^{\text{deg}=2n}(g) = q^n R_n \quad \sum_{i=0}^{2r} C_i^{\text{deg}=n}(g) = T_n,
\]
where
\[
R_n := 1\{g \in \pi^n \text{GL}_2(O_K)\} \quad T_n := 1 \{g \in \text{gl}_2(O_K), |\det(g)|_F = q^{-n} \}.
\]

Therefore, if \( i > r \) we can write \( C_i^{\text{deg}=2r} \) as
\[
C_i^{\text{deg}=2r}(g) = q^i R_i \ast T_{2r-2i}(g) - q^{i+1} R_{i+1} \ast T_{2r-2i-2}(g),
\]
where the symbol \( \ast \) means convolution. Note that
\[
R_i \ast T_{2r-2i}(g) = T_{2r-2i}(\pi^{-i} g),
\]

Therefore we have
\[
\int_{\text{GL}_2(F)}^{\text{deg}=2r} C_i^{\text{deg}=2r}(u^{-1} x u) du = q^i \int_{\text{GL}_2(F)}^{\text{deg}=2r} T_{2r-2i}(\pi^{-i} u^{-1} x u) du - q^{i+1} \int_{\text{GL}_2(F)}^{\text{deg}=2r} T_{2r-2i-2}(\pi^{-i-1} u^{-1} x u) du.
\]

The orbital integral was calculated in Page 411 of [Kot05] when \( |\det(x)|_F = q^{-2r} \) with an odd integer \( 2r \), then
\[
\int_{\text{GL}_2(F)}^{\text{deg}=2r} T_{2r}(u^{-1} x u) du = \frac{q^{2r+1} - 1}{q - 1}.
\]

Therefore
\[
\int_{\text{GL}_2(F)}^{\text{deg}=2r} C_i^{\text{deg}=2r}(u^{-1} x u) du = \frac{q^{2r+1} - q^{i}}{q - 1} - \frac{q^{2r+1} - q^{i+1}}{q - 1} = q^i.
\]

By equation (6.5), we found the left hand side of equation (1.2) can be written by (6.6)
\[
- \frac{1}{2} \ln q \int_{\text{GL}_2(F)}^{\text{deg}=2r} B_1^{\text{deg}=2r}(u^{-1} x u) du = \begin{cases} \sum_{i=0}^{2r-1} iq^{2i-1} + \sum_{i=0}^{2r-1} \left( \frac{2r+1}{2} - i \right) q^{2i} & 2r \equiv 1 \text{ mod } 4 \\ \sum_{i=1}^{2r} iq^{2i-1} + \sum_{i=1}^{2r} \left( \frac{2r+1}{2} - i \right) q^{2i} & 2r \equiv 3 \text{ mod } 4 \end{cases}
\]

Denote this formula by \( \tilde{N}(r) \). This formula can be uniformly written as
\[
\tilde{N}(r) = \sum_{i=0}^{\left\lceil \frac{r}{2} \right\rceil - 1} ((r) - i) q^{2i} + \sum_{i=1}^{\left\lfloor r \right\rfloor - \left\lceil \frac{r}{2} \right\rceil} iq^{2i-1}
\]
Therefore
\[ \tilde{N}(r + 2) - \tilde{N}(r) = \sum_{i=0}^{\left\lceil r/2 \right\rceil} 2q^{2i} + \left( \left\lceil r \right\rceil + 2 - \left\lceil r/2 \right\rceil \right) q^2 \left\lceil r/2 \right\rceil + \left( \left\lceil r \right\rceil - \left\lceil r/2 \right\rceil + 1 \right) q^{2 \left\lceil r \right\rceil - 2 \left\lceil r/2 \right\rceil + 1} \]

So \( \tilde{N}(r + 2) - \tilde{N}(r) \) equals to
\[ 2 \frac{1 - q^2 \left\lceil r/2 \right\rceil}{1 - q^2} + \left( \left\lceil r \right\rceil + 2 - \left\lceil r/2 \right\rceil \right) q^2 \left\lceil r/2 \right\rceil + \left( \left\lceil r \right\rceil - \left\lceil r/2 \right\rceil + 1 \right) q^{2 \left\lceil r \right\rceil - 2 \left\lceil r/2 \right\rceil + 1} \] \hspace{1cm} (6.7)

Note that for \( a = 0 \) or \( a = 1 \), we have an identity
\[ -(q^{-a+1} + q^a)a = \frac{q^{-a} - q^a}{1 - q^{-1}} \] \hspace{1cm} (6.8)

For any \( r \in \frac{1}{2} \mathbb{Z} \), we have
\[ 2 \left\lceil \frac{r}{2} \right\rceil - \left\lceil r \right\rceil = 1 \quad \text{or} \quad 0 \]

Therefore, with the equation (6.8), we have
\[ q^{\left\lceil r \right\rceil} \left( q^{-2 \left\lceil r/2 \right\rceil + \left\lceil r \right\rceil} + q^{2 \left\lceil r/2 \right\rceil - \left\lceil r \right\rceil} \right) \left( 2 \left\lceil \frac{r}{2} \right\rceil - \left\lceil r \right\rceil \right) + q^{\left\lceil r \right\rceil} \frac{q^{-2 \left\lceil r/2 \right\rceil + \left\lceil r \right\rceil} - q^{2 \left\lceil r/2 \right\rceil - \left\lceil r \right\rceil}}{1 - q^{-1}} = 0 \] \hspace{1cm} (6.9)

Add (6.9) to (6.7), we can write \( \tilde{N}(r + 2) - \tilde{N}(r) \) as
\[ q^{-2 \left\lceil r/2 \right\rceil + 2 \left\lceil r \right\rceil} \left( \frac{1}{1 - q^{-1}} + \left\lceil \frac{r}{2} \right\rceil \right) + q^{2 \left\lceil r/2 \right\rceil} \left( \left\lceil \frac{r}{2} \right\rceil - \frac{1}{1 - q^{-1}} \right) + \frac{2 \left( 1 - q^{2 \left\lceil r/2 \right\rceil + 1} \right)}{1 - q^{2}}. \]

Note that this is exactly the same as (5.16). And also \( \tilde{N}(\frac{1}{2}) = 1 = N(\frac{1}{2}), \tilde{N}(\frac{3}{2}) = 2 + q = N(\frac{3}{2}) \). Therefore we proved the Arithmetic Fundamental Lemma for unit Hecke function in \( h = 2 \).

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