Calculus on Fractal Curves in $\mathbb{R}^n$

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Abstract

A new calculus on fractal curves, such as the von Koch curve, is formulated. We define a Riemann-like integral along a fractal curve $F$, called $F^\alpha$-integral, where $\alpha$ is the dimension of $F$. A derivative along the fractal curve called $F^\alpha$-derivative, is also defined. The mass function, a measure-like algorithmic quantity on the curves, plays a central role in the formulation. An appropriate algorithm to calculate the mass function is presented to emphasize its algorithmic aspect.

Several aspects of this calculus retain much of the simplicity of ordinary calculus. We establish a conjugacy between this calculus and ordinary calculus on the real line. The $F^\alpha$-integral and $F^\alpha$-derivative are shown to be conjugate to the Riemann integral and ordinary derivative respectively. In fact, they can thus be evaluated using the corresponding operators in ordinary calculus and conjugacy. Sobolev Spaces are constructed on $F$, and $F^\alpha$-differentiability is generalized. Finally we touch upon an example of absorption along fractal paths, to illustrate the utility of the framework in model making.

1 Introduction

It is now well known that fractals pervade nature $[1,2]$. The geometry of fractals is also well studied $[1,3,4,5,6,7]$. Fractal curves often lack the smoothness properties required by ordinary calculus. For example, observed path of a quantum mechanical particle $[8]$ or Brownian and Fractional Brownian trajectories $[1,3]$ are known to be fractals and are continuous but non-differentiable. A percolating path, just above the percolating phase transition can be considered as an approximate realization of a fractal curve $[9]$. If a long polymer is modeled as a fractal curve, then accumulation of a physical property along the curve would amount to integration on such a curve. This is often carried out using ad hoc procedures.

While there are some remarkable approaches to develop tools for such situations $[10,11,12,13,14]$, much more is desired. This paper aims to formulate a
calculus specifically tailored for fractal curves, in a close analogy with ordinary calculus. In particular, we adopt a Riemann-Stieltjes like approach for defining integrals, because of its simplicity and advantage from algorithmic point of view. Such an approach was conceived in [15] which began with formulation in the terms of Local Fractional derivatives. A new prescription was proposed to give meaning to differential equations on Cantor-like sets which are totally disconnected where the Local Fractional Derivatives do not carry over. The calculus formulated and developed in [16, 17, 18] for fractal subsets of $\mathbb{R}$ fully justifies the prescription in [15]. In particular, an integral and a derivative of order $\alpha$ are defined on Cantor-like totally disconnected subsets of the real line, where $\alpha \in (0, 1]$ is the dimension of $F$. This calculus, called $F^\alpha$-calculus has many results analogous to ordinary calculus and can be viewed as a generalization of ordinary calculus on $\mathbb{R}$. In fact, in [17, 18], a conjugacy between the $F^\alpha$-calculus and ordinary calculus is discussed.

The present paper extends that approach, which was developed for disconnected sets like Cantor-sets, to formulate calculus on fractal curves which are continuous. The organization of the paper is as follows. In Section 2, we define a mass function and integral staircase function. The mass function gives the content of a continuous piece of the fractal curve $F$. The staircase function, more appropriately called the "rise function", is obtained from the mass function and describes the rise of the mass of the curve with respect to the parameter. We emphasize the algorithmic nature of the mass function: by presenting an algorithm to calculate it. In section 3, we show that the mass function allows us to define a new dimension called $\gamma$-dimension, which is algorithmic and finer than the box dimension. In section 4, we discuss the algorithmic nature of mass function and present an algorithm to calculate it. In section 5, the concepts of limits and continuity are adapted to the concepts of $F^\alpha$-limit and $F^\alpha$-continuity. Section 6 is devoted to the discussion of integral on fractal curves called $F^\alpha$-integral. The formulation is analogous to the Riemann integration [19]. The notion of $F^\alpha$-differentiation is introduced in section 7. The fundamental theorems of $F^\alpha$-calculus proved in section 7 state that the $F^\alpha$-integral and $F^\alpha$-derivative are inverses of each other. The conjugacy between $F^\alpha$-calculus on $F$ and ordinary calculus on the real line, discussed in section 6, establishes a relation between the two and gives a simple method to evaluate $F^\alpha$-integrals and $F^\alpha$-derivatives of functions on the fractal $F$. In section 6.2, function spaces of $F^\alpha$-integrable and $F^\alpha$-differentiable functions on the fractal $F$ are explored. In particular Sobolev Spaces are introduced and abstract Sobolev derivatives are constructed. Finally as a simple physical application we briefly touch upon, as an example, a simple model of absorption along a fractal path in section 7. Section 8 is the concluding section.

2 The mass function and the staircase

This paper can be considered as a logical extension of calculus on fractal subsets of the real line developed in [18]. The proofs which are analogous to those in
In this paper we consider fractal curves, i.e. images of continuous functions $f : \mathbb{R} \to \mathbb{R}^n$ which are fractals. To be precise:

Let $[a_0, b_0]$ be a closed interval of the real line.

**Definition 1** A fractal (curve) $F \subset \mathbb{R}^n$ is said to be continuously parametrizable (or just parametrizable for brevity) if there exists a function $w : [a_0, b_0] \to F \subset \mathbb{R}^n$ which is continuous, one-to-one and onto $F$.

In this paper $F$ will always denote such a fractal curve.

**Examples:**

1. A simple example of such a parametrization is the function $w : \mathbb{R} \to \mathbb{R}^2$ defined by $w(t) = (t, W_s^\lambda(t))$ where $W_s^\lambda(t)$ is the well known Weierstrass function [3] given by

$$W_s^\lambda(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin \lambda^k t$$

where $\lambda > 1$ and $1 < s < 2$. The graph of $W_s^\lambda(t)$ is known to be a fractal curve with box-dimension $s$.

2. Our next example constitutes of one important class of parametrizations of self-similar curves in two dimensions (There are other ways of parametrizing fractal curves ; for example see [20]). Let $T_i, i = 0, \ldots, n-1$ be linear operations which are composed of rotation and scaling. Each $T_i$ can be represented by a $2 \times 2$ matrix:

$$T_i = s_i \begin{bmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{bmatrix}.$$

Further, they should satisfy the condition:

$$\sum_{i=0}^{n-1} T_i(v) = v$$

for any vector $v$, and $0 < s_i < 1$ for $i = 0, \ldots, n-1$. The fractal is defined by the limit set [4] of the similarity transformations:

$$S_j(v) = \sum_{i=0}^{j-1} T_i(v_0) + T_j(v), \quad j = 0, \ldots, n-1$$

where $v_0$ is a fixed vector. The limit set will be in the form of a curve because of the way $S_j$ are constructed from $T_i$.

Let $|nt|$ denote the integer part of $nt$. Now, the function $w$ defined implicitly by

$$w(t) = \sum_{i=0}^{\lfloor nt \rfloor} T_i(v_0) + T_{\lfloor nt \rfloor}((w(nt) - \lfloor nt \rfloor)), \quad 0 \leq t \leq 1 \quad (1)$$
parametrizes the above fractal curve. To implement it as an algorithm, we stop the recursion at some appropriate depth. The continuity and invertibility of this parametrization can be numerically verified, when the curve itself is non-self-intersecting.

In particular the von Koch curve is realized by setting all $s_i = 1/3$, $\theta_0 = \theta_3 = 0$, $\theta_1 = -\theta_2 = \pi/3$, and $v_0 = (1,0)$ (the unit vector along $x$ axis).

Hereafter symbols such as $a, b, c$ etc denote numbers in $[a_0, b_0]$ and $\theta, \theta'$ etc denote points of $F$.

Definition 2 For a set $F$ and a subdivision $P_{[a,b]}$, $a < b$, $[a,b] \subset [a_0, b_0]$

$$\sigma^\alpha[F, P] = \frac{\sum_{i=0}^{n-1} |w(t_{i+1}) - w(t_i)|^\alpha}{\Gamma(\alpha + 1)}$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^n$, $1 \leq \alpha \leq n$ and $P_{[a,b]} = \{a = t_0, \ldots, t_n = b\}$.

Next we define the coarsed grained mass function.

Definition 3 Given $\delta > 0$ and $a_0 \leq a \leq b \leq b_0$, the coarse grained mass $\gamma^\delta(F, a, b)$ is given by

$$\gamma^\delta(F, a, b) = \inf_{\{P_{[a,b]}: |P| \leq \delta\}} \sigma^\alpha[F, P]$$

where $|P| = \max_{0 \leq i \leq n-1} (t_{i+1} - t_i)$ for a subdivision $P$.

The mass function is the limit of the coarse-grained mass as $\delta \to 0$:

Definition 4 For $a_0 \leq a \leq b \leq b_0$, the mass function $\gamma^\alpha(F, a, b)$ is given by

$$\gamma^\alpha(F, a, b) = \lim_{\delta \to 0} \gamma^\delta(F, a, b)$$

Remark: Since $\gamma$ is a monotonic function of $\delta$. The limit exists, but could be finite or $+\infty$.

The following properties of the mass function follow easily.

Properties of $\gamma^\alpha(F, a, b)$

- For $a_0 \leq a < b < c \leq b_0$ and $\gamma^\alpha(F, a, c) < \infty$

$$\gamma^\alpha(F, a, c) = \gamma^\alpha(F, a, b) + \gamma^\alpha(F, b, c).$$

- $\gamma^\alpha(F, a, b)$ is increasing in $b$ and decreasing in $a$. 




• If $\gamma^\alpha(F, a, b)$ is finite, $\gamma^\alpha(F, a, t)$ is continuous for $t \in [a, b]$.

Remark: The implication of this result is that no single point has a nonzero mass, or in other words, the mass function is atomless.

• Let $F \subset \mathbb{R}^n$ be parametrizable. Let $\lambda$ be a positive real number, $v \in \mathbb{R}^n$, and let $T$ be a rotation operator. We denote

$$F + v = \{w(t) + v : t \in [a_0, b_0]\}$$

$$\lambda F = \{\lambda w(t) : t \in [a_0, b_0]\}$$

and

$$T F = \{T w(t) : t \in [a_0, b_0]\}$$

Then,

1. Translation :

$$\gamma^\alpha(F + v, a, b) = \gamma^\alpha(F, a, b)$$

2. Scaling :

$$\gamma^\alpha(\lambda F, a, b) = \lambda^\alpha \gamma^\alpha(F, a, b)$$

3. Rotation :

$$\gamma^\alpha(T F, a, b) = \gamma^\alpha(F, a, b)$$

Re-parametrization Invariance of Mass Function

The definitions of $\sigma^\alpha$, $\gamma_\delta^\alpha$, and therefore $\gamma^\alpha$ implicitly involve the particular parametrization $w$. Here we show that although defined through the parametrization, these definitions are invariant under the change of parametrization. In order to be able to unambiguously and explicitly refer to the parametrization, we introduce a temporary change in the notation to explicitly indicate dependence on parametrization. Thus given a parametrization $w : [a, b] \to \mathbb{R}^n$, we use the following notation here:

$$\sigma^\alpha[F, P; w] = \sum_{i=0}^{n-1} \frac{|w(t_{i+1}) - w(t_i)|^\alpha}{\Gamma(\alpha + 1)}$$

$$\gamma_\delta^\alpha(F, a, b; w) = \inf_{|P| \leq \delta} \sigma^\alpha[F, P; w]$$

$$\gamma^\alpha(F, a, b; w) = \lim_{\delta \to 0} \gamma_\delta^\alpha(F, a, b; w)$$

Let $w_1$ and $w_2$ be two parametrizations of the given fractal curve. By our definition of parametrization, $w_1$ and $w_2$ are continuous and one-to-one. Let the domain of $w_1$ be $[a_1, b_1]$, and that of $w_2$ be $[a_2, b_2]$. We further assume that $w_1$ and $w_2$ have the same orientation, i.e. $w_1(a_1) = w_2(a_2)$ and $w_1(b_1) = w_2(b_2)$. 

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Thus, $z = w_2^{-1} \circ w_1 : [a_1, b_1] \rightarrow [a_2, b_2]$ is a continuous, one-to-one and strictly monotonically increasing function.

Now, given $\delta_2 > 0$ and $\epsilon > 0$, there exists a subdivision $P_2$ of $[a_2, b_2]$ such that

$$\sigma^\alpha[F, P_2; w_2] < \gamma^\alpha_{\delta_2}(F, a_2, b_2; w_2) + \epsilon.$$ 

The set of points $P_1 = \{z^{-1}(t) : t \in P_2\}$ forms a subdivision of $[a_1, b_1]$. Then,

$$\sigma^\alpha[F, P_1; w_1] = \sigma^\alpha[F, P_2; w_2]$$

by appropriate substitution. Therefore,

$$\sigma^\alpha[F, P_1; w_1] < \gamma^\alpha_{\delta_1}(F, a_1, b_1; w_1) + \epsilon$$

which implies that

$$\gamma^\alpha_{\delta_1}(F, a_1, b_1; w_1) < \gamma^\alpha_{\delta_2}(F, a_2, b_2; w_2) + \epsilon$$

where $\delta_1 = |P_1|$. Further, since $z$ is continuous, $\lim \delta_1 = 0$, as $\delta_2 \rightarrow 0$, implying that

$$\gamma^\alpha(F, a_1, b_1; w_1) < \gamma^\alpha(F, a_2, b_2; w_2) + \epsilon.$$ 

Since $\epsilon$ is arbitrary, and the same argument remains valid starting with $z^{-1} = w_1^{-1} \circ w_2$, we conclude that

$$\gamma^\alpha(F, a_1, b_1; w_1) = \gamma^\alpha(F, a_2, b_2; w_2).$$

This establishes the fact that the mass function depends only on the fractal curve (i.e., the image of the parametrization), and is independent of the parametrization itself. Since the mass function underlies the calculus developed in the subsequent sections, the calculus is also independent of the particular parametrization chosen.

Now we introduce the integral staircase function for a set $F$ of order $\alpha$.

**Definition 5** Let $p_0 \in [a_0, b_0]$ be arbitrary but fixed. The staircase function $S_{F}^{\alpha} : [a_0, b_0] \rightarrow \mathbb{R}$ of order $\alpha$ for a set $F$ is given by

$$S_{F}^{\alpha}(t) = \begin{cases} \gamma^\alpha(F, p_0, t) & t \geq p_0 \\ -\gamma^\alpha(F, t, p_0) & t < p_0 \end{cases}$$

where $t \in [a_0, b_0]$.

In the rest of this paper we take $p_0 = a_0$ unless stated otherwise.

Here this function may, more appropriately, be described as a rise function. However we retain the name staircase function because in analogous calculus on fractal subsets of the real line this role is played by a staircase.

Throughout the paper we consider only those sets for which $S_{F}^{\alpha}$ is strictly increasing and thus invertible. Further, we define

$$J(\theta) = S_{F}^{\alpha}(w^{-1}(\theta)), \quad \theta \in F$$
Figure 1: $S^\alpha_F$ for von Koch curve. The von Koch curve lies in the $XY$ plane. The vertical lines are drawn to guide the eye (to show how $S^\alpha_F$ rises)

which is the function induced by $S^\alpha_F$ on $F$, and it is also one-to-one.

As an example, figure 1 shows the staircase function for the von koch curve. The curve was parametrized as given in [20].

A log-log graph of the staircase function $S^\alpha_F(t)$ against the Euclidean distance between origin and $w(t)$ for the von-koch curve is shown in fig 2

3 The $\gamma$- Dimension

We now consider the sets $F$ for which the mass function $\gamma^\alpha(F, a, b)$ gives the most useful information. Due to the similarity of the definitions of mass function and the Hausdorff outer measure, the former can be used to define a fractal dimension as follows.

It can be seen that $\gamma^\alpha(F, a, b)$ is infinite up to certain value of $\alpha$, say $\alpha_0$, and jumps down to zero for $\alpha > \alpha_0$. Thus

**Definition 6** The $\gamma$-dimension of $F$, denoted by $\dim_\gamma(F)$, is

$$
\dim_\gamma(F) = \inf\{\alpha : \gamma^\alpha(F, a, b) = 0\} = \sup\{\alpha : \gamma^\alpha(F, a, b) = \infty\}
$$

It follows that the $\gamma$-dimension is finer than the box dimension. Thus

$$
\dim_\gamma(F) \leq \dim_B(F).
$$
Figure 2: log-log graph of Euclidean distance between origin and $w(t)$ (Y-axis) vs $S^t_F(t)$ for $t \in [0, 1]$ for von-Koch curve

$\gamma$-dimension for self-similar curves

Let $\alpha$ denote the $\gamma$-dimension of a self-similar curve, which is made up of $m$ copies of itself, scaled by a factor of $\frac{1}{n}$ and rotated and translated appropriately. Then using the translation, scaling and rotation properties of the mass function, one can see that the mass of the whole curve is given by

$$\gamma^\alpha(F, a_0, b_0) = m\gamma^\alpha\left(\frac{1}{n}F, a_0, b_0\right)$$

$$\gamma^\alpha(F, a_0, b_0) = m\left(\frac{1}{n}\right)^\alpha \gamma^\alpha(F, a_0, b_0)$$

(6)

Hence,

$$\alpha = \log m/\log n$$

(7)

This is same as the Hausdorff dimension of self-similar curves [4].

Thus for self-similar curves

$$\dim_\gamma F = \dim_H F = \dim_B F$$

where $\dim_H F$ denotes the Hausdorff dimension and $\dim_B F$ the box dimension of $F$. 

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4 Algorithmic Nature of the Mass Function

Let us first summarize the definition of mass function:

\[ \gamma^\alpha(F, a, b) = \lim_{\delta \to 0} \inf \{P: |P| \leq \delta\} \sum_{i=0}^{n-1} \frac{|w(t_{i+1}) - w(t_i)|^\alpha}{\Gamma(\alpha + 1)} \]  

One of the main differences between the Hausdorff measure and the mass function is that while the Hausdorff measure is based on sums over a countable covers (composed of arbitrary sets) of the given set \( F \), the mass function is based on finite subdivisions of the parametrization domain. From an algorithmic point of view, the extent of the set of all possible finite subdivisions is much smaller than that of all countable (finite and infinite) covers of a set. This makes the mass function much more amenable to an algorithmic computation.

As in any algorithm which intends to approximate the infimum, we would like to find a subdivision \( P \) such that \( \sigma^\alpha[F, P] \) is close to the infimum. Further, we can consider values of \( \delta \) only as small as practically possible within the reach of numerical calculations. The goal of the algorithm is thus to find a subdivision \( P \) as described above, given a fixed \( \delta \).

However, the set of allowed subdivisions is still large, to explore all of it systematically. Further the constraint \( |P| \leq \delta \) does not restrict the number of points in \( P \), rendering the standard deterministic optimization algorithms either inapplicable or too complex to implement. More appropriate is a Monte Carlo method where a subdivision is modified in a variety of ways randomly but consistently with the constraint \( |P| \leq \delta \), and the change is accepted if the sum \( \sigma^\alpha[F, P] \) decreases due to the modification. The algorithm presented below, is based on this strategy.

A Monte Carlo Algorithm

For the purpose of this algorithm, \([a, b]\) denotes the domain of \( w \). Further, “randomly” means with a uniform probability unless stated otherwise. The symbol \( P \) always indicates the “current” subdivision in consideration.

We begin with a uniform subdivision \( P \) such that \( |P| = \delta/4 \), and iteratively improve it using the following prescription.

1. Choose two numbers \( x, y \in [a, b] \) randomly, and relabel them if necessary so that \( x \leq y \). Then \([x, y] \subset [a, b]\). Let \( P' = \{t_i: 0 \leq i \leq m\} \) denote the set of all points of \( P \cap [x, y] \). We now modify \( P' \) in one of the following ways with equal probability, and denote the resultant by \( P'' \):

   (a) With a probability \( p_c = \min(1, \delta/(y - x)) \), we shift each point \( t_i \) (except \( t_0 \) and \( t_m \)) by a random amount between \([-\delta/2, \delta/2]\), if the resultant subdivision \( P'' \) still satisfies \( |P''| \leq \delta \).

   (b) With a probability \( p_d = \min(1, \delta/(y - x)) \), we remove each point \( t_i \) (except \( t_0 \) and \( t_m \)) from \( P' \), if the resultant subdivision \( P'' \) still satisfies \( |P''| \leq \delta \).
The von Koch curve Image of P

Figure 3: The image (under w) of a numerically computed near-optimal subdivision $P$, for $\delta = 0.05$, superimposed on the von-Koch curve.

(c) With a probability $p_i = \min(1, \delta/(y-x))$, we insert a point between each $t_i$ and $t_{i+1}$ which is chosen randomly from $[t_i, t_{i+1}]$. (However, to avoid accumulating too much of rounding error, we insert the point only if the distance between $t_i$ and $t_{i+1}$ is greater than $\delta/10$.)

2. Form a new subdivision $P_1 = (P \cap [a, x)) \cup P' \cup (P \cap (y, b])$, i.e. the subdivision of which the points belonging to $[x, y]$ are changed by the above procedure. If $\sigma^\alpha[F, P_1] < \sigma^\alpha[F, P]$, then we consider $P_1$ as the “current” subdivision which will be possibly improved further using above steps. Otherwise we consider $P$ again for the purpose.

As the sum $\sigma^\alpha[F, P]$ approaches the infimum, many of the newly formed subdivisions $P'$ are rejected since they sum up higher than $P$. Thus near the infimum, the sum remains constant for many consecutive iterations, and changes only intermittently. Therefore the usual convergence criterion of terminating iteration when the difference between successive iterations or every $K$ iterations ($K$ being a suitable large integer) goes below certain small number, is not useful in this case. Instead, after examining the sum over a large number of iterations, we observe that the sum stops making significant progress between $N' = 1000$ to $N' = 2000$, where $N' = N/n$ is the number of iterations $N$ normalized by the current subdivision size $n$. Further, we need to go through all these iterations
more than once, just to ensure that subdivision is really optimal. Occasionally it may happen that the sum settles a little above the optimal value, getting "trapped" in a "local minimum".

We demonstrate the results of this algorithm as applied on the von Koch curve, parametrized as in equation (1). It turns out that the mass of the entire von Koch curve is a little less than $0.51/\Gamma(\alpha+1)$, $\alpha = \ln(4)/\ln(3)$. The image (under w) of the optimal subdivision found by the algorithm is shown in figure [3] superimposed on the von Koch curve. The evolution of the sum over the normalized number of iterations is shown in figure 4.

The above description assumes that the value of $\alpha$ is the same as the $\gamma$-dimension of the set $F$, say $\alpha_0$. We expect $\delta$-independence in the values of $\sigma^\alpha[F, P(\delta)]$ where $P(\delta)$ denotes the resultant subdivision of the algorithm at the scale $\delta$, since the value of $\gamma^\alpha_\delta$ converges to a finite nonzero value. This is what we observe from the values of $\sigma^\alpha[F, P(\delta)]$ obtained for various values of $\delta$ (figure 4).

Now we would like to consider cases when $\alpha \neq \alpha_0$. Let $0 < \delta_1 < \delta_2$. If $\alpha < \alpha_0$, then $\gamma^\alpha(F, a, b) = \infty$. Therefore we expect that $R(\alpha) = \sigma^\alpha[F, P(\delta_1)]/\sigma^\alpha[F, P(\delta_2)] > 1$. Similarly since $\alpha > \alpha_0$ implies $\gamma^\alpha(F, a, b) = 0$, we expect that $R(\alpha) < 1$. 

![Figure 4: The evolution of $\Gamma(\alpha+1)\sigma^\alpha[F, P]$ over the normalized number of iterations. The evolution is shown only up to $N' = 100$, since the latter part $(100 < N' \leq 2000)$ is almost flat and uninteresting.](image)
This fact can be used to algorithmically calculate the $\gamma$-dimension $\alpha_0$: We need to find the number $\alpha_0$ such that $R(\alpha_0) = 1$. We already know that $\alpha_0 \in [1, m]$, $m$ being the embedding dimension, since $F \in \mathbb{R}^m$ is a curve. Treating this as the initial bracket of values for $\alpha_0$, we just need to use some algorithm such as bisection to shrink this bracket to sufficient accuracy.

5 The $F^\alpha$-Calculus

Most of the proofs which are similar to the proofs in the case of discontinuous sets like Cantor-like sets are omitted.

5.1 $F$-Limit and $F$-Continuity

Now we introduce limits and continuity along a fractal curve.

**Definition 7** Let $F \subset \mathbb{R}^n$ be a fractal curve, and let $f : F \to \mathbb{R}$. Let $\theta \in F$. A number $l$ is said to be the limit of $f$ through points of $F$, or simply $F$-limit, as $\theta' \to \theta$, if given $\epsilon > 0$ there exists $\delta > 0$ such that $\theta' \in F$ and $|\theta' - \theta| < \delta \implies |f(\theta') - l| < \epsilon$

If such a number exists it is denoted by $l = F$-lim$_{\theta' \to \theta} f(\theta')$

**Definition 8** A function $f : F \to \mathbb{R}$ is said to be $F$-continuous at $\theta \in F$ if $f(\theta) = F$-lim$_{\theta' \to \theta} f(\theta')$.

**Definition 9** $f : F \to \mathbb{R}$ is said to be uniformly continuous on $E \subset F$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\theta \in F$ and $\theta' \in E$

$|\theta' - \theta| < \delta \implies |f(\theta') - f(\theta)| < \epsilon$

5.2 $F^\alpha$-Integration

We denote the class of bounded functions $f : F \to \mathbb{R}$ by $B(F)$.

**Definition 10** For $t_1, t_2 \in [a_0, b_0], t_1 \leq t_2$ a section or segment $C(t_1, t_2)$ of the curve is defined as $C(t_1, t_2) = \{w(t') : t' \in [t_1, t_2]\}$

**Definition 11** Let $f : F \to \mathbb{R}$ and $t_1, t_2 \in [a_0, b_0], t_1 \leq t_2$ and let $M[f, C(t_1, t_2)] = \sup_{\theta \in C(t_1, t_2)} f(\theta)$ and $m[f, C(t_1, t_2)] = \inf_{\theta \in C(t_1, t_2)} f(\theta)$
Let $S^F_\alpha(t)$ be finite for $t \in [a, b] \subset [a_0, b_0]$. Let $P = \{t_0, \ldots, t_n\}$ be a subdivision of $[a, b]$. The upper and the lower $F^\alpha$-sum for the function $f$ over the subdivision $P$ are given respectively by

$$U^\alpha[f, F, P] = \sum_{i=0}^{n-1} M[f, C(t_i, t_{i+1})][S^F_\alpha(t_{i+1}) - S^F_\alpha(t_i)], \quad (9)$$

$$L^\alpha[f, F, P] = \sum_{i=0}^{n-1} m[f, C(t_i, t_{i+1})][S^F_\alpha(t_{i+1}) - S^F_\alpha(t_i)]. \quad (10)$$

Now we define the $F^\alpha$-integral

**Definition 12** Let $F$ be such that $S^F_\alpha$ is finite on $[a, b]$. For $f \in B(F)$, the lower and upper $F^\alpha$-integral of the function $f$ respectively on the section $C(a, b)$ are

$$\int_{C(a,b)} f(\theta)d^F_\alpha \theta = \sup_{P_{[a,b]}} L^\alpha[f, F, P] \quad (11)$$

$$\int_{C(a,b)} f(\theta)d^F_\alpha \theta = \inf_{P_{[a,b]}} U^\alpha[f, F, P] \quad (12)$$

If $f \in B(f)$, we say that $f$ is $F^\alpha$-integrable on $C(a, b)$ if

$$\int_{C(a,b)} f(\theta)d^F_\alpha \theta = \int_{C(a,b)} f(\theta)d^F_\alpha \theta$$

and the common value is called the $F^\alpha$-integral, denoted by

$$\int_{C(a,b)} f(\theta)d^F_\alpha \theta.$$

### 5.3 $F^\alpha$-Differentiation

**Definition 13** Let $F$ be a fractal curve. Then the $F^\alpha$-derivative of function $f$ at $\theta \in F$ is defined as

$$(D^F_\alpha f)(\theta) = F\cdot \lim_{\theta' \to \theta} \frac{f(\theta') - f(\theta)}{J(\theta') - J(\theta)} \quad (13)$$

if the limit exists.

**Theorem 14** If $(D^F_\alpha f)(\theta)$ exists for all $\theta \in C(a, b)$, then $f$ is $F$-continuous on $C(a, b)$.

Remark: The $F^\alpha$-derivative $D^F_\alpha(f)$ of a constant function $f : F \to \mathbb{R}$, $f(\theta) = k \in \mathbb{R}$ is zero. This result is to be contrasted with the classical fractional derivative (Riemann-Liouville, and others) of a constant, which is not zero in general \[21, 22, 23, 24].
5.4 Fundamental theorems of $F^\alpha$-calculus

The $F^\alpha$-integration and $F^\alpha$-differentiation are related as inverse operations of each other. The first fundamental theorem states:

**Theorem 15** Let $f \in B(F)$ is an $F$-continuous function on $C(a, b)$, and let $g : f \to \mathbb{R}$ be defined as

$$g(w(t)) = \int_{C(a,t)} f(\theta') d^\alpha_F \theta'$$

for all $t \in [a, b]$. Then

$$(D^\alpha_F g)(\theta) = f(\theta) \quad \text{and} \quad \theta = w(t)$$

The second fundamental theorem says that the $F^\alpha$-integral as a function of upper limit is the inverse of $F^\alpha$-derivative except for an additive constant.

**Theorem 16** Let $f : F \to \mathbb{R}$ be continuously $F^\alpha$-differentiable function and $h : F \to \mathbb{R}$ be $F$-continuous, such that $h(\theta) = (D^\alpha_F f)(\theta)$. Then

$$\int_{C(a,b)} h(\theta) d^\alpha_F \theta = f(w(b)) - f(w(a))$$

Comparison with approaches involving Local Fractional Operators

The local fractional derivative (LFD) operator constructed in [25] was based on the renormalization of Riemann-Liouville differential operator on the real line. It was utilized to establish the relation between the differentiability properties of nowhere differentiable functions, such as Weierstrass function, and the dimension (Holder exponent) of its graph. The domain of these functions is $\mathbb{R}$ and not a fractal. In [15] though the prescription was developed using LFD, it was realized that to make the local fractional Fokker Planck equation causal and dynamically consistent, the evolution had to be restricted to fractal subsets of the real line. Moreover the order of differentiation had to be the dimension of the fractal support. Since Cantor-like sets are totally disconnected, this necessitated a rigorous development of calculus on fractals from first principles, without using standard fractional calculus on $\mathbb{R}$, which was carried out in [16].

The present paper is a logical extension of such a formulation, to fractal curves. While Cantor-like sets considered in [16] are totally disconnected, the von-Koch like fractal curves are continuous but tangentless. Thus all the constructions and proofs have to be carried out keeping in mind this difference of the domain of functions and operators. (Thus for example the notion of 'set of change' and '\(\alpha\)-perfect sets' was crucial for cantor-like sets in [16] which is replaced by the invertibility of $S^\alpha_F$ in case of fractal curves considered in this paper.)
There is multiplicity of approaches leading to the notion of local fractional calculus. Various authors [26], [27], [28] have further developed the notion of local fractional differentiation with different approaches. They provide suitable framework for different classes of problems. The development in [26] is based on approach involving difference operators. In finding derivative the quotient is taken with respect to $h^\alpha$ where $h$ is an increment of the independent variable. This can be contrasted with the use of $S_\alpha F$ in the present approach. This also reflects in the Taylor series where powers of $(S_\alpha F)$ appear (see equation (17) below) rather than powers of $h$ itself as in [26]. Further the domain is $\mathbb{R}$ in [26] whereas it is a fractal curve in the present paper.

In [27] the notion of classical fractional derivative is modified. Again the essential difference mentioned above for Taylor series and the domain functions is also to be noted here. The development in [28] is based on the Weyl Derivative and the domain of functions is $\mathbb{R}$ and not a fractal curve.

We may emphasize that in the present approach the role of the independent variables is delegated to the staircase/rise function $S_\alpha F$, see e.g. equation (13). In this sense our approach is like Stieltjes approach in spirit but with some essential differences as noted in [16]. The function $S_\alpha F$ captures the essence of fractal support, hence its use makes the calculus suitable for fractals.

6 Conjugacy of $F^\alpha$-Calculus and Ordinary Calculus

In this section, we define a map $\phi$ which takes an $F^\alpha$-integrable function $f : F \to \mathbb{R}$ to a Riemann integrable function $g : [S^\alpha F(a_0), S^\alpha F(b_0)] \to \mathbb{R}$ such that their corresponding integrals have equal values. Thus, the map $\phi$ exhibits a conjugacy between the two operations.

First let us define certain classes of functions:

1. $B(F)$ : class of bounded functions $f : F \to \mathbb{R}$.
2. $B([c, d])$ : class of bounded functions $f : [c, d] \to \mathbb{R}$
3. $\mathcal{L}(F)$: set of all functions which are $F^\alpha$-integrable on $C(a_0, b_0)$.
4. The image of $F$ under $S^\alpha F$ is denoted by $K$, i.e $K = [S^\alpha F(a_0), S^\alpha F(b_0)]$, and $B(K)$ denotes the class of functions bounded on $K$.
5. $\mathcal{L}(K)$ denotes the class of functions in $B(K)$ which are Riemann integrable over the interval $K = [S^\alpha F(a_0), S^\alpha F(b_0)]$.

In order to fix the notation, here we briefly review the definition of Riemann integral. Firstly, if $g \in B([c, d])$ and $I \subset [c, d]$ is a closed interval, then we denote $M'[g, I] = \sup_{x \in I} g(x)$ and $m'[g, I] = \inf_{x \in I} g(x)$. Further, the upper and lower sum over a subdivision $P_{[c, d]} = \{y_0, \ldots, y_n\}$ are given by $U'[g, P] = \sum_{i=0}^{n-1} M'[g, [y_i, y_{i+1}]]$ and $L'[g, P] = \sum_{i=0}^{n-1} m'[g, [y_i, y_{i+1}]]$. If the upper and
lower integrals given respectively by $\inf_P U'[g, P]$ and $\sup_P L'[g, P]$ are equal, then $g$ is said to be Riemann integrable, and the Riemann integral

$$\int_c^d g(y)dy$$

is defined to be the common value.

Now we define the above mentioned map $\phi$:

**Definition 17** The map $\phi : B(F) \to B([S^\alpha_F(a_0), S^\alpha_F(b_0)])$ takes $f \in B(F)$ to $\phi[f] \in B([S^\alpha_F(a_0), S^\alpha_F(b_0)])$ such that for each $t \in [a_0, b_0]$,

$$\phi[f](S^\alpha_F(t)) = f(w(t))$$

**Lemma 18** The map $\phi : B(F) \to B(K)$ is one to one and onto.

The proof is straightforward. Thus we are assured that the inverse map $\phi^{-1}$ exists.

The following theorem brings out the conjugacy between $F^\alpha$-integrals of functions along the fractal curve $F$ and the Riemann integrals of their images under $\phi$.

**Theorem 19** A function $f \in B(F)$ is $F^\alpha$-integrable over $C(a, b)$ if and only if $g = \phi[f]$ is Riemann integrable over $[S^\alpha_F(a), S^\alpha_F(b)]$. In other words, a function $f \in B(F)$ belongs to $L^\alpha(F)$ if and only if $g \in L(K)$. Further

$$\int_{C(a,b)} f(\theta)d\theta^\alpha = \int_{S^\alpha_F(a)}^{S^\alpha_F(b)} g(u)du$$

**Proof:** Let $f : F \to R$ be $F^\alpha$-integrable. Then there exists a subdivision $P_{[a,b]} = \{t_0, t_1, \ldots, t_n\}$ such that

$$U^\alpha[f, F, P] - L^\alpha[f, F, P] < \epsilon$$

for any $\epsilon > 0$.

Denote $y_i = S^\alpha_F(t_i)$. Then $Q = \{y_i : 0 \leq i \leq n\}$ is a subdivision of $[S^\alpha_F(a), S^\alpha_F(b)]$.

For any component $[t_i, t_{i+1}]$

$$M[f, C(t_i, t_{i+1})] = \sup_{w \in C(t_i, t_{i+1})} f(w)$$

$$= \sup_{t \in [t_i, t_{i+1}]} f(w(t))$$

$$= \sup_{t \in [t_i, t_{i+1}]} g(S^\alpha_F(t))$$

$$= \sup_{y \in [y_i, y_{i+1}]} g(y)$$

$$= M'[g, [y_i, y_{i+1}]]$$
Therefore,

\[ U^\alpha[f, F, P] = \sum_{i=0}^{n-1} M[f, C(t_i, t_{i+1})][S^\alpha_F(t_{i+1}) - S^\alpha_F(t_i)] \]

\[ = \sum_{i=0}^{n-1} M[f, C(t_i, t_{i+1})]||y_{i+1} - y_i|| \]

\[ = \sum_{i=0}^{n-1} M'[g, ||y_i, y_{i+1}||][y_{i+1} - y_i] \]

\[ = U'[g, Q] \quad (15) \]

Similarly

\[ L^\alpha[f, F, P] = L'[g, Q] \quad (16) \]

then using equations (14), (15) and (16)

\[ U'[g, Q] - L'[g, Q] < \epsilon \]

which implies that \( g \) is Riemann integrable over \( [S^\alpha_F(a), S^\alpha_F(b)] \)

\[ \int_{S^\alpha_F(a)}^{S^\alpha_F(b)} g(u)du = \int_{\theta} f(\theta)d^\alpha_F \theta \]

Conversely if \( g \) is Riemann Integrable, then for given \( \epsilon > 0 \) there exists a subdivision \( Q' = \{v_0, \ldots, v_m\} \) of \( [S^\alpha_F(a), S^\alpha_F(b)] \) such that \( U'[g, Q] - L'[g, Q'] < \epsilon \). Then the converse can be proved by following the above steps in the reverse order.

Let \( f_1 \) denote the indefinite \( F^\alpha \)-integral viz. \( f_1(w(t)) = \int_{C(a,t)} f(\theta)d^\alpha_F \theta \) and let \( g_1 \) denote the ordinary indefinite Riemann integral viz. \( g_1(y) = \int_{S^\alpha_F(a)} g(y')dy' \).

If we further denote the indefinite \( F^\alpha \)-integral operator by \( I^\alpha_F \) and the indefinite Riemann integral operator by \( I \), then the result of theorem (19) can be expressed as

\[ I^\alpha_F = \phi^{-1} I \phi \]

as displayed in the commutative diagram of figure 5.

The following theorem brings out the conjugacy between \( F^\alpha \)-derivative and ordinary derivative.

**Theorem 20** Let \( h \) be a function in \( B(F) \) such that \( g = \phi[h] \) is ordinarily differentiable on \( K = \text{range of } S^\alpha_F \). Then \( D^\alpha_F h(\theta) \) exists for all \( \theta \in F \) and

\[ D^\alpha_F h(\theta) = \left. \frac{dg(v)}{dv} \right|_{v=J(\theta)} \]

**Proof:** Let \( v \in K \). Then by definition

\[ \frac{dg}{dv} = \lim_{u \to v} \frac{g(u) - g(v)}{u - v} \]
i.e given $\epsilon_0 > 0$, there exists $\delta_0 > 0$ such that

$$|u - v| < \delta_0 \implies \left| \frac{dg}{dv} \right| < \epsilon_0$$

Let us recall our assumption that $S_F^\alpha$ is monotonically increasing and one-to-one. Let $t = (S_F^\alpha)^{-1}(v)$, $t' = (S_F^\alpha)^{-1}(u)$. Then $t, t' \in [a_0, b_0], h(t') = g(u)$ and $h(t) = g(v)$. Thus,

$$|S_F^\alpha(t') - S_F^\alpha(t)| < \delta_0 \implies \left| \frac{dg}{dv} \right| < \epsilon_0$$

Since $(w)^{-1}$ and $S_F^\alpha$ are continuous, so is their composition $S_F^\alpha \circ(w)^{-1}$. Therefore, there exists $\delta_1 > 0$ such that

$$|w(t') - w(t)| < \delta_1 \implies |S_F^\alpha(t') - S_F^\alpha(t)| < \delta_0 \implies \left| \frac{dg}{dv} \right| < \epsilon_0.$$

Setting $\theta' = w(t'), \theta = w(t)$, we can rewrite this as

$$|\theta' - \theta| < \delta_1 \implies |J(\theta') - J(\theta)| < \delta_0 \implies \left| \frac{dg}{dv} \right| < \epsilon_0,$$

which by definition of $F$-limit and $D_F^\alpha$ means

$$D_F^\alpha h(\theta) = \lim_{\theta' \to \theta} \frac{h(\theta') - h(\theta)}{J(\theta') - J(\theta)} = \frac{dg}{dv} \bigg|_{v = J(\theta)}.$$

- \textbf{Theorem 21} Let $h \in B(F)$ be an $F^\alpha$-differentiable function at all $\theta \in F$. Further, let $g = \phi[h]$. Then $dg/dv$ exists at $v = J(\theta)$ and

$$\frac{dg}{dv} \bigg|_{v = J(\theta)} = D_F^\alpha h(\theta)$$

Proof: As $g = \phi[h]$, we have $g(S_F^\alpha(t)) = h(w(t))$ for all $t \in [a_0, b_0]$ i.e $g(J(\theta')) = h(\theta)$ for all $\theta \in F$.

By definition and substitution

$$D_F^\alpha h(\theta) = \lim_{\theta' \to \theta} \frac{h(\theta') - h(\theta)}{J(\theta') - J(\theta)} = \lim_{\theta' \to \theta} \frac{g(J(\theta')) - g(J(\theta))}{J(\theta') - J(\theta)}$$

Thus given $\epsilon_0 > 0$ there exists $\delta_0' > 0$ such that

$$|\theta' - \theta| < \delta_0' \implies \left| \frac{g(J(\theta')) - g(J(\theta))}{J(\theta') - J(\theta)} - D_F^\alpha h(\theta) \right| < \epsilon_0.$$
Let \( v = J(\theta) \) and \( u = J(\theta') \), i.e \( \theta = J^{-1}(v) \) and \( \theta' = J^{-1}(u) \). Since \( J^{-1} \) is continuous, there exists a \( \delta > 0 \) such that

\[
|u - v| < \delta \implies |\theta' - \theta| < \delta_0 \implies \left| \frac{g(u) - g(v)}{u - v} - D_F^\alpha h(\theta) \right| < \epsilon_0
\]

Which by definition of ordinary derivative gives

\[
\left. \frac{dg}{dv} \right|_{v=J(\theta)} = \lim_{u \to v} \frac{g(u) - g(v)}{u - v} = D_F^\alpha h(\theta)
\]

This conjugacy can also be expressed as \( D_F^\alpha = \phi^{-1} D\phi \) as shown in the commutative diagram of figure 5.

Figure 5: The relation between \( F^\alpha \)-integral and Riemann integral, also between \( F^\alpha \)-derivative and Ordinary derivative

### 6.1 Taylor Series

One can write a fractal Taylor series for functions on fractal curve \( F \), by using the results of this section.

If \( g = \phi[h] \) be such that the ordinary Taylor series is given by

\[
g(u) = \sum_{n=0}^{\infty} \frac{(u - y)^n}{n!} \frac{d^\alpha g(y)}{dy^n}
\]

is valid for \( u, y \in [S^\alpha_F(a), S^\alpha_F(b)] \), then for \( \theta, \theta' \in F \) it can be seen that

\[
h(\theta) = \sum_{n=0}^{\infty} \frac{(J(\theta) - J(\theta'))^n}{n!} (D_F^\alpha)^n h(\theta')
\]

provided \( h \in B(F) \) is \( F^\alpha \)- differentiable any number of times on \( C(a, b) \) such that \( (D_F^\alpha)^n h \in B(F) \) for any integer \( n > 0 \).
6.2 Function Spaces in $F^\alpha$-Calculus

We introduce the following spaces:

The set of all functions that have $F$-continuous $F^\alpha$-derivatives up to order $k$ can be analysed analogous to [17] and are defined by $C^k(F), k \in \mathbb{N}$: Set of all functions $f : F \to \mathbb{R}$ such that

$$(D_F^\alpha)^n f \in C^0(F) \text{ for all } n \leq k$$

One can define norm on $C^k(F)$ for $F \subset \mathbb{R}^n$, similar to what is defined for $F \subset \mathbb{R}$ using the $F^\alpha$-derivative as follows:

$$||f|| = \sum_{0 \leq n \leq k} \sup_{\theta \in F} |(D_F^\alpha)^n f| \quad f \in C^k(F)$$

$C^k(F)$ are complete with respect to this norm. Unlike in [17] the need for $S^\alpha_F$-concordant functions does not arise since $S^\alpha_F(u) \neq S^\alpha_F(v)$ even if $f(w(u)) = f(w(v))$ for $u \neq v$. It can also be shown that quite easily that $C^k(F)$ is separable.

The spaces of $F^\alpha$-integrable functions and their completion can also be constructed in an analogous manner as is done in [16] for fractal subsets of real line.

Set $L(F)$ of $F^\alpha$-integrable functions is a vector space with usual operations of addition and scalar multiplication. An appropriate norm $N_p$ can be defined for $F^\alpha$ integrable functions which satisfies all the required properties.

$$N_p(f) = ||f||_p = \left[ \int_{C(a,b)} |f(\theta)|^p d_F^\alpha(\theta) \right]^{1/p} \quad 1 \leq p < \infty$$

$N_p$ can be shown to act as a norm on $L'(F)$ where $L'(F)$ is a vector space of equivalent classes of $L(F)$, the class of all $F^\alpha$-integrable functions. $L'_p(F)$ which is $L'(F)$ with specific norm $N_p$ is not complete but can be completed using standard procedure. The complete space is then denoted by $L_p(F)$ and is a Banach space. The spaces $L'_p(F)$ and $L_p(F)$ can also be shown to be separable.

Analogues of abstract Sobolev Spaces can be constructed in exactly the same way as is done in the above cited reference for subsets of Real line.

7 Example: Absorption on fractal curves

Consider the flux of a fluid or of particles moving steadily through and getting absorbed in a percolating cluster or fractured rock. A simple mathematical model of this process for a single branch would be that of particles getting absorbed along a fractal path. The absorption process can then be modelled by the following equation:

$$D_F^\alpha \rho(\theta) = -\kappa \rho(\theta) \quad (18)$$

where $\rho(\theta)$ is the density of fluid at a point $\theta$ of the fractal channel (e.g. backbone of the percolating cluster), $\kappa$ being the coefficient of absorption (which in a simple model is taken as constant), and $D_F^\alpha$ is the $F^\alpha$-derivative.
The left hand side of equation (18) represents the space rate of change of density of particles at a particular position on the fractal curve (or path) varying along the path.

The exact solution of the above equation can be obtained using the conjugacy between $F^\alpha$-derivative and ordinary derivative and applying the corresponding operator on $\rho(\theta)$ as follows:

$$\phi \rho(\theta) = \tilde{\rho}(y) \text{ where } y = J(\theta)$$

then equation (18) becomes

$$\frac{d}{dy} \tilde{\rho}(y) = -\kappa \tilde{\rho}(y)$$

the solution of which is given by

$$\tilde{\rho}(y) = \tilde{\rho}(0) \exp(-\kappa y)$$

Applying the inverse conjugate operator to the above equation we obtain

$$\rho(\theta) = \rho(0) \exp(-\kappa J(\theta))$$

or since $\theta = w(u)$ and $J(\theta) = S^\alpha_R(u)$,

$$\rho(w(u)) = \rho(0) \exp(-\kappa S^\alpha_R(u)) \quad (19)$$

This is like stretched exponential behaviour in view of the relation between euclidean distance and staircase (see fig 2).

8 Conclusion

In this paper we have developed a calculus on parametrizable fractal curves of dimension $\alpha \in [1, n]$. This involves the identification of the important role played by the mass function and the corresponding (rise) staircase function which may be compared with the role played by the independent variable itself in ordinary calculus. The definitions of $F^\alpha$-integral and $F^\alpha$-derivative are specifically tailored for fractal curves of dimension $\alpha$. Further they reduce to Riemann integral and ordinary derivative respectively, when $F = R$ and $\alpha = 1$.

Much of the development of this calculus is carried in analogy with the ordinary calculus. Specifically, we have adopted Riemann-Stieltjes like approach for integration, as it is direct, simple and advantageous from algorithmic point of view. The example of absorption on fractal curves mentioned in section 7 demonstrates the utility of such a framework in modelling. Other applications may include fractal Langevin equation for Brownian motion and Levy processes on such curves, which will follow in future work. This approach may be further useful in dealing with path integrals and other similar applications. Another direction for extension of the considerations in this paper is the extension to crumpled or fractal surfaces which are continuously parametrizable by a finite number of variables.
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