R-Charge Chemical Potential in a 2+1 Dimensional System

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We study probe D5 branes in D3 brane AdS$_5$ and AdS$_5$-Schwarzschild backgrounds as a prototype dual description of strongly coupled 2+1 dimensional quasi-particles. We introduce a chemical potential for a weakly gauged U(1) subgroup of the theory’s global R-symmetry by spinning the D5 branes. The resulting D5 embeddings are complicated by the existence of a region of the space in which the local speed of light falls below the rotation speed. We find regular embeddings through this region and show that the system does not exhibit the spontaneous symmetry breaking that would be needed for a superconductor.

INTRODUCTION

Recently there has been interest in whether the AdS/CFT Correspondence \cite{adscft1,adscft2,adscft3} can be used to understand 2+1 dimensional condensed matter systems (for example \cite{2dcond1,2dcond2,2dcond3,2dcond4,2dcond5,2dcond6}). The typical UV degrees of freedom in these systems are electrons in the presence of a Fermi surface and a gauged U(1), QED. When brought together in certain 2d states they can become relativistic and strongly coupled - possibly such systems might induce superconductivity too by breaking the gauge symmetry. The philosophy, which may be overly naive, is to find relativistic strongly coupled systems that show these behaviours and hope they share some universality with the physical systems. Whether or not that linkage becomes strong, it is interesting to study the AdS duals of 2+1d systems.

In this paper we will study the dynamics of the theory on the world volume of a mixed D3 and D5 brane construction with a 2+1 dimensional intersection\textsuperscript{1}, which has previously been studied at zero temperature in the absence of chemical potential in \cite{mixed1,mixed2,mixed3}. The gravity dual of the D3s, at zero temperature, is AdS$_5 \times S^5$, which is dual to the 3+1 dimensional $\mathcal{N} = 4$ super Yang Mills theory. Here these interactions will be used to loosely represent strongly coupled “phonons”. We will introduce 2+1d “quasi-particles” via D5 branes (with a 2+1d intersection with the D3s) - states connecting the two set of branes should be expected to carry quantum numbers that interact with the D3 brane dynamics and flavour quantum numbers associated with the number of D5 branes - the full field theory can be found in \cite{2dcond6}. We will work in the probe approximation for the D5 branes which corresponds to quenching quasi-particle loops in the phonon background \cite{2dcond7}. At zero chemical potential the theory has $\mathcal{N} = 4$ supersymmetry and at zero quasi-particle mass is conformal \cite{2dcond1,2dcond2,2dcond6}. The system is related to the higher dimensional D3-D7 intersection where the $\mathcal{N} = 4$ gauge theory on the D3 branes has been used to describe gluon dynamics and the D3-D7 strings quarks - some progress in the study of the properties of mesons in 3+1d strongly coupled gauge theories has been achieved \cite{3d3d}. The D3-D5 defect system seems a natural starting point therefore for 2+1 dimensional systems.

The D3-D5 world volume theory has an unbroken SO(3) global symmetry. We will imagine gauging an SO(2)/U(1) subgroup of this to play the role of QED in the solid state system. We will introduce a chemical potential for the quasi-particles with respect to the U(1) - this can be done by simply spinning the D5 branes in the SO(2) plane \cite{spin}. The embedding of the D5 brane is described by a scalar that is charged under this U(1) symmetry so one naively expects to trigger superconductivity in the spirit described in \cite{2dcond7} - but here we would have an explicit understanding of the UV degrees of freedom the scalar describes. Naively one expects the scalar describing the D5 embedding to be destabilized by the presence of a chemical potential which gives the scalar a negative mass squared. We find the minimum area embedding for such spinning probe D5 branes though and find this is not the case.

The crucial physics is that the speed of light decreases as one moves into the centre of AdS - eventually it becomes less than the rotation speed of the D5 brane. We show, following the higher dimensional analysis in \cite{spin,spin2,spin3}, that there are regular D5 embeddings into the interior

\textsuperscript{1} Mea Culpa: in the first preprint version of this paper we advertised our computation as applying to the M2-M5 intersection - this was incorrect since we had dropped a crucial factor of 2 in the $S^7$ radius. We are grateful to Veselin Filev for breaking this to us gently! In fact though the precise zero temperature action studied matches that of the D3-D5 intersection as we now describe in the text - the computations can be so easily translated since the form of the probe D5 action is in this context prescribed by the dimension of the intersection and the search for flat embeddings at zero chemical potential - the supersymmetric D3-D5 intersection matches these conditions. We have updated the thermal computation although the results are qualitatively the same. We hope to return to the more complicated M2-M5 system in the future.

\cite{adscft1,adscft2,adscft3,mixed1,mixed2,mixed3,2dcond1,2dcond2,2dcond3,2dcond4,2dcond5,2dcond6,2dcond7,spin,spin2,spin3}
which have a more complicated embedding structure. The branes bend in the direction of the rotation so that there are two linked scalar fields describing the embedding - this richer theory turns out to not include superconductivity, a subtlety on top of the arguments in [2].

We can introduce mass terms for the quasi-particles that explicitly break the U(1) symmetry and we discuss the embeddings in these cases. There is a first order phase transition when the chemical potential grows above the mass of the quasi-particle bound states - below the transition the quasi-particles exist as deconfined particles whilst above it they are confined into bound states. This transition is analogous to the meson melting transition seen in this system and the D3-D7 system at finite temperature [18, 19, 20, 21]. We also analyze the finite temperature behaviour of these solutions by using the AdS5 Schwarzschild geometry as the background.

**THE D3 THEORY**

We will represent the strong interaction dynamics with the large N \( \mathcal{N} = 4 \) super Yang Mills theory on the surface of a stack of D3 branes. It is described at zero temperature by AdS5 × \( S^5 \)

\[
ds^2 = \frac{(\rho^2 + r^2)}{L^2} dx_{3+1}^2 + \frac{L^2}{(\rho^2 + r^2)} (d\rho^2 + \rho^2 d\Omega_2^2 + dr^2 + r^2 d\Omega_2^2) \tag{1}
\]

where we have written the geometry to display the directions the D3 lie in \((x_{3+1})\), those we will embed the D5 on \((x_{2+1}, \rho \) and \( \Omega_2)\) and those transverse \((r \) and \( \Omega_2)\). \( L \) is the AdS radius.

At finite temperature the description is given by the AdS-Schwarzschild black hole

\[
ds^2 = \frac{u^2}{L^2} (-h(u) dt^2 + dx_3^2) + \frac{L^2}{u^2 h(u)} du^2 + L^2 d\Omega_5^2 \tag{2}
\]

\[
h(u) = 1 - \frac{u_0^4}{u^4} \tag{3}
\]

It is helpful to make the change of variables to isotropic coordinates

\[
\frac{u \ du}{\sqrt{u^4 - u_0^4}} = \frac{dw}{w} \tag{4}
\]

and choose the integration constant such that if \( u_0 = 0 \) the zero-temperature geometry is recovered

\[
2w^2 = u^2 + \sqrt{u^4 - u_0^4} \tag{5}
\]

The metric can now be written as

\[
ds^2 = \frac{1}{L^2} \left( w^2 + \frac{u_0^4}{4w^4} \right) \left( -\frac{w^4 - u_0^4}{w^4 + u_0^4} \right) dt^2 + dx_3^2 + \frac{\frac{L^2}{w}}{w^4 + u_0^4} \left( dp^2 + \rho^2 d\Omega_2^2 + dr^2 + r^2 d\Omega_2^2 \right) \tag{6}
\]

with \( w^2 = \rho^2 + r^2 \), which shares the coordinate structure of (1).

**QUENCHED MATTER FROM A D5 PROBE AT T=0**

We will introduce quenched matter via a probe D5 brane. The underlying brane configuration is as follows:

\[
\begin{array}{ccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\text{D3} & - & - & - & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\text{D5} & - & - & - & \bullet & \bullet & - & - & - & \bullet & \bullet
\end{array}
\]

In polar coordinates the D5 fills the radial direction of AdS5 and is wrapped on a two sphere.

The action for the D5 is just it’s world volume

\[
S \sim T \int d^6 \xi \sqrt{-\det G} \sim \int d\rho \rho^2 \sqrt{1 + r'^2} \tag{7}
\]

where \( T \) is the tension and we have dropped angular factors on the two-sphere.

This is clearly minimized when \( r \) is constant so the D5 lies straight. The value of the constant is the size of the mass gap for the quasi-particles. We will mainly be interested in the conformal case where the constant is zero. Note the general large \( \rho \) solution is of the form

\[
r = m + \frac{c}{\rho} + .. \tag{8}
\]

Here \( m \) is an explicit mass term for the quasi-particles in the Lagrangian and \( c \) the expectation value for a bi-quasi-particle operator - note \( m \) has dimension one and \( c \) dimension two adding to three as required for a Lagrangian term in 2+1d. The solution with non-zero \( c \) is not normalizable in pure AdS5. Note that when \( m = c = 0 \) the theory is conformal. Including a non-zero \( m \) or \( c \) breaks the SO(3) symmetry ie it breaks one transverse SO(2) symmetry. From this it is apparent that \( m \) and \( c \) carry charge under that U(1). Were \( c \) to be non-zero when \( m = 0 \) it would be an order parameter for the spontaneous breaking of the U(1) symmetry.
CHEMICAL POTENTIAL/SPIN

Our theory as yet lacks the relevant perturbation of the Fermi surface and the U(1) of QED. We will associate the U(1) with a subgroup of the SO(3) of the $\Omega_3$ - for concreteness we will use the angle in the $x_7 - x_8$ directions.

To include a chemical potential we will spin the D5 brane in the angular direction $\phi$ of this U(1) with angular speed $\mu$.

The spinning of the D5 branes implies that the quasi-particles see a chemical potential. This is in fact a little bit of a peculiar limit since the background D3 theory also has fields, including scalars, charged under the U(1). We are not allowing that geometry to backreact to the chemical potential. In fact we had better not - the pure D3 theory has a moduli space for separating the D3s in the transverse 6-plane. Were we to set them spinning they would scatter to infinity since there is no central force to support rotation. In the theory on the D3 surface there is a run away Bose-Einstein condensation. We simply wish to switch off this physics - it is not what we are interested in - so we forbid such backreaction. The D3 theory is in an unstable state but will nevertheless provide some strongly coupled interactions for the quasi-particles that do see the chemical potential.

An Overly Naive Ansatz

We first look for solutions where the D5 embedding has $\phi = \mu t$ and we will allow the position $r$ (the radial distance in $x_7 - x_8$) to be a function of $\rho$. The action is

$$ S \sim \int d\rho \, \rho^2 \sqrt{(1 + r')^2(1 - \frac{L^4}{(\rho^2 + r^2)^2} r'^2 \mu^2)} \quad (9) $$

Naively one is expecting the centrifugal force from the spinning to eject the brane from the axis at all but the end points where the boundary conditions hold the brane. This would lead to a spontaneous symmetry breaking or superconducting state. We will see that this is what this naive system tries to achieve.

The equation of motion for $r$ as a function of $\rho$ is easily computed but unrevealing. At large $\rho$ the solutions tend to the no-rotation limit $r \sim m + \frac{c}{\rho}$.

The (pair of) circle(s) in the $(\rho, r)$ plane described by $L^4 \mu^2 r^2 = (\rho^2 + r^2)^2$ is clearly a zero of the action so branes wrapped there provide a solution to the equation of motion. Anything going within the locus described by the two circles is moving faster than the local speed of light and is presumably not physical. This locus is a stationary limit surface - we call it the ergosurface below.

There exist “Karch-Katz” type solutions [12] for D5-branes that do not encounter the stationary limit surface - these solutions essentially lie flat above everything plotted in Fig 1. We want to know what happens to those which have a close encounter with the ergosurface. It turns out that the curves which minimize the action like to hit the surface at a right angle. They then kink onto the surface where they can have zero action.

The relation between $m$ and $c$ for curves impacting on the stationary limit surface in this way is shown in Fig 1 - the presence of non-zero $c$ at $m = 0$ appears to indicate spontaneous breaking of the U(1) symmetry i.e superconductivity. Note there is a first order phase transition between the Karch-Katz embeddings and those hitting the ergosurface - we will discuss this transition further below.

![FIG. 1: Embeddings of D5 branes impacting on the ergosurface in blue and the lowest Karch-Katz embedding in red (top). At the bottom is a plot of $c$ vs $m$ for embeddings (on the left) impacting on the ergosurface and Karch-Katz embeddings (on the right). The solutions oscillate around the value for the lowest Karch-Katz solution as the D5 approaches the very top of the ergosurface.

The problem of course here is that the solutions are singular at the ergosurface where they kink. This is a sign that our ansatz is wrong - none of this is the right physics.

A More Sophisticated Ansatz

We will now try a more sophisticated ansatz where the brane has in addition some profile $\phi(w)$ where $\phi$ is the
angle on which they spin (i.e. $\phi = \mu t + \phi(w)$). The ansatz is inspired by the work in [15] where similar issues are encountered when a magnetic field is switched on on the brane’s world-volume.

We find it numerically convenient to switch coordinates and write the AdS geometry as

$$ds^2 = \frac{w^2}{L^2} dx^2_{3+1} + \frac{L^2}{w^2} \left( dw^2 \right. \left. + w^2 \left( d\theta^2 + \sin^2 \theta d\Omega^2_2 + \cos^2 \theta d\tilde{\Omega}^2_2 \right) \right)$$

the D5 will now be embedded in the $x_{2+1}$, $w$ and $\Omega_2$ directions - the naive solutions above are recovered by looking for solutions that have $\theta(w)$ and $\dot{\phi} = \mu t$ where $\phi$ is the ‘first’ angle of the $\Omega_2$.

In these coordinates the Lagrangian for our more ambitious ansatz for the rotating D5 embedding is

$$L = w^2 \sin^2 \theta \times \sqrt{\frac{1 - \frac{L^4 \mu^2 \cos^2 \theta}{w^2}}{1 + w^2 \theta'^2} + w^2 \cos^2 \theta \phi'^2}$$

If $\phi' \sim \mu$ the two $\mu^2$ terms compete against each other removing the naive intuition about centrifugal force.

Since the action only depends on $\phi'$ and not $\phi$ one can integrate the equation of motion for $\phi'$. One could then substitute back in for $\phi'$ in terms of the integration constant - this though gives an action with a “zero over zero” form at the ergosurface that is hard to work with. Instead, following [15, 16], we Legendre transform to $L' \equiv L - \phi' \frac{\partial L}{\partial \phi'}$. This gives (setting $\frac{\partial L}{\partial \phi'} = J$)

$$L' = \frac{1}{w \cos \theta} \sqrt{\frac{1 - \frac{L^4 \mu^2 \cos^2 \theta}{w^2}}{1 + w^2 \theta'^2}} \times \sqrt{w^6 \sin^4 \theta \cos^2 \theta - J^2}$$

This has a “zero times zero” form at the ergosurface which is simpler to work with numerically.

For a solution that crosses the ergosurface we demand that the action be positive everywhere and this fixes $J$ - the two terms must pass through zero and switch signs together. Having fixed $J$ in this way one can then look at the $\theta$ equation of motion near the ergosurface. Expanding near the surface, and after some algebra, one finds the following consistency equation for the $\theta$ derivative

$$w^2 \theta'^2 + \tan \theta \ w \ \theta' - 1 = 0$$

There are thus two allowed gradients at the ergosurface. In fact numerically we find choosing any gradient focuses on to the same flow both within and outside the ergosurface. We can numerically shoot in and out from a point near the ergosurface in order to generate regular embeddings.

In the three-dimensional $(w, \theta, \phi)$ subspace the ergosurface is the torus given by $L^2 \mu \cos \theta = \pm w$, which in a plane of constant $\phi$ gives two adjacent circles of radius...
Fig. 2 shows a sequence of regular solutions in the \((\rho, r(\rho))\) coordinates of the previous section. To obtain regular solutions one should make an odd continuation to the negative quadrant as shown. We show a full D5 embedding in Fig. 3 with both the \(\theta(w)\) and \(\phi(w)\) dependence plotted - note the D5 rotates at speed \(\mu\) in the \(\phi\) direction (around the axis of the torus).

\[
\frac{\mu L^2}{2} \quad \text{--- THERMAL BEHAVIOUR ---}
\]

One can perform the same analysis in the thermal background. Writing \(b^4 \equiv \frac{\mu L^2}{T^4}\), there is again a torus-like ergosurface given by the equation

\[
L^2 \mu \cos \theta = \pm \frac{1}{w} \frac{w^4 - b^4}{\sqrt{w^4 + b^4}}
\]

and also a spherical horizon at \(w = b\). One finds the horizon always lies within the ergosurface because the local speed of light is zero at the horizon. Note, below we find no phase transition when raising the temperature through the scale of the chemical potential. There would be a transition from a runaway Bose-Einstein condensation to a stable theory were we to allow the chemical potential to backreact on the geometry.

One can form the Legendre-transformed Lagrangian (which recovers the \(T = 0\) case for \(b = 0\))

\[
\mathcal{L} = \frac{1}{w c_{\theta} g} \frac{w^4 + b^4}{w^4 - b^4} \left( 1 - L^4 \mu^2 c_{\theta}^2 g w^2 \frac{(w^4 + b^4)}{(w^4 - b^4)} \right) \left( \sqrt{1 + w^2 \theta'^2} \right) \left( 1 + \sqrt{(1 + w^2 \theta'^2)} \right) \left( \frac{w^4}{w^2} \right) - J^2 \frac{w^4}{w^4 + b^4}
\]

The presence of a non-trivial profile \(\phi(w)\) for the embeddings that penetrate the ergosurface indicates on the field theory side of the duality that there is a vev for the scalar field associated with the phase of the condensate \(c\) - this would be the Goldstone mode if there were spontaneous symmetry breaking. Note that the regular Karch Katz embeddings, away from the ergosurface, have \(\phi(w)\) constant so there is no such vev.

Again we see there is a first order transition between the Karch-Katz type solutions and those that enter the ergosurface region. We plot the values of \(c\) vs \(m\) for these solutions in Fig. 4 - it shows the same spiral structure around the first order transition as we saw with the naive ansatz. We will discuss the meaning of this transition below in the thermal context.

Clearly there is no spontaneous symmetry breaking in these solutions - the solutions smoothly map onto the solution which lies along the axis as the mass parameter \(m\) is taken to zero. In the field theory presumably the conformal symmetry breaking parameter (\(\mu\)) which might trigger symmetry breaking is the same parameter as that telling us there’s a plasma density cutting off the theory - there’s no room for dynamics. This model turns out not to be an example of the behaviour studied in [3].

The embeddings which extremize the action fall into two types - Karch-Katz type embeddings and those which hit the ergosurface. Fluctuations of the former would reveal a bound state spectrum. The latter embeddings inevitably fall onto the event horizon (a selection of these is plotted in Fig. 5 for \(u_0 = 1\)). In addition for these embeddings that pass through the ergosurface \(\theta t\) switches sign on the world volume - the ergosurface acts like a
horizon for the world volume fields \cite{17}. Here fluctuations would have a quasinormal spectrum along the lines of \cite{20}.

There is therefore a first order transition in the behaviour of the theory as the quasi-particle mass goes through the scale of the chemical potential or temperature. This transformation is explored in detail in \cite{17}. Note here it seems the transition is always a meson melting transition at finite temperature. At zero temperature the transition is driven by quantum rather than thermal fluctuations and has been described in terms of a metal-insulator transition in \cite{22}.

THE D3-D7 SYSTEM

Much of the above parallels results already found in the D3-D7 system \cite{14,17}. That system describes an $\mathcal{N} = 2$ 3+1d gauge theory with fundamental matter hypermultiplets in the gauge background of $\mathcal{N} = 4$ super-Yang Mills theory. In \cite{14} an analysis similar to our “naive ansatz” was performed suggesting spontaneous symmetry breaking. Those authors have since refined their analysis in a related system with a background electric field \cite{15} and concluded that if regular embeddings are insisted upon the symmetry breaking is not present (see also \cite{16}). Were they to update \cite{14} they would find embeddings analogous to our D5 embeddings above as they indicate in \cite{17}.

SUMMARY

We have proposed probe D5 branes in D3 brane backgrounds as a plausible dual for a strongly coupled quasi-particle theory in 2+1 dimensions. We introduced a chemical potential (spin) with respect to a global U(1) symmetry of the theory and found the resulting regular D5 embeddings. These embeddings do not display spontaneous symmetry breaking and, indeed, at zero temperature and zero intrinsic mass the theory is essentially indifferent to the chemical potential remaining as a state of conformal quasi-particles. We do show a first order phase transition in the massive theory as the quasi-particle mass crosses the value of the chemical potential - on one side the quasi-particles are confined on the other they are not.

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