Timelike-helicity $B \to \pi\pi$ form factor from light-cone sum rules with dipion distribution amplitudes

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We complete the set of QCD light-cone sum rules for $B \to \pi\pi$ transition form factors, deriving a new sum rule for the timelike-helicity form factor $F_t$ in terms of dipion distribution amplitudes. This sum rule, in the leading twist-two approximation, is directly related to the pion vector form factor. Employing a relation between $F_t$ and other $B \to \pi\pi$ form factors, we obtain also the longitudinal-helicity form factor $F_0$. In this way, all four (axial-)vector $B \to \pi\pi$ form factors are predicted from light-cone sum rules with dipion distribution amplitudes. These results are valid for small dipion masses with large momentum.

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1. INTRODUCTION

The aim of this paper is to present a solution of the problem outlined in Ref. [1], concerning the calculation of the timelike-helicity $B^0 \to \pi^+\pi^-$ form factor $F_t$, from light-cone sum rules. In Ref. [1], the method of QCD light-cone sum rules was applied to calculate the $B \to \pi\pi$ transition form factors, starting from a particular correlation function expanded near the light-cone in dipion distribution amplitudes (DAs). The form factor corresponding to the timelike-helicity of the dilepton and denoted as $F_t$ (for a definition and helicity basis of $B \to \pi\pi$ form factors see e.g. Ref. [2]) could not, however, be consistently obtained from that correlation function because of emerging kinematical singularities.

Here we obtain a new light-cone sum rule for $F_t$ employing a modified correlation function,

$$
\Pi_t(q,k_1,k_2) = i \int d^4 x e^{i q \cdot x} \langle \pi^+(k_1)\pi^0(k_2) | T\{ \bar{u}(x) i m_B \gamma_5 b(x), \bar{b}(0)i m_b \gamma_5 d(0) \} \rangle |0\rangle, \tag{1}
$$

where, instead of the axial-vector $b \to u$ transition current, the pseudoscalar one is used. The immediate advantage of this choice is that the hadronic matrix element of the pseudoscalar current is solely determined by the timelike-helicity $B \to \pi\pi$ form factor we are interested in:

$$
\langle \pi^+(k_1)\pi^0(k_2) | \bar{u} \gamma_5 b | B^0(p) \rangle
= \langle \pi^+(k_1)\pi^0(k_2) | \bar{u} m_b \gamma_5 b | B^0(p) \rangle
= \sqrt{q^2} F_t(q^2,k^2,\zeta). \tag{2}
$$

Here $k^2 = (k_1 + k_2)^2$ is the invariant mass squared of the dipion system, and

$$
2\zeta - 1 = \frac{2q \cdot \bar{k}}{\sqrt{\lambda}} = \beta_\pi(k^2) \cos \theta_\pi, \tag{3}
$$

where $\lambda \equiv m_B^2 + q^2 + 2(m_B^2) - 2m_B^2 k^2 - 2q^2 k^2$ is the Källén function, $\beta_\pi(k^2) = \sqrt{1 - 4m_B^2/k^2}$, $\bar{k} = k_1 - k_2$, and $\theta_\pi$ corresponds to the angle between the three-momenta of the neutral pion and the $B$-meson in the dipion center-of-mass frame. Note that we neglect the light $u$ and $d$ quark masses throughout the paper.

The rest of the derivation follows the procedure explained in detail in Ref. [1]. First we calculate the correlation function (1) to the leading order (zeroth order in $\alpha_s$) and in the lowest twist-two approximation. To this end, we contract the $b$-quark fields in the free propagator and perform the integration in Eq. (1). The result for the correlation function (1), which is by itself an invariant amplitude, reads

$$
\Pi_t(p^2, q^2, k^2, \zeta) = \sqrt{2m_b^2} \int_0^1 du \frac{q \cdot k + uk^2}{(q + uk)^2 - m_b^2} \Phi^{1\rightarrow 1}_{\parallel}(u, \zeta, k^2), \tag{4}
$$

in terms of the isospin-one dipion DAs introduced and defined in [3–6]. Importantly, the above expression—as opposed to the correlation function considered in Ref. [1]—is free from kinematical singularities. Also important is that Eq. (4) only depends on the chiral-even dipion DA:

$$
\langle \pi^+(k_1)\pi^0(k_2) | \bar{u}(x) \gamma_\mu | \pi_\mu(0) \rangle
= -2k_\mu \int_0^1 du e^{in(k \cdot x)} \Phi^{1\rightarrow 1}_{\parallel}(u, \zeta, k^2), \tag{5}
$$

which is normalized to the pion vector form factor in the timelike region:
We also use the double expansion of this DA in partial waves and Gegenbauer polynomials [4]:

\[
\Phi^{(1)}(u, \zeta, k^2) = 6u\tilde{u} \sum_{n=0,2,\ldots} \sum_{\ell=1,3,\ldots} B_{ne}^{(1)}(k^2) C_n^{3/2} \times (u-\tilde{u})\beta_\ell(k^2) P_\ell^{(0)}(\cos \theta_\ell),
\]

(7)

where \( u \equiv 1 - u \), \( P_\ell^{(0)} \) are the associated Legendre polynomials, and the normalization of the DA in Eq. (6) fixes the coefficient \( B_{01}^{(1)}(k^2) = F_\pi(k^2) \).

We now insert the B-meson ground state in the correlation function (1) and, using the definition of \( \bar{F}_t \), write down the hadronic dispersion relation in the variable \( p^2 = (q + k)^2 \), the square of the momentum transferred to the B-meson interpolating current,

\[
\Pi_5(p^2, q^2, k^2, \zeta) = \frac{f_B m_B^2 \sqrt{Q^2} F_t(q^2, k^2, \zeta)}{m_B^2 - p^2} + \cdots,
\]

(8)

where \( f_B \) is the B-meson decay constant and the ellipses denote the contributions of radially excited and continuum states with B-meson quantum numbers. The two remaining steps in the derivation of the light-cone sum rule involve (1) employing the quark-hadron duality approximation with a threshold parameter \( s_0 \) and (2) applying the Borel transformation in the variable \( p^2 \). The resulting sum rule reads

\[
\sqrt{Q^2} F_t(q^2, k^2, \zeta) = -\frac{m_B^2}{\sqrt{2f_B m_B^2}} \int_{u_0}^1 du \frac{u^{-2}(u)}{\sin^3 \theta_\pi} \times (m_B^2 - q^2 + u^2 k^2) \Phi^{(1)}(u, \zeta, k^2),
\]

(9)

where \( s(u) = (m_B^2 - uq^2 + u\tilde{u}k^2)/u \), and \( u_0 \) is the solution to \( s(u_0) = s_0 \).

Using the above LCSR for the form factor \( F_t \), and the sum rule for the form factor \( F_\parallel \) obtained in Ref. [1], together with the relation between three form factors valid to the same twist-two accuracy, we calculate the longitudinal-helicity form factor:

\[
\sqrt{Q^2} F_0(q^2, k^2, \zeta) = \frac{1}{m_B^2 - q^2 - k^2} [\sqrt{\frac{2}{3}} \sqrt{Q^2} F_t(q^2, k^2, \zeta) + 2\sqrt{k^2 q^2 (2\zeta - 1)} F_\parallel(q^2, k^2, \zeta)].
\]

(10)

Thus, all four \( B \to \pi \pi \) form factors in the region of small and intermediate \( q^2 \) and small \( k^2 \) are now accessible from LCSR with dipion DAs.

Applying the partial wave expansion to the two form factors under consideration, we write

\[
F_{0,\ell}(q^2, k^2, \zeta) = \sum_{\ell=0}^{\infty} \sqrt{2\ell + 1} F_{0,\ell}^{(1)}(q^2, k^2) P_\ell^{(0)}(\cos \theta_\ell).
\]

(11)

Substituting Eqs. (7) and (11) into (9) and (10), multiplying both sides by \( P_\ell^{(0)}(\cos \theta_\ell) \) and integrating over \( \cos \theta_\ell \), we obtain the \( \ell \)th partial wave contribution to the \( B^0 \to \pi^- \pi^0 \) form factor (note that only odd partial waves \( \ell = 1, 3, 5, \ldots \) contribute for the isovector dipion state):

\[
\sqrt{Q^2} F_t^{(\ell)}(q^2, k^2) = -\frac{6m_B^2}{\sqrt{2f_B m_B^2} \sqrt{2\ell + 1}} \times \sum_{n=1}^{\infty} B_{ne}^{(1)}(k^2) \int_{u_0}^{1} du \frac{u^{-2}(u)}{\sin^3 \theta_\pi} \sqrt{\frac{2}{3}} \sqrt{Q^2} F_t^{(\ell)}(q^2, k^2)
\]

\[
\times (m_B^2 - q^2 + u^2 k^2) \times C_n^{3/2} (u - \tilde{u}),
\]

(12)

\[
\sqrt{Q^2} F_0^{(\ell)}(q^2, k^2) = \frac{\sqrt{\frac{2}{3}} \sqrt{Q^2} F_t^{(\ell)}(q^2, k^2)}{m_B^2 - q^2 - k^2} \times \sum_{\ell=1}^{\infty} I_{\ell\ell'} F_{\parallel}^{(\ell')}(q^2, k^2).
\]

(13)

where

\[
I_{\ell\ell'} = \sqrt{2\ell + 1} \sqrt{2\ell' + 1} \times \int_{-1}^{1} d\cos \theta_\pi \cos \theta_\pi \frac{P_\ell^{(0)}(\cos \theta_\ell) P_{\ell'}^{(0)}(\cos \theta_{\ell'})}{\sin \theta_\pi}.
\]

(14)

Equations (12) and (13) complement the ones for the partial waves of the form factors \( F_{\parallel,\perp}^{(\ell)} \) obtained in Ref [1].

II. RESONANCE CONTRIBUTION

To assess the dominant \( \rho \)-resonance contribution to the \( P \)-wave of \( B \to \pi \pi \) timelike-helicity form factor, we follow Refs. [1,7] and relate the form factor \( F_t \) to the corresponding \( B \to \rho \) form factor \( A_0(q^2) \) (defining the \( B \to \rho \) form factors as in Refs. [8,9]) by means of a resonance model,

\[
\sqrt{Q^2} F_t^{(\ell = 1)}(q^2, k^2) \]

\[=-\frac{\sqrt{\frac{2}{3}} \beta_\pi(k^2) g_{\rho \pi \pi} m_B A_0(q^2)}{\sqrt{m_B^2 - k^2 - i\sqrt{k^2 \Gamma_\rho(k^2)}}} + \cdots,
\]

(15)

where the ellipsis denotes the contributions of excited resonances, and the energy-dependent width \( \Gamma_\rho(k^2) \) (see definition in Eq. (36) of Ref [1]) effectively takes into account the two-pion mixing with the \( \rho \). For \( A_0(q^2) \) we derive a LCSR in terms of the \( \rho \)-meson DA (in the zero-width approximation) taking for consistency the twist-two
valid only up to power corrections of form factor (e.g., Eq. (25) in Ref. [7]). However, this is necessary details on the $\rho$-meson DAs. We find

$$A_0(q^2) = \frac{m_B^2 f_{B}}{2 f_B m_B^2} \times \int_{0}^{1} \frac{du}{u} \exp \left( \frac{m_B^2 - \bar{u} q^2 + \bar{u} m_B^2}{u M^2} \right) \phi^0(u),$$

containing the chiral-even $\rho$-meson DA $\phi^0(u)$, normalized to the $\rho$-meson decay constant $f_{\rho}$.

Note that if we substitute the LCSRs (12) for $F^{(e=1)}_t(q^2, k^2)$ and (16) for $A_0(q^2)$ to the left-hand and right-hand sides of the one-resonance approximation (15), respectively, and use for simplicity the asymptotic two-pion and $\rho$-meson DAs, that is: $B_{n>0,l} = 0$ and $\phi^0(u) = 6u(1-u)$, the resulting relation will restore $B_{00}(k^2) = F^t_x(k^2)$ in the form of the $\rho$-meson contribution to the pion form factor (e.g., Eq. (25) in Ref. [7]). However, this is valid only up to power corrections of $O(q^2/m_b^2)$, $O(k^2/m_b^2)$ and $O(\Delta/m_b)$, where we rescale the effective threshold as $s_0^B = (m_B + \Delta)^2$. The latter correction originates from a global factor of $1/u$ in the integrand.

III. NUMERICAL ANALYSIS

For the numerical analysis of the sum rule (9), we adopt the same input as in Ref. [1] in particular: the $b$-quark mass, decay constant of $B$, Borel parameter range and the duality threshold. The nonperturbative universal input encoded in the functions $B_{n\ell}^{\pi\pi}(k^2)$ entering the Gegenbauer expansion of the DA are taken from the instanton model used in Ref. [4] and are listed in Eqs. (6.7)–(6.12) there. These estimates are only valid near the dipion threshold

$$k^2 \sim 4m_{\pi}^2.$$  

Hence, we are able to predict the $q^2$-dependence of the form factors $F^{(e=1)}_t(q^2, k^2 \sim 4m_{\pi}^2)$ in the region of large recoil $0 \leq q^2 \leq 10–12$ GeV$^2$, where one can trust the light-cone expansion of the correlation function. The results for the $P$-wave form factors are plotted in Fig. 1, where only the uncertainties from the sum rule parameters $s_0^B$ and $M^2$ are shown.

We find that the higher partial waves are also strongly suppressed in $F_t$ as in the other form factors considered in Ref. [1]. For example, the ratio of $\ell = 3$ to $\ell = 1$ contributions to $F_t(q^2, 4m_{\pi}^2)$ is smaller than 5% at all

1For numerical illustration we will take $k^2 = 0.1$ GeV$^2$, slightly above the two-pion threshold, since the phase-space factor $\beta_{\pi}(k^2)$ in Eq. (12) makes the form factor vanish at threshold.
accessible \( q^2 \). The contribution of nonasymptotic terms is also small, as can bee seen by setting to zero all \( B_{nL}^\parallel \) except \( B_{01}^\parallel \) (see the dashed curves in Fig. 1). This allows us to predict the \( P \)-wave form factor also at larger \( k^2 \), including the \( \rho \)-resonance region and even beyond, provided we use for \( B_{01}^\parallel (k^2) \) the accurate data for the pion form factor \( F_\pi (k^2) \) provided by the Belle collaboration [10]. Our result for \( \lim_{q^2 \to 0} q_2 \sqrt{\pi F_t^{(l=1)}(q^2, k^2)} \), as a function of dipion invariant mass squared is shown in Fig. 2.

We have also calculated the \( B \to \rho \) contribution to the form factor \( F_t^{(l=1)}(q^2, k^2) \) using Eq. (15) and find that the remaining resonant and continuum contributions to this form factor can amount up to 20\% in the small \( k^2 \) region, in agreement with the findings of Refs. [1,7] for the other \( B \to \pi \pi \) form factors.

### IV. CONCLUSION

In conclusion, we have completed the set of LCSRs with dipion DAs for the \( B \to \pi \pi \) form factors. These sum rules are complementary to the ones derived in Ref. [7] in terms of \( B \)-meson DAs and complementary to the derivations from dispersion theory [11] and to the calculations at large \( k^2 \) [12]. The form factor \( F_t \) obtained here, while not contributing to the semileptonic \( B \to \pi \ell \nu \) rate in the massless lepton approximation, plays an important role in the factorization formula for nonleptonic \( B \to \pi \pi \pi \) decays [13–15]. Further improvements of the sum rules presented here and in Refs. [1,7] require the inclusion of higher twists and NLO corrections, but most importantly—for the ones derived here and in Ref. [1]—a better knowledge of dipion DAs and their Gegenbauer coefficients \( B_{nL} \).

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