On Periodical Damping Ratio of a Controlled Dynamical System with Parametric Resonances

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Abstract

This report provides an interpretation on the periodically varying damping ratio of a dynamical system with direct control of oscillation or vibration damping. The principal parametric resonance of the system and a new type of parametric resonance, named "zero-th order" parametric resonance, are investigated by using the method of multiple scales to find approximate, analytical solutions of the system, which provide an interpretation on such damping variations.

Keywords - Method of multiple scales, damping, parametric resonance.

I. INTRODUCTION

This study is motivated by the direct control of oscillation or vibration damping with a weakly-damped dynamical system. The dynamics regarding a specific mode can be approximated by a one-degree-of-freedom (1-DOF) system having a direct damping improvement as modeled by (1):

\[ \ddot{x} + (-2\sigma + u)\dot{x} + (\sigma^2 + \omega^2)x = 0 \]  

where \( x \) is a state. Parameter \( u \) is the increased damping by, e.g., a feedback controller that measures real-time damping ratio and eliminates its error from a reference damping ratio. If \( u = 0 \), the eigenvalue \( \lambda \) is calculated by (2), where \( \zeta \) and \( \omega_n \) are the damping ratio and natural angular frequency, respectively.

\[ \lambda_{1,2} = \sigma \pm j\omega = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \]  

Considering the dynamics of \( u \) over time, approximate \( u \) by a periodic function \( K\cos(\Omega t) \), where \( K \) and \( \Omega \) are its amplitude and angular frequency, respectively. Eq. (1) becomes (3). By using the method of multiple scales (MMS), it will be shown that \( a) \) the principal parametric resonance can be excited when \( \Omega \approx 2\omega \) and \( b) \), the "zero-th order" parametric resonance will be excited when \( \Omega \approx 0 \), which is why it is called "zero-th order".

\[ \ddot{x} + (-2\sigma + K\cos(\Omega t))\dot{x} + (\sigma^2 + \omega^2)x = 0 \]  

In the rest of the report, principal and "zero-th order" parametric resonances of the system are respectively investigated in sections II and III, and conclusions are drawn in section IV.

II. PRINCIPAL PARAMETRIC RESONANCE

When \( \Omega \approx 2\omega \), the principal parametric resonance can be observed in the response of \( x \). For instance, let \( \zeta = 0.0098 \), \( \omega_n = 3.8072 \) rad/s, \( K = 0.5 \), and consequently, \( \sigma = 0.0373 \) and \( \omega = 3.8070 \) rad/s. Then, if let \( \Omega = 6.9115 \) rad/s and let the initial value be \( x = 1, \dot{x} = 0 \), the response of \( x \) is shown in Fig. 1 with its envelop being marked and the damping ratio estimated, e.g., from Prony’s method is given in Fig. 2. The periodic variation in the measured damping ratio shows the parametric resonance due to \( \Omega \approx 2\omega \).

The parametric resonance can be investigated by finding an asymptotic solution via the MMS for (3), through which some properties could be revealed. The basic idea of MMS is to find an asymptotic solution of a perturbed system considering different time scales. The use of MMS follows the methodology in [1], [2], and [3]. First, the periodic parameter \( K\cos(\Omega t) \) is treated as a perturbation by inserting a small dimensionless parameter \( \varepsilon > 0 \) like in (4), where \( K_\varepsilon = K/\varepsilon \).

\[ \ddot{x} + (-2\sigma + \varepsilon K_\varepsilon \cos(\Omega t))\dot{x} + (\sigma^2 + \omega^2)x = 0 \]  

Then, a first order uniform solution of (4) is sought in the form of (5):

\[ x(\varepsilon, T_0, T_1) \approx x_0(T_0, T_1) + \varepsilon x_1(T_0, T_1) \]

\[ T_0 = t, T_1 = \varepsilon t \]  

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where $T_1$ is introduced as a slow-scale time variable, such that Eq. (5) could consider the evolution of the solution over long time-scales of the order $\varepsilon^{-1}$. Note that $x_0$ is exactly the solution of Eq. (4) when $\varepsilon = 0$, and $x_1$ is the part caused by the perturbation. Substitute (5) into (4) and equate the coefficients of powers of $\varepsilon$:

$$D_0^2 x_0 - 2\sigma D_0 x_0 + (\sigma^2 + \omega^2) x_0 = 0$$

(6)

$$D_0^2 x_1 - 2\sigma D_0 x_1 + (\sigma^2 + \omega^2) x_1 = -2D_1 D_0 x_0 - K_\varepsilon \cos(\Omega T_0) D_0 x_0 + 2\sigma D_1 x_0$$

(7)

where $D_n = \partial / \partial T_n$.

The solution of (6) can be expressed in a complex form:

$$x_0(T_0, T_1) = A(T_1) e^{(\sigma + j\omega) T_0} + \bar{A}(T_1) e^{(\sigma - j\omega) T_0}$$

(8)

where the bar denotes the complex conjugate. Substitute (8) into (7):

$$D_0^2 x_1 - 2\sigma D_0 x_1 + (\sigma^2 + \omega^2) x_1 = -2(\sigma + j\omega) e^{(\sigma + j\omega) T_0} D_1 A - 2\xi\omega_n e^{(\sigma + j\omega) T_0} D_1 A$$

$$- \frac{K_\varepsilon}{2} \left[ (\sigma + j\omega) e^{(\sigma + j(\Omega + \omega)) T_0} A + (\sigma - j\omega) e^{(\sigma + j(\Omega - \omega)) T_0} \bar{A} \right] + C.C.$$  

(9)

where C.C. denotes the complex conjugates of the former two terms. Introduce a detuning parameter $\xi$ such that $\Omega = 2\omega + \varepsilon \xi$, (9) can be converted to:

$$D_0^2 x_1 - 2\sigma D_0 x_1 + (\sigma^2 + \omega^2) x_1 = -2(\sigma + j\omega) D_1 A - 2\sigma D_1 A + \frac{K_\varepsilon}{2} \left[ (\sigma - j\omega) e^{i\xi T_0} A \right] e^{(\sigma + j\omega) T_0}$$

$$- \frac{K_\varepsilon}{2} (\sigma + j\omega) e^{(\sigma + j(\Omega + \omega)) T_0} A + C.C.$$  

(10)

It can be verified that the condition (11) should be met to avoid generating secular terms in the solution of $x$. The explanation of secular term can be found in [3]. In this problem it will be a term grows linearly in $t$, such that the identified solution will be unbounded, while in fact the true solution is bounded.

$$2(\sigma + j\omega) D_1 A - 2\sigma D_1 A + \frac{K_\varepsilon}{2} (\sigma - j\omega) e^{i\xi T_0} \bar{A} = 0$$

(11)

With the condition (11), the solution of (10) is given below.
\( x_1(T_0, T_1) = \frac{K_x(\sigma + j\omega)A}{2} \left( \frac{e^{(\sigma+j(\Omega+\omega))T_0}}{\Omega(\Omega + 2\omega)} + \frac{e^{(\sigma-j\omega)T_0}}{2\omega(\Omega + 2\omega)} - \frac{e^{(\sigma+j\omega)T_0}}{2\omega\Omega} \right) + C.C. \) \hspace{1cm} (12)

\( x_1 \) could be ignored compared to \( x_0 \), since usually it is numerically small. Hence, assume \( x(t) \approx x_0(t) \). \( A(T_1) \) can be determined by solving (11). For a weakly-damped system, \( \sigma \) can be assumed to be zero and (11) can be converted to:

\[ \frac{dA}{dT_1} = \frac{K_x}{4} e^{j\xi T_1} \bar{A} \] \hspace{1cm} (13)

The analytical solutions of \( A(T_1) \) and \( x(t) \) can be found for three cases, 1) \((\Omega - 2\omega)^2 - K^2/4 > 0\), 2) \((\Omega - 2\omega)^2 - K^2/4 < 0\), and 3) \((\Omega - 2\omega)^2 = K^2/4 = 0\). For the sake of convenience, define \( \omega_K = \sqrt{|\xi^2 - K^2/4|} = \sqrt{|(\Omega - 2\omega)^2 - K^2/4|/\varepsilon} \), and let \( A_0 = A_{r0} + jA_{im0} \) be the given initial value of \( A(T_1) \).

A. Case 1: \((\Omega - 2\omega)^2 - K^2/4 > 0\)

The solution of \( A(T_1) \) is:

\[ A(T_1) = (C_1 e^{j(\tau_1 - \omega_K T_1)} + C_2 e^{j(\tau_2)} e^{j\frac{\xi + \omega_K}{2} T_1}) \] \hspace{1cm} (14)

\[ C_1 = \sqrt{\frac{\xi + \omega_K}{2\omega_K^2}} \sqrt{\xi(A_{r0}^2 + A_{im0}^2) + K_x A_{r0} A_{im0}} \]

\[ C_2 = \sqrt{\frac{\xi - \omega_K}{2\omega_K^2}} \sqrt{\xi(A_{r0}^2 + A_{im0}^2) + K_x A_{r0} A_{im0}} \]

\[ r_1 = \arctan \left( \frac{2A_{im0}(\xi + \omega_K) + K_x A_{r0}}{2A_{r0}(\xi + \omega_K) + K_x A_{im0}} \right) \]

\[ r_2 = \arctan \left( \frac{2A_{im0}(\xi - \omega_K) + K_x A_{r0}}{2A_{r0}(\xi - \omega_K) + K_x A_{im0}} \right) \] \hspace{1cm} (15)

The solution of \( x(t) \) is obtained by substituting (14) into (5) and ignoring \( x_1 \):

\[ x(t) = 2C_1 e^{\sigma t} \cos(\omega_C t + r_1 - \varepsilon \omega_K t) + 2C_2 e^{\sigma t} \cos(\omega_C t + r_2) \] \hspace{1cm} (16)

\[ \omega_C = \omega + \frac{\xi + \omega_K}{2\varepsilon} \] \hspace{1cm} (17)

At the first glance, \( x(t) \) seems to depend on \( \varepsilon \) and \( \xi \). Actually, such dependence can be immediately removed by substituting \( \xi = (\Omega - 2\omega)/\varepsilon \), \( \omega_K = \sqrt{|(\Omega - 2\omega)^2 - K^2/4|/\varepsilon} \), (15) and (17) into (16). The final expression is not given here for the sake of brevity.

The resulting \( x(t) \) consists of two components. The magnitude of the first component is \( 2C_1 \) and the frequency is \( \omega_C - \varepsilon \omega_K \). The magnitude of the second component is \( 2C_2 \) and the frequency is \( \omega_C \). The validity of the approximated solution can be visualized by the case when \( \Omega = 6.9115 \text{ rad/s} \). The comparison of the true response of \( x(t) \) and the approximated \( x(t) \) from (16) is shown in Fig. 3. The approximated \( x(t) \) is almost the same as the true response.

The principal parametric resonance in this case can be interpreted as follows. Without loss of generality, only consider the case when \( C_2 \) is larger than \( C_1 \). The second component can be more dominant than the first component. Since \( \varepsilon \omega_K \) is much smaller than \( \omega_C \), the term \( \varepsilon \omega_K t \) can be viewed as a slow change of phase of the first component. Then, the change of damping of \( x(t) \) can be interpreted as the periodically phase shift between the two components. When \( r_1 - \varepsilon \omega_K t \approx r_2 + 2m\pi, m = 0, \pm 1, \pm 2, \,..., \) the two components are in-phase and the magnitude of \( x(t) \) is amplified. When \( r_1 - \varepsilon \omega_K t \approx r_2 + 2m\pi + \pi, m = 0, \pm 1, \pm 2, \,..., \) the two components are out-of-phase and the magnitude of \( x(t) \) is reduced. This change is periodical at a frequency equal to \( \varepsilon \omega_K = \sqrt{|(\Omega - 2\omega)^2 - K^2/4|} \), and it makes the response of \( x(t) \) to exhibit a periodically varying damping.

B. Case 2: \((\Omega - 2\omega)^2 - K^2/4 < 0\)

The solution of \( A(T_1) \) is:

\[ A(T_1) = (C_3 e^{j\tau_3} + C_4 e^{j(\tau_4 - \omega_K T_1)}) e^{j\frac{\xi + \omega_K}{2} T_1} \] \hspace{1cm} (18)
Fig. 3. Comparison of true response and approximated solution: \((\Omega - 2\omega)^2 - K^2/4 > 0\).

\[
\begin{aligned}
C_3 &= \sqrt{\frac{K_\epsilon(2K_\epsilon + \omega K)}{\omega K^2}} \begin{vmatrix} A_{re0} + A_{im0} \frac{2\xi}{K_\epsilon + 2\omega K} \end{vmatrix} \\
C_4 &= \sqrt{\frac{K_\epsilon(2K_\epsilon - \omega K)}{\omega K^2}} \begin{vmatrix} A_{re0} + A_{im0} \frac{2\xi}{K_\epsilon - 2\omega K} \end{vmatrix} \\
r_3 &= \arctan \left( \frac{-A_{im0}(K_\epsilon - 2\omega K) - 2\xi A_{re0}}{A_{re0}(K_\epsilon + 2\omega K) + 2\xi A_{im0}} \right) \\
r_4 &= \arctan \left( \frac{A_{im0}(K_\epsilon + 2\omega K) + 2\xi A_{re0}}{-A_{re0}(K_\epsilon - 2\omega K) - 2\xi A_{im0}} \right)
\end{aligned}
\] (19)

The solution of \(x(t)\) is obtained by substituting (18) into (5) and ignoring \(x_1\). Similarly to (16), the solution (20) also does not depend on \(\varepsilon\) and \(\xi\).

\[
x(t) = 2C_3e^{(\sigma+\varepsilon\frac{\omega K}{2})t}\cos \left( (\omega + \frac{\Omega}{2})t + r_3 \right) + 2C_4e^{(\sigma-\varepsilon\frac{\omega K}{2})t}\cos \left( (\omega + \frac{\Omega}{2})t + r_4 \right)
\] (20)

The resulting \(x(t)\) consists of two components. The magnitude of the first component is \(2C_3\) and the magnitude of the second component is \(2C_4\). The frequency of both components is \(\omega + \Omega/2\). The two components of \(x(t)\) have damping coefficient different from \(\sigma\). This validity of the approximated solution can be verified by the case when \(\Omega\) is changed to 7.5524 rad/s. The comparison of the true response of \(x(t)\) and the approximated \(x(t)\) from (20) is shown in Fig. 4. The approximated \(x(t)\) is almost the same as the true response. Note that the response diverges since the damping part of the first component becomes positive.

In this case, the response of \(x(t)\) will not exhibit a periodically varying damping, but might exhibit a time-variant damping. If \(C_3\) is much larger than \(C_4\), the first component will be dominant at the early stage and the response of \(x(t)\) will be damped in a fast pace. Then, the first component will take the dominance after the second component is damped out.

C. Case 3: \((\Omega - 2\omega)^2 - K^2/4 < 0\)

The solution of \(A(T_1)\) is:
parametric resonance here. For instance, let $x$ of the true response of $x$ may cause a time-variant damping. 

Fig. 4. Comparison of true response and approximated solution: $(\Omega - 2\omega)^2 - K^2/4 < 0$.

\[
A(T_1) = A_0 e^{j\frac{K \xi T_1}{\Omega}} + \frac{K \xi}{\Omega} (A_0 - jA_0)T_1 e^{j\frac{K \xi T_1}{\Omega}}, \quad \text{if} \quad \Omega - 2\omega = \frac{K}{2}.
\]

The solution of $x(t)$ is obtained by substituting (21) into (3) and ignoring $x_1$:

\[
x(t) = 2e^{\sigma t} \left[ (A_{re0} + \frac{K}{\Omega} (A_{re0} + A_{im0})t) \cos \left( \frac{\Omega}{2} t \right) - (A_{im0} - \frac{K}{\Omega} (A_{re0} + A_{im0})t) \sin \left( \frac{\Omega}{2} t \right) \right], \quad \text{if} \quad \Omega - 2\omega = \frac{K}{2}.
\]

Each of the resulting $x(t)$ consists of two components at the frequency $\Omega/2$. Note part of the results depends on $t$, which may cause a time-variant damping.

This validity of the approximated solution can be verified by the case when $\Omega$ is changed to $7.3639 \text{ rad/s}$. The comparison of the true response of $x(t)$ and the approximated $x(t)$ from (22) is shown in Fig. 5. The approximated $x(t)$ is almost the same as the true response.

Since the condition $(\Omega - 2\omega)^2 - K^2/4 = 0$ can hardly be met, this case is usually ignored in industrial applications.

III. “ZERO-TH ORDER” PARAMETRIC RESONANCE

When $\Omega \approx 0$, the parametric resonance can also be observed in the response of $x$, which is named as “zero-th order” parametric resonance here. For instance, let $\zeta = 0.0098$, $\omega_n = 3.8072 \text{ rad/s}$, $K = 0.5$, and consequently, $\sigma = 0.0373$ and $\omega = 3.8070 \text{ rad/s}$. Then, if let $\Omega = 0.6283 \text{ rad/s}$ and let the initial value be $x = 1, \dot{x} = 0$, the response is shown in Fig. 6 with the envelop being marked and the measured damping ratio from the Prony’s method is given in Fig. 7. The periodic variation in the measured damping ratio shows the parametric resonance due to $\Omega \approx 0$.

Through MMS, some properties of such a resonance is revealed. Again, consider a small dimensionless parameter $\varepsilon$ as in (4), and take the same derivation as from (4) to (9). By introducing a detuning parameter $\xi$ such that $\Omega = \varepsilon \xi$, (9) can be converted to (23).

\[
D_0^2 x_1 - 2\sigma D_0 x_1 + (\sigma^2 + \omega^2) x_1 = - \left[ 2(\sigma + j\omega)D_1 A - 2\sigma D_1 A + \frac{K\varepsilon}{2} (\sigma + j\omega)e^{j\xi T_0} A \right] e^{(\sigma + j\omega) T_0} A + C.C.
\]

It can be verified that the condition (24) should be met to avoid generating secular terms in the solution of $x(t)$. 

\[
A(T_1) = A_0 e^{-\frac{K \xi T_1}{\Omega}} - \frac{K \xi}{\Omega} (A_0 - jA_0)T_1 e^{-\frac{K \xi T_1}{\Omega}}, \quad \text{if} \quad \Omega - 2\omega = -\frac{K}{2}.
\]
Fig. 5. Comparison of true response and approximated solution: \((\Omega - 2\omega)^2 - K^2/4 = 0\).

Fig. 6. “Zero-th order” parametric resonance: response of \(x\).

Fig. 7. “Zero-th order” parametric resonance: measured damping ratio.

\[
2(\sigma + j\omega)D_1 A - 2\sigma D_1 A + \frac{K_z}{2} (\sigma + j\omega)e^{j\xi T_0} A = 0
\]  

Then, the solution of (23) is given below.

\[
x_1(T_0, T_1) = \frac{K_z(\sigma - j\omega) \tilde{A}}{2} \left( \frac{e^{(\sigma + j(\Omega - \omega)) T_0}}{\Omega(\Omega - 2\omega)} + \frac{e^{(\sigma - j\omega) T_0}}{2\omega\Omega} - \frac{e^{(\sigma + j\omega) T_0}}{2\omega(2\omega - \Omega)} \right) + C.C.
\]
$x_1$ could be ignored compared to $x_0$, since usually it is numerically small. Hence, assume $x(t) \approx x_0(t)$. $A(T_1)$ can be determined by solving (24). First convert (24) to (26). Then, the analytical solution of $A(T_1)$ is shown in (27), where $A_0 = A_{re0} + jA_{im0}$ is the initial value of $A(T_1)$.

$$\frac{dA}{dT_1} = \frac{K_x(-\omega + j\sigma)}{4\omega} e^{j\xi T_1} A$$

(26)

$$A(T_1) = e^{\frac{K_x(\sigma+j\omega)}{4\xi\omega}} e^{j\xi T_1} + A_0 - e^{\frac{K_x(\sigma+j\omega)}{4\xi\omega}}$$

(27)

Substitute (27) into (5) and ignore $x_1$:

$$x(t) = 2e^{\frac{K\sqrt{\sigma^2+\omega^2}}{4\omega} \cos(\Omega t + \theta)} \cos\left(\frac{K\sqrt{\sigma^2+\omega^2}}{4\Omega \omega} \sin(\Omega t + \theta) + \omega t\right)$$

$$+ 2e^{\sigma t} \left(A_{re0}\cos(\omega t) - A_{im0}\sin(\omega t) - e^{\frac{K\sigma}{4\Omega}} \cos(\omega t + \frac{K}{4\Omega})\right)$$

(28)

$$\theta = \arctan\left(\frac{\omega}{\sigma}\right)$$

(29)

The solution includes two components. Note that the first component has a periodically varying parameter $\cos(\Omega t + \theta)$ being added to the original damping part $\sigma t$, which could lead to a periodically varying damping in the response of $x(t)$. The $\sin(\Omega t + \theta)$ term in the first component can be viewed as a periodically varying phase of frequency $\Omega$ that leads to a periodic phase shift relative to the second component. Hence, the solution $x(t)$ will exhibit periodically varying damping ratio, and the frequency is close to $\Omega$.

The validity of the approximated solution can be verified by the case when $\Omega$ is changed to 0.6283 rad/s. The comparison of the true response of $x(t)$ and the approximated $x(t)$ from (28) is shown in Fig. 8. The approximated $x(t)$ is almost the same as the true response.

Fig. 8. Comparison of true response and approximated solution: “zero-th order” parametric resonance.

IV. CONCLUSION

The response of a weakly-damped 1-DOF system with direct control of its damping ratio can exhibit parametric resonances under principal parametric excitation and “zero-th order” parametric excitation. It is shown that the principal parametric resonance can be classified into three cases depending on $(\Omega - 2\omega)^2 - K^2/4 > 0$, $< 0$, or $= 0$. Specifically, when $(\Omega - 2\omega)^2 -$
$K^2/4 > 0$, the magnitude of $x$ periodically variation in time at a frequency close to $\sqrt{(\Omega - 2\omega)^2 - K^2/4}$, which manifests periodical changes of its damping ratio. When “zero-th order” parametric resonance is excited, the magnitude of $x(t)$ can periodically vary in time at a frequency close to $\Omega$.

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