Microscopic statistical description of classical matter around black holes

P. Nicolini a,b,c,e,1 and M. Tessarotto a,d

Abstract

The problem of statistical description of classical matter, represented by a N-body system of relativistic point particles falling into a black hole, is investigated, adopting a classical framework. A covariant microscopic statistical description is developed for the classical particle system by introducing the definition of observer-related N-particle microscopic distribution function and by suitably extending to general relativity the concept of Gibbs microscopic entropy. The corresponding entropy production rate is calculated taking into account the presence of an event horizon and proven to obey, for $N \gg 1$, an exact microscopic H-theorem which holds for arbitrary geometries.

Key words:
Relativistic statistical mechanics, Classical black holes, Boltzmann and Gibbs entropy, Black hole thermodynamics.

1 Introduction

Since its inception the thermodynamical interpretation of black holes (BH) has been the subject of debate. Indeed, the mathematical analogy between the laws of thermodynamics and black hole physics following from classical general relativity still escape a complete and satisfactory interpretation. In particular it is not yet clear whether this analogy is merely

1 Corresponding author: email: Piero. Nicolini@cmfd.univ.trieste.it
formal or leads to an actual identification of physical quantities belonging to apparently unrelated frameworks. The difficulty is related to the very concept of entropy usually adopted in BH theory, based on Boltzmann entropy, which is determined by the number \( W \) of microscopic complexions compatible with the macroscopic state of a physical system

\[
S_{bh} = K \ln W.
\]

Indeed \( S_{bh} \) does not rely on a true statistical description of physical systems, but only on the classification of the microstates of the system, quantal or classical. As a consequence, the evaluation of \( S_{bh} \) requires the knowledge of the internal structure of the BH, a result which obviously cannot be achieved in the context of a purely classical description of BH. Therefore the evaluation of \( S_{bh} \) requires a consistent formulation of quantum theory of gravitation and matter \[3, 5]\. This can be based, for example, on string theory \[7\] and can be conveniently tested in the framework of semiclassical gravity \[8, 9\].

A basic difficulty of quantum theories founded on the concept of Boltzmann entropy, is that up not now they have not leaved up to their expectations since they have not yet achieved their primary goal, i.e., the rigorous proof of an H-theorem and the full consistency with the second law of thermodynamics \( \delta S_{bh} \geq 0 \). Indeed, estimates of the Boltzmann entropy based quantum theory of gravitation \[3, 5\] and yielding \( S_{bh} \equiv \frac{1}{4} k c A G \), being \( A \) the area of the event horizon, are inconsistent from this viewpoint, since as a consequence of the BH radiation effect \[4\] the radius of the BH may actually decrease. Hence, the Boltzmann entropy \( S_{bh} \) cannot be interpreted, in a proper sense, as a physical entropy of the BH. To resolve this difficulty a modified constitutive equation for the entropy was postulated \[1, 2\], in order to include the contribution of the matter in the BH exterior, by setting

\[
S' = S + S_{bh},
\]

\((S' \text{ denoting the so-called Bekenstein entropy})\) where \( S \) represents the correction carried by the matter outside the BH (notice, however, that also \( S \) cannot be interpreted as entropy). As a consequence a generalized second law \( \delta S' \geq 0 \) was proposed \[1, 2\] which can be viewed as nothing more than the ordinary second law of thermodynamics applied to a system containing a BH. However, the precise definition and underlying statistical basis both for \( S \) and potentially also of \( S_{bh} \) remain obscure. Thus a fundamental problem appears their precise estimates based on suitable microscopic models.

On the other hand, if one regards the BH as a classical object in the space-time continuum, provided the surrounding falling matter can be assumed as formed by a suitably large number of particles, the estimate of the BH entropy should be achievable directly in the context of classical statistical mechanics, by adopting the customary concept of statistical entropy, i.e., Gibbs entropy.

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In contrast to the Boltzmann entropy, this is based on a statistical description of physical systems and is defined in terms of the probability distribution of the observable microstates of the system. In fact, as is well-known, its definition coincides with the axiomatic definition of Shannon entropy, yielding the measure of ignorance on statistical ensembles. A first result of this type been presented in previous work by Niccolini and Tessarotto (hereon denoted as Ref.I), where a covariant kinetic theory in the presence of an event horizon was developed for classical matter in the BH exterior, treated as a N-body system (with $N \gg 1$) of classical point particles undergoing, before capture, a purely Hamiltonian dynamics. By introducing a suitable definition for the relativistic kinetic entropy of infalling classical matter described as an ensemble of point particles, an H-Theorem was reached, tanking into account the presence of the black hole event horizon.

The goal of the present paper is to extend the results of Ref.I by developing a microscopic (N-body) statistical description of classical matter in the presence of an event horizon. For this purpose we intend to evaluate the Gibbs (microscopic) entropy of classical matter falling into the BH event horizon. In particular, we intend to prove the validity of an exact H-theorem holding for Gibbs entropy, provided the infalling matter can be described as a suitably large classical system ($N \gg 1$) of point (neutral or charged) particles forming plasma or a rarefied gas which interact mutually only via a mean Hamiltonian field.

The scheme of the presentation is as follows. In Sec. II, first the basic assumptions of the theory are introduced, which include the concept of particle capture domain, assumed to be defined by a smooth hypersurface located suitably close to the event horizon (capture surface). Second, a covariant microscopic statistical description is formulated for a N-body system of classical point particles referred to an arbitrary observer. In particular, in the exterior domain of the BH (outside the capture surface) the particle system is assumed to obey a Hamiltonian dynamics. As a consequence, in the same domain the Liouville theorem for the corresponding phase-flow is readily recovered and, based on the concept of observer-related N-particle microscopic distribution function and probability density, the relativistic N-body Liouville equation is determined. Finally suitable boundary conditions are introduced for the microscopic distribution function on the capture surface.

In Sec. III the definitions of observer-related relativistic Gibbs and kinetic entropies for the infalling classical matter are introduced and the corresponding entropy production rates are determined. The relationship between the two entropies is proven to hold in an elementary way for classical systems of weakly interacting point particles. As a consequence the obvious physical interpretation of the contributions of the entropy production rate are pointed out.
Finally, in Sec. IV, an H-theorem is recovered for the Gibbs entropy, extending the result of Ref. I. The result is proven to hold for BH characterized by an event horizon of arbitrary shape and size. In particular, the result is proven to apply, in contrast to Boltzmann entropy in quantum theory of gravitation [5], also to the case of matter falling in a BH with contracting event horizons, such as BH implosions or slow contractions.

2 The N-body covariant microscopic statistical description

In this section we introduce the basic framework of the theory. The assumptions we are going to make deal with the treatment of non-isolated black hole formed by the collapse of a star and surrounded by matter. The BH, together with its associated event horizon, and of the matter surrounding the BH and falling toward the event horizon are all assumed as classical. It is sufficient for our purposes to assume the following hypotheses:

I) Infalling particles capture is due to the redshift phenomenon occurring near the event horizon for an arbitrary observer located far from the BH (for example in a region where space time is asymptotically flat [13]). As a result particles sufficiently close to the event horizon effectively disappear to the observer (Assumption 1). In the sequel we shall assume that all particle capture events occurs in a subdomain, to be identified with a surface γ of the space-time (capture surface), localized infinitesimally close to the event horizon.

II) The total energy of infalling particles is finite in such a way that local distortions of space-time are negligible (Assumption 2).

The matter outside the BH is described by a system of $N \gg 1$ identical classical particles to be referred to an arbitrary observer $O$. If we assume that the system is Hamiltonian, its evolution is well known and results uniquely determined by the classical equations of motion, defined with respect to the observer $O$. To this purpose let us choose $O$, without loss of generality, in a region where space time is (asymptotically) flat, endowing it with the proper time $\tau$, where $\tau$ is assumed to span the set $I \subseteq \mathbb{R}$ (observer’s time axis). Without loss of generality we can assume that the particles are points, i.e., they are described by the 1-particle canonical states $x_i$ (with $i = 1, N$) spanning the 8-dimensional phase space $\Gamma_i$, where $x_i = (r^\mu_i, p^\mu_i)$. The analogous treatment of particles having higher degree of freedom is straightforward. Therefore, the evolution of the system, described in terms of the N-body canonical state $x \equiv \{x_1, \ldots, x_N\}$, is determined by a suitable relativistic Hamiltonian $H = H(x)$, where each canonical 1-particle state $x_i$ ($i = 1, N$) results parameterized in terms of the $i$-th particle world line arc length $s_i$ (see [12]). As a consequence, requiring that $s_i = s_i(\tau)$ results a strictly monotonic function it follows that,
the particle state can be also parameterized in terms of the observer’s time \(\tau\). Therefore, the particle states are determined by the canonical equations:

\[
\frac{ds_i}{d\tau} = \frac{ds_i}{d\tau} \left[ x_i(\tau), H \right]_{x_i},
\]

where \(\left[ f(x_i(\tau), H) \right]_{x_i} = \delta(f(x_i(\tau)) - \delta(x_i(\tau)) = \delta_{oi}\),

and the previous initial-value problem admits by assumption a unique solution defining a \(C^2(\Gamma^N \times I)\)-solution. It follow that the phase-flow defined by the mapping \(x_o = \{x_{o1}, ..., x_{oN}\} \rightarrow x(\tau) \equiv \{x_1(s_1(\tau)), ..., x_N(s_N(\tau))\}\) satisfies a Liouville theorem, namely

\[
\int_{\Gamma^N} dx p_G^{(N)}(x) = 1.
\]

Notice that the Dirac deltas introduced above must be intended as physical realizability equations. In particular the condition placed on the arc lengths \(s_i\) implies that the \(i\)-th particle of the system is parameterized with respect to \(s_i(\tau)\), i.e., it results functionally dependent on the proper time of the observer; instead the constraints placed on 4-velocity implies that \(u_i^\mu\) must belong to the hypersurface \(\delta_i\) of equation \(\sqrt{u_i^\mu u_i^\mu} = 1\), and hence \(u_i^\mu\) is a tangent vector to a timelike geodesic. In the sequel we adopt also the notation

\[
\hat{p}_G^{(N)}(x) = p_G^{(N)}(x)\left|\left\{ \sqrt{u_{1i}^\mu u_{1i}^\mu} = 1, ..., \sqrt{u_{Ni}^\mu u_{Ni}^\mu} = 1 \right\}\right.,
\]
to denote \( \rho^{(N)}(x) \) evaluated on the intersection of the hypersurfaces \( \delta_1, \ldots, \delta_N \).
The event horizon of a classical BH is defined by the hypersurface \( r_H \) identical for all particles and specified by the equation

\[
R_m = r_H, \tag{7}
\]

where for \( m = 1, N \), \( R_m \) denotes the a suitable curvilinear coordinate for the \( m \)-th particle and coincides with the radial coordinate in the spherically symmetric case. According to a classical point of view, let us now assume that the particle capture surface be defined by the surface \( \gamma \) of equation

\[
R_m(s_m) = r_\varepsilon(s_m), \tag{8}
\]

where \( r_\varepsilon = (1 + \epsilon)r_H \), while \( \epsilon > 0 \) is an infinitesimal which may depend on explicit or hidden parameters (for example, \( \epsilon \) might depend on the detector used by the observer). The presence of the BH event horizon is taken into account by defining suitable boundary conditions for the kinetic distribution function on the hypersurface \( \gamma \). For this purpose we distinguish between incoming and outgoing distributions on \( \gamma \) with respect to the \( i \)-th particle, \( \rho^{(N)(+i)}_G(x) \big|_\gamma \) and \( \rho^{(N)(-i)}_G(x) \big|_\gamma \) corresponding respectively to \( n_\alpha u^\alpha_i > 0 \) and \( n_\alpha u^\alpha_i \leq 0 \), where \( n_\alpha \) is a locally radial outward \( 4 \)-vector. Therefore the boundary conditions on \( \gamma \) are specified as follows

\[
\rho^{(N)(-i)}_G(x) \big|_\gamma \equiv \rho^{(N)}(x) \prod_{i=1,N} \delta(s_i - s_i(\tau)) \prod_{j=1,N} \delta(\sqrt{u_j \mu u^\mu_j} - 1), \tag{9}
\]
\[
\rho^{(N)(+i)}_G(x) \big|_\gamma \equiv 0.
\]

As previously anticipated, these boundary conditions do not actually require the detailed physical model for the particle loss mechanism, since all particles are assumed to be captured on the same hyper-surface \( \gamma \), independent of their state. This provides a classical loss model for the BH event horizon.

3 Covariant Liouville equation

It is now immediate to determine the covariant Liouville equation which in the external domains, i.e. outside the event horizon, advances in time the microscopic distribution function \( \rho^{(N)}_G(x) \) with respect to the observer \( O \). Thanks to Liouville theorem (i.e., phase-space volume conservation in \( \Gamma^N \) in the sense indicated above) and invoking as usual the axiom of conservation of probability for classical systems, it follows that \( \hat{\rho}^{(N)}(x) \) must satisfies the
differential Liouville equation

\[ \frac{ds_i}{d\tau} \left\{ \frac{dr_i^\mu}{ds_i} \frac{\partial \hat{\rho}^{(N)}(x)}{\partial r_i^\mu} + \frac{dp_{i\mu}}{ds_i} \frac{\partial \hat{\rho}^{(N)}(x)}{\partial p_{i\mu}} \right\} = 0, \quad (10) \]

where by assumption

\[ \frac{ds_i(\tau)}{d\tau} > 0 \quad (11) \]

is made for all \( i = 1, N \), and the summation is understood over repeated indexes. This equation resumes the conservation of the probability in the relativistic phase space in the domain external to the event horizon. Invoking the Hamiltonian dynamics (2), the kinetic equation takes the conservative form

\[ \frac{ds_i}{d\tau} \left[ \hat{\rho}^{(N)}(x), H \right]_{x_i} = 0. \quad (12) \]

Let us now introduce the assumption, holding for collisionless of weakly interacting particles, either neutral or charged, that the Hamiltonian \( H(x) \) can be expressed in the form

\[ H(x) = \sum_{i=1}^{N} H_i, \quad (13) \]

where \( H_i = H_i(x_i) \). This implies that the particles of the \( N \)-body system interact mutually only via a mean-field Hamiltonian force. It follows that the Liouville equation (10) admits the factorized solution

\[ \hat{\rho}^{(N)}(x) = N \prod_{i=1}^{N} \hat{\rho}(x_i) \quad (14) \]

where \( \hat{\rho}(x_i) = \hat{\rho}^{(1)}(x_i)/N \) is the kinetic (1-particle) probability density and \( \hat{\rho}(x_i) \) is the related kinetic distribution function, which manifestly obeys the covariant kinetic equation of the form \[10, 22\]

\[ \frac{ds_i}{d\tau} \left[ \hat{\rho}^{(1)}(x_i), H_i \right]_{x_i} = 0. \quad (15) \]

Notice that this equation is independent of \( N \), the number of particles, to be assumed in the sequel as finite.

### 4 Observer-related relativistic Gibbs and kinetic entropy

Let us now introduce the definition for the microscopic entropy \( S(\rho^{(N)}) \) appropriate in the context of the present covariant theory. The definition follows by straightforward generalization of the relativistic kinetic entropy \[10\] and the concept of Gibbs entropy in non-relativistic \[14, 15\] and relativistic \[16, 17, 18\] statistical mechanics. Thus the concept of Gibbs microscopic entropy can be
defined in analogy to Ref. I, with respect to an observer endowed with proper time \( \tau \)

\[
S(\rho^{(N)}) = - \int_{\Gamma_N} d\mathbf{x}(s) \prod_{j=1,N} \delta(s_j - s_j(\tau)) \prod_{k=1,N} \delta(\sqrt{u_{k\mu}u^\mu_k} - 1)\rho^{(N)} \ln \rho^{(N)}, \tag{16}
\]

Here the notation is as follows: \( d\mathbf{x}(s) \equiv \prod_{j=1,N} d\mathbf{x}_j(s)_j \), where for each particle \( j = 1, N \) the state vectors \( \mathbf{x}_i \) are parameterized with respect to \( s_i \), with \( s_i \) denoting the \( s \)-particle arc length. Finally, \( P \) denotes the principal value of the integral introduced in order to exclude from the integration domain the subset in which the distribution function vanishes. It is immediate to obtain the relationship between Gibbs and kinetic entropy, \( S(\rho^{(1)}) \), previously defined in Ref.I and given by the equation

\[
S(\rho^{(N)}) = - \int_{\Gamma} d\mathbf{x}_1(s) \delta(s_1 - s_1(\tau)) \delta(\sqrt{u_{1\mu}u^\mu_1} - 1)\rho^{(1)} \ln \rho^{(1)}, \tag{17}
\]

where \( \Gamma \) denotes the 1-particle phase space. In fact the condition of factorization (14) implies immediately

\[
S(\rho^{(N)}) = NS(\rho^{(1)}). \tag{18}
\]

The kinetic entropy can also be written in the equivalent way

\[
S(\rho^{(1)}) = -P \int_{\Gamma} d\mathbf{x}_1(s) \delta(s_1 - s_1(\tau))\rho^{(1)}_1(\mathbf{x}) \ln \rho^{(1)}(\mathbf{x}), \tag{19}
\]

where \( \rho^{(1)}_1(\mathbf{x}(s)) \) reads

\[
\rho^{(1)}_1(\mathbf{x}(s)) = \Theta(r_1(s_1) - r_\epsilon(s_1)) \delta(\sqrt{u_{1\mu}u^\mu_1} - 1)\rho^{(1)}(\mathbf{x}(s)), \tag{20}
\]

where \( \Theta \) denotes the strong Heaviside function

\[
\Theta(a) = \begin{cases} 
1 & \text{for } a > 0 \\
0 & \text{for } a \leq 0.
\end{cases} \tag{21}
\]

Equations (18) and (19) allow to determine immediately the entropy production rate associated to the Gibbs entropy, which reads

\[
\frac{dS(\rho^{(N)})}{d\tau} = -N \int_{\Gamma^-} d^3r_1 d^3\mathbf{p}_1 F_{rr_1} \delta(r_1 - r_\epsilon) \dot{\rho}^{(1)} \ln \dot{\rho}^{(1)} \equiv \dot{S}_1 + \dot{S}_2, \tag{22}
\]

where \( \Gamma^- \) is the subdomain of phase space in which \( n_\alpha u_\alpha^\alpha \leq 0 \) and \( F_{rr_1} \) is the characteristic integrating factor

\[
F_{r_1r_\epsilon} \equiv \frac{d\mathbf{s}_i(\tau)}{d\tau} \left( \frac{dr_i}{ds_i} - \frac{dr_\epsilon}{ds_\epsilon} \right). \tag{23}
\]
It follows that \( \frac{dS(\rho^{(N)})}{d\tau} \) and can be interpreted as the average of the entropy flux across the capture surface \( \gamma \). Moreover, \( \dot{S}_1 \) and \( \dot{S}_1 \) denote respectively the contributions to entropy production rate

\[
\dot{S}_1 = -NP \int_{\Gamma^-} d^3r_1 d^3p_1 \frac{ds_1(\tau)}{d\tau} \frac{d\tau}{ds_1} \delta (r_1 - r_{e1}) \hat{\rho}^{(1)} \ln \hat{\rho}^{(1)},
\]

\[
\dot{S}_2 = NP \int_{\Gamma^-} d^3r_1 d^3p_1 \frac{ds_1(\tau)}{d\tau} \frac{d\tau}{ds_1} \delta (r_1 - r_{e1}) \hat{\rho}^{(1)} \ln \hat{\rho}^{(1)}.
\]

We stress that here by construction \( \frac{ds_1(\tau)}{d\tau} > 0 \) [see Eq.(11)] and \( \frac{d\tau}{ds_1} < 0, \frac{dr}{ds_1} \) denoting the "radial" velocity of infalling matter on the surface \( \gamma \), while there results \( \frac{dr}{ds_1} < 0, = 0 \) or \( > 0 \), being \( \frac{dr}{ds_1} \) the local velocity of the surface \( \gamma \), respectively for contracting, stationary and expanding event horizons. However, the signs of \( \dot{S}_1 \) and \( \dot{S}_2 \) are generally not defined, unless further assumptions are taken into account. Notice that, in analogy to the Bekenstein position (1), \( \dot{S}_1 \) and \( \dot{S}_2 \) denote the contributions to the entropy flux carried by incoming matter and by the BH due to the motion of the event horizon, therefore they can be at least qualitatively related in the following way:

\[
\dot{S}_1 \rightarrow \dot{S},
\]

\[
\dot{S}_2 \rightarrow \dot{S}_{bh},
\]

being \( S \) and \( S_{bh} \) the contributions to Bekenstein entropy (1). In particular, it interesting to remark that both \( \dot{S}_1 \) and \( \dot{S}_2 \) result by construction proportional to \( A \), the area of the event horizon, a conclusion which appears in qualitative agreement with estimate for the BH Boltzmann entropy given above for \( S_{bh} \).

5 H-theorem for the Gibbs entropy

Let us now introduce the assumption that the total number of particles is finite, but suitable large (\( N \gg 1 \)). In such a case, it is possible to determine the signs of \( \dot{S}_1 \), which contributes to the entropy production rate (22). Indeed it is possible to prove that there results \( \dot{S}_1 > 0 \) while \( \dot{S}_2 \) has not a definite sign. In addition, thanks to the results given in the previous section, in particular the relationship between the Gibbs and kinetic entropies, \( S(\rho^{(N)}) \) and \( S(\rho^{(1)}) \),
specified by Eq. (18), the following H-theorem holds for \( S(\rho^{(N)}) \),

\[
\frac{dS(\rho^{(N)})}{d\tau} \equiv \dot{S}_1 + \dot{S}_2 \geq 0. \tag{27}
\]

Moreover, we notice that the support of the kinetic distribution function, i.e., the subset of \( \Gamma^- \) in which the kinetic distribution function is non negative, results always compact. This condition is as a direct consequence of the Assumption 2 here considered, implying that the energy of the falling particles reaching the surface \( \gamma \) cannot become infinite.

Therefore, denoting \( \Omega \) the subset of \( \Gamma^- \) in which the kinetic distribution function \( \rho^{(1)} \) is non-zero, in the complementary set \( \Gamma^- \setminus \Omega \), the kinetic distribution function \( \rho^{(1)} = N p^{(1)} \) (being \( p^{(1)} \) the kinetic probability density) results identically zero. Thanks to Assumption 2, it follows that such a set results necessarily bounded. Therefore, so that the following majorization holds

\[
\frac{dS(\rho^{(1)})}{d\tau} \geq P \int_{\Omega} d^3r d^3p |F_{rr\epsilon}| \delta (r - r_\epsilon) \left[ N p^{(1)} - 1 \right]. \tag{28}
\]

Thus, letting

\[
M_\delta \equiv \int_{\Omega} d^3r d^3p |F_{rr\epsilon}| \delta (r - r_\epsilon) \tag{29}
\]

and imposing that \( N \gg 1 \) be sufficiently large to satisfy the inequality

\[
\frac{dS(\rho^{(1)})}{d\tau} \geq \dot{S} \equiv N \inf \left\{ \int_{\Omega} d^3r n_0 V_{\text{eff}}^r \right\} - M_\delta \geq 0 \tag{30}
\]

the thesis of the H-theorem is reached provided we assume \( \inf \left\{ \int d^3r n_0 V_{\text{eff}}^r \right\} > 0 \), a condition consistent with the requirement of a non isolated BH surrounded by matter. In the previous equation we have introduced the additional notation

\[
P \int_{\Gamma} d^3r_1 d^3p_1 \frac{ds_1(\tau)}{d\tau} \frac{dr_1}{ds_1} \delta (r_1 - r_\epsilon) \hat{\rho}^{(1)} = N \int_{\Omega} d^3r n_0 V_{\text{eff}}^r, \tag{31}
\]

being \( n_0 \) the number density. We stress that this result generalized the H-theorem given in Ref.I, since it applies also to Gibbs entropy. The result holds for classical BH having, in principle, arbitrary shape of the event horizon and even in the presence of a contracting event horizon (which might by produced, for example, by star implosions). The present theory appear therefore potentially relevant for a realistic detailed analysis of the BH thermodynamical properties.
6 Conclusions

In this paper a macroscopic statistical description has been adopted for classical matter around black holes. Matter in the immediate vicinities of a BH event horizon has been modelled by a weakly interacting relativistic gas $S_N$. Its dynamics results described by the relativistic Liouville equation, while the presence of the BH event horizon is taken into account by treating it as a classical absorbing porous wall.

By assuming that Hamiltonian dynamics takes into account only mean field interactions between particles, the connection with the kinetic treatment of Ref.I can be immediately established. As a consequence, an H-theorem valid for the Liouville equation can be established on rigorous grounds which applies to every space time geometry and to the case of contracting horizon.

7 Acknowledgments

Work developed with the support of Ministero dell’Istruzione, dell’Università e della Ricerca (MIUR), via the Programma PRIN 2004: ”Modelli della teoria cinetica matematica nello studio dei sistemi complessi nelle scienze applicate”.

8 Appendix: relativistic N-body Liouville theorem

Let us assume that the point particles constitute an isolated N-body system (assumption $\alpha$) obeying the relativistic Hamiltonian equations of motion

$$\frac{dx_i(s_i)}{ds_i} = X_i, \quad (32)$$

$$x_i(s_{i0}) = x_{i_0}, \quad (33)$$

which implies that all the vector fields $X_i \ (i = 1, N)$ are conservative, i.e., for $i = 1, N$:

$$\frac{\partial}{\partial x_{(i)}} \cdot X_i = 0. \quad (34)$$

Introducing the parametrization in terms of the observer’s time $\tau$ and requiring that the functions $s_i = s_i(\tau)$ are strictly monotonic, the equations of motion can be written in the symbolic form it follows

$$\frac{dx_i(s_i)}{d\tau} = \frac{ds_{(i)}}{d\tau} \frac{dx_i(s_i)}{ds_i} = \frac{ds_{(i)}}{d\tau} X_i. \quad (35)$$
Here $\mathbf{x} = \{x_1, \ldots, x_N\}_i \equiv \{y_1, \ldots, y_{8N}\}$, $\mathbf{X} = \{X_1, \ldots, X_N\}_i \equiv \{Y_1, \ldots, Y_{8N}\}$, where by assumption $\alpha$ the vector field $\mathbf{X}$ depends only on the (local or retarded) states of the particles forming the $N$-body system. As a consequence, let us denote

$$\mathbf{x}(\tau) \equiv \mathbf{x}(s(\tau)) \equiv \{x_1(s_1(\tau)), \ldots, x_N(s_N(\tau))\}$$

the solution of the initial value problem (32)-(33) and $\mathbf{x}(\tau_o) \equiv \mathbf{x}(s(\tau_o)) \equiv \{x_1(s_1(\tau_o)), \ldots, x_N(s_N(\tau_o))\} = \mathbf{x}_0$ the initial condition, where $\mathbf{x}(\tau)$ and $\mathbf{x}(\tau_o)$ denote the $N$–body system states as seen by the observer $O$, respectively at times $\tau, \tau_o$. The previous assumptions for the phase-mapping $\mathbf{x}_o \rightarrow \mathbf{x}(\tau) = \chi(\mathbf{x}_o, \tau_o, \tau)$ imply the following theorem:

8.0.1 THM. - Relativistic N-body Liouville theorem

For arbitrary $\mathbf{x}_o \in \Gamma$ and $\tau_o, \tau \in I \subseteq \mathbb{R}$ there results

$$\left| \frac{\partial \mathbf{x}(\tau)}{\partial \mathbf{x}_o} \right| = 1.$$ (38)

In fact, for $N > 1$ the time derivative of the Jacobian $\left| \frac{\partial \mathbf{x}(\tau)}{\partial \mathbf{x}_o} \right|$ reads

$$\frac{d}{d\tau} \left| \frac{\partial \mathbf{x}(\tau)}{\partial \mathbf{x}_0} \right| = \sum_{i=1,8N} ds_i \left| \frac{\partial (y_1(\tau), \ldots, \frac{du}{ds_i}, \ldots, y_{8N}(\tau))}{\partial \mathbf{x}_0} \right|. $$ (39)

Hence by the chain rule

$$\frac{d}{d\tau} \left| \frac{\partial \mathbf{x}(\tau)}{\partial \mathbf{x}_0} \right| =$$

$$= \sum_{i=1,8N} \frac{ds_i}{d\tau} \sum_{r=1,8N} \frac{\partial Y_i}{\partial y_r(\tau)} \left| \frac{\partial (y_1(\tau), \ldots, \frac{du}{ds_i}, \ldots, y_{8N}(\tau))}{\partial \mathbf{x}_0} \right| =$$

$$= \left| \frac{\partial \mathbf{x}(\tau)}{\partial \mathbf{x}_0} \right| \sum_{i=1,8N} ds_i \frac{\partial Y_i}{\partial y_i} \equiv \left| \frac{\partial \mathbf{x}(\tau)}{\partial \mathbf{x}_0} \right| \sum_{i=1,8N} ds_i \partial \mathbf{x}_i \cdot \mathbf{X}_i,$$

and thanks to the condition of conservation (34)

$$\frac{d}{d\tau} \left| \frac{\partial \mathbf{x}(\tau)}{\partial \mathbf{x}_0} \right| = 0,$$ (41)

which implies the thesis. c.v.d.
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