SUBDIFFUSIVE FRACTIONAL BROWNIAN MOTION REGIME FOR PRICING CURRENCY OPTIONS UNDER TRANSACTION COSTS

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Abstract. A new framework for pricing European currency option is developed in the case where the spot exchange rate follows a subdiffusive fractional Brownian motion. An analytic formula for pricing European currency call option is proposed by a mean self-financing delta-hedging argument in a discrete time setting. The minimal price of a currency option under transaction costs is obtained as time-step
\[
\Delta t = \left( \frac{\alpha - 1}{\Gamma(\alpha)} \right)^{-1} \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \frac{1}{\sigma^{2H}}
\]
which can be used as the actual price of an option. In addition, we also show that time-step and long-range dependence have a significant impact on option pricing.

1. Introduction

The classical and still most popular model of option pricing is the Black–Scholes (BS) [1]. It is assumed that the price of risky asset \( V(t) \) is governed by a geometric Brownian motion, that is
\[
V(t) = V_0 e^{\mu t + \sigma B(t)}, \quad V(0) = V_0 > 0 \tag{1.1}
\]
where \( \mu \) and \( \sigma \) are fixed and \( B(t) \) is the Brownian motion.

Empirical research show that the BS model cannot capture many of the characteristic features of prices, such as: long-range dependence, heavy-tailed and skewed marginal distributions, the lack of scale invariance, periods of constant values, etc. In 1983, Garman and Kohlhagen \((G-K)\) [2] presented a modified version of the BS model for pricing currency option. However, some scholars have argued that option pricing with utilizing the \( G-K \) model based on Brownian motion, cannot satisfactorily model for pricing currency option because currencies differ from stocks in financial markets. Hence, they have proposed some generalization of the \( G-K \) model to capture the phenomena from stock markets [2, 3]. To capture these non-normal behaviors, many researchers have considered other distributions with fat tails such as the Pareto-stable distribution and the Generalized Hyperbolic Distribution among others. Moreover, self-similarity and long-range dependence have become important concepts in analyzing the financial time series. There is strong evidence that the stock return has little or no autocorrelation. Since fractional Brownian motion (FBM) has two important properties called self-similarity and long-range dependence, it has the ability to capture the typical tail behavior of stock prices or indexes.

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The fractional Brownian motion (FBM) model is an extension of the BS model, which displays the long-range dependence observed in empirical data. The FBM model is given by

\begin{equation}
\hat{V}(t) = \hat{V}_0 \exp\{\mu t + \sigma \hat{B}_H(t)\}, \quad \hat{V}_0 > 0,
\end{equation}

where $B_H(t)$ is a FBM with Hurst parameter $H \in (\frac{1}{2}, 1)$. It has been shown that the FBM model admits arbitrage in a complete and frictionless market \cite{3,4,5,6,7}. Wang \cite{8} resolved this contradiction by giving up the arbitrage argument and examining option replication in the presence of proportional transaction costs in discrete time setting \cite{10}.

Magdziarz \cite{11} applied the subdiffusive mechanism of trapping events to describe properly financial data exhibiting periods of constant values and introduced the subdiffusive geometric Brownian motion

\begin{equation}
V_\alpha(t) = V(T_\alpha(t)),
\end{equation}

as the model of asset prices exhibiting subdiffusive dynamics, where $V_\alpha(t)$ is a subordinated process (for the notion of subordinated processes please refer to Refs. \cite{12,13,14}), in which the parent process $V(\tau)$ is the geometric Brownian motion defined in \cite{11} and $T_\alpha(t)$ is the inverse $\alpha$-stable subordinator defined in the following way

\begin{equation}
T_\alpha(t) = \inf\{\tau > 0 : Q_\alpha(\tau) > t\}, \quad 0 < \alpha < 1,
\end{equation}

$Q_\alpha(t)$ is the $\alpha$-stable subordinator with Laplace transform: $\mathbb{E}\{e^{-\eta Q_\alpha(\tau)}\} = e^{-\tau \eta^\alpha}$, $0 < \alpha < 1$, where $\mathbb{E}$ denotes the mathematical expectation. Assuming that $T_\alpha(t)$ is independent of the Brownian motion $B(t)$. Moreover, he demonstrated that this model is free-arbitrage but is incomplete. In this regard, he presented a new formula for fair prices of European option with the corresponding subdiffusive BS model. For additional information about more models that describe such characteristic behavior, you can see \cite{15,16,17,18,19}.

In this study, in order to capture the long-range dependence of interest rates and to examine option replication in the presence of proportional transaction costs in a discrete time setting, we consider the problem of pricing currency option, where the spot exchange rate is governed by a subdiffusive FBM as follows

\begin{equation}
S_t = \hat{V}(T_\alpha(t)) = S_0 \exp\{\mu T_\alpha(t) + \sigma \hat{B}_H(T_\alpha(t))\}, \quad S_0 = \hat{V}(0) > 0.
\end{equation}

Making the change of variable, $B_H(t) = \frac{\mu t - r_d t}{\sigma} + \hat{B}_H(t)$, then we have

\begin{equation}
S_t = \hat{V}(R_\beta(t)) = S_0 \exp\{(r_d - r_f)(T_\alpha(t) + \sigma B_H(T_\alpha(t)))\}, \quad S_0 = \hat{V}(0) > 0.
\end{equation}

When the price of the underlying stock $S_t$ satisfies Eq. \cite{20,21}, we derive an explicit option pricing formula for the European currency call option. This formula is similar to the Black–Scholes option pricing formula, but with the volatility being different.

We denote the subordinated process $W_{\alpha,H}(t) = B_H(T_\alpha(t))$, where $B_H(\tau)$ is a FBM and $T_\alpha(t)$ is the inverse $\alpha$-subordinator, which are supposed to be independent. The process $W_{\alpha,H}(t)$ called a subdiffusion process. Particularly, when $H = \frac{1}{2}$, it is a subdiffusion process presented in \cite{20,21}.

Fig. \cite{1} shows typically the differences and relationships between the sample paths of the spot exchange rate in the FBM model and the subdiffusive FBM model.
The rest of the paper proceeds as follows: In Section 2 we provide an analytic pricing formula for the European currency option in the subdiffusive \textit{FBM} environment and some Greeks of our pricing model are also obtained. Section 3 is devoted to analyze the impact of scaling and long-range dependence on currency option pricing. Moreover, the comparison of our subdiffusive \textit{FBM} model and traditional models is undertaken in this section. Finally, Section 4 draws the concluding remarks.

2. Pricing model for the European call currency option

In this section we derive a pricing formula for the European call currency option of the subdiffusive \textit{FBM} model under the following assumptions:

(i) We consider two possible investments: (1) a stock whose price satisfies the equation:

\begin{equation}
S_t = S_0 \exp \{ (r_d - r_f) T_\alpha (t) + \sigma W_{\alpha, H}(t) \}, \quad S_0 > 0,
\end{equation}

where $\alpha \in (\frac{1}{2}, 1)$, $H \in [\frac{1}{2}, 1)$, $2\alpha - \alpha H > 1$ and $r_d$, and $r_f$ are the domestic and the foreign interest rates respectively. (2) A money market account:

\begin{equation}
dF_t = r_d F_t dt,
\end{equation}

where $r_d$ shows the domestic interest rate.

(ii) The stock pays no dividends or other distributions and all securities are perfectly divisible. There are no penalties to short selling. It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate. These are the same valuation policy as in the \textit{BS} model.

(iii) There are transaction costs which are proportional to the value of the transaction in the underlying stock. Let $k$ denote the round trip transaction cost per unit dollar of transaction. Suppose $U$ shares of the underlying stock are bought ($U > 0$) or sold ($U < 0$) at the price $S_t$, then the transaction cost is given by $k \frac{|U|}{2} S_t$ in either buying or selling. Moreover, trading takes place only at discrete intervals.

(iv) The option value is replicated by a replicating portfolio $\Pi$ with $U(t)$ units of stock and riskless bonds with value $F(t)$. The value of the option must equal the value of the replicating portfolio to reduce (but not to avoid) arbitrage opportunities and be consistent with economic equilibrium.
The expected return for a hedged portfolio is equal to that from an option. The portfolio is revised every $\Delta t$ and hedging takes place at equidistant time points with rebalancing intervals of (equal) length $\Delta t$, where $\Delta t$ is a finite and fixed, small time-step.

Remark 2.1. From [20, 22], we have $E(T^m_{\alpha}(t)) = \frac{t^{\alpha m}}{\Gamma(\alpha + m)}$. Then, by using $\alpha$-self-similar and non-decreasing sample paths of $T_{\alpha}(t)$, we can obtain that $\alpha$-self-similar and non-decreasing sample paths of $T_{\alpha}(t)$,

$$E(\Delta T_{\alpha}(t)) = \frac{1}{\Gamma(1 + \alpha)} [(t + \Delta t)^{\alpha} - t^{\alpha}] = \frac{t^{\alpha - 1}}{\Gamma(\alpha)} \Delta t. \quad (2.3)$$

and

$$E(\Delta B_{H}(T_{\alpha}(t))^2) = \left[\frac{t^{\alpha - 1}}{\Gamma(\alpha)}\right]^{2H} \Delta t^{2H}. \quad (2.4)$$

Let $C = C(t, S_{t})$ be the price of a European currency option at time $t$ with a strike price $K$ that matures at time $T$. Then, the pricing formula for currency call option is given by the following theorem

Theorem 2.1. $C = C(t, S_{t})$ is the value of the European currency call option on the stock $S_{t}$ satisfied (1.6) and the trading takes place discretely with rebalancing intervals of length $\Delta t$. Then $C$ satisfies the partial differential equation

$$\frac{\partial C}{\partial t} + (r_{d} - r_{f})S_{t} \frac{\partial C}{\partial S_{t}} + \frac{1}{2} \tilde{\sigma}^2 S_{t}^{2} \frac{\partial^2 C}{\partial S_{t}^2} - r_{d} C = 0, \quad (2.5)$$

with boundary condition $C(T, S_{T}) = \max\{S_{T} - K, 0\}$. The value of the currency call option is

$$C(t, S_{t}) = S_{t}e^{-r_{f}(T-t)}\Phi(d_{1}) - Ke^{-r_{d}(T-t)}\Phi(d_{2}), \quad (2.6)$$

and the value of the put currency option is

$$P(t, S_{t}) = Ke^{-r_{d}(T-t)}\Phi(-d_{2}) - S_{t}e^{-r_{f}(T-t)}\Phi(-d_{1}), \quad (2.7)$$

where

$$d_{1} = \frac{\ln(S_{t}/K) + (r_{d} - r_{f})(T-t) + \tilde{\sigma}^{2}(T-t)}{\tilde{\sigma}\sqrt{T-t}}, \quad (2.8)$$

$$d_{2} = d_{1} - \tilde{\sigma}(t)\sqrt{T-t}, \quad (2.9)$$

where $\Phi(.)$ is the cumulative normal distribution function.

In what follows, the properties of the subdiffusive FBM model are discussed, such as Greeks, which summarize how option prices change with respect to underlying variables and are critically important to asset pricing and risk management. The model can be used to rebalance a portfolio to achieve the desired exposure to certain risk. More importantly, by knowing the Greeks, particular exposure can be hedged from adverse changes in the market by using appropriate amounts of other related financial instruments. In contrast to option prices that, can be observed in the market, Greeks can not be observed and must be calculated given a model assumption. The Greeks are typically computed using a partial differentiation of the price formula.
**Theorem 2.2.** The Greeks can be written as follows

\[
\Delta = \frac{\partial C}{\partial S_t} = e^{-rf(T-t)}\Phi(d_1),
\]

\[
\nabla = \frac{\partial C}{\partial K} = -e^{-rf(T-t)}\Phi(d_2),
\]

\[
\rho_{rd} = \frac{\partial C}{\partial r_d} = K(T-t)e^{-rf(T-t)}\Phi(d_2),
\]

\[
\rho_{rt} = \frac{\partial C}{\partial r_f} = -St(T-t)e^{-rf(T-t)}\Phi(d_1),
\]

\[
\Theta = \frac{\partial C}{\partial t} = S_t r_f e^{-r_f(T-t)}\Phi(d_1) - K r_d e^{-rf(T-t)}\Phi(d_2)
+ S_t e^{-r_f(T-t)} \alpha^2 (\alpha - 1) \frac{H(\alpha - 1)^{2H - 1}}{\Gamma(\alpha)} \Delta t^{2H-1} \frac{\Delta t^{2H-1}}{\sqrt{T-t}} 
+ S_t e^{-r_f(T-t)} \frac{2\sqrt{2\pi k \sigma \Delta t^{2H-1}}}{\Gamma(\alpha)} \Delta t^{2H-1} \frac{\Delta t^{2H-1}}{\sqrt{T-t}} 
(2.14)
\]

\[
\Gamma = \frac{\partial^2 C}{\partial S_t^2} = e^{-r_f(T-t)} \frac{\Phi(d_1)}{S_t \sigma \sqrt{T-t}},
\]

\[
\vartheta = \frac{\partial C}{\partial \sigma} = S_t e^{-r_f(T-t)} \sqrt{T-t} \Phi(d_1).
\]

**Remark 2.2.** The modified volatility without transaction costs \((k = 0)\) is given by

\[
\hat{\sigma}^2 = \sigma^2 \left[ \frac{H(\alpha - 1)^{2H-1}}{\Gamma(\alpha)} \Delta t^{2H-1} \right],
\]

specially if \(\alpha \uparrow 1\),

\[
\hat{\sigma}^2 = \sigma^2 \Delta t^{2H-1},
\]

which is consistent with the result in [23].

Furthermore, from Eq. \((2.18)\) if \(H \uparrow \frac{1}{2}\), then \(\hat{\sigma}^2 = \sigma^2\) which is according to the results with the \(G-K\) model [2].

Letting \(\alpha \uparrow 1\), from Eq. \((2.19)\), we obtain

**Remark 2.3.** The modified volatility under transaction costs is given by

\[
\hat{\sigma}^2 = \sigma^2 \left[ \Delta t^{2H-1} + \sqrt{\frac{2k}{\pi \sigma}} \Delta t^{H-1} \right],
\]

that is in line with the findings in [9].
3. Empirical Studies

The aim of this section is to obtain the minimal price of an option with transaction costs and to show the impact of time scaling $\Delta t$, transaction costs $k$, and subordinator parameter $\alpha$ on the subdiffusive FBM model. Moreover, in the last part we compute the currency option prices using our model and make comparisons with the results of the $G - K$ and FBM models.

Since $\frac{1}{2} < \sqrt{\pi}$ often holds (For example: $\sigma = 0.1, k = 0.01$, from Eq. (3.5) we have

$$
\frac{\sigma^2}{\bar{\sigma}^2} = \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right)^{2H} \Delta t^{2H-1} + \sqrt{\frac{2}{\pi} \frac{k}{\sigma}} \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right)^H \Delta t^{H-1}
$$

\begin{align}
&\geq 2 \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right)^H \Delta t^{H+1} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{k}{\sigma}\right)^{1/2},
\end{align}

where $H > \frac{1}{2}$. Then the minimal volatility $\hat{\sigma}_{min}$ is $\sqrt{2\sigma} \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right)^{1/2} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{k}{\sigma}\right)^{1/2}$ as

$$
\Delta t = \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right)^{-1} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{k}{\sigma}\right)^{1/2}.
$$

Thus the minimal price of an option under transaction costs is represented as $C_{min}(t, S_t)$ with $\hat{\sigma}_{min}$ in Eq. (3.8).

Moreover, the option rehedging time interval for traders to take is $\Delta t = \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right)^{-1} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{k}{\sigma}\right)^{1/2}$. The minimal price $C_{min}(t, S_t)$ can be used as the actual price of an option.

In particular, since $\Delta t < 1, \alpha (\frac{1}{2}, 1)$ and $\frac{\partial C}{\partial H} = S_t e^{-r(T-t)} \sqrt{2\pi e^{-\frac{H^2}{2}} > 0},$

$$
\frac{\partial \hat{\sigma}}{\partial H} = \sigma \left[ 2 \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right)^{2H} \Delta t^{2H-1} + \sqrt{\frac{2}{\pi} \frac{k}{\sigma}} \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right)^H \Delta t^{H-1} \right] \ln \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right) + \ln \Delta t
\times \left[ 2 \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right)^{2H} \Delta t^{2H-1} + \sqrt{\frac{2}{\pi} \frac{k}{\sigma}} \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right)^H \Delta t^{H-1} \right]^{-1/2}
\times \frac{\sigma^2 \ln \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right) + \ln \Delta t}{2\bar{\sigma}} < 0,
$$

and $\frac{\partial C}{\partial H} = \frac{\partial C}{\partial \hat{\sigma}} \frac{\partial \hat{\sigma}}{\partial H}$, then we have

$$
\frac{\partial C}{\partial H} < 0 \quad \text{as} \quad H \in [\frac{1}{2}, 1),
$$

which displays that an increasing Hurst exponent comes along with a decrease of the option value (See Fig. 2).

On the other hand, if $H \uparrow \frac{1}{2}$, then

$$
\hat{\sigma}_{min} = \sqrt{2\sigma} \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right)^{1/2} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{k}{\sigma}\right)^{1/2} \to \sqrt{2} \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right)
$$

and if $\alpha \uparrow 1$, then $\hat{\sigma}_{min} \to \sqrt{2\sigma}$ as $H \uparrow \frac{1}{2}$.

In addition, if $H \uparrow \frac{1}{2}$

$$
\Delta t = \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right)^{-1} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{k}{\sigma}\right)^{1/2} \to \left(\frac{\alpha - 1}{\Gamma(\alpha)}\right)^{-1} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{k}{\sigma}\right)^{1/2},
$$

and if $\alpha \uparrow 1$, then $\Delta t \to \left(\frac{2}{\pi}\right) \left(\frac{k}{\sigma}\right)^2$ as $H \uparrow \frac{1}{2}$.
Lux and Marchesi [21] have shown that Hurst exponent \( H = 0.51 \pm 0.004 \) in some cases, so the equations (3.4) and (3.5) have a practical application in option pricing. For example: if \( H \uparrow \frac{1}{2}, \alpha \uparrow 1, k = 2\% \) and \( \sigma = 20\% \), then \( \hat{\sigma}_{\text{min}} \rightarrow \frac{\sigma}{\pi} \), and \( \Delta t \rightarrow \frac{\pi}{20} \); and if \( H \uparrow \frac{1}{2}, \alpha \uparrow 1, k = 0.2\% \) and \( \sigma = 20\% \), then \( \hat{\sigma}_{\text{min}} \rightarrow \frac{\sigma}{\pi} \), and \( \Delta t \rightarrow \frac{\pi}{20} \times 10^{-4} \).

In the following, we investigate the impact of scaling and long-range dependence on option pricing. It is well known that Mantegna and Stanley [25] introduced the method of scaling invariance from the complex science into the economic systems for the first time. Since then, a lot of research for scaling laws in finance has begun. If \( \Delta t \), \( \text{in-the-money} \), \( \text{at-the-money} \), First, the prices of our subdiffusive \textit{FBM} model are investigated for some \( \Delta t \) and \( \sigma \) for different exponent parameters. For the sake of simplicity, we will just consider the out-of-the-money case. Indeed, using the same method, one can also discuss the remaining cases: in-the-money and at-the-money. Next, the prices of our subdiffusive \textit{FBM} model are investigated for some \( \Delta t \) and \( \sigma \) for different exponent parameters. The prices of the currency option versus its parameters \( H, \Delta t, \alpha \) and \( k \) are revealed in Fig. 2. The selected parameters are \( S_0 = 1.4, K = 1.5, \sigma = 0.1, r_d = 0.03, r_f = 0.02, T = 1, t = 0.1, \Delta t = 0.01, k = 0.01, H = 0.8, \alpha = 0.9 \). Fig. 2 indicates that, the option price is an increasing function of \( k \) and \( \Delta t \), while, it is a decreasing function of \( H \) and \( \alpha \).

![Figure 2. Call currency option values](image-url)
For a detailed analysis of our model, the prices calculated by the $G-K$, $FBM$, and subdiffusive $FBM$ models are compared for both out-of-the-money and in-the-money cases. The following parameters are chosen: $S_t = 1.2$, $\sigma = 0.5$, $r_d = 0.05$, $r_f = 0.01$, $t = 0.1$, $\Delta t = 0.01$, $k = 0.001$, and $H = 0.8$, along with time maturity $T \in [0.1, 2]$, strike price $K \in [0.8, 1.19]$ for the in-the-money case and $K \in [1.21, 1.4]$ for the out-of-the-money case. Figs. 3 and 4 show the theoretical values difference by the $G-K$, $FBM$, and our subdiffusive $FBM$ models for the in-the-money and out-of-the-money, respectively. As indicated in these figures, the values computed by our subdiffusive $FBM$ model are better fitted to the $G-K$ values than the $FBM$ model for both in-the-money and out-of-the-money cases. Hence, when compared to these figures, our subdiffusive $FBM$ model seems reasonable.

4. Conclusion

Without using the arbitrage argument, in this paper we derive a European currency option pricing model with transaction costs to capture the behavior of the spot exchange
rate price, where the spot exchange rate follows a subdiffusive FBM with transaction costs. In discrete time case, we show that the time scaling $\Delta t$ and the Hurst exponent $H$ play an important role in option pricing with or without transaction costs and option pricing is scaling-dependent. In particular, the minimal price of an option under transaction costs is obtained.

**Appendix**

**Proof of Theorem 2.1.** The movement of $S_t$ on time interval $[t, t + \Delta t]$ of length $\Delta t$ is

$$
\Delta S_t = S_{t+\Delta t} - S_t = S_t \left( e^{(r_d - r_f)\Delta T_\alpha(t) + \sigma \Delta W_{\alpha,H}(t)} - 1 \right)
$$

$$
= S_t \left( (r_d - r_f)\Delta T_\alpha(t) + \sigma \Delta W_{\alpha,H}(t) \right) + \frac{1}{2} \left( (r_d - r_f)\Delta T_\alpha(t) + \sigma \Delta W_{\alpha,H}(t) \right)^2 + \frac{1}{6} S_te^{\theta((r_d - r_f)\Delta T_\alpha(t) + \sigma \Delta W_{\alpha,H}(t))}
$$

$$(4.1)$$

$$
\times \left( (r_d - r_f)\Delta T_\alpha(t) + \sigma \Delta W_{\alpha,H}(t) \right)^3,
$$

here $\theta = \theta(t, \Delta t) \in (0, 1)$ is a random variable according to process $S_t$.

Note that

$$
\frac{1}{6} S_te^{\theta((r_d - r_f)\Delta T_\alpha(t) + \sigma \Delta W_{\alpha,H}(t))} \leq \frac{1}{6} S_te^{(r_d - r_f)\Delta T_\alpha(T) e^{\sigma|\Delta W_{\alpha,H}(t)|}}
$$

$$(4.2)$$

$$
\leq \frac{1}{6} S_te^{(r_d - r_f)T_\alpha(T) e^{\sigma|\Delta W_{\alpha,H}(t)|} e^{\sigma|\Delta W_{\alpha,H}(t + \Delta t)|}}.
$$

and for $m \in N$,

$$
E \left( e^{m(r_d - r_f)T_\alpha(T)} \right) = \sum_{j=0}^{\infty} \frac{(n(r_d - r_f))^j}{j!} \left( T_\alpha(T) \right)^j
$$

$$
= \sum_{j=0}^{\infty} \frac{(n(r_d - r_f)T_\alpha)^j}{\Gamma(j\alpha + 1)}
$$

$$
= E_\alpha(n(r_d - r_f)T_\alpha) < +\infty,
$$

where $E_\alpha(.)$ is the Mittag-Leffler function.

Based on Lemmas 2.1 and 2.2 in 17 and Eq. (4.3), we have

$$
\Delta t^{2\varepsilon} \cdot \frac{1}{6} S_te^{\theta((r_d - r_f)\Delta T_\alpha(t) + \sigma \Delta W_{\alpha,H}(t))} = o(\Delta t^\varepsilon),
$$

$$
\frac{1}{6} S_te^{\theta((r_d - r_f)\Delta T_\alpha(t) + \sigma \Delta W_{\alpha,H}(t)))} \left( (r_d - r_f)\Delta T_\alpha(t) + \sigma \Delta W_{\alpha,H}(t) \right)^3
$$

$$
= \Delta t^{-2\varepsilon} \cdot o(\Delta t^\varepsilon) \cdot \left( o(\Delta t^{\alpha - \varepsilon}) + o(\Delta t^{\alpha - H - \varepsilon}) \right)^3
$$

$$
= o(\Delta t^{2\alpha - 4\varepsilon}) = o(\Delta t).
$$

Then,

$$
\Delta S_t = (r_d - r_f)S_t \Delta T_\alpha(t) + \sigma S_t \Delta W_{\alpha,H}(t)
$$

$$
+ \frac{1}{2} \sigma^2 S_t(\Delta W_{\alpha,H}(t))^2 + o(\Delta t).
$$

$$
(4.6)$$
By using the Taylor expansion we get

\[
\Delta C(t, S_t) = \frac{\partial C}{\partial t} \Delta t + \frac{\partial C}{\partial S_t} \Delta S_t + \frac{1}{2} \frac{\partial^2 C}{\partial S_t^2} \Delta S_t^2 + \frac{1}{2} \frac{\partial^2 C}{\partial t^2} \Delta t^2 + \frac{\partial^2 C}{\partial S_t \partial t} \Delta t \Delta S_t + o(\Delta t^{3+H-\varepsilon})
\]

\[
= \frac{\partial C}{\partial t} \Delta t + \frac{\partial C}{\partial S_t} \Delta S_t + \frac{1}{2} \frac{\partial^2 C}{\partial S_t^2} \Delta S_t^2 + o(\Delta t)
\]

\[
= \frac{\partial C}{\partial t} \Delta t + (r_d - r_f) S_t \frac{\partial C}{\partial S_t} \Delta T_a(t) + \sigma S_t \frac{\partial C}{\partial S_t} \Delta W_{o,H}(t)
\]

\[
+ \frac{1}{2} \sigma^2 S_t \frac{\partial^2 C}{\partial S_t^2} (\Delta W_{o,H}(t))^2 + o(\Delta t).
\]

(4.7)

From Eq. (4.3), we obtain that \(\frac{\partial^2 C}{\partial S_t^2}, \frac{\partial^2 C}{\partial t^2}, \frac{\partial^2 C}{\partial t \partial S_t}\) is \(o(\Delta t^{1-H+o})\) and

\[
(4.8) \quad \Delta \left( \frac{\partial C}{\partial S_t} \right) = \frac{\partial^2 C}{\partial S_t^2} \Delta t + \frac{\partial^2 C}{\partial S_t \partial t} \Delta S_t + \frac{1}{2} \frac{\partial^3 C}{\partial S_t^3} \Delta S_t^2 + o(\Delta t),
\]

and

\[
(4.9) \quad |\Delta \left( \frac{\partial C}{\partial S_t} \right)|_S_{t+\Delta t} = \sigma S_t^2 \left| \frac{\partial^2 C}{\partial S_t^2} \right| |\Delta W_{o,H}(t)| + o(\Delta t).
\]

Moreover, from assumptions (iii) and (iv), it is found that the change in the value of portfolio \(\Pi_t\) is

\[
\Delta \Pi_t = U_t(\Delta S_t + r_f S_t \Delta t) + \Delta F_t - \frac{k}{2} |\Delta U_t| S_{t+\Delta t}
\]

\[
= U_t(\Delta S_t + r_f S_t \Delta t) + r_d F_t \Delta t
\]

\[
= - \frac{k}{2} |\Delta U_t| S_{t+\Delta t} + o(\Delta t),
\]

(4.10)

where the number of bonds \(U_t\) is constant during time-step \(\Delta t\). From assumption (v), \(C(t, S_t)\) is replicated by portfolio \(\Pi(t)\). Thus, at time points \(\Delta t, 2\Delta t, 3\Delta t, \ldots\), we have \(C(t, S_t) = U_t S_t + F_t\) and \(F_t = \frac{\partial C}{\partial S_t}\). Therefore, according to Eqs. (4.9)-(4.10) we have

\[
\Delta \Pi = \frac{\partial C}{\partial S_t} \left( (r_d - r_f) S_t \Delta T_a(t) + \sigma S_t \Delta W_{o,H}(t) + \frac{1}{2} \sigma^2 S_t (\Delta W_{o,H}(t))^2 + r_f S_t \Delta t \right)
\]

\[
+ r_d F_t \Delta t - \frac{k}{2} |\Delta \left( \frac{\partial C}{\partial S_t} \right)|_S_{t+\Delta t} + o(\Delta t)
\]

\[
= \frac{\partial C}{\partial S_t} \left( (r_d - r_f) S_t \Delta T_a(t) + \sigma S_t \Delta W_{o,H}(t) + \frac{1}{2} \sigma^2 S_t (\Delta W_{o,H}(t))^2 + r_f S_t \Delta t \right)
\]

\[
+ (C(t, S_t) - S_t \frac{\partial C}{\partial S_t}) r_d \Delta t - \frac{k}{2} \sigma S_t^2 \left| \frac{\partial^2 C}{\partial S_t^2} \right| |\Delta W_{o,H}(t)| + o(\Delta t).
\]

(4.11)

Consequently,

\[
\Delta \Pi - \Delta C = \left( r_d C - (r_d - r_f) S_t \frac{\partial C}{\partial S_t} - \frac{\partial C}{\partial t} \right) \Delta t - \frac{1}{2} \sigma S_t^2 \frac{\partial^2 C}{\partial S_t^2} (\Delta W_{o,H}(t))^2
\]

\[
- \frac{k}{2} \sigma S_t^2 \left| \frac{\partial^2 C}{\partial S_t^2} \right| |\Delta W_{o,H}(t)| + o(\Delta t).
\]

(4.12)
The time subscript, \( t \) has been suppressed. As expected, using Eq. (4.12), (iv), Remark 2.1 and [27] we infer

\[
E(\Delta H - \Delta C) = (r_d C - (r_d - r_f) S_t \frac{\partial C}{\partial S_t} - \frac{\partial C}{\partial t}) \Delta t
\]

\[
- \frac{1}{2} \left( \frac{\sigma}{\sqrt{2 \pi}} \right)^{2H} \Delta t^{2H-1} \frac{\partial^2 C}{\partial S_t^2} - \frac{1}{2} \sqrt{\frac{2}{\pi}} k \sigma S_t^{\alpha-1} \left[ \frac{t}{\Gamma(\alpha)} \right]^{H} \Delta t^{H} \left| \frac{\partial^2 C}{\partial S_t^2} \right|
\]

\[
= \left( r_d C - (r_d - r_f) S_t \frac{\partial C}{\partial S_t} - \frac{\partial C}{\partial t} \right) - \frac{1}{2} \left( \frac{\sigma}{\sqrt{2 \pi}} \right)^{2H} \Delta t^{2H-1} \frac{\partial^2 C}{\partial S_t^2} - \frac{1}{2} \sqrt{\frac{2}{\pi}} k \sigma S_t^{\alpha-1} \left[ \frac{t}{\Gamma(\alpha)} \right]^{H} \Delta t^{H} \left| \frac{\partial^2 C}{\partial S_t^2} \right|
\]

(4.13)

\[
\frac{1}{2} \left( \frac{\sigma}{\sqrt{2 \pi}} \right)^{2H} \Delta t^{2H-1} \frac{\partial^2 C}{\partial S_t^2}
\]

Thus, from Eq. (4.13) we can derive

\[
r_d C = (r_d - r_f) S_t \frac{\partial C}{\partial S_t} + \frac{\partial C}{\partial t} + \frac{1}{2} \left( \frac{\sigma}{\sqrt{2 \pi}} \right)^{2H} \Delta t^{2H-1} \frac{\partial^2 C}{\partial S_t^2} - \frac{1}{2} \sqrt{\frac{2}{\pi}} k \sigma S_t^{\alpha-1} \left[ \frac{t}{\Gamma(\alpha)} \right]^{H} \Delta t^{H} \left| \frac{\partial^2 C}{\partial S_t^2} \right|
\]

(4.14)

We define \( \tilde{\sigma}^2(t) \) as follows

\[
\tilde{\sigma}^2 = \sigma^2 \left( \frac{t}{\Gamma(\alpha)} \right)^{2H} \Delta t^{2H-1} - \frac{1}{2} \sqrt{\frac{2}{\pi}} k \sigma S_t^{\alpha-1} \left[ \frac{t}{\Gamma(\alpha)} \right]^{H} \Delta t^{H} \left| \frac{\partial^2 C}{\partial S_t^2} \right|
\]

(4.15)

where \( \frac{\partial^2 C}{\partial S_t^2} \) is ever positive for the ordinary European currency call option without transaction costs, if the same conduct of \( \tilde{\sigma}^2 \) is postulated here and \( \tilde{\sigma}(t) \) remains fixed during the time-step \([t, \Delta t]\). Then, from Eqs. (4.14) and (4.15) we obtain

\[
\frac{\partial C}{\partial t} + (r_d - r_f) S_t \frac{\partial C}{\partial S_t} + \frac{1}{2} \tilde{\sigma}^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} - r_d C = 0.
\]

(4.16)

Followed by

\[
C = C(t, S_t) = S_t e^{-r_f(T-t)} \Phi(d_1) - K e^{-r_d(T-t)} \Phi(d_2),
\]

(4.17)

and

\[
d_1 = \frac{\ln(S_t/K) + (r_d - r_f)(T-t) + \tilde{\sigma}^2(t)(T-t)}{\tilde{\sigma} \sqrt{T-t}},
\]

\[
d_2 = d_1 - \tilde{\sigma} \sqrt{T-t}.
\]

(4.18)

**Proof of Theorem 2.2.** First, we derive a general formula. Let \( y \) be one of the influence factors. Thus

\[
\frac{\partial C}{\partial y} = \frac{\partial S_t e^{-r_f(T-t)} \Phi(d_1)}{\partial y} + S_t e^{-r_f(T-t)} \frac{\partial \Phi(d_1)}{\partial y} - \frac{\partial K e^{-r_d(T-t)} \Phi(d_2)}{\partial y} - K e^{-r_d(T-t)} \frac{\partial \Phi(d_2)}{\partial y}
\]

But
\[
\frac{\partial \Phi(d_2)}{\partial y} = \Phi'(d_2) \frac{\partial d_2}{\partial y}
\]
\[
= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \frac{\partial d_2}{\partial y}
\]
\[
= \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(d_1 - \tilde{\sigma}\sqrt{T-t})^2}{2} \right) \frac{\partial d_2}{\partial y}
\]
\[
= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \exp \left( d_2 \tilde{\sigma}\sqrt{T-t} \right) \exp \left( -\frac{\tilde{\sigma}^2(T-t)}{2} \right) \frac{\partial d_2}{\partial y}
\]
\[
= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \exp \left( \ln \frac{S_t}{K} + (r_d - r_f)(T-t) \right) \frac{\partial d_2}{\partial y}
\]
\[
= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} S_t \exp \left( (r_d - r_f)(T-t) \right) \frac{\partial d_2}{\partial y}.
\]
(4.20)

Then
\[
\frac{\partial C}{\partial y} = \frac{\partial S_t e^{-(r_f)(T-t)}}{\partial y} \Phi(d_1) - \frac{\partial Ke^{-r_d(T-t)}}{\partial y} \Phi(d_2)
\]
\[
+ S_t e^{-r_f(T-t)} \Phi'(d_1) \frac{\partial \tilde{\sigma}\sqrt{T-t}}{\partial y}.
\]
(4.21)

Substituting in (4.21) we get the desired Greeks.

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