New necessary and sufficient conditions in order that the real Jacobian conjecture in $\mathbb{R}^2$ holds *

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Abstract This paper is devoted to investigate the two-dimensional real Jacobian conjecture. This conjecture claims that if $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ is a polynomial map such that $\det DF(x, y)$ is nowhere zero and $F(0, 0) = (0, 0)$, then $F$ is a global injective.

Firstly, we provide some new necessary and sufficient conditions such that the real Jacobian conjecture holds. By Bendixson compactification, an induced polynomial differential system can be obtained from the Hamiltonian system associated to polynomial map $F$. We prove that the following statements are equivalent: (A) $F$ is a global injective; (B) the origin of induced system is a center; (C) the origin of induced system is a monodromic singular point; (D) the origin of induced system has no hyperbolic sectors; (E) induced system has a $C^k$ first integral with an isolated minimum at the origin and $k \in \mathbb{N}^+ \cup \{\infty\}$. The above conditions (B)-(D) are local dynamical conditions.

Secondly, applying the above results we present a sufficient condition for the validity of the real Jacobian conjecture. By definition a criterion function, when its limit at the origin of induced system exists, then $F$ is a global injective. This analytical condition improves the main result of Braun et al [J. Differential Equations 260 (2016) 5250-5258]. Moreover, we use this analytical condition to give a new proof of the known algebraic sufficient condition. In this work, our all proofs are based on qualitative theory of dynamical systems.

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1 Introduction and main results

Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map with the Jacobian determinant $\det DF(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. Obviously, the map $F$ is a local diffeomorphism. However, it is not always global injective in $\mathbb{R}^2$. Actually, one can impose suitable conditions to guarantee that $F$ is a global diffeomorphism, see for example [7, 11, 15, 26] and references therein.

In algebraic geometry, the well-known Jacobian conjecture is to state that if $F$ is a polynomial map with $\det DF$ a non-zero constant, then $F$ is a global injective. This conjecture was first introduced by Keller in 1939, and up to now it is still open problem. For Jacobian conjecture there are many positive partial results, see [12, 28, 30, 35], etc. The investigation of Jacobian conjecture leads to a stream of valuable results concerning polynomial automorphisms, as shown in survey [3] and book [31], etc.

Another famous conjecture, the real Jacobian conjecture claims that if $F$ is a polynomial map with nonvanishing Jacobian determinant, then $F$ is a global injective, see [27]. Unfortunately, this conjecture is false. In 1994, Pinchuk [25] provided a counterexample which is a non-injective polynomial map $F$ with nonvanishing Jacobian determinant. Nevertheless, the real Jacobian conjecture has still attracted the interest of numerous mathematicians, especially exploring conditions

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such that this conjecture holds. In [9, 10], the authors were based on the structure of polynomial maps to give sufficient conditions. Gwoździewicz in [18] obtained that the real Jacobian conjecture holds if the degrees of \( f \) and \( g \) are less than or equal to 3. Braun et al. [4, 8] generalized this result by showing that the conjecture is true if the degree of \( f \) is at most 4, independently of the degree of \( g \). In the above mentioned papers, the main technique relates algebra, analysis and geometry. Consider a two-dimensional autonomous differential system

\[
\dot{x} = P(x, y), \quad \dot{y} = Q(x, y). \tag{1.1}
\]

Denoted by \( \mathcal{X} = (P, Q) \) the vector field associated to system (1.1). The vector field \( \mathcal{X} \) is \( C^k \) with \( k \in \mathbb{N}^+ \cup \{\infty\} \) if \( P(x, y) \) and \( Q(x, y) \) are \( C^k \). Let \( U \) be an open set of \( \mathbb{R}^2 \). A non-locally constant function \( H : U \to \mathbb{R} \) is called a first integral of \( \mathcal{X} \) if it is constant along any solution curve of \( \mathcal{X} \) contained in \( U \).

Let \( q \in \mathbb{R}^2 \) be a singular point of an analytic vector field \( \mathcal{X} \) in \( \mathbb{R}^2 \). We say that the singular point \( q \) is a center if there is a neighborhood of \( q \) which is filled up with periodic orbits. The period annulus of the center \( q \) is the maximal neighbourhood \( U \) of \( q \) such that all the orbits contained in \( U \setminus \{q\} \) are periodic. The center \( q \) is a global center if its period annulus is the whole \( \mathbb{R}^2 \). A singular point \( q \) is called a focus if all orbits in a neighborhood of \( q \) spirally approach this singular point either in forward or in backward time.

By Hadamard’s Theorem, Sabatini [29] proved the following dynamical result.

**Theorem 1.** Let \( F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2 \) be a polynomial map with nowhere zero Jacobian determinant such that \( F(0, 0) = (0, 0) \). Then the following statements are equivalent.

(a) The origin is a global center for the Hamiltonian polynomial vector field

\[
\mathcal{X} \triangleq (-ff_y - gg_y, ff_x + gg_x). \tag{1.2}
\]

(b) \( F \) is a global diffeomorphism of the plane onto itself.

This theorem provides a global dynamical condition such that \( F \) is a global injective. Recently, Braun and Llibre proved in [6] that if the homogeneous terms of higher degree of \( f \) and \( g \) do not have real linear factors in common and \( \deg f = \deg g \), then \( F \) is a global injective. Later on, Braun et al. [5] improved this result in the following theorem.

**Theorem 2.** Let \( F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2 \) be a polynomial map such that \( \det DF(x, y) \) is nowhere zero and \( F(0, 0) = (0, 0) \). If the higher homogeneous terms of the polynomials \( ff_x + gg_x \) and \( ff_y + gg_y \) do not have real linear factors in common, then \( F \) is a global injective.

Itikawa et al. [21] give two new classes of polynomial maps satisfying the real Jacobian conjecture in \( \mathbb{R}^2 \). A new proof of Pinchuk map which is a non-injective can be found in [2]. In these works, their proofs rely only on the qualitative theory of planar differential systems, following ideas inspired by Theorem 1. We note that the essential of their proofs are to characterize the global dynamical behavior of Hamiltonian polynomial vector field \( \mathcal{X} \). As we know, the global dynamical analysis of Hamiltonian polynomial vector field in many cases are hard and tedious, because we need to get the local behavior at the all finite and infinite singular points, and to determine their separatrix configurations. For this reason, it is natural to ask whether there exist local dynamical conditions to ensure that \( F \) is a global injective.

The singular point \( q \) is monodromic if there exists a neighborhood of \( q \) such that the orbits of the vector field turn around \( q \) either in forward or in backward time.

Denote by \( b(\mathcal{X}) \) the Bendixon compactification (see subsection 2.3) of Hamiltonian vector field \( \mathcal{X} \). Our first result of this paper provides new necessary and sufficient conditions for the validity of the real Jacobian conjecture.

**Theorem 3.** Let \( F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2 \) be a polynomial map with nowhere zero Jacobian determinant such that \( F(0, 0) = (0, 0) \). Then the following statements are equivalent.
(a) $F$ is a global diffeomorphism of the plane onto itself.

(b) The origin of the polynomial vector field $b(\mathcal{X})$ is a center.

(c) The origin of the polynomial vector field $b(\mathcal{X})$ is a monodromic singular point.

(d) The origin of the polynomial vector field $b(\mathcal{X})$ has no hyperbolic sectors.

(e) The polynomial vector field $b(\mathcal{X})$ has a $C^k$ first integral with an isolated minimum at the origin, where $k \in \mathbb{N}^+ \cup \{\infty\}$.

Remark 1. According to the condition (b) of Theorem 3, it is enough only to need local center not necessary global center. The above conditions (c) and (d) are also local. The last condition (e) is from the point of view of integrability. The vector field $b(\mathcal{X})$ always exists a first integral (see Section 4), but such a first integral in general cannot be extended to the origin of $b(\mathcal{X})$. Theorem 3 implies that the local dynamical behavior of $b(\mathcal{X})$ at the origin determines fully whether polynomial $F$ is a global injective. So it allows us to investigate real Jacobian conjecture by some elementary dynamical tools which is the local analysis of singular points, for example blow up technique. In fact, the origin of vector field $b(\mathcal{X})$ is degenerate (see proof of Theorem 3), that is, its linear part identically zero. The two classical problems for degenerate singular point are respectively monodromy problem to decide whether it is of focus-center type, and stability problem to distinguish between a center and a focus. For degenerate singular point, these two problems are very complicated in general vector field, see [16, 17, 23]. It is worth to notice that the monodromy and stability problems of vector field $b(\mathcal{X})$ at the origin are completely solved by Theorem 3.

Using Theorem 3, we present an analytical condition such that the real Jacobian conjecture holds. Our second result is described as follows.

Theorem 4. Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map with nowhere zero Jacobian determinant such that $F(0,0) = (0,0)$. Define a rational criterion function as follows:

$$I(x,y) = \frac{2}{f^2 \left( \frac{x}{x^2 + y^2} \cdot \frac{y}{x^2 + y^2} \right) + g^2 \left( \frac{x}{x^2 + y^2} \cdot \frac{y}{x^2 + y^2} \right)}.$$  \hspace{1cm} (1.3)

If the limit

$$\lim_{(x,y) \to (0,0)} I(x,y)$$  \hspace{1cm} (1.4)

exists, then $F$ is a global injective and

$$\lim_{(x,y) \to (0,0)} I(x,y) = 0.$$

The following example [10] shows that Theorem 4 improves Theorem 2.

Example 1. Consider the polynomial map $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ with $f = y + y^3$ and $g = x + xy^2$. Here $\text{det} \ DF = - (1 + y^2)(1 + 3y^2)$. The higher homogeneous terms of

$$ff_x + gg_x = x + 2xy + xy^4, \quad ff_y + gg_y = y + 2y (x^2 + 2y^2) + y^3 (2x^2 + 3y^2)$$

have $y$ as a common factor. This map $F$ does not satisfy the condition of Theorem 2. The criterion function is given by

$$I(x,y) = \frac{2(\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2})^5}{\left( (x^2 + y^2)^2 + y^2 \right)^2} \leq 2 \left( \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} \right)^5.$$

Obviously, $\lim_{(x,y) \to (0,0)} I(x,y) = 0$. So $F$ is a global injective.
The polynomial $R(x_1,\ldots,x_n)$ is quasi-homogeneous of weighted degree $l$ with respect to weight exponents $s = (s_1,\ldots,s_n)$ if there exist positive integers $s_1,\ldots,s_n$ and $l$ such that for arbitrary $\lambda \in \mathbb{R}^+ = \{ \lambda \in \mathbb{R}, \lambda > 0 \}$, $R(\lambda^{s_1}x_1,\ldots,\lambda^{s_n}x_n) = \lambda^l R(x_1,\ldots,x_n)$. To each polynomial $R(x_1,\ldots,x_n)$, it can be written as the sum of its quasi-homogeneous parts $R = \sum_{i=0}^{d} R_i$, where $R_i$ is quasi-homogeneous polynomial of weighted degree $i$ with respect to weight exponents $s$. Moreover, the quasi-homogeneous term $R_d$ is called the higher quasi-homogeneous term of polynomial $R$ with respect to weight exponents $s$, and denote by $R_s$. For a polynomial map $F = (f^1,\ldots,f^n) : \mathbb{R}^n \to \mathbb{R}^n$, we denote $F_s = (f_s^1,\ldots,f_s^n)$.

In the third result, we prove the following algebraic condition such that the real Jacobian conjecture holds.

**Theorem 5.** Let $F = (f,g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map with nowhere zero Jacobian determinant such that $F(0,0) = (0,0)$. If there is a weight exponents $s = (s_1,s_2)$ such that the each component of $F_s$ is nonzero and $F_s(0,0)$ has only the trivial solution $(x,y) = (0,0)$, then $F$ is a global injective.

Indeed, Theorem 5 is a special case of the main result of [10]. With the help of the algebraic skills, Cima et al. in [10] proved that if a polynomial map $F = (f^1,\ldots,f^n) : \mathbb{R}^n \to \mathbb{R}^n$ with nowhere zero Jacobian determinant exists a weight exponents $s \in \mathbb{N}_+^n$ such that $F_s(x) = 0$ has only the trivial solution $x = 0$, then $F$ is a global injective. Here, we must emphasize that our method is completely different from [10]. We provide a very elementary dynamical proof, which only relies on the qualitative theory of dynamical systems.

To illustrate Theorem 2, Braun et al. [5] give the next example, which can also be solved by Theorem 5.

**Example 2.** Consider the polynomial map $F = (f,g) : \mathbb{R}^2 \to \mathbb{R}^2$ with $f = x(3+x^2)/3$, $g = x+y$ and $\det DF = 1 + x^2$. Taking weight exponents $s = (1,3)$, we get $F_s = (x^3/3,y)$. The system $F_s(0,0) = (0,0)$ has only the solution $x = y = 0$. Thus $F$ is a global injective.

The following class of polynomial maps satisfy Theorem 5.

**Example 3.** Consider the polynomial map $F = (f,g) : \mathbb{R}^2 \to \mathbb{R}^2$ with $f = x^{2k+1} + y^{2l+1} + x$, $g = y^{2l+1} + y$ and $k,l \in \mathbb{N}$. The Jacobian determinant is $\det DF = ((2k+1)x^{2k+1})((2l+1)y^{2l+1})$. Taking weight exponents $s = (2l+1,2k+1)$, we have $F_s = (x^{2k+1} + y^{2l+1},y^{2l+1})$. The system $F_s(0,0) = (0,0)$ has only the solution $x = y = 0$. Then $F$ is a global injective. For a special case $k = 1$ and $l = 2$, the higher homogeneous terms of

$$ff_x + gg_y = x + 4x^3 + 3x^5 + y^5 + 3x^2y^5, ff_y + gg_x = y + y^4(5x + 6y) + 5x^3y^4 + 10y^9$$

have $y$ as a common factor, which does not satisfy the condition of Theorem 2.

**Remark 2.** Examples 2 and 3 tell us that Theorem 5 improves Theorem 2. For $s = (1,1)$, Theorem 5 becomes the mentioned result of [6]. Actually, Theorem 5 can be obtained from Theorem 4, see Section 4. Moreover, this means that Theorem 4 improves Theorem 2, and generalizes the main result of [6].

Theorem 5 can be attained from the Theorem 4 and the mentioned Cima’s result [10], respectively. Note that Cima’s result [10] does not imply our Theorem 4. In the above Example 1, $F_s$ for any weight exponents $s = (s_1,s_2)$ is always $(y^2,xy^2)$, which has nontrivial real solutions.

The rest of this paper is organized as follows. In section 2 we present some preliminary results. Section 3 is devoted to prove Theorem 3. The proofs of Theorems 4 and 5 will be given in section 4.

## 2 Preliminary results

In this section, we introduce some preliminary results.
2.1 Limit sets

Let \( \varphi(t,p) \) be the integral curve of system (1.1) passing through the point \( p \in \mathbb{R}^2 \) such that \( \varphi(0,p) = p \). The set \( K \) is positively invariant if for each \( p \in K \), \( \varphi(t,p) \in K \) for all \( t \geq 0 \). We define the following sets

\[
\omega(p) = \{ x_0 \in \mathbb{R}^2 : \text{there exist } \{t_n\} \text{ with } t_n \to \infty \text{ and } \varphi(t_n,p) \to x_0 \text{ when } n \to \infty \}
\]

and

\[
\alpha(p) = \{ x_0 \in \mathbb{R}^2 : \text{there exist } \{t_n\} \text{ with } t_n \to -\infty \text{ and } \varphi(t_n,p) \to x_0 \text{ when } n \to \infty \}.
\]

The sets \( \omega(p) \) and \( \alpha(p) \) are called the \( \omega \)-limit set and the \( \alpha \)-limit set of \( p \), respectively. Note that an \( \alpha \)-limit set of an integral curve \( \varphi(t,p) \) is the \( \omega \)-limit set of the integral curve \( \varphi(-t,p) \), i.e., after the time reversal. For this reason, it is sufficient to study the \( \omega \)-limit sets.

The following theorem characterized the structure of \( \omega \)-limit sets, see [13] or [32].

**Theorem 6 (Poincaré-Bendixson Theorem).** Let \( K \) be a positively invariant compact set for system (1.1) containing a finite number of singular points, and \( p \in K \). Then one of the following statements holds.

(a) \( \omega(p) \) is a singular point.

(b) \( \omega(p) \) is a periodic orbit.

(c) \( \omega(p) \) consists of a finite number of singular points \( p_1, \ldots, p_n \) and a finite number of orbits \( \gamma_1, \ldots, \gamma_n \) such that \( \alpha(\gamma_i) = p_i \), \( \omega(\gamma_i) = p_{i+1} \) for \( i = 1, \ldots, n-1 \), \( \alpha(\gamma_n) = p_n \) and \( \omega(\gamma_n) = p_1 \).

Possibly, some of the singular points \( p_i \) are identified.

2.2 Local structure of isolated singular points

To characterize the local structure of isolated singular point \( q \) of analytic vector field \( \mathcal{X} \), we need the following definitions, see [13, 33, 34] for more details. Let \( U \) be a local region limited by two orbits of \( \mathcal{X} \) inside with vertex singular point \( q \).

The local region \( U \) is a hyperbolic sector of singular point \( q \) if all orbits resemble hyperbolae in \( U \), see (1) of Figure 1.

The local region \( U \) is an elliptic sector of singular point \( q \) if all orbits have the singular point \( q \) as both \( \alpha \) and \( \omega \) limit sets, see (2) of Figure 1.

The local region \( U \) is a parabolic sector of singular point \( q \) if all orbits positively (or all negatively) flow into singular point \( q \), see (3) of Figure 1.

Note that the two boundaries of a hyperbolic sector are separatrices of singular point \( q \).

![Figure 1: Sectors near a singular point \( q \).](image)

The next theorem is proved in [19, 22].

**Theorem 7.** If \( q \) is an isolated singular point of analytic vector field \( \mathcal{X} \), then singular point \( q \) can only be a center, a focus, or be decomposed into a finite number of hyperbolic, elliptic and parabolic sectors.
2.3 Compactification of vector field

If \( P(x, y), Q(x, y) \) are real polynomials in variables \( x \) and \( y \), then system (1.1) is polynomial system. We say that system (1.1) has degree \( d \) if \( d = \max \{ \deg P, \deg Q \} \). In order to investigate the behavior of the trajectories of a planar polynomial vector field \( \mathcal{X} \) near infinity, we need to compactify it. Usually there are two methods to compactify a planar polynomial vector field: Poincaré compactification and Bendixson compactification. Next we introduce these two compactifications, see Chapter 13 of [1] or Chapter 5 of [13] for more details.

We first briefly describe the Poincaré compactification. The (Poincaré) unit sphere is tangent to \( xy \)-plane at the origin \((0, 0)\) in Figure 2. The point \( M(x, y) \) in the \( xy \)-plane connects with the center \(O\) of the sphere through a straight line which intersects the sphere at the two points \( M' \) and \( M'' \). We project the point \( M' \) on the lower hemisphere vertically into the \( xy \)-plane which leads to the point \( M'' \) on the Poincaré disk \( \mathbb{D}^2 = \{(x, y) \mid x^2 + y^2 \leq 1\} \). It is known that the boundary of the disc \( \mathbb{D}^2 \), i.e. the unit circle \( \mathbb{S}^1 = \{(x, y) \mid x^2 + y^2 = 1\} \), corresponds to the infinity of \( \mathbb{R}^2 \), and called the equator. Then the vector field \( \mathcal{X} \) can be extended analytically to Poincaré sphere by the central projection. So the global dynamics of \( \mathcal{X} \) can be characterized on the Poincaré disk, that is, the finite and infinity of \( \mathcal{X} \) respectively corresponding the interior and boundary \( \mathbb{S}^1 \) of \( \mathbb{D}^2 \).

![Figure 2: The Poincaré compactification and Poincaré disc.](image)

Roughly speaking, the construction of the Bendixson compactification is as follows. Let \( \mathbb{S}^2 = \{Y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1/4\} \) (the Bendixson sphere). Assume that \( \mathcal{X} \) is defined with the tangent plane to the sphere \( \mathbb{S}^2 \) at the south pole \( S = (0, 0, -1/2) \), that is, \( xy \)-plane, see Figure 3. The Bendixson compactified vector field \( b(\mathcal{X}) \) associated to \( \mathcal{X} \) is an analytic vector field induced on \( \mathbb{S}^2 \) by the stereographic projection. More precisely, consider the stereographic projection \( p_N \) from the north pole \( N = (0, 0, 1/2) \) to the \( xy \)-plane. Thus the vector field \( \mathcal{X} \) can be induced to \( \mathbb{S}^2 \setminus N \) by the map \( p_N^{-1} \). Obviously, the infinity of the \( xy \)-plane is transformed by \( p_N^{-1} \) into the north pole \( N \).

To simplify the calculations, we take the two local charts on the Bendixson sphere \( \mathbb{S}^2 \) given by

\[
U_N = \mathbb{S}^2 \setminus N, \quad U_S = \mathbb{S}^2 \setminus S
\]

with associated local maps

\[
p_N : U_N \to \mathbb{R}^2, \quad p_S : U_S \to \mathbb{R}^2,
\]

where \( p_S \) is the stereographic projection of \( \mathbb{S}^2 \) from the south pole \( S \) to the \( uv \)-plane given by the equation \( y_3 = 1/2 \). The map \( p_S \circ p_N^{-1} \) from the \( xy \)-plane minus \( S \) to the \( uv \)-plane minus \( N \) is given by

\[
u = \frac{y}{x^2 + y^2}, \quad u = \frac{x}{x^2 + y^2}. \quad (2.1)
\]
After a scaling of the independent variable in the local chart \((U_N, p_N)\) the expression for \(b(\mathcal{X})\) is

\[
\begin{align*}
\dot{u} &= (u^2 + v^2)^d \left( (v^2 - u^2) P \left( \frac{u}{u^2+v^2}, \frac{v}{u^2+v^2} \right) - 2uvQ \left( \frac{u}{u^2+v^2}, \frac{v}{u^2+v^2} \right) \right), \\
\dot{v} &= (u^2 + v^2)^d \left( (u^2 - v^2) Q \left( \frac{u}{u^2+v^2}, \frac{v}{u^2+v^2} \right) - 2uvP \left( \frac{u}{u^2+v^2}, \frac{v}{u^2+v^2} \right) \right).
\end{align*}
\]

Hence the infinity of system (1.1) becomes the origin of system (2.2).

**Remark 3.** By the Bendixson compactification, the infinity of system (1.1) on the Poincaré disk, i.e. the equator \(S^1\), is transformed into the origin of system (2.2). We say in what follows, unless otherwise specified, that the infinity of a vector field is refer to the equator \(S^1\) on the Poincaré disk.

### 2.4 Topological index

The following results are well known, see Chapter 6 of [13] or [34].

**Theorem 8** (Poincaré Index Formula). Let \(q\) be an isolated singular point having the finite sectorial decomposition property. Let \(e\), \(h\) and \(p\) denote the number of elliptic, hyperbolic and parabolic sectors of \(q\), respectively. Then the index of \(q\) is \((e - h)/2 + 1\).

**Proposition 1.** If a vector field \(\mathcal{X}\) has only isolated singular point, then the sum of indices singular points in a region \(D\) enclosed by its any periodic orbit is 1.

The next proposition follows immediately from Theorem 8.

**Proposition 2.** For an analytic vector field \(\mathcal{X}\), the index of a monodromic singular point (i.e., a center or a focus) is 1.

### 2.5 Generalized trigonometric functions

The **generalized trigonometric functions** were originally proposed by Lyapunov [23]. Let \(s = (s_1, s_2)\) with \(s_1\) and \(s_2\) coprime positive integer. The following Cauchy problem

\[
\frac{dx}{d\theta} = -y^{2s_1-1}, \quad \frac{dy}{d\theta} = x^{2s_2-1}, \quad x(0) = s_1^{-\frac{1}{2s_2}}, \quad y(0) = 0,
\]

has an unique solution \(x(\theta) = Cs\theta\) and \(y(\theta) = Sn\theta\). For \((s_1, s_2) = (1, 1)\), the solution of system (2.3) is the classical trigonometric functions \(C\theta = \cos \theta\) and \(S\theta = \sin \theta\). The properties of the generalized trigonometric functions can be summarized in the following lemma, for its proof see [23].
Lemma 1. The generalized trigonometric functions $Sn\theta$ and $Cs\theta$ have the following properties.

(a) $s_1 Cs^{2s_2} \theta + s_2 Sn^{2s_1} \theta = 1$.
(b) $\frac{dSn\theta}{d\theta} = Cs^{2s_2-1} \theta$ and $\frac{dCs\theta}{d\theta} = -Sn^{2s_1-1} \theta$.
(c) $Cs(-\theta) = Cs\theta$ and $Sn(-\theta) = -Sn\theta$.
(d) $Sn\theta$ and $Cs\theta$ are $T$-periodic functions with

$$T = T_{s_1,s_2} = 2s_1^{-\frac{1}{2s_2}} s_2^{-\frac{1}{2s_1}} \frac{\Gamma\left(\frac{1}{2s_1}\right) \Gamma\left(\frac{1}{2s_2}\right)}{\Gamma\left(\frac{1}{2s_1}\right) + \Gamma\left(\frac{1}{2s_2}\right)},$$

where $\Gamma$ denotes the gamma function.

3 Proof of Theorem 3

The main purpose of this section is to prove Theorem 3.

Lemma 2. Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map with nowhere zero Jacobian determinant such that $F(0,0) = (0,0)$. If $\gcd(f,g)$ is the greatest common divisor of $f$ and $g$, then $y \nmid \gcd(f,g)$ and $x \nmid \gcd(f,g)$.

Proof. Suppose that $y \mid \gcd(f,g)$. Then there exist two polynomials $\bar{f}_1(x,y)$ and $\bar{g}_1(x,y)$ such that $f = yf_1$ and $g = yg_1$. This is in contraction with $\det DF(0,0) \neq 0$. Thus, $y \nmid \gcd(f,g)$. By a similar way, one can prove that $x \nmid \gcd(f,g)$ also holds. \hfill $\square$

The following theorem is due to Mazzi and Sabatini [24].

Theorem 9. Assume that system (1.1) is $C^k$ with an isolated singular point $q$ and $k \in \mathbb{N}^+ \cup \{\infty\}$. Then the singular point $q$ is a center if and only if there exists a $C^k$ first integral with an isolated minimum at $q$.

Proposition 3. Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map with nowhere zero Jacobian determinant such that $F(0,0) = (0,0)$. Then $q$ is a singular point of the Hamiltonian vector field (1.2) if only if $F(q) = (0,0)$. In this case, this singular point $q$ is a center of vector field (1.2).

Proof. The Hamiltonian vector field $\mathcal{X}$ can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -f_y & -g_y \\ f_x & g_x \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

Indeed, $q$ is a singular point of $\mathcal{X}$ if and only if

$$\begin{pmatrix} -f_y(q) & -g_y(q) \\ f_x(q) & g_x(q) \end{pmatrix} \begin{pmatrix} f(q) \\ g(q) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The sufficiency is obvious. Since $\det DF(q) \neq 0$, $f(q) = g(q) = 0$. The necessity holds.

Since $\det DF(q) \neq 0$, singular point $q$ is an isolated. The Hamiltonian vector field (1.2) has the Hamiltonian

$$H(x,y) = \frac{f^2(x,y) + g^2(x,y)}{2}. \quad (3.1)$$

Clearly, $q$ is an isolated minimum of $H$, that is, $H(x,y) \geq H(q) = 0$. By Theorem 9, $q$ is a center. This proves the Proposition 3. \hfill $\square$
Lemma 3. Let \( f(x, y) = \sum_{i=1}^{n} f_i(x, y), \ g(x, y) = \sum_{j=1}^{m} g_j(x, y) \) and \( F = (f, g) \) with nowhere zero Jacobian determinant \( \det DF(x, y) \), where \( f_i(x, y) \) and \( g_j(x, y) \) are homogeneous polynomials of degree \( i \) and \( j \), respectively. Then \( f_1(x, y) = g_1(x, y) = 0 \) if and only if \( (x, y) = (0, 0) \).

**Proof.** Since \( f_1(x, y) \) and \( g_1(x, y) = 0 \) are linear, we have
\[
\begin{pmatrix}
    f_1(x, y) \\
    g_1(x, y)
\end{pmatrix} = \begin{pmatrix}
    f_{1x} & f_{1y} \\
    g_{1x} & g_{1y}
\end{pmatrix} \begin{pmatrix}
    x \\
    y
\end{pmatrix} = \begin{pmatrix}
    0 \\
    0
\end{pmatrix}.
\]

The determinant of the coefficient matrix of system (3.2) is \( \det DF(0, 0) \neq 0 \). The lemma holds.

Lemma 4. Let \( \phi(t, p) \) be the trajectory of vector field (1.2) passing through the regular point \( p \in \mathbb{R}^2 \) for \( t \in \mathbb{R} \). Then one of the following statements holds.

(a) \( \omega(p) = \alpha(p) \) is a periodic orbit located at a period annulus.

(b) \( \omega(p) \subset S^1 \) and \( \alpha(p) \subset S^1 \).

Here \( S^1 \) is the infinity of the Poincaré disc.

**Proof.** By Proposition 3, each finite singular point \( q \) of vector field (1.2) is a center. From Theorem 6 it follows that \( \omega(p) = \alpha(p) \) is a periodic orbit, or \( \omega(p) \subset S^1 \) and \( \alpha(p) \subset S^1 \).

The limit sets of vector field \( b(\mathcal{X}) \) is given in next proposition.

Proposition 4. Let \( \psi(t, \bar{p}) \) be the trajectory of vector field \( b(\mathcal{X}) \) passing through the regular point \( \bar{p} \in \mathbb{R}^2 \) for \( t \in \mathbb{R} \). Then one of the following statements holds.

(a) \( \omega(\bar{p}) = \alpha(\bar{p}) \) is a periodic orbit located at a period annulus.

(b) \( \omega(\bar{p}) = \alpha(\bar{p}) \) is the origin of vector field \( b(\mathcal{X}) \).

**Proof.** By Lemma 4, on the Bendixson sphere \( S^2 \) (see Figure 3) the \( \alpha \) and \( \omega \) limit sets of each orbit of the vector field \( b(\mathcal{X}) \) can only be the north pole \( N \) or a periodic orbit located at a period annulus. This proposition is confirmed.

Let \( f(x, y) = \sum_{i=1}^{n} f_i(x, y), \ g(x, y) = \sum_{j=1}^{m} g_j(x, y) \) and \( d = \max\{n, m\} \), where \( f_i(x, y) \) and \( g_j(x, y) \) are homogeneous polynomials of degree \( i \) and \( j \), respectively. From the equation (2.2), the Bendixson compactification of vector field \( \mathcal{X} \) is given by
\[
\begin{cases}
    \dot{u} = \sum_{i=1}^{d} \sum_{j=1}^{d} (u^2 + v^2)^{2d-i-j} [((u^2 - v^2) (f_i(u, v) f_{j2}(u, v) + g_i(u, v) g_{j1}(u, v))] \\
    \quad - 2uv (f_i(u, v) f_{jx}(u, v) + g_i(u, v) g_{jx}(u, v))], \\
    \dot{v} = \sum_{i=1}^{d} \sum_{j=1}^{d} (u^2 + v^2)^{2d-i-j} [((u^2 - v^2) (f_i(u, v) f_{jx}(u, v) + g_i(u, v) g_{jx}(u, v))] \\
    \quad + 2uv (f_i(u, v) f_{jy}(u, v) + g_i(u, v) g_{jy}(u, v))],
\end{cases}
\]

(3.3)

It is easy to check that the origin of system (3.3) is degenerate.

The local dynamical behavior of vector field \( b(\mathcal{X}) \) on the Poincaré disk is given as follows.

Proposition 5. Let \( F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2 \) be a polynomial map such that \( \det DF(x, y) \) is nowhere zero and \( F(0, 0) = (0, 0) \). Then the following statements hold.

(a) The finite singular points of polynomial vector field \( b(\mathcal{X}) \) other than its origin are centers.

(b) For polynomial vector field \( b(\mathcal{X}) \), there are no infinite singular points on the Poincaré disc.

(c) The origin of the polynomial vector field \( b(\mathcal{X}) \) has no parabolic sectors.
Proof. (a) By Proposition 3, statement (a) can be easily proved. 

(b) Taking the Poincaré transformation \( u = 1/\bar{v}, \ v = \bar{u}/\bar{v} \) and rescaling the time \( dt = \bar{v}^{d-1}d\tau \), system (3.3) can be written as

\[
\begin{align*}
\dot{u} &= (1 + \bar{u}^2) \sum_{i=1}^{d} \sum_{j=1}^{d} j\bar{v}^i j^{-2} (1 + \bar{u}^2)^{2-2i-2j} (f_i (1, \bar{u}) f_j (1, \bar{u}) + g_i (1, \bar{u}) g_j (1, \bar{u})), \\
\dot{v} &= -\bar{v} \sum_{i=1}^{d} \sum_{j=1}^{d} \bar{v}^i j^{-2} (1 + \bar{u}^2)^{2-2i-2j} [((1 - \bar{u}^2) (f_i (1, \bar{u}) f_j (1, \bar{u}) + g_i (1, \bar{u}) g_j (1, \bar{u})) \\
& - 2\bar{u} (f_i (1, \bar{u}) f_j (1, \bar{u}) + g_i (1, \bar{u}) g_j (1, \bar{u})]].
\end{align*}
\]

Applying Lemma 3, we have that the system (3.4) has no singular points on the \( \bar{u} \)-axis. 

Using the Poincaré transformation \( u = \bar{u}/\bar{v}, \ v = 1/\bar{v} \) with the scaling \( dt = \bar{v}^{d-1}d\tau \), system (3.3) becomes

\[
\begin{align*}
\dot{u} &= -(1 + \bar{u}^2) \sum_{i=1}^{d} \sum_{j=1}^{d} j\bar{v}^i j^{-2} (1 + \bar{u}^2)^{2-2i-2j} (f_i (\bar{u}, 1) f_j (\bar{u}, 1) + g_i (\bar{u}, 1) g_j (\bar{u}, 1)), \\
\dot{v} &= -\bar{v} \sum_{i=1}^{d} \sum_{j=1}^{d} \bar{v}^i j^{-2} (1 + \bar{u}^2)^{2-2i-2j} [((\bar{u}^2 - 1) (f_i (\bar{u}, 1) f_j (\bar{u}, 1) + g_i (\bar{u}, 1) g_j (\bar{u}, 1)) \\
& + 2\bar{u} (f_i (\bar{u}, 1) f_j (\bar{u}, 1) + g_i (\bar{u}, 1) g_j (\bar{u}, 1))].
\end{align*}
\]

Similarly, the origin is not a singular point of system (3.5). The statement (b) holds.

(c) Assume that the origin of \( b(\mathcal{X}) \) has an attracting parabolic sector \( V \). Let \( \psi (t, \bar{p}) \) be the trajectory of vector field \( b(\mathcal{X}) \) passing through the regular point \( \bar{p} \in V \). Then \( \omega (\bar{p}) \) is the origin of \( b(\mathcal{X}) \). From Proposition 4, \( \alpha (\bar{p}) \) is also the origin of \( b(\mathcal{X}) \), which means that \( V \) is an elliptic sector. This is a contradiction. So statement (c) is confirmed. \( \square \)

Proof of Theorem 3. For clarity, we will split the proof into four steps.

Firstly, we prove that statement (a) is equivalent to statement (b). By Theorem 1, (a) \( \Rightarrow \) (b) is obvious.

(b) \( \Rightarrow \) (a). The origin of the polynomial vector field \( b(\mathcal{X}) \) is a center, which means that every solution trajectory of the polynomial vector field \( \mathcal{X} \) in a neighborhood of infinity (the equator \( S^1 \)) is a closed orbit. Thus all finite singular points of \( \mathcal{X} \) are contained in a periodic orbit. From Propositions 1, 2, and 3, it follows that the \( (0, 0) \) is the unique finite singular point of \( \mathcal{X} \), which is a center. Combining with the behavior of the trajectories of \( \mathcal{X} \) near infinity, we obtain that \( (0, 0) \) is a global center of the vector field \( \mathcal{X} \).

Secondly, we show that statement (b) is equivalent to statement (c). By the definition of monodromic singular point, it is obviously that (b) \( \Rightarrow \) (c).

(c) \( \Rightarrow \) (b). Ilyashenko in [20] and Écalle in [14] prove that a monodromic singular point of an analytic vector field must be either a center or a focus. Assume that the origin of \( b(\mathcal{X}) \) is an attracting focus. Let \( U \) be a sufficiently small neighborhood of the origin. For the all regular point \( \bar{p} \in U \setminus \{ O \}, \omega (\bar{p}) = \{ O \} \). By Proposition 4, \( \alpha (\bar{p}) = \{ O \} \) which is a contradiction. Thus the origin of \( b(\mathcal{X}) \) is a center.

Thirdly, we prove that statement (c) is equivalent to statement (d). The (c) \( \Rightarrow \) (d) is obvious.

(d) \( \Rightarrow \) (c). By Theorem 8, the index of the origin of the polynomial vector field \( b(\mathcal{X}) \) is \( e/2 + 1 \). Since the origin of vector field \( \mathcal{X} \) is a center, on the Bendixson sphere \( S^2 \) (see Figure 3) the south pole \( S \) is also a center. This means that there exists a periodic orbit of \( b(\mathcal{X}) \) such that it contains all finite singular points of \( b(\mathcal{X}) \). Let \( c \) denote the number of finite singular points of \( b(\mathcal{X}) \) other than its origin. By statement (a) of Proposition 5 and Proposition 1, we have \( e/2 + 1 + c = 1 \), that is, \( e = c = 0 \). So, the origin is the unique finite singular point of \( b(\mathcal{X}) \). From statement (c) of Proposition 5 and Theorem 7 it follows that the origin is a monodromic singular point.

Finally, it follows from Theorem 9 that statement (e) is equivalent to statement (b).

We complete the proof of Theorem 3. \( \square \)
4 Proofs of Theorems 4 and 5

Our main aim of this section is to prove Theorems 4 and 5. We first give the proof of Theorem 4.

Proof of Theorem 4. Let \( f(x, y) = \sum_{i=1}^{n} f_i(x, y) \), \( g(x, y) = \sum_{j=1}^{m} g_j(x, y) \) and \( d = \max\{n, m\} \), where \( f_i(x, y) \) and \( g_j(x, y) \) are homogeneous polynomials with degree \( i \) and \( j \), respectively. Then there exists a \( \theta_0 \in [0, 2\pi] \) such that \( f_d^2(\cos \theta_0, \sin \theta_0) + g_d^2(\cos \theta_0, \sin \theta_0) \neq 0 \). Otherwise, \( f_d(x, y) = g_d(x, y) \equiv 0 \). We have

\[
\lim_{r \to 0} I(r \cos \theta_0, r \sin \theta_0) = \lim_{r \to 0} \frac{2}{\frac{f^2(r \cos \theta_0, \sin \theta_0)}{r^d} + \frac{g^2(r \cos \theta_0, \sin \theta_0)}{r^d}} = \lim_{r \to 0} \frac{2r^{2d}}{\sum_{i=1}^{d} \sum_{j=1}^{d} r^{2d-i-j}[f_i(\cos \theta_0, \sin \theta_0)f_j(\cos \theta_0, \sin \theta_0) + g_i(\cos \theta_0, \sin \theta_0)g_j(\cos \theta_0, \sin \theta_0)]} = 0.
\]

Therefore, \( I(x, y) \to 0 \) as \( (x, y) \to (0, 0) \) along the straight line

\[
l := \{(x, y) : x = r \cos \theta_0, y = r \sin \theta_0, r \in \mathbb{R}\}.
\]

Since the limit \( \lim_{(x, y) \to (0, 0)} I(x, y) \) exists, we obtain

\[
\lim_{(x, y) \to (0, 0)} I(x, y) = \lim_{(x, y) \to (0, 0)} I(x, y) = 0. \tag{4.1}
\]

Along straight line \( l \)

The Hamiltonian vector field (1.2) has the Hamiltonian

\[
H(x, y) = \frac{f^2(x, y) + g^2(x, y)}{2}. \tag{4.2}
\]

So the vector field \( b(X) \) has the first integral

\[
I(u, v) = \frac{2}{f^2(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}) + g^2(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2})}.
\]

It is easy to check that

\[
\tilde{I}(u, v) = \begin{cases} I(u, v), & (u, v) \neq (0, 0), \\ 0, & (u, v) = (0, 0), \end{cases} \tag{4.3}
\]

is also first integral of vector field \( b(X) \). Since

\[
\lim_{(u, v) \to (0, 0)} I(u, v) = 0,
\]

\( \tilde{I}(u, v) \) is continuous on \( \mathbb{R}^2 \).

Let \( \gamma(t) = (u(t), v(t)) \) be an orbit of \( b(X) \) tending to the origin as \( t \to +\infty \) (or \( t \to -\infty \)). There exists a constant \( C \) such that \( \tilde{I}(\gamma(t)) = \tilde{I}(u(t), v(t)) = C \). Since \( \tilde{I} \) is a continuous function on \( \mathbb{R}^2 \),

\[
C = \lim_{t \to +\infty} \tilde{I}(\gamma(t)) = \lim_{t \to +\infty} \tilde{I}(u(t), v(t)) = 0.
\]

This means that \( \gamma(t) = (u(t), v(t)) = (0, 0) \). There are no orbits tending or leaving the origin. Thus, the origin of \( b(X) \) is a monodromic singular point. By Theorem 3, \( F \) is a global injective.

The next lemma will be used to prove Theorem 5.

Lemma 5. Consider the map

\[
\Phi : [0, T) \times (0, +\infty) \to \mathbb{R}^2 \setminus \{(0, 0)\}
\]

\[
(\theta, r) \mapsto \left( \frac{r^{2\alpha_1+2\alpha_2}C_\theta}{r^{2\alpha_1+2\alpha_2}C_\theta + r^{2\alpha_2}C_\theta}, \frac{r^{2\alpha_1+2\alpha_2}S_\theta}{r^{2\alpha_1+2\alpha_2}S_\theta + r^{2\alpha_2}S_\theta} \right), \tag{4.4}
\]

where \( T \) define in Lemma 1. Then map \( \Phi \) is invertible and

\[
\lim_{r \to 0} \Phi(\theta, r) = (0, 0).
\]
Proof. Let
\[
(x, y) = \left( \frac{r^{s_1 + 2s_2} C_s \theta}{r^{2s_1} S_n^2 \theta + r^{2s_2} C_s^2 \theta}, \frac{r^{2s_1 + s_2} S_n\theta}{r^{2s_1} S_n^2 \theta + r^{2s_2} C_s^2 \theta} \right).
\]

Using property (a) of Lemma 1, one can get
\[
r = \left( s_1 \left( \frac{x}{x^2 + y^2} \right)^{2s_2} + s_2 \left( \frac{y}{x^2 + y^2} \right)^{2s_1} \right)^{-\frac{1}{2s_1 + 2s_2}}, C_s \theta = \frac{r^{s_1} x}{x^2 + y^2}, S_n \theta = \frac{r^{s_2} y}{x^2 + y^2}.
\]

(4.5)

From the above equations, there is a unique \((\theta, r)\) for each \((x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}\) such that \(\Phi(\theta, r) = (x, y)\). So \(\Phi\) is invertible.

In order to prove \(\lim_{r \to 0} \Phi (\theta, r) = (0, 0)\), we only need to show that
\[
\lim_{r \to 0} x^2 + y^2 = \lim_{r \to 0} \frac{r^{2s_1 + 2s_2}}{r^{2s_1} S_n^2 \theta + r^{2s_2} C_s^2 \theta} = 0.
\]

(4.6)

Without loss of generality, we can assume \(s_1 \geq s_2\). One can obtain that
\[
\frac{r^{2s_1 + 2s_2}}{r^{2s_1} S_n^2 \theta + r^{2s_2} C_s^2 \theta} \leq \frac{s_1 s_{2s_2}}{s_1 r^{2s_1 - 2s_2} - s_2} S_n^{2s_1} \theta + 1
\]

(4.7)

with \(0 \leq S_n^{2s_1} \leq 1/s_2\). Since \(\lim_{r \to 0} s_1 r^{2s_1 - 2s_2} - s_2 = -s_2 < 0\), there exists a \(\delta > 0\) such that \(s_1 r^{2s_1 - 2s_2} - s_2 < 0\) for all \(|r| < \delta\). Then,
\[
\frac{r^{2s_1 + 2s_2}}{r^{2s_1} S_n^2 \theta + r^{2s_2} C_s^2 \theta} \leq \frac{s_1 s_{2s_2}}{(s_1 r^{2s_1 - 2s_2} - s_2) S_n^{2s_1} \theta + 1} \leq s_2 r^{2s_2}
\]

(4.8)

for all \(|r| < \delta\) and \(\theta \in [0, T]\). Consequently,
\[
\lim_{r \to 0} x^2 + y^2 = \lim_{r \to 0} \frac{r^{2s_1 + 2s_2}}{r^{2s_1} S_n^2 \theta + r^{2s_2} C_s^2 \theta} = 0.
\]

The proof is finished. 

\[\square\]

Remark 4. The map \(\Phi\) is similar to polar coordinates transformation. For the particular case \(s = (1, 1)\), \(\Phi\) becomes the classical polar coordinates.

Proof of Theorem 5. By Theorem 4, it is enough to prove that
\[
\lim_{(u, v) \to (0, 0)} I(u, v) = 0.
\]

Let \(f(x, y) = \sum_{i=1}^{n} f_i(x, y)\), \(g(x, y) = \sum_{j=1}^{m} g_j(x, y)\) and \(d = \max\{n, m\}\), where quasi-homogeneous polynomials \(f_i(x, y)\) and \(g_j(x, y)\) with respect to the weight exponent \(s = (s_1, s_2)\) are weighted degree \(i\) and \(j\), respectively. Here, \(F_s = (f_d, g_d)\).

Consider the change of coordinates
\[
u = \frac{r^{s_1 + 2s_2} C_s \theta}{r^{2s_1} S_n^2 \theta + r^{2s_2} C_s^2 \theta}, \quad v = \frac{r^{2s_1 + s_2} S_n \theta}{r^{2s_1} S_n^2 \theta + r^{2s_2} C_s^2 \theta}
\]

Applying Lemma 5, we have
\[
\lim_{(u, v) \to (0, 0)} I(u, v) = \lim_{r \to 0} \frac{2}{f^2 \left( \frac{C_s \theta}{r^{s_1}}, \frac{S_n \theta}{r^{s_2}} \right) + g^2 \left( \frac{C_s \theta}{r^{s_1}}, \frac{S_n \theta}{r^{s_2}} \right)}
\]

\[
= \lim_{r \to 0} \frac{2}{\sum_{i=1}^{d} \sum_{j=1}^{d} r^{2d-i-j} \left[ f_i (C_s \theta, S_n \theta) f_j (C_s \theta, S_n \theta) + g_i (C_s \theta, S_n \theta) g_j (C_s \theta, S_n \theta) \right]}
\]

12
Since the system of equations $f_d(x, y) = g_d(x, y) = 0$ has only the trivial solution $(x, y) = (0, 0)$, there exists a $L > 0$ such that $L \leq f_d^2 (C\theta, S\theta) + g_d^2 (C\theta, S\theta)$ for all $\theta \in [0, T)$. Since

$$\lim_{r \to 0} \frac{1}{\sum_{i=1}^{d} \sum_{j=1}^{d} r^{2d-i-j} [f_i (C\theta, S\theta) f_j (C\theta, S\theta) + g_i (C\theta, S\theta) g_j (C\theta, S\theta)]} \leq \frac{1}{L},$$

there exists a $\delta > 0$ such that

$$0 < \frac{1}{\sum_{i=1}^{d} \sum_{j=1}^{d} r^{2d-i-j} [f_i (C\theta, S\theta) f_j (C\theta, S\theta) + g_i (C\theta, S\theta) g_j (C\theta, S\theta)]} < \frac{2}{L}$$

for all $|r| < \delta$ and $\theta \in [0, T)$. Therefore,

$$\lim_{(u,v) \to (0,0)} I(u, v) = 0.$$

This ends the proof. □

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