Pointwise analog of the Stečkin approximation theorem

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Abstract

We show the pointwise version of the Stečkin theorem on approxima-
tion by de la Vallée-Poussin means. The result on norm approximation is
also derived.

Key words: Pointwise approximation by de la Vallée-Poussin means

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1 Introduction

Let \( L^p (1 \leq p < \infty) [C] \) be the class of all \( 2\pi \)-periodic real-valued functions integrable in the Lebesgue sense with \( p \)-th power \([\text{continuous}]\) over \( Q = [-\pi, \pi] \) and let \( X^p = L^p \) when \( 1 \leq p < \infty \) or \( X^p = C \) when \( p = \infty \).

Let us define the norms of \( f \in X^p \) as

\[
\| f \| = \| f \|_{X^p} = \| f(\cdot) \|_{X^p} := \left\{ \begin{array}{ll}
\int_Q |f(x)|^p \, dx & \text{when } 1 \leq p < \infty \\
\sup_{x \in Q} |f(x)| & \text{when } p = \infty
\end{array} \right.
\]

and

\[
\| f \|_{x,\delta} = \| f \|_{X^p,x,\delta} = \| f(\cdot) \|_{X^p,x,\delta} := \sup_{0<h \leq \delta} \| f(\cdot) \|_{X^p,x,h}
\]

where

\[
\| f \|^{\circ}_{x,\delta} = \| f \|_{X^p,x,\delta} = \| f(\cdot) \|_{X^p,x,\delta}^{\circ} := \left\{ \begin{array}{ll}
\frac{1}{2\pi} \int_{x-h}^{x+h} |f(t)|^p \, dt & \text{when } 1 \leq p < \infty \\
\sup_{0<h \leq \delta} \sup_{0<|t|\leq h} |f(x+t)| & \text{when } p = \infty
\end{array} \right.
\]

We note additionally that

\[
\| f \|_{x,0} = \| f \|^{\circ}_{x,0} = \| f(\cdot) \|_{x,0} = |f(x)|.
\]

Consider the trigonometric Fourier series of \( f \)

\[
Sf(x) = \frac{a_0(f)}{2} + \sum_{k=0}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx)
\]

with the partial sums \( S_k f \).

Let

\[
\sigma_{n,m} f(x) := \frac{1}{m+1} \sum_{k=n-m}^{n} S_k f(x) \quad (m \leq n = 0, 1, 2, \ldots)
\]

As a measure of approximation by the above quantities we use the pointwise
characteristics

\[ w_x f(\delta) = w_{x f}(\delta)_{X^p} := \| \Delta_x f(\cdot) \|_{X^p x, \delta} \]

\[
= \begin{cases} 
\sup_{0 < h \leq \delta} \left\{ \frac{1}{2h} \int_{-h}^h |\Delta_x f(t)|^p dt \right\}^{1/p} & \text{when } 1 \leq p < \infty \\
\sup_{0 < h \leq \delta} \left\{ \sup_{0 < |t| \leq h} |\Delta_x f(t)| \right\} & \text{when } p = \infty
\end{cases}
\]

cf. [1] and

\[ w^o_x f(\delta) = w_{x f}^o(\delta)_{X^p} := \| \Delta_x f(\cdot) \|^o_{X^p x, \delta} \]

\[
= \begin{cases} 
\left\{ \frac{1}{2\delta} \int_{-\delta}^\delta |\Delta_x f(t)|^p dt \right\}^{1/p} & \text{when } 1 \leq p < \infty \\
\sup_{0 < |t| \leq \delta} |\Delta_x f(t)| & \text{when } p = \infty
\end{cases}
\]

and also

\[
\Omega_{x f} \left( \frac{\pi}{n+1} \right) = \Omega_{x f} \left( \frac{\pi}{n+1} \right)_{X^p} := \frac{1}{n+1} \sum_{k=0}^n w_x f(\frac{\pi}{k+1})_{X^p}
\]

\[
\Omega_{x f}^o \left( \frac{\pi}{n+1} \right) = \Omega_{x f}^o \left( \frac{\pi}{n+1} \right)_{X^p} = \frac{1}{n+1} \sum_{k=0}^n w_{x f}^o(\frac{\pi}{k+1})_{X^p},
\]

where \( \Delta_x f(t) := f(x+t) - f(x) \),

constructed on the base of definition of \( X^p - points \) ([Lebesgue points\( L^p\) - points]) or [points of continuity \( C\) - points]). We also use the modulus of continuity of \( f \) in the space \( X^p \) defined by the formula

\[ \omega f(\delta) = \omega f(\delta)_{X^p} := \sup_{0 < |h| \leq \delta} \| \Delta_x f(h) \|_{X^p} \]

and its arithmetic mean

\[
\Omega f \left( \frac{\pi}{n+1} \right) = \Omega f \left( \frac{\pi}{n+1} \right)_{X^p} = \frac{1}{n+1} \sum_{k=0}^n \omega f(\frac{\pi}{k+1})_{X^p}.
\]

We can observe that, for \( f \in X^{\tilde{p}} \) and \( \tilde{p} \geq p \),

\[ ||w \cdot f(\delta)_{X^p}||_C \leq \omega f(\delta)_C, \]

whence

\[ ||\Omega \cdot f(\delta)_{X^p}||_C \leq \Omega f(\delta)_C \]

and

\[ ||w \cdot f(\delta)_{X^p}||_{X^p} \leq \omega f(\delta)_{X^p}, \]
whence
\[ \|\Omega^2 f(\delta)\|_{X^p} \leq \Omega f(\delta)_{X^p}. \]

Let introduce one more measure of pointwise approximation analogical to the best approximation of function \( f \) by trigonometric polynomials \( T \) of the degree at most \( n \) \((T \in H_n)\)

\[ E_n(f)_{X^p} := \inf_{T \in H_n} \{ \| f(\cdot) - T(\cdot) \|_{X^p} \}, \]

namely

\[ E_n(f, x; \delta) = E_n(f, x; \delta)_{X^p} := \inf_{T \in H_n} \{ \| f(\cdot) - T(\cdot) \|_{X^p, x, \delta} \} \]

\[ = \begin{cases} \inf_{T \in H_n} \left\{ \sup_{0<h \leq \delta} \left[ \frac{1}{h} \int_{-h}^{h} |f(x+t) - T(x+t)|^p dt \right]^\frac{1}{p} \right\} & \text{when } 1 \leq p < \infty \\ \inf_{T \in H_n} \left\{ \sup_{0<|h| \leq \delta} |f(x+h) - T(x+h)| \right\} & \text{when } p = \infty \end{cases} \]

and

\[ E_n^o(f, x; \delta) = E_n^o(f, x; \delta)_{X^p} := \inf_{T \in H_n} \{ \| f(\cdot) - T(\cdot) \|_{X^p, x, \delta}^o \}. \]

We will also use its arithmetic mean

\[ F_{n,m}(f, x) = F_{n,m}(f, x)_{X^p} := \frac{1}{m+1} \sum_{k=0}^{m} E_n\left(f, x; \frac{\pi}{k+1}\right)_{X^p} \]

and

\[ F_{n,m}^o(f, x) = F_{n,m}^o(f, x)_{X^p} := \frac{1}{m+1} \sum_{k=0}^{m} E_n^o\left(f, x; \frac{\pi}{k+1}\right)_{X^p}. \]

Denote also

\[ X^p(w_x) = \{ f \in X^p : w_x f(\delta) \leq w_x(\delta) \}, \]

where \( w_x \) is a function of modulus of continuity type on the interval \([0, +\infty)\), i.e. a nondecreasing continuous function having the following properties: \( w_x(0) = 0 \), \( w_x(\delta_1 + \delta_2) \leq w_x(\delta_1) + w_x(\delta_2) \) for any \( 0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \).

Using these characteristics we will show the pointwise version of the Stečkin generalization of the Fejér-Lebesgue theorem. As a corollaries we will obtain the mentioned original result of S. B. Stečkin on norm approximation as well the result of N. Tanović-Miller.

By \( K \) we shall designate either an absolute constant or a constant depending on some parameters, not necessarily the same of each occurrence.
2 Statement of the results

At the begin we formulate the partial solution of the considered problem.

Theorem 1 If \( f \in X^p \) then, for any positive integer \( m \leq n \) and all real \( x \),

\[
|\sigma_{n,m} f(x) - f(x)| \leq \pi^2 E_{n-m}^\circ (f, x, \pi \frac{\pi}{2n-m+1}) X + 6F_{n-m,m}^\circ (f, x)_X + \int_{\pi \frac{\pi}{2n-m+1}}^{\pi m+1} E_{n-m}^\circ (f, x, t)_X \frac{dt}{t} + E_{n-m}^\circ (f, x; 0)_X.
\]

and

\[
|\sigma_{n,m} f(x) - f(x)| \leq (6 + \pi^2) F_{n-m,m}^\circ (f, x)_X \left[ 1 + \ln \frac{n+1}{m+1} \right] + E_{n-m}^\circ (f, x; 0)_X.
\]

Now, we can present the main result on pointwise approximation.

Theorem 2 If \( f \in X^p \) then, for any positive integer \( m \leq n \) and all real \( x \),

\[
|\sigma_{n,m} f(x) - f(x)| \leq K \sum_{\nu=0}^{n} F_{n-m+m+\nu,m}^\circ (f, x)_X + F_{n-m+m+\nu,\nu}^\circ (f, x)_X + E_{2n}^\circ (f, x; 0)_X.
\]

This immediately yields the following result of Stečkin [5].

Theorem 3 If \( f \in C \) then, for any positive integer \( n \) and \( m \leq n \)

\[
||\sigma_{n,m} f(\cdot) - f(\cdot)||_C \leq K \sum_{\nu=0}^{n} E_{n-m+m+\nu}^\circ (f)_C \frac{1}{m+\nu+1}.
\]

Remark 1 Theorem also holds if instead of \( C \) we consider the spaces \( X^p \) with \( 1 \leq p < \infty \). In the proof we need the Hardy-Littlewood estimate of the maximal function.

At every \( X^p - point \) \( x \) of \( f \)

\[
\Omega_x f(\gamma)_{X^p} = o_x(1) \quad \text{as} \quad \gamma \to 0+.
\]

and thus from Theorem 1 we obtain the corollary which state the result of the Tanović-Miller type [6].

Corollary 1 If \( f \in X^p \) then, for any positive integer \( m \leq n \) at every \( X^p - point \) \( x \) of \( f \),

\[
|\sigma_{n,m} f(x) - f(x)| = o_x(1) \left[ 1 + \ln \frac{n+1}{m+1} \right] \quad \text{as} \quad n \to \infty.
\]
3 Auxiliary results

In order to prove our theorems we require some lemmas

**Lemma 1** If $T_n$ is the trigonometric polynomial of the degree at most $n$ of the best approximation of $f \in X^p$ with respect to the norm $\| \cdot \|_{X^p}$ then, it is also the trigonometric polynomial of the degree at most $n$ of the best approximation of $f \in X^p$ with respect to the norm $\| \cdot \|_{X^p, x, \delta}$ for any $\delta \in [0, \pi]$.

**Proof.** From the inequalities

$$\| E_n(f, \cdot, \delta)_{X^p} \|_{X^p} \geq \| E_n^o(f, \cdot, \delta)_{X^p} \|_{X^p} = \| f - T_n, \delta \|_{X^p, \cdot, \delta} \|_{X^p} = \| f - T_n \|_{X^p} = E_n(f)_{X^p},$$

and

$$\| E_n^o(f, \cdot, \delta)_{X^p} \|_{X^p} \leq \| f - T_n \|_{X^p, x, \delta} \|_{X^p} = \| f - T_n \|_{X^p} = E_n(f)_{X^p},$$

where $T_n, \delta$ and $T_n$ are the trigonometric polynomials of the degree at most $n$ of the best approximation of $f \in X^p$ with respect to the norms $\| \cdot \|_{X^p, x, \delta}$ and $\| \cdot \|_{X^p}$ respectively, we obtain relation

$$\| f - T_n, \delta \|_{X^p} = \| f - T_n \|_{X^p} = E_n(f)_{X^p},$$

whence $T_n, \delta = T_n$ for any $\delta \in [0, \pi]$ by uniqueness of the trigonometric polynomial of the degree at most $n$ of the best approximation of $f \in X^p$ with respect to the norm $\| \cdot \|_{X^p}$ (see e.g. [2], p. 96). We can also observe that for such $T_n$ and any $h \in [0, \delta]$

$$\| f - T_n \|_{X^p, x, h} = E_n^o(f, x, h)_{X^p} \leq E_n(f, x, \delta)_{X^p} \leq \| f - T_n \|_{X^p, x, \delta}.$$  

Hence

$$E_n(f, x, \delta)_{X^p} = \| f - T_n \|_{X^p, x, \delta}$$

and our proof is complete. $\blacksquare$

**Lemma 2** If $n \in \mathbb{N}_0$ and $\delta > 0$ then $E_n(f, x; \delta)_{X^p}$ is nonincreasing function of $n$ and nondecreasing function of $\delta$. These imply that for $m, n \in \mathbb{N}$ the function $F_{n, m}(f, x)_{X^p}$ is nonincreasing function of $n$ and $m$ simultaneously.
Proof. The first part of our statement follows from the property of the norm $\| \cdot \|_{x,\delta}$ and supremum. The second part is a consequence of the calculation

$$
\frac{F_{n,m+1}(f,x)_{X^p}}{F_{n,m}(f,x)_{X^p}} = \frac{m+1}{m+2} \left( 1 + \frac{E_n(f,x;\frac{\pi}{m+2})_{X^p}}{\sum_{k=0}^m E_n(f,x;\frac{\pi}{k+1})_{X^p}} \right) 
\leq \frac{m+1}{m+2} \left( 1 + \frac{E_n(f,x;\frac{\pi}{m+1})_{X^p}}{\sum_{k=0}^m E_n(f,x;\frac{\pi}{k+1})_{X^p}} \right) 
= \frac{m+1}{m+2} \left( 1 + \frac{1}{m+1} \right) = 1.
$$

Lemma 3 Let $m, n, q \in \mathbb{N}_0$ such that $m \leq n$ and $q \geq m + 1$. If $f \in X^p$ then

$$
|\sigma_{n+q,m} f(x) - \sigma_{n,m} f(x)| \leq K F_{n-m,m}(f,x)_{X^p} \sum_{\nu=0}^{q-1} \frac{1}{m+\nu+1}.
$$

Proof. It is clear that

$$
\sigma_{n,m} f(x) = \frac{1}{m+1} \sum_{k=n-m}^n \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_k(t) \, dt
= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) V_{n,m}(t) \, dt
$$

where

$$
V_{n,m}(t) = \frac{1}{m+1} \sum_{k=n-m}^n D_k(t) \quad \text{and} \quad D_k(t) = \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}}.
$$

Hence, by orthogonality of the trigonometric system,

$$
\sigma_{n+q,m} f(x) - \sigma_{n,m} f(x)
= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[ f(x+t) - T_{n-m}(x+t) \right] \left( V_{n+q,m}(t) - V_{n,m}(t) \right) \, dt
= \frac{1}{\pi (m+1)} \sum_{k=n-m}^n \int_{-\pi}^{\pi} \left[ f(x+t) - T_{n-m}(x+t) \right] \left( D_{k+q}(t) - D_k(t) \right) \, dt
= \frac{1}{\pi (m+1)} \sum_{k=n-m}^n \int_{-\pi}^{\pi} \left[ f(x+t) - T_{n-m}(x+t) \right] \frac{\sin \frac{(2k+2q+1)t}{2} - \sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} \, dt
= \frac{1}{\pi (m+1)} \sum_{k=n-m}^n \int_{-\pi}^{\pi} \left[ f(x+t) - T_{n-m}(x+t) \right] \frac{\sin \frac{at}{\pi} \cos \frac{(2k+q+1)t}{2}}{\sin \frac{t}{2}} \, dt
$$

7
with trigonometric polynomial $T_{n-m}$ of the degree at most $n-m$ of the best approximation of $f$.

Using the notations

$$I_1 = \left[-\frac{\pi}{q}, \frac{\pi}{q}\right], \quad I_2 = \left[-\frac{\pi}{m+1}, \frac{\pi}{m+1}\right] \cup \left[\frac{\pi}{q}, \frac{\pi}{m+1}\right]$$

$$I_3 = \left[-\pi, -\frac{\pi}{m+1}\right] \cup \left[\frac{\pi}{m+1}, \pi\right]$$

we get

$$\sum = \frac{1}{\pi (m+1)} \sum_{k=n-m}^{n} \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) [f(x+t) - T_{n-m}(x+t)]$$

$$\sin\frac{q\pi}{2} \cos \frac{(2k+q+1)t}{2} \sin \frac{t}{2} dt$$

$$= \sum_1 + \sum_2 + \sum_3.$$ 

and

$$\sum_1 \leq \frac{1}{\pi (m+1)} \sum_{k=n-m}^{n} \int_{I_1} |f(x+t) - T_{n-m}(x+t)| qdt$$

$$= \frac{q}{\pi} \int_{I_1} |f(x+t) - T_{n-m}(x+t)| dt$$

$$\leq 2E_{n-m} \left(f, x; \frac{\pi}{q}\right)_{X^p}$$

We next evaluate the sums $\sum_2$ and $\sum_3$ using the partial integrating and Lemma 1. Thus

$$\sum_2 \leq \int_{I_2} \frac{|f(x+t) - T_{n-m}(x+t)|}{t} dt$$

$$= 2 \left[\frac{1}{2t} \int_{-t}^{t} |f(x+u) - T_{n-m}(x+u)| du\right]_{t=\frac{\pi}{q}}^{t=\frac{\pi}{m+1}}$$

$$+ 2 \int_{\frac{\pi}{q}}^{\frac{\pi}{m+1}} \frac{1}{t} \left[\frac{1}{2t} \int_{-t}^{t} |f(x+u) - T_{n-m}(x+u)| du\right] dt$$

$$\leq 2E_{n-m} \left(f, x; \frac{\pi}{m+1}\right)_{X^p} + 2 \int_{-t}^{t} \frac{1}{t} E_{n-m} \left(f, x; t\right)_{X^p} dt$$

$$\leq 4E_{n-m} \left(f, x; \frac{\pi}{m+1}\right)_{X^p} \left[1 + \ln \frac{q}{m+1}\right]$$

$$\leq 4E_{n-m} \left(f, x; \frac{\pi}{m+1}\right)_{X^p} \left[1 + \sum_{\nu=0}^{q-1} \frac{1}{m+\nu+1}\right]$$

8
\[
\sum_3 \leq \frac{1}{m + 1} \int_{I_3} \left| \frac{f(x + t) - T_{n-m}(x + t)}{t} \right| \left| \sum_{k=n-m}^n \cos \left( kt + \frac{q + 1}{2} t \right) \right| dt
\]
\[
\leq \frac{1}{m + 1} \int_{I_3} \left| \frac{f(x + t) - T_{n-m}(x + t)}{t} \right| \left| \frac{2 \sin \left( \frac{(n+1)t}{2} \right) \cos \left( \frac{2(n-m+q+1)t}{2} \right)}{2 \sin \frac{\pi}{2}} \right| dt
\]
\[
\leq \frac{\pi}{m + 1} \int_{I_3} \left| \frac{f(x + t) - T_{n-m}(x + t)}{t^2} \right| dt
\]
\[
= \frac{\pi}{m + 1} \left\{ 2 \left[ \frac{1}{2t} \int_{-t}^t |f(x + u) - T_{n-m}(x + u)| du \right]_{t=\frac{\pi}{m+1}}^{t=\pi} + 4 \int_{\frac{\pi}{m+1}}^\pi \frac{1}{t^2} \left[ \frac{1}{2t} \int_{-t}^t |f(x + u) - T_{n-m}(x + u)| du \right] dt \right\}
\]
\[
\leq \frac{\pi}{m + 1} \left\{ 2E_{n-m}(f, x; \pi)_{X_0} + 4 \int_{\frac{\pi}{m+1}}^\pi \frac{1}{t^2} \int_{-t}^{\pi} E_{n-m}(f, x; t)_{X_0} dt \right\}
\]
\[
= \frac{2\pi}{m + 1} \left\{ E_{n-m}(f, x; \pi)_{X_0} + 2 \int_{1}^{m+1} \frac{E_{n-m}(f, x; \frac{\pi}{u})_{X_0}}{u^2} du \right\}
\]
\[
= \frac{2\pi}{m + 1} \left\{ E_{n-m}(f, x; \pi)_{X_0} + \frac{2}{\pi} \sum_{k=0}^{m-1} \int_{k+1}^{k+2} E_{n-m}(f, x; \frac{\pi}{u})_{X_0} du \right\}
\]
\[
\leq \frac{2\pi}{m + 1} \left\{ E_{n-m}(f, x; \pi)_{X_0} + \frac{2}{\pi} \sum_{k=0}^{m-1} \int_{k+1}^{k+2} E_{n-m}(f, x; \frac{\pi}{k+1})_{X_0} \right\}
\]
which proves Lemma 2. ■

Before formulating the next lemmas we define a new difference. Let \( m, n \in \mathbb{N}_0 \) and \( m \leq n \). Denote
\[
\tau_{n,m} f(x) := (m + 1) \left\{ \sigma_{n+m+1,m} f(x) - \sigma_{n,m} f(x) \right\}.
\]
Lemma 4 Let \( m, n, \mu \in \mathbb{N}_0 \) such that \( 2\mu \leq m \leq n \). If \( f \in X^p \) then

\[
|\tau_{n,m} f(x) - \tau_{n-\mu,m-\mu} f(x)| \leq K\mu F_{n-\mu+1,\mu-1} (f, x)_{X^p} \ln \frac{m}{\mu}.
\]

Proof. The proof follows by the method of Leindler[3]. Namely

\[
\tau_{n,m} f(x) - \tau_{n-\mu,m-\mu} f(x) = \left( \sum_{k=n+m-2\mu+2}^{n+m+1} - 2 \sum_{k=n-\mu+1}^{n} \right) [S_k f(x) - f(x)]
\]

and

\[
|\tau_{n,m} f(x) - \tau_{n-\mu,m-\mu} f(x)| \leq \left| \left( \sum_{k=n+m-2\mu+2}^{n+m+1} - 2 \sum_{k=n-\mu+1}^{n} \right) [S_k f(x) - f(x)] \right|
\]

\[
= \mu |\sigma_{n+m-\mu+1,\mu-1} f(x) - \sigma_{n-\mu-1} f(x)| + \mu |\sigma_{n+m+1,\mu-1} f(x) - \sigma_{n-\mu-1} f(x)|.
\]

By Lemma 2, for \( 2\mu \leq m \),

\[
|\tau_{n,m} f(x) - \tau_{n-\mu,m-\mu} f(x)| \leq K\mu F_{n-\mu+1,\mu-1} (f, x)_{X^p} \left[ 1 + \ln \left( \frac{n-\mu+1+\mu-1}{\mu} \right) \right]
\]

\[
+ K\mu F_{n-\mu+1,\mu-1} (f, x)_{X^p} \left[ 1 + \ln \left( \frac{m+\mu-1}{\mu} \right) \right]
\]

\[
\leq K\mu F_{n-\mu+1,\mu-1} (f, x)_{X^p} \left[ 1 + \ln \left( \frac{m}{\mu} \right) \right]
\]

and our proof is complete. ■

Lemma 5 Let \( m, n \in \mathbb{N}_0 \) and \( m \leq n \). If \( f \in X^p \) then

\[
|\tau_{n,m} f(x)| \leq K \sum_{k=n-m}^{n} F_{k,k+n+m} (f, x)_{X^p}.
\]

Proof. Our proof runs parallel with the proof of Theorem 1 in [5]. If \( m = 0 \) then

\[
|\tau_{n,0} f(x)| = |\sigma_{n+1,0} f(x) - \sigma_{n,0} f(x)| \leq K F_{n,0} (f, x)_{X^p}.
\]
and if $m = 1$ then

$$|	au_{n,1} f(x)| \leq 2 |\sigma_{n+1,1} f(x) - \sigma_{n,1} f(x)| \leq K F_{n-1,1} (f,x)_{X^p}$$

$$\leq K [F_{n-1,1} (f,x)_{X^p} + F_{n-1,1} (f,x)_{X^p}]$$

by Lemma 2 and Lemma 3.

Next we construct the same decreasing sequence $(m_{s})$ of integers that was given by S. B. Stečkin. Let

$$m_0 = m, \quad m_s = m_{s-1} - \left[ \frac{m_{s-1}}{2} \right] \quad (s = 1, 2, \ldots)$$

where $[y]$ denotes the integral part of $y$. It is clear that there exists a smallest index $t \geq 1$ such that $m_t = 1$ and

$$m = m_0 > m_1 > \ldots > m_t = 1.$$

By the definition of the numbers $m_s$ we have

$$m_s \geq m_s - 1 / 2$$

$$m_{s-1} - m_s = \left[ \frac{m_{s-1}}{2} \right] \geq \left[ \frac{m_{s-1}}{3} \right] \quad (s = 1, 2, \ldots, t)$$

whence

$$m_{t-1} = 2, \quad m_{t-1} - m_t = 1$$

and

$$m_{s-1} - m_s \leq 3 (m_s - m_{s+1}) \quad (s = 1, 2, \ldots, t-1)$$

follow.

Under these notations we get the following equality

$$\tau_{n,m} f(x) = \sum_{s=1}^{t} (\tau_{n-m+m_{s-1},m_s-1} f(x) - \tau_{n-m+m_s,m_s} f(x)) + \tau_{n-m+m_t,m_t} f(x)$$

whence, by $m_t = 1$,

$$|\tau_{n,m} f(x)| \leq \sum_{s=1}^{t} |\tau_{n-m+m_{s-1},m_s-1} f(x) - \tau_{n-m+m_s,m_s} f(x)| + |\tau_{n-m+m_t,m_t} f(x)|$$

follows.

It is easy to see that the terms in the sum $\sum_{s=1}^{t}$, by Lemma 4, with $\mu = m_{s-1} - m_s$ and $m = m_{s-1}$ do not exceed

$$K (m_{s-1} - m_s) F_{n-m+m_{s+1},m_{s+1}-1} (f,x)_{X^p} \ln \frac{m_{s-1}}{m_{s-1} - m_s},$$

where $(s = 1, 2, \ldots, t-1)$.  

11
and by Lemma 3 we get
\[ |\tau_{n-m+1,1} f(x)| \leq 2|\sigma_{n-m+2,1} f(x) - \sigma_{n-m+1,1} f(x)| \leq K F_{n-m,1} (f,x)_{X_p} \]

Thus
\[ |\tau_{n,m} f(x)| \leq K \sum_{s=1}^{t-1} 3(m_s - m_{s+1}) F_{n-m+s+1,m_s} (f,x)_{X_p} \ln 3 \]
\[ + K F_{n-m+2,m-2} (f,x)_{X_p} + K F_{n-m,1} (f,x)_{X_p} \]

whence, by the monotonicity of \( F_{\nu,\mu} (f,x)_{X_p} \),
\[ |\tau_{n,m} f(x)| \]
\[ \leq K \left( \sum_{s=1}^{t-1} \sum_{\nu=m_{s+1}+1}^{m_s} F_{n-m+s+1,\nu} (f,x)_{X_p} + \sum_{\nu=0}^{2} F_{n-m+\nu,m-\nu-1} (f,x)_{X_p} \right) \]
\[ + K F_{n-m,1} (f,x)_{X_p} \]
\[ \leq K \sum_{\nu=0}^{m_{1+1}} F_{n-m+\nu,\nu} (f,x)_{X_p} + K F_{n-m,1} (f,x)_{X_p} \]
\[ \leq K \sum_{\nu=0}^{m} F_{n-m+\nu,\nu} (f,x)_{X_p} + K F_{n-m,1} (f,x)_{X_p} \]
\[ \leq K \sum_{k=n-m}^{n} F_{k,k-n+m} (f,x)_{X_p} + K F_{n-m,1} (f,x)_{X_p} \]

\[ \blacksquare \]

4 Proofs of the results

4.1 Proof of Theorem 2

The proof follows the lines of the proofs of Theorem 4 in [5] and Theorem in [3]. Therefore let \( n > 0 \) and \( m \leq n \) be fixed. Let us define an increasing sequence \( (n_s : s = 0, 1, ..., t) \) of indices introduced by S. B. Stečki in the following way. Set \( n_0 = n \). Assuming that the numbers \( n_0, ..., n_s \) are already defined and \( n_s < 2n \), we define \( n_{s+1} \) as follows: Let \( \nu_s \) denote the smallest natural number such that
\[ F_{n_s-m+\nu_s,\nu} (f,x)_{X_p} \leq \frac{1}{2} F_{n_s-m,\nu} (f,x)_{X_p} \quad (\nu = 0, 1, ..., n). \]

According to the magnitude of \( \nu_s \) we define
\[ n_{s+1} = \begin{cases} 
  n_s - m + 1 & \text{for } \nu_s \leq m, \\
  n_s + \nu_s & \text{for } m + 1 \leq \nu_s < 2n + m - n_s, \\
  2n + m & \text{for } \nu_s \geq 2n + m - n_s.
\end{cases} \]
If \( n_{s+1} < 2n \) we continue the procedure, and if once \( n_{s+1} \geq 2n \) then we stop the construction and define \( t := s + 1 \).

By the above definition of \((n_s)\) we have the following obvious properties:

\[
t \geq 1, \quad n = n_0 < n_1 < \ldots < n_t, \quad 2n \leq n_t \leq 2n + m,
\]

and

\[
n_{s+1} - n_s \geq m + 1 \quad (s = 0, 1, \ldots, t - 1),
\]

and relations

\[
F_{n_{s+1}-m,\nu} (f, x)_{X^p} \leq \frac{1}{2} F_{n_s - m, \nu} (f, x)_{X^p} \quad \text{for } s = 0, 1, \ldots, t - 2,
\]

and

\[
\frac{1}{2} F_{n_s - m, \nu} (f, x)_{X^p} \leq F_{n_{s+1} - m, \nu} (f, x)_{X^p} \quad \text{for } s = 0, 1, \ldots, t - 1
\]

whenever \( n_{s+1} - n_s > m + 1 \).

Let us start with

\[
|\sigma_{n,m} f (x) - f (x)| = \sum_{s=0}^{t-1} \left[ |\sigma_{n_s,m} f (x) - f (x)| - |\sigma_{n_{s+1},m} f (x) - f (x)| \right]
\]

\[
+ |\sigma_{n_t,m} f (x) - f (x)|
\]

\[
\leq \sum_{s=0}^{t-1} |\sigma_{n_{s+1},m} f (x) - \sigma_{n_s,m} f (x)| + |\sigma_{n_t,m} f (x) - f (x)|
\]

\[
= \sum_{s=0}^{t-1} \frac{1}{m + 1} |\tau_{n_s,m} f (x)| + |\sigma_{n_t,m} f (x) - f (x)|.
\]

Using Theorem 1 and that \( 2n \leq n_t \leq 2n + m \) we get

\[
|\sigma_{n,m} f (x) - f (x)| \leq K F_{n_t - m, \nu} (f, x)_{X^p} \left[ 1 + \ln \frac{n_t + 1}{m + 1} \right] + |f (x) - T_{n_t - m} (x)|
\]

\[
\leq K \sum_{\nu=0}^{n} \frac{F_{n-m+m, \nu} (f, x)_{X^p}}{m + \nu + 1} + |f (x) - T_{n_t - m} (x)|
\]

\[
\leq K \sum_{\nu=0}^{n} \frac{F_{n-m+m, \nu} (f, x)_{X^p} + F_{n-m+m, \nu} (f, x)_{X^p}}{m + \nu + 1} + |f (x) - T_{n_t - m} (x)|
\]

\[
\leq K \sum_{\nu=0}^{n} \frac{F_{n-m+m, \nu} (f, x)_{X^p} + F_{n-m+m, \nu} (f, x)_{X^p}}{m + \nu + 1} + E_{n_t - m} (f, x; 0)_{X^p}
\]

\[
\leq K \sum_{\nu=0}^{n} \frac{F_{n-t,0} (f, x)_{X^p} + F_{n-t,0} (f, x)_{X^p}}{m + \nu + 1} + E_{2n} (f, x; 0)_{X^p}.
\]

The estimate of the sum we derive from the following one

\[
\left| \frac{1}{m + 1} \tau_{n_s,m} f (x) \right| \leq K \sum_{\nu=0}^{n_{s+1} - n_s - 1} \frac{F_{n_{s+1} - m, \nu} (f, x)_{X^p} + F_{n_s - m, \nu} (f, x)_{X^p}}{m + \nu + 1}.
\]
The proof of this inequality we split in two parts. If \( n_{s+1} - n_s = m + 1 \), then by Lemma 5,

\[
\frac{1}{m+1} \sum_{s=0}^{n_{s+1} - n_s} F(x) \leq K \frac{1}{m+1} \sum_{k=n_s-m}^{n_s} F_{k,k+n_s+m} (f, x)_{X_\nu} \leq K \sum_{\nu=0}^{n_{s+1} - n_s - 1} \frac{F_{n_s-m+\nu,\nu} (f, x)_{X_\nu}}{m+\nu+1}.
\]

If \( n_{s+1} - n_s > m + 1 \), then, by Lemma 3,

\[
\frac{1}{m+1} \sum_{s=0}^{n_{s+1} - n_s} F(x) \leq K F_{n_s-m,m} (f, x)_{X_\nu} \leq K \sum_{\nu=0}^{n_{s+1} - n_s - 1} \frac{1}{m+\nu+1}.
\]

and since \( \frac{1}{2} F_{n_s-m,m} (f, x)_{X_\nu} \leq F_{n_s+1-m-1,m} (f, x)_{X_\nu} \) we have

\[
\frac{1}{m+1} \sum_{s=0}^{n_{s+1} - n_s} F(x) \leq 2K \sum_{\nu=0}^{n_{s+1} - n_s - 1} \frac{F_{n_s-m+\nu,m} (f, x)_{X_\nu} + F_{n_s-m+\nu,\nu} (f, x)_{X_\nu}}{m+\nu+1}.
\]

Consequently,

\[
\sum_{s=0}^{t-1} \frac{1}{m+1} \sum_{s=0}^{n_{s+1} - n_s} F(x) \leq 2K \sum_{s=0}^{t-1} \sum_{\nu=0}^{n_{s+1} - n_s - 1} \frac{F_{n_s-m+\nu,m} (f, x)_{X_\nu} + F_{n_s-m+\nu,\nu} (f, x)_{X_\nu}}{m+\nu+1}.
\]

Since \( n_{s+1} - n_s \leq 2n + m - n - 1 = n + m - 1 \) for all \( s \leq t - 1 \), changing the order of summation we get

\[
\sum_{s=0}^{t-1} \frac{1}{m+1} \sum_{s=0}^{n_{s+1} - n_s} F(x) \leq 2K \sum_{\nu=0}^{n+m-1} \frac{1}{m+\nu+1} \sum_{s=n_{s+1} - n_s > \nu} [F_{n_s-m+\nu,m} (f, x)_{X_\nu} + F_{n_s-m+\nu,\nu} (f, x)_{X_\nu}].
\]

Using the inequality

\[
F_{n_{s+1}-m,\nu} (f, x)_{X_\nu} \leq \frac{1}{2} F_{n_s-m,\nu} (f, x)_{X_\nu} \quad \text{for} \quad \nu = 0, 1, 2, ..., n_{s+1} - n_s - 1 \quad \text{with} \quad s = 0, 1, 2, ..., t - 2.
\]
we obtain
\[ \sum_{s: n_{s+1}-n_s > \nu} \left[ F_{n_s-m,\nu,m} (f, x)_{X^p} + F_{n_s-m,\nu,\nu} (f, x)_{X^p} \right] = F_{n_p-m,\nu,m} (f, x)_{X^p} + F_{n_p-m,\nu,\nu} (f, x)_{X^p} \]
\[ + \sum_{s \geq p+1: n_{s+1}-n_s > \nu} \left[ F_{n_{s+1}-m,\nu,m} (f, x)_{X^p} + F_{n_{s+1}-m,\nu,\nu} (f, x)_{X^p} \right] \]
\[ \leq F_{n_p-m,\nu,m} (f, x)_{X^p} + F_{n_p-m,\nu,\nu} (f, x)_{X^p} \]
\[ + \sum_{s \geq p+1} F_{n_s-m,m} (f, x)_{X^p} \left[ F_{n_s-m,m} (f, x)_{X^p} + F_{n_s-m,\nu} (f, x)_{X^p} \right] \]
\[ \leq F_{n_p-m,\nu,m} (f, x)_{X^p} + F_{n_p-m,\nu,\nu} (f, x)_{X^p} + 2 \left[ F_{n_{p+1}-m,m} (f, x)_{X^p} + F_{n_{p+1}-m,\nu} (f, x)_{X^p} \right] \]
\[ \leq 3 \left[ F_{n_p-m,\nu,m} (f, x)_{X^p} + F_{n_p-m,\nu,\nu} (f, x)_{X^p} \right], \]
where \( p \) denote the smallest index \( s \) having the property \( n_{s+1}-n_s > \nu \).

Hence
\[ \sum_{s=0}^{t-1} \frac{1}{m+1} r_{n_s,m} f (x) \leq K \sum_{\nu=0}^{n+m-1} \frac{F_{n-m,\nu,m} (f, x)_{X^p} + F_{n-m,\nu,\nu} (f, x)_{X^p}}{m+\nu+1} \]
\[ \leq K \sum_{\nu=0}^{n} \frac{F_{n-m,\nu,m} (f, x)_{X^p} + F_{n-m,\nu,\nu} (f, x)_{X^p}}{m+\nu+1}. \]

and our proof follows.

\[ \square \]

### 4.2 Proof of Theorem 3

The proof follows by the obvious inequality
\[ \|E_n (f, x; \delta)_{C} \|_C \leq E_n (f)_{C}. \]

\[ \square \]

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