THE REGULAR OPEN-OPEN TOPOLOGY FOR FUNCTION SPACES

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ABSTRACT. The regular open-open topology, $T_{roo}$, is introduced, its properties for spaces of continuous functions are discussed, and $T_{roo}$ is compared to $T_{oo}$, the open-open topology. It is then shown that $T_{roo}$ on $H(X)$, the collection of all self-homeomorphisms on a topological space, $(X, T)$, is equivalent to the topology induced on $H(X)$ by a specific quasi-uniformity on $X$, when $X$ is a semi-regular space.

KEY WORDS AND PHRASES. Compact-open topology, admissible topology, open-open topology quasi-uniformity, regular open set, semi-regular space, topological group.

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1. INTRODUCTION.

A set-set topology is one which is defined as follows: Let $(X, T)$ and $(Y, T')$ be topological spaces. Let $U$ and $V$ be collections of subsets of $X$ and $Y$, respectively. Let $F \subseteq Y^X$, the collection of all functions from $X$ into $Y$. Define, for $U \in U$ and $V \in V$, $(U, V) = \{f \in F : f(U) \subseteq V\}$. Let $S(U, V) = \{(U, V) : U \in U$ and $V \in V\}$. If $S(U, V)$ is a subbasis for a topology $T(U, V)$ on $F$ then $T(U, V)$ is called a set-set topology.

Some of the most commonly discussed set-set topologies are the compact-open topology, $T_{co}$, which was introduced in 1945 by R. Fox [1], and the point-open topology, $T_p$. For $T_{co}$, $U$ is the collection of all compact subsets of $X$ and $V = T^*$, the collection of all open subsets of $Y$, while for $T_p$, $U$ is the collection of all singletons in $X$ and $V = T^*$.

In section 2 of this paper, we shall introduce and discuss the regular open-open topology for function spaces. It will be shown which of the desirable properties $T_{roo}$ possesses. In section 3, Pervin and almost-Pervin spaces are explained.

The fact that $T_{roo}$, on $H(X)$, is actually equivalent to the regular-Pervin topology of quasi-uniform convergence will be discussed in section 4 along with the topic of quasi-uniform convergence. The advantage of the regular open-open topology is the set-set notation which provides us with
We assume a basic knowledge of quasi-uniform spaces. An introduction to quasi-uniform spaces may be found in Fletcher and Lindgren's [2] or in Murdeshwar and Naimpally's book [3].

Throughout this paper we shall assume \((X,T)\) and \((Y,T')\) are topological spaces.

2. THE REGULAR OPEN-OPEN TOPOLOGY.

A subset, \(W\), of \(X\) is called a regular open set provided \(W = \text{Int}_X(C\text{li}_X(W))\). If we let \(U\) be the collection of all regular open sets in \(X\) and \(V = T^*\), then \(S_{\text{rooo}} = S(U,V)\) is the subbasis for a topology, \(T_{\text{rooo}}\), on any \(F \subseteq Y^X\), which is called the regular open-open topology.

A topological space, \(X\), is called semi-regular provided that for each \(U \in X\) and each \(x \in U\) there exists a regular open set, \(V\), in \(X\), such that \(x \in V \subseteq U\). One can easily show that if \((X,T)\) is a semi-regular space then \(T_{\text{rooo}} \subseteq T_{\text{ooo}}\), the open-open topology (Porter, [4]) which has as a subbasis the set \(S_{\text{oo}} = \{(U,V) : U \in T \text{ and } V \in T^*\}\).

We now examine some of the properties of function spaces the regular open-open topology possesses. The first two theorems also hold for the open-open topology even even when \(X\) is not semi-regular. The proofs of these two theorems are straightforward and are left to the reader.

**THEOREM 1.** Let \((X,T)\) be a semi-regular space and \(F \subseteq C(X,Y)\). If \((Y,T')\) is \(T_i\) for \(i = 0, 1, 2\), then \((F,T_{\text{rooo}})\) is \(T_i\) for \(i = 0, 1, 2\).

A topology, \(T'\), on \(F \subseteq Y^X\) is called an admissible (Arens [5]) topology for \(F\) provided the evaluation map, \(E: (F,T') \times (X,T) \rightarrow (Y,T^*)\), defined by \(E(f,x) = f(x)\), is continuous.

**THEOREM 2.** If \(F \subseteq C(X,Y)\) and \(X\) is semi-regular, then \(T_{\text{rooo}}\) is admissible for \(F\).

Arens also has shown that if \(T'\) is admissible for \(F \subseteq C(X,Y)\), then \(T'\) is finer than \(T_{\text{oooo}}\). From this fact and Theorem 2, it follows, as it does for \(T_{\text{ooo}}\), that \(T_{\text{oo}} \subseteq T_{\text{rooo}}\) when \(X\) is semi-regular.

**THEOREM 3.** The sets of the form \((U,V)\) where both \(U\) and \(V\) are regular open sets in \(X\) form a subbasis for \((H(X),T_{\text{rooo}})\).

**PROOF.** Let \((U,V)\) be a subbasic open set in \((H(X),T_{\text{rooo}})\). i.e., \(U\) is a regular open set and \(O\) is an open set, not necessarily regular. Let \(f \in (U,O)\). Then \(f(U) \subseteq O\), so \(f \in (U,f(U)) \subseteq (U,O)\) and \(f(U)\) is a regular open set.

Let \((G,\circ)\) be a group such that \((G,T)\) is a topological space, then \((G,T)\) is a topological group provided the following two maps are continuous. \((1)\) \(m: G \times G \rightarrow G\) defined by \(m(g_1,g_2) = g_1 \circ g_2\) and \((2)\) \(\Phi: G \rightarrow G\) defined by \(\Phi(g) = g^{-1}\). If only the first map is continuous, then we call \((G,T)\) a quasi-topological group (Murdeshwar and Naimpally [3]).

Note that \(H(X)\) with the binary operation \(\circ\), composition of functions, and identity element \(e\), is a group. It is not difficult to show that if \((X,T)\) is a topological space and \(G\) is a subgroup of \(H(X)\) then \((G,T_{\text{rooo}})\) is a quasi-topological group. However, \((G,T_{\text{oo}})\) is not always a topological group (Porter, [4]) since \(\Phi\) is not always continuous although \(m\) is always continuous. But we discover the following about the regular open-open topology.
THEOREM 4. Let $X$ be a semi-regular space and let $G$ be a subgroup of $H(X)$. Then $(G, T_{roo})$ is a topological group.

PROOF. Let $X$ be a semi-regular space and let $G$ be a subgroup of $H(X)$. Let $(U, V)$ be a subbasic open set in $T_{roo}$ such that both $U$ and $V$ are regular open sets. Let $(f, g) \in m^{-1}((U, V))$. Then, $f \circ g(U) \subset V$ and $g(U) \subset f^{-1}(V)$. So, $(f, g) \in (g(U), V) \times (U, g(U)) \in T_{roo} \times T_{roo}$. But $(g(U), V) \times (U, g(U)) \subset m^{-1}((U, V))$. Thus, $m$ is continuous.

Note that the inverse map $\Phi : G \to G$ is bijective and that $\Phi^{-1} = \Phi$. Thus, in order to show that $\Phi$ is continuous, it suffices to show that $\Phi$ is an open map. To this end, let $(O, U)$ be a subbasic open set in $T_{roo}$ where $O$ and $U$ are both regular open sets. Clearly, $\Phi((O, U)) = ((X \setminus O), (X \setminus U))$ since we are dealing with homeomorphisms. Note that if $C, K$ are regular closed sets then $Int_X C$, $Int_X K$ are regular open sets. Thus, since $(X \setminus O)$, $(X \setminus U)$ are regular closed sets, $Int_X (X \setminus U)$, $Int_X (X \setminus O)$ are regular open sets. Again, since $G$ is a set of homeomorphisms, $(X \setminus U, X \setminus O) = (Int_X (X \setminus U), Int_X (X \setminus O))$ but this is in $T_{roo}$. Therefore, $\Phi(O, U)$ is an open set in $T_{roo}$. So, $\Phi$ is open and we are done.

3. PERVIN AND ALMOST-PERVIN SPACES.

A topological space, $(X, T)$, is called a Pervin space (Fletcher [4]) provided that for each finite collection, $\mathcal{A}$, of open sets in $X$, there exists some $h \in H(X)$ such that $h \neq e$ and $h(U) \subset U$ for all $U \in \mathcal{A}$. A topological space, $(X, T)$ is called almost-Pervin provided that for each finite collection, $\mathcal{A}$, of regular open sets, there exists some $h \in H(X)$ such that $h \neq e$ and $h(O) \subset O$ for all $O \in \mathcal{A}$.

Topologies are rarely interesting if they are the trivial or discrete topology. We have previously shown (Porter, [4]) that $(H(X), T_{roo})$ is not discrete if and only if $(X, T)$ is a Pervin space. The situation for $T_{roo}$ is similar.

THEOREM 5. $(H(X), T_{roo})$ is not discrete if and only if $(X, T)$ is almost-Pervin.

PROOF. First, assume that $(X, T)$ is an almost-Pervin space. Let $W$ be a basic open set in $T_{roo}$ which contains $e$; i.e. $W = \bigcap_{i=1}^{n}(O_i, U_i)$ where $O_i \subset U_i$ for each $i = 1, 2, 3, \ldots, n$ and $O_i$ and $U_i$ are regular open sets in $X$. $\{O_i : i = 1, 2, 3, \ldots, n\}$ is a finite collection of regular open sets in $X$, and $X$ is an almost-Pervin space, hence, there exists some $h \in H(X)$ such that $h \neq e$ and $h(O_i) \subset O_i \subset U_i$. So, $h \in W$ and $h \neq e$. Therefore, $(H(X), T_{roo})$ is not a discrete space.

Now assume that $(H(X), T_{roo})$ is not discrete. Let $V$ be a finite collection of regular open sets in $X$. Let $O = \bigcap_{U \in V}(U, U)$. Then, $O$ is a basic open set in $(H(X), T_{roo})$ which is not a discrete space.

Hence, there exists $h \in O$ with $h \neq e$. So, $(X, T)$ is almost-Pervin.

The above proof, along with the few needed definitions involving $T_{roo}$, is an example of the simplification that the definition of $T_{roo}$ offers over the quasi-uniform definition and notation.

4. QUASI-UNIFORM CONVERGENCE.

Recall that if $Q$ is a quasi-uniformity on $X$, then the topology, $T_Q$, on $X$, which has as its
neighborhood base at \( x \), \( B_x = \{ U[x] : U \in Q \} \), is called the topology induced by \( Q \). The ordered triple \( (X, Q, T_Q) \) is called a quasi-uniform space. A topological space, \( (X, T) \) is quasi-uniformizable provided there exists a quasi-uniformity, \( Q \), such that \( T_Q = T \). In 1962, Pervin [7] proved that every topological space is quasi-uniformizable by giving the following construction.

Let \((X, T)\) be a topological space. For each \( O \in T \), define the set \( S_O = (O \times O) \cup ((X \setminus O) \times X) \). Let \( S = \{ S_O : O \in T \} \). Then \( S \) is a subbasis for a quasi-uniformity, \( P \), for \( X \), called the Pervin quasi-uniformity and, as is easily shown, \( T_P = T \).

If we use the same basic structure as above but change the subbasis to \( S = \{ S_O : O \) is a regular open set \} then the quasi-uniformity induced will be called the regular-Pervin quasi-uniformity, \( RP \).

If \((X, Q)\) is a quasi-uniform space then \( Q \) induces a topology on \( H(X) \) called the topology of quasi-uniform convergence w.r.t. \( Q \), as follows: For each set \( U \in Q \), let us define \( W(U) = \{ (f, g) \in H(X) \times H(X) : (f(z), g(z)) \in U \text{ for all } z \in X \} \). Then, \( B(Q) = \{ W(U) : U \in Q \} \) is a basis for \( Q^* \), the quasi-uniformity of quasi-uniform convergence w.r.t. \( Q \) (Naimpally [8]). Let \( T_{Q^*} \) denote the topology on \( H(X) \) induced by \( Q^* \). \( T_{Q^*} \) is called the topology of quasi-uniform convergence w.r.t. \( Q^* \).

If \( P \) is the Pervin quasi-uniformity on \( X \), \( T_P \) is the Pervin topology of quasi-uniform convergence and if \( RP \) is the regular-Pervin quasi-uniformity on \( X \), then \( T_{RP} \) is called the regular-Pervin topology of quasi-uniform convergence, \( T_{RP^*} \).

It has been shown that the open-open topology is equivalent to the Pervin topology of quasi-uniform convergence (Porter, [4]). It is also true that the regular open-open topology is equivalent to the regular-Pervin topology of quasi-uniform convergence. The method of two proofs are exactly the same and we leave this one for the reader.

THEOREM 6. Let \((X, T)\) be a topological space and let \( G \) be a subgroup of \( H(X) \). Then, \( T_{oo} = T_{RP^*} \) on \( G \).

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