Computation of transmission eigenvalues by the regularized Schur complement for the boundary integral operators

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Abstract

This paper is devoted to the computation of transmission eigenvalues in the inverse acoustic scattering theory. This problem is first reformulated as a two-by-two boundary system of boundary integral equations. Next, utilizing the Schur complement technique, we develop a Schur complement operator with regularization to obtain a reduced system of boundary integral equations. The Nyström discretization is then used to obtain an eigenvalue problem for a matrix. We employ the recursive integral method for the numerical computation of the matrix eigenvalue. Numerical results show that the proposed method is efficient and reduces computational costs.

Keywords: transmission eigenvalues, inverse scattering, boundary integral equations, Nyström method, Schur complement, spectral projection

1 Introduction

We consider in this paper the calculation of the transmission eigenvalue problems, which play an important role in scattering theory for inhomogeneous media. Transmission eigenvalues are not only related to the validity of the linear sampling method [4], but also give information on the material properties of the scattering object [2, 5]. The transmission eigenvalue problem is a boundary value problem for a coupled pair of partial differential equations in a bounded domain. But that problem is not covered by the standard theory of elliptic partial differential equations since it is neither elliptic nor self-adjoint. Hence its study is widely perceived as a challenging issue. Calculating the transmission eigenvalues requires special effort.

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Research associated with transmission eigenvalues have focused on two main themes. The first concerns are the general theory of these problems such as the solvability, discreteness and existence, and the spectral properties of the transmission eigenvalue problems [2, 5, 19]. The mathematical methods for studying these problems include the variational methods and boundary integral equation methods [7]. Efficient numerical methods to determine transmission eigenvalues is the second topic. The basis for the numerical techniques for solving the transmission eigenvalues are the finite element [10, 21], boundary element methods [3, 7, 12, 18] and radial basis functions [13]. Note that the transmission eigenvalue problem is non-linear and non-selfadjoint. The numerical discretization usually leads to a non-Hermitian and nonlinear matrix eigenvalue problem. That is very challenging in numerical linear algebra. The integral based methods [1, 9] were developed to compute the corresponding matrix eigenvalues. An approximation to the eigenvalue in a given simple closed curve in the complex plane is found by spectral projection using counter integral of the resolvent [9, 23].

In this paper, we develop a new integral equation formulation in terms of the Schur complement to a two by two system of boundary integral equations, which is used to formulate the transmission eigenvalue problem. If one of those boundary integral operators is not invertible, we employ the regularization strategy for modifying the Schur complement. The Nyström method based on trigonometric interpolation is used to the discretization of the integral equations for the domains with smooth boundary. For domains with corners, we replace the uniform mesh to sigmoidal-graded meshes. The nonlinear matrix eigenvalue problem is computed by the recursive integral method. This new method proposed reduces the computational costs and can be used to study the transmission eigenvalues for a more general refractive index and domain.

We organize this paper in five sections. The boundary integral equation formulations for the transmission eigenvalue problems are developed in Section 2. We propose a Schur complement with regularization for the two by two boundary integral equations in Section 3 and introduce the recursive integral method to find the eigenvalues for a bounded operator. We describe a Nyström discretization for the boundary integral operators and spectral projection in Section 4. Finally, numerical results are presented in Section 5 to confirm the effectiveness of the proposed method.

2 Integral equation method

We present in this section the integral equations of the Helmholtz interior transmission problem. The integral equations for that problems are formulated as a $2 \times 2$ system of integral equations, which are derived from Green’s formulas.

We begin with a brief introduction to the transmission eigenvalue problem. Let $\Omega \subset \mathbb{R}^2$ be an open bounded Lipschitz domain. Let $\mu$ be the constant refraction index such that $\mu > 1$. The transmission eigenvalue problem is to find $\kappa \in \mathbb{C}$ such that there exist non-trivial solutions $w \in L^2(\Omega)$ and $u \in L^2(\Omega)$
with \( w - u \in H^2(\Omega) \) satisfying

\[
\begin{align*}
\Delta w + \kappa^2 \mu w &= 0, \text{ in } \Omega, \\
\Delta u + \kappa^2 u &= 0, \text{ in } \Omega, \\
w - u &= 0, \text{ on } \Gamma, \\
\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu} &= 0, \text{ on } \Gamma,
\end{align*}
\]

where \( \Gamma := \partial \Omega \) and \( \nu \) is the unit outward normal to \( \Gamma \). According to [5], any nonzero value \( \kappa \) such that there are nontrivial solutions \( w \) and \( u \) of (1)-(4) is called a transmission eigenvalue.

We first recall the boundary integral operators. Let \( \Phi_\kappa \) be the Green’s function given by

\[
\Phi_\kappa(x, y) := \frac{i}{4} H_0^{(1)}(\kappa|x - y|), \quad x, y \in \mathbb{R}^2,
\]

where \( H_0^{(1)} \) is the Hankel function of the first kind of order zero. The single and double layer potentials are defined by

\[
(SL_\kappa[\phi])(x) := \int_\Gamma \Phi_\kappa(x, y) \phi(y) ds(y), \quad x \in \mathbb{R}^2 \setminus \Gamma,
\]

and

\[
(DL_\kappa[\phi])(x) := \int_\Gamma \frac{\partial \Phi_\kappa(x, y)}{\partial \nu(y)} \phi(y) ds(y), \quad x \in \mathbb{R}^2 \setminus \Gamma,
\]

where \( \phi \) is an integrable function. The interior Dirichlet traces on \( \Gamma \) of the single and double layer potentials are given by

\[
(SL_\kappa[\phi])^- = SL_\kappa[\phi] = \frac{\partial w}{\partial \nu} = \alpha, \\
(DL_\kappa[\phi])^- = DL_\kappa[\phi] - \frac{\phi}{2},
\]

where the single and double layer operators are defined by

\[
(SL_\kappa[\phi])(x) := \int_\Gamma \Phi_\kappa(x, y) \phi(y) ds(y), \quad x \in \Gamma,
\]

and

\[
(DL_\kappa[\phi])(x) := \int_\Gamma \frac{\partial \Phi_\kappa(x, y)}{\partial \nu(y)} \phi(y) ds(y), \quad x \in \Gamma.
\]

We now present the integral formulations for the transmission eigenvalue problem. To this end, we denote that \( \kappa_1 := \kappa \sqrt{\mu} \)

\[
\alpha := \frac{\partial w}{\partial \nu}, \quad \beta := w, \text{ on } \Gamma.
\]

According to the boundary conditions (3)-(4), we have that

\[
\frac{\partial u}{\partial \nu} = \alpha \text{ and } u = \beta, \text{ on } \Gamma.
\]
We then have the following integral representation

\[ w = S\mathcal{L}_{\kappa_1}[\alpha] - D\mathcal{L}_{\kappa_1}[\beta] \quad \text{and} \quad u = S\mathcal{L}_{\kappa}[\alpha] - D\mathcal{L}_{\kappa}[\beta] \quad \text{in} \quad \Omega. \] (8)

This together with (5) and (6) yields that

\[ \beta = S_{\kappa_1}[\alpha] - D_{\kappa_1}[\beta] + \frac{\beta}{2} \quad \text{and} \quad \beta = S_{\kappa}[\alpha] - D_{\kappa}[\beta] + \frac{\beta}{2} \quad \text{on} \quad \Gamma. \] (9)

From (9), we obtain the following 2 \times 2 system of boundary integral equations:

\[ \mathcal{Z}(\kappa) \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \] (10)

where

\[ \mathcal{Z}(\kappa) = \begin{pmatrix} \mathcal{I}/2 + D_{\kappa} & -S_{\kappa} \\ \mathcal{I}/2 + D_{\kappa_1} & -S_{\kappa_1} \end{pmatrix}. \]

and \( \mathcal{I} \) denotes the identity operator. The transmission eigenvalues are \( \kappa \)'s satisfying (10). This implies that \( \kappa \) is a transmission eigenvalue if zero is an eigenvalue of \( \mathcal{Z} \).

We now recall some properties of the aforementioned integral operators.

\textbf{Lemma 2.1} ([11, 20]). Let \( \Gamma \) be of class \( C^{2,1} \). Then we have

(i) The operator \( S_{\kappa} : H^{1/2}(\Gamma) \rightarrow H^{3/2}(\Gamma) \) is Fredholm with index zero.

(ii) The operator \( D_{\kappa} : H^{3/2}(\Gamma) \rightarrow H^{3/2}(\Gamma) \) is compact.

The following theorem follows directly from Lemma 2.1

\textbf{Theorem 2.1.} Let \( \Gamma \) be of class \( C^{2,1} \). Then the operator \( \mathcal{Z}(\kappa) : H^{3/2}(\Gamma) \times H^{1/2}(\Gamma) \rightarrow H^{3/2}(\Gamma) \times H^{3/2}(\Gamma) \) is Fredholm of index zero and analytic on \( \kappa \in \mathbb{C} \setminus \mathbb{R}^- \).

\section{The Schur complement and recursive integral method}

We present in this section the Schur complement with regularization for the system of the boundary integral equations (10). This leads to a nonlinear and non-selfadjoint eigenvalue problem. We use contour integral based on spectral projection to test if zero is an eigenvalue of the corresponding operators.

We first introduce Schur complement for the 2 \times 2 block operator \( \mathcal{Z} \). If zero is an eigenvalue of \( \mathcal{Z}(\kappa) \) for \( \kappa \in \mathbb{C} \setminus \mathbb{R}^- \), we have nontrivial solutions to the equations

\[ (\mathcal{I}/2 + D_{\kappa})\beta - S_{\kappa}\alpha = 0, \] (11)

\[ (\mathcal{I}/2 + D_{\kappa_1})\beta - S_{\kappa_1}\alpha = 0. \] (12)
If we assume that $S_{\kappa_1}$ is invertible, we first solve for $\alpha$ from (12), getting
$$\alpha = S_{\kappa_1}^{-1}(I/2 + D_{\kappa_1})\beta.$$ and substituting this expression for $\alpha$ in the equation (11), we obtain that
$$[(I/2 + D_\kappa) - S_\kappa S_{\kappa_1}^{-1}(I/2 + D_{\kappa_1})]\beta = 0.$$ If $S_{\kappa_1}$ is invertible, we call the operator
$$A(\kappa) := (I/2 + D_\kappa) - S_\kappa S_{\kappa_1}^{-1}(I/2 + D_{\kappa_1})$$ (13)
the Schur complement of $S_{\kappa_1}$ in $Z_1(\kappa)$. We conclude that if zero is an eigenvalue of $A(\kappa)$, $Z(\kappa)$ has an eigenvalue equal to zero. The transmission eigenvalues are $\kappa$’s satisfying $A(\kappa)\beta = 0$.

We remark in passing that the single layer operator $S_{\kappa_1}$ is invertible for $\Omega$ with $C^{2,1}$ boundary, if $\kappa_1^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $\Omega$. For the above approach, we have to exclude the Dirichlet eigenvalue in $\Omega$. To overcome this weakness, we propose schur complement with a regularization for $Z$.

We now develop a regularization technique for solving (11)-(12) for the case $S_{\kappa_1}$ is not invertible. Let $\eta > 0$ be a regularization parameter. Following the idea of the standard Tikhonov regularization, we introduce the operator
$$A^0(\kappa) := (I/2 + D_\kappa) - S_\kappa(\eta I + S_{\kappa_1}^*S_{\kappa_1})^{-1}S_{\kappa_1}^*(I/2 + D_{\kappa_1}),$$ (14)
which is called the regularized Schur complement of $S_{\kappa_1}$ in $Z_1(\kappa)$. We note that $A^0 = A$ for $\eta = 0$. According to Lemma [2.1] we have the following Fredholm property of $A^0$.

**Theorem 3.1.** Let $\Gamma$ be of class $C^{2,1}$ and $\eta \geq 0$. Then the operator $A^0(\kappa) : H^{3/2}(\Gamma) \to H^{3/2}(\Gamma)$ is Fredholm of index zero and analytic on $\kappa \in \mathbb{C}\setminus\mathbb{R}^-$.

In the following part, we present the method to test if zero is an eigenvalue of $A^0(\kappa)$. Note that $A^0(\kappa)\beta = 0$ is a nonlinear integral eigenvalue problem and the discretization for the integral equations by the Nyström method leads to a dense matrices. We shall use the recursive integral method to test that zero is an eigenvalue of $A^0(\kappa)$ or not, and recall that method as follows. We define the resolvent set of $A^0(\kappa)$ by
$$\rho(A^0(\kappa)) := \{z \in \mathbb{C} : (zI - A^0(\kappa))^{-1} \in B(\mathbb{X})\},$$
and its spectrum $\sigma(A^0(\kappa))$, where $\mathbb{X}$ is a Banach space. Let the spectral projection $\mathcal{P}$ associated with $A^0(\kappa)$ and zero denote by
$$\mathcal{P} := \frac{1}{2\pi i} \int_\gamma (zI - A^0(\kappa))^{-1}dz,$$ (15)
where $\gamma$ is a closed rectifiable curve on the complex plane in $\rho(A^0(\kappa))$ enclosing zero, but no any other point in $\sigma(A^0(\kappa))$. If there are no eigenvalues inside $\gamma$, we have that $\mathcal{P}f = 0$ for $f \in \mathbb{X}$. Let $f$ be randomly chosen and $\gamma$ be a circle with small diameter. $\|\mathcal{P}^2f\|$ is used to decide whether zero is an eigenvalue of $A^0(\kappa)$ or not, since $\mathcal{P}^2 = \mathcal{P}$. Hence, we need not to compute the eigenvalues of $A^0(\kappa)$, and compare them with zero.
4 The Nyström discretization

We present in this section the Nyström discretization of the operator \( Z(\kappa) \) and the spectral projection \( P \) for completeness. We refer the readers to [16, 17] for more details on the Nyström discretization for domains with smooth boundary, and [8] for Lipschitz domains.

We first parametrize the boundary integral operators \( S_\kappa \) and \( D_\kappa \). We assume that the boundary curve \( \Gamma \) is described by a \( 2\pi \)-periodic parametric representation of the form

\[
\Gamma := \{ z(t) := (z_1(t), z_2(t))^\top : t \in [0, 2\pi] \}.
\]

Let \( J_n \) denote the Bessel function of the first kind of order \( n \). The parameterized operator \( S_\kappa \) is

\[
(S_\kappa[\phi])(s) = \int_0^{2\pi} K^S(s, t; \kappa) \phi(t) dt, \quad \text{for } s \in [0, 2\pi],
\]

where \( \phi(t) = \phi(z(t)) \) and

\[
K^S(s, t; \kappa) = \frac{i}{4} H^{(1)}_0(\kappa|z(s) - z(t)|) = K^S_1(s, t; \kappa) \ln \left\{ 4 \sin^2 \frac{s - t}{2} \right\} + K^S_2(s, t; \kappa)
\]

with

\[
K^S_1(s, t; \kappa) = -\frac{1}{4\pi} J_0(\kappa|z(s) - z(t)|)z'(t), \quad K^S_2(s, t; \kappa) = K^S(s, t; \kappa) - K^S_1(s, t; \kappa) \ln \left\{ 4 \sin^2 \frac{s - t}{2} \right\}.
\]

Note that

\[
K^S_1(t, t; \kappa) = -\frac{1}{4\pi} |z'(t)| \quad \text{and} \quad K^S_2(t, t; \kappa) = \left( \frac{i}{4} - \frac{\gamma}{2\pi} - \frac{1}{4\pi} \left( \frac{\kappa^2}{4} |z'(t)|^2 \right) \right) |z'(t)|,
\]

with Euler’s constant \( \gamma \).

Recalling the definition [7], we notice that the kernel of \( D_\kappa \) has singularity at the corners and the definition of that integral is understood in the sense of Cauchy principal value integral. We split the kernel into smooth and singular components. The parameterized operator \( D_\kappa \) is

\[
(D_\kappa[\phi])(s) = \int_0^{2\pi} (K^D(s, t; \kappa) - K^D(s, t; 0)) \phi(t) dt
\]

\[
+ \int_0^{2\pi} K^D(s, t; 0)(\phi(t) - \phi(s)) dt
\]

\[
+ \phi(s) \int_0^{2\pi} K^D(s, t; 0) dt, \quad \text{for } s \in [0, 2\pi],
\]

where
\[ K^D(s,t;\kappa) = \frac{i\kappa (z(s) - z(t)) \cdot \nu(z(t))}{4 |z(s) - z(t)|} H_1^{(1)}(\kappa |z(s) - z(t)|) z'(t) \]
\[ = K^D_1(s,t) \ln \left\{ 4 \sin^2 \frac{s - t}{2} \right\} + K^D_2(s,t), \]

with
\[ K^D_1(s,t;\kappa) = -\frac{\kappa (z(s) - z(t)) \cdot \nu(z(t))}{4\pi |z(s) - z(t)|} J_1(\kappa |z(s) - z(t)|) z'(t), \]
\[ K^D_2(s,t;\kappa) = K^D(s,t;\kappa) - K^D_1(s,t;\kappa) \ln \left\{ 4 \sin^2 \frac{s - t}{2} \right\}, \]

and
\[ K^D(s,t;0) = \frac{1}{2\pi} \frac{(z(s) - z(t)) \cdot \nu(z(t))}{|z(s) - z(t)|^2} |z'(t)|. \]

Note that
\[ K^D_2(t,t;\kappa) = K^D(t,t;\kappa) = \frac{1}{4\pi} \frac{z''(t) \cdot \nu(z(t))}{|z'(t)|}. \]

We now approximate the integral operators \( Z(\kappa) \) as follows. For the 2\( \pi \)-periodic integrands we choose an equidistant set of knots \( t_j := \pi j/n \) with \( n \in \mathbb{N} \) and \( j = 0, 1, \ldots, 2n - 1 \). We approximate the operators \( S_\kappa \) and \( D_\kappa \) by the quadratures
\[ \int_0^{2\pi} \varphi(t) \, dt \approx \frac{\pi}{n} \sum_{j=0}^{2n-1} \varphi(t_j), \]
\[ \int_0^{2\pi} \varphi(t) \ln \left\{ 4 \sin^2 \frac{s-t}{2} \right\} \, dt \approx \sum_{j=0}^{2n-1} R_j^{(n)}(s) \varphi(t_j), \]

with
\[ R_j^{(n)}(s) = \frac{2\pi}{n} \sum_{\ell=1}^{n-1} \frac{\ell}{\pi^2} \cos \left( \ell \left( s - t_j \right) \right) - \frac{\pi^2}{n^2} \cos \left( n \left( s - t_j \right) \right). \]

The sequences of numerical integration operators for \( S_{\kappa_1}, S_\kappa, D_{\kappa_1} \) and \( D_\kappa \) are denoted by \( S_{\kappa_1,n}, S_\kappa,n, D_{\kappa_1,n} \) and \( D_{\kappa,n,n} \). We then approximate \( Z(\kappa) \) by a sequence of numerical integration operators
\[ Z_n(\kappa) = \begin{pmatrix} I_n/2 + D_{\kappa,n} & -S_{\kappa,n} \\ I_n/2 + D_{\kappa_1,n} & -S_{\kappa_1,n} \end{pmatrix}. \]

The transmission eigenvalues are approximated by \( \kappa \)'s satisfying
\[ Z_n(\kappa) \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]
In passing, we comment on that the eigenpairs $Z_n$ and $Z$ are related to each other in some sense. We denote the space of trigonometric polynomials by
\[ T_n := \text{span}\{1, \cos nt, \cos mt, \sin mt : m = 1, 2, \ldots, n - 1\}. \]

For $g_n := [g_0, g_1, \ldots, g_{2n-1}]^\top \in \mathbb{C}^{2n}$, there exists a unique trigonometric polynomial $v_n \in T_n$ of the form
\[ v_n(t) := \sum_{j=0}^n a_j \cos j t + \sum_{j=1}^{n-1} b_j \sin j t, \]
satisfying $v_n(t_j) = g_j$ ($j = 0, 1, \ldots, 2n - 1$) and $v_n \in C[0, 2\pi]$. This implies that $Z_n$ and $Z$ are bounded operators on $C[0, 2\pi]$. Applying the pointwise convergence of the Nyström method [14], we obtain the spectral properties of $Z_n$ and $Z$ in $C[0, 2\pi]^2$ in the following theorem.

**Theorem 4.1.** If $\kappa \in \mathbb{C}\setminus\mathbb{R}^-$ is not an eigenvalue of $Z$, then there exists a positive integer $n_0$, such that for all $n \in \mathbb{N}$ with $n > n_0$, $\kappa$ is also not an eigenvalue of $Z_n$.

**Proof.** We assume that $\kappa \in \mathbb{C}\setminus\mathbb{R}^-$ is not an eigenvalue of $Z$. This implies that $Z^{-1}$ is bounded. For any $v \in C[0, 2\pi]^2$ with $\|v\|_\infty = 1$, we define $\xi := v/\|Zv\|_\infty$. Note that $\|\xi\|_\infty \leq \|Z^{-1}\|$. There exists a positive integer $n_0$ such that for all $n \in \mathbb{N}$ with $n > n_0$,
\[
\|Z\xi - Z_n Z\xi\|_\infty \geq \|Z^{-1}(Z - Z_n)Z\xi\|_\infty \\
\leq \|Z^{-1}\| \|(Z - Z_n)\|_2 \|\xi\|_\infty \\
\leq \|Z^{-1}\|_2 \|(Z - Z_n)\|_2 \\
\leq \frac{1}{2}.
\]

For any $v_n \in T_n$ with $\|v_n\|_\infty = 1$, let $\xi_n := v_n/\|Z v_n\|_\infty$. This leads to that $\|Z\xi_n\|_\infty = 1$. We obtain that there exists a positive integer $n_0$ such that for all $n \in \mathbb{N}$ with $n > n_0$,
\[
\|Z^{-1} Z_n Z\xi_n\|_\infty \geq \|Z\xi_n\|_\infty - \|Z\xi_n - Z_n Z\xi_n\|_\infty \geq \frac{1}{2}.
\]
This yields that $\kappa$ is not an eigenvalue of $Z_n$ and completes the proof.

We conclude from Theorem [4.1] that $Z_n$ is free of spurious eigenvalues of $Z$ for $n$ large enough. We determine the transmission eigenvalues by [18]. The Schur complement of $S_{\kappa,1,n}$ in $Z_n$ with regularization is given by
\[ A_0^\eta(\kappa) := (I/2 + D_{\kappa,n}) - S_{\kappa,n}(\eta I + S_{\kappa,n}^* S_{\kappa,n})^{-1} S_{\kappa,n}^* S_{\kappa,n} (I/2 + D_{\kappa,n}), \]
where $\eta > 0$. For the case $S_{\kappa,1,n}$ is invertible, the Schur complement of $S_{\kappa,1,n}$ in $Z_n$ is given by
\[ A_0^\eta(\kappa) := (I/2 + D_{\kappa,n}) - S_{\kappa,n} S_{\kappa,1,n}^{-1} (I/2 + D_{\kappa,n}). \]
We then consider the Nyström discretization for the domain with corners. We assume that the domain $\Omega$ has corners at $z(T_j)$ for $j = 1, 2, \ldots, m$, where $0 \leq T_1 < T_2 < \cdots < T_m < 2\pi$. $z_1$ and $z_2$ are smooth with $|z_1'(t)|^2 + |z_2'(t)|^2 > 0$ on each interval $t \in [T_j, T_{j+1}]$ for $j = 1, 2, \ldots, m - 1$. We replace the equidistant mesh by a sigmoidal-graded mesh through substituting a new variable based on the sigmoid transform \cite{7,15}. Let $T_0 = 0$ and $T_{m+1} = 2\pi$. For $j = 0, 1, 2, \ldots, m$, we define the sigmoid transform by
\[
 w(s) := \frac{T_{j+1}[v(s)]^p + T_j[1-v(s)]^p}{[v(s)]^p + [1-v(s)]^p}, \quad s \in [T_j, T_{j+1}],
\]
\[
v(s) = \left(\frac{1}{p} - \frac{1}{2}\right) \left(\frac{T_j + T_{j+1} - 2s}{T_{j+1} - T_j}\right)^3 + \frac{1}{2} \sum_{j=0}^{m-1} r e^{i\theta_j} x_j,
\]
where $p \geq 2$. By a change of variables $t \mapsto w(t)$, we obtain the new parametrization
\[
 \Gamma := \{z(w(t)) := (z_1(w(t)), z_2(w(t)))^\top : t \in [0, 2\pi]\}.
\]
With substitution of the new parametrization above, we obtain the sequences of numerical integration operators for $S_\kappa$, $S_\kappa$, $D_\kappa$, and $D_\kappa$ based on the equidistant mesh, which like the domain with smooth boundary.

We next approximate the spectral projection $\mathcal{P}$ defined by \cite{15}. We choose $\gamma$ as a small circle centered at the origin with radius $r \in (0, 1)$. We substitute $\gamma := \{z = re^{i\theta} : \theta \in [0, 2\pi]\}$ to obtain that
\[
 \mathcal{P} = \frac{1}{2\pi} \int_0^{2\pi} r e^{i\theta} (r e^{i\theta} \mathcal{I} - \mathcal{A}^\eta(\kappa))^{-1} d\theta.
\]
The quadrature rule \cite{16} is employed for the numerical computation of $\mathcal{P}$. For fixed $m \in \mathbb{N}$, and $f \in \mathbb{C}^{2^m}$, the approximation $\mathcal{P}_m$ is computed by the quadrature rule
\[
 \mathcal{P}_m f := \frac{1}{2m} \sum_{j=0}^{2m-1} r e^{i\theta_j} x_j,
\]
where $\theta_j := \pi j/m$ for $j = 0, 1, \ldots, 2m - 1$ are quadrature points, and $x_j$ are the solutions of the following linear system
\[
 (r e^{i\theta_j} \mathcal{I} - \mathcal{A}^\eta_n(\kappa)) x_j = f.
\]

We finally present the algorithm for testing whether zero is an eigenvalue of $\mathcal{A}^\eta_n$ or not.

5 Numerical results

We shall illustrate in this section the computation of interior transmission eigenvalues by using \cite{13}. The index of refraction is chosen as $\mu = 16$. 

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Algorithm Numerical computation of the transmission eigenvalues
Step 1 Choose $\kappa \in (a, b)$, which is an interval of wavenumbers.
Step 2 For fixed $n \in \mathbb{N}$, calculate $S_{\kappa_1,n}, S_{\kappa,n}, D_{\kappa_1,n}$ and $D_{\kappa,n}$.
Step 3 Choose $\eta \in [0, 1)$ to obtain $A_{\eta}^n(\kappa)$, where the regularization parameter $\eta > 0$ for the case that the condition number of $S_{\kappa_1,n}$ is large.
Step 4 For $m \in \mathbb{N}$, $r \ll 1$ and a random $f \in \mathbb{C}^2$, compute
\[
RIM_m(\kappa) := \left\| P_m \left[ \frac{P_m f}{\|P_m f\|} \right] \right\|.
\]
Step 5 Decide if $\gamma$ contains an eigenvalue and $\kappa$ is a transmission eigenvalue.

We start with an interval $(a, b)$ of wavenumbers and uniformly divide it into $N$ subintervals. For each wavenumber, the boundary integral operators are discretized with $n = 32$, namely, 64 quadrature nodes over $[0, 2\pi]$. We set $m = 64$.

Example 1. Let $\Omega$ be a disk with radius $1/2$. In this case, the exact transmission eigenvalues are $\kappa$’s such that

\[
J_1(\kappa/2)J_0(2\kappa) - 4J_0(\kappa/2)J_1(2\kappa) = 0,
\]

and

\[
J_{m-1}(\kappa/2)J_m(2\kappa) - 4J_m(\kappa/2)J_{m-1}(2\kappa) = 0,
\]

for $m \in \mathbb{N}$. The exact values in $[1.5, 5]$ are given by

$\kappa_1 = 1.9880$, $\kappa_2 = 2.6129$, $\kappa_3 = 3.2267$, $\kappa_4 = 3.7409$, $\kappa_5 = 3.8264$,

$\kappa_6 = 4.2958$, $\kappa_7 = 4.4154$, $\kappa_8 = 4.9418$, $\kappa_9 = 4.9959$.

The numerical results are presented in Figures 1-3. We mark each location of the exact eigenvalues by a red line. We first choose the interval to be $[1.6, 2.2]$ with dividing it into 100 subintervals, where the value of $RIM_m(\kappa)$ is plotted in Figure 1 with radii $r = 0.005, 0.003$ and 0.001. We see that the result for the circle with radius $r = 0.001$ is better than the other two cases. This implies that the effectiveness of recursive integral method is affected by the radius of the circle. We shall choose $r < 0.05$ in the following examples. We choose the interval to be $[2.3, 2.8]$ and $[3.3, 5]$ with dividing it into 100 subintervals in Figure 2. We choose the interval to be $[3.5, 4]$ with dividing it into 100 subintervals, and $[4, 5]$ with dividing it into 200 subintervals in Figure 2. We see that the value of $RIM_m(\kappa)$ is zero except the location near the eigenvalues. This concurs with the theoretical estimate. We finally search for transmission eigenvalues in the complex plane $\mathbb{C}$. The real part is seted in the interval $[4.85, 4.95]$ and divided it.
Figure 1: Plots of $\log RIM_n(\kappa)$ with different radii $r$ in Example 1: (a) $r = 0.005$ (b) $r = 0.003$ (c) $r = 0.001$.

into 20 subinterval. The imaginary part is chosen in the intervals $[0.5, 0.7]$ and $[-0.7, -0.5]$, where each interval is divided into 200 subintervals. The result are presented in Figure 3. We find that there exist a pair of complex eigenvalues around $k = 4.90 \pm 0.58i$.

**Example 2.** We consider in this example a peanut-shaped domain enclosed by the equation

$$\sqrt{0.25 + \cos^2 t}(\cos t, \sin t), \quad t \in [0, 2\pi].$$

We choose the interval to be $[1.3, 1.6]$ and $[1.65, 2]$ with dividing it into 200 subintervals. The results are shown in Figure 3. We compare the results with the eigenvalues computed by finite element methods from [22]. Each location of these eigenvalues is marked by a red line. We conclude that the algorithm proposed in this paper is effective.

In the following four examples, we test the method with regularization for some domains with corners, where the domains are plotted in Figure 5. We note that the location of the eigenvalues computed by finite element methods from [22] is marked by a red line in Example 3 and 4.

**Example 3.** We consider in this example a unit square centered at the origin.
Figure 2: Plots of $\log RIM_n(\kappa)$ in different intervals for Example 1: (a) $[2.3, 2.8]$ (b) $[3, 3.5]$ (c) $[3.5, 4]$ (d) $[4, 5]$.

Figure 3: Plots of $\log RIM_n(\kappa)$ for Example 1 in the complex plane: (a) contour plot (b) surface plot.
Figure 4: Plots of log $\text{RIM}_n(\kappa)$ for Example 2 in different intervals: (a) $[1.3, 1.6]$ (b) $[1.65, 2]$.

Figure 5: Several domains with corners: (a) square (b) triangle (c) L-shape (d) pentagon.
We choose the intervals to be $[1.65, 1.95]$, $[2.25, 2.55]$ and $[2.7, 3]$. Each interval is divided into 100 subintervals. The results are shown in Figures 6-7. We plot the results in Figure 6 for different regularization parameters with $\eta = 10^{-m}$, $m = 2, 3, \ldots, 7$. We see that the result for $\eta = 10^{-2}$ and $\eta = 10^{-7}$ is worst. This implies that the regularization is indeed necessary and effective. According to Figure 6, we shall use $\eta = 10^{-5}$ as the regularization parameter in the following examples. The results for the intervals $[2.25, 2.55]$ and $[2.7, 3]$ are shown in Figure 7.

**Example 4.** We consider in this example a triangle with vertexes $(-\sqrt{3}/2, -1/2)$, $(\sqrt{3}/2, -1/2)$ and $(0, 1)$. We choose the intervals to be $[1.65, 1.95]$, $[2.10, 2.40]$ and $[2.70, 3.00]$. Each interval is divided into 100 subintervals. The results are shown in Figure 5.

**Example 5.** We consider in this example the L-shape domain with vertexes $(\sqrt{2}/2, \sqrt{2}/2)$, $(\sqrt{2}/2, -\sqrt{2}/2)$, $(-\sqrt{2}/2, \sqrt{2}/2)$ and $(0, -\sqrt{2}/2)$. We choose the intervals to be $[1.5, 2]$, $[2.2, 5]$ and $[2.5, 3]$. Each interval is divided into 200 subintervals. The results are shown in Figure 4. The eigenvalues computed in this domains are $1.5541$, $1.8920$, $2.1131$, $2.4937$, $2.5226$, $2.7349$.

**Example 6.** In the last example, we consider a regular pentagon with vertexes $(\cos(2\pi j/n), \sin(2\pi j/n))$, $j = 0, 1, 2, 3, 4$. We choose the intervals to be $[1.5, 2]$, $[2.2, 5]$ and $[2.5, 3]$. Each interval is divided into 200 subintervals. The results are shown in Figure 7. The eigenvalues computed in this domain are $1.8945$, $2.2563$, $2.4724$, $2.6432$, $2.8769$, $2.9146$ and $2.9523$.

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**References**

[1] W-J Beyn. An integral method for solving non-linear eigenvalue problems. *Linear Algebra Appl.*, 436:3839–3863, 2012.
Figure 6: Plots of $\log RIM_n(\kappa)$ for different $\eta$ in Example 3: (a) $\eta = 10^{-2}$ (b) $\eta = 10^{-3}$ (c) $\eta = 10^{-4}$ (d) $\eta = 10^{-5}$ (e) $\eta = 10^{-6}$ (f) $\eta = 10^{-7}$. 
Figure 7: Plots of log RIM$\_n(\kappa)$ for Example 3 in different intervals: (a) [2.25, 2.55] (b) [2.7, 3].

Figure 8: Plots of log RIM$\_n(\kappa)$ for Example 4 in different intervals: (a) [1.65, 1.95] (b) [2.1, 2.4] (c) [2.7, 3].
Figure 9: Plots of $\log \text{RIM}_n(\kappa)$ for Example 5 in different intervals: (a) $[1.5, 2]$ (b) $[2, 2.5]$ (c) $[2.5, 3]$. 
Figure 10: Plots of $\log \text{RIM}_n(\kappa)$ for Example 6 in different intervals: (a) $[1.5, 2]$ (b) $[2, 2.5]$ (c) $[2.5, 3]$. 
[2] F. Cakoni, D. Colton, and H. Haddar. *Inverse Scattering Theory and Transmission Eigenvalues*. Cambridge University Press, SIAM, Philadelphia, 2016.

[3] F. Cakoni and R. Kress. A boundary integral equation method for the transmission eigenvalue problem. *Applicable Analysis*, 96(1):23–38, 2017.

[4] D. Colton and A. Kirsch. A simple method for solving inverse scattering problems in the resonance region. *Inverse Problems*, 12:383–393, 1996.

[5] D. Colton and R. Kress. *Inverse Acoustic and Electromagnetic Scattering Theory*, 4th edition. Springer Nature, Cham, 4 edition, 2019.

[6] D. Colton, P. Monk, and J. Sun. Analytical and computational methods for transmission eigenvalues. *Inverse Problems*, 26(4):045011, 2010.

[7] A. Cossonnière and H. Haddar. Surface integral formulation of the interior transmission eigenvalue problem. *J. Integral Equ. Appl.*, 25:341–376, 2013.

[8] V. Domínguez, M. Lyon, and C. Turc. Well-posed boundary integral equation formulations and Nyström discretizations for the solution of Helmholtz transmission problems in two-dimensional Lipschitz domains. *J. Integral Equ. Appl.*, 28(3):395–440, 2016.

[9] R. Hung, A. Struthers, J. Sun, and R. Zhang. Recursive integral method for transmission eigenvalues. *J. Comput. Phys.*, 327:830–840, 2016.

[10] X. Ji, J. Sun, and T. Turner. A mixed finite element method for helmholtz transmission eigenvalues. *ACM Transaction on Mathematical Software*, 38(4):Algorithm 922, 2012.

[11] A. Kirsch. Surface gradients and continuity properties for some integral operators in classical scattering theory. *Mathematical Methods in the Applied Sciences*, 11(4):789–804, 1989.

[12] A. Kleefeld. A numerical method to computer interior transmission eigenvalues. *Inverse Problems*, 29:104012, 2013.

[13] A. Kleefeld and L. Pieronek. The method of fundamental solutions for computing acoustic interior transmission eigenvalues. *Inverse Problems*, 34(3):035007, 2018.

[14] R. Kress. *Linear Integral Equations*. Springer-Verlag, Berlin Heidelberg New York, 1989.

[15] R. Kress. A Nyström method for boundary integral equations in domains with corners. *Numer. Math.*, 58(2):145–161, 1990.

[16] R. Kress. On the numerical solution of a hypersingular integral equation in scattering theory. *Journal of Computational and Applied Mathematics*, 61:345–360, 1995.
[17] R. Kress. A collocation method for a hypersingular boundary integral equation via trigonometric differentiation. *Journal of Integral Equations and Applications*, 26(2):197–213, 2014.

[18] R. Kress. Nonlocal impedance conditions in direct and inverse obstacle scattering. *Inverse Problems*, 35:024002, 2019.

[19] H. Liu. On local and global structures of transmission eigenfunctions and beyond. *arXiv:2008.03120v1*, 2020.

[20] W. McLean. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, Cambridge, 2000.

[21] J. Sun and A. Zhou. *Finite Element Methods for Eigenvalue Problems*. Chapman and Hall/CRC, Boca. Raton, FL, 2016.

[22] T. Li, W. Huang, W. Lin and J. Liu. On spectral analysis and a novel algorithm for transmission eigenvalue problems. *J. Sci. Comput.*, 59(64):83–108, 2015.

[23] F. Zeng, J. Sun, and L. Xu. A spectral projection method for transmission eigenvalues. *Science China Mathematics*, 59(8):1613–1622, 2016.