ON TOPOLOGICAL COMPLEXITY OF HYPERBOLIC GROUPS

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Abstract. We show that the topological complexity of a finitely generated torsion free hyperbolic group $\pi$ with $\text{cd} \pi = n$ equals $2n$.

1. Introduction

The topological complexity of a space is a numerical invariant defined by M. Farber [F] in his study of motion planning in robotics. The topological complexity $\text{TC}(X)$ of a space $X$ is the minimum of $k$ such that $X \times X$ admits an open cover $U_0, \ldots, U_k$ with the property that over each $U_i$ there is a continuous motion planning algorithm $s_i$, i.e. a continuous map $s_i : U_i \to PX$ to the path space $PX = X^{[0,1]}$ such that $s_i((x, y))(0) = x$ and $s_i((x, y))(1) = y$ for all $(x, y) \in U_i$.

The topological complexity is a homotopy invariant in spirit of the Lusternik-Schnirelmann category $\text{cat}(X)$. Like in the case of the LS-category, generally this invariant is hard to compute. Since $\text{TC}(X)$ is homotopy invariant, one can define $\text{TC}(\pi)$, the topological complexity of a discrete group $\pi$ as $\text{TC}(\pi) = \text{TC}(B\pi)$. This definition works for the LS-category as well. Both group invariants turns to be $\infty$ in the presence of torsions. So we consider torsion free groups only. Eilenberg and Ganea [EG] proved that the LS-category of a discrete group equals its cohomological dimension, $\text{cat}(\pi) = \text{cd}(\pi)$. It turns out that in the case of topological complexity the range of $\text{TC}(\pi)$ is between $\text{cd}(\pi)$ and $2\text{cd}(\pi)$ (see [CP], [Ru], [DS]).

The minimal value of $\text{TC}$ is taken on abelian groups, $\text{TC}(\pi) = \text{cd}(\pi)$. Combining results of [DS] and [Dr1] present the cases of the maximal value $\text{TC}(\pi) = 2\text{cd}(\pi)$ for groups which are free "square", $\pi = H \ast H$, of a geometrically finite group. In this paper we investigate further the question when the topological complexity of a group is maximal. We consider groups $\pi$ with finite classifying complex $B\pi$. In particular, in this paper we prove the equality $\text{TC}(\pi) = 2\text{cd}(\pi)$ for hyperbolic groups. A giant step in computation of $\text{TC}$ for hyperbolic groups was made by Farber and Mescher [FM]. In the light of the formula $\text{cd}(\pi \times \pi) = 2\text{cd}(\pi)$ [Dr1] their result states that for hyperbolic groups with $\text{cd}(\pi) = n$ either $\text{TC}(\pi) = 2n$ or $\text{TC}(\pi) = 2n - 1$.

In the paper we use the following properties of torsion free hyperbolic groups: The centralizer of every nontrivial element is cyclic and a finitely generated torsion free hyperbolic group $\pi$ has finite classifying complex $B\pi$. Our main tool is the characterization of $\text{TC}(\pi)$ given by Farber-Grant-Lupton-Oprea [FGLO].

1.1. Theorem. The topological complexity $\text{TC}(\pi)$ of a group $\pi$ with finite $B\pi$ equals the minimal number $k$ such that the canonical map $f : E(\pi \times \pi) \to E_D(\pi \times \pi)$ admits a $(\pi \times \pi)$-equivariant deformation to the $k$-skeleton $E_D(\pi \times \pi)^{(k)}$. 
Here $E_F G$ is the notations for a classifying space for $G$-action with isotropy groups in the class of subgroups $F$ [Lu]. The class $D$ is defined by conjugations of the diagonal subgroup $\Delta(\pi) \subset \pi \times \pi$ and their intersections.

It turns out to be that the essentiality of $f$ in the top dimension can be detected cohomologically. In the case when $B\pi$ is an oriented manifold this can be done using the induced map of the orbit spaces $\bar{f} : B(\pi \times \pi) \to B_D(\pi \times \pi)$. Generally, our argument uses compactifications of the classifying spaces and the main result of [Dr1].

1.1. Notations and conventions. In this paper a group $\pi$ that admits a finite classifying space $B\pi$ is called geometrically finite. Hyperbolic groups in the paper a Gromov’s hyperbolic [Gr].

The notations for cohomology of a group is $H^*(\pi, M)$ and for cohomology of a space $H^*(X; M)$.

We consider an actions of a discrete group $G$ on CW-complex $X$ that preserve the CW-complex structure in such a way that if an open cell $e$ is taken by an element $g \in G$ to itself then $g$ restricted to $e$ is the identity map. Such a complex is called a $G$-CW-complex.

2. Classifying space $E_D G$

For a discrete group $G$ and a family of its subgroups $F$ which is closed under conjugation and finite intersection a classifying complex $E_F G$ is a $G$-CW-complex whose isotropy subgroups are from $F$ and for each $G$-CW complex with isotropy groups from $F$ there is a unique up to a $G$-homotopy a $G$-equivariant map $f : X \to E_F G$. This implies in particular, that any two such complexes $G$-homotopy equivalent. Note that the universal cover $EG$ of the classifying space $BG$ is a $G$-CW-complex. The corresponding map

$$f : EG \to E_F G$$

is called a canonical map.

It was proven in [Lu] that a $G$-CW-complex $X$ is $E_F G$ if and only if for each group $F \in F$ the fixed point set $X^F$ is contractible.

2.1. The family $D$ for groups with cyclic centralizers. Let $\pi$ be a torsion free discrete group, we set $G = \pi \times \pi$. We denote by $D$ the smallest family of subgroups $H \subset G$ which contains the diagonal $\Delta(\pi) \subset \pi \times \pi$, the trivial subgroup and which is closed under conjugations and finite intersections. A complete description of the family $D$ is given in 4.1 of [FGLO]. Let $D'$ denote the set of all groups in $D$ except the trivial group.

Also it was proven in [FGLO] that for a torsion free group $\pi$ with cyclic centralizers for any two centralizers $Z(a), Z(b)$ of elements $a, b \in \pi$ either $Z(a) = Z(b)$ or $Z(a) \cap Z(b) = \{e\}$. Their description of $D$ in the case of cyclic centralizers can be stated as follows:

2.1. Proposition. For a torsion free group $\pi$ with cyclic centralizers a subgroup $H \subset \pi \times \pi$ belongs to $D'$ if and only if it is of the form

$$(\ast) \quad H_{\gamma, b} = (\gamma, e)\Delta(Z(b))(\gamma^{-1}, e)$$

where $\gamma, b \in \pi$ and $\Delta : \pi \to \pi \times \pi$ is the diagonal map.
Thus, the family $\mathcal{D}'$ for a torsion free group with cyclic centralizers consists of the conjugates of the diagonal subgroup $\Delta(\pi) \subset \pi \times \pi$ and the conjugates of the centralizers of all nontrivial elements in $\Delta(\pi)$.

2.2. Proposition. Given $\gamma, b, a, c \in \pi$, either $H_{\gamma, b} = H_{a, c}$ or $H_{\gamma, b} \cap H_{a, c} = (e, e)$.

Proof. If $(e,e) \neq (x,y) \in H_{\gamma, b} \cap H_{a, c}$, then $c \in Z(b)$ and hence $H_{a, c} = H_{a, b}$. Also, $\gamma b^k \gamma^{-1} = a b^k a^{-1}$ for some $k$. Hence $\alpha^{-1} \gamma \in Z(b)$ and hence, $\gamma = \alpha b^n$ for some $n$. Then $H_{\gamma, b} = H_{a, b}$. \hfill $\square$

We call two elements $a, b \in \pi$ weakly conjugate if there are integers $k$ and $\ell$ such that $a^k$ is conjugate to $b^\ell$. Clearly this is an equivalence relation. Note that for a torsion free group the unit form a separate equivalence class. Let $C$ be the set of the weak conjugacy classes of nontrivial elements in $\pi$. Let $A$ be a section of $C$, i.e. a collection of elements $a_F \in F \in C$ indexed by $C$. We may assume that $a$ is a generator of $Z(a) \cong \mathbb{Z}$. If not, then $Z(a)$ is generated by some $t$ and $a = t^n$ lies in the same weak conjugacy class. So we can choose $t$ instead of $a$.

2.3. Proposition. The set of groups $\mathcal{D}'$ can be presented as the disjoint union

$$\mathcal{D}' = \bigsqcup_{a \in A} \{ g \Delta(Z(a))g^{-1} \mid g \in G \}$$

of families of groups indexed by $a \in A$ where each family consists of all distinct conjugates of $\Delta(Z(a))$.

Proof. Note that every conjugate

$$(x, y)\Delta(Z(a))(x^{-1} y^{-1}) = (xy^{-1}, e)\Delta(yZ(a)y^{-1})(xy^{-1}, e),$$

$(x, y) \in \pi \times \pi = G$, is one of the groups $H_{\gamma, b}$ from Proposition 2.1. Suppose that $b$ belongs to the weak conjugacy class of $a$, $b^k = xa^\ell x^{-1}$. Then

$$(\gamma, e)(b^k, b^\ell)(\gamma^{-1}, e) = (\gamma x, x)(a^\ell, a^\ell)(\gamma x, x)^{-1} \in (\gamma x, x)\Delta(Z(a))(\gamma x, x)^{-1}.$$  

By Proposition 2.2 $H_{\gamma, b}$ coincides with $(\gamma x, x)\Delta(Z(a))(\gamma x, x)^{-1}$. Since $Z(b) = Z(b^k)$, we obtain $H_{\gamma, b} = H_{\gamma, b^k}$ and hence,

$H_{\gamma, b} \in \{ g \Delta(Z(a))g^{-1} \mid g \in G \}$.

Now we show that the families in the union are disjoint. Suppose that

$$g \Delta(Z(a))g^{-1} = h \Delta(Z(b))h^{-1}.$$  

Then $\Delta(Z(a)) = g^{-1} h \Delta(Z(b))h^{-1} g$. Let $g^{-1} h = (x, y)$. Then the condition

$$g^{-1} h \Delta(Z(b))h^{-1} g \subset \Delta(\pi)$$

implies $xb^{-1} = yb^{-1}$. Therefore, $x^{-1} y \in Z(b)$ and, hence, $y = xb^k$ for some $k$. Then

$$(x, y) \Delta(Z(b))(x^{-1}, y^{-1}) = (x, x)(e, b^k)\Delta(Z(b))(e, b^{-k})(x^{-1}, x^{-1}) = \Delta(xZ(b)x^{-1}) = \Delta(Z(xb^{-1})) = \Delta(Z(a)).$$

The last equality implies that $xb^{-1} = a^k$ for some $k$. Thus, $b$ is weakly equivalent to $a$. \hfill $\square$

We note that the unit $e \in \pi$ form a separate conjugacy class with $Z(e) = \pi$. Thus, the summand corresponding to $e \in A$ consists of conjugates of the diagonal subgroup.
2.2. Construction of the classifying space $E_D G$. Assume that a group $G$ acts on a space $X$ and $A$ is a $G$-invariant closed subset. Let $\phi : A \to Y$ be a $G$-equivariant map. Then, clearly, the space $Y \cup_{\phi} X$ obtained from attaching $X$ to $Y$ along $A$ admits a natural $G$-action.

2.4. Definition. Let $X$ be a CW-complex with a $G$-action preserving the CW-complex structure. Let $\phi : \partial D^n \to X^{(n-1)}$ be an attaching map. We say that the complex

$$Z = X \cup_{\bigcup_{g \in G}} (\bigsqcup_{g \in G} D^n)$$

is obtained from $X \cup_{\phi} D^n$ by the $G$-translation. Thus, the action of $G$ on $X$ extends to $Z$.

Let $G = \pi \times \pi$ and let $E\pi$ be the universal covering of a classifying complex $B\pi$ of the group $\pi$. As the above we assume that $\pi$ is torsion free and it has centralizers of all nontrivial elements cyclic. Note that the translates of the diagonal $\Delta = \Delta(E\pi) \subset E\pi \times E\pi$ by elements of $g \in G$ are either disjoint or coincide. We consider the quotient space

$$X = (E\pi \times E\pi) / \{g\Delta\}_{g \in G}$$

obtained from the product $E\pi \times E\pi$ by collapsing to points all translates of the diagonal. We may assume that $E\pi$ has a $G$-equivariant CW-complex structure such that $\Delta$ is a subcomplex. Then $X$ has the quotient CW-complex structure. Note that the distinct translates $g\Delta$ of $\Delta$ can be indexed by elements of $\pi$ in the following way $\{((\gamma, e)\Delta)_{\gamma \in \pi}\}$. We denote the image of of $(\gamma, e)\Delta$ in $X$ by $v_\gamma$ and the image of $\Delta$ by $v = v_e$. Thus, $v_\gamma = (\gamma, e)v$ are vertices in $X$. Note that $X$ is contractible and the action of $G$ on $E\pi \times E\pi$ induces a $G$-action on $X$.

Let $a$ be a generator of the centralizer $Z(a) \cong \mathbb{Z}$, $a \in \pi$. Then the set $\{(a^n, e)v \mid n \in \mathbb{Z}\}$ is the fixed point set for the group $\Delta(Z(a))$ and the set

$$\{g(a^n, e)(v) \mid n \in \mathbb{Z}\}$$

is the fixed point set of the group $g\Delta(Z(a))g^{-1}$ for $g \in G$.

We construct $E_D G$ by attaching to $X$ cells of dimensions 1, 2, and 3.

We consider the natural action of $G$ on the set of unordered pairs $\{\{u, v\} \mid u, v \in Gv\}$.

2.5. Proposition. For any $a \in A$ the stabilizer of the pair $\sigma = \{v, v_a\}$ is the centralizer $\Delta(Z(a))$.

Proof. Suppose that $(x, y)v = \{v, v_a\}$, $x, y \in \pi$. There are two cases: $(x, y)v = v$ and $(x, y)v = v_a$ or $(x, y)v = v_a$ and $(x, y)v = v$.

In the first case, from the equality $(x, y)v = (xy^{-1}, v)v = v$ we obtain that $xy^{-1} = e$ and, hence, $x = y$. From the condition $(x, y)v_a = v$, we obtain $(x, y)v_a = (x, a)(a, e)v = (xa, x)v = (xa, x)(e)v = (a, e)v$ and, hence, $xa^{-1}x = a$. Thus $x \in Z(a)$ and $(x, y) \in \Delta(Z(a))$.

In the second case we obtain the pair $(x^2, y^2)$ satisfies the conditions of the first case. Therefore, $x^2 = y^2$ and $x^2, y^2 \in Z(a)$. The latter implies that $x \in Z(a)$ and $y \in Z(a)$ (see [FGLO]).

2.6. Corollary. For every $n \in \mathbb{Z}$ the stabilizer of the pair $\sigma_n = \{v_{a^n}, v_{a^{n+1}}\}$ is $\Delta(Z(a))$. 

\[\square\]
Proof. We note that $\sigma_n = (a^n, e)\sigma$. Hence the stabilizer of $\sigma_n$ is the conjugate of the stabilizer of $\sigma$,
$$(a^n, e)\Delta(Z(a))(a^{-n}, e) = \Delta(Z(a)).$$
\square

For a fixed $a \in A$ we construct a $G$-equivariant CW complex $X_a$ containing $X$ as an invariant subset. Let the pair $\sigma = \{v, v_a\}$ also denote the interval spanned by $\{v, v_a\}$. We consider such an interval for each pair of points in the orbit $G\{v, v_a\}$. Thus, we obtain a $G$-equivariant family of intervals attached to $X$ such that no two interval have the same set of vertices. We denote a new complex by $X^1_a$. The action of $G$ on $X$ extends to $X^1_a$. Note that in $X^1_a$ the intervals $(a^n, e)\sigma_n, n \in \mathbb{Z}$, form a real line $R_a \cong \mathbb{R}$ attached to $X$ along the integers $\mathbb{Z} \subset \mathbb{R}$ to the vertices $v_a\sigma_n$. Then $R_a$ is the fixed point set for $\Delta(Z(a))$ and it is contractible. Moreover, each group $g\Delta(Z(a))g^{-1}, g \in G$ has the contractible fixed point set $gR_a$.

Next we will make $X^1_a$ simply connected. For that we choose a subcomplex $I \subset X$ homeomorphic to the interval $[0, 1]$ connecting the vertices $v$ and $v_a$ and attach a 2-disk $D$ to the circle $I \cup \sigma$. Then we define $X^2_a$ to be a $G$-complex obtained from $X^1_a \cup D$ by the $G$-translation.

The disks $D$ and $(a, a)D$ together with a contractible space $X$ form in $X^2_a$ a unique up to homotopy 2-sphered $S_n$. We fill it with a 3-ball $B$, using an attaching map for
$$\phi : \partial B \to D \cup (a, a)D \cup X^{(2)}.$$ We define a $G$-complex $X_a$ as the complex obtained from $X^2_a \cup \phi B$ by the $G$-translation.

2.7. Proposition. The CW complex $X_a$ has the following properties:
1. $X_a$ is $G$-equivariant;
2. $X_a$ is contractible;
3. The groups $g\Delta(Z(a))g^{-1}$ have contractible fixed point sets;
4. dim$(X_a \setminus X) = 3$.

Proof. The conditions 1, 3, and 4 are obvious. We prove 2.

Let $Y_a$ be a CW-complex obtained by the above procedure from $X$ by attaching the cells $\sigma$, $D$, and $B$ with the translations by the group $\Delta(Z(a)) \cong \mathbb{Z}$. Since $\sigma$ is fixed by $\Delta(Z(a))$, the complex $Y_a$ has one new 1-cell $\sigma$. In dimensions 2 and 3, it has cells indexed by $\mathbb{Z}$ such that the quotient CW complex $Y_a/(X \cup D)$ isomorphic to the reduced double suspension over the reals $\mathbb{R}$ with $\mathbb{Z}$ as the 0-skeleton. Therefore, $Y_a$ is contractible. Note that
$$X_a = \bigcup_{[g] \in G/\Delta(Z(a))} gY_a$$
with $X$ being the common part of all summands $gY_a$. Hence $X_a$ is contractible. \square

We perform this construction for all $a \in A$ and define
$$X_3 = \bigcup_{a \in A} X_a$$
which is obtained from the disjoint union $X_3 = \bigsqcup_{a \in A} X_a$ by the identification of all copies of $X \subset X_a$. 
2.8. Proposition. For \(a, b \in A, \ a \neq b\), The orbits of the pairs \(G\{v, v_a\}\) never lands in \(R_b\).

Proof. Suppose \(g(v) = v_bk\) and \(g(v_a) = v_bm\). Let \(g = (x, y) \in \pi \times \pi\). The first equality implies that \((x, y)v = (xy^{-1}, e)v = (b^k, e)v\) and, hence, \(x = b^ky\). The second equality implies \((x, y)(a, e)v = (xa, y)v = (xay^{-1}, e)v = (b^m, e)v\) and, therefore, \(xay^{-1} = b^m\). Thus, \(b^kyay^{-1} = b^m\) which implies that \(a\) and \(b\) are weakly conjugate. This contradicts to the choice of \(a\) and \(b\). □

2.9. Proposition. The complex \(X_3\) satisfies all the conditions of \(EDG\).

Proof. The CW complex \(X_3\) is a \(G\)-complex as the union of \(G\)-complexes with a \(G\)-invariant intersection. Note that \(G\) acts freely on \(X_3\setminus (\bigcup_a GR_a)\). Already we know that the vertices of the fixed point complex of \(\Delta(Z(b))\) are those of \(R_b\). In view of Proposition 2.8, no new edges are spanned by vertices in \(R_b\) and hence the fixed point set of \(\Delta(Z(b))\) is just \(R_b\) which is contractible. Therefore, the fixed point sets are contractible for all groups \(H \in D\).

The complex \(X_3\) is contractible as a union of contractible complexes \(X_a\) with a contractible intersection. □

2.3. CW structure of \(B_DG\). Note that \(B_DG = E_DG\) contains \((B\pi \times B\pi)/\Delta(B\pi)\) as a subcomplex with a 3-dimensional complement. Moreover, as a CW complex

\[B_DG = \bigcup_{a \in A} X_a/G = (B\pi \times B\pi)/\Delta(B\pi) \cup \bigcup_{a \in A} (e^1 \cup e^2 \cup e^3)\]

where \(e^1\) is the image of \(\text{Int}(\sigma)\), \(e^2\) is the image of \(\text{Int}(D)\), and \(e^3\) is the image of the interior of the 3-ball \(B\) from \(X_a\). Since \(D\) can be deformed to \(I\), the complex \(B_DG\) is homotopy equivalent to

\[(B\pi \times B\pi)/\Delta(B\pi) \cup \bigcup_{a \in A} e^3\]

where 3-cells are attached to the image of \(I\). Hence the attaching maps are null-homotopic. Thus, the space \(B_DG\) is homotopy equivalent to the bouquet of \((B\pi \times B\pi)/\Delta(B\pi)\) with a wedge of 3-spheres,

\[B_DG \sim (B\pi \times B\pi)/\Delta(B\pi) \vee \bigvee_{a \in A} S^3\]

3. Conditions when \(TC(\pi)\) is maximal

We use the following

3.1. Theorem ([Dr1]). For a geometrically finite group \(\pi\) with \(cd(\pi) = n\),

\[H^{2n}(\pi \times \pi, \mathbb{Z}(\pi \times \pi)) \neq 0.\]

We note that for a group \(G\) with finite \(BG\) there is the equality \([Br]\)

\[H^k(G, \mathbb{Z}G) = H^k_\pi(EG; \mathbb{Z})\]

where \(EG\) is the universal cover of \(BG\) and \(H^*_\pi\) stands for cohomology with compact supports.
Here is our main result.

3.2. Theorem. Suppose that a geometrically finite group $\pi$ has cyclic centralizers and $\text{cd} \, \pi = n \geq 2$. Then $\text{TC}(\pi) = 2n$.

Proof. Let $G = \pi \times \pi$ and let $E = E\pi \times E\pi$ where $E\pi$ is the universal cover of a finite simplicial complex $B\pi = E\pi/\pi$. Let $B = BG = E/G$. Suppose that the canonical map $f : E \to X_3$ can be deformed by a $G$-homotopy $\Phi : E \times I \to X_3$ to a map with the image in $X_3^{(2n-1)}$. Let

$$Z = \bigcup_{g \in G} g\Delta(E\pi) \cup E^{(2)}.$$ 

We use the notation $\alpha Y$ for the one-point compactification of a locally compact space $Y$. Since $\dim \alpha Z = \max\{n, 2\} < 2n - 1$, the exact sequence of pair implies that $H^{2n}(\alpha E) = H^{2n}(\alpha E, \alpha Z)$. Therefore, the collapsing map $\bar{q} : \alpha E \to \alpha E/\alpha Z$ induces an isomorphism of cohomology groups

$$\bar{q}^* : H^{2n}(\alpha E/\alpha Z; \mathbb{Z}) \to H^{2n}(\alpha E; \mathbb{Z}).$$

Consider the commutative diagram

\[
\begin{array}{ccc}
X_3 & \xrightarrow{f} & E \\
\downarrow{\psi} & & \downarrow{q} \\
E/Z & \xrightarrow{\xi} & \alpha E/\alpha Z \\
\end{array}
\]

where $\psi$ is defined by sending $X_3 \setminus X$ to the point in $E/Z$ generated by $Z$. The map $\xi$ is a continuous bijection which is a homeomorphism in the complement of the singular one-point sets defined by $Z$ and $\alpha Z$.

Let $\alpha = \alpha E \setminus \alpha E/\alpha Z$ denote the point at infinity. We claim that sending $\alpha \times I$ to the singular point $z_0 \in \alpha E/\alpha Z$ continuously extends the homotopy

$$\xi \circ \psi \circ \Phi : E \to \alpha E/\alpha Z$$

to a continuous homotopy $\hat{\Phi} : \alpha E \times I \to \alpha E/\alpha Z$.

Assume the contrary: There is an open neighborhood $U$ of $z_0$ and a sequence $x_n \in E$ converging to $\alpha$ such that

$$\xi \psi \Phi(x_n \times I) \cap (\alpha E/\alpha Z) \setminus U \neq \emptyset$$

for all $n$. Then $\psi \Phi(x_n \times I) \cap C \neq \emptyset$ for all $n$ where

$$C = \xi^{-1}((\alpha E/\alpha Z) \setminus U)$$

is a compact subset of $E \setminus Z \subset E/Z$. Fix $t_n \in I$ such that $\psi \Phi(x_n, t_n) \in C$. Let $p : E \to B$ be the universal covering map and let

$$\bar{p} : E/Z \to B/p(Z)$$

be the induced map. We consider a metric $d_E$ on $E$ lifted from a metric on $B$. Let $V$ be an open neighborhood of the singular point $\bar{b}_0 \in B/p(Z)$ such that

$$C \cap \bar{p}^{-1}(V) = \emptyset.$$
where \( \hat{V} \) is the closure of \( V \). We may assume that the quotient map \( \nu : B \to B/p(Z) \) is 1-Lipschitz. Consider the metric \( \mu \) on \( W = E/Z \setminus p^{-1}(V) \) lifted with respect to \( \bar{p} \) from the metric on \( B/p(Z) \setminus V \). Note that the action of \( G \) on \( W \) is proper discontinuous by isometries. Let \( \epsilon = \text{dist}_{\mu}(C, p^{-1}\partial V) > 0 \). The homotopy \( \psi \Phi \) defines a homotopy \( \hat{\Phi} : B \times I \to B/p(Z) \) such that the diagram

\[
\begin{array}{ccc}
E \times I & \xrightarrow{\phi \Phi} & E/Z \\
\downarrow \text{proj} & & \downarrow \bar{p} \\
B \times I & \xrightarrow{\hat{\Phi}} & B/p(Z)
\end{array}
\]

commutes. Then the restriction

\[
\psi \Phi \big|_{W^\prime} : W^\prime = (E \times I) \setminus \nu^{-1}(V) \to W
\]

is Lipschitz.

Compactness of \( B \) and \( I \) implies that there is a subsequence \( (x_{n_k}, t_{n_k}) \) such that \( \{\bar{p}(x_{n_k})\} \) converges to some \( \bar{x} \in B \) and \( \{t_{n_k}\} \) converges to some \( t^* \in I \). Note that \( \Phi(\bar{x}, t^*) \in \bar{p}(C) \). Let \( p(x) = \bar{x} \). Then \( G(x) \times t^* \subset W^\prime \). By the choice of the metric on \( E \) there is a sequence of elements \( g_k \in G, \|g_k\| \to \infty \), such that

\[
d_E(x_{n_k}, g_k x) \to 0.
\]

Since \( \bar{p}(C) \cap \hat{V} = \emptyset \), for sufficiently large \( k \) we have \( Gx \times t_{n_k} \subset W^\prime \). Then the Lipschitz condition of the restriction of \( \psi \Phi \) to \( W^\prime \) implies

\[
d^*(\psi \Phi(x_{n_k}, t_{n_k}), \psi \Phi(g_k(x), t_{n_k})) < \epsilon/2
\]

for sufficiently large \( k \).

Then

\[
d^*(\psi \Phi(g_k x, t^*), C) \leq d^*(\psi \Phi(g_k x, t^*), \psi \Phi(g_k x, t_k)) + d^*(\psi \Phi(g_k x, t_k), \psi \Phi(x_{n_k}, t_k)).
\]

For sufficiently large \( k \) each of the above summands does not exceed \( \epsilon/2 \). Thus, the sequence

\[
g_k(\psi \Phi(x, t^*)) = \psi \Phi(g_k x, t^*)
\]

stays in a bounded distance from a compact set. This contradicts with the properness of the action of \( G \) on \( W \).

Thus, there is a homotopy

\[
\hat{\Phi} : \alpha E \times I \to \alpha E/\alpha Z
\]

of \( \bar{q} \) to a map with the image in the \((2n-1)\)-dimensional compact space \( \alpha E^{(2n-1)}/\alpha Z \). Therefore, \( \bar{q} \) induces zero homomorphism of \( 2n \)-cohomology. Therefore,

\[
H^{2n}(G, Z G) = H^{2n}_c(E; Z) = H^{2n}(\alpha E; Z) = 0.
\]

This contradicts to Theorem 3.3. \( \square \)

4. Maximal value of TC via cohomology of the orbit spaces

In this section we present a shorter proof of the main theorem for Poincaré Duality groups \( \pi \).

4.1. Proposition. Suppose that for a geometrically finite group \( \pi \) with \( \text{cd} \pi = n \geq 2 \) and cyclic centralizers, \( H^{2n}(B\pi \times B\pi; q^* A) \neq 0 \) for some local system of coefficients \( A \) on the quotient space \( (B\pi \times B\pi)/\Delta(B\pi) \) where \( q : B\pi \times B\pi \to (B\pi \times B\pi)/\Delta(B\pi) \) is the quotient map. Then \( \text{TC}(\pi) = 2n \).
Proof. The exact sequence of pairs form a commutative diagram
\[
\begin{array}{ccc}
H^2_n(B \pi \times B \pi; q^* A) & \xrightarrow{\cong} & H^2_n(B \pi \times B \pi, \Delta(B \pi); q^* A) \\
\wedge \quad & & \wedge \\
q^* \uparrow & & q^* \uparrow \\
\cong & & \cong \\
H^2_n(B \pi \times B \pi/ \Delta(\pi); A) & \xrightarrow{\cong} & H^2_n(B \pi \times B \pi/ \Delta(B \pi), pt; A) \\
\end{array}
\]
where the right vertical homomorphism is an isomorphism in view of excision \[\text{Bre}\]. Thus, \(q^*\) is an isomorphism. Therefore \(q\) cannot be deformed to the \(2n-1\)-skeleton in \((B \pi \times B \pi)/\Delta(B \pi)\) as well as in \(B \pi G\). Hence \(\tilde{q} : E \pi \times E \pi \to E \pi G\) cannot be \((\pi \times \pi)\)-equivariantly deformed to the \((2n-1)\)-skeleton. By Theorem \[\text{1.1}\], \(\text{TC}(\pi) \geq 2n\). \[\blacksquare\]

4.2. Corollary. Let a group \(\pi\) be with cyclic centralizers and with the classifying space \(B \pi\) be a manifold. Then \(\text{TC}(\pi) = 2 \text{cd}(\pi)\).

Proof. In the case of orientable manifold \(B \pi\) we apply Proposition \[\text{4.1}\] by taking the integers as coefficients.

If \(B \pi\) is non-orientable, we consider the tensor product \(O \hat{\otimes} O\) of the orientation sheaves coming from different factors. Since the action of the diagonal subgroup is trivial on the stalk \(Z \otimes Z\), it follows that \(O \hat{\otimes} O = q^* A\) for some local system \(\mathcal{A}\) on \((B \pi \times B \pi)/\Delta(B \pi)\) (see \[\text{Dr2}\]). \[\blacksquare\]

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