Integrability of the Gross-Pitaevskii Equation with Feshbach Resonance management

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Abstract

In this paper we study the integrability of a class of Gross-Pitaevskii equations managed by Feshbach resonance in an expulsive parabolic external potential. By using WTC test, we find a condition under which the Gross-Pitaevskii equation is completely integrable. Under the present model, this integrability condition is completely consistent with that proposed by Serkin, Hasegawa, and Belyaeva [V. N. Serkin et al., Phys. Rev. Lett. 98, 074102 (2007)]. Furthermore, this integrability can also be explicitly shown by a transformation, which can convert the Gross-Pitaevskii equation into the well-known standard nonlinear Schrödinger equation. By this transformation, each exact solution of the standard nonlinear Schrödinger equation can be converted into that of the Gross-Pitaevskii equation, which builds a systematical connection between the canonical solitons and the so-called nonautonomous ones. The finding of this transformation has a significant contribution to understanding the essential properties of the nonautonomous solitions and the dynamics of the Bose-Einstein condensates by using the Feshbach resonance technique.

Key words: integrability, WTC test, Gross-Pitaevskii equation, Bose-Einstein condensate, Feshbach resonance

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1 Introduction

Dilute-gas Bose-Einstein condensates (BECs) have been generated by many experiment groups since 1995 (1; 2). It is well-known that the mean-field dynamics of...
the BECs at low temperature can be well described by a three-dimensional Gross-Pitaevskii (GP) equation (3; 4), where the BECs are confined to an expulsive harmonic external potential. In some physically important cases the GP equation can be reduced effectively to a one-dimensional GP equation (5; 6; 7; 8; 9; 10; 11; 12). This simplified equation reads after re-scaling the physical variables, 

\[ i\hbar \frac{\partial}{\partial t}\psi(x, t) = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + g_0 \Gamma(t) |\psi(x, t)|^2 + V(x) \right) \psi(x, t), \quad (1) \]

where \(\psi(x, t)\) denotes the macroscopic wave function of the condensate and \(V(x)\) is the potential confining the condensate. The \(g_0\) represents the nonlinear interaction strength and a time-dependent factor of \(\Gamma(t)\) is controlled by the so-called Feshbach resonance (13; 14). It is very interesting to study the dynamics of the BECs described by Eq. (1) with the Feshbach resonance management.

It should be pointed out that Eq. (1) is a special case of a more generalized nonlinear Schrödinger (NLS) equation with varying in time and space dispersion and nonlinearity. In the literature, this generalized NLS equation was named as the nonautonomous NLS equation (15). The exact solutions of the nonautonomous NLS equation are suggested to name the nonautonomous solitons in order to conceptually distinguish from the canonical solitons introduced by Zabusky and Kruskal (16). Actually, the studies of the integrability of the generalized NLS equation have a long history. As early as in 1976 Chen and Liu (17) went beyond the concept of the canonical soliton and found the exact integrability and the Lax pair of NLS equation with space-varying potentials. More recently, the dynamics of the solitons of these generalized NLS equation has been extensively studied (15; 18; 19) and some novel concepts, i.e., the nonautonomous NLS equation (15) and the nonautonomous soliton (19) have been proposed.

A classical way to study the integrability of the standard NLS equation and the nonautonomous one is the construction of the Lax pair of these NLS equations based on the inverse scattering transform (20). In this paper, we use the WTC test method suggested by Weiss, Tabor and Carnevale (21; 22) in 1983 to study the integrability of the GP equation with Feshbach resonance management. This method is based on the Painlevé test for PDE. Through the WTC test, we get an integrability condition. At the same time, we also find a transformation that can convert the GP equation with Feshbach resonance management into the standard NLS equation, which has been extensively studied and a number of exact solutions including various solitons have been obtained in the literature. From the well-known solutions of the standard NLS equation, we can obtain novel solutions of the integrable GP equation with Feshbach resonance management. This transformation builds a systematical connection between the well-known canonical solitons and the nonautonomous ones and provide a novel way to study the dynamics of the BEC described by the GP equation with Feshbach resonance management. Finally, we also find a new type of solution of the standard NLS equation, which is also a new type
of solution of the GP equation.

The outline of this paper is as follows. In Sec. 2 we explicitly present the WTC test of the GP equation (1) and discuss the condition the equation can pass the WTC test, i.e., the integrability condition of Eq. (1). In Sec. 3 we show a transformation to convert the GP equation (1) into the well-known standard NLS equation, which demonstrates the complete integrability of the GP equation under the condition obtained in the previous section. In Sec. 4, as some applications, we list some solutions of the standard NLS equation, including a new type of solution, from which numerous novel solutions of the integrable GP equation with Feshbach resonance management can be obtained by the transformation.

2 The Painlevé test for the GP equation with Feshbach resonance management

For simplicity, we rewrite Eq. (1) as

\[ i \frac{\partial}{\partial t} u(x, t) + \frac{\partial^2}{\partial x^2} u(x, t) + g(x, t) |u(x, t)|^2 u(x, t) + V_0 x^2 u(x, t) = 0 \]  

(2)

In order to use the Painlevé analysis, it is convenient to introduce a complex function \( v(x, t) = u(x, t)^* \). Thus, Eq. (2) becomes a pair of complex equations,

\[
\begin{cases}
  i \frac{\partial}{\partial t} u(x, t) + \frac{\partial^2}{\partial x^2} u(x, t) + g(x, t) (u(x, t))^2 v(x, t) + V_0 x^2 u(x, t) = 0, \\
  -i \frac{\partial}{\partial t} v(x, t) + \frac{\partial^2}{\partial x^2} v(x, t) + h(x, t) (v(x, t))^2 u(x, t) + V_0 x^2 v(x, t) = 0,
\end{cases}
\]  

(3)

where \( h(x, t) = g(x, t)^* \).

The next step is to seek a solution of Eqs. (3) in the following form

\[
\begin{align*}
  u(x, t) &= \phi(x, t)^{-p} \sum_{i=0}^{\infty} u_i(t) \phi(x, t)^i, \\
  v(x, t) &= \phi(x, t)^{-q} \sum_{j=0}^{\infty} v_j(t) \phi(x, t)^j.
\end{align*}
\]  

(4)

The coefficients \( g(x, t) \) and \( h(x, t) \) can be expanded on the singularity manifold as follows

\[
\begin{align*}
  g(x, t) &= \sum_{k=0}^{\infty} g_k(t) \phi(x, t)^k, \\
  h(x, t) &= \sum_{m=0}^{\infty} h_m(t) \phi(x, t)^m.
\end{align*}
\]  

(5)

Here we use the Kruskal assumption \( \phi(x, t) = x + \psi(t) \), where \( u_i(t), v_j(t), g_k(t), h_m(t) \) are analytic functions in the neighborhood of a non-characteristic singularity manifold defined by \( x = -\psi(t) \).
By the standard procedure, the leading order is given by $p = q = 1$, and one can obtain the following relations

$$
\begin{bmatrix}
Q_1 & g_0(t) (u_0(t))^2 \\
h_0(t) (v_0(t))^2 & Q_2
\end{bmatrix}
\begin{bmatrix}
u_j(t) \\
v_j(t)
\end{bmatrix}
= \begin{bmatrix} F_j \\ G_j \end{bmatrix},
$$

(6)

where $j = 1, 2, \cdots$, and

$$
\begin{aligned}
Q_1 &= (j-1)(j-2) + 2g_0(t)u_0(t)v_0(t), \\
Q_2 &= (j-1)(j-2) + 2h_0(t)u_0(t)v_0(t).
\end{aligned}
$$

Here $g_0(t), h_0(t), u_0(t)$ and $v_0(t)$ obey

$$
g_0(t)u_0(t)v_0(t) = h_0(t)u_0(t)v_0(t) = -2.
$$

(7)

The recursion relations are determined by

$$
F_j = -\frac{d}{dt}u_{j-2}(t) - iu_{j-1}(t) \left( \frac{d}{dt} \psi(t) \right) j + 2iu_{j-1}(t) \frac{d}{dt} \psi(t) - g_j(t)(u_0(t))^2 v_0(t)
\begin{aligned}
-g_0(t) &v_0(t) \sum_{m=1}^{j-1} u_{j-m}(t) u_m(t) - g_0(t) \sum_{m=1}^{j-1} v_{j-m}(t) \sum_{k=0}^{m} u_{m-k}(t) u_k(t) \\
- \sum_{m=1}^{j-1} g_{j-m}(t) &\sum_{l=0}^{m} v_{m-l}(t) \sum_{k=0}^{l} u_{l-k}(t) u_k(t) - V_0x^2u_{j-2}(t),
\end{aligned}
$$

(8)

$$
G_j = -\frac{d}{dt}v_{j-2}(t) + iv_{j-1}(t) \left( \frac{d}{dt} \psi(t) \right) j - 2iv_{j-1}(t) \frac{d}{dt} \psi(t) - h_j(t)(v_0(t))^2 u_0(t)
\begin{aligned}
-h_0(t) &u_0(t) \sum_{m=1}^{j-1} v_{j-m}(t) v_m(t) - h_0(t) \sum_{m=1}^{j-1} u_{j-m}(t) \sum_{k=0}^{m} v_{m-k}(t) v_k(t) \\
- \sum_{m=1}^{j-1} h_{j-m}(t) &\sum_{l=0}^{m} u_{m-l}(t) \sum_{k=0}^{l} v_{l-k}(t) v_k(t) - V_0x^2v_{j-2}(t).
\end{aligned}
$$

(9)

One notes that $g_j(t) = 0$ if $j < 0$. It is also true for $h_j(t), u_j(t), v_j(t)$. The expressions $F_j, G_j$ for a given $j$ depend only on the expansion coefficients of $u_l, v_l$, with $l < j$ and $g_l, h_l$ with $l \leq j$. Therefore the above equations represent the recursion relations of the unknown $u_j, v_j$ ($j > l$) from the known $u_l, v_l$ ($l < j$).

The above recursion relations determine the unknown expansion coefficients uniquely unless the determinant of the matrix in Eq. (6) is zero. Those values of $j$ at which the determinant is equal to zero are called as the resonance points. After some calculations it is found that the resonance points only occur at

$$
j = -1, 0, 3, 4.
$$
The resonance point of \( j = -1 \) corresponds to the arbitrariness of the singular manifold \( \phi(x, t) \).

From the recursion relations, we find the compatibility conditions for the remaining resonance points

\[
\begin{align*}
  j = 0: & \quad g_0(t) u_0(t) v_0(t) = h_0(t) u_0(t) v_0(t) = -2, \quad (10) \\
  j = 3: & \quad v_0(t) F_3 = u_0(t) G_3, \quad (11) \\
  j = 4: & \quad v_0(t) F_4 + u_0(t) G_4 = 0. \quad (12)
\end{align*}
\]

Due to the arbitrariness of \( u_0(t) \) and \( v_0(t) \), Eq. (10) implies \( g(x, t) \equiv h(x, t) \). According to the definition of \( h(x, t) \), one concludes that \( g(x, t) \) must be a real function.

According to Eq. (6), one can obtain uniquely \( u_1(t), v_1(t), u_2(t), v_2(t) \). Inserting them into Eq. (11), after some manipulations one has

\[
g_0(t) \frac{d}{dt} g_1(t) - 2 g_1(t) \frac{d}{dt} g_0(t) = 0. \quad (13)
\]

Solving for \( g_1(t) \), we obtain

\[
g_1(t) = C_1 \left( g_0(t) \right)^2, \quad (14)
\]

where \( C_1 \) is an arbitrary constant.

Inserting Eq. (14) into Eq. (11), one can obtain \( u_3(t) \) when \( v_3(t) \) is arbitrary. Replacing \( u_1(t), v_1(t), u_2(t), v_2(t), u_3(t) \) into Eq. (12), after some simplifications and letting the coefficients of \( \psi(t), \frac{d}{dt} \psi(t), \frac{d^2}{dt^2} \psi(t) \) be zero (due to the arbitrariness of \( \psi(t) \)), one has

\[
C_1 = 0. \quad (15)
\]

Thus Eq. (14) means that

\[
g_1(t) = 0. \quad (16)
\]

According to the definition of \( g_1(t) \), \( g_1(t) = \frac{\partial}{\partial x} g(x, t) \big|_{x=-\psi(t)} \), when \( \psi(t) \) is arbitrary, \( g(x, t) \) must be independent of \( x \), so one has

\[
\begin{align*}
  \begin{cases} 
    g(x, t) = g(t) = g_0(t), \\
    g_1(t) = g_2(t) = \cdots = 0.
  \end{cases} \quad (17)
\end{align*}
\]

Finally, the conditions (15)-(17), together with compatibility condition (12) give the following equation

\[
4 V_0 \left( g_0(t) \right)^2 + \left( \frac{d^2}{dt^2} g_0(t) \right) g_0(t) - 2 \left( \frac{d}{dt} g_0(t) \right)^2 = 0. \quad (18)
\]
From this equation we have
\[ g_0(t) = \frac{e^{\pm 2\sqrt{V_0}t}}{A e^{4\sqrt{V_0}t} - B}, \]
where \( A, B \) are arbitrary constants. For a proper choice of \( A \) and \( B \), \( g_0(t) \) can change its sign at certain time (singularity point), which should be related to the Feshbach resonance.

Thus, one can conclude that only when
\[ g(x, t) = g(t) = \frac{e^{\pm 2\sqrt{V_0}t}}{A e^{4\sqrt{V_0}t} - B}, \]
Eq. (2) can pass the WTC test. We will show below that this condition is sufficient for the integrability of Eq. (2). For convenience, it is also useful to introduce \( \lambda = \pm 2\sqrt{V_0} \), i.e., \( V_0 = \frac{1}{4} \lambda^2 \). Thus \( g(x, t) \) becomes
\[ g(x, t) = \frac{2 g_0 e^{\lambda t}}{A e^{2\lambda t} - B}. \] (19)

It is very interest to note that under the present NLS equation, Eq. (19) is completely satisfied the exact integrability condition proposed by Serkin, Hasegawa, and Belyaeva [ see Eq. (2) in Ref. (19)]. However, from the present analysis, Eq. (19) is a necessary condition for the Eq. (2) to pass the WTC test. A sufficient integrability condition should be shown by further exploring a transformation which converts the GP equation (2) into a standard integrable NLS equation.

3 Integrability

Under the condition Eq. (19), Eq. (2) can be rewritten as
\[ i \frac{\partial}{\partial t} u(x, t) + \frac{\partial^2}{\partial x^2} u(x, t) + \frac{2 g_0 e^{\lambda t}}{A e^{2\lambda t} - B} |u(x, t)|^2 u(x, t) + \frac{1}{4} \lambda^2 x^2 u(x, t) = 0. \] (20)

It should be noted that when \( A = 0, B = -1 \) and \( \lambda > 0 \), Eq. (20) is reduced to the case discussed in Ref. (11).

Now we further prove that Eq. (20) is completely integrable. Our idea is to find a transformation to convert exactly Eq. (20) into the standard NLS equation
\[ i \frac{\partial}{\partial T} Q(X, T) + \varepsilon \frac{\partial^2}{\partial X^2} Q(X, T) + \delta |Q(X, T)|^2 Q(X, T) = 0. \] (21)
In the literature, one often takes $\epsilon = \pm \frac{1}{2}, \delta = \pm 1$ or $\epsilon = \pm 1, \delta = \pm 2$.

In order to reduce Eq. (20) to the standard NLS equation (21), we look for a transformation of the form

$$\psi(x, t) = Q(X(x, t) , T(t)) e^{ia(x,t)+c(t)}.$$ (22)

One also notes that other similarity transformations (23; 24) have been explored to reduce the nonautonomous NLS equation to the standard NLS equation. In comparison to Eq. (22), the form of the similarity transformation in Ref. (23) is quite different from ours. The transformation parameters in Ref. (23) is independent of the space variables and the space dependence of the transformation is completely specialized by the explicit form of the similarity transformation introduced. In Ref. (24), another similarity transformation is introduced. However, this transformation reduces the nonautonomous NLS equation to a stationary NLS equation, as shown in Eq. (2) in Ref. (24).

After some tedious and technical calculations, one can obtain the explicit forms of the transformation 22. In the following we present them in different cases.

**Case I**: $A = 0, B = -1$

In this case, Eq. (20) becomes

$$i \frac{\partial}{\partial t} u(x,t) + \frac{\partial^2}{\partial x^2} u(x,t) + 2g_0 e^{\lambda t} |u(x,t)|^2 u(x,t) + \frac{1}{4} \lambda^2 x^2 u(x,t) = 0,$$ (23)

and the transformation reads

$$\begin{align*}
X(x,t) &= \frac{2g_0 \epsilon e^{\lambda t}}{\delta} \left( x - \frac{2g_0 \epsilon e^{\lambda t}}{\delta} \right), \\
T(t) &= \frac{2g_0 \epsilon^2}{\delta^2 \lambda} \left( e^{2\lambda t} - 1 \right), \\
a(x,t) &= -\frac{1}{4} \left( x^2 - \frac{8g_0 e^{\lambda t}}{\delta} x + \frac{8g_0^2 e^{2\lambda t}}{\delta^2 \lambda^2} \right), \\
c(t) &= \frac{1}{2} \left( t - \frac{1}{\lambda} \ln \frac{\delta}{2g_0 \epsilon} \right).
\end{align*}$$

Using this transformation, it is straightforward to check that Eq. (23) has the form of Eq. (21). Thus, according to a known solution $Q(X,T)$ of Eq. (21), one can obtain a solution of Eq. (23) as follows

$$u(x,t) = \sqrt{\frac{2g_0 \epsilon}{\delta}} \frac{1}{2} Q \left( \frac{2g_0 \epsilon e^{\lambda t}}{\delta} \left( x - \frac{2g_0 \epsilon e^{\lambda t}}{\delta} \right), \frac{2g_0^2}{\delta^2 \lambda} \left( e^{2\lambda t} - 1 \right) \right) e^{\frac{1}{4} \left( x^2 - \frac{8g_0 e^{\lambda t}}{\delta} x + \frac{8g_0^2 e^{2\lambda t}}{\delta^2 \lambda^2} \right)}.$$ (24)

As noted in Ref. (25), this solution has some interesting features since it can take into account the fallout of particles form the BEC for $\lambda < 0$ and the amplification
of a soliton if $\lambda > 0$. To show this, one can define a new variable $Q'(x, t) = \sqrt{2g_0 e^{\frac{1}{2} \lambda t}} u(x, t)$, which obeys the equation

$$i \frac{\partial}{\partial t} Q'(x, t) + \frac{\partial^2}{\partial x^2} Q'(x, t) + |Q'(x, t)|^2 Q'(x, t) + \frac{1}{4} \lambda^2 x^2 Q'(x, t) = \frac{i}{2} \lambda Q'(x, t), \quad (25)$$

which includes an additional complex term at the right-hand side of Eq. (25).

One notes that only when $\varepsilon \delta g_0 > 0$, the above transformation is well-defined. In particular, for $g_0 > 0 (g_0 < 0)$, $\varepsilon = 1, \delta = 2 (\delta = -2)$ or $\varepsilon = \frac{1}{2}, \delta = 1 (\delta = -1)$ Eq. (23) denotes the standard NLS equation, which has been extensively studied in the literature.

**Remark:** From the solution (24), it is obvious that if $\lambda > 0$, the solutions of Eq. (23) can be compressed into very high local matter densities by increasing the absolute value of atomic scattering length. For the bright soliton ($g_0 > 0, \varepsilon \delta > 0$) case, it has been discussed in details in Ref. (11). For the dark soliton ($g_0 < 0, \varepsilon \delta < 0$) case, the situation is similar. In addition, if $\lambda < 0$, the solutions can be expanded into very low local matter densities by decreasing the absolute value of atomic scattering length, as clearly seen from Eq. (25).

**Case II: $A \neq 0$**

In this case, the transformation can be obtained as follows

$$\begin{align*}
X(x, t) &= \frac{2g_0 e^{\lambda t}}{\delta (e^{2\lambda t} A - B)} \left( x - \frac{2g_0 \varepsilon}{\delta \lambda^2 A e^{\lambda t}} \right), \\
T(t) &= -\frac{2g_0^2 \varepsilon}{\lambda \delta^2 A} \left( \frac{1}{e^{2\lambda t} A - B} - \frac{1}{A - B} \right), \\
a(x, t) &= \frac{\lambda (A e^{2\lambda t} + B) x^2 + \frac{8g_0 \varepsilon}{\lambda} x e^{\lambda t} - \frac{8g_0^2 \varepsilon^2}{\lambda^2 e^{2\lambda t}}}{4(e^{2\lambda t} A - B)}, \\
c(t) &= \frac{1}{2} \lambda \left( t - \frac{1}{\lambda} \ln \frac{\delta (e^{2\lambda t} A - B)}{2g_0 \varepsilon} \right).
\end{align*}$$

Likewise, this transformation can convert Eq. (20) into the form of Eq. (21). Note that only when

$$\frac{\delta (e^{2\lambda t} A - B)}{g_0 \varepsilon} > 0,$$

the above transformation is well-defined. The sign of $\varepsilon \delta$ is determined by the signs of $\lambda$ and $g_0$, and the values of $A$ and $B$.

### 4 Explicit solutions of the GP equation

As pointed out above, the solutions of the GP equation (20) can be obtained from the solutions of the standard NLS equation (21). This is a very efficient method
to find new solutions of the GP equation (20) since a number of solutions of the standard NLS equation (21) have been obtained by some classical methods such as the inverse scattering transform, the Lax pairs, the bilinear technique, and the Backlund transformation, and so on.

In the following we first list some well-known solutions of the standard NLS equation (21), which can be transformed into the solutions of the GP equation (20). As an example, we explicitly present the fundamental bright one-soliton solution of the GP equation.

**Fundamental bright one-soliton solution**

\[ Q(X,T) = \text{sech}(X) e^{i \frac{1}{2} T} \]

when \( g_0 > 0, \varepsilon = \frac{1}{2}, \delta = 1 \). According to Eq. (24), the fundamental bright one-soliton solution of the GP equation (22) can be written as

\[
\begin{align*}
  u(x,t) &= \sqrt{g_0} e^{i \frac{1}{2} \lambda t} Q \left[ g_0 e^{\lambda t} \left( x - \frac{g_0}{\lambda} e^{\lambda t} \right), \frac{g_0^2}{\lambda} \left( e^{2\lambda t} - 1 \right) \right] e^{-i \frac{1}{4} \left( x^2 - \frac{4g_0}{\lambda^2} e^{2\lambda t} + \frac{2g_0^2}{\lambda^2} e^{2\lambda t} \right)} \\
&= \sqrt{g_0} e^{i \frac{1}{2} \lambda t} \text{sech} \left[ g_0 e^{\lambda t} \left( x - \frac{g_0}{\lambda} e^{\lambda t} \right) \right] e^{-i \frac{1}{4} \left( x^2 - \frac{4g_0}{\lambda^2} e^{2\lambda t} + \frac{2g_0^2}{\lambda^2} e^{2\lambda t} \right)}.
\end{align*}
\]

For simplicity, below we only list the soliton solutions of the standard NLS equation, the corresponding solutions of the GP equation (20) can be obtained by the same procedure shown above.

**Fundamental dark one-soliton solution**

\[ Q(X,T) = q_0 \tanh(q_0 X) e^{i q_0^2 T} \]

when \( q_0 \) is an arbitrary constant and \( q_0 < 0, \varepsilon = -\frac{1}{2}, \delta = 1 \).

**Envelope solutions of the bright soliton**

\[ Q(X,T) = k_1 \text{sech} \left( k_1 X - k_1 k_2 T \right) e^{-i \left( -k_2 X + \frac{1}{2} (k_2^2 - k_1^2) T \right)} \]

where \( k_1, k_2 \) are arbitrary constants, and \( \varepsilon = \frac{1}{2}, \delta = 1 \).

**Envelope solutions of the dark soliton**

\[ Q(X,T) = k_1 \tanh \left( k_1 X + k_1 k_2 T \right) e^{-i \left( -k_2 X + \frac{1}{2} (k_2^2 + 2k_1^2) T \right)} \]

where \( k_1, k_2 \) are arbitrary constants, and \( \varepsilon = -\frac{1}{2}, \delta = 1 \).

For bright and dark N-soliton solutions of Eq. (21), one can refer to Refs. (26; 27; 28; 29; 30; 31; 32).
Besides the well-known soliton solutions listed above, in the following we discuss a new type exact solution of the standard NLS equation (21) by using the direct truncation method proposed in Ref. (33). These new solutions, of course, also give new solutions of the GP equation (20).

Assume that

\[ Q(X, T) = \eta(k_1 X + \omega_1 T)e^{i(k_2 X + \omega_2)} \]

be a solution of Eq. (21), where \( \eta(.) \) is a real function. Some straightforward calculations give

\[ \omega_1 = -2 \varepsilon k_1 k_2, \]

and

\[ -\varepsilon \eta(\xi) k_2^2 - \eta(\xi) \omega_2 + \varepsilon \left( \frac{d^2}{d\xi^2} \eta(\xi) \right) k_1^2 + \delta (\eta(\xi))^3 = 0, \]

where \( \xi = k_1 X + \omega_1 T \). By the similar procedure proposed in Ref. (33) we set

\[ \eta(\xi) = \frac{a + bf(\xi) + cg(\xi)}{p + qf(\xi) + rg(\xi)}, \]

where \( a, b, c, p, q, r \) are constants to be determined together with \( k_1, k_2 \) and \( \omega_2 \). The functions \( f(\xi) \) and \( g(\xi) \) are assumed to obey

\[
\begin{cases}
  f(\xi)^2 + g(\xi)^2 = 1, \\
  \frac{d}{d\xi} f(\xi) = g(\xi)\sqrt{1 - k^2 f(\xi)^2}, \\
  \frac{d}{d\xi} g(\xi) = -f(\xi)\sqrt{1 - k^2 f(\xi)^2},
\end{cases}
\]

Inserting (27) and (28) into Eq. (26), we get a polynomial of \( f(\xi) \) and \( g(\xi) \). Collecting the coefficients and let them be zero, one obtains a set of algebraic equations of \( a, b, c, p, q, r, k_1, k_2 \) and \( \omega_2 \). Solving for the set of equations one can obtain many exact solutions. Here we only present three representative solutions.

**The solution I:**

\[
Q(X, T) = \sqrt{\frac{2\varepsilon}{\delta}} k k_1 \text{cn} (k_1 X - 2 \varepsilon k_1 k_2 T, k) e^{-i(-k_2 X + \varepsilon (k_1^2 + k_2^2 - 2k_1^2 k_2) T)},
\]

where \( k_1, k_2 \) are arbitrary constants and \( \varepsilon \delta > 0 \). When \( k \to 1 \), we obtain the envelope solutions of bright soliton

\[
Q(X, T) = \sqrt{\frac{2\varepsilon}{\delta}} k k_1 \text{sech} (k_1 X - 2 \varepsilon k_1 k_2 T) e^{-i(-k_2 X + \varepsilon (k_2^2 - k_1^2) T)}.
\]

When \( k \to 0 \), we obtain the period solutions

\[
Q(X, T) = \sqrt{\frac{2\varepsilon}{\delta}} k k_1 \cos (k_1 X - 2 \varepsilon k_1 k_2 T) e^{-i(-k_2 X + \varepsilon (k_1^2 + k_2^2) T)}.
\]
The solution II:

\[ Q(X, T) = \sqrt{-\frac{2\varepsilon}{\delta}} k k_1 \text{sn} \left( k_1 X - 2 \varepsilon k_1 k_2 T, k \right) e^{-i\left(-k_2 X + \varepsilon (k_1^2 + k_2^2) T\right)}, \]

where \( k_1, k_2 \) are arbitrary constants and \( \varepsilon \delta < 0 \). When \( k \to 1 \), we obtain the envelope solutions of dark soliton

\[ Q(X, T) = \sqrt{-\frac{2\varepsilon}{\delta}} k k_1 \text{tanh} \left( k_1 X - 2 \varepsilon k_1 k_2 T \right) e^{-i\left(-k_2 X + \varepsilon (k_1^2 + k_2^2) T\right)}. \]

When \( k \to 0 \), we obtain the period solutions

\[ Q(X, T) = \sqrt{-\frac{2\varepsilon}{\delta}} k k_1 \sin \left( k_1 X - 2 \varepsilon k_1 k_2 T \right) e^{-i\left(-k_2 X + \varepsilon (k_1^2 + k_2^2) T\right)}. \]

The solution III:

\[ Q(X, T) = \sqrt{-\frac{\varepsilon}{2\delta}} k k_1 \text{sn} \left( k_1 X - 2 \varepsilon k_1 k_2 T, k \right) \frac{1}{1 + \text{cn} \left( k_1 X - 2 \varepsilon k_1 k_2 T, k \right)} e^{-\frac{i}{2}\left(-2k_2 X + \varepsilon (2k_2^2 - k_1^2 + 2k_1^2 k_2^2) T\right)}. \]

where \( k_1, k_2 \) are arbitrary constants. When \( k \to 1 \), we obtain the envelope solutions

\[ Q(X, T) = \sqrt{-\frac{\varepsilon}{2\delta}} k k_1 \text{tanh} \left( k_1 X - 2 \varepsilon k_1 k_2 T \right) \frac{1}{1 + \text{sech} \left( k_1 X - 2 \varepsilon k_1 k_2 T \right)} e^{-\frac{i}{2}\left(-2k_2 X + \varepsilon (2k_2^2 + k_1^2) T\right)}. \]

To our knowledge, the solution III is of a new type.

**Remark 1** Set \( k_2 = 0 \) in the solutions I, II, and III, we get the stationary solutions.

**Remark 2** Obviously, when \( \varepsilon \delta > 0 \), the standard NLS equation has bright soliton solutions, and when \( \varepsilon \delta < 0 \), it has dark soliton solutions.

**Remark 3** As mentioned above, only when \( \varepsilon \delta g_0 > 0 \), the transformation is well-defined. According to Remark 2 we know that, when \( g_0 > 0 \), the GP equation (23) has bright soliton solutions and when \( g_0 < 0 \), the GP equation (23) has dark soliton solutions. For the GP equation (20), one can use the Feshbach resonance to control the bright and the dark soliton in the BECs. This is because that if \( \frac{2g_0 e^{\lambda t}}{A e^{2M_B} - B} > 0 \), Eq. (20) is corresponding to Eq. (21) with \( \varepsilon \delta > 0 \), and if \( \frac{2g_0 e^{\lambda t}}{A e^{2M_B} - B} < 0 \), Eq. (20) is corresponding to the case of \( \varepsilon \delta < 0 \). The exact solutions of the GP equation with Feshbach resonance management presented in this paper have a significant contribution to the experimental study of the dynamics of the BECs.

In conclusion, we discuss the integrability of the GP equation with Feshbach resonance management. By WTC test we find a condition under which the GP equation
is completely integrable. Meanwhile, we also find a transformation, which can convert the GP equation with Feshbach resonance management into the standard NLS equation. By this transformation, the well-known exact solutions of the standard NLS equation can be converted into the exact solutions of the GP equation. To our knowledge, this is the first time to give a correspondence from the exact solutions of the standard NLS equation to the exact solutions of the GP equation, and thus we can give abundant exact solutions for the GP equation. These solutions would be important for understanding the dynamical behavior of the BECs by using the Feshbach resonance technique.

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