MULTI-LOOP CALCULATIONS IN THE STANDARD MODEL:
techniques and applications

J. FLEISCHER†, M. TENTYUKOV‡ and O. L. VERETIN§
Fakultät für Physik, Universität Bielefeld
D-33615 Bielefeld, Germany

Abstract: We present a review of the Bielefeld-Dubna activities on
the multiloop calculations. In the first part a C-program DIANA (DIagram
ANAlyser) for the automation of Feynman diagram evaluations is presented,
in the second part various techniques for the evaluation of scalar diagrams
are described, based on the Taylor expansion method and large mass ex-
pansion.

1. Automation of Feynman diagram evaluation

Recent high precision experiments require, on the side of the theory,
high-precision calculations resulting in the evaluation of higher loop dia-
grams in the Standart Model (SM). For specific processes thousands of
multiloop Feynman diagrams do contribute, and it turns out to be im-
possible to perform these calculations by hand. This makes the request for
automation a high-priority task.

Several different packages have been developed with different areas of
applicability. For example, FEYNARTS / FEYNCALC [1] are MATHEMA-
TICA packages convenient for various aspects of the calculation of radiative
corrections in the SM. There are several FORM packages for evaluating mul-
tiloop diagrams, like MINCER [2], and a package [3] for the calculation of
3-loop bubble integrals with a mass. Other automatic packages are GRACE
[4] and COMHPEP [5], which partially perform full calculations, from
the process definition to the cross-section values.

A somewhat different approach is pursued by XLOOPS [6]. A graphical
user interface makes XLOOPS an ‘easy-to-handle’ program package, but is
mainly aimed to the evaluation of single diagrams. To deal with thousands of
diagrams, it is necessary to use special techniques like databases and special
controlling programs. In [7] for evaluating more than 11000 diagrams the

* Presently by J. Fleischer at 21th School of Theoretical Physics, Ustroń, Poland,
  September 1997.
† E-mail: fleischer@physik.uni-bielefeld.de
‡ On leave of absence from Joint Institute for Nuclear Research, 141980 Dubna,
  Moscow Region, Russian Federation. E-mail: tentukov@thsun1.jinr.dubna.su
§ E-mail: veretin@physik.uni-bielefeld.de
¶ Supported by Bundesministerium für Forschung und Technologie under PH/05-
  7B92P 9.
special database-like program MINOS was developed. It calls the relevant FORM programs, waits until they finished, picks up their results and repeats the process without any human interference.

All these packages have different efficiency in different domains. It seems impossible to develop an universal package, which will be effective for all tasks. This point of view motivated us to seek our own way of automatic evaluation of Feynman diagrams.

Our first step is dedicated to the automation of the muons two-loop anomalous magnetic moment (AMM) \( \frac{1}{2} (g - 2) \). For this purpose the package TLAMM was developed \([8]\). The algorithm is implemented as a FORM-based program package. For generating and automatically evaluating any number of two-loop self-energy diagrams, a special C-program has been written. This program creates the initial FORM-expression for every diagram generated by QGRAF \([10]\), executes the corresponding subroutines and sums up the various contributions. In the SM 1832 two-loop diagrams contribute in this case. The calculation of the bare diagrams is finished. For the purpose of demonstration, we have applied TLAMM to a closed subclass of diagrams of the SM which we refer to as “toy” model. Some details of the calculation are presented for this case.

A more general project called DIANA (DIagram ANAlyser) \([9]\) for the evaluation of Feynman diagrams is being finished by our group at present and will also be shortly described below.

### 1.1. The toy model

We considered as a toy example the model involving a light charged spinor \( \Psi \), a photon \( A_\mu \), and a heavy neutral scalar field \( \Phi \). The scalar has triple \( (g) \) and quartic \( (\lambda) \) self-interactions, and the Yukawa coupling \( (y) \) to the spinor. The Lagrangian of this model reads (in Euclidean space-time)

\[
L = \frac{1}{2} \partial^\mu \phi \partial^\mu \phi + \frac{1}{2} M^2 \Phi^2 - \frac{g}{3!} \Phi^3 - \frac{\lambda}{4!} \Phi^4 + \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \\
+ \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + \bar{\Psi} \left( \partial + i e A + m \right) \Psi - y \Phi \bar{\Psi} \Psi
\]

where \( e \) is the electric charge and \( \alpha \) is a gauge fixing parameter.

There are 40 diagrams contributing to the two-loop AMM of the fermion. The calculation of the AMM reduces, after differentiation and contractions with projection operators, to diagrams of the propagator type with external momentum on the fermion mass shell (for details see \([11]\)). After performing the Dirac and Lorentz algebra, all diagrams can be reduced to some set of master integrals.
For the evaluation of these integrals we use the asymptotic expansion in large masses \[\text{[12]}\]. For a given scalar graph $G$ the expansion in large mass is given by the formula

$$
F_G(q, M, m, \epsilon) \overset{M \to \infty}{\sim} \sum_{\gamma} F_{G/\gamma}(q, m, \epsilon) \circ T_{q^\gamma, m^\gamma} F_{\gamma}(q^{\gamma}, M, m^{\gamma}, \epsilon),
$$

(2)

where $\gamma$’s are subgraphs involved in the asymptotic expansion, $G/\gamma$ denotes shrinking of $\gamma$ to a point; $F_{\gamma}$ is the Feynman integral corresponding to $\gamma$; $T_{q^\gamma, m^\gamma}$ is the Taylor operator expanding the integrand in small masses $\{m_{\gamma}\}$ and external momenta $\{q_{\gamma}\}$ of the subgraph $\gamma$; $\circ$ stands for the convolution of the subgraph expansion with the integrand $F_{G/\gamma}$. The sum goes over all subgraphs $\gamma$ which (a) contain all lines with large masses, and (b) are one-particle irreducible w.r.t. light lines.

Individual integrals are specified by the powers of the scalar denominators, called indices of the lines. From the point of view of the asymptotic
expansion method the topology of the diagram is essential. All diagrams of the toy model that contribute to the two-loop AMM can be classified in terms of 9 prototypes (we omit the pure QED diagrams). These prototypes and their corresponding subgraphs involved in the asymptotic expansion, are given in Fig. 1. Almost all integrals occurring in the asymptotic expansion of the muon AMM in the SM can be evaluated analytically using the package SHELL2 [13].

1.2. The piloting C-program TLAMM

For the automatic calculation we have created a special piloting program written in C. This program

1. reads QGRAF output;

2. creates a file containing the complete FORM program for calculating each diagram;

3. executes FORM;

4. reads FORM output, picks out the result of the calculation, and builds the total sum of all diagrams in a single file which can be processed by FORM.

Identifiers for vertices and propagators and the explicit Feynman rules are read from separate files and then inserted into the FORM program. Because the number of identifiers needed for the calculation of all diagrams at once may exceed the FORM capacity, the piloting program retains for each diagram only those involved in its calculation.

All initial settings are defined in a configuration file. The latter contains information about the file names, identifiers of topologies, the distribution of momenta, and the description of the model in terms of the notation that is some extension of QGRAF’s. The program carries out the complete syntax check of all input files except the QGRAF output.

There exist several options which allow one to process only the diagrams

• explicitly listed by number;

• of a given prototype;

• of a specified topology.

There are also some debugging options.

QGRAF generates the diagrams in symbolic form in terms of vertices. Fig. 2 shows one of the diagrams in the toy model with the corresponding QGRAF output.
The Feynman integrand generated by the C-program for this diagram reads:

$$g_{VF}(\mu)=E_2_{H_e2*F_F(-k^2+p,q/2,me2)}*E_2_{H_e2*F_F(k1-k2,q/2,me2)}*E_2_{H_e2*F_F(k1,-q/2,me2)}*E_2_{H_e2*H_H(k1-p,0)*H_H(k2,0)};$$

A demo diskette for evaluating the toy model can be provided on request.

1.3. Project DIANA

At present we have finished the main part of the more general program DIANA (DIagram ANAlyser). For generating Feynman diagrams we use again QGRAF. The program DIANA consists of two parts:

- Analyzer of diagrams.
- Interpreter of a special text manipulating language (TM);

The TM language is a very simple TeX-like language for creating source code and organizing the interactive dialog.

The analyzer reads QGRAF output and passes necessary information to the interpreter. For each diagram the interpreter performs the TM-program, producing input for further evaluation of the diagram like for example (3).

Thus the program:

Reads QGRAF output and for each diagram it:
1. Defines the topology.

2. Looking for it in the table of all known topologies and distributes momenta according to the current topology.

3. Creates a special internal representation of the diagram corresponding to the Feynman integrand.

4. Invokes the interpreter to execute the TM-program and passes to it the necessary data.

Executing the TM-program provides the possibility to calculate each diagram using FORM or another formulae manipulating language, to do some numerical calculation by means of FORTRAN, to create a postscript file for the picture of the current diagram, etc.

If we do not yet know all needed topologies, we may use the program to determine missing topologies that occur in the process.

The main goal of the TM-language is the creation of text files. In principle we could have used one of the existing languages, but we want a very specific language: it should be powerful enough to create arbitrary program texts. On the other hand, it should be very simple and easy-in-use, so that even non-programmers can use it.

Similar to the TeX language, all lines without special escape characters (\") are simply typed to the output file. So, to type “Hello, world!” in the file “hello” we may write down the following program:

\program
\setout(hello)
Hello, world!

Each word whose first character is an escape character will be considered as a command. This feature makes this language very easy-to-use.

At present, we have finished the C-part of this project.

2. Evaluation of scalar diagrams

In this second part methods for the evaluation of scalar three point functions will be discussed. We will concern ourselves with the Taylor expansion with respect to an external momentum squared and the large mass expansion explained above. The efficiency of both approaches will be compared.

2.1. Expansion of three-point functions in terms of external momenta squared
Taylor series expansions in terms of one external momentum squared, \( q^2 \) say, were considered in \([14]\), Padé approximants were introduced in \([15]\) and in Ref. \([16]\) it was demonstrated that this approach can be used to calculate Feynman diagrams on their cut by analytic continuation. The Taylor coefficients are expressed in terms of “bubble diagrams”, i.e. diagrams with external momenta equal zero, which makes their evaluation relatively easy. In the case under consideration we have two independent external momenta in \( d = 4 - 2\varepsilon \) dimensions. The general expansion of (any loop) scalar 3-point function with its momentum space representation \( C(p_1, p_2) \) can be written as

\[
C(p_1, p_2) = \sum_{l,m,n=0}^{\infty} a_{lmn} (p_1^2)^l (p_2^2)^m (p_1 p_2)^n
\]

where the coefficients \( a_{lmn} \) are to be determined from the given diagram.

For many applications it suffices to confine to the case \( p_1^2 = p_2^2 = 0 \), which is e.g. physically realized in the case of the Higgs decay into two photons \((H \to \gamma\gamma)\) with \( p_1 \) and \( p_2 \) the momenta of the photons. In this case only the coefficients \( a_{00n} \) are needed.

In the two-loop case we consider the scalar integral \((k_3 = k_1 - k_2, \text{ see also Fig. 3})\)

\[
\frac{1}{(\pi^2)^2} \int \frac{d^4k_1 d^4k_2}{((k_1 + p_1)^2 - m_1^2)((k_1 + p_2)^2 - m_2^2)((k_2 + p_1)^2 - m_3^2)((k_2 + p_2)^2 - m_4^2)(k_3^2 - m_5^2)(k_4^2 - m_6^2)}.
\]

\( k_t \) in line 4 (with mass \( m_4 \)) depends on the topology: for the planar diagram we have \( k_t = k_2 \) while for the non-planar we have \( k_t = k_3 \).

![Fig. 3. Planar and non-planar scalar vertex diagrams and their kinematics](image)

With obvious abbreviations for the scalar propagators: \( c_i \) the \( i^{th} \) scalar propagator of \((5)\) with \( p_1 = p_2 = 0 \), we can quite generally write for the \( n^{th} \)
Taylor coefficient \(18\):

\[
(i \pi^2)^2 a_{00n} = \frac{2^n}{n+1} \int d^4 k_1 d^4 k_2 F_n \cdot \frac{1}{c_1 c_2 c_3 c_4 c_5 c_6}.
\]

The numerator \(F_n\) can be written as

\[
F_n = \sum_{\nu=0}^{n} c_1^{-(n-\nu)} c_3^{-\nu} \sum_{\nu' = 0}^{n} c_2^{-(n-\nu')} c_4^{-\nu'} \cdot A_{\nu\nu'}^n(k_1, k_2, k_t).
\]

with

\[
A_{\nu\nu'}^n(k_1, k_2, k_t) = \sum_{\mu=0}^{\nu+\nu'} \sum_{\sigma=0}^{n} \sum_{\tau=0}^{\nu'} b_{\nu\nu'}^{n\mu,\sigma\tau} 
\cdot (k_1^2)^{n-(\nu+\nu')} + \mu (k_2^2)^{\sigma} (k_1 k_2)^{\nu' - \mu - \sigma + \tau} (k_1 k_t)^{\nu' - \mu + \sigma - \tau} (k_2 k_t)^{\mu - \sigma - \tau} (k_t^2)\tau,
\]

where the coefficients \(b_{\nu\nu'}^{n\mu,\sigma\tau}\) are given by

\[
b_{\nu\nu'}^{n\mu,\sigma\tau} = (n+1) 2^{\lambda-n-1} (n-\nu)! (n-\nu')! \nu'! \nu'! \Gamma(d-1) \Gamma(n + d - \frac{1}{2}) \Gamma(n + d - 2) \Gamma(n + d - 2 - i)
\]

\[
\sum_{\min(n_i, n-\nu)}^{\min(n_i, n-\nu') \min(n_i, n-\nu')} \sum_{\max(0, n_i-\nu) \max(0, n_i-\nu')}^{2j+k} \frac{2^{2\rho}}{\rho! \sigma! \tau!} \sum_{i=1}^{\frac{\nu}{2}+1} (-4)^{i-1}(i-1)! \Gamma(n + d - \frac{1}{2} - i)
\]

\[
\sum_{j=\max(0, n_i-\nu)}^{2j+k} \sum_{k=\max(0, n_i-\nu')} \frac{1}{\sigma_j! \sigma_j! \tau_k! \tau_k!}
\]

with \(\lambda = \nu + \nu' - 2\mu, \rho = \mu - \sigma - \tau, \nu_j = \nu_i + j, \sigma_j = \nu_i + j, \sigma_j = \nu_i + j, \nu_k = \nu_i + k, \sigma_j = \nu_i + j, \nu_k = \nu_i + k\), and \(\tau_k = \tau - \nu_j - (n-\nu) + (i-1)\) and \(\tau_k = \tau - \nu_j - (n-\nu) + (i-1)\).

For the planar diagram \((k_t = k_2)\) \(8\) simplifies and we have

\[
A_{\nu\nu'}^n(k_1, k_2) = \sum_{\mu=\max(0, \nu+\nu'-n)}^{\nu+\nu'} a_{\nu\nu'}^{n\mu}(k_1^2)^{n-(\nu+\nu')} + \mu (k_2^2)^{\nu'} (k_1 k_2)^{\nu-\nu'-2\mu},
\]

with

\[
a_{\nu\nu'}^{n\mu} = \sum_{\sigma=0}^{\nu} \sum_{\tau=0}^{\nu'} b_{\nu\nu'}^{n\mu,\sigma\tau},
\]
The coefficients $a_{\mu\nu}$ are mass independent and have been calculated with FORM up to order $\varepsilon^2$ ($d = 4 - 2\varepsilon$) and stored for the first 30 Taylor coefficients, i.e. they are given in terms of rational numbers. For the non-planar case the situation is more difficult since the storing of the coefficients $b_{\mu\nu,\sigma\tau}$ with two more indices is practically impossible for high Taylor coefficients. In many cases, however, if the threshold of the diagram under consideration is high, only a few (say 8) Taylor coefficients are sufficient for a high precision calculation of the diagram under consideration and in such a case the direct calculation of the $b$'s in each case causes no problem.

Finally all remaining integrals can be reduced to bubble-integrals of the type

$$V_{\alpha\beta\gamma}(\{m\}) = \int \frac{d^dk_1 d^dk_2}{(k_1^2 - m_1^2)^\alpha (k_2^2 - m_2^2)^\beta ((k_1 - k_2)^2 - m_3^2)^\gamma},$$

or to factorizing one-loop integrals. The genuine two-loop bubble integrals are reduced by means of recurrence relations to $V_{111}(\{m\})$. This can be done numerically for the arbitrary mass case or also analytically for special cases like e.g. only one non-zero mass. For details see [17], [18]. To perform the recursion numerically, it is important to use the multiple precision FORTRAN by D.Bailey ([19]) since tremendous cancellations occur in this case.

We have in some detail presented one approach for the calculation of the Taylor expansion of Feynman diagrams, others were worked out in Refs. [20], [21]. The latter one is particularly suited for programming in terms of a formulae manipulating language like FORM.

### 2.2. The method of analytic continuation

Assume, the following Taylor expansion of a scalar diagram or a particular amplitude is given $C(p_1, p_2, \ldots) = \sum_{m=0}^{\infty} a_m y^m \equiv f(y)$ and the function on the r.h.s. has a cut for $y \geq y_0$. In the above case of $H \rightarrow \gamma\gamma$ one introduces $y = q^2/4m^2$ with $q^2 = (p_1 - p_2)^2$ as adequate variable with $y_0 = 1$.

The method of evaluation of the original series consists in a first step in a conformal mapping of the cut plane into the unit circle and secondly the reexpansion of the function under consideration into a power series w.r.t. the new conformal variable

$$\omega = \frac{1 - \sqrt{1 - y/y_0}}{1 + \sqrt{1 - y/y_0}}.$$  \hspace{1cm} (13)

By this conformal transformation, the $y$-plane, cut from $y_0$ to $+\infty$, is mapped into the unit circle (see Fig.[1]) and the cut itself is mapped on its
Fig. 4. Conformal mapping of the $y = q^2/4m_t^2$-complex plane into the $\omega$-plane.

boundary, the upper semicircle corresponding to the upper side of the cut. The origin goes into the point $\omega = 0$.

After conformal transformation it is suggestive to improve the convergence of the new series w.r.t. $\omega$ by applying the Padé method \[22,23\]. A convenient technique for the evaluation of Padé approximations is the $\varepsilon$-algorithm of \[22\] which allows one to evaluate the Padé approximants recursively.

Historically the first example considered was the two-loop three-point scalar (planar) integral with the kinematics of the decay $H \rightarrow \gamma\gamma$: $m_6 = 0$ and all other masses $m_i = m_t (i = 1, \ldots, 5)$. In this special case all Taylor coefficients can be expressed in terms of $\Gamma$-functions. Later a closed expression for arbitrary coefficients has been found in \[21\].

Table 1. Results on the cut ($q^2 > 4m_t^2$) in comparison with \[24\].

| $q^2/m_t^2$ | \[14/14\]   | \[24\]   |
|-------------|--------------|-----------|
|             | Re | Im    | Re | Im          |
| 4.01        | 11.935 | 12.699 | 11.9347(1) | 12.69675(8) |
| 4.10        | 2.66245 | 9.0955 | 2.66246(2) | 9.0954(2)   |
| 5.0         | -1.985804823 | 2.758626375 | -1.98580(2) | 2.758625(2) |
| 10.0        | -0.7569432708 | -0.0615483234 | -0.756943(1) | -0.061547(1) |
| 40.0        | 0.064585286(7) | -0.0645672604 | 0.064585286(7) | -0.0645672604 |
| 400.0       | 0.00008190 | -0.002167005 | 0.0000819074(3) | -0.002167005(3) |

Results for this kinematics on the cut are given in Table 1. The process $H \rightarrow \gamma\gamma$ was investigated before in Ref. \[24\]. For the master integral under consideration in \[24\] all integrations but one could be performed analytically and only the last one had to be done numerically. Similarly, high precision is obtained on the cut in the approach of \[16\].

Further examples of the efficiency of our method were given in Refs. \[25\] and \[26\]. In the latter case a diagram of particular interest for the $Z \rightarrow bb$
decay has been considered, namely a planar diagram with a low threshold: $m_1 = m_2 = m_5 = m_6 = m_b = 4.5\text{GeV}$ and $m_3 = m_4 = m_Z = 91\text{GeV}$. If this diagram is to be evaluated on the $Z$ mass-shell, a precision of only some four decimals can be obtained with 30 Taylor coefficients, which is hardly good enough. On the other hand it is a very reasonable approximation in this case to put $m_b = 0$ from the very beginning and thus consider diagrams with zero thresholds, which should be evaluated more easily with higher precision. These will be considered in the next Sect.

2.3. Two-loop vertex diagrams with zero thresholds

Concerning the vertex diagrams, there are many different topologies contributing to a 3-point function in the SM. For our purpose of demonstrating the method, we confine ourselves to the planar case shown in Fig. 5a. Figs. 5c,d presents infrared divergent diagrams.

As a typical (and very complicated case) we consider here Fig. 5b with equal non-zero masses [27]. There are two massless cuts so that we shall have the double logarithm in the expansion. The set of subgraphs in this case is given by $\gamma_1 = \Gamma$ and the higher terms with $\gamma \neq \Gamma : \gamma_2 = \{3456\}, \gamma_3 = \{1256\}, \gamma_4 = \{56\}$ (in curly brackets: the line numbers of the subgraphs $\gamma$ in (2), line numbers are shown in Fig. 7). Note that $\gamma_3$ and $\gamma_4$ are disconnected. After summing up all four contributions we see that the double and single poles in $\varepsilon$ cancel as well as the scale parameter $\mu$, with the result ($x = q^2/M^2$)

\[
F_\Gamma(q^2, M^2) = \frac{1}{M^4} \sum_{n=0}^{\infty} \sum_{j=0}^{2} f_{jn} \ln^j(-x) x^n \\
\equiv \frac{1}{M^4} \left\{f_0(x) + f_1(x) \ln(-x) + f_2(x) \ln^2(-x)\right\}, \tag{14}
\]
Table 2. Results for timelike $q^2$ for diagram 5b.

| $\frac{q^2}{M^2}$ | [12/12] | Im     | [15/15] | Im     |
|-----------------|--------|--------|--------|--------|
| 0.05            | +2.948161245 | 20.938528 | +2.948161245 | 20.938528 |
| 0.1             | −1.108116127 | 16.04132127 | −1.108116127 | 16.04132127 |
| 0.5             | −4.820692261 | 5.066080015 | −4.820692261 | 5.066080015 |
| 1.0             | −3.890154 | 1.67549787 | −3.890156 | 1.67549788 |
| 1.5             | −2.904588 | 0.42979 | −2.904581 | 0.429778 |
| 2.0             | −2.18294 | −0.06976 | −2.182981 | −0.069728 |
| 10.0            | −0.191 | −0.208 | −0.194 | −0.215 |

Table 3. The finite part for diagram 5d.

| $\frac{q^2}{m_6^2}$ | [15/15] | numerical results |
|-----------------|--------|--------------------|
| Re             | Im     | Re | Im |
| 0.5            | 81.175601719 | 12.06458720 | 81.1750 | 12.0644 |
| 1.0            | 17.7659 | 19.97799834 | 17.7658 | 19.9779 |
| 2.0            | −0.6047 | 7.3759 | −0.604 | 7.376 |
| 10.0           | −0.572 | 0.023 | −0.570 | 0.025 |

where the $f_{jn}$ are now given in terms of rational numbers and $\zeta(2)$. $f_2(x)$ can be summed analytically, yielding

$$f_2(x) = (\ln|1 + x| - i\pi \theta(1 + x))^2/x^2.$$  \hspace{1cm} (15)

Thus, we have to Padé approximate $f_0$ and $f_1$ only. Close to the second threshold at $q^2 = M^2$ the convergence is indeed excellent (see Table 2, 27). It should be noted that for the physical application we have in mind, i.e. $Z \rightarrow b\bar{b}$, this is just the case of interest. It is worthwhile to note the sharp increase for low $q^2$ due to $\ln^2(-q^2/M^2)$.

Infrared divergent diagrams (Figs. 5c,d) have been successfully considered in [28]. For Fig. 5c the convergence is indeed excellent again. The different mass case, i.e. $m_6 \neq m_1 = m_2$, had to be calculated numerically according to the procedure outlined in Sect. 2.1. For Fig. 5d the convergence is by far not that good: for $q^2/M^2 = 1$ a precision of 6 decimals is still obtained with a [15/15] approximant, while for $q^2/M^2 \simeq 10$ only 2 decimals were obtained (see Table 3) where the “numerical results” were obtained by the method of Ref. [29].

This much slower convergence implies that indeed it is very desirable to have at least in the case of only one non-zero mass a compact representation of the Taylor coefficients in terms of rational numbers, i.e. not to have to go through the machinery of calculating the coefficients as described in Sect. 2.1. Such a possibility will be described in the next section.


2.4. The Differential Equation Method

We saw that to obtain the expansion of a diagram one has to go through a rather combersome machinery. The more coefficients are asked for, the more efforts and machine resources are required. Thus it is very desirable to have analytic expressions for expansion coefficients whenever possible. This can be done with the aid of the Differential Equation Method (DEM) [30] if only one non-zero mass occurs. The DEM allows one to get results for massive diagrams by reducing the problem to diagrams with simpler structure.

Let us introduce a graphical notation for the scalar propagators (in euclidean space-time)

\[
\frac{1}{(q^2)\alpha} = \alpha, \quad \frac{1}{(q^2 + m^2)\alpha} = \frac{\alpha}{m^2}
\]

\(\alpha\) and \(m\) refer to the index and mass of a line. Then one can derive the following recurrence relation for a massive triangle [30]

\[
\begin{align*}
\alpha_2 \alpha_3 \alpha_1 (D - 2\alpha_1 - \alpha_2 - \alpha_3) &= -2m_1^2\alpha_1 \\
+\alpha_2 \left(\frac{\alpha_2 + 1}{\alpha_1 - 1} - \frac{\alpha_2 + 1}{\alpha_1} - (m_1^2 + m_2^2)\right) + (\alpha_2 \leftrightarrow \alpha_3)
\end{align*}
\]

along with some other graphical relations (see details in [30]).

Using this technique we analysed in [31] the class of 3-point two-loop massive graphs. As an example for the diagram of Fig.5c we get

\[
(D - 4)J_{5c} = 2 - 2 - 4m^2 - 2m^2 - 2
\]

In the r.h.s. of (16) the last two terms can be combined, resulting in \(dJ/dm^2\) while for the second term we proceed in turn as

\[
(D - 4) = \frac{2}{2} + \frac{2}{2} - \frac{2}{2}
\]

Thus we are left with simple diagrams (these can be done completely by Feynman parameters) and the derivative of the initial diagram w.r.t. \(m^2\).
The solution of the corresponding differential equation in terms of a series obtained from an integral representation reads

\[
J_{5c} = -\frac{\Gamma^2(1+\varepsilon)}{(q^2)^2(m^2)^2} \sum_{n=1}^{\infty} \frac{(-x)^n\Gamma^2(n)}{\Gamma(2n+1)} \left[ \frac{1}{x^2} - \frac{1}{\varepsilon} \left( \ln(x) + S_1(n-1) \right) - \frac{3}{2}S_2(n-1) \right]
- \frac{15}{2}S_1^2(n-1) + 4S_1(n-1)S_1(2n) - \zeta(2) - \ln(x)S_1(n-1) + \frac{1}{2}\ln^2(x)
\]

where

\[
S_l(n) = \sum_{k=1}^{n} \frac{1}{k^l}
\]

A similar formula was obtained for the diagram of Fig.5d (see [31]).

2.5. Large mass expansion versus small momentum expansion

If there are two or more different masses involved the coefficients \(a_{lmn}\) in (4) are not just numbers any more but complicated functions of mass ratios. In this case one can try to perform a large mass (LM) expansion rather than a small momentum expansion.

Here we just consider one example of a large mass expansion for the 3-point function (a more detailed analysis will be given in [32]). Let us consider the diagram of Fig. 6a with two non-zero masses \(m_W\) and \(m_{top}\), contributing to \(Z\) decay in the SM. Thus we have \(q^2 = m_Z^2\). One can use the procedure described in Sect. 2.1. to expand this diagram in the external momentum squared. Then the actual parameter of expansion is \(q^2/m_{top}^2\) and the radius of convergence is given by the lowest (non-zero) threshold i.e. \(q^2 = (m_{top} + m_W)^2 \ll m_Z^2\). Another possibility is to expand in the ratios \(q^2/m_{top}^2\) and \(m_W^2/m_{top}^2\), though one will see that this expansion does not work that well.

The asymptotic expansion in the limit of the large mass is given by (16). In our case there are 4 subgraphs that contribute, i.e. \(\gamma_1 = \{123456\}\), \(\gamma_2 = \{126\}\) \(\gamma_3 = \{12345\}\) and \(\gamma_4 = \{12\}\) (see Fig.3a-d). By direct evaluation we find that there are induced poles of the order \(1/\varepsilon^3\) in subgraphs while in the the sum they cancel which serves as a good check. For this particular diagram the result of the LM expansion looks like

\[
\text{dia} = \sum_{n=-1}^{\infty} A_n = \left( b_n^{(0)} + b_n^{(1)}L + b_n^{(2)}L^2 + b_n^{(3)}L^3 \right) \left( \frac{1}{m_{top}^2} \right)^n
\]

with \(L = \log(m_{top}^2/\mu^2)\) and \(b_n^{(i)}\)'s being known functions of \(q^2\), \(m_W^2\) and \(\mu^2\).
In Table 4 we give the results of the numerical analysis for the graph at hand. We observed bad behavior of the series (16): namely 10 coefficients give only 3 stable figures. Since the parameters of expansion $q^2/m^2_{\text{top}}$ and $m^2_W/m^2_{\text{top}}$ both are of order 0.25 one would expect better convergence. We have found that this bad convergence remains true for some other graphs of the $Zb\bar{b}$ process [32]. Moreover no analytic properties are known in the $1/m^2$ variable and thus no conformal mapping like (13) is now available to improve the convergence. However we apply the Padé summation. Nevertheless the LM expansion for 3-point functions in such a regime looks very attractive since it can be rather easily implemented in formulæ manipulating languages like FORM even in the presence of more than one non-zero mass. In the worst case one achieves 0.1% accuracy with 10 coefficients. Our programs allow us to get 15-20 coefficients in a reasonable time.

Table 4. Terms in the LM expansion corresponding to formula (16) at $\mu = m_{\text{top}} = 180$, $m_W = 80$, $q^2 = 90^2$. The last two lines give Padé approximants obtained with 10 terms of the LM expansion and 8 Taylor coefficients of the small $q^2$ expansion.

| $n$ | $\text{Re } A_n$ | $\text{Im } A_n$ |
|-----|--------------------|------------------|
| -1  | -6.35040           | 29.98705         |
| 0   | -3.85690           | -6.19256         |
| 1   | -1.74681           | -4.34057         |
| 2   | 0.22329            | -1.30291         |
| 3   | 0.65892            | -0.21984         |
| 4   | 0.47058            | 0.00121          |
| 5   | 0.25376            | 0.01435          |
| 6   | 0.13099            | 0.00492          |
| 7   | 0.07311            | -0.00673         |
| 8   | 0.04517            | 0.00099          |
| 9   | 0.02984            | -0.00099         |
| 10  | 0.02042            | -0.00002         |

|         | $\text{LM}$        | $\text{small-q}$ |
|---------|---------------------|-------------------|
| $\text{Re } A_n$ | -9.9956            | 17.9527           |
| $\text{Im } A_n$ | 17.9527            | -9.9926682590236  |

2.6. Conclusion
Any involved calculation in field theory necessarily consists of two parts: 1) automatic generation of Feynman diagrams and source codes and 2) techniques of evaluating scalar Feynman diagrams. In this paper we presented both. Still some efforts have to be applied to handle the numerators of diagrams. We are working also on the automation of the renormalization procedure.

Acknowledgements

We would like to thank M. Kalmykov for helpful discussion. M.T. and O.V. acknowledge the University of Bielefeld for the warm hospitality.

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