On some birational invariants of hyper-Kähler manifolds

Chenyu Bai

Abstract

We study in this article three birational invariants of projective hyper-Kähler manifolds: the degree of irrationality, the fibering gonality and the fibering genus. We first improve the lower bound in a recent result of Voisin saying that the fibering genus of a Mumford–Tate very general projective hyper-Kähler manifold is bounded from below by a constant depending on its dimension and the second Betti number. We also study the relations between these birational invariants for projective \(K3\) surfaces of Picard number \(1\) and study the asymptotic behaviors of their degree of irrationality and fibering genus.

1 Introduction

In the paper [2], a birational invariant, the degree of irrationality, of projective manifolds is studied, in particular for hypersurfaces. The degree of irrationality \(\text{Irr}(X)\) of a projective manifold \(X\) is defined to be the minimal degree of all possible dominant rational maps from \(X\) to the projective space \(\mathbb{P}^n\), with \(n = \dim X\). The degree of irrationality is a natural generalization of gonality of curves, and it measures to which extent \(X\) is far from being rational, in the sense that \(X\) is rational if and only if \(\text{Irr}(X) = 1\).

The degree of irrationality is now relatively well-understood for very general hypersurfaces in projective spaces [2] [6] [10] and for very general abelian varieties [6] [5], while for \(K3\) surfaces, it is still mysterious. In particular, the following conjecture proposed in [2] remains open and is a main motivation of the second part of this paper.

**Conjecture 1.1** ([2]). Let \(\{(S_d, L_d)\}_{d \in \mathbb{N}}\) be very general polarized \(K3\) surfaces such that \(L_d^2 = 2d - 2\). Then

\[
\limsup_{d \to \infty} \text{Irr}(S_d) = +\infty.
\]

In an attempt to understand and estimate the degree of irrationality, many other birational invariants are proposed among which are the fibering gonality and fibering genus introduced in [9]. As is defined in loc. cit., for a projective manifold \(X\), the fibering gonality \(\text{Fibgon}(X)\) (resp. fibering genus \(\text{Fibgen}(X)\)) of \(X\) is the minimal gonality (resp. geometric genus) of the general fiber of a fibration \(X \dashrightarrow B\) into curves.

In this article, we study these three birational invariants, namely the degree of irrationality, the fibering gonality and the fibering genus, for projective hyper-Kähler manifolds. A projective hyper-Kähler manifold \(X\) is a simply connected complex projective manifold such that \(\mathcal{H}^0(X, \Omega_X^2) = \mathbb{C}\sigma_X\), where \(\sigma_X\) is a nowhere degenerate holomorphic 2-form on \(X\). A projective \(K3\) surface is a 2-dimensional projective hyper-Kähler manifold.
Let \( X \) be a projective hyper-Kähler manifold of dimension \( 2n \). Let \( q_X \) be its Beauville–Bogomolov–Fujiki form. Let \( b_{2,\text{tr}}(X) \) be the dimension of \( H^2(X, \mathbb{Q})_{\text{tr}} \), the transcendental part of \( H^2(X, \mathbb{Q}) \). In [9, Theorem 1.5], Voisin proves the following result on the fibering genus of Mumford–Tate very general projective hyper-Kähler manifolds.

**Theorem 1.2** (Voisin [9]). Assume that the Mumford–Tate group of the Hodge structure on \( H^2(X, \mathbb{Q})_{\text{tr}} \) is maximal. Assume that \( n \geq 3 \) and \( b_{2,\text{tr}}(X) \geq 5 \). Then if \( X \) admits a fibration \( \phi : X \rightarrow B \) with \( \dim B = 2n - 1 \), the general fiber of \( \phi \) has genus \( g \geq \text{Inf}(n + 2, 2\left\lfloor b_{2,\text{tr}}(X) - \frac{3}{2}\right\rfloor) \). In other words, \( \text{Fibgen}(X) \geq \text{Inf}(n + 2, 2\left\lfloor b_{2,\text{tr}}(X) - \frac{3}{2}\right\rfloor) \).

In this theorem, the assumption on the maximality of the Mumford–Tate group is equivalent to saying that it is the special orthogonal group of \( (H^2(X, \mathbb{Q})_{\text{tr}}, q_X) \). This assumption is satisfied by the very general fiber of a complete lattice polarised family of hyper-Kähler manifolds.

Our first result is a slight improvement of this theorem, giving a sharper lower bound of the fibering genus of Mumford–Tate very general projective hyper-Kähler manifolds. This improvement is obtained by a more refined discussion on an orthogonal relation (see Proposition 2.1 (iii) for details) observed in [9].

**Theorem 1.3.** Assume that the Mumford–Tate group of the Hodge structure of \( H^2(X, \mathbb{Q})_{\text{tr}} \) is maximal. Assume that \( b_{2,\text{tr}}(X) \geq 5 \). Then

\[
\text{Fibgen}(X) \geq \min \left\{ n + \left\lfloor -1 + \sqrt{8n - 7} \right\rfloor, 2\left\lfloor b_{2,\text{tr}}(X) - \frac{3}{2}\right\rfloor \right\}.
\]

Our second result is about the relations between the degree of irrationality, the fibering gonality and the fibering genus of a projective K3 surface with Picard number 1.

**Theorem 1.4.** Let \( S \) be a projective K3 surface whose Picard number is 1. Then one of the following two cases holds:

(a) \( \text{Irr}(S) = \text{Fibgon}(S) \);
(b) \( \text{Fibgen}(S)^2 \leq \text{Fibgon}(S)^{21} \).

In particular, if the fibering gonality of \( S \) is very small compared to the fibering genus, then it is equal to the degree of irrationality of \( S \). The fibering genus of a projective K3 surface of Picard number 1 is studied in [3]. We will rely on [3, Theorem B] (stated below as Theorem 1.5 for completeness).

**Theorem 1.5** (Ein–Lazarsfeld [3]). Let \( S \) be a projective K3 surface whose Picard group is generated by a line bundle of degree \( 2d - 2 \). Then the fibering genus of \( S \) is of order \( \sqrt{d} \). More precisely,

\[
\sqrt{\frac{d}{2}} \leq \text{Fibgen}(S) \leq 2\sqrt{2d}.
\]

Using Theorem 1.4 and [3, Theorem B], we can compare the asymptotic behaviors of the degree of irrationality and the fibering gonality of very general polarized K3 surfaces.

**Theorem 1.6.** Let \( \{S_d\}_{d \in \mathbb{N}} \) be projective K3 surfaces such that the Picard group of \( S_d \) is generated by a line bundle with self intersection number \( 2d - 2 \). Then

\[
\limsup_{d \to \infty} \text{Irr}(S_d) = +\infty \iff \limsup_{d \to \infty} \text{Fibgon}(S_d) = +\infty.
\]
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\[9\] Claire Voisin, *Hodge Theory and Complex Algebraic Geometry II*.
2 Fibering genus of very general hyper-Kähler manifolds

Let $X$ be a projective hyper-Kähler manifold of dimension $2n$. Let $f : X \to B$ be a fibration into curves and let $\tau : \tilde{X} \to X$ and $\bar{f} : \tilde{X} \to B$ be a resolution of indeterminacy points. Let $\tilde{X}_b$ be a smooth fiber of $\bar{f}$ over a general point $b \in B$. Contracting with $\tau^* \sigma_X$ gives a morphism of vector bundles $N_{\tilde{X}_b/X} \to \Omega_{\tilde{X}_b}$. Therefore, we have a morphism

$$\sigma_b : T_{B,b} \to H^0(\tilde{X}_b, \Omega_{\tilde{X}_b})$$

defined as the composition of the natural morphisms $T_{B,b} \to H^0(\tilde{X}_b, N_{\tilde{X}_b/X})$ and $\tau^* \sigma_X : H^0(\tilde{X}_b, N_{\tilde{X}_b/X}) \to H^0(\tilde{X}_b, \Omega_{\tilde{X}_b})$. Let $\rho : T_{B,b} \to H^1(\tilde{X}_b, T_{\tilde{X}_b})$ be the Kodaira–Spencer map. In [9], Voisin proves the following

**Proposition 2.1** (Voisin [9]). (i) If $n \geq 2$, then $\tilde{X}_b$ is not hyperelliptic.
(ii) The rank of $\sigma_b$ is $\geq n$.
(iii) Let $I_b$ be image of $\text{Im} \sigma_b \otimes H^0(\tilde{X}_b, K_{\tilde{X}_b})$ under the multiplication map

$$\mu : H^0(\tilde{X}_b, K_{\tilde{X}_b}) \otimes H^0(\tilde{X}_b, K_{\tilde{X}_b}) \to H^0(\tilde{X}_b, K_{\tilde{X}_b}^\otimes 2).$$

Then $\rho(\ker \sigma_b) \subset H^1(\tilde{X}_b, T_{\tilde{X}_b})$ is orthogonal to $I_b$ via the Serre pairing

$$H^1(\tilde{X}_b, T_{\tilde{X}_b}) \otimes H^0(\tilde{X}_b, K_{\tilde{X}_b}^\otimes 2) \to H^1(\tilde{X}_b, K_{\tilde{X}_b}) \cong \mathbb{C}.$$\hspace{1cm}(iv) Assume that that $b_{2,2}(X) \geq 5$ and that the Mumford–Tate group of $H^2(X, \mathbb{Q})_{tr}$ is maximal. If $\rho : T_{B,b} \to H^1(\tilde{X}_b, T_{\tilde{X}_b})$ is not injective, then $g(\tilde{X}_b) \geq 2\left(\frac{b_{2,2}(X)-1}{2}\right)$.

The proofs of (i), (ii) and (iii) are explicitly written in [9]. Although (iv) is not stated in [9] with this generality, it is essentially proved there (see the proof of Lemma 2.13 and Lemma 2.15 in loc. cit.).

2.1 Proof of Theorem 1.3

For $n = 1$, the inequality is obvious, since $K3$ surfaces cannot be fibered into rational curves. From now on, let us assume $n \geq 2$ to be able to assume $\tilde{X}_b$ is not hyperelliptic (see Proposition 2.1(ii)). Let $k$ be the corank of $\sigma_b$, i.e., $\text{rank} \sigma_b = g - k$, where $g$ is the genus of the curve $\tilde{X}_b$. We now prove

**Lemma 2.2.** The notation as above, we have the following inequality

$$\dim \ker \rho \geq 2n - 1 - (g - k) - \frac{k(k + 1)}{2}.$$\hspace{1cm}Proof. We use the notation as in Proposition 2.1(iii). By Serre duality, the orthogonality result in Proposition 2.1(iii) implies that $\dim \rho(\ker \sigma_b) + \dim I_b \leq \dim H^0(\tilde{X}_b, K_{\tilde{X}_b}^\otimes 2)$, that is,

$$\text{codim}(I_b \subset H^0(\tilde{X}_b, K_{\tilde{X}_b}^\otimes 2)) \leq \dim \rho(\ker \sigma_b).$$ (2)
The multiplication map \( \mu : H^0(\bar{X}_b, K_{\bar{X}_b}) \otimes H^0(\bar{X}_b, K_{\bar{X}_b}) \to H^0(\bar{X}_b, K_{\bar{X}_b}^2) \) factors through the symmetric product \( \text{Sym}^2 H^0(\bar{X}_b, K_{\bar{X}_b}) \), that is, we have a commutative diagram

\[
\begin{array}{ccc}
H^0(\bar{X}_b, K_{\bar{X}_b}) \otimes H^0(\bar{X}_b, K_{\bar{X}_b}) & \xrightarrow{\mu} & H^0(\bar{X}_b, K_{\bar{X}_b}^2) \\
\text{pr} & & \mu' \\
\text{Sym}^2 H^0(\bar{X}_b, K_{\bar{X}_b}) & & ,
\end{array}
\]

where \( \text{pr} : H^0(\bar{X}_b, K_{\bar{X}_b}) \otimes H^0(\bar{X}_b, K_{\bar{X}_b}) \to \text{Sym}^2 H^0(\bar{X}_b, K_{\bar{X}_b}) \) is the canonical symmetrization map. Let \( \text{Im} \sigma_b \cdot H^0(\bar{X}_b, K_{\bar{X}_b}) \) denote the image of \( \text{Im} \sigma_b \otimes H^0(\bar{X}_b, K_{\bar{X}_b}) \) under \( \text{pr} \). Then we have

\[
\text{codim}(\text{Im} \sigma_b \cdot H^0(\bar{X}_b, K_{\bar{X}_b})) \subset \text{Sym}^2 H^0(\bar{X}_b, K_{\bar{X}_b}) = \frac{k(k+1)}{2}. \tag{3}
\]

On the other hand, by Max Noether theorem (see [1], Chapter III, §2 or [7]), the multiplication map \( \mu' : \text{Sym}^2 H^0(\bar{X}_b, K_{\bar{X}_b}) \to H^0(\bar{X}_b, K_{\bar{X}_b}^2) \) is surjective, since \( \bar{X}_b \) is not hyperelliptic. Taking into account the fact that \( I_b \) is the image of \( \text{Im} \sigma_b \cdot H^0(\bar{X}_b, K_{\bar{X}_b}) \) under the map \( \mu' \), we have the following inequality

\[
\text{codim}(I_b \subset H^0(\bar{X}_b, K_{\bar{X}_b}^2)) \leq \text{codim}(\text{Im} \sigma_b \cdot H^0(\bar{X}_b, K_{\bar{X}_b}) \subset \text{Sym}^2 H^0(\bar{X}_b, K_{\bar{X}_b})). \tag{4}
\]

Combining (2), (3) and (4), we get \( \dim \rho(\ker \sigma_b) \leq \frac{k(k+1)}{2} \), from which we deduce that

\[
\dim \ker \rho \geq \text{dim} \ker \sigma_b - \frac{k(k+1)}{2} = 2n - 1 - (g-k) - \frac{k(k+1)}{2}.
\]

Proof of Theorem 1.3: Assuming that \( g(\bar{X}_b) < 2 \lceil \frac{1+\sqrt{8n-7}}{2} \rceil \), we have to prove \( g \geq n + \lceil \frac{1+\sqrt{8n-7}}{2} \rceil \). By Proposition 2.1(ii), (iv) and Lemma 2.2, we have the following constraints on \( g \) and \( k \):

\[
\begin{align*}
\left\{ \begin{array}{l}
k \geq 0 \\
g - k - n \geq 0 \\
2n - 1 - (g-k) - \frac{k(k+1)}{2} \leq 0.
\end{array} \right.
\end{align*}
\]

In order to find the minimal possible value of \( g \) under these constraints, we make the following discussion according to the values of \( k \):

- When \( 0 \leq k \leq \frac{1+\sqrt{8n-7}}{2} \), we have \( k - \frac{k(k+1)}{2} + 2n - 1 \geq n + k \). Hence, the minimal possible value of \( g \) in this domain is the minimum of \( k - \frac{k(k+1)}{2} + 2n - 1 \) with \( 0 \leq k \leq \frac{1+\sqrt{8n-7}}{2} \), which is \( n + \frac{1+\sqrt{8n-7}}{2} \).
- When \( k \geq \frac{1+\sqrt{8n-7}}{2} \), we have \( k - \frac{k(k+1)}{2} + 2n - 1 \leq n + k \). Hence, the minimal possible value of \( g \) in this domain is the minimum of \( n + k \) with \( k \geq \frac{1+\sqrt{8n-7}}{2} \), which is \( n + \frac{1+\sqrt{8n-7}}{2} \).
Since $g$ and $k$ are integers, we find $g \geq n + \left\lfloor \frac{-1 + \sqrt{8a - 7}}{2} \right\rfloor$, as desired.

\[\square\]

**Remark.** Our proof relies on the inequality (4) which only uses the surjectivity of the multication map $\mu'$. With more information on the geometry of the canonical embedding, and in particular, on the gonality of the fibers, we could get a better estimate in Theorem 1.3.

3 Relations between birational invariants of $K3$ surfaces

In this section, we are going to prove Theorem 1.4 that relates the three birational invariants, namely, the degree of irrationality, the fibering gonality and the fibering genus, of projective $K3$ surfaces of Picard number 1.

3.1 A factorization

Let $S$ be a smooth projective surface and let $f : S \dashrightarrow B$ be a fibration into curves over a smooth base $B$. After a resolution of indeterminacies of $f$ and replacing $S$ by another birational model, we may assume $f : S \rightarrow B$ is a morphism. Let $d$ be the gonality of the general fiber of $f$. The general fiber $C$ admits a degree $d$ morphism from $C$ to $\mathbb{P}^1$. Standard argument shows that we can spread this morphism into a family up to a generically finite base change.

**Lemma 3.1.** There is a generically finite morphism $\pi : B' \rightarrow B$ and a degree $d$ dominant rational map $\psi : S \times_B B' \dashrightarrow \mathbb{P}^1 \times B'$ over $B'$.

**Proof** Let $B_0$ be the smooth locus of $f : S \rightarrow B$ and let $f_0 : S_0 \rightarrow B_0$ be the restriction of $f$ on smooth locus. Let $p : \text{Pic}^d(S_0/B_0) \rightarrow B_0$ be the degree $d$ relative Picard variety of $f_0 : S_0 \rightarrow B_0$. By the assumption on the general fiber of $f : S \rightarrow B$, the Brill-Noether locus in $\text{Pic}^d(S_0/B_0)$ of linear systems of degree $d$ and dimension 1 is dominant over $B_0$ via the map $p : \text{Pic}^d(S_0/B_0) \rightarrow B_0$. Let $B'_0$ be a general reduced irreducible subscheme of $\text{Pic}^d(S_0/B_0)$ that is dominant and generically finite over $B_0$ by $p$. Let us take $B'$ to be a completion of $B'_0$. Then by construction, the universal line bundle restricted to $S_0 \times_B B'_0$ gives a dominant rational map $\psi : S \times_B B' \dashrightarrow \mathbb{P}^1_{B'}$ of degree $d$, as desired. \[\square\]

It is natural to ask if $\psi : S \times_B B' \dashrightarrow \mathbb{P}^1 \times B'$ over $B'$ descends to a rational map $\psi_B : S \dashrightarrow \mathbb{P}^1 \times B$ over $B$. A moment of thinking will convince us that we are asking too much, because $\pi : B' \rightarrow B$ is in general not a Galois cover. We make the following construction. Let $n$ be the degree of the morphism $\pi : B' \rightarrow B$. Consider the $n$-th self fibred product of $B'$ over $B$: $B' \times_B \ldots \times_B B'$. Define $B''$ to be the closure in $B' \times_B \ldots \times_B B'$ of the set
\[
\{(x_1, \ldots, x_n) \in B' \times_B \ldots \times_B B' : x_1, \ldots, x_n \text{ are distinct}\}.
\]

Then $\pi' : B'' \rightarrow B$ is of degree $n!$ and the symmetric group $S_n$ permuting the components of $B' \times_B \ldots \times_B B'$ acts on an open dense subset of $B''$. The rational map $\psi : S \times_B B' \dashrightarrow \mathbb{P}^1 \times B'$ given in Lemma 3.1 can be extended to a rational map
\[
\psi' : S \times_B B'' \dashrightarrow (\mathbb{P}^1)^n \times B''
\] (5)
over $B''$ in a natural way: let $x \in S$ and let $y = (y_1, \ldots, y_n) \in B''$ be general points, we define $\psi'(x, y) = (\psi(x, y_1), \ldots, \psi(x, y_n), y)$. Moreover, the symmetric group $S_n$ acts canonically on both sides of (5) in the following way. To define the action of $S_n$ on $S \times_B B''$, we let $S_n$ act trivially on $S$ and act as permutations of components of $B''$; and to define the action on $(\mathbb{P}^1)^n \times B''$, we let $S_n$ act as permutations of components for both $(\mathbb{P}^1)^n$ and $B''$. It is clear from the construction that $\psi': S \times_B B'' \rightarrow (\mathbb{P}^1)^n \times B''$ is $S_n$-equivariant. Thus $\psi'$ induces a rational map $S \rightarrow ((\mathbb{P}^1)^n \times B'')/S_n$ over $B$. Let $S'$ be the image of this map. Thus we get a dominant rational map

$$\phi: S \rightarrow S'.$$

**Proposition 3.2.** $S'$ is a surface and the degree of $\phi$ divides $d$. Furthermore, if $n \geq 2$, the general fiber of $S' \rightarrow B$ is of geometric genus $\leq (\frac{d}{\deg \phi} - 1)^2$.

**Proof.** Over the general point $b = (b_1, \ldots, b_n) \in B''$, $\psi'$ is given by the morphism $\psi'_b: C \rightarrow (\mathbb{P}^1)^n$ induced by the $n$ morphisms $C \rightarrow \mathbb{P}^1$ of degree $d$ corresponding to the points $b_i \in B'$, where $C$ is the fiber of $f: S \rightarrow B$ over $\pi'(b) \in B$. Let $C'$ be the image of $\psi'_b$. Then the fiber of $S' \rightarrow B$ over $\pi'(b) \in B$ is $C'$ by construction. Thus $S'$ is a surface, and the degree of $\phi$ is the degree of $C$ over $C'$, which divides $d$. This proves the first statement. To prove the second, we need to prove the geometric genus of $C'$ is $\leq (\frac{d}{\deg \phi} - 1)^2$. Since $C'$ is a curve of degree $d$ in $(\mathbb{P}^1)^n$, we can use the Castelnuovo type lemma below concerning algebraic curves in $(\mathbb{P}^1)^n$.

**Lemma 3.3.** Let $n \geq 2$. Let $C$ be an integral curve in $(\mathbb{P}^1)^n$ of degree $(d, \ldots, d)$. Then the geometric genus of $C$ is less than or equal to $(d - 1)^2$.

**Proof.** We prove by induction on $n$. When $n = 2$, it is the adjunction formula. Now assume that any curve $C' \subset (\mathbb{P}^1)^{n-1}$ of degree $(e, \ldots, e)$ has geometric genus $\leq (e - 1)^2$. Consider the projection $C \rightarrow C'' \subset (\mathbb{P}^1)^{n-1}$ to the first $n - 1$ components. The degree of $C \rightarrow C''$ is of the form $d/e$, for some $e$. Hence, $C'' \subset (\mathbb{P}^1)^{n-1}$ is a curve of degree $(e, \ldots, e)$, hence it has geometric genus $\leq (e - 1)^2$ by induction assumption. Let $\tilde{C}$ and $\tilde{C''}$ be the normalization of $C$ and $C''$ respectively. Then $\tilde{C}$ is birational to its image $C''$ in $\tilde{C}'' \times \mathbb{P}^1$, where the map to the second component is given by the composition map $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^1$. Here, the map $i_n: C \rightarrow \mathbb{P}^1$ is the projection map to the $n$-th component. Note that $C''$ is of degree $(d/e, d)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and note that $NS(C'' \times \mathbb{P}^1) = NS(C') \oplus NS(\mathbb{P}^1)$. Adjunction formula and $g(C'') \leq (e - 1)^2$ give us $p_a(C'') = \frac{d(g(C'))}{e} - d + 1 \leq d(e + d/e - 2) - d + 1 \leq (d - 1)^2$ since $1 \leq e \leq d$. Hence, $p_a(C) = g(\tilde{C}) = p_a(C'') \leq p_a(C'') \leq (d - 1)^2$, as desired.

### 3.2 Proof of Theorem 1.4: The Case (a)

Let $f: S \rightarrow B = \mathbb{P}^1$ be a fibration into curves realizing the fibering gonality of $S$. After a desingularization of indeterminacies of $f$, we get a dominant morphism $\tilde{f}: \tilde{S} \rightarrow B$ whose general fiber is of gonality $d = \text{Fibgon}(S)$. In Section 3.1, we constructed a surface $S'$ that is a fibration over $B$ into curves and a dominant rational map $\phi: \tilde{S} \rightarrow S'$ over $B$ of degree dividing $d$. The Kodaira dimension $\kappa(S')$, the irregularity $q(S')$ and the geometric genus $p_a(S')$ of $S'$ cannot exceed those of $S$ since $S'$ is dominated by $S$. By Enriques-Kodaira classification of algebraic surfaces [4], $S'$ can only be birational to $\mathbb{P}^2$, a K3 surface or an Enriques surface.
If $S'$ is a rational surface, then $\text{Irr}(S) \leq \deg \phi \leq d = \text{Fibgon}(S)$. Here, the inequality $\deg \phi \leq d$ is because of Proposition 3.2. But clearly $\text{Fibgon}(S) \leq \text{Irr}(S)$. We get the equality. This is case (a) of the theorem. We will treat the case when $S'$ is birational to a $K3$ surface in Section 3.4 and we will prove in Proposition 3.7 the inequality given in case (b) of the theorem.

It remains to eliminate the case when $S'$ is birational to an Enriques surface $S''$. After a birational modification of $\tilde{S}$, there is a dominant morphism $g : \tilde{S} \to S''$. Since $\tilde{S}$ is simply-connected, $g$ factors through the universal covering $S''$ of $S''$. $S''$ is a K3 surface of Picard number at least 10. But Lemma 3.4 below shows that the Picard number of $S''$ can only be 1 since it is dominated by $S$. This gives us a contradiction. The proof of Theorem 1.4 is thus concluded when $S'$ is either rational or an Enriques surface.

Lemma 3.4. Let $\phi : X \to X'$ be a dominant rational map between projective hyper-Kähler manifolds of the same dimension. Then $b_{2,1}(X) = b_{2,1}(X')$.

Proof. Let $\tau : \tilde{X} \to X$ and $\tilde{\phi} : \tilde{X} \to X'$ be a resolution of indeterminacy points of $\phi$. Then $\tau^* : H^2(X, \mathbb{Q})_r \to H^2(\tilde{X}, \mathbb{Q})_r$ is an isomorphism and $\tilde{\phi}^* : H^2(X', \mathbb{Q})_r \to H^2(\tilde{X}, \mathbb{Q})_r$ is injective. They are moreover both morphisms of Hodge structures. But $H^2(X, \mathbb{Q})_r$ and $H^2(X', \mathbb{Q})_r$ are simple Hodge structures. This implies that $\tilde{\phi}^* : H^2(X', \mathbb{Q})_r \to H^2(\tilde{X}, \mathbb{Q})_r$ is an isomorphism and hence the result.

The proof of Theorem 1.4 in the case where $S'$ is birational to a K3 surface will be completed in Section 3.4.

### 3.3 Rational maps between $K3$ surfaces

We treat rational maps between $K3$ surfaces in this part. Let $S$ (resp. $S'$) be a projective $K3$ surface whose Picard group is generated by an ample line bundle of degree $2D - 2$ (resp. $2D' - 2$). Let $\phi : S \to S'$ be a dominant rational map.

Proposition 3.5. We have the following inequality

$$\frac{1}{(\deg \phi)^{21}} \leq \frac{D - 1}{D'} \leq (\deg \phi)^{21}.$$  

Proof. Let $\tau : \tilde{S} \to S$ and $\tilde{\phi} : \tilde{S} \to S'$ be a resolution of indeterminacy points of $\phi : S \to S'$. Let $T$ (resp. $T'$) be the lattice $H^2(S, \mathbb{Z})_r$ (resp. $H^2(S', \mathbb{Z})_r$) endowed with the intersection form. For a positive integer $e$, define $T'(e^2)$ to be the lattice $T'$ with the quadratic form multiplied by $e$. For example, with this notation, the sublattice $et'$ of $T'$ is isometric to $T'(e^2)$ as lattices. The image $E$ of the morphism $\tilde{\phi}^* : H^2(S', \mathbb{Z})_r \to H^2(S, \mathbb{Z})_r \cong T$, viewed as a sublattice of $T$, is isometric to $T'(\deg \phi)$. The isomorphism $H^2(S, \mathbb{Z})_r \cong T$ is because $\tau^* : H^2(S, \mathbb{Z})_r \to H^2(\tilde{S}, \mathbb{Z})_r$ is an isomorphism. We thus get the following equalities

$$[T : E]^2 = \frac{\text{disc}(E)}{\text{disc}(T)} = (\deg \phi)^{21} \frac{\text{disc}(T')}{\text{disc}(T)} = (\deg \phi)^{21} \frac{D' - 1}{D - 1}. \quad (6)$$

On the other hand, we have

Lemma 3.6. The morphism of abelian groups $\tilde{\phi} : H^2(\tilde{S}, \mathbb{Z})_r \to H^2(S', \mathbb{Z})_r$ is injective and sends $E$ onto $(\deg \phi)T'$.
Proof. The projection formula shows that \( \tilde{\phi}^* \tilde{\phi}^* = \deg \phi \cdot Id \). Hence, \( \tilde{\phi} \) sends \( E \) onto \( (\deg \phi)T' \). By Lemma 3.4 and the fact that \( \tilde{\phi} \) is surjective with \( \mathbb{Q} \)-coefficients, the kernel of \( \tilde{\phi} \) is of torsion. But \( H^2(\tilde{S}, \mathbb{Z})_{tr} \) is torsion-free, as \( \tilde{S} \) is simply connected. We conclude that the kernel of \( \tilde{\phi} \) is zero, as desired. \( \square \)

Now Lemma 3.6 implies that

\[
1 \leq [T : E] \leq [T' : (\deg \phi)T'] = (\deg \phi)^{21}.
\]

(7)

Proposition 3.5 follows by combining (6) and (7). \( \square \)

Remark. One can similarly prove the more general result on hyper-Kähler manifolds, namely Proposition 1.7. The detailed proof is given in Section 4. A sharper lower bound will also be given there.

3.4 Proof of Theorem 1.4: The Case (b)

Now let us continue the proof of Theorem 1.4. Let \( S \) be a \( K3 \) surface of Picard number 1 and let \( f : S \to B = \mathbb{P}^1 \) be a fibration into curves realizing the fibering gonality. Let \( \phi : S \to S' \) be the rational map constructed as in Section 3.1. Recall that \( S' \) can only be birational to \( \mathbb{P}^2 \), a \( K3 \) surface or an Enriques surface, and we have treated, in Section 3.2, the cases when \( S' \) is birational to \( \mathbb{P}^2 \) or to an Enriques surface. In the rest of this section, we discuss the case when \( S' \) is birational to a \( K3 \) surface. By changing the birational model, we may assume \( S' \) is a \( K3 \) surface. By Lemma 3.4 the Picard number of \( S' \) is also 1. The following proposition shows that in our situation the Case (b) of Theorem 1.4 holds, which concludes the proof of Theorem 1.4.

Proposition 3.7. The following inequality holds:

\[
\text{Fibgen}(S) \leq \text{Fibgon}(S)^{21/2}.
\]

Proof. Let \( D \) and \( D' \) be the degrees of the \( K3 \) surfaces \( S \) and \( S' \), respectively. Let \( C' \) be the general fiber of \( S' \to B \) as in Section 3.1. Then we have the following inequalities

\[
\left( \frac{\text{Fibgon}(S)}{\deg \phi} - 1 \right)^2 \geq p_g(C') \geq \sqrt{\frac{D'}{2}} \geq \sqrt{\frac{D}{2(\deg \phi)^{21}}} \geq \frac{\text{Fibgon}(S)}{4(\deg \phi)^{21/2}}
\]

by Proposition 3.2

by Ein–Lazarsfeld’s theorem (Theorem 1.5)

by Proposition 3.5

by Ein–Lazarsfeld’s theorem (Theorem 1.5).

Note that \( \text{Fibgon}(S) \geq 2 \deg \phi \). Proposition 3.7 follows from these inequalities. \( \square \)

4 Some inequalities about Picard lattices of hyper-Kähler manifolds

We prove in this section Propositions 1.7 and 1.8.
Proof of Proposition [1.7] Let \( \tau : \tilde{X} \to X \) and \( \tilde{\phi} : \tilde{X} \to X' \) be a resolution of indeterminacy points of \( \phi : X \to X' \). Let \( T \) (resp. \( T' \)) be the lattice \( H^2(X, \mathbb{Z})_{tr} \) (resp. \( H^2(X', \mathbb{Z})_{tr} \)) endowed with the Beauville–Bogomolov–Fujiki form. As in the proof of Proposition [3.5] for a positive integer \( e \), define \( T'(e) \) to be the lattice \( T' \) with the quadratic form multiplied by \( e \). We claim that the image \( E \) of the morphism \( \tilde{\phi}^* : H^2(X', \mathbb{Z})_{tr} \to \tau^* H^2(X, \mathbb{Z})_{tr} \cong T' \), viewed as a sublattice of \( T' \), is isometric to \( T'(\deg(\phi)^2) \). This follows from the equalities

\[
[q_X(\tilde{\phi}^* \alpha)]^n = c_X \cdot (f_X \tilde{\phi}^* \alpha^{2n}) = (\deg(\phi) \cdot c_X \cdot (f_X \alpha^{2n}) = (\deg(\phi) \cdot [q_{X'}(\alpha)]^n),
\]

where \( c_X = c_{X'} \) is the Fujiki constant for the deformation class of \( X \). Now the claim implies the following equalities

\[
[T : E]^2 = \frac{\text{disc}(E)}{\text{disc}(T)} = (\deg(\phi) \frac{b_{2,1}(X)}{c_{2,1}(X)}) \cdot \frac{\text{disc}(T')}{\text{disc}(T)} = (\deg(\phi) \frac{b_{2,1}(X)}{c_{2,1}(X)}) \cdot \frac{\text{disc}(\text{Pic}(X'))}{\text{disc}(\text{Pic}(X))}.
\]

(8)

On the other hand, with a similar argument to Lemma [3.6], we prove that the morphism of abelian groups \( \tilde{\phi} : H^2(X, \mathbb{Z})_{tr} \to H^2(X', \mathbb{Z})_{tr} \), injective and sends \( E \) to \( \deg(\phi) T' \). Hence,

\[
1 \leq [T : E] \leq [T' : (\deg(\phi) T')] = (\deg(\phi) b_{2,1}(X)).
\]

(9)

The proposition follows by combining (8) and (9).

Proof of Proposition [1.8] The only thing that needs proving, in the view of Proposition [1.7], is the following inequality

\[
\frac{1}{(\deg(\phi)^{\rho(S)}) \leq \frac{\text{disc}(\text{Pic}(S))}{\text{disc}(\text{Pic}(\tilde{S})))}.
\]

Let \( \tau : \tilde{S} \to S \) and \( \tilde{\phi} : \tilde{S} \to S' \) be a resolution of indeterminacy points of \( \phi : S \to S' \). The morphism \( \tilde{\phi}^* : \text{Pic}(S') \to \text{Pic}(\tilde{S}) \) enlarges the quadratic form by \( \deg(\phi) \) since \( \tilde{\phi}^* \alpha \cup \tilde{\phi}^* \beta = \tilde{\phi}^* (\alpha \cup \beta) = \deg(\phi) \cdot (\alpha \cup \beta) \) for \( \alpha, \beta \in \text{Pic}(S') \). Thus

\[
\text{disc}(\tilde{\phi}^*(\text{Pic}(S'))) = (\deg(\phi)^{\rho(S)}) \text{disc}(\text{Pic}(S')).
\]

(10)

Lemma 4.1. The sublattice \( \ker(\tilde{\phi}_* : \text{Pic}(\tilde{S}) \to \text{Pic}(S')) \) of \( \text{Pic}(\tilde{S}) \) is the orthogonal complement of \( \tilde{\phi}^*(\text{Pic}(S')) \) in \( \text{Pic}(\tilde{S}) \).

Proof Let \( \alpha \in \text{Pic}(\tilde{S}) \). Let us show that \( \tilde{\phi}^* \alpha = 0 \) if and only if for any \( \beta \in \text{Pic}(S') \), we have \( \alpha \cup \tilde{\phi}^* \beta = 0 \) in \( H^4(\tilde{S}, \mathbb{Z}) \). The projection formula gives \( \tilde{\phi}_* (\alpha \cup \tilde{\phi}^* \beta) = (\tilde{\phi}_* \alpha) \cup \beta \in H^4(S', \mathbb{Z}) \). Hence, if \( \tilde{\phi}^* \alpha = 0 \), then \( \tilde{\phi}_* (\alpha \cup \tilde{\phi}^* \beta) = 0 \). But \( \tilde{\phi} : H^4(\tilde{S}, \mathbb{Z}) \to H^4(S', \mathbb{Z}) \) is an isomorphism, we must have \( \alpha \cup \tilde{\phi}^* \beta = 0 \). Conversely, if \( \alpha \cup \tilde{\phi}^* \beta = 0 \) for every \( \beta \in \text{Pic}(S') \), still by the projection formula, we get \( \tilde{\phi}_* \alpha \cup \beta = 0 \), which implies that \( \tilde{\phi}^* \alpha = 0 \) since the intersection product map is nondegenerate on \( \text{Pic}(S') \). 

Taking into account the fact that the intersection map on \( H^2(\tilde{S}, \mathbb{Z}) \) is nondegenerate on \( \tilde{\phi}^* \text{Pic}(S') \), Lemma 4.1 implies that \( \ker(\tilde{\phi}_*) \oplus \tilde{\phi}^*(\text{Pic}(S')) \) is a direct sum and that \( \ker(\tilde{\phi}_*) \oplus \tilde{\phi}^*(\text{Pic}(S')) \) is of finite index in the Abelian group \( \text{Pic}(\tilde{S}) \). Hence,

\[
\frac{\text{disc}(\ker(\tilde{\phi}_*)) \cdot \text{disc}(\tilde{\phi}^*(\text{Pic}(S')))}{\text{disc}(\text{Pic}(\tilde{S}))} = [\text{Pic}(\tilde{S}) : \ker(\tilde{\phi}_*) \oplus \tilde{\phi}^*(\text{Pic}(S'))]^2.
\]

(11)

Since \( \tilde{\phi}_* : \text{Pic}(\tilde{S}) \to \text{Pic}(S') \) sends \( \ker(\tilde{\phi}_*) \oplus \tilde{\phi}^*(\text{Pic}(S')) \) onto \( \deg(\phi) \text{Pic}(S') \), and since the induced morphism

\[
\tilde{\phi} : \text{Pic}(\tilde{S}) / \ker(\tilde{\phi}_*) \to \text{Pic}(S')
\]
is injective, we have

\[
\text{Pic}(\tilde{S}) : (\ker(\tilde{\phi}_* \oplus \tilde{\phi}^*(\text{Pic}(S'))]) = [(\text{Pic}(\tilde{S}) / \ker \tilde{\phi}_*) : \tilde{\phi}^*(\text{Pic}(S'))] \leq [\text{Pic}(S') : (\deg \phi) \text{Pic}(S')] = (\deg \phi) \rho(S),
\]

(12)

where \(\tilde{\phi}^*(\text{Pic}(S'))\) is the image of \(\tilde{\phi}^*(\text{Pic}(S'))\) in \(\text{Pic}(\tilde{S})/\ker \tilde{\phi}_*\). Notice the following Lemma.

**Lemma 4.2.** \(|\text{disc}(\text{Pic}(\tilde{S}))| = |\text{disc}(\text{Pic}(S))|\)

**Proof** \(\tilde{S}\) is obtained by a sequence of blowing-ups of points from \(S\). Therefore,

\[
\text{Pic}(\tilde{S}) = \tau^*\text{Pic}(S) \oplus \bigoplus_i \mathbb{Z}E_i,
\]

where \(E_i\) is the total transform in \(\tilde{S}\) of the exceptional divisor of the \(i\)-th blowing-up. We have the following formula for the intersection numbers of \(E_i\):

\[
E_i \cdot E_j = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}
\]

Hence, \(\text{disc}(\text{Pic}(\tilde{S})) = \text{disc}(\text{Pic}(S)) \cdot \text{disc}(\bigoplus \mathbb{Z}E_i) = \pm \text{disc}(\text{Pic}(S))\), as desired. \(\square\)

The proposition now follows from Lemma 4.2 and from inequalities (10), (11) and (12), noticing that \(\text{disc}(\ker(\tilde{\phi}_*)) \geq 1\). \(\square\)

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Sorbonne Université and Université de Paris, CNRS, IMJ-PRG, F-75005 Paris, France.

Email adress: chenyu.bai@imj-prg.fr.