Mass effects in muon and semileptonic $b \rightarrow c$ decays

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Quantum chromodynamics (QCD) effects in the semileptonic decay $b \rightarrow c \ell \bar{\nu}$ are evaluated to the second order in the coupling constant, $O(\alpha_s^2)$, and to several orders in the expansion in quark masses, $m_q/m_b$. Corrections are calculated for the total decay rate as well as for the first two moments of the lepton energy and the hadron system energy distributions. Applied to the muon decay, they decrease its predicted rate by $-0.43$ ppm.

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Decays of heavy fermions are an abundant source of information about fundamental interactions. Particularly important among them is the muon ($\mu$) decay. Insensitive to strong interactions, it can be very precisely described by the electroweak model. The experiment MuLan at the Paul Scherrer Institute will likely measure the rate of the muon decay with an uncertainty better than 1 ppm and thus improve the determination of the Fermi constant $G_F$ that describes the strength of the charged-current weak interaction \cite{1}. Along with the fine structure constant $\alpha$ and the $Z$-boson mass, $G_F$ is one of the three pillars of electroweak Standard Model tests \cite{2}.

In a separate effort, the TWIST experiment at TRIUMF measures energy and angular distributions of positrons in the $\mu^+$ decay, testing the Standard Model and searching for new interactions, notably new bosons predicted by left-right symmetric models \cite{3,4,5}.

To match this experimental progress, both the rate \cite{6} and the energy distribution \cite{7} have been calculated in quantum electrodynamics (QED) with $O(\alpha^2)$ accuracy. Two-loop weak corrections have also been calculated \cite{8}. In the decay rate studies, the electron mass $m_e$ was assumed negligible in the already small $O(\alpha^2)$ effects.

Here we show that the finite $m_e$ effect decreases the muon decay rate by about half ppm, exceeding previous estimates \cite{10} and approaching the expected MuLan precision.

The final-state fermion mass effects are much larger in the heavy-quark decay $b \rightarrow c \ell \bar{\nu}$. Studied in $B$-factories and the Tevatron, this process provides information about the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $V_{cb}$, as well as about parameters governing heavy-quark dynamics (see \cite{9} for an up to date review and references). Also in this case, theoretical studies at $O(\alpha_s^2)$ are complete only for a massless final-state quark \cite{11}. For the actual massive $c$-quark, $O(\alpha_s^2)$ effects are known in some special cases of kinematics \cite{12,13,14}. So-called Brodsky-Lepage-Mackenzie (BLM) corrections \cite{15} have been obtained for the width \cite{16} and moments of the energy spectrum \cite{17,18}. Also some logarithms of the mass $m_c$ have been determined to all orders in $\alpha_s$ \cite{19}. Most recently, Melnikov calculated numerically the $m_c$ effects for the width and the first two moments of the energy distribution of hadrons and of the charged lepton produced in this decay \cite{20}. In this paper we present corresponding analytical results obtained as an expansion in powers and logarithms of $\rho \equiv m_c/m_b$.

The construction of such mass expansion is illustrated with an $O(\alpha_s)$ example in Fig. 1. Real and virtual corrections are calculated together as cuts of the diagram \cite{11}(a). Depending on virtualities of momenta flowing into and through the charm quark lines, $m_c$ can be treated as small compared to those momenta, or else those momenta can be treated as small compared to the $b$-quark mass in other lines of the diagram. The most interesting case is shown in Fig. 1(d), where two loop momenta are of order $m_c$. This configuration generates odd powers of $\rho$ and will be discussed in some detail below.

At $O(\alpha_s^2)$ the number of diagrams is larger and the analysis of momentum scales more challenging, with as many as 11 regions in some diagrams, but it follows the general pattern outlined above. As a result, even the four-loop diagrams shown in Fig. 2 are calculated as a series in $\rho$ with exact coefficients.

The most challenging contributions arise when all loop momenta are hard ($\sim m_b$). All charm-quark lines are Taylor-expanded in $m_c$ generating a large number of integrals with various powers of denominator factors. The method of Ref. \cite{21} is employed to reduce all integrals to a set of master integrals. Most are known \cite{10}, sufficient to reproduce the $m_c = 0$ limit as well as the $O(\rho^2)$ terms.

FIG. 1: Example of an expansion of a double-scale integral. Thick, thin, and dashed lines correspond to $m_b$, $m_c$, and massless propagators. Loop momenta can be all hard (Taylor expansion in $m_c$ of (a)), and one or more soft ((b), (c), and (d)). Double line in (d) denotes a static propagator.
The one-gluon correction, known exactly \[26\], has an expansion in \(\rho\) starting with

\[ X_1 = \frac{25}{8} - \frac{\pi^2}{2} - (34 + 24 \ln \rho) \rho^2 + 16\pi^2\rho^3 \]
\[ - \left( \frac{273}{2} - 36 \ln \rho + 72 \ln^2 \rho + 8\pi^2 \right) \rho^4 + \ldots \] (3)

The second order correction \(X_2\) is a sum of finite, gauge invariant parts proportional to various color factors,

\[ X_2 = C_F X_A + C_A X_N + T_R (X_C + X_H + N_L X_L) \] . (4)

\(X_L, X_C,\) and \(X_H\) denote contributions of \(c, b,\) and \(N_L = 3\) species of massless quarks. SU(3) color factors are \(T_R = \frac{2}{3}, C_F = 4, C_A = 3\).

Using techniques described above, expansions of \(X_A, X_N, X_C, X_L,\) and \(X_H\) are obtained through \(O(\rho^2)\), sufficient for sub-percent accuracy of \(X_2\) for the physical \(\rho \approx 0.3\). These expressions being rather lengthy, Table II lists numerical values of the coefficients. \(X_N\) and \(X_C\) are shown explicitly for the purpose of subsequent discussion.

\[ X_C = -\frac{1009}{288} + 8 \cdot 3 \zeta_3 + \frac{77}{216} \pi^2 - \frac{5}{4} \pi^2 \rho \]
\[ + \left( \frac{145}{3} + \frac{52}{3} \ln \rho - 8 \ln^2 \rho + \frac{16}{3} \pi^2 \right) \rho^2 \]
\[ + \left( \frac{569}{36} + \frac{64}{3} \ln \rho \right) \pi^2 \rho^3 + \ldots , \]

\[ X_N = \frac{154927}{10368} - 383\zeta_3 + 95\pi^2 - \frac{53}{216} \pi^2 \ln 2 + 101\pi^4 + 1440 \]
\[ + \frac{539\pi^4}{1080} - \frac{1181\pi^2}{216} - \frac{185}{3} \ln \rho + 22 \ln^2 \rho - 43\zeta_3 \]
\[ + 4\pi^2 \ln 2 - \frac{1537}{16} \rho^2 + \frac{556\pi^2}{3} - \frac{1136\pi^2 \ln 2}{3} \]
\[ + \frac{56\pi^3}{3} - \frac{124\pi^2 \ln \rho}{3} + \frac{1777\pi^4}{720} - \frac{23807}{864} \]
\[ + \left( \frac{577}{36} - 39\zeta_3 - \frac{104\pi^2}{3} + 30\pi^2 \ln 2 - 15\pi^2 \right) \ln \rho \]
\[ + \left( 5\pi^2 - 185 \right) \ln^2 \rho + 88 \ln^3 \rho + 4\pi^2 \ln^2 2 \]
\[ + 48L_i \frac{1}{2} - 215\pi^2 \zeta_3 + 727\pi^2 \frac{535\zeta_3}{48} - \frac{615\zeta_3}{4} \]
\[ + 2 \ln^4 2 - \frac{13\pi^2 \ln 2}{2} \rho^4 + \ldots \] (5)

These correction significantly depend on \(\rho\) near its realistic values: \(X_2(0.25) = -6.59\), while \(X_2(0.3) = -4.89\).

In addition to the decay rate, corrections to the first two moments in lepton energy \(\bar{E}_l = E_l/m_b\), and the hadronic system energy \(E_h = E_h/m_b\), have been computed. The average is taken over the whole phase space of decay products. These moments are parameterized by

\[ \frac{1}{\Gamma_0} \langle \bar{E}_l^2 \rangle = \sum_{j=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^j L_j^{(n)} ,\]
and similarly for the moments of $\hat{E}_b$, described by coefficients $H_{2}^{(n)}$.

Table I shows the second-order corrections $L_{2}^{(1,2)}$ and $H_{2}^{(1,2)}$.

Finally, Table I lists $U_{C}$, the c-quark contribution to $\Gamma(b \to u\ell\bar{\nu})$, defined by analogy with $X_{C}$ of (I). This result is useful e.g. in $b \to s\gamma$ studies [27].

Our results can be tested by comparing logarithms of the mass ratio with the values predicted by the renormalization group analysis [19]. That study summed up some of the logarithmic effects to all orders in the coupling constant. When those results are expanded in $\alpha_s$, three terms can be tested in order $\alpha_s^3$, $\rho^2 \ln \rho$, $\rho^3 \ln \rho$, and $\rho^4 \ln \rho$. They are related to the presence in the tree-level decay width (2) of terms $\rho^2$ and $\rho^4 \ln \rho$, and the absence of $\rho^3$ there.

Ref. [19] traced the origin of those terms to operators that can be constructed from the $b$ and $c$ quark fields and determined anomalous dimensions of those operators. The logs in terms $\rho^2$ were found to arise from the running of the $c$-quark mass, and those in $\rho^4$ − from a complicated mixing of a variety of dimension seven operators. Our results fully agree with that analysis.

However, terms $\rho^2 \ln \rho$ turn out to work differently. In [19] they were attributed to the running of $m_c$, with the resulting coefficient that disagrees with our result. We find that those terms originate from the four-quark operator $\bar{t}_b \Gamma_{\mu} c \bar{c} \Gamma^\mu h_b$. Here $h_b$ denotes the static field describing a slow quark $b$. As discussed in Ref. [19], this operator gives rise to terms $\rho^2$ in the tree-level decay in the case of a vector coupling ($\Gamma_{\mu} = \gamma_{\mu}$), but does not contribute at the tree-level in the chiral case ($\Gamma_{\mu} = \gamma_{\mu}(1 - \gamma_5)$). However, we find that at $O(\alpha_s)$ it has a finite matrix element as shown in Fig. (I,d), responsible for the cubic mass term in the decay width [3].

At the next loop level, the effects of the coupling constant running and of the anomalous dimension of that operator generate terms $\alpha_s^3 \rho^3 \ln \rho$. Including charm, $N_C + 1$ quarks contribute to the coupling running. Denoting $\beta_0 \equiv \frac{11}{3} C_A - \frac{4}{3} T_R(N_C + 1)$, one finds

$$-8\beta_0 + \frac{32}{3} T_R - 12 C_A = \frac{32}{3} T_R N_C - \frac{64}{3} T_R - \frac{124}{3} C_A,$$

in agreement with the coefficients of $\pi^2 \rho^3 \ln \rho$ in Eq. (5).

The linear term $-5\pi^2 \rho/4$ in Eq. (5) is noteworthy. As will now be explained, it arises because the on-shell (pole) definition of the $b$-quark mass has been used. Although not suitable for phenomenology, it has been adopted for the ease of comparisons with Ref. [20] and simplicity of presentation. The linear term $m_q/m_b$ arises from a $c$-quark ($m_c \ll m_b$) loop inserted in the gluon propagator in Fig. (I,4). Thus it does not depend on the final-state quark mass and equally affects decays $b \to c\ell \bar{\nu}$ and $b \to u\ell \bar{\nu}$. Arising from the gluons with momentum $O(m_q)$ this effect becomes a problem for the perturbative analysis when $q$ is light, $m_q \lesssim \Lambda_{QCD}$. This illustrates how a linear $\Lambda_{QCD}/m_b$ correction appears even in the total $b$-quark width when the pole mass is used. If a short-distance mass definition is used, such as the $MS$ mass

$$28,$$ such terms are absorbed into the lowest-order decay width and are absent in higher-orders of the $\alpha_s$ expansion [29, 30, 31]. Note the factor five in the coefficient of the linear term, related to the fifth power of $m_b$ in the width formula.

Where the linear correction really counts is the muon decay, whose width is traditionally expressed using the muon pole mass $m_{\mu}$. The previous study of the muon decay [10] neglected the electron mass $m_e$. Its effect on the decay rate was assumed to arise from terms $\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_{\mu}}{m_{\mu}}\right)^2 \ln^2 \left(\frac{m_{\mu}}{m_{\mu}}\right)^2 \simeq 1.5 \ast 10^{-8}$. The theoretical error was estimated by taking the coefficient of this term to be 24. Due to the overlooked linear correction the electron mass effect turns out to be even larger, affecting the determination of the Fermi constant from the anticipated new measurement.

The Fermi constant is determined from the measured muon lifetime using the relation [10]

$$\frac{1}{\tau_\mu} = \frac{G_{\mu}^2 m_{\mu}^5}{192\pi^5} (1 + \Delta q),$$

where $\Delta q$ describes effects of the finite electron mass and radiative corrections in the limit of the four-fermion contact interaction ($m_{\mu} \ll M_W$). The latter have been known exactly in the first order in $\alpha_s$, and in the limit of zero $m_e$ in order $\alpha^2$. Present result extends the knowledge of the $\alpha^2$ correction to the case of the finite $m_e$. The extra shift is $\Delta q(m_e) \simeq -0.43 \ast 10^{-6}$, comparable with the expected precision of the MuLaN result.

Even though the coefficient of the $\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_{\mu}}{m_{\mu}}\right)^2 \ln^2 \left(\frac{m_{\mu}}{m_{\mu}}\right)^2$ term is only $-11$, less than half the value taken in [10] for the purpose of an error estimate, the total electron mass effect, dominated by the linear term $m_e/m_{\mu}$, is larger than expected.

For the decay $b \to c\ell\bar{\nu}$, we confirm numerical results of [20]. The flexible numerical method developed in that reference enables one to impose phase space cuts. Our approach is complementary. It provides analytical results and gives insight into the small-mass region, important for the muon decay. It also facilitates changes of the scheme used for the heavy quark masses. Finally, it reveals the structure of logarithms and highlights the relative importance of various operators, improving on the analysis of [19]. But is it possible to combine the analytical approach with cuts on the lepton energy?

According to [20], effects of cuts for the lepton energy moments can be modeled with the tree-level distribution [32]. For the cut $E_\ell > E_{cut} = 1$ GeV one finds $L_{2, cut}^{(n)} / L_{1, cut}^{(n)} \approx L_{0, cut}^{(n)} / L_{0}^{(n)}$. For the rate $L_{2, cut}^{(0)}$ the error is only $-4\%$, and less than $-2\%$ for $f_{1,2, cut}^{(1,2)}$, indicating smallness of the hard-gluon radiation.

The relative impact of cuts on the hadron energy moments ($\hat{E}_H^{(n)}$) depends on the order $n$ of the moment only very weakly [20]. To within 1%, $H_{2, cut}^{(1)} / H_{2}^{(1)}$ and $H_{2, cut}^{(2)} / H_{2}^{(2)}$ both equal $L_{2, cut}^{(0)} / L_{2}^{(0)}$ and significantly ex-
\[ \rho \text{ shows the analogue of } H \]

\[ B^{564} L_{0,1}^{(1,2)} \]

First-order corrections behave similarly: \( H_{0,1}^{(1)} / H_{0,1}^{(2)} \approx H_{0,1}^{(2)} / H_{0,1}^{(1)} \approx L_{0,1}^{(1)} / L_{0,1}^{(0)} \) within 0.1\%. Thus the effect of cuts \( E_{\text{cut}} \lesssim 1 \text{ GeV} \) on QCD corrections to \( \langle E_{\text{h}}^2 \rangle \) can be approximated by that on the rate.

Our result for the correction to the width exceeds the earlier estimate based on an interpolation of special kinematic results [14]. The full mass dependence being now known, it is clear that the correction varies strongly as a function of the mass ratio and is close to vanishing near the physical value. This indicates cancellations that were not taken into account in [14]. In particular, the non-BLM correction turns out to be about \( 1.7(\alpha_s/\pi)^2 \) [20]. While the full phenomenological analysis requires a fit of the rate and moments, we expect this correction to decrease the value of \( |V_{cb}| \) determined from inclusive \( b \) decays by about one percent, bringing it slightly closer to the exclusive-decay result.

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\[ L_{0,1}^{(2)} \]

\[ H_{0,1}^{(1)} \]

\[ H_{0,1}^{(2)} \]

\[ U_{0,1} \]

\[ X_A \]

\[ X_{\bar{N}} \]

\[ X_C \]

\[ X_L \]

\[ X_H \]

\[ L_{0,1}^{(1)} \]

\[ L_{0,1}^{(2)} \]

\[ H_{0,1}^{(1)} \]

\[ H_{0,1}^{(2)} \]

\[ U_{0,1} \]

\[ \alpha_s \]

\[ \pi \]

\[ \rho \]

\[ \nu \]

\[ \pi \]

\[ \tau \]

\[ \gamma \]

\[ \delta \]

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\[ \theta \]

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