Vaidya Collapse with Nonzero Radial Pressure

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Abstract. We discuss the collapse of a fluid with nonzero pressure within the context of the Vaidya spacetime considering a decaying cosmological parameter. It is found that both black holes and naked singularities can form, depending upon the initial conditions.

1. Introduction

The cosmic censorship hypothesis (CCH) [1] is regarded as one of the most important unsolved problems in classical general relativity theory. If we look at a typical star and ask what happens after it has exhausted its nuclear fuel, there are three possibilities, depending upon the mass of the star. The first two possibilities are, in order of increasing starting mass, a white dwarf or a neutron star. However, if the star is massive enough (typically of the order of about 10 solar masses or so, depending upon whether it rotates or not) after it has exhausted its nuclear fuel, there is nothing to halt the collapse process. According to the Hawking-Penrose singularity theorems [2], it will end up in a singularity. However these theorems do not give information about the nature of this singularity itself.

According to the cosmic censorship hypothesis, the singularity will be a black hole, covered by an event horizon. Despite much investigation, as yet, there is no proof of the hypothesis. However, there is another possibility, and that is that naked singularities can form, which can in principle be observed by a distant observer. Nowadays, there are numerous examples where the hypothesis is violated, although there is still no agreement if these are physically realistic or not [3-4] (and references therein). One of the first examples that were studied to exhibit a naked singularity is the Vaidya solution [5], which represents a spherically symmetric body consisting of null dust that is radiating (or collapsing). The metric is given by

\[ ds^2 = -\left[1 - \frac{2m(v)}{r}\right]dv^2 + 2dvdr + r^2d\Omega^2, \]

where \( v \) is the advanced ingoing time coordinate, \( r \) is the usual radial coordinate and

\( d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \)

represents the metric of the unit two-sphere. Dwivedi and Joshi [6] have given a good treatment of the occurrence of the naked singularity for this spacetime. One of the criticisms that has been levelled against the Vaidya metric is that it consists of null dust which has no pressure. In a realistic gravitational collapse scenario, we expect there to be pressure. The question that arises is whether the introduction of pressure can help to avoid the naked singularity and hence restore the validity of the cosmic censorship hypothesis. Recently, an interesting generalization of the Vaidya spacetime has been given [7] which incorporates nonzero radial pressure:

\[ ds^2 = -\left[1 - \frac{2m(v)}{r}\right]dv^2 + 2dvdr + r^2d\Omega^2 - Pdr^2, \]

where \( P \) is the radial pressure.
there are nowadays many reasons for considering a nonzero cosmological constant. With the recent discovery [8–9] of the accelerated expansion of the universe, the cosmological constant has made a comeback in the currently accepted ΛCDM concordance model of cosmology. Wagh and Maharaj [10] had studied gravitational collapse in the Vaidya spacetime with a cosmological constant, and they found that the introduction of the cosmological constant did not make any difference to the situation as obtained without the constant. Apart from this, there are now strong reasons to consider a decaying cosmological parameter since this can solve the cosmological constant problem, and it is claimed that this can give a better fit to cosmological observations than the usual concordance model [14].

2. Generalised Vaidya Solution

In this work, the gravitational collapse of the Vaidya model is studied with a variable cosmological parameter, to see if the hypothesis is still violated. By assuming some reasonable forms for \( m(v) \) and \( A(v) \), and studying analytically the geodesics of the spacetime following the methods as espoused in [11], we find that both black holes and naked singularities can form, depending upon fairly general initial conditions. We consider the mass function to be a linear function of the time given by

\[
m(v) = \frac{2m(v)}{r} + \frac{A(v)r^2}{3},
\]

where \( A(v) \) is the cosmological parameter, considered variable.

It can easily be verified that there is a singularity at \( r = 0 \), e.g., by examining the Kretschmann scalar \( K = R_{abcd}R^{abcd} \) which works out to be

\[
K = \frac{48m^2(v)}{r^8} + \frac{8A^2(v)}{3}.
\]

This scalar diverges as \( r \to 0 \).

Let us first look at the energy momentum tensor for the problem under discussion. For the metric (2), the complex null vectors may be taken as follows:

\[
l_a = -\delta^i_a, \quad n_a = -\frac{\Delta \delta^i_a}{2r^2} + \delta^a_s, \quad m_a = \frac{r}{\sqrt{2}} \left( \delta^i_s + isin\theta \delta^0_s \right),
\]

where \( \Delta = r^2 + 2m(v) - \frac{A(v)r^4}{3} \). Here \( l_a \) and \( n_a \) are real null vectors, and \( m_a \) is complex. They satisfy the usual normalisation conditions \( l_a n^a = -1 = m_a \bar{m}^a \), with the other products of the null vectors being zero. From Einstein's field equations \( R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab} - \Lambda g_{ab} \), we find that the energy momentum tensor describing the matter content for the spacetime (2) is given by

\[
T_{ab} = \mu l_a l_b + 2\rho l_a n_b - 2pm_{(a} \bar{m}_{b)},
\]

where the coefficients \( \rho, p \) and \( \mu \) are the density, pressure and null density, respectively. These quantities are given by:

\[
\rho = -p = -\Lambda(v), \quad \mu = -\frac{m(v)}{r^2} + \frac{r\Lambda(v)}{3}.
\]

Here, and over dot represents a derivative with respect to \( v \).

3. Existence of naked singularities

Let \( K^a \equiv dx^a/dk = \left( \dot{v}, \dot{r}, \dot{\theta}, \dot{\phi} \right) \) be the tangent vector to the null geodesics, where \( k \) is an affine parameter. The following geodesic equations may then be derived:
\[ \frac{dK^a}{dk} + \frac{1}{r} \left[ \frac{m(v)}{r} - \frac{\Lambda(v)r^2}{3} \right] = 0, \]  
\[ \frac{dK^a}{dk} + \frac{\dot{m}(v)}{r} (K^a)^2 = 0. \]

We introduce [11]

\[ K^a = \frac{P}{r}. \]

From the null condition \( K^a K_a = 0 \), and the geodesic equations we obtain

\[ K^a = \left[ 1 - \frac{2m(v)}{r} - \frac{\Lambda(v)r^2}{3} \right] P \frac{2r}{2r^2}, \]
\[ \frac{dP}{dk} \left[ 1 - \frac{4m(v)}{r} + \frac{\Lambda(v)r^2}{3} \right] \frac{P}{2r^2} = 0. \]

We shall integrate the above geodesic equations for a linear mass function and a variable cosmological parameter. Most \( \Lambda \) decay laws come from phenomenological considerations and a decaying \( \Lambda \) term is proving to provide a better fit to observations than the conventional CDM model [14]. The actual form of the decay law for \( \Lambda \) is likely to be complicated, but for the purposes of this work, we consider a fairly simple form that can allow meaningful conclusions to be drawn from it. We consider a simple generalisation of the commonly assumed form \( \Lambda \alpha \frac{1}{r^2} \):

\[ \Lambda(v) = \frac{3\eta}{v^2}, \]

where \( \eta \) is a constant. With this choice, the spacetime is self-similar, i.e., it admits homothetic Killing vector field \( \xi \):

\[ \xi^a = r \frac{\partial}{\partial r} + v \frac{\partial}{\partial v}, \]

which satisfies the condition \( L \xi g_{ab} = \xi_{a,b} + \xi_{b,a} = 2g_{ab} \), where \( L \) denotes the Lie derivative. It follows that \( \xi^a K_a \) is constant along the radial null geodesics and hence a constant of motion:

\[ \xi^a K_a = rK_r + vK_v = C. \]

Using equations (3), (10), (11) and (13), equation (15) becomes

\[ P = \frac{2yC}{\xi^3 - y^2 + 2y + \eta}, \]

and thus we can determine the geodesics completely. In this equation \( y \) is a similarity variable. It may be seen that \( P \) satisfies the differential equation (12).

For the singularity to be naked there must be a light ray that emerges from the singularity. We will investigate the behaviour of the radial null geodesics near the singularity to determine if outgoing geodesics meet the singularity in the past. By equations (10) and (11), radial null geodesics satisfy the equation

\[ \frac{dr}{dv} = \frac{1}{2} \left[ 1 - \frac{2m(v)}{r} - \frac{\Lambda(v)r^2}{3} \right]. \]

On using equations (3) and (13), equation (17) becomes

\[ \frac{dr}{dv} = \frac{1}{2} \left[ 1 - \xi^3 - \frac{\eta}{y^2} \right]. \]
In order to determine the nature of the limiting value of $y$ at $r = 0, v = 0$ on a singular geodesic, we let
\[ \lim_{r \to 0} y = \lim_{v \to 0} \frac{v}{r}. \]
From equation (18), we find that
\[ \zeta y_0^2 - y_0^2 + 2y_0 + \eta = 0. \tag{19} \]
For the singularity to be naked, equation (19) must exhibit at least one positive root. This will then indicate that there is at least one outgoing geodesic which terminates at the singularity in the past. The absence of positive roots indicates that the singularity is a black hole. This occurrence of positive roots implies that the strong CCH is violated, though not necessarily the weak CCH.

We will discuss the two cases $\eta > 0$ and $\eta < 0$ separately. For $\eta > 0$, in the limit as $\eta \to 0$, equation (19) has positive roots in the range $\zeta \in (0, 1/8]$. For $\zeta = 0$, naked singularities form in this range, and black holes outside. Thus, $\zeta_c = 1/8$ is called a critical value because at this value, a transition occurs and the end state changes from naked singularities to black holes. This is in agreement with earlier work [11]. The weak energy condition is satisfied for $\zeta \geq \frac{2\eta}{y^2}$. We wish to determine the effect of a nonzero $\Lambda$ on the collapse, at least qualitatively. It can be verified that $\zeta_c \to 0$, as $\eta \to \infty$ i.e., the naked singularity spectrum of the collapsing Vaidya region continuously shrinks with increasing $\eta$. Hence, the initial data space $(0, \zeta_c]$ for a naked singularity contracts once $\Lambda$ is taken into account. Thus, at a qualitative level, the presence of $\Lambda$ favours black holes over naked singularities.

For the other case, $\eta < 0$, it can easily be verified that equation (19), being a cubic equation, must have at least one real root. From the theory of such equations, it follows that for $\zeta > 0$ and $\eta < 0$, any real root of equation (19) must be positive. Thus gravitational collapse for $\eta < 0$ always leads to a naked singularity.

4. Conclusion
Thus it appears that the introduction of pressure is not sufficient to halt the occurrence of naked singularities in the Vaidya spacetime with a variable cosmological parameter. It is not too difficult to introduce charge into this model, and this investigation is ongoing. It is also worthwhile to mention that another fertile area for research is that of finding regular black holes, viz., black holes that resemble known ones, such as Schwarzschild, but which are regularly behaved at the origin [12-13]. Of course, all of these regular black holes violate one of the energy conditions, and require a modification of general relativity, e.g., nonlinear electrodynamics or a scalar field.

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