Secondary terms in the asymptotics of moments of $L$-functions

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Abstract
We propose a refined version of the existing conjectural asymptotic formula for the moments of the family of quadratic Dirichlet $L$-functions over rational function fields. Our prediction is motivated by two natural conjectures that provide sufficient information to determine the analytic properties (meromorphic continuation, location of poles, and the residue at each pole) of a certain generating function of moments of quadratic $L$-functions. The number field analogue of our asymptotic formula can be obtained by a similar procedure, the only difference being the contributions coming from the archimedean and even places, which require a separate analysis. To avoid this additional technical issue, we present, for simplicity, the asymptotic formula only in the rational function field setting. This has also the advantage of being much easier to test.

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1 Introduction

Over the past two decades, a great deal of work in analytic number theory focused on understanding various aspects of moments of families of automorphic $L$-functions. In particular, the problem of obtaining asymptotics for the moments of central values of $L$-functions in families received considerable attention, beginning with the work of Hardy and Littlewood [23], who obtained the asymptotic formula for the Riemann zeta-function on the critical line,

$$
\int_0^T |\zeta(\frac{1}{2} + it)|^2 \, dt \sim T \log T \quad (\text{as } T \to \infty).
$$

About eight years later, Ingham [24] obtained the asymptotic formula for the fourth moment

$$
\int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt \sim \frac{1}{2\pi^2} T \log^4 T \quad (\text{as } T \to \infty).
$$

No other asymptotic formula for higher moments of the Riemann zeta-function is known, but it is conjectured that

$$
\int_0^T |\zeta(\frac{1}{2} + it)|^{2k} \, dt \sim C_k T \log^k T \quad (\text{for } k \geq 3).
$$

The precise value of the constant $C_k$ was conjectured by Keating and Snaith [27] using random matrix theory; see also [10] for a more accurate conjecture, where the full main terms in the asymptotic formula are predicted.

Another classical example of interest is furnished by the moments of families of automorphic $L$-functions

$$
M_r(D) := \sum_{\chi \leq D} L(\frac{1}{2}, \chi)^r
$$

the sum being over real primitive Dirichlet characters. In 1981, Jutila [25] computed the first and second moment, and Soundararajan [36], [12], and Young [42] obtained asymptotics for the third moment with different error terms. Under the generalized Riemann hypothesis, Shen [35] obtained an asymptotic formula for the fourth moment (using the method of Soundararajan and Young [37]). It was conjectured [10] that

$$
M_r(D) \sim D Q(\log D) \quad (\text{as } D \to \infty)
$$

for some polynomial $Q$ of degree $r(r+1)/2$. An explicit description of this polynomial is given in [21].

An important feature of the family of quadratic Dirichlet $L$-functions is that its moments admit a perfect function field analogue. In this context, asymptotics are known only for the first four moments by the work of Florea [17, 18, 19], and the corresponding conjectural asymptotic formula for all moments was given in [2]; see also [33].

In [12], the first-named author, Goldfeld and Hoffstein conjectured the existence of additional terms in (1), when $r \geq 3$; if $r = 3$ there is one additional term of size $D^2$, and if $r \geq 4$, we have the following:

**Conjecture 1.1.** — Let $N \geq 1$ be an integer, and take $(N + 1)^{-1} < \Theta < N^{-1}$. Then, as $D \to \infty$, we have

$$
M_r(D) = \sum_{n=1}^N D^{(\frac{r}{2} + \Theta)} Q_{n,r}(\log D) + O(D^{(1+\Theta)/2})
$$

for some polynomials $Q_{n,r}(x)$.

Evidence in favor of this conjecture, both theoretical and numerical, was provided by Zhang [43] and the survey [44], and by Alderson and Rubinstein [1]. When $r = 3$, the existence of the additional term was proved very recently in the function field case by the first-named author [11], and by the first-named author and Whitehead [16], for a smoothed version of $M_3(D)$.

Our aim in this paper is to develop a method to determine the polynomials $Q_{n,r}(x)$ occurring in the asymptotic formula (2). However, when dealing with this problem over the rationals (or its extension to number fields), one is invariably
facing the unpleasant task of computing the contributions to \( Q_{r, n}(x) \) corresponding to the archimedean and even places. To avoid this technical issue, we shall discuss in this paper only the (rational) function field analogue of (2).

To state the function field version of (2), let \( \mathbb{F}_q \) be a finite field of odd characteristic; for simplicity, we assume that the cardinality \( q \) of \( \mathbb{F}_q \) is congruent to 1 modulo 4. For \( d \in \mathbb{F}_q[x] \) monic and square-free, let \( L(s, \chi_d) \) denote the Dirichlet \( L \)-function associated to the quadratic symbol \( \chi_d(m) = (d/m) \).

**Conjecture 1.2.** — Let \( D, N \geq 1 \) and \( r \geq 4 \) be integers. Then, for any \( (N+1)^{-1} < \Theta < N^{-1} \), we have

\[
\sum_{d \text{-monic} \& \, \text{sq. free}} L\left( \frac{1}{2}, \chi_d \right) = \sum_{n \leq N} Q_{r, n}(D, q) q^{\frac{1+\Theta}{2}D} + O_{\Theta, q}\left(q^{D(1+\Theta)/2}\right)
\]

for computable \( Q_{r, n}(D, q) \).

For notational convenience, we shall suppress the dependence of \( Q_{r, n}(D, q) \) upon \( r \). To determine these coefficients, we will investigate the *conjectural* analytic properties (meromorphic continuation, location of poles and the residue at each pole) of the generating function

\[
\mathcal{W}(s_1, \ldots, s_r, \xi) = \sum_{D \geq 0} \left( \sum_{d \text{-monic} \& \, \text{sq. free}} \prod_{k=1}^{r} L(s_k, \chi_d) \right) \xi^D
\]

for fixed \( s_1, \ldots, s_r \in \mathbb{C} \) with \( \Re(s_k) = \frac{1}{2} \). Rather than approaching this function directly (as it does not seem to provide sufficient information), we proceed as in [11], and write

\[
\mathcal{W}(s_1, \ldots, s_r, q^{-\tau+1}) = \sum_{h \text{-monic}} \mu(h) Z(s_1, \ldots, s_r, l; h) \quad \text{(for } \Re(s_r+1) > 1) \quad (3)
\]

where \( Z(s_1, \ldots, s_r, l; h) \) is a finite sum whose terms involve the (suitably normalized) twisted Weyl group multiple Dirichlet series introduced in [13], see 4.4. These multiple Dirichlet series satisfy a group of functional equations isomorphic to the Weyl group \( W \) of the Kac-Moody Lie algebra \( g(A) \) associated to the generalized Cartan matrix \( A = (A_{ij})_{1 \leq i, j \leq r+1} \) whose entries are given by

\[
A_{ij} = \begin{cases} 
2 & \text{if } i = j \\
-1 & \text{if } i \neq j \text{ and either } i = r + 1 \text{ or } j = r + 1 \\
0 & \text{otherwise.}
\end{cases}
\]

We note that, unlike the finite-dimensional case, when \( r \geq 4 \) (\( W \) is infinite in this case), these series are not determined by the group of functional equations alone. In fact, the complexity grows with \( r \), which can be seen by inspecting some low-index coefficients of the multiple Dirichlet series that can be computed using the explicit comparison between the Grothendieck-Lefschetz and Arthur-Selberg trace formulas established by Weissauer [38] and [39]; see also [32].

We conjecture that these twisted Weyl group multiple Dirichlet series admit meromorphic continuation to the interior of a complexified convex cone \( X_0^* \), a transformation of the interior of the complexified Tits cone of \( W \). More precisely, we conjecture that these functions are holomorphic on \( X_0^* \), except for possible simple poles corresponding (via a simple change of variables) to the positive real roots of \( g(A) \). The twisted Weyl group multiple Dirichlet series occurring in the expression of \( Z(s_1, \ldots, s_r+1, l; h) \) are twists of the multiple Dirichlet series constructed in [13], normalized by a certain \( W \)-invariant Euler product with the local factors at the primes dividing \( h \) removed, and whose local factors are supported on squares. The effect of this normalization (which we introduced in order to remove potential singularities corresponding to the imaginary roots of \( g(A) \)) is completely annihilated in (3). We also conjecture that the series in (3) is absolutely convergent for \( \Re(s_i) \geq \frac{1}{2}, i = 1, \ldots, r, \) and \( \Re(s_{r+1}) > \frac{1}{2} \), away from the poles of \( Z(s_1, \ldots, s_r, l; h) \), which combined with the previous conjecture, would provide the information we need to obtain the relevant analytic properties of the function \( \mathcal{W}(s_1, \ldots, s_r, \xi) \). In particular, all singularities of this function (in the variable \( \xi \)) should be simple.
poles at points corresponding to positive real roots of \( g(A) \); if \( \alpha = \sum k_i \alpha_i \) is a positive real root, for a choice of positive simple roots \( \alpha_i \) (\( i = 1, \ldots, r+1 \)), the residue at the corresponding (simple) pole can be computed explicitly, and we shall do so in Section 5.

Our first conjecture is supported, for example, by an extension of the so-called Eisenstein Conjecture (EC) \([5, 6, 7, 9, 28, 30]\) to Kac-Moody groups. This extension of EC predicts that twisted Weyl group multiple Dirichlet series occur as Whittaker coefficients of a minimal parabolic Eisenstein series on a metaplectic cover of a (root datum related) Kac-Moody group; in this respect, a preliminary result has been recently obtained by Patnaik and Puskás \([31]\), who proved a Casselman-Shalika type formula for Whittaker functions on metaplectic covers of Kac-Moody groups over non-archimedean local fields. It is conceivable that the constant term of this minimal parabolic Eisenstein series can be expressed via a generalized Gindikin-Karpelevich formula in terms of a ratio of an \textit{infinite} product of zeta and \( L \)-functions. Presumably, this infinite product is structurally similar to the denominator of the character of an irreducible highest-weight \( g(A) \)-module, which converges on the interior of the complexified Tits cone of \( W \). Thus the constant term would be meromorphic on \( X^+_0 \), and by a well-known principle, the minimal parabolic Eisenstein series should have meromorphic continuation to the same domain. However, developing the Langlands-Shahidi method in the context of Kac-Moody groups is quite problematic at the moment, a serious obstacle in doing so being the lack of an adequate integration theory over the relevant unipotent radicals allowing us to transfer the “meromorphy property” from an Eisenstein series to its Whittaker coefficients; see \([20]\) and the reference therein for specific information in the special case of Eisenstein series on loop groups. Nonetheless, in some special cases, it is possible to obtain the meromorphic continuation of the corresponding Weyl group multiple Dirichlet series more directly (see, for example, \([8, 40]\), and especially the forthcoming manuscripts \([14, 15]\), where the conjecture is proved in the important case of \textit{untwisted} Weyl group multiple Dirichlet series of type \( D_{4n}^{1+1} \) over rational function fields).

Our conjecture on the absolute convergence of (3) can be motivated by an analogue of the Lindel"of hypothesis for the twisted Weyl group multiple Dirichlet series in the twisting parameters combined with a Ramanujan type bound for the coefficients of the \( p \)-parts of the Weyl group multiple Dirichlet series (see Assumption (H1)).

By assuming the two conjectures (and, for consistency, also the Ramanujan bound for the coefficients of the \( p \)-parts) to be true, we show that Conjecture 1.2 holds with \( Q_n(D, q) \) given by a limit of a sum of contributions coming from the singularities of the generating function \( \mathcal{H}(s_1, \ldots, s_r, \xi) \) corresponding to the set of positive real roots of \( g(A) \) with \( k_{r+1} = n \). For example, when \( n = 1 \), the set of roots contributing to \( Q_1(D, q) \) is

\[
\left\{ \sum_{j=1}^r k_j \alpha_j + \alpha_{r+1} : k_j = 0 \text{ or } 1 \right\}
\]

(see \([2]\) for the computation of \( Q_1(D, q) \)), and when \( n = 2 \), the set of corresponding roots is

\[
\left\{ \sum_{j=j_1, j_2, j_3} k_j \alpha_j + \alpha_{r+1} : 1 \leq j_1 < j_2 < j_3 \leq r \text{ and } k_j \in \{0, 2\} \text{ for } j \neq j_1, j_2, j_3 \right\}
\]

and \( Q_2(D, q) \) is a polynomial of degree \((r-3)(r+10)/2 \) in \( D \) with leading coefficient given by

\[
2^{19-7r} \frac{1121! \ldots (r-4)!}{7!9! \ldots (2r-1)!} \left( 1 + 2q^{1/8} + 10q^{1/2} + 7q^{3/4} + 20q + 7q^{5/4} + 10q^{7/4} + q^2 \right) \xi(\frac{1}{2}) \prod_p \left( \frac{1}{\sqrt{|p|}} \right)
\]

\[
+ (-1)^r \left( 1 - q^{1/8} + 10q^{1/2} - 7q^{3/4} + 20q - 7q^{5/4} + 10q^{7/4} - q^2 \right) \xi(\frac{1}{2}) \prod_p \left( \frac{1}{\sqrt{|p|}} \right)
\]

\[
+ iq^{1/8} - 4q^{1/2} + 7iq^{3/4} + 6q - 7iq^{5/4} - 4q^{3/2} + iq^{7/4} + q^2 \right) L(\frac{1}{2}) \prod_p \left( \frac{1}{\sqrt{|p|}} \right)
\]

\[
+ (-i)^r \left( 1 + iq^{1/8} - 4q^{1/2} - 7iq^{3/4} + 6q + 7iq^{5/4} - 4q^{3/2} - iq^{7/4} + q^2 \right) L(\frac{1}{2}) \prod_p \left( \frac{1}{\sqrt{|p|}} \right)
\]

\[.\]
Here $\zeta(\frac{1}{2}) = (1 - \sqrt{7})^{-1}$, $L(\frac{1}{2}) = (1 + \sqrt{7})^{-1}$, and

\[
P_i(t) = (1 - t)(r^2 + 7r - 14) / 2 (1 + t)(r^2 + 7r - 28) / 2 \\
\cdot \left[ (t + r^2) \left( t + 6r^2 + r^3 \right) + \frac{1}{2} (1 + t)^{4-r} + \frac{1}{2} (1 - t)^{-r} \left( 1 + 10t + 20r^2 + 10r^3 + t^4 \right) \right].
\]

In general, the explicit form of the leading coefficient of $Q_n(D, q)$ involves 2n contributions, each corresponding to a 2n-th root of 1. The computation of these contributions is similar, albeit increasingly involved as n grows. By comparison, the coefficient $Q_2(x)$ in (2) is a polynomial of degree $(r-3)(r+10)/2$, and we have

\[
Q_2(x) \sim 2^{10-7r} \frac{0!1!2! \ldots (r-4)!}{7!9!11! \ldots (2r-1)!} c_\infty \zeta(2)^2 \frac{1}{x^{(r-3)(r+10)/2}} \quad \text{as } x \to \infty
\]

where $c_2$ and $c_\infty$ are contributions at 2 and the archimedean place, respectively, and $\zeta(2)^2 (\frac{1}{2})$ is the Riemann zeta-function with the local factor at 2 removed.

1.1 Structure of the paper

In Section 2, we recall the necessary background from the theory of Kac-Moody Lie algebras and quadratic Dirichlet $L$-functions. In Section 3, we introduce a class of formal power series in several variables satisfying certain properties. In Section 4, we introduce the relevant multiple Dirichlet series associated to moments of quadratic Dirichlet $L$-functions, and discuss their general properties. In particular, the group of functional equations satisfied by these multiple Dirichlet series is a consequence of the fact that the local parts (or p-parts) of these series satisfy the properties of the power series introduced in Section 3. We then proceed by formulating two conjectures, giving in particular, the meromorphic continuation of the generating function of moments of quadratic $L$-series. In Section 5, we study the poles and residues of the functions introduced in §4, and in Section 6, we combine the bound (H1) and Conjectures 4.1 and 4.3 with the information obtained in Section 5 to deduce asymptotics for moments of quadratic $L$-functions. Finally, in Section 7, we study in detail the first two terms of the asymptotics obtained in Section 6.

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2 Preliminaries

2.1 Kac-Moody Lie algebras

We first recall some basic facts about root systems of Kac-Moody Lie algebras; for a detailed exposition of the theory of Kac-Moody algebras, the reader may consult the standard reference [26].

Let $A = (A_{ij})_{i,j = 1}^{n}$ be a generalized Cartan matrix, that is, the entries $A_{ij}$ satisfy the following conditions:

(i) $A_{ii} = 2$ for $i = 1, \ldots, n$;

(ii) $A_{ij}$ are non-positive integers for $i \neq j$;

(iii) $A_{ii} = 0$ implies $A_{ij} = 0$.

For simplicity, let us assume throughout that $A = DS$, where $D$ is an $n \times n$ invertible diagonal matrix, and $S$ is a symmetric matrix. Set $I = \{1, \ldots, n\}$, and let $(\mathfrak{h}, \Pi, \Pi')$ be a realization of $A$. Thus $\mathfrak{h}$ is a complex vector space of dimension $2n - \text{rank } A$, and $\Pi = \{\alpha_i : i \in I\} \subset \mathfrak{h}^*$, $\Pi' = \{\alpha^*_i : i \in I\} \subset \mathfrak{h}$ are linearly independent sets satisfying the condition

$\alpha_j(\alpha^*_i) = A_{ij}$ (for all $i, j \in I$).
The Kac-Moody algebra associated to the matrix $A$ (see [26, p. 159]) is then the Lie algebra $\mathfrak{g}(A)$ on generators $e_i, f_i$ ($i \in I$), $\mathfrak{h}$, and the defining relations:

- $[e_i, f_j] = \delta_{ij} \alpha'_i$, where $\delta_{ij}$ is the Kronecker delta;
- $[h, e_i] = \alpha_i(h)e_i$ and $[h, f_i] = -\alpha_i(h)f_i$ (for $h \in \mathfrak{h}$);
- $[h, h'] = 0$ (for $h, h' \in \mathfrak{h}$);
- $\text{ad}(e_i)^{-1}h_j(e_i) = 0$ and $\text{ad}(f_i)^{-1}h_j(f_i) = 0$ (for $i, j \in I$ with $i \neq j$).

One of the fundamental results of the theory of Kac-Moody algebras is the direct sum decomposition

$$\mathfrak{g}(A) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\Delta \subset \sum_{i \in I} \mathbb{Z}\alpha_i \setminus \{0\}$ is the set of all roots of $\mathfrak{g}(A)$, and for each $\alpha \in \Delta$, $\mathfrak{g}_\alpha := \{ x \in \mathfrak{g}(A) \colon [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \}$ is the root space attached to $\alpha$. Letting $\Delta_+ = \Delta \cap \sum_{i \in I} \mathbb{N}\alpha_i$, (resp. $\Delta_-$) denote the set of all positive (resp. all negative) roots, we have

$$\Delta_+ = -\Delta_- \quad \text{and} \quad \Delta = \Delta_+ \cup \Delta_- \quad (4)$$

Moreover, if $m_\alpha := \dim \mathfrak{g}_\alpha \geq 1$ is the multiplicity of $\alpha$, we have $m_\alpha = m_{-\alpha}$ for all $\alpha \in \Delta$. For each $i \in I$, $\alpha_i \in \Delta_+$, $m_{\alpha_i} = 1$ and $\mathbb{Z}\alpha_i \cap \Delta = \{\alpha_i, -\alpha_i\}$. The elements $\alpha_i$ ($i \in I$) are called the simple roots of $\mathfrak{g}(A)$.

We also have the following important fact, see [26, Lemma 1.3]: If $\beta \in \Delta_+ \setminus \{\alpha_i\}$ then $(\beta + \mathbb{Z}\alpha_i) \cap \Delta = \Delta_+$.

For each $i \in I$, we define the fundamental reflection $w_i$ of $\mathfrak{h}^*$ by

$$w_i(\lambda) = \lambda - \lambda(\alpha'_i)(\alpha_i) \quad \text{(for } \lambda \in \mathfrak{h}^*).$$

The subgroup $W := \{w_i \colon i \in I\}$ of GL($\mathfrak{h}^*$) is the Weyl group of $\mathfrak{g}(A)$. The set of roots $\Delta$ is stabilized by $W$, and $m_{w_i(\alpha)} = m_\alpha$ for $\alpha \in \Delta$ and $w \in W$. The elements of the subset $\Delta^\text{re} := \text{W}I$ of $\Delta$ are called real roots, and the elements of $\Delta^\text{im} := \Delta \setminus \Delta^\text{re}$ are called imaginary roots. Both sets $\Delta^\text{re}$ and $\Delta^\text{im}$ split according to (4) as $\Delta^\text{re} = \Delta^\text{re}_+ \cup \Delta^\text{re}_-$, $\Delta^\text{im} = \Delta^\text{im}_+ \cup \Delta^\text{im}_-$, and $\Delta^\text{re}_- = -\Delta^\text{re}_+$, $\Delta^\text{im}_+ = -\Delta^\text{im}_-$.

Recall that the Lie algebra $\mathfrak{g}(A)$ is assumed to be symmetrizable. Thus, by [26, Theorem 2.2], $\mathfrak{g}(A)$ admits a non-degenerate symmetric bilinear $\mathbb{C}$-valued form $\langle \cdot, \cdot \rangle$ – playing the role of a “Killing-Cartan form” when $\mathfrak{g}(A)$ is infinite-dimensional. Using this bilinear form, we have the following characterization of the imaginary roots, see [26, Prop. 5.2, c]: $\alpha \in \Delta$ is an imaginary root if and only if $\langle \alpha, \alpha \rangle = 0$. Thus $\langle \alpha, \alpha \rangle > 0$ if $\alpha \in \Delta^\text{re}$.

The real roots also enjoy the following important property, see [26, Prop. 5.1, c]. For $\alpha \in \Delta^\text{re}$ and $\beta \in \Delta$, let $S(\alpha, \beta) = \{\beta + k\alpha : k \in \mathbb{Z}\} \cap \Delta$ denote the $\alpha$-string through $\beta$. Then there exist $u, v \in \mathbb{N}$ such that

$$S(\alpha, \beta) = \{\beta - u\alpha, \ldots, \beta - \alpha, \beta, \beta + \alpha, \ldots, \beta + v\alpha\} \quad (5)$$

and $u - v = \beta(\alpha')$. Moreover, the reflection $w_{\alpha}(\lambda) = \lambda - \lambda(\alpha')(\alpha) \in \mathfrak{h}^*$, reverses the sequence of elements in $S(\alpha, \beta)$.

For $\alpha = \sum_{i \in I} k_i \alpha_i$, with $k_i \in \mathbb{Z}$, the integer $d(\alpha) = \sum_{i \in I} k_i$ is called the height$^1$ of $\alpha$. The elements of $\Delta$ of height one are $\alpha_i$ ($i \in I$). The following proposition can be used to construct the set $\Delta_+$ inductively.

**Proposition 2.1 (Moody [29]).** — Let $\alpha = \sum_{i \in I} k_i \alpha_i$ with $k_i \in \mathbb{N}$ be of height $d(\alpha) > 1$. Then $\alpha \in \Delta_+$ if and only if either there exists a fundamental reflection $w_i$ ($i \in I$) such that $w_i(\alpha)$ is a root and $d(w_i(\alpha)) < d(\alpha)$, or for all $i$, $d(w_i(\alpha)) \geq d(\alpha)$ and there exists $i \in I$ such that $\alpha - \alpha_i$ is a root. Furthermore, if $\alpha \in \Delta_+$, then $\alpha \in \Delta^\text{re}_+$ if and only if $w_i(\alpha)$ is a real root and $d(w_i(\alpha)) < d(\alpha)$ for some fundamental reflection $w_i$.

$^1$In [26] the height of an element $\alpha$ in the root lattice is denoted by $\mathfrak{h} \alpha$ instead of $d(\alpha)$. 

6
From now on, we shall assume that

\[
A = \begin{pmatrix}
2 & -1 & 
\vdots & 
-1 & 
\end{pmatrix}
\]

where we take \( n = r + 1 \) with \( r \geq 1 \). Thus \( A = (A_{ij})_{1 \leq i, j \leq r + 1} \) is the symmetric matrix whose entries are:

\[
A_{ij} = \begin{cases} 
2 & \text{if } i = j \\
-1 & \text{if } i \neq j \text{ and either } i = r + 1 \text{ or } j = r + 1 \\
0 & \text{otherwise.}
\end{cases}
\]

We have that \( \det A = -2^{r-1}(r-4) \). By [26, Theorem 4.3], one finds at once that \( A \) is of finite, affine, or indefinite type when \( r \leq 3 \), \( r = 4 \), or \( r \geq 5 \), respectively. For example, the Dynkin diagram of \( A \) is \( D_4 \) when \( r = 3 \) and \( D_4 \) when \( r = 4 \); for \( r \geq 5 \), the Dynkin diagram of \( A \) has a star-like shape.

With notation as above, we have:

**Lemma 2.2.** — Let \( \alpha = \sum k_i \alpha_i \in \Delta^+ \) with \( k_{r+1} \geq 1 \). Then \( k_i \leq k_{r+1} \) for all \( i = 1, \ldots, r \).

**Proof.** From the definition of the Cartan matrix \( A \), we have

\[
u_i - v_j = \alpha(\alpha_i^\ast) = 2k_i - k_{r+1}
\]

for all \( 1 \leq i \leq r \). Since \( k_{r+1} > 0 \), it follows from (5) that \( \alpha - u_i \alpha_i \in \Delta^+ \). In particular, \( k_{r+1} - k_i - v_i \geq 0 \), and our assertion follows.

Now let \( \alpha = \sum j k_j \alpha_j \) \( (k_j \in \mathbb{Z}) \) be an element of the root lattice. Since \( w_r(\alpha_j) = \alpha_j - A_{ij} \alpha_i \), the effect of the fundamental reflection \( w_i \) on the coefficients of \( \alpha \) is given by

\[
k_j \rightarrow \begin{cases} 
k_j & \text{if } j \neq i \\
k_{r+1} - k_i & \text{if } j = i \leq r \\
- k_{r+1} + \sum_{l=1}^{r} k_l & \text{if } j = i = r + 1.
\end{cases}
\]

(One will notice that these relations also give the conclusion of Lemma 2.2.) Thus to determine the set of positive roots it suffices to find the elements \( \alpha \in \Delta^+ \) whose coefficients satisfy the condition

\[
k_j \leq k_{r+1}/2 \quad \text{ (for } j = 1, \ldots, r)\]

(*)

It follows at once from Proposition 2.1 that, if \( \alpha \in \Pi \) is a positive real root satisfying the condition (*), then \( d(w_{r+1}(\alpha)) < d(\alpha) \), i.e.

\[
\sum_{j=1}^{r} k_j \leq 2k_{r+1} - 1.
\]

(**)

This gives a simple inductive procedure to determine the set of positive real roots. Another special feature of the set \( \Delta^e \) is expressed in the following lemma:

**Lemma 2.3.** — We have \( \Delta^e = W \alpha_{r+1} \).

**Proof.** This follows at once from the fact that \( w_i w_{r+1}(\alpha_i) = \alpha_{r+1} \) for all \( 1 \leq i \leq r + 1 \).
Denominator formula. Since A has real entries, we can take a realization \((h_R, \Pi, \Pi^*)\) of A over \(\mathbb{R}\). Thus \(h_R\) is a vector space of dimension \(2r + 2 - \text{rank} A\) over \(\mathbb{R}\), so that \((\mathfrak{h}, \Pi, \Pi^*)\) (\(\mathfrak{h} := h_R \otimes_{\mathbb{R}} \mathbb{C}\)) is the realization of A over \(\mathbb{C}\). The set

\[ C = \{ h \in h_R : \alpha_i(h) \geq 0 \text{ for } i = 1, \ldots, r + 1 \} \]

is called the fundamental chamber; and the union

\[ X = \bigcup_{w \in W} w(C) \]

is called the Tits cone. The complexified Tits cone \(X_C\) is dened to be

\[ X_C = \{ x + \sqrt{-1}y : x \in X, y \in h_R \}. \]

Let \(\rho \in \mathfrak{h}^*\) be a fixed element such that \(\rho(\alpha_i^*) = 1\ (i = 1, \ldots, r + 1)\), and, for \(\lambda \in \mathfrak{h}^*\), define \(e^\lambda\) by setting \(e^\lambda(h) := e^{\lambda(h)}\ (h \in \mathfrak{h})\). Since A is a symmetric matrix, hence \(g(A)\) admits a standard invariant bilinear form [26, §2.3], we have the Weyl-Kac denominator formula [26, Theorem 10.4 and (10.4.4)]:

\[ \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha(ma)}) = \sum_{w \in W} e(w)e^{(\rho) - w} \]

where \(e(w) = (-1)^{l(w)}\), \(l(w)\) being the length of \(w\). The natural domain of absolute convergence of both sides of the formula is the interior (in metric topology) of \(X_C\), see [26, Proposition 10.6].

The denominator formula can be used to compute the multiplicities of the imaginary roots of \(g(A)\), see [3].

2.2 Quadratic \(L\)-series

Let \(\mathbb{F}_q\) be a finite field of odd characteristic. For simplicity, we shall assume throughout that \(q \equiv 1 \pmod{4}\). For \(m \in \mathbb{F}_q[x]\), \(m \neq 0\), put \(|m| = q^{\deg m}\), and, for \(d, m \in \mathbb{F}_q[x]\) with \(m\) monic, let \((d/m)\) denote the quadratic symbol; we take \((d/1) = 1\). Since \(q \equiv 1 \pmod{4}\), we have the quadratic reciprocity law

\[ \left( \frac{d}{m} \right) = \left( \frac{m}{d} \right) \quad \text{(for coprime non-constant monic polynomials \(d, m \in \mathbb{F}_q[x]\)).} \]

In addition, if \(b \in \mathbb{F}_q^*\) then \(\left( \frac{b}{m} \right) = \text{sgn}(b)^{\deg m}\) for all non-constant \(m \in \mathbb{F}_q[x]\), where, for \(d(x) = b_0x^r + b_1x^{r-1} + \cdots + b_n \in \mathbb{F}_q[x]\) \((b_0 \neq 0)\), \(\text{sgn}(d) = 1\) if \(b_0 \in (\mathbb{F}_q^*)^2\) and \(\text{sgn}(d) = -1\) if \(b_0 \notin (\mathbb{F}_q^*)^2\).

For \(d\) square-free, put \(\chi_d(m) = (d/m)\). The \(L\)-function associated to \(\chi_d\) is defined by

\[ L(s, \chi_d) = \sum_{m \in \mathbb{F}_q[x]} \chi_d(m)|m|^{-s} = \prod_{\text{p-monic & irreducible}} (1 - \chi_d(p)|p|^{-s})^{-1} \quad \text{(for complex \(s\) with \(\Re(s) > 1\)).} \]

It is well-known that \(L(s, \chi_d)\) is a polynomial in \(q^{-s}\) of degree \(\deg d - 1\) when \(d\) is non-constant; if \(d \in \mathbb{F}_q^*\),

\[ L(s, \chi_d) = \xi(s) = \frac{1}{1 - q^{1-s}} \quad \text{(when \(\text{sgn}(d) = 1\)) and} \quad L(s, \chi_d) = \frac{1}{1 + q^{1-s}} \quad \text{(when \(\text{sgn}(d) = -1\)).} \]

If one defines \(\gamma(s, d)\) by

\[ \gamma(s, d) := q^{\frac{1}{2}(\deg d)(s-2)}(1 - \text{sgn}(d)q^{-s})^{\frac{1}{2}}(1 - \text{sgn}(d)q^{s-1})^{\frac{1}{2}} \]

then \(L(s, \chi_d)\) satisfies the functional equation

\[ L(s, \chi_d) = \gamma(s, d)|d|^{\frac{1}{2-s}}L(1-s, \chi_d). \]
3 Functions attached to the Weyl group \( W \)

We start with a power series

\[
f(z; q) = 1 + \sum_{k=0} a(k; q)z^k
\]

where the sum is over all tuples \( k = (k_1, \ldots, k_r, k_{r+1}) \in \mathbb{N}^{r+1} \setminus \{0\} \), and the coefficients \( a(k; q) \) are functions of odd prime powers \( q \); the power series is assumed to converge absolutely and uniformly in a small open polydisk (depending upon \( q \)) centered at 0. We take this function such that:

1. If we set \( \bar{z} = (z_1, \ldots, z_r) \), \( \bar{k} = (k_1, \ldots, k_r) \) and \( l = k_{r+1} \), then we can write

\[
f(z; q) = \frac{\sum_{\text{l-even}} P_1(\bar{z}; q)z_{r+1}^l}{\prod_{j=1}^r (1 - z_j)} + \sum_{l=0}^{l=\text{odd}} P_1(\bar{z}; q)z_{r+1}^l
\]

\[
= \frac{\sum_{\mathcal{Z}_{\bar{l}}} \mathcal{Q}_1(z_{r+1}; q)}{1 - z_{r+1}} + \sum_{\mathcal{Z}_{\bar{l}}} \mathcal{Q}_1(z_{r+1}; q)\frac{z_{r+1}^l}{1 - z_{r+1}}
\]

where \( P_1(\bar{z}; q) \) and \( Q_1(z_{r+1}; q) \) are polynomials. Here \(|\bar{k}| := k_1 + \cdots + k_r\).

2. The power series obtained by expanding

\[
\sum_{l=0} P_1(\bar{z}; q)z_{r+1}^l
\]

is absolutely convergent for arbitrary \( \bar{z} \in \mathbb{C}^r \), provided \(|z_{r+1}| \) is sufficiently small, and the power series obtained by expanding

\[
\sum_{\mathcal{Z}_{\bar{l}}} \mathcal{Q}_1(z_{r+1}; q)\frac{z_{r+1}^l}{1 - z_{r+1}}
\]

is absolutely convergent for any \( z_{r+1} \in \mathbb{C} \), provided all \(|z_1|, \ldots, |z_r| \) are sufficiently small.

3. We have

\[
P_0(\bar{z}; q) = P_1(\bar{z}; q) = Q_0(z_{r+1}; q) \equiv 1.
\]

4. The polynomials \( P_1(\bar{z}; q) \) are symmetric in \( \bar{z} \), and if \( l \) is odd then \( P_1(\bar{z}; q) \) is even, i.e., \( P_1(\bar{z}; q) = P_1(-\bar{z}; q) \).

5. We have the functional equations

\[
P_1(z_1, z_2, \ldots, z_r; q) = (\sqrt{q}z_1)^{-\delta_1} P_1\left(\frac{1}{q}z_1, \ldots, z_r; q\right) \quad \text{and} \quad Q_1(z_{r+1}; q) = (\sqrt{q}z_{r+1})^{\delta_2} Q_1\left(\frac{1}{q}z_{r+1}; q\right)
\]

(7)

with \( \delta_1 = 0 \) or 1 according as \( n \) is even or odd.

For our purposes, we shall need a specific power series satisfying all these properties, see the next section for definitions.

Put \( f_{\text{even}}(z; q) := (f(z, z_{r+1}; q) + f(z, -z_{r+1}; q))/2 \) and \( f_{\text{odd}}(z; q) := (f(z, z_{r+1}; q) - f(z, -z_{r+1}; q))/2 \), and define

\[
M(u, q) := \begin{pmatrix}
\frac{1 - qu}{qu(1 - u)} & \frac{1}{\sqrt{q}u} \\
\frac{1 + qu}{qu(1 + u)} & \frac{1}{\sqrt{q}u}
\end{pmatrix}
\]

and

\[
f(z; q) := \begin{pmatrix}
f_{\text{even}}(z, z_{r+1}; q) \\
f_{\text{odd}}(z, z_{r+1}; q)
\end{pmatrix}
\]

also, let

\[
U := \begin{pmatrix}
1/2 & 1 & 1/2 \\
1/2 & 0 & -1/2 \\
1/2 & -1 & 1/2
\end{pmatrix}.
\]

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Note that $U^2 = I$, where $I$ is the identity $3 \times 3$ matrix.

For $i = 1, \ldots, r+1$ and $z \in \mathbb{C}^{r+1}$, define $w_i z = z' \in \mathbb{C}^{r+1}$ by

$$z'_j = \begin{cases} 
  z_j & \text{if } i, j \leq r, i \neq j \\
  1/\sqrt{q} z_i & \text{if } i = j \\
  \sqrt{q} z_j & \text{if } i \neq j, \text{and either } i = r + 1 \text{ or } j = r + 1.
\end{cases}$$

The group generated by $w_1, \ldots, w_{r+1}$ is easily seen to be isomorphic to the Weyl group $W$ in 2.1. In what follows, by abuse of notation, we shall still denote the group $(w_i : i = 1, \ldots, r+1)$ by $W$. From the above properties of $f(z; q)$, one checks that

$$f(z; q) = M(z; q)f(w_1 z; q) \quad \text{(for } i = 1, \ldots, r) \quad \text{and} \quad Uf(z; q) = M(z_{r+1}; q)Uf(w_r z; q).$$

Let $M_w(z) = M_w(z; q)$, for $w \in W$, be defined as follows. Put

$$M_w(z) = \begin{cases} 
  M(z; q) & \text{if } 1 \leq i \leq r \\
  UM(z_{r+1}; q)U & \text{if } i = r + 1
\end{cases}$$

and extend this function to $W$ by the 1-cocycle relation:

$$M_{ww'}(z) = M_w(z)M_{w'}(w'z) \quad \text{(for } w, w' \in W).$$

The cocycle $M_w(z; q)$ is well-defined, and by (8), we have

$$f(z; q) = M_w(z; q)f(wz; q)$$

for every $w \in W$.

### 4 Multiple Dirichlet series

In this section, we shall introduce multiple Dirichlet series (MDS) naturally associated with moments of quadratic Dirichlet $L$-functions. The construction of these series, originally given in [13] (see also [34], where more general MDS are discussed), is based upon a set of axioms that uniquely characterize them. By simply adapting the proof of [41, Proposition 2.2.1] in the case of MDS for affine Weyl groups, one can show that the so-called $p$-parts of our MDS satisfy the conditions 1–5 in the previous section. An almost immediate consequence is that the multiple Dirichlet series we consider here satisfy a group of functional equations isomorphic to the Weyl group of the Kac-Moody algebra in 2.1.

#### 4.1 Generating series

Let $F_q$ be a finite field of odd characteristic. We fix throughout an algebraic closure $\overline{F}_q$ of $F_q$. For $s = (s_1, \ldots, s_{r+1}) \in \mathbb{C}^{r+1}$, consider a formal series $Z(s)$ of the form

$$Z(s) = \sum_{m_1, \ldots, m_r, d - \text{monic}} \frac{\chi_{d}(\widehat{m}_i) \cdots \chi_{d}(\widehat{m}_r)}{|m_1|^{s_1} \cdots |m_r|^{s_r} |d|^{s_{r+1}}} \cdot A(m_1, \ldots, m_r, d)$$

where $\widehat{m}_i$ denotes the part of $m_i$ coprime to $d_0$. We assume that the coefficients $A(m_1, m_r, m_3, \ldots, m_r, d)$ are multiplicative, in the sense that

$$A(m_1, \ldots, m_r, d) = \prod_{p \parallel m_i, p^k \parallel d} A(p^{k_1}, \ldots, p^{k_r}, p^k)$$
the product being taken over monic irreducibles \( p \in \mathbb{F}_q[x] \). If we set \( z_i = q^{-k_i} \), for \( 1 \leq i \leq r + 1 \), we can also write (12) as a formal power series
\[
\sum_{k_1, \ldots, k_r, l \geq 0} b(k_1, \ldots, k_r, l; q) z_1^{k_1} \cdots z_r^{k_r} z_{r+1}^l.
\]

We specify the coefficients \( A(p^{k_1}, \ldots, p^{k_r}, p^l) \) and \( b(k_1, \ldots, k_r, l; q) \) by considering formal power series of the form
\[
\sum_k c(k; q) z^k
\]
with coefficients \( c(k; q) \) (\( k \in \mathbb{N}^{r+1} \)) finite sums
\[
c(k; q) = \sum_j P_j^{(k)}(q) \lambda_j
\]
where \( P_j^{(k)}(x) \in \mathbb{Q}[x] \), and \( \lambda_j \) are distinct \( q \)-Weil algebraic integers of weights \( \nu_j \in \mathbb{N} \). We are assuming that:

(i) Every \( q \)-Weil integer \( \lambda_j \) occurs in (14) together with all its complex conjugates.

(ii) If \( \lambda_j \) and \( \lambda_j' \) are conjugates over \( \mathbb{Q} \), then \( P_j^{(k)}(x) = P_j^{(k)}(x) \).

(iii) For each \( j, \deg P_j^{(k)} + \nu_j \leq |k| \) and \( P_j^{(k)}(x)^2 \equiv 0 \pmod{x^{|k|+\nu_j+2}} \) when \( |k| = k_1 + \cdots + k_{r+1} \geq 2 \).

It is proved in [13] that there exists a unique series (12), and a unique series (13), whose coefficients are of the form (14) with all properties (i) – (iii), such that the following conditions are satisfied:

(a) The subseries
\[
\sum_{k_1, \ldots, k_r, l \geq 0} b(k_1, \ldots, k_r, 0; q) z_1^{k_1} \cdots z_r^{k_r} = \frac{1}{\prod_{i=1}^{r+1} \frac{1}{1-qz_i}}
\]
i.e., is a product of \( r \) zeta functions. In addition,

\[
\sum_{k_1, \ldots, k_r \geq 0} b(k_1, \ldots, k_r, 1; q) z_1^{k_1} \cdots z_r^{k_r} = q
\]
and
\[
\sum_{l \geq 0} b(0, \ldots, 0, l; q) z_{r+1}^l = \frac{1}{1-qz_{r+1}}.
\]

In particular, \( A(1, \ldots, 1, 1) = b(0, \ldots, 0, 0; q) = 1 \).

(b) For every \( (k_1, \ldots, k_r, l) \in \mathbb{N}^{r+1} \), the coefficient \( b(k_1, \ldots, k_r, l; q^n) \) corresponding to the series (12) over any finite field extension \( \mathbb{F}_{q^n} \subset \mathbb{F}_q \) of \( \mathbb{F}_q \) is given by
\[
b(k_1, \ldots, k_r, l; q^n) = c(k_1, \ldots, k_r, l; q^n) = \sum_j P_j^{(k_1 \ldots k_r l)}(q^n) \lambda_j^n.
\]
Moreover, the polynomials \( P_j^{(k)} \) are independent of the prime power \( q \), for all \( k \in \mathbb{N}^{r+1} \) and all \( j \).

(c) For every monic irreducible \( p \in \mathbb{F}_q[x] \) of degree \( e \geq 1 \), the coefficients \( A(p^{k_1}, \ldots, p^{k_r}, p^l) \) are given by
\[
A(p^{k_1}, \ldots, p^{k_r}, p^l) = \frac{q^{(k_1 + \cdots + k_r)l}}{q^{k_1 + \cdots + k_r} c(k_1, \ldots, k_r, l; q^{-e})} = \frac{q^{(k_1 + \cdots + k_r + l)}}{q^{k_1 + \cdots + k_r + l} c(k_1, \ldots, k_r, l; q^{-e})} \sum_j P_j^{(k_1 \ldots k_r l)}(q^{-e}) \lambda_j^{-e}.
\]
\[\text{In fact we shall always take an expression (14) in reduced form, by which we mean that } \lambda' = q^n \lambda \text{ for some } n \in \mathbb{N} \text{ if and only if } \lambda' = \lambda.\]
In what follows, we will need a twisted version of (12). To define these series, we fix throughout an element \( \theta_0 \in \mathbb{F}_q^* \backslash \left( \mathbb{F}_q^* \right)^2 \). Let \( c \in \mathbb{F}_q[x] \) be monic and square free, and fix a factorization \( c = c_1c_2c_3 \), with \( c_i \), monic. For \( a_1, a_2 \in \{ 1, \theta_0 \} \), we define \( Z^{(c)}(s; \chi_{a_2c_2}, \chi_{a_1c_1}) \) by

\[
Z^{(c)}(s; \chi_{a_2c_2}, \chi_{a_1c_1}) = \sum_{m_1, \ldots, m_r, d \text{--monic}} \frac{\chi_{a_1c_1}(m_1) \cdots \chi_{a_2c_2}(d_0)}{|m_1|^{s_1} \cdots |m_r|^{s_r} |d|^{s_{r+1}}} \cdot A(m_1, \ldots, m_r, d)
\]

with the same coefficients \( A(m_1, \ldots, m_r, d) \) as before.

**Remark 1.** — The analogue of (15) over \( \mathbb{Q} \) (or over any number field) is constructed similarly from the same (unique) series (13). Concretely, for \( d \in \mathbb{Z} \) non-zero and square free, let \( \chi_d(m) \) be the quadratic character defined by

\[
\chi_d(m) = \begin{cases} 
(\frac{d}{m}) & \text{if } d \equiv 1 \pmod{4} \\
(\frac{d}{m}) & \text{if } d \equiv 2 \pmod{4}.
\end{cases}
\]

Fix an odd, positive, square free integer \( c \). Let \( a_1, a_2 \in \{ \pm 1, \pm 2 \} \), and let \( c_1, c_2 \) divide \( c \). Define

\[
Z^{(c)}(s; \chi_{a_2c_2}, \chi_{a_1c_1}) = \sum_{m_1, \ldots, m_r, d \geq 1} \frac{\chi_{a_1c_1}(m_1) \cdots \chi_{a_2c_2}(d) \chi_{a_1c_1}(d_0)}{|m_1|^{s_1} \cdots |m_r|^{s_r} d^{s_{r+1}}} \cdot A(m_1, \ldots, m_r, d)
\]

with \( A(m_1, \ldots, m_r, d) \) multiplicative, and \( A(p^{k_1}, \ldots, p^{k_r}, p^j) \), for any prime \( p \geq 3 \), given by

\[
A(p^{k_1}, \ldots, p^{k_r}, p^j) = p^{k_1 + \cdots + k_r + j} \sum_j p_j^{(k_1 + \cdots + k_r + j)} (p^{-1})^{j} \lambda_j^{-1}.
\]

This function should (conjecturally) have all the analytic properties that will be shortly discussed in the function-field case.

### 4.2 Functional equations

With the notations introduced in 4.1, let \( a(k; q) \), for \( k \in \mathbb{N}^{r+1} \setminus \{ 0 \} \), be defined by

\[
a(k; q) := q^{|k|} \sum_j p_j^{(k)}(q^{-1}) \lambda_j^{-1} = \sum_j p_j^{(k)}(q^{-1}) q^{\deg p_j^{(k)} - 1}\lambda_j
\]

where \( p_j^{(k)} \) and \( \lambda_j \) refer to the unique sets of polynomials and \( q \)-Weil algebraic integers, respectively, determined by the conditions (a) – (c) in 4.1. Note that by the first condition in (iii), 4.1, \( p_j^{(k)}(x^{-1}) x^{\deg p_j^{(k)} - 1} \in \mathbb{Q}[x] \) when \( |k| > 1 \), and by the second condition in (iii), 4.1, we have

\[
\max_j \left( \frac{1}{r} + \deg p_j^{(k)}(x^{-1}) x^{\deg p_j^{(k)} - 1} \right) \leq \frac{|k|}{r} - 1 \quad (\text{if } |k| > 1).
\]

This leads us to make the following:

**Assumption.** For every small \( \varepsilon > 0 \) and \( |k| \geq 2 \), we have

\[
a(k; q) \ll_{\varepsilon} q^{(\frac{1}{2} + \varepsilon)|k| - 1}. \quad (H1)
\]
Remark 2. — It is conceivable that the method used in [4] to bound the Betti numbers of the moduli space of hyperelliptic curves of any fixed genus can be adapted to prove a more natural bound of the form

$$|a(k; q)| < C^{|k|} q^{|k| - 1} \quad (|k| > 1)$$

for an absolute constant $C$ depending only upon $r$.

We now take the function $f(z; q)$ at the beginning of Section 3 to be

$$f(z; q) := 1 + \sum_{k \neq 0} a(k; q) z^k = 1 + \sum_{k \neq 0} \left( \sum_{j} P_j^{(k)}(q^{-1}) q^{-r} \lambda_j \right) (qz)^k;$$

by (H1), it converges absolutely in the polydisk $|z| < q^{\frac{1}{2}}$ $(i = 1, \ldots, r + 1)$. The same idea as in the proof of [41, Proposition 2.2.1] shows that the function $f(z; q)$ satisfies the conditions 1–5 in Section 3, and thus, the corresponding vector function $f(z; q)$ satisfies the functional equations (11).

The series $Z^{(c)}(s; \chi_{a_2^{(2)}, \chi_{a_1^{(1)}}})$ are constructed from $f(z; q)$ exactly as in [11, Section 3]. By (H1), it is clear that they converge absolutely for $\Re(s) > 1$ $(i = 1, \ldots, r + 1)$, and in this domain we can write, see loc. cit.,

$$Z^{(c)}(s; \chi_{a_2^{(2)}, \chi_{a_1^{(1)}}}) = \sum_{(d, e) = 1 \atop d = d_0, e = e_0} \prod_{i=1}^{r+1} \frac{L^{(c^{(2)})}(s, \chi_{a_1^{(1)}}(d_i)) \cdot \chi_{a_2^{(2)}}(d_0) P_d(s; \chi_{a_1^{(1)}}(d_0))}{|d|^{s+1}}. \quad (16)$$

Here $L^{(c^{(2)})}(s, \chi_{a_1^{(1)}}(d_i)) = L(s, \chi_{a_1^{(1)}}(d_i)) \cdot \prod_{p|d} (1 - \chi_{a_1^{(1)}}(d_i)(p)|p|^{-s})$, the product being over the monic irreducible divisors of $c_2 c_1$, and $P_d(s; \chi_{a_1^{(1)}}(d_0))$, where $s := (s_1, \ldots, s_r)$, is the Dirichlet polynomial defined by

$$P_d(s; \chi_{a_1^{(1)}}(d_0)) = \prod_{l=1}^{r+1} P_l(p^{-s_1}, \ldots, p^{-s_r}; q^{\deg p}) \cdot \prod_{p|d_0} P_l(p^{s_1}, \ldots, p^{s_r}; q^{\deg p}) \cdot \prod_{p|d_0} P_l(p^{s_1}, \ldots, p^{s_r}; q^{\deg p}),$$

where $P_l(s; q)$ are the polynomials defined by the condition 1 in Section 3. Notice that by (7), $P_d(s; \chi_{a_1^{(1)}}(d_0))$ satisfies a functional equation in each of the variables $s_1, \ldots, s_r$. Moreover, it is clear that, for fixed $s_1, \ldots, s_r \in \mathbb{C} \setminus \{1\}$, the series (16) converges absolutely when $s_{r+1}$ has sufficiently large real part.

We can also write

$$Z^{(c)}(s; \chi_{a_2^{(2)}, \chi_{a_1^{(1)}}}) = \sum_{(m_1, \ldots, m_r) = 1 \atop m_1 \cdots m_r = n_0 p_0^2} \frac{L^{(c^{(2)})}(s_{r+1}; \chi_{a_2^{(2)} n_0}) \chi_{a_1^{(1)}}(n_0) Q_m(s_{r+1}; \chi_{a_2^{(2)} n_0})}{|m_1|^{s_1} \cdots |m_r|^{s_r}} \quad (17)$$

where, for $m = (m_1, \ldots, m_r)$, the Dirichlet polynomial $Q_m(s_{r+1}; \chi_{a_2^{(2)} n_0})$ is given by

$$Q_m(s_{r+1}; \chi_{a_2^{(2)} n_0}) = \prod_{p|d_0} Q_p(|p|^{-s_{r+1}}; q^{\deg p}) \cdot \prod_{p|d_0} Q_p(|p|^{-s_{r+1}}; q^{\deg p}).$$

Again, by (7), the polynomials $Q_m(s_{r+1}; \chi_{a_2^{(2)} n_0})$ satisfy a functional equation as $s_{r+1} \to 1 - s_{r+1}$, and for every $s_{r+1} \in \mathbb{C} \setminus \{1\}$, the series (17) converges absolutely as long as all $s_1, \ldots, s_r$ have sufficiently large real parts.
As in [11, 3.1], the expressions (16) and (17) of \( Z^{(c)}(s; \chi_{\Delta_{a_{1}c_{1}}}) \) can be used to show that this family of multiple Dirichlet series satisfies a group of functional equations isomorphic to the Weyl group of the Kac-Moody algebra in 2.1. To state explicitly the functional equations corresponding to the generators \( w_1, \ldots, w_{r+1} \) of \( W \), define \( U_m(s) = 1 \) if \( m = 1 \), and

\[
U_m(s_{r+1}) = \prod_{|p|<m} |p|^{t_r+1} \left( 1 - |p|^{2s_{r+1}} \right)^{-1}
\]

for (non-constant) monic square free \( m \in \mathbb{Z}[x] \), the product being over the monic irreducible divisors of \( m \). If we set

\[
w^i := (s_1, \ldots, 1-s_1, \ldots, s_r, s_r + s_{r+1} - s) \quad \text{for } i \leq r,
\]

\[
w^{r+1} := (s_1 + s_{r+1} - s, s_r + s_{r+1} - s,
\]

then the functional equations of \( L(s, \chi_d) \), \( P_d(s; \chi_{\Delta_{a_1c_1}(0)}) \), and \( Q_m(s_{r+1}; \chi_{\Delta_{a_2c_2}}) \) imply that

\[
Z^{(c)}(s; \chi_{\Delta_{a_2c_2}}, \chi_{\Delta_{a_1c_1}}) = \frac{1}{2} \frac{\varphi(c_2 c_3)}{|c_2 c_3|} \prod_{p|c_2 c_3} \left( 1 - |p|^{2s_{r+1}} \right)^{-1}
\]

\[
\cdot \sum_{\vartheta \in \{1, 0\}} \chi_{\Delta_{a_2c_2}}(c_2) \left( \vartheta'^{(r+1)}(s_{r+1} + d_2) \right) \left( \vartheta'_{s_{r+1}}(s_{r+1}) \right) \sum_{m|c_2 c_3} \chi_{\Delta_{a_2c_2}}(m) U_m(s_{r+1}) Z^{(c)}(w^{r+1}; s; \chi_{\Delta_{a_2c_2}}, \chi_{\Delta_{a_1c_1}})
\]

(18)

and

\[
Z^{(c)}(s; \chi_{\Delta_{a_2c_2}}, \chi_{\Delta_{a_1c_1}}) = \frac{1}{2} \frac{\varphi(c_2 c_3)}{|c_2 c_3|} \prod_{p|c_2 c_3} \left( 1 - |p|^{2s_{r+1}} \right)^{-1}
\]

\[
\cdot \sum_{\vartheta' \in \{1, 0\}} \chi_{\Delta_{a_2c_2}}(c_2) \left( \vartheta'^{(s_1 + d_1)}(s_1) \right) \left( \vartheta'_{s_1}(s_1) \right) \sum_{\ell|c_2 c_3} \chi_{\Delta_{a_2c_2}}(\ell) U_\ell(s_1) Z^{(c)}(w^{s_1}; s; \chi_{\Delta_{a_2c_2}^f}, \chi_{\Delta_{a_1c_1}})
\]

(19)

where \( \varphi \) is Euler’s totient function over \( \mathbb{Z}_q[x] \). Since \( Z^{(c)}(s; \chi_{\Delta_{a_2c_2}}, \chi_{\Delta_{a_1c_1}}) \) is symmetric in the variables \( s_1, \ldots, s_r \), we have similar functional equations for \( s_{r+1}, \ldots, s_{r+1} \) in the variables \( s_{r+1}, \ldots, s_{r+1} \), respectively.

### 4.3 Analytic continuation

For the purpose of the present discussion, it will be convenient to substitute \( s_i = s_i + \frac{s}{2} \), for \( i = 1, \ldots, r + 1 \).

We recall from Section 2.1 that the Weyl-Kac denominator \( \prod_{a \in \Delta_0} (1 - e^{-\alpha})^{m_a} \) (where \( e^\lambda = e^{\lambda h^*} \) and \( h \in h \)), is a holomorphic function on the interior of the complexified Tits cone \( \mathfrak{T}_c \). Substituting \( e^{-\alpha} \cdot \lambda \rightarrow q^{-\lambda} \), for \( i = 1, \ldots, r + 1 \), the Weyl-Kac denominator becomes \( \prod_{a \in \Delta_0} (1 - q^{-2\alpha})^{m_a} \), where, for \( \alpha = \sum k_i \alpha_i \in \Delta \) and \( s = (s_1, \ldots, s_{r+1}) \in \mathbb{C}^{r+1} \), we set \( \alpha(s) := \sum k_i s_i \); it is holomorphic on \( \mathfrak{T}_c^* \) : the interior of the corresponding complexified convex cone. Note that, for \( \varepsilon > 0 \), \( (0, \ldots, 0, \varepsilon) \in X^*_0 \).

Let \( Q^* = \sum \mathbb{N} \alpha \) consist of the elements \( \alpha = \sum k_i \alpha_i \) with both \( k_i + \cdots + k_r \) and \( k_{r+1} \) non-zero even natural numbers. For a real parameter \( t > 1 \), let \( \mathcal{S}_w \) consist of the absolutely convergent series on \( X^*_0 \),

\[
1 + \sum_{\alpha \in Q^* \setminus \{0\}} f_\alpha(t) t^{-\alpha(s)}
\]

with \( r^{(a(t)/2)} f_\alpha(t) \in \mathbb{Q}[t] \) of degree \( \leq (d(\alpha) - 2)/2 \), \( f_\alpha(t) \ll \varepsilon^{-1+ed(\alpha)} \) for every small \( \varepsilon > 0 \), which are \( W \)-invariant. Here \( \alpha(s) = \sum k_i s_i \), for \( \alpha = \sum k_i \alpha_i \). Finally, let \( D^{(c)}(s) \) be defined by

\[
D^{(c)}(s) = \prod_{\alpha \in \mathfrak{T}_c^*} \left( 1 - q^{-2\alpha(s)} \right).
\]
Conjecture 4.1. — Let \( a_1, a_2 \in \{ 1, \Theta \} \), \( c \in \mathbb{F}_q[x] \) monic and square-free, and write \( c = c_1 c_2 c_3 \), with \( c_i, i = 1, 2, 3 \), monic. Then there exists \( I(s, t) \in \mathcal{I}_q \), symmetric in \( s_1, \ldots, s_r \), and non-vanishing on \( X_0^* \) for all \( t > 1 \), such that the function

\[
\prod_{p \mid \text{irr}} I(s, |p|) \cdot D^m(s) Z^{(c)} \left( s + \frac{1}{2}, \chi_{a_2 c_2}, \chi_{a_1 c_1} \right), \quad \frac{1}{2} := \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \in \mathbb{C}^{r+1}
\]

is holomorphic on \( X_0^* \); the product \( \prod_{p \mid \text{irr}} I(s, |p|) \) converges absolutely to a holomorphic function on \( X_0^* \).

Remark 3. — When \( r \geq 5 \), we do not have any prediction for the series \( I(s, t) \) in the conjecture. If \( r = 4 \) and \( c = 1 \), one can take \( I(s, t) \equiv 1 \) (see [15]), and we conjecture that the same should be true for arbitrary \( c \).

From now on we shall assume that Conjecture 4.1 holds. Since \( \prod_{p \mid \text{irr}} I(s, |p|) \) is a Dirichlet series supported on tuples \((m_1, \ldots, m_r, d)\) of monic polynomials with \( m_1 \cdots m_r \) and \( d \) square, we can express \( \prod_{p \mid \text{irr}} I(s - \frac{1}{2}, |p|) \cdot Z^{(c)}(s; \chi_{a_2 c_2}, \chi_{a_1 c_1}) \) as a series of the form (15), for some multiplicative coefficients \( \tilde{A}(m_1, \ldots, m_r, d) \) satisfying

\[
\tilde{A}(p^{k_1}, \ldots, p^{k_r}, p^l) = A(p^{k_1}, \ldots, p^{k_r}, p^l) = 1
\]

for all monic irreducibles \( p \), when either \( k_1 = \cdots = k_r = 0 \) or \( l = 0 \).

In what follows, we shall denote \( \prod_{p \mid \text{irr}} I(s - \frac{1}{2}, |p|) \cdot Z^{(c)}(s; \chi_{a_2 c_2}, \chi_{a_1 c_1}) \) by \( \tilde{Z}^{(c)}(s; \chi_{a_2 c_2}, \chi_{a_1 c_1}) \). Since \( I(s, t) \) is \( W \)-invariant, the functions \( Z^{(c)}(s; \chi_{a_2 c_2}, \chi_{a_1 c_1}) \) and \( \tilde{Z}^{(c)}(s; \chi_{a_2 c_2}, \chi_{a_1 c_1}) \) satisfy the same functional equations.

### 4.4 Generating functions for moments of \( L \)-functions

For \( s = (s_1, \ldots, s_r + 1) \in \mathbb{C}^{r+1} \) with \( \Re(s_i) > 1 \), and \( a_2 \in \{ 1, \Theta \} \), define

\[
Z_0(s, \chi_{a_2}) = \sum_{d \text{ monic} \& \text{sq. free}} \left| I(s, \chi_{a_2}) \right|^2 |d|^{-s_r - 1}
\]

and, for \( h \) square-free monic, put

\[
Z(s, \chi_{a_2}; h) = \sum_{d \text{ monic} \& \text{sq. free}} \frac{\chi_{a_2}(d)}{|d|^{s_r + 1}} \cdot \tilde{A}(m_1, \ldots, m_r, d).
\]

Recalling that the coefficients \( \tilde{A}(m_1, \ldots, m_r, d) \) are multiplicative, \( \tilde{A}(p^{k_1}, \ldots, p^{k_r}, 1) = 1 \), and

\[
\tilde{A}(p^{k_1}, \ldots, p^{k_r}, p) = \begin{cases} 1 & \text{if } k_1 = \cdots = k_r = 0 \\ 0 & \text{otherwise} \end{cases}
\]

for every monic irreducible \( p \), we can write (see [11, Lemma 5.1])

\[
Z_0(s, \chi_{a_2}) = \sum_{h \text{ monic}} \mu(h) Z(s, \chi_{a_2}; h).
\]  

(20)

Here \( \mu(h) \) is the Möbius function defined for non-zero polynomials over \( \mathbb{F}_q \) by \( \mu(h) = (-1)^{\omega(h)} \) if \( h \) is square-free, and \( h \) is a constant times a product of \( \omega(h) \) distinct monic irreducibles, and \( \mu(h) = 0 \) if \( h \) is not square-free; it is understood that \( \mu(h) = 1 \) if \( h \in \mathbb{F}_q^\times \).

We can express \( Z(s, \chi_{a_2}; h) \) in terms of the functions \( \tilde{Z}^{(c)}(s; \chi_{a_2 c_2}, \chi_{a_1 c_1}) \) as follows. Set \( f(z; q) := I(\sqrt{q} z, q) f(z; q) \), where \( I(\sqrt{q} z, q) \) denotes the function \( I(s - \frac{1}{2}, q) \), with \( q^{-s} \) replaced by \( z \), \( i = 1, \ldots, r + 1 \), and \( f(z; q) \) is the power series defined in 3.2. Then \( f(z; q) \) satisfies the conditions 1–5 in Section 3, and if we write

\[
f(z; q) = 1 + \sum_{k=0}^\infty \tilde{a}(k; q) z^k
\]
the coefficients $\tilde{a}(k;q)$ satisfy the estimate (H1). Set $\tilde{f}_{\text{even}}(z;q) = I(\sqrt{q}z,q)f_{\text{even}}(z;q)$, $\tilde{f}_{\text{odd}}(z;q) = I(\sqrt{q}z,q)f_{\text{odd}}(z;q)$, and let $\tilde{f}(z;q) = I(\tilde{f}_{\text{even}}(z;\bar{z},z_{r+1};q),\tilde{f}_{\text{odd}}(z;\bar{z},z_{r+1};q),\tilde{f}_{\text{even}}(-z;\bar{z},z_{r+1};q))$.

We have the following:

**Lemma 4.2.** — For $\sigma > \frac{1}{2}$, and sufficiently small $\varepsilon > 0$, we have

$$\tilde{f}(z;q) = \left( \prod_{j=1}^{r} (1-z_{j})^{-1}, z_{r+1}, \prod_{j=1}^{r} (1+z_{j})^{-1} \right) + O_{\varepsilon,\sigma}(q^{-2\sigma+(4r+2)\varepsilon})$$

in the polydisk $|z_{i}| \leq q^{-\frac{1}{2}+\varepsilon}, i = 1, \ldots, r$, and $|z_{r+1}| \leq q^{-\sigma}$.

**Proof.** Write

$$\prod_{j=1}^{r} (1-z_{j}) \cdot \tilde{f}_{\text{even}}(z;q) = 1 + \sum_{k=1}^{\infty} \tilde{b}(k;q)z^{k} = 1 + \sum_{j=1}^{r} (1-z_{j}) \cdot \sum_{k=1}^{\infty} \tilde{a}(k;q)z^{k}$$

and

$$\tilde{f}_{\text{odd}}(z;q) = z_{r+1} + \sum_{k=1}^{\infty} \tilde{a}(k;q)z^{k}.$$

Since the coefficients $\tilde{a}(k;q)$ satisfy the estimate (H1), $\tilde{b}(k;q) \ll q^{\left(\frac{1}{2}+\varepsilon\right)|k|^{-1}}$. Moreover, since $\prod_{j=1}^{r} (1-z_{j}) \cdot \tilde{f}_{\text{even}}(z;q)$ is invariant under $w_{i}$, for $1 \leq i \leq r$, the coefficients $\tilde{b}(k;q)$ vanish unless $k = (k_{1}, \ldots, k_{r}, k_{r+1})$ is such that $k_{i} \leq k_{r+1}$ for all $i$. Similarly, from the functional equations $\tilde{f}_{\text{odd}}(z;q) = (\sqrt{q}z)^{-1}f_{\text{odd}}(w_{i};q)$, for $1 \leq i \leq r$, we deduce that the coefficients of $f_{\text{odd}}(z;q)$ vanish unless $k = (k_{1}, \ldots, k_{r}, k_{r+1})$ satisfies the condition $k_{i} \leq k_{r+1} - 1$ for all $i = 1, \ldots, r$. Accordingly, for $|z_{i}| \leq q^{-\frac{1}{2}+\varepsilon} (i = 1, \ldots, r), |z_{r+1}| \leq q^{-\sigma}$, and $\varepsilon > 0$ such that $1 - 2\sigma + (4r+2)\varepsilon < 0$,

$$\prod_{j=1}^{r} (1-z_{j}) \cdot \tilde{f}_{\text{even}}(z;q) - 1 \ll q^{-1} \cdot \sum_{k_{r+1} \geq 2} q^{\left(\frac{1}{2} \sigma + (2r+1)\varepsilon\right)} k_{r+1} \sum_{\begin{array}{c} k_{r+1} \geq 1, \text{ k even} \\ |k| \leq k_{r+1} \end{array}} q^{|k|}$$

$$\ll q^{-1} \cdot (1-q^{-2\sigma})^{-r} \cdot \sum_{k_{r+1} \geq 2} q^{\left(\frac{1}{2} \sigma + (2r+1)\varepsilon\right)} k_{r+1}$$

$$\ll q^{-2\sigma+(4r+2)\varepsilon} \left(1-q^{-2\sigma}\right)^{-r} \left(1-q^{-2\sigma+(4r+2)\varepsilon}\right)^{-1}$$

Similarly,

$$|\tilde{f}_{\text{odd}}(z;q) - z_{r+1}| \ll q^{-1} \cdot \sum_{k_{r+1} \geq 3, k_{r+1} \text{ odd}} q^{\left(\frac{1}{2} \sigma + (2r+1)\varepsilon\right)} k_{r+1} \sum_{|k| \leq k_{r+1} - 1} q^{|k|} \ll q^{-2\sigma+(4r+2)\varepsilon}$$

which completes the proof. □

If we now define,

$$F(z;q) := z_{r+1}^{3} \tilde{f}_{\text{odd}}(z_{r+1};q) - z_{r+1}^{2}$$

and, for $a \in \{0,1\}$,

$$G^{(a)}(z;q) := \frac{1}{2} \left\{ \tilde{f}_{\text{even}}(z;\bar{z},z_{r+1};q) - \prod_{k=1}^{r} (1-z_{k})^{-1} \right\} z_{r+1}^{2} + \frac{(-1)^{a+2}}{2} \left\{ \tilde{f}_{\text{even}}(-z;\bar{z},z_{r+1};q) - \prod_{k=1}^{r} (1+z_{k})^{-1} \right\} z_{r+1}^{2}$$

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then, as in [11, Section 5], we can write

\[ Z(s, \chi_h; h) = |h|^{-2s+1} \sum_{h = c_1, c_2} \chi_{c_1, c_2} \mathcal{Z}^{(h)}(s; \chi_{c_1, c_2}) \prod_{p \mid c_1} F(|p|^{-s_1}, \ldots, |p|^{-s_{r+1}}; |p|) \prod_{p \mid c_2} G^{(1)}(|p|^{-s_1}, \ldots, |p|^{-s_{r+1}}; |p|) \prod_{p \mid c_1} G^{(0)}(|p|^{-s_1}, \ldots, |p|^{-s_{r+1}}; |p|). \] (21)

Conjecture 4.1 combined with the following conjecture gives the meromorphic continuation of the function \( Z_0(s, \chi_{a_0}) \).

**Conjecture 4.3.** — The series

\[ \sum_{h \text{--monic}} \mu(h)Z(s, \chi_{a_0}; h) \]

is absolutely convergent for \( \Re(s_i) \geq \frac{1}{2}, \ i = 1, \ldots, r, \) and \( \Re(s_{r+1}) > \frac{1}{2} \), away from the zero-set of \( D^0(s - \frac{1}{2}) \).

In the next sections, we shall see how the asymptotics for moments of L-functions, summed over monic square-free polynomials of fixed degree, can be obtained from Assumption (H1), and Conjectures 4.1, 4.3.

## 5 Poles and residues

Our first goal in this section is to compute the residues at the poles of the functions \( \mathcal{Z}^{(c)}(s; \chi_{a_2, c_2}, \chi_{a_1, c_1}) \) introduced at the end of 4.3. By Conjecture 4.1, the poles of \( \mathcal{Z}^{(c)}(s; \chi_{a_2, c_2}, \chi_{a_1, c_1}) \) are among the zeros of the infinite product \( D^0(s - \frac{1}{2}) \).

In particular, each pole is simple, and corresponds to a positive real root of the Kac-Moody Lie algebra with the generalized Cartan matrix in 2.1.

Taking \( s_1, \ldots, s_r \) with sufficiently large real parts, we see that \( \mathcal{Z}^{(c)}(s; \chi_{a_1, c_1}) \) (i.e., if \( a_2 = a \) and \( c_2 = 1 \)) has a simple pole when \( q^{-s_{r+1}} = \text{sgn}(a) q^{-1} \). The part of \( \mathcal{Z}^{(c)}(s; \chi_{a_1, c_1}) \) that contributes to this pole is

\[ L^{(c)}(s_{r+1}, \chi_{a_1}) \cdot \sum_{m_1, \ldots, m_r = 1}^{\text{even}} \frac{\tilde{Q}_m(s_{r+1}; \chi_{a_1})}{|m_1|^s_{r+1} \cdots |m_r|^s_{r+1}} = L^{(c)}(s_{r+1}, \chi_{a_1}) \prod_{p \mid c} R_p(s, \chi_{a_1}) \] (22)

where \( R_p(s, \chi_{a_1}) \) is given by

\[ R_p(s, \chi_{a_1}) = \sum_{|\vec{a}| \text{--even}} \frac{\tilde{Q}_m(s_{r+1}; \chi_{a_1})}{|\vec{a}|^{s_{r+1}} \cdots |\vec{a}|^{s_r}} \]

\[ = (1 - \chi_{a_1}(p) |p|^{-s_{r+1}})(\tilde{f}(|p|^{-s_1}, \ldots, |p|^{-s_r}, \chi_{a_1}(p) |p|^{-s_{r+1}}; |p|)) \]

In what follows, it will be convenient to express (22) as

\[ \frac{L(s_{r+1}, \chi_{a_1}) R(s, \chi_{a_1})}{L(s_{r+1}, \chi_{a_1}) \prod_{p \mid c} R_p(s, \chi_{a_1})} \]

where we set

\[ R(s, \chi_{a_1}) := \prod_p R_p(s, \chi_{a_1}). \]

Thus we get:

\[ \frac{\mathcal{Z}^{(c)}(s; \chi_{a_0}, \chi_{a_1, c_1})}{L(s_{r+1}, \chi_{a_0})} \bigg|_{q^{-s_{r+1}} = q^{-1}} = \frac{\mathcal{Z}^{(c)}(s; \chi_{a_1}, \chi_{a_1, c_1})}{\zeta(s_{r+1})} \bigg|_{s_{r+1}=1} = \frac{R(s', 1, \chi_1)}{\zeta(1) \prod_{p \mid c} R_p(s', 1, \chi_1)}. \]
where \( s' = (s_1, \ldots, s_r) \), and \( \chi_i \) is the trivial character.

Let \( \alpha = \sum k_i \alpha_i \) (with \( k_i \in \mathbb{N} \)) be a fixed positive real root, and let \( w \in W \) be an element (in reduced form) sending \( \alpha \) to the simple root \( \alpha_{\nu+1} \); we shall denote the length of \( w \) by \( l(w) \). Fix \( a \in \{1, \theta_0\} \), and \( \zeta_{\alpha} \in \mathbb{C} \) such that \( \zeta_{\alpha}^{l(w+1)} = sgn(a) \).

To this data, we attach the residue \( R_w(s'; c, a) \) defined by

\[
R_w(s'; c, a) := \lim_{q^{(d(a)+1-2n(s))/2\nu+1} \rightarrow c_q^{-1}} k_{r+1} \left( 1 - \zeta_{\alpha} q^{d(a)+1-2n(s)}/2\nu+1 \right) \tilde{Z}^{(c)}(s; \chi_{w^2}, \chi_{\theta_{\nu+1}})
\]

where \( \alpha(s) = \sum k_i \alpha_i, \theta \in \{1, \theta_0\} \) and \( \tilde{c} \) is a square-free divisor of \( c \). Here, it is understood that \( s_{r+1} \) is expressed in terms of the other variables. The factor \( k_{r+1} \) should be thought of as the limit

\[
\lim_{q^{(d(a)+1-2n(s))/2\nu+1} \rightarrow c_q^{-1}} \frac{1 - sgn(a)q^{d(a)+1-2n(s)}/2}{1 - \zeta_{\alpha} q^{d(a)+1-2n(s)}/2\nu+1}
\]

and thus

\[
R_w(s'; c, a) = \left. \frac{\tilde{Z}^{(c)}(s; \chi_{w^2}, \chi_{\theta_{\nu+1}})}{L\left( \alpha(s) - \frac{d(a)}{2} + \frac{1}{2}, \chi_{a} \right)} \right|_{q^{-a(s)} = \frac{d(a)}{2} = sgn(a)q^{-1}}
\]

where

\[
\prod_{p \mid c} \tilde{R}_p(s; \chi_{a}) \prod_{p \mid \tilde{c}} R_p(s; \chi_{a}) \right|_{q^{-\epsilon} = \frac{d(a)}{2} = sgn(a)q^{-1}}
\]

Here \( R_p(s; \chi_{a}) \) is defined by (23).

**Proposition 5.1.** — Let notations be as above. Then there exist functions \( \Gamma_w(a_2, a; \zeta_{\alpha}) \), independent of \( c_1, c_2, c_3 \), and \( L_{m,p}(z_1, \ldots, z_r; \zeta_{\alpha}) \), parametrized by the prime divisors of \( c \), such that

\[
\Gamma_w(a_2, a; \zeta_{\alpha}) \prod_{p \mid c} \tilde{R}_p(s; \chi_{a}) R_w(s'; c, a) \prod_{p \mid \tilde{c}} R_p(s; \chi_{a})
\]

where \( \zeta_{\alpha}^{l(w)} k_{r+1} \Gamma_w(a_2, a; \zeta_{\alpha}) \prod_{p \mid \tilde{c}} R_p(s; \chi_{a}) \Gamma_w(s'; c, a) \prod_{p \mid \tilde{c}} R_p(s; \chi_{a})
\]

**Proof.** Set \( Z^{(c)}(s; \chi_{w^2}, \chi_{\theta_{\nu+1}}) := \chi_{a_1}(c_2) \chi_{a_2}(c_1) \left( \frac{s}{c_2} \right) \tilde{Z}^{(c)}(s; \chi_{w^2}, \chi_{\theta_{\nu+1}}) \right). \) Then \( Z^{(c)}(s; \chi_{w^2}, \chi_{\theta_{\nu+1}}) \) satisfies the (somewhat) simpler functional equation

\[
Z^{(c)}(s; \chi_{w^2}, \chi_{\theta_{\nu+1}}) = \frac{1}{2} |c_2| \frac{1}{c_2} \frac{s}{c_2} \frac{1}{c_2} \Phi(c_1c_2) \left[ c_1c_3 \right]^{p_{c_1c_3}} \left( 1 - |p|^{2\nu+1} \right)^{-1} \sum_{\theta \in \{1, \theta_0\}} \Gamma_w(s_{\nu+1}; a_2) \sum_{m \mid c_1c_3} U_m(s_{\nu+1}) Z^{(c)}(s; \chi_{w^2}, \chi_{\theta_{\nu+1}}) \sum_{m \mid c_1c_3} U_m(s_{\nu+1}) Z^{(c)}(s; \chi_{w^2}, \chi_{\theta_{\nu+1}})
\]

We have similar functional equations with respect to all the other transformations \( w_i (i = 1, \ldots, r) \), and hence with respect to all \( w \in W \). Notice that the quadratic characters get absorbed into the functions \( Z^{(c)} \). One should also note that the expression

\[
|c_2| \frac{s}{c_2} \frac{1}{c_2} \Phi(c_1c_2) \left[ c_1c_3 \right]^{p_{c_1c_3}} \left( 1 - |p|^{2\nu+1} \right)^{-1} U_m(s_{\nu+1}) \quad (\text{with } m \mid c_1c_3)
\]

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occurring in the right-hand side of (24), factors over irreducibles dividing $c$.

The functional equation corresponding to $w$ relates $\tilde{Z}_\chi^{(c)}(s; \chi_{a_2}, \chi_{c_1})$ to a sum of functions of the form $\tilde{Z}_\chi^{(c)}(s; \chi_{\theta^i}, \chi_{\alpha c_1})$ with $\theta^i, \theta^j \in \{1, \theta_0\}$, and $\bar{c}_1, \bar{c}_2$ monic square-free and coprime divisors of $c$. The latter functions could have a pole when $q^{-s} = \text{sgn}(a)q^{-i}$ (where $q^{-s}$ has the obvious meaning) only if $\theta^i = a$ and $\bar{c}_2 = 1$. We have

\[
\tilde{Z}_\chi^{(c)}(s; \chi_{\alpha}, \chi_{\alpha c_1}) = \chi_{\alpha}(\bar{c}_1)\tilde{Z}_\chi^{(c)}(s; \chi_{\alpha}, \chi_{\alpha c_1})
\]

and thus

\[
(1 - \zeta_0 q^{-(d(a) + s)/2k_{v+1}})\tilde{Z}_\chi^{(c)}(s; \chi_{\alpha}, \chi_{\alpha c_1}) q^{-a(s)\kappa_{v+1} - \zeta_0^{-1}q^{-(d(a) + s)/2k_{v+1}}} = \frac{\chi_{\alpha}(\bar{c}_1)}{k_{v+1}} R_w(s'; c, a).
\]

The sum of products of the factors involving $Y_q^+$, $Y_q^-$ factors out, giving rise to $\Gamma_w(a_2, a; \zeta_q)$. This function is clearly independent of $c_1, c_2$ and $c_3$. The remaining sum factors over the prime divisors of $c = c_1c_2c_3$. This completes the proof.

We shall use this proposition in order to determine the residues at the poles of the function $Z_w(s, \chi_{a_2})$. For this purpose, it will be convenient to index the family of functions $\{L_{m,p}(z_1, \ldots, z_r; \zeta_a)\}_{p \in \mathbb{P}}^j$ by $L_{m,p}^{(j)}$ with $j = 1, 2, 3$ according as $p$ is a divisor of $c_1$, $c_2$ or $c_3$, respectively.

5.1 Computation of $\Gamma_w(a_2, a; \zeta_q)$

To compute $\Gamma_w(a_2, a; \zeta_q)$, we shall use the fact that this function is independent of $c_1, c_2$ and $c_3$. Indeed, by taking $c = 1$, the functional equation of $Z(s; \chi_{a_2}, \chi_{a_1}) := Z^{(1)}(s; \chi_{a_2}, \chi_{a_1})$ with respect to $w \in \mathbb{W}$ will give, after taking the appropriate residue, the precise relationship between $\Gamma_w$ and $M_w$.

To see this, set $t = q^{-s}$ ($i = 1, \ldots, r + 1$) in $Z(s; \chi_{a_2}, \chi_{a_1})$, and let $Z(t; \chi_{a_2}, \chi_{a_1})$ denote the resulting function. If we define $\hat{Z}(t; q) := \hat{Z}(t(1, 1), Z(t(1, 1), Z(t; \chi_{\theta_0}, 1), Z(t; 1, \chi_{\theta_0}))$ then, by (18), (19) and the fact that $Z(t; \chi_{a_2}, \chi_{a_1})$ is symmetric in the variables $t_1, \ldots, t_r$, we have the functional equations

\[
\hat{Z}(t; q) = B^{-1}M_w(qt; 1/q)B \cdot \hat{Z}(wt; q) \quad (\text{for } w \in \mathbb{W})
\]

where

\[
B := \begin{pmatrix}
1/2 & 1/2 & 0 \\
1/2 & -1/2 & 0 \\
-1/2 & 1/2 & 1
\end{pmatrix}.
\]

Here $M_w$ is the cocycle defined by (9) and (10). We note that the function $Z(t; \chi_{\theta_0}, \chi_{\theta_0})$, which could have been included in $\hat{Z}(t; q)$, can be easily expressed in terms of the other three functions as

\[
Z(t; \chi_{\theta_0}, \chi_{\theta_0}) = -Z(t; 1, 1) + Z(t; \chi_{\theta_0}, 1) + Z(t; 1, \chi_{\theta_0}).
\]

Multiplying (25) by $1 - \zeta_0 q^{-(d(a) + s)/2k_{v+1}}$ and then taking the limit as $q^{-a(s)\kappa_{v+1}} \to \zeta_0^{-1}q^{-(d(a) + s)/2k_{v+1}}$, it follows from Proposition 5.1 that

\[
2^{-l(n)}\Gamma_w(a_2, a; \zeta_q) = (\epsilon^*(a_2), \epsilon^*(a_2), 0) \cdot B^{-1} M_w(q^{-s_1}, \ldots, q^{-s_{r+1}}; 1/q)_{q^{-a(s)\kappa_{v+1}} = \zeta_0^{-1}q^{-(d(a) + s)/2k_{v+1}}} B \cdot (\epsilon^*(a), \epsilon^*(a), \epsilon^*(a))
\]

with $\epsilon^*(\bar{\theta}) = (1 \pm \text{sgn}(\bar{\theta}))/2$ for $\bar{\theta} \in \{1, \theta_0\}$.}

\[3\text{Recall that the functions } Z^{(c)}(s; \chi_{a_2}, \chi_{a_1}) \text{ and } \tilde{Z}^{(c)}(s; \chi_{a_2}, \chi_{a_1}) \text{ satisfy the same functional equations.} \]
5.2 Residues of $Z_0(s, \chi_{\alpha_2})$

Let all notations be as before. By using (20), (21) and Proposition 5.1, it is not hard to check that

$$
\lim_{q^{-(a(s)+1)/(2r+1)} \to \epsilon^{-1}} q^{-\frac{d(a(s)+1)}{2r+1}} (1 - \zeta_0 q^{d(a(s)+1)/(2r+1)}) Z_0(s, \chi_{\alpha_2}) = \frac{\Gamma_n(a, z; \chi_{\alpha_2})}{2(\alpha)} R(s, \chi_{\alpha_2}) \cdot \prod_p S_{p}^{(s)}(\zeta_0, s, \chi_{\alpha_2})
$$

with $S_{p}^{(s)}(\zeta_0, s, \chi_{\alpha_2})$ (w being an element in reduced form of $W$ sending $\alpha$ to $\alpha_{r+1}$) given by

$$
S_{p}^{(s)}(\zeta_0, s, \chi_{\alpha_2}) = \left( L_p(a(s) - \frac{d(a)}{2} + \frac{1}{2}, \chi_{\alpha_2})^{-1} \cdot L_p(a(s) - \frac{d(a)}{2} + \frac{1}{2}, \chi_{\alpha_2})R_p(s, \chi_{\alpha_2}) + L_{\min} G^{(s)}([p]^{-s}, ..., [p]^{-r}; [p]) + L_{\max} G^{(s)}([p]^{-s}, ..., [p]^{-r}; [p]) \right) q^{a(s)/\alpha_r} = \zeta_0 q^{-\frac{d(a(s)+1)}{2r+1}}.
$$

Here $R_p(s, \chi_{\alpha_2})$ (resp. $R(s, \chi_{\alpha_2})$) denotes the function $R_p(s, \chi_{\alpha_2})$ (resp. $R(s, \chi_{\alpha_2})$) with $q^{a(s)/\alpha_r}$ such that $q^{a(s)/\alpha_r} = \zeta_0 q^{-\frac{d(a(s)+1)}{2r+1}}$. Note that

$$
L_p\left(\alpha(s) - \frac{d(a)}{2} + \frac{1}{2}, \chi_{\alpha_2}\right) = 1 - [p]^{-1}
$$

and by (23),

$$
L_p(s_{r+1}, \chi_{\alpha_2})R_p(s, \chi_{\alpha_2}) = \chi_{\alpha_2}(p) f_{\text{odd}}([p]^{-s_{r+1}}, ..., [p]^{-s_r}; [p]) + f_{\text{even}}([p]^{-s_{r+1}}, ..., [p]^{-s_r}; [p]) + f_{\text{even}}([p]^{-s_{r+1}}, ..., [p]^{-s_r}; [p]) / 2.
$$

We replace $L_p(a(s) - \frac{d(a)}{2} + \frac{1}{2}, \chi_{\alpha_2})R_p(s, \chi_{\alpha_2})$ in $S_{p}^{(s)}$ using the identity, and the $F, G^{(s)} (a \in \{0, 1\})$ in $S_{p}^{(s)}$ using the definition of these functions. To $f_{\text{odd}}$ and $f_{\text{even}}$ coming from $L_p(a(s) - \frac{d(a)}{2} + \frac{1}{2}, \chi_{\alpha_2})R_p(s, \chi_{\alpha_2})$, we apply the functional equation corresponding to $\zeta_{r+1}$ to get back in the variables $s_1, ..., s_{r+1}$. Viewing $f_{\text{odd}}([p]^{-s_{r+1}}, ..., [p]^{-s_r}; [p])$ and $f_{\text{even}}([p]^{-s_{r+1}}, ..., [p]^{-s_r}; [p])$ as independent variables, it is then not hard to see that the functions $L_{\min}^{(s)}$, are precisely those that cancel out the $f_{\text{odd}}$'s and the $f_{\text{even}}$'s in the expression of $S_{p}^{(s)}$ when $q^{a(s)/\alpha_r}$ is such that $q^{a(s)/\alpha_r} = \zeta_0 q^{-\frac{d(a(s)+1)}{2r+1}}$.

Concretely, assume, without loss of generality, that $p$ is linear, and set $z_i = q^{-s_i}$ for $1 \leq i \leq r + 1$. Then the contributions of $f_{\text{odd}}(z, z_{r+1}; q)$ and $f_{\text{even}}(z, z_{r+1}; q)$ to $L_p(\cdots, \chi_{\alpha_2})R_p(\cdots, \chi_{\alpha_2})$ is given by

$$
(1/2, sgn(a), 1/2) M_{r+1}(wz, q) \tilde{f}(z; q);
$$

the remaining contribution of these functions is

$$
(L_{\min}^{(1)}, L_{\min}^{(2)}, L_{\min}^{(3)})(1/2, 0, -1/2) \tilde{f}(z; q).
$$

Setting

$$
g(z; q) := \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix} \tilde{f}(z; q)
$$

we must have

$$
(L_{\min}^{(1)}, L_{\min}^{(2)}, L_{\min}^{(3)}) g(z; q) = (1/2, sgn(a), 1/2) M_{r+1}(wz, q) \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} g(z; q).
$$

Here the last component of $z$ is taken in agreement with the relation $q^{-\frac{a(s)}{\alpha_r}} = \zeta_0 q^{-\frac{d(a(s)+1)}{2r+1}}$. Thus the functions $L_{\min}^{(s)}$ are the coefficients of the entries of $g(z; q)$ in the right-hand side of (27). To obtain $L_{\min}^{(s)}([p]^{-s_{r+1}}, ..., [p]^{-s_r}; \chi_{\alpha_2})$ for arbitrary monic irreducible $p$, we just replace $sgn(a)$ by $\chi_{\alpha}(p) = sgn(a)^{deg p}$, $z_i = [p]^{-s_i}$ and $q$ by $[p]$.  

20
By our assumptions, this function is meromorphic in the open disk

$$S_p^w(s', \zeta_0) = (1 - |p|^{-1}) \left( L^{(1)}_{w, p} |p|^{-3s+1} + \frac{L^{(2)}_{w, p} + L^{(3)}_{w, p}}{2 \prod_{k=1}^{2} (1 - |p|^{-k})} + \frac{L^{(3)}_{w, p} - L^{(2)}_{w, p}}{2 \prod_{k=1}^{2} (1 + |p|^{-k})} \right)_{q^{-a(s)/2s+1} = \zeta_0^{-1} q^{-(d(a)+1)/2k_{s+1}}}. \quad (28)$$

As expected, we have the equality

$$R(s', \chi_d) \cdot \prod_p S_p^w(s', \zeta_0) = \prod_p S_p^w(s', \zeta_0)$$

but since the product \( \prod_p S_p^w(s', \zeta_0) \) is divergent on any neighborhood of \( s_1 = \cdots = s_r = \frac{1}{2} \), a (non-trivial when \( k_{r+1} \geq 2 \)) regularization process is required to justify \((29)\); we shall merely discuss the regularization process when \( k_{r+1} = 2 \), see 7.2 – the general case following, essentially, the same argument.

6 Asymptotics of moments

For \( n \geq 1 \), let \( \Phi_n \) denote the subset of \( \Delta^\infty_n \) with \( k_{r+1} = n \). By Lemma 2.2, \( \Phi_n \) is a finite set. For each \( \alpha \in \Phi_n \), fix \( w_\alpha \in W \) in reduced form such that \( w_\alpha^{-1}(\alpha_{r+1}) = \alpha \). Let \( \mathcal{G}_n(z, a_2) \), with \( z = (z_1, \ldots, z_{s+1}) \) \( (z_i = q^{-x}) \) and \( a_2 \notin \{ 1, \theta_0 \} \), be defined by

$$\mathcal{G}_n(z, a_2) := n^{-1} \sum_{a \in \Phi_n} \sum_{a_1 \{1, \theta_0\}} \sum_{\zeta_0 = \text{sgn}(a)} \frac{\Gamma_w(a_2, a; \zeta_0)}{2^{|w_\alpha|}} (1 - \zeta_0 q^{(d(a)+1)/2k_{s+1}})^{-1} \int_p S_p^w(z, \zeta_0)$$

where \( \Gamma_w(a_2, a; \zeta_0) \) and \( S_p^w(z, \zeta_0) \) are obtained from \((26)\) and \((27)\), \((28)\), respectively. It is clear that the expression of \( \mathcal{G}_n(z, a_2) \) is independent of the choice of the elements \( w_\alpha \). Via \((29)\), the function \( \mathcal{G}_n(z, a_2) \) gives (after substituting \( z_i = q^{-x} \), \( i = 1, \ldots, r + 1 \)) the sum of the principal parts of \( Z_0(s, \chi_d) \) at the poles corresponding to the roots in \( \Phi_n \).

Now let \( s_1, \ldots, s_r \) be (fixed) distinct complex numbers with \( \Re(s_i) = \frac{1}{2} \) and consider the function \( Z_0(s_1, \ldots, s_{r+1}, \chi_1) \). For convenience, we replace \( q^{-s_{r+1}} \) in \( Z_0 \) by \( \xi \), and denote the resulting function by \( \mathcal{W}(\xi) \). Thus

$$\mathcal{W}(\xi) = \mathcal{W}(s', \xi) = \sum_{D \geq 0} \left( \sum_{d \text{ monic } \chi_d \text{ free}} \prod_{k=1}^{r} L(s_k, \chi_d) \right) \xi^D.$$

By our assumptions, this function is meromorphic in the open disk \( |\xi| < q^{-1/2} \), with the only possible poles at

$$\xi_{\alpha, \zeta_0} := \zeta_0^{-1} q^{a(\alpha) / 2k_{s+1}},$$

where, for \( \alpha = \sum k_i \alpha_i \in \Delta^\infty_n \), we set \( a(s') := \sum k_i s_i \); to avoid difficulties, we will initially assume that the poles of \( \mathcal{W}(\xi) \) are simple\(^4\). For \( n, D \geq 1 \), let \( Q_n(s'; D, q) \) be defined by

$$Q_n(s'; D, q) = n^{-1} \cdot \prod_{a \in \Phi_n} \left( \frac{\Gamma_w(1, a; \zeta_0)}{2^{|w_\alpha|}} S_n(s', \zeta_0) \right)^{r_D} q^{(d(a)+1)/2k_{s+1}}$$

where \( S_n(s', \zeta_0) := \prod_p S_p^w(z, \zeta_0) \), with \( S_p^w(z, \zeta_0) \) given by \((28)\).

With this notation, our main result is the following:

\(^4\)This happens generically. In fact, two poles corresponding to positive real roots \( \alpha \) and \( \alpha' \) could coincide only if \( \alpha, \alpha' \in \Phi_n \), for some \( n \geq 1 \). Thus, by choosing \( s_1 = \frac{1}{2} + i \), with \( \pi i \log q, \frac{1}{2}, \ldots, \frac{1}{2} \), \( \mathcal{Q} \)-linearly independent, we ensure that the poles of \( \mathcal{W}(\xi) \) are simple.
Theorem 6.1. — Let $D, N \geq 1$ be integers, and suppose that $r \geq 4$. Then, under the Assumption (H1), and Conjectures 4.1, 4.3, we have

$$
\sum_{d \text{-monic \\& sq. free}} \prod_{k=1}^{r} L(s_k, \chi_{d}) = \sum_{n \leq N} Q_n(s'; D, q) + O_{n, q, r} \left( q^{(1+o)(1/2)} \right)
$$

for any $(N+1)^{-1} < \Theta < N^{-1}$.

**Proof.** For $(N+1)^{-1} < \Theta < N^{-1}$, let $\mathcal{A}_\Theta = \{ \xi \in \mathbb{C} : q^{-3} \leq |\xi| \leq q^{-(1+\Theta)^2/2} \}$, and consider the integral

$$
I(D) = \frac{1}{2\pi \sqrt{-1}} \int_{\partial \mathcal{A}_\Theta} \frac{\mathcal{W}(\xi)}{\xi^{D+1}} d\xi,
$$

where $\partial \mathcal{A}_\Theta$ denotes the boundary of $\mathcal{A}_\Theta$. In view of our assumptions, the function $\mathcal{W}(\xi)$ is meromorphic in $|\xi| < q^{-1/2}$, and since $\Re(s_k) = \frac{1}{2}$, for $k = 1, \ldots, r$, its poles are on the circles

$$
|\xi| = |\xi_{k, \alpha}| = q^{\frac{\alpha + 1}{2}} \quad \text{(with } \alpha \in \Phi_n, \xi_{k, \alpha}^2 = 1 \text{ for } n \geq 1). \nonumber
$$

Thus, by our choice of $\Theta$, this function has no poles on the boundary of $\mathcal{A}_\Theta$. On the other hand, we have

$$
\sum_{d \text{-monic \\& sq. free}} \prod_{k=1}^{r} L(s_k, \chi_{d}) = \frac{1}{2\pi \sqrt{-1}} \int_{|\xi| = q^{-(1+\Theta)^2/2}} \frac{\mathcal{W}(\xi)}{\xi^{D+1}} d\xi
$$

and

$$
\frac{1}{2\pi \sqrt{-1}} \int_{|\xi| = q^{-(1+\Theta)^2/2}} \frac{\mathcal{W}(\xi)}{\xi^{D+1}} d\xi \ll_{n, q, r} q^{(1+\Theta)^2/2}
$$

where the implied constant is taken to be the maximum of $|\mathcal{W}(\xi)|$ on the circle $|\xi| = q^{-(1+\Theta)^2/2}$. By applying the residue theorem, our assertion follows at once from (30) and the definition of $Q_n(s'; D, q)$. \hfill $\Box$

**Remark 4.** — It would be interesting to study the analytic properties of the generating series $\sum_{n \geq 1} Q_n(s'; D, q)$, with fixed $D$, as a function of $s_1, \ldots, s_r$, and $q$ (not necessarily a prime power).

In order to get the asymptotic formula at the center of the critical strip, we have to study in detail the behavior of the function $\mathcal{W}(s', \xi)$ in a neighborhood of a pole. To do so, fix $n \geq 1$, $\xi$ a $2n$-th root of 1, and recall that

$$
\prod_{\alpha \in \Phi_n} \left( 1 - q^{\alpha(d+1-2n\xi^2)} \xi^{2k+1} \right) \cdot \mathcal{W}(s', \xi)
$$

where, as before, $\alpha = \sum k_i \alpha_i$, is holomorphic in a neighborhood of the point $s_i = \frac{1}{2}$ $(i = 1, \ldots, r)$, and $\xi = q^{-\frac{n+1}{2}} \xi$; we can eliminate the unnecessary factors of the product, that is, those that do not vanish at this point, and consider instead the function

$$
\mathcal{W}_{n, \xi}(s', \xi) = \prod_{\alpha \in \Phi_n} \left( 1 - q^{\alpha(d+1-2n\xi^2)/2n} \xi^{-1} \right) \cdot \mathcal{W}(s', \xi). \quad (31)
$$

Let $s_i = s_i - \frac{1}{2}$, $u = \xi - q^{-\frac{n+1}{2}} \xi$, and consider the Weierstrass polynomial $W_{n, \xi}(s_i, \ldots, s_r, u)$ defined by

$$
W_{n, \xi}(s_i, \ldots, s_r, u) = \prod_{\alpha \in \Phi_n} \left( u - q^{\frac{2k_i + 1 + 2k_i s_i - n - 1}{2n}} \xi + q^{-\frac{n+1}{2}} \xi \right).
$$

By applying the Weierstrass division theorem (see, for example, [22]), there exist unique functions $R_{n, \xi}, S_{n, \xi}$, both holomorphic in a neighborhood of $0 \in \mathbb{C}^{r+1}$, and $R_{n, \xi}$ polynomial in $u$ with $\deg_u R_{n, \xi} < \deg_u W_{n, \xi}$, such that

$$
\mathcal{W}_{n, \xi}(s_1 + \frac{1}{2}, \ldots, s_r + \frac{1}{2}, u + q^{-\frac{n+1}{2}} \xi) = (S_{n, \xi} \cdot W_{n, \xi})(s_1, \ldots, s_r, u) + R_{n, \xi}(s_1, \ldots, s_r, u).
$$
Dividing this by the product in (31), it follows that we can express \( \mathcal{W}(s', \xi) \) as

\[
\mathcal{W}(s', \xi) = \frac{\tilde{S}_{n, \xi}(s', \xi)}{\prod_{\alpha \in \Phi_n} (1 - q^{(\beta(\alpha) + 1 - 2n(\alpha'))/2n} \xi^{-1})}
\]

with \( \tilde{S}_{n, \xi} \) and \( \tilde{R}_{n, \xi} \) holomorphic in a neighborhood of the point \((s', \xi) = (1, \ldots, 1, q^{-\frac{4n+1}{2}} \xi) \in \mathbb{C}^{r+1}\); the function \( \tilde{R}_{n, \xi}(s', \xi) \) is a polynomial in the variable \( \xi \) of degree smaller than \(|\Phi_n|\).

We have the following:

**Proposition 6.2.** — For \( n \geq 1 \) and \( \xi \) a 2n-th root of 1, let \( \mathcal{S}_{n, \xi}(s', \xi) \) be defined by

\[
\mathcal{S}_{n, \xi}(s', \xi) = \frac{1}{n!} \sum_{\alpha \in \Phi_n} 2^{-\|w_0\|} \prod_{\alpha} (1, \alpha; \xi^{-1}) S_a(s', \xi^{-1})
\]

with \( a = 1 \) or \( \Theta_0 \) according as \( \xi^n = 1 \) or \( -1 \). Then, for \( s' = (s_1, \ldots, s_r) \) in a neighborhood of \((\frac{1}{2}, \ldots, \frac{1}{2}) \in \mathbb{C}^r\), and \( \xi \neq q^{(2n(\alpha') - d(\alpha) - 1)/2n} \xi' \) for all \( \alpha \in \Phi_n \), we have

\[
\mathcal{S}_{n, \xi}(s', \xi) = \frac{\tilde{R}_{n, \xi}(s', \xi)}{\prod_{\alpha \in \Phi_n} (1 - q^{(\beta(\alpha) + 1 - 2n(\alpha'))/2n} \xi^{-1})}.
\]

In particular, the function \( \mathcal{S}_n(q^{-\frac{2n+1}{2}}, \ldots, q^{-\frac{n+1}{2}}, 1) \) can be analytically continued for \( s' = (s_1, \ldots, s_r) \) in a neighborhood of \((\frac{1}{2}, \ldots, \frac{1}{2}) \), and \( \xi \neq q^{(2n(\alpha') - d(\alpha) - 1)/2n} \xi' \) with \( \alpha \in \Phi_n \), and \( \xi \) any 2n-th root of 1.

**Proof.** Let \( D_{1/2} \) be a polydisk centered at \( \frac{1}{2} := (\frac{1}{2}, \ldots, \frac{1}{2}) \in \mathbb{C}^r \) (resp. a disk centered at \( q^{-\frac{4n+1}{2}} \xi \)) such that (32) holds on \( V := D_{1/2} \times D_{n, \xi} \) (away from the singularities of the right-hand side), and \( q^{(2n(\alpha') - d(\alpha) - 1)/2n} \xi \in D_{n, \xi} \) for every \( s' \in D_{1/2} \), and all \( \alpha \in \Phi_n \). Let \( a_1, \ldots, a_r \) be \( \mathbb{Q} \)-linearly independent complex numbers, and put \( a := (a_1, \ldots, a_r) \). If \( p \) is a sufficiently small positive number, then \( s'_p := \frac{1}{2} + pa \in D_{1/2} \), and the numbers \( q^{(\beta(\alpha) - 2n(\alpha'))/2n} \) are mutually distinct; thus, by continuity, this property holds throughout a polydisk \( D_{s'_p} \subset D_{1/2} \) centered at \( s'_p \).

Now, for each \( s' \in D_{s'_p} \), we can write (via a partial fraction decomposition in \( \xi \))

\[
\tilde{R}_{n, \xi}(s', \xi) = \sum_{\alpha \in \Phi_n} \mathcal{C}_{n, \xi; a}(s') \prod_{\alpha} (1 - q^{(\beta(\alpha) + 1 - 2n(\alpha'))/2n} \xi^{-1})
\]

for \( \xi \neq q^{(2n(\alpha') - d(\alpha) - 1)/2n} \xi' \). Substituting this into (32), we find that, for each \( \alpha \in \Phi_n \),

\[
\mathcal{C}_{n, \xi; a}(s') = \lim_{\xi \to q^{(2n(\alpha') - d(\alpha) - 1)/2n} \xi'} (1 - q^{(\beta(\alpha) + 1 - 2n(\alpha'))/2n} \xi^{-1}) \mathcal{W}(s', \xi) = \frac{\Gamma_{n}(1, \alpha; \xi^{-1})}{2^{|w_0|}n!} S_a(s', \xi^{-1}).
\]

This gives our first assertion when \( s' \in D_{s'_p} \); it extends to \( D_{1/2} \) by analytic continuation.

We now have

\[
\mathcal{S}_n(q^{-\frac{2n+1}{2}}, \ldots, q^{-\frac{n+1}{2}}, 1) = \sum_{\xi} \tilde{R}_{n, \xi}(s', \xi) \prod_{\alpha \in \Phi_n} (1 - q^{(\beta(\alpha) + 1 - 2n(\alpha'))/2n} \xi^{-1})
\]

which proves the second assertion, and completes the proof.

**Theorem 6.3.** — Let \( D, N \geq 1 \) be integers, and suppose that \( r \geq 4 \). Then, under the Assumption (H1), and Conjectures 4.1, 4.3, we have

\[
\sum_{d \text{--monic and sq. free}} L\left(\frac{1}{2}, \chi_d\right)^r = \sum_{n \leq N} Q_n(D, q) q^{(\frac{4r+1}{4})D} + O_{r,q}(q^{D(1+\Theta)/2})
\]
for any $(N+1)^{-1} < \Theta < N^{-1}$, where
\[
Q_n(D, q)q^{(\frac{1}{2} - \frac{1}{2})D} = \sum_{\xi \in \Sigma_n, \xi = q^{-\Theta}/\xi} \text{Res} \quad \Sigma_n, \xi \left( \frac{1}{2}, \xi \right) \xi^{-D-1}
\]
for $n \geq 1$.

**Proof.** Our assumptions imply that, for every small positive $\epsilon$, the function $W(\xi) = W(\frac{1}{2}, \xi)$ is meromorphic in the disk $|\xi| < q^{-1/2 + \epsilon}$ with the only possible poles at $\xi = q^{-\frac{\Theta + 1}{2}}$, $n = 1, 2, \ldots$, and for each $n$, $\xi$ is any $2n$-th root of 1. By Proposition 6.2, the principal part of $W(\xi)$ at a pole $\xi = q^{-\frac{\Theta + 1}{2}}$ is given by
\[
\frac{\tilde{R}_n(\frac{1}{2}, \xi)}{(1 - q^{-\Theta + 1}/\xi)} = \Sigma_n, \xi(\frac{1}{2}, \xi).
\]
From here on, the argument proceeds exactly as in the proof of Theorem 6.1. □

**Remark 5.** — The order of the pole of the rational function $\Sigma_n, \xi(\frac{1}{2}, \xi)$ at $\xi = q^{-\frac{\Theta + 1}{2}}$ may be smaller than the maximal limit $|\Phi_n|$, and this happens, for example, when $n = 1$. While, for any given $n \geq 1$, the function $\Sigma_n, \xi(s', \xi)$ (and hence the coefficient $Q_n(D, q)$ in the above asymptotic formula) can, in principle, be computed explicitly, we do not have any prediction, at the moment, for what the order of the pole of $\Sigma_n, \xi(\frac{1}{2}, \xi)$ at $\xi = q^{-\frac{\Theta + 1}{2}}$ should, in general, be.

## 7 Some explicit computations

### 7.1 The case $n = 1$

By Lemma 2.2, each $\alpha \in \Phi_1$ is of the form
\[
\alpha = \sum_{i=1}^r k_i \alpha_i + \alpha_{r+1}
\]
with $k_i = 0$ or 1 for all $i = 1, \ldots, r$. Conversely, any element $\alpha_i + \alpha_{r+1}$ is a root, since
\[
\prod_{i=1}^r w_i(\alpha_{r+1}) = \alpha_i + \alpha_{r+1};
\]
the subgroup $W_0 := \langle w_i : i = 1, \ldots, r \rangle$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$. Taking $w$ to be the identity, we find by (27) that $I_{1, p}^{(1)} = \text{sgn}(a)^{\deg p}$, $I_{1, p}^{(2)} = 0$ and $I_{1, p}^{(3)} = 1$. Thus we get:
\[
S_{p}^{(1)} = \left(1 - \frac{1}{|p|}\right) \left(1 + \frac{1}{2 \prod_{k=1}^{r} (1 - |p|^{-k})}\right).
\]
Similarly, by (26),
\[
\Gamma_{ia}(a_2, a; \xi) = \varepsilon^+(a_2)\varepsilon^-(a) + \varepsilon^+(a_2)\varepsilon^-(a).
\]
Summing now over $a$, we find that the contribution to $\Sigma_{1}(\cdot, a_2)$ corresponding to $\alpha = \alpha_{r+1}$ is
\[
\left(\frac{\varepsilon^+(a_2)}{1 - q^{-\Theta + 1}} + \frac{\varepsilon^-(a_2)}{1 + q^{-\Theta + 1}}\right) \cdot \prod_p \left(1 - \frac{1}{|p|}\right) \left(1 + \frac{1}{2 \prod_{k=1}^{r} (1 - |p|^{-k})}\right).
\]
To compute the other contributions, write $q^{-\Theta} = q^{-1/2} \xi_k$ ($k = 1, \ldots, r$), and put $\xi = q^{-\Theta + 1}$. Set
\[
S_p(\xi_1, \ldots, \xi_r) := \left(1 - \frac{1}{|p|}\right) \left(1 + \frac{1}{2 \prod_{k=1}^{r} (1 - (q^{-1/2} \xi_k)^\deg p)^{-1} + \prod_{k=1}^{r} (1 + (q^{-1/2} \xi_k)^\deg p)^{-1}}\right).
\]

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Let $\alpha = \sum_{k} a_k + a_{r+1}$ ($k = 0$ or 1) be a root in $\Phi_1$, and take $w = w_\alpha = w^k_1 \cdots w^k_r$; then $M_{w^{-1}}(wz; q) = M_w(z; q)^{-1}$ is a diagonal matrix. By (27) and (28), we have

$$S^w_p = \left(1 - \frac{1}{|p|}\right) \left(D^{(1)}_{w,p}(\text{sgn}(a) \xi_1)^{\deg p} + \frac{\alpha}{2 \prod_{k=1}^{r} (1 - (q^{-1/2} \xi_k)^{\deg p})} + \frac{\beta}{2 \prod_{k=1}^{r} (1 + (q^{-1/2} \xi_k)^{\deg p})}\right)_{\xi_k = \sqrt{qz}, \frac{qz}{1 + qz}}$$

where $M_w((q^{-1/2} \xi_1)^{\deg p}, \ldots, (q^{-1/2} \xi_r)^{\deg p}; p|p|) = \text{diag}(D^{(1)}_{w,p}, D^{(2)}_{w,p}, D^{(3)}_{w,p})$. This diagonal matrix can be easily computed by using the cocycle relation (10) and induction on the length of $w$; we have $M_w(z; q)^{-1} = \prod_i M_w(z; q)^{-k_i}$, and by (9),

$$M_w(z; q)^{-1} = \text{diag}\left(-\frac{qz_1(1-z_1)}{1-qz_1}, \sqrt{qz_1}, \frac{qz_2(1+z_2)}{1+qz_2}\right) \quad (i = 1, \ldots, r).$$

Accordingly, $S^w_p(\xi_1, \ldots, \xi_r) = S_p(\xi^{\epsilon_1}_1, \ldots, \xi^{\epsilon_r}_r)$, where $\epsilon_i = 1 - 2k_i$. We also have

$$2^{-\epsilon_0} \Gamma_w(a_2, a; \xi_1) = (1, \text{sgn}(a_2), 0) \cdot M_w\left(q^{1/2} \xi_1, \ldots, q^{1/2} \xi_r, 1/q\right) \cdot (1/2, \text{sgn}(a)/2, 1/2)$$

$$= \frac{1}{2} \prod_{i=1}^{r} \left(1 - \frac{q^{1/2} \xi_1^{1-\epsilon}}{1 - q^{1/2} \xi_i^{1-\epsilon}}\right)^{(1-\epsilon)/2} + \frac{1}{2} \text{sgn}(a) \text{sgn}(a_2) \prod_{i=1}^{r} \xi_i^{(1-\epsilon)/2}.$$ 

The product $\prod_p S_p(\xi_1, \ldots, \xi_r)$ does not converge when $|\xi_k| = 1$, but it has meromorphic continuation. Indeed, letting $A_p(\xi_1, \ldots, \xi_r)$ be defined by

$$A_p(\xi_1, \ldots, \xi_r) = \prod_{1 < i, j < r} \left(1 - \frac{(\xi_i \xi_j)}{|p|}\right) \cdot \left(1 - \frac{2 + \prod_{k=1}^{r} (1 - (q^{-1/2} \xi_k)^{\deg p})^{-1} + \prod_{k=1}^{r} (1 + (q^{-1/2} \xi_k)^{\deg p})^{-1}}{2 (1 + |p|)}\right)$$

it is easy to see that the product $\prod_p A_p(\xi_1, \ldots, \xi_r)$ is absolutely convergent in the polydisk $|\xi_k| < q^{1/2}(k = 1, \ldots, r)$, with sufficiently small positive $\epsilon$. It is clear that

$$\prod_p S_p(\xi_1, \ldots, \xi_r) = \frac{1}{\zeta(2)} \prod_{1 < i, j < r} \left(1 - \frac{\xi_i \xi_j}{q}\right)^{-1} \cdot \prod_p A_p(\xi_1, \ldots, \xi_r).$$

Thus if we write the right-hand side of this as $\zeta(2)^{-1} G(\xi_1, \ldots, \xi_r)$, then we can express

$$\partial_1 = \frac{1}{\zeta(2)} \sum_{k=1}^{r} \prod_{i=1}^{r} \left(1 - \frac{q^{1/2} \xi_1^{1-\epsilon}}{1 - q^{1/2} \xi_i^{1-\epsilon}}\right)^{(1-\epsilon)/2} \cdot \frac{G(\xi^{\epsilon_1}_1, \ldots, \xi^{\epsilon_r}_r)}{1 - q^{1/2} \xi_1^{1-\epsilon} \cdots \xi_r^{1-\epsilon} \xi^2}$$

$$+ \frac{(q^{1/2} \xi_1)^{\deg p}}{\zeta(2)} \sum_{\epsilon_i = \pm 1} \frac{G(\xi^{\epsilon_1}_1, \ldots, \xi^{\epsilon_r}_r)}{1 - q^{1/2} \xi_1^{1-\epsilon} \cdots \xi_r^{1-\epsilon} \xi^2}.$$ 

It turns out that this function can be defined and is analytic in an entire neighborhood of $|\xi_k| = 1 (k = 1, \ldots, r)$. To see this, we need the following lemma – a variant of [10, Lemma 2.5.2].

**Lemma 7.1.** — Let $a_1, \ldots, a_r$ be distinct non-zero complex numbers such that $a_i a_j \neq 1$ for all $1 \leq i, j \leq r$. Suppose $h$ is a symmetric function of $r$ variables, holomorphic on a domain containing $\left(a_1^{\delta_i}, \ldots, a_r^{\delta_i}\right)$ for all $(\delta_i, \ldots, \delta_r) \in \{\pm 1\}^r$. Then

$$\sum_{\delta_i = \pm 1} \frac{h(a_1^{\delta_1}, \ldots, a_r^{\delta_r})}{\prod_{1 < i, j < r} (1 - a_i^{\delta_i} a_j^{\delta_j})} = (-1)^{r(r+1)/2} \prod_{i=1}^{r} \frac{1}{2 \sqrt{\pi}} \int \cdots \int h(z_1, \ldots, z_r) \prod_{1 < i, j < r} (1 - z_i a_j) dz_1 \cdots dz_r$$

where each path of integration encloses the $a_i^{\pm 1}$, but not $z_i = 0$. 

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Proof. The right-hand side is the sum of the residues at the poles (with non-zero coordinates) of the integrand.

Fix a pole at $z_i = a_i^{-1}$ for $i = 1, \ldots, r$. We cannot have $j_i = j_{i'}$ for some $i < i'$, as the factor $(z_i - z^*)^2(1 - z_i z^*)$ in the numerator vanishes when $z_i = a_i^{-1}$ and $z^* = a_{i'}^{-1}$. Thus the assignment $\sigma : i \mapsto j_i$ is an element of the symmetric group $\mathfrak{S}_r$. If we denote the exponent $\{ \pm 1 \}$ of $a_i = a_{\sigma(i)}$ by $\delta(i)$, it follows (upon replacing $z$ by $z_{i'}^{-1}$) that the residue of the integrand at the pole $(a_{\delta(i)}, \ldots, a_{\delta(r)})$ is

$$\frac{h(a_1^{-1}, \ldots, a_r^{-1})}{\prod_{j=1}^r a_j^{-1}} \cdot \prod_{1 \leq i < j \leq r} \frac{(a_i - a_j)^2}{(1 - a_i a_j)} \prod_{k=1}^r \left( \frac{-a_i^2}{1 - a_i^2 a_k^{-1}} \right).$$

For $i < j$, we have

$$\frac{(a_i^2 - a_j^2)}{(1 - a_i a_j)(1 - a_i a_j^{-1})(1 - a_i a_j^{-1})} = -\frac{a_i a_j^{-1}}{1 - a_i a_j^{-1}},$$

and the equality in the lemma follows.

To apply the formula in the lemma, note that in (33) we can express:

$$\prod_{i=1}^r \left( 1 - q^{1/2} \xi_i^{-1} \right)^{-1/2} = \prod_{i=1}^r \frac{1 - q^{1/2} \xi_i^{-1}}{1 - q^{1/2} \xi_i} \quad (\text{for } \xi_i \in \{ \pm 1 \}, i = 1, \ldots, r).$$

Thus if we let

$$\mathcal{H}_1(\xi_1, \ldots, \xi_r, u) = \prod_{i=1}^r \frac{(1 - q^{1/2} \xi_i)}{1 - u \xi_i^{-1}} \cdot A(\xi_1, \ldots, \xi_r),$$

$$\mathcal{H}_2(\xi_1, \ldots, \xi_r, u) = \frac{A(\xi_1, \ldots, \xi_r)}{1 - u \xi_1 \cdots \xi_r},$$

with $A(\xi_1, \ldots, \xi_r) := \prod_{p} A_p(\xi_1, \ldots, \xi_r)$, then we can write

$$\mathfrak{S}_1 = \frac{1}{\xi(2)} \prod_{i=1}^r \left( 1 - q^{1/2} \xi_i \right)^{-1} \cdot \sum_{\xi_1 = \pm 1} \mathcal{H}_1(\xi_1^2, \ldots, \xi_r^2, q^{1/2} \xi_1 \xi_2 \cdots \xi_r) \frac{1}{\prod_{1 \leq i < j \leq r} (1 - q^{1/2} \xi_i \xi_j)}.$$

The sum in the right-hand side can be expressed, using Lemma 7.1, as a multiple integral, and note that the integrand is defined and holomorphic in a neighborhood of $\xi_1 = \cdots = \xi_r = 1$. Thus for each $D \in \mathbb{N}$, the coefficient $Q(D, q)$ can be obtained from the integral

$$\frac{1}{2\pi i} \int_{|\xi| = q^{-2}} \mathfrak{S}_1(q^{-1/2}, \ldots, q^{-1/2}, 1) \frac{d\xi}{\xi_{D+1}}.$$

see for comparison [2, Conjecture 5].

7.2 The case $n = 2$

First the set $\Phi_2$ is computed as follows. Let $\alpha \in \Phi_2$ satisfy the condition $(\ast)$. Then the coefficients of $\alpha$ also satisfy the inequality $(\ast \ast)$, i.e.

$$\sum_{j=1}^r k_j \leq 3.$$
It follows that at most three $k_j$ ($1 \leq j \leq r$) can be non-zero, and hence by (6), $w_{r+1}(\alpha) = \alpha_j + \alpha_{j'}$ (with $1 \leq j_1 < j_2 \leq r$), or $w_{r+1}(\alpha) = \alpha_j + \alpha_{j'} + \alpha_{r+1}$ for some $1 \leq j_1 < j_2 < j_3 \leq r$. However, $w_{r+1}(\alpha_j + \alpha_{j'}) = -\alpha_j + \alpha_{j'} \not\in \Delta$, and so
\[ \alpha = \alpha_j + \alpha_{j'} + \alpha_{r+1}. \]

It is now clear that a positive real root $\alpha$ is in $\Phi_2$ if and only if it is of the form
\[ \alpha = \sum_{j \neq j_1, j_2, j_3} k_j \alpha_j + \alpha_{j_1} + \alpha_{j_2} + 2\alpha_{r+1} \]
for some $1 \leq j_1 < j_2 < j_3 \leq r$, and with $k_j \in \{0, 2\}$ for $j \neq j_1, j_2, j_3$.

For $\alpha \in \Phi_2$, we take the Weyl group element sending $\alpha$ to $\alpha_{r+1}$ to be
\[ w_\alpha := w_j w_{j'} w_{r+1} \prod_{k=1}^r w_j. \]

This element is in reduced form, since the set of positive real roots sent by $w_\alpha$ to negative real roots is
\[ \{ \alpha_j \}_{j \neq j_1, j_2, j_3} \cup \{ \beta, \beta + \alpha_1, \beta + \alpha_2, \beta + \alpha_3 \} \]
where $\beta = \sum \alpha_j + \alpha_{r+1}$, the sum being over all $j \leq r$ such that $k_j = 2$.

To compute $\Sigma_2(z, a_2)$, consider first the case $r = 3$. Thus $\Phi_2 = \{ \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 \}$, and $w_\alpha = w_1 w_2 w_3 w_4$. Let $a \in \{1, \theta_0\}$, and fix $\zeta_0$ such that $\zeta_0^2 = \text{sgn}(a)$. We shall compute
\[ \frac{1}{2^5} \frac{\Gamma_\alpha(a_2, a_1; \zeta_0)}{1 - \zeta_0 q^{1/2} \zeta_0 p} \prod_{\beta} S_{\beta}^\alpha (z, \zeta_0). \]

Using the cocycle relation (10), we have
\[ M_{w_\alpha} (w_\alpha z) = M_{w_\alpha} (z)^{-1} = M_{w_1 w_2 w_3} (w_4 z)^{-1} M_{w_4} (z)^{-1}. \]

Writing $M_{w_1 w_2 w_3} (w_4 z)^{-1} = \text{diag}(d_1, d_2, d_3)$ and $M_{w_4} (z)^{-1} = U \text{diag}(e_1, e_2, e_3) U^*$, it follows from (27) that
\[ L_{w_{1\ldots4}, p}^{(1)} = \frac{1}{2} e_1(d_1 + d_2 + 2 \text{sgn}(a)d_3) - \frac{1}{2} e_1(d_1 - d_2 + 2 \text{sgn}(a)d_3), \]
\[ L_{w_{1\ldots4}, p}^{(2)} = \frac{1}{2} e_1(d_1 + d_2 + 2 \text{sgn}(a)d_3) + \frac{1}{2} e_1(d_1 - d_2 + 2 \text{sgn}(a)d_3) \]
\[ (35) \]
\[ L_{w_{1\ldots4}, p}^{(3)} = \frac{1}{2} e_1(d_1 + d_2 + 2 \text{sgn}(a)d_3) + \frac{1}{2} e_1(d_1 - d_2 + 2 \text{sgn}(a)d_3) - \frac{1}{2} e_1(d_1 - d_2). \]

Here $p$ is a monic linear polynomial. Setting $z_k = q^{-1/2 \zeta_0^k}$ ($k = 1, 2, 3$, and hence $z_k = (q^{-1/2 \zeta_0^k})^{-1}$, we have explicitly
\[ d_1 = \frac{(1 - \zeta_0^1 \zeta_0^2 \zeta_0^3)}{(1 - \zeta_0^1 \zeta_0^2 \zeta_0^3)} \frac{(1 - \zeta_0^2 \zeta_0^3 \zeta_0^1)}{(1 - \zeta_0^2 \zeta_0^3 \zeta_0^1)} \frac{(1 - \zeta_0^3 \zeta_0^1 \zeta_0^2)}{(1 - \zeta_0^3 \zeta_0^1 \zeta_0^2)} \]
\[ d_2 = \frac{(1 + \zeta_0^1 \zeta_0^2 \zeta_0^3)}{(1 + \zeta_0^1 \zeta_0^2 \zeta_0^3)} \frac{(1 + \zeta_0^2 \zeta_0^3 \zeta_0^1)}{(1 + \zeta_0^2 \zeta_0^3 \zeta_0^1)} \frac{(1 + \zeta_0^3 \zeta_0^1 \zeta_0^2)}{(1 + \zeta_0^3 \zeta_0^1 \zeta_0^2)} \]
\[ (36) \]
and
\[ e_1 = \frac{1 - (\zeta_0 q^{1/2 \zeta_0^1 \zeta_0^2 \zeta_0^3})^{-1}}{1 - \zeta_0 q^{-1/2 \zeta_0^1 \zeta_0^2 \zeta_0^3}} \]
\[ e_2 = (\zeta_0 q^{1/2 \zeta_0^1 \zeta_0^2 \zeta_0^3})^{-1} \]
\[ e_3 = \frac{1 + (\zeta_0 q^{1/2 \zeta_0^1 \zeta_0^2 \zeta_0^3})^{-1}}{1 - \zeta_0 q^{-1/2 \zeta_0^1 \zeta_0^2 \zeta_0^3}}. \]
\[ (37) \]
Thus, by (28), we find that

\[
S_p^{\text{w,a}} = \frac{1 - \text{sgn}(a) \ell_0(z_1^{a_1}, z_2^{a_2}, z_3^{a_3}) q^{-3/2} \cdots - \ell_2(z_1^{a_1}, z_2^{a_2}, z_3^{a_3}) q^{-6} + q^{-1}}{\left(1 - \frac{\text{sgn}(a) z_2^{a_2} z_3^{a_3}}{\sqrt{q}}\right) \prod_{1 \leq i \leq s} \left(1 - \frac{\text{sgn}(a) z_i^{a_i}}{\sqrt{q}}\right) \prod_{1 \leq i < j \leq s} \left(1 - \frac{\text{sgn}(a) z_i^{a_i} z_j^{a_j}}{\sqrt{q^2}}\right)}
\]

where \(\ell_0(z_1^{a_1}, z_2^{a_2}, z_3^{a_3})\) \((n = 3, \ldots, 12)\) are symmetric polynomials in the variables \(z_1^{a_1}, z_2^{a_2}, z_3^{a_3}\) (i.e., invariant under the hyperoctahedral group \(S_2 \times S_3\)) with positive integral coefficients. We regularize \(S_p^{\text{w,a}}\) by taking its numerator, which we shall denote by \(S_p^{\text{reg}}\). Note that the denominator of \(S_p^{\text{w,a}}\) is \(R_p\left(\frac{z_1 z_2 z_3}{q^{1/2}}\right)^{-1}\), where

\[
R_p(z_1, z_2, z_3) = \left(1 - q z_1 z_2 z_3\right)^{-1} \prod_{1 \leq i < j \leq 3} (1 - z_i z_j)^{-1}
\]

is the local factor of the “modified” residue at \(z_4 = 1/q\) of the function \(Z(z_1, z_2, z_3, z_4) := Z^{(1)}(s; \chi_0, \chi_1)\), see [11, Section 4, eqn (20)].

**Remark 6.** — It is easy to check that in fact the functions

\[
\left(\frac{L_{\text{w,a}}^{(1)}}{q^{1/4} z_1^{a_1} z_2^{a_2} z_3^{a_3}}\right) R_p\left(\frac{z_1 z_2 z_3}{q^{1/2}}\right) \quad \text{and} \quad \frac{L_{\text{w,a}}^{(1)} - L_{\text{w,a}}^{(2)}}{2 \prod_{j=1}^3 (1 - q^{-1/2 z_j})} \cdot R_p\left(\frac{z_1 z_2 z_3}{q^{1/2}}\right)
\]

computed using (35) – (38), are symmetric in the variables \(z_1^{a_1}, z_2^{a_2}, z_3^{a_3}\). These symmetries will be used in an essential way later in this section.

Now when \(r = 3\), the function \(Z^{(r)}(s; \chi_0, \chi_1)\) is meromorphic on all of \(\mathbb{C}^4\). Using Proposition 5.1, it follows (as in [11]) that the product in (34) is

\[
\prod_{\bar{p}} S_{\bar{p}}^{\text{reg}} = \left(1 - \text{sgn}(a) \sqrt{\frac{q^{1/2} z_1^{a_1} z_2^{a_2} z_3^{a_3}}{z_4}}\right)^{-1} \prod_{1 \leq i < j \leq 3} \left(1 - \frac{\text{sgn}(a) z_i^{a_i} z_j^{a_j}}{\sqrt{q^2}}\right)^{-1} \prod_{\bar{p}} S_{\bar{p}}^{\text{reg}}
\]

where, for a general monic irreducible, \(S_{\bar{p}}^{\text{reg}}\) is obtained by replacing \(\text{sgn}(a) \rightarrow \chi(p)\), \(z_4 \rightarrow q^{\deg p}\), and \(q \rightarrow |p|\). Note that the product over \(\bar{p}\) in the right-hand side converges only in a neighborhood of \(\xi_k = 1\) \((k = 1, 2, 3)\). When \(\xi_1 = \xi_2 = \xi_3\), we have \(S_{\bar{p}}^{\text{reg}} = P\left(\chi(p) \xi_2^{2 \deg p}, |p|^{-1/2}\right)\), with \(P(x, y)\) given by

\[
P(x, y) = (1 - y^2)(1 - xy)(1 - x^{-1} y)
\]

\[
\left[1 + (x + x^{-1})y + (x + x^{-1})^2 y^2 - 4(x + x^{-1})^2 y^3 - 5(x + x^{-1})^3 y^4 + (x + x^{-1})^4 (3 x - x^{-1})(x^{-1} - 3 x) y^5 - (x + x^{-1})^2 (7 + 3 x^2 + 3 x^{-2}) y^7 + (8 + 5 x^2 + 5 x^{-2})^8 - (x + x^{-1})^3 y^9 - y^{10}\right].
\]

In particular,

\[
P(1, y) = (1 - y)^2(1 + y)(1 + 4y + 11y^2 + 10y^3 - 11y^4 + 11y^6 - 4y^7 - y^8)
\]

which is precisely the polynomial occurring in [43] and [11].

By (26), the remaining part of (34) is given by

\[
2^{-5} T_{\text{w}}(a_2; a; \xi_0) = \left[\frac{\psi(z_1, z_2, z_3; q) + \psi(z_1 + z_2, z_3; -z_3; q)}{4 q^{1/2 z_1 z_2 z_3}} + \frac{\text{sgn}(a_2) (\psi(z_1, z_2, z_3; q) - \phi(-z_1, z_2, z_3; -z_3; q))}{4 q^{1/2 z_1 z_2 z_3}} - \frac{\text{sgn}(a)}{2 q^{z_1 z_2 z_3}} \left(1 - \text{sgn}(a_2) q z_4\right)\right]_{z_1 = \frac{\xi_0}{\sqrt{q}}, z_2 = \frac{\xi_0}{\sqrt{q}}, z_3 = \frac{\xi_0}{\sqrt{q}}, z_4 = \frac{1}{\sqrt{q z_0(\xi_0) q}}}
\]

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where the functions $\phi(z_1, z_2, z_3, z_4; q)$ and $\psi(z_1, z_2, z_3, z_4; q)$ are given by

$$
\phi(z_1, z_2, z_3, z_4; q) = \frac{1 - \sqrt{q} z_1 z_4}{1 - \sqrt{q} z_2 z_3} \frac{1 - \sqrt{q} z_1 z_4}{1 - \sqrt{q} z_2 z_3} \frac{1 - q^{1/2} z_1 z_4}{1 - q^{1/2} z_2 z_3} \frac{1 - q^{1/2} z_1 z_4}{1 - q^{1/2} z_2 z_3},
$$

$$
\psi(z_1, z_2, z_3, z_4; q) = -\frac{(1 - q^{3/2} z_4)\phi(z_1, z_2, z_3, z_4; q)}{\sqrt{q - q^2 z_4}}.
$$

(39)

For $\xi_1 = \xi_2 = \xi_3 = 1$, one obtains, up to a factor of $\frac{1}{q}$, the values given in [11, Table1, p. 20].

**The case** $r \geq 4$. We fix a root $\alpha \in \Phi_2$,

$$
\alpha = \alpha_t + \alpha_s + 2\sum_{j \in J} \alpha_j + 2\alpha_{r+1} \quad (\text{where } J \subseteq \{4, \ldots, r\}).
$$

Write $\alpha = \alpha' + \alpha$, where $\alpha' := \alpha_t + \alpha_s + 2\alpha_{r+1} \in \Phi_2$. Then $w_\alpha = w_{\alpha'}w_j$, where $w_j := \prod_{j \in J} w_j$. Applying (10), we see that $M_{w_\alpha}(w_{\alpha'}) = M_{w_{\alpha'}}(w_j)M_{w_j}(z)^{-1}$. Thus, if we let $M_{w_j}(z)^{-1} = \text{diag}(b_1, b_2, b_3, \ldots, b_{r+1})$, and $p$ a monic linear polynomial, then, by (27), we have

$$
L^{(1)}_{w_{\alpha'}, p} = b_2 L^{(1)}_{w_{\alpha'}}, p, \quad L^{(2)}_{w_{\alpha'}, p} = b_1 (L^{(2)}_{w_{\alpha'}}, p + L^{(3)}_{w_{\alpha'}}, p), \quad L^{(3)}_{w_{\alpha'}, p} = L^{(3)}_{w_{\alpha'}}, p
$$

with $L^{(i)}_{w_{\alpha'}, p} (i = 1, 2, 3)$ computed by using (35), (36) and (37). Indeed, the functions $L^{(i)}_{w_{\alpha'}, p}$ are evaluated at

$$
q^{1/2} z_j \equiv w_j(z_1, z_2, z_3, z_{r+1}) = \left( z_1, z_2, z_3, z_{r+1}, \prod_{j \in J} q^{1/2} z_j \right)
$$

subject to the condition that $z^{1/2} = \xi_\alpha^{-1} q^{-(1/4)}$. Accordingly, if we set $z_k = q^{-1/2} \xi_k (k = 1, \ldots, r)$, then we can write

$$
z_{r+1} \prod_{j \in J} q^{1/2} z_j = \frac{1}{q^{-1/4} \xi_1 \xi_2 \xi_3}.
$$

It now follows from (28) that

$$
S_p^{w_\alpha} = (1 - q^{-1}) \left[ \frac{b_2 L^{(1)}_{w_{\alpha'}}, p}{q^{1/4} \xi_1 \xi_2 \xi_3 \prod_{j \in J} \xi_j} + \frac{b_1 (L^{(2)}_{w_{\alpha'}}, p + L^{(3)}_{w_{\alpha'}}, p)}{2 \prod_{j = 1} (1 - q^{-1/2} \xi_j)} + \frac{b_3 (L^{(3)}_{w_{\alpha'}}, p - L^{(2)}_{w_{\alpha'}}, p)}{2 \prod_{j = 1} (1 + q^{-1/2} \xi_j)} \right]
$$

(40)

where the entries of the diagonal matrix $M_{w_\alpha}(z)^{-1}$ are given by

$$
b_1 = \prod_{j \in J} \left( 1 - \frac{1 - q^{-1/2} \xi_j}{1 - q^{-1/2} \xi_j} \right), \quad b_2 = \prod_{j \in J} \xi_j, \quad \text{and} \quad b_3 = \prod_{j \in J} \left( 1 + \frac{1 - q^{-1/2} \xi_j}{1 + q^{-1/2} \xi_j} \right).
$$

Note that the first term in $S_p^{w_\alpha}$ is

$$
\frac{L^{(1)}_{w_{\alpha'}}, p (q^{-1/2} \xi_1, q^{-1/2} \xi_2, q^{-1/2} \xi_3, q^{-1/2} \xi_4; q)}{q^{1/4} \xi_1 \xi_2 \xi_3 \xi_4} = \frac{L^{(1)}_{w_{\alpha'}}, p (q^{-1/2} \xi_1, q^{-1/2} \xi_2, q^{-1/2} \xi_3, q^{-1/2} \xi_4; q)}{q^{1/4} \xi_1 \xi_2 \xi_3 \xi_4} \cdot \prod_{j \in J} \xi_j.
$$

Moreover, $S_p^{w_{\alpha'}}$ is obtained by replacing $\xi_j \mapsto \xi_j^{-1}$ (for $j \in J$) in $S_p^{w_\alpha}$. We regularize $S_p^{w_{\alpha'}}$ by setting

$$
S_p^{w_{\alpha'}} := S_p^{w_{\alpha'}} R_p \left( \frac{\xi_1 \xi_2 \xi_3 \xi_4}{q^{1/4} \xi_1 \xi_2 \xi_3 \xi_4} \right) \left( \frac{\xi_1 \xi_2 \xi_3 \xi_4}{q^{1/4} \xi_1 \xi_2 \xi_3 \xi_4} \right) \left( 1 - \frac{\xi_1 \xi_2 \xi_3 \xi_4}{q} \right) \left( 1 - \frac{\xi_1 \xi_2 \xi_3 \xi_4}{q} \right) \left( 1 - \frac{\xi_1 \xi_2 \xi_3 \xi_4}{q} \right)
$$

(41)
where $R_p^{(3)}(z_1, z_2, z_3) = R_p(z_1, z_2, z_3)$ is the local factor (38). This is a polynomial in $q^{-1/2}$ of the form

$$S_p^{\text{reg}} = 1 + O\left(q^{-\frac{3}{2}}\right).$$

As before, we define $S_p^{\text{reg}}$, for arbitrary $p$, by simply substituting $\text{sgn}(a) \mapsto \chi_a(p)$, $\xi \mapsto \xi^{\deg p}$, and $q \mapsto |p|$ in the above formula. The product $\prod_p S_p^{\text{reg}}$ converges absolutely, as long as each of the first three variables $\xi_1, \xi_2, \xi_3$ is in a small neighborhood of the unit circle, and $|\xi_k| < q^\varepsilon$ ($k = 1, \ldots, r$) for sufficiently small positive $\varepsilon$. Let $N_p(\xi, \chi_a)$, where $\xi := (\xi_1, \ldots, \xi_r)$, be defined by

$$N_p(\xi, \chi_a) = R_p^{(3)} \left( \left( \frac{\xi_1^{\deg p}}{|p|^{1/2}}, \frac{\xi_2^{\deg p}}{|p|^{1/2}}, \frac{\xi_3^{\deg p}}{|p|^{1/2}} \right) \right) \cdot \prod_{i=1}^3 \prod_{j=1}^r \left( 1 - \frac{\xi_i^{2\deg p} \xi_j^{2\deg p}}{|p|} \right)^{-1} \cdot \prod_{4 \leq k \leq l \leq r} \left( 1 - \frac{\xi_k^{2\deg p} \xi_l^{2\deg p}}{|p|} \right)^{-1},$$

and set $N(\xi, \chi_a) := \prod_p N_p(\xi, \chi_a)$. Thus

$$N(\xi, \chi_a) = \left( 1 - \text{sgn}(a) \sqrt{q^e} \xi_1^{2\deg p} \xi_2^{2\deg p} \xi_3^{2\deg p} \right)^{-1} \cdot \prod_{1 \leq i \leq j \leq 3} \left( 1 - \frac{\text{sgn}(a) \xi_i^{2\deg p} \xi_j^{2\deg p} \xi_k^{2\deg p}}{p} \right)^{-1} \cdot \prod_{4 \leq k \leq l \leq r} \left( 1 - \frac{\xi_k^{2\deg p} \xi_l^{2\deg p}}{p} \right)^{-1}.$$

For every monic irreducible $p$, let $R^{(3)}_p(\xi)$ denote the local factor of the (modified) residue at $z_{r+1} = 1/q$ of the function $Z(z) := Z^{(3)}(s, \chi_1, \chi_a)$. For $a \in \{1, \theta_0\}$, define $\mathcal{S}_p^{(3)}(\xi, \chi_a)$ by

$$\mathcal{S}_p^{(3)}(\xi, \chi_a) := R_p^{(3)} \left( \left( \frac{\xi_1^{\deg p}}{|p|^{1/2}}, \frac{\xi_2^{\deg p}}{|p|^{1/2}}, \frac{\xi_3^{\deg p}}{|p|^{1/2}} \right) \right) \cdot \prod_{i=1}^3 \prod_{j=1}^r \left( 1 - \frac{\xi_i^{2\deg p} \xi_j^{2\deg p}}{|p|} \right)^{-1} \cdot \prod_{4 \leq k \leq l \leq r} \left( 1 - \frac{\xi_k^{2\deg p} \xi_l^{2\deg p}}{|p|} \right)^{-1},$$

and put $\mathcal{S}_p^{(3)}(\xi, \chi_a) = \prod_p \mathcal{S}_p^{(3)}(\xi, \chi_a)$. Then the product over $p$ in (30) (in the variables $\xi_1, \ldots, \xi_r$) corresponding to the $(\alpha', \zeta_a)$-term is given by the formula

$$\mathcal{S}^{(3)}(\xi, \chi_a) \prod_p \mathcal{S}_p^{(3)}(\xi, \chi_a),$$

this product converges initially when $\xi_1, \xi_2, \xi_3$ are in a small neighborhood of the unit circle, and $|\xi_k| < q^\varepsilon$ for $k \geq 4$. Thus we cannot immediately simplify $\prod_p \mathcal{S}_p^{(3)}(\xi, \chi_a)$ in (42). To show that this product does indeed cancel $\mathcal{S}_p^{(3)}(\xi, \chi_a)$, and thus obtain a completely explicit formula for (42), we shall need the following lemma.

**Lemma 7.2.** — For sufficiently small $\varepsilon > 0$, the series defining $R_p^{(i)} / R_p^{(3)}$ is absolutely convergent, and

$$\frac{R_p^{(i)}(\xi_1^{\deg p}, \xi_2^{\deg p}, \xi_3^{\deg p})}{R_p^{(3)}(\xi_1^{\deg p}, \xi_2^{\deg p}, \xi_3^{\deg p})} = 1 + O\left(|p|^{-1+\varepsilon}\right)$$

in the polydisk $\Omega : |\xi_k| < q^{-\frac{1}{4}+\frac{1}{2}\varepsilon}$ for $k \leq 3$, and $|\xi_k| < q^{-3-11\varepsilon}$ for $k \geq 4$.

**Proof.** We can obviously assume that $p$ is linear. By (11) and (23), we can write

$$R_p^{(i)}(\xi) = (1 + q^{-1}) \cdot \left( \frac{1}{2}, \frac{1}{2} \right) f(\xi, q^{-1}; q) = (1 - q^{-1}) \cdot \left( \frac{1}{2}, \frac{1}{2} \right) M_{w_{d+1}}(\xi, q^{-1}; q) f(\xi, q^{-1}; q)$$

where $w_{d+1} = w_1 w_2 w_3 w_{r+1}$. The entries of

$$\xi_1^{\deg p}, \xi_2^{\deg p}, \xi_3^{\deg p}(\xi_1, \xi_2, \xi_3)^{-1} \cdot \left( \frac{1}{2}, \frac{1}{2} \right) M_{w_{d+1}}(\xi, q^{-1}; q)$$
are polynomials in $\xi_1, \xi_2, \xi_3$, and $w_{c_i}^{-1}(\xi, q^{-1}) = \left( w_{c_i}^{-1}(\xi, q^{-1}) \right)_k$ for $1 \leq k \leq r+1$ with

$$w_{c_i}^{-1}(\xi, q^{-1})_k = \begin{cases} 
\left( \xi_1, \xi_2, \xi_3 \right)_k / \xi_k & \text{if } k \leq 3 \n q^{\xi_1} \xi_2 \xi_3 q^{-1} & \text{if } 4 \leq k \leq r 
\left( q^{3/2} \xi_1 \xi_2 \xi_3 \right)^{-1} & \text{if } k = r + 1.
\end{cases}$$

Thus $R_{c_i}^{(0)}(\xi, R_{c_i}^{(0)}(\xi, \xi_2, \xi_3))$ is holomorphic for $-\frac{1}{2} - \epsilon < \log_q |\xi_3| < -\frac{1}{2}$, and $\log_q |\xi_3| < -3 - 8 \epsilon$ for $k \geq 4$.

Now, for $i < j \leq r$, let $w_{ij} = w_i w_j w_{ij+1} w_i w_j$, and put $w^* = w_{12} w_{13} w_{23}$. By applying the cocycle relation (10), it is straightforward to check that the function $R_{c_i}^{(0)} / R_{c_i}^{(0)}$ is invariant under $w^*$. Note that this element acts on $\xi_3$ by

$$\xi_3 \mapsto \left( \sqrt[3]{q} \xi_3 \right)^{-1} \text{ if } k \leq 3, \quad q^{3/2} \xi_1 \xi_2 \xi_3 \text{ if } k \geq 4.$$

Accordingly, the function $R_{c_i}^{(0)}(\xi, R_{c_i}^{(0)}(\xi, \xi_2, \xi_3))$ is holomorphic for $-\frac{1}{2} < \log_q |\xi_3| < -\frac{1}{2} + \frac{1}{2} (k \leq 3)$, and $\log_q |\xi_3| < -3 - 11 \epsilon$ for $k \geq 4$; it is clear that in this region the function is represented by the scalar function

$$(1 - q^{-1}) R_{c_i}^{(0)}(\xi, \xi_2, \xi_3)^{-1} \cdot \left( \frac{1}{2}, 1, \frac{1}{2} \right) M_{w^*} \left( \xi, q^{-1} \right) \tilde{f}(w_{c_i}^{-1}(\xi, q^{-1}); q) .$$

Since, by our assumptions, the vector function $\tilde{f}(\xi, \xi_1, \xi_2, \xi_3)$ continues meromorphically to the open convex cone $X^*$, it follows at once that $R_{c_i}^{(0)} / R_{c_i}^{(0)}$ is meromorphic in the polydisk $\Omega_\epsilon$ stated in the lemma.

To see that no pole can in fact occur in this region, we argue as follows. We show that:

$$d(\alpha) - 1 + \sum_{k=1}^{r} 2n_k \log_q |\xi_k| - 2n_{r+1} < 0$$

(43)

for all $\alpha = \sum n_k \alpha_k \in \Delta^c_\epsilon$, and $\xi \in \Omega_\epsilon$. Indeed, since $l(w^*) = 11,$

$$w^*(\alpha) < 0 \iff \alpha \in \left\{ \alpha_i, \alpha_2, \alpha_3 \right\} \cup \left\{ \sum n_k \alpha_k + \alpha_{r+1} : n_k \in \{0, 1\}, \text{not all zero} \right\} \cup \left\{ \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_{r+1} \right\}.$$

All these roots satisfy (43), as the reader can easily verify. If $w^*(\alpha) > 0$, then by computing the coefficients of $\alpha_k$ (for $k \leq 3$) in $w^*(\alpha),$ one finds that

$$n_1 + n_2 + n_3 \leq 6 \sum_{k=4}^{r} n_k .$$

(44)

Thus, if $\log_q |\xi_3| < -\frac{1}{2} + \frac{1}{2}$ for $k \leq 3$, and $\log_q |\xi_3| < -3 - 11 \epsilon$ for $k \geq 4$, then the expression in (43) is smaller than

$$\left( \frac{1}{2} + \epsilon \right) (n_1 + n_2 + n_3) - (5 + 22 \epsilon) \sum_{k=4}^{r} n_k$$

which by (44) is $\leq 0$. It follows that

$$\frac{R_{c_i}^{(0)}(\xi)}{R_{c_i}^{(0)}(\xi_2, \xi_3)} = (1 - q^{-1}) R_{c_i}^{(0)}(\xi, \xi_2, \xi_3)^{-1} \cdot \left( \frac{1}{2}, 1, \frac{1}{2} \right) \tilde{f}(\xi, q^{-1}; q)$$

$$= (1 - q^{-1}) \sum B_{n_1, n_2, \ldots, n_r} \xi_1^{n_1} \xi_2^{n_2} \cdots \xi_r^{n_r}$$

the power series being normally convergent on $\Omega_\epsilon$. Note that $B_{0, \ldots, 0} = (1 - q^{-1})^{-1}$ so that the constant term of $R_{c_i}^{(0)} / R_{c_i}^{(0)}$ is 1.
For fixed \( n_1, \ldots, n_r \), let us temporarily denote the subseries \( \sum_{n_i, n_j, n_k, n_0 \geq 0} B_{n_1, n_2, n_3} \xi_1^{n_1} \xi_2^{n_2} \xi_3^{n_3} \) by \( F_{n_1, n_2, n_3}(\xi_1, \xi_2, \xi_3) \). The invariance of \( R_p^{(r)} \) under \( w^* \) implies that

\[
F_{n_1, n_2, n_3}(\xi_1, \xi_2, \xi_3) = \left( q^{1/2 \xi_1^2 \xi_2^2 \xi_3^2} \right)^{n_1 + \cdots + n_r} F_{n_1, n_2, n_3}\left( \frac{1}{\sqrt{q} \xi_1}, \frac{1}{\sqrt{q} \xi_2}, \frac{1}{\sqrt{q} \xi_3} \right)
\]

when \( |\xi_3| = q^{-1/4} \). Thus by applying Cauchy’s integral formula

\[
B_{n_1, n_2, n_3} = \frac{1}{(2\pi \sqrt{-1})^3} \oint_{|\xi_0|=q^{-1/4}} \oint_{|\xi_1|=q^{-1/4}} \oint_{|\xi_2|=q^{-1/4}} \left( q^{1/2 \xi_0^2 \xi_1^2 \xi_2^2} \right)^{n_1 + \cdots + n_r} F_{n_1, n_2, n_3} \left( \frac{1}{\sqrt{q} \xi_0}, \frac{1}{\sqrt{q} \xi_1}, \frac{1}{\sqrt{q} \xi_2} \right) d\xi_0 d\xi_1 d\xi_2
\]

we find that

\[
B_{n_1, n_2, n_3} = 0 \quad \text{if} \quad 2(n_1 + \cdots + n_r) < \min\{n_1, n_2, n_3\}.
\]

On the other hand, if \( |\xi_3| = q^{1/2-\varepsilon} \) for all \( k \leq r \), we have

\[
|R_p^{(r)}(\xi_1, \xi_2, \xi_3)| - 1 \ll -1 + \sum_{n_i \geq 0} \sum_{n_j \geq 0} \sum_{n_k \geq 0} q\left(\frac{1}{2} - 10\varepsilon\right)(n_1 + \cdots + n_r) \sum_{n_i = 0} 2(n_1 + \cdots + n_r) \sum_{n_j = 0} 2(n_2 + \cdots + n_r) \sum_{n_k = 0} q\left(\frac{1}{2} + 1/6\right)(n_1 + n_2 + n_3)
\]

which completes the proof.

\[
\square
\]

**Proposition 7.3.** — For sufficiently small \( \varepsilon > 0 \), we have

\[
\mathcal{R}^{(r)}(\xi, \chi_a) \prod_p S_p^{w}(\xi, \chi_a) = N(\xi, \chi_a) \prod_p S_p(\xi, \chi_a)
\]

for \( \xi_1, \xi_2, \xi_3 \) in a small neighborhood of the unit circle, and \( |\xi_3| < q^k \) for \( k \geq 4 \).

**Proof.** By definition, \( S_p^{w}(\xi, \chi_a) = N_p(\xi, \chi_a) S_p(\xi, \chi_a) \). Thus, by taking \( \xi = (\xi_1, \ldots, \xi_r) \) as stated in the proposition, we can write

\[
\prod_p N_p(\xi, \chi_a) = \frac{\mathcal{R}^{(r)}(\xi, \chi_a)}{\prod_p \mathcal{R}^{(r)}(\xi, \chi_a)}
\]

the product in the left-hand side converges by our assumptions. To conclude that

\[
\prod_p N_p(\xi, \chi_a) = \frac{N(\xi, \chi_a)}{\mathcal{R}^{(r)}(\xi, \chi_a)}
\]

it suffices to show that

\[
N_p(\xi, \chi_a)^{-1} \mathcal{R}^{(r)}(\xi, \chi_a) = 1 + O(|p|^{-1-\eta_k}) \quad \text{(with } \eta_k \sim \frac{1}{2} \text{ as } \varepsilon \to 0) \]

(45)

uniformly in \( D_\varepsilon := \{ \xi : q^{-k/3} < |\xi_3| < q^{k/2} \} \) for \( k = 1, 2, 3 \) and \( |\xi_0| < q^{k/2} \) for \( k = 4, \ldots, r \); here it is understood that the function \( N_p(\xi, \chi_a)^{-1} \mathcal{R}^{(r)}(\xi, \chi_a) \) is holomorphic in \( D_\varepsilon \). Indeed, assuming (45) for the moment, the product

\[
\prod_p N_p(\xi, \chi_a)^{-1} \mathcal{R}^{(r)}(\xi, \chi_a)
\]
converges absolutely, hence it is a holomorphic function, in \( D_\varepsilon \). Restricting to \(|\xi_3| < q^{-\varepsilon} \) \((k \geq 4)\), we clearly have
\[
\prod_{i=1}^{r} \prod_{j=4}^{r} \left( 1 - \xi_3^{-2} \xi_j^{-2} \right) \cdot \prod_{4 \leq k \leq l \leq r} \left( 1 - \xi_k^{-2} \xi_l^{-2} \right)
\]
\[
= \prod_{p} \left( \prod_{i=1}^{r} \prod_{j=4}^{r} \left( 1 - \frac{\xi_3^{-2} \xi_j^{-2}}{|p|} \right) \left( 1 - \frac{\xi_k^{-2} \xi_l^{-2}}{|p|} \right) \right).
\]
Put \( \xi_1 = \xi_3 q \), \( \xi_2 = \xi_3 q \), \( \xi_3 = \xi_3 q \), \( \xi_4 = \xi_3 q \), and \( \xi_k = \frac{\xi_3 q}{q (3/2) \xi_3} \) for \( k \geq 4 \). Thus
\[
\xi_1 = q^{1/2} \sqrt{\xi_3 / \xi_1}, \quad \xi_2 = q^{1/2} \sqrt{\xi_3 / \xi_2}, \quad \xi_3 = q^{1/2} \sqrt{\xi_3 / \xi_3}, \quad \xi_4 = q^{1/2} \sqrt{\xi_3 / \xi_4}
\]
with the obvious choice of the square root. Note that
\[
q^{-\frac{1}{2} - \varepsilon} < |\xi_3| < q^{-\frac{1}{2} + \frac{1}{2}} \quad \Rightarrow \quad q^{-\frac{1}{2} - \varepsilon} < |\xi_3| < q^{\frac{1}{2}} \quad \text{(for } k \leq 3 \text{) and } |\xi_3| < q^{-\frac{1}{2} + \frac{1}{2}} \quad \Rightarrow \quad |\xi_3| < q^{\frac{1}{2}} \quad \text{(for } k \geq 4 \text{)}.
\]
It follows that the product \( \prod_p (R_p^{(3)}) / (R_p^{(3)}) \), as a function of the variables \( \xi' = (\xi_1', \ldots, \xi_r') \), is holomorphic in the domain \( D_\varepsilon' = \{ (\xi_1', \ldots, \xi_r') : q^{-\frac{1}{2} - \varepsilon} < |\xi_3'| < q^{-\frac{1}{2} + \frac{1}{2}} \text{ for } k \leq 3, \ |\xi_3'| < q^{-\frac{1}{2} - \varepsilon} \text{ for } k \geq 4 \} \). On the other hand, by Lemma 7.2, the (original) series of \( R_p^{(3)} / (R_p^{(3)}) \) is absolutely convergent, and
\[
\frac{R_p^{(3)}(\xi_3 \deg p, \xi_3 \deg p)}{R_p^{(3)}(\xi_3 \deg p, \xi_3 \deg p)} = 1 + O \left( |p|^{-1 - \varepsilon} \right)
\]
when \( |\xi_3'| < q^{-\frac{1}{2} + \frac{1}{2}} \) for \( k \leq 3 \), and \( |\xi_3'| < q^{-1 - \varepsilon} \) for \( k \geq 4 \). If \( |\xi_3'| < q^{-\frac{1}{2} - \varepsilon} \) \((k \leq 3)\),
\[
R_p^{(3)}(\xi_1', \xi_2', \xi_3') = \prod_p R_p^{(3)}(\xi_1', \xi_2', \xi_3')
\]
and thus, subject to (45), the proposition follows by analytic continuation.

It remains to establish the asymptotic formula (45). By applying the functional equation (11), we have
\[
\mathcal{D}_p^{(1)}(\xi, \chi_p) = R_p^{(1)} \left( \left( \frac{\xi_3}{q} \xi_3 \xi_3 \xi_3 \deg p \right) \frac{1}{|p|^{1/2} \deg p} \cdot \frac{1}{|p|^{1/2} \deg p} \cdot \frac{1}{|p|^{1/2} \deg p} \cdot \frac{1}{|p|^{1/2} \deg p} \cdot \frac{1}{|p|^{1/2} \deg p} \cdot \frac{1}{|p|^{1/2} \deg p} \cdot \frac{1}{|p|^{1/2} \deg p} \right)
\]
\[
\left( 1 - |p|^{-1} \cdot \chi_p(p), \frac{1}{2} \right) M_{w_1}(z; |p|)^{-1} \bar{f}(z; |p|) \right) \left| \left( \frac{1}{|p|^{1/2} \deg p} \right)^{1/2} \cdot \frac{1}{|p|^{1/2} \deg p} \cdot \frac{1}{|p|^{1/2} \deg p} \cdot \frac{1}{|p|^{1/2} \deg p} \cdot \frac{1}{|p|^{1/2} \deg p} \right| \right|.
\]
where we recall that \( w_1 = w_1 w_2 w_3 w_4 w_5 + 1 \). By Lemma 4.2, the vector function \( \bar{f}(z; |p|) \) satisfies
\[
\bar{f}(z; |p|) = \left( \prod_{j=1}^{r} \left( 1 - z_j \right)^{-1}, z_r + 1 \prod_{j=1}^{r} \left( 1 + z_j \right)^{-1} \right) + O \left( |p|^{-1 + 4(r+1)\varepsilon} \right)
\]
as long as \( |z| \leq |p|^{-\frac{1}{2} + \varepsilon} (i = 1, \ldots, r) \), and \( |z_{r+1}| \leq |p|^{-\frac{1}{2} + \varepsilon} \). By computing \( \frac{1}{2} \chi_p(p), \frac{1}{2} \right) M_{w_1}(z; |p|)^{-1} \) using (10), one finds that
\[
\left( 1 - |p|^{-1} \right) N_p(\xi, \chi_p)^{-1} \cdot \left( \frac{1}{2} \chi_p(p), \frac{1}{2} \right) M_{w_1}(z; |p|)^{-1} \left( \prod_{j=1}^{r} \left( 1 - z_j \right)^{-1}, z_r + 1 \prod_{j=1}^{r} \left( 1 + z_j \right)^{-1} \right)
\]
\[
= 1 + O \left( |p|^{-1 - \varepsilon} \right)
\]
uniformly for \( \xi \in D_\varepsilon \), with \( \eta_{\varepsilon} \sim \frac{1}{2} \) as \( \varepsilon \to 0 \), and \( z_j (j \leq r + 1) \) as above. This gives (45), which completes the proof. \( \square \)
It remains to compute the factor $2^{-l(w_0)}\Gamma_{w_0}(a_2, a; z_\alpha)$. For ease of notation, let $\mathbf{M}_w(z; q) = M_w(qz; 1/q)$ for $w \in W$. Then 
$$
\mathbf{M}_{w_0}(z; q) = \frac{1}{2q^{y/2}z_0 z_1 z_{r+1}} \left( \begin{array}{l}
\psi(z_1, z_2, z_3, z_{r+1}; q) \\
\phi(z_1, z_2, z_3, z_{r+1}; q) \\
-\psi(-z_1, -z_2, -z_3, -z_{r+1}; q)
\end{array} \right).
$$

(46)

Here $\phi$ and $\psi$ are given by (39). The formula (26) now yields:

$$
2^{-l(w_0)}\Gamma_{w_0}(a_2, a; z_\alpha) = \frac{1}{2} \left( \prod_{j \in J} \frac{1 - z_j^{-1}}{1 - q z_j}, \text{sgn}(a) q^{\ell/2}, \prod_{j \in J} \frac{1}{q^{z_j}}, 0 \right)
$$

$$
\cdot \mathbf{M}_{w_0}(z; q) |_{z_{r+1} = \frac{1}{q^{z_{r+1}}(z_0 z_1 z_{r+1})}} (1, \text{sgn}(a), 1).
$$

The computation of $Q_\alpha(D, q)$. Putting everything together, we can now compute the contribution of the principal parts at the poles corresponding to the roots in $\Phi_2$. Here $S_a(z, \chi_\alpha)$ is defined as follows. Express $\alpha$ as

$$
\alpha = \sum_{j=1}^{3} k_j \alpha_j + \alpha_n + \alpha_r + 2\alpha_{r+1},
$$

for some $1 \leq j_1 < j_2 < j_3 \leq r$, and with $k_j \in \{0, 2\}$ for $j \neq j_1, j_2, j_3$. Write \{1, ..., r\} \setminus \{j_1, j_2, j_3\} = \{j_4, ..., j_r\}, and, for $i \leq r$, define $\delta_i = -1$ or 1 according as $k_i = 2$ or not. If we set, as before, $z_k = q^{z_k/2}$ for $k \leq r$, then, by Proposition 7.3,

$$
S_a(z, \chi_\alpha) := N(\xi_{j_1}, \xi_{j_2}, \xi_{j_3}, \xi_{j_4}, ..., \xi_{j_r}, \chi_\alpha) \prod_p S_p^{\deg}(\xi_{j_1}, \xi_{j_2}, \xi_{j_3}, \xi_{j_4}, ..., \xi_{j_r}, \chi_\alpha)
$$

where $S_p^{\deg}(\xi, \chi_\alpha)$ is given explicitly by (41), (40), and (38), with $\text{sgn}(a) \mapsto \chi_\alpha(p)$, $\xi \mapsto \xi^{\deg(p)}$, and $q \mapsto |p|$. Notice that, due to the singularities of

$$
\prod_{i=1}^{r} \prod_{j \neq i} \left(1 - \xi_j \xi_i\right)^{-1}(1 - \xi_j^{-1} \xi_i)^{-1} \cdot \prod_{4 \leq k \leq r} \left(1 - \xi_k \xi_{k+1}\right)^{-1}
$$

in $N(\xi, \chi_\alpha)$, we cannot take $\xi_\delta = \xi_\delta^{\text{right}}$ for $1 \leq i \neq j \leq r$. To see that $S_\alpha(z, a_2)$ is indeed defined when $z_k = q^{1/2}$, for all $k \leq r$, we first note that $\prod_p S_p^{\deg}(\xi, \chi_\alpha)$ is symmetric in the variables $\xi_{j_1}, \xi_{j_2}, \xi_{j_3}$, and (separately) in $\xi_k$, for $4 \leq k \leq r$. This follows at once from (41), (40), and Remark 6. Moreover, the functions $\mathcal{G}_1(\xi, \zeta_\alpha)$ and $\mathcal{G}_2(\xi, \zeta_\alpha)$ defined by

$$
\mathcal{G}_1(\xi, \zeta_\alpha) := \prod_{j=1}^{r} \left(1 - q^{1/2} \xi_j\right)^{-1} \left(1 - \text{sgn}(a) \sqrt{\xi_{j_1} \xi_{j_2} \xi_{j_3}}\right) \prod_{1 \leq i \leq 3} \left(1 - \frac{\text{sgn}(a) \sqrt{\xi_{j_1} \xi_{j_2} \xi_{j_3}}}{\xi_j}\right)
$$

$$
\cdot (1, 0, 0) \mathbf{M}_{w_0} \left( q^{-1/2} \xi_{j_1}, q^{-1/2} \xi_{j_2}, q^{-1/2} \xi_{j_3}, \frac{1}{q^{1/2} \xi_{j_1} \xi_{j_2} \xi_{j_3}}; q \right) (1, \text{sgn}(a), 1)
$$

and

$$
\mathcal{G}_2(\xi, \zeta_\alpha) := \prod_{j=1}^{r} \frac{\xi_{j_1} \xi_{j_2} \xi_{j_3}}{\xi_j}\n$$

$$
\cdot (0, 1, 0) \mathbf{M}_{w_0} \left( q^{-1/2} \xi_{j_1}, q^{-1/2} \xi_{j_2}, q^{-1/2} \xi_{j_3}, \frac{1}{q^{1/2} \xi_{j_1} \xi_{j_2} \xi_{j_3}}; q \right) (1, \text{sgn}(a), 1)
$$

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Lemma 7.5. and, for polydisk $\mu$-divisors $\xi(z) = 0$ and a small positive $\rho$, where $q(z) = 2$ and a sufficiently small positive $\rho$, we have

$$\mathcal{G}_2(q^{-1/2,2}, z_{r+1}, dz) = \frac{1}{2\pi^2} \prod_{\sigma \leq 1} \left\{ \mathcal{G}_2(z, z_{r+1}, \zeta_{r+1}) \prod_{j=1}^{\rho} (1 - q^{1/2} z_{r+1} - z_{r+1} z_j) \right\}$$

where $q^{-1/2,2} := (q^{-1/2,2}, \ldots, q^{-1/2,2})$. Moreover, for complex $u$ with $|u| < q^{-3/4}(1 - \rho)^{-3}$, we have

$$\sum_{\sigma \leq 1} \sum_{\delta_{o(i)} = \pm 1} \mathcal{G}_2(z, z_{r+1}, u, \zeta_{r+1}) = \frac{(-1)^{(r+1)/2}}{2\pi \sqrt{-1}} \int_{|z| = 1} \mathcal{G}_2(z, z_{r+1}, u, \zeta_{r+1})$$

$$\frac{1}{(1 - z_1 \zeta_{r+1})} \cdot \prod_{1 \leq i < j \leq r} (1 - z_1 \zeta_{r+1}) \cdot \prod_{1 \leq k \leq \ell \leq 3} (1 - z_k \zeta_{r+1}) \cdot \prod_{1 \leq k \leq \ell \leq 3} (1 + z_k \zeta_{r+1}) \cdot \prod_{1 \leq k \leq \ell \leq 3} (1 + z_k \zeta_{r+1}) \cdot \prod_{1 \leq k \leq \ell \leq 3} (1 + z_k \zeta_{r+1}) \cdot \prod_{1 \leq k \leq \ell \leq 3} (1 + z_k \zeta_{r+1}) \cdot \prod_{1 \leq k \leq \ell \leq 3} (1 + z_k \zeta_{r+1})$$

where $e_i = 1$ or 2 according as $i \leq 3$ and $j \geq 4$ or not, and $z' = z'_1 \cdots z'_{r+1}$.

Proof. By Proposition 7.3, the function $\prod_p S_p(z, \zeta_{r+1}) = N(z, \zeta_{r+1})^{-1} N(z, \zeta_{r+1}) \prod_p S_p(z, \zeta_{r+1})$ is holomorphic on a small polydisk $|z| \leq \rho$. As all these functions are symmetric in the variables $z_1^2, z_2^2, z_3^2$, and (separately) symmetric in $z_k$ ($k \geq 4$), our first assertion can be verified by a simple counting argument. To establish the integral formula, we apply the following slight generalization of Lemma 7.1:

Lemma 7.5. Let $a_1, \ldots, a_r$ be distinct elements on the unit circle such that $a_i \neq 1$ and $|a_i - 1| < \varepsilon$, for all $1 \leq i, j \leq r$ and a small positive $\varepsilon$. Suppose $h$ is a function of $r$ complex variables, holomorphic on the polydisk $|z| \leq \varepsilon$, $i = 1, \ldots, r$, and, for $0 \leq m < r$, define $K_m(z)$ by

$$K_m(z) = \prod_{k=1}^{m} \prod_{l=m+1}^r (1 - z_k^2 z_l^2) \prod_{1 \leq k \leq l \leq r} (1 - z_k^2 z_l^2)$$

Then we have

$$\frac{(-1)^{(r+1)/2}}{2\pi \sqrt{-1}} \int_{|z| = 1} \int_{|z| = 1} h(z) \prod_{1 \leq i < j \leq r} (1 - z_i z_j) \prod_{1 \leq k \leq \ell \leq 3} (1 - z_k z_j) \prod_{1 \leq k \leq \ell \leq 3} (1 + z_k z_j) \prod_{1 \leq k \leq \ell \leq 3} (1 + z_k z_j) \cdot \prod_{1 \leq k \leq \ell \leq 3} (1 + z_k z_j) \cdot \prod_{1 \leq k \leq \ell \leq 3} (1 + z_k z_j) \cdot \prod_{1 \leq k \leq \ell \leq 3} (1 + z_k z_j)$$

where $e_i = 1$ or 2 according as $i \leq m$ and $j \geq m + 1$ or not, and $z' := z'_1 \cdots z'_{r+1}$.
Proof. We proceed as in the proof of Lemma 7.1. The poles of the integrand (inside the contour) occur at $z_i = \delta_{a(i)}$, for $\sigma \in \mathbb{S}_r$, and $\delta_{a(i)} \in \{ \pm 1 \}$ for $1 \leq i \leq r$. Writing
\[
\prod_{1 \leq i < j \leq r} (z_i - z_j) = \prod_{1 \leq i < j \leq r} (z_i - z_j)^2
\]
the identity stated in the lemma follows easily from the fact that
\[
\frac{(1 - z_i z_j)}{(1 - z_i z_j)(1 - \overline{z}_i \overline{z}_j)} = \prod_{1 \leq k \leq r} (1 - z_k z_i)\]
with $z_i = \delta_{a(i)}$ for $i = 1, \ldots, r$. □

The second assertion in our theorem follows at once from this lemma by taking $m = 3$,
\[
h(z) = \frac{\mathcal{L}_1(z, \zeta_0) \prod_{p} \mathcal{L}_p(z, \chi_0)}{1 - q^{r/2} \zeta_0 z_1 \cdots z_r u}
\]
and $a_i = \xi_i$ for all $i$. This completes the proof of the theorem. □

One should note that the assumption on $\xi_1, \ldots, \xi_r$ (or $a_1, \ldots, a_r$ in the lemma) being on the unit circle, made solely to simplify the exposition, can be removed by analytic continuation.

**Corollary 7.6.** — For $D \in \mathbb{N}$ and $r \geq 4$, the coefficient $Q_i(D, q)$ is given by
\[
Q_i(D, q) = \frac{1}{2^3 (r-3)!} \sum_{r \in \mathbb{Z}_{\pm 1}^r} \left( \prod_{i=1}^{r} \left( 1 - q^{2r+1} \right)^{-1} \right) \left( \prod_{j=1}^{r} \left( 1 - q^{2j+1} \right)^{-1} \right) \mathcal{L}_1(D, \zeta_0) + \mathcal{L}_2(D, \zeta_0)
\]
where $\mathcal{L}_1, \mathcal{L}_2(D, \zeta_0)$ has the integral representation
\[
\mathcal{L}_1(D, \zeta_0) = \frac{(-1)^{(r+1)/2}}{(2\pi \sqrt{-1})^{r}} \int \cdots \int \frac{\mathcal{L}_1(z, \zeta_0) \prod_{p} \mathcal{L}_p(z, \chi_0)}{z_1^D \cdots z_r^D} \prod_{1 \leq i < j \leq r} (z_i - z_j)^2 (1 - z_i z_j) \prod_{1 \leq k \leq l \leq 3} (1 - z_k z_l) \prod_{1 \leq k \leq l \leq r} (1 - z_k z_l) \frac{dz}{z^r}
\]
the integrals being taken over sufficiently small circles $|z_i - 1| = \rho$, $i = 1, \ldots, r$. Proof. Consider the integral
\[
\frac{1}{2\pi \sqrt{-1}} \int \mathcal{L}_2(q^{-1/2} \xi, 1) \frac{dz}{z^r} \quad (48)
\]
where, for ease of notation, we set $\xi := \xi_{r+1}$, and $q^{-1/2} \xi := (q^{-1/2} \xi_1, \ldots, q^{-1/2} \xi_r)$. Using the identity
\[
\mathcal{L}_2(q^{-1/2} \xi, 1) = \frac{1}{4} \sum_{r \in \mathbb{Z}_{\pm 1}^r} \left( \mathcal{L}_1(\xi, \zeta_0) \prod_{j=1}^{r} (1 - q^{1/2} \xi_j)^{-1} + \frac{1}{\xi_1 \cdots \xi_r} \mathcal{L}_2(\xi, \zeta_0) \right)
\]
and the definition of the functions $\mathcal{L}_1, \mathcal{L}_2(\xi, \zeta_0)$, we can express (48) via (47) as a sum of two multiple integrals involving the rational function
\[
\left( 1 - q^{1/2} \zeta_0 z_1 \cdots z_r \right)^{-1} \prod_{i,j=1}^{r} (1 - z_i \xi_j)^{-1} (1 - z_j \xi_i)^{-1}.
\]
this function exhibits the dependence upon either of $\xi_1, \ldots, \xi_r$ or $\zeta$ in the integrands. Taking $p$ even smaller if necessary so that $q^{-2} < q^{-3/4}(1 - p)^{-3}$ and setting $\xi_1 = \cdots = \xi_r = 1$, our conclusion follows at once by integrating with respect to $\zeta$, and applying the simple formula

$$\frac{1}{2\pi\sqrt{1 - \varepsilon^2}} \int \frac{dz}{z^{d+1}(1 - cz)} = e^d$$

with $c = q^{3/4} \xi_1 \cdots \xi_r$. (Note that $|cz| < 1$ when $|z| < q^{-2}$).

We have the following:

**Proposition 7.7.** — The functions $\tilde{F}_{1,1}(D, \zeta_0)$ are polynomials in $D$ of degree $(r - 3)(r + 10)/2$ with leading terms given by

$$3 \cdot 2^{25-r}(r-3)! \frac{0! 1! 2! \cdots (r-4)!}{7! 9! 11! \cdots (2r-1)!} \cdot \tilde{F}_{1,1}(1, \zeta_0) \prod_p \phi_p(1, \zeta_0) \cdot D^{(r-3)(r+10)/2}$$

where we set $1 := (1, 1, \ldots, 1)$.

**Proof.** The integral $\tilde{F}_{1,1}(D, \zeta_0)$ is the coefficient of $\prod_{i=1}^r (z_i - 1)^{2r-1}$ in the power series expansion of

$$\left(\sum_{i=1}^{2r} \frac{Z_{r-i}}{Z_{r-i}} \cdot 1 \right) \prod_{i=1}^r \prod_{j=1}^r (z_i - z_j)^{2r-1} \cdot \prod_{i=1}^r \prod_{j=1}^r (1 + z_i - z_j)^{2r-1} \cdot \prod_{i=1}^r \prod_{j=1}^r (1 + z_i - z_j)^{z_i - z_j}$$

around $z_i = 1, i = 1, \ldots, r$. Substituting $x_i = z_i - 1$, and using the binomial expansion

$$(1 + x)^D = 1 - Dx + \frac{D(D + 1)}{2} x^2 - \frac{D(D + 1)(D + 2)}{6} x^3 + \ldots$$

it is clear that $\tilde{F}_{1,1}(D, \zeta_0)$ is a polynomial in $D$. To compute the highest power of $D$ in this polynomial, notice that every monomial in the expansion of

$$\prod_{4 \leq i < j \leq r} (z_i - z_j)^{2r-1}(1 - z_i z_j) = (-1)^{(r-3)(r-4)/2} \cdot \prod_{4 \leq i < j \leq r} (x_i - x_j)^2 (x_i x_j + x_i + x_j)$$

has degree at least $3(r-3)(r-4)/2$. Thus the highest power of $D$ occurring in the polynomial is at most

$$(2r-1)(r-3) - \frac{3(r-3)(r-4)}{2} = \frac{(r-3)(r+10)}{2}$$

and the total contribution to $D^{(r-3)(r+10)/2}$ is given by

$$\tilde{F}_{1,1}(1, \zeta_0) \prod_p \phi_p(1, \zeta_0) \cdot \frac{1}{(2\pi\sqrt{1 - \varepsilon^2})} \int \int \int \Delta(x_1, x_2, x_3, x_4, x_5, x_6) e^{a \cdot \Sigma_{i=4}^{r} x_i} dx_1 dx_2 dx_3$$

Here $\Delta(y_1, \ldots, y_n)$ is the Vandermonde determinant

$$\Delta(y_1, \ldots, y_n) = \det(y_i^{i-1})_{1 \leq i, j \leq n} = \prod_{1 \leq i < j \leq n} (y_j - y_i).$$

The first integral equals $-48$, and the second integral can be computed as in [21, Proposition 2.1]. Thus, by substituting $y_i = -Dx_i$, it follows, as in loc. cit., that

$$\frac{1}{(2\pi\sqrt{1 - \varepsilon^2})} \int \int \int \Delta(x_1, x_2, x_3, x_4, x_5, x_6) e^{a \cdot \Sigma_{i=4}^{r} x_i} dx_1 dx_2 dx_3$$

$$= (r-3)! \frac{0! 1! 2! \cdots (r-4)!}{7! 9! 11! \cdots (2r-1)!} \det \left( \begin{array}{cccc} 2r+1 & -2j & \cdots & 0 \\ i-1 & i+1 & \cdots & i+r-3 \end{array} \right)_{1 \leq i, j \leq r-3} \cdot (-D)^{(r-3)(r+10)/2}$$
where \( \binom{n}{k} \) is the binomial coefficient. The determinant equals \((-2)^{(r-5)(r-4)/2}\), which can be seen as follows. The first row is 1, 1, ..., 1, and so, by replacing the \( j \)-th column \( (j = 2, \ldots, r-3) \) by the difference between the \((j-1)\)-th and \( j \)-th columns, we are reduced (after removing the first row and the first column) to the computation of an \((r-4) \times (r-4)\) determinant with the first row 2, 2, ..., 2. Applying the same procedure, we are further reduced to the computation of an \((r-5) \times (r-5)\) determinant with the first row 4, 4, ..., 4. Continuing, it follows by applying the identity

\[
\binom{a+2}{b} - \binom{a+1}{b-1} = \binom{a+1}{b-1} + \binom{a}{b-1}
\]

that at the \( k \)-th step, we get an \((r-k-2) \times (r-k-2)\) determinant with the first row \( 2^{k-1}, 2^{k-1}, \ldots, 2^{k-1} \). Thus

\[
\det\left(\frac{(2r+1-2j)}{i-1}\right)_{1 \leq i, j \leq r-3} = \prod_{k=1}^{r-3} (-1)^{r-k-3} 2^{k-1} = (-2)^{(r-5)(r-4)/2}.
\]

From this, our last assertion follows immediately.

Thus the leading coefficient of \( Q_2(D,q) \) is

\[
2^{19-7r} \cdot \frac{0! 1! 2! \ldots (r-4)!}{7! 9! 11! \ldots (2r-1)!} \sum_{\zeta = \pm 1} \left\{ \frac{(1,1,0) M_{\mu_\omega}(q^{-1/2}, q^{-1/2}, q^{-3/4}; q)}{1 + q^{1/4} + 10 q^{1/2} + 7 q^{3/4} + 20 q + 7 q^{3/4} + 10 q^{3/2} + q^{7/4} + q^2} \right\}
\]

where, by (46) and (39),

\[
(1,1,0) M_{\mu_\omega}(q^{-1/2}, q^{-1/2}, q^{-3/4}; q) = \begin{cases} 1 + q^{1/4} + 10 q^{1/2} + 7 q^{3/4} + 20 q + 7 q^{3/4} + 10 q^{3/2} + q^{7/4} + q^2 & \text{if } \zeta = \text{sgn}(a) = 1 \\
1 - q^{1/4} + 10 q^{1/2} - 7 q^{3/4} + 20 q - 7 q^{3/4} + 10 q^{3/2} - q^{7/4} + q^2 & \text{if } \zeta = -1 \text{ and } \text{sgn}(a) = 1 \\
1 - i q^{1/4} + 4 q^{1/2} - 7 i q^{3/4} + 6 q - 7 i q^{3/4} - 4 q^{3/2} + i q^{7/4} + q^2 & \text{if } \zeta = i \text{ and } \text{sgn}(a) = -1 \\
1 + i q^{1/4} - 4 q^{1/2} - 7 i q^{3/4} + 6 q + 7 i q^{3/4} - 4 q^{3/2} - i q^{7/4} + q^2 & \text{if } \zeta = -i \text{ and } \text{sgn}(a) = -1,
\end{cases}
\]

and, by (35) - (38), (40) and (41), \( S_{\mu_\omega}^2(1, \chi_\omega) = P_r(\chi_\omega(p)/\sqrt{|p|}) \) with

\[
P_r(t) = (1-t)^{(r^2+7r-14)/2} (1+t)^{(r^2+7r-28)/2} \cdot \left( t + r^2 - \frac{1}{2} (1+t)^{4-r} + \frac{1}{2} (1-t)^{-r} \left( 1 + 10r + 20r^2 + 10r^3 + 4r^4 \right) \right).
\]

Note that

\[
P_r(t) = 1 - 14(r-2)t^3 - \frac{9t^4 + 12r^2 + 59r^2 - 696r + 1164}{12} t^4 + O(t^5)
\]

which implies the absolute convergence of the product \( \prod_p S_{\mu_\omega}^2(1, \chi_\omega) \).

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