Nonlinear $q$-voter model with inflexible zealots

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We study the dynamics of the nonlinear $q$-voter model with inflexible zealots in a finite well-mixed population. In this system, each individual supports one of two parties and is either a susceptible voter or an inflexible zealot. At each time step, a susceptible adopts the opinion of a neighbor if this belongs to a group of $q \geq 2$ neighbors all in the same state, whereas inflexible zealots never change their opinion. In the presence of zealots of both parties the model is characterized by a fluctuating stationary state and, below a zealotry density threshold, the distribution of opinions is bimodal. After a characteristic time, most susceptibles become supporters of the party having more zealots and the opinion distribution is asymmetric. When the number of zealots of both parties is the same, the opinion distribution is symmetric and, in the long run, susceptibles endlessly swing from the state where they all support one party to the opposite state. Above the zealotry density threshold, when there is an unequal number of zealots of each type, the probability distribution is single-peaked and non-Gaussian. These properties are investigated analytically and with stochastic simulations. We also study the mean time to reach a consensus when zealots support only one party.

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I. INTRODUCTION

The voter model (VM) [1] is one of the simplest and most influential examples of individual-based systems exhibiting collective behavior. The VM has been used as a paradigm for the dynamics of opinion in socially interacting populations, see e.g. [2, 3] and references therein. The classical, or linear, VM is closely related to the Ising model [4] and describes how consensus results from the interactions between neighboring agents endowed with a discrete set of states (“opinions”). While the VM is one of the rare exactly solvable models in non-equilibrium statistical physics, it relies on oversimplified assumptions such as perfect conformity and lack of self-confidence of all voters. This is clearly unrealistic as it is recognized that members of a society respond differently to stimuli: Many exhibit conformity while some show independence, and this influences the underlying social dynamics [5, 6]. In order to mimic the dynamics of socially interacting agents with different levels of confidence, this author introduced “zealots” in the VM [6, 7]. Originally zealots were agents favoring one opinion [6, 7]. The case of inflexible zealots whose state never changes was then also studied [8], and the influence of committed and/or independent individuals was considered in various models of opinion and social dynamics [9, 10]. Recently, authors have investigated the effect of zealots in naming and cooperation games, and even in theoretical ecology [11].

In recent years, many versions of the VM have been proposed [2]. A particularly interesting variant of the VM is the two-state nonlinear $q$-voter model (qVM) introduced in [12]. In this model $q$ randomly picked neighbors may influence a voter to change its opinion. When $q = 2$, the qVM is closely related to the Sznajd model [13, 14] and to that of Ref. [15]. The properties of the qVM have received much attention and there is a debate on the expression of the exit probability in one dimension [16, 17].

Here, we investigate a generalization of the nonlinear qVM, with $q \geq 2$, in which a well-mixed population consists of inflexible zealots and susceptible voters influenced by their neighbors. As a motivation, this parsimonious model allows to capture three important concepts of social psychology [18] and sociology [19]: (i) conformity/imitation is an important social mechanism for collective actions; (ii) group pressure is known to influence the degree of conformity, especially when a group size threshold is reached [20]; (iii) the degree of conformity can be radically altered by the presence of some individuals that are capable of resisting group pressure [21, 22]. Here, the qVM mimics the process of conformity by imitation with group-size threshold, whereas zealots are independent agents that resist social pressure and can thus prevent to reach unanimity.

In this work, we study the fluctuation-driven dynamics of the two-state qVM with zealots in finite well-mixed populations and shed light on the deviations from the mean field description and from the linear case ($q = 1$). We find that below a zealotry density threshold the probability distribution is bimodal instead of Gaussian and, after a characteristic time, most susceptibles become supporters of the party having more zealots. When both parties have the same small number of zealots, susceptibles endlessly swing from the state where they all support one party to the other with a mean switching time that approximately grows exponentially with the population size.

In the next section we introduce the model. Sections III and IV are dedicated to the mean field description and to the model’s stationary probability distribution. In Secs. V and VI we discuss the long-time dynamics and the mean consensus time when there is one type of zealots. We summarize our findings and conclude in

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The complementary loss processes of A-susceptibles after the event equals -q. This accounts for processes in which the number of A-susceptibles (non-zealot voters in state +1) and others are “susceptibles”. Here, zealots are individuals that never change opinion: they permanently support either party A (A-zealots) or party B (B-zealots). Susceptible voters can change their opinion under the pressure of a group of neighbors. The population thus consists of a number $Z_+$ of A-zealots (pinned in state +1) and $Z_-$ of B-zealots (pinned in state −1), and a total of $S = N - Z_+ - Z_-$ susceptibles agents, of which $n$ are A-susceptibles (non-zealot voters in state +1) and $S - n$ are B-susceptibles (non-zealot voters in state −1). The fraction, or density, of susceptibles in the entire population remains constant and is given by $s = S/N$. For simplicity we assume that all agents have the same persuasion ability.

At each time step, a susceptible voter consults a group of $q$ neighbors (with $q > 1$) and, if there is consensus in the group, the voter is persuaded to adopt the group’s state with rate 1. The dynamics is a generalization of the nonlinear qVM with a finite density of zealots, and consists of the following steps:

1. Pick a random voter. If this voter is a zealot nothing happens.
2. If the picked voter is a susceptible, then pick a group of $q$ neighbors (for the sake of simplicity repetition is allowed, as in Refs. 4, 20). If all $q$ neighbors are in the same state, the selected voter also adopts that state. Nothing happens in the update if there is no consensus among the $q$ neighbors, or if the voter and its $q$ neighbors are already in the same state.
3. Repeat the above steps ad infinitum or until consensus is reached.

The case $q = 1$ corresponds to the classical (linear) voter model 1, 5, 10, and we therefore focus on $q \geq 2$.

For the sake of simplicity, we investigate this model on a complete graph (well-mixed population of size $N$). The state of the population is characterized by the probability $P_n(t)$ that the number of A-susceptibles at time $t$ is $n$. This probability obeys the master equation

$$\frac{dP_n(t)}{dt} = T_{n-1}^+ P_{n-1}(t) + T_{n+1}^- P_{n+1}(t) - (T_n^+ + T_n^-) P_n(t).$$

The first line accounts for processes in which the number of A-susceptibles after the event equals $n$, while the second term accounts for the complementary loss processes.

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symmetric low zealotry \((Z_\pm = Z_-)\), the number of susceptibles first approaches either the state \(n \approx 0\) (all B-susceptibles) or \(n \approx S\) (all A-susceptibles). After a characteristic time (see Sec. V.A), all susceptibles suddenly start switching from one state to the other, see Fig. II(a). A similar feature has been observed in the Sznajd model \((q = 2)\) with anticonformity \[17\]. When \(Z_+ > Z_- > 0\), the majority of susceptibles become A-supporters after a typical time (see Sec. V.B). The fluctuations in the number of A-susceptibles then grow endlessly, see Fig. II(b). An important aspect of this work is to analyze how demographic fluctuations arising in finite populations alter the mean field predictions. In Section V the phenomena illustrated by Fig. IV are studied in large-but-finite populations, and we show that these phenomena are beyond the reach of the next section’s mean field analysis.

III. MEAN FIELD DESCRIPTION

For further reference, it is useful to consider the mean field (MF) limit of an infinitely large population, \(N \to \infty\). In such a setting, demographic fluctuations are negligible and the rates \((2)\) can be written in terms of the density \(x = n/N\) of A-susceptibles, and the densities \(z_\pm = Z_\pm/N\) of zealots of each type: \(T_n^+ \to T^+(x) = (s - x)(x + z_+)^q\) and \(T_n^- \to T^-(x) = x(s + z_- - x)^q\). The MF dynamics is described by the rate equation obtained by averaging \(n/N\) from Eq. \((1)\) (and rescaling time as \(Nt \to t\) \[22\]):

\[
\dot{x} = T^+(x) - T^-(x) = (s - x)(x + z_+)^q - x(s - x + z_-)^q, \tag{3}
\]

where the dot denotes the time derivative and \(s = S/N\).

In the absence of zealotry \((z_+ = 0, s = 1)\), Eq. \((3)\) has two stable absorbing fixed points, \(x = 0\) (all B-supporters) and \(x = 1\) (all A-supporters) corresponding to consensus with either A or B party, separated by an unstable fixed point \(x = 1/2\) (mixture of A- and B-voters) \[17\]. It is worth noting that the dynamics of the qVM without zealots ceases when a consensus is reached and this happens in a finite time when the population size is finite \[14\] \[3\]. However, in the presence of zealots supporting both parties, the population composition endlessly fluctuates \[14\], see, e.g., Fig. IV.

In the presence of zealotry, the interior fixed points of Eq. \((\ref{eq:3})\) satisfy \(T^+(x) = T^-(x)\), which leads to

\[
\left(\frac{s - x}{x + z_+} - \frac{s - x + z_-}{s - x + z_-}\right)^q = 1. \tag{4}
\]

Depending on the values of \(z_\pm\) and \(q\), this equation has either three physical roots, or a single physical solution.

A. The symmetric case \(z_+ = z_- = z\)

When the density of zealots of both types is identical, \(z_+ = z_- = z\) and \(s = 1 - 2z\) with \(0 < z < 1/2\), Eq. \((\ref{eq:3})\) becomes

\[
\dot{x} = (1 - 2z - x)(x + z)^q - x(1 - z - x)^q,
\]

that is characterized by a fixed point \(x^* = s/2\). When \(z\) is sufficiently low, Eq. \((\ref{eq:4})\) has two further fixed points: \(x_+^*\) and \(x_-^* = s - x_+^*\). The analysis for arbitrary \(q > 1\) is unwieldy, but insight can be gained by focusing on \(q = 2\) and \(q = 3\), for which

\[
x_\pm^* = \left\{ \begin{array}{ll}
\frac{1}{2} (s \pm \sqrt{1 - 4z}) & (q = 2) \\
\frac{1}{4} (s \pm \sqrt{1 - 3z}) & (q = 3). 
\end{array} \right.
\]

We readily verify that \(x_\pm^*\) are both stable when \(z < z_c(q)\), with \(z_c(2) = 1/4\) and \(z_c(3) = 1/3\). When \(z > z_c(q)\), the fixed points \(x_\pm^*\) are unphysical and \(x^* = s/2\) is stable. This picture holds for arbitrary finite value of \(q > 1\): \(x_\pm^*\) are stable and the MF dynamics is characterized by bistability below a critical zealotry density \(z_c(q)\), while \(x^* = s/2\) is unstable when \(z < z_c\) and stable when \(z \geq z_c\), see Fig. IV(a,b). By determining when Eq. \((\ref{eq:4})\) has three physical roots, we have found the critical zealotry density \(z_c(q)\):

\[
z_c(q) = \left\{ \begin{array}{ll}
1/4 & (q = 2) \\
1/3 & (q = 3) \\
3/8 & (q = 4) \\
2/5 & (q = 5), 
\end{array} \right.
\]

while \(z_c(1) = 0\) since in the linear VM Eq. \((\ref{eq:3})\) has always one single stable fixed point \[14\]. Hence, the value of \(z_c\) increases with \(q\), while the values of \(x_\pm^*\) get closer to the values 0 (all B-susceptibles) and \(s\) (all A-susceptibles) as \(q\) increases with \(z\) kept fixed.

In this MF picture, the population’s average opinion given by the magnetization \(m(t) = 2x(t) - s\) undergoes a supercritical pitchfork bifurcation at \(z = z_c\) \[23\]: At \(t \to \infty\), the critical value \(z_c\) separates an ordered phase \((z < z_c)\), where a majority of susceptibles supports one party, from a disordered phase \((z > z_c)\) in which each party is supported by half of the susceptibles, see Fig. IV.
is given by the magnetization \( m \) have determined the critical density of zealotry when Eq. (3) has three physical fixed points, we have found that the stationary MF magnetization vanishes when \( z \) and \( z \) and \( z \) and \( z \) and \( z \) at higher zealotry, see Fig. 2(c,d). Hence, the stationary MF magnetization at low zealotry \( z < z_c \) depends on the initial condition and is \( m(\infty) = m^*_a = 2(x^*_a + \delta z) - s \) if \( m(0) > 2(x^* + \delta z) - s \) and \( m(\infty) = m^*_c = 2(x^*_c + \delta z) - s \) if \( m(0) < 2(x^* + \delta z) - s \). When \( z > z_c \) the stationary MF magnetization is \( m(\infty) = m^*_+ \).

C. The absorbing case \( z_+ = \zeta, z_- = 0 \)

When there are only A-zealots, \( z_+ = \zeta > 0 \) and \( z_- = 0 \), Eq. (3) becomes

\[
\dot{x} = (1 - \zeta - x) \left[ (x + \zeta)^q - x(1 - \zeta - x)^{q-1} \right],
\]

and has an absorbing fixed point \( x^*_a = 1 - \zeta \). Below a critical zealotry density \( \zeta_c(q) \), this rate equation admits two other fixed points: \( x^*_a, \) that is stable, and \( x^* \) that is unstable and separates \( x^*_a \) and \( x^*_b, \) see Fig. 3(e,f). When \( \zeta > \zeta_c(q) \), the absorbing state \( x^*_a = 1 - \zeta \) is the only fixed point. For \( q = 2 \) and \( q = 3 \), we explicitly find

\[
x^*_b = \begin{cases} \frac{1}{4} \left( 1 - 3\zeta - \sqrt{1 - (6 - \zeta)\zeta} \right) & (q = 2) \\ \frac{1-2\zeta(1+\zeta)\sqrt{1-k}}{2(2+\zeta)} & (q = 3) \end{cases}
\]

and

\[
x^* = \begin{cases} \frac{1}{4} \left( 1 - 3\zeta + \sqrt{1 - (6 - \zeta)\zeta} \right) & (q = 2) \\ \frac{1-2\zeta(1+\zeta)\sqrt{1+k}}{2(2+\zeta)} & (q = 3) \end{cases}
\]

From these expressions, and more generally by determining when Eq. (3) has three physical fixed points, we have found the critical zealotry density in the absorbing case:

\[
\zeta_c(q) = \begin{cases} 3 - 2\sqrt{2} & (q = 2) \\ 1/4 & (q = 3) \\ 0.295 & (q = 4) \\ 0.326 & (q = 5) \end{cases}
\]

We thus distinguish two regimes:

(i) When \( \zeta < \zeta_c(q) \) both \( x^*_{a,b} \) are stable and the dynamics crucially depends on the initial density \( x_0 \) of A-susceptibles: If \( x_0 > x^* \), the final state is the consensus with party A; whereas the steady state consists of a vast majority of B-party voters when \( x_0 < x^* \). In Sec. VI, we show that random fluctuations drastically alter this picture: In a finite population, \( x_0^* \) is a metastable state when \( \zeta < \zeta_c(q) \) and \( x_0 < x^* \), and we shall see that the A-consensus is reached after a very long transient that scales exponentially with the population size.

(ii) When \( \zeta > \zeta_c(q) \), as well as when \( \zeta = \zeta_c \) and \( x_0 > x^* \), the absorbing state is rapidly reached.

IV. STATIONARY PROBABILITY DISTRIBUTION

In this section, we compute the stationary probability distribution (SPD) of the qVM with zealotry when there
is no absorbing state, and show that it shape generally differs from the Gaussian-like distribution obtained in the linear VM with zealots [4].

The SPD \( P_n^* = \lim_{t \to \infty} P_n(t) \) obeys the following stationary master equation, obtained from Eq. (3):

\[
T_{n-1}^+ P_{n-1}^* + T_{n+1}^- P_{n+1}^* - (T_n^+ + T_n^-) P_n^* = 0.
\]

The exact SPD is uniquely obtained by iterating the detailed balance relation \( T_{n-1}^+ P_{n-1}^* = T_n^- P_n^* \) [23], yielding

\[
P_n^* = P_0^* \prod_{j=0}^{n-1} (T^+_{j+1}/T^-_{j+1})
\]

\[
= P_0^* \prod_{j=0}^{n-1} \left( \frac{S-j}{j+1} \right)^q \left( \frac{j+Z_+}{S+Z_--j-1} \right)^q,
\]

where the normalization \( \sum_{n=0}^{\infty} P_n^* = 1 \) gives \( P_0^* = 1/[1+\sum_{j=0}^{S} T^+_{j}/T^-_{j+1}] \) and \( P_n^* = 1 - P_0^* - \sum_{k=1}^{S} P_k^* \).

Since \( n = N[(m+s)/2-\delta z] \), the stationary magnetization distribution \( Q_m^* \) has the same shape as \( F_n^* \), with

\[
Q_m^* = P_n^* \left[ N[(m+s)/2-\delta z] \right] \left( \frac{S-j}{j+1} \right)^q \left( \frac{j+Z_+}{S+Z_--j-1} \right)^q.
\]

In large populations, a useful approximation of (10) is obtained by writing \( P_n^* = P_0^* \exp \left( \sum_{j=0}^{n-1} \Psi_j \right) \) with \( \Psi_j = \ln \left( T^+_{j}/T^-_{j+1} \right) \) and by using Euler-MacLaurin formula \( \sum_{j=0}^{n-1} \Psi_j = \int_0^n \Psi(x) dx + (\Psi_0 + \Psi_{n+1})/2 \), where we have neglected higher order terms [24].

When \( N \gg 1 \), it is useful to work in the continuum limit with the rates \( T_n^\pm \to T^\pm(x) \), as in Sec. III. By introducing

\[
\Psi(x) = \ln \left[ T^+(x)/T^-(x) \right],
\]

we have \( \sum_{j=0}^{\infty} \Psi_j \approx N \int_0^x \Psi(x) dx \) to leading order in \( N \). Hence, the leading contribution to the SPD when \( N \gg 1 \) is

\[
P_n^* \sim P_0^* \exp \left( N \int_0^x \Psi(y) dy \right) = P^*(x).
\]

The local maxima of \( P^*(x) \) satisfy \( \Psi(x) = 0 \), see (12), and thus coincide with the fixed points of Eq. (3). As a consequence, in large populations \( P_n^* \) is either characterized by a single peak at \( n^* = N x^* \) when \( z > z_c \), or has two peaks at the metastable states \( n^+_z = N x^+ \) when \( z < z_c \). In this case, there is bistability and the amplitudes of the peaks at \( n^+_z \) are in the ratio \( N \gg 1 \)

\[
\frac{P^*_n}{P^*_{n-1}} \sim e^{N \int_{-z}^{+z} \Psi(x) dx}.
\]

The integrals in Eqs. (13) and (14) can be computed, but their expressions are unenlightening. Here, we infer the properties of \( P_n^* \sim P^*(x) \) and \( Q_m^* \) from those of \( \Psi(x) \).

**A. Stationary probability distribution in the symmetric case**

In the symmetric case, \( z_+ = z_ = z \), Eq. (12) becomes

\[
\Psi(x) = \ln \left[ \left( \frac{1 - (x + 2z)}{x} \right)^q \left( \frac{x + z}{1 - (x + z)} \right)^q \right]
\]

and has the symmetry \( \Psi(x) = -\Psi(s-x) \). We distinguish the cases of low and high zealotry density:

(i) When \( z < z_c(q) \), the fixed points \( x^+ \) of Eq. (3) are also the roots of \( \Psi(x) \). Hence, when \( N \gg 1 \), \( P_n^* = P_{S-n}^* \sim P^*(x) \propto e^{N \int_{-z}^{+z} \Psi(x) dx} \) is a symmetric bimodal SPD characterized by two peaks at \( n = n^+ \). As a consequence, \( Q^*_m = Q_{-m}^* \) is an even function.

In Figure 4 (a), we show the exact SPD for \( q = 2 - 4 \) characterized by two peaks of same intensity at \( n = n^+ \) and a local minimum at \( n^* = S/2 \). We remark that when
\[ sP_n^* \text{ vs. } n/s \text{ for } q = 2 - 4 \text{ at high zealotry } z > z_c. \]

Here, \( N = 200 \) and \((q, z) = (2, 0.4) (\diamondsuit), (3, 0.4) (\circ), (3, 0.375) (\Delta), (4, 0.4) (\bigcirc)\). The SPDs have a single peak at \( n/s = N/2 \) and width broadens when \( q \) and \( 1/z \) are increased at fixed \( N \). Inset: \( SP_n^* \text{ vs. } n/S \) for \( q = 2, z = 0.4 \) and different values of \( N \). Here, \( N = 200 (\diamondsuit) \) and \( N = 600 (\bullet) \).

Fig. 5: Rescaled SPD under asymmetric zealotry \( Z \Delta = N(1 \pm \delta)z \). \( a \) \( sP_n^* \text{ vs. } n/s \) from Eq. (10) at low zealotry \((z < z_c)\). Here, \( N = 200 \) and \((q, Z_+, Z_-) = (2, 41, 39) (\diamondsuit), (3, 51, 49) (\circ), (3, 53, 47) (\Delta), (4, 51, 49) (\bigcirc)\). For \( q = 4 \), the range where \( P_n^* \lesssim 10^{-12} \) is not shown. Inset: \( SP_n^* \text{ vs. } n/S \) for \( q = 2 \) and \((N, Z_+, Z_-) = (200, 41, 39) (\bigcirc), (400, 82, 78) \text{ (gray-filled symbols)}\). \( b \) Left-skewed rescaled SPD at high zealotry, with a single peak at \( n_+^* \). Here, \( N = 200 \) and \((q, Z_+, Z_-) = (2.67, 63) (\circ), (3, 72, 68) (\circ), (3, 74, 66) (\Delta), (4, 82, 78) (\bigcirc)\), see text.

B. Stationary probability distribution in the asymmetric case

In the asymmetric case, with zealot densities \( z_\pm = (1 \pm \delta)z \) and \( \delta > 0 \), Eq. (12) is

\[
\Psi(x) = \ln \left[ \frac{(1 - (x + 2z))}{x} \right] \left( \frac{x + z(1 + \delta)}{1 - z(1 + \delta) - x} \right)^q
\]

and has either three or one physical roots:

(i) At fixed \( q \) and \( \delta \), when \( z < z_c(q, \delta) \), the fixed points \( x^* \) and \( x_\pm^* \) of Eq. (13) are the physical roots of \( \Psi(x) \).

Since \( P_n^* \sim P^*(x) \propto e^{N \int_0^x \Psi(y) dy} \) when \( N \gg 1 \), the SPD is again a bimodal distribution peaked at \( n_+^* \). However, \( x_+^* \) has a greater basin of attraction than \( x_-^* \) and \( \int_{x_+^*}^{x_-^*} \Psi(y) dy > 0 \). As a consequence, the SPD is asymmetric, with the peak at \( n_+^* \) being much stronger than the one at \( n_-^* \). The ratio of the peaks is given by \( (14) \), which shows that the asymmetry of \( P_n^* \) grows exponentially with \( N \) and increases with \( q \), see Fig. 6(a).

(ii) When \( z \geq z_c \), the only physical root of \( \Psi(x) \) is \( x^* = s/2 \), as in the classical voter model (10). Hence, \( P_n^* \sim P^*_{S-\infty} \sim P^*(x) \propto e^{N \int_0^{s/2} \Psi(y) dy} \) has a single maximum at \( x = s/2 \) when \( N \gg 1 \). The resulting symmetric Gaussian-like distribution centered at \( n^* = s/2 \) when \( N \gg 1 \), see Fig. 5, is very similar to the SPD obtained in the classical voter model with zealots (13). Fig. 5 inset illustrates that the probability density steepens around \( s/2 \) when the population size is increased.
an asymmetry in the zealotry in the linear VM does not significantly affect the form of the SPD, we here find that in the qVM even a small bias in the zealotry drastically changes the shape of the SPD and leads to marked dominance of the party with more zealots.

In Fig. 4(a), we report the exact SPD for $q = 2 - 4$ and illustrate its asymmetric bimodal nature, with marked peaks of different intensities at $n^*_1$. We notice that the asymmetry in the peaks intensity, given by (14), is stronger when we increase $q$ and $Z_+ - Z_- \propto \delta z$. As in the symmetric case, the SPD decays dramatically away from the peaks and $P_n^* \ll n \ll n^*$ vanishes with $N \gg 1$ and when $q$ is increased. In Fig. 4(a, inset) we show that the SPD remains bimodal when the population size is increased, and the main influence of raising $N$ is to concentrate the probability density near its peak at $x^*_+ = n^*_+ / N$ (when $N \gg 1$).

(ii) At fixed $q$ and $\delta$, when $z > z_c$, the only real root of $\Psi(x)$ is $x = x^*_+$. This lies closer to $x = s$ than to $x = 0$, and hence $\int_0^x \Psi(y)dy$ is an asymmetric function with a single maximum at $x = x^*_+$. Therefore, in large populations $P_n^*$ is an asymmetric left-skewed SPD with a single peak at $n = n^*_+$, as shown in Fig. 5(b) where we see that the SPD broadens when $q$ is increased and that it steepens when $Z_+ - Z_- \propto \delta z$ is increased. As above, the probability density steepens around $x^*_+$ when $N$ is increased.

V. FLUCTUATION-DRIVEN DYNAMICS AT LOW ZEALOTRY

We now study how a small non-zero density of zealots of both parties ($0 < z < z_c$) affects the qVM long-time dynamics. We show that, after a typically long transient, all susceptibles voters switch allegiance from the state $n = 0$ (all B-susceptibles) to state $n = S$ (all A-susceptibles) in a typical switching time. In the symmetric case, there is “swing-state dynamics” with all susceptibles endlessly swinging allegiance. In the asymmetric case where party A has more zealots than party B, the dynamics is characterized by various time-scales and by growing fluctuations around the metastable state $n^*_1$. Below, we show that the long-time qVM dynamics is driven by fluctuations and characterized by a mean switching time that scales (approximately) exponentially with the system size $N$ in large-but-finite populations.

A. Swing-state dynamics and switching time in the case of symmetric zealotry

As illustrated in Fig. 1(a), the long-time dynamics in the symmetric case is characterized by the continuous swinging from states $n \approx 0$ to $n \approx S$ and vice versa. When $Z_+ = Z_-$, all susceptibles thus continuously switch allegiance in the long run. In that regime, the magnetization $m(t) = (2n(t) - S)/N$ is thus characterized by abrupt jumps from $m \approx \pm s$ to $m \approx \mp s$, see Fig. 2 while the stationary ensemble-averaged magnetization $\langle m(\infty) \rangle = \sum_{m=-s}^{s} mQ_m = 0$, since $Q_m$ is even and each agent is as likely to be in one or the opposite state. A similar phenomenon has been found in the Sznajd model ($q = 2$) with anticonformity [17].

This swing-state phenomenon is not captured by the mean field description of Sec. III and is here characterized by the mean time $\tau_0^S$ to switch for the first time from state $n = 0$ to $n = S$. The scaling of $\tau_0^S$ on $N$ allows us to rationalize the data of Fig. 6 where the switching time is found to dramatically increase with the population size.

FIG. 7: Typical evolution of the rescaled magnetization for different values of $q$ and $N$ with initial condition $m(0) = -s$ (all B-susceptibles) on a semi-log scale: (a) Single realization of $m(t)/s$ vs. time for $q = 2, z = z_+ = 0.2 < z_c$ and population size $N = 100$. At time $t \approx 625$, $m = s$ and starts swinging back and forth the values $m = \pm s$. Here, the MF predicts $m^*/s \approx 0.745$ and $\pm m^*/s$ are shown as dashed lines. Inset: $m(t)/s$ vs. time for $q = 3, z = 0.3 < z_c$ and $N = 100$. The system starts swinging between $m = \pm s$ at $t \approx 640$. Here, $\pm m^*/s \approx 0.693$ (dashed). (b) $m(t)/s$ vs. time for the same parameters as in (a) but with $N = 300$. The system’s magnetization switches to $m = s$ only at $t \approx 2 \cdot 10^5$. Inset: $m(t)/s$ vs. time for $q = 3, z = 0.3$ and $N = 400$. The magnetization switches to $m = s$ at $t \approx 3 \cdot 10^5$. The difference in the switching times in (a) and (b) results from the exponential scaling of the mean switching time on $N$, see text.
Clearly, the symmetry implies that the mean switching, or swinging, time $\tau_0^S$ is identical to the mean time $\tau_0^S$ to switch from $n = S$ to $n = 0$.

Finding the mean switching time can be formulated as a first-passage time problem and, when $N \gg 1$, $\tau_0^S$ can be computed using the framework of the backward Fokker-Planck equation (bFPE) \[22]. In this context, the model’s bFPE infinitesimal generator is

$$G_b(x) = [T^+(x) - T^-(x)] \partial_x + \left[\frac{T^+(x) + T^-(x)}{2}\right] \partial_x^2. \quad (15)$$

The mean time $\tau^S(x_0)$ to be absorbed at $x = s$ (all A-susceptibles), starting from the initial state $x = x_0$, with a reflective boundary at $x = 0$ (all B-susceptibles), obeys

$$G_b(x_0) \tau^S(x_0) = -1, \quad (16)$$

with $(d/dx)\tau^S(0) = 0$ and $\tau^S(s) = 0$ (reflective and absorbing boundaries) \[22, 23\]. To obtain the mean switching time $\tau_0^S$ we solve Eq. (16) with $x_0 = 0$ using standard methods \[22\], and obtain

$$\tau_0^S = 2N \int_0^s dy \ e^{-N\phi(y)} \int_y^s e^{N\phi(v)} dv \left[\frac{T^+(v) + T^-(v)}{2}\right], \quad (17)$$

where $\phi(v) = -2 \int_0^v du \ \left\{\frac{T^-(u) - T^+(u)}{T^-(u) + T^+(u)}\right\}$. As with other fluctuation-driven phenomena associated with metastable states, see e.g. \[22, 25, 29\] and below, this result predicts that the mean switching time $\tau_0^S$ grows (approximately) exponentially with the population size $N$. This explains the difference of various orders of magnitude in the switching time observed in Figs. 7(a) and 7(b).

The predictions of \[13\] are reported in Fig. 8 for various values of $z < z_c$. These are in good agreement with the results of numerical simulations (averaged over 1000 samples, each run for $10^6$ simulation steps). When $z$ is lowered well below $z_c$, the peaks of the SPD approach $n = 0$ and $n = S$. In this case, $\tau_0^S$ increases and switching allegiance takes very long. At fixed $z < z_c$, we find that $\tau_0^S$ increases with $q$. Interestingly, we also find that $\tau_0^S$ can exhibit a non-monotonic dependence on $z$ just below $z_c$ when $q$ is kept fixed, as shown in Fig. 8.

B. Time-scale separation and growing fluctuations in the asymmetric case

In the asymmetric case $0 < z_- < z_+$, the party A has more zealot supporters than party B. In this situation, when $z < z_c$ the SPD has a marked peak near $n = S$, see Fig. 3(a). As shown in Fig. 3(b), the long time dynamics is characterized by a large majority of susceptibles becoming A supporters independently of the initial state. The magnetization $m(t) = 2\delta z + [2n(t) - S]/N$ thus fluctuates around its MF value $m_0^F$ before reaching $m = m_{\text{max}} = s + 2\delta z$ when all susceptibles are supporters of party A, see Fig. 3(a). The population composition then endlessly fluctuates, with a majority of susceptibles supporting party A. In this case, with Eq. (11), the stationary ensemble-averaged magnetization $(m(\infty)) = \sum_{m=m_{\text{min}}}^{m_{\text{max}}} m Q_m^* > 0$ is positive.

The qVM dynamics is thus characterized by various regimes not captured by the mean field description. For concreteness, we consider that the initial density of A-susceptibles is $x_0 < x^*$, as in Fig. 3, and distinguish four time scales:

(i) After a mean time of order $\tau_r$, the system quickly relaxes toward the metastable state $n_+^*$ where a random voter has the MF opinion $m_+^*$, see Fig. 3(a).

(ii) After a mean time of order $\tau^+$, almost all realizations suddenly approach the metastable state $n_+^*$ where $m(t) \approx m_+^*$, see Figs. 3(a) and 3(b). The mean transition time $\tau^+$, as well as the average relaxation times, can be estimated using Kramers’ classical escape rate theory \[27\]. The latter gives the mean transition time $\tau_K$ between the two local minima of the double-well potential $U(x)$ in which an overdamped Brownian particle is moving subject to a zero-mean delta-correlated Gaussian white noise force $\xi(t)$. Here, we consider a potential $U(x)$ such that $dU/dx = T^-(x) - T^+(x)$, and the noise correlations $\langle \xi(t)\xi(t') \rangle = \delta(t-t') [T^+(x^*) + T^-(x^*)]/N$. The bFPE generator of this Brownian particle is \[13\] with a constant diffusive term evaluated at $x^*$. Kramer’s formula hence gives \[25, 27\]:

$$\tau^+ \simeq \tau_K = 2\pi \tau_1 \tau_2 e^{2N \int_{x^*}^{x^-} \frac{T^-(u) - T^+(u)}{T^-(u) + T^+(u)} \, du},$$

where $\tau_1 = 1/\sqrt{U''(x^*)}$, and $\tau_2 = 1/\sqrt{U''(x^-)}$ denotes the mean relaxation time from state $n = n^*$ to $n = n_+^*$. (iii) The system then fluctuates around $n_+^*$ before reaching the state $n = S$ (all A-susceptibles) where $m = m_{\text{max}}$, see Fig. 3(a) after a mean time $\tau_0^S$. In the realm of the bFPE, the mean time $\tau^S$ for all susceptibles

\[8\]
is well approximated by Kramer’s formula, yielding

$$\tau_{S} = 2N \int_{x_0}^{x_S} dy \ e^{-N\phi(y)} \int_{0}^{y} \frac{e^{N\phi(v)}}{T^{+}(v) + T^{-}(v)} \ dv.$$  \hspace{1cm} (18)

When $n_{S}$ is close to the state $n = S$, the main contribution to $\tau_{S}$ is given by the mean transition time $\tau_{S}^{\pm}$ that is independent of $x_0 < x^*$, as illustrated by Fig. 9(b). This is well approximated by Kramer’s formula, yielding

$$\tau_{S} \sim \tau_{S}^{\pm} \simeq 2\pi \tau_{T_{1}} \tau_{T_{2}} e^{2N \int_{x_0}^{x_{S}} \frac{\tau^{-(y)} - \tau^{+(y)}}{S^{-(\tau^{+}(y))} + S^{+(\tau^{-(y)})}} \ dy},$$

showing that the mean switching time scales exponentially with the population size.

(iv) The amplitude of the fluctuations around $n \approx S$, where $m(t) \approx m_{\text{max}}$ grows endlessly in time, see Fig. 9(b), and the system eventually returns to the state $n = 0$ (all B-susceptibles). Yet, this occurs after an enormous amount of time, of order $e^{2N \int_{x_0}^{x_{S}} \frac{\tau^{-(y)} - \tau^{+(y)}}{S^{-(\tau^{+}(y))} + S^{+(\tau^{-(y)})}} \ dy}$, that is generally not physically observable when $N \gg 1$.

The predictions (13) and its approximation (17) are reported in Fig. 8(b), where they are in good agreement with the results of stochastic simulations. These results confirm that $\tau_{S}$ grows approximately exponentially with $N$ when $N \gg 1$. In Fig. 9(b), we also see that $\tau_{S}$ increases with $1/z$, and with $q$ when $z$ and $\delta$ are fixed. As illustrated in Fig. 9(a), contrary to the case of symmetric zealotry, there is no “swing-state dynamics”: After a mean time $\tau_{S}$ the population persists near $n \approx S$ where most susceptibles are A supporters and the magnetization is $m \approx m_{\text{max}}$, and there is virtually no switching back to state $n \approx 0$. Hence, a small bias in the zealotry, combined with fluctuations and nonlinearity, can greatly affect the voters’ opinion in the qVM.

VI. MEAN CONSENSUS TIME IN THE PRESENCE OF ONE TYPE OF ZEALOTS

When there are only A-zealots, with $z_{+} = \zeta$ and $z_{-} = 0$, an A-party consensus is always reached. Yet, the dynamics leading to the corresponding absorbing state $n = S$ depends non-trivially on the zealotry density and on the initial density $X_0$ of A-susceptibles.

Here, the fluctuation-driven dynamics is characterized by the mean consensus time (MCT). As illustrated in Fig. 10, the MCT can change by several orders of magnitudes when $\zeta$ and $x_0$ change over a small range: (i) Below the critical zealotry density $\zeta_c$, the MCT grows exponentially with the population size $N$ when $x_0 < x^*$, see Fig. 10(b); (ii) Otherwise the MCT grows logarithmically with $N$, see Fig. 10(inset). These phenomena are analyzed as follows:

(i) When $\zeta \leq \zeta_c$ and $x_0 < x^*$, in line with the MF analysis, the density of A-susceptibles first lingers around the metastable state $x^*$ until a large fluctuation drives the system towards the absorbing state. This large-fluctuation-driven phenomenon is particularly well captured by the WKB theory (25) (30). The essence of this method consists of studying the quasi-stationary probability distribution (QSPD) $\pi_{n}$ obtained by setting $P_{S}(t) \approx \pi_{n} e^{-t/\tau_{c}}$ for $0 \leq n < S$ and $P_{S}(t) \approx 1 - e^{-t/\tau_{c}}$ into the master equation (4). The MCT is the mean decay time $\tau_{c}$ of the QSPD. Since $(d/dt)P_{S} \approx e^{-t/\tau_{c}} / \tau_{c} \approx T_{S-1}^{+} \pi_{S-1} e^{-t/\tau_{c}}$, we indeed find (29) (30)

$$\tau_{c} = (T_{S-1}^{+} \pi_{S-1})^{-1}.$$  \hspace{1cm} (20)

The computation of the MCT therefore requires finding the QSPD. This obeys

$$T_{n-1}^{+} \pi_{n-1} + T_{n+1}^{-} \pi_{n+1} - (T_{n}^{+} + T_{n}^{-}) \pi_{n} = 0, \hspace{1cm} (21)$$

obtained from Eq. (4) upon neglecting an exponentially small term $\pi_{n} / \tau_{c}$. In the limit $N \gg 1$, the density $x = $
n/N is treated as a continuous variable and Eq. (20) yields $\tau_c^{-1} = (\pi(s)/N) \frac{d}{ds} T^+(x)\big|_{x=x_c^+}$. In the continuum limit, Eq. (21) is solved with the WKB Ansatz

$$\pi(x) \simeq \mathcal{A} e^{-N S(x) - S_1(x)},$$

where $S(x)$ is the action, $S_1(x)$ is the amplitude, and $\mathcal{A} \sim e^{N S(x_c^+)}$ is a normalization constant [30]. By substituting (23) into (21), to leading order we find [23] [30]

$$S(x) = -\int_x^\infty \Psi(y) \, dy,$$

where, as in Sec. IV, $\Psi(y) = \ln [T^+(y)/T^-(y)]$.

Hence, when $x_0 < x^*$ and $N \gg 1$, the leading contribution to the MCT is given by the accumulated action $\Delta S$ over the path joining the metastable state $x = x_0^*$ and the unstable steady state $x = x^*$ [23] [30]:

$$\tau_c \sim e^{N[S(x^*) - S(x_0^*)]} = e^{N \Delta S}. \quad (24)$$

The next-to-leading correction arising from $S_1(x)$ is given in Refs. [29] [30], but for our purpose Eq. (24) already provides useful information on the MCT. In fact, for $q = 2$ and $q = 3$, the action (23) explicitly reads

$$-S(x) = \begin{cases} 
(1 - \zeta) \ln (1 - \zeta - x) + 2 \zeta \ln (x + \zeta) \\
+ x \ln \left[ \frac{(x+\zeta^2)}{(1-\zeta-x)} \right] \\
\zeta \ln x + 2 \ln (1 - \zeta - x) \\
+ (x + \zeta) \ln \left[ \frac{(x+\zeta^3)}{(1-\zeta-x)} \right] \end{cases} \quad (q = 2) \quad (25)$$

With these expressions, and with (6) and (7) for $x_0^*$ and $x^*$, the leading contribution to the WKB approximation of the MCT is computed explicitly, and the results reported in Fig. (a) are in excellent agreement with those of stochastic simulations when $N \gg 1$ and confirm that $\tau_c$ grows exponentially with $N$. We can also check that $\Delta S$ is a decreasing function of $\zeta$, which clearly implies that the MCT grows when $\zeta$ is decreased.

(ii) When $\zeta > \zeta_c$, or for any $\zeta > 0$ when the initial density $x_0 > x^*$, the A-party consensus is reached much quicker than in the case (i), typically after a time of order $O(\ln N)$, see Fig. (b) inset. The backward Fokker-Planck formalism is again suitable to derive this result. In such a framework, the MCT obeys Eq. (4) supplemented by reflective and absorbing boundary conditions $\tau_c(0) = 0$ and $\tau(s) = 0$ [23]. Proceeding as in Sec. V, we find again the expression:

$$\tau_c(x_0) = 2N \int_{x_0}^{x^*} dy \, e^{-N \phi(y)} \int_{\phi(y)}^{0} \frac{e^{N \phi(v)}}{T^+(v) - T^-(v)} \, dv. \quad (26)$$

As shown in the inset of Fig. (b) this expression is in good agreement with the results of stochastic simulations and captures the functional dependence of the MCT whose leading contribution grows logarithmically with $N$ and increases with $q$. It is also worth noting that Eq. (26) also provides a meaningful approximation of the MCT in the metastable regime, even though when $N \gg 1$ its predictions are usually less accurate than those of the WKB method, see e.g. [25] [30].

VII. SUMMARY AND CONCLUSION

We have studied the dynamics of the non-linear q-voter model (qVM) in the presence of inflexible zealots in a finite well-mixed population. In this model, voters can support two parties and are either “susceptibles” or “inflexible zealots”. Susceptible voters adopt the opinion of a group of $q \geq 2$ neighbors if they all agree, while zealots are here individuals whose state never changes. The qVM with zealots is introduced as a simple non-trivial model able to capture the essence of important concepts of social psychology and sociology, such as the relevance of conformity and independence as mechanisms for collective actions [1] [6], and the existence of group-size threshold that influences the social impact of conformity [1].
In spite of its simplicity and the fact that the detailed balance is satisfied, the dynamics of the non-linear $q$VM with zealots is rich and characterized by fluctuation-driven phenomena and non-trivial probability distributions. The dynamics is particularly interesting at low level of zealotry, where the stationary distribution is bimodal. In this case, we have found that when one party has more zealots than the other, the intensity of one peak greatly exceeds that of the other. The dynamics is thus characterized by various time scales and growing fluctuations around a state in which a majority of susceptibles support one party to the state where they all support the other party. We have rationalized all these features by computing the exact stationary probability distribution and, within the backward Fokker-Planck formalism, the mean times for all susceptibles to switch allegiance. We have thus found that these mean switching times grow approximately exponentially with the population size, and they increase when the number of zealot decreases at low zealotry. When zealots support only one party, we have shown that a consensus is reached in a mean time that grows either exponentially or logarithmically with the population size, depending on the zealotry density and the initial condition.

Our findings show that the properties of the nonlinear $q$VM with zealots ($q \geq 2$) are dominated by fluctuations, and have revealed that they are sensitive to even a small bias in the zealot densities. Most of the features of the nonlinear $q$VM with inflexible zealots are therefore beyond the reach of a simple mean field analysis and generally deviate from those of the classical linear voter model.

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