ON LIFTING AND MODULARITY OF REDUCIBLE RESIDUAL
GALOIS REPRESENTATIONS OVER IMAGINARY QUADRATIC
FIELDS

TOBIA BERGER\textsuperscript{1} AND KRZYSZTOF KLOSIN\textsuperscript{2}

Abstract. In this paper we study deformations of mod $p$ Galois representations $\tau$ (over an imaginary quadratic field $F$) of dimension 2 whose semi-simplification is the direct sum of two characters $\tau_1$ and $\tau_2$. As opposed to [BK13] we do not impose any restrictions on the dimension of the crystalline Selmer group $H^1_\Sigma(F, \operatorname{Hom}(\tau_2, \tau_1)) \subset \operatorname{Ext}^1(\tau_2, \tau_1)$. We establish that there exists a basis $B$ of $H^1_\Sigma(F, \operatorname{Hom}(\tau_2, \tau_1))$ arising from automorphic representations over $F$ (Theorem 8.1). Assuming among other things that the elements of $B$ admit only finitely many crystalline characteristic 0 deformations we prove a modularity lifting theorem asserting that if $\tau$ itself is modular then so is its every crystalline characteristic zero deformation (Theorems 8.2 and 8.5).

1. Introduction

Let $p$ be an odd prime. Let $F$ be a number field, $\Sigma$ a finite set of primes of $F$ (containing all primes $\mathfrak{p}$ of $F$ lying over $p$) and $G_\Sigma$ the Galois group of the maximal extension of $F$ unramified outside $\Sigma$. Let $E$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathcal{F}$. Let $\tau_1, \tau_2 : G_\Sigma \to \text{GL}_n(\mathcal{F})$ be two absolutely irreducible non-isomorphic representations with $n_1 + n_2 = n$, which we assume lift uniquely to crystalline representations $\tilde{\tau}_i : G_\Sigma \to \text{GL}_n(\mathcal{O})$.

The aim of this article is to study deformations of non-semi-simple continuous crystalline representations $\tau : G_\Sigma \to \text{GL}_n(\mathcal{F})$ whose semi-simplification is $\tau_1 \oplus \tau_2$ in the case $n = 2$ and $F$ is an imaginary quadratic field. We analyzed this deformation problem in [BK13] under the additional assumption that $H^1_\Sigma(F, \operatorname{Hom}(\tau_2, \tau_1))$ is one-dimensional (which is equivalent to saying that there exists only one such $\tau$ up to isomorphism). Here $H^1_\Sigma$ denotes the subgroup of $H^1$ consisting of classes unramified outside $\Sigma$ and crystalline at all $\mathfrak{p} \mid p$. In this paper we do not make any assumption on this dimension. Disposing of the “dim=1” assumption is more than a technicality as in the general case one can no longer expect to be able to identify the universal deformation ring with a Hecke algebra.

This question was studied by Skinner and Wiles for $n = 2$ and totally real fields $F$ in the seminal paper [SW99]. In that paper the authors analyze primes $\mathfrak{q}$ of the (ordinary) universal deformation ring $R_{\tau}$ of $\tau$ and prove that they are ‘pro-modular’ in the sense that the trace of the deformation corresponding to $R_{\tau} \to R_{\tau}/\mathfrak{q}$ occurs in the Hecke algebra $T$. In particular no direct identification of $R_{\tau}$ and $T$ is made.

2010 Mathematics Subject Classification. 11F80, 11F55.

Key words and phrases. Galois representations, Galois deformations, automorphic forms, modularity.

The work of the second author was partially supported by a PSC-CUNY Award, jointly funded by The Professional Staff Congress and The City University of New York.
In this article we take a different approach and work with the reduced universal deformation ring \( R^\text{\text{red}}_\tau \) and its ideal of reducibility. In the “dim=1” case, the authors proved (as a consequence of an \( R = T \)-theorem - Theorem 9.14 in [BK13]) that \( R^\text{\text{red}}_\tau \) is a finitely generated \( \mathbb{Z}_p \)-module. In contrast, if \( \dim H^1_{\text{dR}}(F, \text{Hom}(\tilde{\tau}_2, \tilde{\tau}_1)) > 1 \), while there are only finitely many automorphic representations whose associated Galois representations are deformations of \( \tau \), the ring \( R^\text{\text{red}}_\tau \) may potentially be infinite over \( \mathbb{Z}_p \) (Remark 2.13). This is a direct consequence of the existence of linearly independent cohomology classes inside the Selmer group which can be used to construct non-trivial lifts to \( \text{GL}_2(\mathbb{F}_p[X]) \). The resulting (potentially large) characteristic \( p \) components of \( R^\text{\text{red}}_\tau \) do not arise from automorphic representations and in this paper we will ignore them by considering a certain torsion-free quotient \( R^\text{\text{red}}_\tau / \mathfrak{m}_p \) instead of \( R^\text{\text{red}}_\tau \) itself. It is however possible that by doing so we are excluding some characteristic \( p \) deformations whose traces may be modular in the sense of [CM09] (i.e. arise from torsion Betti cohomology classes).

On the other hand, as opposed to the situation studied in [SW99], over an imaginary quadratic field there are no reducible deformations to characteristic zero which in turn is a consequence of the finiteness of the Bloch-Kato Selmer group \( H^1_{\text{dR}}(F, \text{Hom}(\tilde{\tau}_2, \tilde{\tau}_1)) \otimes \mathbb{Q}_p / \mathbb{Z}_p \) (Lemma 2.19), where \( \tilde{\tau}_1, \tilde{\tau}_2 \) are (unique) lifts to characteristic zero of \( \tau_1 \) and \( \tau_2 \) respectively.

While each \( \tau \) may possess non-modular reducible characteristic \( p \) deformations, the situation is complicated further by the fact that in general many \( \tau \)'s do not admit any modular deformations at all (this phenomenon does not arise in the “dim=1” case). Indeed, first note that two extensions in \( \text{Ext}^1_{G_2}(\tau_2, \tau_1) \) define isomorphic representation of \( G_2 \) if and only if they are (non-zero) scalar multiples of each other. In particular, if \( \dim_{\mathbb{F}_p} \text{Ext}^1_{G_2}(\tau_2, \tau_1) = 1 \), then there is a unique non-semi-simple representation of \( G_2 \) with semi-simplification \( \tau_1 \oplus \tau_2 \). (Similarly, if \( \dim_{\mathbb{F}_p} H^1_{\text{dR}}(F, \text{Hom}(\tau_2, \tau_1)) = 1 \) then there exists a unique crystalline such representation.) However, if \( \dim_{\mathbb{F}_p} \text{Ext}^1_{G_2}(\tau_2, \tau_1) = m \), then there are \( \frac{q^{m-1}}{q-1} \) non-isomorphic such representations where \( q = \# F \). This demonstrates that in general not all reducible representations \( \tau \) can be modular (of a particular level and weight), as the number of such characteristic zero automorphic forms is fixed (in particular it is independent of making a residue field extension). Nevertheless, we are able to prove (see Corollary 1.8) that there exists an \( \mathbb{F} \)-basis \( \mathcal{B} \) of \( H^1_{\text{dR}}(F, \text{Hom}(\tau_2, \tau_1)) \) arising from modular forms. For this we combine a congruence ideal bound for a Hecke algebra with the upper bound on the Selmer group of \( \text{Hom}(\tilde{\tau}_2, \tilde{\tau}_1) \) predicted by the Bloch-Kato conjectures.

Let \( R^\text{tr,0}_\tau \) be the image in \( R^0_{\text{red}} \) of the subalgebra generated by traces of \( R^\text{\text{red}}_\tau \) for \( \tau \) arising from a modular form. As pointed out we can extend the set consisting of \( \tau \) to a modular basis \( \mathcal{B} := \{ \tau^1 = \tau, \tau^2, \ldots, \tau^s \} \) of \( H^1_{\text{dR}}(F, \text{Hom}(\tau_2, \tau_1)) \). Our ultimate goal is to show that it is possible to identify \( R^\text{\text{red}}_{\tau^1,\tau^2} \) with the quotient \( T_\tau \) of a Hecke algebra \( T \). Here the quotient \( T_\tau \) corresponds to automorphic forms for which there exists a lattice in the associated Galois representation with respect to which the mod \( p \) reduction equals \( \tau \).

To prove our main modularity lifting theorem (Theorem 8.2) we work under the following two assumptions. On the one hand we assume that the modular basis \( \mathcal{B} \) is unique in the sense that any other such consists of scalar multiples of the elements of \( \mathcal{B} \). On the other hand we assume that all \( \tau \in \mathcal{B} \) admit only finitely many characteristic zero deformations, which in particular implies that the quotient \( R^0_{\tau} \)
we define is a finitely generated $\mathbb{Z}_p$-module. The first assumption can be replaced with the assumption that the Bloch-Kato Selmer group $H^1_1(F, \text{Hom}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is annihilated by $p$ (Theorem 8.5). This second result is in a sense ‘orthogonal’ to the main results of [BK11] and [BK13], where the same Selmer group is assumed to be cyclic, but of arbitrary finite order.

Our approach relies on simultaneously considering all the deformation problems for representations $\tau^i$ ($i = 1, 2, \ldots, s$). As in [BK13] we first study “reducible” deformations via the quotients $R^{tr, 0}_{\tau^i}/I^{tr, 0}_{\tau^i}$ for the reducibility ideal $I^{tr, 0}_{\tau^i}$ of the trace of the universal deformation into $\text{GL}_2(R^0)$ as defined by Bellaïche and Chenevier. These ideals are the analogues of Eisenstein ideals $J_{\tau^i}$ on the Hecke algebra side.

To relate $\prod R^{tr, 0}_{\tau^i}/I^{tr, 0}_{\tau^i}$ to the order of a Bloch-Kato Selmer group we make use of a lattice construction of Urban (Theorem 1.1 of [Urb01], see Theorem 4.1 in this paper). In fact it is a repeated application of Urban’s theorem (on the Hecke side and on the deformation side) that allows us to prove a modularity lifting theorem. We show that when the upper bound on the Selmer group and the lower bound on the congruence ideal agree (which in many cases is a consequence of the Bloch-Kato conjecture), this implies that every reducible deformation which lifts to characteristic zero of every $\tau^i$ is modular (cf. section 6). It is here that we make use of the assumption on the ‘uniqueness’ of $\mathcal{B}$ to be able to use a result of Kenneth Kramer and the authors [BKK14] on the distribution of Eisenstein-type congruences among various residual isomorphism classes of Galois representations (cf. Section 5). Yet another application of Urban’s Theorem allows us to prove the existence of a deformation to $\text{GL}_2(R^{tr, 0})$ and as a consequence to identify $R^{tr, 0}_{\tau^i}$ with $R^0_{\tau^i}$ (Theorem 6.2). Using the fact that the ideal of reducibility of $R^0_{\tau^i}$ is principal (Proposition 7.1) and applying the commutative algebra criterion (Theorem 4.1 in [BK13]) we are finally able to obtain an isomorphism $R^{red}_{\tau^i} \cong T_{\tau^i}$ and thus a modularity lifting theorem (Theorems 8.2 and 8.5).

Throughout the paper we work in a slightly greater generality than necessary for the imaginary quadratic case to stress that our results apply in a more general context if one assumes some standard conjectures. However, in section 8 we gather all the assumptions in the imaginary quadratic case as well as the statements of the main theorems (Theorems 8.1, 8.2 and 8.3) in this context. Hence the reader may refer directly to that section for the precise (self-contained) statements of the main results of the paper in that case.

We would like to thank Gebhard Böckle and Jack Thorne for helpful comments and conversations related to the contents of this article. We would also like to express our gratitude to the anonymous referee for suggesting numerous improvements throughout the article. The second author was partially supported by a PSC-CUNY Award, jointly funded by The Professional Staff Congress and The City University of New York.

2. Deformation rings

Let $F$ be a number field and $p > 2$ a prime with $p \nmid \# \text{Cl}_F$ and $p$ unramified in $F/\mathbb{Q}$. Let $\Sigma$ be a finite set of finite places of $F$ containing all the places lying over $p$. Let $G_{\Sigma}$ denote the Galois group $\text{Gal}(F_{\Sigma}/F)$, where $F_{\Sigma}$ is the maximal extension of $F$ unramified outside $\Sigma$. For every prime $q$ of $F$ we fix compatible embeddings $\overline{\mathbb{T}} \hookrightarrow \overline{\mathbb{T}}_q \hookrightarrow \mathbb{C}$ and write $D_q$ and $I_q$ for the corresponding decomposition and inertia subgroups of $G_F$ (and also their images in $G_{\Sigma}$ by a slight abuse of notation). Let
which are non-semi-simple (Lemma 2.1. Every representation $r : G_\Sigma \to \text{GL}_m(F)$ is a continuous homomorphism.

We recall from [CHT08] p. 35 the definition of a crystalline representation: Let $\mathfrak{p} \mid p$ and $A$ be a complete Noetherian $\mathbb{Z}_p$-algebra. A representation $\rho : D_\mathfrak{p} \to \text{GL}_n(A)$ is crystalline if for each Artinian quotient $A'$ of $A$, $\rho \otimes A'$ lies in the essential image of the Fontaine-Lafaille functor $G$ (for its definition see e.g. [BK13] Section 5.2.1). We also call a continuous finite-dimensional $G_\Sigma$-representation $V$ over $Q_p$ (short) crystalline if, for all primes $\mathfrak{p} | p$, Fil$^0 D = D$ and Fil$^{p-1} D = (0)$ for the filtered vector space $D = (\text{B}_{\text{cris}} \otimes Q_p, V)^{D_\mathfrak{p}}$ defined by Fontaine (for details see [BK13] Section 5.2.1).

Following Mazur we call two representations $\tilde{\tau}_1, \tilde{\tau}_2 : G_{\Sigma} \to \text{GL}_m(A)$ for $A \in \text{LCN}(E)$ such that $r = \tilde{\tau}_1 = \tilde{\tau}_2$ (mod $m_A$) strictly equivalent if there exists $M \in \ker(\text{GL}_2(A) \to \text{GL}_2(F))$ such that $\tilde{\tau}_1 = M\tilde{\tau}_2 M^{-1}$. A (crystalline) $O$-deformation of $r$ is then a pair consisting of $A \in \text{LCN}(E)$ and a strict equivalence class of continuous representations $\tilde{r} : G_{\Sigma} \to \text{GL}_m(A)$ that are crystalline at the primes dividing $p$ and such that $r = \tilde{r}$ (mod $m_A$), where $m_A$ is the maximal ideal of $A$. (So, in particular we do not impose on our lifts any conditions at primes in $\Sigma \setminus \Sigma_p$.) Later we assume that if $q \in \Sigma$, then $\# k_q \neq 1 \pmod{p}$, which means that all deformations we consider will trivially be “$\Sigma$-minimal”. As is customary we will denote a deformation by a single member of its strict equivalence class.

If $r$ has a scalar centralizer then the deformation functor is representable by $R_r \in \text{LCN}(E)$ since crystallinity is a deformation condition in the sense of [Maz97]. We denote the universal crystalline $O$-deformation by $\rho_r : G_{\Sigma} \to \text{GL}_m(R_r)$. Then for every $A \in \text{LCN}(E)$ there is a one-to-one correspondence between the set of $O$-algebra maps $R_r \to A$ and the set of crystalline deformations $\tilde{r} : G_{\Sigma} \to \text{GL}_m(A)$ of $r$.

For $j \in \{1, 2\}$ let $\tau_j : G_{\Sigma} \to \text{GL}_{n_j}(F)$ be an absolutely irreducible continuous representation. Assume that $\tau_1 \not\cong \tau_2$. Consider the set of isomorphism classes of $n$-dimensional residual (crystalline at all primes $\mathfrak{p} | p$) representations of the form:

$$\tau = \begin{bmatrix} \tau_1 & * \\ \tau_2 \end{bmatrix} : G_{\Sigma} \to \text{GL}_n(F),$$

which are non-semi-simple ($n = n_1 + n_2$).

From now on assume $p \nmid n!$.

**Lemma 2.1.** Every representation $\tau$ of the form (2.1) has scalar centralizer.

**Proof.** This is easy. □

2.2. Pseudo-representations and pseudo-deformations. We next recall the notion of a pseudo-representation (or pseudo-character) and pseudo-deformations (from [BC09] Section 1.2.1 and [Böc11] Definition 2.2.2).

**Definition 2.2.** Let $A$ be a topological ring and $R$ a topological $A$-algebra. A (continuous) $A$-valued pseudo-representation on $R$ of dimension $d$, for some $d \in \mathbb{N}_{>0}$, is a continuous function $T : R \to A$ such that
(i) \( T(1) = d \) and \( d! \) is a non-zero divisor of \( A \);

(ii) \( T \) is central, i.e. such that \( T(xy) = T(yx) \) for all \( x, y \in R \);

(iii) \( d \) is minimal such that \( S_{d+1}(T)(x) = 0 \), where, for every integer \( N \geq 1 \),

\[
S_N(T)(x) = \sum_{\sigma \in \mathcal{S}_N} \epsilon(\sigma) T^\sigma(x),
\]

where for a cycle \( \sigma = (j_1, \ldots, j_m) \) we define \( T^\sigma((x_1, \ldots, x_{d+1})) = T(x_{j_1}, \ldots, x_{j_m}) \),

and for a general permutation \( \sigma \) with cycle decomposition \( \prod_{i=1}^r \sigma_i \) we let \( T^\sigma(x) = \prod_{i=1}^r T^{\sigma_i}(x) \).

In the case when \( R = A[G_S] \) the pseudo-representation \( T \) is determined by its restriction to \( G_\Sigma \) (see [BC09] Section 1.2.1) and we will also call the restriction of \( T \) to \( G_\Sigma \) a pseudo-representation.

We note that if \( \rho : A[G_S] \to M_n(A) \) is a morphism of \( A \)-algebras then \( \rho \) is a pseudo-representation of dimension \( \nu \) (see [BC09] Section 1.2.2).

According to [BC09] Section 1.2.1, if \( T : R \to A \) is a pseudo-representation of dimension \( d \) and \( A' \) an \( A \)-algebra, then \( T \otimes A' : R \otimes A' \to A' \) is again a pseudo-representation of dimension \( d \).

Following [SW99] (see also [Böc11] Section 2.3) we define a pseudo-deformation of \( \tau_1 + \tau_2 \) to be a pair \((T, A)\) consisting of \( A \in \text{LCN}(E) \) and a continuous pseudo-representation \( T : G_\Sigma \to A \) such that \( T = \text{tr} \tau_1 + \text{tr} \tau_2 \) (mod \( m_A \)), where \( m_A \) is the maximal ideal of \( A \).

By the sentence following [SW99] Lemma 2.10 (see also [Böc11] Proposition 2.3.1) there exists a universal pseudo-deformation ring \( R^{ps} \in \text{LCN}(E) \) and we write \( T^{ps} : G_\Sigma \to R^{ps} \) for the universal pseudo-deformation. For every \( A \in \text{LCN}(E) \) there is a one-to-one correspondence between the set of \( \mathcal{O} \)-algebra maps \( R^{ps} \to A \) and the set of pseudo-deformations \( T : G_\Sigma \to A \) of \( \text{tr} \tau_1 + \text{tr} \tau_2 \). Any deformation of a representation \( \tau \) as in (2.1) gives rise (via its trace) to a pseudo-deformation of \( \text{tr} \tau_1 + \text{tr} \tau_2 \), so there exists a unique \( \mathcal{O} \)-algebra map \( R^{ps} \to R_\tau \) such that the trace of the deformation equals the composition of \( T^{ps} \) with \( R^{ps} \to R_\tau \).

We write \( R_\tau^{red} \) for the quotient of \( R_\tau \) by its nilradical and \( \rho_\tau^{red} \) for the corresponding universal deformation, i.e. the composite of \( \rho_\tau \) with \( R_\tau \to R_\tau^{red} \). We further write \( R_\tau^{tr} \subset R_\tau^{red} \) for the closed \( \mathcal{O} \)-subalgebra of \( R_\tau^{red} \) generated by the set

\[
S := \{ \text{tr} \rho_\tau(F \text{Prob}_q) \mid q \notin \Sigma \}.
\]

**Lemma 2.3.** The image of \( R^{ps} \to R_\tau^{red} \) is \( R_\tau^{tr} \) and hence \( R_\tau^{tr} \) is an object in the category \( \text{LCN}(E) \).

**Proof.** This is clear (cf. [CV03] Theorem 3.11) since \( R^{ps} \) is topologically generated by \( T(\text{Prob}_q) \) (and \( R_\tau^{tr} \) is closed). \( \square \)

### 2.3. Selmer groups.

For a crystalline \( p \)-adic \( G_\Sigma \)-module \( M \) (finitely generated or cofinitely generated over \( \mathcal{O} \) - for precise definitions cf. [BK13], section 5) we define the Selmer group \( H^1_c(F, M) \) to be the subgroup of \( H^1_{\text{cont}}(F_\Sigma, M) \) consisting of cohomology classes which are crystalline at all primes \( p \) of \( F \) dividing \( p \). Note that we place no restrictions at the primes in \( \Sigma \) that do not lie over \( p \). For more details cf. [loc.cit.].

We are now going to state our assumptions. The role of the first one is to rigidify the problem of deforming the representations \( \tau_j \) appearing on the diagonal of the
residual representations. The role of the second is to rule out characteristic zero upper triangular deformations.

**Assumption 2.4.** Assume that $R_{\tau_j} = \mathcal{O}$ and denote by $\tilde{\tau}_j$ the unique lifts of $\tau_j$ to $\text{GL}_{n_j}(\mathcal{O})$.

**Assumption 2.5 ("Bloch-Kato conjecture").** One has the following bound:

\[ \# H^1(F, \text{Hom}_{\mathcal{O}}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_{\mathcal{O}} E/\mathcal{O}) \leq \# \mathcal{O}/L, \]

for some non-zero $L \in \mathcal{O}$.

**Remark 2.6.** In applications the constant $L$ will be the special $L$-value at zero of the Galois representation $\text{Hom}_{\mathcal{O}}(\tilde{\tau}_2, \tilde{\tau}_1)$ divided by an appropriate period.

For the remainder of this section we will work under the above two assumptions.

2.4. **Ideal of reducibility.** Let $A$ be a Noetherian Henselian local (commutative) ring with maximal ideal $m_A$ and residue field $F$ and let $R$ be an $A$-algebra. We recall from [BC09] Proposition 1.5.1 the definition of the ideal of reducibility of a (residually multiplicity free) pseudo-representation $T : R \to A$ of dimension $n$, for which we assume that

\[ T = \text{tr} \tau_1 + \text{tr} \tau_2 \mod m_A \]

**Definition 2.7 ([BC09] Proposition 1.5.1 and Definition 1.5.2).** There exists a smallest ideal $I$ of $A$ such that $T \mod I$ is the sum of two pseudo-characters $T_1, T_2$ with $T_i = \text{tr} \tau_i \mod m_A$. We call this smallest ideal the **ideal of reducibility** of $T$ and denote it by $I_T$.

**Definition 2.8.** We will write $I_{ps} \subset R_{ps}$ for the ideal of reducibility of the universal pseudo-deformation $T_{ps} : R_{ps}[G_{\Sigma}] \to R_{ps}$, $I_{ps} \subset R_{ps}$ for the ideal of reducibility of $\text{tr} \rho_r : R_r[G_{\Sigma}] \to R_r$, $I_{ps}^{\text{red}} \subset R_{ps}^{\text{red}}$ for the ideal of reducibility of $\text{tr} \rho_{r, \text{red}} : R_{r, \text{red}}[G_{\Sigma}] \to R_{r, \text{red}}$ and $I_{ps}^{\text{red}}$ for the ideal of reducibility of $\text{tr} \rho_{r, \text{red}} : R_{r, \text{red}}[G_{\Sigma}] \to R_{r, \text{red}}$.

**Lemma 2.9.** Let $I_0$ be the smallest closed ideal of $R_{ps}^{\text{red}}$ containing the set

\[ \{ \text{tr} \rho_{r, \text{red}}(Frob_v) - \text{tr} \tilde{\tau}_1(Frob_v) - \text{tr} \tilde{\tau}_2(Frob_v) \mid v \notin \Sigma \}. \]

Then $I_0$ equals the ideal of reducibility $I_{ps}^{\text{red}} \subset R_{ps}^{\text{red}}$.

**Proof.** By the Chebotarev density theorem we get $\text{tr} \rho_{r, \text{red}} = \text{tr} \tilde{\tau}_1 + \text{tr} \tilde{\tau}_2 \mod I_0$, hence $I_0 \supset I_{ps}^{\text{red}}$. Conversely, we know from the definition of the ideal of reducibility that $I_{ps}^{\text{red}}$ is given by the sum of two pseudo-characters reducing to $\text{tr} \tau_i$. By Assumption 2.4 and Theorems 7.6 and 7.7 of [BK13] (see also [Boc11] Theorem 2.4.1) these two pseudo-characters must equal $\text{tr} \tilde{\tau}_i \mod I_{ps}^{\text{red}}$. This shows that $I_{ps}^{\text{red}} \supset I_0$. \qed

**Corollary 2.10.** The quotient $R_{ps}^{\text{red}}/I_{ps}^{\text{red}}$ is cyclic. \qed

**Remark 2.11.** Combined with Lemma 7.11 of [BK13] this shows that for any pseudo-deformation $T : A[G_{\Sigma}] \to A$ of $\text{tr} \tau_1 + \text{tr} \tau_2$ with ideal of reducibility $I_T$ for which there is a surjection $R_{ps}^{\text{red}} \to A$, the quotient $A/I_T$ is cyclic.

**Proposition 2.12.** The module $R_{r}/I_{r}$ is a torsion $\mathcal{O}$-module.
Proof. Fix σ ∈ ℤ₊ and set S := R/I. Suppose that S is not torsion. Let φ : S → R := S/ωS be the canonical surjection (of O-algebras). Let A := φ(O).

We first claim that A = O/ωO. Clearly ω∈S = 0 in S/ωS, so we just need to prove that ω∈S−1 ∉ ωS. Suppose on the contrary that ω∈S−1 ∈ ωS. Then there exists s ∈ S such that

\[ \omega \in S \text{ in } S. \]  

Since the residue field of S is O/ω = F, we see that ω is not a unit in S, and hence 1 − ωs is a unit in S. Thus (2.2) implies that ω∈S−1 = 0 in S, which leads to a contradiction and hence we have proved that A = O/ωO.

We now use the following lemma.

Lemma 2.13. There exists an O-submodule B ⊂ R such that

\[ R = A ⊕ B \]

as O-modules.

Proof. This follows from the following result.

Lemma 2.14 (Lemma 6.8(ii), p.222 in [Hum80]). Let A′ be a module over a PID R′ such that p^nA′ = 0 and p^n−1A′ ≠ 0 for some prime p ∈ R′ and a positive integer n. Let a be an element of A′ of order p^n. Then there is a submodule C′ of A′ such that A′ = R′a ⊕ C′.

Apply Lemma 2.14 for R′ = O, A′ = R, p = ω, n = σ, a = ψ(1). Then R′a = A.

We now finish the proof of Proposition 2.12. Let e be an O-module generator of A. Write \( \rho_I: G \to GL_n(R) \) for the deformation corresponding to the canonical map \( R/I \to R \). Then we can write

\[ \rho_I = \begin{bmatrix} \hat{\tau}_1 & \alpha e + \beta \\ \hat{\tau}_2 \end{bmatrix}, \]

where \( \alpha: G \to M_{n_1 \times n_2}(O) \) and \( \beta: G \to M_{n_1 \times n_2}(B) \) are maps (here we identify \( \hat{\tau}_j \) with its composition with \( O \to R \)). Define

\[ \rho_I^+: G \to GL_n(A) \quad g \mapsto \begin{bmatrix} \hat{\tau}_1(g) & \alpha(g)e \\ \hat{\tau}_2(g) \end{bmatrix}. \]

We must check that \( \rho_I^+ \) is a homomorphism. This follows easily from the fact that \( \rho_I \) is a homomorphism and the fact that \( A \) is a direct summand of \( R \). Moreover, note that the image of \( \alpha \) is not contained in \( M_{n_1 \times n_2}(\omega O) \) because \( \rho_I \) reduces to \( \tau \) which is not semi-simple.

Note that \( \rho_I^+ \) is an upper-triangular deformation into \( GL_n(O/\omega) \). Moreover, since \( \rho_I^+ \) reduces to \( \tau \), it gives rise to an element in \( H^1_{\text{red}}(F, \text{Hom}_O(\tau, \tau) \otimes E/O) \) which generates an \( O \)-submodule isomorphic to \( O/\omega \). Since \( \sigma \) was arbitrary we conclude that \( H^1_{\text{red}}(F, \text{Hom}_O(\tau, \tau) \otimes E/O) \) must be infinite which contradicts Assumption 2.5. This concludes the proof of Proposition 2.12. \( \square \)

Remark 2.15. If \( \dim_F H^1_{\text{red}}(F, \text{Hom}(\tau, \tau)) = 1 \) then \( R/I \) and \( R^\text{red}/I^\text{red} \) are cyclic \( O \)-modules by Corollary 7.12 in [BK13] which combined with Proposition 2.12 implies finiteness of \( R/I \) and \( R^\text{red}/I^\text{red} \). On the other hand given that \( \dim_F H^1_{\text{red}}(F, \text{Hom}(\tau, \tau)) > 1 \) it is easy to construct an upper-triangular (not necessarily crystalline) lift of \( \tau \) to \( F[[X]] \) which would suggest that in general \( R/I \),
and even $R^\text{red}_\tau/I^\text{red}_\tau$ (since $F[[X]]$ is reduced), may have positive Krull dimension. Indeed, to see this, let $f$ be a cohomology class corresponding to $\tau$ and let $g$ be a cohomology class linearly independent from $f$. Then the representation

$$\rho = \begin{bmatrix} \tau_1 & \tau_2(f + gX) \\ 0 & \tau_2 \end{bmatrix}$$

is a non-trivial lift of $\tau$ to $GL_n(F[[X]])$. In particular there is no guarantee that $R^\text{red}_\tau$ is a finitely generated $O$-module. Since our method of proving modularity relies on that property we will restrict in the following section to the ‘characteristic zero’ part of $R^\text{red}_\tau$ of which we will demand that it is finite over $O$.

2.5. The ring $R^0_\tau$. Set $\mathcal{P}(\tau) := \{p \in \text{Spec}(R_\tau) \mid R_\tau/p = O\}$. For the rest of this article we assume the following:

**Assumption 2.16.** Assume that $\mathcal{P}(\tau)$ is finite.

We then define $R^0_\tau$ to be the image of $R_\tau$ in $\prod_{p \in \mathcal{P}(\tau)} O$. It is clear that $R^0_\tau$ is a finitely generated $O$-module and an object in $LCN(E)$. Note that the canonical surjection $R_\tau \to R^0_\tau$ factors through $R^\text{red}_\tau$. Write $\rho^0_\tau$ for the composition of $\rho_\tau$ with the map $\varphi_\tau : R_\tau \to R^0_\tau$. Write $I^0_\tau$ for the ideal of reducibility of $\text{tr} \rho^0_\tau$. By [BK13, Lemma 7.11], we have $\varphi_\tau(I_\tau^0) \subset I^0_\tau$ (in fact equality holds since the opposite inclusion is obvious) and thus $\varphi_\tau$ induces a surjection $R_\tau/I_\tau \to R^0_\tau/I^0_\tau$.

**Lemma 2.17.** The quotient $R^0_\tau/I^0_\tau$ is finite.

**Proof.** This follows immediately from Proposition 2.12 and the surjectivity of $R_\tau/I_\tau \to R^0_\tau/I^0_\tau$. \hfill $\square$

Define $R^{\text{tr},0}_\tau \subset R^0_\tau$ to be the closed $O$-subalgebra generated by the set

$$S := \{ \text{tr} \rho^0_\tau(\text{Frob}_q) \mid q \notin \Sigma \}.\,$$

**Lemma 2.18.** The image of $R^{\text{tr},0}_\tau$ under $\varphi_\tau : R_\tau \to R^0_\tau$ is $R^{\text{tr},0}_\tau$. Thus $R^{\text{tr},0}_\tau$ is an object in the category $LCN(E)$.

**Proof.** It is clear that $R^{\text{tr},0}_\tau \subset \varphi_\tau(R^{\text{tr},0}_\tau)$. On the other hand $S \subset \varphi_\tau(R^{\text{tr},0}_\tau)$, so the equality holds because $S$ is dense in $R^{\text{tr},0}_\tau$. \hfill $\square$

We will write $I^{\text{tr},0}_\tau \subset R^{\text{tr},0}_\tau$ for the ideal of reducibility of $\text{tr} \rho^0_\tau$. By Lemma 2.18 and Lemma 7.11 in [BK13] we get that $\varphi_\tau(I^{\text{tr},0}_\tau) \subset I^{\text{tr},0}_\tau$ (in fact equality holds) and thus $\varphi_\tau$ induces a surjection $R^{\text{tr},0}_\tau/I^{\text{tr},0}_\tau \to R^{\text{tr},0}_\tau/I^{\text{tr},0}_\tau$. By Remark 2.11 the quotient $R^{\text{tr},0}_\tau/I^{\text{tr},0}_\tau$ is a cyclic $O$-module.

2.6. Generic irreducibility of $\rho^0_\tau$.

**Lemma 2.19.** For any $\tau$ as in (2.4) $\rho^0_\tau \otimes_{R_\tau} \mathcal{F}$ is irreducible. Here $\mathcal{F}$ is any of the fields $\mathcal{F}_s$ in $\mathcal{F}^0 = \prod_s \mathcal{F}_s$, where $\mathcal{F}^0$ is the total ring of fractions of $R^0_\tau$.

**Proof.** First note that since $R^0_\tau$ is a finitely generated $O$-module and since $E$ is assumed to be sufficiently large we may assume that all of the fields $\mathcal{F}_s$ are equal to $E$. If any of the representations $\rho^0_\tau \otimes_{R_\tau} \mathcal{F}$ is reducible write $\rho = \bigoplus_{j=1}^s \rho_j$ for its semi-simplification with each $\rho_j$ irreducible, $j = 1, 2, \ldots, s$. Then by compactness of $G_{\Sigma}$ for each $1 \leq j \leq s$ there exists a $G_{\Sigma}$-stable $O$-lattice inside the representation space of $\rho_j$. This implies that $\text{tr} \rho_j(\sigma) \in O$ for all $\sigma \in G_{\Sigma}$ and all $1 \leq j \leq s$. Hence $\text{tr} \rho$ splits over $O$ into the sum of traces of $\rho_j$. Since $\rho^0_\tau$ is a deformation of $\tau$ we
easily conclude that \( \rho = \rho_1 \oplus \rho_2 \) with \( \rho_j \) (with respect to some lattice) being a deformation of \( \tau_j \) \( (j = 1, 2) \). Using the fact that \( \rho^0 \) is a deformation of \( \tau \) we now deduce that there is an \( \mathcal{O} \)-lattice inside the space of \( \rho^0 \oplus R \mathcal{F} \) with respect to which \( \rho^0 \oplus R \mathcal{F} \) is block-upper-triangular (with correct dimensions) and non-semi-simple. When we reduce it modulo \( \varpi^m \), the upper-right shoulder will give rise to an element of order \( \varpi^m \) in \( H^1( \mathbb{F}, \text{Hom}_\mathcal{O}( \tau_2, \tau_1) \otimes \mathcal{O} E/\mathcal{O}) \). Since \( m \) is arbitrary this contradicts Assumption 2.5. \( \square \)

3. The rings \( T_\tau \)

Let us now define the rings \( T_\tau \) that will correspond to \( R^0 \) on the Hecke side.

**Proposition 3.1.** If \( \rho : G_\Sigma \to \GL_n(\mathbb{E}) \) is irreducible and satisfies

\[
\overline{\rho}^\ss = \tau_1 \oplus \tau_2
\]

then there exists a lattice inside \( \mathbb{E}^n \) so that with respect to that lattice the mod \( \varpi \) reduction \( \overline{\rho} \) of \( \rho \) has the form

\[
\overline{\rho} = \begin{bmatrix}
\tau_1 & * \\
0 & \tau_2
\end{bmatrix}
\]

and is non-semi-simple.

**Proof.** This is a special case of \( \text{Urb01} \), Theorem 1.1, where the ring \( \mathcal{B} \) in \([\text{loc.cit.}]\) is a discrete valuation ring = \( \mathcal{O} \). \( \square \)

For each representation \( \tau \) as in (2.1) let \( \Phi_\tau \) be the set of (inequivalent) characteristic zero deformations of \( \tau \), i.e. crystalline at \( \mathfrak{p} \mid p \) Galois representations \( \rho : G_\Sigma \to \GL_n(\mathcal{O}) \) whose reduction equals \( \tau \). Also, let \( \Phi_{\tau, \mathbb{E}} \) be the set of (inequivalent) crystalline at \( \mathfrak{p} \mid p \) Galois representations \( \rho : G_\Sigma \to \GL_n(\mathbb{E}) \) such that there exists a \( G_\Sigma \)-stable lattice \( L \) in the space of \( \rho \) so that the mod \( \varpi \)-reduction of \( \rho_L \) equals \( \tau \).

The following is a higher-dimensional analogue of Lemma 2.13(ii) from \( \text{SW99} \):

**Proposition 3.2.** One has \( \Phi_{\tau, \mathbb{E}} \cap \Phi_{\tau', \mathbb{E}} = \emptyset \) if \( \tau \not\sim \tau' \).

**Proof.** Let \( \rho : G_\Sigma \to \GL_n(\mathbb{E}) \) be a representation such that \( \overline{\rho}^\ss = \tau_1 \oplus \tau_2 \) and let \( T \) equal its trace. Suppose there exist two lattices \( L_i \) in the representation space of \( \rho \) such that the reductions of the corresponding representations \( \rho_{L_i} \) are given by \( \tau \) and \( \tau' \) with \( \tau \not\sim \tau' \) as in (2.1). We now consider the classes \( c_{L_i} \) of the cocycles corresponding to \( \overline{\rho}_{L_i} \) in \( \text{Ext}_1^{G(\mathbb{G})/\ker T}(\tau_2, \tau_1) \). Using Assumption 2.5 above and Corollary 7.8 in \( \text{BK13} \) we conclude that the quotient \( \mathcal{O}/\mathcal{I}_T \) is finite. Thus arguing as in the proof of Proposition 1.7.4 in \( \text{BC09} \) but using Proposition 3.1 in \( \text{BK13} \) instead of generic irreducibility of \( T \) to conclude that \( \ker T = \ker \rho \) (see \( \text{BC09} \). Proof of Proposition 1.7.2, on how this equality - which follows from Proposition 1.6.4 in \([\text{loc.cit.}]\) in the generically irreducible case - is used) we obtain that the existence of \( \rho_{L_i} \) with trace \( T \) and non-split reduction as in (2.1) implies that \( \text{Ext}_1^{\chi}(\tau_2, \tau_1) \) is 1-dimensional, where \( \chi := (\mathcal{O}(G_{\Sigma})/\ker T)/\varpi(\mathcal{O}(G_{\Sigma})/\ker T) \).

First note that \( X = \mathcal{O}(G_{\Sigma})/\varpi \mathcal{O}(G_{\Sigma})+\ker T \). Second one clearly has that \( \ker(\mathcal{O}(G_{\Sigma}) \to \mathcal{F}(G_{\Sigma})) = \varpi \mathcal{O}(G_{\Sigma}) \). These two facts imply that the map \( \mathcal{O}(G_{\Sigma}) \to X \) factors through \( \mathcal{O}(G_{\Sigma}) \to \mathcal{F}(G_{\Sigma}) \) and that the kernel of the resulting surjection \( \mathcal{F}(G_{\Sigma}) \to X \) equals \( (\ker T)\mathcal{F}(G_{\Sigma}) \). Thus we have \( X = \mathcal{F}(G_{\Sigma})/\ker T \mathcal{F}(G_{\Sigma}) \), so by the above we conclude that \( \text{Ext}_1^{\mathcal{F}(G_{\Sigma})/\ker T}(\tau_2, \tau_1) \) is one-dimensional. This means the corresponding representations of \( \mathcal{F}(G_{\Sigma})/\ker T \) are isomorphic. Since \( \ker T = \)
The following notation will remain in force throughout the paper.

**Notation 3.3.** Write $\mathcal{S}$ for the set of isomorphism classes of residual representations of the form $(2.1)$. Set $\Phi = \bigcup_{\tau \in \mathcal{S}} \Phi_{\tau}$.

**Remark 3.4.** Assumption 2.16 that $\mathcal{P}(\tau)$ is finite is equivalent to assuming that the set $\Phi_{\tau}$ is a finite set.

We now fix subsets $\Pi_{\tau} \subset \Phi_{\tau}$ and $\Pi \subset \Phi$ of deformations. In our later application these will be taken to correspond to all the modular deformations corresponding to cusps of a particular weight and level which are congruent to a fixed Eisenstein series. In particular $\Pi_{\tau}$ may be empty.

Whenever $\Pi_{\tau} \neq \emptyset$ we obtain an $\mathcal{O}$-algebra map $R_{\tau} \to \prod_{\rho \in \Pi_{\tau}} \mathcal{O}$. This induces a map

$$R_{\tau}^{tr} \to \prod_{\rho \in \Pi_{\tau}} \mathcal{O}. \tag{3.2}$$

**Definition 3.5.** We (suggestively) write $T_{\tau}$ for the image of the map (3.2) - note that this also depends on the choice of the set $\Pi_{\tau}$ - and denote the resulting surjective $\mathcal{O}$-algebra map $R_{\tau}^{tr} \to T_{\tau}$ by $\phi_{\tau}$. Also we will write $T$ for the image of $\phi : R^{ps} \to \prod_{\rho \in \Pi} \mathcal{O}$, where $\phi$ is induced from the traces of the deformations $\rho_{\tau}$. Finally we will write $J_{\tau} \subset T_{\tau}$ for the ideal of reducibility of the pseudo-representation $T_{\tau} \otimes_{R_{\tau}^{tr},\phi} \text{tr} \rho_{\tau} : T_{\tau}[G_{\Sigma}] \to T_{\tau}$ and $J \subset T$ for the ideal of reducibility of the pseudo-representation $R^{ps} \otimes_{R_{\tau}^{tr},\phi} T : T[G_{\Sigma}] \to T$.

**Lemma 3.6.** The maps $R_{\tau}^{tr} \to T_{\tau}$ and $R^{ps} \to T$ factor through $R_{\tau}^{tr,0}$ and the image of $R^{ps}$ inside $R_{\tau}^{tr} \otimes_{\mathcal{O}} E$ respectively.

**Proof.** Clearly the kernel of $R_{\tau} \to \prod_{\rho \in \Pi_{\tau}} \mathcal{O}$ contains $\bigcap_{p \in \mathcal{P}(\tau)} p$. Thus the map $R_{\tau} \to \prod_{\rho \in \Pi_{\tau}} \mathcal{O}$ factors through $R_{\tau}^{0}$. Then the claim follows since $\varphi_{\tau}(R_{\tau}^{tr}) = R_{\tau}^{tr,0}$ by Lemma 2.18.

**Lemma 3.7.** The quotient $T_{\tau}/J_{\tau}$ is cyclic and one has $J_{\tau} = \phi_{\tau}(I_{\tau}^{tr})$.

**Proof.** The first part is a consequence of Lemma 2.9 and was already mentioned in Remark 2.11.

By Lemma 7.11 in [BK13] we know that $J_{\tau} \supset \phi_{\tau}(I_{\tau}^{tr})$. For the opposite inclusion we argue as follows. We need to show that $\phi_{\tau} \circ \text{tr} \rho_{\tau} \equiv \Psi_{1} + \Psi_{2} \mod \phi_{\tau}(I_{\tau}^{tr})$ for $\Psi_{1}, \Psi_{2}$ pseudo-representations.

Put $B = T$, $A = R_{\tau}^{tr}$ and write $\varphi$ for $\phi_{\tau} : R_{\tau}^{tr} \to T$ and $T_{B}$ for $T_{\tau} \otimes_{R_{\tau}^{tr},\phi} \text{tr} \rho_{\tau}$. Let $x \in B[G_{\Sigma}]$. Since $\varphi$ is surjective there exists $y \in A[G_{\Sigma}]$ such that $\varphi(y) = x$. Then by definition of $T_{B}$ we have $T_{B}(x) = \varphi \circ T(y) = \varphi(\Psi_{1}(y) + \Psi_{2}(y) + i)$ for some pseudo-representations $\Psi_{1}, \Psi_{2}$ and $i \in I_{\tau}^{tr}$. Now set $\Psi_{j}(x) := \varphi \circ \Psi_{j}(y)$ for $j = 1, 2$.

**Corollary 3.8.** One has $J_{\tau} = \phi_{\tau}(I_{\tau}^{tr,0})$.

**Proof.** By Lemma 3.6 the map $\phi_{\tau}$ factors through $R_{\tau}^{tr,0}$. By abuse of notation we will denote the induced map also by $\phi_{\tau}$ as in the statement of the Corollary. Then since $\varphi_{\tau}(I_{\tau}^{tr}) = I_{\tau}^{tr,0}$ (with $\varphi_{\tau}$ as in section 2.5) we get the corollary.
Lemma 3.9. The quotient $T/J$ is cyclic.

Proof. For this we prove as in Lemma 2.9 that $J$ is equal to the smallest closed ideal of $T$ generated by the set \{$(\phi \circ T^v)(Frob_v) - \text{tr} \tilde{\tau}_1(Frob_v) - \text{tr} \tilde{\tau}_2(Frob_v) | v \notin \Sigma$\}. We note that Assumption 2.4 can again be applied as Definition 3.5 tells us that $\phi$ is induced by the traces of the crystalline deformations $\rho_\pi$. $\square$

4. The lattice $\mathcal{L}$ and modular extensions

We will make a frequent use of the following result that is due to Urban [Urb01]. Let $B$ be a Henselian and reduced local commutative algebra that is a finitely generated $O$-module. Since $O$ is assumed to be sufficiently large and $B$ is reduced we have $B \subset \hat{B} = \prod_{i=1}^s O \subset \prod_{i=1}^s E = B$, where $\hat{B}$ stands for the normalization of $B$ and $\mathcal{F}_B$ for its total ring of fractions. Write $m_B$ for the maximal ideal of $B$. For any finitely generated free $\mathcal{F}_B$-module $M$, any $B$-submodule $N \subset M$ which is finitely generated as a $B$-module and has the property that $N \otimes_B \mathcal{F}_B = M$ will be called a $B$-lattice.

Theorem 4.1 ([Urb01] Theorem 1.1). Let $R$ be a $B$-algebra, and let $\rho$ be an absolutely irreducible representation of $R$ on $\mathcal{F}_B$ (i.e., $\rho$ composed with each of the projections $\mathcal{F}_B \twoheadrightarrow E$ is absolutely irreducible) such that there exist two representations $\rho_i$ for $i = 1, 2$ in $M_n(B)$ and $I$ a proper ideal of $B$ such that

(i) the coefficients of the characteristic polynomial of $\rho$ belong to $B$;
(ii) the characteristic polynomials of $\rho$ and $\rho_1 \oplus \rho_2$ are congruent modulo $I$;
(iii) $\overline{\rho}_1 := \rho_1 \mod m_B$ and $\overline{\rho}_2 := \rho_2 \mod m_B$ are absolutely irreducible;
(iv) $\overline{\rho}_1 \neq \overline{\rho}_2$.

Then there exist an $R$-stable $B$-lattice $\mathcal{L}$ in $\mathcal{F}_B$ and a $B$-lattice $T$ of $\mathcal{F}_B$ such that we have the following exact sequence of $R$-modules:

$$0 \rightarrow \rho_1 \otimes_B T/I T \rightarrow L \otimes_B B/I \rightarrow \rho_2 \otimes_B B/I \rightarrow 0$$

which splits as a sequence of $B$-modules. Moreover, $\mathcal{L}$ has no quotient isomorphic to $\overline{\rho}_1$.

Since we will not only use Theorem 4.1 itself but also the construction of the lattice $\mathcal{L}$ let us briefly summarize how $\mathcal{L}$ is built (for details cf. [loc.cit., p. 490-491]). Let $\rho_i$ be the composition of the representation $\rho$ with the projection $B \twoheadrightarrow O$ onto the $i$th component of $\hat{B}$. Urban shows that we can always conjugate $\rho_i$ (over $E$) so that the mod $\varpi$-reduction of (the conjugate of $\rho_i$ which we will from now on denote by) $\rho_i$ has the form

$$\begin{pmatrix} \overline{\rho}_1 & * \\ 0 & \overline{\rho}_2 \end{pmatrix}.$$
Set \( \rho_B := (\rho_1) \). It is also shown in [loc.cit.] that the matrices \( \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix} \) are in the image of \( \rho_B \). One then defines the lattice \( \mathcal{L} \) to be the \( B \)-submodule of \( \hat{B} \) generated by \( \rho(r)^t \begin{bmatrix} 0,0,\ldots,0,1 \end{bmatrix} \), where \( r \) runs over \( \mathcal{R} \) and set \( \mathcal{L}^1 := \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \mathcal{L} \) and \( \mathcal{L}^2 := \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} \mathcal{L} \).

Let \( \mathcal{T} \) be as in Notation 3.3. Let \( \mathcal{T} \), \( \Pi \subset \Phi \) and \( \Pi_\tau \subset \Phi_\tau \) be as in section 3. Write \( \tilde{\tau} \) as in section 3. Let \( \mathbf{T} = \prod_{T \in \mathcal{T}} \mathbf{O} \) for the normalization of \( \mathbf{T} \). Let \( \rho \) in Theorem 4.1 be \( \rho_{\Pi} = \prod_{T \in \mathcal{T}} \prod_{\rho_\tau \in \Pi_\tau} \rho_\tau \) and \( \rho_\tau = \tilde{\tau}_i \), \( i = 1, 2 \), where \( \tilde{\tau}_i : G\Sigma \to \text{GL}_{n_i}(\mathbf{O}) \) is a fixed crystalline deformation of \( \tau_i \), which we from now on assume exists. (If one works under Assumption [2,4] then the \( \tilde{\tau}_i \)'s are unique, but we do not need this uniqueness for the arguments of this section.) Note that the reduction of \( \rho_\tau \) already has the form (4.1), so we take \( \rho_B \) = \( \rho_{\Pi} \) and define lattices \( \mathcal{L}, \mathcal{L}^1 \) and \( \mathcal{L}^2 \) as above (with \( \mathcal{B} = \mathcal{T}, \mathcal{R} = \mathbf{T} |_{G\Sigma} \)). The \( G\Sigma \)-action on \( \mathcal{L} \) is then via restriction of \( \rho_{\Pi} \) to \( \mathcal{L} \). Write \( \mathbf{m} = \mathbf{m}_{\mathcal{T}} \) for the maximal ideal of the local ring \( \mathbf{T} \) and let \( J \) as in section 3 be its reducibility ideal.

By Theorem 4.1 and Lemmas 1.1 and 1.5(ii) in [Urb01] there exists a \( \mathbf{T} \)-lattice \( \mathcal{T} \) and a short exact sequence of \( \mathbf{T} |_{G\Sigma} \)-modules (which splits as a sequence of \( \mathbf{T} \)-modules):

(4.2) \[ 0 \to \mathcal{L}^1 \otimes_{\mathbf{T}} \mathbf{T}/J \to \mathcal{L} \otimes_{\mathbf{T}} \mathbf{T}/J \to \mathcal{L}^2 \otimes_{\mathbf{T}} \mathbf{T}/J \to 0 \]

with \( \mathcal{L}^\mathbf{T}/J = \tilde{\tau}_1 \otimes_{\mathbf{O}} \mathcal{T}/J \) and \( \mathcal{L}^2 \otimes_{\mathbf{T}} \mathbf{T}/J = \tilde{\tau}_2 \otimes_{\mathbf{O}} \mathbf{T}/J \) as \( \mathbf{T} \)-modules where the \( \mathbf{T} \)-action on \( \tilde{\tau}_1 \otimes_{\mathbf{O}} \mathcal{T}/J \) is via the second factor.

Note that we have the following identification

(4.3) \[ \text{Hom}_\mathbf{O}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_{\mathbf{O}} \mathcal{T}/J \to \text{Hom}_{\mathcal{T}/J}(\mathcal{L}^2 \otimes_{\mathbf{T}} \mathbf{T}/J, \mathcal{L}^1 \otimes_{\mathbf{T}} \mathbf{T}/J). \]

Let \( s : \mathcal{L}^2 \otimes_{\mathbf{T}} \mathbf{T}/J \to \mathcal{L} \otimes_{\mathbf{T}} \mathbf{T}/J \) be a section of \( \mathcal{T}/J \)-modules of (4.2). Using (4.3) as in [Klo09], p.159-160, we define a cohomology class \( c \in H^1(F_{\Sigma}, \text{Hom}_\mathbf{O}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_{\mathbf{O}} \mathcal{T}/J \) by

\[ g \mapsto (\lambda_2 \otimes t \mapsto s(\lambda_2 \otimes t) - g \cdot s(g^{-1} \cdot \lambda_2 \otimes t)). \]

We also define a map \( \iota_J : \text{Hom}_\mathbf{O}(\mathcal{T}/J, E/\mathbf{O}) \to H^1(F_{\Sigma}, \text{Hom}_\mathbf{O}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_{\mathbf{O}} E/\mathbf{O}) \) by \( f \mapsto (1 \otimes f)(c) \).

Let us just briefly remark that \( \iota_J \) is independent of the choice of the section \( s \). From now on we will make the following assumption on the quotient \( \mathbf{T}/J \).

**Assumption 4.2.** One has

\[ \# \mathbf{T}/J \geq \# \mathcal{O}/L \]

with \( L \) as in Assumption 2.3.

**Remark 4.3.** In Section 7 we will describe a particular setup for \( n = 2 \) and \( F \) an imaginary quadratic field under which Assumptions 2.3 and 4.2 are satisfied. However, we expect that these conditions hold also for other CM fields (for \( n = 2 \)), and have therefore presented the results of this and the following sections under these two general assumptions.
Lemma 4.4. If Assumptions 2.5 and 4.2 hold, then the map
\[ \iota_J : \text{Hom}_\mathcal{O}(\mathcal{T}/J\mathcal{T}, E/\mathcal{O}) \to H^1(F, \text{Hom}_\mathcal{O}(\bar{\tau}_2, \bar{\tau}_1) \otimes_{\mathcal{O}} E/\mathcal{O}) \]
is injective and its image equals \( H^1_\Sigma(F, \text{Hom}_\mathcal{O}(\bar{\tau}_2, \bar{\tau}_1) \otimes_{\mathcal{O}} E/\mathcal{O}) \).

Proof. For the injectivity of \( \iota_J \) and for the fact that its image lands in the Selmer group one follows the strategy in [Ber05, p.119-120] which was later spelled out in a higher dimensional case in [Klo09, Lemmas 9.25 and 9.26]. Let us outline the argument here. Let \( f \in \ker \iota_J \) and set \( K_f := (\mathcal{T}/J\mathcal{T})/\ker f, I_f := (E/\mathcal{O})/\text{Im} f \) and \( \tilde{T} := \text{Hom}_\mathcal{O}(\bar{\tau}_2, \bar{\tau}_1) \). Tensoring the exact sequence \( 0 \to K_f \xrightarrow{f} E/\mathcal{O} \to I_f \to 0 \) with \( \otimes_{\mathcal{O}} \tilde{T} \) we get the exactness of the bottom row of the following commutative diagram:
\[
\begin{array}{ccc}
H^0(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} I_f) & \to & H^1(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} K_f) \\
\phi \downarrow & & \downarrow \phi \otimes f \\
H^1(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} \mathcal{T}/J\mathcal{T}) & \to & H^1(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} E/\mathcal{O}).
\end{array}
\]

Clearly \( H^1(1 \otimes f) \circ \phi(c) = 0 \) and since the first term in the bottom row vanishes (as a consequence of absolute irreducibility of \( \bar{\tau}_1 \) and the fact that \( \bar{\tau}_1 \not\sim \bar{\tau}_2 \)) we get \( \phi(c) = 0 \). Assuming \( f \neq 0 \), one constructs an \( \mathcal{O} \)-module \( A \subset \mathcal{T}/J\mathcal{T} \) containing \( \ker f \) such that \( (\mathcal{T}/J\mathcal{T})/A = \mathcal{O}/\pi \). It is easy to show that \( \phi(c) = 0 \) implies the splitting of the following exact sequence of \( \mathcal{T}[G_\Sigma] \)-modules
\[ 0 \to \tau_1 \to (\mathcal{L}/J\mathcal{L})/(\pi \mathcal{L} + \bar{\tau}_1 \otimes_{\mathcal{O}} A) \to \tau_2 \to 0 \]
contradicting the fact that \( \mathcal{L} \) has no quotient isomorphic to \( \tau_1 \) (cf. Theorem 4.1). This proves injectivity of \( \iota_J \).

On the other hand the fact that \( \text{Im}(\iota_J) \) is contained in the Selmer group can be deduced from the fact that each of the representations \( \rho_\pi \) making up \( \rho = \rho_1 = \bigoplus_{\pi \in \Pi} \rho_\pi \) is crystalline because it implies that the cohomology class \( c \) is also crystalline (see [Klo09, proof of Lemma 9.25 for more details]).

Using [Klo09, Lemma 9.21] (which is just a slightly expanded version of Theorem 4.1) we get \( \text{Fitt}_\mathcal{T}(\tilde{T}) = 0 \).

By Lemma 3.9 we know that \( \mathcal{T}/J = \mathcal{O}/\pi^n \) for some \( n \). Recall property 4 from the Appendix in [MWS84]: For an \( R \)-module \( M \) and an ideal \( I \subset R \) we have
\[
\text{Fitt}_{R/I}(M/IM) = \text{Fitt}_R(M) + I \subset R/I.
\]
Since \( \text{Fitt}_\mathcal{T}(\mathcal{T}) = 0 \) this implies that \( \text{Fitt}_{\mathcal{O}/\pi^n}(\mathcal{T}/J\mathcal{T}) = \text{Fitt}_\mathcal{T}(\mathcal{T}/J\mathcal{T}) = 0 \) in \( \mathcal{T}/J = \mathcal{O}/\pi^n \).

Note that \( \pi^n \) annihilates \( \mathcal{T}/J\mathcal{T} \), so using (4.5) again we get \( \text{Fitt}_{\mathcal{O}/\pi^n}(\mathcal{T}/J\mathcal{T}) = \text{Fitt}_{\mathcal{T}/J}\mathcal{T}(\mathcal{T}/J\mathcal{T})/\pi^n) = \text{Fitt}_\mathcal{T}(\mathcal{T}/J) + \pi^n\mathcal{O} \).

Together this shows that \( \text{Fitt}_\mathcal{T}(\mathcal{T}/J) \) maps to the 0-ideal in \( \mathcal{O}/\pi^n \), i.e.
\[ \text{Fitt}_\mathcal{T}(\mathcal{T}/J) \subset \pi^n\mathcal{O} = \text{Fitt}_\mathcal{O}(\mathcal{T}/J). \]

By property 11 in the Appendix of [MWS84] we know that \( \pi^{\text{length}_\mathcal{O}(\mathcal{T}/J\mathcal{T})}\mathcal{O} \subset \text{Fitt}_\mathcal{O}(\mathcal{T}/J\mathcal{T}) \). This, combined with Assumption 4.2 and Assumption 2.5 implies that \( \iota_J \) must in fact surject onto the Selmer group. \( \square \)
Since (4.2) splits as a sequence of $T/J$-modules we can tensor it with $\otimes_{T/J} F$ and obtain an exact sequence of $F[G_\Sigma]$-modules

$$0 \to L^1 \otimes_T F \to L \otimes_T F \to L^2 \otimes_T F \to 0$$

with

$$L^1 \otimes_T F \cong \tau_1 \text{ and } L^2 \otimes_T F \cong \tau_2.$$ 

Arguing as above (with $\mathfrak{m}_T$ instead of $J$) we again obtain an injective map $\iota : \text{Hom}_O(T/JT, F) \to H^1(F_\Sigma, \text{Hom}_F(\tau_2, \tau_1))$.

**Lemma 4.5.** Suppose that Assumptions 2.5 and 4.2 hold. The map $\iota : \text{Hom}_O(T/JT, F) \to H^1(F_\Sigma, \text{Hom}_F(\tau_2, \tau_1))$ is injective and its image equals $H^1_\Sigma(F, \text{Hom}_F(\tau_2, \tau_1))$.

**Proof.** We have the following commutative diagram

$$\begin{array}{ccc}
\text{Hom}_O(T/JT, E/O) & \xrightarrow{\iota_J} & H^1(F_\Sigma, \text{Hom}_O(\hat{\tau}_2, \hat{\tau}_1) \otimes_O E/O) \\
\downarrow & & \downarrow \\
\text{Hom}_O(T/JT, F) & \xrightarrow{\iota} & H^1(F_\Sigma, \text{Hom}_F(\tau_2, \tau_1))
\end{array}$$

Denote the right vertical arrow by $f$. Lemma 4.4 implies that the image of $f \circ \iota$ is contained in the $\varpi$-torsion of $H^1_\Sigma(F, \text{Hom}_O(\hat{\tau}_2, \hat{\tau}_1) \otimes_O E/O)$. Moreover, by Proposition 5.8 in [BK13], we know that the $\varpi$-torsion of $H^1_\Sigma(F, \text{Hom}_O(\hat{\tau}_2, \hat{\tau}_1) \otimes_O E/O)$ coincides with $f(H^1_\Sigma(F, \text{Hom}_F(\tau_2, \tau_1)))$. Since $f$ is injective, this implies that the image of $\iota$ is contained in the Selmer group. Hence it remains to show that $\iota_J$ is an isomorphism on $\varpi$-torsion. But this is clear since $\iota_J$ is an isomorphism by Lemma 4.4. \hfill $\square$

**Lemma 4.6.** Suppose that Assumptions 2.5 and 4.2 hold. Write $\mathcal{E}_\Pi$ for $\prod_{\pi \in \Pi} \mathcal{E}_\pi$. The $F[G_\Sigma]$-module $\mathcal{L} \otimes_T F$ coincides with the $F$-subspace of $\prod_{\pi \in \Pi} F^\pi$ generated by $\mathcal{E}_\Pi(r)e_n$, where $r$ runs over $F[G_\Sigma]$, $e_n$ is a column matrix in $F^n$ whose last entry is 1 and all the other ones are zero.

**Proof.** By definition of $\mathcal{L}$, every element of $\mathcal{L} \otimes_T F$ can be written as $\sum_i t_i \rho_\Pi(g_i)e_n \otimes a_i$ with $t_i \in T$, $a_i \in F$ and $g_i \in G_\Sigma$. Writing $\tilde{t}_i$ for the image of $t_i$ under the canonical map $T \to F$, we can re-write the above sum as $\sum_i \rho_\Pi(g_i)e_n \otimes a_i \tilde{t}_i$ and $a_i \tilde{t}_i \in F$. It suffices to show now that for every $g \in G_\Sigma$ we get $\rho_\Pi(g)e_n \otimes 1 = \mathcal{E}_\Pi(g)e_n \otimes 1$. Write

$$\rho_\Pi(g) = \begin{bmatrix} a_{11}(g) + a'_{11}(g) & a_{12}(g) + a'_{12}(g) \\ a'_{21}(g) & a_{22}(g) + a'_{22}(g) \end{bmatrix},$$

where $a_{11}, a'_{11}$ are $(n - 1 \times (n - 1))$-matrices, $a_{22}, a'_{22}$ are scalars and the other matrices have sizes determined by these two and the entries of $a'_{ij}(g)$ lie in $\varpi O \oplus \varpi O \oplus \cdots \oplus \varpi O$. Thus,

$$\rho_\Pi(g)e_n \otimes 1 = \begin{bmatrix} a_{12}(g) + a'_{12}(g) \\ a_{22}(g) + a'_{22}(g) \end{bmatrix} \otimes 1 = \begin{bmatrix} a_{12}(g) \\ a_{22}(g) \end{bmatrix} \otimes 1 + \begin{bmatrix} a'_{12}(g)/\varpi \\ a'_{22}(g)/\varpi \end{bmatrix} \otimes \varpi,$$

and the latter tensor is zero. This proves the lemma. \hfill $\square$

Let us now turn to the 2-dimensional situation, where every $\tau$ is (up to a twist) of the form

$$\tau = \begin{bmatrix} 1 & * \\ 0 & \chi \end{bmatrix}$$
for a Galois character \( \chi \). Note that \( \text{Hom}_\mathbb{Q}(\mathcal{T}/J\mathcal{T}, \mathbf{F}) = \text{Hom}_\mathbb{Q}(\mathcal{T} \otimes_{\mathcal{T}} \mathcal{T}/J, \mathbf{F}) = \text{Hom}_\mathbb{Q}(\mathcal{T} \otimes_{\mathcal{T}} \mathcal{T}/\mathfrak{m}, \mathbf{F}) = \text{Hom}_\mathbb{Q}(\mathcal{T} \otimes_{\mathcal{T}} \mathbf{F}, \mathbf{F}) \).

**Proposition 4.7.** Suppose that Assumptions 2.5 and 4.2 hold. The image of \( \iota : \text{Hom}(\mathcal{T} \otimes_{\mathcal{T}} \mathbf{F}, \mathbf{F}) \hookrightarrow H^1_{\text{ét}}(F, \chi^{-1}) \) is spanned by extensions \( \tau \) such that \( \Pi_{\tau} \neq \emptyset \).

**Proof.** Let \( \Pi \) be as above and \( \overline{\Pi} \) be a subset \( \Pi \) consisting of representatives of distinct isomorphism classes of residual representations (i.e., one element from every non-empty \( \Pi_{\tau} \)). By Lemma 4.6 the lattice \( \mathcal{L} \otimes_{\mathcal{T}} \mathbf{F} \) is generated by vectors

\[
x = \begin{bmatrix} (1, 1, \ldots, 1) & \chi(g) \alpha(g) \\ (0, 0, \ldots, 0) & (\chi(g), \chi(g), \ldots, \chi(g)) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Let us explain the notation: There are \( r \) elements in \( \overline{\Pi} \) (which we will denote by \( \tau^1, \tau^2, \ldots, \tau^r \)), \( \sigma := \sum_{i=1}^{r} s_i \) elements in \( \Pi \). Moreover, \( \alpha \) is a \( \sigma \)-tuple of functions such that \( \alpha(g) \) equals

\[
(\alpha_{1,1}, f_1(g), \ldots, \alpha_{1,s_1}, f_1(g), \alpha_{2,1}, f_2(g), \ldots, \alpha_{2,s_2}, f_2(g), \ldots, \alpha_{r,1}, f_r(g), \ldots, \alpha_{r,s_r}, f_r(g)) \in \mathbf{F}^r,
\]

where the \( \alpha_{i,j} \) are elements of \( \mathbf{F}^\times \). We get that \( x \) equals

\[
\chi(g) \begin{bmatrix} (\alpha_{1,1}, f_1(g), \ldots, \alpha_{1,s_1}, f_1(g), \alpha_{2,1}, f_2(g), \ldots, \alpha_{2,s_2}, f_2(g), \ldots, \alpha_{r,1}, f_r(g), \ldots, \alpha_{r,s_r}, f_r(g)) \\ (1, 1, \ldots, 1) \end{bmatrix}.
\]

Let \( \alpha^j \) be the \( j \)th entry of \( \alpha \). Then we conclude that \( \mathcal{L} \otimes_{\mathcal{T}} \mathbf{F} \cong \mathcal{T} \otimes_{\mathcal{T}} V = V \), where \( V \) is the subspace of \( (\mathbf{F} \oplus \mathbf{F})^{\#\overline{\Pi}} \) spanned over \( \mathbf{F} \) by the set vectors of the form

\[
\begin{bmatrix} \chi(g) \alpha^1(g) \\ \chi(g) \alpha^2(g) \\ \vdots \\ \chi(g) \alpha^\sigma(g) \end{bmatrix}.
\]

For \( j \in \{1, 2, \ldots, \sigma\} \) define integers \( n(j) \in \{1, 2, \ldots, r\} \) and \( m(j) \in \{1, 2, \ldots, s_{n(j)}\} \) by the equality

\[
\alpha^j(g) = \alpha_{n(j), m(j)} f_{n(j)}(g).
\]

The \( G_\Sigma \)-action on \( V \) is via \( \mathcal{P}_\Pi \); hence \( h \in G_\Sigma \) acts on \( \begin{bmatrix} \chi(g) \alpha^j(g) \\ \chi(g) \end{bmatrix} \) via the \( n(j) \)th residual representation in \( \overline{\Pi} \), i.e., by multiplication by \( \tau^{n(j)}(h) \). In particular all the vectors in \( V \) have the form

\[
v = \begin{bmatrix} [a_1] \quad [a_2] \quad \cdots \quad [a_\sigma] \end{bmatrix}.
\]

By definition we have

\[
\mathcal{L}^2 \otimes_{\mathcal{T}} \mathbf{F} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathcal{L} \otimes_{\mathcal{T}} \mathbf{F} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} V \cong \mathbf{F}(\chi)
\]

as \( \mathbf{F}[G_\Sigma] \)-modules, where we write \( \mathbf{F}(\chi) \) for the one-dimensional \( \mathbf{F} \)-vector space on which \( G_\Sigma \) acts via \( \chi \). The surjective \( \mathbf{F}[G_\Sigma] \)-module map \( V \to \mathbf{F}(\chi) \) is given by sending a vector \( v \) as in \( (4.3) \) to \( a \). Write \( V' \) for the kernel of this map. Identifying \( V' \) with \( \mathcal{L} \otimes_{\mathcal{T}} \mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{L} \otimes_{\mathcal{T}} \mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} V \) provides us with a splitting (only as \( \mathbf{F} \)-vector spaces) of the short exact sequence of \( \mathbf{F}[G_\Sigma] \)-modules

\[
0 \to V' \to V \to \mathbf{F}(\chi) \to 0.
\]
Since the $G_2$-action on $L^1 \otimes_T F$ is trivial, we have $V' = L^1 \otimes_T F = T \otimes_T F$. Clearly, we may assume that the vectors in $V'$ all have the form

$$\tau_0 = \left( \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ a_\sigma \\ 0 \end{bmatrix} \right).$$

Let $\phi_j \in \text{Hom}_F(V', F) = \text{Hom}(T \otimes_T F, F)$ be the homomorphism sending $v_0$ as in \(4.8\) to $a_j$.

Then the map $\iota$ sends $\phi_j$ to the cocycle $\alpha_{n(j)}(j)f_n(j)$, i.e. to the residual representation $\left[ \begin{array}{c} n(j) \\ \chi \end{array} \right]$, which is isomorphic to the residual representation of the $n(j)$th element of $\Pi$. So, this proves that the image of $\iota$ is spanned by modular extensions.

\begin{remark}
Corollary 4.8 does not imply that $\Pi_{\tau} \neq \emptyset$ for all isomorphism classes $\tau \in \mathcal{T}$. In fact, if we replace $F$ by its finite extension $F'$ of degree $m$, then the order of $\mathcal{T}$ increases (since it is given by $\#H^1_2(F, \text{Hom}(\tau_2, \tau_1))/q-1$, where $q$ is the order of the residue field), while the number of modular forms, i.e., $\sum_{\tau \in \mathcal{T}} \#H^1_2(\# \Pi_{\tau}$ remains the same.

5. Bounding the size of $\prod_i T_i/\phi(I_i)$

In this section we keep in force Assumptions 2.5 and 4.2. Moreover, we work in the two-dimensional setup and set $\tau_1 = 1$ and $\tau_2 = \chi$ (which can always be achieved by twisting by a Galois character). Let $\mathcal{B} := \{e_1, \ldots, e_s\}$ be a basis of $H^1_2(F, \text{Hom}(\tau_2, \tau_1)) = H^1_2(F, \chi^{-1})$ consisting of ‘modular’ extensions, i.e., extensions $\tau$ such that $\Pi_\tau \neq \emptyset$ (cf. Corollary 4.8) and write $\tau'$ for the corresponding residual representations. Let us write $T_i$ for $\text{Hom}(T_i)$ and $\Pi_\tau$ for $J_\tau$ and $\Pi_{\tau'}$ respectively. Write $p_i : T \rightarrow T_i$ for the canonical projection. Consider the map $T \rightarrow \prod_{i=1}^s T_i$. Let $K \subset T$ be as in section 8. Set $J_i = p_i(K)$. Note that $J_i$ is an ideal because $p_i$ is surjective.

Let us begin with an observation that there is no reason to expect that the canonical map

$$T/J \rightarrow \prod_{i=1}^s T_i/J_i$$

should in general be injective or surjective. In fact Lemma 8.4 shows that $T/J$ is cyclic over $\mathcal{O}$. However, as we shall see below the orders of both sides are equal provided that the basis $\mathcal{B}$ is unique up to scaling and that all of the ideals $J_i$ are principal.

\begin{proposition}
$\# \prod_{i=1}^s T_i/J_i \leq \#T/J$.
\end{proposition}

\begin{proof}
In this proof we follow mostly the notation of [Klo09], section 9. Let $\Pi_i$ be as before. As in section 8 we use Theorem 4.1 to get a lattice $L_i \subset \prod_{\tau \in \Pi_i} \rho_{\tau}$ and a finitely generated $T_i$-module $\mathcal{T}_i$ such that the following sequence of $T_i/J_i(G_2)$-modules is exact:

$$0 \rightarrow (T_i/J_i) \otimes_{\mathcal{O}} \tilde{\tau_1} \rightarrow L_i/J_i \rightarrow (T_i/J_i) \otimes_{\mathcal{O}} \tilde{\tau_2} \rightarrow 0.$$
Similarly to the situation in section 4, the sequence splits as a sequence of $T_i$-modules, hence after tensoring with $F$ we obtain a short exact sequence:

$$0 \to (T_i/J_i) \otimes_{T_i} \tau_1 \to L_i/J_i \otimes_{T_i} F \to (T_i/J_i) \otimes_{T_i} T_i \to 0.$$  

Fix $i$. As in the proof of Lemma 4.5 the sequence (5.2) gives rise to an injection

$$(5.3) \quad \iota_i : \text{Hom}(T_i/J_i, T_i, F) \to H^1_F(T, \text{Hom}_T(\tau_2, \tau_1)).$$

Arguing exactly as in the proof of Proposition 4.7 we see that the image of $\iota_i$ is one-dimensional and is spanned by the cohomology classes corresponding to the isomorphism class of $\tau^i$, i.e., for every $i = 1, 2, \ldots, s$ one has $\text{Im}(\iota_i) \subset \langle e_i \rangle$. This implies that $\text{Hom}(T_i/J_i, T_i, F)$ is one-dimensional and hence $T_i/J_i$ is a cyclic $\mathcal{O}$-module, and hence a cyclic $T_i/J_i(= \mathcal{O}/\mathfrak{p}^d_i)$-module (cf. Lemma 3.1). On the other hand, again using Lemma 9.21 of [Klo09], we get that $\text{Fit}_{T_i/J_i} = 0$ and this implies (as in the proof of Lemma 4.4) that

$$\text{val}_p(\#T_i/J_i) \leq \text{val}_p(\#T_i/J_i) = 1.$$ 

This combined with the fact that $T_i/J_i$ is a cyclic $T_i/J_i$-module implies that $T_i/J_i \cong T_i/J_i$. In particular this implies that the lattice $L_i/J_i \cong (T_i/J_i)^2$ as $T_i$-modules.

Let $\tilde{\rho}_i : G \to \text{GL}_2(T_i/J_i)$ be the representation given by the short exact sequence $0 \to (T_i/J_i) \otimes \tau_1 \to L_i/J_i \to (T_i/J_i) \otimes \tau_2 \to 0$ (coming from the sequence (5.1) and the fact that $T_i/J_i \cong T_i/J_i$). One has $T_i/J_i = \mathcal{O}/\mathfrak{p}^d_i$ and since $\tilde{\rho}_i$ reduces to $\tau^i$ we must have $d_i \leq r_i$, where $\mathcal{O} A_i \cong \mathcal{O}/\mathfrak{p}^r_i A_i$. So, in particular we get that $\sum d_i \leq \sum r_i$. Combining Assumptions 2.2 with 1.2 we obtain the claim of Proposition 5.1. 

Our goal is now to prove the opposite inequality, which under some additional assumption will follow from a more general commutative algebra result which was proved by the authors and Kenneth Kramer in [BKK14] and which we will now present.

Let $s \in \mathbb{Z}_+$ and let $\{n_1, n_2, \ldots, n_s\}$ be a set of $s$ positive integers. Set $n = \sum_{i=1}^s n_i$. Let $A_i = \mathcal{O}^n$ with $i \in \{1, 2, \ldots, s\}$. Set $A = \prod_{i=1}^s A_i = \mathcal{O}^n$. Let $\varphi_i : A \to A_i$ be the canonical projection. Let $T \subset A$ be a (local complete) $\mathcal{O}$-subalgebra which is of full rank as an $\mathcal{O}$-submodule and let $J \subset T$ be an ideal of finite index. Set $T_i = \varphi_i(T)$ and $J_i = \varphi_i(J)$. Note that each $T_i$ is also a (local complete) $\mathcal{O}$-subalgebra and the projections $\varphi_{i|T}$ are local homomorphisms. Then $J_i$ is also an ideal of finite index in $T_i$.

**Theorem 5.2** [BKK14, Theorem 2.1]. If $\#F^s \geq s - 1$ and each $J_i$ is principal, then $\# \prod_{i=1}^s T_i/J_i \geq \#T/J$.

Let $V$ be a vector space and write $\mathbb{P}^1(V)$ for the set of all lines in $V$ passing through the origin. There is a canonical map $V \setminus \{0\} \to \mathbb{P}^1(V)$ sending a vector $v$ to the line spanned by $v$.

Let $\mathcal{S}$ be the set of all modular bases of $H^1_F(F, \text{Hom}(\tau_2, \tau_1))$, i.e., the set of bases $\mathcal{B}' = \{e'_1, e'_2, \ldots, e'_s\}$ having the property that $\Pi_{i'} \neq \emptyset$, where $e'_i$ is the residual representation corresponding to the extension represented by $e'_i$. The set $\mathcal{S}$ is non-empty as $B \in \mathcal{S}$.

**Definition 5.3.** We will say that $H^1_F(F, \text{Hom}(\tau_2, \tau_1))$ has a *projectively unique modular basis* if the images of all the elements of $\mathcal{S}$ in $\mathbb{P}^1(H^1_F(F, \text{Hom}(\tau_2, \tau_1))$ agree.
In the case when $H^1_s(F, \text{Hom}(\tau_2, \tau_1))$ has this property we will refer to any element of $S$ as the projectively unique modular basis.

Note that it is possible to find $i_0 \in \{1, 2, \ldots, s\}$ such that the set $B' := B \cup \{e_i\} \setminus \{e_{i_0}\}$ is still a basis of $H^1_s(F, \text{Hom}(\tau_2, \tau_1))$ (and one still has that $\Pi_{\tau} \neq \emptyset$ for all $\tau' \in B'$). Hence we can assume without loss of generality that $B = \{e_1, e_2, \ldots, e_s\}$ with $\tau^1 = \tau$. In fact, if $H^1_s(F, \text{Hom}(\tau_2, \tau_1))$ has a projectively unique modular basis, it follows that if $B'$ is another modular basis, then the isomorphism classes of the residual representations corresponding to the elements of $B'$ are the same as the isomorphism classes of the residual representations corresponding to the elements of $B$.

**Proposition 5.4.** Suppose $H^1_s(F, \text{Hom}(\tau_2, \tau_1))$ has a projectively unique modular basis. If for each $i$, the ideal $J_i$ of $T_i$ is principal and $T_i/J_i$ is finite, then $\# \prod_{i=1}^s T_i/J_i \geq \#T/J$.

**Proof.** First note that our assumption that $E$ be sufficiently large allows us to assume that $\# \mathcal{P}_E^X$ satisfies the inequality in Theorem 5.2. Since $T$ is a free $\mathcal{O}$-module of finite rank we set $n$ to be that rank and define $n_i$ to be the rank of $T_i$. The assumption that $B$ be projectively unique guarantees that every $\mathcal{O}$-algebra homomorphism has a corresponding residual Galois representation isomorphic to $\tau^i$ for some $i$. Hence $n = \sum_{i=1}^s n_i$. Finally, note that $T/J$ is finite. Indeed, first note that if we consider $T$ as a (full rank) $\mathcal{O}$-subalgebra of $\prod_{i=1}^s T_i$ then for a sufficiently large exponent $N$, we have $p^N e_i \in T$, where $e_i \in \prod_{i=1}^s T_i$ is the idempotent corresponding to $T_i$. On the other hand because $T_i/J_i$ is finite for each $i$, there exists a positive integer $M$ such that $p^M \in J_i$ for each $i$. Let $x_i \in J_i$ be a preimage of $p^M \in J_i$. Then $p^{N+M} = \sum_{i=1}^s x_i p^N e_i \in J_i$, hence $T/J$ is torsion and thus finite. The Proposition now follows from Theorem 5.2 by taking $T = T_i$. $\square$

**Remark 5.5.** Theorem 5.2 also has consequences for congruences between modular forms. Suppose that $T = T_\Sigma$ is the cuspidal Hecke algebra (acting on the space of automorphic forms over imaginary quadratic fields of weight 2 right invariant under a certain compact subgroup $K_f$) localized at a maximal ideal corresponding to an Eisenstein series, say $E$. Let $J = J_\mathcal{E}$ be the Eisenstein ideal corresponding to $\mathcal{E}$ (see section 7.3 for the details). Let $\mathcal{N}$ be the set of $\mathcal{O}$-algebra homomorphisms $T \rightarrow \mathcal{O}$, i.e. to cuspidal Hecke eigencharacters congruent to the eigencharacter $\chi_0$ of $E$ mod $\varpi$. For $\lambda \in \mathcal{N}$ write $m_\lambda$ for the largest positive integer such that $\lambda(T) \equiv \lambda_0(T)$ mod $\varpi^{m_\lambda}$ for all $T \in T$. Let $e$ be the ramification index of $E$ over $\mathbb{Q}_p$. As a consequence of Theorem 5.2 we get the following inequality (cf. Proposition 4.3 in [BRKK14])

$$\frac{1}{e} \sum_{\lambda \in \mathcal{N}} m_\lambda \geq \text{val}_e(\#T/J).$$

For many more applications of Theorem 5.2 see [loc.cit.].

**Corollary 5.6.** Suppose that Assumptions 2.5 and 4.2 hold. If the modular basis $B$ is projectively unique and if for each $i$, the ideal $J_i$ of $T_i$ is principal and $T_i/J_i$ is finite, then

$$\# \prod_{i=1}^s T_i/J_i = \#T/J \geq \#\mathcal{O}/L.$$
6. Urban’s method applied to $R^\text{tr,0}$

In this section we again set $n = 2$, $\tau_1 = 1$ and $\tau_2 = \chi$ and we fix a residual representation

$$
\tau : G_{\Sigma} \to \text{GL}_2(\mathbb{F}), \quad \tau = \begin{bmatrix} 1 & \chi f \\ 0 & \chi \end{bmatrix}.
$$

Let $\Phi_\tau, \Pi_\tau$ be as in section 3. From now on we will assume that $\Pi_\tau$ is non-empty. Also recall that we make Assumption 2.10 which by Remark 3.3 is equivalent to assuming $\Phi_\tau$ is finite (in particular $R^\text{tr,0}_\tau$ is defined and finitely generated as an $\mathcal{O}$-module). The surjection $\phi : R^\text{tr,0}_\tau \to \mathcal{T}_\tau$ (cf. Definition 3.5 and Lemma 3.6) descends to a surjection

$$
(6.1) \quad R^\text{tr,0}_\tau / I^\text{tr,0}_\tau \to \mathcal{T}_\tau / J_\tau
$$

(since by Lemma 3.3 $\phi_\tau(I^\text{tr,0}_\tau) = J_\tau$). The main goal of this section is to prove that under certain assumptions the map in (6.1) is an isomorphism (Theorem 6.3). Before we state the theorem let us demonstrate several properties of $R^\text{tr,0}_\tau$. In particular we will show that $R^\text{tr,0}_\tau \cong R^0_\tau$ (Theorem 6.2). In this section we also assume that Assumptions 2.4, 2.5 and 4.2 are satisfied.

**Lemma 6.1.** The $\mathcal{O}$-rank of $R^0_\tau$ equals the $\mathcal{O}$-rank of $R^\text{tr,0}_\tau$. In particular the normalizations (and the total rings of fractions) of $R^0_\tau$ and $R^\text{tr,0}_\tau$ coincide.

**Proof.** Write $\rho$ for $\rho^\tau_\sigma$, i.e., $\rho : G_{\Sigma} \to \text{GL}_2(R^0_\tau)$. We claim that we can conjugate $\rho$ so that for every $g \in G_{\Sigma}$ we have $\rho(g) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, c, d \in R^\text{tr,0}_\tau$. Indeed, since the characters on the diagonal of $\tau$ are distinct mod $\varpi$, we can find $\sigma \in G_{\Sigma}$ on which they differ, so that the eigenvalues of $\tau(\sigma)$ lift by Hensel’s lemma to distinct eigenvalues of $\rho(\sigma)$ in $R^0_\tau$, and we can conjugate (over $R^0_\tau$) to have $\rho(\sigma)$ be diagonal with these lifted eigenvalues. For a general element $g \in G_{\Sigma}$, we then compare $\text{tr} \rho(g)$ with $\text{tr} \rho(\sigma g)$ and use that the eigenvalues are distinct mod $\varpi$ to see that the two diagonal entries of $\rho(g)$ lie in $R^\text{tr,0}_\tau$. Similarly we show (cf. the proof of Lemma 3.27 in [DDT97]) that the lower-left entry also lies in $R^\text{tr,0}_\tau$.

Note that since $R^0_\tau$ is a finitely generated $\mathcal{O}$-module (and $\mathcal{O}$ is assumed to be sufficiently large) we get an embedding

$$
R^0_\tau \hookrightarrow \hat{R}^0_\tau \cong \prod_{i=1}^k \mathcal{O},
$$

where $\hat{R}^0_\tau$ is the normalization of $R^0_\tau$. For $i = 1, \ldots, k$, write $\rho_i$ for the composition of $\rho$ with the projection onto the $i$th component of $\hat{R}^0_\tau$. Suppose that the $\mathcal{O}$-rank of $R^\text{tr,0}_\tau$ is strictly smaller than the $\mathcal{O}$-rank of $R^0_\tau$. Then there exist two minimal primes (after possibly renumbering the minimal primes we will call them $p_1, p_2$) of $R^0_\tau$ which contract to the same minimal prime $p$ of $R^\text{tr,0}_\tau$. Hence we get the following
This implies that the corresponding two deformations (to $\mathcal{O}$) $\rho_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $\rho_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ (since their $a$-, $c$- and $d$-entries factor through $R_{\tau}^{tr,0}/p$) must satisfy $a_1 = a_2 =: a$, $c_1 = c_2 =: c$ and $d_1 = d_2 =: d$. In particular their traces are equal. Using Lemma 2.19 we see that both $\rho_1 \otimes \mathcal{O} E$ and $\rho_2 \otimes \mathcal{O} E$ are irreducible and thus by the Brauer-Nesbitt Theorem we conclude that $\rho_1 \otimes \mathcal{O} E \cong \rho_2 \otimes \mathcal{O} E$. Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{GL}_2(E)$ be such that

\begin{equation}
(6.2) \quad M \rho_1 = \rho_2 M.
\end{equation}

Then an easy matrix calculation shows that

\begin{equation}
(6.3) \quad Aa + Bc = aA + b_2 C
\end{equation}

and

\begin{equation}
(6.4) \quad Cb_1 + Dd = cB + dD
\end{equation}

from which we get that $Cb_1 = Cb_2$. Suppose for the moment that there exists $g \in G_{\Sigma}$ such that $b_1(g) \neq b_2(g)$. Since $\mathcal{O}$ is a domain we conclude that $C = 0$. Since the representations $\rho_1$ and $\rho_2$ are irreducible over $E$, the function $c$ cannot be identically zero. Using (6.4) we conclude that $B = 0$. Finally, computing the lower-left entries on both sides of (6.2) we get $Dc = cA$, so again using the fact $c$ is not identically zero and that $\mathcal{O}$ is a domain we get that $A = D$. Thus, $M$ is a non-zero scalar matrix. Hence we get a contradiction to our assumption on the existence of $g$ and we conclude that $b_1 = b_2$. In particular $\rho_1$ and $\rho_2$ are identical deformations of $\tau$ which correspond to distinct minimal primes of $R_{\tau}^0$. Hence $\rho_1$ and $\rho_2$ give rise to two different homomorphisms from $R_{\tau}^{red}$ to $\mathcal{O}$. This contradicts the bijectivity of the correspondence

$$\text{Hom}_{\mathcal{O}_{-\text{alg}}}(R_{\tau}^{red}, \mathcal{O}) \leftrightarrow \{\text{deformations of } \tau \text{ into } \mathcal{O}\}.$$ 

\[\square\]

**Theorem 6.2.** There exists a deformation $\rho_{\tau}^{tr,0} : G_{\Sigma} \to \text{GL}_2(R_{\tau}^{tr,0})$ of $\tau$. The resulting canonical map $R_{\tau}^{red} \to R_{\tau}^{tr,0}$ factors through $R_{\tau}^0$ and induces an isomorphism $R_{\tau}^0 \cong R_{\tau}^{tr,0}$.

**Proof.** We will (once again) apply Theorem 4.1 (due to Urban). In the notation of section 4 we will write $\mathcal{F}_\mathcal{B} = \mathcal{F}$ to be total ring of fractions of $\mathcal{B} = R_{\tau}^{tr,0} \subset \mathcal{F}$. Note that by Lemma 6.1, $\mathcal{F}$ is also the total ring of fractions of $R_{\tau}^0$. Moreover, we take $\mathcal{R} = R_{\tau}^{tr,0}[G_{\Sigma}]$, $\rho = \rho_{\tau} \otimes \mathcal{B} \mathcal{F} : G_{\Sigma} \to \text{GL}_2(\mathcal{F})$ which induces a morphism $\rho : R_{\tau}^{tr,0}[G_{\Sigma}] \to M_2(\mathcal{F})$ of $R_{\tau}^{tr,0}$-algebras. As before, the representations denoted in
Theorem 4.1] by $\rho_1$ and $\rho_2$ are our unique lifts $\tilde{\tau}_1, \tilde{\tau}_2 : G_\Sigma \to \text{GL}_2(O) \hookrightarrow \text{GL}_2(R_{\tau}^{tr,0})$ and we set $I = I_{\tau}^{tr,0}$. Note that conditions (i) and (ii) of Theorem 4.1 are satisfied respectively by the definition of $R_{\tau}^{tr,0}$ and of $I_{\tau}^{tr,0}$ and conditions (iii) and (iv) are satisfied by our assumptions on $\tau_1$ and $\tau_2$. Finally, the condition of irreducibility of $\rho$ is satisfied by Lemma 4.19.

Hence we conclude from Theorem 4.1 that there exists an $R_{\tau}^{tr,0}[G_\Sigma]$-stable lattice $L \subset \mathcal{F}^2$ and a finitely generated $R_{\tau}^{tr,0}$-module $T_{\tau} \subset \mathcal{F}$ such that we have an exact sequence of $R_{\tau}^{tr,0}[G_\Sigma]$-modules:

\[ 0 \to \tilde{\tau}_1 \otimes_{O} T_{\tau} / I_{\tau}^{tr,0} T_{\tau} \to L \otimes_{R_{\tau}^{tr,0}} R_{\tau}^{tr,0} / I_{\tau}^{tr,0} \to \tilde{\tau}_2 \otimes_{O} R_{\tau}^{tr,0} / I_{\tau}^{tr,0} \to 0. \]

It follows from [Ur50] Lemmas 1.1 and 1.5 that $L = T_{\tau} \otimes R_{\tau}^{tr,0}$ as $R_{\tau}^{tr,0}$-modules. We will now show that $T_{\tau} \cong R_{\tau}^{tr,0}$. Indeed, the lattice $L$ is defined as in section 4 but since we only work with a fixed residual representation $\tau$, the representation $\rho$ in Theorem 4.1 equals the $\prod_{\rho_1 \in \Phi_{\tau}} \rho_{\tau}$. Using Lemma 4.4 for this representation (i.e., when we replace $\rho_1$ with $\rho$ as above), we conclude that the $G_\Sigma$-module $L \otimes_{R_{\tau}^{tr,0}} F$ is the subspace of $\prod_{\rho_1 \in \Phi_{\tau}} F^2$ generated by $\mathcal{P}(\nu)e_2$ (with notation as in that lemma). This subspace is clearly isomorphic to $\tau$ as a $G_\Sigma$-module. So, the middle term in (6.5) after tensoring with $F$ is two-dimensional, hence we must have $T_{\tau} / I_{\tau}^{tr,0} T_{\tau} \otimes_{R_{\tau}^{tr,0}} F = T_{\tau} / \mathfrak{m} T_{\tau} = F$, where $\mathfrak{m}$ is the maximal ideal of $R_{\tau}^{tr,0}$. Thus, by Nakayama’s Lemma, we see that $T_{\tau}$ is generated over $R_{\tau}^{tr,0}$ by one element, say $x$. Consider the surjective map $\phi : R_{\tau}^{tr,0} \to T_{\tau}$ given by $r \mapsto rx$. Let $a \in \ker \phi$. Then $a$ annihilates $T_{\tau}$. However, by definition of $T_{\tau}$ and the fact that $R_{\tau}^{tr,0}$ embeds into its ring of fractions $\mathcal{F}$ we consider $x$ and $a$ as elements of $\mathcal{F} = \prod E$. If $a = (a_1, a_2, \ldots, a_s)$ and $x = (x_1, x_2, \ldots, x_s)$ and $a_j \neq 0$, then $x_j = 0$. However, this implies that $T_{\tau} \otimes \mathcal{F} \neq \mathcal{F}$, which contradicts the fact that $T_{\tau}$ is a lattice in $\mathcal{F}$ (cf. Theorem 4.1). Hence $\phi$ is injective and we indeed have $T_{\tau} = R_{\tau}^{tr,0}$. Thus $L = (R_{\tau}^{tr,0})^2$ as an $R_{\tau}^{tr,0}$-module and $\rho$ gives rise to a representation $\rho_{\tau}^{tr,0} : G_\Sigma \to \text{GL}_2(R_{\tau}^{tr,0})$.

By the above it is clear that $\rho_{\tau}^{tr,0}$ reduces to $\tau$. Let us make sure that the resulting representation is crystalline. Indeed, the lattice $L$ lives inside the finite direct product of the representations $\rho_{\tau}$ for $\pi \in \Phi_{\tau}$ and each of the $\rho_{\tau}$ is crystalline. Hence as a submodule of a finite direct sum of crystalline representations $\rho_{\tau}^{tr,0}$ is crystalline. This proves the first assertion.

By universality of $R_{\tau}^{red}$ we obtain an $O$-algebra map $\phi : R_{\tau}^{red} \to R_{\tau}^{tr,0}$. Since $R_{\tau}^{tr,0}$ is a subring of the direct product of finitely many copies of $O$, the map $\phi$ clearly factors through $R_{\tau}^{0}$. Let us denote the induced map $R_{\tau}^{0} \to R_{\tau}^{tr,0}$ also by $\phi$. We claim that $\phi$ is surjective. Indeed, by its definition $R_{\tau}^{tr,0}$ is generated by traces of $\rho_{\tau}^{0}$. So, it is enough to show that the traces of $\rho_{\tau}^{0}$ and $\rho_{\tau}^{tr,0} = \phi \circ \rho_{\tau}^{0}$ coincide. This follows from the construction of the lattice $L$ which is a $G_\Sigma$-stable lattice inside the representation $R_{\tau}^{0} \otimes \mathcal{F}$.

In particular, both representations $\rho_{\tau}^{0}$ and $\rho_{\tau}^{tr,0}$ are isomorphic after tensoring with $\mathcal{F}$, hence they must have equal traces. Since both $R_{\tau}^{0}$ and $R_{\tau}^{tr,0}$ are finitely generated $O$-modules with the same rings of fractions (Lemma 6.1), the kernel of $\phi$ must be a torsion $O$-module. This implies that the kernel must be zero (as $R_{\tau}^{0}$ embeds into $\prod E$). This proves the second assertion.

Suppose that $\tau$ is such that $\Pi_{\tau} \neq \emptyset$. Recall that by Corollary 4.8 there exists a basis $B = \{ e_1, e_2, \ldots, e_s \}$ of $H^2_{\Sigma}(\mathcal{F}, \text{Hom}(\tau_2, \tau_1))$ such that $\Pi_{\tau_i} \neq \emptyset$ for all $i = 1, 2, \ldots, s$, where $\tau_i$ denotes the representation corresponding to the extension $e_i$. It is possible to find $i_0 \in \{1, 2, \ldots, s \}$ such that the set $B' := B \cup \{ e_{i_0} \} \setminus \{ e_{i_0} \}$ is...
still a basis (and one still has that \( \Pi_{\tau'} \neq \emptyset \) for all \( \tau' \in \mathcal{B}' \)). Hence we can assume without loss of generality that \( \mathcal{B} = \{ e_1, e_2, \ldots, e_s \} \) with \( \tau_1 = \tau \).

**Theorem 6.3.** Suppose Assumptions 2.4, 2.5, 2.10 and 4.2 are satisfied. Suppose moreover that the modular basis \( \mathcal{B} \) is projectively unique and that for each \( i = 1, 2, \ldots, s \) the corresponding ideal \( J_{\tau_i} \) is principal. Then the map

\[
R^{\text{tr},0}_{\tau} / I^{\text{tr},0}_{\tau} \to T_{\tau} / J_{\tau}
\]

is an isomorphism.

**Proof.** In the proof of Theorem 6.2 we showed the existence of an \( R^{\text{tr},0}_{\tau} \)-algebra morphism \( p : R^{\text{tr},0}_{\tau}[G_\Sigma] \to M_2(\mathcal{F}) \) and an \( R^{\text{tr},0}_{\tau}[G_\Sigma] \)-stable lattice \( \mathcal{L} \subset \mathcal{F}^2 \) together with a finitely generated \( R^{\text{tr},0}_{\tau} \)-submodule \( \mathcal{T}_\tau \subset \mathcal{F} \) such that we have an exact sequence of \( R^{\text{tr},0}_{\tau}[G_\Sigma] \)-modules:

\[
(6.6) \quad 0 \to \tilde{\tau}_1 \otimes_\mathcal{O} \mathcal{T}_\tau / I^{\text{tr},0}_{\tau} \to \mathcal{L} \otimes R^{\text{tr},0}_{\tau} / I^{\text{tr},0}_{\tau} \to \tilde{\tau}_2 \otimes_\mathcal{O} R^{\text{tr},0}_{\tau} / I^{\text{tr},0}_{\tau} \to 0
\]

which splits as a sequence of \( R^{\text{tr},0}_{\tau} \)-modules. As in section 4 we get a map:

\[
\iota : \text{Hom}_\mathcal{O}(\mathcal{T}_\tau / I^{\text{tr},0}_{\tau} \mathcal{T}_\tau, E/\mathcal{O}) \to H^1(F_\Sigma, \text{Hom}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes_\mathcal{O} E/\mathcal{O}).
\]

The fact that \( \iota \) is injective and that its image is contained in the Selmer group is proved in the same way as Lemma 4.4. Tensoring (6.6) with \( F \) and arguing as in the proof of Proposition 2.11 (this time using that \( R^{\text{tr},0}_{\tau} / I^{\text{tr},0}_{\tau} \) is cyclic by Remark 2.11) we see that \( \mathcal{T}_\tau / I^{\text{tr},0}_{\tau} \mathcal{T}_\tau \) is cyclic and that

\[
(6.7) \quad \text{val}_p(\# \mathcal{T}_\tau / I^{\text{tr},0}_{\tau} \mathcal{T}_\tau) \geq \text{val}_p(\# R^{\text{tr},0}_{\tau} / I^{\text{tr},0}_{\tau}).
\]

Using the above arguments for the rings \( R^{\text{tr},0}_{\tau} \) corresponding to the residual representation arising from \( e_i \), and putting them together we obtain

\[
(6.8) \quad \text{val}_p(\# \prod_{i=1}^s R^{\text{tr},0}_{\tau_i} / I^{\text{tr},0}_{\tau_i}) \leq \text{val}_p(\# \prod_{i=1}^s T_{\tau_i} / I^{\text{tr},0}_{\tau_i} \mathcal{T}_{\tau_i}) \leq \text{val}_p(\# H^1_2(F, \text{Hom}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes E/\mathcal{O}))
\]

(the first inequality comes from (6.7)). The assumption in Corollary 3.6 that \( T_{\tau_i} / J_{\tau_i} \) are finite is satisfied since this is true for \( R^{\text{tr},0}_{\tau_i} / I^{\text{tr},0}_{\tau_i} = R^{\text{tr},0}_{\tau} / I^{\text{tr},0}_{\tau} \) by Theorem 6.2 and Lemma 2.17. Combining Assumptions 2.4, 2.10 with Corollary 5.10 we obtain that

\[
(6.9) \quad \text{val}_p(\# H^1_2(F, \text{Hom}(\tilde{\tau}_2, \tilde{\tau}_1) \otimes E/\mathcal{O})) \leq \text{val}_p(\# \prod_{i=1}^s T_{\tau_i} / J_{\tau_i}).
\]

Combining (6.8) with (6.9) we conclude that the maps

\[
R^{\text{tr},0}_{\tau_i} / I^{\text{tr},0}_{\tau_i} \to T_{\tau_i} / J_{\tau_i}
\]

must be isomorphisms for every \( i = 1, 2, \ldots, s \). This completes the proof of Theorem 6.3. \( \square \)

7. Imaginary quadratic case

For the remainder of the article we specialize to the case \( n = 2 \), \( F \) imaginary quadratic, \( \tau_1 = 1 \) and \( \tau_2 = \chi : G_\Sigma \to \mathbb{F}_2 \) an anticyclotomic character (i.e., we assume that \( \chi(\epsilon g c) = \chi(g^{-1}) \) for all \( g \in G_\Sigma \) and \( c \in G_\mathbb{Q} \) a complex conjugation). Recall again our assumption 2.10 that \( \mathcal{P}(\tau) \) is finite (so that \( R^2 \) is defined and finitely generated as an \( \mathcal{O} \)-module).
7.1. Principality of the ideal of reducibility. Let \( f \in H^1_0(F, \chi^{-1}) \) be non-zero. Set \( \tau = \left[ \begin{array}{c} 1 \\ \chi f \\ \chi \end{array} \right] \) with universal deformation ring \( R_\tau \).

**Proposition 7.1.** Suppose Assumption 7.4 is satisfied and write \( \Psi \) for \( \tilde{\tau}_2 \). Then the ideal of reducibility \( I^0_\tau \) of \( R^0_\tau \) is principal.

We begin with the following lemma.

**Lemma 7.2.** There exists \( \rho^0_{\tau, \text{opp}} : G_\Sigma \to \text{GL}_2(R^0_\tau) \) such that \( \rho^0_{\tau, \text{opp}} = \left[ \begin{array}{c} \chi & * \\ 1 & 1 \end{array} \right] \) and is non-split.

**Proof.** Let \( c \) be the complex conjugation. Define \( \rho'_\tau \) by

\[
\rho'_\tau(g) = \rho^0_\tau(cgc).
\]

Then

\[
\rho'_\tau = \left[ \begin{array}{c} 1 \\ \chi(cgc)f(cgc) \\ \chi(cgc) \end{array} \right] = \left[ \begin{array}{c} 1 \\ \chi^{-1}(g)f(cgc) \\ \chi^{-1}(g) \end{array} \right].
\]

If this is split, then \( f^* \in H^1_0(F, \chi) \) defined by \( f^*(g) = f(cgc) \) is the zero cohomology class. However, \( a \mapsto a^c \) gives an isomorphism of \( H^1_0(F, \chi^{-1}) \) onto \( H^1_0(F, \chi) \) (cf. Lemma 7.15 in [Ber05]), so \( f^* \neq 0 \) since \( f \neq 0 \). Now set \( \rho^0_{\tau, \text{opp}} = \rho'_\tau \otimes \Psi. \)

**Proof of Proposition 7.1.** We first note that the ideal of reducibility of \( tr\rho^0_{\tau, \text{opp}} \) equals \( I^0_\tau \). By Proposition 3.1 in [BK15] we know that \( \ker \rho^0_\tau = \ker tr \rho^0_\tau \) and \( \ker \rho^0_{\tau, \text{opp}} = \ker tr \rho^0_{\tau, \text{opp}} \) since \( R^0_\tau/I^0_\tau \) is finite by Lemma 2.17. Noting that \( \rho^0_{\tau}(R^0_\tau[G_\Sigma]) = \rho^0_{\tau, \text{opp}}(R^0_\tau[G_\Sigma]) \) we therefore have an isomorphism

\[
R^0_\tau[G_\Sigma]/\ker tr \rho^0_\tau \cong R^0_\tau[G_\Sigma]/\ker tr \rho^0_{\tau, \text{opp}}.
\]

This means that applying [BC09] Proposition 1.7.4 with \( \rho^0_\tau \) and \( \rho^0_{\tau, \text{opp}} \) establishes condition (1) in [BC09] Proposition 1.7.5, so we can conclude that \( I^0_\tau \) is principal.

(2) Note that the generic irreducibility assumption in the propositions in [BC09] can be replaced by \( \ker \rho = \ker \tau \).

**Corollary 7.3.** Suppose Assumptions 7.4, 7.5, 7.16 and 4.2 are satisfied. Then the ideal \( J_\tau \) of \( T_\tau \) is principal.

**Proof.** Let \( \phi : R^0_{\tau, \text{opp}} \to T_\tau \) be the canonical surjection. By Corollary 3.3 we know that \( \phi(I^0_{\tau, \text{opp}}) = J_\tau \). The claim follows from combining Theorem 6.2 with Proposition 7.1.

**Remark 7.4.** For other fields \( F \) (e.g. CM fields), principality of the ideal of reducibility would hold for conjugate self-dual representations (see Theorem 2.11 of [BK13]).

7.2. Selmer groups. In this subsection we discuss Assumptions 7.4 and 2.3 for certain characters \( \chi \), for which we will prove our main results.

**Example 7.5.** Fix a prime \( p \) lying over \( p \) and denote by \( i_p \) the fixed embedding \( \mathbb{F} \hookrightarrow \mathbb{F}_p \) and \( i_\infty : \mathbb{F} \hookrightarrow \mathbb{C} \). Let \( \tau_\Sigma = \chi \) be a \( p \)-adic Galois character of the following form: Let \( \phi_1, \phi_2 \) be two Hecke characters of infinity types \( z \) and \( z^{-1} \) respectively, and set \( \phi = \phi_1/\phi_2 \). Assume that \( \Sigma \) contains the set \( S_p \) of primes dividing \( M_1M_2M_3M_4 \), where \( M_i \) denotes the conductor of \( \phi_i \). Let \( \phi_p : G_\Sigma \to \mathcal{O}^* \) denote the \( p \)-adic Galois character corresponding to \( \phi \) defined by \( \phi_p(\text{Frob}_q) = \).
Assume also that if \( q \in \Sigma \), then \( \#k_p \neq 1 \pmod{p} \). Under this assumption Assumption 2.4 will be satisfied by Corollary 9.7 in \([BK13]\).

Let \( L^\int(0, \phi) \) be the special \( L \)-value attached to \( \phi \) as in \([BK09]\). Write \( W \) for \( \Hom_{\mathcal{O}}(\Psi, 1) \otimes E/\mathcal{O} \). In this case we adapt Assumption 2.4 as follows:

**Conjecture 7.6.** \( \#H^1_{\Sigma}(F, W) \leq \#\mathcal{O}/\varpi^m \), where \( m = \text{val}_\varpi(L^\int(0, \phi)) \).

Note that this conjecture implies Assumption 2.4 for \( \Sigma = \Sigma_p = \{ p \mid p \} \). However, our conclusions hold for all sets \( \Sigma \supset \Sigma_p \) for which \( H^1_{\Sigma} = H^1_F \) (see Lemma 5.6 of \([BK13]\)).

**Remark 7.7.** Conjecture 7.6 can in many cases be deduced from the Main conjecture proven by Rubin \([Rub91]\). If \( \phi^{-1} = \psi^2 \) for \( \psi \) a Hecke character associated to a CM elliptic curve, then one can argue as follows. By Proposition 4.4.3 in \([Dee99]\) and using that \( H^1_{\Sigma}(F, W) \cong H^1_{\Sigma}(F, W^c) \), we have \( \#H^1_{\Sigma}(F, W) = \#H^1_{\Sigma}(F, E/\mathcal{O}(\phi_p^{-1})) \). Thus we can use Corollary 4.3.4 in \([Dee99]\) which together with the functional equation satisfied by \( L(0, \phi) \) implies the desired inequality.

### 7.3. Link of rings \( T_\Sigma \) to an actual Hecke algebra

In this section we recall from \([Ber09]\) an Eisenstein ideal bound for a Hecke algebra \( T(\Sigma) \) acting on cuspidal automorphic forms. We also recall results about Galois representations associated to such forms and use this to relate \( T(\Sigma) \) to the ring \( T \) defined in Section 3.

Continuing with the notation of Example \( 7.2 \) we assume \( \phi = \phi_1/\phi_2 \) is unramified. For an ideal \( \mathfrak{m} \) in \( \mathcal{O}_F \) and a finite place \( q \) of \( F \) put \( \mathfrak{m}_q = \mathfrak{m}\mathcal{O}_{F,q} \). We define

\[
U^1(\mathfrak{m}_q) = \{ k \in \text{GL}_2(\mathcal{O}_{F,q}) \mid \det(k) \equiv 1 \pmod{\mathfrak{m}_q} \}.
\]

Now put

\[
K_f := \prod_{q|\mathfrak{m}_1} U^1(\mathfrak{m}_1,q) \subset \text{Res}_{F/Q} \text{GL}_2(A_f).
\]

Denote by \( S_2(K_f) \) the space of cuspidal automorphic forms of \( \text{Res}_{F/Q} \text{GL}_2(A) \) of weight 2, right-invariant under \( K_f \) (for more details see Section 3.1 of \([Urb95]\)).

Put \( \gamma = \phi_1/\phi_2 \) and write \( S_2(K_f, \gamma) \) for the forms with central character \( \gamma \).

From now on, let \( \Sigma \) be a finite set of places of \( F \) containing \( S_\phi = \{ q \mid \mathfrak{m}_1, \mathfrak{m}_2 \} \cup \{ q \mid pd_F \} \).

We denote by \( T(\Sigma) \) the \( \mathcal{O} \)-subalgebra of \( \text{End}_\mathcal{O}(S_2(K_f, \gamma)) \) generated by the Hecke operators \( T_q \) for all places \( q \not\in \Sigma \).

Let \( J(\Sigma) \subset T(\Sigma) \) be the ideal generated by

\[
\{ T_q - \phi_1(\varpi_q) \cdot \#k_q - \phi_2(\varpi_q) \mid q \not\in \Sigma \}.
\]

**Definition 7.8.** Denote by \( m(\Sigma) \) a maximal ideal of \( T(\Sigma) \) containing the image of \( J(\Sigma) \). We set \( T_\Sigma := T(\Sigma)/m(\Sigma) \). Moreover, set \( J_\Sigma := J(\Sigma)T_\Sigma \). We refer to \( J_\Sigma \) as the Eisenstein ideal of \( T_\Sigma \).

**Theorem 7.9 (Berger, Theorem 14).** Let \( p > 3 \) and assume \( \ell \not\equiv \pm 1 \pmod{p} \) for \( \ell | d_F \). Let \( \phi \) be an unramified Hecke character of infinity type \( \phi(\infty)(z) = z^2 \). There exist Hecke characters \( \phi_1, \phi_2 \) with \( \phi_1/\phi_2 = \phi \) such that their conductors are divisible only by ramified primes or inert primes not congruent to \( \pm 1 \pmod{p} \) and such that

\[
\#(T_\Sigma/J_\Sigma) \geq \#(\mathcal{O}/(L^\int(0, \phi))).
\]
The space $S_2(K_f, \psi)$ is isomorphic as a $G(\mathbb{A}_f)$-module to $\bigoplus \pi_f^{K_f}$ for automorphic representations $\pi$ of a certain infinity type (see Theorem 7.10 below) with central character $\psi$. Here $\pi_f$ denotes the restriction of $\pi$ to $GL_2(\mathbb{A}_f)$ and $\pi_f^{K_f}$ stands for the $K_f$-invariants.

For $g \in G(\mathbb{A}_f)$ we have the usual Hecke action of $[K_f g K_f]$ on $S_2(K_f, \psi)$. For primes $q$ such that the $v$th component of $K_f$ is $GL_2(\mathcal{O}_{E,v})$ we define $T_q = [K_f \left[ \frac{\omega_q}{t_1} 1 \right] K_f]$.

Combining the work of Taylor, Harris, and Soudry with results of Friedberg-Hoffstein and Laumon/Weissauer, one can show the following (see [BH07] for general case of cuspforms of weight $k$ and forthcoming work for general CM fields by C. P. Mok (with a similar assumption on the central character) and Harris-Taylor-Thorne-Lan (without such an assumption)):

**Theorem 7.10** ([BH07] Theorem 1.1). Given a cuspidal automorphic representation $\pi$ of $GL_2(A_F)$ with $\pi_{\infty}$ isomorphic to the principal series representation corresponding to

$$\begin{bmatrix} t_1 & * \\ t_2 & \end{bmatrix} \mapsto \left( \frac{t_1}{|t_1|} \right) \left( \frac{t_2}{|t_2|} \right)$$

and cyclotomic central character $\psi$ (i.e., $\psi^c = \psi$), let $\Sigma_{\pi}$ denote the set consisting of the places of $F$ lying above $p$, the primes where $\pi$ or $\pi^c$ is ramified, and the primes ramified in $F/\mathbb{Q}$.

Then there exists a finite extension $E$ of $F_p$ and a Galois representation

$$\rho_\pi : G_{\Sigma_{\pi}} \to GL_2(E)$$

such that if $q \notin \Sigma_{\pi}$, then $\rho_{\pi}$ is unramified at $q$ and the characteristic polynomial of $\rho_{\pi}(\text{Frob}_q)$ is $x^2 - a_q(\pi)x + \psi(\omega_q)(|\mathbb{Z}_q|)$, where $a_q(\pi)$ is the Hecke eigenvalue corresponding to $T_q$. Moreover, $\rho_{\pi}$ is absolutely irreducible.

Regarding the crystallinity of the representations $\rho_{\pi}$ we make the following conjecture (see Section 7.9 for the definition of a short crystalline Galois representation, and note that we assume $p > 3)$:

**Conjecture 7.11.** If $\pi$ is unramified at $q \mid p$ then $\rho_{\pi}|_{G_q}$ is crystalline and short.

This has now been proven in many cases by A. Jorza [Jor10]. Note that for the choice of characters $\phi_i$ as in Theorem 7.9 the cuspforms occurring in $S_2(K_f, \psi)$ are unramified at $q \mid p$.

**Definition 7.12.** Let $\chi$ be the mod $\varpi$ reduction of $\phi_{\pi, \epsilon}$. We now define the subsets $\Pi$ as the set of (strict equivalence classes of Galois) deformations of residual representations of the form $\tau$, one for each $\rho_\pi$ associated to an automorphic representation $\pi$ occurring in $S_2(K_f, \gamma)_{m(\Sigma)}$ and define $\Pi_\tau$ to be the subset with residual representation isomorphic to (a twist by $\phi_{2, p}$ of) $\tau = \begin{bmatrix} 1 & * \\ 0 & \chi \end{bmatrix}$. (Note that $\Pi_\tau \cap \Pi_{\tau'} = \emptyset$ for $\tau \neq \tau'$ by Proposition 3.2).

For every $\tau$ one has the natural surjective map $T \to T_\tau$ resulting from the surjections $R^{\rho_\pi} \to R^{\rho_{\tau}}$.

**Lemma 7.13.** If $\Pi$ is the set of modular deformations defined above then the ring $T \subset \prod_{\rho_\pi \in \Pi} \mathcal{O}$ defined in the previous section can be identified with the Hecke
algebra \( T_\Sigma \). Furthermore, \( T_\tau \) agrees with the quotient of \( T_\Sigma \) acting on the subspace of automorphic forms spanned by eigenforms \( \pi \) such that \( \rho_\pi \in \Pi_\tau \).

**Proof.** We will just prove the first part (concerning \( T \) and \( T_\Sigma \) - the proof for \( T_\tau \) being analogous). We define the following \( \mathcal{O} \)-algebra map:

\[
 f : T_\Sigma \to T \subset \prod_{\rho_\pi \in \Pi} \mathcal{O} : T_\rho \mapsto (a_q(\pi))_{\rho_\pi \in \Pi},
\]

where we use that \( a_q(\pi) = \text{tr} \rho_\pi(\text{Frob}_q) \) and therefore \( (a_q(\pi))_{\rho_\pi \in \Pi} = T(\text{Frob}_q) \).

We check that this map \( f \) is injective: By definition, \( T_\Sigma \hookrightarrow \oplus_{\rho_\pi \in \Pi} \text{End}_\mathcal{O}(V_\rho^{K'}) \), where we denote by \( V_\rho \) the representation space of \( \pi \). Since \( T_\rho \) acts on \( \pi \) by \( a_\rho(\pi) \), the image in each summand is given by the \( \mathcal{O} \)-algebra generated by the \( a_\rho(\pi) \)'s. Hence injectivity follows.

For surjectivity first note that \( f(T_\Sigma) \supset S := \{ T(\text{Frob}_q) \mid q \notin \Sigma \} \). Since \( f \) is injective let us identify \( T_\Sigma \) with \( f(T_\Sigma) \). Clearly \( T_\Sigma \) is local, complete with respect to its maximal ideal \( \mathfrak{m}_\Sigma := \mathfrak{m}(\Sigma)/T_\Sigma \) and \( T \) has the same properties derived from the properties of \( R^{\text{ss}} \). Moreover, looking at the residue fields we see that \( \mathfrak{m}_T \cap T_\Sigma = \mathfrak{m}_\Sigma \), so the \( \mathfrak{m}_\Sigma \)-adic topology on \( T_\Sigma \) is the subspace topology induced from the \( \mathfrak{m}_T \)-adic topology on \( T \). Then using Theorem 8.1 in [Mat89] again we see that the closure of \( T_\Sigma \) in \( T \) equals the completion of \( T_\Sigma \). But since \( T_\Sigma \) is already complete and \( T \) is topologically generated by \( S \) (i.e., the closure of \( S \) in \( T \) equals \( T \)), we conclude that \( T_\Sigma = T \), hence we are done. \( \square \)

### 8. Main result

In this section we will state our main theorems (Theorems 8.1, 8.2 and 8.5) for the two-dimensional Galois representations over imaginary quadratic fields considered in section 7. In this case many of the assumptions introduced throughout the paper can be proven to hold. However, we would like to stress that the conclusions are still valid if instead one assumes Assumptions 2.4, 2.5 and 4.2. To make this section self-contained we will repeat all the assumptions in the case of an imaginary quadratic field which were made in section 7.

Let \( F \) be an imaginary quadratic field, \( p > 3, p \nmid \#\text{Cl}_F d_F \), and assume \( \ell \neq \pm 1 \mod p \) for \( \ell \mid d_F \). Let \( \phi \) be an unramified Hecke character of infinity type \( \phi(\infty)(z) = z^2 \) and write \( \chi \) for the mod \( \varpi \)-reduction of \( \phi \). Furthermore, assume that \( \Sigma \) contains the set of places \( S_\phi \) (containing the primes dividing the conductors of the two auxiliary characters \( \phi_i \) from Theorem 7.9).

Let \( \tau : G_\Sigma \to \text{GL}_2(F) \) be a non-semi-simple representation of the form

\[
\tau = \begin{bmatrix} 1 & 0 \\ 0 & \chi \end{bmatrix}.
\]

**Theorem 8.1.** Suppose Conjecture 7.9 is satisfied (see Remark 7.14). There exists an \( F \)-basis \( B \) of \( H_2^c(F, \chi^{-1}) \) which is modular, i.e., such that if \( b \in B \) and \( f : G_\Sigma \to \text{GL}_2(F(\chi^{-1})) \) is a cocycle representing \( b \), then the residual representation

\[
\rho_f : G_\Sigma \to \text{GL}_2(F), \quad \rho_f(g) = \begin{bmatrix} 1 & f(g) \chi(g) \\ 0 & \chi(g) \end{bmatrix}
\]

is (up to a twist) the reduction (mod \( \varpi \)) of a representation \( \rho_\pi : G_\Sigma \to \text{GL}_2(\mathcal{O}) \) attached to an automorphic representation \( \pi \) of \( \text{GL}_2(\mathcal{A}_F) \).
Theorem 8.2. Assume that the Galois representations \( \rho_\pi \) for \( \pi \) occurring in \( S_2(K_f, \Psi; \kappa(\Sigma)) \) (for notation please see Section 7.3) are crystalline at \( v \mid p \). Also suppose that \( \#H^1_{\text{cr}}(F, E/\mathcal{O}((\phi_\psi)^{-1})) \leq \#\mathcal{O}/\varpi^m \), where \( m = \text{val}_p(L^{\text{disc}}(0, \phi)) \).

Let \( \rho : G_{\Sigma} \to \text{GL}_2(E) \) be a continuous, irreducible representation which is crystalline at \( p \mid p \) and write \( \tau : G_{\Sigma} \to \text{GL}_2(F) \) for its mod \( \varpi \) reduction with respect to some lattice in \( E^2 \). Suppose that \( \tau^{\text{ss}} \cong 1 \oplus \chi \). Assume that the sets \( \Phi_\tau \) for \( \tau^\prime \in \mathcal{B} \) are finite. Then \( \rho \) is modular, i.e., there exists an automorphic representation \( \pi^\prime \) of \( \text{GL}_2(A_F) \) such that \( \rho \cong \rho_\pi \).

Remark 8.3. As discussed in Remark 7.11 and the paragraph following Conjecture 7.11 the first two assumptions should be satisfied in the majority of cases by work of Jorza and Rubin.

Remark 8.4. We note that when \( \text{rk}_\mathcal{O}T = \dim_F H^1_{\text{cr}}(F, \text{Hom}(\tau_2, \tau_1)) \) it is easy to see that \( H^1_{\text{cr}}(F, \text{Hom}(\tau_2, \tau_1)) \) has a projectively unique modular basis. It may be possible to check numerically in specific examples if \( \text{rk}_\mathcal{O}T = \dim_F H^1_{\text{cr}}(F, \text{Hom}(\tau_2, \tau_1)) \) holds, but we have unfortunately not been able to carry this out.

Proof. This is a summary of the arguments carried out so far. As in the proof of Theorem 8.1 we note that Assumption 4.2 is satisfied. Also Assumption 2.4 is satisfied by Corollary 9.7 in [BK13] (see also discussion in Example 7.5). Let \( R^\text{tr,0}_i, T_i \) be as before the \( \mathcal{O} \)-subalgebra of \( R^0_{\tau_i} \) (defined as the image of \( R^0_{\tau_i} \) inside \( \prod_{p \in P \setminus \{p\}} \mathcal{O} \)) generated by traces and the Hecke algebra (respectively) corresponding to \( e_i \). We denote the corresponding ideals of reducibility and the Eisenstein ideal by \( I^\text{tr,0}_i \) and \( J_i \) respectively. We get for every \( i = 1, 2, \ldots, s \) a commutative diagram with surjective arrows:

\[
\begin{array}{ccc}
R^\text{tr,0}_i & \xrightarrow{\phi_i} & T_i \\
\downarrow & & \downarrow \\
R^\text{tr,0}_i / I^\text{tr,0}_i & \rightarrow & T_i / J_i
\end{array}
\]
By Theorem 6.3 and Corollary 7.3 the bottom map is an isomorphism for every $i = 1, 2, \ldots, s$.

By Theorem 6.3 we obtain a canonical map $R_{\text{red}}^{\tau} \to R_{\text{tr}}^{\tau, 0}$ which factors through an isomorphism $R_{\tau}^{\text{red}} \cong R_{\tau}^{\text{tr}, 0}$. We know by Proposition 7.1 that $I_{\tau}^{\text{red}}$ is principal. Using this and diagram (8.1) for $i = \tau$ (and our conclusion that the bottom map in (8.1) is an isomorphism) we can apply Theorem 4.1 from [BK13] to conclude that $\phi_{\tau}$ is an isomorphism. Then any $O$-algebra map $R_{\tau} \to O$ factors through $R_{\tau}^{\text{red}} \cong R_{\tau}^{\text{tr}, 0}$, so any deformation to $O$ is modular. \hfill \Box

Let $\Psi$ be as in example 7.5. The property of $H^1_{\Sigma}(F, \chi^{-1})$ of having a projectively unique modular basis can be replaced by the assumption that $H^1_{\Sigma}(F, \Psi^{-1} \otimes E/O)$ is an $F$-vector space.

**Theorem 8.5.** Let the notation and assumptions be the same as in Theorem 8.2 except that we do not demand that $H^1_{\Sigma}(F, \chi^{-1})$ has a projectively unique modular basis, but instead assume that $\varpi H^1_{\Sigma}(F, \Psi^{-1} \otimes E/O) = 0$. Then, as before, $\rho$ is modular.

**Proof.** We need to reprove Theorem 6.3. Our assumption that $\varpi$ annihilates the Selmer group along with injectivity of $\iota$ in the proof of Theorem 6.3 implies that $\varpi$ annihilates $T_{\tau}/I_{\tau}^{\text{tr}, 0}T_{\tau}$. We showed in the proof of Theorem 6.2 that $T_{\tau} \cong R_{\tau}^{\text{tr}, 0}$. Hence we conclude that $\varpi$ annihilates $R_{\tau}^{\text{tr}, 0}/I_{\tau}^{\text{tr}, 0}$. As discussed towards the end of section 2.5 the module $R_{\tau}^{\text{tr}, 0}/I_{\tau}^{\text{tr}, 0}$ is a cyclic $O$-module, hence we must have $R_{\tau}^{\text{tr}, 0}/I_{\tau}^{\text{tr}, 0} \cong F$. Since $T_{\tau}/J_{\tau}$ is a non-zero $O$-module, this implies that the map $R_{\tau}^{\text{tr}, 0}/I_{\tau}^{\text{tr}, 0} \to T_{\tau}/J_{\tau}$ must be injective. \hfill \Box

**References**

BC09. J. Bellaïche and G. Chenevier, $p$-adic families of Galois representations and higher rank Selmer groups, Astérisque (2009), no. 324.

Ber05. T. Berger, An Eisenstein ideal for imaginary quadratic fields, Thesis, University of Michigan, Ann Arbor, 2005.

Ber09. ______, On the Eisenstein ideal for imaginary quadratic fields, Compos. Math. 145 (2009), no. 3, 603–632.

BH07. T. Berger and G. Harcos, $l$-adic representations associated to modular forms over imaginary quadratic fields, Int. Math. Res. Not. IMRN (2007), no. 23, Art. ID rnm113, 16.

BK09. T. Berger and K. Klosin, A deformation problem for Galois representations over imaginary quadratic fields, Journal de l’Institut de Math. de Jussieu 8 (2009), no. 4, 669–692.

BK11. ______, R=T theorem for imaginary quadratic fields, Math. Ann. 349 (2011), no. 3, 675–703.

BK13. ______, On deformation rings of residually reducible Galois representations and R = T theorems, Math. Ann. 355 (2013), no. 2, 481–518.

BKK14. T. Berger, K. Klosin, and K. Kramer, On higher congruences between automorphic forms, Math. Res. Lett. 21 (2014), no. 1, 71–82.

Böc11. G. Böckle, Deformations of Galois representations (CRM Advanced Course on Modularity), preprint (2011).

CHT08. L. Clozel, M. Harris, and R. Taylor, Automorphy for some $l$-adic lifts of automorphic mod $l$ Galois representations, Publ. Math. Inst. Hautes Études Sci. (2008), no. 108, 1–181, With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras.

CM09. F. Calegari and B. Mazur, Nearly ordinary Galois deformations over arbitrary number fields, J. Inst. Math. Jussieu 8 (2009), no. 1, 99–177.
CV03. S. Cho and V. Vatsal, *Deformations of induced Galois representations*, J. Reine Angew. Math. **556** (2003), 79–98.

DDT97. H. Darmon, F. Diamond, and R. Taylor, *Fermat’s last theorem*, Elliptic curves, modular forms & Fermat’s last theorem (Hong Kong, 1993), Internat. Press, Cambridge, MA, 1997, pp. 2–140.

Dee99. J. Dee, *Selmer groups of Hecke characters and Chow groups of self products of CM elliptic curves*, preprint (1999), arXiv:math.NT/9901155v1.

Hun80. T. Hungerford, *Algebra*, Graduate Texts in Mathematics, vol. 73, Springer-Verlag, New York, 1980, Reprint of the 1974 original.

Jor10. A. Jorza, *Crystalline representations for GL₂ over quadratic imaginary fields*, Ph.D. thesis, Princeton University, 2010.

Klo09. K. Klosin, *Congruences among automorphic forms on U(2, 2) and the Bloch-Kato conjecture*, Annales de l’institut Fourier **59** (2009), no. 1, 81–166.

Mat89. H. Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.

Maz97. B. Mazur, *An introduction to the deformation theory of Galois representations*, Modular forms and Fermat’s last theorem (Boston, MA, 1995), Springer, New York, 1997, pp. 243–311.

MW84. B. Mazur and A. Wiles, *Class fields of abelian extensions of Q*, Invent. Math. **76** (1984), no. 2, 179–330.

Rub91. K. Rubin, *The “main conjectures” of Iwasawa theory for imaginary quadratic fields*, Invent. Math. **103** (1991), no. 1, 25–68.

SW99. C. M. Skinner and A. J. Wiles, *Residually reducible representations and modular forms*, Inst. Hautes Études Sci. Publ. Math. (1999), no. 89, 5–126 (2000).

Urb95. E. Urban, *Formes automorphes cuspidales pour GL₂ sur un corps quadratique imaginaire. Valeurs spéciales de fonctions L et congruences*, Compositio Math. **99** (1995), no. 3, 283–324.

Urb01. _____, *Selmer groups and the Eisenstein-Klingen ideal*, Duke Math. J. **106** (2001), no. 3, 485–525.

---

1School of Mathematics and Statistics, University of Sheffield, Hicks Building, Hounsfield Road, Sheffield S3 7RH, UK, email: tberger@cantab.net

2Department of Mathematics, Queens College, City University of New York, 65-30 Kissena Blvd, Queens, NY 11367, USA, email: kklosin@qc.cuny.edu