ON RELATED VARIETIES TO THE COMMUTING VARIETY OF A SEMISIMPLE LIE ALGEBRA

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ABSTRACT. Let \( g \) be a semisimple Lie algebra of finite dimension. The nullcone \( N \) of \( g \) is the set of \((x, y)\) in \( g \times g \) such that \( x \) and \( y \) are nilpotents and are in the same Borel subalgebra. The main result of this paper is that \( N \) is a closed and irreducible subvariety of \( g \times g \), its normalization has rational singularities and its normalization morphism is bijective.

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1. INTRODUCTION

The basic field \( \mathbb{k} \) is an algebraic closed field of characteristic zero. Let \( g \) be a semisimple Lie algebra of finite dimension and let \( G \) be its adjoint group. We denote by \( b \) a Borel subalgebra of \( g \), \( \mathfrak{h} \) a Cartan subalgebra of \( g \) contained in \( b \), \( b_g \) and \( \text{rk}_g \) the dimensions of \( b \) and \( \mathfrak{h} \) respectively. Let \( W \) be the Weyl group of \( g \) with respect to \( \mathfrak{h} \). The symmetric algebra of \( g \) is denoted by \( S(g) \) and the subalgebra of its \( G \)-invariant elements is denoted by \( S(g)^G \). Let \( B_g \) be the set of \((x, y)\) in \( g \times g \) such that \( x \) and \( y \) are in the same Borel subalgebra. In Section 4, we show the following theorem:

**THEOREM 1.1.**
(i) The variety \( B_g \) is closed and irreducible of dimension \( 3b_g - \text{rk}_g \), but it isn’t normal,
(ii) the algebra of \( W \)-invariant regular functions on \( \mathfrak{h} \times \mathfrak{h} \) is isomorphic to the algebra of \( G \)-invariant regular functions on \( B_g \).

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The variety \( B_\mathfrak{g} \) contains two interesting varieties, the nullcone and the commuting variety. The nullcone of \( \mathfrak{g} \), denoted by \( N \), is defined by the set of zeros of the \( G \)-invariant polynomial functions \( f \) on \( \mathfrak{g} \times \mathfrak{g} \) such that \( f(0) = 0 \). By [25], it can be equivalently defined as the variety of pairs \((x, y)\) in \( B_\mathfrak{g} \) such that \( x \) and \( y \) are nilpotents. It is well-known that the nullcone plays a fundamental role in the theory of invariants and its applications. This cone was introduced and studied by D. Hilbert in his famous paper ”Ueber die vollen Invariantensystem” ([16]). H. Kraft and N. R. Wallach studied in [21] the geometry of the nullcone; they showed that the nullcone of any number of copies of the adjoint representation of \( G \) on \( \mathfrak{g} \) is irreducible and they gave a resolution of singularities of this variety. Moreover, in [22], they studied when the polarizations of a set of invariant functions defining the nullcone of a representation \( V \) define the nullcone of a direct sum of several copies of \( V \). This question was earlier studied by M. Losik, P. W. Michor, and V. L. Popov in [23]. In [27], V. L. Popov gives a general algorithm to determine the irreducible components of maximal dimension of the nullcone using the weights of the representation and their multiplicities. In Section 5 of this paper, we show the following theorem:

**THEOREM 1.2.**

(i) The nullcone \( N \) is closed and irreducible of dimension 
\[3(b_\mathfrak{g} - \text{rk}_G),\]

(ii) the codimension of the set of its singular points is at least four and the
normalization morphism of \( N \) is bijective.

In the last section of this paper, we use the ideas of Wim H. Hesselink (see [15]) for showing the following theorem:

**THEOREM 1.3.** The normalizations of \( N \) and \( B_\mathfrak{g} \) have rational singularities.

The other interesting variety of \( B_\mathfrak{g} \) is the commuting variety. We recall that the commuting variety \( C_\mathfrak{g} \) of \( \mathfrak{g} \) is the set of \((x, y)\) in \( \mathfrak{g} \times \mathfrak{g} \) such that \([x, y] = 0\). We used a result of A. Josef in [19] on \( C_\mathfrak{g} \) to prove that \( k[B_\mathfrak{g}]^G \) (see Notations) and \( S(\mathfrak{h} \times \mathfrak{h})^W \) are isomorphic.

In this paper, we also proved some other results on \( B_\mathfrak{g} \). We used the homogeneous generators \( p_1, \ldots, p_{\text{rk}_G} \) of \( S(\mathfrak{g})^G \), chosen so that the sequence of their degrees \( d_1, \ldots, d_{\text{rk}_G} \) is increasing. By [4], \( d_1 + \ldots + d_{\text{rk}_G} = b_{\text{rk}_G} \). Now, let \( X := \sigma(\mathfrak{h} \times \mathfrak{h}) \), whenever \( \sigma \) is the morphism from \( \mathfrak{g} \times \mathfrak{g} \) to \( \mathbb{k}^{b_\mathfrak{g} + \text{rk}_G} \) defined by

\[\sigma(x, y) = (p_1^{(0)}(x, y), \ldots, p_1^{(d_1)}(x, y), \ldots, p_{\text{rk}_G}^{(0)}(x, y), \ldots, p_{\text{rk}_G}^{(d_{\text{rk}_G})}(x, y)),\]

with \( p_i^{(n)} \) are the 2-order polarizations of \( p_i \) of bidegree \((d_i - n, n)\) (see Notations).

It is shown in Section 3 that there is an isomorphism between \((\mathfrak{h} \times \mathfrak{h})^W \) and the normalization of \( X \), the normalization morphism of \( X \) is bijective, and the codimension of the complement of the set of regular points of \( X \) is at least two. On the other hand, the variety \( \sigma(B_\mathfrak{g}) \) is equal to \( X \) and \( B_\mathfrak{g} \) is an irreducible component of \( \sigma^{-1}(X) \).

Finally, in the appendix, we give some results on the positive roots related to Theorem 1.3.
2. Notations

We consider the diagonal action of $G$ on $g \times g$ and the diagonal action of $W$ on $h \times h$. Let $u$ be the set of the nilpotent elements in $b$, let $R$ be the root system of $h$ in $g$, let $R_+$ be the positive root system defined by $b$ and let $\Pi$ be the set of simple roots of $R_+$. We denote by $B$ the normalizer of $b$ in $G$, $U$ its unipotent radical, $H$ and $N_G(b)$ the centralizer and the normalizer of $h$ in $G$. We use the following notations:

- if $G \times A \to A$ is an action of $G$ on the algebra $A$, denote by $A^G$ the subalgebra of the $G$-invariant elements of $A$,
- for $i = 1, \ldots, \text{rk} g$, the 2-order polarizations of $p_i$ of bidegree $(d_i - n, n)$ denoted by $p_i^{(n)}$ are the unique elements in $(S(g) \otimes \mathbb{C} S(g))^G$ satisfying the following relation
  \[ p_i(ax + by) = \sum_{n=0}^{d_i} a^{d_i-n} b^n p_i^{(n)}(x, y), \]
  for all $a, b \in \mathbb{C}$ and $(x, y) \in g \times g$,
- for $i = 1, \ldots, \text{rk} g$, $\varepsilon_i$ is the element of $S(g) \otimes \mathbb{C} g$ defined by
  \[ \langle \varepsilon_i(x), v \rangle = p'_i(x)(v), \forall x, v \in g, \]
  whenever $p'_i(x)$ is the differential of $p_i$ at $x$ for $i = 1, \ldots, \text{rk} g$,
- for $i = 1, \ldots, \text{rk} g$, the 2-polarizations of $\varepsilon_i$ of bidegree $(m - m - 1, m)$ denoted by $\varepsilon_i^{(m)}$ are the unique elements in $S(g) \otimes \mathbb{C} S(g) \otimes \mathbb{C} g$ satisfying the following relation
  \[ \varepsilon_i(ax + by) = \sum_{n=0}^{d_i-1} a^{d_i-n-1} b^n \varepsilon_i^{(n)}(x, y), \]
  for all $a, b \in \mathbb{C}$ and $(x, y) \in g \times g$,
- $g'$ is the set of regular elements of $g$,
- $h'$ is the subset of regular elements of $g$ belonging to $h$,
- for $x, y$ in $g$, $P_{x,y}$ is the subspace of $g$ generated by $x$ and $y$,
- $\pi$ is the morphism from $h$ to $k^{\text{rk} g}$ defined by $\pi(x) := (p_1(x), \ldots, p_{\text{rk} g}(x))$,
- for $X$ algebraic variety and for $x \in X$, $O_{x,X}$ is the local ring of $X$ at $x$ $\mathfrak{M}_x$ is its maximal ideal and $\hat{O}_{x,X}$ is the completion of $O_{x,X}$ for the $\mathfrak{M}_x$-adic topology,
- $\langle \cdot, \cdot \rangle$ is the Killing form of $g$,
- for $x \in b$, $x = x_0 + x_+$ with $x_0 \in h$ and $x_+ \in u$,
- $h$ is the element of $h$ such that $\beta(h) = 1$, for all $\beta$ in $\Pi$ and $h(t)$ is the one parameter subgroup of $G$ generated by $\text{ad} h$. Then for all $t$ in $b$,
  \[ \lim_{t \to 0} h(t)(x) = x_0. \]

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3. ON THE VARIETY $\sigma(h \times h)$

Denote by $(x, y) \mapsto \overline{(x, y)}$ the canonical map from $h \times h$ to $(h \times h)/W$.

**Proposition 3.1.** The set $X := \sigma(h \times h)$ is closed in $k^{h_r +rk}$. 

**Proof.** For $m := \sup\{d_i, i \in \{i, \ldots, rk\}\}$, let $(t_i)_{i=1, \ldots, m+1}$ be in $k$, pairwise different, let $\alpha$ be the morphism from $h \times h$ to $h^{m+1}$ defined by

$$\alpha(x, y) := (x + t_1y, \ldots, x + t_{m+1}y)$$

and let $\gamma$ be the morphism from $h^{m+1}$ to $k^{(m+1)rk}$ defined by

$$\gamma(x_1, \ldots, x_{m+1}) := (\pi(x_1), \ldots, \pi(x_{m+1})).$$

Let $\beta$ be the morphism from $k^{h_r +rk}$ to $k^{(m+1)rk}$ defined by

$$\beta((z^{(j)}_i)_{1 \leq i \leq rk, 0 \leq j \leq d_i}) := \left( \sum_{j=0}^{d_1} t^{(j)}_1 z^{(j)}_1, \sum_{j=0}^{d_2} t^{(j)}_2 z^{(j)}_2, \ldots, \sum_{j=0}^{d_{rk}} t^{(j)}_{m+1} z^{(j)}_{m+1} \right).$$

Since $(t_i)_{i=1, \ldots, m+1}$ are pairwise different then $\beta$ is an isomorphism from $k^{h_r +rk}$ onto $\beta(k^{h_r +rk})$. Since $\pi$ is a finite morphism, $\gamma$ is too. Hence $\gamma \circ \alpha(h \times h)$ is closed in $k^{(m+1)rk}$. Furthermore, $\gamma \circ \alpha(h \times h)$ is contained in the image of $\beta$. Then $\sigma(h \times h)$ is closed since $\beta \circ \sigma(h \times h) = \gamma \circ \alpha(h \times h)$. 

**Proposition 3.2.** Let $(X, \mu)$ be the normalization of $X = \sigma(h \times h)$.

(i) There exists a bijection $\overline{\sigma}$ from $h \times h/W$ to $X$ such that $\overline{\sigma}(\overline{(x, y)}) = \sigma(x, y)$ for all $(x, y)$ in $h \times h$.

(ii) There exists a unique morphism $\sigma_n$ from $(h \times h)/W$ to $X_n$ such that $\mu \circ \sigma_n = \overline{\sigma}$.

(iii) The morphism $\sigma_n$ is an isomorphism.

(iv) The morphism $\mu$ is bijective.

**Proof.** (i) Since $\sigma(w(x, y)) = \sigma(x, y)$, for all $w$ in $W$ and $(x, y)$ in $h \times h$, $\overline{\sigma}$ is well defined and it is surjective. Let $(x, y), (x', y')$ be in $h \times h$ such that $\overline{\sigma}((x, y)) = \overline{\sigma}((x', y'))$. Then:

$$\overline{\sigma}((x, y)) = \overline{\sigma}((x', y')) \Rightarrow \sigma(x, y) = \sigma(x', y')$$

$$\Rightarrow p_1(x + ty) = p_1(x' + ty'), \forall t \in k, \forall i \in \{1, \ldots, rk\}.$$

Since $W$ is finite then there exist $t_1, t_2$ two different elements of $k$ and $w$ in $W$ such that we have the following system:

$$\begin{cases} x' + t_1y' = w(x) + t_1w(y) \\ x' + t_2y' = w(x) + t_2w(y) \end{cases}$$

then $x' = w(x)$ et $y' = w(y)$, hence $(x', y') = (x, y)$, then $\overline{\sigma}$ is injective and therefore it is bijective.

(ii) The variety $(h \times h)/W$ is a quotient of a smooth variety by a finite group, so it is normal. Since $\overline{\sigma}$ is bijective, it is a dominant morphism. Since $(h \times h)/W$ is normal and since $\overline{\sigma}$ is a dominant morphism, there exists a unique morphism $\sigma_n$
from \((\mathfrak{h} \times \mathfrak{h})/W\) into \(X_n\) such that \(\mu \circ \sigma_n = \bar{\sigma}\) ([14] [ch. II, Ex. 3.8]). Then there is a commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{h} \times \mathfrak{h}/W & \xrightarrow{\sigma_n} & X_n \\
\sigma \downarrow & & \downarrow \mu \\
X & & X
\end{array}
\]

(iii) Let \(\Psi\) be the comorphism of \(\sigma_n\) from \(k[X_n]\) into \(S(\mathfrak{h} \times \mathfrak{h})^W\). Show that \(\Psi\) admits an inverse. Let \(\Phi\) be the map from \(S(\mathfrak{h} \times \mathfrak{h})^W\) to \(k[X_n]\) defined by \(\Phi(Q) = P\), with \(P(z) = Q(x, y)\), whenever \((x, y)\) in \(\mathfrak{h} \times \mathfrak{h}\) and such that \(\sigma_n(x, y) = z\). The map \(\Phi\) is well defined. In fact, let \((x, y)\) and \((x', y')\) be in \(\mathfrak{h} \times \mathfrak{h}\) such that \(\sigma_n(x, y) = \sigma_n(x', y')\). We have to prove:

\[Q(x, y) = Q(x', y')\]

We have:

\[\sigma_n(x, y) = \sigma_n(x', y') \Rightarrow \mu \circ \sigma_n(x, y) = \mu \circ \sigma_n(x', y') \Rightarrow \sigma(x, y) = \sigma(x', y').\]

It has been proved that there exists \(w\) in \(W\) such that \(x' = w(x)\) and \(y' = w(y)\) then

\[Q(x', y') = Q(w(x), w(y)) = Q(x, y).\]

It is clear that the map \(\Phi\) is an algebra homomorphism. We now prove that \(P\) belongs to \(k[X_n]\). Let \(\bar{\Gamma}\) be the graph of \(Q\) and let \(\bar{\Gamma}\) be the image of \(\Gamma\) under the \(\sigma_n \times 1_k\). Then \(\bar{\Gamma}\) is the graph of \(P\) and it is closed since \(\sigma_n \times 1_k\) is a finite morphism and \(\Gamma\) is closed. Let \(\chi\) be the projection of \(\bar{\Gamma}\) onto \(X_n\). Then it is a bijection. Then by Zariski’s main theorem ([26]) \(\chi\) is an isomorphism since \(X_n\) is normal. Hence \(P\) is a regular function on \(X_n\).

Let us show that \(\Phi = \Psi^{-1}\). Let \(P\) be in \(k[X_n]\) and let \(Q\) be in \(S(\mathfrak{h} \times \mathfrak{h})^W\). We have:

\[\Phi \circ \Psi(P) = \Phi(P \circ \sigma_n) = P\]

\[\Psi \circ \Phi(Q) = \Phi(Q) \circ \sigma_n.\]

Now calculate \((\Phi(Q) \circ \sigma_n)(x, y)\), for \((x, y)\) in \(\mathfrak{h} \times \mathfrak{h}\),

\[(\Phi(Q) \circ \sigma_n)(x, y) = \Phi(Q)(\sigma_n(x, y)) = Q(x, y),\]

then \(\Psi \circ \Phi(Q) = Q\). Hence \(\Psi\) is an isomorphism from \(k[X_n]\) to \(S(\mathfrak{h} \times \mathfrak{h})^W\), whence the assertion.

(iv) Since \(\mu \circ \sigma_n = \bar{\sigma}\), since \(\sigma_n\) is an isomorphism and since \(\bar{\sigma}\) is bijective, \(\mu\) is bijective.

\[\square\]

Remark 3.1. By [18], \(\bar{\sigma}\) is an isomorphism if and only if the algebra \(S(\mathfrak{h} \times \mathfrak{h})^W\) is generated by the 2-polarizations of elements of \(S(\mathfrak{h})^W\), i.e. for \(\mathfrak{g}\) of type \(A_n, B_n\) and \(C_n\). For \(\mathfrak{g}\) of type \(D_n\), some generators of \(S(\mathfrak{h} \times \mathfrak{h})^W\) are given, and not all are polarizations. In particular for \(n\) even, all the polarizations of elements of \(S(\mathfrak{h})^W\) have even total degree and \(S(\mathfrak{h} \times \mathfrak{h})^W\) contains elements of odd total degree. So \(k[X_n]\) strictly contains \(k[X]\).

Proposition 3.3. : Let \(\Gamma := \{(x, y) \in \mathfrak{h} \times \mathfrak{h}|P_{x,y} \cap \mathfrak{h}' \neq \emptyset\}\), we have:

(i) \(\text{codim}_X \sigma(\Gamma)^c \geq 2\),


(iii) for all \( z \) in \( \sigma(\Gamma) \), \( z \) is a regular point of \( X \).

Proof. (i) Let \((x, y)\) be in \( \Gamma^c \), We have:

\[
(x, y) \in \Gamma^c \Rightarrow P_{x, y} \cap h' = \emptyset \Rightarrow x, y \in h \setminus h'.
\]

Then \( \Gamma^c \) is included in \( h \setminus h' \times h \setminus h' \). Hence codim\,_{h}\,_{h'\,_{h'}} = 2 \) and since \( \sigma \) is a finite morphism, then

\[
\text{codim}_{X}\,\sigma(\Gamma)^c \geq 2.
\]

(ii) Let \((x, y)\) be an element in \( \Gamma \) and let \((t_{i})_{i=1, \ldots, m+1}\) be in \( k \), pairwise different, chosen such that \( x + t_{i}y \) is regular for all \( i \), with \( m = \sup\{d_{i}, i \in \{1, \ldots, rk_{g}\}\} \). Let \( \beta \) be the morphism from \( k^{(h \times h)} \) to \( k^{(m+1)rk_{g}} \) defined by

\[
\beta((c_{i}^{(j)})_{1 \leq i \leq rk_{g}, 0 \leq j \leq d_{i}}) = \left( \sum_{j=0}^{d_{1}} c_{11}^{(j)} , \sum_{j=0}^{d_{2}} c_{12}^{(j)} , \ldots , \sum_{j=0}^{d_{rk_{g}}} c_{m+1}^{(j)} \right).
\]

Since \((t_{i})_{i=1, \ldots, m+1}\) are pairwise different then \( \beta \) is an isomorphism from \( k^{(h \times h)} \) onto \( k^{(m+1)rk_{g}} \). Moreover, \( \beta(\sigma(h \times h)) \) is equal to \( \tau(h \times h) \), where \( \tau \) is the morphism from \( h \times h \) to \( k^{(m+1)rk_{g}} \) defined by

\[
\tau(x, y) := (\pi(x + t_{1}y), \ldots, \pi(x + t_{m+1}y)).
\]

Let \( \tau_{1, 2} \) be the morphism from \( h \times h \) to \( k^{2rk_{g}} \) defined by

\[
\tau_{1, 2}(x, y) := (\pi(x + t_{1}y), \pi(x + t_{2}y)),
\]

let \( \varphi \) be the projection from \( \tau(h \times h) \) to \( \tau_{1, 2}(h \times h) \) defined by

\[
\varphi((\pi(x + t_{1}y), \ldots, \pi(x + t_{m+1}y)) := (\pi(x + t_{1}y), \pi(x + t_{2}y))
\]

and let \( (\tau_{1, 2})_{*} \), and \( \varphi_{*} \) be the comorphisms of \( \tau_{1, 2} \) and \( \varphi \) respectively. Then there is the following diagram:

\[
\begin{array}{ccc}
\hat{\mathcal{O}}_{T(x, y), \tau(h \times h)} & \xrightarrow{\tau_{*}} & \hat{\mathcal{O}}_{(x, y), h \times h} \\
\varphi_{*} \downarrow & & \downarrow (\tau_{1, 2})_{*}^{-1} \\
\hat{\mathcal{O}}_{\tau_{1, 2}(x, y), \tau_{1, 2}(h \times h)} & & \\
\end{array}
\]

Indeed, since \( x + t_{i}y \) is a regular point, \( \pi \) is an étale morphism at \( x + t_{i}, y \) for \( i = 1, 2 \), hence \( (\tau_{1, 2})_{*} \) is an isomorphism. Let \( \alpha \) be the morphism from \( h \times h \) to \( h^{m+1} \) defined by

\[
\alpha(x, y) := (x + t_{1}y, \ldots, x + t_{m+1}y)
\]

and let \( \gamma \) be the morphism from \( h^{m+1} \) to \( k^{(m+1)rk_{g}} \) defined by

\[
\gamma(x_{1}, \ldots, x_{m+1}) := (\pi(x_{1}), \ldots, \pi(x_{m+1})).
\]

Then there is the following commutative diagram:

\[
\begin{array}{ccc}
h \times h & \xrightarrow{\alpha} & h \times h \\tau_{1, 2} \downarrow & & \downarrow \gamma \\
\tau_{1, 2}(h \times h) \xrightarrow{\varphi} & & \tau(h \times h)
\end{array}
\]
Let $\alpha_*$ and $\gamma_*$ be the comorphisms of $\alpha$ and $\gamma$ respectively. Since $\alpha$ is an isomorphism, $(\alpha_*)^{-1} \circ (\tau_{1,2})$ is an isomorphism. Hence the coordinates on $\alpha_*$ are formal series of coordinates on $\hat{\alpha}$. Let $\varphi$ in the image of the ring, so that $\tau \circ \varphi$ is surjective. Since $\varphi$ is an isomorphism. Hence $\hat{\varphi}$ is a smooth point of $\tau(h \times h)$. So for $(x, y)$ in $\Gamma$, $\varphi(x, y)$ is regular in $\sigma(h \times h)$. □

We recall that the commuting variety $\mathcal{C}_g$ of $g$ is the set of $(x, y)$ in $g \times g$ such that $[x, y] = 0$.

**Proposition 3.4.** : The variety $X$ is the image of the commuting variety $\mathcal{C}_g$ of $g$ by $\sigma$.

**Proof.** Let $(x, y)$ be in $\mathcal{C}_g$ and let $x = x_s + x_n$ and $y = y_s + y_n$ be the Jordan decompositions of $x$ and $y$. Since $[x, y] = 0$, $[x_s, y_s] = 0$. Then $x_s$ and $y_s$ belong to a same Cartan subalgebra of $g$. Since the Cartan subalgebras are conjugate under $G$, there exists $g$ in $G$ such that $g(x_s)$ and $g(y_s)$ are in $h$. Since

$$p_i(x + ty) = p_i(x_s + ty_s), \forall i = 1, \ldots, \text{rk}_g$$

and $\sigma$ is $G$-invariant,

$$\sigma(x, y) = \sigma(g(x_s), g(y_s)).$$

Hence $\sigma(\mathcal{C}_g)$ is contained in $X$. Since $\mathcal{C}_g$ contains $h \times h$, $X$ is equal to $\sigma(\mathcal{C}_g)$. □

4. ON THE SUBVARIETY $\mathcal{B}_g$ OF $g \times g$

We consider the action of $B$ on $G \times B \times B$ given by $b. (g, x, y) = (gb^{-1}, b(x), b(y))$. Let $\gamma'$ be the morphism from $G \times B \times B$ to $g \times g$ defined by

$$\gamma'(g, x, y) = (g(x), g(y))$$

and let $\gamma$ be the morphism from $G \times B \times B$ to $g \times g$ defined through the quotient by $\gamma'$. Let $\mathcal{B}_g$ be the set of $(x, y)$ in $g \times g$ such that $x$ and $y$ are in the same Borel subalgebra. Then $\mathcal{B}_g$ is the image of $G \times B \times B$ by $\gamma$.

**Proposition 4.1.** : The morphism $\gamma$ is a desingularization of $\mathcal{B}_g$. The subvariety $\mathcal{B}_g$ is closed and irreducible of dimension $3b_g - \text{rk}_g$. Moreover it is not normal.

**Proof.** Since $\mathcal{B}_g$ is the image of $G \times B \times B$ by $\gamma$ and since $G/B$ is a projective variety, $\mathcal{B}_g$ is closed and $\mathcal{B}_g$ is irreducible as the image of an irreducible variety. Since $G \times B \times B$ is a vector bundle on the smooth variety $G/B$ then it is a smooth variety. Let $\mathcal{B}$ be the subvariety of the Borel subalgebras of $g$ and let $\tilde{\mathcal{B}}$ be the subvariety of elements $(u, x, y)$ of $\mathcal{B} \times g \times g$ such that $u$ contains $x$ and $y$. Hence there exists an isomorphism $\nu$ from $G \times B \times B$ onto $\mathcal{B}$ such that $\gamma$ is the compound of $\nu$ and the canonical projection of $\tilde{\mathcal{B}}$ on $g \times g$. Whence $\gamma$ is a proper morphism
since $G/B$ is a projective variety. It remains to show that $\gamma$ is a birational morphism. Let be

$$\Omega_g := \{(x, y) \in g \times g | P_{x, y} \setminus \{0\} \subset g', \dim P_{x, y} = 2\}.$$ 

The subset $\Omega_g$ is an open subset of $g \times g$ and $\Omega_g \cap B_g$ is a nonempty open subset of $B_g$. Let $(x, y)$ be an element in $\Omega_g \cap B_g$. According to [1] (Corollary 2) and [20] (Theorem 9), the subspace $V_{x, y}$ generated by $\{\varepsilon_i^{(m)}(x, y), i = 1, \ldots, \text{rk} g, m = 0, \ldots, d_i\}$ is the unique Borel subalgebra which contains $x$ and $y$; hence $\nu^{-1}(V_{x, y}, x, y)$ is the unique point in $G \times B b \times b$ above $(x, y)$. Whence $\gamma$ is birational. Since $\gamma$ is birational, $\dim B_g = \dim(G \times B b \times b) = 3b_g - \text{rk} g$.

Since $\gamma$ is a birational morphism, for $g$ non trivial, if $B_g$ would be normal, by Zariski’s main theorem a fiber of $\gamma$ would have cardinality 1 if it was finite, but $|\gamma^{-1}(x, 0)| = |W|$ for $x$ in $h'$. So, $B_g$ isn’t normal. □

Let $(B_{g, n}, \eta)$ be the normalization of $B_g$.

**Lemma 4.1.** There exists a unique closed immersion $\iota_n$ from $h \times h$ to $B_{g, n}$ such that $\eta \circ \iota_n = \iota$, whenever $\iota$ is the injection of $h \times h$ in $B_g$.

**Proof.** Since $\gamma$ is a dominant morphism from $G \times B b \times b$ to $B_g$ and since $G \times B b \times b$ is normal, there exists a unique morphism $\gamma_n$ from $G \times B b \times b$ to $B_{g, n}$ such that $\eta \circ \gamma_n = \gamma$ ([14] [ch. II, Ex. 3.8]). Then there is a commutative diagram:

$$
\begin{array}{ccc}
G \times B b \times b & \xrightarrow{\gamma_n} & B_{g, n} \\
\gamma \downarrow & & \eta \downarrow \\
B_g & & \\
\end{array}
$$

Let $i$ be the canonical injection of $h \times h$ into $G \times B b \times b$. Hence taking $\iota_n = \gamma_n \circ i$, $\eta \circ \iota_n = \iota$ since $\iota = \gamma \circ i$. Then there is a commutative diagram:

$$
\begin{array}{ccc}
h \times h & \xrightarrow{\iota_n} & \iota_n(h \times h) \\
\iota \downarrow & & \eta \downarrow \\
B_g & & \\
\end{array}
$$

Since $\gamma_n$ and $i$ are closed, $\iota_n$ is closed. □

**Proposition 4.2.** We have the following properties:

(i) $\sigma(B_g)$ is equal to $X$,

(ii) $B_g$ is an irreducible component of $\sigma^{-1}(X)$.

**Proof.** (i) Let $(x, y)$ be in $B_g$. Since $\sigma$ is $G$-invariant, we can assume that $x$ and $y$ are in $b$. We have:

$$p_i(x + ty) = p_i(x_0 + ty_0), \forall i = 1, \ldots, \text{rk} g$$

$$\Rightarrow \sigma(x, y) = \sigma(x_0, y_0),$$
then $\sigma(B_g)$ is contained in $X$, whence the equality since $h \times h$ is contained in $B_g$.

(ii) For $z$ in $k^{b_g + 3rg}$, by [6], the dimension of $\sigma^{-1}(z)$ is $3b_g - 3rg$, then
\[
\dim \sigma^{-1}(X) = 3b_g - rg = \dim B_g,
\]
hence $B_g$ is an irreducible component of $\sigma^{-1}(X)$. \hfill \Box

Let $\tau'$ be the morphism from $G \times b \times b$ to $h \times h$ defined by
\[
\tau'(g, x, y) = (x_0, y_0).
\]
For $b$ in $B$,
\[
\tau'(b, (g, x, y)) = \tau'(gb^{-1}, b(x), b(y)) = (b(x_0), b(y_0)) = (x_0, y_0),
\]
indeed, since $B = UH$, there exists $u$ in $U$ and $h$ in $H$ such that $b = uh$. Then $b(x) = b(x_0 + x_+) = u(x_0) + b(x_+)$, $u(x_0)$ is in $x_0 + u$ and $b(x_+)$ is in $u$, hence $b(x)$ is in $x_0 + u$. Then $\tau'$ is constant on the $B$-orbits. Hence there exists a morphism $\tau$ from $G \times B \times b$ to $h \times h$ defined by
\[
\tau((g, x, y)) = (x_0, y_0).
\]

Lemma 4.2. Let $z$ and $z'$ be in $G \times B \times b$ such that $\gamma(z') = \gamma(z)$. Then there exists $w$ in $W$ such that $\tau(z') = w.\tau(z)$.

Proof. Let $(g, x, y)$ and $(g', x', y')$ be in $G \times b \times b$ two representatives of $z$ and $z'$ respectively. We have
\[
\gamma(z) = \gamma(z') \Rightarrow (g'(x'), g'(y')) = (g(x), g(y)) \Rightarrow (x', y') = g^{-1}.g.(x, y).
\]
Since $G = \bigcup_{w \in W} UwB$, there exists $u$ in $U$, $w$ in $W$, $g_w$ in $N_G(h)$ a representative of $w$ and $b$ in $B$ such that $g^{-1}.g = u g_w b$. Let $x = x_0 + x_+$, $x' = x'_0 + x'_+$, $y = y_0 + y_+$ and $y' = y'_0 + y'_+$, whenever $x_0, x'_0, y_0$ and $y'_0$ are in $h$ and $x_+, x'_+, y_+$ and $y'_+$ are in $u$. We have:
\[
(x', y') = (ug_w b(x), ug_w b(y)) \Rightarrow (u^{-1}(x'), u^{-1}(y')) = (g_w b(x), g_w b(y)).
\]
Since $b(x)$ is in $x_0 + u$, since $b(y)$ is in $y_0 + u$, since $u^{-1}(x')$ is in $x'_0 + u$ and since $u^{-1}(y')$ is in $y'_0 + u$, there exists $v_x, v_y, v_{x'}$ and $v_{y'}$ in $u$, such that
\[
b(x) = x_0 + v_x, b(y) = y_0 + v_y, \quad u^{-1}(x') = x'_0 + v_{x'}, \quad u^{-1}(y') = y'_0 + v_{y'}.
\]
Hence
\[
x'_0 + v_{x'} = g_w(x_0) + g_w(v_x) \quad \text{and} \quad y'_0 + v_{y'} = g_w(y_0) + g_w(v_y).
\]
Since $v_x$ is in $\sum_{\alpha \in \mathbb{R}} g^\alpha$, $g_w(v_x)$ is in $\sum_{\alpha \in \mathbb{R}} g_w^{\alpha + \alpha_1} = \sum_{\alpha \in \mathbb{R}} g^\alpha$. Then $g_w(v_x) - v_{x'}$ is in $\sum_{\alpha \in \mathbb{R}} g^\alpha$ and $g_w(x_0) = x'_0$, similarly $g_w(y_0) = y'_0$. Hence $\tau(z') = g_w.\tau(z)$. \hfill \Box
Proposition 4.3. There exists uniquely defined morphism $\Phi$ from $S(\mathfrak{h} \times \mathfrak{h})^W$ to $k[\mathcal{B}_{g,n}]^G$ such that
\[ \Phi(P) \circ \iota_n = P, \quad \forall P \in S(\mathfrak{h} \times \mathfrak{h})^W. \]

Proof. Let $P$ be in $S(\mathfrak{h} \times \mathfrak{h})^W$, let $\Gamma$ be the graph of $P \circ \tau$ and let $\Gamma'$ be the image of $\Gamma$ by $\gamma_n \times \operatorname{id}_h$. We want to prove that $\Gamma'$ is a graph of an element $Q$ in $k[\mathcal{B}_{g,n}]^G$. Let $(x, t)$ and $(x, t')$ be in $\Gamma'$. Let $z$ and $z'$ be in $G \times B \times b$, such that $(z, t), (z', t') \in \Gamma$ and $\gamma_n(z) = \gamma_n(z') = x$.

By lemma 4.2, there exists $w$ in $W$ such that $\tau(z') = w \tau(z)$ hence $P \circ \tau(z') = P \circ \tau(z)$ and $t = t'$.

Then $\Gamma'$ is a graph of a function $Q$ on $\mathcal{B}_{g,n}$. Since $\Gamma$ and $\gamma_n$ are closed, $\Gamma'$ is closed too. Let $\theta$ be the projection of $\Gamma'$ on $\mathcal{B}_{g,n}$. Since $\theta$ is bijective and since $\mathcal{B}_{g,n}$ is normal then by Zariski’s main theorem $\theta$ is an isomorphism and hence $Q$ is a regular function on $\mathcal{B}_{g,n}$. Since $\gamma_n$ and $\Gamma$ are $G$-invariant, $\Gamma'$ is $G$-invariant and $Q$ is $G$-invariant, then $Q$ is in $k[\mathcal{B}_{g,n}]^G$. Hence we have a morphism of algebra $\Phi$ from $S(\mathfrak{h} \times \mathfrak{h})^W$ to $k[\mathcal{B}_{g,n}]^G$ such that
\[ \Phi(P) \circ \gamma_n = P \circ \tau. \]

Then, for $P$ in $S(\mathfrak{h} \times \mathfrak{h})^W$,
\[
\Phi(P) \circ \iota_n = \Phi(P) \circ \gamma_n \circ i = P \circ \tau \circ i = P,
\]

since $\tau \circ i = \operatorname{id}_{\mathfrak{h} \times \mathfrak{h}}$. \qed

Proposition 4.4. The morphism $\Phi$ is an isomorphism from $S(\mathfrak{h} \times \mathfrak{h})^W$ onto $k[\mathcal{B}_{g}]^G$.

Proof. There are canonical morphisms
\[
k[\mathcal{B}_{g}]^G \rightarrow k[\mathfrak{c}_{\mathfrak{g}}]^G, \quad k[\mathfrak{c}_{\mathfrak{g}}]^G \rightarrow S(\mathfrak{h} \times \mathfrak{h})^W,
k[\mathfrak{g} \times \mathfrak{g}]^G \rightarrow k[\mathcal{B}_{g}]^G \text{ and } S(\mathfrak{g} \times \mathfrak{g})^G \rightarrow k[\mathfrak{c}_{\mathfrak{g}}]^G
\]
given by restrictions. Since $G$ is a reductive group, the two last arrows are surjective. So the first one is surjective and the second one is an isomorphism by ([19], Theorem 2.9). For all $(x, y)$ in $\mathfrak{b} \times \mathfrak{b}$, the intersection of the closure of $B_x, (x, y)$ and $\mathfrak{h} \times \mathfrak{h}$ is nonempty. Indeed, according to Section 2,
\[ \lim_{t \to 0} h(t)(x, y) = (x_0, y_0). \]

So any $G$-orbit in $\mathcal{B}$ has a nonempty intersection with $\mathfrak{c}_{\mathfrak{g}}$. As a consequence, the first arrow is an isomorphism since $G,(\mathfrak{h} \times \mathfrak{h})$ is dense in $\mathfrak{c}_{\mathfrak{g}}$ by [28]. Then the compound of the two first arrows is an isomorphism whose inverse is $\Phi$. \qed
5. On the nullcone

Let $\mathfrak{N}_g$ be the nilpotent cone and let $\mathfrak{N}'_g$ be the set of its regular points. We denote by $\mathcal{N}$ the set of $(x, y)$ in $\mathfrak{B}_g$ such that $x$ and $y$ are nilpotents and $\mathcal{N}'$ the set of its smooth points. Let $\vartheta$ be the restriction of $\gamma$ to $G \times_B u \times u$. Then $\mathcal{N}$ is the image of $G \times_B u \times u$ by $\vartheta$.

**Proposition 5.1.** The morphism $\vartheta$ is a desingularization of $\mathcal{N}$. The variety $\mathcal{N}$ is closed and irreducible of dimension $3(b_g - \text{rk}_g)$.

**Proof.** Since $G \times_B u \times u$ is a vector bundle on the smooth variety $G/B$, it is a smooth variety. There exists an embedding $\epsilon$ from $G \times_B u \times u$ into $\tilde{B}$ such that $\vartheta$ is the compound of $\epsilon$ and the canonical projection of $\tilde{B}$ on $\mathfrak{g} \times \mathfrak{g}$. Whence $\vartheta$ is a proper morphism since $G/B$ is a projective variety.

The set $\mathcal{N}' \cap \mathcal{N}$ is an open subset of $\mathcal{N}$. Let $(x, y)$ be in $\mathcal{N}' \cap \mathcal{N}$. Since $x$ is regular, there exists a unique Borel subalgebra containing $x$, so $\vartheta^{-1}(x, y)$ has a unique point. Hence $\vartheta$ is birational.

Since $\mathcal{N}$ is the image of $G \times_B u \times u$ and since $G/B$ is projective variety, $\mathcal{N}$ is closed. Since $G \times_B u \times u$ is irreducible, $\mathcal{N}$ is irreducible. Since $\vartheta$ is birational,

$$\dim \mathcal{N} = \dim(G \times_B u \times u) = 3(b_g - \text{rk}_g).$$

□

**Lemma 5.1.** Let $(x, y)$ be an element in $\mathcal{N}$ and let $(v, w)$ be in $T_{(x, y)} \mathcal{N}$. Then $v + tw$ is contained in $T_{x+ty} \mathfrak{N}$, for all $t$ in $k$.

**Proof.** Let $p_i'$ be the differential of $p_i$ for $i = 1, \ldots, \text{rk}_g$. Since $\mathfrak{N} = \pi^{-1}(0)$, it suffices to show that $p_i'(x + ty)(v + tw) = 0$ for all $i = 1, \ldots, \text{rk}_g$. We have

$$p_i'(x + ty)(v + tw) = \frac{d}{da} p_i(x + ty + a(v + tw))|_{a=0}$$

$$= \frac{d}{da} p_i(x + av + t(y + aw))|_{a=0}$$

$$= \sum_{m=0}^{d_i} t^m \frac{d}{da} p_i^{(m)}(x + av, y + aw)|_{a=0}.$$ 

Since $(v, w)$ is in $T_{(x, y)} \mathcal{N}$ and since $\mathcal{N} \subset \sigma^{-1}(\{0\})$,

$$\frac{d}{da} p_i^{(m)}(x + av, y + aw)|_{a=0} = 0, \forall m = 0, \ldots, d_i - 1 \text{ and } i = 1, \ldots, \text{rk}_g.$$ 

Then $p_i'(x + ty)(v + tw) = 0, \forall i = 1, \ldots, \text{rk}_g$, whence the lemma. □
Let $t$ be in $k$ and let the morphisms
\begin{align*}
\theta' & : G \times u \longrightarrow \mathfrak{N} \\
(g, u) & \mapsto g(u) \\
\rho & : G \times u \times u \longrightarrow G \times u \\
(g, u, u') & \mapsto (g, u + tu') \\
\tau & : N \longrightarrow \mathfrak{N} \\
\tau(x, y) & \mapsto x + ty.
\end{align*}
Let $\theta'$ be the morphism from $G \times_B u$ to $\mathfrak{N}$ defined through the quotient by $\theta'$ and let $\tau'$ be the morphism from $G \times_B u \times u$ to $G \times_B u$ defined through the quotient by $\rho$. Then we have the commutative diagram:
\[
\begin{array}{ccc}
G \times_B u \times u & \xrightarrow{\theta'} & N \\
\tau' \downarrow & & \downarrow \tau \\
G \times_B u & \xrightarrow{\theta'} & \mathfrak{N}
\end{array}
\]

**Lemma 5.2.** Let $(x, y)$ be in $N$. If the intersection of $P_{x,y}$ and $g'$ is not empty, then $(x, y)$ is regular in $N$.

**Proof.** It suffices to prove the lemma for $x$ regular. Let $(x, y)$ be in $N$ such that $x$ is regular and let $(v, u)$ be in $T_{(x,y)}N$. By Lemma 5.1, $v + tw$ is in $T_{x+ty}\mathfrak{N}$ for all $t$ in $k$. Let $t \neq 0$ be in $k$, such that $x + ty$ is regular. Since $x$ is regular, there exists $(\xi, \omega_1)$ in $T_{(x,x)}G \times_B u$ such that $[\xi, x] + \omega_1 = v$ and since $x + ty$ is regular, there exists $\omega_2$ in $u$ such that $[\xi, x + ty] + \omega_2 = v + tw$. Then, for $\omega'_2 = \frac{1}{t}(\omega_2 - \omega_1)$, $[\xi, y] + \omega'_2 = w$. Hence,
\[
T_{(x,y)}N \subset \{ (v, w) \in g \times g | \exists \xi \in g \text{ and } \omega_1, \omega_2 \in u, \text{ verifying } [\xi, x] + \omega_1 = v \text{ and } [\xi, y] + \omega_2 = w \}.
\]
Let $\mu$ be the morphism from $g \times u \times u$ to $g \times g$ defined by
\[
\mu(\xi, \omega_1, \omega_2) = ([\xi, x] + \omega_1, [\xi, y] + \omega_2).
\]
Then $T_{(x,y)}N$ is contained in the image of $\mu$. The set $\mu^{-1}(\{0\})$ is equal to the set of elements $(\xi, \omega_1, \omega_2)$ such that $[\xi, x]$ and $[\xi, y]$ are in $u$. Let $(\xi, \omega_1, \omega_2)$ be in $\mu^{-1}(\{0\})$. Since $x$ is regular, $[\xi, x]$ is in $u$ if and only if $\xi$ is in $b$. Hence $\dim \mu^{-1}(\{0\}) = b_g$, whence
\[
\dim T_{(x,y)}N \leq \dim g \times u \times u - \dim \mu^{-1}(\{0\}) = 3(b_g - \text{rk}g) = \dim \mathfrak{N}.
\]
As a result, $(x, y)$ is regular in $N$.

**Corollary 5.1.** The codimension of the set of singular points of $N$ is at least four.

**Proof.** By Lemma 5.2 we have
\[
(\mathfrak{N}_g' \times \mathfrak{N}_g \cup \mathfrak{N}_g \times \mathfrak{N}_g') \cap N \subset \mathfrak{N}',
\]
Lemma 5.3. Let \( (x, y) \) be in \( N \) and let \( b \) and \( b' \) be two elements in \( B_{x,y} \). There exist a sequence of projective lines \( (L_i)_{1 \leq i \leq q} \) and a sequence \( (b_i)_{0 \leq i \leq q} \) in \( B_{x,y} \) such that \( b_0 = b \), \( b_q = b' \) and for all \( i = 1, \ldots, q \), \( b_{i-1} \) and \( b_i \) are in \( L_i \).

Proof. The proof is inspired from [17] (6.5). There exists \( g \) in \( G \) such that \( b' = g(b) \) and by Bruhat decomposition there exist \( b, b' \) in \( B \), \( w \) in \( W \) and \( n_\alpha \) in \( N_G(\mathfrak{h}) \) representing \( w \) such that \( g = b n_\alpha b' \). Since \( b' \) is in \( B \), we can assume that \( g = b n_\alpha \).

We will show the Lemma by induction on the length \( l(w) \) of \( w \).

Suppose that \( l(w) = 1 \), then \( w = s_\alpha \), for a simple root \( \alpha \). Let \( u = b^{-1}(x) \) and \( v = b^{-1}(y) \). So \( u \) and \( v \) belong to \( u \cap s_\alpha(u) \), the nilpotent radical of the minimal parabolic subalgebra \( p_\alpha \). Then \( x \) and \( y \) belong too, hence \( x \) and \( y \) are in all Borel subalgebras of \( p_\alpha \), whence

\[ L_\alpha \subset B_{x,y} \quad \text{and} \quad b, b' \in L_\alpha. \]

Suppose the Lemma true for \( l(w) \leq q - 1 \).

Let \( w = s_1 s_2 \ldots s_q \) be a reduced expression, whenever each \( s_i \) is the reflection associated to the simple root \( \alpha_i \). Let \( u = b^{-1}(x) \) and \( v = b^{-1}(y) \), so \( u \) and \( v \) are in \( u \cap w(u) \). We have \( u = \bigoplus_{\gamma \in \mathbb{R}^+} g^\gamma \), then \( w(u) = \bigoplus_{\gamma \in \mathbb{R}^+} g^{w(\gamma)} \). Hence \( w \cap w(u) = \bigoplus_{\gamma \in \mathbb{R}^+, w(\gamma) > 0} g^\gamma \).

Since \( w(\alpha_q) < 0 \), \( u \cap w(u) \subset u \cap s_{\alpha_q}(u) \), the nilpotent radical of \( p_{\alpha_q} \), then \( u \) and \( v \) are in \( u \cap s_{\alpha_q}(u) \), hence \( x \) and \( y \) are too. In particular, \( x \) and \( y \) are in \( s_{\alpha_q}(b) \).

As a result, \( b \) and \( s_{\alpha_q}(b) \) are in \( L_{\alpha_q} \) and \( s_{\alpha_q}(b) \) and \( b m_w(s_{\alpha_q}(b)) \) are in \( B_{x,y} \), with \( w' = s_1 \ldots s_{q-1} \) and \( n_w \in N_G(\mathfrak{h}) \) is a representative of \( w' \). Since \( l(w') = q - 1 \), by induction hypothesis, there exist a sequence of projective lines \( (L_i)_{1 \leq i \leq q - 1} \) and a sequence \( (b_i)_{0 \leq i \leq q - 1} \) in \( B_{x,y} \), such that

\[ b_0 = s_{\alpha_q}(b), \quad b_{q-1} = b m_w(s_{\alpha_q}(b)) \quad \text{and} \quad b_{i-1}, b_i \in L_i, \forall 1 \leq i \leq q - 1. \]

And we have

\[ b, s_{\alpha_q}(b) \in L_{\alpha_q} = L_0, \]

then the sequence of projective lines \( (L_i)_{0 \leq i \leq q - 1} \) and the sequence \( b, b_0, \ldots, b_{q-1} \) verify

\[ b_{q-1} = b', \quad b, b_0 \in L_0 \quad \text{and} \quad b_{i-1}, b_i \in L_i, \forall 1 \leq i \leq q - 1. \]

\[ \square \]
It was to be proven. \[]

**Proposition 5.2.** The set $\mathcal{B}_{x,y}$ is connected.

**Proof.** Let $b$ and $b'$ be two elements in $\mathcal{B}_{x,y}$ and let $X$ be the connected component of $\mathcal{B}_{x,y}$ containing $b$. By Lemma 5.3, there exist a sequence of projective lines $(L_i)_{1 \leq i \leq q}$ and a sequence $(b_i)_{0 \leq i \leq q}$ in $\mathcal{B}_{x,y}$ such that

$$b_0 = b, \quad b_q = b' \quad \text{and} \quad b_{i-1}, b_i \in L_i, \forall i = 1, \ldots, q.$$ 

Since $b$ and $b_1$ are in $L_1$ and since $L_1$ is connected, $X$ contains $L_1$ and $b_1$. By induction on $q$, suppose that $X$ contains $b_{q-1}$. Since $b_{q-1}$ and $b_q$ are in $L_q$ and since $L_q$ is connected, $X$ contains $L_q$ and $b_q = b'$. Then $\mathcal{B}_{x,y}$ is connected. \[]

Let $(N_n, \tau)$ be the normalization of $N$.

**Proposition 5.3.** The morphism $\tau$ is bijective.

**Proof.** Since $G \times_B u \times u$ is normal and since $\vartheta$ is dominant, there exists a unique morphism $\vartheta_n$ from $G \times_B u \times u$ to $N_n$ such that $\vartheta = \tau \circ \vartheta_n$. Then there is a commutative diagram:

$$
\begin{array}{ccc}
G \times_B u \times u & \xrightarrow{\vartheta_n} & N_n \\
\vartheta \downarrow & & \downarrow \tau \\
N & & \\
\end{array}
$$

Let $(x, y)$ be in $N$. So $\vartheta^{-1}(x, y) = \vartheta_n^{-1}(\tau^{-1}(x, y))$. Since $\vartheta$ is a birational proper morphism, $\vartheta_n$ is too. Then $\vartheta^{-1}(x, y)$ is the disjoint union of the fibres of $\vartheta_n$ at the elements of $\tau^{-1}(x, y)$. By Zariski’s main theorem $\vartheta_n^{-1}(z)$ is connected, for all $z$ in $N_n$. Since $\mathcal{B}_{x,y}$ is connected, $\vartheta^{-1}(x, y)$ is too. Whence $|\tau^{-1}(x, y)| = 1$. As a result $\tau$ is bijective since $\tau$ is surjective. \[]

### 6. Rational singularities

In this section, we deeply use the ideas of [15]. For $E$ a finite dimensional $B$-module, we denote by $\mathcal{L}(E)$ the sheaf of local sections of the fiber bundle over $G/B$ defined as the quotient of $G \times E$ under the right action of $B$ given by $(g, v), b := (gb, b^{-1}(v))$. Then $\mathcal{L}$ is a covariant exact functor from the category of finite dimensional $B$-modules to the category of locally free $O_{G/B}$-modules of finite rank. Let $\Delta$ be the diagonal of $G/B \times G/B$ and let $\mathcal{I}_\Delta$ be its ideal of definition in $O_{G/B \times G/B}$. Since $\Delta$ is isomorphic to $G/B$, $O_{G/B}$ is isomorphic to $O_{G/B \times G/B}/\mathcal{I}_\Delta$. Let $(v_i)_{i=1,\ldots,j}$ be a sequence of subalgebras of $b$ containing $u$, whenever $j$ is a positive integer and let $E = \wedge^{m_i} v_1 \otimes \cdots \otimes_k \wedge^{m_j} v_j$, whenever $(m_i)_{i=1,\ldots,j}$ is a sequence of nonnegative integers.

**Lemma 6.1.** Let $E_1, E_2$ be two $B$-modules. For an integer $i$,

$$H^i(G/B \times G/B, \mathcal{L}(E_1) \boxtimes \mathcal{L}(E_2)) = \bigoplus_{l+k=i} H^l(G/B, \mathcal{L}(E_1)) \otimes_k H^k(G/B, \mathcal{L}(E_2)).$$
Proof. Let set \( M := \mathcal{L}(E_1) \boxtimes \mathcal{L}(E_2) \). Let us consider flabby resolutions of \( \mathcal{L}(E_1) \) and \( \mathcal{L}(E_2) \),
\[
0 \to \mathcal{L}(E_1) \to M_0 \to M_1 \to \ldots \\
0 \to \mathcal{L}(E_2) \to N_0 \to N_1 \to \ldots
\]
For \( i \geq 0 \), we denoted by \( M_i \) and \( N_i \) the spaces of global sections of \( M_i \) and \( N_i \) respectively. Then the cohomologies of \( \mathcal{L}(E_1) \) and \( \mathcal{L}(E_2) \) are the cohomologies of the complexes
\[
0 \to \Gamma(G/B, \mathcal{L}(E_1)) \to M_0 \to M_1 \to \ldots \\
0 \to \Gamma(G/B, \mathcal{L}(E_2)) \to N_0 \to N_1 \to \ldots
\]
respectively. From these two complexes we deduce a double complex \( C^{\bullet, \bullet} \) whose underlying space is the direct sum of the spaces \( M_i \otimes_k N_j \) and the cohomology of the simple complex \( C^\bullet \) deduced from \( C^{\bullet, \bullet} \) is the cohomology of \( M \). The term \( E_2^{p,q} \) of the spectral sequence of \( C^{\bullet, \bullet} \) is
\[
E_2^{p,q} = H^p(G/B, \mathcal{L}(E_1)) \otimes_k H^q(G/B, \mathcal{L}(E_2)).
\]
Then
\[
H^i(C^\bullet) = \bigoplus_{l+k=i} H^l(G/B, \mathcal{L}(E_1)) \otimes_k H^k(G/B, \mathcal{L}(E_2)),
\]
whence the Lemma. \( \square \)

**Proposition 6.1.** For \( i > 0 \), \( H^{i+m_1+\ldots+m_j}(G/B, \mathcal{L}(E)) = 0 \).

**Proof.** Let show that
\[
H^{i+m_1+\ldots+m_j}(G/B, \mathcal{L}(E)) = 0, \text{ for } i > 0 \text{ (**)}
\]
by induction on \( (n+1-i, j) \), whenever \( n = \dim G/B \).

For \( n+1-i = 0 \), by [14] (ch.III, Theorem 2.7) \( H^{i+m_1+\ldots+m_j}(G/B, \mathcal{L}(E)) = 0 \) and for \( j = 1 \), by [15] (Theorem B) (***) holds. Suppose that (**) holds for pairs smaller than \( (n+1-i, j) \), show it for \( (n+1-i, j) \).

Let \( E_1 = \wedge^{m_1} v_1 \otimes \ldots \otimes \wedge^{m_{i-1}} v_{i-1} \) and let \( E' = \wedge^{m_j} v_j \). Then \( \mathcal{L}(E_1) \boxtimes \mathcal{L}(E') \) is the sheaf of local sections of the fiber bundle over \( G/B \times G/B \) defined as the quotient of \( G \times G \times E_1 \times E' \) under the right action of \( B \times B \) given by \( (g, g', v, v') \cdot (b, b') = (gb, g'b', b^{-1}(v), b'^{-1}(v')) \). There exists an exact sequence
\[
0 \to J_\Delta \to \mathcal{O}_{G/B \times G/B} \to \mathcal{O}_{G/B} \to 0,
\]
whence the following exact sequence
\[
0 \to J_\Delta \otimes \mathcal{O}_{G/B \times G/B} \mathcal{L}(E_1) \boxtimes \mathcal{L}(E') \to \mathcal{L}(E_1) \boxtimes \mathcal{L}(E') \to \mathcal{L}(E_1) \otimes \mathcal{L}(E') \to 0.
\]
Hence we will have the long exact sequence
\[
\cdots \to H^i(G/B \times G/B, J_\Delta \otimes \mathcal{L}(E_1) \boxtimes \mathcal{L}(E')) \to H^i(G/B \times G/B, \mathcal{L}(E_1) \boxtimes \mathcal{L}(E')) \to \mathcal{L}(E_1) \otimes \mathcal{L}(E') \to \mathcal{L}(E_1) \otimes \mathcal{L}(E') \to \cdots.
\]
By Lemma 6.1,\n\[
H^{i+m_1+\ldots+m_j}(G/B \times G/B, \mathcal{L}(E_1) \boxtimes \mathcal{L}(E')) = \bigoplus_{i+k=i} H^i(G/B, \mathcal{L}(E_1)) \otimes_k H^k(G/B, \mathcal{L}(E'))
\]
then, let \( l = l' + m_1 + \ldots + m_{j-1} \),
• if \( k > m_j \), by [15] (Theorem B)

\[
H^k(G/B, \mathcal{L}(E')) = 0,
\]

• and if \( k \leq m_j \), then

\[
l \geq i + m_1 + \ldots + m_{j-1} \quad \Rightarrow \quad l' \geq i
\]

\[
\Rightarrow \quad n + 1 - l' \leq n + 1 - i
\]

\[
\Rightarrow \quad (n + 1 - l', j - 1) < (n + 1 - i, j).
\]

Hence, by induction hypothesis

\[
H^i(G/B, \mathcal{L}(E_1)) = 0.
\]

Whence

\[
H^{i+m_1+\ldots+m_j}(G/B \times G/B, \mathcal{L}(E_1) \boxtimes \mathcal{L}(E')) = 0, \text{ for } i > 0,
\]

then it remains to show that

\[
H^{i+1+m_1+\ldots+m_j}(G/B \times G/B, \mathfrak{g}_{\Delta} \otimes \mathcal{L}(E_1) \boxtimes \mathcal{L}(E')) = 0, \text{ for } i > 0
\]

Let identify \( G/B \) with \( \Delta \). The sequence \( \mathfrak{g}_{\Delta}^k, k \in \mathbb{N}^* \), is a descending filtration of \( \mathfrak{g}_{\Delta} \).

Let denote by \( \mathcal{G}_{\Delta} \) the associated graded sheaf to this module. Then \( \mathcal{G}_{\Delta} \) is a sheaf of graded algebra over \( G/B \).

Moreover, the sequence \( \mathfrak{g}_{\Delta}^k \otimes \mathcal{O}_{G/B} \otimes \mathcal{L}(E_1) \boxtimes \mathcal{L}(E') \), \( k \in \mathbb{N}^* \), is a descending filtration of \( \mathfrak{g}_{\Delta} \otimes \mathcal{O}_{G/B} \otimes \mathcal{L}(E_1) \boxtimes \mathcal{L}(E') \) and the associated graded sheaf is isomorphic to \( \mathcal{G}_{\Delta} \otimes \mathcal{O}_{G/B} \otimes \mathcal{L}(E_1) \boxtimes \mathcal{L}(E') \) since \( \mathcal{L}(E_1) \boxtimes \mathcal{L}(E') \) is locally free.

According to ([14], Ch. II, Theorem 8.17), since \( \Delta \) is a smooth subvariety of \( G/B \times G/B \), the following short sequence

\[
0 \to \mathfrak{g}_{\Delta} / \mathfrak{g}_{\Delta}^2 \to \Omega^1_{G/B \times G/B} \otimes \mathcal{O}_{G/B} \otimes \mathcal{O}_{G/B} \to \Omega^1_{G/B} \to 0
\]

is exact. Since \( \Omega^1_{G/B} \) is canonically isomorphic to \( \mathcal{L}(u^*) \), \( \mathfrak{g}_{\Delta} / \mathfrak{g}_{\Delta}^2 \) is canonically isomorphic to \( \mathcal{L}(u^*) \). Then, according to ([14], Ch. II, Theorem 8.21A), \( \mathcal{G}_{\Delta} \) is isomorphic to \( \mathcal{L}(\mathcal{S}(u^*)) \) since \( \Delta \) is locally a complete intersection in \( G/B \times G/B \).

As a result, the associated graded module to the filtration on \( \mathfrak{g}_{\Delta} \otimes \mathcal{O}_{G/B} \otimes \mathcal{L}(E_1) \boxtimes \mathcal{L}(E') \) is isomorphic to the \( \mathcal{O}_{G/B} \)-module

\[
\mathcal{L}(\mathcal{S}(u^*)) \otimes \mathcal{O}_{G/B} \mathcal{L}(E_1) \otimes \mathcal{O}_{G/B} \mathcal{L}(E')
\]

Hence it suffice to prove that

\[
H^{i+1+m_1+\ldots+m_j}(G/B, \mathcal{L}(\mathcal{S}(u^*)) \otimes \mathcal{O}_{G/B} \mathcal{L}(E_1) \otimes \mathcal{O}_{G/B} \mathcal{L}(E')) = 0, \text{ for } i > 0.
\]

According to the identification of \( \mathfrak{g} \) with its dual by the killing form, one has a short exact sequence

\[
0 \to \mathfrak{b} \to \mathfrak{g}^* \to u^* \to 0.
\]

From this exact sequence, on deduces the exact Koszul complex

\[
\cdots \to K_2 \xrightarrow{d} K_1 \xrightarrow{d} K_0 \to S(u^*) \to 0,
\]

with

\[
K_p := S(\mathfrak{g}^*) \otimes_k \Lambda^p(\mathfrak{b}),
\]
for all nonnegative integers \( q \)

\[
M_{\square} \quad \text{since } H^* \quad \text{let } \psi
\]

\[
\text{Then for all } p \quad \text{are exact with } \quad \text{are exact}
\]

\[
\text{So that the sequences } \quad \text{is exact}
\]

\[
\text{are exact with } \quad \text{for all } p. \quad \text{Then, since } \mathcal{L} \quad \text{is exact}
\]

\[
\text{for all nonnegative integers } q \quad \text{for any nonnegative integer } i
\]

\[
d(a \otimes (a_0 \land \ldots \land a_p)) := \sum_{i=0}^{p} (-1)^i a_i a \otimes (a_0 \land \ldots \land \hat{a}_i \land \ldots \land a_p).
\]

This complex \( K_* \) is canonically graded by

\[
K_* := \sum_{q} K_*^q, \quad \text{where } K_*^q := S^{q-p}(g^*) \otimes_k \Lambda^p(b).
\]

So that the sequences

\[
\cdots \rightarrow K_*^q \rightarrow K_*^q \rightarrow K_*^0 \rightarrow S^q(u^*) \rightarrow 0
\]

\[
\cdots \rightarrow K_*^q \rightarrow K_*^q \rightarrow K_*^0 \rightarrow S(u^*) \otimes_k E_1 \otimes_k E' \rightarrow 0
\]

are exact with

\[
\tilde{K}_*^q := K_*^q \otimes_k E_1 \otimes_k E',
\]

for all \( p \). Then, since \( \mathcal{L} \) is an exact functor, one deduces the exact sequence of \( \mathcal{O}_{G/B} \)-modules

\[
\cdots \rightarrow \mathcal{L}(\tilde{K}_*^q) \rightarrow \mathcal{L}(\tilde{K}_*^0) \rightarrow \mathcal{L}(\tilde{K}_*^q) \rightarrow \mathcal{L}(S(u^*) \otimes_k E_1 \otimes_k E') \rightarrow 0.
\]

Since \( H^* \) is an exact \( \delta \)-functor, for \( i > m_1 + \ldots + m_j \),

\[
H^{i+1}(G/B, \mathcal{L}(S(u^*)) \otimes_k \mathcal{L}(E_1) \otimes_k \mathcal{L}(E')) = 0, \quad \text{if } H^{i+1+p}(G/B, \mathcal{L}(\tilde{K}_*^q)) = 0,
\]

for all nonnegative integers \( q \) and \( p \), but, by induction hypothesis,

\[
H^{i+1+p}(G/B, \mathcal{L}(\tilde{K}_*^q)) = 0, \quad \text{for } i > m_1 + \ldots + m_j,
\]

whence the Proposition. \( \square \)

Let \( v \) be a subalgebra of \( b \) containing \( u \). For any \( B \)-module \( E \), we denoted by \( \mathcal{M}(E) \) the quotient of \( G \times v \times v \times E \) under the right action of \( B \) given by

\[
(g, x, y, v).b := (gb, b^{-1}(x), b^{-1}(y), b^{-1}(v)).
\]

Then \( \mathcal{M}(E) \) is a fiber bundle over \( G \times_B (v \times v) \).

**Proposition 6.2.** Let \( F \) be the trivial \( B \)-module of dimension 1. Then

\[
H^i(G \times_B (v \times v), \mathcal{M}(F)) = 0, \quad \forall i > 0.
\]

**Proof.** We denote by \( g^* \) and \( v^* \) the duals of \( g \) and \( v \) respectively, and we denote by \( \psi \) the canonical morphism from \( G \times_B (v \times v) \) to \( G/B \). Then \( \mathcal{M}(F) \) is equal to \( \psi^*(\mathcal{L}(F)) \). As it is easy to see that \( \psi^* (\mathcal{O}_{G \times_B (v \times v)}) \) is equal to \( \mathcal{L}(v^* \times v^*) \). So we have the equality

\[
\psi^* \psi^*(\mathcal{L}(F)) = \psi^* (\mathcal{O}_{G \times_B (v \times v)}) \otimes_{\mathcal{O}_{G/B}} \mathcal{L}(F) = \sum_{q \in \mathbb{N}} \mathcal{L}(S^q(v^* \times v^*)).
\]

From this equality, we deduce the equality

\[
H^i(G \times_B (v \times v), \mathcal{M}(F)) = \sum_{q \in \mathbb{N}} H^i(G/B, \mathcal{L}(S^q(v^* \times v^*))), \quad (*)
\]

for any nonnegative integer \( i \).

Let \( E \) be a \( B \)-module which is a direct sum of finite dimensional \( B \)-modules and let \( M_E \) be the \( S(g^*) \)-module

\[
M_E := S(g^*) \otimes_k E.
\]
Since $g$ and $g^*$ are identified by $(.,.)$, the kernel of the canonical projection from $g^*$ to $v^*$ is equal to $\mathfrak{t}$, whenever $\mathfrak{t}$ is the orthogonal complement of $v$ in $g$. Then the Koszul complex,

$$\ldots \rightarrow K_2(E) \rightarrow K_1(E) \rightarrow K_0(E) \rightarrow S(v^*) \otimes_k E \rightarrow 0$$

defined by:

$$K_n(E) := M_{E} \otimes_k \Lambda^n(\mathfrak{t}) \ , \ d : K_{n+1}(E) \rightarrow K_n(E)$$

$$d(m \otimes (a_0 \wedge \ldots \wedge a_n)) = \sum_{i=0}^{n} (-1)^i a_i m \otimes (a_0 \wedge \ldots \wedge \hat{a}_i \wedge \ldots \wedge a_n),$$

is exact. This complex $K_\bullet(E)$ is canonically graded by

$$K_\bullet(E) := \sum_q K^q_\bullet(E), \text{ where } K^q_n(E) := S^{q-n}(g^*) \otimes_k E \otimes_k \Lambda^n(\mathfrak{t}).$$

Thus we have the long exact sequence of $B$-modules

$$\ldots \rightarrow K^q_2(E) \rightarrow K^q_1(E) \rightarrow K^q_0(E) \rightarrow S^q(v^*) \otimes_k E \rightarrow 0.$$
Proof. Since $N_n$ and $B_{g,n}$ are normal, 
\[(\varphi_n)_*(\mathcal{O}_{G \times B(u \times u)}) = \mathcal{O}_{N_n} \quad \text{and} \quad (\gamma_n)_*(\mathcal{O}_{G \times B(b \times b)}) = \mathcal{O}_{B_{g,n}}.\]

By Proposition 6.2, 
\[H^i(G \times B(u \times u), \mathcal{O}_{G \times B(u \times u)}) = H^i(G \times B(u \times u), \mathcal{M}(E(0))) = 0, \quad \text{for} \quad i > 0\]
since $\mathcal{O}_{G \times B(u \times u)} = \psi^*(\mathcal{O}_{G/B}) = \psi^*(\mathcal{L}(E(0)))$, and 
\[H^i(G \times B(b \times b), \mathcal{O}_{G \times B(b \times b)}) = H^i(G \times B(b \times b), \mathcal{M}(E(0))) = 0, \quad \text{for} \quad i > 0\]
since $\mathcal{O}_{G \times B(b \times b)} = \psi^*(\mathcal{O}_{G/B}) = \psi^*(\mathcal{L}(E(0))).$ \(\Box\)

Since $\varphi_n$ and $\gamma_n$ are desingularisation morphisms of $N_n$ and $B_{g,n}$ respectively, the corollary is a consequence of the theorem.

**Corollary 6.1.** The normalizations $N_n$ and $B_{g,n}$ of $N$ and $B_g$ respectively have rational singularities.

**Appendix A**

Recall that $g$ is simple and $R, R_+, \Pi$ are the root data. We denote by $\beta_1, \ldots, \beta_{rk_g}$ the elements of $\Pi$ ordered as in [4] and by $\rho$ the half sum of positive roots. In particular, if $g$ has not type $D_{rk_g}$ or $E$, $\beta_i$ is not orthogonal to $\beta_{i+1}$ for $i = 1, \ldots, rk_g - 1$. When $g$ has type $D_{rk_g}$, $\beta_{rk_g - 1}$ is orthogonal to $\beta_{rk_g}$. When $g$ has type $E$, $\beta_2$ is not orthogonal to $\beta_4$ and $\beta_3$ is not orthogonal to $\beta_1$ and $\beta_4$. For $\beta$ in $\Pi$, the reflexion associated to $\beta$ is denoted by $s_\beta$.

**Lemma A.1.** We suppose $g$ simple. Let $\alpha$ be in $R_+$.

(i) If $\alpha$ is simple, then $s_\alpha(\rho - \alpha)$ is regular dominant.

(ii) If $\alpha$ is not simple, then $\rho - \alpha$ is not regular.

(iii) If $\alpha$ is the biggest root then $\rho + \alpha$ is regular dominant.

(iv) If $\alpha$ is not the biggest root then $\rho + \alpha$ is not regular when $g$ has type $A_{rk_g}$, $E_6, E_7, E_8$.

(v) If $\alpha$ is not the biggest root:
   (a) for $g$ of type $B_{rk_g}, F_4, G_2$, $\rho + \alpha$ is regular for two values of $\alpha$ and when it is not dominant there exists a simple root $\beta$ such that $s_\beta(\rho + \alpha)$ is dominant,
   (b) for $g$ of type $C_{rk_g}, D_{rk_g}$, $\rho + \alpha$ is regular for one value of $\alpha$ and for this value it is dominant.

Proof. (i) Let $\alpha$ be a simple root. Then $s_\alpha(\rho) = \rho - \alpha$. So $s_\alpha(\rho - \alpha)$ is regular dominant since $\rho$ is regular dominant.

(iii) Let $\alpha$ be the biggest root. Then for any positive root $\gamma$, $\langle \alpha, \gamma \rangle$ is nonnegative. Consequently, $\rho + \alpha$ is regular and dominant since $\langle \rho, \beta \rangle$ is positive for any $\beta$ in $\Pi$. 

(ii) If \( \alpha \) is not simple, then \( \rho - \alpha \) is not regular.

If \( \beta_{i_1}, \ldots, \beta_{i_l} \) are pairwise different elements of \( \Pi \) such that \( \beta_{i_j} \) is not orthogonal to \( \beta_{i_{j+1}} \) for \( j = 1, \ldots, l-1 \) and so that

\[
s_{\beta_{i_l}}(\beta_{i_{l-1}}) = \beta_{i_{l-1}} + \beta_{i_l},
\]

then

\[
s_{\beta_{i_l}}(\rho - (\beta_{i_1} + \cdots + \beta_{i_l})) = \rho - (\beta_{i_1} + \cdots + \beta_{i_l}),
\]

since \( s_{\beta_{i_l}}(\rho) = \rho - \beta_{i_l} \). Hence \( \rho - \alpha \) is not regular if \( \alpha \) is the sum of pairwise different simple roots since we can order them so that the last condition is satisfied.

We denote by \( \mathcal{R}'_+ \) the subset of positive roots whose coordinates in the basis \( \Pi \) are not at most to 1. Then it remains to prove that \( \rho - \alpha \) is not regular for any \( \alpha \) in \( \mathcal{R}'_+ \). For that purpose, it will be enough to find \( \beta \) in \( \Pi \) such that \( s_\beta(\rho - \alpha) = \rho - \alpha \).

We then consider the different possible cases.

1) \( \mathfrak{g} \) has type \( B_{rkg} \) and

\[
\alpha = \beta_i + \cdots + \beta_k + 2(\beta_{k+1} + \cdots + \beta_{rkg}),
\]

for \( 1 \leq i \leq k < rk \). If \( i < k \), then

\[
s_{\beta_i}(\rho - \alpha) = \rho - \alpha.
\]

If \( i = k < rk - 1 \) then

\[
s_{\beta_{k+1}}(\rho - \alpha) = \rho - \beta_{k+1} - (\beta_i + \beta_{k+1}) + 2\beta_{k+1} - 2(\beta_{i+1} + \beta_{i+2})
\]

\[-(\alpha - \beta_i - 2\beta_{i+1} - 2\beta_{i+2}) = \rho - \alpha.
\]

If \( \alpha = \beta_{rk-1} + 2\beta_{rk} \), then

\[
s_{\beta_{rk}}(\rho - \alpha) = \rho - \beta_{rk} - (\beta_{rk-1} + 2\beta_{rk}) + 2\beta_{rk}
\]

\[-\beta_{rk-1} + \beta_{rk}.
\]

In this case \( \rho - \alpha \) is not regular since \( \rho - \beta_{rk-1} - \beta_{rk} \) is not regular by which goes before.

2) \( \mathfrak{g} \) has type \( C_{rk} \). Let us suppose

\[
\alpha = \beta_i + \cdots + \beta_k + 2(\beta_{k+1} + \cdots + \beta_{rk-1}) + \beta_{rk},
\]

for \( 1 \leq i \leq k < rk \). If \( i < k \), then

\[
s_{\beta_i}(\rho - \alpha) = \rho - \alpha.
\]

If \( i = k \) then

\[
s_{\beta_{k+1}}(\rho - \alpha) = \rho - \beta_{k+1} - (\beta_i + \beta_{k+1}) + 2\beta_{k+1} - 2(\beta_{i+1} + \beta_{i+2})
\]

\[-(\alpha - \beta_i - 2\beta_{i+1} - 2\beta_{i+2}) = \rho - \alpha.
\]

Let us suppose

\[
\alpha = 2(\beta_k + \cdots + \beta_{rk-1}) + \beta_{rk},
\]

for \( 1 \leq k < rk \). If \( k < rk - 1 \), then

\[
s_{\beta_k}(\rho - \alpha) = \rho - \beta_k - 2(\beta_{k+1} + \cdots + \beta_{rk-1}) - \beta_{rk}.
\]
Hence $\rho - \alpha$ is not regular since the right hand side of the last equality is not regular. If $k = \text{rk} \mathfrak{g} - 1$, then

$$s_{\beta_{\text{rk} \mathfrak{g} - 1}}(\rho - \alpha) = \rho - \beta_{\text{rk} \mathfrak{g} - 1} - \beta_{\text{rk} \mathfrak{g}}.$$  

Hence $\rho - l\alpha$ is not regular since $\rho - \beta_{\text{rk} \mathfrak{g} - 1} - \beta_{\text{rk} \mathfrak{g}}$ is not regular.

3) $\mathfrak{g}$ has type $D_{\text{rk} \mathfrak{g}}$. Let us suppose

$$\alpha = \beta_i + \ldots + \beta_k + 2(\beta_{k+1} + \ldots + \beta_{\text{rk} \mathfrak{g} - 2}) + \beta_{\text{rk} \mathfrak{g}} + \beta_{\text{rk} \mathfrak{g} - 1},$$

for $1 \leq i \leq k < \text{rk} \mathfrak{g} - 2$. If $i < k$, then

$$s_{\beta_i}(\rho - \alpha) = \rho - \alpha.$$  

If $i = k < \text{rk} \mathfrak{g} - 3$ then

$$s_{\beta_{i+1}}(\rho - \alpha) = \rho - \beta_{i+1} - (\beta_i + \beta_{i+1}) + 2(\beta_{i+1} + \beta_{i+2}) - (\alpha - \beta_i - 2\beta_{i+1} - 2\beta_{i+2}) = \rho - \alpha.$$  

If $i = k = \text{rk} \mathfrak{g} - 3$, then

$$s_{\beta_{\text{rk} \mathfrak{g} - 2}}(\rho - \alpha) = \rho - \beta_{\text{rk} \mathfrak{g} - 2} - (\beta_{\text{rk} \mathfrak{g} - 3} + \beta_{\text{rk} \mathfrak{g} - 2}) + 2\beta_{\text{rk} \mathfrak{g} - 2} - (\beta_{\text{rk} \mathfrak{g} - 1} + \beta_{\text{rk} \mathfrak{g} - 2}) = \rho - \alpha.$$  

So in any case, $\rho - \alpha$ is not regular.

4) $\mathfrak{g}$ has type $E$. For $i = 1, \ldots, \text{rk} \mathfrak{g}$, we denote by $J_i$ the subset of $j$ in $\{1, \ldots, \text{rk} \mathfrak{g}\} \setminus \{i\}$ such that $\beta_j$ is not orthogonal to $\beta_i$. Then $|J_i| \leq 2$ if $i \neq 4$ and $|J_4| = 3$. For $i = 1, \ldots, \text{rk} \mathfrak{g}$, we denote by $n_i$ the coordinate of $\alpha$ at $\beta_i$ in the basis $\Pi$ and we set

$$L(\alpha) := [n_2, n_1, n_3, n_4, \ldots, n_{\text{rk} \mathfrak{g}}].$$

If there exists some $i$ such that

$$2n_i - 1 = \sum_{j \in J_i} n_j, \quad (4)$$

then

$$s_{\beta_i}(\rho - \alpha) = \rho - \beta_i + n_i\beta_i - s_{\beta_i}(\sum_{j \in J_i} n_j\beta_j) - \sum_{j \in \cup_{i} J_i} n_j\beta_j = \rho + (n_i - 1)\beta_i - (\sum_{j \in J_i} n_j)\beta_i - \alpha + n_i\beta_i = \rho - \alpha,$$

since all the roots have the same length. Consequently, if $\alpha$ satisfies the equality $(4)$ for some $i$, then $\rho - \alpha$ is not regular. For each case, we will give some $i$ for which $\alpha$ satisfies equality $(4)$. The result will be presented in a table. Its first column gives some values of $i$ and the numbers $j$ which are on the same line in the second column are such that the root $r(j)$ satisfies equality $(4)$ with respect to $i$. The roots are given by the map $\alpha \mapsto L(\alpha)$.

a) We suppose $\mathfrak{g}$ of type $E_6$. The image of $\mathcal{R}^+_\mathfrak{g}$ by the map $\alpha \mapsto L(\alpha)$ is given by

\begin{align*}
  & r(1):=[1,0,1,2,1,0] & r(2):=[1,1,1,2,1,0] & r(3):=[1,0,1,2,1,1] \\
  & r(4):=[1,1,2,2,1,0] & r(5):=[1,1,1,2,1,1] & r(6):=[1,0,1,2,2,1] \\
  & r(7):=[1,1,2,2,1,1] & r(8):=[1,1,1,2,2,1] & r(9):=[1,1,2,2,2,1] \\
  & r(10):=[1,1,2,3,2,1] & r(11):=[2,1,2,3,2,1].
\end{align*}
The corresponding table is given by

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| 2 | 11 |   |   |   |   |   |
| 3 | 4  |   |   |   |   |   |
| 4 | 1,2,10 |   |   |   |   |   |
| 5 | 6,8,9 |   |   |   |   |   |
| 6 | 3,5,7 |   |   |   |   |   |

b) We suppose that \( g \) has type \( E_7 \). Let \( \mathcal{R}'_+ \) be the subset of elements of \( \mathcal{R}'_+ \) such that \( n_7 \neq 0 \). The image of \( \mathcal{R}'_+ \) by the map \( \alpha \mapsto L(\alpha) \) is given by

\[
\begin{align*}
\mathbf{r}(1):= & [1,0,1,2,1,1,1]  & \mathbf{r}(2):= & [1,1,1,2,1,1,1]  & \mathbf{r}(3):= & [1,0,1,2,2,1,1] \\
\mathbf{r}(4):= & [1,1,2,2,1,1,1]  & \mathbf{r}(5):= & [1,1,1,2,2,1,1]  & \mathbf{r}(6):= & [1,0,1,2,2,2,1] \\
\mathbf{r}(7):= & [1,1,2,2,2,1,1]  & \mathbf{r}(8):= & [1,1,1,2,2,2,1]  & \mathbf{r}(9):= & [1,1,2,2,2,2,1] \\
\mathbf{r}(10):= & [1,1,2,3,2,1,1] & \mathbf{r}(11):= & [1,1,2,3,2,2,1]  & \mathbf{r}(12):= & [2,1,2,3,2,1,1] \\
\mathbf{r}(13):= & [1,1,2,3,3,2,1] & \mathbf{r}(14):= & [2,1,2,3,2,2,1]  & \mathbf{r}(15):= & [2,1,2,3,3,2,1] \\
\mathbf{r}(16):= & [2,1,2,4,3,2,1] & \mathbf{r}(17):= & [2,1,3,4,3,2,1]  & \mathbf{r}(18):= & [2,2,3,4,3,2,1].
\end{align*}
\]

The corresponding table is given by

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| 1 | 18 |   |   |   |   |   |   |
| 3 | 17 |   |   |   |   |   |   |
| 4 | 16 |   |   |   |   |   |   |
| 5 | 13,15 |   |   |   |   |   |   |
| 6 | 6,8,9,11,14 |   |   |   |   |   |   |
| 7 | 1,2,3,4,5,7,10,12 |   |   |   |   |   |   |

c) We suppose that \( g \) has type \( E_8 \). Let \( \mathcal{R}'_+ \) be the subset of elements of \( \mathcal{R}'_+ \) such that \( n_8 \neq 0 \). The image of \( \mathcal{R}'_+ \) by the map \( \alpha \mapsto L(\alpha) \) is given by

\[
\begin{align*}
\mathbf{r}(1):= & [1,0,1,2,1,1,1,1]  & \mathbf{r}(2):= & [1,0,1,2,2,1,1,1]  & \mathbf{r}(3):= & [1,1,1,2,1,1,1,1] \\
\mathbf{r}(4):= & [1,0,1,2,2,2,1,1]  & \mathbf{r}(5):= & [1,1,2,2,1,1,1,1]  & \mathbf{r}(6):= & [1,1,1,2,2,1,1,1] \\
\mathbf{r}(7):= & [1,1,2,2,2,1,1,1]  & \mathbf{r}(8):= & [1,1,1,2,2,2,1,1]  & \mathbf{r}(9):= & [1,0,1,2,2,2,1,1] \\
\mathbf{r}(10):= & [1,1,2,3,2,1,1,1] & \mathbf{r}(11):= & [1,1,2,2,2,2,1,1]  & \mathbf{r}(12):= & [1,1,2,2,2,2,2,1] \\
\mathbf{r}(13):= & [2,1,2,3,2,1,1,1] & \mathbf{r}(14):= & [1,1,2,3,2,2,1,1]  & \mathbf{r}(15):= & [1,1,2,2,2,2,2,1] \\
\mathbf{r}(16):= & [2,1,2,3,2,2,1,1] & \mathbf{r}(17):= & [1,1,2,3,3,2,1,1]  & \mathbf{r}(18):= & [1,1,2,3,2,2,2,1] \\
\mathbf{r}(19):= & [2,1,2,3,3,2,1,1] & \mathbf{r}(20):= & [2,1,2,3,2,2,2,1]  & \mathbf{r}(21):= & [1,1,2,3,3,2,2,1] \\
\mathbf{r}(22):= & [2,1,2,4,3,2,1,1] & \mathbf{r}(23):= & [2,1,2,3,3,2,2,1]  & \mathbf{r}(24):= & [1,1,2,3,3,3,2,1] \\
\mathbf{r}(25):= & [2,1,3,4,3,2,1,1] & \mathbf{r}(26):= & [2,1,2,4,3,2,2,1]  & \mathbf{r}(27):= & [2,1,2,3,3,3,2,1] \\
\mathbf{r}(28):= & [2,2,3,4,3,2,1,1] & \mathbf{r}(29):= & [2,1,3,4,3,2,2,1]  & \mathbf{r}(30):= & [2,1,2,4,3,3,2,1] \\
\mathbf{r}(31):= & [2,2,3,4,3,2,2,1] & \mathbf{r}(32):= & [2,1,3,4,3,3,2,1]  & \mathbf{r}(33):= & [2,1,2,4,4,3,2,1] \\
\mathbf{r}(34):= & [2,2,3,4,3,3,2,1] & \mathbf{r}(35):= & [2,1,3,4,4,3,2,1]  & \mathbf{r}(36):= & [2,1,3,5,4,3,2,1] \\
\mathbf{r}(37):= & [2,2,3,4,3,3,2,1] & \mathbf{r}(38):= & [3,1,3,5,4,3,2,1]  & \mathbf{r}(39):= & [2,2,3,5,4,3,2,1] \\
\mathbf{r}(40):= & [3,2,3,5,4,3,2,1] & \mathbf{r}(41):= & [2,2,4,5,4,3,2,1]  & \mathbf{r}(42):= & [3,2,4,5,4,3,2,1] \\
\mathbf{r}(43):= & [3,2,4,6,5,3,2,1] & \mathbf{r}(44):= & [3,2,4,6,5,3,2,1]  & \mathbf{r}(45):= & [3,2,4,6,5,4,2,1] \\
\mathbf{r}(46):= & [3,2,4,6,5,3,3,1] & \mathbf{r}(47):= & [3,2,4,6,5,4,3,2].
\end{align*}
\]
The corresponding table is given by

|   |   |   |   |
|---|---|---|---|
| 1 | 2 | 38,40 |
| 2 | 3 | 41,42 |
| 3 | 4 | 36,39,43 |
| 4 | 5 | 33,35,37,44 |
| 5 | 6 | 24,27,30,32,34,45 |
| 6 | 7 | 9,12,15,18,20,21,23,26,29,31,46 |
| 7 | 8 | 1.8,10,11,13,14,16,17,19,22,25,28,47 |

5) \( g \) has type F. For \( i = 1, \ldots, 4 \), we denote by \( n_i \) the coordinate of \( \alpha \) at \( \beta_i \) in the basis \( \Pi \) and we set

\[
L(\alpha) := [n_1, n_2, n_3, n_4].
\]

We identify \( \alpha \) with \( L(\alpha) \). If \( 2n_2 + n_4 = 2n_3 - 1 \), then

\[
s_{\beta_3}(\rho - \alpha) = \rho - \beta_3 - n_1\beta_1 - n_2\beta_2 - n_4\beta_4 - (2n_2 + n_4 - n_3)\beta_3
= \rho - \alpha.
\]

If \( n_1 + n_3 = 2n_2 - 1 \), then

\[
s_{\beta_2}(\rho - \alpha) = \rho - \beta_2 - n_1\beta_1 - n_3\beta_3 - n_4\beta_4 - (n_1 + n_3 - n_2)\beta_2
= \rho - \alpha.
\]

If \( n_2 = 2n_1 - 1 \), then

\[
s_{\beta_1}(\rho - \alpha) = \rho - \beta_1 - n_2\beta_2 - n_3\beta_3 - n_4\beta_4 - (-n_1 + n_2)\beta_1
= \rho - \alpha.
\]

Analogously, \( s_{\beta_4}(\rho - \alpha) = \rho - \alpha \) if \( n_3 = 2n_4 - 1 \). The image of \( \mathcal{R}_+ \) by the map \( \alpha \mapsto L(\alpha) \) is given by

\[
r(1) := [0,1,2,0] \quad r(2) := [1,1,2,0] \quad r(3) := [0,1,2,1] \quad r(4) := [1,2,2,0] \quad r(5) := [1,1,2,1] \quad r(6) := [0,1,2,2] \quad r(7) := [1,2,2,1] \quad r(8) := [1,1,2,2] \quad r(9) := [1,2,3,1] \quad r(10) := [1,2,2,2] \quad r(11) := [1,2,3,2] \quad r(12) := [1,2,4,2] \quad r(13) := [1,3,4,2] \quad r(14) := [2,3,4,2].
\]

We then deduce the following table

|   |   |
|---|---|
| 1 | 2,5,8,14 |
| 2 | 4,7,10,13 |
| 3 | 3,5,9 |
| 4 | 11 |

From the equalities:

\[
s_{\beta_3}(\rho - r(1)) = \rho - \beta_3 - \beta_2
s_{\beta_4}(\rho - r(6)) = \rho - \beta_4 - \beta_2 - 2\beta_3
= \rho - r(3)
\]
\[
s_{\beta_3}(\rho - r(12)) = \rho - \beta_3 - \beta_1 - 2(\beta_2 + 2\beta_3) + 4\beta_3 - 2\beta_3 - 2\beta_4
= \rho - r(11).
\]

So for \( i = 1, 6, 7 \), \( \rho - r(i) \) is not regular by which goes before.
6) \( g \) has type \( G_2 \). From the equalities:
\[
s_{\beta_1}(\rho - (2\beta_1 + \beta_2)) = \rho - \beta_1 + 2\beta_1 - (\beta_2 + 3\beta_1) = \rho - (2\beta_2 + \beta_1) \\
s_{\beta_2}(\rho - (3\beta_1 + 2\beta_2)) = \rho - \beta_2 + 2\beta_2 - 3(\beta_2 + \beta_1) = \rho - (3\beta_1 + 2\beta_2) \\
s_{\beta_1}(\rho - (3\beta_1 + \beta_2)) = \rho - \beta_1 + 3\beta_1 - (\beta_2 + 3\beta_1) = \rho - (\beta_2 + \beta_1).
\]
So for any \( \alpha \in \mathcal{R}_+ \), \( \rho - \alpha \) is not regular.

(iv) If \( \alpha \) is not the biggest root then \( \rho + \alpha \) is not regular when \( g \) has type \( \text{A}_{\text{rk}g} \), \( \text{E}_6 \), \( \text{E}_7 \), \( \text{E}_8 \).

Let us suppose that \( \alpha \) is not the biggest root and that \( g \) has type \( \text{A}_{\text{rk}g} \), \( \text{E}_6 \), \( \text{E}_7 \), \( \text{E}_8 \). If \( \beta_{i_1}, \ldots, \beta_{i_l} \) are pairwise different elements of \( \Pi \) such that \( \beta_{i_j} \) is not orthogonal to \( \beta_{i_{j+1}} \) for \( j = 1, \ldots, l - 1 \), then
\[
s_{\beta_{i_1}}(\rho + \beta_{i_1} + \cdots + \beta_{i_{l-1}}) = \rho + \beta_{i_1} + \cdots + \beta_{i_{l-1}},
\]
since \( s_{\beta_{i_1}}(\rho) = \rho - \beta_{i_1} \). Hence \( \rho + \alpha \) is not regular if \( \alpha \) is the sum of \( l < \text{rk}g \) pairwise different simple roots. In particular, the statement is proved when \( g \) has type \( \text{A}_{\text{rk}g} \). So we can suppose that \( g \) has type \( E \). We then use the notations of (ii,4). If there exists some \( i \) such that
\[
2n_i + 1 = \sum_{j \in J_i} n_j, \quad (5)
\]
then
\[
s_{\beta_i}(\rho + \alpha) = \rho - \beta_i - n_i \beta_i + s_{\beta_i}(\sum_{j \in J_i} n_j \beta_j) + \sum_{j \notin J_i \cup \{i\}} n_j \beta_j = \rho - (n_i + 1) \beta_i + (\sum_{j \in J_i} n_j) \beta_i + \alpha - n_i \beta_i = \rho + \alpha,
\]
since all the roots have the same length. Consequently, if \( \alpha \) satisfies the equality (5) for some \( i \), then \( \rho + \alpha \) is not regular. If all the coordinates of \( \alpha \) are equal to 1, then \( \alpha \) satisfies equality (5) for \( i = 4 \). Hence \( \rho + \alpha \) is not regular in this case and we have only to consider the cases when \( \alpha \) belongs to \( \mathcal{R}_+ \). For each case, we will give some \( i \) for which \( \alpha \) satisfies equality (5). The result will be presented in a table. Its first column gives some values of \( i \) and the numbers \( j \) which are on the same line in the second column are such that the root \( r(j) \) satisfies equality (5) with respect \( i \). The roots are given by their images by the map \( \alpha \mapsto L(\alpha) \). We recall that the biggest root corresponds to \( r(i) \) for \( i \) the biggest one.

1) \( g \) has type \( \text{E}_6 \). The corresponding table is given by

\[
\begin{align*}
1 & \quad 6 \\
2 & \quad 10 \\
3 & \quad 8 \\
4 & \quad 9 \\
5 & \quad 3,5,7 \\
6 & \quad 1,2,4
\end{align*}
\]
2) \( \mathfrak{g} \) has type \( E_7 \). Let \( R''_+ \) be the subset of elements of \( R'_+ \) such that \( n_7 \neq 0 \). Any element of \( R'_+ \setminus R''_+ \) which satisfies equality (5) with respect to \( i \leq 5 \) and \( E_6 \) satisfy the same equality with respect to \( i \) and \( \mathfrak{g} \) since \( \beta_i \) is orthogonal to \( \beta_7 \). If it satisfies equality (5) with respect to \( 6 \) and \( E_6 \), then it satisfies the same equality with respect to \( i \), \( \beta_i \) is orthogonal to \( \beta_7 \). If it satisfies equality (5) with respect to \( 7 \) and \( E_7 \), then it satisfies the same equality with respect to \( i \) and \( \mathfrak{g} \) since \( n_7 = 0 \). If \( \alpha \) is the biggest root of the subsystem generated by \( \beta_1, \ldots, \beta_6 \), then \( n_6 = 1 \). So \( s_{\beta_7}(\rho + \alpha) = \rho + \alpha \) since \( s_{\beta_7}(\beta_6) = \beta_6 + \beta_7 \). Hence we have only to consider the elements of \( R''_+ \). The corresponding table is given by

\[
\begin{array}{c}
1 & 6,17 \\
2 & 13 \\
3 & 8,16 \\
4 & 9,15 \\
5 & 1,2,4,11,14 \\
6 & 3,5,7,10,12 \\
\end{array}
\]

3) \( \mathfrak{g} \) has type \( E_8 \). Let \( R''_+ \) be the subset of elements of \( R'_+ \) such that \( n_8 \neq 0 \). Any element of \( R'_+ \setminus R''_+ \) which satisfies equality (5) with respect to \( i \leq 6 \) and \( E_7 \) satisfy the same equality with respect to \( i \) and \( \mathfrak{g} \) since \( \beta_i \) is orthogonal to \( \beta_8 \). If it satisfies equality (5) with respect to \( 7 \) and \( E_7 \), then it satisfies the same equality with respect to \( i \) and \( \mathfrak{g} \) since \( n_8 = 0 \). If \( \alpha \) is the biggest root of the subsystem generated by \( \beta_1, \ldots, \beta_7 \), then \( n_7 = 1 \). So \( s_{\beta_8}(\rho + \alpha) = \rho + \alpha \) since \( s_{\beta_7}(\beta_6) = \beta_6 + \beta_7 \). Hence we have only to consider the elements of \( R''_+ \). The corresponding table is given by

\[
\begin{array}{c}
1 & 38 \\
2 & 24,36,41 \\
3 & 9,12,33,39,40 \\
4 & 15,27,35,37,42 \\
5 & 1,3,5,18,20,30,32,34,43 \\
6 & 2,6,7,10,13,21,23,26,29,31,44 \\
7 & 4,8,11,14,16,17,19,22,25,28,45 \\
8 & 16 \\
\end{array}
\]

(v) If \( \alpha \) is not the biggest root:

(a) for \( \mathfrak{g} \) of type \( B_{rkg}, F_4, G_2 \), \( \rho + \alpha \) is regular for two values of \( \alpha \) and when it isn’t dominant there exists a simple root \( \beta \) such that \( s_{\beta}(\rho + \alpha) \) is dominant,

(b) for \( \mathfrak{g} \) of type \( C_{rkg}, D_{rkg} \), \( \rho + \alpha \) is regular for one value of \( \alpha \) and for this value it is dominant.

(a) 1) For \( \mathfrak{g} \) of type \( B_{rkg} \), let be \( \alpha_1 \) be the sum of simple roots. For any positive root \( \gamma \), \( \langle \alpha_1, \gamma \rangle \) is nonnegative, then \( \alpha_1 \) is dominant. Hence \( \rho + \alpha_1 \) is regular dominant. Let be \( \alpha_2 \) be the sum of the simple roots of biggest length. Then \( \alpha_2 = \beta_1 + \ldots + \beta_{rkg-1} \). Since \( s_{\beta_{rkg}}(\rho + \alpha_2) = \rho + \alpha_1, \rho + \alpha_2 \) is regular. It remains to show that \( \rho + \alpha \) is not regular for the other cases, i.e there exists a simple root \( \beta \) such that \( s_{\beta}(\rho + \alpha) = \rho + \alpha \).
The possible values of $\alpha$ are

\[ \alpha_{1,i} = \beta_i + \cdots + \beta_{rk_g}, 1 \leq i \leq rk_g \]
\[ \alpha_{2,i,j} = \beta_i + \cdots + \beta_{j-1}, 1 \leq i < j \leq rk_g \text{ or } 1 < i < j \leq rk_g \]
\[ \alpha_{3,i,j} = \beta_i + \cdots + \beta_{j-1} + 2(\beta_j + \cdots + \beta_{rk_g}), 1 < i < j \leq rk_g \]
\[ \alpha_{4,j} = \beta_1 + \cdots + \beta_j + 2(\beta_{j+1} + \cdots + \beta_{rk_g}), 1 < j \leq rk_g. \]

If $\alpha$ is equal to $\alpha_{1,i}$ or $\alpha_{3,i,j}$,

\[ s_{\beta_{i-1}}(\rho + \alpha) = \rho + \alpha, \]

if $\alpha$ is equal to $\alpha_{2,i,j}$, for $1 \leq i < j \leq rk_g$,

\[ s_{\beta_j}(\rho + \alpha) = \rho + \alpha, \]

for $1 < i < j \leq rk_g$,

\[ s_{\beta_{i-1}}(\rho + \alpha) = \rho + \alpha, \]

if $\alpha$ is equal to $\alpha_{4,j}$,

\[ s_{\beta_j}(\rho + \alpha) = \rho + \alpha. \]

2) For $g$ of type $F_4$, let be $\alpha_1 = \beta_1 + 2\beta_2 + 3\beta_3 + 2\beta_4$. For any positive root $\gamma$, $\langle \alpha_1, \gamma \rangle$ is nonnegative, then $\alpha_1$ is dominant. Hence $\rho + \alpha_1$ is regular dominant. Let be $\alpha_2 = \beta_1 + 2(\beta_2 + 3\beta_3 + 2\beta_4)$. Since $s_{\beta_3}(\rho + \alpha_2) = \rho + \alpha_1$, $\rho + \alpha_2$ is regular.

Let $i = 1, 2, 3, 4$, we denote $n_i$ the coordinate of $\alpha$ at $\beta_i$ in the basis $\Pi$ and we set $L(\alpha) := [n_1, n_2, n_3, n_4]$.

Let $R'_+$ be the set of $\alpha$ in $R_+$ such that $\alpha$ is not the biggest root and it is not in $\{\alpha_1, \alpha_2\}$. The image of $R'_+$ by the map $\alpha \mapsto L(\alpha)$ is given by

\[
\begin{align*}
    r(1) & := [1,0,0,0] & r(2) & := [0,1,0,0] & r(3) & := [0,0,1,0] \\
    r(4) & := [0,0,0,1] & r(5) & := [1,1,0,0] & r(6) & := [0,1,1,0] \\
    r(7) & := [0,0,1,1] & r(8) & := [1,1,1,0] & r(9) & := [0,1,1,1] \\
    r(10) & := [1,1,1,1] & r(11) & := [0,1,2,0] & r(12) & := [1,1,2,0] \\
    r(13) & := [0,1,2,1] & r(14) & := [1,2,2,0] & r(15) & := [1,1,2,1] \\
    r(16) & := [0,1,2,2] & r(17) & := [1,2,2,1] & r(18) & := [1,1,2,2] \\
    r(19) & := [1,2,3,1] & r(20) & := [1,2,4,2] & r(21) & := [1,3,4,2].
\end{align*}
\]

We identify $\alpha$ with $L(\alpha)$. In the following table, the value $i$ in the first column and the values $j$ on the same line verify the equality $s_{\beta_j}(\rho + r(j)) = \rho + r(j)$,

|   | 1  | 2  | 3  | 4  |
|---|----|----|----|----|
| 1 | 2, 6, 9, 11, 13, 16, 21 | 1, 7, 12, 15, 18, 20 | 4, 10, 17 | 3, 8, 19 |

Since

\[ s_{\beta_3}(\rho + r(5)) = \rho + r(8) \]
\[ s_{\beta_4}(\rho + r(14)) = \rho + r(17), \]

for $i = 5, 14$, $\rho + r(i)$ is not regular by which goes before.
3) For $\mathfrak{g}$ of type $G_2$, let be $\alpha = 2\beta_1 + \beta_2$. For any positive root $\gamma$, $\langle \alpha, \gamma \rangle$ is nonnegative, then $\alpha$ is dominant. Hence $\rho + \alpha$ is regular dominant. Since $s_{\beta_1}(\rho + \beta_2) = \rho + \alpha$, $\rho + \beta_2$ is regular.

For $\alpha$ not in $\{\alpha_1, \alpha_2\}$, $\alpha$ is in $\{\beta_1, \beta_1 + \beta_2, 3\beta_1 + \beta_2\}$. Since

$$s_{\beta_2}(\rho + \beta_1) = \rho + \beta_1,$$

$$s_{\beta_1}(\rho + \beta_1 + \beta_2) = \rho + \beta_1 + \beta_2,$$

$$s_{\beta_2}(\rho + 3\beta_1 + \beta_2) = \rho + 3\beta_1 + \beta_2,$$

$\rho + \alpha$ isn’t regular.

(b) 1) $\mathfrak{g}$ has type $C_{\rho k}$, let be $\alpha = \beta_1 + 2(\beta_2 + \cdots + \beta_{\rho k-1}) + \beta_{\rho k}$. For any positive root $\gamma$, $\langle \alpha, \gamma \rangle$ is nonnegative, then $\alpha$ is dominant. Hence $\rho + \alpha$ is regular dominant. The other possible values of $\alpha$ are

$$\alpha_{1,i,j} = \beta_i + \cdots + \beta_{j-1}, 1 \leq i < j \leq \rho k$$

$$\alpha_{2,i,j} = \beta_i + \cdots + \beta_{j-1} + 2(\beta_j + \cdots + \beta_{\rho k-1}) + \beta_{\rho k}, 1 < i < j \leq \rho k$$

$$\alpha_{3,j} = \beta_1 + \cdots + \beta_j + 2(\beta_{j+1} + \cdots + \beta_{k-1}) + \beta_{\rho k}, 1 < j \leq \rho k - 1$$

$$\alpha_{4,i} = 2(\beta_i + \cdots + \beta_{\rho k-1}) + \beta_{\rho k}, 1 < i \leq \rho k.$$

If $\alpha$ is equal to $\alpha_{1,i,j}$,

$$s_{\beta_i}(\rho + \alpha) = \rho + \alpha,$$

if $\alpha$ is equal to $\alpha_{2,i,j}$,

$$s_{\beta_{i-1}}(\rho + \alpha) = \rho + \alpha,$$

if $\alpha$ is equal to $\alpha_{3,j}$,

$$s_{\beta_j}(\rho + \alpha) = \rho + \alpha,$$

if $\alpha$ is equal to $\alpha_{4,i}$,

$$s_{\beta_{i-1}}(\rho + \alpha) = \rho + \alpha + \beta_{i-1}.$$

In this case $\rho + \alpha$ is not regular since $\rho + \alpha + \beta_{i-1} = \rho + \alpha_{2,i-1,i}$.

2) $\mathfrak{g}$ has type $D_{\rho k}$, let be $\alpha = \beta_1 + 2(\beta_2 + \cdots + \beta_{\rho k-2}) + \beta_{\rho k-1} + \beta_{\rho k}$. For any positive root $\gamma$, $\langle \alpha, \gamma \rangle$ is nonnegative, then $\alpha$ is dominant. Hence $\rho + \alpha$ is regular dominant. The other possible values of $\alpha$ are

$$\alpha_{1,i,j} = \beta_i + \cdots + \beta_{j-1}, 1 \leq i < j \leq \rho k$$

$$\alpha_{2,i} = \beta_i + \cdots + \beta_{\rho k}, 1 \leq i \leq \rho k$$

$$\alpha_{3,i,j} = \beta_i + \cdots + \beta_{j-1} + 2(\beta_j + \cdots + \beta_{\rho k-2}) + \beta_{\rho k-1} + \beta_{\rho k}, 1 < i < j \leq \rho k - 1$$

$$\alpha_{4,j} = \beta_1 + \cdots + \beta_j + 2(\beta_{j+1} + \cdots + \beta_{\rho k-2}) + \beta_{\rho k-1} + \beta_{\rho k}, 1 < j < \rho k - 2.$$

If $\alpha$ is equal to $\alpha_{1,i,j}$ for $i \notin \{1, \rho k - 2, \rho k - 1\}$,

$$s_{\beta_{i-1}}(\rho + \alpha) = \rho + \alpha$$

for $i = 1$ and $j \notin \{\rho k - 2, \rho k - 1, \rho k\}$,

$$s_{\beta_j}(\rho + \alpha) = \rho + \alpha$$

for $i = 1$ and $j = \rho k - 2$,

$$s_{\beta_{\rho k-2}}(\rho + \alpha) = \rho + \alpha$$
for $i = 1$ and $j \in \{\text{rk}\, g - 1, \text{rk}\, g\}$,
\[ s_{\beta_{\text{rk} g}}(\rho + \alpha) = \rho + \alpha \]
for $i = \text{rk}\, g - 2$, $\alpha = \beta_{\text{rk} g - 2} + \beta_{\text{rk} g - 1}$ and
\[ s_{\beta_{\text{rk} g}}(\rho + \alpha) = \rho + \alpha, \]
for $i = \text{rk}\, g - 1$, $\alpha = \beta_{\text{rk} g - 1}$ and
\[ s_{\beta_{\text{rk} g - 2}}(\rho + \alpha) = \rho + \alpha, \]
if $\alpha$ is equal to $\alpha_{3,i,j}$,
\[ s_{\beta_{i - 1}}(\rho + \alpha) = \rho + \alpha, \]
if $\alpha$ is equal to $\alpha_{2,i}$ with $i = 1$,
\[ s_{\beta_{\text{rk} g - 2}}(\rho + \alpha) = \rho + \alpha, \]
for $i \not\in \{1, \text{rk}\, g - 1\}$,
\[ s_{\beta_{i - 1}}(\rho + \alpha) = \rho + \alpha, \]
for $i = \text{rk}\, g - 1$,
\[ s_{\beta_{\text{rk} g - 2}}(\rho + \alpha) = \rho + \beta_{\text{rk} g - 2} + \beta_{\text{rk} g - 1} + \beta_{\text{rk} g} \]
and since $\beta_{\text{rk} g - 2} + \beta_{\text{rk} g - 1} + \beta_{\text{rk} g}$ is equal to $\alpha_{2,i}$ with $i = \text{rk}\, g - 2$, $\rho + \beta_{\text{rk} g - 2} + \beta_{\text{rk} g - 1} + \beta_{\text{rk} g}$ is not regular,
if $\alpha$ is equal to $\alpha_{4,j}$,
\[ s_{\beta_{j}}(\rho + \alpha) = \rho + \alpha. \]

\[\Box\]

References

[1] A.V. Bolsinov, Commutative families of functions related to consistent Poisson brackets, Acta Applicandae Mathematicae, Vol. 24, 1991, p. 253-274.
[2] R. Bott and L. W. Tu, Differential forms in algebraic topology, GTM 82, Springer.
[3] N. Bourbaki, Groupes et algèbres de Lie, Chapitre I, Hermann, Paris.
[4] N. Bourbaki, Groupes et algèbres de Lie, Chapitre IV, V, VI, Hermann, Paris.
[5] N. Bourbaki, Groupes et algèbres de Lie, Chapitre VII, VIII, Hermann, Paris.
[6] J-Y. Charbonnel and A. Moreau, Nilpotent bicone and characteristic submodule of a reductive Lie algebra, Transformation Groups, Vol. 14, No. 2, 2009, p. 319-360.
[7] N. Chriss and V. Ginzburg, Representation theory and complex geometry, Birkhäuser.
[8] M. Demazure, une démonstration algébrique d’un théorème de Bott, Inventiones Mathematicae, volume 5, 1968, p. 349-356.
[9] J. Dixmier, Enveloping algebras, Graduate studies in mathematics, American Mathematical Society, 1996.
[10] R. Elkik, Singularités rationnelles et déformations, Inventiones Mathematicae, Vol 47, 1978, p. 139-147.
[11] R. Elkik, Rationalité des singularités canoniques, Inventiones Mathematicae, Vol 64, 1981, p. 1-6
[12] V. Ginzburg, Isospectral commuting variety and the Harish-Chandra D-module, arXiv:1002.3311v2 [math.AG] 25 Feb 2010.
[13] A. Grothendieck and J.A. Dieudonné, Eléments de géométrie algébrique III, Publ. Math. de l'I.H.E.S. 11 (Paris 1961)
[14] R. Hartshorne, Algebraic geometry, GTM 52, Springer.
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[15] Wim H. Hesselink, *Cohomology and the resolution of the nilpotent variety*, Mathematische Annalen, Band 223, 1976, p. 249-252.
[16] D. Hilbert, *Ueber die vollen Invariantensystem*, Math. Ann. 42, 1893, p. 313-373.
[17] James E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Mathematical Surveys and Monographs, volume 43, 1995.
[18] M. Hunziker, *Classical invariant theory for finite reflection groups*, Transformation groups, volume 2, No 2, 1997 p. 147-163.
[19] A. Joseph, *On a Harish-Chandra homomorphism*, C. R. Académie des sciences, t. 324 (1997), p. 759-764.
[20] B. Kostant, *Lie group representations on polynomial rings*, American Journal of Mathematics, 85, 1963, p. 327–404.
[21] H. Kraft, N. R. Wallach, *On the nullcone of representations of reductive groups*, Pacific J. Math. 224, 2006, p. 119-140.
[22] H. Kraft, N. R. Wallach, *Polarizations and nullcone of representations of reductive groups*, Progress in Mathematics, 278, 2009, p. 149-162.
[23] M. Losik, P. W. Michor, V. L. Popov, *On polarizations in invariant theory*, Journal of Algebra, 301, 2006, p. 406-424.
[24] H. Matsumura, *Commutative ring theory*, Cambridge studies in advanced mathematics, no.8, 1986.
[25] D. Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34, 1965.
[26] D. Mumford, *The red book of varieties and schemes*, Lecture Notes in Mathematics, Springer.
[27] V. L. Popov, *The cone of Hilbert Nullforms*, Steklov Mathematical Institute, 241, 2003, p. 177-194.
[28] R. W. Richardson, *Commuting varieties of semisimple Lie algebras and algebraic groups*, Compositio Mathematica, volume 38 (1979), p. 311-322.
[29] P. Tauvel and R. W. T. Yu, *Lie algebras and Algebraic groups*, Monographs in Mathematics, Springer, 2005.

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