TRANSITION PHENOMENA FOR THE ATTRACTOR OF AN ITERATED FUNCTION SYSTEM

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Abstract. Iterated function systems (IFSs) and their attractors have been central to the theory of fractal geometry almost from its inception. And contractivity of the functions in the IFS has been central to the theory of iterated functions systems. If the functions in the IFS are contractions, then the IFS is guaranteed to have a unique attractor. Recently, however, there has been an interest in what occurs to the attractor at the boundary between contractvity and expansion of the IFS. That is the subject of this paper. For a family $F_t$ of IFSs depending on a real parameter $t > 0$, the existence and properties of two types of transition attractors, called the lower transition attractor $A^*$ and the upper transition attractor $A^*$, are investigated. A main theorem states that, for a wide class of IFS families, there is a threshold $t_0$ such that the IFS $F_t$ has a unique attractor $A_t$ for $t < t_0$ and no attractor for $t > t_0$. At the threshold $t_0$, there is an $F_{t_0}$-invariant set $A^*$ such that $A^* = \lim_{t \to t_0} A_t$.

1. Introduction

Iterated function systems (IFSs) and their attractors have been central to the theory of fractal geometry almost from its inception. And contractivity of the functions in the IFS has been central to the theory of iterated functions systems. If the functions in the IFS are contractions, then the IFS is guaranteed to have a unique attractor (see Hutchinson’s seminal Theorem 1.1 below). Recently, however, there has been an interest in what occurs to the attractor at the boundary between contractivity and expansion of the IFS. That is the subject of this paper.

Let $X$ denote a complete metric space with metric $d(\cdot, \cdot)$. A finite iterated function system (IFS) is a set

$$F := \{f_1, f_2, \ldots, f_N\}$$

of $N \geq 2$ continuous functions from $X$ to itself. An IFS is affine if its functions are invertible affine functions on $d$-dimensional Euclidean space $\mathbb{R}^d$, projective if its functions are non-singular projective functions on $d$-dimensional real projective space $\mathbb{RP}^d$, and Möbius if its functions are Möbius transformations on the extended complex plane $\mathbb{C} \cup \{\infty\}$, i.e., on the Riemann sphere. An affine IFS all of whose functions are similarities is referred to as a similarity IFS. An affine IFS all of whose functions are non-singular linear maps is referred to as a linear IFS.

For a function $f : X \to X$, let

$$\text{Lip}(f, d) := \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)}$$

denote the Lipschitz constant of $f$ with respect to the metric $d$. Let

$$\text{Lip}(F, d) := \max_{f \in F} \text{Lip}(f, d).$$

A function $f$ is Lipschitz if $\text{Lip}(f, d) < \infty$, and an IFS $F$ is Lipschitz if $\text{Lip}(F, d) < \infty$. A function $f$ is a contraction with respect to $d$ if $\text{Lip}(f, d) < 1$, and is nonexpansive if $\text{Lip}(f, d) \leq 1$. 

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Definition 1.1. An IFS $F$ on $\mathbb{X}$ is **contractive**, if there is an equivalent metric $d'$ on $\mathbb{X}$, i.e., a metric $d'$ giving the same topology as the original metric $d$, such that $\mathbb{X}$ remains complete with respect to $d'$ and $\text{Lip}(F,d') < 1$.

Allowing metrics topologically equivalent to the original metric is essential, for example, to the validity of Theorem 3.1 below. Also see Example 3.1.

For the collection $\mathcal{K}(\mathbb{X})$ of non-empty compact subsets of $\mathbb{X}$, the classical Hutchinson operator $F : \mathcal{K}(\mathbb{X}) \rightarrow \mathcal{K}(\mathbb{X})$ is given by

$$F(K) := \bigcup_{f \in F} f(K).$$

By abuse of language, the same notation $F$ is used for the IFS, the set of functions in the IFS, and for the Hutchinson operator; the meaning should be clear from the context. A compact set $A \subseteq \mathbb{X}$ is the (strict) **attractor** of $F$ if there is an open neighborhood $U \supseteq A$ such that

- (invariance) $F(A) = A$, and
- (attraction) $A = \lim_{n \to \infty} F^{(n)}(K)$,

where $F^{(n)}$ denotes the $n$-fold composition, the limit is with respect to the Hausdorff metric and is independent of the non-empty compact set $K \subseteq U$. So the attractor is the Banach fixed point of the Hutchinson operator on $\mathcal{K}(U)$. The largest such set $U$ is called the **basin** of $F$.

**Theorem 1.1** (Hutchinson [14]). A contractive IFS on a complete metric space $\mathbb{X}$ has a unique attractor with basin $\mathbb{X}$.

In classical IFS theory, it is assumed that the functions in the IFS are contractions, a natural assumption in light of Hutchinson’s theorem. More recently, however, papers have appeared on IFS attractors assuming average contractivity (see [30] for a survey), on IFSs that are weakly contractive (see, for example, [21]), and on relaxing the definition of an attractor; see, for example, [17] [18] in which the notion of a semiattractor is introduced to explain the nature of supports of invariant measures of average contractive IFSs [4]. This paper is concerned with attractor phenomena at the transition between contractivity and expansion of a one-parameter IFS family, between the existence and non-existence of an attractor. To illustrate this kind of transition phenomena, consider the following family $F_i$ of IFSs that depends on a real parameter $t > 0$, which is based on [33 Example 1.1].

**Example 1.1.** In $\mathbb{R}^3$ let $F_i := \{f_{(i,t)}, 1 \leq i \leq 2\}$ be the one-parameter affine family where $f_{(i,t)}(v) = t \cdot L_i(v - q_i) + q_i$, and where

$$L_1 = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is the rotation by $\pi/4$ about the $z$-axis and $q_1 = (0,0,2)$ is a fixed point of $L_1$ outside the $xy$-plane; $L_2 = 0.4 \cdot L_1$ and $q_2 = (1,0,0)$.

For $t \in (0,1)$, the IFS $F_i$ is contractive and has an attractor $A_t$. Figure 1 shows views of $A_t$ for $t = 0.9$ and $t = 0.96$. For $t \geq 1$, the IFS $F_i$ fails to be contractive and has no attractor. The value $t = 1$ is called a threshold, defined precisely in Definition 1.2 below.

The question arises as to the nature of the transition at the threshold $t = 1$. In this example, intriguing $F_1$-invariant sets occur. We refer to such sets as **transition attractors**, and we consider two types: lower transition attractors, denoted $A_\bullet$, and upper transition attractors, denoted $A^\bullet$. Precise definitions appear in Section 4. The terminology “upper” and “lower” is due to the fact that, for appropriately defined one-parameter families, it is the case that $A_\bullet \subseteq A^\bullet$.

Figure 2 shows the lower transition attractor and Figure 3 shows the upper transition attractor for the IFS family of Example 1.1. The subject of transition attractors, in two guises, was introduced independently in [20] and [33].
Definition 1.2. A one-parameter family is an IFS family

\[ F_t := \{ f_{(1,t)}, f_{(2,t)} \ldots, f_{(N,t)} \} \]

parametrized by a real number \( t \in (0, \infty) \). The intuition is that, the nearer the parameter \( t \) is to 0, the more contractive the functions in the IFS, and as \( t \) increases, the functions in \( F_t \) become less contractive. A real number \( t_0 \) is called the threshold for the existence of an attractor of \( F_t \) if \( F_t \) has an attractor for \( t < t_0 \) but fails to have an attractor for \( t > t_0 \).

Figure 1. The attractor \( A_t \) for the one-parameter affine family \( F_t \) of Example 1.1 for parameter values \( t = 0.9 \) (top line), \( t = 0.96 \) (bottom line); side and bottom view of a fractal "cone". The green and blue colours indicate the image of the attractor under the two maps of the IFS. Note that \( f_{(1,t)}(A) \cap f_{(2,t)}(A) \neq \emptyset \).

The main open question in [33] was the following.

Question 1.1. If \( A_t, t \in (0, t_0) \), denotes the attractor of a one-paramter family \( F_t \) of affine IFSs with threshold \( t_0 \), what conditions on \( F_t \) guarantee the existence of a unique upper transition attractor, i.e., a compact \( F_{t_0} \)-invariant set \( A^* \) such that

\[ A^* = \lim_{t \to t_0} A_t. \]

In [33] certain conditions on a one-parameter family of affine functions were conjectured to guarantee such an upper transition attractor \( A^* \). A main result in this paper is a proof of a strong version of that conjecture in the setting of a real Banach space.
2. Organization - Previous and New Results

The paper is organized as follows.

- (Section 3: Contractivity, Attractors, and Thresholds)

  In this paper we are interested in transitions for one-parameter IFS families $F_t$ at thresholds between the existence and non-existence of an attractor, between contractivity and of non-contractivity of $F_t$. For affine families (Definition 3.1) like that of Example 1.1, it is known that there does exist a single threshold (Theorem 3.2). This is also the case for our main Theorem 5.2.

  From the origin of IFS theory, the existence of an attractor has been associated with the contractivity of the IFS. The precise relationship, however, has not been completely delineated. The issue involves the converse of Hutchinson’s Theorem 1.1. For an IFS
$F$ on a complete metric space, Hutchinson’s theorem states that contractivity of an IFS is a sufficient condition for the existence of a unique attractor. When the IFS contains only one mapping, the converse (which is a converse to the Banach Contraction Mapping Theorem) was proved by Janós [16] and by Leader [19]. A converse is known to hold for affine, projective and Möbius IFSs (Theorem 3.1). In general, however, there are examples of IFSs which admit attractors yet there is no equivalent metric with respect to which the functions in the IFS are contractions.

In Examples 3.2, 3.3, and 3.4, the IFS $F$ admits a unique attractor but $\text{Lip}(F, d) > 1$ for all equivalent metrics $d$ on $X$.

• (Section 4: Lower Transition Attractors, Upper Transition Attractors, and Semiattractors)

If a threshold $t_0$ for the existence of an attractor does exist for a one-parameter family, then the question arises as to what occurs at this threshold. For some one-parameter families $F_t$, there exist intriguing $F_{t_0}$-invariant sets referred to as the lower and upper transition attractors (Definitions 4.1 and 4.3). The existence and some properties of a lower attractor is the subject of Theorem 4.2, Corollary 4.1, and Proposition 4.1. Example 4.2 illustrates these results. The existence and some properties of an upper transition attractor is the subject of Theorem 4.3 and Proposition 4.1. The relationship between the lower and upper transition attractors is the subject of Theorem 4.4.

The lower transition attractor of certain one-parameter families is shown in statement (iv) of Theorem 4.2 to be the semiattractor (Definition 4.2) of an associated single IFS. Properties of semiattractors are contained in Theorem 4.1.

• (Section 5: The Existence of a Unique Upper Transition Attractor)

Theorem 5.2, the main result of the paper, provides an answer to Question 1.1 in the introduction - giving conditions that guarantee a unique upper transition attractor at a threshold for the existence of an attractor. The existence of a unique upper transition attractor was conjectured for a special type of one-parameter similarity family in Euclidean space in [33]. The underlying space in Theorem 5.2 is the more general Banach space, and the one-parameter families are more general than in [33]. Examples 5.1, 5.2, 5.3, and 5.4 show that the assumptions in the hypothesis of Theorem 5.2 are all necessary, at least in the infinite dimensional case. Question 6.2 in Section 6 asks whether the “periodicity” assumption in Theorem 5.2 can be dropped assuming a less exotic space.

• (Section 6: Open Problems)

There remain questions and conjectures about thresholds and transition attractors that remain open. Several are posed in this section.

3. Contractivity, attractors, and thresholds

For an IFS on a complete metric space, the converse of Hutchinson’s Theorem 1.1 does not, in general, hold. Examples 3.2, 3.3, and 3.4 are provided below. These examples notwithstanding, a converse does hold in the affine, Möbius, and projective cases.

**Theorem 3.1** ([2, 5, 32]). An affine, Möbius, or projective IFS can have at most one attractor. Moreover,

1. An affine IFS $F$ has an attractor if and only if $F$ is contractive on $\mathbb{R}^d$.

2. A Möbius IFS $F$ has an attractor $A \neq \mathbb{C} \cup \{\infty\}$ if and only if $F$ is contractive on an open set whose closure is not $\mathbb{C} \cup \{\infty\}$.

3. A projective IFS $F$ has an attractor that avoids some hyperplane if and only if $F$ is contractive on the closure of some open set.

For an IFS $F$ the distinction between all functions in $F$ being contractions and $F$ contractive must be emphasized. See Example 3.1 below.
Example 3.1 (A family of contractive affine IFSs \( F_t \) on \( \mathbb{R}^2 \) such that the functions in \( F_t \) are not contractions with respect to the Euclidean metric cf. [21] Example 6.3.). Define \( F_t := \{ f_{(1,t)}, f_{(2,t)} \} \), where

\[
f_{(1,t)}(v) = \begin{pmatrix} 0 & \kappa_1 t \\ \lambda_1 / t & 0 \end{pmatrix} v,
\]

\[
f_{(2,t)}(v) = \begin{pmatrix} 0 & \kappa_2 t \\ \lambda_2 / t & 0 \end{pmatrix} v + \left( \frac{1}{\lambda_2} - t \frac{1}{1/\kappa_2 - 1/t} \right),
\]

where \( \lambda_i, \kappa_j < 1 \) for \( i, j \in \{1, 2\} \).

The functions in \( F_t \) are contractions with respect to the Euclidean metric only for \( \min\{1/\kappa_1, 1/\kappa_2\} > t > \max\{\lambda_1, \lambda_2\} \). We claim, however, that \( F_t \) admits an attractor \( A_t \) for all \( t > 0 \). Since \( F_t \) consists of affine functions for each \( t > 0 \), it would then follow that the IFS \( F_t \) is contractive for each \( t > 0 \) by Theorem 3.1 part (1).

To see that \( F_t \) has an attractor, consider the second iterate \( F_t^2 := \{ f_{(i,t)} \circ f_{(j,t)} : 1 \leq i, j \leq 2 \} \) of \( F_t \) given by

\[
f_{(i,t)} \circ f_{(j,t)}(v) = \begin{pmatrix} \kappa_i \lambda_j & 0 \\ 0 & \kappa_j \lambda_i \end{pmatrix} v + a(i,t),
\]

where the vectors \( a(i,t), i = 1, 2 \), are readily calculated. The two functions in \( F_t^2 \) are contractions for all \( t \in (0, \infty) \) when \( 0 < \kappa_i \lambda_j < 1 \) for \( i, j \in \{1, 2\} \), and therefore have an attractor for all \( t \in (0, \infty) \). If an attractor exists for one of them, then \( F_t \) and \( F_t^2 \) have the same attractor. Therefore \( F_t \) admits an attractor \( A_t \) for all \( t \). The attractor of \( F_t^2 \) is shown in Figure 4 for three values of \( t \) in the case that \( \lambda_1 = 1/4, \kappa_1 = 3, \lambda_2 = 1/5 \) and \( \kappa_2 = 2 \). The functions in \( F_t \) are contractions with respect to the Euclidean metric only for \( t \in (1/4, 1/3) \), yet the functions in the second iterate \( F_t^2 \) are contractions for all \( t \in (0, \infty) \).

**Figure 4.** The attractor \( A_t \) for the one-parameter affine family \( F_t \) of Example 3.1 for successive parameter values \( t = .5, 1, 5 \).
There exist IFSs that have an attractor but are not contractive. For the examples \( F \) in \([6, 21]\) \( \text{Lip}(F,d) = 1 \). Our counterexamples below are of

(1) an IFS \( F \) on the circle \( S^1 \) that admits a unique attractor but \( \text{Lip}(F,d) > 1 \) for all equivalent metrics \( d \) on \( X \) (Example 3.2).

(2) a stronger counterexample of an IFS \( F \) on \( S^1 \) that admits a unique attractor but \( \text{Lip}(f,d) > 1 \) for all \( f \in F \) and all equivalent metrics \( d \) on \( X \) (Example 3.3), and

(3) an IFS on \( \mathbb{R}^n \) that has an attractor but is not contractive (Example 3.4).

Let \( S^1 \) be the unit circle centered at the origin in the complex plane, and let \( f : S^1 \to S^1 \) be the angle doubling map \( f(z) = z^2 \) (cf. [10]). Let \( \rho : S^1 \to S^1 \) be the rotation map \( \rho(z) = e^{i\alpha}z \) where \( \alpha/2\pi \) is irrational and let \( g(z) = \rho \circ f(z) \). The following proposition is helpful in showing the validity of the two examples. We include a direct proof of this proposition for completeness though it can be obtained from the standard theory of topological dynamics, see Remark 3.1.

**Proposition 3.1.** If \( d \) is any metric on \( S^1 \) inducing the standard topology on \( S^1 \), then \( \text{Lip}(f,d) > 1 \) and \( \text{Lip}(g,d) > 1 \).

**Proof.** Suppose that \( \text{Lip}(f,d) \leq 1 \) for some \( d \). Abbreviate the point \( e^{i\theta} \) by \( z_\theta \). Then for any \( z = z_\phi \neq 1 \) we have

\[
d(1,z) = d(f(1),f(z_\phi/2)) \leq d(1,z_\phi/2) = d(f(1),f(z_\phi/4)) \leq d(1,z_\phi/4) = \cdots.
\]

Therefore \( d(1,z) \leq d(1,z_{2^n}) \) for all \( n \in \mathbb{N} \). Because \( z_{2^n} \to 1 \) as \( n \to \infty \) with respect to the standard topology on \( S^1 \), we have that \( d(1,z) < \epsilon \) for every \( \epsilon > 0 \). Therefore \( d(1,z) = 0 \) and \( z = 1 \), a contradiction.

To see that \( \text{Lip}(g,d) > 1 \), note that the mappings \( f \) and \( g \) are conjugate, specifically \( g = \rho^{-1} \circ f \circ \rho \). If \( d \) is any metric inducing the standard topology on \( S^1 \), then the metric

\[
d'(x,y) := d(\rho^{-1}(x),\rho^{-1}(y))
\]

also induces the standard topology on \( S^1 \). By the paragraph above, there exist \( x',y' \in S^1 \) such that

\[
d'(f(x'),f(y')) = c d'(x',y'),
\]

where \( c > 1 \). If \( x = \rho^{-1}(x') \) and \( y = \rho^{-1}(y') \), then

\[
d(g(x),g(y)) = d(\rho^{-1} \circ f \circ \rho(x),\rho^{-1} \circ f \circ \rho(y)) = d'(f \circ \rho(x),f \circ \rho(y)) = d'(f(x'),f(y'))
\]

\[= c d'(x',y') = c d(\rho^{-1}(x'),\rho^{-1}(y')) = c d(x,y).\]

\[\square\]

**Remark 3.1.** One can easily see that \( f \) and \( g \) are locally distance doubling with respect to the arc metric on \( S^1 \). Therefore they are topologically expanding ([11] chapter 2.2 and [28]). Since the notion of a topologically expanding map on a compact space does not depend on the choice of metric, this proves Proposition 3.1. Moreover, neither \( f \) nor \( g \) are locally nonexpansive at any point under any admissible metrization \( d \) of \( S^1 \).

**Example 3.2** (An IFS on the circle \( S^1 \) having an attractor, but with \( \text{Lip}(F,d) > 1 \) for all admissible metrizations \( d \) of \( S^1 \)). With \( f \) and \( \rho \) as defined above, let \( F := \{ f, \rho, \text{id} \} \), where id is the identity map on \( S^1 \). That \( \text{Lip}(F,d) > 1 \) follows from Proposition 3.1. That \( S^1 \) is the attractor of \( F \) is seen as follows. The invariance \( F(S^1) = S^1 \) is clear since \( h \) is a rotation. That \( \lim_{n \to \infty} F^{(n)}(z) = S^1 \) for any \( z \in S^1 \) can be seen as follows. We have \( \{ \rho^m(z) : 0 \leq m \leq n \} \subseteq F^{(n)}(z) \) and \( \{ \rho^m(z) \}_{m=0}^\infty \) is dense in \( S^1 \), since \( h \) is an irrational rotation.

**Example 3.3** (An IFS on the circle \( S^1 \) having an attractor, but with \( \text{Lip}(f,d) > 1 \) for all \( f \in F \) under any admissible metrization \( d \) of \( S^1 \)). With \( f \) and \( g \) has defined above, let \( F := \{ f, g \} \). Again, that \( \text{Lip}(f,d) > 1 \) and \( \text{Lip}(g,d) > 1 \) follows from Proposition 3.1. That \( S^1 \) is the attractor of \( F \) is seen as follows. The invariance \( F(S^1) = S^1 \) is clear since \( f \) maps \( S^1 \) onto itself. That
lim_{n \to \infty} F^{(n)}(z) = S^1 \text{ for any } z \in S^1 \text{ can be seen as follows. Abbreviate the point } e^{i\theta} \text{ by } z_\theta. \text{ For } (a_1, a_2, \ldots, a_n) \in (\mathbb{Z}_2)^n, \text{ denote by } f_{(a_1, a_2, \ldots, a_n)} : S^1 \to S^1 \text{ the map given by }

f_{(a_1, a_2, \ldots, a_n)}(z_\theta) = z(2 \alpha \theta + \sum_{k=1}^n 2^{k-1} a_k \alpha)

(That is } f_{(a_1, a_2, \ldots, a_n)} = f_{a_n} \circ \cdots \circ f_{a_2} \circ f_{a_1}, \text{ under identification } f_0 := f, f_1 := g.) \text{ Then for any } z = z_\theta \text{ we have }

F^{(n)}(z) = \{f_{(a_1, \ldots, a_n)}(z_\theta) \mid (a_1, \ldots, a_n) \in (\mathbb{Z}_2)^n\} = \{p^{(m)}(z_{2\alpha \theta}) \mid 0 \leq m \leq 2^n\}.

Therefore, for any } \epsilon > 0 \text{ there is an } n \text{ such that there is no arc on } S^1 \text{ of length } \epsilon \text{ not containing a point of } F^{(n)}(z). \text{ Therefore } \lim_{n \to \infty} F^{(n)}(z) = S^1.

**Example 3.4** (An IFS on } \mathbb{R}^n \text{ that has an attractor but is not contractive). Let } A \text{ be a unit cube in } \mathbb{R}^n, \text{ or any other convex compact set in } \mathbb{R}^n, \text{ other than a single point, that is the attractor of an IFS } F. \text{ Then } A \text{ is a retract of } \mathbb{R}^n, \text{ i.e., there exists a continuous map } r : \mathbb{R}^n \to A \text{ such that } r(\mathbb{R}^n) = A \text{ and } r \text{ restricted to } A \text{ is the identity map. (In fact, any set homeomorphic to a convex compact subset of a Banach space } X \text{ is a retract of } X, \text{ cf. [13 Chp. I, Corollary 1.4, Definition 1.7 and Theorem 1.9.1].}) \text{ Since } A \text{ contains more than one point, the map } r \text{ cannot be a contraction with respect to any metric equivalent to the Euclidean metric. Now let } G = F \cup \{r\}. \text{ Then } G \text{ is an IFS with attractor } A. \text{ Indeed, } F^k(S) \subseteq G^k(S) \subseteq F^k(S) \cup A \text{ for any non-empty } S \subset \mathbb{R}^n. \text{ Since } r \text{ cannot be a contraction with respect to any metric equivalent to the Euclidean metric, the IFS } G \text{ is not contractive.}

**Remark 3.2.** The possibility of remetrization of a given IFS } F \text{ by a metric making each map weakly contractive is equivalent to the existence of a coding map } [3, 25].

**Definition 3.1.** A one-parameter family

\[ F_t := \{f_{(i,t)}, f_{(2,t)} \ldots, f_{(N,t)}\} \]

whose functions have the form

\[ f_{(i,t)}(x) = t f_i(x) + q_i, \quad x \in \mathbb{R}^d \]

where

\[ F := \{f_1, f_2, \ldots, f_N\} \quad \text{ and } \quad Q := \{q_1, q_2, \ldots, q_N\} \]

are a set of invertible affine transformations on } \mathbb{R}^d \text{ and a set of vectors in } \mathbb{R}^d, \text{ respectively, is called a one-parameter affine family.

Theorem 3.2 below states that a one-parameter affine family has a threshold for the existence of an attractor. The threshold in Example 3.4 is } t_0 = 1. \text{ See [8, 9, 29] for background on the joint spectral radius.

**Theorem 3.2** ([33]). \textit{For a one-parameter affine family } F_t, \text{ let } t_0 = 1/\rho(F), \text{ where } \rho(F) \text{ is the joint spectral radius of the linear parts of the functions in } F. \text{ Then } F_t \text{ has an attractor for } t < t_0 \text{ and fails to have an attractor for } t > t_0. \text{More can be said for a linear family } F_t, \text{ all of whose maps are of the form } f_t(x) = t L(x), \text{ where } L \text{ is a non-singular linear map. In this case, it immediately follows from Theorem 3.2 that the attractor } A_t \text{ of } F_t \text{ is the origin, a single point, for all } t < t_0, \text{ and there is no attractor for all } t > t_0. \text{ However, the following holds.

**Theorem 3.3** ([7]). \textit{Let } F_t \text{ be an irreducible ( } F \text{ admits no non-trivial invariant subspace), one-parameter linear IFS family on } \mathbb{R}^d \text{ with threshold } t_0. \text{ Then there exists a compact } F_{t_0}-\text{invariant set that is centrally symmetric, star-shaped, and whose affine span is } \mathbb{R}^d.\]
In other words, the attractor evolves with the parameter $t$ from trivial to non-existent, blowing up only at the single threshold value $t = t_0$. An example in $\mathbb{R}^2$ is shown in Figure 5 for $F := \{L_1, L_2\}$ where

$$L_1 = \begin{pmatrix} 0.02 & 0 \\ 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0.0594 & -1.98 \\ 0.495 & 0.01547 \end{pmatrix}. \quad (1)$$

**Figure 5.** A transition attractor for a linear one-parameter family.

### 4. Lower Transition Attractors, Upper Transition Attractors, and Semiattractors

Consider a one-parameter family

$$F_t := \{f_{(i,t)} : 1 \leq i \leq N\}, \quad t \in [0, \infty),$$

consisting of Lipschitz maps defined on a complete metric space $(X, d)$. Let $t_0$ be the threshold for the existence of an attractor as given in Definition 1.2. We say that $t_0$ is a **threshold for contractivity** if $F_t$ is contractive for all $0 < t < t_0$ and

$$\hat{t}_0 = \sup \{t : F_t \text{ is contractive}\}.$$ 

We assume throughout this section that $F_t$ has a finite contractivity threshold. Note that for $t < \hat{t}_0$, the IFS $F_t$ has an attractor; hence

$$\hat{t}_0 \leq t_0$$

if a threshold $t_0$ exists. It is often the case and it is the interesting case when $\hat{t}_0 = t_0$, but we know from the examples in Section 3 that this is not always true. Even when it is not the case, the theorems in this section hold.

It can be assumed without loss of generality that $\hat{t}_0 = 1$. Indeed, we can redefine

$$\tilde{f}_{(i,t)} := f_{(i,tt_0)}$$

and get $\hat{t}_0 = 1$ for

$$\tilde{F}_t := \{\tilde{f}_{(i,t)} : 1 \leq i \leq N\}.$$ 

Therefore we restrict the parameter $t$ to the closed interval $[0, 1]$ in this section and Section 5.

The following conditions on the one-parameter family $F_t$ on the metric space $(X, d)$ appear in the hypotheses of the results in this section.

(H1) The map $t \mapsto f_{(i,t)}(x) \in X$ is continuous for every $x \in X$ and every $i = 1, \ldots, N$;

(H2) $\text{Lip}(F_t, d) < 1$ for all $t \in [0, 1)$;
(H3) $q_i := \lim_{t \to 1^-} q_{i,t}$ exist for each $1 \leq i \leq N$, where $q_{i,t}$ denotes the unique fixed point of $f_{i,t} : [0,1]$. Define $Q := \{q_i : 1 \leq i \leq N\}$.

**Remark 4.1.** If the contractivity threshold for $F_t$ is $\hat{t}_0 = 1$, then for every $t < 1$ there exists an admissible metric $d_t$ such $\text{Lip}(F_t, d_t) < 1$. The somewhat stronger assumption (H2) states that there is a single metric $d$ such that $\text{Lip}(F_t, d) < 1$.

**Remark 4.2.** If (H1) and (H2) hold, then

(a) Each $F_t : \mathcal{K}(X) \to \mathcal{K}(X)$, $t < 1$, is a Banach contraction in the Hausdorff metric.
(b) $\text{Lip}(F, d) \leq 1$. In particular, the Hutchinson operator $F : \mathcal{K}(X) \to \mathcal{K}(X)$ is nonexpansive in the Hausdorff metric.
(c) The limit point $q_i \in Q$ in (H3) is a fixed point of $f_{i,1}$. One should be aware, however, that $q_i$ is not necessarily a unique fixed point of $f_{i,1}$ (just think of the affine one-parameter family $f_{i,1}(x) = tx$; see also Example 4.2).

### 4.1. The Lower Transition Attractor and Semiattractor.

**Definition 4.1.** The lower transition attractor of $F_t$ is the smallest (with respect to inclusion) set $A_\bullet$, which is $(F_1, Q)$-invariant, i.e., $F_1(A_\bullet) = A_\bullet$ and $A_\bullet \supseteq Q$. (Equivalently, $A_\bullet$ is the smallest set with $F_1(A_\bullet) \cup Q = A_\bullet$; see the first part of proof of Theorem 4.2.)

**Definition 4.2.** Let $F$ be an IFS on a metric space $X$. If the intersection is nonempty, then the **semiattractor** of $F$ is

$$A_\bullet := \bigcap_{x \in X} \text{Li}(F^{(n)}(\{x\})),$$

where $\text{Li}(S_n)$ is the lower Kuratowski limit (13) of a sequence of sets $S_n \subseteq X$, i.e.,

$$\text{Li}(S_n) := \{y \in X : \text{there exist points } x_n \in S_n \text{ such that } x_n \to y\}.$$

Note that a semiattractor can be unbounded, e.g., [17]. The following properties of an IFS with semiattractor $A_\bullet$ hold.

**Theorem 4.1.** If $F$ is an IFS on a complete metric space with semiattractor $A_\bullet$, then

1. $F(A_\bullet) = A_\bullet$; moreover $A_\bullet$ is the smallest $F$-invariant set.
2. If $F$ admits an attractor $A$ with a full basin $X$, then $A_\bullet = A$.

The notion of a semiattractor comes into play in [20, 31], where functions that are not contractions are added to an IFS consisting of contractions. This allows for the use of standard methods for computer drawing of the attractor of contractive IFS.

**Theorem 4.2.** Let $F_t$ be a one-parameter family $F_t$, $t \in [0,1]$, on a complete metric space $(X,d)$ that satisfies (H1), (H2) and (H3). Then the lower transition attractor $A_\bullet$ always exists. Moreover $A_\bullet$ obeys the following properties:

1. $A_\bullet = \bigcap \{A \in 2^X : F_1(A) = A \text{ and } Q \subseteq A\}$.
2. $A_\bullet = \bigcup_{n \geq 0} F_1^n(Q)$.
3. $A_\bullet = \bigcup_{n \geq 0} F_1^n(Q')$, where $Q' = \{q_i : i \in J\}$ and $J \neq \emptyset$ is such that $\{i \in \{1,\ldots,N\} : \text{Lip}(f_{i,1}) = 1\} \subseteq J \subseteq \{1,\ldots,N\}$. In other words, $Q' \subseteq Q$ contains at least the fixed point limits of those functions $f_{i,1}$ that are not a contraction.
4. The lower transition attractor $A_\bullet$ is the semiattractor of any IFS of the form $F_1^n := F_1 \cup \{\tilde{q}(x) : q \in Q'\}$, where $\tilde{q}(x) := q$ is the constant map on $X$.
Proof. Clearly, $F_1^\prime(S) = F_1(S) \cup Q'$ for any nonempty $S \subseteq X$, and $F_1(Q') \subseteq Q'$. First note that the set $A$ is the smallest $F_1$-invariant set if and only if $A$ is the smallest $F_1$-invariant set which contains $Q'$. Indeed, $A = F_1^\prime(A) = F_1(A) \cup Q'$ implies $F_1(A) \subseteq A$ and $A = A \cup Q' \supseteq Q'$. Hence $F_1(A) = F_1(A \cup Q') = F_1(A) \cup F_1(Q') \supseteq F_1(A) \cup Q' = A$. In the reverse direction, if $A = F_1(S)$ and $A \supseteq Q'$, then $F_1^\prime(A) = F_1(A) \cup Q' = A \cup Q' = A$.

Second, observe that the subsystem $\{q_i : q \in Q'\} \subseteq F_1^\prime$ consists of contractions and admits a semi-attractor (even attractor), which is $Q'$. Hence, by the Lasota–Myjak criterion ([26] Theorem 6.3), $F_1^\prime$ admits a semi-attractor, denoted $A'$. Furthermore, since $A' \supseteq Q'$ and $(F_1^\prime)^n(Q') = F_1^n(Q')$, we have $A' = \bigcup_{n \geq 0} F_1^n(Q')$ due to the self-regeneration formula in the Lasota–Myjak criterion ([24] Theorem 6.3 eq. (6.9)). In particular, the above is true for $Q' = Q$, in which case we write $A_\ast$ for the semi-attractor.

We have established the existence of a lower transition attractor, which is $A_\ast = A_\ast$, and properties (i) and (ii).

Third, we shall establish that all $A'$ are equal to $A_\ast$. This will give the representation of $A_\ast$ as a semi-attractor of any $F_1^\prime$, and in turn property (iii). Of course $A_\ast \subseteq A_\ast$. Consider $q_i = f_{i,1}(q_i)$ with $i \not\in J$. Since $\{q_i\}$ is the attractor of the subsystem $\{f_{i,1}\} \subseteq F_1$, we have $q_i \in A'$. Overall $Q \subseteq A'$ and $A \subseteq A'$.

Under mild additional conditions on $F_1$, the lower transition attractor is compact. See Corollary 4.1 and Remark 4.3 below. These results require extending some concepts defined in Section 1 to infinite IFSs, e.g., ([23] 23]. Let $F$ be a finite or infinite IFS on a complete metric space $X$. The Hutchinson operator on $X$ induced by $F$ is the operator $F : 2^X \to 2^X$ acting on the power set of $X$ and given by the formula

$$F(S) := \bigcup_{f \in F} f(S)$$

for all $S \subseteq X$. Note that, for a finite IFS, the closure can be dropped if $S$ is compact. An IFS $F$ on $X$ will be called compact if $F(K)$ is compact for every compact set $K \subseteq X$. Clearly, any finite IFS is compact.

Given an IFS $F$ on $X$, the monoid induced by $F$ is

$$M(F) := \{f_1 \circ \cdots \circ f_k : f_1, \ldots, f_k \in F, k \in \mathbb{N}\} \cup \{\text{id}_X\}.$$ 

A monoid can be treated as a new IFS. In particular, we may speak of a compact monoid.

Corollary 4.1. Let $F_1$ be as in Theorem 4.2 and let $J = \{1 \leq i \leq N : \text{Lip}(f_{i,1}, d) = 1\}$. If the monoid $M\{f_{i,1} : i \in J\}$ is compact, then the lower transition attractor $A_\ast$ of $F_1$ is compact.

Proof. The statement follows from Theorem 4.2 (iv) and from ([31] Theorems 4.1).

Remark 4.3. If either of the following two conditions hold, then the monoid $M\{f_{i,1} : i \in J\}$ is compact.

- $J = \{i_\ast\}$ for some $i_\ast \in \{1, \ldots, N\}$, and $f_{i_\ast,1}$ is a periodic isometry, cf. ([20]);
- $X$ is proper and all $f_{i,1}$, $i \in J$, have a common fixed point (not necessarily unique), cf. ([31] Theorem 4.2 (ii), Lemma 2.2 item 3).

The compactness of the lower transition attractor $A_\ast$ in Corollary 4.1 cannot be inferred from (H1), (H2), and (H3) alone. Example 4.1 below is a counterexample.

Example 4.1. [A one-parameter family satisfying (H1), (H2), and (H3) whose lower transition attractor is not compact.]

On $\mathbb{R}$ let $F_t := \{g_t, f_t\}$, where $g_t(x) = -tx$ and $f_t(x) = -tx + t + 1$. For $t \in (0, 1)$ we have $A_t = [-t/(1 - t), 1/(1 - t)]$. In this case $A_\ast = \mathbb{R}$.

Example 4.2 below is a 3-dimensional example illustrating the previous results in this section.
Example 4.2. In $\mathbb{R}^3$ let $F_t = \{f_{i,t}, 1 \leq i \leq 5\}$ be the one-parameter affine family where $f_{i,t}(v) = t L_i(v - q_i) + q_i$, and

$$L_1 = L_2 = L_3 = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

The map $L_4$ is the rotation by $\pi/2$ about $y$-axis, and $L_5$ is the reflection in the $xz$-plane. The fixed points are

$$q_i = \left( \cos \frac{2\pi(i - 1)}{3}, \sin \frac{2\pi(i - 1)}{3}, 0 \right) \text{ for } i = 1, 2, 3, \quad q_4 = (0, 1, 0), \quad q_5 = (0, 0, 1),$$

where $q_1, q_2, q_3$ are the third roots of unity in the $xy$-plane. Note that the attractor of the IFS $\{f_{(i,1)}, 1 \leq i \leq 3\}$ is the Sierpiński triangle in the $xy$-plane with vertices $q_1, q_2, q_3$. For each $1 \leq i \leq 5$, the point $q_i$ is a common fixed point of $f_{(i,t)}$ for $t \in [0,1]$. However, $q_i$ is not the only fixed point of $f_{(i,1)}$ for $i = 4, 5$. More precisely, $f_{(4,1)}$ has the whole $y$-axis as its set of fixed points; $f_{(5,1)}$ has the whole $xz$-plane as its set of fixed points; and $(0,0,0) \neq q_4, q_5$ is the only common fixed point of $f_{(4,1)}$ and $f_{(5,1)}$.

On the left in Figure 6 is the attractor $A_t$ of $F_t$ for $t = 0.8$. By Theorem 4.2 the lower transition attractor $A_\bullet$ for IFS family $F_t$ of Example 4.2 exists; it appears on the right in Figure 6. By Corollary 4.1 $A_\bullet$ is compact, the relevant monoid being finite. Figure 6 was generated using Mekhontsev’s IFSStile program [24]. To draw $A_\bullet$ using this program we have applied part (iv) of Theorem 4.2 which identifies $A_\bullet$ as a semiattractor of a suitable IFS $F_t^\bullet := \{f_{(1,1)}, f_{(2,1)}, f_{(3,1)}, f_{(4,1)}, f_{(5,1)}, q_4, q_5\}$ related to $F_t$. Then the resulting IFS $F_t^\bullet$ was replaced with a contractive IFS according to [31] Theorem 4.1 (B)].

![Figure 6](image)

**Figure 6.** The attractor $A_t$ for the one-parameter affine family $F_t$ of Example 4.2 for parameter value $t = .8$ and the lower transition attractor $A_\bullet$ of $F_t$.

4.2. The Upper Transition Attractor.

**Definition 4.3.** Call a compact set $A^\bullet$ an upper transition attractor of a one-parameter IFS family $F_t := \{f_{i,t}, f_{j,t}, \ldots, f_{N,t}\}$, $t \in [0,1]$, if there is an increasing sequence $t_n \to 1$ such that

$$A^\bullet = \lim_{n \to \infty} A_{t_n}. $$

Theorem 4.3 Theorem 4.4 and Proposition 4.1 below are strong versions of results on upper transition attractors and their relation to the lower transition attractor that were proved in [33] only for special cases of one-parameter similarity families.
Lemma 4.1. Assume that \( C \) is proper, \( f \) satisfies (H2) and, for each \( 1 \leq i \leq N \), the map \([0,1] \ni t \mapsto f_{i,t} \in C(X)\) is continuous with respect to the topology of uniform convergence in \( C(X) \). Then \( F \) admits at least one upper transition attractor.

To prove this theorem we need the following lemma.

Lemma 4.1. Assume that \((f_n)\) is a sequence of contractions on a complete metric space \((X,d)\), uniformly convergent to some function \( f \). Then the set of fixed points of maps \( f_n, n \in \mathbb{N} \), is bounded.

Proof. Let \( d_{\sup}(f,g) := \sup_{x \in X} d(f(x), g(x)) \) for \( f,g : X \to X \). For \( n \in \mathbb{N} \), let \( x_n \) be the fixed point of \( f_n \). Fix an \( n_0 \in \mathbb{N} \) such that \( d_{\sup}(f_n, f) < 1 \) for all \( n \geq n_0 \). For every \( n \geq n_0 \), we have

\[
d(x_n, x_{n_0}) \leq d(f_n(x_n), x_{n_0}) + d(x_{n_0}, f_n(x_{n_0})) \\
\leq d_{\sup}(f_n, f) d(x_n, x_{n_0}) \\
\leq d_{\sup}(f_n, f) + d_{\sup}(f) d(x_n, x_{n_0}).
\]

Hence

\[
d(x_n, x_{n_0}) \leq \frac{d_{\sup}(f_n, f) + d_{\sup}(f) d(x_n, x_{n_0})}{1 - \text{Lip}(f)} \leq \frac{2}{1 - \text{Lip}(f)}.
\]

Therefore

\[
diam\{x_n : n \in \mathbb{N}\} \leq 2 \max\{d(x_1, x_{n_0}), ..., d(x_{n_0-1}, x_{n_0})\} \frac{2}{1 - \text{Lip}(f)} < \infty.
\]

Proof of Theorem 4.3. Suppose that the assertion does not hold. Then we can find a convergent sequence \((t_n) \subset [0,1]\) so that the family \( A_{t_n}, n \in \mathbb{N} \), of attractors of \( F_{t_n} \), are not all included in some bounded set. In other words, the set \( \{A_{t_n} : n \in \mathbb{N}\} \) is not bounded in \( K(X) \). Let \( t = \lim_{n \to \infty} t_n \). Now observe that for every compact set \( K \subset K(X) \), we have

\[
h(F_{t_n}(K), F_t(K)) = h\left( \bigcup_{i=1}^{N} f_{i,t_n}(K), \bigcup_{i=1}^{N} f_{i,t}(K) \right) \\
\leq \max\{h(\bigcup_{i=1}^{N} f_{i,t_n}(K), f_{i,t}(K)) : i = 1, ..., N\} \\
\leq \max\{d_{\sup}(f_{i,t_n}, f_i) : i = 1, ..., N\}
\]

Hence

\[
sup\{h(F_{t_n}(K), F_t(K)) : K \subset K(X)\} \leq \max\{d_{\sup}(f_{i,t_n}, f_i) : i = 1, ..., N\} \to 0.
\]

Therefore the assumptions of Lemma 4.1 are satisfied (for a family of Hutchinson operators) and the family \( \{A_{t_n} : n \in \mathbb{N}\} \) is bounded in \( K(X) \), a contradiction.

The existence of an upper transition attractor in Theorem 4.3 cannot be inferred from (H1), (H2), and (H3) alone; see Example 4.1. Neither can it be inferred from (H1) and the assumption that all maps in \( F_t \) are contractions for all \( t \in [0,1] \); see Example 4.3 below.

Example 4.3. Motivated by the construction in [27 Example 1], we will construct a one-parameter family of IFSs \( F_t, t \in [0,1] \), with the following properties:

(a) \( F_t \) satisfies (H1);
(b) for all \( t \in [0,1] \) all maps in \( F_t \) are contractions, in particular \( F_t \) satisfies (H2);
(c) \( F_t \) has no upper transition attractor.

Let \( \ell^1 \) be the Banach space of absolutely convergent sequences of real numbers. We will construct a function \( f_t : \ell^1 \to \ell^1 \) such that the one-parameter family \( F_t = \{f_t\} \), consisting of a single function, will satisfy the properties (a), (b), (c) above. For each \( t \in [0,1] \) the function \( f_t \) will have the form

\[
f_t(x) = \begin{cases} 
\frac{t}{\phi(t)} \cdot z_t + (1-t)z_t & \text{if } t < 1 \\
0 & \text{if } t = 1,
\end{cases}
\]
where $0$ is the sequence of zeros and the linear functional $\phi_t: \ell_1 \to \mathbb{R}, t \in [0, 1]$, has the form

$$
\phi_t(x) = \begin{cases} 
\alpha(t)x_n + \beta(t)x_{n+1} & \text{if } t < 1 \\
0 & \text{if } t = 1,
\end{cases}
$$

where we use the notation $x = (x_n) \in \ell^1$. It remains to define $z_t \in \ell^1$ for each $t \in [0, 1]$, the functions $\alpha, \beta: [0, 1) \to [0, 1]$, and the integer $n_t$ for all $t < 1$, and to show that properties (a), (b), (c) hold for $F_t$.

To define $\alpha, \beta$ and $n_t$, choose any increasing sequence $(a_n)$ of real numbers tending to 1 and such that $a_0 = 0$. For each $t \in [0, 1)$, find $n_t \in \{0, 1, 2, \ldots\}$ so that $a_{n_t} \leq t < a_{n_t+1}$. Clearly, for any $t \in [0, 1)$, we have that $t \in [a_n, a_{n+1})$ if and only if $n_t = n$. Now choose maps $\alpha, \beta, c: [0, 1) \to [0, 1]$ which satisfy the following conditions:

(i) $\alpha, \beta, c$ are right continuous on $[0, 1)$;
(ii) $\alpha, \beta, c$ are continuous on each interval $(a_n, a_{n+1}), n \in \mathbb{N} \cup \{0\}$;
(iii) for any $n \in \mathbb{N} \cup \{0\}$, we have that
\begin{align*}
(iii_a) \quad & \alpha(a_n) = 1 \text{ and } \lim_{t \to a_{n+1}}^+ \alpha(t) = 0; \\
(iii_b) \quad & \beta(a_n) = 0 \text{ and } \lim_{t \to a_{n+1}^-} \beta(t) = 1; \\
(iii_c) \quad & c(a_n) = 1 \text{ and } \lim_{t \to a_{n+1}^-} c(t) = 0; \\
(iv) \quad & \max\{\alpha(t), \beta(t)\} = 1 \text{ for all } t \in [0, 1); \\
(v) \quad & c(t)\alpha(t) + (1 - c(t))\beta(t) > t \text{ for all } t \in [0, 1).
\end{align*}

The choice of maps $\alpha, \beta, c$ is possible. For example, $\alpha$ can be constant 1 on each interval $[a_n, \frac{1}{2}(a_n + a_{n+1})]$ and affine on $[\frac{1}{2}(a_n + a_{n+1}), a_{n+1})$. Similarly $\beta$ can be affine on each interval $[a_n, \frac{1}{2}(a_n + a_{n+1})]$ and constant 1 on $[\frac{1}{2}(a_n + a_{n+1}), a_{n+1})$. Finally, $c$ can be constant 1 on $[a_n, \frac{1}{2}(a_n + a_{n+1}) - \xi]$, constant 0 on $[\frac{1}{2}(a_n + a_{n+1}) + \xi, a_{n+1}]$ and affine on $[\frac{1}{2}(a_n + a_{n+1}) - \xi, \frac{1}{2}(a_n + a_{n+1}) + \xi]$, where $\xi > 0$ is sufficiently small (for example, $\xi = \frac{1}{2}(a_{n+1} - a_n)(1 - a_{n+1})$).

Graphs of $\alpha, \beta$ and $c$ are illustrated in Figure 7.

![Figure 7. The graphs of $\alpha$, $\beta$ and $c$](image-url)

To define $z_t$, use (iv) and the classical correspondence between linear functionals on $\ell_1$ and the space $\ell_\infty$ to obtain

$$
(4.1) \quad ||\phi_t|| = \left\| \left(0, \ldots, 0, \alpha(t), \beta(t), 0, \ldots \right) \right\|_\infty = \max\{||\alpha(t)||, ||\beta(t)||\} = 1
$$

for all $t \in [0, 1)$. Now fix any $x \in \ell_1$ and define the map $g_x: [0, 1] \to \mathbb{R}$ by

$$
g_x(t) := \phi_t(x).
$$

Next we show that $g_x$ is continuous. By (i) and (ii) we see that $g_x$ is right continuous on the whole interval $[0, 1)$ and continuous on each interval $(a_n, a_{n+1}), n \in \mathbb{N} \cup \{0\}$. Using (iiiia) and (iiib), for $n \geq 1$ we have

$$
\lim_{t \to a_n} g_x(t) = \lim_{t \to a_n} \left(\alpha(t)x_{n-1} + \beta(t)x_n\right) = x_n = g_x(a_n),
$$

where $x_n = x_{n-1} + \alpha(t)x_{n-1} + \beta(t)x_n$. Therefore, $g_x$ is continuous on $[a_n, a_{n+1})$. Now fix $t \in (a_{n+1}, a_{n+2})$ and consider the sequence of partial sums $S_n(t)$ of the series $\sum_{k=n}^{\infty} g_x(\gamma_n) x_k$ where $\gamma_n < t$. For $n \geq 1$, we have

$$
S_n(t) = \sum_{k=n}^{\infty} g_x(\gamma_n) x_k = \sum_{k=n}^{\infty} \left(\alpha(\gamma_n)x_{k-1} + \beta(\gamma_n)x_k\right).
$$

Since $g_x(\gamma_n)$ is continuous at $a_n$, we have

$$
\lim_{n \to \infty} S_n(t) = \lim_{n \to \infty} \sum_{k=n}^{\infty} \left(\alpha(\gamma_n)x_{k-1} + \beta(\gamma_n)x_k\right) = \sum_{k=0}^{\infty} \left(\alpha(t)x_k + \beta(t)x_{k+1}\right) = g_x(t).
$$

Therefore, $g_x$ is continuous on $[0, 1)$.
which gives left continuity at \( a_n \) and, in consequence, its continuity at \( a_n \). Finally, we observe that \( g_\varepsilon \) is continuous at 1:

\[
0 \leq \lim_{t \to 1} |g_\varepsilon(t)| \leq \lim_{n \to \infty} (|x_n| + |x_{n+1}|) = 0 = \phi_1(x).
\]

For \( t \in [0,1) \) define

\[
z_t := c(t)e_{n_t} + (1 - c(t))e_{n_t+1} \in \ell_1
\]

where \( e_n \) is the \( n \)-th unit vector in \( \ell_1 \). Note that, by (v), we have that

\[
\phi_t(z_t) = \alpha(t)c(t) + (1 - c(t))\beta(t) > t.
\]

We now verify statement (a), that

\[
[0, 1] \ni t \mapsto f_t(x)
\]

is continuous. As was shown for \( g_\varepsilon \), using (i), (ii) and (iii) we see that the map

\[
[0, 1] \ni t \mapsto \phi_\varepsilon(z_t) \in \mathbb{R}
\]

is continuous. The continuity of \([0, 1] \ni t \mapsto f_t(x)\) follows easily from the continuity of the maps \( t \mapsto \phi_t(x) \) and \( t \mapsto \phi_\varepsilon(z_t) \) for \( t \in [0, 1) \), which were observed earlier, and the continuity of the map \([0, 1] \ni t \mapsto z_t \in \ell_1 \) that can be proved in a similar way. Furthermore, since \( \phi_\varepsilon(x) \to \phi_\varepsilon(0) = 0 \) and \( 1 - t \to 0 \) when \( t \to 1 \), we have

\[
\|f_\varepsilon(x)\| \leq \frac{t}{\phi_\varepsilon(z_t)} \cdot |\phi_\varepsilon(x)||z_t| + (1 - t)||z_t|| \leq |\phi_\varepsilon(x)| + (1 - t) \to 0,
\]

when \( t \to 1 \). Hence the map \([0, 1] \ni t \mapsto f_t(x)\) is also continuous at 1.

We next verify statement (b), that \( f_t \) is a contraction for \( t < 1 \). We have

\[
\|f_t(x) - f_t(y)\| = \frac{t}{\phi_\varepsilon(z_t)} |\phi_t(x) - \phi_t(y)| \cdot ||z_t|| \leq \frac{t}{\phi_\varepsilon(z_t)} ||\phi_t|| \cdot ||x - y||.
\]

Moreover \( \frac{t}{\phi_\varepsilon(z_t)} < 1 \) from (4.2) and \( ||\phi_t|| = 1 \) from (4.1).

It only remains to check property (c), that \( F_t \) has no upper transition attractor. We have

\[
f_t(z_t) = t \frac{\phi_t(z_t)}{\phi_\varepsilon(z_t)} z_t + (1 - t)z_t = z_t
\]

for \( t \in [0,1) \). Therefore \( z_t \) is a unique fixed point of \( f_t \); in particular, \( A_{F_t} = \{z_t\} \). On the other hand, the unique fixed point of \( f_1 \) is clearly the zero sequence \( 0 \), so \( \{0\} \) is the only candidate for an upper transition attractor of \( F_t \) (see Theorem 4.4 below). However, for \( t < 1 \) we have

\[
||z_t - 0|| = ||z_t|| = 1.
\]

Thus all three properties (a), (b), (c) of our example have been verified.

For Theorem 4.4 below and in Section 5 we will need the following technical lemma.

**Lemma 4.2.** Let \( X \) be a metric space and let \( f_t, \ t \in [0,1], \) be a family of nonexpansive selfmaps of \( X \) such that for every \( x \in X \), the map \([0,1] \ni t \mapsto f_t(x)\) is continuous. Then for every nonempty and compact set \( D \subseteq X \),

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall s, t \in [0,1] \quad (|s - t| < \delta \ \Rightarrow \ \sup_{x \in D} d(f_s(x), f_t(x)) \leq \varepsilon).
\]

In particular, for every nonempty and compact set \( D \subseteq X \), the map

\[
[0, 1] \ni t \mapsto f_t(D) \in \mathcal{K}(X)
\]

is uniformly continuous.
Proof. Assume first that the set $D$ is finite. Fix $\varepsilon > 0$. Then for every $t \in [0,1]$, we can find $\delta_t > 0$ such that for every $s \in [0,1]$ with $|s - t| < \delta_t$ we have
\begin{equation} \label{equation4.3}
\sup_{x \in D} d(f_s(x), f_t(x)) < \frac{\varepsilon}{2}.
\end{equation}
The choice of $\delta_t$ is possible since $D$ is finite and the map $t \mapsto f_t(x)$ is continuous for every $x \in D$. Since $[0,1]$ is compact, we can choose a finite subcover of the open cover $\left( t - \frac{\delta_t}{2}, t + \frac{\delta_t}{2} \right)$, $t \in [0,1]$.

Let $\left( t_i - \frac{\delta_i}{2}, t_i + \frac{\delta_i}{2} \right)$, $i = 1, ..., k$, be this subcover and choose
$$\delta := \frac{1}{2} \min\{\delta_i : i = 1, ..., k\}.$$ 
Now let $s, t \in [0,1]$ be such that $|s - t| < \delta$. By the choice of $t_1, ..., t_k$, we can find $i = 1, ..., k$ so that
$$|t - t_i| < \frac{\delta_i}{2}.$$ 
Then also
$$|s - t_i| \leq |s - t| + |t - t_i| < \delta + \frac{\delta_i}{2} \leq \delta_i,.$$ 
Hence by (4.3), for every $x \in D$, we have
$$d(f_s(x), f_t(x)) \leq d(f_s(x), f_{t_i}(x)) + d(f_{t_i}(x), f_t(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
and
$$\sup_{x \in D} d(f_s(x), f_t(x)) \leq \varepsilon.$$
Now assume that $D$ is nonempty and compact. Take any $\varepsilon > 0$, and find a finite set $D' \subseteq D$ so that the Hausdorff distance $h(D', D) < \frac{\varepsilon}{3}$. By previous considerations, there exists $\delta > 0$ such that if $|s - t| < \delta$, then
$$\sup_{x \in D'} d(f_s(x), f_t(x)) \leq \frac{\varepsilon}{3}.$$ 
If $x \in D$, then we can find $x' \in D'$ so that $d(x, x') < \frac{\varepsilon}{3}$, and thus
$$d(f_s(x), f_t(x)) \leq d(f_s(x), f_{t'}(x')) + d(f_{t'}(x'), f_t(x')) + d(f_t(x'), f_s(x)) \leq 2d(x, x') + \frac{\varepsilon}{3} \leq \varepsilon.$$ 
Therefore
$$\sup_{x \in D} d(f_s(x), f_t(x)) \leq \varepsilon.$$ 

In what follows, we denote the Hausdorff distance by $h$.

**Theorem 4.4.** Let $F_t$, $t \in [0,1]$, satisfy (H1) and (H2). If $A^*$ is any upper transition attractor of $F_t$, then

(i) $F_1(A^*) = A^*$.

If, in addition, $F_t$ satisfies (H3), then

(ii) $A^* \supseteq Q$, in particular $A^* \supseteq A_*$, where $A_*$ is the lower transition attractor of $F_t$ and $Q$ is the set of limit fixed points from (H3).

**Proof.** Let $t_n \to 1$ be such that $A_{t_n} \to A^*$ with respect to $h$ as $n \to \infty$. To establish (i) recall that each $F_{t_n}$ and $F_1$ are nonexpansive with respect to $h$ (part (b) of Remark 4.2). Furthermore, according to Lemma 4.2 we have
\begin{equation} \label{equation4.4}
\max_{1 \leq i \leq N} h(f_{(i,1)}(A^*), f_{(i,t_n)}(A^*)) \to 0.
\end{equation}
Hence, by using $F_t(A_{t_n}) = A_{t_n}$ we get
\[
h(F_1(A^*), A^*) \leq h(F_1(A^*), F_t(A^*)) + h(F_t(A^*), F_t(A_{t_n})) + h(F_{t_n}(A_{t_n}), A^*) \\
\leq h(F_1(A^*), F_t(A^*)) + 2h(A_{t_n}, A^*) \to 0. \]

Now we establish (ii). Observe that $q_{i, t_n} \in A_{t_n} \to A^*$, and $q_{i, t_n} \to q_i \in Q$ as $n \to \infty$, Thus $Q \subseteq A^*$. Hence $A^*$ is $(F_1, Q)$-invariant, and therefore it contains $A_*$. \hfill \Box

**Remark 4.4.** Assuming (H1), (H2) and (H3), $A_*$ is compact whenever $A^*$ exists.

**Proposition 4.1.** Assume that $F_t$, $t \in [0, 1]$, satisfies (H1) and (H2). Let $f_{(i_*, 1)}$ be an isometry for some $i_* \in \{1, \ldots, N\}$.

(a) If there exists an upper transition attractor $A^*$, then it is $f_{(i_*, 1)}$-symmetric, i.e., $f_{(i_*, 1)}(A^*) = A^*$.

(b) If there exists a lower transition attractor $A_*$ that is compact, then it is $f_{(i_*, 1)}$-symmetric.

**Proof.** Observe that $f_{(i_*, 1)}(A^*) \subseteq A^*$. Then the isometry $f_{(i_*, 1)}$ is surjective on compactum $A^*$. Analogously for $A_*$. \hfill \Box

5. **The Existence of a Unique Upper Transition Attractor**

This section addresses Question 1.1 in the introduction. Theorem 5.2 below gives an affirmative answer for a large class of one-parameter IFS families.

We start with Lemma 5.1 below for infinite IFSs, which is already known for finite IFSs. Here the Hutchinson operator $F: 2^X \to 2^X$ is as defined in Section 4.

**Definition 5.1.** For a finite or infinite IFS $F$, a nonempty compact set $A$ is a **Hutchinson attractor** on a complete metric space $X$ if
- (invariance) $F(A) = A$, and
- (attraction) $A = \lim_{n \to \infty} F^n(S)$,

for every nonempty closed and bounded set $S \subseteq X$, the limit with respect to the Hausdorff metric. Note that a Hutchinson attractor, if it exists, is unique.

A generalization of the Hutchinson theorem is the following (see [31] and the references therein):

**Theorem 5.1.** If an IFS $F$ on $(X, d)$ satisfies $\sup_{f \in F} \text{Lip}(f, d) < 1$ and is compact, then it admits a Hutchinson attractor.

Roughly speaking, Lemma 5.1 says that, if compact IFSs $F, G$ are close to each other on a bounded subinvariant set, in the sense that each map $f$ from $F$ has a close neighbour $g \in G$, and vice-versa, then attractors of $F$ and $G$ are also close.

**Lemma 5.1.** Let $G := \{g_i : i \in I\}$ and $H := \{h_j : j \in J\}$ be two compact IFSs on a complete metric space $(X, d)$ such that $\text{Lip}(G, d) < 1$ and $\text{Lip}(H, d) < 1$. Let $B \subseteq X$ be a compact set such that $G(B) \subseteq B$ and $H(B) \subseteq B$, and let $\delta > 0$ satisfy
\[
\forall i \in I \exists j \in J \forall x \in B \ d(g_i(x), h_j(x)) \leq \delta \quad \text{and} \quad \forall j \in J \exists i \in I \forall x \in B \ d(g_i(x), h_j(x)) \leq \delta.
\]
Then
\[
h(A_G, A_H) \leq \frac{\delta}{1 - \min\{\text{Lip}(G, d), \text{Lip}(H, d)\}},
\]
where $A_G$ and $A_H$ are the Hutchinson attractors of $G$ and $H$, respectively.

**Remark 5.1.** Given two compact IFSs $G$ and $H$ with attractors $A_G$ and $A_H$, there always exists a nonempty compact $B \subseteq X$ such that $G(B) \subseteq B$ and $H(B) \subseteq B$. Indeed, since $G$ and $H$
are compact, the IFS $F \cup G$ is also compact, hence admits the attractor $A_{G \cup H}$. Furthermore, for any nonempty compact set $D \subseteq X$, the set
\[ B := \text{cl} \left( D \cup \bigcup_{n \in \mathbb{N}} (G \cup H)^{(n)}(D) \right) = A_{G \cup H} \cup D \cup \bigcup_{n \in \mathbb{N}} (G \cup H)^{(n)}(D) \]
is compact, and $G(B) \cup H(B) \subseteq B$.

**Proof.** (Of Lemma 5.1) By (5.1), we can easily see that for any compact $D \subseteq B$,
\[ h(G(D), H(D)) \leq \delta. \]
Without loss of generality suppose $\alpha = \text{Lip}(G, d) \leq \text{Lip}(H, d)$. We will check inductively that for every $n \in \mathbb{N}$,
\[ h(G^{(n)}(B), H^{(n)}(B)) \leq \delta \sum_{k=0}^{n-1} \alpha^k. \]
The case $n = 1$ of (5.3) is exactly (5.2) for $D := B$. Assume that the inequality (5.3) holds for some $n \in \mathbb{N}$. Then
\[ h(G^{(n+1)}(B), H^{(n+1)}(B)) \leq h(G(G^{(n)}(B)), G^{(n)}(H(B))) + h(G^{(n)}(H(B)), H^{(n)}(B))) \leq \alpha h(G^{(n)}(B), H^{(n)}(B)) + \delta \leq \alpha \delta \sum_{k=0}^{n-1} \alpha^k + \delta = \delta \sum_{k=0}^{n} \alpha^k, \]
where the penultimate inequality follows from (5.2) for $D := H^{(n)}(B)$, and the last inequality uses (5.3) for $n$. Thus (5.3) is true for $n + 1$. Now from (5.3) and the convergence of the Hutchinson iterates to the attractor, we get
\[ h(A_G, A_H) \leq \delta \sum_{k=0}^{\infty} \alpha^k = \frac{\delta}{1 - \alpha}. \]
This completes the proof. \[\square\]

**Lemma 5.2.** Let $X$ be a metric space and $f_t, t \in [0, 1]$, be a family of nonexpansive selfmaps of $X$ such that the map $[0, 1] \ni t \mapsto f_t(x)$ is continuous for every $x \in X$. Then the IFS $F := \{f_t : t \in [0, 1]\}$ is compact.

**Proof.** Take any nonempty and compact set $D \subseteq X$. By Lemma 4.2 the map $[0, 1] \ni t \mapsto f_t(D) \in K(X)$ is continuous. This implies that
\[ F(D) = \bigcup_{t \in [0, 1]} f_t(D) = \bigcup_{t \in [0, 1]} f_t(D) \]
is compact thanks to [13] Corollary 2.20 chap. 2.1 p.42 and Theorem 2.68 chap. 2.2 p. 62]. \[\square\]

Recall that any surjective isometry $g : X \to X$ of a real normed space is of the following form:
\[ g(x) = \hat{g}(x) + b = \hat{g}(x - x_*), \]
where $\hat{g} : X \to X$ is a linear isometry, $b = g(0) \in X$ and $x_* = g^{-1}(0)$ (cf. [11] chap.1.3, Mazur–Ulam theorem).

**Lemma 5.3.** Let $X$ be a real Banach space; let $g : X \to X$ be a surjective isometry; let $x_* = g^{-1}(0)$; and let $\hat{g}$ be the linear part of $g$. For $t \in [0, 1]$, set
\[ g_t(x) := tg(x) + x_*, \ x \in X. \]
The following statements hold:
(a) For every $m \in \mathbb{N}$, $t_1, \ldots, t_m \in [0, 1]$ and for all $x \in X$, we have
\[ g_{t_1} \circ \cdots \circ g_{t_m}(x) = t_1 \cdots t_m \hat{g}^{(m)}(x - x_*) + x_*. \]
(b) $g_1$ is periodic if and only if $\hat{g}$ is periodic, and their periods are the same.
(c) If \( g_1 \) is periodic, then the monoid generated by the IFS \( G := \{ g_t : t \in [0, 1] \} \) is compact.

**Proof.** By the preceding observations concerning surjective isometries, we have
\[
g_{t_1}(x) = t_1 \hat{g}(x - x_*) + x_*
\]
which gives us (a) for \( m = 1 \). Suppose that (a) is true for some \( m \in \mathbb{N} \). Then we have
\[
g_{t_1} \circ \ldots \circ g_{t_m} \circ g_{t_{m+1}}(x) = t_1 \cdots t_m \hat{g}^{(m)}(t_{m+1} \hat{g}(x - x_*) + x_* - x_*) + x_*
\]
so we obtain (a) for \( m + 1 \). This ends the proof of (a).

By (a), for every \( m \in \mathbb{N} \) and \( x \in \mathfrak{X} \), we have
\[
g_{t_1}^{(m)}(x) - x_* = \hat{g}^{(m)}(x - x_*)
\]
Hence if \( g_1^{(m)} = \text{id}_\mathfrak{X} \), then also \( \hat{g}^{(m)} = \text{id}_\mathfrak{X} \), and vice-versa. Thus (b) is true.

Now we prove (c). By (a), each element of the desired monoid \( \mathcal{M}(G) \), distinct from the identity map, is of the form
\[
g_{t_1} \circ \ldots \circ g_{t_m}(x) = t_1 \cdots t_m \hat{g}^{(m)}(x - x_*) + x_* = t_1 \cdots t_m \hat{g}^{(m)}(x - x_*) + x_* = g_{t_1}^{(i)}(x)
\]
for some \( i = 1, \ldots, p \) where \( p \) is the period of \( \hat{g} \) and \( t := \sqrt{t_1 \cdots t_m} \). Hence
\[
\mathcal{M}(G) = \{ g_{t_1}^{(i)} : i = 1, \ldots, p, \; t \in [0, 1] \}
\]
(note that \( g_1^{(i)} = \text{id}_\mathfrak{X} \)). In particular, \( \mathcal{M}(G) \) is the finite union of IFSs \( \{ g_{t_1}^{(i)} : t \in [0, 1] \} \) over \( i = 1, \ldots, p \), which are compact in view of Lemma \ref{lem:compactIFSs}. Thus \( \mathcal{M}(G) \) itself is compact. \( \square \)

We now state the main result of this section, which shows that quite a wide class of IFS families possess a unique upper transition attractor. Note that statement (b) in Lemma \ref{lem:periodicIFS} is intended to clarify the assumption on periodicity of the linear part of \( g \) in Theorem \ref{thm:mainTheorem}.

**Theorem 5.2.** Let \( \mathfrak{X} \) be a real Banach space and let \( g : \mathfrak{X} \to \mathfrak{X} \) be a surjective isometry on \( \mathfrak{X} \) with periodic linear part. Consider the one-parameter family
\[
F_t^g := F_t \cup \{ g_t \}
\]
on \( \mathfrak{X} \) with \( t \in [0, 1] \), where
\[
F_t := \{ f_{(i,t)} : 1 \leq i \leq N \}
\]
and
\[
g_t(x) := tg(x) + x_*,
\]
where \( x_* = g^{-1}(0) \) or, equivalently, \( g_t(x_*) = x_* \) for all \( t \). Assume that \( F_t \) satisfies:

(i) for any \( i = 1, \ldots, N \) and \( x \in \mathfrak{X} \), the map \( [0, 1] \ni t \mapsto f_{(i,t)}(x) \) is continuous, and

(ii) \( \sup \{ \text{Lip}(F_t, || \cdot ||) : t \in [0, 1] \} < 1 \).

Then \( t_0 = 1 \) is a threshold for the one-parameter family \( F_t^g \), and \( F_t^g \) has a unique upper transition attractor.

**Remark 5.2.** Two comments before the proof:

First, Examples \ref{ex:nonThreshold} \ref{ex:threshold} \ref{ex:nonThreshold2} and \ref{ex:threshold2} below show that the assumptions in the hypothesis of Theorem \ref{thm:mainTheorem} are essential.

Second, that the upper transition attractor in \ref{prop:upperTransitionAttractor} Proposition 8.1 is unique is a direct consequence of Theorem \ref{thm:mainTheorem}.

**Proof of Theorem 5.2.** Since all functions in \( F_t^g \) are contracts for \( t < 1 \), the one-parameter family \( F_t^g \) has an attractor \( A_t \) for \( t \in [0, 1) \). Since \( g_t \) is a similarity with ratio greater than 1 for \( t > 1 \), the one-parameter family \( F_t^g \) has no attractor for \( t > 1 \). Therefore \( t_0 = 1 \) is a threshold for \( F_t^g \). In other words \( t_0 = 0 \) = 1.
From [33, Proposition 8.1] the existence of a unique upper transition attractor is equivalent to the uniform continuity of the map

\[ [0, 1) \ni t \mapsto A_t \in \mathcal{K}(X). \]

Hence we will prove that this map is uniformly continuous.

Step 1. Finding a nonempty and compact set \( B \) so that

\[ f_{(i,t)}(B) \subseteq B \text{ and } g_t(B) \subseteq B \]

for all \( t \in [0, 1] \) and \( i = 1, ..., N \).

Consider the IFSs

\[
F := \bigcup_{t \in [0,1]} F_t = \{ f_{(i,t)} : i = 1, ..., N, \ t \in [0,1] \} = \bigcup_{i=1,\ldots,N} \{ f_{(i,t)} : \ t \in [0,1] \}
\]

\[
G := \{ g_t : t \in [0,1] \}.
\]

In view of Lemma 5.2, the IFS \( F \) is a finite union of compact IFSs. Hence \( F \) is compact. Also, in view of Lemma 5.3 (c), the monoid \( \mathbb{M}(G) \) is compact. Moreover, \( G \) consists of nonexpansive maps and by assumption (ii) we have \( \text{Lip}(F, \| \cdot \|) < 1 \). Then using [31, Theorem 4.1] (cf. also [31, Remark 2.2]), we see that the IFS \( F \cup G \) has compact semiattractor \( B \). In particular, \( B = F(B) \cup G(B) \).

Therefore (5.4) holds.

Step 2. An alternative description of the attractor \( A_t \) of \( F_t^q \).

Fix a real value \( t \in [0,1) \). Clearly,

\[
\text{Lip}(F_t^q, \| \cdot \|) \leq \max\{ t, \text{Lip}(F, \| \cdot \|) \} < 1.
\]

Hence \( F_t^q \) generates a unique attractor \( A_t \). Again using [31, Theorem 4.1] for IFSs \( F_t \) and \( \{ g_t \} \), we see that \( A_t \) can be viewed as the attractor of the IFS

\[
M_t := \{ g_t^{(m)} \circ f_{(i,t)} : i = 1, ..., N, \ m = 0, 1, 2, ... \}
\]

where \( g_t^{(0)} = \text{id}_X \). Note that the assumptions of [31, Theorem 4.1] will be satisfied if we observe that the monoid

\[
\mathbb{M}(\{ g_t \}) = \{ g_t^{(m)} : m = 0, 1, 2, ... \}
\]

is compact. This is the case as it is a subset of a compact IFS \( \mathbb{M}(G) \) considered in Step 1. (Alternatively, we can observe that \( \mathbb{M}(\{ g_t \}) \) is compact by using the fact \( \text{Lip}(g_t) \leq t < 1 \).)

Moreover, in view of (5.4), we see that \( A_t \subseteq B \).

Step 3. Uniform continuity of the map \([0, t_0] \ni t \mapsto A_t \), where \( t_0 \in [0,1) \).

Fix any \( t_0 \in [0,1) \). Clearly,

\[
\sup\{ \text{Lip}(F_t^q, \| \cdot \|) : t \in [0, t_0] \} \leq \max\{ t_0, \text{Lip}(F, \| \cdot \|) \} < 1.
\]

Hence the assumptions of [15, Theorem 2.6] are satisfied. This means that the map \([0, t_0] \ni t \mapsto A_t \) is continuous. As \([0, t_0] \) is compact, it is uniformly continuous.

Step 4. Uniform continuity of the map \([0,1) \ni t \mapsto A_t \).

The idea in the proof below is that if both \( t, s < 1 \) are appropriately less than 1, then we make use of uniform continuity proved in Step 3, whereas if \( s, t \) are both sufficiently close to 1, then for each map of the form \( g_t^{(m)} \circ f_{(i,t)} \) we will find sufficiently close neighbour \( g_s^{(k)} \circ f_{(i,s)} \) (where
\( k \) will be appropriately chosen, and vice-versa. Then we will make use of Lemma 5.1.

Let \( \hat{g} \) be the linear part of \( g \). Then by Lemma 5.3, we see that for every \( m \in \mathbb{N} \) and \( x \in X \), we have:

\[
    (5.7) \quad g^{(m)}_i(x) = t^m \hat{g}^{(m)}(x - x_*) + x_*. 
\]

Let \( p \) be the period of \( \hat{g} \). Take any \( \varepsilon > 0 \) and choose \( r \in (0, 1) \) such that

\[
    (5.8) \quad (1 - r^p) \cdot (\text{diam}(B \cup \{0\}) + \|x_*\|) < \frac{\varepsilon}{2}. 
\]

Then choose \( \delta > 0 \) such that:

- (a) for \( s, t \in [0, r] \), if \( |t - s| < \delta \), then \( h(A_t, A_s) < \varepsilon \);
- (b) for \( s, t \in [0, 1] \), if \( |t - s| < \delta \), then
  \[
  \sup\{|f_{(i,t)}(x) - f_{(i,s)}(x)| : i = 1, \ldots, N, x \in B\} < \frac{\varepsilon}{2};
\]
- (c) \( (1 - (r - \delta)^p) \cdot (\text{diam}(B \cup \{0\}) + \|x_*\|) \leq \frac{\varepsilon}{2}. \]

The choice of \( \delta \) is possible by Step 3 (for item (a)), by Lemma 4.2 (for item (b)) and condition (5.8) (for item (c)).

Now choose \( s, t \in [0, 1] \) such that \( |s - t| < \delta \). If \( s, t \leq r \), then \( h(A_t, A_s) \leq \varepsilon \) in view of (a). Hence assume that

\[
    (5.9) \quad \max\{s, t\} \geq r. 
\]

Take any \( i = 1, \ldots, N \) and \( m = 0, 1, 2, \ldots \), and let \( m', l' \) be such that \( m = pm' + l' \), and \( l' = 0, \ldots, p - 1 \). Then let \( k' \) be the least nonnegative integer such that

\[
    sp^{k'} + l' \leq tp^{m'} + l' 
\]

and set \( k := pk' + l' \). We will show that

\[
    (5.10) \quad |t^m - s^k| \leq 1 - (r - \delta)^p. 
\]

Using \( s^k \leq t^m < s^{k-p} \), we have

\[
    |t^m - s^k| = t^m - s^k \leq \min\{1, s^{k-p}\} - s^k = \min\{1 - s^k, s^{k-p}(1 - s^p)\} \\
    \leq \left\{ \begin{array}{ll}
    1 - s^{k'} & \text{if } k' = 0 \\
    s^{k-p}(1 - s^p) & \text{if } k' \geq 1
    \end{array} \right. \\
\]

\[
\leq 1 - s^p \leq 1 - (r - \delta)^p, 
\]

where the last inequality follows from \( r - \delta \leq s \) (thanks to (5.9)). Thus we have shown (5.10).

Now fix \( i = 1, \ldots, L \) and choose any \( x \in B \). Assume that \( m \geq 1 \) (which also implies \( k \geq 1 \)).

Set \( z_t := f_{(i,t)}(x) - x_* \) and \( z_s := f_{(i,s)}(x) - x_* \). Then by (b) and (c) from the choice of \( \delta \), we have

\[
    ||z_t - z_s|| = ||f_{(i,t)}(x) - f_{(i,s)}(x)|| < \frac{\varepsilon}{2}, 
\]

and

\[
    ||z_s|| \leq ||f_{(i,s)}(x) - 0|| + ||x_*|| \leq \text{diam}(B \cup \{0\}) + ||x_*|| \leq \frac{\varepsilon}{2} \cdot (1 - (r - \delta)^p)^{-1}. 
\]
Hence by (5.7) and (5.10), and the fact that \( \hat{g}^p = \text{id}_X \), we have
\[
\|g_t^{(m)} \circ f_{i,t}(x) - g_s^{(k)} \circ f_{i,s}(x)\| = \|t^m \hat{g}^{(m)}(f_{i,t})(x) - x_s + s^k \hat{g}^{(k)}(f_{i,s})(x) - x_s\|
\]
\[
= \|t^m \hat{g}^{(m)}(z_t) - s^k \hat{g}^{(k)}(z_s)\|
\]
\[
= \|t^m \hat{g}(t^m)z_t - s^k \hat{g}(t^k)z_s\|
\]
\[
\leq \|t^m \hat{g}(t^m)z_t - t^m \hat{g}(t^m)z_s\| + \|t^m \hat{g}(t^k)z_s - s^k \hat{g}(t^k)z_s\|
\]
\[
= t^m \cdot \|\hat{g}(t^m)z_t - z_s\| + t^m \cdot \|\hat{g}(t^k)z_s\|
\]
\[
\leq t^m \cdot \|z_t - z_s\| + t^m \cdot \|z_t\|
\]
\[
< \varepsilon.
\]

When \( m = 0 \) (and consequently \( k = 0 \)), we also have
\[
\|g_t^{(m)} \circ f_{i,t}(x) - g_s^{(k)} \circ f_{i,s}(x)\| = \|f_{i,t}(x) - f_{i,s}(x)\| < \frac{\varepsilon}{2}.
\]

Similar reasoning works when the roles of \( s \) and \( t \) are switched. Hence we see that condition (5.4) from Lemma 5.1 is satisfied for IFPs \( M_t \) and \( M_s \), whose attractors are \( A_t \) and \( A_s \), respectively (for definitions of \( M_t \) and \( M_s \), see (5.6)). Thus, using Lemma 5.1 and the fact that
\[
\text{Lip}(M_s, \|\cdot\|), \text{Lip}(M_t, \|\cdot\|) \leq \text{Lip}(F, \|\cdot\|) < 1
\]
(recall definition of \( F \) in (5.5) and notice that \( \text{Lip}(g^{(m)} \circ f) = \text{Lip}(f) \) for \( f \in F \)), we get
\[
h(A_t, A_s) \leq \frac{\varepsilon}{1 - \text{Lip}(F, \|\cdot\|)}.
\]

We conclude that the map \([0, 1) \ni t \mapsto A_t \) is uniformly continuous.

As mentioned in Remark 5.2, Examples 5.1, 5.3, 5.4 show that each of the assumptions in the hypothesis of Theorem 5.2 is necessary. If any assumption is removed, not only does the family \( F_t^g \) not have a unique upper transition attractor, but it may have no upper transition attractor at all. In particular, Example 5.4 provides an infinite dimensional one-parameter family where the function \( g \) is not periodic and the one-parameter family has no upper transition attractor. For a more restricted one-parameter family, however, this assumption may not be necessary; see Question 6.2.

**Example 5.1.** [The assumption that \([0, 1) \ni t \mapsto f_{i,t}(x)\) is continuous is necessary.]

Let \( F_t^g := \{f_t, g_t\} \) be a one-parameter family on \( \mathbb{R} \), where \( f_t(x) = tx + 1/(1-t) \to \infty \) as \( t \to 1 \). Since \( q_t \in A_t \), the limit \( \lim_{t \to 1} A_t \) does not exist.

**Example 5.2.** [The assumption that \( x^* = g^{-1}(0) \) is necessary.]

Let \( F_t^g := \{f_t, g_t\} \), where \( g(x) = x = g_t(x) \) and \( f_t(x) = tx + 1 \). Here \( F_t^g \) satisfies the assumptions of Theorem 5.2 except that \([0, 1) \ni t \mapsto f_t(x)\) is not continuous at \( t = 1 \) for any \( x \). The fixed point \( q_t \) of \( g_t \) is \( q_t = 2/(1-t) \to \infty \) as \( t \to 1 \). Since \( q_t \in A_t \), the limit \( \lim_{t \to 1} A_t \) does not exist.

**Example 5.3.** [The assumption that \( \sup\{\text{Lip}(F_t, \|\cdot\|) : t \in [0, 1]\} < 1 \) is necessary.]

On \( \mathbb{R} \), let \( F_t^g := \{f_t, g_t\} \), where \( g_t(x) = -tx \) and \( f_t(x) = -tx + t + 1 \). (This is Example 4.1 from Section 4.) Here \( F_t^g \) satisfies the assumptions of Theorem 5.2 except that \( \lim_{t \to 1} \text{Lip}(f_t, \|\cdot\|) = 1 \). For \( t \in (0, 1) \) we have \( A_t = [-t/(1-t), 1/(1-t)] \); therefore \( \lim_{t \to 1} A_t \) does not exist.

**Example 5.4.** [The assumption that \( g \) is periodic is necessary.]
Let $\mathbb{X} := \ell^\infty(\mathbb{C})$ denotes the real Banach space of all bounded complex sequences, endowed with the supremum norm. For $k \in \mathbb{N}$, set $\alpha_k := \frac{\pi}{2k}$, and define $g : \mathbb{X} \to \mathbb{X}$ by
\[ g((x_k)) := (x_k e^{i\alpha_k}), \]
that is, each coordinate $x_k$ is rotated around the origin by angle $\alpha_k$. Next define $f : \mathbb{X} \to \mathbb{X}$ by
\[ f((x_k)) := \left(\frac{1}{4}(x_k - 1)\right). \]
Observe that $f(1) = 0$, where 1 and 0 are sequences of ones and zeroes, respectively. For $t \in (0, 1]$, define
\[ g_t((x_k)) := tg((x_k)) = (tx_k e^{i\alpha_k}) \]
and
\[ f((x_k)) := tf((x_k)) + 1 = \left(\frac{t}{4}x_k + 1 - \frac{t}{4}\right). \]
Clearly, the map $[0, 1] \ni t \mapsto f_t(x)$ is continuous for every $x \in \mathbb{X}$, $\text{Lip}(f_t) = \frac{t}{4}$ and $g_t^{-1}(0) = 0$. Hence, setting $F_t := \{f_t, g_t\}$, all assumptions of Theorem 5.2 are satisfied except that $\hat{g}$ (which here coincides with $g$) is not periodic. We will now show that $F_t^g$ does not have any upper transition attractor.

Let
\[ (5.11) \quad D := \{0\} \cup \bigcup_{m=0}^\infty B\left(\left(\frac{3}{4}t^m e^{im\alpha}, 1\right), \frac{1}{4}t^m\right) \]
where $B(\cdot, \cdot)$ denotes the closed ball in $\mathbb{X}$, where the first coordinate is the center and the second coordinate is the radius. We first show that for every $t \in [0, 1]$, the attractor $A_t$ of $F_t^g := \{f_t, g_t\}$ is a subset of $D$. Clearly, the set $D \subseteq \overline{B}(0, 1)$, and it is easy to see that
\[ f_t(\overline{B}(0, 1)) \subseteq \overline{B}\left(\left(\frac{3}{4}, 1\right), \frac{1}{4}\right) \subseteq D, \]
where $\left(\frac{3}{4}, 1\right)$ is the constant sequence whose coordinates equal $\frac{3}{4}$. Hence
\[ f_t(D) \subseteq D. \]
On the other hand, for every $m = 0, 1, 2, \ldots$, we have
\[ g_t\left(\overline{B}\left(\left(\frac{3}{4}t^m e^{im\alpha}, 1\right), \frac{1}{4}t^m\right)\right) = \overline{B}\left(\left(\frac{3}{4}t^{m+1} e^{i(m+1)\alpha}, \frac{1}{4}t^{m+1}\right)\right) \subseteq D \]
and $g_t(0) = 0$; so we also have
\[ g_t(D) \subseteq D. \]
Altogether we have $F_t^g(D) \subseteq D$. As $D$ is closed, we get (5.11). Now since the sequence 1 is the fixed point of $f_t$, it belongs to the attractor $A_t$, and hence also
\[ (5.12) \quad \left(t^m e^{im\alpha}\right) = g_t^{(m)}(1) \in A_t \]
for every $m \in \mathbb{N}$.

We are ready to prove that $(F_t^g)$ does not generate any upper transition attractor, that is, there is no sequence $t_n \in [0, 1)$ with $t_n \to 1$ so that $(A_{t_n})$ converges. First observe that it is enough to prove that
\[ (5.13) \quad \forall s \in (\frac{1}{2}, 1) \exists t_0 < 1 \forall t \in [t_0, 1] \quad h(A_t, A_s) \geq \frac{1}{2}. \]
Indeed, suppose that (5.13) holds, and for some sequence $(t_n) \subseteq [0, 1)$ converging to 1 we have that $(A_{t_n})$ is convergent. Then $(A_{t_n})$ is a Cauchy sequence in $\mathcal{K}(X)$ and we can find $n_0 \in \mathbb{N}$ so that $h(A_{t_n}, A_{t_m}) < \frac{1}{2}$ for all $n \geq n_0$ and $m \geq n_0 \geq \frac{1}{2}$. On the other hand, setting $s := t_{n_0}$ and using (5.13), we can find $n \geq n_0$ with $h(A_{t_n}, A_{t_{n_0}}) \geq \frac{1}{2}$, which gives a contradiction.


We will now prove (5.13). Choose any \( s \in \left[ \frac{1}{4}, 1 \right) \), and find the least \( k_0 \in \mathbb{N} \) such that \( s^{k_0} < \frac{1}{2} \). As \( s \geq \frac{1}{2} \), we see that \( s^{k_0} \geq \frac{1}{4} \). Since \( 1 - s^{k_0} > \frac{1}{2} \), we can find \( t_0 < 1 \) such that for \( t \in [t_0, 1) \) we have

\[
|t^{2k_0} - s^{k_0}| > \frac{1}{2}.
\]

Choose any \((x_k) \in A_*\). By the definition of \( D \) (see (5.11) and the fact that \( A_t \subseteq D \), we can consider three cases:

Case 1. \((x_k) \in B\left(\left( \frac{1}{4}s^m e^{im\alpha_k}, \left( \frac{1}{4}s^m \right) \right) \right)\) for some \( m \leq k_0 \).

Since \( m\alpha_k \leq k_0 \frac{\pi}{k_0} = \frac{\pi}{2} \), we have

\[
t^{2k_0} \geq \left| t^{2k_0} + \frac{3}{4}s^m e^{im\alpha_k} \right| \leq \left| t^{2k_0} + x_{k_0} \right| + \left| -x_{k_0} + \frac{3}{4}s^m e^{im\alpha_k} \right| \leq \left| t^{2k_0} + x_{k_0} \right| + \frac{1}{4},
\]

so by (5.14) we get

\[
\left| t^{2k_0} + x_{k_0} \right| \geq t^{2k_0} - \frac{1}{4} \geq t^{2k_0} - s^{k_0} > \frac{1}{2}.
\]

Case 2. \((x_k) \in B\left(\left( \frac{1}{4}s^m e^{im\alpha_k}, \left( \frac{1}{4}s^m \right) \right) \right)\) for some \( m \geq k_0 \).

Since \( t^{2k_0} > s^{k_0} \geq s^m \), we have

\[
t^{2k_0} - \frac{3}{4}s^k \leq t^{2k_0} - \frac{3}{4}s^m \leq \left| t^{2k_0} + \frac{3}{4}s^m e^{im\alpha_k} \right| \leq \left| t^{2k_0} + x_{k_0} \right| + \left| -x_{k_0} + \frac{3}{4}s^m e^{im\alpha_k} \right| \leq \left| t^{2k_0} + x_{k_0} \right| + \frac{1}{4}s^m \leq \left| t^{2k_0} + x_{k_0} \right| + \frac{1}{4}s^{k_0}.
\]

Thus by (5.14),

\[
\left| t^{2k_0} + x_{k_0} \right| \geq t^{2k_0} - \frac{3}{4}s^{k_0} - \frac{1}{4}s^{k_0} > \frac{1}{2}.
\]

Case 3. \((x_k) = 0\).

In this case

\[
\left| t^{2k_0} + x_{k_0} \right| = t^{2k_0} > \frac{1}{2}.
\]

Summing up, we have that

\[
\left| \left( t^{2k_0} e^{i\frac{2k_0\pi}{k_0}} \right) - (x_k) \right| \geq \left| t^{2k_0} e^{i\frac{2k_0\pi}{k_0}} - x_{k_0} \right| = \left| -t^{2k_0} - x_{k_0} \right| = \left| t^{2k_0} + x_{k_0} \right| > \frac{1}{2}.
\]

By (5.12) we see that \( \left( t^{2k_0} e^{i\frac{2k_0\pi}{k_0}} \right) \in A_t \), so the above shows that

\[
h(A_t, A_*) \geq \inf_{(x_k) \in A_*} \left( \left| \left( t^{2k_0} e^{i\frac{2k_0\pi}{k_0}} \right) - (x_k) \right| \right) \geq \frac{1}{2}
\]

and the proof of (5.13) is complete.

6. Open Problems

Examples 3.2, 3.3, and 3.4 show that an IFS with an attractor need not be contractive. In Example 3.3 no function in the IFS \( F \) is a contraction. In fact, with respect to any equivalent metric \( d \) on the circle, \( \text{Lip}(f, d) > 1 \) for all \( f \in F \). This is not the case in Example 3.4. It can be asked whether such a strong counterexample exists for \( \mathbb{R}^n \).

**Question 6.1.** Is there an example of an IFS \( F \) on \( \mathbb{R}^n \) that has an attractor \( A \) with basin \( \mathbb{R}^n \) but (1) \( A \) is not the attractor of any proper subset of \( F \) and (2) with respect to any metric \( d \) equivalent to the Euclidean metric we have \( \text{Lip}(f, d) > 1 \) for all \( f \in F \).
For a large class of one-parameter IFS families, Theorem 5.2 guarantees the existence of a unique upper transition attractor $A^*$ such that $A^* = \lim_{t \to t_0} A_t$ at a threshold $t_0$. The theorem, however, assumes that the linear part of the special function $g$ is periodic. Example 6.4 shows that, in general, the assumption of periodicity of the linear part cannot be dropped. But the underlying space in that example is a non-separable infinite dimensional space.

**Question 6.2.** Can the assumption of periodicity of the linear part of the function $g$ in Theorem 5.2 be dropped assuming a less exotic space? In particular, can the assumption be dropped for a one-parameter similarity family with threshold $t_0$ satisfying the following properties:

- All $f_t \in F_t$ are contractions for $t \in [0, t_0]$, $g_t$ is a contraction for $t \in [0, t_0)$ and $\text{Lip}(g_{t_0}) = 1$, and
- the unique fixed point of each $f_t \in F_t$ and $g_t$ is independent of $t \in [0, t_0)$.

In [33, Theorem 8.2] relationships between the upper and lower transition attractors are given for a special type of one-parameter family. It can be asked whether the same relationships hold in a more general setting. In particular:

**Conjecture 6.1.** If $F_t$ satisfies properties (H1), (H2), (H3) of Section 4 and if $A_* = A^*$ for some upper transition attractor of $F_t$, then $A^*$ is the unique upper transition attractor of $F_t$ and $A^*$ is an attractor of $F_t$.

Recall that in a metric space $(X, d)$, a *segment* with ends $x, y \in X$ is defined by $[x, y] := \{z \in X : d(x, z) + d(z, y) = d(x, y)\}$. A set $S \subseteq X$ is *metrically convex* if $[x, y] \subseteq S$ for all $x, y \in S$. The *metrically convex hull* of $S \subseteq X$ is $\text{conv}_d S := \bigcup_{x, y \in S} [x, y]$.

**Conjecture 6.2.** If the functions in $F_t$ map metrically convex sets onto metrically convex sets, then the metrically convex hulls of $A_*$ and $A^*$ in $(X, d)$ coincide: $\text{conv}_d A_* = \text{conv}_d A^*$.

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