Galois actions on torsion points of universal one-dimensional formal modules

Matthias Strauch

Department of Pure Mathematics and Mathematical Statistics
Centre for Mathematical Sciences, University of Cambridge
Wilberforce Road, Cambridge, CB3 0WB, United Kingdom
M.Strauch@dpmms.cam.ac.uk

Abstract. Let $F$ be a local non-Archimedean field with ring of integers $\mathfrak{o}$ and uniformizer $\varpi$. Let $X$ be a one-dimensional formal $\mathfrak{o}$-module of $F$-height $n$ over the algebraic closure $\overline{F}$ of the residue field of $\mathfrak{o}$. By the work of Drinfeld, the universal deformation $X$ of $X$ is a formal group over a power series ring $R_0$ in $n-1$ variables over the completion of the maximal unramified extension $\hat{\mathfrak{o}}^{ur}$ of $\mathfrak{o}$. For $h \in \{0, \ldots, n-1\}$ let $U_h \subset \text{Spec}(R_0)$ be the locus where the connected part of the associated $\varpi$-divisible module $X[\varpi^\infty]$ has height $h$. Using the theory of Drinfeld level structures we show that the representation of $\pi_1(U_h)$ on the Tate module of the étale quotient is surjective.

Contents

1. Preliminaries
2. Statement of the result
3. Level structures
4. Proof of the surjectivity
References

1. Preliminaries

Let $F$ be a local field non-Archimedean field with ring of integers $\mathfrak{o}$ and uniformizer $\varpi$. Let $q$ be the cardinality of the residue field of $\mathfrak{o}$, and denote this field by $\mathbb{F}_q$. Fix a one-dimensional formal $\mathfrak{o}$-module $X$ of $F$-height $n$ over the algebraic closure $\overline{F}$ of the residue field of $\mathfrak{o}$. By the work of Drinfeld, cf. [D], the universal deformation $X$ of $X$ is a formal $\mathfrak{o}$-module over a power series ring $R_0$ in $n-1$ variables over $\hat{\mathfrak{o}}^{ur}$, the completion of the maximal unramified extension of $\mathfrak{o}$. We denote by

$$\left[\cdot\right]_X : \mathfrak{o} \rightarrow \text{End}_{R_0}(X)$$

the action of $\mathfrak{o}$ on the universal deformation.
One may choose coordinates $u_1, \ldots, u_{n-1}$ of $R_0$ and a coordinate $T$ on $X$ such that the multiplication by $\varpi$ on $X$ is given by a power series $[\varpi]_X(T) \in R_0[[T]]$ with the property that for all $i = 0, \ldots, n$:

\begin{equation}
[\varpi]_X(T) \equiv u_i T^{q_i} \mod (u_0, \ldots, u_{i-1}), \quad \deg (q_i + 1),
\end{equation}

where we have put $u_0 = \varpi$ and $u_n = 1$ (cf. [Ha], sec. 21.5, [HG], Prop. 5.7).

The subset of $\text{Spec}(R_0)$ where the $\varpi$-divisible $\mathfrak{o}$-module $X[\varpi^\infty]$ is étale is precisely $\text{Spec}(R_0) - V(\varpi)$. Then the subset $U \subset \text{Spec}(R_0/(\varpi))$ where the étale quotient of $X[\varpi^\infty]$ has height $n - 1$ is $\text{Spec}(R_0/(\varpi)) - V(u_1)$. Denote by $\kappa = \mathbb{F}((u_1, \ldots, u_{n-1}))$ the field of fractions of $R_0/(\varpi)$ and put $\eta = \text{Spec}(\kappa)$. Let $\kappa^a$ be an algebraic closure of $\kappa$ and put $\bar{\eta} = \text{Spec}(\kappa^a)$. We consider the Tate module of the étale quotient of $X[\varpi^\infty]$ over $\eta$:

$$T_1 := T_\varpi((X[\varpi^\infty]_\eta)^{\text{ét}}) = \lim_{\longrightarrow} X[\varpi^m]^{\text{ét}}_\eta(\kappa^a)$$

which is a free $\mathfrak{o}$-module of rank $n - 1$. The absolute Galois group $\pi_1(\eta, \bar{\eta})$ of $\kappa$ acts on this $\mathfrak{o}$-module $\mathfrak{o}$-linearly, and this action factors through $\pi_1(U, \bar{\eta})$:

\begin{equation}
\pi_1(\eta, \bar{\eta}) \longrightarrow \pi_1(U, \bar{\eta}) \longrightarrow \text{Aut}_\mathfrak{o}(T_1)^{\times} \simeq GL_{n-1}(\mathfrak{o}).
\end{equation}

In [T], Conj. 1.4, Y. Tian conjectures that this representation is surjective. We give here a simple proof of the surjectivity, using only results about Drinfeld level structures. Tian’s conjecture is more general than our result since it applies to elementary $p$-divisible groups of any slope in $(0, 1)$, not only to one-dimensional groups of slope $\frac{1}{n}$ as is the case considered here. In [T] a proof of the surjectivity is given in the case of an elementary $p$-divisible group of slope $\frac{1}{n}$. In [B2] P. Boyer proves the irreducibility of Igusa varieties in the case of certain ‘simple’ Shimura varieties which had been studied previously in [HT]. The irreducibility of the ‘Igusa varieties of the first kind’ implies the surjectivity of the Galois representation considered here (in fact, it is equivalent to the surjectivity). Boyer derives this result from a detailed study of the cohomology of these Shimura varieties, cf. [B1]. As already stated above, the method used here employs only results about rings representing deformation functors with level structures.

Shortly after a first draft of this paper was written, Eike Lau used the results presented here to prove the surjectivity of the monodromy representation for certain

\footnote{This is the locus where $X[\varpi^\infty]$ is ‘ordinary’, where ‘ordinary’ means in this context that the connected component of $X[\varpi^\infty]$ over $U$ is a Lubin-Tate formal $\varpi$-module of height one.}
Newton strata in the universal deformation space of an arbitrary connected Barsotti-Tate group over an algebraically closed field of characteristic \( p \). Lau’s results cover in particular all cases of Tian’s conjecture, cf. [L].

2. Statement of the result

In this paper we prove a statement which is slightly more general than the surjectivity of (1.2). For \( h \in \{0, \ldots, n-1\} \) put

\[
R_{h,0} = R_0/(\varpi, u_1, \ldots, u_{h-1})
\]

with the convention that \( u_0 = \varpi \), and so \( R_{0,0} = R_0 \). Then the closed reduced subscheme of \( \text{Spec}(R_0) \) where the height of the connected component of \( X[\varpi^\infty] \) is at least \( h \) is equal to \( \text{Spec}(R_{h,0}) \), and the open part of \( \text{Spec}(R_{h,0}) \) where the height of the connected component is equal to \( h \) is

\[
U_h := \text{Spec}(R_{h,0}) - V(u_h).
\]

Hence the scheme \( U \) considered above is equal to \( U_1 \). Let \( \kappa_h \) be the field of fractions of \( R_{h,0} \) and put \( \eta_h = \text{Spec}(\kappa_h) \). Let \( \kappa_h^a \) be an algebraic closure of \( \kappa_h \) and put \( \bar{\eta}_h = \text{Spec}(\kappa_h^a) \). Fix a positive integer \( m \). Denote by

\[
T_{h,m} := X[\varpi^m]_{\bar{\eta}_h}(\kappa_h^a)
\]

the module of \( \kappa_h^a \)-valued points of the group scheme \( X[\varpi^m]_{\bar{\eta}_h} \) over \( \eta_h \). It is a free \( \mathfrak{o}/(\varpi^m) \)-module of rank \( n - h \). The absolute Galois group \( \pi_1(\eta_h, \bar{\eta}_h) \) of \( \kappa_h \) acts \( \mathfrak{o} \)-linearly on \( T_{h,m} \), and we denote this representation by \( \sigma_{h,m} \):

\[
\sigma_{h,m} : \pi_1(\eta_h, \bar{\eta}_h) \longrightarrow \text{Aut}_{\mathfrak{o}}(T_{h,m}) \simeq GL_{n-h}(\mathfrak{o}/(\varpi^m)).
\]

\( \sigma_{h,m} \) clearly factors as \( \pi_1(\eta_h, \bar{\eta}_h) \rightarrow \pi_1(U_h, \bar{\eta}_h) \rightarrow \text{Aut}_{\mathfrak{o}}(T_{h,m}). \)

**Theorem 2.1.** For all \( h \in \{0, \ldots, n-1\} \) and \( m > 0 \) the homomorphism \( \sigma_{h,m} \) is surjective. In particular, denoting by

\[
T_h := \lim_{\longleftarrow m} X[\varpi^m]_{\bar{\eta}_h}(\kappa_h^a)
\]

the Tate module of the étale quotient of \( X[\varpi^\infty]_{\eta_h} \), the resulting representation

\[
\sigma_h : \pi_1(\eta_h, \bar{\eta}_h) \longrightarrow \text{Aut}_{\mathfrak{o}}(T_h) \simeq GL_{n-h}(\mathfrak{o})
\]

is surjective.
3. LEVEL STRUCTURES

Fix an integer \( m > 0 \). Let \( \mathcal{M}_m = \text{Spf}(R_m) \) be the formal scheme over \( \mathfrak{o}' \) which parameterizes deformations of \( X \) equipped with a level-\( m \)-Drinfeld structure. It is proven in [D], Prop. 4.3, that the ring \( R_m \) is a regular local ring and a finite flat \( R_0 \)-algebra. Let

\[
\phi := \phi^\text{univ}_m : (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n \longrightarrow \mathfrak{m}_{R_m}
\]

be the universal level-\( m \)-structure. Here \( \mathfrak{m}_{R_m} \) denotes the maximal ideal of \( R_m \) which is given the structure of an \( \mathfrak{o} \)-module after having fixed a coordinate \( T \) on the universal deformation \( X \). That \( \phi \) is a level-\( m \)-structure means that there is an invertible power series \( \varepsilon_m(T) \in R_m[[T]] \) such that there is an equality

\[
\prod_{a \in (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n} (T - \phi(a)) = \varepsilon_m(T) \cdot [\varpi](T)
\]

of formal power series over \( R_m \), cf. [D], definition before Prop. 4.3. For ease of notation put

\[
A_m = (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n
\]

and let \( e_i = (0, \ldots, 0, \varpi^{-m}, 0, \ldots, 0) \in A_m \), be the \( i \)th standard generator of \( A_m \) as an \( \mathfrak{o}/(\varpi^m) \)-module. By [D], Prop. 4.3, the sequence \( \phi(e_1), \ldots, \phi(e_n) \) is a regular system of parameters for \( R_m \), and the analogous fact is true for any basis of \( A_m \). The group

\[
G_m := \text{Aut}_\mathfrak{o}(A_m) \simeq GL_n(\mathfrak{o}/(\varpi^m))
\]

acts on \( R_m \) via its action on the universal Drinfeld basis. Concretely: \( g \in \text{Aut}_\mathfrak{o}(A_m) \) maps \( \phi(a) \) to \( \phi(g(a)) \) for any \( a \in A_m \). Let \( \kappa_m = \text{Frac}(R_m) \) be the field of fractions of \( R_m \). Then \( \kappa_m \) is a Galois extension of \( \kappa_0 = \text{Frac}(R_0) \) and the action of \( G_m \) on \( \kappa_m \) defines an isomorphism

\[
G_m \xrightarrow{\simeq} \text{Gal}(\kappa_m/\kappa_0).
\]

(For details see [St], Thm. 2.1.2.) This means that the field \( \kappa_m \) generated by the \( \varpi^m \)-torsion points of \( X \) is a Galois extension of \( \kappa_0 \) with Galois group isomorphic to \( G_m \). (This was proven for the case \( \mathfrak{o} = \mathbb{Z}_p \) by a different method in [RZ].)
4. Proof of the Surjectivity

Fix $m > 0$ as above and $h \in \{0, \ldots, n - 1\}$. We consider the prime ideal $p_h = (\varpi, u_1, \ldots, u_{h-1})$ of $R_0$, and determine the prime ideals of $R_m$ lying over $p_h$, as well as the corresponding decomposition and inertia groups. Let

$$p_{h,m} = (\phi_1, \ldots, \phi_h) \subset R_m$$

be the prime ideal generated by the regular sequence $\phi_1, \ldots, \phi_h$, and put $R_{h,m} = R_m/p_{h,m}$, $\kappa_{h,m} = \text{Frac}(R_{h,m})$. Denote by

$$A_{h,m} = o/(\varpi^m)e_1 \oplus \cdots \oplus o/(\varpi^m)e_h.$$

the submodule of $A_m$ generated by the first $h$ standard generators. Define $P_{h,m} \subset G_m$ to be the stabilizer of the submodule $A_{h,m} \subset A_m$ and let $Q_{h,m} \subset P_{h,m}$ be the subgroup which acts trivially on the quotient $A_m/A_{h,m}$.

**Proposition 4.1.** i) The ideal $p_{h,m} \subset R_m$ lies over $p_h$, and the finite field extension $\kappa_{h,m}/\kappa_h$ is normal.

ii) Via the isomorphism 3.2 the group $P_{h,m}$ is the decomposition group of $p_{h,m}$.

iii) The inertia subgroup of $P_{h,m}$ is equal to $Q_{h,m}$, and the canonical map

$$P_{h,m}/Q_{h,m} \longrightarrow \text{Aut}(\kappa_{h,m}/\kappa_h)$$

is a bijection. In particular, if $\kappa_{h,m}^s$ denotes the maximal separable extension of $\kappa_h$ contained in $\kappa_{h,m}$, then $\text{Gal}(\kappa_{h,m}^s/\kappa_h)$ is isomorphic to $GL_{n-h}(o/(\varpi^m))$.

**Proof.** When $h = 0$ the statements of this proposition are equivalent to the assertion that $\kappa_m/\kappa_0$ is a Galois extension with group $G_m$, cf. 3.2 Since this is known ([St], Thm. 2.1.2) we assume from now on $1 \leq h \leq n - 1$.

i) We consider first the reduction of the left side of 4.1 modulo $p_{h,m}$ and see that as polynomials over $R_m/p_{h,m}$ we have

$$(4.2) \quad \prod_{a \in \varpi^{m-1}A_m} (T - \phi(a)) \mod p_{h,m} = \prod_{a' \in \varpi^{m-1}A_{h,m}'} (T - \phi(a'))^{q_h},$$

where $A_{h,m}' = o/(\varpi^m)e_{h+1} \oplus \cdots \oplus o/(\varpi^m)e_n$. Using 1.1 we see that modulo $p_{h,m}$ the coefficients $\varpi, u_1, \ldots, u_{h-1}$ of $[\varpi] \alpha(T)$ vanish, and so $p_h$ is necessarily contained in $p_{h,m} \cap R_0$. Since both prime ideals have height $h$ and $R_m$ is integral over $R_0$, we get that $p_{h,m} \cap R_0 = p_h$. In particular, $u_h$ is not in $p_{h,m}$. Now consider the coefficient of $T^{q_h}$ in 4.2. This coefficient is equal to
\[ \pm \prod_{a' \in \O_{A'} - \{0\}} \phi(a') . \]

Comparing this with the reduction of (4.1) modulo \( \mathfrak{p} \) shows that \( \phi(a') \notin \mathfrak{p}_{h,m} \) for every non-zero \( a' \in \O_{A'} \). A fortiori: \( \phi(a') \notin \mathfrak{p}_{h,m} \) for every non-zero \( a' \in A'_{h,m} \). Since \( \phi(a + a') = \phi(a) + X \phi(a') = \phi(a) + \phi(a') + \phi(a) (\cdots) \), we deduce for later use that

\[ (4.3) \quad \text{For all } a' \in A_{m} - A_{h,m} : \phi(a') \notin \mathfrak{p}_{h,m} \]

By [Bou], Ch. V, §2.2, Thm. 2, the extension \( \kappa_{h,m}/\kappa_h \) is normal.

ii) \( P_{h,m} \) is clearly contained in the decomposition group of \( \mathfrak{p}_{h,m} \). If \( g \in G_{m} \) is not in \( P_{h,m} \), then there is \( i \in \{1, \ldots, h\} \) such that \( g(e_i) \) is not in \( A_{h,m} \). Hence \( g \), considered as Galois automorphism, maps \( \phi_i \) to an element \( \phi(g(e_i)) \) which is not in \( \mathfrak{p}_{h,m} \), by what we have seen above, cf. (4.3). This shows that \( P_{h,m} \) is the decomposition group of \( \mathfrak{p}_{h,m} \).

iii) The ring \( R_{h,m} \) is a regular local ring with regular system of parameters given by the images \( \tilde{\phi}_j \) of \( \phi_j \) for \( j = h + 1, \ldots, n \). The action of an element \( g \in P_{h,m} \) on \( R_{h,m} \) sends \( \tilde{\phi}_j \) to \( \phi(g(e_j)) \mod \mathfrak{p}_{h,m} \). In order for \( g \) to induce the identity on \( R_{h,m} \) it is necessary and sufficient that

\[ \phi(g(e_j)) \mod \mathfrak{p}_{h,m} = \phi(e_j) \mod \mathfrak{p}_{h,m} \text{ for all } j = h + 1, \ldots, n . \]

When we consider both sides of this equation as torsion points of the formal group \( X \) over \( R_{h,m} \) we find that their difference, as an element of the maximal ideal of \( R_{h,m} \) (with the group law induced by \( X \)), is equal to

\[ \phi(g(e_j)) - X \phi(e_j) \mod \mathfrak{p}_{h,m} = \phi(g(e_j) - e_j) \mod \mathfrak{p}_{h,m} . \]

If this difference is zero in \( R_{h,m} \) then, by (4.3), we have \( g(e_j) - e_j \in A_{h,m} \) for all \( j = h + 1, \ldots, n \). This means that \( g \) is in \( Q_{h,m} \). The last assertion is [Bou], Ch. V, §2.3, Prop. 6.

**Proof of Theorem 2.1.** We let \( A'_{h,m} = \O/(\varpi^m) \oplus \cdots \oplus \O/(\varpi^m) \) as above. It follows from (4.3) that the induced map

\[ \tilde{\phi} : A'_{h,m} \longrightarrow R_{h,m} \subset \kappa_{h,m} \]

is injective. Since \( X_{\varpi^m} \) has rank \( m^{n-k} \) we see that \( \kappa_{h,m} \) is obtained by adjoining all \( \varpi^m \)-torsion points of \( X_{\varpi_h} \) to \( \kappa_h \). Denote by \( \kappa'_{h,m} \) the maximal separable extension of \( \kappa_h \) contained in \( \kappa_{h,m} \). Then the canonical maps
are bijections. The connected component \( X[[\varpi]]^{\text{ét}}_{\eta h}(\kappa_{h,m}) \) is infinitesimal of degree \( q^{mh} \), and so the torsion points of \( X[[\varpi]]^{\text{ét}}_{\eta h} \) are exactly the elements \( \bar{\phi}(a') \varpi^{mh} \in \kappa_{h,m} \) which lie actually in the maximal separable sub-extension \( \kappa_{h,m}^s \subset \kappa_{h,m} \). The action of the group

\[
\Aut(\kappa_{h,m}/\kappa_h) = P_{h,m}/Q_{h,m} \simeq \Aut_\mathfrak{o}(A'_{h,m}) \simeq GL_{n-h}(\mathfrak{o}/(\varpi^m))
\]
on the set of torsion points

\[
\{ \bar{\phi}(a') \mid a' \in A'_{h,m} \}
\]
of \( X[[\varpi]]^{\text{ét}}_{\eta h} \) induces an action on the set of torsion points

\[
\{ \bar{\phi}(a')^{q^{mh}} \mid a' \in A'_{h,m} \}
\]
of \( X[[\varpi]]^{\text{ét}}_{\eta h} \). The latter action is again equal to the full automorphism group \( \Aut_\mathfrak{o}(T_{h,m}) \). Hence the map \( \Gal(\kappa_{h,m}^s/\kappa_h) \to \Aut_\mathfrak{o}(T_{h,m}) \) is an isomorphism. \( \Box \)

References

[B1] P. Boyer, Monodromie du faisceau pervers des cycles évanescents de quelques variétés de Shimura et applications. Preprint, arXiv: math.AG/0511531v2.

[B2] P. Boyer, On the irreductibility of some Igusa varieties. Preprint, arXiv: math/0702329.

[Bou] N. Bourbaki, Commutative algebra. Chapters 1-7. Elements of Mathematics. Springer-Verlag, Berlin, 1989.

[D] V. G. Drinfeld, Elliptic modules. English translation: Math. USSR-Sb. 23 (1974), no. 4, 561–592.

[Ha] M. Hazewinkel, Formal groups and applications. Pure and Applied Mathematics, 78. Academic Press, New York-London, 1978.

[HG] M. J. Hopkins, B. H. Gross, Equivariant vector bundles on the Lubin-Tate moduli space. Topology and representation theory (Evanston, IL, 1992), 23–88, Contemp. Math., 158, Amer. Math. Soc., Providence, RI, 1994.

[HT] M. Harris, R. Taylor, The geometry and cohomology of some simple Shimura varieties. With an appendix by Vladimir G. Berkovich. Annals of Mathematics Studies, 151. Princeton University Press, Princeton, NJ, 2001.

[L] E. Lau, p-adic monodromy of some Newton strata in the universal deformation of a p-divisible group. Preprint, 2007.

[RZ] M. Rosen, K. Zimmermann, Torsion points of generic formal groups. Trans. Amer. Math. Soc. 311 (1989), no. 1, 241–253.

[St] M. Strauch, Deformation spaces of one-dimensional formal modules and their cohomology. To appear in: Advances in Mathematics. arXiv: math/0611109.

[T] Y. Tian, p-adic monodromy of the universal deformation of an elementary Barsotti-Tate group. Preprint, arXiv:0708.2022v1.