Abstract: 
In this work, we present a new simple way to encode/decode messages transmitted via a noisy channel and protected against errors by the Hamming method. We also propose a fast and efficient algorithm for the encoding and the decoding process which do not use neither the generator matrix nor the parity-check matrix of the Hamming code.

Keywords: Error-correcting codes, parity-check bits, coding, Hamming code.

MSC: 11B50, 94B05, 94B35.

1 Introduction

The theory of error-correcting codes started in the first half of the last century with the valuable contribution of Claude Shannon [28]. Richard Hamming [9] was also one of the pioneers in the field. As he pointed it out himself at the end of his article, the only work on error-correcting codes before his publication, has been that of Golay [8].

The principle of error-detecting and correcting codes is the redundancy technique. Before sending a message $M$ we code it by adding some bits. These bits are calculated according to some specific mathematical laws. After receiving the sent data, we check again the added bits by the same laws. Therefore, we can detect or even correct occurred errors.

In 1954, Muller [24] applied the Boolean algebra for constructing error-detecting codes. In 1960, Reed and Solomon [26] extended the Hamming scheme and proposed a class of multiple error-correcting codes by exploiting polynomials. Almost at the same year, Bose with Chaudhuri [2] and independently Hocquenghem [12] showed how to design error-correcting codes if we choose in advance the number of errors we would like to correct. The method is based on polynomial roots in finite fields. The invented sets are now known as BCH codes. In 1967, Viterbi [31] devised a convolutionaly error-correcting code based on the principle of the maximum likelihood. In 1978, a cryptosystem based on error-correcting codes theory was proposed by McEliece [23]. It is now considered as a serious
candidate encryption system that will survive quantum computer attacks [3]. Since then, curiously and even with the intensive use of computers and digital data transmission, no revolutionary change happened in the methods of error-detecting and correcting codes until the appearance of Turbo-codes in the early 1990s [1].

Most of the codes studied in modern scientific research on coding theory are linear. They possess a generator matrix for coding and a parity-check matrix for error-correcting. Cyclic codes [10, 19, 13] constitute a particular remarkable class of linear codes. They are completely determined by a single binary polynomial and therefore can be easily implemented by shift registers.

The low-density parity-check codes, (LDPC), first discovered by Gallager in 1962 [7], were brought up to date by MacKay in 1999. These codes have a parity-check matrix whose columns and rows contain a small number of 1’s. Like the Turbo-codes, they achieve information rates up to the Shannon limit [1, 22, 28].

The Hamming code belongs to a family of error-correcting codes whose coding and decoding procedure are easy. This is why it is still widely used today in many applications related to the digital data transmission and communication networks. In 2021, Falcone and Pavone [6] studied the weight distribution problem of the binary Hamming code. In 2001, 2015, 2017 and 2018, attempts to improve the decoding of Hamming codes have been performed [11, 15, 16, 3].

In this work, we present an original and simple way to encode/decode messages transmitted via a noisy channel and protected against errors by the Hamming method. We consequently construct algorithms for the encoding and the decoding procedures which do not use neither the generator matrix nor the parity-check matrix of the Hamming code. To the best of our knowledge, this issue has not been studied before and does not appear in the mathematical or computer science literature on coding theory.

The article is organized as follows. In Section 2, we briefly review the Hamming error-correcting code. Section 3 contains some preliminaries. Our contribution on coding and decoding messages is described in Section 4. We conclude in Section 5.

Throughout this paper, we shall use standard notation. In particular \( \mathbb{N} \) is the set of all positive integers \( 0, 1, 2, 3, \ldots \). If \( a, b, n \in \mathbb{N} \), we write \( a \equiv b \pmod{n} \) if \( n \) divides \( a - b \), and \( a = b \mod{n} \) if \( a \) is the remainder of the Euclidean division of \( b \) by \( n \). The binary representation of \( n \) is noted \( B(n) = \epsilon_{k-1}\epsilon_{k-2}\ldots\epsilon_2\epsilon_1\epsilon_0 \) with \( \epsilon_i \in \{0, 1\} \) and means that
\( n = \sum_{i=0}^{k-1} \epsilon_i 2^i \). The function \( \log \) should be interpreted as logarithm to the base 2. The largest integer which does not exceed the real \( x \) is denoted by \( \lfloor x \rfloor \).

We start, in the next section, by recalling the construction of the Hamming code [9] and some known relevant facts on the associated algorithm.

2 Brief recall on the Hamming code

Assume that a binary message \( M = a_1a_2\ldots a_n \) was transmitted through a noisy channel and that the received message is \( M' = b_1b_2\ldots b_n \). If at most one single error has occurred during the transmission, then with \( \log n \) parity checks, the Hamming algorithm [9] efficiently determines the error position or detect double-bit errors. In this section, we review the main steps when using the Hamming code in digital data communication networks. There is an abundant literature on the subject. For more details, see for instance [9][19] p. 38][30] p. 319][10] Chap. 8][13] p. 29][16] [20] p. 23].

2.1 The coding procedure

In a binary representation, assigning \( k \) bits to the error position, allows to analyze and decode any message of length \( n = 2^k - 1 \). The main idea of Hamming method relies on a logical equivalence. The \( j \)th bit, among the \( k \) possible bits, is 1 if and only if the error has occurred at a bit \( a_i \) of \( M \) whose index \( i \) has 1 in the \( j \)th position in its binary representation.

Hamming defines \( a_{2^j} \) as a parity-check bit and it is equal to the sum modulo 2 of all bits of \( M \) with index having 1 in the \( j \)th position in its binary representation.

Bits \( a_1, a_2, a_2^2, \ldots, a_2^j, \ldots, a_2^{k-1} \) are kept as parity-check bits. All the other bits are for information. Their number is \( m = n - k = 2^k - k - 1 \).

2.2 The decoding procedure

Suppose that the received codeword is \( M' = b_1b_2\ldots b_n \). For each fixed integer \( j \in \{0, 1, 2, \ldots, k - 1\} \), if \( b_{2^j} \) is the sum modulo 2 of all bits of \( M' \) with index having 1 in the \( j \)th position in its binary representation, then the error position has 0 at the \( j \)th place from the right in its binary representation, if not, it is a 1.
Example 2.1: Let us illustrate the technique by un exemple. Suppose that we received a message with 15 bits as it is indicated in Table 1.

| Indexes | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| bits    | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1  | 1  | 0  | 0  | 1  | 1  |

Table 1: The received 15 bits

Let $\epsilon_3 \epsilon_2 \epsilon_1 \epsilon_0$ be the binary representation of the error position. By the Hamming algorithm [9], $\epsilon_0$ is the sum modulo 2 of the bits whose binary representation starts by 1. So:

$\epsilon_0 = (b_1 + b_3 + b_5 + b_7 + b_9 + b_{11} + b_{13} + b_{15}) \mod 2 = 1.$

$\epsilon_1$ is the sum of the bits whose binary representation has 1 in the second position. So:

$\epsilon_1 = (b_2 + b_3 + b_6 + b_7 + b_{10} + b_{11} + b_{14} + b_{15}) \mod 2 = 0.$

$\epsilon_2$ is the sum of the bits whose binary representation has 1 in the third position. So:

$\epsilon_2 = (b_4 + b_5 + b_6 + b_7 + b_{12} + b_{13} + b_{14} + b_{15}) \mod 2 = 1.$

$\epsilon_3$ is the sum of the bits whose binary representation has 1 in the last position. So:

$\epsilon_3 = (b_8 + b_9 + b_{10} + b_{11} + b_{12} + b_{13} + b_{14} + b_{15}) \mod 2 = 0.$

Finally, the error position is $\epsilon_3 \epsilon_2 \epsilon_1 \epsilon_0 = 0 \ 1 \ 0 \ 1$ or 5 in the decimal base. The term $b_5$ must be corrected.

2.3 Complexity of the algorithm

A Hamming code can also be defined by its $k \times n$ generator matrix $G$ and its $(n - k) \times n$ parity-check matrix $H$ [32]. Let $x$ be the message to send. To compute the codeword $xG$, we need $(k - 1)n \simeq n \log n$ binary additions and $kn$ bit multiplications.

The decoding complexity: Let $y$ be the received message. To compute the syndrome $Hy$ we perform $(n - k)(n - 1) \simeq n(n - \log n)$ binary additions and $(n - k)n \simeq n(n - \log n)$ bit multiplications. To locate the error position, in the worst case, we compare the vector $Hy$ to every column, so we need $n - k \simeq n - \log n$ bits comparison. As there are $n$ columns in the matrix $H$, the total number of the bit comparisons is $n(n - k) \simeq n(n - \log n)$.

If the columns of parity-check matrix $H$ are arranged in order of increasing binary numbers from 1 to $n$, we do not need to make comparisons [10] p. 83]. The syndrome $Hy$ is exactly the binary representation of the error position.
3 Preliminaries

Let \( k \in \mathbb{N} \setminus \{0\} \) and \( n = 2^k - 1 \). For every fixed \( j \in \{0, 1, 2, \ldots, k-1\} \) we define the set \( S(j, n) \) as

\[
S(j, n) = \{0 \leq u \leq n \mid \mathcal{B}(u) \text{ contains } 1 \text{ in the } (j+1)\text{th position from the right}\}. \tag{1}
\]

Hence \( S(0, n) = \{1, 3, 5, 7, \ldots, n\} \), \( S(1, n) = \{2, 3, 6, 7, 10, 11, \ldots, n\} \), \( S(2, n) = \{4, 5, 6, 7, 12, 13, 14, 15, \ldots, n\} \), \( S(3, n) = \{8, 9, 10, 11, 12, 13, 14, 15, 24, 25, 26, 27, 28, 29, 30, 31, \ldots, n\} \), ...

Observe that the binary representation of \( n \) is \( \mathcal{B}(n) = 111\ldots 11 \) and so it contains 1 at any position.

**Remark 3.1:** To schematize what is the set \( S(j, n) \), suppose that we have in front of us the line representing all positive integers. From the term \( 2^j \), we keep the \( 2^j \) consecutive integers, we delete the following \( 2^j \) elements, we keep the \( 2^j \) following, we delete the following \( 2^j \) terms, and so on alternately... We repeat this procedure until the part just before the limit \( n \).

**Proposition 3.1:** For any positive integer \( u = (2^{\alpha} + 1)2^j + i \) where \( 0 \leq \alpha \leq 2^{k-j-1} - 1 \) and \( 0 \leq i \leq 2^j - 1 \), we have \( u \in S(j, n) \).

**Proof.** First \( u \) is a positive integer and \( u \leq (2^{2^{k-j-1} - 2} + 1)2^j + 2^j - 1 = 2^k - 1 = n \).

On the other hand, in base 2, elements \( \alpha \) and \( i \) can be written as:

\[
\alpha = \sum_{t=0}^{m} b_t 2^t, \ b_t \in \{0, 1\}, \ m < k - j - 1 \text{ and } i = \sum_{t=0}^{r} c_t 2^t, \ c_t \in \{0, 1\}, \ r < j.
\]

Therefore:

\[
u = \sum_{t=0}^{m} b_t 2^{j+1+t} + 2^j + \sum_{t=0}^{r} c_t 2^t.
\]

As \( i \in \{0, 1, 2, \ldots, 2^j - 1\} \), the binary representation of \( u \) is

\[
\mathcal{B}(u) = b_s b_{s-1} \ldots b_1 b_0 1 0 0 \ldots 0 0 c_r c_{r-1} \ldots c_1 c_0 \tag{2}
\]

As it is easy to see that \( b_0 \) is in the position \( j + 2 \), we deduce that 1 is in the \( (j+1) \text{th position} \) and the proof is achieved.

The next result is essential to the construction of our algorithm.

**Theorem 3.1:** Let \( k \in \mathbb{N} \setminus \{0\} \) and \( n = 2^k - 1 \). For every fixed \( j \in \{0, 1, 2, \ldots, k-1\} \), if
we set $T(j, n) = \{(2\alpha + 1)2^j + i \mid 0 \leq \alpha \leq 2^{k-j-1} - 1$ and $0 \leq i \leq 2^j - 1\}$, then we have:

$$T(j, n) = S(j, n) \quad (3)$$

Proof. Proposition 2.1 shows that $T(j, n) \subset S(j, n)$. Conversely consider an element $u \in S(j, n)$. So the binary representation $B(u)$ contains 1 in the $(j+1)^{th}$ position from the right. Hence: $u = \sum_{t=0}^{s} b_t 2^{j+1+t} + 2^j \sum_{t=0}^{r} c_t 2^t$, $r < j$. By choosing $\alpha = \sum_{t=0}^{s} b_t 2^t$ and $i = \sum_{t=0}^{r} c_t 2^t$, we get $u = (2\alpha + 1)2^j + i$. Since $0 \leq u \leq n$, we can easily verify that $0 \leq \alpha \leq 2^{k-j-1} - 1$, which ends the proof.

**Corollary 3.1:** Let $r = 2^j - 1$ and $s = 2^{k-j-1} - 1$. In the coding Hamming algorithm, the checking bits $a_{2^j}$, which is artificially zero before the calculation, can be computed as:

$$a_{2^j} = \left( \sum_{\alpha=0}^{s} \sum_{i=0}^{r} a_{(2\alpha+1)2^j+i} \right) \mod 2 \quad (4)$$

For the decoding step,

$$\epsilon_j = \left( \sum_{u \in S(j,n)} a_u \right) \mod 2 = \left( \sum_{u \in T(j,n)} a_u \right) \mod 2, \text{ which give relations (4) and (5).} \quad \square$$

**Example 3.1:** In the early 1980s, the Minitel system [27, p. 177,185] [21, p. 110] was a national network in France, precursor to the moderne Internet. A Hamming code with 7 parity-check bits was used to correct single error in messages $M = a_1 a_2 \ldots a_n$, $n = 2^7 - 1$.

Let us see how, by Corollary 3.1, we can compute for instance the parity-check bit $a_{2^4}$ in the coding step.

We have $n = 127$ and $j = 4 \Rightarrow r = 15$ and $s = 3$. By relation (4) we fill the following table:

To determine the term $a_{2^4}$, we need to calculate the sum modulo 2 of all the 64 bits in the second column of Table 2.

If the message $M$ was received, in the decoding step, the sum modulo 2 of all the 64 bits
Values of $\alpha$ bits to add
\begin{tabular}{|c|c|}
\hline
0 & $a_{16} = 0 + a_{17} + a_{18} + \ldots + a_{31}$ \\
1 & $a_{48} + a_{49} + a_{50} + \ldots + a_{63}$ \\
2 & $a_{80} + a_{81} + a_{82} + \ldots + a_{95}$ \\
3 & $a_{112} + a_{113} + a_{114} + \ldots + a_{127}$ \\
\hline
\end{tabular}

Table 2: Computation of the parity-check $a_{24}$

in Table 2, with the real received value of $a_{16}$, gives the 4th bit from the right in the binary representation of the error position.

**Theorem 3.2:** Let $k \in \mathbb{N} - \{0\}$ and $n = 2^k - 1$. For every fixed $j \in \{0, 1, 2, \ldots, k - 1\}$, if we set $U(j, n) = \{2^j + 2i - (i \mod 2^j) \mid 0 \leq i \leq 2^{k-1} - 1\}$, then we have:

$$U(j, n) = T(j, n) \quad (6)$$

**Proof.** Let $u = 2^j + 2i - (i \mod 2^j) \in U(2^j, n)$. Put $i = q2^j + r$ with $q \in \mathbb{N}$ and $0 \leq r < 2^j$. So $i \mod 2^j = r$. Therefore $u = 2^j + q2^{j+1} + 2r - r = q2^{j+1} + 2^j + r$ and then $B(u)$ has 1 in the $(j + 1)$th position from the right. Consequently $u \in S(j, n) = T(j, n)$ by Theorem 3.1.

Conversely let $u = (2\alpha + 1)2^j + i \in T(j, n)$. By the definition of $T(j, n)$, we have $0 \leq \alpha \leq 2^{k-j-1} - 1$ and $0 \leq i \leq 2^j - 1$.

Put $i_1 = \alpha 2^j + i$. So $i_1 \mod 2^j = i$. On the other hand $2^j + 2i_1 - (i_1 \mod 2^j) = 2^j + \alpha 2^{j+1} + 2i - i = (2\alpha + 1)2^j + i$. Moreover $0 \leq i_1 \leq 2^{k-1} - 2^j + i \leq 2^{k-1} - 2^j + (2^j - 1) = 2^{k-1} - 1$.

Conclusion: $u \in U(j, n)$. \hfill \square

**Corollary 3.2:** Let $k \in \mathbb{N} - \{0\}$, $n = 2^k - 1$ and $r = 2^{k-1} - 1$. For every fixed $j \in \{0, 1, 2, \ldots, k - 1\}$, we put $J = 2^j$.

For the coding step:

$$a_J = \left[ \sum_{i=1}^{r} a_{J + 2i - (i \mod J)} \right] \mod 2 \quad (7)$$

For the decoding step:

$$\epsilon_J = \left[ \sum_{i=0}^{r} a_{J + 2i - (i \mod J)} \right] \mod 2 \quad (8)$$

**Proof.** By the Hamming algorithm, for the coding process, we have

$$a_{2^j} = \left( \sum_{u \in S(j, n) - \{2^j\}} a_u \right) \mod 2 \quad (9)$$
But Theorem 3.1 and Theorem 3.2 imply that the three sets $S(j, n)$, $T(j, n)$, $U(j, n)$ are identical, so:

$$a_{2j} = \left( \sum_{u \in U(j, n) - \{2^j\}} a_u \right) \mod 2 = \left( \sum_{i=0, i \neq 2^j}^{2^{k-1}-1} a_{2j+2^i-(i \mod 2^j)} \right) \mod 2$$

$$= \left( \sum_{i=1}^{2^{k-1}-1} a_{2j+2^i-(i \mod 2^j)} \right) \mod 2 = \left( \sum_{i=1}^{r} a_{J+2^i-(i \mod J)} \right) \mod 2,$$

and we get relation (7).

Similar proof for relation (8).

The next result is an other alternative manner for the calculation of the parity-check bits in the Hamming algorithm.

**Corollary 3.3:** With the same hypothesis as in Corollary 3.2., we have:

For the coding step:

$$a_J = \left[ \sum_{i=1}^{r} a_{J([i/J]+i)} \right] \mod 2 \quad (10)$$

For the decoding step:

$$\epsilon_J = \left[ \sum_{i=0}^{r} a_{J([i/J]+i)} \right] \mod 2 \quad (11)$$

**Proof.** for every $i \in \{0, 1, 2, \ldots, r\}$, the Euclidean division of $i$ by $J$ gives $i = J[i/J] + (i \mod J)$, so $J + 2i - (i \mod J) = J + i + ([i/J])J = J(1 + [i/J]) + i$. Thus $a_{J+2i-(i \mod J)} = a_{J(1+[i/J]+i)}$ and by Corollary 3.2, we get relations (10) and (11). □

We now move to the presentation of our coding and decoding algorithms.
4 Our algorithms for the Hamming code

Relation (4) in Corollary 3.1 leads to the following coding algorithm where comments are in italic font and delimited with braces

Algorithm 1 Determination of the parity check symbols

Require: The message $M = a_1 a_2 \ldots a_n$ to code before sending.
Ensure: The computation of all the checking bits $a_{2j}$.

$k \leftarrow 4$ \{$k$ is the number of the checking bits $a_{2j}$.\}

$n \leftarrow 2^k - 1$ \{$n$ is the length of the message $M$.\}

$M \leftarrow [1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1]$ \{$M$ is an example of a message to code.\}

for $j$ in $\{0, 1, \ldots, k - 1\}$ do

$u \leftarrow 2^j$ \{$u$ is the index of the checking symbols $a_{2j}$.\}

$max\_alpha \leftarrow 2^{k-j-1} - 1$ \{The bound $max\_alpha$ is the maximal value of $\alpha$.\}

$S \leftarrow 0$

for $\alpha$ in $\{0, 1, \ldots, max\_alpha\}$ do

$v \leftarrow (2\alpha + 1)u$

for $i$ in $\{0, 1, \ldots, 2^j - 1\}$ do

$w \leftarrow v + i$ \{$w = (2\alpha + 1)2^j + i$ is the index of the bit to add to $S$.\}

$S \leftarrow S + a_w$

end for

end for

$S \leftarrow S - a_{2j}$ \{The term $a_{2j}$ must not be part of the sum $S$.\}

$S \leftarrow S \mod 2$

$\alpha_{2j} \leftarrow S$ \{We assign $S$ to the checking term $a_{2j}$.\}

end for

print($M$) \{$M$ is the final coded message to send.\}
Relation (5) in Corollary 3.1 leads to the following decoding algorithm where comments are in italic font and delimited with braces.

**Algorithm 2** Determination of the error location

**Require:** The received message $M = b_1 b_2 \ldots b_n$ to correct.

**Ensure:** The computation of the error position.

$k ← 4 \{k$ is the number of the checking bits $b_{2j}\}$

$n ← 2^k − 1 \{n$ is the length of the message $M\}$

$M ← [1, 1, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 1] \{M$ is an example of a received message.$\}$

$X ← 0 \{X$ is the error position in decimal base$\}$

for $j$ in $\{0, 1, \ldots, k − 1\}$ do

$u ← 2^j \{u$ is the index of the checking symbols $b_{2j}\}$

$max_\alpha ← 2^{k−j−1} − 1 \{The bound $max_\alpha$ is the maximal value of $\alpha.$\}$

$S ← 0$

for $\alpha$ in $\{0, 1, \ldots, max_\alpha\}$ do

$v ← (2\alpha + 1)u$

for $i$ in $\{0, 1, \ldots, 2^j − 1\}$ do

$w ← v + i \{w = (2\alpha + 1)2^j + i$ is the index of the bit to add to $S.$\}$

$S ← S + b_w$

end for

end for

$S ← S \mod 2 \{S$ is computed in base 2.$\}$

$X ← X + S \ast 2^j \{We find the error position $X$ in base 10.$\}$

end for

print($X$) \{$X$ is the final error position. If $X = 0$ there is no error in the transmission.$\}$

## 5 Conclusion

In this paper, we presented a new simple and effective method for coding/decoding any transmitted message through a noisy channel that is protected against errors by the Hamming scheme. We also implemented practical corresponding algorithms. Our technique constitutes an alternative to the classical use of the generator matrix for coding or the parity-check matrix for decoding.

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