Abstract. Necessary and sufficient conditions for a sequence of positive integers to be the degree sequence of a $k$-connected simple graph are detailed. Conditions are also given under which such a sequence is necessarily $k$-connected.

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1. Introduction

Necessary and sufficient conditions for a sequence of non-negative integers to be connected i.e. the degree sequence of some finite simple connected graph, are implicit in Hakimi [4] and have been stated explicitly by the author in [5] and expounded on in [6]. This note builds upon these conditions of Hakimi and details necessary and sufficient conditions for a sequence of non-negative integers to be $k$-connected. The note concludes with necessary and sufficient conditions for a sequence of non-negative integers to be necessarily $k$-connected i.e. the sequence can only be realised as a $k$-connected graph.

2. Preliminaries

Let $G = (V_G, E_G)$ be a graph where $V_G$ denotes the vertex set of $G$ and $E_G \subseteq [V_G]^2$ denotes the edge set of $G$ (given that $[V_G]^2$ is the set of all 2-element subsets of $V_G$). An edge $\{a, b\}$ is denoted $ab$. A graph is finite when $|V_G| < \infty$ and $|E_G| < \infty$, where $|X|$ denotes the cardinality of the set $X$. The union of graphs $G$ and $H$ is the graph $G \cup H = (V_G \cup V_H, E_G \cup E_H)$ and $G \cup ab$ is understood to be the graph $(V_G, E_G) \cup (\{a, b\}, \{ab\})$. A graph is simple if it contains no loops (i.e. $aa \notin E_G$) or parallel/multiple edges (i.e. $\{ab, ab\} \subsetneq E_G$). The degree of a vertex $v$ in a graph $G$, denoted $\text{deg}(v)$, is the number of edges in $G$ which contain $v$. A graph where all vertices have degree $k$ is called a $k$-regular graph. A path is a graph with $n$ vertices in which two vertices, known as the endpoints, have degree 1 and $n - 2$ vertices
have degree 2. A graph is \textit{connected} if there exists at least one path between every pair of vertices in the graph. Paths $P_1$ and $P_2$, both with endpoints $a$ and $b$, are \textit{internally disjoint} if $P_1 \cap P_2 = \{\{a, b\}, \{\}\}$. A graph $G$ is $k$-\textit{connected} when there exists at least $k$ internally disjoint paths in $G$ between any two vertices in $G$. This characterisation of a graph being $k$-connected is synonymous with Menger’s Theorem. $K_n$ denotes the \textit{complete graph} on $n$ vertices. The \textit{complement} of a simple graph $G$ is the simple graph $\overline{G}$ with vertex set $V_G$ and edge set the pairs of vertices in $V_G$ which are not contained in $E_G$. All basic graph theory definitions can be found in standard texts such as [1], [2] or [3].

3. Degree sequences

A finite sequence $s = \{s_1, ..., s_n\}$ of non-negative integers is called \textit{graphic} if there exists a finite simple graph with vertex set $\{v_1, ..., v_n\}$ such that $v_i$ has degree $s_i$ for all $i = 1, ..., n$. Given a graph $G$ then the degree sequence $d(G)$ is the monotonic non-increasing sequence of degrees of the vertices in $V_G$. This means that every graphical sequence $s$ is equal to the degree sequence $d(G)$ of some graph $G$ (subject to possible rearrangement of the terms in $s$).

Definition 3.1. A finite sequence $s = \{s_1, ..., s_n\}$ of positive integers is called $k$-\textit{connected} if there exists a finite simple $k$-connected graph with vertex set $\{v_1, ..., v_n\}$ such that $\text{deg}(v_i) = s_i$ for all $i = 1, ..., n$.

A finite sequence $s = \{s_1, ..., s_n\}$ of positive integers is called \textit{necessarily} $k$-\textit{connected} if $s$ can only be realisable as a simple $k$-connected graph.

Given a sequence of positive integers $s = \{s_1, ..., s_n\}$ then define the associated pair of $s$, denoted $(\varphi(s), \epsilon(s))$, to be the pair $(n, \frac{1}{2} \sum_{i=1}^{n} s_i)$. Where no ambiguity can arise, $(\varphi(s), \epsilon(s))$ is simply denoted $(\varphi, \epsilon)$.

4. Results

Theorem 4.1. Given a sequence $s = \{s_1, ..., s_n\}$ of positive integers, with the associated pair $(\varphi, \epsilon)$, such that $s_i \geq s_{i+1}$ for $i = 1, ..., n-1$ then $s$ is $k$-connected if and only if

- $\epsilon \in \mathbb{N}$,
- $s_1 \leq \varphi - 1$ and $s_n \geq k$,
- $\frac{k\varphi}{2} \leq \epsilon \leq \left(\varphi\right)\left(\frac{\varphi}{2}\right)$.

Proof. ($\Rightarrow$) Clearly $\epsilon \in \mathbb{N}$ is a necessary condition for any sequence $s$ to be realisable as half the sum of the degrees in any graph is the number of edges in that graph which must be a natural number. The necessity of the condition $s_1 \leq \varphi - 1$ follows from the definition of a simple graph and the need for $s_n \geq k$ is due to the fact that every vertex in a $k$-connected graph has degree
at least $k$.

The necessity of the condition $\epsilon \geq \frac{k\varphi}{2}$ follows from the observation that the minimum possible $\epsilon$ of any $k$-connected sequence is $\epsilon = \frac{k\varphi}{2}$ which is contained in the associated pair of $s = \{k, \ldots, k\}$, the degree sequence of a $k$-regular graph. Note that $\epsilon$ must be even, by definition, and this occurs whenever $\varphi$ and $k$ are not both odd (as there cannot exist a $k$-regular graph, where $k$ is odd and the number of vertices is also odd). If both $\varphi$ and $k$ are odd, then the minimum possible $\epsilon$ of any $k$-connected sequence is $\epsilon = \frac{k\varphi + 1}{2}$ which is contained in the associated pair of $s = \{k + 1, k, \ldots, k\}$. To show the necessity of the condition $\epsilon \leq \left(\frac{\varphi}{2}\right)$, the parity of $\varphi$ is irrelevant. When maximising $\epsilon$ it follows that the maximum possible $\epsilon$ of any $k$-connected sequence is $\epsilon = \left(\frac{\varphi}{2}\right)$ which is contained in the associated pair of $s = \{\varphi - 1, \ldots, \varphi - 1\}$, the degree sequence of the complete graph $K_\varphi$.

It remains to show that, for a fixed $\varphi$, all sequences with $\frac{k\varphi}{2} < \epsilon < \left(\frac{\varphi}{2}\right)$ are realisable. Suppose that $\varphi$ and $k$ are not both odd and $k < \varphi - 1$. To ensure that $\epsilon$ remains even, two terms in $\{k, \ldots, k\}$ must be incremented by one to give $\{k + 1, k + 1, k, \ldots, k\}$. Given that $\{k, \ldots, k\}$ is realisable as a $k$-regular graph $G$ then $\{k + 1, k + 1, k, \ldots, k\}$ is realisable as $G \cup ab$ where $ab \in G$. This incrementing of two terms in a graphic sequence can be continued until the sequence with $\epsilon = \left(\frac{\varphi}{2}\right)$ is reached and this process is summarised in Figure 1.

| $s = \{s_1, \ldots, s_n\}$ | $\epsilon$ | $|E_G|$ |
|-----------------------------|-------------|-------------|
| $\{k, \ldots, k\}$         | $\frac{k\varphi}{2}$ | $\frac{kn}{2}$ |
| $\{k + 1, k + 1, k, \ldots, k\}$ | $\frac{k\varphi + 2}{2}$ | $\frac{kn}{2} + 1$ |
| ...                         | ...         | ...         |
| $\{n - 1, \ldots, n - 1\}$ | $\left(\frac{\varphi}{2}\right)$ | $(\binom{n}{2})$ |

Figure 1. $k$-connected degree sequences when $\varphi$ and $k$ are not both odd.

A similar argument exists when $\varphi$ and $k$ are both odd, with $k < \varphi - 1$, and this argument is summarised in Figure 2.

($\Leftarrow$) Suppose that $s = \{s_1, \ldots, s_n\}$ is $k$-connected. This means that $s$ is the degree sequence of a $k$-connected graph $G$, hence $\sum_{i=1}^{n} \deg(v_i) = 2|V_G|$ and
so $\epsilon \in \mathbb{N}$. As $G$ is $k$-connected then $\deg(v_i) \geq k$ for all $i = 1, \ldots, n$ hence if $G$ is a minimally $k$-connected graph on $n$ vertices then $d(G) = \{k, \ldots, k\}$ with $|E_G| = \frac{kn}{2}$ if $n$ and $k$ are not both odd or $d(G) = \{k+1, k, \ldots, k\}$ with $|E_G| = \frac{kn^2+1}{2}$ if $n$ and $k$ are both odd, hence $s_n \geq k$ and $\epsilon \geq \frac{k\varphi}{2}$. As $G$ is simple then $\deg(v_i) \leq n-1$ for all $i = 1, \ldots, n$ and the maximal simple ($k$-connected) graph on $n$ vertices is the complete graph $K_n$ which has the degree sequence $\{n-1, \ldots, n-1\}$ and $|E_{K_n}| = \binom{n}{2}$, hence $s_1 \leq n-1$ and $\epsilon \leq \binom{n}{2}$. To show that $k$-connected graphs exist for each $|E_G| \in \mathbb{N}$ such that $\frac{kn}{2} < |E_G| < \binom{n}{2}$ refer to the argument showing the existence of graphic sequences with $\frac{k\varphi}{2} < \epsilon < \binom{n}{2}$ detailed above, along with FIGURES 1 and 2.

**Theorem 4.2.** Given a sequence $s = \{s_1, \ldots, s_n\}$ of positive integers, with the associated pair $(\varphi, \epsilon)$, such that $s_i \geq s_{i+1}$ for $i = 1, \ldots, n-1$ then $s$ is necessarily $k$-connected if and only if $s$ is $k$-connected and $\epsilon > \binom{\varphi}{2} + 2k - 1$.

**Proof.** ($\Rightarrow$) Clearly it is necessary for $s$ to be $k$-connected if it is to be necessarily $k$-connected. It is required to show that it is necessary for $\epsilon > \binom{\varphi}{2} + 2k - 1$. Consider a sequence $s = \{s_1, \ldots, s_n\}$ such that $\epsilon = \binom{\varphi}{2} + 2k - 1$.

Observe that one such sequence is $s' = \{n-1, \ldots, n-1, n-3, \ldots, n-3, k, k\}$

where $\varphi(s') = k-1+n-k-1+2 = n$ and $\epsilon(s') = \frac{(n-1)(n-1)+2(n-k-1)+2k}{2} = \frac{(n-2)(n-3)+2(2k-1)}{2} = \binom{n-2}{2} + 2k - 1$.

Observe that $s' = d(G_1)$, see FIGURE 3 where $G_1 = H_1 \cup H_2$ with $H_1 \simeq K_{k+1}$, $H_2 \simeq K_{n-2}$ and $H_1 \cap H_2 \simeq K_{k-1}$. Note that as $G_1 \setminus (H_1 \cap H_2)$ is disconnected and that $H_1 \cap H_2 \simeq K_{k-1}$ (i.e. $|V_{H_1 \cap H_2}| = k-1$) then $G_1$ is $(k-1)$-connected.

However, $s' = \{n-1, \ldots, n-1, n-3, \ldots, n-3, k, k\}$ is, in fact, $k$-connected as $s'$ is also the degree sequence of $G_2$, see FIGURE 3 (noting that $v_iv_j \in E_{G_1}$ but $v_iv_j \notin E_{G_2}$). Therefore, it is required that $\epsilon > \binom{n-2}{2} + 2k - 1$ if $s$ is to
be necessarily $k$-connected.

Note that Figures 4, 5 and 6 are, respectively, the $k = 1, 2$ and 3 versions of Figure 3.

(⇐) It is now required to show that if $s$ is $k$-connected and $\epsilon > \binom{n-2}{2} + 2k - 1$ then $s$ is necessarily $k$-connected. To show this it is required to show that the maximum number of edges in a graph with $n$ vertices which is not $k$-connected is $\binom{n-2}{2} + 2k - 1$. The graph $G_1$ in Figure 3 shows that such a graph exists, so it remains to show that a graph with $\epsilon = \binom{n-2}{2} + 2k - 1$ is maximally $(k - 1)$-connected i.e. adding one edge will always result in a $k$-connected graph.
Observe that any maximally \((k-1)\)-connected graph \(G\) on \(n\) vertices will necessarily contain a cut set \(C\) containing \(k-1\) vertices. This means that \(G \setminus C\) is disconnected. To maximise the number of edges in \(G\) it is clear that \(G \setminus C\) contains two connected components i.e. \(G = H_1 \cup H_2\) where \(V_{H_1} \cap V_{H_2} = C\) with \(H_1 \simeq K_{a+b+1} - C\), \(H_2 \simeq K_{b+1} - C\) and \(H_1 \cap H_2 \simeq K_{a-b}\) (noting that \(a+b = n - |C|\)). So, the task of maximising \(|E_{H_1 \setminus C}| + |E_{H_2 \setminus C}|\) is equivalent to minimising the number of edges in a complete bipartite graph \(K_{a,b}\) as \(K_n \setminus (E_{H_1 \cup H_2}) \simeq K_{a,b}\).

Let \(a + b = n - |C|\), with \(a \leq b\), then \(|E_{K_{a,b}}| = ab\) where \(a, b \in \{1, ..., n - |C| - 1\}\). Note that \(a > 0\) as \(G \setminus C\) is disconnected i.e. \(K_a \neq K_0 = (\emptyset, \emptyset)\). It is straightforward to show that \(ab\) attains its maximum at \(a = b = \frac{n-|C|}{2}\), when \(n\) is even, and at \(a = \lceil \frac{n-|C|}{2} \rceil, b = \lfloor \frac{n-|C|}{2} \rfloor\) when \(n\) is odd. It follows that \(ab\) is minimised when \(a = 1\) and \(b = n - |C| - 1\). However, observe that \(a > 1\) as \(a = 1\) implies that \(H_1 \simeq K_{|C|+1}\) which means that \(d(G)\) contains a term equal to \(|C| = k - 1\), but this contradicts the \(s_n \geq k\) condition. Hence \(|E_{K_{a,b}}|, with \(a + b = n - |C|\), is minimised when \(a = 2\) and \(b = n - |C| - 2\) and so any maximally \((k-1)\)-connected graph on \(n\) vertices is isomorphic to \(H_1 \cup H_2\) where \(H_1 \simeq K_{k+1}, H_2 \simeq K_{n-k-1}\) and \(H_1 \cap H_2 \simeq K_{k-1}\), see \(G_1\) in Figure 6. Notice that the union of \(G_1\) and any edge in \(G_1\) results in a \(k\)-connected graph.

**Corollary 4.3.** All simple graphs with \(n\) vertices and at least \(\frac{n^2 - 5n + 6 + 4k}{2}\) edges are \(k\)-connected.

**Proof.** As shown in Theorem 4.2 a maximally \((k-1)\)-connected graph with \(n\) vertices is isomorphic to the union of \(H_1 \simeq K_{k+1}\) and \(H_2 \simeq K_{n-2}\) where \(H_1 \cap H_2 \simeq K_{k-1}\) and such a graph has \(|E_{K_{k+1}}| + |E_{K_{n-2}}| - |E_{K_{k-1}}| = \binom{n-2}{2} + \binom{k+1}{2} - \binom{k-1}{2} + \binom{k+1}{2} - \frac{(k-1)(k-2)}{2} = \binom{n-2}{2} + \frac{4k-2}{2}\) edges. Hence, any simple graph with \(n\) vertices and at least \(\binom{n-2}{2} + 2k - 1 + 1 = \frac{(n-2)(n-3)+4k}{2}\) edges is \(k\)-connected.

Note that for all \(n, k \in \mathbb{N}\), \(n^2 - 5n\) is even and \(n^2 - 5n + 6 + 4k > 0\), hence \(\frac{n^2 - 5n + 6 + 4k}{2} \in \mathbb{N}\).
References

[1] Bondy, J. A. and Murty, U. S. R., *Graph theory*, Graduate Texts in Mathematics 244 (2008), Springer.

[2] Diestel, R., *Graph theory*, Graduate Texts in Mathematics 173 (2000), Springer-Verlag.

[3] Gould, R., *Graph theory*, (1988), The Benjamin/Cummings Publishing Co. Inc.

[4] Hakimi, S. L., *On realizability of a set of integers as degrees of the vertices of a linear graph. I*, J. Soc. Indust. Appl. Math. 10 (1962), 496–506.

[5] McLaughlin, J., *On connected degree sequences*, [arXiv:1511.09321](https://arxiv.org/abs/1511.09321) (2015).

[6] McLaughlin, J., *On connected simple graphs and their degree sequences*, [arXiv:1512.01377](https://arxiv.org/abs/1512.01377) (2015).

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