About the coordinate time for photons in Lifshitz Space-times

J. R. Villanueva

Departamento de Física y Astronomía, Facultad de Ciencias,
Universidad de Valparaíso, Gran Bretaña 1111,
Valparaíso, Chile, and
Centro de Astrofísica de Valparaíso,
Gran Bretaña 1111, Playa Ancha,
Valparaíso, Chile.

Yerko Vásquez

Departamento de Ciencias Físicas,
Facultad de Ingeniería, Ciencias y Administración,
Universidad de La Frontera,
Avenida Francisco Salazar 01145, Casilla 54-D,
Temuco, Chile.

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In this paper we studied the behavior of radial photons from the point of view of the coordinate time in (asymptotically) Lifshitz space-times, and we found a generalization to the result reported in previous works by Cruz et. al. [Eur. Phys. J. C 73, 7 (2013)], Olivares et. al. [Astrophys. Space Sci. 347, 83-89 (2013)], and Olivares et. al. [arXiv: 1306.5285]. We demonstrate that all asymptotically Lifshitz space-times characterized by a lapse function $f(r)$ which tends to one when $r \to \infty$, present the same behavior, in the sense that an external observer will see that photons arrive at spatial infinity in a finite coordinate time. Also, we show that radial photons in the proper system cannot determine the presence of the black hole in the region $r_+ < r < \infty$, because the proper time results to be independent of the lapse function $f(r)$.

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I. INTRODUCTION

Lifshitz space-times have proven to be important from a holographic point of view. They represent gravity duals of non-relativistic systems and they are of particular interest in the studies of critical exponent theory and phase transitions [1, 2]. In these space-times the spatial and the temporal coordinates scale in different ways, reflecting the symmetries of boundary field theory.

Lifshitz vacua space-times in d-dimensions are represented by the line element

$$ ds^2 = -\frac{r^{2z}}{\ell^2} dt^2 + \frac{\ell^2}{r^2} dr^2 + r^2 d\vec{x}^2, $$

where $z$ is the dynamical exponent, $\ell$ is the only length scale in the geometry and $\vec{x}$ stands for a spatial (d-2)-dimensional vector.

This metric is invariant under the following change of coordinates:

$$ t \to \lambda^z t, \quad x \to \lambda x, \quad r \to r/\lambda. $$

In this way, here $z$ is understood as a measure of the anisotropy between spatial and temporal coordinates. For $z = 1$ we recover the AdS metric in Poincaré coordinates.

Black hole solutions whose asymptotic behavior is given by metric [1] have been studied recently. Lifshitz black holes with a flat transverse section in four dimensions with $z = 2$ have been reported in [3, 4], a topological Lifshitz black hole with $z = 2$ was found in [5], a black hole with spherical horizon with $z = 4$ was found in [6], a three-dimensional Lifshitz black hole with $z = 3$ was reported in [7], black holes of Lovelock theory can be found in [8].
Recently, in [8,11] the authors showed that radial photons in those Lifshitz backgrounds can reach the asymptotic region in a finite coordinate time, as seen by an external observer. Motivated by these results we analyzed Lifshitz space-times with an arbitrary dynamical exponent $z$, and showed that this behavior still holds.

The paper is organized as follows. In section II we obtain the fundamental equations for massless particles in the (asymptotically) Lifshitz space-time, then we analyze the radial motion in terms of the coordinate time. Also, we analyze different classes of lapse function that appear in the literature. In section III we analyze the radial motion in terms of the coordinate time. Finally, in section IV we discuss our results and make conclusions.

II. GENERIC LIFSHITZ BLACK HOLE SPACE-TIME

The metric of a generic Lifshitz space-times can be written as

$$ds^2 = \frac{r^{2z}}{\ell^{2z}} f(r) dt^2 + \frac{\ell^2}{r^2} \frac{dr^2}{f(r)} + \frac{r^2}{\ell^2} d\mathbf{x}^2$$

(3)

where $\mathbf{x}$ is a $(d-2)$-dimensional vector, and the function $f(r)$ depends only on the radial coordinate. Furthermore, this function takes the value $f(r) = 1$ when $r \to \infty$, which means that the metrics are asymptotically Lifshitz. So, the Lagrangian associated with the metric (3) for radial photons is given by [8,12]

$$2L = -\frac{r^{2z}}{\ell^{2z}} f(r) i^2 + \frac{\ell^2}{r^2} \dot{r}^2 f(r) = 0$$

(4)

which is independent of the space-time dimension, and the dot represents a derivative with respect to an affine parameter along the trajectory.

Thus, we can immediately obtain the following quadrature

$$\frac{dr}{dt} = \pm \frac{r^{z+1}}{\ell^{z+1}} f(r),$$

(5)

which will be solved by explicitly giving the radial function $f(r)$. Therefore, assuming that photons are placed in $r = R_0$ when $t = 0$, we can write the above equation as

$$t = \pm \ell^{z+1} \int_{R_0}^{r} \frac{dr'}{r'^{z+1} f(r')}.$$  

(6)

In order to obtain a full description, we will study different space-times characterized by the radial function $f(r)$ separately.

A. Lifshitz Vacua Space-time

This case represents a space-time without black holes, i.e., the case with $f(r) = 1$ [3]. Thus, Eq. (6) is written as

$$t(r) = \pm \ell^{z+1} \int_{R_0}^{r} \frac{dr'}{r'^{z+1}},$$

(7)

so, a straightforward integration yields to

$$T(X) = \pm \left[ \left( \frac{1}{X_0} \right)^z - \left( \frac{1}{X} \right)^z \right],$$

(8)

where $T = t/\ell$ and $X = r/\ell$ are dimensionless variables, and $X_0 = R_0/\ell$. Thus, an external observer sees that photons moving from $X = X_0$ to $X = 0$ need an infinite coordinate time to do so, whereas photons moving from $X_0$ to the spatial infinite needs a finite time, $T_1$, given by

$$T_1 = \lim_{X \to \infty} T(X) = \left( \frac{1}{X_0} \right)^z.$$  

(9)

In FIG. 1 we plot the behavior of the dimensionless function given by Eq. (8).

![FIG. 1: Plot for the dimensionless coordinate time, $T \equiv t/\ell$, of photons generated by the lapse function $f(X) = 1$ given by Eq. (5), where $X \equiv r/\ell$ and $X_0 \equiv R_0/\ell$. Here we presents several curves for different values of the dynamical exponent $z$. This shows that, as seen by a system external to photons, they will fall asymptotically to origin $X = 0$. On the other hand, this external observer will see that photons arrive at the spatial infinite in a finite coordinate time, $T_1$, given by Eq. (9).](image)

We know from the above analysis that for Lifshitz vacua space-times, the integral

$$t_1 = \ell^{z+1} \int_{R_0}^{\infty} \frac{dr}{r^{z+1}},$$

(10)

converges. Now, we would like to know if the following integral converges

$$t_1 = \ell^{z+1} \int_{R_0}^{\infty} \frac{dr}{r^{z+1} f(r)},$$

(11)
for a generic \( f (r) \), with \( f (r) \to 1 \) when \( r \to \infty \).

In order to determine this, we apply the limit comparison test for improper integrals. First, defining \( f_1 (r) = \frac{1}{r^n} \) and \( f_2 (r) = \frac{1}{r^{n+1}} \), both positive definite in the interval \([R_0, \infty)\).

Now, taking the limit we obtain

\[
\lim_{r \to \infty} f_1 (r) = \lim_{r \to \infty} f_2 (r) = 1.
\]

This limit is \( a = 1 \) for asymptotically Lifshitz black holes; therefore, in accordance with the criteria for integral convergence, the integral \([11]\) is convergent. This means that it takes a finite coordinate time to reach the asymptotic region of radial photons. In the next sections we will consider generic functions \( f (r) \) and evaluate explicitly the integrals \([11]\) for arbitrary \( z > 0 \).

**B. Asymptotically Lifshitz black hole I**

Here we will consider the family of asymptotically Lifshitz black holes whose radial function have the form

\[
f(r) = 1 - \frac{r_+^{n}}{r^n}.
\]

Here \( r_+ \) is the so-called event horizon of the Lifshitz black hole, and \( n \) is a real number which may depend (or not) on the dynamical exponent \( z \). Functions of this kind can be found, for example, in \([3, 5, 7, 13]\). Therefore, we can rewrite Eq. \( \text{[6]} \) as

\[
t(r) = \pm \ell^{z+1} \int_{R_0}^{r} \frac{dr'}{r'^{2z-(n-1)}} \left( r'^n - r_+^n \right),
\]

in which case, a straightforward integration leads us to the solution

\[
T(X) = \pm \frac{1}{z} \left[ \Omega(X_0; X_+, z, n) - \Omega(X; X_+, z, n) \right],
\]

where the function \( \Omega \) is given explicitly by

\[
\Omega(y; X_+, z, n) = \frac{1}{y^n} 2 F_1 \left( 1, \frac{z}{n}, \frac{z+n}{n}, \frac{X_+^n}{y^n} \right),
\]

and \( X_+ \equiv r_+/\ell \). In Fig. 2 we plot the behavior of the dimensionless function given by Eq. \( \text{[14]} \).

It can be ascertained here that an external observer will see that photons take an infinite coordinate time to arrive at the event horizon, which is a common fact with Einstein’s space-times (S, SdS, SAdS, etc.). Moreover, again we obtain the situation described in the previous example, i.e., there is a finite coordinate time in which photons arrive at the spatial infinite given by

\[
T_1 = \lim_{X \to \infty} T(X) = \frac{2 F_1 \left( 1, \frac{z}{n}, \frac{n+z}{n}, \frac{X_+^n}{X_0^n} \right)}{z X_0^2}.
\]

Recently, this behavior has been reported in \([9]\) for \( z = 3 \) and \( n = 2 \), in \([10]\) for \( z = 2 \) and \( n = 2 \), and in \([11]\) for \( z = 2 \) and \( n = 4 \).

![Fig. 2: Plot for the dimensionless coordinate time, \( T \equiv t/\ell \), of photons generated by the lapse function \( f(X) = 1 - X_+^n/X^n \) given by Eq. \( \text{[14]} \), where \( X \equiv r/\ell \), \( X_+ \equiv r_+/\ell \) and \( X_0 \equiv R_0/\ell \). Here we present several curves for different values of the dynamical exponent \( z \) and the parameter \( n \). This shows that, as seen by a system external to photons, they will fall asymptotically to event horizon \( X = X_+ \). On the other hand, this external observer will see that photons arrive at the spatial infinite in a finite coordinate time, \( T_1 \), given by Eq. \( \text{[16]} \).](image)

**C. Asymptotically Lifshitz black hole II**

In addition to the case above, let us consider the following function:

\[
f(r) = \left( 1 - \frac{r_+^n}{r^n} \right) \left( 1 + \frac{g}{r^n} \right),
\]

where \( r_+ \) is the event horizon and \( g \) is any real constant. This kind of lapse function can be founded, for example, in \([6]\). Therefore Eq. \( \text{[9]} \) can be written as

\[
t(r) = \pm \ell^{z+1} \int_{R_0}^{r} \frac{dr'}{r'^{2z-(n-1)}} \left( r'^n - r_+^n \right) \left( r^n + g \right),
\]

and its dimensionless solution is given by

\[
T(X) = \pm \frac{\Psi(X_0; X_+, z, n, \tilde{g}) - \Psi(X; X_+, z, n, \tilde{g})}{z (\tilde{g} + X_+^n)},
\]

where the function \( \Psi \) is given explicitly by
the Lagrangian (4) we obtain

$$
\Psi(y, X_+, z, n, \tilde{g}) = \frac{X^n}{y^{2n}} \frac{2F_1}{y^n} \left( 1, \frac{z}{n} \frac{n + z}{n}, X^\frac{n}{y^n} \right) + \frac{\tilde{g}}{y^n} \frac{2F_1}{y^n} \left( 1, \frac{z}{n} \frac{n + z}{n}, \tilde{g} \frac{y^n}{y^n} \right),
$$

(20)

and $\tilde{g} \equiv g/\ell^n$. In FIG. 3 we plot the behavior of the dimensionless function given by Eq. (19). Again, we found the same qualitative behavior as preceding cases. An external observer will see that photons take an infinite coordinate time to arrive at event horizon, while they arrive at the spatial infinite in a finite coordinate time given by

$$
T_1 = \lim_{X \to \infty} T(X) = \frac{\Psi(X_0, X_+, z, n, \tilde{g})}{\tilde{g} + (X^2_+)},
$$

(21)

III. A NOTE ON THE PROPER SYSTEM

Since the metric of a Lifshitz space-time is static, the Lagrangian associated is independent of the temporal coordinate $t$, and thus, its corresponding canonical conjugate momentum is a conserved quantity. So, considering the Lagrangian (1) we obtain

$$
\Pi_t = \frac{\partial L}{\partial \dot{t}} = -\frac{r^{2z}}{\ell^{2z}} f(r) \dot{t} = -\sqrt{E}.
$$

(22)

Therefore, by combining Eq. (5) with Eq. (22) we obtain the quadrature

$$
\frac{dr}{d\tau} = \pm \frac{f^{z-1}}{r^{z-1}} \sqrt{E},
$$

(23)

and its dimensionless solution is given by

$$
\Theta(X) = \pm \frac{X^z - X_0^z}{\sqrt{E}},
$$

(24)

where $\Theta \equiv \tau/\ell$ is the dimensionless proper time. The first observation of the solution (24) is that the proper system no detects the shape of the lapse function $f(r)$, i.e., if photons are in the region $r_+ < r < \infty (X_+ < X < \infty)$, then they cannot determine the presence of black hole, which is in agreement with General Relativity, moreover, the Schwarzschild case is recuperated when $z = 1$. In this sense, massless particles presents the same asymptotic behavior as in Einstein’s space-times:

1. They cross the event horizon in a finite (dimensionless) proper time $\Theta_+$ given by

$$
\Theta_+ \equiv \Theta(X_+) = \frac{X_0^z - X_+^z}{\sqrt{E}},
$$

(25)

and, eventually, arrives at the origin in a finite (dimensionless) proper time given by

$$
\Theta_0 \equiv \Theta(X = 0) = \frac{X_0^z}{\sqrt{E}}
$$

(26)

2. They requires an infinite (dimensionless) proper time to arrive at the spatial infinite, i.e., $\tau \to \infty$ when $r \to \infty$.

IV. SUMMARY

We obtained a generalization for the behavior of massless particles in a Lifshitz space-time reported in previous works [9–11]. This generalization is independent of the dynamical exponent $z$, and the dimension of space-time. We found that space-time looks similar to space-times of general relativity in the sense that an external observer
FIG. 4: Plot for the dimensionless proper time, $\Theta \equiv \tau / \ell$, as a function of the dimensionless radial coordinate $X \equiv r / \ell$, where $X_0 \equiv R_0 / \ell$. Here we present several curves for different values of the dynamical exponent $z$ with the same value of the constant of motion $E$. Notice that the Schwarzschild case is recuperated when $z = 1$. Notice that the Schwarzschild case is recuperated when $z = 1$. Measures an infinite coordinate time to photons traveling to the event horizon (or to the origin, if a Lifshitz vacua space-time is considered). On the other hand, by using of a simple criterion of convergence, we found a general condition over the measure of coordinate time employed by photons to arrive at spatial infinite. Our result is that an external observer will see that photons arrive at the spatial infinite in a finite coordinate time, and the condition is that the lapse function, $f(r)$, tends to one when $r \to \infty$. Obviously, this condition always satisfies the asymptotic Lifshitz space-times, and therefore, our novel result becomes general for all space-times that satisfy the above condition. Additionally, we obtained that, from the point of view of the proper system, massless particles present the same behavior as Einstein’s space-times, because the proper time is independent of the lapse function $f(r)$, therefore, the photon cannot determine the presence of the black hole in the region $r_+ < r < \infty$.

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