Null hypersurfaces and trapping horizons

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Abstract

The purpose in this work is to study (marginally) trapped submanifolds lying in a null hypersurface. Let \((M, g, N) \rightarrow \mathcal{M}(c)\) be a null hypersurface of a Lorentzian space form, endowed with a Screen Integrable and Conformal rigging \(N\). The (Marginally) Trapped Submanifolds we are interested with are particular leaves of the screen distribution according to the sign of their expansions. We prove that if \(M\) is totally umbilical then leaves of the screen distribution are space forms with known sectional curvature. In particular, in a spacetime \(\mathcal{M}(c)\) with constant sectional curvature \(c\), cross-sections of a Non-Expanding Horizon are space forms of the same sectional curvature. We also show that a null Monge hypersurface graph of a function \(F\) is a trapping horizon if and only if \(F\) is harmonic.

Keywords: Monge hypersurface, Null hypersurface, screen distribution, Rigging vector field, Marginally Trapped Submanifold, Trapping Horizon, Black Hole

1 Introduction

Let \((\mathcal{M}, \mathcal{g})\) be a proper semi-Riemannian manifold and \(x : M \rightarrow \mathcal{M}\) be an embedded hypersurface of \(\mathcal{M}\). The pull-back metric \(g = x^*\mathcal{g}\) can be either degenerate or non-degenerate on \(M\). When \(g\) is non-degenerate, one says that \((M, g)\) is a semi-Riemannian hypersurface of \((\mathcal{M}, \mathcal{g})\) and \((M, g)\) is said to be a null (or degenerate, or lightlike) hypersurface of \((\mathcal{M}, \mathcal{g})\) when \(g\) is degenerate. Since any semi-Riemannian hypersurface has a natural transversal vector field, namely the Gauss map, which is anywhere orthogonal to the hypersurface, there is a standard way to study such an hypersurface. Geometry tools of the ambient manifold \(\mathcal{M}\) are projected orthogonally on \(M\) and give new tools which can be used to study the geometry of the hypersurface.

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But, for a null hypersurface \( M \), the normal bundle \( TM^\perp \) is not transversal, since it is tangent to the null hypersurface. Therefore, we need to find other approaches to study the geometry of null hypersurfaces. Many authors, for instance K. L. Duggal, A. Bejancu and M. Gutierrez, B. Olea [1, 2] have proposed some approaches. In [1], it is proved that for any choice of a supplementary distribution \( S(TM) \) (called screen distribution) of \( TM^\perp \) in \( TM \), there exists a unique rank one transversal bundle \( tr(TM) \) to \( M \). The difficulty with this method is that, (1) there may not necessarily exist a section of \( tr(TM) \) globally defined on \( M \) and (2) such a screen distribution is not easy to select explicitly. Studying the geometry of \( M \) using the decomposition provided by this transversal bundle is then local. In [2], authors consider a vector field \( \zeta \) globally defined on \( M \) and anywhere transversal to \( M \) (called a rigging vector field). \( \zeta \) fixes a unique transversal bundle and a unique screen distribution. A rigging vector field may not necessarily exist for any null hypersurface. However for a spacetime (time-oriented Lorentzian manifold) \( \overline{M} \), there exists a timelike vector field globally defined. This timelike vector field can be set as a rigging for any null hypersurface of \( \overline{M} \), since a timelike vector field can’t be tangent to a null hypersurface.

Introduced by Penrose in [3], the concept of \textit{trapped surfaces} plays an important role in general relativity. A spacelike surface \( S \) is said to be a trapped surface if all light rays emitted from the surface locally converge. Nothing can escape, not even the light. It is believed that there will be a marginally trapped surface separating the trapped surfaces from the untrapped ones where the outgoing light rays are instantaneously parallel. Marginally trapped surfaces are intensively study in general relativity [4, 5, 6, 7].

Galileo’s principle according to which all bodies fall equally fast is the equivalent to the Newtonian principle saying that the initial mass (the \( m \) in the fundamental Newton formula \( F = ma \)) and the passive gravitational mass (the mass acted on by a gravitational field) are equal for a given body [8]. Hence for this two theories, gravity is a field present always in the universe which affects all bodies. In general relativity, gravitational field is the manifestation of the curvature of the spacetime which is the consequence of the presence of the matter and no notion of intense gravitational field can be attached to one single spacetime point: a local notion becomes necessary. A normal bundle of a spacelike codimension-two submanifold \( S \) can be spanned by two future-directed null vector fields, say \( N \) and \( \xi^+ \). Since trajectories of light are null geodesics, \( S \) can be taken as initial event for sending two pulses of light: one towards one side of the surface (say inwards) and another towards the other side (say outwards). When the gravitational field is weak, the pulse of light sent outwards will increase its area, while the pulse of light sent inwards will have decreasing area. If the gravitational field near the submanifold is intense and directed inwards, it is possible that the outward light geodesics may bend inwards sufficiently so that the area of the light fronts decreases. This geometric fact may be taken as indicator of the presence.
of strong gravitational field. Spacelike codimension-two submanifolds where this behaviour occurs are called trapped submanifolds and the ones with a behavior borderline between the "normal" situation and the strong gravitational field situation are called marginally trapped. [4, 5, 6, 9, 10, For more physical comment on (marginally) trapped submanifolds.]

Let $x : M \rightarrow \overline{M}$ be a null hypersurface of a $(n + 2)$−dimensional spacetime $(\overline{M}, \langle \cdot , \cdot \rangle)$. Let $N$ be a null vector field defined on $\overline{M}$ and anywhere transversal to $M$: we call $N$ a rigging vector field for $M$.

Let $\xi$ be the unique null vector field on $M$ such that $\langle N, \xi \rangle = 1$ ($\xi$ is called the associated rigged vector field) and $S(N) = ker(x^\ast \langle N, \cdot \rangle)$ the associated screen distribution. The shape operator $A_N$ (resp. $\star A_\xi$) is defined on sections of $TM$ as the projection of the covariant derivative $-\nabla \cdot N$ (resp. $-\nabla \xi$) on the screen distribution along $N$ (resp. $\xi$). $M$ (resp. the screen distribution) is said to be totally umbilical if there is a function $\rho$ (resp. $\lambda$) such that $\star A_\xi = \rho P$ (resp. $A_N = \lambda P$), $P$ being the projection from $TM$ onto $S(N)$ along $\xi$. The vector field $N$ is said to be conformal if there exists a function $\varphi$ such that $A_N = \varphi A_\xi$. A Screen Integrable and Conformal (SIC) rigging is a conformal rigging with integrable associated screen distribution. This paper is organized as follows. This section is labeled Introduction. In the next Section 2, we recall the general setup and notations on null hypersurfaces. Particularly, we mention some useful results on rigged lightlike hypersurfaces and the behavior of some geometrical objects under the change of rigging. Section 3 is devoted to the main results of this paper. We characterize a (marginally) trapped submanifold lying in $M$ by geometric tools of $M$ and show that trapping horizons are minimal null hypersurfaces (null hypersurfaces for which the trace of $A_\xi$ vanishes).

We also prove that cross-sections of a Non-Expanding Horizon in a spacetime of constant sectional curvature are space forms of the same sectional curvature. More generally, let $(M, g, N) \rightarrow \overline{M}(c)$ be a null hypersurface (of a spacetime of constant sectional curvature $c$) endowed with a SIC rigging $N$. We prove that if $M$ is totally umbilical with $A_\xi = \rho P$, then leaves of the screen distribution are space forms with constant sectional curvature $\kappa = c + 2\varphi \rho^2$. We investigate the case of Monge null hypersurfaces in Lorentz-Minkowski space $\mathbb{R}^{n+2}_{\ast}$ and find that they cannot be foliated by trapped submanifold. We also obtain the following results.

**Theorem A**: A null hypersurface $(M, g)$ of a semi-Riemannian space form $(\overline{M}(c), \overline{g})$ is totally geodesic if and only if it is minimal.

**Theorem B**: Let $M$ be a null hypersurface of a semi-Riemannian manifold satisfying the null convergence condition. Then, $M$ is totally geodesic if and only if $M$ is minimal.
2 General setup on null hypersurfaces

Throughout this work, $(\overline{M}, \overline{g})$ is a $(n + 2)$-dimensional $(n \geq 1)$ semi-Riemannian manifold of index $q \geq 1$. From Section 3, we will set $(\overline{M}, \overline{g})$ to be a Lorentzian manifold time-oriented (and then space-oriented) of signature $(-, +, \cdots, +)$, which we call a spacetime. $\nabla$ and $\hat{\nabla}$ will denote respectively the Levi-Civita connection and the Riemannian curvature of $\overline{g}$. The metric $\overline{g}$ will sometime be denoted by $\langle \cdot, \cdot \rangle$. All manifolds are supposed to be smooth and connected. Let $\Sigma$ be a $d$-dimensional manifold with $d \leq n + 2$. If there exists an immersion $x : \Sigma \to \overline{M}$ then, $x(\Sigma)$ is said to be a $d$-dimensional immersed submanifold of $\overline{M}$. If moreover $x$ is injective one says that $x(\Sigma)$ is a $d$-dimensional submanifold of $\overline{M}$. If in addition the inverse map $x^{-1}$ is a continue map from $x(\Sigma)$ to $\Sigma$, $x(\Sigma)$ is a $d$-dimensional embedded submanifold of $\overline{M}$. When $x(\Sigma)$ is an embedded submanifold, one identify $\Sigma$ and $x(\Sigma)$. All submanifolds will be taken as embedded and through the identification, saying that $x : M \to \overline{M}$ is a submanifold will mean that there is an embedding $x : \Sigma \to \overline{M}$ such that $M = x(\Sigma)$. An hypersurface of $\overline{M}$ is a submanifold of $\overline{M}$ of dimension $d = n + 1$. We will said that $x : (M, g) \to (\overline{M}, \overline{g})$ is an isometrically immersed submanifold when, $x : M \to \overline{M}$ is a submanifold of $\overline{M}$ and $\overline{g} = x^* \overline{g}$. An isometrically immersed submanifold $x : (M, g) \to (\overline{M}, \overline{g})$ will said to be a spacelike submanifold (resp. a Lorentzian submanifold) if $(M, g)$ is a spacelike (resp. a Lorentzian) manifold, and a null submanifold (or a lightlike submanifold) if the metric $g$ is degenerate. The latter means that at each point $p \in M$ there exists a nonzero vector $u \in T_pM$ such that $g_p(u, v) = 0$ for any $v \in T_pM$.

Let $x : (M, g) \to (\overline{M}, \overline{g})$ be a isometrically immersed lightlike hypersurface. Then, the normal bundle $TM^\perp$ is a sub-distribution of $TM$ and the orthogonal projection of $T_p\overline{M}$ onto $T_pM$ is not defined. To study the geometry of the null hypersurface $M$, we need to choose smoothly a transversal direction to $T_pM$ and define up to this transversal direction a projection onto $T_pM$. There are infinitely many possibilities to choose a complementarity of $TM^\perp$ in $TM$ and a such complementarity is called screen distribution. Let $\mathcal{S}(TM)$ be a fixed screen distribution. One then has,

$$TM = \mathcal{S}(TM) \oplus_{\text{orth}} TM^\perp.$$  \hfill (2.1)

Firstly we have:

**Theorem 2.1** ([1]). Let $(M, g, \mathcal{S}(TM))$ be a null hypersurface endowed with a chosen screen distribution $\mathcal{S}(TM)$. Then, there exists a unique rank $1$ vector bundle $\text{tr}(TM)$ over $M$, such that for any nonzero section $\xi$ of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section $N$ of $\text{tr}(TM)$ on $\mathcal{U}$ satisfying

$$\overline{g}(\xi, N) = 1, \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \quad \forall W \in \Gamma(\mathcal{S}(TM)|_\mathcal{U}).$$  \hfill (2.2)

One then has the decomposition,

$$TM|_M = TM \oplus \text{tr}(TM) = \mathcal{S}(TM) \oplus_{\text{orth}} (TM^\perp \oplus \text{tr}(TM)).$$  \hfill (2.3)
Secondly, it is noteworthy that the choice of a null transversal vector field \(N\) along \(M\) determines the null transversal vector bundle, the screen distribution \(S(TM)\) and a unique radical vector field, say \(\xi\), satisfying (2.2) and (2.3). Now, to continue our discussion, we need to clarify the concept of rigging for our null hypersurface. In [2], the authors introduce the notion of rigging for null hypersurfaces of Lorentzian manifolds. We show here that this notion can be generally defined when the ambient space is a semi-Riemannian manifold as well as it is the case here.

**Definition 2.1.** A **rigging** for \(M\) is a vector field \(L\) defined on some open set \(\mathcal{O}\) containing \(M\) such that \(L_p \notin T_p M\) for each \(p \in M\).

One can always suppose that \(\mathcal{O} = \overline{M}\). A **null rigging** for \(M\) is a rigging \(L\) such that for all \(p \in M\), \(\overline{\eta}_p(L_p, L_p) = 0\). Let \(L\) be a rigging for \(M\) then \(L\) is a vector field over \(\overline{M}\) and we set \(\theta\) to be the \(1\)-form \(\overline{\eta}\)-metrically equivalent to \(L\). We also set \(\eta = x^*\theta\) and \(\overline{g} = g + \eta \otimes \eta\).

**Lemma 2.1.** \(\overline{g}\) is a non-degenerate metric on \(M\).

*Proof.* Let \(u \in T_p M\) such that \(\overline{g}_p(u, v) = 0\) for every \(v \in T_p M\). In particular, for \(\xi \in TM^\bot\), one has \(0 = \overline{g}_p(u, \xi_p) = \overline{g}_p(\xi_p, L_p)\overline{g}_p(u, L_p)\). Since \(\xi \in \Gamma(TM^\bot)\) and \(\overline{g}\) is non-degenerate, \(\overline{g}_p(\xi_p, L_p) \neq 0\) and then \(\overline{g}_p(u, L_p) = 0\). Joining this with the fact that \(T_p \overline{M}|_M = \text{span}\{L_p\} \oplus T_p M\), one has \(\overline{g}_p(u, v) = 0\) for every \(v \in T_p \overline{M}\), which implies that \(u = 0\), since \(\overline{g}_p\) is a non-degenerate metric. \(\Box\)

The triple \((M, g, L)\) is called a **rigged null hypersurface** and \(\overline{g}\) the **associated metric**, it is a semi-Riemannian metric of index \(q - 1\). One defines the screen distribution associated to the chosen rigging \(L\) by \(S(TM) = \ker(\eta)\).

**Definition 2.2.** The **rigged** vector field of \(L\) is the vector field which is \(\overline{g}\)-metrically equivalent to the \(1\)-form \(\eta\) and it is denoted by \(\xi\).

**Lemma 2.2.** Let \(x : (M, g, L) \rightarrow (\overline{M}, \overline{g})\) be a rigged null hypersurface, \(\xi\) the associated rigged vector field and \(S(TM)\) the associated screen distribution. Then, \(N = L - \frac{1}{2}\overline{g}(L, L)\xi\) is a null rigging and equations in (2.2) and decompositions in (2.3) hold with transversal bundle \(\text{tr}(TM) = \text{span}\{N\}\).

*Proof.* It is easy to see that \(N\) is a null rigging and \(\eta = \overline{g}(N, \cdot)\). By definition, \(\overline{g}(\xi, \xi) = g(\xi, \xi) + \eta(\xi)^2 = \overline{g}(N, \xi)^2 \neq 0\). In another hand since \(\xi\) is the \(\overline{g}\)-metrically equivalent vector to the \(1\)-form \(\eta\), one has \(\overline{g}(\xi, \xi) = \eta(\xi) = \overline{g}(N, \xi)\). It follows that \(\overline{g}(N, \xi) = 1\). Since \(S(TM) = \ker(\eta)\), the other equality in (2.2) holds and decompositions in (2.3) are straightforward. \(\Box\)

Since from any rigging one has a null rigging, in what follows, we will be considering a rigged null hypersurface \(x : (M, g, N) \rightarrow (\overline{M}, \overline{g})\) endowed with a null rigging \(N\). An advantage of a rigging is that the null transverse \(N\) is globally defined on \(M\). But the existence of a global transverse is not always guaranteed for any null hypersurface.
However if \((\mathcal{M}, \mathcal{g})\) is a (time-oriented) spacetime then, there exists a timelike vector field \(L\) globally defined on \(\mathcal{M}\). This timelike vector field is then a rigging for any null hypersurface of \(\mathcal{M}\) (since a timelike vector field can’t belong to the tangent space of a null hypersurface of Lorentzian manifold) and one can use \(L\) to derive a null rigging as above. We said that \((\mathcal{M}, \mathcal{g}, \mathcal{N})\) is a closed rigging null hypersurface when the \(1\)-form \(\eta\) is closed.

**Lemma 2.3.** Any closed rigging null hypersurface \((\mathcal{M}, \mathcal{g}, \mathcal{N})\) is foliated.

**Proof.** Since the associated screen distribution \(S(TM)\) is the kernel of the closed \(1\)-form \(\eta\), then by the Frobenius theorem, the screen distribution \(S(TM)\) is integrable and its leaves foliate \(\mathcal{M}\).

Let \(\nabla\) be the connection on \(\mathcal{M}\) induced from \(\nabla\) through the projection along the transverse bundle \(tr(TM)\). For every section \(U\) of \(TM\), one has \(g(\nabla_U \xi, \xi) = 0\). The Weingarten map is the endomorphism field

\[
\chi: \Gamma(TM) \rightarrow \Gamma(TM) \quad U \mapsto \nabla_U \xi.
\]

The Gauss-Weingarten equations of the immersion \(x: \mathcal{M} \rightarrow \mathcal{N}\) are given by

\[
(2.4) \quad \nabla_U V = \nabla_U V + B(U,V)N,
\]

\[
(2.5) \quad \nabla_U PV = \hat{\nabla}_U PV + C(U, PV)\xi,
\]

\[
(2.6) \quad \nabla_U N = -A_N U + \tau(U)N,
\]

\[
(2.7) \quad \nabla_U \xi = -\hat{\chi}_U U - \tau(U)\xi,
\]

for all \(U, V \in \Gamma(TM)\), where \(\hat{\nabla}\), denotes the connection on the screen distribution \(S(TM)\) induced by \(\nabla\) through the projection morphism \(P\) of \(\Gamma(TM)\) onto \(\Gamma(S(TM))\) with respect to the decomposition \((2.1)\). \(B\) and \(C\) are the local second fundamental forms of \(\mathcal{M}\) and \(S(TM)\) respectively, \(A_N\) and \(\hat{\chi}_U\) are the shape operators on \(TM\) and \(S(TM)\) respectively, and the rotation \(1\)-form \(\tau\) is given by

\[
\tau(U) = g(\nabla_U N, \xi).
\]

Note that in general relativity when \(\mathcal{N}\) is a black hole and \(\mathcal{M}\) its horizon, \(\tau\) is called the rotation \(1\)-form because it is related to the angular momentum of the horizon, and \(\tau(\xi)\) is the non-affinity coefficient because when \(\tau(\xi) \neq 0\), integral lines of \(\xi\) are not geodesic lines (or trajectory of photons). \(\tau\) is called by Hájíček [11, 12] a gravimagnetic field and by Damour [13, 14] a surface momentum density. When \(\mathcal{M}\) is a killing horizon, \(\tau(\xi)\) is called the horizon’s surface gravity. It is easy to check that the Weingarten map and the second fundamental form of \(\mathcal{M}\) are related by

\[
(2.8) \quad B(U, V) = -g(\chi(U), V), \quad \forall U, V \in \Gamma(TM),
\]
which show that the Weingarten map is $g$–symmetric. Also, second fundamental forms and sharp operators are related by

\[(2.9)\quad B(U, V) = g(A_\xi U, V), \quad C(U, PV) = g(A_N U, V) \quad \forall U, V \in \Gamma(TM).\]

One checks that

\[(2.10)\quad B(U, \xi) = 0, \quad A_\xi \xi = 0 \quad \forall U \in \Gamma(TM).\]

Let $\bar{N}$ be another rigging for $M$. There exists a section $\zeta$ of $TM$ and a nowhere vanishing smooth function $\phi$ such that $\bar{N} = \phi N + \zeta$. The following Lemma give relationships between geometrical objects described above.

**Lemma 2.4 ([15]).** Let $N$ and $\bar{N}$ be two rigging for $M$ with $\bar{N} = \phi N + \zeta$, where $\zeta \in \Gamma(TM)$ and $\phi \in C(M)$. Then,

1. $\bar{\xi} = \frac{1}{\phi} \xi$;
2. $2\phi \eta(\zeta) + \langle \zeta, \zeta \rangle = 0$;
3. $\bar{B}^N = \frac{1}{\phi} B^N$;
4. $\bar{\rho} = P - \frac{1}{\phi} g(\zeta, \cdot) \xi$;
5. $\bar{\nabla} = \nabla - \frac{1}{\phi} B(\cdot, \cdot) \zeta$;
6. $\tau^{\bar{N}} = \tau^N + d \ln |\phi| + \frac{1}{\phi} B(\zeta, \cdot)$;
7. $\bar{A}_\xi = \frac{1}{\phi} A_\xi - \frac{1}{\phi^2} B^N(\zeta, \cdot) \xi$;
8. $\bar{A}_N = \phi A_N - \nabla \cdot \zeta + [\tau^N + d \ln |\phi| + \frac{1}{\phi} B(\zeta, \cdot)] \zeta$.

It follows from (2.10) that integral curves of $\xi$ are pregeodesics in both $\bar{M}$ and $M$, as $\nabla \xi = -\tau^N(\xi) \xi$. Using (2.10), one gets that for tangent vectors of the null hypersurface, one has $L_{\xi g} = L_{\xi g}$. Hence, if $\xi$ is a killing vector field for $\bar{M}$ then, $\xi$ is a killing vector field for $M$. Using Gauss-Weingarten equations, it is nothing to check that

\[(2.11)\quad L_{\xi g} = 2B.\]

**Definition 2.3.** A **killing horizon** is a rigged null hypersurface of a spacetime, whose corresponding rigged $\xi$ is a killing vector field.

From the equation above, one deduce the following.

**Proposition 2.1.** A null hypersurface $M$ of a spacetime is a killing horizon if and only if the local second fundamental form (hence the shape operator) of $M$ identically vanishes.

One can also say that a rigged null hypersurface is a killing horizon if and only if the Weingarten map is $TM^\perp$–valued.

A null hypersurface $M$ is said to be **totally umbilical** (resp. **totally geodesic**) if there exists a smooth function $\rho$ on $M$ such that at each $p \in M$ and for all $u, v \in T_p M$, $B^N(\rho)(u, v) = \rho(p) g(u, v)$ (resp. $B^N$ vanishes identically on $M$). These are intrinsic notions on any null
hypersurface in the way that they do not depend on the choice of the rigging (see Lemma 2.4 item 3). The total umbilicity and the total geodesibility conditions for \( M \) can also be written respectively as \( \hat{A}_\xi = \rho P \) and \( \hat{A}_N = 0 \). Also, the screen distribution \( S(TM) \) is totally umbilical (resp. totally geodesic) if \( C^N(U, PV) = \lambda g(U, V) \) for all \( U, V \in \Gamma(TM) \) (resp. \( C^N = 0 \)), which is equivalent to \( A_N = \lambda P \) (resp. \( A_N = 0 \)). It is noteworthy to mention that the shape operators \( \hat{A}_\xi \) and \( A_N \) are \( S(TM) \)-valued.

**Lemma 2.5.** [1] Let \((M, g, N)\) be a totally umbilical rigged null hypersurface of a \((n+2)\)-dimensional pseudo-Riemannian space-form. Then \( \rho \) from the above definition satisfies

\[
\begin{align*}
\xi(\rho) + \rho \tau^N(\xi) - \rho^2 &= 0 \\
PU(\rho) + \rho \tau^N( PU) &= 0,
\end{align*}
\]

for all \( U \in \Gamma(TM) \).

The induced connection \( \nabla \) is torsion-free but not necessarily \( g \)-metric unless \( M \) is totally geodesic. In fact we have for all sections \( U, V, W \) of \( TM \),

\[
(\nabla_U g)(V, W) = \eta(V) B(U, W) + \eta(W) B(U, V).
\]

One defines the mean curvature \( \hat{S}_1 \) of the null hypersurface \( M \) and the mean curvature \( S_1 \) of the screen distribution \( S(TM) \) as the trace of the endomorphism fields \( \hat{A}_\xi \) and \( A_N \) respectively. Thus,

\[
\hat{S}_1 = tr(\hat{A}_\xi) \quad \text{and} \quad S_1 = tr(A_N).
\]

**Definition 2.4.** The null hypersurface (resp. the screen distribution \( S(TM) \)) is said to be **minimal** if \( \hat{S}_1 \) (resp. \( S_1 \)) identically vanishes.

**Lemma 2.6.** For every vector field \( U \in \Gamma(TM) \), one has

\[
tr(\nabla_U \hat{A}_\xi) = U \cdot \hat{S}_1
\]

**Proof.** Since \( \hat{A}_\xi \) is diagonalizable, there exists a quasi-orthonormal frame \( \{\hat{E}_0 = \xi, \hat{E}_1, \ldots, \hat{E}_n\} \) of eigenvectors with corresponding real eigenfunctions \( k_0 = 0, k_1, \ldots, k_n \) such that

\[
(\hat{E}_i, \hat{E}_j) = \epsilon_i \delta_{ij}, \quad \text{with} \quad \epsilon_i = \pm 1, \quad \forall i, j = 1, \ldots, n.
\]

It follows that

\[
tr(\nabla_U \hat{A}_\xi) = \sum_{i=1}^n \epsilon_i (\nabla_U \hat{A}_\xi) \hat{E}_i \hat{E}_i + (\nabla_U \hat{A}_\xi) \xi, N = \sum_{i=1}^n U \cdot \hat{k}_i = U \cdot \hat{S}_1.
\]

\( \square \)
Gauss-Codazzi equations of $(M, g, N)$ are given for $U, V, W \in \Gamma(TM)$ and $X \in \Gamma(S(TM))$ by
\begin{equation}
\langle R(U, V)W, X \rangle = \langle R(U, V)W, X \rangle + B(U, W)C(V, X) - B(V, W)C(U, X) \tag{2.16}
\end{equation}
\begin{equation}
\langle R(U, V)W, N \rangle = \langle R(U, V)W, N \rangle + C(U, X)\tau(V) - C(V, X)\tau(U), \tag{2.17}
\end{equation}
\begin{equation}
\langle R(U, V)W, \xi \rangle = \langle \nabla_V B(V, W) - \nabla_B B(U, W) \rangle \tag{2.18}
\end{equation}
\begin{equation}
\langle R(U, V)\xi, N \rangle = C(\hat{A}_\xi U, PV) - C(U, \hat{A}_\xi PV) - d\tau(U, V), \tag{2.19}
\end{equation}
where $R$ is the Riemann curvature of $\nabla$. As definition of the Ricci tensor of $R$ we use
\begin{equation}
\langle \overline{Rc}(U, V) \rangle = tr(W \mapsto \langle R(W, U)V \rangle),
\end{equation}
for all $U, V \in \Gamma(M)$.

**Lemma 2.7.** The following equation holds
\begin{equation}
\langle \overline{Rc}(\xi, \xi) \rangle = \xi \cdot \hat{S}_1 + \tau(\xi) \hat{S}_1 - tr\left(\hat{A}_\xi^2\right). \tag{2.21}
\end{equation}

**Proof.** Let $\hat{T}_1 := -\hat{S}_1 + \hat{A}_\xi: \Gamma(TM) \to \Gamma(TM)$ be the first Newton transformation of $\hat{A}_\xi$, as defined in [16]. By the Lemma 2.1, $\tilde{g}$ is a non-degenerate metric on $M$ and it is easy to check that the precedent frame $\{\hat{E}_0 = \xi, \hat{E}_1, \ldots, \hat{E}_n\}$ is an $\tilde{g}$-orthonormal frame. Hence,
\begin{equation}
div^\nabla (\hat{T}_1) = tr\left(\nabla \hat{T}_1\right) = \sum_{a=0}^{n} \epsilon_a \left(\nabla \hat{E}_a\right) \hat{E}_a = \sum_{a=0}^{n} \epsilon_a \left[\nabla \hat{E}_a\right] \hat{E}_a = \sum_{a=0}^{n} \epsilon_a \left[\nabla \hat{E}_a\right] \hat{E}_a - \hat{E}_a \cdot \hat{S}_1 \hat{E}_a.
\end{equation}
For $U \in \Gamma(TM)$, using covariant derivative formula (2.14) one has
\begin{equation}
\langle \text{div}^\nabla (\hat{T}_1), U \rangle = \sum_{a=0}^{n} \epsilon_a \langle \nabla \hat{E}_a, \hat{E}_a \rangle - \hat{A}_\xi \cdot \hat{E}_a U - \sum_{a=0}^{n} \epsilon_a \langle \nabla \hat{E}_a, \hat{S}_1 \rangle g(\hat{E}_a, U).
\end{equation}
Using Gauss-Codazzi equation (2.19), one has
\begin{equation}
\langle \hat{E}_a, (\nabla_{\hat{E}_a} \hat{A}_\xi)U \rangle = \langle \hat{E}_a, (\nabla_{\hat{E}_a} \hat{A}_\xi)U \rangle + \langle R(\hat{E}_a, U) \hat{E}_a, \xi \rangle + B(\hat{E}_a, \hat{E}_a) \tau(U) - B(\hat{E}_a, U) \tau(\hat{E}_a).
\end{equation}
Above relation becomes
\begin{equation}
\langle \text{div}^\nabla (\hat{T}_1), U \rangle = \langle \nabla_{AU\hat{A}_\xi}, \xi \rangle + \sum_{a=0}^{n} \epsilon_a \left[\langle R(\hat{E}_a, U) \hat{E}_a, \xi \rangle - B(\hat{E}_a, U) \tau(\hat{E}_a)\right] + \tau(\hat{S}_1) \hat{S}_1 + tr\left(\hat{A}_\xi^2\right) \eta(U) - PU(\hat{S}_1).
\end{equation}
Using the above Lemma and taking $U = \xi$, one obtains (2.21).

Equation (2.21) is called null Raychaudhuri equation. Since $\hat{A}_\xi$ is diagonalizable, and in a space form $\mathcal{Ric}(\xi, \xi) = 0$, the following result holds.

**Theorem 2.2.** A null hypersurface $(M, g)$ of a semi-Riemannian space form $(M(c), g)$ is totally geodesic if and only if it is minimal.

A consequence of this result is that if a semi-Riemannian manifold $\overline{M}$ admits a null hypersurface which is minimal but not totally geodesic then, $\overline{M}$ does not have constant sectional curvature. Recall that a semi-Riemannian manifold satisfies the null (resp. the reverse null) convergence condition if $\mathcal{Ric}(V) \geq 0$ (resp. $\mathcal{Ric}(V) \leq 0$) for any null vector field $V$. More generally, we have the following result.

**Theorem 2.3.** Let $M$ be a null hypersurface of a semi-Riemannian manifold satisfying the null convergence condition. Then, $M$ is totally geodesic if and only if $M$ is minimal.

**Definition 2.5.** A manifold $M$ endowed with a torsion free linear connection $\nabla$ is said to be a **locally flat** manifold if the Riemannian curvature of $\nabla$ identically vanishes.

### 3 Marginally (Outer) Trapped Submanifolds

In what follows, $(\overline{M}, \overline{g})$ is a $(n+2)$-dimensional spacetime, i.e. a time-oriented Lorentzian manifold.

Let $S$ be a spacelike codimensional-two submanifold of the spacetime $\overline{M}$. (Some authors call $S$ a surface.) Let $N^+$ and $\xi^+$ be two future-directed lightlike vector fields of $\overline{M}$ normalized by $(N^+, \xi^+) = -1$ and such that $TS^+ = \text{span}\{N^+, \xi^+\}$. We set $\xi^+$ to be in the outgoing direction. For all sections $X, Y$ of the tangent bundle $TS$, Gauss and Weingarten formulas of the immersion $S \rightarrow \overline{M}$ are given by

\begin{align}
\nabla_X Y &= \hat{\nabla}_X Y + \Pi(X, Y) \\
\nabla_X N^+ &= -A^+_N X + \hat{\nabla}_X N^+ \\
\nabla_X \xi^+ &= -A^+_\xi X + \hat{\nabla}_X \xi^+,
\end{align}

where $\hat{\nabla}$ is the Levi-Civita connection of $S$, $\Pi$ is the second fundamental form, $A^+_N$ and $A^+\xi$ are the shape operators with respect to $N^+$ and $\xi^+$ respectively.

In general relativity, the expansions of $S$ with respect to $N^+$ and $\xi^+$ are defined as the traces $\theta_{N^+} = -\text{tr}(A^+_N)$, $\theta_{\xi^+} = -\text{tr}(A^+\xi)$. The mean curvature vector is given by $H = -\text{tr}(\Pi) = -\theta_{N^+} \xi^+ - \theta_{\xi^+} N^+$. Let $N$ be a compactly supported normal vector to $S$ an $(\phi^N_\epsilon)_{\epsilon \in I}$ the associated one parameter group of diffeomorphisms of $\overline{M}$. For each $\epsilon$, ...
$S_t = \phi^t_N(S)$ is called a Lie dragging of $S$ along $N$. We then have $S_0 = S$. Let $|S_\epsilon|$ denote the area of $S_\epsilon$. One has [17]

\begin{equation}
\frac{d}{d\epsilon} |S_\epsilon| = \int_{S_\epsilon} \langle H, N \rangle \eta_g, \tag{3.4}
\end{equation}

where $\eta_g$ is the metric form on $S$. Let us set $\delta_N|S_\epsilon| = \frac{d}{d\epsilon} [|S_\epsilon|];$ then, the first order variation of the area of $S$ along the deformation vector $N$ is $\delta_N|S|$. By taking $N = N^+$ in (3.4), one has

$$\delta_N|S| = \int_{S_\epsilon} \theta_{\xi^+} \eta_g.$$

Hence when $\theta_{\xi^+} < 0$, the area of $S$ decrease when $S$ is dragging along $N^+$; this is taken as a clear signal of the presence of a strong gravitational field, and $S$ is call a weakly future trapped surface.

Introduced by Penrose in [3], the concept of trapped surface plays an important role in general relativity. The surface $S$ is said to be a trapped surface if all light rays emitted from the surface locally converge. Nothing can escape, not even light. It is believed that there will be a marginally trapped surface, separating trapped surfaces from the untrapped ones, where the outgoing light rays are instantaneously parallel. It is prove that, $S$ is a trapped surface if and only if the two expansions are of the same sign, and marginally trapped if and only if (at least) one of the expansions vanishes, which is equivalent to say that the mean curvature vector field is lightlike or zero.

**Definition 3.1.** A codimension-two spacelike submanifold of a Lorentzian manifold is said to be a future

- **Trapped Submanifold (TS)** if $\theta_{\xi^+} < 0$ and $\theta_{N^+} < 0$.
- **Marginally Trapped Submanifold (MTS)** if $\theta_{\xi^+} = 0$ and $\theta_{N^+} \leq 0$.
- **Trapped Outer Submanifold (TOS)** if $\theta_{\xi^+} < 0$.
- **Marginally Outer Trapped Submanifold (MOTS)** if its mean curvature vector is lightlike or zero.

The outgoing direction depends on the choice and when the expansion in one direction is zero one takes this direction as the outgoing direction. In others words, a MOTS is a submanifold for which the expansion in the outgoing direction is zero.

**Definition 3.2.**

- **The trapped region** of the Lorentzian manifold $\overline{M}$ is the set of points belonging to some trapped submanifold $S \subset \overline{M}$. Its boundary is the trapping boundary.
- A trapping horizon of $\overline{M}$ is (the closure of) a hypersurface $M$ foliated by (closed) MOTSs.
- A trapping horizon $M$ is said to be future (respectively, past) if for each MOTS leaf of $M$, there exists $k$ and $l$ (as above) such that $\theta_{N^+} < 0$ (respectively, $\theta_{N^+} > 0$).
A trapping horizon \( M \) is said to be outer (respectively, inner) if for each MOTS leaf of \( M \), there exists \( k \) and \( l \) (as above) such that \( \delta_N + \theta_\xi < 0 \) (respectively, \( \delta_N + \theta_\xi > 0 \)).

Recall that for any covariant tensor \( \Gamma \) on \( S \), the first order variation of \( \Gamma \) along a vector field \( N \) is defined by

\[
\delta_N(\Gamma) = \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} (\varphi^\epsilon_\star(\Gamma_\epsilon)) |_{\epsilon = 0},
\]

where \( \Gamma_\epsilon \) is the analogous of \( \Gamma \) on \( S_\epsilon = \varphi^\epsilon_\star(S) \).

**Definition 3.3.** A quasi-local black hole horizon is a Future Outer Trapping Horizon (FOTH).

In general relativity, a trapping horizon is always foliated by closed (compact without boundary) MOTS. In this work a trapping horizon is foliated by MOTS not necessarily closed.

### 3.1 Null Trapping Horizon

Let \( x : (M, g, N) \to (\overline{M}, \overline{\gamma}) \) be a rigged null hypersurface of the spacetime \( (\overline{M}, \overline{\gamma}) \), with \( N \) future-directed. (Recall that since \( \overline{M} \) is a time-oriented spacetime, any null hypersurface \( M \to \overline{M} \) has a null rigging vector field.) Let’s assume that the rigging \( N \) is with an integrable screen distribution \( S(TM) = \ker(x_\star \langle N, \cdot \rangle) \) and we denote by \( S \) a generic leaf. It is a well known fact that \( S \) is a codimensional-two spacelike submanifold of \( (\overline{M}, \overline{\gamma}) \) and the normal bundle \( TS \perp \) of the immersion \( S \to \overline{M} \) is spanned by \( N \) and \( \xi \). We set \( N^+ = N \) and \( \xi^+ = -\xi \) to be the future-directed null normals spanning the normal bundle of \( S \), with \( \xi^+ \) the outgoing direction.

Using equations (2.4) to (2.7), we derive Gauss and Weingarten formulas of the immersion \( S \to \overline{M} \) as

\[
\nabla_X Y = \nabla^*_X Y + B(X, Y)N + C(X, Y)\xi = \nabla^*_X Y + h(X, Y),
\]

\[
\nabla_X N = -A_N X + \tau(X)N = -A_N X + \nabla^*_X N,
\]

\[
\nabla_X \xi = -A_\xi X - \tau(X)\xi = -A_\xi X + \nabla^*_X \xi,
\]

for any \( X, Y \in \Gamma(S(TM)) \), where \( h(X, Y) \) is the second fundamental form, and \( \nabla^*_\perp \) is the normal connection. Hence the screen connection \( \hat{\nabla} \) is the Levi-Civita connexion of \( S \), and the shape operators of the null hypersurface \( M \) and the ones of the immersion \( S \to \overline{M} \) (defined by (3.2) and (3.3)) are related on \( S \) by

\[
A_N = A^+_N \quad \text{and} \quad A_\xi = -A^+_\xi.
\]

Hence, expansions of \( S \) and mean curvatures of \( M \) are related by

\[
\theta_{N^+} = -\text{tr}(A^+_N) = -\text{tr}(A_N) = -S_1
\]

\[
\theta_{\xi^+} = -\text{tr}(A^+_\xi) = \text{tr}(A_\xi) = \hat{S}_1.
\]

Thus the mean curvature of the lightlike hypersurface is the expansion of leaves of the screen distribution in the outgoing direction. Hence, a
leaf \( S \) is a trapped submanifold if and only if \( \dddot{x}_1 < 0 \) and \( S_1 > 0 \); \( S \) is a marginally trapped submanifold if and only if \( \dddot{x}_1 = 0 \) and \( S_1 \geq 0 \); \( S \) is a trapped outer submanifold if and only if \( \dddot{x}_1 < 0 \); \( S \) is a marginally outer trapped submanifold if and only if \( \dddot{x}_1 = 0 \); and \( M \) is a null quasi-local black hole horizon if and only if \( \dddot{x}_1 = 0 \), \( S_1 > 0 \) and \( \delta_N \dddot{x}_1 < 0 \). The mean curvature vector is given by

\[
(3.9) \quad H = -\text{tr}(h) = -S_1 \xi - \dddot{x}_1 N = -\theta_N^+ \xi^+ - \theta_N^+ N^+.
\]

**Example 1.** We consider the 6-dimensional spacetime \( (\overline{M} = \mathbb{R}^6, \overline{g}) \) endowed with the metric

\[
\overline{g} = -(dx^0)^2 + (dx^1)^2 + \exp 2x^0[(dx^2)^2 + (dx^3)^2] + \exp 2x^1[(dx^4)^2 + (dx^5)^2],
\]

\((x^0, \ldots, x^5)\) being the usual cartesian coordinates on \( \mathbb{R}^6 \). The hypersurface \( M \) of \( \overline{M} \) defined by

\[
M = \{(x^0, \ldots, x^5) \in \mathbb{R}^6; x^0 + x^1 = 0 \}
\]

is a lightlike hypersurface of \( (\overline{M}, \overline{g}) \) and the vector field \( N = -\frac{1}{2} \left( \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1} \right) \) is a null rigging for \( M \) with corresponding rigged vector field 

\[
\xi = \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1} \quad \text{and integrable screen distribution} \ S(TM) = \text{span} \{ \dddot{x}_1, \dddot{x}_2, \dddot{x}_3, \dddot{x}_4 \}
\]

with 

\[
\dddot{x}_1 = e^{-2x^0} \frac{\partial}{\partial x^2}, \quad \dddot{x}_2 = e^{-2x^0} \frac{\partial}{\partial x^3}, \quad \dddot{x}_3 = e^{-2x^1} \frac{\partial}{\partial x^4}, \quad \dddot{x}_4 = e^{-2x^1} \frac{\partial}{\partial x^5}.
\]

By direct computations, one sees that \( M \) is not totally geodesic but is minimal (hence \( \overline{M} \) doesn’t have constant sectional curvature) and \( S_N = \text{tr}(A_N) = 2 > 0 \). Hence leaves of the screen distribution given by \( S = \{x^0 = \text{cste}, x^1 = \text{cste}\} \) are marginally trapped submanifolds of \( (\overline{M}, \overline{g}) \).

**Example 2.** Let us consider the Schwarzschild spacetime, whose metric \( g \) is given in the Schwarzschild coordinates \((t, r, \theta, \varphi)\) by

\[
(3.10) \quad g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).
\]

This is the first non-trivial solution of Einstein’s equations, found by the astrophysicist Karl Schwarzschild in the end of 1915. It is the metric outside a spherical body of mass \( m \) and radius \( r = 2m \). The singularity presented by this metric on \( r = 2m \) is a apparent singularity due to the bad choice of coordinate systems. Let us set

\[
t = t_s + 2m \ln \left| \frac{r}{2m} - 1 \right|.
\]

Then, \((t, r, \theta, \varphi)\) is a new coordinate systems called ingoing Eddington-Finkelstein coordinates, and in which metric (3.10) becomes

\[
g = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{4m}{r} dt dr + \left(1 + \frac{2m}{r}\right) dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).
\]

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It is nothing to see that the hypersurface

\[ M : \ r = 2m, \]

is a null hypersurface, called the event horizon. One can rig this light-like hypersurface by the following future directed null rigging and associated rigged:

\[ N = \frac{r}{2m} \partial_t - \frac{r}{2m} \partial_r, \quad \xi = -\partial_t. \]

Corresponding screen distribution is given by \( S(TM) = \text{span}(\hat{E}_1, \hat{E}_2) \) with

\[ \hat{E}_1 = \partial_\theta, \quad \hat{E}_2 = \partial_\phi. \]

This screen is integrable and leaves are spheres \( \{ t = \text{cste}, r = 2m \} \), which foliate \( M \). A direct computation gives

\[ \star S_1 = 0, \quad S_1 = m \geq 0. \]

Hence, \( M \) is totally geodesic, the screen distribution is totally umbilical with \( \lambda = 2m \). It follows that

\[ \delta N \star S_1 = 0, \quad S_1 = m \geq 0. \]

Then, spheres \( t = \text{cste}, r = 2m \) are marginally trapped surfaces. Hence, \( M \) is a null future trapping horizon. Also, \( M \) is a non-expanding horizon and spheres \( \{ t = \text{cste}, r = 2m \} \) are cross sections.

Moreover, the one parameter group of diffeomorphisms (just the flow) of \( N \) starting at \( (t_0, r_0, \theta_0, \phi_0) \) is given by

\[ \phi_\epsilon = (t_0 + (1 - \exp(-\epsilon/2m))r_0, r_0 \exp(-\epsilon/2m), \theta_0, \phi_0). \]

The image of a sphere \( S = \{ t = t_0, r = 2m \} \) by \( \phi_\epsilon \) (Lie dragging of \( S \) along \( N \)) is

\[ S_\epsilon := \phi_\epsilon(S) = \{ t = t_0 + 2m(1 - \exp(-\epsilon/2m)), r = 2m \exp(-\epsilon/2m) \}. \]

These spheres are spacelike surfaces and corresponding normalized pairs are given by

\[ N_\epsilon = \frac{r \exp(-\epsilon/2m)}{2m} (1, -1, 0, 0), \]

\[ \xi_\epsilon = \left( -\frac{m \exp(\epsilon/2m)}{x} - \frac{2m^2 \exp(\epsilon/m)}{x^2}, -\frac{m \exp(\epsilon/2m)}{x} + \frac{2m^2 \exp(\epsilon/m)}{x^2}, 0, 0 \right). \]

By a direct calculation, one finds

\[ \star S_{\epsilon 1} = \frac{2m \exp(\epsilon/2m)(-r + 2m \exp(\epsilon/2m))}{r^3}. \]

Hence,

\[ \delta N \star S_1 = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \star S_{\epsilon 1} = \frac{r - 4m}{r^3} < 0. \]

\( M \) is then a null quasi-local black hole horizon.
Let $\bar{R}, R, \hat{R}$ be the Riemannian curvatures of $\nabla$, $\bar{\nabla}$ and $\hat{\nabla}$ respectively. It is straightforward to check that the Codazzi and Ricci equations of the immersion of the immersion $S \to \bar{M}$ are given by

$$(\bar{R}(X,Y)Z,T) = \langle \hat{R}(X,Y)Z,T \rangle$$

(3.11)

$$+ \langle A_{h(X,Z)}Y,T \rangle - \langle A_{h(Y,Z)}X,T \rangle,$$

(3.12)

$$\langle \bar{R}(X,Y)Z,\eta \rangle = \langle [\nabla_X h](Y,Z),\delta \rangle - \langle [\nabla_Y h](X,Z),\delta \rangle,$$

(3.13)

$$\langle \bar{R}(X,Y)\delta,\eta \rangle = \langle R^+(X,Y)\delta,\eta \rangle - \langle [A_{\delta}, A_{\eta}]X,Y \rangle,$$

for all $X,Y,Z,T \in \Gamma(S(TM))$ and for any $\delta, \eta \in \Gamma(TS^\perp)$. Here, $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}Xh)(Y,Z) = \hat{\nabla}_X h(Y,Z) - h(\hat{\nabla}_Y Z, h(\hat{\nabla}_X Z).$$

It is straightforward to show that

(3.14)

$$\bar{\nabla}X h = (\nabla_X B + \tau(X)B) N + (\nabla_X C - \tau(X)C) \xi.$$

With the tools of the null hypersurface $M$, the Codazzi and Ricci equations (3.11)-(3.13) give

$$(\bar{R}(X,Y)Z,T) = \langle R(X,Y)Z,T \rangle$$

(3.15)

$$+ B(X,Z)C(Y,T) - B(Y,Z)C(X,T)$$

$$= \langle \hat{R}(X,Y)Z,T \rangle$$

$$+ B(X,Z)C(Y,T) - B(Y,Z)C(X,T)$$

$$+ C(X,Z)B(Y,T) - C(Y,Z)B(X,T)$$

$$(\bar{R}(X,Y)Z,N) = (\nabla_X C)(Y,Z) - (\nabla_Y C)(X,Z)$$

(3.16)

$$+ C(X,Z)\tau(Y) - C(Y,Z)\tau(X),$$

$$\langle \bar{R}(X,Y)Z,\xi \rangle = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)$$

(3.17)

$$+ B(Y,Z)\tau(X) - B(X,Z)\tau(Y),$$

(3.18)

$$\langle \bar{R}(X,Y)\xi,N \rangle = C(\hat{A}_\xi X,Y) - C(X,\hat{A}_\xi Y) - d\tau(X,Y).$$

We say that $S$ has a parallel mean curvature vector when $\hat{\nabla}H = 0$. $S$ is called a parallel submanifold when $\bar{\nabla}h = 0$. One can check that parallel submanifolds have parallel mean curvature vector.

**Definition 3.4.** A *Non-Expanding Horizon* (NEH) of a $(n+2)$-dimensional Lorentzian manifold is a null hypersurface $M$ foliated by MOTSs.

A NEH is then a null hypersurface $M$ foliated by the so-called cross-sections $S$ with null expansions in the outgoing direction. Notice that, a NEH is a null trapping horizon.

**Proposition 3.1.** Let $(M,g,N)$ be a closed rigged null hypersurface of a Lorentzian manifold $(\bar{\nabla},\bar{M})$, and $S$ a generic leaf of the screen distribution. Then,

1. $M$ is a null trapping horizon if and only if $M$ is minimal;
\( \therefore S \) has parallel mean curvature vector if and only if expansions satisfy \( d\theta^+_N = \theta^+_N \tau \) and \( d\theta^+_\xi = -\theta^+_\xi \tau \) on \( S \).

**Proof.** Taking normal derivative in (3.9), one obtains

\[
\nabla H = (d\theta^+_N - \theta^+_N \tau) \xi - (d\theta^+_\xi + \theta^+_\xi \tau) N
\]

and the second item follows. If \( M \) is minimal then by Definition 2.4 and (3.8), the outgoing expansion of \( S \) identically vanishes and then, \( M \) is a null trapping horizon. Conversely, if \( M \) is a null trapping horizon then, \( M \) is foliated by MOTSs. Since This MOTSs are spacelike, the distribution associated to this foliation is a screen distribution for \( M \). Let \((N, \xi)\) be a normalizing pair associated with this screen distribution as given into Theorem 2.1. The second equality in (3.8) give \( tr(\hat{A}_\xi) = \theta^+_\xi = 0 \), which by Definition 2.4 leads to : \( M \) is minimal.

A non-expanding horizon is then a null trapping horizon and cross-sections are MOTS leaves. In general relativity, it is proved using null Raychaudhuri equation and null dominant energy condition, that any NEH of a spacetime solution of Einstein’s equations, has vanishing second fundamental form \( B \). In Theorem 2.2, we prove the same result in a space form not necessarily satisfying Einstein’s equations. Recall that the spacetime \((M, \tilde{g})\) is an Einstein spacetime without cosmological constant if and only if for all local sections \( U, V \) of the bundle \( TM \), one has

\[
\text{Ric}(U,V) - \frac{1}{2} \langle U, V \rangle \mathcal{R} = 8\pi T(U,V),
\]

where \( \mathcal{R} \) is the scalar curvature and \( T \) is the total energy-momentum. For \( U = V = \xi \), one obtains from the above equation \( \text{Ric}(\xi, \xi) = 8\pi T(\xi, \xi) \). Null Raychaudhuri equation (2.21) gives

\[
8\pi T(\xi, \xi) = \xi \cdot ^*S_1 + \tau(\xi) ^*S_1 - tr(\hat{A}_\xi^2).
\]

Hence if the null dominant energy condition holds, then \( T(\xi, \xi) \geq 0 \) and a NEH has vanishes shape operator \( \hat{A}_\xi \).

**Theorem 3.1.** In a spacetime with constant sectional curvature \( c \), cross-sections of a NEH are space forms of the same constant sectional curvature \( c \).

**Proof.** Since a NEH is totally geodesic, equation (3.15) leads to

\[
\langle \text{R}(X,Y)Z, T \rangle = \langle \hat{R}(X,Y)Z, T \rangle
\]

which completes the proof. \( \square \)
3.2 SIC-normalized null hypersurfaces

Let \((M, g, N) \rightarrow (\overline{M}, \overline{g})\) be a rigged null hypersurface of a \((n+2)\)-dimensional spacetime. The rigging \(N\) is said to be with conformal screen distribution if the two shape operators are conformal. This means that there exists a function \(\varphi\) such that \(A_N = \varphi \hat{A}_\xi\).

**Definition 3.5.** A Screen Integrable and Conformal (SIC) rigging is one for which the screen distribution is integrable and conformal.

When conformal factor \(\varphi\) is 1 and the rigging is closed, \(N\) is called a UCC (Unitary Conformally Closed) rigging. Hence, UCC rigging defined in [18], is a particular case of SIC rigging. Since the screen distribution \(S(TM) = \ker(\eta)\) then from the two last items of Lemma 2.4, it follows that if \(N\) is a SIC-rigging then any change of rigging \(\tilde{N} = \varphi N\) is also a SIC-rigging.

**Lemma 3.1.** Let \(x : (M, g, N) \rightarrow (\overline{M}, \langle \cdot, \cdot \rangle)\) be a rigged null hypersurface.

1. If \(N\) is a closed rigging with conformal screen distribution then the rotation 1-form \(\tau^N\) vanishes on the screen distribution.
2. If \(N\) is a rigging with conformal screen distribution and vanishing rotation 1-form then, the 1-form \(\eta = x^*\langle N, \cdot \rangle\) is closed.

**Proof.** Assume \(\eta\) to be closed and let \(U, V\) be tangent vector fields to \(M\). The condition \(d\eta(U, V) = U \cdot \eta(V) - V \cdot \eta(U) - \eta([U, V]) = 0\) is equivalent to \(\langle \nabla_U N, V \rangle = \langle \nabla_V N, U \rangle\). Then by the weingarten formula, we get

\[
\langle -A_N U, V \rangle + \tau^N(U)\eta(V) = \langle -A_N V, U \rangle + \tau^N(V)\eta(U).
\]

In this relation, take \(V = \xi\) to get

\[
\tau^N(U) = -\langle A_N \xi, U \rangle + \tau^N(\xi)\eta(U)
\]

From here if \(N\) is with conformal screen then \(A_N \xi = \varphi \hat{A}_\xi \xi = 0\) and the first item is prove. If \(\tau^N\) identically vanishes and \(N\) is with conformal screen distribution then \(A_N\) is symmetric and equation (3.22) holds, which is equivalent to \(\eta\) is closed.

A consequence of the second item in the Lemma above is that a rigging with conformal (resp. unitary conformally) screen distribution and vanishing rotation 1–form is a SIC-rigging (resp. UCC-rigging). The following is to show that there exists null hypersurfaces with SIC-rigging.

Let \(M\) be the Monge hypersurface of the Lorentz-Minkowski space \(\mathbb{R}^{n+2}\) given by

\[
M = \{ x = (x^0 = F(x^1, \ldots, x^{n+1}), x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+2} \},
\]
where $F : D \to \mathbb{R}$ is a smooth function defined on an open subset $D$ of $\mathbb{R}^{n+1}$. For a vector field $X = X^A \frac{\partial}{\partial x^A} \in \mathbb{R}^{n+2}$ a necessary and sufficient condition to be tangent to $M$ is that $X^0 = X^1 F'_{x^1} + \cdots + X^1 F'_{x^{n+1}}$. Then $\delta = \frac{\partial}{\partial x^0} + \sum_{a=1}^{n+1} F'_a \frac{\partial}{\partial x^a}$ is normal to $M$. The later is a null hypersurface if and only if $\delta$ is a null vector field. This is equivalent to

$$\sum_{a=1}^{n+1} (F'_a)^2 = ||\nabla F||^2 = 1,$$

where $\nabla F$ is the gradient of $F$ with respect to the Euclidean structure $|| \cdot ||$ of $\mathbb{R}^{n+1}$.

Let us assume that $M$ is a Monge null hypersurface and consider the null rigging

$$\mathcal{M}_F = \frac{1}{\sqrt{2}} \left[ - \frac{\partial}{\partial x^0} + \sum_{a=1}^{n+1} F'_a \frac{\partial}{\partial x^a} \right] = \frac{1}{\sqrt{2}} (-1, \nabla F),$$

with corresponding rigged vector field

$$\xi_F = \frac{1}{\sqrt{2}} \left[ \frac{\partial}{\partial x^0} + \frac{1}{\sqrt{2}} \sum_{a=1}^{n+1} F'_a \frac{\partial}{\partial x^a} \right] = \frac{1}{\sqrt{2}} (1, \nabla F).$$

Let us consider the natural (global) parametrization of $M$ given by

$$\begin{cases}
x^0 = F(u^1, \ldots, u^{n+1}) \\
x^a = u^a (u^1, \ldots, u^{n+1}) \\
end{cases} \quad (u^1, \ldots, u^{n+1}) \in D.$$

Then $\Gamma(TM)$ is spanned by $\{ \frac{\partial}{\partial u^a} \}_a$ with

$$\frac{\partial}{\partial u^a} = F'_u \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^a}.$$

Then, taking partial derivative of (3.23) with respect to $x^b$ ($1 \leq b \leq n+1$) leads to

$$\sum_{a=1}^{n+1} F'_{x^a} F''_{x^a x^b} = 0.$$

Now take a covariant derivative by the flat connection $\nabla$ and using (3.28),

$$\nabla_{\frac{\partial}{\partial u^a}} \xi_F = \frac{1}{\sqrt{2}} \sum_{b=1}^{n+1} F''_{u^a u^b} \frac{\partial}{\partial x^b}$$

$$= \frac{1}{\sqrt{2}} \sum_{b=1}^{n+1} \left( -F''_{u^a u^b} F'_{u^b} \frac{\partial}{\partial x^0} + F''_{u^a u^b} \frac{\partial}{\partial x^b} \right)$$

$$\nabla_{\frac{\partial}{\partial u^a}} \xi_F = \frac{1}{\sqrt{2}} \sum_{b=1}^{n+1} F''_{u^a u^b} \frac{\partial}{\partial u^b} = \nabla_{\frac{\partial}{\partial u^a}} \mathcal{M}_F,$$
which is belong to $\Gamma(S(TM))$ (one proves this by using (3.28)) and shows that the rotation 1–form identically vanishes and

$$A_{a} \left( \frac{\partial}{\partial u^{a}} \right) = A^{*}_{\ell} \left( \frac{\partial}{\partial u^{a}} \right) = -\frac{1}{\sqrt{2}} \sum_{b=1}^{n+1} F''_{u^{a}u^{b}} \frac{\partial}{\partial u^{b}}.$$ 

Thus, the screen distribution is conformal with conformal factor $\phi = 1$. Hence by Lemma 3.1, $\mathcal{M}_{F}$ is a UCC-rigging. $\mathcal{M}_{F}$ is called the generic UCC-rigging of the Monge null hypersurface $M$. This proves the existence of so many null hypersurfaces of $\mathbb{R}^{n+2}$ equipped with a UCC-rigging. Rescaling $\mathcal{M}_{F}$ by a nowhere zero smooth function $\phi$ to have $N = \phi \mathcal{M}_{F}$, it is easy to check that when $\phi$ is not constant on the screen distribution, $N$ is a SIC-rigging which is not a UCC-rigging.

The matrix of $A^{*}_{\ell}$ with respect to the basis $\left\{ \frac{\partial}{\partial u^{a}} \right\}_{a}$ is given by

$$A^{*}_{\ell} = -\frac{1}{\sqrt{2}} \begin{pmatrix} F''_{u^{1}u^{1}} & \cdots & F''_{u^{1}u^{n+1}} \\ \vdots & \ddots & \vdots \\ F''_{u^{n+1}u^{1}} & \cdots & F''_{u^{n+1}u^{n+1}} \end{pmatrix} = -\frac{1}{\sqrt{2}} \text{Hess}(F)$$

and by using (3.8), it follows that expansions are given by,

$$-\theta_{N^{+}} = \theta_{\ell^{+}} = \text{tr}(A^{*}_{\ell}) = -\frac{1}{\sqrt{2}} \sum_{b=1}^{n+1} F''_{u^{a}u^{b}} = -\frac{1}{\sqrt{2}} \Delta F,$$

Which shows that expansions are of different signs. This is a general fact for all null hypersurfaces in the Lorentz-Minkowski space $\mathbb{R}^{n+2}$. Hence in Lorentz-Minkowski space, a null hypersurface cannot be foliated by trapped submanifolds. From the above equation, one derives the following result.

**Proposition 3.2.** Let $x : (M, g, \mathcal{M}_{F}) \rightarrow \mathbb{R}^{n+2}$ be a Monge null hypersurface endowed with its generic UCC-rigging $\mathcal{M}_{F}$. Then, the screen distribution is integrable with leaves the level sets of the function $F$, and a leaf $S$ is a Marginally (Outer) Trapped Submanifold if and only if $F$ is harmonic on $S$.

**Theorem 3.2.** A Monge null hypersurface $M \rightarrow \mathbb{R}^{n+2}$ graph of a function $F$ is a trapping horizon if and only if $F$ is harmonic.

**Proof.** If $M$ is a trapping horizon then, there exists a foliation of $M$ by MOTSs of $\mathbb{R}^{n+2}$. The distribution of this foliation can be set as well as a screen distribution on $M$. By Proposition 3.1 and Definition 2.4, $M$ is minimal and it follows from equality (3.30) that $F$ is harmonic. Conversely if $F$ is harmonic then endowed $M$ with the generic UCC-rigging, it follows from Proposition 3.2 that the screen distribution is integrable and leaves are MOTSs. Meaning that $M$ is a trapping horizon.

From now on, $(M(c), \bar{g})$ is a $(n + 2)$–dimensional spacetime with constant sectional curvature $c \in \mathbb{R}$, and $x : (M, g, N) \rightarrow (M(c), \bar{g})$ is
a SIC-rigged null hypersurface. From here, Gauss-Codazzi equations (3.15)-(3.18) become

\[
\langle R(X,Y)Z,T \rangle = c \left( \langle Y,Z \rangle \langle X,T \rangle - \langle X,Z \rangle \langle Y,T \rangle \right) 
+ \varphi \left( B(Y,Z)B(X,T) - B(X,Z)B(Y,T) \right),
\]

(3.31)

\[
\langle \hat{R}(X,Y)Z,T \rangle = c \left( \langle Y,Z \rangle \langle X,T \rangle - \langle X,Z \rangle \langle Y,T \rangle \right) 
+ 2 \varphi \left( B(Y,Z)B(X,T) - B(X,Z)B(Y,T) \right),
\]

(3.32)

\[
(\nabla_X B)(Y,Z) + B(Y,Z)\tau(X) = (\nabla_Y B)(X,Z) + B(X,Z)\tau(Y),
\]

(3.33)

for all \( X,Y,Z,T \in \Gamma(S(TM)) \). From equations (3.31) and (3.32) above, the following result is straightforward.

**Proposition 3.3.** Let \( x : (M,g,N) \to (\mathcal{M},\tilde{g}) \) be a SIC-rigged null hypersurface of a locally flat spacetime (thus \( c = 0 \)). Then, \( (M,\nabla) \) is locally flat if and only if \( (S,\hat{\nabla}) \) is locally flat, for any leaf \( S \) of the screen distribution.

**Theorem 3.3.** Let \( x : (M,g,N) \to (\mathcal{M}(c),\tilde{g}) \) be a UCC-rigged null hypersurface, and \( S \) a generic leaf of the screen distribution. Then,

1. \( S \) is parallel if and only if \( B \) is parallel;
2. \( S \) has parallel mean curvature if and only if the two expansions \( \theta_N \) and \( \theta_\xi \) are closed;
3. if \( M \) is totally umbilical with \( B = \rho g \), then \( \rho \) is constant on each (connected) leaf of the screen distribution. In addition if \( \tau(\xi) = \rho \) then \( \rho \) is constant on \( M \).

**Proof.** Notice that the rotation 1–form \( \tau \) vanishes on the screen distribution since the rigging is UCC (Lemma 3.1). Notice also that \( \varphi = 1 \) and \( B = C \). Hence, the first (resp. second) item follows from the equality (3.14) (resp. (3.19)). From equation (3.33), one has

\[
(X \cdot \rho)g(Y,Z) + \rho( \nabla_X g)(Y,Z) = (Y \cdot \rho)g(X,Z) + \rho( \nabla_Y g)(X,Z).
\]

Since \( \nabla \) is \( g \)–metric on sections of the screen distribution (Proof: equation (2.14)), the latter equation leads to \( (X \cdot \rho)Y = (Y \cdot \rho)X \), for all \( X,Y \in \Gamma(S(TM)) \). This show that \( (X \cdot \rho) = 0 \), for all \( X \in \Gamma(S(TM)) \). This completes the proof of item 3.

**Theorem 3.4.** Let \( x : (M,g,N) \to (\mathcal{M}(c),\tilde{g}) \) be a SIC-rigged null hypersurface of a spacetime with constant sectional curvature \( c \). If \( M \) is totally umbilical (or geodesic) then, each (connected) leaf of the screen distribution is a space form.

**Proof.** Assume that \( M \) is totally umbilical with \( B = \rho g \). Then equation (3.32) gives

\[
\langle \hat{R}(X,Y)Z,T \rangle = (c + 2\varphi \rho^2) \left( \langle Y,Z \rangle \langle X,T \rangle - \langle X,Z \rangle \langle Y,T \rangle \right),
\]
Which show that sectional curvature of $S$ is constant $\kappa = c + 2\varphi \rho^2$ for all sections $\sigma = \text{span}(X,Y)$. It is then known that $\kappa$ is a constant function. This prove that each connected leaf of the screen distribution has constant sectional curvature

$$\kappa = c + 2\varphi \rho^2.$$  

Hence for a SIC-rigged totally umbilical null hypersurface $(M, g, N) \to (\mathbb{M}(c), \overline{g})$ with $B = \rho g$ and $A_N = \varphi \hat{A}_\xi$, the product $\varphi \rho^2$ is constant on each leaf of the screen distribution. It is noteworthy that a closed rigging with conformal screen distribution is a SIC-rigging.

**Corollary 3.1.** Let $x : (M, g, N) \to (\mathbb{M}(c), \overline{g})$ be a null hypersurface endowed with a closed rigging with conformal screen distribution with $A_N = \varphi \hat{A}_\xi$. If $M$ is totally umbilical with $B = \rho g$ then, $\rho$ and $\varphi$ are constants on each (connected) leaf of the screen distribution.

**Proof.** By Lemma 3.1, the rotation 1–form vanishes on the screen distribution and equation (3.33) become

$$(\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z).$$

Now, if $M$ is totally umbilical then by using the above equation, one shows that $\rho$ is constant on each leaf of the screen distribution. It follows that $\varphi$ also is constant on each leaf of the screen distribution, since it is the case for $\varphi \rho^2$.

The following is a direct consequence of the above Theorem.

**Corollary 3.2.** Let $x : (M, g, N) \to \mathbb{R}^{n+2}_1$ be a UCC-rigged null hypersurface. If $M$ is totally umbilical with $B = \rho g$ then, each (connected) leaf of the screen distribution is a space form of positive constant sectional curvature $c = 2\rho^2$ and parallel mean curvature vector.

**Example 3.** Let $x : \Lambda_0^{n+1} \to \mathbb{R}^{n+2}_1$, $p = (x^1, \ldots, x^{n+1}) \mapsto x = (x^0 = F(x^1, \ldots, x^{n+1}), x^1, \ldots, x^{n+1})$ be the future null cone, which is the graph of the function

$$F = \left(\sum_{a=1}^{n+1}(x^a)^2\right)^{1/2}.$$  

This is a totally umbilical null hypersurface in $\mathbb{R}^{n+2}_1$ and the generic UCC-rigging (3.24) becomes

$$\mathcal{N}_F = -\frac{1}{\sqrt{2}} \frac{\partial}{\partial x^0} + \frac{1}{x^0 \sqrt{2}} \sum_{a=1}^{n+1} (x^a) \frac{\partial}{\partial x^a},$$

and the corresponding rigged vector field

$$\xi_F = -\frac{1}{\sqrt{2}} \frac{\partial}{\partial x^0} - \frac{1}{x^0 \sqrt{2}} \sum_{a=1}^{n+1} (x^a) \frac{\partial}{\partial x^a}.$$
For this rigging, the screen distribution is integrable and leaves of the screen distribution are sections of the future lightcone by hyperplanes $x^0 = \text{cste}$. These are spheres of radius $x^0$ centered at $(x^0, 0, \ldots, 0) \in \mathbb{R}^{n+2}$. (See [18] for a proof.) All the principal curvatures are given by

$$\rho = \frac{1}{x^0 \sqrt{2}},$$

which are constants on each leaf of the screen distribution, and this agrees with Theorem 3.3. By the Theorem above, each leaf of the screen distribution has positive constant sectional curvature $c = 2\rho^2 = \frac{1}{(x^0)^2}$, which is really the sectional curvature of a sphere of radius $x^0$.

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