A Simple Explicit Construction of an $n^{\tilde{O}(\log n)}$-Ramsey Graph

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Abstract

We show a simple explicit construction of an $2^{\tilde{O}(\sqrt{\log n})}$ Ramsey graph. That is, we provide a poly(n)-time algorithm to output the adjacency matrix of an undirected $n$-vertex graph with no clique or independent set of size $2^{\tilde{O}(\sqrt{\log n} \log \log n)}$ for every $\epsilon > 0$.

Our construction has the very serious disadvantage over the well-known construction of Frankl and Wilson [FW81] that it is only explicit and not very explicit, in the sense that we do not provide a poly-logarithmic time algorithm to compute the neighborhood relation. The main advantage of this construction is its extreme simplicity. It is also somewhat surprising that even though we use a completely different approach we get a bound which essentially equals the bound of [FW81]. This construction is quite simple and was obtained independently by others but as far as we know has not been published elsewhere.

1 The Construction

As mentioned above, we prove the following proposition:

Proposition 1.1. Let $\epsilon > 0$ be some constant. There is a polynomial-time algorithm $A$ that on input $n$ outputs the adjacency matrix for a graph $H$ on $n$ vertices with no clique or independent set of size $2^{\epsilon \sqrt{\log n}}$.

Proof. We will need to recall the notion of the Abbott product of two graphs: if $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are graphs, then the Abbott product of $G$ and $H$, denoted by $G \otimes H$ is the graph with vertex set $V_H \times V_H$ (where $\times$ denotes Cartesian product) and where $(u, v), (u', v')$ is an edge if either $(u, u')$ is an edge in $G$ or $u = u'$ and $(v, v')$ is an edge in $H$. One can think of $G \otimes H$ as obtained by replacing each node of $G$ with an entire copy of $H$ (both vertices and edges), where each two different copies of $H$ have either all the edges between them or none of the edges between them, depending on whether the corresponding vertices in $G$ are neighbors. We let $G^l$ denote $G \otimes G \otimes \cdots \otimes G$ ($l$ times).

We let $\omega(G)$ be the clique number of $G$ (i.e., the size of the largest clique in $G$) and $\alpha(G)$ be the independence number of $G$ (i.e., the size of the largest independent set in $G$). The basic fact we need about the Abbott product is that $\omega(G \otimes H) = \omega(G) \cdot \omega(H)$ and $\alpha(G \otimes H) = \alpha(G) \cdot \alpha(H)$.

We can now specify our construction. Given $\epsilon > 0$, the algorithm $A'$ will choose a constant $c > 1$ (the exact choice of $c$ will be specified later), and let $k = 2^{\sqrt{\log n}}$ and using $k^{O(\log k)} = n^{O(1)}$ running time construct a graph $G$ on $k$ vertices such that $\omega(G), \alpha(G) < 3 \log k$. 

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1P. Pudlak, personal communications, July 2004.
Constructing such a graph can be done using well-known techniques: as a first observation note that in time \( k^{O(\log k)} \) it can be verified that a graph \( G \) satisfies \( \omega(G), \alpha(G) < 3 \log k \). Thus, it is enough to show an explicit family of \( k^{O(\log k^2)} \) graphs, where one of which satisfies this condition. A graph can be represented as a string of length \((\binom{k}{2})\). We claim that if we choose this string from a sample space that is \( 2^{−5 \log^2 k} \)-close to being \( 5 \log^2 k \)-wise independent then with high probability the graph will satisfy \( \omega(G), \alpha(G) < 2 \log k \), once we prove this then we’ll be done since explicit sample spaces with cardinality \( k^{O(\log k)} \) were given by Naor and Naor \cite{NN93}. However, this follows by the same reason that a random graph satisfies this property: that every set of \( 4 \cdot 2^{−5 \log^2 k}k \) edges has probability at most \( 2 \cdot 2^{−8.5 \log^2 k}k + 2^{−5 \log^2 k}k \ll 1/(\binom{k}{3 \log k}) \) to be identically zero or identically one.\(^2\)

Now, the algorithm will compute the graph \( H = G_{\frac{1}{c}}^{\sqrt{\log n}} \). This graph has \( n \) vertices, but

\[
\omega(H), \alpha(H) < (3 \log k)^{\frac{1}{c}}^{\sqrt{\log n}} = (3c^{\sqrt{\log n}})^{\frac{1}{c}}^{\sqrt{\log n}} < 2^{\log c + \frac{2}{2c} \log \log n^{\sqrt{\log n}}}
\]

we choose \( c \) large enough such that the constant expression in the exponent will be smaller than \( \epsilon \).

\[\Box\]

References

\cite{FW81} P. Frankl and R. M. Wilson. Intersection theorems with geometric consequences. Combinatorica., 4(1):357–368, 1981.

\cite{NN93} J. Naor and M. Naor Small-Bias Probability Spaces: Efficient Constructions and Applications SIAM J. Comput., 22(4): 838-856, 1993.

\(^2\)Another approach that may work is derandomization using the method of conditional expectations.