Bose-Einstein Condensation May Occur in a Constant Magnetic Field

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Abstract
Bose-Einstein condensation of charged scalar and vector particles may actually occur in presence of a constant homogeneous magnetic field, but there is no critical temperature at which condensation starts. The condensate is described by the statistical distribution. The Meissner effect is possible in the scalar, but not in the vector field case, which exhibits a ferromagnetic behavior.

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It was pointed out long ago by Schafroth [1] that for a non-relativistic boson gas, Bose-Einstein condensation (BEC) does not take place in the presence of a constant magnetic field. The problem was studied afterwards by May [2], who investigated the condensation in the non-relativistic case for an arbitrary dimension $d$ and showed that it occurs for $d \geq 5$, and later by Daicic et al. [3], [4], who extended the considerations made by [2] to the relativistic high temperature case and found new magnetization properties. More recently Toms [5] has proved that BEC in presence of a constant magnetic field does not occur in any number of spatial dimensions, and Elmfors et al. [6] consider that in the 3$d$ case, although a true condensate is not formed, the Landau ground state can accommodate a large charge density. The last paper discuss also the magnetization of the relativistic scalar gas and shows that the Meissner effect occurs in the low temperature case, in analogy to the non-relativistic case studied by [1]. All these papers are characterized by very careful calculations and their discrepancies, when arised, are in general related to subtle points. One essential argument in all of them is that condensation cannot occur since the condition for condensation, taken from the symmetric (zero field) case as $\mu = \sqrt{Mc^2 \pm eB\hbar c}$ for the relativistic case ($\mu' = \mu - Mc^2 \pm eB\hbar/mc = 0$ in the nonrelativistic limit), when applied to the case in which there is an external magnetic field, leads to a divergent behaviour of the density in terms of $\mu'$.

There are two different ideas which usually are considered to be the same, concerning what is to be understood as BEC: 1) The existence a finite fraction of the total particle density in the ground state at some temperature $T > 0$. 2) The existence of a critical temperature $T_c > 0$ for which $\mu' = 0$, such that for $T < T_c$ some significant amount of particles start to condense in the ground state.

The present author’s point of view is that i) the condition for condensation in the magnetic field case corresponds to 1), i.e. it cannot be extrapolated as the same one of the standard zero field case, (which is 2)), since in the magnetic field case there are different physical conditions: explicit spatial symmetry breakings, discrete Landau states; ii) since the chemical potential is not an independent thermodynamical variable, but
depends on the density, temperature and magnetic field, there is no divergence problem at all, iii) as different from the zero field case, in presence of the magnetic field the expression for the density contains the ground state contribution.

Let us remind the origin of the critical quantities $\mu_c$, $T_c$ in the standard theory of BEC (with zero magnetic field). The chemical potential $\mu' = f(N, T)$ is a decreasing function of temperature at fixed density $N$, and for $\mu' = 0$ one gets an equation defining $T_c = f_c(N)$. For temperatures $T < T_c$, as $\mu' = 0$, the expression for the density gives values $N'(T) < N$, and the difference $N - N' = N_0$ is interpreted as the density of particles in the condensate. This is to be expected since the density of states, proportional to $4\pi p^2$, cancels the infrared divergence of the Bose-Einstein distribution $n(p, T)$ for $\mu = 0$. The expression for $N$ contain only the particles in excited states. Thus, for $\mu' \to 0$, taking $p_0 \sim \sqrt{-2M\mu'}$, the amount of particles in a neighbourhood of the ground state of amplitude $p_0$ is $N_0 = 4\pi \int_0^{p_0} p^2 n(p, T) dp \sim MT p_0/\hbar^3$. In the magnetic field case, the infrared divergence of the Bose-Einstein distribution resulting from taking $\mu' = 0$ is not cancelled by the density of states in momentum space at $T \neq 0$. In other words, for a given constant density, we are not allowed to put $\mu' = 0$ keeping $T \neq 0$. Now the amount of particles in a small neighbourhood of the ground state of amplitude $p_0$, as we shall see below, is $N_0 \sim eB \int_0^{p_0} n(p_3, T) dp_3 \sim eBM^{1/2} T/\sqrt{-\mu'}$, which is large for small $\mu'$ and leads to conclude that in the magnetic field case the expression for the density accounts for the contribution of the population in all quantum states.

Thus, in the magnetic field case the problem of finding a macroscopic ground state density is not conditioned to have $\mu' = 0$.

We shall investigate in more detail the problem of BEC in the physical $d = 3$ space, at large densities and strong magnetic fields. Under these conditions BEC occurs, but without having a definite critical temperature $T_c > 0$, i.e. the phase transition is diffuse. The concept of diffuse phase transitions, as those not having a definite critical temperature, but an interval of $T$, was introduced long ago by Smolenski and Isupov [7] in their investigation of the phase transition which occur in certain ferroelectric materials.
By assuming as in [6] a constant microscopic magnetic field $B$ along the $p_3$ axis (the external field is $H^{\text{ext}} = B - 4\pi M(B)$, where $M(B)$ is the magnetization), the thermodynamic potential for a gas of scalar particles placed in it is

$$\Omega_s = \frac{eB}{4\pi^2\hbar^2c^3} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dp_3 \left[ \ln(1 - e^{-(\epsilon_q - \mu)\beta}) (1 - e^{-(\epsilon_q + \mu)\beta}) + \beta \epsilon_q \right]$$

(1)

where $\epsilon_q = \sqrt{p_3^2c^2 + M^2c^4 - 2eBhc(n + \frac{1}{2})}$, the last term in (1) accounts for the vacuum energy and $\mu$ is the chemical potential. We shall use $\beta = 1/T$, where $T = kT$ in what follows, $k$ being the Boltzmann constant and $t$ the absolute temperature in Kelvins.

For a vector field the one-loop thermodynamic potential is

$$\Omega_v = \frac{eB}{4\pi^2\hbar^2c^3} \int_{-\infty}^{\infty} dp_3 \left[ \ln(1 - e^{-(\epsilon_0 - \mu)\beta}) (1 - e^{-(\epsilon_0 + \mu)\beta}) + \beta \epsilon_0 \right]$$

$$+ \frac{eB}{4\pi^2\hbar^2c^3} \sum_{n=0}^{\infty} \beta_n \int_{-\infty}^{\infty} dp_3 \left[ \ln(1 - e^{-(\epsilon_q - \mu)\beta}) (1 - e^{-(\epsilon_q + \mu)\beta}) + \beta \epsilon_q \right]$$

(2)

where $\beta_n = 3 - \delta_{0n}$, $\epsilon_0 = \sqrt{p_3^2c^2 + M^2c^4 - eBhc}$, $\epsilon_q = \sqrt{p_3^2c^2 + M^2c^4 + 2eBhc(n + \frac{1}{2})}$.

The mean density of particles minus antiparticles (average charge divided by $e$) is given by $N_{s,v} = -\partial \Omega_{s,v} / \partial \mu$.

We have explicitly

$$N_s = \frac{eB}{4\pi^2\hbar^2c^3} \sum_{n=0}^{\infty} \left[ \int_{-\infty}^{\infty} dp_3 (n_p^+ - n_p^-) \right].$$

(3)

where $n_p^\pm = \left[ \exp(\epsilon_q \mp \mu) \beta - 1 \right]^{-1}$. For the vector field case we have,

$$N_v = \frac{eB}{4\pi^2\hbar^2c^3} \int_{-\infty}^{\infty} dp_3 (n_{0p}^+ - n_{0p}^-)$$

$$+ \frac{eB}{4\pi^2\hbar^2c^3} \sum_{n=0}^{\infty} \beta_n \int_{-\infty}^{\infty} dp_3 (n_p^+ - n_p^-)$$

(4)

where $n_{0p}^\pm = \left[ \exp(\epsilon_q \mp \mu) \beta - 1 \right]^{-1}$, $\epsilon_q = \sqrt{p_3^2c^2 + M^2c^4 - eBhc}$, and $\beta_{0n} = 3 - \delta_{0n}$.

For $T \to 0$, $\mu \to M_{\pm}c^2$ (we named $M_{\pm} = \sqrt{M^2 \pm eBh/c^3}$), the population in Landau quantum states other than $n = 0$ vanishes (this is shown explicitly in the Appendix) and the density for the $n = 0$ state is infrared divergent. We expect then most of the
population to be in the ground state, since for small temperatures \( n_{0p} \) is vanishing small and \( n_{0p}^+ \) is a bell-shaped curve with its maximum at \( p_3 = 0 \). We will proceed as in [8] and call \( p_0 \gg \sqrt{-2M\mu'} \) some characteristic momentum. We have then, by assuming \(-\mu' \ll T\), for the density in a small neighbourhood of \( p_3 = 0\),

\[
N_{0s,v} = \frac{eBT}{2\pi^2\hbar^2c} \int_0^{p_0} \frac{dp_3}{\sqrt{p_3^2 c^2 + M^2 c^4 + eBhc - \mu}}
\]

\[
= \frac{eBT}{2\pi^2\hbar^2c} \int_0^{p_0} \frac{dp_3}{p_3^2/2M_\pm - \mu'}
\]

\[
= \frac{eBT}{4\pi\hbar^2c} \sqrt{\frac{2M_\pm}{-\mu'}}
\]

(5)

where \( N_{s,v} = N_{0s,v} + \delta N_{s,v} \) and \( \delta N_{s,v} \) is the density in the interval \([p_0, \infty]\). Actually \( \delta N_{s,v} \) is negligibly small and \( N_{0s,v}^0 \simeq N_{s,v} \).

The expression (5) was obtained in [8] and more recently in [3]. Formally it indicates a divergent behaviour of \( N_{0s,v} \) for \( \mu' \to 0 \), but we must be careful in doing that interpretation. Actually it means nothing more that in a neighbourhood of the ground state, at constant temperature, \( N_{0s,v} \sim (\mu')^{-\frac{1}{2}} \), and for constant \( N_{0s,v} \), \( \mu' \sim T^2 \). This comes from the fact that \( \mu' \) is not, and cannot be taken, as an independent thermodynamic variable (see i.e. [4], chapter 3), but depends on the quantities \( N_{s,v}^0, T, B \) and it is enough that \( \mu' \sim T^2 f(N_{0s,v}) \) for \( T \to 0 \) (\( f \) being a finite function) to avoid the divergence. Thus (5) simply means that

\[
\mu' = -\frac{e^2B^2T^2M_\pm}{8\pi^2N_{0s,v}^2\hbar^4c^2}.
\]

(6)

We observe that \( \mu' \) is a decreasing function of \( T \) and vanishes for \( T = 0 \), where the "critical" condition \( \mu = M_\pm c^2 \) is reached. As shown below, in that limit the Bose-Einstein distribution degenerates in a Dirac \( \delta \) function, which means to have all the system in the ground state \( p_3 = 0 \). To see this, we shall rewrite the momentum density of particles
around the ground state $p_3 = 0$, $n = 0$ approximately as

$$n_0(p_3) = \frac{T}{\frac{p_3^2}{2M_\pm} + \frac{e^2B^2T^2M_\pm}{8\pi^2\hbar^4c^2N_{0s,v}}}$$

$$= \frac{4\pi\hbar^2cN_{0s,v}}{eB} \cdot \frac{\gamma}{p_3^2 + \gamma^2}$$

(7)

where

$$\gamma = \frac{eBTM_\pm}{2\pi\hbar^2cN_{0s,v}} = \frac{p_T}{v^3N_{0s,v}} = \sqrt{-2M_\pm\mu'}$$

where $p_T = \sqrt{2\pi M_\pm T}$ is the thermal momentum, $v^3 = hc\lambda/eB$ the elementary volume cell, $\lambda = h/p_T$ being the De Broglie thermal wavelength. We have approximated the Bose-Einstein distribution by one proportional to a Cauchy distribution, having its maximum $\sim \gamma^{-1}$ for $p_3 = 0$. We have that $\gamma \to 0$ for $T \to 0$, but for small fixed $T$, $\gamma$ also decreases as $v^3N_{0s,v}$ increases. We remind that in the zero field case, the condensation condition demands $N\lambda^3 > 2.612$.

One can write then

$$\frac{1}{2} N_{0s,v} = \frac{eB}{4\pi^2\hbar^2c} \int_{-\gamma}^{\gamma} n_0(p_3) dp_3.$$  

(8)

Thus, approximately one half of the total density is concentrated in the narrow strip of width $2\gamma$ around the $p_3 = 0$ momentum. It results that for densities and magnetic fields large enough, if we choose an arbitrary small neighbourhood of the ground state, of momentum width $2p_{30}$, one can always find a temperature $T > 0$ small enough such that $\gamma \ll p_{30}$ and (8) and (9) are satisfied. The condensate appears and it is described by the statistical distribution.

We have also

$$\lim_{\gamma \to 0} n_0(p_3) = 4\pi^2\hbar^2c\frac{N_{s,v}}{eB} \delta(p_3).$$

(9)

and obviously

$$N_{s,v} = \lim_{\gamma \to 0} \frac{eB}{4\pi^2\hbar^2c} \int_{-\infty}^{\infty} n_0(p_3) dp_3.$$  

(10)
Thus, if $T = 0$, all the density $N_{s,v}$ lies in the condensate, as occur in the zero field case, but here the total density is described explicitly by the integral in momentum space.

Obviously, one cannot fix any (small) value for $\gamma$ from which the distribution starts to have a manifest $\delta(p_3)$ behavior; and there is no critical temperature for condensation to start, which we interpret as a diffuse phase transition. But when $\gamma$ and $1/\Delta$ decrease enough (we define $\Delta = \omega h / T$, and $\omega = eB/Mc$. If $\Delta \ll 1$, the system is confined to the $n = 0$ Landau state, see below), the conditions for condensation mentioned above are satisfied.

In the non-relativistic case, the above results can be derived even from Schafroth’s formulae. We can write the non-relativistic limit for $N_s$ (which is a very good approximation for the relativistic case if $Mc^2 \gg T$, since the main contribution to the integrals comes from values of $p_3 \leq Mc$ ), as

$$N_s = \frac{eB}{2\pi^2\hbar^2c}\sqrt{\frac{\pi MT}{2}} \sum_{m=1}^{\infty} \frac{e^{-\mu_1 m\beta}}{m^{1/2}} \frac{1}{1 - e^{-2\omega h m\beta}}$$

$$= \frac{eB}{2\pi^2\hbar^2c}\sqrt{\frac{\pi MT}{2}} \sum_{m=1}^{\infty} \frac{e^{-\mu_1 m\beta}}{m^{1/2}} \left[ 1 + \frac{e^{-2\omega h m\beta}}{1 - e^{-2\omega h m\beta}} \right]. \quad (11)$$

where the unity in angular brackets accounts for the Landau $n = 0$ state and the second term for all the set of excited states. We have used $\mu_1 = Mc^2 + \omega h - \mu$ and $\omega = eB/Mc$. Now, if the parameter $\Delta = \omega h / T \gg 1$, and if $\mu_1 \to 0$, even at large temperatures, we can neglect the second term in angular brackets in (11), and approximate the resulting sum by an integral. This is done by introducing the continuous variable $x = Tm$ and by writing $T \sum_{n=1}^{\infty} \to \int_0^{\infty} dx$. After integrating, we obtain back (5), with $\mu_1, M$ in place of $\mu', M_{\pm}$.

Neutron stars, where strong magnetic fields and very high densities are assumed to exist, may provide conditions for the occurrence of BEC. It has been conjectured that superfluid and superconductive effects are produced (see i.e. [10], [11] and references therein).

As a simplified model of the gas of kaons in a neutron star, we take $N_s \sim 10^{44}$ cm$^{-3}$,
\( t \simeq 10^{8}\)K, and local magnetic fields \( B \sim 10^{14} \) G (fields of order \( 10^{15} \) G have been estimated inside hadrons [122]), the conditions for BEC can be also satisfied, if the medium provide also screening mechanisms for the very large electric fields which also arise. In this case \( \gamma = 10^{-30} \). If we suppose the star of dimensions \( \sim 10^7\) cm, the discrete momentum states would be spaced by an amount \( \delta p \sim 10^{-34}\) g cm/s. This means to have one half of the total density distributed in these \( 2\gamma/\delta p = 10^4 \) quantum states. The ground state density, with strictly zero momentum \( p_3 \), can thus be estimated as \( 10^{40} \) cm\(^{-3}\). We observe in this case the dimensionless phase space density \( \mathcal{N} v^3 \sim 10^{12} \).

We turn now to the magnetization problem. From (1), we get that for \( T \to 0 \), the magnetization of the scalar field is

\[
\mathcal{M}_s = -\frac{\partial \Omega}{\partial B}
\]

\[
= -\frac{e}{4\pi^2\hbar} \sum_0^\infty \int_{-\infty}^\infty dp_3 \frac{e(n+\frac{1}{2})}{\epsilon_q} (n_q^+ + n_q^-).
\]

(12)

In the condensation limit we get

\[
\mathcal{M}_s = -\frac{eN_s\hbar}{2M_c}.
\]

(13)

We see that the magnetization is opposed to the external field and the system behaves as a perfect \textit{dia-ferromagnetic} (this was first pointed out by Schafroth [1]).

If we take the \( B(> T) \to 0 \) limit of (13), one can discuss the critical conditions for the arising of the Meissner effect in the low temperature relativistic case. The condition (obtained for \( T \neq 0 \) in ref. [6]), is \( H_{c}^{ext} = -\mathcal{M}_s(0) \). In particular, we agree with [6] that in the relativistic case the Meissner effect occurs in analogy with the non-relativistic one, and that it is not necessarily connected with high temperature pair creation processes, as suggested in [3].

For the vector field case, the magnetization in the condensation limit is positive since all the system is in the Landau \( n = 0 \) state, (see eq. (2)), and

\[
\mathcal{M}_v = \frac{eN_v\hbar}{2M_c}.
\]

(14)
and we have that the condensate of vector particles behave as a true ferromagnet. In particular, we can write \( 0 = H_c^{\text{ext}} = B - \mathcal{M}(B) \) as the condition for spontaneous magnetization to occur.

In concluding, it is important to remind that for the first time true BEC has been recently observed \([13]\) in evaporated \(^{87}\text{Rb}\) atoms confined in a magnetic trap creating an ellipsoidal potential. The system is far from being an isotropic noninteracting gas, and although it also differs from our present model, it has, however, some analogies. In particular, the anisotropy of the confining potential leads to a larger velocity spread in the axial direction as compared to the radial one, which is the analog of the \( p^3 \) spread of the \( n(p_3) \) distribution in the magnetic field case, confined radially to the \( n = 0 \) Landau quantum state.

The author thanks R. Baquero, J. Hirsch, K. Kirsten, D. B. Lichtenberg, O. Perez-Martinez and D. Quesada for comments. He is especially indebted to A. E. Shabad for a discussion long ago, from which some of the basic ideas presented in this paper arose.

**Appendix**

We will present a demonstration concerning the vanishing of the density for excited Landau states if \( \Delta \gg 1, M\beta \gg 1 \). We use in this section units \( \hbar = c = 1 \), and consider only the scalar field case. Let us call \( N_e \) the density corresponding to excited states. We can write, by using the \( K_2 \) Bessel function representation in terms of dimensionless quantities \( \bar{M}_+ = M_+\beta, \bar{M}_+ = M_+\beta \) and \( \bar{\mu} = \mu\beta \), after summing over Landau quantum numbers from 1 to \( \infty \),

\[
N_e = \frac{eB}{8\pi^2} \sum_{m=1}^{\infty} m \sinh m\bar{\mu} \int_0^\infty \frac{dt}{t^2} e^{-\frac{m^2}{4t} - \bar{M}_+^2 t} \cdot \frac{e^{-2eB\beta^2t}}{1 - e^{-2eB\beta^2t}}.
\]

We will establish an upper bound to this quantity. Let us introduce \( x = \bar{M}_+ t \) and cut the integral in two parts by the point \( x_0 = \theta / (\bar{M}_+ + \Delta) \), where \( \theta \ll 1 \). Let us call also \( \xi \) to some point in between \( x_0 \) and \( \infty \). We have
\[ N_e \leq \frac{M^2}{8\pi^2} \sum_{m=1}^{\infty} m \sinh m\bar{\mu} \left[ \frac{16}{m^4 M^2} e^{\frac{m^2 \bar{\mu}^2}{\bar{\mu}}} + \frac{2\omega h e^{-m^2 \bar{M}/\bar{\xi}}}{\xi^2} \cdot \frac{e^{-2\Delta \xi}}{1 - e^{-2\Delta \xi}} \cdot \frac{e^{-(\bar{M}+x_0)}}{\bar{M}+x_0} \right] \quad (16) \]

Both series can be made to converge to an arbitrary small number as \( \theta \to 0 \) and \( 1/\Delta \to 0 \), provided that \( \bar{M} \gg 1 \).

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