NECESSARY CONDITIONS FOR A WEAK MINIMUM IN A GENERAL OPTIMAL CONTROL PROBLEM WITH INTEGRAL EQUATIONS ON A VARIABLE TIME INTERVAL

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ABSTRACT. We study an optimal control problem with a nonlinear Volterra-type integral equation considered on a nonfixed time interval, subject to endpoint constraints of equality and inequality type, mixed state-control constraints of inequality and equality type, and pure state constraints of inequality type. The main assumption is the linear–positive independence of the gradients of active mixed constraints with respect to the control. We obtain first-order necessary optimality conditions for an extended weak minimum, the notion of which is a natural generalization of the notion of weak minimum with account of variations of the time. The conditions obtained generalize the corresponding ones for problems with ordinary differential equations.

1. Introduction. It is commonly known that, for problems with ordinary differential equations (ODEs), the theory of first order necessary optimality conditions including the Pontryagin maximum principle is now completely developed. It covers problems both on a fixed and a nonfixed time intervals containing pure state and mixed state-control constraints, as well as different types of integral and endpoint constraints. A challenging question arises about similar complete theory for problems with control system given by Volterra-type integral equations. Such equations could be considered as a close generalization of ODEs, since, in some respects, they possess similar properties. However, this similarity is not complete, and the integral equations have also some essential specificity that differ them from the ODEs. Nevertheless, like the calculus of variations and optimal control for ODEs, the corresponding theory for Volterra-type equations should desirably include necessary conditions both for the strong and weak local minima.

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First results on optimality conditions for problems with Volterra-type equations appeared soon after publication of the famous monograph by Pontryagin and his collaborators [16], and concerned the maximum principle. These results were due to Vinokurov [17], Bakke [1], Neustadt [15], Kamien and Muller [12], Hartl and Sethi [11], and Carlson [4]. All of them considered problems on a fixed time interval, without mixed or state constraints. Later, De la Vega [5] considered a time optimal control problem, which was the first publication concerning problems with integral equations on a nonfixed time interval. After that the interest to problems with integral equations arose again, and new results, including second-order conditions for state constrained problems, were obtained by Bonnans, Dupuis, and De la Vega [3]. In [8], we considered a general problem with state and mixed constraints on a fixed time interval and obtained necessary conditions for the weak minimum. If the time interval is nonfixed, the notion of weak minimum should be slightly generalized to the notion of extended weak minimum, which takes into account also the variations of time. Necessary conditions for this type of minimum, in problems without state or mixed constraints were obtained in our recent paper [9].

In the present paper we consider a general problem combining both a nonfixed time interval and state and mixed constraints, and obtain first order conditions for an extended weak minimum. They are given in Theorem 3.1. As far as we know, such conditions for problems with integral equations on a variable time interval subject to state and mixed constraints were not obtained up to now. Note that they are not a direct combination of the results in [8, 9]. Based on those former results it is hardly possible to give even the proper formulation of the present results. Their novelty, as compared with those for problems on a fixed time interval is that the costate equation and the terminal transversality condition with respect to \( t \) involve nonstandard terms that are absent in problems with ODEs. Along with this, the presence of state and mixed constraints in problems on a nonfixed time interval (unlike in problems on a fixed interval) leads to additional delicacy not only in the proofs, but even in the formulations.

As was already mentioned in [8], necessary conditions for the weak (or extended weak) minimum in optimal control problems constitute an important stage in derivation of any further necessary optimality condition, including maximum principle or higher order conditions, and thus, they deserve a separate thorough study for each specific class of problems, like it is done in the classical calculus of variations. This is why we focus on these conditions. Following the tradition, we call them stationarity conditions (or local maximum principle).

The paper is organized as follows. In Section 2 we formulate a general optimal control problem with integral equations on a variable time interval, called Problem A. We also define in this section the notion of extended weak minimum. Section 3 is devoted to formulation of the main result of the paper — the first order necessary condition for an extended weak minimum in Problem A (Theorem 1).

The proof of Theorem 1 is based on a reduction of Problem A to a problem of new type B on a fixed time interval, stated in Section 4. The control system in Problem B has a more general type than that in Problem A. In Theorem 2 of this section we present the Lagrange multipliers rule (LMR) for a class of abstract nonsmooth optimization problems which contains Problem B as a special case. In Section 5, we apply LMR to Problem B, and then, in Section 6, performing some analysis of LMR, obtain the local maximum principle in Problem B. This important intermediate result is given in Theorem 3. Finally, in Section 7, we finish the proof.
of Theorem 1 by reducing Problem A to a specific problem of type B and applying the obtained local maximum principle to the reduced problem.

2. General optimal control problem with integral equations on a variable time interval (Problem A). We consider the following control system of Volterra-type integral equations on a variable time interval \([t_0, t_1]\):

\[
x(t) = x(t_0) + \int_{t_0}^{t} f(t, s, x(s), u(s)) \, ds,
\]

where \(x(\cdot)\) is a continuous \(n\)-dimensional and \(u(\cdot)\) a measurable essentially bounded \(r\)-dimensional vector-function on \([t_0, t_1]\). As usual, we call \(x\) the state variable and \(u\) the control variable (or simply the control). We assume for simplicity that the function \(f\) is defined and \textit{twice continuously differentiable} on an open set \(\mathcal{R} \subset \mathbb{R}^{2+n+r}\).

The problem is to minimize the endpoint cost functional

\[
J = \varphi_0(t_0, x(t_0), t_1, x(t_1)) \rightarrow \min
\]
on the set of solutions of system (1) satisfying the endpoint constraints

\[
\eta_j(t_0, x(t_0), t_1, x(t_1)) = 0, \quad j = 1, \ldots, d(\eta),
\]

\[
\varphi_i(t_0, x(t_0), t_1, x(t_1)) \leq 0, \quad i = 1, \ldots, d(\varphi),
\]

the mixed state-control constraints

\[
F_i(t, x(t), u(t)) \leq 0 \quad \text{for a.e. } t \in [t_0, t_1], \quad i = 1, \ldots, d(F),
\]

\[
G_j(t, x(t), u(t)) = 0 \quad \text{for a.e. } t \in [t_0, t_1], \quad j = 1, \ldots, d(G),
\]

and the pure state constraints

\[
\Phi_k(t, x(t)) \leq 0 \quad \text{for all } t \in [t_0, t_1], \quad k = 1, \ldots, d(\Phi),
\]

where the functions \(\varphi_0, \varphi_i, \eta_j\) are assumed to be defined and continuously differentiable on an open set \(\mathcal{P} \subset \mathbb{R}^{2n+2}\), and the functions \(F_i, G_j, \Phi_k\) are defined and continuously differentiable on an open set \(\mathcal{Q} \subset \mathbb{R}^{1+n+r}\). (The \(\Phi_k\) can be formally considered as functions of three variables \(t, x, u\).) The notation \(d(\varphi), d(\eta), d(F), d(G)\), etc. stand for the numbers of these functions.

Moreover, we impose the following

Assumption RMC (on the regularity of mixed constraints). The mixed constraints (5)-(6) are regular in the following sense: at any point \((t, x, u) \in \mathcal{Q}\) satisfying relations \(F_i \leq 0 \ \forall i \) and \(G_j = 0 \ \forall j\), the system of vectors

\[
F'_{iu}(t, x, u), \quad i \in I(t, x, u), \quad G'_{ju}(t, x, u), \quad j = 1, \ldots, d(G),
\]
is positively-linearly independent, where \(I(t, x, u) = \{i \mid F_i(t, x, u) = 0\}\) is the set of active indices of mixed inequality constraints at the given point.

Recall that a system consisting of two tuples of vectors \(p_1, \ldots, p_m\) and \(q_1, \ldots, q_k\) in the space \(\mathbb{R}^r\) is said to be positively-linearly independent if there does not exist a nontrivial tuple of multipliers \(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_k\) with all \(\alpha_i \geq 0\) such that

\[
\sum_i \alpha_i p_i + \sum_j \beta_j q_j = 0.
\]

The problem (1)-(7) will be called Problem A, and the relations (2)-(4) its endpoint block.
Note that the function $f$ explicitly depends on two time variables, $t$ and $s$, the roles of which are essentially different. Conventionally, the variable $s$ will be called inner, while $t$ will be called outer, time variable, and one should carefully distinguish between them in further considerations. Among the four arguments of the function $f$ and its derivatives, the first argument will always be the outer and the second one the inner time variable, no matter by which letters they will be denoted.

Note also that the integral equation (1) is equivalent to the following integro-differential equation:

$$
\dot{x}(t) = f(t,x(t),u(t)) + \int_{t_0}^{t} f(t,s,x(s),u(s)) \, ds, \quad (8)
$$

where the last integral shows, in a sense, how “far” we are from an ordinary differential equation. (Here $f_t$ means the partial derivative of the function $f(t,s,x,u)$ with respect to the first, outer time variable $t$.) If $f$ does not depend on the outer time $t$, i.e., $f = f(s,x(s),u(s))$, then this integral disappears, and Problem A becomes a standard optimal control problem with the ODE $\dot{x}(t) = f(t,x(t),u(t))$.

Obviously, each pair $(x(t),u(t))$ under consideration must “lie” in the domain $R$ of the function $f(t,s,x,u)$, i.e.,

$$(t,s,x(s),u(s)) \in R \quad \text{for a.e.} \quad (t,s) \in \Delta[t_0,t_1],
$$

where $\Delta[t_0,t_1] = \{(t,s) : t_0 \leq s \leq t \leq t_1\}$. We will need even a stronger condition.

Definition 2.1. A pair of functions $w(t) = (x(t),u(t))$ defined on an interval $t \in [t_0,t_1]$ (with continuous $x(t)$ and measurable essentially bounded $u(t)$) will be called a process in Problem A if it satisfies (1) and its “extended graph”

$$
\Gamma(w) = \{(t,s,x(s),u(s)) : (t,s) \in \Delta[t_0,t_1]\}
$$

lies in the set $R$ with some “margin”, i.e.,

$$
\text{dist} \left((t,s,x(s),u(s)), \partial R\right) \geq \text{const} > 0 \quad \text{for a.a.} \quad (t,s) \in \Delta[t_0,t_1], \quad (9)
$$
or equivalently, there exists a compact set $\Omega \subset R$ such that $(t,s,x(s),u(s)) \in \Omega$ for a.a. $(t,s) \in \Delta[t_0,t_1]$. A process in problem A is called admissible if it satisfies all the constraints of the problem.

Like in any problem on a nonfixed time interval, the notion of weak minimum in Problem A needs a modification.

Definition 2.2. We will say that an admissible process $w^0(t) = (x^0(t),u^0(t))$, $t \in [\hat{t}_0,\hat{t}_1]$ provides the extended weak minimum if there exists an $\varepsilon > 0$ such that, for any Lipschitz continuous bijective mapping $\rho : [\hat{t}_0,\hat{t}_1] \rightarrow [t_0,t_1]$ satisfying the conditions $|\rho(t) - t| < \varepsilon$ and $|\dot{\rho}(t) - 1| < \varepsilon$, and for any admissible process $w(t) = (x(t),u(t))$, $t \in [t_0,t_1]$, satisfying the conditions

$$
|x(\rho(t)) - x^0(t)| < \varepsilon \quad \forall t, \quad \text{and} \quad |u(\rho(t)) - u^0(t)| < \varepsilon \quad (\forall) \quad t,
$$

the following inequality holds: $J(w) \geq J(w^0)$. (Notation $(\forall)$ conveniently means “for almost all”.)
The conditions on \( \rho \) imply \( \rho(t_0) = \hat{t}_0 \) and \( \rho(t_1) = \hat{t}_1 \) with \( |\hat{t}_0 - t_0| < \varepsilon \) and \( |\hat{t}_1 - t_1| < \varepsilon \). If the interval \([t_0, t_1]\) is fixed and we take \( \rho(t) = t \), then relations (11) describe the usual uniform closeness between the processes \( w^0 \) and \( w \) both in the state and control variables. However, since \( \rho(t) \) is variable, relations (11) extend the set of “competing” processes by allowing perturbations of the time, and thus, even for a fixed time interval, the extended weak minimum is, in general, stronger than the usual weak minimum.

3. **Local maximum principle in Problem A.** Let a process (10) provide the extended weak minimum in Problem A. We assume that the endpoints of the reference state \( x^0(t) \) do not lie on the boundary of state constraints. To be more precise, that

\[
\Phi_k(\hat{t}_0, x^0(\hat{t}_0)) < 0, \quad \Phi_k(\hat{t}_1, x^0(\hat{t}_1)) < 0, \quad k = 1, \ldots, d(\Phi).
\]

To formulate optimality conditions, let us introduce a tuple of Lagrange multipliers corresponding to all the constraints and the cost of Problem A:

\[
(\alpha, \beta, \psi_x(t), \psi_t(t), h_i(t), m_j(t), \mu_k(t)),
\]

\( i = 1, \ldots, d(F), \quad j = 1, \ldots, d(G), \quad k = 1, \ldots, d(\Phi), \)

where \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d(\varphi)) \in \mathbb{R}^{d(\varphi) + 1} \) with \( \alpha_i \geq 0 \) \( \forall i \) (for short, we will simply write \( \alpha \geq 0 \)), and \( \beta = (\beta_1, \ldots, \beta_d(\eta)) \in \mathbb{R}^{d(\eta)} \) are vectors,

\[ \psi_x : [\hat{t}_0, \hat{t}_1] \to \mathbb{R}^n \] and \( \psi_t : [\hat{t}_0, \hat{t}_1] \to \mathbb{R} \)

are functions of bounded variation (\( \psi_x \) a row \( n \)-vector), continuous at \( \hat{t}_0 \) and left continuous at every point \( t \in (\hat{t}_0, \hat{t}_1) \),

\[ \mu_k : [\hat{t}_0, \hat{t}_1] \to \mathbb{R}, \quad k = 1, \ldots, d(\Phi), \]

are nondecreasing functions, continuous at \( \hat{t}_0 \) and left continuous at every point \( t \in (\hat{t}_0, \hat{t}_1) \) (without loss of generality we assume that \( \mu_k(\hat{t}_0) = 0 \) for all \( k \)),

\[ h_i : [\hat{t}_0, \hat{t}_1] \to \mathbb{R}^+, \quad i = 1, \ldots, d(F), \quad \text{and} \quad m_j : [\hat{t}_0, \hat{t}_1] \to \mathbb{R}, \quad j = 1, \ldots, d(G), \]

are measurable bounded functions. We denote by \( d\psi_x \), \( d\psi_t \), \( d\mu_k \) the Lebesgue–Stieltjes measures which correspond to the functions \( \psi_x, \psi_t, \mu_k \), respectively. These measures have no atoms at the points \( \hat{t}_0 \) and \( \hat{t}_1 \), and obviously, \( d\mu_k \geq 0 \), \( k = 1, \ldots, d(\Phi) \). By \( \hat{\mu}_k(t) \) we denote the generalized derivative with respect to \( t \) of monotone nondecreasing function \( \mu_k(t) \), hence \( \hat{\mu}_k(t) dt = d\mu_k(t) \). Similarly, \( \hat{\psi}_x(t) \) and \( \hat{\psi}_t(t) \) denote generalized derivatives of functions \( \psi_x(t) \) and \( \psi_t(t) \), respectively.

In what follows, all pointwise relations involving continuous functions hold for any \( t \), and those involving measurable functions hold for almost all \( t \).

Further, introduce the modified Pontryagin function

\[
H(t, s, x, u) = \psi_x(t)f(t, s, x, u) + \int_{s}^{t} \psi_x(\tau) f_i(\tau, s, x, u) d\tau
\]

(here, \( \psi_x f \) is the product of the row and column \( n \)-vectors), and the augmented (or extended) modified Pontryagin function

\[
\overline{H}(t, s, x, u) = H(t, s, x, u)
\]

\[ - \sum_i h_i(t) F_i(s, x, u) - \sum_j m_j(t) G_j(s, x, u) - \sum_k \hat{\mu}_k(t) \Phi_k(s, x). \]
Also, introduce the endpoint Lagrange function

$$l(t_0, x_0, t_1, x_1) = \left( \sum_{i=0}^{d(\varphi)} \alpha_i \varphi_i + \sum_{j=1}^{d(\eta)} \beta_j \eta_j \right) (t_0, x_0, t_1, x_1).$$  

(16)

Both these functions refer to the tuple (13).

In view of equation (8), define the function

$$R(t) = \int_{t_0}^{t} f(t, s, x^0(s), u^0(s)) \, ds.$$  

(17)

The functions $H, \overline{H}, l$, and $R$ will be used in formulation of optimality conditions.

In what follows, we will need the expression of partial derivative of the functions $H(t, s, x, u)$ with respect to the second, inner variable $s$ along the process $x^0(t), u^0(t)$:

$$H_s(t, s, x(t), u(t)) = \psi_x(t) f_x(t, s, x^0(t), u^0(t)) + \int_{t}^{t_1} \psi_x(\tau) f_{xs}(\tau, t, x^0(t), u^0(t)) \, d\tau$$  

(here $f_s$ is the partial derivative of the function $f(t, s, x, u)$ with respect to the second, inner variable $s$, and $f_{xs}$ is its second mixed partial derivative), and therefore

$$\overline{H}_s(t, t, x(t), u(t)) = H_s(t, t, x(t), u(t))$$

$$- \sum_i h_i(t) F_{it}(t, x(t), u(t)) - \sum_j m_j(t) G_{jt}(t, x(t), u(t))$$

$$- \sum_k \mu_k(t) \Phi_{kt}(t, x(t)),$$

because, in order to take the derivative of $F_i(s, x, u)$ w.r.t. $s$, one should take its derivative w.r.t. its first argument, i.e. $F_{it}$, and similar for $G_j(s, x, u)$ and $\Phi_k(s, x, u)$.

For the process (10) and tuple (13) with the specified properties, let us formulate the conditions of local maximum principle (or the stationarity conditions):

a) the nonnegativity conditions

$$\alpha_i \geq 0, \quad h_i(t) \geq 0, \quad i = 1, \ldots, d(F), \quad d\mu_k \geq 0, \quad k = 1, \ldots, d(\Phi),$$

(18)

b) the nontriviality condition

$$|\alpha| + |\beta| + \sum_k \mu_k(t_1) + \sum_i \int_{t_0}^{t_1} h_i(t) \, dt > 0,$$

(19)

c) the endpoint complementary slackness conditions

$$\alpha_i \varphi_i(t_0, x^0(t_0), \hat{t}_1, x^0(\hat{t}_1)) = 0, \quad i = 1, \ldots, d(\varphi),$$

(20)

d) the pointwise complementary slackness conditions

$$d\mu_k(t) \Phi_k(t, x^0(t)) \equiv 0, \quad k = 1, \ldots, d(\Phi),$$

(21)

$$h_i(t) F_i(t, x^0(t), u^0(t)) = 0 \quad \text{a.e. on } [\hat{t}_0, \hat{t}_1], \quad i = 1, \ldots, d(F),$$

(22)

e) the adjoint equation in $x$

$$-d\psi_x(t) = \frac{d}{ds}(\overline{H}_s(t, s, x^0(t), u^0(t))) \, dt,$$

or equivalently,

$$-d\psi_x(t) = \left( \psi_x(t) f_x(t, s, x^0(t), u^0(t)) + \int_{t}^{t_1} \psi_x(\tau) f_{xs}(\tau, t, x^0(t), u^0(t)) \, d\tau \right) \, dt$$
\[-\left( \sum_i h_i(t) F_{ix}(t, x^0(t), u^0(t)) + \sum_j m_j(t) G_{jx}(t, x^0(t), u^0(t)) \right) dt - \sum_k d\mu_k(t) \Phi_{kx}(t, x^0(t)), \quad (23)\]

f) the adjoint equation in \( t \)
\[-d\psi_t(t) = \Pi_s(t, t, x^0(t), u^0(t)) dt - d\psi_x(t) R(t),\]
or equivalently,
\[-d\psi_t(t) = \left( \psi_x(t) f_s(t, t, x^0(t), u^0(t)) + \int_t^{t_1} \psi_x(\tau) f_{is}(\tau, t, x^0(t), u^0(t)) d\tau \right) dt
- \left( \sum_i h_i(t) F_{ix}(t, x^0(t), u^0(t)) + \sum_j m_j(t) G_{jx}(t, x^0(t), u^0(t)) \right) dt
- \sum_k d\mu_k(t) \Phi_{kx}(t, x^0(t)) - d\psi_x(t) R(t), \quad (24)\]
g) the transversality conditions in \( x \),
\[\psi_x(\hat{t}_0) = l_{x_0}, \quad -\psi_x(\hat{t}_1) = l_{x_1}, \quad (25)\]
h) the transversality conditions in \( t \),
\[\psi_t(\hat{t}_0) = l_{t_0}, \quad -\psi_t(\hat{t}_1) = l_{t_1} - \psi_x(\hat{t}_1) R(\hat{t}_1), \quad (26)\]
i) the stationarity condition of the extended Pontryagin function with respect to the control
\[\Pi_s(t, t, x^0(t), u^0(t)) = 0 \quad \text{a.e. on} \quad [\hat{t}_0, \hat{t}_1],\]
i.e.,
\[\psi_x(t) f_{su}(t, t, x^0(t), u^0(t)) + \int_t^{t_1} \psi_x(\tau) f_{isu}(\tau, t, x^0(t), u^0(t)) d\tau
- \sum_i h_i(t) F_{isu}(t, x^0(t), u^0(t)) + \sum_j m_j(t) G_{jisu}(t, x^0(t), u^0(t)) = 0, \quad (27)\]
k) and the "energy evolution law"
\[H(t, t, x^0(t), u^0(t)) + \psi_t(t) = 0 \quad \text{a.e. on} \quad [\hat{t}_0, \hat{t}_1]. \quad (28)\]

We call it in this way, since together with (24) it gives the equation for evolution of the function \( H(\psi_x(t), t, x^0(t), u^0(t)) = -\psi_t(t) \), which is often (especially in mechanical problems) regarded as the total energy of the system:
\[\dot{H} = \mathcal{P} - \dot{\psi}_x R.\]

(If the state and mixed constraints are absent and the dynamics does not explicitly depend on time: \( f = f(x,u) \), then \( \mathcal{P} = H, \ R = 0, \) and we get the convenient "energy conservation law": \( H = \text{const} \) along the optimal process.)

Note that using the generalized derivatives of functions of bounded variation, one can represent the adjoint equation in \( x \) and \( t \) in an easy-to-remember form:
\[-\frac{d\psi_x(t)}{dt} = H_x(t, t, x^0(t), u^0(t)) - \sum_i h_i(t) F_{ix}(t, x^0(t), u^0(t))
- \sum_j m_j(t) G_{jx}(t, x^0(t), u^0(t)) - \sum_k \frac{d\mu_k(t)}{dt} \Phi_{kx}(t, x^0(t)), \quad (29)\]
\[
- \frac{d\psi(t)}{dt} = H_s(t, x^0(t), u^0(t)) - \sum_i h_i(t) F_{ik}(t, x^0(t), u^0(t))
- \sum_j m_j(t) G_{jk}(t, x^0(t), u^0(t)) - \sum_k \frac{d\mu_k(t)}{dt} \Phi_{kt}(t, x^0(t)) - \frac{d\psi_s(t)}{dt} R(t).
\] (30)

The main result of the paper is the following

**Theorem 3.1.** If a process \( w^0(t) = (x^0(t), u^0(t)) \), \( t \in [t_0, t_1] \) provides the extended weak minimum in Problem A and satisfies assumption (12), then there exists a tuple of multipliers \((\alpha, \beta, \psi_x, \psi_t, h_i, m_j, \mu_k)\) satisfying the specified above properties and such that conditions a)–k) of the local maximum principle hold true.

Like in our previous paper [9], in order to prove Theorem 3.1, we reduce Problem A to an auxiliary problem on a fixed time interval by the change of time variable \( t = t(\tau) \), where \( dt/d\tau = v(\tau) \) and \( v(\tau) > 0 \). The obtained auxiliary problem is a problem of type B described in the next section. We will derive optimality conditions in Problem B, apply them to the auxiliary problem, and rewrite the obtained conditions in terms of the original Problem A. Let us pass to the realization of this plan.

4. **Problem B on a fixed time interval.** Consider a system of the following form on a fixed interval \([t_0, t_1]\):

\[
x(t) = x(t_0) + \int_{t_0}^t g(t, s, y(t), y(s), x(s), u(s)) \, ds,
\] (31)

\[
y(t) = y(t_0) + \int_{t_0}^t h(t, s, y(s), u(s)) \, ds,
\] (32)

where \( x(t) \) and \( y(t) \) are continuous functions of dimensions \( n \) and \( m \) respectively, \( u(t) \) is a measurable and essentially bounded function on \([t_0, t_1]\). Here we still denote the time by \( t \). Like before, the data functions \( g \) and \( h \) are assumed to be twice continuously differentiable on an open set \( \mathcal{R} \subset \mathbb{R}^{2+2m+n+r} \) (if \( h \) is formally considered as a function of six variables \( t, s, y(t), y(s), x, u \)).

Note, that this system does not fall into the framework of equation (1), since the integrand of the first equation depends on \( y(t) \), which can be regarded as the outer state variable, while \( y(s) \) is the inner state. Thus, we have to study a new, broader than (1), class of integral control systems.

Adding to the obtained system the mixed constraints, the state constraints and the terminal block, we obtain the following **Problem B on a fixed interval** \([t_0, t_1]\):

on the set of solutions \( w = (y, x, u) \) to system (31)–(32) satisfying the constraints

\[
F_i(t, y(t), x(t), u(t)) \leq 0 \quad \text{for a.e. } t \in [t_0, t_1], \quad i = 1, \ldots, d(F),
\] (33)

\[
G_j(t, y(t), x(t), u(t)) = 0 \quad \text{for a.e. } t \in [t_0, t_1], \quad j = 1, \ldots, d(G),
\] (34)

\[
\Phi_k(t, y(t), x(t)) \leq 0 \quad \text{for all } t \in [t_0, t_1], \quad k = 1, \ldots, d(\Phi),
\] (35)

\[
\eta_j(y(t_0), x(t_0), y(t_1), x(t_1)) = 0, \quad j = 1, \ldots, d(\eta),
\] (36)

\[
\varphi_i(y(t_0), x(t_0), y(t_1), x(t_1)) \leq 0, \quad i = 1, \ldots, d(\varphi),
\] (37)

one has to minimize the endpoint cost functional

\[
J = \varphi_0(y(t_0), x(t_0), y(t_1), x(t_1)) \rightarrow \min.
\] (38)

Like before, the functions \( \eta_j, \varphi_i, \varphi_0 \) are continuously differentiable on an open set \( \mathcal{P} \subset \mathbb{R}^{2n+2m} \), and the functions \( F_i, G_j, \Phi_k \) continuously differentiable on an
open set $\mathcal{Q} \subset \mathbb{R}^{1+m+n+r}$ (considering $\Phi_k$ as functions of four variables $t, y, x, u$). Moreover, we assume that the mixed constraints (33) and (34) are regular in the same sense as in Problem A. By definition, an *admissible process* in Problem B is any triple $(y(t), x(t), u(t))$, $t \in [t_0, t_1]$, satisfying all the constraints of this problem, whose extended graph lies in $\mathcal{R}$ with some margin.

We consider Problem B as a particular case of an abstract nonsmooth problem in a Banach space, hence we can apply the well known abstract Lagrange multipliers rule for nonsmooth problems. For the reader’s convenience, we recall its formulation.

**Lagrange multipliers rule for an abstract nonsmooth problem.** Let $X, Y$, and $Z_i$, $i = 1, \ldots, \nu$, be Banach spaces, $\mathcal{D} \subset X$ an open set, $K_i \subset Z_i$ $i = 1, \ldots, \nu$, closed convex cones with nonempty interiors, $f_0 : \mathcal{D} \rightarrow \mathbb{R}$, $b_i : \mathcal{D} \rightarrow Z_i$, $i = 1, \ldots, \nu$, and $g : \mathcal{D} \rightarrow Y$ given mappings. Consider the following problem

$$f_0(x) \rightarrow \min, \quad b_i(x) \in K_i, \quad i = 1, \ldots, \nu, \quad g(x) = 0. \quad (39)$$

We study the question of a local minimum at an admissible point $x^0 \in \mathcal{D}$. Assume that the cost $f_0$ and the mappings $b_i$ are Fréchet differentiable at $x^0$, the operator $g$ is strictly differentiable at $x^0$, and the image of $g'(x^0)$ is closed (the weak regularity of equality constraint).

Let $K^*_i$ be the dual cone and $K_i^0 = -K^*_i$ the polar cone to $K_i$, $i = 1, \ldots, \nu$. The following theorem holds [8].

**Theorem 4.1.** Let $x^0$ provide a local minimum in problem (39). Then there exist Lagrange multipliers $\alpha_0 \geq 0$, $\zeta^*_i \in K^*_i$, $i = 1, \ldots, \nu$, and $y^* \in Y^*$, not all equal to zero, satisfying the complementary slackness conditions

$$\langle \zeta^*_i, b_i(x^0) \rangle = 0, \quad i = 1, \ldots, \nu,$$

and such that the Lagrange function

$$L(x) = \alpha_0 f_0(x) + \sum_{i=1}^{\nu} \langle \zeta^*_i, b_i(x) \rangle + \langle y^*, g(x) \rangle$$

is stationary at $x^0$: $L'(x^0) = 0$.

5. **Lagrange multipliers rule for problem B.** Let us apply Theorem 4.1 to Problem B which we represent in the form (39). In this problem, the role of $X$ is played by the space

$$W = C([t_0, t_1], \mathbb{R}^n) \times C([t_0, t_1], \mathbb{R}^n) \times L_\infty([t_0, t_1], \mathbb{R}^r)$$

with elements $w = (y, x, u)$ and the norm $||w|| = ||y||_C + ||x||_C + ||u||_\infty$. The local minimum in this norm is exactly the weak minimum.

The corresponding open set $\mathcal{D} \subset W$ is defined by the open sets $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{R}}$.

In what follows, $w^0 = (y^0, x^0, u^0) \in W$ is a point of weak minimum in Problem B. Note that because of (31)–(32), the functions $x^0$ and $y^0$ are Lipschitz continuous.

The smoothness assumptions on the data functions are obviously fulfilled in $\mathcal{D}$. Indeed, the cost functional (38) and the endpoints inequalities (37) are smooth, the mixed inequalities (33) have the form

$$a_i(w) := F_i(\cdot, y(\cdot), x(\cdot), u(\cdot)) \in K, \quad i = 1, \ldots, d(F),$$
where $K$ is the cone $L^-([t_0,t_1],\mathbb{R})$ of nonpositive functions in the Banach space $L^\infty([t_0,t_1],\mathbb{R})$ (one and the same for all $i$), and the state inequalities (35) have the form

$$b_k(w) := \Phi_k(\cdot, y(\cdot), x(\cdot)) \in \Omega, \quad k = 1, \ldots, d(\Phi),$$

where $\Omega$ is the cone $C^-([t_0,t_1],\mathbb{R})$ of nonpositive functions in the Banach space $C([t_0,t_1],\mathbb{R})$ (one and the same for all $k$). The mappings $a_i(w)$ and $b_k(w)$ are obviously smooth.

**Weak regularity of the equality constraints.** So, we only have to check the weak regularity assumption for the equality constraints, i.e. that the derivative of corresponding operator has a closed image. We will need the following property.

Let an $n \times n$ matrix $Q(t,s)$ defined for $(t,s) \in [t_0,t_1] \times [t_0,t_1]$ be measurable and bounded in $s$, and continuous in $t$ uniformly with respect to $s$. Consider the linear integral operator $P : C_0([t_0,t_1],\mathbb{R}^n) \to C_0([t_0,t_1],\mathbb{R}^n)$ defined by

$$x(t) \mapsto x(t) - \int_{t_0}^t Q(t,s) x(s) \, ds.$$

**Lemma 5.1.** The operator $P$ is surjective.

This fact is well-known, see e.g. [8]. We will use it below.

Define the spaces

$$Z_1 = C_0([t_0,t_1],\mathbb{R}^n), \quad Z_2 = C_0([t_0,t_1],\mathbb{R}^m),$$

$$Z_3 = L^\infty([t_0,t_1],\mathbb{R}^{d(G)}), \quad Z_4 = \mathbb{R}^{d(n)}.$$

The operator of equality constraints consists of four components:

$$T(w) = (T_1(w), T_2(w), T_3(w), T_4(w)), \quad T_i : W \to Z_i, \quad i = 1,2,3,4,$$

$$T_1(w) := x(t) - x(t_0) - \int_{t_0}^t g(t,s,y(t),y(s),x(s),u(s)) \, ds = z_1(t),$$

$$T_2(w) := y(t) - y(t_0) - \int_{t_0}^t h(t,s,y(s),u(s)) \, ds = z_2(t),$$

$$T_3(w) := g(t,y(t),x(t),u(t)) = z_3(t),$$

$$T_4(w) := \eta(y(t_0),x(t_0),y(t_1),x(t_1)) = z_4,$$

where $G = (G_1,\ldots,G_{d(G)})$. Thus, $T$ is a mapping $W \to Z$, where

$$Z = Z_1 \times Z_2 \times Z_3 \times Z_4 = C_0([t_0,t_1],\mathbb{R}^n) \times C_0([t_0,t_1],\mathbb{R}^m) \times L^\infty([t_0,t_1],\mathbb{R}^{d(G)}) \times \mathbb{R}^{d(n)}.$$

The equality constraints of Problem B can be represented as $T(w) = 0$.

The derivative of this operator at the point $w^0$ has the form:

$$T'(w^0)w = (T'_1(w^0)w, T'_2(w^0)w, T'_3(w^0)w, T'_4(w^0)w),$$

where

$$T'_1(w^0)w = x(t) - x(t_0)$$

$$- \int_{t_0}^t [g_{y_1}(t,s)y(t) + g_{y_2}(t,s)y(s) + g_x(t,s)x(s) + g_u(t,s)u(s)] \, ds = z_1(t),$$

$$T'_2(w^0)w = y(t) - y(t_0)$$

$$- \int_{t_0}^t [h_{y_1}(t,s)y(t) + h_{y_2}(t,s)y(s) + h_x(t,s)x(s) + h_u(t,s)u(s)] \, ds = z_2(t),$$

$$T'_3(w^0)w = \eta(y(t_0),x(t_0),y(t_1),x(t_1)) = z_4,$$
Here and in what follows we use the shortened notation $y$ for the inner state variables where $g$ is an element of $u$.

The operator $L$ is bounded. Hence it is possible to express $R$ always denote the derivatives with respect to the outer and inner state variables $y$ (i.e., the third and fourth arguments) of $g$, respectively.

Note that (40) can be rewritten as

$$x(t) - x(t_0) = R^B(t) y(t),$$

where

$$R^B(t) := \int_{t_0}^t g_y(t, s) ds = \int_{t_0}^t g_y(t, s, y^0(t), y^0(s), x^0(s), u^0(s)) ds.$$

Denote by $\Psi$ the linear operator

$$\Psi : w \in W \rightarrow (z_1, z_2, z_3) \in Z_1 \times Z_2 \times Z_3,$$

where

$$z_1 = T'_1(w^0)_w, \quad z_2 = T'_2(w^0)_w, \quad z_3 = T'_3(w^0)_w.$$  \hfill (47)

**Lemma 5.2.** The operator $\Psi$ is surjective.

**Proof.** Let us take any $z_1 \in Z_1, z_2 \in Z_2, z_3 \in Z_3$ and try to find $w = (y, x, u) \in W$ satisfying (47). We will seek for $u$ in the form $u(t) = G^*_u(t) v(t)$, where $v(\cdot)$ is an element of $L_\infty([t_0, t_1], \mathbb{R}^{d(G^*)})$ and $G^*_u(t)$ is the transposed matrix. Then the third relation in (47), defined by (42), becomes

$$G_y(t) y(t) + G_x(t) x(t) + G_u(t) G^*_u(t) v(t) = z_3(t).$$

The matrix $G_u(t) G^*_u(t)$ is nondegenerate, and moreover, its inverse $(G_u(t) G^*_u(t))^{-1}$ is bounded. Hence it is possible to express $v(t)$ as a function of $x(t)$ and $y(t)$ from the latter relation:

$$v(t) = \left( G_u(t) G^*_u(t) \right)^{-1} \left( z_3(t) - G_y(t) y(t) - G_x(t) x(t) \right).$$

Consequently,

$$u(t) = G^*_u(t) v(t) = M(t) \left( z_3(t) - G_y(t) y(t) - G_x(t) x(t) \right).$$  \hfill (48)
where \( M(t) := G_u^*(t)(G_u(t)G_u^*(t))^{-1} \). Using this expression in the second relation in (47) with account of (41), we get

\[
y(t) - y(t_0) - \int_{t_0}^{t} [A(t, s) y(s) + B(t, s) x(s)] ds = \tilde{z}_2(t), \tag{49}
\]

where

\[
A(t, s) = h_{u_1}(t, s) - h_{u_2}(t, s) M(s) G_y(s), \quad B(t, s) = -h_{u_2}(t, s) M(s) G_x(s),
\]

\[
\tilde{z}_2(t) = z_2(t) + \int_{t_0}^{t} h_{u_2}(t, s) M(s) z_3(s) ds.
\]

Note that \( \tilde{z}_2(t_0) = 0 \), i.e., \( \tilde{z}_2 \in Z_2 \).

Consider the first relation in (47) given in detail by (45). Substituting (49) into (48) and taking into account (42), we obtain

\[
x(t) - x(t_0) - R^B(t)x(t_0) - \int_{t_0}^{t} [C(t, s) y(s) + D(t, s) x(s)] ds = \tilde{z}_1(t),
\]

where

\[
C(t, s) = R^B(t) B(t, s) + g_{u_2}(t, s) - g_{u_2}(t, s) M(s) G_y(s),
\]

\[
D(t, s) = R^B(t) A(t, s) + g_{u_1}(t, s) - g_{u_2}(t, s) M(s) G_x(s),
\]

\[
\tilde{z}_1(t) = z_1(t) + R^B(t) \tilde{z}_2(t) + \int_{t_0}^{t} g_{u_2}(t, s) M(s) z_3(s) ds.
\]

Note again that \( \tilde{z}_1(t_0) = 0 \), i.e., \( \tilde{z}_1 \in Z_1 \). Setting \( x(t_0) = 0 \) and \( y(t_0) = 0 \), we come to a system of the form

\[
\begin{align*}
\begin{cases}
x(t) - \int_{t_0}^{t} [C(t, s) y(s) + D(t, s) x(s)] ds = \tilde{z}_1(t), \\
y(t) - \int_{t_0}^{t} [A(t, s) y(s) + B(t, s) x(s)] ds = \tilde{z}_2(t),
\end{cases}
\tag{50}
\end{align*}
\]

where \( \tilde{z}_1 \in Z_1 \) and \( \tilde{z}_2 \in Z_2 \) are given functions.

Consider the operator \( \Gamma : Z_1 \times Z_2 \rightarrow Z_1 \times Z_2 \) mapping a pair \((x(t), y(t))\) to the pair \((\tilde{z}_1(t), \tilde{z}_2(t))\) defined by (50). According to Lemma 5.1, this operator is surjective. Hence, for any given \((\tilde{z}_1(t), \tilde{z}_2(t))\) in the space \( Z_1 \times Z_2 \) one can find \((x(t), y(t))\) in the same space satisfying (50). Defining \( u(t) \) as in (48), we obtain a solution to (47). 

Since the linear operator \( T'_1(u^0) \) is finite dimensional, the surjectivity of operator \( \Psi = (T'_1(u^0), T'_2(u^0), T'_3(u^0)) \) implies that the operator \( T'(u^0) = (\Psi, T'_1(u^0)) \) has a closed image. Hence the equality constraints of Problem B are weakly regular at the point \( u^0 \).

Observe now that the dual (conjugate) space \( C_0^*([t_0, t_1], \mathbb{R}^n) \) to the Banach space \( C_0([t_0, t_1], \mathbb{R}^n) \) consists of all Radon measures having no atoms at zero, i.e., measures defined by functions \( \psi(t) \) of bounded variation that are continuous at \( t = t_0 \). Moreover, it is convenient to assume that the functions \( \psi(t) \) have a right limit value \( \psi(t_1^+) \) at the point \( t_1 \), and hence is defined the jump \( \Delta \psi(t_1) := \psi(t_1^+) - \psi(t_1^-) \) at this point. (The jump at \( t_0 \) vanishes since \( \psi \) is continuous at \( t_0 \).) The norm of \( \psi \) is its total variation \( \text{Var} \psi \).

The elements of conjugate space to \( Z \) are the quadruples \((\psi_x, \psi_y, m, \beta)\), where \( \psi_x(t) : [t_0, t_1] \rightarrow \mathbb{R}^n \) and \( \psi_y(t) : [t_0, t_1] \rightarrow \mathbb{R}^m \) are functions of bounded variation continuous at \( t_0 \), \( m \in L_\infty([t_0, t_1], \mathbb{R}^d) \), and \( \beta \in \mathbb{R}^d \). It is again convenient
to assume that $\psi_x$ and $\psi_y$ have right limit values $\psi_x(t_1 + 0)$ and $\psi_y(t_1 + 0)$, and hence there defined the jumps at the point $t_1$

$$\Delta \psi_x(t_1) := \psi_x(t_1 + 0) - \psi_x(t_1 - 0), \quad \Delta \psi_y(t_1) := \psi_y(t_1 + 0) - \psi_y(t_1 - 0).$$

It is also convenient to assume that the functions $\psi_x$ and $\psi_y$ are left continuous at any point $t \in (t_0, t_1]$.

**The inequality constraints.** First, let us turn to the state inequality constraints. Note that, for any point $\omega^0$ of the cone $\Omega = C^\preceq([t_0, t_1], \mathcal{R})$, an arbitrary element of the supporting cone at this point is given by a nonnegative Riemann–Stieltjes measure $d\mu(t)$ generated by a nondecreasing function $\mu(t)$ satisfying the condition $d\mu(t)\omega^0(t) \equiv 0$. For any $k = 1, \ldots, d(\Phi)$, we will use this fact for the function

$$\omega^0(t) = \Phi_k(t, y^0(t), x^0(t)).$$

It implies that the corresponding measure $d\mu_k(t)$ satisfies the complementary slackness condition $d\mu_k(t)\Phi_k(t, y^0(t), x^0(t)) \equiv 0$, i.e. is concentrated on the set

$$\{ t : \Phi_k(t, y^0(t), x^0(t)) = 0 \}.$$

Note also, that in view of assumption (12), the measures $d\mu_k$ have no atoms at the ends of the interval, i.e. the functions $\mu_k$ are continuous at $t = t_0$ and $t = t_1$. Without loss of generality, we can assume that $\mu_k$ are left continuous on $(t_0, t_1)$, and $\mu_k(t_0) = 0$. In view of this, $\text{Var} \mu_k = \mu_k(t_1)$.

Now consider the mixed inequality constraints. Note that the description of the dual cone to $K = L_\infty^\preceq([t_0, t_1], \mathcal{R})$ is more complicated than to the above cone $\Omega$. We will need the following fact (see, e.g., [14]).

**Lemma 5.3.** Let $v^0 \in K$ (i.e., $v^0(t) \leq 0$ a.e.). Then the conditions $p \in K^0$ and $\langle p, v^0 \rangle = 0$ (i.e., $p$ is a support functional to $K$ at the point $v^0$) are equivalent to the conditions: $p \geq 0$ and $p$ is concentrated on each of the sets

$$M^\delta = \{ t : v^0(t) \geq -\delta \}, \quad \delta > 0.$$

For any $i$ we will apply this lemma to the function $v^0(t) = F_i(t, y^0(t), x^0(t), u^0(t))$. It implies that the corresponding functional $p_i$ is nonnegative and concentrated on the set

$$M^\delta = \{ t \mid F_i(t, y^0(t), x^0(t), u^0(t)) \geq -\delta \} \quad \text{for any} \quad \delta > 0.$$

Recall that, according to the Yosida–Hewitt theorem [13], any functional $p \in L_\infty^\preceq([t_0, t_1], \mathcal{R})$ can be represented in the form $p = p_a + p_s$, where $p_a \in L^1([t_0, t_1], \mathcal{R})$ is an absolutely continuous (regular) component, and $p_s$ is a singular component. Moreover, $||p|| = ||p_a|| + ||p_s||$. If $p \geq 0$, then also $p_a \geq 0$ and $p_s \geq 0$.

A functional $\xi \in L_\infty^\preceq([t_0, t_1], \mathcal{R})$ is called singular if there exists a sequence of measurable sets $E_k \subset [t_0, t_1]$ with $\text{mes} E_k \to 0$, such that $\xi$ is concentrated on each $E_k$, i.e., $\forall k, \forall u \in L_\infty([t_0, t_1], \mathcal{R})$, one has

$$\langle \xi, u \rangle = \langle \xi, \chi_{E_k} u \rangle,$$

where $\chi_{E_k}(t)$ is the characteristic function of the set $E_k$ (equal to 1 on $E_k$ and 0 outside of $E_k$). If needed, one can also assume that the sequence $E_k$ is decreasing (telescopic), i.e. $E_k \supset E_{k+1}$ for all $k$.

If a functional $p$ is regular and concentrated on each of the sets $E_k$, then obviously it is concentrated on their intersection. For singular functionals, this is not true.
**Application of Theorem 4.1 to Problem B.** Now, all assumptions of the abstract nonsmooth problem (39) are satisfied for Problem B, and we can apply Theorem 4.1. According to this theorem, if \( w^0 \) provides the weak minimum, then there exist vectors \( \alpha = (\alpha_0, \ldots, \alpha_{d(\varphi)}) \in \mathbb{R}^{1+d(\varphi)}, \) \( \alpha \geq 0, \) and \( \beta = (\beta_1, \ldots, \beta_{d(\eta)}) \in \mathbb{R}^{d(\eta)}, \) nondecreasing functions \( \mu_k(t), \) \( k = 1, \ldots, d(\Phi), \) with \( \mu_k(t_0) = 0, \) functions of bounded variation \( \psi_x(t) \) and \( \psi_y(t) \) of dimensions \( n \) and \( m, \) respectively, continuous at \( t_0 \) and left continuous at any \( t \in (t_0, t_1], \) nonnegative functionals \( h_i \in L^*_\infty([t_0, t_1], \mathbb{R}), \) \( i = 1, \ldots, d(F), \) where each \( h_i \) is concentrated on the set \( M_i^t \) for any \( t = \delta > 0, \) and functionals \( m_j \in L^*_\infty([t_0, t_1], \mathbb{R}), \) \( j = 1, \ldots, d(G), \) such that the following conditions hold:

the nontriviality condition

\[
|\alpha| + |\beta| + \text{Var} \psi_x + \text{Var} \psi_y + \sum_k \mu_k(t_1) + \sum_i ||h_i|| + \sum_j ||m_j|| > 0, \quad (51)
\]

the endpoints complementary slackness conditions

\[
\alpha_i \varphi_i(y^0(t_0), x^0(t_0), y^0(t_1), x^0(t_1)) = 0, \quad i = 1, \ldots, d(\varphi), \quad (52)
\]

the pointwise complementary slackness conditions

\[
d\mu_k(t) \Phi_k(t, y^0(t), x^0(t)) \equiv 0, \quad k = 1, \ldots, d(\Phi), \quad (53)
\]

and such that the Lagrange function

\[
\mathcal{L}(w) = \left( \sum_{i=0}^{d(\varphi)} \alpha_i \varphi_i + \sum_{j=1}^{d(\eta)} \beta_j \eta_j \right)(y(t_0), x(t_0), y(t_1), x(t_1))
\]

\[
+ \int_{t_0}^{t_1} d\psi_x(t) \left( -x(t) + x(t_0) + \int_{t_0}^{t} g(t, s, y(t), y(s), x(s), u(s)) \, ds \right)
\]

\[
+ \int_{t_0}^{t_1} d\psi_y(t) \left( -y(t) + y(t_0) + \int_{t_0}^{t} h(t, s, y(s), u(s)) \, ds \right)
\]

\[
+ \sum_i \langle h_i, F_i(t, y(t), x(t), u(t)) \rangle + \sum_j \langle m_j, G_j(t, y(t), x(t), u(t)) \rangle
\]

\[
+ \sum_k \int_{t_0}^{t_1} d\mu_k(t) \Phi_k(t, y(t), x(t))
\]

is stationary at the point \( w^0: \quad \mathcal{L}'(w^0) = 0, \quad \text{i.e.,} \quad \mathcal{L}'(w^0) \bar{w} = 0 \quad \forall \bar{w} = (\bar{y}, \bar{x}, \bar{u}) \in W. \quad (54)
\]

For notational convenience, define the **endpoint Lagrange function**

\[
l(y_0, x_0, y_1, x_1) = \left( \sum_{i=0}^{d(\varphi)} \alpha_i \varphi_i + \sum_{j=1}^{d(\eta)} \beta_j \eta_j \right)(y_0, x_0, y_1, x_1). \quad (55)
\]
6. Analysis of the Euler–Lagrange equation for Problem B. Let us analyze stationarity condition (54). It means that for all test functions \( \bar{x} \in C([t_0, t_1], \mathbb{R}^n) \), all \( \bar{y} \in C([t_0, t_1], \mathbb{R}^m) \), and all \( \bar{u} \in L_\infty([t_0, t_1], \mathbb{R}^r) \) there holds

\[
\int_{t_0}^{t_1} \left( g_{yy}(t, s) \bar{y}(s) + g_x(t, s) \bar{x}(s) + g_u(t, s) \bar{u}(s) \right) \, ds 
+ \int_{t_0}^{t_1} \left( h_y(t, s) \bar{y}(s) + h_u(t, s) \bar{u}(s) \right) \, ds 
+ \sum_i \{ h_i, F_{iy} \bar{y} + F_{ix} \bar{x} + F_{iu} \bar{u} \} 
+ \sum_j \{ m_j, G_{3y} \bar{y} + G_{3x} \bar{x} + G_{3u} \bar{u} \}
\]

where \( g_{yy}(t, s), g_x(t, s), F_{iu}(s), l_{x_0}, \) etc., are defined by (44) and \( R^B(t) \) by (46) along the reference process \( w^0(t) \).

First, let us simplify the notriiviality condition.

**Lemma 6.1.** Condition (51) is equivalent to the following one

\[
|\alpha| + |\beta| + \sum_k \mu_k(t_1) + \sum_i ||h_i|| > 0. \tag{57}
\]

**Proof.** Indeed, if the left hand side of (57) vanishes, then the stationarity condition (56) means, in view of relations (40)–(42), that

\[
\int_{t_0}^{t_1} \left( d\psi_x(t) \bar{z}_1(t) + d\psi_y(t) \bar{z}_2(t) \right) + \sum_j \{ m_j, \bar{z}_3 \} = 0,
\]

where, due to Lemma 5.2, \( \bar{z}_1, \bar{z}_2, \) and \( \bar{z}_3 \) are arbitrary elements of \( Z_1, Z_2, \) and \( Z_3, \) respectively. Therefore, \( d\psi_x \equiv 0, \ d\psi_y \equiv 0, \) and all \( m_j = 0. \)

Note that the functions \( \psi_x \) and \( \psi_y \) in the Euler–Lagrange equation (56) are defined up to arbitrary constants. It will be convenient to assume that

\[
\psi_x(t_1 + 0) = 0, \quad \psi_y(t_1 + 0) = 0.
\]

Recall that we assume \( \psi_x \) and \( \psi_y \) to be left continuous at any point \( t \in (t_0, t_1], \) in particular, at the point \( t_1. \) Hence, the jumps of these functions at \( t_1 \) are

\[
\Delta \psi_x(t_1) = -\psi_x(t_1), \quad \Delta \psi_y(t_1) = -\psi_y(t_1).
\]

Recall also that both \( \psi_x \) and \( \psi_y \) are continuous at the point \( t_0. \)

Condition (56) decomposes into three independent conditions with respect to \( \bar{x}, \bar{y} \) and \( \bar{u} \) separately.
The \( u \)-component of the Euler–Lagrange equation. Set \( \bar{x} = 0 \) and \( \bar{y} = 0 \) in (56). Then for all \( \bar{u} \in L_\infty([t_0, t_1], \mathbb{R}^r) \) we have
\[
\int_{t_0}^{t_1} d\psi_x(t) \left( \int_{t_0}^{t} g_u(t, s)\bar{u}(s) \, ds \right) + \int_{t_0}^{t_1} d\psi_y(t) \left( \int_{t_0}^{t} h_u(t, s)\bar{u}(s) \, ds \right) + \sum_i \langle h_i, F_{iu} \bar{u} \rangle + \sum_j \langle m_j, G_{ju} \bar{u} \rangle = 0.
\]
Changing the order of integration in the first two summands, we get
\[
\int_{t_0}^{t_1} ds \left( \int_s^{t_1} d\psi_x(t) g_u(t, s)\bar{u}(s) \right) + \int_{t_0}^{t_1} ds \left( \int_s^{t_1} d\psi_y(t) h_u(t, s) \right) + \sum_i \langle h_i, F_{iu} \bar{u} \rangle + \sum_j \langle m_j, G_{ju} \bar{u} \rangle = 0. \tag{58}
\]
Define the functions
\[
\sigma_x(s) = \int_s^{t_1} d\psi_x(t) g_u(t, s) = \int_s^{t_1} d\psi(t) g_u(t, s) + \Delta \psi_x(t_1) g_u(t_1, s), \tag{59}
\]
\[
\sigma_y(s) = \int_s^{t_1} d\psi_y(t) h_u(t, s) = \int_s^{t_1} d\psi(t) h_u(t, s) + \Delta \psi_y(t_1) h_u(t_1, s). \tag{60}
\]
Here and in what follows we agree for definiteness, that all integrals with respect to measures \( d\psi_x \), \( d\psi_y \) are taken on the half-open interval \([t_0, t_1)\), and possible jumps of \( \psi_x \) and \( \psi_y \) at the point \( t_1 \) are selected as separate summands. (As we know, these functions have no jumps at \( t_0 \).) Observe that functions \( \sigma_x \) and \( \sigma_y \) are measurable and bounded. Equation (58) then becomes
\[
\int_{t_0}^{t_1} (\sigma_x(s) + \sigma_y(s)) \bar{u}(s) \, ds + \sum_i \langle h_i, F_{iu} \bar{u} \rangle + \sum_j \langle m_j, G_{ju} \bar{u} \rangle = 0. \tag{61}
\]
Here, all summands can be considered as functionals of \( \bar{u} \in L_\infty([t_0, t_1], \mathbb{R}^r) \). The first of them is absolutely continuous, and the two last ones satisfy the assumption of positive-linear independence (Sec. 2). Our nearest goal is to show that all the functionals \( h_i, m_j \) are also absolutely continuous, i.e., belong to \( L_1([t_0, t_1], \mathbb{R}) \). We will do it similarly to [8].

For any \( \delta > 0 \), introduce the set of “almost active” indices corresponding to the mixed inequality constraints:
\[
I_\delta(s) = \{ i \mid F_i(s) \geq -\delta \},
\]
where, according to the notation (44), \( F_i(s) = F_i(s, y^0(s), x^0(s), w^0(s)) \).

**Lemma 6.2.** (see [8]). Let assumption RMC be satisfied. Then there exist \( \delta > 0 \) and \( c > 0 \) such that for almost all \( s \in [t_0, t_1] \), for any numbers \( \alpha_i \geq 0 \) and \( \beta_j \), the following estimate holds:
\[
\left| \sum_{i \in I_\delta(s)} \alpha_i F_{iu}(s) + \sum_{j=1}^{d(G)} \beta_j G_{ju}(s) \right| \geq c \left( \sum_{i \in I_\delta(s)} \alpha_i + \sum_{j=1}^{d(G)} |\beta_j| \right). \tag{62}
\]

Fix some \( \delta > 0 \) and \( c > 0 \) from Lemma 6.2, and recall that each functional \( h_i \) is concentrated on the set \( M^\delta_i \). Let us split the interval \([t_0, t_1]\) into a finite number of subsets \( E_1, \ldots, E_N \) of positive measure, on each of which the set of “almost active” indices \( I_k = \{ i \mid F_i(s) \geq -\delta \} \) is constant. (This can be done, e.g., as follows. Enumerate all possible nonempty subsets of indices contained in
\[ \{1, \ldots, d(F)\} \text{ and denote them by } I_k, k = 1, \ldots, \hat{k}. \text{ For each } k, \text{ let } E_k \text{ be the set of all those } s \in [t_0, t_1] \text{ for which the set of \textquotedblleft almost active\textquotedblright\ indices coincide with } I_k, \text{ i.e., } \{ i \mid F_i(s) \geq -\delta \} = I_k. \text{ Among all } E_k, \text{ select the sets of positive measure. Let these sets be } E_1, \ldots, E_N. \text{ They form a required partition of the interval } [t_0, t_1]. \]

Now, fix any \( k \in \{1, \ldots, N\} \) and consider the system of vector-functions

\[ p_i(s) = F_{iu}(s), \quad i \in I_k, \quad q_j(s) = G_{ju}(s), \quad j = 1, \ldots, d(G). \]  

(63)

Lemma 6.2 implies that, for almost all \( s \in E_k \), for any numbers \( \alpha_i \geq 0 \) and \( \beta_j \), the following estimate holds:

\[ \left| \sum_{i \in I_k} \alpha_i p_i(s) + \sum_{j=1}^{d(G)} \beta_j q_j(s) \right| \geq c \left( \sum_{i \in I_k} \alpha_i + \sum_{j=1}^{d(G)} |\beta_j| \right). \]  

(64)

In this case, we say that the system of vector-functions \( p_i(s), q_j(s) \) is uniformly positively-linearly independent (UPLI) on the set \( E_k \).

Then we use the following theorem, concerning any system of vector-functions \( p_i(s), q_j(s) \) considered on any set \( E \) of positive measure. (See [7, Lemma 1] or [14, Theorem 45] or [8, Theorem 7.1].)

**Theorem 6.3. (on the absence of singular components).** Let a system of \( r \)-vector-functions \( p_i(s), q_j(s) \) be UPLI on a set \( E \) of positive measure, and let elements \( h_i, m_j \in L^*_\infty(E, \mathbb{R}) \) and \( \sigma \in L_1(E, \mathbb{R}^r) \) be such that all \( h_i \geq 0 \), and

\[ \sum_i \langle h_i, p_i(s) \bar{u}(s) \rangle + \sum_j \langle m_j, q_j(s) \bar{u}(s) \rangle = \int_E \sigma(s) \bar{u}(s) \, ds \]  

(65)

for all \( \bar{u} \in L^*_\infty(E, \mathbb{R}^r) \). Then all \( h_i, m_j \) are elements of \( L_1(E, \mathbb{R}) \) either, i.e., they have no singular components.

Now, for any \( k \), applying this theorem to system (63) on the set \( E_k \subset [t_0, t_1] \) with \( \sigma = -(\sigma_x + \sigma_y) \), we get in view of relation (61) that the functionals \( h_i, m_j \) are elements of \( L_1(E_k, \mathbb{R}) \). But then, on the whole interval \( [t_0, t_1] \), they are elements of \( L_1([t_0, t_1], \mathbb{R}) \), and relation (61)=(65) becomes

\[ \int_{t_0}^{t_1} \left( \sigma_x(s) + \sigma_y(s) \right) \bar{u}(s) \, ds \]

\[ + \sum_i \int_{t_0}^{t_1} h_i(s) F_{iu}(s) \bar{u}(s) \, ds + \sum_j \int_{t_0}^{t_1} m_j(s) G_{ju}(s) \bar{u}(s) \, ds = 0, \]

Since this relation holds for all \( \bar{u} \in L^*_\infty([t_0, t_1], \mathbb{R}^r) \), we obtain

\[ \sigma_x(s) + \sigma_y(s) + \sum_i h_i(s) F_{iu}(s) + \sum_j m_j(s) G_{ju}(s) = 0 \quad (\forall) s \in (t_0, t_1). \]  

(66)

This is the stationarity condition with respect to \( u \).

By Lemma 5.3, each \( h_i \) is concentrated on the set \( M^0_i \), i.e., the following complementary slackness conditions hold:

\[ h_i(t) F_i(t, y^0(t), x^0(t), u^0(t)) = 0 \quad \text{a.e. on } [t_0, t_1], \quad i = 1, \ldots, d(F). \]  

(67)
Moreover, setting in (64) \( \alpha_i = h_i(s) \) and \( \beta_j = m_j(s) \), we obtain for every \( k \) the estimate
\[
|\sigma(s)| \geq c \left( \sum_{i \in I_k} h_i(s) + \sum_{j=1}^{d(G)} |m_j(s)| \right) \tag{68}
\]
a.e. on \( E_k \), whence, the boundedness of \( \sigma(s) \) implies that the functions \( h_i(s), m_j(s) \) are also essentially bounded (not just integrable).

Now, let us take into account that the functions \( g \) and \( h \) are twice continuously differentiable. In this case, the integrals in (59) and (60) can be taken by parts. Note preliminarily that, for any fixed \( s \),
\[
\frac{d}{dt} g_u(t,s) = g_{tu}(t,s) + g_{yu}(t,s) \cdot \dot{y}^0(t), \quad \frac{d}{dt} h_u(t,s) = h_{tu}(t,s),
\]
where, like before, the partial derivatives of \( g \) and \( h \) are evaluated along the reference process \( u^0(t) \) according to (44). Then, using the left-continuity of the functions \( \psi_x, \psi_y \) and the agreement \( \psi_x(t_1 + 0) = \psi_y(t_1 + 0) = 0 \), we can write
\[
\sigma_x(s) = \int_s^{t_1} d\psi_x(t) g_u(t,s)
\]
\[
\sigma_y(s) = \int_s^{t_1} d\psi_y(t) h_u(t,s)
\]
and similarly,
\[
\sigma_y(s) = \int_s^{t_1} d\psi_y(t) g_u(s,s) = -\psi_y(s) h_u(s,s) - \int_s^{t_1} \psi_y(t) h_{tu}(t,s) dt. \tag{69}
\]
Then condition (66) of stationarity in the control becomes
\[
\psi_x(s) g_u(s,s) + \psi_y(s) h_u(s,s)
\]
\[
+ \int_s^{t_1} \left[ \psi_x(t) \left( g_{tu}(t,s) + g_{yu}(t,s) \dot{y}^0(t) \right) + \psi_y(t) h_{tu}(t,s) \right] dt
\]
\[
- \sum_i h_i(s) F_i(s) - \sum_j m_j(s) G_j(s) = 0 \quad (\forall) \ s \in (t_0,t_1). \tag{70}
\]
Let us introduce the function
\[
H(s,y,x,u) = \psi_x(s) g(s,s,y^0(s),y,x,u) + \psi_y(s) h(s,s,y,u)
\]
\[
+ \int_s^{t_1} \left[ \psi_x(t) \left( g_{tu}(t,s,y^0(t),y,x,u) + g_{yu}(t,s,y^0(t),y,x,u) \dot{y}^0(t) \right) \right.
\]
\[
\left. + \psi_y(t) h_{tu}(t,s,y,u) \right] dt, \tag{71}
\]
which we call the modified Pontryagin function. Then condition (70) becomes
\[
H_u(s,y^0(s),x^0(s),u^0(s))
\]
\[
- \sum_i h_i(s) F_i(s) - \sum_j m_j(s) G_j(s) = 0 \quad (\forall) \ s \in (t_0,t_1). \tag{72}
\]
Let us also introduce the function
\[
\Pi(s,y,x,u) = H(s,y,x,u)
\]
\[
- \sum_i h_i(s) F_i(s,y,x,u) - \sum_j m_j(s) G_j(s,y,x,u) - \sum_k \mu_k(s) \Phi_k(s,y,x), \tag{73}
\]
which we call the augmented (or extended) modified Pontryagin function. Then relation (72) takes a simpler, easy to remember form:

\[ \Pi_u(s, y^0(s), x^0(s), u^0(s)) = 0 \quad (\forall) \ s \in [t_0, t_1]. \]  

(74)

The \( x \)-component of the Euler–Lagrange equation. Now, set \( \bar{y} = 0 \) and \( \bar{u} = 0 \) in equality (56). Then \( \forall \bar{x} \in C([t_0, t_1], \mathbb{R}^n) \)

\[
\begin{align*}
&l_{x_0}\bar{x}(t_0) + l_{x_1}\bar{x}(t_1) + \int_{t_0}^{t_1} d\psi_x(t) \left(-\bar{x}(t) + \bar{x}(t_0)\right) \\
&+ \int_{t_0}^{t_1} d\psi_x(t) \int_{t}^{t} g_x(t, s, \bar{x}(s)) \, ds \\
&+ \int_{t_0}^{t_1} \left(\sum_i h_i(s) F_{ix}(s) + \sum_j m_j(s) G_{jx}(s)\right) \bar{x}(s) \, ds \\
&+ \int_{t_0}^{t_1} \sum_k d\mu_k(s) \Phi_{kx}(s) \bar{x}(s) = 0.
\end{align*}
\]  

(75)

Changing the order of integration in the second integral, we write

\[
\int_{t_0}^{t_1} d\psi_x(t) \int_{t_0}^{t} g_x(t, s) \bar{x}(s) \, ds = \int_{t_0}^{t_1} ds \left(\int_s^{t} d\psi_x(t) g_x(t, s)\right) \bar{x}(s).
\]

In view of the accepted agreement, we represent the integrals with respect to the measure \( d\psi_x \) as follows:

\[
\int_{t_0}^{t_1} d\psi_x(t) \left(-\bar{x}(t) + \bar{x}(t_0)\right)
\]

\[
= \int_{t_0}^{t_1-0} d\psi_x(s) (\bar{x}(t_0) - \bar{x}(s)) + \Delta\psi_x(t_1) (\bar{x}(t_0) - \bar{x}(t_1))
\]

and

\[
\int_{s}^{t_1} d\psi_x(t) g_x(t, s) = \int_{s}^{t_1-0} d\psi_x(t) g_x(t, s) + \Delta\psi_x(t_1) g_x(t_1, s).
\]

Note that the function

\[
\zeta(s) = \int_{s}^{t_1} d\psi_x(t) g_x(t, s)
\]

(76)

is measurable and bounded, since the function \( g_x(t, s) \) is continuous in \( t \) and measurable and bounded in \( s \). Consequently, equation (75) becomes

\[
\begin{align*}
&l_{x_0}\bar{x}(t_0) + l_{x_1}\bar{x}(t_1) + \int_{t_0}^{t_1} d\psi_x(s) (\bar{x}(t_0) - \bar{x}(s)) \\
&+ \Delta\psi_x(t_1) (\bar{x}(t_0) - \bar{x}(t_1)) + \int_{t_0}^{t_1} \zeta(s) \bar{x}(s) \, ds \\
&+ \int_{t_0}^{t_1} \left(\sum_i h_i(s) F_{ix}(s) + \sum_j m_j(s) G_{jx}(s)\right) \bar{x}(s) \, ds \\
&+ \int_{t_0}^{t_1} \sum_k d\mu_k(s) \Phi_{kx}(s) \bar{x}(s) = 0.
\end{align*}
\]  

(77)
Gathering the terms containing \( \bar{x}(t_0) \) and \( \bar{x}(t_1) \), respectively, and also the terms with the integral of \( \bar{x}(s) \) over interval \((t_0, t_1)\), we get:

\[
\left( t_{x_0} + \int_{t_0}^{t_1} d\psi_x(s) + \Delta \psi_x(t_1) \right) \bar{x}(t_0) + \left( t_{x_1} - \Delta \psi_x(t_1) \right) \bar{x}(t_1)
- \int_{t_0}^{t_1} d\psi_x(s) \bar{x}(s)
+ \int_{t_0}^{t_1} (\zeta(s) + \sum_i h_i(s) F_{ix}(s) + \sum_j m_j(s) G_{jx}(s)) \bar{x}(s) \, ds
+ \int_{t_0}^{t_1} \sum_k d\mu_k(s) \Phi_{kx}(s) \bar{x}(s) = 0. \tag{78}
\]

Recall that the measures \( d\mu_k \) have no atoms at the points \( t_0 \) and \( t_1 \), i.e. the functions \( \mu_k \) are continuous at these points.

Since the function \( \bar{x}(t) \) on \((t_0, t_1)\) and its values \( \bar{x}(t_0), \bar{x}(t_1) \) can be chosen “almost independently” of each other (see Lemma A.2 in [8] or Lemma 6.3 in [9]), the coefficients at them must vanish. Thus, we obtain the transversality conditions

\[
l_{x_0} + \int_{t_0}^{t_1} d\psi_x(s) + \Delta \psi_x(t_1) = 0, \tag{79}
\]

\[
l_{x_1} - \Delta \psi_x(t_1) = 0, \tag{80}
\]

and the adjoint equation in \( x \) as an equality between measures:

\[
- d\psi_x(s) + \left( \zeta(s) + \sum_i h_i(s) F_{ix}(s) + \sum_j m_j(s) G_{jx}(s) \right) ds
+ \sum_k d\mu_k(s) \Phi_{kx}(s) = 0, \tag{81}
\]

where \( \zeta(s) \) is defined in (76). Recall that \( \Delta \psi_x(t_1) = -\psi_x(t_1) \), \( \psi_x(t_1) - 0 = \psi_x(t_1) \), and hence the boundary conditions (79) and (80) write:

\[
\psi_x(t_0) = l_{x_0}, \quad \psi_x(t_1) = -l_{x_1}. \tag{82}
\]

(We also use here the continuity of \( \psi_x \) at \( t_0 \).)

Now, take again into account that the function \( g \) is twice continuously differentiable and rewrite the function \( \zeta \), taking by parts the integrals in its definition (76). Note preliminarily that, for any \( t, s \),

\[
\frac{d}{dt} g_x(t, s) = g_{tx}(t, s) + g_{yx}(t, s) \cdot \dot{y}^0(t).
\]

Then

\[
\zeta(s) = \int_{s}^{t_1} d\psi_x(t) g_x(t, s)
= -\psi_x(s) g_x(s, s) - \int_{s}^{t_1} \psi_x(t) \left( g_{tx}(t, s) + g_{yx}(t, s) \cdot \dot{y}^0(t) \right) dt. \tag{83}
\]

Substituting this expression into (81), we obtain another form of the adjoint equation in \( x \) :

\[
- d\psi_x(s) = \psi_x(s) g_x(s, s) \, ds
+ \left( \int_{s}^{t_1} \psi_x(t) \left( g_{tx}(t, s) + g_{yx}(t, s) \cdot \dot{y}^0(t) \right) dt \right)
\]
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\[- \sum_i h_i(s) F_i(x(s)) - \sum_j m_j(s) G_j(x(s)) \, ds - \sum_k d\mu_k(s) \Phi_kz(s). \quad (84)\]

Using the generalized derivatives of functions of bounded variation, we can rewrite this equation as

\[- \frac{d\psi_x}{ds} = \psi_x(s) g_x(s, s) + \int_s^{t_1} \psi_x(t) \left( g_{ix}(t, s) + g_{iy}(t, s) \cdot \gamma^0(t) \right) \, dt \]

\[- \sum_i h_i(s) F_i(s) - \sum_j m_j(s) G_j(s) - \sum_k \frac{d\mu_k(s)}{ds} \Phi_kz(s), \quad (85)\]

or shortly,

\[- \frac{d\psi_x}{ds} = \overline{H}_x(s, y^0(s), x^0(s), u^0(s)), \quad (86)\]

where \( \overline{H} \) is defined in (73).

**The \( y \)-component of the Euler–Lagrange equation.** Finally, set \( \bar{x} = 0 \) and \( \bar{u} = 0 \) in equation (56). Then \( \forall \bar{y} \in C([t_0, t_1], \mathbb{R}^m) \)

\[ l_{y_0} \bar{y}(t_0) + l_{y_1} \bar{y}(t_1) + \int_{t_0}^{t_1} d\psi_y(t) \left( R^0(t) \bar{y}(t) + \int_{t_0}^{t} g_{y_1}(t, s) \bar{y}(s) \, ds \right) \]

\[ + \int_{t_0}^{t_1} d\psi_y(t) \left( - \bar{y}(t) + \bar{y}(t_0) + \int_{t_0}^{t} h_y(t, s) \bar{y}(s) \, ds \right) \]

\[ + \int_{t_0}^{t_1} \left( \sum_i h_i(t) F_{iy}(t) + \sum_j m_j(t) G_{iy}(t) \right) \bar{y}(t) \, dt \]

\[ + \int_{t_0}^{t_1} \sum_k d\mu_k(t) \Phi_{ky}(t) \bar{y}(t) = 0. \quad (87)\]

Gathering similar terms, we get

\[ \left( l_{y_0} + \int_{t_0}^{t_1} d\psi_y(t) \right) \bar{y}(t_0) + l_{y_1} \bar{y}(t_1) + \int_{t_0}^{t_1} \left[ d\psi_x(t) R^0(t) - d\psi_y(t) \right] \]

\[ + \left( \sum_i h_i(t) F_{iy}(t) + \sum_j m_j(t) G_{iy}(t) \right) \bar{y}(t) \, dt + \sum_k d\mu_k(t) \Phi_{ky}(t) \bar{y}(t) \]

\[ + \int_{t_0}^{t_1} d\psi_x(t) \int_{t_0}^{t} g_{y_1}(t, s) \bar{y}(s) \, ds + \int_{t_0}^{t_1} d\psi_y(t) \int_{t_0}^{t} h_y(t, s) \bar{y}(s) \, ds = 0. \quad (88)\]

Changing the order of integration in the double integrals and using the above agreement about the integrals in \( d\psi_x, d\psi_y \), we obtain

\[ \left( l_{y_0} - \psi_y(t_0) \right) \bar{y}(t_0) + l_{y_1} \bar{y}(t_1) \]

\[ + \int_{t_0}^{t_1} \left( d\psi_x(t) R^0(t) - d\psi_y(t) \right) \bar{y}(t) + \left( \Delta \psi_x(t_1) R^0(t_1) - \Delta \psi_y(t_1) \right) \bar{y}(t_1) \]

\[ + \int_{t_0}^{t_1} \left( \sum_i h_i(t) F_{iy}(t) + \sum_j m_j(t) G_{iy}(t) \right) \bar{y}(t) \, dt \]

\[ + \int_{t_0}^{t_1} \left( \sum_k d\mu_k(t) \Phi_{ky}(t) \right) \bar{y}(t) + \int_{t_0}^{t_1} Q(s) \bar{y}(s) \, ds = 0, \quad (89)\]
where the function
\[ Q(s) := \int_s^{t_1} \left( d\psi_x(t) g_{y_x}(t, s) + d\psi_y(t) h_y(t, s) \right) \]
\[ = \int_s^{t_1-0} \left( d\psi_x(t) g_{y_x}(t, s) + d\psi_y(t) h_y(t, s) \right) \]
\[ + \Delta \psi_x(t_1) g_{y_x}(t_1, s) + \Delta \psi_y(t_1) h_y(t_1, s) \] (90)
is measurable and bounded (since the functions \( g_{y_x}(t, s) \) and \( h_y(t, s) \) are continuous in \( t \) and measurable and bounded in \( s \); see (44)).

Recall that \( \Delta \psi_y(t_1) = -\psi_y(t_1) \). Changing the variables \( t \leftrightarrow s \) in the last integral of (89) (to unify it with the previous integrals), and gathering similar terms, we get
\[
\begin{aligned}
&\left( t_{y_0} - \psi_y(t_0) \right) \bar{y}(t_0) + \left( t_{y_1} + \psi_y(t_1) - \psi_x(t_1) R^B(t_1) \right) \bar{y}(t_1) \\
&+ \int_{t_0}^{t_1} \left[ d\psi_x(t) R^B(t) - d\psi_y(t) \right] \\
&+ \left( \sum_i h_i(t) F_{iy}(t) + \sum_j m_j(t) G_{jy}(t) + Q(t) \right) dt \\
&+ \sum_k d\mu_k(t) \Phi_{ky}(t) \right] \bar{y}(t) = 0.
\end{aligned}
\]

Using again the arguments of “almost independency” of the function \( \bar{y}(t) \) on \((t_0, t_1)\) and its values \( \bar{y}(t_0), \bar{y}(t_1) \), we infer that the coefficients at them must vanish, hence we get the transversality conditions for \( \psi_y \):
\[
\psi_y(t_0) = t_{y_0}, \quad \psi_y(t_1) = -t_{y_1} + \psi_x(t_1) R^B(t_1),
\] (91)
and an equality between measures:
\[
d\psi_y(t) = d\psi_x(t) R^B(t) \\
+ \left( \sum_i h_i(t) F_{iy}(t) + \sum_j m_j(t) G_{jy}(t) + Q(t) \right) dt + \sum_k d\mu_k(t) \Phi_{ky}(t).
\] (92)

Now, recall again that the function \( g \) is twice continuously differentiable and rewrite the function \( Q \), taking by parts the integrals in its definition and using the following relations
\[
\begin{aligned}
\frac{d}{dt} g_{y_x}(t, s) &= g_{t y_x}(t, s) + g_{y x y_x}(t, s) \cdot \dot{y}^0(t), \\
\frac{d}{dt} h_y(t, s) &= h_{t y}(t, s).
\end{aligned}
\]

We then have
\[
\begin{aligned}
Q(s) &= \int_s^{t_1} \left( d\psi_x(t) g_{y_x}(t, s) + d\psi_y(t) h_y(t, s) \right) \\
&= -\psi_x(s) g_{y_x}(s, s) - \psi_y(s) h_y(s, s) \\
&- \int_s^{t_1} \left( \dot{\psi}_x(t) \left( g_{y_x}(t, s) + g_{y x y_x}(t, s) \cdot \dot{y}^0(t) \right) + \psi_y(t) h_{t y}(t, s) \right) dt.
\end{aligned}
\]

Hence, adjoint equation (92) becomes
\[
-d\psi_y(s) = -d\psi_x(s) R^B(s) \\
+ \left( \psi_x(s) g_{y_x}(s, s) + \psi_y(s) h_y(s, s) \right) ds
\]
Thus, the control system has the form

\[ g \]

Problem B in which the function \( g \) does not explicitly depend on \( t, s \), nor \( h \) on \( t, s, y \). Thus, the control system has the form

\[
- \frac{d\psi_y(s)}{ds} = - \frac{d\psi_x(s)}{ds} R^B(s) + \psi_x(s) g_y(s, s) + \psi_y(s) h_y(s, s)
\]

or, shortly,

\[
- \frac{d\psi_y(s)}{ds} = \mathcal{H}_y(s, y^0(s), x^0(s), u^0(s)) - \frac{d\psi_x(s)}{ds} R^B(s).
\]

Local maximum principle in Problem B. Thus, we proved the following

**Theorem 6.4.** If a process \( u^0 = (y^0, x^0, u^0) \) provides the weak minimum in Problem B, then there exist vectors \( \alpha = (\alpha_0, \ldots, \alpha_{d(\varphi)}) \in \mathbb{R}^{1+d(\varphi)}, \ \alpha \geq 0, \) and \( \beta = (\beta_1, \ldots, \beta_{d(G)}) \in \mathbb{R}^{d(\varphi)} \), nondecreasing functions \( \mu_k(t), \ k = 1, \ldots, d(\Phi), \) continuous at \( t_0, t_1 \) and left-continuous on \( (t_0, t_1) \) with \( \mu_k(t_0) = 0, \) and functions of bounded variations \( \psi_x(t), \psi_y(t) \) of dimensions \( n \) and \( m \), respectively, continuous at \( t_0, t_1 \) and left-continuous on \( (t_0, t_1) \), nonnegative functions \( h_i \in L_\infty([t_0, t_1], \mathbb{R}), \ i = 1, \ldots, d(F), \) and functions \( m_j \in L_\infty([t_0, t_1], \mathbb{R}), \ j = 1, \ldots, d(G), \) such that the following holds: nontriviality condition (97), complementary slackness conditions (52), (53), and (67), adjoint equations (86) and (94) (with \( H, \mathcal{H}, R^B \) and \( l \) as in (6), (73), (46), and (55)), respectively, transversality conditions (82) and (91), and stationarity condition with respect to the control (74).

**Problem C.** For our main purpose, we will need the following particular case of Problem B in which the function \( g \) does not explicitly depend on \( t, s \), nor \( h \) on \( t, s, y \). Thus, the control system has the form

\[
x(t) = x(t_0) + \int_{t_0}^t g(y, y(s), x(s), u(s)) \, ds,
\]

\[
y(t) = y(t_0) + \int_{t_0}^t h(u(s)) \, ds.
\]

Also, let the functions \( F_i, G_j, \Phi_k \) do not depend on \( t \), i.e. the mixed and state constraints have the form:

\[
F_i(y, x, u) \leq 0, \quad i = 1, \ldots, d(F),
\]

\[
G_j(y, x, u) = 0, \quad j = 1, \ldots, d(G),
\]

\[
\Phi_k(y, x) \leq 0, \quad k = 1, \ldots, d(\Phi).
\]

Then, the expressions for the functions \( H, \mathcal{H}, \) and \( R \) in Problem C are a bit more simple than those in Problem B:

\[
H(s, y, x, u) = \psi_x(s) g(y^0(s), y, x, u) + \psi_y(s) h(u)
\]
that the value \( v \) by variations of the control \( \tau \) and rewrite it as a problem of type C on a fixed interval of a new time variable \( \tau \) and control \( \tilde{\tau} \), and the terminal block now involves only the endpoints of state variables: 

\[ R^C(t) = \int_{t_0}^{t} g_{y_1}(y_0^0(t), y_0^0(s), x_0^0(s), u_0^0(s)) ds. \] 

The terminal block in Problem C and the function \( l \) are the same as in Problem B. This case will be used in the next section.

7. **Proof of Theorem 3.1.** Now, let us get back to our main Problem A of Sec. 2 and rewrite it as a problem of type C on a fixed interval of a new time variable \( \tau \in [\tau_0, \tau_1] \) by setting 

\[ t(\tau) = t(\tau_0) + \int_{\tau_0}^{\tau} v(\sigma) d\sigma. \] 

Now \( t(\tau) \) is regarded as a new state variable with *a priori* nonfixed endpoints \( t(\tau_0) = t_0 \) and \( t(\tau_1) = t_1 \), while \( v \) is a new control variable satisfying \( v > 0 \). Note that the value \( t(\tau_0) = t_0 \) is variable by definition, and \( t(\tau_1) = t_1 \) can be varied by variations of the control \( v(\tau) \). The corresponding state variable \( \tilde{x}(\tau) = x(t(\tau)) \) and control \( \tilde{u}(\tau) = u(t(\tau)) \) satisfy the equation 

\[ \tilde{x}(\tau) = \tilde{x}(\tau_0) + \int_{\tau_0}^{\tau} f(t(\tau), t(\sigma), \tilde{x}(\sigma), \tilde{u}(\sigma)) v(\sigma) d\sigma. \] 

Thus, here the state variables are \( \tilde{x}, t, \) the control variables are \( \tilde{u}, v, \) and the domain of \((t, \tilde{x}, \tilde{u}, v)\) is \( \mathcal{R} = \mathcal{R} \times \{v > 0\} \). The first variable \( t(\tau) \) of function \( f \) plays the role of *outer state* and the second one \( t(\sigma) \) the role of *inner state*. The control system comprises two equations \((103)\) and \((104)\), the mixed and state constraints have the form 

\[ F_i(t(\tau), \tilde{x}(\tau), \tilde{u}(\tau)) \leq 0 \quad (\forall) \quad \tau \in [\tau_0, \tau_1], \quad i = 1, \ldots, d(F), \] 

\[ G_j(t(\tau), \tilde{x}(\tau), \tilde{u}(\tau)) = 0 \quad (\forall) \quad \tau \in [\tau_0, \tau_1], \quad j = 1, \ldots, d(G), \] 

\[ \Phi_k(t(\tau), \tilde{x}(\tau), \tilde{u}(\tau)) \leq 0 \quad (\forall) \quad \tau \in [\tau_0, \tau_1], \quad k = 1, \ldots, d(\Phi), \] 

and the terminal block now involves only the endpoints of state variables: 

\[ \eta_j(t(\tau_0), \tilde{x}(\tau_0), t(\tau_1), \tilde{x}(\tau_1)) = 0, \quad j = 1, \ldots, d(\eta), \] 

\[ \varphi_i(t(\tau_0), \tilde{x}(\tau_0), t(\tau_1), \tilde{x}(\tau_1)) \leq 0, \quad i = 1, \ldots, d(\varphi), \] 

\[ J = \varphi_0(t(\tau_0), \tilde{x}(\tau_0), t(\tau_1), \tilde{x}(\tau_1)) \rightarrow \min. \] 

The obtained problem \((103)\)–\((110)\) is a particular case of Problem C stated at the end of Sec. 6 (if one redefines \( t(\tau) = y(\tau), \quad \tilde{x}(\tau) = x(\tau), \quad \tilde{u}(\tau) = u(\tau), \quad (u, v) = (u_1, u_2), \) and further \( \tau = t, \sigma = s \), where \( g = f(t, s, x, u) v \) and \( h = v \). It will be called *Problem D*. We emphasize again that it is considered on a fixed interval \([\tau_0, \tau_1]\).

Note that the function \( g \) does not involve explicitly the outer time variable \( \tau \), while its first variable \( t(\tau) \) is the *outer state*. The functions \( F_i, G_j, \Phi_k \) are now time-independent.

Now, let a process \( w^0(t) = (x^0(t), u^0(t)), \quad t \in [t_0, t_1], \) provide the extended weak minimum in Problem A. To this process, there corresponds any process 

\[ \tilde{w}^0(\tau) = (\tilde{x}^0(\tau), \tilde{u}^0(\tau), \tilde{v}^0(\tau)), \quad \tau \in [\tau_0, \tau_1], \] 

while its first variable \( \tilde{t}(\tau) \) is the *inner state*.
of Problem D satisfying the only condition that \( \int_{\tau_0}^{\tau_1} \psi^0(\sigma) \, d\sigma = \tilde{t}_1 - \tilde{t}_0 \).

One can easily show that the new process \( \tilde{w}^0(\tau) \) provides the weak minimum in Problem D. For convenience, we choose \([\tau_0, \tau_1] = [\tilde{t}_0, \tilde{t}_1] \) and \( v^0(\tau) \equiv 1 \).

Our aim is to apply Theorem 6.4 to the process \( \tilde{w}^0 \) of Problem D, taking into account the specific form of functions \( \tau, \sigma \) instead of \( x \) and \( t \).

According to (100), the modified Pontryagin function for Problem D (we equip it with the superscript \( D \)) is

\[
H^D(\sigma, t, x, v, u) = \psi_x(\sigma) f(t^0(\sigma), t, x, u) v + \psi_t(\sigma) v + \int_{t}^{\tau_1} \psi_x(\tau) f_t(t^0(\tau), t, x, u) v \cdot v^0(\tau) \, d\tau.
\]

(Since \( g = f(t, s, x, u, v) \), its derivative with respect to the outer state variable is \( g_{y_t} = f_t v \), where \( f_t \) always denotes the derivative with respect to the first, outer time variable.)

Since \( t^0(\tau) \equiv t \) and \( v^0(\tau) \equiv 1 \), we have

\[
H^D(\sigma, t, x, u) = \psi_x(\sigma) f(\sigma, t, x, u) v + \psi_t(\sigma) v + \int_{t}^{\tau_1} \psi_x(\tau) f_t(\tau, t, x, u) v \, d\tau = \left( H(\sigma, t, x, u) + \psi_t(\sigma) \right) v,
\]

where the function

\[
H(\sigma, t, x, u) := \psi_x(\sigma) f(\sigma, t, x, u) + \int_{t}^{\tau_1} \psi_x(\tau) f_t(\tau, t, x, u) \, d\tau
\]

coincides with the modified Pontryagin function (14) for Problem A if \( t \) in the latter is changed by \( \sigma \) and \( s \) is changed by \( t \). It follows that

\[
H^D(\sigma, t^0, x^0, v^0) = H_\sigma(\sigma, x^0, u^0),
\]

\[
H^D(\sigma, t^0, x^0, v^0) = H(\sigma, x^0, u^0) + \psi_t(\sigma),
\]

\[
H^D(\sigma, t^0, x^0, v^0) = H(\sigma, x^0, u^0),
\]

\[
H^D(\sigma, t^0, x^0, v^0) = H(\sigma, x^0, u^0).
\]

Explain that the left hand side of equation (116) is the partial derivative of the function \( H^D \) (as in (111)) with respect to the state variable \( t \), while the right hand side of this equation is the partial derivative of the function \( H \) (as in (14)) with respect to the inner (second) time variable \( s \).

Introduce also the augmented modified Pontryagin function for Problem A, according to (15). If we interchange \( t \leftrightarrow \sigma \), we obtain

\[
H(\sigma, t, x, u) = H(\sigma, t, x, u)
\]

\[
- \sum_i h_i(\sigma) F_i(t, x, u) - \sum_j m_j(\sigma) G_j(t, x, u) - \sum_k \mu_k(\sigma) \Phi_k(t, x).
\]

According to (46), the function \( R \) for Problem D (considered in the variables \( \tau, \sigma \) instead of \( t, s \)) is

\[
R^D(\tau) = \int_{t_0}^{\tau} f_1(t^0(\tau), x^0(\sigma), u^0(\sigma)) v^0(\sigma) \, d\sigma = \int_{t_0}^{\tau} f_1(\tau, x^0(\sigma), u^0(\sigma)) v^0(\sigma) \, d\sigma.
\]
Changing the roles of $\tau$ and $\sigma$ for further convenience, we get
\[ R^g(\sigma) = \int_{t_0}^{t_1} f_\mu(\sigma, \tau, x^0(\tau), u^0(\tau)) \, d\tau = R(\sigma), \] (118)
where $R$ is defined in (17). Finally, note that the endpoint Lagrange function has the usual definition (16).

Since the process $(t^0(\tau), x^0(\tau), u^0(\tau), v^0(\tau))$ provides the weak minimum in Problem D, Theorem 6.4 asserts that there exist vectors $\alpha \in \mathbb{R}^{1+d(\sigma)}$, $\alpha \geq 0$, and $\beta \in \mathbb{R}^{d(\alpha)}$, nondecreasing functions $\mu_k(\sigma)$, $k = 1, \ldots, d(\Phi)$, with $\mu_k(\tau_0) = 0$, and functions of bounded variations $\psi_x(\tau)$ and $\psi_y(\tau)$ of dimensions $n$ and $m$, respectively, both continuous at $\tau_0$, $\tau_1$ and left-continuous on $(\tau_0, \tau_1)$, non-negative functions $h_i \in L_\infty([\tau_0, \tau_1], \mathbb{R})$, $i = 1, \ldots, d(F)$, and functions $m_j \in L_\infty([\tau_0, \tau_1], \mathbb{R})$, $j = 1, \ldots, d(G)$, such that the following conditions are satisfied:

(i) nontriviality condition (see (57))
\[ |\alpha| + |\beta| + \sum_k \mu_k(\tau_1) + \sum_i \|h_i\|_{\infty} > 0. \] (119)

(ii) complementary slackness conditions
\[ \alpha_i \varphi_i(t^0(\tau_0), x^0(\tau_0), t^0(\tau_1), x^0(\tau_1)) = 0, \quad i = 1, \ldots, d(\varphi), \]
\[ d\mu_k(\tau) \Phi_k(t^0(\tau), x^0(\tau)) \equiv 0, \quad k = 1, \ldots, d(\Phi), \]
\[ h_i(\tau) F_i(t^0(\tau), x^0(\tau), u^0(\tau)) = 0 \quad \text{a.e. on } [\tau_0, \tau_1], \]

(iii) adjoint equation in $x$ (see (85), (86), and (115))
\[ -\frac{d\psi_x}{d\sigma} = H_x(\sigma, \sigma, x^0(\sigma), u^0(\sigma)) \]
\[ -\sum_i h_i(\sigma) F_{ix}(\sigma) - \sum_j m_j(\sigma) G_{jx}(\sigma) - \sum_k \frac{d\mu_k(\sigma)}{d\sigma} \Phi_{kx}(\sigma), \] (120)

(iv) adjoint equation in $t$ (see (93), (94), and (116))
\[ -\frac{d\psi_t}{d\sigma} = H_t(\sigma, \sigma, x^0(\sigma), u^0(\sigma)) - \frac{d\psi_x}{d\sigma} R^B(\sigma) \]
\[ -\sum_i h_i(\sigma) F_{it}(\sigma) - \sum_j m_j(\sigma) G_{jx}(\sigma) - \sum_k \frac{d\mu_k(\sigma)}{d\sigma} \Phi_{kt}(\sigma), \] (121)

where $R^B(\sigma)$ is as in (118),

(v) transversality conditions in $x$ (see (82))
\[ \psi_x(\tau_0) = l_{x_0}, \quad \psi_x(\tau_1) = -l_{x_1}, \] (122)

(vi) transversality conditions in $t$ (see (91))
\[ \psi_t(\tau_0) = l_{t_0}, \quad \psi_t(\tau_1) = -l_{t_1} + \psi_x(\tau_1) R^B(\tau_1). \] (123)

(vii) stationarity condition with respect to the control $u$ (see (72), (74), and (113))
\[ H_u(\sigma, \sigma, x^0(\sigma), u^0(\sigma)) = 0 \quad (\forall) \sigma \in [\tau_0, \tau_1], \] (124)

(viii) and stationarity condition with respect to the control $v$ (see (74) and (114))
\[ H(\sigma, \sigma, x^0(\sigma), u^0(\sigma)) + \psi_t(\sigma) = 0 \quad (\forall) \sigma \in [\tau_0, \tau_1]. \] (125)

Now, let us change, in the obtained conditions, the interval $[\tau_0, \tau_1]$ by $[\hat{t}_0, \hat{t}_1]$. Also, in the complementary slackness conditions, change $\tau$ by $t$, and in all further
conditions change $\sigma$ by $t$. Also recall that $R^D = R$. Then we obtain all conditions of the local maximum principle given in Theorem 3.1. Thus, Theorem 3.1 is completely proved.

**Comments to the conditions on $\psi_t$.** To explain the presence of additional terms in the conditions on $\psi_t$, let us consider the case when the dependence of the function $f$ on the outer time is multiplicative, i.e. $f = c(t) g(s, x, u)$, where $c$ is a smooth scalar and $g$ a smooth $n$-vector function. In this case, equation (1) has the form

$$x(t) = x(t_0) + c(t) \int_{t_0}^t g(s, x(s), u(s)) \, ds. \quad (126)$$

Suppose for simplicity that the constraints (4)–(7) are absent, the initial time $t_0 = 0$ and state $x(0) = x_0$ are fixed, and one simply has to minimize the terminal cost

$$J = \varphi(t_1, x(t_1)) \to \min. \quad (127)$$

This is our Problem A.

Now, introduce a new state variable $z$ which is subject to equation

$$\dot{z}(t) = g(t, x(t), u(t)), \quad z(0) = 0. \quad (128)$$

Then $x(t) = x_0 + c(t) z(t)$, the initial constraints are $t_0 = 0$, $z(t_0) = 0$, the new control system is

$$\dot{z}(t) = g(t, x_0 + c(t) z(t), u(t)), \quad (129)$$

and the cost now is

$$J = \varphi(t_1, x_0 + c(t_1) z(t_1)) \to \min. \quad (130)$$

We get a new Problem E in the variables $t, z, u$, which is a standard Lagrange problem with ODEs.

If a pair $(x^0(t), u^0(t))$ defined on an interval $[0, \hat{t}_1]$ provides the extended weak minimum in Problem A, then obviously the pair $(z^0(t), u^0(t))$ defined on the same interval $[0, \hat{t}_1]$, where $z^0(t)$ is defined by (128), provides the extended weak minimum in Problem E. The converse is as well true.

According to (14) and (126), for Problem A we have

$$H^A(t, s, x, u) = \psi_x(t) c(t) g(s, x, u) + \left( \int_s^{t_1} \psi_x(\tau) \dot{c}(\tau) d\tau \right) g(s, x, u)$$

$$= \xi(t) g(s, x, u),$$

where $\xi(t) = \psi_x(t) c(t) + \int_t^{t_1} \psi_x(\tau) \dot{c}(\tau) d\tau$.

The function $R(t) = c(t) \int_0^{t} g(s, x^0(s), u^0(s)) \, ds = \dot{c}(t) z^0(t)$. In view of (23) and (24), the costate variables $\psi_x$ and $\psi^A_x$ satisfy equations

$$-\dot{\psi}_x(t) = \xi(t) g_x(t, x^0(t), u^0(t)), \quad -\dot{\psi}^A_x(t) = \xi(t) g_x(t, x^0(t), u^0(t)) - \dot{\psi}_x(t) R(t) \quad (131)$$

(here we use a shortened notation $g_x(t) = g_x(t, x^0(t), u^0(t))$, etc.), and, in view of (25) and (26), the transversality conditions are
\(-\psi_x(\hat{t}_1) = \varphi_{x_1}(\hat{t}_1, x^0(\hat{t}_1)),\)
\[-\psi^A_x(\hat{t}_1) = \varphi_{t_1}(\hat{t}_1, x^0(\hat{t}_1)) + \varphi_{x_1}(\hat{t}_1, x^0(\hat{t}_1)) R(\hat{t}_1) \tag{132}\]

\(= \varphi_{t_1}(\hat{t}_1) + \varphi_{x_1}(\hat{t}_1) \hat{c}(\hat{t}_1) z^0(\hat{t}_1).\)

Integrating by parts the expression for \(\xi(t)\), we get

\[
\xi(t) = \psi_x(\hat{t}_1) c(\hat{t}_1) - \int_t^{\hat{t}_1} \dot{\psi}_x(\tau) c(\tau) \, d\tau = -\varphi_{x_1}(\hat{t}_1) c(\hat{t}_1) - \int_t^{\hat{t}_1} \dot{\psi}_x(\tau) c(\tau) \, d\tau, \tag{133}\]

hence

\[
\dot{\xi}(t) = \dot{\psi}_x(t) c(t) = -\xi(t) g_x(t) c(t), \quad \xi(\hat{t}_1) = -\varphi_{x_1}(\hat{t}_1) c(\hat{t}_1).\]

For Problem E with an ODE we can apply the standard stationarity conditions, so we have

\[
H^E(t, z, u) = \psi_z g(t, x_0 + c(t) z, u),
\]

the costate variables \(\psi_z\) and \(\psi^E_t\) satisfy equations

\[
\dot{\psi}_z(t) = \psi_z(t) g_x(t) c(t),
\]

\[
\dot{\psi}^E_z(t) = \psi_z(t) \left( g_x(t) + g_x(t) \hat{c}(t) z^0(t) \right), \tag{134}\]

and the transversality conditions

\[
\dot{\psi_z}(\hat{t}_1) = \varphi_{x_1}(\hat{t}_1) c(\hat{t}_1),
\]

\[
\dot{\psi}^E_z(\hat{t}_1) = \varphi_{t_1}(\hat{t}_1) + \varphi_{x_1}(\hat{t}_1) \hat{c}(\hat{t}_1) z^0(\hat{t}_1). \tag{135}\]

Now, let us find relations between \(\psi_z, \psi^A_z\) and \(\psi_z, \psi^E_z\). Suppose there are \(\psi_z, \psi^A_z\) satisfying the conditions (131)–(132) and \(H^A_u = 0\). Set \(\psi_z = \xi\) and \(\psi^E_z = \psi^A_z\). Then

\[
\dot{\psi}_z = \dot{\xi} = -\xi g_x c = -\psi_z g_x c \quad \text{and} \quad \dot{\psi}_z(\hat{t}_1) = \xi(\hat{t}_1) = -\varphi_{x_1}(\hat{t}_1) c(\hat{t}_1),
\]

so we obtain the first relations in (134)–(135), while the second relations simply coincide with those in (131)–(132). Since \(\psi_z = \xi\), then \(H^A(t, t, x^0(t), u) = H^E(t, z^0(t), u)\), so the stationarity in \(u\) for both the problems are equivalent. Thus, \(\psi_z\) and \(\psi^E_z\) satisfy all the required conditions.

Consider the reverse, more interesting passage. Suppose there are \(\psi_z, \psi^E_z\) satisfying (134)–(135) and \(H^E_u = 0\). Then

\[
\dot{\psi}_z(t) = -\varphi_{x_1}(\hat{t}_1) c(\hat{t}_1) + \int_t^{\hat{t}_1} \psi_z(\tau) g_x(\tau) c(\tau) \, d\tau.
\]

Set \(\psi_z(t) = -\varphi_{x_1}(\hat{t}_1) c(\hat{t}_1) + \int_t^{\hat{t}_1} \psi_z(\tau) g_x(\tau) \, d\tau\) and \(\psi^A_z = \psi^E_z\). The transversality conditions are then satisfied.

Define \(\xi(t)\) by (133). Since \(\dot{\psi}_z = -\psi_z g_x\), we get, in view of (134)–(135):

\[
\xi(t) = -\varphi_{x_1}(\hat{t}_1) c(\hat{t}_1) - \int_t^{\hat{t}_1} \dot{\psi}_x(\tau) c(\tau) \, d\tau
\]

\[
= -\varphi_{x_1}(\hat{t}_1) c(\hat{t}_1) + \int_t^{\hat{t}_1} \psi_z(\tau) g_x(\tau) c(\tau) \, d\tau = \psi_z(\hat{t}_1) - \int_t^{\hat{t}_1} \dot{\psi}_x(\tau) \, d\tau = \psi_z(t).
\]

Therefore, \(\dot{\psi}_z = -\xi g_x\), i.e. we obtain the first equation in (131), while the second one coincides with that in (134).
Thus, the stationarity conditions for Problem A are in the complete accordance with those for Problem E. The appearance of additional terms in the conditions for $\psi_t$ in Problem A is caused by the additional dependence of the control system (129) and the cost (130) in Problem E on the time variable $t$, since they explicitly involve the function $c(t)$. This time variable appears in Problem A as the outer time and generates the function $R$.

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