Normality of Necessary Optimality Conditions for Calculus of Variations Problems with State Constraints

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Abstract

We consider non-autonomous calculus of variations problems with a state constraint represented by a given closed set. We prove that if the interior of the Clarke tangent cone of the state constraint set is non-empty (this is the constraint qualification that we suggest here), then the necessary optimality conditions apply in the normal form. We establish normality results for (weak) local minimizers and global minimizers, employing two different approaches and invoking slightly diverse assumptions. More precisely, for the local minimizers result, the Lagrangian is supposed to be Lipschitz with respect to the state variable, and just lower semicontinuous in its third variable. On the other hand, the approach for the global minimizers result (which is simpler) requires the Lagrangian to be convex with respect to its third variable, but the Lipschitz constant of the Lagrangian with respect to the state variable might now depend on time.

Keywords: Calculus of Variations · Constraint qualification · Normality · Optimal Control · Neighboring Feasible Trajectories

1 Introduction

We consider the following non-autonomous calculus of variations problem subject to a state constraint

\[
\begin{align*}
\text{minimize} & \quad \int_{S}^{T} L(t, x(t), \dot{x}(t)) \, dt \\
\text{over arcs } & \quad x(.) \in W^{1,1}([S, T], \mathbb{R}^{n}) \text{ satisfying} \\
& \quad x(S) = x_0, \\
& \quad x(t) \in A \quad \text{for all } t \in [S, T],
\end{align*}
\]

for a given Lagrangian \( L : [S, T] \times \mathbb{R}^{n} \times \mathbb{R}^{n} \rightarrow \mathbb{R} \), an initial datum \( x_0 \in \mathbb{R}^{n} \) and a closed set \( A \subset \mathbb{R}^{n} \).

Briefly stated, the problem consists in minimizing the integral of a time-dependent Lagrangian \( L \) over admissible absolutely continuous arcs \( x(.) \) (that is, the left end-point and the state constraint of the problem (CV) are satisfied). We say that an admissible arc \( \bar{x} \) is a \( W^{1,1} \)-local minimizer if there exists \( \varepsilon > 0 \) such that

\[
\int_{S}^{T} L(t, \bar{x}(t), \dot{\bar{x}}(t)) dt \leq \int_{S}^{T} L(t, x(t), \dot{x}(t)) dt,
\]

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for all admissible arcs $x$ satisfying
\[ \| x(\cdot) - \bar{x}(\cdot) \|_{W^{1,1}} \leq \epsilon. \]

The purpose of this paper is to derive necessary optimality conditions in the normal form for problems like (CV); that is, when trajectories satisfy a set of necessary conditions with a nonzero cost multiplier. Indeed, in the pathological situation of abnormal extrema, the objective function to minimize does not intervene in the selection of the candidates to be minimizers. To overcome this difficulty, new additional hypotheses have to be imposed, known as constraint qualifications, that permit to identify some class of problems for which normality is guaranteed. There has been a growing interest in the literature to ensure the normality of necessary optimality conditions for state-constrained calculus of variations problems. For instance, [10] deals with autonomous Lagrangian, studied for $W^{1,1}$-local minimizers and with a state constraint expressed in terms of an inequality of a twice continuously differentiable function. The constraint qualification referred to these smooth problems imposes that the gradient of the function representing the state constraint set is not zero at any point on the boundary of the state constraint set. The result in [10] has been successively extended in [13] to the nonsmooth case, for $L^\infty$-local minimizers, imposing a constraint qualification which makes use of some hybrid subgradients to cover situations in which the function, which defines the state constraint set, is not differentiable. More precisely the idea of the constraint qualification in [13] is the following: the angle between any couple of (hybrid) subgradients of the function that defines the state constraint set is ‘acute’. A useful technique employed in [10] and [13] in order to derive optimality conditions in the normal form consists in introducing an extra variable which reduces the reference calculus of variations problem to an optimal control problem with a terminal cost (this method is known as the ‘state augmentation’).

In this paper, the state constraint set is given in the intrinsic form (i.e. it is a given closed set) and the constraint qualification we suggest is to assume that the interior of the Clarke tangent cone to the state constraint set is nonempty. We emphasize that our constraint qualification generalizes the ones discussed in [13] and [11] and can be therefore applied to a broader class of problems. This is clarified in Proposition 3.4 and Example 3.5 where the constraint qualifications reported in [13] and [11] imply the one that we suggest in our paper.

We propose two main theorems which establish the normality of the necessary optimality conditions for $W^{1,1}$-local minimizers and for global minimizers. For these two cases we identify two different approaches, both based on a state augmentation technique, but combined with:

1) either a construction of a suitable control and a normality result for optimal control problems (this is for the $W^{1,1}$-local minimizers case);

2) or a ‘distance estimate’ result coupled with a standard maximum principle in optimal control (for the global minimizers case).

More precisely, in the first result approach (considering the case of $W^{1,1}$-local minimizers), we invoke some stability properties of the interior of the Clarke tangent cone. This allows to select a particular control which pushes the dynamic of the control system inside the state constraint more than the reference minimizer. In such circumstances, necessary conditions in optimal control apply in the normal form: we opt here for the broader version (i.e. covering $W^{1,1}$-local minimizers) of normality for optimal control problem established in [13] (which is concerned merely with $L^\infty$-local minimizers). The normality of the associated optimal control problem yields the desired normality property for our reference calculus of variations problem, considering $W^{1,1}$-local minimizers. This result is valid for Lagrangians $L = L(t, x, v)$ which are Lipschitz w.r.t. $x$ and lower semicontinuous w.r.t. $v$. The second result approach (which deals with global minimizers) requires to impose slightly different assumptions on the data: here the Lipschitz constant of the Lagrangian w.r.t. $x$ might be an integrable function depending on the time variable, but $v \mapsto L(t, x, v)$ has to be convex. It is also necessary to impose a constraint qualification which is slightly stronger than that one considered for the first result (but
still in the same spirit). The proof in this case is much shorter, simpler, and it employs a neighboring feasible trajectories result satisfying $L^\infty$–linear estimates (cf. [19] and [2]). This permits to find a global minimizer for an auxiliary optimal control problem to which we apply a standard maximum principle. The required normality form of the necessary conditions for the reference problem in calculus of variations can be therefore derived.

This paper is organized as follows. In a short preliminary section, we provide some of the basic notions and results of the nonsmooth analysis that are used throughout the paper. Section 3 provides the main results of this paper: necessary optimality conditions in a normal form for non-autonomous calculus of variations problems, for $W^{1,1}$–local and global minimizers. The last two sections are devoted to prove our main results making use of a normality result for optimal control problems (for the case of $W^{1,1}$–local minimizers) and a neighboring feasible trajectory result (for the case of global minimizers).

**Notation.** In this paper, $|.|$ refers to the Euclidean norm. $B$ denotes the closed unit ball. Given a subset $X$ of $\mathbb{R}^n$, $\partial X$ is the boundary of $X$, $\text{int }X$ is the interior of $X$ and $\text{co }X$ is the convex hull of $X$. The Euclidean distance of a point $y$ from the set $X$ (that is $\inf_{x \in X} |x - y|$) is written $d_X(y)$. Given a measure $\mu$ on the time interval $[S,T]$, we write $\text{supp}(\mu)$ for the support of the measure $\mu$. $W^{1,1}([S,T], \mathbb{R}^n)$ denotes the space of absolutely continuous functions defined on $[S,T]$ and taking values in $\mathbb{R}^n$.

## 2 Some basic nonsmooth analysis tools

We introduce here some basic nonsmooth analysis tools which will be employed in this paper. For further details, we refer the reader for instance to the books [11], [5], [7], [8], [18] and [20].

Take a closed set $A \subset \mathbb{R}^n$ and a point $\bar{x} \in A$. The **proximal normal cone** of $A$ at $\bar{x}$ is defined as

$$N^P_A(\bar{x}) := \{ \eta \in \mathbb{R}^n : \text{there exists } M > 0 \text{ such that } \eta \cdot (x - \bar{x}) \leq M \|x - \bar{x}\|^2 \text{ for all } x \in A \} .$$

The **limiting normal cone** of $A$ at $\bar{x} \in A$ is defined to be

$$N_A(\bar{x}) := \{ \eta \in \mathbb{R}^n : \text{there exists sequences } x_i \xrightarrow{A} \bar{x} \text{ and } \eta_i \rightarrow \eta \text{ such that } \eta_i \in N^P_A(x_i) \text{ for all } i \}.$$  

Here $x_i \xrightarrow{A} \bar{x}$ means that $x_i \rightarrow \bar{x}$ and $x_i \in A$, for all $i$.

Given a lower semicontinuous function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{ \infty \}$, the **limiting subdifferential** of $f$ at a point $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) < +\infty$ can be expressed in terms of the limiting normal cone to the epigraph of $f$ as follows:

$$\partial f(\bar{x}) := \{ \eta : (\eta, -1) \in N_{\text{epi } f}(\bar{x}, f(\bar{x})) \} .$$

We also make use of the **hybrid subdifferential** of a Lipschitz continuous function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ on a neighborhood of $\bar{x}$, defined as:

$$\partial^h h(\bar{x}) := \{ \gamma \in \mathbb{R}^n : \text{there exists } x_i \xrightarrow{h} \bar{x} \text{ such that } h(x_i) > 0, \text{ for all } i \text{ and } \nabla h(x_i) \rightarrow \gamma \} ,$$

where $x_i \xrightarrow{h} \bar{x}$ denotes $x_i \rightarrow \bar{x}$ and $h(x_i) \rightarrow h(\bar{x})$ for all $i$. Moreover, we make reference to the Clarke tangent cone to the closed set $A$ at $\bar{x} \in A$, defined as follow:

$$T_A(\bar{x}) := \{ \xi \in \mathbb{R}^n : \text{ for all } x_i \xrightarrow{A} \bar{x} \text{ and } t_i \downarrow 0, \text{ there exist } \{a_i\} \text{ in } A \text{ such that } t_i^{-1}(a_i - x_i) \rightarrow \xi \} .$$

We present also the following result:

**Proposition 2.1.** (i) $\partial^h d_A(a) \subset \text{co } (N_A(a) \cap \partial B)$ for a closed set $A \subset \mathbb{R}^n$ and $a \in \partial A$.

(ii) If in addition $\text{co } N_A(a)$ is pointed, then $0 \notin \text{co } (N_A(a) \cap \partial B)$.

(We recall that $\text{co } N_A(a)$ is ‘pointed’ if for any nonzero elements $d_1, d_2 \in \text{co } N_A(a)$, $d_1 + d_2 \neq 0$.)

**Proof.** The proof of (i) follows directly from the definition of the hybrid subdifferential of the distance function $d_A(.)$, and from the Caratheodory Theorem; while (ii) is proved by contradiction. \qed
Main Results

This section provides the main results of this paper: necessary optimality conditions in a normal form for non-autonomous calculus of variations problems in the form of (CV). More precisely, we are interested in providing normality conditions for problems in which the state constraint is given in an implicit form: $A$ is just a closed set. Using the distance function, the state constraint $x(t) \in A$ can be equivalently written as a pathwise functional inequality

$$d_A(x(t)) \leq 0 \quad \text{for all } t \in [S,T].$$

We provide first a normality result for $W^{1,1}$-local minimizers. We assume therefore that for a given reference arc $\bar{x} \in W^{1,1}([S,T], \mathbb{R}^n)$:

(CV1) $L(t, x, v)$ is measurable in $(t, v)$, bounded on bounded sets; and there exist $\epsilon' > 0$, $K_L > 0$ such that for all $t \in [S,T]$ and $x, x' \in \bar{x}(t) + \epsilon' B$

$$|L(t, x, v) - L(t, x', v)| \leq K_L |x - x'| \quad \text{uniformly on } v \in \mathbb{R}^n.$$

(CV2) $v \mapsto L(t, \bar{x}(t), v)$ is lower semicontinuous for all $t \in [S,T]$.

**Theorem 3.1 (W^{1,1}-Local Minimizers).** Let $\bar{x}$ be a $W^{1,1}$-local minimizer for (CV), and assume that hypotheses (CV1)-(CV2) are satisfied. Suppose also that $\bar{x}$ is Lipschitz continuous and

(CQ) $\text{int } T_A(z) \neq \emptyset$, for all $z \in \bar{x}([S,T]) \cap \partial A$.

Then, there exist $p(\cdot) \in W^{1,1}([S,T], \mathbb{R}^n)$, a function of bounded variation $\nu(\cdot) : [S,T] \to \mathbb{R}^n$, continuous from the right on $(S,T)$, such that: for some positive Borel measure $\mu$ on $[S,T]$, whose support satisfies

$$\text{supp}(\mu) \subset \{t \in [S,T] : \bar{x}(t) \in \partial A\},$$

and some Borel measurable selection

$$\gamma(t) \in \partial d_A(\bar{x}(t)) \quad \mu - \text{a.e. } t \in [S,T]$$

we have

(a) $\nu(t) = \int_{[S,t]} \gamma(s) d\mu(s)$ for all $t \in (S,T]$;

(b) $\dot{p}(t) \in \text{co } \partial_x L(t, \bar{x}(t), \dot{\bar{x}}(t))$ and $q(t) \in \text{co } \partial_z L(t, \bar{x}(t), \dot{\bar{x}}(t))$ a.e. $t \in [S,T]$;

(c) $q(T) = 0$;

where

$$q(t) = \begin{cases} p(S) & t = S \\ p(t) + \int_{[S,t]} \gamma(s) d\mu(s) & t \in (S,T] \end{cases}.$$

The following theorem concerns global minimizers, establishing the same necessary optimality conditions (in the normal form) as Theorem 3.1. Here we invoke slightly different assumptions: we impose the convexity of the Lagrangian w.r.t. $v$, and the stronger constraint qualification (CQ) below (valid for any point belonging to the boundary of $A$); however, we allow that the Lipschitz constant of $L$ w.r.t. $x$ might depend on $t$. More precisely, we assume that for some $R_0 > 0$,
Theorem 3.2

Remark 3.3. We observe that the validity of Theorem 3.1 and Theorem 3.2 requires that the minimizer (local or global) \( \bar{x}(\cdot) \) is Lipschitz. This assumption is not restrictive, indeed, not only covers previous results in this framework (as \([13]\) and \([10]\)), but earlier work shows that Lipschitz regularity of minimizers can be obtained (for both cases of autonomous and non-autonomous Lagrangian) under unrestrictive assumptions on the data. This is not a peculiar property of some problems in the absence of state constraints (see for instance \([9]\), \([4, \text{Theorem 4.5.2 and Theorem 4.5.4}]\), \([6, \text{Corollary 3.2}]\)). It is well-known also in the framework of state-constrained calculus of variations problems, see \([5, \text{Corollary 16.19}]\), \([20, \text{Theorem 11.5.1}]\) (for the autonomous case), and more recently in \([5, \text{Theorem 5.2}]\) (for the nonautonomous case).

We recall that the problem of normality of necessary conditions in calculus of variations has been previously investigated in \([10]\), for the case of autonomous Lagrangian and smooth state constraint expressed in terms of a scalar inequality function \( \{ x : h(x) \leq 0 \} \) (that is, the function \( h \) is of class \( C^2 \)). This result has been successively extended to the case where the scalar function \( h \) is nonsmooth in \([13]\), always in the framework of autonomous Lagrangian. In both papers \([13]\) and \([10]\), the Lagrangian is supposed to be Lipschitz w.r.t. \( x \) and convex w.r.t. \( v \). In our paper we deal with a non-autonomous Lagrangian and with a constraint qualification of different nature when we consider a state constraint condition in terms of a merely closed set \( A \).

In the following proposition and example, it is shown that the constraint qualification invoked in \([13]\) (when we consider the state constraint to be expressed in terms of a distance function, that is \( h(.) := d_A(.) \)) implies our constraints qualifications \([CQ]\) and \([CQ]\), but the reverse implication is not true. Accordingly, \([CQ]\) and \([CQ]\) can be applied to a larger class of problems.
Proposition 3.4. Consider a closed set \( A \subset \mathbb{R}^n \) and assume that the distance function to \( A \), \( d_A(\cdot) \), satisfies the following condition: if for any given \( y \in \partial A \) there exists \( c > 0 \) such that
\[
\gamma_1 \cdot \gamma_2 > c , \quad \text{for all} \quad \gamma_1 , \gamma_2 \in \partial^> d_A(y) , \tag{3.1}
\]
then, \( \text{int } T_A(y) \neq \emptyset . \)

(Condition (3.1) is a slightly weaker version of the constraint qualification considered in [12].)

Proof of Proposition 3.4 Since the hypothesis (3.1) considered here clearly implies that \( 0 \notin \partial^> d_A(y) \), then the proof uses the same ideas of [12, Proposition 2.1] which consists in showing that the cone \( \mathbb{R}^+(\partial^> d_A(y)) \) is closed and pointed and that it is also equal to \( \text{co } N_A(y) \). That is the convex hull of the limiting normal cone to \( A \) is pointed and, consequently, its polar, \( T_A(y) \) has a nonempty interior. \( \square \)

Example 3.5. Consider the set \( A := \{(x,y) : h(x,y) \leq 0\} \), where \( h : \mathbb{R}^2 \to \mathbb{R} \) such that
\[
h(x,y) = |y| - x.
\]

It is straightforward to check that \( \partial^> h(0,0) = \text{co } \{\gamma_1, \gamma_2\} \), where \( \gamma_1 := (-1, +1) \), and \( \gamma_2 := (-1, -1) \). Therefore, at the point \( (0,0) \), condition (3.1) is violated when a minimizer \( \bar{x} \) is such that \( (0,0) \in \bar{x}([S,T]) \). This is because we could find two vectors \( \gamma_1 \) and \( \gamma_2 \) such that \( \gamma_1 \cdot \gamma_2 = 0 \). However, \( \text{int } T_A(0,0) \neq \emptyset \), and more in general \( \text{int } T_A(y) \neq \emptyset \) for all \( y \in \partial A \). Then, (CQ) (and also (\( \tilde{CQ} \)) is always satisfied.

This is a simple example which shows that we can find a state constraint set \( A \) defined by a functional inequality (for some Lipschitz function \( h \)) such that (CQ) (and also (\( \tilde{CQ} \))) is always verified but (3.1) fails to hold true when a minimizer goes in a region where \( A \) is nonsmooth.

4 Proof of Theorem 3.1 (\( W^{1,1} \)–Local Minimizers)

Along this section, we identify a class of optimal control problems whose necessary optimality conditions apply in the normal form, under some constraint qualifications. The main purpose of introducing such problems is that calculus of variations problems can be regarded as an optimal control problem (owing to the ‘state augmentation’ procedure). Therefore, the results on optimal control problems will be used to establish normality of optimality conditions for the reference calculus of variations problem (CV).

4.1 Normality in Optimal Control Problems

We recall that ‘normality’ means that the Lagrangian multiplier associated with the objective function — here written \( \lambda \) — is different from zero (it can be taken equal to 1).

Consider the fixed left end-point optimal control problem \( \text{(P)} \) with a state constraint set \( A \subset \mathbb{R}^n \) which is merely a closet set:

\[
\begin{align*}
\text{minimize} & \quad g(x(T)) \\
\text{over } & \quad x \in W^{1,1}([S,T],\mathbb{R}^n) \text{ and measurable functions } u \text{ satisfying} \\
& \quad \dot{x}(t) = f(t,x(t),u(t)) \quad \text{a.e. } t \in [S,T] \\
& \quad x(S) = x_0 \\
& \quad x(t) \in A \quad \text{for all } t \in [S,T] \\
& \quad u(t) \in U(t) \quad \text{a.e. } t \in [S,T] .
\end{align*}
\]

The data for this problem comprise functions \( g : \mathbb{R}^n \to \mathbb{R}, f : [S,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \), an initial state
$x_0 \in \mathbb{R}^n$, and a multifunction $U(.) : [S,T] \rightharpoonup \mathbb{R}^m$. The set of control functions for $[P]$, denoted by $\mathcal{U}$, is the set of all measurable functions $u : [S,T] \rightarrow \mathbb{R}^m$ such that $u(t) \in U(t)$ a.e. $t \in [S,T]$.

We say that an admissible process $(\bar{x}, \bar{u})$ is a $W^{1,1}$-local minimizer if there exists $\epsilon > 0$ such that

$$g(\bar{x}(T)) \leq g(x(T)),$$

for all admissible processes $(x, u)$ satisfying

$$\|x(.) - \bar{x}(.)\|_{W^{1,1}} \leq \epsilon.$$

There follows a ‘normal’ version of the maximum principle for state constrained problems. For a $W^{1,1}$-local minimizer $(\bar{x}, \bar{u})$ and a positive scalar $\delta$, we assume the following:

(H1) The function $(t, u) \mapsto f(t, x, u)$ is measurable for each $x \in \mathbb{R}^n$. There exists a measurable function $k(t, u)$ such that $t \mapsto k(t, \bar{u}(t))$ is integrable and

$$|f(t, x, u) - f(t, x', u)| \leq k(t, u)|x - x'|$$

for $x, x' \in \bar{x}(t) + \delta \mathbb{B}$, $u \in U(t)$, a.e. $t \in [S,T]$. Furthermore there exist scalars $K_f > 0$ and $\epsilon' > 0$ such that

$$|f(t, x, u) - f(t, x', u)| \leq K_f|x - x'|$$

for $x, x' \in \bar{x}(S) + \delta \mathbb{B}$, $u \in U(t)$, a.e. $t \in [S, S + \epsilon']$.

(H2) Gr $U(.)$ is measurable.

(H3) The function $g$ is Lipschitz continuous on $\bar{x}(T) + \delta \mathbb{B}$.

Reference is also made to the following constraint qualifications. There exist positive constants $K, \bar{\epsilon}, \bar{\beta}, \bar{\rho}$ and a control $\hat{u} \in \mathcal{U}$ such that

(CQ1) \begin{equation}
|f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), \hat{u}(t))| \leq K, \quad \text{for a.e. } t \in (\tau - \bar{\epsilon}, \tau) \cap [S,T]
\end{equation}

and

$$\eta \cdot [f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), \hat{u}(t))] < -\bar{\beta},$$

for all $\eta \in \partial_x f\bar{A}(\bar{x}(s))$, a.e. $s, t \in (\tau - \bar{\epsilon}, \tau) \cap [S,T]$ and for all $\tau \in \{\sigma \in [S,T] : \bar{x}(\sigma) \in \partial A\}$.

(CQ2) If $x_0 \in \partial A$, then for a.e. $t \in [S, S + \bar{\epsilon}]$

$$|f(t, x_0, \bar{u}(t))| \leq K, \quad |f(t, x_0, \bar{u}(t))| \leq K,$$

and

$$\eta \cdot [f(t, x_0, \bar{u}(t)) - f(t, x_0, \hat{u}(t))] < -\bar{\beta}$$

for all $\eta \in \partial_x f\bar{A}(x)$, $x \in (x_0 + \bar{\rho} \mathbb{B}) \cap \partial A$.

(CQ3) \begin{equation}
\text{co } N_A(\bar{x}(t)) \text{ is pointed for each } t \in [S,T].
\end{equation}

**Theorem 4.1.** Let $(\bar{x}, \bar{u})$ be a $W^{1,1}$-local minimizer for $[P]$. Assume that hypotheses (H1)-(H3) and the constraint qualifications (CQ1)-(CQ3) hold. Then, there exist $p(.) \in W^{1,1}([S,T], \mathbb{R}^n)$, a Borel measure $\mu(.)$ and a $\mu-$integrable function $\gamma(.)$ such that

(i) $-\hat{p}(t) \in \text{co } \partial_2 q(t) \cdot f(t, \bar{x}(t), \bar{u}(t))$ a.e. $t \in [S,T],$

(ii) $-q(T) \in \partial g(\bar{x}(T)),$
Theorem 4.1 might seem to be a particular case of [12, Theorem 3.2] when the optimal trajectory has an outward pointing velocity. Moreover in [12] relations with previous considerations in our paper. Indeed, in (4.1) and (4.2), the boundedness of the dynamics is considered and at the initial data $x_0$ and at the optimal trajectory points $\bar{x}(t)$ only for times $t < \tau$ at which the trajectory hits the boundary of the state constraint of a possibly nonsmooth function. This result was extended in [12] (considering always the inward pointing condition has just to be satisfied for almost all times at which the optimal trajectory points $\bar{x}(t)$) is considered to be a local minimizer (by adding eventually an extra variable $\bar{y}(t) = | f(t, x(t), u(t)) - \dot{x}(t) |$) and for a state constraint expressed in terms of a closed set $A$.

Indeed, let $(\bar{x}, \bar{u})$ be merely a $W^{1,1}$—local minimizer for $[P]$. Then there exists $\alpha > 0$ such that $(\bar{x}, \bar{y} \equiv 0, \bar{u})$ is a $L^\infty$—local minimizer for

$$
\begin{cases}
\text{minimize } g(x(T)) \\
\text{over } x \in W^{1,1}([S,T], \mathbb{R}^n) \text{ and measurable functions } u \text{ satisfying}
\end{cases}
$$

(P1)

$$
\begin{align*}
&\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [S,T] \\
&\dot{y}(t) = | f(t, x(t), u(t)) - \dot{x}(t) | \quad \text{a.e. } t \in [S,T] \\
&(x(S), y(S), y(T)) \in \{ x_0 \} \times \{ 0 \} \times \alpha \mathbb{B} \\
&x(t) \in A \quad \text{for all } t \in [S,T] \\
&u(t) \in U(t) \quad \text{a.e. } t \in [S,T].
\end{align*}
$$

(It suffices to prove it by contradiction.) Therefore we can apply [13, Theorem 4.2] in its normal formulation to the augmented problem [P1] with reference to the $L^\infty$—local minimizer $(\bar{x}, \bar{y} \equiv 0, \bar{u})$. After expliciting the corresponding necessary optimality conditions, we notice that the adjoint arc $r$ associated with the state variable $y$ is constant because the right hand side of the dynamics does not depend on $y$. Moreover, $r = 0$ owing to the transversality condition since $\bar{y}(T) \in \alpha \mathbb{B}$ is inactive. Hence, we deduce the required necessary optimality conditions as represented in Theorem 4.1 covering the case of $W^{1,1}$—local minimizers.

$\square$

Remark 4.2. Normality of the necessary optimality conditions for state constrained optimal control problems has been well studied and many results exist in the literature where each result requires some regularity assumptions on the problem data (cf. [13], [15], [14], [12] etc.). In [13], normality is established for free right-endpoint optimal control problems having $L^\infty$—local minimizers. The constraint qualifications assumed on the data are in the form of (CQ1) and (CQ2) but considered for an inequality state constraint of a possibly nonsmooth function. This result was extended in [12] (considering always the case of $L^\infty$—local minimizers) to cover a larger class of problems where the state constraint is given in terms of a closed set and the right-endpoint (denoted by the set $K_1$ in [12]) is considered to be a closed subset of $\mathbb{R}^n$. New (and weaker) constraint qualifications are discussed in [12] to guarantee normality when $\bar{x}(T) \in \text{int } K_1$: the inward pointing condition has just to be satisfied for almost all times at which the optimal trajectory has an outward pointing velocity. Moreover in [12] relations with previous constraint qualifications are examined.

Theorem 4.1 might seem to be a particular case of [12, Theorem 3.2] when $K_1 = \mathbb{R}^n$ (eventually after considering the augmented problem in order to cover the case of $W^{1,1}$—local minimizers). However, looking closely to assumption (H3) in [12], we see that (H3) is a stronger assumption than the one considered in previous papers (see for instance [11], [17], [13]) and in assumptions (4.1) and (4.2) considered in our paper. Indeed, in (4.1) and (4.2), the boundedness of the dynamics is considered only for the optimal control $\bar{u}(\cdot)$ and the constructed control $\dot{u}(\cdot)$ at the initial data $x_0$ and at the optimal trajectory points $\bar{x}(t)$ only for times $t < \tau$ at which the trajectory hits the boundary of the state.
constraint, however in [12], the boundedness concerns all controls $u \in U(t)$ and all $x$ near the optimal trajectory $\bar{x}(\cdot)$, which is not implied by the assumptions of our paper.

We underline the fact that to derive the desired necessary conditions stated in Theorem 3.1 above, a crucial requirement is the possibility to apply normality results for optimal control problems with dynamics which might be unbounded (cf. Remark 4.5).

4.2 Technical Lemmas

We invoke now two technical lemmas which are crucial for establishing the proof of Theorem 3.1. The first lemma says that one can select a particular bounded control $v(\cdot)$ which pulls the dynamics inward the state constraint set more than the reference minimizer.

Lemma 4.3. Let $\bar{x}(\cdot)$ be a Lipschitz $W^{1,1}$-local minimizer for $\{CV\}$ for which $\{CV1\}$ and $\{CQ\}$ are satisfied. Then, there exist positive constants $\varepsilon, \rho, \beta, C, C_1$ and a measurable function $v(\cdot)$ such that:

(i) $\|v - \dot{\bar{x}}\|_{L^\infty} \leq C$ and $\|v\|_{L^\infty} \leq C_1$.

(ii) For all $\tau \in \{\sigma \in [S,T] : \bar{x}(\sigma) \in \partial A\}$,
\[
\sup_{\eta \in \text{co}(N_A(\bar{x}(t)) \cap \partial B)} (v(t) - \dot{\bar{x}}(t)) \cdot \eta < -\beta, \quad \text{a.e. } s, t \in (\tau - \varepsilon, \tau).
\]

(iii) If $x_0 = \bar{x}(S) \in \partial A$, then
\[
\sup_{\eta \in \text{co}(N_A(x) \cap \partial B)} (v(t) - \dot{\bar{x}}(t)) \cdot \eta < -\beta, \quad \text{a.e. } t \in [S, S + \varepsilon).
\]

Proof of Lemma 4.3. Indeed, by defining the function $v(\cdot) : [S, T] \to \mathbb{R}^n$ as follows
\[
v(t) := \dot{\bar{x}}(t) + \sum_{j=1}^{k} \chi_{z_j + \frac{\epsilon_j}{2}\text{int } B}(\bar{x}(t)) \zeta_j, \quad t \in [S, T]
\]
where $z_j, \epsilon_j$ and $\zeta_j$ are respectively the points, positive numbers and vectors of Lemma 4.4. $\chi_Y$ denotes the characteristic function of the subset $Y \subset \mathbb{R}^n$, and owing to the assertions of Lemma 4.4 we can prove that the constructed $v(\cdot)$ verifies the statement of Lemma 4.3. (We refer the reader to [16] for full details of the proof.)
4.3 Proof of Theorem 3.1

We first employ a standard argument, the so-called ‘state augmentation’, which allows to write the problem of calculus of variations (CV) as an optimal control problem of type \((P')\). Indeed, it is enough to add an extra absolutely continuous state variable

\[
z(t) = \int_{S}^{t} L(s, x(s), \dot{x}(s)) ds
\]

and consider the dynamics \(\dot{x} = u\). We notice that \(z(S) = 0\) and \(U(t) = \mathbb{R}^n\).

Then, the problem (CV) can be written as the optimal control problem \((P')\):

\[
\begin{align*}
\text{minimize} & \quad z(T) \\
\text{over} & \quad W^{1,1} - \text{arcs} \ (x(\cdot), z(\cdot)) \text{ and measurable functions } u(\cdot) \in \mathbb{R}^n \text{ satisfying} \\
& (\dot{x}(t), \dot{z}(t)) = (u(t), L(t, x(t), u(t))) \quad \text{a.e. } t \in [S, T] \\
& (x(S), z(S)) = (x_0, 0) \\
& x(t) \in A \quad \text{for all } t \in [S, T] \\
& u(t) \in U(t) = \mathbb{R}^n \quad \text{a.e. } t \in [S, T].
\end{align*}
\]

We set \(w(t) := \begin{pmatrix} x(t) \\ z(t) \end{pmatrix} \) and \(\tilde{f}(t, w(t), u(t)) := \begin{pmatrix} u(t) \\ L(t, x(t), u(t)) \end{pmatrix}\). Here the set of controls \(U\) is the set of dynamics \(\dot{x}(t) \in \mathbb{R}^n\) for a.e. \(t \in [S, T]\).

It is easy to prove that if \(\bar{x}\) is a \(W^{1,1}\)-local minimizer for the reference calculus of variations problem (CV), then \((\tilde{w}(t) = \begin{pmatrix} \bar{x}(t) \\ \bar{z}(t) \end{pmatrix}, \bar{u})\) is a \(W^{1,1}\)-local minimizer for \((P')\) where \(\tilde{z}(t) = L(t, \bar{x}(t), \bar{u}(t))\) and \(\bar{u} = \bar{x}\).

The proof of Theorem 3.1 is given in three steps. The first step is devoted to show that the constraint qualifications (CQ1)-(CQ3) of Theorem 4.1 are mainly implied by (CQ) of Theorem 3.1. In step 2, we verify that hypotheses (H1)-(H3) of Theorem 4.1 can be deduced from hypothesis (CV1) of Theorem 3.1. In step 3, we apply Theorem 4.1 to \((P')\) in order to obtain the assertions of Theorem 3.1.

**Step 1.** Prove that the constraint qualifications (CQ1)-(CQ3) of Theorem 4.1 are mainly implied by (CQ) of Theorem 3.1.

The constraint qualifications (CQ1) and (CQ2) for the optimal problem \((P')\) become as follows: there exist positive constants \(K, \bar{\epsilon}, \beta, \bar{\rho}\) and a control \(\hat{u} \in U\) such that

(CQ1')

\[
|\tilde{f}(t, (\bar{x}(t), \bar{z}(t)), \hat{u}(t)) - \tilde{f}(t, (\bar{x}(t), \bar{z}(t)), \bar{u}(t))| \leq K, \quad \text{a.e. } t \in (\tau - \bar{\epsilon}, \tau) \cap [S, T] \quad (4.3)
\]

and

\[
\bar{\eta} \cdot [\tilde{f}(t, (\bar{x}(t), \bar{z}(t)), \hat{u}(t)) - \tilde{f}(t, (\bar{x}(t), \bar{z}(t)), \bar{u}(t))] < -\bar{\rho}, \quad (4.4)
\]

for all \(\bar{\eta} \in \partial^*_w d_A(\bar{x}(s))\) a.e. \(s, t \in (\tau - \bar{\epsilon}, \tau) \cap [S, T]\) and for all \(\tau \in \{s \in [S, T] : \bar{x}(\tau) \in \partial A\}\). Here \(\partial^*_w d_A(\bar{x}(s)) := \text{co} \{ (a, b) : \text{there exists } s_i \rightarrow s \text{ such that } d_A(\bar{x}(s_i)) > 0 \text{ for all } i, d_A(\bar{x}(s_i)) \rightarrow d_A(\bar{x}(s)) \text{ and } \nabla_w d_A(\bar{x}(s_i)) \rightarrow (a, b) \}\).

(CQ2') If \(x_0 \in \partial A\) then for a.e. \(t \in [S, S + \bar{\epsilon})\)

\[
|\tilde{f}(t, (x_0, 0), \hat{u}(t))| \leq K \quad \text{and} \quad |\tilde{f}(t, (x_0, 0), \bar{u}(t))| \leq K, \quad (4.5)
\]
By the Lipschitz continuity of the minimizer $\bar{x}$, we start with the proof of condition (4.5).

1. We state with the proof of condition (4.5). By the Lipschitz continuity of the minimizer $\bar{x}$, we take $R := |x_0| + \|\dot{x}\|_{L^\infty}(1 + T - S) > 0$. Then $\bar{x}([S, T]) \subset R\mathbb{B}$ and $|\bar{x}(t)| \leq R$ for a.e. $t \in [S, T]$. Take $x_0 \in \partial A$. Since the function $L(\cdot, \cdot, \cdot)$ is bounded on bounded sets (owing to (CV1)), we obtain

$$|\bar{f}(t, (x_0, 0), \bar{u}(t))| = \left|\frac{\bar{u}(t) = \dot{\bar{x}}(t)}{L(t, x_0, \bar{u}(t))}\right| \leq K \quad \text{a.e. } t \in [S, T], \text{ for some } K > 0.$$  

Moreover, under (CQ), Lemma 4.3 ensures the existence of a measurable function $v(.)$ and positive constants $C$ and $C_1$ such that $\|v\|_{L^\infty} \leq C_1$ and $\|v - \dot{\bar{x}}\|_{L^\infty} \leq C$. Therefore, by choosing a control $\hat{u}$ such that $\hat{u}(t) = v(t)$ a.e. $t$, we deduce easily that

$$|\hat{f}(t, (x_0, 0), \hat{u}(t))| \leq K \quad \text{a.e. } t \in [S, T] \quad \text{for some } K > 0.$$  

Condition (4.5) is therefore satisfied.

2. Condition (4.5) follows also (for $K$ big enough) from the particular choice of $\hat{u}(t)$ to be the bounded control $v(.)$ of Lemma 4.3 and from (CV1).

3. To prove condition (4.6), we consider the same choice of the control function $\hat{u}(\cdot)$, that is $\hat{u}(t) := v(t)$ for $t \in [S, S + \varepsilon]$, where $\varepsilon$ is the positive number and $v(t)$ is the measurable function which satisfy the properties of Lemma 4.3. Condition (iii) of Lemma 4.3 implies that there exist positive constants $\rho, \beta$ such that

$$\eta \cdot (\hat{u}(t) - \dot{x}(t)) < -\beta \quad \text{for a.e. } t \in [S, S + \varepsilon]$$  

for all $\eta \in \text{co}(N_A(x) \cap \partial \mathbb{B})$ and for all $x \in (x_0 + \bar{\rho}\mathbb{B}) \cap \partial A$. In particular for all $\eta \in \partial^2_A d_A(x)$ (owing to Proposition 2.1(i)) and by choosing $\bar{\varepsilon} \in (0, \varepsilon)$, $\bar{\rho} \in (0, \rho]$ and $\bar{\beta} = \beta$. Take now any $\tilde{\eta} \in \partial^2_A d_A(x)$ where $x \in (x_0 + \bar{\rho}\mathbb{B}) \cap \partial A$. By definition of $\partial^2_A d_A(x)$

$$\tilde{\eta} \in \left(\text{co} \{ a : \text{there exists } x_i \xrightarrow{d_A} x \text{ such that } d_A(x_i) > 0 \text{ for all } i, \text{ and } \nabla_A d_A(x_i) \to a\}, 0 \right).$$

This is because $\nabla_A d_A(x_i) = 0$. We conclude that $\tilde{\eta}$ can be written as:

$$\tilde{\eta} := (\eta, 0) \quad \text{where } \eta \in \partial^2_A d_A(x).$$

It follows that, owing to (4.7):

$$(\eta, 0) \cdot [\hat{f}(t, (x_0, 0), \hat{u}(t)) - \bar{f}(t, (x_0, 0), \bar{u}(t))] = \eta \cdot (\hat{u}(t) - \bar{u}(t)) < -\bar{\beta},$$

for all $\eta \in \partial^2_A d_A(x)$ a.e. $t \in [S, S + \bar{\varepsilon}]$ for all $x \in (x_0 + \bar{\rho}\mathbb{B}) \cap \partial A$. Condition (4.6) is therefore confirmed.

4. For the proof of (4.4), we take any $\tau \in \{ \sigma \in [S, T] : \bar{x}(\sigma) \in \partial A \}$. Consider again the positive constants $\varepsilon$ and $\beta$ and the selection $v(.)$ provided by Lemma 4.3. Property (iii) of the latter lemma implies that for a.e. $s, t \in (\tau - \varepsilon, \tau]$.

$$\eta \cdot (v(t) - \dot{x}(t)) < -\beta$$  

for all $\eta \in \text{co}(N_A(\bar{x}(s)) \cap \partial \mathbb{B})$. In particular for all $\eta \in \partial^2_A d_A(\bar{x}(s))$ (owing to Proposition 2.1(i)) and for $\bar{\varepsilon} \in (0, \varepsilon]$ and $\bar{\beta} = \beta$ and by considering the control function $\hat{u}(t) := v(t)$ for $t \in (\tau - \varepsilon, \tau]$ (and
eventually for \( t \in (\tau - \varepsilon, \tau) \). Now we take an element \( \tilde{\eta} \) in \( \partial^d A(\bar{x}(s)) \) for \( s \in (\tau - \varepsilon, \tau] \cap [S, T] \). It follows that,

\[
\tilde{\eta} \in \left( \co \{ a : \text{there exists } t_i \rightarrow s \text{ s.t. } d_A(\bar{x}(s_i)) > 0 \text{ for all } i, d_A(\bar{x}(s_i)) \rightarrow d_A(\bar{x}(s)) \text{ and } \nabla_x d_A(\bar{x}(s_i)) \rightarrow a \} \right)
\]

which is equivalent to write \( \tilde{\eta} \) as

\[
\tilde{\eta} := (\eta, 0) \quad \text{where } \eta \in \partial^d A(\bar{x}(s))
\]

and \( s \in (\tau - \varepsilon, \tau] \cap [S, T] \). Making use of (4.8), it follows that

\[
(\eta, 0) \cdot [\dot{f}(t, (\bar{x}(t), \bar{z}(t)), \dot{u}(t)) - \dot{\bar{f}}(t, (\bar{x}(t), \bar{z}(t)), \bar{u}(t))] = \eta \cdot (\dot{u}(t) - \bar{u}(t)) < -\tilde{\beta},
\]

for all \( \eta \in \partial^d A(\bar{x}(s)) \) for a.e. \( s, t \in (\tau - \varepsilon, \tau] \cap [S, T] \) and for all \( \tau \in \{ \sigma \in [S, T] : \bar{x}(\sigma) \in \partial A \} \). We conclude that condition (4.4) is satisfied.

5. Finally, it is clear that [CQ1] implies that [CQ3] is satisfied.

**Step 2.** It is an easy task to prove that hypotheses [H1]-[H3] adapted to the optimal control problem \( [P'] \) are satisfied. This is a direct consequence of hypothesis (CV1).

**Step 3.** Apply Theorem 4.1 to \( [P'] \) and then obtain the assertions of Theorem 3.1. Since \( \bar{x} \) is a \( W^{1,1} \) local minimizer for the reference calculus of variations problem (CV), then \( ((\bar{x}, \bar{z}), \dot{\bar{x}} = \bar{u}) \) is a \( W^{1,1} \) local minimizer for the optimal control problem \( [P'] \). Theorem 4.1 can be therefore applied to the problem \( [P'] \). Namely, there exist a couple of absolutely continuous functions \( (p_1, p_2) \in W^{1,1}([S, T], \mathbb{R}^n) \times W^{1,1}([S, T], \mathbb{R}) \), a Borel measure \( \mu(.) \) and a \( \mu \)-integrable function \( \gamma(.) \), such that:

\[
\begin{align*}
\text{(i)}' & - (\dot{p}_1(t), \dot{p}_2(t)) \in \co \partial_{(x,z)} [(q_1(t), q_2(t)) \cdot \dot{f}(t, (\bar{x}(t), \bar{z}(t)), \bar{u}(t))] \text{ a.e. } t \in [S, T], \\
\text{(ii)}' & - (q_1(T), q_2(T)) \in \partial_{x,z} \bar{z}(T), \\
\text{(iii)}' & (q_1(t), q_2(t)) \cdot \dot{f}(t, (\bar{x}(t), \bar{z}(t)), \bar{u}(t)) = \max_{u \in U(t)} (q_1(t), q_2(t)) \cdot \dot{f}(t, (\bar{x}(t), \bar{z}(t)), u), \\
\text{(iv)}' & \gamma(t) \in \partial^d A(\bar{x}(t)) \quad \text{and} \quad \text{supp}(\mu) \subset \{ t \in [S, T] : \bar{x}(t) \in \partial A \},
\end{align*}
\]

where

\[
q_1(t) = \begin{cases} 
  p_1(S) & t = S \\
  p_1(t) + \int_{[S,t]} \gamma(s) d\mu(s), & t \in (S,T]
\end{cases}
\]

and

\[
q_2(t) = p_2(t), \quad \text{for } t \in [S,T].
\]

The transversality condition [(ii)'] ensures that \( q_1(T) = 0 \). This implies that condition (c) of Theorem 3.1 is satisfied. Furthermore, \( q_2(T) = -1 = p_2(T) \). The adjoint system [(i)'] ensures, by expanding it and applying a well-known nonsmooth calculus rule, that:

\[
-(\dot{p}_1(t), \dot{p}_2(t)) \in \co \partial_{x,z} (q_1(t) \cdot \bar{u}(t) + q_2(t) L(t, \bar{x}(t), \bar{u}(t))).
\]

By the Lipschitz continuous of \( L(t, ., u) \) on a neighborhood of \( \bar{x}(t) \), we obtain

\[
-(\dot{p}_1(t), \dot{p}_2(t)) \in q_2(t) \co \partial_y L(t, \bar{x}(t), \bar{u}(t)) \times \{0\}.
\]
Therefore, \( \dot{p}_2(t) = 0 \) and \(-\dot{p}_1(t) \in q_2(t) \co \partial_x L(t, \bar{x}(t), \bar{u}(t)) \). We deduce that \( p_2(t) = q_2(t) = -1 \) and \( \dot{p}_1(t) \in \co \partial_x L(t, \bar{x}(t), \bar{u}(t)) \) a.e. \( t \).

The maximization condition \( (\text{iii}) \) is equivalent to:

\[
q_1(t) \cdot \bar{u}(t) - L(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in \mathbb{R}^n} \{ q_1(t) \cdot u - L(t, \bar{x}(t), u) \}.
\]

Deriving the expression above in terms of \( u \), and evaluating it at \( u = \dot{x} \), we obtain

\[
0 \in \co \partial_x \{ q_1(t) \cdot \bar{u}(t) - L(t, \bar{x}(t), \bar{u}(t)) \}.
\]

Making use of the lower semicontinuity property of \( v \mapsto L(t, \bar{x}(t), v) \) (cf. \( \text{(CV2)} \)) and using the sum rule [20, page 45], we have

\[
0 \in \co \{ \partial_x (q_1(t) \cdot \dot{x}(t)) + \partial_x (-L(t, \bar{x}(t), \bar{u}(t))) \} \quad \text{a.e.}
\]

Moreover, since \( \co \partial (-f) = -\co \partial (f) \), we deduce that

\[
0 \in q_1(t) - \co \partial_x (L(t, \bar{x}(t), \bar{u}(t))).
\]

Equivalently,

\[
q_1(t) \in \co \partial_x L(t, \bar{x}(t), \bar{u}(t)) \quad \text{a.e.} \quad t.
\]

Condition \( (\text{iii}) \) of Theorem 3.1 is therefore satisfied. Condition \( (\text{iv}) \) is straightforward owing to \( (\text{iv})' \). By consequence, the proof of Theorem 3.1 is complete. \( \square \)

**Remark 4.5.** One may expect that the proof of Theorem 3.1 follows directly from results like [12, Theorem 3.2] for the particular case of free right-hand point constraint (eventually after expressing \( \text{(CV)} \) as an optimal control problem and by subsequently considering the augmented problem to cover the case of \( W^{1,1} \)-local minimizers). However, [12, Theorem 3.2] cannot be applied in our paper. Indeed, by inspecting [12], we notice that in order to apply the normality result [12, Theorem 3.2], some regularity assumptions on the data should be satisfied, coupled with constraint qualifications (obviously weaker than the ones proposed in our paper \( \{ \text{CQ1, \ldots, CQ3} \} \), and those discussed in the series of papers [13, 17, 14]). In particular, [12] assumes the following boundedness strong condition on the dynamics (referred as \( \text{(H3)} \) in [12]): for a given \( \delta > 0 \) and a local minimizer \( \bar{x}(\cdot) \),

\[
\text{there exists } C_u \geq 0 \text{ such that } |f(t, x, u)| \leq C_u \text{ for } x \in \bar{x}(t) + \delta \mathbb{B}, \ u \in U(t), \ \text{and } t \in [S, T].
\]

It is clear that this regularity assumption is stronger than the one considered in our paper (cf. inequalities \( (4.1) \) and \( (4.2) \) above), and it cannot be satisfied for our problem \( (P') \). Indeed, the dynamic \( \tilde{f} \) in problem \( (P') \) is the following

\[
\tilde{f}(t, (x(t), z(t)), u(t)) := \begin{pmatrix} u(t) \\ L(t, x(t), u(t)) \end{pmatrix} \quad \text{where } u(t) \in \mathbb{R}^n.
\]

The dynamic \( \tilde{f} \) satisfies assumption \( \text{(H3)} \) of [12] only when \( u(\cdot) \) and \( x(\cdot) \) are bounded (owing to the boundedness of \( L \) on bounded sets by \( \text{(CV)} \)). By contrast, nothing guarantees, under the assumptions considered in our paper, that \( \tilde{f} \) satisfies \( \text{(H3)} \) of [12] for any \( x \) near \( \bar{x}(\cdot) \) and for any \( u \in U(t) \) which eventually has to coincide with the whole space \( \mathbb{R}^n \), to derive the correct necessary conditions (in the normal form) for the class of problems considered in Theorem 3.1.
5 Proof of Theorem 3.2 (Global Minimizers)

In this subsection we give details of a shorter proof based on a simple technique using the neighboring feasible trajectory result with linear estimate (initially introduced in [19]), while regarding the calculus of variations problems as an optimal control problem with final cost. The result we aim to prove is valid for global minimizers and under the stronger constraint qualification

\[
\text{(CQ)} \quad \text{int } T_A(z) \neq \emptyset , \text{ for all } z \in \partial A.
\]

We consider \( \bar{x}(.) \) to be a global minimizer for the calculus of variations problem \( (\text{CV}) \). We now employ the same standard argument (known as state augmentation) previously stated in the first proof technique. This allows to write the problem of calculus of variations \( (\text{CV}) \) as an optimal control problem. Indeed, by adding an extra absolutely continuous state variable

\[ z(t) = \int_{S}^{t} L(s, x(s), \dot{x}(s)) \, ds \]

and by considering the dynamics \( \dot{x} = u \), the problem \( (\text{CV}) \) can be written as the optimal control problem \( (\tilde{P}) \):

\[
(\tilde{P}) \quad \begin{cases}
\text{minimize} & z(T) \\
\text{over } W^{1,1} & \text{arcs } (x(.), z(.)) \text{ satisfying}
\end{cases}
\]

\[ (\dot{x}(t), \dot{z}(t)) \in F(t, x(t)) \quad \text{a.e } t \in [S, T] \]

\[ (x(S), z(S)) = (x_0, 0) \]

\[ (x(t), z(t)) \in A \times \mathbb{R} \quad \text{for all } t \in [S, T] . \]

where

\[ F(t, x) := \{ (u, L(t, x, u)) \mid u \in MB \} \quad (5.1) \]

for any \( M > 0 \) large enough (for instance \( M \geq 2||\bar{u}(.)||_{L^\infty} \)).

It is easy to prove that if \( \bar{x}(.) \) is a global minimizer for \( (\text{CV}) \), then \( (\bar{x}(.), \bar{z}(.) ) \) is a global minimizer for \( (\tilde{P}) \) where \( \bar{z}(t) = \int_{S}^{t} L(s, \bar{x}(s), \dot{x}(s)) \, ds \).

The proof of Theorem 3.2 is given in three steps. In Step 1, we show that the neighboring feasible trajectory theorem in [2 Theorem 2.3] holds true for our velocity set \( F(., .) \) and our constraint qualification \( A \times \mathbb{R} \). In Step 2, we combine a penalization method with the \( L^\infty \) linear estimate provided by [2 Theorem 2.3] which will permit to derive a minimizer for a problem involving a penalty term (in terms of the state constraint) in the cost. Step 3 is devoted to apply a standard Maximum Principle to an auxiliary problem and deduce the assertions of Theorem 3.2 in their normal form.

**Step 1.** In this step, we prove that \( (F(., .), A \times \mathbb{R}) \) satisfies the hypothesis of [2 Theorem 2.3]. Indeed, the set of velocities is nonempty, of closed values owing to the Lipschitz continuity of \( u \to L(t, x, u) \) (see [8 Proposition 2.2.6]), and \( F(., x) \) is Lebesgue measurable for all \( x \in \mathbb{R}^n \). Moreover, the set \( F(t, x) \) is bounded on bounded sets, owing to the boundedness of \( L(., .) \) on bounded set, and to the boundedness of \( \bar{u}(t) \) a.e. \( t \in [S, T] \) (we recall that the minimizer \( \bar{x}(.) \) is assumed to be Lipschitz). The Lipschitz continuity of \( F(t, .) \) and the absolute continuity from the left of \( F(., x) \) is a direct consequence of assumption \( (\text{CV1}) \). Finally, we observe that

\[ F(t, x) \cap \text{int } T_{A \times \mathbb{R}}(x, z) = (MB \cap \text{int } T_A(x)) \times \mathbb{R} . \]
The constraint qualification \( \text{CQ} \) that we suggest
\[
\text{int } T_A(x) \neq \emptyset \quad \text{for all } x \in \partial A ,
\]
and the boundedness of \( \|\bar{u}\|_{L^\infty} = \|\dot{x}\|_{L^\infty} \) guarantee that for \( M > 0 \) chosen large enough
\[
F(t, x) \cap \text{int } T_{A \times \mathbb{R}}(x, z) \neq \emptyset .
\]
Therefore, \([2, \text{Theorem 2.3}]\) is applicable.

**Step 2.** In this step, we combine a penalization technique with the linear estimate given by \([2, \text{Theorem 2.3}]\). We obtain that \((\bar{x}, \bar{z})\) is also a global minimizer for a new optimal control problem, in which an extra penalty term intervenes in the cost:

**Lemma 5.1.** Assume that all hypotheses of Theorem \([3, \text{Theorem 2.3}]\) are satisfied. Then, \((\bar{x}, \bar{z})\) is a global minimizer for the problem:

\[
\begin{aligned}
&\text{minimize} \quad z(T) + K \max_{t \in [S,T]} d_{A \times \mathbb{R}}(x(t), z(t)) =: J(x(\cdot), z(\cdot)) \\
&\text{over arcs} \quad (x, z) \in W^{1,1}([S,T], \mathbb{R}^n \times \mathbb{R}) \text{ satisfying} \\
&(\dot{x}(t), \dot{z}(t)) \in F(t, x(t)) \quad \text{a.e. } t \in [S,T] \\
&(x(S), z(S)) = (x_0, 0), \quad x(S) \in A .
\end{aligned}
\]

Here, \( K \) is the constant provided by \([2, \text{Theorem 2.3}]\).

**Proof.** Suppose that there exists a global minimizer \((\hat{x}(\cdot), \hat{z}(\cdot))\) for \((P_1)\) such that
\[
J(\hat{x}(\cdot), \hat{z}(\cdot)) < J(\bar{x}(\cdot), \bar{z}(\cdot)) .
\]

Denote by \( \hat{\varepsilon} := \max_{t \in [S,T]} d_{A \times \mathbb{R}}(\hat{x}(t), \hat{z}(t)) \), the extent to which the reference trajectory \((\hat{x}(\cdot), \hat{z}(\cdot))\) violates the state constraint \( A \times \mathbb{R} \). By the neighboring feasible trajectory result (with \(L^\infty\)-estimates) \([2, \text{Theorem 2.3}]\), there exists an \( F \)-trajectory \((x(\cdot), z(\cdot))\) and \( K > 0 \) such that

\[
\begin{aligned}
&(x(S), z(S)) = (x_0, 0) \\
&(x(t), z(t)) \in A \times \mathbb{R} \quad \text{for all } t \in [S,T] \\
&\| (x(\cdot), z(\cdot)) - (\hat{x}(\cdot), \hat{z}(\cdot)) \|_{L^\infty(S,T)} \leq K \hat{\varepsilon} .
\end{aligned}
\]

In particular
\[
|z(T) - \hat{z}(T)| \leq K \hat{\varepsilon} .
\]

Therefore,
\[
z(T) \leq \hat{z}(T) + K \hat{\varepsilon} = J(\bar{x}(\cdot), \bar{z}(\cdot)) < J(\hat{x}(\cdot), \hat{z}(\cdot)) = \bar{z}(T) .
\]

But this contradicts the minimality of \((\bar{x}, \bar{z})\) for \((P_1)\). The proof is therefore complete. \(\square\)

A consequence of Lemma 5.1 is the following:

**Lemma 5.2.** \( \bar{X}(\cdot) := (\bar{x}(\cdot), \bar{z}(\cdot)) = \int_S^T L(s, \bar{x}(s), \bar{x}(s)) \, ds, \bar{w}(\cdot) \equiv 0 \) is a global minimizer for

\[
\begin{aligned}
&\text{minimize} \quad g(X(T)) \\
&\text{over} \quad X(\cdot) = (x(\cdot), z(\cdot), w(\cdot)) \in W^{1,1}([S,T], \mathbb{R}^{n+2}) \text{ satisfying} \\
&\dot{X}(t) = (\dot{x}(t), \dot{z}(t), \dot{w}(t)) \in G(t, X(t)) \quad \text{a.e. } t \in [S,T] \\
&h(X(t)) \leq 0 \quad \text{for all } t \in [S,T] \\
&(x(S), z(S), w(S)) \in \{x_0\} \times \{0\} \times \mathbb{R}^+, \quad x(S) \in A .
\end{aligned}
\]
The cost function $g$ is defined by  

$$g(X(T)) := z(T) + Kw(T).$$

The multivalued function is defined by  

$$G(t, X(t)) := \{(u, L(t, x, u), 0) : u \in MB\}$$

and the function $h : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (which provides the state constraint in terms of a functional inequality) is given by:  

$$h(X) = h(x, z, w) := d_A(x) - w$$

**Proof.** By contradiction, suppose that there exists a state trajectory $X(.) := (x(.), z(.), w(.))$ satisfying the state and dynamic constraints of the problem, such that  

$$g(X(T)) < g(\bar{X}(T)).$$

Observe that $w(.) \equiv w \geq 0$, and the state constraint condition is equivalent to  

$$\max_{t \in [S, T]} d_A(x(t)) \leq w.$$  

Then, we would obtain  

$$J(x(.), z(.)) = z(T) + K \max_{t \in [S, T]} d_A(x(t))$$

$$\leq z(T) + Kw$$

$$= g(X(T)) < g(\bar{X}(T)) = J(\bar{x}(.), \bar{z}(.)).$$

This contradicts the fact that $(\bar{x}(.), \bar{z}(.))$ is a global minimizer for $[P1]$.

**Step 3.** In this step we apply known necessary optimality conditions: there exist costate arcs $P(.) = (p_1(.), p_2(.), p_3(.)) \in W^{1,1}([S, T], \mathbb{R}^{n+2})$ associated with the minimizer $(\bar{x}(.), \bar{z}(.), \bar{w} \equiv 0)$, a Lagrange multiplier $\lambda \geq 0$, a Borel measure $\mu(.) : [S, T] \to \mathbb{R}$ and a $\mu$-integrable function $\gamma(.) = (\gamma_1(.), \gamma_2(.), \gamma_3(.))$ such that:

(i) $(\lambda, P(.), \mu(.)) \neq (0, 0, 0),$  

(ii) $-\dot{P}(t) \in \co \partial_{(x, z, w)} \left(Q(t) \cdot (\dot{x}(t), L(t, \bar{x}(t), \dot{x}(t)), 0)\right),$  

(iii) $(P(S), -Q(T)) \in \lambda \partial g(\bar{x}(T), \bar{z}(T), \bar{w} \equiv 0) + (\mathbb{R} \times \mathbb{R} \times \mathbb{R}^-) \times \{\{0\} \times \{0\} \times \{0\})$,  

(iv) $Q(t) \cdot (\dot{x}(t), L(t, \bar{x}(t), \dot{x}(t)), 0) = \max_{u \in MB} Q(t) \cdot (u, L(t, \bar{x}(t), u), 0)$  

(v) $\gamma(t) \in \partial_{(x, z, w)}^+ h(\bar{x}(t), \bar{z}(t), \bar{w} \equiv 0) \quad \mu$-a.e. and $\supp(\mu) \subset \{t : h(\bar{x}(t), \bar{z}(t), \bar{w} \equiv 0) = 0\}$.

Here, $Q(.) : [S, T] \to \mathbb{R}^{n+2}$ is the function  

$$Q(t) := P(t) + \int_{[S, t]} \gamma(s) \, d\mu(s) \quad \text{for } t \in (S, T].$$

Note that by condition $[V]$, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t)) \in \partial_{x}^+ d_A(\bar{x}(t)) \times \{0\} \times \{-1\} \quad \mu$-a.e.

From the conditions above, we derive the following:

(i)' $(\lambda, p_1(.), p_2(.), p_3(.), \mu(.)) \neq (0, 0, 0, 0, 0),$
(ii)’ $-\dot{p}_1(t) \in p_2(t)co \partial_\cdot L(t, \bar{x}(t), \dot{x}(t))$, a.e. $t \in [S, T]$ and $p_2 = \bar{p}_3 \equiv 0$,

(iii)’ $-(p_1(t) + \int_S^T \gamma_1(s) \, d\mu(s)) = 0$, $p_3(S) = \bar{p}_3 \leq 0$, $-(p_3(T) - \int_{[S, T]} d\mu(s)) = \lambda K$ and $p_2(T) = p_2 = -\lambda$,

(iv)’ $(p_1(t) + \int_S^t \gamma_1(s) \, d\mu(s)) \cdot \dot{x}(t) - \lambda L(t, \bar{x}(t), \dot{x}(t)) = \max_{u \in A} \{(p_1(t) + \int_S^t \gamma_1(s) \, d\mu(s)) \cdot u - \lambda L(t, \bar{x}(t), u)\}$

(v)’ $\gamma_1(t) \in \partial^\ast_d A(\bar{x}(t))$ $\mu$-a.e. and $\text{supp}(\mu) \subset \{ t : h(\bar{x}(t), \dot{z}(t), \bar{w} \equiv 0) = 0 \}$

We prove that conditions (ii)’ (v)’ apply in the normal form (i.e. with $\lambda = 1$). Indeed, suppose that $\lambda = 0$, then

$$-p_3 + \int_{[S, T]} d\mu(s) = 0 \quad p_1 = 0 \quad p_2 = 0 \quad p_3 = 0.$$ 

But this contradicts the nontriviality condition (i)’ Therefore, the relations (ii)’ (v)’ apply in the normal form. Moreover, notice that the convexity of $L(t, x, \cdot)$ yields that the maximality condition (iv)’ is verified globally (i.e. for all $u \in \mathbb{R}^n$), and by deriving it w.r.t. $u = \dot{x}$, and making use of the max rule ([20] Theorem 5.5.2) we deduce that

$$p_1(t) + \int_{[S, T]} \gamma_1(s) \, d\mu(s) \in \text{co} \partial_\cdot L(t, \bar{x}(t), \dot{x}(t)) \quad \text{a.e.} \quad t \in [S, T].$$

This permits to conclude the necessary optimality conditions in the normal form of Theorem 3.2.

Acknowledgments: The authors are thankful to the reviewer of the first version of the paper who suggested to use the $L^\infty$—linear distance estimate approach. Moreover, helpful comments and suggestions by Piernicola Bettiol are gratefully acknowledged.

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