QUASI-FREDHOLM AND SAPHAR SPECTRUMS FOR THE 
$\alpha$-TIMES INTEGRATED SEMIGROUPS

A. TAJMOUATI, A. EL BAKKALI, M.B. MOHAMED AHMED AND H. BOUA

Abstract. We continue to study $\alpha$-times integrated semigroups. Essentially, we characterize the different spectrums of $\alpha$-times integrated semigroups by the spectrums of their generators. Particulary quasi-Fredholm, Kato, essentially Kato, Saphar and essentially Saphar spectrums are defined by $\sigma(T)$, where $\sigma(T) = \{\lambda \in \mathbb{C} \mid \lambda - T \text{ is not bijective}\}$ and $\sigma_p(T) = \{\lambda \in \mathbb{C} \mid \lambda - T \text{ is not one to one}\}$. The function resolvent of $T \in \mathcal{B}(X)$ is defined for all $\lambda \in \rho(T)$ by $R(\lambda, T) = (\lambda - T)^{-1}$. An operator $T$ is called Kato, in symbol $T \in D(X)$, if $R(T)$ is closed and $N(T) \subseteq R^\infty(T)$. An operator $T$ is called essentially Kato, in symbol $T \in eD(X)$, if $R(T)$ is closed and $N(T) \subseteq eR^\infty(T)$. An operator $T$ is called relatively regular if there exists $S$ such that $TST = T$.

For the subspaces $M$ and $N$ of $X$ we write $M \subseteq_e N$ if there exists a finite-dimensional subspace $F \subseteq X$ such that $M \subseteq N + F$. We can choose $F$ satisfying $F \subseteq M$ and $F \cap N = \emptyset$. An operator $T$ is called Saphar, in symbol $T \in S(X)$, if $T$ is relatively regular and $N(T) \subseteq eR^\infty(T)$. An operator $T$ is called essentially Saphar, in symbol $T \in eS(X)$, if $T$ is relatively regular and $N(T) \subseteq eR^\infty(T)$. The Kato, essentially Kato, Saphar and essentially Saphar spectrums are defined by

$$
\sigma_K(T) = \{\lambda \in \mathbb{C} \mid \lambda - T \in D(X)\};
\sigma_{eK}(T) = \{\lambda \in \mathbb{C} \mid \lambda - T \in eD(X)\};
\sigma_S(T) = \{\lambda \in \mathbb{C} \mid \lambda - T \in S(X)\};
\sigma_{eS}(T) = \{\lambda \in \mathbb{C} \mid \lambda - T \in eS(X)\};
$$

The degree of stable iteration $\text{dis}(T)$ of an operator $T$ is defined by

$$
\text{dis}(T) = \inf\{n \in \mathbb{N} \mid \forall m \geq n, R(T^n) \cap N(T) = R(T^m) \cap N(T)\}.
$$

An operator $T$ is called quasi-Fredholm, in symbol $T \in q\Phi(X)$, if there exists $d \in \mathbb{N}$ such that $R(T^n)$ and $R(T) + N(T^n)$ are closed for all $n \geq d$ and $\text{dis}(T) = d$. The quasi-Fredholm spectrum is defined by

$$
\sigma_{qe}(T) = \{\lambda \in \mathbb{C} \mid \lambda - T \notin q\Phi(X)\}.$$

2010 Mathematics Subject Classification. 47D62, 47A10.
Key words and phrases. $\alpha$-times integrated semigroup, quasi-Fredholm, Kato, Saphar, essentially Kato and Saphar.
Let $\alpha \geq 0$ and let $A$ be a linear operator on a Banach space $X$. We recall that $A$ is the generator of an $\alpha$-times integrated semigroup $(S(t))_{t \geq 0}$ on $X$ if $\omega, +\infty \subseteq \rho(A)$ for some $\omega \in \mathbb{R}$ and there exists a strongly continuous mapping $S : [0, +\infty[ \to \mathcal{B}(X)$ satisfying
\[
\|S(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0 \text{ and some } M > 0;
\]
\[
R(\lambda, A) = \lambda^\alpha \int_0^{+\infty} e^{-\lambda t} S(t)ds \text{ for all } \lambda > \max\{\omega, 0\},
\]
in this case, $(S(t))_{t \geq 0}$ is called an $\alpha$-times integrated semigroup and the domain of its generator $A$ is defined by
\[
D(A) = \{x \in X / \int_0^t S(s)Axds = S(t)x - \frac{t^\alpha x}{\Gamma(\alpha + 1)}\},
\]
where $\Gamma$ is the Euler integral giving by
\[
\Gamma(\alpha + 1) = \int_0^{+\infty} x^\alpha e^{-x}dx.
\]
We know that $(S(t))_{t \geq 0} \subseteq \mathcal{B}(X)$ is an $\alpha$-times integrated semigroup if and only if
\[
S(t+s) = \frac{1}{\Gamma(\alpha)} \int_0^{t+s} (t+s-r)^{\alpha-1}S(r)xdr - \int_0^t (t+s-r)^{\alpha-1}S(r)xdr
\]
for all $x \in X$ and all $t, s \geq 0$.

In [12], the authors have studied the different spectrums of the 1-times integrated semigroups. In our paper [10], we have studied descent, ascent, Drazin, Fredholm and Browder spectrums of an $\alpha$-times integrated semigroup. Also in [11], we have investigated essential ascent and descent, upper and lower semi-Fredholm and semi-Browder spectrums of an $\alpha$-times integrated semigroup. In this paper, we continue to study the $\alpha$-times integrated semigroups for all $\alpha > 0$. We investigate the relationships between the different spectrums of the $\alpha$-times integrated semigroups and their generators, precisely quasi-Fredholm, Kato, essentially Kato, Saphar and essentially Saphar spectrums.

### 2. Main results

**Lemma 2.1.** [4, Proposition 2.4] Let $A$ be the generator of an $\alpha$-times integrated semigroup $(S(t))_{t \geq 0} \subseteq \mathcal{B}(X)$ where $\alpha \geq 0$. Then for all $x \in D(A)$ and all $t \geq 0$ we have

1. $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$.
2. $S(t)x = \frac{t^\alpha x}{\Gamma(\alpha + 1)} x + \int_0^t S(s)Axds$.

Moreover, for all $x \in X$ we get $\int_0^t S(s)xds \in D(A)$ and
\[
A \int_0^t S(s)xds = S(t)x - \frac{t^\alpha x}{\Gamma(\alpha + 1)} x.
\]

We begin by the lemmas.

**Lemma 2.2.** Let $A$ be the generator of an $\alpha$-times integrated semigroup $(S(t))_{t \geq 0}$ with $\alpha > 0$. Then for all $\lambda \in \mathbb{C}$ and all $t \geq 0$
(1) \((\lambda - A)D_\lambda(t)x = \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)x, \, \forall x \in X\) where

\[D_\lambda(t)x = \int_0^t e^{\lambda(t-r)} S(r)dr;\]

(2) \(D_\lambda(t)(\lambda - A)x = \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)x, \, \forall x \in D(A).\)

Proof. (1) By Lemma 2.1, we know that for all \(x \in D(A)\)

\[S(s)x = \frac{s^\alpha}{\Gamma(\alpha + 1)} x + \int_0^s S(r)Axdr.\]

Then, since \(\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)\), we obtain

\[S'(s)x = \frac{s^{\alpha-1}}{\Gamma(\alpha)} x + S(s)Ax.\]

Therefore, we conclude that

\[D_\lambda(t)Ax = \int_0^t e^{\lambda(t-s)} S(s)Axds\]

\[= \int_0^t e^{\lambda(t-s)} [S'(s)x - \frac{s^{\alpha-1}}{\Gamma(\alpha)} x]ds\]

\[= \int_0^t e^{\lambda(t-s)} S'(s)xd - \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} xds\]

\[= S(t)x + \lambda D_\lambda(t)x - \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} xds\]

Finally, we obtain for all \(x \in D(A)\)

\[D_\lambda(t)(\lambda - A)x = \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) \right]x.\]

(2) Let \(\mu \in \rho(A)\). From proof of Lemma 2.1 we have for all \(x \in X\)

\[R(\mu, A)S(s)x = S(s)R(\mu, A)x.\]

Hence, for all \(x \in X\) we conclude

\[R(\mu, A)D_\lambda(t)x = R(\mu, A) \int_0^t e^{\lambda(t-s)} S(s)xds\]

\[= \int_0^t e^{\lambda(t-s)} R(\mu, A)S(s)xds\]

\[= \int_0^t e^{\lambda(t-s)} S(s)R(\mu, A)xds\]

\[= D_\lambda(t)R(\mu, A)x.\]
Therefore, we obtain for all $x \in X$

$$D_{\lambda}(t)x = \int_0^t e^{\lambda(t-s)} S(s)x ds$$

$$= \int_0^t e^{\lambda(t-s)} S(s)(\mu - A)R(\mu, A)x ds$$

$$= \mu \int_0^t e^{\lambda(t-s)} S(s)R(\mu, A)x ds - \int_0^t e^{\lambda(t-s)} S(s)AR(\mu, A)x ds$$

$$= \mu \int_0^t e^{\lambda(t-s)} R(\mu, A)S(s)x ds - \int_0^t e^{\lambda(t-s)} S(s)AR(\mu, A)x ds$$

$$= \mu R(\mu, A) \int_0^t e^{\lambda(t-s)} S(s)x ds - \int_0^t e^{\lambda(t-s)} S(s)AR(\mu, A)x ds$$

$$= \mu R(\mu, A) D_{\lambda}(t)x - D_{\lambda}(t)AR(\mu, A)x$$

$$= \mu R(\mu, A) D_{\lambda}(t)x - [S(t)R(\mu, A)x + \lambda D_{\lambda}(t)R(\mu, A)x - \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} R(\mu, A) x ds]$$

$$= \mu R(\mu, A) D_{\lambda}(t)x - [R(\mu, A)S(t)x + \lambda R(\mu, A)D_{\lambda}(t)x - R(\mu, A) \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} x ds]$$

$$= R(\mu, A)[(\mu - \lambda) D_{\lambda}(t)x - S(t)x + \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} x ds]$$

Therefore, for all $x \in X$ we have $D_{\lambda}(t)x \in D(A)$ and

$$(\mu - A)D_{\lambda}(t)x = (\mu - \lambda)D_{\lambda}(t)x + \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} x ds - S(t)x.$$ 

Finally, for all $x \in X$ and all $\lambda \in \mathbb{C}$ we obtain

$$(\lambda - A)D_{\lambda}(t)x = \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) \right] x.$$

\[\square\]

**Lemma 2.3.** Let $A$ be the generator of an $\alpha$-times integrated semigroup $(S(t))_{t \geq 0}$ with $\alpha > 0$. Then for all $\lambda \in \mathbb{C}$, all $t \geq 0$ and all $x \in X$

1. We have the identity

$$(\lambda - A)L_{\lambda}(t) + \varphi_{\lambda}(t)D_{\lambda}(t) = \phi_{\lambda}(t)I,$$

where $L_{\lambda}(t) = \int_0^t e^{-\lambda s} D_{\lambda}(s)ds$, $\varphi_{\lambda}(t) = e^{\lambda t}$ and $\phi_{\lambda}(t) = \int_0^t \int_0^s e^{-\lambda r} \frac{s^{\alpha-1}}{\Gamma(\alpha)} dr d\tau$.

Moreover, the operator $L_{\lambda}(t)$ is commute with each one of $D_{\lambda}(t)$ and $(\lambda - A)$.

2. For all $n \in \mathbb{N}^*$, there exists an $L_{\lambda,n}(t) \in \mathcal{B}(X)$ such that

$$(\lambda - A)L_{\lambda,n}(t) + [\varphi_{\lambda}(t)]^n [D_{\lambda}(t)]^n = [\phi_{\lambda}(t)]^n I.$$

Moreover, the operator $L_{\lambda,n}(t)$ is commute with each one of $D_{\lambda}(t)$ and $\lambda - A$.

3. For all $n \in \mathbb{N}^*$, there exists an operator $D_{\lambda,n}(t) \in \mathcal{B}(X)$ such that

$$(\lambda - A)^n [L_{\lambda}(t)]^n + D_{\lambda,n}(t)D_{\lambda}(t) = [\phi_{\lambda}(t)]^n I.$$

Moreover, the operator $D_{\lambda,n}(t)$ is commute with each one of $D_{\lambda}(t)$, $L_{\lambda}(t)$ and $\lambda - A$.
(4) For all \( n \in \mathbb{N}^* \), there exists an operator \( K_{\lambda,n}(t) \in B(X) \) such that

\[
(\lambda - A)^n K_{\lambda,n}(t) + [D_\lambda(t)]^n [D_{\lambda,n}(t)]^n = [\phi_\lambda(t)]^{n^2} I,
\]

Moreover, the operator \( K_{\lambda,n}(t) \) is commute with each one of \( D_\lambda(t), D_{\lambda,n}(t) \) and \( \lambda - A \).

**Proof.**

(1) Let \( \mu \in \rho(A) \). By Lemma 2.2, for all \( x \in X \) we have \( D_\lambda(s)x \in D(A) \) and hence

\[
L_\lambda(t)x = \int_0^t e^{-\lambda s} D_\lambda(s)xds
\]

\[
= \int_0^t e^{-\lambda s} R(\mu,A)(\mu - A)D_\lambda(s)xds
\]

\[
= R(\mu,A)[\mu \int_0^t e^{-\lambda s} D_\lambda(s)xds - \int_0^t e^{-\lambda s} AD_\lambda(s)xds]
\]

\[
= R(\mu,A)[\mu L_\lambda(t)x - \int_0^t e^{-\lambda s} AD_\lambda(s)xds]
\]

Therefore for all \( x \in X \), we have \( L_\lambda(t)x \in D(A) \) and

\[
(\mu - A)L_\lambda(t)x = \mu L_\lambda(t)x - \int_0^t e^{-\lambda s} AD_\lambda(s)xds.
\]

Thus

\[
AL_\lambda(t)x = \int_0^t e^{-\lambda s} AD_\lambda(s)xds.
\]

Hence, we conclude that

\[
(\lambda - A)L_\lambda(t)x = \lambda L_\lambda(t)x - \int_0^t e^{-\lambda s} AD_\lambda(s)xds
\]

\[
= \lambda L_\lambda(t)x - \int_0^t e^{-\lambda s} [AD_\lambda(s)x - \int_0^s e^{\lambda(s-r)} \frac{r^{\alpha-1}}{\Gamma(\alpha)} xdr + S(s)x]ds
\]

\[
= \lambda L_\lambda(t)x - \lambda \int_0^t e^{-\lambda s} D_\lambda(s)xds + \int_0^t e^{-\lambda s} \int_0^s e^{\lambda(s-r)} \frac{r^{\alpha-1}}{\Gamma(\alpha)} xdrds - \int_0^t e^{-\lambda s} S(s)xds
\]

\[
= \lambda L_\lambda(t)x - \lambda L_\lambda(t)x + \int_0^t \int_0^s e^{-\lambda r} \frac{r^{\alpha-1}}{\Gamma(\alpha)} xdrds - e^{-\lambda t} \int_0^t e^{\lambda(t-s)} S(s)xds
\]

\[
= \int_0^t \int_0^s e^{-\lambda r} \frac{r^{\alpha-1}}{\Gamma(\alpha)} xdrds - e^{-\lambda t} D_\lambda(t)x
\]

\[
= [\phi_\lambda(t)I - \varphi_\lambda(t)D_\lambda(t)]x,
\]

where \( \phi_\lambda(t) = \int_0^t \int_0^s e^{-\lambda r} \frac{r^{\alpha-1}}{\Gamma(\alpha)} drds \) and \( \varphi_\lambda(t) = e^{-\lambda t} \).

Therefore, we obtain

\[
(\lambda - A)L_\lambda(t) + \varphi_\lambda(t)D_\lambda(t) = \phi_\lambda(t)I.
\]
Since $S(s)S(t) = S(t)S(s)$ for all $s, t \geq 0$, then $D_\lambda(s)S(t) = S(t)D_\lambda(s)$.
Hence

\[
D_\lambda(t)D_\lambda(s) = \int_0^t e^{\lambda(t-r)}S(r)D_\lambda(s)dr
= \int_0^t e^{\lambda(t-r)}S(r)D_\lambda(s)dr
= \int_0^t e^{\lambda(t-r)}D_\lambda(s)S(r)dr
= D_\lambda(s)\int_0^t e^{\lambda(t-r)}S(r)dr
= D_\lambda(s)D_\lambda(t).
\]

Thus, we deduce that

\[
D_\lambda(t)L_\lambda(t) = D_\lambda(t)\int_0^t e^{-\lambda s}D_\lambda(s)ds
= \int_0^t e^{-\lambda s}D_\lambda(t)D_\lambda(s)ds
= \int_0^t e^{-\lambda s}D_\lambda(s)D_\lambda(t)ds
= \int_0^t e^{-\lambda s}D_\lambda(s)dsD_\lambda(t)
= L_\lambda(t)D_\lambda(t).
\]

Since for all $x \in X$ $AL_\lambda(t)x = \int_0^t e^{-\lambda s}AD_\lambda(s)xds$ and for all $x \in D(A)$ $AD_\lambda(s)x = D_\lambda(s)Ax$, then we obtain for all $x \in D(A)$

\[
(\lambda - A)L_\lambda(t)x = \lambda L_\lambda(t)x - AL_\lambda(t)x
= \lambda L_\lambda(t)x - \int_0^t e^{-\lambda s}AD_\lambda(s)xds
= \lambda L_\lambda(t)x - \int_0^t e^{-\lambda s}AD_\lambda(s)xds
= \lambda L_\lambda(t)x - \int_0^t e^{-\lambda s}D_\lambda(s)Axds
= \lambda L_\lambda(t)x - L_\lambda(t)Ax
= L_\lambda(t)(\lambda - A)x.
\]

(2) Since $(\lambda - A)L_\lambda(t) + \varphi_\lambda(t)D_\lambda(t) = \phi_\lambda(t)I$, then for all $n \in \mathbb{N}^*$ we obtain

\[
[\varphi_\lambda(t)D_\lambda(t)]^n = [\phi_\lambda(t)I - (\lambda - A)L_\lambda(t)]^n
= \sum_{i=0}^n C_n^n [\phi_\lambda(t)]^{n-i}[-(\lambda - A)L_\lambda(t)]^i
= [\phi_\lambda(t)]^n I - (\lambda - A) \sum_{i=1}^n C_n^n [\phi_\lambda(t)]^{n-i}[-(\lambda - A)]^{i-1}[L_\lambda(t)]^i
= [\phi_\lambda(t)]^n I - (\lambda - A)L_\lambda, n(t),
\]
where
\[ L_{\lambda,n}(t) = \sum_{i=1}^{n} C_{n}^{i} [\phi_{\lambda}(t)]^{n-i} [-\lambda A]^{i-1}[L_{\lambda}(t)]^{i}. \]

Therefore, we have
\[ (\lambda - A)L_{\lambda,n}(t) + [\varphi_{\lambda}(t)]^{n}[D_{\lambda}(t)]^{n} = [\phi_{\lambda}(t)]^{n}I. \]

Finally, for commutativity, it is clear that \( L_{\lambda,n}(t) \) commute with each one of \( D_{\lambda}(t) \) and \( \lambda - A \).

(3) For all \( n \in \mathbb{N}^{*} \), we obtain
\[ [(\lambda - A)L_{\lambda}(t)]^{n} = [\phi_{\lambda}(t)I - \varphi_{\lambda}(t)L_{\lambda}(t)]^{n} = \sum_{i=0}^{n} C_{n}^{i} [\phi_{\lambda}(t)]^{n-i}[-\varphi_{\lambda}(t)L_{\lambda}(t)]^{i} \]
\[ = [\phi_{\lambda}(t)]^{n}I - D_{\lambda}(t) \sum_{i=1}^{n} C_{n}^{i} [\phi_{\lambda}(t)]^{n-i}[-\varphi_{\lambda}(t)L_{\lambda}(t)]^{i-1} \]
\[ = [\phi_{\lambda}(t)]^{n}I - D_{\lambda}(t)D_{\lambda,n}(t), \]

where
\[ D_{\lambda,n}(t) = \sum_{i=1}^{n} C_{n}^{i} [\phi_{\lambda}(t)]^{n-i}[-\varphi_{\lambda}(t)L_{\lambda}(t)]^{i-1}. \]

Therefore, we have
\[ (\lambda - A)^{n}[L_{\lambda}(t)]^{n} + D_{\lambda}(t)D_{\lambda,n}(t) = [\phi_{\lambda}(t)]^{n}I. \]

Finally, for commutativity, it is clear that \( D_{\lambda,n}(t) \) commute with each one of \( D_{\lambda}(t), L_{\lambda}(t) \) and \( \lambda - A \).

(4) Since we have \( D_{\lambda}(t)D_{\lambda,n}(t) = [\phi_{\lambda}(t)]^{n}I - (\lambda - A)^{n}[L_{\lambda}(t)]^{n} \), then for all \( n \in \mathbb{N} \)
\[ [D_{\lambda}(t)D_{\lambda,n}(t)]^{n} = \left[ [\phi_{\lambda}(t)]^{n}I - (\lambda - A)^{n}[L_{\lambda}(t)]^{n} \right]^{n} \]
\[ = [\phi_{\lambda}(t)]^{n^{2}}I - \sum_{i=1}^{n} C_{n}^{i} \left[ [\phi_{\lambda}(t)]^{n-i}[(\lambda - A)^{n-i}][L_{\lambda}(t)]^{n-i} \right]^{i} \]
\[ = [\phi_{\lambda}(t)]^{n^{2}}I - (\lambda - A)^{n} \sum_{i=1}^{n} C_{n}^{i} [\phi_{\lambda}(t)]^{n(n-i)(\lambda - A)^{(i-1)}}[L_{\lambda}(t)]^{n^{2}i} \]
\[ = [\phi_{\lambda}(t)]^{n^{2}}I - (\lambda - A)^{n} K_{\lambda,n}(t), \]

where \( K_{\lambda,n}(t) = \sum_{i=1}^{n} C_{n}^{i} [\phi_{\lambda}(t)]^{n(n-i)}(\lambda - A)^{(i-1)}[L_{\lambda}(t)]^{n^{2}i} \). Hence we obtain
\[ [D_{\lambda}(t)]^{n}[D_{\lambda,n}(t)]^{n} + (\lambda - A)^{n} K_{\lambda,n}(t) = [\phi_{\lambda}(t)]^{n^{2}}I. \]

Finally, the commutativity is clear.

Now, we prove this result.

**Proposition 2.1.** Let \( A \) be the generator of an \( \alpha \)-times integrated semigroup \((S(t))_{t \geq 0}\) with \( \alpha > 0 \). For all \( \lambda \in \mathbb{C} \) and all \( t \geq 0 \), if \( R \left[ \int_{0}^{t} e^{\lambda(t-s)} S_{s}^{\alpha-1}ds - S(t) \right]^{n} \) is closed, then \( R(\lambda - A)^{n} \) is also closed.
Proof. Let \((y_n)_{n \in \mathbb{N}} \subseteq X\) such that \(y_n \to y \in X\) and there exists \((x_n)_{n \in \mathbb{N}} \subseteq D(A)\) satisfying
\[
(\lambda - A)^m x_n = y_n.
\]
By Lemma 2.3, we obtain
\[
(\lambda - A)^m [L_\lambda(t)]^m y_n + G_{\lambda,m}(t)D_\lambda y_n = [\phi_\lambda(t)]^m y_n.
\]
Hence, we have
\[
[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^m G_{\lambda,m}(t)x_n = [D_\lambda(t)]^m (\lambda - A)^m G_{\lambda,m}(t)x_n; \\
n = G_{\lambda,m}(t) [D_\lambda(t)]^m (\lambda - A)^m x_n; \\
n = G_{\lambda,m}(t) [D_\lambda(t)]^m y_n; \\
n = [\phi_\lambda(t)]^m y_n - (\lambda - A)^m [L_\lambda(t)]^m y_n.
\]
Then,
\[
[\phi_\lambda(t)]^m y_n - (\lambda - A)L_\lambda(t)y_n \in R \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^m.
\]
Since \(R [\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^m\) is closed, hence \(G_{\lambda,m}(t)\) is bounded linear and \([\phi_\lambda(t)]^m y_n - (\lambda - A)^m [L_\lambda(t)]^m y_n\) converges to \([\phi_\lambda(t)]^m y - (\lambda - A)^m [L_\lambda(t)]^m y\).
Therefore, we conclude that
\[
[\phi_\lambda(t)]^m y - (\lambda - A)^m [L_\lambda(t)]^m y \in R \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^m.
\]
Then, there exists \(z \in X\) such that
\[
[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^m z = [\phi_\lambda(t)]^m y - (\lambda - A)^m [L_\lambda(t)]^m y.
\]
Hence for all \(t \neq 0\), we have \(\phi_\lambda(t) \neq 0\) and
\[
y = \frac{1}{[\phi_\lambda(t)]^m} [\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]^m z + (\lambda - A)^m [L_\lambda(t)]^m y
\]
\[
\]
\[
\]
Finally, we obtain
\[
y \in R(\lambda - A)^m.
\]

\[\Box\]

**Proposition 2.2.** Let \(A\) be the generator of an \(\alpha\)-times integrated semigroup \((S(t))_{t \geq 0}\) with \(\alpha > 0\). Then for all \(\lambda \in \mathbb{C}\) and all \(t \geq 0\), we have

1. If \(\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)\) is relatively regular, then \(\lambda - A\) is also;
2. If \(N[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)] \subseteq R^\infty [\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]\), then \(N(\lambda - A) \subseteq R^\infty (\lambda - A)\).
3. If \(N[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)] \subseteq c R^\infty [\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]\), then \(N(\lambda - A) \subseteq c R^\infty (\lambda - A)\).
Proof. (1) Suppose that 
\[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) | \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) = \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t). \]

Using Lemma 2.3, we obtain 
\[ \phi_\lambda(t)(\lambda - A) = [(\lambda - A)L_\lambda(t) + G_\lambda(t)D_\lambda(t)](\lambda - A); \]
\[ = (\lambda - A)L_\lambda(t)(\lambda - A) + \psi_\lambda(t)[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]; \]
\[ = (\lambda - A)L_\lambda(t)(\lambda - A) + \psi_\lambda(t)[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) | \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]; \]
\[ = (\lambda - A)L_\lambda(t)(\lambda - A) + \psi_\lambda(t)[[\lambda - A]D_\lambda(t) | T(t)[D_\lambda(t)(\lambda - A)]]; \]
\[ = (\lambda - A)[L_\lambda(t) + \psi_\lambda(t)D_\lambda(t)T(t)D_\lambda(t)](\lambda - A). \]

Therefore, \( \lambda - A \) is relatively regular.

(2) It is automatic by 
\[ N(\lambda - A) \subseteq N[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]; \]
\[ \subseteq R^\infty[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]; \]
\[ \subseteq R^\infty(\lambda - A). \]

(3) It is automatic by 
\[ N(\lambda - A) \subseteq N[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]; \]
\[ \subseteq R^\infty[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]; \]
\[ \subseteq R^\infty(\lambda - A). \]

\( \square \)

The following result discusses the Kato and Saphar spectrum.

**Theorem 2.1.** Let \( A \) be the generator of an \( \alpha \)-times integrated semigroup \( \langle S(t) \rangle_{t \geq 0} \) with \( \alpha > 0 \). Then for all \( t \geq 0 \)
\[ (1) \int_0^t e^{(t-s)\sigma_K(A)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \subseteq \sigma_K(S(t)); \]
\[ (2) \int_0^t e^{(t-s)\sigma_S(A)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \subseteq \sigma_S(S(t)); \]
\[ (3) \int_0^t e^{(t-s)\sigma_{\sigma_K(A)}(A)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \subseteq \sigma_{\sigma_K}(S(t)); \]
\[ (4) \int_0^t e^{(t-s)\sigma_{\sigma_S(A)}(A)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \subseteq \sigma_{\sigma_S}(S(t)). \]
Proof. \( (1) \) Suppose that \( \int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \notin \sigma_K(S(t)) \), then we have
\[
R[\int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]
\]
closed and
\[
N\left[\int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)\right] \subseteq R^\infty\left[\int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)\right].
\]
Thus by Propositions 2.1 and 2.2, we obtain \( R(\lambda - A) \) is closed and
\[
N(\lambda - A) \subseteq R^\infty(\lambda - A).
\]
Therefore \( \lambda - A \) is Kato and hence
\[
\lambda \notin \sigma_K(A).
\]
\( (2) \) Suppose that \( \int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \notin \sigma_S(S(t)) \), then we have
\[
\int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)
\]
relatively regular and
\[
N\left[\int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)\right] \subseteq R^\infty\left[\int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)\right].
\]
Thus by Propositions 2.1 and 2.2, we obtain \( \lambda - A \) is relatively regular and
\[
N(\lambda - A) \subseteq R^\infty(\lambda - A).
\]
Therefore \( \lambda - A \) is Saphar and hence
\[
\lambda \notin \sigma_S(A).
\]
\( (3) \) Suppose that \( \int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \notin \sigma_K(S(t)) \), then we have
\[
R[\int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)]
\]
closed and
\[
N\left[\int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)\right] \subseteq R^\infty\left[\int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)\right].
\]
Thus by Propositions 2.1 and 2.2, we obtain \( R(\lambda - A) \) is closed and
\[
N(\lambda - A) \subseteq R^\infty(\lambda - A).
\]
Therefore \( \lambda - A \) is essentially Kato and hence
\[
\lambda \notin \sigma_{eK}(A).
\]
\( (4) \) Suppose that \( \int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \notin \sigma_S(S(t)) \), then we have
\[
\int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)
\]
relatively regular and
\[
N\left[\int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)\right] \subseteq R^\infty\left[\int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)\right].
\]
Thus by Propositions 2.1 and 2.2, we obtain \( \lambda - A \) is relatively regular and
\[
N(\lambda - A) \subseteq R^\infty(\lambda - A).
\]
Therefore \( \lambda - A \) is essentially Saphar and hence
\[
\lambda \notin \sigma_{eS}(A).
\]
\( \square \)

**Proposition 2.3.** Let \( A \) be the generator of an \( \alpha \)-times integrated semigroup \((S(t))_{t \geq 0}\) with \( \alpha > 0 \). Then for all \( \lambda \in \mathbb{C} \) and all \( t \geq 0 \), we have
\[
dis \left[\int_{0}^{t} e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)\right] = n, \text{ then } dis(A - \lambda) \leq n.
\]
Proof. Since \( \text{dis} \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right] = n \), then for all \( m \geq n \), we have
\[
R\left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right] \cap N\left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right] = \]
\[
R\left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right] \cap N\left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right].
\]
Let \( m \geq n \) and \( y \in R(\lambda - A)^m \cap N(\lambda - A) \), then there exists \( x \in X \) such that
\[
y = (\lambda - A)^m x.
\]
Using Lemma 2.3 and since \( y \in N(\lambda - A) \), we obtain
\[
[\phi_\lambda(t)]^m y = [\phi_\lambda(t)]^m y = (\lambda - A)L_{\alpha,m}(t)y + [\varphi_\lambda(t)]^m [D_\lambda(t)]^m y = L_{\alpha,m}(t)(\lambda - A)y + [\varphi_\lambda(t)]^m [D_\lambda(t)]^m (\lambda - A)^m x = [\varphi_\lambda(t)]^m [D_\lambda(t)(\lambda - A)]^m x = [\varphi_\lambda(t)]^m \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right]^m x = \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right]^m [\varphi_\lambda(t)]^m x.
\]
Then
\[
y \in R\left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right]^m.
\]
Hence, since \( y \in N(\lambda - A) \subseteq N\left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right] \), then
\[
y \in R\left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right] \cap N\left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right].
\]
Therefore
\[
y \in R\left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right] \cap N\left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right].
\]
Then there exists \( z \in X \) satisfying
\[
y = \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right]^m z = (\lambda - A)^n [D_\lambda(t)]^n z.
\]
So \( y \in R(\lambda - A)^n \), and therefore
\[
dis(\lambda - A) \leq n.
\]

\[\square\]

**Proposition 2.4.** Let \( A \) be the generator of an \( \alpha \)-times integrated semigroup \( (S(t))_{t \geq 0} \) with \( \alpha > 0 \). For all \( \lambda \in \mathbb{C} \) and all \( t \geq 0 \), if \( R\left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right] + N\left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, ds - S(t) \right] \) is closed, then \( R(\lambda - A) + N(\lambda - A)^n \) is also.

**Proof.** Let \( (y_n)_{n \in \mathbb{N}} \subseteq X \) such that \( y_n \to y \in X \) and there exist \( (x_n)_{n \in \mathbb{N}} \subseteq D(A) \) and \( (z_n)_{n \in \mathbb{N}} \subseteq N(\lambda - A)^m \) satisfying
\[
y_n = (\lambda - A)x_n + z_n.
\]
By Lemma 2.3, we obtain
\[
[D_\lambda(t)]^m y_n = [D_\lambda(t)]^m (\lambda - A)x_n + [D_\lambda(t)]^m z_n;
\]
\[
= \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) \right] [D_\lambda(t)]^m x_n + [D_\lambda(t)]^m z_n;
\]
Since
\[
\left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) \right] [D_\lambda(t)]^m z_n = [D_\lambda(t)]^m [D_\lambda(t)]^m (\lambda - A) z_n = 0,
\]
we conclude that
\[
[D_\lambda(t)]^m y_n \in R \left[ \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) \right] [D_\lambda(t)]^m + N \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) \right]^m \right].
\]
Moreover, we have
\[
R \left[ \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) \right] [D_\lambda(t)]^m + N \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) \right]^m \right]
\]
is closed and \([D_\lambda(t)]^m y_n\) converges to \([D_\lambda(t)]^m y\), then there exist \(x \in X\) and \(z \in N \left[ \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) \right] \right] \) such that
\[
[D_\lambda(t)]^m y = \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) \right] x + z.
\]
Hence, we have
\[
[D_\lambda(t)]^{2m} y = [D_\lambda(t)]^m [D_\lambda(t)]^m y;
\]
\[
= [D_\lambda(t)]^m \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) \right] x + [D_\lambda(t)]^m z,
\]
Therefore, using Lemma 2.3, we obtain
\[
[\phi_\lambda(t)]^{2m} = (\lambda - A) L_{\lambda,2m}(t) y + [\phi_\lambda(t)]^{2m} [\varphi_\lambda(t)]^{2m} [D_\lambda(t)]^{2m} y;
\]
\[
= (\lambda - A) L_{\lambda,2m}(t) y + [\varphi_\lambda(t)]^{2m} \left[ [D_\lambda(t)]^m \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) \right] x + [D_\lambda(t)]^m z \right];
\]
\[
= (\lambda - A) L_{\lambda,2m}(t) y + [\varphi_\lambda(t)]^{2m} [D_\lambda(t)]^m D_\lambda(t) (\lambda - A) x + [D_\lambda(t)]^m z;
\]
\[
= (\lambda - A) [L_{\lambda,2m}(t) y + [\varphi_\lambda(t)]^{2m} [D_\lambda(t)]^m D_\lambda(t) (\lambda - A) x + [D_\lambda(t)]^m z].
\]
Since
\[
(\lambda - A)^m [\varphi_\lambda(t)]^{2m} [D_\lambda(t)]^m z = \varphi_\lambda(t)^{2m} \left[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t) \right] z = 0,
\]
we deduce that
\[
y \in R(\lambda - A) + N(\lambda - A)^m.
\]

The following theorem examines the quasi-Fredholm spectrum.

**Theorem 2.2.** Let \(A\) be the generator of an \(\alpha\)-times integrated semigroup \((S(t))_{t \geq 0}\) with \(\alpha > 0\). Then for all \(t \geq 0\), we have
\[
\int_0^t e^{(t-s)\sigma_{eq}(A)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \subseteq \sigma_{eq}(S(t)).
\]
Proof. Suppose that
\[ \int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \notin \sigma_{qe}(S(t)). \]
Then there exists \( d \in \mathbb{N} \) such that for all \( n \geq d \)
\( R[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)] \) and
\( R[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)] + N[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)] \) are closed and
\[ \text{dis}[\int_0^t e^{\lambda(t-s)} \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds - S(t)] = d. \]
Using Propositions [2.1, 2.3 and 2.4] we obtain for all \( n \geq d \)
\( R[\lambda - A] \) and \( R[\lambda - A] + N[\lambda - A] \) are closed and \( \text{dis}(\lambda - A) \leq d \). Therefore, \( \lambda - A \) is quasi-Fredholm
and hence
\[ \lambda \notin \sigma_{qe}(A). \]
\[ \square \]

References

[1] P. Aiena, Fredholm and Local Spectral Theory with Applications to Multipliers, Kluwer Acad. Press, 2004.
[2] W. Arendt, Vector-valued Laplace Transforms and Cauchy Problems, Israel J. Math, 59 (3) (1987), 327-352.
[3] A. Elgoutri and M. A. Taoudi, Spectral Inclusions and stability results for strongly continuous semigroups, Int. J. of Math. and Mathematical Sciences, 37 (2003), 2379-2387.
[4] M. Hieber, Laplace transforms and \( \alpha \)-times integrated semigroups, Forum Math. 3 (1991), 595-612.
[5] C. Kaiser, Integrated semigroups and linear partial differential equations with delay, J. Math Anal and Appl. 292 (2) (2004), 328-339.
[6] J.J. Koliha and T.D. Tran, The Drazin inverse for closed linear operators and asymptotic convergence of \( C_0 \)-semigroups, J.Oper.Theory. 46 (2001), 323Ű336.
[7] C. Miao Li and W. Quan Zheng, \( \alpha \)-times integrated semigroups: local and global, Studia Mathematica 154 (3) (2003), 243-252.
[8] V. Müller, Spectral theory of linear operators and spectral systems in Banach algebras 2nd edition, Oper.Theo.Adva.Appl, 139 (2007).
[9] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, Springer-Verlag, New York 1983.
[10] A. Tajmouati, A. El Bakkali and M.B. Mohamed Ahmed, Spectral inclusions between \( \alpha \)-times integrated semigroups and their generators, Submitted.
[11] A. Tajmouati, A. El Bakkali and M.B. Mohamed Ahmed, Semi-Fredholm and semi-Browder spectrums for the \( \alpha \)-times integrated semigroups, Submitted.
[12] A. Tajmouati and H. Boua, Spectral theory for integrated semigroups, Inter Journal of Pure and Appl Math, 104 (4) (2016), 847-860.
[13] A.E. Taylor and D.C. Lay, Introduction to Functional Analysis, 2nd ed. New York: John Wiley and Sons, 1980.

A. Tajmouati, M.B. Mohamed Ahmed and H. Boua
Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar Al Mahraz, Fez, Morocco.
E-mail address: abdelaziz.tajmouati@usmba.ac.ma
E-mail address: bbaba2012@gmail.com
E-mail address: hamid12boua@yahoo.com

A. El Bakkali
Department of Mathematics University Chouaïb Doukkali, Faculty of Sciences. 24000, El Jadida, Morocco.
E-mail address: aba0101q@yahoo.fr