ON LIE GROUPS AND THE THEORY OF COMPLEX VARIABLES

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ABSTRACT. In this note we envisage the relation existing between the Lie Groups and the Theory of Complex Variables. In particular, it is shown that the dimensions of the irreducibles representations of $SU(N)$ may be written in terms of the Eisenstein integers and an identity is built up between the imaginary parts of the dimensions of the irreducible representations of the Lie Groups $SU(3)$ and $Sp(4)$.

1. INTRODUCTION

In a work dealing with the classes of binary quadratic forms with complex integral coefficients, G. Eisenstein [D] introduced the numbers $a + b\omega$, where $a$ and $b$ are real integers and $\omega$ is an imaginary cube root of unity.

D. Speiser [S, Fig.7 and Fig. 22] noticed a curious connection between Lie Groups of rank 2 and the theory of complex variables. In particular, he pointed out that in the lattice formed by the dimension function of $SU(3)$, the values are arithmetical means of their closed neighbors, fact which is reminiscent of harmonic functions. Consequently, he proposed the following identity for the dimension $\text{Dim}(p_1, p_2)$ of an irreducible representation $(p_1, p_2)$ of $SU(3)$\footnote{To obtain equation (1.1) use must be made of $z = (p_1 + \frac{a}{p_2}) + ip_2\frac{\sqrt{3}}{2}$.}:

$$\text{Dim}(p_1, p_2) = \frac{1}{2}p_1p_2(p_1 + p_2) = \frac{1}{3\sqrt{3}}\text{Im}z^3. \quad (1.1)$$

In Section 2 we shall clarify the meaning of this equation and subsequently we shall write the irreducible representation of $SU(N)$ in terms of the Eisenstein numbers. In Section 3 an identity is built up between the imaginary parts of the dimensions of the irreducible representations of the Lie Groups $SU(3)$ and $Sp(4)$.

2. EISENSTEIN INTGERS AND THE DIMENSIONS OF THE IRREDUCIBLE REPRESENTATIONS OF THE UNITARY GROUPS

The Eisenstein integers numbers are $a + b\omega$, where $\omega = \frac{-1 + \sqrt{3}i}{2}$ is one of the cubic roots of unity, and the others are 1 and $\omega^2 = \frac{-1 - \sqrt{3}i}{2}$. These numbers form a triangular lattice [CG, Fig 8.10 (a), p. 221] and [C, pp. 2–5].

Lemma 2.1. The dimensions of the irreducible representations $(p_1, p_2)$ of $SU(3)$ may be written through Eisenstein numbers as:

$$\text{Dim}(p_1, p_2) = \frac{\text{Im}(a + b\omega)^3}{3\sqrt{3}}. \quad (2.1)$$
The imaginary part of \((a + b\omega)^3\) is:

\[
\text{Im}(a + b\omega)^3 = 3ab(a\omega + b\omega^2) = i3\sqrt{3}\frac{(a-b)ab}{2}.
\] (2.2)

The number \(N(a, b) = \frac{1}{3\sqrt{3}}\text{Im}(a + b\omega)^3 = \frac{(a-b)ab}{2}\) may be assigned to each Eisenstein lattice point. If we set \(a = p_1 + p_2\) and \(b = p_2\), we recover \(D(p_1, p_2)\).

Long ago H. Weyl [W] obtained the branching law for the groups of linear transformations. Hereafter we shall restrict his result to the case of unitary groups. For briefness we omit the details and we state the reduction

\[
\text{SU}(2) \rightarrow \text{SU}(N-1);
\]

the \text{SU}(N) irreducible representation \((p_1, p_2, \ldots, p_{N-1})\) reduces into \text{SU}(N-1) irreducible representations according to the formula

\[
[(p_1, p_2, \ldots, p_{N-1})] = \sum_{k_{1,1}=1}^{p_1} \sum_{k_{1,2}=1}^{p_2} \cdots \sum_{k_{1,N-1}=1}^{p_{N-1}} (p_1 - k_{1,1} + k_{1,2}, p_2 - k_{1,2} + k_{1,3}, \ldots, p_{N-2} - k_{1,N-2} + k_{1,N-1}).
\] (2.3)

In order to write the dimension of \((p_1, p_2, \ldots, p_{N-1})\) in terms of Eisenstein numbers the reduction chain \text{SU}(N) \rightarrow \text{SU}(N-1) \rightarrow \cdots \rightarrow \text{SU}(3)\) must be considered, i.e. the \text{SU}(3) content of \((p_1, p_2, \ldots, p_{N-1})\) must be displayed. As an example, let us examine the case of an irreducible representation \text{SU}(5). Taking into account the chain \text{SU}(5) \rightarrow \text{SU}(4) \rightarrow \text{SU}(3),\) the dimension of \((p_1, p_2, \ldots, p_{N-1})\) may be expressed as follows:

\[
\text{Dim}(p_1, p_2, p_3, p_4) = \sum_{k_{1,1}=1}^{p_1} \sum_{k_{1,2}=1}^{p_2} \sum_{k_{1,3}=1}^{p_3} \sum_{k_{1,4}=1}^{p_4} \sum_{k_{2,1}=1}^{p_1-k_{1,1}+k_{1,2}} \sum_{k_{2,2}=1}^{p_2-k_{1,2}+k_{1,3}} \sum_{k_{2,3}=1}^{p_3-k_{1,3}+k_{1,4}} \frac{1}{3\sqrt{3}} I_m \left( (a-b-k_{1,1}+k_{1,2}-k_{2,1}+k_{2,2}) + \frac{1}{2}(b-k_{1,2}+k_{1,3}-k_{2,2}+k_{2,3}) + i\sqrt{3}/2(b-k_{1,1}+k_{1,3}-k_{2,2}+k_{2,3}) \right)^3.
\]

Although such identities acquire an involved aspect their underlying structure is transparent. The general result is:
Lemma 2.2. The dimension of the irreducible representation \((p_1, p_2, \ldots, p_{N-1})\) of \(SU(N)\) may be written in terms if the Eisenstein numbers as:

\[
Dim(p_1, p_2, p_3, p_4) = \sum_{k_1=1}^{p_1} \cdots \sum_{k_{N-1}=1}^{p_{N-1}} p_{N-1} - k_{1,1} + k_{1,2}, \quad p_1 - k_{1,1} + k_{1,2} + k_{2,1}, \quad \ldots, \quad p_{N-1} - k_{1,1} + k_{1,2} + k_{2,1} + k_{2,2} + k_{2,3},
\]

where \(p_1, p_2, \ldots, p_{N-1}\) are positive integers.

Remark that, for \(N > 3\), this formula consists of \(\frac{1}{2}(N + 2)(N - 3)\) summations.

3. Concerning an identity on the complex plane

In this section our purpose is to build up an identity between the imaginary parts of the dimensions of the irreducible representations of the Unitary Group \(SU(3)\) and the dimensions of the irreducible representations of the Symplectic Group \(Sp(4)\). These Groups are subgroups of the Unitary Group \(SU(4)\). The Group \(SU(4)\) has rank 3, hence its lattice is 3-dimensional and to each lattice point \((p_1, p_2, p_3)\) corresponds an irreducible representation whose dimension is given by:

\[
Dim(p_1, p_2, p_3) = \frac{1}{2 \sqrt{3}} \pi p_1 p_2 p_3 (p_1 + p_2) (p_2 + p_3) (p_1 + p_2 + p_3)
\]

where \(p_1, p_2, p_3\) are positive integers.

In order to construct such an identity, we shall follow a procedure whose main steps are:

(a) The branching rule for the reduction \(SU(4) \to SU(3)\).

(b) The branching rule for the reduction \(SU(4) \to Sp(4)\) and the geometrical transformation which allows us to express the \(Sp(4)\) lattice point \((q_1, q_2)\) by means of the complex variable \(z'\).

The final move is nothing but an adequate combination of (a) and (b).

For \(N = 4\), Weyl’s branching rule 2.3 may be written as:

\[
[(p_1, p_2, p_3)] = \sum_{a_1=1}^{p_1} \sum_{a_2=1}^{p_2} \sum_{a_3=1}^{p_3} (p_1 - a_1 + a_2, p_2 - a_2 + a_3),
\]

where \((p_1, p_2, p_3)\) corresponds to an irreducible representation of \(SU(4)\) in the Group lattice. The square brackets indicate the \(SU(3)\) content of \((p_1, p_2, p_3)\).

At this point it seems in order to recall that the Symplectic Group \(Sp(4)\) consists of the subset of unitary matrices in 4 dimensions \((4 = 2l)\) which leaves invariant a skew symmetric bilinear form:

\[
\sum_{i=1}^{2} (x_i y_{i+2} - x_{i+2} y_i).
\]
The existence of a non-degenerate skew symmetric form requires an even number of dimensions. Besides, let us remark that the reduction \( SU(4) \to Sp(4) \) has been solved using a geometrical method [I]. This reduction admits three cases which take into account the \( Sp(4) \) lattice diagram symmetry:

\[
p_1 < p_3, \quad [(p_1, p_2, p_3)] = \sum_{\rho=0}^{p_1-1} \sum_{\lambda=0}^{p_2-1} (p_1 + p_3 - 1 - 2\rho, 1 + \rho + \lambda) \tag{3.3}
\]

\[
p_1 = p_3, \quad [(p_1, p_2, p_3)] = \sum_{\rho=0}^{p_1-1} \sum_{\lambda=0}^{p_2-1} (2p_1 - 1 - 2\rho, 1 + \rho + \lambda) \tag{3.4}
\]

(\text{or the same expression with } p_3 \text{ instead of } p_1). \]

\[
p_1 > p_3, \quad [(p_1, p_2, p_3)] = \sum_{\rho=0}^{p_1-1} \sum_{\lambda=0}^{p_2-1} (p_1 + p_3 - 1 - \rho, 1 + \lambda + \rho). \tag{3.5}
\]

A geometric consideration of the 2–dimensional lattice of \( Sp(4) \) is crucial to find a transformation which gives room to an identity between the irreducible representations of \( Sp(4) \) and the imaginary part of a complex expression. To achieve our goal let us envisage the \( Sp(4) \) lattice point \((q_1, q_2)\) in such a manner that the \( q_1 \)-axis coincides with the \( x \)-axis of the complex plane. We get:

\[
z' = \frac{q_1}{2} + i \left( \frac{q_1}{2} + q_2 \right). \tag{3.6}
\]

From this transformation and the dimension formula for the irreducible representations of \( Sp(4) \)

\[
\text{Dim}(q_1, q_2) = \frac{1}{6} q_1 q_2 (q_1 + q_2)(q_1 + 2q_2). \tag{3.7}
\]

we deduce that

\[
\text{Dim}(q_1, q_2) = \text{Im} \frac{z'^4}{6}. \tag{3.8}
\]

Let \([(p_1, p_2, p_3)]_{SU(3)}\) and \([(p_1, p_2, p_3)]_{Sp(4)}\) denote respectively \(3.2\), \(3.3\),\(3.4\) and \(3.5\). We may state the identity on the complex plane in the symbolic form:

**Lemma 3.1.**

\[
\text{Im} \frac{z^3}{3\sqrt{3}} [(p_1, p_2, p_3)]_{SU(3)} = \text{Im} \frac{z'^4}{6} [(p_1, p_2, p_3)]_{Sp(4)}.
\]

where to each resulting term of the decompositions must be applied the corresponding coordinate transformation.

To illustrate Lemma 3.1, let us work out the reduction corresponding to the irreducible representation \((5, 3, 2)\) of dimension 1000 of \( SU(4) \):

\[
\text{Im} \frac{z^3}{3\sqrt{3}} \left( \sum_{a_1=1}^{5} \sum_{a_2=1}^{3} \sum_{a_3=1}^{2} (5 - a_1 + a_2, 3 - a_2 + a_3) \right)^3 = \text{Im} \frac{z'^4}{6} \left( \sum_{\rho=0}^{1} \sum_{\lambda=0}^{2} (6 - 2\rho, 1 + \lambda + \rho) \right)^4.
\]

Through algebraic manipulations, finally, we obtain:
\[
\frac{1}{3\sqrt{3}} \text{Im}\left[\left(\frac{5}{2} + i\frac{\sqrt{3}}{2}\right)^3 + 2\left(\frac{7}{2} + i\frac{\sqrt{3}}{2}\right)^3 + 2\left(\frac{9}{2} + i\frac{\sqrt{3}}{2}\right)^3 + 2\left(\frac{11}{2} + i\frac{\sqrt{3}}{2}\right)^3 + 2\left(\frac{13}{2} + i\frac{\sqrt{3}}{2}\right)^3 + 2\left(\frac{15}{2} + i\frac{\sqrt{3}}{2}\right)^3 \right]
+ \left[\left(3 + i2\sqrt{3}\right)^3 + \left(4 + i2\sqrt{3}\right)^3 + \left(5 + i2\sqrt{3}\right)^3 + \left(6 + i2\sqrt{3}\right)^3 + \left(7 + i2\sqrt{3}\right)^3 \right]
+ \left[\left(3 + i\sqrt{3}\right)^3 + 2\left(4 + i\sqrt{3}\right)^3 + 2\left(5 + i\sqrt{3}\right)^3 + 2\left(6 + i\sqrt{3}\right)^3 + 2\left(7 + i\sqrt{3}\right)^3 \right]
+ \left[\left(8 + i\sqrt{3}\right)^3 + \left(\frac{7}{2} + i\frac{\sqrt{3}}{2}\right)^3 + \left(\frac{9}{2} + i\frac{\sqrt{3}}{2}\right)^3 + \left(\frac{11}{2} + i\frac{\sqrt{3}}{2}\right)^3 + \left(\frac{13}{2} + i\frac{\sqrt{3}}{2}\right)^3 \right]
+ \left(\frac{15}{2} + i\frac{\sqrt{3}}{2}\right)^3 \right] = \frac{\text{Im}}{6} (\left((4i - 3)^4 + (5i - 3)^4 + (6i - 3)^4 + (4i - 2)^4 + (5i - 2)^4 + (6i - 2)^4 \right)).
\]

Dimensional verification:

\[
\text{Dim}[(5, 3, 2)]_{SU(3)} = [6 + 30 + 54 + 84 + 120 + 81] + [10 + 24 + 42 + 64 + 90] + [8 + 30 + 48 + 70 + 96 + 63] + [6 + 10 + 15 + 21 + 28].
\]

\[
\text{Dim}[(5, 3, 2)]_{Sp(4)} = [56 + 160 + 324 + 64 + 140 + 256].
\]

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