Gauge Fields Emerging From Extra Dimensions
— a Born-Oppenheimer approach —

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Abstract

We propose a dynamical mechanism to induce gauge fields in four dimensional space-time from a single scalar field or a spinor field in higher dimensions. The Born-Oppenheimer treatment of the extra dimensions is an essential ingredient in our approach. A possible applications of the idea to low dimensional condensed matter systems and high temperature field theory are also pointed out. This paper is an extended version of our previous unpublished work (SUNY-NTG-89-48, Jan. 1990).
1. Introduction. One of the ultimate goals in physics is to understand the fundamental law of interactions acting between elementary particles. Presently, it is generally believed that the gauge theory, which is based on the principle of the local gauge invariance, provides a fundamental form of interactions. This strong requirement settles the form of interactions between matter and field in almost unique way.

However, it seems that there is still a room to look for the origin of the gauge symmetry without imposing it from the outset. A possible way is to extract the symmetry a posteriori as a result of some simple dynamical mechanism. In what follows, we shall develop a theory along with this thought. The basic idea has been already reported in our unpublished/unsubmitted work [1], and the present paper is its extended version. We will derive a massless gauge field only through the scalar field or the Dirac field: The central point is that the gauge field emerges as an effective field as a result of the smearing out the degrees of freedom associated with the “extra dimensions”. The idea may somewhat share the spirit with Kaluza-Klein theories [2]. However, the gauge field is induced dynamically in our case, while it is already assumed from the outset in the Kaluza-Klein theories [3]. The induced gauge field in our approach has formal analogy with that obtained by the Born-Oppenheimer approximation in quantum mechanical systems such as the diatomic molecules [4], which we will discuss more.

Let us start with a complex scalar field living in the (4+N)-dimensional space-time labeled by $x^M = (x^\mu, \theta^i)$, ($\mu = 0 \sim 3$, $i = 5 \sim 4 + N$) with the metric $g^{MM'} = \text{diag.}(1, -1, \cdot \cdot \cdot, -1)$. $x^\mu$ and $\theta^i$ denote the 4-dim. space-time coordinate and the extra N-dimensional space coordinate, respectively. Assume that the lagragian of the system has a simple form:

$$\mathcal{L}(x, \theta) = -\Phi^\dagger \partial_M \partial^M \Phi - [c_1 \Phi^\dagger \Phi + c_2 (\Phi^\dagger \Phi)^2], \quad (1)$$

and we will consider the case where $c_{1,2} \geq 0$ for simplicity. By using the standard Hubbard-Stratonovich transformation (the gaussian trick in the path integral) [5], (1) is made equivalent with

$$\mathcal{L}(x, \theta) = -\Phi^\dagger [\partial_x^2 + \partial_\theta^2 + c_1 + \sigma(x, \theta)] \Phi + \frac{1}{4c_2} \sigma(x, \theta)^2, \quad (2)$$
where $\sigma(x, \theta)$ is an independent auxiliary-field to be integrated out in the path integral together with $\Phi$.

As far as the internal space is compactified, the following discussions do not depend on its topology nor the dimensinality $N$. Therefore, we will take $N = 1$ and adopt a circle $S^1$ for the internal space in this section to demonstrate the essential idea. In this choice, $\Phi(x, \theta)$ obeys the boundary condition

$$\Phi(x^\mu, \theta + 2\pi R) = \Phi(x^\mu, \theta),$$

where $R$ is the “radius” of the internal space and has a size of $O(1/(\text{Plank mass}))$. We will assume that $R$ is independent of $x^\mu$ for simplicity unless otherwise is stated; this assumption is not essential for the following discussions.

Let us adopt here the Born-Oppenheimer (BO) approach and assume that the fluctuation of $\Phi$ by the change of $x^\mu (\theta)$ is “slow” (“fast”). This corresponds to neglecting $\partial_x$ compared to $\partial_\theta$ when one solves the dynamics in the internal space. This assumption is justified if the size of the internal space is small enough compared to the typical scale in the 4-dim. world. In the BO approximation, $\Phi$ at each space-time point $x^\mu$ is expanded as

$$\Phi(x, \theta) = \sum_n \phi_n(x) f_n(\theta; x).$$

Here $f_n(\theta; x)$ is a complete set of functions in the internal space and is a solution of the BO equation where $\partial_x$ is neglected by definition:

$$H_{BO} f_n(\theta; x) = \lambda_n(x) f_n(\theta; x) \quad \text{with} \quad H_{BO} = \partial_\theta^2 + c_1 + \sigma(x, \theta),$$

where $\lambda_n(x)$ is an eigenvalue at each point $x^\mu$. Note that $x$ and $\theta$ do not enter separately in $f_n$, because $\sigma$ is a function of both variables. $f_n$ satisfies the normalization and completeness relations

$$\int d\theta f_n^*(\theta; x) f_m(\theta; x) = \delta_{nm}, \quad \sum_n f_n^*(\theta; x) f_m(\theta'; x) = \delta(\theta - \theta').$$

The effective Lagrangian $\hat{L}$ in the ordinary space-time is defined as
\[
\hat{\mathcal{L}}(x) = \int d\theta \ L(x, \theta) \ .
\]  

(7)

Substituting the decomposition (4) into (1) and using (6), one obtains

\[
\hat{\mathcal{L}} = \left[ (\partial^\mu - iA^\mu_{nm})\phi_n \right]^\dagger (\partial_\mu - iA^{\dagger}_m)\phi_m - \int d\theta \ [\lambda_n \phi_n \phi_n - \frac{1}{4c^2} \sigma^2(x, \theta)],
\]

(8)

with

\[
A^\mu_{nm}(x) = i \int d\theta \ f_n^*(\theta; x) \partial^\mu f_m(\theta; x),
\]

(9)

and \(\partial^\mu = \partial/\partial x_\mu\). Because of the decomposition (4), \(\hat{\mathcal{L}}\) in (8) has an obvious local U(1) gauge invariance, namely,

\[
\phi_n \to \phi_n e^{i\alpha}, \quad f_n \to f_n e^{-i\alpha} \quad \text{or} \quad A^\mu_{nm} \to A^\mu_{nm} + \delta_{nm} \partial^\mu \alpha.
\]

(10)

Note here that \(\sigma\) is gauge-singlet by definition. From the above transformation property, \(A^\mu_{nm}\) can be identified with a U(1) gauge field induced by the existence of the internal space.

There are several ways to the non-abelian generalization of the above result. Let us present two possible examples [7]. (i) If there exists \(k\)-fold degeneracy for particular mode \(n\) in the BO equation (3), the rotation among the degenerate modes with the same eigenvalue \(\lambda_n\) induces \(U(k)\) gauge symmetry and associated \(U(k)\) Yang-Mills (YM) field \(A^{ab}_\mu(x)\) with \(a, b = 1 \sim k\) (\(n, m\) indices are suppressed here.) This is easily seen by the decomposition with an explicit label \(a\); \(\Phi = f^{(a)}(\theta; x)\phi^{(a)}(x)\) (\(a = 1 \sim k\) and the label \(n\) is not shown explicitly here). The local gauge invariance reads \(\phi \to U \phi, f \to fU^\dagger\) where \(U\) acts on the suffix \(a\). (ii) Suppose we introduce a scalar field \(\Phi^{ab}(x, \theta)\) \((a, b = 1 \sim k)\) transforming as \(U_{\text{global}}^{\dagger}\Phi U_{\text{global}}\) under the \(U(k)\) global rotation. Then, the BO decomposition for the \(k \times k\) matrix \(\Phi^{ab} = \sum_n \sum_c f^{ac}_n(\theta; x)\phi^{cb}_n(x)\) naturally induces a local \(U(k)\) gauge invariance \(\phi \to U_{\text{local}} \phi, f_n \to f_n U_{\text{local}}^\dagger\) where \(U_{\text{local}}\) acts on the suffix \(c\). Associated \(U(k)\) YM field transforms \(A^\mu \to U_{\text{local}}(A^\mu + i\partial^\mu)U_{\text{local}}^\dagger\). In the latter example, one extra-dimension is enough to accomodate the YM field, which is in contrast to the Kaluza-Klein theories where higher extra-dimensions are necessary [3].
Among a series of the scalar fields $\phi_n(x)$, the one corresponding to the lowest eigenvalue $\lambda_0$ will survive at low energies. Thus, retaining only $n = m = 0$, one arrives at a simply effective lagrangian

$$\hat{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - \int d\theta [\lambda \phi^\dagger \phi - \frac{1}{4\epsilon_2} \sigma^2],$$

where $D_\mu \equiv \partial_\mu - iA_\mu$ and we have suppressed the label $n = m = 0$. The functional integral over $\sigma$ should still be carried out to get a true low energy action, which will be discussed in the next section.

A comparison of our approach with the BO treatment of the diatoms is worth mentioning here. Our field variables $f_n$ and $\phi$ correspond to the electronic wave function and the nuclear wave function for the diatomic molecule, respectively. In the BO approximation of the diatom, the electronic wave function is solved as if the nuclear coordinate $X$ is an external parameter, which induces the “gauge field” in the effective hamiltonian for the nuclear motion [4]. This is quite analogous to our BO treatment of the internal space. A slight difference is that the object we are concerned with is the field variables, while it is the wave functions in the case of diatoms.

3. Non-triviality of the induced gauge field. Up to now, we have neither given an explicit form of $f_n(\theta; x)$ nor discussed whether the induced $A_\mu$ has nontrivial field strength $F_{\mu\nu}$. To answer these questions, let us go back to the BO equation (5). If $\sigma$ is zero (i.e. $c_2 = 0$), $f_n$ is readily solved as

$$f_n^{(0)}(\theta; x) = \frac{1}{\sqrt{2\pi R}} e^{in/R} \quad \text{with} \quad \lambda_n^{(0)} = c_1 + \left(\frac{n}{R}\right)^2.$$  

From this, one observes the following facts: (i) If $R$ is not a function of $x$, the induced gauge field $A_\mu$ is trivially zero and so does the field strength $F_{\mu\nu}$. (ii) Even if $R$ is assumed to be $x$ dependent quantity, $F_{\mu\nu}$ is still zero since $A_\mu$ can be written as a total divergence.

When $c_2 \neq 0$, the gauge field becomes non-trivial even if $R$ is $x$ independent. This can be seen explicitly by calculating $f_n$ in the first order perturbation with respect to $\sigma$;
where we have used abbreviated notations such as \( f_n^{(0)} = |n\rangle \) and \( \langle l | \sigma | n\rangle = \int d\theta f_l^{(0)*} \sigma(x, \theta) f_n^{(0)} \). The induced gauge field corresponding to (13) becomes e.g.

\[
A_{\mu n}(x) = -i \sum_l \frac{\langle n | \sigma | l\rangle \langle l | \partial_\mu \sigma | n\rangle}{(\lambda_n^{(0)} - \lambda_l^{(0)})^2},
\]

which produces \( F_{\mu\nu} \neq 0 \).

So far, \( A_\mu \) is given as an implicit function of the “background” field \( \sigma(x, \theta) \). To treat the induced gauge field as a dynamical one, we have to make a functional integration over \( \sigma \).

For this purpose, let us insert an unity \( 1 = \int [dB_\mu(x)] \delta(B_\mu - A_\mu) \) into the partition function with the lagrangian (11), which results in

\[
Z = \int [dB_\mu] [d\phi d\phi^*] e^{i \int d^4x (D_\mu \phi)^2 \int [d\sigma] \delta(B_\mu - A_\mu) e^{i \int d^4x d\theta (-\lambda^\dagger \phi + \frac{1}{4c_2} \sigma^2)}}.
\]

Here \( D_\mu \) is redefined as \( \partial_\mu - iB_\mu \). (13) has a gauge symmetry with simultaneous changes \( \phi \rightarrow \exp[i\alpha] \phi, A_\mu \rightarrow A_\mu + \partial_\mu \alpha \) and \( B_\mu \rightarrow B_\mu + \partial_\mu \alpha \). After the \( \sigma \)-integration, \( A_\mu \) disappears from \( Z \), thus the gauge symmetry carried by \( \phi \) and \( B_\mu \) remains as a remnant of the original symmetry carried by \( \phi \) and \( A_\mu \) in (10). This implies that only the gauge invariant terms composed of \( \phi \) and \( B_\mu \) appears in \( Z \) after the \( \sigma \)-integration, i.e.

\[
Z = \int [dB_\mu] [d\phi d\phi^*] e^{i \int d^4x ((D_\mu \phi)^2 - c_3 \phi^\dagger \phi - c_4 F_{\mu\nu}^2 + \cdots)},
\]

with constants \( c_3, c_4, \ldots \) and \( F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \). The exponent in (16) is the final low energy effective action where \( \phi \) and \( B_\mu \) are the independent fields. The kinetic term of \( B_\mu \) is induced by the quantum fluctuation of \( \sigma \sim \Phi^\dagger \Phi \). The situation is similar to that in the \( \text{CP}^n \) model \[8\] where the kinetic term of the composite gauge field is induced dynamically. The analogy can be seen more closely by using \( \delta(B_\mu - A_\mu) = \int [dC_\mu] \exp(iC_\mu(B_\mu - A_\mu)) \) in (15). A major difference from the \( \text{CP}^n \) model is that our composite gauge field \( A_\mu \) is quite an implicit function of \( \sigma \), while the gauge field in the \( \text{CP}^n \) model is simply proportional to \( z^\dagger \partial_\mu z \) with \( z \) being the original \( \text{CP}^n \) field. This make the evaluation of the coefficients \( c_3, c_4, \ldots \) more involved and we will not pursue it here except for the general consideration.
4. Gauge field induced by a spinor. We now turn to the gauge field induced by a Dirac field. Since most of the manipulations are the same with the boson case, we will just show the outline of the derivation here. Let’s start with a massless Dirac field in 4+N dimensions with vector-type self interaction as an example;

\[ L = \bar{\Psi} i\Gamma^M \partial_M \Psi - c_2(\bar{\Psi} \Gamma_M \Psi)^2, \]  

where \( \Gamma_M \) is the 4+N dimensional gamma matrices satisfying \( \{ \Gamma^M, \Gamma^{M'} \} = 2g_{MM'} \). After the Hubbard-Stratonovich transformation, the equivalent lagrangian reads,

\[ L = \bar{\Psi} i\Gamma^M (\partial_M - i\omega_M) \Psi - \frac{1}{4c_2} \omega_M^2(x, \theta). \]  

\( \omega_\mu \) here is an auxiliary field and has nothing to do with the induced gauge field we are looking for.

Since \( \tilde{\Gamma}^M \equiv i\Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \cdot \Gamma^M \) has a property \( \{ \tilde{\Gamma}^\mu, \tilde{\Gamma}^\nu \} = 4g^{\mu\nu}, \{ \tilde{\Gamma}^i, \tilde{\Gamma}^j \} = 4g^{ij}, [\tilde{\Gamma}^\mu, \tilde{\Gamma}^i] = 0 \) with \( \mu, \nu = 0 \sim 3 \) and \( i, j = 5 \sim 4 + N \), one can choose a product representation as usual, \( \tilde{\Gamma}^\mu = \gamma_\mu^{(4)} \otimes I^{(N)} \) and \( \tilde{\Gamma}^i = I^{(4)} \otimes \gamma_i^{(N)} \) where \( \gamma^{(L)} \) and \( I^{(L)} \) are the Dirac matrices and unit matrix in \( L \)-dimensions, respectively. The Born-Oppenheimer equation corresponding to (5) reads

\[ i\gamma^i(\partial_i - i\omega_i)f_n(x_\mu, \theta) = \lambda_n f_n(x_\mu, \theta), \]  

with \( \Psi = \psi_n(x) \otimes f_n(\theta; x) \), and the effective lagrangian is written as

\[ \hat{L} = \bar{\psi}_n i\gamma^\mu[\delta_{nm}(\partial_\mu - i\omega_\mu) - iA^{nm}_\mu]\psi_m - \int d\theta (\lambda_n \bar{\Psi}_n \Psi - \omega^2_\mu), \]  

with

\[ A^{nm}_\mu(x) = i \int d\theta f_n^\dagger \partial_\mu f_m. \]  

The local gauge invariance \( \psi_n \rightarrow \exp(i\alpha) \psi_n, A^{nm}_\mu \rightarrow A^{nm}_\mu + \delta_{nm} \partial_\mu \alpha \) is obvious in the above formula. Final low energy effective lagrangian \( L_{eff} \) is obtained by taking \( n = m = 0 \) and integrating out \( \omega_\mu \);
\[
\mathcal{L}_{\text{eff}} = \bar{\psi} \gamma^{\mu} i \gamma_\mu D^\mu \psi - c_3 \bar{\psi} \psi - c_4 F_{\mu\nu}^2 + \cdots,
\]

(22)

with \( D_{\mu} \equiv \partial_{\mu} - i B_{\mu} \) and \( F_{\mu\nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu} \).

5. Further directions.

(I) We have so far concentrated on the vector type gauge fields. However, by generalizing the local field \( \Psi(x, \theta) \) to the string field \( \Psi(x(\sigma), \theta) \) or the membrane field \( \Psi(x(\sigma_1, \sigma_2), \theta) \), one may induce antisymmetric tensor fields such as \( A_{\mu\nu} \) and \( A_{\mu\nu\lambda} \) which are the gauge fields associated with the motion of the string or the membrane as is discussed in ref. [9].

(II) Field theories at high temperature \( (T) \) is a possible place to apply the idea in this paper. In fact, in the imaginary-time formulation of the finite \( T \) field theories, the size of the compactified “time” direction is \( 1/T \). Thus, at high \( T \), emergence of an induced gauge field is expected after eliminating the modes in the time-direction through the BO procedure. This may provide us with a new method to construct high \( T \) effective field theories [10].

(III) Application of our idea to the low-dimensional condensed matter system is also interesting. Instead of introducing the hypothetical extra dimensions, let us imagine to curl up a two-dimensional sheet (on which electrons are living) to a tube. If the diameter of the tube is small enough, the system can be treated almost like a one-dimensional system “except” for the gauge field induced by the curling-up procedure. This gauge field is a remnant of the wave functions of the electrons moving in the compactified dimension, and it will affect the dynamics of the electrons moving along the uncompactified direction. The carbon nanometer tube, which was recently discovered in the laboratory [11], may be used to study such a novel phenomenon. Further works along these lines line are under investigation and will be reported elsewhere [12].

6. Summary. In summary, we proposed a dynamical mechanism to induce the spin-one gauge fields from the scalar or spinor fields in higher dimensions. The Born-Oppenheimer
treatment for the compact internal space naturally induces gauge fields, which is analogous to the similar phenomena in quantum mechanical systems. The idea could possibly be applied also to the low dimensional electron systems and the high $T$ field theories.

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