Which Causality?
Differences between Trajectory and Copenhagen Analyses of an Impulsive Perturbation

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Abstract

The trajectory and Copenhagen representations render different predictions for impulsive perturbations. The different predictions are due to the different roles that causality plays in the trajectory and Copenhagen interpretations. We investigate a small perturbing impulse acting on the ground state of an infinitely deep square well. For the two representations, the first-order perturbation calculations for the temporal change in energy differ. This temporal change in energy for the trajectory representation is dependent upon the microstate of the wave function. We show that even under Copenhagen epistemology, the two representations predict different theoretical results.

PACS Numbers: 3.65.Bz; 3.65.Ca

Key words: causality, trajectory interpretation, Copenhagen interpretation, quantum epistemology.

Suggested short running title: Which Causality?

1 INTRODUCTION

The trajectory interpretation has strong causality by precept. Bohr, the chief founder of the Copenhagen interpretation, held that causality had to be dropped for a consistent quantum theory.\(^1\) Born, another founder of the Copenhagen interpretation, asserted “in the quantum theory it is the principle of causality, or more accurately that of determination, which must be dropped” (emphases in original).\(^2\) More recent champions of the Copenhagen interpretation have construed “causality” to be the temporal evolution of the Schrödinger wave function with time.\(^3,4\) (We shall use herein this evolved usage of “causality” with regard to the Copenhagen interpretation.) This causality for the Copenhagen interpretation is a weak causality. The different roles that causality plays in the trajectory and Copenhagen interpretations are fundamental and manifest different theoretical predictions.

The Copenhagen interpretation of quantum mechanics has been consistent with observations of large ensembles. But it only renders a probability of outcomes for individual events. Heretofore, the mainstream of physics has considered strong causal theories of quantum mechanics to be only alternative representations that rendered nothing new and, likewise, only predicted probabilities for outcomes. For example, Bohm’s stochastic causal theory of quantum mechanics, which purports to be consistent with the probability amplitude of the wave function, introduces chance by precept.\(^5\)

On the other hand, the trajectory representation is not based upon chance.\(^6\)–\(^8\) The trajectory itself is strongly causal and deterministic. The trajectory representation has shown that the Born postulate of the Copenhagen interpretation, which attributes a probability amplitude to the Schrödinger wave function, is
unnecessary. The trajectory representation has shown that the Schrödinger wave function of the Copenhagen interpretation is not an exhaustive description of quantum phenomenon because the trajectory representation distinguishes microstates, each specifying a distinct trajectory, of the wave function. Each trajectory or microstate is sufficient by itself to specify the Schrödinger wave function. We need not invoke an ensemble of trajectories to get the Schrödinger wave function. The trajectory representation renders deterministic predictions for an individual particle viza-viz the probability predictions of the Copenhagen representation for an ensemble of particles.

Herein, we investigate a situation where the type of causality distinguishes the trajectory interpretation from the Copenhagen interpretation. We investigate a perturbing impulse acting on a particle in the ground state of an infinitely deep square well. First-order perturbation calculations for the trajectory and Copenhagen interpretations predict different results for the change in energy. A perturbing impulse acts on the particle wherever it is in its trajectory at the instant of impulse. In the Copenhagen interpretation, the perturbing impulse acts upon the Schrödinger wave function as it is described at the instant of impulse. There exists a quantum operator, the Hamiltonian, for making the Copenhagen prediction regarding energy. We recognize that the Copenhagen interpretation epistemologically restricts in principle what can be measured. We structure this investigation to address those concerns of the Copenhagen school. While the trajectory is completely determined by a necessary and sufficient set of initial conditions, we may relax our knowledge of initial conditions to satisfy Copenhagen epistemology and still show different predictions. This renders a counter example where a strong causal theory, the trajectory representation, can predict something new and different from the Copenhagen interpretation even with Copenhagen epistemology. An investigation of the particle’s energy during perturbation is sufficient to establish this counter example.

We study a perturbing potential that has an impulse in time described by a δ-function and that is spatial symmetric, linear and limited to a small but finite domain of the square well. This perturbing potential has been chosen for five reasons. First, it accentuates the difference in the temporal behavior of energy for the two representations. Second, first-order perturbation theory suffices for exhibiting differences between the two representations. Third, the physics of how this perturbation temporally effects energy is easily understood. Fourth, we can spatially confine the perturbation to the neighborhood of the nodes of the ground state (at the well’s edge) of the infinitely deep square well while we concurrently confine the perturbation to the neighborhood of nodes of latent excited states. This ensures that all perturbation matrix elements will be small. And fifth, it is mathematically tractable as our computations use algebraic expressions or elementary functions. Where necessary, approximations can be done by reasonable algebraic expressions that do not lose any physics.

We make it explicit that the trajectory representation considered herein differs with pilot-wave representations. They have different equations of motion. The trajectory representation does not invoke a ψ-field. The equations of motion for the trajectory representation are determined by the Hamilton-Jacobi transformation equation while Bohmian mechanics assumes that the conjugate momentum should be the mechanical momentum. This leads to different results that have been discussed elsewhere. For a bound state of the square well, Bohmian mechanics predicts that the bound particle always stands still. We also note that the trajectories consider here are different from the trajectories for the Feynman representation. The Feynman representation uses a classical generator of the motion as its propagator while the trajectory representation uses a generalized Hamilton’s characteristic function as its generator of the motion.

The trajectory representation and the Schrödinger representation without the Copenhagen interpretation mutually imply each other. The Born postulate of the Copenhagen interpretation assigns a probability amplitude to the Schrödinger wave function. Lest we forget, Schrödinger opposed the Copenhagen interpretation of his wave function. Nothing herein implies that time-dependent perturbation theory for a trajectory representation is easier than that for contemporary wave mechanics. It is much more difficult. It is even more difficult in general than classical canonical perturbation theory. But then, we ask so much more from trajectory perturbations.

The rest of this paper is organized as follows. Section 2 describes the trajectory representation of a particle in an infinitely deep square well in the unperturbed state. Section 3 addresses the differences between the time-dependent perturbation of energy for the two representations. Section 4 describes how the investigation...
is structured to comply with Copenhagen epistemology. A brief discussion is given in Section 5 regarding our findings for ensemble averages and any subsequent experimental verification.

2 BEFORE PERTURBATION

Unperturbed System: Let us consider initially that a particle is in the ground state of an infinitely deep square well whose (unperturbed) potential is given by

\[ V(x) = \begin{cases} \infty, & |x| > q \\ 0, & |x| \leq q \end{cases} \]  

where \( q \) is finite positive.

The trajectory representation is based upon a generalized Hamilton-Jacobi formulation. The time-independent generalized Hamilton-Jacobi equation for quantum mechanics for a particle is given for one-dimensional motion in the \( x \)-direction by

\[ \frac{(\partial W/\partial x)^2}{2m} + V - E = -\frac{\hbar^2}{4m} (W; x) \]  

where \( \partial W/\partial x \) is conjugate momentum, \( E \) is the energy for the particle, \( m \) is mass of the particle, and \( \hbar = h/(2\pi) \) where in turn \( h \) is Planck’s constant. The term \( (W; x) \) in Eq. (2) is the Schwarzian derivative of Hamilton’s characteristic function, \( W \), with respect to \( x \). The Schwarzian derivative is given by

\[ \langle W; x \rangle = \left[ \frac{\partial^3 W/\partial x^3}{\partial W/\partial x} - \frac{3}{2} \left( \frac{\partial^2 W/\partial x^2}{\partial W/\partial x} \right)^2 \right] . \]

The left side of Eq. (2) manifests the classical Hamilton-Jacobi equation while the Schwarzian derivative on the right side manifests the higher order quantum effects. The general solution for \( \partial W/\partial x \) is given by

\[ \partial W/\partial x = \pm (2m)^{1/2} (a\phi^2 + b\theta^2 + c\phi\theta)^{-1} \]  

where \( (a, b, c) \) is a set of real coefficients such that \( a, b > 0 \), and \( (\phi, \theta) \) is a set of normalized independent solutions of the associated time-independent one-dimensional Schrödinger equation,

\[ -\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + (V - E)\psi = 0 \]

where \( \psi \) is the Schrödinger wave function. The independent solutions \( (\phi, \theta) \) are normalized so that their Wronskian, \( W(\phi, \theta) = \phi \frac{d\theta}{dx} - d\phi/dx \theta \), is scaled to give \( W^2(\phi, \theta) = 2m/|\hbar^2(ab - c^2/4)| > 0 \). (The nonlinearity of the generalized Hamilton-Jacobi equation induces this normalization upon \( W \).) This ensures that \( (a\phi^2 + b\theta^2 + c\phi\theta) > 0 \). We note for completeness that a particular set \( (\phi, \theta) \) of independent solutions of the Schrödinger equation may be chosen by the superposition principle so that the coefficient \( c \) is zero.

The motion in phase space is specified by Eq. (3). This phase-space trajectory is a function of the set of coefficients \( (a, b, c) \). The \( \pm \) sign in Eq. (3) designates that the motion may be in either \( x \)-direction. The corresponding solution for the generalized Hamilton’s characteristic function, \( W \), is given by

\[ W = \hbar \arctan \left( \frac{b(\theta/\phi) + c/2}{(ab - c^2/4)^{1/2}} \right) + K \]  

where \( K \) is an integration constant, which we may set to zero herein.

The unperturbed energy, \( E \), and action variable, \( J \), for the ground state are quantized by \( E = \hbar^2 \pi^2/(8mq^2) = J^2/(32mq^2) \). The relationship between \( E \) and \( J \) is consistent with classical mechanics where \( E_{\text{classical}} = J^2/(32mq^2) \). The quantization of \( E \) and \( J \) are independent of the coefficients \( a, b \) and \( c \). Subsequently, the wave number for the ground state is specified to be \( k = \pi/(2q) \).
The set of independent solutions \((\phi, \theta)\) for this unperturbed square well is chosen such that \(\phi\) represents the symmetric bound state given by

\[
\phi = \left( \frac{2m}{\hbar^2k^2(ab - c^2/4)} \right)^{1/4} .
\]

\[
\begin{align*}
\lim_{V \to \infty} \frac{(E/V)^{1/2}}{\cos(kx)}, & \quad \text{if } x > q \\
\lim_{V \to \infty} \frac{(E/V)^{1/2}}{\cos(kx)}, & \quad -q \leq x \leq q \\
\lim_{V \to \infty} \frac{(E/V)^{1/2}}{\cos(kx)}, & \quad x < -q
\end{align*}
\]

where \(V \to \infty\) manifests an infinitely deep square well, where \(E\) is a function of \(V\) given implicitly by the quantizing equation \(E^{1/2} = \hbar(2m)^{-1/2}q^{-1}\arctan\left\{ \left\{ (V - E)/E \right\}^{1/2} \right\}\) for the finite square well, and where \(\kappa\) is a function of \(V\) given by \(\kappa = [2m(V - E)]^{1/2}/\hbar\). [Note that \(\lim_{V \to \infty} \frac{E}{V} \to h^2\pi^2/(8mq^2) = E\). The other solution, \(\theta\), is unbound and is not unique as any amount of \(\phi\) may be added to it. While \(\phi\) represents the symmetric bound state, the corresponding \(\theta\) that we have chosen is antisymmetric. We present this unbound solution as

\[
\theta = \left( \frac{2m}{\hbar^2k^2(ab - c^2/4)} \right)^{1/4} .
\]

\[
\begin{align*}
\lim_{V \to \infty} \frac{(V - E)^{1/2}}{ \sin(kx)}, & \quad \text{if } x > q \\
\lim_{V \to \infty} \frac{(V - E)^{1/2}}{ \sin(kx)}, & \quad -q \leq x \leq q \\
\lim_{V \to \infty} \frac{(V - E)^{1/2}}{ \sin(kx)}, & \quad x < -q
\end{align*}
\]

We have retained the limiting process as an intermediate step in describing \(\phi\) and \(\theta\) in the classically forbidden region, \(|x| > q\), to facilitate evaluating the Wronskian there. The corresponding Wronskian gives \(W^2(\phi, \theta) = 2m/\hbar^2(ab - c^2/4)\geq 0\) everywhere as expected. The (unperturbed) ground state Schrödinger wave function is well known to be

\[
\psi = \begin{cases} 
0, & x > q \\
q^{-1/2} \cos(kx), & -q \leq x \leq q \\
0, & x < -q.
\end{cases}
\]  

(5)

**Microstates:** For bound states in general and the ground state in particular, microstates of the Schrödinger wave function exist where the particular choice of the set of coefficients \((a, b, c)\) specifies a unique trajectory in phase space for a given quantized energy \(E\) or quantized action variable \(J\) as developed elsewhere. Each microstate is consistent with the bound-state Schrödinger wave function for the bound state can also be expressed by

\[
\psi = \frac{(2m)^{1/4} \cos(W/\hbar)}{(a - c^2/(4b))^{1/2} (\partial W / \partial x)^{1/2}} = \frac{(a\theta^2 + b\theta^2 + c\theta)^{1/2}}{(a - c^2/(4b))^{1/2}} \cos \left[ \arctan \left( \frac{b\theta + c/2}{(ab - c^2/4)^{1/2}} \right) \right] = \phi.
\]

Thus, before the perturbing impulse acts on the particle, each microstate of the ground state has the same quantized energy as the Schrödinger wave function.

Hamilton’s characteristic function is a generator of motion. The equation of motion in the domain \([x, t]\) is given by the Hamilton-Jacobi transformation equation for a constant coordinate (often called Jacobi’s theorem). The procedure simplifies for coordinates whose conjugate momenta are separation constants. For stationarity, \(E\) is a separation constant for time. Thus, the equation of motion for the trajectory time, \(t\), relative to its constant coordinate \(\tau\), is given as a function of \(x\) by

\[
t - \tau = \partial W / \partial E
\]

(7)
where the trajectory for a given energy, $E$, is a function of a set of coefficients $(a, b, c)$ and $\tau$ specifies the epoch.

We see that a Hamilton-Jacobi development of the trajectory representation does not return us to the classical representation because we must consider microstates, which do not occur in an infinitely deep, classical square well. The trajectory associated with a microstate of given energy $E$ may be specified by the set of coefficients $(a, b, c)$ or by the set of initial conditions $(x_o, \dot{x}_o, \ddot{x}_o)$.

On the other hand, classical motion in one dimension is simpler for it is specified for a given $E$ by the initial condition $x_o$. Nevertheless, the particular microstate, $a = b$ and $c = 0$, does correspond to the contemporary concept of classical motion.

### 3 DURING PERTURBATION

**Perturbing System:** Let us now apply a spatially dependent impulse at time $\gamma$ to the quantum particle in the ground state. The impulse is spatially dependent and, where occurring, is toward the center of the well. The subsequent time-dependent perturbing potential, $F\Delta V(x, t)$, is a spatially dependent impulse, which is symmetric, given by

$$F\Delta V(x, t) = \delta(t - \gamma)F \cdot \begin{cases} 
0, & |x| > q \text{ or } |x| \leq q - \epsilon \\
-x - q + \epsilon, & -q \leq x < -q + \epsilon \\
x - q + \epsilon, & q - \epsilon < x \leq q 
\end{cases} \tag{8}$$

where $F$ is a factor with units of force and where $0 < \epsilon \ll q$. The spatial dependence of $\Delta V$ is intentionally symmetric to simplify our investigation.

Let us now calculate the first-order changes in energy induced by $F\Delta V(x, t)$. We compare the calculated changes both for a Copenhagen interpretation based upon the standard time-dependent perturbation theory for the Schrödinger representation and for a trajectory representation based upon a canonical, generalized Hamilton-Jacobi perturbation theory. Both theories are based upon variation of parameters. Different first-order predictions in $E$ are exhibited for the two perturbation methods. First-order changes in $E$ are sufficient for our comparison because we are free to make $\epsilon$ and $F$ as small as necessary.

We first apply the standard wave-function time-dependent perturbation theory. The variation in the coefficient (parameter) of the ground state, $C_0$ (the coefficient of the eigenfunction that, by itself, specified the particle before time $\gamma$) is well known and given by

$$i\hbar \dot{C}_0 = F(\Delta V)_{00}C_0 + O(F^2)$$

where

$$(\Delta V)_{00} = \langle 0|\Delta V|0 \rangle = \left[ \frac{\epsilon^2}{2} - \frac{1}{4k^2} + \frac{\cos(2k\epsilon)}{4k^2} \right]$$

Thus, $C_0$ varies to first order in $F$. Therefore, as is well known, the complex behavior of $C_0(t)$, to first order in $F$, changes in angular frequency due to a finite $(\Delta V)_{00}$. Ergo, the first-order change in energy, $E_1$ of an ensemble of particles to first order in $F$ is given by

$$E_1 = F(\Delta V)_{00} = F \left[ \frac{\epsilon^2}{2} - \frac{1}{4k^2} + \frac{\cos(2k\epsilon)}{4k^2} \right]. \tag{9}$$

We now apply a canonical perturbation expansion, based upon a generalized Hamilton-Jacobi theory to the trajectory representation. To do this, we allow the parameters $E$ and $\tau$, which were constants of the motion in the unperturbed equation of motion, Eq. (7), to be varied as a function of time by the perturbation. (Energy is no longer a separation constant for a time-dependent Hamiltonian; consequently, the “constant coordinate”, $\tau$, is also no longer a constant coordinate.) We expand $E$ and $\tau$ in a power series in $F$. As we are only interested in the first order term of energy, we shall compute this term only.
The generalized Hamilton-Jacobi equation, Eq. (2) manifests the same canonical transformation that generates the classical Hamilton-Jacobi transformation. Hence, the equations of motion for the variables \((E, \tau)\) for the trajectory representation are the same as those for classical perturbation theory. These well known equations of motion are given by

\[
\dot{E} = F \partial V(E, \tau, t)/\partial \tau \quad \text{and} \quad \dot{\tau} = -F \partial V(E, \tau, t)/\partial E.
\]

If the perturbation is small [we are free to make \((F, \epsilon)\) as small as required], then we can produce a first-order approximation \((E_1, \tau_1)\) to the time variation of \((E, \tau)\) by using the constant, unperturbed values \((E_0, \tau_0)\) on the right sides of the above equations. For the perturbation expansion of interest (the expansion of \(E\)), we must solve

\[
\dot{E}_1 = -F \partial V(E_0, \tau_0, t)/\partial \tau_0.
\]  

Reversion of \(t(x)\) to \(x(t)\): The unperturbed equation of motion is given by the Hamilton-Jacobi transformation equation, Eq. (7), which renders

\[
t - \tau = \frac{\pm 2 \hbar k (ab - c^2/4)^{1/2}}{a + b + (a - b) \cos(2kx) + c \sin(2kx)}
\]  

where the \(\pm\) sign indicates that motion may be in either direction. The trajectory is dependent upon the coefficients \((a, b, c)\), which specify the particular microstate. If \(a \neq b\) or \(c \neq 0\), then Eq. (11) is a transcendental equation that cannot be turned inside out to solve for \(x\) in closed form. But the spatial dependence of \(\Delta V(x, t)\) is non-zero only over two small regions: \(-q \leq x < -q + \epsilon\) and \(q - \epsilon < x \leq q\). In these two regions we may expand the trigonometric functions in a Maclaurin series about \(-q\) and \(q\) respectfully. For example, we may express Eq. (11) for a particle traveling in the +x-direction in the domain \(-q \leq x < -q + \epsilon\) as

\[
t - \tau = \frac{\hbar k (ab - c^2/4)^{1/2}}{b - ck(x + \epsilon) + (a - b)k^2(x + \epsilon)^2} + O((x + \epsilon)^3), \quad 0 \leq x + \epsilon \leq \epsilon \ll 1.
\]  

In Eq. (12), keeping the Maclaurin series expansion of the trigonometric functions only up to second order still captures the physical nuances of the microstates as specified by the coefficients \((a, b, c)\) (again, \(\epsilon\) may be made as small as necessary). If we ignore the \(O((x + \epsilon)^3)\) terms in Eq. (12), we may now turn the equation of motion inside out to approximate \((x + \epsilon)\) as a quadratic solution to Eq. (12). An approximate \((x + \epsilon)\) as a function of \((E, \tau, t)\) may be given for a particle traveling in the +x-direction as

\[
x + \epsilon \approx \left(\frac{\hbar k}{\hbar k (t - \tau)} + \frac{m}{\hbar k (t - \tau)^2}\right) - \left(\frac{\hbar k}{\hbar k (t - \tau)} + \frac{m}{\hbar k (t - \tau)^2}\right)^2 - 4 \frac{a - b}{b} \frac{k^2}{m} \left(1 + \frac{m}{\hbar k (t - \tau)}\right)^{1/2}, \quad 0 \leq x + \epsilon \leq \epsilon \ll 1
\]  

where \(G = b/(ab - c^2/4)^{1/2}\). While we use \(G\) to make Eq. (13) less cumbersome, we note that \(G\) is a constant of the motion in its own right. This follows for \(G\) can be expressed entirely by other constants of the motion and other physical constants. We have \(G = (2m)^{1/2}/(\hbar W)\) where \(W\) is, as previously given, the Wronskian for the set of independent solutions \((\phi, \theta)\) and \(I\) is the Ermakov invariant given by \(I = [a - c^2/(4b)]^{-1}\). For completeness, the constant of the motion \(G\) by Eq. (12) can be related to the periodicity of the microstate by
Perturbation Effects: We may now formulate $\Delta V(E_0, \tau_0, t)$ for a particle traveling in the $+x$-direction in the domain $-q \leq x < -q + \epsilon$ from Eqs. (8) and (13) as

$$
\Delta V \approx -\delta(t-\gamma)\epsilon \left\{ \frac{m}{\hbar k (\tau_0 - \tau)} \left( \frac{\hbar k}{\hbar k (\tau_0 - \tau)} + \frac{\hbar k}{m} \right) - \left[ \left( \frac{\hbar k}{\hbar k (\tau_0 - \tau)} \right)^2 - 4 \frac{a-b}{b} k^2 \left( 1 + \frac{m}{\hbar k (\tau_0 - \tau)} \right) \right]^{1/2} \right\}
$$

(14)

where $E_0$ is manifested by $k$ for $E_0 = \hbar^2 k^2/(2m) = \hbar^2 \pi^2/(8mq^2)$.

We now have the wherewithal to determine the right side of Eq. (10). From Eqs. (10) and (14), $\dot{E}_1$ is given by

$$
\dot{E}_1 \approx \frac{\delta(t-\gamma)F}{2a-b} \left\{ \frac{-m}{\hbar k (\gamma - \tau_0)^2} \left( \frac{\hbar k}{\hbar k (\gamma - \tau_0)^2} \right)^2 - 4 \frac{a-b}{b} k^2 \left( 1 + \frac{m}{\hbar k (\gamma - \tau_0)} \right) \right\}
$$

$$
- \frac{mg}{\hbar k} < \gamma - \tau_0 \leq - \frac{m(q-\epsilon)}{\hbar k (1 - \frac{b}{a} + \frac{b^2}{a} k^2 \epsilon^2)}.
$$

We may now integrate the above over the $\delta$-function duration to get

$$
E_1 \approx \frac{T b F}{2(a-b)k^2} \left\{ \frac{-m}{\hbar k (\gamma - \tau_0)^2} \left( \frac{\hbar k}{\hbar k (\gamma - \tau_0)^2} \right)^2 - 4 \frac{a-b}{b} k^2 \left( 1 + \frac{m}{\hbar k (\gamma - \tau_0)} \right) \right\}
$$

(15)

$$
- \frac{mg}{\hbar k} < \gamma - \tau_0 \leq - \frac{m(q-\epsilon)}{\hbar k (1 - \frac{b}{a} + \frac{b^2}{a} k^2 \epsilon^2)}
$$

where the “$T$” in “$T b F$” in the above is a unit measure in time. This “$T$” manifests that $\delta$-function impulse has been integrated over a time domain that includes the instant that the $\delta$-function acts. If the particle is between $-q$ and $-q + \epsilon$ at time $\gamma$, then the particle and the system causing the perturbing force will transfer energy between each other. The amount of energy transferred to first order in $F$ is given by the magnitude of $\dot{E}_1$. The direction of transfer is given by the sign of $E_1$. From Eq. (15), $E_1$ is a function of the particular microstate as specified by the coefficients $(a, b, c)$. The energy for the particle is $E = E_0 + E_1 + O(F^2)$.

Let us simplify $E_1$ by considering the particular microstate where the coefficients are specified by $a = b$ and $c = 0$. Then, Eq. (13) becomes exact, and Eq. (15) becomes exactly

$$
E_1 \bigg|_{a=b,c=0} = +F \frac{\hbar k}{m} T, \quad \frac{mg}{\hbar k} < \gamma - \tau_0 \leq - \frac{m(q-\epsilon)}{\hbar k}
$$

(16)

This simplified case has an intuitive physical interpretation. We see from the plus sign on the right side of Eq. (16) that the perturbing system does work on the particle by transferring to first order the energy $E_1$ to it. The amount of energy is proportional to the force factor, $F$, and the distance that the particle transits against this force during the perturbation’s duration. The transit distance for $a = b$ and $c = 0$ is given by the product of particle velocity, $\hbar k/m$, and one unit of time denoted by “$T$” in Eq. (16). This is also what we would intuitively expect from classical perturbation theory and is a manifestation that the trajectory representation is strongly causal.
We report that if the particle had been traveling in the +x-direction and had been located between \( q - \epsilon \) and \( q \) at time \( \tau \), then \( E_1 \) would be given by

\[
E_1 \approx \frac{\tau bF}{2(a-b)k^2} \left\{ \frac{m}{\hbar k G(\tau_0 - \gamma)^2} - \frac{m}{\hbar k G(\tau_0 - \gamma)^2} \left( \frac{m}{\hbar k G(\tau_0 - \gamma)} \right)^2 - 4a^{-1}k^2 \frac{m}{\hbar k G(\tau_0 - \gamma)^2} \right\},
\]  

(17)

For \( a = b \) and \( c = 0 \), we have

\[
E_1 \bigg|_{a=b,c=0} = -F \frac{\hbar k}{m}, \quad \frac{m(q-\epsilon)}{\hbar k} < \gamma - \tau_0 < \frac{mq}{\hbar k}.
\]

where the particle does work on the perturbing system by transferring to first order the energy \( E_1 \) to it.

Now we consider the effect of the perturbation upon a particle travelling in the −x-direction. We arbitrarily assume that the cycle of interest is one for which particle motion in the +x-direction occurs first. When the particle reflects at the well boundary \( x = q \), the trajectory in phase space rounds a singular point and jumps from the Riemann sheet for +x motion for that cycle to the Riemann sheet for −x motion for that cycle. Innate to this jumping of Riemann sheets is a shift in the epoch \( \tau \). As \( t \mid_{x=q} \) must be the same for both Riemann sheets, we have from Eq. (7) that \( \tau_0 = \tau_0 + 2\eta q/(\hbar k G) \) where the subscript of \( \tau \) denotes the direction of motion. Unless needed, the sign subscript for \( \tau \) will not be made explicit.

We report that if the particle had been traveling in the −x-direction and had been located between \( q \) and \( q - \epsilon \) at time \( \tau \), then \( E_1 \) would be given by

\[
E_1 \approx \frac{\tau bF}{2(a-b)k^2} \left\{ \frac{-m}{\hbar k G(\tau_0 - \gamma)^2} + \frac{m}{\hbar k G(\tau_0 - \gamma)^2} \left( \frac{m}{\hbar k G(\tau_0 - \gamma)} \right)^2 - 4a^{-1}k^2 \frac{m}{\hbar k G(\tau_0 - \gamma)^2} \right\},
\]  

(18)

For \( a = b \) and \( c = 0 \), we now have

\[
E_1 \bigg|_{a=b,c=0} = +F \frac{\hbar k}{m}, \quad \frac{m(q-\epsilon)}{\hbar k} < \gamma - \tau_0 < \frac{mq}{\hbar k}.
\]

For motion in the −x-direction, we see that the perturbing system does work on the particle as expected.

We report that if the particle had been traveling in the −x-direction and had been located between −\( q + \epsilon \) and −\( q \) at time \( \gamma \), then \( E_1 \) would be given by

\[
E_1 \approx \frac{\tau bF}{2(a-b)k^2} \left\{ \frac{-m}{\hbar k G(\tau_0 - \gamma)^2} - \frac{m}{\hbar k G(\tau_0 - \gamma)^2} \left( \frac{m}{\hbar k G(\tau_0 - \gamma)} \right)^2 - 4a^{-1}k^2 \frac{m}{\hbar k G(\tau_0 - \gamma)^2} \right\},
\]  

(19)

For \( a = b \) and \( c = 0 \), we again have

\[
E_1 \bigg|_{a=b,c=0} = -F \frac{\hbar k}{m}, \quad \frac{m(q-\epsilon)}{\hbar k} < \gamma - \tau_0 < \frac{mq}{\hbar k}
\]

as expected.

If the particle is located between −\( q + \epsilon \) and \( q - \epsilon \) at time \( \gamma \), then
the trajectory interpretation, while the Copenhagen interpretation finds that Copenhagen interpretations predict different first-order perturbation energies. We must use ensemble averages. Even with ensemble averages, we may still show that the trajectory and investigation, we shall assume that we do not know the necessary and sufficient initial conditions and that differences. We are free to apply a trajectory analysis with incomplete information. For the sake of this edge of the initial conditions attendant with strong causality, although sufficient, is not necessary to show our findings to any distribution of microstates so long as the sub-ensemble for each microstate would tend, with increasing sub-ensemble size, toward a relative balance between particles going in each direction along the x-axis. This relative balance is so because each microstate’s orbit in phase space has mirror symmetry about the configuration x-axis in accordance with Eq. (3). Hence, we may generalize our findings to any distribution of microstates so long as the sub-ensemble of each microstate is sufficiently large. In such case, we still have that \( \langle E_1 \rangle_{\text{average}} = 0 \) for the trajectory interpretation while the Copenhagen interpretation finds that \( E_1 \) is finite as given by Eq. (9).

One is cautioned not to infer from \( \langle E_1 \rangle_{\text{average}} = 0 \) in the preceding paragraph for a uniform distribution of \( \tau \) that the expectation value of \( E_1 \) in the trajectory representation would be given to first order by the time average of the perturbing potential evaluated in the unperturbed state. For perturbations of long duration, \( E_1 \) in contemporary quantum mechanics is given to first order by the expectation value of the perturbing potential in the unperturbed state, which in classical mechanics corresponds to the time average of the perturbation evaluated in the unperturbed system.\(^{15}\) But we have here an impulsive perturbation. Even for classical mechanics, \( \langle E_1 \rangle_{\text{average}} \), as generated by an impulsive perturbation with a uniform distribution of \( \tau \), would be zero and is not equivalent to the time average of the perturbing potential evaluated in the unperturbed system. Thus, classical mechanics offers a precedent for impulsive perturbation. Usually, the orbital period is assumed to be shorter than the duration of the perturbation in classical systems. As noted in the preceding paragraph, this assumptions would allow us to simplify by averaging over the orbit. But we consider here the reverse situation with an impulsive perturbation. Immaterial of how much we know, what happens by strong causality is determined by the particle’s position at the time of the perturbing impulse and what it is doing then [i.e., \( \dot{x}(\gamma), \dot{\psi}(\gamma) \) and \( \ddot{x}(\gamma) \)] as described by a particular microstate of the unperturbed energy. These initial conditions are more extensive then those needed for

\[
E_1 = 0, \quad |\tau - \gamma| \leq \frac{m(q - \epsilon)}{\hbar k G (1 - \frac{q}{c} k \epsilon + \frac{\hbar^2}{\gamma} k^2 \epsilon^2)}. \tag{20}
\]

But we note that the first order \( E_1 \) is zero here for any microstate of the trajectory representation because the particle is not located in a space-time location where the perturbation acts.

Under the trajectory interpretation, we do know where the particle is in trajectory theory: the trajectory representation is strongly causal. The time dependence of the perturbing system is a \( \delta \)-function. Ergo, we can predict individual results, \( E_1 \), in trajectory theory by Eqs. (15) and (17)–(20). These trajectory predictions for \( E_1 \) differ with the predictions, Eq. (9), of the Copenhagen interpretation that are based upon ensemble averages! Furthermore, the trajectory predictions are a function of the particular microstate.

4 COPENHAGEN EPISTEMOLOGY

Copenhagen epistemology rejects in principle sufficient knowledge of the initial conditions to specify an individual trajectory (microstate). The Copenhagen school denies strong causality for describing a particle’s progress along a trajectory but rather limits causality to the evolution of \( \psi \). Fortunately, complete knowledge of the initial conditions attendant with strong causality, although sufficient, is not necessary to show the differences. We are free to apply a trajectory analysis with incomplete information. For the sake of this investigation, we shall assume that we do not know the necessary and sufficient initial conditions and that we must use ensemble averages. Even with ensemble averages, we may still show that the trajectory and Copenhagen interpretations predict different first-order perturbation energies.

Had we not known the particle’s position at time \( \gamma \) due to some practical limitation and, consistent with the trajectory interpretation, not due to some limitation in principle, then if we assume a uniform distribution of \( \tau \) over the duration of one cycle, the average \( E_1 \) as determined by Eqs. (15) and (17)–(20), would be given by \( \langle E_1 \rangle_{\text{average}} = 0 \). Lest we forget, each microstate (trajectory) by itself is sufficient to deduce \( \psi \). Physically, averaging \( E_1 \) for a uniform distribution of \( \tau \) manifests that, to first order, the perturbing system is as likely to do work on the particle as the particle is likely to do work on the perturbing system. We do not need to know the microstate of any particular particle of the ensemble. It is sufficient that the ensemble be large enough so that the sub-ensemble for each microstate would tend, with increasing sub-ensemble size, toward a relative balance between particles going in each direction along the x-axis. This relative balance is so because each microstate’s orbit in phase space has mirror symmetry about the configuration x-axis in accordance with Eq. (3). Hence, we may generalize our findings to any distribution of microstates so long as the sub-ensemble of each microstate is sufficiently large. In such case, we still have that \( \langle E_1 \rangle_{\text{average}} = 0 \) for the trajectory interpretation while the Copenhagen interpretation finds that \( E_1 \) is finite as given by Eq. (9).

One is cautioned not to infer from \( \langle E_1 \rangle_{\text{average}} = 0 \) in the preceding paragraph for a uniform distribution of \( \tau \) that the expectation value of \( E_1 \) in the trajectory representation would be given to first order by the time average of the perturbing potential evaluated in the unperturbed state. For perturbations of long duration, \( E_1 \) in contemporary quantum mechanics is given to first order by the expectation value of the perturbing potential in the unperturbed state, which in classical mechanics corresponds to the time average of the perturbation evaluated in the unperturbed system.\(^{15}\) But we have here an impulsive perturbation. Even for classical mechanics, \( \langle E_1 \rangle_{\text{average}} \), as generated by an impulsive perturbation with a uniform distribution of \( \tau \), would be zero and is not equivalent to the time average of the perturbing potential evaluated in the unperturbed system. Thus, classical mechanics offers a precedent for impulsive perturbation.
classical perturbation theory. In classical mechanics, the unperturbed energy, the applicable Riemann sheet of $W_{\text{classical}}$ and the particle’s position, $x(\gamma)$, at the time of perturbing impulse suffice for classical causality.

5 DISCUSSION

The predictions for an ensemble average $E_1$ of the trajectory and Copenhagen interpretations differ. In this ensemble, we need not know the distribution of microstates nor the exact distribution of the constant coordinate, $\tau$. Hence, this comparison of ensemble averages between the trajectory and Copenhagen interpretation complies with Copenhagen epistemology. The underlying reason for the difference in the predictions between the Copenhagen interpretation and the trajectory representation is due to their difference regarding causality. The trajectory interpretation is strongly causal for individual trajectories while the Copenhagen interpretation is weakly causal for ensembles while only giving probabilities for individual events. Any subsequent experimental verification of our results is actually a test of which causality quantum mechanics observes.

The particular problem that we have examined here was chosen to simplify the mathematics in our exposition. We need not be constrained to such ideal conditions in any experimental verification. We do advise that the duration of any perturbation be much shorter than the orbital period of the test particle. (Had we examined a small perturbation of long duration relative to the particles orbital period, then the perturbing potential would produce a change in energy given by the time average of the perturbing potential evaluated by the unperturbed system which would correspond to the expected first-order change in energy in wave mechanics.)\textsuperscript{15} A perturbation of very short duration with spatial dependence would generate predictions that accentuate the different roles that causality plays in the trajectory and Copenhagen interpretations.
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ERRATA: WHICH CAUSALITY? DIFFERENCES BETWEEN TRAJECTORY AND COPENHAGEN ANALYSES OF AN IMPULSIVE PERTURBATION,

*Int. J. Mod. Phys.* A 14, 1111 (1999)

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Errata published: *Int. J. Mod. Phys.* A 16 (2001) 2447

**Abstract:** The evaluation of the matrix element $(\Delta V)_{00}$ is corrected. This correction increases the differences between the trajectory and Copenhagen analyses.

The unnumbered displayed equation preceding Eq. (9) is corrected by including the factor $\delta(t - \tau)/q$ so that $(\Delta V)_{00}$ becomes

$$(\Delta V)_{00} = \langle 0 | \Delta V | 0 \rangle = \frac{\delta(t - \gamma)\epsilon^2}{q} \left[ \frac{1}{4k^2} - \frac{\cos(2k\epsilon)}{4k^2} \right]$$

Consequently, the correct first-order change in energy, $E_1$, of an ensemble of particles is given by correcting Eq. (9) to read

$$E_1 = F(\Delta V)_{00} = \frac{F\delta(t - \gamma)}{q} \left[ \frac{\epsilon^2}{2} - \frac{1}{4k^2} + \frac{\cos(2k\epsilon)}{4k^2} \right].$$

Including this correcting factor $\delta(t - \tau)/q$ increases the magnitude of $E_1$ during perturbation for the Copenhagen interpretation. Meanwhile, the average $E_1$ of the ensemble of particles for the trajectory representation under Copenhagen epistemology remains $\langle E_1 \rangle_{\text{average}} = 0$ during perturbation.$^1$ This strengthens our findings that trajectory and Copenhagen analyses differ regarding perturbing impulses.

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