Entangling the vibrational modes of two massive ferromagnetic spheres using cavity magnomechanics

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Abstract
We present a scheme to entangle the vibrational phonon modes of two massive ferromagnetic spheres in a dual-cavity magnomechanical system. In each cavity, a microwave cavity mode couples to a magnon mode (spin wave) via the magnetic dipole interaction, and the latter further couples to a deformation phonon mode of the ferromagnetic sphere via a nonlinear magnetostrictive interaction. We show that by directly driving the magnon mode with a red-detuned microwave field to activate the magnomechanical anti-Stokes process a cavity–magnon–phonon state-swap interaction can be realized. Therefore, if the two cavities are further driven by a two-mode squeezed vacuum field, the quantum correlation of the driving fields is successively transferred to the two magnon modes and subsequently to the two phonon modes, i.e., the two ferromagnetic spheres become remotely entangled. Our work demonstrates that cavity magnomechanical systems allow to prepare quantum entangled states at a more massive scale than currently possible with other schemes.

1. Introduction
Preparing entangled states of macroscopic, massive objects is of significance to many fundamental studies, e.g., probing the boundary between the quantum and classical worlds [1–3], tests of decoherence theories at the macro scale [4–6], and gravitational quantum physics [7], among many others. Over the past decade, significant progress has been made in the field of cavity optomechanics [8] in preparing entangled states of massive objects, with experimental realizations of entanglement between a mechanical oscillator and an electromagnetic field [9, 10], as well as between two mechanical oscillators [11–13]. All those entangled states were created and detected by utilizing the radiation pressure interaction, or, more specifically, the optomechanical two-mode squeezing and beamsplitter (state-swap) interactions, realized by driving the cavity with a blue- and red-detuned electromagnetic field, respectively, and optimally working in the resolved sideband limit.

In analogy to cavity optomechanics, in recent years cavity magnomechanics (CMM) [14] has received increasing attention, owing to its potential for realizing quantum states at a more macroscopic scale [15–17] and possible applications in quantum information processing and quantum sensing [18]. In these systems, a magnon mode (spin wave) of a ferromagnetic yttrium-iron-garnet (YIG) sphere couples to a microwave (MW) cavity field [19–24], and simultaneously couples to the vibrational phonon mode (deformation mode) of the sphere via the magnetostrictive force [25]. Owing to the high spin density and the low damping rate of YIG, the interaction between the MW cavity field and the magnon mode can easily enter the strong coupling regime [19–24], thus providing an excellent platform for the study of strong interaction between light and matter. Many interesting phenomena have been explored in the context of cavity magnonics, such as a magnon gradient memory [26], exceptional points [27], the manipulation of distant...
spin currents [28], level attraction [29], nonreciprocity [30], among others. In the tripartite system of CMM, the phonon mode is typically of low frequency due to the large size of the sphere. The magnomechanical interaction is a radiation pressure-like, dispersive interaction [14, 31] and the phonon mode, as depicted in figure 1. The magnon and phonon modes are supported by the YIG sphere, which has a typical diameter in the 100 μm range [14]. The magnon mode is embodied by the collective motion of a large number of spins in the YIG sphere, and the phonon mode is the deformation mode of the sphere caused by the magnetostrictive force [25]. In each cavity, the magnon mode couples to the MW cavity mode via the magnetic dipole interaction, and to the phonon mode via the nonlinear radiation pressure-like magnomechanical interaction. In our scheme, each magnon mode is directly driven through a strong red-detuned MW field, realized, e.g., driving the YIG sphere with a small loop antenna at the end of a superconducting MW line [33, 34], which enhances the magnomechanical coupling strength, cools the phonon mode [15], and activates the magnon-phonon state-swap interaction [16].

The Hamiltonian of the system is given by

\[ H/h = \sum_{j=1,2} \left\{ \omega_{mj} a_j^\dagger a_j + \omega_{mj} m_j^+ m_j + \omega_{bj} b_j^\dagger b_j + g_j \left( a_j^\dagger m_j + a_j m_j^+ \right) + G_0 m_j^+ m_j \left( b_j^\dagger + b_j \right) \right\} + \sum_{j=1,2} \omega_{mj} a_j^\dagger a_j, \]

where \( a_j, m_j, \) and \( b_j \) (\( \omega_{mj}, \omega_{mj}, \) and \( \omega_{bj} \)) are the annihilation operators (resonance frequencies) of the cavity, magnon and phonon modes, respectively, satisfying \( [O_j, O_j^\dagger] = 1 \) (\( O = a, m, b \)), with \( j = 1, 2 \). The magnon frequency \( \omega_{mj} \) can be adjusted by varying the external bias magnetic field \( H_j \) via \( \omega_{mj} = \gamma_a H_j \), where the
gyromagnetic ratio for YIG $\gamma_0/2\pi = 28$ GHz T$^{-1}$ $g_i$ is the cavity–magnon coupling rate, which can be much larger than the dissipation rates of the two modes, $g_i > \kappa_{m,j}, \kappa_j$ [19–24]. $G_0$ is the bare magnon–phonon coupling rate, which is usually quite small, but can be enhanced by driving the magnon mode with a strong MW field. The Rabi frequency $\Omega_j = \sqrt{2} g_j \sqrt{N_j} B_{0j}$ [15] denotes the coupling rate between the magnon mode and its driving magnetic field with frequency $\omega_{0j}$ and amplitude $B_{0j}$, while $N_j = \rho V_j$ is the total number of spins, with $\rho = 4.22 \times 10^{27}$ m$^{-3}$ the spin density of YIG and $V_j$ is the volume of the spheres. Note that for the magnon modes, we have expressed the collective spin operators in terms of Boson (oscillator) operators via the Holstein–Primakoff transformation [51] under the condition of low-lying excitations, $\langle m_j^+ m_j \rangle \ll 2N_j$ (for simplicity we assume the two spheres to be of the same size and thus of the same total number of spins $N$), where $s = \frac{3}{2}$ is the spin number of the ground state Fe$^{3+}$ ion in YIG.

We now assume the two cavities to be driven by a continuous, two-mode squeezed vacuum MW input field with frequency $\omega_{0j}$ and each cavity to be resonant with the squeezed drive as well as the magnon mode, such that $\omega_{0j} = \omega_{m,j} = \omega_{0j}$, or $\Delta_{0j} = \Delta_{m,j} = \Delta_{0j} = \Delta_j (j = 1, 2)$, where the detunings $\Delta_{0j} = \omega_{0j} - \omega_{0j}$ ($O = a, m, s$) are with respect to the magnon drive frequency $\omega_{0j}$, see figure 1(b). This situation is easily realized as all three frequencies are tunable, and the resonant case also corresponds to the optimal situation for transferring squeezing from the driving field to the magnon mode [16, 46]. Note that $\Delta_1 = \Delta_2$ is however not required as each should match the frequency of the phonon mode of the respective YIG sphere, i.e., $\Delta_j \simeq \omega_{0j}$. This corresponds to the magnon mode being resonant with the blue mechanical sideband (see figure 1(b)), which is required for realizing the magnomechanical state-swap interaction in each sphere, such that the squeezing can further be transferred from the magnon mode to the phonon mode.

The quantum Langevin equations (QLEs) for describing the cavity, magnon, and phonon modes are given by (in the frame rotating at the magnon drive frequency $\omega_{0j}$)

$$\dot{a}_j = - (i \Delta_j + \kappa_{m,j}) a_j - i g_j m_j + \sqrt{2 \kappa_{m,j}} a_j^{in},$$
$$\dot{m}_j = - (i \Delta_j + \kappa_{m,j}) m_j - ig_0 a_j - i G_0 m_j (b_j^+ + b_j) + \Omega_j + \sqrt{2 \kappa_{m,j}} m_j^{in},$$
$$\dot{b}_j = - (i \omega_{0j} + \gamma_j) b_j - i G_0 m_j^+ m_j + \sqrt{2 \gamma_j} b_j^{in},$$

(2)
where \( \gamma_j \) are the mechanical damping rates, and \( \hat{a}_i^{\dagger} \), \( \hat{m}_i^0 \) and \( \hat{b}_i^0 \) are input noise operators for the cavity, magnon, and phonon modes, respectively. Owing to the injection of a two-mode squeezed vacuum field, which shapes the noise properties of two MW cavity fields, the input noise of the two cavities \( \hat{a}_i^{\dagger} \), become quantum correlated and possess the correlation functions

\[
\langle \hat{a}_i^{\dagger}(t) \hat{a}_i^{\dagger}(t') \rangle = (N' + 1) \delta (t - t'), \\
\langle \hat{a}_i^{\dagger}(t) \hat{a}_i^{\dagger}(t') \rangle = N \delta (t - t'), \\
\langle \hat{a}_j^{\dagger}(t) \hat{a}_i^{\dagger}(t') \rangle = M e^{-i(\Delta_1 + \Delta_2)} \delta (t - t'), \\
\langle \hat{a}_j^{\dagger}(t) \hat{a}_j^{\dagger}(t') \rangle = M' e^{i(\Delta_1 + \Delta_2)} \delta (t - t'), \quad (j \neq k = 1, 2)
\]

where \( N' = \sinh^2 r, M = \sinh r \cos h r \). Here \( r \) is the squeezing parameter of the two-mode squeezed vacuum field, which is typically produced by a Josephson parametric amplifier (JPA) [52], a Josephson mixer [53], or the combination of a JPA and an MW beamsplitter [54, 55]. Note that the phase factors in the vacuum field, which is typically produced by a Josephson parametric amplifier (JPA) [52], a Josephson second-order fluctuation terms. As a result, the QLEs (2) are separated into two sets of equations for the magnomechanical interaction. This frequency shift is typically small because of a small correlation as

\[
\langle m_j \rangle = \frac{1}{g_j^2 + (i\Delta_j + \kappa_m)} \left( i\Delta_j + \kappa_m \right),
\]

where \( \Delta_j = \Delta_j + 2G_{ij} \text{Re} \langle b_j \rangle \) is the effective magnon-drive detuning including the frequency shift caused by the magnomechanical interaction. This frequency shift is typically small because of a small \( G_{ij} \) [14], \( |\Delta_j - \Delta_j| \ll \Delta_j \approx \omega_{bj} \), and thus hereafter we can safely assume \( \Delta_j \approx \Delta_j \). When \( \Delta_j \approx \omega_{bj} \gg \kappa_j, \kappa_m \), which is easily satisfied [14], equation (5) takes a simple approximate form \( \langle m_j \rangle \approx i\Delta_j \Omega_j/(g_j^2 - \Delta_j^2) \), which is a pure imaginary number. The solutions of \( \langle a_j \rangle \) and \( \langle b_j \rangle \) can then be obtained by \( \langle a_j \rangle = iG_{ij} \langle m_j \rangle/(i\Delta_j + \kappa_m) \), and \( \langle b_j \rangle = -iG_{ij} |\langle m_j \rangle|^2/(i\omega_{bj} + \gamma_j) \approx -G_{ij} |\langle m_j \rangle|^2/\omega_{bj} \), taking into account the mechanical Q factor is typically high, \( \omega_{bj}/\gamma_j \gg 1 \). The average \( \langle b_j \rangle \) is therefore a real number, implying that the average of mechanical momentum, \( \langle p_j \rangle = \sqrt{2} \text{Im} \langle b_j \rangle \), is zero in the steady state.

The QLEs for the quantum fluctuations are given by

\[
\delta \dot{a}_j = - \left( i\Delta_j + \kappa_m \right) \delta a_j - ig_j \delta m_j + \sqrt{2\kappa_m} \delta a_j^{\dagger}, \\
\delta \dot{m}_j = - \left( i\Delta_j + \kappa_m \right) \delta m_j - ig_j \delta a_j - G_j \left( \delta b_j + \delta b_j^{\dagger} \right) + \sqrt{2\kappa_m} \delta m_j^{\dagger}, \\
\delta \dot{b}_j = - (i\omega_{bj} + \gamma_j) \delta b_j - G_j \left( \delta m_j^{\dagger} - \delta m_j \right) + \sqrt{2\gamma_j} \delta b_j^{\dagger},
\]

where \( G_j = iG_{0j} |\langle m_j \rangle| \) is the effective magnomechanical coupling rate. We now move to a reference frame rotating at frequency \( \Delta_j = \omega_{bj} \) by introducing the slowly moving operators \( \tilde{a}_j, \tilde{m}_j = \delta a_j e^{-i\Delta_j t}, \delta m_j = \delta m_j e^{-i\Delta_j t}, \delta b_j = \delta b_j e^{-i\omega_{bj} t} \), where \( \delta \dot{a}_j, \delta \dot{m}_j \) and \( \delta \dot{b}_j \) are defined in the new reference frame. We make the same transformation for the input noise operators, and obtain noise correlations in the new frame, which remain the same as in equations (3) and (4) but without the phase factors in equation (3), as we are now in a frame that is resonant with the squeezed drive field. By substituting the above
transformations into the QLEs (6), and neglecting fast oscillating non-resonant terms, we obtain the following QLEs

\[
\begin{align*}
\dot{\delta \hat{a}} &= -\kappa_{\delta a} \delta \hat{a} - ig_{\delta} \delta \hat{m}_{j} + \sqrt{2\kappa_{\delta a}} \hat{b}_{j}^{\text{in}}, \\
\dot{\delta \hat{m}_{j}} &= -\kappa_{\delta m} \delta \hat{m}_{j} - ig_{\delta} \delta \hat{a} - G_{j} \delta \hat{b}_{j} + \sqrt{2\kappa_{\delta m}} \hat{m}_{j}^{\text{in}}, \\
\dot{\delta \hat{b}_{j}} &= -\gamma_{\delta} \hat{b}_{j} + G_{j} \delta \hat{m}_{j} + \sqrt{2\gamma_{\delta}} \hat{b}_{j}^{\text{in}},
\end{align*}
\]

(7)

which are a good approximation if the condition \(\Delta_{j} = \omega_{\delta j} \gg G_{j}, g_{\delta}, \kappa_{\delta a}, \kappa_{\delta m}, \gamma_{\delta}\) is satisfied. The QLEs (7) clearly reveal a beamsplitter interaction in the cavity–magnon and magnon–phonon subsystems, which allows for cooling the phonon modes and the transfer of two-mode squeezing from the driving fields to the two cavity modes, then to the two magnon modes, and finally to the two phonon modes of the two spatially separated YIG spheres.

3. Entanglement of two YIG spheres

We now proceed to study the entanglement of the two phonon modes. We rewrite the QLEs (7) in terms of quadrature fluctuations, which can be cast in the following form

\[
\dot{u}(t) = Au(t) + n(t),
\]

(8)

where \(u = (\delta X_{1}, \delta Y_{1}, \delta X_{2}, \delta Y_{2}, \delta X_{1}, \delta Y_{1}, \delta X_{2}, \delta Y_{2}, \delta p_{1}, \delta q_{2}, \delta p_{2})^{T}\), and the quadrature fluctuation operators are defined as \(\delta X_{j} = (\delta \hat{a}_{j} + \delta \hat{a}_{j}^{\dagger})/\sqrt{2}\), \(\delta Y_{j} = i(\delta \hat{a}_{j}^{\dagger} - \delta \hat{a}_{j})/\sqrt{2}\), \(\delta X_{j} = (\delta \hat{m}_{j} + \delta \hat{m}_{j}^{\dagger})/\sqrt{2}\), \(\delta Y_{j} = i(\delta \hat{m}_{j}^{\dagger} - \delta \hat{m}_{j})/\sqrt{2}\), and \(\delta p_{j} = i(\delta \hat{b}_{j} - \delta \hat{b}_{j}^{\dagger})/\sqrt{2}\). Similarly, we can define the quadratures of the input noise \(O_{j}^{\text{in}}\) (\(O = x, y, X, Y, q, p\)). For simplicity, we have removed the tilde signs for the quadrature operators. \(n = \left(\sqrt{2\kappa_{\delta a}} \hat{b}_{1}^{\text{in}}, \sqrt{2\kappa_{\delta m}} \hat{m}_{1}^{\text{in}}, \sqrt{2\gamma_{\delta}} \hat{b}_{1}^{\text{in}}, \sqrt{2\gamma_{\delta}} \hat{m}_{1}^{\text{in}}, \sqrt{2\gamma_{\delta}} \hat{b}_{2}^{\text{in}}, \sqrt{2\gamma_{\delta}} \hat{m}_{2}^{\text{in}}\right)^{T}\) is the vector of input noise, and the drift matrix \(A\) is large and its specific form is provided in appendix A.

Owing to the linearized dynamics and the Gaussian nature of input noise, the system preserves Gaussian states for all times. The steady state of the system is a six-mode Gaussian state, which is fully characterized by a \(12 \times 12\) covariance matrix (CM) \(C\), whose entries are defined as \(C_{jk}(t) = \frac{1}{2}(\langle u_{j}(t)u_{k}(t') + u_{k}(t')u_{j}(t)\rangle\) (\(s, k = 1, 2, \ldots, 12\)). The stationary CM \(C\) can be obtained by directly solving the Lyapunov equation [56, 57]

\[
AC + CA^{T} = -D,
\]

(9)

where \(D\) is the diffusion matrix defined by \(D_{jk}(t - t') = \frac{1}{2}(n_{j}(t)n_{k}(t') + n_{k}(t')n_{j}(t))\). It can be written in the form of a direct sum, \(D = D_{a} \oplus D_{m} \oplus D_{b}\), where \(D_{a}\) is related to the squeezed input noise of the two cavity modes

\[
D_{a} = \begin{pmatrix}
\kappa_{\delta a}(2N+1) & 0 & \sqrt{\kappa_{\delta a}} \sqrt{\kappa_{\delta m}} (-M + M^{*}) & \sqrt{\kappa_{\delta a}} \sqrt{\kappa_{\delta m}} (M - M^{*}) \\
0 & \kappa_{\delta a}(2N+1) & \sqrt{\kappa_{\delta a}} \sqrt{\kappa_{\delta m}} (M + M^{*}) & \sqrt{\kappa_{\delta a}} \sqrt{\kappa_{\delta m}} (-M - M^{*}) \\
\sqrt{\kappa_{\delta a}} \sqrt{\kappa_{\delta m}} (-M + M^{*}) & \sqrt{\kappa_{\delta a}} \sqrt{\kappa_{\delta m}} (M - M^{*}) & \kappa_{\delta a}(2N+1) & 0 \\
\sqrt{\kappa_{\delta a}} \sqrt{\kappa_{\delta m}} (M + M^{*}) & \sqrt{\kappa_{\delta a}} \sqrt{\kappa_{\delta m}} (-M - M^{*}) & 0 & \kappa_{\delta a}(2N+1)
\end{pmatrix},
\]

(10)

and \(D_{m}\) (\(D_{b}\)) is associated with the thermal input noise for two magnon (phonon) modes, \(D_{m} = \text{diag} [\kappa_{\delta m}(2N_{m}+1), \kappa_{\delta m}(2N_{m}+1), \kappa_{\delta m}(2N_{m}+1), \kappa_{\delta m}(2N_{m}+1)]\), and \(D_{b} = \text{diag} \left[\gamma_{1}(2N_{b}+1), \gamma_{1}(2N_{b}+1), \gamma_{2}(2N_{b}+1), \gamma_{2}(2N_{b}+1)\right]\). Once the CM of the system is obtained, one can then extract the state of the two phonon modes and calculate their entanglement property. We adopt the logarithmic negativity [58] to quantify the entanglement of the Gaussian states, which definition is provided in appendix B.

We present our main result of the steady-state entanglement between two YIG spheres in figure 2. The stability is guaranteed by the negative eigenvalues (real parts) of the drift matrix \(A\). We have adopted experimentally feasible parameters [14]: \(\omega_{a} = \omega_{m} = \omega_{B} = 2\pi \times 10\ \text{GHz}, \omega_{b} = 2\pi \times 10\ \text{MHz}, \omega_{b_{2}} = 1.2\omega_{b_{1}}, \gamma = 2\times 10\ \text{Hz}, \kappa_{a} = 2\pi \times 3\ \text{MHz}, \kappa_{m} = \kappa_{a}/5\), and \(T = 10\ \text{mK}\). Note that, in our model the linewidth of the magnon (cavity) mode is defined as \(2\kappa_{\delta a}\). Here we take \(2\kappa_{\delta a} = 1.2\ \text{MHz}\), which is larger than the magnon intrinsic dissipation (typically of the order of 1 MHz), as well as the demonstrated value 1.12 MHz [14]. For simplicity, we have assumed equal frequencies for the two cavity (magnon) modes, and squeezed driving fields, \(\omega_{\delta 1} = \omega_{\delta 2} \equiv \omega_{\delta} (O = a, m, s) [59]\), due to their flexible tunability, but generally different frequencies for the two phonon modes. This means that the frequencies of the two magnon drive fields are also different because \(\omega_{\delta j} = \omega_{m} - \omega_{b_{j}}\). For convenience, we have also assumed equal
dissipation rates for all pairs of modes of the same type. In figure 2(a), we show the mechanical entanglement versus two coupling rates \( g_1 = g_2 \equiv g \) and \( G_1 = G_2 \equiv G \), and consider \( g, G \ll \kappa_a \ll \omega_{b,1,2} \), in order to meet the condition used for deriving equation (7). Figure 2(b) shows that in the steady state the two cavity/magnon/phonon modes are all entangled, and the entanglement increases with larger \( r \). The mechanical entanglement is even stronger than the magnon entanglement when \( r > \sim 0.2 \), although the former is transferred from the latter. This is possible because the cavities are continuously driven, and the total entanglement is distributed among the three different subsystems with steady-state bipartite entanglement. We use a relatively larger cavity decay rate \( \kappa_a \gg \kappa_m \), which has been shown to be an optimal condition for obtaining magnon entanglement [46], which is a pre-requisite for phonon entanglement in our protocol. We would like to note that, for the parameters of figure 2(b), the entanglement of any two modes of different types are either negligibly small or zero.

In figure 3, we show the entanglement as a function of bath temperature for a two-mode squeezed vacuum of \( r = 0.4 \). This corresponds to a logarithmic negativity \( E_N = 0.8 \) [60] of the driving field, which has been experimentally demonstrated in reference [53]. With such a driving field, we obtain mechanical entanglement \( E_N = 0.54 \) for \( T = 10 \) mK, and the entanglement survives up to 118 mK.

Lastly, we would like to discuss how to detect the entanglement. The generated entanglement of two YIG spheres can be verified by measuring the CM of the two phonon modes [12, 13]. The mechanical quadratures can be measured by coupling each sphere to an additional optical cavity which is driven by a weak red-detuned laser. This yields an optomechanical state-swap interaction which maps the phonon state onto the cavity output field [61]. By homodyning this field, the mechanical quadratures can be measured, based on which the CM can be reconstructed.
4. Validity of the model

We now discuss the validity of the approximations that were made in our model. For the magnon modes, we have assumed low-lying excitations, \( \langle m \rangle m \rangle \ll 2N_0 \) in order to express the collective spin operators in terms of Boson operators. For a 250-\( \mu \)m-diameter YIG sphere, \( N \approx 3.5 \times 10^{16} \), and the coupling \( G = 0.2\epsilon_0 N \approx 2\pi \times 0.6 \) MHz used in figures 2(b) and 3 corresponds to \( \langle m \rangle \approx 1.2 \times 10^2 \) for \( G_0/2\pi = 50 \) MHz. Therefore, \( \langle m \rangle m \rangle \approx 1.4 \times 10^{14} \ll 2N_0 = 1.7 \times 10^{17} \), which is well satisfied.

We have also assumed the magnon frequency shift caused by the magnomechanical interaction to be negligible, i.e., \( \Delta \approx \Delta \). While in the numerical study we have considered two phonon modes of close frequencies, for simplicity we assume equal frequencies \( \omega_{b(1,2)}/2\pi = 10 \) MHz for a brief estimation. We obtain \( \langle b \rangle = G_0\langle m \rangle \approx 7.2 \times 10^3 \), and the frequency shift \( 2G_0 \langle b \rangle \approx 4.5 \times 10^3 \) Hz, which is much smaller than \( \Delta = \omega_b \approx 6.3 \times 10^7 \) Hz, and thus can be safely neglected.

We have further adopted strong pumps for the magnon modes, which may bring in unwanted nonlinearities owing to the Kerr nonlinear term \( K m m m m \) in the Hamiltonian [34], where \( K \) is the Kerr coefficient. For a 250-\( \mu \)m-diameter sphere, \( K/2\pi \approx 6.4 \) nHz [15]. In order to keep the Kerr effect negligible, \( K \langle m \rangle \ll \Omega \) must be guaranteed. With the parameters used in the plots in figures 2(b) and 3, we obtain a Rabi frequency \( \Omega \approx \langle m \rangle (\Delta^2 - g^2)/\Delta = 6.9 \times 10^{14} \) Hz (corresponding to the drive magnetic field \( B_0 \approx 3.8 \times 10^{-3} \) T and drive power \( P = 8.3 \) mW [62]), and we thus have \( K \langle m \rangle \approx 6.9 \times 10^{13} \) Hz \( \ll \Omega \). Therefore, the Kerr nonlinearity can also be safely neglected in our linearized model.

5. Conclusions

We have presented a protocol to entangle the vibrational modes of two massive ferromagnetic spheres in a hybrid cavity–magnon–phonon system. The cavity–magnon subsystem has an intrinsic state-swap interaction, whereas the magnon–phonon subsystem is coupled by a nonlinear magnetostrictive interaction. This allows for the successive transfer of quantum correlations from a two-mode squeezed driving field to two cavity modes, then to two magnon modes, and finally to two phonon modes. We further analyze the validity of the model in detail by confirming the conditions of the approximations that have been made, and the feasibility of the protocol by considering realistic parameters, as well as experimentally accessible squeezing in MW sources. Our work studies quantum entanglement between two truly massive objects and may find applications in the study of macroscopic quantum mechanics and gravitational quantum physics.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files). The source data for the figures is available at 10.5281/zenodo.4446839.

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Appendix A. Drift matrix

Here we provide the specific form of the drift matrix \( A \) used in equation (8), which can be constructed in the form of

\[
A = \begin{pmatrix}
A_c & A_{cm} & 0_4 \\
A_{cm} & A_m & A_{mb} \\
0_4 & -A_{mb} & A_b
\end{pmatrix},
\]

where \( 0_4 \) is the 4 \times 4 zero matrix, \( A_c = -\text{diag}(\delta_{a_1}, \delta_{a_2}, \delta_{a_2}, \delta_{a_2}) \), \( A_m = -\text{diag}(\delta_{m_1}, \delta_{m_2}, \delta_{m_2}, \delta_{m_2}) \), \( A_b = -\text{diag}(\gamma_{1}, \gamma_{1}, \gamma_{2}, \gamma_{2}) \), and \( A_{cm} \) and \( A_{mb} \) are the coupling matrices for the cavity–magnon and...
The entanglement of two-mode Gaussian states can be quantified by the logarithmic negativity \[58\], which is defined as \[63\]

\[
E_N := \max \left\{ 0, -\ln 2 \tilde{\nu} - \right\},
\]

where \(\tilde{\nu} = \min \text{eig}(|\Omega_2 C_\nu|\) (with the symplectic matrix \(\Omega_2 = \bigoplus_{j=1}^2 \sigma_j \) and the \(y\)-Pauli matrix \(\sigma_j\)) is the minimum symplectic eigenvalue of the CM \(C_\nu\), \(C_\nu\) the CM of two phonon modes, which is obtained by removing in \(C\) the rows and columns related to the cavity and magnon modes, and \(\mathcal{P} = \text{diag}(1, -1, 1, 1)\) is the matrix that performs partial transposition on CMs \[64\]. In the same way, we can calculate the logarithmic negativity of two cavity/magnon modes.

**Appendix B. Entanglement measure-logarithmic negativity**

The entanglement of two-mode Gaussian states can be quantified by the logarithmic negativity \[58\], which is defined as \[63\]

\[
E_N := \max \left\{ 0, -\ln 2 \tilde{\nu} - \right\},
\]

where \(\tilde{\nu} = \min \text{eig}(|\Omega_2 C_\nu|\) (with the symplectic matrix \(\Omega_2 = \bigoplus_{j=1}^2 \sigma_j \) and the \(y\)-Pauli matrix \(\sigma_j\)) is the minimum symplectic eigenvalue of the CM \(C_\nu\), \(C_\nu\) the CM of two phonon modes, which is obtained by removing in \(C\) the rows and columns related to the cavity and magnon modes, and \(\mathcal{P} = \text{diag}(1, -1, 1, 1)\) is the matrix that performs partial transposition on CMs \[64\]. In the same way, we can calculate the logarithmic negativity of two cavity/magnon modes.

**References**

[1] Leggett A J 1980 *Prog. Theor. Phys. Suppl*. 69 80

[2] Leggett A J 2002 J. Phys.: Condens. Matter 14 R415

[3] Fröwis F, Sekatski P, Dürr W, Gisin N and Sangouard N 2018 *Rev. Mod. Phys.* 90 025004

[4] Bassi A, Lochan K, Satin S, Singh T P and Ulbricht H 2013 *Rev. Mod. Phys.* 85 471

[5] Zhang J, Zhang T and Li J 2017 *Phys. Rev. A* 95 012141

[6] Weaver M J, Newsom D, Luna F, Löfler W and Bouwmeester D 2018 *Phys. Rev. A* 97 063832

[7] Marletto C and Vedral V 2017 *Phys. Rev. Lett.* 119 240402

[8] Aspelmeyer M, Kippenberg T J and Marquardt F 2014 *Rev. Mod. Phys.* 86 1391

[9] Palomaki T A, Teufel J D, Simmonds R W and Lehnert K W 2013 *Science* 342 710

[10] Riedinger R, Hong S, Norte A, Slater J A, Shang J, Krause A G, Anant V, Aspelmeyer M and Gröblacher S 2016 *Nature* 530 313

[11] Riedinger R, Wallucks A, Marinković I, Löschnauer C, Aspelmeyer M, Hong S and Gröblacher S 2018 *Nature* 556 473

[12] Ockeloen-Korppi C F, Damskäg E, Pirkkalainen J-M, Asjad M, Clerk A A, Massel F, Woolley M J and Sillanpää M A 2018 *Nature* 556 478

[13] Kotler S et al 2020 arXiv:2004.05515

[14] Zhang X, Zou C-L, Jiang L and Tang H X 2016 *Sci. Adv.* 2 e1501286

[15] Li J, Zhu S-Y and Agarwal G S 2018 *Phys. Rev. Lett.* 121 203601

[16] Li J, Zhu S-Y and Agarwal G S 2019 *Phys. Rev. A* 99 021801(R)

[17] Li J and Zhu S-Y 2019 *New J. Phys.* 21 085001

[18] Lachance-Quirion D, Tabuchi Y, Gloppe A, Usami K and Nakamura Y 2019 *Appl. Phys. Express* 12 070101

[19] Huebl H, Zollitsch C, Lotze J, Hocke F, Greifenstein M, Marx A, Gross R and Goennenwein S T B 2013 *Phys. Rev. Lett.* 111 127003

[20] Tabuchi Y et al 2014 *Phys. Rev. Lett.* 113 083603

[21] Zhang X, Zou C-L, Jiang L and Tang H X 2014 *Phys. Rev. Lett.* 113 156401

[22] Goryachev M, Farr W G, Creedon D L, Fan Y, Kostylev M and Tobar M E 2014 *Phys. Rev. Appl.* 2 054002

[23] Bai L et al 2015 *Phys. Rev. Lett.* 114 227201

[24] Zhang D et al 2015 *Npj Quantum Inf.* 1 15014

[25] Kittel C 1998 *Phys. Rev.* 110 836

[26] Zhang X et al 2015 *Nat. Commun.* 6 8914

[27] Zhang D et al 2017 *Nat. Commun.* 8 1368

[28] Bai L, Harder M, Hyde P, Zhang Z, Hu C-M, Chen Y P and Xiao J Q 2017 *Phys. Rev. Lett.* 118 217201

[29] Harder M, Yang Y, Yao B M, Yu C H, Rao J W, Gu J Y, Stamps R I and Hu C-M 2018 *Phys. Rev. Lett.* 121 132703

[30] Wang Y-P et al 2019 *Phys. Rev. Lett.* 123 127202

[31] Gonzalez-Ballestero C, Hümmer D, Gieseler J and Romero-Iñart O 2020 *Phys. Rev. B* 101 125404

[32] Ullah K, Tahir Nassem M and Moustacchiou O K 2020 *Phys. Rev. A* 102 033721

[33] Yu M, Shen H and Li J 2020 *Phys. Rev. Lett.* 124 213604

[34] Wang Y-P, Zhang G-Q, Zhang D, Li T-F, Hu C-M and You J Q 2018 *Phys. Rev. Lett.* 120 057202

[35] Tan H 2019 *Phys. Rev. Res.* 1 033161

[36] Kong C, Wang B, Liu Z-X, Xiong H and Wu Y 2019 *Opt. Express* 27 5544
[37] Ding M-S, Zheng L and Li C 2019 Sci. Rep. 9 15723
[38] Potts C A, Bittencourt V A S V, Viola Kusminskiy S and Davis J P 2020 Phys. Rev. Appl. 13 064001
[39] Huai S-N, Liu Y-L, Zhang J, Yang L and Liu Y-X 2019 Phys. Rev. A 99 043803
[40] Wang M, Zhang D, Li X-H, Wu Y-Y and Sun Z-Y 2019 IEEE Photon. 11 5300108
[41] Wang L et al 2020 Ann. Phys. 532 2000028
[42] Zhang Z, Scully M O and Agarwal G S 2019 Phys. Rev. Res. 1 023021
[43] Yuan H Y, Zheng S, Ficek Z, He Q Y and Yang M-H 2020 Phys. Rev. B 101 014419
[44] Elyasi M, Blanter Y M and Bauer G E W 2020 Phys. Rev. B 101 054402
[45] Nair J M P and Agarwal G S 2020 Appl. Phys. Lett. 117 084001
[46] Yu M, Zhu S-Y and Li J 2020 J. Phys. B: At. Mol. Opt. Phys. 53 065402
[47] Luo D-W, Qian X-F and Yu T 2020 arXiv:2006.06132
[48] Zhang J, Peng K and Braunstein S L 2003 Phys. Rev. A 68 031808
[49] Mazzola L and Paternostro M 2011 Phys. Rev. A 83 062335
[50] Paternostro M, Mazzola L and Li J 2012 J. Phys. B: At. Mol. Opt. Phys. 45 154010
[51] Holstein T and Primakoff H 1940 Phys. Rev. 58 1098
[52] Eichler C et al 2011 Phys. Rev. Lett. 107 113601
[53] Flurin E, Roch N, Mallet F, Devoret M H and Huard B 2012 Phys. Rev. Lett. 109 183901
[54] Menzel E P et al 2012 Phys. Rev. Lett. 109 230502
[55] Pogorzalek S et al 2019 Nat. Commun. 10 2604
[56] Vitali D, Gigan S, Ferreira A, Boehm H R, Tombesi P, Guererro A, Vedral V, Zeilinger A and Aspelmeyer M 2007 Phys. Rev. Lett. 98 030405
[57] Parks P C and Hahn V 1993 Stability Theory (Englewood Cliffs, NJ: Prentice- Hall)
[58] Eisert J 2001 PhD Thesis University of Potsdam, Potsdam, Germany
[59] Vidal G and Werner R F 2002 Phys. Rev. A 65 032314
Plenio M B 2005 Phys. Rev. Lett. 95 090503
[60] Reference [53] reported entangled MW fields of logarithmic negativity $E_N = 1.15$ with the base of 2, which corresponds to $E_N = 0.8$ in our definition with the base of natural constant.
[61] Our model is, however, not limited to this special case, because the QLEs (7) and (8) are derived by assuming only that, in each cavity, the cavity and magnon modes are resonant with the squeezed drive field and the blue mechanical sideband, i.e., $\omega_aj = \omega_mj = \omega sj$, and $\Delta_j $ $\approx \omega bj$ ($j = 1, 2$).
[62] Reference [53] reported entangled MW fields of logarithmic negativity $E_N = 1.15$ with the base of 2, which corresponds to $E_N = 0.8$ in our definition with the base of natural constant.
[63] Li J, Gröblacher S, Zhu S-Y and Agarwal G S 2018 Phys. Rev. A 98 011801(R)
[64] The drive magnetic field $B_0$ is related to the power $P$ via $B_0 = 1R_2 P \mu_0 c$ [15], with $R$ being the radius of the sphere, $c$ the speed of an electromagnetic wave propagating in vacuum, and $\mu_0$ the vacuum magnetic permeability.
[65] Adesso G and Illuminati F 2007 J. Phys. A: Math. Theor. 40 7821
[66] Simon R 2000 Phys. Rev. Lett. 84 2726