Poisson-Lie Structures and Quantisation with Constraints

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Abstract

We develop here a simple quantisation formalism that make use of Lie algebra properties of the Poisson bracket. When the brackets \( \{ H, \varphi_i \} \) and \( \{ \varphi_i, \varphi_j \} \), where \( H \) is the Hamiltonian and \( \varphi_i \) are primary and secondary constraints, can be expressed as functions of \( H \) and \( \varphi_i \) themselves, the Poisson bracket defines a Poisson-Lie structure. When this algebra has a finite dimension a system of first order partial differential equations is established whose solutions are the observables of the theory. The method is illustrated with a few examples.

1. The quantisation of systems with constraints is old as the quantum mechanics itself. The first such problem brilliantly solved was the finding of the hydrogen atom spectrum by Pauli in 1926 [1]. Enforcing the constraints in classical mechanics has a satisfactory solution [2, 3], but this is no more true in quantum mechanics. The constraints, i.e. a set of functions

\[ \varphi_i(q, p) = 0 \quad , \quad i = 1, 2, \ldots, m \quad (1.1) \]

restrict the motion of the classical system to a manifold embedded in the initial Euclidean phase space and in consequence the canonical quantisation rules

\[ [q_i, p_j] = \hbar \delta_{ij} \]

are no more sufficient for the quantum description of the physical system.

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In general quantisation is not a well-defined procedure existing today a
variety of methods which sometimes give different results when applied to
physical problems, although the starting points are similar from the classi-
cal point of view. We mention only the people who derive the Schrödinger
equation by Feynman’s path integral method; see for example [4, 5, 6], who
find an extra energy term proportional to the Riemann scalar curvature of
the manifold, even for the simple case of the motion of a particle on the
n-dimensional sphere.

The most successful method for imposing quantum constraints is that
found by Dirac [7], however nowadays there are some voices who reject it
claiming that the resulting energy spectrum is incorrect even for simple
systems [8, 9]. The mechanism found by Dirac was the introduction of a
new symplectic structure, the Dirac bracket, to handle the second-class con-
straints and the using of the Legendre multiplicators for finding the true
Hamiltonian.

The purpose of this paper is to look at the problem of quantisation with
constraints from a slightly modified point of view and to show that the new
proposal leads to correct results.

When one studies constrained systems one starts with a Hamiltonian,
\( H(q, p) \), and a number of relations of the form (1.1), called primary con-
straints, which at their turn generate secondary constraints. Let suppose
that after a finite number of steps the process closes, i.e. no new secondary
constraints are generated. In the most simple cases one obtains a Poisson
algebra of the form

\[
\{ H(q, p), \varphi_i(q, p) \} = C^j_i \varphi_j(q, p) \\
\{ \varphi_i(q, p), \varphi_j(q, p) \} = C^k_{ij} \varphi_k(q, p)
\]  

(1.2)

where \( C^j_i \) and \( C^k_{ij} \) are constant structure coefficients. In our opinion this
Poisson structure is the basic structure for the quantisation procedure. Since
the Poisson algebra (1.2) transforms by quantisation into a Lie algebra the
physical observables of the model will be given by the Casimir operators;
this means that no one of the initial operators transform into a veritable
observable.

We applied this idea to the motion of a particle on the \( n \)-dimensional
sphere and we have found that the "Hamiltonian", i.e. the Casimir of the
 corresponding algebra is a quadratic function in the old Hamiltonian and the
constraints [10]. This quantity is the square of the angular momentum, a result which everybody expected to be so.

We want to extend this method to more general situations than those given by Eqs. (1.2) by developing a formalism which makes use of the Lie algebra properties of the Poisson bracket. We hope that this formalism will solve at least a part of problems encountered in quantisation with constraints.

More precisely let \((u_1, u_2, \ldots, u_r)\) denote \(r\) functions of \(2n\) independent variables \((q_1, \ldots, q_n, p_1, \ldots, p_n)\) and suppose that all Poisson brackets \(\{u_i, u_j\}\) can be expressed as functions of \((u_1, u_2, \ldots, u_r)\). In this case these functions form a Poisson-Lie structure and any function of \((u_1, u_2, \ldots, u_r)\) belongs to this algebra. This kind of structure was first introduced by S. Lie who use the name of function group [13]. The full phase space is \(\mathbb{R}^{2n}\) with generic point \((q, p)\) and the usual Poisson algebra \(\mathcal{P} = (C^\infty(\mathbb{R}^{2r}), \{\cdot, \cdot\})\) is the setting for the problem.

The systems with constraints are good candidates for such structures since we start with a Hamiltonian and a number of primary constraints of the form (1.1). By taking the Poisson brackets \(\{H, \varphi_i\}\) and \(\{\varphi_i, \varphi_j\}\) they generate secondary constraints. Let suppose that this process closes and at the end we obtain a finite number of independent dynamical variables \((u_1, \ldots, u_r)\) which describe the dynamics of the constrained system. These dynamical variables satisfy a system of equations of the following form

\[
\{u_i, u_j\} = f_{ij}(u_1, \ldots, u_r) \tag{1.3}
\]

with \(f_{ij} = -f_{ji}\) skew-symmetric functions. If \(f_{ij}\) has a power series expansion this may have the form

\[
f_{ij}(u_1, \ldots, u_r) = a_{ij} + b_{ij}^k u_k + c_{ij}^{kl} u_k u_l + \ldots
\]

In this approach we make no distinction between Hamiltonian, primary and secondary constraints, first or second class constraints, all of them are simply dynamical variables living in a democratic society, the rules on which they obey being the system of equations (1.3). Now, because the Hamiltonian is only one of the pairs, we have to solve the problem of integrals of motion for a dynamical system governed by Eqs. (1.3). It seems natural to extend the classical solution, \(F\) is an integral of motion if \(\{F, H\} = 0\), to the new
context by requiring that $F$ is an integral of motion if

$$\{F, u_i\} = 0, \quad i = 1, 2, \ldots, r$$

We remind that the same condition was used by Dirac too [7], but only in the new symplectic structure, the Dirac bracket, $\{\cdot, \cdot\}_D$, and not in the canonical Poisson structure as we do here. Taking into account the Poisson-Lie structure defined by Eqs. (1.3) the above equation is equivalent to the following system of first order partial differential equations

$$\sum_{i=1}^{r} \frac{\partial F}{\partial u_i} \{u_i, u_j\} = \sum_{i=1}^{r} \frac{\partial F}{\partial u_i} f_{ij}(u_1, \ldots, u_r) = 0$$

(1.4)

$$j = 1, \ldots, r$$

Being a homogeneous system a necessary condition for the existence of a non-trivial solution, $F \neq ct$, is

$$\det |\{u_i, u_j\}| = \det |f_{ij}(u_1, \ldots, u_r)| = 0$$

(1.5)

The solution(s) of the system (1.4) will depend in general on all dynamical variables and will play the rōle of the classical Hamiltonian for non-constrained systems, they being the conserved physical quantities of the dynamical system. The classical theory of first order partial differential equations tell us that if the rank of the system (1.4) is $p$ then (1.4) may have up to $n = r - p$ independent solutions and the easiest way to obtain them is by using the characteristic method [14, 15]. The simplest solutions of the system (1.4) are called elementary solutions [14, 15], the general solution being an arbitrary continuous and derivable function of these elementary solutions $G = G(F_1, F_2, \ldots)$.

By quantisation $\{\cdot, \cdot\}$ goes into $\frac{1}{i\hbar} [\cdot, \cdot]$ and the observables of the theory will be the solutions of the system (1.4). When the algebra (1.3) reduces to that of a semi-simple Lie algebra the solutions $F_k$ will be the Casimir operators of this algebra and if the respective algebra has rank $l$ there will be $l$ Casimir operators by the well-known result of Racah [16]. Thus Eqs. (1.3)-(1.4) represent a generalisation of the known powerful machinery of representation theory of Lie algebras and give us a method for finding the maximal set of commuting observables for a given physical system. Finding the physically relevant operators and their spectra is one of the goals of any quantum theory.
2. In the following we shall illustrate the new method with a few examples to show that its content is not void.

2.1 We consider first the motion of a particle on a \( n - 1 \)-dimensional sphere which is the toy model for testing quantum constrained dynamics \[8, 10, 11, 12\]. The free Hamiltonian is

\[ H = \frac{1}{2} \langle p, p \rangle = \frac{p^2}{2} \]

where \( \langle p, p \rangle \) denotes the Euclidean scalar product in the \( n \)-dimensional space, i.e., \( \sum_{i=1}^{n} p_i^2 \). The primary constraint is usually written as

\[ \varphi = (q, q) - R^2 = r^2 - R^2 = 0 \]

The Eqs. (1.3) take the form

\[
{\{ \varphi, H \} = 2V \quad \{ V, H \} = 2H \\
\{ \varphi, V \} = 2(\varphi + R^2) = 2r^2}
\]

where \( V = (q, p) \) is the secondary constraint. The system of differential equations is

\[
-2V \frac{\partial F}{\partial H} - 2(\varphi + R^2) \frac{\partial F}{\partial V} = 0 \\
2V \frac{\partial F}{\partial \varphi} - 2H \frac{\partial F}{\partial V} = 0 \\
2(\varphi + R^2) \frac{\partial F}{\partial \varphi} - 2H \frac{\partial F}{\partial H} = 0
\]

(2.1)

The condition (1.5) is satisfied the dimension of the matrix being odd. By applying the characteristic method \[14, 15\] we have from the last equation

\[ \varphi'(t) = \varphi + R^2 \quad H'(t) = -H \]

The solution is

\[ \varphi + R^2 = e^t \quad H = e^{-t} \]

By eliminating \( t \) we find that the solution has the form

\[ F = (\varphi + R^2)H + g(V) \]
If we use this form in the second equation we get \( g(V) = -V^2/2 \). Thus an elementary solution of the system (2.1) is

\[
F = UH - V^2/2
\]

where \( U = \varphi + R^2 = r^2 = (q, q) \).

The "Hamiltonian" will be

\[
H = UH - V^2/2 = \frac{1}{2} \left( \sum_{i=1}^{n} q_i^2 \sum_{j=1}^{n} p_j^2 - \left( \sum_{i=1}^{n} p_i q_i \right) \right) = \frac{1}{2} \sum_{i<j}^{n} (q_i p_j - q_j p_i)^2 = \frac{1}{2} \sum_{i<j}^{n} L_{ij}^2 = \frac{1}{2} L^2
\]

Thus the quantum observable is the square of the angular momentum [10, 11, 12]. Let show that \( H \) is the good classical Hamiltonian of the problem. The Hamilton equations

\[
\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}
\]

have the form

\[
\dot{q}_j = p_j (q, q) - q_j (q, p)
\]

\[
\dot{p}_j = -q_j (q, q) + p_j (q, p)
\]

Multiplying the first equation by \( p_j \), the second by \( q_j \) and taking the sum we get

\[
\dot{q}_j p_j + q_j \dot{p}_j = \frac{d}{dt} (q, p) = \frac{dV}{dt} = 0
\]

Similarly multiplying the first equation by \( q_j \) we obtain

\[
\dot{q}_j q_j = \frac{1}{2} \frac{dU}{dt} = (q, p)(q, q) - (q, q)(q, p) = 0
\]

which shows that \( U \) and \( V \) are constant in time and if the constraints are fulfilled at the initial time they will be fulfilled at any time. We consider the last two relations as a consistency check of the formalism.

2.2 We consider now a more complicated structure, the functions \( f_{ij} \) entering Eqs. (1.4) being quadratic functions. One of the first such a structure
is that introduced by Sklyanin in connection with the Yang-Baxter equations [17]. The eqs. (1.4) have the following form

\[
\begin{align*}
\{u_2, u_1\} &= b_1 u_3 u_4 \quad \{u_3, u_1\} = b_2 u_2 u_4 \\
\{u_4, u_1\} &= b_3 u_2 u_3 \quad \{u_3, u_2\} = a_1 u_1 u_4 \\
\{u_4, u_2\} &= a_2 u_1 u_3 \quad \{u_4, u_3\} = a_3 u_1 u_2
\end{align*}
\]

\(a_i\) and \(b_i\), \(i = 1, 2, 3\) being arbitrary complex numbers. The case considered by Sklyanin was \(a_1 = -a_2 = a_3\) and \(b_1 + b_2 + b_3 = 0\). The system (1.5) has the form

\[
\begin{align*}
b_1 u_3 u_4 \frac{\partial F}{\partial u_2} + b_2 u_2 u_4 \frac{\partial F}{\partial u_3} + a_2 u_1 u_3 \frac{\partial F}{\partial u_4} &= 0 \\
-b_1 u_3 u_4 \frac{\partial F}{\partial u_1} + a_3 u_1 u_4 \frac{\partial F}{\partial u_3} + b_3 u_2 u_3 \frac{\partial F}{\partial u_4} &= 0 \tag{2.2} \\
-b_2 u_2 u_4 \frac{\partial F}{\partial u_1} - a_3 u_1 u_4 \frac{\partial F}{\partial u_2} + a_1 u_1 u_2 \frac{\partial F}{\partial u_4} &= 0 \\
-b_3 u_2 u_3 \frac{\partial F}{\partial u_1} - a_2 u_1 u_3 \frac{\partial F}{\partial u_2} - a_1 u_1 u_2 \frac{\partial F}{\partial u_3} &= 0
\end{align*}
\]

The condition (1.5) is equivalent to

\[
a_1 b_1 - a_2 b_2 + a_3 b_3 = 0 \tag{2.3}
\]

so in the following we suppose that (2.3) holds. From the first equation we have

\[
u_2'(t) = b_1 u_3 u_4, \quad u_3'(t) = b_2 u_2 u_4, \quad u_4'(t) = b_3 u_2 u_3
\]

From the first two relations we have that \(b_1 u_3^2 - b_2 u_2^2 = ct\). This suggest to look for a solution of the form

\[
F(u_1, u_2, u_3, u_4) = b_1 u_3^2 - b_2 u_2^2 + g(u_1)
\]

independent of \(u_4\). The substitution of this \(F\) in the second equation (2.2) gives \(g(u_1) = a_3 u_1^2\) and the first Casimir has the form

\[
C_1 = a_3 u_1^2 - b_2 u_2^2 + b_1 u_3^2
\]

In the same manner one finds the second solution which is

\[
C_2 = a_1 u_1^2 - b_3 u_3^2 + b_2 u_4^2
\]
2.3 Another quadratic algebra is found in ref. [18] used to describe the kinematical symmetry of a spin chain on a one dimensional lattice. It has the form

\[
\{u_1, u_2\} = -\frac{a}{2} u_2^2 \quad \{u_1, u_3\} = au_1 \quad (2.4)
\]

\[
\{u_2, u_3\} = au_2 \quad \{u_4, u_i\} = 0 \quad i = 1, 2, 3
\]

Since \( u_4 \) commutes with the other generators a solution of the eqs. (1.4)) is of the form \( f(u_4) \) with \( f \) an arbitrary derivable function. The other Casimir is

\[
C = u_1 u_2^{-1} - \frac{1}{2} u_3^2
\]

If we perturb the second equation (2.4) to the following form

\[
\{u_1, u_3\} = au_1 + bu_2 \quad (2.2)
\]

obtaining a Poisson-Lie structure on the 2-dimensional Galilei algebra [19], the Casimir is more complicated and cannot be guessed simply. The characteristic method gives

\[
C = au_1 u_2^{-1} - \frac{1}{2} \log u_2 - \frac{a}{2} u_3
\]

2.4 An other interesting example appears in the construction of Wess-Zumino-Witten models on non semi-simple groups [20]. The algebra has the following structure

\[
\{J, P_i\} = \epsilon_{ij} P_j, \quad \{P_i, P_j\} = \epsilon_{ij} T, \quad \{T, J\} = \{T, P_i\} = 0, i = 1, 2
\]

In general, given a Lie algebra to define a WZW model one needs a bilinear form in the generators of the algebra, form which is symmetric, invariant and non-degenerate. Usually for semi-simple groups one takes \( \text{Tr} u_i u_i \), with the trace taken in the adjoint representation of the group. For nonsemisimple groups this quadratic form is degenerate. By applying our formalism one finds easily the two Casimirs

\[
C_1 = P_1^2 + P_2^2 + 2JT, \quad C_2 = g(T)
\]
where \( g(T) \) is an arbitrary derivable function of \( T \). Thus the most general bilinear form is

\[
\Omega = a(P_1^2 + P_2^2 + 2JT) + bT^2
\]

where \( a \) and \( b \) are two arbitrary constants, which is the result of Nappi and Witten.

2.5 Now we want to show that finding the spectrum of the hydrogen atom is also a problem of quantisation with constraints. The classical Hamiltonian is

\[
H = \frac{p^2}{2m} - \frac{\kappa}{r}
\]

where \( m \) is the reduced mass and \( \kappa = Ze^2 \). \( H \) and the angular momentum \( L = r \times p \) are constants of the motion. But these quantities are not enough to make the orbit to be closed, and not enough for having a discrete spectrum. We quote from Schiff’s book [21]

"The rotational symmetry of \( H \) is enough to cause the orbit to lie in some plane through \( O \), but is not enough to require the orbit to be closed. A small deviation of the potential energy from the Newtonian form \( V(r) = -(\kappa/r) \) causes the major axis \( PA \) of the ellipse to precess slowly, so that the orbit is not closed. This suggests that there is some quantity, other than \( H \) and \( L \), that is a constant of the motion and that can be used to characterise the orientation of the major axis in the orbital plane."

Such a quantity, which we see as a constraint, is the Laplace-Runge-Lenz vector. It is proportional with the Di-polar momentum of the orbit and has the form

\[
M = \frac{p \times L}{m} - \frac{\kappa}{r} r
\]

These constraints generate the first quadratic algebra in quantum physics. Indeed we have

\[
\{M_i, M_j\} = -\frac{2}{m} \epsilon_{ijk} HL_k \quad i, j = 1, 2, 3
\]

where \( H \) and \( L \) are the energy and, respectively, the angular momentum. The energy commutes with all the other quantities

\[
\{H, M_i\} = \{H, L_i\} = 0 \quad i = 1, 2, 3
\]
and we have also

\[ \{M_i, L_j\} = \epsilon_{ijk} M_k, \quad \{L_i, L_j\} = \epsilon_{ijk} L_k, \quad i = 1, 2, 3 \]

The Eqs. (1.4) have the form

\[
\begin{align*}
-L_3 \frac{\partial F}{\partial L_2} + L_2 \frac{\partial F}{\partial L_3} - M_3 \frac{\partial F}{\partial M_2} + M_2 \frac{\partial F}{\partial M_3} &= 0 \\
L_3 \frac{\partial F}{\partial L_1} - L_1 \frac{\partial F}{\partial L_3} + M_3 \frac{\partial F}{\partial M_1} - M_1 \frac{\partial F}{\partial M_3} &= 0 \\
-L_2 \frac{\partial F}{\partial L_1} + L_1 \frac{\partial F}{\partial L_2} - M_2 \frac{\partial F}{\partial M_1} + M_1 \frac{\partial F}{\partial M_2} &= 0 \\
-M_3 \frac{\partial F}{\partial L_2} + M_2 \frac{\partial F}{\partial L_3} + \alpha L_3 \frac{\partial F}{\partial M_2} - \alpha L_2 \frac{\partial F}{\partial M_3} &= 0 \\
M_3 \frac{\partial F}{\partial L_1} - M_1 \frac{\partial F}{\partial L_3} - \alpha L_3 \frac{\partial F}{\partial M_1} + \alpha L_1 \frac{\partial F}{\partial M_3} &= 0 \\
-M_2 \frac{\partial F}{\partial L_1} + M_1 \frac{\partial F}{\partial L_2} + \alpha L_2 \frac{\partial F}{\partial M_1} - \alpha L_1 \frac{\partial F}{\partial M_2} &= 0
\end{align*}
\] (2.5)

where \( \alpha = 2H/m \). Since \( H \) commutes with all the other quantities it is in the centre of algebra and, such as in the previous example, it will be a Casimir, i.e. an observable in the quantum theory. Thus \( C_1 = H = E \) is a good quantum number.

One can easily see that \( L^2 \) and \( M^2 \) are separately solutions of the first three equations (2.5), but none of them satisfies the last three equations. We look for a solution of the form

\[ F = a L^2 + b M^2 \]

with \( a \) and \( b \) some constants. We find from the fourth equation that \( b/a = -1/\alpha \) and the second Casimir is

\[ C_2 = a(L^2 - \frac{M^2}{\alpha}) \]

The third is

\[ C_3 = L \cdot M \]
If we use the quantum form of $M^2$, i.e.

$$M^2 = \frac{2E}{m} (L^2 + \hbar^2) + \kappa^2$$

and take $a = 1/4$ in the second Casimir, we find the known form of the energy levels

$$E = -\frac{m\kappa^2}{2\hbar^2(2c + 1)^2}$$

where $c = 0, 1/2, 1, \ldots$ are the eigenvalues of $C_2$. In conclusion the hydrogen atom has a symmetry group, but this is a nonsemisimple one, its Lie algebra has dimension seven, and, more important, for explaining the discrete spectrum is not necessary to invent a dynamical symmetry like $O(4)$.

3. The Dirac quantum theory was patterned after the classical theory, the ”observables” representing constraints must have zero expectation values. This requirement is not consistent with the fact that the Poisson brackets between Hamiltonian and constraints and between constraints themselves may not vanish such as Eqs. (1.3) show. In this paper we have shown that this inconsistency disappears if we postulate that the observables are the Casimir operators of the algebra (1.3). A consequence of this postulate is the following, starting with a Dirac form Hamiltonian

$$H_D = H + \mu_i \varphi_i$$

may be misleading and cause troubles when using it for the description of physical systems, the true Hamiltonians being more complicated functions of both the old Hamiltonian and the constraints together, as the above examples suggest. The lesson to be learnt is that for constrained systems almost no one of the initial dynamical variables transforms into an observable. In this respect the hydrogen atom is an exception, the reason being that the classical Hamiltonian commutes with all the constraints, being in the centre of the Poisson-Lie group.

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