TAME EXTENSION OF ALMOST O-MINIMAL STRUCTURE

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Abstract. We consider an almost o-minimal expansion of an ordered group $M = (M, <, +, 0, \ldots)$ and its tame extension $N = (N, <, +, 0, \ldots)$. We demonstrate that the subset $\{ x \in M^n \mid N \models \Phi(x, a) \}$ of $M^n$ defined by a formula $\Phi(x, y)$ with $M$-bounded parameters $a$ in $N$ is $M$-definable. We also introduce its corollaries.

1. Introduction

The notion of tame pairs was initially introduced by Marker and Steinhorn in [7] (using different terms). Pillay also considered the same problem in [9]. It was further developed by van den Dries and Lewenberg in [3, 2]. Using Marker and Steinhorn’s result on definable types [7], we get that the subset $\{ x \in M^n \mid N \models \Phi(x, a) \}$ of $M^n$ defined by a formula $\Phi(x, a)$ with parameters $a$ in $N$ is $M$-definable [2, Theorem 1.1]. Here, $M = (M, <, +, 0, \ldots)$ is an o-minimal structure and $N = (N, <, +, 0, \ldots)$ is its tame extension. The corollaries of this result together with the outputs of [3] are discussed in [2]. They are used for the study of limit sets of definable families in [4].

The author introduced the notion of almost o-minimality in [6], which satisfies a weaker finiteness condition than o-minimality. It is a generalization of a locally o-minimal expansion of the set of reals and admits uniform local definable cell decomposition [6, Theorem 1.7]. We anticipate that assertions which hold true in o-minimal structures also hold true in almost o-minimal structures under reasonable additional assumptions. In this paper, we investigate a tame extension of an almost o-minimal structure. We demonstrate that the subset of $M^n$ defined by a formula with $M$-bounded parameters in $N$ is $M$-definable for an almost o-minimal structure $M$ and its tame extension $N$.

Let us recall the definitions.

Definition 1.1 ([6]). An expansion $M = (M, <, \ldots)$ of densely linearly ordered set without endpoints is almost o-minimal if any bounded definable set in $M$ is a finite union of points and open intervals.

Definition 1.2 ([3]). Let $L$ be a language containing a predicate $<$. Let $M = (M, <, \ldots)$ be an $L$-expansion of a dense linear order without endpoints, and $N = (N, <, \ldots)$ be its extension. An element $y \in N$ is $M$-bounded if $x_1 \leq y \leq x_2$ for some $x_1, x_2 \in M$. An $N$-definable set is called parameterized by $M$-bounded parameters if there exists a finite subset $A$ of $M$-bounded elements in $N$, and it is defined by an $L(M \cup A)$-formula. When the extension $M \subseteq N$ is elementary, we say that the extension $M \subseteq N$ is tame or $M$ is tame in $N$ (in [7], $M$ is called...
Dedekind complete in \( \mathcal{N} \) if for each \( \mathcal{M} \)-bounded \( y \in \mathcal{N} \), there is an \( x \in \mathcal{M} \) such that either

(1) \( x = y \), or
(2) \( x < y \) and there are no \( x' \in \mathcal{M} \) with \( x < x' < y \), or
(3) \( y < x \) and there are no \( x' \in \mathcal{M} \) with \( y < x' < x \).

Such an \( x \) is uniquely determined by \( y \). We call \( x \) the standard part of \( y \) relative to \( \mathcal{M} \) and write \( x = \text{st}_\mathcal{M}(y) \). We omit the subscript \( \mathcal{M} \) when it is clear from the context. An \( n \)-tuple \( (y_1, \ldots, y_n) \in \mathcal{N}^n \) is \( \mathcal{M} \)-bounded if each coordinate \( y_i \) is \( \mathcal{M} \)-bounded. If \( \mathcal{M} \subseteq \mathcal{N} \) is a tame extension and \( (y_1, \ldots, y_n) \in \mathcal{N}^n \) is \( \mathcal{M} \)-bounded, we put \( \text{st}_\mathcal{M}(y_1, \ldots, y_n) = (\text{st}_\mathcal{M}(y_1), \ldots, \text{st}_\mathcal{M}(y_n)) \).

Our main result is as follows:

**Theorem 1.3.** Let \( \mathcal{L} \) be a language. Let \( \mathcal{M} = (\mathcal{M}, <, +, 0, \ldots) \) be an almost o-minimal \( \mathcal{L} \)-expansion of an ordered group and \( \mathcal{M} \subseteq \mathcal{N} = (\mathcal{N}, <, +, 0, \ldots) \) be a tame extension. Let \( a \in \mathcal{N}^n \) be an \( \mathcal{M} \)-bounded tuple and \( \Phi(x,y) \) be \( \mathcal{L}(\mathcal{M}) \)-formula, where \( x \) and \( y \) are \( m \)-tuple and \( n \)-tuple of free variables, respectively. Then, the set

\[
S = \{ x \in \mathcal{M}^m \mid \mathcal{N} \models \Phi(x,a) \}
\]

is \( \mathcal{M} \)-definable. In other words, the intersection \( S \cap \mathcal{M}^m \) of an \( \mathcal{N} \)-definable subset \( S \) of \( \mathcal{N}^m \) parameterized by \( \mathcal{M} \)-bounded parameters with \( \mathcal{M}^m \) is \( \mathcal{M} \)-definable.

This paper is organized as follows: We use several results obtained in previous studies. We recall them in Section 2. Section 3 is the main body of this paper, and it is devoted to the proof of the theorem. We introduce the corollaries of the theorem in Section 4.

In the last of this section, we summarize the terms and notations used in this paper. When \( \mathcal{M} \subseteq \mathcal{N} \) is an elementary extension and \( S \) is an \( \mathcal{M} \)-definable set, the notation \( S^\mathcal{N} \) denotes the \( \mathcal{N} \)-definable subset defined by the formula defining the \( \mathcal{M} \)-definable set \( S \). Since \( \mathcal{M} \subseteq \mathcal{N} \) is an elementary extension, \( S^\mathcal{N} \) is independent of the choice of the formula \( \phi \).

When a first-order structure in consideration is clear from the context, the term ‘definable’ means ‘definable in the structure with parameters.’ We call it \( \mathcal{M} \)-definable when we emphasize the structure \( \mathcal{M} \). The notation \( f|_A \) denotes the restriction of a map \( f : X \to Y \) to a subset \( A \) of \( X \). Consider a linearly ordered set without endpoints (\( \mathcal{M}, < \)). An open interval is a nonempty set of the form \( \{ x \in \mathcal{M} \mid a < x < b \} \) for some \( a, b \in \mathcal{M} \cup \{ \pm \infty \} \). It is denoted by \( (a,b) \) in this paper. An open box is the Cartesian product of open intervals. The closed interval is defined similarly and denoted by \([a,b] \). When an expansion \( \mathcal{M} = (\mathcal{M}, <, \ldots) \) of a dense linear order without endpoints is given, the set \( \mathcal{M} \) equips the order topology induced from the order \( < \). The space \( \mathcal{M}^n \) equips the product topology of the order topology. We consider these topologies. The space \( \mathcal{M}^0 \) is a singleton with the trivial topology.

**2. Preliminary**

We recall the results in \([6]\) and \([5]\) in this section. We first recall the definition of dimension of a set definable in a structure.

**Definition 2.1** (\([5]\)). Consider an expansion of a densely linearly order without endpoints \( \mathcal{M} = (\mathcal{M}, <, \ldots) \). Let \( X \) be a nonempty definable subset of \( \mathcal{M}^n \). The dimension of \( X \) is the maximal nonnegative integer \( d \) such that \( \pi(X) \) has a nonempty
interior for some coordinate projection $\pi : M^n \to M^d$. We set $\dim(X) = -\infty$ when $X$ is an empty set.

We also need the following definition:

**Definition 2.2 (Local monotonicity).** A function $f$ defined on an open interval $I$ is locally constant if, for any $x \in I$, there exists an open interval $J$ such that $x \in J \subseteq I$ and the restriction $f|_J$ of $f$ to $J$ is constant. A function $f$ defined on an open interval $I$ is locally strictly increasing if, for any $x \in I$, there exists an open interval $J$ such that $x \in J \subseteq I$ and $f$ is strictly increasing on the interval $J$. We define a locally strictly decreasing function similarly.

The author developed the dimension theory for sets definable in a definably complete locally o-minimal structure satisfying the property (a) which is defined in [5, Definition 1.1]. We do not give the definitions of definably complete structures, locally structures and property (a) here. Their definitions and their references are found in [5]. The important fact is that an almost o-minimal expansion of an ordered group is a definably complete locally o-minimal structure satisfying the property (a) thanks to [6, Corollary 2.12, Lemma 4.6] and [5, Proposition 2.13]. Using this fact, we get the following proposition:

**Proposition 2.3.** Let $M = (M, <, +, 0, \ldots)$ be an almost o-minimal expansion of an ordered group. The following assertions hold true:

(1) **(Strong local monotonicity)** Let $I$ be an interval and $f : I \to M$ be a definable function. There exists a mutually disjoint definable partition $I = X_d \cup X_c \cup X_+ \cup X_-$ satisfying the following conditions:
   (i) the definable set $X_d$ is discrete and closed;
   (ii) the definable set $X_c$ is open and $f$ is locally constant on $X_c$;
   (iii) the definable set $X_+$ is open and $f$ is locally strictly increasing and continuous on $X_+$;
   (iv) the definable set $X_-$ is open and $f$ is locally strictly decreasing and continuous on $X_-.$

(2) Let $X_1$ and $X_2$ be definable subsets of $M^n$. Set $X = X_1 \cup X_2$. Assume that $X$ has a nonempty interior. At least one of $X_1$ and $X_2$ has a nonempty interior.

(3) A definable set is of dimension zero if and only if it is discrete. When it is of dimension zero, it is also closed.

(4) Let $X$ and $Y$ be definable subsets of $M^n$. We have

$$\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}.$$ 

(5) Let $f : X \to M^n$ be a definable map. We have $\dim(f(X)) \leq \dim X$.

(6) Let $f : X \to M^n$ be a definable map. The notation $\mathcal{D}(f)$ denotes the set of points at which the map $f$ is discontinuous. The inequality $\dim(\mathcal{D}(f)) < \dim X$ holds true.

(7) Let $X$ be a definable set. The notation $\partial X$ denotes the frontier of $X$ defined by $\partial X = \overline{X} \setminus X$. We have $\dim \partial X < \dim X$.

(8) Let $\varphi : X \to Y$ be a definable surjective map whose fibers are equi-dimensional; that is, the dimensions of the fibers $\varphi^{-1}(y)$ are constant. The equalities $\dim X = \dim Y + \dim \varphi^{-1}(y)$ hold true for all $y \in Y$.

(9) Let $X$ be a definable subset of $M^{m+n}$. The notation $\pi : M^{m+n} \to M^n$ denotes the projection onto the last $n$ coordinates. There exists a definable
map $\varphi : \pi(X) \to X$ such that the composition $\pi \circ \varphi$ is the identity map on $\pi(X)$.

Proof. (1) [5, Theorem 2.11(ii)]; (2) [5, Theorem 2.11(iii)]; (3) [5, Proposition 3.2]; (4)-(7) [5, Theorem 3.8(4)-(7)]; (8) [5, Theorem 3.14]; (9) Well-known. \hfill \Box

The following lemma asserts that an almost o-minimal structure $\mathcal{M}$ has an o-minimal structure $\mathcal{R}$ such that any bounded $\mathcal{M}$-definable set is $\mathcal{R}$-definable.

**Lemma 2.4.** Let $\mathcal{M} = (M, <, +, 0, \ldots)$ be an almost o-minimal expansion of an ordered group. There exists an o-minimal expansion $\mathcal{R} = (M, <, +, 0, \ldots)$ of the ordered group satisfying the following conditions:

(i) Any set definable in $\mathcal{R}$ is $\mathcal{M}$-definable.

(ii) Any bounded $\mathcal{M}$-definable set is definable in $\mathcal{R}$.

Proof. [6, Theorem 2.13] \hfill \Box

Almost o-minimal expansions of ordered groups admit partition into multi-cells and uniform local definable cell decomposition. We first recall the definition of semi-definability.

**Definition 2.5.** Let $\mathcal{R} = (M, <, \ldots)$ be an o-minimal structure. A subset $X$ of $M^n$ is semi-definable in $\mathcal{R}$ if the intersection $U \cap X$ is definable in $\mathcal{R}$ for any bounded open box $U$ in $M^n$.

A semi-definable subset $X$ of $M^n$ is semi-definably connected if there are no non-empty proper semi-definable closed and open subsets $Y_1$ and $Y_2$ of $X$ such that $Y_1 \cap Y_2 = \emptyset$ and $X = Y_1 \cup Y_2$. For any $x \in X$, there exists a maximal semi-definably connected semi-definable subset $Y$ of $X$ containing the point $x$ by [6, Theorem 3.6]. The set $Y$ is called the semi-definably connected component of $X$ containing the point $x$.

Let $\mathcal{M} = (M, <, +, 0, \ldots)$ be an almost o-minimal expansion of an ordered group. Let $\mathcal{R}$ be the o-minimal structure given in Lemma 2.4. Any set definable in $\mathcal{M}$ is simultaneously semi-definable in $\mathcal{R}$.

We next recall the definitions of multi-cells given in [6, Definition 4.19].

**Definition 2.6.** Consider an almost o-minimal expansion of an ordered group $\mathcal{M} = (M, <, +, \ldots)$. Let $n$ be a positive integer. A definable subset $X$ of $M^n$ is a multi-cell if it satisfies the following conditions:

- If $n = 1$, either $X$ is a discrete definable set or all semi-definably connected components of the definable set $X$ are open intervals.
- When $n > 1$, let $\pi : M^n \to M^{n-1}$ be the projection forgetting the last coordinate. The projection image $\pi(X)$ is a multi-cell and, for any semi-definably connected component $Y$ of $X$, $\pi(Y)$ is a semi-definably connected component of $\pi(X)$ and $Y$ is one of the following forms:

  - $Y = \pi(Y) \times M$,
  - $Y = \{(x, y) \in \pi(Y) \times M \mid y = f(x)\}$,
  - $Y = \{(x, y) \in \pi(Y) \times M \mid y > f(x)\}$,
  - $Y = \{(x, y) \in \pi(Y) \times M \mid y < g(x)\}$ and
  - $Y = \{(x, y) \in \pi(Y) \times M \mid f(x) < y < g(x)\}$
for some semi-definable continuous functions $f$ and $g$ defined on $\pi(Y)$ with $f < g$.

We obtain the following theorem:

**Theorem 2.7.** A set definable in an almost o-minimal expansion of an ordered group is partitioned into finitely many multi-cells.

**Proof.** [6, Theorem 1.7] \[ \square \]

Let us review the definition of cells.

**Definition 2.8** (Definable cell decomposition). Consider an expansion of dense linear order without endpoints $\mathcal{M} = (M, <, \ldots)$. Let $(i_1, \ldots, i_n)$ be a sequence of zeros and ones of length $n$. $(i_1, \ldots, i_n)$-cells are definable subsets of $M^n$ defined inductively as follows:

- A $(0)$-cell is a point in $M$ and a $(1)$-cell is an open interval in $M$.
- An $(i_1, \ldots, i_n, 0)$-cell is the graph of a definable continuous function defined on an $(i_1, \ldots, i_n)$-cell. An $(i_1, \ldots, i_n, 1)$-cell is a definable set of the form $(x, y) \in C \times M$ if $f(x) < y < g(x)$, where $C$ is an $(i_1, \ldots, i_n)$-cell and $f$ and $g$ are definable continuous functions defined on $C$ with $f < g$.

A cell is an $(i_1, \ldots, i_n)$-cell for some sequence $(i_1, \ldots, i_n)$ of zeros and ones. The sequence $(i_1, \ldots, i_n)$ is called the type of an $(i_1, \ldots, i_n)$-cell. An open cell is a $(1, \ldots, 1)$-cell. The dimension of an $(i_1, \ldots, i_n)$-cell is defined by $\sum_{j=1}^{n} i_j$.

We inductively define a **definable cell decomposition** of an open box $B \subseteq M^n$.

For $n = 1$, a definable cell decomposition of $B$ is a partition $B = \bigcup_{i=1}^{m} C_i$ into finitely many cells. For $n > 1$, a definable cell decomposition of $B$ is a partition $B = \bigcup_{i=1}^{m} C_i$ into finitely many cells such that $\pi(B) = \bigcup_{i=1}^{m} \pi(C_i)$ is a definable cell decomposition of $\pi(B)$, where $\pi : M^n \to M^{n-1}$ is the projection forgetting the last coordinate. Consider a finite family $\{A_\lambda\}_{\lambda \in \Lambda}$ of definable subsets of $B$. A **definable cell decomposition of $B$ partitioning $\{A_\lambda\}_{\lambda \in \Lambda}$** is a definable cell decomposition of $B$ such that the definable sets $A_\lambda$ are unions of cells for all $\lambda \in \Lambda$.

The following theorem is the uniform local decomposition theorem for almost o-minimal structures.

**Theorem 2.9** (Uniform local definable cell decomposition). Consider an almost o-minimal expansion of an ordered group $\mathcal{M} = (M, <, 0, +, \ldots)$. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a finite family of definable subsets of $M^{m+n}$. Take an arbitrary positive element $R \in M$ and set $B = (-R, R)^n$. Then, there exists a partition into finitely many definable sets

$$M^m \times B = X_1 \cup \ldots \cup X_k$$

such that $B = (X_1)_b \cup \ldots \cup (X_k)_b$ is a definable cell decomposition of $B$ for any $b \in M^m$ and either $X_i \cap A_\lambda = \emptyset$ or $X_i \subseteq A_\lambda$ for any $1 \leq i \leq k$ and $\lambda \in \Lambda$. Furthermore, the type of the cell $(X_i)_b$ is independent of the choice of $b$ with $(X_i)_b \neq \emptyset$. Here, the notation $S_b$ denotes the fiber of a definable subset $S$ of $M^{m+n}$ at $b \in M^m$.

**Proof.** [6, Theorem 1.7] \[ \square \]

3. **Proof of the main theorem**

This section is devoted to the proof of Theorem 1.3. We prove it and the following lemma simultaneously. Our proof is partially inspired by the geometric proof of a similar assertion for o-minimal structures in [9].
Lemma 3.1. Let $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ and $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ be as in Theorem 1.3. Assume that $n > 0$ and there are no $\mathcal{L}(\mathcal{M})$-formula $\phi(y)$ such that $\mathcal{N} \models \phi(a)$ and $\dim(\{y \in \mathbb{N}^n \mid \mathcal{M} \models \phi(y)\}) < n$. Set $\alpha = (a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1}$ and $\beta = a_n \in \mathbb{N}$.

Let $\pi : \mathbb{N}^{m+n-1} \to \mathbb{N}^m$ be the coordinate projection onto the first $m$-coordinates. We consider $\mathcal{M}$-definable subsets $T$ and $C$ of $\mathbb{N}^m$ and $\mathbb{N}^{m+n-1}$, respectively, with $T = \pi(C)$. Let $f : C \to \mathbb{N}$ be a bounded $\mathcal{M}$-definable function. Then, the set

$$\Xi(T, C, f) = \{t \in T \mid \alpha \in (C^N)_t, \ f(t, \alpha) < \beta\}$$

is $\mathcal{M}$-definable, where $(C^N)_t$ denotes the fiber of the $\mathcal{N}$-definable set $C^N$ at $t$.

Proof of Theorem 1.3 and Lemma 3.1. By Lemma 2.4, there exists an o-minimal universe $\mathcal{M}$ having the same universe $\mathcal{R}$ such that (i) any $\mathcal{R}$-definable set is definable in $\mathcal{M}$ and (ii) any bounded $\mathcal{M}$-definable set is definable in $\mathcal{R}$. We fix such an o-minimal structure $\mathcal{R}$ in the proof.

In the proof of Theorem 1.3, we set

$$X = \{(x, y) \in \mathbb{N}^m \times \mathbb{N}^m \mid \mathcal{M} \models \Phi(x, y)\}$$

and

$$T = \{x \in \mathbb{N}^m \mid \mathcal{M} \models \exists y \Phi(x, y)\}.$$

Let $C$ be the image of $X$ under the coordinate projection of $\mathbb{N}^{m+n}$ forgetting the last coordinate in the proof of Theorem 1.3. The same symbols $C^i$ and $T$ are used in the statement of the lemma, but this abuse of symbols will not confuse the readers.

Let $\text{len}(a)$ denote the length of the tuple $a$. We prove Theorem 1.3 and Lemma 3.1 by induction on $(\text{len}(a), \text{dim}(T))$ in the lexicographic order simultaneously. Theorem 1.3 obviously holds true when $n = 0$. In the rest of the proof, we assume that $n > 0$. We set $\alpha = (a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1}$ and $\beta = a_n \in \mathbb{N}$. We first show that Lemma 3.1 implies Theorem 1.3.

Claim 1. We may assume that $X_t$ is a bounded cell for any $t \in T$.

Set $b = \text{st}(a)$ and fix a bounded open box $B$ in $\mathbb{N}^m$ containing the point $b$. There is a partition

$$X \cap (\mathbb{N}^m \times B) = X_1 \cup \ldots \cup X_k$$

such that either the fiber $(X_i)_t$ of $X_i$ at $t$ is empty or a cell for any $1 \leq i \leq k$ and $t \in T$ by Theorem 2.7. Let $S_i = \{x \in \mathbb{N}^m \mid (x, a) \in X_i^N\}$. Since $S = \bigcup_{i=1}^k S_i$, the set $S$ is $\mathcal{M}$-definable if $S_i$ are $\mathcal{M}$-definable for all $1 \leq i \leq k$. Considering $S_i$ instead of $S$, we may assume that $X_i$ is a bounded cell for any $t \in T$.

Claim 2. We may assume that there are no $\mathcal{L}(\mathcal{M})$-formula $\phi(y)$ such that $\mathcal{N} \models \phi(a)$ and $\dim(\{y \in \mathbb{N}^n \mid \mathcal{M} \models \phi(y)\}) < n$.

Assume that such a formula $\phi(y)$ exists. Set $V = \{y \in \mathbb{N}^n \mid \mathcal{M} \models \phi(y)\}$. We may assume that $V$ is bounded considering $V \cap B$ instead of $V$ if necessary. The $\mathcal{M}$-definable set $V$ is definable in $\mathcal{R}$. Applying the definable cell decomposition theorem for o-minimal structures [1], we can get a cell decomposition $\{C_i\}_{i=1}^j$ partitioning $V$. Note that the $\mathcal{R}$-definable sets $C_i$ are also definable in $\mathcal{M}$. We have $a \in C_j^N$ for some $1 \leq j \leq k$. We may assume that $V$ is a bounded cell definable in $\mathcal{R}$ considering the cell $C_j$ instead of $V$. Set $\ell = \text{dim}(V)$. Let $p : \mathbb{N}^m \to \mathbb{N}^{m-\ell}$ be the projection onto the first $\ell$ coordinates. Since $V$ is a cell, we may assume that $p(V)$ is open and $V$ is the graph of an $\mathcal{R}$-definable continuous map $\varphi : p(V) \to \mathbb{N}^{n-\ell}$ by permuting the coordinates if necessary. Consider the $\mathcal{L}(\mathcal{M})$-formula $\psi(x, z) = (z \in p(V)) \land \Phi(x, (a_1, \ldots, a_\ell))$. We obviously have $S = \{x \in \mathbb{N}^m \mid \mathcal{N} \models \psi(x, (a_1, \ldots, a_\ell))\}$.
When \( \ell < n \), the set \( S \) is \( \mathcal{M} \)-definable by the induction hypothesis. We have succeeded in the reduction.

We are now ready to demonstrate that Lemma 3.1 implies Theorem 1.3. By Claim 1 and the definition of cells, there exist bounded \( \mathcal{M} \)-definable functions \( f, g : C \to M \) such that, if \( f \prec g \) on their domain, the functions \( f(t, \cdot) \) and \( g(t, \cdot) \) defined on \( C_t \) are continuous for any \( t \in T \), and we have either

\[
X = \{(t, x, y) \in T \times M^{n-1} \times M \mid x \in C_t, \ f(t, x) < y < g(t, x)\}
\]

or

\[
X = \{(t, x, y) \in T \times M^{n-1} \times M \mid x \in C_t, \ f(t, x) = y \},
\]

where \( C_t \) denotes the fiber of \( C \) at \( t \). In the second case, the set \( S \) is an empty set. In fact, if \( S \) is not empty, there exists \( t \in M^n \) with \( \beta = f(t, \alpha) \), which contradicts Claim 2. By the definition of \( S \) and \( X \), we have

\[
S = \{t \in M^n \mid t \in T^N, \ f(t, \alpha) < \beta < g(t, \alpha)\}
\]

\[
= \{t \in T \mid \alpha \in (C^N)_t, \ f(t, \alpha) < \beta < g(t, \alpha)\}
\]

because \( N \) is an elementary extension of \( \mathcal{M} \). We therefore have

\[
S = \{t \in T \mid \alpha \in (C^N)_t, \ f(t, \alpha) < \beta\}
\]

\[
\setminus (\{t \in T \mid \alpha \in (C^N)_t, \ g(t, \alpha) < \beta\} \cup \{t \in T \mid \alpha \in (C^N)_t, \ g(t, \alpha) = \beta\}).
\]

Note that Claim 2 implies that the tuple \( a \) satisfies the assumption in Lemma 3.1. The first and second sets in the right hand of the equality are \( \mathcal{M} \)-definable by Lemma 3.1. The third set is an empty set; otherwise, it contradicts Claim 2. We have demonstrated that Lemma 3.1 implies Theorem 1.3.

The remaining task is to demonstrate Lemma 3.1. The induction hypothesis implies that we may assume that \( \mathcal{T}(T, C, f) \) is of a simpler form.

**Claim 3.** We may assume that \( \alpha \in (C^N)_t \) for any \( t \in T \) and

\[
(1) \quad \mathcal{T}(T, C, f) = \{t \in T \mid f(t, \alpha) < \beta\}.
\]

In fact, by the induction hypothesis, Theorem 1.3 holds for \( \alpha \). Therefore, the set

\[
D = \{t \in T \mid \alpha \in (C^N)_t\} = \{t \in M^n \mid t \in T^N, \ \alpha \in (C^N)_t\}
\]

is an \( \mathcal{M} \)-definable set. We obviously have \( \mathcal{T}(T, C, f) = \{t \in D \mid f(t, \alpha) < \beta\} = \mathcal{T}(D, C \cap (D \times M^{n-1}), f|_{C \cap (D \times M^{n-1})}) \), where the notation \( f|_{C \cap (D \times M^{n-1})} \) denotes the restriction of \( f \) to \( C \cap (D \times M^{n-1}) \). The tuple \( (D, C \cap (D \times M^{n-1}), f|_{C \cap (D \times M^{n-1})}) \) satisfies the equality (1). Therefore, we may assume that the conditions in Claim 3 hold true, considering the tuple \( (D, C \cap (D \times M^{n-1}), f|_{C \cap (D \times M^{n-1})}) \) instead of the tuple \( (T, C, f) \).

Set \( d = \dim T \). We are now ready to show Lemma 3.1 when \( d = 0 \). We reduce to the case in which \( f \) is defined on \( T \times M^{n-1} \). In fact, take an element \( c \in M \) larger than \( \beta \). It is possible because \( \beta \) is \( \mathcal{M} \)-bounded. Consider the \( \mathcal{M} \)-definable function \( f' : T \times M^{n-1} \to M \) which coincides with \( f \) on \( C \) and is constantly \( c \) elsewhere. We have

\[
\mathcal{T}(T, C, f) = \{t \in T \mid f(t, \alpha) < \beta\} = \{t \in T \mid f'(t, \alpha) < \beta\}
\]

\[
= \mathcal{T}(T, T \times M^{n-1}, f').
\]

Replacing \( f \) with \( f' \), we may assume that \( f \) is defined on \( T \times M^{n-1} \).
Consider the \( \mathcal{M} \)-definable set
\[
E = \{ (x, y) \in M^{n-1} \times M \mid \mathcal{M} \models \exists t \in T \ f(t, x) < y \}.
\]
When \( a = (\alpha, \beta) \notin E^N \), the set \( \exists(T, C, f) \) is an empty set, and it is obviously \( \mathcal{M} \)-definable. Hence, we may assume that \( a = (\alpha, \beta) \in E^N \). By the assumption of the lemma, the definable set \( E \) has a nonempty interior. Consider the map \( \lambda : E \to M \) defined by
\[
\lambda(x, y) = \sup \{ f(t, x) \mid t \in T, \ f(t, x) < y \}.
\]
We also consider the \( \mathcal{M} \)-definable set given by
\[
Z = \{ (t, x, y) \in T \times M^{n-1} \times M \mid f(t, x) < y, \ f(t, x) = \lambda(x, y) \}.
\]
The set \( \{ f(t, x) \mid t \in T, \ f(t, x) < y \} \) is of dimension zero for any fixed \( (x, y) \in E \) by Proposition 2.3(5) because \( T \) is of dimension zero. It is closed and discrete by Proposition 2.3(3). It means that, for any \( (x, y) \in E \), the fiber of the set \( Z \) at \( (x, y) \) given by
\[
Z(x, y) = \{ t \in T \mid f(t, x) < y, \ f(t, x) = \lambda(x, y) \}
\]
is not an empty set. It immediately implies the inequality
\[
(2) \quad \lambda(x, y) < y.
\]
We can construct an \( \mathcal{M} \)-definable map \( \tau : E \to T \) so that \( (\tau(x, y), x, y) \in Z \) by Proposition 2.3(9).

The set of points at which \( \tau \) is discontinuous is of dimension smaller than \( \dim E = n \) by Proposition 2.3(6). In particular, it has an empty interior. Therefore, the interior \( U \) of the set
\[
\{ x \in E \mid \tau \text{ is continuous at } x \}
\]
has a nonempty \( \mathcal{M} \)-definable subset of \( M^n \) with \( \dim(E \setminus U) < n \) by Proposition 2.3(2) and (7). By the assumption on the tuple \( a \) in the lemma, we get \( a = (\alpha, \beta) \in U^N \).

Take a bounded open box \( B \) in \( M^n \) containing the point \( b = \text{st}(a) \). We immediately get \( a \in B^N \). The intersection \( U \cap B \) is definable in \( R \). Apply the definable cell decomposition theorem for \( o \)-minimal structures [1, Chapter 3, Theorem 2.11]. We can partition \( U \cap B \) into \( R \)-definable cells. Let \( U \cap B = U_1 \cup \ldots \cup U_l \) be such a partition. Note that \( U_i \) are all \( \mathcal{M} \)-definable for all \( 1 \leq i \leq l \). Since \( a \in (U \cap B)^N \), we have \( a \in (U_i)^N \) for some \( 1 \leq i \leq l \). We may assume that \( i = 1 \) without loss of generality. Recall that the map \( \tau \) is continuous and \( U_1 \) is definably connected because \( U_1 \) is a cell. Therefore, the image \( \tau(U_1) \) is definably connected. Since \( \tau(U_1) \) is a subset of the discrete set \( T \), the set \( \tau(U_1) \) is a singleton. Let \( u \in M^m \) be the unique point contained in \( \tau(U_1) \). We have
\[
(3) \quad f(u, x) = \lambda(x, y)
\]
for all \( (x, y) \in U_1 \) because \( \tau(x, y) \in Z(x, y) \).

We want to show that
\[
(4) \quad \mathcal{M} \models \forall t \in T \ (f(t, x) < y \iff f(t, x) \leq f(u, x))
\]
for all \( (x, y) \in U_1 \). Fix arbitrary \( (x, y) \in U_1 \) and \( t \in T \). Assume first that \( f(t, x) < y \). We have \( f(t, x) \leq \lambda(x, y) \) by the definition of the function \( \lambda \). We then get \( f(t, x) \leq f(u, x) \) by the equality (3). We next prove the opposite implication.
Assume that \( f(t, x) \leq f(u, x) \). The inequality (2) and the equality (3) immediately imply the inequality \( f(t, x) \leq y \). We have shown the equivalence (4). We now get

\[ N \models \forall t \in T^N, \ (f(t, \alpha) < \beta \leftrightarrow f(t, \alpha) \leq f(u, \alpha)) \]

because \( a = (\alpha, \beta) \in (U_1)^N \). In particular, we obtain

\[ \exists(T, C, f) = \{ t \in T \mid f(t, \alpha) \leq f(u, \alpha) \}. \]

The right hand of the equality is \( M \)-definable by the induction hypothesis. We have demonstrated Lemma 6.1 when \( d = 0 \).

We finally prove Lemma 6.1 for \( d > 0 \). Let \( p : M^m \to M^{m-1} \) be the coordinate projection forgetting the last coordinate. We reduce to a simpler case.

**Claim 4.** We may assume the following:

- For any \( u \in p(T) \) and \( w \in M^{n-1} \), the set \( I_{u,w}(C) = \{ v \in M \mid (u, v, w) \in C \} \)
  is either an empty set or the union of open intervals, and the map \( f_{u,w} : I_{u,w}(C) \to M \) given by \( f_{u,w}(v) = f(u, v, w) \) is locally strictly increasing and continuous;
- We have \( \alpha \in (C^N)_t \) for any \( t \in T \).

We prove Claim 4. We first reduce to the case in which (*) the fiber \( T_u = \{ v \in M \mid (u, v) \in T \} \) is of dimension one for any \( u \in p(T) \). Let \( \pi_i : M^m \to M \) be the coordinate projection onto the \( i \)-th coordinate. Set \( d' = \min\{1 \leq i \leq m \mid \exists u \ \text{dim} \pi_i^{-1}(u) \cap T = 1\} \). We prove it by reverse induction on \( d' \). When \( d' < m \), permute the \( i \)-th and \( m \)-th coordinate. We can reduce to the case in which \( d' = m \).

We set \( P_i = \{ u \in p(T) \mid \text{dim} T_u = i \} \) for \( i = 0, 1 \). Set \( T_i = p^{-1}(P_i) \cap T \). The set \( T_1 \) satisfies the condition (*). When \( \text{dim} T_0 = d \), the lemma is true for \( T_0 \) by the induction hypothesis on \( d \). We can reduce to the case in which the condition (*) is satisfied when \( \text{dim} T_0 = d \) by the induction hypothesis on \( d' \). We have \( \dim p(T) = d - 1 \) by Proposition 2.3(8) when the condition (*) is satisfied.

We next reduce to the case in which \( T \) is a multi-cell and \( \dim p(T) = d - 1 \).

We may assume that the fiber \( T_u = \{ v \in M \mid (u, v) \in T \} \) is of dimension one for some \( u \in p(T) \) by permuting the coordinates if necessary. Apply Theorem 2.7. Let \( T = \bigcup_{i=1}^l D_i \) be a partition into multi-cells. Since we have \( \exists(T, C, f) = \bigcup_{i=1}^l \exists(D_i, C \cap (D_i \times M^{m-1}), f|_{C \cap (D_i \times M^{m-1})}) \), the set \( \exists(T, C, f) \) is \( M \)-definable if \( \exists(D_i, C \cap (D_i \times M^{m-1}), f|_{C \cap (D_i \times M^{m-1})}) \) are \( M \)-definable for all \( 1 \leq i \leq l \). Therefore, we may assume that \( T \) is a multi-cell without loss of generality.

For simplicity, we assume that \( C = T \times M^{n-1} \). In fact, the same proof as the case in which \( d = 0 \) justifies this assumption. We consider the sets

\[
B_1 = \{ (u, v, w) \in T \times M^{m-1} \mid \exists s_1, s_2, s_1 < v < s_2, \{ u \} \times (s_1, s_2) \times \{ w \} \subseteq C, \\
f_{u,w} \text{ is strictly increasing and continuous on } (s_1, s_2) \},
\]
\[
B_2 = \{ (u, v, w) \in T \times M^{m-1} \mid \exists s_1, s_2, s_1 < v < s_2, \{ u \} \times (s_1, s_2) \times \{ w \} \subseteq C, \\
f_{u,w} \text{ is strictly decreasing and continuous on } (s_1, s_2) \}
\]
\[
B_3 = \{ (u, v, w) \in T \times M^{m-1} \mid \exists s_1, s_2, s_1 < v < s_2, \{ u \} \times (s_1, s_2) \times \{ w \} \subseteq C, \\
f_{u,w} \text{ is constant on } (s_1, s_2) \},
\]

where \( u, v \) and \( w \) are elements in \( M^{n-1}, M \) and \( M^{n-1} \), respectively. Set \( B_4 = (T \times M^{n-1}) \setminus (B_1 \cup B_2 \cup B_3) \). By Proposition 2.3(1), the set \( \{ v \in M \mid (u, v, w) \in B_4 \} \)
is of dimension not greater than zero for any fixed \( u \in p(T) \) and \( w \in M^{n-1} \). We get \( \dim B_4 < d + n - 1 \) by Proposition 2.38. We also have

\[
\mathfrak{T}(T, C, f) = \{ t \in T \mid f(t, \alpha) < \beta \} = \bigcup_{i=1}^{4} \{ t \in T \mid \alpha \in (B_i^N)_t, \ f(t, \alpha) < \beta \}.
\]

Set \( T_i = \{ t \in T \mid \alpha \in (B_i^N)_t \} \) for \( 1 \leq i \leq 4 \). The sets \( T_i \) are \( \mathcal{M} \)-definable by the induction hypothesis. By the definition of \( T_i \), the condition that \( \alpha \in (B_i^N)_t \) is satisfied for any \( t \in T_i \). We also set \( C_i = (T_i \times M^{n-1}) \cap B_t \) for all \( 1 \leq i \leq 4 \). For any \( t \in T_i \), we have \( (B_i)_t = (C_i)_t \) and we get \( (B_i^N)_t = (C_i^N)_t \). Therefore, we have \( \alpha \in (C_i^N)_t \) for any \( t \in T_i \). We get

\[
\{ t \in T \mid \alpha \in (B_i^N)_t, \ f(t, \alpha) < \beta \} = \{ t \in T_i \mid f(t, \alpha) < \beta \} = \mathfrak{T}(T_i, C_i, f|_{C_i}).
\]

We have shown \( \mathfrak{T}(T, C, f) = \bigcup_{i=1}^{4} \mathfrak{T}(T_i, C_i, f|_{C_i}) \). Therefore, we have only to prove that \( \mathfrak{T}(T_i, C_i, f|_{C_i}) \) is \( \mathcal{M} \)-definable for each \( 1 \leq i \leq 4 \). As for the case in which \( i = 4 \), we have \( \dim(T_4) < d \) by Proposition 2.38 because the fiber \( (B_4)_t \) is of dimension \( n - 1 \) for each \( t \in T_4 \) because \( (B_4^N)_t \) contains the point \( \alpha \). The set \( \mathfrak{T}(T_4, C_4, f|_{C_4}) \) is \( \mathcal{M} \)-definable by the induction hypothesis. We consider the case in which \( i = 1 \). The tuple \( (T_1, C_1, f|_{C_1}) \) satisfies the conditions in the claim. The set \( \mathfrak{T}(T_1, C_1, f|_{C_1}) \) is also \( \mathcal{M} \)-definable by the assumption of the claim. The case in which \( i = 2 \) is also easy. Set

\[
\widehat{T}_2 = \{(u, v) \in M^{m-1} \times M \mid (u, -v) \in T_2 \} \quad \text{and} \quad \widehat{C}_2 = \{(u, v, w) \in M^{m-1} \times M \times M^{n-1} \mid (u, -v, w) \in C_2 \}
\]

The \( \mathcal{M} \)-definable function \( \widehat{f} : \widehat{C}_2 \to M \) is given by \( \widehat{f}(u, v, w) = f(u, v, w) \). The tuple \( (\widehat{T}_2, \widehat{C}_2, \widehat{f}) \) satisfies the assumption of the claim. The set \( \mathfrak{T}(\widehat{T}_2, \widehat{C}_2, \widehat{f}) \) is \( \mathcal{M} \)-definable. The set \( \mathfrak{T}(T_2, C_2, f|_{C_2}) \) is also \( \mathcal{M} \)-definable because \( \mathfrak{T}(T_2, C_2, f|_{C_2}) = \{(u, v) \in M^{m-1} \times M \mid (u, -v) \in \mathfrak{T}(\widehat{T}_2, \widehat{C}_2, \widehat{f}) \} \).

We finally consider the case in which \( i = 3 \). We set \( T = T_3, C = C_3 \) and \( f = f|_{C_3} \) for the simplicity of notations. The following conditions are satisfied:

- For any \( u \in p(T) \) and \( w \in M^{n-1} \), the set \( I_{u, w}(C) \) is either an empty set or the union of open intervals, and the map \( f_{u, w} : I_{u, w}(C) \to M \) is locally constant;
- We have \( \alpha \in (C_i^N)_t \) for any \( t \in T \).

We have only to demonstrate that \( \mathfrak{T}(T, C, f) \) is \( \mathcal{M} \)-definable in this case. We construct an \( \mathcal{M} \)-definable map \( \rho : T \to T \) as follows: Fix an arbitrary element \( u \in p(T) \) and \( v \in M \) with \( (u, v) \in T \). We define \( r_1(u, v) \) and \( r_2(u, v) \) by

\[
r_1(u, v) = \inf \{ v' \in M \mid \forall v'' \ (v'< v'' < v) \to (u, v'') \in T \} \quad \text{and} \quad r_2(u, v) = \sup \{ v' \in M \mid \forall v'' \ (v<v'' < v') \to (u, v'') \in T \}.
\]

Take a positive element \( c \in M \). We set \( \rho(u, v) \) as follows:

\[
\rho(u, v) = \begin{cases} 
(u, 0) & \text{if } r_1(u, v) = -\infty \text{ and } r_2(u, v) = +\infty, \\
(u, r_2(u, v) - c) & \text{if } r_1(u, v) = -\infty \text{ and } r_2(u, v) < \infty, \\
(u, r_1(u, v) + c) & \text{if } r_1(u, v) > -\infty \text{ and } r_2(u, v) = \infty, \\
(u, (r_1(u, v) + r_2(u, v))/2) & \text{otherwise.}
\end{cases}
\]

It is easy to demonstrate that, for any \( u \in p(T) \), the fiber \( \rho(T)_u \) of \( \rho(T) \) at \( u \) does not contain an interval. It means that \( \dim \rho(T)_u = 0 \). We get \( \rho(T) < d \) by Proposition
because \( \dim \mathfrak{p}(T) = d - 1 \). The restriction of \( \rho \) to \( \rho(T) \) is an identity map by the definition. A locally constant function definable in a definably complete structure defined on an open interval is constant. Therefore, the restriction of \( f_{u,v} \) to an open interval is constant. It implies the equality
\[
f(u, v, w) = f(\rho(u, v), w)
\]
for any \((u, v, w) \in C\). In particular, we obtain
\[
f(u, v, \alpha) = f(\rho(u, v), \alpha)
\]
for any \((u, v) \in T\). Set \( C' = (\rho(T) \times M^{n-1}) \cap C \). The map \( f' \) is defined as the restriction of \( f \) to \( C' \). The set \( \Sigma(\rho(T), C', f') \) is an \( M \)-definable set by the induction hypothesis because \( \dim \rho(T) < d \). On the other hand, we get
\[
\Sigma(T, C, F) = \{(u, v) \in T \mid f(u, v, \alpha) < \beta\} = \{(u, v) \in T \mid f(\rho(u, v), \alpha) < \beta\}
\]
\[
\rho^{-1}\{(u, v) \in \rho(T) \mid f(u, v, \alpha) < \beta\} = \rho^{-1}(\Sigma(\rho(T), C', f')).
\]
It implies that the set \( \Sigma(T, C, F) \) is an \( M \)-definable set. We have completed the proof for the case in which \( i = 3 \), and we also have demonstrated Claim 4.

The maps \( \rho : T \to T \), \( r_1 : T \to M \cup \{-\infty\} \) and \( r_2 : T \to M \cup \{+\infty\} \) are the maps defined in the proof of Claim 4. Set \( T' = \rho(T) \). The maps \( \kappa_1 : T' \to M \cup \{-\infty\} \) and \( \kappa_2 : T' \to M \cup \{+\infty\} \) are the restrictions of \( r_1 \) and \( r_2 \) to \( T' \), respectively. We use these notations in the rest of the proof. The first condition of Claim 4 is expressed by a first-order formula. Therefore, the univariate function \( f(u, \cdot, \alpha) \) is also strictly increasing and continuous in the interval \((r_1(u, v), r_2(u, v))\) for all \((u, v) \in T\). We use this fact without notice.

**Claim 5.** We may further assume that, for any \( u \in \mathfrak{p}(T) \) and any maximal interval \( I \) contained in the fiber \( T_u \) of \( T \) at \( u \), there exists a unique \( d \in I^N \) such that
\[
f(u, d, \alpha) = \beta.
\]

We demonstrate Claim 5. Since \( N \) is an elementary extension of \( M \), \( f(u, \cdot, \alpha) \) is strictly increasing in \( I^N \) by Claim 4. Therefore, there is at most one element \( d \in I^N \) satisfying the condition in Claim 5. Set
\[
W_1 = \{(u, v) \in T' \mid \forall v' \in N, \ (\kappa_1(u, v) < v' < \kappa_2(u, v)) \to (f(u, v', \alpha) < \beta)\}.
\]

Theorem 1.3 holds true for \( T' \) by the induction hypothesis. Therefore, the set \( W_1 \) is \( M \)-definable. Consider the set
\[
\widehat{W}_1 = \{(u, v) \in T \mid \forall v' \in N, \ (r_1(u, v) < v' < r_2(u, v)) \to (f(u, v', \alpha) < \beta)\}.
\]
We have \( \widehat{W}_1 = \rho^{-1}(W_1) \). It implies that \( \widehat{W}_1 \) is also \( M \)-definable. We put
\[
\widehat{W}_2 = \{(u, v) \in T \mid \forall v' \in N, \ (r_1(u, v) < v' < r_2(u, v)) \to (f(u, v', \alpha) > \beta)\}.
\]
It is also \( M \)-definable similarly.

Set \( \bar{T} = T \setminus (\widehat{W}_1 \cup \widehat{W}_2) \), \( \bar{C} = C \cap (\bar{T} \times M^{n-1}) \) and \( \bar{f} = f|_{\bar{T}} \). It is obvious that \( \Sigma(\bar{T}, \bar{C}, \bar{f}) = \widehat{W}_1 \cup \Sigma(\bar{T}, \bar{C}, \bar{f}) \). We may assume that the condition in Claim 5 is satisfied considering the tuple \((\bar{T}, \bar{C}, \bar{f})\) instead of the tuple \((T, C, f)\). We have proven Claim 5.
Thanks to Claim 5, for any \((u, v) \in T'\), we can find the unique \(d \in N\) satisfying \(\kappa_1(u, v) < d < \kappa_2(u, v)\) and \(f(u, d, \alpha) = \beta\). We denote such \(d\) by \(\delta(u, v)\). It induces a map \(\delta: T' \to N\). Consider the \(\mathcal{L}(M)\)-formula:

\[
\Theta(u, v, v', w, z) = ((u, v) \in T') \land ((u, v, w, z) \in C) \land (\kappa_1(u, v) < v' < \kappa_2(u, v)) \land (f(u, v', w) = z).
\]

Note that the graph of \(\delta\) is given by \(\{(u, v, v') \in T' \times N \mid \mathcal{N} \models \Theta(u, v, v', \alpha, \beta)\}\).

We consider the following two \(\mathcal{L}(M \cup \{\alpha\})\)-formulas:

\[
\bar{\epsilon}_1(u_1, v_1, v', u_2, v_2) = ((u_1, v_1) \in T) \land ((u_2, v_2) \in T) \land (r_1(u_1, v_1) < v' < r_2(u_1, v_1)) \land (f(u_2, v_2, \alpha) \geq f(u_1, v', \alpha)),
\]

\[
\bar{\epsilon}_2(u_1, v_1, v', u_2, v_2) = ((u_1, v_1) \in T) \land ((u_2, v_2) \in T) \land (r_1(u_1, v_1) < v' < r_2(u_1, v_1)) \land (f(u_2, v_2, \alpha) \leq f(u_1, v', \alpha)).
\]

There is an \(\mathcal{L}(M)\)-formula \(\epsilon_i(u_1, v_1, v', u_2, v_2)\) such that

\[
\mathcal{N} \models \bar{\epsilon}_i(u_1, v_1, v', u_2, v_2) \iff \mathcal{M} \models \epsilon_i(u_1, v_1, v', u_2, v_2)
\]

for any \(i = 1, 2\) and \((u_1, v_1, v', u_2, v_2) \in T \times M \times T\) because Theorem \ref{thm:main} holds true for \(\alpha\) by the induction hypothesis. We set

\[
\epsilon_1(u_1, v_1, u_2, v_2) = \forall v' \ (r_1(u_1, v_1) < v' < r_2(u_1, v_1) \land v' < v_1) \to \epsilon'_1(u_1, v_1, v', u_2, v_2),
\]

\[
\epsilon_2(u_1, v_1, u_2, v_2) = \forall v' \ (r_1(u_1, v_1) < v' < r_2(u_1, v_1) \land v' > v_1) \to \epsilon'_2(u_1, v_1, v', u_2, v_2) \text{ and }
\]

\[
\epsilon(u_1, v_1, u_2, v_2) = \epsilon_1(u_1, v_1, u_2, v_2) \land \epsilon_2(u_1, v_1, u_2, v_2).
\]

We consider the set

\[
L(u, v) = \{(u_2, v_2) \in T \mid \mathcal{M} \models \epsilon(u, \text{st}(\delta(u, v)), u_2, v_2)\}
\]

for any \((u, v) \in T\). The set \(L(u, v)\) is an \(\mathcal{M}\)-definable set for any fixed \((u, v) \in T\). We consider two cases separately.

**Case A.** There exists \((\tilde{u}, \tilde{v}) \in T\) such that \(\dim L(\tilde{u}, \tilde{v}) < d\).

Fix \((\tilde{u}, \tilde{v}) \in T\) such that \(\dim L(\tilde{u}, \tilde{v}) < d\). We consider the following three sets:

\[
\mathfrak{T}_1(T, C, f) = \{(u, v) \in T \mid \mathcal{N} \models \{(u, v) \in L(\tilde{u}, \tilde{v}) \land f(u, v, \alpha) < \beta\}\},
\]

\[
\mathfrak{T}_2(T, C, f) = \{(u, v) \in T \mid \mathcal{N} \models \{\neg \epsilon_1(\tilde{u}, \text{st}(\delta(\tilde{u}, \tilde{v})), u, v) \land f(u, v, \alpha) < \beta\}\},
\]

\[
\mathfrak{T}_3(T, C, f) = \{(u, v) \in T \mid \mathcal{N} \models \{\neg \epsilon_2(\tilde{u}, \text{st}(\delta(\tilde{u}, \tilde{v})), u, v) \land f(u, v, \alpha) < \beta\}\}.
\]

We obviously have \(\mathfrak{T}(T, C, f) = \bigcup_{i=1}^{3} \mathfrak{T}_i(T, C, f)\). We have only to show that \(\mathfrak{T}_i(T, C, f)\) is \(\mathcal{M}\)-definable for each \(1 \leq i \leq 3\). Since \(\dim L(\tilde{u}, \tilde{v}) < d\) by our case hypothesis, the set \(\mathfrak{T}_1(T, C, f)\) is \(\mathcal{M}\)-definable by the induction hypothesis.

We next consider \(\mathfrak{T}_2(T, C, f)\). Fix an arbitrary \((u, v) \in T\) with

\[\mathcal{N} \models \neg \epsilon_1(\tilde{u}, \text{st}(\delta(\tilde{u}, \tilde{v})), u, v).\]

Note that the equalities \(r_i(\tilde{u}, \text{st}(\delta(\tilde{u}, \tilde{v})))) = r_i(\tilde{u}, \tilde{v})\) hold true for \(i = 1, 2\) by the definition of \(\delta\). There exists \(v' \in M\) such that \(r_1(\tilde{u}, \tilde{v}) < v' < r_2(\tilde{u}, \tilde{v})\) and \(f(u, v, \alpha) < f(\tilde{u}, v', \alpha)\). We have \(v' < \delta(\tilde{u}, \tilde{v})\) because \(v' \in M\). Since \(f(u, v, \alpha)\) is strictly increasing on \((r_1(\tilde{u}, \tilde{v}), r_2(\tilde{u}, \tilde{v}))\), we get \(f(\tilde{u}, v', \alpha) < f(\tilde{u}, \delta(\tilde{u}, \tilde{v}), \alpha) = \beta\).

We finally obtain \(f(u, v, \alpha) < \beta\). Therefore, we get \(\mathfrak{T}_2(T, C, f) = \{(u, v) \in T \mid \mathcal{N} \models \)
\[\neg \epsilon_1(\langle \bar{u}, \text{st}(\delta(\bar{u}, \bar{v})), u, v \rangle), \text{ which is } M\text{ -definable by the induction hypothesis. We can prove that, if } \mathcal{N} \models \neg \epsilon_2(\langle \bar{u}, \text{st}(\delta(\bar{u}, \bar{v})), u, v \rangle, \text{ we have } f(u, v, \alpha) > \beta \text{ in the same manner. The set } \mathcal{T}_3(T, C, f) \text{ is an empty set. We have demonstrated Lemma 3.1 in Case A.}

Case B. The equality \( \dim L(\langle \bar{u}, \bar{v} \rangle) = d \) holds true for each \( \langle \bar{u}, \bar{v} \rangle \in T \).

We consider the \( M\text{-definable set } \Lambda \text{ defined by} \)
\[
\Lambda = \{(u, v, v') \in T \times M \mid r_1(u, v) < v' < r_2(u, v) \wedge \dim(\{(u_2, v_2) \in T \mid M \models \epsilon(u, v', u_2, v_2) = d\}) \}
\]
It is a definable set by the definition of dimension. Our case hypothesis implies that
\[
(u, v, \text{st}(\delta(u, v))) \in \Lambda
\]
for each \((u, v) \in T\). We demonstrate that the fiber \( \Lambda_{(u, v)} \) of \( \Lambda \) at \((u, v) \in T\) is of dimension zero. Fix \((u, v) \in T\) for a while. Consider the \( M\text{-definable set} \)
\[
\mathfrak{Z} = \{(u_2, v_2, v') \in T \times M \mid M \models \epsilon(u, v', u_2, v_2) \wedge (u, v, v') \in \Lambda)\}
\]
Let \( q_1 : M^{m+1} \rightarrow M^m \) and \( q_2 : M^{m+1} \rightarrow M \) be the coordinate projections onto first \( m \) coordinates and onto the last coordinate, respectively. The projection image \( q_2(\mathfrak{Z}) \) coincides with \( \Lambda_{(u, v)} \) and the fiber \( \mathfrak{Z} \cap q_2^{-1}(v') \) is of dimension \( d \) for any \( v' \in \Lambda_{(u, v)} \) by the definition of the set \( \Lambda \). We get
\[
\dim \mathfrak{Z} = \dim \Lambda_{(u, v)} + d
\]
by Proposition \( \ref{prop:dim} \). On the other hand, for any \((u_2, v_2, v') \in \mathfrak{Z}\), we obtain \( M \models \epsilon(u, v', u_2, v_2) \). It implies that, for all \( r_1(u, v') < v'' < r_2(u, v') \), we get \( f(u, v'', \alpha) \leq f(u_2, v_2, \alpha) \) if \( v'' < v' \) and \( f(u, v'', \alpha) \geq f(u_2, v_2, \alpha) \) if \( v'' > v' \). Since \( f(u, \cdot, \alpha) \) is continuous on \((r_1(u, v'), r_2(u, v'))\), the equality
\[
f(u, v', \alpha) = f(u_2, v_2, \alpha)
\]
holds true. When we fix \((u_2, v_2) \in q_1(\mathfrak{Z})\), at most one \( v' \) satisfies the above equality because \( f(u, \cdot, \alpha) \) is strictly increasing. It means that the fiber \( q_1^{-1}(u_2, v_2) \cap \mathfrak{Z} \) is a singleton. We therefore get
\[
\dim \mathfrak{Z} = \dim q_1(\mathfrak{Z}) \leq \dim T = d
\]
by Proposition \( \ref{prop:dim} \). We have demonstrated \( \dim \Lambda_{(u, v)} = 0 \). In particular, the fiber \( \Lambda_{(u, v)} \) is discrete and closed by Proposition \( \ref{prop:dim} \).

The point \( \text{st}(\delta(u, v)) \) is the closest point in the \( M\text{-definable subset} \)
\[
\{v' \in M \mid (u, v, v') \in \Lambda\}
\]
of \( M \) to \( \delta(u, v) \) because \( \text{st}(\delta(u, v)) \in \Lambda_{(u, v)} \) and \( \Lambda_{(u, v)} \) is discrete and closed for any \((u, v) \in T'\). In other word, the set
\[
\Gamma = \{(u, v, v') \in (T' \times M) \cap \Lambda \mid \mathcal{N} \models \forall v'' ((u, v, v'') \in \Lambda
\]
\[
\rightarrow |\delta(u, v) - v'| \leq |\delta(u, v) - v''|)\}
\]
\[
= \{(u, v, v') \in (T' \times M) \cap \Lambda \mid \mathcal{N} \models \forall v'' \forall w (\Theta(u, v, w, \alpha, \beta)
\]
\[
\wedge (u, v, v'') \in \Lambda) \rightarrow (|w - v'| \leq |w - v''|)\}
\]
is the graph of the composition \( \text{st} \circ \delta \). The definition of the formula \( \Theta(u, v, v', w, z) \) is found in the equality \( \ref{eq:Theta} \). It is \( M\text{-definable by the induction hypothesis because} \)
dim(T' × M) ∩ Λ ≤ dim T' < d by Proposition [2.3] (8). We have demonstrated that the composition st ◦ δ is $\mathcal{M}$-definable.

We consider the set
\[ Q_1 = \{(u, v) ∈ T' | st(δ(u, v)) < δ(u, v)\} = \{(u, v) ∈ T' | T |= \forall v', θ(u, v', α, β) ∧ (st(δ(u, v)) < v')\}. \]

It is $\mathcal{M}$-definable because Theorem [1.3] holds true for $T'$. The set
\[ P_1 = \{(u, v) ∈ T | st(δρ(u, v))) < δ(ρ(u, v))\} = ρ^{-1}(Q_1) \]
is also $\mathcal{M}$-definable. The set
\[ P_2 = \{(u, v) ∈ T | st(δ(ρ(u, v))) ≥ δ(ρ(u, v))\} \]
is also $\mathcal{M}$-definable for the same reason. We then have
\[ Ξ(T, C, f) = \{(u, v) ∈ T | f(u, v, α) < β\} = \{(u, v) ∈ P_1 | r_1(u, v) < v ≤ st(δ(ρ(u, v)))\} \]
∪ \{(u, v) ∈ P_2 | r_1(u, v) < v < st(δ(ρ(u, v)))\} because $f(u, , α)$ is strictly increasing on $(r_1(u, v), r_2(u, v))$. Therefore, $Ξ(T, C, f)$ is $\mathcal{M}$-definable because the composition $st ◦ δ$ is $\mathcal{M}$-definable. We have demonstrated Lemma [3.1] \(□\)

4. Corollaries of the main theorem

Using Theorem [1.3], we can get the following corollaries in the same manner as [2]. Let $\dim_{\mathcal{M}} S$ denote the dimension of an $\mathcal{M}$-definable set $S$. We also define $\dim_{\mathcal{N}} S$ in the same manner.

**Corollary 4.1.** Let $L$, $\mathcal{M}$ and $\mathcal{N}$ be as in Theorem [1.3]. For any $\mathcal{N}$-definable subset $S$ of $\mathcal{N}^n$ parameterized by $\mathcal{M}$-bounded parameters, we have $\dim_{\mathcal{N}} S ≥ dim_{\mathcal{M}} S ∩ \mathcal{N}^n$.

**Proof.** We prove the lemma by induction on $(n, k = dim_{\mathcal{N}}(S))$ under the lexicographic order. Set $T = S ∩ \mathcal{M}^n$. When $k = 0$, the set $S$ is discrete and closed by Proposition [2.3] (3). Since $T$ is a subset of $S$, it is discrete or an empty set. Therefore, we have $dim_{\mathcal{M}}(T) ≥ 0$ by Proposition [2.3] (3).

The lemma is obvious when $n = k$. The lemma has been demonstrated when $n = 1$.

We consider the case in which $k > 0$. Let $P_k$ be the set of all coordinate projections from $\mathcal{N}^n$ onto $\mathcal{N}^k$. It is a finite set and we fix a linear order on the set $P_k$. The image of $S$ under some coordinate projection in $P_k$ has a nonempty interior by the definition of dimension. Let $π_S : \mathcal{N}^n → \mathcal{N}^k$ be the largest element in $P_k$ under which the image of $S$ has a nonempty interior. The same notation $π_S$ also denotes the coordinate projection $\mathcal{N}^n → \mathcal{N}^k$. This abuse of notations will not confuse the readers. Set $S_1 = \{x ∈ S | dim_{\mathcal{N}}(S ∩ π_S^{-1}(π_S(x))) > 0\}$ and $T_1 = S_1 ∩ \mathcal{N}^n$. Either $S_1$ is of dimension smaller than $k$ or $π_{S_1}$ is smaller than $π_S$ in $P_k$. We have $dim_{\mathcal{N}} S_1 ≥ dim_{\mathcal{M}} T_1$ by the induction hypothesis.

Set $S_2 = S \setminus S_1$. By the definition of $S_2$, we have $dim_{\mathcal{N}} S_2 ∩ π_S^{-1}(π_S(x)) = 0$ for all $x ∈ S_2$ by the definition of $S_2$. Set $T_2 = S_2 ∩ \mathcal{N}^n$. We immediately get $dim_{\mathcal{M}} T_2 ∩ π_S^{-1}(π_S(x)) ≤ 0$ for all $x ∈ S_2$ for the same reason as the case in which $k = 0$. Since $π_S(T_2) ⊆ π_S(S_2) ∩ \mathcal{M}^{n-1}$ and $dim_{\mathcal{N}} π_S(S_2) = dim_{\mathcal{N}} S_2 ≤ k$
by Proposition 2.3, we get \( \dim_M \pi_S(T_2) \leq \dim_M \pi_S(S_2) \cap M^{n-1} \leq k \) by the induction hypothesis. We get \( \dim_M T_2 \leq k \) by Proposition 2.3. We immediately get \( \dim_M T = \max\{ \dim_M T_1, \dim_M T_2 \} \leq k \) by Proposition 2.3. \( \square \)

Remark 4.2. The inequality in Corollary 4.1 may be strict. For instance, consider a singleton \( S \) defined by an \( M \)-bounded element in \( N \setminus M \). We obviously have \( S \cap M = \emptyset \) and \( 0 = \dim_N S > \dim_M (M \cap S) = -\infty \).

Corollary 4.3. Let \( L, M \) and \( N \) be as in Theorem 1.3. Let \( V \) be the set of \( M \)-bounded elements in \( N \). Consider an \( N \)-definable subset \( S \) of \( N^n \) parameterized by \( M \)-bounded parameters. The set \( st(S \cap V^n) \) is \( M \)-definable.

Proof. Let \( x, y \), and \( z \) denote \( n \)-tuples of elements in a set, and the notations \( x_i, y_i \), and \( z_i \) denote the \( i \)-th element, respectively. Consider the set \( T = \{(x, y) \in M^n \times M^n \mid N \models \exists z \in S, x_i < z_i < y_i \text{ for all } 1 \leq i \leq n\} \). It is \( M \)-definable by Theorem 1.3. We obviously have

\[
\begin{align*}
st(S \cap V^n) &= \{ z \in M^n \mid M \models (\forall x_1, \ldots, \forall x_n, \forall y_1, \ldots, \forall y_n, \\
&\quad (x_1 < z_1 < y_1) \land \cdots \land (x_n < z_n < y_n) \rightarrow (x, y) \in T) \}.
\end{align*}
\]

It means that \( \text{st}(S \cap V^n) \) is \( M \)-definable. \( \square \)

Corollary 4.4. Let \( L, M \) and \( N \) be as in Theorem 1.3. Consider a \( N \)-definable function \( f : N^n \to N \) parameterized by \( M \)-bounded parameters. The three sets

\[
\begin{align*}
D_{-\infty} &= \{ x \in M^n \mid f(x) < y \ (\forall y \in M) \}, \\
D_{\infty} &= \{ x \in M^n \mid f(x) > y \ (\forall y \in M) \} \text{ and} \\
D &= M^n \setminus (D_{-\infty} \cup D_{\infty})
\end{align*}
\]

and the map \( g : D \to M \) given by \( x \mapsto \text{st}(f(x)) \) are all \( M \)-definable.

Proof. Consider the set \( X = \{(x, y) \in M^n \times M \mid f(x) < y\} \). It is \( M \)-definable by Theorem 1.3. We have \( D_{-\infty} = \{ x \in M^n \mid \forall y (x, y) \in X \} \). It is obviously \( M \)-definable. We can show that \( D_{\infty} \) is \( M \)-definable, similarly. The \( M \)-definability of \( D \) is now trivial.

We next consider the set \( Y = \{(x, y_1, y_2) \in M^n \times M \times M \mid y_1 < f(x) < y_2 \} \). It is \( M \)-definable by Theorem 1.3. The graph of \( g \) is given by \( \{(x, y) \in M^n \times M \mid M \models \forall y_1, \forall y_2, y_1 < y < y_2 \rightarrow (x, y_1, y_2) \in Y \} \). We have shown that \( g \) is \( M \)-definable. \( \square \)

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