Homogenization in fractional elasticity – One spatial dimension

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In this note we treat the equations of fractional elasticity in one spatial dimension. After establishing well-posedness, we use an abstract result in the theory of homogenization to derive effective equations in fractional elasticity with highly oscillating coefficients. The approach also permits the consideration of non-local operators (in time and space).

1 Introduction

As general references for the homogenization of (stationary) elasticity we refer to [2–4, 8] and [1, 6]. For an account of the methods mentioned here we refer to [9] with extensions in [10–12]. The equation of (fractional) elasticity can be found in [7, Section 3.1.2.5] or [5, pp. 102]. A Kelvin-Voigt type model for one spatial dimension (see also [12, Example 2.6]) may be written as follows. For $(t, x) \in \mathbb{R} \times [0, 1]$ consider

$$\partial_0 \mu(t,x) \partial_0 u(t,x) - \partial_1 T(t,x) = F(t,x),$$

$$(C(x) + \partial_0^\alpha D(x)) \partial_1 u(t,x) = T(t,x),$$

where $u$ is the displacement field, $T$ denotes the stress tensor, $\mu$ is the mass density, $C, D$ are material parameters and $F$ is a given external forcing term. The operators $\partial_0$ and $\partial_1$ denote the derivatives with respect to the temporal and the spatial variable, respectively. The scalar $\alpha \in [1/2, 1]$ is given and the fractional derivative $\partial_0^\alpha$ is understood in the sense of a functional calculus for $\partial_0$ (see [7, Section 6.1, p.427]) to be specified below. Of course the unknowns $u$ and $T$ are subject to certain boundary and initial conditions. For simplicity, we shall assume homogeneous initial conditions for $u$ and $T$ and homogeneous Dirichlet boundary conditions for the displacement field $u$. In the following, we will sketch how to prove well-posedness for the above system in an appropriate Hilbert space setting. Moreover, we address the question, whether the sequence of solutions $(u_n, T_n)_n$ of the system

$$\partial_0 \mu(n x) \partial_0 u_n(t,x) - \partial_1 T_n(t,x) = F(t,x),$$

$$(C(n x) + \partial_0^\alpha D(n x)) \partial_1 u_n(t,x) = T_n(t,x),$$

converges. Assuming periodicity of $\mu$, $C$ and $D$, we show the weak convergence of $(u_n, T_n)_n$ and derive an equation satisfied by the respective limit. A more detailed account of the results mentioned can be found in [12]. In particular, we refer to [12, Example 2.6, Example 3.5 and Remark 3.6].

2 Functional analytic framework and well-posedness

For a Hilbert space $H$ and $\nu > 0$ we denote the space of square-integrable functions on $\mathbb{R}$ with respect to the weighted Lebesgue-measure $\exp(-2\nu)\lambda$ by $L^2_\nu(\mathbb{R}; H)$. The space of weakly differentiable $L^2_\nu$-functions with weak derivative in $L^2_\nu$ is denoted by $H^1_\nu(\mathbb{R}; H) := \{ f \in L^2_\nu(\mathbb{R}; H); f' \in L^2_\nu(\mathbb{R}; H) \}$. We define the (time-)derivative $\partial_0 : H^1(\mathbb{R}; H) \subseteq L^2(\mathbb{R}; H) \to L^2(\mathbb{R}; H), f \mapsto f'$. It turns out that $0 \in \rho(\partial_0)$, $\|\partial_0^{-1}\| \leq \frac{1}{\nu}$. Moreover, there is an explicit spectral representation $\mathcal{L}_\nu$, the Fourier-Laplace transform, defined as the unitary extension of the mapping

$$L^2_\nu(\mathbb{R}; H) \subseteq L^2(\mathbb{R}; H) \to L^2(\mathbb{R}; H), f \mapsto \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} e^{-ix(-\nu)\xi} f(x)dx,$$

where $L^2_\nu$ denotes the set of compactly supported $L^2$-functions. Realizing that $\partial_0 = \mathcal{L}_\nu^\ast (im + \nu) \mathcal{L}_\nu$, where $m$ denotes the multiplication-by-argument-operator in $L^2(\mathbb{R}; H)$ with maximal domain, we define the following functional calculus for $\partial_0^{-1}$:

$$M(\partial_0^{-1}) := \mathcal{L}_\nu^\ast M\left(\frac{1}{im + \nu}\right) \mathcal{L}_\nu,$$

where $M \in \mathcal{H}(\mathbb{B}(r, r); L(H)) := \{ M : \mathbb{B}(r, r) \to L(H); M \text{ bounded, analytic}\}$, endowed with the sup-norm $\|\cdot\|_\infty$. For $\beta \in \mathbb{R}$ we define $\partial_0^{[\beta]} := \partial_0^{[\beta]} \partial_0^{-[\beta]} = \partial_0^{[\beta]} (\partial_0^{-1})^{[\beta] - \beta}$. We have $\|\partial_0^{[\beta]}\| \leq \nu^{\beta}$ for all $\beta \leq 0$.

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In the application we have in mind the spatial Hilbert space $H$ is given by $L^2(0, 1)$. The spatial derivatives are thus defined on $L^2(0, 1)$:

$$\partial_t : H_1(0, 1) \subseteq L^2(0, 1) \rightarrow L^2(0, 1), f \mapsto f',$$

where $H_1(0, 1)$ is the space of weakly differentiable $L^2(0, 1)$-functions with derivative in $L^2(0, 1)$. We denote by $\partial_{1,0}$ the restriction of $\partial_1$ to functions vanishing at 0 and 1. It is evident that $\partial_1^* = -\partial_{1,0}$. The well-posedness theorem can be formulated as follows:

**Theorem 2.1** (12, Theorem 2.1) Let $\alpha \in [1/2, 1]$, $K > 0$. Let $\mu, C, D \in L(L^2(0, 1))$. We assume $\mu, D$ to be selfadjoint and that there exists $c > 0$ such that $\mu \geq c$ and $D \geq c$. Then there exists $\nu_0 > 0$ and such that for all $\nu \geq \nu_0$ and $(F, G) \in L^2_y(\mathbb{R}; L^2(0, 1)^2)$ there is a unique $(u, T) \in H^\nu(\mathbb{R}; L^2(0, 1)) \oplus L^2_y(\mathbb{R}; L^2(0, 1))$ with

$$
\begin{pmatrix}
\partial_0 (\mu) & 0 \\
(C + D\partial_0^\alpha)^{-1} & 0 \\
\end{pmatrix} 
- 
\begin{pmatrix}
0 \\
\partial_{1,0} \\
\partial_1 \\
0 \\
\end{pmatrix} 
= 
\begin{pmatrix}
F \\
G \\
\end{pmatrix}.
$$

Moreover, the estimate $\| (\partial_0 u, T) \|_{L^2_y(\mathbb{R}; L^2(0, 1)^2)} \leq K \| (F, G) \|_{L^2_y(\mathbb{R}; L^2(0, 1)^2)}$ is satisfied.

### 3 The homogenized equation

Our derivation of the limit equation is based on the following theorem.

**Theorem 3.1** (abstract homogenization result, 12, Theorem 4.1) Let $H$ be a Hilbert space, $A : D(A) \subseteq H \rightarrow H$ skew-selfadjoint with compact resolvent, $\nu > 0$, $r > 1/(2\nu)$. Let $(M_n)_n$ be a $\| \cdot \|_\infty$-bounded sequence in $H^\infty(B(r, r); L(H))$ satisfying the properties

- $\exists c > 0 \forall n \in \mathbb{N} \forall z \in B(r, r) : Re z^{-1} M_n(z) \geq c$, and
- $(M_n(\partial_1^{-1}))_n$ converges in the weak operator topology $\tau_\omega$ of $L(L^2_y(\mathbb{R}; H))$ to some $M (\iff : M_n(\partial_1^{-1}) \rightharpoonup M)$.

Then $(\partial_0 M_n(\partial_0^{-1}) + A)^{-1} \rightharpoonup (\partial_0 M + A)^{-1}$.

Now, let $\mu, C, D \in L^\infty(\mathbb{R})$ be 1-periodic and such that there exists $c > 0$ with $\mu, D \geq c$. Then, by the above theorem

$$\left(\partial_0 \begin{pmatrix} \mu(n) \\ 0 \end{pmatrix} (C(n) + D(n)\partial_0^\alpha)^{-1} - \begin{pmatrix} 0 \\ \partial_{1,0} \\ \partial_1 \\
0 \\
\end{pmatrix} \right)^{-1} \rightharpoonup \left(\begin{pmatrix} \int_0^1 \mu(x) \frac{dx}{x} \\ 0 \\
0 \\
\partial_0^{-\alpha} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( -\partial_0^{-\alpha} \right)_{x=1} \right) = \left(\begin{pmatrix} 0 \\ \partial_{1,0} \\ \partial_1 \\
0 \\
\end{pmatrix} \right)^{-1},$$

which can be verified by a Neumann series expansion and using that for 1-periodic mappings $b$ we have $b(n \cdot) \rightarrow \int_0^1 b$ with respect to the $\sigma(L^\infty(\mathbb{R}), L^1)$-topology (see e.g. [3, Theorem 2.6]). The resulting Kelvin-Voigt model can then be written as

$$\begin{pmatrix}
\int_0^1 \mu(x) \frac{dx}{x} \\
\partial_0^{-\alpha} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( -\partial_0^{-\alpha} \right)_{x=1} \int_0^1 \left( \frac{C(x)^{\ell}}{\ell!} \right) \frac{dx}{x} \\
\partial_0^{-\alpha} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( -\partial_0^{-\alpha} \right)_{x=1} \int_0^1 \left( \frac{C(x)^{\ell}}{\ell!} \right) \frac{dx}{x} \\
\partial_0^{-\alpha} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( -\partial_0^{-\alpha} \right)_{x=1} \int_0^1 \left( \frac{C(x)^{\ell}}{\ell!} \right) \frac{dx}{x} \\
\end{pmatrix} k \partial_{1,0} u.$$  

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