The Unfolding Semantics of Functional Programs

José M. Rey Poza and Julio Mariño

School of Computer Science,
Universidad Politécnica de Madrid
josem.rey@gmail.com, jmarino@fi.upm.es

Abstract. The idea of using unfolding as a way of computing a program semantics has been applied successfully to logic programs and has shown itself a powerful tool that provides concrete, implementable results, as its outcome is actually source code. Thus, it can be used for characterizing not-so-declarative constructs in mostly declarative languages, or for static analysis. However, unfolding-based semantics has not yet been applied to higher-order, lazy functional programs, perhaps because some functional features absent in logic programs make the correspondence between execution and unfolding not as straightforward. This work presents an unfolding semantics for higher-order, lazy functional programs and proves its adequacy with respect to a given operational semantics. Finally, we introduce some applications of our semantics.

Keywords: Semantics, Unfolding, Functional Programming.

1 Introduction

The broad field of program semantics can be classified according to the different meanings intended to be captured or the various techniques employed. Thus, traditionally, the term denotational semantics is used when a high-level, implementation independent description of the behaviour of a program is pursued, while operational semantics usually refers to descriptions intended to capture more implementation-related properties of the execution of a program, which can then be used to gather resource-aware information, or as a blueprint for actual language implementations.

The inability of those denotational semantics to capture certain aspects of logic programs (such as the computed answer semantics) and the introduction of “impure” constructs in Prolog, led to a considerable amount of proposals for alternative semantics of logic programs during the 80’s and 90’s. One of the most remarkable proposals is the so-called s-semantics approach [4] which explores the possibility of using syntactic denotations for logic programs. In other words, programs in a very restricted form are the building blocks of the denotation, and program transformation (e.g. via unfolding) takes the role of interpretation transformers in traditional constructions. Being closer to the source code facilitates the treatment of the less declarative aspects.
Prolog code / s-semantics unfolding

\begin{align*}
\text{add}(\text{zero}, X, X). \\
\text{add}(\text{suc}(X), Y, \text{suc}(Z)) &\leftarrow \text{add}(X, Y, Z).
\end{align*}

Functional code / Funct. unfolding

\begin{align*}
\text{add} \text{ Zero } x &= x \\
\text{add} \text{ (Suc } x) y &= \text{Suc (add x y)}
\end{align*}

\begin{align*}
S_1 &= \{\text{add(Zero, x, x)}\} \\
S_2 &= \{\text{add(Zero, x, x),} \\
&\quad \text{add(suc(Zero), x, suc(x))}\}
\end{align*}

\begin{align*}
I_0 &= \emptyset \\
I_1 &= \{\text{add Zero x = x}\} \\
I_2 &= \{\text{add Zero x = x,} \\
&\quad \text{add (Suc Zero) y = (Suc y)}\}
\end{align*}

\textbf{Fig. 1.} Logic and functional versions of a simple program, and their unfoldings.

However, in spite of the fact that unfolding is a technique equally applicable to functional programs, little attention has been paid to its use as a semantics tool. Investigating how unfolding can be applied to find the semantics of functional programs is the goal of this paper.

\subsection{1.1 Unfolding Semantics}

The process of unfolding is conceptually simple: replace any function or predicate invocation by its definition. In logic programming this amounts to unifying some literal in the body of a rule with the head of some piece of knowledge that has already been calculated, and placing the corresponding body instance where the literal was.

The previous paragraph mentions two important concepts: the first is that of piece of knowledge generated by unfolding program rules according to all current pieces of knowledge. Every piece of knowledge (called a fact) is valid source code. The set of facts may increase with every iteration. A set of facts is called an interpretation. In addition, the second concept hinted in the paragraph above is that of initial interpretation.

\textbf{Unfolding in Logic Programming.} As an example, the left half of Fig. 1 shows a predicate called \text{add} that adds two Peano Naturals. This part shows the source code (upper side) together with the corresponding unfolding results (lower side).

The general unfolding procedure can be easily followed in the example, where the first two clause sets are generated ($S_1$ and $S_2$).

\textbf{Unfolding in Functional Programming.} Unfolding in functional programming (FP) follows the very same idea of unfolding in logic programming: any function invocation is replaced by the right side of any rule whose head matches the invocation.

Consider the right half of Fig. 1 as the functional version of the previous example, written in our model language. Some differences and analogies between
both paradigms can be spotted: In FP, unfolding generates rules (equations) as pieces of knowledge, instead of clauses which appeared in logic programming. The starting seed is also different: bodyless rules are used in logic programming while the empty set is used in functional programming.

Finally, observe that both unfoldings (logic and functional) produce valid code and analogous results, being \( I_n \) equivalent to \( S_n \). This fact provides a clue into two of the main reasons to define an unfolding semantics: first they are implementable as the procedure above shows and, second, they are also a clear point between denotational and operational semantics in proving the equivalence between a semantics of each type.

### 1.2 Extending Unfolding Semantics to Suit Functional Programs

Section 1.1 showed that the ideas of unfolding semantics in logic programming can also be applied to FP. However, some features of FP (e.g. higher-order, laziness) render the unmodified procedures invalid.

Consider the function `filter` in Fig. 2. It takes a list of values and returns those values in the list that satisfy a predicate passed as its first argument.

Applying naïve unfolding to `filter` is impossible since `ite` (short for if-then-else) demands a boolean value but both \( p \) and \( x \) are unknown at unfold time (i.e. before execution).

In order to overcome this limitation, we have developed a technique capable of generating facts in the presence of incomplete information. In this case we generate conditional facts (the last four facts in \( I_2 \)). The function `match` checks whether a given term `matches` an expression that cannot be evaluated at unfolding time (here, \( (p \ x) \)). Observe that `match` must be ready to deal with infinite values in its second argument.

Note that, in automatically-generated code, such as the unfolded code shown in Fig. 2 and the figures to come, variables are most often renamed and that our unfolding implementation uses tuples to represent curried expressions.
1.3 Related Work

One of the earliest usages of unfolding in connection to semantics is due to Scott [9], who used it to find the denotation of recursive functions, even though the word *unfolding* was not used at the time.

Concerning logic programming, our main inspiration source is s-semantics [4], which defines an unfolding semantics for pure logic programs that is defined as the set of literals that can be successfully derived by using the program given.

In addition, fold and unfold have been used in connection to many other problems in the field of logic programming. For example [7] describes a method to check whether a given logic program verifies a logic formula. It does this by applying program transformations that include fold and unfold.

Partial evaluation of logic programs has also been tackled by means of unfolding but it usually generates huge data structures even for simple programs.

As in logic programming, fold/unfold transformations have been used extensively to improve the efficiency of functional programs [5], but not as a way of constructing a program’s semantics.

Unfolding has also been applied to functional-logic programming [1]. However, that paper is not oriented towards finding the meaning of a program but to unfold it partially to achieve some degree of partial evaluation. Besides, it is restricted to first order, eager languages.

*Paper Organization* Section 2 presents preliminary concepts. Section 3 describes the unfolding semantics itself, the core of our proposal. Section 4 presents the formal meaning we want to assign to the core language that we will be using. Section 5 points out some applications of unfolding semantics. Section 6 concludes.

2 Preliminaries

*Notation* Substitutions will be denoted by $\sigma, \rho, \mu$. $\sigma(e)$ or just $\sigma e$ will denote the application of substitution $\sigma$ to $e$. The empty substitution will be denoted by $\varepsilon$. $e \equiv e'$ will denote that the expressions $e$ and $e'$ have the same syntax tree.

Given a term $t$, a position within $t$ is denoted by a dot-separated list of integers. $t|_o$ denotes the content of position $o$ within $t$. Replacement of the content at position $o$ within a term $t$ by some term $t'$ is denoted by $t[t']|_o$. The set of positions within an expression $e$ will be denoted by $\text{Pos}(e)$.

$k, d$ will be used to denote constructors while $c$ will denote guards.

The *auxiliary* functions $\text{fst} : a \times b \rightarrow a$ and $\text{snd} : a \times b \rightarrow b$ extract the first and second element of a tuple, respectively. Boolean conjunction and disjunction are denoted by $\land$ and $\lor$. $\text{mgu}(f t_1 \ldots t_n, f e_1 \ldots e_n)$ (where the $t_j$ are terms and $e_i$ do not have user-defined functions) denotes its most general unifier. The conditional operator will denoted by $\triangleright$, which has type $(\triangleright) : \text{Bool} \rightarrow a \rightarrow a$ and is defined as: $(\text{True} \triangleright a) = a$, $(\text{False} \triangleright a) = \perp$.

Regarding types, $A \to B$ denotes a partial function from domain $A$ to domain $B$. The type of $\pi$-interpretations (sets of facts) is noted by $\text{Set}(\mathcal{F})$. $\mathcal{F}$
is intended to denote the domain from which facts are drawn. The projection of an interpretation $I$ to predefined functions only is denoted as $I_p$. Lack of information is represented by $\perp$ in unfolding interpretations and by the well-known symbol $\perp$ when it is related to the minimum element of a Scott domain. Lastly, HNF stands for head normal form. An expression is said to be in head normal form if it is a variable or its root symbol is a constructor. Normal form (NF) terms are those in HNF, and whose subterms are in NF.

2.1 Core Language. Abstract Program Syntax

The language$^1$ that will be the base to develop this work is a functional language with guarded rules. Guards (which are optional) are boolean expressions that must be rewritten to $\text{True}$ in order for the rule that contains it to be applicable.

Note that the language we are using is a purely functional language (meaning that it uses pattern matching, higher-order features and referential transparency).

Let us consider a signature $\Sigma = \langle V_\Sigma, DC_\Sigma, FS_\Sigma, PF_\Sigma \rangle$ where $V$ is the set of variables, $DC$ is the set of Data Constructors that holds at least $\perp$ and a tuple-building constructor, $FS$ holds the user-defined functions and $PF$ denotes the set of predefined functions that holds at least a function $\text{match}$, a function $\text{nunif}$ and a function $\circ$ that applies an expression $e$ to a list of expressions (that is, $e\circ[e_1, \ldots, e_n]$ represents $(e e_1 \ldots e_n)$). $PF$ and $FS$ are disjoint.

Some of the sets above depend on the program $P$ under study, so they should be denoted as, e.g., $FS_P$ but we will omit that subscript if it is clear from the context. All these sets are indexed by arity. The domains for a program are:

\[
\begin{align*}
\text{(VARIABLES)} & \quad V ::= x, y, z, w \ldots \\
\text{(TERMS)} & \quad T ::= v \mid k t_1 \ldots t_k \quad v \in V, k \in DC, t_i \in T \\
\text{(EXPRESSIONS)} & \quad E ::= t \mid f' e_1 \ldots e_k \quad f' \in FS \cup PF, t \in T, e_i \in E \\
\text{(PATTERNS)} & \quad \text{Pat} ::= f t_1 \ldots t_k \quad f \in FS, t_i \in T \\
\text{(RULES)} & \quad \text{Rule} ::= l \mid g = r \quad l \in \text{Pat}, g \in E, r \in E \\
\text{(PROGRAMS)} & \quad \mathcal{P} ::= \text{Set} (\text{Rule})
\end{align*}
\]

Terms are built with variables and constructors only. Expressions comprise terms and those constructs that include function symbols (predefined or not).

Note that the description corresponding to expressions ($E$) does not allow for an expression to be applied to another expression but we still want our language to be a higher order one. We manage higher order by means of partial applications, written by using the predefined function $\circ$. Thus, an application like $e_1 e_2$ (where $e_1$ and $e_2$ are arbitrary expressions) is represented in our setting by $e_1 \circ[e_1]$ (or by $\circ(e_1, [e_2])$ in prefix form).

To ensure that programs defined this way constitute a confluent rewriting system, these restrictions will be imposed on rules [6]: linear rule patterns, no free variables in rules (a free variable is one that appears in the rule body but

\[\text{We assume the core language to be typed although we do not develop its typing discipline here because of lack of space.}\]
not in the guard or the pattern) and finally, no superposition among rules (i.e. given a function application, at most a rule must be applicable).

The core language does not include local declarations (i.e. let, where) but this does not remove any power from the language since local declarations can be translated into additional rules by means of lambda lifting.

3 Unfolding Semantics for the Core Language

3.1 Interpretations

Definition 1 (Fact and $\pi$-Interpretation). We will use the word fact to denote any piece of proven knowledge that can be extracted from a program $P$ and which conforms to the following restrictions: (i) They have shape $h \mid c = b$, (ii) $b$ and $c$ include no symbols belonging to FS, (iii) Predefined functions are not allowed inside $b$ or $c$ unless the subexpression headed by a symbol in PF cannot be evaluated further (e.g. $x + 1$ would be allowed in $b$ or $c$ but $1 + 1$ would not, $2$ should be used instead) and (iv) The value of $c$ can be made equal to True (by giving appropriate values to its variables). The type of facts is denoted $\mathcal{F}$. Facts can be seen as rules with a restricted shape.

In addition, a $\pi$-interpretation is any set of valid facts that can be generated by using the signature of a given program $P$. The concept of $\pi$-interpretation has been adapted from the concept with the same name in s-semantics.

The reason for imposing these restrictions on facts is to have some kind of canonical form for interpretations. Even with this restrictions, a program does not have a unique interpretation, but we intend to be as close to a canonical form for interpretations as possible.

3.2 Defining the Unfolding Operator

The process we are about to describe is represented in pictorial form in the Appendix, Sect. A in order to help understand the process as a whole.

The unfolding operator relies on a number of auxiliary functions that are described next, together with the operator itself. A full example aimed at clarifying how these functions work can be found in the Appendix (Example 3).

Evaluation of Predefined Functions The function $\text{eval}$ (Fig. 3) is in charge of finding a value for those expressions which do not contain any full application of user-defined functions. Since predefined functions do not have rules, their appearances cannot be rewritten, just evaluated. Only predefined functions are evaluated; all the other expressions are left untouched. Note that $\text{eval}$ requires the interpretation in order to know how to evaluate predefined functions.
\[ \text{eval} : \text{Set}(\mathcal{F}) \times E \to E \]

\[
\begin{align*}
\text{eval}(I, x) &= x & x &\in V \\
\text{eval}(I, (k \ e_1 \ldots e_n)) &= (k \ e'_1 \ldots e'_n) & (k &\in DC, n \geq 0, \text{arity}(k) = n, \text{eval}(I, e_i) = e'_i) \\
\text{eval}(I, (k e_1, \ldots, e_n)) &= (k e'_1, \ldots, e'_n) & (k &\in DC, n \geq 0, n < \text{arity}(k)) \\
\text{eval}(I, (p e_1 \ldots e_n)) &= (I_p e'_1 \ldots e'_n) & (p &\in PF, (p e'_1, \ldots, e'_n) \text{ can be evaluated to NF without error. It is left untouched otherwise.}) \text{eval}(I, e_i) = e'_i, p \in PF - \{\text{match}\}) \\
\text{eval}(I, (c_1 \land \ldots \land c_{i-1} \land c_{i+1} \land \ldots \land c_n) \triangleright e') &= \sigma(c_1 \land \ldots \land c_{i-1} \land b \land c_{i+1} \land \ldots \land c_n \triangleright e') & \text{if } \text{match}(p, e) = (\sigma, b). \\
\text{eval}(I, (x e_1 \ldots e_m)) &= (x e_1 \ldots e_m) & x &\in V \\
\text{eval}(I, (p e_1 \ldots e_m)) &= (p e_1 \ldots e_m) & p &\in PF_o, m < o. \\
\text{eval}(I, (f e_1 \ldots e_m)) &= (f e_1 \ldots e_m) & f &\in FS_n, m \leq n \\
\end{align*}
\]

Fig. 3. Evaluation of predefined functions

**Housekeeping the Fact Set** Every time a step of unfolding takes place, new facts might be added to the interpretation. These new facts may overlap with some existing facts (that is, be applicable to the same expressions as the existing ones). Although overlapping facts do not alter the meaning of the program, they are redundant and removing them makes the interpretation smaller and more efficient. The function \texttt{clean} removes those redundancies. We believe this cleaning step is a novel contribution in the field of unfolding semantics for functional languages (see [2], where a semantics for logic functional programs is presented but where interpretations are not treated to remove any possible overlapping).

Given an interpretation, the function \texttt{clean} removes the overlapping pairs in order to erase redundant facts. Before defining \texttt{clean}, some definitions are needed.

**Definition 2 (Overlapping Facts).** A fact \( h \mid c = b \) overlaps with some other fact \( h' \mid c' = b' \) if the following two conditions are met:

- There exists a substitution \( \mu \) such that: \( h \equiv \mu(h') \) and
- The condition \( c \land \mu(c') \) is satisfiable\(^2\).

Intuitively, two facts overlap if there is some expression that can be rewritten by using any of the facts.

What \texttt{clean} does is to remove any overlapping between facts from the interpretation it receives as argument. It does this by conserving the most specific fact of every overlapping fact set untouched while restricting the other facts of the set so that the facts do not overlap any more. This restriction is accomplished by adding new conditions to the fact’s guard.

In order to be able to state that a fact is more specific than some other, we need an ordering:

\(^2\) Note that satisfiability is undecidable in general. This means that there might be cases where clean is unable to remove overlapping facts.
Definition 3 (Term and Fact Ordering). Let us define \( t \preceq t' (t, t' \in E) \), \((t, t')\) linear or \( \mu(t) = \sigma(t') \) for some substitutions \( \mu, \sigma \):

- \( \perp^S \preceq t \quad t \in T \)
- \( \preceq x \quad t \in T, x \in V \)
- \( t_1 \ldots t_n \preceq t'_1 \ldots t'_n \) if and only if \( t_i \preceq t'_i \forall i : 1..n \)
- \( (k \ t_1 \ldots t_n) \preceq (k \ t'_1 \ldots t'_n) \) if and only if \( t_i \preceq t'_i \forall i : 1..n \), \( k \in DC \cup PF \)

Now, this ordering can be used to compare facts.

Given two overlapping facts \( F \equiv f \ t_1 \ldots t_n \ | \ c = b \) and \( F' \equiv f \ t'_1 \ldots t'_n \ | \ c' = b' \), it is said that \( F' \) is more specific than \( F \) if and only if at least one of the following criteria is met:

- \( t'_1 \ldots t'_n \preceq t_1 \ldots t_n \) or
- If \( t'_1 \ldots t'_n \) and \( t_1 \ldots t_n \) are a variant of each other (i.e., they are the same term with variables renamed), the fact that is more specific than the other is the one with the most restrictive guard (a guard \( c' \) is more restrictive than another guard \( c \) if and only if \( c' \) entails \( c \) but not viceversa).
- If two facts are such that their patterns are a variant of each other and their guards entail each other, the fact that is more specific than the other is the one with the greatest body according to \( \preceq \).

Remember that facts’ bodies do not contain full applications of user-defined functions, so \( \preceq \) will never be used to compare full expressions. However, \( \preceq \) may be used to compare expressions with partial applications or with predefined functions. In these cases, function symbols (both from \( FS \) or from \( PF \)) must be treated as constructors. Note that, in a program without overlapping rules, the bodies of two overlapping facts are forced to be comparable by means of \( \preceq \).

Definition 4 (Function clean).

Given a fact \( F \) belonging to an interpretation \( I \), let us define the set \( S^F_I = \{ F_i \equiv f t_1 \ldots t_n \ | \ c_i = b_i \in I \text{ such that } F \text{ overlaps with } F_i \text{ and } F_i \text{ is more specific than } F (i : 1..m) \} \).

Considering the set \( S^F_I \) for every fact \( F \in I \), we can define clean (whose type is \( \text{Set}(F) \rightarrow \text{Set}(F) \)) as:

\[
\begin{align*}
\text{clean}(I) &= I - I^\perp - I^\perp \cup \\
&= \bigcup_{F'\equiv f t_1 \ldots t_n \ | \ c=\perp} \left[ \forall F_i \in S^F_I \big( \mu((t_1, \ldots, t_n), (t_1, \ldots, t_n)) \vee \neg(c_i) = b \big) \right] \big( F_i \equiv f t_1 \ldots t_n | c_i = b_i \big)
\end{align*}
\]

where \( - \) stands for set subtraction and:

- \( I^\perp = \{ l = \perp^s \text{ such that } (l = \perp^s) \in I \} \). clean removes all the facts that are identically \( \perp \).
\[ \text{nunif}: T \times T \rightarrow \text{Bool} \]

\[
\begin{align*}
\text{nunif}(x, t) &= \text{nunif}(t, x) = \text{False} & t \in T, x \in V \\
\text{nunif}(k, d) &= \text{True} & k, d \in \text{DC}, k \neq d \\
\text{nunif}((p_1, p_2), (p_3, p_4)) &= \text{nunif}(p_1, p_3) \lor \text{nunif}(p_2, p_4) & \text{Tuples} \\
\text{nunif}(k(\ldots), k'(\ldots)) &= \text{True} & k \neq k' \\
\text{nunif}(k(p_1, \ldots, p_n), k(p_1', \ldots, p_n')) &= \text{nunif}(p_1, p_1') \lor \ldots \lor \text{nunif}(p_n, p_n') & k \in \text{DC}
\end{align*}
\]

**Fig. 4.** Lack of unification between patterns: function \text{nunif}.

\[ I^O = \{ F' \in I \text{ such that } S^F_I \neq \emptyset \}. \text{ All the facts } F' \text{ in } I \text{ which are overlapped by some more specific fact are removed from } I \text{ and replaced by the amended fact shown above which does not overlap with any fact in } S^F_I. \]

The function \text{nunif} (Fig. 4) denotes lack of unification between its arguments.

Under some conditions \text{clean} will not add new facts to the given interpretation. This will happen if the guards for the facts under the big \( \cup \) in Eq. 1 are unsatisfiable. If the program under analysis meets certain properties, this is sure to happen. Two definitions are needed to define those properties:

**Definition 5 (Complete Function Definition).** A function definition for function \( f \) written in the core language is said to be complete if and only if for any well-typed, full application \( f t_1 \ldots t_n \) of \( f \), where the \( t_i \) are terms there is a rule \( f p_1 \ldots p_n | g = r \) that can be used to unfold that application (that is, there exists a substitution \( \sigma \) such that \( \sigma(t_1, \ldots, t_n) = (p_1, \ldots, p_n) \) and \( \sigma(g) \) satisfiable).

**Definition 6 (Productive Rule).** A program rule is said to be productive if at least one fact which is not equal to the unguarded bottom (\( \bot^* \)) is generated by unfolding that rule at some interpretation \( I_m \) (in finite).

\text{clean} will not add new facts if all the function definitions in the program are complete and all the rules in the program are productive. The following Lemma states this. Note that the conditions mentioned are sufficient but not necessary.

**Lemma 1 (When Can \text{clean} Drop Facts).** Let \( P \) be a program without overlapping rules, whose function definitions are all complete and whose rules are all productive. Then:

For every fact \( H \equiv f t_1 \ldots t_n \mid c = b \in I_n \) which is a result of unfolding the rule \( A \equiv f s_1 \ldots s_n \mid g = r \), there exist in \( I_{n+1} \) some facts which are also the result of unfolding \( A \) which cover all the invocations of \( f \) covered by \( H \).

The proof for this Lemma can be found in the Appendix.

We will be using the simplified version of \text{clean} whenever the program under analysis meets the criteria that have been just mentioned.

To finish this section, let us state a result that justifies why it is legal to use \text{clean} to remove overlapping facts.
\[
\text{match} : T \times E \to (V \to E) \times E
\]

\[
\begin{align*}
\text{match}(x, e) &= \{(x \leftarrow e), \text{True}\} & x &\in V \\
\text{match}(t, \bot^*) &= \{\}, \text{False} \\
\text{match}(t, (f e_1 \ldots e_n)) &= (\sigma_h', c_h' \land b_h') & t &\in T, f \in FS \cup PF, \ hnf((f e_1 \ldots e_n)) = c_h' \triangleright c_h, \text{match}(t, e_h') = (\sigma_h', b_h') \\
\text{match}(k, k) &= \{\}, \text{True} & k &\in DC \\
\text{match}((k \ldots), (k' \ldots)) &= \{\}, \text{False} & (k, k' &\in DC, k \neq k') \\
\text{match}((k_1 \ldots k_n), (k e_1 \ldots e_n)) &= (\sigma_1 \circ \cdots \circ \sigma_n, b_1 \land \ldots \land b_n) & (k_i &\in T, k \in DC, \text{match}(t, e_i) = (\sigma_i, b_i)) \\
\text{match}(t, (f e_1 \ldots e_n)) &= \{(\}, \text{False} & (f &\in FS_m \cup PF_m, m > n) \\
\text{match}(t, x) &= \{(x \leftarrow t), \text{True}\} & (t &\in T, x \in V)
\end{align*}
\]

Fig. 5. Function match.

\[
\text{umatch} : T \times E \to (V \to E) \times E
\]

\[
\begin{align*}
\text{umatch}(t, e) &= (\sigma, \text{True}) & \text{if there exists some unifying } &\sigma \text{ such that } \sigma(t) \equiv e . \\
\text{umatch}(t, e) &= (\sigma, \text{snd(match}(t |_o, e |_o))) \land e & \text{if } e \text{ and } t \text{ do not unify because there is at least a position } &o \text{ such that } e |_o \text{ is headed by a symbol of } PF \text{ (including } \theta) \text{ and } t |_o \text{ is not a variable}. \text{ umatch}(t, e |_o) = (\sigma, c). \\
\text{umatch}(t, e) &= (e, \text{False}) & \text{if } e \text{ and } t \text{ do not unify but this is not due to a predefined function symbol in } e.
\end{align*}
\]

Fig. 6. umatch: Generation of matching conditions.

**Lemma 2 (Programs without Overlapping).** The fixpoint interpretation (namely, \(I_\omega = U^\infty P(I_\bot)\) where \(U\) is the unfolding operator that will be presented later) of any program \(P\) without overlapping rules cannot have overlappings. \(I_\bot\) is the empty interpretation.

The proof for this Lemma can be found in the Appendix.

**Lazy Matching of Facts and Rules** The unfolding process rewrites user-defined function applications but predefined functions (including partial application) will be left unaltered by the unfolding steps since there are no rules for them. This means that when a match is sought to perform an unfolding step, the arguments to the user-defined functions may include predefined functions that must be evaluated before it is known whether they match some pattern. Such applications may also generate infinite values. Thus, we need a function \textit{match}\(^3\) that lazily matches a pattern to an expression.

Recall Fig. 2. The unfolding operator generates facts containing \textit{match} whenever it finds a subexpression headed by a symbol in \(PF\) that needs to be matched against some rule pattern. These special facts can be thought as imposing assumptions on what the pattern must be like before proceeding.

\(^3\) Note that \textit{match} is similar to operator \(=:<=\) proposed in [3].
Those assumptions are included inside the fact’s guard. Two functions are needed in connection to those assumptions: umatch (Fig. 6) \(^4\) generates them as a conjunction of calls to match (Fig. 5) which performs the matches at runtime.

umatch and match must be distinguished: umatch fits facts’ heads into expressions for unfolding while match is not an unfolding function; it is a function used to check (at runtime) whether certain conditions are met in evaluated expressions. umatch does not call match: umatch generates code that uses match.

The function hnf, used in the definition for match, receives an expression and returns that expression evaluated to Head Normal Form. hnf has type $E \rightarrow E$.

In the result of umatch, $\sigma$ is a list of assignments assigning values to variables inside the arguments passed to umatch and the right part of the result is a condition of the form $\bigwedge_i \text{snd}(\text{match}(p_i, e_i))$ where the $p_i$ are patterns and the $e_i$ are expressions without symbols of $FS$ (they have been removed by unfolding).

The function match returns whether that matching was possible and a list of assignments from variables to expressions. The rules of match are tested from the first to the last, applying the first suitable one only.

Both lists of assignments (the ones returned by umatch or match) are not exactly substitutions because variables can be assigned to full expressions (not just terms) but they behave as such.

Two remarks must be made about match: (i) The first element of the pair returned by match is never used inside the definitions given in this paper because it is only used in order to bind variables at runtime (not at unfolding time). Those bindings will occur when a guard containing calls to match is evaluated. (ii) Therefore, match is not purely functional (i.e., it is not a side effect-free).

Example 1. (How umatch works.) Code that generates a situation like the one described is the one in Fig. 7 left. Part of its unfolding appears in Fig. 7 right \(^5\).

When the rule for app_first is unfolded, it is found that $(f n)$ cannot be unfolded any more but it still does not match $(x:xs)$ (the pattern in first’s rule). Therefore, the second rule for umatch imposes the assumption in the resulting fact that $(f n)$ must match $(x:xs)$ if the rule for app_first is to be applied. Note that $f@[n]$ ($f$ applied to variable $n$) generates an infinite term in this case. This is why match cannot be replaced by strict equality. Example 2 in the Appendix (Sect. C) shows how unfolding behaves when infinite structures are generated.

Unfolding Operator Operator $U(I)$ (short form for $U_P(I)$) where $I$ is a $\pi$-interpretation is defined as shown in Fig. 8.

Given a program $P$, its meaning is given by the least fixed point of $U_P$ or by $I_\omega(=U_P^\infty(I_\bot))$ if the program has infinite semantics.

The auxiliary function unfold, that unfolds a rule using the facts in an interpretation, is defined in Fig. 9. The behaviour of unfold can be described as

\(^4\) Observe that a function like umatch is not needed in pure Prolog since every atom is guaranteed to have a rule and lack of instantiation will cause a runtime error.

\(^5\) The variables in the unfolder’s output have been renamed to ease understanding.
a) Code that Needs Matching | b) Unfolding of the Source Code

\[
\begin{align*}
\text{from}\_n &: \mathbb{Int} \rightarrow \mathbb{Int} \\
\text{from}\_n n &= n: (\text{from}\_n(n+1)) \\
\text{first} &: \mathbb{a} \rightarrow \mathbb{a} \\
\text{first} (x:xs) &= x \\
\text{app}\_\text{first} &: (\mathbb{a} \rightarrow \mathbb{b}) \rightarrow \mathbb{a} \rightarrow \mathbb{b} \\
\text{app}\_\text{first} f n &= \text{first}(f n) \\
\text{main} &: \mathbb{Int} \rightarrow \mathbb{Int} \\
\text{main} n &= \text{app}\_\text{first} \text{from}\_n n
\end{align*}
\]

Note: Any code preceded by * in every line has been generated by our Prolog-based unfold. The unfold uses Prolog terms to represent functional applications. That is why the unfold uses tuples to represent curried applications.

Fig. 7. Lazy matching of terms and rules.

\[
\begin{align*}
I_0 &= I_0^+ = \emptyset \\
I_{m+1} &= U(I_m) = \text{clean}(I_{m+1}^\top) \\
I_{m+1}^\top &= \bigcup_{\Lambda \in \text{Rules}} (\text{unfold}(\Lambda, I_m \cup I_m^\bot)) \cup I_m \\
I_{m+1} &= \{l = \bot^* \text{ such that } (l = \bot^*) \in I_{m+1}^\top\}
\end{align*}
\]

Fig. 8. Unfolding operator.

follows: \textit{unfold} receives a (partially) unfolded rule (a \textit{pseudofact}) which is unfolded by means of recursive calls. When the input to \textit{unfold} has no invocations of user defined functions, it is just returned as it is (case 1). Otherwise, the pseudofact is unfolded by considering all the facts and positions \(o\) which hold an invocation of a user-defined function (Case 2a). Those positions occupied by user-defined function calls which cannot be unfolded are replaced by \(\bot^*\) (case 2b). \textit{unfold} returns all the possible facts obtained by executing this procedure.

When performing the unfolding of a program, \textit{unfold} behaves much like the rewriting process in a TRS (i.e., it tries all the possible pairs \((\text{position} \ o, \ \text{fact})\)). To summarize, \(\bot^*\) and \textit{match} are the two enhancements required to write valid code for unfolding functional programs. If eager evaluation is used, these enhancements would not be necessary but naïve unfolding would still fail to work.

4 Operational Semantics

The operational semantics that describes how ground expressions written in the kernel language are evaluated is shown in Fig. 10. The semantics defines a small step relationship denoted by \(\leadsto\). The notation \(e \leadsto e'\) means that the expression \(e\) can be rewritten to \(e'\). The reduction relation \((p \ e_1 \ldots e_n) \leadsto^p t \ (p \in \text{PF})\) states that \(p \ e_1 \ldots e_n\) can be rewritten to \(t\) by using the definition of the predefined function \(p\).

The unfolding and operational semantics are equivalent in the following sense for any ground expression \(\text{goal}\):

\[
\text{goal} \leadsto^* e' \leftrightarrow e' \in \text{ueval}(I_\infty, \text{goal})
\]

where \(\leadsto^*\) is
With declarative debugging, the debugger and test environment unfold. Where:

\[ \left\{ \begin{array}{l}
\{ \langle h'' \mid c'' = b'' \rangle \text{ such that } (h'' \mid c'' = b'') \in \text{unfold}(l) \mid \text{eval}(I_m, g) = \text{eval}(I_m, r) \} \text{ if } g \text{ and } r \text{ have no total apps. of user funcs.} \\
\forall o \in \text{Pos}(r) \cup \text{Pos}(g), \text{resp. } g \mid_o = f \ e_1 \ldots e_n, f \in FS_n \\
\forall (f \ t_1 \ldots t_m \mid c_j = b_j) \in I_m \text{ such that} \\
\text{unmatch((t_1, \ldots, t_m), (e'_1, \ldots, e'_n))} = (\sigma, c'_m) \text{ and } c'_m \text{ satisfiable} \\
\end{array} \right. \]

A position is unfoldable:

- Case 2a): Some facts fit position \( o \)
- Case 2b): No facts fit position \( o \)

where:

- \( e'_i = \text{eval}(I_m, e_i) \) \( \forall i : 1 \ldots n \)
- \( g(t) \mid_o = g[t] \mid_o \) if \( o \in \text{Pos}(g) \) and \( g[t] \mid_o = g \) otherwise.
- \( r(t) \mid_o = r[t] \mid_o \) if \( o \in \text{Pos}(r) \) and \( r[t] \mid_o = r \) otherwise.

Fig. 9. Unfolding of a program rule using a given interpretation

The transitive and reflexive closure of \( \Rightarrow \) and \( e' \) is in normal form according to \( \Rightarrow \), \( \text{ueval} \) is a function that evaluates expressions by means of unfolding and \( I_\infty \) is the limit of the interpretations found by repeatedly unfolding the program. This equivalence is proved in the Appendix, Sect. B.3.

Note that this semantics is fully indeterministic; it is not meant to be used in any kind of implementation and its only purpose is to serve as a pillar for the demonstration of equivalence between the unfolding and an operational semantics. Therefore, the semantics is not lazy or greedy in itself. It is the choice of reduction positions where the semantics’ rules are applied what will make a certain evaluation lazy or not.

5 Some Applications of the Unfolding Semantics

Declarative Debugging With declarative debugging, the debugger consults the internal structure of source code to find out what expressions depend on other expressions and turns this information into an Execution Dependence Tree (EDT).

\footnote{The listings of unfolded code provided in this paper have been generated by our unfolder. Source at \text{http://www.github.com/josem-rey/unfolder} and test environment at \text{https://babel.ls.fi.upm.es/~jmrey/online_unfolder/unfolding.html}}
The debugger uses this information as well as answers from the user to blame an error on some rule. We have experimentally extended the unfolder to collect intermediate results as well as the sequence of rules that leads to every fact. This additional information allows our unfolder to build the EDT for any program run. Consider for example this buggy addition:

A1: addb Zero n = n
A2: addb Suc(Zero) n = Suc(n)
A3: addb Suc(Suc(m)) n = Suc(addb m n)
M24: main24 = addb Suc(Suc(Suc(Zero))) Suc(Zero)

We can let the program unfold until main24 is fully evaluated. This happens in I₃, which contains the following fact for the main function (after much formatting):

root:main24 = Suc(Suc(Suc(Zero))) <M24>
   n1: addb(Suc(Suc(Suc(Zero)))<M24>),Suc(Zero))=Suc(Suc(Suc(Zero)))<A3>
   n2: addb(Suc(Zero),Suc(Zero)) = Suc(Suc(Zero)) <A2>

Now, following the method described in [8], we can think of the sequence above as a 3-level EDT in which the root and node n₁ contain wrong values while the node n₂ is correct, putting the blame on rule A3.

The main reason that supports the use of unfolding for performing declarative debugging is that it provides a platform-independent environment to test complex programs. This platform independence can help check the limitations of some implementations (such of unreturned answers due to endless loops).

Test Coverage for a Program It is said that a test case for a program covers those rules that are actually used to evaluate the test case. We would like to reach full code coverage with the smallest test set possible. The unfolder can be a valuable tool for finding such a test set if it is enhanced to record the list of rules applied to reach every fact.
What must be done with the enhanced unfolder is to calculate interpretations until all the rules appear at least once in the rule list associated to the facts that do not contain any $\bot$ and then apply a minimal set coverage algorithm to find the set of facts that will be used as the minimal test set. For example:

\begin{verbatim}
R1: rev [] = []  // List inversion
R2: rev (x:xs) = append (rev xs) [x]
A1: append [] x = x
A2: append (x:xs) ys = x:(append xs ys)
\end{verbatim}

The first interpretation contains:

\begin{itemize}
  \item $\text{rev}(\text{Nil}) = \text{Nil}$ \textless R1\textgreater \\
  \item $\text{append}(\text{Nil},b) = b$ \textless A1\textgreater \\
  \item $\text{append}(\text{Cons}(b,c),d) = \text{Cons}(b,\text{Bot})$ \textless A2\textgreater \\
\end{itemize}

So, appending the empty list to any other achieves 50% coverage of \texttt{append}. Reversing the empty list uses 1 rule for \texttt{rev}: the coverage rate is 50% too. $I_3$ has:

\begin{itemize}
  \item $\text{append}(\text{Cons}(b,\text{Nil}),c) = \text{Cons}(b,c)$ \textless A2, A1\textgreater \\
  ... \\
  \item $\text{rev}(\text{Cons}(b,\text{Cons}(c,\text{Nil}))) = \text{Cons}(c,\text{Cons}(b,\text{Nil}))$ \textless R2, R2, R1, A1, A2, A1\textgreater \\
\end{itemize}

This shows that the minimal test set to test \texttt{append} must consist of appending a one element list to any other list. Meanwhile, reversing a list with 2 elements achieves a 100% coverage of the code: all the rules are used.

To close this section, we would like to mention that Abstract Interpretation can be used along with unfolding to find properties of the programs under study such as algebraic or demand properties. See examples 4, 5, 6 in the Appendix.

## 6 Conclusion and Future Work

We have shown that unfolding can be used as the basis for the definition of a semantics for lazy, higher-order functional programs written in a kernel language of conditional equations. This is done by adapting ideas from the s-semantics approach for logic programs, but dealing with the aforementioned features was not trivial, and required the introduction of two ad-hoc primitives to the kernel language: first, a syntactic representation of the undefined and second, a matching operator that deals with partial information.

Effort has also been devoted to simplifying the code produced by the unfolder, by erasing redundant facts and constraining the shape of acceptable facts. We have provided a set of requirements for programs that ensure the safety of these simplification procedures. We have also proven the equivalence of the proposed unfolding semantics with an operational semantics for the kernel language.

We have implemented an unfolder for our kernel language. Experimenting with it supports our initial claims about a more “implementable” semantics.

Regarding future work, we want to delve into the applications that have been just hinted here, particularly declarative debugging and abstract interpretation.
Finally, we are working on a better characterization of the necessary conditions that functional programs must meet in order for different optimized versions of the clean method to work safely.

References

1. Alpuente, M., Falaschi, M., Vidal, G.: Narrowing-driven Partial Evaluation of Functional Logic Programs. In: Proc. ESOP’96. LNCS, vol. 1058. Springer (1996)
2. Alpuente, M., Falaschi, M., Moreno, G., Vidal, G.: Safe folding/unfolding with conditional narrowing. In: Proc. ALP’97. pp. 1–15. Springer LNCS (1997)
3. Antoy, S., Hanus, M.: Declarative programming with function patterns. In: Proc. of LOPSTR’05. pp. 6–22. Springer LNCS (2005)
4. Bossi, A., Gabbrielli, M., Levi, G., Martelli, M.: The s-semantics approach: Theory and applications. Journal of Logic Programming 19/20, 149–197 (1994)
5. Burstall, R.M., Darlington, J.: A transformation system for developing recursive programs. J. ACM 24(1), 44–67 (Jan 1977)
6. Hanus, M.: The integration of functions into logic programming: From theory to practice. Journal of Logic Programming pp. 583–628 (1994)
7. Pettorossi, A., Proietti, M.: Perfect model checking via unfold/fold transformations. In: Computational Logic, LNCS 1861. pp. 613–628. Springer (2000)
8. Pope, B., Naish, L.: Buddha - A declarative debugger for Haskell (1998)
9. Scott, D.: The lattice of flow diagrams (Nov 1970)
APPENDIX

This appendix is not part of the submission itself and is provided just as supplementary material for reviewers. It pursues the following goals:

1. To provide a pictorial representation of the functions involved in the unfolding process, which hopefully helps in grasping how the whole process works (Sect. A).
2. To describe in what sense the unfolding and the operational semantics are equivalent and to prove such equivalence (Sect. B).
3. To present a larger example that intends to clarify how the functions that have been used actually work as well as additional examples (Sect. C).
4. To establish some results that support the validity of the code generated by the unfolder (Sect. D).

A Pictorial Representation of the Unfolding Process

Throughout Sect. 3 a number of auxiliary functions were presented. These functions are depicted in Fig. 11. The figure can be explained as follows:

The starting point is $U$. $U$ does nothing but to call $unfold$ and remove the redundant facts by calling $clean$. It is then up to the user to call $U$ again to perform another step in the unfolding process.

The second level of the figure shows $unfold$, which takes a program rule and unfolds it as much as possible. $unfold$ calls itself with the output of its previous execution until no more positions are left to unfold (arrow pointing downwards). If $unfold$ receives an input where at least one position is unfoldable, it calls $eval$ on the arguments of the unfoldable expression and then calls $umatch$ to perform the actual fitting between the unfoldable position and the head of some fact.
The last level of the figure (below the dashed line) represents the execution of the unfolded code. This part is not related with the definition of the unfolding operator, but with the execution of the unfolded code. The code is made of the output of \textit{unfold} whose guards are (possibly) extended with $c'_m$, the output from \textit{umatch}, which contains the invocations to \textit{match}. Observe that the output from \textit{umatch} goes to the generated code only, not to the unfolding process.

To the best of our knowledge, this unfolding process is a first effort to formulate an unfolding operator beyond naive unfolding.

B Equivalence between the Unfolding Semantics and the Operational Semantics

B.1 Unfolding of an Expression

Let us define a function $\textit{ueval}$ that finds what is the normal form for a given expression by means of unfolding. In short, what $\textit{ueval}$ does is to evaluate a given (guarded) expression by unfolding it according to a given interpretation.

The function $\textit{ueval}$ has type $\textit{ueval} : \textit{Set}(F) \times E \rightarrow \textit{Set}(E)$ and is defined as shown in Fig. 12. Note that any expression $e$ is equivalent to $(\textit{True} \triangleright e)$.

B.2 Trace of a Fact or an Expression

Given a fact $F$, belonging to any interpretation $I$, its trace is the list of pairs $(A_i, o)$ where $A_i$ is a rule in $\text{rule}(P) \cup \{A \perp_s, f = f \, x_1 \ldots x_n = \perp^s\}$ $\forall f \in FS_n$ and $o$ is a position within the expression to which the next rule in the trace is to be applied. This position indicates what subexpression within the current expression is to be replaced by the body of the rule applied.

Let us define the function that returns all the traces associated to all the facts derivable from a single rule ($+$ denotes list of lists concatenation that returns the list of lists resulting from appending every list in the first argument to every list in the second argument):

$$\textit{tr}' : \textit{Set}(F) \times R \rightarrow [\tau]$$

where $R$ is the type of program rules and $\tau$ is the type of traces.

- $\textit{tr}'(I, f \, t_1 \ldots t_n|c = b) = [\ ]$ if $f \, t_1 \ldots t_n|c = b$ is a valid input for the case 1 of $\textit{unfold}$.
- $\textit{tr}'(I, f \, t_1 \ldots t_n|c = b) = o.\textit{tr}(I, F) + + \textit{tr}'(I, F')$ if the case 2a) of $\textit{unfold}$ can be applied to $f \, t_1 \ldots t_n|c = b$ using fact $F \in I$ at position $o$. $F'$ is the result of unfolding $f \, t_1 \ldots t_n|c = b$ as it is done in the aforementioned case of $\textit{unfold}$.
- $\textit{tr}'(I, f \, t_1 \ldots t_n|c = b) = [[(A \perp_s, f', o)]] + + \textit{tr}'(I, F')$ if the case 2b) of $\textit{unfold}$ can be applied to $f \, t_1 \ldots t_n|c = b$ at position $o$. $F'$ is the result of unfolding $f \, t_1 \ldots t_n|c = b$ as it is done in case 2b) of $\textit{unfold}$.
ueval : \text{Set}(\mathcal{F}) \times E \rightarrow \text{Set}(E)

\begin{align*}
\text{ueval}(I, e) &= \{e\} \text{ if no rule from evalAux applies to any position of } e \\
\text{ueval}(I, e) &= \bigcup_o \text{ueval}(I, e|\text{uevalAux}(I, e|o)||o) \\
& \quad \forall o \text{ such that a rule of } \text{uevalAux} \text{ is applicable to } e|o.
\end{align*}

\text{uevalAux} : \text{Set}(\mathcal{F}) \times E \rightarrow E

\begin{align*}
\text{uevalAux}(I, p e_1 \ldots e_n) &= t \quad \text{if } p e_1 \ldots e_n \sim^p t \quad (p \in PF_n) \\
\text{uevalAux}(I, (c_1 \land \ldots \land c_{i-1} \land \text{snd}(\text{match}(p,e)) \land c_{i+1} \land \ldots \land c_n \triangleright e')) &= \sigma(c_1 \land c_{i-1} \land b \land c_{i+1} \land \ldots \land c_n \triangleright e') \\
& \quad \text{if } \text{match}(p,e) = (\sigma, b) \\
\text{uevalAux}(I, \text{True} \land \text{True}) &= \text{True} \\
\text{uevalAux}(I, b_1 \land b_2) &= \text{False} \quad b_1, b_2 \in \text{Bool}, b_1 = \text{False} \text{ or } b_2 = \text{False} \\
\text{uevalAux}(I, \text{True} \triangleright e) &= e \\
\text{uevalAux}(I, \text{False} \triangleright e) &= \bot^s \\
\text{uevalAux}(I, f e_1 \ldots e_n) &= \sigma'(e \triangleright b) \quad \text{if } f \in FS_n, \exists f t_1 \ldots t_n|e = b \in I. \sigma' = \text{mgu}((t_1, \ldots, t_n), (e_1, \ldots, e_n)) \text{ with } \sigma'(e) = \text{True} \\
\text{uevalAux}(I, f e_1 \ldots e_n) &= \bot^s \quad \text{if } f \in FS_n, \exists f t_1 \ldots t_n|e = b \in I. \sigma' = \text{mgu}((t_1, \ldots, t_n), (e_1, \ldots, e_n)) \text{ with } \sigma'(e) = \text{True}
\end{align*}

Fig. 12. The \text{ueval} function: Evaluating expressions by means of unfolding

The composition of a position \( o \) and a trace (denoted by \( o.tr(\ldots) \) above) is defined (for every trace in a given list) as:

\begin{itemize}
  \item \( o.[\ ] = [\ ] \)
  \item \( o.[(\Lambda, o')|xs] = [(\Lambda, o')|o.xs] \)
\end{itemize}

The list of traces for a fact \( F \) with respect to an interpretation \( I \) (\( tr(I, F) \)) relies on \( tr' \):

\[ tr : \text{Set}(\mathcal{F}) \times \mathcal{F} \rightarrow [\tau] \]

\[ tr(I, F) = [[(A_F, \{\})]] + \{\tau \text{ such that } \tau \in tr'(I, A_F) \land unfold(A_F, I) \text{ generates } F \text{ according to the steps given by the trace } \tau} \]}
where $\Lambda_F$ is the only program rule that can generate $F$.

Note that $tr$ and $tr'$ are mutually recursive.

The list of traces of an expression $e$ according to interpretation $I$ (denoted $Tr(I, e)$) is defined as the tail of all the lists in $tr(I, goal' = e)$ where $goal'$ is a new function name that does not appear in the program $P$ and the tail of a list is the same list after removing its first element.

**B.3 Equivalence between the Unfolding and Operational Semantics**

This section will show that the unfolding semantics and the operational semantics are equivalent in the following sense for any ground expression $goal$:

$$goal \sim^* e' \iff e' \in \text{ueval}(I_\infty, goal)$$

where $\sim^*$ is the transitive and reflexive closure of $\sim$ and $e'$ is in normal form according to $\sim$.

Given a program $P$, $I_\infty$ is the limit of the following sequence:

- $I_0 = \emptyset$
- $I_{n+1} \ (n \geq 0) = \bigcup_{\Lambda \in \text{rule}(P)} \text{unfold}(I_n, \Lambda)$

**B.4 Proof of Equivalence**

We are now proving that Eq. 2 holds.

This part of the double implication will be proven by induction on the number of $\sim$-steps that an expression requires to reach normal form.

**Base case (n=0):** If $goal \sim^0 e'$, then $goal = e'$, which means that $goal$ is in normal form already. Therefore, $goal$ has no full applications of symbols in $PF \cup FS$. In that case, $\text{ueval}(I, goal) = \{goal\} \ \forall I \in \text{Set}(F)$.

**Induction step:**

Let us take as induction hypothesis that any expression $goal$ such that $goal \sim^0 e'$ (where $e'$ is in normal form) then $e' \in \text{ueval}(I_\infty, goal)$.

Let $e^{n+1}$ be an expression that requires $n+1 \sim$-steps in order to reach normal form. Then there must exist (at least) one expression $e^n$ such that:

$$e^{n+1} \sim e^n \sim^n e'$$

where $e'$ is in normal form. Now, if we prove that both $e^{n+1}$ and $e^n$ unfold to the same values (that is, $\text{ueval}(I_\infty, e^{n+1}) = \text{ueval}(I_\infty, e^n)$), then we can apply the induction hypothesis to $e^n$ to state that $e^n \sim^n e' \rightarrow e' \in \text{ueval}(I_\infty, e^{n+1}) = \text{ueval}(I_\infty, e^n)$.

Let us check all the rules in the operational semantics for the single $\sim$ step going from $e^{n+1}$ to $e^n$.

**Rule rule**
In this case, \( e^{n+1} = f \ e_1 \ldots e_a \ (f \in FS_a) \) and \( e^n = \sigma(r) \) (assuming that the rule for \( f \) within program \( P \) is \( A \equiv f \ t_1 \ldots t_n | g = r \) and \( \sigma = mgu((t_1, \ldots, t_n), (e_1, \ldots, e_n)) \)).

By *reductio ad absurdum* let us assume now that \( ueval(I, e^{n+1}) \neq ueval(I, e^n) \). Then,

\[ ueval(I, f \ e_1 \ldots e_a) \neq ueval(I, \sigma(g \triangleright r)) \]

However, note that \( \sigma(A) \) is equal to the rule instance \( f \ e_1 \ldots e_a | \sigma(g) = \sigma(r) \), which states exactly the opposite of the equation above. We have reached a contradiction, which means that our initial hypothesis (namely, \( ueval(I, e^{n+1}) \neq ueval(I, e^n) \)) is false.

**Rule rulebot**

In this case, \( e^{n+1} = f \ e_1 \ldots e_a \ (f \in FS_a) \) and \( e^n = \bot \). If there is no rule in \( P \) whose pattern can unify with \( e^{n+1} \) while at the same time having a satisfiable guard, it is sure that no fact in any interpretation derived from \( P \) will be such that its head unifies with \( e^{n+1} \) while at the same time having a satisfiable guard (which forces \( uevalAux \) to use its last case). That means that \( e^{n+1} \) cannot be reduced to anything different from \( \bot \). The same happens with \( e^n \) (which is already equal to \( \bot \)). Therefore, \( ueval(I, e^{n+1}) = ueval(I, e^n) \) as we wanted to prove.

**Rule predef**

In this case, \( e^{n+1} = p \ e_1 \ldots e_a \) and \( e^n = t \ (t \in T) \) where \( p \in PF \) and \( p \ e_1 \ldots e_a \) has value \( t \) according to the predefined functions known to the environment being used.

Also in this case \( ueval(I, p \ e_1 \ldots e_a) = \{t\} \) and \( ueval(I, t) = \{t\} \) for any interpretation \( I \). This case simply evaluates predefined functions.

**Rule andfalse**

In this case, \( e^{n+1} = e_1 \wedge e_2 \) and \( e^n = False \) when either \( e_1 \leadsto^* False \) or \( e_2 \leadsto^* False \). Let us assume without loss of generality that \( e_1 \leadsto^* False \).

Since \( e^{n+1} \) requires \( n+1 \) \( \leadsto^* \) steps to reach normal form, then \( e_1 \) must take at most \( n \) steps to reach its normal form. This means that the induction hypothesis is applicable to \( e_1 \) and therefore \( ueval(I, e_1) \supset \{False\} \). This in turn means that \( ueval(I, e_1 \wedge e_2) \supset \{False\} \) as we wanted to prove (assuming that the logical connector \( \wedge \) is defined as lazy conjunction in \( eval \)).

The remaining rules (*ANDTRUE, IFTENTRUE, IFTHENFALSE*) are proven in a similar way.

Let us proceed now to the reverse implication.

\( \leftarrow \)

The proof will be driven by structural induction on the shape of the expression to be evaluated (\( goal \)).

Let \( goal \) be an expression that has no full applications of any symbol of \( FS \) or \( PF \). Then, \( ueval(I, goal) = \{goal\} \) and \( \leadsto \) cannot apply any rewriting, so \( goal \leadsto^* goal \), as we wanted to prove.

Next, let \( goal = p \ e_1 \ldots e_a \ (p \in PF_n) \) and no \( e_i \) has any full application of any symbol in \( FS \cup PF \). Then, \( ueval(I, goal) = \{t\} \) if \( goal \leadsto^p t \). \( \leadsto \) will apply...
PREDEF, IFTHENTRUE, IFTHENFALSE, ANDTRUE or ANDFALSE to evaluate the same predefined function and reach the same $t$.

Next, let $goal = f\ e_1\ldots\ e_n$ ($f \in FS_n$). If $ueval(I_\infty, goal)$ includes $e'$ that is because $goal$ has a trace (since $e'$ is in normal form) That is, $[(A_1, o_1), \ldots, (A_k, o_k)] \in Tr(I_\infty, goal)$ for some $k > 0$. We are now going to prove that:

$$e' \in ueval(I_\infty, goal) \wedge [(A_1, o_1), \ldots, (A_k, o_k)] \in Tr(I_\infty, goal) \rightarrow goal \sim^* e' (\text{using the exact sequence of rules given below})$$

Specifically, it will be proven that every trace element $(A, o)$ is equivalent to the following $\sim^*$-sequence at position $o$ of the expression input for the trace element:

1. The rules dealing with predefined functions (namely, PREDEF, IFTHENTRUE, IFTHENFALSE, ANDTRUE, ANDFALSE) will be applied to the expressions being rewritten as many times as possible.
2. RULE or RULEBOT: The rule RULE will be applied if $A \neq A_{\pperp, f}$. RULEBOT will be applied otherwise.
3. The rules dealing with predefined functions (namely, PREDEF, IFTHENTRUE, IFTHENFALSE, ANDTRUE, ANDFALSE) will be applied to the expression returned by the previous step as many times as possible.

The proof will be driven by induction on the length of the trace for $goal$.

**Base case:** $([(A = f\ p_1\ldots\ p_a | g = r, o = \{}))] \in Tr(I_\infty, goal))$. If $goal$ can be rewritten to normal form $e'$ by just using the rule $A$, that means that $A \neq A_{\pperp, f}$ (since $e'$ is assumed to be in normal form) and that $I_\infty$ must contain a fact $f\ t_1\ldots\ t_a | c = b$, according to the definition for $tr'$ (second case), such that $\exists \sigma' = mgu((t_1, \ldots, t_a), (e_1^+, \ldots, e_n^+))$ and $eval(I_\infty, \sigma'(c)) = True$ and $eval(I_\infty, \sigma'(b)) = e'$, $(e_1^+ = eval(I_\infty, e_1))$.

On the other hand, $\sim$ can apply RULE to $goal = f\ e_1\ldots\ e_n$. This rule rewrites $goal$ to:

$$\sigma(g \triangleright r)$$

where $\sigma = mgu((p_1, \ldots, p_a), (e_1^+, \ldots, e_n^+))$. Note that the application of $eval$ here is equivalent to the application of all the cases of $uevalAux$ except the last two ones (let us call these first cases $uevalAux_{predf}$) which in turn have the same effect than the $\sim$-rules PREDEF, IFTHENTRUE, IFTHENFALSE, ANDTRUE, ANDFALSE as many times as necessary to evaluate any predefined function that appears in any of the arguments to $f$. Let us remark that $match$ is unnecessary in $\sim$ since all the expressions handled by $\sim$ are ground and the substitution returned by $match$ generates the same ground expressions than $\sigma$ in rule RULE. We also assume that $match$ cannot be used in normal programs.

The same cases for predefined functions ($uevalAux_{predf}$) can be applied to the expression above to get $\sigma(eval(I_\infty, g) \triangleright eval(I_\infty, r))$.

By construction, we know that $\sigma'(t_1, \ldots, t_a) = \sigma(p_1, \ldots, p_a) = (e_1^+, \ldots, e_n^+)$.

Since any valid program in our setting can only have at most one rule that matches the ground expression $fe_1\ldots e_n$, then:
\[
- \text{eval}(I_\infty, \sigma'(b)) = \text{eval}(I_\infty, \sigma(r)) = e'
- \text{eval}(I_\infty, \sigma'(c)) = \text{eval}(I_\infty, \sigma(g)) = \text{True}
\]

Therefore, both unfolding and \(\sim\) have used a fact and a rule which were syntactically identical (once predefined functions have been evaluated) to find the same answer for \(\text{goal}\). This proves the base case.

**Induction step:** Now the length of the trace for \(\text{goal}\) is equal to \(l + 1\) \((l > 0)\).

Let us assume, as induction hypothesis that:

\[e' \in \text{ueval}(I_\infty, \text{goal}) \rightarrow \text{goal} \sim^* e'\]

provided that the trace for \(\text{goal}\) has exactly \(l\) elements and using the sequence of \(\sim\)-rules given earlier for that trace.

Let us prove the equation for goals with trace of length \(l + 1\). In order to do that let us consider that \([\{A_1, o_1\}, \ldots, \{A_{l+1}, o_{l+1}\}] \in \text{Tr}(I_\infty, \text{goal})\) and an intermediate expression \(e'\) whose trace is the same as before except for the first element.

Two cases have to be looked at here: One in which \(\text{rulebot}\) can apply to \(e'\) and then:

\[- \text{goal}|_{o_1} = f'' e''_1 \ldots e''_t \quad (f'' \in FS_f)\]
\[- \exists(F'' \in I_\infty) \text{such that } F'' = f'' t''_1 \ldots t''_r | e'' = b'' \text{ and}\]
\[- \sigma' = \text{mgu}((t''_1, \ldots, t''_t), (e''_1, \ldots, e''_r)) \text{ where } e''_1 = \text{eval}(I_\infty, e''_j)\]

and \(\text{True} \in \text{ueval}(I_\infty, \sigma'(e''))\).

- If \(F''\) does not exist that means that no rule for \(f''\) originates a fact like \(F''\) which in turn means that no rule unifies with \((e''_1, \ldots, e''_r)\) either. This forces \(\sim\) to apply \(\text{rulebot}\) to \(e'|_{o_1}\) and the same does \text{ueval} (last rule).

Note that the sequence of \(\sim\) rules applied in this case was \((\text{predef, ifthentrue, ifthenfalse, andtrue, andfalse})^*, \text{rulebot}\).

Since the trace for \(\text{goal}|_{o_1}\) has length \(l\), the induction hypothesis is suitable for it and then:

\[e' \in \text{ueval}(I_\infty, e') \rightarrow e' \sim^* e'\]

by using the sequence of \(\sim\)-rules given earlier for \(e'\)

and since the normal form for \(\text{goal}\) is the same as the normal form for \(e'\) by definition of unfolding, we have that the trace for \(\text{goal}\) has the required shape.

Lastly, consider some goal whose trace is again

\([\{A_1, o_1\}, \ldots, \{A_{l+1}, o_{l+1}\}]\) but where \(A_1 \neq A_{l+1, f}\). Let us apply \(\text{uevalAux}_{\text{predef}}\) (or, equivalently, \(\text{predef, ifthentrue, ifthenfalse, andtrue, andfalse}\)) as many times as possible to the arguments of \(\text{goal}|_{o_1}\) to get \(f'' e''_{1+} \ldots e''_{r+}\) and then \(\text{rule}\) (the only rule that \(\sim\) can apply to \(f'' e''_{1+} \ldots e''_{r+}\)) by using the rule
\[ A_1 \equiv f'' \ p'_1 \ldots p'_f \ | g'' = r'' \] and the unifier
\[ \sigma = \text{mgu}((p''_1, \ldots, p''_f), (e''_1, \ldots, e''_f)). \] Now, we can rewrite \( \text{goal} \) to:

\[ \text{goal}[\sigma(g'' \triangleright r'')]_{\sigma_1} \]

given that according to \( tr'' \)'s definition (second case), the rule \( A_1 \) is applicable to \( \text{goal}_{\sigma_1} \) (since the trace of any fact begins by the rule which originates the fact and \( \text{unfold} \) has been able to apply a fact derived from \( A_1 \) to \( \text{goal}_{\sigma_1} \)). This means that the expression above can be rewritten to \( \text{goal}[\text{eval}(I_\infty, \sigma(g'' \triangleright r''))]_{\sigma_1} \) which is the new \( e' \). Note that the \( \text{eval} \) in the former expression is equivalent to \((\text{predef}, \text{ifthentrue}, \text{ifthenfalse}, \text{andtrue}, \text{andfalse})^* \) (which in turn is equivalent to applying the cases of \( \text{uevalAux}_{predef} \) as many times as necessary).

What we know now is:

- The normal form for \( \text{goal} \) is the same as the normal form for \( e' \) since \( \sim \) conserves the semantics by definition.
- The rewriting sequence from \( \text{goal} \) to its normal form is the same as the rewriting sequence for \( e' \) preceded by the rules applied above (i.e. \((\text{predef}, \text{ifthentrue}, \text{ifthenfalse}, \text{andtrue}, \text{andfalse})^*, \text{rule}, (\text{predef}, \text{ifthentrue}, \text{ifthenfalse}, \text{andtrue}, \text{andfalse})^*)\)).
- Since the length of the trace for \( e' \) is equal to \( l \), the induction hypothesis is applicable to it and then:

\[ e' \in \text{ueval}(I_\infty, e') \rightarrow e' \sim^* e' \]

by using the sequence of \( \sim \)-rules given earlier for \( e' \)

Since we have proved that, given an unfolding sequence, we are able to find a precise sequence of \( \sim \)-rules that provokes the very same effect to any given expression, we have also proved the \( \leftarrow \) implication of the theorem.

**B.5 Example**

Let us present an example that may clarify some of the concepts involved in the proof of equivalence. Consider this program:

\[
\begin{align*}
f(x) &= g(x+1) \\
g(x) &= h(x+2) \\
h(x) &= x+3 \\
j(5) &= 6 \\
f2(x, y) &= x+y \\
goal &= f(4) \\
goal2 &= f2(f(4), f(4)) \\
goal3 &= K(j(5))
\end{align*}
\]
What we have here are a constructor (K), some test goals (goal1, goal2 and goal3) and some functions. A first function group (f, g, h) is such that f requires the evaluation of g and h. We will see that the trace for f’s facts reflects this issue. Next, we have two more unrelated functions. j and goal3 will be used to demonstrate the usage of RULEBOT while f2 and goal2 will show why non-unique traces exist.

The first interpretation generated by the unfolder is:

* h(b) = b+3 <H>
* j(5) = 6 <J>
* f2(b,c) = b+c <F2>
* goal3 = K(Bot) <Goal3>

The sequences enclosed between < and > is the trace of every fact. It can be seen, however, that the unfolder does not display the position of every step and it does not display the usage of $\Lambda_s$, j, rules either (see the last fact).

So, the whole trace for the last fact would be

[([Goal3,{}], [Lambda_Bot,{}])].

All the other facts have a trace of length 1. It can be seen that they are identical to their respective program rules. This property supports the proof for the base case of the induction since applying the rule to perform rewriting is exactly the same as applying a fact with a trace length of one. Realize that even though these facts cannot be unfolded any further, they require eval to reach a normal form (since the addition needs to be evaluated after its arguments have been bound to ground values).

Further interpretations provide more interesting traces. This is $I_4$:

* h(b) = b+3 <H>
* j(5) = 6 <J>
* f2(b,c) = b+c <F2>
* g(b) = b+2+3 <G,H>
* goal3 = K(6) <Goal3,J>
* f(b) = b+1+2+3 <F,G,H>
* goal = 10 <Goal,F,G,H>
* goal2 = 20 <Goal2,F2,F,G,H,F,G,H>

Take a look at how the fact regarding goal reflects the dependency between f, g and h. That fact has a trace of length 4 (it is very easy to follow how goal is evaluated by looking at its trace). Removing the first element of its trace (as needed in the induction step) yields the trace <F,G,H> for which there is a fact (the fact for f). This means that in this case, the induction step says that evaluating goal is the same as applying RULE to find f(4) (eval is not needed in this step) and then applying the induction hypothesis to f(4) whose trace is an element shorter than that of goal.

Finally, consider how goal2 can be evaluated to normal form in multiple orders. Since f2 demands both arguments, both of them must be taken to normal form but the order in which is done is irrelevant. Since our unfolder does not show the positions for the reduction steps all the traces for goal2 look the same but more than one trace would appear if positions were taken into account.
B.6 Lemma: Applicability of Every Step Inside a Fact’s Trace

We have seen that the proof of equivalence between the operational semantics and the unfolding semantics relies on the fact that every element inside a fact’s trace is applicable to the expression resulting from applying all the trace steps that preceded the steps under consideration. How can we be sure that a trace step is always applicable to the expression on which that step must operate? This lemma states that this will always happen. Intuitively, what the lemma says is that the trace of a fact is nothing else that the sequence of rules that unfold has applied to get from a program rule to a valid fact.

Lemma 3 (Applicability of Every Step Inside a Fact’s Trace). Let \( e \) be an expression and let \( [(A_1, o_1), \ldots, (A_m, o_m)] \in \text{Tr}(I_\infty, e) \). If \( e \xrightarrow{\star} e_{m'} \quad (m' \geq 0, m' < m) \) and \( [(A_{m'+1}, o_{m'+1}), \ldots, (A_m, o_m)] \in \text{Tr}(I_\infty, e_{m'}) \) then the following assertions are true:

1. If \( A_{m'+1} = \bot s_f t_1 . . . t_n | g = r' \) such that \( \exists \sigma = \text{mgu}(f t_1 . . . t_n, \text{eval}(I_\infty, e_{m'} | o_{m'+1})) \) and \( \text{True} \in \text{ueval}(I_\infty, \sigma(g')) \)
2. If \( A_{m'+1} \neq \bot s_f t_1 . . . t_n | g = r \) and \( \exists \sigma = \text{mgu}(f t_1 . . . t_n, \text{eval}(I_\infty, e_{m'} | o_{m'+1})) \) and \( \text{True} \in \text{ueval}(I_\infty, \sigma(g')) \)

C Additional Examples

Example 2 (Lazy Evaluation).

Think of the following code and its first interpretations:

\[
\begin{align*}
\text{first} & : [a] \to a \\
\text{first} \ (x:xs) & = x \\
\text{ones} & : [\text{Int}] \\
\text{ones} & = 1:\text{ones} \\
\text{main} & : \text{Int} \\
\text{main} & = \text{first} \ \text{ones}
\end{align*}
\]

The semantics for this program is infinite: every step adds a 1 to the list generated by \text{ones}.

Consider the step from \( I_1 \) to \( I_2 \): when unfolding \text{ones}, a fact matching \text{ones} is found in \( I_1 \) (namely, \( \text{ones}=1:\bot s \)) so this last value is replaced in the right side of the rule. Since the new value for \( \text{ones} \) is greater than the existing fact and both heads are a variant of each other, the function \text{clean} can remove the old fact. The fact \( \text{ones}=1:\bot s \) can now be used to evaluate \text{main}. Since the new fact \( \text{main}=1 \) is greater than the fact \( \text{main}=\bot s \) it replaces the existing one. The fact for \text{first} remains unaltered.

Example 3 (Larger Example).

Let us present an example that intends to describe how all the functions and concepts that we have seen throughout the paper work. Think of the following program:
ite : Bool → a → a → a
ite(True, t, e) = t
ite(False, t, e) = e

gen : Int → [Int]
gen n = n : (gen (n+1))

senior : Int → Bool
senior age = ite(age > 64, True, False)

map : (a → b) → [a] → [b]
map(f, []) = []
map(f, (x:xs)) = (f x) : (map(f, xs))

main50 : [Bool]
main50 = map(senior, gen(64))

Let us see how this program is unfolded.

First, the initial interpretation ($I_0$) is empty, by definition. At this point, the function unfold is applied to every rule in turn, using $I_0$ as the second argument. This produces the following interpretation ($I_1$):

- The two rules for ite have no applications of user-defined functions, so nothing has to be done to them in order to reach a fact in normal form. That is why they appear in $I_1$ right away.
- The rule for gen is a little bit different since this rule does have an application to a user-defined function. However, since $I_0$ contains nothing about those functions, all that unfold can do is to replace that invocation by the special symbol Bot (represented by $\bot$ in formulas) to represent that nothing is known about the value of gen(b+1).
- The function senior has no facts inside $I_1$ since the function clean removes any unguarded fact with a body equal to $\bot$. This is precisely what has happened since $I_0$ contains no information about ite, so the resulting new fact for senior would be * senior age=Bot

How did we get here?. When a rule is applied to unfold, every full application of a symbol in FS is replaced by the value assigned to the application in the interpretation also applied to unfold. The actual matching between an expression and some rule head is performed by umatch, which is called by unfold every time an expression needs to be unfolded. In most cases, umatch behaves as a simple unifier calculator but higher order brings complexity into this function (in s-semantics, where higher order does not exist, simple unification is used in the place of umatch). In this case, the interpretation applied was the empty one, so, the following has happened to every rule:

- The two rules for ite have no applications of user-defined functions, so nothing has to be done to them in order to reach a fact in normal form. That is why they appear in $I_1$ right away.
- The rule for gen is a little bit different since this rule does have an application to a user-defined function. However, since $I_0$ contains nothing about those functions, all that unfold can do is to replace that invocation by the special symbol Bot (represented by $\bot$ in formulas) to represent that nothing is known about the value of gen(b+1).
- The function senior has no facts inside $I_1$ since the function clean removes any unguarded fact with a body equal to $\bot$. This is precisely what has happened since $I_0$ contains no information about ite, so the resulting new fact for senior would be * senior age=Bot
– The first rule for map is left untouched since it has no full applications of user-defined functions (as it happened with ite).

– The second rule for map generates the fact
  \[ * \text{map}(b,\text{Cons}(c,d)) = \text{Cons}(\text{b@[c]},\text{Bot}) \]
  where the Bot denotes that the value for \text{map}(f,xs) is not contained inside \text{I}_0.

– And, finally, there is not any fact for main50 since the whole application of map that appears at the root of the body is unknown, so it gets replaced by Bot, which is in turn eliminated by clean (and moved into \text{I}^\bot).

Since we saw that two facts were removed by clean because they did not have a guard and their body was equal to Bot, \text{I}^1, has the following content:

* \text{senior(age)} = \text{Bot}
* \text{main50} = \text{Bot}

These two facts will be reinjected into the factset when \text{I}_2 is calculated but in this case, they do not have a noticeable effect on the results, so we will not insist on them any more.

One more iteration of the unfolding operator generates \text{I}_2:

* \text{ite(True,}b,c\text{)} = b
* \text{ite(False,}b,c\text{)} = c
* \text{map}(b,\text{Nil}) = \text{Nil}
* \text{gen}(b) = \text{Cons}(b,\text{Cons}(b+1,\text{Bot}))
* \text{senior}(b) \mid \text{snd(match(True,b>64))} = \text{True}
* \text{senior}(b) \mid \text{snd(match(False,b>64))} = \text{False}
* \text{map}(b,\text{Cons}(c,\text{Nil})) = \text{Cons}(\text{b@[c]},\text{Nil})
* \text{map}(b,\text{Cons}(c,\text{Cons}(d,e)))) = \text{Cons}(\text{b@[c]},\text{Cons}(\text{b@[d]},\text{Bot}))
* \text{main50} = \text{Cons}(\text{Bot},\text{Bot})

Remember that \text{I}_2 has been calculated by taking \text{I}_1 \cup \text{I}^\bot as the relevant interpretation. By definition of the unfolding operator, \text{I}_2 includes all the facts that were already present inside \text{I}_1 (unless they are removed by clean).

Remember also that we are using the optimized version of clean (the one that removes subsumed facts instead of enlarging the constraints of the subsuming facts). Once these aspects have been settled, the calculations that lead to the formation of \text{I}_2 can be explained as follows:

– The two facts for ite are transferred directly from \text{I}_1 into \text{I}_2. This is so since they cannot be unfolded any further and besides, they are not overlapped by any fact. The same happens with the first fact for map.

– The fact for gen is much more interesting: There are not two facts for gen in \text{I}_2. There is only one. This is due to the application of clean in unfold. What has happened here is that clean has compared the old fact (* \text{gen}(b)=\text{Cons}(b,\text{Bot})) to the new one (* \text{gen}(b)=\text{Cons}(b,\text{Cons}(b+1,\text{Bot}))) and has removed the old one. The reason for this is that both facts clearly overlap but the newest fact has a body that is greater (according to \sqsubseteq) than that of the old fact. Given that the optimized version of clean is being used (all the functions here are complete and the rules are productive), the old fact is removed.
One more point of interest here: Note that the expression \( b+1 \) cannot be further unfolded since the value for \( b \) is unknown at unfolding time. We will see the opposite case later.

– The explanation for \textit{senior} will be detailed later.

– The two facts for \textit{map} have become three. This has happened as follows:

  – The second rule for \textit{map}, when unfolded using \( I_1 \cup I_1^\perp \) generates two facts:

    * \( \text{map}(b, \text{Cons}(c, \text{Nil})) = \text{Cons}(b@[c], \text{Nil}) \)
    * \( \text{map}(b, \text{Cons}(c, \text{Cons}(d, e))) = \text{Cons}(b@[c], \text{Cons}(b@[d], \text{Bot})) \)

    Those two facts overlap with the old fact

    * \( \text{map}(b, \text{Cons}(c, d)) = \text{Cons}(b@[c], \text{Bot}) \)

    which brings us to the count of three facts for \textit{map}.

– \textit{main50} has progressed slightly: The invocation of \textit{map} within the body of \textit{main50} has been replaced by the body of the second fact for \textit{map} in \( I_1 \cup I_1^\perp \) generating \( \text{Cons}(\text{senior}@[b], \text{Bot}) \). Since nothing is known about \textit{senior} in \( I_1 \cup I_1^\perp \), the final result is \( \text{Cons}(\text{Bot}, \text{Bot}) \).

The unfolding of \textit{senior} requires special attention: In order to unfold the only rule for this function, the call to \textit{ite} is unfolded. However, the first argument for \textit{ite} must be fully known before proceeding. This is impossible at unfolding time since \textit{age} will receive its value later, at runtime. The only way to go in cases like this is to assume certain hypotheses and to generate facts that record those hypotheses. In this example, we are forced to assume that \textit{age} \( > 64 \) is \textit{True} when the first rule for \textit{ite} is unfolded while \textit{age} \( > 64 \) is assumed to be \textit{False} when the second rule for \textit{ite} is unfolded. These hypotheses are recorded in the guards for the facts corresponding to \textit{senior}.

The function responsible for generating these hypotheses is \textit{umatch} (more specifically, its second rule). This rule is used when an expression rooted by a predefined function (here, \texttt{<}) has to be matched to some pattern term which is not a variable (here, \textit{True} and then \textit{False}). In this case, \textit{umatch} extends the new fact's guard by adding the new condition (here \( \text{snd}(\text{match}(\text{True}, b > 64)) \)) (resp. \textit{False}) and then proceeds as if the PF-rooted expression matched the given pattern in order to continue generating hypotheses. In this case, \textit{umatch} would call itself with \( \text{umatch}(\text{True}, \text{True}) \), (resp. \textit{False}) which is solved by using \textit{umatch}'s first rule which generates no more conditions or variable substitutions.

Unfolding once again yields \( I_5 \):

* \textit{ite}(\text{True}, b, c) = b
* \textit{ite}(\text{False}, b, c) = c
* \text{map}(b, \text{Nil}) = \text{Nil}
* \textit{senior}(b) \mid \text{snd}(\text{match}(\text{True}, b > 64)) = \text{True}
* \textit{senior}(b) \mid \text{snd}(\text{match}(\text{False}, b > 64)) = \text{False}
* \text{map}(b, \text{Cons}(c, \text{Nil})) = \text{Cons}(b@[c], \text{Nil})
* \text{gen}(b) = \text{Cons}(b, \text{Cons}(b+1, \text{Cons}(b+1+1, \text{Bot})))
* \text{map}(b, \text{Cons}(c, \text{Cons}(d, \text{Nil}))) = \text{Cons}(b@[c], \text{Cons}(b@[d], \text{Nil}))
* \text{map}(b, \text{Cons}(c, \text{Cons}(d, \text{Cons}(e, f)))) = \text{Cons}(b@[c], \text{Cons}(b@[d], \text{Cons}(b@[e], \text{Bot})))
* \textit{main50} \mid \text{snd}(\text{match}(\text{True}, 65 > 64)) = \text{Cons}(\text{True}, \text{Cons}(\text{True}, \text{Bot}))
We are not repeating all the details above. Instead, we just want to point out some interesting aspects of this interpretation:

– The reader might have expected to find expressions like $64 > 64$ fully reduced (that is, replaced by $False$). That would be correct but boolean operators are not evaluated due to a limitation in the implementation of our unfolder. In this example, this limitation is a blessing in disguise since those expressions are needed to understand the origin of some facts.

– An expression like $b+1+1$ has not been reduced to $b+2$ since it stands for $(b+1)+1$. The function eval has returned the same expression that it is given since it cannot be further evaluated.

– The combinatory explosion of facts denotes that the unfolder tries all possible unfolding alternatives (in particular, those facts with less than two conditions in the guard are the result of unfolding senior before ite, so the result for senior cannot be other than an unguarded $\bot$).

– Note that our Prolog implementation does not have an underlying constraint solver, so the entailment condition of the guards that is used to sort overlapping facts is not checked. That is why the unfolder has generated facts that should have been removed, such as $main50 = Cons(Bot, Cons(Bot, Bot))$.

– A value of $65$ appears whenever the function eval has been applied to evaluate $64+1$.

Example 4 (Unfolding and Abstract Interpretation). This example will show how unfolding can be used to synthesize an abstract interpreter of a functional program. Think of the problem of the parity of addition. The sum of Peano naturals can be defined as shown in Fig. 1 (right).

We also know that the successor of an even number is an odd number and viceversa. The abstract domain (the domain of parities) can be written as:

```
data Nat# = Suc c# Nat# | Even# | Odd#
```

Now, the user would define the abstract version for \( \text{add} \) together with the properties of \( \text{Suc} \) regarding parity:

\[
\begin{align*}
\text{add}^# &: \text{Nat}^# \to \text{Nat}^# \to \text{Nat}^# \\
\text{add}^# \text{ Even}^# m &= m \\
\text{add}^# (\text{Suc}_c^# n) m &= \text{Suc}_{f^#} (\text{add}^# n m)
\end{align*}
\]

\[
\begin{align*}
\text{Suc}_{f^#} &: \text{Nat}^# \to \text{Nat}^# \\
\text{Suc}_{f^#} \text{ Even}^# &= \text{Odd}^# \\
\text{Suc}_{f^#} \text{ Odd}^# &= \text{Even}^#
\end{align*}
\]

In order to enforce the properties of the successor in the abstract domain, a catamorphism \(^7\) linking \( \text{Suc}_{f^#} \) to \( \text{Suc}_c^# \) will be used:

\[
\begin{align*}
\text{C}^# s &: \text{Nat}^# \to \text{Nat}^# \\
\text{C}^# s \text{ Even}^# &= \text{Even}^# \\
\text{C}^# s \text{ Odd}^# &= \text{Odd}^# \\
\text{C}^# s (\text{Suc}_c^# n) &= \text{Suc}_{f^#} (\text{C}^# s n)
\end{align*}
\]

Then, the unfolding process that has been described must be slightly modified: after every normal unfolding step, every abstract term in a pattern must be replaced by the term returned by the catamorphism. By doing this, the unfolding of the previous program reaches a fixed point at \( I_2 \) \(^8\):

\[
\begin{align*}
* \text{add}^#(\text{Even}^#, m) &= m \\
* \text{Suc}_{f^#}(\text{Even}^#) &= \text{Odd}^# \\
* \text{Suc}_{f^#}(\text{Odd}^#) &= \text{Even}^# \\
* \text{add}^#(\text{Odd}^#, \text{Odd}^#) &= \text{Even}^# \\
* \text{add}^#(\text{Odd}^#, \text{Even}^#) &= \text{Odd}^#
\end{align*}
\]

**Example 5 (Addition of Parities Revisited).**

As an interesting point of comparison, consider this alternative version for \( \text{add}^# \):

\[
\begin{align*}
\text{addr}^# &: \text{Nat}^# \to \text{Nat}^# \to \text{Nat}^# \\
\text{addr}^# \text{ Even}^# m &= m \\
\text{addr}^# (\text{Suc}_c^# n) m &= \text{addr}^# n (\text{Suc}_c^# m)
\end{align*}
\]

The fixed point for this new function is as follows (also in \( I_2 \)):

\[
\begin{align*}
* \text{addr}^#(\text{Even}^#, b) &= b \\
* \text{addr}^#(\text{Odd}^#, b) &= \text{Suc}^#(b) \\
* \text{Suc}_{f^#}(\text{Even}^#) &= \text{Odd}^# \\
* \text{Suc}_{f^#}(\text{Odd}^#) &= \text{Even}^#
\end{align*}
\]

**Example 6 (Demand Analysis).**

The following example shows how abstraction can help to find program properties. This particular example investigates how to find demand properties for the functions in a program. By **demand properties** we mean the level of definition

\(^7\) A catamorphism takes a term an returns the term after replacing constructors by a corresponding operator.

\(^8\) The rules for the catamorphism do not take part in unfolding.
that a function requires in its arguments in order to return a result strictly more
defined than $\bot$.

For the sake of simplicity, we are limiting our analysis to top-level positions
within the arguments although the method can be easily extended to cope with
deeper positions.

As before, we begin by defining the abstract domain. This example will run
on Peano Naturals, so the new domain reflects what elements are free variables
and what others are not:

```
data NatDemand# = Z# | S# NatDemand# | FreeNat#
```

As an example, we will use the well known function $\text{leq}$. $\text{leq} \ x \ y$ returns whether
$x$ is lesser or equal than $y$. The standard (unabstracted) version of $\text{leq}$ is as follows:

```
leq : Nat \to Nat \to Bool
leq Zero y = True
leq (Suc x) Zero = False
leq (Suc x) (Suc y) = leq x y
```

The abstracted version, which is useful for finding the demand properties for $\text{leq}$
at the top level positions of its arguments is as follows:

```
data Bool# = True# | False# | DontCareBool#
```

```
leq# :: NatDemand# \to NatDemand# \to Bool#
leq# Zero# FreeNat# = DontCareBool#
leq# (Suc# x) Zero# = DontCareBool#
leq# (Suc# x) (Suc y) = leq# x y
```

Observe that those rule bodies that do not influence the demand properties
of the function have been abstracted to $\text{DontCareBool}$ (and not to $\text{True}$ and $\text{False}$
in order to get an abstract representation that is as simple as possible while not
losing any demand information). Note that $\text{FreeNat}$ represents that a certain
argument is not demanded. This abstraction transformation can be mechanised:
Any singleton variable in a rule is sure not to be demanded so it is abstracted
to $\text{FreeNat}$. The rest of variables are left as they are.

What we need next is to define the functions that assert when a term is not
free (that is, demanded when it appears as a function argument). We need one
such function for every data constructor of type $\text{NatDemand}$:

```
FreeNatZ# : NatDemand#
FreeNatZ# = FreeNat#
```

```
S#FreeNat# : NatDemand# \to NatDemand#
S#FreeNat# = S# FreeNat#
```

```
S#FreeNat# Z# = S# Z#
S#FreeNat# (S# _) = S# FreeNat#
```

We also need the catamorphisms that link the functions above to the constructors
belonging to the type $\text{NatDemand}$:
As we did in the previous example, we now have to apply the following steps to a program composed of the rules for `leq#`, `freeNat_f#`, `Z_f#` and `S_f#`:

- Apply an unfolding iteration.
- Apply the catamorphisms to the heads of the resulting facts.
- Evaluate the resulting head expressions.

The fixed point is reached at the second iteration ($I_2$). It contains the following:

$\ast$ \texttt{leq}(Z#, \texttt{FreeNat#}) = \texttt{DontCareBool#}$

$\ast$ \texttt{leq}(S#(\texttt{FreeNat#}), Z#) = \texttt{DontCareBool#}$

$\ast$ \texttt{leq}(S#(\texttt{FreeNat#}), S#(\texttt{FreeNat#})) = \texttt{DontCareBool#}$

$\ast$ \texttt{z_f#} = Z#

$\ast$ \texttt{s_f#}(\texttt{FreeNat#}) = S#(\texttt{FreeNat#})

$\ast$ \texttt{s_f#}(Z#) = S#(\texttt{FreeNat#})

$\ast$ \texttt{s_f#}(S#(b)) = S#(\texttt{FreeNat#})

$\ast$ \texttt{freeNat_f#} = \texttt{FreeNat#}$

That means that \texttt{leq#} does not demand its second argument if the first one is \texttt{Z#} (since \texttt{FreeNat#} represents no demand at all). However, \texttt{leq#} demands its second argument if the first one is headed by \texttt{S#}. Note that we are considering top level positions for the arguments only but that deeper positions can be easily considered by just extending \texttt{s_f#}.

### D Validity of the Unfolded Code

The lemma below supports the validity of the code generated by the unfolding process:

#### D.1 Proof of Lemma 1

Let $H$ be a fact generated by unfolding rule $\Lambda$ and belonging to interpretation $I_n$. Let $\{S_i \ (i : 1..m)\}$ be the set of facts that belong to $I_{n+1}$, that have been generated by unfolding $\Lambda$ and which overlap with $H$.

By \textit{reductio ad absurdum}, let us think that, even in the conditions stated, the $S_i$ do not cover all the cases that $H$ covers. Then, it must be possible to build at least one fact $S'$ that overlaps with $H$ but that does not overlap with any fact $S_i$.

In order to build a fact like $S'$, the following options can be taken:
1. Choose $H$ such that its pattern and/or guard does not match with any of the rules for $f$.
2. When unfolding $H$, use a fact that has not been used when calculating the facts $\{S_i \mid i : 1..m\}$.

However, condition 1 is impossible since all the function definitions are assumed to be complete (i.e. there is no fact for $f$ which does not match a rule) and to have only generating rules. In addition, condition 2 is also impossible since $unfold$ uses all the existing facts by definition.

Note that the condition which requires that the rules be generative cannot be dropped since a complete function having one or more non-generative rules would have some facts removed from $In_{n+1}$ by $clean$, which would render the function definition incomplete in that interpretation.

Therefore, no fact like $S'$ can exist. We have reached a contradiction and thus we have proved that under the conditions stated for $P$, $clean$ can always get rid of the most general fact.

### D.2 Proof of Lemma 2

If program $P$ does not have overlapping rules then any pair of rules $l \mid g = r$ and $l' \mid g' = r'$ must meet one of the following conditions:

1. There is no unifier between $l$ and $l'$.
2. If a substitution $\sigma$ is such that $\sigma = mgu(l, l')$, then the constraint $g \land \sigma(g')$ is unsatisfiable.

At every application, the $unfold$ function takes a rule and applies a substitution to its pattern as well as a (possible) conjunction to its guard. Now:

1. If the two rules given do not overlap because $l$ and $l'$ cannot be unified, applying any substitution to them makes them even less unifiable.
2. If the two rules given do not overlap because $l$ and $l'$ can be unified but the conjunction of their guards cannot be satisfied, adding a conjunction to either guards makes their combined satisfiability even less likely.

Up to this point, we have shown that the unfoldings of any two non-overlapping rules cannot give rise to overlapping facts but the facts generated by the unfolding of a single rule may still contain overlapping pairs. In order to prove that the unfoldings of a single rule from a program $P$ can be written without overappings, we need to use the function $ueval$ that was defined in Sect. B.1.

We now want to prove that, for any single rule $R$ belonging to a program $P$ without overlapping rules, the unfoldings of $R$ carry the same meaning with or without the cleaning phase. That is, let us call $Pr = unfold(R, In_{\infty})$:

$$ueval(Pr, c \triangleright e) = ueval(clean(Pr), c \triangleright e) \quad \forall c \triangleright e \in E$$

We will prove that Equation 3 holds by induction on the number of full applications of symbols of $FS$ held in $c$ and $e$ combined.
Base case: If neither \( c \) nor \( e \) have any full application of symbols of \( FS \), then both \( c \) and \( e \) are expressions (terms which may include calls to predefined functions) and therefore cannot be unfolded any more. Their value (as computed by \( ueval \)) does not depend on the interpretation used, so Equation 3 trivially holds.

Induction step: Let us assume that Equation 3 holds if \( c \) and \( e \) have a combined total of \( n \) full applications of symbols of \( FS \) and let us try to prove that Equation 3 holds when \( c \) and \( e \) have a combined total of \( n + 1 \) full applications of symbols of \( FS \).

In order to do that, let us define an expression \( e' \) which has exactly one more application of symbols of \( FS \) than \( e \) (the reasoning over \( c \) would be analogous). Let us define \( e' = e[f t_1 \ldots t_n]|o \) where \( e|_o \in E \) which no full invocations of symbols of \( FS \), \( f \in FS, t_i \in T \). This guarantees that \( e' \) has one more full application of symbols of \( FS \) than \( e \). Since the induction hypothesis holds for \( c \triangleright e \), all we have to prove is:

\[
ueval(P_R, True \triangleright f t_1 \ldots t_n) = ueval(clean_P(P_R), True \triangleright f t_1 \ldots t_n) \tag{4}
\]

Now, if \( P_R \) does not contain overlapping facts or does not contain facts about \( f \) at all, the Equation above trivially holds since the interpretations \( P_R \) and \( clean_P(P_R) \) are the same by definition of \( clean \).

Let us now assume that \( P_R \) contains (maybe among others), the following facts:

\[ F \equiv f p_1 \ldots p_n | c = b \]
\[ F_i \equiv \sigma_i(f p_1 \ldots p_n | c \land c_i = b) \quad (i : 1..m) \]

That is, the facts \( F_i \) overlap \( F \) and \( F_i \) are more specific than \( F \). Then, by definition of \( clean \), \( clean_P(P_R) \) will hold the facts \( F_i \) together with a new fact:

\[ F' \equiv f p_1 \ldots p_n | c \land \bigwedge_i (\text{numif}(p_1, \ldots, p_n), \sigma_i(p_1, \ldots, p_n)) \lor \neg(\sigma_i(c_i)) = b \]

Let \( \chi = f t_1 \ldots t_n \). The following cases can occur:

\[ F \] is not unfoldable by \( F \), then it is not unfoldable by any of the more specific facts (the \( F_i \) and \( F' \)), so Equation 4 holds.

\[ F \] is unfoldable by \( F \) but not by any of the \( F_i \), then \( \chi \) is unfoldable by \( F' \), which returns the same result as \( F \).

\[ F \] is unfoldable by \( F \) and one of the \( F_i \), then the left side of Equation 4 returns two values (let them be \( e^F \triangleright e^F \) and \( e_i^F \triangleright e_i^F \)) which verify \( e^F \triangleright e^F \subseteq e_i^F \triangleright e_i^F \). Since all the functions have to be well-defined, the value for \( \chi \) has to be the greatest of the two mandatorily. The right side of Equation 4 returns only the value \( e_i^F \triangleright e_i^F \) by definition of \( clean \) (which will have removed \( F \) from \( P_R \) and replaced it by \( F' \) which will not be usable to unfold \( \chi \)).