Reduction groups and related integrable difference systems of NLS type

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Abstract

We extend the reduction group method to the Lax-Darboux schemes associated with nonlinear Schrödinger type equations. We consider all possible finite reduction groups and construct corresponding Lax operators, Darboux transformations, hierarchies of integrable differential-difference equations, integrable partial difference systems and associated scalar partial difference equations.

1 Introduction

In the theory of integrable systems the connections between partial differential equations, differential-difference and partial difference systems are well known. A clear and seminal account of these connections can be found in [5], [7]. They can be formulated in the frame of a Lax-Darboux scheme, where

- the Lax structure (Lax representation, also known as zero curvature representation) is associated with partial differential equations (PDEs) and their symmetries [26, 1];
- Darboux transformations, which are automorphisms of the Lax structure, lead to Bäcklund transformations which can be regarded as integrable differential-difference equations (DΔEs) [17, 16, 7];
- Bianchi permutability of the Darboux transformations yields integrable partial difference equations (PΔEs) whose symmetries are the former DΔEs [28, 2].

In this paper we extend the reduction group method [21] to Lax-Darboux schemes for nonlinear Schrödinger type equations. More precisely, we study Lax operators of the form

\[ \mathcal{L} = D_x + U(p, q; \lambda), \]  

where the $2 \times 2$ matrix $U$ belongs to the Lie algebra $\mathfrak{sl}_2(\mathbb{C}(\lambda))$. Matrix $U(p, q; \lambda)$ depends implicitly on $x$ through two potentials $p, q$, and is a rational function in the spectral parameter $\lambda$. Imposing the invariance of operator $\mathcal{L}$ under the action of a reduction group, which is a finite subgroup of the group of automorphisms of $\mathfrak{sl}_2(\mathbb{C}(\lambda))$, we construct systematically the Lax operators corresponding to deep reductions. In this case there is a complete classification of finite reduction groups [9, 19, 20] and corresponding reduced Lax operators [9]. Namely, in the $\mathfrak{sl}_2(\mathbb{C}(\lambda))$ case there are only five distinct cases:

(i) the trivial reduction group (no reductions);
(ii) $\mathbb{Z}_2$ group with a degenerate orbit;
(iii) $\mathbb{Z}_2$ group with a generic orbit;

(iv) $\mathbb{Z}_2 \times \mathbb{Z}_2$ group with a degenerate orbit;

(v) $\mathbb{Z}_2 \times \mathbb{Z}_2$ group with a generic orbit.

In the cases (i)-(iv) we construct an invariant Lax operator, a corresponding PDE, invariant Darboux transformations, corresponding integrable D$\Delta$Es and P$\Delta$Es. The simplest case (i) has been studied in detail in [7]. We present it here for completeness, in order to illustrate all elements of the corresponding Lax-Darboux scheme, such as dressing chains (also known as Bäcklund transformations) and their first integrals; to give a detailed derivation of associated integrable P$\Delta$Es, and to discuss possible initial-value problems for these P$\Delta$Es. The case (v) can be studied by the methods presented in the paper but leads to cumbersome expressions, and we have decided to omit it in order to keep our results presentable.

Darboux transformations are automorphisms of the Lax structure and discrete symmetries of the corresponding PDEs. With each Darboux transformation we associate an infinite lattice and a map. If there are two Darboux transformations, then the condition of their commutativity (the Bianchi permutability) yields an integrable system of P$\Delta$Es.

Although the theory of Darboux transformations is rather well developed and has a long history, there are a few important problems which require further research. One of the problems is to give a complete description of all possible Darboux transformations for a given Lax operator. In the case of the Schrödinger operator the solution is known: there is one Darboux transformation (depending on a parameter) and any other Darboux transformation can be represented as a composition of such transformations and their inverses for a certain choice of the parameters [5]. However the description of all possible Darboux transformations associated with a given Lax operator is still an open problem.

The paper is organised as follows. In the following section, we introduce our notation and give the general scheme of these considerations. In the next four sections we consider the Lax operators related to the nonlinear Schrödinger equation (Section 3), and operators derived from the reduction group method, [20, 9], namely $\mathbb{Z}_2$ reduction (Sections 4 and 5) and dihedral group reduction (Section 6).

2 Lax-Darboux scheme

In this section, we explain our terminology by describing the Lax-Darboux scheme. We present the class of Lax operators under consideration and discuss our general assumptions for the construction of Darboux matrices. Moreover, we introduce the notation we use throughout the paper.

With the single term Lax-Darboux scheme we describe several structures which are related to each other and all of them are related to integrability. To be more precise, the Lax-Darboux scheme incorporates Lax operators, corresponding Darboux matrices and Darboux transformations, as well as the Bianchi permutability of the latter transformations.

- Lax operators are linear operators of the form $\mathcal{L} = D_x + U$, where the $N \times N$ matrix $U$ is an element of a specific Lie algebra. As it was described in the previous section, in this paper we consider only the case where $U(p, q; \lambda)$ is a $2 \times 2$ matrix belonging to the Lie algebra $\mathfrak{sl}_2(\mathbb{C}(\lambda))$, and its dependence on the continuous variable $x$ is implicit through the potentials $p$ and $q$.

- Darboux transformations $\mathcal{S}$ are automorphisms of the Lax operator $\mathcal{L}$. They map $\mathcal{L}$ to $\tilde{\mathcal{L}}$ by updating potentials $p$ and $q$. In other words,

\[
\mathcal{S} : \mathcal{L} \mapsto \tilde{\mathcal{L}}, \quad \text{where} \quad \mathcal{L} = D_x + U(p, q; \lambda), \quad \tilde{\mathcal{L}} = D_x + U(\tilde{p}, \tilde{q}; \lambda),
\]
with $\tilde{p}, \tilde{q}$ denoting the updated potentials.

Darboux transformations consist of Darboux matrices $M$ along with corresponding dressing chains or Bäcklund transformations.

• A Darboux matrix $M$ maps a fundamental solution of the equation $L(Ψ) = 0$ to a fundamental solution $\tilde{Ψ}$ of $\tilde{L}(\tilde{Ψ}) = 0$ according to $\tilde{Ψ} = MΨ$. In general, matrix $M$ is invertible and depends on $p, q$, their updates $\tilde{p}, \tilde{q}$, the spectral parameter $λ$, and some auxiliary functions.

• Dressing chains are sets of differential equations relating the potentials and the auxiliary functions involved in $L$ and $\tilde{L}$. They can be regarded as integrable systems of DΔEs. This follows from the interpretation of the corresponding Darboux transformation as defining a shift on the lattice according to the sequence

$$\cdots \overset{S}{\rightarrow} (p, q) \overset{S}{\rightarrow} (p, q) \overset{S}{\rightarrow} (\tilde{p}, \tilde{q}) \overset{S}{\rightarrow} \cdots.$$

• If the Lax operator admits two commuting Darboux transformations $S$ and $T$, then they define a two-dimensional lattice for which we adopt the multi-index notation $(p_{ij}, q_{ij}) = S^i T^j (p, q)$, where $i, j \in \mathbb{Z}$. This interpretation allows us to derive systems of integrable PΔEs by considering the Bianchi permutability of the corresponding transformations.

In order to implement the above scheme, firstly we construct Darboux transformation $S$. From the definition of Darboux matrix $M$ follows that

$$MLM^{-1} = \tilde{L}, \quad (2)$$

or, denoting the updated potentials with $p_{10}, q_{10}$ and matrix $U(p_{10}, q_{10}; λ)$ with $U_{10}$, we can rewrite equation (2) explicitly as

$$D_x M + U_{10}M − MU = 0. \quad (3)$$

For a given Lax operator $L$, the above equation can be used to determine $M$, as well as the corresponding dressing chain. Moreover, since matrices $U$ and $U_{10}$ are traceless, it follows from Abel’s theorem that the determinant of $M$ is a first integral of the dressing chain. For the Lax operators we consider here, it is natural to assume that matrix $M$ depends rationally on the spectral parameter $λ$, and inherits the reduction group symmetries of the corresponding operator $L$.

The interpretation of the Darboux transformation $S$ as defining a lattice direction allows us to think the updated potentials in $\tilde{L}$ as shifts of the original ones $p, q$ in that particular lattice direction. In this semi-discrete setting, the corresponding dressing chain can be seen as an integrable differential-difference equation [17] deriving from the compatibility condition of the Lax-Darboux pair (also referred to as semi-discrete Lax pair)

$$D_x Ψ = −U(p, q; λ)Ψ, \quad Ψ_{10} = M(p, q, p_{10}, q_{10}; λ)Ψ.$$

In this discrete interpretation, the Bianchi permutability of two different Darboux transformations yields an integrable system of PΔEs in two discrete variables. Employing the standard notation for difference equations, we denote the two discrete variables with $n$ and $m$, and interpret $S$ and $T$ as the corresponding shift operators defined by

$$S^i T^j (h(n, m)) = h(n + i, m + j) \equiv h_{ij}.$$
In particular, when \( i = j = 0 \), we will omit the index “00”, i.e. \( h = h(n, m) \).

Now the shift operators \( S \) and \( T \) act on a fundamental solution \( \Psi \) as

\[
\begin{align*}
S : \Psi & \mapsto \Psi_{10} = M(p, q, p_{10}, q_{10}; f; \lambda)\Psi \equiv M\Psi, \\
T : \Psi & \mapsto \Psi_{01} = K(p, q, p_{01}, q_{01}; g; \lambda)\Psi \equiv K\Psi,
\end{align*}
\]

where \( M \) and \( K \) are the corresponding Darboux matrices with \( f \) and \( g \) denoting any auxiliary (vector) functions. The Bianchi permutability of (4) according to Figure 1 allows us to compute \( \Psi_{11} \) in two different ways. This yields the consistency condition

\[
T(M)K - S(K)M = 0,
\]

which is nothing else but the compatibility condition of the Darboux pair (also referred to as fully discrete Lax pair)

\[
S(\Psi) = M(p, q, p_{10}, q_{10}; f; \lambda)\Psi, \quad T(\Psi) = K(p, q, p_{01}, q_{01}; g; \lambda)\Psi.
\]

The resulting condition (5) yields a set of polynomial equations for \( p, q, f, g \) and their shifts. This set may have two branches of solutions. One of them leads to a trivial system, cf. (17) below, whereas the other branch yields a non-trivial integrable system of partial difference equations. Symmetries and first integrals for the non-trivial system follow from the dressing chain and the first integrals of the corresponding Darboux transformations.

For some of these discrete systems, we employ first integrals and conservation laws to reduce the number of dependent variables and derive integrable scalar equations of Toda type. The form of these systems allows us to formulate a Cauchy problem on a single or a double staircase.

In our derivations, we find more than one Darboux transformation for each Lax operator we consider. We would like to emphasise here that the interpretation of any pair of Darboux matrices as a discrete Lax pair as described above does not always lead to a non-trivial discrete system. In the following sections we present only the pairs of Darboux matrices which lead to genuinely non-trivial discrete integrable systems.

3 Nonlinear Schrödinger equation

In order to illustrate our approach, we consider a well known operator

\[
L = D_x + U(p, q; \lambda) = D_x + \lambda \sigma_3 + \begin{pmatrix} 0 & 2p \\ 2q & 0 \end{pmatrix}, \quad \sigma_3 = \text{diag}(1, -1),
\]

Figure 1: Bianchi commuting diagram

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which is the spatial part of the Lax pair for the nonlinear Schrödinger equation \[32\]

\[ p_t = p_{xx} + 4p^2q, \quad q_t = -q_{xx} - 4pq^2. \]  

(7)

It is straightforward to verify that the constant matrix

\[ M = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha \beta \neq 0, \]

is a Darboux transformation for this operator corresponding to the scaling symmetry of (7).

\[ p_{10} = \alpha \beta^{-1} p, \quad q_{10} = \beta \alpha^{-1} q. \]

The simplest $\lambda$-dependent Darboux matrix one may consider is

\[ M = \lambda M_1 + M_0. \]  

(8)

Substituting (6) and (8) into the compatibility condition (3), the coefficient of $\lambda^2$ implies that matrix $M_1$ must be diagonal. Additionally, from the diagonal part of the coefficient of $\lambda$ we conclude that $M_1$ must be constant. Hence, $M_1 = \text{diag}(c_1, c_2)$. We could choose either $c_1 = 1, c_2 = 0$, or $c_1 = 0, c_2 = 1$ or $c_1 = c_2 = 1$. Since the first two choices are gauge equivalent and the third one can be given as a composition of two suitable Darboux matrices with $c_1c_2 = 0$, we choose

\[ M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \]

Moreover, the off-diagonal part of the coefficient of $\lambda$ implies that the $(2, 2)$ element of $M_0$ is constant and, hence, we have to consider two distinct cases.

The first case corresponds to

\[ M = \lambda \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) + \left( \begin{array}{cc} f & a \\ b & 0 \end{array} \right), \]

i.e. the $(2, 2)$ entry of $M_0$ is zero. In this case, equation (3) is equivalent to the system

\[ a = p, \quad b = q_{10}, \quad f_x = 2(aq - bp_{10}), \quad a_x = 2fp, \quad b_x = -2fq_{10}. \]  

(9)

The first two equations determine functions $a$ and $b$, while the last two implies that $pq_{10} = \gamma$, where $\gamma$ is a non-zero constant (since det $M = \gamma \neq 0$). Without any loss of generality we can set $\gamma = 1$ and thus we have

\[ q_{10} = \frac{1}{p}, \quad p_{10} = p \left( pq - \frac{1}{2}f_x \right), \quad f = \frac{px}{2p}. \]  

(10)

Finally, the Darboux matrix is given by

\[ M(p, f) = \lambda \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) + \left( \begin{array}{cc} f & p \\ \frac{1}{p} & 0 \end{array} \right), \]  

(11)

and the dressing chain (the Bäcklund transformation (10)) can be rewritten in the form of the Toda lattice in a new variable $\phi = \log p$

\[ \phi_{xx} = 4e^{\phi - \phi_{10}} - 4e^{\phi_{10} - \phi}. \]  

5
In this case the Darboux transformation \((p, q) \rightarrow (p_{10}, q_{10})\) is explicit
\[
p_{10} = p \left( pq - \frac{1}{4} \left( \frac{p_x}{p} \right)_x \right), \quad q_{10} = \frac{1}{p}.
\]

Alternatively, we can choose the \((2, 2)\) element of \(M_0\) to be non zero and, without loss of generality set it to 1, i.e.
\[
M_0 = \left( \begin{array}{cc} f & a \\ b & 1 \end{array} \right).
\]

Now, it follows from (3) that
\[
a = p, \quad b = q_{10}, \quad \partial_x f = 2(pq - p_{10} q_{10}), \quad \partial_x p = 2(pf - p_{10}), \quad \partial_x q_{10} = 2(q - q_{10} f).
\]

A first integral of the above system is provided by the determinant of \(M\), \(\det M = \lambda + f - pq_{10}\)
\[
\partial_x (f - pq_{10}) = 0.
\]

Hence, matrix \(M\) has the following form
\[
M(p, q_{10}, f) = \lambda \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) + \left( \begin{array}{cc} f & p \\ q_{10} & 1 \end{array} \right)
\]
and (12) is the corresponding dressing chain.

### 3.1 Derivation of discrete systems

Having derived two Darboux matrices for operator (6), we focus on the generic one given in (14) and consider the following Darboux pair
\[
\Psi_{10} = M(p, q_{10}, f)\Psi, \quad \Psi_{01} = M(p, q_{01}, g)\Psi,
\]
which explicitly reads as follows.
\[
\Psi_{10} = \left( \lambda \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) + \left( \begin{array}{cc} f & p \\ q_{10} & 1 \end{array} \right) \right)\Psi, \quad \Psi_{01} = \left( \lambda \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) + \left( \begin{array}{cc} g & p \\ q_{01} & 1 \end{array} \right) \right)\Psi.
\]

The compatibility condition of (15) results to
\[
\begin{align*}
& f_{01} - f - (g_{10} - g) = 0, \\
& f_{01} g - f_{01} g_{10} - p_{10} q_{10} + p_{01} q_{01} = 0, \\
& p (f_{01} - g_{10}) - p_{10} + p_{01} = 0, \\
& q_{11} (f - g) - q_{01} + q_{10} = 0.
\end{align*}
\]

This system can be solved either for \((p_{01}, q_{01}, f_{01}, g)\) or for \((p_{10}, q_{10}, f, g_{10})\). It has two branches of solutions. A trivial one
\[
p_{10} = p_{01}, \quad q_{10} = q_{01}, \quad f = g, \quad g_{10} = f_{01}.
\]
corresponds to \( M(p, q_{10}, f) = M(p, q_{01}, g) \), and a non-trivial solution given by

\[
\begin{align*}
p_{01} &= \frac{q_{10}p^2 + (g_{10} - f)p + p_{10}}{1 + p q_{11}}, \\
f_{01} &= \frac{q_{11}(p_{10} + p q_{10}) + f - p q_{10}}{1 + p q_{11}}, \\
q_{01} &= \frac{p_{10} q_{11}^2 + (f - g_{10}) q_{11} + q_{10}}{1 + p q_{11}}, \\
g &= \frac{q_{11}(p f - p_{10}) + g_{10} + p q_{10}}{1 + p q_{11}}.
\end{align*}
\]

Some properties of the above system follow immediately from the derivation of the corresponding Darboux transformations. First of all, it admits two first integrals, cf. relation (13), namely

\[
(T - 1)(f - p q_{10}) = 0 \quad \text{and} \quad (S - 1)(g - p q_{01}) = 0.
\]

We can interpret functions \( f \) and \( g \) as being given on the edges of the quadrilateral where system (18) is defined, and, consequently, consider system (18) as a vertex-bond system [11]. System (18) admits the conservation law

\[
(T - 1)f = (S - 1)g
\]

which is the first equation in (16).

Moreover, relations (12) imply that system (18) admits one generalised symmetry generated by the differential-difference equations

\[
\begin{align*}
\partial_x p &= fp - p_{10} = gp - p_{01}, \\
\partial_x q &= q_{-10} - f_{-10}q = q_{0,-1} - g_{0,-1}q, \\
\partial_x f &= pq - p_{10}q_{10}, \\
\partial_x g &= pq - p_{01}q_{01}.
\end{align*}
\]

Our choice to solve system (16) for \( p_{01}, q_{01}, f_{01} \) and \( g \) is motivated by the initial value problem related to system (18). Suppose that initial values for \( p \) and \( q \) are given at the vertices along the solid staircase as shown in Figure 2. Functions \( f \) and \( g \) are given on the edges of this initial value configuration in a consistent way with the first integrals (22). In particular, horizontal edges carry the initial values of \( f \) and vertical edges the corresponding ones of \( g \). With these initial conditions, the values of \( p \) and \( q \) can be uniquely determined at every vertex of the lattice, while \( f \) and \( g \) on the corresponding edges. This is obvious from the rational expressions (18) defining the evolution above the staircase, cf. Figure 2. For the evolution below the staircase, one has to use

\[
\begin{align*}
\begin{align*}
p_{10} &= \frac{q_{10}p^2 + (f_{01} - g)p + p_{01}}{1 + p q_{11}}, \\
p_{01} &= \frac{p_{10} q_{11}^2 + (f - g_{01}) q_{11} + q_{10}}{1 + p q_{11}}, \\
f &= \frac{q_{11}(p g - p_{01}) + f_{01} + p q_{01}}{1 + p q_{11}}, \\
g &= \frac{q_{11}(p_{01} + p f_{01}) + g - p q_{01}}{1 + p q_{11}}.
\end{align*}
\end{align*}
\]

Figure 2: Initial value problem and direction of evolution
which uniquely defines the evolution below the staircase as indicated in Figure 2.

We could consider more general initial value configurations of staircases of lengths $\ell_1$ and $\ell_2$ in the $n$ and $m$ lattice direction, respectively. Such initial value problems are consistent with evolutions (18), (21) determining the values of all fields uniquely at every vertex and edge of the lattice.

It follows from (19) that

$$f - p q_{10} = \alpha(n) \quad \text{and} \quad g - p q_{01} = \beta(m).$$

and we can use these relations to eliminate $f$ and $g$ from (21). This results to a non-autonomous partial difference system for $p$ and $q$ only

$$\begin{align*}
p_{01} &= p_{10} - \frac{\alpha(n) - \beta(m)}{1 + p q_{11}} p, \\
q_{01} &= q_{10} + \frac{\alpha(n) - \beta(m)}{1 + p q_{11}} q_{11}.
\end{align*}$$

(23)

Symmetries of this system can be derived directly from corresponding symmetries of system (18) by taking into account (22). In particular, it follows from (20) that

$$\partial_x p = 2(p^2 q_{10} + \alpha(n)p - p_{10}), \quad \partial_x q = 2(q_{-10} - p_{-10} q^2 - \alpha(n - 1)q)$$

is a symmetry of (23).

3.1.1 Derivation of the discrete Toda equation

Returning now to the construction of a discrete Lax pair, we employ matrix $M(p, f)$, given in (11), and matrix $M(p, q_{01}, g)$, in (14). That is, we consider the following system

$$\Psi_{10} = \left( \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \left( \frac{f}{p} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \right) \right) \Psi, \quad \Psi_{01} = \left( \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \left( \frac{g}{q_{01}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \right) \Psi.$$

The compatibility condition of the above system implies that

$$p = \frac{1}{q_{10}}, \quad g = \alpha(m) + \frac{q_{01}}{q_{10}},$$

as well as

$$f = \frac{q_{01}}{q_{10}} - \frac{q_{10}}{q_{11}} + \alpha(m), \quad f_{01} = \frac{q_{11}}{q_{20}} - \frac{q_{10}}{q_{11}} + \alpha(m).$$

From the consistency of the latter equations and setting $q = \exp(-w_{-1,-1})$, we derive the fully discrete Toda equation

$$e^{w_{01}-w} - e^{w_{0,0}-1} + e^{w_{1,-1}-w} - e^{w_{-1,1}-1} = \alpha(m + 1) - \alpha(m),$$

(24)

along with its generalised symmetry

$$\partial_x w = e^{w_{0,1}-w} - e^{w_{1,-1}-w} - \alpha(m).$$

Moreover, a conserved form of Toda equation is

$$(T - 1) \left( e^{w_{0,-1}-w - 10} - e^{w_{-1,0}-1} + \alpha(m) \right) = (S - 1) e^{w_{0,-1}-w - 10}.$$

It is worth noting that a staircase initial value problem for the Toda equation (24) involves the points $w_{i,-1}$ and $w_{i,-1}$, i.e. a staircase which is the reflection of the one shown in Figure 2 with respect to a vertical or horizontal line of the discrete plane.
4 \( \mathbb{Z}_2 \) reduction group: Degenerate orbit

Let us now consider an operator \( \mathcal{L}(\lambda) \) which is invariant under the transformation

\[
s_1(\lambda): \mathcal{L}(\lambda) \to \sigma_3 \mathcal{L}(-\lambda) \sigma_3.
\]

The above involution generates the reduction group [21] which is isomorphic to the \( \mathbb{Z}_2 \) group. The invariant operator corresponding to this orbit can be taken in the form

\[
\mathcal{L} = D_x + \lambda^2 \sigma_3 + \lambda \left( \begin{array}{cc} 0 & 2p \\ 2q & 0 \end{array} \right),
\]

and it is the spatial part of the Lax pair for the derivative nonlinear Schrödinger equation [13]

\[
p_t = p_{xx} + 4 (p^2 q)_x, \quad q_t = -q_{xx} + 4 (p q^2)_x.
\]

It can be easily verified that the constant matrix

\[
M = \left( \begin{array}{cc} \alpha & 0 \\ 0 & \beta \end{array} \right), \quad \alpha \beta \neq 0,
\]

is a Darboux matrix for operator (26) corresponding to the scaling symmetry of system (27).

\[p_{10} = \alpha \beta^{-1} p, \quad q_{10} = \beta \alpha^{-1} q.\]

Considering Darboux matrix \( M \) with the same symmetry, i.e. \( M(\lambda) = \sigma_3 M(-\lambda) \sigma_3 \), we find after some analysis that the simplest \( \lambda \)-dependent Darboux matrix can be written in the form

\[
M = \lambda^2 M_2 + \lambda M_1 + M_0,
\]

where matrices \( M_2 \) and \( M_0 \) are diagonal and matrix \( M_1 \) is off-diagonal. Additionally, from the compatibility condition (3) follows that \( M_0 \) is a constant matrix. Moreover, following an argument similar to the one we used in the previous section, we consider only the case \( \text{rank}(M_2) = 1 \). Hence, summarizing the above analysis, we choose

\[
M_2 = \left( \begin{array}{cc} f & 0 \\ 0 & 0 \end{array} \right), \quad M_1 = \left( \begin{array}{cc} 0 & a \\ b & 0 \end{array} \right) \quad \text{and} \quad M_0 = \left( \begin{array}{cc} c_1 & 0 \\ 0 & c_2 \end{array} \right), \quad c_1, c_2 \in \mathbb{C}.
\]

With these choices, equation (3) firstly determines functions \( a, b \) in terms of \( f, p, q_{10} \). In particular we find that

\[a = f p, \quad b = f q_{10}.\]

In terms of these relations, Darboux matrix becomes

\[
M(p, q_{10}, f; c_1, c_2) = \lambda^2 \left( \begin{array}{cc} f & 0 \\ 0 & 0 \end{array} \right) + \lambda \left( \begin{array}{cc} 0 & f p \\ f q_{10} & 0 \end{array} \right) + \left( \begin{array}{cc} c_1 & 0 \\ 0 & c_2 \end{array} \right),
\]

and we derive the Bäcklund transformation

\[
\partial_x p = 2 p (p_{10} q_{10} - p q) - 2 \frac{c_2 p_{10} - c_1 p}{f}, \quad \partial_x q_{10} = 2 q_{10} (p_{10} q_{10} - p q) - 2 \frac{c_1 q_{10} - c_2 q}{f},
\]

\[
\partial_x f = 2 f (p q - p_{10} q_{10}).
\]
A first integral of the above system, which also guarantees that the determinant of matrix (29) is independent of \( x \), is given by
\[
\partial_x \left( f^2 p q_{10} - c_2 f \right) = 0. \tag{31}
\]

It is apparent that if constants \( c_1, c_2 \) are not zero, we can always set them to 1 by composing Darboux matrix (29) with an appropriate Darboux matrix (28). Hence, we can impose without loss of generality that these constants are either 0 or 1. There are two particular sets of values for these constants at which differential-difference equations (30) can be brought to polynomial form.

1. First we consider the case when \( c_1 = c_2 = 0 \). It follows from equations (30) that \( f = 1/p \) and \( q_{10} = p \), in view of which matrix \( M \) degenerates to
\[
M(p) = \lambda^2 \begin{pmatrix} 1/p & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{32}
\]

The corresponding Bäcklund transformation becomes
\[
q_{10} = p, \quad \partial_x p = 2 p^2 (p_{10} - q) \tag{33}
\]
and the first integral (31) holds identically. The resulting differential-difference equations (33) are the modified Volterra chain.

2. When \( c_1 = 1 \) and \( c_2 = 0 \), the Darboux matrix becomes
\[
M(p, q_{10}, f) = \lambda^2 \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & fp \\ f_{q10} & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{34}
\]
the Bäcklund transformation simplifies to
\[
\partial_x p = 2 p (p_{10} q_{10} - p q) + \frac{2p}{f}, \quad \partial_x q_{10} = 2 q_{10} (p_{10} q_{10} - p q) - \frac{2q_{10}}{f}, \quad \partial_x f = 2 f (p q - p_{10} q_{10}) \tag{35}
\]
and the first integral (31) becomes
\[
\partial_x \left( f^2 p q_{10} \right) = 0. \tag{36}
\]
In the context of differential-difference equations, if we make the point transformation
\[
p = u^2, \quad q = v_{-10}^2,
\]
and subsequently, using the first integral (36), set
\[
f^2 u^2 v^2 = 1 \iff f = \frac{\pm 1}{u v},
\]
system (35) can be written in a polynomial form as
\[
\partial_x u = u(u_{10}^2 v^2 - u^2 v_{-10}^2) \pm u^2 v, \quad \partial_x v = v(u_{10}^2 v^2 - u^2 v_{-10}^2) \mp uv^2.
\]
4.1 Derivation of discrete systems

Now we consider the difference Lax pair

$$\Psi_{10} = M(p, q_{10}, f; c_1, c_2) \Psi, \quad \Psi_{01} = M(p, q_{01}, g; 1, 1) \Psi,$$

where matrix $M$ is given in (29) and at least one of the constants $c_1$, $c_2$ is different from 0. It follows from the above system that

$$f g_{10} - g f_{01} = 0, \quad (38a)$$
$$f_{01} q_{11} - f q_{10} - c_1 g_{10} q_{11} + c_2 g_{01} = 0, \quad (38b)$$
$$f_{01} p_{01} - f p - c_2 g_{10} p_{10} + c_1 g_{p} = 0, \quad (38c)$$
$$f_{01} - f - c_1 (g_{10} - g) - f g_{10} p_{10} q_{10} + g f_{01} p_{01} q_{01} = 0. \quad (38d)$$

We can solve equations (38) for $p_{01}$, $q_{01}$, $f_{01}$ and $g$ (or for $p_{10}$, $q_{10}$, $f$ and $g_{10}$). If $c_1 = c_2 = 1$, we derive two sets of solutions, as in the case of the nonlinear Schrödinger. Specifically, the first branch is the singular solution already given in (17), while the second branch involves rational expressions of the remaining variables. When either $c_1$ or $c_2$ is equal to 0, then system (38) admits a unique non-trivial solution. This solution is given by

$$p_{01} = \frac{A}{f B^2} (f^2 p^2 q_{10} + c_2 f p (g_{10} p_{10} q_{10} - 1) - c_2^2 g_{10} p_{10} + c_1 c_2 g_{10} p), \quad f_{01} = f \frac{B}{A}, \quad (39a)$$
$$q_{01} = \frac{B}{g_{10} A^2} (f (q_{11} - q_{10} + g_{10} p_{10} q_{10} q_{11}) + c_1 g_{10} q_{11} (g_{10} p_{10} q_{11} - 1)), \quad g = g_{10} \frac{A}{B}, \quad (39b)$$

where $A := f p q_{11} + c_2 (g_{10} p_{10} q_{11} - 1)$ and $B := f p q_{10} + c_1 g_{10} p q_{11} - c_2$.

In this discrete context, the first integrals of the Bäcklund transformation given in (31) become first integrals for system (39), i.e.

$$(T - 1) (f^2 p q_{10} - c_2 f) = 0, \quad (S - 1) (g^2 p q_{01} - g) = 0. \quad (40)$$

Moreover, a generalised symmetry of the latter system follows from (30) and it is generated by the differential-difference equations

$$\partial_x p = p (p_{10} q_{10} - p q) - \frac{c_2 p_{10} - c_1 p}{f} = p (p_{01} q_{01} - p q) - \frac{p_{01} - p}{g},$$
$$\partial_x q = q (p q - p_{-10} q_{-10}) - \frac{c_1 q - c_2 q_{-10}}{f_{-10}} = q (p q - p_{0,-1} q_{0,-1}) - \frac{q - q_{0,-1}}{g_{0,-1}},$$
$$\partial_x f = f (p q - p_{10} q_{10}), \quad \partial_x g = g (p q - p_{01} q_{01}).$$

4.1.1 First integrals and a seven point scalar difference equation

Let us consider now system (38) with $c_1 = c_2 = 1$ and try to implement the first integrals (40) so that to reduce the number of functions involved in this system by setting

$$f^2 p q_{10} - f = \alpha(n), \quad g^2 p q_{01} - g = \beta(m). \quad (41)$$

One option is to use the above relations to replace $f$, $g$ in terms of $p$ and $q$. In this case, we must solve equations (41), which are quadratic in $f$ and $g$, and hence introduce square roots, and finally derive a system of non-polynomial equations (correspondences) for $p$ and $q$. 

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Another option is, instead of eliminating $f$ and $g$, to use relations (41) to replace the shifts of $q$. In this case, equations (41) imply
\[ q_{10} = \frac{\alpha(n) + f}{f_p}, \quad q_{01} = \frac{\beta(m) + g}{g_p}. \] (42)
Moreover, equation (38a) suggests to introduce a potential $u$ through the relations
\[ f = \frac{u_{10}}{u}, \quad g = \frac{u_{01}}{u}. \] (43)
Additionally, we introduce $v$ by $v := p/u$ for convenience.

Applying all the above substitutions to system (38), we derive a system for $v$ and $u$, namely
\[ u_{11} + \alpha(n)u_{01} - \frac{u_{11} + \beta(m)u_{10}}{v_{01}} = 0, \quad u_{11}(v_{10} - v_{01}) + v(u_{10} - u_{01}) = 0, \] (44)
while a symmetry for this system is generated by
\[ \partial_x u = \frac{-v}{v_{-10}}(u + \alpha(n - 1)u_{-10}), \quad \partial_x v = \frac{u}{u_{10}}(v + \alpha(n)v_{10}) = \frac{u}{u_{01}}(v + \beta(m)v_{01}). \] (45)

From equations (44) we can derive a higher order scalar equation either for $u$ or for $v$, namely
\[ (S - 1) \log X(n, u, u_{-10}) - (T - 1) \log Y(m, u, u_{01}) + (ST^{-1} - 1) \log Z(n, m, u, u_{-11}) = 0, \] (46)
and
\[ (S - 1) \log X(n, v_{-10}, v) - (T - 1) \log Y(m, v_{01}, v) + (ST^{-1} - 1) \log Z(n, m, v_{-11}, v) = 0, \] (47)
where
\[ X(n, u, x) = 1 + \alpha(n - 1) \frac{x}{u}, \quad Y(m, u, y) = 1 + \beta(m - 1) \frac{y}{u}, \quad Z(n, m, u, z) = \frac{z - u}{\alpha(n - 1)z - \beta(m)u}. \] (48)
A symmetry for equation (46) follows from (45) and it is generated by
\[ \partial_x u = u X(n, u, u_{-10}) Y(m + 1, u_{01}, u) Z(n, m, u, u_{-11}) \]
while
\[ \partial_x v = v X(n, v_{-10}, v) Y(m + 1, v, v_{01}) Z(n, m, v_{-11}, v) \]
generates a symmetry for equation (47).

Equations (46), (47) are similar and have the same properties. They are defined on a stencil of seven points and can be solved uniquely with respect to any $u_{ij}$ and $v_{ij}$ except $u$ and $v$, respectively. Because of this feature, if initial data are given along a double staircase, then these equations uniquely determine the evolution above and below this initial configuration as it is shown in Figure 3.

Remark. When $\alpha(n)$, $\beta(m)$ are constants, i.e. $\alpha(n) = \alpha$, $\beta(m) = \beta$, equations (46) and (47) are related to the discrete Toda equation
\[ (S - 1) \log(e^{w_{-10}} - 1) + (T - 1) \log(e^{w_{01}} - 1) + (ST - 1) \log \frac{e^{w_{-11}} + \gamma}{e^{w_{-11}} + 1} = 0, \quad \gamma := \frac{\alpha}{\beta}, \] (49)
The two quadrilaterals with one common vertex where equation is defined

The initial value problem

Figure 3: The stencil of seven points and the initial value problem

i.e. equation (H) in [3]. This relation is made evident if we first reverse the $m$ direction, i.e. change indices $(i, j)$ to $(i, -j)$ and operator $\mathcal{T}$ to its inverse $\mathcal{T}^{-1}$ in both equations (46), (47), and then make the point transformation

$$u = (-1)^m \alpha^n \beta^m e^{-w} \quad \text{and} \quad v = (-1)^m \alpha^{-n} \beta^{-m} e^w,$$

to each equation, respectively. In this context, system (44) defines the self-duality transformation for the Toda equation (49) [3]. In particular, if we make the above change of variables to system (44), then it will become

$$e^{\tilde{w}_{10}} - \tilde{w} = \frac{e^{w_{10} - w_{0,-1}} + \gamma}{e^{w_{10} - w_{0,-1}} + 1} (e^{w_{1,-1} - w_{10} - 1}) , \quad e^{\tilde{w}_{0,-1}} - \tilde{w} = \frac{1}{\gamma} \frac{e^{w_{10} - w_{0,-1}} + \gamma}{e^{w_{10} - w_{0,-1}} + 1} (e^{w_{1,-1} - w_{0,-1} + 1}),$$

(50)

where $w$ and $\tilde{w}$ are two different solutions of equation (49).

4.1.2 Lax pair with matrix (32) and a six point difference equation

Let us consider now the Lax pair

$$\Psi_{10} = M(p)\Psi, \quad \Psi_{01} = M(p, q_{01}, g; 1, 1)\Psi,$$

where matrix $M(p)$ is given in (32) and $M(p, q_{01}, g; 1, 1)$ in (29). It follows from the compatibility condition of the above pair that

$$q_{10} = p, \quad g = \frac{p_{01} - p}{p(p_{01}p_{-11} - pp_{10})}, \quad g_{10} = \frac{p}{p_{01}} g,$$

and finally we arrive at the six point difference equation

$$\frac{p_{11} - p_{10}}{p_{10}(p_{11}p_{01} - pp_{10})} = \frac{p_{01} - p}{p_{01}(p_{01}p_{-11} - pp_{10})}. \quad (51)$$

We also find a first integral and a symmetry of this equation, which are given by

$$(S - 1) \frac{(p - p_{01})(p_{10} - p_{-11})}{(p_{01}p_{-11} - pp_{10})^2} = 0 \quad \text{and} \quad \partial_x p = p^2 (p_{10} - p_{-10}).$$
It is worth noting that equation (51) can be uniquely solved with respect to any value of $p$ except $p_{10}$ and $p_{01}$. If initial data are given along a double staircase as it is shown in Figure 4, which must be consistent with the first integral, then the evolution of these data is uniquely determined above and below the double staircase by equation (51).

Remark. If we set the value of the above first integral to $\alpha(m)$ and, subsequently, make the change of independent variables $(n, m) \mapsto (k, l) := (n + m, m)$, then we will arrive at the following quadrilateral equation for $\tilde{p}(k, l) = p(n, m)$.

$$(\tilde{p} - \tilde{p}_{11})(\tilde{p}_{10} - \tilde{p}_{01}) = \alpha(l)(\tilde{p}_{10} - \tilde{p}_{01} \tilde{p}_{11})^2.$$  

5 $\mathbb{Z}_2$ reduction group: Generic orbit

A $\mathbb{Z}_2$ invariant Lax operator with simple poles in the generic orbit can be taken in the form

$$L = Dx + \frac{1}{\lambda - 1} S - \frac{1}{\lambda + 1} \sigma_3 S \sigma_3, \quad S := \frac{1}{p - q} \begin{pmatrix} p + q & -2pq \\ 2 & -p - q \end{pmatrix}. \quad (52)$$

The corresponding NLS type equation is

$$p_t = p_{xx} - \frac{2p_x^2}{p - q} + \frac{8pq - 4p^2q_x}{(p - q)^2}, \quad q_t = -q_{xx} - \frac{2q_x^2}{p - q} + \frac{8pq - 4q^2p_x}{(p - q)^2},$$

which is actually equation (m) in [23].

The Darboux matrix for the above Lax operator is derived in the same way as in the previous section and three distinct cases occur.

1. The first Darboux matrix is

$$M = \lambda \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} + \begin{pmatrix} b_1u & 0 \\ 0 & b_2v \end{pmatrix}, \quad uv = 1, \quad a_i, b_i \in \mathbb{C} \quad \text{and} \quad |a_1a_2|^2 + |b_1b_2|^2 \neq 0, \quad (53)$$

and the Bäcklund transformation is given by

$$p_{10} = \frac{a_1 + b_1p u}{b_2v + a_2p}, \quad q_{10} = \frac{a_1 + b_1q u}{b_2v + a_2q}, \quad \partial_x u = 4u \left( \frac{p_{10}}{p_{10} - q_{10}} - \frac{p}{p - q} \right). \quad (54)$$
This transformation contains as particular subcases two Darboux transformations related to point symmetries, namely scalings \((a_1 = a_2 = 0, u = v = 1)\) and inversions \((b_1 = b_2 = 0, a_1 = a_2 = 1)\).

2. The second Darboux matrix is

\[
M(p, q_{10}) = \frac{1}{\lambda - 1} \begin{pmatrix} q_{10} & 1 & -p \\ 1 & -p & q_{10} \end{pmatrix} - \frac{1}{\lambda + 1} \begin{pmatrix} q_{10} & -1 & -p \\ 1 & -p & q_{10} \end{pmatrix},
\]

and the Bäcklund transformation is given by

\[
q_{10} = \frac{-1}{p}, \quad \partial_x p = 4p \left( \frac{p}{p - q} - \frac{p_{10}}{p_{10} - q_{10}} \right).
\]

3. The last Darboux matrix is given by

\[
M(p, q_{10}, f; c_1, c_2) = \frac{f}{\lambda - 1} \begin{pmatrix} q_{10} & -p & q_{10} \\ 1 & -p & -q_{10} \end{pmatrix} - \frac{f}{\lambda + 1} \begin{pmatrix} q_{10} & p & q_{10} \\ 1 & -p & -q_{10} \end{pmatrix} + \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_1 & c_2 \end{pmatrix},
\]

where \(c_1, c_2\) are constants such that \(|c_1|^2 + |c_2|^2 \neq 0\) and, without loss of generality, we can set these constants equal to 0 or 1. The derivatives of \(p, q_{10}\) and \(f\) are given by the following relations

\[
\begin{align*}
\partial_x p &= 4p \left( \frac{p_{10}}{q_{10} - p_{10}} - \frac{p}{q - p} \right) + \frac{2}{f} \frac{c_2 p_{10} - c_1 p}{q_{10} - p_{10}}, \\
\partial_x q_{10} &= -4q_{10} \left( \frac{p_{10}}{q_{10} - p_{10}} - \frac{p}{q - p} \right) + \frac{2}{f} \frac{c_2 q_{10} - c_1 q}{q - p}, \\
\partial_x f &= \frac{2c_1}{q_{10} - p_{10}} - \frac{2c_2}{q - p}.
\end{align*}
\]

Function

\[
\Phi(c_1, c_2) = (2fp + c_2)(2fq_{10} - c_1)
\]

defines a first integral for equations \([58]\), i.e. \(D_x \Phi(c_1, c_2) = 0\) on solutions of the latter system.

### 5.1 Derivation of discrete systems

The first discrete Lax pair to consider is

\[
\Psi_{10} = M(p, q_{10}, f; c_1, c_2) \Psi, \quad \Psi_{01} = M(p, q_{01}, g; 1, 1) \Psi,
\]

where matrix \(M\) is given in \([57]\) and for constants \(c_1, c_2\) one may consider three distinct cases : (i) \(c_1 = c_2 = 1\), (ii) \(c_1 = 1, c_2 = 0\) and (iii) \(c_1 = 0, c_2 = 1\).

In this generic setting, the compatibility condition of system \([60]\) results to

\[
\begin{align*}
f_{01} - & f - c_1 g_{10} + c_2 g = 0, \\
f_{01} p_{01} q_{11} - & f p q_{10} - c_2 g_{10} p_{10} q_{11} + c_1 g p q_{01} = 0, \\
f_{01} q_{11} - & f q_{10} - c_1 g_{10} q_{11} + c_1 g q_{01} - 2q_{11}(g f_{01} q_{01} - g_{10} f q_{10}) = 0, \\
f_{01} p_{01} - & f p - c_2 g_{10} p_{10} + c_2 g p + 2p(g f_{01} p_{01} - g_{10} f p_{10}) = 0.
\end{align*}
\]

This system can be solved for either \((p_{01}, q_{01}, f_{01}, g)\) or \((p_{10}, q_{10}, f, g_{10})\). When \(c_1 = c_2 = 1\), then it leads to a solution with two branches: one branch is the trivial solution \([17]\), while the non-trivial branch
involves rational expressions of the remaining variables. In the other two cases \((c_1 = 1, c_2 = 0\) or \(c_1 = 0, c_2 = 1\)), system \(\text{(61)}\) admits a unique non-trivial solution. In any case, the non-trivial branch can be easily found, but is omitted here because of its length, and we consider it as a difference system. For this system, it can be verified that it admits two first integrals, 
\[
(T - 1) (2fp + c_2) (2fq_{10} - c_1) = 0, \quad (S - 1) (2gp + 1) (2gq_{01} - 1) = 0,
\]
and a symmetry generated by
\[
\begin{align*}
\partial_p p &= 2p \left( \frac{p_{10}}{q_{10} - p_{10}} - \frac{p}{q - p} \right) + \frac{1}{f} \frac{c_2 p_{10} - c_1 p}{q_{10} - p_{10}} = 2p \left( \frac{p_{01}}{q_{01} - p_{01}} - \frac{p}{q - p} \right) + \frac{1}{g} \frac{p_{01} - p}{q_{01} - p_{01}}, \\
\partial_q q &= -2q \left( \frac{p}{q - p} - \frac{p_{10}}{q_{10} - p_{10}} \right) + \frac{1}{f} \frac{c_2 q - c_1 q_{10}}{q_{10} - p_{10}} = -2q \left( \frac{p}{q - p} - \frac{p_{01}}{q_{01} - p_{01}} \right) + \frac{1}{g} \frac{q_{01} - q_{01}}{q_{01} - p_{10}}, \\
\partial_q f &= \frac{c_1}{q_{10} - p_{10}} - \frac{c_2}{q - p}, \quad \partial_q g = \frac{1}{q_{01} - p_{01}} - \frac{1}{q - p}.
\end{align*}
\]

We can use the two first integrals \(\text{(62)}\) to reduce the number of dependent variables involved in system \(\text{(61)}\). In particular, we have two different options. The first option is to use the first integrals to remove function \(q\) from the system and a conservation law to replace \(f\) and \(g\) with a potential \(u\), as we did in the previous section. The second option is to consider particular values for these integrals so that to eliminate \(f\) and \(g\). These considerations are presented in the following two subsections.

**5.1.1 First integrals and a seven point scalar equation**

Let us consider the case \(c_1 = c_2 = 1\) for system \(\text{(61)}\) and its integrals \(\text{(62)}\). Choosing the values of the latter,
\[
(2fp + 1) (2fq_{10} - 1) = \alpha(n) - 1, \quad (2gp + 1)(2gq_{01} - 1) = \beta(m) - 1,
\]
we can express \(q_{10}\) and \(q_{01}\) in terms of \(p, f\) and \(g\) as
\[
q_{10} = \frac{1}{2f} \frac{2fp + \alpha(n)}{2fp + 1}, \quad q_{01} = \frac{1}{2g} \frac{2gp + \beta(m)}{2gp + 1}.
\]

Moreover, the first equation of \(\text{(61)}\) for \(c_1 = c_2 = 1\) has the form of a conservation law, suggesting the introduction of a potential \(u\) via the relations
\[
f = u_{10} - u, \quad g = u_{01} - u.
\]

We use now relations \(\text{(65)}\), \(\text{(66)}\) to eliminate \(q, f\) and \(g\) from system \(\text{(61)}\) and derive the following system for \(p\) and \(u\).
\[
\begin{align*}
2p_{10} &= \frac{\alpha(n) - \beta(m)}{u_{10} - u_{01}} - \frac{\beta(m)}{u_{11} - u_{10}} - \frac{2(\alpha(n) - 1)p}{1 + 2p(u_{10} - u)}, \\
2p_{01} &= \frac{\alpha(n) - \beta(m)}{u_{10} - u_{01}} - \frac{\alpha(n)}{u_{11} - u_{01}} - \frac{2(\beta(m) - 1)p}{1 + 2p(u_{01} - u)}.
\end{align*}
\]

A symmetry of this system easily follows from \(\text{(63)}\) by using substitutions \(\text{(65)}, \text{(66)}\) but it is omitted here because of its length. Equations \(\text{(67)}\) can be solved uniquely either for the pair \((p_{10}, u_{10})\) or for \((p_{01}, u_{01}),\)
but here we present it in this form because it is more elegant and convenient. Moreover it makes apparent the invariance of the system under the involution $(p_{ij}, u_{ij}, \alpha(n), \beta(m)) \leftrightarrow (p_{ji}, u_{ji}, \beta(m), \alpha(n))$. Regarding the Cauchy problem, initial values along a staircase are compatible with the evolution defined by the above system.

Equations (67) can be decoupled to a scalar equation for $u$. Indeed, the compatibility condition $\mathcal{T}(p_{10}) = \mathcal{S}(p_{01})$ implies that $u$ must obey the equation

$$(S - 1) \frac{\alpha(n - 1)}{u - u_{-10}} - (T - 1) \frac{\beta(m - 1)}{u - u_{0-1}} + (ST^{-1} - 1) \frac{\beta(m) - \alpha(n - 1)}{u - u_{-11}} = 0,$$  

which, up to point transformations, is the non-autonomous version of the Toda-type equation (A) in [3], cf. also [4]. A symmetry of this equations follows from the symmetry of system (67) and is generated by

$$\partial_x u = \frac{(u_{10} - u)(u_{0,-1} - u)(u_{1,-1} - u)}{F_{0,-1}} = - \frac{(u_{-10} - u)(u_{01} - u)(u_{-11} - u)}{F_{-10}},$$

where

$$F_{00} := \alpha(n)(u - u_{01})(u_{10} - u_{11}) - \beta(m)(u - u_{10})(u_{01} - u_{11}).$$

Equation (68) is defined on a stencil of seven points and can be solved uniquely with respect to any $u_{ij}$ except $u$. Because of this property, if initial data are given along a double staircase, then equation (68) uniquely determine the evolution above and below this initial configuration as it is shown in Figure 3.

**Remark.** Equation (68) is the Euler-Lagrange equation for the Lagrangian

$$\mathcal{L} = \alpha(n - 1) \log(u - u_{-10}) - \beta(m - 1) \log(u - u_{0-1}) - (\alpha(n - 1) - \beta(m - 1)) \log(u_{-10} - u_{0-1}),$$

which is also considered as a Lagrangian for the discrete Schwarzian KdV or Q10 [18], the form of which is $F_{00} = 0$ [24, 6], where $F$ is given in (70).

### 5.1.2 First integrals and a five point scalar equation

Now we consider the case $c_1 = c_2 = 1$ and two particular values for the first integrals given in (62). More precisely, let us consider that

$$(2fp + 1)(2fq_{10} - 1) = 0, \quad (2gp + 1)(2gq_{01} - 1) = -1,$$  

from which we can express $f$ and $g$ in terms of $p$ and $q$ rationally. While the second equation determines $g$ uniquely, the first equation admits two different solutions and we choose

$$f = -\frac{1}{2p}, \quad g = \frac{1}{2} \left( \frac{1}{q_{01}} - \frac{1}{p} \right).$$

Then, for $c_1 = c_2 = 1$ and in view of substitutions (72), system (61) and its symmetry (63) reduce to

$$p_{10} - p = q_{10} - q_{01}, \quad \frac{1}{p_{10}} - \frac{1}{p_{01}} = \frac{1}{q_{11}} - \frac{1}{q_{01}}$$

$$(73a)$$

\[1\] The solution $g = 0$ is not considered since along it system (61) and its symmetry (63) degenerate.

\[2\] The second choice for $f$ leads to a system related to (73) by a point transformation.
\[
\partial_x p = p^2 \left( \frac{1}{q_{10} - p_{10}} - \frac{1}{q - p} \right), \quad \partial_x q = \frac{pq}{p - q} - \frac{p - 10q - 10}{p_{10} - q_{10}}, \quad (73b)
\]
respectively.

It can be readily verified that the above discrete system for \( p \) and \( q \) can be written in a conserved form as
\[
(S - 1)(q - p) = (T - 1)q, \quad (S - 1) \left( \frac{1}{p} - \frac{1}{q_{01}} \right) = (T - 1) \frac{1}{p},
\]
We can use either of these conserved forms to introduce a potential and then derive an equation only for the potential. In either of the cases, we end up actually with the same scalar equation. Here, we introduce potential \( w \) employing the first conservation law and, in particular, we set
\[
p = w_{0,-1} - w_{1,0}, \quad q = w_{0,1} - w_{1,-1}.
\]
The substitution of the above expressions into equations (73) results to a scalar equation for potential \( w \),
\[
\frac{1}{w - w_{10}} + \frac{1}{w - w_{-10}} = \frac{1}{w - w_{1,-1}} + \frac{1}{w - w_{-11}}, \quad (74)
\]
and a symmetry of this equation is generated by
\[
\partial_x w = \frac{(w - w_{-10})(w - w_{-11})}{w_{-10} - w_{-11}}.
\]
A staircase initial value problem for equation (74) is similar to the one we considered for the Toda equation in the previous section. That is, initial data can be given at points \( w_{i,-i} \) and \( w_{i,-i-1} \) from which a solution can be uniquely determined on the whole lattice.

**Remark.** By the change of independent variables \((n, m) \mapsto (k, l) := (n + m, m)\), equation (74) can be written as “the missing identity of Frobenius” for the function \( \tilde{w}(k, l) = w(n, m) \),
\[
\frac{1}{\tilde{w} - \tilde{w}_{10}} + \frac{1}{\tilde{w} - \tilde{w}_{-10}} = \frac{1}{\tilde{w} - \tilde{w}_{01}} + \frac{1}{\tilde{w} - \tilde{w}_{0,-1}},
\]
which appears in the theory of Padé approximants \([10]\), as well as in relation with the discrete KdV equations H1, H3 \([24, 6]\) and the \( \epsilon \)-algorithm \([27]\).

### 5.1.3 A Lax pair with matrix (55) and a six point difference equation

Now we consider the discrete Lax pair
\[
\Psi_{10} = M(p, q_{10})\Psi, \quad \Psi_{01} = M(p, q_{01}, g_{1,1})\Psi,
\]
where \( M(p, q_{10}) \) is given in (55) and \( M(p, q_{01}, g_{1,1}) \) in (57). The compatibility condition of this system implies
\[
q_{10} = \frac{-1}{p}, \quad g = \frac{p_{-11}(p_{01} - p)(1 + pp_{10})}{2pp_{10} - p_{01}p_{-11}}, \quad g_{10} = \frac{(p_{01} - p)(1 + p_{01}p_{-11})}{2(pp_{10} - p_{01}p_{-11})},
\]
If we used the second conservation law to introduce the potential, then the resulting equation would be related to (74) by interchanging \( n \) and \( m \), i.e. changing indices \((ij)\) to \((ji)\).
which subsequently leads to the scalar difference equation

\[ p_{10}p_{01} \{ (p_{11} + p_{-11}) (p_{10}p_{20} + p_{01}) - (p + p_{20})(p_{-11}p_{11} + p_{10}) \} + (1 - p_{10}p_{01}) (p_{10}^2p_{20} - p_{-11}p_{01}^2) = 0. \]  

(75)

This equation admits the first integral

\[ \Phi := \frac{pp_{10}p_{01}p_{-11}}{(pp_{10} - p_{01}p_{-11})^2} \left( p + \frac{1}{p_{10}} - p_{01} - \frac{1}{p_{-11}} \right) \left( \frac{1}{p} + p_{10} - \frac{1}{p_{01}} - p_{-11} \right). \]  

(76)

Moreover, a generalised symmetry of (75) is generated by

\[ \partial_x p = p \left( \frac{1}{1 + pp_{10}} - \frac{1}{1 + pp_{-10}} \right). \]

Finally, it can be easily shown that a non-autonomous symmetry of equation (75) is generated by

\[ \partial_t p = \left( \frac{n}{1 + pp_{10}} - \frac{n - 1}{1 + pp_{-10}} - \frac{1}{2} \right) p. \]

Equation (75) is defined on a stencil of six points, cf. Figure 4 and can be uniquely solved with respect to any value of \( p \) except \( p_{10} \) and \( p_{01} \). Initial data for equation (75) can be given along a double staircase as it is shown in Figure 4.

Remark. If we set the value of the first integral (76) to \( \alpha(m) \) and, subsequently, make the change of independent variables \((n, m) \mapsto (k, l) := (n + m, m)\), then equation \( \Phi = \alpha(m) \) will become a quadrilateral equation (correspondence) for \( \tilde{p}(k, l) = p(n, m) \), namely

\[ \tilde{p}p_{10}\tilde{p}_{01}\tilde{p}_{11}H(\tilde{p})H(\tilde{p}^{-1}) = \alpha(l)(\tilde{p}p_{10} - \tilde{p}_{01}\tilde{p}_{11})^2, \quad H(\tilde{p}) := \tilde{p} + \frac{1}{\tilde{p}_{10}} - \frac{1}{\tilde{p}_{01}} - \tilde{p}_{11}. \]  

(77)

Obviously if we set \( \alpha(l) = 0 \), the above equation reduces to Hirota’s discrete KdV equation [12] either in the form \( H(\tilde{p}) = 0 \) or \( H(\tilde{p}^{-1}) = 0 \). Hence, we consider equation (77) as a quadratic Hirota KdV equation. This relation allowed us to derive the non-autonomous symmetry of equation (75) from the corresponding symmetries of Hirota’s KdV equation [31].

6 Dihedral reduction group: Degenerate orbit

We now consider Lax operators which are invariant with respect to the following transformations

\[ s_1(\lambda) : \mathcal{L}(\lambda) \to \sigma_3 \mathcal{L}(\lambda) \sigma_3, \quad s_2(\lambda) : \mathcal{L}(\lambda) \to \sigma_1 \mathcal{L}(\lambda^{-1}) \sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]  

(78)

Here, the reduction group is generated by the above set of involutions and it is isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{D}_2 \). The invariant Lax operator corresponding to the degenerate orbit can be taken in the form

\[ \mathcal{L} = D_x + \lambda^2 \sigma_3 + \lambda \begin{pmatrix} 0 & 2q \\ 2p & 0 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & 2q \\ 2p & 0 \end{pmatrix} - \frac{1}{\lambda^2} \sigma_3. \]  

(79)

This operator corresponds to the following deformation of the derivative NLS equation [23]

\[ p_t = p_{xx} + 8 (p^2 q)_x - 4 q_x, \quad q_t = -q_{xx} + 8 (pq^2)_x - 4 p_x. \]  

(80)
It is simple to check that matrix $\sigma_3$ is a Darboux matrix for operator (79) and corresponds to the discrete symmetry $(p, q) \mapsto (-p, -q)$ of system (80). A $\lambda$-dependent Darboux matrix for operator (79) is

$$M(p, q_{10}, f; u) = u \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} + \lambda \begin{pmatrix} 0 & p \\ q_{10} & 0 \end{pmatrix} + f \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & q_{10} \\ p & 0 \end{pmatrix}$$

and the corresponding Bäcklund transformation is given by

$$\partial_x p = 2 \left( (p_{10} q_{10} - p q) p + (p - p_{10}) f + q - q_{10} \right),$$

$$\partial_x q_{10} = 2 \left( (p_{10} q_{10} - p q) q_{10} + p - p_{10} + (q - q_{10}) f \right),$$

$$\partial_x f = 2 \left( (p_{10} q_{10} - p q) f + (p - p_{10}) p + (q - q_{10}) q_{10} \right),$$

$$\partial_x u = -2 (p_{10} q_{10} - p q) u.$$ (82a)

It is straightforward to show that these differential equations admit two first integrals $\partial_x \Phi^{(i)} = 0$, $i = 1, 2$, where

$$\Phi^{(1)} = u^2 (f - p q_{10}), \quad \Phi^{(2)} = u^2 (f^2 + 1 - p^2 - q^2_{10}),$$

which imply that matrix $M$ has constant determinant since

$$\det M = \left( \lambda^2 + \frac{1}{\lambda^2} \right) \Phi^{(1)} + \Phi^{(2)}.$$

### 6.1 Derivation of discrete systems

We introduce the discrete Lax pair

$$\Psi_{10} = M(p, q_{10}, f; u) \Psi, \quad \Psi_{01} = M(p, q_{01}, g; v) \Psi,$$

where matrix $M$ is given in (81). The compatibility condition of this Lax pair leads to a set of equations for $p, q, f$ and $g,$

$$f_{01} - f - g_{10} + g + p_{01} q_{01} - p_{10} q_{10} = 0,$$

$$(f_{01} - g_{10}) p + g_{01} q_{01} - f_{10} p_{10} - q_{10} = 0,$$

$$(f - g) q_{11} + g_{10} q_{10} - f_{01} q_{01} - p_{01} + p_{10} = 0,$$

$$f_{01} g - f g_{10} + p (p_{01} - p_{10}) + q_{11} (q_{01} - q_{10}) = 0,$$

and an equation solely for $u$ and $v,$

$$u_{01} v - v_{10} u = 0.$$ (86)

Functions $u, v$ are apparently redundant since they are completely separated from the remaining ones and are involved only in equation (86). Taking the value of the first integral $\Phi^1$ in (83) to be 1, then we can set

$$u^2 = \frac{1}{f - p q_{10}}, \quad v^2 = \frac{1}{g - p q_{01}}.$$ (87)

In view of this substitution, equation (86) becomes

$$(T - 1) \ln (f - p q_{10}) = (S - 1) \ln (g - p q_{01}),$$
which can be easily verified to be a conservation law for equations (85).

Equations (85) can be easily solved with respect to \((p_{01}, q_{01}, f_{01}, g_{01})\) or \((p_{10}, q_{10}, f_{10}, g_{10})\) leading to a solution with two branches: the trivial branch (17) and the non-trivial one which we consider as a system of difference equations. For the latter system it can be easily verified that it admits two first integrals

\[
\left( T - 1 \right) \frac{f - p q_{10}}{f^2 - \left( p^2 + q_{10}^2 \right) + 1} = 0, \quad \left( S - 1 \right) \frac{g - p q_{01}}{g^2 - \left( p^2 + q_{01}^2 \right) + 1} = 0,
\]

a conservation law

\[
\left( T - 1 \right) \left( f + p q \right) = \left( S - 1 \right) \left( g + p q \right),
\]

and a symmetry given by

\[
\partial_x p = \left( p_{10} q_{10} - p q \right) p + \left( p - p_{10} \right) f + q - q_{10},
\]
\[
\partial_x q = \left( p q - p_{10} q - p \right) q + p_{10} - p + q - q_{10} \left( f - q \right),
\]
\[
\partial_x f = \left( p_{10} q_{10} - p q \right) f + \left( p - p_{10} \right) p + (g - q_{10}) q_{10},
\]
\[
\partial_x g = \left( p_{01} q_{10} - p q \right) g + (p - p_{01}) p + (q - q_{10}) q_{10}.
\]

Now, we will consider two particular values for the first integrals (87) which allow us to reduce the number of functions involved in system (85) by expressing \(f, g\) polynomially in terms of \(p\) and \(q\).

### 6.1.1 First reduction and a Toda type equation

Let us first consider for the first integrals the values

\[
\left( T - 1 \right) \left( f - p q_{10} \right) - \left( g - p q_{01} \right) = 0,
\]

which imply that

\[
f - p q_{10} = 0, \quad (g - p + q_{10} - 1)(g + p - q_{10} - 1) = 0.
\]

From these algebraic equations, we choose the solution

\[
f = p q_{10}, \quad g = p - q_{01} + 1.
\]

If we substitute these expressions into system (85), its conservation laws and symmetry and then make the point transformation \((p, q) = (\tilde{p} - 1, \tilde{q} - 1)\), we will come up with the system

\[
\tilde{p}_{01} = \tilde{p}_{10} \tilde{q}_{10} \tilde{q}_{11}, \quad \tilde{q}_{01} = \frac{(\tilde{p} - 2)(\tilde{q}_{10} - 2)\tilde{q}_{11}}{\tilde{p}_{10} \tilde{q}_{10} - 2\tilde{q}_{11}} + 2,
\]

along with its conservation laws

\[
\left( T - 1 \right) (\tilde{p} - 1)(\tilde{q}_{10} + \tilde{q} - 2) = \left( S - 1 \right)(\tilde{q} (\tilde{p} - 1) - \tilde{q}_{01}), \quad \left( T - 1 \right) \ln \tilde{p} \tilde{q}_{10} = \left( S - 1 \right) \ln \tilde{p}
\]

and its symmetry

\[
\partial_x \tilde{p} = \tilde{p} (\tilde{p} - 2)(\tilde{q}_{10} - \tilde{q}), \quad \partial_x \tilde{q} = \tilde{q} (\tilde{q} - 2)(\tilde{p} - \tilde{p}_{-10}).
\]

**Remark.** Using the second conservation law above to introduce a potential \(w\) by

\[
\tilde{p} = \exp \left( w - w_{0,-1} \right), \quad \tilde{q} = \exp \left( w_{0,-1} - w_{-10} \right),
\]
we derive the scalar equation
\[ e^{u_0 - w} - e^{w - u_0 - 1} - e^{u_1 - 1 - w} + e^{w - u_1 - 11} = \frac{1}{2} \left( e^{u_0 - w - 11} - e^{u_1 - 1 - u_0 - 1} \right) \] (91)
and its symmetry
\[ \partial_x w = e^{w - u_0 - 1} - e^{u_1 - 1 - w} - \frac{1}{2} e^{u_1 - 1 - u_0 - 1}. \]

6.1.2 Second reduction and a seven point scalar equation

Another choice for the values of the first integrals [87] is
\[ \frac{f - p q_{10}}{f^2 - (p^2 + q_{10}^2) + 1} = -\frac{1}{2}, \quad \frac{g - p q_{10}}{g^2 - (p^2 + q_{10}^2) + 1} = \frac{1}{2}, \]
or, equivalently,
\[ (f + p + q_{10} + 1)(f - p - q_{10} + 1) = 0, \quad (g - p + q_{10} - 1)(g + p - q_{10} - 1) = 0. \] (92)
The above equation has four solutions of solutions and we choose
\[ f = p + q_{10} - 1, \quad g = p - q_{10} + 1. \] (93)
As before, the substitution of (93) into equations (89) and the point transformation \((p, q) \mapsto (\hat{p} + 1, \hat{q} + 1)\) result to
\[ \hat{p}_{01} = \hat{p}_{10} - \hat{q}_{11} + 2 + \frac{\hat{p}_{10}(\hat{q}_{10} - 2)}{\hat{p}}, \quad \hat{q}_{01} = \hat{p} \frac{\hat{p}_{10} \hat{q}_{10} - 2 \hat{q}_{11}}{\hat{p}_{10} (\hat{p} + \hat{q}_{10} - 2) - \hat{p} \hat{q}_{11}}. \] (94)

Similarly, we find two conservation laws
\[ (T - 1) \hat{p} \hat{q} = (S - 1)(\hat{p} \hat{q} - 2 \hat{q}_{01}), \quad (T - 1) \ln(\hat{p} - 2)(\hat{q}_{10} - 2) = (S - 1) \ln \hat{p}(\hat{q}_{01} - 2), \] (95)
and a symmetry\footnote{These differential-difference equations are related to the relativistic Volterra lattice [30].}
\[ \partial_x \hat{p} = (\hat{p} - 2)(\hat{p}_{10}(\hat{q}_{10} - 2) - \hat{p}(\hat{q} - 2)), \quad \partial_x \hat{q} = (\hat{q} - 2)(\hat{q}(\hat{p} - 2) - \hat{q}_{10}(\hat{p} - 10)). \] (96)

Remark. Using the first conservation law in (95), we can introduce a potential \(y\) by
\[ \hat{p} = 2 \frac{y - y_{-10}}{y_{0, -1} - y_{-10}}, \quad \hat{q} = 2(y_{0, -1} - y_{-10}), \] (97)
and derive the Toda-type equation\footnote{If we reverse the \(m\) direction, then this equation will become equation (D) in [3], cf. also [41].}
\[ (S - 1) \log (y - y_{-10}) - (T - 1) \log (y_{0, -1} - y) + (ST^{-1} - 1) \log \left(1 - \frac{1}{y - y_{-11}}\right) = 0 \] (98)
A conserved form of equation (98) is given by
\[(S - 1) \ln \left( \frac{(y_0 - 1 - y)(y_1 - 1 - y - 1)}{y_{-10} - y_{0, -1}} \right) = (T - 1) \ln \left( \frac{(y_{-10} - y)(y_{-11} - y + 1)}{y_{-10} - y_{0, -1}} \right),\]
while the differential-difference equation
\[\partial_x y = \left( y_{10} - y \right) \left( \frac{1}{y_{1, -1} - y} - 1 \right) \equiv -\left( y_{01} - y \right) \left( y_{-10} - y \right) \left( \frac{1}{y_{-11} - y} + 1 \right)\]
defines a symmetry of this equation.

**Remark.** If we use the second conservation law in (95) to introduce a potential \( \phi \) by the relations
\[
\hat{p} = \frac{2}{1 + e^{\phi_{0, -1} - \phi_{-10}}}, \quad \hat{q} = 2 \left( 1 + e^{\phi_{0, -1} - \phi_{-10}} + e^{\phi_{-10} - \phi_{-11}} \right),
\]
then system (94) will reduce to equation 6
\[(S - 1) e^{\phi - \phi_{-10}} + (T - 1) e^{\phi_{-1} - \phi_{0, -1}} - (ST - 1) \frac{1}{1 + e^{\phi - \phi_{-11}}} = 0.\]
A symmetry of this equation is generated by
\[\partial_x \phi = e^{\phi_{0, -1} - \phi} - e^{\phi_{-10} - \phi} + \frac{1}{1 + e^{\phi - \phi_{-10}}} \equiv 1 + e^{\phi_{0, -1} - \phi} - e^{\phi_{10} - \phi} - \frac{1}{1 + e^{\phi - \phi_{-11}}}.\]

**Remark.** Combining transformations (97) and (99), we derive the duality transformation 3
\[y_{10} - y = e^{\phi_{0, -1}} + \frac{1}{1 + e^{\phi_{1, -1} - \phi}}, \quad y_{01} - y = -e^{\phi_{-10} - \phi} - \frac{1}{1 + e^{\phi_{-1} - \phi}},\]
which connect solutions of equations (98), (100).

### 7 Concluding remarks

In this paper we discussed the Darboux-Lax scheme for Lax operators related to nonlinear NLS type equations. We derived integrable systems of differential-difference and partial difference equations, and discussed several reductions to scalar Toda-type equations. The results of this paper have already been employed in other works. For some of the systems presented here the symmetry structure and recursion operators were studied in [14], whereas the connection of our systems with Yang-Baxter maps was explored in [15]. Lax-Darboux schemes corresponding to other Lie algebras are discussed in [8, 22].

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6If we reverse the \( m \) direction, the resulting equation will be equation (E) in [3].
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