On general \((\alpha, \beta)\)-metrics of weak Landsberg type

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Abstract

In this paper, we study general \((\alpha, \beta)\)-metrics which \(\alpha\) is a Riemannian metric and \(\beta\) is an one-form. We have proven that every weak Landsberg general \((\alpha, \beta)\)-metric is a Berwald metric, where \(\beta\) is a closed and conformal one-form. This show that there exist no generalized unicorn metric in this class of general \((\alpha, \beta)\)-metric. Further, We show that \(F\) is a Landsberg general \((\alpha, \beta)\)-metric if and only if it is weak Landsberg general \((\alpha, \beta)\)-metric, where \(\beta\) is a closed and conformal one-form.

Keywords: Finsler geometry; general \((\alpha, \beta)\)-metrics; Weak Landsberg metric; generalized Unicorn problem.

1 Introduction

A Finsler manifold \((M, F)\) is a \(C^\infty\) manifold equipped with a Finsler metric which is a continuous function \(F : TM \to [0, \infty)\) with the following properties:

(1) Smoothness: \(F(x, y)\) is \(C^\infty\) on \(TM - \{0\}\).

(2) Positive homogeneity: \(F(x, \lambda y) = \lambda F(x, y)\) for all \(\lambda > 0\).

(3) Strong convexity: The fundamental tensor \((g_{ij}(x, y))\) is positive definite at all \((x, y) \in TM - \{0\}\), where

\[
g_{ij}(x, y) := \frac{1}{2} [F^2]_{y^i y^j}(x, y).
\]

There are a lot of non-Riemannian metrics in Finsler geometry. Randers metric is the simplest non-Riemannian Finsler metric, which was introduced by G. Randers in [6]. As a generalization of Randers metric, \((\alpha, \beta)\)-metric is defined by the following form

\[
F = \alpha \phi(s), \quad s = \frac{\beta}{\alpha},
\]

which \(\alpha\) is a Riemannian metric, \(\beta\) is a one-form and \(\phi(s)\) is a \(C^\infty\) positive function. A more general metric class called general \((\alpha, \beta)\)-metric was first

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introduced by C. Yu and H. Zhu in [9]. By definition, it is a Finsler metric expressed in the following form

\[ F = \alpha \phi(b^2, s), \quad s = \frac{\beta}{\alpha}. \]

This class of Finsler metrics include some Finsler metrics constructed by Bryant (see [3], [4] and [5]).

In Finsler geometry, there are several classes of metrics, such as Berwald metric, Landsberg metric, and weak Landsberg metric. We know that Berwald metric is a bit more general than Riemannian and Minkowskian metric. However, every Berwald metric is not only a Landsberg metric but also a weak Landsberg metric. For \((\alpha, \beta)\)-metrics, by definitions, we have the following relations

\[
\{ \text{Riemannian} \} \& \{ \text{locally Minkowskian} \} \subset \{ \text{Berwald} \} \subset \{ \text{Landsberg} \},
\]

and

\[
\{ \text{Landsberg metrics} \} \subset \{ \text{weak Landsberg metrics} \}.
\]

The pivotal question is, is there a Landsberg metric that is not Berwaldian? This problem was called ”unicorn” by D. Bao [2]. Moreover, is there weak Landsberg metric that is not Berwaldian? This problem was called ”generalized unicorn” by Zou and Cheng [12].

In 2009, Z. Shen has proved that a regular \((\alpha, \beta)\)-metric \(F = \alpha \phi(\beta/\alpha)\)-metric on \(M\) of dimension \(n > 2\) is a Landsberg metric if and only if \(F\) is a Berwald metric [8]. On the other hand, Z. Shen and G.S. Asanov have constructed almost regular \((\alpha, \beta)\)-metrics which are Landsberg metrics but not Berwald metrics respectively (see [1] and [8]). In 2014, Y. Zou and Cheng have showed that if \(\phi = \phi(s)\) is a polynomial in \(s\), then \(F\) is a weak Landsberg metric if and only if \(F\) is a Berwald metric. They generalized the main theorem on unicorn problem for regular \((\alpha, \beta)\)-metrics in [12].

In general \((\alpha, \beta)\)-metrics, Zohrehvand and Maleki in [11] showed that hunting for an unicorn cannot be successful in the class of metrics where \(\beta\) is a closed and conformal one-form, i.e. \(b_{ij} = c\alpha_{ij}\), where \(b_{ij}\) is the covariant derivation of \(\beta\) with respect to \(\alpha\) and \(c = c(x)\) is a scalar function on \(M\). In this paper, we show that Landsberg metric and weak Landsberg metric are equivalent in the class of general \((\alpha, \beta)\)-metrics where \(\beta\) is a closed and conformal one-form and thus hunting for an generalized unicorn cannot be successful.

**Theorem 1.** Let \(F = \alpha\phi(b^2, \frac{\beta}{\alpha})\) be a non-Riemannian general \((\alpha, \beta)\)-metric on an \(n\)-dimensional manifold \(M\) and \(\beta\) satisfies

\[ b_{ij} = c\alpha_{ij}, \]  \hspace{1cm} (1)

where \(b_{ij}\) is the covariant derivation of \(\beta\) with respect to \(\alpha\) and \(c = c(x)\) is a scalar function on \(M\). Then \(F\) is a weak Landsberg metric if and only if it is Landsberg metric.
Corollary 1. Let \( F = \alpha \phi(b^2, \beta) \) be a non-Riemannian general \((\alpha, \beta)\)-metric on an \( n \)-dimensional manifold \( M \) and \( \beta \) is closed and conformal. Then \( F \) is a weak Landsberg metric if and only if it is Berwald metric.

Thus, under the certain condition, the generalized unicorn’s problem cannot be successful in the class of general \((\alpha, \beta)\)-metrics. In this case \( F \) can be expressed by

\[
F = \alpha \Phi \left( \frac{s^2}{e^\int \left( \frac{1}{b^2} - \frac{1}{b^2} \right) db^2} + s^2 \int \theta e^\int \left( \frac{1}{b^2} - \frac{1}{b^2} \right) db^2 \right) e^\int \left( \frac{1}{b^2} \theta - \frac{1}{b^2} \right) db^2 \quad s
\]

where \( \Phi(.) \) is any positive continuously differentiable function and \( \theta \) is a smooth function of \( b^2 \).

2 Preliminaries

For a given Finsler metric \( F = F(x, y) \), the geodesic of \( F \) satisfies the following differential equation:

\[
\frac{d^2 x^i}{dt^2} + 2 \Theta^i \left( x, \frac{dx}{dt} \right) = 0,
\]

where \( \Theta^i = \Theta^i(x, y) \) are called the geodesic coefficients defined by

\[
\Theta^i = \frac{1}{4} g^{il} \left\{ [F^2]_{xm} y^m y^l - [F^2]_{xl} \right\}.
\]

For a tangent vector \( y := y^i \partial / \partial x^i \in T_x M \), the Berwald curvature \( B^i := B^i_{jkl} \partial / \partial x^i \otimes dx^j \otimes dx^k \otimes dx^l \), can be expressed by

\[
B^i_{jkl} := \frac{\partial \Theta^i}{\partial y^j \partial y^k \partial y^l}.
\]

Thus, a Finsler metric \( F \) is a Berwald metric if and only if \( B = 0 \).

The Landsberg curvature can be expressed by

\[
L_{jkl} := -\frac{1}{2} y^m g_{iml} B^i_{jkl}
\]

A Finsler metric \( F \) is a Landsberg metric if and only if \( L_{jkl} = 0 \), i.e.

\[
y^m g_{iml} B^i_{jkl} = 0.
\]

Thus, Berwald metrics are always Landsberg metrics.

There is a weaker non-Riemannian quantity than the Landsberg curvature \( L \) in Finsler geometry, \( J = J_k dx^k \), where

\[
J_k := g^{ij} L_{ijk},
\]

(2)
and \((g^{ij}) = (g_{ij})^{-1}\). We call \(\mathbf{J}\) the mean Landsberg curvature of Finsler metric \(F\). A Finsler metric \(F\) is called weak Landsberg metric if its mean Landsberg curvature \(\mathbf{J}\) vanishes \([7]\).

Let \(F\) be a Finsler metric on a manifold \(M\). \(F\) is called a general \((\alpha, \beta)\)-metrics, if \(F\) can be expressed as the form

\[
F = \alpha \phi(b^2, s), \quad s = \frac{\beta}{\alpha}, \quad b^2 := \|\beta\|_\alpha^2,
\]

where \(\alpha\) is a Riemannian metric and \(\beta := b_i(x)y^i\) is an one-form with \(\|\beta\|_\alpha < b_0\) for every \(x \in M\). The function \(\phi = \phi(b^2, s)\) is a positive \(C^\infty\) function satisfying

\[
\phi - s\phi_2 > 0, \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0
\]

when \(n \geq 3\) or

\[
\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0
\]

when \(n = 2\), where \(s\) and \(b\) are arbitrary numbers with \(|s| \leq b < b_0\), for some \(0 < b_0 \leq +\infty\). In this case, the fundamental tensor is given by \([9]\)

\[(g_{ij}) = \rho a_{ij} + \rho_0 b_i b_j + \rho_1 (b_i \alpha_j + b_j \alpha_i) - s \rho_1 \alpha_i \alpha_j,
\]

where

\[
\rho := \phi(\phi - s\phi_2), \quad \rho_0 := \phi\phi_{22} + \rho_2 \rho_2, \quad \rho_1 = (\phi - s\phi_2)\phi_2 - s\phi\phi_{22}.
\]

Moreover,

\[
\det(g_{ij}) = \phi^{n+1}(\phi - s\phi_2)^{n-2}(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})\det(a_{ij}),
\]

\[
g^{ij} = \frac{1}{\rho} \{a^{ij} + \eta b^i b^j + \eta_0 \alpha^{-1}(b^i y^j + b^j y^i) + \eta_1 \alpha^{-2} y^i y^j\},
\]

where \((g^{ij}) = (g_{ij})^{-1}\), \((a^{ij}) = (a_{ij})^{-1}\), \(b^i = a^{ij} b_j\),

\[
\begin{align*}
\eta &= \frac{\phi_{22}}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}}, \quad \eta_0 = \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{\phi(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \quad \\
\eta_1 &= \frac{(s\phi + (b^2 - s^2)\phi_{22})(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{\phi^2(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}.
\end{align*}
\]

Not that, we use the indices 1 and 2 as the derivation with respect to \(b^2\) and \(s\), respectively \([9]\).

Let \(b_{ij}\) denote the coefficients of the covariant derivative of \(\beta\) with respect to \(\alpha\). Let \([9]\)

\[
r_{ij} := \frac{1}{2} (b_{ij} + b_{ji}), \quad r_{00} := r_{ij} y^i y^j, \quad r_i := b^i r_{ji}, \quad r_0 := r_{ij} y^i, \quad r^i := a^{ij} r_j, \quad r := b^i r_i
\]

\[
s_{ij} := \frac{1}{2} (b_{ij} - b_{ji}), \quad s^i_0 := a^{ik} s_{kj} y^j, \quad s_i := b^i s_{ji}, \quad s_0 := s_{ij} y^i, \quad s^i := a^{ij} s_j.
\]
\( \beta \) is a closed one-form if and only if \( s_{ij} = 0 \), and it is a conformal one-form with respect to \( \alpha \) if and only if \( b_{ij} + b_{ji} = ca_{ij} \), where \( c = c(x) \) is a nonzero scalar function on \( M \). Thus, \( \beta \) is closed and conformal with respect to \( \alpha \) if and only if \( b_{ij} = ca_{ij} \), where \( c = c(x) \) is a nonzero scalar function on \( M \).

For a general \((\alpha, \beta)\)-metric, its spray coefficients \( G^i \) are related to the spray coefficients \( G^i_\alpha \) of \( \alpha \) by \[ G^i = G^i_\alpha + c\alpha^2 \{ \Theta(1 + 2Rb^2) + s\Omega \} l^i + \{ \Psi(1 + 2Rb^2) + s\Pi - R \} b^i \] where \( l^i := y^i / \alpha \) and

\[ Q := \frac{\phi_2}{\phi - s\phi_2}, \quad R := \frac{\phi_1}{\phi - s\phi_2}, \]
\[ \Theta := \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{2\phi(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \quad \Psi := \frac{\phi_{22}}{2(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \]
\[ \Pi := \frac{(\phi - s\phi_2)\phi_{12} - s\phi\phi_{22}}{(\phi - s\phi_2)(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \quad \Omega := \frac{2\phi_1}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi} \Pi \]

When \( \beta \) is closed and conformal one-form, i.e. satisfies (1), then

\[ r_{oo} = ca^2, \quad r_0 = c\beta, \quad r = cb^2, \quad r^i = cb^i, \quad s^i_0 = s_0 = s^i = 0 \]

Substituting this into (9) yields (10)

\[ G^i := G^i_\alpha + ca^2 \{ \Theta(1 + 2Rb^2) + s\Omega \} l^i + ca^2 \{ \Psi(1 + 2Rb^2) + s\Pi - R \} b^i \]

If we have

\[ E := \frac{\phi_2 + 2s\phi_1}{2\phi} - H s\phi + (b^2 - s^2)\phi_2 \]
\[ H := \frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \]

then from (10)

\[ G^i := G^i_\alpha + ca^2 El^i + ca^2 Hb^i. \]

The Berwald curvature of a general \((\alpha, \beta)\)-metric, when \( \beta \) is a closed and conformal one-form, is computed in (10):

**Proposition 1.** Let \( F = \alpha\phi(b^2, s) \), \( s = \beta / \alpha \), be a general \((\alpha, \beta)\)-metric on an \( n \)-dimensional manifold \( M \). Suppose that \( \beta \) satisfies (1), then the Berwald curvature of \( F \) is given by

\[ B^i_{jkl} = \frac{c}{\alpha} U^i_{jkl} \]
where 

\[ U^i_{jkl} := \{(E - sE_2)a_{kl} + 2E_2 b_k b_l|\delta^i_j + s(3E_{22} + sE_{222})l_j b_k l^i \]
- \{(E_2 + sE_{222})b_{l_j} l_j b_{l^i} \}(k \rightarrow l \rightarrow j \rightarrow k) 
- \{sE_{22} [a_{jl} b_k l^i + (l_k b_l + l^i b_l) \delta_j^i]\}
+ \{(E - sE_2 - s^2E_{22})(a_{jl} l^i + l^i \delta_j^i) l_k \}(k \rightarrow l \rightarrow j \rightarrow k) 
+ \{(3E - 3sE_2 - 6^2E_{22} - s^3E_{222})l_j l_k l^i + E_{222}b_k b_l b_j l^i \}
+ \{(H_2 - sH_{22})(b_j - s l_j) a_{kl} - (H_2 - sH_{22} - s^2H_{222})b_{l_j} l_l + E_{222}b_k b_l b_j \}
+ \{s(3H_2 - 3sH_{22} - s^2H_{222})l_j l_k l_l + H_{222}b_k b_l b_j \} \delta^i_j,

where \(E\) and \(H\) is defined in [11] and [12], \(l_j := a_{ij} l^i\) and \((k \rightarrow l \rightarrow j \rightarrow k)\) denotes cyclic permutation.

3 The mean Landsberg curvature of General \((\alpha, \beta)\)-metrics

By use of Proposition\([1]\) M. Zohrehvand and H. Maleki calculated the Landsberg curvature of a general \((\alpha, \beta)\)-metric in [11], when \(\beta\) is a closed and conformal one-form:

**Proposition 2.** Let \(F = \alpha \phi(b^2, s), s = \beta/\alpha, \) be a general \((\alpha, \beta)\)-metric on an \(n\)-dimensional manifold \(M\). Suppose that \(\beta\) satisfies [1], then the Landsberg curvature of \(F\) is given by

\[
L_{jkl} = -\frac{c}{2} \phi V_{jkl}
\]

where

\[
V_{jkl} := \{(E - sE_2)a_{kl} + 2E_2 b_k b_l|\phi l_j + \phi_2(b_j - s l_j)\}
- \{(E_2 + sE_{222})(b_l - s l_l - 2E_{22} s l_j) \phi l_j \}(k \rightarrow l \rightarrow j \rightarrow k) 
- \{sE_{22} [a_{jl} b_k \phi + (l_k b_l + s l_l) \phi l_j + \phi_2(b_j - s l_j)]\}
+ \{(E - sE_2 - s^2E_{22})(a_{jl} l^i + (s l_j + \phi_2(b_j - s l_l)) l_k l^i \}(k \rightarrow l \rightarrow j \rightarrow k)
+ \{(3E - 3sE_2 - 6^2E_{22} - s^3E_{222})l_j l_k l^i + E_{222}b_k b_l b_j \phi l^i \}
+ \{(H_2 - sH_{22})(a_{jl} b_l - s l_j) - b_l l_k l^i \}
- sH_{222} l_j b_k - s l_j \}(s l_j + (b^2 - s^2) \phi_2 \phi) \phi \phi_2 \}(k \rightarrow l \rightarrow j \rightarrow k)
+ \{s(3H_2 - 3sH_{22} - s^2H_{222})l_j l_k l_l + H_{222}b_k b_l b_j \}(s l_j + (b^2 - s^2) \phi_2 \phi_2).
\]

By use of Proposition [2] and Maple, we can calculate the mean Landsberg curvature of a general \((\alpha, \beta)\)-metric, when \(\beta\) is a closed and conformal one-form:
Proposition 3. Let $F = \alpha \phi(b^2,s)$, $s = \beta/\alpha$, be a general $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$. Suppose that $\beta$ satisfies $[\ref{eq:beta}]$, then the Landsberg curvature of $F$ is given by

$$J_j = -\frac{c\phi}{2\rho}W_j,$$

where

$$W_j := \{(E - sE_2)(n + 1)\phi_2 + 3E_{22}\phi_2(b^2 - s^2) - sE_{22}(n + 1)\phi + E_{222}\phi(b^2 - s^2)
+ \{(H_2 - sH_{22})(n + 1) + H_{222}(b^2 - s^2)\}(s\phi + (b^2 - s^2)\phi_2)
+ 3\eta(E - sE_2)\phi_2(b^2 - s^2) + 3\eta E_{22}\phi_2(b^2 - s^2)^2 - 3s\eta E_{222}\phi(b^2 - s^2)
+ \eta E_{222}(b^2 - s^2)^2\phi + \eta[3(H_2 - sH_{22})(b^2 - s^2) + H_{222}(b^2 - s^2)^2]
\times (s\phi + (b^2 - s^2)\phi_2)\}(b_j - s\ell_j)$$

(18)

where $\rho$ and $\eta$ is defined in $[\ref{eq:rho}]$ and $[\ref{eq:eta}]$ and $l_j := a_{ij}l^i$.

Proof. To compute the mean Landsberg curvature $J_j := g^{kl}L_{ijkl}$ using $[\ref{eq:Lijkl}]$, we need to $[\ref{eq:Lijkl}]$. Using $L_{ijkl}y^k = 0$, we get

$$J_j := \frac{1}{\rho}\{a^{kl} + \eta b^k b^l\}L_{ijkl} = \frac{c\phi}{2\rho}\{a^{kl} + \eta b^k b^l\}V_{jkl}.$$

(19)

We need

$$a^{kl}V_{jkl} = \{(E - sE_2)(n + 1)\phi_2 + 3E_{22}\phi_2(b^2 - s^2) - sE_{22}(n + 1)\phi + E_{222}\phi(b^2 - s^2)
+ \{(H_2 - sH_{22})(n + 1) + H_{222}(b^2 - s^2)\}(s\phi + (b^2 - s^2)\phi_2)\}(b_j - s\ell_j),$$

and

$$\eta b^k b^l V_{jkl} = \{3\eta(E - sE_2)\phi_2(b^2 - s^2) + 3\eta E_{22}\phi_2(b^2 - s^2)^2 - 3s\eta E_{222}\phi(b^2 - s^2)
+ \eta E_{222}(b^2 - s^2)^2\phi + \eta[3(H_2 - sH_{22})(b^2 - s^2) + H_{222}(b^2 - s^2)^2]
\times (s\phi + (b^2 - s^2)\phi_2)\}(b_j - s\ell_j).$$

Substituting these in $[\ref{eq:Lijkl}]$, we obtain $[\ref{eq:Lijkl}]$. 

Now, we can obtain the necessary and sufficient conditions for a general $(\alpha, \beta)$-metric to be weak Landsbergian.

Proposition 4. Let $F = \alpha \phi(b^2,s)$, $s = \beta/\alpha$, be a general $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$. Suppose that $\beta$ satisfies $[\ref{eq:beta}]$, then $F$ is weak Landsberg metric if and only if the following equations hold:

$$E_{22} = 0, \quad H_{222} = 0,$$

(20)

$$E - sE_2)\phi_2 + (H_2 - sH_{22})(s\phi + (b^2 - s^2)\phi_2 = 0$$

(21)
Proof. Let $F = \alpha \phi(b^2, s)$, $s = \beta/\alpha$, be a weak Landsberg metric, where $\beta$ is a closed and conformal one-form. From Proposition 2, it concluded

$$W_j = 0,$$

where $W_j$ is defined in (18). We can rewrite (22) as following:

$$[1 + n + 3(b^2 - s^2)\eta][(E - sE_2)\phi_2 + (H_2 - sH_{22})(s\phi + (b^2 - s^2)\phi_2)]$$

$$+ (b^2 - s^2)[1 + (b^2 - s^2)\eta][s\phi + (b^2 - s^2)\phi_2]H_{222}$$

$$+ 3(b^2 - s^2)[1 + (b^2 - s^2)\eta]\phi_2 - [1 + n + 3(b^2 - s^2)\eta]s\phi \{E_{22} + (b^2 - s^2)[1 + (b^2 - s^2)\eta]\phi E_{222} = 0. \quad (23)$$

Thus

$$[1 + n + 3(b^2 - s^2)\eta][(E - sE_2)\phi_2 + (H_2 - sH_{22})(s\phi + (b^2 - s^2)\phi_2)] = 0 \quad (24)$$

$$+ (b^2 - s^2)[1 + (b^2 - s^2)\eta][s\phi + (b^2 - s^2)\phi_2]H_{222} = 0 \quad (25)$$

$$+ 3(b^2 - s^2)[1 + (b^2 - s^2)\eta]\phi_2 - [1 + n + 3(b^2 - s^2)\eta]s\phi \{E_{22} = 0 \quad (26)$$

$$(b^2 - s^2)[1 + (b^2 - s^2)\eta]\phi E_{222} = 0. \quad (27)$$

Since $1 + n + 3(b^2 - s^2)\eta \neq 0$, we have from (24)

$$(E - sE_2)\phi_2 + (H_2 - sH_{22})(s\phi + (b^2 - s^2)\phi_2) = 0. \quad (28)$$

From (25), we know $(b^2 - s^2)[1 + (b^2 - s^2)\eta] \neq 0$, then $s\phi + (b^2 - s^2)\phi_2 = 0$ or $H_{222} = 0$. If $s\phi + (b^2 - s^2)\phi_2 = 0$, we have

$$\phi = \sigma(b^2)\sqrt{s^2 - b^2}$$

where $\sigma(b^2)$ is any positive smooth function. This is a Riemannian case and we have

$$H_{222} = 0. \quad (29)$$

In (26), we have $3(b^2 - s^2)[1 + (b^2 - s^2)\eta]\phi_2 - [1 + n + 3(b^2 - s^2)\eta]s\phi \neq 0$ and

$$E_{22} = 0. \quad (30)$$

Proof of Theorem 1. Since in (11) is obtained exactly (20) and (21) for Landsberg metrics, according to [4] it is obvious.

Proof of Corollary 1. Here, by (11), we have $E - sE_2 = 0$ and $H_2 - sH_{22} = 0$ also. In (10), it is proved that a general $(\alpha, \beta)$-metric where $\beta$ is closed and conformal one-form, is a Berwald metric if and only if

$$E - sE_2 = 0, \quad H_2 - sH_{22} = 0$$

and in this case $F$ can be expressed by

$$F = \alpha \varphi \left(\frac{s^2}{e^f(\frac{1}{2}\theta - b^2\theta)db^2 + s^2 \int \frac{1}{2}e^f(\frac{1}{2}\theta - b^2\theta)db^2} \right) e^f(\frac{1}{2}b^2\theta - \frac{1}{2}\theta)db^2 \quad (31)$$

where $\varphi(.)$ is any positive continuously differentiable function and $\theta$ is a smooth function of $b^2$ [10]. This complete the proof.
4 Examples

In this section, we will explicitly construct some new examples.

Example 1. Take $\varphi(t) := 1 + \xi t$ and $\theta(b^2) := 1$, then by (31)

$$\phi(b^2, s) = \frac{(s^2(\xi e^{b^4} - 1) + b^2)e^{b^4}s}{b^4(b^2 - s^2)}.$$ 

We can see that $\phi(b^2, s)$ satisfies in (20) and (21). Moreover, the corresponding general $(\alpha, \beta)$-metrics

$$F := \alpha \phi \left( b^2, \frac{\beta}{\alpha} \right)$$

are Landsberg and weak Landsberg metrics, i.e. $L_{jkl} = 0$ and $J_j = 0$.

Example 2. Take $\varphi(t) := \frac{s}{1+\xi t}$ and $\theta(b^2) := \frac{\varepsilon}{1+\xi b^2}$ in (31), we have

$$\phi(b^2, s) := \frac{\mu(b^2 - s^2)e^{\frac{\varepsilon}{s^2(b^2 - s^2)}}s}{(s^2b^2\xi e^{\frac{s^2b^2}{b^2 - s^2} - s^2ln(s^2 + 1) + b^2 - s^2})b^2}.$$ 

The corresponding general $(\alpha, \beta)$-metrics are Landsberg and weak Landsberg metrics.

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