Degenerate $q$-Euler polynomials

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Abstract

Recently, some identities of degenerate Euler polynomials arising from $p$-adic fermionic integrals on $\mathbb{Z}_p$ were introduced in Kim and Kim (Integral Transforms Spec. Funct. 26(4):295-302, 2015). In this paper, we study degenerate $q$-Euler polynomials which are derived from $p$-adic $q$-integrals on $\mathbb{Z}_p$.

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1 Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $v_p$ be the normalized exponential valuation in $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p}$. Let $q$ be an indeterminate in $\mathbb{C}_p$ such that $|1 - q|_p < p^{-\frac{1}{2}}$. The $q$-extension of $x$ is defined as $[x]_q = \frac{1 - q^x}{1 - q}$. Note that $\lim_{q \to 1} [x]_q = x$. For $f \in C(\mathbb{Z}_p) = \{f \mid f$ is a $\mathbb{C}_p$-valued continuous function on $\mathbb{Z}_p\}$, the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[pN]_q} \sum_{x=0}^{pN-1} f(x)(-q)^x \quad \text{(see [1, 2])},$$

where $[x]_{-q} = \frac{1 - (-q)^x}{1 - q}$.

By (1.1), we easily get

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0) \quad (f_1(x) = f(x + 1)), \quad (1.2)$$

and

$$q^n I_{-q}(f_n) + (-1)^n I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l) \quad (n \in \mathbb{N}), \quad (1.3)$$

where $f_n(x) = f(x + n)$ (see [1–16]).

The ordinary fermionic $p$-adic integral on $\mathbb{Z}_p$ is defined as

$$\lim_{q \to 1} I_{-q}(f) = I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{pN-1} f(x)(-1)^x \quad \text{(see [2])}. \quad (1.4)$$
The degenerate Euler polynomials of order \(r (\in \mathbb{N})\) are defined by the generating function to be

\[
\left( \frac{2}{(1 + \lambda t)^{\frac{1}{2}} + 1} \right)^r (1 + \lambda t)^{\frac{1}{2}} = \sum_{n=0}^{\infty} E_n^{(r)}(x \mid \lambda) \frac{t^n}{n!} \quad (\text{see } [5, 6, 10]), \tag{1.5}
\]

where \(\lambda, t \in \mathbb{Z}_p\) such that \(|\lambda t|_p < p^{-\frac{1}{p-1}}\).

From (1.5), we have

\[
\sum_{n=0}^{\infty} \lim_{\lambda \to 0} E_n^{(r)}(x \mid \lambda) \frac{t^n}{n!} \\
= \lim_{\lambda \to 0} \left( \frac{2}{(1 + \lambda t)^{\frac{1}{2}} + 1} \right)^r (1 + \lambda t)^{\frac{1}{2}} \\
= \left( \frac{2}{e^t + 1} \right)^r e^{xt} \\
= \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}, \tag{1.6}
\]

where \(E_n^{(r)}(x)\) are the higher-order Euler polynomials.

Thus, by (1.6), we get

\[
\lim_{\lambda \to 0} E_n^{(r)}(x \mid \lambda) = E_n^{(r)}(x) \quad (n \geq 0). \tag{1.7}
\]

When \(x = 0\), \(E_n^{(r)}(\lambda) = E_n^{(r)}(0 \mid \lambda)\) are called the higher-order degenerate Euler numbers, while \(\lim_{\lambda \to 0} E_n^{(r)}(\lambda) = E_n^{(r)}\) are called the higher-order Euler numbers.

In [10], it was shown that

\[
E_n^{(r)}(x \mid \lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + x_2 + \cdots + x_r + x \mid \lambda)_n \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r), \tag{1.8}
\]

where \((x)_n = x(x-1) \cdots (x-n+1)\) and \(n \in \mathbb{Z}_{\geq 0}\).

In this paper, we study \(q\)-extensions of the degenerate Euler polynomials and give some formulae and identities of those polynomials which are derived from the fermionic \(p\)-adic \(q\)-integrals on \(\mathbb{Z}_p\).

### 2 Some identities of \(q\)-analogues of higher-order degenerate Euler polynomials

In this section, we assume that \(\lambda, t \in \mathbb{Z}_p\) with \(|\lambda t|_p < p^{-\frac{1}{p-1}}\). From (1.2), we have

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \lambda t)^{[x_1 + \cdots + x_r]+\lambda} \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\
= \left( \frac{[2]_q}{q(1 + \lambda t)^{\frac{1}{2}} + 1} \right)^r (1 + \lambda t)^{\frac{1}{2}}. \tag{2.1}
\]
Now, we define a \( q \)-analogue of degenerate Euler polynomials of order \( r \) as follows:

\[
\left( \frac{[2]_q}{q(1 + \lambda t)^{1/\lambda} + 1} \right)^r (1 + \lambda t)^\frac{x}{\lambda} = \sum_{n=0}^{\infty} E^{(r)}_{n,q}(x \mid \lambda) \frac{t^n}{n!}. \tag{2.2}
\]

Thus, by (2.2), we easily get

\[
\sum_{n=0}^{\infty} \lim_{\lambda \to 0} E^{(r)}_{n,q}(x \mid \lambda) \frac{t^n}{n!} = \lim_{\lambda \to 0} \left( \frac{[2]_q}{q(1 + \lambda t)^{1/\lambda} + 1} \right)^r (1 + \lambda t)^\frac{x}{\lambda} = \left( \frac{[2]_q}{qe^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E^{(r)}_{n,q}(x) \frac{t^n}{n!}, \tag{2.3}
\]

where \( E^{(r)}_{n,q}(x) \) are called the higher-order \( q \)-Euler polynomials (see [15–17]). Thus, by (2.3), we get

\[
\lim_{\lambda \to 0} E^{(r)}_{n,q}(x \mid \lambda) = E^{(r)}_{n,q}(x) \quad (n \geq 0).
\]

For \( \lambda \in \mathbb{C} \) with \( \lambda \neq 1 \), the Frobenius-Euler polynomials of order \( r \) are defined by the generating function to be

\[
\left( \frac{1 - \lambda}{e^t - \lambda} \right)^r e^{xt} = \sum_{n=0}^{\infty} H^{(r)}_{n}(x \mid \lambda) \frac{t^n}{n!} \quad \text{(see [3, 18])}. \tag{2.4}
\]

By replacing \( \lambda \) by \(-q^{-1}\), we get

\[
\left( \frac{1 + q^{-1}}{e^t + q^{-1}} \right)^r e^{xt} = \sum_{n=0}^{\infty} H^{(r)}_{n}(x \mid -q^{-1}) \frac{t^n}{n!}. \tag{2.5}
\]

Now, we define the degenerate Frobenius-Euler polynomials of order \( r \) as follows:

\[
\left( \frac{1 - u}{(1 + \lambda t)^{1/\lambda} - u} \right)^r (1 + \lambda t)^\frac{x}{\lambda} = \sum_{n=0}^{\infty} h^{(r)}_{n}(x, u \mid \lambda) \frac{t^n}{n!}. \tag{2.6}
\]

From (2.6), we note that

\[
\sum_{n=0}^{\infty} \lim_{\lambda \to 0} h^{(r)}_{n}(x, u \mid \lambda) \frac{t^n}{n!} = \lim_{\lambda \to 0} \left( \frac{1 - u}{(1 + \lambda t)^{1/\lambda} - u} \right)^r (1 + \lambda t)^\frac{x}{\lambda} = \left( \frac{1 - u}{e^t - u} \right)^r e^{xt} = \sum_{n=0}^{\infty} H_{n}(x \mid u) \frac{t^n}{n!}. \tag{2.7}
\]
Thus, by (2.7), we get

$$\lim_{\lambda \to 0} j^{(r)}_n(x, u | \lambda) = H_n(x | u) \quad (n \geq 0).$$

By (2.2) and (2.6), we get

$$\mathcal{E}^{(r)}_{n,q}(x | \lambda) = h^{(r)}_n(x, -q^{-1} | \lambda) \quad (n \geq 0). \quad (2.8)$$

From (2.1) and (2.2), we have

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} \left( \frac{x_1 + \cdots + x_r + x}{\lambda} \right)^n \frac{d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r)}{n!} \lambda^n \frac{\mu^n}{n!} = \sum_{n=0}^{\infty} \mathcal{E}^{(r)}_{n,q}(x | \lambda) \frac{\mu^n}{n!}. \quad (2.9)$$

Now, we define

$$(x | \lambda)_n = x(x-\lambda) \cdots (x-(n-1)\lambda) \quad (n > 0),$$

$$(x | \lambda)_0 = 1. \quad (2.10)$$

By (2.9) and (2.10), we get

$$\int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} (x + x_1 + \cdots + x_r | \lambda)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) = \mathcal{E}^{(r)}_{n,q}(x | \lambda) \quad (u \geq 0). \quad (2.11)$$

Therefore, by (2.6) and (2.11), we obtain the following theorem.

**Theorem 2.1** For $n \geq 0$, we have

$$\mathcal{E}^{(r)}_{n,q}(x | \lambda) = \int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} (x_1 + \cdots + x_r + x | \lambda)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r)$$

$$= h^{(r)}_n(x, -q^{-1} | \lambda) \quad (n \geq 0),$$

where $h^{(r)}_n(x, u | \lambda)$ are called the degenerate Frobenius-Euler polynomials of order $r$.

It is not difficult to show that

$$(x_1 + \cdots + x_r + x | \lambda)_n$$

$$= (x_1 + \cdots + x_r + x)(x_1 + \cdots + x_r + x - \lambda) \cdots (x_1 + \cdots + x_r + x - (n-1)\lambda)$$

$$= \lambda^n \left( \frac{x_1 + \cdots + x_r + x}{\lambda} \right)_n$$

$$= \lambda^n \sum_{l=0}^{n} S_1(n, l) \left( \frac{x_1 + \cdots + x_r + x}{\lambda} \right)^l$$

$$= \sum_{l=0}^{n} \lambda^{n-l} S_1(n, l) (x_1 + \cdots + x_r + x)^l, \quad (2.12)$$

where $S_1(n, l)$ is the Stirling number of the first kind.
We observe that
\[
\int_{Z} \cdots \int_{Z} e^{i x_1 + \cdots + x_r + x t} d\mu_{\lambda}(x_1) \cdots d\mu_{\lambda}(x_r) = \left( \frac{[2]_q}{q e^t + 1} \right)^r.
\]  
(2.13)

Thus, by (2.13), we get
\[
\sum_{n=0}^{\infty} \int_{Z} \cdots \int_{Z} (x_1 + \cdots + x_r + x)^n d\mu_{\lambda}(x_1) \cdots d\mu_{\lambda}(x_r) \frac{t^n}{n!} = \left( \frac{[2]_q}{q e^t + 1} \right)^r e^t = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{n!}.
\]  
(2.14)

By comparing the coefficients on both sides of (2.14), we get
\[
E_{n,q}^{(r)}(x) = \int_{Z} \cdots \int_{Z} (x_1 + \cdots + x_r + x)^n d\mu_{\lambda}(x_1) \cdots d\mu_{\lambda}(x_r).
\]  
(2.15)

From Theorem 2.1, (2.12) and (2.15), we note that
\[
h_{n}^{(r)}(x, q^{-1} | \lambda) = \int_{Z} \cdots \int_{Z} (x_1 + \cdots + x_r + x | \lambda)^n d\mu_{\lambda}(x_1) \cdots d\mu_{\lambda}(x_r)
\]
\[
= \sum_{l=0}^{n} \lambda^{n-l} S_1(n, l) \int_{Z} \cdots \int_{Z} (x_1 + \cdots + x_r + x)^l d\mu_{\lambda}(x_1) \cdots d\mu_{\lambda}(x_r)
\]
\[
= \sum_{l=0}^{n} \lambda^{n-l} S_1(n, l) E_{l,q}^{(r)}(x)
\]
\[
= \sum_{l=0}^{n} \lambda^{n-l} S_1(n, l) H_{l}^{(r)}(x | -q^{-1}).
\]  
(2.16)

Therefore, by (2.16), we obtain the following theorem.

**Theorem 2.2** For \( n \geq 0 \), we have
\[
h_{n}^{(r)}(x, q^{-1} | \lambda) = \sum_{l=0}^{n} \lambda^{n-l} S_1(n, l) H_{l}^{(r)}(x | -q^{-1}).
\]

In particular,
\[
E_{n,q}^{(r)}(x | \lambda) = \sum_{l=0}^{n} \lambda^{n-l} S_1(n, l) E_{l,q}^{(r)}(x).
\]

By replacing \( t \) by \( (e^t - 1)/\lambda \) in (2.2), we get
\[
\left( \frac{[2]_q}{q e^t + 1} \right)^r e^t
\]
\[
= \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x | \lambda) \frac{1}{n!} \frac{1}{\lambda^n} (e^t - 1)^n
\]
\[
\sum_{n=0}^{\infty} E_{n,q}^{(r)}(x \mid \lambda) \frac{1}{\lambda^n} \sum_{m=n}^{\infty} S_2(m,n) \frac{\lambda^m}{m!} t^m
\]

\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} E_{n,q}^{(r)}(x \mid \lambda) \lambda^{m-n} S_2(m,n) \right) \frac{t^m}{m!},
\]

where \( S_2(m,n) \) is the Stirling number of the second kind.

Thus, by (2.17), we obtain the following theorem.

**Theorem 2.3** For \( m \geq 0 \), we have

\[
H_m^{(r)}(x \mid -q^{-1}) = \sum_{n=0}^{m} h_n^{(r)}(x, -q^{-1} \mid \lambda) \lambda^{m-n} S_2(m,n).
\]

In particular,

\[
E_{m,q}^{(r)}(x) = \sum_{n=0}^{m} E_{n,q}^{(r)}(x \mid \lambda) \lambda^{m-n} S_2(m,n).
\]

When \( r = 1 \), \( E_{n,q}(x \mid \lambda) = E_{n,q}^{(1)}(x \mid \lambda) \) are called the degenerate \( q \)-Euler polynomials. In particular, \( x = 0 \), \( E_{n,q}(\lambda) = E_{n,q}(0 \mid \lambda) \) are called the degenerate \( q \)-Euler numbers. \( h_n(x, u \mid \lambda) = h_n^{(1)}(x, u \mid \lambda) \) are called the degenerate Frobenius-Euler polynomials. When \( x = 0 \), \( h_n(u \mid \lambda) = h_n(0, u \mid \lambda) \) are called the degenerate Frobenius-Euler numbers.

From (1.2), we have

\[
\int_{\mathbb{Z}_p} (1 + \lambda t)^{x_t} d\mu_{-q}(x_1)
\]

\[
= \left( \frac{[2]_q}{q(1 + \lambda t)^{\frac{1}{2}} + 1} \right) (1 + \lambda t)^{\frac{x}{2}}
\]

\[
= \left( \frac{1 + q^{-1}}{(1 + \lambda t)^{\frac{1}{2}} + q^{-1}} \right) (1 + \lambda t)^{\frac{x}{2}}
\]

\[
= \sum_{n=0}^{\infty} h_n(x, -q^{-1} \mid \lambda) \frac{t^n}{n!}.
\]

Thus, by (2.18), we get

\[
h_n(x, -q^{-1} \mid \lambda)
\]

\[
= \int_{\mathbb{Z}_p} (x_1 + x \mid \lambda)_n d\mu_{-q}(x_1)
\]

\[
= \lambda^n \int_{\mathbb{Z}_p} \left( \frac{x_1 + x}{\lambda} \right)_n d\mu_{-q}(x_1)
\]

\[
= \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} \int_{\mathbb{Z}_p} (x_1 + x)^l d\mu_{-q}(x_1)
\]

\[
= \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} H_l(x \mid -q^{-1})
\]
and

\[ h_n(-q^{-1} | \lambda) = \sum_{l=0}^{d} S_1(n, l) \lambda^{-l} H_l(-q^{-1}). \] (2.20)

For \( d \in \mathbb{N} \), by (1.3), we get

\[ q^d \int_{\mathbb{Z}_p} (x_1 + d \mid \lambda) d \mu_{-q}(x_1) = [2]_q \sum_{l=0}^{d-1} (-1)^l q^l (l \mid \lambda). \] (2.21)

Let \( d \equiv 1 \) (mod 2). Then we have

\[ [2]_q \sum_{l=0}^{d-1} (-1)^l q^l (l \mid \lambda) = q^d h_n(d, -q^{-1} \mid \lambda) + h_n(-q^{-1} \mid \lambda). \] (2.22)

For \( d \in \mathbb{N} \) with \( d \equiv 0 \) (mod 2), we get

\[ [2]_q \sum_{l=0}^{d-1} (-1)^l q^l (l \mid \lambda) = q^d h_n(d, -q^{-1} \mid \lambda) - h_n(-q^{-1} \mid \lambda). \] (2.23)

Therefore, by (2.22) and (2.23), we obtain the following theorem.

**Theorem 2.4** Let \( d \in \mathbb{N} \) and \( n \geq 0 \).

(i) For \( d \equiv 1 \) (mod 2), we have

\[ q^d h_n(d, -q^{-1} \mid \lambda) + h_n(-q^{-1} \mid \lambda) = [2]_q \sum_{l=0}^{d-1} (-1)^l q^l (l \mid \lambda). \]

(ii) For \( d \equiv 0 \) (mod 2), we have

\[ q^d h_n(d, -q^{-1} \mid \lambda) - h_n(-q^{-1} \mid \lambda) = [2]_q \sum_{l=0}^{d-1} (-1)^l q^l (l \mid \lambda). \]

**Corollary 2.5** Let \( d \in \mathbb{N} \) and \( n \geq 0 \).

(i) For \( d \equiv 1 \) (mod 2), we have

\[ q^d E_{n,q}(d \mid \lambda) + E_{n,q}(\lambda) = [2]_q \sum_{l=0}^{d-1} (-1)^l q^l (l \mid \lambda). \]
(ii) For \( d \equiv 0 \pmod{2} \), we have
\[
q^d E_{n,q}(d \mid \lambda) - E_{n,q}(\lambda) = [2]_q \sum_{l=0}^{d-1} (-1)^{l-1} q^l (l \mid \lambda)_n.
\]

From (1.1), we note that
\[
\int_{\mathbb{Z}_p} f(x) \, d\mu_{-q}(x) = \frac{[2]_q}{[2]_q^d} \sum_{l=0}^{d-1} (-q)^l \int_{\mathbb{Z}_p} f(a + dx) \, d\mu_{-q^d}(x),
\]
where \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \).

By (2.24), we get
\[
\int_{\mathbb{Z}_p} f(x) \, d\mu_{-q}(x) = \frac{[2]_q}{[2]_q^d} \sum_{l=0}^{d-1} (-q)^l \int_{\mathbb{Z}_p} f(a + dx) \, d\mu_{-q^d}(x),
\]
where \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \) and \( n \geq 0 \).

Therefore, by (2.25), we obtain the following theorem.

**Theorem 2.6** For \( n \geq 0 \), \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \), we have
\[
E_{n,q}(\lambda) = d^n \frac{[2]_q}{[2]_q^d} \sum_{a=0}^{d-1} (-q)^a E_{n,q^d} \left( \frac{a + x}{d} \mid \frac{\lambda}{d} \right).
\]

Moreover,
\[
E_{n,q}(x \mid \lambda) = d^n \frac{[2]_q}{[2]_q^d} \sum_{a=0}^{d-1} (-q)^a E_{n,q^d} \left( \left. \frac{a + x}{d} \right\rvert \frac{\lambda}{d} \right).
\]

Now, we consider the degenerate \( q \)-Euler polynomials of the second kind as follows:
\[
\hat{E}_{n,q}(x \mid \lambda) = \int_{\mathbb{Z}_p} (- (x_1 + x) \mid \lambda)_n \, d\mu_{-q}(x_1) \quad (n \geq 0).
\]

From (2.26), we note that
\[
\sum_{n=0}^{\infty} \hat{E}_{n,q}(x \mid \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \lambda^n \int_{\mathbb{Z}_p} \left( \frac{x_1 + x}{n} \right) d\mu_{-q}(x_1) t^n.
\]
\[(1 + \lambda t)^{-x/\lambda} \int_{\mathbb{Z}_p} (1 + \lambda t)^{-x_1/\lambda} d\mu_q(x_1)
= \frac{[2]_q}{(1 + \lambda t)^{1/\lambda} + q} (1 + \lambda t)^{(1-x)/\lambda}. \quad (2.27)\]

When \( x = 0 \), \( \hat{E}_{n,q}(\lambda) = \hat{E}_{n,q}(0 \mid \lambda) \) are called the degenerate \( q \)-Euler numbers of the second kind.

By (2.26), we get

\[\hat{E}_{n,q}(x \mid \lambda)
= \lambda^n \int_{\mathbb{Z}_p} \left( -\frac{x_1 + x}{\lambda} \right)_n d\mu_q(x)
= \lambda^n \sum_{l=0}^{n} S_1(n, l) \frac{(-1)^l}{\lambda^l} \int_{\mathbb{Z}_p} (x_1 + x)^l d\mu_q(x)
= \sum_{l=0}^{n} S_1(n, l) \lambda^{n-l} (-1)^l E_{l,q}(x). \quad (2.28)\]

Thus, from (2.28), we have

\[(-1)^n \hat{E}_{n,q}(x \mid \lambda)
= \sum_{l=0}^{n} (-1)^{n-l} S_1(n, l) \lambda^{n-l} E_{l,q}(x)
= \sum_{l=0}^{n} \left| S_1(n, l) \right| \lambda^{n-l} E_{l,q}(x). \quad (2.29)\]

We observe that

\[
\sum_{n=0}^{\infty} E_{n,q}^{-1}(1-x)^n \frac{t^n}{n!}
= \frac{1 + q^{-1}}{q^{-1}e^t + 1} e^{(1-x)t}
= \frac{1 + q}{qe^t + 1} e^{-xt}
= \frac{[2]_q}{qe^t + 1} e^{-xt}
= \sum_{n=0}^{\infty} (-1)^n E_{n,q}(x) \frac{t^n}{n!}. \quad (2.30)
\]

From (2.30), we have

\[E_{n,q}^{-1}(1-x) = (-1)^n E_{n,q}(x) \quad (n \geq 0). \quad (2.31)\]

By replacing \( t \) by \( e^{x_1-1} \) in (2.27), we get

\[
\sum_{n=0}^{\infty} \hat{E}_{n,q}(x \mid \lambda) \frac{1}{n!} \lambda^n (e^{x_1} - 1)^n
= \frac{1 + q}{e^t + q} e^{(1-x)t}
\]
\[
\sum_{n=0}^{\infty} E_{n,q}^{-1}(1-x) \frac{t^n}{n!}, \quad (2.32)
\]

On the other hand, we have
\[
\sum_{m=0}^{\infty} \hat{E}_{m,q}(x | \lambda) \frac{1}{m!} \lambda^m (e^{\lambda t} - 1)^m = \sum_{m=0}^{\infty} \left( \sum_{m=0}^{n} \hat{E}_{m,q}(x | \lambda) S_2(n, m) \lambda^{n-m} \right) \frac{t^n}{n!}, \tag{2.33}
\]

From (2.32) and (2.33), we note that
\[
(-1)^n E_{n,q}^{-1}(x) = \sum_{m=0}^{n} \hat{E}_{m,q}(x | \lambda) S_2(n, m) \lambda^{n-m} \tag{2.34}
\]

Therefore, by (2.29) and (2.34), we obtain the following theorem.

**Theorem 2.7**  
For \( n \geq 0 \), we have
\[
(-1)^n \hat{E}_{n,q}(x | \lambda) = \sum_{l=0}^{n} S_1(n, l) \lambda^{n-l} E_{l,q}(x)
\]

and
\[
(-1)^n E_{n,q}^{-1}(x) = \sum_{l=0}^{n} S_2(n, l) \lambda^{n-l} \hat{E}_{l,q}(x | \lambda).
\]

It is easy to show that
\[
\binom{x + y}{n} = \sum_{l=0}^{n} \binom{x}{l} \binom{y}{n-l} \quad (n \geq 0). \tag{2.35}
\]

From (2.35), we have
\[
\frac{(-1)^n E_{n,q}(\lambda)}{n!} = \frac{(-1)^n}{n!} \int_{\mathbb{Z}_p} (x_1 | \lambda)_n d\mu_{-q}(x_1)
\]
\[
= \lambda^n \int_{\mathbb{Z}_p} \left( \frac{-\lambda}{ \kappa + n - 1} \right)^n d\mu_{-q}(x_1)
\]
\[
= \lambda^n \sum_{l=0}^{n} \binom{n-1}{n-l} \int_{\mathbb{Z}_p} \left( \frac{-\lambda}{ \kappa + l} \right)^n d\mu_{-q}(x_1)
\]
\[ \lambda^{n} \sum_{l=1}^{n} \binom{n-1}{l-1} \frac{1}{l!} \int_{\mathbb{Z}_p} (-x_1 | \lambda) \, d\mu_{\lambda q}(x_1) \]

\[ = \sum_{l=1}^{n} \binom{n-1}{l-1} \lambda^{n-l} \frac{1}{l!} \mathcal{E}_{l,q}(\lambda) \]  

(2.36)

and

\[ \frac{(-1)^n}{n!} \mathcal{E}_{n,q}(\lambda) = \sum_{l=1}^{n} \binom{n-1}{l-1} \lambda^{n-l} \frac{1}{l!} \mathcal{E}_{l,q}(\lambda). \]  

(2.37)

**Competing interests**
The authors declare that they have no competing interests.

**Authors’ contributions**
All authors contributed equally to this work. All authors read and approved the final manuscript.

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