Improved Gradient Projection Algorithm for Deblurred Image Application

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Abstract. For the purpose of reducing the noise of deblurred image, an improved Gradient Projection method by sparse coding Reconstruction (GPSR) algorithm is proposed. Different with the traditional Gradient Projection method for sparse reconstruction code (GPSR) method, the obvious distinctions between the original image and deblurred image are projected onto dimensional projection image value. Thus the picture noise are deleted. The classical Gradient Sparse Projection computing method, searching direction changes time by time. Now we propose the new method by running with fix step. The constraints of searching method is guarantee. We prove the new computing method in theory, and at the same time, we running the method by classical picture. We compare the running results of classic method and the fix step hunt method (FXHM), the running results of two methods are shown in the article, the performance of fix step hunt method (FXHM) is higher than the classic method as possible as it can.

1. Introduction
Based on the Nyguist sampling theory, when the traditional image collection system is worked, the higher image data are produced because of the higher resolution [1, 2]. At the same time, the hardware system and technology are difficult to meet the data requirement [3~5]. The Compressed Sensing (CS) theory shows that, at the time of the signal is sparse or can be compressed, the Nyquist sampling theory can works effectively [6].

The Compressed Sensing (CS) theory is consist of three parts: the first one is the sparse base signal values, the second part is the observation matrix and the observation values, and the last part is the construction method of the signals. The sign of sparse signal is a researching problem. The best searching path or the computing method is what we research. Now, we introduce the Compressed Sensing (CS) theory, For the purpose of lowest sampling scale, instead of a highest sample scales of the compression method. In the CS theory, the signal or the image which are input into the computing method, the data can be sample in less rate and frequency, compared with the Nyquist sample scale, and at the same time the condense data can be restore into the original data of signals by the computing method.

The classic Gradient Projection method search path changes the time by time, in this way the computing time is wasting, the search speed is long. Now we propose the new fix step hunt method (FXHM), the computing time is reduced, the search speed is higher than the classic computing method. The constraints of searching method is guarantee. We prove the new computing method by theory, and at the same time, we running the method by classic image. We compare the running results of classic...
method and the fix step hunt method (FXHM), the running results of two methods are shown in the article, the performance of fix step hunt method (FXHM) is higher than the classic method.

2. Step of Classical CS Method
The classic CS method are run as follow steps.
Firstly, the input signals (data or image) will be sparse. If the signals (data or image) are compact, this classic CS method will not worked. After that the signals are represented by formula or transform.
The signals (data or image) are sign by the signals $s \in R^N$, the array or the matrix is the $W$. The signals (data or image) can be shown in the following formula (1):

$$s = Wx$$  \hspace{1cm} (1)

The symbol of input signals (data or image) are shown in expression (2):

$$\|x\|_0 \leq K$$ \hspace{1cm} (2)

Here the non-zero matrix of $\|x\|_0$ symbol is expression as $l_0$ symbol, now we change the formula (2) into expression (3)

$$y = \Phi x = \Phi W x = Ax$$ \hspace{1cm} (3)

Here, the target vector $s$ is the $s \in R^M$, we know the input signals (data or image) are sparse signals, if the signals (data or image) are compact, the expression (4) shows the computing method and searching step.

$$\text{minimum} \|x\|_0 \hspace{0.5cm} s.t. y = Ax$$ \hspace{1cm} (4)

After this step, we get the coefficient vectors, we can restore the input signals (data or image) $s$ by the following $s = Wx$ formula.

We know that the formula (4) is a NP-hard problem, the NP problem will not be run in limited time. We will resolve the problem by line program method. The crucial reason is that the signals must be sparse, thus we can use the wavelet transform method change the signals. And we will construct the observation matrix. Then we will project the signals (data or picture) onto low dimensional space.

There are two key factor in CS Theory.
First factor is that the signals are not sparse in most conditions, we will use some wavelet transform or Fourier transform method, thus the transformed signals can satisfied the sparse.
Second factor is that the observation matrix will satisfied some conditions, thus the information will not loss or loss a little, and the signals (data or picture) can be restored by the reconstruction method.

The running process the CS theory is that the signals (data or picture) are projected onto the lower dimensional space, the observation matrix will reduce the dimension of the signals (data or picture) and without the information loss.

3. Improved Fixed Step Hunt Method (FXHM)
The original input signals (data or image) are $s \in R^{m\times n}$ symbol, the random projection process is follows. $y = \Phi s$, $\Phi \in R^{m\times n}$ is the random projection matrix, where $s \in R^{m\times n}$.

The running process of compression sensing reconstruction is that the sparse of image translation, we get the original signals (data or image) form m symbol.

Now the sparse matrix is shown by $l_0$ symbol. It is difficult to get $l_0$ symbol directly, thus we can get $l_1$ symbol firstly.

For the two dimension image inputs signals, we propose new method and signs show the compression sensing method. We show the expression (4) with following formula (5):
\[ \hat{\alpha} = \arg \min_{\alpha} \frac{\|y - \Phi s\|^2}{2} + \tau \|\alpha\| \]  
\[ (5) \]

In formula (5), the \( \tau \) symbol is a positive or zero value, because \( \alpha = \Psi^T s \), now we put \( s = \Psi^T \alpha \) in expression (5), we get the expression (6) as followings:

\[ \hat{\alpha} = \arg \min_{\alpha} \frac{\|y - \Phi \Psi^T \alpha\|^2}{2} + \tau \|\alpha\| \]  
\[ (6) \]

Now in formula (6), we replace the signals \( A = \Phi \Psi^T, x = \alpha \) symbol, then we show the expression (6) as formula (7):

\[ \hat{x} = \arg \min_{x} \frac{\|y - Ax\|^2}{2} + \tau \|x\| \]  
\[ (7) \]

In classic GPRS method, we show the expression (7) with formula (8),

\[ \min_{u,v} \frac{\|y - A(u - vv)\|^2}{2} + \tau l_n^x uu + \tau l_n^x vv \]  
\[ s.t. uu \geq 0, vv \geq 0 \]  
\[ (8) \]

Now, in expression (8), signal \( uu \) and signal \( vv \) are the negative (minus) and positive (right) part of signal \( xx \):

\[ xx = uu - vv, uu \geq 0, vv \geq 0 \]  
\[ (9) \]

In two-times constraint form, the formula (8) is shown as formula (10):

\[ \min_{u,v} c^T z + \frac{1}{2} z^T B z = F(z) \]  
\[ (10) \]

Now in, \[ z = \begin{bmatrix} uu \\ vv \end{bmatrix}, b = A^T y, c = \tau l_{2n}, B = \begin{bmatrix} A^T A, -A^T A \\ -A^T A, A^T A \end{bmatrix} \], signal \( uu \) and signal \( vv \) are the negative (minus) and positive (right) parts of the matrix \( xx \), and \( xx = uu - vv, uu \geq 0, vv \geq 0 \).

The gradient of \( F(z) \) is express as followings:

\[ \nabla F(z) = c + B z \]  
\[ (11) \]

The step of GPSR is that:

\[ w^{(kk)} = (z^{(kk)} - \alpha^{(kk)}) \nabla f(z^{(kk)}) \]  
\[ (12) \]

\[ z^{(kk+1)} = z^{(kk)} - \chi^{(kk)} (w^{(kk)} - z^{(kk)}) \]  
\[ (13) \]

Followings are the steps of GPRS, the first step is computing the set \( z^{(0)} \) symbol, where

\[ \beta \in (0,1), \mu \in (0,1/2), kk = 0 \]  
\[ (14) \]

Next step is that we will compute \( \alpha_0 \) symbol with the following formula (15):

\[ \alpha_0 = \frac{(g^{(kk)})^T g^{(kk)}}{(g^{(kk)})^T B g^{(kk)}} \]  
\[ (15) \]
The running step is computing the $\chi^{(kk)}$ time by time with the formula (16)

$$\delta^{(kk)} = (z^{(kk)}) - \Delta f(z^{(kk)}) + \alpha^{(kk-1)} \delta^{(kk-1)} - z^{(kk)}$$  \hspace{1cm} (16)

In formula (16), the $z^{(kk+1)}$ symbol is the result of computing method, $z^{(kk+1)}$ is the reconstructed image (signals or data). Assumed that, we don’t get the $z^{(kk+1)}$ symbol, we will hunt $\chi^{(kk)}$ continue.

In classic GPSR, the researching orientation is alternately between negative (minus) and positive (right) orientation. In this way the computing time is wasting, the search speed is long. Now we propose the new fix step hunt method (FXHM), the computing time is reduced and the search speed is higher than the classic computing method.

We propose the support set is $\Psi_{kk}$ in $kk$ step, and the reconstruction vertex is $H_{kk}$ symbol, and the signal difference is $r_{kk}$ symbol, thus the target function is formula (7).

We compute the partial derivative of formula (7), then we get the formula (17).

$$d_{kk} = \psi_{kk}^T(y - \hat{H}_{kk})/2 - \tau d_{kk} ||\hat{H}_{kk}||/2 - \tau w$$  \hspace{1cm} (17)

Where $w = \begin{cases} 1 & \text{where } (\hat{H}_{kk} > 0) \\ -1 & \text{where } (\hat{H}_{kk} < 0) \end{cases}$, the formula (15) is update to formula (18):

$$\alpha_0 = \frac{\langle (g_{kk}^{(kk)})^T g_{kk}^{(kk)} \rangle}{\langle (g_{kk}^{(kk)})^T B g_{kk}^{(kk)} \rangle} = \frac{\|g_{kk}^{(kk)}\|^2}{\|\Psi_{kk} g_{kk}^{(kk)}\|^2}$$  \hspace{1cm} (18)

4. Image Running Results and Analysis Method

The image construction results will be estimate by the PSNR, we will get the image MSN by following formula (19). We assume that the picture size is $MM \times NN$.

$$\text{Mse} = \sum_{MM \times NN} [I_1(mm,nn) - I_2(mm,nn)]^2/MM \times NN$$  \hspace{1cm} (19)

We will get the PSNR (Peak Signal to Noise Ratio) results by formula (20).

$$\text{Psnr} = 10 \log_{10}(\frac{\text{MAX}_1^2}{\text{MSE}}) = 20 \log_{10}(\frac{\text{MAX}_1}{\sqrt{\text{MSE}}})$$  \hspace{1cm} (20)

We run both the classic GPSR method and the fix step hunt method (FXHM) with two classic testing image BRIDGE and BOAT picture.
In Fig.1, the figure (a) is the original BRIDGE picture, figure (b) is the blurred classical BRIDGE picture, the blurred ratio is variance four, figure (c) is the running results with fix step hunt method (FXHM), figure (d) is the running results of traditional MALLAT method. The PSNR of fix step hunt method (FXHM) is 16.74, and the PSNR of ALLAT method PSNR is 15.34. The fix step hunt method (FXHM) improves the PSNR 1.4 points and improves the picture quality.
In Fig. 2, the figure (a) is the classical BOAT picture, figure (b) is the blurred classical BOAT picture, the blurred ratio is variance four, figure (c) is the running results with fix step hunt method (FXHM), figure (d) is the running results of traditional MALLAT method. The PSNR of fix step hunt method (FXHM) is 23.48 points, and the PSNR of ALLAT method PSNR is 21.75 points. The fix step hunt method (FXHM) improves the PSNR 1.73 points and improves the picture quality.

5. Conclusions
The classical Gradient Projection method searching path changes time by time at each step, in this way the computing time is wasting, the search speed is long. Now we propose the new fix step hunt method (FXHM), the computing time is reduced, and the search speed is higher than the classic CS theory computing method. The constraints of searching method must be guarantee. We prove the new fix step hunt method (FXHM) in theory, and at the same time, we running the method by classical picture. We compare the running results of classical method and the fix step hunt method (FXHM), the running results of two methods are shown in the article, the performance of fix step hunt method (FXHM) is higher than the classic CS theory computing method as possible as it can.

6. References
[1] Candès E J, Romberg J, Tao T. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information [J]. IEEE Trans on Information Theory, 2006, 52 (2): 489-510.
[2] DONOHO. Compressed Sensing [J]. IEEE Transactions on Information Theory, 2006, 52: 1289-1307.
[3] TROPP J A. Computational methods for sparse solution of linear inverse problems[C] Proceedings of the 2009 IEEE Special Issue on Applications of Sparse Representation and Compressive Sensing, 2010, 98(6): 948-958.

[4] Riofrio C A, Gross D, Flammia S T, et al. Experimental quantum compressed sensing for a seven-qubit system [J]. Nature Communications, 2017, 8.

[5] Birgin E G, Martinez J M, Raydan M. Spectral projected gradient methods: review and perspectives [J]. J. Stat. Softw., 2014, 60 (3): 539-559.

[6] DONOHO D L, TSAIG Y, DRORI I, et al. Sparse solution of underdetermined systems of linear equations by stage wise orthogonal matching pursuit [J]. IEEE Transactions on Information Theory, 2012, 58(2): 1094-1121.