Generating Functions with $\tau$-Invariance and Vertex Representations of Quantum Affine Algebras $U_{r,s}(\hat{g})$
(I): Simply-laced Cases

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Abstract. We put forward the exact version of two-parameter generating functions with $\tau$-invariance, which allows us to give a unified and inherent definition for the Drinfeld realization of two-parameter quantum affine algebras for all the untwisted types. As verification, we first construct their level-one vertex representations of $U_{r,s}(\hat{g})$ for simply-laced types, which in turn well-detect the effectiveness of our definitions both for $(r,s)$-generating functions and $(r,s)$-Drinfeld realization in the framework of establishing the two-parameter vertex representation theory.

1. Introduction

In 2000, Benkart and Witherspoon revitalized the research of two-parameter quantum groups. They studied the structures of two-parameter quantum groups $U_{r,s}(g)$ for $g = gl_n$, or $sl_n$ in [BW1] previously obtained by Takeuchi [T], and the finite-dimensional representations and Schur-Weyl duality for type $A$ in [BW2], and obtained some new finite-dimensional pointed Hopf algebras $u_{r,s}(sl_n)$ in [BW3], which possess new ribbon elements under some conditions (may yield new invariants of 3-manifolds). Since 2004, Bergeron, Gao and Hu [BCH1] further presented the structures of two-parameter quantum groups $U_{r,s}(g)$ for $g = so_{2n+1}$, $sp_{2n}$, $so_{2n}$, and investigated the environment condition on the Lusztig symmetries’ existence for types $A$, $B$, $C$, $D$ in [BCH1] and type $G_2$ in [HS]. A generalization of this fact to the multi-parameter cases arising from Drinfeld doubles of bosonizations of Nichols algebras of diagonal type has been obtained by Heckenberger [H], which provides an explicit realization model for the abstract concept “Weyl groupoid” playing a key role in the classification of both Nichols algebras of diagonal type (cf. [H1]) and finite-dimensional pointed Hopf algebras with abelian group algebras as the coradicals [AS]. The finite-dimensional weight representation theory for type $A$ ([BW2]), types $B$, $C$, $D$ ([BCH2]), $E$ ([BH]) was done.

A unified definition for any types and the explicit formulae of two-parameter quantum groups as two kind of 2-cocycle deformations of one-parameter quantum groups.
groups with double group-like elements in generic case, and a general treatment for the deformed finite-dimensional representation category have been intrinsically described by Hu-Pei \cite{HP1, HP2}. (Furthermore, for a multiparameter version, see Pei-Hu-Rosso \cite{PHR}). Recently, this important observation on the explicit formula for 2-cocycle deformation serves as a crucial point for categorifying two-parameter quantum groups in generic case. In roots of unity case, the small quantum groups structure $u_{r,s}(\mathfrak{g})$, together with the convex PBW-type Lyndon bases was studied explicitly for types $B$, $G_2$ \cite{HW1, HW2}, $C$ \cite{CH}, $F_4$ \cite{CxH}. Especially, Isomorphism Theorem for $u_{r,s}(\mathfrak{g})$'s depending on parameter-pairs $(r,s)$ established by Hu-Wang \cite{HW1, HW2} was used to distinguish the iso-classes of new finite-dimensional pointed Hopf algebras in type $A$ for small order of roots of unity (see Benkart at al \cite{BPW}). Surprisingly, the study of two-parameter small quantum groups brings us new examples of non-semisimple and non-pointed Hopf algebras with non-pointed duals when García \cite{G} studied the quantum subgroups of two-parameter quantum general linear group $GL_{\alpha,\beta}(n)$.

On the other hand, Hu-Rosso-Zhang first studied two-parameter quantum affine algebras associated to affine Lie algebras of type $A_{n-1}^{(1)}$, and gave the descriptions of the structure and Drinfeld realization of $U_{r,s}(\hat{\mathfrak{g}})$ (see \cite{HRZ}). The discussions for the other affine cases of untwisted types and the corresponding vertex operators constructions for all untwisted types have been done in \cite{HZ1, HZ2, Z}. Recently, using a combinatorial model of Young diagrams, Jing-Zhang \cite{JZ1} gave a fermionic realization of the two-parameter quantum affine algebra of type $A_{n-1}^{(1)}$; while \cite{JZ2} provided a group-theoretic realization of two-parameter quantum toroidal algebras using finite subgroups of $SL_2(\mathbb{C})$ via McKay correspondence.

In the present paper, we will study two-parameter quantum affine algebras for untwisted types from a uniform approach via working out the exact two-parameter version of generating functions with $\tau$-invariance, and construct the level-one vertex operator representations for simply-laced types, for the other types will be discussed in forthcoming work.

The paper is organized as follows. We first give the definition of two-parameter quantum affine algebras $U_{r,s}(\hat{\mathfrak{g}})$ ($\mathfrak{g}$ is of $ADE$-type) in the sense of Hopf algebra in Section 2. Two-parameter quantum affine algebras $U_{r,s}(\hat{\mathfrak{g}})$ are characterized as Drinfeld doubles $D(\hat{\mathfrak{B}}, \hat{\mathfrak{B}}')$ of Hopf subalgebras $\hat{\mathfrak{B}}$ and $\hat{\mathfrak{B}}'$ with respect to a skew-dual pairing we give. In Section 3, we obtain the two-parameter Drinfeld realization via the generating functions we define in the two-parameter setting. It is worthy to mention that a similar 2-cocycle deformation between two-parameter and one-parameter quantum affine algebras doesn’t work yet for Drinfeld generators, even though it indeed exists when one works for Chevalley-Kac-Lusztig generators (see \cite{HP2, PHR}). From this point of view, there’s the rub that explicit formulae for defining the Drinfeld realization in the two-parameter setting are nontrivial. Note that by comparison with \cite{HRZ}, the definition has been slightly revised, where the canonical central element $c$ of affine Lie algebras involved plays a well-connected role in the definition such as a result of the product for the doubled group-like elements $\gamma$ and $\gamma'$ keeps group-like (see Definitions 2.2 (X1), & 3.1 (3.2)), this also behaves well when one considers especially those vertex representations of higher levels. While in the case $r = q = s^{-1}$, this phenomenon degenerates and is invisible in one-parameter case. In order to recover those inherent features in
the two-parameter setting, to some extent, representation theory serves as a nice sample here to help us to achieve some further necessary insights into the algebra structure itself. In Section 4, we prove the Drinfeld Isomorphism from the two-parameter quantum affine algebra $U_{r,s}(\hat{g})$ (g is of type $D$ and $E$) of Drinfeld-Jimbo type towards the two-parameter Drinfeld realization $U_{r,s}(\hat{g})$ using the quantum calculations for $(r,s)$-brackets as developed in [HRZ] for type $A_n^{(1)}$. Here we afford an alternative proof of the homomorphism to be injective. In Section 5, we start from the two-parameter quantum Heisenberg algebra to obtain the Fock space, and construct the level-one vertex representations of two-parameter quantum affine algebras for simply-laced cases, which are irreducible. Also, these constructions in turn well-detect the effectiveness of the $(r,s)$-Drinfeld realization we defined. We include details of some proofs of a few Lemmas as an appendix, through which readers can see the quantum calculations for $(r,s)$-brackets how to work effectively in the two-parameter setting.

2. Quantum affine algebras $U_{r,s}(\hat{g})$ and Drinfeld double

2.1. Notations and preliminaries. From now on, denote by $g$ the finite-dimensional simple Lie algebras of simply-laced types with rank $n$. Let $K \supset \mathbb{Q}(r,s)$ denote an algebraically closed field, where the two-parameters $r, s$ are nonzero complex numbers satisfying $r^2 \neq s^2$. Let $E$ be a Euclidean space $\mathbb{R}^n$ with an inner product $(\cdot, \cdot)$ and an orthonormal basis $\{\epsilon_1, \ldots, \epsilon_n\}$.

Let $(a_{ij})_{ij \in I}$ ($I = \{1, 2, \ldots, n\}$) be a Cartan matrix of simple Lie algebra $g$ with Cartan subalgebra $h$. Let $\Phi$ be a root system of $g$ and $\alpha_i$ ($i \in I$) be the simple roots. It is possible to regard $\Phi$ as a subset of a Euclidean space $E$. Denote by $\theta$ the highest root of $\Phi$.

Let $\hat{g}$ be an affine Lie algebra associated to simple Lie algebra $g$ with Cartan matrix $(a_{ij})_{ij \in I_0}$, where $I_0 = \{0\} \cup I$. In the following, we list the affine Dynkin diagrams of simply-laced types, the labels on vertices fix an identification between $I_0$ and $\{0, 1, \ldots, n\}$ such that $I$ corresponds to $\{1, 2, \ldots, n\}$.

![Diagram of $A_n^{(1)}$ and $D_n^{(1)}$ affine Dynkin diagrams](attachment:diagram.png)
Let $\delta$ denote the primitive imaginary root of $\hat{g}$, Take $\alpha_0 = \delta - \theta$, then $\Pi' = \{\alpha_i \mid i \in I_0\}$ is a base of simple roots of affine Lie algebra $\hat{g}$. Denote by $c$ the canonical central element, and $h$ the Coxeter number of affine Lie algebra $\hat{g}$. We need the following data on (prime) root systems.

**Type $A_{n-1}$:**
\[
\Pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n\},
\Psi = \{\pm(\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq n + 1\},
\theta = \alpha_1 + \cdots + \alpha_n,
\alpha_0 = \delta - \theta = \delta - \epsilon_1 + \epsilon_n,
\Pi' = \{\alpha_0, \alpha_1, \cdots, \alpha_n\}.
\]

**Type $D_n$:**
\[
\Pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i < n\} \cup \{\alpha_n = \epsilon_{n-1} + \epsilon_n\},
\Psi = \{\pm \epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq n\},
\theta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n,
\alpha_0 = \delta - \theta = \delta - \epsilon_1 - \epsilon_2,
\Pi' = \{\alpha_0, \alpha_1, \cdots, \alpha_n\}.
\]

**Type $E_6$:**
\[
\Pi = \{\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8 - (\epsilon_2 + \cdots + \epsilon_7)), \alpha_2 = \epsilon_1 + \epsilon_2, \\
\alpha_3 = \epsilon_2 - \epsilon_1, \alpha_4 = \epsilon_3 - \epsilon_2, \alpha_5 = \epsilon_4 - \epsilon_3, \alpha_6 = \epsilon_5 - \epsilon_4\},
\Psi = \{\pm \epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq n = 5\} \cup \left\{\frac{1}{2} \sum_{i=1}^{5} (-1)^{k(i)} \epsilon_i - \frac{1}{2} (\epsilon_6 + \epsilon_7 - \epsilon_8) \mid k(i) = 0, 1, \text{ add up to an odd integer}\right\},
\theta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6,
\alpha_0 = \delta - \theta = \delta - \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8),
\Pi' = \{\alpha_0, \alpha_1, \cdots, \alpha_6\}.
\]

### 2.2. Two-parameter quantum affine algebras $U_{r,s}(\hat{g})$

In this paragraph, we give the definition of the two-parameter quantum affine algebras $U_{r,s}(\hat{g})$ (see [HRZ] for $\hat{g} = A_n^{(1)}$).

Assigned to $\Pi'$, there are two sets of mutually-commutative symbols $W = \{\omega_i^{r+1} \mid 0 \leq i \leq n\}$ and $W' = \{\omega_i^{r+1} \mid 0 \leq i \leq n\}$. Define a pairing $\langle . , . \rangle$:
$W' \times W \rightarrow \mathbb{K}$ as follows

$$(1_{A_{n-1}}) \quad J = (\langle \omega'_i, \omega_j \rangle) = (i, j) \quad \text{for } A^{(1)}_{n-1},$$

where $J = \begin{pmatrix} rs^{-1} & r^{-1} & 1 & \cdots & 1 & s \\ s & rs^{-1} & r^{-1} & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & rs^{-1} & r^{-1} \\ r^{-1} & 1 & 1 & \cdots & s & rs^{-1} \end{pmatrix}$.

$$(1_{D_n}) \quad J = (\langle \omega'_i, \omega_j \rangle) = (i, j) \quad \text{for } D^{(1)}_{n},$$

where $J = \begin{pmatrix} rs^{-1} & (rs)^{-1} & r^{-1} & \cdots & 1 & (rs)^2 \\ rs & rs^{-1} & r^{-1} & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & rs^{-1} & (rs)^{-1} \\ (rs)^{-2} & 1 & 1 & \cdots & rs & rs^{-1} \end{pmatrix}$.

$$(1_{E_6}) \quad J = (\langle \omega'_i, \omega_j \rangle) = (i, j) \quad \text{for } E^{(1)}_{6},$$

where $J = \begin{pmatrix} rs^{-1} & (rs)^{-1} & r^{-2}s^{-1} & (rs)^{-1} & rs & rs \\ rs & rs^{-1} & 1 & r^{-1} & 1 & 1 \\ rs^2 & 1 & rs^{-1} & 1 & r^{-1} & 1 \\ rs & s & 1 & rs^{-1} & r^{-1} & 1 \\ (rs)^{-1} & 1 & s & s & rs^{-1} & r^{-1} \\ (rs)^{-1} & 1 & 1 & 1 & s & rs^{-1} \end{pmatrix}$.

(2) $\langle \omega'_i, \omega_j \rangle = \langle \omega'^{\mp 1}_i, \omega_j \rangle^{-1} = \langle \omega'_i, \omega_j \rangle^\mp 1$, for any $\hat{\mathfrak{g}}$.

**Remark 2.1.** The above structure constant matrix $J$ is said the two-parameter quantum affine Cartan matrix, which is the generalization of the classical quantum affine Cartan matrix under the condition $r = s^{-1} = q$.

**Definition 2.2.** Let $U_{r,s}(\hat{\mathfrak{g}})$ be the unital associative algebra over $\mathbb{K}$ generated by the elements $e_i, f_j, \omega^\pm_1, \omega^\pm_j (j \in I_0), \gamma^\pm_1, \gamma^\pm_2, D^\pm_1, D^\pm_2$, satisfying the following relations (where $c$ is the canonical central element of $\hat{\mathfrak{g}}$):

(X1) $\gamma^\pm_1, \gamma^\pm_2$ are central with $\gamma = \omega^\pm_1, \gamma' = \omega^\pm_2, \gamma \gamma' = (rs)^c$, such that $\omega_i \omega_i^{-1} = \omega'_i \omega^{-1}_i = 1 = DD^{-1} = D'D^{-1}$, and

$$[\omega^\pm_1, \omega^\pm_j] = [\omega^\pm_j, D^\pm_1] = [\omega'^\pm_1, D'^\pm_1] = \omega'^\pm_1, D'^\pm_1 = 0,$$

$$[\omega^\pm_1, \omega'^\pm_j] = [\omega'^\pm_1, D'^\pm_1] = [D'^\pm_1, D'^\pm_1] = [\omega'^\pm_1, \omega'^\pm_j].$$

(X2) For $i, j \in I_0$,

$$D e_i D^{-1} = r^\delta_{ij} e_i,$$

$$\omega_j e_i \omega_j^{-1} = \langle \omega'_i, \omega_j \rangle e_i,$$

$$D f_i D^{-1} = r^{-\delta_{ij}} f_i,$$

$$\omega_j f_i \omega_j^{-1} = \langle \omega'_i, \omega_j \rangle^{-1} f_i.$$

(X3) For $i, j \in I_0$,

$$D' e_i D'^{-1} = s^\delta_{ij} e_i,$$

$$\omega'_j e_i \omega'_j^{-1} = \langle \omega'_i, \omega'_j \rangle^{-1} e_i,$$

$$D' f_i D'^{-1} = s^{-\delta_{ij}} f_i,$$

$$\omega'_j f_i \omega'_j^{-1} = \langle \omega'_i, \omega'_j \rangle f_i.$$
(X4) For \( i, j \in I_0 \), we have
\[
[e_i, f_j] = \frac{\delta_{ij}}{r-s} (\omega_i - \omega'_j).
\]

(X5) For any \( i \neq j \), we have the \((r,s)\)-Serre relations
\[
(ad_i e_i)^{1-a_{ij}} (e_j) = 0,
\]
\[
(ad_r f_i)^{1-a_{ij}} (f_j) = 0,
\]
where the definitions of the left-adjoint action \( ad_i e_i \) and the right-adjoint action \( ad_r f_i \) are given in the following sense:
\[
ad_i a (b) = \sum_{(a)} a_{(1)} b S(a_{(2)}), \quad ad_r a (b) = \sum_{(a)} S(a_{(1)}) b a_{(2)}, \quad \forall a, b \in U_{r,s}(\hat{g}),
\]
where \( \Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \) is given by Proposition 2.3 below.

2.3. Hopf algebra and Drinfeld double. The following is straightforward.

**Proposition 2.3.** The algebra \( U_{r,s}(\hat{g}) \) \((\hat{g} = A_{n-1}^{(1)}, D_n^{(1)} \text{ and } E_6^{(1)})\) is a Hopf algebra under the comultiplication, the counit and the antipode defined below \((0 \leq i \leq n)\)
\[
\Delta(\omega_i^{\pm 1}) = \omega_i^{\mp 1} \otimes \omega_i^{\pm 1}, \quad \Delta(\omega_i') = \omega_i' \otimes \omega_i',
\]
\[
\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega_i',
\]
\[
\varepsilon(\omega_i^{\pm 1}) = \varepsilon(\omega_i') = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0,
\]
\[
S(\omega_i^{\pm 1}) = \omega_i^{-1}, \quad S(\omega_i') = \omega_i'^{-1},
\]
\[
S(e_i) = -\omega_i^{-1} e_i, \quad S(f_i) = -f_i \omega_i'^{-1}.
\]

**Remark 2.4.** When \( r = s^{-1} = q \), Hopf algebra \( U_{r,s}(\hat{g}) \) modulo the Hopf ideal generated by the elements \( \omega_i^{\pm 1} - \omega_i'^{-1} \) \((0 \leq i \leq n)\), is just the quantum group \( U_q(\hat{g}) \) of Drinfel’d-Jimbo type.

**Definition 2.5.** A skew-dual pairing of two Hopf algebras \( \mathcal{A} \) and \( \mathcal{U} \) is a bilinear form \( \langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{A} \rightarrow K \) such that
\[
\langle f, 1_{\mathcal{A}} \rangle = \varepsilon_{\mathcal{U}}(f), \quad \langle 1_{\mathcal{U}}, a \rangle = \varepsilon_{\mathcal{A}}(a),
\]
\[
\langle f, a_1 a_2 \rangle = \langle \Delta_{op}^{\mathcal{U}}(f), a_1 \otimes a_2 \rangle, \quad \langle f_1 f_2, a \rangle = \langle f_1 \otimes f_2, \Delta_{\mathcal{A}}(a) \rangle,
\]
for all \( f, f_1, f_2 \in \mathcal{U} \), and \( a, a_1, a_2 \in \mathcal{A} \), where \( \varepsilon_{\mathcal{U}} \) and \( \varepsilon_{\mathcal{A}} \) denote the counits of \( \mathcal{U} \) and \( \mathcal{A} \), respectively, and \( \Delta_{\mathcal{U}} \) and \( \Delta_{\mathcal{A}} \) are their respective comultiplications.

Let \( \mathcal{B} = \hat{B}(\hat{g}) \) (resp. \( \mathcal{B}' = \hat{B}'(\hat{g}) \)) denote the Hopf subalgebra of \( \hat{U} = U_{r,s}(\hat{g}) \) generated by \( e_j, \omega_j^{\pm 1} \) (resp. \( f_j, \omega_j'^{\pm 1} \)) with \( 0 \leq j \leq n \) for \( \hat{g} = A_{n-1}^{(1)} \), and with \( 0 \leq j \leq n \) for \( \hat{g} = D_n^{(1)} \), respectively. The following result was obtained for the type \( A_{n-1}^{(1)} \) case by [HRZ].
Proposition 2.6. There exists a unique skew-dual pairing \( \langle \cdot, \cdot \rangle^{\ast} : \hat{B}^\circ \times \hat{B}^\circ \rightarrow \mathbb{K} \) of the Hopf subalgebras \( \hat{B} \) and \( \hat{B}^\circ \) in \( U_{r,s}(\hat{g}) \) such that \( \langle f_i, e_j \rangle^{\ast} = \frac{\delta_{ij}}{s_i - r_j} \), and the conditions (1X) and (2) are satisfied, and all other pairs of generators are 0. Moreover, we have \( \langle S(a), S(b) \rangle = (a, b) \) for \( a \in \hat{B}^\circ, b \in \hat{B} \).

Definition 2.7. For any two skew-paired Hopf algebras \( A \) and \( U \) by a skew-dual pairing \( \langle \cdot, \cdot \rangle \), one may form the Drinfeld’s double \( D(A, U) \) as in [KS, 8.2], which is a Hopf algebra whose underlying coalgebra is \( A \otimes U \) with the tensor product coalgebra structure, and whose algebra structure is defined by

\[
(a \otimes f)(a' \otimes f') = \sum \langle S_U(f_{(1)}), a'_{(1)} \rangle \langle f_{(3)}, a'_{(3)} \rangle a_{(2)} \otimes f_{(2)} f',
\]

for \( a, a' \in A \) and \( f, f' \in U \). The antipode \( S \) is given by

\[
S(a \otimes f) = (1 \otimes S_U(f))(S_A(a) \otimes 1).
\]

Clearly, both mappings \( A \ni a \mapsto a \otimes 1 \in D(A, U) \) and \( U \ni f \mapsto 1 \otimes f \in D(A, U) \) are injective Hopf algebra homomorphisms. Let us denote the image \( a \otimes 1 \) (resp. \( 1 \otimes f \)) of \( a \) (resp. \( f \)) in \( D(A, U) \) by \( \hat{a} \) (resp. \( \hat{f} \)). By (3), we have the following cross commutation relations between elements \( \hat{a} \) (for \( a \in A \)) and \( \hat{f} \) (for \( f \in U \)) in the algebra \( D(A, U) \):

\[
\hat{f} \hat{a} = \sum \langle S_U(f_{(1)}), a_{(1)} \rangle \langle f_{(3)}, a_{(3)} \rangle a_{(2)} \hat{f}_{(2)},
\]

\[
\sum \langle f_{(1)}, a_{(1)} \rangle \hat{f}_{(2)} a_{(2)} = \sum a_{(1)} \hat{f}_{(1)} \langle f_{(2)}, a_{(2)} \rangle.
\]

In fact, as an algebra the double \( D(A, U) \) is the universal algebra generated by the algebras \( A \) and \( U \) with cross relations (4) or, equivalently, (5).

Theorem 2.8. The two-parameter quantum affine algebra \( U = U_{r,s}(\hat{g}) \) is isomorphic to the Drinfeld’s quantum double \( D(\hat{B}, \hat{B}^\circ) \).

2.4. Triangular decomposition of \( U_{r,s}(\hat{g}) \). Let \( U_0 = \mathbb{K}[\omega_0^{\pm 1}, \cdots, \omega_n^{\pm 1}], U' = \mathbb{K}[\omega_0^{\pm 1}, \cdots, \omega_n^{\pm 1}, \sigma] \), and \( U_0' = \mathbb{K}[\omega_0^{\pm 1}, \cdots, \omega'_n, \sigma] \) denote the Laurent polynomial subalgebras of \( U_{r,s}(\hat{g}) \), \( \hat{B} \), and \( \hat{B}^\circ \) respectively. Clearly, \( U_0 = U_0' \) and \( \hat{B} \) are generated by \( e_i \) (resp. \( f_i \)) for all \( i \in I_0 \). By definition, \( \hat{B} = U_{r,s}(\hat{g}) \otimes U_0' \), \( \hat{B}^\circ = U_0'^\circ \otimes U_{r,s}(\hat{g}) \), so that the double \( D(\hat{B}, \hat{B}^\circ) \) is generated \( U_{r,s}(\hat{g}) \otimes U_0'^\circ \otimes U_{r,s}(\hat{g}) \), as vector spaces. On the other hand, if we consider \( \langle \cdot, \cdot \rangle^\circ : \hat{B}^\circ \times \hat{B} \rightarrow \mathbb{K} \) by \( \langle b', b \rangle^\circ := \langle S(b'), b \rangle \), the convolution inverse of the skew-dual paring \( \langle \cdot, \cdot \rangle \) in Proposition 2.6, the composition with the flip mapping \( \sigma \) then gives rise to a new skew-dual paring \( \langle | \rangle^\circ : \hat{B}^\circ \times \hat{B} \rightarrow \mathbb{K} \), given by \( \langle b | b' \rangle^\circ = \langle S(b'), b \rangle \). As a byproduct of Theorem 2.8 (see [BGH1, Coro. 2.6]), we get the standard triangular decomposition of \( U_{r,s}(\hat{g}) \).

Corollary 2.9. \( U_{r,s}(\hat{g}) \cong U_{r,s}(\hat{g}) \otimes U_0'^\circ \otimes U_{r,s}(\hat{g}) \), as vector spaces.

Definition 2.10. (Prop. 3.2 [HRZ]) Let \( \tau \) be the \( \mathbb{Q} \)-algebra anti-automorphism of \( U_{r,s}(\hat{g}) \) such that \( \tau(s) = s, \tau(e_i) = e_i, \tau(\omega_i) = \omega'_i, \tau'(\omega'_i) = \omega_i, \tau'(\omega_j) = \gamma, \tau'(\gamma') = \gamma, \tau(D) = D', \tau(D') = D \), and

\[
\tau(e_i) = f_i, \quad \tau(f_i) = e_i, \quad \tau(\omega_i) = \omega'_i, \quad \tau'(\omega'_i) = \omega_i,
\]

\[
\tau(\gamma) = \gamma', \quad \tau'(\gamma') = \gamma, \quad \tau(D) = D', \quad \tau(D') = D.
\]

Then \( \hat{B}^\circ = \tau(\hat{B}) \) with those induced defining relations from \( \hat{B} \), and those cross relations in (X2)–(X4) are antisymmetric with respect to \( \tau \).
3. Drinfeld Realization via Generating Functions with \( \tau \)-invariance

3.1. Generating functions with \( \tau \)-invariance and Drinfeld realization.

In order to obtain the intrinsic definition of Drinfeld realization of the two-parameter quantum affine algebra \( U_{r,s}(\mathfrak{g}) \), we need first to construct the generating functions \( g_{ij}^{\pm}(z) \) \((1 \leq i, j \leq n)\) with \( \tau \)-invariance, which is due to the first author defined as follows (This was initially motivated in part by section 1.1 in [Gr] in the one-parameter setting regardless of \( \tau \)-invariance there).

Let \( \alpha_i, \alpha_j \in \Delta \), we set \( g_{ij}^{\pm}(z) = \sum_{n \in \mathbb{Z}_+} \pm c_{ij}^{(n)} z^n := \sum_{n \in \mathbb{Z}_+} \pm c_{ij}^{(n)} z^n \), a formal power series in \( z \), where the coefficients \( \pm c_{ij}^{(n)} \) are determined from the Taylor series expansion in the variable \( z \) at \( 0 \in \mathbb{C} \) of the function

\[
\sum_{n \in \mathbb{Z}_+} \pm c_{ij}^{(n)} z^n = g_{ij}^{\pm}(z) = \frac{G_{ij}^{\pm}(z, 1)}{F_{ij}^{\pm}(z, 1)},
\]

where we got some observations from the discussions for type \( A_n^{(1)} \) we did in [HRZ] to define \( F_{ij}^{\pm}(z, w) \), \( G_{ij}^{\pm}(z, w) \) as follows.

\[
F_{ij}^{\pm}(z, w) := z - (\langle i, j \rangle (j, i))^{\pm\frac{1}{2}} w, \quad G_{ij}^{\pm}(z, w) := (j, i)^{\pm 1} z - (\langle j, i \rangle (i, j)^{-1})^{\pm\frac{1}{2}} w.
\]

We have a uniform formula for \( g_{ij}^{\pm}(z) \) as below

\[
g_{ij}^{\pm}(z) = \frac{(j, i)^{\pm 1} z - (\langle j, i \rangle (j, i))^{\pm\frac{1}{2}}}{z - (\langle i, j \rangle (j, i))^{\pm\frac{1}{2}}}
\]

\[= ((j, i)(i, j)^{-1})^{\pm\frac{1}{2}} ((\langle i, j \rangle (j, i))^{\pm\frac{1}{2}} z - 1)
\]

(3.1)

Then in both cases (whenever \( i = j \) or \( i \neq j \)), we have a uniform expansion formula for \( g_{ij}^{\pm}(z) = \sum_{k \geq 0} \pm c_{ij}^{(k)} z^k \) with \( \pm c_{ij}^{(0)} = (i, j)^{\mp 1} \) and \( \pm c_{ij}^{(k)} = \pm c_{ij}^{(0)} \frac{(k-1)!}{k!} (\langle i, i \rangle^{\pm \frac{2k}{2}} - (i, i)^{\pm \frac{2k}{2}}) \), for \( k > 0 \).

Define the other generating functions in a formal variable \( z \) (also see [HRZ]):

\[\delta(z) = \sum_{n \in \mathbb{Z}} z^n, \quad x_i^{\pm}(z) = \sum_{k \in \mathbb{Z}} x_i^{\pm}(k) z^{-k},\]

\[\omega_i(z) = \sum_{m \in \mathbb{Z}_+} \omega_i(m) z^{-m} = \omega_i \exp \left( (r-s) \sum_{\ell \geq 1} a_i(\ell) z^{-\ell} \right),\]

\[\omega_i'(z) = \sum_{n \in \mathbb{Z}_+} \omega_i'(m) z^{-m} = \omega_i' \exp \left( (r-s) \sum_{\ell \geq 1} a_i(\ell) z^{\ell} \right).\]

The following property for the generating functions \( g_{ij}^{\pm}(z) \) is rather crucial for deriving the inherent definition of Drinfeld realization in the two-parameter version.

**Proposition 3.1. (The \( \tau \)-invariance of generating functions)** Assume \( \tau(r) = s, \tau(s) = r, \tau((i, j)) = (j, i)^{-1}, \tau(z) = z^{-1}, \tau(x_i^{\pm}(k)) = x_i^{\mp}(-k), \tau(\omega_i(m)) = \omega_i'(-m), \tau(\omega_i'(m)) = \omega_i(m) \), for \( m \in \mathbb{Z}_+ \) with \( \omega_i(0) = \omega_i, \omega_i'(0) = \omega_i' \), then

(i) \( \tau(g_{ij}^{\pm}(z)) = g_{ji}^{\mp}(z) \), and \( g_{ij}^{\pm}(z^{-1}) = g_{ij}^{\mp}(z) \), \( g_{ij}^{\pm}(z) = g_{ij}^{\mp}(z) \).

(ii) \( \tau(\delta(z)) = \delta(z), \tau(x_i^{\pm}(z)) = x_i^{\mp}(z), \tau(\omega_i(z)) = \omega_i'(z) \) and \( \tau(\omega_i'(z)) = \omega_i(z) \).
Now let us formulate the inherent definition of Drinfeld realization for two-parameter quantum affine algebra $U_{r,s}(\hat{g})$ via our generating functions with $\tau$-invariance. Set $r_i = r^{d_i}$, $s_i = s^{d_i}$, where $A = DB$, $D = \text{diag}\{d_0, \cdots, d_n\}$ and $B$ is symmetric.

**Definition 3.2. (Theorem.)** The $(r, s)$-Drinfeld realization $U_{r,s}(\hat{g})$ associated to the two-parameter quantum affine algebra $U_{r,s}(\hat{g})$ is the associative algebra with unit 1 and generators $\{x_i^\pm(k), \omega_i(m), \omega'_i(-n), \gamma^{\pm\frac{1}{2}}, \gamma^{\pm\frac{1}{2}}, D^{\pm1}, D'^{\pm1} \mid i \in I, k \in \mathbb{Z}, m, n \in \mathbb{Z}_+\}$ satisfying the relations below with $\tau$-invariance, written in terms of $\tau$-invariant generating functions of formal variables $z, w$ with $\mathbb{Q}$-anti-involution $\tau$ such that $\tau(\gamma) = \gamma', \tau(\gamma') = \gamma$ (where $c$ is the canonical central element of $\hat{g}$ and $\tau$ is defined as in Proposition 3.1, set $g_{ij}(z) := g_{ij}^e(z)$):

\[(3.2) \quad \gamma^{\pm\frac{1}{2}}, \gamma'^{\pm\frac{1}{2}} \text{ are central and mutually inverse such that } \gamma \gamma' = (rs)^c,\]
\[(3.3) \quad \omega_i(0)^{\pm1}, \omega'_i(0)^{\pm1} \text{ mutually commute, where } \omega_i(0) = \omega_i, \omega'_i(0) = \omega'_i,\]
\[(3.4) \quad \omega_i(z) \omega_j(w) = \omega_j(w) \omega_i(z), \quad \omega'_i(z) \omega'_j(w) = \omega'_j(w) \omega'_i(z),\]
\[(3.5) \quad g_{ij}(zw^{-1}(\gamma \gamma')^{\frac{1}{2}}) \omega_i(z) \omega_j(w) = g_{ij}(zw^{-1}(\gamma \gamma')^{\frac{1}{2}}) \omega'_i(w) \omega'_j(z),\]
\[(3.6) \quad Df(z)D^{-1} = f\left(\frac{z}{w}\right), D'f(z)D'^{-1} = f\left(\frac{z}{s}\right), \text{ for } f(z) = x_i^\pm(z), \omega_i(z), \omega'_i(z),\]
\[(3.7) \quad \omega_i(z)x_j^\pm(w)\omega_i(z)^{-1} = g_{ij}\left(\frac{z}{w}(\gamma \gamma')^{\frac{1}{2}}\right)^{\pm1} x_j^\pm(w),\]
\[(3.8) \quad \omega_i(z)x_j^\pm(w)\omega_i(z)^{-1} = g_{ji}\left(\frac{z}{w}(\gamma \gamma')^{\frac{1}{2}}\right)^{\mp1} x_j^\pm(w),\]
\[(3.9) \quad [x_i^\pm(z), x_j^\pm(w)] = \frac{\delta_{ij}}{r_i - s_i} \left(\delta(zw^{-1} \gamma')\omega_i(w \gamma') - \delta(zw^{-1} \gamma)\omega'_i(z \gamma'^{-\frac{1}{2}})\right),\]
\[(3.10) \quad F_{ij}^\pm(z, w)x_j^\pm(z)x_i^\pm(w) = G_{ij}^\pm(z, w)x_j^\pm(z)x_i^\pm(w),\]
\[(3.11) \quad x_i^\pm(z)x_j^\pm(w) = \delta_{ij}^{-1} x_j^\pm(z)w_i^\pm(z), \text{ for } a_{ij} = 0,\]
\[(3.12) \quad \text{Sym}_{z_1, \cdots, z_n} \sum_{k=0}^{n-1} (-1)^k (r_i s_i)^{\frac{k(k-1)}{2}} \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array}\right] x_i^\pm(z_1) \cdots x_i^\pm(z_k) x_j^\pm(w) \times x_i^\pm(z_{k+1}) \cdots x_i^\pm(z_n) = 0, \text{ for } a_{ij} < 0, 1 \leq i < j < n,\]
\[(3.13) \quad \text{Sym}_{z_1, \cdots, z_n} \sum_{k=0}^{n-1} (-1)^k (r_i s_i)^{\frac{k(k-1)}{2}} \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array}\right] x_i^\pm(z_1) \cdots x_i^\pm(z_k) x_j^\pm(w) \times x_i^\pm(z_{k+1}) \cdots x_i^\pm(z_n) = 0, \text{ for } a_{ij} < 0, 1 \leq j < i < n,\]

where Sym$_{z_1, \cdots, z_n}$ denotes symmetrization w.r.t. the indices $(z_1, \cdots, z_n)$. In particular, $\tau$ keeps each term among the relations (3.2)—(3.6), (3.9)—(3.11); but interchanges the relations between (3.7) and (3.8), the ones between (3.12) and (3.13).

**Remark 3.3.** (1) When $r = q = s^{-1}$, $g_{ij}(z)$ is the same as that of the one-parameter quantum affine algebras (cf. [Gr]).
(2) When $r = q = s^{-1}$, the algebra $U_{q^{-1}}(\hat{g})$ modulo the ideal generated by the set $\{\omega'_i - \omega_i^{-1} (i \in I), \gamma' - \gamma^{-\frac{1}{2}}\}$, is the usual Drinfeld realization $U_q(\hat{g})$. 

(3) Denote $\pm z^{(k)} := \pm c^{(k)} / c^{(0)}$, $t := r-s$. (3.7) is equivalent to the following
\[
\exp(-t \sum_{\ell>0} a_{\ell}(t) z^\ell) \cdot x_j^\pm(w) \cdot \exp(t \sum_{\ell>0} a_{\ell}(t) z^\ell) = \sum_{k \geq 0} \pm z^{(k)} \gamma^\pm \left(\frac{(\gamma')^{\pm \frac{1}{2}} z}{w}\right)^k x_j^\pm(w).
\]
(3.8) is equivalent to the following
\[
\exp\left(t \sum_{\ell>0} a_{\ell}(\ell) z^\ell\right) \cdot x_j^\pm(w) \cdot \exp\left(-t \sum_{\ell>0} a_{\ell}(\ell) z^\ell\right) = \sum_{k \geq 0} \pm z^{(k)} \gamma^\pm \left(\frac{(\gamma')^{\pm \frac{1}{2}} w}{z}\right)^k x_j^\pm(w).
\]
Both expansions give rise to the relations (D61) and (D62) below.

**Definition 3.4. (Equivalent Definition.)** The unital associative algebra $U_{r,s}(\mathfrak{g})$ over $K$ is generated by the elements $x^\pm(k)$, $a_i(t)$, $\omega_i^1$, $\omega_i^2$, $\gamma^\pm$, $\gamma^\pm\delta$, $D^\pm$, $D_i^\pm$, $(i \in I, k, k' \in Z, \ell \in Z \setminus \{0\})$, subject to the following defining relations:

(D1) $\gamma^\pm$, $\gamma^\pm \delta$ are central such that $\gamma \gamma' = (rs)^c$, $\omega_i \omega_i^{-1} = \omega_i' \omega_i'^{-1} = 1$ $(i \in I)$, and for $i, j \in I$, one has
\[
\left[\omega_i^{\pm 1}, \omega_j^{\pm 1}\right] = \left[\omega_i^{\pm 1}, D_i^{\pm 1}\right] = \left[\omega_j^{\pm 1}, \omega_i^{\pm 1}\right] = \left[\omega_j^{\pm 1}, D_i^{\pm 1}\right] = 0
\]
\[
= \left[\omega_i^{\pm 1}, \omega_j^{\mp 1}\right] = \left[\omega_j^{\pm 1}, \omega_i^{\mp 1}\right] = \left[D_i^{\pm 1}, D_i^{\pm 1}\right] = \left[D_i^{\pm 1}, \omega_i^{\pm 1}\right] = \left[D_i^{\pm 1}, \omega_j^{\pm 1}\right].
\]

(D2) $[a_i(t), a_j(t')] = \delta_{t+t',0} \left(\gamma^{\ell+1}(i,j)\frac{\ell_{xj} \ell_{tx} - \ell_{tx} \ell_{jx}}{\ell t (r_i - s_i)}\right) \cdot \gamma_{t+1} - \gamma_{t-1}$. 

(D3) $[a_i(t), \omega_j^{\pm 1}] = [a_i(t), \omega_j^{\mp 1}] = 0$.

(D4) $\left. D x_i^\pm(k) D^{-1} = r^k x_i^\pm(k), \quad D x_i^\pm(k) D^{-1} = s^k x_i^\pm(k), \quad D a_i(t) D^{-1} = r^t a_i(t), \quad D a_i(t) D^{-1} = s^t a_i(t). \right.$

(D5) $\omega_i x_j^\pm(k) \omega_i^{-1} = \langle \omega_j', \omega_i \rangle^{1 \pm 1} x_j^\pm(k)$, $\omega_i' x_j^\pm(k) \omega_i'^{-1} = \langle \omega_j', \omega_i \rangle^{1 \pm 1} x_j^\pm(k)$.

(D61) $[a_i(t), x_j^\pm(k)] = \pm \left(\gamma^{\ell+1}(i,j)\frac{\ell_{xj} \ell_{tx} - \ell_{tx} \ell_{jx}}{\ell t (r_i - s_i)}\right) \gamma^{\pm \delta} x_j^\pm(k+1)$, for $\ell > 0$.

(D62) $[a_i(t), x_j^\pm(k)] = \pm \left(\gamma^{\ell+1}(i,j)\frac{\ell_{xj} \ell_{tx} - \ell_{tx} \ell_{jx}}{\ell t (r_i - s_i)}\right) \gamma^{\pm \delta} x_j^\pm(k-1)$, for $\ell < 0$.

(D7) $x_i^{\pm}(k+1) x_j^\pm(k') = \langle j, i \rangle^{\pm}(+1) x_j^\pm(k') x_i^\pm(k+1)$
\[
= -\frac{\delta_{ij}}{r_i - s_i} \left(\gamma^{\ell-k} \gamma^{-k/2} \omega_i^{(k+k'')} - \gamma^{k'} \gamma^{-k/2} \omega_i^{(k+k'')}\right).
\]

(D8) $\left[x_i^\pm(k), x_j^{-\pm}(k')\right] = \frac{\delta_{ij}}{r_i - s_i} \left(\gamma^{\ell-k} \gamma^{-k/2} \omega_i^{(k+k'')} - \gamma^{k'} \gamma^{-k/2} \omega_i^{(k+k'')}\right)$,
where $\omega_i(m)$, $\omega_i'(m)$ $(m \in \mathbb{Z}_{\geq 0})$ such that $\omega_i(0) = \omega_i$ and $\omega_i'(0) = \omega_i'$ are defined as below:

$$\sum_{m=0}^{\infty} \omega_i(m) z^{-m} = \omega_i \exp \left(\sum_{\ell=1}^{\infty} a_{\ell}(\ell) z^-\ell\right), \quad (\omega_i(0) = 0, \quad \forall m > 0);$$

$$\sum_{m=0}^{\infty} \omega_i'(m) z^{-m} = \omega_i' \exp \left(-\sum_{\ell=1}^{\infty} a_{\ell}(\ell) z^-\ell\right), \quad (\omega_i'(0) = 0, \quad \forall m > 0).$$
\( \tau \)-\textit{invariant generating functions, vertex representations of \( U_{r,s}(\widehat{\mathfrak{g}}) \) \( 11 \) 

(D9_1) \( x_i^\pm (m)x_j^\pm (k) = \langle j, i \rangle ^\pm x_j^\pm (k)x_i^\pm (m), \) \( \text{for } a_{ij} = 0, \)

(D9_2) \( \text{Sym}_{m_1, \cdots, m_n} \sum_{k=0}^{n=1-a_{ij}} (-1)^k (r_i, s_i) \frac{k(k-1)}{2} \frac{1-a_{ij}}{k} \times x_i^\pm (m_k) \cdots x_i^\pm (m_1) = 0, \) \( \text{for } a_{ij} \neq 0, 1 \leq i < j < n, \)

(D9_3) \( \text{Sym}_{m_1, \cdots, m_n} \sum_{k=0}^{n=1-a_{ij}} (-1)^k (r_i, s_i) \frac{k(k-1)}{2} \frac{1-a_{ij}}{k} \times x_i^\pm (m_1) \cdots x_i^\pm (m_k) = 0, \) \( \text{for } a_{ij} \neq 0, 1 \leq j < i < n, \)

where \( [m]_{\pm i} = \frac{r^{i-m-s-m}}{r_i-s_i}, [m]_{\pm i} = [m]_{\pm i} \cdots [2]_{\pm i}[1]_{\pm i}, \) \( \frac{m}{n} \pm i = \frac{[m]_{\pm i}}{[n]_{\pm i}! [m-n]_{\pm i}!}; \)

\( \text{Sym}_{m_1, \cdots, m_n} \) denotes symmetrization w.r.t. the indices \( (m_1, \cdots, m_n). \)

As one of crucial observations of considering the compatibilities of the defining system above, we have

**PROPOSITION 3.5.** \( \text{There exists the Q-algebra antiautomorphism } \tau \text{ of } U_{r,s}(\widehat{\mathfrak{g}}) \text{ such that } \tau(r) = s, \tau(s) = r, \tau(\omega'_i, \omega_j)^\pm 1 = (\omega'_j, \omega_i)^\mp 1 \) and

\[ \tau(\omega_i) = \omega'_i, \quad \tau(\omega'_i) = \omega_i, \quad \tau(\gamma) = \gamma', \quad \tau(\gamma') = \gamma, \quad \tau(D) = D', \quad \tau(D') = D, \]

\[ \tau(x_i^\pm (m)) = x_i^\mp (m), \quad \tau(a_i(\ell)) = a_i(-\ell), \]

\[ \tau(\phi_i(m)) = \varphi_i(-m), \quad \tau(\varphi_i(-m)) = \phi_i(m), \]

and \( \tau \) preserves each defining relation \( (Dn) \) in Definition 3.1 for \( n = 1, \cdots, n. \)

**3.2. Quantum Lie bracket.** In this paragraph, we first establish an algebraic isomorphism between the two realizations of two-parameter quantum affine algebras \( U_{r,s}(\widehat{\mathfrak{g}}) \) in the above, which is called Drinfeld isomorphism in one-parameter quantum affine algebras. We need to make some preliminaries on the definition of quantum Lie bracket that appears to be regardless to degrees of relative elements (see the properties (3.16) & (3.17) below). This a bit generalized quantum Lie bracket compared to the one used in the usual construction of the quantum Lyndon basis (for definition, see [R2]), which is consistent with the cases when adding the bracketing on those corresponding Lyndon words, is crucial to our proving later on.

**DEFINITION 3.6.** For \( q_i \in \mathbb{K}^* = \mathbb{K}\setminus\{0\} \) and \( i = 1, 2, \cdots, s - 1, \) The quantum Lie brackets \( [a_1, a_2, \cdots, a_s]_{(q_1, q_2, \cdots, q_{s-1})} \) and \( [a_1, a_2, \cdots, a_s]_{(q_1, q_2, \cdots, q_{s-1})} \) are defined inductively by

\[ [a_1, a_2]_{q_1} = a_1a_2 - q_1a_2a_1, \]

\[ [a_1, a_2, \cdots, a_s]_{(q_1, q_2, \cdots, q_{s-1})} = [a_1, [a_2, \cdots, a_s]_{(q_2, \cdots, q_{s-1})}]_{q_1}, \]

\[ [a_1, a_2, \cdots, a_s]_{(q_1, q_2, \cdots, q_{s-1})} = [a_1, \cdots, a_{s-1}]_{(q_1, \cdots, q_{s-2})} a_s]_{q_{s-1}}, \]

By consequences of the above definitions, the following identities follow

\[ [a, bc]_v = [a, b]_q c + q [a, c]_q^v b, \]

\[ [ab, c]_v = a [b, c]_q + q [a, c]_q^v b, \]

\[ [a, [b, c]_u]_v = [a, b]_q c \frac{u}{v} + q [a, c]_q^v b \frac{u}{v}, \]

\[ [[a, b], c]_v = [a, b]_q c \frac{u}{v} + q [a, c]_q^v b \frac{u}{v}. \]
In particular, we get immediately,
\begin{align}
(3.18) \quad [a, [b_1, \ldots, b_k]_{(v_1, \ldots, v_{k-1})}] &= \sum_i [b_1, \ldots, [a, b_i, \ldots, b_k]_{(v_1, \ldots, v_{k-1})}], \\
(3.19) \quad [a, a, b]_{(u, v)} &= a^2b - (u+v)aba + (uv)ba^2 = (uv)[b, a, a]_{(u-1, v-1)}, \\
(3.20) \quad [a, a, b]_{(u^2, uv, v^2)} &= a^3b - [3]_{u,v} a^2ba + (uv)[3]_{u,v}aba^2 - (uv)^3ba^3, \\
(3.21) \quad [a, a, b, b]_{(u^3, u^2v, uv^2, v^3)} &= a^4b - [4]_{u,v} a^3ba + uv \frac{4}{2} a^2ba^2 \\
&\quad - (uv)^3[4]_{u,v} aba^3 + (uv)^6ba^4,
\end{align}

where \([n]_{u,v} = \frac{n - u}{v}, [n]_{u,v}! := [n]_{u,v} \cdots [2]_{u,v} [1]_{u,v}, [n]_{u,v}^\Delta := \frac{[n]_{u,v}!}{[m]_{u,v}!}\frac{[n]_{u,v}^\Delta}{[m]_{u,v}^\Delta}.

By the definition above, the formula (D7) will take the convenient form as
\begin{align}
(3.22) \quad [x_i^+(k), x_j^+(k'+1)]_{(i,j)=1} = - (j,i) (i,j)^{-1} \frac{1}{2} [x_i^+(k'), x_i^+(k+1)]_{(j,i)=1}.
\end{align}

3.3. Quantum root vectors. Furthermore, for each \(\alpha = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_n} \in \Delta^+, \alpha_{i_n} := \alpha_{i_{2n}} + \cdots + \alpha_{i_n} \in \Delta^+, \) by [R2], we can construct the quantum root vector \(x_{\alpha}^+(0)\) as a \((r, s)\)-bracketing in an inductive fashion, for more details, see [HRZ]:
\begin{align}
x_{\alpha}^+(0) &:= [x_{\alpha_{i_1} + \cdots + \alpha_{i_{n-1}}}^+(0), x_{\alpha_{i_n}}^-(0)]_{\omega_{\alpha_{i_1} + \cdots + \alpha_{i_{n-1}}}^\Delta, \omega_{\alpha_{i_n}}^\Delta}^{-1} \\
&= [\cdots [x_{\alpha_1}^+(0), x_{\alpha_2}^+(0)]_{(i_1, i_2)^{-1}}^{-1}, \cdots, x_{\alpha_n}^+(0)]_{\omega_{\alpha_{i_1} + \cdots + \alpha_{i_{n-1}}}^\Delta, \omega_{\alpha_{i_n}}^\Delta}^{-1}.
\end{align}

Applying \(\tau\) to (*), we can obtain the definition of quantum root vector \(x_{\alpha}^-(0)\) as below:
\begin{align}
x_{\alpha}^-(0) &:= [x_{\alpha_{i_n}}^-(0), x_{\alpha_{i_{n-1}}}^-(0)]_{\omega_{\alpha_{i_1} + \cdots + \alpha_{i_{n-1}}}^\Delta, \omega_{\alpha_{i_n}}^\Delta}^{-1} \\
&= [x_{\alpha_{i_n}}^-(0) \cdots [x_{\alpha_2}^-(0), x_{\alpha_1}^-(0)]_{(i_2, i_1)^{-1}}^{-1}, \cdots, x_{\alpha_1}^-(0)]_{\omega_{\alpha_{i_1} + \cdots + \alpha_{i_{n-1}}}^\Delta, \omega_{\alpha_{i_n}}^\Delta}^{-1}.
\end{align}

**Definition 3.7.** (see Definition 3.9 in [HRZ]) For \(\alpha = \alpha_{i_1} + \cdots + \alpha_{i_n} \in \Delta^+, \) we define the quantum affine root vectors \(x_{\alpha}^\pm(k)\) of nontrivial level \(k\) by
\begin{align}
x_{\alpha}^+(k) := [\cdots [x_{i_1}^+(k), x_{i_2}^+(0)]_{(i_1, i_2)^{-1}}^{-1}, \cdots, x_{i_n}^+(0)]_{\omega_{\alpha_{i_1} + \cdots + \alpha_{i_{n-1}}}^\Delta, \omega_{\alpha_{i_n}}^\Delta}^{-1}, \\
x_{\alpha}^-(k) := [x_{i_n}^-(0), \cdots, [x_{i_2}^-(0), x_{i_1}^-(0)]_{(i_2, i_1)^{-1}}^{-1}, \cdots, x_{\alpha_{i_1}}^-(0)]_{\omega_{\alpha_{i_1} + \cdots + \alpha_{i_{n-1}}}^\Delta, \omega_{\alpha_{i_n}}^\Delta}^{-1},
\end{align}
where \(\tau(x_{\alpha}^\pm(\pm k)) = x_{\alpha}^\pm(\mp k)\).

**Remark 3.8.** Using the definition, we fix an ordering of the maximal root \(\theta\), and give the maximal quantum root vectors \(x_\theta^+(1)\) and \(x_\theta^-(1)\) as follows.
For the case of \(A_n^{(1)}\), we fix the maximal root \(\theta = \alpha_1 + \alpha_2 + \cdots + \alpha_n\), and
\begin{align}
x_\theta^+(1) &= [x_n^+(0), x_{n-1}^+(0), \cdots, x_2^+(0), x_1^+(1)]_{(s, \ldots, s)}, \\
x_\theta^+(1) &= [x_1^+(1), x_2^+(0), \cdots, x_n^+(0)]_{(r, \ldots, r)}.
\end{align}
For the case of \(D_n^{(1)}\), we fix the maximal root \(\theta = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} + \alpha_n + \alpha_{n-1} + \cdots + \alpha_2\), and
\begin{align}
x_\theta^+(1) &= [x_n^+(0), \cdots, x_n^+(0), x_{n-2}^+(0), \cdots, x_2^+(0), x_1^+(1)]_{(s, \ldots, s, r-1, \ldots, r-1)}, \\
x_\theta^+(1) &= [x_1^+(1), x_2^+(0), \cdots, x_n^+(0), x_2^+(0), \cdots, x_2^+(0)]_{(r, \ldots, r, s-1, \ldots, s-1)},
\end{align}
For the case of $E^{(1)}_6$, we fix the maximal root

$$\theta = \alpha_1 + \alpha_3 + \cdots + \alpha_6 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_4 + \alpha_2,$$

and

$$x_\theta^- (1) = x_{\alpha_{13456243542}}^- (1) = [x_2^-(0), x_{\alpha_{13456243542}}^- (1)]_{r^{-2}s^{-1}} = \cdots$$

$$= [x_2^-(0), x_4^-(0), x_5^-(0), x_1^-(0), x_4^-(0), x_2^-(0), x_6^-(0), \cdots],$$

$$x_\theta^+ (-1) = x_{\alpha_{13456243542}}^+ (-1) = [x_{\alpha_{13456243542}}^+ (-1), x_2^+ (0)]_{r^{-1}s^{-1}}$$

$$= [x_1^+ (-1), x_3^+ (0), \cdots, x_6^+ (0), x_2^+ (0), x_4^+ (0)],$$

$$x_\theta^+ (0), x_6^+ (0), x_4^+ (0), x_2^+ (0)]_{(r^{-2}s^{-1})}.$$

### 3.4. Two-parameter Drinfeld’s Isomorphism Theorem

We state the main theorem as follows.

**Theorem 3.9.** Given a simple Lie algebra $\mathfrak{g}$ of simply-laced type, let $\theta = \alpha_{i_1} + \cdots + \alpha_{i_{h-1}}$ be the maximal root with respect to a chosen prime root system $\Pi$. Then there exists an algebra isomorphism $\Psi : U_{r,s}(\hat{\mathfrak{g}}) \rightarrow U_{r,s}(\hat{\mathfrak{g}})$ defined by: for $i \in I$,

- $\omega_i \rightarrow \omega_i'$
- $\omega_i' \rightarrow \omega_i''$
- $\omega_0 \rightarrow \gamma'^{-1} \omega_0^{-1}$
- $\omega'_0 \rightarrow \gamma'^{-1} \omega'_0^{-1}$
- $\gamma^{\pm \frac{1}{2}} \rightarrow \gamma^{\pm \frac{1}{2}}$
- $D^{\pm 1} \rightarrow D^{\pm 1}$
- $D'^{\pm 1} \rightarrow D'^{\pm 1}$
- $e_i \rightarrow x_i^+(0)$
- $f_i \rightarrow x_i^-(0)$
- $e_0 \rightarrow x_\theta^- (1) \cdot (\gamma'^{-1} \omega_0^{-1})$
- $f_0 \rightarrow a \tau (x_\theta^- (1) \cdot (\gamma'^{-1} \omega_0^{-1})) = a (\gamma^{-1} \omega_0^{-1}) \cdot x_\theta^+ (-1)$

where $\omega_i = \omega_i \cdots \omega_{i_{h-1}}, \omega'_0 = \omega'_i \cdots \omega'_{i_{h-1}}$, and

$$a = \begin{cases} 1, & \text{for type } A^{(1)}; \\ (rs)^{n-2}, & \text{for type } D^{(1)}; \\ (rs)^{4}, & \text{for type } E_6^{(1)}. \end{cases}$$

**Remark 3.10.** Let $E_i, F_i$ ($i \in I_0$) and $\omega_0, \omega'_0$ denote the images of $e_i, f_i$ ($i \in I_0$) and $\omega_0, \omega'_0$ in the algebra $U_{r,s}(\hat{\mathfrak{g}})$, respectively. Denote by $U'_{r,s}(\hat{\mathfrak{g}})$ the subalgebra of $U_{r,s}(\hat{\mathfrak{g}})$ generated by $E_i, F_i, \omega_i^{\pm 1}, \omega'_i^{\pm 1}$ ($i \in I_0$), $\gamma^{\pm \frac{1}{2}}, \gamma'^{\pm \frac{1}{2}}, D^{\pm 1}, D'^{\pm 1}$, that is,

$$U'_{r,s}(\hat{\mathfrak{g}}) := \left< E_i, F_i, \omega_i^{\pm 1}, \omega'_i^{\pm 1}, \gamma^{\pm \frac{1}{2}}, \gamma'^{\pm \frac{1}{2}}, D^{\pm 1}, D'^{\pm 1} \mid i \in I_0 \right>.$$

Thereby, to prove the Drinfeld isomorphism theorem (Theorem 3.9) is equivalent to prove the following three Theorems:

**Theorem A.** $\Psi : U_{r,s}(\hat{\mathfrak{g}}) \rightarrow U'_{r,s}(\hat{\mathfrak{g}})$ is an epimorphism.
Theorem 5. \( U_{r,s}^l(\hat{g}) = U_{r,s}(\hat{g}) \).

Theorem 6. There exists a surjective \( \Phi : U_{r,s}^l(\hat{g}) \rightarrow U_{r,s}(\hat{g}) \) such that \( \Psi \Phi = 1 \).

4. Proof of Drinfeld Isomorphism Theorem

For completeness, we check Theorem 4 for types \( D_n^{(1)} \) and \( E_6^{(1)} \) (similarly for types \( E_7^{(1)} \) and \( E_8^{(1)} \)), since we have proved it for type \( A_n^{(1)} \) in [HRZ].

4.1. Proof of Theorem 4 for \( U_{r,s}(D_n^{(1)}) \). We have to check relations (X1)—(X5) in Definition 2.2 for type \( D_n^{(1)} \). It is easy to verify relations (X1)—(X3), which are similar to the case of \( A_n^{(1)} \) ([HRZ]).

To check relation (X4), we first consider that when \( i \neq 0 \),
\[
[E_0, F_i] = [x_\theta^- (1) (\gamma^{-1} \omega_0^{-1}), x_i^- (0)] = -[x_i^- (0), x_\theta^- (1)]_{\omega_0^-} \cdot (\gamma^{-1} \omega_0^{-1}).
\]

Thus, relation (X4) follows from immediately from the following Lemma 4.1 in the case of \( i \neq 0 \).

Lemma 4.1. Let \( i \in \{1, \ldots, n\} \), then we have
\[
[x_i^- (0), x_\theta^- (1)]_{\omega_0^-} = 0.
\]

To show Lemma 4.1, the following Lemmas will play a crucial role which will be proved in the appendix.

For our purpose, we need some notations: For \( 1 \leq i < j \leq n - 1 \),
\[
x_{\alpha_{i,j}} (1) = [x_i^- (0), \ldots, x_2^- (0), x_1^- (1)]_{\gamma^{-1} \omega_0^{-1}},
\]
\[
x_{\beta_{i,j}} (1) = [x_i^- (0), \ldots, x_n^- (0), x_{n-2}^- (0), \ldots, x_{j+1}^- (0), x_j^- (1)]_{\gamma^{-1} \omega_0^{-1}}.
\]

Consequently, we get \( x_\theta^- (1) = x_{\beta_{1,2}}^- (1) \).

Lemma 4.2. \( [x_{i-1}^- (0), x_{\beta_{i-1,i+1}}^- (1)]_{\gamma^{-1} \omega_0^{-1}} = 0, \) for \( 1 < i < n \).

Lemma 4.3. \( [x_i^- (0), x_{\beta_{i,i+1}}^- (1)]_{\gamma^{-1} \omega_0^{-1}} = 0, \) for \( 1 \leq i \leq n - 1 \).

Lemma 4.4. \( [x_2^- (0), x_{\beta_{1,4}}^- (1)] = 0. \)

Lemma 4.5. \( [x_i^- (0), x_{\beta_{1,i+2}}^- (1)] = 0, \) for \( 3 \leq i \leq n - 2 \).

The following Lemmas can be verified directly.

Lemma 4.6. \( [x_i^- (0), x_{\beta_{i,i}}^- (1)]_{\gamma^{-1} \omega_0^{-1}} = 0, \) for \( 3 \leq i \leq n - 1 \).

Lemma 4.7. \( [x_n^- (0), x_{\beta_{1,n}}^- (1)]_{\gamma^{-1} \omega_0^{-1}} = 0. \)

Now using the above Lemmas, we turn to prove Lemma 4.1.

Proof of Lemma 4.1. (I) When \( i = 1, \) \( \langle \omega_1^+, \omega_0 \rangle = rs \) and \( \langle \omega_1^+, \omega_\theta \rangle = (rs)^{-1} \). It follows from Lemma 4.3 for the case of \( i = 1 \).
(II) When $i = 2$, $\langle \omega'_2, \omega_0 \rangle = s$, that is, $\langle \omega'_2, \omega_0 \rangle = s^{-1}$. Let us first consider

$$x_{\overline{g}}(1) = x_{\overline{\beta_{1,2}}}(1) \quad \text{(by definition)}$$

$$= [x_{\overline{2}}(0), x_{\overline{3}}(0), x_{\overline{\beta_{1,4}}}(1)]_{r^{-1}, s^{-1}} \quad \text{(by (3.10))}$$

$$= [[x_{\overline{2}}(0), x_{\overline{2}}(0), x_{\overline{3}}(0)]_{r^{-1}, s^{-1}}, x_{\overline{\beta_{1,4}}}(1)]_{r^{-1}} \quad (= 0 \text { by Lemma 4.4})$$

$$+ r^{-1}[x_{\overline{3}}(0), x_{\overline{2}}(0), x_{\overline{\beta_{1,4}}}(1)]_{r^{-1}} \quad (= 0 \text { by Lemma 4.4})$$

$$= 0.$$

(III) When $3 \leq i \leq n-1$, $\langle \omega'_i, \omega_0 \rangle = 1$, that is to say, $\langle \omega'_i, \omega_0 \rangle = 1$. We may use (3.10) and (D9$_1$) to show that

$$[x_{\overline{i}}(0), x_{\overline{g}}(1)] \quad \text{(by definition)}$$

$$= [x_{\overline{i}}(0), [x_{\overline{2}}(0), \ldots, x_{\overline{i-2}}(0), x_{\overline{\beta_{1,i-1}}}(1)]_{r^{-1}, \ldots, r^{-1}}]$$

$$= [x_{\overline{2}}(0), \ldots, x_{\overline{i-2}}(0), [x_{\overline{i}}(0), x_{\overline{\beta_{1,i-1}}}(1)]_{r^{-1}, \ldots, r^{-1}}].$$

For this purpose, it suffices to check that $[x_{\overline{i}}(0), x_{\overline{\beta_{1,i-1}}}(1)] = 0$. It is now straightforward to verify that

$$[x_{\overline{i}}(0), x_{\overline{\beta_{1,i-1}}}(1)]_{r^{-1}, s} \quad \text{(by definition)}$$

$$= [x_{\overline{i}}(0), [x_{\overline{i-1}}(0), x_{\overline{i-1}}(0), x_{\overline{\beta_{1,i+1}}}(1)]_{r^{-1}, r^{-1}}]_{r^{-1}, s} \quad \text{(by (3.10))}$$

$$= [x_{\overline{i}}(0), [x_{\overline{i-1}}(0), x_{\overline{i-1}}(0)]_{r^{-1}, r^{-1}}, x_{\overline{\beta_{1,i+1}}}(1)]_{r^{-1}, r^{-1}, s} \quad \text{(by (3.10))}$$

$$+ r^{-1}[x_{\overline{i}}(0), x_{\overline{i}}(0), [x_{\overline{i-1}}(0), x_{\overline{\beta_{1,i+1}}}(1)]_{r^{-1}, r^{-1}, s}] \quad (= 0 \text { by Lemma 4.5})$$

$$= [[x_{\overline{i}}(0), x_{\overline{i-1}}(0), x_{\overline{i}}(0)]_{r^{-1}}, x_{\overline{\beta_{1,i+1}}}(1)]_{r^{-2}} \quad (= 0 \text { by (D9$_2$))}$$

$$+ s[[x_{\overline{i-1}}(0), x_{\overline{i}}(0)]_{r^{-1}}, [x_{\overline{i}}(0), x_{\overline{\beta_{1,i+1}}}(1)]_{r^{-1}}]_{r^{-1}, r^{-1}, s} \quad (= 0 \text { by (3.17))}$$

$$= s[x_{\overline{i-1}}(0), x_{\overline{i}}(0), x_{\overline{\beta_{1,i+1}}}(1)]_{r^{-1}} \quad (= 0 \text { by Lemma 4.5})$$

$$+ [x_{\overline{i-1}}(0), x_{\overline{\beta_{1,i+1}}}(1)]_{r^{-1}} \quad (= 0 \text { by definition})$$

$$= [x_{\overline{\beta_{1,i+1}}}(1), x_{\overline{i}}(0)]_{r^{-1}, s},$$

which implies that $(1 + r^{-1}s)[x_{\overline{i}}(0), x_{\overline{\beta_{1,i-1}}}(1)] = 0$. That is to say, $r \neq -s$, $[x_{\overline{i}}(0), x_{\overline{\beta_{1,i-1}}}(1)] = 0$. 


(IV) When \( i = n, \( \langle \omega'_n, \omega_0 \rangle = (rs)^{-2} \), that is to say \( \langle \omega'_n, \omega_0 \rangle = (rs)^2 \). By using (3.16) and (D9), we see that

\[
[x^n_0(0), x^n_δ(1)]_{(rs)^2} = \left[ x^n_0(0), x^n_δ(0), \ldots, x^n_{r-3}(0), x^n_{\beta_1,n-2}(1) \right]_{(r^{-1}, \ldots, r^{-1})}(rs)^2
\]

Hence, if we have verified that \( [x^n_0(0), x^n_{\beta_1,n-2}(1)]_{(rs)^2} = 0 \), then we would get the desired conclusion. We actually have

\[
x^n_{\beta_1,n-2}(1) = [x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(1)]_{(s, r^{-1}, r^{-1})} \quad \text{(by (3.16))}
\]

\[
= [x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(1)]_{(rs)^{-1}, s} + (rs)^{-1} \left[ x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(1) \right]_{(rs)^2} \quad \text{(by (3.16))}
\]

\[
= (rs)^{-1} \left[ x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(1) \right]_{rs^2}
\]

\[
+ r^{-2}s^{-1} \left[ x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(1) \right]_{rs^2} \quad \text{(by (3.16) & (D9))}
\]

It can be proved from the above steps that

\[
x^n_{\beta_1,n-2}(1) = (rs)^{-1} \left[ x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(1) \right]_{rs^2, r^{-1}}.
\]

Hence, we may easily check that

\[
[x^n_0(0), x^n_{\beta_1,n-2}(1)]_{rs^3} = (rs)^{-1} \left[ x^n_0(0), x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(0) \right]_{r^{-1}, s} \quad \text{(by (3.16))}
\]

\[
= (rs)^{-1} \left[ x^n_0(0), x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(0) \right]_{(r^{-1}, s), s} + r^{-1} \left[ x^n_0(0), x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(0) \right]_{rs^2} \quad \text{(by (3.17))}
\]

\[
= r^{-1} \left[ x^n_0(0), x^n_0(0), x^n_{\beta_1,n-2}(1) \right]_{rs^2, r^{-1}} \quad \text{(by Lemma 4.7)}
\]

\[
+ rs \left[ x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(0), x^n_{\beta_1,n-2}(1) \right]_{rs^2, r^{-1}, r^{-1}} \quad \text{(by definition)}
\]

So, we have \( (1 + r^{-1}s) [x^n_0(0), x^n_{\beta_1,n-2}(1)]_{(rs)^2} = 0 \). Since \( r \neq -s \), then we get \( [x^n_0(0), x^n_{\beta_1,n-2}(1)]_{(rs)^2} = 0 \). This completes the proof of Lemma 4.1. \( \Box \)

We are now ready to prove relation (X4) for \( i = j = 0 \), that is,

**Proposition 4.8.** \( [E_0, F_0] = \gamma^{-1} \omega_{a_0}^{-1} - \gamma^{-1} \omega_a'^{-1} \).

**Proof.** Note that the construction of \( E_0 \) and \( F_0 \), we check the statement step by step. First, using (D1) and (D5), we have

\[
[E_0, F_0] = (rs)^{-2} \left[ x_0^+(1), \gamma^{-1} \omega_{a_0}^{-1}, \gamma^{-1} \omega_{a_0}^{-1} x_0^+(1) \right] = (rs)^{-2} \left[ x_0^+(1), x_0^+(1) \right] \cdot (\gamma^{-1} \gamma^{-1} \omega_{a_0}^{-1} \omega_a'^{-1}).
\]
We may now use the result of the case of $A_{n-1}^{(1)}$
\[ [x_{\alpha_1,n-1}^{-}(1), x_{\alpha_1,n-1}^{+}(-1)] = \gamma \omega_{\alpha_1,n-1}' - \gamma' \omega_{\alpha_1,n-1}. \]

Applying the above result, it is now straightforward to verify that
\[ [x_{\beta_1,n}^{-}(1), x_{\beta_1,n}^{+}(-1)] \]
\[ = [[x_{\alpha_1,n-1}^{-}(0), x_{\alpha_1,n-1}^{-}(1)], x_{\alpha_1,n-1}^{-}(1), x_{\alpha_1,n-1}^{+}(0)]_s \]
\[ + [[x_{\alpha_1,n-1}^{-}(0), x_{\alpha_1,n-1}^{+}(1)], x_{\alpha_1,n-1}^{+}(1), x_{\alpha_1,n-1}^{-}(0)]_s \]
\[ = \gamma \omega_{\alpha_1,n-1}' \frac{\omega_n - \omega_{n-1}}{r - s} + \gamma \omega_{\alpha_1,n-1}' - \gamma' \omega_{\alpha_1,n-1} \frac{\omega_n}{r - s}. \]

We have then by repeating the above step
\[ [x_{\beta_1,n-1}^{-}(1), x_{\beta_1,n-1}^{+}(-1)] \]
\[ = [[x_{\beta_1,n-1}^{-}(0), x_{\beta_1,n-1}^{-}(1)], x_{\beta_1,n-1}^{-}(1), x_{\beta_1,n-1}^{+}(0)]_s \]
\[ + [[x_{\beta_1,n-1}^{-}(0), x_{\beta_1,n-1}^{+}(1)], x_{\beta_1,n-1}^{+}(1), x_{\beta_1,n-1}^{-}(0)]_s \]
\[ = (r s)^{-1} \gamma \omega_{\beta_1,n} \frac{\omega_n - \omega_{n-1}}{r - s} + (r s)^{-1} \gamma \omega_{\beta_1,n}' - \gamma' \omega_{\beta_1,n} \frac{\omega_n}{r - s}. \]

Furthermore, it follows from the above results
\[ [x_{\beta_1,n-2}^{-}(1), x_{\beta_1,n-2}^{+}(-1)] \]
\[ = [[x_{\beta_1,n-2}^{-}(0), x_{\beta_1,n-2}^{-}(1)], x_{\beta_1,n-2}^{-}(1), x_{\beta_1,n-2}^{+}(0)]_s \]
\[ + [[x_{\beta_1,n-2}^{-}(0), x_{\beta_1,n-2}^{+}(1)], x_{\beta_1,n-2}^{+}(1), x_{\beta_1,n-2}^{-}(0)]_s \]
\[ = (r s)^{-2} \gamma \omega_{\beta_1,n-1}' \frac{\omega_{n-2} - \omega_{n-2}}{r - s} + (r s)^{-2} \gamma \omega_{\beta_1,n-1}' - \gamma' \omega_{\beta_1,n-1} \frac{\omega_{n-2}}{r - s}. \]
By the same way, we get at last

\[ [x_{\beta_1,2}^-(1), x_{\beta_1,2}^+(1)] = (rs)^2\gamma_{\beta_1,2} - \gamma'_{\beta_1,2}. \]

As a consequence, we obtain the required result

\[ [E_0, F_0] = \frac{\gamma'_{-1} - \gamma'_{0}}{r - s}. \]

For the rest of this subsection, we will focus on checking the Serre relations of \( U_{r,s}(D_n^{(1)}) \).

**Lemma 4.9.**

1. \( E_nE_0 = (rs)^2 E_0E_n \)
2. \( E_0E_2^2 - (r + s)E_0E_2 + rsE_2^3E_0 = 0 \)
3. \( E_2E_2 - (r + s)E_0E_2 + rsE_2^2E_0 = 0 \)
4. \( F_0F_n = (rs)^2 F_nF_0 \)
5. \( F_0F_0^2 - (r + s)F_0F_2 + rsF_2^2F_0 = 0 \)
6. \( F_2F_0 - (r + s)F_2F_2 + rsF_0F_2^2 = 0 \)

**Proof.** Relations (4)—(6) follow from the action of \( \tau \) on relations (1)—(3). To be precise, let us just consider the second and third relation.

1. For the second equality, it is easy to see that

\[
[E_2, x_0^-(1)] = [([x_0^+(0), x_0^-(0)], x_{\beta_1,3}^-(1))]_{r^{-1}} \quad \text{(by \( \Box \))}
\]

\[ = \frac{\gamma'_{-1}}{r - s} \quad \text{(by (D8) & (D5))}
\]

\[ + [x_0^-(0), \ldots, x_{r-2}^-(0), x_{r-1}^+(0), x_0^-(0), x_0^+(0), x_{r-1}^+(1)]_{(s, \ldots, r-1, \ldots, r^{-1})}
\]

\[ = (0 \text{ by (D8), (D5) & (D9)})
\]

\[ = (rs)^{-1} x_{\beta_1,3}^+(1) \omega_2.
\]

By the above observation, it can be proved in a straightforward manner that

\[
E_0E_2^2 - (r + s)E_0E_2 + rsE_2^2E_0
\]

\[ = rs(E_2x_0^+(1) - (1 + r^{-1}s)E_0x_0^+(1)E_2 + r^{-1}s x_0^+(1)E_2^2)(\gamma'_{-1} \omega_0^{-1})
\]

\[ = rs[E_2, x_0^+(1)]_{r^{-1}} (\gamma'_{-1} \omega_0^{-1})
\]

\[ = [E_2, x_{\beta_1,3}^+(1)] \omega_2 (\gamma'_{-1} \omega_0^{-1}) \quad \text{(by \( \Box \) & (D8))}
\]

\[ = 0.
\]

2. Note that the formula of \[ E_2, x_0^-(1) \] obtained in (1), we actually have

\[
E_0E_2 - (r + s)E_0E_2 + (rs)E_2E_0
\]

\[ = (rs) [x_0^-(1), x_0^+(1), E_2]_{(1, r^{-1})} (\gamma'_{-2} \omega_0^{-2})
\]

\[ = - [x_0^-(1), x_{\beta_1,3}^+(1)\omega_2]_{(r^{-1})} (\gamma'_{-2} \omega_0^{-2})
\]

\[ = - [x_0^-(1), x_{\beta_1,3}^+(1)]_{(r^{-1})} \omega_2 (\gamma'_{-2} \omega_0^{-2}) \quad \text{(by Lemma \( \Box \))}
\]

\[ = 0.
\]
LEMMA 4.10. \([x_{\beta_1,3}^{-1}(1), x_{\beta_1,2}^{-1}(1)]_s = 0, \text{ for } r \neq -s.\)

Proof. See the appendix.

4.2. Proof of Theorem \(A\) for \(U_{r,s}(E_6^{(1)})\). As usual, we need to verify some critical relations of Theorem \(A.\)

Note that the highest root of simple Lie algebra \(E_6\) is
\[
\theta = \alpha_{13456243542} = \alpha_1 + \alpha_3 + \cdots + \alpha_6 + \alpha_2 + \alpha_4 + \alpha_3 + \alpha_5 + \alpha_4 + \alpha_2.
\]
The maximal quantum root vectors \(x_{\theta}^{-}(1)\) and \(x_{\theta}^{+}(1)\) are defined as follows
\[
x_{\theta}^{-}(1) = x_{\alpha_{13456243542}}^{-}(1) = [x_{\alpha_{13456243542}}^{-}(0), x_{\alpha_{13456243542}}^{-}(1)]_{r,-2s-1} = \cdots
\]
\[
= [x_{2}^{-}(0), x_{4}^{-}(0), x_{5}^{-}(0), x_{6}^{-}(0), x_{7}^{-}(0), x_{8}^{-}(0), x_{9}^{-}(0), x_{10}^{-}(0), \cdots,
\]
\[
, x_{3}^{-}(0), x_{1}^{-}(1)]_{(s, \cdots, s-r^{-1}, s-r^{-1}, s, s, r^{-2}s-1)},
\]
\[
x_{\theta}^{+}(1) = x_{\alpha_{13456243542}}^{+}(1) = [x_{\alpha_{13456243542}}^{+}(1) - (1), x_{\alpha_{13456243542}}^{+}(0)]_{r-1s-2}
\]
\[
= [x_{1}^{+}(-1), x_{3}^{+}(0), \cdots, x_{6}^{+}(0), x_{7}^{+}(0), x_{8}^{+}(0), x_{9}^{+}(0), x_{10}^{+}(0),
\]
\[,
\]
Similarly, relation (X4) holds due to Lemma 4.11 below in the case of \(i \neq 0.\)

LEMMA 4.11. \([x_{i}^{-}(0), x_{\theta}^{-}(1)]_{(\omega_{i}, \omega_{\theta})} = 0, \text{ for } i = 1, 2, \cdots, 6.\)

To verify Lemma 4.11 the following three Lemmas, which we will check in the appendix, will play a crucial role. To be precise, let \(x_{i_1i_2\cdots i_n}(k) = x_{\alpha_{i_1,i_2,\cdots, i_n}}^{\pm}(k).\)

We may easily check that

LEMMA 4.12. \([x_{i}^{-}(0), x_{\alpha_{13456243542}}^{-}(1)]_{(rs)^{-1}} = 0.\)

More generally, we have

LEMMA 4.13.
\[
[x_{4}^{-}(0), x_{\alpha_{13456243542}}^{-}(1)]_{r} = 0,
\]
\[
[x_{3}^{-}(0), x_{\alpha_{13456243542}}^{-}(1)]_{(rs)^{-1}} = 0.
\]

LEMMA 4.14. \([x_{1}^{-}(0), x_{\alpha_{13456243542}}^{-}(1)]_{(rs)^{-1}} = 0.\)

Proof of Lemma 4.11. We may now use the previous Lemmas to show that

(1) When \(i = 1, \langle \omega_{i}, \omega_{\theta} \rangle = rs\) and \(\langle \omega_{i}, \omega_{\theta} \rangle = (rs)^{-1},\)
\[
[x_{1}^{-}(0), x_{\theta}^{-}(1)]_{(rs)^{-1}} = 0.
\]

\[
= [x_{1}^{-}(0), x_{2}^{-}(0), x_{\alpha_{13456243542}}^{-}(1)]_{(r-2s-1,(rs)^{-1})} = 0 \text{ (by definition)}
\]
\[
= [x_{1}^{-}(0), x_{2}^{-}(0), x_{\alpha_{13456243542}}^{-}(1)]_{r-3s-2} = 0 \text{ (by (3.16))}
\]
\[
= [x_{2}^{-}(0), x_{\alpha_{13456243542}}^{-}(1)]_{(rs)^{-1}} = 0 \text{ (by Lemma 4.13)}
\]
\[
= 0.
\]
(II) When $i = 2$, $\langle \omega'_2, \omega_0 \rangle = rs^2$ and $\langle \omega'_1, \omega_0 \rangle = r^{-1}s^{-2}$.

\[
[x_2^-(0), x_0^-(1)]_{r^{-1}s^{-2}} = [x_2^-(0), x_2^-(0), x_4^-(0), x_{134562435}(1)]_{(s, r^{-2}s^{-1}, r^{-1}s^{-2})} = [x_2^-(0), [x_2^-(0), x_4^-(0)]_{r^{-1}}, x_{134562435}(1)]_{(r^{-1}, r^{-1}s, s^{-2})} + r^{-1}[x_2^-(0), x_4^-(0), [x_2(0), x_{134562435}(1)]_{(r^{-1})}]_{(r^{-1}, r^{-1}s^{-2})} = 0 \text{ by Lemma 4.12}
\]

\[
= [[[x_2^- (0), x_2^-(0), x_4^-(0)]_{(r^{-1}, s^{-1})}, x_{134562435}(1)]_{r^{-2}s^{-2}}} + s^{-1}[[x_2^- (0), x_4^- (0), [x_2(0), x_{134562435}(1)]_{(r^{-1})}]_{r^{-1}s^{-1}} = 0 \text{ by Lemma 4.12}
\]

(III) When $i = 3$, $\langle \omega'_3, \omega_0 \rangle = rs$ and $\langle \omega'_3, \omega_0 \rangle = (rs)^{-1}$. We may easily check that

\[
[x_3^-(0), x_0^-(1)]_{(rs)^{-1}} = [x_3^-(0), x_2^-(0), x_{1345624354}(1)]_{(r^{-2}s^{-1}, (rs)^{-1})} = [x_3^-(0), [x_2^-(0), x_{1345624354}(1)]_{r^{-3}s^{-2}} + [x_2^-(0), [x_3^- (0), x_{1345624354}(1)]_{(rs)^{-1}}]_{r^{-2}s^{-1}} = 0 \text{ by Lemma 4.13}
\]

(IV) When $i = 4$, $\langle \omega'_4, \omega_0 \rangle = (rs)^{-1}$ and $\langle \omega'_4, \omega_0 \rangle = rs$. We have

\[
[x_4^-(0), x_0^-(1)]_{s^2} = [x_4^-(0), x_2^-(0), x_{134562435}(1)]_{(s, r^{-2}s^{-1}, s^2)} = [x_4^-(0), [x_2^-(0), x_4^- (0)]_{r^{-1}}, x_{134562435}(1)]_{(r^{-1}, s^2)} + r^{-1}[x_2^-(0), x_4^- (0), [x_2(0), x_{134562435}(1)]_{(r^{-1})}]_{(r^{-1}, r^{-1}s, s^2)} = 0 \text{ by Lemma 4.12}
\]

\[
= [[[x_4^- (0), x_2^-(0), x_4^- (0)]_{(r^{-1}, s)}, x_{134562435}(1)]_{r^{-1}s} + s[[x_2^- (0), x_4^- (0)]_{r^{-1}}, [x_4^- (0), x_{134562435}(1)]_{s}]_{(rs)^{-1}} = 0 \text{ by Lemma 4.13}
\]

\[
+ rs[[x_2^- (0), x_{1345624354}(1)]_{r^{-2}s^{-1}}, x_4^- (0)]_{r^{-2}} = 0 \text{ by definition}
\]

which implies that $(1 + r^{-1}s)[x_4^- (0), x_0^- (0)]_{rs} = 0$. That is to say, when $r \neq -s$,

\[
[x_4^- (0), x_0^- (0)]_{rs} = 0.
\]

(V) The proof of the case $i = 5$ or 6 is similar to that of the case $i = 1$ or 3, which are left to the readers. \qed

We would like to point out that the proof of relation $[E_0, F_0] = \omega_0^0 - \omega_0^0$ is the same as that of the case of $D_n^{(1)}$. We now proceed to show the Serre relations for $E_6^{(1)}$. 

PROPOSITION 4.15. We have the following Serre relations

1. \( E_0 E_2^2 - (rs)(r + s)E_2 E_0 E_2 + (rs)^3 E_2^3 E_0 = 0 \),
2. \( E_2^3 E_2 - (rs)(r + s)E_2 E_0 E_2 + (rs)^3 E_2 E_0 E_2 = 0 \),
3. \( F_2^2 F_0 - (rs)(r + s)F_2 F_0 F_2 + (rs)^3 F_0 F_2^2 = 0 \),
4. \( F_2 F_0^2 - (rs)(r + s)F_2 F_0 F_2 + (rs)^3 F_0^2 F_2 = 0 \).

PROOF. Here we only give the proof of the first \((r, s)\)-Serre relation, and the others are left to the readers.

Let us first to prove that

\[
\left[ E_2, x_0^-(1) \right] = \left[ \left[ x_2^+(0), x_2^- (0) \right], x_{1345624354}^- (1) \right]_{r^{-2}s^{-1}} \text{ (by definition & (3.16))}
\]

\[
= \left[ x_2^+(0), x_2^- (0), x_{1345624354}^- (1) \right]_{r^{-2}s^{-1}} \text{ (by (D8), (D5))}
\]

\[
+ \left[ x_2^- (0), x_4^- (0), x_4^- (0), x_4^- (0), x_2^+ (0), x_2^- (0) \right],
\]

\[
\left[ x_{13456}^- (1) \right]_{r^{-1}s^{-1}r^{-1}s^{-1}r^{-1}s^{-1}} \text{ (=0 by (D8), (D5) & (D9))}
\]

\[
= (rs)^{-2} x_{1345624354}^- (1) \omega_2
\]

By the above result, we get that

\[
E_0 E_2^2 - (rs)(r + s)E_2 E_0 E_2 + (rs)^2 E_2^2 E_0
\]

\[
= (rs)^{-2} \left( E_2^2 x_0^-(0) - (1 + r^{-1}s^{-1}) E_2 x_0^-(1) E_2 + (r^{-1}s^{-1}) E_2^2 \right) \gamma \omega_2^{-1}
\]

\[
= (rs)^{-2} \left[ E_2, x_0^-(1) \right]_{r^{-1}s^{-1}} \gamma \omega_2^{-1}
\]

\[
= (rs)^{-2} \left[ E_2, x_0^-(1) \right] \omega_2
\]

\[
= \left[ x_4^- (0), x_5^- (0), x_4^- (0), x_4^- (0), x_5^- (0) \right]
\]

\[
\left[ x_{13456}^- (1) \right]_{r^{-1}s^{-1}r^{-1}s^{-1}r^{-1}s^{-1}} \text{ (=0 by (D8), (D5) & (D9))}
\]

\[
= 0.
\]

\[ \square \]

4.3. Proof of Theorem B. This is similar to that of the case \( A_{n-1}^{(1)} \) [HRZ].

We shall show that the algebra \( \mathcal{U}_{r,s}(\hat{\mathfrak{g}}) \) is generated by \( E_i, F_i, \omega_i^{\pm 1}, \omega_i^{\pm 1}, \gamma_i^{\pm \frac{1}{2}}, \gamma_i^{\pm \frac{3}{2}} \) (\( i \in I_0 \)). More explicitly, any generators of the algebra \( \mathcal{U}_{r,s}(\hat{\mathfrak{g}}) \) are in the subalgebra \( \mathcal{U}_{r,s}'(\hat{\mathfrak{g}}) \). To do so, we also need the following two Lemmas, which can be similarly checked like in [HRZ].

**LEMMA 4.16.** (1)

\[ x_i^- (1) \]

\[ = \left[ E_2, E_3, \cdots, E_{n-2}, E_n, \cdots, E_2, E_0 \right] \gamma \omega_1 \in \mathcal{U}_{r,s}'(D_n^{(1)}), \]

then for any \( i \in I \), \( x_i^- (1) \in \mathcal{U}_{r,s}'(D_n^{(1)}) \).

(2)

\[ x_i^+ (-1) \]

\[ = \tau \left[ E_2, E_3, \cdots, E_{n-2}, E_n, \cdots, E_2, E_0 \right] \gamma \omega_1 \]

\[ = \gamma \omega_1' [ F_0, F_2, \cdots, F_{n-2}, F_n, \cdots, F_2, F_0 ] \in \mathcal{U}_{r,s}'(D_n^{(1)}), \]

then for any \( i \in I \), \( x_i^+ (-1) \in \mathcal{U}_{r,s}'(D_n^{(1)}) \).

\[ \square \]
Lemma 4.17. (1) 
\[ x^{-1}_{i} \] 
\[ = [E_3, E_6, E_2, E_4, E_3, E_5, E_4, E_2, E_0]_{(r^{-1}s^{-2}, r, r, s^{-1}, r, r^{-1}, s^{-1}, r)} \gamma' \omega_1 \] 
\[ \in \mathcal{U}_{r,s}(E_6^{(1)}) \], 
then for any \( i \in I \), \( x^{-1}_{i} \in \mathcal{U}_{r,s}(E_6^{(1)}) \).
(2) 
\[ x^{+1}_{i}(-1) \] 
\[ = \tau\left([E_3, \cdots, E_6, E_2, E_4, E_3, E_5, E_4, E_2, E_0]_{(r^{-1}s^{-2}, r, r, s^{-1}, r, r^{-1}, s^{-1}, r)} \gamma' \omega_1\right) \] 
\[ = \gamma \omega'_1 [F_0, F_2, F_4, F_5, F_3, F_4, F_2, F_6, \cdots, F_3]_{(s, \cdots, s, r^{-1}, s, s, r^{-2}s^{-1})} \] 
\[ \in \mathcal{U}_{r,s}(E_6^{(1)}) \], 
then for any \( i \in I \), \( x^{+1}_{i}(-1) \in \mathcal{U}_{r,s}(E_6^{(1)}) \). \[ \square \]

Furthermore, applying the above results and combining (D8) with (D6), we get the following

Lemma 4.18. (1) \( a_i(l) \in \mathcal{U}_{r,s}^{r,s}(\hat{g}) \), for \( l \in \mathbb{Z}\{0\} \).
(2) \( x^\pm_{i}(k) \in \mathcal{U}_{r,s}^{r,s}(\hat{g}) \), for \( k \in \mathbb{Z} \). \[ \square \]

Therefore, by induction, all generators are in the subalgebra \( \mathcal{U}_{r,s}^{r,s}(\hat{g}) \). So, this finishes the proof of Theorem B. \[ \square \]

4.4. Proof of Theorem C. This subsection focuses on showing Theorem C.

Theorem C. There exists a surjective \( \Phi : \mathcal{U}_{r,s}^{r,s}(\hat{g}) \longrightarrow \mathcal{U}_{r,s}(\hat{g}) \) such that \( \Psi \Phi = \Phi \Psi = 1 \).

Proof. We define \( \Psi \) on the generators as follows. For \( i \in I_0 \),
\[ \Psi(E_i) = e_i, \quad \Psi(F_i) = f_i, \quad \Psi(\omega_i) = \omega_i, \quad \Psi(\omega'_i) = \omega'_i, \] 
\[ \Psi(\gamma) = \gamma, \quad \Psi(\gamma') = \gamma', \quad \Psi(D) = D, \quad \Psi(D') = D'. \] 
Consequently, it is not difficult to see that \( \Psi \Phi = \Phi \Psi = 1 \). \[ \square \]

Up to now, we prove the Drinfeld Isomorphism Theorem for the two parameter quantum affine algebras.

5. Vertex Representations

In the last section, we turn to construct the level-one vertex representations of two-parameter quantum affine algebras \( \mathcal{U}_{r,s}(\hat{g}) \) for types \( X^{(1)}_n \) (where \( X = ADE \)). More precisely, in our construction we can take \( c = 1 \) in the Drinfeld relations in this section.
5.1. Two-parameter quantum Heisenberg algebra.

**Definition 5.1.** Two-parameter quantum Heisenberg algebra $U_{r,s}(\hat{\mathfrak{h}})$ is the subalgebra of $U_{r,s}(\hat{\mathfrak{g}})$ generated by $\{a_j(l), \gamma^\pm \frac{1}{2}, \gamma'^\pm \frac{1}{2} | l \in \mathbb{Z}\setminus\{0\}, j \in I \}$, satisfying the following relation, for $m, l \in \mathbb{Z}\setminus\{0\}$

$$[a_i(m), a_j(l)] = \delta_{m+l,0} \frac{(rs)_{|m|} (r_i s_i - m a_{ij}) \gamma^{|m|} - \gamma'|^{|m|}}{|m|} \cdot \frac{\gamma^{|m|} - \gamma'|^{|m|}}{m \cdot j - s_j}.$$  

We denote by $U_{r,s}(\hat{\mathfrak{h}}^+) \ (\text{resp. } U_{r,s}(\hat{\mathfrak{h}}^-))$ the commutative subalgebra of $U_{r,s}(\hat{\mathfrak{h}})$ generated by $a_j(l) \ (\text{resp. } a_j(-l))$ with $l \in \mathbb{Z}^{>0}, j \in I$. In fact, we have $U_{r,s}(\hat{\mathfrak{h}}^-) = S(\hat{\mathfrak{h}}^-)$, where $S(\hat{\mathfrak{h}}^-)$ is the symmetric algebra associated to $\hat{\mathfrak{h}}^-$. Then $S(\hat{\mathfrak{h}}^-)$ is a $U_{r,s}(\hat{\mathfrak{h}})$-module with the action defined by

$$\gamma^\pm \frac{1}{2} \cdot v = r^\pm \frac{1}{2} v, \quad \gamma'^\pm \frac{1}{2} \cdot v = s^\pm \frac{1}{2} v,$$

$$a_i(l) \cdot v = a_i(-l) v,$$

$$a_i(l) \cdot v = \sum_j (rs)_{l_j} (r_i s_i - m a_{ij}) \cdot \frac{r^l - s^l}{r_j - s_j} \cdot \frac{dv}{da_j(-l)}.$$

for any $v \in S(\hat{\mathfrak{h}}^-)$, $l \in \mathbb{Z}^{>0}$ and $i \in I$.

5.2. Fock space. Let $Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ be the root lattice of $\mathfrak{g}$ with the Killing form $(\cdot | \cdot)$, one can form a group algebra $\mathbb{K}[Q]$ with base elements of the form $e^\beta \ (\beta \in Q)$, and the product

$$e^\beta e^\beta' = e^{\beta + \beta'}, \quad \beta, \beta' \in Q.$$

We define the Fock space as

$$\mathcal{F} := S(\hat{\mathfrak{h}}^-) \otimes \mathbb{K}[Q],$$

and make it into a $U_{r,s}(\hat{\mathfrak{h}})$-module (where $U_{r,s}(\hat{\mathfrak{h}})$ is generated by $\{a_i(\pm l), \omega_i^{\pm 1}, \omega_i^{\pm 1}, \gamma^\pm \frac{1}{2}, \gamma'^\pm \frac{1}{2} | i \in I, l \in \mathbb{Z}^{>0} \}$) via extending the action of $U_{r,s}(\hat{\mathfrak{h}}^-)$ to the Fock space $\mathcal{F}$. Let $z$ be a complex variable and add the action of $\alpha_i(0)$ as follows:

$$z^{\alpha_i(0)}(v \otimes e^\beta) = z^{(\alpha_i | \beta)}(v \otimes e^\beta),$$

$$e^\alpha(v \otimes e^\beta) = v \otimes e^{\alpha + \beta},$$

$$a_i(\pm l) \cdot (v \otimes e^\beta) = (a_i(\pm l) \cdot v) \otimes e^\beta, \quad l \in \mathbb{Z}\setminus\{0\},$$

$$\varepsilon_i(0) \cdot (v \otimes e^\beta) = (\varepsilon_i, \beta) v \otimes e^\beta,$$

such that

$$\omega_i \cdot (v \otimes e^\beta) = (\beta, i) v \otimes e^\beta,$$

$$\omega'_i \cdot (v \otimes e^\beta) = (i, \beta) v \otimes e^\beta.$$

5.3. Vertex operators. Let $\epsilon_0(\cdot, \cdot) : Q \times Q \to \mathbb{K}^*$ be the cocycle such that

$$\epsilon_0(\alpha, \beta + \theta) = \epsilon_0(\alpha, \beta) \epsilon_0(\alpha, \theta),$$

$$\epsilon_0(\alpha + \beta, \theta) = \epsilon_0(\alpha, \theta) \epsilon_0(\beta, \theta),$$

$$\epsilon_0(\alpha, \beta) = \epsilon_0(\beta, \alpha).$$
we construct such a cocycle directly by
\[ \epsilon_0(\alpha_i, \alpha_j) = \begin{cases} (-r_i s_i)^{z_{ij}}, & i > j, \\ (r s)^{\frac{1}{2}}, & i = j, \\ 1, & i < j. \end{cases} \]

For \( \alpha, \beta \in Q \), define \( \mathbb{K} \)-linear operators as
\[ \epsilon_0(v \otimes e^\beta) = \epsilon_0(\alpha, \beta) v \otimes e^\beta, \]
\[ D(r)(v \otimes e^\beta) = r^\beta v \otimes e^\beta, \quad D(s)(v \otimes e^\beta) = s^\beta v \otimes e^\beta. \]

We can now introduce the main vertex operators.
\[ E_\pm(\alpha_i, z) = \exp \left( \pm \sum_{n=1}^{\infty} \frac{s^{\pm n/2}}{[n]} a_i(-n)z^n \right), \]
\[ E_\pm(\alpha_i, z) = \exp \left( \mp \sum_{n=1}^{\infty} \frac{s^{\mp n/2}}{[n]} a_i(n)z^{-n} \right), \]
where \([n] = \frac{r^n - s^n}{r - s}\) for \( n \in \mathbb{Z}^>0 \), \( \alpha_i \in Q \).

**Theorem 5.2.** For the simply-laced cases, we have the vertex representation (of level-1) \( \pi \) of \( U_{r,s}(\mathfrak{g}) \) on the Fock space \( \mathcal{F} \) as follows:
\[ \gamma^{\pm 1} \mapsto r^{\pm 1}, \quad \gamma'^{\pm 1} \mapsto s^{\pm 1}, \]
\[ D \mapsto D(r), \quad D' \mapsto D(s), \]
\[ x_j^+(z) \mapsto X_j^+(z) = E_j^+(\alpha_j, z) E_j^+(\alpha_j)z^{\alpha_j(0) + 1} r_{\alpha_j}^{-\frac{1}{2}} e_{\alpha_j}, \]
\[ x_j^-(z) \mapsto X_j^-(z) = E_j^-(\alpha_j, z) E_j^+(\alpha_j)z^{-\alpha_j(0) + 1} r_{\alpha_j}^{\frac{1}{2}} e_{\alpha_j}, \]
\[ \omega_j(z) \mapsto \Phi_j(z) = \omega_j \exp \left( - (r - s) \sum_{k>0} a_j(-k)z^k \right), \]
\[ \omega_j(z) \mapsto \Psi_j(z) = \omega_j \exp \left( (r - s) \sum_{k>0} a_j(k)z^{-k} \right). \]

**5.4. Proof of the theorem 5.2.** We have to show that the operators \( X_j^\pm(z), \Phi_j(z) \) and \( \Psi_j(z) \) satisfy all the relations of Drinfeld’s realization with \( \gamma = r, \gamma' = s \). More explicitly, we want to show that \( X_j^\pm(z), \Phi_j(z) \) and \( \Psi_j(z) \) satisfy relations (3.2)–(3.13) in Definition 3.1. It is clear that (3.2)–(3.5) follow from the construction of vertex operators \( \Phi_j(z) \) and \( \Psi_j(z) \). We are going to divide the proof into several steps.

It is easy to see that \( \Psi_i, \Phi_i, e^{\pm \alpha_i} \) have the following commutative relations:
\[ \Phi_i(z) e^{\pm \alpha_j} = \langle i, j \rangle^{\mp 1} e^{\pm \alpha_j} \Phi_i(z), \]
\[ \Psi_i(z) e^{\pm \alpha_j} = \langle j, i \rangle^{\pm 1} e^{\pm \alpha_j} \Psi_i(z). \]

Note that the action of \( c \) is the identity, since here we construct a level-one representation. (3.6) follows from the following lemma.

**Lemma 5.3.** For \( i, j \in I \), we have
\[ g_{ij} \left( \frac{z}{w} (rs)^{\frac{1}{2}} \right) \Phi_i(z) \Psi_j(w) = g_{ij} \left( \frac{z}{w} (rs)^{\frac{1}{2}} \right) \Psi_j(w) \Phi_i(z). \]
**Proof:** When $a_{ij} = 0$, the proof is trivial, so we only check the relation for the case of $a_{ij} \neq 0$ (i.e., $a_{ij} = -1$ and $a_{ii} = 2$).

$$
\Phi_i(z)\Psi_j(w) \\
= \omega_i \exp \left( - (r-s) \sum_{k > 0} a_i(-k) z^k \right) \cdot \omega_j \exp \left( (r-s) \sum_{k > 0} a_j(k) w^{-k} \right) \\
= \Psi_j(w)\Phi_i(z) \exp \left( - (r-s)^2 \sum_{k > 0} (a_i(-k), a_j(k)) \left( \frac{z}{w} \right)^k \right) \\
= \Psi_j(w)\Phi_i(z) \exp \left( - \sum_{k > 0} (rs)^k \frac{k}{2} \frac{\langle i, i \rangle - \langle i, i \rangle}{k} \cdot [k] \left( \frac{z}{w} \right)^k \right) \\
= \Psi_j(w)\Phi_i(z) \exp \left\{ \begin{array}{ll}
(i, j)^{-\frac{1}{2}} (\frac{z}{w} + rs) - 1 \left( \frac{i, i}{\frac{z}{w} + rs} - 1 \right), & i \neq j, \\
(i, i)^{-\frac{1}{2}} (\frac{z}{w} + rs) - 1 \left( \frac{i, i}{\frac{z}{w} + rs} - 1 \right), & i = j,
\end{array} \right.
\right.

$$

where we used the formal identity $\log(1 - z) = - \sum_{n=1}^{\infty} \frac{z^n}{n}$. \hfill \Box

Furthermore, the following Lemma 5.4 follows from the relations (3.7) and (3.8).

**Lemma 5.4.** For $i, j \in I$, we have

$$
\begin{align*}
\Phi_i(z) X_i^\pm(w) \Phi_i(z)^{-1} &= g_{ij} \left( \frac{z}{w} (rs)^{\frac{k}{2} + \frac{1}{2}} \right)^{\pm 1} X_j^\pm(w), \\
\Psi_i(z) X_i^\pm(w) \Psi_i(z)^{-1} &= g_{ji} \left( \frac{w}{z} (rs)^{\frac{k}{2} + \frac{1}{2}} \right)^{\pm 1} X_j^\pm(w).
\end{align*}
$$

**Proof:** Naturally we only check (5.2) and (5.3) for the case of $a_{ij} \neq 0$. The vertex operator is a product of two exponential operators and a middle term operator. So we first consider

$$
\Phi_i(z) E_i^\pm(\alpha_j, w) \\
= \exp \left( - (r-s) \sum_{k > 0} a_i(-k) z^k \right) r_i^{\varepsilon_i+(0)} s_i^{\varepsilon_i(0)} \exp \left( + \sum_{k > 0} \frac{r_i^+}{k} a_j(k) w^{-k} \right) \\
= E_i^\pm(\alpha_j, w) \Phi_i(z) \exp \left( \pm (r-s) \sum_{k > 0} \frac{r_i^+}{k} (a_i(-k), a_j(k)) \left( \frac{z}{w} \right)^k \right) \\
= E_i^\pm(\alpha_j, w) \Phi_i(z) \exp \left( \pm \sum_{k > 0} (rs)^k \frac{k}{2} \frac{\langle i, i \rangle - \langle i, i \rangle}{k} \cdot [k] \left( \frac{z}{w} \right)^k \right) \\
= E_i^\pm(\alpha_j, w) \Phi_i(z) \exp \left\{ \begin{array}{ll}
(i, j)^{-\frac{1}{2}} (\frac{z}{w} + rs) - 1 \left( \frac{i, i}{\frac{z}{w} + rs} - 1 \right), & i \neq j, \\
(i, i)^{-\frac{1}{2}} (\frac{z}{w} + rs) - 1 \left( \frac{i, i}{\frac{z}{w} + rs} - 1 \right), & i = j,
\end{array} \right.
\right.
$$

$$
\begin{align*}
= E_i^\pm(\alpha_j, w) \Phi_i(z) g_{ij} \left( \frac{z}{w} (rs)^{\frac{k}{2} + \frac{1}{2}} \right)^{\pm 1} (i, j)^{\pm 1}.
\end{align*}
$$
We would proceed in the same way with the following result,

\[
\Psi_i(z) E^\pm_-(\alpha_j, w) = \exp \left( (r - s) \sum_{k>0} a_i(k) z^{-k} \right) r_i^{(0)} s_i^{(0)} \exp \left( \pm \sum_{k>0} \frac{s^\pm}{k} a_j(-k) w^k \right) E^\pm_-(\alpha_j, w) \Psi_i(z) \exp \left( \pm (r - s) \sum_{k>0} \frac{s^\pm}{k} [a_i(k), a_j(-k)] \left( \frac{w}{z} \right)^k \right)
\]

Applying (5.1) and (5.2), we would arrive at the required results. Thus the proof of the lemma is complete. \(\square\)

Before checking the relations (3.9) and (3.10) and the quantum Serre-relations, we need to introduce a useful notion—normal ordering, which plays an important role in the theory of ordinary vertex operator calculus. Define

\[
: \alpha_i(n) \alpha_j(-n) : = \alpha_j(-n) \alpha_i(n) = \alpha_j(-n) \alpha_i(n),
\]

\[
: \alpha_i(0) \alpha_j : = \alpha_j \alpha_i(0) = \frac{1}{2} (\alpha_i(0) a_j + a_j \alpha_i(0)).
\]

We can extend the notion to the vertex operators. For example, we define

\[
: X^\pm_i(z) X^\pm_j(w) : = E^\pm_+(\alpha_i, z) E^\pm_+(\alpha_j, w) E^\pm_+(\alpha_i, z) E^\pm_+(\alpha_j, w)
\]

\[
\cdot \exp \left( \pm (r_1 \pm r_2) \sum_{k>0} \frac{s^\pm}{k} a_i(0) a_j(0) + \epsilon_{\pm \alpha_i(0)} \epsilon_{\pm \alpha_j} \right)
\]

\[
: X^\pm_i(z) X^\pm_j(w) : = E^\pm_+(\alpha_i, z) E^\pm_+(\alpha_j, w) E^\pm_+(\alpha_i, z) E^\pm_+(\alpha_j, w)
\]

\[
\cdot \exp \left( \pm (r_1 \pm r_2) \sum_{k>0} \frac{s^\pm}{k} a_i(0) a_j(0) + \epsilon_{\pm \alpha_i(0)} \epsilon_{\pm \alpha_j} \right)
\]

Therefore, the following formulas hold

\[
: X^\pm_i(z) X^\pm_j(w) = : X^\pm_j(w) X^\pm_i(z) :,
\]

\[
: X^\pm_i(z) X^\pm_j(w) = : X^\pm_j(w) X^\pm_i(z) :.
\]

Using the above notation, we give the operator product expansions as follows
Lemma 5.5. For $i \neq j$ in $I$ such that $a_{ij} \neq 0$, we have the following operator product expansions.

\[
X_i^\pm(z)X_j^\pm(w) =: X_i^\pm(z)X_j^\pm(w) : \left(\frac{z}{w^{1/2}} - (rs)^{1/2}\left(\frac{w}{z}\right)\right)\delta_0(\alpha_i, \alpha_j)^{\mp 1}
\]

\[
X_i^\pm(z)X_j^\pm(w) =: X_i^\pm(z)X_j^\pm(w) : \left(1 - (r^{-1}s)^{1/2}\left(\frac{w}{z}\right)\right)^{-1}\left(\frac{w}{z}\right)^{1/2}\delta_0(\alpha_i, \alpha_j)^{\mp 1}
\]

\[
X_i^\pm(z)X_i^\pm(w) =: X_i^\pm(z)X_i^\pm(w) : \left(1 - (rs)^{1/2}(rs)^{1/2}\left(\frac{w}{z}\right)^{-1}\right)^{-1}\left(\frac{w}{z}\right)^{1/2}\delta_0(\alpha_i, \alpha_i)^{\mp 1}
\]

where the two factors are called contraction factors, denoted as $X_i^\pm(z)X_j^\pm(w)$ and $X_i^\pm(z)X_i^\pm(w)$ respectively in the sequel.

**Proof.** By the definition of normal ordering, the first and third formula follow from the following equalities.

\[
E_\pm^+(\alpha_i, z)E_\pm^+(\alpha_j, w)
\]

\[
= \exp \left( \mp \sum_{k>0} \frac{r^{1/2}}{|k|} a_i(k)z^{-k} \right) \exp \left( \mp \sum_{k>0} \frac{s^{1/2}}{|k|} a_j(-k)w^k \right)
\]

\[
= E_\pm^+(\alpha_j, w)E_\pm^+(\alpha_i, z) \exp \left( \sum_{k>0} \frac{(rs)^{1/2}}{|k|^2} \left[ a_i(k), a_j(-k) \right] \left(\frac{w}{z}\right)^k \right)
\]

\[
= E_\pm^+(\alpha_j, w)E_\pm^+(\alpha_i, z) \exp \left( \sum_{k>0} \frac{(rs)^{1/2}}{|k|} \frac{(i, i)^{a_{ij}k}}{k(r-s)} \left(\frac{w}{z}\right)^k \right),
\]

\[
= E_\pm^+(\alpha_j, w)E_\pm^+(\alpha_i, z)
\]

\[
\begin{cases} 
(1 - (rs)^{1/2}\left(\frac{w}{z}\right))^{-1}, & i \neq j, \\
(1 - (rs)^{1/2}(r^{-1}s)^{1/2}\left(\frac{w}{z}\right))^{-1}, & i = j.
\end{cases}
\]

It remains to deal with the rest two formulas, which hold from the following results.

\[
E_\pm^+(\alpha_i, z)E_\pm^+(\alpha_j, w)
\]

\[
= \exp \left( \mp \sum_{k>0} \frac{r^{1/2}}{|k|} a_i(k)z^{-k} \right) \exp \left( \pm \sum_{k>0} \frac{s^{1/2}}{|k|} a_j(-k)w^k \right)
\]

\[
= E_\pm^+(\alpha_j, w)E_\pm^+(\alpha_i, z) \exp \left( - \sum_{k>0} \frac{(r^{-1}s)^{1/2}}{|k|^2} \left[ a_i(k), a_j(-k) \right] \left(\frac{w}{z}\right)^k \right)
\]

\[
= E_\pm^+(\alpha_j, w)E_\pm^+(\alpha_i, z) \exp \left( - \sum_{k>0} \frac{(r^{-1}s)^{1/2}}{|k|} \frac{(i, i)^{a_{ij}k}}{k(r-s)} \left(\frac{w}{z}\right)^k \right)
\]

\[
= E_\pm^+(\alpha_j, w)E_\pm^+(\alpha_i, z)
\]

\[
\begin{cases} 
(1 - (r^{-1}s)^{1/2}\left(\frac{w}{z}\right))^{-1}, & i \neq j, \\
(1 - (r^{-1}s)^{1/2}(r^{-1}s)^{1/2}\left(\frac{w}{z}\right))^{-1}(1 - (r^{-1}s)^{1/2}(r^{-1}s)^{1/2}\left(\frac{w}{z}\right))^{-1}, & i = j.
\end{cases}
\]
where we have used the formula: \( \log(1 - x) = -\sum_{n>0} \frac{x^n}{n} \).

Now we turn to check the relation (3.10).

**Lemma 5.6.** The vertex operators satisfy the following

\[
F_{ij}^\pm(z, w) X_j^\pm(z) X_i^\pm(w) = G_{ij}^\pm(z, w) X_j^\pm(w) X_i^\pm(z),
\]

where \( F_{ij}^\pm(z, w) \) and \( G_{ij}^\pm(z, w) \) are defined as definition 3.1.

**Proof.** Similarly, if \( a_{ij} = 0 \), it is trivial, So we only show the Lemma for \( a_{ij} \neq 0 \).

First we notice that

\[
\frac{G_{ij}^\pm(z, w)}{F_{ij}^\pm(z, w)} = \frac{(j, i)^{\pm 1}z - ((j, i)(i, j))^{\pm \frac{1}{2}}w}{z - (i, j)(j, i)^{\pm \frac{1}{2}}w} = \begin{cases} 
(rs)^{\pm \frac{1}{2}}(r^{-1}su)^{\pm \frac{1}{2}}z - w, & i > j; \\
(rs)^{\pm \frac{1}{2}}(r^{-1}sv)^{\pm \frac{1}{2}}z - w, & i < j; \\
(rs^{-1})^{\pm \frac{1}{2}}z - w, & i = j,
\end{cases}
\]

On the other hand, we have

\[ : X_i^\pm(z) X_j^\pm(w) := X_j^\pm(w) X_i^\pm(z) : \]

Thus we get the required statement immediately. \(\square\)

Furthermore, before verifying the relation (3.9), we need the following lemma.

**Lemma 5.7.** We claim that

\[ : X_i^+(z) X_i^-(zr) := \Phi_i(zs^{-\frac{1}{2}})(rs)^{-\frac{1}{2}}, \]

\[ : X_i^+(ws^{-1}) X_i^-(w) := \Psi_i(wr^{\frac{1}{2}})(rs)^{-\frac{1}{2}}, \]

which can be easily verified directly.

**Proof.** Continuing to use the above notations, we get immediately,

\[ : X_i^+(z) X_i^-(zr) : \]

\[
= \exp \left( \sum_{n=1}^{\infty} \frac{a_i(-n)(z)^n}{n} \right) \exp \left( -\sum_{n=1}^{\infty} \frac{a_i(-n)(zr)^n}{n} \right) \\
= \exp \left( -r^{-n/2} a_i(n)(z)^{-n} \right) \exp \left( \sum_{n=1}^{\infty} \frac{a_i(n)(zr)^{-n}}{n} \right) \\
= \exp \left( - (r - s) \sum_{n>0} a_i(-n)(zs^{-\frac{1}{2}})^{n} \right) r^{\varepsilon_i(0)} s^{-\frac{1}{2}} \\
= \Phi_i(zs^{-\frac{1}{2}})(rs)^{-\frac{1}{2}}.
\]
Similarly, we get for \( i > j \).\]

\[
\begin{align*}
[X_i^+(z), X_j^-(w)] &= \frac{\delta_{ij}}{r_i - s_i} \left( \delta(z w^{-1} s) \phi_i(w r^{-\frac{1}{2}}) - \delta(z w^{-1} r) \phi_i(z s^{-\frac{1}{2}}) \right).
\end{align*}
\]

**Proof.** Firstly, observe that

\[
X_i^+(z) X_i^-(w) = X_i^+(z) X_i^-(w).
\]

If \( i > j \), it is easy to see that

\[
[X_i^+(z), X_j^-(w)]
= X_i^+(z) X_j^-(w) - X_j^-(w) X_i^+(z)
= X_i^+(z) X_j^-(w) - \left( (z w^{-1} s)^{\frac{1}{2}} (w z^{-1} s)^{-\frac{1}{2}} (w z^{-1} r)^{\frac{1}{2}} - (w z^{-1} s)^{\frac{1}{2}} (w z^{-1} r)^{-\frac{1}{2}} (w z^{-1} s)^{-\frac{1}{2}} (w z^{-1} r)^{-\frac{1}{2}} \right)
= 0.
\]

Similarly, we get for \( i < j \),

\[
X_i^+(z) X_j^-(w)
= X_i^+(z) X_j^-(w) - X_j^-(w) X_i^+(z)
= X_i^+(z) X_j^-(w) - \left( \left( (z w^{-1} s)^{\frac{1}{2}} - (w z^{-1} s)^{\frac{1}{2}} (w z^{-1} r)^{-\frac{1}{2}} (w z^{-1} s)^{-\frac{1}{2}} (w z^{-1} r)^{-\frac{1}{2}} \right) + (w z^{-1} s)^{\frac{1}{2}} (w z^{-1} r)^{\frac{1}{2}} \right)
= 0.
\]

It suffices to show the case of \( i = j \). Firstly, we get directly,

\[
X_i^+(z) X_i^-(w) = (r s)^{-\frac{1}{2}} \left( \frac{1}{1 - r^{-1} w} \frac{1}{1 - s^{-1} w} \right) w z
= (r s)^{-\frac{1}{2}} \left( \frac{r w}{1 - r^{-1} w} - \frac{s w}{1 - s^{-1} w} \right).
\]

On the other hand, it is clear to see that

\[
X_i^+(w) X_i^-(z) = (r s)^{\frac{1}{2}} \left( \frac{r w}{1 - r^{-1} w} - \frac{s w}{1 - s^{-1} w} \right).
\]
As a consequence, the bracket becomes

\[ [X^+_i(z), X^-_i(w)] =: X^+_i(z)X^-_i(w) : (rs)^{-\frac{1}{2}} \left( \frac{r \frac{w}{z}}{1 - s \frac{w}{z}} - \frac{s \frac{w}{z}}{1 - r \frac{w}{z}} \right) \left( \frac{r^2 s \frac{w}{z}}{1 - r \frac{w}{z}} - \frac{r s^2 \frac{w}{z}}{1 - s \frac{w}{z}} \right) \]

\[ =: X^+_i(z)X^-_i(w) : \left( \sum_{n \geq 0} s^{-n} r \left( \frac{w}{z} \right)^{n+1} \right) \]

\[ - (r-s) \left( \sum_{n \geq 0} r^{-n} s \left( \frac{w}{z} \right)^{n+1} - \sum_{n \geq 0} r^{n+2} s \left( \frac{w}{z} \right)^{n+1} + \sum_{n \geq 0} s^{n+2} \left( \frac{w}{z} \right)^{n+1} \right) \]

\[ =: X^+_i(z)X^-_i(w) : (rs)^{-\frac{1}{2}} \left( \delta \left( \frac{w}{z} \right) - \delta \left( \frac{z}{w} \right) \right) \]

Applying the above Lemma 5.7, then we arrive at

\[ [X^+_i(z), X^-_i(w)] =: X^+_i(z)X^-_i(w) : \left( rs \right)^{-\frac{1}{2}} \left( \delta \left( \frac{w}{z} \right) - \delta \left( \frac{z}{w} \right) \right) \]

\[ = \frac{1}{r-s} \left( \Phi_i(wr^{\frac{1}{2}})\delta \left( \frac{z}{w} \right) - \Phi_i(zs^{-\frac{1}{2}})\delta \left( \frac{w}{z} \right) \right), \]

where we have used the property of \( \delta \)-function:

\[ f(z_1, z_2)\delta \left( \frac{z_1}{z_2} \right) = f(z_1, z_2)\delta \left( \frac{z_2}{z_1} \right) = f(z_2, z_1)\delta \left( \frac{z_1}{z_2} \right). \]

Lastly, we are left to show the quantum Serre-relations (3.11)—(3.13).

For simplicity, we only check the “+” case of \( a_{ij} = -1, 1 \leq i < j < n \), i.e.,

\[ X^+_i(z_1)X^+_j(z_2)X^+_j(w) - (r+s)X^+_i(z_1)X^+_j(w)X^+_j(z_2) \]

\[ + (rs)X^+_j(w)X^+_i(z_1)X^+_j(z_2) \{ z_1 \leftrightarrow z_2 \} = 0. \]

The others can be obtained similarly.

Let us review the following formulas for further reference.

\[ X^+_j(w)X^+_i(z) =: X^+_j(w)X^+_i(z) : \left( \frac{z}{w} \right)^{\frac{1}{2}} (1 - (r^{-1}s)\frac{z}{w})^{-1} \epsilon_0(\alpha_j, \alpha_i), \]

\[ X^+_i(z_1)X^+_i(z_2) =: X^+_i(z_1)X^+_i(z_2) : \frac{z_2}{z_1}^{-1} (1 - \frac{z_2}{z_1})^{-1} (r^{-1}s)\frac{z_2}{z_1} \epsilon_0(\alpha_i, \alpha_i). \]

By the properties of normal ordering, we then have

\[ X^+_j(w)X^+_i(z_1)X^+_i(z_2) \]

\[ =: X^+_j(w)X^+_i(z_1)X^+_i(z_2) : \frac{z_1}{w} \frac{z_2}{z_1}^{-1} \left( rs \right)^{-1/2} \frac{z_2}{z_1}^{-1/2} (1 - (r^{-1}s)\frac{z_2}{z_1}) \frac{z_2}{z_1} \]
fractions, we can express the contraction factors as follows.

\[
X_i^+(z_1)X_j^+(w)X_i^+(z_2) =: X_i^+(z_1)X_j^+(w) : \left( \frac{z_2}{z_1} \right)^{\frac{1}{2}} \left( \frac{(1 - r^{-1}s)z_2}{1 - r^{-1}s} \right) \left( \frac{(1 - r^{-1}s)z_1}{1 - r^{-1}s} \right)^{\frac{1}{2}} ,
\]

\[
X_i^+(z_1)X_i^+(z_2)X_j^+(w) =: X_i^+(z_1)X_i^+(z_2)X_j^+(w) : \left( \frac{w}{z_1} \right)^{\frac{1}{2}} \left( \frac{(r^{-1}s)^{1/2}(1 - r^{-1}s)}{1 - r^{-1}s} \right)^{\frac{1}{2}} ,
\]

Note that

\[
: X_j^+(w)X_i^+(z_1)X_i^+(z_2) : = : X_i^+(z_1)X_j^+(w)X_i^+(z_2) : = : X_i^+(z_1)X_i^+(z_2)X_j^+(w) : .
\]

Let us proceed to show this formal series identity. Both as formal series and fractions, we can express the contraction factors as follows.

\[
\frac{(z_1 - z_2)(1 - r^{-1}s)z_1}{(1 - r^{-1}s)^{\frac{3}{2}}(z_1)} \left( \frac{(1 - r^{-1}s)^{\frac{1}{2}}}{1 - r^{-1}s} \right) \frac{1 - r^{-1}s}{1 - r^{-1}s} - \frac{1}{w(1 - r^{-1}s)^{\frac{1}{2}}(z_1)} ,
\]

\[
\frac{(z_1 - z_2)(1 - r^{-1}s)z_1}{(1 - r^{-1}s)^{\frac{3}{2}}(z_1)} \left( \frac{(1 - r^{-1}s)^{\frac{1}{2}}}{1 - r^{-1}s} \right) \frac{1 - r^{-1}s}{1 - r^{-1}s} = \frac{1}{w(z_1 - z_2)} \left( \frac{(1 - r^{-1}s)^{\frac{1}{2}}}{z_1(1 - r^{-1}s)^{\frac{3}{2}}(z_1)} + \frac{1}{w(1 - r^{-1}s)^{\frac{1}{2}}(z_1)} ,
\]

\[
\frac{(z_1 - z_2)(1 - r^{-1}s)z_1}{(1 - r^{-1}s)^{\frac{3}{2}}(z_1)} \left( \frac{(1 - r^{-1}s)^{\frac{1}{2}}}{1 - r^{-1}s} \right) \frac{1 - r^{-1}s}{1 - r^{-1}s} = (1 - r^{-1}s)^{\frac{1}{2}}(z_1) \left( \frac{1}{1 - r^{-1}s)^{\frac{3}{2}}(z_1)} - \frac{z_1}{1 - r^{-1}s)^{\frac{1}{2}}(z_1} .
\]

Changing the positions of \(z_1\) and \(z_2\), we obtain the other three expressions for the part \(\{z_1 \leftrightarrow z_2\}\) in the quantum Serre relation. Substituting the above expressions into the right hand side of (5.6) and pulling out the common factors, we obtain that

\[
(z_1z_2)^{-\frac{1}{2}} \left\{ (z_1 - (r^{-1}s)z_2) \left( \frac{(1 - r^{-1}s)^{\frac{1}{2}}}{1 - r^{-1}s)^{\frac{3}{2}}(z_1)} - \frac{(1 - r^{-1}s)^{\frac{1}{2}}}{1 - r^{-1}s)^{\frac{3}{2}}(z_1)} \right) \right. \\
\left. + (r + s)(r^{-1}s)^{1/2}w(z_1 - z_2) \left( \frac{1 - r^{-1}s)^{\frac{1}{2}}}{z_1(1 - r^{-1}s)^{\frac{3}{2}}(z_1)} + \frac{1}{w(1 - r^{-1}s)^{\frac{1}{2}}(z_1)} \right) \\
+ \frac{w}{z_2}(1 - r^{-1}s)^{\frac{1}{2}} \left( \frac{z_1}{1 - r^{-1}s)^{\frac{3}{2}}(z_1)} - \frac{z_1}{1 - r^{-1}s)^{\frac{1}{2}}(z_1)} \right) + (z_1 \leftrightarrow z_2) \right\} ,
\]

where \(\frac{1}{1 - r^{-1}s)^{\frac{1}{2}}(z_1)}\) stands for \(\sum_{n=0}^{\infty} (r^{-1}s)^n(z_1)^n\) and similar fraction for other formal power series.
Collecting the factors \(\frac{1}{1-(r^{-1}s)^{\frac{1}{2}}} \cdot \frac{1}{1-(r^{-1}s)^{\frac{1}{2}}} \cdot \frac{1}{1-(r^{-1}s)^{\frac{1}{2}}} \cdot \frac{1}{1-(r^{-1}s)^{\frac{1}{2}}}\), we get
\[
\frac{1}{1-(r^{-1}s)^{\frac{1}{2}}} (r^{-1}s - \frac{1}{2} (z_1 - (r^{-1}s)z_2) - (r^{-1}s)^{-\frac{1}{2}} (z_2 - (r^{-1}s)z_1) + (r+s) (zs) - \frac{1}{2} (z_2 - z_1) ) + \frac{1}{1-(r^{-1}s)^{\frac{1}{2}}} z_2 \frac{w}{z_1} (- (r^{-1}s)^{-\frac{1}{2}} (z_1 - (r^{-1}s)z_2) + (r^{-1}s)^{-\frac{1}{2}} (z_2 - (r^{-1}s)z_1) ) - \frac{1}{1-(r^{-1}s)^{\frac{1}{2}}} z_2 \frac{w}{z_2} (w(1 - (r^{-1}s)\frac{z_1}{z_2}) + w(\frac{z_1}{z_2} - (r^{-1}s)) + (r+s)(rs) - \frac{1}{2} (r^{-1}s)^{\frac{1}{2}} w(1 - \frac{z_2}{z_1}) )
\]
where each term is zero.

Consequently, we complete the proof of Theorem 5.2.

\[\square\]

6. Appendix: Proofs of Some Lemmas via Quantum Calculations

Here, we would provide more details for some Lemmas’ proofs, through which readers can see the quantum calculations of \((r, s)\)-brackets how to work.

**Proof of Lemma 4.2.** It is easy to get that
\[
x_{\beta_{i-1,i+1}}^-(1) = \left[ x_{i+1}^-(0), \cdots, x_n^-(0), x_{n-2}^-(0), \cdots, x_{i+1}^-(0),
\right.
\left.\left[ x_{i}^-(0), x_{i-1}^-(1) \right]_{s, \cdots, s, r^{-1}, \cdots, r^{-1}}\right]
\]
by \((3.22)\)

\[
= -(rs)^{\frac{1}{2}} \left[ x_{i+1}^-(0), \cdots, x_n^-(0), x_{n-2}^-(0), \cdots, x_{i+1}^-(0),
\right.
\left.\left[ x_{i-1}^-(0), x_{i}^-(1) \right]_{s^{-1}} \right]_{s, \cdots, s, r^{-1}, \cdots, r^{-1}}
\]
by \((3.16)\) & \((D9_1)\)

\[
= -(rs)^{\frac{1}{2}} \left[ x_{i+1}^-(0), \cdots, x_n^-(0), x_{n-2}^-(0), \cdots, x_{i+1}^-(0),
\right.
\left.\left[ x_{i-1}^-(0), x_{i}^-(1) \right]_{(r^{-1}, s^{-1})} \right]_{s, \cdots, s, r^{-1}, \cdots, r^{-1}}
\]
\((= 0 \text{ by } (D9_3))\)

\[
= 0.
\]

\[\square\]

**Proof of Lemma 4.3.** Use induction on \(i\). For the case of \(i = n-1\), by definition, it is easy to see that
\[
[x_{n-1}^-(0), x_{\beta_{n-1,n}}^- (1) ]_{(rs)^{-1}} = [x_{n-1}^-(0), x_n^-(1) ]_{(rs)^{-1}} = 0.
\]
Suppose Lemma 4.3 is true for the case of \( i \), then for the case of \( i - 1 \), we note that

\[
x_{\beta_{i-1},i}(1) = x_{i-1}(0), \cdots, x_n(0), x_{n-2}(0), \cdots, x_{i-1}(1)_{(s, \cdots, s, r^{-1}, \cdots, r^{-1})} \text{ (by (3.22))} = r^{-1}s \left[ x_{i-1}(0), \cdots, x_n(0), x_{n-2}(0), \cdots, [x_{i-1}(1), x_{i-1}(0)]_r \right]_{(s, \cdots, s, r^{-1}, \cdots, r^{-1})} \text{ (by (3.16) & (D9_1))}
\]

\[
\cdots = r^{-1}s \left[ x_{i-1}(0), [x_{i-1}(0), x_{i-1}(0)]_{r,r^{-1}} \right]_{(r,r^{-1})} \text{ (by definition)}
\]

\[
\cdots = r^{-1}s \left[ x_{i-1}(0), x_{\beta_{i-1},i+1}(1), x_{i-1}(0) \right]_{(r,r^{-1})} \text{ (by (3.16))}
\]

\[
r^{-1}s \left[ [x_{i-1}(0), x_{\beta_{i-1},i+1}(1)]_{(r,s^{-1}), i-1}, [x_{i-1}(0)]_s \right]_{rs} = 0 \text{ by inductive hypothesis}
\]

\[
r^{-2} \left[ x_{\beta_{i-1},i+1}(1), [x_{i-1}(0), x_{i-1}(0)]_s \right]_{rs}
\]

Using the above identity, then we have

\[
[x_{i-1}(0), x_{\beta_{i-1},i}(1)]_{r^{-2}} = r^{-2} \left[ x_{i-1}(0), [x_{\beta_{i-1},i+1}(1), [x_{i-1}(0), x_{i-1}(0)]_s \right]_{rs} \text{ (by (3.16))}
\]

\[
r^{-2} \left[ [x_{i-1}(0), x_{\beta_{i-1},i+1}(1)]_{r^{-1}}, [x_{i-1}(0), x_{i-1}(0)]_s \right]_{rs}
\]

\[
+ r^{-3} \left[ x_{\beta_{i-1},i+1}(1), [x_{i-1}(0), x_{i-1}(0), x_{i-1}(0)]_{(s,r^{-1})} \right]_{(r,s)^2} = 0 \text{ by (D9_3)}
\]

\[
r^{-2} \left[ [x_{i-1}(0), x_{\beta_{i-1},i+1}(1)]_{r^{-1}}, [x_{i-1}(0), x_{i-1}(0)]_s \right]_{rs}
\]

At the same time, we can also get

\[
[x_{i-1}(0), x_{\beta_{i-1},i+1}(1)]_{r^{-1}} = [x_{i-1}(0), x_{i+1}(0), \cdots, x_n(0), x_{n-2}(0), \cdots,
\]

\[
\cdot x_{i+1}(1)_{(s, \cdots, s, r^{-1}, \cdots, r^{-1})}]_{r^{-1}} \text{ (by (D9_1))}
\]

\[
[x_{i-1}(0), x_{i+1}(0), \cdots, x_n(0), x_{n-2}(0), \cdots, x_{i+1}(0),
\]

\[
\cdot x_{i-1}(1)_{(s, \cdots, s, r^{-1}, \cdots, r^{-1})}]_{r^{-1}} \text{ (by (3.22) and definition)}
\]

\[
r^{-2} \left[ x_{i-1}(0), \cdots, x_n(0), x_{n-2}(0), \cdots, x_{i+1}(0),
\]

\[
\cdot x_{i-1}(1)]_s \text{ (by definition)}
\]

\[
= - (rs)^{-\frac{1}{2}} x_{\beta_{i-1},i+1}(1).
\]
Expanding the two sides of the above identity, one gets
\[ [x_{\beta_{i-1}}(0), x_{\beta_{i-1},1}(1)]_{r^{-2}} \]
\[ = -r^{-\frac{i}{2}}s^{-\frac{i}{2}}[x_{\beta_{i-1},i+1}(1), [x_{i-1}^-(0), x_{i-1}^-(0)]_s]_{rs} \quad \text{(by (3.16))} \]
\[ = -r^{-\frac{i}{2}}s^{-\frac{i}{2}}[x_{\beta_{i-1},i+1}(1), x_{i-1}^-(0)]_r, x_{i-1}^-(0)]_s^2 \quad \text{(by definition)} \]
\[ - r^{-\frac{i}{2}}s^{-\frac{i}{2}}[x_{i-1}^-(0), [x_{\beta_{i-1},i+1}(1), x_{i-1}^-(0)]_s]_{r^{-1}s} \quad (=0 \text{ by Lemma 4.2}) \]
\[ = (rs)^{-1}[[x_{\beta_{i-1},1}(1), x_{i-1}^-(0)]_s^2. \]

Expanding the two sides of the above identity, one gets
\[ (1 + r^{-1}s)[x_{i-1}^-(0), x_{\beta_{i-1},1}(1)]_{(rs)^{-1}} = 0, \]
which implies that if \( r \neq -s \), then \([x_{i-1}^-(0), x_{\beta_{i-1},1}(1)]_{(rs)^{-1}} = 0. \) Thus we have checked Lemma 4.3 for the case of \( i - 1 \). Consequently, Lemma 4.3 has been proved by induction. \( \square \)

**Proof of Lemma 4.4.** Firstly, note that
\[ [x_{2}^-(0), x_{\alpha_{1},4}(1)] \quad \text{(by definition)} \]
\[ = [x_{2}^-(0), [x_{4}^-(0), \cdots, x_{n}^-(0), x_{n-2}^-(0), \cdots, x_{4}^-(0), x_{\alpha_{1},4}(1)]_{(s, \cdots, s, r^{-1}, \cdots, r^{-1})}] \quad \text{(by (3.16) & (D9_1))} \]
\[ = [x_{4}^-(0), \cdots, x_{n}^-(0), x_{n-2}^-(0), \cdots, x_{4}^-(0), [x_{2}^-(0), x_{\alpha_{1},4}(1)]_{(s, \cdots, s, r^{-1}, \cdots, r^{-1})}. \]

So it suffices to check the relation \([x_{2}^-(0), x_{\alpha_{1},4}(1)] = 0. \)

In fact, it is easy to see that
\[ [x_{2}^-(0), x_{\alpha_{1},4}(1)]_{r^{-1}s} \quad \text{(by definition)} \]
\[ = [x_{2}^-(0), [x_{2}^-(0), x_{2}^-(0), x_{1}^-(1)]_{(s, s)}]_{r^{-1}s} \quad \text{(by (3.16))} \]
\[ = [x_{2}^-(0), [x_{2}^-(0), x_{2}^-(0)], x_{1}^-(1)]]_{(s, r^{-1}s)} \quad \text{(by (3.16))} \]
\[ + s[x_{2}^-(0), x_{2}^-(0), x_{1}^-(1)]_{(1, r^{-1}s)} \quad (=0 \text{ by (D9_1)}) \]
\[ = [[x_{2}^-(0), [x_{2}^-(0), x_{2}^-(0)]_s]_{r^{-1}}, x_{1}^-(1)]_{s^2} \quad (=0 \text{ by (D9_3))} \]
\[ + r^{-1}[[x_{3}^-(0), x_{2}^-(0)]_s, [x_{2}^-(0), x_{1}^-(1)]_s]_{rs} \quad \text{(by definition & (3.17))} \]
\[ = r^{-1}[x_{3}^-(0), [x_{2}^-(0), x_{2}^-(0), x_{1}^-(1)]_s]_{s^2} \quad (=0 \text{ by (D9_2))} \]
\[ + [[x_{3}^-(0), x_{2}^-(0), x_{1}^-(1)]_{(s, s)}, x_{2}^-(0)]_{r^{-1}s} \quad \text{(by definition)} \]
\[ = [x_{\alpha_{1},4}(1), x_{2}^-(0)]_{r^{-1}s}. \]

Then, we obtain \((1 + r^{-1}s)[x_{2}^-(0), x_{\alpha_{1},4}(1)] = 0. \) When \( r \neq -s \), we arrive at our required conclusion \([x_{2}^-(0), x_{\alpha_{1},4}(1)] = 0. \) \( \square \)
Proof of Lemma 4.6. Repeatedly using (3.16), it is easy to get that

\[
[x_i^- (0), x_{\beta_1,1}^- (1)]_{s-1} = [x_i^- (0), [x_i^- (0), x_{i+1}^- (0), x_{\beta_1,2}^- (1)]_{(r-1),s-1}]
\]

(by definition)

\[
= [x_i^- (0), [x_i^- (0), x_{i+1}^- (0)]_{s-1}, x_{\beta_1,2}^- (1)]_{(r-1),s-1}
\]

(by (3.16))

\[
+ r^{-1} [x_i^- (0), x_{i+1}^- (0), x_i^- (0), x_{\beta_1,2}^- (1)]_{(1,s-1)}
\]

(=0 by Lemma 4.5)

\[
= [\left( x_i^- (0), x_{i+1}^- (0) \right)_{s-1}, x_{\beta_1,2}^- (1)]_{r-1}
\]

(=0 by (D9))

\[
+ s^{-1} [\left( x_i^- (0), x_{i+1}^- (0) \right)_{r-1}, x_{\beta_1,2}^- (1)]_{r-1}
\]

(=0 by Lemma 4.5)

\[
= 0.
\]

\[
\square
\]

Proof of Lemma 4.7. By direct calculation, one has

\[
[x_n^- (0), x_n^- (0), x_{\alpha_1,1}^- (1)]_{(r,s^2,rs^2)}
\]

(by definition)

\[
= [x_n^- (0), x_n^- (0), x_{n-1}^- (0), x_{\alpha_1,-1}^- (1)]_{(s,rs^2,rs^2)}
\]

(by (3.16))

\[
= [x_n^- (0), [x_n^- (0), x_{n-1}^- (0)]_{rs}, x_{\alpha_1,-1}^- (1)]_{(s,rs^2)}
\]

(=0 by (D9))

\[
+ rs [x_n^- (0), x_{n-1}^- (0), [x_n^- (0), x_{\alpha_1,-1}^- (1)]_{s-1}]_{(r,s^2)}
\]

(by (3.10))

\[
= rs [x_n^- (0), x_{n-1}^- (0), x_n^- (0), x_{\alpha_1,-1}^- (1)]_{s-1}
\]

(=0 by (D9))

\[
+ (rs)^2 [x_{n-1}^- (0), x_n^- (0), x_{\alpha_1,-1}^- (1)]_{(s,r)}
\]

(by (3.10))

\[
= (rs)^2 [x_{n-1}^- (0), x_n^- (0), x_{\alpha_1,-1}^- (1)]_{(s,r)}
\]

(\text{by (3.10)})

Therefore, it suffices to show the relation \([x_n^- (0), x_n^- (0), x_{\alpha_1,-1}^- (1)]_{(s,r)} = 0.\)

Indeed, it is easy to get the following conclusion

\[
[x_n^- (0), x_n^- (0), x_{\alpha_1,-1}^- (1)]_{(s,r)}
\]

(by definition)

\[
= [x_n^- (0), x_n^- (0), x_{n-2}^- (0), x_{\alpha_1,-2}^- (1)]_{(s,s)}
\]

(by (3.16))

\[
= [x_n^- (0), [x_n^- (0), x_{n-2}^- (0)]_{s}, x_{\alpha_1,-2}^- (1)]_{(s,r)}
\]

(by (3.16))

\[
+ s [x_n^- (0), x_{n-2}^- (0), [x_n^- (0), x_{\alpha_1,-2}^- (1)]_{(1,r)}
\]

(=0 by (3.16) & (D9))

\[
= [[x_n^- (0), [x_n^- (0), x_{n-2}^- (0)]_{s}], x_{\alpha_1,-2}^- (1)]_{s}
\]

(=0 by (D9))

\[
+ r [[x_n^- (0), x_{n-2}^- (0)]_{s}, [x_n^- (0), x_{\alpha_1,-2}^- (1)]_{r^{-1}s}
\]

(=0 by (3.18) & (D9))

\[
= 0.
\]

\[
\square
\]

Proof of Lemma 4.10. First, we shall check that

\[
[x_i^- (1), x_{\beta_1,3}^- (1)]_{r-1} = 0.
\]
In fact, we get directly
\[
[x_1^-(1), x_{\beta_1,3}^{-1}]_{r-1} \\
= [x_1^-(1), [x_{\beta_1,4}^-(0), x_{\beta_1,4}^{-1}(1)]_{r-1}]_{r-1} \\
= [[x_1^-(1), x_{\beta_1,4}^-(0)], x_{\beta_1,4}^{-1}(1)]_{r-2} \\
\quad + [x_3^-(0), [x_1^-(1), x_{\beta_1,4}^{-1}(1)]_{r-1}]_{r-1} \\
\quad \text{(repeating by (3.10) & (D9))}
\]
\[
= [x_3^-(0), \cdots, x_n^-(0), x_{n-2}^-(0), \cdots, x_1^-(1), x_3^-(0)], \\
\quad [x_2^-(0), x_1^-(1)]_{s, \cdots, s, r^{-1}, \cdots, r^{-1}} \\
\quad + [x_3^-(0), \cdots, x_n^-(0), x_{n-2}^-(0), \cdots, x_3^-(0), \\
\quad [x_1^-(1), [x_2^-(0), x_1^-(1)]_{s, \cdots, s, r^{-1}, \cdots, r^{-1}}]_{r-1}]_{s, \cdots, s, r^{-1}, \cdots, r^{-1}} \\
\quad \text{(=0 by (D9))}
\]
\[
= 0.
\]

Using the above fact, we would like to show that
\[
[x_2^-(0), [x_1^-(1), x_\theta^-(1)]_{r-2}]_1 = 0.
\]

Indeed, notice that
\[
[x_2^-(0), [x_1^-(1), x_\theta^-(1)]_{r-2}]_{r-1} \\
= [x_2^-(0), [x_1^-(1), x_2^- (0)]_{r-1}]_{r-1} \\
\quad + [x_2^-(0), [x_1^-(1), x_{\beta_1,3}^-(1)]_{(r-1, r-1)}]_{(r-1, r-1)} \\
\quad \text{(using (3.16))}
\]
\[
= [[x_2^-(0), x_1^-(1)], x_2^- (0)]_{r-1} \\
\quad + s[[x_1^-(1), x_2^- (0)]_{r-1}, [x_2^- (0), x_{\beta_1,3}^-(1)]_{r-1}]_{r-2 s-1} \\
\quad \text{(by definition)}
\]
\[
= s[x_1^-(1), [x_2^- (0), x_\theta^-(1)]_{s-1}]_{r-3} \\
\quad + [[x_1^-(1), x_\theta^-(1)]_{r-2}, x_2^- (0)]_{r-1} \\
= [[x_1^-(1), x_\theta^-(1)]_{r-2}, x_2^- (0)]_{r-1}.
\]

The above result means that if \( r \neq -s \), then
\[
[x_2^-(0), [x_1^-(1), x_\theta^-(1)]_{r-2}]_1 = 0,
\]
which will be used in the sequel.

Using the above result, we get easily
\[
[[x_2^-(0), x_1^-(1)]], x_\theta^-(1)]_{r-2 s-1} \\
= [x_2^-(0), [x_1^- (1), x_\theta^- (1)]_{r-2}]_1 \\
\quad + r^{-2}[[x_2^- (0), x_\theta^- (1)]_{s-1}, x_1^- (1)]_{r-2 s} \\
\quad \text{(=0 by Lemma 4.1)}
\]
\[
= 0.
\]
Applying the above statement, we are ready to derive that

\[
\frac{[x_{\beta_1,\alpha}^{-}(0), x_{\theta}^{-}(1)]}{s} = [x_{\beta_1,\alpha}^{-}(0), x_{\theta}^{-}(1)] \quad \text{(by definition & (3.16))}
\]

\[
= [x_{\beta_1,\alpha}^{-}(0), x_{\theta}^{-}(1), x_{\theta}^{-}(0)]_s \quad \text{(repeating by (3.16) & Lemma 4.1)}
\]

\[
+ \left[ [x_{\beta_1,\alpha}^{-}(0), x_{\theta}^{-}(1), x_{\theta}^{-}(1)]_s^2 \right. = 0 \quad \text{(by Lemma 4.1)}
\]

\[
= \cdots
\]

\[
= [x_{\beta_1,\alpha}^{-}(0), \cdots, x_n^{-}(0), x_{n-2}^{-}(0), \cdots, x_2^{-}(0),
\]

\[
\left[ [x_2^{-}(0), x_1^{-}(1)]_s, x_{\theta}^{-}(1)_r-x^{-1}(s, s, s, s, r, r, \cdots, r, r^{-1}) \right. = 0 \quad \text{(by the above result)}
\]

\[
= 0.
\]

**Proof of Lemma 4.12.** Firstly, we need to consider

\[
x_{13456}^{-}(1)
\]

\[
= [x_{\theta}^{-}(0), x_5^{-}(0), x_4^{-}(0), x_3^{-}(0), x_1^{-}(1)]_{(\alpha, s, s, s)} \quad \text{(by (3.16))}
\]

\[
= [x_{\theta}^{-}(0), x_5^{-}(0), x_4^{-}(0), x_3^{-}(0), x_1^{-}(1)]_{(s, s, s, s)}
\]

\[
+ [x_{\theta}^{-}(0), x_5^{-}(0), x_4^{-}(0), x_3^{-}(0), x_1^{-}(1)]_{(s, s, s, s)} = 0 \quad \text{(by (3.16) & (D9))}
\]

\[
= [x_{\theta}^{-}(0), x_5^{-}(0), x_4^{-}(0), x_3^{-}(0), x_1^{-}(1)]_{(s, s, s, s)}.
\]

Using the above result, now we turn to check

\[
[x_4^{-}(0), x_{13456}^{-}(1)]_{r^{-1}},
\]

\[
= [x_{\theta}^{-}(0), x_5^{-}(0), x_4^{-}(0), x_3^{-}(0), x_1^{-}(1)]_{(s, r^{-1})} \quad \text{(by (3.16))}
\]

\[
+ r^{-1}[x_{\theta}^{-}(0), x_5^{-}(0), x_4^{-}(0), x_3^{-}(0), x_1^{-}(1)]_{(s, s)} \quad \text{(by (3.17))}
\]

\[
= r^{-1}[x_{\theta}^{-}(0), x_5^{-}(0), x_4^{-}(0), x_3^{-}(0), x_1^{-}(1)]_{(s, s, s)} \quad \text{(by (D9))}
\]

\[
+ [x_{\theta}^{-}(0), x_5^{-}(0), x_4^{-}(0), x_3^{-}(0), x_1^{-}(1)]_{(s, s, s, s)} \quad \text{(by (3.16))}
\]

\[
= [x_{\theta}^{-}(0), x_5^{-}(0), x_4^{-}(0), x_3^{-}(0), x_1^{-}(1)]_{(s, s, s, s, s)} \quad \text{(by (D9))}
\]

\[
+ s[x_{\theta}^{-}(0), x_5^{-}(0), x_4^{-}(0), x_3^{-}(0), x_1^{-}(1)]_{(s, s, s, s)} \quad \text{(by (D9))}
\]

\[
= -r^{-1}s[x_{\theta}^{-}(0), x_{13456}^{-}(1)]_{r^{-1}},
\]

this implies

\[
(1 + r^{-1}s)[x_4^{-}(0), x_{13456}^{-}(1)]_{1} = 0.
\]

So, when \( r \neq -s \), we have

\[
[x_4^{-}(0), x_{13456}^{-}(1)]_{1} = 0.
\]
To get our required conclusion, we also need to deal with
\[ x_{134562}^{-1}(0), x_{134562}^{-1}(1), x_{134562}^{-1}(1), x_{134562}^{-1}(1), x_{134562}^{-1}(1) \]
(by definition)
\[ = [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(1)]_{(r^{-1}, s^{-1})} \]
(by (3.10))
\[ = [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(0)]_{(r^{-1}, s^{-1})} \]
(by (3.10))
\[ + [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(1)]_{(r^{-1}, s^{-1})} \]
(= 0 by (D9))
\[ + [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(1)]_{(r^{-1}, s^{-1})} \]
(= 0 by (3.16) & (D9))
\[ = 0. \]

Combining the definition of quantum root vector with the above relation, one has
\[ x_{134562}^{-1}(1) \]
(by definition)
\[ = [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(1)]_{(r^{-1}, s^{-1})} \]
(by (3.10))
\[ = [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(1)]_{(r^{-1}, s^{-1})} \]
(by (3.10))
\[ + [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(1)]_{(r^{-1}, s^{-1})} \]
(= 0 by the above result)
\[ = [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(1)]_{(r^{-1}, s^{-1})} \]
(= 0 by the above result)

Therefore, by definition, (3.16) and Serre relations, we arrive at
\[ x_{134562}^{-1}(1)_{(r^{-1}, s^{-1})} \]
(by definition)
\[ = [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(1)]_{(r^{-1}, s^{-1})} \]
(= 0 by (D9))

So, to show Lemma 4.12 it suffices to check that \( x_{134562}^{-1}(0), x_{134562}^{-1}(1)_{(r^{-1}, s^{-1})} = 0 \).

Actually,
\[ x_{134562}^{-1}(0), x_{134562}^{-1}(1), x_{134562}^{-1}(1), x_{134562}^{-1}(1), x_{134562}^{-1}(1) \]
(by definition)
\[ = [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(1)]_{(r^{-1}, s^{-1})} \]
(by (3.16))
\[ = [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(1)]_{(r^{-1}, s^{-1})} \]
(by (3.16))
\[ + [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(1)]_{(r^{-1}, s^{-1})} \]
(= 0 by the above result)
\[ + [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(1)]_{(r^{-1}, s^{-1})} \]
(by definition)
\[ = [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(1)]_{(r^{-1}, s^{-1})} \]
(by (3.16))
\[ = [x_{134562}^{-1}(0), x_{134562}^{-1}(0), x_{134562}^{-1}(1)]_{(r^{-1}, s^{-1})} \]
(by (3.16))

Expanding the two sides of the above relation, we get immediately
\[ (1 + r^{-1}s) x_{134562}^{-1}(0), x_{134562}^{-1}(1)_{(r^{-1}, s^{-1})} = 0. \]

Hence, we obtain our required conclusion.

\[ \square \]

**Proof of Lemma 4.13.** To get the first relation, we notice that
\[ x_{134562}^{-1}(1) = [[x_{134562}^{-1}(0), x_{134562}^{-1}(0)], x_{134562}^{-1}(1)]. \]
Then we get directly
\[
[x_4^- (0), x_{1345624}^- (1)]_r
= [ [x_4^- (0), x_4^- (0), x_{2}^- (0)](s, r), x_{1345624}^- (1)]_{r-1}
= 0 \quad \text{(by (D9))}
\]
\[+ r[x_4^- (0), x_2^- (0)]_s [x_4^- (0), x_{1345624}^- (1)]_{r-2}
= 0 \quad \text{(by the proof of Lemma 4.12)}
\]
\[= 0.
\]
To check the second relation, we obtain easily
\[
[x_5^- (0), x_{134562435}^- (1)]_{s^{-2}}
= [ [x_5^- (0), x_4^- (0), x_{134562435}^- (1)](s, s^{-2})
= [ [x_5^- (0), x_4^- (0)]_{s^{-1}}, x_{134562435}^- (1)]_1
\]
\[+ s^{-1}[x_4^- (0), x_5^- (0), x_{134562435}^- (1)]_{s^{-1}^{-2}}
= 0 \quad \text{(by the below conclusion)}
\]
\[= [ [x_5^- (0), x_4^- (0)]_{s^{-1}}, [x_5^- (0), x_{1345624}^- (1)](s, r^{-1})]
= [ [x_5^- (0), x_4^- (0)]_{s^{-1}}, x_5^- (0), x_{1345624}^- (1)]_1
\]
\[+ r[x_5^- (0), [ [x_5^- (0), x_4^- (0)]_{s^{-1}}, [x_5^- (0), x_{1345624}^- (1)](s, r^{-1})]]_{r^{-2}}
= r[x_5^- (0), [x_5^- (0), x_4^- (0), x_5^- (0), x_{1345624}^- (1)](s, 1)]_{(rs)^{-1}r^{-2}}
\]
\[= 0 \quad \text{(by (D9))}
\]
\[+ r[x_3^- (0), [x_3^- (0), x_5^- (0), x_{1345624}^- (1)](s, r^{-1}), x_4^- (0)]_{s^{-1}r^{-2}}
= r[x_3^- (0), x_5^- (0), x_{134562435}^- (1)]_{s^{-1}r^{-2}}
\]
\[= -rs^{-1}[x_3^- (0), x_{134562435}^- (1)]_r
= -rs^{-1}[x_3^- (0), x_{134562435}^- (1)]_{r^{-2}}.
\]
This implies
\[(1 + rs^{-1})[x_3^- (0), x_{134562435}^- (1)]_{(rs)^{-1}r^{-1}} = 0.
\]
So, when \(r \neq -s\),
\[x_3^- (0), x_{134562435}^- (1)]_{(rs)^{-1}r^{-1}} = 0.
\]
Up to now, the proof of Lemma 4.13 is to show that
\[x_3^- (0), x_{134562435}^- (1)]_{s^{-1}r^{-1}} = 0,
\]
and
\[x_4^- (0), x_5^- (0), x_{1345624}^- (1)]_{s, 1} = 0.
\]
Before giving the proof of the above two conclusions, we also need the following claims, whose proofs are easy and left to the reader.

(1) \[x_3^- (0), x_{1345624}^- (1)]_1 = 0,
(2) \[x_5^- (0), x_{1345624}^- (1)]_1 = 0,
(3) \[x_4^- (0), x_{1345624}^- (1)]_r = 0.

Therefore, using the above claims, one has

\[
[x_3^{-1}(0), x_{13456243}^{-1}(1)]_{s^{-1}} \quad \text{(by definition & (3.16))}
\]

\[
= [x_3^{-1}(0), x_4^{-1}(0), x_5^{-1}(0)]_{r^{-1}}, x_{134562}^{-1}(1)]_{s_{s^{-1}}} \quad \text{(by (3.16))}
\]

\[
+ r^{-1}[x_3^{-1}(0), x_4^{-1}(0), x_5^{-1}(0)]_{r_{s^{-1}}}, x_{134562}^{-1}(1)]_{s} \quad (= 0 \text{ by (1))}
\]

\[
= [\ldots]_{s} \quad (= 0 \text{ by (D9)}_{s})
\]

\[
+ s^{-1}[x_3^{-1}(0), x_4^{-1}(0)]_{r^{-1}}, x_{134562}^{-1}(1)]_{s^2} \quad (= 0 \text{ by (1))}
\]

\[
= 0.
\]

For the last conclusion, we also get obviously

\[
[x_4^{-1}(0), x_5^{-1}(0), x_{1345624}^{-1}(1)]_{s, r^{-1}s} \quad \text{(by definition & (3.16))}
\]

\[
= [x_4^{-1}(0), x_6^{-1}(0)]_{s} \quad \text{(by (3.16))}
\]

\[
+ r[x_4^{-1}(0), x_5^{-1}(0), x_6^{-1}(0)]_{r_{s}}, x_{134562}^{-1}(1)]_{s_{s^{-1}}} \quad (= 0 \text{ by (2))}
\]

\[
= [\ldots]_{s} \quad (= 0 \text{ by (D9}_{s})
\]

\[
+ r^{-1}[x_5^{-1}(0), x_6^{-1}(0)]_{r_{s}}, x_{134562}^{-1}(1)]_{s_{s}} \quad (= 0 \text{ by (3))}
\]

\[
= r^{-1}[x_5^{-1}(0), x_6^{-1}(0)]_{r_{s}}, x_{1345624}^{-1}(1)]_{s} \quad (= 0 \text{ by (3))}
\]

\[
= -r^{-1}s[x_4^{-1}(0), x_5^{-1}(0), x_{1345624}^{-1}(1)]_{s, s^{-1}}.
\]

Expanding the two sides of the above relation, we obtain

\[
(1 + r^{-1}s)x_4^{-1}(0), x_5^{-1}(0), x_{1345624}^{-1}(1)]_{s, 1} = 0.
\]

So, \([x_4^{-1}(0), x_5^{-1}(0), x_{1345624}^{-1}(1)]_{s, 1} = 0\) under the condition \(r \neq -s\).

**Proof of Lemma 4.14.** For simplicity, we introduce some notations

\[
A \doteq [x_4^{-1}(0), x_6^{-1}(0), x_6^{-1}(0), x_4^{-1}(0), x_3^{-1}(1)]_{s, s, s^{-1}, r^{-1}, s},
\]

and

\[
B \doteq [x_2^{-1}(0), x_5^{-1}(0), x_6^{-1}(0), x_4^{-1}(1)]_{s, s, r^{-1}}.
\]

Note that

\[
[x_1^{-1}(0), x_{1345624345}^{-1}(1)]_{(rs)^{-1}} \quad \text{(by (3.16))}
\]

\[
= [x_4^{-1}(0), x_5^{-1}(0), x_4^{-1}(0), x_{1345624345}^{-1}(1)]_{(rs)^{-1}}_{s, s}.
\]

As a consequence, it suffices to verify that \([x_1^{-1}(0), x_{13456243}^{-1}(1)]_{(rs)^{-1}} = 0.\)
In fact, we observe that
\[ x_{13456243}^{-1}(1) \]
\[ = [x_3^{-1}(0), x_4^{-1}(0), x_5^{-1}(0), x_6^{-1}(0), x_7^{-1}(0), x_8^{-1}(1)]_{s,s,s,r^{-1},s,r^{-1}} \]  
(by (D7) & (D9))
\[ = [x_3^{-1}(0), x_1^{-1}(0), x_4^{-1}(0), x_5^{-1}(0), x_6^{-1}(0), x_8^{-1}(0), x_7^{-1}(1)]_{s,s,s,r^{-1},s,r^{-1},r^{-1}} \]  
(by (3.10))
\[ = [[x_3^{-1}(0), x_1^{-1}(0), A]_{r^{-2} s^{-1}} + [x_1^{-1}(0), x_3^{-1}(0), A]_{(rs)^{-1}}]_{r^{-1}} \]
\[ = [[x_3^{-1}(0), x_1^{-1}(0), A]_{(rs)^{-1}}]_{r^{-1}} \]
\[ = [x_3^{-1}(0), x_1^{-1}(0), A]_{(rs)^{-1}}. \]

Applying the result, we obtain immediately
\[ [x_1^{-1}(0), x_{13456243}^{-1}(1)]_{s^{-2}} \]  
(by definition)
\[ = [x_1^{-1}(0), [x_7^{-1}(0), x_1^{-1}(0)]_s, A]_{(r^{-2} s^{-1}, r^{-2})} \]  
(by (3.16))
\[ = [[x_1^{-1}(0), x_3^{-1}(0), x_1^{-1}(0)]_{s,r^{-1}}, A]_{r^{-3} s^{-1}} \]  
(=0 by (D9))
\[ + r^{-1}[[x_1^{-1}(0), x_3^{-1}(0)]_s, x_1^{-1}(0), A]_{(rs)^{-1}} \]  
(by (3.17))
\[ = r^{-1}[x_1^{-1}(0), [x_1^{-1}(0), x_3^{-1}(0), A]_{(r^{-1} s^{-1})}]_{r^{-1} s} \]  
(=0 by (D9))
\[ + (rs)^{-1}[[x_1^{-1}(0), x_1^{-1}(0), A]_{(r^{-1} s^{-1}, x^{-1})}]_{s^{-2}} \]
\[ = -r^{-1} s[x_1^{-1}(0), x_{13456243}^{-1}(1)]_{s^{-2}}. \]

This leads to \((1 + r^{-1} s)[x_1^{-1}(0), x_{13456243}^{-1}(1)]_{(rs)^{-1}} = 0.\)
So, under the condition \(r \neq -s\), we get
\[ [x_1^{-1}(0), x_{13456243}^{-1}(1)]_{(rs)^{-1}} = 0. \]

Finally, we are left to verify
\[ [x_3^{-1}(0), A]_{(rs)^{-1}} = 0. \]

Indeed, it is easy to see that
\[ [x_3^{-1}(0), A]_{r^{-2}} \]  
(by definition)
\[ = [x_3^{-1}(0), [x_4^{-1}(0), x_3^{-1}(0)]_s, B]_{(r^{-1}, r^{-2})} \]  
(by (3.16))
\[ = [[[x_3^{-1}(0), x_4^{-1}(0), x_3^{-1}(0)]_{s,r^{-1} s^{-1}}, B]_{r^{-2}} \]  
(=0 by (D9))
\[ + r^{-1}[[x_3^{-1}(0), x_3^{-1}(0)]_s, x_3^{-1}(0), B]_{r^{-1}} \]  
(by (3.17))
\[ = r^{-1}[x_3^{-1}(0), [x_3^{-1}(0), x_3^{-1}(0), B]_{(r^{-1} s^{-1})}]_{s^{-2}} \]  
(=0 by (D9))
\[ + (rs)^{-1}[[x_3^{-1}(0), x_3^{-1}(0), B]_{(r^{-1} s^{-1}, x^{-1})}]_{s^{-2}} \]
\[ = -r^{-1} s[x_3^{-1}(0), A]_{s^{-2}}. \]

This means \((1 + r^{-1} s)[x_3^{-1}(0), A]_{(rs)^{-1}} = 0.\)
So, under the condition \(r \neq -s\), we get \([x_3^{-1}(0), A]_{(rs)^{-1}} = 0.\)

Therefore, this completes the proof. \(\square\)
Added in proof. Part of the work started initially 10 years ago when the first author visited l’DMA, l’Ecole Normale Supérieure de Paris from October to November, 2004, the Fachbereich Mathematik der Universität Hamburg from November 2004 to February 2005. It was not until his visit to ICTP (Trieste, Italy) from March to August, 2006 that Hu found out the explicit formula of the generating function $g_{ij}(z)$ with $\tau$-invariance which leads to the inherent definition for the Drinfeld realization in two-parameter setting. A reason for preventing the submission of this work for 10 years is the newly found and amending constraint: $\gamma\gamma' = (rs)^c$. The original constraint $\gamma\gamma' = rs$ is apparently dissatisfied since product of two group-likes should be still group-like. The current change was realized and made by the first author when the second author tried to construct level-two vertex operator representation and found that the constraint should be $\gamma\gamma' = (rs)^2$ in that module. Finally, the authors would like to express their thanks to Dr. Yunnan Li, who pointed out a necessary revision for the definition of $\gamma, \gamma'$ in Definition 2.2, which is crucial for our main Theorem 3.9.

ACKNOWLEDGMENT

Hu is supported in part by the NNSF of China (No. 11271131), the RSFDLP from the MOE of China. Zhang would like to thank the support of NSFC grant (No. 11101258) and Shanghai Leading Academic Discipline Project (J50101). Both authors are indebted to Marc Rosso for his recommending Grossé’s arXiv-preprint earlier in 2004, from where the initial motivation stemmed.

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