Correlational properties of two-dimensional solvable chaos on the unit circle

Aki-Hiro Sato and Ken Umeno
Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Yoshida-Honcho, Sakyo-ku, 606-8501, Kyoto JAPAN

Abstract
This article investigates correlational properties of two-dimensional chaotic maps on the unit circle. We give analytical forms of higher-order covariances. We derive the characteristic function of their simultaneous and lagged ergodic densities. We found that these characteristic functions are described by three types of two-dimensional Bessel functions. Higher-order covariances between x and y and those between y and y show non-positive values. Asymmetric features between cosine and sine functions are elucidated.

1 Introduction
Knowledge on solvable chaos is useful for designing random number generators [1, 2, 3, 4] and Monte Carlo integration [5]. The idea of applying chaos theory to randomness has produced important works recently [6, 7, 8, 9]. Geisel and Fairen analyzed statistical properties of Chebyshev maps [10]. They showed the mixing properties and higher order moments with higher-order characteristic functions. González and Pino proposed a pseudo random number generator based on logistic maps [11]. Collins et al. [12] have applied the logit transformation to the logistic map variable for producing a sequence with a near Gaussian distribution. These solvable chaotic properties enable us to design and employ chaos for application purposes.

First, let us consider maps in the form of Chebyshev polynomials of degree $k$

$$x_{t+1} = T_k(t),$$

(1)

which map the interval $[-1, 1]$ onto the same interval. The first few polynomials are explicitly $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, and $T_3(x) = 4x^3 - 3x$. Since, there is permutability of the Chebyshev polynomials, $T_k(T_l(x)) = T_{kl}(x)$, Eq. (1) can be expressed as

$$x_t = T_{k^t}(x_0).$$

(2)
It was shown by Adler and Rivlin that Chebyshev maps with \( k \geq 2 \) are ergodic and strongly mixing. This map dynamics has the invariant measure \( \mu(dx) = \frac{dx}{\pi \sqrt{1-x^2}} \). Geisel and Fairen shows that the characteristic function of the Chebyshev maps can be expressed as Bessel function \([10]\). They further considered the higher-order characteristic function. Following their strategy, we consider the characteristic function of two-dimensional solvable chaotic maps on a unit circle. We further calculate the higher-order covariance based on the characteristic function.

This article is organized as follows. In Sec. 2, we introduce two-dimensional chaotic maps on a unit circle. In Sec. 3, we show that simultaneous covariance among two variables is independent. In Sec. 4, we derive an analytical form of higher-order covariance among two variables. In Sec. 5, we compute higher-order covariance among two variables with lags. Sec. 6 is devoted to concluding remarks.

## 2 Two-dimensional solvable chaos

In this article, we consider two-dimensional maps on a unit circle. Suppose that \( z_t = x_t + \sqrt{-1}y_t \) denotes a complex number, where \( x_t \) is a real number and \( y_t \) is an imaginary part at step \( t \) (\( t = 0, 1, \ldots \)). Then, we define the complex dynamics as

\[
z_{t+1} = z_t^k,
\]

where \( k \) is an integer. We can also express Eq. (3) as

\[
\begin{align*}
x_{t+1} &= P_k(x_t, y_t) \\
y_{t+1} &= Q_k(x_t, y_t),
\end{align*}
\]

where \( P_k(x, y) \) and \( Q_k(x, y) \) are defined as

\[
(x + \sqrt{-1}y)^k = P_k(x, y) + \sqrt{-1}Q_k(x, y),
\]

\[
x^2 + y^2 = 1.
\]

The first few polynomials are explicitly given by \( P_1(x, y) = x, \) \( Q_1(x, y) = y, \) \( P_2(x, y) = x^2 - y^2, \) \( Q_2(x, y) = 2xy, \) \( P_3(x, y) = x^3 - 3xy^2, \) \( Q_3(x, y) = 3x^2y - y^3, \) \( P_4(x, y) = x^4 - 6x^2y^2 + y^4, \) \( Q_4(x, y) = 4x^3y - 4xy^3, \) \( P_5(x, y) = x^5 - 10x^3y^2 + 5xy^4, \) \( Q_5(x, y) = 5x^4y - 10x^2y^3 + y^5, \) \( P_6(x, y) = x^6 - 15x^4y^2 + 15x^2y^4 - y^6, \) \( Q_6(x, y) = 6x^5y - 20x^3y^3 + 6xy^5, \) \( P_7(x, y) = x^7 - 21x^5y^2 + 35x^3y^4 - 7xy^6, \) \( Q_7(x, y) = 7x^6y - 35x^4y^3 + 21x^2y^5 - x^7. \)

In general, \( P_k(x, \pm \sqrt{1-x^2}) = T_k(x) \) is satisfied. Specifically, \( Q_k(x, y) \) for odd ordered \( k \) is equivalent to \( Q_k(\pm \sqrt{1-y^2}, y) = -T_k(y). \)

If we set an initial condition \( z_0 = x_0 + \sqrt{-1}y_0 \) on the unit circle \( |z_0| = 1, \) \( z_t \) is also mapped on the unit circle. In this case, Eq. (5) can be rewritten as

\[
\exp(\sqrt{-1}\theta)^k = \exp(k\theta\sqrt{-1}),
\]

\[
\exp(\sqrt{-1}\theta)^k = \exp(k\theta\sqrt{-1}),
\]
where \( \theta \) denotes the argument of \((x, y)\) on the two-dimensional plane. It is convenient to represent the polynomial \( P_k(x, y) \) and \( Q_k(x, y) \) in the form

\[
\begin{aligned}
P_k(\cos \theta, \sin \theta) &= \cos(k\theta) \\
Q_k(\cos \theta, \sin \theta) &= \sin(k\theta)
\end{aligned}
\]  

(8)

Fig. 1 shows a trajectory of \((x_t, y_t)\) for \( k = 2 \). The value at each step stands on the unit circle.

Figure 1: 800 steps of a trajectory of the two-dimensional chaotic map for \( k = 2 \). The initial value is given by \((x_0, y_0) = (-0.820000, 0.572364)\).

By introducing \( \theta_t \) as the argument of \( z_t \), we have

\[ \theta_{t+1} = k\theta_t. \]  

(9)

The solution of Eq. (9) can be written as

\[ \theta_t = k^t\theta_0, \]  

(10)

by using \( \theta_0 \), denoted as the argument of \( z_0 \). Therefore, \( z_t = x_t + \sqrt{-1}y_t \) is rewritten as

\[ z_t = \cos(k^t\theta_0) + \sqrt{-1}\sin(k^t\theta_0) = \exp(k^t\theta_0\sqrt{-1}). \]  

(11)

Eq. (11) is ergodic and has the constant invariant density \( \rho_{\Theta}(\theta) = \frac{1}{2\pi} \) \((0 \leq \theta \leq 2\pi)\) since Eq. (9) is a Bernoulli map on mod 2\( \pi \).

Transforming the orthogonal coordinate \((x, y)\) into the polar coordinate \((r, \theta)\) by \( x = r \cos \theta \) and \( y = r \sin \theta \), we have \( \rho_R(r) = \delta(r - 1) \). Therefore, the joint
invariant density of \( x \) and \( y \) can be described as

\[
\rho_{XY}(x, y) = \rho_\Theta(\theta) \rho_R(r) \left| \frac{\partial(\theta, r)}{\partial(x, y)} \right| = \frac{\delta(\sqrt{x^2 + y^2} - 1)}{2\pi \sqrt{x^2 + y^2}},
\]

(12)

where \( \delta(\cdot) \) represents Dirac’s \( \delta \)-function. The marginal density in terms of \( x \) is given by

\[
\rho_X(x) = \int_{-1}^{1} \rho_{XY}(x, y) dy = \frac{1}{2\pi} \int_{-1}^{1} \frac{\delta(\sqrt{x^2 + y^2} - 1)}{\sqrt{x^2 + y^2}} dy
\]

\[
= \frac{1}{\pi} \int_{|x| - 1}^{\sqrt{x^2 + 1} - 1} \frac{\delta(t)}{\sqrt{(t + 1)^2 - x^2}} dt
\]

\[
= \frac{1}{\pi \sqrt{1 - x^2}}.
\]

In the same way, we obtain

\[
\rho_Y(y) = \int_{-1}^{1} \rho_{XY}(x, y) dx = \frac{1}{\pi \sqrt{1 - y^2}}.
\]

(13)

Note that \( \rho_X(x) \) and \( \rho_Y(y) \) are the same as the ergodic density of the Chebyshev maps.

### 3 Simultaneous covariance

Next, let us consider auto-correlations of \( x \) and \( y \) and cross-correlation between \( x \) and \( y \). Obviously, mean values of \( x \) and \( y \) are given as zero.

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t = \int_{-1}^{1} x \rho_X(x) dx = \int_{-1}^{1} \frac{x}{\pi \sqrt{1 - x^2}} dx = 0,
\]

(14)

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} y_t = \int_{-1}^{1} y \rho_Y(y) dy = \int_{-1}^{1} \frac{y}{\pi \sqrt{1 - y^2}} dy = 0.
\]

(15)

We shall introduce four types of correlations:

\[
c_{XX}(\tau) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t x_{t+\tau} = \int_{-1}^{1} dx \int_{-1}^{1} dy x P_k \circ \cdots \circ P_k(x, y) \rho_{XY}(x, y)
\]

(16)

\[
c_{YY}(\tau) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} y_t y_{t+\tau} = \int_{-1}^{1} dx \int_{-1}^{1} dy y Q_k \circ \cdots \circ Q_k(x, y) \rho_{XY}(x, y)
\]
\(c_{XY}(\tau) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t y_{t+\tau} = \int_{-1}^{1} dx \int_{-1}^{1} dy \frac{Q_k \circ \cdots \circ Q_k(x,y)\rho_{XY}(x,y)}{T} \)  

(17)

\(c_{YX}(\tau) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} y_t x_{t+\tau} = \int_{-1}^{1} dx \int_{-1}^{1} dy \frac{P_k \circ \cdots \circ P_k(x,y)\rho_{XY}(x,y)}{T} \)  

(18)

Transforming the orthogonal coordinate \((x, y)\) into the polar coordinate \((r, \theta)\), we can calculate Eqs. (16) to (19) as

\[c_{XX}(\tau) = \frac{1}{2\pi} \int_{0}^{2\pi} \cos \theta \cos k\theta d\theta = \frac{1}{2} \delta_{1,k} \]

(20)

\[c_{YY}(\tau) = \frac{1}{2\pi} \int_{0}^{2\pi} \sin \theta \sin k\theta d\theta = \frac{1}{2} \delta_{1,k}\]

(21)

\[c_{XY}(\tau) = \frac{1}{2\pi} \int_{0}^{2\pi} \cos \theta \sin k\theta d\theta = 0 \]

(22)

\[c_{YX}(\tau) = \frac{1}{2\pi} \int_{0}^{2\pi} \sin \theta \cos k\theta d\theta = 0 \]

(23)

These are extensions of Chebyshev maps derived by Geisel and Fairen to the two-dimensional map [10]. Therefore, the auto-correlations of \(x\) and \(y\) decay 0 for \(\tau \geq 1\), and the cross-correlations between \(x\) and \(y\) are zero. Furthermore, the correlation between \(z_t\) and \(z_{t+\tau}\), where \(\cdot\) is denoted as the complex conjugate of \(\cdot\), is also zero,

\[\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} z_t z_{t+\tau} = c_{XX}(\tau) - c_{YY}(\tau) + \sqrt{-1}(c_{XY}(\tau) + c_{YX}(\tau)) = 0. \]  

(24)

Note that Eqs. (20) to (24) are derived by means of the permutability of \(z^k\) and the orthogonality between \(P_k(x,y)\) and \(Q_k(x,y)\). Clearly, from Eq. (3) we can prove the permutability of \(z^k\) such as \((z^k)^l = z^{kl}\). For \(k \geq 1\) and \(l \geq 1\), we also have the orthogonal relations among \(P_k(x,y)\) and \(Q_k(x,y)\)

\[\int_{-1}^{1} dx \int_{-1}^{1} dy P_k(x,y) P_l(x,y)\rho_{XY}(x,y) = \frac{1}{2\pi} \int_{0}^{2\pi} \cos(k\theta) \cos(l\theta) d\theta = \frac{1}{2} \delta_{k,l}, \]

(25)

\[\int_{-1}^{1} dx \int_{-1}^{1} dy Q_k(x,y) Q_l(x,y)\rho_{XY}(x,y) = \frac{1}{2\pi} \int_{0}^{2\pi} \sin(k\theta) \sin(l\theta) d\theta = \frac{1}{2} \delta_{k,l}, \]

(26)

\[\int_{-1}^{1} dx \int_{-1}^{1} dy Q_k(x,y) P_l(x,y)\rho_{XY}(x,y) = \frac{1}{2\pi} \int_{0}^{2\pi} \sin(k\theta) \cos(l\theta) d\theta = 0 \]

(27)
4 Simultaneous higher order covariance

Let us consider the characteristic function of the simultaneous joint density \( \rho_{XY}(x,y) \), defined as

\[
\Phi(u,v) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} e^{\sqrt{-1}(ux_t+vy_t)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\sqrt{-1}(ux+vy)} \rho_{XY}(x,y) \, dx \, dy.
\]  

(28)

Inserting Eq. (12) into Eq. (28), we have

\[
\Phi(u,v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\sqrt{-1}(ux+vy)} \delta(\sqrt{x^2+y^2}-1) \, dx \, dy
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^{\infty} r \, \delta(\sqrt{u \cos \theta + v \sin \theta} - (v-1)) \, r \, dr = J_{1,1}^{1,1}(u,v),
\]

(29)

where \( J_{p,q}^{n}(u,v) \) is defined as

\[
J_{n}^{p,q}(u,v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\sqrt{-1}(u \sin(\rho \theta) + v \sin(q \theta) - n \theta)} \, d\theta.
\]  

(30)

This is similar to the two-dimensional Bessel function which was studied by Korsch et al. [13], however, it is a bit different from it. They define the two-dimensional Bessel functions with three integer indices \( n, p, \) and \( q \) as

\[
\hat{J}_{n}^{p,q}(u,v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\sqrt{-1}(u \sin(\rho \theta) + v \sin(q \theta) - n \theta)} \, d\theta
\]  

(31)

In his definition, the two-dimensional Bessel function consists of two sine functions. However, in our definition this consists of cosine and sine functions.

Clearly, both the two-dimensional Bessel functions satisfy

\[
J_{0}^{1,1}(u,0) = J_{0}(u), \quad J_{0}^{1,1}(0,v) = J_{0}(v),
\]  

(32)

\[
\hat{J}_{0}^{1,1}(u,0) = J_{0}(u), \quad \hat{J}_{0}^{1,1}(0,v) = \hat{J}_{0}(v),
\]  

(33)

where \( J_{n}(u) \) is the Bessel function defined as

\[
J_{n}(u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\sqrt{-1}n \theta - u \sin \theta} \, d\theta.
\]  

(34)

In the one-dimensional case, Eq. (29) is equivalent to the characteristic function of Chebyshev polynomials, which is derived by Geisel and Fairen [10].
We can further expand $\Phi(u, v)$ in terms of $u$ and $v$,

$$
\Phi(u, v) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(\sqrt{-1})^n}{n!} \int_{0}^{2\pi} (u \cos \theta + v \sin \theta)^n d\theta
$$

$$
= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(\sqrt{-1})^n}{n!} \sum_{m=0}^{n} \binom{n}{m} u^m v^{n-m} \int_{0}^{2\pi} \cos^m \theta \sin^{n-m} \theta d\theta.
$$

Therefore, we have

$$
\langle X^m Y^{n-m} \rangle = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} x^m_t y^{n-m}_t
$$

$$
= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy x^m y^{n-m} \rho_{XY}(x, y)
$$

$$
= \frac{1}{2\pi} \int_{0}^{2\pi} \cos^m \theta \sin^{n-m} \theta d\theta. \quad (0 \leq m \leq n). \quad (35)
$$

We also have the equality

$$
\int_{0}^{\pi/2} \cos^{2p-1} \theta \sin^{2q-1} \theta d\theta = \frac{1}{2} B(p, q) = \frac{1}{2} \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},
$$

where $B(a, b)$ denotes the beta function, defined as

$$
B(a, b) = \int_{0}^{1} \tau^{a-1}(1-\tau)^{b-1} d\tau,
$$

and $\Gamma(a)$ represents the gamma function, defined as

$$
\Gamma(a) = \int_{0}^{\infty} e^{-\tau} \tau^{a-1} d\tau.
$$

Inserting Eq. (36) into $p = m/2 + 1/2$ and $q = (n-m)/2 + 1/2$ and using symmetry of cosine and sine functions and $\Gamma(n+1) = n!$, we obtain

$$
\langle X^m Y^{n-m} \rangle = \begin{cases} 
\frac{2^{n-m+1} \Gamma(m+1)^2}{2\pi \Gamma(n+1)^2} \frac{(m-1)!(n-m-1)!}{n!!} & (n, m : \text{even}) \\
0 & \text{(otherwise)}
\end{cases} \quad (39)
$$

Hence, the characteristic function of $\rho_{XY}(x, y)$ is described as

$$
\Phi(u, v) = \sum_{n=0}^{\infty} (-1)^n \sum_{m=0}^{n} \frac{(u^2)^m (v^2)^{n-m}}{(2m)!!(2n-2m)!!(2n)!!}.
$$

(40)

This is a natural extension of the Bessel function of degree 0 to the two-dimensional case,

$$
J_0(z) = \sum_{r=0}^{\infty} \frac{(-z^2)^r}{(2r)!!(2r)!!}.
$$

(41)
Since we can further calculate the $m$-th order moment of $x_t$ and the $n-m$-th order moment of $y_t$ as

$$
\langle X^m \rangle = \int_{-1}^{1} dx \int_{-1}^{1} dy x^m \rho_{XY}(x, y)
= \frac{1}{2\pi} \int_{0}^{2\pi} \cos^m \theta d\theta = \begin{cases}
\frac{(m-1)!!}{m!!} & (m : \text{even}) \\
0 & (m : \text{odd})
\end{cases},
$$

and

$$
\langle Y^{n-m} \rangle = \int_{-1}^{1} dx \int_{-1}^{1} dy y^{n-m} \rho_{XY}(x, y)
= \frac{1}{2\pi} \int_{0}^{2\pi} \sin^{n-m} \theta d\theta = \begin{cases}
\frac{(n-m-1)!!}{(n-m)!!} & (n-m : \text{even}) \\
0 & (n-m : \text{odd})
\end{cases},
$$

where $m!! = 2 \cdot 4 \cdot 6 \cdots m$ for even $m$ and $m!! = 1 \cdot 3 \cdot 5 \cdots m$ for odd $m$, we get

$$
\text{Cov}[X^m, Y^{n-m}] = \langle X^m Y^{n-m} \rangle - \langle X^m \rangle \langle Y^{n-m} \rangle
= \begin{cases}
\frac{(m-1)!!(n-m-1)!!}{m!!(n-m)!!} & (m, n : \text{even}) \\
0 & \text{(otherwise)}
\end{cases},
$$

Here, we consider the negativity of even ordered moments. Hammersley suggested that antithetic variables are effective for variance reduction in Monte Carlo integrations [14]. The antithetic-variates method permits estimates through the use of negative correlated random variables faster than independent random variables. Let us confirm the sign of Eq. (44). We get

$$
1 - \frac{n!!}{m!!(n-m)!!} = 1 - \frac{(\frac{n}{2})!}{(\frac{n}{2})!(\frac{n-m}{2})!}
= 1 - \left( \frac{n}{2} \right) \leq 0,
$$

since from the definition of combination, we have

$$
\binom{\frac{n}{2}}{\frac{m}{2}} = \frac{(\frac{n}{2})!}{(\frac{m}{2})!(\frac{n-m}{2})!} \geq 1.
$$

The equality is satisfied if and only if $m = 0$ or $m = n$. Note that Eq. (44) is independent of a value of $k$.

Therefore, Eq. (44) implies that $x_t$ and $y_t$ do not have any correlations for the odd-ordered moments, however, do have a negative covariance for the even-ordered moments. Fig. 2 shows the relationship between $n$ and Cov[$X^m, Y^{n-m}$]. It is confirmed that the covariance monotonically increases and approaches to zero as $n$ increasing.
Furthermore, we calculate covariance between $x^m_t$ and $x^{n-m}_t$, and between $y^m_t$ and $y^{n-m}_t$. From Eqs. (42) and (43), we have

$$\operatorname{Cov}[X^m, X^{n-m}] = \operatorname{Cov}[Y^m, Y^{n-m}]$$

$$= \begin{cases} \frac{1}{2n} \left[ \left( \frac{n}{2} \right) - \left( \frac{m}{2} \right) \left( \frac{n-m}{2} \right) \right] & \text{if } n, m \text{ are even} \\ 0 & \text{otherwise} \end{cases} \geq 0 \quad (n, m : \text{even})$$

(47)

The non-negativity of Eq. (47) is proven as follows. Let us consider the case that $n$ is even. From

$$(1 + x)^n = \left\{ (1 + x)^\frac{n}{2} \right\}^2,$$

one has

$$\sum_{m=0}^{n} \binom{n}{m} x^m = \left( \sum_{m=0}^{\frac{n}{2}} \binom{n}{2m} x^m \right)^2$$

(49)

Comparing $x^m$’s coefficient, we get the following inequality

$$\binom{n}{m} \geq \binom{\frac{n}{2}}{\frac{m}{2}}^2.$$

(50)

Therefore, we obtain

$$\frac{1}{2n} \left[ \left( \frac{n}{2} \right) - \left( \frac{m}{2} \right) \left( \frac{n-m}{2} \right) \right] = \frac{1}{2n} \binom{n}{\frac{m}{2}} \left[ \left( \frac{n}{m} \right) - \left( \frac{\frac{m}{2}}{\frac{m}{2}} \right) \right]^2 \geq 0.$$

(51)

5 Higher order covariance with lags

More generally, we can introduce a characteristic function of the joint density between $x^m_{t+p}$ and $y^{n-m}_{t+q}$.

$$\Psi_{XY}(u, v) \equiv \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} e^{\sqrt{-1}(u x_{t+p} + v y_{t+q})}$$

$$= \left\langle \exp(\sqrt{-1}(u P_k \circ \cdots \circ P_k(x,y) + v Q_k \circ \cdots \circ Q_k(x,y))) \right\rangle$$

$$= \int_{-1}^{1} dx \int_{-1}^{1} dy \exp(\sqrt{-1}(u P_k \circ \cdots \circ P_k(x,y) + v Q_k \circ \cdots \circ Q_k(x,y))) \rho_{XY}(x,y)$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} e^{\sqrt{-1}(u \cos(k\theta) + v \sin(k\theta))} d\theta = J_{0}^{k_p,k_q}(u, v).$$

(52)
Similarly to $\Phi(u, v)$, from the expansion in terms of $u$ and $v$, we obtain

$$
\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t^m y_{t+q}^{n-m} = \int_{-1}^{1} dx \int_{-1}^{1} dy \underbrace{P_k \circ \cdots \circ P_k}_{p}(x, y) \underbrace{Q_k \circ \cdots \circ Q_k}_{q}(x, y) \rho_{XY}(x, y)
$$

$$
= \frac{1}{2\pi} \int_{0}^{2\pi} \cos^{m}(k^p\theta) \sin^{n-m}(k^q\theta) \, d\theta. \quad (53)
$$

By using

$$
\cos^{m}(k^p\theta) \sin^{n-m}(k^q\theta) = \frac{1}{2^m} \left( e^{\sqrt{1-k^p}\theta} + e^{-\sqrt{1-k^p}\theta} \right)^m = \frac{1}{(2\sqrt{-1})^{n-m}} \left( e^{\sqrt{1-k^q}\theta} - e^{-\sqrt{1-k^q}\theta} \right)^{n-m}
$$

$$
= \frac{1}{2^n (\sqrt{-1})^{n-m}} \sum_{r=0}^{m} \sum_{s=0}^{n-m} (-1)^{n-m-s} \frac{m!}{r!(m-r)!} \frac{(n-m)!}{s!(n-m-s)!} e^{\sqrt{1}[(2r-m)k^p + (2s-n+m)k^q] \theta},
$$

and

$$
\frac{1}{2\pi} \int_{0}^{2\pi} e^{\sqrt{1} k^q \alpha \theta} \, d\theta = \delta_{0,\alpha}, \quad (55)
$$

we obtain

$$
\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t^m y_{t+q}^{n-m}
$$
Since we further have

\[
\langle X^m_{t+p} \rangle = \int_{-1}^{1} dx \int_{-1}^{1} dy \left[ P_k \circ \cdots \circ P_k (x, y) \right]^m \rho_{XY}(x, y) \\
= \frac{1}{2\pi} \int_0^{2\pi} \cos^m (k_p \theta) d\theta \\
= \begin{cases} \\
\frac{(m-1)!!}{m!!} (m \text{ : even}) \\
0 \text{ (m : odd)} \end{cases}, \\
\tag{57}
\]

and

\[
\langle Y^{n-m}_{t+q} \rangle = \int_{-1}^{1} dx \int_{-1}^{1} dy \left[ Q_k \circ \cdots \circ Q_k (x, y) \right]^{n-m} \rho_{XY}(x, y) \\
= \frac{1}{2\pi} \int_0^{2\pi} \cos^{n-m} (k_q \theta) d\theta \\
= \begin{cases} \\
\frac{(n-m-1)!!}{(n-m)!!} (n-m \text{ : even}) \\
0 \text{ (n-m : odd)} \end{cases}, \\
\tag{58}
\]

we get

\[
\text{Cov}[X^m_{t+p}, Y^{n-m}_{t+q}] = \langle X^m_{t+p} Y^{n-m}_{t+q} \rangle - \langle X^m_{t+p} \rangle \langle Y^{n-m}_{t+q} \rangle \\
= \begin{cases} \\
\frac{(-1)^{n-m}}{2\pi} \sum_{r=0}^{m} \sum_{s=0}^{n-m} \binom{m}{r} \binom{n-m}{s} (-1)^{-s} \delta_{0,(2r-m)k^p+(2s-n+m)k^q} \\
(2r-m)k^p + (2s-n+m)k^q = 0, \quad (0 \leq r \leq m; 0 \leq s \leq n-m) \tag{59}
\end{cases}
\]

Kohda et al. showed that the higher-order covariance of Chebyshev maps have no correlation \[15\]. We use their derivation in our case. According to Kac’s statistical independence \[16\] when in Eq. \[50\]

\[
(2r-m)k^p + (2s-n+m)k^q = 0, \quad (0 \leq r \leq m; 0 \leq s \leq n-m) \tag{60}
\]

holds for any \( k^p \) and \( k^q \) if and only if \( r = m/2 \) and \( s = (n-m)/2 \), \( k^p \) and \( k^q \) are called linearly independent. Then \( x^m_{t+p} \) and \( y^{n-m}_{t+q} \) are statistically independent \[15\].

Let consider the case that \( m \) and \( n \) are even. From elementary facts about the theory of numbers, we know that

\[
N = k^e + r \quad (0 \leq r < k), \tag{61}
\]
where \( N \) is a natural number, and \( k, e \) and \( r \) are non-negative integers. In the case that \( 2r - m > 0, 2s - n + m < 0 \), and \( p < q \) we have

\[
(2r - m)k^p + (2s - n + m)k^q = \{(2r - m) + (2s - n + m)k^{q-p}\}k^p
\]

\[
= \{(k^{e_1} + r') - (k^{e_2} + s')k^{q-p}\}k^p
\]

\[
= \{(k^{e_1} + r') - k^{e_2+q-p} - s'k^{q-p}\}k^p. \tag{62}
\]

Therefore, if \([(2r - m)/k] = 0, [(2s - n + m)/k] = 0, \) and \( e_1 = e_2 + q - p \) hold, then \((2r - m)k^p + (2s - n + m)k^q = 0\) is satisfied for integers \( r \) and \( s \) other than \( r = m/2 \) and \( s = (n - m)/2 \). When \( m > k \), and \( n - m > k \), we have \([(2r - m)/k] = 0 \) and \([(2s - n + m)/k] = 0 \). Therefore, \( m \geq k^{e_1} \) and \( n - m \geq k^{e_2} \) would be satisfied. Namely, when \( n < k^{e_1} + k^{e_2} = k^{e_2}(k^{q-p} + 1), x_{t+p}^m \) and \( y_{t+q}^{n-m} \) are statistically independent. This implies that \( q - p \) goes infinity, \( x_{t+p}^m \) and \( y_{t+q}^{n-m} \) become statistically independent in an exponential manner.

Fig. \( 8 \) shows \( \text{Cov}[X_{t+p}^m, Y_{t+q}^{n-m}] \) for \((p, q) = (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), \) and \((0, 6)\). As shown in figures, we found that the covariances decrease as \(|p - q|\) increasing. The range of the covariances approach to zero as \( q \) increasing.

Obviously, Eq. \( (60) \) has solutions \( r = m/2 \) and \( s = (n - m)/2 \). A sum of the contributions for \( r = m/2 \) and \( s = (n - m)/2 \) in Eq. \( (62) \) is equivalent to \( \frac{(m-1)!!(n-m-1)!!}{m!!(n-m)!!} \). Since \( \text{Cov}[X_{t+p}^m, Y_{t+q}^{n-m}] \) is less than zero from the numerical simulation, for solutions other than \( r = m/2 \) and \( s = (n - m)/2 \) of Eq. \( (60) \), it should satisfy that a sum of negative contributions is greater than a sum of positive contributions.

We may consider two types of second-order characteristic functions with lags. Note that Geisel and Fürein \( 10 \) considered a similar second-order characteristic function for the Chebyshev maps. Their characteristic function corresponds to \( \Psi_{XX}(u, v) \) in our definition.

\[
\Psi_{XX}(u, v) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} e^{i(ux_{t+p} + vy_{t+q})}
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} e^{-i(u\cos(k^p\theta) + v\cos(k^q\theta))} d\theta \tag{63}
\]

\[
\Psi_{YY}(u, v) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} e^{i(uy_{t+p} + vy_{t+q})}
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} e^{i(u\sin(k^p\theta) + v\sin(k^q\theta))} d\theta \tag{64}
\]

Similarly to \( \Psi_{XY}(u, v) \), from the expansion in terms of \( u \) and \( v \), we obtain

\[
\Psi_{XX}(u, v) = \sum_{n=0}^{\infty} \left( \frac{-1}{n!} \right)^2 \sum_{m=0}^{n} \binom{n}{m} \langle X_{t+p}^m X_{t+q}^{n-m} \rangle u^m v^{n-m}, \tag{65}
\]

\[
\Psi_{YY}(u, v) = \sum_{n=0}^{\infty} \left( \frac{-1}{n!} \right)^2 \sum_{m=0}^{n} \binom{n}{m} \langle Y_{t+p}^m Y_{t+q}^{n-m} \rangle u^m v^{n-m}, \tag{66}
\]
Figure 3: Scatter plots of \( \text{Cov}[X_{m+t}^n, Y_{n-m}^{n-q}] \) in terms of \( n \) (0 \( \leq m \leq n \)) at \( k = 2 \) and \( p = 0 \), (a) \( q = 1 \), (b) \( q = 2 \), (c) \( q = 3 \), (d) \( q = 4 \), (e) \( q = 5 \), and (f) \( q = 6 \). Filled squares represent theoretical values, and filled circles values obtained from numerical integration.

where

\[
\langle X_{m+t}^n X_{n-m}^{n-m} \rangle = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} x_{m+t}^n x_{n-m}^{n-m} = \frac{1}{2\pi} \int_0^{2\pi} \cos^m(k^p\theta) \cos^{n-m}(k^q\theta) d\theta,
\]

\[
\langle Y_{m+t}^n Y_{n-m}^{n-m} \rangle = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} y_{m+t}^n y_{n-m}^{n-m} = \frac{1}{2\pi} \int_0^{2\pi} \sin^m(k^p\theta) \sin^{n-m}(k^q\theta) d\theta.
\]
By using
\[
\cos^m(k\theta) \cos^{n-m}(k\theta) = \frac{1}{2^m} (e^{\sqrt{-1}k\theta} + e^{-\sqrt{-1}k\theta})^m = \frac{1}{2^n (n-m)!} \sum_{r=0}^{n-m} \frac{m!}{r!(m-r)!} \frac{(n-m)!}{s!(n-m-s)!} e^{\sqrt{-1}[(2r-m)k^p + (2s-n+m)k^q] \theta},
\]
\[
\sin^m(k\theta) \sin^{n-m}(k\theta) = \frac{1}{2^{-m-
}} (e^{\sqrt{-1}k\theta} - e^{-\sqrt{-1}k\theta})^m = \frac{1}{2^m} \sum_{r=0}^{n-m} \frac{r!(m-r)!}{m!(n-m-r)!} \delta_0,(2r-m)k^p + (2s-n+m)k^q \theta,
\]
therefore, we have
\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t^m y_t^{n-m} = \begin{cases} \frac{1}{m} \sum_{r=0}^{m} \frac{n-m}{s!} \frac{m!}{r!(m-r)!} \frac{(n-m)!}{s!(n-m-s)!} \delta_0,(2r-m)k^p + (2s-n+m)k^q \theta & (m, n : \text{even}) \\ 0 & \text{(otherwise)} \end{cases}
\]
(67)
\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} y_t^m y_t^{n-m} = \begin{cases} \frac{(-1)^m}{2^n} \sum_{r=0}^{m} \frac{n-m}{s!} \frac{m!}{r!(m-r)!} \frac{(n-m)!}{s!(n-m-s)!} \delta_0,(2r-m)k^p + (2s-n+m)k^q \theta & (m, n : \text{even}) \\ 0 & \text{(otherwise)} \end{cases}
\]
(68)
We further have
\[
\Cov[X_t^m, X_t^{n-m}] = (X_t^m X_t^{n-m}) - (X_t^m X_t^{n-m}) = \begin{cases} \frac{1}{2^m} \sum_{r=0}^{m} \frac{n-m}{s!} \frac{m!}{r!(m-r)!} \frac{(n-m)!}{s!(n-m-s)!} \delta_0,(2r-m)k^p + (2s-n+m)k^q \theta & (m, n : \text{even}) \\ 0 & \text{(otherwise)} \end{cases}
\]
(69)
\[
\Cov[Y_t^m, Y_t^{n-m}] = (Y_t^m Y_t^{n-m}) - (Y_t^m Y_t^{n-m}) = \begin{cases} \frac{(-1)^m}{2^n} \sum_{r=0}^{m} \frac{n-m}{s!} \frac{m!}{r!(m-r)!} \frac{(n-m)!}{s!(n-m-s)!} \delta_0,(2r-m)k^p + (2s-n+m)k^q \theta & (m, n : \text{even}) \\ 0 & \text{(otherwise)} \end{cases}
\]
(70)
A sum of contributions for $r = m/2$ and $s = (n-m)/2$ in Eq. (70) is equivalent to $\frac{(-1)^m}{m!}$. If Eq. (69) has other solutions than $r = m/2$ and $s = (n-m)/2$, then the covariance positively increases. Therefore, we could prove $\Cov[X_t^m, X_t^{n-m}] \geq 0$.

We also have
\[
\Cov[Y_t^m, Y_t^{n-m}] = (Y_t^m Y_t^{n-m}) - (Y_t^m Y_t^{n-m}) = \begin{cases} \frac{(-1)^m}{2^n} \sum_{r=0}^{m} \frac{n-m}{s!} \frac{m!}{r!(m-r)!} \frac{(n-m)!}{s!(n-m-s)!} \delta_0,(2r-m)k^p + (2s-n+m)k^q \theta & (m, n : \text{even}) \\ 0 & \text{(otherwise)} \end{cases}
\]
(71)

Fig. 5 shows covariance between $X_{t+p}^m$ and $X_{t+q}^{n-m}$, and between $Y_{t+p}^m$ and $Y_{t+q}^{n-m}$. It is found that Cov[$X_{t+p}^m, X_{t+q}^{n-m}$] shows non-negative values, and that Cov[$Y_{t+p}^m, Y_{t+q}^{n-m}$] shows non-positive values. We found that Cov[$X_{t+p}^m, Y_{t+q}^{n-m}$] takes the same non-positive value as Cov[$Y_{t+p}^m, Y_{t+q}^{n-m}$] for $p \neq q$ from Figs. 3 and 5. The reason is because $\cos^m(k^p\theta)\sin^{n-m}(k^q\theta)$ and $\sin^m(k^p\theta)\sin^{n-m}(k^q\theta)$ have the same area to the x-axis, but $\cos^m(k^p\theta)\cos^{n-m}(k^q\theta)$ is different from them as shown in Fig. 4.

A sum of the contributions for $r = m/2$ and $s = (n - m)/2$ in Eq. (71) is equivalent to $\left(\frac{(n-m-1)!!}{(n-m)!!}\right)^2$. Since Cov[$Y_{t+p}^m, Y_{t+q}^{n-m}$] is less than zero from the numerical simulation, for solutions other than $r = m/2$ and $s = (n - m)/2$ of Eq. (60), it should satisfy that a sum of negative contributions is greater than a sum of positive contributions.

\[ J_{cc}^{p,q}(u,v) = \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{1}{2}(u \cos(p\theta) + v \cos(q\theta))} d\theta, \quad (72) \]
\[ J_{sc}^{p,q}(u,v) = \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{1}{2}(u \sin(p\theta) + v \cos(q\theta))} d\theta, \quad (73) \]
\[ J_{ss}^{p,q}(u,v) = \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{1}{2}(u \sin(p\theta) + v \sin(q\theta))} d\theta. \quad (74) \]

Therefore, it is suggested that $\Psi_{XX}(u,v) \neq \Psi_{YY}(u,v) \neq \Psi_{XY}(u,v)$ for $q \neq p$ from numerical simulation. This also implies that three types of two-dimensional Bessel functions are not equivalent;

![Figure 4: The wave forms of $\cos^m(k^p\theta)\sin^{n-m}(k^q\theta)$, $\sin^m(k^p\theta)\sin^{n-m}(k^q\theta)$, and $\cos^m(k^p\theta)\cos^{n-m}(k^q\theta)$ for $p = 0$, $q = 1$, $n = 10$, and $m = 4$.](image-url)
6 Conclusion

We studied two-dimensional chaotic maps on the unit circle, which is an extension of the Chebyshev maps to two-dimensional map on the unit circle. We examined correlational properties of this two-dimensional chaotic map. We gave

Figure 5: Scatter plots of Cov\[X_{t+p}^m, X_{t+q}^{n-m}\] and Cov\[Y_{t+p}^m, Y_{t+q}^{n-m}\] in terms of \(n\) at \(k = 2\) and \(p = 0\), (a) \(q = 1\), (b) \(q = 2\), (c) \(q = 3\), (d) \(q = 4\), (e) \(q = 5\), and (f) \(q = 6\). Unfilled squares represent theoretical values of Cov\[X_{t+p}^m, X_{t+p}^{n-m}\], filled squares numerical values of Cov\[X_{t+p}^m, X_{t+p}^{n-m}\], unfilled circles theoretical values of Cov\[Y_{t+p}^m, Y_{t+p}^{n-m}\], and filled circles numerical values of Cov\[Y_{t+p}^m, Y_{t+p}^{n-m}\].
analytical forms of higher-order moments. Furthermore, we derived the characteristic function of both simultaneous and lagged ergodic densities. We found that these characteristic functions are given by three types of two-dimensional Bessel functions. We proved four theorems and proposed two conjectures as follows:

**Theorems:**

1. The higher-order covariances between $x_t$ and $y_t$ shows non-positive values for integers $n$ and $m$ ($0 \leq m \leq n$):
   \[
   \text{Cov}[X^m, Y^{n-m}] \leq 0. \tag{75}
   \]

2. The higher-order covariance between $x_t$ and $x_t$ shows non-negative values for integer $n$ and $m$ ($0 \leq m \leq n$):
   \[
   \text{Cov}[X^m, X^{n-m}] \geq 0. \tag{76}
   \]

3. The higher-order covariance between $y_t$ and $y_t$ shows non-negative values for $n$ and $m$ ($0 \leq m \leq n$):
   \[
   \text{Cov}[Y^m, Y^{n-m}] \geq 0. \tag{77}
   \]

4. The higher-order covariance between $x_{t+p}$ and $x_{t+q}$ ($p \neq q$) shows non-negative values for integer $n$ and $m$ ($0 \leq m \leq n$):
   \[
   \text{Cov}[X^m_{t+p}, X^{n-m}_{t+q}] \geq 0. \tag{78}
   \]

**Conjectures:**

1. The higher-order covariances between $x_{t+p}$ and $y_{t+q}$ ($p \neq q$) shows non-positive values for integers $n$ and $m$ ($0 \leq m \leq n$):
   \[
   \text{Cov}[X^m_{t+p}, Y^{n-m}_{t+q}] \leq 0. \tag{79}
   \]

2. The higher-order covariance between $y_{t+p}$ and $y_{t+q}$ ($p \neq q$) shows non-positive values for $n$ and $m$ ($0 \leq m \leq n$):
   \[
   \text{Cov}[Y^m_{t+p}, Y^{n-m}_{t+q}] \leq 0. \tag{80}
   \]

Therefore, we can generate antithetic sequences as $x_0, y_0, x_1, y_1, \ldots, x_t, y_t, \ldots$ or $y_0, y_1, y_2, \ldots, y_t, \ldots$ obtained from Eq. (4). Asymmetric features between cosine and sine functions were elucidated. Using the proposed two-dimensional chaotic map, we can generate antithetic pseudo random sequences for Monte Carlo integration.
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