VIRTUALLY FREE FINITE-NORMAL-SUBGROUP-FREE GROUPS ARE STRONGLY VERBALLY CLOSED

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Any virtually free group $H$ containing no non-trivial finite normal subgroup (e.g., the infinite dihedral group) is a retract of any finitely generated group containing $H$ as a verbally closed subgroup.

0. Introduction

A subgroup $H$ of a group $G$ is called verbally closed [MR14] (see also [Rom12], [RKh13], [Mazh17], [KlMa18], [Mazh18]) if any equation of the form

$$w(x_1, x_2, \ldots) = h,$$

where $w$ is an element of the free group $F(x_1, x_2, \ldots)$ and $h \in H$,

having a solution in $G$ has a solution in $H$. If each finite system of equations with coefficients from $H$

$$\{w_1(x_1, x_2, \ldots) = 1, \ldots, w_m(x_1, x_2, \ldots) = 1\},$$

having a solution in $G$ has a solution in $H$, then the subgroup $H$ is called algebraically closed.

Surely, the algebraic closedness is stronger than the verbal closedness. However, these properties turn out to be equivalent in many cases. A group $H$ is called strongly verbally closed [Mazh18] if it is algebraically closed in any group containing $H$ as a verbally closed subgroup. (Thus, verbal closedness is a property of a subgroup, while the strong verbal closedness is a property of an abstract group.) For example, the following groups are strongly verbally closed:

- all abelian groups [Mazh18];
- all free groups [KlMa18];
- the fundamental groups of all connected surfaces, except possibly the Klein bottle [Mazh18].

The main result of this paper can be stated as follows.

**Theorem 1.** The following groups are strongly verbally closed:

1) all virtually free group containing no non-trivial finite normal subgroups;
2) all free products $\ast_{i \in I} H_i$, where the set $I$ is finite or infinite, $|I| > 1$, and $H_i$ are nontrivial groups satisfying nontrivial laws.

A large part of Theorem 1 was known earlier: in [Mazh18], Assertion 2) was proved for all non-dihedral groups under the additional condition that $I$ is finite; in [KlMa18], the strong verbal closedness was proved for all infinite virtually free non-dihedral groups containing no infinite abelian noncyclic subgroups. Paper [KlMa18] contains also examples of virtually free groups that are not strongly verbally closed.

Actually, most of this paper is devoted to the proof of the following particular case of (both assertions of) Theorem 1:

**The infinite dihedral group is strongly verbally closed.**

For all non-dihedral groups, Theorem 1 is relatively easily derived from known facts (in Section 1).

The difficulty with the infinite dihedral group is that it is “too abelian” to apply sophisticated tools based on the Lee words [Lee02] (see [MR14], [Mazh17], [KlMa18], [Mazh18]); on the other hand, it is “too nonabelian” to apply simple arguments (see [KlMa18], [Mazh18]). Of course, the dihedral group is metabelian and this is the basis of our approach. In Section 3, we give an “explicit” criterion for an infinite dihedral subgroup to be verbally (and algebraically) closed. This criterion is similar (in some sense) to the following simple fact about abelian subgroups, which can be easily derived from a result proved in [Mazh18]:

an abelian subgroup $H$ is verbally (and algebraically) closed in a group $G$ if and only if its intersection with the commutator subgroup $G'$ of $G$ is trivial and the image $H$ in the quotient group $G/G'$ is pure (servant).

The notion of algebraic closedness can be characterised in structural language if the group $H$ is equationally Noetherian, i.e. any system of equations over $H$ with finitely many unknowns is equivalent to its finite subsystem. Namely, the algebraic closedness in this case is equivalent to the “local retractness” (see Section 1):

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An equationally Noetherian subgroup $H$ of a group $G$ is algebraically closed in $G$ if and only if it is a retract of each finitely generated over $H$ subgroup of $G$ (i.e. a subgroup of the form $\langle H, X \rangle$, where $X \subseteq G$ is a finite set).

All virtually free groups (including the infinite dihedral group) are equationally Noetherian [KlMa18]. Therefore, Theorem 1 implies that

\[
\text{each virtually free group } H \text{ containing no finite non-trivial normal subgroup is a retract of every finitely generated group containing } H \text{ as a verbally closed subgroup.}
\]

In Section 1, we prove some auxiliary facts that allow us to reduce the proof of the main theorem to the case, where the group $G$ containing a verbally closed dihedral subgroup $H$ is an extension of an abelian group $Q$ by an elementary abelian 2-group $C$; and so $Q$ is a $C$-module. Section 2 contains general information about such modules.

In Section 3, we state and prove a criterion for an infinite dihedral subgroup to be algebraically (and verbally) closed.

In the last section, we consider an example illustrating the main step of the proof. This is essentially the simplest example of the situation, where algebraic unclosedness is almost obvious, while the proof of verbal unclosedness requires a nontrivial argument. We tried to make the last section independent; so, readers may read this section first. We warn these readers however that the example considered there shows some, but not all, difficulties we face.

**Notation.** which we use, is mainly standard. Note only that, if $X$ is a subset of a group, then $|X|$, $(X)$, and $C(X)$ is the cardinality of $X$, the subgroup generated by $X$, and the centraliser of $X$, respectively. The letters $\mathbb{Z}$ and $\mathbb{Q}$ denote the sets of integers and rationals. The cyclic group of order $k$ generated by an element $x$ is denoted $\langle x \rangle_k$. The free group with a basis $x_1, \ldots, x_n$ is denoted $F(x_1, \ldots, x_n)$ (or $F_n$).

### 1. Auxiliary lemmata

**Proposition 1.** An equationally Noetherian subgroup $H$ is algebraically closed in a group $G$ if and only if $H$ is a retract of each finitely generated over $H$ subgroup of $G$.

**Proof.** Suppose that $H$ is algebraically closed in $G$. Then $H$ is algebraically closed in any subgroup $\bar{G}$ of $G$ containing $H$. If $\bar{G}$ is finitely generated over $H$, then $H$ is a retract of $\bar{G}$ by virtue of the following fact [MR14]:

An equationally Noetherian algebraically closed subgroup $H$ of a finitely generated over $H$ group is a retract.

Now, suppose that $H$ is a retract of each finitely generated over $H$ subgroup of $G$ and a system of equations $S = \{w_1(x_1, \ldots, x_n) = 1, \ldots, w_m(x_1, \ldots, x_n) = 1\}$, where $w_i \in F(x_1, \ldots, x_n) * H$, has a solution $g_1, \ldots, g_n$ in $G$.

There exists a retraction $\rho: \langle H, g_1, \ldots, g_n \rangle \to H$. Therefore, $\rho(g_1), \ldots, \rho(g_n)$ is a solution to $S$ in $H$ as required.

**Corollary.** Assertion 2) of Theorem 1 holds for all non-dihedral groups.

**Proposition 2.** If each finite subset of a group $H$ is contained in a strongly verbally closed subgroup which is also verbally closed in $H$, then $H$ is strongly verbally closed.

**Proof.** Suppose that the group $H$ is verbally closed in an overgroup $G$, and some finite system of equations $S$ with coefficients from $H$ has a solution in $G$. The set of coefficients of $S$ is contained in some subgroup $H_1 \subseteq H$ which is strongly verbally closed and verbally closed in $H$. The latter means that $H_1$ is verbally closed in $G$ (because verbal closeness is a transitive property). Now, strong verbal closedness of $H_1$ implies solvability of the system $S$ in $H_1 \subseteq H$ as required.

**Proposition 3.** Assertion 1) of Theorem 1 holds for all non-dihedral groups.

**Proof.** If a group $H$ is virtually cyclic, then it contains a finite normal subgroup $K$ such that the quotient $H/K$ is either trivial, infinite cyclic, or infinite dihedral [Sta71]. The normal finite subgroup $K$ must be trivial by the condition of Theorem 1. All abelian groups are strongly verbally closed [Mazh18] and we are done in this case.

If $H$ is not virtually cyclic, then it contains a non-abelian free normal subgroup $F$ of a finite index $n$. This means that the centraliser of the verbal subgroup $V = \{h^n \mid h \in H\} \subseteq F$ is normal and finite (because the centraliser of any nonabelian free subgroup in a virtually free group is finite). Hence, $C(V) = \{1\}$ by the condition of Theorem 1. Therefore, the centraliser of some finitely generated subgroup $V_1 \subseteq V$ is trivial (because again the centraliser of a
nonabelian free subgroup in a virtually free group is finite and a descending chain of finite subgroups stabilises). It remains to apply the following fact ([Mazh18], Corollary 1):

- if a verbal subgroup \( V \) of a group \( H \) is free nonabelian and the centraliser of a finitely generated subgroup of \( V \) is trivial, then \( H \) is strongly verbally closed.

The rest of this paper is devoted to the proof that the infinite dihedral group is strongly verbal closed.

**Lemma 1** ([RKh13], Lemma 1.1). If \( V(G) \) is a verbal subgroup of a group \( G \), and \( H \) is a verbally closed subgroup of \( G \), then the image of \( H \) under the natural homomorphism \( G \to G/V(G) \) is verbally closed.

2. Commuting involutions on abelian groups

Suppose that a finite elementary abelian 2-group \( C \) (a finite direct power of the two-element group) acts by automorphisms on a finitely generated abelian group, i.e., \( Q \) is a \( C \)-module. Let \( X \) be the set of all homomorphisms (characters) \( \chi: C \to \{\pm 1\} \) and

\[
Q_\chi = \{ q \in Q \mid cq = \chi(c)q \text{ for all } c \in C \}.
\]

There is a natural homomorphism from \( Q \) to the additive group of the vector space (over \( \mathbb{Q} \)) \( \mathbb{Q} \otimes Q \). The kernel of this homomorphism is the torsion part \( T(Q) \) of \( Q \). The action of \( C \) on \( Q \) extends naturally to a linear representation \( C \to \text{GL}(\mathbb{Q} \otimes Q) \). This representation is completely reducible and the irreducible representations are one-dimensional (the characters of \( C \)). Thus, \( \mathbb{Q} \otimes Q = \bigoplus_{\chi \in X} (\mathbb{Q} \otimes Q_\chi) \). The natural projection \( Q \otimes Q \to \mathbb{Q} \otimes Q_\chi \) is denoted by \( p_\chi \). The vectors \( x \in Q_\chi \) are called \( \chi \)-components of the vector \( v \in Q \otimes Q \) and denoted by \( v_\chi \). Clearly \( \bigoplus_{\chi \in X} (1 \otimes Q_\chi) \subseteq 1 \otimes Q \subseteq \bigoplus_{\chi \in X} p_\chi (1 \otimes Q) \). If one of these inclusions is an equality, then the other is an equality; in this case, we say that the \( C \)-module \( 1 \otimes Q \) is decomposable. In the general case, the \( C \)-module \( \bigoplus_{\chi \in X} p_\chi (1 \otimes Q) \) can be called the decomposable closure of the module \( 1 \otimes Q \).

We need the following simple formula valid for any character \( \chi \) and any \( q \in Q \):

\[
1 \otimes \left( \prod_{c \in C} (1 + \chi(c)c) \right) \cdot q = (2^{\vert C \vert} \otimes q)_\chi \quad \text{and} \quad |T(Q)| \cdot \prod_{c \in C} (1 + \chi(c)c) \cdot q = (2^{\vert C \vert} \cdot |T(Q)| \cdot q)_\chi,
\]

(\( x_\chi \in Q_\chi \) in the right-hand side of the latter equality are components of an element \( x \in \bigoplus Q_\chi \subseteq Q \)). Here, the second equality follows from the first one, because the kernel of the homomorphism \( q \mapsto 1 \otimes q \) is the torsion part of \( Q \) (and the first equality shows that the module \( 2^{\vert C \vert} \cdot |T(Q)| \cdot q \) is contained in the direct sum of \( \chi \)-components of \( Q \)). To prove the first equality note that the \( \chi \)-component of the element \( 1 \otimes q \) in the left-hand side is multiplied by two \( |C| \) times. As for the other components, they vanish, because, for each character \( \chi \neq \chi' \), there exists \( c \in C \) such that \( \chi(c) = -\chi'(c) \). Formula (1) implies that, for all \( q \in Q \), we have the equality

\[
1 \otimes \left( \sum_{\chi \in X} \left( \prod_{c \in C} (1 + \chi(c)c) \right) \cdot q \right) = 2^{\vert C \vert} \otimes q \quad \text{and} \quad |T(Q)| \cdot \sum_{\chi \in X} \left( \prod_{c \in C} (1 + \chi(c)c) \right) \cdot q = |T(Q)| \cdot 2^{\vert C \vert} \cdot q.
\]

We also need the following simple generalisation of formula (1): If \( \varphi: C \to \widehat{C} \) is an epimorphism from one finite elementary abelian 2-group onto another and \( \widehat{q} \in \widehat{Q} \) is an element of a decomposable \( \widehat{C} \)-module \( \widehat{Q} \), then, for any character \( \chi \) of \( C \),

\[
1 \otimes \left( \prod_{c \in C} (1 + \chi(c)\varphi(c)) \right) \cdot \widehat{q} = \begin{cases} 2^{\vert C \vert} \otimes \widehat{q}_\chi, & \text{if } \chi = \widehat{\chi} \circ \varphi; \\ 0, & \text{if } \chi \neq \widehat{\chi} \circ \varphi \text{ for any character } \widehat{\chi}: \widehat{C} \to \{\pm 1\}. \end{cases}
\]

We call an element \( q \in Q \) simple if, for some character \( \chi \), the \( \chi \)-component \( (1 \otimes q)_\chi = p_\chi (1 \otimes q) \) is a primitive element of the free abelian group \( p_\chi (1 \otimes Q) \), i.e. \( p_\chi (1 \otimes q) \not\in k \cdot p_\chi (1 \otimes Q) = p_\chi (k \otimes Q) \) for \( k \in \mathbb{Z} \setminus \{\pm 1\} \).

**Simple-element Lemma.** An element \( q \in Q \) is simple if and only if its order is infinite and the group \( Q \) decomposes into a direct sum \( Q = \langle q \rangle \oplus M \), where the subgroup \( M \subseteq Q \) is a \( C \)-submodule, i.e. \( cm \in M \) for all \( c \in C \) and \( m \in M \).

**Proof.** Let \( q \) be a simple element, i.e. \( (1 \otimes q)_\chi \) is a primitive element of the group \( p_\chi (1 \otimes Q) \). Clearly, this implies that the order of \( q \) is infinite (otherwise \( 1 \otimes q = 0 \)). Moreover, \( p_\chi (1 \otimes Q) = \langle (1 \otimes q)_\chi \rangle \oplus D \), for some subgroup \( D \) (because a primitive element of a free abelian group is contained in some basis). This means that \( 1 \otimes Q = \langle (1 \otimes q) \rangle \oplus (p^{-1}_\chi(D) \cap (1 \otimes Q)) \).
Moreover, the group $A = p^{-1}_x(D)$ is a $C$-submodule. Therefore, $Q = \langle q \rangle \oplus \psi^{-1}(A)$, where $\psi: Q \to Q \otimes Q$ is the natural homomorphism sending $x$ to $1 \otimes x$.

The proof of the other direction is left to readers as an exercise (we are not going to use it).

**Example.** Let the group $C = \langle c \rangle_2$ act on $Q = \mathbb{Z} \oplus \mathbb{Z}$ by permutations of coordinates. There are two characters: $X = \{\chi_+, \chi_-\}$, where $\chi_+(c) = 1$ and $\chi_-(c) = -1$. Moreover, $Q_{\chi_+} = \langle (1,1) \rangle$ and $Q_{\chi_-} = \langle (1,-1) \rangle$. The subgroup $Q_{\chi_+} \oplus Q_{\chi_-}$ has index two in $Q$. The decomposable closure is

$$\overline{Q} = \left\langle \left( \frac{1}{2}, \frac{1}{2} \right) \right\rangle \oplus \left\langle \left( \frac{1}{2}, -\frac{1}{2} \right) \right\rangle = \{(x, y) \in \mathbb{Q}^2 \mid x + y \in \mathbb{Z} \iff x = y\}.$$  

The projections $p_{\chi_\pm}: \overline{Q} \to \langle Q \rangle_{\chi_\pm}$ are $p_{\chi_+(x, y)} = (\frac{x+y}{2}, \frac{x-y}{2})$ and $p_{\chi_-}(x, y) = (\frac{x-y}{2}, \frac{x+y}{2})$. The element $(2, 5)$ is not simple, because $2 \cdot (\frac{3}{2}, \frac{1}{2}) = 3 \cdot (\frac{1}{2}, -\frac{1}{2})$. Actually, it is easy to show that an element $(x, y) \in Q$ is simple if and only if $x \pm y = \pm 1$ for some choice of signs.

### 3. Algebraically closed infinite dihedral subgroups

Consider a finitely generated group $G$ whose subgroup $Q = \langle \{g^2 \mid g \in G\} \rangle$ generated by squares of all elements is abelian. The finite elementary abelian 2-group $C = G/Q$ acts on $Q$ by automorphisms

$$(gQ) \circ q \overset{\text{def}}{=} gqq^{-1} \quad \text{(this is well-defined, because $Q$ is abelian)}.$$  

So, $Q$ is a $C$-module and we can apply the results of the previous section. Note only that now we stick to multiplicative notation, i.e. we write, e.g., $cq_1^{-1}q_2^2$ instead of $cq_1 + 2q_2$. To simplify formulae, we set $\tilde{q} \overset{\text{def}}{=} q^{T(\chi)}$. Formula (1) takes the form

$$w_\chi(\bar{q}) \overset{\text{def}}{=} f_\chi(x_1, f_\chi(x_2, \ldots, f_\chi(x_c, \ldots))) = \left( \tilde{q}_2^{\chi_1} \right)^x, \quad \text{where } \{c_1, \ldots, c_\mid c\} = C \quad (1')$$

and $f_\chi(gQ, x) \overset{\text{def}}{=} \chi(x)(gQ)g^{-1}$ is a “skew commutator” (which is well defined, i.e. does not depend on the choice of the representative $g$ of the coset $c = gQ$).

An analogue of formula (2) takes the form

$$\prod_{\chi \in X} w_\chi(\bar{q}) = \left( \tilde{q}^{2^{\mid c\}} \right) \quad \text{for all } q \in Q. \quad (2')$$

The strong verbal closedness of the infinite dihedral group follows immediately from Proposition 1 and the following theorem.

**Theorem 2.** If $H = \langle b \rangle_2 \lt \langle a \rangle_\infty$ is an infinite dihedral subgroup of a finitely generated group $G$, then the following conditions are equivalent:

1) $H$ is verbally closed in $G$;
2) $H$ is algebraically closed in $G$;
3) $H$ is a retract of $G$ (i.e. the image of an endomorphism $\rho$ such that $\rho \circ \rho = \rho$);
4) $aQ'$ is a simple element of the $G/Q$-module $Q/Q'$, where $Q = \langle \{g^2 \mid g \in G\} \rangle$.

**Proof.** To prove the implication 4) $\implies$ 3), first note that $H \cap Q' = \{1\}$ (otherwise $aQ'$ is of finite order in $Q/Q'$ and is not simple).

Now Simple-element Lemma provides us with a normal in $G/Q'$ subgroup $M \subset Q/Q'$ such that $Q/Q' = \langle aQ' \rangle \times M$. The composition of the natural homomorphisms $G \to G/Q' \to (G/Q')/M = G_1$ is injective on $H$; the obtained group $G_1$ is virtually cyclic (the subgroup generated by squares of all its elements is generated by element $a^2$). It is well known that any virtually cyclic group contains a finite normal subgroup $N$ such that the quotient group is either cyclic or dihedral (see, e.g., [Sta71]). Therefore, the composition of homomorphisms $G \to G/Q' \to (G/Q')/M = G_1 \to G_1/N$ is the required retraction onto $H$ (here, we use that the infinite dihedral group has no finite normal subgroups; and, in $G_1$, the subgroup generated by squares of all its elements is generated by $a^2$).

The implications 3) $\implies$ 2) $\implies$ 1) are general facts valid for any groups (see Introduction).

It remains to prove the implication 1) $\implies$ 4). By Lemma 1, we can assume that $G$ satisfies the law $[x^2, y^2] = 1$, i.e. the subgroup $Q$ generated by squares of all elements of $G$ is abelian (and finitely generated, because it has finite index in a finitely generated group $G$). Indeed, taking the quotient group of $G$ by the commutator subgroup of the subgroup generated by squares of all elements affects neither $H$, nor condition 1), (by Lemma 1), nor condition 4).
Suppose that the element \( a^2 \) is not simple. This means that

\[
\widetilde{(a^2)} \overset{\text{def}}{=} a^{2|T(Q)|} = \prod_{\chi \in X} q(\chi)^{k_\chi}, \quad \text{for some } k_\chi \in \mathbb{Z} \setminus \{ \pm 1 \} \text{ and } q(\chi) \in Q
\]

(where \( x_\chi \) are components of \( x \in |T(Q)|Q \subseteq Q \otimes Q = \bigoplus (Q \otimes Q_\chi) \)). Formula (2') gives the equality

\[
\prod_{\chi \in X} w_\chi ((q(\chi))^{k_\chi}) = (a^2)^{2|C|}. \quad (3)
\]

Now, we decompose the finite elementary abelian 2-group \( G/Q \) into the direct product of order-two groups:
\( G/Q = \langle d_1Q \rangle_2 \times \ldots \times \langle d_mQ \rangle_2 \) and consider the words \( v_\chi(x_1, \ldots, x_m, y) \in F(x_1, \ldots, x_m, y) \) (in the free group) obtained from \( w_\chi \) (see formula (1')) by substitution \( c_i \) with their expressions via generators \( d_1, \ldots, d_m \) and substitution \( q \) with a new letter \( y \):

\[
v_\chi(d_1, \ldots, d_m, q) = w_\chi(q).
\]

Formula (3) shows that the equation

\[
\prod_{\chi \in X} \left( v_\chi \left( x_1, \ldots, x_m, \prod_{i=1}^{n} y_{\chi,i}^{2|T(Q)|} \right)^{k_\chi} \right) = (a^2)^{2|C|} \quad (4)
\]

where \( n \) is a number such that each element of \( Q \) is a product of \( n \) squares (e.g., \( n = \text{rk}Q + 1 \), has a solution in \( G \):

\[
x_j = d_j, \quad y_{\chi,i} = g_{\chi,i}, \quad \text{where } g_{\chi,i} \in G \text{ are such that } \prod_{i=1}^{n} g_{\chi,i}^2 = q(\chi).
\]

It remains to show that equation (4) has no solutions in the dihedral group \( H = \langle b \rangle_2 \rtimes \langle a \rangle_\infty \).

A substitution \( x_j = b^i \cdot a^{k_j} \) naturally defines a homomorphism \( \varphi: C \to H/\langle a^2 \rangle \) and a character \( \chi': C \to \{ \pm 1 \} \):

\[
\chi'(d_jQ) = (-1)^{j_i}, \quad \text{i.e. } \chi' = \widetilde{\chi} \circ \varphi, \text{ where } \widetilde{\chi}: H/\langle a^2 \rangle \to \{ \pm 1 \} \text{ is the character of the action of } H/\langle a^2 \rangle \text{ on } \langle a^2 \rangle.
\]

Let us substitute the variables \( y_{\chi,i} \) by elements \( h_{\chi,i} \in H \) and note that

\[
\prod_{i=1}^{n} y_{\chi,i}^2 = \prod_{i=1}^{n} h_{\chi,i}^2 = a^{2l_{\chi}} \quad \text{for some } l_{\chi} \in \mathbb{Z}.
\]

Then the multiplicative analogue of (\( * \)) gives the equality

\[
v_\chi \left( x_1, \ldots, x_m, \prod_{i=1}^{n} y_{\chi,i}^{2|T(Q)|} \right) = v_\chi \left( x_1, \ldots, x_m, a^{2l_{\chi}|T(Q)|} \right) = \begin{cases} (a^{2l_{\chi}})^{2|T(Q)|}, & \text{if } \chi = \chi'; \\ 1, & \text{if } \chi \neq \chi'. \end{cases}
\]

Therefore, this substitution transforms the left-hand side of equation (4) into

\[
a^{2l_{\chi} \cdot 2|C| \cdot |T(Q)|} \cdot k_\chi \neq a^{2 \cdot 2|C| \cdot |T(Q)|}, \quad \text{because } k_\chi \neq \pm 1,
\]

and (4) has no solutions in \( H \) as required.

4. An example

Consider the following example:

\[
G = D_\infty \times D_\infty = \langle (b_1)_2 \rtimes \langle a_1 \rangle_\infty \rangle \times \langle (b_2)_2 \rtimes \langle a_2 \rangle_\infty \rangle \quad \text{and } \quad G \supset H = \langle b \rangle_2 \rtimes \langle a \rangle_\infty \simeq D_\infty, \text{ where } b = b_1b_2 \text{ and } a = a_1^3a_2^5.
\]

In this case, \( Q = \langle \{ g^2 \mid g \in G \} \rangle = \langle a_1^2 \rangle_\infty \times \langle a_2^2 \rangle_\infty \simeq \mathbb{Z} \oplus \mathbb{Z} \) and \( C = G/Q = \langle a_1Q \rangle_2 \times \langle b_1Q \rangle_2 \times \langle a_2Q \rangle_2 \times \langle b_2Q \rangle_2 \). Thus, there are \( 2^4 = 16 \) different characters \( C \to \{ \pm 1 \} \), and only for two of them, \( \alpha \) and \( \beta \), the subgroups \( Q_\chi \) are nontrivial:

\[
\alpha : b_1 \mapsto -1, \quad a_1, a_2, b_2 \mapsto 1, \quad \beta : b_2 \mapsto -1, \quad a_1, b_1, a_2 \mapsto 1.
\]

Clearly, the element \( a^2 \) is not simple and we have to prove that the subgroup \( H \) is not verbally closed.
The lengthy word \( v_\chi(x_1, x_2, x_3, x_4, y) \) is the composition (in an arbitrary order) of the following 16 words (as functions of \( y \)):

\[
\begin{align*}
&f_\chi(1, y), \quad f_\chi(x_1, y), \quad f_\chi(x_1 x_2, y), \quad f_\chi(x_1 x_2 x_3, y), \quad f_\chi(x_1 x_2 x_3 x_4, y), \quad f_\chi(x_1 x_3, y), \quad f_\chi(x_1 x_3 y), \quad f_\chi(x_1 x_4, y), \quad f_\chi(x_2, y), \quad f_\chi(x_2 x_3, y), \quad f_\chi(x_2 x_3 x_4, y), \quad f_\chi(x_3, y), \quad f_\chi(x_3 x_4, y), \quad f_\chi(x_4, y).
\end{align*}
\]

Here, the first arguments are all expressions of the form \( x_1^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_3} x_4^{\varepsilon_4} \), where \( \varepsilon_i \in \{0, 1\} \), and

\[
f_\chi(x_1^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_3} x_4^{\varepsilon_4}, y) = y \cdot x_1^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_3} x_4^{\varepsilon_4} \cdot y(x_1^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_3} x_4^{\varepsilon_4} \cdot y(x_1^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_3} x_4^{\varepsilon_4} \cdot \cdots)^{-1}.
\]

For example, \( f_\alpha(x_1^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_3} x_4^{\varepsilon_4}, y) = y \cdot x_1^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_3} x_4^{\varepsilon_4} \cdot y^{-1} a_k^{-1} \cdot y(x_1^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_3} x_4^{\varepsilon_4})^{-1} \). We see that, in the dihedral group,

\[
f_\alpha \left( (a^{k_1} b_1)^{\varepsilon_1} (a^{k_2} b_2)^{\varepsilon_2} (a^{k_3} b_3)^{\varepsilon_3} (a^{k_4} b_4)^{\varepsilon_4}, a^{2k} \right) = \begin{cases} 
  a^{4k}, & \text{if } \varepsilon_2 + \sum_{i=1}^{4} \delta_i \varepsilon_i \text{ is even;}
  1, & \text{if } \varepsilon_2 + \sum_{i=1}^{4} \delta_i \varepsilon_i \text{ is odd;}
\end{cases}
\]

where \( k_i \in \mathbb{Z} \) and \( \delta_i, \varepsilon_i \in \{0, 1\} \).

Thus, if we take \( \delta_2 = 1 \) and \( \delta_1 = \delta_3 = \delta_4 = 0 \), then \( f_\alpha \left( (a^{k_1} b_1)^{\varepsilon_1} (a^{k_2} b_2)^{\varepsilon_2} (a^{k_3} b_3)^{\varepsilon_3} (a^{k_4} b_4)^{\varepsilon_4}, a^{2k} \right) \) becomes equal to \( a^{4k} \) for any choice of \( \varepsilon_i \); if we take any other tuple of \( \delta_i \in \{0, 1\} \), then at least one of 16 expressions (**) becomes 1 after a substitution \( x_i \to a^{k_i} b_i^{-1} \) and \( y \to a^{2k} \). This means that, for the composition \( v_\alpha \) of expressions (**), we have

\[
v_\alpha \left( (a^{k_1} b_1)^{\varepsilon_1} (a^{k_2} b_2)^{\varepsilon_2} (a^{k_3} b_3)^{\varepsilon_3} (a^{k_4} b_4)^{\varepsilon_4}, a^{2k} \right) = \begin{cases} 
  a^{2^{17k}}, & \text{if } (\delta_1, \delta_2, \delta_3, \delta_4) = (0, 1, 0, 0); 
  1, & \text{otherwise.}
\end{cases}
\]

All other characters behave similarly. For instance,

\[
v_{\alpha \beta} \left( (a^{k_1} b_1)^{\varepsilon_1} (a^{k_2} b_2)^{\varepsilon_2} (a^{k_3} b_3)^{\varepsilon_3} (a^{k_4} b_4)^{\varepsilon_4}, a^{2k} \right) = \begin{cases} 
  a^{2^{17k}}, & \text{if } (\delta_1, \delta_2, \delta_3, \delta_4) = (0, 1, 0, 1); 
  1, & \text{otherwise.}
\end{cases}
\]

Equations (4) has the form

\[
\left( v_\alpha \left( x_1, x_2, x_3, x_4, y_{x_1}^2 \right) \right)^3 \cdot \left( v_\beta \left( x_1, x_2, x_3, x_4, y_{x_2}^2 \right) \right)^5 \cdot \prod_{\chi \neq \alpha, \beta} \left( v_\chi \left( x_1, x_2, x_3, x_4, y_{x_3}^2 \right) \right)^{2018} = a^{2^{17}}.
\]

(We take the exponent 2018 to emphasize that we can put any number here, except \( \pm 1 \); of course, the simplest choice is to replace 2018 with 0.)

What is said means that, for any substitution \( x_i \to a^{k_i} b_i^{-1} \), the left-hand side of the equation takes a value in

\[
\begin{align*}
  &\left< a^{2^{17}}, 3 \right>, \text{ if } (\delta_1, \delta_2, \delta_3, \delta_4) = (0, 1, 0, 0); \\
  &\left< a^{2^{17}}, 5 \right>, \text{ if } (\delta_1, \delta_2, \delta_3, \delta_4) = (0, 0, 0, 1); \\
  &\left< a^{2^{17}}, 2018 \right>, \text{ in all other cases.}
\end{align*}
\]

Thus, equation (4) has no solutions in the dihedral group. On the other hand, in \( G \), there is a solution: \( x_1 = a_1, x_2 = b_1, x_3 = a_2, x_4 = b_2, y_{\alpha, 1} = a_1, y_{\beta, 1} = a_2, y_{\chi, 1} = 1 \) for \( \chi \notin \{\alpha, \beta\} \).

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