THE SZLENK INDEX OF $L_p(X)$ AND $A_p$

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Abstract. Given a Banach space $X$, a $w^*$-compact subset of $X^*$, and $1 < p < \infty$, we provide an optimal relationship between the Szlenk index of $K$ and the Szlenk index of an associated subset of $L_p(X)^*$. As an application, given a Banach space $X$, we prove an optimal estimate of the Szlenk index of $L_p(X)$ in terms of the Szlenk index of $X$. This extends a result of Hájek and Schlumprecht to uncountable ordinals. More generally, given an operator $A : X \to Y$, we provide an estimate of the Szlenk index of the "pointwise $A$" operator $A_p : L_p(X) \to L_p(Y)$ in terms of the Szlenk index of $A$.

1. Introduction

Throughout this work, $X$ will be a fixed Banach space and $K \subset X^*$ will be a $w^*$-compact, non-empty subset. For $1 < p < \infty$, we let $K_p$ denote the $w^*$-closure in $L_p(X)^*$ of all functions of the form $gh \in L_q(X^*) \subset L_p(X)^*$, where $g : [0, 1] \to K$ is simple and Lebesgue measurable, and $h \in B_{L_q}$. Recall that these functions act on $L_p(X)$ by $(gh, f) = \int_0^1 \langle g(\varpi), f(\varpi) \rangle h(\varpi) d\varpi$ for $f \in L_p(X)$. Note that if $R \geq 0$ is such that $K \subset RB_{X^*}$, $K_p \subset RB_{L_p(X)^*}$, so that $K_p$ is also $w^*$-compact. If $K = B_{X^*}$, $K_p = B_{L_p(X)^*}$ by the Hahn-Banach theorem. If $A : X \to Y$ is an operator, then there exists a “pointwise $A$” operator $A_p : L_p(X) \to L_p(Y)$ given by $(A_p f)(\varpi) = A(f(\varpi))$ for all $\varpi \in [0, 1]$. Then if $K = A^*B_{Y^*}$, $K_p = (A_p)^*B_{L_p(Y)^*}$, which follows from the Hahn-Banach theorem. Thus it is natural to examine what relationship exists between $K$ and $K_p$. In particular, one may ask what relationship exists between the Szlenk indices of these sets. To that end, we obtain the optimal relationship. In what follows, $\omega$ denotes the first infinite ordinal.

Theorem 1. Fix $1 < p < \infty$. Suppose that $\xi$ is an ordinal such that $Sz(K) \leq \omega^\xi$. Then $Sz(K_p) \leq \omega^{1+\xi}$. If $K$ is convex, $Sz(K_p) \leq \omega Sz(K)$. If $K$ is convex and $Sz(K) \geq \omega^\omega$, $Sz(K) = Sz(K_p)$.

Using the facts stated in the introduction that $K_p = (A_p)^*B_{L_p(Y)^*}$ if $K = A^*B_{Y^*}$, we immediately deduce the following from Theorem 1.

Corollary 2. Fix $1 < p < \infty$. If $A : X \to Y$ is an operator and $K = A^*B_{Y^*}$, then $Sz(A_p) \leq \omega Sz(A)$, and if $Sz(A) \geq \omega^\omega$, $Sz(A_p) = Sz(A)$. In particular, $A_p$ is Asplund if and only if $A$ is.

Applying Corollary 2 to the identity of a Banach space, we extend the result of Hájek and Schlumprecht from [8] to uncountable ordinals.
We recall that $K$ is said to be $w^*$-fragmentable if for any non-empty subset $L$ of $K$ and any $\varepsilon > 0$, there exists a $w^*$-open subset $U$ of $X^*$ such that $L \cap U \neq \emptyset$ and $\text{diam}(L \cap U) < \varepsilon$. We recall that $K$ is $w^*$-dentable if for any non-empty subset $L$ of $K$ and any $\varepsilon > 0$, there exists a $w^*$-open slice $S$ of $X^*$ such that $L \cap U \neq \emptyset$ and $\text{diam}(L \cap S) < \varepsilon$. We recall that a $w^*$-open slice is a subset of $X^*$ of the form $\{x^* \in X^* : \text{Re } x^*(x) > a\}$ for some $x \in X$ and $a \in \mathbb{R}$. As mentioned in [5], a consequence of Corollary 2 is that if $Sz(K) \leq \omega^\xi$, then $Sz(K) \leq Dz(K) \leq \omega^{1+\xi}$, where $Dz(K)$ denotes the $w^*$-dentability index of $K$. Thus Corollary 2 implies that $K$ is $w^*$-dentable if and only if it is $w^*$-fragmentable.

In addition to considering the Szlenk index of a set, one may consider the $\xi$-Szlenk power type $p_\xi(L)$ of the set $L$, which is important in $\xi$-asymptotically uniformly smooth renormings of Banach spaces and operators. The concept of a $\xi$-asymptotically uniformly smooth operator was introduced in [6], and further sharp renorming results regarding the $\xi$-Szlenk power type of an operator were established in [4]. To that end, we have the following.

**Theorem 3.** For any ordinal $\xi$ and any $1 < p < \infty$, if $1/p + 1/q = 1$, $p_{1+\xi}(K_p) \leq \max\{q, p_\xi(K)\}$.

In the case that $\xi \geq \omega$ and $p_\xi(K) \leq p$, $p_\xi(K) = p_\xi(K_p)$, in showing that Theorem 3 is sharp in some cases.

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2. $L_p(X)$, Trees, Szlenk Index, Games

2.1. Trees, $\Gamma_{\xi,n}$, $\mathbb{P}_{\xi,n}$, and stabilization results. Given a set $\Lambda$, we let $\Lambda^{<\mathbb{N}}$ denote the finite, non-empty sequences in $\Lambda$. Given two members $s, t$ of $\Lambda^{<\mathbb{N}}$, we let $s \triangleleft t$ denote the concatenation of $s$ and $t$, $|s|$ denotes the length of $s$, $s \preceq t$ means $s$ is an initial segment of $t$, and $s|_i$ denotes the initial segment of $s$ having length $i$. Given $t \in \Lambda^{<\mathbb{N}}$, we let $[\preceq t] = \{s \in \Lambda^{<\mathbb{N}} : s \preceq t\}$.

Any subset $T$ of $\Lambda^{<\mathbb{N}}$ which contains all non-empty initial segments of its members will be called a $B$-tree. We define by transfinite induction the derived $B$ trees of $T$. We let $MAX(T')$ denote the $\preceq$-maximal members of $T$ and $T' = T \setminus MAX(T)$. We then define $T^0 = T$, $T^{\xi+1} = (T^{\xi})'$, and if $\xi$ is a limit ordinal, $T^\xi = \cap_{\gamma < \xi} T^\gamma$. We let $o(T)$ denote the smallest ordinal $\xi$ such that $T^\xi = \emptyset$, provided such an ordinal exists. If no such ordinal exists, we write $o(T) = \infty$. We say $T$ is well-founded if $o(T)$ is an ordinal, and $T$ is ill-founded if $o(T) = \infty$. For convenience, we agree to the convention that if $\xi$ is an ordinal $\xi < \infty$, and that $\omega_\infty = \infty$.

Given a $B$-tree $T$ and a Banach space $Y$, we let $TY = \{(\zeta_i, Z_i)_{i=1}^k : (\zeta_i)_{i=1}^k \in T, Z_i \in \text{codim}(Y)\}$, where codim($Y$) denotes the closed subspaces of $Y$ having finite codimension in $Y$. We let $C$ denote the norm compact subsets of $B_X$ and

$$T.X.C = \{(\zeta_i, Z_i, C_i)_{i=1}^k : (\zeta_i)_{i=1}^k, Z_i \in \text{codim}(X), C_i \in C\}.$$
We note that \( T.Y \) and \( T.X.C \) are \( B \)-trees. Furthermore, for any ordinal \( \gamma \), \((T.Y)\gamma = T^\gamma.Y\) and \((T.X.C)^\gamma = T^\gamma.X.C\). In particular, \( T.Y \) and \( T.X.C \) have the same order as \( T \).

Given a \( B \)-tree \( T \), a Banach space \( Y \), and a collection \((x_i)_{i \in T.Y} \subset Y\), we say \((x_i)_{i \in T.Y}\) is \textit{normally weakly null} provided that for any \( t = (\zeta_i, Z_i)_{i=1}^\kappa \in T.Y \), \( x_i \in Z_k \). Given another \( B \)-tree \( S \) and a function \( \sigma : S.Y \to T.Y \), we say \( \sigma \) is a \textit{pruning} provided that for every \( s, s_1 \in S.Y \) with \( s \prec s_1 \), \( \sigma(s) \prec \sigma(s_1) \), and if \( s_1 = s^\gamma(\zeta, Z) \) and \( \sigma(s_1) = t^\gamma(\mu, W) \) for some \( t \in T.Y \), \( W \leq Z \). If \( \sigma : S.Y \to T.Y \) is a pruning and \( \tau : \text{MAX}(S.Y) \to \text{MAX}(T.Y) \) is such that for every \( s \in \text{MAX}(S.Y), \sigma(s) \preceq \tau(s) \), we say the pair \((\sigma, \tau)\) is an \textit{extended pruning}, and denote this by \((\sigma, \tau) : S.Y \to T.Y \).

For every \( \xi \in \mathbb{N} \) and \( n \in \mathbb{N} \), a \( B \)-tree \( \Gamma_{\xi,n} \) was defined in \( [3] \) so that \( o(\Gamma_{\xi,n}) = \omega^\xi.n \). Furthermore, a function \( \mathbb{P}_{\xi} : \Gamma_{\xi} \to [0,1] \) was defined so that for every \( t \in \text{MAX}(\Gamma_{\xi}) \), \( \sum_{s \leq t} \mathbb{P}_{\xi}(s) = 1 \). Furthermore, \( \Gamma_{\xi+1} \) is the disjoint union of \( \Gamma_{\xi,n}, n \in \mathbb{N} \). For convenience, we define \( \mathbb{P}_{\xi,n} : \Gamma_{\xi,n} \to [0,n] \) by \( \mathbb{P}_{\xi,n}(s) = n\mathbb{P}_{\xi+1}(s) \). It follows from the definitions that \( \Gamma_{\xi,1} = \Gamma_{\xi} \) and \( \mathbb{P}_{\xi,1} = \mathbb{P}_{\xi} \). For every \( \xi \) and every \( n \in \mathbb{N} \), there exist disjoint subsets \( \Lambda_{\xi,n,1}, \ldots, \Lambda_{\xi,n,n} \) of \( \Gamma_{\xi,n} \) such that \( \Gamma_{\xi,n} = \uplus_{i=1}^n \Lambda_{\xi,n,i} \). It follows from the facts regarding \( \mathbb{P}_{\xi+1} \) discussed in \( [3] \) that, with these definitions, for every ordinal \( \xi \), every \( n \in \mathbb{N} \), \( 1 \leq i \leq n \), and every \( t \in \text{MAX}(\Gamma_{\xi,n}), \sum_{s \leq t} \mathbb{P}_{\xi,n}(s) = 1 \). For any Banach space \( Y \), we may define \( \mathbb{P}_{\xi,n} \) on \( \Gamma_{\xi,n}.Y \) and \( \Gamma_{\xi,n}.X.C \) by letting

\[
\mathbb{P}_{\xi,n}((\zeta_i, Z_i)_{i=1}^k) = \mathbb{P}_{\xi,n}((\zeta_i)_{i=1}^k)
\]

and

\[
\mathbb{P}_{\xi,n}((\zeta_i, Z_i, C_i)_{i=1}^k) = \mathbb{P}_{\xi,n}((\zeta_i)_{i=1}^k).
\]

We say an extended pruning \((\sigma, \tau) : \Gamma_{\xi,n}.X \to \Gamma_{\xi,n}.X \) is \textit{level preserving} provided that for every \( 1 \leq i \leq n \), \( \sigma(\Lambda_{\xi,n,i}) \subset \Lambda_{\xi,n,i} \).

The following theorem collects results from Theorem 3.3, Propositions 3.2, 3.3, and Lemma 3.4 of \( [4] \).

**Theorem 4.** Suppose \( \xi \) is an ordinal and \( n \) is a natural number.

(i) If \( f : \Pi(\Gamma_{\xi,n}.X) \to \mathbb{R} \) is bounded and \( \lambda \in \mathbb{R} \) is such that

\[
\lambda < \inf_{t \in \text{MAX}(\Gamma_{\xi,n}.X)} \sum_{s \leq t} \mathbb{P}_{\xi,n}(s) f(s,t),
\]

then there exist a level preserving extended pruning \((\sigma, \tau) : \Gamma_{\xi,n}.X \to \Gamma_{\xi,n}.X \) and real numbers \( b_1, \ldots, b_n \) such that \( \lambda < \sum_{i=1}^n b_i \) and for every \( 1 \leq i \leq n \) and every \( \Lambda_{\xi,n,i} \ni s \leq t \in \text{MAX}(\Gamma_{\xi,n}.X), b_i \leq f(\sigma(s), \tau(t)) \).

(ii) If \((M,d)\) is a compact metric space and \( f : \Pi(\Gamma_{\xi,n}.X) \to M \) is any function, then for any \( \delta > 0 \), there exist \( x_1, \ldots, x_n \in M \) and a level preserving extended pruning \((\sigma, \tau) : \Gamma_{\xi,n}.X \to \Gamma_{\xi,n}.X \) such that for every \( 1 \leq i \leq n \) and every \( \Lambda_{\xi,n,i} \ni s \leq t \in \text{MAX}(\Gamma_{\xi,n}.X), d(x_i, f(\sigma(s), \tau(t))) < \delta \).
(iii) If $F$ is a finite set and $f : \text{MAX}(\Gamma_{\xi,n}, X) \to F$ is any function, there exists a level preserving extended pruning $(\sigma, \tau) : \Gamma_{\xi,n}, X \to \Gamma_{\xi,n}, X$ such that $f \circ \tau|_{\text{MAX}(\Gamma_{\xi,n}, X)}$ is constant.

(iv) For any natural numbers $k_1 < \ldots < k_r \leq n$, there exists an extended pruning $(\sigma, \tau) : \Gamma_{\xi,r}, X \to \Gamma_{\xi,n}, X$ such that for every $1 \leq i \leq r$, $\sigma(\Lambda_{\xi, n, i}) \subset \Lambda_{\xi, n, k_i}$.

2.2. The Szlenk index, Szlenk power type. Given a $w^*$-compact subset $L$ of $X^*$ and $\varepsilon > 0$, we let $s_\varepsilon(K)$ denote the set consisting of those $x^* \in L$ such that for every $w^*$-neighborhood $V$ of $x^*$, $\text{diam}(L \cap V) > \varepsilon$. We define the transfinite derivations

\[
s^0_\varepsilon(L) = L,
\]

\[
s^{\xi+1}_\varepsilon(L) = s_\varepsilon(s^\xi_\varepsilon(L)),
\]

and if $\xi$ is a limit ordinal,

\[
s_\varepsilon(\xi)(L) = \bigcap_{\zeta < \xi} s_\varepsilon(\zeta)(L).
\]

If there exists an ordinal $\xi$ such that $s^\xi_\varepsilon(L) = \emptyset$, we let $Sz(L, \varepsilon)$ be the minimum such ordinal. Otherwise we write $Sz(L, \varepsilon) = \infty$. Since $s^\xi_\varepsilon(L)$ is $w^*$-compact, we deduce that $Sz(L, \varepsilon)$ cannot be a limit ordinal. We agree to the conventions that $\omega\infty = \infty$ and $\xi < \infty$ for any ordinal $\xi$. We let $Sz(L) = \sup_{\varepsilon > 0} Sz(L, \varepsilon)$. If $B : Z \to W$ is an operator, we let $Sz(B, \varepsilon) = Sz(B^*B_{W*}, \varepsilon)$, $Sz(B) = Sz(B^*B_{W*})$. If $Z$ is a Banach space, $Sz(Z, \varepsilon) = Sz(I_Z, \varepsilon)$ and $Sz(Z) = Sz(I_Z)$.

We recall that a set $L \subset X^*$ is called $w^*$-fragmentable if for any $\varepsilon > 0$ and any $w^*$-compact, non-empty subset $M$ of $L$, $s_\varepsilon(M) \subset M$. This is equivalent to $Sz(L) < \infty$. We say an operator $B : Z \to W$ is Asplund if $B^*B_{W*}$ is $w^*$-fragmentable, which happens if and only if $Sz(B) < \infty$. We say a Banach space $Z$ is Asplund if $I_Z$ is Asplund. These are not the original definitions of Asplund spaces and operators, but they are equivalent to the original definitions (see [1]).

If $Sz(K) < \omega^{\xi+1}$, then for any $\varepsilon > 0$, $Sz(K, \varepsilon) \leq \omega^\xi n$ for some $n \in \mathbb{N}$. We let $Sz_\xi(K, \varepsilon)$ be the smallest $n \in \mathbb{N}$ such that $Sz(K, \varepsilon) \leq \omega^\xi n$. We define the $\xi$-Szlenk power type $p_\xi(K)$ of $K$ by

\[
p_\xi(K) = \limsup_{\varepsilon \to 0^+} \frac{\log Sz_\xi(K, \varepsilon)}{|\log(\varepsilon)|}.
\]

This value need not be finite. By convention, we let $p_\xi(K) = \infty$ if $Sz(K) > \omega^{\xi+1}$. We let $p_\xi(A) = p_\xi(A^*B_{Y*})$ and $p_\xi(X) = p_\xi(B_{X*})$. The quantities $p_\xi(X)$, $p_\xi(A)$ are important for the renorming theorem of $\xi$-asymptotically uniformly smooth norms with power type modulus.

Given a $w^*$-compact subset $L$ of $X^*$ and $\varepsilon > 0$, we let $\mathcal{H}_\varepsilon^L$ denote the set of Cartesian products $\prod_{i=1}^n C_i$ such that $C_i \in \mathcal{C}$ for each $1 \leq i \leq n$ and such that there exist $(x_i)_{i=1}^n \in \prod_{i=1}^n C_i$ and $x^* \in K$ such that for each $1 \leq i \leq n$, $\text{Re} \ x^*(x_i) \geq \varepsilon$. 


2.3. The Szlenk index of $K_p$. Recall that for $1 < p < \infty$, $L_p(X)$ denotes the space of equivalence classes of Bochner integrable functions $f : [0, 1] \to X$ such that $\int \|f\|^p < \infty$, where $[0, 1]$ is endowed with its Lebesgue measure. Recall also that if $1 < q < \infty$, $L_q(X^*)$ is isometrically included in $L_p(X^*)$ by the action

$$f \mapsto \int \langle g, f \rangle,$$

for $g \in L_q(X^*)$. We also recall that if $\varphi : X \to \mathbb{R}$ is any Lipschitz function, then for any $f \in L_p(X)$, $\varphi \circ f \in L_p$.

We note that the Szlenk index and the $\xi$ Szlenk power type of $K$ are unchanged by scaling $K$ by a positive scalar or by replacing $K$ with its balanced hull. Moreover, for a positive scalar $c$, $(cK)_p = cK_p$, which has the same Szlenk index and $\xi$-Szlenk power type as $K_p$. If $\mathcal{T}K$ is the balanced hull of $K$, $K_p \subset (\mathcal{T}K)_p$ and $Sz(K) = Sz(\mathcal{T}K)$ ([3, Lemma 2.2]) so that Theorem 1, Corollary 2, and Theorem 3 hold in general if they hold under the assumption that $K \subset B_X$ is balanced. Therefore we can and do assume throughout that $K \subset B_{X^\ast}$ and $K$ is balanced.

Let $\varphi : X \to \mathbb{R}$ be given by $\varphi(x) = \max_{x^* \in K} \text{Re } x^*(x)$. Since we have assumed $K$ is balanced, $\varphi(x) = \max_{x^* \in K} |x^*(x)|$. It is easy to see that for any $1 < p < \infty$ and any $f \in L_p(X)$, $\|\varphi(f)\|_{L_p} = \max_{f^* \in K} \text{Re } f^*(f)$. Combining this fact with [4, Corollary 2.4] and the proof of that corollary, we obtain the following.

**Theorem 5.** Fix $1 < p, \alpha < \infty$.

(i) If for every $B$-tree $T$ with $o(T) = \omega^{1+\xi}$ and every normally weakly null $(f_t)_{t \in T} \subset B_{L_p(X)}$,

$$\inf \{\|\varphi(f)\|_{L_p} : t \in T, f \in \text{co}(f_s : \emptyset < s \leq t)\} = 0,$$

then $Sz(K_p) \leq \omega^{1+\xi}$.

(ii) If there exists a constant $C$ such that for every $n \in \mathbb{N}$, every $B$-tree $T$ with $o(T) = \omega^{1+\xi} n$, and every normally weakly null collection $(f_t)_{t \in T} \subset B_{L_p(X)}$,

$$\inf \{\|\varphi(f)\|_{L_p} : t \in T, f \in \text{co}(f_s : \emptyset < s \leq t)\} \leq Cn^{-1/\alpha},$$

then $p_{1+\xi}(K_p) \leq \alpha$.

**Proposition 6.** Suppose $T$ is a non-empty $B$-tree. Suppose also that $(C_s)_{s \in T.X} \subset C$ is fixed and for $s = (\zeta_i, Z_i)^{1 \leq i \leq n} \in T.X$, let $\lambda(s) = Z_k \cap C_s$. Suppose that $S$ is a non-empty, well-founded $B$-tree and $\theta : S.X \to T.X$ is a pruning. For $s \in S.X$, let $s(s) = \prod_{i=1}^n \lambda(\theta(s_i))$. If $\varepsilon > 0$ is such that for every $t \in S.X$, $s(s) \in \mathcal{H}^K_{\varepsilon} \neq \emptyset$, then for any $0 < \delta < \varepsilon$, any $0 \leq \gamma < o(S)$, and any $s \in S^\gamma.X$, $s(s) \in \mathcal{H}_{\varepsilon}(K) \neq \emptyset$. Moreover, for any $0 < \delta < \varepsilon$, $Sz(K, \delta) > o(S)$.

**Proof.** We induct on $\gamma$. The base case is the hypothesis. Assume $\gamma + 1 < o(S)$ and the result holds for $\gamma$. Assume $s \in S^{\gamma+1}.X$, which means there exists $\zeta$ such that $s^{-}(\zeta, Z) \in S^\gamma.X$ for all $Z \in \text{codim}(X)$. Then for every $Z \in \text{codim}(X)$, there exists $Z \geq W_Z \in \text{codim}(X)$
such that \( \mathfrak{f}(s^-((\zeta, Z))) \subset \mathfrak{f}(s) \times B_{W_Z} \). From this and the inductive hypothesis, for every \( Z \in \text{codim}(X) \), we fix \( x_Z \in B_{W_Z}, \ (x_Z^{|s_i|})_{i=1}^{|s_i|} \in \mathfrak{f}(s) \), and \( x^*_Z \in s^\gamma_\delta(K) \) such that \( \text{Re } x^*_Z(x_Z) \geq \varepsilon \) and \( \text{Re } x^*_Z(x^*_Z) \geq \varepsilon \) for each \( 1 \leq i \leq |s| \). By compactness of \( \mathfrak{f}(s) \times K \) with the product topology, where \( \lambda(\theta(|s|)) \) has its norm topology and \( K \) has its \( w^* \)-topology,

\[
\emptyset \neq \bigcap_{Z \in \text{codim}(X)} \{ (x^1_Y, \ldots, x^{|s|}_Y) : Z \ni Y \in \text{codim}(X) \} \subset \mathfrak{f}(s) \times K.
\]

Fix \((x_1, \ldots, x_{|s|}, x^*)\) lying in this intersection. Obviously \( x^* \in s^\gamma_\delta(K) \). Moreover, for any \( w^* \)-neighborhood \( V \) of \( x^* \), there exists \( Z \in \text{codim}(X) \) such that \( \ker(x^*) \subset Z \) and \( x^*_Z \in V \), whence

\[
\text{diam}(s^\gamma_\delta(K) \cap V) \geq \| x^*_Z - x^* \| = \text{Re } (x^*_Z - x^*)(x_Z) = \text{Re } x^*_Z(x_Z) \geq \varepsilon > \delta.
\]

This implies \( x^* \in s^{\gamma+1}_\delta(K) \). It is obvious that \( \text{Re } x^*(x_i) \geq \varepsilon \) for all \( 1 \leq i \leq |s| \). This shows that \( \mathfrak{f}(s) \in \mathcal{H}^{s^{\gamma+1}_\delta(K)} \) and completes the successor case.

Finally, assume \( \gamma < \omega(S) \) is a limit ordinal and the result holds for all ordinals less than \( \gamma \). Fix \( s \in S^\gamma \times X \) and let \( \mathfrak{f}(s) \times K \) be topologized as in the successor case. By the inductive hypothesis, for all \( \beta < \gamma \), there exists \((x_{\beta}, \ldots, x^\beta_{|s|}, x^\beta_\delta) \in \mathfrak{f}(s) \times K \) such that \( x^\beta_\delta \in s^\gamma_\delta(K) \) and for all \( 1 \leq i \leq |s| \), \( \text{Re } x^\beta_\delta(x_i) \geq \varepsilon \). By compactness of \( \left( \prod_{i=1}^{|s|} \lambda(\theta(|s_i|)) \right) \times K \),

\[
\bigcap_{\beta < \gamma} \{ (x^1_\mu, \ldots, x^{|s|}_\mu, x^*_\mu) : \mu \ni \beta \} \neq \emptyset.
\]

Clearly any \((x_1, \ldots, x_{|s|}, x^*)\) lying in this intersection is such that \( x^* \in s^\gamma_\delta(K) \) and for any \( 1 \leq i \leq |s| \), \( \text{Re } x^*(x_i) \geq \varepsilon \). This shows that \( \mathfrak{f}(s) \in \mathcal{H}^{s^\gamma_\delta(K)} \) and completes the induction.

We have shown that for any \( 0 < \delta < \varepsilon \), \( S\zeta(K, \delta) > \omega(S) \). If \( \omega(S) \) is a limit ordinal, we deduce that \( S\zeta(K, \delta) > \omega(S) \) since \( S\zeta(K, \delta) \) cannot be a limit ordinal. If \( \omega(S) \) is a successor, say \( \omega(S) = \xi + 1 \), then there exists a length 1 sequence \((\zeta) \in S^\xi \). For every \( Z \in \text{codim}(X) \), \( \mathfrak{f}((\zeta, Z)) = W_Z \cap C_{\theta((\zeta, Z))} \) for some \( W \subset Z \). The first part of the proof yields that for each \( Z \in \text{codim}(X) \), there exists \( x_Z \in W_Z \cap C_{\theta((\zeta, Z))} \subset W_Z \cap B_X \) and some \( x^*_Z \in s^\xi_\delta(K) \) such that \( \text{Re } x^*_Z(x_Z) \geq \varepsilon \). Arguing as in the successor case, we deduce that any \( w^* \)-limit of a subnet of \((x^*_Z)_{Z \in \text{codim}(X)} \) lies in \( s^{\xi+1}_\delta(K) \), whence \( S\zeta(K, \delta) > \xi + 1 = \omega(S) \).

\[\square\]

### 2.4. Games

Suppose \( T \subset \Lambda^{<\mathbb{N}} \) is a well-founded, non-empty \( B \)-tree and \( \mathcal{E} \subset \text{MAX}(T \times \mathcal{C}) \) is some subset. We define the game on \( T \times \mathcal{C} \) with target set \( \mathcal{E} \). Player I first chooses \((\zeta_1, Z_1) \in \Lambda \times \text{codim}(X) \) such that \((\zeta) \in T \) and Player II then chooses \( C_1 \in \mathcal{C} \). Assuming \((\zeta_i, Z_i)_{i=1}^n \in T \times \mathcal{C} \) and \( C_1, \ldots, C_n \in \mathcal{C} \) have been chosen, the game terminates if \((\zeta_i, Z_i)_{i=1}^n \in \text{MAX}(T \times X) \). Otherwise Player I chooses \((\zeta_{n+1}, Z_{n+1}) \in \Lambda \times \text{codim}(X) \) such that \((\zeta)_{i=1}^{n+1} \in T \) and Player II chooses \( C_{n+1} \in \mathcal{C} \). Since \( T \) is well-founded, this game must terminate after finitely many steps. Suppose that the resulting choices are \((\zeta_i, Z_i, C_i)_{i=1}^n \in \mathcal{E} \), and Player I wins otherwise.
A strategy for Player I for the game on $T.X.C$ with target set $E$ is a function $\psi : T.X.C \cup \{\varnothing\} \to \Lambda \times \text{codim}(X)$ such that if $\psi((\zeta_i, Z_i, C_i)_{i=1}^{n-1}) = (\zeta_n, Z_n)$, $(\zeta_i)_{i=1}^n \in T$. We say $\psi$ is a winning strategy for Player I provided that for any sequence $(\zeta_i, Z_i, C_i)_{i=1}^n \in \text{MAX}(T.X.C)$ such that $(\zeta_i, Z_i) = \psi((\zeta_j, Z_j, C_j)_{j=1}^{i-1})$ for every $1 \leq i \leq n$, $(\zeta_i, Z_i, C_i)_{i=1}^n \notin E$.

A strategy for Player II for the game on $T.X.C$ with target set $E$ is a function $\psi$ defined on the set

$$\{((\zeta_i, Z_i, C_i)_{i=1}^{n-1}, (\zeta_n, Z_n)) : (\zeta_i, Z_i, C_i)_{i=1}^n \in \{\varnothing\} \cup T.X.C, (\zeta_n, Z_n) \in \Lambda \times \text{codim}(X), (\zeta_i)_{i=1}^n \in T\}$$

and taking values in $C$. We say $\psi$ is a winning strategy for Player II provided that for any sequence $(\zeta_i, Z_i, C_i)_{i=1}^n \in \text{MAX}(T.X.C)$ such that $C_i = \psi((\zeta_j, Z_j, C_j)_{j=1}^{i-1}, (\zeta_i, Z_i))$ for all $1 \leq i \leq n$, $(\zeta_i, Z_i, C_i)_{i=1}^n \in E$.

**Proposition 7.** [3 Proposition 3.1] For any non-empty, well-founded B-tree $T$ and any $E \subset T.X.C$, either Player I or Player II has a winning strategy for the game on $T.X.C$ with target set $E$.

**Proposition 8.** Suppose that Player II has a winning strategy for a game on $T.X.C$ with target set $E$. Then there exists $(C_s)_{s \in T.X} \subset C$ such that for every $t = (\zeta_i, Z_i)_{i=1}^k \in \text{MAX}(T.X)$, $(\zeta_i, Z_i, C_{t_i})_{i=1}^k \in E$.

**Proof.** Fix a winning strategy $\psi$ for Player II in the game. We define $C_s$ by induction on $|s|$. We let $C_{(\zeta, Z)} = \psi(\varnothing, (\zeta, Z))$. If $|s| = k + 1$, $C_{s|k}$ has been defined for every $1 \leq i \leq k$, and $s = s|k(\zeta, Z)$, we let $C_s = \psi(s|k, (\zeta, Z))$.

For the next proposition, if $h \in L_p(X)$ is a simple function, we let $\overline{h}$ be the function in $L_p(X)$ such that $\overline{h}(\varnothing) = 0$ if $h(\varnothing) = 0$ and $\overline{h}(\varnothing) = h(\varnothing)/||h(\varnothing)||$ otherwise.

**Proposition 9.** Let $\xi$ be an ordinal, $n$ a natural number, and let $T$ be a B-tree with $o(T) \geq \omega^{1+\xi n}$. If $\psi$ is a strategy for Player I for some game on $\Gamma_{\xi,n}.X.C$, then for any $1 < p < \infty$, any $\delta > 0$, and any normally weakly null $(f_i)_{i \in T.L_p(X)} \subset B_{L_p(X)}$, there exist $s = (\zeta_i, Z_i)_{i=1}^k \in \text{MAX}(\Gamma_{\xi,n}.X)$, $\varnothing = t_0 < t_1 < \ldots < t_k \in T.L_p(X)$, $g_i \in \text{co}(f_u : t_{i-1} < u \leq t_i)$, $h_i \in B_{L_p(X)}$, and $C_i \in \mathcal{C}$ such that for every $1 \leq i \leq k$,

1. $h_i$ is simple,
2. $\text{range}(\overline{h_i}) = C_i \subset B_{Z_i},$
3. $\|g_i - h_i\|_{L_p(X)} < \delta,$
4. $(\zeta_i, Z_i) = \psi((\zeta_j, Z_j, C_j)_{j=1}^{i-1}).$

**Remark 10.** For a B-tree $S$ on $\Lambda$ and $s \in S$, we let $S(s)$ denote those non-empty sequences $u \in \Lambda^{<\mathbb{N}}$ such that $s \triangleleft u \in S$. An easy induction argument yields that for any ordinals $\xi, \zeta$, $S^\xi(s) = (S(s))^{\xi}$ for any ordinal $\xi$. From this it follows that $s \in S^\xi$ if and only if $o(S(s)) \geq \xi$. 

THE SZLENK INDEX OF $L_p(X) AND A_p$
Furthermore, another easy induction yields that if \((S^k)^{\xi} = S^{\xi + \zeta}\), from which it follows that if \(o(S) \geq \xi + \zeta\), \(o(S^k) \geq \zeta\). Therefore if \(s \in S^{\xi + \omega}\), \(o(S^k(s)) \geq \omega\).

**Proof of Proposition 3.1.** We first note that if \(Z \in \text{codim}(X)\), \(L_p(X)/L_p(Z)\) is either the zero vector space or isomorphic to \(L_p\), and therefore has Szlenk index not exceeding \(\omega\). As explained in [5], this means that for any \(B\)-tree \(T\) with \(o(T) \geq \omega\), any \(\delta > 0\), and any normally weakly null \((f_t)_{t \in T}.L_p(X) \subset B_{L_p(X)}\), there exist \(t \in T.\omega\), \(g \in \text{co}(f\_s : \emptyset \prec s \leq t\), and \(h \in B_{L_p(Z)}\) such that \(\|g - h\|_{L_p(X)} < \delta\). Moreover, by the density of simple functions, we may assume this \(h\) is simple.

Let \(\psi\) be a strategy for Player I for a game on \(\Gamma_{\xi,n}.X.C\). Let \(T\) be a \(B\)-tree with \(o(T) = \omega^{1+\xi_0}\) and define \(\gamma : \Gamma_{\xi,n}.X \cup \{\emptyset\} \rightarrow [0, \omega^{\xi_0}]\) by letting \(\gamma(t) = \max\{\mu \leq \omega^{\xi_0} : t \in (\Gamma_{\xi,n}.X)^\mu\}\) for \(t \in \Gamma_{\xi,n}.X\) and \(\gamma(\emptyset) = \omega^{\xi_0}\). Let \(s_0 = t_0 = \emptyset\). Now assume that for some \(k \in \mathbb{N}\) and all \(1 \leq i < k\), \(s_i \in \Gamma_{\xi,n}.X\), \(\zeta_i \in [0, \omega^{\xi_0}]\), \(Z_i \in \text{codim}(X)\), \(t_i \in T.\omega\), \(g_i, h_i \in B_{L_p(X)}\), and \(C_i \in C\) have been chosen such that for all \(1 \leq i < k\),

\[(i)\quad h_i \text{ is simple},
(ii)\quad s_i = (\zeta_j, Z_j)_{j=1}^i,
(iii)\quad t_0 < t_1 < \ldots < t_{k-1},
(iv)\quad t_i \in (T.\omega)(X))^{\gamma(s_i)},
(v)\quad (\zeta_i, Z_i) = \psi((\zeta_j, Z_j, C_j)_{j=1}^{k-1}),
(vi)\quad g_i \in \text{co}(f_u : t_{i-1} < u \leq t_i),
(vii)\quad \|g_i - h_i\|_{L_p(X)} < \delta,
(viii)\quad \text{range}(h_i) = C_i \subset B_{Z_i}.
\]

If \(s_{k-1}\) is maximal in \(\Gamma_{\xi,n}.X\), we let \(s = s_{k-1}\), and one easily checks that the conclusions are satisfied. Otherwise let \((\zeta_k, Z_k) = \psi((\zeta_j, Z_j, C_j)_{j=1}^{k-1})\) and \(s_k = s_{k-1}(\zeta_k, Z_k)\). Let \(u_{k-1}\) be the sequence of first members of the pairs of \(t_{k-1}\) and let \(U\) denote the proper extensions of \(u_{k-1}\) in \(T.\omega\). Then \(f_{\gamma(u_{k-1})} \in U.\omega\) \(B_{L_p(X)}\) is normally weakly null and \(o(U) \geq \omega\) by the remark preceding the proof, so that the previous paragraph yields the existence of some \(u' \in U.\omega\) \(B_{L_p(X)}\), \(g_k \in \text{co}(f_u : t_{k-1} < u \leq t_{k-1}^* u')\), and some simple function \(h_k \in L_p(Z_k)\) such that \(\|g_k - h_k\|_{L_p(X)} < \delta\). Let \(t_k = t_{k-1}^* u'\). In order to apply the remark before the proof, we note that since \(s_{k-1} \prec s_k\), \(\gamma(s_{k-1}) \geq \gamma(s_k) + 1\). Since \(\omega\gamma(s_{k-1}) \geq \omega(\gamma(s_k) + 1) = \omega\gamma(s_k) + \omega\), the remark preceding the proof applies. Note that \(C_k := \text{range}(h_k) \subset B_{Z_k}\). This completes the recursive construction. Since \(\Gamma_{\xi,n}.X\) is well-founded, eventually this process terminates. The resulting \(s = (\zeta_i, Z_i)_{i=1}^k \in MAX(\Gamma_{\xi,n}.X)\) clearly satisfies the conclusions.

\(\square\)

### 3. Definition of an Associated Space and Two Games

#### 3.1. The associated space and its properties

If \(E\) is a vector space with seminorm \(\| \cdot \|\), we say a sequence \((e_i)_{i=1}^n \in E\) is 1-\textit{unconditional} provided that for any scalars \((a_i)_{i=1}^n\)
and any \((\varepsilon_i)_{i=1}^n \in \{\pm 1\}^n\), \(\|\sum_{i=1}^n \varepsilon_i a_i e_i\| = \|\sum_{i=1}^n a_i e_i\|\). Recall that for \(1 < p < \infty\), a vector space \(E\) with seminorm \(\| \cdot \|\) which is spanned by the 1-unconditional basis \((e_i)_{i=1}^n\) is called \(p\)-concave provided there exists a constant \(C\) such that for any \((f_i)_{i=1}^n \subset L_p\),

\[ \|\sum_{i=1}^n f_i e_i\|_{L_p(E)} \leq C \|\sum_{i=1}^n f_i\|_{L_p} \|e_i\|_E. \]

The smallest such constant \(C\) is denoted by \(M_p(E)\).

Given \(x \in \text{span}(e_i : 1 \leq i \leq n)\), where \((e_i)_{i=1}^n\) is a Hamel basis for the seminormed space \(E\), we write \(x = \sum_{i=1}^n a_i e_i\) and \(\text{supp}(x) = \{i \leq n : a_i \neq 0\}\). We say the vectors \(x_1, \ldots, x_n \in \text{span}(e_i : 1 \leq i \leq n)\) are disjointly supported if the sets \(\text{supp}(x_1), \ldots, \text{supp}(x_n)\) are pairwise disjoint.

For \(1 < \beta < \infty\), we say that an unconditional Hamel basis \((e_i)_{i=1}^n\) for a seminormed space \(E\) satisfies an 1-lower \(\ell_\beta\) estimate provided that for any \(m \in \mathbb{N}\) and any disjointly supported elements \((x_i)_{i=1}^m \subset E\),

\[ \left(\sum_{i=1}^m \|x_i\|^{\beta}\right)^{1/\beta} \leq \|\sum_{i=1}^m x_i\|. \]

**Theorem 11.** [7] Theorem 1.f.7 Fix \(1 < \beta < p < \infty\). There exists a constant \(C' = C'(\beta, p)\) such that if \((e_i)_{i=1}^n\) is a 1-unconditional basis for the seminormed space \(E\) which satisfies a 1-lower \(\ell_\beta\) estimate, then \(E\) is \(p\)-concave and \(M_p(E) \leq C'\).

For the remainder of this section, \(T\) is a fixed, non-empty \(B\)-tree.

For a non-empty set \(J\), we let \(c_0(J)\) be the span of the canonical Hamel basis \((e_j)_{j \in J}\) in the space of scalar-valued functions on \(J\), where \(e_j\) is the indicator of the singleton \(\{j\}\). We let \(e_j^*\) denote the coordinate functional to \(e_j\). Given \(x \in c_0(J)\), we may write \(x = \sum_{j \in J} a_j e_j\). Then we define \(|x|\) to be \(\sum_{j \in J} |a_j| e_j\). A suppression projection is an operator \(P\) from \(\text{span}(e_j^* : j \in J)\) into itself such that there exists a subset \(F\) of \(J\) such that \(P \sum_{j \in J} a_j e_j^* = \sum_{j \in F} a_j e_j^*\).

For \(0 < \phi < \theta < 1\), let

\[ N_{\theta, \phi, T} = \{0\} \cup \left\{ \theta \sum_{i=1}^k e_{j_i}^* : t = (\zeta_i, Z_i, C_i)_{i=1}^{|t|} \in T.X.C, 1 \leq j_1 < \cdots < j_k \leq |t|, \prod_{i=1}^k Z_{j_i} \cap C_{j_i} \in \mathcal{H}_\phi^K \right\} \subset \text{span}(e_j^* : t \in T.X.C). \]

For \(0 < \phi < \theta < 1\) and \(1 < \alpha < \infty\), let

\[ M_{\theta, \phi, \alpha, T} = \left\{ \sum_{i=1}^k a_i g_i : g_i \in \bigcup_{n=1}^\infty N_{\theta^n, \phi^n, T}, a_i \geq 0, \sum_{i=1}^k a_i^\alpha \leq 1, \text{supp}(g_i) \text{ are pairwise disjoint} \right\}. \]

Note that the set \(M_{\theta, \phi, \alpha, T}\) is closed under suppression projections.

We define the seminorm \(\| \cdot \|_{\theta, \phi, \alpha, T}\) on \(c_0(T.X.C)\) by

\[ \|x\|_{\theta, \phi, \alpha, T} = \sup \{ f(|x|) : f \in M_{\theta, \phi, \alpha, T} \}. \]
Claim 12. Fix $1 < \alpha < \infty$ and $0 < \phi < \theta < 1$. For any $t \in T.X.C$, $(e_{t_i})_{i=1}^{|t|}$ is 1-unconditional and satisfies a 1-lower $\ell_\beta$ estimate in its span, where $1/\alpha + 1/\beta = 1$.

Proof. Note that 1-unconditionality is obvious. Fix $x_1, \ldots, x_n \in \text{span}(e_{t_i} : 1 \leq i \leq |t|)$ with disjoint supports. That is, there exist pairwise disjoint subsets $S_1, \ldots, S_n$ of $\{1, \ldots, |t|\}$ such that $x_i \in \text{span}(e_{t_j} : j \in S_i)$. Then there exist $g_1, \ldots, g_n \in M_{\theta,\phi,\alpha,T}$ such that for each $1 \leq i \leq n$, $g_i(|x_i|) = ||x_i||_{\theta,\phi,\alpha,T}$. Since $M_{\theta,\phi,\alpha,T}$ is closed under suppression projections, we may assume that $\text{supp}(g_i) \subset S_i$ for each $1 \leq i \leq n$. Then if $(a_i^n)_{n=1}^i$ are such that $\sum_{i=1}^n a^n_i = 1$, $a_i \geq 0$, and $\sum_{i=1}^n a_i ||x_i||_{\theta,\phi,\alpha,T} = (\sum_{i=1}^n ||x_i||_{\theta,\phi,\alpha,T}^\beta)^{1/\beta}$, $g := \sum_{i=1}^n a_i g_i \in M_{\theta,\phi,\alpha,T}$ and
\[
\| \sum_{i=1}^n x_i \|_{\theta,\phi,\alpha,T} \geq g \left( \left\| \sum_{i=1}^n x_i \right\| \right) = \sum_{i=1}^n a_i g_i (\|x_i\|) = \left( \sum_{i=1}^n \|x_i\|_{\theta,\phi,\alpha,T}^\beta \right)^{1/\beta}.
\]

\hfill \square

Claim 13. Fix $1 < \alpha < \infty$ and $0 < \phi < \theta < 1$. For any $t = (\zeta, Z_i, C_i)_{i=1}^k \in T.X.C$, any sequence $(x_i)_{i=1}^k \in \prod_{i=1}^k Z_i \cap C_i$, and any sequence $(a_i)_{i=1}^k$ of non-negative scalars,
\[
\vartheta \left( \sum_{i=1}^k a_i x_i \right) \leq \frac{1}{\theta - \phi} \left\| \sum_{i=1}^k a_i e_{t_i} \right\|_{\theta,\phi,\alpha,T}.
\]

Proof. We recall that if $C \in \mathcal{C}$, $C \subset B_X$ by the definition of $\mathcal{C}$. With $t$, $(x_i)_{i=1}^k \in \prod_{i=1}^k Z_i \cap C_i$, and $(a_i)_{i=1}^k$ as in the statement, fix $x^* \in K$ such that $\text{Re} x^* (\sum_{i=1}^k a_i x_i) = \vartheta (\sum_{i=1}^k a_i x_i)$. For all $j \in \mathbb{N}$, let $B_j = \{ i \leq k : \text{Re} x^* (x_i) \in (\phi^j, \phi^{j-1}] \}$. Note that for every $j \in \mathbb{N}$, $\theta^j \sum_{i \in B_j} e_{t_i} \in N_{\theta^j,\phi^j,\alpha,T}$, so
\[
\phi^{j-1} \sum_{i \in B_j} a_i = \phi^{-1} (\phi/\theta)^j (\theta^j \sum_{i \in B_j} e_{t_i}) (\sum_{i=1}^k a_i e_{t_i}) \leq \phi^{-1} (\phi/\theta)^j \left\| \sum_{i=1}^k a_i e_{t_i} \right\|_{\theta,\phi,\alpha,T}.
\]

Then
\[
\vartheta \left( \sum_{i=1}^k a_i x_i \right) \leq \sum_{j=1}^\infty \sum_{i \in B_j} a_i \text{Re} x^* (x_i) \leq \sum_{j=1}^\infty \phi^{j-1} \sum_{i \in B_j} a_i \leq \sum_{j=1}^\infty \phi^{-1} (\phi/\theta)^j \left\| \sum_{i=1}^k a_i e_{t_i} \right\|_{\theta,\phi,\alpha,T} = \frac{1}{\theta - \phi} \left\| \sum_{i=1}^k a_i e_{t_i} \right\|_{\theta,\phi,\alpha,T}.
\]

\hfill \square

Corollary 14. Fix $1 < p, \alpha, \beta < \infty$ with $1/\alpha + 1/\beta = 1$ and $\beta < p$. Let $C' = C'(\beta, p)$ be the constant from Theorem [11]. Suppose that $\xi$ is an ordinal, $n$ is a natural number, $\varepsilon > 0$, and $0 < \phi < \theta < 1$ are such that Player I has a winning strategy in the game with target set
\[
\left\{ t \in \text{MAX}(\Gamma_{\xi,n.X.C}) : \left\| \sum_{s \leq t} P_{\xi,n}(s) e_s \right\|_{\theta,\phi,\alpha,T_{\xi,n}} > \varepsilon \right\}.
\]
Then for any B-tree $T$ with $\sigma(T) \geq \omega^{1+\xi n}$ and any normally weakly null $(f_t)_{t \in T,L_p(X)} \subset B_{L_p(X)}$,

$$\inf \{ \| \varrho(f) \|_{L_p} : t \in T,L_p(X), f \in \text{co}(f_s : \emptyset \prec s \preceq t) \} \leq \frac{C'\varepsilon}{n(\theta - \phi)}.$$ 

**Proof.** Recall for the proof that for a simple function $h \in L_p(X)$, $\overline{h}$ is the function in $L_p(X)$ such that $\overline{h}(\omega) = 0$ if $h(\omega) = 0$ and $\overline{h}(\omega) = h(\omega)/\|h(\omega)\|$ otherwise.

Fix a winning strategy $\psi$ for Player I in the game with the indicated target set. Fix $\delta > 0$. By Proposition [9] there exist $s = (\zeta_i, Z_i)_{i=1}^k \in MAX(\Gamma_{\xi,n,X})$, $\emptyset = t_0 \prec \ldots \prec t_k$, $g_i \in \text{co}(f_u : t_{i-1} \preceq u \preceq t_i)$, simple functions $h_i \in B_{L_p(X)}$, and $C_i \in C$ such that $\|g_i - h_i\|_{L_p(X)} < \delta$, range$(\overline{h}_i) = C_i \subset B_{Z_i}$, and $(\zeta_i, Z_i) = \psi((\zeta_j, Z_j, C_j)_{j=1}^{i-1})$. This means that for any $\omega \in [0,1]$, $(\overline{h}_i(\omega))_{i=1}^k \in \prod_{i=1}^k Z_i \cap C_i$, whence by Claim [13] for any non-negative scalars $(a_i)_{i=1}^k$,

$$\varrho(\sum_{i=1}^k a_i h_i(\omega)) = \varrho(\sum_{i=1}^k a_i \|h_i(\omega)\| \overline{h}_i(\omega)) \leq \frac{1}{\theta - \phi} \sum_{i=1}^k a_i \|h_i(\omega)\| e_{a_i} ~\varrho_{\theta,\phi,\Gamma_{\xi,n}}.$$ 

Since by Claim [13] $(e_u)_{u \preceq s}$ satisfies a lower $\ell_\beta$ estimate in its span, we deduce that

$$\| \varrho(\sum_{i=1}^k n^{-1} P_{\xi,n}(s_i)h_i) \|_{L_p} = \left( \int_0^1 \left| \varrho(\sum_{i=1}^k n^{-1} P_{\xi,n}(s_i) \|h_i(\omega)\| \overline{h}_i(\omega)) \right|^p \text{d}\omega \right)^{1/p} \leq \frac{1}{n(\theta - \phi)} \left( \int_0^1 \left\| \sum_{u \preceq s} P_{\xi,n}(u) \|h_u(\omega)\| e_u \|_{\theta,\phi,\Gamma_{\xi,n}} \right|^p \text{d}\omega \right)^{1/p} \leq \frac{C'}{n(\theta - \phi)} \sum_{u \preceq s} P_{\xi,n}(u) \|h_u(\omega)\|_{L_p(X)} e_u \|_{\theta,\phi,\Gamma_{\xi,n}} \leq \frac{C'\varepsilon}{n(\theta - \phi)}.$$ 

Here we have used 1-unconditionality, $\|h_i\|_{L_p(X)} \leq 1$ for each $1 \leq i \leq k$, and the fact that since $\psi$ is a winning strategy for Player I,

$$\| \sum_{u \preceq s} P_{\xi,n}(u) \|h_u(\omega)\|_{L_p(X)} e_u \|_{\theta,\phi,\Gamma_{\xi,n}} \leq \| \sum_{u \preceq s} P_{\xi,n}(u) e_u \|_{\theta,\phi,\Gamma_{\xi,n}} \leq \varepsilon.$$ 

Let $g = n^{-1} \sum_{u \preceq s} P_{\xi,n}(u)g_i \in \text{co}(f_u : u \preceq t_k)$ and $h = n^{-1} \sum_{u \preceq s} P_{\xi,n}(u)h_i$. Since $\varrho$ is 1-Lipschitz, it follows that $\| \varrho(g) - \varrho(h) \|_{L_p} \leq \|g - h\|_{L_p(X)} < \delta$, so that

$$\| \varrho(g) \|_{L_p} \leq \delta + \| \varrho(h) \|_{L_p} \leq \delta + \frac{C'\varepsilon}{n(\theta - \phi)}.$$ 

Since $\delta > 0$ was arbitrary, we are done.
3.2. Particular games on $\Gamma_{\xi, n}X.C$. The statement of Proposition 6 is notationally cumbersome. We isolate the following result as a way of using Proposition 6.

Lemma 15. Fix $0 < \phi < \theta < 1$. Suppose that $\xi$ is an ordinal, $m, n$ are natural numbers, $(C_s)_{s \in \Gamma_{\xi, n}X} \subset C$, and $(\sigma, \tau) : \Gamma_{\xi, n}X \to \Gamma_{\xi, n}X$ is an extended pruning. For $t = (\zeta_i, Z_i)_{i=1}^k \in \Gamma_{\xi, n}X$, let $r(t) = (\zeta_i, Z_i, C_{t(i)})_{i=1}^k$. If $\nu \in \mathbb{N}$ is such that for every $s \in \text{MAX}(\Gamma_{\xi, m}X)$, there exists a functional $h_s \in \cup_{s=1}^\nu N_{\theta', \phi', \Gamma_{\xi, n}}$ such that $\cup_{t \leq s} r(t) \subset \text{supp}(h_s)$, then $Sz(K, \phi'/2) > \omega^k m$.

Proof. For $s = (\zeta_i, Z_i)_{i=1}^k \in \Gamma_{\xi, n}X$, let $\lambda(s) = Z_k \cap C_s$. For $s \in \Gamma_{\xi, m}X$, let $s(s) = \prod_{i=1}^{|s|} \lambda(\sigma(s_i))$.

Fix $s \in \text{MAX}(\Gamma_{\xi, m}X)$ and let $h_s \in \cup_{s=1}^\nu N_{\theta', \phi', \Gamma_{\xi, n}}$ be as in the statement of the lemma and fix $1 \leq l \leq \nu$ such that $h_s \in N_{\theta', \phi', \Gamma_{\xi, n}}$. We will prove that $s(s) \in \mathcal{H}_\phi^K$. Since for any $1 \leq m \leq k$ and any $C_1, \ldots, C_k \in C$ such that $\prod_{i=1}^k C_i \in \mathcal{H}_\phi^K$, $\prod_{i=1}^m C_i \in \mathcal{H}_\phi^K$, this will show that for any non-empty initial segment $s_1$ of $s, s(s_1) \in \mathcal{H}_\phi^K$. From here, an appeal to Proposition 6 will finish the proof.

Fix $u = (\mu_i, W_i, C_i)_{i=1}^{|u|} \in \Gamma_{\xi, n}X.C$ and $1 \leq j_1 < \ldots < j_\mu \leq |u|$ such that $h_{s_i} = \theta' \sum_{i=1}^m e_{u,i}$ and $\prod_{i=1}^m W_{j_i} \cap C_{j_i} \in \mathcal{H}_\phi^K$. Let $\tau(s) = t = (\zeta_i, Z_i)_{i=1}^{|s|}$. For each $1 \leq i \leq |s|$, let $l_i = |\sigma(s_i)|$. Note that for all $1 \leq i \leq |s|$, $r(\sigma(s_i)) = (\zeta_j, Z_j, C_{t(j)})_{j=1}^{l_i}$ and $s(s) = \prod_{j=1}^{|s|} Z_{t(j)} \cap C_{t(j)}$. By hypothesis,

$$\{(\zeta_j, Z_j, C_{t(j)})_{j=1}^{l_i} : 1 \leq i \leq |s|\} = \{r(\sigma(s_i)) : 1 \leq i \leq |s|\} \subset \text{supp}(h_s) = \{(\mu_j, W_j, C_j)_{j=1}^{l_i} : 1 \leq i \leq \mu\}.$$  

From this it follows that there exist $m_1 < \ldots < m_{|s|}$ such that for every $1 \leq i \leq |s|$, $r(\sigma(s_i)) = u_{j_{m_i}}$. Choose $(x_i)_{i=1}^m \in \prod_{i=1}^m W_{j_i} \cap C_{j_i}$ such that there exists $x^* \in K$ so that $	ext{Re } x^*(x_i) \geq \phi'$ for each $1 \leq i \leq \mu$, which exists because $\prod_{i=1}^m W_{j_i} \cap C_{j_i} \in \mathcal{H}_\phi^K$. Since $Z_{i_1} = W_{j_{m_1}}$ and $C_{t(i_1)} = C_{j_{m_1}}$, $(x_{m_1})_{i=1}^{|s|} \in \prod_{i=1}^{|s|} Z_{j_i} \cap C_{t(i_1)}$, which shows that $s(s) \in \mathcal{H}_\phi^K$. Since $l \leq \nu$, $\mathcal{H}_\phi^K \subset \mathcal{H}_\phi^K$, so that $s(s) \in \mathcal{H}_\phi^K$.

\[ \square \]

Lemma 16. Fix $1 < \alpha < \infty$ and $0 < \phi < \theta < 1$. If $Sz(K) \leq \omega^k$, then for any $\varepsilon > 0$, Player I has a winning strategy in the game with target set

$$\{t \in \text{MAX}(\Gamma_{\xi, n}X.C) : \lVert \sum_{s \leq t} P_{\xi}(s)e_{s}\rVert_{\theta, \phi, \alpha, \Gamma_{\xi}} > \varepsilon\}.$$  

Proof. Suppose not. Then by Proposition 8, there exist $\varepsilon > 0$ and $(C_s)_{s \in \Gamma_{\xi, n}X} \subset C$ such that $\varepsilon < \inf \{\lVert \sum_{s \leq t} P_{\xi}(s)e_{r(s)}\rVert_{\theta, \phi, \alpha, \Gamma_{\xi}} : t \in \text{MAX}(\Gamma_{\xi, n}X)\}$.

For $s = (\zeta_i, Z_i)_{i=1}^k \in \Gamma_{\xi, n}X$, let $r(s) = (\zeta_i, Z_i, C_{t(i)})_{i=1}^k$. For every $t \in \text{MAX}(\Gamma_{\xi, n}X), f_t \in M_{\theta, \phi, \alpha, \Gamma_{\xi}}$ such that $\text{supp}(f_t) \subset [\leq r(t)]$ and $f_t(\sum_{s \leq t} P_{\xi}(s)e_{r(s)}) = \lVert \sum_{s \leq t} P_{\xi}(s)e_{r(s)}\rVert_{\theta, \phi, \alpha, \Gamma_{\xi}}.$
Define $F : \Pi(\Gamma_{\xi,X}) \to \mathbb{R}$ by letting $F(s,t) = f_t(e_{r(s)})$. By Theorem 4, there exists an extended pruning $(\sigma, \tau) : \Gamma_{\xi,X} \to \Gamma_{\xi,X}$ such that

$$
\varepsilon < \inf_{(s,t) \in \Pi(\Gamma_{\xi,X})} F(\sigma(s), \tau(t)).
$$

Fix $\nu \in \mathbb{N}$ such that $\varepsilon > \theta^\nu$ and for each $t \in \text{MAX}(\Gamma_{\xi,X})$, write $f_{\tau(t)} = \sum_{i=1}^{k_t} a_{i,t} g_{i,t}$ where $a_{i,t} \geq 0$, $\sum_{i=1}^{k_t} a_{i,t} \leq 1$, and $g_{i,t} \in \sum_{n=1}^{\infty} N_{\theta^n,\phi^n,\Gamma_{\xi}}$ have pairwise disjoint supports. For each $t \in \text{MAX}(\Gamma_{\xi,X})$, let

$$
R_t = \{i \leq k_t : a_{i,t} > \varepsilon\}.
$$

Since $\sum_{i=1}^{k_t} a_{i,t} \leq 1$, $|R_t| \leq \lfloor 1/\varepsilon^\alpha \rfloor =: k_0$. Note that since $\varepsilon < f_{\tau(t)}(e_{r(\sigma(s))})$ for any $\emptyset < s \leq t$, $\sigma(s) \in \cup_{i \in R_t} \text{supp}(g_{i,t})$. We write $\sum_{i \in R_t} a_{i,t} g_{i,t} = \sum_{i=1}^{l_t} b_{i,t} h_{i,t}$ where $l_t \leq k_0$, $(b_{i,t})_{i=1}^{l_t}$ is an enumeration of $(a_{i,t})_{i \in R_t}$, and $(h_{i,t})_{i=1}^{l_t}$ is the corresponding enumeration of $(g_{i,t})_{i \in R_t}$. Define $\kappa : \Pi(\Gamma_{\xi,X}) \to \{1, \ldots, k_0\}$ by letting $\kappa(\sigma, \tau)$ be the unique $i \leq l_t$ such that $r(\sigma(s)) \in \text{supp}(h_{i,t})$. By Theorem 4(ii), there exists an extended pruning $(\sigma', \tau') : \Gamma_{\xi,X} \to \Gamma_{\xi,X}$ and $1 \leq l \leq k_0$ such that $\kappa(\sigma'(s), \tau'(t)) = l$ for all $(s,t) \in \Pi(\Gamma_{\xi,X})$. We now note that for any $s \in \text{MAX}(\Gamma_{\xi,X})$, $h_{l,\tau'(s)} \in \sum_{n=1}^{\infty} N_{\theta^n,\phi^n,\Gamma_{\xi}}$ is such that

$$
\{r(\sigma \circ \sigma'(u)) : \emptyset < u \leq s\} \subset \text{supp}(h_{l,\tau'(s)}),
$$

and an appeal to Lemma 15 yields that $Sz(K, \phi^\nu/2) > \omega^2$. This contradiction finishes the proof. To see that $h_{l,\tau'(s)} \in \sum_{n=1}^{\infty} N_{\theta^n,\phi^n,\Gamma_{\xi}}$, we note that if $h_{l,\tau'(s)} \in N_{\theta^n,\phi^n,\Gamma_{\xi}}$,

$$
\varepsilon \leq h_{l,\tau'(s)}(e_{r(\sigma \circ \sigma'(s))}) \leq \|h_{l,\tau'(s)}\|_\infty \leq \theta^\varepsilon.
$$

This shows that $i \leq \nu$ by our choice of $\nu$.

\[\square\]

**Lemma 17.** Fix $1 < \alpha, \beta < \infty$ and $0 < \phi < 2^{-1/\alpha}$ and assume that $1/\alpha + 1/\beta = 1$. Assume that for some $C \geq 1$ and all $i \in \mathbb{N}$, $Sz_{\xi}(K, \phi^i/2) \leq C2^i$. Let $\theta = 2^{-1/\alpha}$. Then for any $n \in \mathbb{N}$ and any $C_1 > C$, Player I has a winning strategy in the game with target set

$$
\{t \in \text{MAX}(\Gamma_{\xi,n,X}) : \sum_{s \leq t} \mathbb{P}_{\xi,n}(s)e_s\|\theta,\phi,\alpha,\Gamma_{\xi,n} > C_1 n^{1/\beta}\}.
$$

**Proof.** Suppose not. Then for some $n \in \mathbb{N}$, there exist $(C_s)_{s \in \Gamma_{\xi,n,X}} \subset \mathcal{C}$ and

$$(f_t)_{t \in \text{MAX}(\Gamma_{\xi,n,X})} \subset M_{\theta,\phi,\alpha,\Gamma_{\xi,n}}$$

such that

$$
C n^{1/\beta} < \inf_{t \in \text{MAX}(\Gamma_{\xi,n,X})} f_t\left(\sum_{s \leq t} \mathbb{P}_{\xi,n}(s)e_{r(s)}\right).
$$

We may assume as in Lemma 16 that $\text{supp}(f_t) \subset [\leq r(t)]$ for each $t \in \text{MAX}(\Gamma_{\xi,n,X})$. Then by Theorem 4(i), there exist a level preserving extended pruning $(\sigma, \tau) : \Gamma_{\xi,n} \to \Gamma_{\xi,n}$ and numbers, $b_1, \ldots, b_n$ such that $C n^{1/\beta} < \sum_{i=1}^{n} b_i$ and for all $1 \leq i \leq n$ and all $\Lambda_{\xi,n,i} \ni s \leq t \in \text{MAX}(\Gamma_{\xi,n})$, $f_{\tau(t)}(e_{r(\sigma(s))}) \geq b_i$. Fix $\delta > 0$ such that $C n^{1/\beta} + n\delta < \sum_{i=1}^{n} b_i$. Let

$$
R = \{i \leq n : b_i \geq \delta\}.
$$
Sublemma 18. There exist a level preserving extended pruning \((\sigma_0, \tau_0) : \Gamma_{\xi,n}.X \to \Gamma_{\xi,n}.X, l, w \in \mathbb{N}, (a_i)_{i=1}^l \in B_{\Gamma_{\xi,n}}^l, (k_i)_{i \in R} \subset \{1, \ldots, l\}, (w_i)_{i=1}^l \subset \{1, \ldots, w\}, \) and \((g_i)_{i \in \text{MAX}(\Gamma_{\xi,n}.X)} \subset M_{\theta, \phi, \sigma, \tau_{\xi,n}}^l\) such that

(i) for each \(t \in \text{MAX}(\Gamma_{\xi,n}.X)\), \(\|g_t - f_{\tau_0(t)}\|_\infty < \delta\),

(ii) for any \(t \in \text{MAX}(\Gamma_{\xi,n}.X)\), there exist disjointly supported functionals \(h_{1,t}, \ldots, h_{l,t}\) such that \(h_{i,t} \in N_{\theta_{w_i}, \phi_{w_i}, \Gamma_{\xi,n}}^l\) and \(g_t = \sum_{i=1}^{l} a_i h_{i,t}\),

(iii) for \(i \in R\) and \(\Lambda_{\xi,n,i} \ni s \leq t \in \text{MAX}(\Gamma_{\xi,n}.X)\), \(r(\sigma \circ \sigma_0(s)) \in \text{supp}(h_{k_i,t})\).

We first finish the proof of the lemma and then return to the proof of the sublemma. Note that item (iii) of the sublemma implies that for \(i \in R\) and \(\Lambda_{\xi,n,i} \ni s \leq t \in \text{MAX}(\Gamma_{\xi,n})\),

\[b_i \leq g_t(e_{r(\sigma \circ \sigma_0(s))}) + \delta = a_{k_i} \theta_{w_{k_i}} + \delta.\]

From this and our choice of \(\delta\) we deduce that

\[C \cdot n^{1/\beta} \leq \sum_{i \in R} b_i \leq \delta n + \sum_{i \in R} a_{k_i} \theta_{w_{k_i}}.\]

Partition \(R\) into sets \(R_1, \ldots, R_l\), where \(R_j = \{i \in R : k_i = j\}\), so that

\[C \cdot n^{1/\beta} \leq \sum_{i \in R} a_{k_i} \theta_{w_{k_i}} = \sum_{j=1}^{l} a_j \theta_{w_j} |R_j|.|J|.\]

We claim that for each \(j\), \(|R_j| \leq C \cdot 2^{w_j}\). Indeed, suppose \(|R_j| > C \cdot 2^{w_j}\) for some \(j\). By Theorem 4(iv), if \(R_j = \{r_1, \ldots, r_m\}\), with \(r_1 < \ldots < r_m\), there exists extended pruning \((\sigma', \tau') : \Gamma_{\xi,m}.X \to \Gamma_{\xi,n}.X\) such that \(\sigma'(\Lambda_{\xi,m,i}) \subset \Lambda_{\xi,n,r_i}\). We now use Lemma 15 to deduce that \(SZ_{\xi}(K, \phi_{w_j}/2) > C \cdot 2^{w_j}\), which is a contradiction. Thus we deduce that \(|R_j| \leq C \cdot 2^{w_j}\) for each \(i\). This means that for each \(1 \leq j \leq l\),

\[\theta_{w_j} = (2^{-1/\alpha})^{w_j} = (2^{w_j})^{-1/\alpha} \leq C^{1/\alpha} |R_j|^{-1/\alpha} \leq C |R_j|^{-1/\alpha}.\]

Then

\[\sum_{j=1}^{l} a_j \theta_{w_j} |R_j| \leq C \sum_{j=1}^{l} a_j |R_j|^{-1/\alpha} \leq C \sum_{j=1}^{l} a_j |R_j|^{1/\beta} \leq C \left( \sum_{j=1}^{l} |a_j|^{\alpha} \right)^{1/\alpha} \left( \sum_{j=1}^{l} |R_j| \right)^{1/\beta} \]

\[\leq C |R|^{1/\beta} \leq C n^{1/\beta}.\]

Thus we reach a contradiction.

We now return to the proof of the sublemma. First fix \(w \in \mathbb{N}\) such that \(\theta^w < \delta\). For each \(t \in \text{MAX}(\Gamma_{\xi,n})\), write \(f_{\tau(t)} = \sum_{i=1}^{k_i} a_i f_i,t\), for some disjointly supported \(f_i,t \in \cup_{\sigma \in \mathbb{N}} N_{\theta, \phi, \Gamma_{\xi,n}}\) and \(a_i,t \geq 0\) such that \(\sum_{i=1}^{k_i} a_{i,t}^{\alpha} \leq 1\). Let \(S_t = \{i \leq k_t : \|a_i,t f_i,t\|_\infty \geq \delta\}\). Note that since \(\sum_{i=1}^{k_t} a_{i,t}^{\alpha} \leq 1\), \(|S_t| \leq [1/\delta^{\alpha}] =: k_0\). As in the previous lemma, we write \(\sum_{i \in S_t} a_{i,t} f_i,t = \sum_{i=1}^{k_t} a_{i,t}' f_i,t\), for some \(l_t \leq k_0\). Considering the function from \(\text{MAX}(\Gamma_{\xi,n}.X)\) given by \(t \mapsto l_t \in \{1, \ldots, k_0\}\), we use Theorem 4(iii) to obtain \(l \in \mathbb{N}\) and a level preserving extended pruning \((\sigma', \tau') : \Gamma_{\xi,n}.X \to \Gamma_{\xi,n}.X\) such that for all \(t \in \text{MAX}(\Gamma_{\xi,n}.X)\), \(l_{\tau(t)} = l\). Note that since \(\|a_{i,t}' f_i,\tau(t)'\|_\infty \geq \delta\) for every \(1 \leq i \leq l\) and \(t \in \text{MAX}(\Gamma_{\xi,n})\), if \(f_{i,t}' \in N_{\theta', \phi', \Gamma_{\xi,n}}\).
This implies that for any $j \leq w$. Let $w_i,\tau'(t)$ be the value $j \in \{1, \ldots, w\}$ such that $f_i,\tau'(t) \in N_{\theta_i,\phi_i,\Gamma_{\xi,n}}$. By considering the map from $\text{MAX}(\Gamma_{\xi,n},X)$ into $B_{L_p} \times \{1, \ldots, w\}$ given by
\[
t \mapsto ((a_i,\tau(t))_{i=1}^l, (w_i,\tau'(t))_{i=1}^l),
\]
we use Theorem 4(iii) again to find another level preserving extended pruning $(\sigma'',\tau'') : \Gamma_{\xi,n},X \to \Gamma_{\xi,n},X, (a_i)_{i=1}^l \in B_{L_p}$ and $(w_i)_{i=1}^l \subset \{1, \ldots, w\}$ such that for all $t \in \text{MAX}(\Gamma_{\xi,n}),$
\[
\|((a_i,\tau\tau(t))_{i=1}^l) - (a_i)_{i=1}^l\|_{\ell^\alpha} < \delta\]
and for all $1 \leq i \leq l$, $f_i,\tau',\tau''(t) \in N_{\theta_i,\phi_i,\Gamma_{\xi,n}}$. Note that for all $t \in \text{MAX}(\Gamma_{\xi,n},X),$
\[
\|f_i,\tau',\tau''(t) - \sum_{i=1}^l a_i f_i,\tau',\tau''(t)\|_\infty < \delta.
\]
This implies that for any $i \in R$ and any $\Lambda_{\xi,n,i} \ni s \leq t$, since
\[
\delta \leq b_i \leq f_i,\tau',\tau''(e_r(\phi_0,\sigma_0,\sigma''(s))),
\]
r$(\sigma \circ \sigma' \circ \sigma''(s)) \in \bigcup_{i=1}^l \supp(f_i,\tau',\tau''(t))$. Thus we may let $\kappa(s,t)$ be the unique $j \in \{1, \ldots, l\}$ such that $r(\sigma \circ \sigma' \circ \sigma''(s)) \in \supp(f_j,\tau',\tau''(t))$ if $s \in \bigcup_{i \in R} \Lambda_{\xi,n,i}$, and $\kappa(s,t) = 0$ otherwise. Applying Theorem 4(ii), we deduce the existence of $(k_i)_{i \in R} \subset \{1, \ldots, l\}$ and a level preserving extended pruning $(\sigma'',\tau'') : \Gamma_{\xi,n},X \to \Gamma_{\xi,n},X$ such that setting $\sigma_0 = \sigma' \circ \sigma'' \circ \sigma'', \tau_0 = \tau' \circ \tau'' \circ \tau'''$, $h_{i,t} = f_i,\tau_0(t)$, and $g_t = \sum_{i=1}^l a_i h_{i,t}$ finishes the proof.

4. Proof of the Main Results

Proof of Theorem 7. Let $\phi = 1/3$ and $\theta = 2/3$, so that $\frac{1}{\theta_p} = 3$. Fix $1 < p < \infty$. Fix any $1 < \alpha, \beta < \infty$ such that $\beta < p$ and $1/\alpha + 1/\beta = 1$. Let $C'' = C''(\beta, p)$ be the constant from Theorem 11. Fix $\varepsilon > 0$. By Lemma 16, Player I has a winning strategy in the game with target set
\[
\left\{ t \in \text{MAX}(\Gamma_{\xi},X,C) : \| \sum_{s \leq t} \mathbb{P}_\xi(s)e_s\|_{\theta,\phi,\alpha,\Gamma_{\xi}} > \varepsilon \right\}.
\]
By Corollary 14, for any $B$-tree $T$ with $o(T) = \omega^{1+\xi}$ and any normally weakly null collection $(f_i)_{i \in T,L_p(x)} \subset B_{L_p(x)}$,\[
\inf \left\{ \| \varrho(f)\|_{L_p} : t \in T,L_p(X), f \in \text{co}(f_s : \emptyset < s \leq t) \right\} \leq 3C''\varepsilon.
\]
We deduce $S\omega(K_p) \leq \omega^{1+\xi}$ by Theorem 5(i).

It is clear that $S\omega(K) \leq S\omega(K_p)$ for any $1 < p < \infty$. If $K$ is convex, then either $S\omega(K) = \infty$, in which case $S\omega(K_p) = \infty = \omega\infty = S\omega(K)$, or there exists an ordinal $\xi$ such that $S\omega(K) = \omega^\xi$ [2 Proposition 4.2]. We deduce that $S\omega(K_p) \leq \omega^{1+\xi} = \omega S\omega(K)$ by the previous paragraph. In the case that $\xi \geq \omega, 1 + \xi = \xi$.

\[\square\]
Proof of Theorem 3: If $p_ξ(K) = ∞$, there is nothing to show, so assume $p_ξ(K) < ∞$. Fix $1 < p, q < ∞$ with $1/p + 1/q = 1$. Fix $1 < α, β, γ < ∞$ such that $\max\{p_ξ(K), q\} < γ < α$ and $1/α + 1/β = 1$. Let $C' = C'(β, p)$ be the constant from Theorem 11. Let $φ = 2^{−1/γ}$ and note that $\sup_{i ∈ N} ε^{γ} S_{ξ}(K, φ^i/2)/2^i < ∞$. By Lemma 17, with $θ = 2^{−1/α}$, there exists a constant $C_1$ such that for every $n ∈ N$, Player I has a winning strategy in the game with target set $\left\{t ∈ MAX(Γ_ξ,n,X.C) : \left\|\sum_{s ≤ t} P_ξ,n(s)e_s\right\|_θ,φ,α,Γ_ξ,n > C_1/n^{1/β}\right\}$.

By Corollary 14, for every $n ∈ N$, every $B$-tree $T$ with $o(T) = ω^{1+ξ}n$, and every normally weakly null $(f_t)_{t ∈ T.L_p(X)} ⊂ B_{L_p(X)}$, $\inf\left\{\left\|g(f)\right\|_{L_p} : t ∈ T.L_p(X), f ∈ co(∅ ⪯ s ≤ t)\right\} \leq \frac{C_1C'}{n(θ − φ)}n^{1/β} = \frac{C_1C'}{n^{1/α}θ − φ}$.

By Theorem 5(ii), $p_{1+ξ}(K_p) ≤ α$. Since $α > \max\{p_ξ(K), q\}$ was arbitrary, we deduce that $p_{1+ξ}(K_p) ≤ \max\{p_ξ(K), q\}$.

□

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