One parameter family of an integrable $sp\ell(2|1)$ vertex model: Algebraic Bethe ansatz and ground state structure

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Abstract

We formulate in terms of the quantum inverse scattering method the exact solution of a $sp\ell(2|1)$ invariant vertex model recently introduced in the literature. The corresponding transfer matrix is diagonalized by using the algebraic (nested) Bethe ansatz approach. The ground state structure is investigated and we argue that a Pokrovsky-Talapov transition is favored for certain value of the 4-dimensional $sp\ell(2|1)$ parameter.

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1 Introduction

Recently a novel solution of the Yang-Baxter equation invariant under the $spl(2|1)$ symmetry has been obtained in the literature by Bracken et al and Maassaran [1, 2, 3]. Its peculiar feature is the presence of an additional non-additive parameter whose origin goes back to the continuous 4-dimensional irreducible representation of the algebra $spl(2|1)$ [4]. The physical meaning of this solution is that the associated quantum one-dimensional Hamiltonian [3] can be interpreted as an exactly solved model of strongly correlated electrons possessing extra fine-tuned hopping terms and electron pair interactions besides the typical correlations appearing in the Hubbard model. Most of the Bethe ansatz results concerning this model were obtained either by using the coordinate Bethe ansatz method directly on the fermionic Hamiltonian [6, 7] or by taking advantage of the fusion construction of the 4-dimensional $spl(2|1)$ $R$-matrix. [2].

In this last case, we note that the eigenvalues of the corresponding transfer matrix and the Bethe ansatz equations have appeared only as reasonable conjectures. We also remark that the authors [9] managed to present the phenomenological analytical Bethe ansatz approach for a special point of the continuous parameter in which the $R$-matrix can be seen as a braid-monoid invariant. However, for this interesting system, a more unified approach such as the quantum inverse scattering framework [10, 11, 12] has not yet succeeded to be formulated. We remark that the quantum inverse scattering approach makes it possible to present the Bethe ansatz results in an elegant way: the Bethe ansatz equations and the eigenvalues of the transfer matrix appear as a consequence of systematic algebraic manipulations of the creation and annihilation operators acting on a certain pseudovacuum. The main purpose of this paper is to fill this gap by presenting a detailed formulation of the quantum inverse scattering method for such new $R$-matrix invariant by the superalgebra $spl(2|1)$. For sake of simplicity, we are going to describe our formulation for the rational limit of the $spl(2|1)$ $R$-matrix. Some numerical and analytical results concerning the ground state structure of this model are also presented.

This paper is organized as follow. The next section is concerned with the presentation of

1We remark that the $R$-matrix in the context of the fundamental representation of $OSP(2|2)$ algebra has been previously discussed in ref. [5]
the Boltzmann weights, the associated Hamiltonian and the basic properties of the $spl(2|1)$ rational $R$-matrix. In section 3 we formulate the quantum inverse scattering method and in section 4 we obtain several important commutation relations. In section 5 we elaborate on the construction of the eigenstates and it is shown that the eigenvalues depend on an extra auxiliary problem of diagonalization. The corresponding nested Bethe ansatz equations and the eigenvalues are then determined in section 6. In section 7 we discuss the ground state structure of this model and we present some evidence that a Pokrovsky-Talapov transition may occur for a special value of the non-additive parameter of the $spl(2|1)$ $R$-matrix. Section 8 is reserved for our conclusions and remarks concerning the application of our formulation to other integrable systems. In appendices $A$, $B$ and $C$ we have collected some useful relations concerning the $spl(2|1)$ algebra, the two-particle state, and the three-particle state, respectively.

## 2 The $spl(2|1)$ vertex model and its properties

We start this section by describing the Boltzmann weights of the $spl(2|1)$ $R$-matrix recently proposed in refs. [1, 2, 3]. We shall consider the rational limit of the $spl(2|1)$ $R$-matrix, avoiding certain extra mathematical manipulations typical of trigonometric weights. We note, however, that the vertex structure is basically the same for both rational and trigonometrical cases and therefore many of the ideas described in this paper are quite general. Here we will adopt the notations of the appendix of ref. [2]. The rational $spl(2|1)$ $R$-matrix can be written in terms of certain combination of projectors $P_1(b)$ and $P_3(b)$ [2] of the $spl(2|1)$ algebra as follows:

$$R(\lambda, b) = I - \frac{4\lambda}{1 - 2b + 2\lambda} P_1(b) - \frac{4\lambda}{1 + 2b + 2\lambda} P_3(b)$$

where $\lambda$ is the spectral parameter, $b$ characterizes the continuous 4-dimensional representation of the $spl(2|1)$ algebra [3] and $I$ is the identity operator. For completeness we present the explicit

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2 We remark that at points $b = \pm 1/2$ the corresponding one-dimensional Hamiltonian is singular. Here we exclude of our analysis such atypical representation.
matrix expression of the projectors $P_1(b)\ e\ P_3(b)$ in Appendix A. Using these formulas one can see that the rational $spl(2|1)$ $R$-matrix consists of 36 non-null Boltzmann weights. We have schematized them in figure 1. We choose to represent each bond of the lattice with the variables $f_i$ and $b_i\ (i = 1, 2)$ in order to represent the fermionic $(f_1, f_2)$ and the bosonic $(b_1, b_2)$ degrees of freedom used by Maassarani [4]. More precisely, the $spl(2|1)$ $R$-matrix can be considered as a matrix acting on the tensor product of the two 4-dimensional auxiliary space $C^4 \times C^4$ and can be arranged as a $16 \times 16$ matrix which in the $fbbf$ grading possesses the following form

$$R(\lambda, b) = \begin{pmatrix}
  l(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & m(\lambda) & 0 & 0 & f(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & m(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & n(\lambda) & 0 & 0 & -\sigma(\lambda) & 0 & 0 & \sigma(\lambda) & 0 & 0 & p(\lambda) & 0 & 0 & 0 \\
  0 & f(\lambda) & 0 & 0 & m(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

In the expression (2) the functional dependence of the Boltzmann weights is given by

$$l(\lambda) = \frac{1 - 2b - 2\lambda}{1 - 2b + 2\lambda}, \quad r(\lambda) = \frac{1 + 2b - 2\lambda}{1 + 2b + 2\lambda}, \quad n(\lambda) = \frac{(1 - 2b)(1 + 2b)}{(1 - 2b + 2\lambda)(1 + 2b + 2\lambda)}$$

$$m(\lambda) = \frac{1 - 2b}{1 - 2b + 2\lambda}, \quad q(\lambda) = \frac{1 + 2b}{1 + 2b + 2\lambda}, \quad p(\lambda) = \frac{-4\lambda(\lambda + 1)}{(1 - 2b + 2\lambda)(1 + 2b + 2\lambda)}$$

$$f(\lambda) = \frac{2\lambda}{1 - 2b + 2\lambda}, \quad g(\lambda) = \frac{2\lambda}{1 + 2b + 2\lambda}, \quad t(\lambda) = \frac{4\lambda^2}{(1 + 2b + 2\lambda)(1 - 2b + 2\lambda)}$$

3
\[
\sigma(\lambda) = \frac{2\lambda[(1-2b)(1+2b)]^{1/2}}{(1+2b+2\lambda)(1-2b+2\lambda)}, \quad s(\lambda) = \frac{1-4b^2+4\lambda}{(1+2b+2\lambda)(1-2b+2\lambda)}
\] (3)

A general feature of the \( R \)-matrix (2) is that many of their Boltzmann weights are not invariant under the charge \( 1 \leftrightarrow 2 \) symmetry for same specie of bosonic and fermionic index, namely \( f_1 \leftrightarrow f_2 \) and \( b_1 \leftrightarrow b_2 \). For instance, from figure 1 and expressions (3), these are the cases of the following pairs of weights \( \{l(\lambda), r(\lambda)\}, \{m(\lambda), q(\lambda)\} \) and \( \{f(\lambda), g(\lambda)\} \). In such case, a standard crossing symmetry cannot be implemented, since it is not possible to find common crossing factor for the weights \( l(\lambda) \) and \( r(\lambda) \), for example. In fact, under the \( 1 \leftrightarrow 2 \) symmetry the parameter \( b \) is reflected to \(-b\). Hence, the remaining invariance is the reflection \( b \rightarrow -b \), and the physical properties derived either of \( R(\lambda, b) \) or of \( R(\lambda, -b) \) should in fact be the same.

We remark, however, that the point \( b = 0 \) is clearly an exception of the discussion we have made above. For this special point, the \( 1 \leftrightarrow 2 \) symmetry is present, usual crossing relations are then possible to be established and consequently a interpretation of \( R(\lambda, b = 0) \) in terms of a factorizable \( S \)-matrix turns out to be possible \[9\].

Further properties of the \( R \)-matrix (2) can be seen by using certain relations satisfied by the projectors \( P_1(b) \) and \( P_3(b) \). We have collected such important relations in Appendix A. For example, one can show that the quasi-classical \( r \)-matrix originated from the \( R \)-matrix (1) encodes the \( spl(2|1) \) symmetry in a standard way, namely in terms of the \( spl(2|1) \) Casimir operator. In fact, making the redefinition \( \lambda \rightarrow \frac{\lambda}{\eta} \) and expanding around \( \eta = 0 \) we find

\[
R(\lambda, b, \eta) \sim P^g \left[ 1 + \frac{\eta(C(b) - I)}{2\lambda} \right]
\] (4)

where \( \eta \) is the classical parameter, \( C(b) \) is the Casimir operator of \( spl(2|1) \) and \( P^g \) is the graded permutation operator. Details of this calculation can be found in Appendix A. Another interesting feature of such \( R \)-matrix is its braid-monoid representation at the special point \( b = 0 \). In particular, a Temperley-Lieb operator \( E \) (\textit{monoid}) can be defined in terms of the projectors \( P_1(b) \) and \( P_3(b) \) as follows

\[
E = \lim_{b \to 0} 4b[P_1(b) - P_3(b)]
\] (5)

which together with the permutation operator \( P^g \) satisfy the braid-monoid relation \[9\]. In this
sense, it seems very interesting to find the underlying algebraic structure for the points \( b \neq 0 \). After Baxterization such general structure should produce the \( spl(2\,|\,1) \) \( R \)-matrix (2). A precise answer to this question has eluded us so far.

## 3 The quantum inverse scattering approach

The main purpose of this section is to begin the formulation of the eigenvalue problem of the corresponding transfer matrix \( T(\lambda) \) of the \( spl(2\,|\,1) \) vertex system on a square lattice of size \( L \times L \). The diagonalization problem

\[
T(\lambda) |\Phi\rangle = \Lambda(\lambda) |\Phi\rangle \tag{6}
\]

can be solved by using an algebraic construction \([10, 11, 12]\) based on the Yang-Baxter algebra of monodromy matrices \( \mathcal{T}(\lambda) \)

\[
R(\lambda - \mu) \mathcal{T}(\lambda) \otimes \mathcal{T}(\mu) = \mathcal{T}(\mu) \otimes \mathcal{T}(\lambda) R(\lambda - \mu) \tag{7}
\]

where the matrix \( \mathcal{T}(\lambda) \) acts on the tensor product of an auxiliary space and a quantum space \( C^4 \otimes C^{4L} \) and is given in terms of the product of vertex operators \( L(\lambda) \) by

\[
\mathcal{T}(\lambda) = L_{oL}(\lambda) L_{oL-1}(\lambda) \ldots L_{o1}(\lambda) \tag{8}
\]

where the index ‘\( o \)’ stands for the \( 4 \times 4 \) auxiliary space, and as usual the transfer matrix \( T(\lambda) \) is obtained as a trace of the monodromy matrix \( \mathcal{T}(\lambda) \) over such auxiliary space. The elements of the vertex operator \( L_{ab}^{cd}(\lambda) \) are related to those of the \( spl(2\,|\,1) \) \( R \)-matrix (2) by a permutation on the \( C^4 \times C^4 \) tensor space

\[
L_{ab}^{cd}(\lambda) = R_{ba}^{ed}(\lambda) \tag{9}
\]

The corresponding quantum one-dimensional Hamiltonian can be obtained as the logarithmic derivative of the transfer matrix \( T(\lambda) \) at the regular point \( \lambda = 0 \). After some algebraic manipulation (see Appendix A), the associated spin chain can be only written in terms of the Casimir operator of the \( spl(2\,|\,1) \) algebra by the following expression

\[
H = -\frac{2}{(2b + 1)^2(1 - 2b)^2} \sum_{i=1}^{L} \left[ 2(1 + 2b^2)I - (1 + 4b^2)C_{i,i+1}(b) - C_{i,i+1}^2(b) \right] \tag{10}
\]
where we assumed standard periodic boundary conditions. As a consequence, we can see that the reflection symmetry \( b \rightarrow -b \) is explicitly exhibited by expression (10) since there exists an isomorphism between \( C_{i,i+1}(b) \) and \( C_{i,i+1}(-b) \) (see end of appendix A).

Before going on, we remark that the \( R \)-matrix (2) is a null-parity (Grassmann) braid operator, and after some sign definitions\(^3\) produces a vertex operator which solves the graded Yang-Baxter equation [13]. In this case one has to use the supersymmetric formalism developed in refs. [8] by basically changing standard properties such as trace and tensor product by their analogs on the graded spaces. However, a graded formulation does not simplify the original problem of diagonalization of the transfer matrix. On the contrary, in terms of practical calculations with the corresponding Hamiltonian, one has to be very careful to keep track of the fermions signs appearing on the tensor product of the Hilbert space. Here we would like to stress that the diagonalization of the corresponding Hamiltonian for small lattice sizes is extremely important as a guideline, giving us a correct insight of basic properties such as the ground state structure. For that reason we stick with the standard formalism, in which such job can be performed in a more direct and safe way. Anyhow, the basic difference between the standard and graded formulation will be the presence of extra phase factors on the Bethe ansatz equations. However, such phase-factors can be accomplished as general boundary conditions similarly as has been done before for the graded \( OSP(1|2) \) model [14].

After presenting the basic definitions and discussions concerning the diagonalization problem (6), we are going to turn our attention to the construction of the eigenstates \( |\Phi\rangle \) and eigenvalues \( \Lambda(\lambda) \), respectively. We start our discussion by solving the commutation relations which follows as a consequence of the Yang-Baxter algebra.

\(^3\)The graded \( \hat{R} \)-matrix is related to the standard one (2) by \( \hat{R}_{ba}^{cd}(\lambda) = (-1)^{p(a)p(b)} R_{ab}^{cd}(\lambda) \), where \( p(i) \) denotes the Grassmann parity of \( i = 1, \cdots, 4 \).
4 The fundamental commutation relations

The proper way to work out the intertwining relations (7) depends, to some extent, on the properties of the vertex operator $L(\lambda)$ when it acts on a given reference state. Let us consider as the reference state the usual ferromagnetic pseudovacuum given by

$$|0\rangle_i = \prod_{i=1}^{L} \otimes |0\rangle_i, \quad |0\rangle_i = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(11)

By using equations (2) and (9) we find that the vertex operator $L(\lambda)$ acting on the state $|0\rangle_i$ has the important triangular form

$$L(\lambda) |0\rangle_i = \begin{pmatrix} l(\lambda) & * & * & * \\ 0 & f(\lambda) & 0 & * \\ 0 & 0 & f(\lambda) & * \\ 0 & 0 & 0 & p(\lambda) \end{pmatrix} |0\rangle_i$$

(12)

Now, if we write the monodromy matrix $T(\lambda)$ as a $4 \times 4$ matrix having the following particular form

$$T(\lambda) = \begin{pmatrix} B(\lambda) & B_1(\lambda) & B_2(\lambda) & F(\lambda) \\ C_1(\lambda) & A_{11}(\lambda) & A_{12}(\lambda) & E_1(\lambda) \\ C_2(\lambda) & A_{21}(\lambda) & A_{22}(\lambda) & E_2(\lambda) \\ C_3(\lambda) & C_4(\lambda) & C_5(\lambda) & D(\lambda) \end{pmatrix}$$

(13)

the problem of diagonalization of the transfer matrix becomes

$$[B(\lambda) + \sum_{a=1}^2 A_{aa}(\lambda) + D(\lambda)] |\Phi\rangle = A(\lambda) |\Phi\rangle$$

(14)

Moreover, as a consequence of definition (8), we find that the following diagonal relations are also satisfied

$$B(\lambda) |0\rangle = [l(\lambda)]^L |0\rangle, \quad D(\lambda) |0\rangle = p(\lambda)^L |0\rangle, \quad A_{aa}(\lambda) |0\rangle = f(\lambda)^L |0\rangle, \quad a = 1, 2$$

(15)

as well as the annihilation properties

$$C_i(\lambda) |0\rangle = 0 \, (i = 1, \cdots, 5), \quad A_{ab}(\lambda) |0\rangle = 0 \, (a \neq b = 1, 2)$$

(16)
In particular for the eigenstate $|0\rangle$ the eigenvalue $\Lambda(\lambda)$ is determined to be

$$\Lambda(\lambda) = [l(\lambda)]^L + \sum_{a=1}^{2}[f(\lambda)]^L + [p(\lambda)]^L$$  \hspace{1cm} (17)

In order to construct other eigenvalues one has to find the commutation rules between the operators appearing in definition (13). Unlike to what happens to the 6-vertex model [10, 11, 12] and its multi-state generalizations [13, 16], some of the ‘nice’ commutation relations of the $spl(2|1)$ vertex model are more complicated and require additional work. In many cases one needs to combine in a special way certain relations in order to get the appropriate commutation rule. Let us illustrate the main idea for the particular case of the commutation relations between the operators $A_{ab}(\lambda)$ and $B_{c}(\lambda)$. As usual, we substitute the form of the monodromy matrix (13) in the intertwining equation (7) and by using the Boltzmann weights of the $R$-matrix (2) we find the relation

$$A_{ab}(\lambda)B_{c}(\mu) = \frac{1}{f(\lambda - \mu)}\tilde{r}^{bc}_{ed}(\lambda - u)B_{c}(\mu)A_{ad}(\lambda) - \frac{m(\lambda - \mu)}{f(\lambda - \mu)}B_{b}(\lambda)A_{ac}(\mu)$$

$$-\xi_{bc}\frac{\sigma(\lambda - \mu)}{f(\lambda - \mu)}\{B(\mu)E_{a}(\lambda) + F(\mu)C_{a}(\lambda)\}, \ a, b, c = 1, 2$$  \hspace{1cm} (18)

where here and in the following repeated indices denote the sum operation. The elements of the vector $\vec{\xi}$ are

$$\vec{\xi} = (0 \ 1 \ -1 \ 0)$$  \hspace{1cm} (19)

and the matrix $\tilde{r}(\lambda)$ has the structure

$$\tilde{r}(\lambda) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & s(\lambda) & t(\lambda) & 0 \\
0 & t(\lambda) & s(\lambda) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  \hspace{1cm} (20)

Previous experience with the utilization of commutation relations for other systems [10, 11, 12, 15, 16] suggests us that the term $B(u)E_{a}(\lambda)$ has a wrong order in the commutation rule (18). This can be disentangled by commuting the operators $B(\mu)$ and $E_{a}(\lambda)$ with the help of
the following additional commutation relations

\[ B(\mu)E_a(\lambda) = \frac{f(\lambda - \mu)}{p(\lambda - \mu)}E_a(\lambda)B(\mu) + \frac{m(\lambda - \mu)}{p(\lambda - \mu)}F(\lambda)C_a(\mu) \]

\[ -\frac{\sigma(\lambda - \mu)}{p(\lambda - \mu)}\xi_{bc}B_b(\mu)A_{ac}(\lambda) - \frac{n(\lambda - \mu)}{p(\lambda - \mu)}F(\mu)C_a(\lambda), \ a = 1, 2 \]  

(21)

obtaining as a final result the expression

\[ A_{ab}(\lambda)B_c(\mu) = \frac{1}{f(\lambda - \mu)}r_{e_b}^{le}(\lambda - \mu)B_c(\mu)A_{ad}(\lambda) - \frac{m(\lambda - \mu)}{f(\lambda - \mu)}B_b(\lambda)A_{ac}(\mu) - \xi_{bc}\sigma(\lambda - \mu) \]

\[ \left\{ \frac{f(\lambda - \mu)}{p(\lambda - \mu)}E_a(\lambda)B(\mu) + \frac{m(\lambda - \mu)}{p(\lambda - \mu)}F(\lambda)C_a(\mu) - \frac{1}{p(\lambda - \mu)}F(\mu)C_a(\lambda) \right\}, \ a, b, c = 1, 2 \]  

(22)

where now the matrix \( r \) has the following form

\[ r(\lambda) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & b(\lambda) & a(\lambda) & 0 \\
0 & a(\lambda) & b(\lambda) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \]  

(23)

with

\[ a(\lambda) = \frac{\lambda}{\lambda + 1}, \ b(\lambda) = \frac{1}{\lambda + 1} \]  

(24)

It is remarkable that such \( r \)-matrix is precisely the one appearing on the isotropic 6-vertex (or the XXX Heisenberg) model. In some sense the Boltzmann weights of the \( spl(2|1) \) \( R \)-matrix (2) conspire together in the commutation rules in order to give us as a hidden symmetry the 6-vertex structure. We think that this is the algebraic reason why the bare scattering matrix of the equivalent fermionic problem (in the coordinate Bethe ansatz approach) \[3, 4\] was previously determined to have the 6-vertex form. The other commutation rules concerning the operators \( B_a(\lambda) \) which are important in the diagonalization problem (6) can be obtained by basically following the method described above. They are given by

\[ B(\lambda)B_a(\mu) = \frac{l(\mu - \lambda)}{f(\mu - \lambda)}B_a(\mu)B(\lambda) - \frac{m(\mu - \lambda)}{f(\mu - \lambda)}B_a(\lambda)B(\mu), \ a = 1, 2 \]  

(25)

\(^4\)Here we have used the identity \( n(x) - p(x) = 1 \).
The commutation rules (22,25-27) form the basis of the algebraic Bethe ansatz for the creation operator \( B_a(\lambda) \). It turns out, however, that \( F(\lambda) \) also works as a creation operator, and, as we shall see in next section, it plays an important role in the eigenstate construction. Therefore, the commutation relations with the operator \( F(\lambda) \) are also necessary and they are given by the expressions

\[
A_{aa}(\lambda) F(\mu) = \frac{g(\lambda - \mu)}{f(\lambda - \mu)} \left[ 1 - \frac{q^2(\lambda - \mu)}{g^2(\lambda - \mu)} \right] F(\mu) A_{aa}(\lambda) + \frac{q(\lambda - \mu)}{g(\lambda - \mu)} E_a(\lambda) B_a(\mu) \\
- \frac{m(\lambda - \mu)}{f(\lambda - \mu)} B_a(\lambda) E_a(\mu) + \frac{m(\lambda - \mu) q(\lambda - \mu)}{g(\lambda - \mu) f(\lambda - \mu)} F(\lambda) A_{aa}(\mu), \ a = 1, 2
\] (28)

\[
B(\lambda) F(\mu) = \frac{l(\mu - \lambda)}{p(\mu - \lambda)} F(\mu) B(\lambda) - \frac{n(\mu - \lambda)}{p(\mu - \lambda)} F(\lambda) B(\mu) - \frac{\sigma(\mu - \lambda)}{p(\mu - \lambda)} \xi_{ab} B_a(\lambda) B_b(\mu)
\] (29)

\[
D(\lambda) F(\mu) = \frac{r(\lambda - \mu)}{p(\lambda - \mu)} F(\mu) D(\lambda) - \frac{n(\lambda - \mu)}{p(\lambda - \mu)} F(\lambda) D(\mu) + \frac{\sigma(\lambda - \mu)}{p(\lambda - \mu)} \xi_{ab} E_a(\lambda) E_b(\mu)
\] (30)

Other necessary commutation relations are mentioned in the appendices \( B \) and \( C \). In the next section we shall use all of them in the construction of the eigenstates \( |\Phi\rangle \).

5 The construction of the eigenstates and the eigenvalues

We now have almost the complete machinery to start the construction of the eigenstates of the \( \mathfrak{sp}(2|1) \) transfer matrix. The eigenstates can be obtained by acting the creation operators \( B_a(\lambda) \) and \( F(\lambda) \) over the ferromagnetic pseudovacuum \(|0\rangle\). Instead of presenting the general
solution, we find more illuminating first to discuss our method of construction for the lowest
eigenstates. The first excitation over the pseudovacuum $|0\rangle$, i.e. the one-particle state, is given
in terms of a linear combination between the operators $B_a(\lambda)$ by

$$|\Phi_1(\lambda_1)\rangle = B_a(\lambda_1)F^a|0\rangle$$  \(31\)

The diagonalization problem (6) for the one-particle eigenstate (31) is solved by making
use of the commutation rules (22,25-27) and of properties (15,16), and we find the following
important relations

$$B(\lambda)|\Phi_1(\lambda_1)\rangle = \frac{l(\lambda_1 - \lambda)}{f(\lambda_1 - \lambda)}[l(\lambda)]^L|\Phi_1(\lambda_1)\rangle - \frac{m(\lambda_1 - \lambda)}{f(\lambda_1 - \lambda)}[l(\lambda_1)]^LB_a(\lambda)F^a|0\rangle$$  \(32\)

$$D(\lambda)|\Phi_1(\lambda_1)\rangle = \frac{g(\lambda_1 - \lambda_1)}{p(\lambda_1 - \lambda_1)}[p(\lambda)]^L|\Phi_1(\lambda_1)\rangle + \frac{\sigma(\lambda_1 - \lambda_1)}{p(\lambda_1 - \lambda_1)}[f(\lambda_1)]^L\xi_{ab}E_a(\lambda)F^b|0\rangle$$  \(33\)

$$\sum_{a=1}^{2} A_{aa}(\lambda)|\Phi_1(\lambda_1)\rangle = \frac{1}{f(\lambda - \lambda_1)}[1 + a(\lambda - \lambda_1)][f(\lambda)]^L|\Phi_1(\lambda_1)\rangle$$

$$- \frac{m(\lambda_1 - \lambda_1)}{f(\lambda - \lambda_1)}[f(\lambda_1)]^LB_a(\lambda)F^a|0\rangle - \frac{\sigma(\lambda_1 - \lambda_1)}{p(\lambda_1 - \lambda_1)}[l(\lambda_1)]^L\xi_{ab}E_a(\lambda)F^b|0\rangle$$  \(34\)

The terms proportional to the eigenstate $|\Phi_1(\lambda_1)\rangle$ are denominated ‘wanted terms’ and
contribute for the eigenvalue $\Lambda(\lambda, \lambda_1)$. The remaining ones are the so called ‘unwanted terms’
and they must be canceled out. From expressions (32-34) we can see that this is the case,
provided that

$$\left[\frac{l(\lambda_1)}{f(\lambda_1)}\right]^L = 1$$  \(35\)

where we have used the reflection property

$$\frac{m(\lambda)}{f(\lambda)} = - \frac{m(-\lambda)}{f(-\lambda)}$$  \(36\)

The two-particle state $|\Phi_2(\lambda_1, \lambda_2)\rangle$ depends both of the operators $B_a(\lambda)$ and $F(\lambda)$. This
becomes clear if we consider the commutation rule (27), suggesting the following ansatz

$$|\Phi_2(\lambda_1, \lambda_2)\rangle = B_a(\lambda_1)B_b(\lambda_2)F^{ba}|0\rangle + h(\lambda_1, \lambda_2)[l(\lambda_2)]^LF(\lambda_1)\xi_{ba}F^{ba}|0\rangle$$  \(37\)

\(^5\) Here it is interesting to point out that condition (35) does not impose any further restriction on the
constants $F^a$ appearing in the linear combination (31). As a consequence, the corresponding eigenvalue is
double degenerated (see also section (7)).
The state $|\Phi_2(\lambda_1, \lambda_2)\rangle$ degenerates several unwanted terms if one tries to solve the corresponding eigenvalue with the help of the commutation rules (22,25-27) and (28-30) for the operators $B_a(\lambda)$ and $F(\lambda)$, respectively. There exists some unwanted terms which can be automatically canceled out by an appropriate choice of the function $h(\lambda_1, \lambda_2)$. For instance, the unwanted terms

$$[l(\lambda_2)]^L \xi_{ab} E_a(\lambda) E_b(\lambda_1), \quad [l(\lambda_2)]^L B_a(\lambda) E_a(\lambda_1)$$

(38)

are all of them proportional to $(F_{12}^1 - F_{21}^2)$ and can be excluded by fixing the following form for the function $h(\lambda_1, \lambda_2)$

$$h(\lambda_1, \lambda_2) = h(\lambda_1 - \lambda_2) = -\frac{\sigma(\lambda_1 - \lambda_2)}{p(\lambda_1 - \lambda_2)}$$

(39)

The other remaining unwanted terms need a further restriction in order to be canceled out. For example, by collecting the contributions concerning the term of kind $B_a(\lambda) B_b(\lambda_2)$ we find that

$$-\frac{m(\lambda - \lambda_1)}{f(\lambda - \lambda_1)} \left\{ \frac{1}{f(\lambda_1 - \lambda_2)} [f(\lambda_1)]^L r_{ba}^{lm} (\lambda_1 - \lambda_2) F^{ml} - \frac{1}{f(\lambda_2 - \lambda_1)} [l(\lambda_1)]^L l(\lambda_2 - \lambda_1) F^{ba} \right\} B_a(\lambda) B_b(\lambda_2) |0\rangle$$

(40)

Considering the identity

$$\frac{l(\lambda)}{f(\lambda)} = -\frac{1}{f(-\lambda)}$$

(41)

the term given in equation (40) is canceled by imposing the following restriction

$$\left[ \frac{l(\lambda_i)}{f(\lambda_i)} \right]^L F^{ba}_{i} = -r_{ba}^{lm} (\lambda_i - \lambda_j) F^{ml}, \quad i \neq j$$

(42)

In fact, in Appendix $B$ we show that all unwanted terms of many different kinds can be canceled out by using such restrictions for both $\lambda_1$ and $\lambda_2$. We remark that conditions (42) are particular cases of the general Bethe ansatz equations which are going to be discussed in the next section. Before going on, it is important to notice that a certain recurrence relation can be established between the one and the two-particle states. In order to see this, it is convenient to write our results in a more compact way. Let us define the $n$-particle state as

$$|\Phi_n(\lambda_1, \cdots, \lambda_n)\rangle = \Phi_n(\lambda_1, \cdots, \lambda_n).\vec{F} |0\rangle$$

(43)
where $\vec{\Phi}_n(\lambda_1, \ldots, \lambda_n)$ and $\vec{F}$ are vectors with $2^n$ components. Here we shall denote the components of vector $\vec{F}$ by $F_{2^n \cdots \cdot a_1}$. Considering the bilinear vector $\vec{B}(\lambda)$ as

$$
\vec{B}(\lambda) = (B_1(\lambda) \quad B_2(\lambda)) \tag{44}
$$

and taking into account our previous results (31, 37) and (39), the corresponding two-particle vector $\vec{\Phi}_2(\lambda_1, \lambda_2)$ is then written as

$$
\vec{\Phi}_2(\lambda_1, \lambda_2) = \vec{B}(\lambda) \otimes \vec{\Phi}_1(\lambda_2) + \left[l(\lambda_2)\right]^L \frac{\sigma(\lambda_1 - \lambda_2)}{p(\lambda_1 - \lambda_2)} F(\lambda_1) \vec{\xi} \otimes \vec{\Phi}_0 \tag{45}
$$

where $\vec{\Phi}_0$ is the unitary constant. A remarkable property present in equation (45) is the symmetry under the exchange $\lambda_1 \leftrightarrow \lambda_2$. More precisely one can show that

$$
\vec{\Phi}_2(\lambda_1, \lambda_2) = \vec{\Phi}_2(\lambda_2, \lambda_1) \frac{r_{12}(\lambda_1 - \lambda_2)}{l(\lambda_1 - \lambda_2)} \tag{46}
$$

where we have used the commutation rules (27) and the identity

$$
\frac{h(\lambda)}{h(-\lambda)} = r_{12}^{21}(\lambda) - r_{21}^{12}(\lambda) \tag{47}
$$

The exchange symmetry between the variables $\lambda_i$ is always a welcome feature in the algebraic Bethe ansatz analysis (see e.g. refs. [12, 17]). Such property can be used to cancel several kinds of unwanted terms differing under the permutation on variables $\{\lambda_i\}$. By using this property we can avoid extra cumbersome mathematical analysis, like that appearing on the direct proof we gave in Appendix B for the variable $\lambda_2$ entering in the two-particle state. This last discussion and the recurrence relation (45) serve as a motivation for us go on and to search for the three-particle eigenstate. As before, we start with an ansatz which is able to collect together the ‘easy’ unwanted terms. We find that the three-particle state has the following structure

$$
\vec{\Phi}(\lambda_1, \lambda_2, \lambda_3) = \vec{B}(\lambda_1) \otimes \vec{\Phi}_2(\lambda_2, \lambda_3) + \left[l(\lambda_2)\right]^L F(\lambda_1) \frac{\sigma(\lambda_1 - \lambda_2)}{p(\lambda_1 - \lambda_2)} \vec{\xi} \otimes \vec{\Phi}_1(\lambda_3) \vec{F}_2(\lambda_2, \lambda_3) + \left[l(\lambda_3)\right]^L F(\lambda_1) \frac{\sigma(\lambda_1 - \lambda_3)}{p(\lambda_1 - \lambda_3)} \vec{\xi} \otimes \vec{\Phi}_1(\lambda_2) \vec{F}_3(\lambda_2, \lambda_3) \tag{48}
$$
A simple way of determining the unknown function $\hat{F}_j(\lambda_2, \lambda_3)$ is by combining the exchange symmetry property with the direct cancellation of the simplest unwanted terms. From the permutation symmetry we find the relations

$$\hat{F}_2(\lambda_2, \lambda_3) = \frac{l(\lambda_3 - \lambda_2)}{f(\lambda_3 - \lambda_2)} I$$

(49)

$$\hat{F}_3(\lambda_2, \lambda_3) = \frac{r_{23}(\lambda_2 - \lambda_3)}{f(\lambda_2 - \lambda_3)}$$

(50)

For completeness, in Appendix C we have presented some details of this analysis as well as how certain unwanted terms are easily canceled out. Now, after this detailed exposition of our method, the general $n$-particle eigenstate can be, then, obtained as an induction of the expressions (45) and (48). As a final result, we obtain that the $n$-particle state satisfies the following recurrence relation

$$\vec{\Phi}_n(\lambda_1, \ldots, \lambda_n) = B(\lambda_1) \otimes \vec{\Phi}_{n-1}(\lambda_2, \ldots, \lambda_n) + F(\lambda_1)\vec{\xi} \otimes \sum_{j=2}^{n} \frac{l(\lambda_j)}{l(\lambda_1 - \lambda_j)} \sum_{j=2}^{n} \frac{r_{j,j+1}(\lambda_k - \lambda_j)}{l(\lambda_k - \lambda_j)}$$

(51)

where the index under the $r$-matrix denotes its action position on the tensor product $\otimes \cdots \otimes \otimes \cdots \otimes$. Under a consecutive permutation $\lambda_j \leftrightarrow \lambda_{j+1}$, the $n$-particle eigenstate satisfies the symmetry relation

$$\vec{\Phi}_n(\lambda_1, \ldots, \lambda_j, \lambda_{j+1}, \ldots, \lambda_n) = \vec{\Phi}_n(\lambda_1, \ldots, \lambda_{j+1}, \lambda_j, \ldots, \lambda_n)$$

(52)

Here we would like to make some remarks. It is interesting to notice that the recurrence relation (51) resembles in some way the one found by Tarasov [17] in the algebraic solution of the Izergin-Korepin model ($A_2^2$ system [18]). Of course, our case is much more involved due to the necessity of vectorial notation and the presence of a hidden symmetry enhanced by the vector $\vec{\xi}$ and the 6-vertex $r$-matrix. Disregarding the obvious subtleties of our construction, one is tempted to interpret (51) as the simplest vectorial generalization of the one found by Tarasov for the $A_2^2$ model. In fact, the 6-vertex model is one of the simplest system possessing
a non-diagonal and factorizable \(r\)-matrix. In this sense, we believe that our recurrence relation (51) is an important guideline for further generalizations concerning the presence of other non-diagonal \(r\)-matrix.

In order to close this section, we turn to the restriction condition on the variables \(\{\lambda_i\}\). An important unwanted term comes from the structure \(B_{a_1}(\lambda_1)B_{a_2}(\lambda_2)\cdots B_{a_n}(\lambda_n)\) when one of the \(\lambda_i\) is exchanged with \(\lambda\), producing for example the \(B_{a_1}(\lambda)B_{a_2}(\lambda_2)\cdots B_{a_n}(\lambda_n)\) unwanted term. Such term is generated by the action of the operator \(\sum_a A_{aa}(\lambda)\) and \(B(\lambda)\). By using the commutation rules (25) the term coming from \(B(\lambda)\) is

\[
\frac{m(\lambda - \lambda_1)}{f(\lambda - \lambda_1)} [l(\lambda_1)]^L \prod_{j=2}^{n} \frac{l(\lambda_j - \lambda_1)}{f(\lambda_j - \lambda_1)} F^{a_{n-a_1}} B_{a_1}(\lambda) B_{a_2}(\lambda_2) \cdots B_{a_n}(\lambda_n) |0\rangle \tag{53}
\]

and under action of \(\sum_a A_{aa}(\lambda)\) we get

\[
- \frac{m(\lambda - \lambda_1)}{f(\lambda - \lambda_1)} [f(\lambda_1)]^L \prod_{j=2}^{n} \frac{1}{f(\lambda_j - \lambda)} r^{a_{1\lambda_1}}_{c_1d_1}(\lambda - \lambda_2) r^{d_1a_2}_{c_2d_2}(\lambda - \lambda_3) \cdots r^{d_{n-2a_n}}_{c_{n-1}d_{n-1}}(\lambda - \lambda_n) \nonumber \\
F^{a_{n-a_1}} B_{d_{n-1}}(\lambda) B_{c_1}(\lambda_2) \cdots B_{c_{n-1}}(\lambda_n) |0\rangle \tag{54}
\]

One can write this last term in a more compact form by defining an auxiliary transfer matrix associated to the problem of an inhomogeneous 6-vertex system as

\[
T^{(1)}(\lambda, \{\lambda_i\})^{a_{1\cdots a_n}}_{b_1\cdots b_n} = r^{c_1a_1}_{b_1d_1}(\lambda - \lambda_1) r^{d_1a_2}_{b_2d_2}(\lambda - \lambda_2) \cdots r^{d_{n-1a_n}}_{b_{n-1}d_{n-1}}(\lambda - \lambda_n) \tag{55}
\]

and equation (54) becomes

\[
- \frac{m(\lambda - \lambda_1)}{f(\lambda - \lambda_1)} [f(\lambda_1)]^L \prod_{j=2}^{n} \frac{1}{f(\lambda_j - \lambda)} T^{(1)}(\lambda = \lambda_i, \{\lambda_j\})^{a_{1\cdots a_n}}_{b_1\cdots b_n} F^{a_{n-a_1}} B_{b_1}(\lambda) B_{b_2}(\lambda_2) \cdots B_{b_n}(\lambda_n) |0\rangle \tag{56}
\]

Here we remark that the same kind of reasoning can be done for any \(\lambda_i\), namely if \(B_{a_i}(\lambda)\) replaces \(B_{a_i}(\lambda_i)\). Now one has to use the property (52) and to perform cyclic permutations until one gets the variable \(\lambda_i\) on the first place. The basic difference is that now we get a string of ordered \(r\)-matrices multiplying the components \(F^{a_{n-a_1}}\) of vector \(\vec{F}\). This argument is commonly used in many other algebraic constructions \[12, 17\] and it stresses the importance of the exchange symmetry (52). Taking into account such discussion and collecting together equations (53) and (56), we find that the unwanted term \(B_{a_1}(\lambda_1) \cdots B_{a_n}(\lambda) \cdots B_{a_n}(\lambda_n)\) is canceled.
provided that
\[
\frac{[l(\lambda_i)]^L}{f(\lambda_i)} \prod_{j \neq i}^{n} \frac{f(\lambda_i - \lambda_j)l(\lambda_j - \lambda_i)}{f(\lambda_j - \lambda_i)} F^{a_1 \cdots a_n} = T^{(1)}(\lambda = \lambda_i, \{\lambda_j\})_{a_1 \cdots a_n}^{b_1 \cdots b_n} F^{b_1 \cdots b_n} \tag{57}
\]

which for \( n = 1, 2 \) reproduce our early conditions (35) and (42), respectively.

Other crucial unwanted term is that made by replacing one \( B_a(\lambda) \) by \( E_a(\lambda) \), such as \( E_{a_1}(\lambda)B_{a_2}(\lambda_2) \cdots B_{a_n}(\lambda_n) \). These terms come out the contribution of \( \sum_a A_{aa}(\lambda) \) and \( D(\lambda) \). The piece which comes from \( \sum_a A_{aa}(\lambda) \) has the form
\[
\frac{\sigma(\lambda - \lambda_1)}{p(\lambda - \lambda_1)} [l(\lambda_1)]^L \prod_{j = 2}^{n} \frac{l(\lambda_j - \lambda_1)}{f(\lambda_j - \lambda_1)} (-1)^{a_1} F^{a_1 \cdots a_n} E_{a_1}(\lambda)B_{a_2}(\lambda) \cdots B_{a_n}(\lambda_n) |0\rangle \tag{58}
\]

and the contribution of \( D(\lambda) \) is
\[
- \frac{\sigma(\lambda - \lambda_1)}{p(\lambda - \lambda_1)} [f(\lambda_1)]^L \prod_{j = 2}^{n} \frac{1}{f(\lambda_1 - \lambda_j)} r_{c_1d_1}(\lambda_1 - \lambda_2) r_{c_2d_2}(\lambda_1 - \lambda_3) \cdots r_{c_{n-1}d_{n-1}}(\lambda_1 - \lambda_n) F^{a_1 \cdots a_n} \xi k E_k(\lambda)B_{c_1}(\lambda_2) \cdots B_{c_{n-1}}(\lambda_n) |0\rangle \tag{59}
\]

By combining these two terms together and by using definition (55) one concludes that such unwanted terms are canceled out by the same restriction (57), as it should be. Of course we have several others non-trivial unwanted terms. Although we do not have a systematic proof that such terms vanish, the checks we performed so far have been rather exhaustive (see e.g. appendices B and C), and seem to us to establish beyond reasonable doubt that the restriction (57) is the unique condition to be imposed on the \( n \)-particle eigenstate in order to cancel the unwanted terms.

Finally, the eigenvalue \( \Lambda(\lambda, \{\lambda_i\}) \) of the \( n \)-particle eigenstate can be calculated by keeping only the terms proportional to the eigenstate \( |\Phi_n(\lambda_1, \cdots, \lambda_n)\rangle \). For instance, that proportional to the vector \( B_{a_1}(\lambda_1)B_{a_2}(\lambda_2) \cdots B_{a_n}(\lambda_n) \) is determined by the extensive use of the first terms of the commutation rules (22,25-27), and we find that
\[
\Lambda(\lambda, \{\lambda_i\}) = [f(\lambda)]^L \prod_{i = 1}^{n} \frac{1}{f(\lambda - \lambda_i)} \Lambda^{(1)}(\lambda, \{\lambda_i\}) + [l(\lambda)]^L \prod_{i = 1}^{n} \frac{l(\lambda_i - \lambda)}{f(\lambda_i - \lambda)} + [p(\lambda)]^L \prod_{i = 1}^{n} \frac{g(\lambda - \lambda_i)}{f(\lambda - \lambda_i)} \tag{60}
\]

\(^\text{6}\)Some trivial unwanted terms come in pairs and are automatically eliminated such as we have exemplified for the cases of the two and three-particle states.
where $\Lambda^{(1)}(\lambda, \{\lambda_i\})$ is the eigenvalues of the auxiliary problem related to the transfer matrix of the inhomogeneous 6-vertex model, i.e.

$$T^{(1)}(\lambda, \{\lambda_i\})^{b_1 \cdots b_n}_{a_1 \cdots a_n} F^{b_n \cdots b_1} = \Lambda^{(1)}(\lambda, \{\lambda_i\}) F^{a_n \cdots a_1}$$  \hspace{1cm} (61)

In conclusion, the results of this section show us that the computation of the eigenstates and the eigenvalues of the $spl(2|1)$ model is still dependent of an additional eigenvalue problem for the inhomogeneous transfer matrix $T^{(1)}(\lambda, \{\lambda_i\})$. We shall discuss this matter in the next section.

6 The nested Bethe ansatz equations

The auxiliary problem (61) can still be solved by an algebraic approach since the Yang-Baxter algebra (7) is also valid for inhomogeneous transfer matrices. The corresponding monodromy matrix has the form

$$T^{(1)}(\lambda, \{\lambda_i\}) = L^{(1)}(\lambda - \lambda_n) L^{(1)}(\lambda - \lambda_{n-1}) \cdots L^{(1)}(\lambda - \lambda_1)$$ \hspace{1cm} (62)

where $L^{(1)}(\lambda)$ is the vertex operator of the isotropic 6-vertex model and its matrix elements on the space $C^2 \times C^2$ are given by

$$L^{(1)}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(\lambda) & b(\lambda) & 0 \\ 0 & b(\lambda) & a(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$  \hspace{1cm} (63)

In order to go on, we basically have to adapt the well known algebraic results for the homogeneous 6-vertex model of Faddeev et al \cite{10, 11, 12} in the case of an irregular lattice. We recall that this problem has appeared in many different contexts in the literature \cite{15, 16}, but for sake of completeness we present the main results. Taking the monodromy matrix as

$$T^{(1)}(\lambda, \{\lambda_i\}) = \begin{pmatrix} A^{(1)}(\lambda, \{\lambda_i\}) & B^{(1)}(\lambda, \{\lambda_i\}) \\ C^{(1)}(\lambda, \{\lambda_i\}) & D^{(1)}(\lambda, \{\lambda_i\}) \end{pmatrix}$$  \hspace{1cm} (64)
and defining the reference state as

\[ |0^{(1)}\rangle = \prod_{i=1}^{n} |0^{(1)}\rangle_i, \quad |0^{(1)}\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]  

we find the following properties

\[ A^{(1)}(\lambda, \{\lambda_i\}) |0^{(1)}\rangle = |0^{(1)}\rangle, \quad D^{(1)}(\lambda, \{\lambda_i\}) |0^{(1)}\rangle = \prod_{i=1}^{n} a(\lambda - \lambda_i) |0^{(1)}\rangle, \quad C^{(1)}(\lambda, \{\lambda_i\}) |0^{(1)}\rangle = 0 \]  

(66)

The algebraic Bethe ansatz developed by Faddeev et al \[ \text{[10, 11, 12]} \] says that all eigenvectors of the transfer matrix \( A^{(1)}(\lambda, \{\lambda_i\}) + D^{(1)}(\lambda, \{\lambda_i\}) \) can be written as

\[ |\Phi^{(1)}(\mu_1, \cdots, \mu_m)\rangle = \prod_{i=1}^{m} B^{(1)}(\mu_i, \{\lambda_j\}) |0^{(1)}\rangle \]  

(67)

for some additional restriction on the variables \( \{\mu_i\} \). Moreover, the Yang-Baxter algebra (7) for the monodromy matrix (62) and the 6-vertex \( r \)-matrix (23) yields the following commutation relations

\[ A^{(1)}(\lambda, \{\lambda_i\}) B^{(1)}(\mu, \{\lambda_i\}) = \frac{1}{a(\mu - \lambda)} B^{(1)}(\mu, \{\lambda_i\}) A^{(1)}(\lambda, \{\lambda_i\}) - \frac{b(\mu - \lambda)}{a(\mu - \lambda)} B^{(1)}(\lambda, \{\lambda_i\}) A^{(1)}(\mu, \{\lambda_i\}) \]  

(68)

\[ D^{(1)}(\lambda, \{\lambda_i\}) B^{(1)}(\mu, \{\lambda_i\}) = \frac{1}{a(\lambda - \mu)} B^{(1)}(\mu, \{\lambda_i\}) D^{(1)}(\lambda, \{\lambda_i\}) - \frac{b(\lambda - \mu)}{a(\lambda - \mu)} B^{(1)}(\lambda, \{\lambda_i\}) D^{(1)}(\mu, \{\lambda_i\}) \]  

(69)

\[ \left[ B^{(1)}(\mu, \{\lambda_i\}), B^{(1)}(\lambda, \{\lambda_i\}) \right] = 0 \]  

(70)

By using the commutation relations (68-70) we can carry \( A^{(1)}(\lambda, \{\lambda_i\}) + D^{(1)}(\lambda, \{\lambda_i\}) \) through all \( B^{(1)}(\mu_i, \{\lambda_j\}) \), and we find that the eigenvalue is

\[ \Lambda^{(1)}(\lambda, \{\lambda_i\}, \{\mu_j\}) = \prod_{j=1}^{m} \frac{1}{a(\mu_j - \lambda)} + \prod_{i=1}^{n} a(\lambda - \lambda_i) \prod_{j=1}^{m} \frac{1}{a(\lambda - \mu_j)} \]  

(71)

if the numbers \( \{\mu_j\} \) satisfy the following system of equations

\[ \prod_{i=1}^{n} a(\mu_j - \lambda_i) = - \prod_{k=1}^{m} a(\mu_j - \mu_k), \quad j = 1, \cdots, m \]  

(72)
Hence, by using the expression for $\Lambda^{(1)}(\lambda, \{\lambda_i\}, \{\mu_j\})$ in equation (60), we finally find that the eigenvalues of the spl$(2|1)$ model has the following general form

$$\Lambda(\lambda, \{\lambda_i\}, \{\mu_j\}) = [f(\lambda)]^L \prod_{i=1}^{n} \frac{1}{f(\lambda - \lambda_i)} \left\{ \prod_{j=1}^{m} \frac{1}{a(\mu_j - \lambda)} + \prod_{i=1}^{n} a(\lambda - \lambda_i) \right\} \prod_{j=1}^{m} \frac{1}{a(\lambda - \mu_j)} + [l(\lambda)]^L \prod_{i=1}^{n} \frac{l(\lambda_i - \lambda)}{f(\lambda_i - \lambda)} + [p(\lambda)]^L \prod_{i=1}^{n} \frac{g(\lambda - \lambda_i)}{f(\lambda - \lambda_i)}$$

(73)

and the restriction condition (57) for the eigenstates becomes

$$\left[ \frac{l(\lambda_i)}{f(\lambda_i)} \right]^L = -(-1)^n \prod_{i=1}^{n} \frac{1}{a(\mu_i - \lambda_i)}$$

(74)

where we have used the identity given in equation (41).

The set of coupled equations (72) and (74) are usually denominated nested Bethe ansatz equations. They can be written in a more symmetric way if we perform the transformation

$$\lambda_i \rightarrow \frac{\lambda_i}{2} - \frac{b - 1/2}{2}, \quad \mu_j \rightarrow \frac{\mu_j}{2} - \frac{b + 1/2}{2}$$

(75)

and afterwards by performing the rescaling

$$\lambda_i \rightarrow \frac{\lambda_i}{i}, \quad \mu_j \rightarrow \frac{\mu_j}{i}$$

(76)

we get

$$\left[ \frac{\lambda_i - (b - 1/2)i}{\lambda_i + (b - 1/2)i} \right]^L = (-1)^{L-n} \prod_{i=1}^{n} \frac{\lambda_i - \mu_i + i}{\lambda_i - \mu_i - i}, \quad i = 1, \ldots, n$$

$$\prod_{i=1}^{m} \frac{\mu_k - \mu_i - 2i}{\mu_k - \mu_i + 2i} = -\prod_{i=1}^{n} \frac{\mu_k - \lambda_i - i}{\mu_k - \lambda_i + i}, \quad k = 1, \ldots, m$$

(77)

The eigenvalues $\Lambda(\lambda, \{\lambda_i\}, \{\mu_i\})$, after performing the transformations (75) and (76), can be written as

$$\Lambda(\lambda, \{\lambda_i\}, \{\mu_i\}) = \left[ \frac{\lambda}{(1/2 - b)i + \lambda} \right]^L \left\{ \prod_{i=1}^{n} \frac{\lambda_i + (b - 1/2)i - 2\lambda}{\lambda_i - (b - 1/2)i - 2\lambda} \prod_{i=1}^{m} \frac{\mu_i + (3/2 - b)i - 2\lambda}{\mu_i - (b + 1/2)i - 2\lambda} + \prod_{i=1}^{n} \frac{\lambda_i + (b - 1/2)i - 2\lambda}{\lambda_i - (b + 3/2)i - 2\lambda} \prod_{i=1}^{m} \frac{\mu_i - (b + 5/2)i - 2\lambda}{\mu_i - (b + 1/2)i - 2\lambda} \right\} + \left\{ \frac{1/2 - b}{i + \lambda} \right\}^L \prod_{i=1}^{n} \frac{\lambda_i + (b - 1/2)i - 2\lambda}{\lambda_i - (b - 1/2)i - 2\lambda}$$

$$+ \left\{ \frac{1/2 - b}{i + \lambda} \right\}^L \prod_{i=1}^{n} \frac{\lambda_i + (b - 1/2)i - 2\lambda}{\lambda_i - (b + 3/2)i - 2\lambda}$$

(78)
In particular, the eigenenergies $E(L)$ of the corresponding Hamiltonian (10) can be calculated by taking the logarithmic derivative of $\Lambda(\lambda, \{\lambda_i\}, \{\mu_j\})$ at the regular point $\lambda = 0$. More precisely, by performing the operation $i\frac{d\ln \Lambda(\lambda, \{\lambda_i\}, \{\mu_j\})}{d\lambda}\bigg|_{\lambda=0}$ we find

$$E(L) = \sum_{i=1}^{n} \frac{4(b - 1/2)}{(\lambda_i^1)^2 + (b - 1/2)^2} + \frac{2L}{b - 1/2}$$

(79)

In order to conclude this section we would like to make the following remarks. First of all, our analytical result (78) can be confronted with the conjecture made by Maassarani [2] concerning the structure of the eigenvalues of the $spl(2|1)$ model. Here, one first has to disregard the presence of certain phase factors, which is the basic difference between the standard and the supersymmetric formulation. We conclude that our analytical result for the eigenvalue $\Lambda(\lambda, \{\lambda_i\}, \{\mu_j\})$ is in agreement with the conjecture made by Maassarani [2], if one takes the rational limit of equation (73) of ref. [4]. Our second remark is concerned with the extension of the Bethe ansatz results obtained in this section for a more general classes of twisted boundary conditions. These boundary conditions correspond to the introduction of a seam with different Boltzmann weights along the infinite direction on the cylinder. Such weights depend on two
angles $\theta_1$ and $\theta_2$ and we represent them by the operator $\hat{L}(\lambda)$ given by

$$
\hat{L}(\lambda) = \begin{pmatrix}
I(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \omega_1 f(\lambda) & 0 & 0 & m(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \omega_2 f(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & m(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega_1 \omega_2 p(\lambda) & 0 & 0 & -\omega_2 \sigma(\lambda) & 0 & 0 & \omega_1 \sigma(\lambda) & 0 & 0 & n(\lambda) & 0 & 0 & 0 \\
0 & m(\lambda) & 0 & 0 & f(\lambda) / \omega_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \omega_1 \omega_2 t(\lambda) & 0 & 0 & s(\lambda) & 0 & 0 & \omega_1 \sigma(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & m(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_2 g(\lambda) & 0 & 0 & 0 & q(\lambda) & 0 & 0 \\
0 & 0 & 0 & 0 & \omega_1 \sigma(\lambda) & 0 & 0 & s(\lambda) & 0 & 0 & t(\lambda) / \omega_1 \omega_2 & 0 & 0 & \omega_1 \sigma(\lambda) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

(80)

where $\omega_{1,2} = e^{-i\theta_{1,2}}$. The integrability is still preserved and the nested Bethe ansatz equations (77) are modified in the following way

$$
\left[ \frac{\lambda_i - (b - 1/2) i}{\lambda_i + (b - 1/2) i} \right]^L = (-1)^{L-n} e^{-i\theta_1} \prod_{i=1}^m \frac{\lambda_i - \mu_i + i}{\lambda_i - \mu_i - i}, \quad i = 1, \ldots, n
$$

$$
\prod_{i=1}^m \frac{\mu_k - \mu_i - 2i}{\mu_k - \mu_i + 2i} = -e^{-i(\theta_1 - \theta_2)} \prod_{j=1}^n \frac{\mu_k - \lambda_j + i}{\mu_k - \lambda_j - i}, \quad k = 1, \ldots, m
$$

(81)

and the eigenenergy equation (78) remains unchanged. One interesting case of twisted boundary conditions is when $\theta_1 = -\theta_2 = \vartheta/2$. In this case the $sl(2|1)$ algebra is preserved and is accomplished in terms of the $sl(2|1)$ generators as follows. The odd generators transform as

$$
V_\pm \rightarrow e^{\pm \vartheta/2} V_\pm, \quad \nabla_\pm \rightarrow e^{\pm \vartheta/2} \nabla_\pm
$$

(82)

and the even generators behave as

$$
S_\pm \rightarrow e^{\pm \vartheta} S_\pm, \quad S_3 \rightarrow S_3, \quad B \rightarrow B
$$

(83)

\[ ^7 \text{In the fermionic version of the corresponding Hamiltonian (10) this means that the fermions with spins up } c_\uparrow(i) \text{ and down } c_\downarrow(i) \text{ satisfy the boundary conditions } c_{\uparrow,\downarrow}(1) = e^{i \theta_{1,2}} c_{\uparrow,\downarrow}(L + 1), \quad c_{\uparrow,\downarrow}^\dagger(1) = e^{-i \theta_{1,2}} c_{\uparrow,\downarrow}^\dagger(L + 1) \]
This special boundary condition is quite useful, because by varying the angle $\vartheta$ until $\vartheta = 2\pi/8$ (in the sector where $(L - n)$ is even) we can study the spectrum of the system in the supersymmetric formulation. In this sense, both the standard and the supersymmetric formulation may be related by performing in determined sectors of theory a twisted boundary condition. This idea has been used with success to study the spectrum of the $OSP(1|2)$ spin chain [14].

The final remark consists of a discussion concerning the reflection symmetry $b \rightarrow -b$ in terms of our algebraic formulation. As has already been noticed by Maassarani [2], the $R$-matrix structure (2) allowed an additional reference state such as

$$|\tilde{0}\rangle = \prod_{i=1}^{L} \otimes |\tilde{0}\rangle_i, \quad |\tilde{0}\rangle_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$  \hspace{1cm} (84)

The triangular form of the vertex operator $\mathcal{L}(\lambda)$ is now given by

$$\mathcal{L}(\lambda) |\tilde{0}\rangle_i = \begin{pmatrix} p(\lambda) & 0 & 0 & 0 \\ * & g(\lambda) & 0 & 0 \\ * & 0 & g(\lambda) & 0 \\ * & * & * & r(\lambda) \end{pmatrix} |\tilde{0}\rangle_i$$ \hspace{1cm} (85)

which can be related to the earlier triangular property (12) by $b \rightarrow -b$ and $f_1 \leftrightarrow f_2$. Such relation becomes even more rigorous if one looks for the commutation rules coming from the Yang-Baxter algebra (7). In this case, we find that it is more convenient to start with the following matrix form of the monodromy $\tilde{T}(\lambda)$

$$\tilde{T}(\lambda) = \begin{pmatrix} \tilde{D}(\lambda) & \tilde{C}_5(\lambda) & \tilde{C}_4(\lambda) & \tilde{C}_3(\lambda) \\ \tilde{E}_2(\lambda) & \tilde{A}_{22}(\lambda) & \tilde{A}_{21}(\lambda) & \tilde{C}_2(\lambda) \\ \tilde{A}_1(\lambda) & \tilde{A}_{12}(\lambda) & \tilde{A}_{11}(\lambda) & \tilde{C}_1(\lambda) \\ \tilde{F}(\lambda) & \tilde{B}_2(\lambda) & \tilde{B}_1(\lambda) & \tilde{B}(\lambda) \end{pmatrix}$$ \hspace{1cm} (86)

*Notice that this corresponds to a periodic (anti-periodic) boundary condition for the bosonic (fermionic) degrees of freedom.
By using the Yang-Baxter algebra with this structure, the commutation rules of the diagonal
terms with operator \( B_a(\lambda) \), for example, are given by

\[
\tilde{A}_{ab}(\lambda) \tilde{B}_c(\mu) = \frac{1}{g(\lambda - \mu)} \rho_{ed}^{bc}(\lambda - \mu) \tilde{B}_e(\mu) \tilde{A}_{ad}(\lambda) - \frac{q(\lambda - \mu)}{g(\lambda - \mu)} \tilde{B}_b(\lambda) \tilde{A}_{ac}(\mu) + \frac{\sigma(\lambda - \mu)}{g(\lambda - \mu)} \xi_{bc} \left\{ \frac{g(\lambda - \mu)}{p(\lambda - \mu)} E_a(\lambda) B(\mu) + \frac{q(\lambda - \mu)}{p(\lambda - \mu)} \tilde{F}(\lambda) \tilde{C}_a(\mu) - \frac{1}{p(\lambda - \mu)} \tilde{F}(\mu) \tilde{C}_a(\lambda) \right\}, \quad a, b, c = 1, 2
\]  

\[ (87) \]

\[
\tilde{B}(\lambda) \tilde{B}_a(\mu) = \frac{r(\mu - \lambda)}{g(\mu - \lambda)} \tilde{B}_a(\mu) \tilde{B}(\lambda) - \frac{q(\mu - \lambda)}{g(\mu - \lambda)} \tilde{B}_a(\lambda) \tilde{B}(\mu)
\]  

\[ (88) \]

\[
\tilde{D}(\lambda) \tilde{B}_a(\mu) = \frac{f(\lambda - \mu)}{p(\lambda - \mu)} \tilde{B}_a(\mu) \tilde{D}(\lambda) + \frac{m(\lambda - \mu)}{p(\lambda - \mu)} \tilde{F}(\mu) \tilde{C}(\lambda)_{a+3} - \frac{n(\lambda - \mu)}{p(\lambda - \mu)} \tilde{F}(\lambda) \tilde{C}(\mu)_{a+3} + \frac{\sigma(\lambda - \mu)}{p(\lambda - \mu)} \xi_{cb} \tilde{E}_b(\lambda) \tilde{A}_{ca}(\mu)
\]

\[ (89) \]

Such commutation relations are the same of that given in equations (22,25,26) by perform-
ing the reflection symmetry \( b \rightarrow -b \) and \( \sigma \rightarrow -\sigma \). We have checked that similar situation
appears for the other commutations rules. Since \( \sigma \rightarrow -\sigma \) can be considered as a canonical
transformation, we conclude that the eigenvalues parametrized by equations (77) and (78) ei-
ther with \( +b \) or with \( -b \), should produce the same spectrum for the \( spl(2|1) \) model. From the
Bethe ansatz point of view, one would think that this sounds a bit strange, since the topology
of the nested Bethe ansatz equations are very different for \( \pm b \). The correct interpretation is as
follows. They in fact can produce the same eigenvalues, but of course with different structure
of zeros. Actually, we have checked this fact by solving numerically the nested Bethe ansatz
equation for some values of \( \pm b \) with a finite lattice size \( L \).

7 The ground state structure

The purpose of this section is to study both numerically and analytically the behaviour of the
ground state of the \( spl(2|1) \) Hamiltonian (10). Due to the reflection symmetry \( b \rightarrow -b \), it is
enough to investigate the regimes \( 0 \leq b < 1/2 \) and \( 1/2 < b < \infty \). We begin our analysis by
investigating the behaviour around the reference states with ferromagnetic feature. In order
to keep our previous notation (section 2) we define the four basic states acting on the $i$-th site of the one-dimensional lattice by

$$
\begin{align*}
  f_1(i) &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},
  b_1(i) &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},
  b_2(i) &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},
  f_2(i) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\end{align*}
$$

(90)

The low-lying excitations over the ferromagnetic state

$$
\begin{align*}
  |\phi_0^{f_1}\rangle &= |f_1(1) \cdots f_1(L)\rangle
\end{align*}
$$

(91)

can be constructed if one takes the linear combination of the states made by replacing one state $f_1(k)$ by $b_{1,2}(k)$, namely

$$
\begin{align*}
  |\phi_{1,2}^{f_1}\rangle &= \sum_{k=1}^L \alpha(k) |f_1(1) \cdots b_{1,2}(k) \cdots f_1(L)\rangle
\end{align*}
$$

(92)

While the eigenvalue of the ferromagnetic state is trivially determined to be

$$
E_0^{f_1}(b) = -\frac{2L}{1/2 - b}
$$

(93)

that of the low-lying excitation $|\phi_{1,2}^{f_1}\rangle$ is related to the solution of the equation

$$
E_{1,2}^{f_1} \alpha(k) = -\frac{2}{1/2 - b} (L - 1) \alpha(k) + \frac{1}{1/2 - b} [\alpha(k + 1) + \alpha(k - 1)]
$$

(94)

This relation can be solved by taking $\alpha(k) = Ae^{ikp}$ and the dispersion relation is

$$
E_{1,2}^{f_1}(p) = -\frac{2L}{1/2 - b} + \frac{4}{1/2 - b} \cos^2(p/2)
$$

(95)

where under periodic boundary condition the momenta $p$ assumes the values

$$
p = \frac{2\pi n}{L}; \quad n = 1, \cdots, L - 1
$$

(96)

Analogously, by repeating the same reasoning to the other ferromagnetic state

$$
\begin{align*}
  |\phi_0^{f_2}\rangle &= |f_2(1) \cdots f_2(L)\rangle
\end{align*}
$$

(97)
we find the following results
\[ E_{0}^{f_{2}}(b) = -\frac{2L}{1/2 + b} \]
\[ E_{1/2}^{f_{2}}(p) = -\frac{2L}{1/2 + b} + \frac{4}{1/2 + b}\cos^{2}(p/2) \]

(98)
(99)

In the regime \(1/2 < b < \infty\) we have \(E_{0}^{f_{2}}(b) < 0\) and \(E_{0}^{f_{1}}(b) > 0\). Moreover, the sign in the dispersion relation of \(E_{1/2}^{f_{2}}(p)(E_{1/2}^{f_{1}}(p))\) is positive (negative). This suggests that the ground state is given by \(E_{0}^{f_{2}}(b)\) and the excitations over it grows until reaching the upper-bound \(E_{0}^{f_{1}}(b)\).

Remarkably enough, the statement concerning the ground state can be put in a more rigorous ground. In fact, following Bader and Schilling \[20\] the ground state satisfies the relation
\[ E_{0} \geq LE_{0}^{2} \]

(100)

where \(E_{0}^{2}\) is the lowest eigenvalue of the two-body Hamiltonian \(H_{i,i+1}\). In the regime \(0 < b < 1/2\) we find that
\[ E_{0}^{2} = -\frac{2}{1/2 + b} \]

(101)

On the other hand, the expectation value of the \(spd(2|1)\) Hamiltonian in the ferromagnetic state \(|f_{2}(1) \cdots f_{2}(L)\rangle\) is \(-\frac{2L}{1/2+b}\), and by the variational principle we have
\[ E_{0} \leq -\frac{2L}{1/2 + b} \]

(102)

By comparing equations (100) and (102) we conclude that in the regime \(1/2 < b < \infty\) the ground state energy is \(E_{0}^{f_{2}}(b)\). Actually, by analysing the dispersions relation \(E_{1/2}^{f_{2}}(p)\) around \(p \sim \pi\) we find that it is three-fold degenerated. The rest of the statement, i.e. that the spectrum satisfies \(-\frac{2}{1/2+b} \leq E/L \leq -\frac{2}{1/2-b}\), has been confirmed by a numerical diagonalization of the Hamiltonian (10) up to \(L \leq 12\).

In the regime \(0 \leq b < 1/2\) rigorous results become more involved since the Hamiltonian (10) is not hermitian \[9\]. For instance, such ‘nice’ argument of Bader and Schilling \[20\] needs further elaboration. In this case, the ferromagnetic state \(|\phi_{0}^{f_{1}}\rangle\) has lower energy than the state

\[9\] Nevertheless, we have numerically checked up to \(L \leq 12\) that the ground state is real.
\[ |\phi_0^{f_2}\rangle \] and the sign on the dispersion relation of \( E_{1,2}^{f_1}(p) \) is now positive. This may suggest that there exist states with much lower energy. Indeed, for \( b = 0 \) we have found previously that the ground state has an antiferromagnetic structure and in the thermodynamic limit the energy per site \( e_\infty \) was determined to be \( e_\infty = -8\ln(2) \simeq -5.5452 < E_0^{f_1,f_2}(b = 0) = -4 \). Furthermore, the numerical diagonalization of Hamiltonian (10) for \( b \neq 0 \) \( (0 \leq b < 1/2) \) shows that the ground state jumps into the many possible \( U(1) \) sectors of the theory if one varies the lattice size \( L \). Let us illustrate this from the Bethe ansatz point of view. As an example, let us assume that for sufficient small \( b \) some states have the zeros structure similar to that previously found for the ground state at \( b = 0 \) \[9\], namely

\[
\lambda_j = \xi_j + i + O(e^{-aL}), \quad \mu_j = \xi
\]  

By substituting such structure of zeros in the nested Bethe ansatz equations, (see ref. \[9\]) we find that the density of variables \( \xi_j, \rho(\xi) \), in the thermodynamic limit, satisfies the following integral equation

\[
\psi'(\xi) + 2\pi \rho(\xi) = \int_{-\infty}^{+\infty} \varphi'(\xi - u)\rho(u)du
\]  

where the prime symbol stands for the derivative and the functions \( \psi(\xi) \) and \( \varphi(\xi) \) are given by

\[
\psi(\xi) = 2[\arctan(\frac{\xi}{3/2 - b}) - \arctan(\frac{\xi}{b + 1/2})], \quad \varphi(\xi) = 2\arctan(\xi/2)
\]

This integral equation is solved by Fourier techniques and we find that

\[
\rho(\xi) = \frac{\sin[\pi(1/2 - b)]}{\cosh(\pi \xi) + \cos[(1/2 - b)\pi]}
\]

Since the \( spl(2|1) \) Hamiltonian can be interpreted as spin-3/2 chain, the magnetization per site is then given by

\[
\frac{M_{ag}}{L} = 3 - \frac{n + m}{L}
\]

where the integers \( n \) and \( m \) (see Bethe ansatz equation) are the number of variables \( \lambda_j \) and \( \mu_l \), respectively. From equation (106) we find that \( \frac{n}{L} = 2(1/2 - b) \) and \( \frac{m}{L} = (1/2 - b) \), and as
consequence of equation (107) we have

\[
\frac{M_{ag}}{L} = 3b
\] (108)

It is also possible to verify that the energy per particle associated with the structure of zeros (103) is lower than \(E_{f1}^0(b)\). In summary, we believe that our results lead to the following picture. Strictly at \(b = 0\) the system has zero magnetization and presents an antiferromagnetic behaviour. As soon as we turn \(b > 0\), the ground state gets a finite magnetization and is partially ferromagnetic ordered. We believe that this picture remains in the whole regime \(0 < b < 1/2\). After the singular point \(b = 1/2(b > 1/2)\), the system is then fully ordered in the ferromagnetic state \(|\phi_{f2}^0\rangle\). In this sense the parameter \(b\) plays the role of the incommensurability such as a chemical potential or a magnetic field indicating that at \(b = 1/2\) the system presents a phase-transition of Pokrovsky-Talapov type \([21]\). In fact, the typical quadratic form of the dispersion relation appears in \(E_{f2}^{1,2}(b,p)\) for the low-lying excitation around \(p \sim \pi\). It seems interesting to understand this picture in terms of the particles and anti-particles of a factorizable \(S\)-matrix. We suspect that the point \(b = 1/2\) behaves as the threshold for the mass of the solitons and the anti-solitons present in the theory.

8 Conclusions

We have shown that the one parameter family of an integrable \(spl(2|1)\) vertex model is exactly solved by the algebraic Bethe ansatz. The eigenstates have been formulated in terms of the creation operators through the recurrence relation (51). The eigenvalues of the corresponding transfer matrix is computed by solving an auxiliary problem related to an inhomogeneous 6-vertex model. We have discussed how the reflection symmetry \(b \rightarrow -b\) can be encoded in terms of the commutation rules. The ground state picture has been discussed and we have presented arguments that the system has a commensurable/incommensurable phase transition of Pokrovski-Talapov type.

\[\text{It is remarkable that although we have considered } b << 1/2, \text{ the limit } b \rightarrow 1/2 \text{ in equation (108) reproduces the ferromagnetic structure found for } b > 1/2.\]
We believe that the formulation described in this paper is by no means only particular to the isotropic $spl(2|1)$ vertex model. As we have already commented, the whole construction can be generalized for the anisotropic model (trigonometric case) almost directly. The study of the phase-diagram of the anisotropic model and in particular the Pokrovski-Talapov transition, seems to us to be a very interesting problem. Besides such direct generalization, we have reasons to think that our formulation is the cornerstone to solve, by the algebraic Bethe ansatz approach, certain integrable models related with the symmetry $C_n$. This should be the case of the isotropic $Sp(2n)$ and $OSP(2|2n − 2)$ vertex models [22]. Other model that is still waiting for an algebraic solution is the Hubbard model. Since this system also possesses a hidden 6-vertex symmetry, we strongly believe that our formalism can be adapted in order to give the algebraic Bethe ansatz solution of the Hubbard model [23]. We hope to report on these problems in a future publications.

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Appendix A: The $spl(2|1)$ algebra and the R-matrix properties

The algebra $spl(2|1)$ [4] consists of four even generators $\{S_\pm, S_3, B\}$ and four odd generators $\{V_\pm, \overline{V}_\pm\}$. Following refs. [4, 2] they satisfy the commutation rules

$$[S_3, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 2S_3, \quad \{V_\pm, \overline{V}_\pm\} = \pm 1/2S_\pm$$

(A.1)

$$\{V_+, \overline{V}_-\} = -1/2P_+, \quad \{\overline{V}_+, V_-\} = -1/2P_-, \quad [P_\pm, \overline{V}_\pm] = \pm \overline{V}_\pm, \quad [P_\mp, V_\pm] = \pm V_\pm$$

(A.2)

$$[P_\pm, V_\pm] = [P_\mp, \overline{V}_\pm] = 0; \{V_i, V_j\} = \{\overline{V}_i, \overline{V}_j\} = 0, i, j = \pm$$

(A.3)

where the symbols $[,]$ and $\{,\}$ denote the commutator and the anti-commutator, respectively. We also have the identity $P_\pm = S_3 \pm B$. The 4-dimensional representation [4, 2] of these
generators in the \(bf fb\) grading possesses the following matrix representations

\[
V_+ = \begin{pmatrix} 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad V_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \end{pmatrix}, \quad P_+ = \begin{pmatrix} 1/2 - b & 0 & 0 & 0 \\ 0 & 1/2 - b & 0 & 0 \\ 0 & 0 & -1/2 - b & 0 \\ 0 & 0 & 0 & -1/2 - b \end{pmatrix},
\]

\[
\bar{V}_+ = \begin{pmatrix} 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{V}_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 1/2 + b & 0 & 0 & 0 \\ 0 & -1/2 + b & 0 & 0 \\ 0 & 0 & 1/2 + b & 0 \\ 0 & 0 & 0 & -1/2 + b \end{pmatrix},
\]

\(S_+ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 \end{pmatrix}\)

where \(4\alpha \gamma = 1 + 2b\) and \(4\beta \epsilon = 1 - 2b\). The Casimir operator is written in terms of these generators as

\[
C_{i,i+1}(b) = 2\{S_+ \otimes S_- + S_- \otimes S_+\} + 2\{P_+ \otimes P_- + P_- \otimes P_+\} + 4\{V_- \otimes V_+ + V_- \otimes V_+ - V_+ \otimes V_- - V_+ \otimes V_-\} + 4b^2 I
\]

In equation (A.7) the symbol \(\otimes\) stands for the supertensor product. More precisely the elements of \(A^\otimes B\) are

\[
(A^\otimes B)_{ij}^{ab} = (-1)^{p(i)p(j)+p(a)p(b)+p(i)p(B)}A_{ai}B_{bj}
\]

where \(p(f)\) is the Grassmann parity of the object \(f\).

As it has been observed by Maassarani [2] the projectors \(P_i, i = 1, 2, 3\) of the \(spl(2|1)\) algebra play a fundamental role in the construction of the \(R\)-matrix. In the isotropic limit, following ref. [2], we find that the \(16 \times 16\) matrices expressions for the projectors \(P_i\) in the
$$P_1(b) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{\alpha}(b) & 0 & 0 & \tilde{\gamma}(b) & 0 & 0 & -\tilde{\alpha}(b) & 0 & 0 & \tilde{\gamma}(b) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\tilde{\gamma}(b) & 0 & 0 & \tilde{\beta}(b) & 0 & 0 & -\tilde{\beta}(b) & 0 & 0 & -\tilde{\gamma}(b) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{\gamma}(b) & 0 & 0 & -\tilde{\beta}(b) & 0 & 0 & \tilde{\beta}(b) & 0 & 0 & \tilde{\gamma}(b) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{\alpha}(b) & 0 & 0 & \tilde{\gamma}(b) & 0 & 0 & -\tilde{\gamma}(b) & 0 & 0 & \tilde{\alpha}(b) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

(A.9)
where $P_2(b) = I - (P_1(b) + P_3(b))$ and \( \tilde{\alpha}(b) = (2b + 1)/8b \), \( \tilde{\beta}(b) = (2b - 1)/8b \) and \( \tilde{\gamma}(b) = \sqrt{1 - 4b^2}/8b \). Besides the usual projectors identities, \( P_1(b)P_3(b) = \delta_{i,j}P_3(b) \), the operators \( P_1(b) \) and \( P_3(b) \) satisfy the following useful relations

\[
P^g = I - 2[P_1(b) + P_3(b)], \quad P^g P_1(b) = -P_1(b), \quad P^g P_3(b) = -P_3(b)
\]  

(A.11)

where \( (P^g)_{ij}^{lk} = (-1)^{p(i)p(j)}\delta_{i,k}\delta_{j,l} \) defines the graded permutation operator. By using definitions (A.7,A.8) and expressions (A.9,A.10), the Casimir invariant can be connected to the projectors \( P_1(b) \) and \( P_3(b) \) as

\[
C_{i,i+1}(b) = I - 2[P_1(b) + P_3(b)] + 4b[P_1(b) - P_3(b)]
\]  

(A.12)

Such relations are important in the study of the classical limit of the \textit{spl}(2|1) \textit{R}-matrix. In fact, by introducing the quasi-classical parameter \( \eta \) in equation (1) as \( \lambda \to \frac{4}{\eta} \), we have

\[
R(\lambda, b, \eta) = I - \frac{4\lambda}{(1 - 2b)\eta + 2\lambda}P_1(b) - \frac{4\lambda}{(1 + 2b)\eta + 2\lambda}P_3(b)
\]  

(A.13)
By expanding this last expression around $\eta = 0$ we find

$$ R(\lambda, b, \eta = 0) = I - 2[P_1(b) + P_3(b)] = P^a \quad (A.14) $$

and

$$ \frac{\partial R(\lambda, b, \eta)}{\partial \eta} \big|_{\eta=0} = \frac{1}{\lambda} \{P_1(b) + P_3(b) - 2b[P_1(b) - P_3(b)]\} \quad (A.15) $$

As a consequence, we find that equation (4) of section 2 follows from the expressions (A.14,A.15) and the identity (A.12). Analogously, the corresponding Hamiltonian can be also written in terms of the Casimir operator. Considering that the two-body Hamiltonian $H_{i,i+1}$ is determined as the derivative of the $R$-matrix at the regular point $\lambda = 0$, we obtain

$$ H_{i,i+1} = -\frac{4}{1-2b}P_1(b) - \frac{4}{1+2b}P_3(b) \quad (A.16) $$

Now if we consider the identities (A.11,A.12) the square of the Casimir operator is

$$ C^2_{i,i+1}(b) = I + 8b[P_3(b) - P_1(b)] + 16b^2[P_3(b) + P_1(b)] \quad (A.17) $$

and now by solving equations (A.12) and (A.17) for the operators $P_1(b)$ and $P_3(b)$ we find

$$ P_1(b) \frac{1}{1 + 2b} = \frac{1}{16b(1 - 2b)(1 + 2b)}[(4b + 1)I - 4bC_{i,i+1}(b) - C^2_{i,i+1}(b)] \quad (A.18) $$

$$ P_3(b) \frac{1}{1 - 2b} = \frac{1}{16b(1 - 2b)(1 + 2b)}[(4b - 1)I - 4bC_{i,i+1}(b) + C^2_{i,i+1}(b)] \quad (A.19) $$

These last relations are then used in equation (A.16) in order to reproduce the Hamiltonian expression (10) presented in section 3.

Finally, we remark that the $spl(2|1)$ algebra is invariant under the following isomorphic transformation

$$ S_\pm \rightarrow S_\pm, \quad S_3 \rightarrow S_3, \quad P_\pm \rightarrow P_\mp, \quad V_\pm \rightarrow \overline{V}_\pm, \quad \overline{V}_\pm \rightarrow V_\pm \quad (A.20) $$

which in terms of the parameter $b$ means the reflection symmetry $b \rightarrow -b$. This can be easily seen from the matrix representations (A.4,A.5,A.6) of the $spl(2|1)$ generators, provided we also perform the canonical transformation $f_1 \leftrightarrow f_2$. 

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Appendix B : The two-particle state

The main purpose of this Appendix is to give extra details that the two-particle state (45) we have constructed is in fact an eigenstate under the Bethe ansatz restriction (42). In order to collect the unwanted terms we have to use the annihilation property (16) and the following additional commutation rules

\[
C_a(\lambda)B_b(\mu) = B_b(u)C_a(\lambda) - \frac{m(\lambda - \mu)}{p(\lambda - \mu)} [B(\lambda)A_{ab}(\mu) - B(\mu)A_{ab}(\lambda)], \quad a = 1, 2 \tag{B.1}
\]

\[
C_{a+3}(\lambda)B_b(\mu) = \frac{s(\lambda - \mu)}{p(\lambda - \mu)} B_a(\mu)C_{b+3}(\lambda) + \frac{t(\lambda - \mu)}{p(\lambda - \mu)} B_b(\mu)C_{a+3}(\lambda)
- \xi_{ab} \frac{\sigma(\lambda - \mu)}{p(\lambda - \mu)} \{F(\mu)C_3(\lambda) + B(\mu)D(\lambda)\} - \frac{n(\lambda - \mu)}{p(\lambda - \mu)} B_a(\lambda)C_{b+3}(\mu)
+ \frac{\sigma(\lambda - \mu)}{p(\lambda - \mu)} \xi_{im} A_{ta}(\lambda) A_{mb}(\mu), \quad a = b = 1, 2 \tag{B.2}
\]

\[
E_a(\lambda)B_b(\mu) = \frac{q(\lambda - \mu)}{f(\lambda - \mu)} B_b(\mu)E_a(\lambda) - \frac{m(\lambda - \mu)}{f(\lambda - \mu)} F(\lambda)A_{ab}(\mu)
+ \frac{q(\lambda - \mu)}{f(\lambda - \mu)} F(\mu)A_{ab}(\lambda), \quad a = b = 1, 2 \tag{B.3}
\]

By combining such commutation rules together with those mentioned in section 4 and property (15) we find that the non-trivial unwanted terms have the following structures

\[
B_a(\lambda)B_b(\lambda_i); \quad E_a(\lambda)B_b(\lambda_i); \quad F(\lambda)(F^{12} - F^{21}) \tag{B.4}
\]

For the first two cases in equation (B.4) it is enough to fix \(\lambda_i = \lambda_1\), since the other possibility \((\lambda_i = \lambda_2)\) has already been discussed in the main text (see section 5). We now summarize the functional forms which multiply the terms (B.4), and discuss how they are canceled out.

1. The \(B_a(\lambda)B_a(\lambda_1)\) term, for a given index \(a\), appears as

\[
\left\{ \frac{m(\lambda - \lambda_1) m(\lambda_1 - \lambda_2)}{f(\lambda_1 - \lambda) f(\lambda_2 - \lambda_1)} - \frac{m(\lambda - \lambda_2) r^{aa}_{a}(\lambda_1 - \lambda)}{f(\lambda_2 - \lambda)} \right\} F^{aa}[f(\lambda_2)]^L +
\left\{ \frac{m(\lambda - \lambda_1) m(\lambda_1 - \lambda_2)}{f(\lambda_1 - \lambda) f(\lambda_1 - \lambda_2)} - \frac{m(\lambda - \lambda_2) r^{aa}_{a}(\lambda - \lambda_1) r^{aa}_{a}(\lambda_1 - \lambda_2)}{f(\lambda_2 - \lambda)} \right\} F^{aa}[f(\lambda_2)]^L \tag{B.5}
\]
and by using the identities (36) and (41), we find
\[
\begin{align*}
\left\{ -\frac{m(\lambda_2 - \lambda)}{f(\lambda_2 - \lambda) f(\lambda_1 - \lambda)} + \frac{m(\lambda_1 - \lambda) m(\lambda_2 - \lambda_1)}{f(\lambda_1 - \lambda) f(\lambda_2 - \lambda_1)} \right\} [l(\lambda_2)]^L + [f(\lambda_2)]^L \right\} F^{aa} (B.6)
\end{align*}
\]
which is null by taking \( a = b \) in the Bethe ansatz equation (42).

2. The \( B_a(\lambda) B_b(\lambda_1) \) term for \( a \neq b \) is more involved. Here the cases \( B_1(\lambda) B_2(\lambda_2) \) and \( B_2(\lambda) B_1(\lambda_2) \) are equivalent. The functional structure which comes from the creation operator \( F(\lambda_1) \) of the two-particle eigenstate is
\[
\frac{\sigma(\lambda_1 - \lambda) \sigma(\lambda_1 - \lambda_2)}{p(\lambda_1 - \lambda) p(\lambda_1 - \lambda_2)} [l(\lambda_2)]^L (F^{12} - F^{21}) \quad (B.7)
\]
and that coming from \( B_a(\lambda_1) B_b(\lambda_2) \) are
\[
\begin{align*}
\left\{ \frac{m(\lambda_1 - \lambda) m(\lambda_2 - \lambda_1)}{f(\lambda_1 - \lambda) f(\lambda_2 - \lambda_1)} F^{21} - \frac{m(\lambda_2 - \lambda)}{f(\lambda_2 - \lambda)} \left[ \frac{r_{12}^{12}(\lambda_1 - \lambda)L^{F^{21}} + r_{12}^{21}(\lambda_1 - \lambda)L^{F^{12}}}{f(\lambda_1 - \lambda)} \right] \right\} [l(\lambda_2)]^L + \\
\left\{ \frac{m(\lambda_1 - \lambda)}{f(\lambda_1 - \lambda)} \frac{m(\lambda_1 - \lambda_2)}{f(\lambda_1 - \lambda_2)} - \frac{m(\lambda_1 - \lambda)}{f(\lambda_1 - \lambda)} \frac{1}{f(\lambda_1 - \lambda_2) f(\lambda_1 - \lambda_2) l(\lambda_1 - \lambda)} \right\} F^{12} [f(\lambda_2)]^L \quad (B.8)
\end{align*}
\]

The simplest way to see that this term is null is to use the Bethe ansatz equations (42) for \( \lambda_1 = \lambda_2 \) in order to have only terms proportional to \( F^{21} [f(\lambda_2)]^L \) and \( F^{12} [f(\lambda_2)]^L \). In particular we have
\[
[l(\lambda_2)]^L (F^{12} - F^{21}) = -[f(\lambda_2)]^L \left\{ r_{12}^{12}(\lambda_2 - \lambda_1) - r_{12}^{21}(\lambda_2 - \lambda_1) \right\} (F^{21} - F^{12}) \quad (B.9)
\]

By making such manipulations and collecting the term proportional to \( F^{21} [f(\lambda_2)]^L \) of expressions (B.7) and (B.8) we have
\[
\begin{align*}
\left\{ \frac{-m(\lambda_1 - \lambda) m(\lambda_2 - \lambda_1)}{f(\lambda_1 - \lambda) f(\lambda_2 - \lambda_1)} r_{12}^{12}(\lambda_2 - \lambda_1) - \frac{\sigma(\lambda_1 - \lambda) \sigma(\lambda_1 - \lambda_2)}{p(\lambda_1 - \lambda) p(\lambda_1 - \lambda_2)} [r_{12}^{12}(\lambda_2 - \lambda_1) - r_{12}^{21}(\lambda_2 - \lambda_1)] \right\} F^{21} [f(\lambda_2)]^L + \\
+ \frac{m(\lambda_2 - \lambda)}{f(\lambda_2 - \lambda)} \frac{1}{f(\lambda_1 - \lambda)} \left[ r_{12}^{12}(\lambda_1 - \lambda)L^{r_{21}^{12}(\lambda_2 - \lambda_1)} + r_{12}^{21}(\lambda_1 - \lambda)L^{r_{12}^{12}(\lambda_2 - \lambda_1)} \right] \right\} F^{21} [f(\lambda_2)]^L \quad (B.10)
\end{align*}
\]
which is in fact null if one uses the identity (47) and the factorization relation
\[
-\frac{\sigma(\lambda_1 - \lambda) \sigma(\lambda_2 - \lambda_1)}{p(\lambda_1 - \lambda) p(\lambda_2 - \lambda_1)} (\lambda_2 - \lambda_1 + 1) = \frac{m(\lambda_1 - \lambda) m(\lambda_2 - \lambda_1)}{f(\lambda_1 - \lambda) f(\lambda_2 - \lambda_1)} (\lambda_2 - \lambda_1)
\]
we finally find that the term proportional to $\frac{1}{f(\lambda_2 - \lambda)} (\lambda_2 - \lambda)$

\[
\frac{-m(\lambda_2 - \lambda)}{f(\lambda_2 - \lambda)} \frac{1}{f(\lambda_1 - \lambda)} (\lambda_2 - \lambda)
\]  

(B.11)

Analogously, the terms proportional to $F^{12}[f(\lambda_2)]^L$ are

\[
\left\{ -\frac{m(\lambda_1 - \lambda)}{f(\lambda_1 - \lambda)} \frac{m(\lambda_2 - \lambda_1)}{f(\lambda_2 - \lambda_1)} r_{21}^{12}(\lambda_2 - \lambda_1) + \frac{m(\lambda - \lambda_1)}{f(\lambda - \lambda_1)} \frac{m(\lambda_2 - \lambda_2)}{f(\lambda_2 - \lambda_2)} + \frac{m(\lambda_2 - \lambda)}{f(\lambda_2 - \lambda)} \frac{1}{f(\lambda_1 - \lambda)} 
\right\}

\[
[r_{12}^{12}(\lambda_1 - \lambda) r_{21}^{12}(\lambda_2 - \lambda_1) + r_{12}^{12}(\lambda_1 - \lambda) r_{21}^{12}(\lambda_2 - \lambda_1)] - \frac{m(\lambda - \lambda_2)}{f(\lambda - \lambda_2)} \frac{1}{f(\lambda - \lambda_1) l(\lambda_1 - \lambda)}
\]

\[
+ \frac{\sigma(\lambda - \lambda_1) \sigma(\lambda_1 - \lambda_2)}{p(\lambda - \lambda_1) p(\lambda_1 - \lambda_2)} [r_{12}^{12}(\lambda_1 - \lambda) - r_{21}^{12}(\lambda_2 - \lambda_1)] \right\} F^{12}[f(\lambda_2)]^L
\]  

(B.12)

Now, by noticing that the first two terms in (B.12) can be simplified as

\[
\frac{m(\lambda_1 - \lambda)}{f(\lambda_1 - \lambda)} \frac{m(\lambda_2 - \lambda_1)}{f(\lambda_2 - \lambda_1)} r_{12}^{12}(\lambda_2 - \lambda_1)
\]  

(B.13)

one can proceed as we did before. If we use the simplification (B.11) and the last relation (B.13) we finally find that the term proportional to $F^{12}[f(\lambda_2)]^L$ is also null.

3. The $E_a(\lambda) B_b(\lambda_2)$ term for $a \neq b$ appears as

\[
\left\{ \frac{\sigma(\lambda - \lambda_2) f(\lambda - \lambda_1)}{p(\lambda - \lambda_2) p(\lambda - \lambda_1)} - \frac{\sigma(\lambda - \lambda_1) m(\lambda_2 - \lambda_2)}{p(\lambda - \lambda_1) f(\lambda_2 - \lambda_2)} \right\} F^{aa}[f(\lambda_2)]^L +
\]

\[
\left\{ \frac{\sigma(\lambda - \lambda_1) m(\lambda_2 - \lambda_1)}{p(\lambda - \lambda_1) f(\lambda_2 - \lambda_1)} - \frac{\sigma(\lambda - \lambda_2) r_{21}^{12}(\lambda - \lambda_1)}{p(\lambda - \lambda_1) g(\lambda - \lambda_1)} \right\} F^{aa}[l(\lambda_2)]^L
\]  

(B.14)

and by using the identity

\[
\frac{f(\lambda)}{p(\lambda)} = -\frac{r_{21}^{12}(\lambda)}{g(\lambda)}
\]  

(B.15)

we are able to simplify equation (B.14) as

\[
\left\{ \frac{\sigma(\lambda - \lambda_2) f(\lambda - \lambda_1)}{p(\lambda - \lambda_2) p(\lambda - \lambda_1)} + \frac{\sigma(\lambda - \lambda_1) m(\lambda_2 - \lambda_1)}{p(\lambda - \lambda_1) f(\lambda_2 - \lambda_1)} \right\} F^{aa} \left([l(\lambda_2)]^L + [f(\lambda_2)]^L \right)
\]  

(B.16)

which is automatically null by taking $a = b$ in the Bethe ansatz equations (42).

4. The $E_a(\lambda) B_a(\lambda_2)$ term has a contribution from both creation operators: $B_i(\lambda_1) B_j(\lambda_2)$ and $F(\lambda_1)$. The contribution coming from $F(\lambda_1)$ is

\[
- \frac{q(\lambda - \lambda_1) \sigma(\lambda_1 - \lambda_2)}{g(\lambda - \lambda_1) p(\lambda_1 - \lambda_2)} [l(\lambda_2)]^L (F^{12} - F^{21})
\]  

(B.17)
and those coming from $B_i(\lambda_1)B_j(\lambda_2)$ are

$$\left\{ \begin{array}{c}
\sigma(\lambda - \lambda_1) m(\lambda_2 - \lambda_1) \\
p(\lambda - \lambda_1) f(\lambda_2 - \lambda_1)
\end{array} \right\} F^{12} - \left\{ \begin{array}{c}
\sigma(\lambda - \lambda_2) [r^{111}_{11}(\lambda - \lambda_1)F^{21} - r^{122}_{12}(\lambda - \lambda_1)F^{12}] \\
p(\lambda - \lambda_2) g(\lambda - \lambda_1)
\end{array} \right\} [l(\lambda_2)]^L +
\left\{ \begin{array}{c}
\sigma(\lambda - \lambda_1) m(\lambda_1 - \lambda_2) \\
p(\lambda - \lambda_1) f(\lambda_1 - \lambda_2)
\end{array} \right\} F^{21}[f(\lambda_2)]^L
$$

(B.18)

In order to show that all these terms together are in fact null, one can follow the same steps of the procedure which we have used for the term $B_i(\lambda_1)B_j(\lambda_2)$ ($i \neq j$). However, the crucial factorization relation here is a bit different, namely

$$\sigma(\lambda - \lambda_1) m(\lambda_2 - \lambda_1) f^{21}_1(\lambda_2 - \lambda_1) = -g(\lambda - \lambda_1) \sigma(\lambda_2 - \lambda_1)
\frac{q(\lambda - \lambda_1)}{p(\lambda - \lambda_1) p(\lambda - \lambda_2) g(\lambda - \lambda_1) (\lambda - \lambda_2 + 1)} (\lambda - \lambda_2 + 1)
+ \frac{1}{p(\lambda - \lambda_2) g(\lambda - \lambda_1) (\lambda_2 - \lambda_1 + 1)(\lambda - \lambda_1 + 1)}
$$

(B.19)

5. To collect all the non-trivial unwanted terms proportional to $F(\lambda)(F^{12} - F^{21})$ is a very cumbersome job. The main reason is that all the diagonal operators $\sum_{a=1}^{2} A_{aa}(\lambda), B(\lambda)$ and $D(\lambda)$ give non-trivial contributions which are proportional to the many combinations of $[l(\lambda_1)]^L \{ \cdots [f(\lambda_2)]^L + \cdots \} [l(\lambda_2)]^L$ and $[f(\lambda_1)]^L \{ \cdots [f(\lambda_2)]^L + \cdots \} [l(\lambda_2)]^L$. For instance, the terms proportional to $[l(\lambda_1)]^L$ are

$$\left\{ \begin{array}{c}
- \frac{\sigma(\lambda - \lambda_1) m(\lambda_2 - \lambda_1)}{p(\lambda - \lambda_1) f(\lambda_2 - \lambda_1)} \frac{1}{f(\lambda - \lambda_1) l(\lambda - \lambda_1)} + \frac{\sigma(\lambda - \lambda_1) m(\lambda_1 - \lambda) m(\lambda_1 - \lambda_2)}{p(\lambda - \lambda_1) f(\lambda - \lambda_1)} \frac{1}{f(\lambda_1 - \lambda_2)}
\end{array} \right\} [f(\lambda_2)]^L +
\left\{ \begin{array}{c}
\frac{\sigma(\lambda_1 - \lambda_2) n(\lambda_1 - \lambda)}{p(\lambda_1 - \lambda_2) p(\lambda_1 - \lambda)} + \frac{\sigma(\lambda_1 - \lambda)}{p(\lambda_1 - \lambda)} m(\lambda_2 - \lambda)
\end{array} \right\} [l(\lambda_2)]^L
$$

(B.20)

while those proportional to $[f(\lambda_1)]^L$ are

$$\left\{ \begin{array}{c}
- \frac{\sigma(\lambda - \lambda_2) m(\lambda - \lambda_1)}{p(\lambda - \lambda_2) p(\lambda - \lambda_1)} + \frac{\sigma(\lambda_1 - \lambda_2) n(\lambda - \lambda_1)}{p(\lambda_1 - \lambda_2) p(\lambda - \lambda_1)}
\end{array} \right\} [f(\lambda_2)]^L +
\left\{ \begin{array}{c}
\frac{m(\lambda - \lambda_1)}{f(\lambda - \lambda_1) g(\lambda - \lambda_1)} \frac{1}{p(\lambda - \lambda_2)} \sigma(\lambda - \lambda_2)
\end{array} \right\} [l(\lambda_2)]^L -
\frac{\sigma(\lambda - \lambda_1) m(\lambda - \lambda_1) m(\lambda_1 - \lambda_2)}{p(\lambda - \lambda_1) p(\lambda - \lambda_1) f(\lambda_1 - \lambda_2)} - 2 \frac{\sigma(\lambda_1 - \lambda_2) m(\lambda - \lambda_1) g(\lambda - \lambda_1)}{p(\lambda_1 - \lambda_2) g(\lambda - \lambda_1) f(\lambda - \lambda_1)}
$$

(B.21)
In order to cancel these expressions one has to use the Bethe ansatz identity (42) for both \( \lambda_1 \) and \( \lambda_2 \). For instance, we first use such relation for \( \lambda_2 \) in equations (B.20,B.21), and as a result we only have terms proportional to \([f(\lambda_2)]^L\{\cdots\}[f(\lambda_1)]^L + (\cdots)[l(\lambda_1)]^L\}. Now, we perform the same operation for \( \lambda_1 \) in order to eliminate the \([l(\lambda_1)]^L\} terms. The final expression is very complicated, but we have checked that it is null with the helping of the Mathematica\textsuperscript{TM} software.

This finishes our analysis concerning the unwanted terms, and the results of this Appendix together with those of section 5 show that the Bethe ansatz equations (42) are sufficient conditions to cancel all of them out.

For sake of completeness let us also discuss the wanted terms. They are responsible for the eigenvalues of the transfer matrix. The wanted terms are constituted by two kinds of creation operators and from the two-particle eigenstate expression (45) we have

\[
B_a(\lambda_1)B_b(\lambda_2)F^{ba}, \quad l(\lambda_2)^L \frac{\sigma(\lambda_1 - \lambda_2)}{p(\lambda_1 - \lambda_2)} F(\lambda_1)
\] (B.22)

The contribution proportional to the first term of equation (B.22) can be obtained directly from the commutation relations (22,25-27), and it is easy to get

\[
\Lambda(\lambda, \{\lambda_i\}) = l(\lambda)^L \prod_{i=1}^2 \frac{l(\lambda_i - \lambda)}{f(\lambda_i - \lambda)} + f(\lambda)^L \prod_{i=1}^2 \frac{\Lambda^{(1)}(\lambda, \{\lambda_i\})}{f(\lambda - \lambda_i)} + p(\lambda)^L \prod_{i=1}^2 \frac{g(\lambda - \lambda_i)}{p(\lambda - \lambda_i)}
\] (B.23)

where \( \Lambda^{(1)}(\lambda, \{\lambda_i\}) \) is the eigenvalue of the inhomogeneous 6-vertex system with two sites (see equation (61) of section 5). Of course the contribution coming from the second term has to be precisely the same as that we have got in equation (B.23). Such calculation is a bit more elaborated, since the action of the diagonal operators \( \sum_a A_{aa}(\lambda) \), \( B(\lambda) \) and \( D(\lambda) \) on the term \( B_a(\lambda_1)B_b(\lambda_2)F^{ba} \) can produce many terms of type \( F(\lambda_1)(F^{12} - F^{21}) \). The analysis is as follows.

The terms coming from \( D(\lambda) \) are

\[
p(\lambda)^L l(\lambda)^L (F^{12} - F^{21}) \left[ \frac{g(\lambda - \lambda_1) \sigma(\lambda - \lambda_2)}{p(\lambda - \lambda_1) p(\lambda - \lambda_2)} - \frac{r(\lambda - \lambda_1) \sigma(\lambda_1 - \lambda_2)}{p(\lambda - \lambda_1) p(\lambda_1 - \lambda_2)} \right]
\] (B.24)

and if we use the factorization identity

\[
\frac{g(\lambda - \lambda_1) \sigma(\lambda - \lambda_2)}{r(\lambda - \lambda_1) - p(\lambda_1 - \lambda_2)} + r(\lambda - \lambda_1) p(\lambda - \lambda_2) = \prod_{i=1}^2 g(\lambda - \lambda_i)
\] (B.25)
we find that the operator $D(\lambda)$ contributes to the eigenvalue with

$$p(\lambda)^L \prod_{i=1}^{2} \frac{g(\lambda - \lambda_i)}{p(\lambda - \lambda_i)}$$

(B.26)

Those coming form $B(\lambda)$ are

$$l(\lambda)^L l(\lambda_2)^L (F^{12} - F^{21}) \left[ \frac{-\sigma(\lambda_1 - \lambda)l(\lambda_1 - \lambda)m(\lambda_2 - \lambda)}{p(\lambda_1 - \lambda)f(\lambda_1 - \lambda)f(\lambda_2 - \lambda)} - \frac{l(\lambda_1 - \lambda)\sigma(\lambda_1 - \lambda_2)}{p(\lambda_1 - \lambda)p(\lambda_1 - \lambda_2)} \right]$$

(B.27)

and by using the relation

$$\frac{\sigma(\lambda_1 - \lambda)m(\lambda_2 - \lambda)}{\sigma(\lambda_1 - \lambda_2)}f(\lambda_1 - \lambda)f(\lambda_2 - \lambda) = p(\lambda_1 - \lambda)l(\lambda_2 - \lambda)$$

(B.28)

we find that the contribution of $B(\lambda)$ is

$$l(\lambda)^L \prod_{i=1}^{2} \frac{l(\lambda_i - \lambda)}{f(\lambda_i - \lambda)}$$

(B.29)

Finally, the operator $\sum_a A_{aa}(\lambda)$ generates the terms which carry the hidden 6-vertex structure. They have been collected as

$$f(\lambda)^L l(\lambda_2)^L (F^{12} - F^{21}) - \frac{\sigma(\lambda_1 - \lambda_2)}{p(\lambda_1 - \lambda_2)} \left\{ \frac{2g(\lambda - \lambda_1)}{f(\lambda - \lambda_1)} \left[ 1 - \frac{g^2(\lambda - \lambda_1)}{g^2(\lambda - \lambda_1)} \right] + \frac{\sigma(\lambda - \lambda_1)g(\lambda - \lambda_1)p(\lambda_1 - \lambda_2)[r_{11}^{11}(\lambda - \lambda_1) + r_{21}^{21}(\lambda - \lambda_1)]}{p(\lambda - \lambda_2)f(\lambda - \lambda_1)\sigma(\lambda_1 - \lambda_2)} - \frac{\sigma(\lambda - \lambda_1)m(\lambda - \lambda_2)p(\lambda_1 - \lambda_2)}{f(\lambda - \lambda_1)p(\lambda - \lambda_1)f(\lambda - \lambda_2)\sigma(\lambda_1 - \lambda_2)} \right\}$$

(B.30)

Remarkably enough the terms on the bracket of equation (B.30) can be factorized only in terms of the 6-vertex $r$-matrix as

$$\prod_{i=1}^{2} \frac{1}{f(\lambda - \lambda_i)} \left\{ r_{11}^{11}(\lambda - \lambda_1) r_{21}^{12}(\lambda - \lambda_2) + r_{12}^{21}(\lambda - \lambda_1) r_{22}^{22}(\lambda - \lambda_2) - r_{21}^{21}(\lambda - \lambda_1) r_{12}^{12}(\lambda - \lambda_2) \right\}$$

(B.31)

However, by using the auxiliary eigenvalue equation (61) for two sites, one is able to establish the following identity

$$\Lambda^{(1)}(\lambda, \{\lambda_i\})(F^{12} - F^{21}) = \left\{ r_{11}^{11}(\lambda - \lambda_1) r_{21}^{12}(\lambda - \lambda_2) + r_{12}^{21}(\lambda - \lambda_1) r_{22}^{22}(\lambda - \lambda_2) - r_{21}^{21}(\lambda - \lambda_1) r_{12}^{12}(\lambda - \lambda_2) \right\} (F^{12} - F^{21})$$

(B.32)
and as a consequence of the expressions (B.31) and (B.32) we have that the eigenvalue contribution of $\sum_a A_{aa}(\lambda)$ is

$$f(\lambda)^L \prod_{i=1}^2 \frac{\Lambda^{(1)}(\lambda, \{\lambda_i\})}{f(\lambda - \lambda_i)}$$

(B.33)

Hence, this completes the proof that the two-particle state (45) is the eigenstate of the $spl(2|1)$ model.

**Appendix C : The three particle state**

This appendix is mainly concerned with the symmetrization property (52) of the three particle state. We begin our discussion with the $\lambda_2 \leftrightarrow \lambda_3$ permutation. In this case, the three particle vector (see equation (48) of section 5) becomes

$$\vec{\Phi}_3(\lambda_1, \lambda_2, \lambda_3) = \vec{B}(\lambda_1) \otimes \vec{B}(\lambda_2) \otimes \vec{B}(\lambda_3) + [l(\lambda_3)]^L \frac{\sigma(\lambda_2 - \lambda_3)}{p(\lambda_2 - \lambda_3)} \vec{B}(\lambda_1) \otimes F(\lambda_3) \vec{\xi}$$

$$+ [l(\lambda_3)]^L \frac{\sigma(\lambda_1 - \lambda_3)}{p(\lambda_1 - \lambda_3)} F(\lambda_1) \vec{\xi} \otimes \vec{B}(\lambda_2) \hat{F}_2(\lambda_3, \lambda_2) + [l(\lambda_2)]^L \frac{\sigma(\lambda_1 - \lambda_2)}{p(\lambda_1 - \lambda_2)} F(\lambda_1) \vec{\xi} \otimes \vec{B}(\lambda_3) \hat{F}_3(\lambda_3, \lambda_2)$$

(C.1)

In order to relate this vector with the symmetric one $\vec{\Phi}_3(\lambda_1, \lambda_2, \lambda_3)$, we turn $\vec{B}(\lambda_2)$ over $\vec{B}(\lambda_3)$ with the help of the commutation rule (27) and as a result we get

$$\vec{B}(\lambda_1) \otimes \vec{B}(\lambda_2) \otimes \vec{B}(\lambda_3) = \left\{ \vec{B}(\lambda_1) \otimes \vec{B}(\lambda_3) \otimes \vec{B}(\lambda_2) + \frac{\sigma(\lambda_3 - \lambda_2)}{p(\lambda_3 - \lambda_2)} [l(\lambda_2)]^L \vec{B}(\lambda_1) \otimes F(\lambda_3) \vec{\xi} \right\} \frac{r_{23}(\lambda_2 - \lambda_3)}{l(\lambda_2 - \lambda_3)}$$

$$- [l(\lambda_3)]^L \frac{\sigma(\lambda_2 - \lambda_3)}{p(\lambda_2 - \lambda_3)} \vec{B}(\lambda_1) \otimes F(\lambda_2) \vec{\xi}$$

(C.2)

The last term of this identity cancels out the second term in the vector $\vec{\Phi}_3(\lambda_1, \lambda_2, \lambda_3)$ and the symmetrization rule

$$\vec{\Phi}_3(\lambda_1, \lambda_2, \lambda_3) = \vec{\Phi}_3(\lambda_1, \lambda_3, \lambda_2) \frac{r_{23}(\lambda_2 - \lambda_3)}{l(\lambda_2 - \lambda_3)}$$

(C.3)

is valid provided the functions $\hat{F}_2(\lambda_2, \lambda_3)$ and $\hat{F}_3(\lambda_2, \lambda_3)$ satisfy the following equations

$$\hat{F}_3(\lambda_3, \lambda_2) = \hat{F}_2(\lambda_2, \lambda_3) l(\lambda_2 - \lambda_3) r_{23}^{-1}(\lambda_2 - \lambda_3)$$

(C.4)
\[ \hat{F}_2(\lambda_3, \lambda_2) = \hat{F}_3(\lambda_2, \lambda_3) l(\lambda_2 - \lambda_3) r^{-1}_{23}(\lambda_2 - \lambda_3) \]  
\hline

but in fact they are equivalent since we have the following inversion properties

\[ l(x)l(-x) = 1, \quad r_{23}(x)r_{23}(-x) = I_{23} \]  
\hline

Similar reasoning can be implemented for the permutation \( \lambda_1 \leftrightarrow \lambda_2 \). In this case however, besides turning \( \vec{B}(\lambda_1) \) over \( \vec{B}(\lambda_2) \), we also have to turn \( \vec{B}(\lambda_1) \) over \( F(\lambda_2)\vec{\xi} \). By using the following commutation rule

\[ F(\lambda)B_a(\mu) = \frac{q(\lambda - \mu)}{l(\lambda - \mu)} F(\mu)B_a(\lambda) + \frac{g(\lambda - \mu)}{l(\lambda - \mu)} B_a(\mu)F(\lambda), \quad a = 1, 2 \]  
\hline

This leads us with the term

\[ l(\lambda_2) L \sigma(\lambda_1 - \lambda_2) \frac{F(\lambda_1)\vec{\xi} \otimes \vec{B}(\lambda_3)\hat{F}_2(\lambda_2, \lambda_3)}{p(\lambda_1 - \lambda_2)} \]  
\hline

which appears on the vector \( \vec{\Phi}_3(\lambda_1, \lambda_2, \lambda_3) \) and has to be canceled out by

\[ - l(\lambda_2) L \sigma(\lambda_1 - \lambda_2) \frac{F(\lambda_1)\vec{\xi} \otimes \vec{B}(\lambda_3) \frac{l(\lambda_3 - \lambda_2)}{f(\lambda_3 - \lambda_2)}}{p(\lambda_1 - \lambda_2)} \]  
\hline

and therefore we find that the function \( \hat{F}_2(\lambda_2, \lambda_3) \) satisfies

\[ \hat{F}_2(\lambda_2, \lambda_3) = \frac{l(\lambda_3 - \lambda_2)}{f(\lambda_3 - \lambda_2)} I \]  
\hline

Hence, equations (C.4) and (C.10) are able to fix the functions \( \hat{F}_2(\lambda_2, \lambda_3) \) and \( \hat{F}_3(\lambda_2, \lambda_3) \), and as a consequence follows the expressions (49) and (50) of section 5. We have also checked the consistency of these results by verifying that all other terms satisfy the condition of symmetry (51) of section 5. Such calculation involves additional properties, since we need to turn twice the operators \( \vec{B}(\lambda_i) \). Here we list some extra identities which are extremely useful to prove the symmetry under \( \lambda_1 \leftrightarrow \lambda_2 \),

\[ [\vec{\xi} \otimes \vec{B}(y)] . r_{12}(x) = \frac{1 - x}{1 + x} \vec{\xi} \otimes \vec{B}(y), \quad [\vec{B}(y) \otimes \vec{\xi}] . r_{23}(x) = \frac{1 - x}{1 + x} \vec{B}(y) \otimes \vec{\xi} \]  
\hline
\[ [\vec{B}(y) \otimes \vec{\xi}] \cdot r_{12}(x) = \vec{B}(y) \otimes \vec{\xi} + r^{12}_{21}(x) \vec{\xi} \otimes \vec{B}(y), \quad [\vec{\xi} \otimes \vec{B}(y)] \cdot r_{23}(x) = \vec{\xi} \otimes \vec{B}(y) + r^{12}_{21}(x) \vec{B}(y) \otimes \vec{\xi} \]

(C.12)

Finally, another useful test is to look for certain unwanted terms which must be automatically canceled out. For instance, this is the case of the terms

\[ [l(\lambda_2)]^L B_a(\lambda) E_a(\lambda_1) B_a(\lambda_3), \quad [l(\lambda_3)]^L E_a(\lambda) E_b(\lambda_1) B_c(\lambda_2) \]

(C.13)

It is direct to verify that the first term in eq.(C.13) is canceled out with the help of \( \tilde{F}_2(\lambda_2, \lambda_3) \) while the second term depends on the functional form of the function \( \tilde{F}_3(\lambda_2, \lambda_3) \).

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Figures

Figure 1. The 36 nonvanishing Boltzmann weights of the rational $spl(2|1)$ model.