Scattering from infinity for semilinear models of Einstein’s equations satisfying the weak null condition

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Abstract

We show global existence backwards from scattering data for models of Einstein’s equations in wave coordinates satisfying the weak null condition. The data is in the form of the radiation field at null infinity recently shown to exist for the forward problem in Lindblad [26]. Our results are sharp in the sense that we show that the solution has the same spatial decay as the radiation field does at null infinity, as for the forward problem.

1 Introduction

1.1 Global existence for Einstein’s equations close to Minkowski space

Einstein’s equations in wave coordinates become a system of nonlinear wave equations

\[ \Box_g g_{\mu\nu} = F_{\mu\nu}(g)(\partial g, \partial g), \quad \text{where} \quad \Box_g = g^{\alpha\beta} \partial_\alpha \partial_\beta, \]

for a Lorentzian metric \( g_{\alpha\beta} \). Here \( F_{\mu\nu} \) is quadratic form in \( \partial g \) with coefficients depending on \( g \). The global initial value problem is to show global existence of solutions for all positive times given asymptotically flat initial data close to Minkowski metric \( m_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \)

\[
g_{ij}|_{t=0} = (1 + Mr^{-1}) \delta_{ij} + o(r^{-1-\gamma}), \quad \partial_t g_{ij}|_{t=0} = o(r^{-2-\gamma}), \quad r = |x|, \quad 0 < \gamma < 1, \quad M > 0.
\]  

(1.2)

Christodoulou-Klainerman [3] proved global existence for Einstein’s equations \( R_{\mu\nu} = 0 \). Their proof avoids using coordinates since it was believed the metric in wave coordinates may blow up. John [10] [11] had noticed that solutions to some nonlinear wave equations blow up for small data, whereas Klainerman [13] [14], see also [4], came up with the null condition, that guaranteed global existence. However Einstein’s equations do not satisfy the null condition. Instead Lindblad and Rodnianski in [18] showed that Einstein’s equations in wave coordinates satisfy a weak null condition and in [20] [25] used it to prove global existence.

Here we consider the backward problem of for models of Einstein’s equations in wave coordinates finding a solution that has given scattering data as time tend to infinity. For Einstein’s equations in higher dimensions this was studied by Wang [28]. (In the context of black hole spacetimes a scattering construction was given by Dafermos-Holzegel-Rodnianski [6].) However, the three dimensional case is more delicate and requires a description of the
asymptotic behaviour of solutions of the forward problem given recently in Lindblad [26]. We propose some models that we will argue capture some of the main difficulties and prove scattering for these.

1.2 The semilinear weak null condition model of Einstein’s equations

In order to unravel the effect of the quasilinear terms in Einstein’s equations one can change to characteristic coordinates as in [3], but this loses regularity and is not explicit since it depends on the unknown solution. Instead [26] use the asymptotics of the metric to determine the characteristic surfaces asymptotically and use this to construct coordinates. Due to the wave coordinate condition the outgoing light cones of a solution with asymptotically flat data approach those of the Schwarzschild metric with the same mass $M$. The outgoing light cones for the Schwarzschild metric satisfy $t \sim r^*-q^*$, where $r^* = r + M \ln r$. One can therefore make the change of variables $x = r \omega \to \tilde{x} = r^* \omega$, for large $r$, and the wave operator $\tilde{\Box} = \Box^\ast$ in the $(t, x^\ast)$ coordinates.

In [26] it was shown that modulo lower order terms decaying faster in these coordinates the metric satisfy the semilinear model system for Einstein’s equations

$$\Box h_{\mu\nu} = F_{\mu\nu}(m)(\partial h, \partial h).$$

Here $F_{\mu\nu}(m)(\partial h, \partial h)$ is a sum of classical nullforms which we denote by $Q_{\mu\nu}(\partial h, \partial h)$ and

$$P(\partial h, \partial h), \quad \text{where} \quad P(D, E) = D^\alpha E^\beta /4 - D^{\alpha\beta} E_{\alpha\beta}/2.$$ (1.4)

In [18] it was observed that in a null frame $N$, $L = \partial_t - \partial_r$, $L = \partial_t + \partial_r$, $S_1, S_2 \in S^2$, $\langle S_i, S_j \rangle = \delta_{ij}$ (1.5) the semilinear model satisfy a weak null condition. In fact, it is well known that for solutions of wave equations derivatives tangential to the outgoing light cones $\bar{\partial} \in T = \{L, S_1, S_2\}$ decay faster. Modulo tangential derivatives $\bar{\partial} h$ we have $\partial h \sim L_{\mu} \partial_{q} h$, where $\partial_q = (\partial_r - \partial_t)/2$, $L_{\mu} = m_{\mu\nu} L^\nu$, and modulo quadratic terms with at least one tangential derivative

$$\Box h_{\mu\nu} \sim L_{\mu} L_\nu P(\partial_q h, \partial_q h).$$ (1.6)

Here in a null frame

$$P(D, E) = -\tilde{D}^{AB} \tilde{E}_{AB}/2, \quad A, B \in S = \{S_1, S_2\},$$ (1.7)

is the norm of the traceless part of the angular components, $\tilde{D}^{AB} = D^{AB} - \delta^{AB} \text{tr}_S D$, $\text{tr}_S D = \delta_{AB} D^{AB}$. In [18] it was noticed that the system (1.6) has a weak null structure in the sense that $P$ only depend on the angular components but the angular components of the right hand side of (1.6) vanish due to that $L_{\mu} A^\mu = 0$.

We therefore propose to study scattering for the following simplified semilinear model

$$\Box \psi = Q(\partial \psi, \partial \psi)$$ (1.8)

$$\Box \phi = (\partial \psi)^2$$ (1.9)

that has a similar weak null structure as (1.3).
1.3 Asymptotics for Einstein’s equations in wave coordinates

In [26] it was shown that in the \((t, r^*\omega)\) coordinates the metric satisfy (1.3) modulo faster decaying terms. This was then used to obtain the asymptotics for Einstein’s equations. It was shown that asymptotically the metric is Minkowski metric \(m_{\mu\nu}\) plus

\[
h_{\mu\nu}(t, r\omega) \sim \frac{H_{\mu\nu}(r^*-t, \omega)}{t+r} + \frac{K_{\mu\nu}(\frac{(t+r^*)}{(r-t)}, \omega, r^*-t)}{t+r}, \quad r^* \sim r + M \ln r, \quad \omega = \frac{x}{|x|},
\]  

(1.10)

where \(\langle q \rangle = (1 + q^2)^{1/2}\). Here \(H\) is concentrated close to the outgoing light cones \(r^* - t\) constant, and \(K\) to leading order is homogeneous of degree 0 with a log singularity at the light cone for the nontangential components:

\[
|H(q^*, \omega)| \lesssim (1 + |q^*|)^{-\gamma'},
\]  

(1.11)

\[
|K(s, \omega, q)| \lesssim \ln s
\]  

(1.12)

for any \(\gamma' < \gamma\) in (1.2). \(H\) is the radiation field of the free curved wave operator. \(K\) has two parts, the backscattering of the wave operator with a source term \(F_{\mu\nu} \sim P_S(\partial_\mu h, \partial_\nu h)\), where \(P_S\) is the norm of the components tangential to the spheres, and term coming from the long range effect of the mass of initial data. In the wave zone when \(|t - r^*| \ll t + r^*\)

\[
K_{\mu\nu}(\frac{(t+r^*)}{(r-t)}, \omega, r^*-t) \sim L_\mu(\omega)L_\nu(\omega) \frac{1}{2} \ln \left(\frac{(t+r^*)}{(t-r^*)}\right) \int_{r^* - t}^{\infty} n(q^*, \omega) dq^* + 2M \delta_{\mu\nu},
\]  

where \(L_\mu = m_{\mu\nu}L^\nu, L = (1, \omega)\), and

\[
n(q^*, \omega) = -P(\partial_{q^*} H, \partial_{q^*} H)(q^*, \omega),
\]  

(1.13)

What is particularly important is the additional decay of \(H\) in (1.11). Here \(\gamma'\) is any number less than \(\gamma\) in the asymptotic flatness condition (1.2). In the interior of the light cone \(r^* < t\) this was very difficult to prove for the forward problem, see [26].

For the backward problem we simply give the radiation fields \(H\) satisfying (1.11) and we want to prove that there is a solution to Einstein’s equations with the given asymptotics (1.10) which initially satisfy the asymptotic flatness condition (1.2) for any \(\gamma < \gamma'\). For the backward problem this is particularly difficult to prove in the exterior of the light cone \(r^* > t\), c.f. Section 1.4-1.5 below.

1.4 Scattering with spacial decay for the simplified semilinear models

We note that the semilinear model problem (1.3) to highest order has exactly the same asymptotics as the full Einstein’s equations (1.1), since the asymptotics above was derived from reducing Einstein’s equation to this system. Moreover the simplified system (1.8)-(1.9) will have the same kind of asymptotics. Therefore we study scattering for the simplified system in this paper with particular emphasis on obtaining the right decay of initial data from the decay of the radiation field. To make things more clear we will separate the model (1.8)-(1.9) into two models.
The first model is an equation satisfying the classical null condition

$$\Box u = Q(\partial u, \partial u)$$  \hfill (1.14)

for which we give in Section 4 asymptotic data as $t \to \infty$

$$u \sim u_0$$  \hfill (1.15)

in the form of a solution to the homogeneous equation $\Box u_0 = 0$.

The second is a simpler model system

$$\Box \psi = 0$$  \hfill (1.16)

$$\Box \phi = (\partial_t \psi)^2$$  \hfill (1.17)

for which give data in the form of radiation fields. In Section 5 we first study just the linear equation (1.16) with data in the form of a radiation field

$$\psi \sim \frac{F_0(t - t, \omega)}{r}.$$  \hfill (1.18)

We will assume that for some $1/2 < \gamma < 1$:

$$\|F_0\|_{N,1/2}^2 = \sum_{|\alpha| + k \leq N} \int_{\mathbb{R}} \int_{\mathbb{S}^2} |\langle q \rangle^k \partial_q \partial_\omega F_0(q, \omega)|^2 \langle q \rangle^{2\gamma - 1} dS(\omega) dq \leq C$$  \hfill (1.19)

and as a consequence for some smaller $N'$

$$\|F_0\|_{N',1}^2 = \sum_{|\alpha| + k \leq N'} \sup_{q \in \mathbb{R}} \sup_{\omega \in \mathbb{S}^2} \|\langle q \rangle^k \partial_q \partial_\omega F_0(q, \omega)\| \langle q \rangle^\gamma \leq C.$$  \hfill (1.20)

Our main theorem for the linear homogeneous equation is

**Theorem 1.1.** The wave equation $\Box \psi = 0$ has a solution with radiation field $F_0$ as in (1.18) that for any $s < \gamma + 1/2$ satisfy

$$\|\psi(t, \cdot)\|_{k,s-1} := \sum_{|I| \leq k} \|\langle t - r \rangle^{s-1}(Z^I \psi)(t, \cdot)\|_{L^2(\mathbb{R}^3)} \lesssim \|F_0\|_{2+k,1/2}.$$  \hfill (1.21)

Here $Z^I$ is any combination of $|I| \leq k$ of the vector fields $Z$, that commute with $\Box$, i.e. the translations $\partial_t$, $\partial_i$, rotations $x^i \partial_j - x^j \partial_i$ and the boosts $x^i \partial_i + t \partial_t$, and the scaling vector field $t \partial_t + x^i \partial_i$.

For corresponding statement for the nonlinear system (1.16)-(1.17) see Proposition 6.1. As a consequence of the theorem and Klainerman-Sobolev inequality with weights we have

**Corollary 1.1.** Let $\psi$, $\gamma$ and $F_0$ be as in as in Theorem 1.1. Then for any $\gamma' < \gamma$ and $|I| \leq k - 2$ we have

$$|Z^I \psi(t, x)| \lesssim \frac{\|F_0\|_{2+k,1/2}}{\langle t + r \rangle \langle t - r \rangle^{\gamma'}}.$$  \hfill (1.22)
The solution $\psi$ is constructed by giving data for $\psi$ when $t = T$, in the form of the following approximate solution which is supported in the wave zone, away from the origin,

$$
\psi_0 = \frac{F_0(r - t, \omega)}{r} \chi\left(\frac{(t-r)}{r}\right), \quad (1.23)
$$

where $\chi$ is a smooth decreasing function such that $\chi(s) = 1$, when $s \leq 1/8$ and $\chi(s) = 0$, when $s \geq 1/4$, and then showing that the limit as $T \to \infty$ exists.

In Section 6 we then finally estimate also the equation for $\phi$ in the same norms we use for $\psi$. However first we need to subtract off an approximate solution picking up the source term. We define $\phi_0$ to be the solution of

$$
\Box \phi_0 = (\psi_0')^2, \quad \text{where} \quad \psi_0' = -\frac{F_0'(r - t, \omega)}{r} \chi\left(\frac{(t-r)}{r}\right), \quad (1.24)
$$

with vanishing data at $-\infty$. In [16, 26] there was a formula for the solution

$$
\Phi^2[n](t, r\omega) = \int_{t-r}^{\infty} \frac{1}{4\pi} \int_{S^2} \frac{n(q, \sigma)\chi\left(\frac{\langle q \rangle}{\rho}\right)}{t - r + q + r(1 - \langle \omega, \sigma \rangle)} dS(\sigma) dq, \quad (1.25)
$$

where $n(q, \omega) = F_0'(q, \omega)^2$, and an estimate for the solution of the form

$$
|\Phi^2[n](t, r\omega)| \lesssim \frac{1}{2r} \ln \left(\frac{\langle t + r \rangle}{\langle t - r \rangle}\right) \quad (1.26)
$$

1.5 The fractional Conformal Morawetz energy estimate from infinity

The main technical tool of the paper used to obtain the right spacial decay is a new energy estimate from infinity with strong weights:

**Theorem 1.2.** Let $t_1 \leq t_2$, then for $s \geq 1$

$$
\|\phi(t_1, \cdot)\|_{1+s,1} \lesssim \|\phi(t_2, \cdot)\|_{1+s,1} + \int_{t_1}^{t_2} \|\langle t + r \rangle^s \Box \phi(t, \cdot)\|_{L^2} dt \quad (1.27)
$$

provided that $\lim_{r \to \infty} \sup_{\omega \in S^2} r^{s+1/2}|\phi(t, r\omega)| = 0$. Here

$$
\|\phi(t, \cdot)\|_{1+s,1}^2 := \int_{\mathbb{R}^3} \langle t + r \rangle^{2s} \left( \langle \partial_t + \partial_r \rangle (r\phi) \right)^2 + |\nabla (r\phi)|^2 \right) \right. \\
+ \langle t - r \rangle^{2s} \left( \langle \partial_t - \partial_r \rangle (r\phi) \right)^2 + \phi^2 + \frac{(r\phi)^2}{(t-r)} \frac{dx}{r^2} \quad (1.28)
$$

is a norm such that

$$
\|\phi(t_1, \cdot)\|_{1+s,1} := \sum_{|I| \leq 1} \|\langle t - r \rangle^{s-1}(Z^I \phi)(t, \cdot)\|_{L^2} \lesssim \|\phi(t_1, \cdot)\|_{1+s,1}. \quad (1.29)
$$
If $s = 1$ this reduces to the classical conformal Morawetz energy estimate which hold in both the forward and direction $t_1 > t_2$ (with the boundaries of integration switched) and the backward direction $t_1 < t_2$. We remark that in the forward direction $t_1 > t_2$ it was proven in Lindblad-Sterbenz [23] that this estimate holds if $1/2 \leq s \leq 1$. The weight in $\langle t - r \rangle$ is what gives the extra spacial decay. These estimates are also related to the $r^p$-weighted estimates of Dafermos and Rodnianski [7, 27]. While these are derived from multipliers of the form $r^p(\partial_t + \partial_r)$, the multipliers here are $(t + r)^s(\partial_t + \partial_r) + (t - r)^s(\partial_t - \partial_r)$, which have the advantage of providing immediate decay rates in $\langle t - r \rangle$.

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Note added. In the course of the preparation of this manuscript a new proof of the stability of Minkowski space has appeared in [9], with some related results.

Notation. $m$: Minkowski metric on $\mathbb{R}^{3+1}$.
$q_\pm$: For any $q \in \mathbb{R}$, $q_+ := \max\{q, 0\}$, and $q_- := -\min\{q, 0\}$.
$Z^I$: $Z$ denotes a commuting vectorfield, $I$ is a multi-index of length $|I|$.
$dS$: $dS = r^2 dS(\omega)$ denotes the area element on the sphere of radius $r$.

2 Energy and decay estimates

2.1 Weighted space-time energy estimates from infinity

We start by deriving several weighted energy estimates for the equation

$$\Box \phi = F$$

which are relevant both to the forward/backward problems.

For any given $t_2 > t_1$, $R > (t_2 - t_1)$ let us denote by

$$\mathcal{D} = \bigcup_{t_1 \leq \tau \leq t_2} \Sigma^R_{\tau}(\tau - t_2)$$

where $\Sigma^R_{\tau} = \{(t, x) : \tau = \tau, |x| \leq R_+\}$. Note that $\partial \mathcal{D} = \Sigma^R_{t_1}(t_2 - t_1) \cup \Sigma^R_{t_2} \cup C_R$, where $C_R = \{(t, x) : t - |x| = t_2 - R, t_1 < t < t_2\}$ is a (truncated) outgoing null hypersurface. $\mathcal{D}$ is the backward domain of dependence of a ball of radius $R$ in $t = t_2$, see Fig. 1.

In the following we will also denote by

$$|\partial_\phi|^2 := (\partial_t \phi)^2 + |\nabla \phi|^2 \quad \text{and} \quad |\overline{\partial} \phi|^2 := (\partial_t \phi + \partial_r \phi)^2 + |\nabla \phi|^2$$

meaning that $\overline{\partial}$ denote derivatives tangential to the lightcone.

Proposition 2.1. Let $\phi$ be a solution to (2.1) on $\mathcal{D}$. Then

$$\int_{\Sigma^R_{t_2}} |\partial_\phi|^2 w(q) \, dx + \int_{\mathcal{D}} 2F \partial_\phi \phi w(q) \, dx dt + \int_{\mathcal{D}} \frac{1}{4} (|\overline{\partial} \phi|^2 + |\nabla \phi|^2) w'(q) \, dx dt$$

$$= \int_{\Sigma^R_{t_1}(t_2 - t_1)} |\partial_\phi|^2 w(q) \, dx + \int_{t_1}^{t_2} \int_{\partial \Sigma^R_{\tau}(t_2 - t)} |\overline{\partial} \phi|^2 w(q) \, dS dt$$

where $w(q)$ is an arbitrary function of $q = r - t$. 

Proof. Set $W = w(q)\partial_t$ where $w$ is only a function of $q = r - t$, and

$$J_\mu = T_{\mu\nu}W^\nu$$

then

$$\nabla^\mu J_\mu = F(W\phi) + T_{\mu\nu}\pi^{\mu\nu}$$

where

$$\pi_{\mu\nu} = \frac{1}{2}L_WM_{\mu\nu}$$

It makes sense to write this out in a null frame ($L = \partial_t + \partial_r, \bar{L} = \partial_t - \partial_r, e_A : A = 1, 2$)

$$T_{\mu\nu}\pi^{\mu\nu} = m^{LL}m^{\bar{L}\bar{L}}T(L, \bar{L})\pi(L, L) + g^{LL}g^{\bar{L}\bar{L}}T(L, L)\pi(L, L)$$

where we used that the only non-vanishing components of the deformation tensor of $W$ are

$$\pi(L, L) = \frac{1}{2}(Lw)m(\partial_t, L) = \frac{1}{2}\partial_tw$$

$$\pi(L, \bar{L}) = (Lw)m(\partial_t, \bar{L}) = \partial_t w$$

Thus

$$T_{\mu\nu}\pi^{\mu\nu} = \frac{1}{4}(\partial_tw)(|\nabla\phi|^2 + \frac{1}{8}(\partial_tw)(L\phi)^2).$$

Finally the boundary terms are

$$T(W, \partial_t) = w(q)T(\partial_t, \partial_t) = \frac{1}{2}w(q)((\partial_t\phi)^2 + |\nabla\phi|^2)$$

$$T(W, L) = w(q)T(\partial_t, L) = \frac{1}{2}w(q)((L\phi)^2 + |\nabla\phi|^2)$$

and thus we obtain by Stokes theorem

$$\int_D F(W\phi) + \frac{1}{4}(\partial_tw)|\nabla\phi|^2 + \frac{1}{8}(\partial_tw)(L\phi)^2 \, dx dt =$$

$$= \int_D \nabla^\mu J_\mu \, dx dt = \int_D d^*J = \int_{\partial D} *J = \int_{\Sigma_{t_2}^R - (t_2 - t_1)} J^0 - \int_{\Sigma_{t_1}^R} J^0 + \int_{C_R} *J$$

$$= - \int_{\Sigma_{t_2}^R} T(W, \partial_t) + \int_{\Sigma_{t_1}^R - (t_2 - t_1)} T(W, \partial_t) + \int_{t_2}^{t_1} \int_{\Sigma^2} T(W, L) \, d\mu_\gamma dt$$

which completes the proof.
The following two corollaries correspond to different choices of the function \( w \).

**Corollary 2.2.** Let \( R > 0, t_2 > t_1 \), and \( \phi \) be a solution to \((2.1)\) on \( \mathcal{D} \). Then for all \( \mu > 0 \),

\[
\| \partial \phi \|_{L^2(\Sigma_t^{R-(t_2-t_1)})} + \left( \int_D |\partial \phi|^2 \frac{\mu \, dx \, dt}{(1 + |q|)^{1+2\mu}} \right)^{1/2} + \left( \sup_{R' \leq R} \int_{C_{R'}} |\partial \phi|^2 \, dS \right)^{1/2} \lesssim \| \partial \phi \|_{L^2(\Sigma_t^{R+(t_2-t_1)})} + \int_{t_1}^{t_2} \| F \|_{L^2(\Sigma_t^{R+(t-1_2)})} \, dt \quad (2.4)
\]

**Proof.** Let us choose \( w = w_0 \), where

\[
w_0(q) = \begin{cases} 
1 + (1 + q)^{-2\mu} & q > 0 \\
2 + 2\mu \int_q^0 (1 + |s|)^{-1-2\mu} \, ds & q < 0
\end{cases}
\]

then \( 1 \leq w_0(q) \leq 3 \), and in both cases

\[
w'_0(q) = -2\mu(1 + |q|)^{-1-2\mu}
\]

Note also that since \( \mathcal{D} \) lies to the future of \( C_R \), we have \( q < R - t_2 \).

Therefore by Prop. \((2.1)\) we obtain

\[
\int_{\Sigma_t^{R-(t_2-t_1)}} \frac{1}{2} w_0(q)|\partial \phi|^2 \, dx + \int_{C_R} \frac{1}{2} w_0(R - t_2)|\partial \phi|^2 \, dS + \int_{D} \frac{\mu}{4} \frac{|\partial \phi|^2}{(1 + |q|)^{1+2\mu}} \, dx \, dt \\
\leq \int_{\Sigma_t^{R}} \frac{1}{2} w_0(q)|\partial \phi|^2 \, dx + \int_{D} F w_0(q) \partial_t \phi \, dx \, dt
\]

Also by the standard energy estimate (corresponding to choosing \( w \equiv 1 \) in Prop. \((2.1)\))

\[
\left| \int_{D} F \partial_t \phi \, dx \, dt \right| \lesssim \int_{t_1}^{t_2} \| \partial \phi \|_{L^2(\Sigma_t^{R+(t_2-t_1)})} \| F \|_{L^2(\Sigma_t^{R+(t_2-t_1)})} \, dt \\
\leq C \left( \| \partial \phi \|_{L^2(\Sigma_t^{R})} + \int_{t_1}^{t_2} \| F \|_{L^2(\Sigma_t^{R+(t_2-t_1)})} \, dt \right)^2
\]

The two inequalities combined give the estimate for the second term in the statement of the Corollary, and the standard energy estimate the first term.

**Corollary 2.3.** For any \( \mu \geq 0 \) and \( \gamma \geq -\frac{1}{2} \), we have

\[
\int_{\Sigma_t} |\partial \phi|^2 \, w_\gamma \, dx + \int_D |\partial \phi|^2 \left( \frac{\mu/2}{(1 + |q|)^{1+2\mu}} + \frac{1+2\gamma(1 + q_\gamma)^{2\gamma}}{4} \right) \, dx \, dt + \sup_{R' \leq R} \int_{C_{R'}} |\partial \phi|^2 \, w_\gamma \, dS \\
\leq \int_{\Sigma_t} |\partial \phi|^2 \, w_\gamma \, dx + 2 \int_{D} F \partial_t \phi \, w_\gamma \, dx \, dt
\]

where

\[
w_\gamma = (1 + q_\gamma)^{1+2\gamma}
\]

\[2.7]
Proof. For $\mu \geq 0$ and $\gamma \geq -\frac{1}{2}$ let

$$w(q) = \begin{cases} 1 + (1 + |q|)^{-2\mu} & q > 0 \\ 1 + (1 + |q|)^{1+2\gamma} & q < 0 \end{cases}$$ (2.8)

in which case

$$w'(q) = \begin{cases} -2\mu(1 + |q|)^{-1-2\mu} & q > 0 \\ -(1+2\gamma)(1 + |q|)^{2\gamma} & q < 0 \end{cases}$$ (2.9)

For the application of Prop. 2.1 let us now distinguish between the exterior and interior by introducing the notation:

$$\Sigma^e_t = \{ (t, x) : |x| \leq t \} \quad \Sigma^i_t = \Sigma_t \setminus \Sigma^e_t$$ (2.10)

$$D^i = \{ (t, x) \in D : |x| \leq t \} \quad D^e = D \setminus D^i$$ (2.11)

We obtain

$$\int_{\Sigma^e_t} |\partial \phi|^2 dx + \int_{\Sigma^i_t} |\partial \phi|^2 w(q) dx + \int_D \frac{1+2\gamma}{4}(1 + |q|)^{2\gamma} |\partial \phi|^2 dxdt + \int_{D^e} \frac{|\partial \phi|^2 \mu / 2}{(1 + |q|)^{1+2\mu}} dx dt$$

$$\leq \int_{\Sigma^i_t} w(q)|\partial \phi|^2 dx + 2 \int_{\Sigma^e_t} |\partial \phi|^2 dx + 2 \int_D Fw(q) \partial_t \phi dx dt$$

which implies the statement of the Corollary. \qed

Remark 2.1. The choice of $w(q)$ is inferred from the forward problem with the role of the exterior $q > 0$ and the interior $q < 0$ interchanged $[25]$.  

2.2 Conformal energy estimates from infinity

In this section we prove a “fractional Morawetz estimate” for the backward problem.

For any $s \geq 1$ consider the following norm on $\Sigma_t$,

$$\|\phi(t, \cdot)\|_{1+s, -1}^2 := \int_{\Sigma_t} (t + r)^{2s} \left( |(\partial_t + \partial_r) (r \phi)|^2 + |\nabla (r \phi)|^2 \right)$$

$$+ (t - r)^{2s} \left( |(\partial_t - \partial_r) (r \phi)|^2 + \phi^2 + \frac{(r \phi)^2}{(t-r)^2} \right) \frac{dx}{r^2}$$ (2.12)

and the corresponding flux through the cones $C_R$:

$$F^s_R(t_1, t_2) := \int_{t_1}^{t_2} \int_{\partial \Sigma_t} (t + r)^{2s} |(\partial_t + \partial_r) (r \phi)|^2 + (t - r)^{2s} |\nabla (r \phi)|^2 dS(\omega) dt$$ (2.13)

Proposition 2.4 (Fractional Morawetz estimate from infinity). Let $t_1 \leq t_2$, then for $s \geq 1$

$$\|\phi(t_1, \cdot)\|_{1+s, -1} + \sup_{R} F^s_R(t_1, t_2)^{1/2} \leq \|\phi(t_2, \cdot)\|_{1+s, -1} + \int_{t_1}^{t_2} \|\phi(t_2, \cdot)^{s} \Box \phi(t_1, \cdot)\|_{L^2} dt$$ (2.14)

provided that $\lim_{R \to \infty} \sup_{\partial \Sigma^{\infty}_{t_1}} r^{s+1/2} |\phi| = 0$. Moreover

$$\|\phi(t_1, \cdot)\|_{1+s, -1} := \sum_{|I| \leq 1} \|\phi(t_1, \cdot)^{s-1} (Z^I \phi)(t_1, \cdot)\|_{L^2} \lesssim \|\phi(t_1, \cdot)\|_{1+s, -1}. \quad (2.15)$$
This result is central to this paper and follows from the following generalisation of the classical conformal Morawetz estimate for the wave equation on Minkowski space.

Consider the following “conformal energy”

\[
E_R^s(t) := \int_{\Sigma^t_R} \langle t+r \rangle^{2s} |(\partial_t + \partial_r)(r \phi)|^2 + \langle (t+r)^{2s} + (t-r)^{2s} \rangle |\nabla (r \phi)|^2 + \langle (t-r)^{2s} \rangle |(\partial_t - \partial_r)(r \phi)|^2 \frac{dx}{r^2}
\]

(2.16)

**Proposition 2.5.** For \( s \geq 1 \) let \( E_R^s \) and \( F_R^s \) be defined by (2.16), and (2.13) respectively. Then for \( t_1 \leq t_2 \)

\[
E_R^s(t_2-t_1)(t_1) + F_R^s(t_1, t_2) = E_R^s(t_2) + \int_{t_1}^{t_2} \int_{\Sigma^t_R(t_2-t)} r^{-1} X(r \phi) \Box \phi - r \partial_r \Omega |\nabla \phi|^2 dx dt
\]

(2.17)

where \( X = \langle t+r \rangle^{2s}(\partial_t + \partial_r) + \langle (t-r)^{2s} \rangle |(\partial_t - \partial_r)| \) and \( \Omega = \langle (t+r)^{2s} - (t-r)^{2s} \rangle / r \).

For the proof, we consider a more general class of energy identities associated to the following generalisation of the following multipliers \( X \), and functions \( f \in C^3(\mathbb{R}) \):

\[
X = L_+ + L_-, \quad L_\pm = f(t \pm r)(\partial_t \pm \partial_r), \quad \partial_r = \omega^t \partial_t.
\]

(2.18)

The classical conformal Morawetz vectorfield corresponds to the choice \( f(v) = v^2 \), while for the fractional Morawetz estimates we will choose

\[
f(v) = \frac{1}{a}(1 + v^2)^{a/2}, \quad a \geq 2.
\]

(2.19)

For this section also see [23] where the forward version of the above theorem was proved. Let \( T_{\alpha \beta} \) be an energy momentum tensor for the linear wave equation,

\[
T_{\alpha \beta} = \phi_\alpha \phi_\beta - \frac{1}{2} m_{\alpha \beta} m^{\mu \nu} \phi_\mu \phi_\nu, \quad \phi_\alpha = \partial_\alpha \phi, \quad \partial_\alpha T_{\alpha \beta} = (\Box \phi) \phi_\beta, \quad T_{\alpha} = -m^\alpha_\beta \phi_\alpha \phi_\beta,
\]

where \( \Box = m^\alpha_\beta \partial_\alpha \partial_\beta \). Then for any vectorfield \( X \),

\[
\partial_\alpha P_{\alpha} = (\partial_\alpha T_{\alpha \beta}) X^\beta + \frac{1}{2} T_{\alpha \beta}^\alpha (\nabla_\alpha \phi_\beta), \quad P_{\alpha} = T_{\alpha \beta} X^\beta \quad \text{and} \quad (\nabla_\alpha \phi_\beta) = \partial_\alpha X_\beta + \partial_\beta X_\alpha
\]

**Lemma 2.6.** With \( X \) defined by (2.18) we have the identity

\[
\frac{(\nabla_\alpha \phi_\beta)}{2} T_{\alpha \beta} = \left( \frac{f(t+r) - f(t-r)}{r} - f'(t+r) - f'(t-r) \right) |\nabla \phi|^2 - \frac{f(t+r) - f(t-r)}{r} m^{\mu \nu} \phi_\mu \phi_\nu
\]

Proof. With \( L = \partial_t + \partial_r \) and \( L = \partial_t - \partial_r \) we compute

\[
(L_+) \pi_{\alpha \beta} = f(t+r) (\partial_\alpha L_\beta + \partial_\beta L_\alpha) - f'(t+r) (L_\alpha L_\beta + L_\beta L_\alpha)
\]

\[
(L_-) \pi_{\alpha \beta} = f(t-r) (\partial_\alpha L_\beta + \partial_\beta L_\alpha) - f'(t-r) (L_\alpha L_\beta + L_\beta L_\alpha)
\]
where we used that $L_\alpha = -L^\alpha$. Moreover, recall that $m_{\alpha\beta} = -(L_\alpha L_\beta + L_\beta L_\alpha)/2 + \eta_{\alpha\beta}$, where $\eta^{\alpha\beta} \phi_\alpha \phi_\beta = (\delta^{ij} - \omega^i \omega^j) \phi_i \phi_j = |\nabla \phi|^2$ is the angular gradient. Furthermore $\partial_\alpha L_\beta = \partial_\beta L_\alpha = 0$ and $\partial_\alpha L_j = \partial_\alpha \omega_j = (\delta_{ij} - \omega^i \omega_j)/r = \eta^{ij}/r$. Hence $\partial_\alpha L_\beta = \eta^{\alpha\beta}$ and

$$\frac{(L_\pm)_{\alpha\beta}}{2} = \pm \frac{f(t \pm r)}{r} \eta_{\alpha\beta} - f'(t \pm r)(\eta_{\alpha\beta} - m_{\alpha\beta})$$

Here

$$\frac{(\eta_{\alpha\beta} - m_{\alpha\beta}) T^{\alpha\beta}}{2} = \frac{(L_\alpha L_\beta + L_\beta L_\alpha)}{2} T^{\alpha\beta} = |\nabla \phi|^2$$

It follows that

$$\frac{(L_\pm)_{\alpha\beta}}{2} T^{\alpha\beta} = \left( \pm \frac{f(t \pm r)}{r} - f'(t \pm r) \right) |\nabla \phi|^2 + \frac{f(t \pm r)}{r} m^{\mu\nu} \phi_\mu \phi_\nu$$

which proves the lemma. □

We can rewrite the identity of the previous Lemma as

$$\frac{(X)_{\alpha\beta}}{2} T^{\alpha\beta} = (-r \partial_r \Omega)|\nabla \phi|^2 - \Omega m^{\mu\nu} \phi_\mu \phi_\nu,$$

where $\Omega(t, r) = \frac{f(t + r) - f(t - r)}{r}$ (2.20)

and thus

$$\partial_\alpha P^\alpha = (\Box \phi) X^\alpha \phi_\alpha - r(\partial_r \Omega)|\nabla \phi|^2 - \Omega m^{\mu\nu} \phi_\mu \phi_\nu$$

Moreover, observe that $\Box \Omega = 0$, and hence

$$\partial_\alpha \tilde{P}^\alpha = (\Omega \phi + X^\alpha \phi_\alpha) \Box \phi + (-r \partial_r \Omega)|\nabla \phi|^2,$$

where $\tilde{P}^\alpha = P^\alpha + \Omega \phi \phi_\alpha - \frac{1}{2} \phi^2 \Omega^a$ (2.22)

Also note that $Xr = f(t + r) - f(t - r)$, hence $(\Omega + X^\alpha \partial_\alpha) \phi = r^{-1} X^\alpha \partial_\alpha (r \phi)$.

Let us summarize the divergence properties of the modified current $\tilde{P}_\alpha$ as follows:

**Lemma 2.7.** Let the current $\tilde{P}_\alpha$ be defined by (2.22), where $X$ is chosen as in (2.18), and $\Omega$ defined by (2.20), then we have

$$\partial^\alpha \tilde{P}_\alpha = r^{-1} X(r \phi) \Box \phi - r \partial_r \Omega|\nabla \phi|^2$$

Let us now turn to the flux terms

$$\tilde{P}_\alpha \partial_\alpha = T(X, \partial_t) + \Omega \phi \partial_t \phi - \frac{1}{2} \phi^2 \partial_t \Omega$$

(2.23)

$$\tilde{P}_\alpha L^\alpha = T(X, L) + \Omega \phi L \phi - \frac{1}{2} \phi^2 L \Omega$$

(2.24)

**Lemma 2.8.** Let $C^+_R$ the outgoing null hypersurface from $\partial \Sigma^R_{t_1}$, truncated at $t = t_2$, then for any differentiable function $f$, we have with $X$, $\Omega$ and $\tilde{P}$ as defined above:

$$\frac{1}{2} \int_{\Sigma^R_{t_1}} \left[ f(t + r)(L(r \phi))^2 + f(t - r)(\bar{L}(r \phi))^2 + (f(t + r) + f(t - r)) r^2|\nabla \phi|^2 \right] \frac{dx}{r^2}$$

$$+ \int_{C^+_R} \left[ f(t + r)(L(r \phi))^2 + f(t - r) r^2|\nabla \phi|^2 \right] \frac{d\omega dv}{r^2} =$$

$$= \int_{\Sigma^R_{t_1}} \tilde{P} \cdot \partial_t dx + \int_{C^+_R} \star \tilde{P}^\alpha + \frac{1}{2} \int_{S^2} r (f(t - r) + f(t + r)) \phi^2 d\omega |_{t = t_2, r = R + (t_2 - t_1)}$$
Proof. First $\partial_t = (L + L)/2$, and with $X = f(t + r)L + f(t - r)L$ we have

$$T(X, L) = f(t + r)(L\phi)^2 + f(t - r)|\nabla \phi|^2$$

$$T(X, L) = f(t + r)|\nabla \phi|^2 + f(t - r)(L\phi)^2$$

$$T(X, \partial_t) = \frac{1}{2} f(t + r)(L\phi)^2 + \frac{1}{2} \left(f(t + r) + f(t - r)\right)|\nabla \phi|^2 + \frac{1}{2} f(t - r)(L\phi)^2$$

Secondly note that

$$\Omega \partial_t \phi = \frac{f(t + r) - f(t - r)}{r} \partial_t \phi$$

$$= \frac{f(t + r)}{r} L\phi - \frac{f(t - r)}{r} L\phi - \frac{f(t + r) + f(t - r)}{r} \partial_r \phi$$

Now on one hand

$$- \int_{\Sigma_t} \frac{f(t + r) + f(t - r)}{r} \phi \partial_r \phi \, dx = - \frac{1}{2} \int_{\Sigma_t} r(f(t + r) + f(t - r)) \partial_r \phi^2 \, d\omega dr$$

$$= \frac{1}{2} \int_{\Sigma_t} \left[ \frac{1}{r} (f'(t + r) - f'(t - r)) + \frac{f(t + r) + f(t - r)}{r^2} \right] \phi^2 \, dx - \frac{1}{2} \int_{\Sigma_t} \partial_r \left(r(f(t + r) + f(t - r))\phi^2\right) \, d\omega dr$$

and on the other hand

$$\partial_t \Omega = \frac{f'(t + r) - f'(t - r)}{r}$$

Therefore

$$\int_{\Sigma_t} \tilde{P}_r \partial_t^r + \frac{1}{2} \int_{\Sigma_t} \partial_r \left(r(f(t + r) + f(t - r))\phi^2\right) \, d\omega dr$$

$$= \frac{1}{2} \int_{\Sigma_t} \frac{1}{r^2} \left[ (f(t + r)(rL\phi)^2 + f(t - r)(rL\phi)^2 + (f(t + r) + f(t - r))r^2|\nabla \phi|^2 \right.$$

$$+ 2f(t + r)r\phi L\phi - 2f(t - r)r\phi L\phi + (f(t + r) + f(t - r))\phi^2 \right] \, dx$$

$$= \frac{1}{2} \int_{\Sigma_t} \frac{1}{r^2} \left[ (f(t + r)(rL\phi + \phi)^2 + f(t - r)(rL\phi - \phi)^2 + (f(t + r) + f(t - r))r^2|\nabla \phi|^2 \right]$$

which proves the formula:

$$\frac{1}{2} \int_{\Sigma_t} \left[ (f(t + r)(L(r\phi))^2 + f(t - r)(L(r\phi))^2 + (f(t + r) + f(t - r))r^2|\nabla \phi|^2 \right] \frac{dx}{r^2}$$

$$= \int_{\Sigma_t} \tilde{P} \cdot \partial_t \, dx + \frac{1}{2} \int_{\partial \Sigma_t} r(f(t + r) + f(t - r))\phi^2 d\omega \quad (2.25)$$

Regarding the null flux, we proceed similarly by writing

$$\Omega L\phi = \frac{2f(t + r)}{r} L\phi - \frac{f(t + r) + f(t - r)}{r} L\phi$$

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and integrating by parts, on the null hypersurface $C_R$ truncated by $t = t_2$, and $t = t_1$,

$$
- \frac{1}{2} \int_{C_R} \frac{f(t + r) + f(t - r)}{r} L(\phi^2) = - \frac{1}{2} \int_{v_1}^{v_2} \frac{f(2v) + f(2u)}{r} \partial_v (\phi^2) r^2 d\omega dv
$$

$$
= - \frac{1}{2} \int_{S^2} r(f(t - r) + f(t + r)) \phi^2 d\omega |_{v_1}^{v_2} + \frac{1}{2} \int_{C_R} \left[ \frac{f(t + r) + f(t - r)}{r^2} + \frac{2f'(t + r)}{r} \right] \phi^2 d\omega
$$

where we denoted by $2u = t - r$, $2v = t + r$, hence $L = \partial_v$, and $r = v - u$. Since

$$
L \Omega = \frac{2f'(t + r)}{r} - \frac{f(t + r) - f(t - r)}{r^2}
$$

we obtain

$$
\int_{C_R} \ast \tilde{P} = \int_{v_1}^{v_2} \tilde{P} \cdot L r^2 d\omega dv = \int_{v_1}^{v_2} \left\{ f(t + r)(L\phi)^2 + f(t - r)|\nabla \phi|^2
\right. \\
+ \frac{2f(t + r)}{r} \phi L \phi + \left. \frac{f(t + r)}{r^2} \phi^2 \right\} r^2 d\omega dv - \frac{1}{2} \int_{S^2} r(f(t - r) + f(t + r)) \phi^2 d\omega |_{v_1}^{v_2}
$$

which proves the identity:

$$
\int_{v_1}^{v_2} \left[ f(t + r)(L\phi)^2 + f(t - r)r^2|\nabla \phi|^2 \right] d\omega dv =
$$

$$
= \int_{C_R} \ast \tilde{P} + \frac{1}{2} \int_{S^2} r(f(t - r) + f(t + r)) \phi^2 d\omega |_{v_1}^{v_2} \quad (2.26)
$$

Adding the formulas (2.25) and (2.26) we see that the boundary terms at $\partial \Sigma_1^R$ exactly cancel, and we obtain the statement of the Lemma. □

We can now prove the identity (2.17).

**Proof of Proposition 2.5.** By Stokes theorem applied to the 1-form $\tilde{P}$ on the domain $D$ we have

$$
\int_{\Sigma_2^R} \tilde{P} \cdot d\xi + \int_{t_1}^{t_2} \int_{\Sigma_{t}^{R-(t_2-t)}} \partial^\alpha \tilde{P} d\alpha dt = \int_{\Sigma_{t_1}^{R-(t_2-t_1)}} \tilde{P} \cdot \partial^\alpha d\alpha + \int_{C_R} \ast \tilde{P}
$$

A formula for the divergence of $\tilde{P}$ is given in Lemma 2.7. Moreover the right hand side of the above identity is computed in Lemma 2.8, see also (2.25) for the flux term on the left hand side, and note thus that the boundary terms at $\partial \Sigma_{t_2}^R$ cancel.

Now set $f(v) = \langle v \rangle^2$, $s \geq 1$ as in (2.19) with $a = 2s$. Then the statement of the Proposition follows. □

To prove Proposition 2.4 we exploit that with our choice of $f$ the “bulk” has the correct sign for the backward problem.
Lemma 2.9. Suppose that \( f \) is a function such that
\[
f''(v) + f''(-v) \geq 0, \quad \text{and} \quad f'''(v) \geq 0, \quad \text{for} \quad v \geq 0. \tag{2.27}
\]
Then for \( t \geq 0 \) and \( r \geq 0 \)
\[
\frac{1}{r}(f(t+r) - f(t-r)) - (f'(t+r) + f'(t-r)) \leq 0 \tag{2.28}
\]
In particular, this condition \((2.27)\) is true for \( f(v) = (1+v^2)^{a/2} \), if \( a \geq 2 \).

Proof. We have
\[
f(t+r) - f(t-r) = \int_{t-r}^{t+r} f'(v) \, dv = (v-t)f'(v)igg|_{t-r}^{t+r} - \int_{t-r}^{t+r} (v-t)f''(v) \, dv \tag{2.29}
\]
and it follows that
\[
\frac{1}{r}(f(t+r) - f(t-r)) - (f'(t+r) + f'(t-r)) = -\frac{1}{r} \int_{t-r}^{t+r} (v-t)f'''(v) \, dv. \tag{2.30}
\]
Integrating by parts once more we see that
\[
\frac{1}{r}(f(t+r) - f(t-r)) - (f'(t+r) + f'(t-r)) = \int_{t-r}^{t+r} \frac{(v-t)^2 - r^2}{2r} f'''(v) \, dv \tag{2.31}
\]
Since \((v-t)^2 - r^2 \leq 0\), when \( t-r \leq v \leq t+r \), \((2.28)\) follows directly if \( t \geq r \). When \( t \leq r \) we divide the domain of integration up into two parts. In the region \([t-r, t+r]\) we use that \( f'''(v) \geq 0 \) as before. In the region \([t-r, r-t]\) we will use both inequalities in \((2.27)\). We have
\[
\int_{r-t}^{t-r} 2r \, dv = \int_{0}^{r-t} 2r \, f'''(v) \, dv + \int_{0}^{t-r} 2r \, f'''(v) \, dv
\]
\[
= \int_{0}^{r-t} 2r \, f'''(v) \, dv + \int_{0}^{t-r} 2r \, f'''(v) \, dv \leq 0. \tag{2.32}
\]
If \( f(v) = (1+v^2)^{a/2}/a \) then \( f'(v) = v(1+v^2)^{a/2-1}/a \)
\( f''(v) = (1+v^2)^{a/2-1} + (1+v^2)^{a/2-2}v^2(a-2) \)
and \( f'''(v) = (a-2)3v(1+v^2)^{a/2-2} + (1+v^2)^{a/2-3}(a-2)(a-4)v^3 \). Taking out a common factor we get \( f'''(v) = (a-2)v(1+v^2)^{a/2-3}(3(1+v^2) + (a-4)v^2) \). Hence
\[
f'''(v) = (a-2)v(1+v^2)^{a/2-3}((a-1)v^2 + 3) \geq 0, \quad \text{if} \quad v \geq 0 \quad \text{and} \quad a \geq 2.
\]
If \( f(-v) = f(v) \) then \( f'''(-v) = -f'''(v) \) so \((2.27)\) hold for \( f(v) = (1+v^2)^{a/2}/a \), if \( a \geq 2 \) \( \square \).

In Proposition \(2.4\) we also control the zeroth order term which we obtain by a weighted Hardy inequality.

Lemma 2.10. Let \( f \geq 0 \) be an even twice differentiable function on \( \mathbb{R} \), such that \( f'(v) \geq 0 \) for \( v \geq 0 \), and \( f''(v) \leq C f'''(v) f(v) \). Then
\[
\int_{\Sigma^2} f''(t-r) \phi^2 \, dx \leq \int_{\Sigma^t} \left[ f(t+r) (Lr\phi)^2 + f(t-r) (Lr\phi)^2 \right] \frac{dx}{r^2} + \int_{\partial \Sigma^2} |f| \phi^2 r^2 \, dw
\]
Proof. With $\psi = r\phi$ we have

$$\int_{\Sigma_t^R} f''(t-r)\phi^2 \, dx = \int_0^R \int_{S^2} f''(t-r)\psi^2 \, dr \, d\omega$$

and

$$\int_0^R f''(t-r)\psi^2 \, dr = -\int_0^R \frac{d}{dr} [f'(t-r)] \psi^2 \, dr = -f'(t-r)\psi^2 \bigg|_{r=0}^R + \int_0^R f'(t-r)2\psi \, dr$$

$$\leq |f'(t-R)|\psi^2(R) + \sqrt{\int_0^R \frac{f'(t-r)^2}{f(t-r)} \psi^2 \, dr} \sqrt{\int_0^R f(t-r)\psi^2 \, dr}$$

By the assumption on $f$, namely $f'(q)/f(q) \leq Cf''(q)$, it follows that

$$\int_0^R f''(t-r)\psi^2 \, dr \lesssim |f'(t-R)|\psi^2(R) + \int_0^R f(t-r)\psi^2 \, dr$$

and hence

$$\int_{\Sigma_t^R} f''(t-r)\phi^2 \, dx \lesssim \int_{\partial\Sigma_t^R} |f'|\phi^2 r^2 \, d\omega + \int_{\Sigma_t^R} f(t-r)(\partial_r(r\phi))^2 \frac{dr}{r^2}$$

By assumption $f$ is even and increasing on $\mathbb{R}_+$, hence $f(t+r) \geq f(t-r)$ and

$$\frac{1}{2} \int_{\Sigma_t^R} \left[f(t+r)(L(r\phi))^2 + f(t-r)(L(r\phi))^2\right] \frac{dx}{r^2} \geq \int_{\Sigma_t^R} f(t-r)(\partial_r(r\phi))^2 \frac{dx}{r^2}$$

Lemma 2.11. We have

$$\int_{\Sigma_t} (t-r)^{2s} \phi^2 \frac{dx}{r^2} \lesssim \int_{\Sigma_t} (t-r)^{2s-2} \phi^2 \, dx + \int_{\Sigma_t} (t-r)^{2s}(\partial_r(r\phi))^2 \frac{dx}{r^2}$$

provided $\lim_{R \to \infty} \sup_{\partial\Sigma_t^R} (t-r)^{2s} r \phi^2 = 0$.

Proof. We have $\phi^2 = 2\phi \partial_r(r\phi) - \partial_r(r\phi^2)$ and hence

$$\int_0^R (t-r)^{2s} \phi^2 \, dr = \int_0^R 2(t-r)^{2s} \phi \partial_r(r\phi) + (\partial_r(t-r)^{2s}) r \phi^2 \, dr - (t-r)^{2s} r \phi^2 \bigg|_0^R$$

By assumption the boundary term vanishes, and after integrating over the sphere we obtain by Cauchy-Schwarz that

$$\int_0^\infty \int_{S^2} (t-r)^{2s} \phi^2 dS(\omega) \, dr \lesssim \left( \int \int (t-r)^{2s} \phi^2 dS(\omega) \, dr \right)^{\frac{1}{2}} \times$$

$$\times \left( \int \int (t-r)^{2s}(\partial_r(r\phi))^2 + (t-r)^{2s-2}r \phi^2 dS(\omega) \, dr \right)$$

where we used that $|\partial_r(t-r)^{2s}| \leq 2s(t-r)^{2s-1}$. This proves the Lemma. \qed
Proof of Proposition 2.4. Recall that we have chosen \( f(v) = \langle v \rangle^{2s} \), \( s \geq 1 \) to obtain (2.17). Therefore by Lemma 2.9 \((-r \partial_t \Omega) \leq 0\), and thus

\[
E_{R-\langle t_2-t_1 \rangle}(t_1) + F_{R}^{s}(t_1, t_2) \leq E_{R}^{s}(t_2) + \int_{t_1}^{t_2} \int_{\Sigma_{t}} \int_{S} f \big( r \partial_{t} \phi \big) \frac{dx}{t^2} dt
\]

Moreover,

\[
\int_{\Sigma_{t}} \frac{1}{r} \big( X(r\phi) \big) \frac{\partial r}{t} \frac{dx}{r^2} \leq \left( \int_{\Sigma_{t}} \langle t+r \rangle^{2s} (L(r\phi)) \frac{2 \frac{dx}{r^2}} {r^2} \right)^{\frac{1}{2}} \| \langle t+r \rangle^{s} \partial_{r} \phi \|_{L^{2}(\Sigma_{t})} \]

\[
+ \left( \int_{\Sigma_{t}} \langle t-r \rangle^{2s} (L(r\phi)) \frac{2 \frac{dx}{r^2}} {r^2} \right)^{\frac{1}{2}} \| \langle t-r \rangle^{s} \partial_{r} \phi \|_{L^{2}(\Sigma_{t})} \leq \left( E_{R}^{s}(t) \right)^{\frac{1}{2}} \| \langle t+r \rangle^{s} \partial_{r} \phi \|_{L^{2}(\Sigma_{t})}
\]

Thus to infer the estimate in Prop. 2.5 from (2.17), when \( R \to \infty \), it remains to control the zeroth order term. Note that with our choice of \( f \), \( f''(v) \geq 2s(v)^{2s-2} \), and \( |f'(v)| \leq 2s(v)^{2s-1} \). Hence by Lemma 2.10

\[
\int_{\Sigma_{t}} \langle t-r \rangle^{2s-2} \phi^{2} \frac{dx}{r^2} \leq E_{R}^{s}(t) + \int_{\partial \Sigma_{t}} \phi^{2} \langle t-r \rangle^{s-1} r \frac{dx}{r^2} dt
\]

Furthermore, in the limit \( R \to \infty \), the boundary term vanishes by assumption, and by Lemma 2.11

\[
\int_{\Sigma_{t}} \langle t-r \rangle^{2s-2} \phi^{2} \frac{dx}{r^2} \leq \int_{\Sigma_{t}} \langle t-r \rangle^{2s-2} \phi^{2} \frac{dx}{r^2} + \int_{\Sigma_{t}} \langle t-r \rangle^{2s} (\partial_{r}(r\phi))^{2} \frac{dx}{r^2}
\]

Here the last term is also controlled by \( E_{R}^{s}(t) \), as \( R \to \infty \), as demonstrated in the last step of the proof of Lemma 2.10. This shows the first estimate of the Proposition.

Finally let us turn to the bound (2.14). The estimate is obviously true for \( |I| = 0 \). For \( |I| = 1 \) we have \( Z \in \{ \partial_{t}, \partial_{t} \big( \Omega_{ij} \big), S, \Omega_{0i} \} \). For \( Z = \partial_{t} \) a time translation we simply have \( \partial_{t} \phi = \partial_{t}(r\phi)/r \), hence the inequality holds for \( Z = \partial_{t} \), in fact

\[
\| \langle t-r \rangle^{s-1} r \partial_{t} \phi \|^{2} \leq \| \phi \|^{2}_{1,+,s-1}
\]

For \( Z \) a rotation note that \( |\Omega_{ij} \phi|^{2} \leq |\nabla(r\phi)|^{2} \), hence the inequality holds true for \( Z = \Omega_{ij} \), in fact

\[
\| \langle t-r \rangle^{s-1} t \Omega_{ij} \phi \|^{2} \leq \int \langle t-r \rangle^{2s} \frac{t^{2}}{(t-r)^{2}} |\nabla(r\phi)|^{2} \frac{dx}{r^2} \leq \| \phi \|^{2}_{1,+,s-1}
\]

Moreover the inequality holds for scalings, \( Z = S = t \partial_{t} + r \partial_{r} \) because

\[
\int_{\Sigma_{t}} \langle t-r \rangle^{2s-2} (S\phi)^{2} \frac{dx}{r^2} \leq \int_{\Sigma_{t}} \langle t-r \rangle^{2s} \left( \frac{t^{2}}{(t-r)^{2}} (\partial_{t}(r\phi))^{2} + \frac{r^{2}}{(t-r)^{2}} (\partial_{r}(r\phi))^{2} \right) \frac{dx}{r^2} + \langle t-r \rangle^{2s-2} \phi^{2} \frac{dx}{r^2}
\]
Also note that
\[
\| (t-r)^{s-1}(t+r)\mathcal{L}\phi \|^2 + \| (t-r)^{s-1}(t-r)\mathcal{L}\phi \|^2 \lesssim \\
\lesssim \int (t+r)^{2s}(L(\phi))^2 + (t-r)^{2s}(\mathcal{L}(\phi))^2 + (t-r)^{2s}(\mathcal{L}(\phi))^2 \mathcal{L} \phi \frac{dx}{r^2} \lesssim \| \phi \|^2_{1+s-1}
\]
Since we can expand the translations \( Z = \partial_i = (\partial_t, \partial_r, \partial_r) + \nabla_i \) we have
\[
\int (t-r)^{2s-2}(\partial_i \phi)^2 dx \lesssim \int (t-r)^{2s-2}\{(\partial_t \phi)^2 + |\nabla \phi|^2\} dx \lesssim \\
\lesssim \int (t-r)^{2s-2}(S \phi)^2 + (t-r)^{2s-2}t^2 r^2 (\partial_t (r \phi))^2 + (t-r)^{2s-2} |\nabla (r \phi)|^2 \mathcal{L} \phi \frac{dx}{r^2}
\]
hence the estimate also holds for \( Z = \partial_i \). Finally we can express the boosts \( \Omega_{0i} = t \partial_i + x^i \partial_t \) as
\[
\Omega_{0i} = \frac{x^i}{r} (t \partial_r + r \partial_t) + t \nabla_i = \frac{1}{2} \frac{x^i}{r} ((t + r)L - (t-r)L) + t \nabla_i
\]
and conclude that also \( Z = \Omega_{0i} \) satisfies the inequality:
\[
\| (t-r)^{s-1} \Omega_{0i} \phi \| \lesssim \| (t-r)^{s-1}(t+r)\mathcal{L} \phi \| + \| (t-r)^{s-1}(t-r)\mathcal{L} \phi \| \\
+ \| (t-r)^{s-1} \frac{t}{r} |\nabla (r \phi)|\| \lesssim \| \phi \|_{1+s-1}
\]
This proves (2.14) for \( |I| = 1 \), and thus the Proposition.

\[
\square
\]

3 Decay estimates

For the examples of semilinear wave equations considered in this paper, we will separate the solution
\[
u = u_0 + v
\]
in a linear, or more generally known or prescribed part \( u_0 \), and a remainder \( v \). (For example in Section 4 \( u \) satisfies the classical wave equation with null condition, and \( u_0 \) is a solution to the linear wave equation with the same data at infinity.) Moreover for scattering from infinity, initial data for \( u \) is prescribed only from \( u_0 \). We then derive an equation for \( v = u - u_0 \), and consider the problem with trivial initial data on \( \Sigma_T \) as \( T \to \infty \); in particular we will assume that
\[
\lim_{T \to \infty} \int_{\Sigma_T} |\partial v|^2 w dx = 0 .
\]
The known solution \( u_0 \) has non-trivial initial data at \( t = T \), and we will assume that
\[
D_k := \lim_{T \to \infty} \sum_{|I| \leq k} \int_{\Sigma_T} |\partial Z^I u_0|^2 w dx < \infty
\]
for some \( k \geq 5 \), where \( Z^I \) denotes a string of commuting vectorfields of length \( |I| \), and \( w = w(|x| - t) \) is a given weight function. As part of the proof we will show that in fact the energy for \( v \) decays
\[
\int_{\Sigma_T} |\partial v|^2 w dx \lesssim \frac{\varepsilon}{t^{\delta}},
\]
and this can be used to get improved decay estimates for \( v \) compared to \( u_0 \).
3.1 Classical decay estimates

Here \( w \equiv 1 \), and \( u_0 \) is a solution to the linear wave equation. Then the norms in (3.3) are of course preserved, and as a result of the Klainerman Sobolev inequality we have

\[
(1 + t + r)(1 + |t - r|)^{\frac{3}{2}}|\partial u_0| \leq CD_2
\] (3.5)

Another standard application of the Klainerman Sobolev inequality is the improved pointwise bound on the tangential derivatives to the outgoing cone.

**Lemma 3.1.** Suppose that \( v(t, x) \to 0 \) as \( |x| \to \infty \). Then

\[
\sup_{\Sigma_t} |v| \lesssim \frac{1}{(1 + t + r)^{\frac{3}{2}}} \sum_{|I| \leq 2} \|Z^I v\|_{L^2(\Sigma_t)}, \quad \sup_{\Sigma_t} |\bar{\partial} v| \lesssim \frac{1}{(1 + t + r)^{\frac{3}{2}}} \sum_{|I| \leq 3} \|\partial Z^I v\|_{L^2(\Sigma_t)}.
\] (3.6)

**Proof.** By the Klainerman-Sobolev inequality

\[
|v(t, r\omega)| \leq \int_r^\infty |\bar{\partial} v(t, s\omega)| ds \leq C_S \sum_{|I| \leq 2} \|Z^I \bar{\partial} v\|_{L^2(\Sigma_t)} \int_r^\infty \frac{ds}{(1 + t + s)(1 + |t - s|)^{\frac{3}{2}}} \leq \frac{C_S}{(1 + t + r)^{\frac{3}{2}}} \sum_{|I| \leq 2} \|\partial Z^I v\|_{L^2(\Sigma_t)}
\]

and thus by a well-known pointwise identity for \( \bar{\partial} v \) in terms of \( Z v \),

\[
|\bar{\partial} v| \leq C \sum_{|I| = 1} \frac{|Z^I v|}{1 + t + r} \leq \frac{CC_S}{(1 + t + r)^{\frac{3}{2}}} \sum_{|I| \leq 3} \|\partial Z^I v\|_{L^2(\Sigma_t)}.
\]

\[\square\]

When we construct a solution from data on \( \Sigma_T \), and take \( T \to \infty \), we cannot a priori assume that the solution is compactly supported on all \( \Sigma_t, t \leq T \). A variant of the above proof applies even if the solution is compactly supported only on \( \Sigma_T \).

**Lemma 3.2.** Let \( T > 0 \) and \( u \in C^\infty(\mathbb{R}^{3+1}) \), and assume that \( u \) is compactly supported on \( \Sigma_T \), while for all \( t \leq T \)

\[
\sum_{|I| \leq 3} \|\partial Z^I u\|_{L^2(\Sigma_t)} \leq C
\] (3.7)

Then for all \( t \leq T \)

\[
\sup_{\Sigma_t} |\bar{\partial} u| \lesssim \frac{C}{(1 + t)^{\frac{3}{2}}}
\] (3.8)

**Proof.** We integrate from the point \( (t, r\omega) \) along the outward directed line

\[
\gamma(\lambda) : \lambda \mapsto (t + \lambda, (r + \sigma\lambda)\omega)
\]

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for some $\sigma > 1$, until it hits $t = T$; then for $|I| = 1$,

\[
|Z^I u(t, r\omega)| \leq |Z^I u(\gamma(T - t)) + \int_0^{T-t} \sigma|\partial Z^I u(\gamma(\lambda))|d\lambda
\]

\[
\leq CC_S \int_0^\infty \frac{\sigma d\lambda}{(1 + t + r + (1 + \sigma)\lambda)(1 + r - t + (\sigma - 1)\lambda)^{1/2}} \leq CC_S \frac{1}{(1 + t)^{3/2}}
\]

and therefore

\[
|\overline{\partial u}| \leq C \sum_{|I| = 1} \frac{|Z^I v|}{1 + t + r} \leq CC_S \frac{1}{(1 + t + r)(1 + t)^{3/2}}.
\]

\[\square\]

### 3.2 Weighted interior decay estimates

Another way that gives improved interior decay — if we have the stronger interior energy estimate — is to use that for any $R$,

\[
\int_{t_1}^{t_2} \int_{\partial \Sigma^I_{t_1} \cap (t_2 - t)} |\overline{\partial} Z^I v|^2 w_\gamma^2 dS dt \leq \frac{C'}{t_1^{2\gamma}}
\]

(3.9)

If we pick $R = t_2 - t_1$ then this gives us an estimate for $v(t_1, 0)$. Once we have an estimate for $v$ at the origin we can integrate from the origin and get a better decay estimate everywhere in the interior.

**Lemma 3.3.** Suppose that $v = \partial_t v = 0$ when $t = t_2$. Then

\[
t_1^{1/2} |w_\gamma^{1/2}|v(t_1, x)| \lesssim \sum_{|I| \leq 3} \left( \int_{t_1}^{t_2} \int_{\partial \Sigma^I_{t_1} \cap (t_2 - t)} |\overline{\partial} Z^I v|^2 w_\gamma^2 dS dt + \int_{\Sigma_{t_1}} |\partial Z^I v|^2 w_\gamma^2 dx \right)^{1/2}.
\]

(3.10)

We first give the estimate at the origin.

**Lemma 3.4.** Suppose that $v = \partial_t v = 0$ for $t = t_2$, $t_2 > t_1$. Then

\[
t_1^{1+\gamma}|v(t_1, 0)| \lesssim \sum_{|I| \leq 3} \left( \int_{t_1}^{t_2} \int_{\partial \Sigma^I_{t_1} \cap (t_2 - t)} |\overline{\partial} Z^I v|^2 w_\gamma^2 dS dt \right)^{1/2}.
\]

(3.11)

**Proof.** First note that on $C_R$, $R = t_2 - t_1$, we have $w_\gamma = (1 + t_1)^{1+2\gamma}$, so it is enough to prove it for $\gamma = -1/2$.

Now, on one hand we have with $V_+(\xi) := (\partial_t + \partial_\xi)(rv)(t_1 + \xi, \xi \omega)$ that

\[
|v(t_1, 0)| = |V_+(0)| \leq \int_0^{t_2-t_1} |\partial_\xi V_+| d\xi
\]

\[
\leq \left( \int_0^{t_2-t_1} |(t_2 + \xi)\partial_\xi V_+|^2 d\xi \right)^{1/2} \leq \frac{1}{t_1^{1/2}} \left( \int_0^{t_2-t_1} |(t_2 + \xi)\partial_\xi V_+|^2 d\xi \right)^{1/2}
\]
Here \((t + \xi)\partial_\xi V_\perp(r) = (t + r)(\partial_t + \partial_r)(\partial_t + \partial_r)(rv)\) so taking the supremum over angles and using the Sobolev lemma on the sphere,

\[
t_1^{1/2}|v(t_1, 0)| \lesssim \sum_{|I| \leq 2} \left( \int_{t_1}^{t_2} \int_{\partial \Sigma_{t_1}} |\Omega^I(t + r)(\partial_t + \partial_r)(\partial_t + \partial_r)(rv)|^2 dS(\omega) dr \right)^{1/2}
\]

The stated inequality then follows from the identity \(\int_0^t |\partial_\xi (\xi w)|^2 d\xi = \int_0^t |\partial_\xi w|^2 \xi d\xi + tw(t)^2\), obtained from expanding the derivative and integrating by parts. \(\Box\)

### 3.3 Decay estimates from the conformal energy

Finally we show a weighted version of the Klainerman Sobolev inequality that is adapted to the energy introduced in Section 2.2.

**Lemma 3.5.** For all \(\phi \in H^2(\Sigma_t)\) we have

\[
\sup_{\Sigma_t} \left( (t + r)(t - r)^{1/2 + s - 1} |\phi| \right) \lesssim \sum_{|I| \leq 2} \| (t - r)^{s-1} Z^I \phi(t, \cdot) \|_{L^2(\Sigma_t)}
\]

**Proof.** For \(s = 1\) this is the standard Klainerman Sobolev inequality. For \(s > 1\), let us then apply the inequality in the case \(s = 1\) to the function \((t - r)^{-1}\phi:\)

\[
(t + r)(t - r)^{1/2}(t - r)^{s-1} |\phi| \lesssim \sum_{|I| \leq 2} \| Z^I ( (t - r)^{-1}\phi) (t, \cdot) \|_{L^2(\Sigma_t)}
\]

Since

\[
Z(t - r)^{s-1} \leq (s - 1)(t - r)^{s-2}|Z(t - r)| \lesssim (t - r)^{s-1}
\]

and hence \(|Z^I (t - r)^{s-1}| \lesssim (t - r)^{s-1}\), for \(|I| \leq 2\), the statement of the Lemma follows. \(\Box\)

### 4 The classical null condition model

Let us start with a simple equation of the form

\[
\Box u = Q(\partial u, \partial u) \quad (4.1)
\]

where \(Q\) is a null form.

We wish to construct a solution with given data on \(t = T\). Let \(u = v + u_0\), where \(u_0\) is the linear solution which assumes the given data at \(t = T\). Then

\[
\Box v = Q(\partial(v + u_0), \partial(v + u_0))
= Q(\partial v, \partial v) + Q(\partial v, \partial u_0) + Q(\partial u_0, \partial v) + Q(\partial u_0, \partial u_0) \quad (4.2)
\]

and \(v\) has trivial data at \(t = T\).

Now the higher order variations of \(v\) satisfy

\[
\Box Z^I v = F_I(\partial v, \partial u_0) \quad (4.3)
\]

where \(F_I\) is quadratic in the \(Z\)-derivatives of \(\partial v\), and \(\partial u_0\) with the following property:
Lemma 4.1.

\[ |F_I| \lesssim \sum_{|I| \leq |J|+1, K \leq |I|} \left( |\partial Z^I v| (|\overline{\partial Z^K v}| + |\partial Z^K u_0|) + |\partial Z^J u_0| (|\overline{\partial Z^K u_0}| + |\overline{\partial Z^K v}|) \right) \]  \hspace{1cm} (4.4)

**Proof.** This follows from standard properties of null forms, namely that if \( Q \) is a null from then \( ZQ(\partial u, \partial v) = Q(\partial Z u, \partial v) + Q(\partial u, \partial Z v) + \tilde{Q}(\partial u, \partial v) \) where \( \tilde{Q} \) is another null form, and that for all null forms we have the pointwise bound: \(|Q(\partial u, \partial v)| \lesssim |\partial u||\partial v| + |\partial u||\partial v|\). \( \square \)

**Proposition 4.2.** Let \((u_0, \partial_t u_0)|_{t=T} \in C^\infty_0(\mathbb{R}^3)\) be initial data for (4.1) at \( t = T \), and assume that

\[ D := \lim_{T \to \infty} \sum_{|I| \leq 5} \left( \|\nabla Z^I u_0\|_{L^2(\Sigma_T)} + \|Z^I \partial_t u_0\|_{L^2(\Sigma_T)} \right) < \infty \]  \hspace{1cm} (4.5)

Then for any \( \mu > 0 \), there exists \( t_0 > 0 \) and a smooth solution \( u \) to (4.1) on \([t_0, \infty) \times \mathbb{R}^3\), of the form

\[ u = u_0 + v \]  \hspace{1cm} (4.6)

where \( u_0 \) is a linear wave with the prescribed scattering data, and \( v \) satisfies

\[ \sum_{|I| \leq 5} \|\partial Z^I v\|_{L^2(\Sigma_t)} \lesssim \frac{1}{t_0^{\frac{1}{2} - \mu}} \quad (t \geq t_0) \]  \hspace{1cm} (4.7)

**Proof.** We set

\[ u = u_0 + v \]

where \( u_0 \) is a solution to the linear equation with the given data on \( \Sigma_T \), and \( v \) is a solution to (4.2) with trivial data on \( \Sigma_T \).

Therefore by Cor. 2.2, for any \( \mu > 0 \), we have that

\[ \sum_{|I| \leq 5} \|\partial Z^I u_0\|_{L^2(\Sigma_t)} + \sum_{|I| \leq 5} \left( \int_t^T \int_{\mathbb{R}^3} \frac{\mu}{4} \frac{|\overline{\partial Z^I u_0}|^2}{(1 + |q|)^{1+2\mu}} \, dx \, dt \right)^{\frac{1}{2}} \leq \sum_{|I| \leq 5} \|\partial Z^I u_0\|_{L^2(\Sigma_T)} \]

and

\[ \sum_{|I| \leq 5} \|\partial Z^I v\|_{L^2(\Sigma_t)} + \sum_{|I| \leq 5} \left( \int_t^T \int_{\mathbb{R}^3} \frac{\mu}{4} \frac{|\overline{\partial Z^I v}|^2}{(1 + |q|)^{1+2\mu}} \, dx \, dt \right)^{\frac{1}{2}} \leq C \sum_{|I| \leq 5} \int_t^T \|F_I\|_{L^2(\Sigma_t)} \]

where \( F_I \) is estimated in Lemma 4.1 above.

Lemma 3.2 trivially applies to the linear solution and we have that

\[ (1 + |q|)^{\frac{1}{2}} \sum_{|I| \leq 3} |\partial Z^I u_0| \leq \frac{C_S D}{t} \quad \sum_{|I| \leq 2} |\overline{\partial Z^I u_0}| \leq \frac{C_S D}{t^\frac{1}{2}} \]

Now assume, that for some \( \delta > 0 \),

\[ \sum_{|I| \leq 5} \|\partial Z^I v\|_{L^2(\Sigma_t)} + \sum_{|I| \leq 5} \left( \int_t^T \int_{\mathbb{R}^3} \frac{\mu}{4} \frac{|\overline{\partial Z^I v}|^2}{(1 + |q|)^{1+2\mu}} \, dx \, dt \right)^{\frac{1}{2}} \leq \frac{C_0}{t^\delta} \]  \hspace{1cm} (*)
Then by the Klainerman-Sobolev inequality

$$(1 + |t - r|)^{\frac{1}{2}} \sum_{|I| \leq 3} |\partial Z^I v| \leq \frac{C_0 C_S}{t^{1+\delta}}$$

and by Lemma 3.2 (recall that the data for $v$ on $\Sigma_T$ is trivial)

$$\sum_{|I| \leq 2} |\partial Z^I v| \leq \frac{C_S C_0}{t^{2+\delta}}$$

Consider the first string of terms in $F_1$:

$$\sum_{|J|+|K| \leq 5} \|\partial Z^J v||\overline{\partial Z^K v}\|_{L^2(\Sigma_t)} \leq \sum_{|J| \leq 2, |K| \leq 5} \|\partial Z^J v||\overline{\partial Z^K v}\|_{L^2(\Sigma_t)} + \sum_{|J| \leq 5, |K| \leq 2} \|\partial Z^J v||\overline{\partial Z^K v}\|_{L^2(\Sigma_t)}$$

We will estimate these terms given the improved decay rates of $\overline{\partial v}$, over $\partial v$, first in the $L^\infty$, then in the $L^2$-sense, as expressed in Cor. 2.2 and Lemma 3.2 (and the bootstrap assumptions) above.

Firstly, using the $L^\infty$ estimate,

$$\sum_{|J| \leq 5} \int_t^T \|\partial Z^J v||\overline{\partial Z^K v}\|_{L^2(\Sigma_t)} dt \leq \sum_{|J| \leq 5} \int_t^T \|\overline{\partial Z^K v}\|_{L^\infty(\Sigma_t)} \|\partial Z^J v\|_{L^2(\Sigma_t)} dt$$

$$\leq \sum_{|J| \leq 5} \int_t^T \frac{C_S C_0}{(1 + t)^{\frac{1}{2}+\delta}} \|\partial Z^J v\|_{L^2(\Sigma_t)} dt \leq \frac{C_S^2 C_0}{t^{2+\delta}} \int_t^\infty \frac{1}{(1 + t)^{\frac{1}{2}+\delta}} dt$$

Secondly, using the $L^2$ estimate,

$$\sum_{|J| \leq 2} \int_t^T \|\partial Z^J v||\overline{\partial Z^K v}\|_{L^2(\Sigma_t)} dt \leq \sum_{|J| \leq 5} \int_t^T \left( \int_{\mathbb{R}^3} |\partial Z^J v|^2 |\overline{\partial Z^K v}|^2 dx \right)^{\frac{1}{2}} dt$$

$$\leq \sum_{|J| \leq 2} \int_t^T \sup_{\mathbb{R}^3} \left[ (1 + (|x| - t))^{\frac{1}{2}+\mu} |\partial Z^J v| \right] \left( \int_{\mathbb{R}^3} \frac{|\overline{\partial Z^K v}|^2}{(1 + |x| - t)^{1+2\mu}} dx \right)^{\frac{1}{2}} dt$$

$$\leq \sum_{|J| \leq 2} \left( \int_t^T \left[ \sup_{\mathbb{R}^3} (1 + (|x| - t))^{\frac{1}{2}+\mu} |\partial Z^J v|^2 \right]^2 dx dt \right)^{\frac{1}{2}} \sum_{|J| \leq 5} \left( \int_t^T \int_{\mathbb{R}^3} \frac{|\overline{\partial Z^K v}|^2}{(1 + |x| - t)^{1+2\mu}} dx dt \right)^{\frac{1}{2}}$$

$$\leq \frac{2 C_0 C_S}{\sqrt{\mu}} \left( \int_t^\infty \frac{1}{(1 + t)^{2+2\delta - 2\mu}} dt \right)^{\frac{1}{2}} \frac{C_0}{t^{\delta}} \leq \frac{2}{\sqrt{\mu(1 + \delta - 2\mu)}} \frac{C_0^2 C_S}{t^{2+\delta}}$$

Now we proceed similarly for the other terms in $F_1$. Indeed, for the purely linear terms, the above estimates formally apply if we replace $v$ by $u_0$, and set $\delta = 0$. This yields:

$$\int_t^T \sum_{|I|+|J| \leq 5} \|\partial Z^J u_0||\overline{\partial Z^K u_0}\|_{L^2(\Sigma_t)} dt \leq \frac{C_S}{t^{\frac{3}{2}-\mu}} \sum_{|I| \leq 5} \|\partial Z^I u_0\|_{L^2(\Sigma_T)}$$
Similarly,
\[
\sum_{|I|+|J|\leq 5} \int_t^T \| |\partial Z^I u_0| |\partial Z^J v| \|_{L^2(\Sigma_t)} dt \leq \frac{C_0 C_S}{t^{\frac{1}{2}+\delta-\mu}} \sum_{|I|\leq 5} \| \partial Z^I u_0 \|_{L^2(\Sigma_T)}; \tag{4.8}
\]
and
\[
\sum_{|I|+|J|\leq 5} \int_t^T \| |\partial Z^I v| |\partial Z^J u_0| \|_{L^2(\Sigma_t)} dt \leq \frac{C_0 C_S}{t^{\frac{1}{2}+\delta-\mu}} \sum_{|I|\leq 5} \| \partial Z^I u_0 \|_{L^2(\Sigma_T)} \tag{4.9}
\]
In conclusion,
\[
\sum_{|I|\leq 5} \| \partial Z^I v \|_{L^2(\Sigma_T)} + \sum_{|I|\leq 5} \left( \int_t^T \int_{\mathbb{R}^3} \frac{\mu}{4} \frac{|\partial Z^I v|^2}{(1+|q|)^{1+2\mu}} dx dt \right)^{\frac{1}{2}} \leq C \sum_{|I|\leq 5} \int_t^T \| F_I \|_{L^2(\Sigma_t)} \leq \frac{CC_S}{t^{2-\mu}} \left( \frac{C_0}{t^5} + \sum_{|I|\leq 5} \| \partial Z^I u_0 \|_{L^2(\Sigma_T)} \right)^2,
\]
which recovers the bootstrap assumption \(\Box\), for \(t\) sufficiently large, provided \(0 < \delta < \frac{1}{2} - \mu\). In fact, we could also choose \(\delta = \frac{1}{2} - \mu\), as long as we choose \(C_0 > D^2\) sufficiently large.

5 Scattering and weighted energies for the homogeneous solution

In this section we want to solve the linear homogeneous wave equation \(\Box \psi = 0\) from infinity with “radiation data” given in the form of the asymptotics
\[
\psi \sim \frac{F_0(r-t, \omega)}{r} + \psi_e, \tag{5.1}
\]
Here \(F_0\) is the Friedlander radiation field, and
\[
\psi_e = M \chi_e(r-t)/r, \tag{5.2}
\]
is an explicit exact solution picking up the mass term in the exterior, where \(\chi_e(s) = 1\), for \(s \geq 2\), and \(\chi_e(s) = 0\) for \(s \leq 1\) is smooth function.

We will assume that for some \(1/2 < \gamma < 1\):
\[
\| F_0 \|_{N, \gamma-1/2}^2 = \sum_{|a|+k \leq N} \int_{\mathbb{R}} \int_{S^2} |(q| \partial_q)^k \partial_\omega^a F_0(q, \omega) |^2 \langle q \rangle^{2\gamma-1} dS(\omega) dq \leq C \tag{5.3}
\]
and as a consequence for some smaller \(N'\)
\[
\| F_0 \|_{N', \infty, \gamma} = \sum_{|a|+k \leq N'} \sup_{q \in \mathbb{R}} \sup_{\omega \in S^2} |(q| \partial_q)^k \partial_\omega^a F_0(q, \omega) | \langle q \rangle^\gamma \leq C. \tag{5.4}
\]
In the following we will also need a second term in the “asymptotic expansion”
\[
\psi \sim \frac{F_0(r-t, \omega)}{r} + \frac{F_1(r-t, \omega)}{r^2} + \psi_e, \quad 2F'_1(q, \omega) = \triangle_\omega F_0(q, \omega), \quad F_1(0, \omega) = 0. \tag{5.5}
\]
(The formula for $F_1$ in terms of $F_0$ is obtained in the next section in order to cancel the leading order term of $\Box$ applied to this expression.)

We wish to solve this problem by giving data for $\psi - \psi_e$ when $t = T$, in the form of the following approximate solution which is supported in the wave zone, away from the origin,

$$\psi_0 = \frac{F_0(r-t,\omega)}{r}\chi\left(\frac{(t-r)}{r}\right), \quad (5.6)$$

where $\chi$ is a smooth decreasing function such that $\chi(s) = 1$, when $s \leq 1/8$ and $\chi(s) = 0$, when $s \geq 1/4$. We will need to show that as $T \to \infty$, this limit exists in the norms used below, or rather the limit constructed from the second order approximation

$$\psi_{01} = \left(\frac{F_0(r-t,\omega)}{r} + \frac{F_1(r-t,\omega)}{r^2}\right)\chi\left(\frac{(t-r)}{r}\right), \quad 2F_1'(q,\omega) = \triangle_\omega F_0(q,\omega), \quad F_1(0,\omega) = 0. \quad (5.7)$$

Here $F_1$ is not unique without the last condition.

5.1 The asymptotics of the approximate solutions

Let us quantify the error by which the approximations $\psi_0$ and $\psi_{01}$ fail to be solutions of the homogeneous wave equation.

Lemma 5.1. We have

$$\Box(Z^T\psi_0) \lesssim \frac{\chi\left(\frac{(t-r)}{2r}\right)}{(t+r)^3} \sum_{k+|\alpha| \leq 2+|I|} \left|\langle q\rangle \partial_\alpha \partial_\omega F_0(q,\omega)\right| \quad (5.8)$$

$$\Box(Z^T\psi_{01}) \lesssim \frac{\chi\left(\frac{(t-r)}{2r}\right)}{(t+r)^4} \sum_{|\alpha|+k \leq 2+|I|} \langle q\rangle \left|\langle q\rangle \partial_\alpha \partial_\omega F_0(q,\omega)\right| + \left|\langle q\rangle \partial_\alpha \partial_\omega F_1(q,\omega)\right|. \quad (5.9)$$

Proof. First consider the case $|I| = 0$. We have

$$\Box\psi_0 = \frac{1}{r^3} \triangle_\omega F_0(r-t,\omega)\chi\left(\frac{(t-r)}{r}\right) - \frac{2}{r} F_0'(r-t,\omega)\chi'\left(\frac{(t-r)}{r}\right) \frac{1}{r^2}$$

$$- \frac{1}{r} F_0(r-t,\omega)(\partial_r - \partial_t)\left(\chi'\left(\frac{(t-r)}{r}\right)\frac{1}{r^2}\right),$$

where $F_0'(q,\omega) = \partial_q F_0(q,\omega)$, and the first estimate follows, because $t \sim r$ in the support of $\chi(\langle t-r\rangle/r)$. Moreover

$$\Box(\psi_{01}) = - \frac{2F_0'(r-t,\omega)}{r} \chi'\left(\frac{(t-r)}{r}\right) \frac{1}{r^3} - \frac{F_0(r-t,\omega)}{r} (\partial_r - \partial_t) \left(\chi'\left(\frac{(t-r)}{r}\right)\frac{1}{r^2}\right)$$

$$+ \frac{\triangle_\omega F_1(r-t,\omega)}{r^4} \chi\left(\frac{(t-r)}{r}\right) - \frac{2F_1'(r-t,\omega)}{r} \chi'\left(\frac{(t-r)}{r}\right) \frac{1}{r^3}$$

$$- \frac{F_1(r-t,\omega)}{r} (\partial_r - \partial_t) \left(\chi'\left(\frac{(t-r)}{r}\right)\frac{1}{r^2} + \chi(\langle t-r\rangle/r)\right).$$

Since $\langle r-t \rangle \sim \langle t+r \rangle$ in the support of $\chi'(\langle t-r\rangle/r)$ the second estimate follows. The general case $|I| \geq 1$ is straightforward. \qed
Given that in the above asymptotic expansion $F_1$ is related to $F_0$ by a propagation equation we infer that:

**Lemma 5.2.** We have

\[ \|F_1\|_{N-2,\gamma-3/2} \lesssim \|F_0\|_{N,\gamma-1/2}, \quad \text{and} \quad \|F_1\|_{N-2,\infty,\gamma-1} \lesssim \|F_0\|_{N,\infty,\gamma} \]  

(5.10)

**Proof.** This follows from using that with $f(q) = \int_q^\infty \langle q'' \rangle^{\gamma-3} dq'' dq'$ we have $f'(q)^2 \leq C f''(q)f(q)$ and since $F_1(0) = 0$

\[ \int_0^\infty F_1(q)^2 f''(q) dq = f'(q)F_1(q)^2\bigg|_0^\infty - 2\int_0^\infty f'(q)F_1(q)F'_1(q) dq \]

\[ \lesssim \sqrt{\int_0^\infty F_1(q)^2 f''(q) dq} \sqrt{\int_0^\infty F'_1(q)^2 f(q) dq} \]

and $|F'_1| \leq |\partial_\omega^2 F_0|, |\alpha| \leq 2$. \qed

In view of the energy estimates in Section 2.2, we need to control $\Box \psi_{01}$ with the relevant weights in $L^2(\Sigma_t)$.

**Lemma 5.3.** We have, for all $s \geq 0$, if $\gamma \leq 3/2$,

\[ \|\langle t + r \rangle^s \Box \psi_{01}(t, \cdot)\|_{L^2_x} \lesssim \frac{\|F_0\|_{4+|I|,\gamma-1/2}}{\langle t \rangle^{3/2+\gamma-s}} \]  

(5.11)

**Proof.** Consider the case $|I| = 0$. First by Lemma 5.1 we have

\[ \|\langle t + r \rangle^s \Box \psi_{01}(t, \cdot)\|_{L^2_x}^2 \lesssim \int \langle t + r \rangle^{2s} \left( \frac{\lambda}{\langle t + r \rangle^4} \right)^2 \sum_{|\alpha|+|k| \leq 2} \left( \|q\|^2 \|q \partial_q^k \partial_\omega^2 F_0(q, \omega)\|^2 + \|q \partial_q^k \partial_\omega^2 F_1(q, \omega)\|^2 \right) dx \]

Hence

\[ \|\langle t + r \rangle^s \Box \psi_{01}(t, \cdot)\|_{L^2_x}^2 \lesssim \int_0^R \int_{\mathbb{R}^2} \frac{r^2\langle q \rangle^{3-2\gamma}}{(t + r)^{8-2s}} \left( \|q\|^{2\gamma-1} \|q \partial_q^k \partial_\omega^2 F_0(q, \omega)\|^2 + \langle q \rangle^{2\gamma-3} \|q \partial_q^k \partial_\omega^2 F_1(q, \omega)\|^2 \right) dS(\omega) dr \]

and if $\gamma \leq 3/2$, we have by Lemma 5.2

\[ \|\langle t + r \rangle^s \Box \psi_{01}(t, \cdot)\|_{L^2_x}^2 \lesssim \frac{\|F_0\|_{2,\gamma-1/2}^2}{\langle t \rangle^{3+2\gamma-2s}} + \frac{\|F_1\|_{2,\gamma-3/2}^2}{\langle t \rangle^{3+2\gamma-2s}} \lesssim \frac{\|F_0\|_{4,\gamma-1/2}^2}{\langle t \rangle^{3+2\gamma-2s}}. \] \qed
5.2 Energy bounds for the linear homogeneous scattering problem

Let us prove an energy bound using the second order asymptotics. We write
\[ \psi = v + \psi_{01} + \psi_e \]
for a solution to \( \Box \psi = 0 \), and since \( \Box \psi_e = 0 \) we obtain the following equation for \( v \):
\[ \Box v = -\Box \psi_{01} \quad (5.12) \]

With vanishing data at \( T \to \infty \), we get by the standard energy estimate in view of Lemma 5.3:
\[ \| \partial (Z^I v)(t, \cdot) \|_{L^1} \lesssim \int_t^\infty \| \Box (Z^I v)(t, \cdot) \| \, dt \lesssim \| F_0 \|_{4 + |I|, \gamma - 1/2} \langle t \rangle^{1/2 + \gamma} \quad (5.13) \]

Moreover by Lemma 3.1,
\[ |Z^I v(t, x)| \lesssim \frac{1}{\langle t + r \rangle^{1/2}} \sum_{|J| \leq 2 + |I|} \| \partial (Z^J v) \|_{L^2(\Sigma_t)} \lesssim \frac{\| F_0 \|_{6 + |I|, \gamma - 1/2}}{\langle t + r \rangle^{1/2} \langle t \rangle^{1/2 + \gamma}} \quad (5.14) \]

**Remark 5.1.** One can also prove an energy bound just using the first order asymptotics. The advantage is then there is no loss in regularity and one can prove that the map from the radiation field to initial data is \( H^1 \) to \( H^1 \) by a regularizing procedure.

5.3 Weighted conformal energy bounds for the homogenous scattering problem

Here we derive weighted energy bounds using the second order asymptotics and the results of Section 2.2.

**Proposition 5.4.** Suppose \( \psi = v + \psi_{01} \) is a solution to \( \Box \psi = 0 \), where \( \psi_{01} \) is an approximate solution of the form (5.7), and \( v \) has trivial initial data at \( t = T \). If \( s < \gamma + 1/2 \), then as \( T \to \infty \),
\[ \| \psi(t, \cdot) \|_{1, s - 1} \lesssim \| F_0 \|_{2, \gamma - 1/2}, \quad (5.15) \]

and
\[ |\psi(t, x)| + |\psi_{01}(t, x)| + |v(t, x)| \lesssim \frac{\| F_0 \|_{6, \gamma - 1/2}}{\langle t + r \rangle^{s-1/2} \langle t - r \rangle^{s-1/2}}. \quad (5.16) \]

**Remark 5.2.** The result trivially extends to higher order derivatives, i.e. one has
\[ \| \psi(t, \cdot) \|_{k, s - 1} := \sum_{|I| \leq k} \| (t - r)^{s-1} (Z^I \psi)(t, \cdot) \|_{L^2(\Sigma_t)} \lesssim \| F_0 \|_{2 + k, \gamma - 1/2}, \quad (5.17) \]

by virtue of the good commutation properties of Lemma 5.7, and hence corresponding point-wise estimates by Lemma 3.5. For simplicity we here just discuss the case \( k = 1 \).

Consider a solution to the homogeneous equation \( \Box \psi = 0 \), and write \( \psi = v + \psi_{01} \), then \( v \) satisfies the equation
\[ \Box v = -\Box \psi_{01}. \quad (5.18) \]
We construct a solution \( v \) by a limiting procedure with vanishing data when \( t = T \to \infty \). Now recall Proposition 2.4 which gives us a bound on

\[
\|v(t, \cdot)\|_{1,s-1} = \sum_{|I| \leq 1} \|\langle t-r \rangle^{s-1}(Z^I v)(t, \cdot)\|_{L^2} \tag{5.19}
\]

Indeed, by (2.14), we have for \( s > 1 \),

\[
\|v(t, \cdot)\|_{1,s-1} \lesssim \|v(t, \cdot)\|_{1,s-1} \lesssim \|T\|_{L^2} dt \tag{5.20}
\]

Note here that \( \square v = -\square \psi_01 \) vanishes when \( r > 2t \) and since we pose vanishing data for \( v \) at \( t = T \) it follows by finite speed of propagation that the solution \( v \) vanishes when \( r > 2T + (T-t) \); hence for any fixed \( t \) also the boundary condition in Prop. 2.4 is verified.

Furthermore by Lemma 5.5 we have

\[
\int_t^T \|\langle t+r \rangle^s \square v(t, \cdot)\|_{L^2} dt \leq \int_t^T \frac{\|F_0\|_{1,\gamma-1/2}}{(t\gamma-s)^{1/2+\gamma-s}} dt \lesssim \frac{\|F_0\|_{1,\gamma-1/2}}{(t\gamma-s)^{1/2+\gamma-s}}, \quad \text{if} \quad s < \gamma + 1/2. \tag{5.21}
\]

When combined with the decay estimates of Section 3.3 this yields:

**Lemma 5.5.** Let \( v \) be a solution to (5.18) with vanishing data at \( t = T \). Then for \( s < \gamma+1/2 \), as \( T \to \infty \),

\[
\sum_{|I| \leq k} \|\langle t-r \rangle^{s-1}(Z^I v)(t, \cdot)\|_{L^2} \lesssim \frac{\|F_0\|_{1,\gamma-1/2}}{(t\gamma-s)^{1/2+\gamma-s}}
\]

and moreover

\[
\|(Z^I v)(t, x)\| \lesssim \frac{\|F_0\|_{6+|I|,\gamma-1/2}}{(t+r)^{1/2+\gamma-s}(t-r)^{s-1/2}}.
\]

**Proof.** Above we have already shown that

\[
\|v(t, \cdot)\|_{1,s-1} \lesssim \frac{\|F_0\|_{4,\gamma-1/2}}{(t\gamma-s)^{1/2+\gamma-s}}, \quad \text{if} \quad s < \gamma + 1/2.
\]

and the first bound then follows from (2.15). Moreover by Lemma 3.5

\[
\langle t+r \rangle \langle t-r \rangle^{1/2+s-1}|v| \lesssim \sum_{|I| \leq 2} \|\langle t-r \rangle^{s-1}Z^I v(t, \cdot)\|_{L^2(\Sigma_t)}
\]

and \( \|\square(Z^I v)\| \lesssim |Z^I \square(\psi_01)| + |\square(\psi_01)| \) so the second bound follows. \( \square \)

We can also apply (2.15) directly to the function \( \psi_01 \), and obtain

\[
\|\psi_01(t, \cdot)\|_{1,s-1} \lesssim \|\psi_01(t, \cdot)\|_{1,s-1} \lesssim \frac{\|F_0\|_{2,\gamma-1/2}}{(t+r)^{\gamma-s}}, \quad \text{if} \quad s < \gamma + 1/2, \tag{5.22}
\]

and again by Lemma 3.5

\[
|\psi_01(t, x)| \lesssim \frac{\|F_0\|_{4,\gamma-1/2}}{(t+r)^{\gamma-s}}, \tag{5.23}
\]

which implies the statement of the Proposition.
5.4 Asymptotics for derivatives of the solution and the exterior part picking up the mass

Given a solution $\psi$ to the homogeneous equation $\Box \psi = 0$, the derivatives $\partial_t \psi$, $\partial_i \psi$ are themselves solutions of the wave equation and have a radiation field. The asymptotics of the time derivative of the radiation field (5.11) are

$$\partial_t \psi \sim -\frac{F'_0(r-t,\omega)}{r} + \psi'_e,$$

(5.24)

where $F'_0(q,\omega) = \partial_q F_0(q,\omega)$ and $\psi'_e = -M \chi'(r-t)/r$. The corresponding formula for the second order expression (5.7) is

$$\partial_t \psi \sim -\frac{F'_0(r-t,\omega)}{r} - \frac{F'_1(r-t,\omega)}{r^2} + \psi'_e, \quad 2F'_1(q,\omega) = \Delta \omega F_0(q,\omega), \quad F_1(0,\omega) = 0.$$  

(5.25)

For other derivatives derivatives the formulas are a bit more involved but are obtained simply by differentiating the expressions. We define

$$\psi'_{01} := \psi'_0 + \psi'_i,$$

where $\psi'_0 := -\frac{F'_0(r-t,\omega)}{r} \chi\left(\frac{r-t}{r}\right)$ and $\psi'_i := -\frac{F'_1(r-t,\omega)}{r^2} \chi\left(\frac{r-t}{r}\right)$

(5.26)

Lemma 5.6.

$$\left| (\partial_t \psi'_{01} + \partial_i \psi'_e)^2 - (\psi'_0 + \psi'_e)^2 \right| \lesssim \frac{\chi'(\frac{t-r}{r})}{(t+r)^4} \left( \langle q \rangle |F_0(q,\omega)| + |F_1(q,\omega)| \right) \times$$

$$\times \left( |F'_0(q,\omega)| + \langle q \rangle^{-1} |F'_1(q,\omega)| + \langle q \rangle^{-1} |F_0(q,\omega)| + \langle q \rangle^{-2} |F_1(q,\omega)| \right).$$

(5.27)

Proof. We have

$$\partial_t \psi'_{01} + \partial_i \psi'_e = \psi'_0 + \psi'_i + \left( \frac{F'_0(r-t,\omega)}{r} + \frac{F'_1(r-t,\omega)}{r^2} \right) \chi'\left(\frac{r-t}{r}\right) \frac{t-r}{(t-r)r}.$$ 

Hence

$$\left| \partial_t \psi'_{01} + \partial_i \psi'_e - \psi'_0 - \psi'_e \right| \lesssim \frac{\chi'(\frac{t-r}{r})}{(t+r)^3} \left( \langle q \rangle |F_0(q,\omega)| + |F_1(q,\omega)| \right).$$

and the inequality follows.

□

5.5 Existence of the limit as $T$ tend to infinity

Let us discuss the proof that the solution $\psi$ constructed in Section 5.3 converges as $T \to \infty$. Let $T_2 > T_1$ and for $i = 1, 2$ let $\psi_i$ be two solutions with data at $T_i$ given by $\psi_{01} + \psi_e$ and let $v_i = \psi_i - \psi_{01} - \psi_e$. Then $v_i$ has vanishing data at $T_i$ for $i = 1, 2$. We need to show that the difference $v = v_2 - v_1$ tends to 0 as $T_2 > T_1 \to \infty$ in the norms we used above. It suffices to show that this is true for one time and it will automatically be true for all smaller times because $\Box v = \Box v_2 - \Box v_1 = 0$ hence the energy at smaller times can be bounded by the energy at larger times. However $v = v_2$ at time $T_1$ so it suffices to show that the norms of $v_2$ at time $T_1$ tend to 0 as $T_2 > T_1 \to \infty$, but this is just what we have proven above.
6 A simple system satisfying the weak null condition

First let us consider the following system of wave equations with a semi-linear structure that appears in the Einstein equations:

\[
\begin{align*}
\Box \psi &= 0 \quad (6.1a) \\
\Box \phi &= (\partial_t \psi)^2 \quad (6.1b)
\end{align*}
\]

6.1 Asymptotic scattering data at infinity for the simple model system

We prescribe asymptotic data for the system (6.1) as follows. First we prescribe asymptotic data for (6.1a) of the form

\[
\psi \sim \frac{F_0(r-t, \omega)}{r} \chi\left(\frac{\langle t-r \rangle}{r}\right) + \psi_e, \quad (6.2)
\]

where we assume that \(F_0\) satisfy the decay assumption (5.3) and \(\chi_e\) is the mass term of the form (5.2). Then we define \(\phi_0\) to be the solution of

\[
\Box \phi_0 = (\psi_0' + \psi_e')^2, \quad \psi_0' = -\frac{F_0'(r-t, \omega)}{r} \chi\left(\frac{\langle t-r \rangle}{r}\right), \quad \psi_e' = -M \chi_e'(r-t)/r \quad (6.3)
\]

with vanishing data at past infinity, \(t = -\infty\). Here \(F_0'(q, \omega) = \partial_q F_0(q, \omega)\) and \(\psi_0'\) is the radiation field for \(\partial_t \psi\), c.f. the discussion in Section 5.4 so we expect

\[
\Box (\phi - \phi_0) = (\partial_t \psi)^2 - (\psi_0' + \psi_e')^2 \quad (6.4)
\]

to be decaying sufficiently fast, so that \(\phi - \phi_0\) will approach a solution to the linear homogeneous wave equation and hence have a radiation field. After that we prescribe asymptotic data for (6.1b) of the form

\[
\phi - \phi_0 \sim \frac{G_0(r-t, \omega)}{r} \chi\left(\frac{\langle t-r \rangle}{r}\right),
\]

where we assume that also the radiation field \(G_0\) satisfy the decay assumption (5.3).

6.2 Energy bounds for the simple model system

We have argued that for the system (6.1) above \(\psi\) and \(\phi - \phi_0\) have radiation fields, and can be approximated by \(\psi_0\) and \(\phi_0\), of the form (5.6). However, as in Section 5.3 for the scattering construction it is necessary to replace these linear homogeneous parts by the improved approximations \(\psi_{01}\) and \(\phi_{01}\) according to the prescription discussed at the beginning of Section 5 see in particular (5.7). The key is that in view of Lemma 5.1 the next order approximations allows us to apply Proposition 2.4 with \(s \geq 1\).

**Proposition 6.1.** Let \((\psi = v + \psi_{01} + \psi_e, \phi = w + \varphi_{01} + \phi_{01})\) be a solution to the system (6.1), where \(\psi_{01}\) and \(\phi_{01}\) are the approximate linear solutions (5.7), with radiation fields \(F\), and
that the $L^\infty$ and by (5.27) we have

\[ \| Z^I w(t, \cdot) \|_{1,s-1} \lesssim \frac{\| F_0 \|_{I+7,\gamma-1/2} + \| F_0 \|_{I+7,\gamma-1/2}^2 + \| G_0 \|_{I+7,\gamma-1/2}}{(t)^{1/2+s-\gamma}}, \quad (6.5) \]

\[ |Z^I w(t, x)| \lesssim \frac{\| F_0 \|_{I+9,\gamma-1/2} + \| F_0 \|_{I+9,\gamma-1/2}^2 + \| G_0 \|_{I+9,\gamma-1/2}}{(t + r)(t - r)^{s-1/2}}, \quad (6.6) \]

and for $v$ the conclusions of Lemma 5.3 hold true.

We will just do the proof for $|I| = 0$ since higher $|I|$ follows in the same way. Recall the definition of $\varphi_0$ in (6.3). Since we improved the asymptotics for $\psi$ we also need to improve the asymptotics for $\varphi_0$, and are led to the definition of $\varphi_{01}$ to be the solution of

\[ \Box \varphi_{01} = (\psi_0' + \psi_e' + \psi_1')^2 \quad (6.7) \]

with vanishing data at past infinity; see also Section 5.4 for the relevant notation.

As indicated in the statement of the Proposition, we write

\[ \psi = v + \psi_{01} + \psi_e, \quad (6.8) \]

\[ \phi = w + \varphi_{01} + \phi_{01}, \quad (6.9) \]

and solve with trivial data for $v$ and $w$ at $t = T$, and let $T \to \infty$.

We have

\[ \Box w = \Box \phi - \Box \varphi_{01} - \Box \phi_{01} \]

\[ = (\partial_t \psi)^2 - (\partial_t \psi_{01} + \partial_t \psi_e)^2 + ((\partial_t \psi_{01} + \partial_t \psi_e)^2 - (\psi_0' + \psi_e' + \psi_1')^2) - \Box \phi_{01} \quad (6.10) \]

\[ = (\partial_t v + 2\partial_t \psi_{01} + 2\partial_t \psi_e)\partial_t v + ((\partial_t \psi_{01} + \partial_t \psi_e)^2 - (\psi_0' + \psi_e' + \psi_1')^2) - \Box \phi_{01} \]

and by (5.27) we have

\[ \left| (\partial_t \psi_{01} + \partial_t \psi_e)^2 - (\psi_0' + \psi_e' + \psi_1')^2 \right| \lesssim \frac{\chi'\left(\frac{q}{2}\right)}{(t + r)^4} \sum_{i=0,1} \langle q \rangle^{1-i} \left( |F_i(q, \omega)| + \langle q \rangle |F_i'(q, \omega)| \right) \times \sum_{i=0,1} \langle q \rangle^{-1-i} \left( |F_i(q, \omega)| + \langle q \rangle |F_i'(q, \omega)| \right). \quad (6.11) \]

We want to calculate

\[ \int_t^T \| (t + r)^4 \Box w(t, \cdot) \|_{L^4} \, dt \]

for $1 \leq s < \gamma + 1/2$ and $s = 0$.

Consider first the second term in the r.h.s. of (6.10). When taking (6.11) in $L^2$, we note that the $L^\infty$ norm of the last factor is bounded by Lemma 5.2 using the assumption (5.4),

\[ \sum_{i=0,1} \langle q \rangle^{-1-i} \left( |F_i(q, \omega)| + \langle q \rangle |F_i'(q, \omega)| \right) \|_{\infty} \lesssim \langle q \rangle^{-1-\gamma} \| F_0 \|_{3, \infty, \gamma} \quad (6.12) \]
and the $L^2$ norm of the first factor can be treated analogously to the homogeneous case, which yields

\[
\| (t + r)^s |\partial_t \psi_0 + \partial_t \psi_v|^2 - \langle \psi_0' + \psi_v' + \psi_v^2 \rangle (t, \cdot) \|_{L^2_t} \lesssim \frac{\| F_0 \|_{H^{4, \frac{1}{2}}}}{(t)^{3/2 + \gamma}} \| F_0 \|_{H^{3, \infty, \gamma}} \tag{6.13}
\]

Moreover as in (5.21) we have the homogeneous estimate for the last term in the r.h.s. of (6.10),

\[
\| (t + r)^s \Delta \phi_0(t, \cdot) \|_{L^2_t} \lesssim \frac{\| G_0 \|_{H^{4, \frac{1}{2}}}}{(t)^{3/2 + \gamma}}. \tag{6.14}
\]

Furthermore for the first term in (6.10) we can apply Prop. 5.4 and Lemma 5.5 to infer that

\[
\langle t + r \rangle^s |\partial_t v + 2 \partial_t \psi_0 + 2 \partial_t \psi_v| |\partial_t v| \lesssim \frac{\langle t + r \rangle^s \| F_0 \|_{H^{7, \frac{1}{2}}}}{(t + r)^{\frac{1}{2} + \gamma}} |\partial_t v| \tag{6.15}
\]

hence by (5.13)

\[
\| (t + r)^s |\partial_t v + 2 \partial_t \psi_0 + 2 \partial_t \psi_v| |\partial_t v| \|_{L^2_t} \lesssim \frac{\| F_0 \|_{H^{7, \frac{1}{2}}}}{(t)^{3/2 + \gamma}} \tag{6.16}
\]

Note that the above inequality holds also for $s = 0$ since one can then use the decay estimate for $s \geq 1$. Therefore

\[
\int_t^T \| (t + r)^s |\partial_t v + 2 \partial_t \psi_0 + 2 \partial_t \psi_v| |\partial_t v| \|_{L^2_t} \| (t + r)^{s-1} |\partial_t v| \|_{L^2_t} dt \lesssim \frac{\| F_0 \|_{H^{7, \frac{1}{2}}}}{(t)^{1/2 + \gamma}} \tag{6.17}
\]

and hence it follows by Proposition 2.4 that

\[
\| w(t, \cdot) \|_{H^{1, s-1}} \lesssim \frac{\| F_0 \|_{H^{7, \frac{1}{2}}} + \| F_0 \|_{H^{7, \frac{1}{2}}}}{(t)^{1/2 + \gamma}} \tag{6.18}
\]

and

\[
\| \partial w(t, \cdot) \| \lesssim \frac{\| F_0 \|_{H^{7, \frac{1}{2}}} + \| F_0 \|_{H^{7, \frac{1}{2}}}}{(t)^{1/2 + \gamma}} \tag{6.19}
\]

Finally as for the linear case we get by Lemma 3.5 the decay estimates

\[
| w(t, x) | \lesssim \frac{\| F_0 \|_{H^{7, \frac{1}{2}}} + \| F_0 \|_{H^{7, \frac{1}{2}}}}{(t + r)(t - r)^{s-1}} \tag{6.20}
\]

6.3 Existence of the limit as $T \to \infty$

In the case when the equations separate as above one can simply reduce the argument to the linear homogeneous case as in Section 5.5. First by the previous section the limit of $\psi$ and hence $v$ exist as $T \to \infty$, where $T$ is the time when data for $v$ vanish. Then one lets $\psi$ be the fixed limiting solution and with $\psi$ given solve the equation for $\phi$ with data for $\phi$ at $T'$ and hence data for $w$ vanishing at $T'$. As in previous section let $T' > T_1$ and let $\phi_1$ and $w_1$ be the corresponding solutions and let $w = w_2 - w_1$ then again $w = w_2$ when $t = T_1$ and $\Box w = 0$. The rest of the argument is identical to Section 5.5.
6.4 The asymptotics with sources

In this section we get estimates and formulas for the solution of

$$\Box \varphi_{01} = (\psi'_0 + \psi'_\sigma + \psi'_t)^2$$  \hspace{1cm} (6.21)

with vanishing data at minus infinity. It can be written as a linear combination of solutions of the following problems.

For 2, 3, 4 let $\Phi[n]$ be the solution of

$$\Box \Phi^k[n](t, \sigma) = n(r - t, \sigma) r^{-k} \chi (\frac{r - t}{r})^2,$$  \hspace{1cm} (6.22)

with vanishing data at $-\infty$, where we assume that for some $a \geq 0$

$$\|n\|_{N,1,\infty,a} = \sum_{|\alpha| + k \leq N} \int_{R} \sup_{\omega \in S^2} |(\langle q \rangle \partial_q)^{k} \partial^\alpha n(q, \omega)| (q_+)^a dq \leq C.$$  \hspace{1cm} (6.23)

**Lemma 6.2.** For $|I| \leq N$

$$|Z^I \Phi^2[n](t, \sigma)| \lesssim \frac{1}{2} \ln \left( \frac{\langle t + r \rangle}{\langle t - r \rangle} \right) \frac{\|n\|_{N,1,\infty,a}}{\langle t - r \rangle^a}$$  \hspace{1cm} (6.24)

and for $k = 3, 4$

$$|Z^I \Phi^k[n](t, \sigma)| \lesssim \frac{1}{(t + r)^{(k-2)}} \frac{\|n\|_{N,1,\infty,a}}{\langle t - r \rangle^a}$$  \hspace{1cm} (6.25)

**Proof.** The proof follows by first taking the supremum over the angles and then using the formula for the fundamental solution in the radial case as in the proof of Lemma 9 in [26] \hfill \Box

In [16, 26] there was a formula for the solution

$$\Phi^2[n](t, \sigma) = \int_{r-t}^{\infty} \frac{1}{4\pi} \int_{S^2} \frac{n(q, \sigma) \chi (\frac{\langle q \rangle}{\rho}) dS(\sigma) dq}{t + r + q + r(1 - \langle \omega, \sigma \rangle)},$$  \hspace{1cm} (6.26)

where

$$\rho = \frac{1}{2} \frac{(t + r + q)(t - r + q)}{t - r + q + r(1 - \langle \omega, \sigma \rangle)}.$$  \hspace{1cm} (6.27)

and

$$\Phi^3[n](t, \sigma) = \int_{r-t}^{\infty} \frac{1}{4\pi} \int_{S^2} \frac{n(q, \sigma) \chi (\frac{\langle q \rangle}{\rho}) dS(\sigma)}{(t - r + q)(t + r + q)} dq,$$  \hspace{1cm} (6.28)

and

$$\Phi^4[n](t, \sigma) = \int_{r-t}^{\infty} \frac{1}{4\pi} \int_{S^2} \frac{n(q, \sigma) \chi (\frac{\langle q \rangle}{\rho}) (t - r + q + r(1 - \langle \omega, \sigma \rangle)) dS(\sigma) dq}{(t - r + q)^2(t + r + q)^2},$$  \hspace{1cm} (6.29)

Close to the light cone as $r \geq t/2$

$$\Phi^2[n](t, \sigma) \sim \Phi^2_0[n](\frac{t-r}{t+r}, r-t, \omega) + \Phi^2_1[n](\frac{t-r}{t+r}, r-t, \omega),$$  \hspace{1cm} (6.30)

where

$$\Phi^2_0[n](\frac{t-r}{t+r}, r-t, \omega) = \frac{1}{2r} \ln (\frac{\langle t + r \rangle}{\langle t - r \rangle}) \int_{r-t}^{\infty} n(q, \omega) dq,$$  \hspace{1cm} (6.31)

and

$$|\Phi^2(\frac{t-r}{t+r}, q, \omega)| \lesssim \frac{1}{(t + r)^{-a}}.$$  \hspace{1cm} (6.32)
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