The Lewis Correspondence for submodular groups

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1 Introduction

Let $G$ denote the group $\text{PSL}_2(\mathbb{R}) = \text{SL}_2(\mathbb{R})/\{\pm 1\}$. Let $\Gamma = \text{PSL}_2(\mathbb{Z})$ be the modular group. A submodular group is a subgroup of $\Gamma$ of finite index. It is the aim of this note to extend the Lewis Correspondence \cite{5, 6, 7} from $\Gamma$ to submodular groups. Since any submodular group $\Lambda$ contains a submodular subgroup which is normal in $\Gamma$ we will first assume that $\Lambda$ is normal and only later move from $\Lambda$ an arbitrary subgroup containing $\Lambda$. Let $H^+ = \{ x + iy \in \mathbb{C} : y > 0 \}$ be the upper half plane in $\mathbb{C}$. The group $G$ acts on $H^+$ by linear fractions, $\frac{az + b}{cz + d}$ This action preserves the hyperbolic geometry given by the Riemannian metric $\frac{1}{y^2} (dx^2 + dy^2)$ so it commutes with the hyperbolic Laplace operator $\Delta = -y^2 \left( (\frac{\partial}{\partial x})^2 + (\frac{\partial}{\partial y})^2 \right)$ and preserves the hyperbolic volume form $dx dy / y^2$. A Maaß form for $\Lambda$ is a function $f \in L^2(\Lambda \setminus H^+)$ which is an eigenfunction of $\Delta$. We also define $H^-$ to be the lower half plane in $\mathbb{C}$.

The Lewis Correspondence attaches a certain “period function” to a given Maaß form for $\Gamma$. To extend it to $\Lambda \neq \Gamma$ we have to start with Maaß forms for $\Lambda$. These form a module under the finite group $\Gamma / \Lambda$ under the regular representation and so Maaß forms for $\Lambda$ are related to Maaß forms for $\Gamma$ twisted by a finite dimensional representation $(\eta, V_\eta)$ by the following mechanism:

Let $W$ be a $\mathbb{C}[\Gamma]$-module, which is finite dimensional as $\mathbb{C}$-vector space and trivially acted upon by $\Lambda$. Under the action of the finite group $\Gamma / \Lambda$ the module $W$ decomposes into isotypic components,

$$ W = \bigoplus_{\eta \in \hat{\Gamma}/\Lambda} W(\eta), $$

where $\hat{\Gamma}/\Lambda$ denotes the set of isomorphism classes of irreducible unitary representations of $\Gamma / \Lambda$, i.e., the unitary dual of this finite group. For $\eta \in \hat{\Gamma}/\Lambda$ let $\check{\eta}$ denote its dual representation. There is a natural isomorphism

$$ \text{ev}: (W \otimes \eta)^\Gamma \otimes \check{\eta} \rightarrow W(\check{\eta}) $$

given by $\text{ev}(\sum_j (w_j \otimes \alpha_j) \otimes \beta) := \sum_j \langle \alpha_j, \beta \rangle w_j$. On the other hand, the inclusion $W(\check{\eta}) \subset W(\eta)$ induces an isomorphism $(W \otimes \eta)^\Gamma \cong (W(\check{\eta}) \otimes \eta)^\Gamma$. 

and the projection map $\text{Pr}$ from $W \otimes \eta$ to $(W \otimes \eta)^\Gamma$ is explicitly given by

$$\text{Pr}(w \otimes \alpha) = \frac{1}{|\Gamma:/\Lambda|} \sum_{\gamma \in \Gamma:/\Lambda} \gamma.w \otimes \gamma.\alpha.$$  

Finally, elementary character theory shows that the canonical projection $\mathcal{P}_\eta: W \to W(\tilde{\eta})$ given by the decomposition \( (\text{Id}) \) equals

$$\mathcal{P}_\eta w = \frac{d_\eta}{|\Gamma:/\Lambda|} \sum_{\gamma \in \Gamma:/\Lambda} \text{tr} \eta(\gamma) (\gamma \cdot w), \quad (3)$$

where $d_\eta$ is the degree of $\eta$ and $\tilde{\eta}$. Here we have used the convention that we write the space of a representation with the same symbol as the representation itself. Occasionally, to put emphasis on the space rather than the representation, we will also write $V_\eta$ for the representation space of $\eta$. In order to describe $W$ we decompose it into isotypic components and each such component is described by $(W \otimes \eta)^\Gamma$. We will in particular apply this to the space of Maaß forms for $\Lambda$ with a given Laplace eigenvalue. But we also can retrieve Maaß forms of an arbitrary submodular group $\Sigma$. For this let $\Lambda \subset \Sigma \subset \Gamma$ be a submodular group which is normal in $\Gamma$, and let $W$ be the space of $\Lambda$-Maaß forms. Then

$$W \cong \bigoplus_{\eta \in \hat{\Lambda}/\Gamma} (W \otimes \eta)^\Gamma \otimes \tilde{\eta}.$$  

The space of $\Sigma$-Maaß forms is just the space of $\Sigma$-invariants herein, i.e., the space

$$W^\Sigma \cong \bigoplus_{\eta \in \hat{\Lambda}/\Gamma} (W \otimes \eta)^\Gamma \otimes \tilde{\eta}^\Sigma.$$  

So $W^\Sigma$ is described by the spaces $(W \otimes \eta)^\Gamma$ and the dimensions $\dim(\tilde{\eta}^\Sigma)$ for $\eta \in \hat{\Lambda}/\Gamma$. This applies in particular to the congruence subgroups $\Lambda = \Gamma(N)$ and $\Sigma = \Gamma_0(N)$. So we fix an irreducible representation $\eta$ of $\Gamma$ with finite image.

We fix the following notation for the canonical generators of $\Gamma$:

$$S = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
Then $S^2 = 1 = (ST)^3$, and $T$ is of infinite order. Let $\mathcal{F}_\eta$ be the space of holomorphic functions $f: \mathbb{C} \setminus \mathbb{R} \to V_\eta$ with

$$f(z + 1) = \eta(T)f(z),$$

$$f(z) = O(1) \text{ as } |\text{Im}(z)| \to \infty,$$

$$0 = f(i\infty) + f(-i\infty).$$

The last condition needs explaining. Since $\eta$ has finite image, there is a smallest $N := N_\eta \in \mathbb{N}$ such that $\eta(T^N)$ equals the identity. It follows that $f$ has a Fourier expansion

$$f(z) = \sum_{k \in \mathbb{Z}} e^{2\pi i k/N} z^k, \quad v^+_k \in V_\eta,$$

in $\mathbb{H}^+$ and similarly with $v^-_k \in V_\eta$ in $\mathbb{H}^-$. Condition (6) leads to $v^+_k = v^-_k = 0$ for every $k \in \mathbb{N}$. Thus the limits do exist and satisfy $f(i\infty) = v^+_0$ and $f(-i\infty) = v^-_0$.

Consider the space $\mathcal{F}_{\nu,\eta}$ of all $f \in \mathcal{F}_\eta$ for which the map

$$z \mapsto f(z) - z^{-2\nu - 1} \eta(S)f\left(\frac{-1}{z}\right)$$

extends holomorphically to $\mathbb{C} \setminus (-\infty,0]$ and the space $\Psi_{\nu,\eta}$ of all holomorphic functions $\psi: \mathbb{C} \setminus (-\infty,0] \to V_\eta$ satisfying

$$\eta(T)\psi(z) = \psi(z + 1) + (z + 1)^{-2\nu - 1} \eta(ST^{-1})\psi\left(\frac{z}{z + 1}\right)$$

and

$$0 = e^{-\pi i \nu} \lim_{\text{Im}(z) \to \infty} \left(\psi(z) + z^{-2\nu - 1} \eta(S)\psi\left(\frac{-1}{z}\right)\right) + e^{\pi i \nu} \lim_{\text{Im}(z) \to -\infty} \left(\psi(z) + z^{-2\nu - 1} \eta(S)\psi\left(\frac{-1}{z}\right)\right),$$

where both limits exist. We call (8) the Lewis equation.

Let $\pi_\nu$ be the principal series representation of $G$ associated with the parameter $\nu \in \mathbb{C}$ and $\pi^{-\omega}_\nu$ the corresponding space of hyperfunction vectors. As a crucial tool we will use the space

$$A_{\nu,\eta}^{-\omega} = (\pi^{-\omega}_\nu \otimes \eta)^\Gamma = H^0(\Gamma, \pi^{-\omega}_\nu \otimes \eta).$$
and call it the space of \( \eta \)-automorphic hyperfunctions.

Generalizing results of Bruggeman (see [1], Prop. 2.1 and Prop. 2.3), we will show in Proposition 2.2 that there is a linear isomorphism \( A_{\nu,\eta}^{-\omega} \to F_{\nu,\eta} \) and (using this) establish in Proposition 2.3 a linear map \( B: A_{\nu,\eta}^{-\omega} \to \Psi_{\nu,\eta} \), which we call the Bruggeman transform. It turns out to be bijective unless \( \nu \in \frac{1}{2} + \mathbb{Z} \).

Recall that a Maaß wave form for a subgroup \( \Lambda \) of \( \Gamma \) (not necessarily normal) and parameter \( \nu \in \mathbb{C} \) is a function \( u \) on \( \mathbb{H}^+ \) which is twice continuously differentiable and satisfies

\[
\begin{align*}
  u(\gamma z) &= u(z) \quad \text{for every } \gamma \in \Gamma, \\
  \infty > & \int_{\Gamma \backslash \mathbb{H}^+} |u(z)|^2 \, dz, \\
  \Delta u &= \left( \frac{1}{4} - \nu^2 \right) u.
\end{align*}
\]

By the regularity of solutions of elliptic differential equations the last condition implies that \( u \) is real analytic. Let \( \mathcal{M}_\nu = \mathcal{M}_\nu^\Lambda \) be the space of all Maaß wave forms for \( \Lambda \).

If \( \Lambda \) is normal of finite index in \( \Gamma \) the finite group \( \Gamma / \Lambda \) acts on this space, and as in (1) we get an isotypic decomposition,

\[
\mathcal{M}_\nu = \bigoplus_{\eta \in \hat{\Gamma} / \Lambda} \mathcal{M}_{\nu}(\eta),
\]

and for each \( \eta \),

\[
\mathcal{M}_{\nu}(\eta) \cong V_{\eta}^* \otimes (V_{\eta} \otimes \mathcal{M}_\nu)^{\Gamma / \Lambda}.
\]

We set \( \mathcal{M}_{\nu,\eta} \) equal to \( (V_{\eta} \otimes \mathcal{M}_\nu)^{\Gamma / \Lambda} \). Then \( \mathcal{M}_{\nu,\eta} \) can be viewed as the space of all functions \( u: \mathbb{H}^+ \to V_{\eta} \) which are twice continuously differentiable and satisfy

\[
\begin{align*}
  u(\gamma z) &= \eta(\gamma)u(z) \quad \text{for every } \gamma \in \Gamma, \\
  \infty > & \int_{\Gamma \backslash \mathbb{H}^+} \|u(z)\|^2 \, dz, \\
  \Delta u &= \left( \frac{1}{4} - \nu^2 \right) u.
\end{align*}
\]
We define the space $S_{\nu,\eta}$ of Maaß cusp forms to be the space of all $u \in M_{\nu,\eta}$ such that

$$\int_0^N u(z + t) \, dt = 0$$

(18)

for every $z \in \mathbb{H}^+$. Here, as before, $N$ is the order of $\eta(T)$, so that in particular $\eta(T)^N = 1$ and $u(z + N) = u(z)$.

For $\text{Re}\nu > -\frac{1}{2}$ consider the space $\Psi^o_{\nu,\eta}$ of all $\psi \in \Psi_{\nu,\eta}$ satisfying

$$\psi(z) = O(\min\{1, |z|^{-C}\}) \quad \text{for } z \in \mathbb{C} \setminus (-\infty, 0],$$

(19)

for some $0 < C < 2\text{Re}\nu + 1$. We call the elements of $\Psi^o_{\nu,\eta}$ period functions. In Lemma 3.1 we establish for $\text{Re}\nu > -\frac{1}{2}$ two linear maps $S_{\nu,\eta} \rightarrow F_{\nu,\eta}$ and $S_{\nu,\eta} \rightarrow \Psi^o_{\nu,\eta}$.

Let $\tilde{M}_{\nu,\eta}$ be the space of all functions $u$ satisfying only (15) and (17). So there is no growth restriction on elements of $\tilde{M}_{\nu,\eta}$. For an automorphic hyperfunction $\alpha \in A_{\nu,\eta}^{-\omega}$ we consider the function $u : G \rightarrow V_\eta$ given by

$$u(g) := \langle \pi_{-\nu}(g) \varphi_0, \alpha \rangle.$$

Then $u$ is right $K$-invariant, hence can be viewed as a function on $\mathbb{H}^+$. As such it lies in $\tilde{M}_{\nu,\eta}$ since $\alpha$ is $\Gamma$-equivariant and the Casimir operator on $G$, which induces $\Delta$, is scalar on $\pi_\nu$ with eigenvalue $\frac{1}{4} - \nu^2$. The transform $P : \alpha \mapsto u$ is called the Poisson transform. It follows from [8], Theorem 5.4.3, that the Poisson transform

$$P : A_{\nu,\eta}^{-\omega} \rightarrow \tilde{M}_{\nu,\eta}$$

(20)

is an isomorphism for $\nu \notin \frac{1}{2} + \mathbb{Z}$.

For $\nu \notin \frac{1}{2} + \mathbb{Z}$ we finally define the Lewis transform as the map $L : M_{\nu,\eta} \rightarrow \Psi_{\nu,\eta}$, given by

$$L := B \circ P^{-1}.$$  

(21)

Our first main result (see Theorem 3.3) is a generalization of [7], Thm. 1.1, and says that the Lewis transform for $\nu \notin \frac{1}{2} + \mathbb{Z}$ and $\text{Re}\nu > -\frac{1}{2}$ is a linear isomorphism between the space of Maaß cusp forms $S_{\nu,\eta}$ and the space $\Psi^o_{\nu,\eta}$ of period functions.

A holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$ is uniquely determined by its values in $\mathbb{R}^+ := (0, \infty)$. Thus, in principle, it is possible to describe the period functions as a space of real analytic functions on the positive halfline.
Following ideas from [7], Chap. III, in this section we show how this can be done in an explicit way.

Consider the space $\Psi^R_{\nu,\eta}$ of all real analytic functions $\psi$ from $(0, \infty)$ to $V_\eta$ satisfying

\begin{align}
\eta(T)\psi(x) &= \psi(x + 1) + (x + 1)^{-2\nu - 1}\eta(ST^{-1})\psi\left(\frac{x}{x + 1}\right) \\
\psi(x) &= o(1/x), \quad \text{as } x \to 0, x > 0, \\
\psi(x) &= o(1), \quad \text{as } x \to +\infty, x \in \mathbb{R}.
\end{align}

Our second main result (see Theorem 4.4) is a generalization of [7], Thm. 2, and says that for $\text{Re}\nu > -\frac{1}{2}$ we have $\Psi^R_{\nu,\eta} = \{\psi|_{(0,\infty)} : \psi \in \Psi^o_{\nu,\eta}\}$.

We summarize the various spaces and mappings considered so far in one diagram:

2 Automorphic hyperfunctions

Let $A$ denote the subgroup of $G$ consisting of diagonal matrices and let $N$ be the subgroup of upper triangular matrices with $\pm 1$ on the diagonal. Let $P = AN$ be the group of upper triangular matrices. Finally, let $K = \text{PSO}(2) = \text{SO}(2)/\{\pm 1\}$ be the canonical maximal compact subgroup of $G$. The group $G$ then as a manifold is a direct product $G = ANK = PK$. For $\nu \in \mathbb{C}$ and $a = \pm \text{diag}(t, t^{-1})$, $t > 0$, let $a'' = t^{2\nu}$. We insert the factor 2 for compatibility reasons. Let $(\pi_\nu, V_{\pi_\nu})$ denote the principal series representation of $G$ with parameter $\nu$. The representation space $V_{\pi_\nu}$ is the Hilbert space of all
functions \( \varphi: G \to \mathbb{C} \) with \( \varphi(\alpha x) = a^{\nu+\frac{1}{2}}\varphi(x) \) for \( a \in A, n \in \mathbb{N}, x \in G \), and \( \int_K |\varphi(k)|^2 \, dk < \infty \) modulo nullfunctions. The representation is \( \pi_\nu(x)\varphi(y) = \varphi(yx) \). There is a special vector \( \varphi_0 \) in \( V_{\pi_\nu} \) given by

\[
\varphi_0(\alpha n x) = a^{\nu+\frac{1}{2}}.
\]

This vector is called the basic spherical function with parameter \( \nu \). The group \( G \) acts on the complex projective line \( \mathbb{P}_1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \) by linear fractions. This action has three orbits: the upper half plane \( \mathbb{H}^+ \), the lower half plane \( \mathbb{H}^- \) and the real projective line \( \mathbb{P}_1(\mathbb{R}) = \mathbb{R} \cup \{\infty\} \). The upper half plane can be identified with \( G/K \) via \( gK \mapsto g.i \) and \( \mathbb{P}_1(\mathbb{R}) \) can be identified with \( \mathbb{P}_1 \backslash G \) via

\[
P \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [c : d].
\]

Our embedding of \( \mathbb{R} \) into \( \mathbb{P}_1(\mathbb{R}) \) is via \( x \mapsto [1 : x] \), which can be viewed as the map

\[
N \to \mathbb{P}_1 \backslash PwN
\]

\[
\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto P \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}
\]

with the Weyl group element \( w = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Note that \( V_{\pi_\nu} \) can also be viewed as a space of sections of a line bundle over \( \mathbb{P}_1 \backslash G \). For this bundle the above embedding provides a trivialization over \( \mathbb{R} \). Using the corresponding Bruhat decomposition

\[
\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} c^{-1} & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & d \end{pmatrix}
\]

for \( c \neq 0 \) we obtain a realization of \( V_{\pi_\nu} \) on \( L^2(\mathbb{R}, \frac{1}{\pi} (1 + x^2)^{2\nu} \, dx) \) with the action

\[
\pi_\nu \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) f(x) = (cx - a)^{-2\nu-1} f \left( \frac{dx - b}{cx - a} \right).
\]

Transferring the action to \( L^2(\mathbb{R}, \frac{1}{\pi} \frac{dx}{1+x^2}) \) then yields the action

\[
\pi_\nu \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \varphi(x) = \left( \frac{1 + x^2}{(cx - a)^2 + (dx - b)^2} \right)^{\nu+\frac{1}{2}} \varphi \left( \frac{dx - b}{-cx + a} \right).
\]
used in \[1\]. This is the realization of the principal series we shall work
with. Note that in this realization the basic spherical function is simply the
constant function 1.

Let \( \pi_{\nu}^\omega \subset \pi_{\nu}^{-\omega} \) be the sets of analytic vectors and hyperfunction vectors,
respectively. For any open neighbourhood \( U \) of \( \mathbb{P}_1(\mathbb{R}) \) inside \( \mathbb{P}_1(\mathbb{C}) \) the space \( \pi_{\nu}^{-\omega} \) can be identified with the space
\[
O(U \setminus \mathbb{P}_1(\mathbb{R}))/O(U),
\]
where \( O \) denotes the sheaf of holomorphic functions. This space does not
depend on the choice of \( U \). For \( U \subseteq \mathbb{C} \) this follows from Lemma 1.1.2 of
\[8\] and generally by subtracting the Laurent series at infinity. The
\( G \)-action
is given by the above formula, where \( x \) is replaced by a complex variable \( z \).

Note that any hyperfunction \( \alpha \) on \( \mathbb{P}_1(\mathbb{R}) \) has a restriction to \( \mathbb{R} \) which can be
represented by a holomorphic function on \( \mathbb{C} \setminus \mathbb{R} \).

**Proposition 2.1** (Symmetry of gluing conditions) For \( f \in \mathcal{F}_\eta \) the following
conditions are equivalent:

1. \( z \mapsto f(z) - z^{-2\nu-1} \eta(S) f\left(\frac{-1}{z}\right) \) extends holomorphically to \( \mathbb{C} \setminus (-\infty, 0] \).
2. \( z \mapsto (1 + z^{-2})^{\nu + \frac{1}{2}} f\left(\frac{-1}{z}\right) \) and \( z \mapsto (1 + z^{2})^{\nu + \frac{1}{2}} \eta(S) f(z) \) define the same
   hyperfunction on \( \mathbb{R} \setminus \{0\} \).

**Proof:** “(2)⇒(1)” Suppose that

\[
(1 + z^{-2})^{\nu + \frac{1}{2}} f\left(\frac{-1}{z}\right) = (1 + z^{2})^{-\nu - \frac{1}{2}} \eta(S) f(z) + q(z)
\]

with a function \( q \) that is holomorphic in a neighborhood of \( \mathbb{R} \setminus \{0\} \). For
\( \text{Re} z > 0 \) we can divide the equation by \( (1 + z^{2})^{\nu + \frac{1}{2}} \) and obtain

\[
z^{-2\nu-1} f\left(\frac{-1}{z}\right) = \eta(S) f(z) + (1 + z^{2})^{-\nu - \frac{1}{2}} q(z).
\]

Since \( \eta(S) = \eta(S)^{-1} \), this implies the claim.

“(1)⇒(2)” If (1) holds, by the same calculation as above we see that for
\( \text{Re} z > 0 \) the function

\[
z \mapsto (1 + z^{2})^{\nu + \frac{1}{2}} \eta(S) f(z) - (1 + z^{-2})^{\nu + \frac{1}{2}} f\left(\frac{-1}{z}\right)
\]
extends holomorphically to the entire right half plane. But then the symmetry of this expression under the transformation \( z \mapsto -\frac{1}{z} \) yields the holomorphic extendability also on the left half plane which proves (2).

Recall the space \( A_{\nu,\eta}^{-\omega} = (\pi_{\nu}^{-\omega} \otimes \eta)^{\Gamma} = H^0(\Gamma, \pi_{\nu}^{-\omega} \otimes \eta) \) of \( \eta \)-automorphic hyperfunctions from [10].

**Proposition 2.2** (cf. [1], Prop. 2.1) There is a bijective linear map

\[
A_{\nu,\eta}^{-\omega} \rightarrow F_{\nu,\eta}
\]

\[
\alpha \mapsto f_\alpha
\]

such that the function \( z \mapsto (1 + z^2)^{\nu + \frac{1}{2}} f_\alpha(z) \) represents the restriction \( \alpha|_\mathbb{R} \).

**Proof:** The space \( A_{\nu,\eta}^{-\omega} = (\pi_{\nu}^{-\omega} \otimes \eta)^{\Gamma} \) can be viewed as the space of all \( V_\eta \)-valued hyperfunctions \( \alpha \) in \( \mathbb{P}_1(\mathbb{R}) \) satisfying the invariance condition

\[
\pi_{\nu}(\gamma^{-1})\alpha = \eta(\gamma)\alpha
\]

for every \( \gamma \in \Gamma \). Pick a representative \( f \) for \( \alpha \). The \( V_\eta \)-valued function \( F: z \mapsto (1 + z^2)^{-\nu - \frac{1}{2}} f(z) \) is holomorphic on \( 0 < |\text{Im}(z)| < \varepsilon \) for some \( \varepsilon > 0 \).

Note that the invariance of \( \alpha \) under \( T \) implies that for some function \( q \), holomorphic on a neighbourhood of \( \mathbb{R} \), we have

\[
\eta(T) f(z) + q(z) = (\pi_{\nu}(T^{-1}) f)(z)
\]

\[
= \left( \frac{1 + z^2}{1 + (z + 1)^2} \right)^{\nu + \frac{1}{2}} f(z + 1)
\]

\[
= (1 + z^2)^{\nu + \frac{1}{2}} F(z + 1),
\]

so that

\[
F(z + 1) = \eta(T) F(z) + (1 + z^2)^{-(\nu + \frac{1}{2})} q(z).
\]

Therefore \( F \) represents a hyperfunction on \( \mathbb{P}_1(\mathbb{R}) = \mathbb{R} \cup \{\infty\} \) which is invariant under the translation \( z \mapsto z + N \). This hyperfunction has a representative which is holomorphic in \( \mathbb{P}_1(\mathbb{C}) \setminus \mathbb{P}_1(\mathbb{R}) \). The freedom in this representative is an additive constant. So there is a unique representative \( f_\alpha \) of the form

\[
f_\alpha(z) = \begin{cases} 
\frac{1}{2} v_0 + \sum_{k=1}^{\infty} e^{2\pi i \frac{k}{N} z} v_k^+, & z \in \mathbb{H}^+, \\
-\frac{1}{2} v_0 - \sum_{k=1}^{\infty} e^{-2\pi i \frac{k}{N} z} v_k^-, & z \in \mathbb{H}^-.
\end{cases}
\]
So $f_\alpha \in \mathcal{F}_\eta$ and $(1 + z^2)^{\nu + \frac{1}{2}} f_\alpha(z)$ represents $\alpha|_\mathbb{R}$. To show the injectivity of the map in the Proposition assume that $f_\alpha = 0$. Then $\alpha$ is supported in $\{\infty\}$. Since latter set is not $\Gamma$-invariant, $\alpha$ must be zero. To see that $f_\alpha$ lies in $\mathcal{F}_{\nu, \eta}$, recall that the invariance of $\alpha$ under $S$ implies that

$$(1 + z^{-2})^{\nu + \frac{1}{2}} f_\alpha \left( \frac{1}{-z} \right) = (1 + z^2)^{\nu + \frac{1}{2}} \eta(S) f_\alpha (z) + \tilde{q}(z)$$

with $\tilde{q}(z)$ holomorphic on a neighbourhood of $\mathbb{R} \smallsetminus \{0\}$. Thus Proposition 2.1 shows that $f_\alpha$ satisfies (7) and hence $f_\alpha \in \mathcal{F}_{\nu, \eta}$. To finally show surjectivity, let $f \in \mathcal{F}_{\nu, \eta}$. Then the function

$$z \mapsto (1 + z^2)^{\nu + \frac{1}{2}} f(z)$$

represents a hyperfunction $\beta_0$ on $\mathbb{R}$ that satisfies $\pi_\nu (T^{-1}) \beta_0 = \eta(T) \beta_0$. Let $\beta_\infty := (\pi_\nu \otimes \eta)(S) \beta_0$. Then $\beta_\infty$ is a hyperfunction on $\mathbb{P}_1(\mathbb{R}) \smallsetminus \{0\}$ with representative $z \mapsto (1 + z^{-2})^{\nu + \frac{1}{2}} \eta(S) f \left( \frac{1}{z^2} \right)$. According to Proposition 2.1 the restrictions of $\beta_0$ and $\beta_\infty$ to $\mathbb{P}_1(\mathbb{R}) \smallsetminus \{0, \infty\}$ agree. Thus $\beta_0$ and $\beta_\infty$ are restrictions of a hyperfunction $\beta$ on $\mathbb{P}_1(\mathbb{R})$ which is then easily seen to be $S$-invariant. Using $\beta_0$ we see that the support of

$$(\pi_\nu \otimes \eta)(T) \beta - \beta$$

is contained in $\{\infty\}$. Using $\beta_\infty$ we see that for $|z| > 2$, $z \notin \mathbb{R}$, this hyperfunction is represented by

$$\left( \frac{1 + z^2}{1 + (z-1)^2} \right)^{\nu + \frac{1}{2}} (1 + (z-1)^{-2})^{\nu + \frac{1}{2}} \eta(TS) f \left( \frac{-1}{z-1} \right)$$

$$- (1 + z^{-2})^{\nu + \frac{1}{2}} \eta(S) f \left( \frac{-1}{z} \right)$$

$$= (1 + z^2)^{\nu + \frac{1}{2}} (z-1)^{-2\nu - 1} \eta(TS) f \left( \frac{-1}{z-1} \right)$$

$$- (1 + z^{-2})^{\nu + \frac{1}{2}} \eta(S) f \left( \frac{-1}{z} \right)$$

$$= (1 + z^{-2})^{\nu + \frac{1}{2}} \times$$

$$\left( \frac{z}{z-1} \right)^{2\nu + 1} \eta(TST^{-1}) f \left( \frac{z-2}{z-1} \right) - \eta(ST^{-1}) f \left( \frac{z-1}{z} \right)$$
Since \( f(z) \) is holomorphic around \( z = 1 \) it follows that this function is holomorphic around \( z = \infty \). Hence \( \beta \) is invariant under \( T \). Now the claim follows because the elements \( S \) and \( T \) generate \( \Gamma \). \( \square \)

**Proposition 2.3** (Bruggeman transform; cf. [1], Prop. 2.3) For \( \alpha \in A_{\nu, \eta}^{-\omega} \) put

\[
\psi_\alpha(z) := f_\alpha(z) - z^{-2\nu-1}\eta(S)f_\alpha\left(\frac{-1}{z}\right),
\]

with \( f_\alpha \) as in Proposition 2.2. Then the Bruggeman transform \( B: \alpha \mapsto \psi_\alpha \) maps \( A_{\nu, \eta}^{-\omega} \) to \( \Psi_{\nu, \eta} \). It is a bijection if \( \nu \notin \frac{1}{2} + \mathbb{Z} \).

**Proof:** Let \( \alpha \in A_{\nu, \eta}^{-\omega} \) and define \( \psi_\alpha \) as in the Proposition. By Proposition 2.2 the map \( \psi_\alpha \) extends to \( \mathbb{C} \smallsetminus (-\infty, 0] \). We compute

\[
\psi_\alpha(z + 1) + (z + 1)^{-2\nu-1}\eta(ST^{-1})\psi_\alpha\left(\frac{z}{z + 1}\right) = \]

\[
= f_\alpha(z + 1) - (z + 1)^{-2\nu-1}\eta(S)f_\alpha\left(\frac{-1}{z + 1}\right) + (z + 1)^{-2\nu-1}\eta(ST^{-1}) \times
\]

\[
\times \left(f_\alpha\left(\frac{z}{z + 1}\right) - \left(\frac{z}{z + 1}\right)^{-2\nu-1}\eta(S)f_\alpha\left(\frac{-1}{z + 1}\right)\right).
\]

Since \( \frac{z}{z + 1} = 1 - \frac{1}{z + 1} \) and \( f_\alpha(1 - \frac{1}{z + 1}) = \eta(T)f_\alpha\left(\frac{-1}{z + 1}\right) \) we see that the two middle summands cancel out. It remains

\[
\eta(T)f_\alpha(z) - z^{-2\nu-1}\eta(ST^{-1}S)f_\alpha\left(\frac{-z - 1}{z}\right) = \]

\[
= \eta(T)f_\alpha(z) - z^{-2\nu-1}\eta(ST^{-1}ST^{-1})f_\alpha\left(\frac{-1}{z}\right) \]

\[
= \eta(T)f_\alpha(z) - z^{-2\nu-1}\eta(S)f_\alpha\left(\frac{-1}{z}\right) \]

\[
= \eta(T)\psi_\alpha(z).
\]

Here we have used \( ST^{-1}ST^{-1} = TS \). This proves that \( \psi_\alpha \) satisfies the functional equation (8).

Next, if \( \nu \notin \frac{1}{2} + \mathbb{Z} \), then one sees that \( \psi_\alpha(z) + z^{-2\nu-1}\eta(S)\psi_\alpha\left(\frac{-1}{z}\right) \) equals zero and so \( \psi_\alpha \) lies in \( \Psi_{\nu, \eta} \). If \( \nu \notin \frac{1}{2} + \mathbb{Z} \) then recall that we take the standard
branch of the logarithm to define $z^{-2\nu-1}$. For $\psi(-1/z)$ one then takes a complimentary branch and one gets the inversion formula
\begin{equation}
f_\alpha(z) = \frac{1}{1 + e^{\pm 2\pi i \nu}} \left( \psi_\alpha(z) + z^{-2\nu-1} \eta(S) \psi_\alpha \left( \frac{-1}{z} \right) \right) \tag{25}\end{equation}
for $z \in \mathbb{H}^\pm$. This proves $B_\alpha \in \Psi_{\nu,\eta}$ and it only remains to show that the Bruggeman transform is surjective. But a simple calculation, similar to the one given above shows that for a holomorphic function $\psi: \mathbb{C} \setminus (-\infty, 0] \to V_\eta$ satisfying $\mathfrak{N}$ the function $f: \mathbb{C} \setminus \mathbb{R} \to V_\eta$, defined from $\psi$ via the inversion formula (25), satisfies (4). If $\psi$ satisfies (3), then $f$ satisfies (5) and (6). In view of Proposition 2.2 this, finally, proves the claim.

\[\square\]

### 3 Maaß wave forms

Recall the space $S_{\nu,\eta}$ of Maaß cusp forms from [18] and consider a $u$ in $S_{\nu,\eta}$. Because of $u(z + N) = u(z)$ the function $u$ has a Fourier series
\[u(z) = u(x + iy) = \sum_{k \in \mathbb{Z}} A_k(y) e^{2\pi i \frac{k}{N} x} v_k\]
for some $v_k \in V_\eta$. The differential equation $\Delta u = \left( \frac{1}{4} - \nu^2 \right) u$ implies a differential equation for $A_k(y)$ which implies that it must be a linear combination of $I$ and $K$-Bessel functions. The fact that $u$ is square integrable rules out the $I$-Bessel functions, so
\[A_k(y) = \sqrt{y} K_\nu \left( 2\pi \frac{|k|}{N} y \right)\]
times a constant which we can assume to be 1 by multiplying it to $v_k$. By Theorem 3.2 of [4] it follows that the norms $\|v_k\|$ are bounded as $|k| \to \infty$. The functional equation $u(z + 1) = \eta(T) u(z)$ is reflected in the fact that the $v_k$ are eigenvectors of $\eta(T)$, since we get $\eta(T)v_k = e^{2\pi i \frac{k}{N}} v_k$. Now set
\[f_u(z) := \begin{cases} \sum_{k>0} k^\nu e^{2\pi i \frac{k}{N} z} v_k, & \text{Im}(z) > 0, \\ -\sum_{k<0} |k|^\nu e^{2\pi i \frac{k}{N} z} v_k, & \text{Im}(z) < 0. \end{cases} \tag{26}\]
From the construction it is clear that $f_u$ satisfies (4) - (6), i.e. $f_u \in \mathcal{F}_\eta$. It will play the role of our earlier $f_\alpha$ (cf. Proposition 2.2), so we define

$$
\psi_u(z) := f_u(z) - z^{-2\nu-1}\eta(S)f_u\left(\frac{-1}{z}\right). \tag{27}
$$

**Lemma 3.1** For $\text{Re}\nu > -\frac{1}{2}$ the equation (27) and (27) define linear maps

$$
S_{\nu,\eta} u \mapsto f_u \quad \text{and} \quad S_{\nu,\eta} u \mapsto \psi_u.
$$

**Proof:** To prove that $f_u \in \mathcal{F}_{\nu,\eta}$ we will need the following two Dirichlet series. For $\varepsilon = 0, 1$ set

$$
L_\varepsilon(u, s) := \sum_{k\neq 0} \text{sign}(k)\varepsilon \left(\frac{N}{|k|}\right)^s v_k.
$$

We will relate $L_0$ and $L_1$ to $u$ by the Mellin transform. For this let

$$
u_0(y) = \frac{1}{\sqrt{y}}u(iy), \quad u_1(y) = \frac{\sqrt{y}}{2\pi i}u_x(iy), \tag{28}
$$

where $u_x = \frac{\partial}{\partial x}u$. Next define

$$
\hat{L}_\varepsilon(u, s) := \int_0^\infty u_\varepsilon(y)y^s dy.
$$

Plugging in the Fourier series of $u$ and using the fact that

$$
\int_0^\infty K_\nu(2\pi y)y^s dy = \Gamma_\nu(s) := \frac{1}{4\pi^s} \Gamma\left(\frac{s - \nu}{2}\right) \Gamma\left(\frac{s + \nu}{2}\right),
$$

we get

$$
\hat{L}_0(u, s) = \Gamma_\nu(s)L_0(u, s),
$$

and similarly,

$$
\hat{L}_1(u, s) = \Gamma_\nu(s + 1)L_1(u, s).
$$

On the other hand, the usual process of splitting the Mellin integral and using the functional equations

$$
u_\varepsilon\left(\frac{1}{y}\right) = (-1)^\varepsilon y \eta(S)\nu_\varepsilon(y), \quad \varepsilon = 0, 1,
$$

we obtain
(which can be checked using the Taylor series of \( u \)), one gets that \( \hat{L}_\varepsilon \) extends to an entire function and satisfies the functional equation,

\[
\hat{L}_\varepsilon(u, s) = (-1)^\varepsilon \eta(S) \hat{L}_\varepsilon(u, 1-s).
\]

With a similar, even easier computation one gets

\[
\int_0^\infty y^\frac{s}{2} (f_u(iy) - (-1)^\varepsilon f_u(-iy)) \frac{dy}{y} = \frac{\Gamma(s) N^\nu}{(2\pi)^s} L_\varepsilon(u, s - \nu).
\]

This implies that the Mellin transforms \( M^\pm f(s) := \int_0^\infty y^s f(\pm iy) \frac{dy}{y} \) can be calculated as

\[
M^\pm f(s) = \pm \frac{\Gamma(s) N^\nu}{(2\pi)^s} \left( L_0(u, s - \nu) \pm L_1(u, s - \nu) \right)
\]

\[
= \pm N^\nu \pi^{-\nu - \frac{s+1}{2}} \Gamma \left( \frac{s+1}{2} \right) \Gamma \left( \frac{2\nu - s}{2} \right) \sin \pi \left( \nu + 1 - \frac{s}{2} \right) \hat{L}_0(u, s - \nu)
\]

\[
+ N^\nu \pi^{-\nu - \frac{s-1}{2}} \Gamma \left( \frac{s-1}{2} \right) \Gamma \left( \frac{2\nu + s}{2} \right) \sin \pi \left( \nu + \frac{1}{2} - \frac{s}{2} \right) \hat{L}_1(u, s - \nu).
\]

The last identity follows from the standard equations

\[
\Gamma \left( \frac{x+1}{2} \right) \Gamma \left( \frac{x}{2} \right) = \Gamma(x) 2^{1-x} \sqrt{\pi}, \quad \Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin \pi x}.
\]

Thus the Mellin transform \( M^\pm f(s) \) is seen to be holomorphic for \( \text{Re}(s) > 0 \) and rapidly decreasing on any vertical strip. The Mellin inversion formula yields for \( C > 0 \),

\[
f_u(\pm iy) = \frac{1}{2\pi i} \int_{\text{Re}(s) = C} y^{-s} M^\pm f_u(s) \, ds.
\]

This extends to any \( z \in \mathbb{C} \setminus \mathbb{R} \) to give

\[
f_u(z) = \frac{1}{2\pi i} \int_{\text{Re}(s) = C} e^{\pm \frac{s+1}{2} is} z^{-s} M^\pm f_u(s) \, ds
\]

for \( z \in \mathbb{H}^\pm \). For \( 0 < C < 2\text{Re}\nu + 1 \) (here we need \( \text{Re}\nu > -\frac{1}{2} \)) it follows that

\[
\psi_u(z) = \frac{1}{2\pi i} \int_{\text{Re}(s) = C} \left( e^{\pm \frac{2}{2} is} z^{-s} - e^{\pm \frac{2}{2} is} z^{-2\nu - 1} z^s \eta(S) \right) M^\pm f_u(s) \, ds.
\]

Writing this as the difference of two integrals, substituting \( s \) in the second integral with \( 2\nu + 1 - s \) and shifting the contour we arrive at the formula

\[
\frac{1}{2\pi i} \int_{\text{Re}(s) = C} \left( e^{\pm \frac{2}{2} is} z^{-s} M^\pm f_u(s) - e^{\pm \frac{2}{2} is} z^{2\nu + 1 - s} \eta(S) M^\pm f_u(2\nu + 1 - s) \right) \, ds.
\]
for $\psi_u$. Using the identities
\[
\pm e^{\pm \frac{\pi}{2} i s} \cos \left( \nu + \frac{1}{2} - \frac{s}{2} \right) \mp e^{\pm \frac{\pi}{2} i (2\nu+1-s)} \cos \frac{s}{2} = i \sin \left( \nu + \frac{1}{2} \right),
\]
\[
e^{\pm \frac{\pi}{2} i s} \sin \left( \nu + \frac{1}{2} - \frac{s}{2} \right) \mp e^{\pm \frac{\pi}{2} i (2\nu+1-s)} \sin \frac{s}{2} = \sin \left( \nu + \frac{1}{2} \right),
\]
and the functional equation of $\hat{L}_\varepsilon$ we see that the integrand of (30) equals
\[
z^{-s} N^\nu \sin \left( \nu + \frac{1}{2} \right) \left[ \pi^{-\nu-\frac{3}{2}} \Gamma \left( \frac{s+1}{2} \right) \Gamma \left( \frac{2\nu+2-s}{2} \right) i \hat{L}_0(u,s-\nu) + \pi^{-\nu-\frac{3}{2}} \Gamma \left( \frac{s}{2} \right) \Gamma \left( \frac{2\nu+1-s}{2} \right) \hat{L}_1(u,s-\nu) \right].
\]
(31)

Since this expression is independent of whether $z$ lies in $\mathbb{H}^+$ or $\mathbb{H}^-$, it follows that $f_u(z) - z^{-2\nu-1} \eta(S) f_u \left( \frac{1}{z} \right)$ extends to a holomorphic function on $\mathbb{C} \smallsetminus (-\infty,0]$, i.e., the function $f_u$ indeed lies in the space $\mathcal{F}_{\nu,\eta}$. The linearity of the map is clear.

It remains to show that $\psi_u \in \Psi^\nu_{\nu,\eta}$. Note that in view of $f_u \in \mathcal{F}_{\nu,\eta}$ Proposition 2.2 shows that the function $z \mapsto (1 + z^2)^{\nu+\frac{1}{2}} f_u(z)$ represents a hyperfunction $\alpha_u \in A_{\nu,\eta}$. Then, according to Proposition 2.3 we have $\psi_u = B(\alpha_u)$ so $\psi_u$ satisfies (9). The asymptotic property (19) now follows from the integral representation (30) with the $C$ chosen there. More precisely, the bound $O(|z|^{-C})$ follows directly from (31) since the integrant divided by $z^{-s}$ is of $\pi$-exponential decay, whereas the $O(1)$-bound is obtained by moving the contour slightly to the left of the imaginary axis picking up the residue at 0 which is proportional to 1 (see [7], §1.4 for more details on this type of argument).

\[\square\]

**Lemma 3.2** For $0 \neq k \in \mathbb{Z}$ let $\alpha_k$ be the hyperfunction on $\mathbb{P}_1(\mathbb{R})$ represented by $(1 + z^2)^{\nu+\frac{1}{2}} f_k(z)$ with
\[
f_k(z) = \begin{cases} \text{sign}(k) \cdot e^{2\pi i \frac{z}{\sqrt{b}}} & \text{for } \text{sign}(k) \cdot \text{Im}(z) > 0 \\ 0 & \text{for } \text{sign}(k) \cdot \text{Im}(z) < 0. \end{cases}
\]

Then we have that
\[
\left\langle \pi_{-\nu} \left( \begin{array}{c} \sqrt{\alpha} \\ \sqrt{\alpha} \end{array} \right) \mathcal{F}_0, \alpha_k \rightangle
\]
equals
\[ 2 \text{sign}(k) \left( \frac{N}{|k|} \right)^\nu \frac{\pi^{-\nu-\frac{1}{2}}}{\Gamma(\frac{1}{2} - \nu)} \sqrt{b} K_\nu \left( \frac{2\pi |k|}{N} b \right) e^{2\pi i \frac{k}{N} a}, \]
where is \( K_\nu \) the \( K \)-Bessel function with parameter \( \nu \).

**Proof:** For this we will need the following identity (cf. [1], §4 or [2], p.136)
\[ \int_{-\infty}^{\infty} \left( \frac{1}{y^2 + (\tau - x)^2} \right)^{\frac{1}{2} - \nu} e^{2\pi i k \tau} \, dt = 2\pi^{\frac{1}{2} - \nu} |k|^{-\nu} \frac{|k| - \nu}{\Gamma(\frac{1}{2} - \nu)} e^{2\pi i k x}. \] (32)

Note that \( g = \left( \begin{array}{c} \sqrt{b} \frac{a}{\sqrt{b}} \\ 0 \sqrt{b} \end{array} \right) \) satisfies \( g \cdot i = a + ib \). Therefore, by abuse of notation, we write \( P(\alpha_k)(a + ib) \) for \( \left\langle \pi^{-\nu} \left( \begin{array}{c} \sqrt{b} \frac{a}{\sqrt{b}} \\ 0 \sqrt{b} \end{array} \right) \varphi_0, \alpha_k \right\rangle \). According to [1], §4, we can calculate
\[
P(\alpha_k)(a + ib) = \left\langle \left( \begin{array}{c} 1 + x^2 \\ b + \left( \frac{x}{\sqrt{b}} - \frac{a}{\sqrt{b}} \right)^2 \end{array} \right)^{-\nu + \frac{1}{2}} , \alpha_k \right\rangle \\
= b^{-\nu + \frac{1}{2}} \left\langle \left( \frac{1 + x^2}{b^2 + (x - a)^2} \right)^{-\nu + \frac{1}{2}}, (1 + z^2)^{\nu + \frac{1}{2}} f_k(z) \right\rangle \\
= \text{sign}(k) b^{-\nu + \frac{1}{2}} \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{b^2 + (x - a)^2} \right)^{-\nu + \frac{1}{2}} e^{2\pi i \frac{k}{N} x} \, dx \\
= 2 \text{sign}(k) \left( \frac{N}{|k|} \right)^\nu \frac{\pi^{-\nu-\frac{1}{2}}}{\Gamma(\frac{1}{2} - \nu)} \sqrt{b} K_\nu \left( \frac{2\pi |k|}{N} b \right) e^{2\pi i \frac{k}{N} a},
\]
where in the last step we have used (32). \( \square \)

**Theorem 3.3** (Lewis transform; cf. [7], Thm. 1.1) For \( \nu \notin \frac{1}{2} + \mathbb{Z} \) and \( \text{Re} \nu > -\frac{1}{2} \) the Lewis transform is a bijective linear map from the space of Maaß cusp forms \( S_{\nu,\eta} \) to the space \( \Psi_{\nu,\eta}^o \) of period functions.

**Proof:** We begin by showing that the Lewis transform is injective on \( S_{\nu,\eta} \). This will be done by proving that we can recover \( u \) from \( \psi_u \), where we use
the notation from Lemma 3.1. The hypothesis $\nu \notin \frac{1}{2} + \mathbb{Z}$ guarantees that we can recover $f_u$ from $\psi_u$ via a simple algebraic manipulation (cf. the proof of Proposition 2.3). Thus it suffices to express $u$ in terms of $\alpha_u$. But applying Lemma 3.2 to the summands in the defining formula (26) for $f_u$, we obtain

$$P\alpha_u(a + ib) = \frac{2^{\nu}}{\pi^{\nu + \frac{1}{2}}} b \sqrt{\sum_{k \neq 0} K_{\nu} \left(2\pi \frac{|k|}{N} b \right) e^{2\pi i \frac{k}{N} a} v_k}$$

$$= \frac{2^{\nu}}{\pi^{\nu + \frac{1}{2}}} u(a + ib). \quad (33)$$

It remains to show that $L(S_{\nu, \eta}) = \Psi_o^{\nu, \eta}$. To do this pick $\psi \in \Psi_o^{\nu, \eta}$. According to Propositions 2.2 and 2.3 we can find a hyperfunction $\alpha \in A_{\Gamma}^{\nu, \eta}$ represented by the function $(1 + z^2)^{\nu + \frac{1}{2}} f$ with $f \in F_{\nu, \eta}$ such that

$$\psi(z) = f(z) - z^{-2\nu - 1} \eta(S) f \left( \frac{1}{z} \right),$$

$$f(z) = \frac{1}{1 + e^{\pm 2\pi iv}} \left( \psi(z) + z^{-2\nu - 1} \eta(S) \psi \left( \frac{1}{z} \right) \right)$$

for $z \in \mathbb{H}^\pm$. The function $f$ admits a Fourier expansion of the form

$$f(z) = \begin{cases} 
\frac{1}{2} v_0 + \sum_{k=1}^{\infty} e^{2\pi i \frac{k}{N} \alpha} v_k, & z \in \mathbb{H}^+, \\
-\frac{1}{2} v_0 - \sum_{k=1}^{\infty} e^{-2\pi i \frac{k}{N} \alpha} v_k, & z \in \mathbb{H}^-.
\end{cases}$$

The asymptotic property (19) of $\psi$ implies that

$$z^{-2\nu - 1} \eta(S) \psi \left( \frac{1}{z} \right) = O(|z|^{-2\nu - 1})$$

for $\zeta \in \mathbb{C} \setminus (-\infty, 0]$. Since $2\nu + 1 > 0$ this implies that there is a constant $\epsilon > 0$ such that

$$f(x + iy) = O(|y|^{-\epsilon})$$

locally uniformly in $x$. Since $f$ is periodic, this shows $v_0 = 0$. Note that $K_{\nu}(t) \sim e^{-t} \sqrt{\frac{2t}{\pi}}$. Therefore we have

$$A_k(y) = \sqrt{y} K_{\nu} \left(2\pi \frac{|k|}{N} y \right) \sim e^{-2\pi \frac{|k|}{N} y} \sqrt{\frac{N}{4|k|}} \quad (34)$$
uniformly in $k$, which implies that
\[ u(z) := u(x + iy) := \sum_{k \in \mathbb{Z}, k \neq 0} A_k(y) e^{2\pi i k \frac{x}{N}} v_k \]
defines a smooth function on $\mathbb{H}^+$. Taking the derivatives termwise, we see that $u$ satisfies (17), i.e. is contained in the range of the Poisson transform. Now Lemma 3.2 shows that
\[ \langle \pi_{-\nu} \left( \begin{array}{c} \sqrt{b} \\ \frac{a}{\sqrt{b}} \end{array} \right) \varphi_0, \alpha \rangle = u \]
and (20) implies $u \in \tilde{M}_{\nu, \eta}$. Note that (18) is a consequence of $v_0 = 0$. Thus in order to show that $\psi \in L(S_{\nu, \eta})$, it only remains to show that $u$ satisfies (16). But (34) implies that $u$ rapidly decreases towards the cusp and hence the finite volume of the fundamental domain proves the square integrability of $u$. □

As a consequence of this proof we see that for $\eta$ the trivial representation, our Lewis transform coincides with $\frac{1}{2} \pi^{\nu + \frac{1}{2}} \Gamma \left( \frac{1}{2} - \nu \right)$ times the one given in [7].

4 Characterizing period functions on $\mathbb{R}^+$

Let $T' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (STS^{-1})^{-1}$.

Lemma 4.1 (cf. [7], §III.3) If a smooth function $\psi: (0, \infty) \to V_\eta$ satisfies (22) with $\nu \not\in \frac{1}{2} + \mathbb{Z}$, then it has the following asymptotic expansions:
\[ \psi(x) \sim_{x \to 0} x^{-2\nu - 1} Q_0 \left( \frac{1}{x} \right) + \sum_{m=-1}^{\infty} C_m^* x^m, \]
\[ \psi(x) \sim_{x \to \infty} x^{-2\nu - 1} Q_\infty \left( \frac{1}{x} \right) + \sum_{m=-1}^{\infty} (-1)^m C_m^* x^{-m-2\nu-1}, \]
where the $Q_0, Q_\infty: \mathbb{R} \to \mathbb{C}$ are smooth functions with
\[ Q_0(x + 1) = \eta(T') Q_0(x), \]
\[ Q_\infty(x + 1) = \eta(T) Q_\infty(x), \]
and the $C^*_m$ can be calculated from the Taylor coefficients $C_m := \frac{1}{m!} \psi^{(m)}(1) \in V_\eta$ of $\psi$ in $1$ via

$$C^*_m = \frac{1}{m + 2\nu + 1} \sum_{k=0}^{M} (-1)^m B_k \left( \frac{m + 2\nu + 1}{k} \right) C_{m-1-k}. \quad (35)$$

Here the $B_k$ are the Bernoulli numbers. If $\psi$ is real analytic, then so are $Q_0$ and $Q_\infty$.

**Proof:** For $\Re \nu > 0$ set

$$Q_0(x) := x^{-2\nu-1} \psi \left( \frac{1}{x} \right) - \sum_{n=0}^{\infty} (n + x)^{-2\nu-1} \eta (T(T')^n)^{-1} \psi \left( 1 + \frac{1}{n + x} \right)$$

and

$$Q_\infty(x) := \psi(x) - \sum_{n=1}^{\infty} (n + x)^{-2\nu-1} \eta (T'T^n)^{-1} \psi \left( 1 - \frac{1}{n + x} \right).$$
Then we have

\[ Q_0(x + 1) - \eta(T')Q_0(x) = \]

\[ = (x + 1)^{-2\nu - 1} \psi \left( \frac{1}{x + 1} \right) - x^{-2\nu - 1} \eta(T') \psi \left( \frac{1}{x} \right) \]

\[ - \sum_{n=0}^{\infty} (n + 1 + x)^{-2\nu - 1} \eta(T(T')^n) \! \! \psi \left( 1 + \frac{1}{n + 1 + x} \right) \]

\[ + \sum_{n=0}^{\infty} (n + x)^{-2\nu - 1} \eta(T') \! \eta(T(T')^n) \! \! \psi \left( 1 + \frac{1}{n + x} \right) \]

\[ = (x + 1)^{-2\nu - 1} \psi \left( \frac{1}{x + 1} \right) - x^{-2\nu - 1} \eta(T') \psi \left( \frac{1}{x} \right) \]

\[ - \sum_{n=1}^{\infty} (n + x)^{-2\nu - 1} \eta(T(T')^n)^{-1} \! \! \psi \left( 1 + \frac{1}{n + x} \right) \]

\[ + \sum_{n=0}^{\infty} (n + x)^{-2\nu - 1} \eta(T(T')^n)^{-1} \! \! \psi \left( 1 + \frac{1}{n + x} \right) \]

\[ = (x + 1)^{-2\nu - 1} \psi \left( \frac{1}{x + 1} \right) - x^{-2\nu - 1} \eta(T') \psi \left( \frac{1}{x} \right) \]

\[ + x^{-2\nu - 1} \eta(T(T')^{-1}) \! \! \psi \left( \frac{1}{x} \right) \]

\[ - \left( 1 + \frac{1}{x} \right)^{-2\nu - 1} \eta(ST^{-1}) \! \! \psi \left( \frac{1}{1 + \frac{1}{x}} \right) \]

\[ = 0, \]
since $T^{-1}ST^{-1} = (T')^{-1}$. Similarly we calculate

\[
Q_∞(x + 1) - \eta(T)Q_∞(x) = \psi(x + 1) - \eta(T)\psi(x)
\]

\[
= \psi(x + 1) - \eta(T)\psi(x)
\]

\[
= \psi(x + 1) - \eta(T)\psi(x)
\]

\[
= \psi(x + 1) - \eta(T)\psi(x)
\]

\[
= \psi(x + 1) - \eta(T)\psi(x)
\]

\[
= 0.
\]

For general $\nu$ we write

\[
Q_0(x) := x^{-2\nu - 1}\psi\left(\frac{1}{x}\right) - \sum_{m=0}^{M} C_m \zeta(m + 2\nu + 1, x)
\]

\[
- \sum_{n=0}^{\infty} (n + x)^{-2\nu - 1} \eta(T(T')^{n})^{-1} \left( \psi\left(1 + \frac{1}{n + x}\right) - \sum_{m=0}^{M} \frac{C_m}{(n + x)^{m}} \right)
\]

\[
Q_∞(x) := \psi(x) - \sum_{m=0}^{M} (-1)^m C_m \zeta(m + 2\nu + 1, x + 1)
\]

\[
- \sum_{n=1}^{\infty} (n + x)^{-2\nu - 1} \eta(T(T')^{n})^{-1} \left( \psi\left(1 - \frac{1}{n + x}\right) - \sum_{m=0}^{M} \frac{C_m}{(n + x)^{m}} \right)
\]

with the Hurwitz zeta function $\zeta(a, x) := \sum_{n=0}^{\infty} \frac{1}{(n + x)^{a}}$. Since the Hurwitz zeta function satisfies

\[
\zeta(a, x) \sim \frac{1}{a - 1} \sum_{k \geq 0} (-1)^k B_k \left(\frac{k}{2}\right) x^{-a-k+1} \quad (36)
\]
we find
\[
\psi(x) = x^{-2\nu - 1}Q_0(x^{-1}) + \sum_{m=0}^{M} C_m \zeta(m + 2\nu + 1, x^{-1}) x^{-2\nu - 1} \\
+ \sum_{n=0}^{\infty} (x^{-1} + n)^{-2\nu - 1} \left( \psi \left( 1 + \frac{1}{n + x^{-1}} \right) - \sum_{m=0}^{M} \frac{C_m}{(n + x^{-1})^n} \right). \\
= O(x^{2\nu + 1 + M})
\]

From this one derives the asymptotics for \(x \to 0\) using (36), see [7], §III.3 for details. The asymptotics for \(x \to \infty\) is shown analogously and the last claim is obvious. □

**Remark 4.2**

(i) If \(\psi(x) = o(x^{\min(1, 2\Re \nu + 1)})\) for \(x \to 0\), then \(Q_0 = 0\) by periodicity, i.e., \(\psi\) is an eigenfunction for the transfer operator

\[
\mathcal{L}_0 \psi(x) := x^{-2\nu - 1} \sum_{n=0}^{\infty} (n + x^{-1})^{-2\nu - 1} \eta(T(T')^n)^{-1} \psi \left( 1 + \frac{1}{n + x^{-1}} \right).
\]

Moreover we have \(C^*_{-1} = 0\).

(ii) If \(\psi(x) = o(x^{\min(0, \Re \nu)})\) for \(x \to \infty\), then \(Q_\infty = 0\), i.e., \(\psi\) is an eigenfunction for the transfer operator

\[
\mathcal{L}_\infty \psi(x) := \sum_{n=1}^{\infty} (n + x)^{-2\nu - 1} \eta(T(T')^n)^{-1} \psi \left( 1 + \frac{1}{n + x} \right).
\]

Moreover we have \(C^*_{-1} = 0\).

(iii) If \(C^*_0 = 0\), then \(C_0 = 0\), and if \(Q_0 = Q_\infty = 0\) we have the equations

\[
\psi(x) = x^{-2\nu - 1} \sum_{n=0}^{\infty} (n + x^{-1})^{-2\nu - 1} \eta(T(T')^n)^{-1} \psi \left( 1 + \frac{1}{n + x^{-1}} \right) \quad (37) \\
\psi(x) = \sum_{n=1}^{\infty} (n + x)^{-2\nu - 1} \eta(T(T')^n)^{-1} \psi \left( 1 + \frac{1}{n + x} \right). \quad (38)
\]

In this case, we can analytically extend \(\psi\) to \(\mathbb{C} \smallsetminus (-\infty, 0]\) via

\[
\psi(z) := \sum_{\gamma \in Q_n} (\psi|_{\nu, \eta \gamma})(z),
\]
where $Q$ is the semigroup generated by $T$ and $T'$, $Q_n$ is the set of $T$–$T'$-words of length $n$ in $Q$, and

$$(\psi|_{\nu,\eta})_n(z) := (cz + d)^{-2\nu - 1}\eta(\gamma)^{-1}\psi(\gamma \cdot z)$$

is a well defined right semigroup action (cf. [3], § 3, and [7], § III.3). The analytically continued function $\psi$ still satisfies (37) and (38). Therefore we can mimick the proof of Lemma 4.1 and use the Taylor expansion in 1 to find

$$\psi(z) = \sum_{m=1}^M C_m \zeta(m+2\nu+1, z^{-1}) z^{-2\nu-1} + O(|\zeta(2\nu+M+2, z^{-1})|)$$

for $|z| \to 0$ and

$$\psi(z) = \sum_{m=1}^M (-1)^m C_m \zeta(m+2\nu+1, z+1) + O(|\zeta(2\nu+M+2, z)|),$$

for $|z| \to \infty$. Now we use the following version of (36) which can be found in [2], § 1.18:

$$\zeta(a, z) = z^{1-a} \frac{\Gamma(a-1)}{\Gamma(a)} + \frac{1}{2} z^{-a} + \sum_{n=1}^N B_{2n} \frac{\Gamma(a+2n-1)}{\Gamma(a)(2n)!} z^{1-2n-a}$$

$$+ O(|z|^{-2N-1-a})$$

for $\text{Re} a > 1$ and $z \in \mathbb{C} \smallsetminus (-\infty, 0]$. Then (40) and (41) result in

$$\psi(z) = O(1) \text{ for } |z| \to 0$$

and

$$\psi(z) = O(|z|^{-2\nu-1}) \text{ for } |z| \to \infty.$$  

Remark 4.3 One can use the slash action (39) to rewrite the real version (22) of the Lewis equation in the form

$$\psi = \psi|_{\nu,\eta} T + \psi|_{\nu,\eta} T'.$$
Theorem 4.4 (cf. [7], Thm. 2) Suppose that \( \text{Re}\nu > -\frac{1}{2} \). Then
\[
\Psi_{\nu,\eta}^R = \{ \psi|_{(0,\infty)} : \psi \in \Psi_{\nu,\eta}^o \}.
\]

Proof: Note first that property (19) of \( \psi \in \Psi_{\nu,\eta}^o \) trivially implies (23) and (24) for \( \psi|_{(0,\infty)} \). Therefore it only remains to show that each element of \( \Psi_{\nu,\eta}^R \) occurs as the restriction of some \( \psi \in \Psi_{\nu,\eta}^o \). To this end we fix a \( \tilde{\psi} \in \Psi_{\nu,\eta}^R \). Since (23) and (24) hold for \( \tilde{\psi} \), we can apply Remark 4.2 to it. Thus \( \tilde{\psi} \) has an analytic continuation to \( \mathbb{C} \setminus (-\infty,0] \) (still denoted by \( \tilde{\psi} \)) and the asymptotics (43) and (44) shows that \( \tilde{\psi} \) indeed satisfies (19).

\[ \square \]

5 A Converse Theorem

Theorem 5.1 Let \( v_k \in V_\eta \) for \( k \in \mathbb{Z} \setminus \{0\} \) such that \( \eta(T)v_k = e^{2\pi i \frac{k}{N}}v_k \) and that the two Dirichlet series
\[
L_\varepsilon(s) = \sum_{k \neq 0} \text{sgn}(k)^\varepsilon \left( \frac{N}{|k|} \right)^s v_k, \quad \varepsilon = 0, 1
\]
converge for \( \text{Re}(s) >> 0 \). Assume that \( \hat{L}_\varepsilon(s) = \Gamma_\nu(s+\varepsilon)L_\varepsilon(s) \) extends to an entire function with
\[
\hat{L}_\varepsilon(s) = (-1)^\varepsilon \eta(S) \hat{L}_\varepsilon(1-s).
\]
Then the function \( u \) given by
\[
u(z) = \sum_{k \neq 0} \sqrt{y}K_\nu \left( \frac{|k|}{2\pi \frac{|k|}{N}}y \right) e^{2\pi i \frac{k}{N}x} v_k
\]
lies in \( S_{\nu,\eta} \).

Proof: The Dirichlet series give rise to an inverse Mellin transform \( f \) as in Section 3. Now follow the argumentation of that section. \( \square \)
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