SYMMETRIC SOLUTIONS TO NONLINEAR VECTORIAL 2ND ORDER ODE’S

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Abstract. It is proven that second order vectorial nonlinear differential systems $y'' = f(y)$ possess a continuum of symmetric solutions. They are shown to possess a continuum of even solutions. If $f(y)$ is an odd function of $y$, then $y'' = f(y)$ is shown also to possess a continuum of odd solutions. The results apply to a significant family of second order vectorial nonlinear differential systems that are not dissipative. This family of differential equations includes the celebrated $N$ body problem of celestial mechanics and other central force problems.

1. Introduction

The purpose of this article is to study symmetric solutions to second order nonlinear ordinary vectorial differential systems. It was motivated by the question whether or not the celebrated equations of celestial mechanics possess solution in which the future evolution of trajectories of the $N$ heavenly bodies is a perfect reflection of their past. It is shown, under appropriate conditions that nonlinear second order ordinary vectorial differential systems $y'' = f(y)$ possess a continuum of even solutions about any initial point $t_0$. Namely, for any $t_0 \in \mathbb{R}$, there exist a continuum of solutions $y[(t-t_0)]$ such that

$$y[(t-t_0)] = y[(-t-t_0)], \quad (t-t_0) \in [-L, L], \quad L > 0.$$  (1.1)

If $f(y)$ is an odd function of $y$, namely if $-f(y) = f(-y)$, then $y'' = f(y)$ is shown to also possesses a continuum of odd solutions about any point $t_0 \in \mathbb{R}$. Namely, for any $t_0 \in \mathbb{R}$, there exist a continuum of solutions $y[(t-t_0)]$ such that

$$-y[(t-t_0)] = y[(-(t-t_0))], \quad (t-t_0) \in [-L, L], \quad L > 0.$$  (1.2)

In contrast, non constant symmetric solutions to first order differential systems $z' = H(z)$ are “scarce”. We will elaborate on this scarcity in the sequel at the end of section 3.

In what follows we propose a result that applies to systems $y'' = f(y)$ where $y$ is a column vector with $n$ scalar components. The proposed theorem applies to a significant list of equations that include central force problems and the pendulum equation. The existence of a continuum of even solutions to the celebrated $N$ body problem, that is given inhere, is a manifestation of the predictive powers of Newton’s equations of celestial mechanics. The method of successive approximations is

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adapted as a main tool that yield the desired results. As a byproduct we obtain new estimates on the interval of existence.

We could not find this result in the voluminous literature of ordinary differential equations. Compare with \cite{1,2,6,7,8,12,10,9}. Neither did we come across the existence of even solutions to the $N$ body problem.

These results could be useful to modeling of physical phenomenon. They could be also useful to the numerical approximations of solutions and to phase space analysis. Given a scalar ordinary differential equation $y'' = f(y)$, let $(y,y')$ be the phase plane. Then, the even solutions orbits intercept the $y$ axis and the odd solutions orbits, if any, intercept the $y'$ axis.

The order of presentation is as follows. Section 2 is dedicated to preliminary definitions and lemmas that are instrumental in proving our main Theorem 11. We study symmetry properties of integrals and derivatives of odd and even functions. In section 3 we specialize the method of successive approximations to second order nonlinear vectorial ordinary differential equations $y'' = f(y)$. We prove the main result of this article that is Theorem 11. In section 4 we list various second order nonlinear differential systems of equations that occur in applications and to which our main Theorem 11 applies. The distinguished $N$ body problem of celestial mechanics is on this list. This means, that the $N$ body problem possesses a continuum of celestial mechanics solutions in which the future is a perfect reflection of the past.

2. Definite Integrals and Derivatives of Odd and Even Functions.

In preparation to proving the existence of even and odd solutions to vectorial nonlinear differential systems of the form $y'' = f(y)$, we need some notation and preparatory lemmas. In what follows we may suppress the notation $(t)$ in the function notation $y(t)$ and replace it by $y$. This, when the meaning is clear. We denote $y'(t) = \frac{dy(t)}{dt} = y'$.

**Lemma 1.** Let $L > 0$. Denote by $I$ an open interval $(-L,L)$ or a closed interval $[-L,L]$.

i) Assume that $-\phi(t) = \phi(-t)$, where $\phi(t) \in C(I)$ is an odd function on $I$. Then, $G(u)$ defined below

$$ G(u) := \int_{0}^{u} \phi(s)ds, $$

is an even function of $u \in I$.

ii) Assume that $\phi(t) = \phi(-t)$, where $\phi(t) \in C(I)$, is an even function on $I$. Then, $G(u)$ defined below

$$ G(u) := \int_{0}^{u} \phi(s)ds, $$

is an odd function on $I$.

iii) Assume that $\phi(t) \in C^{1}(I)$ is an even function. Then the derivative $\Psi(t) := \frac{d\phi(t)}{dt}$ is an odd function of $t \in I$.

iv) Assume that $\phi(t) \in C^{1}(I)$ is an odd function. Then the derivative $\Psi(t) := \frac{d\phi(t)}{dt}$ is an even function of $t \in I$. 

Proof. We first prove i). Consider

\[ G(-u) = \int_0^{-u} \phi(s)ds. \]

Make the following change of variables

\[ s = -v \Rightarrow ds = -dv, \quad \{s = 0 \Rightarrow v = 0\}, \quad \{s = -u \Rightarrow v = u\}. \]

Then,

\[ G(-u) = \int_0^{-u} \phi(s)ds = -\int_0^{u} \phi(-v)dv = \int_0^{u} \phi(v)dv = G(u). \]

as desired. The proof for ii) is similar. Apply the change of variables (2.4) with the assumption \( \phi(s) = \phi(-s) \). Then,

\[ G(-u) = \int_0^{-u} \phi(s)ds = -\int_0^{u} \phi(-v)dv = -\int_0^{u} \phi(v)dv = -G(u). \]

as desired.

iii) We focus on the quotient below with \( t \in (-L, L) \) and with \( h \neq 0 \) and \( h \) arbitrarily small.

\[ Q(t, h) := \frac{\phi(t + h) - \phi(t)}{h}. \]

Since \( \phi(t) \) is an even function then

\[ \phi(t + h) = \phi(-t - h), \quad \phi(t) = \phi(-t). \]

Substitute from (2.8) into the right hand side of (2.7) to obtain

\[ Q(t, h) = -Q(-t, -h) = \frac{-\phi(-t - h) - \phi(-t)}{-h}. \]

Put \( \psi(t) := \frac{d\phi(t)}{dt} \). Take the limit as \( h \to 0 \) in (2.9) and obtain

\[ Q(t, h) := \frac{d\phi(t)}{dt} = \lim_{h \to 0} Q(t, h) = -\lim_{h \to 0} Q(-t, -h) = -\psi(-t). \]

If \( I \) is the closed interval \([-L, L]\), one must exercise caution with signs of \( h \) at the end points of \( I \). Assume that \( t = L \). Then in (2.7) we must assume that \( h < 0 \) in order to make \( \phi(L+h) \) well defined. Notice then that \( \phi(-L-h) \) is also well defined and consequently \( Q(-t, -h) \) in (2.9) is also well defined so that \( \frac{d\phi(L)}{dt} = -\frac{d\phi(-L)}{dt} \) exist as a one sided limit with \( h < 0 \). An analogous argument holds at \( t = -L \) with \( h > 0 \).

iv) We focus again on the quotient

\[ Q(t, h) := \frac{\phi(t + h) - \phi(t)}{h}. \]

Since \( \phi(t) \) is an odd function then

\[ \phi(t + h) = -\phi(-t - h), \quad \phi(t) = -\phi(-t). \]

Substitute from (2.12) into the right hand side of (2.11) to obtain

\[ Q(t, h) = Q(-t, -h) = \frac{\phi(-t - h) - \phi(-t)}{-h}. \]

Take the limit as \( h \to 0 \) in (2.13) and obtain
\[(2.14) \quad \Psi(t) = \lim_{h \to 0} Q(t, h) = \lim_{h \to 0} Q(-t, -h) = \Psi(-t).\]

Assume that \(I\) is the closed interval \([-L, L]\), and \(t = L\) or \(t = -L\). Then, similar arguments as in the proof of iii) hold and the proof is completed. \(\square\)

**Lemma 2.** Let \(L > 0\). Denote by \(I\) an open interval \((-L, L)\) or a closed interval \([-L, L]\). Assume that \(\phi(t) \in C^3(I)\). Then

a) \(\phi(t)\) is an even function on \(I\) iff its derivative is an odd function on \(I\).

b) \(\phi(t)\) is an odd function on \(I\) iff its derivative is an even function on \(I\).

c) Assume that \(\phi(t) \in C^0(I)\). Let \(0 \leq s \leq u \leq t\) or let \(t \leq u \leq s \leq 0\). Put

\[(2.15) \quad F(t) := \int_0^t \int_0^t \phi(s)dsdu.\]

Then, \(F(t) \in C^2(I)\) and \(F(t) = F(-t)\) is an even function on \(I\) iff \(\phi(t) \in C^0(I)\) and \(\phi(t) = \phi(-t)\) is an even function on \(I\).

\(F(t) \in C^2(I)\) and \(-F(t) = F(-t)\) is an odd function on \(I\) iff \(\phi(t) \in C^0(I)\) and \(-\phi(t) = \phi(-t)\) is an odd function on \(I\).

**Proof.** The proof follows easily from Lemma 1. \(\square\)

We clarify now what is an even and an odd function of a scalar function \(u = \text{H}(y_1, y_2, \cdots, y_N)\) of several variables. To this end we denote the transposed column vector \(y^T = (y_1, y_2, \cdots, y_N)\) and we put \(u = \text{H}(y_1, y_2, \cdots, y_N) = H(y)\).

**Definition 3.** Denote by \(\text{REG}\) an open connected set in \(\mathbb{R}^N\). We say that \(H(y)\) is an even function of \(y\) in \(\text{REG}\) if

\[(2.16) \quad H(y) = H(-y), \quad y \in \text{REG}.\]

We say that \(H(y)\) is an odd function of \(y\) in \(\text{REG}\) if

\[(2.17) \quad -H(y) = H(-y), \quad y \in \text{REG}.\]

This definition is different then requiring that \(u = \text{H}(y_1, y_2, \cdots, y_N) = H(y)\) be an even or an odd function in each individual coordinate \(y_j\). In order to bring out the difference we add the following.

**Definition 4.** Denote by \(\text{REG}\) an open connected set in \(\mathbb{R}^N\). We say that \(H(y)\) is an even function of \(y\) in \(\text{REG}\) in the strict sense if

\[(2.18) \quad H(y_1, y_2, \cdots, y_j, \cdots, y_N) = H(y_1, y_2, \cdots, -y_j, \cdots, y_N), \quad y \in \text{REG}, \quad j = 1, 2, \cdots, N.\]

We say that \(H(y)\) is an odd function of \(y\) in \(\text{REG}\) in the stricter sense if

\[(2.19) \quad -H(y_1, y_2, \cdots, y_j, \cdots, y_N) = H(y_1, y_2, \cdots, -y_j, \cdots, y_N), \quad y \in \text{REG}, \quad j = 1, 2, \cdots, N.\]

Consider the following functions

\[(2.20) \quad H(y_1, y_2) := y_1^5 y_2^3, \quad L(y_1, y_2) = y_1^{10} y_2^6.\]

Evidently, \(H(y_1, y_2) := y_1^3 y_2^3\) is an even function in \(\text{REG} := \mathbb{R}^2\). However, it is an odd function in the strict sense in \(\text{REG} := \mathbb{R}^2\). Evidently, \(L(y_1, y_2) = y_1^{10} y_2^6\) is an even function in \(\text{REG} := \mathbb{R}^2\) and it is also an even function in the strict sense in \(\text{REG} := \mathbb{R}^2\).
Remark 5. The reader may want to consider a multinomial in the \((r+w)\) independent variables \(y_1, y_2, \cdots, y_j, \cdots, y_r, y_{r+1}, y_{r+2}, \cdots, y_{r+w}\)

\[
H(y) = [y_1^{2c_1+1}] y_2^{2c_2+1} \cdots y_j^{2c_r+1} \cdots y_{r+w}^{2c_{w+1}}
\]

where \(c_1, c_2, \cdots, c_r, c_1, c_2, \cdots, c_w \in \mathbb{N}_0\), \(r, w \in \mathbb{N}\). Formulation of necessary and sufficient conditions on the powers occurring in \(H(y)\) such that a) \(H(y)\) is an even multivariate function b) \(H(y)\) is an even multivariate function in the strict sense c) \(H(y)\) is an odd multivariate function d) \(H(y)\) is an odd multivariate function in the strict sense, could further clarify the difference between these two types of symmetry.

Next we formulate an analog to Lemma 1 for multivariate functions.

**Lemma 6.** Let \(H(y) \in C^1(REG)\).

i) Assume that \(H(y) = H(-y), y \in REG\). Then the partial derivatives

\[
\Psi_j(y) := \frac{\partial H(y)}{\partial y_j}, \ j = 1, 2, \cdots, N
\]

are odd function in \(REG\).

ii) Assume that \(-H(y) = H(-y), y \in REG\). Then the partial derivatives

\[
\Psi_j(y) := \frac{\partial H(y)}{\partial y_j}, \ j = 1, 2, \cdots, N
\]

are even functions in \(REG\).

iii) Assume \(f(y)\) to be a column vector function, \(f^T(y) := [f_1(y), f_2(y), \cdots, f_n(y)]\), where \(f_j(y), j = 1, 2, \cdots, n\) are the scalar component of \(f(y)\) such that \(f_j(y) \in C^1(REG)\). Then,

\[
f(y) = f(-y), y \in REG \implies \Psi(y) := \frac{\partial f(y)}{\partial y_j} = -\Psi(-y) := -\frac{\partial f(-y)}{\partial y_j}, j = 1, 2, \cdots, N.
\]

Moreover,

\[
f(y) = f(-y), y \in REG \implies \Psi(y) := \frac{\partial f(y)}{\partial y_j} = -\Psi(-y) := -\frac{\partial f(-y)}{\partial y_j}, j = 1, 2, \cdots, N.
\]

[2000] We first prove i) and we focus on the quotient below: with \(y \in REG\); with \(h \neq 0\); and with \(h\) arbitrarily small.

\[
Q_j(y, h) := \frac{H(y_1, y_2, \cdots, y_{j-1}, y_j + h, y_{j+1}, \cdots, y_N) - H(y_1, y_2, \cdots, y_{j-1}, y_j, y_{j+1}, \cdots, y_N)}{h}.
\]

Put as short hand notation

\[
\hat{H}(y_j + h) := H(y_1, y_2, \cdots, y_{j-1}, y_j + h, y_{j+1}, \cdots, y_N),
\]

\[
\hat{H}(-y_j - h) := H(-y_1, -y_2, \cdots, -y_{j-1}, -y_j - h, -y_{j+1}, \cdots, -y_N).
\]

**Proof.** Since \(H(y)\) is an even function then

\[
H(y_1, y_2, \cdots, y_{j-1}, y_j + h, y_{j+1}, \cdots, y_N) = H(-y_1, -y_2, \cdots, -y_{j-1}, -y_j - h, -y_{j+1}, \cdots, -y_N)
\]
Put for a certain continuous function
\[ D := \{ y \in \mathbb{R}^n \mid y - \phi_0(t) \leq b, \ b \geq 0 \} . \]

Assume that
\[ \phi_0(t) \in C(I); \ \phi(t) \in C(I); \ \phi(t) \in D; \ f(y) \in C(D). \]

We say that \( \phi(t) \) is a solution of the integral equation (2.37) if \( \phi(t) \) satisfies the identity relation
\[ \phi(t) \equiv \phi_0(t) + \int_0^t \int_0^u f(\phi(s))dsdu, \ t \in I. \]
The next stage we show how the solution of the initial value problem \( (2.38) \) below is equivalent to the solution of the integral equation \( (2.37) \). To this end we need the following definition.

**Definition 8.** We say that \( y = \phi(t) \) is a solution to the initial value problem \( (2.38) \)

\[
y'' = \frac{d^2y}{dt^2} = f(y), \ y(0) = y_0, \ \frac{dy}{dt}(0) = \eta
\]

in \( D \) on the interval \( I \) if:

\[
y = \phi(t) \in C^2(I) ; \ y = \phi(t) \in D ; \ f(y) \in C(D)
\]

and \( \phi(t) \) satisfies the following identity together with the initial conditions below.

Namely,

\[
\phi''(t) \equiv f(\phi(t)), \ t \in I, \ \phi(0) = y_0, \ \phi'(0) = \eta.
\]

**Remark 9.** We note that at the end points of the interval \( I \) the condition \( y = \phi(t) \in C^2(I) \) means that the following limits exist

\[
\phi'(L) := \lim_{h \to 0^+} \frac{\phi(L-h) - \phi(L)}{h}, \ \phi'(-L) := \lim_{h \to 0^+} \frac{\phi(-L+h) - \phi(-L)}{h},
\]

(2.42)

\[
\phi''(L) := \lim_{h \to 0^+} \frac{\phi'(L-h) - \phi'(L)}{h}, \ \phi''(-L) := \lim_{h \to 0^+} \frac{\phi'(-L+h) - \phi'(-L)}{h}.
\]

Next we prove

**Lemma 10.** Assume:

\[
I = [-L, L]; \ y = \phi(t) \in C^2(I) ; \ y = \phi(t) \in D ; \ \phi_0(t) = y_0 + t\eta.
\]

Then, \( \phi(t) \) is a solution of \( (2.38) \) iff it is a solution of \( (2.37) \).

**Proof.** Assume \( y = \phi(t) \in C^2(I) \) is a solution to the initial value problem \( (2.38) \).

Then, we may integrate both sides of \( (2.38) \) to yield

\[
\phi'(t) = \eta + \int_0^t f(\phi(s))ds.
\]

An integration of \( (2.44) \) yields

\[
\phi(t) - \phi_0(t) = \int_0^t \int_0^u f(\phi(s))dsdu.
\]

Thus, \( \phi(t) \) is a solution of \( (2.37) \). We proceed with the proof of the converse statement. If \( \phi(t) \in C(I) \) and \( \phi(t) \in D \), then \( f(\phi(s)) \) is well defined on \( I \), \( f(\phi(s)) \in C(I) \) as a composition of continuous functions on \( I \) and \( \int_0^t \int_0^u f(\phi(s))dsdu \in C^2(I) \).

Consequently, the right hand side of \( (2.37) \) is twice continuously differentiable on the interval \( I \). Consequently, \( \phi(t) \) on the left hand side of \( (2.37) \), is twice continuously differentiable on \( I \). Hence, \( \phi(0) = y_0 \) and \( \phi'(0) = \eta \) and \( \phi(t) \) is a solution of \( (2.38) \) as desired yield \( (2.38) \) with the desired properties. \( \square \)
3. Symmetric solutions via successive approximations

We are ready to formulate and prove the main theorem.

**Theorem 11.** Assume that:

\[ t \in I; \ y_0, \eta,y, f(y) \in \mathbb{R}^n, \ n \in \mathbb{N}, \phi_0(t) = y_0 + t\eta, \ f(y) \in D. \]

Assume that \( f(y) \) satisfies the Lipchitz condition in \( D \). Namely, there is a constant \( K \geq 0 \) such that for any \( \tilde{y}, \bar{y} \in D \) the following holds

\[ |f(\tilde{y}) - f(\bar{y})| \leq K |\tilde{y} - \bar{y}|. \]

Assume also that

\[ f(y), y(0) = y_0, y'(0) = \eta, \]

\[ D = \{ y, \| y - \phi_0(t) \| \leq b \} \implies \| f(y) \| \leq M. \]

Then, the initial value problem

\[ y'' = f(y), \ y(0) = y_0, y'(0) = \eta, \]

possesses a unique solution \( y(t) \) for \( |t| \leq \sqrt{\frac{b}{M}} \).

The initial value problem

\[ y'' = f(y), \ y(0) = 0, y'(0) = \eta, \]

possesses a unique solution \( y(t) \) for \( |t| \leq \sqrt{\frac{2b}{M}} \) such that \( y(t) \equiv y(-t) \).

Moreover: for any \( t_0 \in \mathbb{R} \), the initial value problem

\[ y'' = f(y), \ y(t_0) = y_0, y'(t_0) = 0, \]

possesses a unique solution \( y(t) \) for \( |t - t_0| \leq \sqrt{\frac{b}{M}} \) such that

\[ y(t - t_0) \equiv y(-t - t_0). \]

If in addition \( -f(y) = f(-y) \) holds for \( y \in D \), then for any \( t_0 \in \mathbb{R} \), the initial value problem

\[ y'' = f(y), \ y(t_0) = y_0, y'(t_0) = \eta, \]

possesses a unique solution \( y(t) \) for \( |t - t_0| \leq \sqrt{\frac{2b}{M}} \) such that

\[ -y(t - t_0) \equiv y(-t - t_0). \]

Furthermore, assume that \( f(y) \) is defined in every disk \( D \) in \( \mathbb{R}^n \), and that for all \( b > 0 \) there exists \( M > 0 \) such that \( |f(y)| \leq M \). Then, all solutions to all initial values above exist globally on \( (-\infty, \infty) \).

**Proof.** We utilize the method of successive approximations e.g. [9] Ch. 1 and [2] Ch. 3 and we adapt and specialize it to the differential system \( y'' = f(y) \). Denote by \( \phi_j(t) \) the successive approximations defined by \( (3.9) \) below.

\[ \phi_{j+1}(t) - \phi_0(t) = \int_0^t \int_0^u f(\phi_j(s))ds du; \ j = 0,1,2,\ldots, \]
We proceed to show by induction the following three properties of the successive approximations $\phi_j(t)$, $j = 0, 1, 2, \ldots$. We show:

i) All $\phi_j(t) \in D$. Namely, that

$$0 \leq s \leq t \leq \sqrt{\frac{2b}{M}} \text{ or } -\sqrt{\frac{2b}{M}} \leq t \leq u \leq s \leq 0.$$  \hspace{1cm} (3.10)

We proceed to show by induction the following three properties of the successive approximations $\phi_j(t)$, $j = 0, 1, 2, \ldots$. We show:

ii) We show that

$$|\phi_j(t) - \phi_0(t)| \leq b, \text{ if } t \in I := [-\sqrt{\frac{2b}{M}}, \sqrt{\frac{2b}{M}}].$$  \hspace{1cm} (3.11)

iii) We show that if $f(y)$ is an odd function, namely

$$f(y) \equiv f(-y), \text{ } y \in D,$$  \hspace{1cm} (3.12)

then

$$-\phi_0(t) \equiv \phi_0(-t) \implies -\phi_j(t) \equiv \phi_j(-t), \text{ } j = 1, 2, \ldots, t \in I.$$  \hspace{1cm} (3.13)

We proceed with the proofs. Evidently, for $j = 0$

$$\phi_0(t) \in D \text{ because } |\phi_0(t) - \phi_0(t)| \equiv 0 \leq b.$$  \hspace{1cm} (3.14)

For $\phi_0(t)$ all three conditions i), ii) and iii) hold trivially. Assume by induction that $\phi_j(t)$ satisfies (3.11). We desire to prove that

$$|\phi_{j+1}(t) - \phi_0(t)| \leq b, \text{ if } t \in I := [-\sqrt{\frac{2b}{M}}, \sqrt{\frac{2b}{M}}].$$  \hspace{1cm} (3.16)

By definition (3.9) of $\phi_{j+1}$, we have with $0 \leq s \leq u \leq t$ the desired estimate

$$|\phi_{j+1}(t) - \phi_0(t)| \leq \int_0^t \int_0^u Mdsdu \leq \frac{1}{2}Mt^2 \leq \frac{1}{2}M(\frac{2b}{M}) = b.$$  \hspace{1cm} (3.17)

If $t \leq u \leq s \leq 0$ then substitute in the double integral $I_j := \int_0^t \int_0^u f(\phi_j(s))dsdu$

$$s = -v_1 \implies ds = -dv_1, \text{ and } \{s = 0 \iff v_1 = 0\}, \text{ and } \{s = u \implies v_1 = -u\}$$

and obtain

$$I_j := \int_0^t \left[ \int_0^u f(\phi_j(s))ds \right]du = \int_0^t \left[ \int_0^{-u} f(\phi_j(-v_1))(-dv_1) \right]du.$$  \hspace{1cm} (3.19)

Change again variables in (3.19) as follows

$$u = -v_2 \implies du = -dv_2, \text{ and } \{u = 0 \iff v_2 = 0\}, \text{ and } \{u = t \implies v_2 = -t\}.$$  \hspace{1cm} (3.20)

Then,

$$I_j := \int_0^t \left[ \int_0^u f(\phi_j(s))ds \right]du = \int_0^t \left[ \int_0^{-u} f(\phi_j(-v_1))(-dv_1) \right]du = \int_0^{-t} \left[ \int_0^{-v_2} f(\phi_j(-v_1))(-dv_1) \right](-dv_2).$$  \hspace{1cm} (3.21)

In sum

$$-\sqrt{\frac{2b}{M}} \leq t \leq u \leq s \leq 0 \iff 0 \leq v_1 \leq v_2 \leq -t \leq \sqrt{\frac{2b}{M}}.$$  \hspace{1cm} (3.22)
By virtue of two changes of variables we obtain two non negative upper limits in (3.21) . This permits us to estimate the norm,

\[
|I_j| = |\phi_j(t) - \phi_0(t)| = \left| \int_0^t \left[ \int_0^u f(\phi_j(s))ds \right] du \right| \leq \int_0^t \left| \int_0^u f(\phi_j(v))dv \right| du_1 dv_2 \leq \frac{1}{2} M(-t)^2 \leq \frac{1}{2} M(\frac{2b}{M}) = b,
\]

as desired. In order to complete the proof for ii) and iii) above, we observe that if \( \phi(s) = \phi(-s) \) for \( s \in I \) and \( \phi(s) \in D \) then \( f(\phi(s)) = f(\phi(-s)) \) . Hence \( f(\phi(s)) \) is also an even function of \( s \) for \( s \in I \). If \( \phi_0(s) = \phi_0(-s) \) for \( s \in I \) then by Lemma 2, all functions in (3.21) are even functions for \( s \in I \) as desired. Namely,

\[
I_j := \int_0^t \int_0^u f(\phi_0(s))ds|du, \quad \phi_j(s) := \phi_0(s) + \int_0^t \int_0^u f(\phi_{j-1}(s))ds|du, \quad j = 1, 2, \ldots.
\]

We turn to successive approximations that are odd functions. Notice that

\[
- \phi(s) = \phi(-s) \quad s \in I, \quad -f(y) = f(-y), \quad \phi(s), y \in D \Rightarrow -f(\phi(s)) = -f(\phi(-s)) = f(\phi(s)).
\]

Therefore, if \( -\phi_0(s) = \phi_0(-s) \) for \( s \in I \) , then by Lemma 2, all expressions in (3.26) are odd functions for \( s \in I \) as desired. Namely,

\[
\phi_j(s) := \phi_0(s) + \int_0^t \int_0^u f(\phi_{j-1}(s))ds|du, \quad j = 1, 2, \ldots.
\]

In the upcoming discussion we will show that the successive approximations \( \phi_j(t) \) converge absolutely and uniformly to a continuous function \( \phi(t) \) on the interval \( I \).

The method of proof requires special consideration of \( \phi_0(t) \) and of \( \phi_1(t) - \phi_0(t) \). The rest of the differences \( \phi_{j+1}(t) - \phi_j(t) : j = 1, 2, \ldots \) follow similar patterns. We assume that in the double integrals below the upper bounds in the integrals obey the order \( 0 \leq s \leq u \leq t \leq L \). Recall that the successive approximations are defined so that

\[
|\phi_1(t) - \phi_0(t)| \leq \int_0^t \int_0^u |f(\phi_0(s))| dsdu \Rightarrow |\phi_1(t) - \phi_0(t)| \leq \int_0^t \int_0^u Mdsdu = \frac{1}{2} Mt^2,
\]

\[
\phi_2(t) - \phi_1(t) = \int_0^t \int_0^u [f(\phi_1(s)) - f(\phi_0(s))]dsdu \Rightarrow |\phi_2(t) - \phi_1(t)| \leq \int_0^t \int_0^u |f(\phi_1(s)) - f(\phi_0(s))| dsdu.
\]

Insert the Lipchitz condition \( (3.2) \) in (3.29) and obtain

\[
|\phi_2(t) - \phi_1(t)| \leq \int_0^t \int_0^u K |\phi_1(s) - \phi_0(s)| dsdu.
\]
Now use the special estimate on the right hand side of \((3.28)\) to obtain

\[
\phi_2(t) - \phi_1(t) \leq \int_0^t \int_0^s K \frac{1}{2!} M s^2 ds du = \frac{M}{4!} K t^4 = \frac{M}{4!} (K)^2 t^4.
\]

The reader can easily verify that

\[
\phi_3(t) - \phi_2(t) \leq \int_0^t \int_0^s K \frac{1}{4!} K M s^4 ds du = \frac{M}{6!} K (K)^3 t^6.
\]

The formulas \((3.31)\) and \((3.32)\) make it easy to guess an induction pattern that is

\[
\phi_j(t) - \phi_{j-1}(t) \leq \frac{M}{(2j)!/K} (K)^{j} t^{2j}, \quad j = 1, 2, \ldots.
\]

Without loss of generality we may assume that the expressions \(\sum_{j=1}^{\infty} \frac{M}{(2j)!/K} (K)^{j} t^{2j}\) are well defined for \(K = 0\) as well. This makes \(f(y)\) by virtue of \((3.2)\) a constant vector and the solution \(\phi(t) \equiv \phi_0(t)\).

In order to aver the induction pattern we need to show that \((3.33)\) implies

\[
\phi_{j+1}(t) - \phi_j(t) \leq \frac{M}{(2(j+1))!/K} (K)^{j+1} t^{2(j+1)}, \quad j = 1, 2, \ldots.
\]

Indeed, insert the induction inequality \((3.33)\) into

\[
\phi_{j+1}(t) - \phi_j(t) \leq \int_0^t \int_0^s K |\phi_j(s) - \phi_{j-1}(s)| ds du 
\]

and obtain the desired conclusion \((3.34)\). It is easily verified by the ratio test that the majorant series \(\sum_{j=1}^{\infty} \frac{M}{(2j)!/K} (K)^{j} t^{2j}\) are absolutely and uniformly convergent on any closed interval \([-r, r]\), \(r > 0\). This is verified by the ratio test

\[
\lim_{j \to \infty} \frac{\frac{M}{(2(j+1))!/K} (K)^{j+1} t^{2(j+1)}}{\frac{M}{(2j)!/K} (K)^{j} t^{2j}} = \lim_{j \to \infty} \frac{K t^2}{(2j + 2)(2j + 1)} = 0.
\]

The series define a function \(\psi(t)\) such that for \(t \in I = [-\sqrt{\frac{2b}{M}}, \sqrt{\frac{2b}{M}}]\) we have

\[
\psi(t) := \sum_{j=1}^{\infty} \frac{M}{(2j)!/K} (K)^{j} t^{2j},
\]

\[
\sum_{j=1}^{\infty} |\phi_j(t) - \phi_{j-1}(t)| \leq \psi(t) := \sum_{j=1}^{\infty} \frac{M}{(2j)!/K} (K)^{j} t^{2j} \leq \psi(\sqrt{\frac{2b}{M}}).
\]

The relation \((3.38)\) make the series

\[
\phi_0(t) + \sum_{j=1}^{\infty} [\phi_j(t) - \phi_{j-1}(t)],
\]
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absolutely and uniformly convergent for \( t \in I = [-\sqrt{2bM}, \sqrt{2bM}] \). Hence, a function \( \phi(t) \) on \( [-\sqrt{2bM}, \sqrt{2bM}] \) exists such that

\[
\phi(t) := \lim_{j \to \infty} \phi_{j+1}(t) = \phi_0(t) + \lim_{j \to \infty} \int_0^t \int_0^u f(\phi_j(s))dsdu = \\
\phi_0(t) + \int_0^t \int_0^u \lim_{j \to \infty} f(\phi_j(s))dsdu = \\
\phi_0(t) + \int_0^t \int_0^u f(\lim_{j \to \infty} \phi_j(s))dsdu = \\
\phi_0(t) + \int_0^t \int_0^u f(\phi(s))dsdu.
\]

If we choose \( \phi_0(t) \equiv y_0 \) then the solution \( \phi(t) := \lim_{j \to \infty} \phi_j(t) \) will be an even solution. If we choose \( \phi_0(t) \equiv t\eta \) then \( \phi(t) := \lim_{j \to \infty} \phi_j(t) \) will be an odd function on condition that \( -f(y) = f(-y) \).

In order to obtain solutions to (3.7) and (3.8), together with the estimates \(|t - t_0| \leq \sqrt{2bM} \) on the interval of existence, we consider the initial value problem

\[
d^2\tilde{y} \frac{d\tau}{d\tau^2} = f(\tilde{y}), \quad \tilde{y}(0) = y_0, \quad \frac{d\tilde{y}}{d\tau}(0) = \eta,
\]

where \( \tau \) is a new independent variable and \( \tilde{y}(\tau) \) is the unique solution of (3.41). Theorem 11 applies to the existence of solutions \( \tilde{y}(\tau) \) with the desired properties of symmetry. Observe that we have

\[
\tau = t - t_0 \implies \{ \tau = 0 \iff t = t_0 \}.
\]

Put

\[
y(t) := \tilde{y}(\tau) = \tilde{y}(t - t_0) \implies y(t) := \tilde{y}(0), \quad \frac{dy(t_0)}{dt} = \frac{d\tilde{y}(0)}{d\tau}.
\]

It is now easily verified that

\[
\frac{dy}{dt} = \frac{d\tilde{y}}{d\tau} \frac{d\tau}{dt} = \frac{d\tilde{y}}{d\tau} \implies \frac{d^2y}{dt^2} = \frac{d^2\tilde{y}}{d\tau^2}.
\]

Since the vector function of \( f(y) \) is independent of \( t \) we conclude that the existence of solutions to the initial value problem (3.41), implies the existence of corresponding solutions to the initial value problem (3.45)

\[
\frac{d^2y}{dt^2} = y'' = f(y), \quad y(0) = y_0, y'(0) = \eta.
\]

Conversely, the existence of solutions to the initial value problem (3.45), implies the existence of corresponding solutions to the initial value problem (3.41).

Last but not least, the existence of global solutions to (3.45) under the conditions stated follows by letting \( b \to \infty \) in (3.10). \( \square \)

\textit{Remark 12.} It is easily verified that the Lipchitz condition (3.2) renders the function \( f(y) \) continuous in \( D \).

Notice also
Remark 13. If \( f(y) \) is defined for all \( y \in \mathbb{R}^n \) and satisfies a sub-linearity condition
\[
|f(y)| \leq M_1 |y| + M_2, \quad M_1, M_2 \geq 0,
\]
and \( M_1, M_2 \) are certain constants independent of \( y \), then the Gronwall Lemma e.g. [2] Chapter 1, implies that there exists a constant \( M_3 > 0 \) for a given initial value problem, such that
\[
|f(y)| \leq M, \quad M := M_1 M_3 + M_2, \quad M_1, M_2, M_3 \geq 0.
\]
The condition (3.46) applies to the pendulum equation \( y'' = -\sin(y) \) with \( M_1 = 0 \) and \( M = M_2 = 1 \) for all initial value problems.

Consider also

Remark 14. How are symmetric solutions to first order systems \( z' = H(z) \) scarce? In order to better understand this scarcity, consider an initial value problem
\[
z' = H(z), \quad z(t_0) = z_0, \quad z, H(z) \in \mathbb{R}^m, \quad m \in \mathbb{N}.
\]
Assume that \( m \neq 2n \) or assume in case that \( m = 2n \) that \( z' = H(z) \) does not originate from \( z^T = (y, y') \) where \( H(z)^T = (y', f(y)) \). Also assume that \( H(z) \in C^1(D) \) where \( D \) is some region of \( \mathbb{R}^m \). If (3.48) possesses an even solution then \( z'(t_0) = \vec{0} = H(z_0) \). Thus, \( z(t) \equiv z_0 \) must be the unique constant even solution. If (3.48) possesses an odd non constant vector solution then the initial value problem (3.48) must obey \( z_0 = \vec{0} \). Denote by \( \overline{JH(z)} \) the Jacobian of \( H(z) \). Then, we must have \( z'' = [JH(z(t_0))]z'(t_0) = [JH(\vec{0})]H(\vec{0}) = \vec{0} \). Thus the Jacobian evaluated at \( \vec{0} \) must have an eigenvalue zero corresponding to the eigenvector \( H(\vec{0}) \neq 0 \). The requirement that some or all even derivatives \( z^{(2k)}(0) = \vec{0}, k = 1, 2, \ldots \), impose further restrictions on \( H(z) \). It is in this sense that first order systems normally do not possess non constant odd solutions.

4. EXAMPLES OF \( y'' = f(y) \) OCCURRING IN APPLICATIONS

In this section we provide examples of second order differential systems and scalar equations to which Theorem 11 applies. These nonlinear differential systems and equations are autonomous and \( f(y) \) is independent of \( y' \). Normally, these are conservative non dissipative systems or systems without damping.

Noteworthy is the \( N \) body problem. Adopt the following notation: \( m_1, m_2, \ldots, m_N \) are the masses of the \( N \) bodies; \( t \in \mathbb{R} \) is the time variable; \( y_j \in \mathbb{R}^3 \) and \( 1 \leq j \leq N \), are column position arrows of the \( N \) bodies, respectively; \( T \) stands for transposition of a vector or a matrix; \( y^T = [y_1, y_2, \ldots, y_N] \) and \( f(y)^T := [f_1(y), f_2(y), \ldots, f_N(y)]^T \) are respectively rows of blocks of column vectors with
\[
y_{k} = f_k(y) := \sum_{j \neq k} \frac{G m_j (y_j - y_k)}{||y_j - y_k||^2} = \frac{1}{m_k} \nabla y_k U, \quad U := \sum_{j < k} \frac{m_j m_k}{||y_j - y_k||} > 0.
\]
\( U \) is the gravitational potential; \( \nabla y_k U \) is the gradient of \( U \) with respect to the components of \( y_k; ||y|| \) is the Euclidean norm, \( G \) is the gravitational constant.

Consider the initial value problem for the \( N \) body problem in celestial mechanics
\[
y'' = f(y), \quad y(t_0) = y_0, y'(t_0) = \eta, \quad y_k(t_0) \neq y_j(t_0), \quad k \neq j, \quad k, j = 1, 2, \ldots, N.
\]
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Compare with [11]. Choosing the velocity vector \( \eta = \mathbf{0} \), then by Theorem 11, the initial value problem

\[
y'' = f(y), \quad y(t_0) = y_0, \quad y'(t_0) = \mathbf{0}, \quad y_k(t_0) \neq y_j(t_0), \quad k \neq j, \quad k, j = 1, 2, \ldots, N,
\]

possesses a continuum of even solutions by varying the position vector \( y_0 \) according to (4.3). Observe that Newton's equations of celestial mechanics satisfy

\[
y'' = -\sum_{j \neq k} \frac{Gm_j(y_j - y_k)}{\|y_j - y_k\|^3} \Rightarrow -f(y) = f(-y).
\]

However, we cannot solve by Theorem 11 for odd solutions an initial value problem with a condition \( y(t_0) = \mathbf{0} \). This, because \( y(t_0) = \mathbf{0} \) means that all of the celestial point masses are in state of mutual collision. Then, the equations in (4.3) contain undetermined and unbounded terms which render the equations invalid.

Past, Present and future of the planetary motion and the Universe are of great interest. Compare with [3]. Choose in (4.2) \( \eta = \mathbf{0} \) and obtain even solutions about \( t_0 \) where \( y[(t-t_0)] \equiv y[-(t-t_0)] \). This means that with initial velocities zero, the trajectories of the \( N \) point masses in the future \( t > t_0 \) are a perfect reflection of the past \( t < t_0 \).

We selected from the textbooks [1, 2, 7, 12, 10, 9, 11] and from certain journals, more examples. A substantial number of differential equations are obtained from a certain potentials \( U(y) \) as follows

\[
y'' = f(y), \text{ with } \quad y \in \mathbb{R}^3, \quad f(y) = -\nabla U(y).
\]

In a series of papers published between 1924 and 1930, G. Manev studied a modification to the Newtonian potential \( U(y) := -\frac{\gamma}{\|y\|} \) where \( \gamma = G(m_1 + m_2) \). With \( \epsilon > 0 \) being a certain parameter Manev studied the potential.

\[
U(y) := -\frac{\gamma}{\|y\|} - \frac{\epsilon}{\|y\|^2}.
\]

For a detailed discussion of the history and applications of the Manev potential we refer the reader to [4, 5]. Another differential system \( y'' = f(y) \) originates from the Kepler anisotropic potential.

\[
U(y) = -\frac{1}{\sqrt{y_1^2 + y_2^2}} - \frac{b}{(\mu y_1^2 + y_2^2)^{\beta/2}} \quad \beta \geq 2, \quad \mu \geq 1, \quad b > 0.
\]

Compare with [6]. The list of equations of scalar nonlinear equations \( x'' = f(x) \) below is taken from [12]. Notice that in the first six differential equations listed below, \( f(x) \) is an odd function of \( x \). Consequently, the first six equations possess a continuum of even solutions and a continuum of odd solutions.

In what follows, \( x, \alpha, a, \omega, \theta, \lambda, g, u, m, F, h, \phi, c \), are real values. A few comments may accompany the equations below.

The equation

\[
\frac{d^2x}{dt^2} = -x - \alpha x^3.
\]

The equation

\[
\frac{d^2x}{dt^2} = -9x,
\]
Two stars, each with gravitational mass \( \mu \), are orbiting each other under their mutual gravitational forces in such away that their orbits are circles of radius \( a \). A satellite of relatively negligible mass is moving on a straight line through the mass center \( G \) such that the line is perpendicular to the plane of the mutual orbits of this binary system. The equation of motion is then given by

\[
\frac{d^2 x}{dt^2} = -\frac{2\mu x}{(a^2 + x^2)^{\frac{3}{2}}}
\]

The Pendulum equation is given by

\[
\frac{d^2 x}{dt^2} = -\omega^2 \sin(x).
\]

It may be approximated for moderate amplitudes by the equation

\[
\frac{d^2 x}{dt^2} = -\omega^2(x - \frac{1}{6}x^3),
\]

where \( x \) is the inclination.

A Pendulum of length \( a \) has a bob of mass \( m \) which is subject to a horizontal force \( m\omega^2a\sin \theta \), where \( \theta \) is the inclination to the downward vertical. The mathematical model governs the motion is given by:

\[
\frac{d^2 \theta}{dt^2} = \omega^2(\cos \theta - \lambda) \sin \theta.
\]

A particle is attached to a fixed point \( O \) on a smooth horizontal plane by an elastic string, \( x \) being the displacement from \( O \). When unstretched, the length of the string is \( 2a \). The equation of motion of the particle, which is constrained to move on a straight line through \( O \), is given by:

\[
\frac{d^2 x}{dt^2} = -x + a \operatorname{sgn}(x), \quad \text{if } |x| > a, \quad \frac{d^2 x}{dt^2} = 0, \quad \text{if } |x| \leq a.
\]

The equation

\[
\frac{d^2 x}{dt^2} = x^4 - x^2.
\]

The equation

\[
\frac{d^2 x}{dt^2} = (x - \lambda)(x^2 - \lambda).
\]

A Pendulum with a magnetic bob oscillates in a vertical plane over a magnet, which repels the bob according to the inverse square law, the equation of motion is then given by:

\[
ma^2\frac{d^2 \theta}{dt^2} = -mg \sin \theta + F \sin \phi,
\]

where \( h > a \) and

\[
F = \frac{c}{a^2 + h^2 - 2ah \cos \theta}.
\]

The equation

\[
\frac{d^2 x}{dt^2} = -\lambda - x^3 + x.
\]
The equation
\[
\frac{d^2 x}{dt^2} = a - \exp(x).
\]

The equation
\[
\frac{d^2 x}{dt^2} = a + \exp(x).
\]

The equation of motion of a conservative system is in more general form studied in various text books. See also [2].

\[
\frac{d^2 x}{dt^2} = -g(x),
\]

where \( g(0) = 0 \), \( g(x) \) is strictly increasing for all \( x \), and

\[
\int_0^x g(u)du \rightarrow \infty \text{ as } x \rightarrow \pm \infty.
\]

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