On Stability of Delay Equations with Positive and Negative Coefficients with Applications

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Abstract. We obtain new explicit exponential stability conditions for linear scalar equations with positive and negative delayed terms

\[ \dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) - \sum_{k=1}^{l} b_k(t)x(g_k(t)) = 0 \]

and its modifications, and apply them to investigate local stability of Mackey–Glass type models

\[ \dot{x}(t) = r(t) \left[ \beta \frac{x(g(t))}{1 + x^n(g(t))} - \gamma x(h(t)) \right] \]

and

\[ \dot{x}(t) = r(t) \left[ \beta \frac{x(g(t))}{1 + x^n(h(t))} - \gamma x(t) \right]. \]

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1. Introduction

The Mackey–Glass equation with two delays and non-monotone feedback

\[ \dot{x}(t) = r(t) \left[ \beta \frac{x(g(t))}{1 + x^n(g(t))} - \gamma x(h(t)) \right] \] (1.1)
generalizes the classical blood production model \cite{17}
\[ \dot{x}(t) = \beta \frac{x(t - \tau)}{1 + x^n(t - \tau)} - \gamma x(t). \tag{1.2} \]

Similarly to (1.2), if \( \gamma < \beta \) then equation (1.1) has a positive equilibrium \( x^* = \left( \frac{\beta}{\gamma} - 1 \right)^{\frac{1}{n}} \). The linearization of (1.1) around \( x^* \) has the form
\[ \dot{x}(t) + r(t) \left[ \gamma x(h(t)) - \gamma \left( 1 - n + \frac{\gamma n}{\beta} \right) x(g(t)) \right] = 0. \tag{1.3} \]

Another equation generalizing (1.2), with two delays involved in the production function
\[ \dot{x}(t) = s(t) \left[ \frac{\beta x(p(t))}{1 + x^n(q(t))} - x(t) \right], \quad t \geq t_0, \tag{1.4} \]
was recently considered in \cite{9}. It has a positive equilibrium \( x^* = (\beta - 1)^{\frac{1}{n}} \) for \( \beta > 1 \), and its linearization about \( x^* \) has the form
\[ \dot{x}(t) + s(t) [x(t) + \alpha x(q(t)) - x(p(t))] = 0, \tag{1.5} \]
where
\[ \alpha = \frac{n(\beta - 1)}{\beta}. \]

This motivates us to investigate stability of linear equations with several positive and negative terms.

To this end, we obtain new explicit exponential stability conditions for the scalar delay differential equation with positive and negative coefficients
\[ \dot{x}(t) + a(t)x(h(t)) - b(t)x(g(t)) = 0, \quad a(t) \geq b(t) \geq 0, \quad h(t) \leq t, \quad g(t) \leq t, \tag{1.6} \]
and for some generalizations of this equation, including equations with several delays, integro-differential equations and equations with distributed delays.

Most stability results for linear delay differential equations were obtained for equations with positive coefficients, see, for example, \cite{13,14,16,19}. There are only few results for equations of type (1.6). The paper \cite{8} involves a review of stability tests for equation (1.6). Most of these results are obtained for the equation
\[ \dot{x}(t) + a(t)x(h(t)) - b(t)x(t) = 0 \tag{1.7} \]
and usually have a complicated form. In \cite{4}, various stability results \cite{12,18,20,24} for equation (1.7) are compared using the test equation
\[ \dot{x}(t) + \sin^2 t [0.6x(t - 2) - bx(t)] = 0, \tag{1.8} \]
where $b < 0.6$ is a positive constant. This equation was considered first in the paper [22] for $b = \frac{1}{15}$, and it was shown that the equation is asymptotically stable. The best known condition $b < 0.26$ for exponential stability of equation (1.8) was obtained in [4]. In the present paper we improve all known stability tests for equation (1.8), getting the estimate $b < 0.34$ for exponential stability.

Equation (1.8) is a special case of the equation with positive and negative coefficients and a non-delay term

$$\dot{x}(t) + r(t)[ax(h(t)) - bx(t)] = 0, \quad (1.9)$$

which is also a particular case of the equation with two delays

$$\dot{x}(t) + r(t)[ax(h(t)) - bx(g(t))] = 0, \quad (1.10)$$

where $a > b > 0$ are constant, $r(t) \geq 0$, while $h$ and $g$ are delayed arguments.

Equations (1.9) and (1.10) appear as linearizations for many mathematical models including Mackey–Glass equation (1.1) and its modifications. In the present paper we obtain explicit exponential stability conditions for (1.9) and (1.10), which are easy to verify. In particular, under some natural additional conditions, equation (1.9) is uniformly exponentially stable if

$$\limsup_{t \to \infty} \int_{h(t)}^{t} r(s) ds < 1 + \frac{1}{e}. \quad (1.11)$$

Applying stability condition (1.11) to equation (1.8), we obtain the estimate $b < 0.34$, as was mentioned above.

We apply our results to Mackey–Glass models with two delays (1.1) and (1.4) deducing explicit local exponential stability (LES) conditions for the positive equilibrium and illustrate these results with numerical simulations.

To obtain stability results for linearized equations, we apply the following tools:

- Bohl–Perron theorem which reduces the exponential stability problem to the norm estimation for linear operators in some functional spaces on semi-axes;
- various transformations of a given equation including a transformation of the independent variable $t$;
- properties of equations with a positive fundamental function;
- a priori estimates of solutions and their derivatives.

The paper is organized as follows. Section 2 contains a review of auxiliary results which are instrumental in the future proofs. In Section 3 the main stability results of the paper are obtained. Section 4 involves an extension of these results to some more general models, such as equations with several positive and negative delayed terms, integro-differential equations and equations
with a distributed delay. In Section 5 we apply the results obtained to Mackey–Glass type equations. Section 6 presents a brief discussion of the results, as well as suggests some projects for future research.

2. Preliminaries

We consider equation (1.6) under the following conditions:

(a1) $a(t) \geq b(t) \geq 0$;

(a2) the functions $h$ and $g$ are Lebesgue measurable on $[0, \infty)$, and $0 \leq t - h(t) \leq \tau$, $0 \leq t - g(t) \leq \delta$ for some finite constants $\tau$ and $\delta$.

Together with equation (1.6), we consider for any $t_0 \geq 0$ the initial value problem

$$\dot{x}(t) + a(t)x(h(t)) - b(t)x(g(t)) = f(t), \quad t \geq t_0; \quad x(t) = \varphi(t), \quad t \leq t_0, \quad (2.1)$$

where

(a3) the right-hand side $f$ is a Lebesgue measurable essentially bounded function on $[t_0, \infty)$, the initial function $\varphi : [t_0 - \max\{\delta, \tau\}, t_0] \rightarrow \mathbb{R}$ is a Borel measurable and bounded function.

**Definition 2.1.** The solution of problem (2.1) is a locally absolutely continuous on $[t_0, \infty)$ function satisfying the equation almost everywhere (a.e.) for $t \geq t_0$ and the initial conditions for $t \in [t_0 - \max\{\delta, \tau\}, t_0]$. The fundamental function $X(t, s)$ is a solution of the problem

$$\dot{x}(t) + a(t)x(h(t)) - b(t)x(g(t)) = 0, \quad t \geq s; \quad x(t) = 0, \quad t < s, \quad x(s) = 1.$$

The following result incorporates the solution representation.

**Lemma 2.2 (Theorem 4.3.1).** The solution of problem (2.1) exists, is unique and has the form

$$x(t) = X(t, t_0)\varphi(t_0) + \int_{t_0}^{t} X(t, s)f(s)ds - \int_{t_0}^{t} X(t, s)[a(s)\varphi(h(s)) - b(s)\varphi(g(s))]ds,$$

where we assume $\varphi(t) = 0$ for $t > t_0$.

**Definition 2.3.** We will say that equation (1.6) is uniformly exponentially stable if there exist positive numbers $M$ and $\gamma$ such that the solution of problem (2.1) with $f \equiv 0$ and an arbitrary $t_0 \geq 0$ has the estimate

$$|x(t)| \leq Me^{-\gamma(t-t_0)} \sup_{t \in [t_0 - \max\{\delta, \tau\}, t_0]} |\varphi(t)|, \quad t \geq t_0, \quad (2.2)$$
where $M$ and $\gamma$ do not depend on either $t_0$ or $\varphi$. The fundamental function $X(t, s)$ of equation (1.6) has an exponential estimate if it satisfies

$$|X(t, s)| \leq M_0 e^{-\gamma_0 (t-s)}, \quad t \geq s \geq 0$$

for some positive numbers $M_0 > 0$ and $\gamma_0 > 0$.

We will say that an equilibrium solution of either (1.1) or (1.4) is locally exponentially stable (LES) if the linearized equation around this equilibrium is uniformly exponentially stable.

Existence of an exponential estimate for the fundamental function is equivalent [8] to the exponential stability for equations with bounded delays (see (a2)). Evidently we can shift the initial point to some $t_1 \in (0, t_0)$ and obtain an exponential estimate for $X(t, s)$ with some other constants $M_1 > 0, \gamma_1 > 0$.

The following result is usually referred to as the Bohl–Perron principle.

**Lemma 2.4** ([2, Theorem 4.7.1]). Assume that the solution of problem (2.1), where $\varphi(t) = 0$, $t \leq t_0$, is bounded on $[t_0, \infty)$ for any $f$ which is essentially bounded on $[t_0, \infty)$. Then equation (1.6) is uniformly exponentially stable.

**Remark 2.5.** The Bohl–Perron principle was stated above for equation (1.6) with two delays, but it is valid for linear equations with an arbitrary number of delays, for integro-differential equations, and for equations with a distributed delay.

**Lemma 2.6** ([7, Lemmas 9,10]). Suppose that exponential estimate (2.2) is valid for a solution of (2.1), with $t_1$ instead of $t_0$ and $M_1$ instead of $M$, and we have a bounded solution growth, i.e. for any $t_0 \leq t_1$, there is a number $A \geq 1$ not dependent on either $t_0$ or $t_1$ such that

$$\sup_{t \in [t_0, t_1]} x(t) \leq A \sup_{t \in [t_0 - \max\{\delta, \tau\}, t_0]} |\varphi(t)|,$$  \hspace{1cm} (2.3)

Then there exists $M > 0$ such that a solution of (2.1) satisfies (2.2) with the same $\gamma$.

**Remark 2.7.** It follows from Lemma 2.6 that in Lemma 2.4 we can consider boundedness of solutions not for all $f$ which are essentially bounded on $[t_0, \infty)$ but only for those that vanish on $[t_0, t_1)$ for any fixed $t_1 > t_0$.

Further, in addition to a possible shift of the initial point, we apply an argument transformation in the exponential estimate.
Lemma 2.8. Let \( p : [0, \infty) \to [0, \infty) \) be a continuous increasing function,
\[
\lim_{t \to \infty} p(t) = \infty \tag{2.4}
\]
and \( t_0 \geq \max\{\delta, \tau\} \). Consider a set of continuous functions
\[
x : [p(t_0 - \max\{\delta, \tau\}), \infty) \to \mathbb{R}
\]
satisfying
\[
|x(p(t))| \leq M_1 e^{-\gamma_1(p(t) - p(t_0))} \sup_{t \in [p(t_0 - \max\{\delta, \tau\}), p(t_0)]} |x(t)|, \quad t \geq t_0, \tag{2.5}
\]
where \( M_1 > 0 \) and \( \gamma_1 > 0 \) are independent of \( t_0 \geq 0 \) and of the values of \( x(t) \) for \( t \in [p(t_0 - \max\{\delta, \tau\}), p(t_0)] \).

Then, for any \( t_1 \geq p(t_0) \) and \( \sigma > 0 \), there exists \( \sigma_1 > 0 \) such that
\[
\sup_{t \leq t_1} |x(t)| < \sigma_1 \text{ implies } |x(t)| < \sigma \text{ for } t \geq t_1, \text{ and } \lim_{t \to \infty} x(t) = 0.
\]

If, in addition, there exist \( T > 0 \) and \( A > 0 \) such that
\[
\limsup_{t, u \to \infty, t > u} \frac{p(t) - p(u)}{t - u} < A, \quad \liminf_{t \to \infty} [p(t + T) - p(t)] > 0 \tag{2.6}
\]
then \( x(t) \) satisfies (2.2) with some positive constant instead of \( \max\{\delta, \tau\} \), for some \( M > 0, \gamma > 0 \) not depending on \( t_0 \geq \max\{\delta, \tau\} \) and on the values \( \varphi(t) \) of \( x(t) \) for \( t \leq t_0 \).

Proof. Let us fix \( \sigma > 0 \), choose \( t_1 \geq p(t_0) \), assume that (2.4) and (2.5) hold, and also that \( \sup_{t \in [p(t_0 - \max\{\delta, \tau\}), p(t_1)]} |x(t)| < \sigma_1 \), where \( \sigma_1 = \frac{\sigma}{M_1} \). Then
\[
\sup_{t \in [p(t_0 - \max\{\delta, \tau\}), p(t_0)]} |x(t)| \leq \sup_{t \leq [p(t_0 - \max\{\delta, \tau\}), t_1]} |x(t)| < \sigma_1,
\]
so inequality (2.5) implies
\[
|x(p(t))| \leq M_1 e^{-\gamma_1(p(t) - p(t_0))} \sup_{t \in [p(t_0 - \max\{\delta, \tau\}), p(t_0)]} |x(t)| < M_1 \sigma_1 = \sigma, \quad t \geq t_0.
\]

We have \( |x(p(t))| < \sigma \) for \( t \geq t_0 \), where by (2.4) and continuity, the function \( p(t) \) takes all the values in \([p(t_0), \infty)\). The function \( p \) is monotone increasing, therefore \( t \geq t_0 \) yields that \( p(t) \geq p(t_0) \), and \( |x(t)| < \sigma \) for any \( t \geq p(t_0) \), in particular, for \( t \geq t_1 \). Further, \( \lim_{t \to \infty} e^{-\gamma_1(p(t) - p(t_0))} = 0 \) and (2.4) imply \( \lim_{t \to \infty} x(t) = 0 \).

Next, let (2.6) hold. By the second inequality in (2.6), there exist \( t_1 > 0 \) and \( \varepsilon > 0 \) such that \( p(t + T) - p(t) > \varepsilon \) for \( t \geq t_1 \). Hence
\[
p(t) - p(t_1) \geq \left[ \frac{t - t_1}{T} \right] \varepsilon > \varepsilon \left( \frac{t - t_1}{T} - 1 \right) = \frac{\varepsilon}{T}(t - t_1) - \varepsilon,
\]
where \([t]\) is the integer part of \(t\).

Without loss of generality, using the first inequality in (2.6), we assume

\[
\frac{p(t) - p(u)}{t - u} \leq A, \quad t > u \geq t_1.
\]

Substituting \(u = t_1\), we have \(p(t) \leq p(t_1) + A(t - t_1)\). Denoting \(w = A(t - t_1)\), we notice that \(p(t) - p(t_1) \geq \frac{\varepsilon}{AT}w - \varepsilon\), thus estimate (2.5) implies

\[
x(p(t_1) + w) \leq M_1 \exp \left\{ -\gamma_1 \left( \frac{\varepsilon}{AT}w - \varepsilon \right) \right\} \sup_{t \in [p(t_1 - \max\{\delta, \tau\}), p(t_1)]} |x(t)|, \quad w > 0,
\]

the inequality holds if the initial point \(p(t_1)\) is substituted by any \(t_2 \geq p(t_1)\). Note that \(p(t_1 - \max\{\delta, \tau\}) \geq p(t_1) - A \max\{\delta, \tau\}\). Thus

\[
|x(t)| \leq Me^{-\gamma (t - t_2)} \sup_{t \in [t_2 - A \max\{\delta, \tau\}, t_2]} |x(t)|,
\]

where

\[
\gamma = \frac{\varepsilon}{AT} \gamma_1, \quad M = M_1 e^{-\gamma_1}, \quad t_2 \geq p(t_1).
\]

Further, (2.5) yields that (2.3) holds for \(t_1 = t_2\) and \(A = M_1\). Thus, the fact that we can shift \(t_2\) to any point \(t_0 < t_2\) follows from Lemma 2.6.

The following example illustrates the significance of condition (2.6) in Lemma 2.8.

**Example 2.9.** The equation

\[
\dot{x}(t) + \frac{1}{t} x(t) = 0, \quad t \geq 1
\]

has the solution \(x(t) = \frac{x(1)}{t}\) which tends to zero but has no exponential estimate. However, after the substitution \(p(t) = \ln t\), for which the second inequality in (2.6) fails, we get the equation \(\dot{x}(s) + x(s) = 0\), its solution has the exponential estimate

\[
x(s) = x(s_0) e^{-(s - s_0)}.
\]

All assumptions, definitions and results formulated above for equation (1.6) are naturally extended to other scalar linear delay differential equations investigated in the paper, without additional discussion.

Consider now a linear equation with a single delay and a non-negative coefficient

\[
\dot{x}(t) + a(t) x(h_0(t)) = 0, \quad a(t) \geq 0, \quad 0 \leq t - h_0(t) \leq \tau_0, \quad (2.7)
\]

and denote by \(X_0(t, s)\) the fundamental function of equation (2.7).
Lemma 2.10 ([1] Theorem 2.21]). Assume that $X_0(t, s) > 0$, $t \geq s \geq t_0$. Then
\[
\int_{t_0+\tau_0}^{t} X_0(t, s)a(s)ds \leq 1.
\]

Lemma 2.11 ([5] Theorem 2.7, [15] Theorem 3.1.1]). If for some $t_0 \geq 0$
\[
\int_{\min\{t_0, h_0(t)\}}^{t} a(s)ds \leq \frac{1}{e}, \quad t \geq t_0
\]
then $X_0(t, s) > 0$, $t \geq s \geq t_0$.

If in addition $a(t) \geq a_0 > 0$ then equation (2.7) is uniformly exponentially stable.

To extend stability results obtained for equation (1.6) to equations with more general delays, we will need the following three “transformation” results reducing terms with either distributed or several concentrated delays to a single term with a concentrated delay.

Lemma 2.12 ([3] Lemma 5]). Assume that $a_k(t), h_k(t), k = 1, \ldots, m$ are measurable functions, $a_k(t) \geq 0$, $h_k(t) \leq t$, $k = 1, \ldots, m$ and $x$ is continuous on $[t_0, \infty)$. Then there exists a measurable function $h_0$ satisfying
\[
h_0(t) \leq t, \quad \min_k h_k(t) \leq h_0(t) \leq \max_k h_k(t)
\]
such that
\[
\sum_{k=1}^{m} a_k(t)x(h_k(t)) = \left(\sum_{k=1}^{m} a_k(t)\right)x(h_0(t)).
\]

Lemma 2.13 ([6]). Assume that $B(t, s)$ is a measurable non-decreasing in $s$ function, $h(t)$ is a measurable function, $h(t) \leq t$, and $x$ is continuous on $[t_0, \infty)$. Then there exists a measurable function $h_0$, $h(t) \leq h_0(t) \leq t$ such that
\[
\int_{h(t)}^{t} x(s)d_s B(t, s) = \left(\int_{h(t)}^{t} d_s B(t, s)\right)x(h_0(t)).
\]

As a particular case of Lemma 2.13 we obtain the following result.

Lemma 2.14. Assume that $A(t, s) \geq 0$, $h(t) \leq t$, and $x$ is continuous on $[t_0, \infty)$. Then there exists a measurable function $h_0$, $h(t) \leq h_0(t) \leq t$ such that
\[
\int_{h(t)}^{t} A(t, s)x(s)ds = \left(\int_{h(t)}^{t} A(t, s)ds\right)x(h_0(t)).
\]
3. Main results

We assume in this section that conditions (a1)–(a3) hold for equation (1.6), and corresponding conditions are satisfied for equations (1.9) and (1.10).

Equation (1.6) is well studied for the case \( h(t) \equiv t \). In particular, the following result is known.

**Theorem 3.1** ([5, Corollary 2.4], [8, Corollary 3.13]). Assume that \( a(t) \geq a_0 > 0 \), \( h(t) \equiv t \) and

\[
\limsup_{t \to \infty} \frac{b(t)}{a(t)} < 1.
\]

Then equation (1.6) is uniformly exponentially stable.

Let us fix an interval \( I = [t_0, t_1] \), \( t_1 > t_0 \geq 0 \), and for any essentially bounded on \( [t_0, \infty) \) function define

\[
|f|_I = \text{ess sup}_{t \in I} |f(t)|, \quad \|f\|_{[t_0, \infty)} = \text{ess sup}_{t \geq t_0} |f(t)|,
\]

\[
a^+ = \max\{a, 0\}. \quad \text{We are in a position to state and prove the first main result of the present paper.}
\]

**Theorem 3.2.** Assume that

\[
a(t) - b(t) \neq 0 \text{ almost everywhere (a.e.), } \int_0^\infty [a(s) - b(s)] ds = \infty, \quad (3.1)
\]

and for some \( t_0 \geq 0 \)

\[
\left\| \left( \int_{h(t)}^t [a(s) - b(s)] ds - \frac{1}{e} \right)^+ \right\|_{[t_0, \infty)} + 2 \left\| \frac{b}{a - b} \right\|_{[t_0, \infty)} \left\| \int_{h(t)}^t [a(s) - b(s)] ds \right\|_{[t_0, \infty)} < 1. \quad (3.2)
\]

Then equation (1.6) is asymptotically stable. If in addition there exists \( T > 0 \) such that

\[
\liminf_{t \to \infty} \int_t^{t + T} [a(s) - b(s)] ds > 0 \quad (3.3)
\]

then (1.6) is uniformly exponentially stable.

**Proof.** Rewrite equation (1.6) as

\[
\dot{x}(t) + [a(t) - b(t)]x(h(t)) + b(t)[x(h(t)) - x(g(t))] = 0 \quad (3.4)
\]

and denote

\[
s = p(t) := \int_{t_0}^t [a(\zeta) - b(\zeta)] d\zeta,
\]

where \( p(t_0) = 0 \). By the assumptions of the theorem, \( p \) is a strictly monotone increasing function satisfying \( \lim_{t \to \infty} p(t) = \infty \). Let us make the substitution \( t = p^{-1}(s) \), \( x(t) = y(s) \), then

\[
\dot{x}(t) = [a(p^{-1}(s)) - b(p^{-1}(s))]|\dot{y}(s), \quad x(h(t)) = y(h_0(s)), \quad x(g(t)) = y(g_0(s)),
\]
and
\[ s - h_0(s) = \int_{h(t)}^t [a(\zeta) - b(\zeta)]d\zeta, \quad s - g_0(s) = \int_{g(t)}^t [a(\zeta) - b(\zeta)]d\zeta. \]

Denote
\[ \tau_0 = \text{ess sup}_{t \geq t_0} \int_{h(t)}^t [a(\zeta) - b(\zeta)]d\zeta, \quad \delta_0 = \text{ess sup}_{t \geq t_0} \int_{g(t)}^t [a(\zeta) - b(\zeta)]d\zeta. \]

Hence \( s - h_0(s) \leq \tau_0 < \infty, \) \( s - g_0(s) \leq \delta_0 < \infty. \) Equation (3.4) has the form
\[
\dot{y}(s) + y(h_0(s)) + \frac{b(p^{-1}(s))}{a(p^{-1}(s)) - b(p^{-1}(s))} \int_{h_0(s)}^{\dot{h}_0(s)} \dot{y}(\zeta)d\zeta = 0. \tag{3.5}
\]

To prove asymptotic stability of equation (3.5), consider the initial value problem
\[
\dot{y}(s) + y(h_0(s)) + \frac{b(p^{-1}(s))}{a(p^{-1}(s)) - b(p^{-1}(s))} \int_{h_0(s)}^{\dot{h}_0(s)} \dot{y}(\zeta)d\zeta = f(s), \quad s \geq 0, \tag{3.6}
\]
\[
y(s) = \dot{y}(s) = 0, \quad s \leq 0,
\]
where \( f \) is an essentially bounded function on \([0, \infty)\) such that
\[
f(s) = 0, \quad s \leq s_0 = \max \left\{ \tau_0, \delta_0, \frac{1}{\varepsilon} \right\}. \tag{3.7}
\]

Conditions (3.6) and (3.7) imply that the solution of (3.6) satisfies \( y(s) = \dot{y}(s) = 0, \) \( s \leq s_0. \)

Denote
\[
r_0(s) = \begin{cases} h_0(s), & h_0(s) \geq s - \frac{1}{\varepsilon}, \\ s - \frac{1}{\varepsilon}, & h_0(s) < s - \frac{1}{\varepsilon}, \end{cases}
\]
then \( s - r_0(s) \leq \frac{1}{\varepsilon}. \) We can rewrite equation (3.6) as
\[
\dot{y}(s) + y(r_0(s)) = -\int_{r_0(s)}^{h_0(s)} \dot{y}(\zeta)d\zeta - \frac{b(p^{-1}(s))}{a(p^{-1}(s)) - b(p^{-1}(s))} \int_{r_0(s)}^{h_0(s)} \dot{y}(\zeta)d\zeta + f(s). \tag{3.8}
\]

Let \( Y_0(s, \zeta) \) be the fundamental function of the equation
\[
\dot{y}(s) + y(r_0(s)) = 0. \tag{3.9}
\]

By Lemma 2.11 we have \( Y_0(s, \zeta) > 0, \) and equation (3.9) is uniformly exponentially stable.
Therefore, from (3.8) and Lemma 2.10, we have

$$y(s) = -\int_0^s Y_0(s, \zeta) \left[ \int_{r_0(\zeta)}^{\beta(\zeta)} \dot{y}(\xi) d\xi + \frac{b(p^{-1}(\zeta))}{a(p^{-1}(\zeta)) - b(p^{-1}(\zeta))} \int_{\gamma(\zeta)}^{\beta(\zeta)} \dot{y}(\xi) d\xi \right] d\zeta$$

$$+ f_1(s),$$

(3.10)

where $f_1(s) = \int_0^s Y_0(s, \zeta) f(\zeta) d\zeta$. Since $Y_0(s, \tau)$ has an exponential estimate, $\|f_1\|_{[0, \infty)} < \infty$.

Since the right-hand side of (3.8) is equal to zero for $s \leq s_0$, where $s_0 \geq \frac{1}{e}$, the zero lower bound in the first integral in (3.10) can be replaced with $s_0$.

In the following, up to the end of the proof, we omit the index $[t_0, \infty)$ in the norm of the functions on $[t_0, \infty)$. Let us fix an interval $I = [0, s_1]$. By Lemma 2.10 we have $0 \leq \int_{s_0}^{s} Y_0(s, \zeta) d\zeta \leq 1$, thus

$$|y|_I \leq \left[ \left( \int_{h(t)}^{t} [a(\zeta) - b(\zeta)] d\zeta - \frac{1}{e} \right)^+ \right] \| \dot{y} \| + \|f_1\|,$$

$$+ \frac{b}{a - b} \left\| \int_{h(t)}^{g(t)} [a(\zeta) - b(\zeta)] d\zeta \right\| \| \dot{y} \| + \|f_1\|.$$  

(3.12)

From equality (3.6) and the last part of (3.11), we have

$$|\dot{y}|_I \leq |y|_I + \frac{b}{a - b} \left\| \int_{h(t)}^{g(t)} [a(\zeta) - b(\zeta)] d\zeta \right\| |\dot{y}|_I + \|f\|.$$  

Therefore

$$|\dot{y}|_I \leq \frac{1}{1 - \frac{b}{a - b} \left\| \int_{h(t)}^{g(t)} [a(\zeta) - b(\zeta)] d\zeta \right\|} |y|_I + M_1,$$

(3.13)
where the denominator is positive by (3.2) and
\[
M_1 = \frac{\|f\|}{1 - \|\frac{b}{a-b}\| \left\| \int_{h(t)}^{g(t)} [a(\zeta) - b(\zeta)] d\zeta \right\|}.
\]

Inequalities (3.12) and (3.13) imply
\[
|y|_I \leq \left( \left( \int_{h(t)}^{g(t)} [a(\zeta) - b(\zeta)] d\zeta - \frac{1}{x} \right)^+ \right)^2 + \|\frac{b}{a-b}\| \left\| \int_{h(t)}^{g(t)} [a(\zeta) - b(\zeta)] d\zeta \right\| \frac{M_1}{1 - \|\frac{b}{a-b}\| \left\| \int_{h(t)}^{g(t)} [a(\zeta) - b(\zeta)] d\zeta \right\|}
\]
\[
= \frac{\left( \left( \int_{h(t)}^{g(t)} [a(\zeta) - b(\zeta)] d\zeta - \frac{1}{x} \right)^+ \right)^2 + \|\frac{b}{a-b}\| \left\| \int_{h(t)}^{g(t)} [a(\zeta) - b(\zeta)] d\zeta \right\|}{M_1} \left( 1 - \|\frac{b}{a-b}\| \left\| \int_{h(t)}^{g(t)} [a(\zeta) - b(\zeta)] d\zeta \right\| \right)
\]
\[
\leq \frac{M_1}{M_2} |y|_I + M_2, \quad \text{(3.14)}
\]

where
\[
M_2 = \frac{\left( \left( \int_{h(t)}^{g(t)} [a(\zeta) - b(\zeta)] d\zeta - \frac{1}{x} \right)^+ \right)^2 + \|\frac{b}{a-b}\| \left\| \int_{h(t)}^{g(t)} [a(\zeta) - b(\zeta)] d\zeta \right\|}{M_1} \left( 1 - \|\frac{b}{a-b}\| \left\| \int_{h(t)}^{g(t)} [a(\zeta) - b(\zeta)] d\zeta \right\| \right)
\]

Inequality (3.14) has the form $|y|_I \leq \alpha |y|_I + M_2$, where the numbers $\alpha > 0$ and $M_2 > 0$ do not depend on the interval $I$. Inequality (3.2) implies $\alpha < 1$ and thus $|y|_I \leq \frac{M_2}{1-\alpha}$. Therefore for any essentially bounded function $f$ on $[t_0, \infty)$ (vanishing on $[t_0, t_1]$ for some $t_1 > t_0$), the solution of problem (3.6) is bounded on $[t_0, \infty)$. Thus by Lemma 2.4 equation (3.5) is uniformly exponentially stable.

Hence for the fundamental function $Y(s, t)$ of equation (3.5) there exist $\lambda > 0$ and $M > 0$ such that
\[
|Y(s, t)| \leq Me^{-\lambda(s-t)}, \quad s \geq t \geq 0,
\]
and its solution $y(s)$ has an exponential estimate. Since $y(s) = x(p(t))$, by (3.1) and Lemma 2.8 equation (1.6) is asymptotically stable. Also, by Lemma 2.8 under (3.3), where (3.3) and global essential boundedness of $a$ and $b$ imply (2.6), equation (1.6) is also uniformly exponentially stable.

**Corollary 3.3.** Assume that (3.1) is satisfied, and for some $t_0 \geq 0$ one of the following two conditions holds:

1) \[
\text{ess sup}_{t \geq t_0} \int_{h(t)}^{t} [a(s) - b(s)] ds \leq \frac{1}{e}, \quad (3.15)
\]
\[
\left\| \frac{b}{a-b} \right\|_{[t_0, \infty)} \left\| \int_{h(t)}^{g(t)} [a(\zeta) - b(\zeta)] d\zeta \right\|_{[t_0, \infty)} \leq \frac{1}{2};
\]
Then equation (1.6) is asymptotically stable. If in addition there exists $T > 0$ such that (3.3) holds, (1.6) is uniformly exponentially stable.

**Proof.** Since for any essentially bounded function $f$ and a number $c \geq 0$ we have

$$\text{ess sup}_{t \geq t_0} (f(t) - c) = \begin{cases} 0, & \text{if } \text{ess sup}_{t \geq t_0} f(t) \leq c, \\ \text{ess sup}_{t \geq t_0} f(t), & \text{if } \text{ess sup}_{t \geq t_0} f(t) > c, \end{cases}$$

we obtain the statement of the corollary by applying (3.2) in Theorem 3.2 in the two cases.

**Corollary 3.4.** Suppose that $g(t) \equiv t$, (3.1) is satisfied, and for some $t_0 \geq 0$ one of the following two assumptions holds:

1) condition (3.15) and

$$\left| \int_{h(t)} b(s) ds \right| < \frac{1}{2};$$

2) condition (3.16) and

$$\left( 1 + 2 \left| \int_{h(t)} b(s) ds \right| \right) < 1 + \frac{1}{e}.$$

Then equation (1.6) is asymptotically stable. If in addition there exists $T > 0$ such that (3.3) holds, (1.6) is uniformly exponentially stable.

Consider now equation (1.10), where

$$a > b > 0, \quad r(t) \geq 0, \quad \int_0^\infty r(s) ds = \infty, \quad r(t) \neq 0 \text{ a.e.} \quad (3.17)$$

**Corollary 3.5.** Assume that (3.17) and for some $t_0 \geq 0$ one of the following conditions hold:

1) $(a - b) \text{ess sup}_{t \geq t_0} \int_{h(t)}^t r(s) ds \leq \frac{1}{e}; \quad b \text{ess sup}_{t \geq t_0} \int_{h(t)}^t r(s) ds < \frac{1}{2};$

2) $(a - b) \text{ess sup}_{t \geq t_0} \int_{h(t)}^t r(s) ds > \frac{1}{e}; \quad (a - b) \text{ess sup}_{t \geq t_0} \int_{h(t)}^t r(s) ds + 2b \text{ess sup}_{t \geq t_0} \int_{h(t)}^t r(s) ds < 1 + \frac{1}{e}.$
Then equation (1.10) is asymptotically stable. If in addition there exists $T > 0$ such that
\[ \liminf_{t \to \infty} \int_t^{t+T} r(s) \, ds > 0 \] (3.18)
then (1.10) is uniformly exponentially stable.

**Corollary 3.6.** Assume that (3.17) is satisfied, $g(t) \equiv t$, and for some $t_0 \geq 0$ one of the following conditions holds:

1) $\esssup_{t \geq t_0} \int_{h(t)}^{t} r(s) \, ds \leq \frac{1}{e(a-b)}$;  
2) $\esssup_{t \geq t_0} \int_{h(t)}^{t} r(s) \, ds > \frac{1}{e(a-b)}$,  

Then equation (1.9) is asymptotically stable. If in addition there exists $T > 0$ such that (3.18) holds then (1.9) is uniformly exponentially stable.

**Example 3.7.** Consider test equation (1.8). We will estimate the values of the parameter $b$ for which the condition 2) of Corollary 3.6 holds. Here $a = 0.6$ and $r(t) = \sin^2 t$. We have $\int_t^{t+2} \sin^2 s \, ds \geq \frac{1}{2}$, hence condition (3.18) holds with $T = 2$. Also
\[ \int_t^{t+2} \sin^2 s \, ds = 1 + \frac{1}{4} [\sin(2t) - \sin(2(t-2))] = 1 + \frac{1}{2} \sin 2 \cos(2(t-1)) \leq 1 + \frac{1}{2} \sin 2 \approx 1.4547. \]

We easily verify that Part 1 of Corollary 3.6 cannot be applied. The first inequality in Part 2 of Corollary 3.6 is
\[ b < 0.6 - \frac{1}{1.4547e} \approx 0.3471, \]
while the second inequality $(0.6 + b)1.4547 < 1 + \frac{1}{e}$ implies
\[ b < \frac{1 + \frac{1}{e}}{1.4547} - 0.6 \approx 0.3403. \]

Hence equation (1.8) is uniformly exponentially stable for $b < 0.34$. We recall that the best known estimate [4] was $b < 0.26$.

Below we present the next main result of the paper.

**Theorem 3.8.** Assume that
\[ a(t) \neq 0 \text{ a.e., } \int_0^\infty a(s) \, ds = \infty \] (3.19)
and for some $t_0 \geq 0$
\[ \esssup_{t \geq t_0} \frac{b(t)}{a(t)} < 1, \quad \esssup_{t \geq t_0} \left| \int_{h(t)}^{g(t)} a(s) \, ds \right| < 1, \] (3.20)
Then equation \eqref{1.6} is asymptotically stable. If in addition there exists \( T > 0 \) such that the condition
\[
\liminf_{t \to \infty} \int_{t+T}^{t+T} a(s) ds > 0
\]
holds then \eqref{1.6} is uniformly exponentially stable.

**Proof.** We proceed similarly to the proof of Theorem 3.2. Denote
\[
s = p(t) := \int_{t_0}^{t} a(\zeta) d\zeta, \quad p(t_0) = 0.
\]
By the conditions on \( a \), the function \( p \) is strictly monotone increasing and \( \lim_{t \to \infty} p(t) = \infty \). After the substitution \( t = p^{-1}(s), \ x(t) = y(s) \) we have
\[
\dot{x}(t) = a(p^{-1}(s)) \dot{y}(s), \quad x(h(t)) = y(h_0(s)), \quad x(g(t)) = y(g_0(s)),
\]
where
\[
s - h_0(s) = \int_{h(t)}^{t} a(\zeta) d\zeta, \quad s - g_0(s) = \int_{g(t)}^{t} a(\zeta) d\zeta,
\]
and equation \eqref{1.6} has the form
\[
\dot{y}(s) + y(h_0(s)) - \frac{b(p^{-1}(s))}{a(p^{-1}(s))} y(g_0(s)) = 0
\]
In order to prove asymptotic stability of equation \eqref{3.23}, consider the initial value problem
\[
\dot{y}(s) + y(h_0(s)) - \frac{b(p^{-1}(s))}{a(p^{-1}(s))} y(g_0(s)) = f(s), \quad s \geq 0,
\]
\[
y(s) = \dot{y}(s) = 0, \quad s \leq 0,
\]
where \( f \) is an essentially bounded function on \([s_0, \infty)\) such that
\[
f(s) = 0, \quad s \leq s_0 = \max \left\{ \tau_0, \delta_0, \frac{1}{\epsilon} \right\},
\]
\[
\tau_0 = \text{ess sup}_{t \geq t_0} \int_{h(t)}^{t} a(\zeta) d\zeta, \quad \delta_0 = \text{ess sup}_{t \geq t_0} \int_{g(t)}^{t} a(\zeta) d\zeta.
\]
Denote
\[
r_0(s) = \left\{ \begin{array}{ll} h_0(s), & h_0(s) > s - \frac{1}{\epsilon}, \\
 s - \frac{1}{\epsilon}, & h_0(s) \leq s - \frac{1}{\epsilon}, \end{array} \right.
\]
We have $s - r_0(s) \leq \frac{1}{e}$. Equation (3.24) can be rewritten as

$$\dot{y}(s) + y(r_0(s)) = \int_{h_0(s)}^{r_0(s)} \dot{y}(\zeta) d\zeta + \frac{b(p^{-1}(s))}{a(p^{-1}(s))} y(g_0(s)) + f(s).$$  \hspace{1cm} (3.25)

Let $Y_0(s, \zeta)$ be the fundamental function of the equation

$$\dot{y}(s) + y(r_0(s)) = 0.$$  \hspace{1cm} (3.26)

By Lemma 2.11, we get that $Y_0(s, \tau) > 0$ and equation (3.26) is uniformly exponentially stable. From (3.25) and Lemma 2.2, we have

$$y(s) = \int_{s_0}^{s} Y_0(s, \zeta) \left[ \int_{h_0(\zeta)}^{r_0(\zeta)} \dot{y}(\xi) d\xi + \frac{b(p^{-1}(\zeta))}{a(p^{-1}(\zeta))} y(g_0(\zeta)) \right] d\zeta + f_1(s),$$

where $f_1(s) = \int_{s_0}^{s} Y_0(s, \zeta) f(\zeta) d\zeta$. Since $Y_0(s, \zeta)$ has an exponential estimate, $\|f_1\|_{[s_0, \infty]} < \infty$.

Further we omit the index $[t_0, \infty)$ in the norm of the functions on $[t_0, \infty)$ and assume $I = [s_0, s_1]$, where $s_1 > s_0$ is fixed. By Lemma 2.10

$$|y|_I \leq \|(r_0 - h_0)^+\| |\dot{y}|_I + \left\| \frac{b}{a} \right\| |y|_I + \|f\|_I. \hspace{1cm} (3.27)$$

Also,

$$(r_0(s) - h_0(s))^+ = (s - h_0(s)) - (s - r_0(s))^+ = \left( \int_{h(t)}^{t} a(\zeta) d\zeta - \frac{1}{e} \right)^+,$$

$$g_0(s) - h_0(s) = (s - h_0(s)) - (s - g_0(s)) = \int_{h(t)}^{t} a(\zeta) d\zeta - \int_{h(t)}^{g(t)} a(\zeta) d\zeta = \int_{h(t)}^{g(t)} a(\zeta) d\zeta.$$

Therefore

$$|y|_I \leq \|(r_0 - h_0)^+\| |\dot{y}|_I + M_1, \hspace{1cm} \text{where} \hspace{0.5cm} M_1 := \left\| \frac{b}{a} \right\| \left( 1 - \frac{\|f\|_I}{\|a\|_I} \right). \hspace{1cm} (3.28)$$

Rewriting equation (3.24) as $\dot{y}(s) = \int_{h_0(s)}^{g_0(s)} \dot{y}(\zeta) d\zeta - \left( 1 - \frac{b(p^{-1}(s))}{a(p^{-1}(s))} \right) y(g_0(s)) + f(s)$ implies

$$|\dot{y}| \leq \|h_0 - g_0\| |\dot{y}|_I + \left( 1 - \frac{b}{a} \right) \|y|_I + \|f\|_I.$$

Then

$$|\dot{y}| \leq \frac{\|1 - \frac{b}{a}\|}{1 - \|h_0 - g_0\|} |\dot{y}|_I + M_2, \hspace{1cm} \text{where} \hspace{0.5cm} M_2 := \frac{\|f\|_I}{1 - \|h_0 - g_0\|}. \hspace{1cm} (3.29)$$
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Inequalities (3.27) and (3.28) yield that

$$|y_I| \leq \left\| \frac{(r_0 - h_0)^+}{1 - \left\| \frac{h_0 - g_0}{h_0 - g_0} \right\|} \right\| |y_I| + M, \text{ where } M := \left\| \frac{(r_0 - h_0)^+}{1 - \left\| \frac{h_0 - g_0}{h_0 - g_0} \right\|} \right\| M_2 + M_1. \quad (3.29)$$

We have

$$\left\| (r_0(s) - h_0(s))^+ \right\| = \left\| \left( \int_{h(t)}^{\cdot} a(s) ds - \frac{1}{e} \right)^+ \right\|, \quad \left\| h_0 - g_0 \right\| = \left\| \int_{h(t)}^{g(t)} a(s) ds \right\|. \quad (3.30)$$

Inequality (3.21) implies

$$\left\| (r_0(s) - h_0(s))^+ \right\| = \left\| \frac{(r_0 - h_0)^+}{1 - \frac{b}{a}} \right\| \left\| \frac{1 - \frac{b}{a}}{1 - \left\| h_0 - g_0 \right\|} \right\| < 1. \quad \text{From (3.29) we have}$$

$$|y_I| \leq \frac{M}{1 - \left\| \frac{(r_0 - h_0)^+}{1 - \frac{b}{a}} \right\| \left\| \frac{1 - \frac{b}{a}}{1 - \left\| h_0 - g_0 \right\|} \right\|} \quad (3.30)$$

The right-hand side of (3.30) does not depend on the interval $I$. Hence for any bounded on $[s_0, \infty)$ function $f$, the solution of problem (3.24) is a bounded on $[s_0, \infty)$ function. Then, by Lemma 2.4 equation (3.23) is uniformly exponentially stable.

However $y(s) = x(p(t))$, therefore (3.19) and Lemma 2.8 yield that equation (1.6) is asymptotically stable.

Since (3.22), together with boundedness of $a$, implies (2.6), by Lemma 2.8 under (3.22) equation (1.6) is also uniformly exponentially stable.

\[ \square \]

**Corollary 3.9.** Assume that (3.19) and (3.20) are satisfied and for some $t_0 \geq 0$ one of the following conditions holds:

1) $\text{ess sup}_{t \geq t_0} \int_{h(t)}^{t} a(s) ds \leq \frac{1}{e}$;
2) $\text{ess sup}_{t \geq t_0} \int_{h(t)}^{t} a(s) ds > \frac{1}{e}$ and

$$\text{ess sup}_{t \geq t_0} \int_{h(t)}^{t} a(s) ds < \frac{1 - \text{ess sup}_{t \geq t_0} \frac{b(t)}{a(t)}}{1 - \text{ess inf}_{t \geq t_0} \frac{b(t)}{a(t)}} \left( 1 - \text{ess sup}_{t \geq t_0} \int_{h(t)}^{g(t)} a(s) ds \right) + \frac{1}{e}.$$ 

Then equation (1.6) is asymptotically stable. If in addition (3.22) holds for some $T > 0$, (1.6) is uniformly exponentially stable.

**Corollary 3.10.** Assume that $g(t) \equiv t$, (3.19) is satisfied, for some $t_0 \geq 0$ and one of the following conditions holds:

$$\text{ess sup}_{t \geq t_0} \frac{b(t)}{a(t)} < 1, \quad \text{ess sup}_{t \geq t_0} \int_{h(t)}^{t} a(s) ds < 1,$$
1) \( \text{ess sup}_{t \geq t_0} \int_{h(t)}^{t} a(s) \, ds \leq \frac{1}{e} \);

2) \( \text{ess sup}_{t \geq t_0} \int_{h(t)}^{t} a(s) \, ds > \frac{1}{e} \) and

\[
\text{ess sup}_{t \geq t_0} \int_{h(t)}^{t} a(s) \, ds < \frac{1}{e} + \text{ess inf}_{t \geq t_0} \int_{h(t)}^{t} a(s) \, ds.
\]

Then equation (1.6) is asymptotically stable. If in addition (3.22) holds for some \( T > 0 \), (1.6) is uniformly exponentially stable.

Consider now equation (1.10), where (3.17) holds.

Corollary 3.11. Let (3.17) be satisfied, for some \( t_0 \geq 0 \)

\[
\text{ess sup}_{t \geq t_0} \left| \int_{h(t)}^{g(t)} r(s) \, ds \right| < \frac{1}{a}.
\]

and one of the following conditions holds:

1) \( a \text{ ess sup}_{t \geq t_0} \int_{h(t)}^{t} r(s) \, ds \leq \frac{1}{e} \);

2) \( \frac{1}{e} < a \text{ ess sup}_{t \geq t_0} \int_{h(t)}^{t} r(s) \, ds < 1 \) and

\[
a \left( \text{ess sup}_{t \geq t_0} \int_{h(t)}^{t} r(s) \, ds + \text{ess sup}_{t \geq t_0} \left| \int_{h(t)}^{g(t)} r(s) \, ds \right| \right) < 1 + \frac{1}{e}.
\]

Then equation (1.10) is asymptotically stable. If in addition there exists \( T > 0 \) such that (3.18) holds then (1.10) is uniformly exponentially stable.

Corollary 3.12. Assume that for some \( t_0 \geq 0 \) one of the following conditions holds:

1) \( \text{ess sup}_{t \geq t_0} \int_{h(t)}^{t} r(s) \, ds \leq \frac{1}{ae} \);

2) \( \frac{1}{ae} < \text{ess sup}_{t \geq t_0} \int_{h(t)}^{t} r(s) \, ds < \frac{1}{2a} (1 + \frac{1}{e}) \).

Then equation (1.9) is asymptotically stable. If in addition there exists \( T > 0 \) such that (3.18) holds then (1.9) is uniformly exponentially stable.

Next, let us compare Theorems 3.2 and 3.8.

Example 3.13. Consider the equation

\[
\dot{x}(t) + x(t - 1) - 0.3x(t) = 0,
\]

which is (1.9) with \( a = 1, b = 0.3, r(t) \equiv 1, h(t) = t - 1, g(t) \equiv t \). Condition 2) of Corollary 3.6 holds, hence equation (3.31) is uniformly exponentially stable. Conditions of Corollary 3.12 are not satisfied.
Consider the equation
\[ \dot{x}(t) + 0.4x(t - 1) - 0.35x(t - 3) = 0 \quad (3.32) \]
which is (1.10) with \( a = 0.4, b = 0.35, r(t) \equiv 1, h(t) = t - 1, g(t) = t - 3. \) Condition 2) of Corollary 3.11 holds, hence equation (3.32) is uniformly exponentially stable. Conditions of Corollary 3.5 are not satisfied.

Hence Theorems 3.2 and 3.8 are independent.

Example 3.14. Consider the equation
\[ \dot{x}(t) + \sin^2 t (ax(t - 2) - bx(t)) = 0. \quad (3.33) \]
For \( a = 0.6 \) and \( b < 0.34, \) Corollary 3.6 implies uniform exponential stability. For this \( a, \) Corollary 3.12 fails to establish stability of (3.33). However, Corollary 3.12 can be applied for
\[ a < \frac{1}{1.4546} \cdot \frac{1}{2} \left( 1 + \frac{1}{e} \right) \approx 0.47 \]
and any \( b < a, \) and for these values of \( a, \) Corollary 3.6 also implies uniform exponential stability.

4. Some generalizations

In this section we consider differential equations with several delays, integro-differential equations and equations with a distributed delay. We will only present generalizations of Theorems 3.1–3.8 to these equations. All the corollaries of the generalized theorems can be obtained similarly to the corollaries of Theorems 3.2 and 3.8.

4.1. Equations with Several Delays. Consider an equation with several delays and positive and negative coefficients
\[ \dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) - \sum_{k=1}^{l} b_k(t)x(g_k(t)) = 0, \quad (4.1) \]
where for the parameters of equation (4.1) the following conditions hold:
(b1) \( a_k \) and \( b_k \) are Lebesgue measurable essentially bounded functions on \([0, \infty), a_k(t) \geq 0, b_k(t) \geq 0, \sum_{k=1}^{m} a_k(t) \geq \sum_{k=1}^{l} b_k(t);\)
(b2) the functions \( h_k \) and \( g_k \) are Lebesgue measurable on \([0, \infty), \) and \( 0 \leq t - h_k(t) \leq \tau, 0 \leq t - g_k(t) \leq \delta \) for some finite constants \( \tau \) and \( \delta. \)
Denote
\[
a(t) = \sum_{k=1}^{m} a_k(t), \quad b(t) = \sum_{k=1}^{l} b_k(t), \quad h(t) = \min_{k} h_k(t), \quad H(t) = \max_{k} h_k(t),
\]
and
\[
g(t) = \min_{k} g_k(t), \quad G(t) = \max_{k} g_k(t),
\]
\[
r(t) = \min \{ h(t), g(t) \}, \quad R(t) = \max \{ H(t), G(t) \}.
\]
Then (b1), (b2) imply that (a1), (a2) hold for \( a, b, h, r, R \) where
\[
\text{Suppose that (3.3) exist the delayed arguments} \ h
\]
therefore \( \dot{x}(t) = f(t) \), (4.4)
\[
x(t) = 0, \quad t \leq t_0,
\]
where \( f \) is an essentially bounded function on \([t_0, \infty)\). By Lemma 2.12 there exist the delayed arguments \( h_0(t) \leq t, h(t) \leq h_0(t) \leq H(t) \), and \( g_0(t) \leq t, g(t) \leq g_0(t) \leq G(t) \) such that
\[
\sum_{k=1}^{m} a_k(t)x(h_k(t)) = a(t)x(h_0(t)), \quad \sum_{k=1}^{l} b_k(t)x(g_k(t)) = b(t)x(g_0(t)),
\]
therefore \( \dot{x}(t) + a(t)x(h_0(t)) - b(t)x(g_0(t)) = f(t) \). Consider now the delay differential equation
\[
\dot{y}(t) + a(t)y(h_0(t)) - b(t)y(g_0(t)) = 0.
\]
We have
\[
\left( \int_{h_0(t)}^{t} [a(s) - b(s)] ds - \frac{1}{e} \right)^+ \leq \left( \int_{h(t)}^{t} [a(s) - b(s)] ds - \frac{1}{e} \right)^+, \quad \int_{h_0(t)}^{g_0(t)} [a(s) - b(s)] ds \leq \int_{h(t)}^{G(t)} [a(s) - b(s)] ds.
\]
By Theorem 3.2 equation (4.5) is uniformly exponentially stable. Hence by Lemma 2.2 the function $x$ which is a solution of a uniformly exponentially stable equation with an essentially bounded right-hand side $f$, is also an essentially bounded function. Thus for any essentially bounded $f$, the solution of problem (4.4) is an essentially bounded function. Lemma 2.4, Remark 2.5 and Lemma 2.8 imply the statement of the theorem.

**Theorem 4.2.** Assume that for some $t \geq t_0$ $a(t) \geq a_0 > 0$, $a(t) \geq \sum_{k=1}^{l} b_k(t)$, 

$$\text{ess sup}_{t \geq t_0} \left( \frac{1}{a(t)} \sum_{k=1}^{l} b_k(t) \right) < 1.$$  \hspace{1cm} (4.6)

Then the equation

$$\dot{x}(t) + a(t)x(t) - \sum_{k=1}^{l} b_k(t)x(g_k(t)) = 0$$

is uniformly exponentially stable.

**Proof.** In this and the next theorem, we follow the scheme of the proof of Theorem 4.1.

Suppose $x(t), t \geq t_0$ is a solution of the initial value problem

$$\dot{x}(t) + a(t)x(t) - \sum_{k=1}^{l} b_k(t)x(g_k(t)) = f(t), \quad t \geq t_0,$$

$$x(t) = 0, \quad t \leq t_0,$$

where $f$ is an essentially bounded function on $[t_0, \infty)$. By Lemma 2.12 there exists the delayed argument $g_0(t) \leq t, g(t) \leq g_0(t) \leq G(t)$ such that

$$\sum_{k=1}^{l} b_k(t)x(g_k(t)) = b(t)x(g_0(t)),$$

therefore $\dot{x}(t) + a(t)x(t) - b(t)x(g_0(t)) = f(t)$. Consider now the delay differential equation

$$\dot{y}(t) + a(t)y(t) - b(t)y(g_0(t)) = 0.$$  \hspace{1cm} (4.7)

Inequality (4.6) and Theorem 3.1 imply that equation (4.7) is uniformly exponentially stable. The proof is concluded similarly to the end of the proof of Theorem 4.1. \qed
Theorem 4.3. Assume that condition \((3.19)\) holds, for some \(t_0 \geq 0\)

\[
\text{ess sup}_{t \geq t_0} \frac{b(t)}{a(t)} < 1, \quad \text{ess sup}_{t \geq t_0} \int_{r(t)}^{R(t)} a(s) ds < 1,
\]

(4.8)

and for some \(t_0 \geq 0\) we have

\[
\left\| \left( \int_{h(-)} a(s) ds - \frac{1}{e} \right)^{+} \right\|_{[t_0, \infty)} \leq 1 - \left\| \frac{b}{a} \right\|_{[t_0, \infty)} \left( 1 - \left\| \int_{r(-)}^{R(-)} a(s) ds \right\|_{[t_0, \infty)} \right),
\]

(4.9)

where \(a, b, h, r, R\) are defined in \((4.2)\) and \((4.3)\). Then equation \((4.1)\) is asymptotically stable. If in addition \((3.22)\) holds, equation \((4.1)\) is uniformly exponentially stable.

Proof. Suppose \(x(t), t \geq t_0\) is a solution of the initial value problem \((4.4)\), where \(f\) is an essentially bounded function on \([t_0, \infty)\). By Lemma 2.12, there exist the delayed arguments \(h_0(t) \leq t, h(t) \leq h_0(t) \leq H(t)\) and \(g_0(t) \leq g(t) \leq g_0(t) \leq G(t)\) such that

\[
\sum_{k=1}^{m} a_k(t)x(h_k(t)) = a(t)x(h_0(t)), \quad \sum_{k=1}^{l} b_k(t)x(g_k(t)) = b(t)x(g_0(t)),
\]

therefore \(\dot{x}(t) + a(t)x(h_0(t)) - b(t)x(g_0(t)) = f(t)\). Consider now equation \((4.5)\). We have

\[
\left( \int_{h_0(t)}^{t} a(s) ds - \frac{1}{e} \right)^{+} \leq \left( \int_{h(t)}^{t} a(s) ds - \frac{1}{e} \right)^{+},
\]

\[
\left| \int_{h_0(t)}^{g_0(t)} a(s) ds \right| \leq \int_{r(t)}^{R(t)} a(s) ds.
\]

By \((4.8)\), \((4.9)\) and Theorem 3.8 equation \((4.5)\) is uniformly exponentially stable. The rest of the proof is the same as in the proof of Theorem 4.1. \(\Box\)

4.2. Equations with Distributed Delays. Consider the equation with distributed delays

\[
\dot{x}(t) + a(t) \int_{h(t)}^{t} x(s) ds A(t, s) - b(t) \int_{g(t)}^{t} x(s) ds B(t, s) = 0,
\]

(4.10)

where \(a, b, h, g\) satisfy \((a1), (a2)\), \(A(t, s), B(t, s)\) are measurable on \([0, \infty) \times [0, \infty)\), \(A(t, \cdot)\) and \(B(t, \cdot)\) are left continuous non-decreasing functions for almost
Stability of Equations with Variable Delays

all \( t, A(\cdot, s) \) and \( B(\cdot, s) \) are locally integrable for any \( s, A(t, h(t)) = B(t, g(t)) = 0 \), and \( A(t, t^+) = B(t, t^+) = 1 \). Then

\[
\int_{h(t)}^{t} d_x A(t, s) = \int_{g(t)}^{t} d_x B(t, s) = 1.
\]

Denote

\[
u(t) = \min\{h(t), g(t)\}, \quad U(t) = \max\{h(t), g(t)\}.
\]

**Theorem 4.4.** Assume that condition (3.1) holds and for some \( t_0 \geq 0 \)

\[
\left\| \left( \int_{h(t)}^{t} [a(s) - b(s)] ds - \frac{1}{e} \right)^{+} \right\|_{[t_0, \infty)} + 2 \left\| \frac{b}{a-b} \right\|_{[t_0, \infty)} \left\| \int_{U(t)}^{t} [a(s) - b(s)] ds \right\|_{[t_0, \infty)} < 1,
\]

Then equation (4.10) is asymptotically stable. If in addition (3.3) holds, (4.10) is uniformly exponentially stable.

**Proof.** Suppose that for \( t \geq t_0, x \) is a solution of the initial value problem

\[
\dot{x}(t) + a(t) \int_{h(t)}^{t} x(s) d_x A(t, s) - b(t) \int_{g(t)}^{t} x(s) d_x B(t, s) = f(t), \quad t \geq t_0, \quad x(t) = 0, \quad t \leq t_0,
\]

where \( f \) is an essentially bounded function on \([t_0, \infty)\). By Lemma 2.13 there exist functions \( h_0(t) \leq t, h(t) \leq h_0(t) \leq t \) and \( g_0(t) \leq t, g(t) \leq g_0(t) \leq t \) such that

\[
\int_{h(t)}^{t} x(s) d_x A(t, s) = x(h_0(t)), \quad \int_{g(t)}^{t} x(s) d_x B(t, s) = x(g_0(t)),
\]

hence \( x \) satisfies the equation

\[
\dot{x}(t) + a(t)x(h_0(t)) - b(t)x(g_0(t)) = f(t).
\]

By Theorem 3.2, equation (4.12) is uniformly exponentially stable. Lemma 2.2 yields that the solution \( x \) of a uniformly exponentially stable equation with an essentially bounded right-hand side \( f \) is an essentially bounded function. Thus, for any essentially bounded function \( f \), the solution of problem (4.11) is essentially bounded. By Lemma 2.4, Remark 2.5 and Lemma 2.8, equation (4.10) is asymptotically stable. Also, by Lemma 2.8 under (3.3) equation (4.10) is uniformly exponentially stable.

The proofs of the following two theorems are similar to the proofs of Theorems 4.2 and 4.3 and thus are omitted.
Theorem 4.5. Assume that \( a(t) \geq a_0 > 0 \) and for some \( t_0 \geq 0 \)
\[
\text{ess sup}_{t \geq t_0} \frac{b(t)}{a(t)} < 1.
\]
Then the equation
\[
\dot{x}(t) + a(t)x(t) - b(t) \int_{g(t)}^{t} x(s)dsB(t, s) = 0
\]
is asymptotically stable, and, if in addition (3.3) holds, uniformly exponentially stable.

Theorem 4.6. Assume that condition (3.19) holds, for some \( t_0 \geq 0 \)
\[
\text{ess sup}_{t \geq t_0} \frac{b(t)}{a(t)} < 1, \quad \text{ess sup}_{t \geq t_0} \int_{a(t)}^{U(t)} a(s)ds < 1,
\]
\[
\left\| \left( \int_{h(t)}^{t} a(s)ds - \frac{1}{e} \right)^+ \right\|_{l_0, \infty} < \frac{1 - \| b \|_{l_0, \infty}}{1 - \frac{b}{a} \| U \|_{l_0, \infty}} \left( 1 - \frac{\int_{U}^{l_0} a(s)ds}{\| U \|_{l_0, \infty}} \right).
\]
Then equation (4.10) is asymptotically stable and, if in addition (3.22) holds, uniformly exponentially stable.

4.3. Integro-differential Equations. The integro-differential equation
\[
\dot{x}(t) + \int_{h(t)}^{t} K(t, s)x(s)ds - \int_{g(t)}^{t} P(t, s)x(s)ds = 0, \tag{4.13}
\]
where \( K(t, s) \) and \( P(t, s) \) are Lebesgue measurable locally integrable on \([0, \infty) \times [0, \infty)\) functions, \( K(t, s) \geq 0, P(t, s) \geq 0, \) is a particular case of (4.10). After denoting
\[
a(t) = \int_{h(t)}^{t} K(t, s)ds, \quad b(t) = \int_{g(t)}^{t} P(t, s)ds,
\]
\[
A(t, s) = \begin{cases} \frac{1}{a(t)} \int_{h(t)}^{s} K(t, \zeta) \, d\zeta, & a(t) > 0, \\ 0, & a(t) = 0, \end{cases}
\]
\[
B(t, s) = \begin{cases} \frac{1}{b(t)} \int_{g(t)}^{s} P(t, \zeta) \, d\zeta, & b(t) > 0, \\ 0, & b(t) = 0, \end{cases}
\]
equation (4.13) has the form of (4.10).

Assume that for the functions \( a, b, h, g \) conditions (a1), (a2) hold. Denote \( u(t) = \min\{h(t), g(t)\} \), \( U(t) = \max\{h(t), g(t)\} \). The following theorems are corollaries of Theorems 4.4, 4.6.
Theorem 4.7. Assume that condition (3.1) holds and for some \( t_0 \geq 0 \)
\[
\left\| \left( \int_{h(t)} [a(s) - b(s)] ds - \frac{1}{e} \right)^+ \right\|_{[t_0, \infty)} + 2 \left\| \frac{b}{a-b} \right\|_{[t_0, \infty)} \left\| \int_{u(t)} [a(s) - b(s)] ds \right\|_{[t_0, \infty)} < 1,
\]
Then equation (4.13) is asymptotically stable and, if in addition (3.3) holds, uniformly exponentially stable.

Theorem 4.8. Assume that \( a(t) \geq a_0 > 0 \), for some \( t_0 \geq 0 \) \( \text{ess sup}_{t \geq t_0} \frac{b(t)}{a(t)} < 1 \). Then the equation
\[
\dot{x}(t) + a(t)x(t) - \int_{g(t)}^{t} P(t, s)x(s)ds = 0
\]
is asymptotically stable and, if in addition (3.3) holds, uniformly exponentially stable.

Theorem 4.9. Assume that condition (3.19) holds, and for some \( t_0 \geq 0 \)
\[
\text{ess sup}_{t \geq t_0} \frac{b(t)}{a(t)} < 1, \quad \text{ess sup}_{t \geq t_0} \int_{u(t)}^{U(t)} a(s)ds < 1,
\]
\[
\left\| \left( \int_{h(t)} a(s)ds - \frac{1}{e} \right)^+ \right\|_{[t_0, \infty)} < \left( 1 - \frac{b}{a} \right)_{[t_0, \infty)} \left( 1 - \frac{1}{a} \right)_{[t_0, \infty)} \left( 1 - \int_{u(t)}^{U(t)} a(s)ds \right)_{[t_0, \infty)}.
\]
Then equation (4.13) is asymptotically stable and, if in addition (3.22) holds, uniformly exponentially stable.

5. Mackey–Glass equations

We recall that an equilibrium of a nonlinear delay differential equation is locally exponentially stable (LES) if the linearized equation is uniformly exponentially stable. In this section we consider Mackey–Glass equation (1.1) under the following conditions which are assumed without further mentioning them:

(c1) \( r \) is a measurable essentially bounded function on \([t_0, \infty)\) function, \( r(t) \geq 0 \), \( r(t) \neq 0 \) a.e., and for some \( T > 0 \) (3.22) holds;
(c2) \( h, g \) are measurable functions, and there exist \( \tau > 0 \) and \( \sigma > 0 \) such that \( 0 \leq t - h(t) \leq \tau \) and \( 0 \leq t - g(t) \leq \sigma \);
(c3) \( n, \beta, \gamma \) are positive constants.

Equation (1.4) is considered satisfying (c1), (c2) with \( r, h, g \) replaced by \( s, p, q \) and \( \beta > 1, n > 0 \).

We will obtain LES conditions for the positive equilibrium. LES conditions for the trivial equilibrium can be obtained similarly.
Theorem 5.1. Assume that
\[ \beta > \gamma, \ n \leq 1 \quad \text{or} \quad n > 1, \ 1 < \frac{\beta}{\gamma} < 1 + \frac{1}{n-1} \] (5.1)
and at least one of the conditions of either Corollary 3.5 or 3.11 holds, where
\[ a = \gamma, \ b = \gamma(1-n+\frac{2n}{\beta}). \] Then the positive equilibrium of equation (1.1) is LES.

Proof. Linearized equation for (1.1) around the positive equilibrium has form (1.3). Condition (5.1) implies that in (1.3) \( a > b \). Corollaries 3.5 and 3.11 imply that equation (1.3) is uniformly exponentially stable. Then the positive equilibrium of equation (1.1) is LES.

Remark 5.2. If \( 1 - n + \frac{2n}{\beta} < 0 \) then (1.3) is an equation with two positive coefficients. Explicit exponential stability conditions for such equations with measurable parameters can be found in [4, 8, 12]. Global stability of (1.1) with \( g(t) \equiv t \) was studied in [10].

Next, we present LES conditions for the positive equilibrium \( x^* = (\beta - 1)^{\frac{1}{\gamma}} \) of equation (1.4). As (1.5) involves more than two terms, we will apply Theorems 4.1 and 4.3 for equations with several delays.

Theorem 5.3. Assume that for equation (1.4) either
\[ \left\| \left( \alpha \int_{q(t)} s(\tau) d\tau - \frac{1}{e} \right)^+ \right\|_{[t_0, \infty)} + 2 \left\| \int_{q(t)}^{\max\{p(t), q(t)\}} s(\tau) d\tau \right\|_{[t_0, \infty)} < 1 \] (5.2)
is satisfied, or both inequalities below hold:
\[ (1 + \alpha) \left\| \int_{q(t)}^{\max\{p(t), q(t)\}} s(\tau) d\tau \right\|_{[t_0, \infty)} < 1, \] (5.3)
\[ \left\| \left(1 + \alpha \right) \int_{q(t)}^{\max\{p(t), q(t)\}} s(\tau) d\tau - \frac{1}{e} \right\|_{[t_0, \infty)} < 1 - \left\| \int_{q(t)}^{\max\{p(t), q(t)\}} (1 + \alpha) s(\tau) d\tau \right\|_{[t_0, \infty)} \] (5.4)
where \( \alpha = \frac{n(\beta - 1)}{\beta} \). Then the positive equilibrium of equation (1.4) is LES.

Proof. The linearized equation for (1.4) about the equilibrium has form (1.5). In order to apply Theorem 4.1 to equation (1.5), denote
\[ a(t) = (1 + \alpha)s(t), \quad b(t) = s(t), \quad h(t) = p(t), \quad g(t) = q(t), \quad r(t) = \min\{p(t), q(t)\}, \quad R(t) = \max\{p(t), q(t)\}. \]
If (5.2) holds then conditions of Theorem 4.1 are satisfied for (1.5), and so the positive equilibrium of equation (1.4) is LES.

By Theorem 4.3 inequalities (5.3) and (5.4) imply uniform exponential stability for the zero solution of (1.5), thus the positive equilibrium of equation (1.4) is LES. \( \square \)
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Example 5.4. Consider a particular case of (1.1)

$$\dot{x}(t) = r \sin^2(\pi t) \left[ \beta \frac{x(t - \sigma)}{1 + x^2(t - \sigma)} - x(t - 1) \right], \quad (5.5)$$

with $\beta > 1$, its linearized about $x^* = \sqrt{\beta - 1}$ version is

$$\dot{x}(t) + r \sin^2(\pi t) \left[ x(t - 1) - \left( \frac{2}{\beta} - 1 \right) x(t - \sigma) \right] = 0.$$ 

Let $\beta = 1.25$. For (5.5), LES is guaranteed if any of the conditions of Corollaries 3.5 or 3.11 are satisfied for $a = 1, b = 0.6, h(t) = t - 1, g(t) = t - \sigma$. Here $a - b = 0.4, r(t) = r \sin^2(\pi t)$,

$$\text{ess sup}_{t \geq t_0} \int_{h(t)}^{t} r(s)ds = \frac{r}{2}.$$ 

For $\sigma = 1.1$,

$$\left| \int_{h(t)}^{g(t)} r(s)ds \right| \leq r \left( 0.05 + \frac{\sin(0.1)}{2} \right) \approx 0.0999917 r < 0.1r,$$

as $|\cos(t)| \leq 1$. Condition 1) in Corollary 3.5 is satisfied for $r \leq \frac{5}{e} \approx 1.8393972$, while 2) holds if $r > \frac{5}{e}$ and (0.32)$r < 1 + \frac{1}{e}$, or $r \leq r_1 \approx 4.27$. Note that Corollary 3.11 here gives a worse estimate, as it involves $r < 2$.

In Figure 1 left, $r = 4 < 4.27$, thus the conditions of Corollary 3.5 are satisfied, and we observe stability as predicted. For $r = 8.5 > 4.27$ in Figure 1 right, the conditions are not satisfied, and we see sustainable oscillations.

For $\sigma = 1.5$ we have

$$\limsup_{t \to \infty} \left| \int_{h(t)}^{g(t)} r(s)ds \right| = \frac{r}{4}.$$
The conditions of Corollary 3.5 hold for $r < 2 + \frac{2}{e} \approx 2.73576$ and, similarly, Corollary 3.11 gives a worse estimate. However, here we illustrate that for $r = 3$ where the conditions of Corollary 3.5 are no longer satisfied, in Figure 2 left, we still observe stability. However, for $r = 3.2$ (Figure 2 right) the positive equilibrium is no longer stable. Let us note that the predicted and obtained in simulations instability bounds are not very far from each other.

Next, let us compare LES conditions for equation (1.4) with known local stability tests.

**Example 5.5.** Consider the equation

$$\dot{x}(t) = 0.1 \sin^2(\pi t) \left[ \frac{2x(t-3)}{1 + x^n(t-6)} - x(t) \right], \quad t \geq 0,$$

(5.6)

Here $\alpha = \frac{n}{2}$. By the first part of Theorem 5.3 equation (5.6) is LES if

$$\left( 0.1 \cdot \frac{n}{2} \cdot 3 - \frac{1}{e} \right)^+ + 2 \cdot \frac{3}{2} \cdot 0.1 < 1,$$

which is satisfied for $n < 7.119$. The second part of Theorem 5.3 requires

$$0.1 \cdot \frac{3}{2} \left( 1 + \frac{n}{2} \right) < 1,$$

which holds for $n < 11.333$, and in addition

$$\left( 0.1 \cdot \frac{3}{2} \left( 1 + \frac{n}{2} \right) - \frac{1}{e} \right)^+ + 0.1 \cdot \frac{3}{2} < 1,$$

which is satisfied for $n < 14.2$. Overall, Theorem 5.3 implies LES of the equilibrium $x^* = 1$ of (1.4) for $n < 11.333$. Figure 3 illustrates solutions for $n = 11$ where we observe stability of $x^*$ and $n = 13$ where sustainable oscillations about $x^*$ are observed.
The assumptions of [9, Theorem 3.5] are satisfied if \( n < \frac{8}{3} \). Further, conditions (a) and (b) of [9, Theorem 3.8] imply \( n < 4 \), while (c) includes \( 0.6(2 + \frac{n}{2}) < 1 \), which cannot be satisfied, while (d) implies \( n < \frac{7}{3} \). We see that the results of Theorem 5.3 for \( n \in [4,11] \), establishes LES of (5.6) while the tests of [9] fail. We are not aware of other stability conditions that can be applied to (5.6).

![Figure 3: A solution of equation (5.6) with (left) \( x(0) = 0.6, \varphi(t) = 0.4, t < 0 \), \( n = 11 \) and (right) \( x(0) = 0.98, \varphi(t) = 0.98, t < 0 \) \( n = 13 \).](image)

### 6. Discussion and Open Problems

Investigation of local exponential stability for Mackey–Glass type models considered in the paper, or for nonlinear models with harvesting, leads to delay differential equations with positive and negative coefficients (with or without a non-delay term) as linearized equations. However, even if such a non-delay term exists, it usually does not dominate over the other terms. For such equations only few explicit stability conditions are known. The present paper fills the gap.

All the equations considered are in the most general setting: the parameters are measurable, and solutions are absolutely continuous functions. We obtain an exponential stability condition for test equation (1.10), which is sharper than other known stability results. However, for some equations known stability results can be better, for example, stability results obtained for autonomous equations by the direct investigation of the roots of quasi-polynomials should outperform general results applied to this class of equations.

In the present paper we obtained local exponential stability results for Mackey–Glass type equation (1.1). Similarly, we can consider an integro-differential Mackey–Glass type equation

\[
\dot{x}(t) = r(t) \left( \beta \int_{h(t)}^{t} K(t,s)x(s)ds - \gamma x(g(t)) \right)
\]

or a Mackey–Glass type equation with a distributed delay

\[
\dot{x}(t) = r(t) \left( \beta \int_{h(t)}^{t} x(s)d_{s}R(t,s) - \gamma x(g(t)) \right).
\]
One of open problems is to obtain explicit instability results for both linear and Mackey–Glass equations considered in the paper. Some other open problems and topics for future research are listed below.

1. It would be interesting to obtain new exponential stability conditions for (1.6), which would allow to deduce the sharp exponential stability result for test equation (1.8).

2. Suppose that for equation (1.6) conditions (a1), (a2) hold, \(a(t) - b(t) \geq a_0 > 0\) and this equation is non-oscillatory. Prove or disprove that (1.6) is uniformly exponentially stable.

3. Consider (1.6) with \(a(t) - b(t)\) being an oscillatory function. The equation

\[
\dot{x}(t) + \sum_{k=1}^{n} a_k(t)x(h_k(t)) = 0,
\]

where all coefficients \(a_k\) or part of them are eventually oscillatory functions, was recently investigated in the papers [7, 11]. Extend the results of [7, 11] to the equations

\[
\dot{x}(t) + \int_{h(t)}^{t} x(s)K(t, s)ds = 0
\]

with an oscillatory kernel \(K(t, s)\) and

\[
\dot{x}(t) + \int_{h(t)}^{t} x(s)R(t, s)ds = 0,
\]

where \(R(t, s)\) can be both increasing and decreasing in the second argument.

4. Consider all equations in the paper without the assumption that delay functions are bounded in condition (a2). Is it possible to deduce asymptotic stability conditions? Note that Lemma 2.4 which is one of the main tools in our investigation assumes boundedness of delays. However, an analogue of Lemma 2.4 exists in the case of infinite but “uniformly exponentially decaying” memory [2].

5. All the stability conditions for equation (4.1) are obtained using the reduction to the equation with two delays which in the proof of Theorem 3.8 would correspond to the change of the variable

\[
s = \int_{t_0}^{t} \sum_{k=1}^{m} a_k(\zeta) d\zeta.
\]

Is it possible to obtain different (and, in some cases, sharper) conditions, with one of the choices for the change of the variable

\[
s = \int_{t_0}^{t} \sum_{k \in I} a_k(\zeta) d\zeta, \quad \text{where } I \subset \{1, 2, \ldots, k\}?
\]
6. In the present paper we derived explicit uniform exponential stability conditions for delay differential equations, integro-differential equations and equations with distributed delays. Obtain explicit uniform exponential stability conditions for mixed type equations as corollaries of Theorems 4.4-4.6, for example, the following ones:

\[ \dot{x}(t) + a(t)x(h(t)) - \int_{g(t)}^{t} K(t, s)x(s)ds = 0, \]

\[ \dot{x}(t) + \int_{h(t)}^{t} K(t, s)x(s)ds - b(t)x(g(t)) = 0, \]

\[ \dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) - \sum_{k=1}^{l} \int_{g_k(t)}^{t} x(s)d_sT_k(t, s) = 0. \]

7. Investigate global exponential stability for nonlinear equations considered in the paper. Is it possible to claim (at least, under certain additional conditions) that local stability implies existence of a global solution and global stability (certainly, for positive initial conditions)?

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