CRYSTAL BASES OF THE FOCK SPACE REPRESENTATIONS AND STRING FUNCTIONS

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Abstract. Let $U_q(g)$ be the quantum affine algebra of type $A^{(1)}_n$, $A^{(2)}_{2n-1}$, $A^{(2)}_{2n}$, $B^{(1)}_n$, $D^{(1)}_n$ and $D^{(2)}_{n+1}$, and let $F(\Lambda)$ be the Fock space representation for a level 1 dominant integral weight $\Lambda$. Using the crystal basis of $F(\Lambda)$ and its characterization in terms of abacus, we construct an explicit bijection between the set of weight vectors in $F(\Lambda)^{\lambda-m\delta}$ ($m \geq 0$) for a maximal weight $\lambda$ and the set of certain ordered sequences of partitions. As a corollary, we obtain the string function of the basic representation $V(\Lambda)$.

1. Introduction

Let $g$ be a classical affine Kac-Moody algebra of type $A^{(1)}_n$, $A^{(2)}_{2n-1}$, $A^{(2)}_{2n}$, $B^{(1)}_n$, $D^{(1)}_n$ and $D^{(2)}_{n+1}$, and let $\Lambda$ be a dominant integral weight of level 1. The weight multiplicities of the basic representation $V(\Lambda)$ can be explained in terms of the string functions. By using, for example, the representations of Virasoro algebras or modular forms, they are given as well-known functions which arise naturally in combinatorics and number theory (cf. [4]).

The purpose of this paper is to understand the combinatorics which lies behind the string functions, that is, to interpret them in a combinatorial way. To this end, we will use the representations of the corresponding quantum affine algebra $U_q(g)$ and their crystal bases. Also, instead of the basic representation, we will use the Fock space representation of $U_q(g)$ with a nice combinatorial realization of its crystal basis.

When $g = A^{(1)}_n$, it can be explained very nicely by the Misra and Miwa’s Fock space representation [13]. They introduced a $U_q(g)$-module $F(\Lambda)$ which is spanned by the set of all partitions. The submodule generated by the empty partition is isomorphic to $V(\Lambda)$ whose crystal is given by the set of $n$-reduced (or $n$-restricted) partitions. We observe that a partition has a weight $\Lambda-m\delta$ ($m \geq 0$), as a crystal element of $F(\Lambda)$, if and only if it has empty $n$-core with weight $m$. Therefore, a weight vector of $F(\Lambda)^{\lambda-m\delta}$ is uniquely determined by its $n$-quotient, an $n$-tuple of partitions whose sum is $m$. From this correspondence and the decomposition of $F(\Lambda)$ into irreducible highest weight modules, we obtain the associated string function of $V(\Lambda)$ immediately.

Generalizing the notion of partitions in case of $A^{(1)}_n$, Kang introduced an abstract crystal $Z(\Lambda)$, the set of proper Young walls which are collections of finite number of blocks added on the ground state wall [5]. Then he showed that the connected component $Y(\Lambda)$ of the ground state wall is isomorphic to the crystal $B(\Lambda)$ of the basic representation. Motivated by the work of Misra and Miwa, in [6, 7], we constructed a $U_q(g)$-module $F(\Lambda)$ having $Z(\Lambda)$
as its crystal, which decomposes as follows:

\[ F(\Lambda) \simeq \bigoplus_{m \geq 0} V(\Lambda - \epsilon m \delta)^{\oplus p(m)}, \]

where \( p(m) \) is the number of partitions of \( m \) and \( \epsilon = 2 \) if \( g = D^{(2)}_{n+1} \) and \( \epsilon = 1 \) otherwise. We call the \( F(\Lambda) \) the Fock space representation of \( U_q(g) \). This can be seen as a combinatorial realization of the Fock space representation by Kashiwara, Miwa, Petersen and Yung for level 1 case [10], and it has a natural analogue of the Lascoux-Leclerc-Thibon’s algorithm for computing global bases element of \( V(\Lambda) \) [11]. In this paper, we will not define the module structure on \( F(\Lambda) \) since we need only the crystal graph of it. But we will give another proof for the decomposition of the crystal \( Z(\Lambda) \).

Therefore, we reduce our problem to characterizing the proper Young walls in \( Z(\Lambda)_{\lambda - m \delta} \) \((m \geq 0)\) for a maximal weight \( \lambda \). The main result in this paper is the construction of a bijection between \( Z(\Lambda)_{\lambda - m \delta} \) and a set of certain ordered sequences of partitions whose generating function allows us to recover the associated string function of \( V(\Lambda) \). Also, the bijection is given more explicitly when we take a particular maximal weight (for example, \( \Lambda = \lambda \)), and it is obtained by modifying the method of abacus which were used when \( g = A_1^{(1)} \). We remark that it might be possible to give a similar characterization of \( Y(\Lambda)_{\lambda - m \delta} \).

In fact, for \( A_1^{(1)}, A_2^{(2)} \) and \( D^{(2)}_{n+1} \), there exists a bijection between \( Y(\Lambda)_{\lambda - m \delta} \) and a set of ordered sequences of partitions (see [11] for \( A_1^{(1)} \)). Then, however, the bijections should include Weyl group actions even in the case of \( \lambda = \Lambda \), and hence become more complicated than those for \( Z(\Lambda)_{\lambda - m \delta} \). This is one of the reason we prefer the Fock space representation rather than the basic representation.

This paper is organized as follows: in Section 2, we recall the notion of abstract crystals and proper Young walls. We refer the reader to [8] for a general exposition on crystal bases and abstract crystals, and [9] for a detailed description and more examples of proper Young walls. In Section 3, we review the results for \( A_1^{(1)} \) which we mentioned before. Then in the following sections, we define the abacus for each type of \( g \), and then characterize the proper Young walls of weight \( \Lambda - m \delta \) from their bead configurations in the abacus to obtain a bijection (Section 4 for \( A_1^{(1)}, A_2^{(2)}, D_{n+1}^{(2)} \), Section 5 for \( A_2^{(2)}, D_{n+1}^{(1)} \) and Section 6 for \( B_n^{(1)} \)).

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2. AFFINE CRYSTALS AND YOUNG WALLS

Let \( I = \{0, 1, \ldots , n\} \) be an index set and let \( (A, P^\vee, P, \Pi^\vee, \Pi) \) be an affine Cartan datum where

1. \( A = (a_{ij})_{i,j \in I} \) is a generalized Cartan matrix of affine type,
2. \( P^\vee = \mathbb{Z}h_0 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d \) is the dual weight lattice,
3. \( \mathfrak{h} = \mathbb{Q} \otimes \mathbb{Z} P^\vee \) is the Cartan subalgebra
4. \( P = \{ \lambda \in \mathfrak{h}^* | \lambda(P^\vee) \subseteq \mathbb{Z} \} \) is the weight lattice,
5. \( \Pi^\vee = \{ h_i | i \in I \} \) is the set of simple coroots,
6. \( \Pi = \{ \alpha_i | i \in I \} \) is the set of simple roots.

Let \( g \) be the corresponding affine Kac-Moody algebra. We denote by \( Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i \) the root lattice, and set \( Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \), \( Q_- = -Q_+ \).
Definition 2.1. An (affine) crystal associated with \((A,P',\Pi',\Pi')\) is a set \(B\) together with the maps \(wt: B \to P\), \(\varphi_i: B \to \mathbb{Z} \cup \{-\infty\}\), \(\bar{\varphi}_i: B \to \mathbb{Z} \cup \{-\infty\}\), and \(\bar{f}_i: B \to B \cup \{0\}\) satisfying the following conditions:

(i) for all \(i \in I\), \(b \in B\), we have

\[
\varphi_i(b) = \varepsilon_i(b) + wt(b)(h_i),
\]

\[
wt(\bar{\varepsilon}_i b) = wt(b) + \alpha_i,
\]

\[
wt(\bar{f}_i b) = wt(b) - \alpha_i,
\]

(ii) if \(\bar{e}_i b \in B\), then

\[
\varepsilon_i(\bar{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\bar{e}_i b) = \varphi_i(b) + 1,
\]

(iii) if \(\bar{f}_i b \in B\), then

\[
\varepsilon_i(\bar{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\bar{f}_i b) = \varphi_i(b) - 1,
\]

(iv) \(\bar{f}_i b = b'\) if and only if \(b = \bar{e}_i b'\) for all \(i \in I\), \(b,b' \in B\),

(v) if \(\varepsilon_i(b) = -\infty\), then \(\bar{e}_i b = \bar{f}_i b = 0\).

From now on, we assume that \(g\) is of type \(A_n^{(1)}, A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)}, A_{2n}^{(2)}\) or \(D_n^{(2)}\). Suppose that we are given the following three kinds of blocks:

| shape | width | thickness | height |
|-------|-------|-----------|--------|
| ![block1] | 1     | 1         | 1      |
| ![block2] | 1     | 1         | 1/2    |
| ![block3] | 1     | 1/2       | 1      |

and we give a coloring on them by the index set \(I\).

Given \(g\) and a dominant integral weight \(\Lambda\) of level 1, we fix a frame \(Y_\Lambda\) called the ground state wall of weight \(\Lambda\), and we stack the above blocks on \(Y_\Lambda\) following the pattern depending on \(g\) and \(\Lambda\), which will be given in later sections. A collection of finite number of colored blocks added on \(Y_\Lambda\) is called a Young wall on \(Y_\Lambda\) if the heights of its columns are weakly decreasing from right to left. We often write \(Y = (y_k)_{k=1}^\infty\) as an infinite sequence of its columns where the columns are enumerated from right to left. We define \(|Y| = (|y_k|)_{k=1}^\infty\) to be the sequence, where \(|y_k|\) is the number of blocks in the \(k\)th column of \(Y\) (except the one in \(Y_\Lambda\)), and call it the associated partition of \(Y\).

A column of a Young wall is called a full column if the block at the top is a unit cube. For type \(A_n^{(1)}\), every Young wall is defined to be proper. For type \(A_{2n-1}^{(2)}, A_{2n}^{(2)}, B_n^{(1)}, D_n^{(1)}\) and \(D_{n+1}^{(2)}\), a Young wall is said to be proper if none of the full columns have the same heights. We denote by \(\mathcal{Z}(\Lambda)\) the set of all proper Young walls on \(Y_\Lambda\).

Let \(\delta = d_0\alpha_0 + \cdots + d_n\alpha_n\) be the null root of \(g\), and set \(a_i = d_i\) if \(g \neq D_{n+1}^{(2)}\), \(a_i = 2d_i\) if \(g = D_{n+1}^{(2)}\). The part of a column with \(a_i\)-many \(i\)-blocks (or \(i\)-colored block) for each \(i \in I\) in some cyclic order is called a \(\delta\)-column. A \(\delta\)-column in a proper Young wall is called removable if it can be removed to yield another proper Young wall. A proper Young wall \(Y\) is said to be reduced if none of its columns contain a removable \(\delta\)-column. We denote by \(\mathcal{Y}(\Lambda)\) the set of all reduced proper Young walls on \(Y_\Lambda\).

Example 2.2. If \(g = A_n^{(2)}\) and \(\Lambda = \Lambda_0\), then the Young wall \(Y\) given below is a proper Young wall in \(\mathcal{Z}(\Lambda_0)\). It is reduced since it contains no removable \(\delta\)-column.
**Definition 2.3.** Let $Y$ be a proper Young wall on $Y_\Lambda$.

1. An $i$-block in $Y$ is called a **removable $i$-block** if $Y$ remains a proper Young wall after removing the block.
2. A place in $Y$ is called an **admissible $i$-slot** if one may add an $i$-block to obtain another proper Young wall.
3. A column in $Y$ is said to be $i$-removable (resp. $i$-admissible) if there is a removable $i$-block (resp. an admissible $i$-slot) in that column.

We now define the operators $\tilde{e}_i$, $\tilde{f}_i$ on $Z(\Lambda)$ as follows. Fix $i \in I$ and let $Y = (y_k)_{k=1}^\infty$ be a proper Young wall on $Y_\Lambda$.

1. To each column $y_k$ of $Y$, we assign
   \[
   \begin{cases}
   - & \text{if } y_k \text{ is twice } i\text{-removable}, \\
   - & \text{if } y_k \text{ is once } i\text{-removable but not } i\text{-admissible}, \\
   -+ & \text{if } y_k \text{ is once } i\text{-removable and once } i\text{-admissible}, \\
   + & \text{if } y_k \text{ is once } i\text{-admissible but not } i\text{-removable}, \\
   ++ & \text{if } y_k \text{ is twice } i\text{-admissible}, \\
   \cdot & \text{otherwise}
   \end{cases}
   \]

   (2) From this sequence of $+$’s and $-$’s, we cancel out every $(+, -)$-pair to obtain a finite sequence of $-$’s followed by $+$’s, reading from left to right. This finite sequence ($- \cdots - , + \cdots +$) is called the **$i$-signature** of $Y$.

   (3) We define $\tilde{e}_i Y$ to be the proper Young wall obtained from $Y$ by removing the $i$-block corresponding to the right-most $-$ in the $i$-signature of $Y$. We define $\tilde{e}_i Y = 0$ if there is no $-$ in the $i$-signature of $Y$.

   (4) We define $\tilde{f}_i Y$ to be the proper Young wall obtained from $Y$ by adding an $i$-block to the column corresponding to the left-most $+$ in the $i$-signature of $Y$. We define $\tilde{f}_i Y = 0$ if there is no $+$ in the $i$-signature of $Y$.

   We also define
   \[
   \text{wt}(Y) = \Lambda - \sum_{i \in I} k_i \alpha_i \in P,
   \]
   \[
   \varepsilon_i(Y) = \text{the number of } -\text{'s in the } i\text{-signature of } Y,
   \]
   \[
   \varphi_i(Y) = \text{the number of } +\text{'s in the } i\text{-signature of } Y,
   \]
   where $k_i$ denotes the number of $i$-blocks in $Y$ that have been added to $Y_\Lambda$. For a set $S$ consisting of some blocks in $Y$, we define $\text{cont}(S) = \sum_{i \in I} k_i \alpha_i \in Q_+$ where $k_i$ denotes the number of $i$-blocks in $S$, and call it the **content** of $S$. For example, $\text{cont}(Y) = \Lambda - \text{wt}(Y)$.

**Theorem 2.4.** ([5])

1. The set $Z(\Lambda)$ together with $\tilde{e}_i$, $\tilde{f}_i$, $\text{wt}$, $\varepsilon_i$ and $\varphi_i$ ($i \in I$), is an affine crystal.
(2) The set \( \mathcal{Y}(\Lambda) \) is an affine subcrystal of \( \mathcal{Z}(\Lambda) \) and isomorphic to \( B(\Lambda) \), where \( B(\Lambda) \) is the crystal of the basic representation \( V(\Lambda) \).

□

Set \( \epsilon = 2 \) if \( g \) is of type \( D^{(2)}_{n+1} \), and \( \epsilon = 1 \) if otherwise. Then we have

Corollary 2.5. There exists an isomorphism of affine crystals

\[
\mathcal{Z}(\Lambda) \simeq \bigoplus_{m \geq 0} B(\Lambda - \epsilon m\delta)^{\oplus p(m)},
\]

where \( B(\Lambda - m\delta) \) is the crystal of the highest weight module \( V(\Lambda - m\delta) \) and \( p(m) \) is the number of partitions of \( m \).

Proof. For convenience, we assume that \( \epsilon = 1 \), or \( g \neq D^{(2)}_{n+1} \). Let \( \lambda \) be a partition of \( m \), that is, a non-increasing sequence of positive integers \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \) whose sum is \( m \). Let \( Y_{\Lambda,\lambda} \) be the proper Young wall which is obtained by adding \( \lambda_k \) many \( \delta \)-columns on the \( k \)th column of \( Y_\Lambda \). Then we can check that \( \text{wt}(Y_{\Lambda,\lambda}) = \Lambda - \epsilon m\delta \) and \( \tilde{e}_i Y_{\Lambda,\lambda} = 0 \) for all \( i \in I \) (see \([7]\)).

For each \( Y \in \mathcal{Y}(\Lambda) \), define \( S_\lambda(Y) \) to be the proper Young wall which is obtained by adding the \( k \)th column of \( Y \) (except the block in \( Y_{\Lambda,\lambda} \)) on the \( k \)th column of \( Y_{\Lambda,\lambda} \). Set

\[
\mathcal{Y}(\Lambda,\lambda) = \{ S_\lambda(Y) \mid Y \in \mathcal{Y}(\Lambda) \}.
\]

Then it is not difficult to see that

\[
\mathcal{Z}(\Lambda) = \bigsqcup_{\lambda} \mathcal{Y}(\Lambda,\lambda),
\]

where the union is taken over all partitions \( \lambda \). Finally, for \( i \in I \), the map \( S_\lambda : \mathcal{Y}(\Lambda) \to \mathcal{Z}(\Lambda) \) commutes with \( \tilde{e}_i \) and \( \tilde{f}_i \), which can be checked directly from the definitions of \( \tilde{e}_i \) and \( \tilde{f}_i \). Therefore, we conclude that \( \mathcal{Y}(\Lambda,\lambda) \) is the connected component of \( Y_{\Lambda,\lambda} \), and hence it is isomorphic to \( B(\Lambda - m\delta) \). This completes the proof. □

Remark 2.6. In \([7]\), we constructed a \( U_q(g) \)-module

\[
\mathcal{F}(\Lambda) = \bigoplus_{Y \in \mathcal{Z}(\Lambda)} \mathbb{Q}(q) Y
\]

called the Fock space representation, and it was shown that \( \mathcal{Z}(\Lambda) \) is the crystal of \( \mathcal{F}(\Lambda) \).

From this, the decomposition given in Corollary 2.5 can be obtained directly by verifying that

\[
\{ Y \in \mathcal{Z}(\Lambda) \mid \tilde{e}_i Y = 0 \text{ for all } i \in I \} = \{ Y_{\Lambda,\lambda} \mid \lambda : \text{partition} \}.
\]

Let \( \lambda \in P \) be a maximal weight in \( \mathcal{Z}(\Lambda) \), i.e. \( |\mathcal{Z}(\Lambda)_{\lambda}| \neq 0 \) but \( |\mathcal{Z}(\Lambda)_{\lambda+\delta}| = 0 \). We define

\[
\Sigma^\Lambda_\lambda(q) = \sum_{m \geq 0} |\mathcal{Z}(\Lambda)_{\lambda-m\delta}| q^m,
\]

and call it the string function of \( \mathcal{F}(\Lambda) \) with respect to \( \Lambda \) and \( \lambda \). By Corollary 2.5, we have

\[
|\mathcal{Z}(\Lambda)_{\lambda-m\delta}| = \sum_{s \geq 0} p(s) |B(\Lambda - \epsilon s\delta)_{\lambda-m\delta}| = \sum_{\epsilon s + t = m} p(s) |B(\Lambda)_{\lambda-t\delta}|.
\]

This implies that

\[
\Sigma^\Lambda_\lambda(q) = \frac{1}{(q^\epsilon)_{\infty}} \sigma^\Lambda_\lambda(q),
\]
where \( (q)_\infty = \prod_{m \geq 1} (1 - q^m) \) and \( \sigma^1(q) = \sum_{m \geq 0} \mid B(\lambda)_{\lambda - m\delta} \mid q^m \) is the string function of \( V(\Lambda) \) with respect to \( \Lambda \) and \( \lambda \) (cf.\[4\]). By using the Weyl group action on the crystal \( Z(\Lambda) \) (cf.\[9\]), we have a bijection

\[
w : Z(\Lambda)_{\lambda - m\delta} \rightarrow Z(\Lambda)_{w\lambda - m\delta},
\]

where \( w \) is a Weyl group element. Thus when we consider the weight multiplicities of the basic representation \( V(\Lambda) \), it is enough to compute

\[
\begin{cases}
\sum^{A^1(q)}_{\Lambda} & \text{when } \mathfrak{g} = A^{(1)}_n, A^{(2)}_{2n-1}, A^{(2)}_{2n}, D^{(1)}_n, D^{(2)}_{n+1}, \\
\sum^{A_{nc}}_{\Lambda}(q), \sum^{A_{n1}}_{\Lambda}(q) & \text{when } \mathfrak{g} = B^{(1)}_n,
\end{cases}
\]

(see \[4\]).

3. \( A^{(1)}_{n-1} \)-case (\( n \geq 2 \))

This section is based on the arguments in \[3\], and we will rewrite them following our notations. First, let us recall some basic terminologies (cf.\[12\]). A partition is a non-increasing sequence of non-negative integers, \( \lambda = (\lambda_k)_{k \geq 1} \) such that all but a finite number of its terms are zero. Each \( \lambda_k \) is called a part of \( \lambda \) and the number of the non-zero parts is called the length of \( \lambda \), denoted by \( \ell(\lambda) \). We also write \( \lambda = (1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots) \), where \( m_i \) is the number of the parts of \( \lambda \) equal to \( i \). We say that \( \lambda \) is a partition of \( m \) (\( m \geq 0 \)) if \( \sum_{k \geq 1} \lambda_k = m \) and write \( |\lambda| = m \). When all non-zero parts of \( \lambda \) are distinct, \( \lambda \) is said to be strict. For each \( m \geq 0 \), let \( \mathcal{P}(m) \) be the set of partitions of \( m \) and set \( \mathcal{P} = \bigcup_{m \geq 0} \mathcal{P}(m) \). We denote by \( p(m) \) the number of partitions of \( m \) with \( p(0) = 1 \) by convention.

A partition \( \lambda = (\lambda_k)_{k \geq 1} \) is identified with a Young diagram which is a collection of boxes stacked from the bottom with \( \lambda_k \) boxes in each \( k \)th column. We will enumerate the columns of a Young diagram from right to left so that the number of boxes are weakly decreasing from right to left. Note that the boxes are stacked from the south-east corner in our definition (cf.\[12\]).

Let \( \lambda \) be a Young diagram and let \( b_1, \ldots, b_r \) be the boxes in the main diagonal of \( \lambda \), which are enumerated from the south-east corner. Let \( \lambda_k' \) (\( 1 \leq k \leq r \)) be the number of boxes lying in the same row of \( b_k \) and to the left of \( b_k \), and let \( \lambda_k'' \) (\( 1 \leq k \leq r \)) be the number of boxes lying in the same column of \( b_k \) and above \( b_k \). Then we have a pair of strict partitions \( \lambda' = (\lambda_1' > \cdots > \lambda_r' \geq 0) \) and \( \lambda'' = (\lambda_1'' > \cdots > \lambda_r'' \geq 0) \) which are uniquely determined by \( \lambda \). We may write \( \lambda = (\lambda'|\lambda'') \), which is called the Frobenius notation of \( \lambda \). For example, \( (1^3, 3, 4^2) = (1^3)(2, 3) \).

Let \( \mathfrak{g} \) be of type \( A^{(1)}_{n-1} \) (\( n \geq 2 \)). Fix a dominant integral weight \( \Lambda \) of level 1. The pattern for \( Z(\Lambda) \) is given as follows:

\[
\begin{array}{ccccccc}
\Lambda = \Lambda_j & | & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
\end{array}
\]

\[
\begin{array}{ccccccc}
0 & 1 & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
\end{array}
\]

\[
\begin{array}{ccccccc}
n & 0 & 1 & 2 & 3 & \cdots & i+1 \\
n-1 & n & 0 & 1 & 2 & \cdots & i+1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & 2 & 3 & \cdots & i+1 \\
\end{array}
\]
For a given $Y = (y_k)_{k \geq 1} \in \mathcal{Z}(\Lambda)$, we may identify $y_k$ with the number of blocks in the $k$th column of $Y$. Then the set $\mathcal{Z}(\Lambda)$ can be identified with $\mathcal{P}$, where $Y_{\Lambda}$ corresponds to the empty partition.

We define the abacus of type $A^{(1)}_{n-1}$ to be the arrangement of positive integers in the following way:

\[
\begin{array}{cccccc}
1 & 2 & \cdots & n-1 & n \\
n+1 & n+2 & \cdots & 2n-1 & 2n \\
\vdots & \vdots & \ddots & \vdots & \vdots
\end{array}
\]

Let $R_k$ ($1 \leq k \leq n$) be the set of integers $s \equiv k \pmod{n}$ and call it the $k$th runner. Each positive integer is called a position. Then we can put a bead, denoted by $\bullet$, at each position and move a bead along the runner which it belongs to. We suppose that there is at most one bead at each position. So, we can move a bead at $s$ one position up (resp. down) along the runner only when there is no bead at $s-n$ (resp. $s+n$).

For a proper Young wall $Y = (y_k)_{k \geq 1} \in \mathcal{Z}(\Lambda)$, choose an $r$ such that $y_k = 0$ for all $k \geq r$. Consider the set of $r$ distinct positive integers $\{z_k | z_k = y_k + r - k + 1, 1 \leq k \leq r\}$. The $(r)$-bead configuration of $Y$ is the set of $r$ beads assigned at the position $z_k$ ($1 \leq k \leq r$). Conversely, a given set of $r$ beads in the abacus represents the $r$-bead configuration of a unique proper Young wall in $\mathcal{Z}(\Lambda)$.

Consider a bead configuration of a proper Young wall $Y \in \mathcal{Z}(\Lambda)$. Suppose that a bead $b$ is movable one position up. Let $Z$ be the proper Young wall obtained by moving $b$ one position up. Then we observe that $Y/Z$ forms a border strip of length $n$; that is, $Y/Z$ is a skew Young diagram consisting of $n$ boxes which is connected and contains no $2 \times 2$-collection of boxes. Furthermore, if we consider a content of $Y/Z$ following the pattern for $\mathcal{Z}(\Lambda)$, then it is $\delta$. Therefore, we have $\text{wt}(Z) = \text{wt}(Y) + \delta$. Note that this process of moving a bead one position up (equivalently, removing a border strip of length $n$ from a Young diagram) is reversible.

Let $\tilde{Y}$ be a proper Young wall obtained by applying the above processes until there is no bead movable up along the runner. Note that $\tilde{Y}$ is uniquely determined since $\tilde{Y}$ does not depend on the order in which we move up the beads. We call $\tilde{Y}$ the $n$-core of $Y$ and denote it by $\text{core}_n(Y)$. The total number of movements of beads to obtain $\text{core}_n(Y)$ from $Y$ is called the $n$-weight of $Y$ and denoted by $\text{wt}_n(Y)$.

**Example 3.1.** Suppose $Y = (1, 3, 5, 6)$ and $n = 4$. Then the bead configuration of $Y$ with 5 beads is as follows:

\[
Y = \begin{array}{cccc}
\text{core}_4(Y) = & \text{wt}_4(Y) = 3.
\end{array}
\]

Note that two proper Young walls in $\mathcal{Z}(\Lambda)$ have the same $n$-core if and only if they have the same weight (or content) (see [3]). For example, suppose that the weight of $Y \in \mathcal{Z}(\Lambda)$
is $\Lambda - m\delta$ for some $m \geq 0$. On the other hand, consider $Z = (1^m) \in Z(\Lambda)$ as a partition. Since \( \text{wt}(Z) = \Lambda - m\delta \) and \( \text{core}_n(Z) = Y_\Lambda \) (or (0)), we have \( \text{core}_n(Y) = Y_\Lambda \). Conversely, any $Y \in Z(\Lambda)$ with \( \text{core}_n(Y) = Y_\Lambda \) has weight $\Lambda - m\delta$ for some $m \geq 0$.

Therefore, for $m \geq 0$, we have

$$Z(\Lambda)_{\Lambda - m\delta} = \{ Y \in Z(\Lambda) \mid \text{core}_n(Y) = Y_\Lambda, \text{wt}_n(Y) = m \}. \tag{3.1}$$

Fix $m \geq 0$. Let $Y$ be a proper Young wall in $Z(\Lambda)_{\Lambda - m\delta}$. Set $M = mn$. Consider the $M$-bead configuration of $Y$. Since \( \text{core}_n Y = Y_\Lambda \), it is easy to see that there are $m$ beads in each runner $R_k$ ($1 \leq k \leq n$). For each $1 \leq k \leq n$, let $\{ b_1^{(k)}, \ldots, b_m^{(k)} \}$ be the set of $m$ beads in $R_k$ enumerated from the bottom. Suppose that $b_i^{(k)}$ is located at $N_i^{(k)}$. Put $p_i^{(k)} = \frac{N_i^{(k)} - k}{n}$. Then $p_i^{(k)} \geq m - i$ and $\{ p_i^{(k)} - m + i \mid 1 \leq i \leq m \}$ forms a unique partition $\lambda^{(k)}$ whose sum is the number of all possible movements of beads in $R_k$ to obtain the core of $Y$. Hence, we obtain an $n$-tuple of partitions $\pi(Y) = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ with $\sum_{i=1}^n |\lambda^{(i)}| = m$, which is called the $n$-quotient of $Y$. Conversely, for a given $n$-tuple of partitions $(\lambda^{(1)}, \ldots, \lambda^{(n)})$ with $\sum_{i=1}^n |\lambda^{(i)}| = m$, we can place $m$ beads in each runner $R_k$ whose corresponding partition is $\lambda^{(k)}$ ($1 \leq k \leq n$). Then the resulting unique proper Young wall $Y$ is in $Z(\Lambda)_{\Lambda - m\delta}$ with $\pi(Y) = (\lambda^{(1)}, \ldots, \lambda^{(n)})$. Therefore, we obtain

**Theorem 3.2.** \( \boxed{3} \) For $m \geq 0$, the map

$$\pi : Z(\Lambda)_{\Lambda - m\delta} \rightarrow \mathcal{P}^{(n)}(m) \tag{3.2}$$

is a bijection where

$$\mathcal{P}^{(n)}(m) = \{ (\lambda^{(1)}, \ldots, \lambda^{(n)}) \mid \lambda^{(i)} \in \mathcal{P}, \sum_{i=1}^n |\lambda^{(i)}| = m \}. \tag{3.3}$$

\( \square \)

**Remark 3.3.** In \( \boxed{3} \), the bijection in the above theorem is given in a more general form, that is, a bijection between $\mathcal{P}^{(n)}(m)$ and the set of all partitions with a given $n$-core and an $n$-weight $m$. This implies that a partition is uniquely determined by its $n$-core and $n$-quotient.

By Theorem \( \boxed{3} \) and \( \boxed{8} \), we obtain $\sigma_\Lambda^\lambda(q) = 1/(q)^{n-1}$, or

$$\dim V(\Lambda)_{\Lambda - m\delta} = \sum_{\sum_{i=1}^n m_i = m} p(m_1) \cdots p(m_{n-1}) \tag{3.3}$$

for $m \geq 0$ (cf. \( \boxed{4} \)).

A partition is called $n$-reduced (or $n$-restricted) if the difference of any two adjacent columns is less than $n$. Note that $Y(\Lambda)$ is the set of all $n$-reduced partitions. In particular, we denote by $\mathcal{P}_0(m)$ the set of all $2$-reduced partitions whose $2$-core is empty and $2$-weight is $m$ (or the set of all strict partitions with empty $2$-core and $2$-weight $m$), and set $\mathcal{P}_0 = \bigcup_{m \geq 0} \mathcal{P}_0(m)$. By \( \boxed{3, 3} \), we have

$$|\mathcal{P}_0(m)| = p(m). \tag{3.4}$$

**Remark 3.4.** The set of $n$-cores is in one-to-one correspondence with $W/W_\Lambda$ where $W$ is the Weyl group of type $A_{n-1}^{(1)}$ and $W_\Lambda$ is the stabilizer subgroup of $\Lambda$. In fact, the bijection is given by $\text{wt}(Y) = w\Lambda$. Moreover, using this fact and the Weyl group action on the crystal $Z(\Lambda)$, Lascoux, Leclerc and Thibon described a bijection between $Y(\Lambda)_{\Lambda - m\delta}$ and $\mathcal{P}^{(n-1)}(m)$ \( \boxed{11} \).
4. $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$-case

Suppose that $g$ is of type $A_{2n}^{(2)}$ ($n \geq 1$) or $D_{n+1}^{(2)}$ ($n \geq 2$) and $\Lambda$ is a dominant integral weight of level 1. The patterns for $Z(\Lambda)$ are given as follows:

$A_{2n}^{(2)}$,

$\Lambda = \Lambda_0$:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
\vdots & & & \\
n & n & n & n \\
\vdots & & & \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
\end{array}
\]

$D_{n+1}^{(2)}$,

$\Lambda = \Lambda_0$:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
\vdots & & & \\
n & n & n & n \\
\vdots & & & \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
\end{array}
\]

$\Lambda = \Lambda_1$:

\[
\begin{array}{cccc}
n-1 & n-1 & n-1 & n-1 \\
n & n & n & n \\
\vdots & & & \\
n & n & n & n \\
\vdots & & & \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
\vdots & & & \\
n-1 & n-1 & n-1 & n-1 \\
\end{array}
\]

Set

\[
\ell = \begin{cases} 
2n + 1 & \text{if } g = A_{2n}^{(2)}, \\
n + 1 & \text{if } g = D_{n+1}^{(2)}
\end{cases} \quad L = \begin{cases} 
\ell & \text{if } g = A_{2n}^{(2)}, \\
2\ell & \text{if } g = D_{n+1}^{(2)}
\end{cases}
\]

\[
\epsilon = L/\ell
\]

(4.1)

Note that $L$ is the number of blocks in a $\delta$-column. We define the *abacus of type* $A_{2n}^{(2)}$ (*resp.* $D_{n+1}^{(2)}$) to be the arrangement of positive integers in the following way:

$A_{2n}^{(2)}$
more than one bead at each position in $R_A$ type abacus. The rules of placing and moving beads in $R$ of all integers $|Y|$ are distinct except when $\ell < L$. We will describe an algorithm in the abacus which is an analogue of removing a border strip in the abacus of type $A^{(1)}_{n-1}$, and then characterize $Z(\Lambda)_{\Lambda - m\delta}$ ($m \geq 0$) in terms of bead configurations.

Let $R_k$ ($1 \leq k < L, k \neq \ell$) be the set of all integers $s \equiv k \pmod{L}$ and let $R_\ell$ be the set of all integers $s \equiv 0 \pmod{\ell}$. We call $R_k$ the $k$th runner. There are $2n + 1$ runners in each abacus. The rules of placing and moving beads in $R_k$ ($k \neq \ell$) are the same as in the case of type $A^{(1)}_{n-1}$, and we say that $R_k$ is of type I. On the other hand, we suppose that there can be more than one bead at each position in $R_\ell$. We denote $k$ beads at $s$ by $\bigcirc_k$. Moreover, if $b$ is a bead at $m(\neq \ell)$ in $R_\ell$, then we can always move up (resp. down) $b$ along the runner by increasing the number of beads at $m - \ell$ (resp. $m + \ell$) by one, and decreasing the number of beads at $m$ by one. We say that $R_\ell$ is of type II.

For $Y \in Z(\Lambda)$, let $|Y| = (|y_k|)_{k \geq 1}$ be its associated partition. Let $\{ |y_1|, \ldots, |y_r| \}$ be the set of all non-zero parts in $|Y|$. Then by definition of $Z(\Lambda)$, the numbers $|y_k|$'s ($1 \leq k \leq r$) are distinct except when $|y_k| \equiv 0 \pmod{\ell}$. We define the bead configuration of $Y$ to be the set of $r$ beads $b_1, \ldots, b_r$ in the above abacus where $b_k$ is placed at $|y_k|$. Note that $Y$ is uniquely determined by its associated partition, and hence by its bead configuration.

**Example 4.1.** Suppose that $g = A^{(2)}_4$. Then

$$
Y = \begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5
\end{bmatrix}
$$

We will describe an algorithm in the abacus which is an analogue of removing a border strip in the abacus of type $A^{(1)}_{n-1}$, and then characterize $Z(\Lambda)_{\Lambda - m\delta}$ ($m \geq 0$) in terms of bead configurations.

**Lemma 4.2.** Let $Y$ be a proper Young wall in $Z(\Lambda)$ and let $Y'$ be the proper Young wall obtained by applying one of the following processes to the bead configuration of $Y$:

(B1) if $b$ is a bead at $s$ in a runner of type I and there is no bead at $s - \ell$, then move $b$ one position up,

(B2) if $b$ is a bead at $s$ in $R_\ell$ and $s \neq \ell$, then move $b$ one position up,

(B3) if $b$ and $b'$ are beads at $s$ and $L - s$ ($1 \leq s \leq n$), respectively, then remove $b$ and $b'$ simultaneously,

(B4) if $b$ is a bead at $\ell$, then remove $b$.
Then we have
\[
\text{wt}(Y'') = \begin{cases} 
\text{wt}(Y) + \epsilon \delta & \text{if } Y'' \text{ is obtained by } (B_i) \ (i = 1, 2, 3), \\
\text{wt}(Y) + \delta & \text{if } Y'' \text{ is obtained by } (B_4).
\end{cases}
\]

**Proof.** If we apply \((B_i) \ (i = 1, 2, 3)\) to \(Y\), then \(Y''\) is obtained by removing some \(L\) blocks from \(Y\), say \(\{b_1, \cdots, b_L\}\). Let \(i_k \ (1 \leq k \leq L)\) be the color of \(b_k\). It follows directly from the pattern for \(Z(\Lambda)\) that \(\sum_{k=1}^L \alpha_{i_k} = \epsilon \delta\) and therefore \(\text{wt}(Y'') = \text{wt}(Y) + \epsilon \delta\). The proof is similar when \(Y''\) is obtained by \((B_4)\).

**Example 4.3.** Let \(Y\) be the proper Young wall in Example 4.1.

1. Apply \((B_1)\) to \(\heartsuit\). This means that we remove a \(\delta\)-column in the 2nd column of \(Y\) and then shift the blocks which are placed in the left of the \(\delta\)-column, to the right as far as possible.

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Therefore, we have
\[
Y'' = \quad \begin{array}{cccccc}
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit \\
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\quad \leftrightarrow \quad \begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

2. Similarly, if we apply \((B_3)\) to \(\heartsuit\) and \(\heartsuit\), then we have

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\quad \leftrightarrow \quad \begin{array}{cccccc}
1 & 2 & 3 & 4 & \heartsuit & \heartsuit \\
6 & 7 & 8 & 9 & \heartsuit & \heartsuit \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

Let \(\tilde{Y}\) be the proper Young wall which is obtained from \(Y\) by applying \((B_i) \ (i = 1, 2, 3, 4)\) until there is no bead movable up or removable. Note that \(\tilde{Y}\) does not depend on the order of steps, hence is uniquely determined.

**Lemma 4.4.** Let \(Y\) be a proper Young wall in \(Z(\Lambda)\). Then \(Y \in Z(\Lambda)_{\Lambda - m\delta}\) for some \(m \geq 0\) if and only if \(\tilde{Y} = Y\).
configuration is as follows: Note that $\lambda = n$ at $\pi(4.6)$. Suppose that $\mu = r_k$ for $1 \leq k \leq n$. We assume that $g$ is of type $A_{2n}^{(2)}$ (the proof for $D_{n+1}^{(2)}$ is similar).

By considering the content of the blocks corresponding to each bead, it is straightforward to check that

\begin{equation}
\text{cont}(Y) = \sum_{i=0}^{n} c_i \alpha_i + M \delta,
\end{equation}

for some $M \geq 0$, where

\begin{equation}
c_i = \begin{cases} 
\sum_{j=1}^{2n} r_j & \text{if } i = 0, \\
\sum_{j=1}^{2n-i} r_j + \sum_{j=2n-i+1}^{2n} 2r_j & \text{if } 1 \leq i \leq n-1, \\
\sum_{j=n+1}^{2n} r_j & \text{if } i = n.
\end{cases}
\end{equation}

Since $\text{cont}(Y) = m \delta$ and $\delta = \sum_{i=0}^{n-1} 2\alpha_i + \alpha_n$, it follows that $c_0 = c_1 = \cdots = c_{n-1} = 2c_n$. From the equations $c_{i-1} = c_i$ for $1 \leq i \leq n-1$ and $c_{n-1} = 2c_n$, we obtain $r_i = r_{2n-i+1}$ and $r_n = r_{n+1}$, respectively.

Conversely, it is clear by Lemma 4.3 that $\bar{Y} = Y_\lambda$ implies that $\text{wt}(Y) = Y - m \delta$ for some $m \geq 0$.

Fix $m \geq 0$. Let $Y$ be a proper Young wall in $\mathcal{Z}(A_{\lambda-m\delta})$. Consider its bead configuration. Let $R_k$ be a runner of type I and $r_k$ the number of beads in $R_k$. Let $b_i^{(k)} (1 \leq i \leq r_k)$ be the beads in $R_k$ enumerated from the bottom to top. Suppose that $b_i^{(k)} (1 \leq i \leq r_k)$ is located at $N^{(k)}_i$ and put $p_i^{(k)} = \frac{N^{(k)}_i - L^{(k)}}{L^{(k)}}$. Then $p_i^{(k)} \geq r_k - i$ and the sequence $p_i^{(k)} (1 \leq i \leq r_k)$ forms a strict partition, say $\mu^{(k)}$. By Lemma 4.4, $r_k = r_{L^{(k)}}$ for $1 \leq k \leq n$ and the pair of $\mu^{(k)}$ and $\mu^{(L^{(k)})}$ determines a unique partition $\lambda^{(k)} = (\mu^{(k)}|\mu^{(L^{(k)})})$ whose sum is the number of all possible moving and removing steps in $R_k$ and $R_{L^{(k)}}$ to obtain $Y$.

In $R_k$, suppose that there are $m_k$ beads at $k \ell$ ($k \geq 1$). Set $\lambda^{(0)} = (1^{m_1}, 2^{m_2}, \cdots)$. We define

\begin{equation}
\pi(Y) = (\lambda^{(0)}, \cdots, \lambda^{(n)}).
\end{equation}

Note that $|\lambda^{(0)}| + \epsilon \sum_{i=1}^{n} |\lambda^{(i)}| = m$. Conversely, for a given $(n+1)$-tuple of partitions $(\lambda^{(0)}, \cdots, \lambda^{(n)})$ with $|\lambda^{(0)}| + \epsilon \sum_{i=1}^{n} |\lambda^{(i)}| = m$, we can associate a unique $Y \in \mathcal{Z}(A_{\lambda-m\delta})$ by reversing the construction of $\pi$. Then $\pi(Y) = (\lambda^{(0)}, \cdots, \lambda^{(n)})$ and it follows that $\pi$ is a bijection.

Summarizing the above argument, we obtain

**Theorem 4.5.** For $m \geq 0$, the map

\begin{equation}
\pi : \mathcal{Z}(A_{\lambda-m\delta}) \rightarrow \bigsqcup_{m_0 + cm_1 = m} \mathcal{P}(m_0) \times \mathcal{P}^{(n)}(m_1)
\end{equation}

is a bijection.

**Example 4.6.** Suppose that $g = A_4^{(2)}$. Let $Y$ be a proper Young wall in $\mathcal{Z}(A_0)$ whose bead configuration is as follows:
Then we have $\tilde{Y} = Y_{\Lambda_0}$. Hence, $Y \in \mathcal{Z}(\Lambda_0)_{\Lambda_0 - 32\delta}$ and $\pi(Y) = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)})$, where

\[
\begin{align*}
\lambda^{(0)} &= (2^4, 3^2), \\
\lambda^{(1)} &= ((1, 3)| (0, 3)) = (1, 2^2, 4), \\
\lambda^{(2)} &= ((0, 2)| (2, 3)) = (1, 4^2).
\end{align*}
\]

Now, we recover the formula for the string functions in [4].

**Corollary 4.7.** We have $\Sigma_A^\lambda(q) = \frac{1}{(q)_{\infty} (q^3)_{\infty}}$.

**Remark 4.8.** If $\mathfrak{g}$ is of type $A_{2n}^{(2)}$, then $\mathcal{Y}(\Lambda)$ can be identified with the set of partitions satisfying the conditions:

1. only parts divisible by $\ell$ may be repeated,
2. the smallest part is smaller than $\ell$,
3. the difference between successive parts is at most $\ell$ and strictly less than $\ell$ if either part is divisible by $\ell$.

In [14], Yamada also gave another combinatorial description of weight vectors for the basic representation $V(\Lambda)$. By using vertex operator construction, he showed that the weight vectors for $V(\Lambda)$ can be parametrized by the set of strict partitions whose parts are not divisible by $\ell$, say $\mathcal{P}'$. Then he described the bead configurations of elements in $\mathcal{P}'$ and computed the weight multiplicities of $V(\Lambda)$ in a similar way.

On the other hand, in [2], Bessenrodt constructed a certain bijection between two kinds of partition sets generalizing the Andrews-Olsson partition identity. As a particular case of her result, we can establish an explicit weight-preserving bijection between $\mathcal{Y}(\Lambda)$ and $\mathcal{P}'$. But, unlike $\mathcal{Y}(\Lambda)$, it seems to be difficult to describe a crystal graph structure on $\mathcal{P}'$.

## 5. $A_{2n-1}^{(2)}$, $D_{n+1}^{(1)}$-Case

Suppose that $\mathfrak{g}$ is of type $A_{2n-1}^{(2)}$ or $D_{n+1}^{(1)}$ $(n \geq 3)$, and $\Lambda$ is a dominant integral weight of level 1. The patterns for $\mathcal{Z}(\Lambda)$ are given as follows:

$A_{2n-1}^{(2)}$ $(n \geq 3)$,
\[
\Lambda = \Lambda_0 : \quad \begin{array}{c|cccc}
\hline
& 2 & 2 & 2 & 2 \\
\hline
1 & 0 & 1 & 0 & 1 \\
2 & 2 & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots \\
\hline
n & n & n & n & n \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 \\
1 & 0 & 1 & 0 \\
\hline
\end{array}
\quad \begin{array}{c|cccc}
\hline
& 2 & 2 & 2 & 2 \\
\hline
1 & 0 & 1 & 0 & 1 \\
2 & 2 & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots \\
\hline
n & n & n & n & n \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 \\
0 & 1 & 0 & 0 \\
\hline
\end{array}
\]

\[
\Lambda = \Lambda_1 : \quad \begin{array}{c|cccc}
\hline
& 2 & 2 & 2 & 2 \\
\hline
1 & 0 & 1 & 0 & 1 \\
2 & 2 & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots \\
\hline
n & n & n & n & n \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 \\
0 & 1 & 0 & 0 \\
\hline
\end{array}
\quad \begin{array}{c|cccc}
\hline
& 2 & 2 & 2 & 2 \\
\hline
1 & 0 & 1 & 0 & 1 \\
2 & 2 & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots \\
\hline
n & n & n & n & n \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 \\
0 & 1 & 0 & 0 \\
\hline
\end{array}
\]

\[
D_{n+1}^{(1)} \quad (n \geq 3),
\]

\[
\Lambda = \Lambda_0 : \quad \begin{array}{c|cccc}
\hline
& 2 & 2 & 2 & 2 \\
\hline
1 & 0 & 1 & 0 & 1 \\
2 & 2 & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots \\
\hline
n-1 & n-1 & n-1 & n-1 \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 \\
1 & 0 & 1 & 0 \\
\hline
\end{array}
\quad \begin{array}{c|cccc}
\hline
& 2 & 2 & 2 & 2 \\
\hline
1 & 0 & 1 & 0 & 1 \\
2 & 2 & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots \\
\hline
n-1 & n-1 & n-1 & n-1 \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 \\
0 & 1 & 0 & 0 \\
\hline
\end{array}
\]

\[
\Lambda = \Lambda_1 : \quad \begin{array}{c|cccc}
\hline
& 2 & 2 & 2 & 2 \\
\hline
2 & 2 & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots \\
\hline
n-1 & n-1 & n-1 & n-1 \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 \\
0 & 1 & 0 & 0 \\
\hline
\end{array}
\quad \begin{array}{c|cccc}
\hline
& 2 & 2 & 2 & 2 \\
\hline
2 & 2 & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots \\
\hline
n-1 & n-1 & n-1 & n-1 \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 \\
0 & 1 & 0 & 0 \\
\hline
\end{array}
\]

\[
\Lambda = \Lambda_n : \quad \begin{array}{c|cccc}
\hline
& 2 & 2 & 2 & 2 \\
\hline
n-1 & n-1 & n-1 & n-1 \\
\vdots & \vdots & \vdots & \vdots \\
\hline
n & n & n & n \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 \\
1 & 0 & 1 & 0 \\
\hline
\end{array}
\quad \begin{array}{c|cccc}
\hline
& 2 & 2 & 2 & 2 \\
\hline
n-1 & n-1 & n-1 & n-1 \\
\vdots & \vdots & \vdots & \vdots \\
\hline
n & n & n & n \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 \\
1 & 0 & 1 & 0 \\
\hline
\end{array}
\]

\[
\Lambda = \Lambda_{n+1} : \quad \begin{array}{c|cccc}
\hline
& 2 & 2 & 2 & 2 \\
\hline
n-1 & n-1 & n-1 & n-1 \\
\vdots & \vdots & \vdots & \vdots \\
\hline
n & n & n & n \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 \\
1 & 0 & 1 & 0 \\
\hline
\end{array}
\quad \begin{array}{c|cccc}
\hline
& 2 & 2 & 2 & 2 \\
\hline
n-1 & n-1 & n-1 & n-1 \\
\vdots & \vdots & \vdots & \vdots \\
\hline
n & n & n & n \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 \\
1 & 0 & 1 & 0 \\
\hline
\end{array}
\]

Set

\[
\ell = \begin{cases} 2n-1 & \text{if } g = A_{2n-1}^{(2)}, \\ n & \text{if } g = D_{2n+1}^{(1)}, \end{cases} \quad L = \begin{cases} \ell & \text{if } g = A_{2n-1}^{(2)}, \\ 2\ell & \text{if } g = D_{2n+1}^{(1)}, \end{cases} \\
\epsilon = L/\ell.
\]

(5.1)
Note that $L$ is the number of blocks in a $\delta$-column or the Coxeter number. Then we define the *abacus of type* $A^{(2)}_{2n-1}$ (resp. $D^{(1)}_{n+1}$) to be the arrangement of positive integers in the following way:

$$A^{(2)}_{2n-1}$$

\begin{array}{cccccc}
1 & 2 & \cdots & \ell - 1 & \ell \\
\ell + 1 & \ell + 2 & \cdots & 2\ell - 1 & 2\ell \\
\vdots & \vdots & & \vdots & \vdots \\
\end{array}

$$D^{(1)}_{n+1}$$

\begin{array}{cccccc}
1 & \cdots & \ell - 1 & \ell + 1 & \cdots & 2\ell - 1 & 2\ell \\
2\ell + 1 & \cdots & 3\ell - 1 & 3\ell + 1 & \cdots & 4\ell - 1 & 4\ell \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}

Let $R_k$ ($1 \leq k < L$, $k \neq \ell$) be the set of all integers $s \equiv k \pmod{L}$ and let $R_\ell$ be the set of all integers $s \equiv 0 \pmod{\ell}$. For $1 \leq k < L$ ($k \neq \ell$), we assume that the $k$th runner $R_k$ is of type I (cf. Section 5). In $R_\ell$, we may put two kinds of beads, white $\bigcirc$ and gray $\bigcirc$, and place more than one bead with the same color at each position. We call $R_\ell$ a *runner of type III*. But we will not move any bead in $R_\ell$.

For $Y \in \mathcal{Z}(\Lambda)$, let $|Y| = (|y_k|)_{k \geq 1}$ be its associated partition. Let $\{|y_1|, \ldots, |y_r|\}$ be the set of all non-zero parts in $|Y|$. Then by definition of $\mathcal{Z}(\Lambda)$, the numbers $|y_k|$s ($1 \leq k \leq r$) are distinct except when $|y_k| \equiv 0 \pmod{\ell}$. We define the *bead configuration* of $Y$ to be the set of $r$ beads $b_1, \ldots, b_r$ placed in the above abacus, where $b_k$ is at $|y_k|$ and the color $c(b_k)$ of $b_k$ is determined by

\[
(5.2) \quad c(b_k) = \begin{cases}
\text{white} & \text{if the block at the top of } y_k \text{ is } \bigtriangleup, \\
\text{gray} & \text{if the block at the top of } y_k \text{ is } \bigtriangleup, \\
\text{white} & \text{otherwise}.
\end{cases}
\]

Then it is easy to see that $Y$ is uniquely determined by its bead configuration.

**Example 5.1.** (1) Suppose that $g = A^{(2)}_{5}$ and $\Lambda = \Lambda_0$.

(2) Suppose that $g = D^{(1)}_{4}$ and $\Lambda = \Lambda_0$. 

![Diagram of abacus configurations](attachment:abacus.png)
Let us describe the algorithm of moving and removing beads in the abacus.

**Lemma 5.2.** Let $Y$ be a proper Young wall in $\mathcal{Z}(\Lambda)$ and let $Y'$ be the proper Young wall obtained by applying one of the following processes to the bead configuration of $Y$:

$(B_1)$ if $b$ is a bead at $s$ in a runner of type I and there is no bead at $s-L$, then move $b$ one position up and change the color of the beads at $k$ ($s-L < k < s$) in $R_k$,

$(B_2)$ if $b$ and $b'$ are beads at $s$ and $L-s$ ($1 \leq s \leq n-1$), respectively, then remove $b$ and $b'$ simultaneously. Also, if $s < \ell < L-s$, then change the color of the beads at $\ell$.

Then we have $\text{wt}(Y') = \text{wt}(Y) + \delta$.

**Proof.** Note that $Y'$ is obtained by removing some $L$ blocks from $Y$, say $\{b_1, \ldots, b_L\}$. Let $i_k$ $(1 \leq k \leq L)$ be the color of $b_k$. It follows directly from the pattern for $\mathcal{Z}(\Lambda)$ that $\sum_{k=1}^{L} \alpha_{i_k} = \delta$ and therefore $\text{wt}(Y') = \text{wt}(Y) + \delta$. \qed

**Example 5.3.**

(1) Let $Y$ be the proper Young wall in Example 5.1 (1). Apply $(B_1)$ to $\square$. This means that we remove a $\delta$-column in the first column of $Y$ and shift the blocks which are placed in the left of the $\delta$-column, to the right as far as possible.

Therefore, we have

$$Y' = \begin{array}{cccc}
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 \\
0 & 1 & 0 & 0
\end{array}$$

(2) Let $Y$ be the proper Young wall in Example 5.1 (2). Similar to (1), if we apply $(B_1)$ to $\square$, then we have

$$Y' = \begin{array}{cccc}
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 \\
0 & 1 & 0 & 0
\end{array}$$
Let \( \bar{Y} \) be the proper Young wall obtained from \( Y \) by applying \((B_1)\) and \((B_2)\) until there is no bead movable up or removable. Note that \( \bar{Y} \) does not depend on the order of steps, hence is uniquely determined.

Set
\[
Z(\Lambda)' = \{ Y = (y_k)_{k \geq 1} \in Z(\Lambda) \mid \|y_k\| \equiv 0 \pmod{\ell} \},
\]
(5.3)
\[
Z(\Lambda)'_{\lambda} = Z(\Lambda)' \cap Z(\Lambda)_{\lambda} \quad \text{for } \lambda \leq \Lambda.
\]
Note that for \( Y \in Z(\Lambda), \ Y \in Z(\Lambda)' \) if and only if there exists no bead in a runner of type I in its bead configuration.

**Lemma 5.4.** Let \( Y \) be a proper Young wall in \( Z(\Lambda) \). Then \( Y \in Z(\Lambda)_{\Lambda - m\delta} \) for some \( m \geq 0 \) if and only if \( \widetilde{Y} \in Z(\Lambda)'_{\Lambda - m'\delta} \) for some \( m' \geq 0 \).

**Proof.** Let \( R_k \) be a runner of type I and let \( r_k \) be the number of beads in \( R_k \) occurring in \( Y \). Note that \( \widetilde{Y} \in Z(\Lambda)' \) if and only if \( r_k = r_{L_k} \) for \( 1 \leq k \leq n - 1 \).

We assume that \( g \) is of type \( A^{(2)}_{2n-1} \) (the proof for \( D^{(1)}_{n+1} \) is similar). Suppose that \( \mathrm{wt}(Y) = \Lambda - m\delta \) for some \( m \geq 0 \). By considering the content of each bead, we see that \( \mathrm{cont}(Y) = \sum_{i=0}^{n} c_i \alpha_i + M\delta \) for some \( M \geq 0 \) where
\[
c_0 + c_1 = \sum_{j=1}^{2n-2} r_j,
\]
(5.4)
\[
c_i = \begin{cases} \frac{\sum_{j=i}^{2n-i-1} r_j + \sum_{j=2n-i}^{2n-2} 2r_j}{2} & \text{if } 2 \leq i \leq n - 1, \\ \frac{\sum_{j=n}^{2n-2} r_j}{2} & \text{if } i = n.
\end{cases}
\]
Since \( \mathrm{cont}(Y) = m\delta \) and \( \delta = \alpha_0 + \alpha_1 + \sum_{i=2}^{n-1} 2\alpha_i + \alpha_n \), it follows that \( c_0 + c_1 = c_2 = \cdots = c_{n-1} = 2c_n \), and hence \( r_i = r_{2n-i-1} \) for \( 1 \leq i \leq n - 1 \). By Lemma 5.2, we have that \( \widetilde{Y} \in Z(\Lambda)'_{\Lambda - m'\delta} \) for some \( m' \geq 0 \).

The converse is clear from Lemma 5.2. \( \square \)

Fix \( m \geq 0 \). Let \( Y \) be a proper Young wall in \( Z(\Lambda)_{\Lambda - m\delta} \). Consider its bead configuration. Let \( R_k \) be a runner of type I and let \( r_k \) be the number of beads in \( R_k \). By Lemma 5.3, \( r_k = r_{L_k} \) for \( 1 \leq k \leq n - 1 \) and as in the case of \( A^{(2)}_{2n} \) or \( D^{(2)}_{n+1} \), we can associate a unique partition \( \lambda^{(k)} \) from the beads in \( R_k \) and \( R_{L_k} \) (1 \leq k \leq n - 1) using Frobenius notation.

We define
\[
P_0(Y) = (\lambda^{(1)}, \cdots, \lambda^{(n-1)}).
\](5.5)
Note that \( \mathrm{cont}(\bar{Y}) = m'\delta \) where \( m' = m - \sum_{i=1}^{n-1} |\lambda^{(i)}| \) and \( Y \) is uniquely determined by \( P_0(Y) \) and \( \bar{Y} \).

Conversely, suppose that we are given an \( (n-1) \)-tuple of partitions \( (\lambda^{(1)}, \cdots, \lambda^{(n-1)}) \) and \( Z \in Z(\Lambda)'_{\Lambda - m'\delta} \) with \( m = m' + \sum_{i=0}^{n-1} |\lambda^{(i)}| \). For \( 1 \leq k \leq n - 1 \), by applying the inverse steps in Lemma 5.2 to \( Z \), we can place beads at \( R_k \) and \( R_{L_k} \) whose corresponding partition is \( \lambda^{(k)} \). Then it is easy to see that the resulting proper Young wall \( Y \) satisfies (i) \( Y \in Z(\Lambda)_{\Lambda - m\delta} \), (ii) \( P_0(Y) = (\lambda^{(1)}, \cdots, \lambda^{(n-1)}) \) and (iii) \( \bar{Y} = Z \).

Summarizing the above arguments, we obtain

**Proposition 5.5.** For \( m \geq 0 \), the map
\[
\psi : Z(\Lambda)_{\Lambda - m\delta} \longrightarrow \bigcup_{m_1 + m_2 = m} \mathcal{P}(n-1)(m_1) \times Z(\Lambda)'_{\Lambda - m_2\delta}
\]
(5.6)
defined by \( \psi(Y) = (P_0(Y), \bar{Y}) \) is a bijection. \( \square \)
Before characterizing \( Z(\Lambda)_{\mathbb{A} - m \delta} \), we recall some notions of two-colored partitions. Let \( \mathbb{N} \) be the set of positive integers. We say that an element in \( \mathbb{N} \) is colored \textit{white}. Let \( \mathbb{N} = \{1, 2, 3, \ldots \} \) be the set of positive integers colored \textit{gray}. Set \( \mathbb{N} = \mathbb{N} \cup \mathbb{N} \). For \( a \) and \( b \) \( \in \mathbb{N} \), we define \( a + b \) to be the colored integer whose value is given by the ordinary sum of their values, and whose color is white if they have the same color, gray if not. For example, \( 1 + 2 = 3 \) and \( 2 + 3 = 5 \). Also, we assume that \( a + 0 = 0 + a = a \) for all \( a \in \mathbb{N} \).

A two-colored partition is a sequence \( \lambda = (\lambda_k)_{k \geq 1} \) of elements in \( \mathbb{N} \cup \{0\} \) such that (i) all but a finite number of \( \lambda_k \)'s are zero (ii) \( \lambda_k \geq \lambda_{k+1} \) as ordinary integers (\( \lambda \) does not depend on the order of the same-valued integers). We also write \( \lambda = (1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots) \) where \( m_k \) is the multiplicity of \( k \in \mathbb{N} \) in \( \lambda \).

We say that \( \lambda \) is a two-colored partition of \( m \) \((m \geq 0)\) if \( \sum_{k \geq 1} k(m_k + m_{k+1}) = m \), and write \(|\lambda| = m \).

Let \( \mathcal{P}' \) be the set of all two-colored partitions such that for each \( k \geq 1 \), both \( k \) and \( k \) do not appear as a part simultaneously. Note that \( \mathcal{P}' \) is closed under addition, i.e. \( \lambda + \mu = (\lambda_k + \mu_k)_{k \geq 1} \in \mathcal{P}' \) for \( \lambda, \mu \in \mathcal{P}' \). For \( Y \in Z(\Lambda)' \), consider its bead configuration where all the beads lie in \( R_\ell \). Define \( \lambda_Y \) to be the two-colored partition in \( \mathcal{P}' \), where the multiplicity of \( k \) (resp. \( k \)) is determined by the number of white (resp. gray) beads at position \( k\ell \). Note that either one of \( m_k \) and \( m_{k+1} \) is zero from the patterns of proper Young walls. Then the map \( Y \mapsto \lambda_Y \) for \( Y \in Z(\Lambda)' \) is a bijection between \( Z(\Lambda)' \) and \( \mathcal{P}' \).

Also, we may identify \( \lambda \in \mathcal{P}' \) with a two-colored Young diagram as follows: for each part \( k \in \mathbb{N} \) of \( \lambda \), we associate a column with \( k \) boxes, and for each part \( k \in \mathbb{N} \) of \( \lambda \), we associate a column with \( k \) boxes where the top box is colored with gray.

Example 5.6. (1) If \( \lambda = (1, 2, 3^2, 4, 5^3) \), then the corresponding two-colored Young diagram is

(2) Suppose that \( g = D_{4(1)} \) and \( \Lambda = \Lambda_0 \).

In general, for a proper Young wall \( Y \in Z(\Lambda)' \), \( \lambda_Y \) can be obtained by identifying

\[
\vcenter{\hbox{\includegraphics{example_diagram}}}
\]

Definition 5.7. For \( \lambda = (\lambda_k) \in \mathcal{P}' \), we define \( \lambda^0 = (\lambda^0_k) \) and \( \lambda^1 = (\lambda^1_k) \) to be the unique partitions satisfying the following conditions:

(i) \( \lambda^0 \in \mathcal{P} \) and \( \lambda^1 \in \mathcal{P}' \).
\( \lambda = \lambda^0 + \lambda^1, \)
\( \lambda^1 = (1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots) \) with \( m_k \neq 0 \), and it is 2-reduced.

Note that the set of all partitions in \( P' \) satisfying the condition (3) can be identified with the set of all 2-reduced partitions since the way of coloring a number is unique.

**Example 5.8.** If \( \lambda = (1, 2, 2, 4, 5) \) given in Example 5.6 (1), then \( \lambda^0 = (1^2, 2^4) \) and \( \lambda^1 = (1^3, 2, 3) \). That is, \( \lambda^0 = \) and \( \lambda^1 = \).

Let \( \lambda \in P' \) be a two-colored partition (or Young diagram). For a box \( b \) in \( \lambda \), suppose that \( b \) lies in the \( p \)th column and the \( q \)th row. We define the residue of \( b \) to be \( p + q \) (mod 2) if \( b \) is white, and \( p + 1 \) (mod 2) if \( b \) is gray. For \( \epsilon = 0, 1 \), we define \( r_\epsilon \) (resp. \( r_\epsilon \)) to be the number of white (resp. gray) boxes in \( \lambda \) with residue \( \epsilon \).

**Lemma 5.9.** Let \( Y \) be a proper Young wall in \( Z(\Lambda)' \) and let \( \lambda = \lambda_Y \) be the corresponding 2-colored partition in \( P' \).

(1) Suppose that \( g \) is of type \( A^{(2)}_{2n-1} \). Then \( Y \in Z(\Lambda)'_{\Lambda - m \delta} \) for some \( m \geq 0 \) if and only if \( \lambda_0 = \lambda_1 \).

(2) Suppose that \( g \) is of type \( D^{(1)}_{n+1} \). Then \( Y \in Z(\Lambda)'_{\Lambda - m \delta} \) for some \( m \geq 0 \) if and only if \( r_0 = r_1 \) and \( \lambda_0 = \lambda_1 \).

**Proof.** (1) Under the correspondence between \( Y \) and \( \lambda_Y \), we observe from the pattern for \( Z(\Lambda) \) that the content of a white box is \( \delta \) and the content of two gray boxes with different residues is \( 2\delta \). This proves (1).

(2) Similarly, we see that (i) the content of two boxes of the same color with different residues is \( 2\delta \) and (ii) no pair of two boxes of different color makes a content which is a multiple of \( \delta \). This proves (2). \( \square \)

For \( Y \in Z(\Lambda)'_{\Lambda - m \delta} \), we define
\[
\pi_1(Y) = ((\lambda_Y)^0, (\lambda_Y)^1),
\]
where we view \( (\lambda_Y)^1 \) as an ordinary 2-reduced partition.

**Proposition 5.10.**

(1) If \( g \) is of type \( A^{(2)}_{2n-1} \), then for \( m \geq 0 \), the map
\[
\pi_1 : Z(\Lambda)'_{\Lambda - m \delta} \rightarrow \bigsqcup_{m_1 + 2m_2 = m} P(m_1) \times P_0(m_2)
\]
is a bijection.

(2) If \( g \) is of type \( D^{(1)}_{n+1} \), then for \( m \geq 0 \), the map
\[
\pi_1 : Z(\Lambda)'_{\Lambda - m \delta} \rightarrow \bigsqcup_{m_1 + m_2 = m} P_0(m_1) \times P_0(m_2),
\]
is a bijection where \( P_0(k) \) is the set of all partitions with empty 2-core and 2-weight \( k \).
Proof. For \( Y \in \mathcal{Z}(\lambda)^{\prime}_{A_{n-\delta}} \), let \( \lambda_Y \) be the two-colored partition in \( \mathcal{P}' \) corresponding to \( Y \). Then there exist unique proper Young walls \( Y_0 \) and \( Y_1 \in \mathcal{Z}(\lambda)' \) such that \( \lambda_{Y_0} = (\lambda_Y)^0 \) and \( \lambda_{Y_1} = (\lambda_Y)^1 \). And we can check that

\[
(5.10) \quad \text{cont}(Y) = \text{cont}(Y_0) + \text{cont}(Y_1).
\]

If \( g \) is of type \( A_{2n-1}^{(2)} \), then each white box in \( \lambda_{Y_0} = (\lambda_Y)^0 \) corresponds to a \( \delta \)-column in \( Y_0 \), which implies that \( \text{cont}(Y_0) \in Z_{ \geq 0} \delta \). Suppose that \( g \) is of type \( D_{n+1}^{(1)} \) and consider the residues of the white boxes in \( \lambda_{Y_0} \) and \( \lambda_{Y_1} \). Since there exist even number of white boxes in each column in \( \lambda_{Y_1} \), the number of white boxes in \( \lambda_{Y_1} \) with residue 0 is equal to the number of white boxes in \( \lambda_{Y_1} \) with residue 1. By Lemma 5.9 (2) and 5.8, the number of white boxes in \( \lambda_{Y_0} \) with residue 0 is equal to the number of white boxes in \( \lambda_{Y_0} \) with residue 1, which implies that the 2-core of \( \lambda_{Y_0} \) is empty and \( \text{cont}(Y_0) \in Z_{ \geq 0} \delta \).

Note that for each box \( b \) in \( \lambda_{Y_1} \), white or gray, the residue of \( b \) is equal to \( \rho + q \) (mod 2) if it is placed in the \( p \)th row and the \( q \)th column. By Lemma 5.9, we have

\[
(5.9) \quad \text{cont}(Y_1) \in Z_{ \geq 0} \delta \quad \text{if and only if} \quad \text{core}_2(\lambda_{Y_1}) = \emptyset.
\]

Therefore, \( \pi_1(Y) = ((\lambda_Y)^0, (\lambda_Y)^1) \in \mathcal{P}(m_1) \times \mathcal{D} \mathcal{P}_0(m_2) \) (resp. \( \mathcal{P}_0(m_1) \times \mathcal{D} \mathcal{P}_0(m_2) \)) for \( m_1 + 2m_2 = m \) (resp. \( m_1 + m_2 = m \)) if \( g \) is of type \( A_{2n-1}^{(2)} \) (resp. \( D_{n+1}^{(1)} \)). Also, \( Y \) is uniquely determined by \( \pi_1(Y) \) by definition.

On the other hand, for a given \( (\lambda^{(0)}, \lambda^{(1)}) \in \mathcal{P}(m_1) \times \mathcal{D} \mathcal{P}_0(m_2) \) (or \( \mathcal{P}_0(m_1) \times \mathcal{D} \mathcal{P}_0(m_2) \)), put \( \lambda = \lambda^{(0)} + \lambda^{(1)} \), where we view \( \lambda^{(1)} \) as a two-colored partition with the color of even (resp. odd) part white (resp. gray). Let \( Y \) be the unique proper Young wall in \( \mathcal{Z}(\lambda)' \) corresponding to \( \lambda \). By the same argument, we have \( \text{cont}(Y) = m \delta \), where \( m_1 + 2m_2 = m \) (resp. \( m_1 + m_2 = m \)) if \( g \) is of type \( A_{2n-1}^{(2)} \) (resp. \( D_{n+1}^{(1)} \)). This correspondence is the inverse map of \( \pi_1 \), and hence it is a bijection.

Now, for each \( m \geq 0 \) and \( Y \in \mathcal{Z}(\Lambda)^{\prime}_{A_{n-\delta}} \), we define

\[
(5.10) \quad \pi(Y) = (\pi_0(Y), \pi_1(Y)).
\]

Then by Proposition 5.9 and 5.10 we obtain

**Theorem 5.11.**

1. If \( g \) is of type \( A_{2n-1}^{(2)} \), then for \( m \geq 0 \), the map

\[
\pi : \mathcal{Z}(\Lambda)^{\prime}_{A_{n-\delta}} \rightarrow \bigcup_{m_1 + 2m_2 = m} \mathcal{P}^{(n)}(m_1) \times \mathcal{D} \mathcal{P}_0(m_2)
\]

is a bijection.

2. If \( g \) is of type \( D_{n+1}^{(1)} \), then for \( m \geq 0 \), the map

\[
\pi : \mathcal{Z}(\Lambda)^{\prime}_{A_{n-\delta}} \rightarrow \bigcup_{\sum_{i=1}^{3} m_i = m} \mathcal{P}^{(n-1)}(m_1) \times \mathcal{P}_0(m_2) \times \mathcal{D} \mathcal{P}_0(m_3)
\]

is a bijection.

Example 5.12. Suppose that \( g = A_5^{(2)} \). Consider the following proper Young wall in \( \mathcal{Z}(\Lambda) \).

\[
Y = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 9 & 0 \\
11 & 12 & 13 & 14 & 15 \\
16 & 17 & 18 & 19 & 20 \\
& & & & \\
& & & & 
\end{array}
\]
Then we have
\[ \tilde{Y} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \pi_0(Y) = (\lambda^{(1)}, \lambda^{(2)}), \quad \text{where} \]
\[ \lambda^{(1)} = ((1, 3)|(0, 2)) = (1, 2^2, 3), \]
\[ \lambda^{(2)} = ((2)|(1)) = (1^2, 2). \]
(5.11)

Also, the 2-colored partition in \( P' \) corresponding to \( \tilde{Y} \) is \( \lambda_{\tilde{Y}} = (1, 4, 2, 4, 2, 3, 3, 4, 3) \) with \( \lambda_{\tilde{Y}}^{(0)} = (1^4) \) and \( |\lambda_{\tilde{Y}}^{(1)}| = 20 \). Since the 2-core of \( \lambda_{\tilde{Y}}^{(1)} \) is empty and \( |\lambda_{\tilde{Y}}^{(1)}| = 20 \), we have \( \tilde{Y} \in \mathcal{Z}(\Lambda)_{\Lambda - 20\delta}^{(1)} \) and
(5.12)
\[ \pi(Y) = ((1, 2^2, 3), (1^2, 2), (1^4), (1^5, 2^4, 3)). \]

Hence, we recover the formulas for the string functions in [4] by a new combinatorial way.

**Corollary 5.13.**

1. If \( g \) is of type \( A^{(2)}_{2n-1} \), then we have
\[ \Sigma_{u}^{A\Lambda\Lambda}(q) = \frac{1}{(q)^{n}(q^2)^{\infty}}. \]
2. If \( g \) is of type \( D^{(1)}_{n+1} \), then we have
\[ \Sigma_{u}^{A\Lambda\Lambda}(q) = \frac{1}{(q)^{n+2}}. \]

**Proof.** By Theorem 3.2 and (3.4), we have
(5.13)
\[ \sum_{m \geq 0} |\mathcal{P}_0(m)| q^m = \frac{1}{(q)^{n}}, \quad \sum_{m \geq 0} |\mathcal{P}_1(m)| q^m = \frac{1}{(q)^{n+2}}. \]

Thus, we obtain the desired formula for \( \Sigma_{u}^{A\Lambda\Lambda}(q) \).

**6. \( B^{(1)}_n \)-CASE**

Suppose that \( g \) is of type \( B^{(1)}_n \), and \( \Lambda \) is a dominant integral weight of level 1. The patterns for \( \mathcal{Z}(\Lambda) \) are given as follows:

\[ \Lambda = \Lambda_0: \]
\[
\begin{array}{cccc}
2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 \\
2 & 2 & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots \\
n-1 & n-1 & n-1 & n-1 \\
n & n & n & n \\
n & n & n & n \\
n-1 & n-1 & n-1 & n-1 \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 \\
0 & 1 & 0 & 0 \\
\end{array}
\]

\[ \Lambda = \Lambda_1: \]
\[
\begin{array}{cccc}
2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 \\
2 & 2 & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots \\
n-1 & n-1 & n-1 & n-1 \\
n & n & n & n \\
n & n & n & n \\
n-1 & n-1 & n-1 & n-1 \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 \\
0 & 1 & 0 & 0 \\
\end{array}
\]
Lemma 6.1. Then the above abacus where \( b \) which is obtained by applying one of the following processes to the bead configuration of \( n \) \((\text{mod} \ k)\) assume that the \( n \) is uniquely determined by its bead configuration.

Set \( \ell = 2n \), which is the number of blocks in a \( \delta \)-column. We define the abacus of type \( B_n^{(1)} \) as follows:

\[
\begin{array}{cccccccc}
\Lambda = \Lambda_n : \\
 & n-1 & n-1 & n-1 & n-1 \\
0 & 0 & 0 & 0 & 0 \\
& n-1 & n-1 & n-1 & n-1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 & 2 \\
& 2 & 2 & 2 & 2 \\
& \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

For \( 1 \leq k < \ell \), let \( R_k \) be the set of all integers \( s \equiv k \) \((\text{mod} \ \ell)\). For \( 1 \leq k < \ell \) \((k \neq n)\), we assume that the \( k \)th runner \( R_k \) is of type I. We also assume that \( R_n \) is of type III (resp. II) and \( R_\ell \) is of type II (resp. III) if \( \Lambda = \Lambda_n \) (resp. \( \Lambda = \Lambda_0, \Lambda_1 \)) (cf. Section 5 and 6).

For \( Y \in \mathcal{Z}(\Lambda) \), let \( \{ |y_1|, \ldots, |y_r| \} \) be the set of all non-zero parts in \( |Y| \). Then by definition of \( \mathcal{Z}(\Lambda) \), the numbers \( |y_k| \)'s \( (1 \leq k \leq r) \) are distinct except when \( |y_k| \equiv 0 \) \((\text{mod} \ n)\). We define the bead configuration of \( Y \) to be the set of \( r \) beads \( b_1, \ldots, b_r \) placed in the above abacus where \( b_k \) is placed at \( |y_k| \), and the color \( c(b_k) \) of \( b_k \) is determined by \((5.2)\).

Then \( Y \) is uniquely determined by its bead configuration.

The algorithms of moving and removing beads in the abacus are as follows (the proof is similar to those of Lemma (5.2) and Lemma (5.2)).

**Lemma 6.1.** Let \( Y \) be a proper Young wall in \( \mathcal{Z}(\Lambda) \) and let \( Y' \) be the proper Young wall which is obtained by applying one of the following processes to the bead configuration of \( Y \):

\[ (B_1) \text{ if } b \text{ is a bead at } s \text{ in a runner of type I and there is no bead at } s - \ell, \text{ then move } b \text{ one position up and change the color of the beads at } k \text{ (} s - \ell < k < s \text{) in the runner of type III}, \]

\[ (B_2) \text{ if } b \text{ is a bead at } s \text{ in the runner of type II, then move } b \text{ one position up along the runner and change the color of the beads at } k \text{ (} s - \ell < k < s \text{) in the runner of type III}, \]

\[ (B_3) \text{ if } b \text{ and } b' \text{ are beads at } s \text{ and } \ell - s \text{ (} 1 \leq s \leq n - 1 \text{) respectively, then remove } b \text{ and } b' \text{ simultaneously. Also if } R_n \text{ is of type III, then change the color of the beads at } n, \]

\[ (B_4) \text{ if there exists at least one bead at } \ell \text{ and } R_\ell \text{ is of type II, then remove one bead at } \ell \text{ and change the color of the beads at } n, \]

\[ (B_5) \text{ if there exists at least two bead at } n \text{ and } R_n \text{ is of type II, then remove two beads at } n. \]

Then we have \( \text{wt}(Y') = \text{wt}(Y) + \delta \). \( \square \)
Example 6.2. Suppose that $g = B_3^{(1)}$.

(1) Let $Y$ be a proper Young wall in $Z(\Lambda_3)$ given below.

If we apply $(B_3)$ to $\circled{1}$ and $\circled{5}$, then

(2) Let $Y$ be a proper Young wall in $Z(\Lambda_0)$ given below.

If we apply $(B_5)$ to $\circled{3}$ then

6.1. Characterization of $Z(\Lambda_{\Lambda-m\delta})$. For $Y \in Z(\Lambda)$, let $\tilde{Y}$ be the proper Young wall obtained from $Y$ by applying $(B_i)$ ($1 \leq i \leq 5$) until there is no bead movable up or removable. We set

$$Z(\Lambda)' = \{ Y = (y_k)_{k \geq 1} \in Z(\Lambda) \mid |y_k| \equiv r \pmod{\ell} \},$$

(6.1)

$$Z(\Lambda)'_{\Lambda} = Z(\Lambda)' \cap Z(\Lambda_{\lambda}) \quad \text{for } \lambda \leq \Lambda,$$

where $r = 0$ if $\Lambda = \Lambda_0$ or $\Lambda_1$, and $r = n$ if $\Lambda = \Lambda_n$. Note that for $Y \in Z(\Lambda)$, $Y \in Z(\Lambda)'$ if and only if there exists no bead in a runner of type I and II in its bead configuration.

Lemma 6.3. Let $Y$ be a proper Young wall in $Z(\Lambda)$. Then $Y \in Z(\Lambda_{\Lambda-m\delta})$ for some $m \geq 0$ if and only if $\tilde{Y} \in Z(\Lambda)'_{\Lambda-m'\delta}$ for some $m' \geq 0$.

Proof. We give a proof for the case $\Lambda = \Lambda_0$ or $\Lambda_1$ (the proof for the case $\Lambda = \Lambda_n$ is similar). Let $R_k$ be a runner of type I and let $r_k$ be the number of beads in $R_k$ occurring in $Y \in Z(\Lambda)$. Note that $\tilde{Y} \in Z(\Lambda)'$ if and only if (i) $r_k = r_{\ell-k}$ for $1 \leq k \leq n-1$, and (ii) there are even number of beads in $R_n$. 

Let $Y$ be a proper Young wall in $Z(\Lambda)_{\Lambda - m\delta}$ for some $m \geq 0$. By considering the content of each bead, we see that $\text{cont}(Y) = \sum_{i=0}^{n} c_i \alpha_i + M\delta$ for some $M \geq 0$, where

$$c_0 + c_1 = \sum_{j=1}^{2n-1} r_j,$$

$$c_i = \sum_{j=i}^{2n-i} r_j + \sum_{j=2n-i+1}^{2n-1} 2r_j \quad \text{if } 2 \leq i \leq n-1. \tag{6.2}$$

Since $\text{cont}(Y) = m\delta$ and $\delta = \alpha_0 + \alpha_1 + \sum_{i=2}^{n} 2\alpha_i$, it follows that $c_0 + c_1 = c_2 = \cdots = c_n$, and hence $r_i = r_{2n-i}$ for $1 \leq i \leq n-1$. Thus, $\bar{Y}$ has no bead in a runner of type I. Suppose that there exists at least one bead in $R_n$ in the bead configuration of $\bar{Y}$. Then by $(B_2)$ and $(B_5)$, there exists only one bead $b$ at $n$ in $R_n$. On the other hand, if we consider the content of the blocks corresponding to all the beads in $R_\ell$, we see that the coefficient of $\alpha_n$ is even. Also, the content of the blocks corresponding to $b$ is $\alpha_\ell + \alpha_2 + \alpha_3 + \cdots + \alpha_n$ ($\epsilon = 0, 1$). This contradicts the fact that the coefficient of $\alpha_n$ in $\text{cont}(\bar{Y}) = m'\delta$ ($m' \geq 0$) is even. Therefore, we have $\bar{Y} \in Z(\Lambda)'_{\Lambda - m'\delta}$ for some $m' \geq 0$. The converse is clear from Lemma 6.11. \qed

Fix $m \geq 0$. Let $Y$ be a proper Young wall in $Z(\Lambda)_{\Lambda - m\delta}$. Consider its bead configuration. Let $r_k$ be the number of beads in $R_k$ of type I. By Lemma 6.3 $r_k = r_{k-1}$ for $1 \leq k \leq n-1$. Hence, we can associate a unique partition $\lambda^{(k)}$ from the beads in $R_k$ and $R_{k-1}$ ($1 \leq k \leq n-1$) using Frobenius notation. Next, we define a partition $\lambda^{(0)}$ as follows: If $\Lambda = \Lambda_n$, we set $\lambda^{(0)} = (1^{m_1}, 2^{m_2}, \cdots)$, where $m_k$ is the number of beads at $k\ell$ in $R_\ell$. If $\Lambda = \Lambda_0$ or $\Lambda_1$, then we set $\lambda^{(0)} = (1^{m_1}, 3^{m_3}, 5^{m_5}, \cdots)$, where $m_{2k-1}$ is the number of beads at $(2k-1)n$ in $R_n$. In the latter case, $\ell(\lambda^{(0)})$ is even since the number of beads in $R_n$ is even by Lemma 6.3. Also, we see that $\ell(\lambda^{(0)})$ is even if and only if $|\lambda^{(0)}|$ is even since $\lambda^{(0)}$ is a partition with each part odd.

We define

$$\pi_0(Y) = (\lambda^{(0)}, \cdots, \lambda^{(n-1)}). \tag{6.3}$$

**Proposition 6.4.** For $Y \in Z(\Lambda)$, we define $\psi(Y) = (\pi_0(Y), \bar{Y})$.

1. If $\Lambda = \Lambda_0$ or $\Lambda_1$, then for $m \geq 0$, the map

$$\psi : Z(\Lambda)_{\Lambda - m\delta} \rightarrow \bigcup_{\sum_{i=1}^{3} m_i = m} \mathcal{O}(2m_1) \times \mathcal{P}(n-1)(m_2) \times Z(\Lambda)'_{\Lambda - m_2\delta}$$

is a bijection, where $\mathcal{O}(k)$ is the set of all partitions of $k$ with odd parts.

2. If $\Lambda = \Lambda_n$, then for $m \geq 0$, the map

$$\psi : Z(\Lambda)_{\Lambda - m\delta} \rightarrow \bigcup_{m_1 + m_2 = m} \mathcal{P}(n)(m_1) \times Z(\Lambda)'_{\Lambda - m_2\delta}$$

is a bijection. \qed

It remains to characterize $Z(\Lambda)'_{\Lambda - m\delta}$ ($m \geq 0$). First, suppose that $\Lambda = \Lambda_0$ or $\Lambda_1$. For each $Y \in Z(\Lambda)'$, let $\lambda_Y$ be the two-colored partition in $\mathcal{P}'$, where the multiplicity of $k$ (resp. $\ell$) is the number of white (resp. gray) beads at position $k\ell$ in $Y$. We observe that the map $Y \mapsto \lambda_Y$ is a bijection between $Z(\Lambda)'$ and $\mathcal{P}'$. We define

$$\pi_1(Y) = ((\lambda_Y)^0, (\lambda_Y)^1), \tag{6.4}$$

(see Definition 6.7), where we view $(\lambda_Y)^1$ as an ordinary 2-reduced partition.
Proposition 6.5. Suppose that $\Lambda = \Lambda_0$ or $\Lambda_1$. Then for $m \geq 0$, the map

$$\pi_1 : \mathcal{Z}(\Lambda)'_{\Lambda-m\delta} \rightarrow \bigsqcup_{m_1+2m_2=m} \mathcal{P}(m_1) \times \mathcal{D}_0(m_2)$$

is a bijection.

Proof. The proof is almost the same as in the case of $A_{2n-1}^{(2)}$ (see Proposition 5.10 (1)).

Next, suppose that $\Lambda = \Lambda_n$. For each $Y \in \mathcal{Z}(\Lambda)'$, let $\lambda \gamma = \lambda$ be the partition in $\mathcal{P}$, where the multiplicity of $k$ (resp. $k$) is the number of white (resp. gray) beads at the position $(2k - 1)n$ in the bead configuration of $Y$. Then the map $Y \mapsto \lambda \gamma$ is a bijection between $\mathcal{Z}(\Lambda_n)'$ and $\mathcal{P}$. Fix $Y \in \mathcal{Z}(\Lambda)'_{\Lambda-m\delta}$ ($m \geq 0$). Let $\mu$ be a unique partition in $\mathcal{P}$ satisfying

(i) $\lambda \gamma = \mu + \nu$ for some $\nu \in \mathcal{P}$, and

(ii) the proper Young wall in $\mathcal{Z}(\Lambda)'$ corresponding to $\mu$ is reduced. Note that $\ell(\lambda) = \ell(\mu)$ and $\mu$ is 2-reduced which has one of the following forms

$$0, (1^{m_1}, 2^{m_2}, 3^{m_3}, \cdots) (m_1 \neq 0), (1^{m_1}, 2^{m_2}, 3^{m_3}, \cdots) (m_1 \neq 0).$$

We define

$$\pi_1(Y) = (\mu, \nu, c),$$

where we view $\mu$ as an ordinary partition and

$$c = \begin{cases} 0 & \text{if } \mu = (0) \text{ or } \mu = (1^{m_1}, 2^{m_2}, 3^{m_3}, \cdots) \text{ with } m_1 \neq 0, \\ 1 & \text{if } \mu = (1^{m_1}, 2^{m_2}, 3^{m_3}, \cdots) \text{ with } m_1 \neq 0. \end{cases}$$

Lemma 6.6. Under the above hypothesis, we have

1. $\mu \in \mathcal{D}_0$ and $\ell(\lambda) = \ell(\mu)$ is even,

2. $|\mu| + |\nu| - \ell(\mu)/2 = m$.

Proof. (1) Since $\text{cont}(Y) = m\delta$, the number of $n$-blocks in $Y$ is even. This implies that $\ell(\lambda)$ is even. So it suffices to show that the 2-core of $\mu$ is empty. Let $Z$ be the reduced proper Young wall in $\mathcal{Z}(\Lambda)'$ corresponding to $\mu$. Then we have

$$\text{cont}(Y) = \text{cont}(Z) + |\nu|\delta.$$ 

By definition of $\mu$, it is 2-reduced and $\ell(\lambda) = \ell(\mu)$. Consider the two-colored Young diagram of $\mu$. From the pattern for $\mathcal{Z}(\Lambda_n)$, we see that $\text{cont}(Z) = m\delta$ for some $m' \geq 0$ if and only if the number of gray boxes with residue 0 is equal to the number of gray boxes with residue 1. Hence, by a similar argument as in Proposition 5.10 we conclude that the 2-core of $\mu$ is empty.

(2) Suppose that $\text{cont}(Z) = m'\delta$ for some $m' \geq 0$. Note that $2m'$ is the number of $n$-blocks in $Z$ (except the ones in $Y_{\lambda}$). For each part $k$ (or $k$) in $\mu$, the number of $n$-blocks in the corresponding column of $Z$ is $2k - 1$. Hence,

$$2m' = \sum (2k - 1)(m_k + m_k) = 2|\mu| - \ell(\mu),$$

where $m_k$ (resp. $m_k$) is the multiplicity of $k$ (resp. $k$) in $\mu$. Since $m = m' + |\nu|$ by (6.7), we get (2).

For $m \geq 0$, let $\mathcal{D}(m)$ be the set of triples $(\mu, \nu, c)$ such that

(i) $\mu \in \mathcal{D}_0$ with $\ell(\mu)$ even,

(ii) $\nu \in \mathcal{P}$ with $\ell(\nu) \leq \ell(\mu)$ and $|\mu| + |\nu| - \ell(\mu)/2 = m$,

(iii) $c = 0$ or 1, and if $\mu = (0)$, then $c = 0$.

Lemma 6.7. For each $m \geq 0$, the number of elements in $\mathcal{D}(m)$ is the coefficients of $q^m$ in $(q^2)_{\infty}/(q^2)_{\infty}$. 


Proof. Let us fix some notations. For variables \(a\) and \(q\), set \((a : q)_k = (1 - a)(1 - aq) \cdots (1 - aq^{k-1}) (k \geq 1)\), and \((a : q)_{\infty} = \prod_{k=1}^{\infty} (1 - aq^{k-1})\). In particular, set \((q)_k = (q : q)_k\) and \((q)_\infty = (q : q)_\infty\).

Let \((\mu, \nu, c)\) be an element in \(\mathcal{G}(m)\) for some \(m \geq 1\). Let \(\mu'\) be the unique 2-reduced partition such that \(\mu = \mu' + (1^{2k})\) where \(\ell(\mu) = 2k\). Then \(|\mu| - \ell(\mu)/2 = |\mu'| + k\), and \(\ell(\mu') < 2k\) since \(\mu\) is 2-reduced. Note that \(\mu\) is uniquely determined by \(\mu'\) and \(k\). By considering the bead configurations of the partitions with empty 2-core, the generating functions for the 2-reduced partitions with empty 2-core and length less than 2\(k\) is

\[
(q^2)_{2k-1} \over (q^2)_k(q^2)_{k-1}(q^{2k})
\]  

(6.9)

(we leave the proof to the readers as an exercise. see Section 3). Also the generating functions for the partitions with length less than or equal to 2\(k\) is \(1/(q^{2k})\). Therefore, the generating functions for \(\mathcal{G}(m)\) \((m \geq 0)\) is given by

\[
1 + 2 \sum_{k \geq 1} (q^2)_{2k-1} \over (q^2)_k(q^2)_{k-1}(q^{2k})q^k.
\]

(6.10)

Note that 2 appears in the above sum because of \(c = 0, 1\) in \((\mu, \nu, c) (\mu \neq (0))\). By elementary computation, we have

\[
(q^2)_{2k-1} \over (q^2)_k(q^2)_{k-1}(q^{2k}) = \prod_{i=1}^{2k-1} (1 + q^i) \over \prod_{j=1}^{2k-1} (1 - q^2)^2
\]

(6.11)

\[
= (-q : q)_k(-q^2 : q^2)_{k-2} \over (q^2)_k
\]

\[
= (-q : q)_k(-1 : q^2)_k \over 2(q^2)_k.
\]

Therefore, the generating function for \(\mathcal{G}(m)\) is

\[
\sum_{k \geq 0} (-q : q)_k(-1 : q^2)_k \over (q^2)_k.
\]

(6.12)

If we apply a Heine’s identity (see Corollary 2.4 in [1]) by replacing \(a, b, c, q\) by \(-q, -1, q^2, q^2\), respectively, we obtain

\[
\sum_{k \geq 0} (-q : q)_k(-1 : q^2)_k \over (q^2)_k = \frac{(-q : q)_\infty(-q^2 : q^2)_\infty}{(q : q^2)_\infty(q^2)_\infty} = \frac{(-q : q)_\infty}{(q)_\infty} = \frac{(q^2)_\infty}{(q^2)_\infty},
\]

(6.13)

which proves our claim. \(\square\)

Proposition 6.8. Suppose that \(\Lambda = \Lambda_n\). For \(m \geq 0\), the map

\[
\pi_1 : \mathcal{Z}(\Lambda)'_{\Lambda - m \delta} \rightarrow \mathcal{G}(m)
\]

is a bijection.

Proof. By Lemma 6.6, \(\pi_1\) is well-defined. Note that \(Y\) is uniquely determined by \(\pi_1(Y)\). Suppose that \((\mu, \nu, c) \in \mathcal{G}(m)\) is given. If \(c = 0\), then we view \(\mu\) as a two-colored partition where the color of an even (resp. odd) part is white (resp. gray). If \(c = 1\), then we view \(\mu\) as a two-colored partition where the color of an even (resp. odd) part is gray (resp. white). Set \(\lambda = \mu + \nu \in \mathcal{P}'\). Then there exists a unique \(Y \in \mathcal{Z}(\Lambda)'\) corresponding to \(\lambda\). By the same
argument in Lemma 6.14 we can check that $Y \in \mathcal{Z}(\Lambda)_{\Lambda - m\delta}^{\prime}$ with $|\mu| + |\nu| - \ell(\mu)/2 = m$, and $\pi_1(Y) = (\mu, \nu, c)$, which defines the inverse map of $\pi_1$. 

**Example 6.9.** Let $Y$ be a proper Young wall in $\mathcal{Z}(\Lambda_n)'$ such that the corresponding partition is $\lambda_Y = (1^2, 2, 3^2, 4, 6, 7)$. If we take

$$\mu = (1^2, 2^4, 3^2), \quad \nu = (1^2, 2, 3, 4),$$

then $\lambda_Y = \mu + \nu$ where $\mu$ is the unique 2-colored partition whose corresponding proper Young wall in $\mathcal{Z}(\Lambda_n)'$ is reduced. Also $\mu$ and $\nu$ satisfy the conditions in Lemma 6.6. Since $|\mu| + |\nu| - \ell(\mu)/2 = 23$, we have $Y \in \mathcal{Z}(\Lambda_n)'_{\Lambda_n - 2\delta}$ and

$$\pi_1(Y) = ((1^2, 2^4, 3^2), (1^2, 2, 3, 4), 1).$$

Now, for $m \geq 0$ and $Y \in \mathcal{Z}(\Lambda)_{\Lambda - m\delta}$ ($\Lambda = \Lambda_0, \Lambda_1, \Lambda_n$), we define

$$\pi(Y) = (\pi_0(Y), \pi_1(Y)).$$

Then by Proposition 6.4, 6.5 and 6.8 we obtain

**Theorem 6.10.**

(1) If $\Lambda = \Lambda_0$ or $\Lambda_1$, then for $m \geq 0$, the map

$$\pi : \mathcal{Z}(\Lambda)_{\Lambda - m\delta} \rightarrow \bigcup_{m_1 + m_2 + 2m_3 = m} \mathcal{P}(2m_1) \times \mathcal{P}(m_2) \times \mathcal{P}(m_3)$$

is a bijection.

(2) If $\Lambda = \Lambda_n$, then for $m \geq 0$, the map

$$\pi : \mathcal{Z}(\Lambda)_{\Lambda - m\delta} \rightarrow \bigcup_{m_1 + m_2 = m} \mathcal{P}(m_1) \times \mathcal{P}(m_2)$$

is a bijection. 

**Example 6.11.** (1) Suppose that $\Lambda = \Lambda_0$.

| Y          | $\bar{Y}$ |
|------------|-----------|
| 1 2 3 4 5 6 | 0         |
| 7 8 9 10 11 |           |
| 12          |           |
| 13 14 15 16 |           |
| 17 18       |           |
| 19 20 21 22 23 |   |
| 24          |           |

and $\pi_0(Y) = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)})$, where

$$\lambda^{(0)} = (1^2, 5^3, 7),$$

$$\lambda^{(1)} = ((3)(1)) = (1^3, 2),$$

$$\lambda^{(2)} = ((0)(2)) = (3).$$

Note that $\lambda_{17} = (1, 2^4, 4^2)$ with $(\lambda_{17})^0 = (1^5, 2^2)$ and $(\lambda_{17})^1 = (1^4, 2^2)$. Since the 2-core of $(\lambda_{17})^1$ is empty and $|\lambda_{17}| = 17$, we have $Y \in \mathcal{Z}(\Lambda)_{\Lambda - 17\delta}$ by Proposition 6.14 (1), and $\pi_1(Y) = ((1^5, 2^2), (1^4, 2^2))$. Hence, we have $Y \in \mathcal{Z}(\Lambda)_{\Lambda - 37\delta}$ and

$$\pi(Y) = ((1^2, 5^3, 7), (1^3, 2), (3), (1^5, 2^2), (1^4, 2^2)).$$
(2) Suppose that $\Lambda = \Lambda_n$.

\[
\begin{array}{cccccc}
1 & 2 & 3^2 & 4 & 5 & 6 \\
7 & 8 & 9^2 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & 17 & 18 \\
19 & 20 & 21 & 22 & 23 & 24 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

If $Y = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, then $\tilde{Y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

and $\pi_0(Y) = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)})$, where

\[
\lambda^{(0)} = (2, 3^3, 4),
\]
\[
\lambda^{(1)} = ((2) | (1)) = (1^2, 2),
\]
\[
\lambda^{(2)} = ((0) | (3)) = (4).
\]

Also, $\lambda_\gamma = (1^2, 2^2, 3^3, 4)$. If we take $\mu = (1^4, 2^4)$ and $\nu = (1^5, 2)$, then $\lambda_\gamma = \mu + \nu$ and the proper Young wall in $\mathcal{Z}(\Lambda_n)$ corresponding to $\mu$ is reduced. Since $((1^4, 2^4), (1^5, 2), 1) \in \mathcal{Q}(15)$, we have $Y \in \mathcal{Z}(\Lambda_n)_{\Lambda_n - 15\delta}$ by Proposition 6.8 and $\pi_1(\tilde{Y}) = ((1^4, 2^4), (1^5, 2), 1)$. Hence, we have $Y \in \mathcal{Z}(\Lambda_{\Lambda_n - 38\delta})$ and

\[
\pi(Y) = ((2, 3^3, 4), (1^2, 2), (4), (1^4, 2^4), (1^5, 2), 1).
\]

Therefore, we obtain a new combinatorial proof of the formulas in [H].

**Corollary 6.12.**

(1) If $\Lambda = \Lambda_0$ or $\Lambda_1$, then we have

\[
\Sigma_\Lambda^4(q) = \frac{1}{2} \left( \frac{1}{(q^2)_{\infty}} \frac{1}{(q^2)_{\infty}} \frac{1}{(q^2)_{\infty}} \frac{1}{(q^2)_{\infty}} \right).
\]

(2) If $\Lambda = \Lambda_n$, then we have $\Sigma_\Lambda^4(q) = \frac{(q^2)_{\infty}}{(q^2)_{\infty}}$.

**Proof.** (1) Let $\Theta(m)$ be the number of the partitions of $m$ with odd parts. Then we have $O(q) = \sum_{m > 0} \Theta(m)q^m = (-q : q)_{\infty} = 1/(q : q^2)_{\infty}$ ($|\Theta(0)| = 1$). Hence the number of the partitions in $\Theta(2m)$ is the coefficient of $q^m$ in

\[
\frac{1}{2} \left( O(q^2) + O(q^2) \right) = \frac{1}{2} \left( \frac{(q^2)_{\infty}}{(q^2)_{\infty}} \frac{(q^2)_{\infty}}{(q^2)_{\infty}} \right).
\]

Hence, we obtain $\Sigma_\Lambda^4(q)$ by multiplying $\frac{1}{(q^2)_{\infty}}$.

(2) This follows from Lemma 6.7. $\square$

6.2. **Characterization of $\mathcal{Z}(\Lambda_0)_{\Lambda_1 - m\delta}$.**

**Lemma 6.13.** For $Y \in \mathcal{Z}(\Lambda_0)$, let $r_k$ be the number of beads in $R_k$ in the bead configuration of $Y$ ($1 \leq k < \ell, k \neq n$). If $Y \in \mathcal{Z}(\Lambda_0)_{\Lambda_1 - m\delta}$ for some $m \geq 0$, then $r_k = r_{\ell-k}$ for $1 \leq k \leq n-1$.

**Proof.** We see that $\text{cont}(Y) = \sum_{i=0}^{n} c_i \alpha_i + M \delta$ for some $M \geq 0$ where $c_i$ is given by (6.2). On the other hand, $\text{cont}(Y) = \gamma + m\delta = (m+1)\alpha_0 + m\alpha_1 + (2m+1)\sum_{i=2}^{n} \alpha_i$, where $\gamma = \alpha_0 + \sum_{i=2}^{n} \alpha_i$. This implies that $c_0 + c_1 = c_2 = \cdots = c_n$, and hence $r_k = r_{\ell-k}$ for $1 \leq k \leq n-1$. $\square$
Suppose that \( Y \in \mathcal{Z}(\Lambda_0)_{\Lambda_1 - m\delta} \) is given. By Lemma 6.13, we can associate a unique partition \( \lambda^{(k)} \) \((1 \leq k \leq n - 1)\) from the beads in \( R_k \) and \( R_{\ell - k} \) using Frobenius notation.

Let \( Y' \) be the proper Young wall obtained by applying \((B_1)\) and \((B_3)\) to \( Y \) until there is no bead in the runners of type I. Note that \( Y' \in \mathcal{Z}(\Lambda_0)_{\Lambda_1 - m\delta} \), where \( m' = m - \sum_{i=1}^{n-1} |\lambda^{(i)}| \).

Set \( \lambda^{(0)} = (1^{m_1}, 3^{m_3}, 5^{m_5}, \ldots) \), where \( m_{2k-1} \) is the number of the beads at \((2k-1)n\). Since \( \text{cont}(Y') = \gamma + m'\delta \), the number of \( n \)-blocks in \( Y' \) (except the ones in \( Y_\lambda \)) is odd and hence the number of beads in \( R_n \) is odd (or \(|\lambda^{(0)}|\) is odd). We define

\[
(6.22) \quad \pi_0(Y) = (\lambda^{(0)}, \ldots, \lambda^{(n-1)}).
\]

Next, consider \( \tilde{Y}' = \tilde{Y} \). Note that \( \tilde{Y} \in \mathcal{Z}(\Lambda_0)_{\Lambda_1 - m''\delta} \), where \( m'' = m' - \frac{|\lambda^{(0)}| - 1}{2} \). From the beads of \( \tilde{Y} \) in \( R_\ell \), we define \( \mu \) to be the partition in \( \mathcal{P}' \), where the multiplicity of \( k \) (resp. \( k' \)) is given by

\[
(6.23) \quad m_k = \begin{cases} 
\text{the number of white beads at } k\ell & \text{if } \ell(|\tilde{Y}|) \text{ is odd}, \\
\text{the number of gray beads at } k\ell & \text{if } \ell(|\tilde{Y}|) \text{ is even}, 
\end{cases} \\
(6.24) \quad m_{k'} = \begin{cases} 
\text{the number of gray beads at } k\ell & \text{if } \ell(|\tilde{Y}|) \text{ is odd}, \\
\text{the number of white beads at } k\ell & \text{if } \ell(|\tilde{Y}|) \text{ is even}, 
\end{cases}
\]

We define

\[
(6.24) \quad \pi_1(\tilde{Y}) = (\mu^0, \mu^1),
\]

where we view \( \mu^1 \) as an ordinary 2-reduced partition.

**Lemma 6.14.** Under the above hypothesis, we have

(1) \( \mu^1 \in \mathcal{P}_0' \),
(2) \(|\mu^0| + |\mu^1| = m'' \), where \( \tilde{Y} \in \mathcal{Z}(\Lambda_0)_{\Lambda_1 - m''\delta} \).

**Proof.** First, note that in \( \tilde{Y} \), there exists exactly one bead \( b \) in \( R_n \) and it is placed at \( n \) since the number of the beads in \( R_n \) in the bead configuration of \( Y' \) is odd. Since there is an one-to-one correspondence between \( \mathcal{P}' \) and \( \mathcal{Z}(\Lambda)^{'} (\Lambda = \Lambda_0, \Lambda_1) \), there exist unique proper Young walls \( Z \in \mathcal{Z}(\Lambda_r)^{'} \) corresponding to \( \mu \) where \( r \equiv \ell(|\tilde{Y}|) + 1 \text{ (mod 2)} \). In fact, \( Z \) can be obtained in the following way:

(i) if \( \ell(|\tilde{Y}|) \text{ is odd} \), then remove the left-most column of \( \tilde{Y} \) whose content is \( \gamma \). The resulting proper Young wall is \( Z \in \mathcal{Z}(\Lambda_0)^{'} \).

(ii) if \( \ell(|\tilde{Y}|) \text{ is even} \), then shift all the blocks to the left column by one position following the pattern, except the first \( n \) blocks (except the one in \( Y_\lambda \)) from the bottom. There are \( n \) blocks left in the first column whose content is \( \gamma \). If we cut out this first column, then the resulting proper Young wall is \( Z \in \mathcal{Z}(\Lambda_1)^{'} \).

From the above facts, we have

\[
(6.25) \quad \text{cont}(\tilde{Y}) = \text{cont}(Z) + \gamma,
\]

where \( \gamma = \alpha_0 + \sum_{i=2}^{n} \alpha_i \). Since \( \text{cont}(\tilde{Y}) = m''\delta + \gamma \), we have \( \text{cont}(Z) = m''\delta \), i.e. \( Z \in \mathcal{Z}(\Lambda_r)^{'}_{\Lambda_r - m''\delta} \). Hence from Proposition 6.15, \( \mu \) satisfies the conditions in (1) and (2). \( \square \)

Now, for each \( m \geq 0 \) and \( Y \in \mathcal{Z}(\Lambda_0)_{\Lambda_1 - m\delta} \), we define

\[
(6.26) \quad \pi(Y) = (\pi_0(Y), \pi_1(\tilde{Y})).
\]

Then, we obtain
Theorem 6.15. For each $m \geq 0$, the map
\[
\pi : \mathcal{Z}(\Lambda_0)_{\Lambda_1 - m\delta} \to \bigcup_{m_1 + m_2 + 2m_3 = m} \mathcal{O}(2m_1 + 1) \times \mathcal{P}^{(n)}(m_2) \times \mathcal{D}(m_3)
\]
is a bijection.

Proof. By Lemma 6.14, $\pi$ is well-defined. The inverse map can be defined by reversing the construction of $\pi$ naturally. \hfill \square

Example 6.16. Consider the following proper Young wall in $\mathcal{Z}(\Lambda_0)$.

\[
Y = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & 17 & 18 \\
19 & 20 & 21 & 22 & 23 & 24 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

Then we have $\tilde{Y} = \begin{array}{cccc}
5 & 6 \\
9 & 12 \\
15 & 18 \\
21 & 24 \\
\vdots & \vdots \\
\end{array}$ and $\pi_0(Y) = (\lambda(0), \lambda(1), \lambda(2))$, where
\[
\lambda(0) = (1^2, 5^3, 7), \\
\lambda(1) = (3)|1) = (1^3, 2), \\
\lambda(2) = (0)|2) = (3).
\]

Note that $\ell(|\tilde{Y}|)$ is odd. Following the rule in (6.28), we get $\mu = (1^2, 4^3)$, where $\mu^0 = (1^5, 2^3)$ and $\mu^1 = (1^4, 2^3)$. Since $\mu^1$ has an empty 2-core, we have $\tilde{Y} \in \mathcal{Z}(\Lambda_0)_{\Lambda_1 - 21\delta}$ and $\pi_1(\tilde{Y}) = ((1^5, 2^3), (1^3, 2^3))$. Therefore, $Y \in \mathcal{Z}(\Lambda_0)_{\Lambda_1 - 38\delta}$ and
\[
\pi(Y) = ((1^2, 5^2, 7), (1^3, 2), (3), (1^5, 2^3), (1^4, 2^3)).
\]

Therefore, we also obtain another proof of the formula in (4).

Corollary 6.17.

\[
\Sigma_{\Lambda_1}(q) = \frac{1}{2q^{\frac{1}{2}}} \left( \frac{1}{(q^2)_{\infty} (q^3)_{\infty} (q^7)_{\infty} - (q^\frac{7}{2})_{\infty}} - \frac{(q^\frac{7}{2})_{\infty}}{(q^\frac{3}{2})_{\infty}} \right).
\]

Proof. As in Corollary 6.12 (1), the number of partitions in $\mathcal{O}(2m + 1)$ ($m \geq 0$) is the coefficient of $q^m$ in
\[
\frac{1}{2q^{\frac{1}{2}}} \left( O(q^\frac{7}{2}) - O(-q^\frac{7}{2}) \right) = \frac{1}{2q^{\frac{1}{2}}} \left( \frac{(q)_{\infty}}{(q^2)_{\infty}} - \frac{(q^\frac{3}{2})_{\infty}}{(q^2)_{\infty}} \right).
\]

Hence, we obtain the result. \hfill \square
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