NONLINEAR DIFFUSIVE SHOCK ACCELERATION WITH MAGNETIC FIELD AMPLIFICATION

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ABSTRACT

We introduce a Monte Carlo model of nonlinear diffusive shock acceleration that allows for the generation of large-amplitude magnetic turbulence, i.e., \( \Delta B \gg B_0 \), where \( B_0 \) is the ambient magnetic field. The model is the first to include strong wave generation, efficient particle acceleration to relativistic energies in nonrelativistic shocks, and thermal particle injection in an internally self-consistent manner. We find that the upstream magnetic field \( B_0 \) can be amplified by large factors and show that this amplification depends strongly on the ambient Alfvén Mach number. We also show that, in the nonlinear model, large increases in \( B \) do not necessarily translate into a large increase in the maximum particle momentum a particular shock can produce, a consequence of high-momentum particles diffusing in the shock precursor where the large amplified field converges to the low ambient value. To deal with the field growth rate in the regime of strong fluctuations, we extend to strong turbulence a parameterization that is consistent with the resonant quasi-linear growth rate in the weak turbulence limit. We believe our parameterization spans the maximum and minimum range of the fluctuation growth, and within these limits we show that the nonlinear shock structure, acceleration efficiency, and thermal particle injection rates depend strongly on the yet to be determined details of wave growth in strongly turbulent fields. The most direct application of our results will be to estimate magnetic fields amplified by strong cosmic-ray modified shocks in supernova remnants.

Subject headings: acceleration of particles — cosmic rays — MHD — radio continuum: ISM — supernova remnants — turbulence — X-rays: ISM

1. INTRODUCTION

Recent observations and modeling of several young supernova remnants (SNRs) suggest the presence of magnetic fields at the forward shock (i.e., the outer blast wave) well in excess of what is expected from simple compression of the ambient circumstellar field, \( B_{\text{CSM}} \). These large fields are inferred from (1) spectral curvature in radio emission (e.g., Reynolds & Ellison 1992; Berezhko et al. 2002), (2) broadband fits of synchrotron emission between radio and nonthermal X-rays (e.g., Berezhko et al. 2003; Völk et al. 2005; see also Cowssik & Sarkar 1980), and (3) sharp X-ray edges (e.g., Vink & Laming 2003; Bamba et al. 2003; Völk et al. 2005; Ellison & Cassam-Chenaï 2005). While these methods are all indirect, fields greater than 500 \( \mu \)G are inferred in Cas A, and values of at least several 100 \( \mu \)G are estimated in the Tycho, Kepler, SN 1006, and G347.3-0.5 remnants. If \( B_{\text{CSM}} \sim 3-10 \mu \text{G} \), amplification factors of 100 or more may be required to explain the fields immediately behind the forward shocks, and this is likely the result of a nonlinear amplification process associated with the efficient acceleration of cosmic-ray (CR) ions via diffusive shock acceleration (DSA). The magnetic field strength is a critical parameter in DSA and also strongly influences the synchrotron emission from shock-accelerated electrons. Since shocks are expected to accelerate particles in diverse astrophysical environments and since synchrotron emission is often an important emission process (e.g., radio jets), quantifying the magnetic field amplification has become an important problem in particle astrophysics and has relevance beyond CR production in SNRs.

If CR production is as efficient as expected in theories of nonlinear DSA (e.g., Blandford & Eichler 1987; Jones & Ellison 1991; Malkov & Drury 2001), the CR pressure gradient in the shock precursor can do work on the incoming plasma and, in principle, place a large amount of energy in magnetic turbulence. In fact, if the DSA process is to work at all, magnetic turbulence must be self-generated on all resonant scale lengths to provide the scattering necessary to drive the particle distribution to isotropy. If the turbulence remains weak, i.e., \( \Delta B/B \ll 1 \), the self-generation of turbulence by CR streaming in the shock precursor can be adequately described using quasi-linear theory (e.g., Skilling 1975; McKenzie & Völk 1982). For large amplitude turbulence, however, this analytic description becomes questionable, and less rigorous approximations must be made.

In principle, a complete description of nonlinear DSA, including magnetic field amplification, is possible with particle-in-cell (PIC) simulations. In reality, however, these simulations are still far too computationally demanding to produce realistic models of astrophysical sources such as supernova remnants. Approximate methods describing nonlinear DSA must be used. Here, we have developed a method that incorporates a phenomenological model of particle diffusion and turbulence growth (similar to Bell & Lucek 2001; Amato & Blasi 2006) in a fully nonlinear Monte Carlo model of DSA (e.g., Jones & Ellison 1991; Ellison et al. 1996; Berezhko & Ellison 1999). While not a first-principles description of the plasma physics, the computational efficiency of our approach gives it a significant advantage over PIC simulations and semianalytic techniques based on the diffusion approximation in that we can determine the shock structure, the injection of thermal particles, the acceleration of these particles to relativistic energies, and the magnetic turbulence, all self-consistently in a fully nonlinear steady state model.

An important advantage of the Monte Carlo simulation is that, in effect, it solves a more basic set of equations governing the...
shock structure and DSA than do techniques based on diffusion-convection equations (e.g., Ellison & Eichler 1984). There is no assumption of isotropy for particle distributions, and this allows an internally self-consistent treatment of thermal particle injection. Injection is still phenomenological and depends on the assumptions made for the particle pitch-angle scattering, but these assumptions are applied equally to all particles. Since there is only one set of scattering assumptions, the Monte Carlo technique eliminates a free “injection parameter,” which is present in all models based on the diffusion approximation and is used to set the injection efficiency. As we show, the strong feedback between injection, shock structure, and magnetic field amplification makes this property of the Monte Carlo technique particularly important.

Our preliminary results show that magnetic field amplification causes an increase in the maximum CR momentum, \( p_{\text{max}} \), that a given shock system can produce compared to the case without amplification. For the case we consider, acceleration truncated by a finite-size shock, this increase is not, however, as large as the increase in \( B \) in the shocked region, since fields on scales of the upstream diffusion length of the highest energy particles strongly influence \( p_{\text{max}} \). Furthermore, the fact that \( B \) ranges smoothly from the unshocked value far upstream, \( B_0 \), to the amplified value, \( B_2 \), downstream from the shock will have consequences for electrons, since radiation losses will be greatest in the downstream region where particles spend a large fraction of their time. Thus, the maximum momentum protons and electrons obtain may be determined by two very different field strengths.

We also show that the amplification factor, \( B_2/B_0 \), increases with the Alfvén Mach number; that is, for a given shock speed and ambient density, a small \( B_0 \) will be amplified more than a larger \( B_0 \). The ability to dramatically increase very low ambient fields may make it possible for reverse shocks in young SNRs to accelerate particles and produce relativistic electrons and radio synchrotron emission (see Ellison et al. 2005b for a discussion of DSA at reverse shocks in SNRs). The injection and acceleration of electrons in amplified fields will be considered in subsequent work; here we consider only proton acceleration.

In this paper, we demonstrate the feasibility of applying Monte Carlo techniques to DSA with field amplification even though we are forced to make a number of approximations regarding the details of wave generation and to neglect wave damping. An important advantage of the Monte Carlo method is that it can be generalized to include more realistic descriptions of nonresonant wave growth, linear and nonlinear damping, and the calculation of the momentum and the space-dependent diffusion coefficient from the turbulent energy density; work along these lines is in progress. While a more accurate description of the plasma physics will influence all aspects of the acceleration process, we expect that \( p_{\text{max}} \) will be most strongly determined by the plasma physics details and thus will remain a critical problem for understanding the origin of Galactic cosmic rays for some time.

2. MODEL

2.1. Assumptions for Magnetic Turbulence Generation

Consider a steady state collisionless shock propagating along a uniform component \( B_0 \) of a stochastic magnetic field \( B \). We only consider parallel shocks where \( B_0 \) is along \( x \) and the shock face is perpendicular to \( B_0 \). As the unshocked plasma approaches the shock and experiences the pressure gradient in the CR precursor, the energetic particles backstreaming from the shock cause fluctuations of the field, \( \Delta B \), to grow. In the linear regime, \( \Delta B \) is perpendicular to \( B_0 \) and the local rate of growth of energy in waves is proportional to the particle pressure gradient. The plasma motion associated with these field fluctuations is initially Alfvénic (transverse and incompressible), and it will remain as long as \( \Delta B \ll B_0 \).

As the perturbations grow and reach \( \Delta B \gtrsim B_0 \), however, it is likely that waves with wavevectors \( k \) not aligned with \( B_0 \) will be generated, due to local CR pressure gradients along the total \( B = B_0 + \Delta B \). With \( \Delta B \gtrsim B_0 \), it becomes impossible to predict the average value of the transverse pressure gradients and the resulting magnetic field structure without knowing the relative phases of different wave harmonics. The problem is further complicated by the fact that this longitudinal, compressible turbulence may produce a strong second-order Fermi particle acceleration effect that, in turn, can damp the longitudinal fluctuations (see, for example, Schlickeiser et al. 1993).

These complications place a precise description of plasma turbulence beyond current analytic capabilities and lead to our goal of obtaining a realistic approximation that includes the essential nonlinear effects from efficient DSA but still allows particle acceleration over a large dynamic range. In order to accomplish this, we consider two limiting cases. The first assumes there is no longitudinal turbulence, in which case the wave growth rate is determined by the Alfvén speed in the unamplified field \( B_0 \). This gives a lower bound to the growth rate. The upper limit assumes that the turbulence is isotropic, in which case the growth rate is determined by the Alfvén speed in the much larger amplified field \( B_{\text{eff}} \) (defined below). The real situation should lie between these two cases, and while we consider these limits, we do not explicitly include second-order Fermi acceleration in our calculations.

To get an initial estimate of the shock structure, we use a Monte Carlo simulation of DSA, assuming that the motion of particles can be described as pitch-angle scattering in the Bohm limit. The Monte Carlo simulation provides the distribution of fast particles at all positions relative to the viscous subshock and allows for a calculation of the particle pressure gradient, \( \partial P_p(x, p)/\partial x \), which drives the amplification process in the shock precursor. Then, we use \( \partial P_p(x, p)/\partial x \) to calculate the total energy density in magnetic turbulence, \( U_{\text{tot}}(x) \), as a function of position across the shock.

This calculation uses a semiclassical expression for wave-dependent growth of magnetic turbulent energy density \( U(x, k) \), which is similar to the well-known equation for Alfvén wave growth (e.g., eq. [B.8] in McKenzie & Völk 1982). The complete expression is presented below as equations (11) and (12). The turbulence amplification term in them, which represents the streaming instability, is

\[
\frac{d}{dt} U(x, k)_{\text{stream}} = V_G \left[ \frac{\partial P_p(x, p)}{\partial x} \frac{dp}{dk} \right]_{p=\tilde{p}(k)}, \tag{11}
\]

where \( p \) is the particle momentum, \( U_{\text{tot}}(x) = \int U(x, k) dk \), \( \tilde{p}(k) \) is the momentum of particles resonant with wavevector \( k \), and much of the complicated plasma physics is contained in \( V_G \), an unknown growth-rate coefficient with dimensions of speed. Once the wave energy density is determined (as described below), it is

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1 We consistently use the subscript 0 for far upstream values, the subscript 2 for downstream values, and the subscript \( \text{sub} \) for values immediately upstream from the subshock. Thus, the overall compression ratio is \( \gamma_{\text{tot}} = u_2/u_1 \), and the subshock compression ratio is \( \gamma_{\text{sub}} = u_1/u_2 \), where \( u \) is the bulk flow speed.

2 It is important to note that while we speak of “thermal” and “superthermal” particles, and may refer to superthermal particles as cosmic rays, the Monte Carlo simulation makes no distinction between thermal and superthermal particles. The same assumptions are applied to particles regardless of their energy.
used to calculate an “effective” amplified magnetic field, \( B_{\text{eff}}(x) \equiv \sqrt{[4\pi U_{\text{tot}}(x)]^2} \), and this field is then used to calculate the particle diffusion coefficient, \( D(x,p) \), as a function of position and momentum.

For the growth of Alfvén waves in quasi-linear theory, \( V_G = v_a \), where \( v_a = B_0/[4\pi\rho(x)]^{1/2} \) is the Alfvén speed calculated with the unamplified field and \( \rho(x) \) is the matter density at position \( x \). As mentioned above, taking \( V_G = v_a \) ignores the transverse gradients of pressure and only the component of the pressure gradient along the initial magnetic field \( B_0 \) is accounted for. This choice of \( V_G \) provides a lower limit on the amplification rate and was used in Amato & Blasi (2006). If, on the contrary, we define the unamplified field and lie between the two extremes for gradient along the initial magnetic field \( B_0 \), i.e.,

\[
D_{\text{eff}} = \frac{4\pi U_{\text{tot}}}{B^2} \frac{\partial U}{\partial x} \frac{B_0}{\rho(x)} \frac{m_p}{e},
\]

as mentioned above, taking \( V_G = \frac{\rho(x)^2 \partial B}{\partial x} \), becomes a “smooth” function of \( x \), i.e., an upstream precursor with \( P_{\rho} \sim \rho u^2 \) forms. In addition to shock smoothing, the overall compression ratio of the shock, \( \rho_\text{tot} \), will increase when the acceleration is efficient (e.g., Eichler 1984; Jones & Ellison 1991), and \( \rho_\text{tot} \) is obtained by iteration as well. Below, we show how our code has been generalized to include the modification of the diffusion coefficient by the buildup of magnetic turbulence.

2.3. Magnetic Field Amplification

To calculate the effect of the pressure gradient of particles on magnetic field fluctuations, we start with an equation similar to equation (B.8) in McKenzie & Völk (1982):

\[
\frac{\partial}{\partial t} U_w + \frac{\partial}{\partial x} F_w = u \frac{\partial}{\partial x} P_w - v_u \frac{\partial P_u}{\partial x} - \bar{L},
\]

where \( U_w = (\Delta B)^2/4\pi \) is the energy density of the waves (assuming that the kinetic energy density of shear wave motion equals the magnetic energy density), \( F_w = (3u/2 - v_u)U_w \) is the wave energy flux (the 3/2 represents the sum of the Poynting flux and the flux associated with the transverse motion of plasma in Alfvén waves), \( P_w = U_w/2 \) is the magnetic pressure of waves acting on the plasma flow, \( \bar{L} \) represents wave energy losses (or gains) due to processes other than compression and amplification by the considered instability, and \( \partial P_w/\partial x \) is the pressure gradient of cosmic rays exciting the Alfvén waves. This equation describes the growth of the Alfvén waves through their instability in the presence of CR streaming, and it assumes that all the waves are moving upstream with respect to the plasma at speed \( v_u \), so \( v_u \). Now, following Bell & Lucek (2001), we separate the turbulence into downstream- and upstream-moving structures, and define the energy density of these structures per wavenumber interval as \( U_+(x,k) \) and \( U_-(x,k) \), respectively, so that the total energy density of turbulence is

\[
U_{\text{tot}} = \int_0^\infty [U_+(x,k) + U_-(x,k)] \, dk.
\]

We also define the partial pressure of particles with momentum \( p \) per unit momentum interval as \( P_p(x,p) \) so that the total pressure in particles, including thermal ones, is

\[
P_{p,\text{tot}} = \int_0^\infty P_p(x,p) \, dp.
\]

To derive equations for \( U_+(x,k) \) and \( U_-(x,k) \), we apply a steady state version of equation (3) to waves with wavenumber \( k \) in the interval \( \Delta k \). For the energy density of these waves in the first order of \( \Delta k \) we substitute

\[
U_w = |U_+(x,k) + U_-(x,k)| \Delta k.
\]

3 Parallel shocks are those where the shock normal is parallel to the magnetic field direction and oblique shocks are ones where the field makes some angle to the normal. The Monte Carlo simulation has been generalized for plane, oblique shocks, in which case \( B \) and \( D \) vary with \( x \) as the strength and angle the magnetic field makes with the shock normal vary (e.g., Ellison et al. 1996, 1999; Ellison & Double 2004).
For the energy flux, we write

\[ F_w = \left\{ \frac{3}{2} u(x) - V_G \right\} U_-(x, k) + \left\{ \frac{3}{2} u(x) + V_G \right\} U_+(x, k) \right\} \Delta k, \] (7)

and for the pressure of particles interacting with these waves,

\[ P_{\text{eff}} = P_{p}(x, p) \Delta p. \] (8)

Substituting these expressions into equation (3), ignoring the energy loss (gain) term \( \tilde{L} \), then dividing both sides by \( \Delta k \) and taking the limit \( \Delta k \to 0 \), we get

\[ \frac{\partial}{\partial x} \left\{ \frac{3}{2} u(x) - V_G \right\} U_-(x, k) + \frac{3}{2} u(x) + V_G \right\} U_+(x, k) \right\} = u(x) \frac{\partial}{\partial x} \left[ \frac{1}{2} U_-(x, k) + \frac{1}{2} U_+(x, k) \right] - v_{\text{wet}} \frac{\partial P_{\text{eff}}(x, p)}{\partial x} \frac{dp}{dk}, \] (9)

where we have replaced \( v_{\text{wet}} \) with a weighted wave speed

\[ v_{\text{wet}}(x, k) \equiv V_G \frac{U_+(x, k) - U_-(x, k)}{U_+(x, k) + U_-(x, k)}, \] (10)

in the driving term. Using \( v_{\text{wet}} \) is a logical extension of the definition of an “average” wave speed by Bell & Lucek (2001) applied to a narrow wavenumber range \( \Delta k \). This, again, is justified as long as \( \Delta B \ll B_0 \), but becomes less clear for strong turbulence. In these equations, \( V_G \) is defined as in equation (2).

The coefficient \( |dp/dk| \) that the driving term has acquired is necessary to relate the interval of wavenumbers, \( \Delta k \), of amplified waves to the interval of momenta of particles interacting with these waves, \( \Delta p \). As mentioned above, the relationship between \( p \) and \( k \) (i.e., the resonant condition) is assumed to be \( k = 1/r_{sg,0} \), where \( r_{sg,0} = cp/(eB_0) \), and \( B_0 \) is the far upstream magnetic field. One may argue that the field that the particles “feel” as uniform in the nonlinear regime, \( \Delta B \gg B_0 \), is the field carried by all long-wavelength (relative to the particle gyroradius) harmonics, rather than \( B_0 \). While this may be true in general, our present model is insensitive to the choice of the resonance condition due to the simplified “Bohm” form of the particle mean free path we assume.

Equation (9) expresses energy conservation and accounts for the amplification of turbulence by the streaming instability with the growth rate determined from kinetic theory. Furthermore, we assume, as did Bell & Lucek (2001) that interactions between the forward- and backward-moving waves drive them to isotropy on a timescale \( \sim r_{sg,0}/V_G \). In order to account for this interaction, we write equation (9) as the sum of the following two equations for \( U_\pm(x, k) \),

\[ \left[ u(x) - V_G \right] \frac{\partial}{\partial x} U_- + U_+ \frac{d}{dx} \left[ \frac{3}{2} u(x) - V_G \right] = \frac{U_-}{U_+ + U_-} V_G \frac{\partial P_{\text{eff}}(x, p)}{\partial x} \frac{dp}{dk} - \frac{V_G}{r_{sg,0}} (U_- - U_+), \] (11)

and

\[ \left[ u(x) + V_G \right] \frac{\partial}{\partial x} U_+ + U_- \frac{d}{dx} \left[ \frac{3}{2} u(x) + V_G \right] = \frac{U_+}{U_+ + U_-} V_G \frac{\partial P_{\text{eff}}(x, p)}{\partial x} \frac{dp}{dk} + \frac{V_G}{r_{sg,0}} (U_- - U_+), \] (12)

which are solved iteratively in the Monte Carlo simulation. We note that equations (11) and (12) are consistent with equation (9), with or without the relaxation terms on the right-hand sides of both equations, but these terms may become important in cases with small shock velocities.

Equations (11) and (12) are generalizations of those introduced by Bell & Lucek (2001). The generalization has two essential improvements. First, it accounts for the spatial dependence of flow speed \( u(x) \) due to nonlinear effects of efficient DSA and, consequently, treats “compression” of the amplified field adequately. To illustrate this effect, consider equations (11) and (12), neglecting the CR pressure gradient term and \( v_{\text{wet}} \), i.e., taking \( V_G \ll u \). Adding the two equations then results in

\[ u(x) \frac{\partial}{\partial x} (U_- + U_+) + \frac{3}{2} (U_- + U_+) \frac{d}{dx} u(x) = 0, \] (13)

which can be easily integrated to give \( U_- + U_+ \sim u^{-3/2} \) or \( B_{\text{eff}} \sim u^{-3/4} \). Consequently, in a shock with a total compression ratio \( r_{\text{tot}} \approx 10 \), for example, the stochastic magnetic field gets a boost in amplification by a factor of about \( r_{\text{tot}}^{-3/2} \approx 6 \) solely through the compression of the plasma. This compressional effect is especially important at the subshock, and the change in magnetic turbulence energy density across the subshock will influence the subshock compression ratio \( r_{\text{sub}} \). This in turn will have a strong effect on the injection efficiency in the Monte Carlo model.

Second, we generalized the equations to describe the whole spectrum of turbulence, \( U_\pm(x, k) \), rather than a single wave band with \( \Delta k = k \). We solve this system with a finite-difference method, integrating from far upstream \( (x \to -\infty) \) to \( x \). The quantities \( u(x) \) and \( P_{\text{eff}}(x, p) \) are obtained from the Monte Carlo simulation, as described in section 2.2. For simplicity in this initial presentation of our model, and because the shocks we are mainly concerned with have high Alfven Mach numbers \( v_{\text{wet}} \ll u \), we have neglected \( V_G \) compared to \( u \) in the first and second terms in equations (11) and (12) in the numerical results we present here.\(^4\)

We do, however, account for the wave or scattering center speed relative to the bulk flow speed in determining the energy change a particle receives as it scatters in the converging flow. In each interaction, we replace \( u(x) \to u(x) + v_{\text{wet}} \), and since \( v_{\text{wet}} \) is generally negative in the upstream region (i.e., \( U_- > U_+ \)), a finite \( v_{\text{wet}} \) dampens particle acceleration. Downstream from the shock, we take \( v_{\text{wet}} = 0 \).

The initial condition for our equations is the far upstream magnetic turbulence spectrum \( U_\pm(x \to -\infty, k) \). We take

\[ U_-(x \to -\infty, k) = U_+(x \to -\infty, k) = \begin{cases} A(k/k_c)^{-\alpha}, & k_c < k < k_m, \\ 0, & k < k_c \text{ or } k > k_m, \end{cases} \] (14)

\(^4\) In all of the examples in this paper, \( V_G < 0.2u(x) \) in the shock precursor and \( V_G < 0.5u_2 \) in the postshock region.
where the limits \( k_m \) and \( k_e \) are chosen to encompass the range between the inverse gyroradii of thermal particles and the most energetic particles in the shock, respectively. For concreteness we take \( \alpha = 1 \), but none of our results depend in any substantial way on \( \alpha, k_m, \) or \( k_e \). Using our definition of the effective, amplified magnetic field,

\[
B_{\text{eff}}^2(x) = 4\pi \int_0^\infty [U_-(x,k) + U_+(x,k)] \, dk,
\]

(15)

the normalization constant \( A \) in equation (14) is determined by requiring that

\[
4\pi \int_0^\infty [U_-(-\infty,k) + U_+(-\infty,k)] \, dk = B_0^2.
\]

(16)

2.4. Particle Scattering

Once the amplified magnetic field is determined from equation (15), the scattering mean free path is set equal to the gyro-radius of the particle in this field, i.e.,

\[
\lambda(x,p) = \frac{cp}{eB_{\text{eff}}(x)}.
\]

(17)

Equation (17) is essentially the Bohm limit and, as such, is a crude approximation. In a strong collisionless shock, modified by efficient DSA, a significant fraction of the energy is contained in high-momentum particles. These particles have long mean free paths and will resonantly produce turbulence where long-wavelength harmonics contain most of the wave energy. Consequently, to low-momentum particles, the strong long-wavelength turbulence appears approximately as a uniform field and equation (17) is justified. For the highest energy particles, however, equation (17) will overestimate the scattering strength, since for these particles most of the harmonics of the magnetic field appear as short-scale fluctuations that are not very efficient at changing the particle’s momentum. In this case, there is no reason to assume that the scattering is resonant, i.e., there is no simple resonance relation between \( k \) and \( p \). This is a critical point, since the form for \( \lambda(x,p) \) at high \( p \) determines the maximum momentum that can be produced in a given shock system, and this is one of the important unsolved problems for DSA.

Clearly, more physically realistic models for both the diffusion coefficient and the resonance condition are required for future work. An approach for determining the mean free path of particles in strongly turbulent fields is described in Bykov & Toptygin (1992), where nonresonant scattering and diffusive transport of particles in large-scale fluctuations are taken into account. The model presented here, where the calculation of the power spectrum of turbulence \( U_\lambda(x,k) \) is coupled to the nonlinear shock structure, will be generalized to include more physically realistic wave-particle interactions in future work.

2.5. Momentum and Energy Conservation

The total energy flux in turbulence is

\[
F_{w,\text{tot}}(x) = \int_0^\infty \left[ \left( \frac{3}{2} u - V_G \right) U_-(x,k) + \left( \frac{3}{2} u + V_G \right) U_+(x,k) \right] \, dk,
\]

(19)

or

\[
F_{w,\text{tot}}(x) \approx \frac{3}{2} u(x) U_{\text{tot}}(x),
\]

(20)

in the limit \( V_G(x) \ll u(x) \).

In order to determine the bulk velocity profile, \( u(x) \), consistent with the back-reaction of accelerated particles and turbulence on the flow, the Monte Carlo simulation solves the following equations expressing the conservation of mass and momentum fluxes,

\[
\rho(x) u(x) = \rho_0 u_0,
\]

(21)

\[
\rho(x) u(x)^2 + P_{p,\text{tot}}(x) + P_{w,\text{tot}}(x) = \rho_0 u_0^2 + P_\rho + P_w \equiv P_0,
\]

(22)

where \( P_{p,\text{tot}}(x) \) is the pressure produced by all particles, thermal and superthermal, and \( P_0 \) is the far upstream momentum flux. The particle pressure is related to \( P_p(x,p) \) by equation (5) and is calculated in the simulation directly from the trajectories of individual particles. The energy flux conservation relation is

\[
\frac{\rho(x) u(x)^3}{2} + F_{p,\text{tot}}(x) + F_{w,\text{tot}}(x) + q_{\text{esc}} = \frac{\rho_0 u_0^3}{2} + F_{\rho} + F_w \equiv F_0,
\]

(23)

where \( F_{p,\text{tot}}(x) \) is the energy flux in all particles, \( q_{\text{esc}} \) is the energy flux lost by particles leaving the system at the upstream free escape boundary (FEB), and \( F_0 \) is the far upstream energy flux. In parallel shocks, \( u(x) \) can be determined from equations (21) and (22) alone. The relation between \( r_{\text{tot}} \) and \( d_{\text{esc}} \) (i.e., eq. [10] in Ellison et al. 1990b) allows equation (23) to be used to check the consistency of the simulation results, as we show with the examples below.

3. RESULTS

In all of the following examples, we set the shock speed \( u_0 = 5000 \text{ km s}^{-1} \), the unshocked proton number density \( n_p = 1 \text{ cm}^{-3} \), and the unshocked proton temperature \( T_0 = 10^6 \text{ K} \). For simplicity, the electron temperature is set to zero and the electron contribution to the jump conditions is ignored. With these parameters, the sonic Mach number \( M_s \simeq 43 \) and the Alfvén Mach number \( M_{\text{Alf}} \simeq 2300(1 \mu \text{G}/B_0) \).

3.1. With and Without Magnetic Field Amplification

In order to obtain a solution that conserves momentum and energy fluxes, both the shock structure, i.e., \( u(x) \), and the overall compression ratio, \( r_{\text{tot}} \), must be obtained self-consistently. Figure 1 shows results for four shocks where \( u(x) \) and \( r_{\text{tot}} \) have been determined with the iterative method described above. In all examples in this section, \( \alpha_{\text{Alf}} = 0 \). First, we compare the results
amplification (heavy solid curves), $r_{\text{tot}} \simeq 11$. This difference in overall compression results from the wave pressure $P_{w,\text{tot}}$ in equation (22) being much larger in the field amplified case, making the plasma less compressible.

Two effects cause $r_{\text{tot}}$ to increase above the test-particle limit of 4 for strong shocks. The first is the production of relativistic particles, which produce less pressure for a given energy density than nonrelativistic particles, making the plasma more compressible. The second, and most important, is the escape of energetic particles at the FEB. As indicated in the energy flux panels of Figure 1, the energy flux drops abruptly at $x \sim d_{\text{FEB}}$ as energetic particles diffuse past the FEB and escape the system. The energy lost from the escaping particles is analogous to radiation losses in radiative shocks and results in an increase in compression ratio. For high Mach number parallel shocks, $1/r_{\text{tot}} \sim (5 - (9 + 16q_{\text{esc}}/F_0)^{1/2})/8$ (Ellison et al. 1990b), so for the dotted curves $r_{\text{tot}} = 22$ and $q_{\text{esc}} \simeq 0.8F_0$, and for the heavy solid curves $r_{\text{tot}} = 11$ and $q_{\text{esc}} \simeq 0.5F_0$, consistent with the energy fluxes shown in Figure 1.

The momentum flux is also conserved, but the escaping momentum flux is a much smaller fraction of the upstream value than for energy (Ellison 1985). Any departures from the far upstream momentum flux seen in Figure 1 are less than the statistical uncertainties in the simulation. As mentioned above, we only include adiabatic heating in the upstream precursor. If wave damping and heating were included, it is expected that $r_{\text{tot}}$ would be reduced from what we see with wave amplification alone.

For a given shock, an increase in $r_{\text{tot}}$ must be accompanied by a decrease in the subshock compression ratio, $r_{\text{sub}}$ (see, e.g., Berezhko & Ellison 1999 for a discussion of this effect). A large $r_{\text{sub}}$ means that high-energy particles with long diffusion lengths get accelerated very efficiently and, therefore, the fraction of particles injected must decrease accordingly to conserve energy. The shock structure adjusts so that weakened injection (i.e., a small $r_{\text{sub}}$) just balances the more efficient acceleration produced by a large $r_{\text{tot}}$. Since $r_{\text{sub}}$ largely determines the plasma heating, the more efficiently a shock accelerates particles, causing $r_{\text{tot}}$ to increase, the less efficiently the plasma is heated. For the cases shown in Figure 1, $r_{\text{sub}} \simeq 3.8$ for the amplified field case (heavy solid curves) and $r_{\text{sub}} \simeq 2.4$ for the case with no $B$-field amplification (heavy dotted curves).

In Figure 2 we show the phase-space distributions, $f(p)$, for the shocks shown in Figure 1. For the two cases with the same parameters except field amplification, we note that the amplified field case (heavy solid curve) obtains a higher $p_{\text{max}}$ and has a higher shocked temperature (indicated by the shift of the “thermal” peak and caused by the larger $r_{\text{sub}}$) than the case with no field amplification (heavy dotted curves). It is significant that the increase in $p_{\text{max}}$ is modest even though $B$ increases by more than a factor or 30 with field amplification. We emphasize that $p_{\text{max}}$ as such is not a parameter in this model; $p_{\text{max}}$ is determined self-consistently once the size of the shock system, i.e., $d_{\text{FEB}}$, and the other environmental parameters are set.

In order to show the effect of changing $d_{\text{FEB}}$, we include in Figures 1 and 2 field amplification shocks with the same parameters, except that $d_{\text{FEB}}$ is changed to $-100r_{\text{sub}}(u_0)$ (dashed curves) and $-10^2r_{\text{sub}}(u_0)$ (light solid curves). From Figure 2, it is clear that $p_{\text{max}}$ scales approximately as $d_{\text{FEB}}$ and that the concave nature of $f(p)$ is more pronounced for larger $p_{\text{max}}$. The field...

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5 See Berezhko & Ellison (1999) for a discussion of how very large $r_{\text{sub}}$-values can result in high Mach number shocks if only adiabatic heating is included in the precursor. The uncertainty on the compression ratios for the examples in this paper is typically $\pm 10\%$. 

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Fig. 1.—Shock structure including momentum and energy fluxes (in units of far upstream values) and the effective magnetic field vs. $x$. Note that the horizontal scale has units of $r_{\text{sub}}(u_0) = m_\text{p}u_0/eB_0$ and is divided at $x = -5r_{\text{sub}}(u_0)$ between a linear and log scale. In all panels, the heavy dotted curves show results without amplification, and all other curves are with amplification. The curves showing the energy flux drop sharply at the upstream FEB, which is at $-10^2r_{\text{sub}}(u_0)$ for the heavy solid and dotted curves, at $-1000r_{\text{sub}}(u_0)$ for the dashed curves, and at $-10^3r_{\text{sub}}(u_0)$ for the light solid curves, as particles freely leave the system. When this escaping flux is included, energy and momentum are conserved to within $\pm 10\%$.

The solution without $B$-field amplification (Fig. 1, dotted curves) has a considerably larger $r_{\text{tot}}$ than the one with amplification, i.e., for no $B$-field amplification, $r_{\text{tot}} \simeq 22$, and with $B$-field

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amplification also increases with $p_{\text{max}}$, but the increase between the $d_{\text{FEB}} = -1000r_g(u_0)$ and $d_{\text{FEB}} = -10^5r_g(u_0)$ cases is less than a factor of 2 (bottom panels of Fig. 1).

In Figure 3 we show the energy density in magnetic turbulence, $U_+(x,k) + U_-(x,k)$; the diffusion coefficient, $D(x,p)$; and particle distributions as functions of $k$ and $p$ at three different positions in the shock. All of these plots are for the example shown with dashed curves in Figures 1 and 2. The bottom panel shows how $f(x,p)$ varies in the precursor, where particles must diffuse upstream against the incoming plasma. The upstream diffusion length, $[D(x,p)u(x)]_{\text{avg}}$, is some weighted average that is determined directly in the Monte Carlo simulation. The diffusion coefficients, shown in the middle panel, are determined from $B_{\text{eff}}$ (eq. [15]), where the $U(x,k)$-values (top panel) are determined by solving equations (11) and (12) self-consistently with the shock structure, using the particle pressure gradients determined from $f(x,p)$. The decrease in $D$ by $\sim 40$ between far upstream and downstream from the shock corresponds to the increase in $B_{\text{eff}}$ shown with the dashed curve in the bottom panel of Figure 1. The top panel clearly shows the spread in $k$ where resonant interactions produce wave growth at the different $x$-positions, including the portion from thermal particles at high $k$.

The efficiency of the shock acceleration process is shown in Figure 4. The curves on the left give the number density of particles with momentum greater than $p$, i.e., $N(>p)$, and the curves on the right give the energy density in particles with momentum greater than $p$, i.e., $E(>p)$. Both sets of curves are determined solely from the downstream particle distributions (calculated in the shock reference frame) shown in Figures 1 and 2 with heavy solid and dotted curves. Thus they ignore escaping particles and the energy in magnetic turbulence. Nevertheless, these curves indicate that the shocks are extremely efficient accelerators that have $>50\%$ of the energy density in $f(p)$ placed in relativistic particles (i.e., $p \geq m_p c$). The actual energy efficiencies are considerably higher since the escaping particles carry away a larger fraction of the total energy than is placed in magnetic turbulence. With $q_{\text{esc}}$ included, well over $50\%$ of the total shock energy is placed in relativistic particles. Despite this high energy efficiency, the fraction of particles that become relativistic is small, i.e., $N(>p = m_p c) \sim 10^{-5}$ in both cases.

The effect of magnetic field amplification on the number of particles injected is evident in the left-hand curves. The larger $r_{\text{sub}}$ (Fig. 4, solid curve) results in more downstream particles being injected into the Fermi mechanism with amplification than without. While it is hard to see from Figure 4, when the escaping energy flux is included, the shock with $B$-field amplification puts a considerably smaller fraction of energy in relativistic particles than the shock without amplification. Again, injection depends in a nonlinear fashion on the shock parameters and the subshock strength will adjust to ensure that just the right amount of injection occurs so that momentum and energy are conserved.
3.2. Alfvén Mach Number Dependence

In Figure 5, we show three examples where $B_0$ has been varied from 0.3 to 30 $\mu$G; all other input parameters are kept constant including the physical distance to the FEB and $f_{Alf}=0$. For these examples, $|d_{FEB}| = 1.7 \times 10^{10}$ m = $5.6 \times 10^{-7}$ pc. In units of $r_g(u_0) = m_p u_0 / (e B_0)$, the units used for the x-coordinates in Figures 1 and 5, this corresponds to $|d_{FEB}| = 1000$, 100, and 10 $r_g(u_0)$, for $B_0 = 0.3$, 3, and 30 $\mu$G, respectively. The top four panels, showing $u(x)/u_0$ and energy flux, have the same format as the corresponding panels in Figure 1. As in Figure 1, $q_{esc}$ is significant and $r_{tot} > 7$ in all cases. The magnetic field panels differ from Figure 1 in that here we plot $B(x)/B_0$ and it is clear that the amplification of $B$ is greatest for the lowest $B_0$, i.e., $B(x)/B_0$ increases with increasing $M_{Alf}$. For the examples shown here, $B_2/B_0 \simeq 400$ for $B_0 = 0.3$ $\mu$G, $B_2/B_0 \simeq 150$ for $B_0 = 3$ $\mu$G, and $B_2/B_0 \simeq 30$ for $B_0 = 30$ $\mu$G. In the bottom panels, we show the pressure in magnetic turbulence, $P_{w,tot}$, divided by the total far upstream momentum flux, $P_0$. For these examples, $P_{w,tot}/P_0 < 0.1$, and the magnetic pressure stays well below equipartition with the gas pressure.

In Figure 6a we show the distribution functions corresponding to the shocks shown in Figure 5. As expected, the shock with the highest $B_0 = 30$ $\mu$G yields the highest $p_{max}$. This $p_{max}$, however, is only about a factor of 5 greater than that for $B_0 = 0.3$ $\mu$G, obtained in a shock system of the same physical size. In Figure 6b we show the distribution functions for three cases for which we have kept the upstream FEB at the same number of gyroradii, i.e., $d_{FEB} = -100 r_g(u_0)$. In this case, the situation with $p_{max}$ is reversed, with the $B_0 = 0.3$ $\mu$G shock obtaining the highest $p_{max}$.

This is a combination of the fact that the physical size of the shock system is largest and that the field amplification is greatest for $B_0 = 0.3$ $\mu$G. The shock structure results are not shown but are similar to those in Figure 5.

3.3. Wave Amplification Factor, $f_{Alf}$

All of the examples shown so far have used the minimum amplification factor $f_{Alf} = 0$ (eq. [2]). We now investigate the effects of varying $f_{Alf}$ between 0 and 1, so that $V_G$ varies between $u_g(x)$ and $B_{had}(x)[4\pi \rho(x)]^{1/2}$. The other shock parameters are the same as used for the dashed curves in Figure 1, i.e., $u_0 = 5000$ km s$^{-1}$, $B_0 = 30$ $\mu$G, and $d_{FEB} = -1000 r_g(u_0)$.

Figure 7 shows $u(x)/u_0$ and $B_{had}(x)/B_0$ for $f_{Alf} = 0, 0.1, 0.5$, and 1, as indicated. The top panels show that increasing the growth rate (increasing $f_{Alf}$ and therefore $V_G$) produces a large change in the shock structure and causes the overall shock compression ratio, $r_{tot}$, to decrease. The decrease in $r_{tot}$ signifies a decrease in the acceleration efficiency and a decrease in the fraction of energy
that escapes at the FEB, and the subshock compression adjusts to ensure conservation of momentum and energy. The values for $r_{\text{tot}}$ and $r_{\text{sub}}$ are given in Table 1, and it is interesting to note that $r_{\text{sub}}$ increases as $r_{\text{tot}}$ decreases and becomes greater than 4 for $f_{\text{Alf}} \geq 0.5$. In contrast to the strong modification of $u(x)$, there is little difference in $B_{\text{eff}}(x)/B_0$ (bottom panels of Fig. 7) and little change in $p_{\text{max}}$ (Fig. 8), between these examples.

The fact that increasing the wave growth rate decreases the acceleration efficiency shows the nonlinear nature of the wave generation process and comes about for two reasons. The most important is that the magnetic pressure term in equation (22), $P_{\text{tot}}/{C_0^2}$, becomes significant compared to $\psi(m_p c)^{\text{VMax}}$ are listed in Table 1. As we have emphasized, the injection efficiency, i.e., the fraction of particles that enter the Fermi process, must adjust to conserve momentum and energy, and the low-momentum peaks shift as a result of this. The filled circles in Figure 8 roughly indicate the injection point separating “thermal” and “superthermal” particles for the two extreme cases of $f_{\text{Alf}} = 0$ and 1. What is illustrated by this is that the injection point shifts, relative to the postshock distribution, when $f_{\text{Alf}}$ is varied. This implies that, if injection is parameterized,

We note that the smooth emergence of a superthermal tail has been seen in spacecraft observations of the quasi-parallel Earth bow shock (i.e., Ellison et al. 1990b) and at interplanetary traveling shocks (i.e., Baring et al. 1997).

| $f_{\text{Alf}}$ | $r_{\text{tot}}$ | $r_{\text{sub}}$ | $B_2/B_0$ | $\psi(m_p c)^{\text{VMax}}$ |
|----------------|----------------|----------------|-----------|------------------|
| 0.0             | 9              | 3.5            | 30        | $3.8 \times 10^{-3}$ |
| 0.1             | 8              | 3.7            | 40        | $4.4 \times 10^{-3}$ |
| 0.5             | 6              | 4.1            | 50        | $6.3 \times 10^{-3}$ |
| 1.0             | 5              | 4.3            | 40        | $8.0 \times 10^{-3}$ |

Note.—The errors on $r_{\text{tot}}$ and $r_{\text{sub}}$ are typically ±10%.
the parameterization must somehow be connected to modifications in the shock structure.

4. SUMMARY AND COMPARISON WITH ALTERNATIVE MODELS

We have generalized a steady state Monte Carlo model of efficient DSA to include magnetic wave growth, allowing the wave amplitude to become large compared to the ambient field, i.e., $\Delta B/B_0 \gg 1$. The model uses a phenomenological treatment of wave generation by applying the linear growth rate formalism in the nonlinear regime, but couples the nonlinear shock structure, the injection rate of thermal particles, the magnetic field amplification, and the determination of the maximum momentum particles obtained from a physical constraint, i.e., the size of the shock system, in an internally self-consistent manner.

Important limitations of our model are the simplified treatment of wave growth and the neglect of wave damping. We also assume a Bohm-like expression for the scattering mean free path [i.e., $\lambda(x,p) = p/(eB_{\text{eff}})]$ rather than calculating $D(x,p)$ from more fundamental relations that account for particle motions in strongly anisotropic turbulence. More physically realistic models for turbulence generation and damping and particle scattering are certainly necessary and may well modify the results we present here. However, it is not straightforward to include these processes in nonlinear models and, to our knowledge, no models have yet been presented with as extensive nonlinear coupling as we calculate. We believe the approximations we make are an important intermediate step and that the results we present are indicative of what more complete models will show. The fact that the Monte Carlo technique can handle anisotropic particle distributions will be essential for including more precise plasma physics in future generalizations.

4.1. Particle-in-Cell Simulations

The difference between our Monte Carlo simulation and a PIC plasma simulation is that the pitch-angle diffusion is treated phenomenologically in the Monte Carlo simulation with an assumed scattering mean free path, whereas the PIC plasma simulation calculates particle trajectories in the self-generated magnetic field obtained directly from Maxwell’s equations. This approximation is essential in order to make the Monte Carlo technique fast enough to model acceleration over a wide dynamic range. We emphasize that while PIC simulations, in principle, can solve the shock acceleration problem completely, their computational requirements are extreme and they are not yet capable of modeling acceleration over a dynamic range large enough to model cosmic sources such as SNRs. The most important constraint on PIC simulations is that they must be done fully in three dimensions (3D). As shown by Jokipii et al. (1993) and Jones et al. (1998), PIC simulations with one or more ignorable dimensions artificially confine particles to field lines and eliminate cross-field diffusion, an essential ingredient in DSA, particularly in oblique shocks. The 3D requirement means that all three box dimensions must be increased to accommodate high-energy particles with long diffusion length scales, so computing requirements become insurmountable for a large enough dynamic range. To model SNR shocks, they must be able to accelerate particles from thermal to highly relativistic energies and, if electrons are to be modeled, they must simultaneously include electron and proton steps. While PIC simulations will be able to investigate important problems, particularly those concerning injection, they will not be able to model the shock acceleration of electrons and protons to the energies necessary to produce broadband radiation with parameters similar to those of SNRs in the foreseeable future. Until then, progress can be made with approximate methods.

4.2. Semianalytic Models with B-Field Amplification

Besides the limited results from PIC simulations, the only models of magnetic field generation with $\Delta B \gg B_0$ in nonlinear DSA that we are aware of are the semianalytic results of Bell & Lucek (2001), Bell (2004, 2005), Ptuskin & Zirakashvili (2003), and Amato & Blasi (2006). As we have mentioned, our work uses the same basic wave generation formalism as Bell & Lucek (2001) with an extensive generalization to include injection, nonlinear shock structure and particle distributions, and $p_{\text{max}}$.

More recently, Bell (2004, 2005) has attempted to improve the plasma physics by calculating amplified fields from a so-called directly driven mode of wave instability. In this scheme, upstream energetic particles standing in the shock frame produce a macroscopic current, and magnetic field, in the upstream plasma frame. This extended the Bell & Lucek (2001) analysis to wave modes other than resonant Alfvénic ones and emphasized that turbulence is likely the result of strongly driven, nonresonant modes at shorter wavelengths rather than Alfvén waves. A recent application of this idea to relativistic shocks expected in GRBs is given Milosavljevic & Nakar (2006). While this is a promising idea for a nonresonant mechanism for wave amplification in strong

7 We note that the computational requirements for the relativistic shocks expected in $\gamma$-ray bursts may actually be less stringent than those for the non-relativistic shocks in SNRs. In shocks with large Lorentz factors, particles start offrelativistic (in the shock frame) and can gain a great deal of energy in just a few shock interactions. It is also possible that electron-positron plasmas dominate the $\gamma$-ray burst fireball, so that important results can be obtained without simultaneously covering electron-proton scales. In the nonrelativistic shocks present in SNRs, however, both electrons and protons must be accelerated over a wide dynamic range from eV to TeV energies by crossing the shock many times.

8 Lucek & Bell (2000) and Bell (2004, 2005) also performed PIC simulations coupled to a 3D MHD model of the background plasma. These results clearly showed that seed $B$ fields can be amplified by orders of magnitude, but they were limited in dynamic range and did not self-consistently model the particle acceleration process.
shocks, careful consideration of the plasma return current must be made (e.g., F. C. Jones 2006, private communication).

Another nonresonant mechanism for long-wavelength magnetic field fluctuation amplification in the vicinity of shock waves was studied by Bykov & Toptygin (2005). They demonstrated that a small fraction of neutral atoms can reduce the plasma transverse conductivity and result in a CR current instability. This instability was shown to produce long-wavelength magnetic fluctuations.

An important attempt at describing turbulence with $\Delta B/B > 1$ was made by Ptuskin & Zirakashvili (2003, 2005). These authors assumed a Kolmogorov-type nonlinear cascade and damping of self-generated turbulence by ion-neutral collisions. Most importantly, they obtained estimates for $p_{\text{max}}$ in a time-dependent analytic calculation. Nonlinear particle acceleration was assumed, but the analytic method required a number of approximations including the spatial distribution of energetic particles in the shock precursor and the spectral form of the energetic particles. Nevertheless, this work showed that nonlinear particle acceleration, combined with $B$-field amplification, may strongly influence $p_{\text{max}},$ one of the most important parameters in DSA.

In our estimation, the most highly developed model of non-linear DSA with magnetic field amplification is that of Amato & Blasi (2006); work based on a series of papers by Blasi and coworkers (i.e., Blasi 2002, 2004; Blasi et al. 2005; Amato & Blasi 2005).9 We now give a detailed comparison between that model and ours. The underlying assumptions for DSA are the same in both models, i.e., particles are driven to isotropy by interactions with magnetic turbulence in the background plasma. Particles gain energy when they diffuse in the converging flow, and neither model includes second-order Fermi acceleration. Both models are also for plane-parallel, steady state shocks.

The most important difference revolves around the method of solution. We use a Monte Carlo technique, while the works of Blasi et al. solve a transport, or diffusion-convection, equation. In the Blasi et al. work, macroscopic quantities (pressure, energy flux, etc.) are derived as moments of the particle distribution function $f(x,p),$ which is assumed to be isotropic in the shock reference frame. In contrast, the Monte Carlo simulation traces the stochastic motion of individual particles as they pitch-angle scatter off the background turbulence and calculates $f(x,p)$ and its moments directly from the particle trajectories, without making any assumptions about the isotropy of $f(x,p).$ In effect, the Monte Carlo simulation solves the more fundamental Boltzmann equation (e.g., Ellison & Eichler 1984). Of course, the semianalytic technique is much faster computationally than the Monte Carlo technique and this will be important for building models in many applications.

Injection can be treated self-consistently in the Monte Carlo simulation because once the pitch-angle scattering assumptions are made, they are applied equally to all particles, and the number and energy of injected particles is fully determined. No distinction is made between thermal and superthermal particles, and the viscous subshock is assumed to be transparent, so there is a nonzero probability for any downstream particle with $v > v_{\text{t2}}$ to be injected. The diffusion approximation, on the other hand, with its assumption of isotropy, forces additional assumptions if injection is to be modeled. Blasi et al. treat the subshock as having a finite thickness comparable to a thermal particle’s gyroradius, and the injection rate is parameterized such that only those particles that have a gyroradius large enough to span the subshock get injected. While this adds an additional parameter independent of the diffusion properties, it has been shown that parameters can easily be determined so that the Monte Carlo and semianalytic models give similar results (see Ellison et al. 2005a).

Another important difference is that Amato & Blasi (2006) have included a phenomenological description of turbulent $B$-field heating, similar in implementation to that used in Berezhko & Ellison (1999), in addition to adiabatic heating. We only include adiabatic heating. The amplified turbulence may be dissipated through collisional and/or collisionless mechanisms and these include (1) linear and nonlinear Landau damping (e.g., Akhiezer et al. 1975; Achterberg & Blandford 1986; Kulots 1978; Vainshtein et al. 1993; Zirakashvili 2000), (2) particle trapping (e.g., Medvedev 1999), and (3) ion-neutral wave damping (e.g., Drury et al. 1996; Bykov & Toptygin 2005). It is important to note that even though the damping rates tend to be smaller than the wave growth rates for long-wavelength fluctuations in turbulent plasmas with a substantial nonthermal component (e.g., Bykov & Toptygin 2005), the generation of magnetic energy density can be approximately balanced by the conversion of that turbulence through the shock (as was assumed here) and MHD cascade processes. The heating of the precursor plasma by dissipation of small scale fluctuations modifies the subshock Mach number (e.g., Ellison et al. 2000), and this in turn modifies injection. The overall acceleration efficiency and, of particular importance for X-ray observations, the temperature of the shocked plasma (e.g., Decourchelle et al. 2000; Hughes et al. 2000; Ellison et al. 2004) will depend on wave dissipation. We plan to include physically realistic models of wave damping in the Monte Carlo simulation in future work.

The two most important elements in these nonlinear models is the turbulence generation and the diffusion coefficient that is derived from it. The assumptions for wave generation by the streaming instability are essentially identical in the Blasi et al. model and ours.10 The most important difference in the models lies in the calculation of the mean free path and diffusion coefficient. Here, we assume Bohm-like diffusion with $B_{\text{eff}}$ (eq. [17]), while Amato & Blasi (2006) use a mean free path due to resonant scattering that, in our notation, would be

$$\lambda_{\text{res}}(x,p) = \frac{c p}{e B_0} \frac{1}{2 \pi^2 k_{\text{res}}(x,k_{\text{res}}) + U_+(x,k_{\text{res}})} \beta^2,$$

where

$$k_{\text{res}} = \frac{1}{r_{p,0}} = \frac{e B_0}{c p}$$

is the resonant wavenumber. Both of these assumptions are approximations. Our Bohm-like expression is just a phenomenological recipe for particle diffusion in strong turbulence, while $\lambda_{\text{res}}$ is only formally valid for weak turbulence but is applied to diffusion in strong, anisotropic turbulence. An important step in advancing the state-of-the-art of nonlinear shock acceleration must center on improving the connection from wave generation to diffusion.

That important aspects of DSA are sensitive to the assumptions made for diffusion can be seen by comparing the self-generated $D(x,p)$ from these two models. Our use of equation (17) forces Bohm-like diffusion with $D(x,p) \propto v p,$ and we show this dependence in Figure 3. Nevertheless, we have argued in § 2.4 that equation (17) overestimates the scattering strength for the highest momentum particles because most of the harmonics appear as short-scale fluctuations to these particles. This would

9 We note that Amato & Blasi (2005, 2006) use a different technique from the previous Blasi et al. work; the new technique allows an exact solution for an arbitrary choice of both the spatial and momentum dependent diffusion coefficient.

10 There is a minor difference in that Amato & Blasi (2006) assume that all waves generated by the streaming instability move upstream, whereas we consider the interaction between waves moving upstream and downstream.
suggest that the actual $p$ dependence of $D(x, p)$ increases with $p$, likely as $D(x, p) \propto p^2$ at the highest energies. In Amato & Blasi (2006), on the other hand, the momentum dependence of $D(x, p)$ is shown to weaken at high $p$ to the point where the scattering becomes strong enough that $D(x, p)$ becomes independent of $p$ in high sonic Mach number shocks.

It is worth noting that the Amato & Blasi (2006) result indicates that the magnetic turbulence spectrum has a slope approaching $\propto k^{-2}$. For this turbulence spectral slope, the “quasi-linear model” predicts energy-independent diffusion (in the weak fluctuation limit, of course). Interestingly enough, the magnetic fluctuation spectrum we obtain with Bohm diffusion has, over a wide wavelength range, a spectral slope roughly corresponding to $\propto k^{-2}$ (see Fig. 3). So, if one formally calculated the quasi-linear diffusion from equation (24) with our fluctuation spectrum, the resulting diffusion coefficient would have only a weak energy dependence. This could mean that the magnetic fluctuation spectral shape is not very sensitive to the choice of diffusion model in the nonlinear calculations. Nevertheless, in this preliminary work, we prefer to use the Bohm diffusion model in the strong fluctuation limit. Since the behavior of $D(x, p)$ at high $p$ determines $p_{\text{max}}$ as well as influencing all other aspects of the shock acceleration process, it is clear that much work remains to be done on this difficult problem.

4.3. Other Models of Nonlinear Shock Acceleration without $B$-Field Amplification

There are a number of nonlinear models of DSA besides those discussed above. However, none of these models include $B$-field amplification.

In a series of papers, Berezhko and coworkers have applied a model of nonlinear DSA to broadband observations of several young SNRs (e.g., Berezhko et al. 1996, 2002). They used a time-dependent solution of the CR transport equation coupled to the gas dynamic equations in spherical symmetry and calculated the superthermal proton distribution, from the forward shock, at all positions in the remnant. From this they can determine the overall contribution a single SNR makes to the Galactic CR proton flux. Furthermore, since they included the acceleration of superthermal electrons, they calculated the photon emission from synchrotron and inverse Compton (IC) processes, as well as from proton-proton interactions and pion decay. This allowed them to fit the broadband photon observations and constrain the model parameters. Of particular importance is their emphasis that magnetic fields much greater than typical ISM values are required to match broadband photon observations. Equally important is their prediction, based on their fits, that TeV photon emission from several SNRs is most likely the result of pion decay from protons rather than IC from electrons (e.g., Berezhko et al. 2003).

This model has been extremely successful and has added to our understanding of SNRs, but it does make important approximations and simplifications. The model assumes Bohm diffusion, parameterizes the number of injected particles, and does not include wave amplification.11

Other nonlinear shock models include those of Kang and Jones and coworkers (e.g., Kang et al. 2002; Kang & Jones 2006), Malkov and coworkers (e.g., Malkov 1997; Malkov & Drury 2001; Malkov et al. 2002; Malkov & Diamond 2006), and the cosmic ray–hydrodynamical (CR-hydro) model of Ellison and coworkers (e.g., Ellison et al. 2004). These models all involve different computational techniques, and all have their particular strengths and weaknesses. They have also been shown to produce similar nonlinear effects to those of Berezhko et al., and have been shown to be in quantitative agreement with our Monte Carlo model (before the addition of $B$-amplification). Since none of these models yet include $B$-field amplification we do not discuss them further.

5. CONCLUSIONS

We have introduced a model of diffusive shock acceleration that couples thermal particle injection, nonlinear shock structure, magnetic field amplification, and the self-consistent determination of the maximum particle momentum. This is a first step toward a more complete solution, and in this preliminary work we make a number of approximations dealing mainly with the plasma physics of wave growth. Keeping in mind that our results are subject to the validity of our approximations, we reach a number of interesting conclusions.

First, our calculations find that efficient shock acceleration can amplify ambient magnetic fields by large factors and are generally consistent with the large fields believed to exist at blast waves in young SNRs, although we have not attempted a detailed fit to SNR observations in this paper. While the numerical values we obtain depend on the particular parameters for our examples, and we will investigate in detail how the amplification depends on sonic Mach number, age, and size of the shock system, etc., in future work, amplification factors of several 100 are clearly possible.

More specifically, we find that the amplification, in terms of the downstream-to-upstream field ratio $B_2/B_0$ is a strong function of Alfvén Mach number, with weak ambient fields being amplified more than strong ones. For the range of examples shown in Figure 5, $B_2/B_0 \sim 30$ for $M_{\text{Alf}} \sim 80$ and $B_2/B_0 \sim 400$ for $M_{\text{Alf}} \sim 8000$. Qualitatively, a strong correlation between amplification and $M_{\text{Alf}}$ should not depend strongly on our approximations and may have important consequences. In young SNRs, the expansion of the ejecta material will drastically reduce any original, pre-SN circumstellar magnetic field. In fact, for any conceivable progenitor, the magnetic field inside of the reverse shock will drop to values too low to support the acceleration of electrons to radio emitting energies only a few years after the explosion (e.g., Ellison et al. 2005b). Evidence for radio emission at reverse shocks in SNRs has been reported (see, Gotthelf et al. 2001 for example), and the strong amplification of low fields we see here may make it possible for reverse shocks in young SNRs to accelerate electrons to relativistic energies and produce radio synchrotron emission. If similar effects occur in relativistic shocks, these large amplification factors will be critical for the internal shocks presumed to exist in $\gamma$-ray bursts (GRBs). Even if large $B$-field amplification is confined to nonrelativistic shocks, which tend to be more efficient accelerators than relativistic ones, amplification will be important for understanding GRB afterglows since the expanding fireball will slow as it moves through the interstellar medium and will always go through transrelativistic and nonrelativistic phases.

As expected, amplifying the magnetic field leads to a greater maximum particle momentum, $p_{\text{max}}$, a given shock can produce. Quantifying $p_{\text{max}}$ is one of the outstanding problems in shock physics because of the difficulty in obtaining parameters for

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11 That $B$-field amplification is not included in the Berezhko et al. model can easily be missed, particularly since the titles of some of these papers, e.g., “Confirmation of Strong Magnetic Field Amplification and Nuclear Cosmic-Ray Acceleration in SN 1006” (Berezhko et al. 2003), might suggest that the model does contain $B$-field amplification. The model of Berezhko et al. is a parallel shock model where the magnetic field is not explicitly included in the convection-diffusion equations. There is no $B$-field amplification and the large downstream fields that Berezhko et al. infer from matching the observations are obtained by an ad hoc compression of the upstream field (see, for example, the discussion before eq. [8] in Volk et al. 2002). To obtain $300 \mu$G downstream, for example, they must start with an unshocked field of $\sim 50 \mu$G, which is then compressed by the shock with a ratio typically $r_{\text{sh}} \sim 6$ for their models.
typical SNRs that allow the production of cosmic rays to energies at and above the CR knee near 10^{15} eV. Assuming that acceleration is truncated by the size of the shock system, we determine $p_{\text{max}}$ from a physical constraint: the relevant parameter is the distance to the free escape boundary in diffusion lengths. This means that the limit on acceleration feeds back on the shock structure and also mimics, in terms of the spectral shape, what happens in actual shocks where $f(x, p)$ must turn over smoothly at the highest energies (as in Fig. 6). The spectral shape will be particularly important if the model is applied to the knee of the CR spectrum or to nonthermal X-ray emission in SNRs.

Our results show that $p_{\text{max}}$ does increase when field amplification is included, but the increase is considerably less than the amplification factor at the shock $B_{\|} / B_0$ (compare the heavy dotted and heavy solid curves in Fig. 2). The main reason for this is that high-momentum particles have long diffusion lengths and the precursor magnetic field well upstream from the subshock strongly influences $p_{\text{max}}$. We calculate the spatial structure of the amplified magnetic field (Figs. 1 and 5), and, as expected, field amplification is greatest near the subshock and $B_{\|}$ merges into the ambient field far upstream. The diffusion length that determines $p_{\text{max}}$ (i.e., $|D(x, p_{\text{max}})/u(x)|_{\text{avg}}$) is some weighted average over the varying $u(x)$ and $B_{\|}(x)$ and is considerably greater than that estimated from $B_2$ alone. If the shock size, in our case $d_{\text{FB}}$, limits acceleration, $p_{\text{max}}$ will be considerably less than crude estimates using a spatially independent $B_2$.

On the other hand, particles spend a large fraction of their time downstream from the shock, where the field is high and collision times are short. If shock age limits acceleration rather than size, we expect the increase in $p_{\text{max}}$ from the amplified field to be closer to the amplification factor, $B_{\|} / B_0$. Thus, the spatial structure of the precursor field and $u(x)$, in addition to the overall amplification of $B$, will determine the relative time spent upstream versus downstream and will determine $p_{\text{max}}$ for a given shock system. This will be even more critical for electrons than for ions, since electrons experience synchrotron and inverse Compton losses, which will mainly occur downstream. Again, the qualitative nature of the above conclusions should not depend on the particular parameters and approximations we make here. We leave more detailed quantitative work for future study.

Finally, it is well known that DSA is inherently efficient. Field amplification reduces the fraction of shock ram kinetic energy that is placed in relativistic particles, but, at least for the limited examples we show here, the overall acceleration process remains extremely efficient. Even with large increases in $B_{\|}(x)$, well over 50% of the shock energy can go into relativistic particles (Fig. 4). As in all self-consistent calculations, the injection efficiency must adjust to conserve momentum and energy. In comparing shocks with and without field amplification, we find that field amplification lowers $r_{\text{ann}}$, and therefore individual energetic particles are, on average, accelerated less efficiently. In order to conserve momentum and energy, this means that more thermal particles must be injected when amplification occurs. The shock accomplishes this by establishing a strong subshock that not only injects a larger fraction of particles, but also more strongly heats the downstream plasma. This establishes a nonlinear connection between the field amplification, the production of cosmic rays, and the X-ray emission from the shocked heated plasma.

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