Adaptive Influence Maximization under General Feedback Models

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Abstract
This note provides some analysis for the adaptive influence maximization problem under general Feedback Models with or without a time constraint.

1 Adaptive AIM Problem and This Note

In a social network, an information cascade is initiated by a seed set and it then spreads from users to users. Given a budget \( k \in \mathbb{Z}^+ \), the classic influence maximization (IM) problem asks for a seed set with at most \( k \) nodes such that the influence of the cascade can be maximized. This problem was proposed by Kempe, Kleinberg and Tardos in Kempe et al. (2003) and it has drawn tremendous attentions.

In the classic IM problem, the seed nodes are decided before the start of the diffusion process, while in its adaptive version we can select more seed nodes after having observed some diffusion results. An adaptive seeding strategy is therefore a policy that specifies the next seed node to be selected according to the current observation. This paper considers the adaptive influence maximization (AIM) problem.

We have the following results.

- We introduce the concept of regret ratio which characterizes the trade-off between seeding and waiting.
- We show that, without a time constraint, the performance of the greedy policy under general feedback models can be bounded by the regret ratio.
- For the time-constrained AIM problem, we provide a hardness result for comparing seeding patterns and design an ideal algorithm which is a heuristic for the general case but provides the best performance bound for special cases which allow approximation algorithms. Based on the ideal algorithm, we provide a practical algorithm with polynomial-time implementations.

2 Definitions

To formally analyze the AIM problem, we introduce a collection of definitions some of which are extended from that in Golovin and Krause (2011). For the convenience of readers, the references of the notations are given in Table 1.

2.1 IC Model and Adaptive Seeding Process

Definition 1 (IC model). A social network is represented by a directed graph \( G = (V, E) \). For each edge \((u, v)\), we say \( u \) is an in-neighbor of \( v \), and \( v \) is an out-neighbor of \( u \). An instance of the IC model is given by a directed graph \( G = (V, E) \) and the probability \( p_e \) on each edge \( e \in E \).
Symbol | Keyword | Reference
--- | --- | ---
\(G = (V, E)\) and \(p_e \in [0, 1]\) |  | Def. 1
\(\phi = (L(\phi), D(\phi)) \in 2^E \times 2^E = \Phi\) | realization | Def. 6
\(\Pr[\phi] \in [0, 1]\) |  |  
\(\phi_1 \prec \phi_2\) | sub-realization | Def. 8
\(\phi_0\) | empty realization | Def. 9
\(\phi_1 \sim \phi_2\) | compatible | Def. 10
\(\phi_1 \oplus \phi_2\) | concatenation | Def. 11
\(U = (\mathcal{S}(U), \mathcal{\dot{U}}(U)) \in 2^V \times 2^E = \Phi\) | status | Def. 12
\(\mathcal{U}_d(U) \in 2^E\) | d-round-status | Def. 14
\(U_1 \cup U_2\) | status union | Def. 15
\(\pi : 2^V \times \Phi \rightarrow 2^V\) | policy | Def. 16
\(T\) | decision tree | Def. 17
\(T|U\) | decision tree conditioned on \(U\) | Def. 19
\(T_1 \oplus T_2\) | concatenation of trees \(T_1\) and \(T_2\) | Def. 20
\(S^T_i\) | the tree-nodes in \(T\) in level \(i\) | Def. 18
\(U^T_i\) | the tree-edges in \(T\) from level \(i\) to \(i + 1\) | Def. 18
\(\mathcal{\dot{S}}_{\text{end}}(U)\) | the endpoint of a tree-edge \(U\) | Def. 18
\(\mathcal{\dot{U}}_{\text{out}}(S)\) | out edges of a tree-node \(S\) | Def. 18

Table 1: Notations.

Definition 2 (Round). In one round, each node \(v\), which is a new seed node or is activated in the last round, attempts to activate each of \(v\)'s inactive neighbors. We assume the observations are made round by round.

An adaptive seeding process can be described as follows.

Definition 3 (Adaptive Seeding Process).

- Repeat the following process until no seed node will be selected.
  - (seeding-step) Select and activate some seed nodes.
  - (observing-step) Observe the diffusion for a certain number of rounds.
- Wait for the diffusion to complete and output the number of active nodes.

Definition 4 (Convenient Set Notations). For an element \(v\) and a set \(S\), we use \(v\) and \(v + S\) in replace of \(\{v\}\) and \(\{v\} \cup S\), respectively.

2.2 Realizations and Status

The definitions in this section are used to describe an intermediate stage during a diffusion process.

Definition 5 (States of Edges). Following Kempe et al. (2003), we speak of each edge \((u, v)\) as being live or dead to indicate that if \(u\) can activate \(v\) once \(u\) becomes active.

Definition 6 (Realization). A realization \(\phi = (L(\phi), D(\phi)) \in 2^E \times 2^E\) is an ordered two-tuple where \(L(\phi) \subseteq E\), \(D(\phi) \subseteq E\), and \(L(\phi) \cap D(\phi) = \emptyset\), specifying the states of the edges that have been observed. In particular, \(e \in L(\phi)\) (resp., \(e \in D(\phi)\)) means \(e\) is a live (resp., dead) edge, and, the state of an edge \(e\) in unknown in \(\phi\) if \(e \notin L(\phi) \cup D(\phi)\). We use \(\Phi\) to denote the set of all realizations. For each realization \(\phi\), we define \(\Pr[\phi]\) as \(\Pr[\phi] := \prod_{e \in L(\phi)} p_e \prod_{e \in D(\phi)} (1 - p_e)\), which is probability that \(\phi\) can be realized (i.e., sampled).
We use the following concepts to describe the observations during the seeding process. When there is no limit on the length of the path, we use the notation \( \infty \)-live-path.

**Definition 8 (Sub-realization).** For two realizations \( \phi_1 \) and \( \phi_2 \), we say \( \phi_1 \) is a sub-realization of \( \phi_2 \) if \( L(\phi_1) \subseteq L(\phi_2) \) and \( D(\phi_1) \subseteq D(\phi_2) \), and denote it as \( \phi_1 \prec \phi_2 \). If \( \phi_1 \prec \phi_2 \), we say \( \phi_2 \) is a super-realization of \( \phi_1 \). Intuitively, \( \phi_2 \) is one possible outcome if we continuous to observe the states of the edges after observing \( \phi_1 \). We use \( \Pr[\phi_2|\phi_1] \) to denote the probability that \( \phi_2 \) can be realized conditioned on \( \phi_1 \), and therefore we have

\[
\Pr[\phi_2|\phi_1] = \prod_{e \in L(\phi_2) \setminus L(\phi_1)} p_e \prod_{e \in D(\phi_2) \setminus D(\phi_1)} (1 - p_e).
\]

**Definition 9 (Partial and Full Realization).** We say \( \phi \) is a full realization if \( L(\phi) \cup D(\phi) = E \), which indicates that all the edges have been observed. We use \( \Psi \) to denote the set of all full realizations. Note that \( \Psi \) also represents the basic event space of the IC model. We use \( \phi_0 = (\emptyset, \emptyset) \) to denote the empty realization. A realization is a partial realization if it is not a full realization.

**Definition 10 (Realization Compatibility).** For two realizations \( \phi_1, \phi_2 \in \Phi \), we say they are compatible if \( L(\phi_1) \cap D(\phi_2) = \emptyset \) and \( D(\phi_1) \cap L(\phi_2) = \emptyset \). We denote this relationship by \( \phi_1 \sim \phi_2 \).

We use the following concepts to describe the observations during the seeding process.

**Definition 11 (Realization Concatenation).** For a set of finite realizations, \( \phi_1, \phi_2, \ldots, \phi_m \), where \( (L(\phi_1) \cup \ldots \cup L(\phi_m)) \cap (D(\phi_1) \cup \ldots \cup D(\phi_m)) \), we define \( \phi_1 \oplus \ldots \oplus \phi_m \) as a new realization with \( L(\phi_1 \oplus \ldots \oplus \phi_m) = L(\phi_1) \cup \ldots \cup L(\phi_m) \) and \( D(\phi_1 \oplus \ldots \oplus \phi_m) = D(\phi_1) \cup \ldots \cup D(\phi_m) \).

**Definition 12 (Status).** A status \( U \) is a two-tuple \( U = (\hat{S}(U), \hat{\phi}(U)) \in 2^V \times \Phi \), where \( \hat{S}(U) \subseteq V \) is the set of the current active nodes and \( \hat{\phi}(U) \in \Phi \) indicates the state of the edges that have been observed. We use \( \hat{U}_0 := (\emptyset, \phi_0) \) to denote the status where there is no active node and no edge has been observed.

**Remark 1.** We use \( \phi \) to denote a realization while use \( \hat{\phi}() \) with an over-dot to denote a realization associated with some object. For example, \( \hat{\phi}(U) \) is a realization associated with a status \( U \). The similar rule of the use of over-dot applies to other notations in this paper.

**Definition 13 (Final Status).** We say a status \( U \) is final if there is no live path from any node in \( \hat{S}(U) \) to \( V \setminus \hat{S}(U) \) in any full realization \( \psi \) where \( \hat{\phi}(U) \prec \psi \).

**Remark 2.** A status is not final if and only if it is possible to have more active nodes in the future rounds even if we do not select any new seed node. When the diffusion process terminates, it reaches a final status.

**Definition 14 (d-round Status).** For a status \( U \) and \( d \in \mathbb{Z}^+ \), we use \( \hat{U}_d(U) \) to denote the set of the possible status after \( d \) rounds when the current status is \( U \). In addition, we use \( \hat{U}_\infty(U) \) to denote the case when \( d \geq n - 1 \).

**Definition 15 (Status Union).** For two statuses \( U_1 \) and \( U_2 \) where \( \hat{\phi}(U_1) \) and \( \hat{\phi}(U_2) \) are compatible, we define \( U_1 \cup U_2 \) as a new status with \( \hat{S}(U_1 \cup U_2) = \hat{S}(U_1) \cup \hat{S}(U_2) \) and \( \hat{\phi}(U_1 \cup U_2) = \hat{\phi}(U_1) \oplus \hat{\phi}(U_2) \).

**Remark 3.** Note that a realization together with the current active nodes determines the rest of the diffusion process. A realization \( \phi \) cannot solely do this because there can be some nodes which are activated by being selected as seed nodes.

### 2.3 Policy and Decision Tree

The definitions in this subsection are used to describe an adaptive seeding strategy.

**Definition 16 (Policy).** A policy \( \pi : 2^V \times \Phi \rightarrow 2^V \) is a function, mapping a status \( U \) to a node-set \( S \subseteq V \), which represents that \( \pi \) will select \( S \) as the next seed set if the current status is \( U \).

**Remark 4.** Following Remark 3, a policy makes decisions according to the current status rather than the current realization.

An adaptive seeding process can be viewed as a decision tree.
We denote the resulted decision tree as $T$. We only modify the status but do not change the tree-node. When $T$ is an adaptive seeding process, level $i$ shows all possible scenarios in the $i$-th seeding step. For each pair $U$ and $\hat{S}_{\text{end}}(U)$, it means that $\hat{S}_{\text{end}}(U)$ is selected by $\pi$ as a seed set when $U$ is observed. The edges out of a tree-node $S$ indicate the possible observations after selecting $S$. For each sequence of status $(U_1, U_2, ...) from the root to a leaf, we have $\hat{\phi}(U_1) \preceq \hat{\phi}(U_{i+1})$ and $\hat{S}_{\text{end}}(U_{i}) \subseteq \hat{S}_{\text{end}}(U_{i+1})$.

The analysis of the policy requires to measure the effect of the union of two policies. In [Golovin and Krause (2011)], it was stated as: running one policy to completion and then running another policy as if from a fresh start, ignoring the information gathered during the running of the first policy. To make this concept more mathematically rigorous, we employ the following definitions.

**Definition 17 (Decision tree).** A decision tree $T$ of an adaptive seeding process is an arborescence, where each tree-edge is associated with a status $U$ which corresponds to a observing-step showing what have been observed, and each tree-node is a node-set $S \subseteq V$ which corresponds to a seeding-step showing the seed nodes that are selected and activated according to the policy. An example is shown in Fig. 1.

**Definition 18 (Decision Tree Notations).** For each tree-node $S$, let $U_{\text{out}}(S)$ be the set of the tree-edges out of $S$, showing possible different observations after selecting $S$. Since two edges cannot have the same realization, by abusing the notation, we use $U$ to denote the tree-edge of which the status is $U$. For a tree-edge $U$, we use $\hat{S}_{\text{end}}(U) \subseteq V$ to denote the end-node of $U$. The tree-nodes can be divided into groups by levels. For a decision tree $T$ and $i \in \{1, 2, 3, \ldots\}$, we use $S_i^T \in 2^V$ to denote the set of the tree-nodes in the $i$-th level. For $i \in \{2, 3, \ldots\}$, we use $U_i^T \subseteq \Phi$ to denote the set of the status of the tree-edges from the tree-nodes in $S_i^T$ to those in $S_{i-1}^T$, and define $U_1^T$ as $\{U_0\}$. In addition, we use $S_{\infty}^T$ to denote the set of the nodes in the the lowest level, i.e., the leaves, and use $U_{\infty}^T$ to denote the set of the tree-edges connecting to the leaves.

**Remark 5.** When a decision tree represents an adaptive seeding process, level $i$ shows all possible scenarios in the $i$-th seeding step. For each pair $U$ and $\hat{S}_{\text{end}}(U)$, it means that $\hat{S}_{\text{end}}(U)$ is selected by $\pi$ as a seed set when $U$ is observed. The edges out of a tree-node $S$ indicate the possible observations after selecting $S$. For each sequence of status $(U_1, U_2, ...) from the root to a leaf, we have $\hat{\phi}(U_1) \preceq \hat{\phi}(U_{i+1})$ and $\hat{S}_{\text{end}}(U_{i}) \subseteq \hat{S}_{\text{end}}(U_{i+1})$.

**Definition 19 (Decision Tree Conditioned on a Status).** Given a decision tree $T$ and a status $U$, we construct another decision tree by modifying $T$ as follows. For each tree-edge $U_a$ in $T$ such that $\hat{\phi}(U_a)$ is not compatible with $\hat{\phi}(U)$, we replace the tree-edge $U_a$ as well as the tree-node $\hat{S}_{\text{end}}(U_a)$. For each tree-edge $U_a$ such that $\hat{\phi}(U_a)$ is compatible with $\hat{\phi}(U)$, we replace the status $U_a$ by $U \cup U_a$. We denote the resulted decision tree as $T|U$ named as the decision tree of $T$ conditioned on status $U$.

**Remark 6.** One can see that if we remove one tree-edge $U$, we must also remove all of its following tree-edges due to Remark 5. In such a tree, each realization is a super-realization of $\hat{\phi}(U)$. Note that we only modify the status but do not change the tree-node. When $T$ is an adaptive seeding process of a certain policy, the tree $T|U$ shows the adaptive seeding process when the states of the edges in $L(\hat{\phi}(U)) \cup D(\hat{\phi}(U))$ have been fixed and the nodes in $\hat{S}(U)$ are activated, but the policy does not have such information.

**Definition 20 (Decision Tree Concatenation).** Given two decision trees $T_1$ and $T_2$, we constructed another decision tree by modifying $T_1$, as follows. For each tree-edge $U$ in $T_1$ where $\hat{S}_{\text{end}}(U)$ is a
leaf, we replace \( \hat{S}_{\text{end}}(U) \) by the tree \( T_2|U \). We denote the new tree as \( T_1 \oplus T_2 \). An example is shown in Fig. 3.

Figure 3: Concatenation of two decision trees. Suppose we have two trees \( T_1 \) and \( T_2 \) where \( T_1 \) has two tree-edges \( U_1 \) and \( U_2 \) connecting to the leaves. Supposing the trees \( T_2|U_1 \) and \( T_2|U_2 \) are as those given in the figure, we have the tree \( T_1 \oplus T_2 \) as shown therein.

Remark 7. In our context, a decision tree can be constructed by: (a) an adaptive seeding process, and (b) a concatenation of two decision trees. The notations in Def. 18 also apply to a concatenation of two decision trees.

Decision tree has the following properties.

Property 1. For each tree-edge \( U_1 \) where \( \hat{S}_{\text{end}}(U_1) \) is not a leaf, we have

\[
\sum_{U_2 \in U_{\text{out}}(\hat{S}_{\text{end}}(U_1))} \Pr[\hat{\phi}(U_2)|\hat{\phi}(U_1)] = 1. \tag{1}
\]

This is because (1) \( \Pr[\hat{\phi}(U_2)|\hat{\phi}(U_1)] \) is the probability that \( \hat{\phi}(U_2) \) happens conditioned on \( U_1 \) and (2) \( U_{\text{out}}(\hat{S}_{\text{end}}(U_1)) \) is the set of all possible observations after observing \( U_1 \) and selecting \( \hat{S}_{\text{end}}(U_1) \) as a seed node. Note that this is also true for a tree which a concatenation of other two trees.

Definition 21 \((A_t(S, \psi))\). For each \( S \in 2^V \) and \( \psi \in \Psi \), we use \( A_t(S, \psi) \) to denote the number of the active nodes in \( \psi \) after \( t \) diffusion rounds when \( S \) is set of the current active nodes. We use \( t = \infty \) for the case when the diffusion completes (i.e., after \( n - 1 \) rounds). For a subset \( S_1 \subseteq V \), a node-set \( S_1 \subseteq V \), a realization \( \phi \) and a full realization \( \psi \), we define that

\[
\Delta_t(S, S_1, \psi) := |A_t(S \cup S_1, \psi)| - |A_t(S, \psi)|, \tag{2}
\]

and we use

\[
\Delta f_t(S, S_1, \phi) = \sum_{\phi \prec \psi, \psi \in \Psi} \Pr[\psi|\phi] \cdot \Delta_t(S, S_1, \phi). \tag{3}
\]
to denote the marginal profit after $t$ diffusion rounds if $S_1$ is selected as the seed node and the current status is $(S, \phi)$.

**Lemma 1 (Kempe et al. (2003)).** For a full-realization $\psi$, according to Kempe et al. (2003), $v \in A_t(S, \psi)$ iff there exists a t-live-path in $\psi$ from a node $u \in S$ to $v$.

### 3 Regret Ratio

Suppose that the current status is $U$, the budget is one, and we aim at maximizing the number of active nodes after $t$ rounds. Let us consider two ways to deploy this seed node.

- **Method 1:** We select the seed node immediately based on $U$. The best marginal profit we can achieve is $\max_v \Delta f_t(\hat{S}(U), v, \hat{\phi}(U))$, by selecting $\arg \max_v \Delta f_t(\hat{S}(U), v, \hat{\phi}(U))$ as the seed node.

- **Method 2:** we wait for $d$ rounds of diffusion and then select the seed node. After $d$ rounds, for each possible status $U_* \in \mathcal{U}_d(U)$, the best marginal profit would be $\max \Delta f_{t-d}(\hat{S}(U_*), v, \hat{\phi}(U_*))$. Thus, the total marginal profit would be $\sum_{U_* \in \mathcal{U}_d(U)} \Pr[\hat{\phi}(U_*)] \Delta f_{t-d}(\hat{S}(U_*), v, \hat{\phi}(U_*))$.

**Definition 22 (Regret ratio).** For a status $U$ and two integers $t, d$, we define the regret ratio as

$$\alpha_{t,d}(U) = \frac{\sum_{U_* \in \mathcal{U}_d(U)} \Pr[\hat{\phi}(U_*)] \Delta f_{t-d}(\hat{S}(U_*), v, \hat{\phi}(U_*))}{\max_v \Delta f_t(\hat{S}(U), v, \hat{\phi}(U))}, \quad (4)$$

which measures the ratio of the marginal profits resulted by those two methods.

When there is no time constraint (i.e., $t = \infty$), we denote it as

$$\alpha_{\infty,d}(U) = \frac{\sum_{U_* \in \mathcal{U}_d(U)} \Pr[\hat{\phi}(U_*)] \Delta f_{\infty}(\hat{S}(U_*), v, \hat{\phi}(U_*))}{\max_v \Delta f_{\infty}(\hat{S}(U), v, \hat{\phi}(U))}, \quad (5)$$

Furthermore, if we wait for the diffusion to complete before selecting the next seed node (i.e., $d = \infty$), we have

$$\alpha_{\infty,\infty}(U) = \frac{\sum_{U_*} \Pr[\hat{\phi}(U_*)] \Delta f_{\infty}(\hat{S}(U_*), v, \hat{\phi}(U_*))}{\max_v \Delta f_{\infty}(\hat{S}(U), v, \hat{\phi}(U))}. \quad (6)$$

Some discussions are given below.

**Remark 8.** We can see that if $U$ is a final status, then the ratio $\alpha_{t,d}(U)$ is always no larger than 1, indicating that the first method is a dominant strategy. Furthermore, if $U$ is a final status and $t = \infty$, we have $\alpha_{\infty,d}(U) = 1$ for each $d$.

**Remark 9.** When $t = \infty$, the ratio $\alpha_{\infty,d}(U)$ is always no less than 1, and therefore the second method is a dominant strategy. In this case, the $\alpha_{\infty,d}$ shows the penalty incurred by not waiting the diffusion to complete before determining the seed node. Furthermore, $\alpha_{\infty,d}$ will not decrease with the increase of $d$. That is, the more rounds we wait for, the better profit we can have. We will use $\alpha_{\infty,\infty}$ to bound the approximation ratio for the case when there is no time constraint.

**Remark 10.** When $t < n - 1$, there is a trade-off determined by the number of the rounds $d$ we would wait for. If we waited for more rounds, we would have more observations and had a better chance to find high quality seed nodes but the same time we would lose more diffusion rounds.

### 4 Unconstrained AIM

In this section, we consider the AIM problem without any time constraint. In particular, we consider the following problem.

**Problem 1 ((k, d)-AIM Problem).** Given a budget $k \in \mathbb{Z}^+$ and an integer $d \in \mathbb{Z}^+$, we consider the seeding strategy that selects one seed node after every $d$ rounds of diffusion, until $k$ seed nodes are selected. We aim to design a policy for selecting nodes under this pattern such that the influence can be maximized.
Remark 11. When \( d \geq n - 1 \), it is equivalent that we wait for the diffusion process to complete before selecting the next seed node, and in this case, we denote it as the \((k, \infty)\)-AIM Problem which reduces to the problem studied in Golovin and Krause (2011).

The adaptive seeding process in the \((k, d)\)-AIM problem under a policy \( \pi \) is described as follows:

**Definition 23** ((\( \pi, k, d \))-process).  
- Set \((S, \phi)\) as \((\emptyset, \phi)\). Repeat the following process for \( k \) times.
  - (seeding-step) Select and activate the node \( \pi(S, \phi) \).
  - (observing-step) Observe the diffusion for \( d \) rounds. Update \((S, \phi)\) by setting \( S \) as the set of the current active nodes and \( \phi \) as the current realization.
- Wait for the diffusion to complete and output the number of active nodes.

We define that we can wait for one round of diffusion even if there is no node can activate their neighbors, which conceptually allows us to wait for any number of rounds. The output of the above diffusion process is a random variable since the diffusion process is stochastic. We use \( F(\pi, k, d) \) to denote the expected number of the active nodes produced by the \((\pi, k, d)\)-process under policy \( \pi \).

**Problem 1.** Given \( k \) and \( d \), find a policy \( \pi \) such that \( F(\pi, k, d) \) is maximized.

For the purpose of analysis, let us consider the following process.

**Definition 24** (L-\((\pi, k, d)\)-process). Set \((S, \phi)\) as \((\emptyset, \phi)\).

- **Step 1.** Repeat the following process for \( k - 1 \) times.
  - (seeding-step) Select and activate the node \( \pi(S, \phi) \).
  - (observing-step) Observe the diffusion for \( d \) rounds. Update \((S, \phi)\) by setting \( S \) as the set of the current active nodes and \( \phi \) as the current realization.
- **Step 2.**
  - (seeding-step) Decide the seed node \( v^* = \pi(S, \phi) \), but do not activate \( v^* \). This is the \( k \)-th seeding step.
  - (observing-step) Wait for the diffusion to complete.
- **Step 3.**
  - (seeding-step) Activate the node \( v^* \). This is the \((k+1)\)-th seeding step.
- Wait for the diffusion to complete and output the number of active nodes.

Let \( F_L(\pi, k, d) \) be the expected number of the active nodes output by the L-\((\pi, k, d)\)-process.

**Lemma 2.** \( F(\pi, k, d) = F_L(\pi, k, d) \).

**Proof.** In the \((\pi, k, d)\)-process and L-\((\pi, k, d)\)-process, the seed nodes are always the same and the only difference is that the last seed in the L-\((\pi, k, d)\)-process may be seeded with a delay. However, with respect to the number of active nodes, it does not matter when we make the seed node activated as long as we allow the diffusion process to finally complete. The idea of lazy seeding was also seen early in Mossel and Roch (2007).

**Definition 25** (L-\((\pi, k, d)\)-tree). We denote the decision tree of the L-\((\pi, k, d)\)-process as the L-\((\pi, k, d)\)-tree. In an L-\((\pi, k, d)\)-process, there are totally \( k \) observing-steps and \( k + 1 \) seeding-steps. Note that in the \( k \)-th seeding step no node is activated and therefore we label the tree-node by a special character \( \epsilon \). An illustration is given in Fig. 4.

**Corollary 1.** The status in \( U_T^\infty \) where \( T \) is an L-\((\pi, k, d)\)-process are final status.
**Definition 26 (Decision Tree Profit).** For a decision tree $T$, we define that

$$F(T) := \sum_{U \in U^T} \Pr[\phi(U)] \sum_{\phi(U) \prec \psi} \Pr[\psi|\phi(U)] \cdot |A_{\infty}(\hat{S}(U) + \hat{S}_{\text{end}}(U), \psi)|$$

(7)

and

$$= \sum_{U \in U^T} \sum_{\phi(U) \prec \psi} \Pr[\psi] \cdot |A_{\infty}(\hat{S}(U) + \hat{S}_{\text{end}}(U), \psi)|$$

(8)

as the profit of the decision tree.\(^2\)

**Remark 12.** As one can see, if $T$ is the decision tree of a certain adaptive seeding process, then $F(T)$ is the expected number of active users resulted by the process. Thus, we have $F(\pi, k, d) = F_L(\pi, k, d) = F(T)$ if $T$ is the $L$-$(\pi, k, d)$-tree.

Note that $f(T_1 \oplus T_2)$ is also well-defined for a concatenation of two trees and we have the following result.

**Lemma 3.** $F(T_1 \oplus T_2) \geq F(T_2)$

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\(^2\)We slightly abuse the notation $F$ by allowing it to have different definitions for different types of inputs.
Proof.

\[ F(T_1 \oplus T_2) \]
\[
\begin{align*}
&= \sum_{U_1 \in U_2^T} \sum_{U_2 \in U_2^T} \Pr[\psi] \cdot |A_{\infty}(\hat{S}(U) + \hat{S}_{\text{end}}(U), \psi)|. \\
&\text{(By Eq. 7)} \\
&= \sum_{U_1 \in U_2^T} \sum_{U_2 \in U_2^T} \Pr[\psi] \cdot |A_{\infty}(\hat{S}(U_1 \cup U_2) + \hat{S}_{\text{end}}(U_2), \psi)|. \\
&\text{(By Def. 20)} \\
&\geq \sum_{U_2 \in U_2^T} \sum_{U_1 \in U_2^T} \Pr[\psi] \cdot |A_{\infty}(\hat{S}(U_2) + \hat{S}_{\text{end}}(U_2), \psi)|. \\
&\text{(By Eq. 7)} \\
&= F(T_2).
\end{align*}
\]

The third equation follows from the fact that the collection of \( \{ \psi \in \Psi : \phi(U_1 \cup U_2) \prec \psi \} \) among all possible pairs \( U_1 \in U_2^T \) and \( U_2 \in U_2^T \) such that \( \phi(U_1) \sim \phi(U_2) \) forms a partition of \( \Psi \). The fourth equation follows from the fact that \( A_{\infty}(S, \psi) \) is monotone with respect to \( S \). The fifth equation follows from the fact that \( \bigcup_{U_1 \in U_2^T} \{ \psi \in \Psi : \phi(U_1 \cup U_2) \prec \psi \} = \{ \psi \in \Psi : \phi(U_2) \prec \psi \} \) for each \( U_2 \in U_2^T \).

\[ \square \]

Remark 13. According to Remark 12, Eq. 1 gives an explicit formula of \( F_L(\pi, k, d) \). However, this formula is only used for analysis and it is not feasible to compute Eq. 4 directly as there are exponential number of terms to sum. If \( d = \infty \), \( L(\pi, k, d) \)-tree is the same as \((\pi, k, d)\)-tree.

Definition 27 (Greedy Policy). Given a status \((S, \phi)\), the greedy policy \( \pi_g \) always select the node that can maximize the marginal gain conditioned on \((S, \phi)\), and therefore,

\[ \pi_g(S, \phi) = \arg \max_{\psi} \Delta f_{\infty}(S, v, \phi) \]

(9)

For a decision tree, we define that \( \alpha(T) := \max_{U \in \bigcup T^T} \alpha_{\infty, \infty}(U) \) which is largest \( \alpha_{\infty, \infty}(U) \) among all the status in the tree.

We have the following result.

Theorem 1. For each policy \( \pi_\ast \), \( F(\pi_g, k, d) \geq (1 - e^{-1/\alpha(T_g)}) \cdot F(\pi_\ast, k, d) \), where \( T_g \) is the decision tree of the \((\pi_g, k, d)\)-process

Proof. The proof is obtained by integrating the regret ratio into the proof of Golovin and Krause in [Golovin et al., 2010] along with the standard submodular analysis. Please see the appendix for the complete proof which is more representationally complicated but more elementary. \[ \square \]

Remark 14. The \((k, d)\)-AIM problem has been studied for the case \( d = \infty \). An early discussion was given by Golovin and Krause in [Golovin et al., 2010] where it was confirmed that the greedy policy provides a \((1 - 1/e)\)-approximation to the \((k, \infty)\)-AIM problem based on the concept of adaptive submodularity. However, as shown in [Golovin et al., 2010] and [Vaswani and Lakshmanan, 2016], when \( d < n - 1 \), the \((k, d)\)-AIM problem is not adaptive submodular and therefore the existing techniques are not applicable. According to Remark 8, \( \alpha(T_g) = 1 \) when \( d = \infty \), because all the status in \( T_g \) are now final status, and therefore we have \( 1 - 1/e \) again for this special case.

Remark 15. In [Tong et al., 2017], we claimed that the greedy algorithm gives a \((1 - 1/e)\)-approximation for Problem 1 when \( d < n - 1 \). We now retract that claim.
5 Time-constrained AIM

In some applications, time (i.e., the number of the rounds) is important and one may need to maximize the influence before a certain time point. For example, when advertising a certain event, one needs to maximize the influence before the event starts; when maximizing the profit for the current month, one needs to maximize the influence by the end of the month; when deploying the positive cascade to spread against rumor, one needs to maximize the influence as soon as possible. In this section, we consider the time-constrained AIM problem, defined as follows.

Problem 3 ((k, t)-time-AIM Problem). Given a budget $k \in \mathbb{Z}^+$ and an integer $t \in \mathbb{Z}^+$, we aim to design a policy for selecting at most $k$ nodes such that the nodes activated by round $t$ can be maximized.

The whole process is described as follows.

Definition 28 (Time-constrained Adaptive Seeding Process).

- Repeat the following process for $t$ times.
  - (seeding-step) Select a set of seed nodes and activate it.
  - (observing-step) Observe the diffusion for one round.

- Output the number of active nodes.

Remark 16. At each seeding-step, we need to decide the size of the seed set as well as the nodes in the seed set, which is different from that in the $(k, d)$-AIM problem where it is fixed that we only select one node at each seeding-step. An critical issue is the trade off between the size of the seed set and the number of the round allowed in the rest of the diffusion. As aforementioned, waiting for one round offers more observations but meanwhile leads to the loss of one diffusion round. As extreme cases, when there is only one diffusion round left, one should immediately use all the budget; when there is no round limit, the optimal way is to activate one seed node each seeding-step, as mentioned in [Tong et al. (2017)].

A seeding pattern specifies how much budget will be used in each seeding-step and it can be either fixed before the seeding process or construed dynamically during the process. Given the budget $k$ and the time constraint $t$, a fixed seeding pattern is a sequence $(a_1, ..., a_t)$ of non-negative integers with $\sum a_i \leq k$ where $a_i$ indicates the number of seed nodes selected in the $i$-th selecting step. Given a policy $\pi$ and two patterns $p_1$ and $p_2$, we say $h_1$ is better than $h_2$ under $\pi$ if $\pi$ results in a larger influence under $h_1$ than that under $h_2$. To solve Problem 3, we may start by considering that which seeding pattern would yield the best strategy. Unfortunately, the following result shows comparing seeding patterns is a hard problem.

Problem 4. Given two fixed seeding patterns $h_1$ and $h_2$ and a policy $\pi$, decide if $h_1$ is better than $h_2$ under $\pi$.

Theorem 2. There is no polynomial-time algorithm for Problem 4 unless $NP = P$. 

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5.1 Special cases

Problem $[3]$ may be hard in general but some of its special cases can be relatively tractable. When there is only one diffusion round allowed (i.e., $t = 1$), one should utilize all the budget in the first seeding-step, and this problem becomes the time-constrained non-adaptive influence maximization problem which is monotone and submodular and therefore a $(1 - 1/e)$-approximation is obtainable. When $t = \infty$, a $(1 - 1/e)$-approximation is obtainable which the $(\pi, k, \infty)$-process under the greedy policy $\pi$ ($\text{Tong et al. (2017)}$). Ideally, a good heuristic algorithm for the general case should have the following basic properties.

Remark 17.

- When $t = 1$ or $t = \infty$, it is able to produce a $(1 - 1/e)$-approximation.
- When $k = 1$, it is able to produce the best solution.

We are looking for the rule able to dealing with the trade-off discussed in Remark $[16]$ while making these properties satisfied in a natural way.

5.2 An Ideal Algorithm

In this section, we present an ideal algorithm.

Let us first introduce some preliminaries. Given a status $U$, and two integers $t$ and $k$, finding $\arg \max_{S \subseteq V, |S| = k} \Delta f_t(S(U), S, \phi(U))$ is a NP-hard problem but it is monotone and submodular. We use $\text{Greedy}(U, t, k)$ to denote its $1 - 1/e$-approximation obtained by the greedy algorithm. For a status $U$ and an integer $k$, we define $U_{k,t} = (S(U) \cup \text{Greedy}(U, k, t), \phi(U))$ as a new status showing the scenario when we select $\text{Greedy}(U, k, t)$ as the new seed set. For a status $U$ and three integers $t, k_1$ and $k_2$ with $k_2 \leq k_1$, we define that

$$\beta(U, k_1, k_2, t) := \sum_{U_s \in U_{1}(U_{k_2,t})} \Pr[\phi(U_s) | \phi(U)] \cdot \max_{|S| = k_1-1} \Delta f_{t-1}(S(U_s), S, \phi(U_s)), \quad (10)$$

and

$$K(U, k_1, t) := \arg \max_{k_2 \in \{\ldots, k_1\}} \beta(U, k_1, k_2, t), \quad (11)$$

where the tie breaking in Eq. $\text{(11)}$ follows the rule that (1) if $U$ is final, 0 is has the lowest priority and, for other numbers, the smaller number has the higher priority, and (2) if $U$ is not final, 0 has the highest priority.

Now we present the algorithm. In each seeding-step when the current status is $U$, the budget left is $k$ and the diffusion rounds left is $t$, we first need to decide how much budget will be used in this step. For each $k_s$, if we select $k_s$ seed nodes in this seeding-step by the greedy algorithm and select optimal $k - k_s$ seed nodes after one diffusion round, the total marginal profit would be $\beta(U, k, k_s, t)$. We use this value to estimate the profit if we select $k_s$ seed nodes in the current step, and, select the $k_s = K(U, k, t)$ that can maximize this value. After the size $K(U, k, t)$ of the seed set is decided, we select and activate the seed set $\text{Greedy}(U, K(U, k, t), t)$ by greedy algorithm. The whole algorithm is described as follows, denoted as the ideal time-constrained greedy (ITCG) policy.

Definition 29 (Adaptive Seeding Process under ITCG policy).

- Set $U_* = U_0, k_* = k$ and $t_* = t$. Repeat the following process for $t$ times.
  - (seeding-step) Compute $K(U_*, k_*, t_*)$. Select and activate $\text{Greedy}(U_*, K(U_*, k_*, t_*), t_*)$.
  - (observing-step) Observe the diffusion for one round. Update $U_*$ as the current status, and set $k_* = k_* - K(U_*, k_*, t_*)$ and $t_* = t_* - 1$.

- Output the number of active nodes.

ITCG policy has the following desired characteristic.

Lemma 4. ITCG policy satisfies the properties given in Remark $[17]$.
Algorithm 1: Compute $K(U, k, t)$

1: Input $(U, k_*, t_*, L)$
2: $\text{max} = -1, \text{maxIndex} = 0, \text{maxSet} = \emptyset$
3: for $k_* = 0 : k$ do
4: Compute $S(U, t, k_*)$
5: $A = 0$
6: for $i = 1 : L$ do
7: Simulate $U_{k_*, i}$ for one round and obtain the current status $U_i$;
8: Compute Greedy$(U_i, t - 1, k - k_*)$
9: Simulate $U_{k_*, i}$ for $t - 1$ round and denote the total number of active nodes as $a$
10: $A = A + a$
11: if $A \geq \text{max}$ then
12: $\text{max} = A$
13: $\text{maxIndex} = k_*$
14: $\text{maxSet} = S(U, t, k_*)$
15: if $A < \text{max}$ then
16: Return $(\text{maxIndex}, \text{maxSet})$

Proof. First, ITCG policy is a $(1 - 1/e)$-approximation when $t = 1$, because $K(U, k_*, 1) = k_*$ and the nodes are selected by the greedy algorithm.

Second, the ITCG is a $(1 - 1/e)$-approximation when $t = \infty$. First, at a certain seeding-step when the status $U$ is not final, we have $\beta(U, k_1, \infty) \leq \beta(U, k, 0, 1)$ for each $k_1 \leq k$, which follows from the fact that waiting the diffusion to complete is always the optimal if there is no time constraint. Therefore, by the tie-breaking rule, the ITCG policy will always wait for the diffusion to complete before selecting the next seed set. Second, at a certain seeding-step when the status $U$ is final, we have $\beta(U, k_1, \infty) \leq \beta(U, k, 1, \infty)$ for $k_1 > 1$, and therefore, again by the tie-breaking rule, we have $K(U, k_1, \infty) = 1$. As a result, the ITCG follows the optimal pattern and selects each seed node in a greedy manner, which yields a $(1 - 1/e)$-approximation.

Remark 18. The ITCG policy is not feasible, in terms of polynomial time computability, as computing $\max_{|S| = k - k_1} \Delta f_{t - 1}(\hat{S}(U_1), S, \hat{\phi}(U_1))$ is NP-hard.

5.3 A Feasible Algorithm

Based on the ITCG policy, we propose a policy with feasible implementations. To overcome the hardness in computing $\max_{|S| = k - k_1} \Delta f_{t - 1}(\hat{S}(U_1), S, \hat{\phi}(U_1))$, we use its $(1 - 1/e)$-approximation as an estimate. Furthermore, We use simulations to estimate $\beta(U, k_1, k_*, t)$. Based on such implementation, the approach for computing $K(U, k, t)$ is shown in Alg. 1 where $L$ is the number of simulations used for estimating. We denote the new policy as practical time-constrained greedy (PTCG) policy of which the seeding process is given as follows.

Definition 30 (Adaptive Seeding Process under PTCG policy).

- Set $U_* = U_{g_0}, k_* = k$ and $t_* = t$. Repeat the following process for $t$ times.
  - (seeding-step) Let $(K, S)$ be the result of Alg. 1 with input $(U, k_*, t_*, L)$. Select and activate $S$. Set $k_* = k_* - K$
  - (observing-step) Observe the diffusion for one round. Update $U_*$ as the current status, and set $t_* = t_* - 1$.

- Output the number of active nodes.

A Proof of Theorem

In the rest of this section, we assume $\pi_*, d$ and $k$ are fixed. For each $i \in \{1, \ldots, k\}$, we use $T_{g_i}$ to denote the decision tree of the L-($\pi_*, i, d$)-process, and use $T_{g_i}$ to denote the decision tree of the
With Lemma 5, the rest of the proof follows the standard analysis of submodular maximization. Therefore, we define that
\[ F(T_g^k) \geq (1 - e^{-1/\alpha(T_g)}) \cdot F(T_*^k). \] (12)

For two integers \( i, j \in \{1, \ldots, k\} \), let us consider the decision tree \( F(T_g^i \oplus T_g^j) \). In addition, we define that \( T_g^0 \oplus T_g^1 := T_g^1 \) and \( T_g^0 \oplus T_*^0 := T_g^1 \). For conciseness, we denote \( T_i^1 \oplus T_j^1 \) by \( T_{i,j} \). Furthermore, we define that \( F(T_{0,0}) := 0 \) but do not define the tree \( T_{0,0} \). By Lemma 3, we have
\[ F(T_{i,k}) \geq F(T_{0,k}). \] (13)

Now let us consider the following lemma which will be proved later.

Lemma 5. \( F(T_{i-1,l}) - F(T_{i-1,l-1}) \leq \alpha(T_g) \cdot (F(T_{0,i}) - F(T_{0,i-1})) \) for each \( l \in \{1, \ldots, k\} \).

With Lemma 5, the rest of the proof follows the standard analysis of submodular maximization. Summing the inequality in 5 over \( l \in \{1, \ldots, k\} \), we have \( F(T_{i-1,k}) - F(T_{i-1,0}) \leq k \cdot \alpha(T_g) \cdot (F(T_{0,i}) - F(T_{0,i-1})) \cdot (\Delta_0 := F(T_{0,k}) - F(T_{1,0}), \text{ and we therefore have } \Delta_i := \Delta_{i-1} \geq \Delta_1 \geq \Delta_1 \cdot \exp \left( \frac{-1}{\alpha(T_g)} \cdot \Delta_1 \right) \cdot \Delta_1. \) That is, \( F(T_{0,k}) - F(T_{k,0}) \leq \exp \left( \frac{-1}{\alpha(T_g)} \right) \cdot (F(T_{0,k}) - F(T_{0,0})) \), and therefore, \( F(T_{0,k}) \geq (1 - e^{-1/\alpha(T_g)}) \cdot F(T_{k,0}) \). It remains to prove Lemma 5.

A.1 Proof of Lemma 5

By Def. 26, we have
\[ F(T_{0,i-1}) = \sum_{U \in U_i} \sum_{\phi(U)^{\Delta_v} \prec v} \Pr[\psi] \cdot |A_{\infty}(\hat{S}(U), \psi)| \] (14)
and,
\[ F(T_{0,i}) = \sum_{U \in U_i} \sum_{\phi(U)^{\Delta_v} \prec v} \Pr[\psi] \cdot |A_{\infty}(\hat{S}(U) + \hat{S}_{\text{end}}(U), \psi)|. \] (15)

Therefore, \( F(T_{0,i}) - F(T_{0,i-1}) \) is equal to
\[ \sum_{U \in U_i} \sum_{\phi(U)^{\Delta_v} \prec v} \Pr[\psi] \cdot \Delta_{\infty}(\hat{S}(U), \hat{S}(U), \psi) \]
\[ = \sum_{U \in U_i} \Pr[\hat{\phi}(U)] \cdot \sum_{\phi(U)^{\Delta_v} \prec v} \Pr[\psi | \hat{\phi}(U)] \cdot \Delta_{\infty}(\hat{S}(U), \hat{S}_{\text{end}}(U), \psi) \]
\{By the greedy policy\}
\[ = \sum_{U \in U_i} \Pr[\hat{\phi}(U)] \cdot \max_{v} \left( \sum_{\phi(U)^{\Delta_v} \prec v} \Pr[\psi | \hat{\phi}(U)] \cdot \Delta_{\infty}(\hat{S}(U), \hat{S}_{\text{end}}(U), \psi) \right) \] (16)

Similarly, \( F(T_{i-1,l}) - F(T_{i-1,l-1}) \) is equal to
\[ \sum_{U \in U_{i-1,l}} \sum_{\phi(U)^{\Delta_v} \prec v} \Pr[\psi] \cdot \Delta_{\infty}(\hat{S}(U), \hat{S}_{\text{end}}(U), \psi) \] (17)

\(^{1}\)One can imagine \( T_{0,0} \) as an empty tree.
Grouping the realizations in $U_{i-1+1}^{T_{i}}$ by that if they are the super-realizations of that in $U_{i}^{T_{i}}$, we have

$$
\sum_{U \in U_{i-1+1}^{T_{i}}} \sum_{\phi(U) \prec \psi} \Pr[\psi] \cdot \Delta_{\infty}(\hat{S}(U), \hat{S}_{end}(U), \psi)
\quad = \sum_{U_{1} \in U_{i}^{T_{i}}} \sum_{\phi(U_{1}) \prec \phi(U_{2})} \sum_{\phi(U_{2}) \prec \psi} \Pr[\psi] \cdot \Delta_{\infty}(\hat{S}(U_{2}), \hat{S}_{end}(U_{2}), \psi)
$$

Now we further group the super-realizations of $U \in U_{i}^{T_{i}}$ by that if they are the super-realizations of that in $U \in U_{i+1}^{T_{i}}$, obtaining that

$$
\sum_{U_{1} \in U_{i}^{T_{i}}} \sum_{U_{2} \in U_{i+1}^{T_{i}}} \sum_{\phi(U_{2}) \prec \psi} \Pr[\psi] \cdot \Delta_{\infty}(\hat{S}(U_{2}), \hat{S}_{end}(U_{2}), \psi)
\quad = \sum_{U_{1} \in U_{i}^{T_{i}}} \sum_{U_{2} \in U_{i+1}^{T_{i}}} \sum_{\phi(U_{1}) \prec \phi(U_{2})} \sum_{\phi(U_{2}) \prec \psi} \Pr[\psi] \cdot \Delta_{\infty}(\hat{S}(U_{3}), \hat{S}_{end}(U_{3}), \psi)
$$

Let consider the following lemma which will be proved later.

**Lemma 6.** For each $U_{2} \in U_{i+1}^{T_{i}}$ and $U_{3} \in U_{i-1+1}^{T_{i}}$ such that $\hat{\phi}(U_{2}) \prec \hat{\phi}(U_{3})$, consider two functions over $V$:

$$
g_{1}(v) := \sum_{\phi(U_{2}) \prec \psi} \Pr[\psi] \cdot \Delta_{\infty}(\hat{S}(U_{2}), v, \psi)
\quad (18)
$$

and

$$
g_{2}(v) := \sum_{\phi(U_{3}) \prec \psi} \Pr[\psi] \cdot \Delta_{\infty}(\hat{S}(U_{3}), v, \psi).
\quad (19)
$$

We have $g_{1}(v) \geq g_{2}(v)$. 

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Supposing Lemma 6 is true, we have,
\[
F(T_{i-1,i}) - F(T_{i-1,i-1}) = \sum_{U_1 \in U_i^{T^+_i}} \Pr[\hat{\phi}(U_1)] \sum_{U_2 \in U_i^{T^+_i}} \Pr[\hat{\phi}(U_2) | \hat{\phi}(U_1)] \sum_{U_3 \in U_i^{T^+_i-1,1_i}} \Pr[\hat{\phi}(U_3) | \hat{\phi}(U_2)] \sum_{\phi(U_3) < \psi} \Pr[\psi | \hat{\phi}(U_3)] \cdot \Delta_\infty(\hat{S}(U_3), \hat{S}_\text{end}(U_3), \psi)
\]

{By Lemma 6}
\[
\leq \sum_{U_1 \in U_i^{T^+_i}} \Pr[\hat{\phi}(U_1)] \sum_{U_2 \in U_i^{T^+_i}} \Pr[\hat{\phi}(U_2) | \hat{\phi}(U_1)] \sum_{U_3 \in U_i^{T^+_i-1,1_i}} \Pr[\hat{\phi}(U_3) | \hat{\phi}(U_2)] \sum_{\phi(U_3) < \psi} \Pr[\psi | \hat{\phi}(U_3)] \cdot \Delta_\infty(\hat{S}(U_2), \hat{S}_\text{end}(U_3), \psi)
\]

{Since \( |\hat{S}_\text{end}(U_3)| = 1 \)}
\[
\leq \sum_{U_1 \in U_i^{T^+_i}} \Pr[\hat{\phi}(U_1)] \sum_{U_2 \in U_i^{T^+_i}} \Pr[\hat{\phi}(U_2) | \hat{\phi}(U_1)] \sum_{U_3 \in U_i^{T^+_i-1,1_i}} \Pr[\hat{\phi}(U_3) | \hat{\phi}(U_2)] \cdot \left( \max_{v} \sum_{\phi(U_3) < \psi} \Pr[\psi | \hat{\phi}(U_3)] \cdot \Delta_\infty(\hat{S}(U_2), v, \psi) \right)
\]

{Since \( \sum_{\phi(U_3) < \psi} \Pr[\psi | \hat{\phi}(U_3)] \cdot \Delta_\infty(\hat{S}(U_2), v, \psi) \) is independent of \( U_3 \)}
\[
= \sum_{U_1 \in U_i^{T^+_i}} \Pr[\hat{\phi}(U_1)] \sum_{U_2 \in U_i^{T^+_i}} \Pr[\hat{\phi}(U_2) | \hat{\phi}(U_1)] \cdot \left( \max_{v} \sum_{\phi(U_3) < \psi} \Pr[\psi | \hat{\phi}(U_3)] \cdot \Delta_\infty(\hat{S}(U_2), v, \psi) \right)
\]

{By the definition of \( \alpha(T_{i,i}) \)}
\[
\leq \sum_{U_1 \in U_i^{T^+_i}} \Pr[\hat{\phi}(U_1)] \cdot \alpha(T_{i,i}) \cdot \max_{v} \left( \sum_{\phi(U_3) < \psi} \Pr[\psi | \hat{\phi}(U_3)] \cdot \Delta_\infty(\hat{S}(U_1), v, \psi) \right)
\]
\[
= \alpha(T_{i,i}) \cdot \left( F(T_{i,i}) - F(T_{i-1,i-1}) \right)
\]

It remains to prove Lemma 6.

A.2 Proving Lemma 6

Lemma. For each \( U_2 \in U_i^{T^+_i-1,1_i} \) and \( U_3 \in U_i^{T^+_i-1,1_i} \) such that \( \hat{\phi}(U_2) \prec \hat{\phi}(U_3) \), consider two functions
\[
g_1(v) := \sum_{\phi(U_2) < \psi} \Pr[\psi | \hat{\phi}(U_2)] \cdot \Delta_\infty(\hat{S}(U_2), v, \psi) \quad (20)
\]
and
\[
g_2(v) := \sum_{\phi(U_3) < \psi} \Pr[\psi | \hat{\phi}(U_3)] \cdot \Delta_\infty(\hat{S}(U_3), v, \psi). \quad (21)
\]
We have \( g_1(v) \geq g_2(v) \).

Since \( \hat{\phi}(U_2) \prec \hat{\phi}(U_3) \), we have (1) \( \{ \psi : \hat{\phi}(U_3) \prec \psi \} \subseteq \{ \psi : \hat{\phi}(U_2) \prec \psi \} \), (2) \( L(\hat{\phi}(U_2)) \subseteq L(\hat{\phi}(U_3)) \) and (3) \( D(\hat{\phi}(U_2)) \subseteq D(\hat{\phi}(U_3)) \). Let us consider the edge set \( E_2 := E \setminus (L(\hat{\phi}(U_2)) \cup D(\hat{\phi}(U_2))) \) which consists of the edges that are to be determined in a full-realization \( \hat{\phi}(U_3) \prec \psi \). Similarly, let us define \( E_1 := E \setminus (L(\hat{\phi}(U_2)) \cup D(\hat{\phi}(U_2))) \) with respect to \( U_2 \). Since \( E_2 \subseteq E_1 \), the edge set \( E \) can be partitioned into three parts, \( \{ L(\hat{\phi}(U_2)) \cup D(\hat{\phi}(U_2)) \}, E_1 \setminus E_2 \) and \( E_2 \). Let \( \Phi_2 := \{ \phi \in \Phi : L(\phi) \cup D(\phi) = E_2 \} \) and \( \Phi_1 \setminus \Phi_2 := \{ \phi \in \Phi : L(\phi) \cup D(\phi) = E_1 \setminus E_2 \}. \) With Def. 11 we have
\[
\{ \psi : \hat{\phi}(U_3) \prec \psi \} = \{ \hat{\phi}(U_3) \oplus \phi_1 : \phi_1 \in \Phi_1 \setminus \Phi_2 \}, \quad (22)
\]
and
\[
\{ \psi : \hat{\phi}(U_2) \prec \psi \} = \{ \hat{\phi}(U_2) \oplus \phi_1 \oplus \phi_2 : \phi_1, \phi_2 \in \Phi_2 \}. \quad (23)
\]
With these notations, we have

\[
g_1(v) = \sum_{\tilde{\phi}(U_2) \in \tilde{\Psi}} \Pr[\tilde{\phi}(U_2)] \cdot \Delta_{\infty}(\tilde{S}(U_2), v, \tilde{\psi})
\]

\[
= \sum_{\tilde{\phi}_2 \in \tilde{\Psi}_2} \sum_{\phi_1 \in \phi_{1/2}} \Pr[\tilde{\phi}(U_2) \oplus \phi_1 \oplus \phi_2] \cdot \Delta_{\infty}(\tilde{S}(U_2), v, \tilde{\phi}(U_2) \oplus \phi_1 \oplus \phi_2)
\]

\[
= \sum_{\tilde{\phi}_2 \in \tilde{\Psi}_2} \Pr[\tilde{\phi}_2] \sum_{\phi_1 \in \phi_{1/2}} \Pr[\tilde{\phi}_1] \cdot \Delta_{\infty}(\tilde{S}(U_2), v, \tilde{\phi}(U_2) \oplus \phi_1 \oplus \phi_2)
\]

and

\[
g_2(v) = \sum_{\tilde{\phi}(U_3) \in \tilde{\Psi}} \Pr[\tilde{\phi}(U_3)] \cdot \Delta_{\infty}(\tilde{S}(U_3), v, \tilde{\psi})
\]

\[
= \sum_{\tilde{\phi}_2 \in \tilde{\Psi}_2} \Pr[\tilde{\phi}(U_3) \oplus \phi_2] \cdot \Delta_{\infty}(\tilde{S}(U_3), v, \tilde{\phi}(U_3) \oplus \phi_2)
\]

\[
= \sum_{\tilde{\phi}_2 \in \tilde{\Psi}_2} \Pr[\tilde{\phi}_2] \cdot \Delta_{\infty}(\tilde{S}(U_3), v, \tilde{\phi}(U_3) \oplus \phi_2)
\]

Since \(\sum_{\phi_1 \in \phi_{1/2}} \Pr[\tilde{\phi}_1] = 1\), to prove \(g_1(v) \geq g_2(2)\), it suffices to prove the following lemma.

**Lemma 7.** \(\Delta_{\infty}(S(U_2), v, \tilde{\phi}(U_2) \oplus \phi_1 \oplus \phi_2) \geq \Delta_{\infty}(S(U_3), v, \tilde{\phi}(U_3) \oplus \phi_2)\) for each \(\phi_1 \in \phi_{1/2}\) and \(\phi_2 \in \phi_2\).

**Proof.** Because

\[
\Delta_{\infty}(S(U_2), v, \tilde{\phi}(U_2) \oplus \phi_1 \oplus \phi_2) = |A_{\infty}(S(U_2) + v, \phi(U_2) \oplus \phi_1 \oplus \phi_2) \setminus A_{\infty}(S(U_2), \phi(U_2) \oplus \phi_1 \oplus \phi_2)|
\]

and

\[
\Delta_{\infty}(S(U_3), v, \tilde{\phi}(U_3) \oplus \phi_2) = |A_{\infty}(S(U_3) + v, \phi(U_3) \oplus \phi_2) \setminus A_{\infty}(S(U_3), \phi(U_3) \oplus \phi_2)|,
\]

it is sufficient to prove that, for each \(u \in V\), \(u\) is in

\[
A_{\infty}(S(U_2) + v, \phi(U_2) \oplus \phi_1 \oplus \phi_2) \setminus A_{\infty}(S(U_2), \phi(U_2) \oplus \phi_1 \oplus \phi_2)
\]

if \(u\) is in

\[
A_{\infty}(S(U_3) + v, \phi(U_3) \oplus \phi_2) \setminus A_{\infty}(S(U_3), \phi(U_3) \oplus \phi_2).
\]

Suppose that \(u\) is in \(A_{\infty}(S(U_3) + v, \phi(U_3) \oplus \phi_2) \setminus A(S(U_3), \phi(U_3) \oplus \phi_2)\). It means \(u\) is in \(A_{\infty}(S(U_3) + v, \phi(U_3) \oplus \phi_2)\) but not in \(A_{\infty}(S(U_3), \phi(U_3) \oplus \phi_2)\). That is, in the full realization \(\phi(U_3) \oplus \phi_2\),

- (a) there exists a \(\infty\)-live-path from \(S(U_3) + v\) to \(u\), and
- (b) there is no \(\infty\)-live-path from \(S(U_3)\) to \(u\).

To prove \(u\) is in \(A_{\infty}(S(U_2) + v, \phi(U_2) \oplus \phi_1 \oplus \phi_2) \setminus A_{\infty}(S(U_2), \phi(U_2) \oplus \phi_1 \oplus \phi_2)\). We have to prove that \(u\) is in \(A_{\infty}(S(U_2) + v, \phi(U_2) \oplus \phi_1 \oplus \phi_2)\) and \(u\) is not in \(A_{\infty}(S(U_2), \phi(U_2) \oplus \phi_1 \oplus \phi_2)\).

First, we prove that \(u\) is in \(A_{\infty}(S(U_2) + v, \phi(U_2) \oplus \phi_1 \oplus \phi_2)\). Since \(u\) is not in \(A_{\infty}(S(U_3), \phi(U_3) \oplus \phi_2)\), we have \(u \in V \setminus S(U_3)\). By (a), there is a \(\infty\)-live-path from \(v\) to \(u\) in \(\phi(U_3) \oplus \phi_2)\). Furthermore, by (b), this path cannot use any edge in \(L(\phi(U_3))\), and therefore this path only uses the edges in \(L(\phi_2)\). Thus, we have this live path in \(\phi(U_2) \oplus \phi_1 \oplus \phi_2\) as well, and therefore, \(u \in A(S(U_2) + v, \phi(U_2) \oplus \phi_1 \oplus \phi_2)\).

Second, since \(U_2\) is a final status and \(u \notin \hat{S}(U_2)\), we have \(u \notin A_{\infty}(S(U_2), \phi(U_2) \oplus \phi_1 \oplus \phi_2)\).
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