Functional models up to similarity and $a$-contractions

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Abstract
We study the generalization of $m$-isometries and $m$-contractions (for positive integers $m$) to what we call $a$-isometries and $a$-contractions for positive real numbers $a$. We show that an operator satisfying a certain inequality in hereditary form is similar to $a$-contraction. This improvement of [9, Theorem I] is based on some Banach algebras techniques. We show that our operator classes are closely connected with fractional finite differences. Using this techniques, we get that, given $0 < b < a$, an $a$-contraction need not to be a $b$-contraction in general, but is a $b$-contraction if a natural additional requirement is imposed.

Keywords Dilation · Functional model · Operator inequality · Similarity · Fractional finite difference

Mathematics Subject Classification 47A63 · 47A20, 47A35 · 47A45

1 Introduction
Let $\alpha(t)$ be a function representable by power series $\sum_{n=0}^{\infty} a_n t^n$ in the unit disc $\mathbb{D} := \{ |t| < 1 \}$. If $T \in L(H)$ is a bounded linear operator on a separable Hilbert space $H$, we put...
whenever this series converges in the strong operator topology SOT in $L(H)$.

Since Sz.-Nagy and Foiaş developed their spectral theory for contractions (see [28]) based on the construction of a functional model, an intensive research has been done on obtaining a functional model for operators $T$ such that $\alpha(T^*, T) \geq 0$ for distinct types of functions $\alpha$. Note that when $\alpha(t) = 1 - t$ the operator inequality $\alpha(T^*, T) \geq 0$ means that $T$ is a contraction. The usual approach, which goes back to Agler (see [3]), is based on the assumption that the function $k(t) := 1/\alpha(t)$ defines a reproducing kernel Hilbert space. In particular, it is assumed that $\alpha$ does not vanish on the unit disc $\mathbb{D}$. We refer the reader to the introduction of our recent paper [1] and the references therein for more details.

In [9], the last two authors consider functions $\alpha$ in the Wiener algebra $A_W$ of analytic functions in the unit disc $\mathbb{D}$ with summable sequence of Taylor coefficients. In that paper, the so called admissible functions $\alpha$ have the form $\alpha(t) = (1 - t)\overline{\alpha}(t)$, where $\overline{\alpha}$ belongs to $A_W$, has real Taylor coefficients, and is positive on the interval $[0, 1]$. Therefore the admissible functions $\alpha$ may have zeroes in $\mathbb{D} \setminus [0, 1)$. In [9, Theorem I] it was proved that whenever $\alpha(T^*, T) \geq 0$, $T$ is similar to a Hilbert space contraction. In Theorem 1.1 we generalize this result.

This paper should be seen as a second part of [1], where we focused on unitarily equivalent models for operators $T$ satisfying $\alpha(T^*, T) \geq 0$ for certain functions $\alpha$. We tried to make our exposition independent of [1].

We need to introduce some notation. Given a function $\alpha$ in the Wiener algebra $A_W$ and an operator $T$ in $L(H)$ with spectrum $\sigma(T) \subset \overline{\mathbb{D}}$, if the series $\sum |\alpha_n|T^{*n}T^n$ converges in SOT, then we say that $\alpha \in A_T$. According to the notation of [1], this means precisely that $T \in \text{Adm}_{\alpha}^w$. Depending on whether we want to fix the operator $T$ or the function $\alpha$, we will use one notation or the other. In the same way, if the series $\sum |\alpha_n|T^{*n}T^n$ converges in the uniform operator topology on $L(H)$, then we say that $\alpha \in A_T^0$, or equivalently $T \in \text{Adm}_{\alpha}^w$. Hence,

$$A_T^0 \subset A_T \subset A_W, \quad \text{and} \quad \text{Adm}_{\alpha} \subset \text{Adm}_{\alpha}^w \subset L(H).$$

We will see later that $A_T$ is a Banach algebra (the norm is given by (2.2)) and $A_T^0$ is a separable closed subalgebra of $A_T$.

We denote by $A_{T, \mathbb{R}}^0$, $A_{T, \mathbb{R}}$, and $A_{W, \mathbb{R}}$ the subsets of functions in $A_{T, \mathbb{R}}$, $A_T$, and $A_W$, respectively, whose Taylor coefficients are real. It turns out (see Proposition 2.13) that if $\alpha \in A_{T, \mathbb{R}}$, then the operator $\alpha(T^*, T)$ is well-defined (that is, the series (1.1) converges in SOT).

**Theorem 1.1** Let $\hat{T}$ be an operator in $L(H)$ with spectrum $\sigma(\hat{T}) \subset \overline{\mathbb{D}}$, and let $\gamma$ be a function of the form

$$\gamma(t) = \alpha(t)\overline{\gamma}(t),$$

where $\alpha(t)$ is a function in $A_{T, \mathbb{R}}^0$ for which $\alpha(T^*, T) \geq 0$. Then $\overline{\gamma}(t)$ is a function in $A_{\hat{T}, \mathbb{R}}^0$ with $\overline{\gamma} \in A_{\hat{T}}^0$. In particular, if $\gamma(t) \in A_{T, \mathbb{R}}$ is a reproducing kernel, then $\overline{\gamma}(t) \in A_{\hat{T}, \mathbb{R}}$ is a reproducing kernel, and if $\gamma(t) \in A_{T, \mathbb{R}}^0$ is a reproducing kernel, then $\overline{\gamma}(t) \in A_{\hat{T}}^0$ is a reproducing kernel.
where \( \alpha \in A_{T^*, \mathbb{R}^+} \) and \( \tilde{\gamma} \in A_{T^*, \mathbb{R}^+}^0 \) is positive on the interval \([0, 1]\). If \( \gamma(\tilde{T}^*, \tilde{T}) \geq 0 \), then \( \tilde{T} \) is similar to an operator \( T \in L(H) \) such that \( \alpha(T^*, T) \geq 0 \).

In other words, this theorem permits one to leave out (in terms of similarity) the factor \( \tilde{\gamma} \). Therefore, the question of obtaining a model for \( \tilde{T} \) up to similarity reduces to obtaining a model for \( T \) (as above). An important case is

\[
\alpha(t) := (1 - t)^{\alpha} \quad \text{for } a > 0. \text{ If } (1 - t)^{\alpha}(T^*, T) \geq 0, \text{ then we say that } T \text{ is an } a\text{-contraction, and if } (1 - t)^{\alpha}(T^*, T) = 0, \text{ we say that } T \text{ is an } a\text{-isometry.}
\]

The next result is an obvious consequence of Theorem 1.1.

**Corollary 1.2** Let \( \tilde{T} \) be an operator in \( L(H) \) with spectrum \( \sigma(\tilde{T}) \subset \overline{\mathbb{D}} \). Suppose that \( \gamma \) has the form

\[
\gamma(t) = (1 - t)^{\alpha} \tilde{\gamma}(t),
\]

for some \( a > 0 \), where \((1 - t)^{\alpha} \in A_{T^*, \mathbb{R}^+} \), and \( \tilde{\gamma} \in A_{T^*, \mathbb{R}^+}^0 \) is positive on the interval \([0, 1]\). If \( \gamma(\tilde{T}^*, \tilde{T}) \geq 0 \), then \( \tilde{T} \) is similar to an \( a\)-contraction.

The case \( a = 1 \) in the above corollary corresponds to [9, Theorem I]. However, in that theorem we assumed that the series \( \sum_n |\tilde{\gamma}_n| \|T^n\|^2 \) converges, which implies the convergence of the series (1.1) in the uniform operator topology.

The consideration of the strong operator topology here is not a mere generalization, but it turns out to be the appropriate topology for this setting. Indeed, using this topology, the results of [1] about the existence of a model for operators \( T \) satisfying that \( \alpha(T^*, T) \geq 0 \) are characterizations (i.e., “if and only if” statements). See for instance Theorem 1.3 below.

We say that a function \( \alpha(t) = \sum_{n=0}^{\infty} a_n t^n \) is of Nevanlinna-Pick type if \( a_0 = 1 \) and \( a_n \leq 0 \) for \( n \geq 1 \). Alternatively, in this case \((1/\alpha)(t)\) is called a Nevanlinna-Pick kernel. Whenever \( k(t) = \sum_{n=0}^{\infty} k_n t^n \) has positive Taylor coefficients \( k_n \), we denote by \( \mathcal{H}_k \) the weighted Hilbert space of power series \( f(t) = \sum_{n=0}^{\infty} f_n t^n \) with finite norm

\[
\|f\|_{\mathcal{H}_k} := \left( \sum_{n=0}^{\infty} |f_n|^2 k_n \right)^{1/2}. \tag{1.2}
\]

Let \( B_k \) be the backward shift on \( \mathcal{H}_k \), defined by

\[
B_k f(t) = \frac{f(t) - f(0)}{t}. \tag{1.3}
\]

If \( a > 0 \) and \( k(t) = (1 - t)^{-\alpha} \), we denote the space \( \mathcal{H}_k \) by \( \mathcal{H}_a \), and the backward shift \( B_k \) by \( B_a \) to emphasize the dependence on \( a \).

It is known since Agler [3] that under certain additional conditions, the inequalities \( \alpha(t) \neq 0 \) for \( |t| < 1 \) and \( \alpha(T^*, T) \geq 0 \) imply the existence of a certain unitarily equivalent model of \( T \). As a consequence, in this situation, \( \tilde{T} \) (as in Theorem 1.1) has a model up to similarity.
In particular, in [13, Theorem 1.3], Clouâtre and Hartz showed the following result. Let \( \alpha \) be a function of Nevanlinna-Pick type. Suppose that \( k(t) := 1/\alpha(t) \) has radius of convergence 1, its Taylor coefficients \( k_n \) are positive and satisfy \( k_n/k_{n+1} \to 1 \) as \( n \to \infty \). Then \( \alpha(T^*, T) \geq 0 \) if and only if \( T \) is unitarily equivalent to a part of an operator of the form \( (B_k \otimes I_E) \oplus S \), where \( I_E \) is the identity operator on a Hilbert space \( E \) and \( S \) is an isometry on another auxiliary Hilbert space. By a part of an operator we mean its restriction to an invariant subspace. In fact, the result by Clouâtre and Hartz applies to tuples of commuting operators.

In [1, Theorem 1.5], we complement this result by obtaining other family of functions \( \alpha \), which includes non-Nevanlinna-Pick cases, so that all operators \( T \in \mathcal{L}(\mathcal{H}) \) satisfying \( \alpha(T^*, T) \geq 0 \) are modeled by parts of operators of the form \( B_k \otimes I_E \). Moreover, we give explicit models (that is, give an explicit space \( E \) and an explicit isometry \( S \)) based on the defect operator \( D \) and the defect space \( \mathcal{D} \), given by

\[
D : \mathcal{H} \to \mathcal{H}, \quad D := (\alpha(T^*, T))^{1/2}, \quad \mathcal{D} := \overline{D\mathcal{H}}. \tag{1.4}
\]

In [1], we also discuss the uniqueness of this model. This depends on whether \( \alpha(1) = 0 \) or \( \alpha(1) > 0 \).

We will discuss the following result.

**Theorem 1.3** If \( 0 < a < 1 \), then the following statements are equivalent.

(i) \( T \) is an \( a \)-contraction.

(ii) There exists a separable Hilbert space \( E \) such that \( T \) is unitarily equivalent to a part of an operator \( (B_a \otimes I_E) \oplus S \), where \( S \) is a Hilbert space isometry.

In that case, one can take for \( E \) the space \( \mathcal{D} \).

Notice that the function \( \alpha(t) = (1 - t)^a \), with \( a > 0 \), is of Nevanlinna-Pick type if and only if \( 0 < a < 1 \). Therefore, this theorem is known and follows using [13, Theorem 1.3]. The statement about the possibility to take \( E = \mathcal{D} \) follows from the explicit model obtained in [1]. The result of Clouâtre and Hartz relies on the study of reproducing kernel Hilbert spaces through the representation theory of their algebras of multipliers. Instead of this method involving \( C^* \)-algebras, here we give an alternative direct proof of Theorem 1.3 using approximation in Besov spaces.

**Corollary 1.4** Suppose that \( \tilde{T} \) satisfies the hypotheses of Corollary 1.2, where \( 0 < a < 1 \). Then \( \tilde{T} \) is similar to a part of an operator of the form \( (B_a \otimes I_E) \oplus S \), where \( S \) is a Hilbert space isometry.

There are many papers on \( m \)-isometries for positive integers \( m \). In the first place, we can mention seminal works by Helton [20], Ball and Helton [8], and Richter [25] where the concepts of 2 and 3-isometry appear. A known trilogy about \( m \)-isometries on Hilbert spaces was published by Agler and Stankus, see [4–6]. In these papers, interesting connections with differential operators, conjugate point theory and nonstationary stochastic processes were revealed. Other works on the topic are
by Bermúdez and coauthors, [16] by Gu, and [26] by Rydhe. In [22], more facts about 2-isometries are established. The recent work [18] discusses \( m \)-isometric tuples of operators on a Hilbert space. The \( m \)-contractions appear as particular cases of the families of operators considered by Gu in [17].

The study of \( a \)-contractions and \( a \)-isometries for non-integer \( a > 0 \) seems to be new. In [1] we discuss some ergodic properties of \( a \)-contractions when \( 0 < a < 1 \).

The topic of \( a \)-contractions and \( a \)-isometries is closely related with the topic of finite differences. Given a sequence of real numbers \( \Lambda = \{\Lambda_n\}_{n \geq 0} \), we denote by \( \nabla \Lambda \) the sequence whose \( n \)-th term is given by

\[
(\nabla \Lambda)_n = \Lambda_{n+1} - \Lambda_n,
\]

whenever the series on the right hand side converges for every \( n \geq 0 \). In particular, for the functions \((1 - t)^a\), with \( a \in \mathbb{R} \), we put

\[
(1 - \nabla)^a \Lambda_n = \sum_{j=0}^{\infty} k^{-a}(j)\Lambda_{j+n}.
\]

Here \( \{k^{-a}(j)\} \) are the so-called Cesàro numbers; i.e., \( k^{-a}(j) \) is the \( j \)-th Taylor coefficient (at the origin) of the function \((1 - t)^a\). The above formula is the forward finite difference of order \( a \) of the sequence \( \Lambda \). For instance, for \( a = 1 \) we get the first order finite difference \((1 - \nabla) \Lambda_n = \Lambda_{n+1} - \Lambda_n \). We address the following two questions.

**Question A** Determine for which \( a, b > 0 \) the inequality \((1 - \nabla)^a \Lambda_n \geq 0 \) (for every \( n \geq 0 \)) implies \((1 - \nabla)^b \Lambda_n \geq 0 \) (for every \( n \geq 0 \)).

**Question B** Given \( a > 0 \), determine the space of solutions \( \Lambda \) of the equation \((1 - \nabla)^a \Lambda = 0\).

We answer to Question A in Theorem 4.4, and to Question B in Theorem 4.6. These two theorems rely strongly on results by Kuttner in [21].

The basic relation between finite differences and \( a \)-contractions is as follows: an operator \( T \in L(H) \) is an \( a \)-contraction if and only if for any \( x \in H \) and any \( n \geq 0 \),

\[
(1 - \nabla)^a \Lambda_n (x) \geq 0, \quad \text{where } \Lambda_n (x) = \|T^nx\|^2.
\]

As an immediate consequence, we derive from Theorems 4.4 and 4.6 the following two results on \( a \)-contractions and \( a \)-isometries.

**Theorem 1.5** Let \( 0 < b < a \), where \( b \) is not an integer. If \( T \) is an \( a \)-contraction and \( T \in \text{Adm}^y_{(1-t)^b} \), then \( T \) is a \( b \)-contraction.
Notice that in general, the requirement \( T \in \text{Adm}_{(1-t)^{\phi}}^{w} \) can fail to hold for an \( a \)-contraction \( T \), and therefore cannot be omitted. For instance, Proposition 4.14 gives a source of examples of this fact.

**Theorem 1.6** Let \( a > 0 \), and let the integer \( m \) be defined by \( m < a \leq m + 1 \). Then the following statements are equivalent.

(i) \( T \) is an \( a \)-isometry.

(ii) \( T \) is an \((m + 1)\)-isometry.

(iii) For each vector \( h \in H \), there exists a polynomial \( p \) of degree at most \( m \) such that \( \|T^n h\|^2 = p(n) \) for every \( n \geq 0 \).

The contents of the paper are the following. In Section 2 we prove Theorem 1.1 using some Banach algebras techniques. In Section 3 we give a direct proof of Theorem 1.3 based on approximation in Besov spaces. In Section 4 we obtain answers to the above questions A and B about finite differences, and prove Theorems 1.5 and 1.6. Finally, in Sect. 5 we discuss the general form of a unitarily equivalent model for \( a \)-contractions with \( a > 1 \) and show that a natural conjecture about it is false. The form of such model and its construction remain open.

**2 Similarity results**

Recall that for a fixed operator \( T \in L(H) \) with \( \sigma(T) \subset \overline{D} \), we put

\[
A_T := \left\{ \alpha \in A_W : \sum_{n=0}^{\infty} |\alpha_n| T^n T^* n \text{ converges in SOT} \right\}. \tag{2.1}
\]

If \( X \) and \( Y \) are two quantities (typically non-negative), then \( X \preceq Y \) (or \( Y \succeq X \)) will mean that \( X \leq CY \) for some absolute constant \( C > 0 \). If the constant \( C \) depends on some parameter \( p \), then we write \( X \preceq_p Y \). We put \( X \asymp Y \) when both \( X \preceq Y \) and \( Y \preceq X \).

The main goal of this section is to prove Theorem 1.1. We will use the following known fact.

**Lemma 2.1** ([19, Problem 120]) If an increasing sequence \( \{A_n\} \) of selfadjoint Hilbert space operators satisfies \( A_n \preceq CI \) for all \( n \), where \( C \) is a constant, then \( \{A_n\} \) converges in the strong operator topology.

Using this lemma, in [1] we obtained the following fact.

**Proposition 2.2** ([1, Proposition 2.3]) Let \( \alpha \in A_W \). Then the following statements are equivalent.
(i) \( \alpha \in A_T \).

(ii) \( \sum_{n=0}^{\infty} |\alpha_n| \| T^n x \|^2 < \infty \) for every \( x \in H \).

(iii) \( \sum_{n=0}^{\infty} |\alpha_n| \| T^n x \|^2 \lesssim \| x \|^2 \) for every \( x \in H \).

For any \( \beta \in A_T \), define

\[
\beta_n := \| \beta(T^n, T) \|_{L(H)} + \| \beta \|_{A_W} \quad (2.2)
\]

(here \( |\beta|(t) = \sum_{n=0}^{\infty} |\beta_n| t^n \)). By Proposition 2.2, if \( \beta \in A_T \), then there exists a constant \( C \) such that

\[
\sum_{n=0}^{N} |\beta_n| \| T^n x \|^2 \leq C \| x \|^2 \quad (2.3)
\]

for every integer \( N \) and every \( x \in H \).

The following estimate is immediate.

**Proposition 2.3** If \( \beta \) is a function in \( A_T \), then for every vector \( x \in H \) we have

\[
\sum_{n=0}^{\infty} |\beta_n| \| T^n x \|^2 \leq \| \beta \|_{A_T} \| x \|^2.
\]

**Notation 2.4** If \( f(t) = \sum_{n=0}^{\infty} f_n t^n \) and \( g(t) = \sum_{n=0}^{\infty} g_n t^n \), we use the notation \( f \geq g \) when \( f_n \geq g_n \) for every \( n \geq 0 \), and the notation \( f \succ g \) when \( f \geq g \) and \( f_0 > g_0 \). For any non-negative integer \( N \), we denote by \( [f]_N \) the truncation \( \sum_{n=0}^{N} f_n t^n \).

**Theorem 2.5** For every operator \( T \) in \( L(H) \) with spectrum contained in \( \overline{D} \), \( A_T \) is a Banach algebra with norm given by (2.2).

**Proof** Let us prove that \( A_T \) has the multiplicative property of algebras and its completeness (the rest of properties for being a Banach algebra are immediate).

Let \( \beta \) and \( \gamma \) belong to \( A_T \). We want to prove that their product \( \delta \) also belongs to \( A_T \). Note that \( \delta \in A_W \). By Proposition 2.2, we just need to prove the existence of a constant \( C > 0 \) such that

\[
\sum_{n=0}^{N} |\delta_n| \| T^n x \|^2 \leq C \| x \|^2 \quad (2.4)
\]

for every non-negative integer \( N \) and every vector \( x \in H \).

So take any \( x \in H \), and let \( N \geq 0 \). Put

\[
|\beta|(t) := \sum_{n=0}^{\infty} |\beta_n| t^n, \quad |\gamma|(t) := \sum_{n=0}^{\infty} |\gamma_n| t^n, \quad \tilde{\delta}(t) = |\beta|(t) \cdot |\gamma|(t).
\]

Hence
\[ \tilde{\delta}_n = \sum_{j=0}^{n} |\beta_j| |\gamma_{n-j}| \geq |\delta_n|. \]

Therefore (2.4) will follow if we prove the existence of a positive constant \( C \) such that

\[ \sum_{n=0}^{N} \tilde{\delta}_n \|T^n x\|^2 \leq C \|x\|^2. \]  

(2.5)

Using that \(|\beta|_N \cdot |\gamma| \geq [\tilde{\delta}]_N\) (recall Notation 2.4), we have

\[ \sum_{n=0}^{N} \tilde{\delta}_n \|T^n x\|^2 = \left< [\tilde{\delta}]_N(T^n, T)x, x \right> \leq \left< (|\beta|_N \cdot |\gamma|)(T^n, T)x, x \right> \]

\[ = \sum_{n=0}^{N} \langle |\gamma|(T^n, T)|\beta_n|T^n x, T^n x \rangle \]

\[ \leq \| |\gamma|(T^n, T)\| \sum_{n=0}^{N} |\beta_n| \|T^n x\|^2 \]

\[ \leq \| |\gamma|(T^n, T)\| \|\beta\|_{A_T} \|x\|^2. \]

Note that the operator \(|\gamma|(T^n, T)\) belongs to \(L(H)\) because \(\gamma \in A_T\). Now we can take \( C = \| |\gamma|(T^n, T)\| \|\beta\|_{A_T} \) (which depends neither on \(N\) nor on \(x\)), and (2.5) follows.

Let us prove now the completeness of \(A_T\). Let \(\{\beta^{(k)}\}_{k \geq 0}\) be a Cauchy sequence in \(A_T\). In other words,

\[ \left\| \beta^{(k)} - \beta^{(\ell)} \right\|_{A_T} \to 0 \quad \text{when } k, \ell \to \infty. \]  

(2.6)

We want to prove the existence of a function \(\beta\) in \(A_T\) such that the sequence \(\{\beta^{(k)}\}_{k \geq 0}\) converges to \(\beta\) in the norm \(\| \cdot \|_{A_T}\). Since \(\| \cdot \|_{A_T} \geq \| \cdot \|_{A_W}\) and \(A_W\) is complete, there exists a function \(\beta\) in \(A_W\) such that \(\beta^{(k)}\) converge to \(\beta\) in the norm of \(A_W\). Now fix \(\varepsilon > 0\). Then by (2.6), there exists an integer \(M\) such that

\[ \left\| \beta^{(k)} - \beta^{(\ell)} \right\|_{A_T} < \varepsilon \]

if \(k, \ell \geq M\). In other words, for every \(N\) we have

\[ \left\| \sum_{n=0}^{N} |\beta^{(k)}_n - \beta^{(\ell)}_n|T^n x \right\|_{L(H)} + \left\| \beta^{(k)} - \beta^{(\ell)} \right\|_{A_W} < \varepsilon. \]

Now taking the limit when \(\ell \to \infty\) above, we obtain
Recall that in the Introduction we defined $A_0^T$ as

$$A_0^T = \{ \beta \in A_T : \sum_{n=0}^{\infty} |\beta_n| T^{*n} T^n \text{ converges in norm of } L(H) \}. \quad (2.7)$$

**Proposition 2.6** $A_0^T$ is the closure of the polynomials in $A_T$. In particular, it is a separable closed subalgebra of $A_T$.

**Proof** Let us denote provisionally by $CP$ the closure of the polynomials in $A_T$. Let $\beta \in CP$ and fix $\varepsilon > 0$. There exists a polynomial $p$ such that

$$\| \beta - p \|_{A_T} < \frac{\varepsilon}{2}.$$

Let $N$ be an integer larger than the degree of $p$. Since $p = [p]_N$,

$$\left\| \sum_{n=N+1}^{\infty} |\beta_n| T^{*n} T^n \right\| \leq \| \beta - [\beta]_N \|_{A_T} \leq \| \beta - p \|_{A_T} + \| [\beta - p]_N \|_{A_T},$$

$$\leq 2 \| \beta - p \|_{A_T} < \varepsilon.$$

Hence $\sum_{n=0}^{\infty} |\beta_n| T^{*n} T^n$ converges uniformly in $L(H)$. This proves the inclusion $CP \subset A_0^T$.

The inclusion $A_0^T \subset CP$ is immediate. Indeed, any $\beta \in A_0^T$ can be approximated in $A_T$ by the truncations $[\beta]_N$.

**Proposition 2.7** Let $\beta$ be a function in $A_T$.

(i) If $|\gamma_n| \leq |\beta_n|$ for every $n$, then $\gamma$ also belongs to $A_T$ and moreover $\| \gamma \|_{A_T} \leq \| \beta \|_{A_T}$.

(ii) If $\gamma_n = \beta_n \tau_n$, where $\{ \tau_n \} \subset \mathbb{C}$ and $\tau_n \to 0$, then $\gamma$ also belongs to $A_0^T$.

**Proof** (i) is immediate. For the proof of (ii), put

$$C_N := \max_{n \geq N} |\tau_n|,$$

for each positive integer $N$. Then
\[
\left\| \sum_{n=N}^{\infty} |\gamma_n| T^{n+1} \right\| \leq C_N \left\| \sum_{n=N}^{\infty} |\beta_n| T^{n+1} \right\| \leq C_N \|\beta\|_{\mathcal{A}_T} \to 0,
\]

and therefore \( \gamma \) belongs to \( \mathcal{A}_T^0 \).

**Proposition 2.8** The characters of \( \mathcal{A}_T^0 \) are precisely the evaluation functionals at points of \( \overline{D} \).

**Proof** Let \( \chi \) be a character of \( \mathcal{A}_T^0 \) (i.e., it is a multiplicative bounded linear functional on \( \mathcal{A}_T^0 \) that satisfies \( \chi(1) = 1 \)). For the function \( t \) in \( \mathcal{A}_T^0 \), let us put

\[
\lambda := \chi(t) \in \mathbb{C}.
\]

Therefore \( \chi \) sends every polynomial \( p(t) \) into the number \( p(\lambda) \). Let us prove now that \( |\lambda| \leq 1 \). Using the obvious fact

\[
|\lambda| = (|\lambda|^n)^{1/n} = |\chi(t^n)|^{1/n}
\]

we obtain

\[
|\lambda| = \limsup_{n \to \infty} |\chi(t^n)|^{1/n} \leq \limsup_{n \to \infty} \|t^n\|_{\mathcal{A}_T^0}^{1/n} \leq 1,
\]

where we have used that \( \|\chi\| = 1 \) (since it is a character) and that the spectral radius of \( T \) is less or equal than 1.

By the continuity of \( \chi \) and the density of the polynomials in \( \mathcal{A}_T^0 \) we obtain that \( \chi \) sends every function \( f(t) \) in \( \mathcal{A}_T^0 \) to \( f(\lambda) \).

**Corollary 2.9** If \( \beta \in \mathcal{A}_T^0 \) does not vanish on \( \overline{D} \), then \( 1/\beta \in \mathcal{A}_T^0 \).

**Proof** By Proposition 2.8, the condition on \( \beta \) means that \( \chi(\beta) \neq 0 \) for every character \( \chi \in \mathcal{A}_T^0 \). Hence the result follows using Gelfand Theory.

**Theorem 2.10** Let \( T \in L(H) \) with \( \sigma(T) \subset \overline{D} \) and let \( f \in \mathcal{A}_{T}^0 \). If \( f(t) > 0 \) for every \( t \in [0, 1] \), then there exists a function \( g \in \mathcal{A}_{T}^0 \) such that \( g > 0 \) and \( fg > 0 \).

As an immediate consequence of this theorem, we obtain the following result.

**Corollary 2.11** If \( f \in A_{W,R} \) satisfies \( f(t) > 0 \) for every \( t \in [0, 1] \), then there exists a function \( g \in A_{W,R} \) such that \( g > 0 \) and \( fg > 0 \).

**Proof** Take as \( T \) any power bounded operator. Then \( \sigma(T) \subset \overline{D} \) and \( \mathcal{A}_T^0 \) is precisely the Wiener algebra \( A_W \). Then apply Theorem 2.10.

For the proof of Theorem 2.10 we need the following result. We denote by \( \mathcal{H}(\overline{D}) \) the set of all analytic functions in a neighborhood of \( \overline{D} \).
Lemma 2.12 ([9, Lemma 2.1]) If \( q \) is a real polynomial such that \( q(t) > 0 \) for every \( t \in [0, 1] \), then there exists a rational function \( u \in \mathcal{H}(\mathbb{D}) \) such that \( u > 0 \) and \( uq > 0 \).

**Proof of Theorem 2.10** By hypothesis, there exists a positive number \( \epsilon \) such that \( f(t) > \epsilon > 0 \) for every \( t \in [0, 1] \).

**Claim** There exists a positive integer \( N \) such that

\[
\left\| \sum_{n=N+1}^{\infty} f_n t^n \right\|_{A_T^0} < \frac{\epsilon}{2}.
\]

Indeed, since \( f \in A_{W,R} \) we deduce that there exists a positive integer \( N_1 \) such that for every \( M \geq N_1 \) we have

\[
\sum_{n=M+1}^{\infty} |f_n| < \frac{\epsilon}{4}.
\]

Using Proposition 2.6 we obtain the existence of a positive integer \( N_2 \) such that for every \( M \geq N_2 \) we have

\[
\left\| \sum_{n=M+1}^{\infty} |f_n| T^n t^n \right\|_{L(H)} < \frac{\epsilon}{4}.
\]

Hence, taking \( N := \max\{N_1, N_2\} \) the claim follows.

In particular, we get

\[
\sum_{n=N+1}^{\infty} |f_n| < \frac{\epsilon}{2}.
\]

Put

\[
f_N(t) := \sum_{n=0}^{N} f_n t^n - \frac{\epsilon}{2}, \quad h(t) := \frac{\epsilon}{2} + \sum_{n \geq N+1, f_n < 0} f_n t^n.
\]

Note that \( f_N \) is a polynomial. It is easy to see that \( h \in A_T^0 \). Since

\[
f_N(t) > \epsilon - \frac{\epsilon}{2} - \frac{\epsilon}{2} = 0
\]

for every \( t \in [0, 1] \), we can use Lemma 2.12 to obtain a function \( u \in \mathcal{H}(\mathbb{D}) \) such that \( u > 0 \) and \( uf_N > 0 \). Note that it is immediate that all the functions in \( \mathcal{H}(\mathbb{D}) \) also belong to the algebra \( A_T^0 \).

For every \( t \in \mathbb{D} \) we have

\[
\left| \sum_{n \geq N+1, f_n < 0} f_n t^n \right| \leq \sum_{n=N+1}^{\infty} |f_n| < \frac{\epsilon}{2}.
\]
Hence $h$ does not vanish on $\overline{D}$, and therefore using Corollary 2.9 we obtain

\[ v := h^{-1} \in A^0_T. \]

Note that $v > 0$. Indeed, if $\sum_{n=0}^{\infty} a_n r^n$ and $\sum_{n=0}^{\infty} b_n t^n$ are two formal power series whose product is 1 with $a_0 > 0$ and $a_n \leq 0$ for all $n \geq 1$, then $b_0 > 0$ and $b_n \geq 0$ for all $n \geq 1$ (use induction, for example).

Finally, put $g := uv$. Obviously $g > 0$, and since $f \succcurlyeq f_N + h$ we have

\[ gf \succcurlyeq g(f_N + h) = vuf_N + u > 0, \]

as we wanted to prove.

Proposition 2.13 Let $f \in A_{T,R}$ and let $B \in L(H)$ be a non-negative operator. Then the operator series

\[ \sum_{n=0}^{\infty} f_n T^n BT^n \]  \hspace{1cm} (2.8)

converges in the strong operator topology in $L(H)$.

Proof First we observe that the above series converges in the weak operator topology. Indeed, for every pair of vectors $x, y \in H$ we have

\[
\left| \left( \sum_{n=N}^{M} f_n T^n BT^n \right) x, y \right| = \left| \sum_{n=N}^{M} f_n \langle BT^n x, T^n y \rangle \right| \\
\leq \sum_{n=N}^{M} |f_n| \|B\| \|T^n x\| \|T^n y\| \\
\leq \sum_{n=N}^{M} |f_n| \|B\| (\|T^n x\|^2 + \|T^n y\|^2).
\]

Since the last sum tends to 0 when $N, M \to \infty$, the statement follows. Next, put

\[ f^+_n := \max\{f_n, 0\}, \quad f^-_n := \max\{-f_n, 0\}. \]

By the above, the series

\[ \sum_{n=0}^{\infty} f^+_n T^n BT^n \quad \text{and} \quad \sum_{n=0}^{\infty} f^-_n T^n BT^n \]

converge in the weak operator topology in $L(H)$. Hence the norms of partial sums of these series are uniformly bounded. Then Lemma 2.1 gives the convergence in SOT of these series. Since $f_n = f^+_n - f^-_n$, we also obtain the convergence in SOT of the series $\sum f_n T^n BT^n$. \qed
**Definition 2.14** As a consequence of Proposition 2.13, for every \( f \in \mathcal{A}_{T,\mathbb{R}} \) and every non-negative operator \( B \in L(H) \) we can define

\[
 f(T^*, T)(B) := \sum_{n=0}^{\infty} f_n T^{*n} BT^n,
\]

where the convergence is in SOT. In particular, when \( B \) is the identity operator in \( L(H) \),

\[
 f(T^*, T) = f(T^*, T)(I) = \sum_{n=0}^{\infty} f_n T^{*n} T^n.
\]

**Remark 2.15** Observe that, by applying Proposition 2.3 to the vector \( x = B^{1/2}h \), we get

\[
 \|f(T^*, T)(B)\| \lesssim \|B\| \|f\|_{\mathcal{A}_T}.
\]  

**Lemma 2.16** Let \( B \in L(H) \) be a non-negative operator and let \( f, g \in \mathcal{A}_{T,\mathbb{R}} \). Put \( h := fg \). Then

(i) \( h(T^*, T)(B) = g(T^*, T)(f(T^*, T)(B)) \);

(ii) \( h(T^*, T) = g(T^*, T)(f(T^*, T)) \).

**Proof** Note that (ii) is just an application of (i) for \( B = I \). Let us start proving (i) for the case where all the coefficients \( f_n \) and \( g_n \) are non-negative. In this case, both parts of (i) are well-defined by Proposition 2.13. Then

\[
 g(T^*, T)(f(T^*, T)(B)) = \sum_{n=0}^{\infty} g_n T^{*n} \left( \sum_{m=0}^{\infty} f_m T^{*m} BT^m \right) T^n
\]

\[
 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g_n f_m T^{*n+m} BT^{n+m}
\]

\[
 = \sum_{k=0}^{\infty} \left( \sum_{n+m=k} f_m g_n \right) T^{*k} BT^k = h(T^*, T)(B),
\]

where in (\( \ast \)) all the series are understood in the sense of the SOT convergence. To justify it, it suffices to pass to quadratic forms and to use that \( f_m \) and \( g_n \) are all non-negative. Finally, the general case \( f, g \in \mathcal{A}_T \) can be derived from the previous one by linearity and using the decompositions

\[
 f = f^+ - f^- \quad \text{and} \quad g = g^+ - g^-,
\]

where \( f^+, f^-, g^+ \) and \( g^- \) have non-negative Taylor coefficients.

**Proof of Theorem 1.1** By Theorem 2.10, there exists a function \( g \in \mathcal{A}_{T,\mathbb{R}}^0 \) such that \( g > 0 \) and \( h := \gamma g > 0 \). Then \( \gamma(t)g(t) = h(t)\alpha(t) \) and, by Lemma 2.16 (ii), we get
Define an operator $B > 0$ by $B^2 := h(\hat{T}^*, \hat{T}) \geq \varepsilon I > 0$ (for some $\varepsilon > 0$). Then we have
\[
\sum_{n=0}^{\infty} \alpha_n \hat{T}^{*n} h(\hat{T}^*, \hat{T}) \hat{T}^n = \sum_{n=0}^{\infty} g_n \hat{T}^{*n} y(\hat{T}^*, \hat{T}) \hat{T}^n \geq 0.
\] (2.10)

In particular this proves Corollary 1.2. The case $a = 1$ in this corollary can be compared with [23, Theorem 3.10]. Notice, however, that in that theorem by Müller it is assumed that $T$ is a contraction.

### 3 A direct proof of Theorem 1.3

Our proof will be based on approximation in Besov spaces.

**Proof of Theorem 1.3** The implication (ii) $\Rightarrow$ (i) is straightforward, so we focus on the implication (i) $\Rightarrow$ (ii). Let $T$ be an $a$-contraction. We divide the proof into three steps. First, we define the operator $S$, then we prove that $S$ is an isometry, and finally we prove that $T$ is unitarily equivalent to a part of $(B_a \otimes I_{\mathcal{H}}) \oplus S$.

We put $\alpha(t) := (1 - t)^a$ and $k(t) := (1 - t)^{-a} > 0$. Recall that $D$ is the non-negative square root of the operator $(1 - t)^a(T^*, T) \geq 0$.

**Step 1** Note that
\[
\|Dx\|^2 = \sum_{j=0}^{\infty} \alpha_j \|T^j x\|^2 \quad (\forall x \in H).
\] (3.1)

Changing $x$ by $T^n x$ in (3.1) we obtain a more general formula:
\[
\|DT^n x\|^2 = \sum_{j=0}^{\infty} \alpha_j \|T^{j+n} x\|^2 \quad (\forall x \in H, \forall n \geq 0).
\] (3.2)
Multiplying (3.2) by \(k_n\) and summing for \(n = 0, 1, \ldots, N\) (for some fixed \(N \in \mathbb{N}\)), we get the following equation

\[
\|x\|^2 = \sum_{n=0}^{N} k_n\|DT^n x\|^2 + \sum_{m=N+1}^{\infty} \|T^m x\|^2 \rho_{N,m} \quad (\forall x \in H),
\]  

(3.3)

where

\[
\rho_{N,m} = \sum_{n=N+1}^{m} k_n \alpha_{m-n} = -\sum_{j=0}^{N} k_j \alpha_{m-j} \quad (1 \leq N + 1 \leq m).
\]  

(3.4)

Since \(0 < a < 1\), we have \(\alpha_n < 0\) for every \(n \geq 1\) (and \(\alpha_0 = 1\)). Note that in the last sum in (3.4) all the \(k_j\)'s are positive and all the \(\alpha_n\)'s are negative, because \(\alpha_0\) does not appear there. We obtain that \(\rho_{N,m} > 0\). Therefore, by (3.3) we have

\[
\|x\|^2 \geq \sum_{n=0}^{N} k_n\|DT^n x\|^2.
\]

Hence the series with positive terms \(\sum k_n\|DT^n x\|^2\) converges, and taking limits in (3.3) when \(N \to \infty\) we obtain

\[
\exists \lim_{N \to \infty} \sum_{m=N+1}^{\infty} \|T^m x\|^2 \rho_{N,m} = \|x\|^2 - \sum_{n=0}^{\infty} k_n\|DT^n x\|^2 \geq 0.
\]  

(3.5)

We are going to define a new semi-inner product on our Hilbert space \(H\) via

\[
[x, y] := \lim_{N \to \infty} \sum_{m=N+1}^{\infty} \langle T^m x, T^m y \rangle \rho_{N,m}.
\]  

(3.6)

By (3.5), \([x, x]\) is correctly defined, and \([x, x] = \langle Ax, x \rangle\), where \(A\) is a self-adjoint operator with \(0 \leq A \leq I\). Hence, by the polarization formula \([x, y]\) is correctly defined for any \(x, y \in H\), and \([x, y] = \langle Ax, y \rangle\).

Let \(E := \{x \in H : [x, x] = 0\}\). It is a closed subspace of \(H\). Put \(\widehat{H} := H/E\). For any vector \(x \in H\), we denote by \(\widehat{x}\) its equivalence class in \(\widehat{H}\). Note that \(\widehat{H} := H/E\) is a new Hilbert space with norm \(\|\cdot\|\) given by

\[
\|\widehat{x}\|^2 = \lim_{N \to \infty} \sum_{m=N+1}^{\infty} \|T^m x\|^2 \rho_{N,m} = \|x\|^2 - \sum_{n=0}^{\infty} k_n\|DT^n x\|^2.
\]  

(3.7)

We set \(S\) to be the operator on \(\widehat{H}\), given by \(\widehat{S}\widehat{x} := \widehat{T}x\) for every \(x \in H\).

**Step 2** Let us see now that the equality \(\|\widehat{T}x\| = \|\widehat{x}\|\) holds for every \(x \in H\).

Observe that this will imply, in particular, that \(S\) is well-defined.

Indeed, note that
\[ \sum_{n=0}^{\infty} k_n \| DT^n x \|^2 = \lim_{r\to1} \sum_{n=0}^{\infty} r^n k_n \| DT^n x \|^2 \quad (3.8) \]

since the RHS is an increasing function of \( r \). Using (3.1), (3.7) and (3.8) we obtain that

\[ ||\hat{x}||^2 = \|x\| - \lim_{r\to1} \sum_{n=0}^{\infty} r^n k_n \| DT^n x \|^2 \]

\[ \quad = \|x\| - \lim_{r\to1} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} r^n k_n \alpha_j \| T^{n+j} x \|^2 \quad (3.9) \]

\[ \quad = \|x\| - \lim_{r\to1} \sum_{m=0}^{\infty} \| T^m x \|^2 u_r(m), \]

where

\[ u_r(m) := \sum_{n+j=m, n, j \geq 0} r^n k_n \alpha_j. \quad (3.10) \]

For each \( m \), \( u_r(m) \) is a continuous function of \( r \in [0, 1] \). We also have \( u_1(m) = 0 \) for \( m \geq 1 \) and \( u_r(0) = 1 \) for \( r \in [0, 1] \). Moreover, since \( u_r(1) = a(1-r) \), we have \( u_r(1) \to 0 \) as \( r \to 1^- \). Therefore

\[ ||\hat{T}x||^2 - ||\hat{x}||^2 = \lim_{r\to1} \sum_{m=2}^{\infty} \| T^m x \|^2 [u_r(m) - u_r(m-1)]. \quad (3.11) \]

We have to prove that it is zero for any \( x \in H \).

**Claim** *There exists a constant \( C > 0 \) independent of \( r \) and \( m \) such that*

\[ |u_r(m) - u_r(m-1)| \leq \frac{C}{m^{1+a}} \quad (\forall m \geq 2). \quad (3.12) \]

Indeed, it is easy to check that

\[ \sum_{m=0}^{\infty} u_r(m) t^m = \frac{(1-t)^a}{(1-rt)^a} \quad (|r| < 1 \text{ and } 0 < r < 1). \]

Multiplying by \((1-t)\), we get

\[ 1 + \sum_{m=1}^{\infty} [u_r(m) - u_r(m-1)] t^m = \frac{(1-t)^{i+1}}{(1-rt)^a} =: f_r(t). \]

Hence (3.12) is equivalent to

\[ \hat{f}_r(n) \leq \frac{C}{(n+1)^{1+a}} \quad (\forall n), \quad (3.13) \]
where the constant $C$ has to be independent of $r$ and $n$. Equivalently, we want to prove that the Fourier coefficients of

$$\sum_{n=0}^{\infty} (1 + n)^{1+a} \hat{f}_r(n) z^n = I_{1-a} f_r$$

(3.14)

are uniformly bounded, where for $\beta \in \mathbb{R}$,

$$I_{\beta} h(z) := \sum_{j=0}^{\infty} (1 + j)^{-\beta} \hat{h}(j) z^j,$$

as in [24, p. 737]. We will prove the even stronger result

$$\left\| I_{1-a} f_r'' \right\|_{H^1} \leq C,$$

(3.15)

where the constant $C$ does not depend on $r$. Here, $H^p$ denotes the classical Hardy space of the unit disc. Since for any $\beta > 0$

$$I_{\beta} h \in H^1 \iff \int_{\mathbb{T}} \left( \int_{0}^{1} |h(\rho \xi)|^2 (1 - r)^{2\beta - 1} d\rho \right)^{1/2} |d\xi| < \infty \quad (3.16)$$

(see [24, p. 737]), we obtain that (3.15) is equivalent to

$$\sup_{r \in [0,1]} \int_{\mathbb{T}} \left[ \int_{0}^{1} |f_r''(\rho \xi)|^2 (1 - \rho)^{1-2a} d\rho \right]^{1/2} |d\xi| \leq C. \quad (3.17)$$

It is immediate to check that $f_r''$ can be represented as

$$f_r''(t) = \sum_{j=0}^{2} c_j(r)(1 - t)^{a-1+j}(1 - rt)^{-a-j},$$

where the $c_j$’s are bounded functions. Using that $|1 - rt| \geq M |1 - t|$ for a certain constant $M$ (for $|t| < 1$ and $0 < r < 1$), we obtain that

$$|f_r''(t)| \leq C_1 |1 - t|^{-1}.$$

Therefore we just need to prove that

$$\int_{\mathbb{T}} \left[ \int_{0}^{1} |1 - \rho \xi|^{-2}(1 - \rho)^{1-2a} d\rho \right]^{1/2} |d\xi| < \infty \quad (3.18)$$

because now we do not have dependence on $r$. By (3.16), this is equivalent to

$$I_{1-a} g \in H^1, \quad \text{where} \quad g(t) = (1 - t)^{-1}.$$

Hence it remains to check that
For the sake of an easier notation, we will prove the equivalent statement
\[
\sum_{n=1}^{\infty} n^{a-1}r^n \in H^1.
\] (3.19)

Recall that \(k(t) = (1 - t)^{-a}\). Then, by [29, Vol. I, p. 77 (1.18)] we have
\[
k_n = \frac{1}{\Gamma(a)} n^{a-1} \left(1 + O\left(\frac{1}{n}\right)\right).
\]

Therefore
\[
n^{a-1} = \Gamma(a)k_n - n^{a-1}v_n, \quad \text{where } |v_n| \lesssim n^{-1}.
\]

Since \(0 < a < 1\), we know that \(\sum k_n r^n = (1 - t)^{-a}\) belongs to \(H^1\). Since the function \(\sum n^{a-1}v_n r^n\) belongs to \(H^2\), (3.19) follows. This finishes the proof of the claim.

Finally, since \(T\) is an \(a\)-contraction, we know that
\[
\sum_{n=1}^{\infty} \frac{1}{n^{1+a}} \|Tx\|^2 \sim \sum_{n=0}^{\infty} |\alpha_n| \|T^n x\|^2 \quad \text{converges for every } x \in H.
\] (3.20)

By the claim, we can estimate the series in the right hand side of (3.11) as
\[
\sum_{m=2}^{\infty} \|T^m x\|^2 |u_r(m) - u_r(m - 1)| \lesssim \sum_{m=2}^{\infty} \frac{\|T^m x\|^2}{m^{1+a}} < \infty.
\]

Hence, Lebesgue’s Dominated Convergence Theorem allows us to exchange the limit with the sign of sum in (3.11). Using that \(u_1(j) = 0\) for every \(j \geq 1\), we obtain that \(\|\hat{T}x\|^2 = \|\hat{x}\|^2\) for any \(x \in H\). Hence \(S\) is well-defined and it is an isometry.

**Step 3** As usual, \(\mathfrak{D}\) is the closure of \(DH\). Let
\[
G : H \to (\mathcal{H}_a \otimes \mathfrak{D}) \oplus \hat{H}, \quad Gx := (\{Dx, DTx, DT^2x, \ldots\}, \hat{x}).
\]

By (3.7), \(G\) is an isometry. It is immediate that
\[
((B_a \otimes I_\mathfrak{D}) \oplus S)G = GT.
\]

Hence we get that \(T\) is unitarily equivalent to a part of \((B_a \otimes I_\mathfrak{D}) \oplus S\).

In Sect. 5, we will see that an analogue of Theorem 1.3 is false if \(a > 1\).
4 Inclusions for classes of \(\alpha\)-contractions and \(\alpha\)-isometries

In this section we prove Theorems 1.5 and 1.6. Let us begin with the following three results given in [21] by Kuttner.

**Theorem 4.1** ([21, Theorem 3]) Let \(\sigma > -1, \sigma\) not an integer. Let \(\Lambda = \{\Lambda_n\}_{n \geq 0}\) be a sequence of real numbers. Suppose that \((1 - \nabla)^{\sigma}\Lambda\) exists. If

\[
\begin{align*}
  s &\geq \sigma, & r + s &> \sigma, & \tau &\geq \sigma - r, & \tau &\geq 0, & (4.1)
\end{align*}
\]

or if

\[
\begin{align*}
  s &\geq \sigma, & r + s &= \sigma, & \tau &> \sigma - r, & \tau &\geq 0, & (4.2)
\end{align*}
\]

then

\[
(1 - \nabla)^{\tau - s}[((1 - \nabla)^{\sigma}\Lambda)_n] = \sum_{j=0}^{\infty} k^{-j}(\sum_{m=0}^{\infty} k^{-s}(m)\Lambda_{m+j+n})
\]

is summable \((C, \tau)\) to \((1 - \nabla)^{r+s}\Lambda_n\).

Recall that a sequence \(\{s_n\}_{n \geq 0}\) is summable \((C, \tau)\) to a limit \(s\) if the sequence of its Cesàro means of order \(\tau\)

\[
C_{\tau}s_n := \frac{1}{k^{\tau+1}(n)} \sum_{j=0}^{n} k^{\tau}(n-j)s_j
\]

converges to \(s\) as \(n\) goes to infinity. Summability \((C, 0)\) is ordinary convergence, and summability \((C, \tau)\) of a sequence implies summability \((C, \tau')\) to the same limit for any \(\tau' > \tau\).

**Theorem 4.2** ([21, Theorem A]) Let \(s > -1 \text{ and } r \geq 0. \text{ If } \Lambda = \{\Lambda_n\}_{n \geq 0} \text{ is a sequence of real numbers, then}

\[
(1 - \nabla)^{r+s}\Lambda_n = (1 - \nabla)^{\tau}[((1 - \nabla)^{\sigma}\Lambda)_n]
\]

for every \(n \geq 0, \) whenever the right hand side exists.

**Theorem 4.3** ([21, Theorem B]) Let \(s > -1, r + s > -1 \text{ and } r + s \text{ be non-integer. If}

\[
\Lambda = \{\Lambda_n\}_{n \geq 0} \text{ is a sequence of real numbers, then}
\]

\[
(1 - \nabla)^{r+s}\Lambda_n = (1 - \nabla)^{\tau}[((1 - \nabla)^{\sigma}\Lambda)_n]
\]

for every \(n \geq 0, \) whenever both sides above exist.

Now we give an answer to Question A from the Introduction. As a consequence, we will derive Theorem 1.5.
Theorem 4.4 Let \( \Lambda = \{ \Lambda_n \}_{n \geq 0} \) be a sequence of real numbers, and let \( 0 < b < a \), where \( b \) is not an integer. If \((1 - \nabla)^b \Lambda_n \geq 0 \) for every \( n \geq 0 \) and \((1 - \nabla)^b \Lambda \) exists, then \((1 - \nabla)^b \Lambda_n \geq 0 \) for every \( n \geq 0 \).

Proof Let \( 0 < b < a \), and let \( \Lambda = \{ \Lambda_n \}_{n \geq 0} \) be a sequence of real numbers such that \((1 - \nabla)^a \Lambda_n \geq 0 \) for every \( n \geq 0 \), and \((1 - \nabla)^b \Lambda \) exists. Putting
\[
\sigma = b, \quad s = a, \quad r = b - a, \quad \tau = \lfloor a \rfloor + 1
\]
in (4.2) (where \( \lfloor a \rfloor \) denotes the biggest integer less than or equal to \( a \)), we obtain that the series
\[
(1 - \nabla)^{b-a}[(1 - \nabla)^a \Lambda]_n = \sum_{j=0}^{\infty} k^{a-b}(j) \left( \sum_{m=0}^{\infty} k^{-a}(m) \Lambda_{m+j+n} \right)
\]
is summable \((C, [a] + 1)\) to \((1 - \nabla)^b \Lambda_n \).

Note that \( k^{a-b}(m) \geq 0 \) for every \( m \geq 0 \) (because \( a - b > 0 \)) and the series in parenthesis in the right hand part of (4.3) are non-negative for every \( j \geq 0 \). We deduce that \((1 - \nabla)^b \Lambda_n \geq 0 \) for every \( n \geq 0 \), as we wanted to prove. \(\square\)

Proof of Theorem 1.5 Let \( 0 < b < a \), where \( b \) is not an integer, and let \( T \) be an \( a \)-contraction such that \( T \in \text{Adm}_b^w \). Fix \( x \in H \) and put \( \Lambda_n := \|T^n x\|^2 \), for \( n \geq 0 \). If we show that
\[
(1 - \nabla)^b \Lambda_n = \sum_{j=0}^{\infty} k^{b}(j) \Lambda_{j+n} = \sum_{j=0}^{\infty} k^{b}(j) \|T^j x\|^2 \geq 0,
\]
for every \( n \geq 0 \), since \( x \in H \) is arbitrary, then \( T \) would be a \( b \)-contraction. But (4.4) follows immediately from the previous theorem, since \((1 - \nabla)^a \Lambda_n \geq 0 \) for every \( n \geq 0 \) (because \( T \) is an \( a \)-contraction) and \((1 - \nabla)^b \Lambda \) exists (since \( T \in \text{Adm}_b^w \)). \(\square\)

We need to recall the following asymptotic behavior of the Cesàro numbers \( k^a(n) \).

Proposition 4.5 If \( a \in \mathbb{C} \setminus \{0, -1, -2, \ldots \} \), then
\[
k^a(n) = \frac{\Gamma(n + a)}{\Gamma(a) \Gamma(n + 1)} = \binom{n + a - 1}{a - 1} \quad \forall n \geq 0,
\]
where \( \Gamma \) is Euler’s Gamma function. Therefore
\[
k^a(n) = \frac{n^{a-1}}{\Gamma(a)} (1 + O(1/n)) \quad \text{as} \ n \to \infty.
\]
Moreover, if \( 0 < a \leq 1 \), then
\[
\frac{(n + 1)^{a-1}}{\Gamma(a)} \leq k^a(n) \leq \frac{n^{a-1}}{\Gamma(a)} \quad \forall n \geq 1.
\]
Proof See [29, Vol. I, p. 77, eq. (1.18)] and [14, eq.(1)]. The last inequality follows from the Gautschi inequality (see [15, eq. (7)]). \qed

The next statement will be used in the proof of Theorem 1.6.

**Theorem 4.6** Let \( a > 0 \), and let the integer \( m \) be defined by \( m < a \leq m + 1 \). If \( \Lambda = \{\Lambda_n\}_{n \geq 0} \) is a sequence of real numbers, then the following statements are equivalent:

(i) \((1 - \nabla)^a \Lambda \equiv 0\) (i.e., all the terms of the sequence \((1 - \nabla)^a \Lambda\) are 0);

(ii) \((1 - \nabla)^{m + 1} \Lambda \equiv 0\);

(iii) There exists a polynomial \( p \) of degree at most \( m \) such that \( \Lambda_n = p(n) \) for every \( n \geq 0 \).

Proof The equivalence (ii) \( \Leftrightarrow \) (iii) is a well known fact (see [11, Theorem 2.1]). Suppose that (i) is true. Let us see that

\[
(1 - \nabla)^{m + 1} \Lambda_n = (1 - \nabla)^{m + 1 - a} [(1 - \nabla)^a \Lambda]_n
\]

(4.6)

for every \( n \geq 0 \). Indeed, the RHS of (4.6) is obviously 0 by assumption. Then we can apply Theorem 4.2 with \( s = a > -1 \) and \( r = m - a \geq 0 \), and (4.6) follows. Therefore we obtain that (i) \( \Rightarrow \) (ii).

Suppose now that (ii) is true. Hence, we also have (iii). Obviously, if \( a = m + 1 \) we obtain (i). Now let us prove (i) for \( m < a < m + 1 \) (so \( a \) is non-integer). We will see that

\[
(1 - \nabla)^a \Lambda_n = (1 - \nabla)^{a - m - 1} [(1 - \nabla)^{m + 1} \Lambda]_n
\]

(4.7)

for every \( n \geq 0 \). Indeed, fix \( n \geq 0 \). Then, by (iii) and (4.5), we have

\[
\Lambda_{j+n} \lesssim (j+1)^{m}.
\]

Therefore

\[
\sum_{j=0}^{\infty} |k^{-a}(j)| \Lambda_{j+n} \lesssim \sum_{j=0}^{\infty} (j+1)^{-a-1}(j+1)^{m} = \sum_{j=0}^{\infty} (j+1)^{m-a-1} < \infty
\]

since \( m < a \). Hence, the series

\[
\sum_{j=0}^{\infty} k^{-a}(j) \Lambda_{j+n}
\]

converges for every \( n \geq 0 \). Thus the LHS of (4.7) exists. The RHS of (4.7) obviously exists since we are assuming (ii). Therefore, taking \( s = m + 1 \) and \( r + s = a \) in Theorem 4.2 we obtain that indeed (4.7) holds, and hence we obtain (i). \qed
Proof of Theorem 1.6 The equivalence between statements (ii) and (iii) is well-known. Suppose that (i) is true. Then, fixing \( h \in H \) and taking \( \Lambda_n := \|T^n h\| \), note that (ii) follows immediately by applying Theorem 4.6.

Suppose now that we have (ii), that is, \( T \) is an \((m+1)\)-isometry. Then, \( \|T^n\|^2 \lesssim (n+1)^m \). Therefore

\[
\sum_{n=0}^{\infty} (n+1)^{-a-1} \|T^n h\|^2 \lesssim \sum_{n=0}^{\infty} n^{m-a-1},
\]

for every \( h \in H \). The last series above converges since \( m < a \). This means that \( T \in \text{Adm}^w_a \), and then (i) follows using Theorem 4.6 again.

\( \square \)

Remark 4.7 In [7, Proposition 8], for any positive integer \( m \), Athavale gives an example of an operator \( T \) (a unilateral weighted shift), which is an \((m+1)\)-isometry but not \( n \)-isometry for any positive integer \( n \leq m \).

We also have the following result.

Theorem 4.8 Let \( 0 < c < b < a \) where \( c \) is not an integer. If \( T \) is an \( a \)-contraction and \( (1-t)^c \) belongs to \( \mathcal{A}_T \), then \( T \) is a \( b \)-contraction.

Proof Fix \( x \in H \) and let \( \Lambda_n := \|T^n x\|^2 \). Taking

\[
\sigma = c, \quad s = a, \quad r = b - a, \quad \tau = [a - b + c] + 1
\]

in (4.1), the statement follows.

\( \square \)

In fact, the only new case in this theorem is when \( b \) is an integer, otherwise it follows from Theorem 1.5.

Let us study now some properties relating weighted shifts with \( a \)-contractions. Given \( s > 0 \), recall that the space \( \mathcal{H}_s \) and the backward shift \( B_s \) on it have been defined in the Introduction, see (1.2), (1.3) and a comment following these formulas. The forward shift \( F_s \) is defined on \( \mathcal{H}_s \) by

\[
F_s f(t) = tf(t).
\]

(4.8)

The asymptotic behavior of the norms of the powers of \( B_s \) and \( F_s \) is

\[
\|B_s^n\|^2 \approx (m + 1)^{\max\{1-s,0\}} \quad \text{and} \quad \|F_s^n\|^2 \approx (m + 1)^{\max\{s-1,0\}}
\]

(4.9)

(see for example [1, eq. (7.6)]).

Theorem 4.9 ([1, Theorem 7.2]) Let \( a \) and \( s \) be positive numbers. Then:

(i) \( B_s \in \text{Adm}^w_a \);

(ii) \( B_s \) is an \( a \)-contraction if and only if \( a \leq s \).
To obtain the corresponding result for the forward shift $F_s$, we need to introduce the following sets:

$$J_{\text{even}} := \bigcup_{j \in \mathbb{Z}_{\geq 0}} (2j, 2j + 1), \quad J_{\text{odd}} := \bigcup_{j \in \mathbb{Z}_{\geq 0}} (2j + 1, 2j + 2). \quad (4.10)$$

Then $\{J_{\text{even}}, J_{\text{odd}}, \mathbb{N}\}$ is a partition of the interval $(0, \infty)$.

**Theorem 4.10** Let $a$ and $s$ be positive numbers. Then $F_s \in \text{Adm}_a^w$ if and only if $s < a + 1$ or $a$ is integer.

**Proof** If $a$ is a positive integer, then obviously any operator in $L(H)$ belongs to $\text{Adm}_a^w$, since in this case $(1 - t)^a$ is just a polynomial. Suppose now that $a$ is not an integer. We will use the notation of [1, Theorem 2.15]. Now $x(t) = (1 - t)^{-s}$ and $\alpha(t) = (1 - t)^a$. Therefore

$$\beta(\nabla)x_m = \sum_{n=0}^{\infty} |k^{-a}(n)| \|F_s^n e_m\|^2 \simeq \sum_{n=0}^{\infty} (n + 1)^{-a-1}(n + m + 1)^{s-1}. $$

If $F_s \in \text{Adm}_a^w$, then the above series must converge for every $m \geq 0$. For each $m$, this series indeed behaves as $\sum n^{s-a-2}$. Hence it implies that $s < a + 1$.

Reciprocally, suppose now that $s < a + 1$. Then

$$\sum_{n=0}^{\infty} (n + 1)^{-a-1}(n + m + 1)^{s-1} \simeq (m + 1)^{s-1} + (m + 1)^{s-1} \simeq x_m. $$

Therefore [1, Theorem 2.15 (i)] implies that $F_s \in \text{Adm}_a^w$. \hfill \Box

**Theorem 4.11** Let $a$ and $s$ be positive numbers and $s < a + 1$. Then:

(i) $(1 - t)^a(F_s^m, F_s) \geq 0$ (that is, $F_s$ is an $a$-contraction) if and only if $s \in J_{\text{even}} \cup \mathbb{N}$;

(ii) $(1 - t)^a(F_s^m, F_s) \leq 0$ if and only if $s \in J_{\text{odd}} \cup \mathbb{N}$;

(iii) $(1 - t)^a(F_s^m, F_s) = 0$ (that is, $F_s$ is an $a$-isometry) if and only if $s \in \mathbb{N}$.

**Proof** By [1, Theorem 2.15], $(1 - t)^a(F_s^m, F_s) \geq 0$ if and only if $(1 - \nabla)^a k^s(m) \geq 0$ for all $m \geq 0$, and similar assertions hold in the context of (ii) and (iii). So we need to study the signs of

$$(1 - \nabla)^a k^s(m) = \sum_{n=0}^{\infty} k^{-a}(n)k^s(n + m).$$

We assert that for any $s < a + 1$,
(1 − ∇)^a k^s(m) = \frac{\sin(\pi s) \Gamma(1 - s + a) \Gamma(s + m)}{\pi \Gamma(m + a + 1)}. \quad (4.11)

Suppose first that \(a\) is a positive integer. Then, by [2, Example 3.4 (ii)], we have

\[(1 − ∇)^a k^s(m) = (-1)^s k^{s-a}(m + a),\]

for every non-negative integer \(m\), and the statement follows easily.

Suppose that \(s \in (0, 1)\) and let \(a\) be any real number with \(a > s - 1\). Using the expression for \(k^s(m)\) given in Proposition 4.5 and applying the idea of the proof of [2, Lemma 1.1], we have

\[
(1 − ∇)^a k^s(m) = \frac{1}{\Gamma(s) \Gamma(1 - s)} \sum_{l=0}^{\infty} k^{-a}(l) \frac{\Gamma(1 - s) \Gamma(s + m + l)}{\Gamma(m + l + 1)}
\]

\[
= \frac{1}{\Gamma(s) \Gamma(1 - s)} \sum_{l=0}^{\infty} k^{-a}(l) \int_0^1 x^{-s}(1 - x)^{s+m+l-1} \, dx
\]

\[
= \frac{1}{\Gamma(s) \Gamma(1 - s)} \int_0^1 x^{a-s}(1 - x)^{s+m-1} \, dx
\]

\[
= \frac{\Gamma(1 - s + a) \Gamma(s + m)}{\Gamma(s) \Gamma(1 - s) \Gamma(m + a + 1)}.
\]

If \(s = 1\), it is immediate that \((1 − ∇)^a k^s(m) = 0\). This gives (4.11) for \(s \in (0, 1]\).

Next, assume that \(s > 1\). The summation by parts formula gives

\[
\sum_{n=0}^{N} k^{-a}(n) k^s(n + m)
\]

\[
= k^s(m) + \sum_{n=1}^{N} (k^{-a+1}(n) - k^{-a+1}(n - 1)) k^s(n + m)
\]

\[
= k^{-a+1}(N) k^s(N + m) + \sum_{n=0}^{N-1} (k^s(n + m) - k^s(n + m + 1)) k^{-a+1}(n)
\]

\[
= k^{-a+1}(N) k^s(N + m) - \sum_{n=0}^{N-1} k^{s-1}(n + m + 1) k^{-a+1}(n).
\]

By passing to the limit as \(N \to \infty\) and using that \(a > s - 1\), we obtain that

\[(1 − ∇)^a k^s(m) = -(1 − ∇)^{a-1} k^{s-1}(m + 1).\]

This implies that whenever (4.11) holds for a pair \((a - 1, s - 1)\) (for all \(m\)), it also holds for the pair \((a, s)\) and for all \(m\). Therefore, the case of an arbitrary pair \((a, s)\) reduces to the case of the pair \((a - n, s - n)\), where \(n < s \leq n + 1\), for which (4.11) has been checked already. This proves our formula for the general case. The sign of \(\sin(\pi s)\) depends on whether \(s \in J_{\text{even}}\), \(s \in J_{\text{odd}}\), or \(s\) is integer, whereas for \(a, s > 0\), all values of \(\Gamma\) in (4.11) are positive. This gives our statements. \(\Box\)
As an obvious consequence of Theorem 4.9, we obtain that it is not possible in general to pass from \( b \)-contractions to \( a \)-contractions when \( 0 < b < a \).

**Proposition 4.12** Let \( 0 < b < s < a \). Then \( B_s \) is a \( b \)-contraction, but not an \( a \)-contraction.

We also have the following result.

**Proposition 4.13** Let \( 0 < a \leq s < 1 \). Then \( B_s \) is an \( a \)-contraction, which is not similar to a contraction.

**Proof** Since \( a \leq s \), Theorem 4.9 gives that \( B_s \) is an \( a \)-contraction, and since \( s < 1 \), (4.9) gives that \( B_s \) is not power bounded. \( \square \)

Moreover, passing from \( a \)-contractions to \( b \)-contractions (when \( 0 < b < a \)) is neither possible in general, as the following statement shows. It is an immediate consequence of Theorems 4.10 and 4.11.

**Proposition 4.14** Let \( 1 < a \leq 2 \) and \( 0 < b < a \). If \( \max\{2, b+1\} < s < a + 1 \), then \( F_s \) is an \( a \)-contraction, but does not belong to \( \text{Adm}_b^w \) (so in particular, \( F_s \) is not a \( b \)-contraction).

**Proposition 4.15** Let \( 0 < s < 1 \). If \( 0 < a \leq \min\{s, 1-s\} \) then \( B_s \) is an \( a \)-contraction, but the series \( \sum \limits_{k \geq 0} k^{-a(n)} \|B_s^n\|^2 \) does not converge.

**Proof** Since \( a \leq s \), Theorem 4.9 gives that \( B_s \) is an \( a \)-contraction. Moreover, using that \( a \leq 1-s \) and (4.9), it is immediate that

\[
\sum_{n=0}^{\infty} k^{-a(n)} \|B_s^n\|^2 = \infty,
\]

and the statement follows. \( \square \)

**Theorem 4.16** Let \( m \) be a positive integer.

(i) If \( T \) is a \((2m + 1)\)-contraction, then \( T \) is a \(2m\)-contraction and

\[
\|T^n x\|^2 \lesssim (n + 1)^{2m} \quad (\forall x \in H).
\]

(ii) If \( T \) is a \((2m)\)-contraction and

\[
\|T^n x\|^2 = o(n^{2m-1}) \quad (\forall x \in H),
\]

then \( T \) is a \((2m - 1)\)-contraction.
Remark 4.17

(a) The fact that \((2m + 1)\)-contractions are \(2m\)-contractions was already proved by Gu in [17, Theorem 2.5]. Here we give an alternative proof, which also works for (ii).

(b) In general, \(2m\)-contractions are not \((2m - 1)\)-contractions. For example, the forward weighted shift \(F_2\) is a 2-isometry (see Theorem 4.11 (iii)), but it is not a contraction. Indeed, it is not power bounded (see (4.9)).

Proof of Theorem 4.16

Fix \(x \in H\) and put \(\Lambda_n := \| T^n x \|_2^2\), for every \(n \geq 0\). It is easy to see (for instance, by induction on \(k\)) that

\[
\Lambda_n = \sum_{j=0}^{k-1} (-1)^j \binom{n}{j} (I - \nabla)\Lambda_0 + (-1)^k \sum_{j=0}^{n-k} \binom{n-1-j}{k-1} (I - \nabla)^k \Lambda_j
\]

(4.12)

for \(n \geq k\). (Note that this formula is a discrete analogue of the Taylor formula with the rest in the integral form.)

Let us prove (i). Suppose that \(T\) is a \((2m + 1)\)-contraction. Taking \(k = 2m + 1\) (and \(n\) sufficiently large) in (4.12), we obtain that \((II) \leq 0\), since \((I - \nabla)^{2m+1} \Lambda_j \geq 0\). Therefore

\[
\Lambda_n \leq \Lambda_0 - \binom{n}{1} (I - \nabla)\Lambda_0 + \cdots + \binom{n}{2m} (I - \nabla)^{2m}\Lambda_0.
\]

(4.13)

If \((I - \nabla)^{2m} \Lambda_0 < 0\), then the RHS of (4.13) is a polynomial in \(n\) of degree \(2m\) whose main coefficient is negative. Hence, it is negative for \(n\) sufficiently large. This contradicts the fact that \(\Lambda_n \geq 0\) for every \(n\). Therefore, \((I - \nabla)^{2m} \Lambda_0 \geq 0\). Since the vector \(x \in H\), fixed at the beginning of the proof, was arbitrary, this means that \(T\) is a \(2m\)-contraction. We have also obtained that \(\Lambda_n \leq (n + 1)^{2m}\). This completes the proof of (i).

Assume the hypotheses of (ii). Now taking \(k = 2m\) (and \(n\) sufficiently large) in (4.12), we obtain that \((II) \geq 0\), since \((I - \nabla)^{2m} \Lambda_j \geq 0\). Therefore

\[
\Lambda_n \geq \Lambda_0 - \binom{n}{1} (I - \nabla)\Lambda_0 + \cdots - \binom{n}{2m-1} (I - \nabla)^{2m-1}\Lambda_0.
\]

(4.14)

If \((I - \nabla)^{2m-1} \Lambda_0 < 0\), then the RHS of (4.13) is a polynomial in \(n\) of degree \(2m - 1\) whose main coefficient is positive. But this contradicts the hypothesis \(\Lambda_n = O(n^{2m-1})\), hence it must be \((I - \nabla)^{2m-1} \Lambda_0 \geq 0\). Since the vector \(x \in H\), fixed at the beginning of the proof, was arbitrary, this means that \(T\) is a \((2m - 1)\)-contraction. \(\square\)
5 Remarks on the models for $a$-contractions with $a > 1$

In Theorem 3.51 of his thesis [27], Schillo proves that a commutative operator tuple belongs to certain classes if and only if it can be modeled as a compression of the tuple of multiplication operators by coordinates on some natural Bergman-type spaces of the unit ball. Specialized to the case of one operator, this result implies that for $a \geq 1$, the following statements are equivalent:

1. $T$ is a contraction and an $a$-contraction;
2. there exists a separable Hilbert space $E$ such that $T$ is unitarily equivalent to a part of an operator $(B_a \otimes I_E) \oplus U$, where $U$ is an unitary operator.

Here we discuss the models of $a$-contractions, without extra assumptions. In view of Theorem 1.3, which gives a model for $a$-contractions when $0 < a < 1$, it is natural to ask whether for $a > 1$, the statements

(a) $T$ is an $a$-contraction,
(b) there exists a separable Hilbert space $E$ such that $T$ is unitarily equivalent to a part of an operator $(B_a \otimes I_E) \oplus S$, where $S$ is an $m$-isometry,

are equivalent (here $m$ is the integer defined by $m - 1 < a \leq m$).

It turns out that one implication is true, but the other is false in general.

**Theorem 5.1** Let $a > 1$ and let $m$ be the positive integer such that $m - 1 < a \leq m$. Then any part of $(B_a \otimes I_E) \oplus S$, where $S$ is an $m$-isometry and $E$ is an auxiliary Hilbert space, is an $a$-contraction.

For the proof of this theorem we reproduce the following lemma from [1].

**Lemma 5.2** ([1, Proposition 2.6]) Let $a > 0$.

(i) If $T$ is an $a$-contraction, then any part of $T$ is also an $a$-contraction.
(ii) If $T_1$ and $T_2$ are $a$-contractions, then $T_1 \oplus T_2$ is also an $a$-contraction.
(iii) If $T$ is an $a$-contraction, then $T \otimes I_E$ (where $I_E$ is the identity operator on some Hilbert space $E$) is also an $a$-contraction.

**Proof of Theorem 5.1** By Theorem 4.9, $B_a$ is an $a$-contraction, and, by Theorem 1.6, $S$ is also an $a$-contraction. Therefore the previous lemma implies that $(B_a \otimes I_E) \oplus S$ is an $a$-contraction. \(\square\)

**Proposition 5.3** Let $a$ belong to the set

$$A := \bigcup_{j \geq 1} (2j - 1, 2j] \subset \mathbb{R}.$$
Let $m$ be the positive integer such that $m - 1 < a \leq m$, and take $s \in (m, a + 1)$ (such $s$ exists). Then the forward weighted shift $F_s$ is an $a$-contraction that cannot be modeled by a part of $(B_a \otimes I_E) \oplus S$, where $S$ is an $m$-isometry.

Proof Assume all the hypothesis of the statement. Since $s < a + 1$ and $s \in J_{\text{even}}$, Theorem 4.11 (i) gives that $F_s$ is an $a$-contraction. The second part of the statement follows by comparison of operator norms. By (4.9), $\| F^n_s \|^2 \approx (n + 1)^s - 1$, for every $n \geq 0$. On the other hand, $B_a$ is a contraction, and $\| S^n \|^2 \lesssim (n + 1)^{m - 1}$ since $S$ is an $m$-isometry. (This last asymptotics is well-known. For instance, it follows immediately from [11, Theorem 2.1].) Therefore
$$\|(B_a \otimes I_E) \oplus S^n\|^2 \lesssim (n + 1)^{m - 1}.$$ Since $m - 1 < s - 1$, we get that $F_s$ cannot be modeled by a part of $(B_a \otimes I_E) \oplus S$. \hfill \Box

Remark 5.4 It remains open whether (a) implies (b) when $a > 1$ belongs to $\mathbb{R} \setminus A$.

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