Non-Abelian topological superconductors from topological semimetals and related systems under the superconducting proximity effect

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Abstract
Non-Abelian topological superconductors are characterized by the existence of zero-energy Majorana fermions bound in the quantized vortices. This is a consequence of the nontrivial bulk topology characterized by an odd Chern number. It is found that in topological semimetals with a single two-band crossing point all the gapped superconductors are non-Abelian ones. Such a property is generalized to related but more generic systems which will be useful in the search for non-Abelian superconductors and Majorana fermions.

1. Introduction

Since the discovery of the quantum Hall effect [1], efforts devoted to understanding various topological states of matter and their phase transitions have greatly enriched the study of condensed matter physics [2–4]. One of the significant aspects is that two topologically distinct states, which cannot be adiabatically connected to each other, can have the same symmetry. This breaks down the Landau–Ginzburg paradigm of phase transitions. Besides, exotic excitations obeying non-Abelian statistics have been found in genuine and model systems [4–8]. In one known class the non-Abelian topological orders are closely related to the fermionic superconducting (or superfluid) pairing states with odd Chern numbers [9, 10]. Protected by the topology there is a zero-energy Majorana fermion in each quantized vortex or on the boundary between the system and a normal (topologically trivial) system. The quantum degeneracy of the ground states with 2N quantized vortices (far away from each other) is $2^N$. Winding between these vortices induces a unitary transformation in the $2^N$-dimensional Hilbert space which leads to the non-Abelian statistics. It has been proposed that non-Abelian excitations such as Majorana fermions can be exploited for the topologically protected quantum computations [11–14].

Besides the known non-Abelian topological orders in fractional quantum Hall systems [5], spin liquids [4, 6], $^3$He films [7] and Sr$_2$RuO$_4$ [8], recently there have been theoretical proposals for non-Abelian topological orders on the surface of topological insulators [15] and in spin–orbit coupled two-dimensional electron/hole systems [16–18] under the superconducting proximity effect as well as in ultracold atomic gases [19]. Time-reversal symmetry breaking is necessary for the nonzero Chern number which can be realized by the magnetic field or via time-reversal symmetry breaking superconducting order or Zeeman type interactions. On the surface of a topological insulator under the s-wave superconducting proximity effect there is no time-reversal symmetry breaking, but the effective ‘vacuum’ of the system is a massive Dirac electron system which breaks the time-reversal symmetry. The Chern number difference between the system and the vacuum (as will also be shown later in this work) is ±1 which protects the zero-energy Majorana fermion in each vortex or on the boundary.
between the system and the effective vacuum [20]. In fact to detect the Majorana fermions in such systems a Zeeman type interaction is usually invoked to induce the effective ‘vacuum’ somewhere [20]. Besides the vortices are also usually induced by an external magnetic field. More recently signatures of Majorana fermions have been observed in spin–orbit coupled one-dimensional quantum wires proximate to superconductors [21, 22]. Inspired by the search for Majorana fermions [23], in this work we study the topological properties of superconducting states in generic semimetals and related systems under general superconducting proximity conditions.

The topological semimetals studied here are systems consisting of two-band crossing points (TBCPs) around the Fermi level which can be viewed as k-space vortices [7, 24]. Away from the TBCP the two bands do not overlap unless through other TBCPs. In two dimensions the TBCP has co-dimension two and carries an integer winding number which can be computed through the Berry phase [7, 25]:

\[ N_w = \frac{1}{\pi} \oint_{C} d\mathbf{k} \cdot \langle \Psi(\mathbf{k}) | i \nabla_{\mathbf{k}} | \Psi(\mathbf{k}) \rangle. \]  

(1)

Here \( C \) is an anti-clockwise path enclosing the TBCP and \( \Psi(\mathbf{k}) \) is the wavefunction (single-valued and continuous) in the band with energy above (or below) the TBCP. Concrete examples are Dirac cones and quadratic band crossings [25] (see figures 1(a) and (b)) where the winding numbers are \( N_w = \pm 1 \) and \( \pm 2 \), respectively. Here the integer \( N_w \) is only defined for the band crossing (whenever the band crossing is gapped \( N_w \) will no longer be an integer). The winding number \( N_w \) characterizes the TBCP [7, 24] and will be used to classify different situations in this study. The discussions hereafter will be split into two cases: (A) when \( N_w \) is even, and (B) when \( N_w \) is odd. Time-reversal \( T \) symmetry is imposed for both cases. For concreteness, case A is restricted to systems with zero angular momentum where the time-reversal operator is \( T = K \) (\( K \) is complex conjugation), whereas case B is for spin-half systems where \( T = K \sigma_0 \).

The findings in this work are as follows. (i) When there is a single TBCP, for both cases A and B, all gapped superconducting states are non-Abelian. Namely, the bulk Chern number is odd. (ii) The same conclusion holds when the TBCP are gapped and deformed (such that the two bands eventually evolve in the same direction in energy), given that only one band crossing the Fermi level. (iv) The discussion is further extended to situations where the time-reversal symmetry is broken and those where there are multiple such TBCPs.

The paper is organized as follows. In section 2 we discuss the situation with a single TBCP for cases A and B. In section 3 the situations when the TBCP is gapped (and deformed) are studied. In section 4 we develop more generalizations. We conclude in section 5. All the discussion are restricted to the weak pairing regime which is relevant to proximity induced pairing orders.

2. Topological semimetals with a single TBCP

2.1. Case A

In spinless (or spin-polarized) many-fermion systems in two-dimensional lattices with multiple orbits in a unit cell with inversion symmetry, there can be TBCPs with even winding numbers. Around such a TBCP the Hamiltonian can be generally written as

\[ H_0(\mathbf{k}) = h_0(\mathbf{k})\sigma_0 + h_x(\mathbf{k})\sigma_x + h_z(\mathbf{k})\sigma_z. \]  

(2)

Here the Pauli matrices, \( \sigma_0 \) and \( \sigma_x \) act on the Wannier orbits (pseudo-spins), and \( \sigma_0 \) is the \( 2 \times 2 \) identity matrix. Due to time-reversal symmetry, the TBCP can only be at a time-reversal invariant momentum \( \mathbf{K} \) when there is only a single such TBCP. \( \mathbf{k} \) is the wavevector measured from \( \mathbf{K} \). \( h_v(\mathbf{k}) = h_v(\mathbf{k}) \) for \( v = 0, x, z \) and \( h_v(\mathbf{k}) = 0 \) due to time-reversal and inversion symmetry. The spectrum is \( \epsilon_{\mathbf{k}} = h_0(\mathbf{k}) \pm \sqrt{h_x^2(\mathbf{k}) + h_z^2(\mathbf{k})} \) for semimetals, \( |h_0(\mathbf{k})| < \sqrt{h_x^2(\mathbf{k}) + h_z^2(\mathbf{k})} \) and \( h_v = 0 \) at \( k = |\mathbf{k}| = 0 \). The eigenstates of \( H_0(\mathbf{k}) \) are

\[ |\nu_+(\mathbf{k})\rangle = \frac{1}{2}[|e^{-i\phi_{\mathbf{k}}} + 1\rangle |\uparrow\rangle + i|e^{i\phi_{\mathbf{k}}} - 1\rangle |\downarrow\rangle], \]

\[ |\nu_-(\mathbf{k})\rangle = \frac{1}{2}[|e^{i\phi_{\mathbf{k}}} - 1\rangle |\uparrow\rangle - i|e^{-i\phi_{\mathbf{k}}} + 1\rangle |\downarrow\rangle], \]

with \( \phi_{\mathbf{k}} = \text{Arg}[h_x(\mathbf{k}) + ih_z(\mathbf{k})] \). The winding number of the TBCP is calculated through equation (1) as

\[ N_w = \frac{1}{2\pi} \oint_{C} d\phi_{\mathbf{k}}. \]  

(4)

This has a transparent physical meaning: \( N_w 2\pi \) is the winding angle of \( \mathbf{h} \) and that of the pseudo-spin direction. The winding number can only be an even integer as \( h_v(\mathbf{k}) = h_v(\mathbf{k}) \).
One example of such TBCP systems is the quadratic band crossing in the checkerboard lattices [25–27], where in the vicinity of \( \mathbf{k} = (\pi, \pi) \), \( h_0(\mathbf{k}) = t_0 k^2 \), \( h_y(\mathbf{k}) = 2t_x k_x k_y \), \( h_z(\mathbf{k}) = t_z (k_x^2 - k_y^2) \), and \( h_y(\mathbf{k}) \equiv 0 \) with \( t_0 \), \( t_x \), and \( t_y \) being the band parameters. The system is a semimetal with winding number \( N_w = 2 \sgn(t_z t_x) = \pm 2 \) when \( |t_0| < |t_x|, |t_z| \).

The general form of the Bogoliubov–de Gennes (BdG) Hamiltonian for the system is

\[
H = \sum_{\mathbf{k}} \Psi^\dagger_{\mathbf{k}} \mathbf{\hat{H}}_k \Psi(\mathbf{k}) \quad \text{and} \quad \Psi(\mathbf{k}) = \begin{pmatrix} \psi^\dagger_1(\mathbf{k}), \psi_1^T(-\mathbf{k}), \psi^T_+(-\mathbf{k}) \end{pmatrix}^T \quad \text{and} \quad \mathbf{\hat{H}}_k = \begin{bmatrix} \hat{H}_0(\mathbf{k}) - \mu & -\hat{\Delta}(\mathbf{k}) \\ -\hat{\Delta}^*(\mathbf{k}) & -\hat{H}_0^*(\mathbf{k}) + \mu \end{bmatrix},
\]

where \( \hat{\Delta}(\mathbf{k}) = i\Delta_x(\mathbf{k})\hat{\sigma}_x + \Delta_z(\mathbf{k})\hat{\sigma}_y + i\Delta_y(\mathbf{k})\hat{\sigma}_z \) is the general form of the superconducting pairing interaction when the Cooper pair have zero angular momentum. \( \Delta_0 \) and \( \Delta_v \) (\( v = x, y, z \)) represent the singlet and triplet pairings, respectively.

In the weak pairing regime, \( |\Delta_v| \ll |\mu| \), only the pairing interaction between nearly degenerate states is important, whereas that between states far away can be ignored. The pairing properties can then be studied by projecting the original Hamiltonian into the subspace spanned by the band that crosses the Fermi level. To the leading order, the projected BdG Hamiltonian is

\[
H_{\text{PBdG}} = \sum_{\mathbf{k}} \Psi^\dagger_{\mathbf{k}} \mathbf{\hat{H}}_k \Psi(\mathbf{k}) \quad \text{and} \quad \mathbf{\hat{H}}_k = \begin{bmatrix} \hat{H}_0(\mathbf{k}) - \mu & \hat{\Delta}(\mathbf{k}) \\ -\hat{\Delta}^*(\mathbf{k}) & -\hat{H}_0^*(\mathbf{k}) + \mu \end{bmatrix}.
\]

Here the \( \pm \) indices are for the \( \mu > 0 \) and \( \mu < 0 \) cases, respectively, and

\[
\Delta_{\text{eff}}(\mathbf{k}) = e^{i\mathbf{k}} \left[ i\Delta_v - \frac{1}{2} \sgn(\mu) \sum_{\pm} (\Delta_x \pm i\Delta_y)e^{\pm i\mathbf{k}} \right].
\]

The eigenstates can then be obtained by directly diagonalizing the above Hamiltonian. The Chern number is given by [28]

\[
N_C \equiv \frac{1}{2\pi} \int d\mathbf{k} e_\mathbf{k} \cdot \left[ \mathbf{V}_k \times \langle \Psi(\mathbf{k}) | \mathbf{V}_k | \Psi(\mathbf{k}) \rangle \right],
\]

where \( \Psi_n \) are the wavefunctions of the occupied bands. Direct calculation yields (for details see the appendix)

\[
N_C = \frac{\sgn(\mu)}{2\pi} \int_{FS}^\pi \frac{dk_y}{2\pi} \frac{dk_x}{2\pi} \theta_\mathbf{k} \theta_\mathbf{\Delta} (\mathbf{k}), \quad \text{(9)}
\]

where \( \theta_\mathbf{k} = \text{Arg}[e^{i\mathbf{k}} \Delta_{\text{eff}}(\mathbf{k})] \). That is, the Chern number is nothing but the winding number of \( e^{i\mathbf{k}} \Delta_{\text{eff}}(\mathbf{k}) \) at the Fermi surface (denoted as ‘FS’ above). Physically this is due to the fact that the superconducting gap is only opened at the Fermi surface in the weak pairing regime [29]. From equations (7) and (9), the effect of the Fermi surface Berry phase on the Chern number is clearly visible.

The error of the eigenstates obtained from the projected Hamiltonian is of the order of \( O(D/|\mu|) \). However, this induces no error in the calculated Chern number due to its topological nature. Namely one always can adiabatically tune the pairing interaction \( \Delta_v \rightarrow \alpha \Delta_v \), via one scaling factor \( \alpha \), to a sufficiently small value to reduce the error without closing the superconducting gap. As the gap is not closed, the Chern number does not change. Hence the error of the calculated Chern number can be infinitesimally small when \( \alpha \rightarrow 0 \). Note that equation (9) does not depend on \( \alpha \). Therefore, there is no error in the Chern number calculated via the projected BdG Hamiltonian in the weak pairing regime.

A crucial observation is that the winding number of \( e^{i\mathbf{k}} \Delta_{\text{eff}}(\mathbf{k}) \) at the Fermi surface can only be odd when it is well-defined. This is because the winding number of \( \Delta_v \) (\( v = x, y, z \)) is odd while that of \( e^{i\mathbf{k}} \Delta_{\text{eff}} \) are even. Hence the Chern number \( N_C \) can only be odd. Therefore all the gapped superconducting states in case A are non-Abelian ones.

2.2. Case B

TBCPs with an odd winding number, such as Dirac cones, can appear in spin-half fermionic systems [3]. The fermion doubling theorem states that in 2D lattice systems there can only be an even number of such TBCPs [30]. However, at the surface of strong topological insulators there can be an odd number of such TBCPs. The concerned systems have a single such TBCP at a time-reversal invariant momentum \( \mathbf{k} \) due to time-reversal symmetry. Rather than \( N_w = \pm 1 \) for a Dirac cone, \( N_w \) can be any odd integer here. The general Hamiltonian around such a TBCP is

\[
H(\mathbf{k}) = h_0(\mathbf{k}) \sigma_0 + h(\mathbf{k}) \cdot \hat{\sigma}.
\]

where the Pauli matrices now denote true-spin and \( \mathbf{k} \) measured from \( \mathbf{k} \) where \( |h_0| < |h| \) and \( |h| = 0 \) at \( \mathbf{k} = 0 \) (we also set \( h_0(\mathbf{k}) = 0 \)) so that the system is a semimetal. We choose the coordinates so that \( h_x(\mathbf{k}) = 0 \), i.e. the winding axis is along the \( z \)-direction. The spectrum is \( \varepsilon_{\pm \mathbf{k}} = h_0(\mathbf{k}) \mp \sqrt{h_0^2 + h_y^2} \) and the eigenstates are \( |\pm \mathbf{k}(\mathbf{k}) \rangle = \frac{1}{\sqrt{2}} (e^{i\mathbf{k}} |\uparrow\rangle \mp |\downarrow\rangle) \) with \( \psi_k = \text{Arg}[h_x(\mathbf{k}) + ih_y(\mathbf{k})] \). Using equation (1) one finds

\[
N_w = \frac{1}{2\pi} \int d\psi_k.
\]

The winding number can only be an odd integer as \( h(-\mathbf{k}) = -h(\mathbf{k}) \) according to time-reversal symmetry.

Following the argument in the previous section, in the weak pairing regime one can study the topological property of the system via the projected BdG Hamiltonian (equation (6)). Here

\[
\Delta_{\text{eff}}(\mathbf{k}) = e^{i\mathbf{k}} \left[ \sgn(\mu) \Delta_0 + \frac{1}{2} \sum_{\pm} (\Delta_x \mp i\Delta_y)e^{\pm i\mathbf{k}} \right].
\]

The Chern number \( N_C \) is the winding number of \( e^{i\mathbf{k}} \Delta_{\text{eff}}(\mathbf{k}) \) at the Fermi surface as in equation (9). It is noted that the winding number of \( e^{i\mathbf{k}} \Delta_{\text{eff}}(\mathbf{k}) \) can only be even. Therefore the Chern number can only be even.

As the system concerned exists only on the boundary of two three-dimensional systems with distinct \( Z_2 \) topology, it does not have well-defined edges [3]. One way to circumvent this problem is to circulate the superconducting state with a ferromagnetic insulating state with the same \( H_0(\mathbf{k}) \) but
with a magnetization along the z-direction, $M \sigma_z$ [15]. When $|M| > |\mu|$, the quasi-particles cannot propagate into the ferromagnetic region. On the boundary between the superconducting region and the ferromagnetic one, there are gapless Majorana edge states. The ferromagnetic insulating state is topologically equivalent to a superconducting massive Dirac fermion system with $|M| > |\mu|$. It has a Chern number of $\text{sgn}(M)N_w$. For instance, there may be $n_c$ clockwise moving edge states and $n_a$ anti-clockwise moving edge states. According to bulk-edge correspondence, $n_c - n_a = N_C - \text{sgn}(M)N_w$. The difference $n_c - n_a$ is fixed by topology and is always odd as $N_C - \text{sgn}(M)N_w$ is odd. Therefore the total number of edge states $N_{\text{edge}} = n_c + n_a$ is definitely odd.

The above analysis can also be applied to the Majorana bound states in the core of a quantized vortex, which can be viewed as edge states occurring in the small circular edge of the vortex with a vacuum at the center [9]. As the boundary condition at the center does not affect the existence of the zero-energy Majorana bound state, it can be tuned such that the vacuum at the center is a superconducting massive Dirac fermion with $|M| > |\mu|$. Therefore there are $N_{\text{edge}}$ Majorana states in the core of a quantized vortex. In reality, there is inevitable mixing between those states (for instance, due to disorder) and interactions between the Majorana fermions, which lift the degeneracy. However, the particle–hole symmetry guarantees the existence of one zero-energy Majorana bound state when $N_{\text{edge}}$ is odd. Accordingly, all the gapped superconducting pairing states here are non-Abelian ones since $N_{\text{edge}}$ is definitely odd. This argument (essentially the same as that in [9]), verifies the existence of the Majorana zero modes in the vortex core without explicitly solving the Schrödinger equation for the quasi-particle spectrum in a vortex as such a property is essentially dictated by the bulk topology [9]. Recent theories [31] also present additional proof of such a relation of the number of Majorana zero modes to the Chern number.

### 3. Systems with a single gapped/deformed TBCP

The general Hamiltonian is given by equation (10) with all the $h_r(k)$ ($v = 0, x, y, z$) being nonzero. The spectrum is $\epsilon_{k\pm} = h_0(k) \pm |\mathbf{h}|$ with $|\mathbf{h}| = \sqrt{h_0^2(k) + h_1^2(k) + h_2^2(k)}$. For case A the eigenstates are

$$
\begin{align*}
|u_+(k)\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} \cos \frac{\eta_k}{2} e^{-i\phi_k} + \sin \frac{\eta_k}{2} e^{-i\phi_k} \\ i \cos \frac{\eta_k}{2} e^{-i\phi_k} - i \sin \frac{\eta_k}{2} e^{-i\phi_k} \end{pmatrix}, \\
|u_-(k)\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} i \sin \frac{\eta_k}{2} e^{-i\phi_k} - i \cos \frac{\eta_k}{2} e^{-i\phi_k} \\ -\cos \frac{\eta_k}{2} e^{-i\phi_k} - \sin \frac{\eta_k}{2} e^{-i\phi_k} \end{pmatrix},
\end{align*}
$$

where $\eta_k = \text{Arg}[h_x + i\sqrt{h_1^2 + h_2^2}]$ and $\phi_k = \text{Arg}[h_z + i\eta_k]$. At $k = 0$, $h_x = h_z = 0$ whereas $h_y \neq 0$. At large $k$, $|h_y| \ll \sqrt{h_1^2 + h_2^2}$ and $\eta_k \to \pi/2$. We restrict the discussion to the situations where $h_y(k) = h_z(k)$. Hence $\eta_k$ is an even function of $k$. For case B the eigenstates are $|u_{\pm}(k)\rangle = \cos \frac{\phi_k}{2} e^{-i\eta_k} |\uparrow\rangle + \sin \frac{\phi_k}{2} e^{-i\eta_k} |\downarrow\rangle$ and $|u_-(k)\rangle = \sin \frac{\phi_k}{2} e^{i\eta_k} |\downarrow\rangle - \cos \frac{\phi_k}{2} e^{i\eta_k} |\uparrow\rangle$, where $\xi_k = \text{Arg}[h_x + i\sqrt{h_1^2 + h_2^2}]$ and $\psi_k = \text{Arg}[h_y + i\xi_k]$. Now at $k = 0$, $h_x = h_y = 0$ and $h_z \neq 0$, whereas at large $k$, $|h_z| \ll \sqrt{h_1^2 + h_2^2}$ and $\eta_k \to \pi/2$. We also assume $h_y(-k) = h_z(k)$ so that $\xi_k$ is an even function of $k$.

Let us first consider case A. When $|h_z(0)| < |\mu|$ and $|h_0| < |\mathbf{h}|$ (i.e. only one band crosses the Fermi level) (e.g. see figure 2(a)), using the previous technique to obtain the projected BdG Hamiltonian (6) one finds

$$
\Delta_{\text{eff}}^\pm(k) = e^{i\phi_k} \{ \Delta_z |\sin \phi_k \mp i \cos \phi_k \cos \eta_k| \pm i \Delta_y \sin \eta_k - \Delta_z |\cos \phi_k \mp i \sin \phi_k \cos \eta_k| \},
$$

where $+ \pm$ are for the higher and lower bands, respectively. Direct calculation yields that the Chern number is still given by equation (9). And the property that all the gapped superconductors are non-Abelian ones still holds since the winding numbers of $e^{i\phi_k}$ are always even and $\eta_k$ is an even function of $k$. This is consistent with the picture that opening a gap below or above the Fermi level does not affect the topological properties. A nontrivial situation is when the BCP is both gapped and deformed so that $|h_0| > |\mathbf{h}|$ at large $k$. In this situation the two bands evolve in the same direction at large $k$ (e.g. see figure 2(b)). When $|\mu| < |h_z(0)|$, only the lower band crosses the Fermi level and the Chern number is

$$
N_C = \int_0^{2\pi} \frac{d\phi}{2\pi} \theta_{\Delta^0} \theta_{\Delta^0} \langle k |_{\text{FS}} + \text{sgn}(h_z(0))N_w. \tag{15}
$$

Here $\theta_{\Delta^0} = \text{Arg}[e^{-i\phi_k}\Delta_{\text{eff}}^0(k)]$. It is seen that as $N_w$ is even, the Chern number is again odd for all the gapped states. Therefore
the system still has the nontrivial property that all the gapped superconducting pairing states are non-Abelian ones. When the Fermi level is so high that both bands cross it, the total Chern number is

\[
N_C = \sum_{\pm} \int_{0}^{2\pi} \frac{dk}{2\pi} \frac{d\theta_{\Delta}^{\pm}(k)}{\epsilon_{FS}}. \tag{16}
\]

Hence the total Chern number becomes even (trivial or Abelian topological superconductors) when the two bands cross the Fermi level.

Now we turn to case B. When the TBCP is gapped the effective superconducting pairings in the two bands are

\[
\Delta_{\pm}^{\text{eff}}(k) = e^{i\theta_0} \left[ \cos^2 \left( \frac{k}{2} \right) (\Delta_x \pm i \Delta_y) e^{\mp i\nu_k} \mp \Delta_0 \sin(\xi_k) + \sin^2 \left( \frac{\xi_k}{2} \right) (\Delta_x \mp i \Delta_y) e^{\pm i\nu_k} \right]. \tag{17}
\]

Here + and − again indicate the higher and lower bands, respectively. Again the Chern number is the same as that at \( h_z = 0 \) when \( |\mu| > |h_z(0)| \) and \( |h_0(0)| < |h| \) (see figure 2(c)). When the TBCP is gapped and deformed \( |h_0(0)| > |h| \) at large \( k \) (e.g. see figure 2(d)). Such a system can exist as a two-dimensional lattice system without violating the fermion doubling theorem. Examples in real life are the spin–orbit coupled two-dimensional electron (hole) systems under a Zeeman (or exchange) field \( h_z \). When \( |\mu| < |h_z(0)| \), only the lower band crosses the Fermi level, and one finds that

\[
N_C = \int_{0}^{2\pi} \frac{dk}{2\pi} \frac{d\theta_{\Delta}^{\pm}(k)}{\epsilon_{FS}} + \text{sgn}(h_z(0))N_w \tag{18}
\]

with \( \theta_{\Delta}^{\pm} = \text{Arg}[e^{-i\nu_k} \Delta_{\text{eff}}^{\pm}(k)] \). Note that the winding number of \( e^{-i\nu_k} \Delta_{\text{eff}}^{\pm}(k) \) can only be even as the winding number of \( e^{\pm i\nu_k} \) is always odd and \( \xi_k \) is an even function of \( k \). Therefore the Chern number can only be odd. When the Fermi level is higher, so that both bands cross it, the total Chern number is given by equation (16), which is always even.

4. More generalizations

In this section we explore further generalizations of the results obtained. The first generalization is for systems with multiple TBCPs (no matter whether they are gapped or deformed). Whenever the superconducting pairing interaction is within each TBCP and there are an odd number of bands crossing the Fermi level, the property that all the gapped superconductors are non-Abelian ones should also hold. This is because the total Chern number is the summation of the contribution from each TBCP. Such situations can appear when every TBCP is located at a time-reversal invariant momentum.

There is a possibility that when the time-reversal symmetry is broken but the inversion symmetry is not the spin-half system can have a single TBCP with an even winding number. A general Hamiltonian for such systems near the TBCP in the form of (10) is

\[
\begin{align*}
    h_0(k) &= M_0 k^2, \quad &h_x(k) &= M_x - \beta k^2, \\
    h_y(k) &= \gamma (k_x^2 - k_y^2), \quad &h_z(k) &= 2\delta k_x k_y.
\end{align*} \tag{19}
\]

where \( M_0, M_x, \beta, \gamma \) and \( \delta \) are band parameters. The TBCP exists when \( |h_0| < |h| \) with \( h_z \equiv 0 \). It is gapped when \( M_x \neq 0 \) and gapped and deformed when \( M_x \neq 0 \) and \( |h_0| > |h| \) at large \( k \). This situation is essentially the same as for case A. One can easily find that the Chern number can only be odd in such systems when there is only one band crossing the Fermi level. This results can be further generalized to systems with multiple such TBCPs.

5. Candidate physical systems

Beside the systems already found in the literature in the search of Majorana fermions, such as systems with single Dirac cone and semiconductor quantum wells with Rashba spin–orbit coupling, there are many unexplored candidate systems which fit into the above discussions. Below we list some candidates which have not yet attracted researchers’ attention.

- **Semiconductor nanostructures with Zeeman (or exchange) splitting and arbitrary spin–orbit coupling.** This is essentially case A with a gapped and deformed TBCP. Given that the winding number \( N_w \) of the TBCP is odd the system supports Majorana fermions in the vortex. This is a direct generalization of the studies in the literature [16]. Specific examples are as follows. (i) A two-dimensional electron system with both Rashba and Dresselhaus spin–orbit couplings. For III–V semiconductor quantum wells with growth direction [001] when Rashba (Dresselhaus) spin–orbit coupling is dominant \( N_w = 1 \) \( (N_w = -1) \). (ii) A two-dimensional heavy hole system where the cubic spin–orbit coupling leads to \( N_w = \pm 3 \) [32]. When there are multiple sub-bands crossing the Fermi level. The total winding number is the summation of the winding number of each sub-band. If the total winding number is odd then all the gapped superconductor phases are non-Abelian topological superconductors, especially when the system is in proximity to an s-wave superconductor.

- **Thin films of topological Weyl semimetals.** In [33], it is found that in the thin film of topological Weyl semimetal HgCr2Se4 the Chern number depends on the thickness of the film. There is a quadratic band crossing point at \( k = 0 \) (Γ point) when the thickness is equal to the critical value. Such a TBCP with \( N_w = 2 \) is a consequence of the topological phase transition from normal insulator to a quantum anomalous Hall insulator with Chern number 2 in the system. Around the critical thickness the quadratic band crossing is gapped. The low energy Hamiltonian is given by equation (19). All the gapped superconductor states are non-Abelian ones, when there is a single band crossing the Fermi level.

- **Optical lattices with a single quadratic band crossing.** Examples are checkerboard lattices near half-filling and kagome lattices above one-third filling (or below two-thirds filling, depending on the sign of the hopping) [27]. When spin-polarized ultracold fermions are filled into
the optical lattices, all the gapped superconductor (or superfluid) phases are non-Abelian ones in the weak pairing regime $|\Delta_{\nu}| \ll |\mu|$.

6. Conclusion and discussions

In this work we have studied the superconducting proximity effect on topological semimetals and related systems with the aim of searching for Majorana fermions and non-Abelian statistics. The non-Abelian superconductors are characterized in the bulk by an odd Chern number which, according to bulk-edge correspondence, guarantees the existence of one Majorana fermion in each quantized vortex. By studying the superconducting proximity effects under general situations, we find that for two cases A and B, where a single TBCP carries an integer winding number, all the superconducting pairing states are non-Abelian ones. We further generalize this property to systems: (i) where such a TBCP is gapped due to time-reversal symmetry breaking but inversion symmetric perturbations; (ii) when the TBCP are gapped and deformed given that only one band crosses the Fermi level; (iii) when there are multiple such TBCPs with an odd number of bands crossing the Fermi level if the superconducting pairing interaction is within each TBCP; (iv) when the TBCP system breaks time-reversal symmetry yet has inversion symmetry. As a consequence of those findings we give several candidate physical systems which can support the Majorana fermions that have not attracted the attention of scientists.

It is noted from equation (9) that the Chern number changes sign when the chemical potential moves across the TBCP, which indicates that there is a topological phase transition in the strong pairing regime. For superconductor and superfluid phases that emerge due to continuous phase transition driven by attractive interaction, the gapped pairing states usually reduce the Ginzburg–Landau free energy more than the nodal ones [34]. Hence the special property found in this work may also imply that the non-Abelian pairing states are energetically favored as the spontaneous symmetry broken phases. This is indeed confirmed in a subsequent work [27].

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Appendix. Details of the derivation of the Chern number

Consider, e.g., systems with a single TBCP carrying an even winding number when $\mu > 0$. There are two occupied bands of the BdG Hamiltonian: one from the band crossing the Fermi level, and the other one from the band below the Fermi level. In the weak pairing regime, $|\Delta_{\nu}| \ll |\mu|$, one can ignore pairing between states separated far away. One can then obtain the wavefunctions of the two bands under the approximation

\[
\Psi_{\nu}(k) = e^{-i\phi_{\nu}^k/2} \begin{pmatrix} \sin \frac{\xi_{\nu}}{2} \\ -\cos \frac{\xi_{\nu}}{2} \end{pmatrix}.
\]

Here $\xi_{\nu} = \text{Arg}[\epsilon_{k+\nu} - \mu + i|\Delta_{\nu}(k)|]$ and $\theta_{\Delta} = \text{Arg}[e^{-i\phi_{\nu}} \Delta_{\nu}(k)]$. An important property is that the Chern number $N_{C}$ does not change without closing the gap. One can then simplify the calculation of $N_{C}$ by adiabatically tuning the system. It is noted that the gap is determined by $|\Delta_{\nu}(k)|$ at the Fermi surface. One can then adiabatically tune the system so that $|\Delta_{\nu}(k)|$ is nonzero only in the vicinity of the Fermi surface [29]. The angular dependence $|\Delta_{\nu}(k, \theta_{\Delta})|$ (here $k_\nu = k \cos \theta_{\Delta}$ and $k_\nu = k \sin \theta_{\Delta}$) at each energy contour can also be adiabatically tuned to be identical to that on the Fermi surface. The Chern number is the integration of the Berry-curvature in the first Brillouin zone.

\[
N_{C} = \sum_{\nu=0,\nu} \int \frac{dk}{2\pi} |\Psi_{\nu}(i\mathbf{\nabla}_{k} \times [\mathbf{\Psi}_{\nu}^* |\mathbf{\nabla}_{k}|\mathbf{\Psi}_{\nu}]|.
\]

One can divide the contribution of the integration into two parts: one from integration over small $k$ (with a cut-off $\Lambda$), another from integration over large $k$ region. Since there is no band-gap closing in the large $k$ region, the Chern number is determined in the small $k$ region, especially, in the vicinity of the Fermi surface where the pairing gap evolves, as shown in [29]. Inserting equation (A.1), one can show that the Chern number due to the $\nu$ band is zero. $N_{C}$ is then solely determined by the $\nu$ band. Direct calculation yields

\[
N_{C} = -\frac{1}{2\pi} \int_{0}^{2\pi} d\theta_{\Delta} \int_{0}^{\Lambda} dk |\mathbf{\nabla}_{k} \times [\mathbf{\Psi}_{\nu}^* |\mathbf{\nabla}_{k}|\mathbf{\Psi}_{\nu}]|.
\]

(A.3)

which results in equation (9). Other expressions for the Chern number $N_{C}$ in the main text can be derived similarly.

References

[1] Klitzing K V, Dorda G and Pepper M 1980 Phys. Rev. Lett. **45** 494

[2] Thouless D J, Kohmoto M, Nightingale M P and den Nijs M 1982 Phys. Rev. Lett. **48** 1559

[3] Hasan M Z and Kane C L 2010 Rev. Mod. Phys. **82** 3045

[4] Wen X G 2004 Quantum Field Theory of Many-Body Systems (Oxford: Oxford University Press)
