Role of dimensionality in complex networks: Connection with nonextensive statistics

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Deep connections are known to exist between scale-free networks and non-Gibbsian statistics. For example, typical degree distributions at the thermodynamical limit are of the form \( P(k) \propto e^{-k/\kappa} \), where the q-exponential form \( e_q^z \equiv [1 + (1 - q)z]^{1/(1-q)} \) optimizes the nonadditive entropy \( S_q \). We introduce and study here \( d \)-dimensional geographically-located networks which grow with preferential attachment involving Euclidean distances through \( r_{ij}^{-\alpha_A} (\alpha_A \geq 0) \). Revealing the connection with q-statistics, we numerically verify (for \( d = 1, 2, 3 \) and \( 4 \)) that the q-exponential degree distributions exhibit, for both \( q \) and \( \kappa \), universal dependences on the ratio \( \alpha_A/d \). Moreover, the \( q = 1 \) limit is rapidly achieved by increasing \( \alpha_A/d \) to infinity.

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Networks emerge spontaneously in many natural, artificial and social systems. Their study is potentially important for physics, biology, economics, social sciences, among other areas. For example, many empirical studies have identified peculiar properties in very different networks such as the Internet and online social networks, to quote but a few. An ubiquitous class of such networks is constituted by the scale-free ones (more precisely, asymptotically scale-free). As we shall soon verify, these networks can be seen as a particular application of nonextensive statistical mechanics, based on the nonadditive entropy \( S_q = k \frac{1}{q-1} \sum_i p_i^q \) (\( q \in \mathbb{R} \); \( S_1 = S_{BG} = -k \sum_i p_i \ln p_i \), where \( BG \) stands for Boltzmann-Gibbs). This current generalization of the BG entropy and corresponding statistical mechanics has been widely successful in clarifying the foundations of thermal statistics as well as for applications in complex systems in high-energy collisions at LHC/CERN (CMS, ALICE, ATLAS and LHCb detectors) and at RHIC/Brookhaven (PHENIX detector), cold atoms, dusty plasmas, spin-glasses, trapped ions, astrophysical plasma, biological systems, type-II superconductors, granular matter (see [12]).

The deep relationship between scale-free networks and q-statistics started being explored in 2005 [16–18], and is presently active [19, 20, 23]. The basic connection comes (along the lines of the BG canonical ensemble) from the fact that, if we optimize the functional \( S_q[P(k)] = k \frac{1}{q-1} \langle dk k P(k) \rangle \) with the constraint \( \langle k \rangle \equiv \int dk k P(k) = \text{constant} \) or analogous (\( k \) being the degree of a generic site, i.e., the number of links that arrive to a given site; \( P(k) \) denotes the degree or connectivity distribution), we straightforwardly obtain \( P(k) = P(0) e^{-k/\kappa} = P(0)/[1 + (q-1)k/\kappa]^{1/q} \), which turns out to be the generic degree distribution for virtually all kinds of scale-free networks. The q-exponential function is defined as \( e_q^z \equiv [1 + (1 - q)z]^{1/(1-q)} (e_1^z = e^z) \). We verify that, for \( q > 1 \) and \( k \to \infty \), \( P(k) \sim 1/k^\gamma \) with \( \gamma = 1/(q-1) \). The classical result \( \gamma = 3 \) [24] corresponds to \( q = 4/3 \).

In the present work we address the question of how universal such results might be, and more specifically, how \( P(k) \) varies with the dimension \( d \) of the system?

Our growing model starts with one site at the origin. We then stochastically locate a second site (and then a third, a fourth, and so on up to \( N \)) through the \( d \)-dimensional isotropic distribution

\[
p(r) \propto \frac{1}{r^{d+\alpha_G}} \quad (\alpha_G > 0; \ d = 1, 2, 3, 4),
\]

where \( r \geq 1 \) is the Euclidean distance from the newly arrived site to the center of mass of the pre-existing system (in one dimension, \( r = |x| \); in two dimensions, \( r = \sqrt{x^2 + y^2} \); in three dimensions \( r = \sqrt{x^2 + y^2 + z^2} \), and so on); \( p(r) \) is zero for \( 0 \leq r < 1 \); the subindex \( G \) stands for growth. We consider \( \alpha_G > 0 \) so that the distribution \( P(r) \) is normalizable; indeed, \( \int_1^\infty dr r^{d-1} r^{-\alpha_G} = \int_1^\infty dr 1/r^{1+\alpha_G} \), which is finite for \( \alpha_G > 0 \), and diverges otherwise. See Fig. 1.

Every new site which arrives is then attached to one of the pre-existing cluster. The choice of the site to be linked with is done through the following preferential attachment probability:

\[
\Pi_{ij} = \frac{k_i r_{ij}^{-\alpha_A}}{\sum k_i r_{ij}^{-\alpha_A}} \in [0, 1] \quad (\alpha_A \geq 0),
\]

where \( k_i \) is the connectivity of the \( i \)-th pre-existing site (i.e., the number of sites that are already attached to site
\( r_{ij} \) is the Euclidean distance from site \( i \) to the newly arrived site \( j \); subindex \( A \) stands for attachment.

For \( \alpha_A \) approaching zero and arbitrary \( d \), the physical distances gradually lose relevance and, at the limit \( \alpha_A = 0 \), all distances becomes irrelevant in what concerns the connectivity distribution, and we therefore recover the Barabási-Albert (BA) model \([24]\), which has topology but no metrics.

\[
P(k) = \frac{c_k}{k^{\alpha_a - 1}}
\]

for \( \alpha_a = 2.0, \alpha_G = 0.0, \) and \( d = 1, 2, 3 \).

\[
P(k) = \frac{c_k}{k^{\alpha_a - 1}}
\]

\( \alpha_a \) for \( 0 \leq \alpha/d \leq \) (long-range interactions, e.g., gravitational and dipole-monopole interactions) and converges for \( \alpha/d > 1 \) (short-range interactions, e.g., Lennard-Jones interaction), and the internal energy per particle is, in the thermodynamical limit, constant for short-range interactions whereas it diverges like \( N^{1-\alpha/d} \) for long-range interactions, \( N \) being the total number of particles.

If all these meaningful scalings are put together, we obtain a highly plausible scenario for the respective domains of validity of the Boltzmann-Gibbs (additive) entropy and associated statistical mechanics, and that of the nonadditive entropies \( S_q \) (with \( q \neq 1 \)) and associated statistical mechanics.

Finally, we notice in Fig. 6 that both \( q \) and \( \kappa \) approach quickly their BG limits (\( q = 1 \)) for \( \alpha_A/d \to \infty \). Moreover, the same exponential \( e^{-\alpha/d} \) appears in both heuristic expressions for \( q \) and \( \kappa \). Consequently, the following linear relation can be straightforwardly established:

\[
\kappa \approx 4.90 - 3.45q.
\]

In fact, this simple relation is numerically quite well satisfied as can be seen in Fig. 4. Its existence reveals an interesting peculiarity of the nature of \( q \)-statistics. If in the celebrated BG factor \( e^{-\text{energy}/kT} \), corresponding to \( q = 1 \), we are free to consider an arbitrary value for \( T \), how come in the present problem, \( \kappa \) is not a free parameter but has instead a fixed value for each specific model that we are focusing on? This is precisely what occurs in the high-energy applications of \( q \)-statistics, e.g., in quark-gluon soup \([29]\) where \( q = 1.114 \) and \( T = 135.2 \text{MeV} \), as well as in all the LHC/CERN and RHIC/Brookhaven experiments \([4]\). Another example which is reminiscent
of this type of behavior is the sensitivity to the initial conditions at the edge of chaos (Feigenbaum point) of the logistic map; indeed, the inverse $q$-generalized Lyapunov exponent satisfies the linear relation $1/\lambda_q = 1 - q$ \cite{30}. The cause of this interesting and ubiquitous feature comes from the fact that $q$-statistics typically emerges at critical-like regimes and is deeply related to an hierarchical occupation of phase space (or Hilbert space or Fock space), which in turn points towards asymptotic power-laws (see also \cite{31}). In other words, $\kappa$ plays a role analogous to a critical temperature, which is of course not a free parameter but is instead fixed by the specific model.

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FIG. 5. $q$ and $\kappa$ for $d = 1, 2, 3, 4$. For $\alpha_A = 0$ and $\forall d$, we recover the Barabási-Albert universality class $q = 4/3$ (hence $\gamma = 3$) [24], which has no metrics.

FIG. 6. $q$ and $\kappa$ versus $\alpha_A/d$ (same data as in Fig. 5). We see that $q = 4/3$ for $0 \leq \alpha_A/d \leq 1$, and a nearly exponential behavior emerges for $\alpha_A/d > 1$ ($\forall d$); similarly for $\kappa$. These results exhibit the universality of both $q$ and $\kappa$. The red dot indicates the Barabási-Albert (BA) universality class $q = 4/3$ [24].

FIG. 7. All the values of $q$ and $\kappa$ for the present $d = 1, 2, 3, 4$ models follow closely the linear relation Eq. (3) (continuous straight line). The upmost value of $q$ is $4/3$, yielding $\kappa \simeq 0.3$ ($\forall d$).

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In his words: “In treating of the canonical distribution, we shall always suppose the multiple integral in equation (92) [the partition function, as we call it nowadays] to have a finite value, as otherwise the coefficient of probability vanishes, and the law of distribution becomes illusory. This will exclude certain cases, but not such apparently, as will affect the value of our results with respect to their bearing on thermodynamics. It will exclude, for instance, cases in which the system or parts of it can be distributed in unlimited space […]. It also excludes many cases in which the energy can decrease without limit, as when the system contains material points which attract one another inversely as the squares of their distances. […]. For the purposes of a general discussion, it is sufficient to call attention to the assumption implicitly involved in the formula (92).”