Poincaré group operators with 4-vector position

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Abstract
We present a new set of massless Poincaré group operators Hermitian with respect to the $1/r$ inner product space, which have quasi-plane wave energy-momentum eigenfunctions having velocity $c$ along their axis of propagation. These are constructed by means of a unitary transformation from a known set of massless Poincaré group operators of helicity $s = 0, \pm \frac{1}{2}, \pm 1 \ldots$. The position vector $\mathbf{r}$ is the space part of a null 4-vector.

I. Introduction

The standard relativistic quantization procedure [1] involves the integral operator

$$\omega \equiv \sqrt{m^2 + p^2} = \sqrt{m^2 - \nabla^2}$$

in units with $\hbar = c = 1$, then

$$H \equiv p^0 = \omega, \quad p = -i\nabla$$

which together with the Lorentz generators of boosts and rotations

$$K = \frac{1}{2}(\mathbf{r}H + H\mathbf{r}), \quad J = -i\mathbf{r} \times \nabla$$

make up the ten Poincaré group operators. There are a number of well-known difficulties with this procedure, of which we mention the following. 1) The conserved inner product in configuration space is [2]

$$\langle \phi | \psi \rangle = \int d^3\mathbf{r} \left[ \phi^* \omega \psi + (\omega \phi^*) \psi \right],$$

but the quantity $\left[ \phi^* \omega \psi + (\omega \phi^*) \psi \right]$ can be negative even for superpositions of positive energy solutions satisfying $i\partial_t \psi = +\omega \psi$. And 2) The position operator $\mathbf{r}$ is not Hermitian, and must be replaced by the Newton-Wigner [3] position operator $\mathbf{r}_{NW}$ the eigenfunctions of which are not delta functions.

We set out in this paper an alternative quantization procedure which allows the ‘natural’ position operator $\mathbf{r}$. Our aims are

(A) to find an alternative set of Poincaré group operators: while the momentum operator will not have the simple form $(-i\nabla)$, we require that

(B) the eigenfunctions of the new energy-momentum operators $P^\lambda$ represent waves which propagate at velocity $c$ along their propagation axis,

(C) the Hamiltonian $P^0$ will be a positive operator,

(D) the Poincaré group operators will be Hermitian with respect to a inner product space $\mathcal{L}$ such that $\langle \psi | \psi \rangle$ is the integral of a positive definite density, and

(E) the position operator $\mathbf{r}$ must be Hermitian with respect to $\mathcal{L}$ and be part of a 4-vector position operator $r^\lambda$.

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In this paper we consider only mass zero representations with helicity \( s = 0, \pm \frac{1}{2}, \pm 1, \ldots \), leaving the case with mass to a following paper. We start from a known [4] set of Poincaré group operators \((a^\lambda, J^{\lambda \mu})\), the space part of \(a^\lambda\) being closely related to the Runge-Lenz vector from the theory of the Schrodinger hydrogen atom. These operators are Hermitian with respect to the \(1/r\) inner product space of (7) below, and furthermore the \(a^0\) operator is positive. As the eigenfunctions of \(a^\lambda\) are waves of variable velocity, we proceed in the next section to find a unitary transformation such that the transformed eigenfunctions have plane wave character - in that their velocity along their propagation axis is \(c\).

The operators [4]

\[
a^0 = -r \nabla^2, \quad a = -2(\partial_r r) \nabla + r \nabla^2 \quad (4)
\]

\[
K \equiv (J^{10}, J^{20}, J^{30}) = -i r \nabla, \quad J \equiv (J^{23}, J^{31}, J^{12}) = -i r \times \nabla \quad (5)
\]

obey the necessary Poincaré group commutation properties:

\[
[J^{\lambda \mu}, a^\nu] = i(\eta^{\mu \nu} a^\lambda - \eta^{\lambda \nu} a^\mu), \quad [a^\lambda, a^\mu] = 0, \quad (6)
\]

\[
[J^{\lambda \mu}, J^{\nu \rho}] = i(\eta^{\lambda \rho} J^{\mu \nu} - \eta^{\lambda \nu} J^{\mu \rho} - \eta^{\mu \rho} J^{\lambda \nu}).
\]

These operators \(a^\lambda, J^{\lambda \mu}\) are perhaps better known as components of a SO(4,2) group representation, see for example [5]. Also

\[
a^\lambda \cdot a_\lambda = 0
\]

so that \(J^{\lambda \mu}, a^\nu\) of (4,5) are a massless spin-zero set of Poincaré group operators. The \(a^0\) is a positive operator. They are Hermitian with respect to the \(1/r\) inner product space \(L_{1/r}\)

\[
\langle \phi | \psi \rangle_{1/r} \equiv \int \frac{1}{r} \overline{\psi}(r) \psi(r) d^3r = \int \frac{1}{r} \overline{\psi}(r, \theta, \phi) \psi(r, \theta, \phi) r dr d\Omega \quad (7)
\]

which is Lorentz-invariant [6]. These \(J^{\lambda \mu}, a^\nu\) are to our knowledge the simplest (non-trivial) set of configuration space Poincaré operators containing only local (differential) operators. The density \(\rho = \frac{1}{r} \overline{\psi} \psi\) is conserved provided that the \(\psi\) are solutions of \(i \partial_t \psi = a^0 \psi\), as then

\[
\partial_t(\frac{1}{r} \overline{\psi} \psi) = \nabla \cdot i [\overline{\psi} \nabla \psi - (\nabla \overline{\psi}) \psi].
\]

Orthonormal eigenfunctions of \(a^\lambda\) with respective eigenvalues \(k^\lambda \equiv (|k|, k) \equiv (k, k)\) are the Bessel functions ([4] formula (92)):

\[
a^\lambda u_k = k^\lambda u_k
\]

where

\[
u_k(r) = \frac{1}{4\pi} J_0\left(\frac{\sqrt{-2k^\lambda r_k}}{2}\right) = \frac{1}{4\pi} J_0\left(\frac{\sqrt{2kr + 2k \cdot r}}{2}\right) \quad (8)
\]

which may be proved by direct differentiation. The quantity \((kr + k \cdot r)\) is non-negative by Schwarz’s rule. The orthogonality and completeness relations are [4,7]

\[
\int \frac{d^3r}{r} u_k(r) u_{k'}(r) = k \delta(k - k'), \quad (9)
\]

\[
\int \frac{d^3k}{k} u_k(r) u_{k'}(r') = r \delta(r - r'). \quad (10)
\]
Helicity $s$ representations of the Poincaré group which reduce to (4,5) when $s = 0$ were found by Derrick [8]. The operators $a_s^0$, $a_s$, $K_s$, $J_s$ which also satisfy the the Poincaré group algebra (6) are

\begin{align*}
a_s^0 &= -r \nabla^2 + 2s \frac{i}{r-z} \partial_\phi + 2s^2 \frac{1}{r-z}, \\
a_s &= -2 (\partial_r r) \nabla + r \nabla^2 - s 4i (W \times \nabla) - s^2 2 e_3 \frac{1}{r-z} \\
K_s &= -i r \nabla - s (\hat{r} \times W) , \\
J_s &= -i \hat{r} \times \nabla + s W
\end{align*}

where

\begin{equation}
\textbf{W} = \left( \frac{x}{r-z}, \frac{y}{r-z}, -1 \right), \quad \partial_\phi = (r \times \nabla)_3
\end{equation}

with $s = 0, \pm \frac{1}{2}, \pm 1, \ldots$. The eigenfunctions of $a_s^\lambda$ are [8]

\begin{equation}
u_{s,k}(r) = \frac{1}{4\pi} e^{2isf(\hat{r}, \hat{k})} J_{2s}(\sqrt{2kr + 2k \cdot r})^{1/2}.
\end{equation}

We give the angular phase factor $e^{2isf(\hat{r}, \hat{k})}$ for reference in the appendix. And $e^{2isf(\hat{r}, \hat{k})} \to e^{2is \phi}$ when $k = (0, 0, \pm k)$, where $e^{i \phi} \equiv (\hat{r} + i \hat{\imath}r)/(\hat{r}^2 + \hat{\imath}r^2)^{1/2}$. Then the eigenfunction (15) is

\begin{equation}
u_{s(0,0,\pm k)}(r) = \frac{1}{4\pi} e^{2is \phi} J_{2s}(\sqrt{2kr \pm 2kz})^{1/2}.
\end{equation}

The eigenfunctions $u_{s,k}$ have the same orthogonality and completeness relations (9,10) as their helicity zero counterparts.

2. The unitary operator $V$

The eigenfunctions $u_{s,k}$ are essentially waves in parabolic coordinates. As massless waves proceed at the velocity of light $c$ (which is unity in our units), we look for a unitary transformation of the operators $a_s^\lambda, J_s^{\lambda \mu}$ defining new operators $P_s^\lambda \equiv V a_s^\lambda V^{-1}, \textbf{F}_s^{\lambda \mu} \equiv V J_s^{\lambda \mu} V^{-1}$, such that the resulting eigenfunctions $V u_{s,k}$ of the $P_s^\lambda$ energy-momentum operators represent waves of velocity $c$ along their propagation axis.

Consider $\psi(u r')$ with $-\infty < u < \infty$, which is the wavefunction along the line through the origin which includes the point $r'$. The unitary operators $V$ of (21) below map $\psi(u r')$ onto itself, which is clearly seen by inspection of (21d). In the appendix we show how $V$ can be constructed from the parity $\mathcal{P}$, inversion $\mathcal{N}$, and Fourier transform operators $\mathcal{F}$:

\begin{align*}
\mathcal{P} f(r, \theta, \phi) &\equiv f(r, \pi - \theta, \phi + \pi) \\
\mathcal{N} f(r, \theta, \phi) &\equiv \frac{1}{r^2} f\left(\frac{1}{r}, \theta, \phi \right) \\
\mathcal{F}^\pm f(r, \theta, \phi) &\equiv \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos(r t) \sin(r t) f(t, \theta, \phi) dt \\
\mathcal{F} \equiv (\mathcal{F}_c \pm i \mathcal{F}_s).
\end{align*}

The operators $V, \mathcal{V}^{-1}$ are

\begin{align*}
\mathcal{V}^{-1} g(r, \theta, \phi) &= \frac{1}{2} \left[ \left( \frac{1}{\sqrt{r}} \mathcal{F}_+ \sqrt{r} \right) - \left( \frac{1}{\sqrt{r}} \mathcal{F}_- \sqrt{r} \right) \mathcal{P} \right] \mathcal{N} g(r, \theta, \phi) \\
\mathcal{V} f(r, \theta, \phi) &= \frac{1}{2} \mathcal{N} \left[ \left( \frac{1}{\sqrt{r}} \mathcal{F}_- \sqrt{r} \right) - \left( \frac{1}{\sqrt{r}} \mathcal{F}_+ \sqrt{r} \right) \mathcal{P} \right] f(r, \theta, \phi) \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{r^3}} \int_0^{\infty} dt \left[ e^{-it/r} \sqrt{r} f(t, \theta, \phi) - e^{it/r} \sqrt{r} f(t, \pi - \theta, \phi + \pi) \right]
\end{align*}

which can be written
In this section we find the eigenfunctions of $P$ then from (21e) and the integrals (1.13.25), (2.13.27) of [10] can be simplified to (corresponding to the form (21d) for $F$)

$$V^{-1}g(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du \, \frac{e^{i/u}}{\sqrt{|u|}} \text{sgn}(u) \, g(ur).$$

(20b)

The kernel $\{e^{-i/u} \sqrt{|u|} \text{sgn}(u)\}$ and its derivatives are continuous at $u = 0$. We see that $(Vf)(r')$ is essentially a Fourier transform of $\sqrt{r} f(r)$ along the line $r = ur'$, $(-\infty < u < \infty)$. The $V^{-1}$ operator can be simplified to (corresponding to the form (21d) for $V$)

$$V^{-1}g(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du \, \frac{e^{i/u}}{\sqrt{|u|}} \text{sgn}(u) \, g(ur)$$

(20b)

If $\sqrt{r} f(r)$ does not converge at infinity, the integral of (21) does not exist. For this case the definitions of $F_{\pm}$ of (19) must be extended (see for example [9]) to the so called generalized cosine (sine) transform. In effect we write $e^{\pm i u} = \mp i \partial_u e^{\pm i u}$ in (21c) and then integrate by parts while discarding the surface terms at infinity.

$$V' f(r) = -\frac{i}{\sqrt{2\pi}} \int_0^\infty 2 \left( e^{-i u} \partial_u \sqrt{u} f(u r) + e^{i u} \partial_u \sqrt{u} f(-u r) \right)$$

(21c)

$$= -\frac{i}{\sqrt{2\pi}} \int_0^\infty du \left( \frac{e^{-i u}}{\sqrt{u}} \sqrt{u} \partial_u \sqrt{u} f(u r) + \frac{e^{i u}}{\sqrt{u}} \sqrt{u} \partial_u \sqrt{u} f(-u r) \right)$$

$$= -\frac{i}{\sqrt{2\pi}} \sqrt{r} \partial_r \left\{ \sqrt{r} \int_0^\infty du \left( e^{-i u} \sqrt{u} f(u r) - e^{i u} \sqrt{u} f(-u r) \right) \right\}.$$  

(21e)

For the last line we have used the identity $\sqrt{u} \partial_u \sqrt{u} f(\pm ur) = \pm \sqrt{r} \partial_r \sqrt{r} f(\pm ur)$. The operator $V' f(r)$ is equal to $V f(r)$ whenever the latter exists, so from now on we drop the prime on $V$.

3. The eigenfunctions $V u_{sk}$

In this section we find the eigenfunctions of $P_s^\lambda = V u_{sk}^\lambda V^{-1}$ which we call

$$u_{sk} = V u_{sk}$$

(22)

and show that these represent waves which have velocity $c$ along their propagation axis.

The $V$ operator sees through any angular variable, so recalling (15) we need to evaluate $V J_{2s}(2[kr + k \cdot r]^{1/2})$ where $2s$ is zero or an integer. We will specialize to the case when $k = (0, 0, k)$ and the eigenfunction is

$$u_{s(0,0,k)}(r) = \frac{1}{4\pi} e^{2i s \phi} J_{2s} \left[ 2kr + 2kz \right]^{1/2},$$

(23)

then from (21e) and the integrals (1.13.25), (2.13.27) of [10]

$$w_{s(0,0,k)}(r) = V u_{s(0,0,k)}(r)$$

$$= -\frac{i e^{2i s \phi}}{4\pi \sqrt{2\pi}} \sqrt{r} \partial_r \left\{ \sqrt{r} \int_0^\infty du \left( \frac{e^{-i u}}{\sqrt{u}} J_{2s} \left[ 2ukr + 2ukz \right]^{1/2} - \frac{e^{i u}}{\sqrt{u}} J_{2s} \left[ 2ukr + 2ukz \right]^{1/2} \right) \right\}$$

$$= -\frac{e^{2i s \phi}}{4\pi \sqrt{2}} \left( \sqrt{r} \partial_r \sqrt{r} \right) \left\{ e^{i(1-2s)\pi/4} \exp \left\{ i \left( \frac{kr + k z}{4} \right) J_s \left( \frac{kr + k z}{4} \right) \right\} J_{s} \left( \frac{kr - k z}{4} \right) 
-e^{-i(1-2s)\pi/4} \exp \left\{ -i \left( \frac{kr - k z}{4} \right) J_s \left( \frac{kr - k z}{4} \right) \right\} \right\}. \quad (24a)
Omitting the \( e^{2i s \phi} \) phase factor, the \( w_{s(0,0,k)} \) are everywhere continuous, and are zero at the origin except for the spin zero case \( s = 0 \). The \( w_{s(0,0,k)} \) have similar asymptotic behaviour for various \( s \), and are of simpler form when \( s \) is half integer. When for example \( s = 1/2 \)

\[
w_{s(0,0,k)}(r) = \frac{e^{i \phi}}{4 \sqrt{\pi}} \left( [kr + k\varepsilon]^{1/2} \exp(i[kr + k\varepsilon]/2) - [kr - k\varepsilon]^{1/2} \exp(-i[kr - k\varepsilon]/2) \right)
\]

(25)

which is proportional to \( \exp(i k\varepsilon) \) along both halves of the \( z \) axis, so that \( \{e^{-ikt} w_{s(0,0,k)}\} \) is a unidirectional wave proceeding in the +\( z \) direction at velocity \( c \). For large \( r \), the \( w_{s(0,0,k)} \) are of order \( \sqrt{r} \), the ‘extra’ factor \( \sqrt{r} \) accounted for by the \( 1/r \) inner product space. For reference we write out the \( w_{s(0,0,k)} \) of (24a) in full below

\[
w_{s(0,0,k)}(r) = \frac{e^{2i s \phi}}{4 \pi \sqrt{2}} \left[ e^{i(1-2s)r} \exp\{ik\lambda/2\} \left\{ (2s-1) J_s(k\lambda/2) - k\lambda \left( J_{s-1}(k\lambda/2) + i J_s(k\lambda/2) \right) \right\} \right.
\]

\[
- e^{-i(1-2s)r} \exp\{-i k\mu/2\} \left\{ (2s-1) J_s(k\mu/2) - k\mu \left( J_{s-1}(k\mu/2) - i J_s(k\mu/2) \right) \right\} \right]
\]

(24b)

where \( \lambda = (r + z)/2, \mu = (r - z)/2 \). A contour plot of these functions reveals the planar nature of the wave fronts.

4. The Lorentz operator \( \mathbf{V} \mathbf{K} \mathbf{V}^{-1} \) simplified

We are interested to see how the Hermitian position operator \( \mathbf{r} \) transforms under boosts and rotations, so we must evaluate the commutators \( \left[ \mathbf{J}_s, \mathbf{r}^b \right], \left[ \mathbf{K}_s, \mathbf{r}^b \right] \) where

\[
\mathbf{J}_s \equiv \mathbf{V} \mathbf{J}_s \mathbf{V}^{-1}, \quad \mathbf{K}_s \equiv \mathbf{V} \mathbf{K}_s \mathbf{V}^{-1}.
\]

We will first simplify the \( \mathbf{J}_s, \mathbf{K}_s \) operators, and here we will only consider the \( s = 0 \) helicity zero operators \( \mathbf{J}, \mathbf{K} \), because the extra helicity components commute with the position operator.

We first need to define the operators \( \mathbf{U}, \mathbf{U}^{-1} \) (we also write out again the \( \mathbf{V}, \mathbf{V}^{-1} \) of (21) for comparison):

\[
\mathbf{U} \equiv \frac{1}{2} \mathcal{N} \frac{1}{\sqrt{r}} \left[ \mathcal{F}_+ + \mathcal{F}_- \mathbf{P} \right] \sqrt{r} \mathcal{N}, \quad \mathbf{U}^{-1} = \frac{1}{2} \frac{1}{\sqrt{r}} \left[ \mathcal{F}_+ + \mathcal{F}_- \mathbf{P} \right] \sqrt{r} \mathcal{N},
\]

(26)

Also

\[
\left[ \mathbf{K}, (\mathcal{N} \frac{1}{\sqrt{r}} \mathcal{F}_\pm \sqrt{r}) \right] = 0, \quad \left[ \mathbf{J}, (\mathcal{N} \frac{1}{\sqrt{r}} \mathcal{F}_\pm \sqrt{r}) \right] = 0,
\]

(27)

\[
\mathbf{K} \mathcal{P} = -\mathcal{P} \mathbf{K}, \quad \mathbf{J} \mathcal{P} = \mathcal{P} \mathbf{J}
\]

(28)

\[
\mathbf{K} \mathbf{V}^{-1} = \mathbf{U}^{-1} \mathbf{K}, \quad \mathbf{V} \mathbf{K} = \mathbf{K} \mathbf{U}
\]

(29)

\[
\left[ \mathbf{J}, \mathbf{V} \right] = \left[ \mathbf{J}, \mathbf{V}^{-1} \right] = 0.
\]

(30)

The first relation (27) becomes apparent when we write out \( \left( \mathcal{N} \frac{1}{\sqrt{r}} \mathcal{F}_\pm \sqrt{r} \right) \psi(\mathbf{r}) \) in the form

\[
\left( \mathcal{N} \frac{1}{\sqrt{r}} \mathcal{F}_\pm \sqrt{r} \right) \psi(\mathbf{r}) \equiv \sqrt{\frac{2}{\pi}} \int_0^\infty du e^{\pm i u} \sqrt{u} \psi(u \mathbf{r}),
\]

then one can see that \( \mathbf{K}, \mathbf{J} \) only act on the \( \mathbf{r} \) in the argument of \( \psi \). And (29) follows from (27), (28). The relation (30) means that

\[
\mathbf{J} \equiv \mathbf{V} \mathbf{J} \mathbf{V}^{-1} = \mathbf{J}.
\]
Following from (29)
\[
\mathbf{K} \equiv \mathcal{V} \mathbf{K} \mathcal{V}^\dagger = (\mathcal{V} \mathcal{U}^\dagger) \mathbf{K} = \mathbf{K} (\mathcal{U} \mathcal{V}^\dagger).
\tag{31}
\]

We can simplify the $(\mathcal{V} \mathcal{U}^\dagger)$, $(\mathcal{U} \mathcal{V}^\dagger)$ unitary operators of (31) above which are adjoints of each other. We need the further identities (referred to in the appendix)
\[
\mathcal{F}_+ \mathcal{F}_+ = -i (\mathcal{H}_e - \mathcal{H}_o), \quad \mathcal{F}_- \mathcal{F}_- = i (\mathcal{H}_e - \mathcal{H}_o),
\]
\[
\mathcal{F}_+ \mathcal{F}_- = 2 - i (\mathcal{H}_e + \mathcal{H}_o), \quad \mathcal{F}_- \mathcal{F}_+ = 2 + i (\mathcal{H}_e + \mathcal{H}_o)
\tag{32}
\]
where $\mathcal{H}_e, \mathcal{H}_o$ are the Hilbert transforms of even, odd functions:
\[
\mathcal{H}_e f(r, \theta, \phi) \equiv -\frac{2r}{\pi} \int_0^\infty \frac{f(t, \theta, \phi)}{r^2 - t^2} dt, \quad \mathcal{H}_o f(r, \theta, \phi) \equiv \frac{2r}{\pi} \int_0^\infty \frac{t f(t, \theta, \phi)}{r^2 - t^2} dt,
\tag{33}
\]
and also we note that
\[
\mathcal{N}(\frac{1}{\sqrt{r}} \mathcal{H}_e \sqrt{r})\mathcal{N} = (\frac{1}{\sqrt{r}} \mathcal{H}_o \sqrt{r}), \quad \mathcal{N}(\frac{1}{\sqrt{r}} \mathcal{H}_o \sqrt{r})\mathcal{N} = (\frac{1}{\sqrt{r}} \mathcal{H}_e \sqrt{r}).
\tag{34}
\]
Then with the aid of (32), (34)
\[
(\mathcal{V} \mathcal{U}^\dagger) = \frac{1}{4} \mathcal{N} \frac{1}{\sqrt{r}} \left[ \mathcal{F}_- - \mathcal{F}_+ \mathcal{P} \right] \left[ \mathcal{F}_+ + \mathcal{F}_- \mathcal{P} \right] \sqrt{r} \mathcal{N}
\]
\[
= \frac{1}{2} \mathcal{N} \frac{1}{\sqrt{r}} \left[ i (\mathcal{H}_e + \mathcal{H}_o) + i (\mathcal{H}_e - \mathcal{H}_o) \mathcal{P} \right] \sqrt{r} \mathcal{N}
\]
\[
= \frac{1}{2} \frac{1}{\sqrt{r}} \left[ (\mathcal{H}_e + \mathcal{H}_o) - (\mathcal{H}_e - \mathcal{H}_o) \mathcal{P} \right] \sqrt{r} \equiv i \frac{1}{\sqrt{r}} \mathcal{G}_- \sqrt{r}
\tag{35}
\]
and similarly
\[
(\mathcal{U} \mathcal{V}^\dagger) = \frac{1}{2} \frac{1}{\sqrt{r}} \left[ (\mathcal{H}_e + \mathcal{H}_o) + (\mathcal{H}_e - \mathcal{H}_o) \mathcal{P} \right] \sqrt{r} \equiv i \frac{1}{\sqrt{r}} \mathcal{G}_+ \sqrt{r}
\tag{36}
\]
where
\[
\mathcal{G}_\pm \equiv \frac{1}{2} \left[ (\mathcal{H}_e + \mathcal{H}_o) \pm (\mathcal{H}_e - \mathcal{H}_o) \mathcal{P} \right]
\tag{37a}
\]
\[
\mathcal{G}_\pm f(r) = \frac{1}{\pi} \int_0^\infty \left( \frac{f(u r)}{u - 1} \mp \frac{f(-u r)}{u + 1} \right) du.
\tag{37b}
\]
Finally substituting (35), (36) into (31) we have the transformed boost operator
\[
\mathbf{\overline{K}} \equiv \mathcal{V} \mathbf{K} \mathcal{V}^\dagger = \mathbf{K} \left( i \frac{1}{\sqrt{r}} \mathcal{G}_+ \sqrt{r} \right) = (i \frac{1}{\sqrt{r}} \mathcal{G}_- \sqrt{r}) \mathbf{K}
\tag{38}
\]
where $\mathbf{K} \equiv (-i \mathbf{r} \nabla)$. Note that $\mathbf{\overline{K}}$ is not a local (differential) operator like $\mathbf{K}$.

5. The position 4-vector operator
The simplification (38) allows us to calculate the commutator $[\mathbf{\overline{K}}^a, r^b]$. For the position operator $\mathbf{r}$ to be the space part of a 4-vector $r^\lambda \equiv (r^0, \mathbf{r})$, then $r^0$ must satisfy both of the following:
\[
[r^\lambda, r^\mu] = i \delta^{ab} r^0, \quad [\mathbf{\overline{K}}, r^0] = i \mathbf{r}.
\tag{39}
\]
To calculate \( \overline{K}^a, r^b \) we must first evaluate \([G_\pm, r]\). Recalling (37b) then
\[
G_\pm (r f(r)) = \frac{1}{\pi} \int_0^\infty \left( \frac{ur f(ur)}{u-1} \pm \frac{ur f(-ur)}{u+1} \right) du
\]
\[
= \frac{1}{\pi} r \int_0^\infty \left( \frac{(u-1+1)f(ur)}{u-1} \pm \frac{(u+1-1)f(-ur)}{u+1} \right) du
\]
\[
= r G_\pm f(r) + \frac{1}{\pi} r \int_0^\infty (f(ur) \pm f(-ur)) du,
\]
and multiplying (37b) from the left by \( r \) and then subtracting from (40) yields the operator identity
\[
[G_\pm, r] = r Z_\pm,
\]
where
\[
Z_\pm f(r) = \frac{1}{\pi} \int_0^\infty (f(ur) \pm f(-ur)) du.
\]

Also with similar methods to the above we find
\[
G_\pm (r f(r)) = r G_\pm f(r) + \frac{1}{\pi} r \int_0^\infty (f(ur) \mp f(-ur)) du
\]
or
\[
G_\pm r = r G_\pm + r Z_\pm.
\]

With the commutator (41) we can now evaluate \([\overline{K}^a, r^b]\) :
\[
[\overline{K}^a, r^b] = [K^a (i \frac{1}{\sqrt{r}} G_+ \sqrt{r}), r^b]
\]
\[
= [K^a, r^b] (i \frac{1}{\sqrt{r}} G_+ \sqrt{r}) + K^a (i \frac{1}{\sqrt{r}} [G_+, r^b] \sqrt{r})
\]
\[
= -i \delta^{ab} r (i \frac{1}{\sqrt{r}} G_+ \sqrt{r}) + K^a (\frac{1}{\sqrt{r}} r^b Z_+ \sqrt{r}) \to -i \delta^{ab} r (i \frac{1}{\sqrt{r}} G_+ \sqrt{r})
\]
where for the last line we put \( Z_+ \sqrt{r} = 0 \). This is a boundary condition on the wavefunction requiring that
\[
Z_+ (\sqrt{r} \psi(r)) \equiv \frac{1}{\pi} \sqrt{r} \int_0^\infty du \sqrt{u} [\psi(ur) + \psi(-ur)] = 0.
\]
The above is automatically satisfied if \( \psi(r) \) is odd under the parity transformation. When \( \psi(r) \) is even under parity then (45) is equivalent to \( \int_0^\infty dr \sqrt{r} \psi(r, \theta, \phi) = 0 \).

Recalling (39), the formula (44) suggests that
\[
r^0 = -r (i \frac{1}{\sqrt{r}} G_+ \sqrt{r})
\]
\[
= - (i \frac{1}{\sqrt{r}} G_+ \sqrt{r}) r + i r (i \frac{1}{\sqrt{r}} Z_+ \sqrt{r}) = - (i \frac{1}{\sqrt{r}} G_- \sqrt{r}) r
\]
where the second line results from (43) and (45). So we can write two equivalent forms for \( r^0 \) :
\[
r^0 = -r (i \frac{1}{\sqrt{r}} G_+ \sqrt{r}) = - (i \frac{1}{\sqrt{r}} G_- \sqrt{r}) r.
\]

We must still check the commutator \([\overline{K}, r^0]\), recalling from (38) and (46) that both the operators \( \overline{K}, r^0 \) can be written in either of two ways:
\[
[\overline{K}, r^0] = \overline{K} r^0 - r^0 \overline{K}
\]
\[
= -K (i \frac{1}{\sqrt{r}} G_+ \sqrt{r}) (i \frac{1}{\sqrt{r}} G_- \sqrt{r}) r + r (i \frac{1}{\sqrt{r}} G_+ \sqrt{r}) (i \frac{1}{\sqrt{r}} G_- \sqrt{r}) \overline{K}
\]
\[
= -K r + r \overline{K} = ir
\]
which is the required result, and we have used the identity
\[
\left( \frac{i}{\sqrt{r}} G_- \sqrt{r} \right) \left( \frac{i}{\sqrt{r}} G_+ \sqrt{r} \right) = (V U^\dagger) (U V^\dagger) = 1
\] (48)
recalling (35), (36). The results (44), (47) show the 4-vector property of
\[
r^\lambda = \left( - r \left( \frac{1}{\sqrt{r}} G_+ \sqrt{r} \right), r \right)
\]
only assuming (45). Furthermore we see from (46) that
\[
(r^0)^2 = r \left( \frac{1}{\sqrt{r}} G_+ \sqrt{r} \right) (i \frac{1}{\sqrt{r}} G_- \sqrt{r}) r = r^2
\] (49)
so that \( r^\lambda \) is a null 4-vector. Finally the \( r^0 \), \( r \) components of \( r^\lambda \) commute as
\[
[r^0, r] = - r \left( \frac{1}{\sqrt{r}} [G_+, r] \sqrt{r} \right) = - r \left( \frac{1}{\sqrt{r}} r Z_+ \sqrt{r} \right) = 0
\]
with (45).

5. Discussion

We have achieved our aims (A) to (E) of the Introduction. And we have shown that the position operator \( r \) has covariant meaning, in that it is the space part of a null 4-vector (as discussed in the last section, this 4-vector property of \( r^\lambda \) requires the wavefunction to be subject to the boundary condition (45)). The difficulties with the position operator have been well known since the inception of relativistic quantum mechanics: the eigenfunctions of the Newton-Wigner \([3]\) operator \( (r_{NW}) \) are smeared out in space, which renders problematic the exact meaning of \( r \) when, for example, potential terms are included in the Hamiltonian. From our viewpoint the difficulties in the usual theory arise from the inner product (3), which is a direct consequence of the momentum operator being \( p = (-i \nabla) \).

Then the inner product (3) disallows the ‘natural’ position operator \( r \).

Our approach has been to allow other possibilities for the momentum operator, only requiring that it is the space part of a 4-vector and that the eigenfunctions of the energy-momentum operator have plane wave character. Because the momentum operators presented here are Hermitian with respect to the simple \( 1/r \) inner product space of (7), this in turn allows for the natural position operator \( r \). The ‘cost’ we have to pay for this transparency of the position operator is that the momentum operators as well as their eigenfunctions are more complicated.
Appendix

The angular phase factor $e^{2i s f(\hat{r}, \hat{k})}$

In spherical coordinates

\[ \hat{r} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \]
\[ \hat{k} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \]

then we give Derrick’s formula ((39) of [8])

\[
e^{i f(\hat{r}, \hat{k})} = \left[ 2/(1 + \hat{k} \cdot \hat{r}) \right]^{1/2} (e^{i\phi} \cos \frac{1}{2} \theta \cos \frac{1}{2} \theta_k + e^{i\phi_k} \sin \frac{1}{2} \theta \sin \frac{1}{2} \theta_k) \tag{A1} \]

and in the particular case when $\hat{k} = (0, 0, 1)$ so that $\theta_k = 0$, then

\[
e^{i f(\hat{r}, \hat{k})} \rightarrow [2/(1 + \hat{r} \cdot \hat{k})]^{1/2} (e^{i\phi} \cos \theta) = e^{i\phi} \tag{A2} \]

The unitary operators $V$

The unitary operators $V$ of (21) can be constructed from the parity $\mathcal{P}$, inversion $\mathcal{N}$, and Fourier transform operators $\mathcal{F}$:

\[
\mathcal{P} f(r, \theta, \phi) \equiv f(r, \pi - \theta, \phi + \pi) \tag{A3} \\
\mathcal{N} f(r, \theta, \phi) \equiv \frac{1}{r^2} f\left(\frac{1}{r}, \theta, \phi\right) \tag{A4} \\
\mathcal{F}_c f(r, \theta, \phi) \equiv \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(rt) \sin(\theta t) \, dt \\
\mathcal{F}_s f(r, \theta, \phi) \equiv \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(rt) \, dt \tag{A5} \\
\mathcal{F}_\pm \equiv (\mathcal{F}_c \pm i \mathcal{F}_s) \tag{A6}
\]

with the adjoint properties

\[
\mathcal{P}^\dagger = \mathcal{P}, \quad \mathcal{N}^\dagger = \mathcal{N}, \quad \left(\frac{1}{\sqrt{\mathcal{P}}} \mathcal{F}_c \sqrt{\mathcal{P}}\right)^\dagger = \left(\frac{1}{\sqrt{\mathcal{P}}} \mathcal{F}_c \sqrt{\mathcal{P}}\right) \tag{A7}
\]

(The self-adjoint property $\mathcal{N}^\dagger = \mathcal{N}$ can be shown by a change of variables $r \rightarrow 1/u$ within the scalar product $L_1/r$, and the adjoint property of $(1/\sqrt{\mathcal{P}}) \mathcal{F}_c \sqrt{\mathcal{P}}$ follows by changing the order of integration within the scalar product.) Also

\[
\mathcal{P} \mathcal{P} = \mathcal{N} \mathcal{N} = \mathcal{F}_c \mathcal{F}_c = \mathcal{F}_s \mathcal{F}_s = 1 \tag{A8}
\]

For the latter property $\mathcal{F}_c (\mathcal{F}_c f(r)) = \mathcal{F}_s (\mathcal{F}_s f(r)) = f(r)$ to hold always, the definitions of $\mathcal{F}_c \mathcal{F}_s$ must be extended [9] to the so called generalized cosine (sine) transform when the integral of (A5) does not exist. Then

\[
\mathcal{F}_c \rightarrow -\frac{1}{r} \partial_r, \quad \mathcal{F}_s \rightarrow \frac{1}{r} \partial_r, \quad \mathcal{F}_\pm \rightarrow \pm \frac{i}{r} \partial_r \tag{A9}
\]

which is effectively an integration by parts within the integral (A5) while discarding the surface terms. When the conventional cosine (sine) transform of (A5) does exist, it agrees with the extended definition (A9). The extension procedure of (A9) can be repeated. Then the operators $\left(\frac{1}{\sqrt{\mathcal{P}}} \mathcal{F}_c \sqrt{\mathcal{P}}\right)$, $\left(\frac{1}{\sqrt{\mathcal{P}}} \mathcal{F}_s \sqrt{\mathcal{P}}\right)$ as well as $\mathcal{P}$, $\mathcal{N}$, are unitary.

The operators $\mathcal{F}_c, \mathcal{F}_s$ do not commute but combine as follows [11]:

\[
\mathcal{F}_s \mathcal{F}_c = -\mathcal{H}_e, \quad \mathcal{F}_c \mathcal{F}_s = \mathcal{H}_o \tag{A10}
\]
where $\mathcal{H}_e, \mathcal{H}_o$ are the Hilbert transforms of even, odd functions:

$$
\mathcal{H}_e f(r, \theta, \phi) \equiv -\frac{2r}{\pi} \int_0^\infty \frac{f(t, \theta, \phi)}{r^2 - t^2} \, dt, \quad \mathcal{H}_o f(r) = -\frac{2}{\pi} \int_0^\infty \frac{f(\lambda r)}{1 - \lambda^2} \, d\lambda,
$$

$$
\mathcal{H}_o f(r, \theta, \phi) \equiv -\frac{2}{\pi} \int_0^\infty \frac{t f(t, \theta, \phi)}{r^2 - t^2} \, dt, \quad \mathcal{H}_o f(r) = -\frac{2}{\pi} \int_0^\infty \lambda f(\lambda r) \, d\lambda.
$$

(A11)

Then it follows that

$$
\mathcal{F}_+ \mathcal{F}_+ = -i (\mathcal{H}_e - \mathcal{H}_o), \quad \mathcal{F}_- \mathcal{F}_- = i (\mathcal{H}_e - \mathcal{H}_o),
$$

$$
\mathcal{F}_+ \mathcal{F}_- = 2 - i (\mathcal{H}_e + \mathcal{H}_o), \quad \mathcal{F}_- \mathcal{F}_+ = 2 + i (\mathcal{H}_e + \mathcal{H}_o).
$$

(A12)

The identities (A12) allow us to construct the unitary operator $\mathcal{V}$ and its adjoint:

$$
\mathcal{V} = \frac{1}{2} \mathcal{N} \left[ \mathcal{F}_- - \mathcal{F}_+ \mathcal{P} \right] \sqrt{\mathcal{F}}, \quad \mathcal{V}^{-1} = \frac{1}{2} \mathcal{N} \left[ \mathcal{F}_+ - \mathcal{F}_- \mathcal{P} \right] \sqrt{\mathcal{F}} \mathcal{N},
$$

(A13)

then the necessary properties $\mathcal{V} \mathcal{V}^{-1} = 1$ etc follow from (A8), as the parity operator $\mathcal{P}$ commutes with $\mathcal{F}_\pm$ and $\mathcal{N}$, also $\mathcal{P}^2 = N^2 = 1$. Writing out the operator $\mathcal{V} f(r, \theta, \phi)$:

$$
\mathcal{V} f(r, \theta, \phi) \equiv \frac{1}{2} \mathcal{N} \left[ \left( \frac{1}{\sqrt{\mathcal{P}}} \mathcal{F}_- \sqrt{\mathcal{P}} \right) - \left( \frac{1}{\sqrt{\mathcal{P}}} \mathcal{F}_+ \sqrt{\mathcal{P}} \right) \right] f(r, \theta, \phi)
$$

(A14a)

$$
= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{r^3} \sqrt{\mathcal{P}}} \int_0^\infty dt \left[ e^{-it/r} \sqrt{t} f(t, \theta, \phi) - e^{it/r} \sqrt{t} f(t, \pi - \theta, \phi + \pi) \right]
$$

(A14b)

$$
= \frac{1}{\sqrt{2\pi}} \int_0^\infty du \left[ e^{-iu} \sqrt{u} f(u r, \theta, \phi) - e^{iu} \sqrt{u} f(u r, \pi - \theta, \phi + \pi) \right]
$$

(A14c)

where we have substituted $t = ur$. Then changing to cartesian coordinates we can write

$$
\mathcal{V} f(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty du \ e^{-iu} \sqrt{|u|} \sgn(u) f(u r)
$$

(A14d)

which is (21d). Then with the operator substitutions of (A9)

$$
\mathcal{V} f(r) \rightarrow -\frac{i}{\sqrt{2\pi}} \int_0^\infty du \left( e^{-iu} \partial_u[\sqrt{|u|} f(u r)] + e^{iu} \partial_u[\sqrt{|u|} f(-ur)] \right)
$$

which is (21e).

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