The Stark effect in the charge-dyon system
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Abstract

The linear Stark effect in the MIC-Kepler problem describing the interaction of charged particle with Dirac’s dyon is considered. It is shown that constant homogeneous electric field completely removes the degeneracy of the energy levels on azimuth quantum number.

In February 2003 Professor Valery Ter-Antonyan: an intellectual, brilliant pedagogue and the expert in quantum mechanics, passed away. During the last years of his life he took a great interest in the study of the Coulomb problem in the presence of topologically nontrivial objects. The two of us had the honour of collaborating with him in this sphere. This modest work is a tribute to his memory.

1 Introduction

The integrable system MIC-Kepler was constructed by Zwanziger \cite{1} and rediscovered by McIntosh and Cisneros \cite{2}. This system is described by the Hamiltonian

\begin{equation}
H_0 = \frac{\hbar^2}{2\mu} (i\nabla + sa)^2 + \frac{\hbar^2r^2}{2\mu r^2} - \frac{\gamma}{r}, \quad \text{where} \quad \text{rot}A = \frac{r}{r^3}.
\end{equation}

(1.1)

Its distinctive peculiarity is the hidden symmetry given by the following constants of motion \cite{14}

\begin{equation}
I = \frac{\hbar}{2\mu} [(i\nabla + sa) \times J - J \times (i\nabla + sa)] + \frac{r}{r}, \quad J = -\hbar (i\nabla + sa) \times r + \frac{hsr}{r}.
\end{equation}

(1.2)

These constants of motion, together with the Hamiltonian, form the quadratic symmetry algebra of the Coulomb problem. The operator \( J \) defines the angular momentum of the system, while the operator \( I \) is the analog of the Runge-Lenz vector. For the fixed negative values of energy the constants of motion form the \( so(4) \) algebra, whereas for positive values of energy - the \( so(3,1) \) one. Due to the hidden symmetry the MIC-Kepler problem could be factorized in few coordinate systems, e.g. in the spherical and parabolic ones. Hence, the MIC-Kepler system is a natural generalization of the Coulomb problem in the presence of Dirac monopole. The monopole number \( s \) satisfies the Dirac’s charge quantization rule, \( s = 0, \pm 1/2, \pm 1, \ldots \).

The MIC-Kepler system could be constructed by the reduction of the four-dimensional isotropic oscillator by the use of the so-called Kustaanheimo-Stiefel transformation both on classical and quantum mechanical levels \cite{3}. In the similar way, reducing the two- and eight-dimensional isotropic oscillator, one can obtain the two- \cite{4} and five-dimensional \cite{5} analogs of MIC-Kepler system. An infinitely thin solenoid providing the system by the spin 1/2, plays the role of monopole in two-dimensional case, whereas in the five-dimensional case this role is performed by the \( SU(2) \) Yang monopole \cite{6}, endowing the system by the isospin. All the above-mentioned systems have Coulomb symmetries and are solved in spherical and parabolic coordinates both in discrete and continuous parts of energy spectra \cite{7}. There are generalizations of MIC-Kepler systems on three-dimensional sphere \cite{8} and hyperboloid \cite{9} as well.

For integer values \( s \) the MIC-Kepler system describes the relative motion of the two Dirac dyons (charged magnetic monopoles). For this we should place

\begin{equation}
s = \frac{eG - Qg}{\hbar c}, \quad \gamma = eQ + gG,
\end{equation}

(1.3)

where \((e,g)\) and \((Q,G)\) are electric and magnetic charges of the first and the second dyons respectively. Parameter \( \mu \) plays the role of the reduced mass, while vector \( r \) determines the position of the second
dyon with respect to the first one. For half-integer s the presence of the magnetic field of infinitely thin solenoid, endowing the system with the spin 1/2, is also supposed (compare with ). It should be noted that the MIC-Kepler system describes also the relative motion of two Dirac dyons as well as that of the two well-separated BPS monopoles/dyons \(^{1}\). The wavefunction of the ground state of the MIC-Kepler problem is of the form:

\[
\psi_{m,\pm|s|} = \text{const} \cdot r^{s} e^{-r/a(|s|+1)} (\cos \frac{\theta}{2})^{s} \pm m \sqrt{\bar{s}m} e^{im\varphi}, \quad m = -|s|, -|s|+1, \ldots, |s| - 1, |s|. \tag{1.5}
\]

As is seen, it is degenerated with respect to the quantum number \(m\) and is not spherically symmetric: the system has a non-zero dipole momentum and in the presence of the external electric field the linear Stark effect is possible. Hence, it seems to be interesting to study the behavior of the MIC-Kepler bound system behaviour in the constant uniform electric field with the purpose to investigate the Stark effect and to calculate the dipole momentum. The aim of the present paper is the study of these issues.

For convenience, we will consider a special case of the charge moving in the field of the Dirac dyon, i.e. the system most similar to the hydrogen atom. In other words, we assume

\[
G = 0, \tag{1.6}
\]

so that the perturbative correction to the MIC-Kepler Hamiltonian \(^{11}\), which is responsible for the interaction of the charge-dyon system with the external constant uniform electric field has the same form as in the case of the hydrogen atom,

\[
\mathcal{H}_{S} = e\varepsilon r. \tag{1.7}
\]

The paper is organized as follows.

In Section 2 we bring wave functions of the MIC-Kepler bound system in spherical and parabolic coordinates.

In Section 3 the behaviour of the charge-dyon bound system in the constant uniform electric field is investigated. The assumption of the presence of the linear Stark effect in the system is confirmed and its dipole momentum is calculated. It is shown that the external electric field completely removes the degeneracy on azimuth quantum number.

In Conclusion we summarize the obtained results discuss their possible generalizations.

## 2 Wavefunctions and spectrum

Let us consider a spectral problem describing the MIC-Kepler problem with the energy \(E^{(0)}\), with the angular momentum \(j\) and with the \(x_3\)-component of the angular momentum \(m\):

\[
\mathcal{H}_0 \psi = E^{(0)} \psi, \quad J^2 \psi = \hbar^2 j(j+1) \psi \quad J_3 \psi = \hbar m \psi \tag{2.1}
\]

where \(\mathcal{H}_0, J\) are defined by the expressions \(^{11}\) with the Dirac’s monopole vector potential with the singularity line directed along the positive semiaxis \(x_3\)

\[
A = \frac{1}{r (r - x_3)} (x_2, -x_1, 0) \tag{2.2}
\]

The solution to this system is as follows:

\[
\psi_{njm}(r; s) = \left( \frac{2j + 1}{8\pi^2} \right)^{1/2} R_{n,j}(r/a) d_{ms}^{\mu}(\theta) e^{im\varphi}, \tag{2.3}
\]

\(^{1}\)The system describing the relative motion of the two well-separated BPS dyons \(^{10}\) is specified by the Hamiltonian

\[
\mathcal{H}_{BPS} = \frac{1}{4\mu} \left( 1 - \frac{2\mu}{r} \right)^{-3/2} (\bar{s} A + e^r) \left( 1 - 2\mu \right)^{1/2} (iv + s A) + \frac{e^r}{r} \tag{1.4}
\]

where \(e\) denote the relative electric charge of the BPS monopoles. As is seen the Schrödinger equation of this Hamiltonian could be immediately transformed to the Schrödinger equation of the MIC-Kepler problem. While the Coulomb symmetry in this system is well-known one, its equivalence to the MIC-Kepler problem might have been noticed in \(^{11}\).
where $d_{ms}$ is the Wigner d-function \cite{12}, and $R_{nj}(r/a)$ is defined by the expression

$$R_{nj}(r) = \frac{2j+1}{n^{j+2}(2j+1)!} \sqrt{\frac{(n+j)!}{(n-j-1)!}} r^j e^{-r/n} F(j-n+1,2j+2,\frac{2r}{n}). \quad (2.4)$$

Here $a = \hbar^2/\mu e^2$ is Bohr radius. The spectrum of the system is specified by the conditions:

$$E_{s}^{(0)} = -\frac{\mu^2}{2\hbar^2 n^2}, \quad n = |s|+1,|s|+2,\ldots \quad (2.5)$$

$$j = |s|,|s|+1,\ldots,n-1; \quad m = -j,-j+1,\ldots,j-1,j. \quad (2.6)$$

The quantum numbers $j,m$ characterize the total momentum of the system and its projection on the axis $x_3$. For the (half)integer $s$, $j,m$ are (half)integers. Performing the identity transformation $\varphi \to \varphi + 2\pi$, one can see, that the wavefunction of the system is single-valued for the integer $s$ and changes its signs for the half-integer $s$. In the latter case the ambiguity of the wave function can be interpreted as the presence of the magnetic field of the infinitely thin solenoid (directed along the axis $x_3$) providing the system by spin 1/2. In the ground state ($n = 1$) we have $j = |s|$, so that the wavefunction is of the form \cite{15}.

Let us consider the MIC-Kepler system on the parabolic basis. In the parabolic coordinates $\xi, \eta \in [0, \infty), \varphi \in [0, 2\pi)$, defined by the formulae

$$x_1 + ix_2 = \sqrt{\xi \eta} e^{i\varphi}, \quad x_3 = \frac{1}{2}(\xi - \eta), \quad (2.7)$$

the differential elements of length and volume read

$$d\xi = \frac{\xi + \eta}{4} \left( d\xi^2 + d\eta^2 \right) + \xi \eta d\varphi^2, \quad dV = \frac{1}{4} (\xi + \eta) d\xi d\eta d\varphi, \quad (2.8)$$

while the Laplace operator looks as follows

$$\Delta = \frac{4}{\xi + \eta} \left[ \frac{\partial}{\partial \xi} \left( \xi \frac{\partial}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \eta \frac{\partial}{\partial \eta} \right) \right] + \frac{1}{\xi \eta} \frac{\partial^2}{\partial \varphi^2}. \quad (2.9)$$

The substitution

$$\psi(\xi, \eta, \varphi) = \Phi_1(\xi) \Phi_2(\eta) e^{i m \varphi} \sqrt{2\pi}. \quad (2.10)$$

separates the variables in the Schrödinger equation and we arrive to the following system

$$\frac{d}{d\xi} \left( \xi \frac{d\Phi_1}{d\xi} \right) + \left[ \mu E_{s}^{(0)} \xi - \frac{(m-s)^2}{4\xi} + \frac{\mu}{2\hbar} \beta + \frac{1}{2a} \right] \Phi_1 = 0, \quad (2.11)$$

$$\frac{d}{d\eta} \left( \eta \frac{d\Phi_2}{d\eta} \right) + \left[ \mu E_{s}^{(0)} \eta - \frac{(m+s)^2}{4\eta} - \frac{\mu}{2\hbar} \beta + \frac{1}{2a} \right] \Phi_2 = 0, \quad (2.12)$$

where $\beta$ – is the separation constant, being the eigenvalue of the $x_3$-component the Runge-Lenz vector $\mathbf{I}$.

For $s = 0$ these equations coincide with the equations of the hydrogen atom in the parabolic coordinates \cite{16}. Thus, we get

$$\psi_{n_1 n_2 m s}(\xi, \eta, \varphi) = \frac{\sqrt{2}}{n^{2a\beta/2}} \Phi_{n_1 m-s}(\xi) \Phi_{n_2 m+s}(\eta) e^{i m \varphi} \sqrt{2\pi}, \quad (2.13)$$

where

$$\Phi_{pq}(x) = \frac{1}{|q|!} \frac{1}{p^1} \sqrt{(p+|q|)!} e^{-x/2an} \left( \frac{x}{an} \right)^{|q|/2} \frac{1}{\Gamma(1-p; |q|+1)} \left( \frac{x}{an} \right), \quad (2.14)$$

Here $n_1$ and $n_2$ are non-negative integers

$$n_1 = \frac{|m-s|+1}{2} + \frac{\mu}{2k\hbar} \beta + \frac{1}{2ak}, \quad n_2 = \frac{|m+s|+1}{2} - \frac{\mu}{2k\hbar} \beta + \frac{1}{2ak} \quad (2.15)$$
where $\kappa = \sqrt{-2\mu E/\hbar}$. From the last relations, taking account \textsuperscript{2.15}, we get that the parabolic quantum numbers $n_1$ and $n_2$ are connected with the principal quantum number $n$ as follows

$$n = n_1 + n_2 + \frac{|m-s| + |m+s|}{2} + 1. \quad (2.16)$$

Thus we have solved the spectral problem

$$\mathcal{H}_0 \psi = E^{(0)} \psi, \quad I_3 \psi = \hbar^2 \beta \psi \quad J_3 \psi = \hbar m \psi, \quad (2.17)$$

where $\mathcal{H}_0, I_3, J_3$ are defined by the expressions \textsuperscript{1.1}, \textsuperscript{1.2}.

### 3 The Stark effect

The Hamiltonian of the MIC-Kepler system in the external constant uniform electric field is of the form

$$\mathcal{H} = \frac{\hbar^2}{2\mu} (i\nabla + sA)^2 + \frac{\hbar^2 s^2}{2\mu r^2} - \frac{\gamma}{r} + |e|\varepsilon z, \quad (3.1)$$

We have assumed that the electric field $\varepsilon$ is directed along positive $x_3$-semiaxes, and the force acting the electron is directed along negative $x_3$-semiaxes.

Since the Hamiltonian \textsuperscript{3.1} possesses axial symmetry, it is convenient to consider the Schrödinger equation of the charge-dyon system in the external constant uniform electric field in the parabolic coordinates. Thus, for convenience, the parabolic wavefunctions \textsuperscript{2.13} of the MIC-Kepler system are chosen as nonperturbed ones for the calculation of the matrix elements of transitions between the mutually degenerated states.

We are mostly interested the matrix elements of the transitions $n_1 n_2 m \rightarrow n_1' n_2' m'$ for the fixed value of the principal quantum number $n$. The perturbation operator in parabolic coordinates reads

$$\hat{V} = |e|\varepsilon z = |e|\varepsilon (\xi - \eta)/2.$$

According to the perturbation theory the first-order corrections to the energy eigenvalue $E_n$ \textsuperscript{2.5} are of the form

$$E_n^{(1)} = \int \psi_{n_1 n_2 m}(\xi, \eta, \varphi) \hat{V} \psi_{n_1 n_2 m}(\xi, \eta, \varphi) d\varphi. \quad (3.2)$$

Hence, taking into account \textsuperscript{2.13}, we get

$$E_n^{(1)} = \frac{|e|\varepsilon}{4a^2 n^4} (I_{n_1, m-s} I_{n_2, m+s} - I_{n_1, m-s} I_{n_2, m+s}), \quad (3.3)$$

where

$$I_{pq} = \int_0^\infty |\Phi_{pq}(x)|^2 dx, \quad \mathcal{I}_{pq} = \int_0^\infty x^2 |\Phi_{pq}(x)|^2 dx. \quad (3.4)$$

Later on, we will make use the formulae (see, e.g. \textsuperscript{13})

$$\int_0^\infty e^{-\lambda z} z^\nu \! F_1 (\alpha; \gamma; k z) \, dz = \Gamma(\nu + 1) \lambda^{-\nu - 1} 2F_1 (\alpha, \nu + 1; \gamma; \frac{k}{\lambda}), \quad (3.5)$$

and

$$2F_1 (\alpha; \beta; \gamma; 1) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)}. \quad (3.6)$$

As a result we will obtain the following expressions

$$I_{pq} = an, \quad \mathcal{I}_{pq} = 2(an)^3 \left[ 3p(\nu + |q| + 1) + \frac{|q|}{2} (|q| + 3) + 1 \right]. \quad (3.7)$$
Now, using these formulae, we could get the first-order correction to the energy eigenvalue

\[ E_{n}^{(1)} = \frac{3\hbar^2|e|\varepsilon}{2\mu\gamma} \left[ n \left( n_1 - n_2 + \frac{|m-s| - |m+s|}{2} \right) + \frac{ms}{3} \right]. \tag{3.8} \]

Similar to the hydrogen atom, the linear term on \( n \) is proportional to the \( x_3 \)-component of the Runge-Lenz vector. However, there is an extra correction linear on \( m \), which removes the degeneracy on the \( x_3 \)-component of the angular momentum.

Thus, in the “charge-Dirac dyon” system there is the linear Stark effect, completely removing the degeneracy on azimuth quantum number \( m \).

For the fixed \( s \), according to the formula \( 2.16 \), the two extreme components of the split energy level correspond to the following values of the parabolic quantum numbers: \( n_1 = n - |s| - 1 \), \( n_2 = 0 \) and \( n_1 = 0 \), \( n_2 = n - |s| - 1 \). The distance between these levels is

\[ \Delta E_n = \frac{3\hbar^2|e|\varepsilon}{\mu\gamma} n (n - |s| - 1), \tag{3.9} \]

i.e. the complete splitting of the level is proportional to \( n^2 \), as on the case of the hydrogen atom.

The presence of the linear Stark effect means that in the unperturbed state the charge-dyon bound system has a dipole momentum with a mean value

\[ \overline{d}_z = -\frac{3\hbar^2|e|}{2\mu\gamma} \left[ n \left( n_1 - n_2 + \frac{|m-s| - |m+s|}{2} \right) + \frac{ms}{3} \right]. \tag{3.10} \]

From the expression for the mean dipole momentum it is natural to give the following definition for the dipole momentum’s operator

\[ d = -\frac{3\hbar^2|e|}{2\mu\gamma} \left[ n^2 \mathbf{1} + \frac{8\mathbf{J}}{3} \right] = |e| \left[ \frac{3\gamma}{4E^{(0)}} \mathbf{1} - \frac{\hbar^2 s}{2\mu\gamma} \mathbf{J} \right]. \tag{3.11} \]

It is mentioned that all the formulae obtained for \( s = 0 \) yields the corresponding formulae for the hydrogen atom.

\section{Conclusion}

We have considered the Stark effect in the MIC-Kepler problem describing the interaction of charge with the Dirac dyon. We have found that in spite of the deep similarity of the charge-dyon system with the hydrogen atom, the relation of the former system to the Stark effect is qualitatively differs from the latter one. Namely, in the MIC-Kepler problem there is the linear Stark effect which completely removes the degeneracy on azimuth quantum number \( m \).

It deserves to be mentioned, that the Stark effect for the charge-dyon system cannot be naively transferred on to the system of two dyons. In the latter case the interaction of the external field with the magnetic charge have to be to be taken into consideration. Hence, the Stark effect in the system of two Dirac dyons will be equivalent to the superposition of should be the superposition of the Stark and Zeeman effects in the charge-dyon system.

Undoubtedly, the existence in the charge-dyon system of the linear Stark effect, as well as of the nonzero dipole momentum is due to the presence of magnetic monopole. It is clear, the system of two well-separated BPS dyons will have similar properties, thought the transformation of the Schrödinger equation corresponding to the Hamiltonian \( 3.1 \) yields the Schrödinger equation of the system of two well-separated BPS dyons, perturbed by a nonlinear electric field. Besides, the formal equivalence of the MIC-Kepler system and of the two well-separated BPS dyons supposes the dependence of the coupling constant of the latter system both on the coupling constant and of the energy of the MIC-Kepler.

It seems to be interesting consider the Stark effect in the MIC-Kepler system on the sphere and hyperboloid, with the aim of the revealing its dependence on the space curvature. The study of the Stark effect on the five-dimensional generalization of the MIC-Kepler problem (the so-called Yang-Coulomb
or $SU(2)$-Kepler problem) could be even more instructive, due to the presence of isospin degrees of freedom.

Acknowledgements. The authors are grateful to Valery Ter-Antonyan for useful discussions and remarks. This work was partially supported by ANSEF No: PS81 and INTAS 00-00262 grants.

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