What is the Regularized Casimir Vacuum Energy Density?

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Abstract: The regularized total Casimir energy in spacetimes with boundaries is not in general equal to the integral of the regularized energy density. This paradoxical phenomenon is most transparently analyzed in the simple example of a massless scalar field in 1+1 dimensions confined to a line element of length $L$ and obeying Dirichlet boundary conditions.

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The quantum vacuum energy density due to zero-point fluctuations of a field in Minkowski space is in general a divergent quantity. In the absence of gravity it will not have any physical consequences and is thus usually ignored. However, if the quantum field is made to satisfy non-trivial boundary conditions, the vacuum energy density will still be divergent, but in a different way from the previous case. The difference between these two results is found to be finite and is called the Casimir vacuum energy density. The subtraction of two divergent quantities to obtain a finite result is an example of a regularization method.

Due to the presence of boundaries one expects in general that the Casimir energy density is position dependent. On the other hand, in homogeneous spaces as on the surface of a sphere, the energy density must be constant because of symmetry. In either case, one would expect that the integrated energy density equals the total Casimir energy, i.e. the regularized value of the infinite sum

$$E_0 = \frac{1}{2} \sum_n \omega_n$$

in the case of non-interacting fields. Here $\omega_n$ are the eigenfrequencies of the wave equation for the given boundary conditions. We will in the following point out that this expectation does not in general hold. This was first realized by Deutsch and Candelas who discovered that the field fluctuations and thus the vacuum energy density in general diverges near confining boundaries.

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Consider first a free Maxwell field. The vacuum energy density is then
\[ u = \frac{1}{2}(\langle E^2 \rangle + \langle B^2 \rangle) \] (2)
and is thus given by the sum \( u = u_E + u_B \) due to fluctuations in the electric and magnetic fields. The vacuum expectation values are most easily obtained in the simplest case of an electromagnetic field in a half-space bounded by an infinite plane. If \( \mathbf{n} \) is the normal to the plane in the \( z \)-direction, the standard, metallic boundary conditions are then \( \mathbf{n} \times \mathbf{E} = \mathbf{n} \cdot \mathbf{B} = 0 \). The regularized field fluctuations are found to be
\[ \langle E^2 \rangle = \frac{3}{16\pi^2 z^4}, \quad \langle B^2 \rangle = -\frac{3}{16\pi^2 z^4} \] (3)
where \( z \) denotes the distance from the plane. The Casimir energy density outside the plane is thus exactly zero.

The result (3) represent the leading terms in the vacuum fluctuations when you approach any bounding surface as shown by Deutsch and Candelas [1]. In this limit the surface will be seen to be locally flat. For instance, the vacuum energy density inside a spherical, metallic shell has been calculated by Olaussen and Ravndal [2] and has this behaviour when one approaches the shell from the inside. Again the leading divergences cancel, but the sub-leading terms do not and gives a divergent result for the energy density and thus also for the integrated Casimir energy. On the other hand, the total Casimir energy obtained by Bender and Hays [4] from the infinite mode sum (1) is finite.

While the calculation of the field fluctuations within a shell requires some numerical work, closed expressions have been obtained by Lütken and Ravndal[5] for the generic Casimir system of two parallel metallic plates with separation \( L \). The regularized results for the fluctuations at a distance \( z \) from the left plane can be written as
\[ \langle E^2 \rangle = -\frac{\pi^2}{16L^4} \left( \frac{1}{45} - F(\theta) \right), \quad \langle B^2 \rangle = -\frac{\pi^2}{16L^4} \left( \frac{1}{45} + F(\theta) \right) \] (4)
where \( \theta = z\pi/L \) is the scaled distance and the function
\[ F(\theta) = -\frac{1}{2} \frac{d^3}{d\theta^3} \cot \theta = \frac{3}{\sin^4 \theta} - \frac{2}{\sin^2 \theta} \] (5)
gives the position dependence of the fluctuations. In the limit \( L \to \infty \) we have essentially only one plate and the result (3) is recovered. We see that although both the electric part \( u_E \) of the vacuum energy density and the magnetic part \( u_B \) diverge near the plates, their sum \( u = -\pi^2/720L^4 \) is constant and just equal to the average of the total Casimir energy[1].

Virtual electrons in the vacuum affects the photons at low energies via the effective Euler-Heisenberg interaction[7]
\[ \mathcal{L}_{EH} = \frac{2\alpha^2}{45m^4} \left[ (E^2 - B^2)^2 + 7(E \cdot B)^2 \right] \] (6)
where $\alpha$ is the fine structure constant and $m$ the electron mass. Its effect on the propagation of light between two parallel metallic plates have been investigated by Scharnhorst and Barton. It will also give a correction to the above free result for the vacuum energy density. Then there is no longer any exact cancellation between the divergences in the electric and magnetic contributions near the plates so that the regulated energy density will diverges there. When integrating it over the volume between the plates one thus gets a divergent result. This is physically meaningless and similar to what was found for free photons confined within a spherical shell.

In order to analyze this problem in a more transparent way, let us instead simplify it by ignoring the degrees of freedom parallel to the walls. Effectively, we can then instead consider a massless scalar field $\phi(t, z)$ confined on the $z$-axis between $z = 0$ and $z = L$. In the free Lagrangian $L = (1/2)[(\partial_t \phi)^2 − (\partial_z \phi)^2]$ we can consider $\partial_t \phi$ to be the electric field and $\partial_z \phi$ to be the magnetic field. We can thus continue to talk about the electric $u_E = (1/2)\langle (\partial_t \phi)^2 \rangle$ and magnetic $u_B = (1/2)\langle (\partial_t \phi)^2 \rangle$ contributions to the vacuum energy density. Imposing the Dirichlet condition that the field vanishes at the boundaries, the normalized eigenmodes of the field are

$$\phi_n(z) = \frac{\sqrt{2}}{L} \sin \left( \frac{\pi n}{L} z \right)$$

where $\omega_n = \pi n/L$ with $n = 1, 2, 3, \ldots$. The total vacuum energy is then

$$E_0 = \frac{\pi}{2L} \sum_{n=1}^{\infty} n = -\frac{\pi}{24L}$$

when we use zeta-function regularization where $\zeta(-1) = -1/12$. Assuming a constant vacuum energy density, it is therefore $u = -\pi/24L^2$.

Let us now consider the partial contributions to the vacuum energy density. The part coming from fluctuations in the electric field is given by the divergent sum

$$u_E = \frac{1}{2L} \sum_{n=1}^{\infty} \omega_n \sin^2 (\omega_n z) = \frac{1}{4L} \sum_{n=1}^{\infty} \omega_n (1 − \cos 2\omega_n z)$$

Again using zeta-function regularization, the first sum is given by $\zeta(-1) = -1/12$ while the second sum becomes

$$\sum_{n=1}^{\infty} n \cos(2\theta n) = \frac{1}{4} \frac{d}{d\theta} \cot \theta = -\frac{1}{4 \sin^2 \theta}$$

These finite results can also be obtained by using other regularization schemes, e.g. a high-frequency cutoff or splitting the positions of the two fields in the correlator. The regulated result for this part of the vacuum energy is therefore

$$u_E = -\frac{\pi}{16L^2} \left( \frac{1}{3} − \frac{1}{\sin^2 \theta} \right)$$
The constant first term is just one half of the mean energy density \( u = -\pi/24L^2 \). The other half will be provided by a similar term in the magnetic contribution. All the position dependence lies in the second term which is positive definite and diverges near the endpoints of the confining region. When it is integrated up it gives an infinite contribution. That is not a real problem yet because there will be a corresponding infinite contribution from the magnetic part which will come with an opposite sign so that they cancel. But the seemingly paradoxical situation arises when we integrate the electric energy density on the form (12) before regularization. Then the each mode in the second, infinite sum gives zero because of the boundary conditions,

\[
\int_0^L dz \, u_E = \frac{1}{4L} \sum_{n=1}^{\infty} \omega_n \left( L - \frac{1}{2\omega_n} \sin (2\omega_n L) \right) = -\frac{\pi}{48L}
\]  

(12)

So the position-dependent part of the vacuum energy density contributes nothing to the total Casimir energy. This is a concrete example of the realization by Deutsch and Candelas\cite{1} that the operations of integration and regularization do not commute in general. The same obviously also holds for the magnetic energy density

\[
u_B = \frac{1}{2L} \sum_{n=1}^{\infty} \omega_n \cos^2 (\omega_n z) = -\frac{\pi}{16L^2} \left( \frac{1}{3} + \frac{1}{\sin^2 \theta} \right)
\]  

(13)

which diverges with opposite sign when approaching the endpoints of the interval.

The structure of these two contributions to the vacuum energy density is exactly the same as for the full Maxwell field in (4). Only the first, position-independent part will contribute to the total Casimir energy for this configuration of two parallel plates. On the other hand, for a spherical shell the position-dependent terms in \( u_E \) and \( u_B \) don’t cancel \cite{3} and it is therefore not completely clear what the physical significance should be given to the resulting total energy density which now diverges when one approaches the shell. The corresponding correlators \( \langle E^2 \rangle \) and \( \langle B^2 \rangle \) can in principle be measured. For instance, they will affect the energy levels of an atom placed in the vicinity of such a metallic boundary \cite{3 \[1\]}\cite{11}. By moving the atom around one can then map the spatial variation of the vacuum fluctuations by spectroscopic methods.

Scharnhorst\cite{8} and Barton\cite{9} could show that by extending the Maxwell theory of light by including the non-linear Euler-Heisenberg interaction \cite{3}, the refractive index is changed in the vacuum between two plates. This is again due to the non-zero values of \( \langle E^2 \rangle \) and \( \langle B^2 \rangle \) in (4). Somewhat surprisingly, the position dependence in both of these correlators cancel out in the net result for the refractive index. Again it is only the constant terms which contribute to the physics.

One can emulate the Euler-Heisenberg interaction in our one-dimensional system by considering the scalar Lagrangian

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{\alpha}{m^2} (\partial_\mu \phi)^4
\]  

(14)
Here $\alpha$ is some small dimensionless constant and $m$ a heavy mass. The Lagrangian is invariant under the field transformation $\phi \to \phi + const$ and thus describes massless particles. Treating the interaction in lowest order perturbation theory, one easily finds the resulting vacuum energy density after regularization to be

$$u = -\frac{\pi}{24L^2} - \frac{\alpha \pi^2}{8m^2 L^4} \left( \frac{1}{18} + \frac{1}{\sin^4 \theta} \right)$$

(15)

Now it diverges near the endpoints where $\theta \to 0$ and the integrated Casimir energy is infinite. However, if we integrate the energy density and then regularize as we did in (12), we get the finite result

$$E_0 = -\frac{\pi}{24L} - \frac{\alpha \pi^2}{144m^2 L^3}$$

(16)

The potentially divergent part vanishes now vanishes since $\zeta(-2) = 0$. We see that the total Casimir energy is furnished by just the constant part of energy density (15). Again there is no contribution from the position-dependent terms.

We have similarly calculated the total Casimir energy for the photons between two plates interacting via the Euler-Heisenberg interaction[10]. The regulated energy density diverges near the walls while the integrated energy is again finite after regularization and equals

$$E_0 = -\frac{\pi^2}{720L^3} - \frac{11\alpha^2 \pi^4}{2^7 3^5 5^3 m^4 L^7}$$

(17)

This new quantum correction to the standard Casimir effect is obviously much too small to be detected by present methods.

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