STATISTICAL ANALYSIS OF TWO-PARAMETER GENERALIZED BIRNBAUM-SAUNDERS CAUCHY DISTRIBUTION*

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Abstract The image features of density function and failure rate function are studied in detail for two-parameter generalized Birnbaum-Saunders Cauchy fatigue life distribution. The logarithmic moment estimation and other two point estimations of parameters are proposed under full sample, and the precisions of point estimations are investigated by Monte-Carlo simulations. The approximate interval estimations of parameters are given by using Taylor expansion, and the precisions of approximate interval estimations are investigated by Monte-Carlo simulations. Finally, several examples show the feasibility of the methods.

Keywords Two-parameter generalized Birnbaum-Saunders Cauchy fatigue life distribution, shape parameter, scale parameter, point estimation, approximate interval estimation.

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1. Introduction

Birnbaum-Saunders model is an important failure distribution model in the probabilistic physical method, which is deduced by Birnbaum and Saunders in 1969 when they studied on the material failure process caused by dominant crack propagation. It is widely applied in the study of mechanical production reliability, mainly used in the study of fatigue failure. Besides, it has the important application in the failure analysis of electronic products performance degradation.

Suppose that \( T \) follows two-parameter Birnbaum-Saunders fatigue life distribution \( BS(\alpha, \beta) \), its distribution function and density function are respectively

\[
F(t) = \Phi \left[ \frac{1}{\alpha} \left( \sqrt{\frac{t}{\beta}} - \sqrt{\frac{1}{t}} \right) \right],
\]

\[
f(t) = \frac{1}{2\alpha\sqrt{\beta}} \left( \frac{1}{\sqrt{t}} + \frac{\beta}{t\sqrt{t}} \right) \phi \left[ \frac{1}{\alpha} \left( \sqrt{\frac{t}{\beta}} - \sqrt{\frac{1}{t}} \right) \right], t > 0
\]

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where $\alpha > 0$ is a shape parameter, $\beta > 0$ is a scale parameter and $\varphi(x), \Phi(x)$ are respectively density function and distribution function of standard normal distribution, that is, $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $\Phi(x) = \int_{-\infty}^{x} \varphi(y)dy$.

Birnbaum-Saunders fatigue life distribution is derived from basic characteristics of fatigue process, so it is more suitable for describing the life regularity of several fatigue failure products than common life distributions such as Weibull distribution and log-normal distribution. Besides, it has become one of common distributions in reliability statistical analysis.

Due to the relations between the two-parameters $BS$ distribution $BS(\alpha, \beta)$ and standard normal distribution $N(0,1)$ which is a symmetric distribution, if $N(0,1)$ is replaced by other symmetric distribution, then the obtained distribution is called generalized $BS$ distribution. For example, if $N(0,1)$ is replaced by standard Laplace distribution, that is, $\frac{1}{\alpha} \left( \sqrt{\frac{T}{\beta}} - \sqrt{\frac{T}{\beta}} \right) \sim L(0,1)$, then it is called $T \sim GBS - Laplace(\alpha, \beta)$; if $N(0,1)$ is replaced by standard Cauchy distribution, that is, $\frac{1}{\alpha} \left( \sqrt{\frac{T}{\beta}} - \sqrt{\frac{T}{\beta}} \right) \sim t(1)$, then it is called $T \sim GBS - Cauchy(\alpha, \beta)$ or $T \sim GBS - Student(\alpha, \beta)$; if $N(0,1)$ is replaced by standard Logistic distribution, that is, $\frac{1}{\alpha} \left( \sqrt{\frac{T}{\beta}} - \sqrt{\frac{T}{\beta}} \right) \sim Logistic(1)$, then it is called $T \sim GBS - Logistic(\alpha, \beta)$.

The researches of this kind of generalized $BS$ distributions see reference [1,3–8,12]. They mostly refer to image features of density function and failure rate function, numerical characteristics and the discussion on MLE of parameter. Xiaojun Zhu and Balakrishnan [12] made a further analysis on $GBS - Laplace(\alpha, \beta)$ in 2015, and proved that MLE is existent and unique, but the process of proof is not perfect. Ronghua Wang [10] proposed the test statistics of fitting test and two new approximate interval estimation methods of environmental factor for two-parameter BS fatigue life distribution.

The image features of density function and failure rate function are studied in detail for two-parameter generalized Birnbaum-Saunders Cauchy fatigue life distribution $GBS - Cauchy(\alpha, \beta)$ in this paper. The logarithmic moment estimation and other two point estimations of parameters are proposed under full sample, and the precisions of point estimations are investigated by Monte-Carlo simulations. The approximate interval estimations of parameters are given by using Taylor expansion, and the precisions of approximate interval estimations are investigated by Monte-Carlo simulations. Finally, several examples show the feasibility of the methods.

### 2. Image Features of Density Function and Failure Rate Function for Two-parameter $GBS - Cauchy (\alpha, \beta)$ Distribution

Suppose non-negative continuous random variable $X$ follows two-parameter Birnbaum-Saunders Cauchy fatigue life distribution $GBS - Cauchy(\alpha, \beta)$, its distribution func-
Suppose non-negative continuous random variable $\phi$ where the density function is $f$. The density function of Birnbaum-Saunders Cauchy fatigue life distribution is given by:

$$f(x) = \frac{1}{2\alpha x} \left[ 1 + \frac{1}{\alpha x} \left( \frac{x}{\beta} - \sqrt{\frac{x^2}{\beta^2} - \frac{\beta}{\pi}} \right) \right].$$

$x > 0, \alpha > 0, \beta > 0$

where $\varphi(t) = \frac{1}{\pi (1 + t^2)}$, $\Phi(t) = \int_{-\infty}^{t} \varphi(y)dy = \frac{1}{2} + \frac{1}{\pi} \arctan t, -\infty < t < +\infty$.

2.1. Image Feature of Density Function $f_X(x)$

**Theorem 2.1.** Suppose non-negative continuous random variable $X$ follows two-parameter Birnbaum-Saunders Cauchy fatigue life distribution $GBS - Cauchy(\alpha, \beta)$, $f(x)$ is firstly strictly monotonic decreasing, then strictly monotonic increasing, and finally strictly monotonic decreasing again for $\alpha \leq 0.910721$; $f(x)$ is strictly monotonic decreasing for $\alpha > 0.910721$.

**Proof.** The density function is

$$f_X(x) = \frac{1}{2\alpha x} \left[ 1 + \frac{1}{\alpha x} \left( \frac{x}{\beta} - \sqrt{\frac{x^2}{\beta^2} - \frac{\beta}{\pi}} \right) \right] = \frac{\alpha \sqrt{\beta}}{2\pi} \frac{x + \beta}{\sqrt{2} \left( x^2 + (\alpha^2 - 2)\beta \right)}.$$

It is obvious that $\lim_{x \to 0} f_X(x) = +\infty$, $\lim_{x \to +\infty} f_X(x) = 0$.

Since $\beta$ is a scale parameter, we choose $\beta = 1$ without loss of generality. Then the density function is $f_X(x) = \frac{\alpha}{\pi} \frac{1}{\sqrt{2} \left( x^2 + (\alpha^2 - 1) \right)}$, and its derivative is

$$f_X'(x) = \frac{\alpha}{\pi} \frac{-3\alpha^2 x^2 - (3\alpha^2 - 2)x - 1}{\sqrt{2} \left( x^2 + (\alpha^2 - 1) \right)^{3/2}}.$$

Let the function be $g(x) = 3\alpha^2 x^2 + (3\alpha^2 - 2)x + 1, x > 0$, and we know

$$\lim_{x \to 0} g(x) = 1, g(1) = 4\alpha^2, \lim_{x \to +\infty} g(x) = +\infty, g'(x) = 9\alpha^2 x + 2(\alpha^2 - 3)x + 3\alpha^2 - 7.$$

Then let the function be $g_1(x) = 9\alpha^2 + 2(\alpha^2 - 3)x + 3\alpha^2 - 7, x > 0$, and we know

$$\lim_{x \to 0} g_1(x) = 3\alpha^2 - 7, g_1(1) = 5\alpha^2 + 8, \lim_{x \to +\infty} g_1(x) = +\infty.$$

$$\Delta = 4(\alpha^2 + 3)^2 - 36(3\alpha^2 - 7) = 4(\alpha^2 - 21\alpha^2 + 72)$$

$$= 4 \left( \frac{\alpha^2 - 21 - \sqrt{153}}{2} \right) \left( \frac{\alpha^2 - 21 + \sqrt{153}}{2} \right)$$

$$= 4 \left( \alpha + \sqrt{\frac{21 - \sqrt{153}}{2}} \right) \left( \alpha + \sqrt{\frac{21 + \sqrt{153}}{2}} \right)$$

$$\times \left( \alpha - \sqrt{\frac{21 - \sqrt{153}}{2}} \right) \left( \alpha - \sqrt{\frac{21 + \sqrt{153}}{2}} \right).$$
Since \( \sqrt{21 - \sqrt{153}} = 2.07734, \sqrt{21 + \sqrt{153}} = 4.08469 \), we have \( \Delta > 0 \) for \( \alpha < \sqrt{21 - \sqrt{153}} \); \( \Delta \leq 0 \) for \( \sqrt{21 - \sqrt{153}} \leq \alpha \leq \sqrt{21 + \sqrt{153}} \); and \( \Delta > 0 \) for \( \alpha > \sqrt{21 + \sqrt{153}} \).

1. When \( \alpha < \sqrt{21 - \sqrt{153}} \), we have \( \Delta > 0 \).

   (1) If \( \alpha \leq \sqrt{7/3} = 1.52753 \), then the equation \( g_1(x) = 0 \) has unique positive root
   \[
   x_1 = -\left(\alpha^2 + 3\right) + \sqrt{\alpha^4 - 21\alpha^2 + 72}.
   \]

   Hence \( g(x) \) has the minimum value at the point \( x = x_1 \), and the minimum value is
   \[
   g(x_1) = \frac{1}{243} \left[ 2\alpha^6 + 42\alpha^2 \sqrt{\alpha^4 - 21\alpha^2 + 72} - 144 \left( -6 + \sqrt{\alpha^4 - 21\alpha^2 + 72} \right) \right] -\alpha^4 \left( 63 + 2\sqrt{\alpha^4 - 21\alpha^2 + 72} \right).\]

   Let the function be \( h(x) = g(x), x \leq \sqrt{7/3}, \) then the image of \( h(x) \) in the interval \((0, \sqrt{7/3})\) is shown as Figure 1. The root of the equation \( h(x) = 0 \) is 0.910721. Then we have \( h(x) \leq 0 \) for \( \alpha \leq 0.910721 \), and \( h(x) > 0 \) for \( 0.910721 < \alpha \leq \sqrt{7/3} \).

\[
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image}
\caption{Image of \( h(x) \) in the interval \((0, \sqrt{7/3})\).}
\end{figure}
\]

(i) When \( \alpha \leq 0.910721 \), we have \( h(x) \leq 0 \), and there are \( x_{01}, x_{02}, x_{02} > x_{01} > 0 \) that satisfy \( g(x_{01}) = g(x_{02}) = 0 \). If \( x < x_{01}, g(x) > 0, f'(x) < 0 \); if \( x_{01} < x < x_{02}, g(x) < 0, f'(x) > 0 \); if \( x > x_{02}, g(x) > 0, f'(x) < 0 \). Then \( f(x) \) is strictly monotonic decreasing, then strictly monotonic increasing, and finally strictly monotonic decreasing again.

(ii) When \( 0.910721 < \alpha \leq \sqrt{7/3} \), we have \( h(x) > 0, g(x) > 0, f'(x) < 0 \). Then \( f(x) \) is strictly monotonic decreasing.

(2) If \( \alpha > \sqrt{7/3} \), that is, \( \sqrt{7/3} < \alpha \leq \sqrt{21 - \sqrt{153}} \), then the equation \( g_1(x) = 0 \) has no positive root. We have \( g_1(x) > 0, g'(x) > 0, g(x) > 0, f'(x) < 0 \), and then \( f(x) \) is strictly monotonic decreasing.

2. When \( \sqrt{21 - \sqrt{153}} \leq \alpha \leq \sqrt{21 + \sqrt{153}} \), we have \( \Delta \leq 0 \). Then we know \( g_1(x) \geq 0, g'(x) \geq 0, g(x) > 0, f'(x) < 0 \), and \( f(x) \) is strictly monotonic decreasing.

3. When \( \alpha > \sqrt{21 + \sqrt{153}} \), we have \( \Delta > 0 \), and the equation \( g_1(x) = 0 \) has no positive root. Then we know \( g_1(x) > 0, g'(x) > 0, g(x) > 0, f'(x) < 0 \), and \( f(x) \) is strictly monotonic decreasing.

\( \square \)
For a given scale parameter $\beta = 1$, we choose the shape parameter $\alpha$ respectively as $0.25, 0.5, 0.75, 0.8, 0.85, 0.9, 0.95, 1, 1.5, 2$, and the images of the corresponding density function $f(x)$ are shown from Figure 2 to Figure 11.

2.2. Discussion on images of failure rate functions

**Lemma 2.1** ([2]). Suppose $T$ is a non-negative continuous random variable, its second order derivation of density function $f(t)$ exists. Let $\eta(t) = -\frac{f''(t)}{f(t)}$, and we have the following conclusions:

(i) If $\eta'(t) > 0$, that is, $\eta(t)$ is a strictly monotonic increasing function, then $\lambda(t)$ is strictly monotonic increasing;
Figure 8. $\alpha = 0.95$

Figure 9. $\alpha = 1$

Figure 10. $\alpha = 1.5$

Figure 11. $\alpha = 2$

(ii) If $\eta'(t) < 0$, that is, $\eta(t)$ is a strictly monotonic decreasing function, then $\lambda(t)$ is strictly monotonic decreasing;

(iii) If there is $t_0, t_0 > 0$ that satisfies $\eta'(t_0) = 0$, and $\eta(t)$ is strictly monotonic increasing and then strictly monotonic decreasing, that is, inverse-bathtub shape, then $\lambda(t)$ may be inverse-bathtub shaped, or strictly monotonic decreasing;

(iv) If there is $t_0, t_0 > 0$ that satisfies $\eta'(t_0) = 0$, and $\eta(t)$ is strictly monotonic decreasing and then strictly monotonic increasing, that is, bathtub shape, then $\lambda(t)$ may be bathtub shaped, or strictly monotonic increasing.

**Theorem 2.2.** Suppose non-negative continuous random variable $X$ follows two-parameter Birnbaum-Saunders Cauchy fatigue life distribution $\text{GBS-Cauchy}(\alpha, \beta)$, the failure rate function $\lambda(x)$ is strictly monotonic decreasing for $\alpha \geq 1.59643$.

**Proof.** The failure rate function is

$$
\lambda(x) = \frac{f(x)}{1 - F(x)}
$$

$$
= \alpha \sqrt{\beta} \frac{x + \beta}{\sqrt{x^2 + (\alpha^2 - 2)x + \beta^2}} \left\{ \pi - 2 \arctan \left[ \frac{1}{\alpha} \left( \sqrt{\beta} - \sqrt{\frac{x}{\beta}} \right) \right] \right\}^{-1}
$$

and we have $\lambda(0) = +\infty, \lambda(+\infty) = \lim_{x \to +\infty} \frac{f(x)}{1 - F(x)} = \lim_{x \to +\infty} \frac{f'(x)}{f(x)} = 0$.

Since $\beta$ is the scale parameter, we choose $\beta = 1$ without loss of generality. Then we have

$$
\eta(x) = - \frac{f'(x)}{f(x)} = \frac{\alpha}{4\pi} \frac{3x^3 + (\alpha^2 + 3)x^2 + (3\alpha^2 - 7)x + 1}{x \sqrt{\alpha^2 + (\alpha^2 - 2)x^2 + 1}}
$$

$$
\left[ \frac{\sqrt{x^2 + (\alpha^2 - 2)x + 1}}{\alpha} \right] \frac{2\pi \sqrt{x^2 + (\alpha^2 - 2)x + 1}}{x + 1}
$$
Let the function be
\[
h(x) = 3x^6 + 2(a^2 + 3)x^5 + (4a^4 + 8a^2 - 21)x^4 + 2(3a^4 - 10a^2 + 6)x^3 + (3a^4 - 8a^2 + 1)x^2 + 2(a^2 - 1)x + 1.
\]
and we have \( h(0) = 1, h(1) = 2a^2(5a^2 - 8), h(\infty) = +\infty, \)
\[
h'(x) = 18x^5 + 10(a^2 + 3)x^4 + 4(a^4 + 8a^2 - 21)x^3 + 6(3a^4 - 10a^2 + 6)x^2 + 2(3a^4 - 8a^2 + 1)x + 2(a^2 - 1)
\]
\[
= 2 \left[ 9x^5 + 5(a^2 + 3)x^4 + 2(a^4 + 8a^2 - 21)x^3 + 3(3a^4 - 10a^2 + 6)x^2 + (3a^4 - 8a^2 + 1)x + (a^2 - 1) \right].
\]
Let the function be
\[
h_1(x) = 9x^5 + 5(a^2 + 3)x^4 + 2(a^4 + 8a^2 - 21)x^3 + 3(3a^4 - 10a^2 + 6)x^2 + (3a^4 - 8a^2 + 1)x + (a^2 - 1), x > 0,
\]
and we have \( h_1(0) = a^2 - 1, h_1(0) \leq 0 \) for \( a \leq 1; h_1(0) > 0 \) for \( a > 1, \)
\[
h_1(\infty) = +\infty, \quad h_1(1) = 2a^2(7a^2 - 8),
\]
\[
h'_1(x) = 45x^4 + 20(a^2 + 3)x^3 + 6(a^4 + 8a^2 - 21)x^2 + 6(3a^4 - 10a^2 + 6)x + (3a^4 - 8a^2 + 1).
\]
Let the function be
\[
h_2(x) = 45x^4 + 20(a^2 + 3)x^3 + 6(a^4 + 8a^2 - 21)x^2 + 6(3a^4 - 10a^2 + 6)x + (3a^4 - 8a^2 + 1), x > 0
\]
and we have
\[
h_2(0) = 3a^4 - 8a^2 + 1 = 3 \left( a^2 - \frac{4 - \sqrt{13}}{3} \right) \left( a^2 - \frac{4 + \sqrt{13}}{3} \right)
\]
\[
= 3(\alpha + 0.362606)(\alpha + 1.59223)(\alpha - 0.362606)(\alpha - 1.59223),
\]
where \( \sqrt{\frac{4 - \sqrt{13}}{3}} = 0.362606, \sqrt{\frac{4 + \sqrt{13}}{3}} = 1.59223. \)
Then we have \( h_2(0) \geq 0 \) for \( \alpha \leq 0.362606; h_2(0) < 0 \) for \( 0.362606 < \alpha < 1.59223; \)
\( h_2(0) \geq 0 \) for \( \alpha \geq 1.59223, \) and \( h_2(\infty) = +\infty, h_2(1) = 27a^4 + 16, \)
\[
h_2'(x) = 45 \cdot 4x^3 + 60(a^2 + 3)x^2 + 12(a^4 + 8a^2 - 21)x + 6(3a^4 - 10a^2 + 6)
\]
\[
= 6 \left[ 30x^3 + 10(a^2 + 3)x^2 + 2(a^4 + 8a^2 - 21)x + (3a^4 - 10a^2 + 6) \right].
\]
Let the function be
\[ h_3(x) = 30x^3 + 10(\alpha^2 + 3)x^2 + 2(\alpha^4 + 8\alpha^2 - 21)x + (3\alpha^4 - 10\alpha^2 + 6), x > 0 \]
and we have
\[
h_3(0) = 3\alpha^4 - 10\alpha^2 + 6 = 3 \left( \alpha^2 - \frac{5 - \sqrt{7}}{3} \right) \left( \alpha^2 - \frac{5 + \sqrt{7}}{3} \right)
\]
\[ = 3(\alpha + 0.885861)(\alpha + 1.59643)(\alpha - 0.885861)(\alpha - 1.59643) \]
where \( \sqrt{\frac{5 - \sqrt{7}}{3}} = 0.885861, \sqrt{\frac{5 + \sqrt{7}}{3}} = 1.59643. \)
Then we have \( h_3(0) \geq 0 \) for \( \alpha \leq 0.885861; h_3(0) < 0 \) for \( 0.885861 < \alpha < 1.59643; \) \( h_3(0) \geq 0 \) for \( \alpha \geq 1.59643, \) and \( h_3(+\infty) = +\infty, h_3(1) = 5\alpha^4 + 16\alpha^2 + 24, \)
\[
h'_3(x) = 90x^2 + 20(\alpha^2 + 3)x + 2(\alpha^4 + 8\alpha^2 - 21) = 2 \left[ 45x^2 + 10(\alpha^2 + 3)x + (\alpha^4 + 8\alpha^2 - 21) \right].
\]
Let the function be \( h_4(x) = 45x^2 + 10(\alpha^2 + 3)x + (\alpha^4 + 8\alpha^2 - 21), x > 0 \), and we have
\[
h_4(0) = \alpha^4 + 8\alpha^2 - 21 = \left( \alpha^2 + (4 + \sqrt{37}) \right) \left( \alpha^2 - (4 + \sqrt{37}) \right)
\]
\[ = \left( \alpha^2 + (4 + \sqrt{37}) \right) (\alpha + 1.44318)(\alpha - 1.44318), \]
where \( \sqrt{4 + \sqrt{37}} = 1.44318. \)
Then we have \( h_4(0) \leq 0 \) for \( \alpha \leq 1.44318; h_4(0) > 0 \) for \( \alpha > 1.44318, \) and
\[
h_4(+\infty) = +\infty, h_4(1) = \alpha^4 + 18\alpha^2 + 54.
\]
For \( h_4(x), \) we know
\[
\Delta = 100(\alpha^2 + 3)^2 - 4 \cdot 45(\alpha^4 + 8\alpha^2 - 21) = -40(2\alpha^4 + 21\alpha^2 - 117)
\]
\[ = -80 \left[ \alpha^2 + \frac{3}{4}(7 + 3\sqrt{17}) \right] \left[ \alpha^2 - \frac{3}{4}(7 + 3\sqrt{17}) \right]
\]
\[ = -80 \left[ \alpha^2 + \frac{3}{4}(7 + 3\sqrt{17}) \right] (\alpha + 2.00674)(\alpha - 2.00674), \]
where \( \sqrt{\frac{3}{4}(7 + 3\sqrt{17})} = 2.00674. \)
Then we have \( \Delta \geq 0 \) for \( \alpha \leq 2.00674 \); \( \Delta < 0 \) for \( \alpha > 2.00674. \) The value of \( \alpha \) is divided into eight situations, which is shown in Table 1. Then we only discuss Situation 7 and Situation 8.

| Situation 1 | Situation 2 | Situation 3 | Situation 4 | Situation 5 | Situation 6 | Situation 7 | Situation 8 |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0.362966    | 0.362606    | 0.885861    | 1.44318     | 1.59643     | 2.00674     | 2.00674     | 2.00674     |
| h_1(0) < 0  | h_1(0) < 0  | h_1(0) < 0  | h_1(0) > 0  | h_1(0) > 0  | h_1(0) > 0  | h_1(0) > 0  | h_1(0) > 0  |
| h_2(0) < 0  | h_2(0) < 0  | h_2(0) < 0  | h_2(0) < 0  | h_2(0) < 0  | h_2(0) < 0  | h_2(0) < 0  | h_2(0) < 0  |
| h_3(0) < 0  | h_3(0) < 0  | h_3(0) < 0  | h_3(0) < 0  | h_3(0) < 0  | h_3(0) < 0  | h_3(0) < 0  | h_3(0) < 0  |
| h_4(0) < 0  | h_4(0) < 0  | h_4(0) < 0  | h_4(0) < 0  | h_4(0) < 0  | h_4(0) < 0  | h_4(0) < 0  | h_4(0) < 0  |
| \Delta > 0  | \Delta > 0  | \Delta > 0  | \Delta > 0  | \Delta > 0  | \Delta > 0  | \Delta > 0  | \Delta < 0  |
(1) When $\alpha > 2.00674$, we have $h_1(0) > 0, h_2(0) > 0, h_3(0) > 0, h_4(0) > 0, \Delta < 0$. It is obvious that $h_4(x) > 0, h_3'(x) > 0, h_2(x) > 0, h_2'(x) > 0, h_1(x) > 0, h'(x) > 0, h(x) > 0, g(x) < 0, \eta'(x) < 0$. Then the image of $\lambda(x)$ is strictly monotonic decreasing.

(2) When $1.59643 \leq \alpha \leq 2.00674$, we have $h_1(0) > 0, h_2(0) > 0, h_3(0) \geq 0, h_4(0) > 0, \Delta \geq 0$. It is obvious that $h_4(x) > 0, h_3'(x) > 0, h_2(x) > 0, h_2'(x) > 0, h_1(x) > 0, h'(x) > 0, h(x) > 0, g(x) < 0, \eta'(x) < 0$. Then the image of $\lambda(x)$ is strictly monotonic decreasing.

Remark 2.1. By drawing the images of $\lambda(x)$, it can be concluded that the image of $\lambda(x)$ is firstly strictly monotonic decreasing and then strictly monotonic increasing and finally strictly monotonic decreasing again for $\alpha \leq 1.1$; while the image of $\lambda(x)$ is gradually strictly monotonic decreasing for $\alpha > 1.1$ . That is, with the increase of shape parameter $\alpha$, the image of $\lambda(x)$ gradually changes from firstly strictly monotonic decreasing, then strictly monotonic increasing and finally strictly monotonic decreasing again to strictly monotonic decreasing.

Let scale parameter be $\beta = 1$ and shape parameter $\alpha$ be $0.1(0.1)2, 1.15, 2.5, 3, 5$, then the images of failure rate function $\lambda(x)$ are shown from Figure 12 to Figure 35.
Statistical analysis of two-parameter...
3. Point Estimations of Parameters for Two-parameter Distribution $GBS - Cauchy(\alpha, \beta)$

Suppose that $X_1, X_2, \cdots, X_n$ is a simple random sample from two-parameter Birnbaum-Saunders Cauchy fatigue life distribution $GBS - Cauchy(\alpha, \beta)$ with sample size
Statistical analysis of two-parameter...

3.1. Quantile estimations and maximum likelihood estimations of parameters

Since $F(\beta) = \Phi(0) = 0.5$, the point estimation $\hat{\beta}_1$ of scale parameter $\beta$ can be the sample median, that is,

$$\hat{\beta}_1 = \begin{cases} \frac{1}{2} (X_{(n/2)} + X_{(n/2+1)}), & \text{when } n \text{ is an even number,} \\ X_{((n+1)/2)}, & \text{when } n \text{ is an odd number.} \end{cases}$$

Since $f(x) = \frac{1}{2\alpha x} \left( \sqrt{\frac{x}{\beta}} + \sqrt{\frac{\beta}{x}} \right) \varphi \left[ \frac{1}{\alpha} \left( \sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}} \right) \right] = \frac{\alpha}{2\pi x} \frac{\sqrt{\frac{x}{\beta}} + \sqrt{\frac{\beta}{x}}}{\alpha^2 + \left( \sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}} \right)^2}$, the likelihood function is

$$L(\alpha, \beta) = \prod_{i=1}^{n} \frac{\alpha}{2\pi x_i} \frac{\sqrt{\frac{x_i}{\beta}} + \sqrt{\frac{\beta}{x_i}}}{\alpha^2 + \left( \sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}} \right)^2}$$

$$= (2\pi)^{-n} \left( \prod_{i=1}^{n} x_i \right)^{-1} \alpha^{n} \prod_{i=1}^{n} \frac{\sqrt{x_i}}{\sqrt{\beta}} + \frac{\sqrt{\beta}}{x_i} \prod_{i=1}^{n} \alpha^2 + \left( \frac{\sqrt{x_i}}{\sqrt{\beta}} - \frac{\sqrt{\beta}}{x_i} \right)^2 \right)^{-1}.$$
\[
\ln L(\alpha, \beta) = -n \ln(2\pi) - \sum_{i=1}^{n} \ln x_i + n \ln \alpha + \sum_{i=1}^{n} \ln \left( \frac{x_i}{\beta} + \sqrt{\frac{\beta}{x_i}} \right)^2
\]
\[\partial \ln L(\alpha, \beta) \frac{\partial}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \frac{2\alpha}{\alpha^2 + \left( \sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}} \right)^2} \cdot \]

Let \( \frac{\partial \ln L(\alpha, \beta)}{\partial \alpha} = 0 \), and we get the function \( \frac{n}{\alpha} - \sum_{i=1}^{n} \frac{2\alpha}{\alpha^2 + \left( \sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}} \right)^2} = 0 \).

After simplifying, we have \( \frac{n}{2} - \sum_{i=1}^{n} \left[ 1 - \frac{\left( \sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}} \right)^2}{\alpha^2 + \left( \sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}} \right)^2} \right] = 0 \), and the point estimation \( \hat{\alpha}_1 \) of shape parameter \( \alpha \) is the root of the following equation

\[
\sum_{i=1}^{n} \frac{\left( \sqrt{\frac{x_i}{\hat{\beta}_1}} - \sqrt{\frac{\hat{\beta}_1}{x_i}} \right)^2}{\alpha^2 + \left( \sqrt{\frac{x_i}{\hat{\beta}_1}} - \sqrt{\frac{\hat{\beta}_1}{x_i}} \right)^2} = \frac{n}{2}.
\]

3.2. Regression estimations of parameters

Let \( Q(\beta) = \sum_{i=1}^{n} \left( \sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}} \right)^2 \), and we have

\[
\frac{dQ(\beta)}{d\beta} = -\frac{1}{\beta} \sum_{i=1}^{n} \left( \frac{x_i}{\beta^2} + \sqrt{\frac{\beta}{x_i}} \right) \left( \sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}} \right).
\]

Let \( \frac{dQ(\beta)}{d\beta} = 0 \), and we have the equation \( \sum_{i=1}^{n} \left( \sqrt{\frac{x_i}{\beta^2}} + \sqrt{\frac{\beta}{x_i}} \right) \left( \sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}} \right) = 0 \), that is,

\[
\frac{1}{\beta} \sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} \frac{1}{x_i} = 0.
\]

Then we have \( \hat{\beta}_2 = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^{-1}} \), and the point estimation \( \hat{\alpha}_2 \) of shape parameter \( \alpha \) is the root of the following equation

\[
\sum_{i=1}^{n} \frac{\left( \sqrt{\frac{x_i}{\hat{\beta}_2}} - \sqrt{\frac{\hat{\beta}_2}{x_i}} \right)^2}{\alpha^2 + \left( \sqrt{\frac{x_i}{\hat{\beta}_2}} - \sqrt{\frac{\hat{\beta}_2}{x_i}} \right)^2} = \frac{n}{2}.
\]
3.3. Logarithmic moment estimations of parameters

Let \( Y = \ln X, Y_i = \ln X_i, i = 1, 2, \cdots, n \), and \( \mu = \ln \beta \), then we have

\[
F_Y(y) = P(Y \leq y) = P(\ln X \leq y) = P(X \leq e^y) = \Phi \left[ \frac{1}{\alpha} \left( \sqrt{\frac{e^y}{e^\mu}} - \sqrt{\frac{e^\mu}{e^y}} \right) \right].
\]

Let \( Z = \frac{Y - \mu}{2} \), and we have \( F_Z(z) = \Phi \left[ \frac{1}{\alpha} (e^z - e^{-z}) \right] = \int_{-\infty}^{z} \frac{1}{\pi} \frac{1}{1 + e^{\alpha t}} dt, \)

\[
f_Z(z) = \frac{1}{\alpha} (e^z + e^{-z}) \varphi \left[ \frac{1}{\alpha} (e^z - e^{-z}) \right] = \frac{1}{\alpha} (e^z + e^{-z}) \frac{1}{\pi} \frac{1}{1 + \frac{\alpha^2 (e^z - e^{-z})^2}{1 + \alpha^2 (e^z - e^{-z})^2}}.
\]

Since \( f_Z(-z) = \frac{1}{\alpha} (e^{-z} + e^z) \frac{1}{\pi} \frac{1}{1 + \frac{\alpha^2 (e^{-z} - e^z)^2}{1 + \alpha^2 (e^{-z} - e^z)^2}} = f_Z(z) \), \( f_Z(z) \) is an even function.

Therefore when \( k \) is an odd number, we know \( E(Z^k) = 0 \).

When \( k \) is an even number, we know

\[
E(Z^k) = \int_{-\infty}^{+\infty} z^k \frac{1}{\alpha} (e^z + e^{-z}) \varphi \left[ \frac{1}{\alpha} (e^z - e^{-z}) \right] dz = 2 \int_{0}^{+\infty} \left( \frac{\ln \phi + \sqrt{\alpha^2 t^2 + 4}}{2} \right)^k \varphi(t) dt = \frac{2}{\pi} \int_{0}^{+\infty} \left( \frac{\ln \phi + \sqrt{\alpha^2 t^2 + 4}}{2} \right)^k \frac{1}{1 + t^2} dt.
\]

Since \( Y = \mu + 2Z, E(Y) = \mu \), the point estimation of parameter \( \mu \) can be \( \hat{\mu} = \bar{Y} \).

Then the point estimation of scale parameter \( \beta \) can be \( \hat{\beta}_3 = \left( \prod_{i=1}^{n} X_i \right)^{1/n} \).

Since \( D(Y) = 4D(Z) = 4E(Z^2) \), the point estimation \( \hat{\alpha}_3 \) of parameter \( \alpha \) is the root of the following equation

\[
8 \pi \int_{0}^{+\infty} \left( \frac{\ln \phi + \sqrt{\alpha^2 t^2 + 4}}{2} \right)^2 \frac{1}{1 + t^2} dt = Y^2 - \bar{Y}^2,
\]

that is,

\[
\int_{0}^{+\infty} \left( \frac{\ln \phi + \sqrt{\alpha^2 t^2 + 4}}{2} \right)^2 \frac{1}{1 + t^2} dt = \frac{\pi}{8} (Y^2 - \bar{Y}^2).
\]

The above equation is an integral equation, and it is complex to solve it. Then we prove that it has unique positive root.

**Lemma 3.1** ([9]). *Suppose that \( g(x) \) is a non-negative function in the interval \([a, +\infty)\), it is integrable in \([a, b]\) for any \( b > a \). If \( \lim_{x \to +\infty} \frac{\ln g(x)}{\ln x} = a \) is convergent for \(-\infty \leq p < -1\), while \( \int_{a}^{+\infty} g(x) dx \) is divergent for \(-1 < p \leq +\infty\).*

**Lemma 3.2.** *The equation \( \int_{0}^{+\infty} \left( \ln \frac{\ln \phi + \sqrt{\alpha^2 t^2 + 4}}{2} \right)^2 \frac{1}{1 + t^2} dt = \frac{\pi}{8} (Y^2 - \bar{Y}^2) \) has unique positive root with respect to \( \alpha \).*
Proof. Let the function be \( g(\alpha) = \int_0^{+\infty} \left( \ln \frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2} \right)^2 \frac{1}{1+t^2} \, dt, \alpha > 0. \) Firstly we prove that the function \( g(\alpha) \) is convergent. Since

\[
\lim_{t \to +\infty} \frac{1}{t} \ln \left[ \left( \ln \frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2} \right)^2 \right] = 2 \lim_{t \to +\infty} \frac{1}{t} \ln \left( \ln \frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2} \right)
\]

\[
= 2 \lim_{t \to +\infty} \left( \frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2} \right)^{-1} \frac{2}{\alpha t + \sqrt{\alpha^2 t^2 + 4}} \frac{1}{2} \left( \alpha + \frac{\alpha^2 t}{\sqrt{\alpha^2 t^2 + 4}} \right) t
\]

\[
= 2 \lim_{t \to +\infty} \left( \frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2} \right)^{-1} \frac{\alpha t \sqrt{\alpha^2 t^2 + 4} + \alpha^2 t^2}{\alpha t \sqrt{\alpha^2 t^2 + 4} + \alpha^2 t^2 + 4} = 0,
\]

\[
\lim_{t \to +\infty} \frac{\ln(1 + t^2)}{\ln t} = \lim_{t \to +\infty} \frac{2t^2}{1 + t^2} = 2,
\]

we have \( \lim_{t \to +\infty} \frac{1}{t} \left\{ \ln \left[ \left( \ln \frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2} \right)^2 \right] - \ln(1 + t^2) \right\} = -2, \) that is,

\[
\lim_{t \to +\infty} \frac{1}{t} \left\{ \ln \left[ \frac{1}{1 + t^2} \left( \ln \frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2} \right)^2 \right] \right\} = -2.
\]

Then according to Lemma 3.1, we know that the function \( g(\alpha) \) is convergent. Next we prove that the equation \( g(\alpha) = 0 \) has unique positive root. Since

\[
\lim_{\alpha \to 0} g(\alpha) = 0, \quad \lim_{\alpha \to +\infty} g(\alpha) = +\infty,
\]

\[
g'(\alpha) = 2 \int_0^{+\infty} \left( \ln \frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2} \right) \frac{2}{\alpha t + \sqrt{\alpha^2 t^2 + 4}} \frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2\sqrt{\alpha^2 t^2 + 4} + 1} \frac{1}{1+t^2} \, dt
\]

\[
= 2 \int_0^{+\infty} \frac{1}{2} \left( \ln \frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2} \right) \frac{t}{\sqrt{\alpha^2 t^2 + 4} + 1} \frac{1}{1+t^2} \, dt > 0,
\]

and \( \bar{Y}^2 - \bar{Y}^2 > 0, \) we know that the equation \( g(\alpha) = 0 \) has unique positive root. \( \square \)

Lemma 3.3 ([11]). Suppose that \( X_1, X_2, \cdots, X_n \) is a simple random sample from the population \( X \) with sample size \( n \), it is denoted by \( E(X) = \mu; D(X) = \sigma^2 < +\infty, \) and the forth central moment \( \nu_4 = E( (X - E(X))^4 ) \) of population \( X \) is limited. If the forth derivative of the function \( h(x) \) is existent and limited, then we have

\[
E[h(X)] = h(\mu) + \frac{1}{2n} h''(\mu) \sigma^2 + O(n^{-2}),
\]

\[
D[h(X)] = \frac{1}{n} [h'(\mu)]^2 \sigma^2 + \frac{1}{n^2} \left\{ h'(\mu) h''(\mu) \nu_3 + \frac{1}{2} [h''(\mu)]^2 \sigma^4 + h'(\mu) h'''(\mu) \sigma^4 \right\} + O(n^{-3}).
\]

Theorem 3.1. \( \hat{\beta}_3 \) is asymptotic unbiased estimation and consistent estimation of \( \beta. \)

Proof. Since \( Y = 2Z + \mu, E(Y) = \mu, Y - E(Y) = 2Z, \) we know that the first to fourth central moment of \( Y \) are \( \nu_1 = E[Y - E(Y)] = 0, \nu_3 = E[Y - E(Y)]^3 = \)
8E(Z^3) = 0,
\[
\nu_2 = E[Y - E(Y)]^2 = 4E(Z^2) = \frac{8}{\pi} \int_0^{+\infty} \left( \ln \alpha t + \sqrt{\alpha^2 t^2 + 4} \right) \frac{1}{1 + t^2} dt,
\]
\[
\nu_4 = E[Y - E(Y)]^4 = 16E(Z^4) = \frac{32}{\pi} \int_0^{+\infty} \left( \ln \alpha t + \sqrt{\alpha^2 t^2 + 4} \right)^4 \frac{1}{1 + t^2} dt.
\]

Let the function be \( h(x) = e^x \), and any order derivation of \( h(x) \) is still \( e^x \). Then we have
\[
E(\hat{\beta}_3) = E(e^{\hat{Y}}) = e^\mu + \frac{1}{2n} e^\mu \cdot 4E(Z^2) + O(n^{-2}) = \beta + \frac{2}{n} \beta E(Z^2) + O(n^{-2}),
\]
\[
D(\bar{\beta}_3) = D(e^{\bar{Y}}) = \frac{1}{n} e^{2\mu} \cdot 4E(Z^2) + \frac{1}{n^2} \left\{ \frac{1}{2} e^{2\mu} \cdot 16\left[E(Z^2)\right]^2 + e^{2\mu} \cdot 16\left[E(Z^2)\right]^2 \right\} + O(n^{-3})
\]
\[
= \frac{4}{n^2} \beta^2 E(Z^2) + \frac{24}{n^2} \beta^2 \left[E(Z^2)\right]^2 + O(n^{-3}).
\]

It is obvious that \( \lim_{n \to +\infty} E(\hat{\beta}_3) = \beta, \lim_{n \to +\infty} D(\bar{\beta}_3) = 0. \)

Therefore \( \hat{\beta}_3 \) is asymptotic unbiased estimation and consistent estimation of \( \beta \).

\[
\text{Table 2. Simulation comparisons of point estimations}
\]

| Sample size | Parameters | Quantile estimation and MLE \( \hat{\alpha}_1, \hat{\beta}_1 \) | Regression estimation \( \hat{\alpha}_2, \hat{\beta}_2 \) | Logarithmic moment estimation \( \hat{\beta}_3 \) |
|-------------|------------|------------------------------------------------|------------------------------------------------|----------------------------------|
|             |            | Mean value | Mean square error | Mean value | Mean square error | Mean value | Mean square error |
| 10          | \( \beta \) | 1.2018     | 0.7515            | 4.6318    | 597.548           | 1.3321     | 1.3688            |
|             | \( \alpha \) | 0.9626     | 0.2692            | 1.9327    | 5.3221            | -          | -                 |
| 15          | \( \beta \) | 1.1145     | 0.4255            | 4.6632    | 758.174           | 1.2107     | 0.8028            |
|             | \( \alpha \) | 0.9792     | 0.1668            | 2.1033    | 16.9944           | -          | -                 |
| 20          | \( \beta \) | 1.0556     | 0.1778            | 3.2599    | 146.272           | 1.1261     | 0.4219            |
|             | \( \alpha \) | 0.9889     | 0.1110            | 1.94312   | 8.5407            | -          | -                 |
| 25          | \( \beta \) | 1.0526     | 0.1301            | 3.3713    | 165.071           | 1.0990     | 0.2721            |
|             | \( \alpha \) | 0.9930     | 0.0877            | 1.9720    | 7.2284            | -          | -                 |
| 30          | \( \beta \) | 1.0431     | 0.1060            | 3.4698    | 476.644           | 1.0740     | 0.2405            |
|             | \( \alpha \) | 0.9872     | 0.0745            | 1.9224    | 6.7699            | -          | -                 |

In order to compare the precisions of various point estimations of parameters \( \alpha, \beta \), we choose the sample size \( n = 10(5)30 \) and the truth values of parameters \( \alpha = 1, \beta = 1 \). Then we obtain the samples from \( GBS - Cauchy(\alpha, \beta) \) by 1000 Monte-Carlo simulations, and calculate mean values and mean square errors of various point estimations for parameters \( \alpha, \beta \). The results are shown in Table 2. It can be concluded that quantile estimation and maximum likelihood estimation \( \hat{\alpha}_1, \hat{\beta}_1 \) are best.

Remark 3.1. Since logarithmic moment estimation \( \hat{\alpha}_3 \) of parameter \( \alpha \) refers to solving complex integral equation, it is not compared in the simulations here.
4. Approximate Interval Estimations of Parameters for Two-parameter Distribution $GBS-Cauchy(\alpha, \beta)$

Let the parameter be $\mu = \ln \beta$, and it is denoted by $Y = \ln X, Y_i = \ln X_i, i = 1, 2, \cdots, n$. Then $Y_1, Y_2, \cdots, Y_n$ is a simple random sample from the distribution function $F_Y(y) = \Phi \{ \frac{1}{\alpha} [\exp \frac{y - \mu}{2} - \exp \left( -\frac{y - \mu}{2} \right) ] \}$ with sample size $n$, and its order statistics are denoted by $Y(1), Y(2), \cdots, Y(n)$.

The first order Taylor expansion of $\frac{1}{\alpha} [\exp \frac{y - \mu}{2} - \exp \left( -\frac{y - \mu}{2} \right) ]$ at the point $y = \mu$ is

$$
\frac{1}{\alpha} \left[ \exp \left( \frac{y - \mu}{2} \right) - \exp \left( -\frac{y - \mu}{2} \right) \right] \approx \frac{y - \mu}{\alpha}.
$$

Then $F_Y(y) = \Phi \{ \frac{1}{\alpha} [\exp \frac{y - \mu}{2} - \exp \left( -\frac{y - \mu}{2} \right) ] \}$ is approximately

$$
F_Y(y) \approx \Phi \left( \frac{y - \mu}{\alpha} \right) = \frac{1}{2} + \frac{1}{\pi} \text{arctan} \left( \frac{y - \mu}{\alpha} \right).
$$

That is, $Y = \ln X$ can be approximately regarded as two-parameter Cauchy distribution with location-scale parameters.

Let $Z = \frac{Y - \mu}{\alpha}, Z_i = \frac{Y_i - \mu}{\alpha}, i = 1, 2, \cdots, n$, and then $Z$ approximately follows standard Cauchy distribution $C(0, 1)$. $Z_1, Z_2, \cdots, Z_n$ follow the same distribution as a simple random sample from standard Cauchy distribution $C(0, 1)$ with sample size $n$, and it is sorted from small to large, which is denoted by $Z(1), Z(2), \cdots, Z(n)$.

4.1. Approximate interval estimations of parameter $\beta$

The approximate interval estimation of parameter $\mu$ is obtained firstly, and then it is easy to obtain the approximate interval estimation of parameter $\beta$.

(1) When $n$ is an even number, it is denoted by $j = \frac{n}{2}, n-j = \frac{n}{2}, A = \frac{1}{2} \sum_{i=j+1}^{n} Y(i)$.

Then we have $\sum_{i=j+1}^{n} Y(i) - \sum_{i=1}^{j} Y(i) > 0$.

Let the function be $F(\mu) = \frac{\sum_{i=j+1}^{n} Y(i) - \sum_{i=1}^{j} Y(i)}{\sum_{i=1}^{n} (Y(i) - \mu)}$, $-\infty < \mu < +\infty$, and we know

$$
F(\mu) = \frac{\sum_{i=j+1}^{n} (Y(i) - \mu) - \sum_{i=1}^{j} (Y(i) - \mu)}{\sum_{i=1}^{n} (Y(i) - \mu)} = \frac{\sum_{i=j+1}^{n} \frac{Y(i) - \mu}{\alpha} - \sum_{i=1}^{j} \frac{Y(i) - \mu}{\alpha}}{\sum_{i=1}^{n} \frac{Y(i) - \mu}{\alpha}} = \frac{\sum_{i=j+1}^{n} Z(i) - \sum_{i=1}^{j} Z(i)}{\sum_{i=1}^{n} Z(i)}.
$$

Then $F(\mu)$ is a pivot that only contains parameter $\mu$. Besides, $F(\mu)$ is a strictly monotonic increasing function of $\mu$, and we know

$$
\lim_{\mu \to +\infty} F(\mu) = 0^+, \lim_{\mu \to -\infty} F(\mu) = +\infty, \lim_{\mu \to A^-} F(\mu) = -\infty, \lim_{\mu \to A^+} F(\mu) = 0^-.
$$
Hence, for a given significance level \( \alpha' \) \( \leq \) the upper \( 1 - \alpha'/2, \alpha'/2 \) quantiles of the pivot \( F(\mu) \) are denoted by \( F_{1-\alpha'/2} \) and \( F_{\alpha'/2} \). Then it is obvious that the approximate interval estimation of parameter \( \mu \) at the confidence level \( 1 - \alpha' \) is

\[
\left( \frac{F_{1-\alpha'/2} - 1}{\sum_{i=j+1}^{n} Y(i)} + \frac{j}{j F_{1-\alpha'/2}} \right), \quad \left( \frac{F_{\alpha'/2} - 1}{\sum_{i=1}^{n} Y(i)} + \frac{j}{j F_{\alpha'/2}} \right).
\]

Furthermore, the approximate interval estimation of parameter \( \beta \) at the confidence level \( 1 - \alpha' \) is \( [\beta_L, \beta_U] \), here

\[
\beta_L = \exp \left\{ \frac{1}{j F_{1-\alpha'/2}} \left( \frac{F_{1-\alpha'/2} - 1}{\sum_{i=1}^{n} Y(i)} + \frac{j}{j F_{1-\alpha'/2}} \right) \right\},
\]

\[
\beta_U = \exp \left\{ \frac{1}{j F_{\alpha'/2}} \left( \frac{F_{\alpha'/2} - 1}{\sum_{i=1}^{n} Y(i)} + \frac{j}{j F_{\alpha'/2}} \right) \right\}.
\]

(2) When \( n \) is an odd number, it is denoted by \( j = \frac{n+1}{2}, n - j + 1 = \frac{n+1}{2}, A = \frac{1}{j} \sum_{i=j}^{n} Y(i) \). Then we have \( \sum_{i=j}^{n} Y(i) - j Y(i) > 0 \).

Let the function be \( F(\mu) = \frac{\sum_{i=j}^{n} Y(i) - j Y(i)}{\sum_{i=1}^{n} Y(i) - \mu} \), \( -\infty < \mu < +\infty \), and we know

\[
F(\mu) = \frac{\sum_{i=j}^{n} Y(i) - j Y(i)}{\sum_{i=1}^{n} Y(i) - \mu} = \frac{\sum_{i=j}^{n} Y(i) - \mu}{\sum_{i=1}^{n} \frac{Y(i) - \mu}{\alpha}},
\]

\[
= \frac{\sum_{i=j}^{n} Z(i) - j Z(i)}{\sum_{i=1}^{n} Z(i)}.
\]

Then \( F(\mu) \) is a pivot that only contains parameter \( \mu \). Besides, \( F(\mu) \) is a strictly monotonic increasing function of \( \mu \), and we know

\[
\lim_{\mu \to A} F(\mu) = 0^+, \quad \lim_{\mu \to A} F(\mu) = +\infty, \quad \lim_{\mu \to A} F(\mu) = -\infty, \quad \lim_{\mu \to A} F(\mu) = 0^-.
\]

Hence, for a given significance level \( \alpha' \), the upper \( 1 - \alpha'/2, \alpha'/2 \) quantiles of the pivot \( F(\mu) \) are denoted by \( F_{1-\alpha'/2} \) and \( F_{\alpha'/2} \). Then it is obvious that the approximate interval estimation of parameter \( \mu \) at the confidence level \( 1 - \alpha' \) is

\[
\left( \frac{F_{1-\alpha'/2} - 1}{\sum_{i=j}^{n} Y(i)} + \frac{j}{j F_{1-\alpha'/2}} \right), \quad \left( \frac{F_{\alpha'/2} - 1}{\sum_{i=1}^{n} Y(i)} + \frac{j}{j F_{\alpha'/2}} \right).
\]
and we know

\[ T(\alpha) = \sum_{i=2}^{n} (n - i + 1) (Y(i) - Y(i-1)), \alpha > 0, \]

and we know

\[ T(\alpha) = \sum_{i=2}^{n} (n - i + 1) \left( \frac{Y(i) - \mu}{\alpha} - \frac{Y(i-1) - \mu}{\alpha} \right) = \sum_{i=2}^{n} (n - i + 1)(Z(i) - Z(i-1)). \]
Then $T(\alpha)$ is a pivot that only contains parameter $\alpha$, and $T(\alpha)$ is a strictly monotonic decreasing function of $\alpha$.

For a given significance level $\alpha'$ the upper $1 - \alpha'/2, \alpha'/2$ quantiles of the pivot $T(\alpha)$ are denoted by $T_{1-\alpha'/2}$ and $T_{\alpha'/2}$. Then the interval estimation of parameter $\alpha$ at the confidence level $1 - \alpha'$ is

$$\frac{\sum_{i=2}^{n} (n - i + 1)(Y_i - Y_{i-1})}{T_{\alpha'/2}} < \alpha < \frac{\sum_{i=2}^{n} (n - i + 1)(Y_i - Y_{i-1})}{T_{1-\alpha'/2}}.$$  

Let the sample size be $n = 3(1)30$, and through 10000 Monte-Carlo simulations, the upper 0.99, 0.975, 0.95, 0.90, 0.85, 0.15, 0.10, 0.05, 0.025, 0.01 quantiles of $T(\alpha)$ are shown in Table 4.

| $n$ | 0.01 | 0.025 | 0.05 | 0.10 | 0.15 | 0.25 | 0.50 | 0.95 | 0.975 | 0.99 |
|-----|------|------|------|------|------|------|------|------|------|
| 3   | 6.145| 2.028 | 1.231| 0.754| 0.472| 0.306| 0.185| 0.139| 0.104| 0.074|
| 4   | 6.657| 2.123| 1.310| 0.857| 0.524| 0.356| 0.210| 0.157| 0.122| 0.088|
| 5   | 7.180| 2.227| 1.414| 0.960| 0.593| 0.422| 0.265| 0.203| 0.164| 0.125|
| 6   | 7.707| 2.330| 1.518| 1.063| 0.662| 0.546| 0.316| 0.251| 0.210| 0.171|
| 7   | 8.235| 2.433| 1.621| 1.164| 0.731| 0.669| 0.364| 0.297| 0.255| 0.216|
| 8   | 8.764| 2.536| 1.722| 1.266| 0.800| 0.792| 0.410| 0.341| 0.298| 0.259|
| 9   | 9.293| 2.638| 1.823| 1.366| 0.869| 0.913| 0.452| 0.380| 0.338| 0.300|
| 10  | 9.822| 2.740| 1.923| 1.466| 0.939| 0.944| 0.491| 0.417| 0.365| 0.327|

In order to investigate the precisions of approximate $\alpha, \beta$ interval estimations of parameters, we choose the sample size $n = 10(1)15$ and the truth values of parameters $\alpha = 1, \beta = 1$. Then we obtain the samples from $GBS - Cauchy(\alpha, \beta)$ by 1000 Monte-Carlo simulations, and calculate mean lower limit, mean upper limit, mean interval length and the number of intervals that contain the truth values of approximate interval estimations for parameters $\alpha$ at the confidence level 0.95. The results are shown in Table 5.
Simulation results of approximate interval estimations

R. E. Glaser, A simple random sample is generated from the distribution

N. Balakrishnan, V. Leiva and J. Lopez, H. W. Gomez, J. F. Olivares-Pacheco and H. Bolfarine, A simple random sample is generated from the distribution

| Sample capacity | approximate interval estimation of $\beta$ | approximate interval estimation of $\alpha$ |
|----------------|----------------------------------------|----------------------------------------|
|                | mean lower limit | mean upper limit | mean interval length | number contains truth value | mean lower limit | mean upper limit | mean interval length | number contains truth value |
| 10             | 0.2716          | 6.8005           | 6.5289               | 955                        | 0.0289          | 2.7562           | 2.7273               | 953                        |
| 11             | 0.3014          | 5.5689           | 5.2675               | 964                        | 0.0271          | 2.5952           | 2.5680               | 954                        |
| 12             | 0.3042          | 6.2001           | 5.8958               | 973                        | 0.0251          | 2.5004           | 2.4753               | 954                        |
| 13             | 0.3150          | 5.1164           | 4.8014               | 971                        | 0.0248          | 2.4190           | 2.3951               | 945                        |
| 14             | 0.3059          | 5.7695           | 5.4636               | 974                        | 0.0239          | 2.3348           | 2.3109               | 946                        |
| 15             | 0.3190          | 4.8452           | 4.5262               | 978                        | 0.0217          | 2.2528           | 2.2323               | 941                        |

5. Simulation Examples

Example 5.1. A simple random sample is generated from the distribution $GBS – Cauchy(0.5, 2)$ with sample size 20 by Monte-Carlo simulations: 0.242714, 1.86936, 1.52568, 2.68807, 2.67345, 1.73983, 5.99763, 1.76747, 0.234591, 8.21912, 2.30862, 2.1332, 3.95174, 3.53211, 4.08302, 1.37021, 1.80273, 6.28822, 1.12606, 0.803841. By using the proposed methods in this paper, the point estimations of parameters $\alpha, \beta$ are respectively

$$\hat{\alpha}_1 = 0.4262, \hat{\beta}_1 = 2.0013, \hat{\alpha}_2 = 0.4367, \hat{\beta}_2 = 1.7878, \hat{\alpha}_3 = 0.1338, \hat{\beta}_3 = 1.9597.$$  

At the confidence level $1 - \alpha' = 0.95$, approximate interval estimation of parameter $\alpha$ is $[0.0086, 0.8603]$, and approximate interval estimation of parameter $\beta$ is $[1.0708, 3.6027]$.

Example 5.2. A simple random sample is generated from the distribution $GBS – Cauchy(1, 1)$ with sample size 10 by Monte-Carlo simulations: 7.04891, 0.0138007, 1.14273, 0.455985, 0.281722, 0.25294, 0.804234, 0.924005, 3.2357, 11.0822. By using the proposed methods in this paper, the point estimations of parameters $\alpha, \beta$ are respectively

$$\hat{\alpha}_1 = 1.0096, \hat{\beta}_1 = 0.8641, \hat{\alpha}_2 = 1.0353, \hat{\beta}_2 = 0.5421, \hat{\alpha}_3 = 0.5696, \hat{\beta}_3 = 0.7913.$$  

At the confidence level $1 - \alpha' = 0.95$, approximate interval estimation of parameter $\alpha$ is $[0.0319, 3.0387]$, and approximate interval estimation of parameter $\beta$ is $[0.1677, 2.8417]$.

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