Logarithmic integrals, zeta values, and tiered binomial coefficients

Michael E. Hoffman1 · Markus Kuba2

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Abstract
We study logarithmic integrals of the form \( \int_0^1 x^i \ln^n(x) \ln^m(1-x) \, dx \). They are expressed as a rational linear combination of certain rational numbers \((n, m)_i\), which we call tiered binomial coefficients, and products of the zeta values \(\zeta(2), \zeta(3), \ldots\). Various properties of the tiered binomial coefficients are established. They involve, amongst others, the binomial transform, truncated multiple zeta and multiple zeta star values, as well as special functions. We present extensions to generalized Nielsen polylogarithms. As an application we revisit the limit law of the number of comparisons of the Quicksort algorithm: we reprove that the moments of the limit law are rational polynomials in the zeta values. Properties of the cumulants of the Quicksort limit law are also discussed.

Keywords Multiple zeta values · Logarithmic integrals · Tiered binomial coefficients · Binomial transform · Quicksort

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Markus Kuba
kuba@technikum-wien.at

Michael E. Hoffman
meh@usna.edu

1 U.S. Naval Academy, Annapolis, MD 21402, USA
2 Department Applied Mathematics and Physics, FH - Technikum Wien, Höchstädtplatz 5, 1200 Wien, Austria
1 Introduction

The multiple zeta values [15,42] are defined by

$$\zeta(i_1, \ldots, i_k) = \sum_{\ell_1 \geq \cdots \geq \ell_k \geq 1} \frac{1}{\ell_1^{i_1} \cdots \ell_k^{i_k}},$$

with admissible indices \((i_1, \ldots, i_k)\) satisfying \(i_1 \geq 2, i_j \geq 1\) for \(2 \leq j \leq k\). Their truncated counterparts, often also called multiple harmonic sums, are defined by

$$\zeta_n(i_1, \ldots, i_k) = \sum_{n \geq \ell_1 \geq \cdots \geq \ell_k \geq 1} \frac{1}{\ell_1^{i_1} \cdots \ell_k^{i_k}}.$$ 

We refer to \(i_1 + \cdots + i_k\) as the weight of this multiple zeta value, and \(k\) as its depth. For an overview as well as a great many pointers to the literature we refer to the articles [17,19,41].

An important variation of the (truncated) multiple zeta values are the so-called (truncated) multiple zeta star values. Here, equality of the indices is allowed:

$$\zeta^\star(i_1, \ldots, i_k) = \sum_{\ell_1 \geq \cdots \geq \ell_k \geq 1} \frac{1}{\ell_1^{i_1} \cdots \ell_k^{i_k}}$$

and

$$\zeta^\star_n(i_1, \ldots, i_k) = \sum_{n \geq \ell_1 \geq \cdots \geq \ell_k \geq 1} \frac{1}{\ell_1^{i_1} \cdots \ell_k^{i_k}}.$$ 

In this work we study the logarithmic integrals \(I^{(i)}_{n,m}\) and their properties, with \(I^{(i)}_{n,m}\) defined as

$$I^{(i)}_{n,m} := \int_0^1 x^i \cdot \ln^n(x) \cdot \ln^m(1 - x) \, dx,$$  \hspace{1cm} (1)

with \(i \geq -1, m \in \mathbb{N}, n \in \mathbb{N}_0 = \{1, 2 \ldots\}\) or \(i, m, n \in \mathbb{N}_0\). We note that the boundary cases \(i = -1, m = 0\) and \(n \in \mathbb{N}_0\) all lead to divergent integrals. Particular instances of such integrals have been studied previously by Köhlig [26–28] and also by Laurenzi [31], who discussed special cases of the instance \(i = 0\) (without obtaining a closed formula). Xu [40] studied, amongst others, the case \(i = -1\) and related it to multiple zeta values. We note here that our results also cover for \(i, m, n \in \mathbb{N}_0\) the variation \(\int_0^1 x^i \cdot \ln^n(x) \cdot \frac{\ln^m(1-x)}{1-x} \, dx\) of the integral, due to

$$\int_0^1 x^i \cdot \ln^n(x) \cdot \frac{\ln^m(1-x)}{1-x} \, dx = \frac{1}{m+1} \left( i \cdot I^{(i-1)}_{n,m+1} + n \cdot I^{(i-1)}_{n-1,m+1} \right).$$
1.1 Main results

Let $S_{n,m}^{(i)}$ denote the normalized values

$$S_{n,m}^{(i)} := \frac{(-1)^{n+m}}{n!m!} \cdot I_{n,m}^{(i)}.$$

Here and throughout this work we often the shorthand notation $\zeta(a,\{1\}_b) = \zeta(a, 1, \ldots, 1)$, where $\{i\}_b$ means $i$ repeated $b$ times. We summarize our main results: an expansion of the normalized logarithmic integrals $S_{n,m}^{(i)}$ for $i \geq 0$ into multiple zeta values. For the sake of completeness we also collect the known result for the special case $i = -1$.

**Theorem 1** For $i, m, n \in \mathbb{N}_0$ the normalized logarithmic integrals $S_{n,m}^{(i)}$ are given by

$$S_{n,m}^{(i)} = (n, m)_i - \sum_{1 \leq a \leq n, 1 \leq b \leq m} (n - a, m - b)_i \cdot \zeta(a + 1, \{1\}_b).$$

Here the values $(n, m)_i$ denote certain rational numbers, called binomial coefficients of tier $i$, given by

$$(n, m)_i = \frac{1}{i+1} \sum_{k=0}^{n} (-1)^k \binom{n-k+m}{m} \zeta_i(\{1\}_k) \zeta_i^*(\{1\}_{n-k+m}).$$

In the special case $i = -1$, $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we have

$$S_{n,m}^{(-1)} = \zeta(n + 2, \{1\}_{m-1}).$$

Various explicit expressions of $(n, m)_i$ in terms of truncated multiple zeta values $\zeta_n([1]_k)$ and star values $\zeta_i^*(\{1\}_k)$ are later on established. As a byproduct of our study we establish the following.

**Corollary 1** For $i \geq -1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ or $i, m, n \in \mathbb{N}_0$ the (normalized) logarithmic integrals $S_{n,m}^{(i)}$ are rational polynomials in the ordinary zeta values $\zeta(2), \zeta(3), \ldots$.

This will follow directly from earlier results in the literature. Borwein, Bradley and Broadhurst proved that for all positive integers $n, m$ the multiple zeta value $\zeta(m + 1, \{1\}_n)$ is a rational polynomial in the $\zeta(i)$ [3, Eq. (10)]:

$$\sum_{m,n \geq 0} \zeta(m + 2, \{1\}_n) x^{m+1} y^{n+1} = 1 - \exp \left( \sum_{k \geq 2} \frac{x^k + y^k - (x + y)^k}{k} \zeta(k) \right).$$

(2)
We mention a recently obtained explicit expression by Kaneko and Sakata [24]:

$$\zeta(m + 1, (1)_{n-1}) = \min(m, n) \sum_{i=1}^{\min(m, n)} (-1)^{i-1} \sum_{\substack{\text{wt}(m) = m, \text{wt}(n) = n \\text{dep}(m) = \text{dep}(n) = i}} \zeta(m + n),$$

where for two indices $m = (m_1, \ldots, m_i)$ and $n = (n_1, \ldots, n_i)$ with weights $\text{wt}(m) = \sum_j m_j = m$ and $\text{wt}(n) = \sum_j n_j = n$, respectively, which have the same depth $\text{dep}(m) = \text{dep}(n) = i$, the sum $m + n$ denotes $(m_1 + n_1, \ldots, m_i + n_i)$.

After our main results, we also turn to extensions. We generalize some results for $S_{n,m}$ to a generalized version of the Nielsen polylogarithm. Moreover, we analyze the logarithmic integrals with negative powers of $i$ and obtain the following extensions of Corollary 1.

**Corollary 2** For $i \in \mathbb{Z} \setminus \mathbb{N}_0$, $m \geq -i$ and $n \in \mathbb{N}_0$ the (normalized) logarithmic integrals $S_{n,m}^{(i)}$ are rational polynomials in the ordinary zeta values $\zeta(2), \zeta(3), \ldots$.

### 1.2 Structure and notation

This article is structured as follows. First, we turn to the integrals $I_{n,m}^{(i)}$ and $S_{n,m}^{(i)}$ and derive various properties of them. We use special Hurwitz zeta values to obtain a recurrence relation for $S_{n,m}^{(i)}$. This recurrence relation is then translated into a recurrence for the the binomials coefficients of tier $i$. The enumeration of $(n, m)_i$ is then solved using generating functions. Various additional properties of these numbers are then given. They involve, amongst others, the binomial transform, truncated multiple zeta and multiple zeta star values, as well as Euler polynomials and Legendre polynomials. Then, we discuss extensions of our results: we discuss negative values of $i$, as well as an extension to a generalization of the Nielsen polylogarithm function. In the final section we discuss applications of our results to the limit law for the number of comparisons in the Quicksort algorithm.

A basic ingredient of our computations are the signless Stirling numbers $[n \atop k]$ of the first kind, also called Stirling cycle numbers. They count the number of permutations of $n$ elements with $k$ cycles [10] and appear as coefficients in the expansions

$$x_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^k = \sum_{k=0}^{n} s(n, k)x^k,$$

relating ordinary powers $x^n$ to the so-called falling factorials $x_n = x(x-1)\ldots(x-(n-1))$, for integers $n \geq 1$, and $x^0 = 1$; here $s(n, k)$ denote the signed Stirling numbers. The definition can be extended to negative integers via $x^{-n} = \frac{1}{(x+n)x}$, $n \geq 1$. Moreover, we denote with $a^{-n} = a(a+1)\ldots(a+n-1)$ the rising factorials$^1$.

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$^1$ The rising factorials are often called Pochhammer symbols and denoted by $(a)_n = a^{(n)}$. Due to the similarity to our binomial coefficients of tier $i$ we opted to use a notation popularized by Graham, Knuth and Patashnik [10].
We denote with $H_n = \sum_{k=1}^{n} \frac{1}{k}$ the $n$-th harmonic numbers and with $H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r}$ the $n$-th generalized harmonic number of order $r \in \mathbb{N}$, which are special instances of truncated multiple zeta values: $H_n^{(r)} = \zeta_n(r)$.

2 Evaluations of the logarithmic integrals

We start with special instances of the integral, which are well known.

**Lemma 1** (Boundary values - Case $m = 0$) For $m = 0$ and $n, i \geq 0$ we have

$$I^{(i)}_{n,0} := \int_{0}^{1} x^i \cdot \ln^n(x) \, dx = \frac{(-1)^n n!}{(i+1)^{n+1}}. \quad (3)$$

The standard proof of this folklore result uses repeated integrated by parts and induction. Alternatively, this can be obtained by using the Beta integral $B(u + 1, v + 1)$, with

$$B(u + 1, v + 1) := \int_{0}^{1} x^u (1 - x)^v \, dx = \frac{\Gamma(u+1)\Gamma(v+1)}{\Gamma(u+v+2)}.$$

We note that the following folklore argument already appears in [13], and has been rediscovered and used several times [9,31].

**Proof** Let $E_{u=a, v=b} = E_{u=a}E_{v=b}$ denote the evaluation operator at $u = a$ and $v = b$. Then,

$$I^{(i)}_{n,m} = E_{u=i, v=0} \frac{\partial^{n+m}}{\partial u^n \partial v^m} B(u + 1, v + 1)$$

Thus, for $m = 0$ we get

$$I^{(i)}_{n,0} = E_{u=i, v=0} \frac{\partial^n}{\partial u^n} B(u + 1, v + 1) = E_{u=i} \frac{\partial^n}{\partial u^n} \frac{\Gamma(u+1)}{\Gamma(u+2)}$$

$$= E_{u=i} \frac{\partial^n}{\partial u^n} \frac{1}{u+1} = \frac{(-1)^n n!}{(i+1)^{n+1}}.$$

\[\square\]

2.1 Symmetry of the integral and boundary values

The symmetry of the logarithmic integrals $I^{(i)}_{n,m}$ involves the binomial transform, as introduced by Knuth [29].
**Definition 1** (Binomial transform) The binomial transform $\mathcal{B}JN$ of a sequence $(a_i)_{i \geq 0}$ is a sequence $(s_i)_{i \geq 0}$, defined by

$$s_i = \mathcal{B}JN(a_i) = \sum_{k=0}^{i} (-1)^k \binom{i}{k} a_k.$$

Note that the binomial transform is evidently a linear operator and an involution: $\mathcal{B}JN \circ \mathcal{B}JN = id$. We refer the reader to the article of the first author [17] for algebraic properties and to Prodinger [35] for additional properties concerning generating functions.

Additionally, given a double sequence $(a_{n,m})_{n,m \geq 0}$, let $\mathcal{C}$ denote the operator which exchanges the indices:

$$\mathcal{C}(a_{n,m}) = a_{m,n},$$

such that $\mathcal{C} \circ \mathcal{C} = id$.

**Proposition 1** (Generalized Symmetry of logarithmic integrals) The logarithmic integrals $I_{n,m}^{(i)}$ satisfy the generalized symmetry relation

$$I_{n,m}^{(i)} = \sum_{j=0}^{i} \binom{i}{j} (-1)^j \cdot I_{m,n}^{(j)}.$$

Equivalently, the binomial transform $\mathcal{B}JN$ with respect to the variable $i$, equals the operator $\mathcal{C}$ with respect to $n$ and $m$:

$$\mathcal{C}(I_{n,m}^{(i)}) = \mathcal{B}JN(I_{n,m}^{(i)}).$$

**Proof** We use the substitution $x = 1 - u$ and readily obtain

$$I_{n,m}^{(i)} = \int_{0}^{1} x^i \cdot \ln^{n}(x) \cdot \ln^{m}(1 - x) \, dx = \int_{0}^{1} (1 - u)^i \cdot \ln^{m}(u) \cdot \ln^{n}(1 - u) \, du.$$

Expansion of the term $(1 - u)^i = \sum_{j=0}^{i} \binom{i}{j} (-1)^j u^j$ gives the stated result. \qed

A direct byproduct of the generalized symmetry is the evaluation of $I_{0,m}^{(i)}$.

**Theorem 2** (Boundary values - case $n = 0$) For $n = 0$ and $m, i \geq 0$ the values $I_{0,m}^{(i)}$ are given by truncated zeta star series,

$$I_{0,m}^{(i)} = (-1)^m m! \cdot \frac{1}{i+1} \cdot \zeta_{i+1}^{*}(1,m).$$

Before we turn to the proof we collect a well known result [4,8,30].
Lemma 2 (Truncated multiple zeta values $\zeta^*_n([1]_k))$ For positive integers $n \geq k$, the values $\zeta^*_n([1]_k)$ can be expressed as

$$\zeta^*_n([1]_k) = \sum_{j=1}^{n} \binom{n}{j} \frac{(-1)^{j-1}}{jk}.$$ 

**Proof** By the symmetry relation,

$$I_{0,m}^{(i)} = \sum_{j=0}^{i} \binom{i}{j} (-1)^j \cdot I_{m,0}^{(j)}.$$ 

Using (3), we get

$$I_{0,m}^{(i)} = \sum_{j=0}^{i} \binom{i}{j} (-1)^j \cdot \frac{(-1)^m m!}{(j+1)^{m+1}}.$$ 

Since $\binom{i}{j} = \frac{i+1}{i+1} \binom{i+1}{j+1}$ we get

$$I_{0,m}^{(i)} = (-1)^m m! \cdot \frac{1}{i+1} \cdot \sum_{j=0}^{i} \binom{i+1}{j+1} (-1)^j \cdot \frac{1}{(j+1)^m}$$
$$= (-1)^m m! \cdot \frac{1}{i+1} \sum_{j=1}^{i+1} \binom{i+1}{j} (-1)^{j+1} \cdot \frac{1}{j^m}.$$ 

Using the Lemma stated before we obtain the claimed result. \(\square\)

2.2 A representation using Stirling numbers

Our aim is to prove the following presentation.

**Proposition 2** (Logarithmic integrals and truncated zeta values) The logarithmic integrals $I_{n,m}^{(i)}$ satisfy

$$I_{n,m}^{(i)} = n! m! (-1)^{m+n} \sum_{\ell=1}^{\infty} \frac{\zeta_{\ell-1}([1]_{m-1})}{\ell (\ell + i + 1)^{n+1}}.$$ 

In particular, for $i = -1$ we have

$$I_{n,m}^{(-1)} = n! m! (-1)^{m+n} \zeta(n + 2, [1]_{m-1}).$$
We have the alternative expression

\[ I_{n,m}^{(i)} = n! m (-1)^{n-1} \sum_{\ell=1}^{\infty} B_{m-1} \left( -0! H_{\ell-1}^{(1)}, -1! H_{\ell-1}^{(2)}, \ldots, -(m-2)! H_{\ell-1}^{(m-1)} \right) \ell (\ell + i + 1)^{n+1}, \]

where the \( B_k \) denote the Bell polynomials.

In order to evaluate the logarithmic integrals we proceed similar to [21]. We can interpret the power \((\log(1-x))^k\) (see [10, p. 351]) as the row generating function of the Stirling numbers of the first kind.

\[ (\log(1-x))^k = k! \sum_{\ell=1}^{\infty} \frac{(-1)^k x^\ell}{\ell!} \left[ \begin{array}{c} \ell \\ k \end{array} \right]. \tag{4} \]

We use a well-known basic relation between Stirling numbers of the first kind and truncated multiple zeta values; see for example [21], or [30] for more properties of \([n]_k\) and simple proofs.

**Lemma 3** (Truncated multiple zeta values \(\zeta_n([1]_k)\)) For positive integers \( n \geq k \) the Stirling numbers of the first kind are related to truncated multiple zeta values:

\[ \left[ \begin{array}{c} n \\ k \end{array} \right] = (n-1)! \zeta_{n-1}([1]_{k-1}), \]

where \([1]_m\) means 1 repeated \( m \) times. Moreover, \(\zeta_n([1]_k)\) is given in terms of Bell polynomials and generalized Harmonic numbers,

\[ \zeta_n([1]_k) = \frac{(-1)^k}{k!} B_k (-0! H_n^{(1)}, -1! H_n^{(2)}, \ldots, -(k-1)! H_n^{(k)}) \]

\[ = \sum_{m_1+2m_2+\ldots=k} \frac{(-1)^{m_2+m_4+\ldots}}{m_1! m_2! \ldots} \left( \frac{H_n^{(1)}}{1} \right)^{m_1} \left( \frac{H_n^{(2)}}{2} \right)^{m_2} \ldots. \]

Now we can rewrite the logarithmic integral as follows:

\[ I_{n,m}^{(i)} = \int_0^1 x^i \ln^n(x) \cdot \ln^m (1-x) \, dx = m! (-1)^m \sum_{\ell=1}^{\infty} \left[ \begin{array}{c} \ell \\ m \end{array} \right] \int_0^1 x^\ell i \ln^n(x) \, dx. \]

We use the known special values (3) as well as Lemma 3 to get the desired representation

\[ I_{n,m}^{(i)} = m! (-1)^m \sum_{\ell=1}^{\infty} \left[ \begin{array}{c} \ell \\ m \end{array} \right] \frac{(-1)^n n!}{(\ell + i + 1)^{n+1}} \]

\[ = n! m! (-1)^{m+n} \sum_{\ell=1}^{\infty} \frac{\zeta_{\ell-1}([1]_{m-1})}{\ell (\ell + i + 1)^{n+1}}. \]
The special case \( i = -1 \) follows easily from the definition of the multiple zeta values.

### 2.3 Partial fraction decomposition and recurrence relations

In the following we further study the normalized values \( S_{n,m}^{(i)} \), given by

\[
S_{n,m}^{(i)} = \sum_{\ell=1}^{\infty} \frac{\zeta_{\ell-1}([1]_{m-1})}{\ell(\ell + i + 1)^{n+1}}.
\]

We introduce a variant \( T_{n,m}^{(i)} \) of the values \( S_{n,m}^{(i)} \), defined by

\[
T_{n,m}^{(i)} = \sum_{\ell=1}^{\infty} \frac{\zeta_{\ell-1}([1]_{m-1})}{(\ell + i + 1)^{n+1}},
\]

for \( n, m \geq 1 \) and \( i \geq -1 \). Its initial values are given by

\[
T_{n,m}^{(-1)} = S_{n-1,m}^{(-1)} = \zeta(n + 1, [1]_{m-1}).
\]

**Proposition 3** (Relations between \( S \) and \( T \) values) The normalized logarithmic integrals \( S_{n,m}^{(i)} \) and the values \( T_{n,m}^{(i)} \) are related by

\[
S_{n,m}^{(i)} = T_{n,m+1}^{(i-1)} - T_{n,m}^{(i)},
\]

for \( m, n, i \geq 0 \). Consequently, for \( m \geq 1 \) we have

\[
T_{n,m}^{(i)} = -\sum_{j=0}^{i} S_{n,m-1}^{(j)} + \zeta(n + 1, [1]_{m-1}).
\]

On the other hand, for \( i \geq 0 \), \( n \geq 1 \) there is also the relation

\[
S_{n,m}^{(i)} = \frac{1}{i+1} \cdot S_{n-1,m}^{(i)} - \frac{1}{i+1} \cdot T_{n,m}^{(i)}.
\]

**Remark 1** The multiple Hurwitz zeta values are defined by

\[
\zeta_H(i_1, \ldots, i_k; p_1 \ldots, p_k) = \sum_{\ell_1 > \cdots > \ell_k \geq 1} \frac{1}{(\ell_1 + p_1)^{i_1} \cdots (\ell_k + p_k)^{i_k}}.
\]

Evidently, they are related to the ordinary multiple zeta values by

\[
\zeta_H(i_1, \ldots, i_k; 0 \ldots, 0) = \zeta(i_1, \ldots, i_k).
\]
The \( T \)-values introduced before are special instances of the multiple Hurwitz zeta values:

\[
T_{n,m}^{(i)} = \sum_{\ell=1}^{\infty} \frac{\zeta_{\ell-1}(\{1\}_{m-1})}{(\ell + i + 1)^{n+1}} = \zeta_{H}(n + 1, \{1\}_{m-1}; i + 1(0)_{m-1}). \tag{5}
\]

**Proof** The truncated zeta values \( \zeta_{\ell}(\{1\}_{m}) \) satisfy the recurrence relation

\[
\zeta_{\ell}(\{1\}_{m}) = \sum_{k=1}^{\ell} \frac{\zeta_{k-1}(\{1\}_{m-1})}{k} = \zeta_{\ell-1}(\{1\}_{m}) + \frac{1}{\ell} \zeta_{\ell-1}(\{1\}_{m-1}).
\]

Hence,

\[
\frac{1}{\ell} \zeta_{\ell-1}(\{1\}_{m-1}) = \zeta_{\ell}(\{1\}_{m}) - \zeta_{\ell-1}(\{1\}_{m}).
\]

This implies that

\[
S_{n,m}^{(i)} = \sum_{\ell=1}^{\infty} \frac{\zeta_{\ell}(\{1\}_{m})}{(\ell + i + 1)^{n+1}} - T_{n,m+1}^{(i)} = T_{n,m+1}^{(i-1)} - T_{n,m+1}^{(i)}.
\]

The stated result for \( T_{n,m}^{(i)} \) follows by summation.

For the second recurrence relation we use the partial fraction decomposition

\[
\frac{1}{\ell(\ell + i + 1)} = \frac{1}{i + 1} \cdot \left[ \frac{1}{\ell} - \frac{1}{\ell + i + 1} \right],
\]

leading directly to the stated result

\[
S_{n,m}^{(i)} = \frac{1}{i + 1} \left[ S_{n-1,m}^{(i)} - T_{n,m}^{(i)} \right].
\]

Next we obtain a recurrence relation for the \( S \)-values.

**Proposition 4** (Recurrence relation for \( S \)-values) The normalized logarithmic integrals \( S_{n,m}^{(i)} \) satisfy for \( n, m > 0 \) and \( i \geq 0 \) the recurrence relation

\[
S_{n,m}^{(i)} = \frac{1}{i + 1} \cdot \left[ S_{n-1,m}^{(i)} + \sum_{j=0}^{i} S_{n,m-1}^{(j)} - \zeta(n + 1, \{1\}_{m-1}) \right],
\]

with initial values \( S_{0,m}^{(i)} = \frac{1}{i+1} \cdot \zeta_{i+1}(\{1\}_{m}) \) and \( S_{n,0}^{(i)} = \frac{1}{(i+1)^{n+1}} \).
Proof  We simply plug the expression for $T_{n,m}^{(i)}$,

$$T_{n,m}^{(i)} = - \sum_{j=0}^{i} S_{n,m-1}^{(j)} + \zeta(n + 1, \{1\}_{m-1}).$$

into

$$S_{n,m}^{(i)} = \frac{1}{i+1} \cdot S_{n-1,m}^{(i)} - \frac{1}{i+1} \cdot T_{n,m}$$

(6)

and get the stated recurrence relation by iteration of (6). The initial values are known and given in (3) and Theorem 2.

3 Tiered binomial coefficients and small values of $i$

We will obtain a compact representation of the normalized logarithmic integrals by introducing special rational numbers. We define the $i$th tier binomial coefficients as the coefficients of the expansion

$$S_{n,m}^{(i)} = (n,m)_i - \sum_{1 \leq a \leq n, 1 \leq b \leq m} (n-a,m-b)_i \zeta(a+1,\{1\}_{b-1}).$$

According to the initial conditions $S_{0,m}^{(i)} = \frac{1}{i+1} \cdot \zeta_{i+1}^{*} (\{1\}_{m})$ and $S_{n,0}^{(i)} = \frac{1}{(i+1)^{n+1}}$ it holds

$$\langle 0, m \rangle_i = S_{0,m}^{(i)} = \frac{1}{i+1} \cdot \zeta_{i+1}^{*} (\{1\}_{m})$$

and

$$\langle n, 0 \rangle_i = S_{n,0}^{(i)} = \frac{1}{(i+1)^{n+1}}.$$

Proposition 5  (Recurrence relation for binomial coefficients of tier $i$) The coefficients $(n,m)_i$ satisfy the recurrence relation

$$(n,m)_i = \frac{(n-1,m)_i}{i+1} + \frac{1}{i+1} \sum_{j=0}^{i} (n,m-1)_j, \quad n,m > 0, \quad i \geq 0,$$

with initial values $(0,m)_i = \frac{1}{i+1} \cdot \zeta_{i+1}^{*} (\{1\}_{m})$ and $(n,0)_i = \frac{1}{(i+1)^{n+1}}$.

Remark 2  Note that the recurrence relation is actually valid for $n,m \geq 0$, excluding $n=m=0$. It is sufficient to define $(-1,m)_i = 0$ and $(n,-1)_i = 0$. 

Proof  By the recurrence relation in Proposition 4 we obtain

\[
(n, m)_i = \sum_{\substack{1 \leq a \leq n \\ 1 \leq b \leq m}} (n - a, m - b)_i \zeta(a + 1, (1)_{b-1})
\]

\[
= \frac{(n - 1, m)_i}{i + 1} - \sum_{\substack{1 \leq a \leq n-1 \\ 1 \leq b \leq m}} \frac{(n - 1 - a, m - b)_i}{i + 1} \zeta(a + 1, (1)_{b-1})
\]

\[
+ \frac{1}{i + 1} \sum_{j=0}^{i} [(n, m - 1)_j - \sum_{\substack{1 \leq a \leq n \\ 1 \leq b \leq m-1}} (n - a, m - 1 - b)_j \zeta(a + 1, (1)_{b-1})]
\]

\[
- \frac{1}{i + 1} \zeta(n + 1, (1)_{m-1}).
\]

First we check the boundary cases. For \(a = n\) and \(b = m\) we get

\[-(0,0)_i = - \frac{1}{i + 1}.\]

For \(a = n\) and \(b < m\) we get

\[-(0, m - b)_i = - \frac{1}{i + 1} \sum_{j=0}^{i} (0, m - 1 - b)_j.\]

Due to

\[
\frac{1}{i + 1} \cdot \zeta^*_{i+1}((1)_{m-b}) = \frac{1}{i + 1} \sum_{j=0}^{i} \frac{1}{j + 1} \cdot \zeta^*_{j+1}((1)_{m-1-b}),
\]

this is compatible with the initial conditions. For \(a < n\) and \(b = m\) we have

\[-(n - a, 0)_i = - \frac{(n - 1 - a, 0)_i}{i + 1},\]

which is true. For the value \((n, m)_i\) we directly get the stated recurrence relation. For the general case \(a < n\) and \(b < m\) we get

\[-(n - a, m - b)_i = - \frac{(n - 1 - a, m - b)_i}{i + 1} - \frac{1}{i + 1} \sum_{j=0}^{i} (n - a, m - 1 - b)_j,
\]

which is simply the shifted recurrence relation. \(\square\)
3.1 A generating functions approach

In the following we derive the generating functions of the $i$th tier binomial coefficients. We begin with two instructive examples and then turn to the general case. For the reader’s convenience we also collect the corresponding OEIS links.

**Example 1** *(Zero tier - binomial coefficients; A007318)* For $i = 0$ the coefficients $(n, m)_0$ satisfy the recurrence relation

$$(n, m)_0 = (n - 1, m)_0 + (n, m - 1)_0$$

with initial conditions $(0, m)_0 = (n, 0)_0 = 1$. One readily obtains, either by guess and prove or by generating functions, the solution

$$(n, m)_0 = \binom{n + m}{n}.$$ 

We present the simple generating functions proof, which can be regarded as a toy example for small $i$. We introduce the generating function $f(x, y) = f_0(x, y)$,

$$f(x, y) = \sum_{n, m \geq 0} (n, m)_0 x^n y^m.$$ 

Multiplication of the recurrence relation with $x^n y^m$ and summation over $n, m \geq 0$, excluding $n = m = 0$, gives

$$f(x, y) - 1 = x \cdot f(x, y) + y \cdot f(x, y).$$ 

Consequently,

$$f(x, y) = \frac{1}{1 - x - y} = \frac{1}{(1 - x)(1 - \frac{y}{1-x})},$$ 

such that

$$[x^n y^m]f(x, y) = [x^n] \frac{1}{(1 - x)^{m+1}} = \binom{n + m}{m}.$$ 

**Example 2** *(First tier; A308737)* For $i = 1$ the coefficients $(n, m)_1$ satisfy the recurrence relation

$$(n, m)_1 = \frac{1}{2}(n - 1, m)_1 + \frac{1}{2}(n, m - 1)_1 + \frac{1}{2}(n, m - 1)_0.$$ 

We introduce the generating function $f_1(x, y)$,

$$f_1(x, y) = \sum_{n, m \geq 0} (n, m)_1 x^n y^m.$$
Multiplication of the recurrence relation with $x^n y^m$ and summation over $n, m \geq 0$, excluding $n = m = 0$, gives

$$f_1(x, y) - \frac{1}{2} = \frac{x}{2} \cdot f_1(x, y) + \frac{y}{2} \cdot f_1(x, y) + \frac{y}{2} f_0(x, y).$$

Consequently,

$$f_1(x, y) = \frac{\frac{1}{2} + \frac{y}{2} f_0(x, y)}{1 - \frac{x}{2} - \frac{y}{2}} = \frac{1}{2(1 - \frac{x}{2} - \frac{y}{2})} + \frac{y}{2(1 - x - y)(1 - \frac{x}{2} - \frac{y}{2})}$$

$$= \frac{1 - x}{(1 - x - y)(2 - x - y)}.$$

Taylor expansion around $x = y = 0$ gives

$$f_1(x, y) = 1 + \frac{3}{4} y + \frac{7}{8} y^2 + \frac{15}{16} y^3 + \frac{31}{32} y^4 + \ldots.$$

In order to extract coefficients we rewrite the generating function

$$f_1(x, y) = 1 + \frac{y}{1 - x - y} - \frac{1 - x}{2(1 - \frac{x}{2} - \frac{y}{2})}.$$

Consequently, we obtain

$$(n, m)_1 = \left( \begin{array}{c} n + m - 1 \\ m - 1 \end{array} \right) - \frac{1}{2^{n+m}} \left[ \frac{1}{2} \left( \begin{array}{c} n + m \\ n \end{array} \right) - \left( \begin{array}{c} n + m - 1 \\ n - 1 \end{array} \right) \right].$$

3.2 General case—explicit expressions

Introducing the generating function $f_i(x, y)$, defined by

$$f_i(x, y) = \sum_{n,m \geq 0} (n, m)_i x^n y^m.$$

We multiply the recurrence relation from Proposition 5 with $x^n y^m$ and sum over $n, m \geq 0$, excluding $n = m = 0$. This gives a full history recurrence relation for the generating functions $f_i(x, y)$:

$$f_i(x, y) - \frac{1}{i+1} = \frac{x}{i+1} \cdot f_i(x, y) + \frac{y}{i+1} \cdot f_i(x, y) + \sum_{j=0}^{i-1} \frac{y}{i+1} f_j(x, y).$$ (7)
This implies that
\[(i + 1) \left(1 - \frac{x}{i+1} - \frac{y}{i+1}\right) f_i(x, y) = 1 + y \sum_{j=0}^{i-1} f_j(x, y).\]

Consequently, taking differences lead to an ordinary recurrence relation
\[f_i(x, y) - f_{i-1}(x, y) = yf_{i-1}(x, y).\]

This implies that
\[f_i(x, y) = \frac{i - x}{(i + 1 - x - y)} f_{i-1}(x, y), \quad i \geq 1. \quad (8)\]

Substituting the result for \(f_0(x, y)\) of our first example leads to the following theorem.

**Theorem 3** The generating function \(f_i(x, y) = \sum_{n, m \geq 0} (n, m) x^n y^m\) of the \(i\)th tier binomial coefficients is given by
\[f_i(x, y) = \frac{1}{(i + 1 - x - y)} \cdot \frac{(i-x)^i}{(i-x-y)} = \frac{1}{1-x-y} \cdot \frac{(x-1)}{(x+y-2)}^{i}, \quad i \geq 0.
\]

Alternative expressions for the generating function with falling and rising factorials are
\[f_i(x, y) = \frac{1}{1-x-y} \cdot \frac{(1-x)^i}{(2-x-y)^i} = \frac{(i-x)^i}{(i+1-x-y)^{i+1}}.
\]

**Example 3** (Second tier binomial coefficients) The theorem above gives
\[f_2(x, y) = \frac{(2-x)(1-x)}{(3-x-y)(2-x-y)(1-x-y)}.\]

Taylor expansion around \(x = y = 0\) gives
\[f_2(x, y) = \frac{1}{3} + \frac{11 \cdot y + 2 \cdot x}{18} + \frac{85 \cdot y^2 + 71 \cdot y \cdot x + 4 \cdot x^2}{108} + \frac{575 \cdot y^3 + 960 \cdot y^2 \cdot x + 393 \cdot y \cdot x^2 + 8 \cdot x^3}{648} + \frac{3661 \cdot y^4 + 9469 \cdot y^3 \cdot x + 7971 \cdot y^2 \cdot x^2 + 2179 \cdot y \cdot x^3 + 16 \cdot x^4}{3888} + \ldots\]
**Proposition 6** The binomial coefficients \((n, m)_i\) of tier \(i\) are given by formulas

\[
(n, m)_i = \frac{1}{i+1} \sum_{k=0}^{n} (-1)^k \binom{n-k+m}{m} \zeta_i([1]_k) \zeta_{i+1}^\ast([1]_{n-k+m}),
\]
as well as

\[
(n, m)_i = i! \sum_{\ell=1}^{i+1} \sum_{k=0}^{n} \frac{(-1)^{\ell+k-1} \zeta_i([1]_k)}{(\ell-1)! (i+1-\ell)!} \cdot \frac{1}{\ell^{n-k+m+1}} \binom{n-k+m}{m}.
\]

**Example 4** (Case \(m = n\), central tiered binomial coefficients) In the special case of \(m = n\) the binomial coefficients \((n, n)_i\) of tier \(i\) are given by

\[
(n, n)_i = \frac{1}{i+1} \sum_{k=0}^{n} (-1)^k \binom{2n-k}{n} \zeta_i([1]_k) \zeta_{i+1}^\ast([1]_{2n-k}).
\]

In order to obtain a closed form expressions for \((n, m)_i\) we record first a basic partial fraction decomposition of the polynomial in the denominator.

**Lemma 4** (Partial fraction decomposition)

\[
\frac{1}{(r-w)^2} = \sum_{\ell=1}^{r} \frac{(-1)^{\ell-1}}{(\ell-1)! (r-\ell)!} \cdot \frac{1}{r-w}.
\]

**Proof Of Proposition 6** The lemma stated before implies that

\[
f_i(x, y) = \sum_{\ell=1}^{i+1} \frac{(-1)^{\ell-1}}{(\ell-1)! (i+1-\ell)!} \cdot \frac{(i-x)^i}{\ell-x-y}.
\]

By definition of the truncated multiple zeta values we have

\[
(i-x)^i = i! \prod_{j=1}^{i} (1 - \frac{x}{j}) = i! \sum_{k \geq 0} (-1)^k \zeta_i([1]_k) x^k.
\]

Hence,

\[
[x^n y^m]f_i(x, y) = [x^n y^m] \sum_{\ell=1}^{i+1} \frac{(-1)^{\ell-1}}{(\ell-1)! (i+1-\ell)!} \frac{(i-x)^i}{\ell-x-y}
= i! \sum_{\ell=1}^{i+1} \sum_{k=0}^{n} \frac{(-1)^{\ell+k-1} \zeta_i([1]_k)}{(\ell-1)! (i+1-\ell)!} \cdot [x^{n-k} y^m] \frac{1}{\ell-x-y}.
\]
\[
= i! \sum_{\ell=1}^{i+1} \sum_{k=0}^{n} \frac{(-1)^{\ell+k-1} \zeta_i([1]_k)}{(\ell-1)! (i+1-\ell)!} \cdot \frac{1}{\ell^{n-k+m+1}} \binom{n-k+m}{m}.
\]

On the other hand, the generating function of truncated zeta star values is
\[
\zeta^*_n([1]_k) = [q^k] \frac{1}{(1 - \frac{q}{1}) (1 - \frac{q}{2}) \ldots (1 - \frac{q}{n})}. \tag{10}
\]

This implies the following expansion:
\[
\frac{1}{(i+1-x-y)^{i+1}} = \frac{1}{(i+1)!} \sum_{j \geq 0} \zeta^*_i([1]_j)(x+y)^j. \tag{11}
\]

Thus,
\[
f_i(x, y) = \left( \sum_{j \geq 0} \zeta^*_i([1]_j)(x+y)^j \right) \left( \sum_{k \geq 0} (-1)^k \zeta_i([1]_k)x^k \right) \end{array}
\]

Extraction of coefficients leads to the stated result. \( \square \)

### 3.3 Additional properties of tiered binomial coefficients

In this subsection we establish various properties of the binomial coefficients of tier \( i \). We discuss a generalized symmetry relation. Then, we obtain several different expressions for the row sums \( N_i = \sum_{m+n=N} (n, m)_i \), involving the Legendre polynomials. Using the Euler polynomials, we show that all central tiered binomial coefficients \( (n, n)_i \), as given in Example 4, can be written in terms of those in even tiers, i.e., with \( i \) even.

Furthermore, results for the alternating infinite sums \( \sum_{n \geq 0} (-1)^n \cdot (n, m)_i \) as well as \( \sum_{m \geq 0} (-1)^m \cdot (n, m)_i \) are given. We also study finite sums \( \sum_{i=0}^{N} (n, m)_i \) with respect to the tier \( i \). Finally, we relate the complete generating function \( \sum_{n, m, i \geq 0} (n, m)_i x^n y^m z^i \) to the Gauss hypergeometric series
\[
_2F_1(a, b, c; z) = \sum_{n \geq 0} \frac{a^n b^n}{c^n} \cdot \frac{z^n}{n!}.
\]

**Proposition 7** (Generalized symmetry for binomial coefficients of tier \( i \)) *The binomial coefficients of tier \( i \) satisfy the generalized symmetry relation*
\[
(n, m)_i = \sum_{j=0}^{i} \binom{i}{j} (-1)^j (m, n)_j.
\]
Equivalently, concerning the binomial coefficients of tier $i$, the binomial transform $\mathcal{B}N$ with respect to the variable $i$, equals the operator $\mathcal{C}$ with respect to $n$ and $m$:

$$\mathcal{C}((n, m)_i) = \mathcal{B}N((n, m)_i).$$

**Proof** We offer two proofs. First, we note that tiered binomial coefficients can be expressed as linear combinations of the sums $I^{(i)}_{n,m}$, due to

$$(n, m)_i = n! m! (-1)^{n+m} I^{(i)}_{n,m} + \sum_{\substack{1 \leq a \leq n \leq \mu \leq \nu \leq \mu}} (n - a, m - b)_i \cdot \zeta(a + 1, (1)_{b})$$

and iteration. The generalized symmetry relation in Proposition 1 holds for all $I^{(i)}_{n,m}$ and consequently also for $(n, m)_i$.

Alternatively, we can use the binomial theorem for the falling factorials to obtain

$$(i - x)^{\frac{i}{j}} = (i + 1 - y - x + (y - 1))^{\frac{i}{j}} = \sum_{j=0}^{i} \binom{i}{j} (i + 1 - y - x)^{\frac{i-j}{j}} (y - 1)^{\frac{j}{j}}.$$

Consequently,

$$f_i(x, y) = \frac{(i - x)^{\frac{i}{j}}}{(i + 1 - x - y)^{\frac{i+1}{j+1}}} = \sum_{j=0}^{i} \binom{i}{j} \frac{(y - 1)^{\frac{j}{j}}}{(j + 1 - y - x)^{\frac{j+1}{j+1}}}$$

$$= \sum_{j=0}^{i} (-1)^{j} \binom{i}{j} \frac{(j - y)^{\frac{j}{j}}}{(j + 1 - y - x)^{\frac{j+1}{j+1}}} = \sum_{j=0}^{i} (-1)^{j} \binom{i}{j} f_j(y, x).$$

$$\Box$$

**Proposition 8** The central tiered binomial coefficients $(n, n)_i$ satisfy for $n \geq 0$

$$(n, n)_{2k+1} = -\sum_{j=0}^{k} E_{2k+1,2j} (n, n)_{2j},$$

where $E_m(x) = \sum_{j=0}^{m} e_{m,j} x^j$ is the $m$th Euler polynomial.

**Remark 3** We note that the generating function of Euler polynomials is given by

$$\mathcal{E}(x, t) = \sum_{m \geq 0} E_m(x) \frac{t^m}{m!} = \frac{2e^{xt}}{e^t + 1}$$
and that \( E_m(x) \) can be explicitly expressed in terms of the Euler numbers \( E_k \):

\[
E_m(x) = \sum_{j=0}^{m} e_{m,j} x^j = \sum_{k=0}^{m} \left( \frac{m}{k} \right) E_k \left( x - \frac{1}{2} \right)^{m-k}.
\]

**Proof** We use induction with respect to \( k \). From Example 2 it follows that \( (n, n)_1 = \frac{1}{2} \binom{2n}{n} \) and thus that the conclusion holds for \( k = 0 \). Now suppose the conclusion holds through \( k - 1 \), where \( k > 0 \). From Proposition 7 we get

\[
2(n, n)_{2k+1} = \sum_{j=0}^{2k} \left( \frac{2k + 1}{j} \right) (-1)^j (n, n)_j.
\]

Hence, by splitting the sum into odd and even \( j \) we get

\[
2(n, n)_{2k+1} = \sum_{j=0}^{k} \left( \frac{2k + 1}{2j} \right) (n, n)_{2j} - \sum_{\ell=0}^{k-1} \left( \frac{2k + 1}{2\ell + 1} \right) (n, n)_{2\ell+1}
\]

\[
= \sum_{j=0}^{k} \left( \frac{2k + 1}{2j} \right) (n, n)_{2j} + \sum_{\ell=0}^{k-1} \left( \frac{2k + 1}{2\ell + 1} \right) \sum_{j=0}^{\ell} e_{2\ell+1,2j} (n, n)_{2j}
\]

\[
= \sum_{j=0}^{k} \left( \frac{2k + 1}{2j} \right) (n, n)_{2j} + \sum_{j=0}^{k-1} (n, n)_{2j} \sum_{\ell=j}^{k-1} \left( \frac{2k + 1}{2\ell + 1} \right) e_{2\ell+1,2j},
\]

so the conclusion holds if

\[
-2e_{2k+1,2j} = \left( \frac{2k + 1}{2j} \right) + \sum_{\ell=j}^{k-1} \left( \frac{2k + 1}{2\ell + 1} \right) e_{2\ell+1,2j}.
\]

By the basic relation for the Euler polynomials [1]

\[
(-1)^{n+1} E_n(-x) = E_n(x) - 2x^n,
\]

we can rewrite the right hand side as

\[
-2e_{2k+1,2j} = \sum_{p=2j}^{2k} \left( \frac{2k + 1}{p} \right) e_{p,2j}.
\]  

(12)

Now use the Euler-polynomial identities [1, 23.1.6, 23.1.7]

\[
E_{2k+1}(x+1) + E_{2k+1}(x) = 2x^{2k+1}, \quad E_{2k+1}(x+1) = \sum_{p=0}^{2k+1} \left( \frac{2k + 1}{p} \right) E_{p}(x)
\]
to get
\[ \sum_{p=0}^{2k+1} \binom{2k+1}{p} E_p(x) + E_{2k+1}(x) = 2x^{2k+1}. \]

Extract the coefficient of \( x^{2j} \), \( j \leq k \), to obtain
\[ \sum_{p=2j}^{2k+1} \binom{2k+1}{p} e_{p,2j} + e_{2k+1,2j} = 0, \]
from which (12) follows. \( \Box \)

**Proposition 9** (Row sums of binomial coefficients of tier \( i \)) The row sums \( N_i := \sum_{m+n=N} \binom{n}{m} \) are given by
\[
N_i = \frac{1}{i+1} \sum_{\ell=0}^{N} (-1)^\ell 2^{N-\ell} \zeta_i([1]_\ell) \zeta_{i+1}^*([1]_{N-\ell}).
\]

Alternatively, we have an expression in terms of the Legendre polynomials \( P_n(x) = \sum_{j=0}^n a_{n,j} x^j \):
\[
N_i = 2^{N-i} \sum_{\ell=0}^i a_{i,i-\ell} \cdot \frac{1}{(2\ell + 1)^{N+1}}.
\]

Moreover, an expression with Bell polynomials is
\[
N_i = \frac{1}{N! \cdot (i+1)} B_N(a_1, \ldots, a_N),
\]
where \( a_k = a_k(i) = (k-1)! \left( f_k^i \right) = 2^k \left( \frac{1}{(i+1)} \right) \).

**Proof** Evaluation of \( f_i(x, y) \) at \( x = y = z \) gives the row sum generating function
\[
f_i(z, z) = \sum_{N \geq 0} z^N \sum_{k=0}^{N} (N-k)_i = \sum_{N \geq 0} N_i \cdot z^N.
\]
Hence, \( N_i = [z^N] f_i(z, z) \) and

\[
N_i = [z^N] \frac{(i - z)^i}{(i + 1 - 2z)^{i+1}} = \frac{1}{i+1} \sum_{\ell=0}^{N} \left( [z^\ell] \prod_{j=1}^{i} \left( 1 - \frac{z}{j} \right) \right) \left( [z^{N-\ell}] \prod_{j=1}^{i+1} \left( 1 - \frac{2z}{j} \right)^{-1} \right).
\]

The expansions (9) and (11) give the stated result.

For the second representation we collect the known formula

\[
P_n(x) = \sum_{j=0}^{n} a_{n,j} x^j = \frac{1}{2^n} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j \binom{n}{j} \binom{2n-2j}{n} x^{n-2j}.
\]

We can rewrite the conclusion as

\[
N_i = \sum_{m+n=N} (n, m)_i = 2^{N-2i} \sum_{j=0}^{\left\lfloor \frac{i}{2} \right\rfloor} (-1)^j \binom{2n-2j}{j, n-j, n-2j} (2j+1)^{-N-1}.
\] (13)

We use the partial fraction decomposition from Lemma 4 and obtain

\[
f_i(z, z) = (i - z)^i \sum_{\ell=1}^{i+1} \frac{(-1)^{\ell+1}}{(\ell - 1)! (i + 1 - \ell)!} \cdot \frac{1}{\ell - 2z}.
\]

We split the sum into even and odd values of \( \ell \) to get

\[
f_i(z, z) = -\sum_{j=1}^{\left\lfloor \frac{i+1}{2} \right\rfloor} \frac{(i - z)^i}{(2j-1)! (i + 1 - 2j)!} \cdot \frac{1}{2j-2z} + \sum_{j=1}^{\left\lfloor \frac{i+1}{2} \right\rfloor+1} \frac{(i - z)^i}{(2j-2)! (i + 2 - 2j)!} \cdot \frac{1}{2j-1 - 2z}.
\] (14)

In the first sum we cancel the denominator with a factor of the numerator, leading to a polynomial of degree \( i - 1 \). For the second sum we derive the Laurent series around the poles \((2j - 1)/2\). Symbolically, we apply the binomial theorem for the falling factorials

\[
(i - z)^i = \left( i - z + \frac{1}{2} - (j - \frac{1}{2}) \right)^i = \sum_{k=0}^{i} \binom{i}{k} \left( j - \frac{1}{2} - z \right)^k \left( i - j + \frac{1}{2} \right)^{i-k}.
\]
The first summand gives \( (i - j + \frac{1}{2}) \frac{1}{2} \). Thus, we get

\[
f_i(z, z) = \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor + 1} \frac{(i - j + \frac{1}{2}) \frac{1}{2}}{(2j - 2)! (i + 2 - 2j)!} \cdot \frac{1}{2j - 1 - 2z} + p_i(z),
\]

where \( p_i(z) \) is a polynomial of degree at most \( i - 1 \), including the first sum of (14) and the remaining summands of the expansion of \( (i - z) \frac{1}{2} \). We claim that \( p_i(z) = 0 \). Avoiding more involved combinatorial reasoning, we argue as follows: assume that \( p_i(z) \neq 0 \). Then,

\[
\lim_{z \to \infty} f_i(z, z) = \lim_{z \to \infty} p_i(z) \in \{\pm \infty\} \cup \mathbb{R}\setminus\{0\}.
\]

However, by definition, the degree of the denominator of \( f_i(z, z) \) is \( i + 1 \), bigger than the degree of the numerator \( i \), so

\[
\lim_{z \to \infty} f_i(z, z) = 0,
\]

a contradiction. This implies that

\[
f_i(z, z) = \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor + 1} \frac{(i - j + \frac{1}{2}) \frac{1}{2}}{(2j - 2)! (i + 2 - 2j)!} \cdot \frac{1}{2j - 1 - 2z}.
\]

Shifting the index, simplification of \( (i - j + \frac{1}{2}) \frac{1}{2} \) and extraction of coefficients then directly leads to (13).

On the other hand, we can use the exp − log representation to get

\[
f_i(z, z) = \frac{1}{i + 1} \cdot \exp \left( \sum_{j=1}^{i} \ln(1 - \frac{z}{j}) - \sum_{j=1}^{i+1} \ln(1 - \frac{2z}{j}) \right).
\]

Series expansion of the logarithm functions give

\[
f_i(z, z) = \frac{1}{i + 1} \cdot \exp \left[ \sum_{k=1}^{\infty} \frac{z^k}{k} \left( \frac{1}{2k - 1} H_k^{(i)} + \frac{2^k}{(i + 1)k} \right) \right].
\]

Thus, it follows that \( N_i \) can be expressed in terms of the complete Bell polynomials \( B_n(x_1, \ldots, x_n) \), which are defined via

\[
\exp \left( \sum_{\ell \geq 1} \frac{x_\ell}{\ell!} z^\ell \right) = \sum_{j \geq 0} \frac{B_j(x_1, \ldots, x_j)}{j!} z^j,
\]

evaluated at \( x_k = a_k(i) = (k - 1)! \left( (2^k - 1) H_k^{(i)} + \frac{2^k}{(i + 1)^k} \right) \).
Proposition 10 The infinite alternating sums \( \sum_{m \geq 0} (n, m)_i \cdot (-1)^m \) satisfy

\[
\sum_{m \geq 0} (n, m)_i \cdot (-1)^m = \frac{i + 1}{(i + 2)^{n+1}} - \frac{i}{(i + 1)^{n+1}}.
\]

The infinite alternating sums \( \sum_{n \geq 0} (n, m)_i \cdot (-1)^n \) are given by

\[
\sum_{n \geq 0} (n, m)_i \cdot (-1)^n = \frac{1}{i + 2} \left( \zeta_{i+2}^*([1]_m) - \zeta_{i+2}^*([1]_{m-1}) \right).
\]

Proof The generating function of \( \sum_{m \geq 0} (n, m)_i \cdot (-1)^m \) is

\[
f_i(x, -1) = \frac{(i - x)^i}{(i + 2 - x)^{i+1}} = \frac{1 - x}{(i + 2 - x)(i + 1 - x)} = \frac{i + 1}{i + 2 - x} - \frac{i}{i + 1 - x}.
\]

Extraction of coefficients gives the stated result. Similarly, the generating function of \( \sum_{n \geq 0} (n, m)_i \cdot (-1)^n \) is

\[
f_i(-1, y) = \frac{(i + 1)!}{(i + 2 - y)^{i+1}} = \frac{1 - y}{(i + 2) \prod_{j=1}^{i+2}(1 - \frac{y}{j})}.
\]

Extraction of coefficients, using the generating function of the truncated zeta star values (10), gives the stated result. \( \square \)

Next we turn to the finite sums \( \sum_{i=0}^{N} (n, m)_i \) with respect to the tier \( i \). We use the following lemma, which can easily be proven using induction.

Lemma 5 Let \( u \) and \( v \) denote variables with \( u \neq v + 1 \) and \( v \notin \mathbb{N} \). Then, for \( n \geq 0 \)

\[
\sum_{i=1}^{n} \frac{\binom{u}{i}}{\binom{v}{i}} = \frac{(v - n)(u)}{(u - v - 1)(v)} = \frac{u}{u - v - 1}.
\]

Proposition 11 The sums \( \sum_{i=0}^{N} (n, m)_i \) of higher tier binomial coefficients over tier \( i \) are given by

\[
\sum_{j=0}^{N} (-1)^j \binom{n + m + 1 - j}{m + 1} \zeta_{N+1}^*([1]_j) \zeta_{N+1}^*([1]_{n+m+1-j}).
\]

Proof The generating functions of the finite sums is given by

\[
\sum_{i=0}^{N} f_i(x, y) = \frac{1}{1 - x - y} \left( 1 + \sum_{j=1}^{n} \frac{(x-1)}{(x+y-2)} \right).
\]

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Thus, we can apply the Lemma stated before and get
\[
\sum_{i=0}^{N} f_i(x, y) = \frac{1}{1-x-y} \cdot \left( 1 + \frac{x + y - N - 2}{-y} \right) \cdot \left( \frac{x-1}{N+1} \right)^{y} = x - 1.
\]
Simplifications give
\[
\sum_{i=0}^{N} f_i(x, y) = \frac{1}{y} \left( \frac{x-1}{N+1} - 1 \right).
\]
Extraction of coefficients give
\[
\sum_{i=0}^{N} (n, m)_i [x^n y^m] \sum_{i=0}^{N} f_i(x, y) = [x^n y^{m+1}] \prod_{j=1}^{N+1} \frac{1 - x^j}{1 - x + y}.
\]
Proceeding as in the extraction of coefficients of \( f_i(x, y) \) leads to the result. \( \Box \)

Finally, we derive the complete generating function.

**Proposition 12** The complete generating function \( f(x, y, z) = \sum_{n,m,i \geq 0} (n, m)_i x^n y^m z^i \) is given by a hypergeometric function:
\[
f(x, y, z) = 2F1(1-x, 1, 2-x-y; z).
\]

**Proof** We have
\[
f(x, y, z) = \sum_{i \geq 0} f_i(x, y) z^i = \frac{1}{1-x-y} \sum_{i \geq 0} \frac{(1-x)^{1-i} \cdot 1^{1-i} z^i}{(2-x-y)^{1-i} i!},
\]
which is the stated Gauss hypergeometric series. \( \Box \)

### 4 Extensions

A natural question is if our results for the (normalized) logarithm integrals \( S_{n,m}^{(i)} \) can be extended to different families of integrals. For example, one can replace the logarithms \( \log^m(1-x) \) by \( \log^m(1+x) \), leading to alternating infinite sums. On the other, the range of the parameter \( i \) can be extended to negative \( i \) in a certain range. In the following we discuss first the logarithmic integrals with negative \( i \). Then, it turns out, that there is a natural generalization of \( S_{n,m}^{(i)} \), closely relation to a well known
special function [28,33]: Nielsen’s polylogarithm \( S_{n,m}(z) \) is defined here\(^2\) by

\[
S_{n,m}(z) = \frac{(-1)^{n+m}}{n!m!} \int_0^1 \frac{\log^n(x) \log^m(1-zx)}{x} \, dx.
\]

Setting \( z = -1 \) leads to alternating multiple zeta values. For \( z = 1 \) we obtain the special instance \( i = -1 \) of the logarithmic integrals analyzed before:

\[
S_{n,m}(1) = S_{n,m}^{(-1)}.
\]

Let \( L_{i_1,...,i_k}(z) \) denote the multiple polylogarithm function

\[
L_{i_1,...,i_k}(z) = \sum_{\ell_1 > \cdots > \ell_k \geq 1} \frac{z^{\ell_1}}{\ell_1^{i_1} \cdots \ell_k^{i_k}}.
\]

It is known that \( S_{n,m}(z) \) is a special multiple polylogarithm function: \( S_{n,m}(z) = L_{n+2,(1)_{m-1}}(z) \), such that \( S_{n,1}(z) = L_{n+2}(z) \) is the ordinary polylogarithm function.

### 4.1 Logarithmic integrals and negative \( i \)

In the following we look at the logarithmic integrals \( S_{n,m}^{(-i)} \) with \( 2 \leq i \leq m \).

**Lemma 6** (Boundary values - case \( n = 0 \)) The logarithmic integrals \( S_{0,m}^{(-i)} \) with \( 2 \leq i \leq m \) are given by

\[
S_{0,m}^{(-i)} = \frac{1}{(i-1)!} \sum_{r=0}^{i-1} \binom{i-1}{r} (i-2)^{i-1-r} \times \sum_{j=0}^r (-1)^{r-j} (r-1)! \zeta_{r-1}([1]_{j-1}) \zeta(m+1-j).
\]

**Proof** We use the substitution \( x = 1 - u \) to obtain

\[
S_{0,m}^{(-i)} = \frac{(-1)^m}{m!} \int_0^1 \frac{\log^m(u)}{(1-u)^i} \, du.
\]

---

\(^2\) In the standard convention, the number \( n \) has to be replaced by \( n - 1 \) on the right hand side of the definition. We opted to shift by one to be more consistent with our earlier definition.
Expansion of $1/(1-u)^i$ into a power series around $u = 0$ gives

$$S_{0,m}^{(-i)} = \frac{(-1)^m}{m!} \sum_{\ell \geq 0} \binom{i-1 + \ell}{i-1} \frac{u^\ell \log^m(u)}{\ell!} du.$$  

The last integral is readily evaluated by our previous result for $S_{0,m}^{(i)}$ in Lemma 3. Thus, we obtain

$$S_{0,m}^{(-i)} = \sum_{\ell \geq 0} \binom{i-1 + \ell}{i-1} \frac{1}{(\ell+1)^{m+1}}.$$  

We use the binomial theorem for the falling factorials

$$\binom{i-1 + \ell}{i-1} = \frac{(i-1 + \ell)!}{(i-1)!} = \frac{((\ell + 1 + i - 2)!}{(i-1)!}$$

$$= \frac{1}{(i-1)!} \sum_{r=0}^{i-1} \binom{i-1}{r} (\ell + 1)\zeta(r-1)i-2} \frac{i-1-r}{r}.$$

Then, we convert the falling factorials into ordinary powers using the Stirling numbers of the first kind, or in other words the truncated multiple zeta values of Lemma 3:

$$(\ell + 1)^x = \sum_{j=0}^{r} (-1)^{r-j}(r-1)!\zeta_{r-1}([\{1\}_{j-1}])(\ell + 1)^j.$$  

Finally, we get the stated expression by changing summations. \hfill \square

Next we state a recurrence relation similar to Proposition 4.

**Proposition 13** The logarithmic integrals $S_{n,m}^{(-i)}$ with $2 \leq i \leq m$ satisfy the recurrence relation

$$S_{n,m}^{(-i)} = \frac{1}{i-1} \left( \sum_{j=2}^{i-1} S_{n,m-1}^{(-j)} - S_{n-1,m}^{(-i)} + \zeta(n+1,\{1\}_{m-1}) + \zeta(n+2,\{1\}_{m-2}) \right).$$  

with initial values given by $S_{0,m}^{(-i)}$ as given in Lemma 6.

The proof is analogous to the result for positive $i$ and omitted. The result of Corollary 2 follows directly using induction and the result of Borwein, Bradley and Broadhurst [3, Eq. (10)], as mentioned in the introduction.
We generalize Nielsen’s polylogarithm to include a power of \( x \),

\[
S_{n,m}^{(i)}(z) = \frac{(-1)^{n+m}}{n!m!} \int_0^1 x^i \log^n(x) \log^m(1-zx) \, dx,
\]

with \( n, m \geq 0 \) and \( i \geq -1 \). For \( i = -1 \) we reobtain the ordinary Nielsen polylogarithm: \( S_{n,m}^{(-1)}(z) = S_{n,m}(z) \). For \( z = 1 \) we get the logarithmic integrals: \( S_{n,m}^{(i)}(1) = S_{n,m}^{(i)} \). The special case \( z = 0 \) leads to \( S_{n,m}(0) = S_{n,0}^{(i)} \), which was collected before in Lemma 3. Thus, we assume in the following that \( z \neq 0 \).

It will turn out that a structurally similar recurrence relation to the logarithmic integrals \( S_{n,m}^{(i)} \) also holds for \( S_{n,m}(z) \). Interestingly, it turns out that the case \( z = 1 \) treated before is very special. Only for \( z = 1 \) the generalized symmetry relations in Propositions 1 and 7 hold.

First, we turn to the boundary values. Obviously, we have \( S_{n,0}(z) = S_{n,0}^{(i)} \). We turn to the remaining case of \( n = 0 \).

**Lemma 7** (Boundary values - case \( n = 0 \)) For \( z \neq 0 \) the generalized Nielsen polylogarithms \( S_{0,m}^{(i)}(z) \) with \( i \geq 0 \) are given by

\[
S_{0,m}^{(i)}(z) = \frac{(-1)^m}{m!z^i} \sum_{j=0}^i \binom{i}{j} (-1)^j \left[ (1-z)^{j+1} \sum_{\ell=0}^{m-1} (-1)^\ell+1 \cdot \frac{m^\ell}{z(j+1)^{\ell+1}} \log^{m-\ell}(1-z) \right. \\
+ (-1)^{m+1} \frac{m!}{z(j+1)^{m+1}} (1-z)^{j+1} - 1 \bigg].
\]

**Proof** We write \( x = (1 - (1 - zx))/z \) and use the binomial theorem to get

\[
S_{0,m}^{(i)}(z) = \frac{(-1)^m}{m!z^i} \sum_{j=0}^i \binom{i}{j} (-1)^j \int_0^1 (1-zx)^j \log^m(1-zx) \, dx.
\]

We evaluate the remaining integrals by establishing a recurrence relation. For \( m \geq 1 \):

\[
\int_0^1 (1-zx)^j \log^m(1-zx) \, dx = - \frac{(1-z)^{j+1}}{(j+1)z} \log^m(1-z) \\
- \frac{m}{j+1} \int_0^1 (1-zx)^j \log^{m-1}(1-zx) \, dx.
\]

This implies that

\[
\int_0^1 (1-zx)^j \log^m(1-zx) \, dx = (1-z)^{j+1} \sum_{\ell=0}^{m-1} \frac{(-1)^\ell+1 \cdot m^\ell}{z(j+1)^{\ell+1}} \log^{m-\ell}(1-z)
\]
Next we state an expression similar to Proposition 2.

**Proposition 14** The generalized Nielsen’s polylogarithms $S_{n,m}^{(i)}(z)$ satisfy

$$S_{n,m}^{(i)}(z) = \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell(\ell + i + 1)^{n+1}} \cdot \zeta_{\ell-1}([1]_{m-1}).$$

The proof is identical to the proof of Proposition 2 and left to the reader.

Finally, we turn to a recurrence relation in the style of Propositions 4 and 13.

**Proposition 15** For non-zero $z$ and $n, m \geq 1$ the generalized Nielsen’s polylogarithms $S_{n,m}^{(i)}(z)$ satisfy the recurrence relation

$$S_{n,m}^{(i)}(z) = \frac{1}{i+1} \left[ S_{n-1,m}^{(i)}(z) + \sum_{j=0}^{i} \frac{1}{z^{i-j}} S_{n,m-1}^{(j)}(z) - \frac{1}{z^{i+1}} L_{n+1,(i)_{m-1}}(z) \right],$$

with initial values $S_{n,0}^{(i)}(z) = S_{n,0}^{(i)}$ as given in Lemma 3 and initial values $S_{0,m}^{(i)}(z)$ as stated in Proposition 7.

**Remark 4** We note that the boundary values $S_{0,m}^{(i)}(z)$ also satisfy for $m \geq 1$ a recurrence relation of this form:

$$S_{0,m}^{(i)}(z) = \frac{1}{i+1} \left[ L_{(i)_{m}}(z) + \sum_{j=0}^{i} \frac{1}{z^{i-j}} S_{0,m-1}^{(j)}(z) - \frac{1}{z^{i+1}} L_{(i)_{m}}(z) \right].$$

**Proof** We follow closely the proofs of Lemma 3 and Proposition 4. Our starting point is Proposition 14 stated before. We obtain the recurrence relation in two different ways: first, by partial fraction decomposition and second, by the recurrence relation for $\zeta_{\ell-1}([1]_{m-1})$. Introducing generalized T-values (5) with $i \geq -1$ and $n, m \geq 0$,

$$T_{n,m}^{(i)}(z) = \sum_{\ell=1}^{\infty} \frac{z^\ell}{(\ell + i + 1)^{n+1}} \zeta_{\ell-1}([1]_{m-1}),$$

with initial values

$$T_{n,m}^{(-1)}(z) = L_{n+1,(1)_{m-1}}(z),$$

we observe that by partial fraction decomposition

$$S_{n,m}^{(i)}(z) = \frac{1}{i+1} \left[ S_{n-1,m}^{(i)} - T_{n,m}^{(i)}(z) \right].$$
\( n \geq 0 \) and \( m \geq 1 \). By the recurrence relation for the truncated zeta values we get

\[
S_{n,m}^{(i)}(z) = \frac{1}{z} T_{n,m+1}^{(i-1)}(z) - T_{n,m+1}^{(i)}(z), \quad i \geq 0.
\]

Thus,

\[
T_{n,m+1}^{(i)}(z) = -\sum_{j=0}^{i} \frac{1}{z^{i-j}} S_{n,m}^{(i)}(z) + \frac{1}{z^{i+1}} T_{n,m+1}^{(-1)}(z).
\]

This leads to the stated result. \( \square \)

5 Applications—moments of the quicksort limit law

5.1 Quicksort algorithm

Quicksort is a famous sorting algorithm invented by Hoare [14]. It has been analyzed in a great many papers under the so-called uniform random model. This means that the input is a random permutation of size \( n \). One of the most popular cost measures is the number of comparison \( C_n \) required to sort a list of length \( n \). Under the uniform random model the number \( C_n \) becomes a random variable, satisfying the stochastic recurrence relation

\[
C_n \overset{(d)}{=} C_{i_n} + C_{n-i_n}^* + n - 1,
\]

\( n \geq 2 \) with initial values \( C_0 = 0 \) and \( C_1 = 0 \). Here, the random variables \( C_{n}^* \) denote independent copies of the \( C_n \), and the random variable \( I_n \) is a discrete uniformly distributed random variable on the set \( [n] = \{1, \ldots, n\} \), independent of the \( C \) and \( C^* \).

The expected value and the variance are readily obtained by taking expectations, and go back to Knuth [25]:

\[
E(C_n) = 2(n+1)H_n - 4n,
\]

\[
V(C_n) \sim (7 - 4\zeta(2))n^2 = \left(7 - \frac{2}{3}\pi^2\right)n^2.
\]

See Zeilberger [5] for a modern computer algebra approach on exact and asymptotic expressions for the moments.

Régnier [36] used martingale theory to prove that the normalized and centered random variable

\[
Z_n := \frac{C_n - E(C_n)}{n+1}
\]

converges to a non-degenerate limit \( Z_\infty \), both almost surely and in \( L_p \) for all \( 1 \leq p < \infty \). Hennequin [12,13] calculated all the moments of \( Z_\infty \) and characterized
its cumulants, also for a great many variants of the quicksort algorithm. Rösler [37] constructed a random variable satisfying a distributional equation

\[ Z^{(d)} = U \cdot Z_1 + (1 - U) \cdot Z_2 + C(U), \]

where the random variables \( Z_1, Z_2 \) are independent copies of \( Z \), \( U \) is a standard uniformly distributed random variable, independent of the \( Z \) variables, and the toll function \( C(x) \) is given by the entropy function

\[ C(x) = 1 + 2x \ln(x) + 2(1 - x) \ln(1 - x). \]

Here the random variable \( Z \) has the same distribution as \( Z_{\infty} \); Bindjeme and Fill [2] showed that \( Z_{\alpha, s} = Z_{\infty} \). The moments of \( C_n \), as well as many related stochastic recurrence relations, have been studied by Hwang and Neininger [22]. For the sake of completeness, we mention that the limit law of \( C_n \) has been refined by Neininger [32] (see also Fuchs [9], Grübel and Kabluchko [11] and Sulzbach [39]):

\[ \sqrt{n} \, n^{1/2} \left( Z_n - Z_{\infty} \right) \rightarrow N(0, 1). \]

Properties of the Quicksort limit \( Z_{\infty} \) are still of interested, see for example the recent works of Janson [23] and also Fill and Hung [6, 7].

Our goal is to obtain additional structural information about the limit law \( Z = Z_{\infty} \). We complement Hennequin’s recurrence relation for the cumulants [13] by adding a new one involving the binomial coefficients \( \binom{n}{m} \) of tier \( i \).

By taking the \( s \)th power, \( s \geq 1 \), we have

\[ Z^s = \sum_{k_1 + k_2 + k_3 = s} \binom{s}{k_1, k_2, k_3} U^{k_1}(1 - U)^{k_2} C^{k_3}(U) \cdot Z_1^{k_1} \cdot Z_2^{k_2}. \]

Let \( \mu_s = \mathbb{E}(Z^s) \). We have \( \mu_0 = 1 \), \( \mu_1 = 0 \) and the recurrence relation

\[ \mu_s = \sum_{k_1 + k_2 + k_3 = s} \binom{s}{k_1, k_2, k_3} \mu_{k_1} \mu_{k_2} \int_0^1 x^{k_1}(1 - x)^{k_2} C^{k_3}(x) \, dx, \]

such that

\[ \mu_s = \frac{s + 1}{s - 1} \sum_{k_1 + k_2 + k_3 = s} \binom{s}{k_1, k_2, k_3} \mu_{k_1} \mu_{k_2} \int_0^1 x^{k_1}(1 - x)^{k_2} C^{k_3}(x) \, dx, \quad s > 1. \]

By our previous results on \( I_{n, m}^{(i)} \) we reobtain the following result of Hennequin [13] directly from the distributional equation of \( Z \).
Theorem 4  The moments of the quicksort limit law $Z_\infty$ are rational polynomials in the ordinary zeta values.

Proof  We use induction with respect to $s$. The statement is true for $s = 1, 2$: $\mu_1 = 0$ and $\mu_2 = 7 - 4\zeta(2)$. Assuming that the statement holds for $1 \leq k \leq s - 1$ the recurrence relation (17) leads to the result, since $\int_0^1 x^{k_1} (1 - x)^{k_2} C_{k_3}(x) \, dx$ can be decomposed a sum of logarithmic integrals $I_{n, m}^{(i)}$, which are by Corollary 1 always rational polynomials in the zeta values. \hfill $\Box$

Let $c_s = CT(\mu_s)$ denote the constant term in the expression of $\mu_s$ in terms of rational polynomials in zeta values. We obtain the following recurrence relation for the sequence $(c_s)_{s \in \mathbb{N}}$.

Proposition 16 (Constant term - Quicksort limit law) The sequence $c_s$ satisfies $c_0 = 1$, $c_1 = 0$ and for $s \geq 1$

$$c_s = \frac{s + 1}{s - 1} \sum_{k_1, k_2 < s \atop k_1 + k_2 + k_3 = s} \binom{s}{k_1, k_2, k_3} c_{k_1} c_{k_2} \sum_{n + m + p = k_3} \binom{k_3}{n, m, p} \frac{(m + k_2)}{s! 2^m} (1 - 1)^j 2^n m! n! (n, m)_{n + k_1 + j},$$

with $(n, m)_i$ denote the binomial coefficients of tier $i$.

Remark 5 (Normalized constant terms) The recurrence relation for $c_s$ suggests to look at the normalized sequence $\tilde{c}_s$, defined by

$$\tilde{c}_s = (-1)^s \frac{c_s}{s! 2^s},$$

with initial values $\tilde{c}_0 = c_0 = 1$ and $\tilde{c}_1 = c_1 = 0$. The recurrence relation takes a particularly simple form:

$$\tilde{c}_s = \frac{s + 1}{s - 1} \sum_{k_1, k_2 < s \atop k_1 + k_2 + k_3 = s} \tilde{c}_{k_1} \tilde{c}_{k_2} \sum_{n + m + p = k_3} \frac{(-1)^p}{p! 2^p} \binom{m + k_2}{j} \cdot (1 - 1)^j (n, m)_{n + k_1 + j}.$$ 

Proof By (2) we have $CT(\zeta(a + 1, \{1\}_b)) = 0$. Consequently, Theorem 1 implies that $CT(S_{n, m}^{(i)}) = (n, m)_i$. We expand $C_{k_3}(x)$ using the multinomial theorem, and then expand again $(1 - x)^{m + k_2}$ using the binomial theorem. Application of the constant term operator to (17) gives the stated result. \hfill $\Box$
5.2 Cumulants

The cumulants of a random variable $Z$ are given by the expansion of the logarithm of the moment generating function

$$M(t) = \mathbb{E}(e^{Zt}) = \sum_{s \geq 0} \frac{\mu_s}{s!} t^s, \quad K(t) = \log(M(t)) = \sum_{s \geq 1} \frac{\kappa_s}{s!} t^s.$$  

The $s$th cumulant is homogeneous of degree $s$; furthermore cumulants of order greater than one are shift invariant. Due to the relation

$$M(t) = \exp(K(t)),$$

the cumulants are related to the ordinary moments by the complete Bell polynomials

$$\mu_s = B_s(\kappa_1, \ldots, \kappa_s).$$

Likewise, the cumulants are given in terms of the moments as

$$\kappa_s = \sum_{j=1}^{s} (-1)^{j-1}(j-1)!B_{s,j}(\mu_1, \ldots, \mu_{s-j+1}).$$

Here, the $B_{s,j}(\mu_1, \ldots, \mu_{s-j+1})$ denote the partial or incomplete Bell polynomials.

**Example 5** (Gumbel distribution - cumulants) The Gumbel distribution is an extreme value distribution with density function

$$f(x) = e^{-x} - e^{-x}, \quad x \in \mathbb{R}.$$  

Its expected value $\mathbb{E}(X) = \gamma \approx 0.5772 \ldots$ is given by the Euler-Mascheroni constant. The cumulants have a particularly appealing form: $\kappa_1(X) = \mathbb{E}(X) = \gamma$, $\kappa_s(X) = (s-1)!\zeta(s)$ for $s > 1$. Consequently, the centered and scaled Gumbel-distributed random variable $G = -2(X - \gamma)$ satisfies

$$\kappa_1(G) = 0, \quad \kappa_s(G) = (-1)^s \cdot 2^s \cdot (s-1)!\zeta(s), \quad s > 1.$$  

Hennequin [13] obtained the cumulants of the quicksort limit law.

**Theorem 5** (Cumulants of Quicksort limit) The cumulants $\kappa_s = \kappa_s(Z)$ of the limit law $Z$ of the Quicksort comparisons satisfy $\kappa_1 = 0$ and

$$\kappa_s = a_s + (-1)^{s+1} \cdot 2^s \cdot (s-1)!\zeta(s), \quad s \geq 2.$$  

Here the rational values $a_s$ are determined by a certain recurrence relation [13].
Let \( Z^* = Z + G \) denote the shifted limit law of the Quicksort comparisons, with \( G = -2(X - \gamma) \), \( X \) a Gumbel-distributed random variable, independent of \( Z \). By the properties of the cumulants of independent random variables

\[
\kappa_s(Z^*) = \kappa_s(Z + G) = \kappa_s(Z) + \kappa_s(G) = \alpha_s \in \mathbb{Q}
\]

with \( \kappa_1(Z^*) = 0 \), \( \kappa_2(Z^*) = 7 \) and so on.

Equivalently, in terms of moments, let \( \mu_s^* = \mathbb{E}((Z^*)^s) \). Then,

\[
\mu_s^* = \sum_{j=0}^{s} \binom{s}{j} \mu_{j,Z} \cdot \mu_{s-j,G} \in \mathbb{Q},
\]

with \( \mu_j = \mu_{j,Z} \) given by (17) and

\[
\mu_{s,G} = B_s(0, 2^2 \cdot 1! \zeta(2), \ldots, (-1)^s 2^s \cdot (s - 1)! \zeta(s)).
\]

We note first that the moments \( \mu_{s,G} \) are rational polynomials in the zetas with constant term given by the Kronecker delta: \( \text{CT}(\mu_{s,G}) = \delta_{0,s} \). Hence, the rational number \( \mu_s^* \) is given by

\[
\mu_s^* = \text{CT}\left(\sum_{j=0}^{s} \binom{s}{j} \mu_{j,Z} \cdot \mu_{s-j,G}\right) = \text{CT}(\mu_{s,Z}) = \mathcal{C}_s.
\]

This constant term may not be unique, since we do not know that rational polynomials in the zeta values are uniquely determined. In fact, not much is known about the odd zeta values \( \zeta(2m+1) \), \( m \in \mathbb{N} \), and their linear dependence over \( \mathbb{Q} \).

Proposition 17 (Cumulants of the shifted Quicksort limit) Let \( Z^* = Z + G \) denote the shifted limit law of the Quicksort comparisons, with \( G = -2(X - \gamma) \), \( X \) a Gumbel-distributed random variable, independent of \( Z \). Then, \( \alpha_s = \kappa_s(Z^*) \in \mathbb{Q} \).

On the other hand, a possibly different rational number \( \hat{\alpha}_s \) is given by

\[
\hat{\alpha}_s = \sum_{j=1}^{s} (-1)^{j-1} (j - 1)! B_{s,j}(\mathcal{C}_1, \ldots, \mathcal{C}_{s-j+1}),
\]

where the constant terms \( \mathcal{C}_s \) are stated in Proposition 16, involving the binomial coefficients of tier \( i \).

One can check numerically that \( \alpha_s = \hat{\alpha}_s \) for small values of \( s \). A possible way of proving the actual equality \( \alpha_s = \hat{\alpha}_s \) would be to study in more detail the involved recurrence relation for \( \alpha_s \), given by Hennequin [13].
6 Conclusion and acknowledgments

We studied the logarithmic integral $I^{(i)}_{n,m} := \int_{0}^{1} x^i \cdot \ln^n(x) \cdot \ln^m(1-x) \, dx$ and the normalized values $S^{(i)}_{n,m}$ for $i \geq -1$. We determined a recurrence relation for $S^{(i)}_{n,m}$, leading to an expansion into elements $\zeta(n + 1, \{1\}_{m-1})$. Additionally, this gave an evaluation of $S^{(i)}_{n,m}$ into ordinary zeta values, involving binomial coefficients $(n, m)_i$ of tier $i$. Various expressions and properties of $(n, m)_i$ are established. We also considered variants and extensions of the logarithmic integrals, providing recurrence relations, which may serve as a starting point for expansions similar to the one provided here in our main theorem. As an application of our results, we revisited the moments and cumulants of the number of comparisons in quicksort, relating them to binomial coefficients $(n, m)_i$ of tier $i$. An open problem is to show that the cumulants $\alpha_s$ of the shifted quicksort limit law coincide with the corresponding values $\hat{\alpha}_s$, obtained from the constant term in the representation of the moments as polynomials in the zeta values.

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The authors can confirm that all relevant data are included in the article.

References

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions, U. S. Government Printing Office, Washington, DC (1964). reprinted by Dover, New York, 1972
2. Bindjeme, P., Fill, J.A.: Exact $L^2$-distance from the Limit for QuickSort Key Comparisons (Extended Abstract). In: Broutin, Nicolas, Devroye, Luc (eds) DMTCS Proceedings: 23rd International Meeting on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA’12), pp. 339–348 (2012)
3. Borwein, J.M., Bradley, D.M., Broadhurst, D.J.: Evaluation of k-fold Euler/Zagier sums: a compendium of results for arbitrary k. Electron. J. Combin. 4(2) (1997). res. art. 5 (21 pp.)
4. Dilcher, K.: Some q-series identities related to divisor functions. Discrete Math. 145, 83–93 (1995)
5. Ekhad, S.B., Zeilberger, D.: A Detailed Analysis of Quicksort Running Time. Personal Journal of Shalosh B. Ekhad and Doron Zeilberger (2019)
6. Fill, J.A., Hung, W.-C.: On the tails of the limiting QuickSort density. Electron. Commun. Probab. 24 (2019). paper no. 7, 11 pages
7. Fill, J.A., Hung, W.-C.: QuickSort: improved right-tail asymptotics for the limiting distribution, and large deviations. In: 2019 Proceedings of the Meeting on Analytic Algorithmics and Combinatorics (ANALCO) (2019)
8. Flajolet, P., Sedgewick, R.: Mellin transforms and asymptotics: Finite differences and Rice’s integrals. Theoretical Computer Science 144, 101–124 (1995)
9. Fuchs, M.: A note on the quicksort asymptotics. Random Structures and Algorithms 46(4), 677–687 (2015)
10. Graham, R.L., Knuth, D.E., Patashnik, O.: Concrete Mathematics, 2nd edn. Addison-Wesley, New York (1994)
11. Grübel, R., Kabluchko, Z.: A functional central limit theorem for branching random walks, almost sure weak convergence, and applications to random trees. Annals of Applied Probability 26(6), 3659–3698 (2016)
12. Hennequin, P.: Combinatorial analysis of quicksort algorithm, Informatique théorique et applications. tome 23(3), 317–333 (1989)
13. Hennequin, P.: Analyse en Moyenne d’Algorithmes, Tri Rapide at Arbres de Recherche, Ph.D. Thesis, Ecole Politechnique, Palaiseau (1991)
14. Hoare, C.A.R.: Quicksort. Comput. J. 5, 10–15 (1962)
15. Hoffman, M.E.: Multiple harmonic series. Pacific J. Math. 152, 275–290 (1992)
16. Hoffman, M.E.: Quasi-shuffle products. Journal of Algebraic Combinatorics 11, 49–68 (2000)
17. Hoffman, M.E.: Algebraic aspects of multiple zeta values. In: Aoki, T., et al. (eds.) Zeta Functions, Topology and Quantum Physics (Developments in Mathematics vol. 14), pp. 51–74. Springer, New York (2005)
18. Hoffman, M.E.: Harmonic-number summation identities, symmetric functions, and multiple zeta values. Ramanujan J. 42, 501–515 (2017)
19. Hoffman, M.E.: Quasi-shuffle algebras and applications. In: Chapoton, F., et al. (eds.) Algebraic Combinatorics, Resurgence, Moulds and Applications, Vol. 2 (IRMA Lectures in Mathematics and Theoretical Physics vol. 32), pp. 327–348. European Math. Soc. Publ. House, Berlin (2020)
20. Hoffman, M.E.: An odd variant of multiple zeta values. Commun. Number Theory Phys 13, 529–567 (2019). arXiv:1612.05232
21. Hoffman, M.E., Kuba, M., Levy, M., Louchard, G.: An Asymptotic Series for an Integral. Ramanujan Journal 53, 1–25 (2020). arXiv:1802.09214
22. Hwang, H.K., Neininger, R.: Phase change of limit laws in the quicksort recurrence under varying toll functions. SIAM Journal on Computing 31(6), 1687–1722 (2002)
23. Janson, S.: On the tails of the limiting Quicksort distribution. Electron. Commun. Probab. 20(81), 7 pages (2015)
24. Kaneko, M., Sakata, M.: On multiple zeta values of extremal height. Bull. Aust. Math. Soc. 93, 186–193 (2016)
25. Knuth, D.E.: The Art of Computer Programming, Volume 3: Sorting and Searching. Addison-Wesley, (1973)
26. Kölbig, K.S.: Closed expressions for $\int_0^1 t^{-1} \log^{n-1} t \log^p (1 - t) \, dt$. Math. Comp. 39, 647–654 (1982)
27. Kölbig, K.S.: Explicit evaluation of certain definite integrals involving powers of logarithms. J. Symbolic Computation 1, 109–114 (1985)
28. Kölbig, K.S.: Nielsen’s generalized polylogarithms. SIAM J. Math. Anal. 17, 1232–1258 (1986)
29. Knuth, D.E.: The Art of Computer Programming, vol. 3. Addison-Wesley, Reading, MA (1973)
30. Kuba, M., Panholzer, A.: A Note on Harmonic number identities. Stirling series and multiple zeta values, International journal of number theory 15(07), 1323–1348 (2019)
31. Laurenzi, B.J.: Logarithmic integrals, polylogarithmic integrals and Euler sums, preprint. arXiv:1010.6229
32. Neininger, R.: Refined Quicksort asymptotics. Random Structures and Algorithms 46, 346–361 (2015)
33. Nielsen, N.: Der Eulersche Dilogarithmus und seine Verallgemeinerungen. Nova Acta Leopoldina 90(3), 123–211 (1909)
34. Petkovsek, M., Wilf, H.S., Zeilberger, D.: A=B. AK Peters. Wellesley, Mass (1996)
35. Prodinger, H.: Some information about the binomial transform. The Fibonacci Quarterly 32, 412–415 (1994)
36. Régnier, M.: A limiting distribution for Quicksort. RAIRO Inform. Théor. Appl. 23, 335–343 (1989)
37. Rösler, U.: A limit theorem for “Quicksort”. RAIRO Inform. Théor. Appl. 25, 85–100 (1991)
38. Sloane, N.J.A.: The On-Line Encyclopedia of Integer Sequences (OEIS) (2019). https://oeis.org
39. Sulzbach, H.: On martingale tail sums for the path length in random trees. Random Structures & Algorithms 50, 493–508 (2017)
40. Xu, C.: Multiple zeta values and Euler sums. J. Number Theory 177, 443–478 (2017)
41. Zudilin, W.: Algebraic relations for multiple zeta values. Russian Math. Surveys 58(1), 1–29 (2003)
42. Zagier, D.: Values of zeta functions and their applications. In: Joseph, A., et al. (eds.) First European Congress of Mathematics (Paris, 1992), vol. II. Progr. Math., vol. 120, pp. 497–512. Birkhäuser, Boston (1994)

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