On the convergence analysis of DCA

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Abstract In this paper, we propose a clean and general proof framework to establish the convergence analysis of the Difference-of-Convex (DC) programming algorithm (DCA) for both standard DC program and convex constrained DC program. We first discuss suitable assumptions for the well-definiteness of DCA. Then, we focus on the convergence analysis of DCA, in particular, the global convergence of the sequence \( \{x^k\} \) generated by DCA under the Lojasiewicz subgradient inequality and the Kurdyka-Lojasiewicz property respectively. Moreover, the convergence rate for the sequences \( \{f(x^k)\} \) and \( \{\|x^k - x^*\|\} \) are also investigated. We hope that the proof framework presented in this article will be a useful tool to conveniently establish the convergence analysis for many variants of DCA and new DCA-type algorithms.

Keywords Convergence analysis · DC program · DCA · Lojasiewicz subgradient inequality · Kurdyka-Lojasiewicz property

Mathematics Subject Classification (2020) 90C26 · 90C30 · 65K05

1 Introduction

Consider the standard Difference-of-Convex (DC) program defined by

\[
\inf \{ f(x) = g(x) - h(x) : x \in \mathbb{R}^n \}, \quad (P)
\]

under the convention \( \infty - \infty = \infty \) and the assumptions

Assumption A:

- \( g \) and \( h \) are functions of \( \mathcal{F}_0(\mathbb{R}^n) \) (the set of functions \( \mathbb{R}^n \rightarrow (-\infty, \infty] \) proper, closed (or lower semi-continuous) and convex);
- the solution set of \( (P) \) is non-empty (implying that \( \emptyset \neq \text{dom}g \subset \text{dom}h \)).

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Let $C \subset \mathbb{R}^n$ be nonempty closed and convex (could be identical to $\mathbb{R}^n$). The convex constrained DC program

$$\inf \{ g(x) - h(x) : x \in C \}. \quad (P_C)$$

can be formulated as a standard DC program as

$$\min \{ (g + \chi_C)(x) - h(x) : x \in \mathbb{R}^n \}, \quad (P')$$

where $\chi_C$ is the indicator function of $C$ defined by $\chi_C(x) = 1$ if $x \in C$ and $\infty$ otherwise. Clearly, both $g + \chi_C$ and $h$ belong to $\Gamma_0(\mathbb{R}^n)$.

DC programming is an active field in nonconvex and nonsmooth optimization [6, 39, 7, 34, 8, 16, 31]. The most renowned solution algorithm for DC programming is DCA (described in Algorithm 1), introduced by Pham Dinh Tao in 1985 as an extension of the subgradient method [39], and extensively developed by Le Thi Hoai An and Pham Dinh Tao since 1994 (see e.g., [34, 35, 36, 16]). The basic idea of DCA applied to the standard DC program (P) is linearizing $h$ at the current iterate $x^k$ to obtain a global convex majorization (cf. surrogate) of $f$, which is minimized to get the next iterate $x^{k+1}$. This idea coincides with the Majorization-Minimization (MM) algorithm proposed by Hunter and Lange in [9], but DCA provides a specific way to create a convex surrogate of $f$ thanks to its DC structure. Due to the standard formulation (P') for the convex constrained DC program, the corresponding convex subproblem $(P_k)$ reads

$$x^{k+1} \in \arg\min \{ g(x) - \langle y^k, x \rangle : x \in \mathbb{R}^n \}. \quad (P_k)$$

DCA has proven to be a promising approach in many large-scale real-world applications, such as sparsity learning [14], clustering [12], molecular conformation [15], portfolio optimization [38, 37, 28], bilinear matrix inequality [26], natural language processing [30], image denoising [45], trust region subproblem [35], mixed-integer optimization [11, 25, 22, 29] and eigenvalue complementarity problems [14, 27, 23, 24], to name a few. There are several variants of DCA including the proximal DCA and the linearized proximal DCA [42, 33, 43, 20], the Boosted DCA [1, 28], the inertial DCA [32, 45], the ADCA [40] and the Stochastic DCA [13]. These methods incorporate with several optimization techniques (e.g., regularization via proximal term, prox-linearization, backtracking line search, Nesterov’s extrapolation, and stochastic approximation) to enhance the overall performance of DCA for structured DC optimization problems. Note that some classical convex/nonconvex optimization algorithms can be viewed as DCA with a special DC decomposition,

Algorithm 1 DCA for problem (P)

| Input: | Initial point $x^0 \in \text{dom} \partial h$; |
|-------|--------------------------------|
| 1:    | for $k = 0, 1, \ldots$, do |
| 2:    | Compute $y^k \in \partial h(x^k)$; |
| 3:    | Solve the convex subproblem: |
|       | $x^{k+1} \in \arg\min \{ g(x) - \langle y^k, x \rangle : x \in \mathbb{R}^n \}$. |
| 4:    | end for |
such as the Goldstein–Levitin–Polyak projection algorithm, the proximal point algorithm, the Expectation-Maximization algorithm, the concave–convex procedure, the iterative shrinkage-thresholding algorithm, and the forward-backward splitting algorithm. See [16] and the references therein for a comprehensive introduction about DC programming, DCA and applications.

DCA may not be well-defined if inappropriate DC decomposition is used. Assumption A is not enough to guarantee the well-definiteness of DCA, see Example 1. As far as we know, appropriate assumptions are not sufficiently discussed in existing works which motivates us to investigate proper assumptions for DC programming and DCA, which should not be too large to allow the use of the arsenal of powerful tools in convex and nonsmooth analysis for a theoretical guarantee of the well-definiteness and the convergence of DCA, but wide enough to cover most real-world nonconvex and nonsmooth optimization problems. Convergence analysis of DCA under appropriate assumptions within a general and extensible proof framework is very important. It is often observed in the literature that authors apply classical convergence results of DCA in their applications without appropriate assumptions, or duplicate existing proofs with minor modification to adopt the particular structure and additional assumptions in their DC formulations and DCA. This motivates us to propose a clean, general and extensible proof framework for establishing convergence analysis of DCA and its variants. In particular, the global convergence of the sequence \( \{x^k\} \) generated by DCA and many variants does not hold in general, which strongly depends on the nature of the DC decomposition and the choice of \( y^k \in \partial h(x^k) \), must also be adapted within the proof framework.

In this paper, we first discuss suitable assumptions for the well-definiteness of DCA in Section 2, and propose Assumption B to complete Assumption A in DC program for guaranteeing the well-definiteness of DCA. A counterexample reveals the importance of a suitable DC decomposition to the well-definiteness of DCA. Then, in Section 3, we focus on a comprehensive study of the convergence analysis of DCA within a clean and general proof framework for convex constrained DC program, where the standard DC program is considered as a special case by taking \( C = \mathbb{R}^n \). These results include non-increasing and convergence of \( \{f(x^k)\} \), sufficiently descent property, square summable property, \( O(1/\sqrt{N}) \) convergence rate, and subsequential convergence of \( \{x^k\} \) to DC critical points under some basic assumptions. Concerning the global (cf. sequential) convergence of the whole sequence \( \{x^k\} \), we propose a general theorem to guarantee the convergence of \( \{x^k\} \) by verifying three assumptions, namely Lyapunov assumption (H1), sufficiently descent assumption (H2), and regularity assumption (H3). Then, we demonstrate how DCA can generate a convergence sequence \( \{x^k\} \) verifying assumptions (H1)-(H3) under two regularity conditions, Lojasiewicz subgradient inequality and Kurdyka-Lojasiewicz property (a more general form of the Lojasiewicz subgradient inequality), respectively. Once the global convergence of \( \{x^k\} \) is guaranteed, the convergence rates of \( \{x^k\} \) and \( \{f(x^k)\} \) are established in Section 4 under the Kurdyka-Lojasiewicz property. We obtain three convergence rates (the finite convergence, the linear convergence and the sublinear convergence) depending on the Lojasiewicz exponent. Note that the major contribution of this article is the proposal of a general proof framework to establish the convergence analysis of DCA, which is hopefully to be a useful tool to easily analyze the convergence of some variants of DCA and new DCA-type algorithms.
2 Well-definiteness of DCA

The well-definiteness of DCA is a crucial question and highly depending on the intrinsic nature of the DC decomposition. Be caution that the assumption $\emptyset \neq \text{dom} g \subset \text{dom} h$ is not sufficient to guarantee the well-definiteness of DCA. Let’s start with a counterexample:

**Example 1** Consider the problem

$$\min_{x \in \mathbb{R}^+} \frac{x^2}{2} + \sqrt{x},$$

(1)

whose minimizer is $x^* = 0$. This problem is DC with a DC representation

$$\min_{x \in \mathbb{R}} \{ g(x) - h(x) \},$$

where

$$g(x) = \frac{x^2}{2} + \chi_{\mathbb{R}^+}(x) \quad \text{and} \quad h(x) = \begin{cases} -\sqrt{x}, & \text{if } x \geq 0, \\ \infty, & \text{otherwise}. \end{cases}$$

(2)

Obviously, **Assumption A** is verified, where both $g$ and $h$ belong to $\Gamma_0(\mathbb{R})$ with $\text{dom} g = \text{dom} h = \mathbb{R}^+$, $\partial h(0) = \emptyset$, and $\partial h(0) = \{0\}$. Applying DCA with $x^0 = 1$, we have $y^0 \in \partial h(x^0) = \{\partial h'(1)\} = \{-1/2\}$, then $x^1 = \arg\min \{ \frac{x^2}{2} + \chi_{\mathbb{R}^+}(x) - y^0 x \} = \emptyset$, which is not included in $\text{dom} \partial h$. Hence, $y^1$ can not be generated by DCA so that DCA is not well defined. This example demonstrates that even using a DC decomposition with $\emptyset \neq \text{dom} g \subset \text{dom} h$, DCA may not be well defined. Moreover, the minimizer $x^* = 0$ is not a DC critical point since $\partial h(0) = \emptyset$, so that the DC criticality defined by $\partial g(x^*) \cap \partial h(x^*) \neq \emptyset$ may not be a necessary optimality condition for (P).

**Remark 1** One may wonder if there is a DC representation for problem (1) in Example 1 such that DCA is well defined? The answer to this question is YES. Consider the DC representation with

$$g(x) = \begin{cases} \frac{x^2}{2} - \sqrt{x}, & \text{if } x \geq 0, \\ \infty, & \text{otherwise}. \end{cases} \quad \text{and} \quad h(x) = \begin{cases} -2\sqrt{x}, & \text{if } x \geq 0, \\ \infty, & \text{otherwise}. \end{cases}$$

(3)

Again **Assumption A** is verified and both $g$ and $h$ belong to $\Gamma_0(\mathbb{R})$ with $\text{dom} g = \text{dom} h = \mathbb{R}^+$, $\partial h(0) = \emptyset$, and $\partial h(0) = \emptyset$. Applying DCA with any point $x^k > 0$, we get $y^k \in \partial h(x^k) = \{-1/\sqrt{x^k}\}$, then solving $x^{k+1} = \arg\min \{ \frac{x^2}{2} - \sqrt{x} - y^k x : x \geq 0 \}$ by Cardano formula, we obtain the update formulation of $x^{k+1}$ as

$$x^{k+1} = \left( \sqrt{\frac{1}{4} + \Delta_k} + \sqrt{\frac{1}{4} - \Delta_k} \right)^2 > 0,$$

(4)

where $\Delta_k := \frac{1}{4} + \frac{1}{(\sqrt{x^k})^2} > 0$ whenever $x^k > 0$. One can easily check that the sequence $\{x^k\}$ generated by DCA (i.e., through (4)) from any initial point $x^0 > 0$ is well defined, decreasing and convergent to the minimizer $x^* = 0$. ☐
Example 1 indicates that we need more refined assumptions on the DC decomposition to guarantee the well-definiteness of DCA. A straightforward answer is: DCA is well defined if and only if, for all $k \in \mathbb{N}$, $x^k \in \text{dom} h$ and $(P_k)$ has an optimal solution. Next, we propose Assumption B to guarantee the well-definiteness of DCA.

**Assumption B:**
- $\emptyset \neq \text{dom} \partial g \subset \text{dom} \partial h$.
- all subproblems $(P_k)$ have optimal solutions (may not be unique).

**Proposition 1** Under Assumption A and Assumption B, DCA is well defined for $(P)$ from an initial point $x^0 \in \text{dom} h$.

**Proof** Given $x^k \in \text{dom} h \neq \emptyset$, the non-emptiness of the solution set of $(P_k)$ implies that $y^k \in \partial g(x^{k+1})$.

Hence $x^{k+1} \in \text{dom} g$. Then by the inclusion $\text{dom} \partial g \subset \text{dom} \partial h$, we get $x^{k+1} \in \text{dom} h$.

By induction, given $x^0 \in \text{dom} h$, we deduce that $x^k \in \text{dom} \partial h, \forall k \in \mathbb{N}$.

Together with the non-emptiness of the solution set of $(P_k)$, we prove the well-definiteness of DCA. \qed

We can see that Assumption B is not satisfied in the DC decomposition (2) since $\text{dom} \partial g = [0, \infty) \not\subset \text{dom} \partial h = (0, \infty)$; while the DC decomposition (3) satisfies Assumption B.

**Remark 2** The non-emptiness assumption of the solution set of $(P_k)$ in Assumption B is automatically verified in many cases such as $g$ is strongly convex, or $C$ is nonempty compact and convex.

**Remark 3** The non-emptiness assumption of the solution set of $(P)$ in Assumption A is often unknown and unverifiable in practice. Without this assumption, DCA may still be well defined. For example, let $g(x) = x^2/2, h(x) = e^x$. Then the solution set of $(P)$ is empty, whereas DCA starting from any initial point $x^0 \in \mathbb{R}$ generates a well-defined sequence $\{x^k\}$ by $x^{k+1} = e^{x^k}$ so that $x^k \to \infty$ and $f(x^k) \to -\infty$. There are some alternative assumptions to replace the non-emptiness of the solution set of $(P)$, such as $f$ is lower bounded and coercive (i.e., $f(x)/\|x\| \to \infty$ as $\|x\| \to \infty$) or level-bounded (i.e., for any $\alpha \in \mathbb{R}$, the level set $\{x : f(x) \leq \alpha\}$ is bounded). Note that there are two common mistakes in the literature:
- Only supposing the lower boundedness of $f$ to assert the non-emptiness of the solution set. For instance, let $g(x) = e^x, h(x) = 0$. Then $f$ is lower bounded with the optimal value 0, but the set of optimal solutions is empty. In this example, $f$ is neither coercive nor level-bounded, and DCA is not well defined because the subproblem $(P_k)$ (namely $\min \{e^x : x \in \mathbb{R}\}$) has no optimal solution.
- Only using the level-boundedness of $f$ to assert the non-emptiness of the solution set, or the lower boundedness of $f$, or the boundedness of the sequence $\{y^k\}$. For example, let $f(x) = \ln(x), \forall x \in (0, \infty)$ and $\infty$ otherwise. Then, $f$ is level-bounded, but the problem $\min \{f(x)\}$ has no solution and $f$ is not lower bounded on $\mathbb{R}$. Moreover, Example 1 with DC representation (3) shows that $f$ is level-bounded but $\{y^k\}$ is unbounded even $\{x^k\}$ is bounded.
3 Convergence analysis of DCA

In this section, we will focus on the convergence analysis of DCA for the convex constrained DC program $(P_C)$. The standard DC program $(P)$ is treated as a special case by taking $C = \mathbb{R}^n$.

Throughout the section, we suppose that the sequence $\{x^k\}$ generated by DCA starting from an initial point $x^0 \in \text{dom} \partial h$ for $(P_C)$ under Assumption A is well-defined.

**Lemma 1** (non-increasing of $\{f(x^k)\}$) The sequence $\{f(x^k)\}$ is non-increasing and convergent.

*Proof* For every $k = 0, 1, \ldots$, DCA generates the sequence $\{x^k\}$ as

$$x^{k+1} \in \text{argmin}\{g(x) - \langle y^k, x \rangle \mid x \in C\},$$

where $y^k \in \partial h(x^k)$. By the convexity of $h$ and $y^k \in \partial h(x^k)$, we have

$$h(x^{k+1}) \geq h(x^k) + \langle x^{k+1} - x^k, y^k \rangle.$$

Then, for every $k \in \mathbb{N}$, we have

$$f(x^{k+1}) = g(x^{k+1}) - h(x^{k+1})$$

$$\leq g(x^{k+1}) - \left( h(x^k) + \langle x^{k+1} - x^k, y^k \rangle \right)$$

$$\leq \min\{g(x) - \langle x, y^k \rangle : x \in \mathbb{C}\} - h(x^k) + \langle x^k, y^k \rangle$$

$$\leq g(x^k) - h(x^k) = f(x^k)$$

which implies that $\{f(x^k)\}$ is non-increasing. The non-emptiness of the solution set of $(P_C)$ implies that $f$ is lower bounded over $\mathbb{C}$. Then the convergence of $\{f(x^k)\}$ is followed by the non-increasing and the lower boundedness of $\{f(x^k)\}$. \qed

**Lemma 2** If either $g$ or $h$ is strongly convex over $\mathbb{C}$ (i.e., $\rho_g + \rho_h > 0$), then

- *(sufficiently descent property)*

$$f(x^k) - f(x^{k+1}) \geq \frac{\rho_g + \rho_h}{2} \|x^k - x^{k+1}\|^2, \forall k = 1, 2, \ldots.$$ (7)

- *(square summable property)*

$$\sum_{k \geq 0} \|x^{k+1} - x^k\|^2 < \infty,$$

hence $\|x^{k+1} - x^k\| \to 0$ as $k \to \infty$.

*Proof* (Sufficiently descent property): For every $k = 0, 1, \ldots$, by the first order optimality condition to the convex problem

$$x^{k+1} \in \text{argmin}\{g(x) - \langle y^k, x \rangle \mid x \in \mathbb{C}\}$$
where $y^k \in \partial h(x^k)$, we have
$$0 \in \partial g(x^{k+1}) + N_C(x^{k+1}) - y^k.$$ 

Thus
$$y^k \in \partial h(x^k) \cap (\partial g(x^{k+1}) + N_C(x^{k+1})).$$

By the $\rho_h$-convexity of $h$ and $y^k \in \partial h(x^k)$, then
$$h(x^{k+1}) \geq h(x^k) + \langle x^{k+1} - x^k, y^k \rangle + \frac{\rho_h}{2} \| x^{k+1} - x^k \|^2.$$ (8)

By the $\rho_g$-convexity of $g$, we have
$$g(x^k) \geq g(x^{k+1}) + \langle x^k - x^{k+1}, w \rangle + \frac{\rho_g}{2} \| x^{k+1} - x^k \|^2$$ (9)

for all $w \in \partial g(x^{k+1})$. Taking $v \in N_C(x^{k+1})$ and $w = y^k - v$, then (9) turns to
$$g(x^k) + \langle x^{k+1} - x^k, y^k - v \rangle - \frac{\rho_g}{2} \| x^{k+1} - x^k \|^2 \geq g(x^{k+1}).$$ (10)

By $v \in N_C(x^{k+1})$, then
$$\langle x - x^{k+1}, v \rangle \leq 0, \forall v \in C.$$ For every $k = 1, 2, \ldots$, we have $x^k \in C$. Taking $x = x^k$ in the above inequality, we have
$$\langle x^k - x^{k+1}, v \rangle \leq 0.$$ (11)

It follows that for all $k = 1, 2, \ldots$

$$f(x^{k+1}) = g(x^{k+1}) - h(x^{k+1})$$

$\leq g(x^{k+1}) - \left( h(x^k) + \langle x^{k+1} - x^k, y^k \rangle + \frac{\rho_h}{2} \| x^{k+1} - x^k \|^2 \right)$

$\leq g(x^k) - h(x^k) + \langle x^k - x^{k+1}, v \rangle - \frac{\rho_g + \rho_h}{2} \| x^{k+1} - x^k \|^2$

$\leq f(x^k) - \frac{\rho_g + \rho_h}{2} \| x^{k+1} - x^k \|^2,$

which leads to the required inequality.

(Square summable property): Suming (7) for $k$ from 1 to $N \geq 1$, we have
$$\sum_{k=1}^{N} \frac{\rho_g + \rho_h}{2} \| x^k - x^{k+1} \|^2 \leq \sum_{k=1}^{N} f(x^k) - f(x^{k+1})$$

$$= f(x^1) - f(x^{N+1})$$

$$\leq f(x^1) - \min \{ f(x) : x \in C \},$$

where the last inequality holds for any $N \geq 1$.

Taking $N \to \infty$, and by the lower boundedness of $f$ over $C$, then
$$\sum_{k \geq 1} \frac{\rho_g + \rho_h}{2} \| x^k - x^{k+1} \|^2 < \infty.$$

It follows immediately by $\rho_g + \rho_h > 0$ and the finiteness of $\| x^0 - x^1 \|$ that
$$\sum_{k \geq 0} \| x^k - x^{k+1} \|^2 < \infty,$$

and as a consequence, $\| x^k - x^{k+1} \| \to 0$ as $k \to \infty$. \hspace{1cm} \Box
Remark 4 \( \|x^k - x^{k+1}\| \to 0 \) can be also derived from the sufficiently descent property and the convergence of \( \{f(x^k)\} \) as

\[
0 \leq \frac{\rho_g + \rho_h}{2} \|x^k - x^{k+1}\| \leq f(x^k) - f(x^{k+1}) \xrightarrow{k \to \infty} 0.
\]

Remark 5 Without the assumption \( \rho_g + \rho_h > 0 \), the sequence \( \{\|x^k - x^{k+1}\|\} \) may not converge to 0. Consider the example:

Example 2 Let \( g(x) = \sup\{-x, 0, x - 1\} \), \( h(x) = \sup\{-x, 0\} \) and \( C = \mathbb{R} \). The functions \( g \) and \( h \) are piecewise linear and convex (cf. polyhedral convex), but neither strongly convex nor strictly convex. Starting DCA from an initial point \( x^0 \in (0, 1) \), we get \( \partial h(x^0) = \{0\} \), and \( x^1 \in \text{argmin}\{g(x)\} = [0, 1] \). Hence, DCA could generate a sequence \( \{x^k\} \subseteq (0, 1) \) such as \( \{x^k\} = \{0.1, 0.9, 0.1, 0.9, \ldots\} \). Hence \( \{\|x^k - x^{k+1}\|\} \) is a constant sequence \( \{0, 0, 0, \ldots\} \) whose limit is nonzero, and thus we don’t have the square summable property. Note that the sequence \( \{f(x^k)\} \) is the constant zero-sequence which is convergent but without verifying the sufficiently descent property.

Remark 6 In some variants of DCA, it is not the sequence \( \{f(x^k)\} \) satisfying the sufficiently descent property, but the sequence of the auxiliary function (cf. Lyapunov function or energy function). This is often encountered in non-monotone variants of DCA, e.g., the proximal DCA with extrapolation (pDCAe) proposed in \cite{43} is a variant of DCA by introducing the Nesterov’s extrapolation into DCA using a proximal DC decomposition. The sequence \( \{f(x^k)\} \) is non-monotone, but we can prove in a similar way as in Lemma 2 that the sequence \( \{\Phi(k^k) := f(x^k) + \frac{\rho}{2}\|x^k - x^{k-1}\|^2\} \) is monotone and enjoys the sufficiently descent property, which helps to guarantee the convergence of the non-monotone sequence \( \{f(x^k)\} \) and the square summable property.

Corollary 1 \( (O\left(\frac{1}{\sqrt{N}}\right) \text{ convergence rate}) \) If either \( g \) or \( h \) is strongly convex over \( C \) (i.e., \( \rho_g + \rho_h > 0 \)), then, after \( N \) iterations of DCA, we have

\[
\frac{1}{N+1} \sum_{k=0}^{N} \|x^{k+1} - x^k\|^2 \leq \frac{2(f(x^0) - f^*)}{(\rho_g + \rho_h)(N+1)} = O\left(\frac{1}{\sqrt{N}}\right),
\]

where \( f^* = \lim_{k \to \infty} f(x^k) \), and

\[
\min_{0 \leq k \leq N} \|x^{k+1} - x^k\| \leq \sqrt{\frac{2(f(x^0) - f^*)}{(\rho_g + \rho_h)(N+1)}} = O\left(\frac{1}{\sqrt{N}}\right).
\]

Proof We get from Lemma 2 that

\[
\sum_{k=0}^{N} \frac{\rho_g + \rho_h}{2} \|x^k - x^{k+1}\|^2 \leq f(x^0) - \min\{f(x) : x \in C\} = f(x^0) - f^*.
\]

Hence,

\[
\sum_{k=0}^{N} \|x^k - x^{k+1}\|^2 \leq \frac{2(f(x^0) - f^*)}{\rho_g + \rho_h}.
\]
It follows immediately that
\[
\frac{1}{N+1} \sum_{k=0}^{N} \|x^{k+1} - x^k\|^2 \leq \frac{2(f(x^0) - f^*)}{(\rho_g + \rho_h)(N+1)} = O\left(\frac{1}{N}\right)
\]
and
\[
\min_{0 \leq k \leq N} \|x^k - x^{k+1}\| \leq \sqrt{\frac{2(f(x^0) - f^*)}{(\rho_g + \rho_h)(N+1)}} = O\left(\frac{1}{\sqrt{N}}\right).
\]

Remark 7 Corollary 1 indicates that the convergence rate depends on \(f(x^0) - f^*\) (i.e., the initial point) and \(\rho_g + \rho_h\) (i.e., the quality of the DC decomposition). A smaller \(f(x^0) - f^*\) and a larger \(\rho_g + \rho_h\) lead to a faster convergence. The convergence rate is sublinear. Note that it is possible to have linear convergence rate under some stronger assumptions. We will provide a better convergence rate under the Kurdyka-Lojasiewicz property in Section 4.

Remark 8 Regarding Corollary 1, employing \(\|x^{k+1} - x^k\| \leq \varepsilon\) for some given tolerance \(\varepsilon > 0\) to terminate DCA (a commonly used stopping criterion for DCA) is extremely risky because there is no guarantee in the optimality of the computed solution. For instance, given a small tolerance \(\varepsilon\), if we use a DC decomposition such that \(\rho_g + \rho_h > \frac{f(x^0) - f^*}{\varepsilon^2}\), then Corollary 1 implies that DCA will end in one iteration without any guarantee in the optimality of the computed solution. The same issue occurs with another commonly used stopping criterion \(|f(x^{k+1}) - f(x^k)| \leq \varepsilon\) as well. Hence, it is very important to propose a rigorous stopping criterion to guarantee the optimality, which is beyond the scope of this manuscript. Throughout the paper, we will only consider DCA without stopping condition.

**DC criticality of the cluster point of \(\{x^k\}\)**

**Theorem 1 (subsequential convergence of \(\{x^k\}\))** Let \(\{x^k\}\) and \(\{y^k\} \subset \partial h(x^k)\) be two well-defined and bounded sequences generated by DCA starting from an initial point \(x^0 \in \text{dom}\partial h\) for DC program \((P_C)\) under Assumption A. If either \(g\) or \(h\) is strongly convex (i.e., \(\rho_g + \rho_h > 0\)) over \(C\), then any cluster point of the sequence \(\{x^k\}\) is a DC critical point of \((P_C)\).

**Proof** By the boundedness of \(\{x^k\}\), there exists a convergent subsequence of \(\{x^k\}\) (Bolzano–Weierstrass theorem), denoted \(\{x^{k_j}\}_j\), converging to a limit point \(x^*\). If either \(g\) or \(h\) is strongly convex over \(C\), then we get \(\|x^k - x^{k+1}\| \to 0\) by Lemma 2, which yields
\[
x^{k_j} \to x^* \text{ and } x^{k_j+1} \to x^*.
\]
The first order optimality condition for \((P_k)\) at \(k = k_j\) reads
\[
y^{k_j} \in \partial g(x^{k_j+1}) \cap \partial h(x^{k_j}).\tag{12}
\]
By the boundedness of the sequence \(\{y^{k_j}\}\), the set of the cluster points of \(\{y^{k_j}\}\) is non-empty. Without loss of generality, we suppose that the sequence \(\{y^{k_j}\}\) is
convergent. Taking limit in (12), we get from the closedness of the graphs of \( \partial g \) and \( \partial h \) that the limit point of \( \{y^k\} \) is included in \( \partial g(x^*) \cap \partial h(x^*) \). Thus \( \partial g(x^*) \cap \partial h(x^*) \neq \emptyset \), i.e., \( x^* \) is a DC critical point. \( \square \)

**Remark 9** The boundedness of the sequences \( \{x^k\} \) and \( \{y^k\} \) are necessary in the proof of Theorem 1. Example 1 with DC representation (3) is not a counterexample since the sequence \( \{y^k\} \) is unbounded as \( y^k \to -\infty \). So DCA generates a sequence \( \{x^k\} \) whose limit point \( x^* = 0 \) is not a DC critical point, but \( x^* \) is still an optimal solution of problem (1). Note that an optimal solution of a DC program may not be a DC critical point and vice versa. But for any optimal solution \( x^* \) such that both \( g \) and \( h \) are subdifferentiable at \( x^* \), then we must have \( \emptyset \neq \partial h(x^*) \subset \partial g(x^*) \), which is called a strongly DC critical point. It is known that the strong DC criticality often coincides with the classical d(irectional)-stationarity under some technical assumptions (e.g., \( \text{ri}(\text{dom} g) \cap \text{ri}(\text{dom} h) \neq \emptyset \)).

**Remark 10** The DC criticality is a weak optimality condition. Next example shows its weakness.

**Example 3** Consider

\[
\begin{align*}
g(x) &= \max\{0, x\} + \chi_{\{x \geq -1\}}(x), \\
h(x) &= \max\{0, -x\},
\end{align*}
\]

where both Assumption A and Assumption B are verified. We will show that \( x^* = 0 \) is a DC critical point since \( \partial g(0) = [0, 1], \partial h(0) = [-1, 0] \), and

\[
0 \in \partial g(0) - \partial h(0) = [0, 2].
\]

Starting DCA from the initial point \( x^0 = 0 \) could generate the zero-sequence by taking \( y^k = 0 \in \partial h(0) \) and \( x^{k+1} = 0 \in \text{argmin}\{g(x)\} = [-1, 0], \forall k \in \mathbb{N} \). Clearly, 0 is not a minimizer of \( \text{min}\{g(x) - h(x)\} \).

This example also demonstrates that a good DC decomposition and how we choose \( y^k \in \partial h(x^k) \) are very important to the quality of an obtained DC critical point by DCA.

- Consider the DC decomposition for Example 3 as:

\[
\begin{align*}
g(x) &= x + \chi_{\{x \geq -1\}}(x), \\
h(x) &= 0.
\end{align*}
\]

Then \( \partial g(0) = \{1\} \) and \( \partial h(0) = \{0\} \). Hence 0 is not a DC critical point anymore, but we have a DC critical point at \(-1\) since \( \partial g(-1) = (-\infty, 1] \) and \( \partial h(-1) = \{0\} \). Starting DCA from any initial point \( x^0 \in \text{dom} \partial h = \mathbb{R} \), we will get \( x^k \in \text{argmin}\{g(x)\} = \{-1\}, \forall k \geq 1 \). Therefore, DCA generates the sequence \( \{x^0, -1, -1, \ldots\} \) whose limit point is \(-1\), the optimal solution.

- Choosing a suitable \( y^k \in \partial h(x^k) \) for non-differentiable \( h \) is a crucial question. In the DC decomposition of Example 3, \( \partial h(0) = [-1, 0] \). If we choose \( y^k \in [-1, 0] \) (i.e., with \( y^k \neq 0 \)), then \( x^{k+1} = \text{argmin}\{g(x) - y^k x\} = \{-1\} \), and hence the sequence \( \{x^k\} \) converges to the optimal solution \(-1\).

**Remark 11** In general, the sequence \( \{x^k\} \) generated by DCA is not necessarily convergent despite that the sequence \( \{f(x^k)\} \) is so (Example 2 is a convincing example). Therefore, we need more assumptions to ensure the convergence of \( \{x^k\} \).
such as the most commonly used assumptions in nonconvex and nonsmooth analysis are the Lojasiewicz subgradient inequality and the Kurdyka-Lojasiewicz property. These assumptions provide some kind of regularities, so that the generated sequence \( \{x^k\} \) by some optimization algorithms (including DCA) will demonstrate better convergence property (e.g., global convergence and better convergence rate). In the next two subsections, we will study the global convergence of \( \{x^k\} \) with these two assumptions respectively. The convergence rate will be investigated in Section 4.

3.1 A general theorem for global convergence of \( \{x^k\} \)

**Theorem 2** Let \( \{x^k\} \) be a well-defined and bounded sequence with \( x^k \neq x^{k+1}, \forall k \in \mathbb{N} \). Suppose that

1. **(Lyapunov assumption)** there exists a (Lyapunov) function \( \Psi : \mathbb{R}^n \to [-\infty, \infty] \) such that the sequence \( \{\Psi(x^k)\} \) is well-defined, non-negative and converging to 0.
2. **(sufficiently descent assumption)** for large enough \( k \), there exists a (Lyapunov) function \( \Psi \) such that
   \[
   \Psi(x^k) - \Psi(x^{k+1}) \geq D\|x^k - x^{k+1}\|^2.
   \]
3. **(regularity assumption)** for large enough \( k \), there exist a differentiable concave function \( \varphi : [0, \eta) \to \mathbb{R}_+ \) (for some \( \eta > \Psi(x^k), \eta \leq \Psi(0) \)) with \( \varphi(0) = 0 \) and \( \varphi' > 0 \) over \( (0, \eta) \), and two non-negative constants \( C_1 \) and \( C_2 \) with \( C_1 + C_2 > 0 \) such that
   \[
   \varphi'(\Psi(x^k)) \times (C_1\|x^k - x^{k+1}\| + C_2\|x^{k-1} - x^k\|) \geq 1.
   \]

Then, for large enough \( k \), we have the inequality
\[
\frac{3}{4}\|x^k - x^{k+1}\| \leq \frac{1}{4}\|x^{k-1} - x^k\| + \frac{\max\{C_1, C_2\}}{D}\left(\varphi(\Psi(x^k)) - \varphi(\Psi(x^{k+1}))\right),
\]
and the sequence \( \{x^k\} \) is convergent.

**Proof** For large enough \( k \), by the concavity of \( \varphi \), we get
\[
\varphi(\Psi(x^k)) - \varphi(\Psi(x^{k+1})) \geq \varphi'(\Psi(x^k))(\Psi(x^k) - \Psi(x^{k+1}))
\]
\[
\geq \frac{D}{C_1}\|x^k - x^{k+1}\|^2 \varphi'(\Psi(x^k))
\]
\[
\geq \frac{D}{C_1}\|x^k - x^{k+1}\|^2
\]
\[
\geq \frac{D}{\max\{C_1, C_2\}}\|x^k - x^{k+1}\| + \|x^{k-1} - x^k\|
\]

It follows by the Young’s inequality \((a \leq a^2/b + b/4 \text{ with } a, b > 0)\) that
\[
\frac{3}{4}\|x^k - x^{k+1}\| - \frac{1}{4}\|x^{k-1} - x^k\| \leq \frac{\|x^k - x^{k+1}\|^2}{\|x^k - x^{k+1}\| + \|x^{k-1} - x^k\|}
\]
Hence, for large enough \( k \) (say \( k \geq N \)), we have
\[
\frac{3}{4}\|x^k - x^{k+1}\| \leq \frac{1}{4}\|x^{k-1} - x^k\| + \frac{\max\{C_1, C_2\}}{D}\left(\varphi(\Psi(x^k)) - \varphi(\Psi(x^{k+1}))\right).
\]
Then, summing for $k$ from $N$ to $\infty$ and using the fact that $\varphi(\Psi(x^k)) \to 0$ as $k \to \infty$, we get

\[
\frac{1}{2} \sum_{k=N}^{\infty} \|x^k - x^{k+1}\| \leq \frac{1}{4} \|x^{N-1} - x^N\| + \frac{\max\{C_1, C_2\}}{D} \varphi(\Psi(x^N)) < \infty,
\]

that is

\[
\sum_{k=0}^{\infty} \|x^k - x^{k+1}\| < \infty,
\]

which implies that the sequence $\{x^k\}$ is Cauchy, thus convergent. \hfill \Box

Remark 12 The regularity assumption (H3) is quite often an intrinsic property of $\Psi$, which can be derived from some error bounds. We will see this connection later in the proof of Lemma 5.

Remark 13 In some variants of DCA, the term $D\|x^k - x^{k+1}\|^2$ in the sufficiently descent assumption (H2) could be given in form of $D\|x^{k-1} - x^k\|^2$. The convergence of $\{x^k\}$ still holds with a similar proof.

Remark 14 Theorem 2 provides a general framework to establish the global convergence of an algorithm if the assumptions (H1) – (H3) are verified. The global convergence analysis of DCA in the next two subsections will be entirely based on this theorem.

3.2 Global convergence of $\{x^k\}$ under the Lojasiewicz subgradient inequality

The key ingredient in this subsection is the well-known Lojasiewicz subgradient inequality established by Bolte-Daniilidis-Lewis [4, Theorem 3.1] as follows:

**Theorem 3 (Lojasiewicz subgradient inequality, [4])** Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a subanalytic function with closed domain, and assume that $f$ is continuous over its domain. Let $x^* \in \mathbb{R}^n$ be a limiting-stationary point of $f$ (i.e., $0 \in \partial^L f(x^*)$). Then there exists an exponent $\theta \in [0, 1)$, a finite constant $M > 0$, and a neighbourhood $V$ of $x^*$ such that

\[
|f(x) - f(x^*)|^\theta \leq M\|y\|, \forall x, y \in \partial^L f(x),
\]

where the convention $0^0 = 1$ is adopted.

In Theorem 3, the notions and properties of the subanalytic function/set are classical and can be found in [18, 19, 3, 4]. The class of subanalytic sets (resp. functions) contains all analytic sets (resp. functions). Subanalytic sets and subanalytic functions enjoy interesting properties. For instance, the class of subanalytic sets is closed under locally finite unions/intersections, relative complements, and the usual projection. The distance function to a subanalytic set is subanalytic; the sum/difference of continuous and subanalytic functions is also subanalytic.

The notation $\partial^L f(x)$ stands for the limiting-subdifferential whose definition is based on the Fréchet subdifferential (both classical subdifferentials for nonsmooth functions). See Rockafellar-Wets [41] and Mordukhovich [21] for a comprehensive introduction about them. We recall their definitions and some useful properties for this paper as follows:
Definition 1 (Fréchet subdifferential) The Fréchet-subdifferential of a proper closed function \( f : \mathbb{R}^n \to (-\infty, \infty] \) at \( x \in \text{dom} f \) is defined by
\[
\partial^F f(x) := \left\{ y \in \mathbb{R}^n \mid \liminf_{z \to x} \frac{f(z) - f(x) - \langle y, z - x \rangle}{\|z - x\|} \geq 0 \right\}.
\]
If \( x \not\in \text{dom} f \), then \( \partial^F f(x) = \emptyset \).

Definition 2 (Limiting subdifferential) The limiting-subdifferential of a proper closed function \( f : \mathbb{R}^n \to (-\infty, \infty] \) at \( x \in \mathbb{R}^n \) is defined by
\[
\partial^L f(x) := \{ y \in \mathbb{R}^n \mid \exists (x^k \to x, f(x^k) \to f(x), y^k \in \partial^F f(x^k)) \text{ and } y^k \to y \}.
\]

It is known that \( \partial^F f(x) \subset \partial^L f(x) \) where \( \partial^F f(x) \) is a closed convex set and \( \partial^L f(x) \) is closed (may not be convex). Both dom\( \partial^F f \) and dom\( \partial^L f \) are dense in dom\( f \). If \( f \) is differentiable at \( x \), then \( \partial^F f(x) = \{ \nabla f(x) \} \) and \( \nabla f(x) \in \partial^L f(x) \). Note that \( \nabla f(x) \) may not be the unique point in \( \partial^F f(x) \) even for differentiable \( f \), e.g., the function \( f(x) = x^2 \sin(1/x) \), \( \forall x \neq 0 \) and \( f(0) = 0 \) is differentiable everywhere (including 0), and we have \( \partial^F f(0) = \{ 0 \} \subset \partial^L f(0) = [-1, 1] \). If \( f \) is of class \( C^1 \), then
\[
\partial^F f(x) = \partial^L f(x) = \{ \nabla f(x) \}, \forall x \in \mathbb{R}^n.
\]

If \( f \) is convex (without differentiability) and \( x \in \text{ri}(\text{dom} f) \), then
\[
\partial f(x) = \partial^F f(x) = \partial^L f(x),
\]
where \( \partial f(x) \) stands for the classical (convex) subdifferential of \( f \) at \( x \), i.e., \( \partial f(x) = \{ y \in \mathbb{R}^n : f(z) \geq f(x) + \langle y, z - x \rangle, \forall z \in \mathbb{R}^n \} \).

In general, let \( h \) be continuously differentiable around \( x \) and \( g \) be finite at \( x \). Then we have the summation rules:
\[
\partial^F (g + h)(x) = \partial^F g(x) + \nabla h(x); \quad \partial^L (g + h)(x) = \partial^L g(x) + \nabla h(x).
\]

For DC function \( g - h \) where \( g \) and \( h \) belong to \( \Gamma_0(\mathbb{R}^n) \), we have
\[
\partial^F (g - h)(x) \subset \partial^L (g - h)(x) \subset \partial g(x) - \partial h(x),
\]
whenever \( h \) is continuous at \( x \). Especially, if \( h \) is differentiable at \( x \), then
\[
\partial^F (g - h)(x) = \partial^L (g - h)(x) = \partial g(x) - \nabla h(x).
\]

However, if \( g \) is differentiable at \( x \), we only have
\[
\partial^L (g - h)(x) = \nabla g(x) + \partial^L (-h)(x) \subset \nabla g(x) - \partial h(x),
\]
where the inclusion is often strict due to \( \partial^L (-h)(x) \subset -\partial h(x) \) whenever \( h \) is non-differentiable at \( x \), e.g., for \( h(x) = |x| \), we get \( \partial^L (-h)(0) = \{-1, 1\} \subset [-1, 1] = -\partial h(0) \).
Global convergence of \( \{x^k\} \) for standard DC program

**Theorem 4** Consider the standard DC program (P) under Assumption A. Let \( \{x^k\} \) and \( \{y^k\} \subset \partial h(x^k) \) be two well-defined and bounded sequences generated by DCA starting from an initial point \( x^0 \in \text{dom} h \) for problem (P). Moreover, suppose that
- \( f \) is continuous over \( \text{dom} f \);
- \( \Phi(x) := f(x) \) satisfies the Łojasiewicz subgradient inequality at any cluster point of \( \{x^k\} \);
- either \( g \) or \( h \) is strongly convex (i.e., \( \rho_g + \rho_h > 0 \));
- \( h \) has locally Lipschitz continuous gradient.

Then the sequence \( \{x^k\} \) is convergent.

**Proof** If there exists \( j > 0 \) such that \( x^j = x^{j+1} \), then the sequence \( \{x^k\} \) converges to \( x^j \) in finitely many iterations. Otherwise (for all \( j > 0 \), \( x^j \neq x^{j+1} \)), the convergence of \( \{x^k\} \) is an immediate consequence of Theorem 2 whose assumptions (H1)-(H3) will be verified in Lemma 3.

**Lemma 3** Under assumptions in Theorem 4. If the sequence \( \{x^k\} \) does not converge in finitely many iterations (i.e., \( x^k \neq x^{k+1}, \forall k \in \mathbb{N} \)), then the assumptions (H1)-(H3) required in Theorem 2 hold.

**Proof** (H1): Denote the set of cluster points of \( \{x^k\} \) by \( \omega(x^0) \), which is bounded since \( \{x^k\} \) is bounded. The convergence of the sequence \( \{f(x^k)\} \) and the continuity of \( f \) over its domain imply that \( f(\omega(x^0)) \) is a constant, denoted \( f^* \). Let us define the Lyapunov function

\[
\Psi(x) := \Phi(x) - f^* = f(x) - f^*.
\]

It follows by the non-increasing and the convergence of the sequence \( \{f(x^k)\} \) (Lemma 1) that

\[
\Psi(x^k) = f(x^k) - f^* \geq 0, \forall k = 1, 2, \ldots,
\]

and

\[
\lim_{k \to \infty} \Psi(x^k) = \Psi(\omega(x^0)) = 0.
\]

Hence, (H1) is verified.

(H2): The sufficiently descent property (Lemma 2) reads

\[
\Psi(x^k) - \Psi(x^{k+1}) = f(x^k) - f(x^{k+1}) \geq \rho_g + \rho_h \|x^k - x^{k+1}\|^2, \forall k = 1, 2, \ldots.
\]

Hence, (H2) is verified with \( D = (\rho_g + \rho_h)/2 > 0 \).

(H3): Theorem 1 says that any point in \( \omega(x^0) \) is a DC critical point. Due to the relations

\[
\partial^L \Psi(x^*) = \partial^L \Phi(x^*) \subset \partial g(x^*) - \partial h(x^*), \forall x^* \in \omega(x^0),
\]

for any point \( x^* \in \omega(x^0) \), there are two possible cases (0 \( \notin \partial^L \Phi(x^*) \) and 0 \( \notin \partial^L \Phi(x^*) \)). If 0 \( \notin \partial^L \Phi(x^*) \), then the Łojasiewicz subgradient inequality is trivially satisfied for \( \Psi \) at \( x^* \). Otherwise, by Theorem 3, \( \Psi \) satisfies the Łojasiewicz subgradient inequality at \( x^* \) as well. Hence, there exist a Łojasiewicz component \( \theta \in [0, 1) \), a constant \( M > 0 \) and \( \epsilon > 0 \) such that

\[
|\Psi(x) - \Psi(\omega(x^0))|_{\theta} \leq M\|y\|, \forall x \in \bigcup_{z \in \omega(x^0)} B(z, \epsilon), \forall y \in \partial^L \Psi(x),
\]
where $B(z, \epsilon) := \{ x \in \mathbb{R}^n : ||x - z|| < \epsilon \}$. It follows by $\Psi(\omega(x^0)) = 0$ that
\[ |\Psi(x)|^\theta \leq M\|y\|, \forall x \in \bigcup_{z \in \omega(x^0)} B(z, \epsilon), \forall y \in \partial^I \Psi(x). \tag{20} \]
Moreover, $\omega(x^0)$ is the set of cluster points of the sequence $\{x^k\}$ implies that
\[ \exists N > 0, \forall k \geq N, x^k \in \bigcup_{z \in \omega(x^0)} B(z, \epsilon). \tag{21} \]
Combining (17), (20) and (21), we get
\[ \exists (N > 0, M > 0, \theta \in [0, 1]), \forall k \geq N, \forall y \in \partial^I \Psi(x^k), |\Psi(x^k)|^\theta \leq M\|y\|. \tag{22} \]
Now, consider the function $\varphi(t) = t^{1-\theta}$ defined on $[0, \infty)$ with $\theta \in [0, 1)$ being the aforementioned Lojasiewicz component. Clearly, $\varphi$ verifies the assumptions that $\varphi$ is differentiable concave from $[0, \infty)$ to $\mathbb{R}$ verifying $\varphi(0) = 0$ and $\varphi' > 0$ over $(0, \infty)$. By the concavity of $\varphi$ and $\Psi(x^k) \geq \Psi(x^{k+1})$, then
\[ \varphi(\Psi(x^k)) - \varphi(\Psi(x^{k+1})) \geq \varphi'(\Psi(x^k))(\Psi(x^k) - \Psi(x^{k+1})). \tag{23} \]
Due to the boundedness of the sequence $\{x^k\}$ and the set $\omega(x^0)$, there exists a bounded open set $D$ containing the whole sequence $\{x^k\}$ and $\omega(x^0)$. Since $h$ has locally Lipschitz continuous gradient over $D$, then
\[ \exists L > 0, \forall k \geq 1, ||\nabla h(x^k) - \nabla h(x^{k+1})|| \leq L||x^k - x^{k+1}||. \tag{24} \]
The first order optimality condition for the convex subproblem (P_k) gives
\[ \nabla h(x^k) \in \partial g(x^{k+1}). \]
Thus
\[ \nabla h(x^k) - \nabla h(x^{k+1}) \in \partial g(x^{k+1}) - \nabla h(x^{k+1}) = \partial^I \Psi(x^{k+1}), \]
that is
\[ \forall k \geq 1, \nabla h(x^k) - \nabla h(x^{k+1}) \in \partial^I \Psi(x^{k+1}). \tag{25} \]
It follows from (22) and (25) that
\[ \exists (N > 0, M > 0, \theta \in [0, 1]), \forall k \geq N, \|
\Psi(x^{k+1})
\|	heta \leq M\|
\nabla h(x^k) - \nabla h(x^{k+1})\|. \]
Combining with (24), we get
\[ \exists (N > 0, M > 0, \theta \in [0, 1], L > 0), \forall k \geq N, \|
\Psi(x^{k+1})
\|	heta \leq ML\|x^k - x^{k+1}\|. \tag{26} \]
Hence, for all $k \geq N + 1$,
\[ \varphi'(\Psi(x^k)) = (1 - \theta)\Psi(x^k)^{-\theta} \geq \frac{1 - \theta}{ML\|x^{k+1} - x^k\|}, \tag{26} \]
which implies that
\[ \varphi'(\Psi(x^k)) \geq \frac{ML}{1 - \theta}\|x^{k+1} - x^k\| \geq 1, \]
that is \((H3)\) with $C_1 = 0$ and $C_2 = ML/(1 - \theta)$. \hfill \square
It follows from (P0) that is

The first order optimality condition for the convex subproblem (P_k) gives

which implies that

where

Combining with (28), then

Hence, for all \(k \geq N\), we get

which implies that

that is (H3) with \(C_1 = ML/(1-\theta)\) and \(C_2 = 0\).

Remark 16 If for all \(k \geq 1\), either \(g\) or \(h\) has locally Lipschitz continuous gradient around \(x^k\). Then, by combining (25) and (29), we have

Hence, for all \(k \geq N+1\),

which implies that

i.e., (H3) is verified with \(C_1 = C_2 = ML/(1-\theta)\).
Global convergence of \( \{ x^k \} \) for convex constrained DC program

**Theorem 5** Consider the convex constrained DC program \((P_C)\) under **Assumption A**. Let \( \{x^k\} \) and \( \{y^k\} \subset \partial h(x^k) \) be two well-defined and bounded sequences generated by DCA starting from an initial point \( x^0 \in \text{dom} h \) for problem \((P_C)\). Moreover, suppose that

- \( f \) is continuous over \( \text{dom} f \);
- \( \Phi(x) := f(x) + \chi_C(x) \) satisfies the Lojasiewicz subgradient inequality at any cluster point of \( \{ x^k \} \);
- either \( g \) or \( h \) is strongly convex over \( C \) (i.e., \( \rho_g + \rho_h > 0 \));
- \( h \) has locally Lipschitz continuous gradient over \( C \).

Then the sequence \( \{ x^k \} \) is convergent.

Due to the equivalent standard DC formulation of \((P_C)\) by introducing the indicator function \( \chi_C \), the proof of Theorem (5) is thus the same as in Theorem 4. We only need to verify the assumptions \((H1)-(H3)\) required in Theorem 2, which will be discussed in Lemma 4.

**Lemma 4** Under assumptions in Theorem 5. If the sequence \( \{x^k\} \) does not converge in finitely many iterations, then the assumptions \((H1)-(H3)\) required in Theorem 2 hold.

**Proof** Let \( \Psi(x) := \Phi(x) - f^* = f(x) + \chi_C(x) - f^* \), \((H1)\) and \((H2)\) can be verified exactly in the same way as in Lemma 3. Now, we will only show the differences in the proof of \((H3)\). We first get the formulas \((22), (23)\), and show that there exists a bounded open set \( D \subset C \) including the sequence \( \{x^k\} \) and the bounded set \( \omega(x^0) \), over which \( h \) has locally Lipschitz continuous gradient, i.e.,

\[
\exists L > 0, \forall k \geq 1, \| \nabla h(x^k) - \nabla h(x^{k+1}) \| \leq L \| x^k - x^{k+1} \|. \tag{31}
\]

The first order optimality condition for the convex subproblem \((P_k)\) (under constraint \( C \)) gives

\[
\nabla h(x^k) \in \partial (g + \chi_C)(x^{k+1}).
\]

Thus

\[
\nabla h(x^k) - \nabla h(x^{k+1}) \in \partial (g + \chi_C)(x^{k+1}) - \nabla h(x^{k+1}) = \partial^L \Psi(x^{k+1}).
\]

Hence,

\[
\forall k \geq 1, \nabla h(x^k) - \nabla h(x^{k+1}) \in \partial^L \Psi(x^{k+1}). \tag{32}
\]

Then, we get again for each \( k \geq N + 1 \) that

\[
\varphi' (\Psi(x^k)) = \frac{ML}{1-\theta} \| x^{k-1} - x^k \| \\geq 1,
\]

that is \((H3)\) with \( C_1 = 0 \) and \( C_2 = ML/(1-\theta) \).

**Remark 17** More assumptions are required for guaranteeing the global convergence of \( \{ x^k \} \) when \( g \) has locally Lipschitz continuous gradient in the convex constrained DC program. Basically, we have to choose \( y^k \in \partial h(x^k) \) and \( x^{k+1} \in \text{argmin}\{g(x) - \langle y^k, x \rangle : x \in C \} \) verifying \( -\nabla g(x^{k+1}) \in \partial^L (\chi_C - h)(x^k) \), which is not easy to check in practice.
3.3 Global convergence of \( \{x^k\} \) under the Kurdyka-Lojasiewicz property

Now, we study the global convergence of the sequence \( \{x^k\} \) under the Kurdyka-Lojasiewicz property defined as follows:

**Definition 3 (Kurdyka-Lojasiewicz property)** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a locally Lipschitz function. We say that \( f \) satisfies the KL property at \( x^* \in \mathbb{R}^n \) if there exist \( \eta \in (0, \infty) \), a neighborhood \( V \) around \( x^* \), and a concave function \( \varphi : [0, \eta) \to \mathbb{R}_+ \) such that

- \( \varphi(0) = 0 \) and \( \varphi \in C^1((0, \eta), \mathbb{R}_+) \);
- \( \varphi' > 0 \) on \( (0, \eta) \);
- \( \forall x \in V \) with \( f(x) - f(x^*) \in (0, \eta) \), we have the KL property
  \[
  \varphi'(f(x) - f(x^*)) \times \text{dist}(0, \partial^L f(x)) \geq 1.
  \]  

We say that \( f \) is a KL function if it satisfies the KL inequality over \( \text{dom} \partial^L f \).

The KL property is originally developed by Lajasiwicz [17] (cf. Lajasiwicz inequality) for differentiable subanalytic functions, generalized by Kurdyka [10] to definable (cf. tame [5]) functions, then extended to nonsmooth regime by Bolte et al. [4, 2] where gradient is replaced by limiting-subdifferential. A remarkable aspect of KL functions is that they are ubiquitous in applications, for example, semialgebraic, subanalytic and log-exp are KL functions (see [10, 4, 2] and the references therein).

In practice, we often take \( \varphi(t) = Mt^{1-\theta} \) for some \( M > 0 \) and \( \theta \in [0, 1) \) where \( \theta \) is called the Lajasiwicz exponent. Then \( \forall x \in V \) with \( f(x) - f(x^*) \in (0, \eta) \), we have

\[
(f(x) - f(x^*))^{\theta} \leq M(1-\theta)\text{dist}(0, \partial^L f(x)),
\]

which is exactly the Lojasiewicz subgradient inequality (given in Theorem 3) by taking \( \text{dist}(0, \partial^L f(x)) := \min\{\|y\| : y \in \partial^L f(x)\} \) and letting \( 0 \in \partial^L f(x^*) \). Hence, the Lojasiewicz subgradient inequality can be viewed as a special case of the KL property, so that the corresponding proofs for the global convergence of \( \{x^k\} \) under the KL property are similar to Theorems 4 and 5.

**Theorem 6** Consider the standard DC program (P) (resp. convex constrained DC program \( (P_C) \)) under Assumption A. Let \( \{x^k\} \) and \( \{y^k\} \subset \{\partial h(x^k)\} \) be two well-defined and bounded sequences generated by DCA starting from an initial point \( x^0 \in \text{dom} \partial h \) for problem (P) (resp. \( (P_C) \)). Moreover, suppose that

- \( f \) is continuous over \( \text{dom} f \);
- \( \Phi(x) := f(x) \) (resp. \( \Phi(x) := f(x) + \chi_C(x) \)) is a KL function.
- either \( g \) or \( h \) is strongly convex (i.e., \( \rho_g + \rho_h > 0 \));
- \( h \) has locally Lipschitz continuous gradient.

Then, the sequence \( \{x^k\} \) is convergent.

Again, Theorem 6 can be proved by using Theorem 2, whose assumptions (H1)-(H3) will be verified in Lemma 5.

**Lemma 5** Under assumptions in Theorem 6. If the sequence \( \{x^k\} \) does not converge in finitely many iterations, then the assumptions (H1)-(H3) required in Theorem 2 hold.
Proof (H1) and (H2) does not depend on the KL property, so they are proved exactly as in Lemma 3. Now, we only need to prove (H3) as follows: Let \( C = \mathbb{R}^n \) for standard DC program. Then for both standard and convex constrained DC programs, \( \Psi := \Phi - f^* \) is a KL function since \( \Phi \) is so, which implies that there exist \( \eta \in (0, \infty], \epsilon > 0 \), a neighborhood of \( \omega(x^0) \) as \( V := \cup_{z \in \omega(x^0)} B(z, \epsilon) \), and a concave function \( \varphi : [0, \eta) \to \mathbb{R}_+ \) with \( \varphi(0) = 0, \varphi \in C^1((0, \eta), \mathbb{R}_+) \), \( \varphi' > 0 \) on \( (0, \eta) \), such that \( \forall x \in V \) and \( \Psi(x) \in (0, \eta) \) we have
\[
\varphi'(\Psi(x)) \times \text{dist}(0, \partial^L \Psi(x)) \geq 1.
\] (34)
Since \( \omega(x^0) \) is the set of limit points of the sequence \( \{x^k\} \), then
\[
\exists N > 0, \forall k \geq N, x^k \in V \text{ and } \Psi(x^k) \in (0, \eta).
\] (35)
Combining (34) and (35), we get
\[
\exists N > 0, \forall k \geq N, x^k \in V \text{ and } \varphi'(\Psi(x^k)) \times \text{dist}(0, \partial^L \Psi(x^k)) \geq 1.
\] (36)
Due to the boundedness of the sequence \( \{x^k\} \) and the set \( \omega(x^0) \), there exists a bounded open set in \( \mathbb{C} \), denoted \( D \), containing the whole sequence \( \{x^k\} \) and \( \omega(x^0) \). Then, the convex function \( h \) has locally Lipschitz continuous gradient over \( D (\subset \mathbb{C}) \), which implies that
\[
\exists L > 0, \forall k \geq 1, \|\nabla h(x^k) - \nabla h(x^{k+1})\| \leq L\|x^k - x^{k+1}\|.
\] (37)
Then the first order optimality condition for the convex subproblem (P_k) (under constraint \( C \)) gives
\[
\nabla h(x^k) \in \partial(g + \chi C)(x^{k+1}).
\]
Thus
\[
\nabla h(x^k) - \nabla h(x^{k+1}) \in \partial(g + \chi C)(x^{k+1}) - \nabla h(x^{k+1}) = \partial^L \Psi(x^{k+1}),
\]
that is the error bound
\[
\forall k \geq N + 1, \text{dist}(0, \partial^L \Psi(x^k)) \leq \|\nabla h(x^{k-1}) - \nabla h(x^k)\|.
\] (38)
It follows from (36), (37) and (38) that
\[
\exists N > 0, \forall k \geq N + 1, x^k \in V \text{ and } \varphi'(\Psi(x^k)) \times L\|x^{k-1} - x^k\| \geq 1.
\] (39)
which is (H3) with \( C_1 = 0 \) and \( C_2 = L \). \( \square \)

Remark 18 We can get similar results as in Lemmas 3 and 4 for the case where \( g \) has locally Lipschitz continuous gradient, whose verification will be omitted.

4 Convergence rate

In this section, we will focus on the convergence rate of DCA concerning the sequences \( \{f(x^k)\} \) and \( \{\|x^k - x^*\|\} \) for DC programs under KL property. Same results will be held under the Lojasiewicz subgradient inequality (as a special case of the KL property).
4.1 Some useful formulas for convergence rate analysis

First, we recall some fundamental results and useful formulas to the convergence rate of a nonnegative sequence \( \{r_k\} \). If \( r_k \leq ck^{-p} \) with \( p > 0 \) and \( c > 0 \), then we have sub-linear convergence \( O(\varepsilon^{-\frac{1}{p}}) \); if \( r_k \leq cq^k \) with \( 0 < q < 1 \) and \( c > 0 \), then we have linear convergence \( O(\ln \varepsilon^{-1}) \); if \( r_{k+1} \leq cr_k^2 \) with \( 0 < cr_0 < 1 \), then we have quadratic convergence \( O(\ln \ln \varepsilon^{-1}) \).

The next three lemmas (cf. Lemma 7, Lemma 6 and Lemma 8) are particularly important to establish the convergence rate of DCA in our analysis.

**Lemma 6** Let \( \{r_k\} \) be a nonincreasing and nonnegative sequence converging to 0. Suppose that there exist two positive constants \( \alpha \) and \( \beta \) such that for all large enough \( k \), we have

\[
r_k^{\alpha} \leq \beta(r_k - r_{k+1}).
\]  

Then

(i) if \( \alpha = 0 \), then the sequence \( \{r_k\} \) converges to 0 in a finite number of steps;

(ii) if \( \alpha \in [0, 1) \) and \( r_k > 0, \forall k \in \mathbb{N} \), then

\[
r_k \leq O\left(\frac{\beta}{1 + \beta}\right)^k, \text{ as } k \to \infty;
\]

i.e., the sequence \( \{r_k\} \) converges linearly to 0 with rate \( \frac{\beta}{1 + \beta} \).

(iii) if \( \alpha > 1 \) and \( r_k > 0, \forall k \in \mathbb{N} \), then

\[
r_k \leq O\left(k^{-\frac{1}{\alpha}}\right), \text{ as } k \to \infty;
\]

i.e., the sequence \( \{r_k\} \) converges sublinearly to 0.

**Proof**

(i) If \( \alpha = 0 \), then (40) implies that for large enough \( k \) (i.e., there exists \( N > 0, \forall k \geq N \), we have

\[
0 \leq r_{k+1} \leq r_k - \frac{1}{\beta}.
\]

It follows by \( r_k \to 0 \) and \( \frac{1}{\beta} > 0 \) that \( \{r_k\} \) converges to 0 in a finite number of steps, and we can estimate the number of steps as:

\[
0 \leq r_{k+1} \leq r_k - \frac{1}{\beta} \leq r_{k-1} - \frac{2}{\beta} \leq \cdots \leq r_N - \frac{k-N+1}{\beta}.
\]

Hence

\[
k \leq \beta r_N + N - 1.
\]

(ii) If \( \alpha \in [0, 1) \) and \( r_k > 0, \forall k \in \mathbb{N} \). Since \( r_k \to 0 \), we have that \( r_k < 1 \) for large enough \( k \). Thus, \( r_{k+1} \leq r_k < 1 \), and it follows by (40) that

\[
r_{k+1} \leq r_k^{\alpha} \leq \beta(r_k - r_{k+1})
\]

for large enough \( k \). Hence there exists \( N > 0 \) such that \( \forall k \geq N \)

\[
r_{k+1} \leq \left(\frac{\beta}{1 + \beta}\right)r_k.
\]
On the convergence analysis of DCA

So that
\[ r_k \leq \left( \frac{\beta}{1 + \beta} \right)^{k-N} r_N = O \left( \left( \frac{\beta}{1 + \beta} \right)^k \right), \text{ as } k \to \infty. \]

That is, \( \{r_k\} \) converges linearly to 0 with rate \( \frac{\beta}{1+\beta} \) for large enough \( k \).

(iii) If \( \alpha > 1 \) and \( r_k > 0 \) for all \( k \in \mathbb{N} \). Let \( \phi(t) = t^{-\alpha} \) and \( \tau > 1 \).

\( \triangleright \) Suppose that \( \phi(r_{k+1}) \leq \tau \phi(r_k) \). By the decreasing of \( \phi(t) \) and \( r_k \geq r_{k+1} \), we have
\[
\phi(r_k)(r_k - r_{k+1}) \leq \int_{r_{k+1}}^{r_k} \phi(t) \, dt = \frac{1}{1 - \alpha} (r_k^{1-\alpha} - r_{k+1}^{1-\alpha}).
\]

It follows from (40) that
\[
\frac{1}{\beta} \leq \phi(r_{k+1})(r_k - r_{k+1}) \leq \tau \phi(r_k)(r_k - r_{k+1}) \leq \frac{\tau}{\alpha - 1} (r_k^{1-\alpha} - r_{k+1}^{1-\alpha}).
\]

Hence
\[ r_k^{1-\alpha} - r_{k+1}^{1-\alpha} \geq \frac{\alpha - 1}{\beta \tau}. \tag{41} \]

\( \triangleright \) Suppose that \( \phi(r_{k+1}) \geq \tau \phi(r_k) \). Taking \( q := \tau^{-1} - 1 \in (0,1) \), then
\[ r_{k+1} \leq qr_k. \]

Hence
\[ r_{k+1}^{1-\alpha} \geq q^{1-\alpha} r_k^{1-\alpha}. \]

It follows that \( \exists N > 0, \forall k \geq N \):
\[ r_{k+1}^{1-\alpha} - r_k^{1-\alpha} \geq (q^{1-\alpha} - 1)r_k^{1-\alpha} \geq (q^{1-\alpha} - 1) r_N^{1-\alpha}. \]

In both cases, there exists a constant \( \zeta := \min \{ (q^{1-\alpha} - 1) r_N^{1-\alpha}, \frac{\alpha - 1}{\beta \tau} \} \) such that for large enough \( k \), we have
\[ r_{k+1}^{1-\alpha} - r_k^{1-\alpha} \geq \zeta. \]

Summing up for \( k \) from \( N \) to \( M - 1 \geq N \), we have
\[ r_M^{1-\alpha} - r_N^{1-\alpha} \geq \zeta(M - N). \]

Then,
\[ r_M \leq \left( r_N^{1-\alpha} + \zeta(M - N) \right)^{\frac{1}{1-\alpha}} = O \left( M^{\frac{1}{1-\alpha}} \right). \]

Hence,
\[ r_k \leq O \left( k^{\frac{1}{1-\alpha}} \right), \text{ as } k \to \infty, \]
i.e., there exists some \( \eta > 0 \) such that
\[ r_k \leq \eta k^{\frac{1}{1-\alpha}} \]
for large enough \( k \), which completes the proof. \( \square \)

A similar result of Lemma 7 for \( r_k^\alpha \leq \beta(r_k - r_{k+1}) \) is stated below.
Lemma 7 Let \( \{r_k\} \) be a nonincreasing and nonnegative sequence converging to 0. Suppose that there exist two positive constants \( \alpha \) and \( \beta \) such that for all large enough \( k \), we have
\[
r_k^\alpha \leq \beta (r_k - r_{k+1}). \tag{42}
\]

Then

(i) if \( \alpha = 0 \), then the sequence \( \{r_k\} \) converges to 0 in finitely many iterations;
(ii) if \( \alpha \in [0,1] \) and \( r_k > 0, \forall k \in \mathbb{N} \), then
\[
r_k \leq O((1 - \frac{1}{\beta})^k), \quad \text{as } k \to \infty;
\]
i.e., the sequence \( \{r_k\} \) converges linearly to 0 with rate \( 1 - \frac{1}{\beta} \).
(iii) if \( \alpha > 1 \) and \( r_k > 0, \forall k \in \mathbb{N} \), then
\[
r_k \leq O(k^{1/\alpha}), \quad \text{as } k \to \infty;
\]
i.e., the sequence \( \{r_k\} \) converges sublinearly to 0.

Proof (i) If \( \alpha = 0 \), then (42) implies that for large enough \( k \) (i.e., there exists \( N > 0, \forall k \geq N \)), we have
\[
0 \leq r_{k+1} \leq r_k - \frac{1}{\beta}.
\]
It follows by \( r_k \to 0 \) and \( \frac{1}{\beta} > 0 \) that \( \{r_k\} \) converges to 0 in a finite number of steps, and we can estimate the number of steps as:
\[
0 \leq r_{k+1} \leq r_k - \frac{1}{\beta} \leq r_{k-1} - \frac{2}{\beta} \leq \cdots \leq r_N - \frac{k-N+1}{\beta}.
\]
Hence
\[
k \leq \beta r_N + N - 1.
\]
(ii) If \( \alpha \in [0,1] \) and \( r_k > 0, \forall k \in \mathbb{N} \), then we get from \( r_k \to 0 \) that \( r_k < 1 \) for large enough \( k \). Thus, it follows by (42) that
\[
r_k \leq r_k^\alpha \leq \beta (r_k - r_{k+1})
\]
for large enough \( k (k \geq N) \). Hence
\[
r_{k+1} \leq (1 - \frac{1}{\beta})r_k, \quad \text{with } \beta > 1,
\]
so that
\[
r_k \leq (1 - \frac{1}{\beta})^k r_N = O((1 - \frac{1}{\beta})^k), \quad \text{as } k \to \infty,
\]
i.e., \( \{r_k\} \) converges linearly to 0 with rate \( 1 - \frac{1}{\beta} \) for large enough \( k \).
(iii) If \( \alpha > 1 \) and \( r_k > 0 \) for all \( k \in \mathbb{N} \). By the decreasing of \( \phi(t) = t^{-\alpha} \) and \( r_k \geq r_{k+1} \), we have
\[
\phi(r_k)(r_k - r_{k+1}) \leq \int_{r_{k+1}}^{r_k} \phi(t) \, dt.
\]
Then
\[
\frac{1}{\beta} \leq \phi(r_k)(r_k - r_{k+1}) \leq \int_{r_{k+1}}^{r_k} \phi(t) \, dt = \frac{r_k^{1-\alpha} - r_{k+1}^{1-\alpha}}{\alpha - 1}.
\]
Hence,
\[ r_{k+1}^{1-\alpha} - r_k^{1-\alpha} \geq \frac{\alpha - 1}{\beta}, \forall k \geq N. \]

Summing up for \( k \) from \( N \) to \( M - 1(\geq N) \), we have
\[ r_M^{1-\alpha} - r_N^{1-\alpha} \geq \frac{\alpha - 1}{\beta}(M - N). \]

Then,
\[ r_M = \left( r_N^{1-\alpha} + \frac{\alpha - 1}{\beta}(M - N) \right) \leq O(M^{1-\alpha}). \]

Hence,
\[ r_k \leq O(k^{\frac{1}{1-\alpha}}), \text{ as } k \to \infty. \]

That is, there exists some \( \eta > 0 \) such that
\[ r_k \leq \eta k^{\frac{1}{1-\alpha}} \]
for large enough \( k \), which completes the proof. \( \square \)

The next lemma is also very useful in our convergence analysis.

**Lemma 8** Let \( \{r_k\} \) be a nonincreasing and nonnegative sequence converging to 0. Suppose that there exist three constants \( a > 0, c > 0 \) and \( b > 0 \) such that for \( k \) large enough, we have
\[ r_k \leq c(r_{k-1} - r_k) + a(r_k - r_{k+1})^b. \] (43)

Then
(i) if \( b \geq 1 \), then \( \exists q \in (0,1) \) such that
\[ r_k \leq O(q^k) \text{ as } k \to \infty; \]
i.e., the sequence \( \{r_k\} \) converges linearly.
(ii) if \( b \in (0,1) \), then
\[ r_k \leq O\left(k^{\frac{b}{1-b}}\right) \text{ as } k \to \infty; \]
i.e., the sequence \( \{r_k\} \) converges sublinearly.

**Proof** The basic idea is to reduce the two terms on the right hand side to one term by asymptotic behavior of the residuals.
(i) If \( b \geq 1 \), as \( r_k - r_{k+1} \to 0 \), then for large enough \( k \), we have
\[ (r_k - r_{k+1})^b \leq r_k - r_{k+1}. \]

Then (43) is reduced to
\[ r_k \leq c(r_{k-1} - r_k) + a(r_k - r_{k+1}) \text{ as } k \to \infty. \]

Now, consider two cases:
\( \triangleright \) If \( r_{k-1} - r_k \sim_{k \to \infty} O(r_k - r_{k+1}) \) (clearly including the case \( r_{k-1} - r_k \sim_{k \to \infty} r_k - r_{k+1} \)), then
\[ r_k \leq O(r_k - r_{k+1}) + a(r_k - r_{k+1}) = O(r_k - r_{k+1}) \text{ as } k \to \infty. \]
Hence, there exists $\beta$ large enough (let $\beta > 1$) such that
\[ r_k \leq \beta(r_k - r_{k+1}) \text{ as } k \to \infty. \]

Then, we get :
\[ r_{k+1} \leq \frac{\beta - 1}{\beta} r_k \text{ as } k \to \infty. \]

Hence, $\exists q := \frac{\beta - 1}{\beta} \in (0, 1)$ such that
\[ r_k \leq O(q^k) \text{ as } k \to \infty. \]

> If $r_k - r_{k+1} \sim_{k \to \infty} O(r_{k-1} - r_k)$, then we get in a similar way that $\exists \beta > 0$ such that
\[ r_k \leq \beta(r_{k-1} - r_k) \text{ as } k \to \infty. \]

Hence, there exists $q := \frac{\beta}{1 + \beta} \in (0, 1)$ such that
\[ r_k \leq O(q^k) \text{ as } k \to \infty. \]

In both cases, we have the linear convergence of the sequence $\{r_k\}$.

(ii) If $b \in (0, 1)$, we can use a similar technique to simplify the right hand side.

> If $r_{k-1} - r_k \sim_{k \to \infty} O(r_k - r_{k+1})$, then
\[ r_k \leq a(r_k - r_{k+1})^b + O((r_k - r_{k+1})^b) \text{ as } k \to \infty. \]

Hence, there exists $C > 0$ such that
\[ r_k \leq C(r_k - r_{k+1})^b, \]
that is
\[ r^b_k \leq C^b(r_k - r_{k+1}). \]

Thus, $\exists \alpha := \frac{1}{b} > 1, \beta := C^b > 0$ such that
\[ r^\alpha_k \leq \beta(r_k - r_{k+1}) \text{ as } k \to \infty. \]

> If $r_k - r_{k+1} \sim_{k \to \infty} O(r_{k-1} - r_k)$, then
\[ r_k \leq O((r_{k-1} - r_k)^b) \text{ as } k \to \infty. \]

Hence, $\exists \alpha := \frac{1}{b} > 1, \beta > 0$ such that
\[ r^\alpha_k \leq \beta(r_{k-1} - r_k) \text{ as } k \to \infty. \]

In both cases, we apply Lemmas 7 (iii) and 6 (iii) respectively to prove that
\[ r_k \leq O(k^\frac{1}{\alpha+b}) = O\left(k^{\frac{1}{b-1}}\right) \text{ as } k \to \infty. \]

Hence, the sequence $\{r_k\}$ converges sublinearly.  \qed
4.2 Convergence rate of DCA

Now, we are ready to establish the convergence rate for DCA with respect to the sequences \( \{f(x^k)\} \) and \( \{\|x^k - x^*\|\} \) under the KL property.

**Theorem 7 (convergence rate of \( \{f(x^k)\} \))** Under assumptions in Theorem 6 and let \( f(x^k) \rightarrow f^* \). Suppose that \( \Psi \) verifies the KL property with the concave function \( \varphi(t) = Mt^{1-\theta} \) for some \( M > 0 \) and \( \theta \in [0, 1) \). Then we have:

(i) if \( \theta = 0 \), then \( \{f(x^k)\} \) converges to \( f^* \) in finitely many iterations;

(ii) if \( \theta \in (0, \frac{1}{2}] \), then \( \exists q \in (0, 1) \) such that

\[
f(x^k) - f^* \leq O\left(k^{\frac{1}{2-\theta}}\right), \quad \text{as } k \to \infty;
\]

i.e., the sequence \( \{f(x^k)\} \) converges linearly to \( f^* \);

(iii) if \( \theta \in (\frac{1}{2}, 1) \), then

\[
f(x^k) - f^* \leq O\left(k^{-\frac{1}{\theta}}\right), \quad \text{as } k \to \infty;
\]

i.e., the sequence \( \{f(x^k)\} \) converges sublinearly to \( f^* \).

**Proof** By the definition of \( \varphi \), we get

\[
\varphi'(\Psi(x^k)) = M(1-\theta)\psi(x^k)^{-\theta}.
\]

Replacing \( \varphi'(\Psi(x^k)) \) in (39) and squaring on both sides to get for large enough \( k \) that

\[
\psi(x^k)^{2\theta} \leq M^2 L^2 (1-\theta)^2 \|x^{k-1} - x^k\|^2 \leq \frac{2M^2 L^2 (1-\theta)^2}{D}\psi(x^{k-1}) - \psi(x^k)),
\]

where \( D = \rho_g + \rho_h > 0 \) as given in (19). Now, taking \( \alpha = 2\theta > 0, \beta = 2M^2 L^2 (1-\theta)^2/(\rho_g + \rho_h) > 0 \) and \( r_k = \psi(x^k) \), then we obtain the desired convergence rate by Lemma 6. \( ~\square \)

**Remark 19** Similar convergence rates can be obtained for the case where \( g \) has locally Lipschitz continuous gradient using Lemma 7 and Lemma 8.

**Theorem 8 (convergence rate of \( \{\|x^k - x^*\|\} \))** Under assumptions in Theorem 6 and let \( x^k \rightarrow x^* \). Then we have:

(i) if \( \theta = 0 \), then \( \{\|x^k - x^*\|\} \) converges to 0 in finitely many iterations;

(ii) if \( \theta \in (0, \frac{1}{2}] \) and \( \|x^k - x^*\| > 0, \forall k \in \mathbb{N} \), then \( \exists q \in (0, 1) \) such that

\[
\|x^k - x^*\| \leq O(q^k), \quad \text{as } k \to \infty;
\]

i.e., the sequence \( \{\|x^k - x^*\|\} \) converges linearly to 0;

(iii) if \( \theta \in (\frac{1}{2}, 1) \) and \( \|x^k - x^*\| > 0, \forall k \in \mathbb{N} \), then

\[
\|x^k - x^*\| \leq O\left(k^{\frac{1}{2-\theta}}\right), \quad \text{as } k \to \infty;
\]

i.e., the sequence \( \{\|x^k - x^*\|\} \) converges sublinearly to 0.
Proof. (i) If $\theta = 0$, we get from Theorem 7 that the sequence $\{f(x^k)\}$ converges to $f^*$ in a finite number of iterations (says $T$ iterations). Then we conclude from the sufficiently descent property (7), i.e.,

$$f(x^k) - f(x^{k+1}) \geq \frac{\rho_g + \rho_h}{2} \|x^k - x^{k+1}\|^2, \forall k \in \mathbb{N}$$

that

$$\|x^k - x^{k+1}\| = 0, \forall k \geq T.$$  

Hence $x^k = x^T, \forall k \geq T$, implying that $\{x^k\}$ converges to $x^* = x^T$ in $T$ iterations. Otherwise ($\theta \neq 0$), then we must have $\|x^k - x^\ast\| > 0, \forall k \in \mathbb{N}$.

(ii) - (iii) Consider the residual $R_t := \sum_{k \geq t} \|x^k - x^{k+1}\|, \forall t \in \mathbb{N}$. The sequence $\{R_t\}$ is non-negative, non-increasing and converging to 0. Then we get from

$$\|x^t - x^{t+1}\| = R_t - R_{t+1}$$  \hspace{1cm} (47)

and the triangle inequality that

$$\|x^t - x^\ast\| = \|x^t - x^{t+1} + x^{t+1} - x^{t+2} \ldots - x^\ast\| \leq \sum_{k \geq t} \|x^k - x^{k+1}\| = R_t, \forall t \in \mathbb{N}. \hspace{1cm} (48)$$

Now, recall the relation (15) that $\exists N > 0, \forall k \geq N + 1$:

$$\frac{3}{4} \|x^k - x^{k+1}\| \leq \frac{1}{4} \|x^{k-1} - x^k\| + \frac{\max\{C_1, C_2\}}{D} (\phi(x^k) - \phi(x^{k+1})).$$

Then, summing for $k$ from $t(\geq N + 1)$ to $\infty$ and using the fact that $\phi(x^k) \rightarrow 0$ as $k \rightarrow \infty$, we get

$$\forall t \geq N + 1, \frac{1}{2} R_t \leq \frac{1}{4} \|x^{t-1} - x^t\| + \frac{\max\{C_1, C_2\}}{D} \phi(x^t). \hspace{1cm} (49)$$

By the definition of $\phi(s) = Ms^{1-\theta}$ for some $M > 0$ and $\theta \in [0, 1)$. Then

$$\phi'(x^k) = M(1 - \theta) \phi(x^k)^{-\theta}. \hspace{1cm} (50)$$

Recall the relation (39) that, for $k$ large enough, we have

$$\phi'(x^k) \times L \|x^{k-1} - x^k\| \geq 1. \hspace{1cm} (51)$$

Then, we get from (50) and (51) that

$$\phi(x^k)^\theta \leq ML(1 - \theta) \|x^k - x^{k-1}\|. \hspace{1cm} (52)$$

Hence

$$\phi(x^k) = M \phi(x^k)^{1-\theta} \hspace{1cm} \leq M \left(M(1 - \theta)L \|x^k - x^{k-1}\|\right)^{1-\theta} \hspace{1cm} \overset{(52)}{=} M \left(M(1 - \theta)L\right)^{1-\theta} (R_{t-1} - R_t)^{1-\theta}. \hspace{1cm} (47)$$

Injecting this inequality and (47) into (49), we get for $t$ large enough that

$$\frac{1}{2} R_t \leq \frac{1}{4} (R_{t-1} - R_t) + \frac{\max\{C_1, C_2\}}{D} M (M(1 - \theta)L)^{1-\theta} (R_{t-1} - R_t)^{1-\theta} \hspace{1cm} (53)$$
Let \( a := \max\{C_1, C_2\} M (M(1 - \theta)L)^{\frac{1-\theta}{\theta}} \) > 0 and \( b := \frac{1-\theta}{\theta} \).

If \( \theta \in (0, \frac{1}{2}] \), then \( b \geq 1 \) and

\[
(R_{t-1} - R_t)^b = O(R_{t-1} - R_t), \quad \text{as} \ t \to \infty.
\]

Then, it follows from (53) that

\[
R_t \leq O(R_{t-1} - R_t), \quad \text{as} \ t \to \infty.
\]

Hence \( \exists C > 0 \) such that

\[
R_t \leq C(R_{t-1} - R_t), \quad \text{as} \ t \to \infty,
\]

implying that \( \exists q := \frac{C}{1+b} \in (0, 1) \) such that

\[
R_t \leq O\left(q^t\right), \quad \text{as} \ t \to \infty. \tag{54}
\]

If \( \theta \in \left(\frac{1}{2}, 1\right) \), then \( b \in (0, 1) \) and

\[
R_{t-1} - R_t = O((R_{t-1} - R_t)^b), \quad \text{as} \ t \to \infty.
\]

It follows from (53) that

\[
R_t \leq O((R_{t-1} - R_t)^b), \quad \text{as} \ t \to \infty.
\]

Hence \( \exists C > 0 \) such that

\[
R_t^{\frac{1}{b}} \leq C(R_{t-1} - R_t), \quad \text{as} \ t \to \infty,
\]

with \( \frac{1}{b} > 1 \). Using Lemma 6 (iii), we get

\[
R_t \leq O\left(t^{\frac{1}{1-\theta}}\right) = O\left(t^{\frac{1-\theta}{\theta}}\right), \quad \text{as} \ t \to \infty.
\]

Hence

\[
R_t \leq O\left(t^{\frac{1-\theta}{\theta}}\right), \quad \text{as} \ t \to \infty. \tag{55}
\]

Combining (48) \( \|x^t - x^*\| \leq R_t \) with (54) and (55), we get that \( \{\|x^k - x^*\|\} \) converges to 0 linearly if \( \theta \in (0, \frac{1}{2}] \) and sublinearly if \( \theta \in \left(\frac{1}{2}, 1\right) \).

\( \square \)

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**Conflict of interest**

The author declares that there is no conflict of interest.
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