Macrosopic quantum tunneling of the Bose-Einstein condensate trapped in cylindrically symmetric potential

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Abstract

We investigate the macroscopic quantum tunneling of the attractive Bose-Einstein condensate. Within the effective Lagrangian framework, we find bounce solutions and explicitly calculate the decay rate of the condensate trapped in a cylindrically symmetric potential. In particular, in the case where the number of condensed bosons is slightly below a certain critical number, we present a detailed analysis of the bounce solutions and discuss the approximations employed in our calculations. The effects of finite temperatures and the shape of the trapping potential are evaluated.

1 Introduction

Macroscopic quantum tunneling is an interesting subject in many areas of physical sciences including low-temperature physics, atomic physics and nuclear physics. Recent realization of the Bose-Einstein condensate of trapped alkali atoms may provide a good testing ground for the investigation of this problem [1].

In this paper we will discuss the macroscopic quantum tunneling of the Bose-Einstein condensate with attractive interactions. The dynamics of the condensate is successfully described by the Gross-Pitaevskii (GP) equation [2, 3]. The s-wave scattering length \( a \) entering the GP equation can be positive or negative, its sign and magnitude depending crucially on the details of the atom-atom interaction. In the case of \(^7\text{Li}\), the interaction is attractive and the scattering length is known to be \( a = -1.45 \pm 0.04 \text{ nm} \) [4, 5]. The attractive interaction causes the condensate to collapse upon itself. When the trapping potential is included, however, the destabilizing influence of the interaction is balanced by the zero-point kinetic energy, thereby allowing a metastable condensate to form [6, 7, 8]. Pérez-García et al. [8] have investigated the GP equation by using a time-dependent variational ansatz for the condensate wave function. Their results reproduce quite accurately the low energy excitation spectrum of the condensate obtained by numerical simulations of the GP equation. We will apply this variational technique to the macroscopic tunneling of the metastable condensate of \(^7\text{Li}\). When the trapping is spherically symmetric, Ueda and Leggette have evaluated the tunneling decay rate at zero temperature [9] (see also [6, 10]). In this paper we develop their analysis and
explicitly write down the decay rate in the case of a cylindrically symmetric trapping potential and further finite temperatures.

In Section 2, according to [6], we derive an effective Lagrangian describing the Bose-Einstein condensate of $^7$Li, and summarize the data of the ground state energy that we shall need in the calculations of tunneling. In Section 3, we present a detailed analysis of bounce solutions. Using the effective Lagrangian and with the help of numerical simulations, we find the bounce solutions. We next consider the special situation, where the number of condensed bosons is slightly below a certain critical number. Then the effective Lagrangian reduces to a simple one-dimensional Lagrangian by appropriate approximations. We present an analytic solution for the bounce within this situation, and explicitly calculate the decay rate of the metastable condensate. We also evaluate the decay rate at finite temperatures and predict a critical temperature, where the rate crosses over from quantum tunneling to thermal hopping. Section 4 is devoted to the summary of our findings.

2 Model

We consider gases of $^7$Li atoms trapped in a cylindrically symmetric harmonic potential

$$V(x, y, z) = \frac{1}{2}mv^2(x^2 + y^2 + \lambda^2z^2),$$

(1)

where $\lambda$ represents the asymmetry parameter of the trapping potential. The dynamics of the condensate is described by the GP Lagrangian

$$\mathcal{L} = \frac{i\hbar}{2} \left( \psi \frac{\partial \psi^*}{\partial t} - \frac{\partial \psi}{\partial t} \psi^* \right) - \frac{\hbar^2}{2m} \nabla \psi^2 - V|\psi|^2 - \frac{2\pi\hbar^2a}{m}|\psi|^4.$$

(2)

In order to obtain the evolution of the condensate wave function, we assume the Gaussian form for the wave function according to [6]:

$$\psi(x, y, z, t) = A(t) \prod_{a=x,y,z} \exp \left[ -\frac{(x_a - \eta_a(t))^2}{2W_a(t)^2} + ix_a\alpha_a(t) + ix_a^2\beta_a(t) \right].$$

(3)

This trial function includes the time-dependent variational parameters, $\eta = (\eta_x, \eta_y, \eta_z)$ (center coordinate), $W = (W_x, W_y, W_z)$(width) and the phase parameters $\alpha = (\alpha_x, \alpha_y, \alpha_z)$, $\beta = (\beta_x, \beta_y, \beta_z)$ which correspond to the canonically conjugate “momentums” to $\eta$ and $W$. The wave function $\psi$ is normalized by the number of condensed bosons $N = \int |\psi|^2d^3x$, so that the parameter $A$ (amplitude) is given by

$$A = \frac{1}{\pi^{3/4}} \sqrt{\frac{N}{W_xW_yW_z}}.$$

(4)

Substituting (3) into (2) and further integrating the GP Lagrangian over space coordinates, one obtains an effective quantum mechanical Lagrangian

$$\mathcal{L}_{eff} = \sum_{a=x,y,z} (p_a\dot{\eta}_a + K_a\dot{W}_a) - \mathcal{H}_{eff}(\eta_a, W_a, p_a, K_a),$$

(5)
where $p_a$ and $K_a$ are the canonically conjugate momentums to $\eta_a$ and $W_a$ defined by
\begin{align}
p_a &= \hbar N(\alpha_a + 2\eta_a\beta_a), \quad (6) \\
K_a &= \hbar N\beta_a W_a. \quad (7)
\end{align}

The Hamiltonian $H_{\text{eff}} = H_0 + H$ consists of two parts: the first part $H_0$ simply describes the harmonic oscillation of the center of the condensate
\begin{equation}
H_0 = \sum_{a=x,y,z} \frac{1}{2mN}p_a^2 + \frac{Nm\nu^2}{2}(\eta_x^2 + \eta_y^2 + \lambda^2\eta_z^2), \quad (8)
\end{equation}
and the remaining part $H$ describes the evolution of the widths of the condensate
\begin{equation}
H = \sum_{a=x,y,z} \frac{1}{mN}K_a^2 + \hat{U}(W) \quad (9)
\end{equation}
with
\begin{equation}
\hat{U}(W) = \frac{mN\nu^2}{4}(W_x^2 + W_y^2 + \lambda^2 W_z^2) + \frac{\hbar^2 N}{4m} \left( \frac{1}{W_x^2} + \frac{1}{W_y^2} + \frac{1}{W_z^2} \right) + \frac{\hbar^2 N^2}{\sqrt{2\pi m}} \frac{1}{W_x W_y W_z}. \quad (10)
\end{equation}

It is convenient to introduce the scales characterizing the trapping potential: (a) length scale $a_0 = \sqrt{\hbar/m\nu}$, (b) energy scale $e_0 = \hbar\nu/2$, (c) time scale $\nu^{-1}$. By using these units we define dimensionless quantities, $\xi = a_0^{-1}\eta, X = a_0^{-1}W$ and $\tau = \nu t$. Then the Lagrangian (5) is rescaled as follows:
\begin{equation}
L_{\text{eff}} = e_0^{-1}L_{\text{eff}} = L_0 + L, \quad (11)
\end{equation}
where
\begin{equation}
L_0 = N \left( \frac{d\xi}{d\tau} \right)^2 - N(\xi_x^2 + \xi_y^2 + \lambda^2\xi_z^2), \quad (12)
\end{equation}
and
\begin{equation}
L = \frac{N}{2} \left( \frac{dX}{d\tau} \right)^2 - NU(X), \quad (13)
\end{equation}
\begin{equation}
U(X) = \frac{1}{2}(X^2 + Y^2 + \lambda^2 Z^2) + \frac{1}{2} \left( \frac{1}{X^2} + \frac{1}{Y^2} + \frac{1}{Z^2} \right) + \frac{P}{XYZ}. \quad (14)
\end{equation}

with $P = \sqrt{2/\pi Na/a_0} < 0$. We now focus our attention on the ground state energy of the condensed Bose system. Under the present analysis, the ground state energy can be calculated by finding the critical points of $U$ and the eigenvalues of the Hessian matrix $H_{ab} = \partial^2 U/\partial X_a \partial X_b$ evaluated on the critical points.
Critical points

The critical points are given by the solutions to

\begin{align}
X &= Y, \\
\frac{1}{Z^4} + \frac{P}{X^2Z^3} &= \lambda^2, \\
\frac{1}{X^4} + \frac{P}{X^4Z} &= 1.
\end{align}

(15) (16) (17)

The solutions are classified by the critical value \( P^* \) of the parameter \( P \); when \( |P| > |P^*| \), there are no critical points. When \( |P| < |P^*| \), there are two critical points, one stable (Morse index = 0) and the other unstable (Morse index = 1) [6]. The critical value \( P^* \) satisfies in addition to the equations (15) (16) (17) also

\[
\frac{P}{X^2Z^3} + \frac{1}{2} \frac{P^2}{X^6Z^4} = 4\lambda^2,
\]

(18)

which can be derived from the condition \( \epsilon_T = 0 \) (see (22)). Thus, \( P^* \) and the corresponding coordinate \( X^* = (X^*, Y^*, Z^*) \) are uniquely determined as a function of the asymmetry parameter \( \lambda \). Indeed we have \( P^* = -4/5^{5/4} \), \( X^* = 5^{-1/4}(1, 1, 1) \) for \( \lambda = 1 \), and general solutions are provided in Fig.1. It should be noticed that for \( P \to P^* \) the stable critical point \( X_s \) and the unstable critical point \( X_u \) take the following asymptotic forms:

\[
X_{s,u} = X^* \pm k(1 - P/P^*)^{1/2} E + \mathcal{O}(1 - P/P^*),
\]

(19)

where \( E = (-P^*_{32}, -P^*_{32}, 1) \) and the coefficient \( k \) is given by

\[
k = \sqrt{\frac{2}{3}} \left( \frac{P^*_{21}(2P^*_{41} - 1)}{2\lambda^2(1 - P^*_{41}) - P^*_{23}} \right)^{1/2},
\]

(20)

Figure 1: \( \lambda \)-dependence of \( P^* \) and \( X^* = (X^*, Y^*, Z^*) \).
with $P^*_{ij} = P^*/4(X^*)^i(Z^*)^j$.

**Eigenvalues of Hessian matrix**

For the eigenvalue problem of the Hessian matrix evaluated on the critical points

$$He_A = e_A^2 e_A \quad (A = T, N, B),$$

we have the following results [6]:

(a) $T$-direction

$$\epsilon_T^2 = 2 \left( \lambda^2 + 1 - P_{23} - \sqrt{8(P_{32})^2 + (1 - \lambda^2 + P_{23})^2} \right),$$

$$e_T = \frac{1}{\Delta_T}(T, T, P_{32}),$$

with

$$T = \frac{1}{4} \left( -\lambda^2 + 1 + P_{23} - \sqrt{8(P_{32})^2 + (1 - \lambda^2 + P_{23})^2} \right).$$

(b) $N$-direction

$$\epsilon_N^2 = 2 \left( \lambda^2 + 1 - P_{23} + \sqrt{8(P_{32})^2 + (1 - \lambda^2 + P_{23})^2} \right),$$

$$e_N = \frac{1}{\Delta_N}(N, N, P_{32}),$$

with

$$N = \frac{1}{4} \left( -\lambda^2 + 1 + P_{23} + \sqrt{8(P_{32})^2 + (1 - \lambda^2 + P_{23})^2} \right).$$

(c) $B$-direction

$$\epsilon_B^2 = 4(1 - 2P_{41}).$$

$$e_B = \frac{1}{\sqrt{2}}(1, -1, 0).$$

Here we used the notation

$$P_{ij} = \frac{P}{4X^iZ^j} \quad (|P| \leq |P^*|),$$

and

$$\Delta_{T,N}^2 = 2(P_{32})^2 + \frac{1}{4} \left[ (P_{23} + 1 - \lambda^2)^2 \mp (P_{23} + 1 - \lambda^2)\sqrt{8(P_{32})^2 + (1 - \lambda^2 + P_{23})^2} \right]$$

by the normalization $e_A \cdot e_B = \delta_{AB}$. It should be noticed that the eigenvalue $\epsilon_T^2$ is positive (negative) for the stable (unstable) critical point and the other eigenvalues are all positive. These results imply the following ground state energy

$$e_0^{-1} E_0 = NU(X_s) + (2 + \lambda) + \sum_{A=T,N,B} \epsilon_A(X_s).$$

The first term is the potential energy evaluated on the stable critical point $X_s$, and the second-third terms represent the zero-point energy coming from collective excitations of the condensate.
3 Macroscopic quantum tunneling

In this section we argue the macroscopic quantum tunneling of the Bose condensate using the Lagrangian (13). The stable critical point \( X_s \) of the potential \( U(X) \) represents a metastable condensate since the parameter \( P \) in \( U(X) \) is negative, and so the ground state energy will have an (exponentially small) imaginary part in addition to (32) if we take account of the tunneling. The decay rate of the metastable condensate is determined from

\[
\Gamma = \frac{2}{\hbar} \text{Im} E_g.
\] (33)

We will calculate the decay rate by using the WKB approximation. Since the Lagrangian (13) includes a macroscopic quantity \( N \) representing the number of condensed bosons, we must be careful for the choice of a small parameter \( h \) controlling the validity of the WKB approximation. The precise value of \( h \) is given by (52), and the decay rate is of the form

\[
\Gamma \simeq A \exp \left( -\frac{S_{cl}}{h} \right),
\] (34)

where \( S_{cl} \) is the Euclidean action evaluated at bounce solution and \( A \) the square root of the determinant of the second variation around the bounce solution, with the zero - mode removed.

Zero temperature

We start with the Euclidean action:

\[
\frac{S_E}{\hbar} = \frac{N}{2} \int_{-\infty}^{\infty} d\tau \left( \frac{1}{2} \left( \frac{dX}{d\tau} \right)^2 + U(X) \right),
\] (35)

where \( U(X) \) is the potential given by (14). The bounce solution is the classical solution to the equations of motion

\[
\frac{d^2}{d\tau^2} X - X + \frac{1}{X^3} + P \frac{1}{X^2YZ} = 0,
\] (36)

\[
\frac{d^2}{d\tau^2} Y - Y + \frac{1}{Y^3} + P \frac{1}{XY^2Z} = 0,
\] (37)

\[
\frac{d^2}{d\tau^2} Z - \lambda^2 Z + \frac{1}{Z^3} + P \frac{1}{XYZ^2} = 0,
\] (38)

subject to the boundary condition

\[
\lim_{\tau \to \pm \infty} X(\tau) = X_s \quad \text{(stable critical point)}.
\] (39)

In Fig.2, we show the behavior of bounce solutions obtained using numerical simulations.

Let us investigate analytically the system (33) by choosing a parameter \( P \) near the critical value \( P^* \). Then the bounce solution \( X_b(\tau) \) is restricted in the neighborhood of the stable critical point \( X_s \). Indeed, the equation (19) gives the estimation, \( |X_b(\tau) - X_s| \sim |X_s - X_u| \sim O((1 - P/P^*)^{1/2}) \). In the following text we will assume

\[
\delta = 1 - P/P^* \sim 10^{-3}, \quad |P| < |P^*|.
\] (40)
Figure 2: Behavior of the bounce solution. The bold-faced curve connecting the two points, \( X_s \) (stable critical point) and \( X_t \) (turning point) corresponds to the bounce solution. The solid curves represent the contours of the potential \( U(X) \). Parameter values: \( \delta = 1 - P/P^* = 0.144 \) for (a) and \( \delta = 0.135 \) for (b).

This parameter region is particularly interesting; as seen later on the value \( S_E/\hbar \) is of the order one, in this region, though the prefactor \( N \) in the action is very large (the number of atoms used in the experiment at Rice University is of order \( 10^3 \) \([11, 12]\)). Thus we can expect to observe the macroscopic quantum tunneling by experiments.

We now introduce a new coordinate \( \mathbf{x} = (x_T, x_N, x_B) \) around \( X_s \)

\[
\mathbf{X} = X_s + \sum_{A=T,N,B} x_A \mathbf{e}_A
\]  

(41)

and expand the potential

\[
U(\mathbf{X}) = U(X_s) + \frac{1}{2} \sum_{A=T,N,B} \epsilon_A^2 x_A^2 + \sum_{n+m+l=3} c_{nml} x_T^n x_N^m x_B^l + \cdots.
\]  

(42)

It should be noticed that the eigenvalue \( \epsilon_T \) approaches to zero for \( \delta \to 0 \). Indeed, we can evaluate the behavior of \( \epsilon_T \) near \( P^* \) using the exact formula (22):

\[
\epsilon_T = \alpha \delta^{1/4} + \mathcal{O}(\delta^{3/4}),
\]  

(43)

where

\[
\alpha^2 = \frac{4}{\lambda^2 + 1 - P_{23}^*} \sqrt{(-6P_{23}^*)(1 - 2P_{41}^*)(2\lambda^2(1 - P_{41}^*) - P_{23}^*)}.
\]  

(44)

On the other hand, eigenvalues \( \epsilon_N \) and \( \epsilon_B \) can be approximated by (25) and (28) evaluated on \( P^* \), and these values become extremely large compared with \( \epsilon_T \) when the parameter \( \delta \) approaches to
zero. This means that the direction of the initial (infinitesimal) velocity of the bounce solution is given by the eigenfunction $e_T$. Thus the trajectory of the bounce solution is mainly described by $x_T(\tau)$, i.e. $T$-component of the coordinate $x(\tau)$, and remaining components $x_N(\tau)$ and $x_B(\tau)$ give higher order corrections. More precisely, using (42) and (43), we can evaluate the bounce solution as $x_T(\tau) \sim O(\delta^{1/2})$, $x_N(\tau) \sim O(\delta)$ and $x_B(\tau) = 0$ by the symmetry of the equations of motion (if we specialize to the spherically symmetric trapping potential, the $N$-component $x_N(\tau)$ exactly vanishes). We now approximate (35) by one-dimensional quantum mechanical action:

$$S_E \approx \frac{N}{2} \int_{-\infty}^{\infty} d\tau \left( \frac{1}{2} \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} \epsilon_T^2 x^2 + \frac{c}{3!} x^3 \right).$$  \tag{45}$$

From (23) and (24), the coefficient $c(<0)$ is given by

$$c = -12(2P_{24} + 4P_{41}P_{24} - 2\lambda^2 P_{42})(1 + 2(P_{32}^2)^{-3/2}) + O(\delta^{1/2}).$$ \tag{47}$$

It is convenient to introduce new scales characterizing the quantum tunneling: according to Fig.3 we define

(a) length scale $R_0 = \frac{3a_0\epsilon_T^2}{|c|} = \frac{3a_0\alpha^2}{|c|} \delta^{1/2}(1 + O(\delta^{1/2}))$,

(b) energy scale $U_0 = \frac{Nh\nu e_T^6}{3c^2} = \frac{Nh\nu c^6}{3c^2} \delta^{3/2}(1 + O(\delta^{1/2}))$.

Then we have a natural time scale

$$T_0 = \frac{R_0}{(2U_0/Nm)^{1/2}} = \frac{\omega_0}{\nu \alpha \delta^{-1/4}(1 + O(\delta^{1/2}))}, \quad \omega_0 = \sqrt{\frac{27}{2}}.$$ \tag{50}$$

representing the “tunneling time”.

Now the action (45) is of the form

$$S_E \approx \frac{1}{h} \int_{-\infty}^{\infty} ds \left( \frac{1}{2} \left( \frac{dq}{ds} \right)^2 + \tilde{U}(q) \right), \quad \tilde{U}(q) = \frac{1}{2} \omega_0^2 q^2(1-q),$$ \tag{51}$$

where we have used rescaled quantities, $q = (a_0/R_0)x$ and $s = (1/\nu T_0)\tau$. The prefactor (effective Plank constant)

$$h = \frac{\hbar}{U_0 T_0} = \frac{2\omega_0 c^2}{9N\alpha^5} \delta^{-5/4}(1 + O(\delta^{1/2}))$$ \tag{52}$$

is a dimensionless parameter controlling the validity of the WKB approximation. The equations of motion can be easily integrated yielding the well known bounce solution

$$q_b(s) = \text{sech}^2 \left( \frac{\omega_0 s}{2} \right).$$ \tag{53}$$
Figure 3: Potential profile for $\delta \to 0$. The potential $\hat{U}(W)$ given by (10) is approximated by a one-dimensional potential $\hat{U}(R) = e_0N((c^2/2)x^2 + (c/3!x^3)$ with $R = a_0x$. The potential $\hat{U}(R)$ has a metastable minimum at $R = 0$ and a barrier of height $U_0 = \hat{U}(R_m), R_m = 2a_0c^2/|c|$.

Using the WKB approximation [13], we obtain the decay rate

$$\Gamma_0 = \left(4\sqrt{\frac{\omega_0^2}{\pi \hbar}} \exp\left(-\frac{S_{cl}}{\hbar}\right)\right)(1 + \mathcal{O}(\hbar))T_0^{-1} \tag{54}$$

with the bounce action $S_{cl} = \frac{8}{15}\omega_0$. It follows from (50) and (52) that for $\delta \to 0$ the leading contribution to $\Gamma_0$ is given by

$$\frac{\Gamma_0}{\nu} \simeq A\sqrt{N}\delta^{7/8} \exp\left(-BN\delta^{5/4}\right). \tag{55}$$

Here, the coefficients $A$ and $B$ are functions of the asymmetry parameter $\lambda$:

$$A = 4\sqrt{\frac{9}{2\pi}}\frac{\alpha^{7/2}}{|c|}, \quad B = \frac{12\alpha^5}{5c^2}, \tag{56}$$

which can be calculated by (44) and (47). Fig.4 shows the $\lambda$-dependence of these coefficients. The spherically symmetric trapping potential ($\lambda = 1$) minimizes the function $B$ and its value is 4.58 in excellent agreement with the result of [9]. The functions $A, B$ remain relatively constant for $\lambda < 1$ but they grow for $\lambda > 1$. For $\delta \to 0$ the tunneling exponent and the prefactor vanish according to $\delta^{5/4}$ and $\delta^{7/8}$, respectively [4, 10]. We find that this scaling law is universal, independently of the shape of the harmonic trapping potential.

**Finite temperature**

In the case of finite temperature $\beta^{-1}$, the bounce solution is given by a periodic solution, i.e. the classical solution in the potential $-\hat{U}(q)$ with energy $-E$ ($0 < E < 1$). From Fig.5 the solution takes the form [14],

$$q_b(s) = q_2 - (q_2 - q_1)\text{sn}^2\left(\frac{\omega_0}{2\sqrt{q_2 - q_0s}}; m\right) \tag{57}$$
with the elliptic modulus \( m = \sqrt{q_2 - q_0} \) and the period \( h\beta \) is given by the complete elliptic integral of the first kind:

\[
h\beta = \frac{4}{\omega_0 \sqrt{q_2 - q_0}} K(m), \quad K(m) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - mx^2)}}.
\] (58)

This solution reduces, of course, to the previous solution (53) for \( E = 0 \). The corresponding bounce action is evaluated as

\[
S_{\text{cl}} = \int_0^{h\beta} ds \left( \frac{1}{2} \left( \frac{dq_b}{ds} \right)^2 + \tilde{U}(q_b) \right) = W + h\beta E,
\] (59)

where

\[
W = \frac{4\omega_0}{15} \sqrt{q_2 - q_0} \left[ 2(q_0^2 + q_1^2 + q_2^2 - q_0q_1 - q_0q_2 - q_1q_2)E(m) + (q_1 - q_0)(2q_0 - q_1 - q_2)K(m) \right]
\] (60)

(\( E(m) \) is the complete elliptic integral of the second kind). The fluctuation modes about the bounce solution include a zero mode \( \phi_1(s) = \dot{q}_b(s) \). Then the determinant factor \( A \) in (34) is calculated from the Gelfand-Yaglom formula [13, 16] :

\[
A(\beta) = \frac{1}{\sqrt{\pi h}} \left. \frac{\phi_1(s)}{\phi_2(s)} \sinh (\omega_0 s) \right|_{s = \beta h/2}, \quad \phi_2(s) = \phi_1(s) \int_0^s \frac{ds'}{\phi_1(s')^2}.
\] (61)

Thus we obtain the finite temperature decay rate due to quantum tunneling:

\[
\Gamma(\beta) = \left( A(\beta) \exp \left( \frac{-S_{\text{cl}}}{h} \right) \right) (1 + \mathcal{O}(h)) T_0^{-1},
\] (62)

Figure 4: \( \lambda \)-dependence of the functions \( A \) and \( B \). (b) shows the details of \( B(\lambda) \) in the region of \( \lambda = 1 \).
Figure 5: Turning points in the potential \( \tilde{U}(q) = \frac{1}{2} \omega_0^2 q^2 (1 - q), \omega_0 = \sqrt{27/2} \). The “energy” \( E(0 < E < 1) \) is determined as a function of \( \beta \) by requiring that the motion between the turning points \( q_1 \) and \( q_2 \) is periodic, with period \( \beta \hbar \).

where

\[
A(\beta) = \sqrt{\frac{\omega_0^3 (q_2 - q_0)^{3/4}(q_2 - q_1)(1 - m^2)}{2\pi \hbar (a(m)E(m) + b(m)K(m))^{1/2}}} \sinh \left( \frac{\omega_0 \beta \hbar}{2} \right)
\]

with

\[
a(m) = 2(m^4 - m^2 + 1), \quad b(m) = (1 - m^2)(m^2 - 2).
\]

For \( E \to 0 \), we have \((1 - m^2) \sinh(\omega_0 \beta \hbar / 2) \to 8, a(m)E(m) + b(m)K(m) \to 2 \) and \( q_0, q_1 \to 0, q_2 \to 1 \), so that \( A(\beta) \to 4\sqrt{\omega_0^3 / \pi \hbar} \), which reproduces the zero-temperature decay rate \( \Gamma_0 \). Let us turn now to the limit \( E \to 1 \), where the period behaves as

\[
\beta \hbar = \frac{2\pi}{\omega_0} \left( 1 + \frac{5}{36} (1 - E) + \cdots \right).
\]

The leading term gives a crossover temperature \( \beta_c^{-1} = \hbar \omega_0 / 2\pi \) \[9\], i.e. for \( \beta^{-1} > \beta_c^{-1} \) the decay rate is given by the familiar Arrhenius-Kramers formula \[18\]. On the other hand, for \( \beta^{-1} < \beta_c^{-1} \) the macroscopic tunneling through the barrier becomes more probable, and the decay rate is given by \[72\]. Recalling the energy unit \( U_0 \) defined by \[49\], and \[52\] we find

\[
\beta_c^{-1} = \frac{\hbar \omega_0}{2\pi} \left( \frac{U_0}{k_B} \right) = \frac{\hbar \nu_\alpha}{2\pi k_B} \delta^{3/4}(1 + O(\delta^{1/2})�).
\]

For small \( (\beta - \beta_c)/\beta_c > 0 \), from \[73\] and \[63\], we obtain the bounce action

\[
\frac{S_{cl}}{\hbar} \approx \beta - \frac{18}{5} \beta_c \left( \frac{\beta - \beta_c}{\beta_c} \right)^2,
\]
and

\[ A(\beta) \simeq \sqrt{\frac{8\omega_0^3}{15\hbar^2\pi^2}} \sinh\left(\frac{\omega_0\beta \hbar}{2}\right) \left(1 - \frac{77}{20} \left(\frac{\beta - \beta_c}{\beta_c}\right) + \frac{20867}{2400} \left(\frac{\beta - \beta_c}{\beta_c}\right)^2\right)\].

(69)

Figure 6: The decay rate \( \Gamma_0 \) as a function of the asymmetry parameter \( \lambda \) for \( \nu = 953 \, \text{sec}^{-1} \), \( a_0/a = -2.13 \times 10^3 \) and \( \delta = 5.0 \times 10^{-3} \).

4 Conclusion

In this paper we have investigated the macroscopic tunneling of the metastable condensate of \(^7\text{Li}\). When the number of particles in the condensate exceeds a critical value \( N^* = \sqrt{\pi/2} P^* a_0/a \), the metastable condensate no longer exists and the equation giving critical points has no solutions. In a region extremely close to \( N^* \), i.e. \( \delta = 1 - P/P^* \ll 1 \), we have shown that the action takes a rather simple form (51), and explicitly calculated the decay rate of the metastable condensate using the WKB approximation.

Finally we make some remarks on our results. In order to justify the WKB approximation, we should choose the effective Plank constant \( h \) to satisfy the condition \( h \ll 1 \). On the other hand, for very small \( h \), it is impossible to observe the macroscopic tunneling; the formula (54) provides an estimate of the tunneling decay rate, \( \Gamma_0 \sim O(e^{-1/h}) \). This implies rather severe conditions on the parameter \( \delta \) through the equation (52). If we use the experimental data at Rice University for the trapping potential \([11, 12]\): \( \lambda \simeq 0.867 \), \( a_0/a \simeq -2.13 \times 10^3 \) and \( \nu \simeq 953 \, \text{sec}^{-1} \), then the conditions are given by \( h \simeq 2.92 \times 10^{-4} \delta^{-5/4} \ll 1 \) and \( \Gamma_0 \simeq 8.17 \times 10^5 \delta^{7/8} \exp(-6.72 \times 10^3 \delta^{5/4}) \sim O(1) \, \text{sec}^{-1} \). Consequently, we have a typical region \( 3.0 \times 10^{-3} < \delta < 7.0 \times 10^{-3} \). Temperature effects on the tunneling decay rate are estimated by using the equations (68) and (69): \( \Gamma(\beta) \) is monotone decreasing for \( \beta > \beta_c \) and hence \( \Delta \Gamma = \Gamma(\beta) - \Gamma_0 < \Gamma(\beta_c) - \Gamma_0 \). For instance, for \( \delta = 5.0 \times 10^{-3} \) the decay rate at zero temperature is \( \Gamma_0 \simeq 1.03 \, \text{sec}^{-1} \) and \( \Delta \Gamma < 2.79 \, \text{sec}^{-1} \). The crossover temperature is then given by \( \beta_c^{-1} \simeq 1.02 \, \text{nK} \), which may be a realizable temperature in the experiments. The details of
a crossover region have been discussed in [17]; there is a narrow crossover region of $O(h^{3/2})$, where the decay rate is given by

$$
\Gamma(\beta)T_0 \approx \sqrt{8\omega_0^3 \over 15\pi^2} \sinh \left( {\omega_0 \beta h \over 2} \right) \text{erf} \left[ \sqrt{36 \over 5\beta_c} (\beta - \beta_c) \right] \exp \left[ -\beta + 18\beta_c \over 5 \left( {\beta - \beta_c \over \beta_c} \right)^2 \right],
$$

(70)

with \( \text{erf}(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} dy \exp(-y^2/2) \). For very small \( h \ll 10^{-2} \), this formula matches smoothly onto (62) and \( \Gamma(\beta)T_0 = (\omega_0/2\pi)[\sinh(\omega_0\beta h/2)/\sin(\omega_0\beta h/2)]\exp(-\beta) \) (Arrhenius-Kramers formula) near \( \beta_c \). However, we can not apply the formula to the macroscopic tunneling since the value of \( h \) in our situation is too large. We leave the issue of crossover region for future research. The shape of the trapping potential also has some effect on the behavior of the decay rate \( \Gamma_0 \): as shown in Fig.\ref{fig:6}, the effect is significant for the disk-shaped potential \( (\lambda > 1) \), although it is rather small for the cigar-shaped potential \( (\lambda \ll 1) \) and \( \Gamma_0 \) is of order \( 10^{-3} \text{ sec}^{-1} \), independently of \( \lambda \).

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