Relevance of complex branch points for partial wave analysis

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A central issue in hadron spectroscopy is to deduce—and interpret—resonance parameters, namely, pole positions and residues, from experimental data, for those are the quantities to be compared to lattice QCD or model calculations. However, not every structure in the observables derives from a resonance pole: the origin might as well be branch points, either located on the real axis (when a new channel composed of stable particles opens) or in the complex plane (when at least one of the intermediate particles is unstable). In this paper we demonstrate first the existence of such branch points in the complex plane and then show on the example of the \( \pi N \) \( P_{11} \) partial wave that it is not possible to distinguish the structures induced by the latter from a true pole signal based on elastic data alone.

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I. INTRODUCTION

The second and third resonance region of baryonic excited states is currently under intense experimental investigation at various laboratories such as ELSA, MAMI, and JLab [1–4]. Many resonances overlap at these energies, and usually partial wave analyses in different frameworks, such as \( K \)-matrix approaches or dynamical coupled-channel models [5–23], are necessary to disentangle the resonance content. Furthermore, many resonances may couple only weakly to the \( \pi N \) channel, and the investigation of different initial and final states in hadronic reactions is mandatory [19]. Also, at these energies multipion intermediate and final states are becoming increasingly important and should be included in the analysis of the \( S \) matrix. For the corresponding \( T \) matrix, channels with stable particles like \( \eta N \) induce a branch point at the threshold energy \( (\sqrt{s} = m_\eta + M_N) \), which may be visible as a cusp in the amplitude [24,25].

For effective multipion channels with one unstable and one stable particle, such as \( \rho N \), the analytic structure is more complicated. In comparison to the branch points on the real \( s \) axis and the first and second sheet poles, the third type of allowed singularities is given by branch points within the complex energy plane. They emerge when amongst groups of particles of an at least three-body decay there exists a strong correlation between two particles. For example, a significant fraction of \( \pi^+ \pi^- \) intermediate and final states typically goes through the \( \rho \) meson. The resulting line shapes are discussed in Ref. [26]. Branch points in the complex plane also emerge in the recently developed complex-mass scheme for baryonic resonances [27].

Known theoretically for a long time [28,29], these branch points are present in several modern approaches, such as the GWU/SAID analysis [5,6], the Jülich [15–19] and EBAC [21,22] approaches, and the Bonn-Gatchina [14] analysis. It is the goal of this study to demonstrate the model-independent character of these complex branch points. To do so, we employ general properties of the \( S \) matrix only. In a particular example it is then shown that the branch points are of relevance in partial wave analyses: if the theoretical partial wave does not include them, their absence can easily be simulated by resonance poles. This, of course, distorts the extracted baryon spectrum. Branch points in the complex plane are thus important for the reliable extraction of resonance parameters.

The paper is organized as follows: in Sec. II the existence of branch points in the complex plane is derived from three-body phase space, in Sec. II A the properties of the branch points are determined, and in Sec. III it is shown that these branch points are relevant in the extraction of the resonance content of partial waves.

II. ANALYTIC STRUCTURE OF THE S MATRIX AND COMPLEX BRANCH POINTS

Every channel opening introduces a new branch point and with it a new sheet to the \( S \) matrix, located at \( s = (\sum m_i)^2 \), with \( m_i \) being the masses of the stable particles in that channel. The first sheet is always the physical one, i.e., where the physical amplitude is situated. The only singularities allowed on the first sheet are poles on the real \( s \) axis below the lowest threshold (= bound states) or branch points on the real axis. On other sheets, poles and branch points can be located anywhere. Poles on the second sheet are called resonances if their real part is located above the lowest threshold, and they are called virtual states if they are located below the threshold, but on the real axis. It is also possible to have poles on the second sheet inside the complex plane with a real part lower than the threshold [30] or on other hidden sheets that are often referred to as shadow poles.

In this study we are interested in branch points on the second sheet in the complex plane, i.e., on the same sheet on which the resonance poles are situated. To prove the emergence of these branch points, let us start from the optical
where $T(j \rightarrow i) = T^+(j \rightarrow i)$ is the $T$ matrix connecting channels $i$ and $j$, and $d\Phi_j$ denotes the phase space of channel $j$. To simplify the argument we assume that the $T$ matrix is in a particular partial wave; below we focus on the singularities of the function that stem from the unitarity cuts only. Singularities like the left-hand cuts, the short nucleon cut [18,31], or the circular cut, induced by the partial wave projection, are ignored in the following for they are irrelevant for the argument given.

To be specific we use the normalization of phase space as proposed by the Particle Data Group [32]. Then we have the following for the $n$-particle phase space,

$$d\Phi_n(P; p_1, \ldots, p_n) = \delta^{(4)} \left( P - \sum_i p_i \right) \prod_i \frac{d^3p_i}{(2\pi)^2E_i},$$

where $P$ is the overall center-of-mass (c.m.) four-momentum.

To avoid complications, which are irrelevant for the validity of the present argument, we now focus on the diagonal channel $i = j$. To be concrete we assume $i = N$. To further simplify the argument, in addition, we focus on $f = N$ as the only relevant intermediate $\pi N$ channel. The latter assumption allows us to write

$$T(\pi N \rightarrow \pi N) = i W(m_{\pi N}^2)D(m_{\pi N}^2)T(\pi N \rightarrow N),$$

where $D(m_{\pi N}^2)$ denotes the physical $\rho$ propagator as a function of the $\pi N$ invariant mass $m_{\pi N}$ and $W(m_{\pi N}^2)$ is the partial wave projected decay vertex, which contains also the information on the orbital angular momentum $\ell$ of the decay into $\pi N$. In the following we abbreviate $m \equiv m_{\pi N}$.

One can decompose the three-body phase space into two subspaces [32],

$$d\Phi_3(P; P_1, \ldots, P_n) = d\Phi_j(q; p_1, \ldots, p_j) \times d\Phi_{n-j+1}(P; q, p_{j+1}, \ldots, p_n)(2\pi)^3d^3m^2.$$  

For the example of the $\rho[\pi\pi]$ $N$ system considered here, the first factor $d\Phi$ refers to the $\pi\pi$ phase space in the $\rho$ subsystem at four-momentum $q$ (note that $m^2 = q^2$), the second is the $\rho N$ phase space at four-momentum $P$, and $n = 3$ and $j = 2$. With this decomposition,

$$\int d\Phi_j(q; p_1, \ldots, p_j)|D(m^2)|W(m^2)|^2 = \frac{1}{\pi} \text{Im}\{D(m^2)\} = \rho(m^2),$$

where $\rho(m^2)$ denotes the spectral density for the resonance normalized via

$$\int_{4m^2_\rho}^\infty dm^2 \rho(m^2) = 1.$$  

We get for the discontinuity of the $\pi N$ amplitude from the $\pi\pi N$ channel

$$\frac{1}{i}[T(\pi N \rightarrow \pi N) - T^+(\pi N \rightarrow \pi N)] = (2\pi)^7 \int \frac{d^2\rho(m^2)}{256\pi^6} \sqrt{s} \frac{d\Omega}{\sqrt{\rho}} \times |T(\pi N \rightarrow N)(s, m^2)|^2 + \ldots,$$

where the ellipses denote contributions from the other channels omitted here. The two-body phase space can be calculated explicitly. One finds

$$d\Phi_2(P; q, p_3) = \frac{1}{256\pi^6} \frac{p(\sqrt{s}, m_3)}{\sqrt{s}} \, d\Omega,$$

with

$$p(\sqrt{s}, m_3) = \frac{1}{2\sqrt{s}} \sqrt{(s - (m_3 + m^2))(s - (m_3 - m^2))}$$

for the c.m. momentum of the nucleon (particle 3) and the pion pair with invariant mass $m_3$.

Using Eq. (7) we may thus express the $T$ matrix through a dispersion integral and obtain

$$T(\pi N \rightarrow \pi N) = \frac{1}{4} \int_{4m^2_\rho}^\infty d\rho(m^2) \rho(m^2) p(\sqrt{s}, m_3) \times \int d\Omega \left| \frac{T(\pi N \rightarrow N)(s', m^2)}{s' - s + i\epsilon} \right|^2 + \ldots,$$

where now the ellipses stand for the unitarity cut contributions from other channels as well as left-hand cut contributions. First of all, there is the three-body cut, which drives the inelasticity of the $T$ matrix. To be concrete, we may write

$$\rho(m^2) = \frac{N}{\pi} \text{Im} \frac{1}{m^2 - m_\rho^2 + im_\rho \Gamma},$$

where $p_0$ is the three-momentum at the nominal resonance mass and $N$ is a normalization factor so that Eq. (6) is fulfilled. The factor $(\rho/p_0)^{2\ell+1}$ accounts for the centrifugal barrier and $p$ is the pion momentum in the rest frame. Note, $\rho = p(m, m_\pi, m_\pi)$ at the $\pi N$ threshold ($\sqrt{s} = 2m_\pi + M_N$); i.e., the p is at rest and the $\rho$ rest frame and the overall rest frame coincide. Note also that the explicit form of Eq. (11) is only for illustration. The $m$ dependence of the denominator is more complicated in general (see, e.g., the Appendix), but the only property needed in the following is the presence of poles in the spectral function.

Indeed, the spectral function $\rho(m^2)$ of Eq. (11) contains a pair of poles located at $m^2 = m_0^2$, where $m_0$ denotes the pole position of the $\rho$ meson, located in the complex plane. We may write $m_0 = m_\rho \pm i\Gamma/2$, where $\Gamma$ denotes the width of the $\rho$ meson.

For the existence of branch points in the complex plane, it is sufficient to consider the imaginary part of Eq. (10) in
the following, or, more correctly, we consider the analytic function $\delta T$, which is $\delta T = \text{Im} \, T$ for $\sqrt{s} \in \mathbb{R}$, but of course $\delta T \neq \text{Im} \, T$ for $\sqrt{s} \notin \mathbb{R}$ (e.g., $\delta T$ develops an imaginary part for complex $\sqrt{s}$, whereas $\text{Im} \, T$ does not). The function $\delta T$ can be straightforwardly evaluated,

$$\delta T = -\frac{\pi}{4\sqrt{s}} \int_{4m_{\rho}^2}^{(\sqrt{s} - m_N)^2} dm^2 \rho(m^2) \times p(\sqrt{s}, m, M_N) p(\sqrt{s}, m, M_N)^2 L g(\sqrt{s}, m),$$

for which the lower integration limit coincides with the branch point $z_b$, whereas $\text{Im} \, T$ develops an imaginary part for complex $\sqrt{s}$, whereas $\text{Im} \, T$ does not. The function $\delta T$ can be straightforwardly evaluated,

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for which the lower integration limit coincides with the branch point $z_b$, whereas $\text{Im} \, T$ develops an imaginary part for complex $\sqrt{s}$, whereas $\text{Im} \, T$ does not. The function $\delta T$ can be straightforwardly evaluated,
of \( m^2 \) will be needed, and thus the (nonanalytic) function \( \text{Im} \) in Eq. (11) needs to be evaluated to obtain a meromorphic expression,

\[
\rho(m^2) = \frac{N}{\pi} \frac{m_p \tilde{\Gamma}}{(m^2 - m_{\rho}^2)^2 + m_p^2 \delta^2} \left( \frac{m - m_0}{m^2 - m_0^2} \right)^{\frac{m-m_0}{m^2 - m_0^2}} \Gamma_1(m^2) .
\]

(15)

The right-hand side shows the behavior of \( \rho(m^2) \) close to the pole at \( m = m_0 \); the function \( \Gamma_1 \) does not contain any poles or zeros close to \( m_0 \) and thus does not influence the threshold behavior. In particular, \( \tilde{\rho}^{2L+1} \), which appears in the numerator [cf. Eq. (11)], has no zero close to \( m_0 \) and can be absorbed in \( \Gamma_1 \). Thus the threshold behavior of the branch points \( z_{b1} \) and \( z_{b2} \) does not depend on \( \ell \), which may appear to be a surprising result.

To obtain the threshold behavior of the branch points \( z_{b1} \) and \( z_{b2} \) in the complex plane, one inserts the right-hand side of Eq. (15) into Eq. (12),

\[
\delta T \sim \int_{4m_\pi^2}^{(\sqrt{s} - M_N)^2} \frac{dm^2}{\Gamma_1(m^2)} \left( \frac{m^2 - m_0^2}{m^2 - m_0^2} \right)^{\frac{m-m_0}{m^2 - m_0^2}} h_2(m^2) .
\]

(16)

where we have expanded the argument of the square root of the \( p \) factors of Eq. (12) in \( m^2 \), at the point \( p(\sqrt{s} = z_{b} = M_N + m_0, m, M_N) \) to obtain the power of the leading zero from these factors. The function \( h_2 \) is again analytic, free of zeros close to \( m = m_0 \), and does not influence the threshold behavior. The integral may now be evaluated setting this numerator and \( h_2 \) constant. The result for the threshold behavior of the branch points \( z_{b1} \) and \( z_{b2} \) is [see also Eq. (13)]

\[
\delta T(z_{b1}, z_{b2}) \sim \left( \sqrt{s} - z_{b1}, z_{b2} \right)^\frac{1}{2L+1} \sim \rho(\sqrt{s}, m_0, M_N)^{2L+1} .
\]

(17)

In Fig. 3 we show the branch point \( z_{b2} \) in the upper \( \sqrt{s} \) half plane [see Fig. 2] for a realistic \( \rho N \) intermediate state and \( L = 0 \). The branch point is clearly visible, together with the cut that in this picture is chosen in the positive Re \( \sqrt{s} \) direction.

To obtain the threshold behavior for the third branch point at \( z_{b3} = 2m_\pi + M_N \) (see Fig. 2), we inspect again Eq. (15). As discussed following Eq. (11), close to \( \sqrt{s} = z_{b3} = 2m_\pi + M_N \) [see Eq. (14)] the \( p \) c.m. frame coincides with the overall c.m. frame, i.e., \( \tilde{p} = \rho(m, m_\pi, m_\pi) \), and thus the \( \ell \)-wave decay in the \( \rho \) subsystem is also an \( \ell \)-wave decay in the overall c.m. system. For \( \sqrt{s} \) in the vicinity of \( z_{b3} \), the denominator of Eq. (15) is free of zeros; however, in contrast to the case of \( z_{b1} \) and \( z_{b2} \), the numerator \( \tilde{\Gamma} \) \( \sim \rho^{2L+1} \) \( \sim \rho^{2L+1} \) \( \sim (m^2 - 4m_\pi^2)^{2L+1/2} \) does have a zero that contributes to the threshold behavior. Inserting Eq. (15) (including this factor) in Eq. (12) and expanding the arguments of the square roots of the \( p \) factors around zero [cf. Eq. (9)], one obtains

\[
\delta T \sim \int_{4m_\pi^2}^{(\sqrt{s} - M_N)^2} \frac{dm^2}{\Gamma_1(m^2)} \left( \frac{m^2 - 4m_\pi^2}{m^2 - 4m_\pi^2} \right)^{\frac{m-m_0}{m^2 - m_0^2}} h_3(m^2) .
\]

(18)

with a function \( h_3 \) free of zeros and poles in the vicinity of \( z_{b3} \). Integration leads now to the threshold behavior of \( z_{b3} \). Thus the threshold behavior of the branch points \( z_{b1} \) and \( z_{b2} \) does not depend on \( \ell \), which may appear to be a surprising result.

B. The limit of vanishing width

It is instructive to study the limit of a vanishing width of the \( \rho \) meson in Eq. (7). Then

\[
\rho(m^2) \longrightarrow \delta \left( m^2 - m_{\rho}^2 \right) \quad \text{for} \quad \Gamma \rightarrow 0 .
\]

This allows us to perform the \( m^2 \) integration to get

\[
\frac{1}{\Gamma} \left[ \Gamma \left( \pi N \rightarrow \pi N \right) - \Gamma \left( \pi N \rightarrow \pi N \right) \right]
\]

\[
= \left( 2\pi \right)^4 \Theta \left( \sqrt{s} - M_N \right)^2 - m_{\rho}^2 \int d\Phi_2(P; q, p_3)
\]

\[
\times \left| T(\pi N \rightarrow \rho N)(s, m_\rho) \right|^2 + \ldots ,
\]

(20)

such that Eq. (10) reduces to the dispersion integral over the standard two-body cut

\[
T(\pi N \rightarrow \pi N) \rightarrow \int_0^\infty \frac{dM_\pi}{\sqrt{s}} \rho(\sqrt{s}, m_\rho, M_N)
\]

\[
\times \int d\Omega \left| T(\pi N \rightarrow \rho N)(s', m_\rho) \right|^2 + \ldots
\]

(21)

The imaginary part, which is given by

\[
\delta T_{\Gamma \rightarrow 0} = -\pi/(4\sqrt{s}) \rho(\sqrt{s}, m_\rho, M_N)^{2L+1} g(\sqrt{s}, m_\rho) .
\]

(22)
has a branch point at $\sqrt{s} = m_\rho + M_N$, which is simply the ordinary two-body threshold on the real $\sqrt{s}$ axis. As $\Gamma \to 0$, the two branch points $z_{b1}$ and $z_{b2}$ in the complex plane move toward the real $\sqrt{s}$ axis until they coincide and form this single branch point at $\sqrt{s} = m_\rho + M_N$. Note that there is a factor of $\Gamma$ in the numerator of Eq. (16), but in the limit $\Gamma \to 0$, another factor $\sim \Gamma$ appears in the denominators from the two poles moving to the real axis, which cancels the $\Gamma$ of the numerator. Thus, indeed the branch point persists in the limit $\Gamma \to 0$ with the result given in Eq. (22).

For the third branch point at $z_{b3} = 2m_\pi + M_N$, Eq. (18) shows that there are no poles that can prevent the term from disappearing in the limit $\Gamma \to 0$; thus, as $\Gamma \to 0$, the third branch point fades away. In other words, $\Gamma \to 0$ means that the $\rho$ decouples from $\pi \pi$ and, thus, in our example, the $\pi \pi N$ channel decouples from $\pi N$.

### III. THE RELEVANCE OF BRANCH POINTS IN THE COMPLEX PLANE

As shown in the previous section, whenever there is a multi-particle intermediate state with pairwise strong correlations, unavoidably branch points show up in the complex plane. As we demonstrate on a particular example in this section, their influence on the data might well be visible. However, as will be also shown, it is in general not possible to deduce the origin of such a structure from elastic data only.

The first model we use is the so-called Jülich model [15–19]. It is a coupled-channel meson-exchange model including the channels $\pi N$, $\eta N$, $K \Lambda$, and $K \Sigma$ as well as three effective $\pi \pi N$ channels, namely, $\pi \Delta$, $\sigma N$, and $\rho N$. All these two-pion channels show the mentioned kind of branch points [18]. In the Appendix we show the connection of the formalism of the Jülich model to the one of the previous section. The Jülich model allows for a good description of the available $\pi N$ data in all partial waves with $j \leq 3/2$ up to an energy of 1.8 GeV and has been recently extended to higher energies, partial waves, and additional reactions [19].

To be specific we focus here on the $P_{11}$ partial wave and the region around $\sqrt{s} \sim 1.7$ GeV. In this energy region, around 300 MeV above the Roper resonance, signals for another resonance, $N(1710)P_{11}$, have been found in several analyses [32]. It is, however, remarkable that in recent analyses of the GWU/SAID group [6], there is no sign for this resonance any more. Like the GWU/SAID analysis, the Jülich model contains explicitly the branch points $z_{b1}$ and $z_{b2}$ in the complex plane at $\sqrt{s} = M_N + m_\rho \pm i\Gamma_{1/2} \sim 1700 \pm 75i$ MeV. However, there are no poles around these energies (the only genuine pole $\sim z_b$ the two branch points

![FIG. 4. (Color online) Fit of the CMB Zagreb model (solid lines) to the $P_{11}$ amplitude provided by the Jülich model (dashed lines). The “data” points represent the single energy solution of the GWU/SAID group [6].](image)
of $K^0\Lambda$ data to further clarify the role of the $N(1710)P_{11}$ (see also Ref. [36]).

Thus, the aim of the present short exercise is not to discard the existence of the much-debated $N(1710)P_{11}$ as such. Rather, we have shown that branch points in the complex plane are relevant; in their absence, resonances may be needed to simulate them; and, thus, the extracted baryon spectrum can be easily distorted.

IV. CONCLUSIONS

Using only general properties of the $S$ matrix we have shown the existence and determined the position of three branch points induced by intermediate quasi-two-body states. Those are three-body states in which two particles are so strongly correlated that the scattering amplitude of this subsystem has a pole. A pole in the subsystem necessarily leads to the appearance of branch points in the complex $\sqrt{s}$ plane of the overall $N$ amplitude. This result is model independent because it does not depend on any particular parametrization, but only on analyticity and general properties of the three-body phase space. We have also determined the threshold behavior of all branch points, which depends on the orbital angular momenta of the two decay processes involved. Finally, on the example of the $P_{11}$ partial wave, it has been shown that branch points in the complex plane are relevant in partial wave analysis: if a theoretical amplitude does not contain the branch points, false resonance signals may be obtained. To allow for a reliable extraction of the baryon spectrum, it is thus mandatory to include also these branch points in the analysis.

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APPENDIX: SPECTRAL REPRESENTATION OF THE JULICH MODEL

In this appendix the connection of the field theoretical formalism, used in the Jülich model of hadron exchange, to the formalism used in this study is outlined, up to overall normalization factors. For further details of the formalism used in the Jülich model, we refer to Ref. [18]. For the example of the $\rho N$ propagator that is considered here, the propagator on the real axis is given by

$$g_{\rho N}(\sqrt{s}, k) = \frac{1}{\sqrt{s} - E_N(k) - E_{\rho}^0(k) - \Sigma[z_\rho(\sqrt{s}, k), k]},$$  \hspace{1cm} \text{(A1)}$$

where $E_N$ is the nucleon energy, $E_{\rho}^0$ is the $\rho$ energy using the bare $\rho$ mass, and $\Sigma$ is the $\rho$ self-energy, where $z_\rho(\sqrt{s}, k)$ is the boosted energy for the $\rho$ subsystem. The explicit form of $z_\rho(\sqrt{s}, k)$ is quoted in Ref. [18] but for the present discussion the only needed property is that $z_\rho(\sqrt{s}, k = 0) = \sqrt{s} - M_N$. The propagator $g_{\rho N}$ is iterated in the multichannel scattering equation, but to investigate the analytic structure it is sufficient to consider the one-loop amplitude

$$G_{\rho N}(\sqrt{s}) = \int_0^\infty dk \, k^2 \, g_{\rho N}(\sqrt{s}, k),$$  \hspace{1cm} \text{(A2)}$$

where for simplicity we have omitted the form factors that regularize this divergent expression. One can rewrite the Dyson-Schwinger representation of Eq. (A1) with the spectral function

$$S(\omega, k) = -\frac{1}{\pi} \text{Im} \frac{1}{\omega - E_N(k) - E_{\rho}^0(k) - \Sigma[z_\rho(\omega, k), k]},$$  \hspace{1cm} \text{(A3)}$$

resulting in the Lehmann representation

$$g_{\rho N}(\sqrt{s}, k) = \int_{2m_\pi + M_N}^{\infty} d\omega \frac{S(\omega, k)}{\sqrt{s} - \omega + i\epsilon}. \hspace{1cm} \text{(A4)}$$

For the imaginary part of the $\rho N$ loop $G_{\rho N}(\sqrt{s})$, one obtains

$$\text{Im} \, G_{\rho N}(\sqrt{s}) = \text{Im} \int_0^\infty dk \, k^2 \, g_{\rho N}(\sqrt{s}, k) = -\pi \int_0^{k_1} dk \, k^2 \, S(\sqrt{s}, k) = -\pi \int_0^{k_1} dk \, k^2 \, S(\sqrt{s}, k). \hspace{1cm} \text{(A5)}$$

The last equality shows that the integration can be cut at $k = k_1$ as for $k > k_1$ the spectral function is zero because then $z_\rho(\sqrt{s}, k) < 2m_\pi$. In particular, $k_1$ is given by $z_\rho(\sqrt{s}, k_1) = 2m_\pi$. Note that the explicit evaluation of the integration limits as done here is necessary if one wants to use the spectral representation in the complex $\sqrt{s}$ plane. This has been shown recently in the context of Feynman parametrized loops [37]: the integration limits have to be analytically continued for complex $\sqrt{s}$ to obtain the analytic continuation of the loop itself, and for this they need to be known explicitly.

Equation (A5) can be rewritten as

$$\text{Im} \, G_{\rho N}(\sqrt{s}) = \int_m^{\sqrt{s} - m_\pi} dm \, \frac{S(\sqrt{s}, k^\text{on}(m), m)}{E_m^\text{on} \times (-\pi) \frac{k^\text{on}(m)}{E_m^\text{on} \sqrt{s}}}, \hspace{1cm} \text{(A6)}$$

with

$$k^\text{on}(m) = p(\sqrt{s}, m, m_\pi), \quad E_m^\text{on} = \sqrt{m_\pi^2 + (k^\text{on})^2}, \quad E_m^\text{on} = \sqrt{m^2 + (k^\text{on})^2},$$  \hspace{1cm} \text{(A7)}$$

and $p$ from Eq. (9). The lower integration limit $m_1$ is given as the solution of $z_\rho[\sqrt{s}, k^\text{on}(m_1)] = 2m_\pi$. The second fraction in Eq. (A6) can be compared to the imaginary part of the well-known [18] propagator of two stable particles, $M$
Thus, the imaginary part of a loop with one stable and one unstable particle can be expressed as an integral over a distribution of imaginary parts of the form of Eq. (A8). Comparing Eq. (A6) to Eq. (12), one sees the formal similarity: there is an integral of a spectral function, which has poles [cf. Eq. (A3)], together with the factor \( k^\text{on}(m) = p(\sqrt{s}, m, m_\pi) \), and both ingredients produce the three branch points \( z_{b1} \), \( z_{b2} \), and \( z_{b3} \) as has been shown in the main text (we have omitted here the additional 2L powers of \( p \) for simplicity). There is a difference in the chosen parametrization in terms of the spectral function [compare \( p(\sqrt{s}, m, m_\pi) \) in Eq. (A7) vs \( p(\sqrt{s'}, m, M_N) \) in Eq. (12)], but this does not change the position of the branch points.

Indeed, \( k^\text{on} = 0 \) for the upper integration limit \( m = \sqrt{s} - m_\pi \) and thus \( z_\rho(\sqrt{s}, 0) = \sqrt{s} - M_N \). The poles of the \( \rho \) resonance in the spectral function \( S \) are located at the complex \( z_\rho = z_\rho^0 \) and consequently the integration limit equals the pole position for \( \sqrt{s} \equiv z_{b1,2} = M_N + z_\rho^0 \), which is indeed Eq. (13). The singularity at \( k^\text{on} = 0 \), coming from the factor \( k^\text{on}(m) \) in Eq. (A6), is also reached if \( m = \sqrt{s} - m_\pi \). It is easy to show that this \( m \) equals the lower integration limit \( m_1 \) for \( \sqrt{s} = 2m_\pi + M_N \) and thus Eq. (A6) indeed provides also the third branch point \( z_{b3} \) from Eq. (14).

In Ref. [18], the amplitude has been analytically continued to the complex plane using contour deformation. In fact, one could use the representation of Eq. (A6) for the same purpose in principle. As shown in the main text, for this, one has to respect the analyticity of the spectral function, i.e., the Im function in Eq. (A3) needs to be explicitly evaluated like in Eq. (15). Second, and this is an additional complication, the self-energy \( \Sigma \) itself has a two-sheet structure and the corresponding cut needs to be rotated as specified in Ref. [18]. This cut in \( \Sigma \) induces the cut of branch point \( z_{b3} \) in the overall \( \pi N \) amplitude. Apart from this and a carefully chosen integration path for the \( m \) integration of Eq. (A6), there are no additional complications, and the spectral representation allows for an alternative method of analytic continuation.

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