LINEAR PROGRAMMING AND THE INTERSECTION OF 
SUBGROUPS IN FREE GROUPS

S. V. IVANOV

Abstract. We study the intersection of finitely generated subgroups of free groups by utilizing the method of linear programming. We prove that if $H_1$ is a finitely generated subgroup of a free group $F$, then the Walter Neumann coefficient $\sigma(H_1)$ of $H_1$ is rational and can be computed in deterministic exponential time of size of $H_1$. This coefficient $\sigma(H_1)$ is a minimal nonnegative real number such that, for every finitely generated subgroup $H_2$ of $F$, it is true that $\bar{r}(H_1 \cap H_2) \leq \sigma(H_1)\bar{r}(H_1)\bar{r}(H_2)$, where $\bar{r}(H) := \max(\bar{r}(H) - 1, 0)$ is the reduced rank of $H$, $r(H)$ is the rank of $H$, and $\bar{r}(H_1, H_2)$ is the reduced rank of a generalized intersection of $H_1, H_2$.

1. Introduction

Let $F$ be a finitely generated free group, let $r(F)$ denote the rank of $F$ and let $\bar{r}(F) := \max(\bar{r}(F) - 1, 0)$ denote the reduced rank of $F$. Let $H_1, H_2$ be finitely generated subgroups of $F$. Hanna Neumann [18] proved that $\bar{r}(H_1 \cap H_2) \leq 2\bar{r}(H_1)\bar{r}(H_2)$ and conjectured that $\bar{r}(H_1 \cap H_2) \leq \bar{r}(H_1)\bar{r}(H_2)$.

These result and conjecture of Hanna Neumann were strengthened by Walter Neumann [19] by considering a generalized intersection of $H_1, H_2$. Let $S(H_1, H_2)$ denote a set of representatives of those double cosets $H_1 tH_2$ of $F, t \in F$, that have the property $H_1 \cap tH_2t^{-1} \neq \{1\}$. Walter Neumann [19] proved that the set $S(H_1, H_2)$ is finite, that reduced rank $\bar{r}(H_1, H_2)$ of a generalized intersection of $H_1, H_2$ satisfies

$$\bar{r}(H_1, H_2) := \sum_{s \in S(H_1, H_2)} \bar{r}(H_1 \cap sH_2s^{-1}) \leq 2\bar{r}(H_1)\bar{r}(H_2),$$

and conjectured that

$$\bar{r}(H_1, H_2) = \sum_{s \in S(H_1, H_2)} \bar{r}(H_1 \cap sH_2s^{-1}) \leq \bar{r}(H_1)\bar{r}(H_2).$$

This strengthened version of the Hanna Neumann conjecture was proved by Friedman [7] and Mineyev [17], see also Dicks’s proof [3] and a proof of [13].

Now suppose that $H_1$ is a fixed finitely generated subgroup of $F$. We will say that a real number $\sigma(H_1) \geq 0$ is the Walter Neumann coefficient for $H_1$, or, briefly, the WN-coefficient for $H_1$, if, for every finitely generated subgroup $H_2$ of $F$, we have

$$\bar{r}(H_1, H_2) \leq \sigma(H_1)\bar{r}(H_1)\bar{r}(H_2)$$

2010 Mathematics Subject Classification. Primary 20E05, 20E07, 20F65, 68Q25, 90C90.

Key words and phrases. Free groups, intersection of subgroups, rank, linear programming.

Supported in part by the NSF under grant DMS 09-01782.
and \( \sigma(H_1) \) is minimal with this property. It is clear that if \( H_1 \) is noncyclic then
\[
\sigma(H_1) = \sup \{ \bar{r}(H_1, H_2) \}
\]
over all finitely generated noncyclic subgroups \( H_2 \) of \( F \) and that \( \sigma(H_1) \leq 1 \) as follows from the strengthened Hanna Neumann conjecture being true.

In this article, we are concerned with algorithmic computability of the WN-coefficient \( \sigma(H_1) \) for a given finitely generated subgroup \( H_1 \) of \( F \) and with other properties of this number \( \sigma(H_1) \). Utilizing the method of linear programming, we will prove the following.

**Theorem 1.1.** Suppose that \( F \) is a free group of finite rank and \( H_1 \) is a finitely generated noncyclic subgroup of \( F \). Then the following are true.

(a) There exists a linear programming problem (LP-problem) associated with \( H_1 \)
\[
\mathcal{P}(H_1) = \max \{ cx \mid Ax \leq b \}
\]
with integer coefficients whose solution is equal to \( -\sigma(H_1)\bar{r}(H_1) \).

(b) There is a finitely generated subgroup \( H_2^* \) of \( F \), \( H_2^* = H_2^*(H_1) \), which corresponds to a vertex solution of the dual problem
\[
\mathcal{P}^*(H_1) = \min \{ b^\top y \mid A^\top y = c^\top, y \geq 0 \}
\]
of the primal LP-problem \([13]\) such that \( \bar{r}(H_1, H_2^*) = \sigma(H_1)\bar{r}(H_1, H_2^*) \). In particular, the WN-coefficient \( \sigma(H_1) \) of \( H_1 \) is rational and satisfies \( 1/\bar{r}(H_1) \leq \sigma(H_1) \leq 1 \).

Furthermore, if \( \Gamma(H_1), \Gamma(H_2^*) \) denote Stallings graphs representing subgroups \( H_1, H_2^* \), resp., \( |\text{ET}| \) denotes the number of oriented edges in a graph \( \Gamma \), and \( m - 1 \) is the rank of \( F \), then
\[
|\text{ET}(H_2^*)| < 2^{2|\text{ET}(H_1)|/4+2 \log_2 m}.
\]

(c) Assume that \( H_1 \) is given by a finite generating set or by a Stallings graph. Then, in deterministic exponential time of size of input, one can write down and solve the LP-problem \([13]\) associated with \( H_1 \). In particular, the WN-coefficient \( \sigma(H_1) \) of \( H_1 \) is computable in deterministic exponential time of size of input. Moreover, in deterministic double exponential time of size of input, one can construct a Stallings graph \( \Gamma(H_2^*) \) of the subgroup \( H_2^* \) of part (b).

Note that results similar to the results of Theorem \([13]\) are also obtained by the author \([14]\) for factor-free subgroups of free products of finite groups. However, arguments of \([14]\) do not apply to free products of infinite groups and here we develop different techniques suitable for free groups. For a generalization of the conjecture \([12]\) to subgroups of free products of groups and relevant results, the reader is referred to \([5, 6, 11, 13, 14]\).

Similarly to \([14]\), the correspondence \( H_2 \rightarrow y(H_2) \) between subgroups \( H_2 \) and vectors \( y(H_2) \) of the feasible polyhedron \( \{ y \mid A^\top y = c^\top, y \geq 0 \} \) of the dual LP-problem \( \mathcal{P}^*(H_1) \), as indicated in part (b) of Theorem \([13]\) plays an important role in proofs and is reminiscent of the correspondence between (resp. almost) normal surfaces in 3-dimensional manifolds and their (resp. almost) normal vectors in the Haken theory of normal surfaces and its generalizations, see \([5, 9, 10, 12, 15]\). In particular, the idea of a vertex solution works equally well both in the context of almost normal surfaces \([12]\), see also \([9, 15]\), and in the context of subgroups of a free group, providing in either situation both the connectedness of
the corresponding object represented by a vertex solution and an upper bound on size of the corresponding object.

In view of Theorem [1.1], it is of interest to look at two properties of finitely generated subgroups of free groups introduced by Dicks and Ventura [4]. Recall that a finitely generated subgroup \( H \) of a free group \( F \) is called \textit{compressed}, see [4], if, for every subgroup \( K \) of \( F \), we have \( \bar{r}(H) \leq \bar{r}(K) \). A finitely generated subgroup \( H \) of a free group \( F \) is called \textit{inert}, see [4], if for every subgroup \( K \) of \( F \), one has \( \bar{r}(H \cap K) \leq \bar{r}(K) \). It is immediate from the definitions that every inert subgroup is compressed. The problem whether every compressed subgroup is inert is stated by Dicks and Ventura [4] and it is still unresolved.

We say that a finitely generated subgroup \( H \) of a free group \( F \) is \textit{strongly inert} if, for every subgroup \( K \) of \( F \), we have \( \bar{r}(H, K) \leq \bar{r}(K) \). Clearly, a strongly inert subgroup is inert. It would be of interest to find an example, if it exists, to distinguish between these two classes of inert and strongly inert subgroups and, more generally, to find a finitely generated subgroup \( H \) of \( F \) such that \( \sup\{ \bar{r}(H \cap K)/\bar{r}(H) \bar{r}(K) \} < \sigma(H) \), where the supremum, as before, is taken over all finitely generated noncyclic subgroups \( K \) of \( F \). Another natural question is to find an algorithm that computes this number \( \sup\{ \bar{r}(H \cap K)/\bar{r}(H) \bar{r}(K) \} \), which could be called the Hanna Neumann coefficient of \( H \).

While we are not able to distinguish between these three classes of compressed, inert, and strongly inert subgroups of \( F \), our algorithms and their running times, that recognize two of these classes, are quite different.

**Proposition 1.2.** Suppose that \( F \) is a free group of finite rank and \( H \) is a finitely generated noncyclic subgroup of \( F \) given by a finite generating set or by its Stallings graph. Then the following hold true.

(a) There is an algorithm that decides, in deterministic exponential time of size of \( H \), whether \( H \) is strongly inert.

(b) There is an algorithm that decides, in nondeterministic polynomial time of size of \( H \), whether \( H \) is not compressed.

Summarizing, we see that the decision problem that inquires whether a finitely generated subgroup \( H \) of \( F \) is strongly inert is in \textit{EXP}, the decision problem that asks whether \( H \) is inert is not known to be decidable, and the decision problem that asks whether \( H \) is compressed is in \textit{coNP} (for the definition of computational complexity classes \textit{EXP}, \textit{coNP} see [1] or [20]).

2. Preliminaries

Suppose that \( X \) is a graph. Let \( VX \) denote the set of vertices of \( X \) and let \( EX \) be the set of oriented edges of \( X \). If \( e \in EX \) then \( e^{-1} \) denotes the edge with the opposite to \( e \) orientation, \( e^{-1} \neq e \).

For \( e \in EX \), let \( e_- \) and \( e_+ \) denote the initial and terminal, respectively, vertices of \( e \). A path \( p = e_1 \ldots e_k \) is called reduced if, for every \( i = 1, \ldots, k-1 \), \( e_i \neq e_{i+1}^{-1} \). The length of a path \( p = e_1 \ldots e_k \) is \( k \), denoted \( |p| = k \). The initial vertex of \( p \) is \( p_- = (e_1)_- \) and the terminal vertex of \( p \) is \( p_+ = (e_k)_+ \). A path \( p \) is closed if \( p_- = p_+ \). If \( p = e_1 \ldots e_k \) is a closed path, then a cyclic permutation \( \bar{p} \) of \( p \) is any path of the form \( e_{i+1} e_{2+i} \ldots e_{k+i} \), where \( i = 1, \ldots, k \) and the indices are considered mod \( k \). The subgraph of \( X \)
that consists of edges of all closed paths \( p \) of \( X \) such that \( |p| > 0 \) and every cyclic permutation of \( p \) is reduced, is called the core of \( X \), denoted \( \text{core}(X) \).

Let \( U \) be a finite connected graph such that \( \text{core}(U) = U \), let \( o \in VU \) and let \( F = \pi_1(U, o) \) be the fundamental group of \( U \) at \( o \). Then \( F \) is a free group of rank \( r(F) = |EU|/2 - |VU| + 1 \), where \(|A|\) is the cardinality of a set \( A \), and the elements of \( F \) can be thought of as reduced closed paths in \( U \) starting at \( o \).

Following Stallings \[22\], see also \[10\], \[2\], with every (finitely generated) subgroup \( H \) of \( F_U = \pi_1(U, o) \), we can associate a (resp. finite) graph \( Y = Y(H) \) and a map \( \beta : Y \to U \) of graphs so that \( H \) is isomorphic to \( \pi_1(Y, o_Y) \), where \( o_Y \in VY \), \( \beta(o_Y) = o \), and a reduced path \( p \in \pi_1(U, o) \) belongs to \( H \) if and only if there is a reduced path \( p_H \in \pi_1(Y, o_Y) \) such that \( \beta(p_H) = p \). In addition, we may assume that \( \beta \) is a locally injective map of graphs, i.e., the restriction of \( \beta \) on a regular neighborhood of every vertex of \( U \) is injective. We call a locally injective map of graphs an immersion. Since \( \beta \) is an immersion, it follows that every reduced path in \( H \subseteq \pi_1(U, o) \) has a unique preimage in \( Y \).

Consider two finitely generated subgroups \( H_1, H_2 \) of the free group \( F_U = \pi_1(U, o) \). Pick a set \( S(H_1, H_2) \) of representatives of those double cosets \( H_1 g H_2 \), \( g \in F_U \), for which the intersection \( H_1 \cap g H_2 g^{-1} \) is nontrivial.

Let \( Y_1, Y_2 \) be Stallings graphs of the subgroups \( H_1, H_2 \) and let \( Y_1 \times Y_2 \) denote the pullback of the maps \( \beta_i : Y_i \to U, i = 1, 2 \). Recall that

\[
\begin{align*}
V(Y_1 \times Y_2) & = \{ (v_1, v_2) \mid v_i \in VY_i, \beta_1(v_1) = \beta_2(v_2) \}, \\
E(Y_1 \times Y_2) & = \{ (e_1, e_2) \mid e_i \in EY_i, \beta_1(e_1) = \beta_2(e_2) \},
\end{align*}
\]

and \((e_1, e_2)_- = ((e_1)_-, (e_2)_-), (e_1, e_2)_+ = ((e_1)_+, (e_2)_+)\).

According to Walter Neumann \[19\], the set \( S(H_1, H_2) \) is finite and the nontrivial intersections \( H_1 \cap s H_2 s^{-1} \), where \( s \in S(H_1, H_2) \), are in bijective correspondence with connected components \( W_s \) of the core \( W := \text{core}(Y_1 \times Y_2) \). Moreover, for every \( s \in S(H_1, H_2) \), we obtain

\[
\bar{r}(H_1 \cap s H_2 s^{-1}) = \bar{r}(W_s) = |EW_s|/2 - |VW_s|.
\]

Hence,

\[
\sum_{s \in S(H_1, H_2)} \bar{r}(H_1 \cap s H_2 s^{-1}) = \bar{r}(W) = |EW|/2 - |VW|.
\]

Let \( \alpha'_i \) denote the projection map \( Y_1 \times Y_2 \to Y_i, i = 1, 2 \), i.e., \( \alpha'_i((e_1, e_2)) = e_i \) and \( \alpha'_i((v_1, v_2)) = v_i \). Restricting \( \alpha'_i \) on \( W \subseteq Y_1 \times Y_2 \), we obtain the map \( \alpha_i : W \to Y_i, i = 1, 2 \). In this notation, we have a commutative diagram depicted in Fig. 1.
In particular, if $X \in \{Y_1, Y_2, W, U\}$, then there is a canonical immersion $\varphi : X \to U$, where $\varphi = \beta_i$ if $X = Y_i$, $i = 1, 2$, $\varphi = \text{id}_U$ if $X = U$ and $\varphi = \beta_o$ if $X = W$. More generally, we will say that $X$ is a $U$-graph if $X$ is equipped with a graph map $\varphi : X \to U$. A $U$-graph $X$ is reduced if $\varphi$ is an immersion. For example, $Y_1, Y_2, W, U$ are reduced $U$-graphs. If $x \in VX \cup EX$, then $\varphi(x) \in Vu \vee Eu$ is called the label of $x$.

It will be convenient to work with the graph $U$ of a special form which we denote $U_m$. The graph $U_m$ contains two vertices $o_1, o_2$, $VU_m := \{o_1, o_2\}$, and the vertices $o_1, o_2$ are connected by $m \geq 3$ nonoriented edges so that the oriented edges $a_1, \ldots, a_m \in EU_m$ start at $o_1$ and end in $o_2$, see Fig. 2, where the case $m = 3$ is depicted.

![Figure 2](image)

From now on we will be considering $U_m$-graphs, where $m \geq 3$, unless it is stated otherwise. Since $U_m$ is fixed, we will be writing $Y_1 \times Y_2$ in place of $Y_1 \times Y_2$.

Denote $A := \{a_1, \ldots, a_m\}$.

Let $X$ be a $U_m$-graph. A vertex $v \in VX$ is called an $i$-vertex if $\varphi(v) = o_i$, $i = 1, 2$. Clearly, if $e_+$ is a 2-vertex then $\varphi(e) \in A = \{a_1, \ldots, a_m\}$ and if $e_+$ is a 1-vertex then $\varphi(e) \in A^{-1} = \{a_1^{-1}, \ldots, a_m^{-1}\}$.

An edge $e \in EX$ is called a $b$-edge, where $b \in A$, if $\varphi(e) = b$. The set of all $b$-edges of $X$ is denoted $E_bX$. Clearly, $\sum_{e \in A} |E_bX| = |EX|/2$, where $|S|$ is the cardinality of a set $S$.

3. The system of linear inequalities $\mathrm{SLI}[Y_1]$

Suppose that $Y_1$ is a finite reduced $U_m$-graph such that $Y_1 = \text{core}(Y_1)$ and

$$f(Y_1) := -\chi(Y_1) = |EY_1|/2 - |VY_1| > 0,$$

where $\chi(Y_1)$ is the Euler characteristic of $Y_1$ (since $EY_1$ is the set of oriented edges of $Y_1$, we use $|EY_1|/2$ in $\chi(Y_1)$). This graph $Y_1$ will be held fixed throughout Sects. 3-4.

Let $(A_1, \ldots, A_m)$ be an $m$-tuple of sets $A_j$ such that $A_j \subseteq E_aY_1$, $j = 1, \ldots, m$. Let $e, f \in \cup_{j=1}^m A_j$. We say that the edges $e, f$ are $i$-related, written $e \sim_i f$, if $e_+ = f_-$ in $Y_1$ when $i = 1$ or $e_+ = f_+$ in $Y_1$ when $i = 2$. Note that it follows from $A_j \subseteq E_aY_1$ and $Y_1$ being a $U_m$-graph that $e_-, f_-$ are 1-vertices while $e_+, f_+$ are 2-vertices. Clearly, $\sim_i$ is an equivalence relation on the set $\cup_{j=1}^m A_j$.

Let $[e]_{\sim_i}$ denote the equivalence class of an edge $e \in \cup_{j=1}^m A_j$ relative to this equivalence relation $\sim_i$ and let $|[e]_{\sim_i}|$ denote the cardinality of $[e]_{\sim_i}$.

We will say that an $m$-tuple $(A_1, \ldots, A_m)$ is $i$-admissible, where $i = 1, 2$ is fixed, if the union $\cup_{j=1}^m A_j$ is not empty and, for every $e \in \cup_{j=1}^m A_j$, we have $|[e]_{\sim_i}| > 1.$
Then it is clear that, for every \( e \in \bigcup_{j=1}^{m} A_j \),
\[
2 \leq |[e]_{\sim_i}| \leq k \leq m,
\] (3.1)
where \( k \) is the number of nonempty sets \( A_j \) in the tuple \((A_1, \ldots, A_m)\).

If \((A_1, \ldots, A_m)\) is an \( i \)-admissible tuple, we define the number \( N_i(A_1, \ldots, A_m)\) to be the sum \( \sum |[e]_{\sim_i}| - 2 \) over all equivalence classes \([e]_{\sim_i}\) of the equivalence relation \( \sim_i \) on \( \bigcup_{j=1}^{m} A_j \). Hence,
\[
N_i(A_1, \ldots, A_m) := \sum_{[e]_{\sim_i}} (|[e]_{\sim_i}| - 2).
\] (3.2)

Since \( \text{core}(Y_1) = Y_1 \), it follows that
\[
\bar{r}(Y_1) = |EY_1|/2 - |VV_1| = \frac{1}{2} \sum_{u \in VV_1} (\deg u - 2),
\]
where \( \deg u \) is the degree of a vertex \( u \in VV_1 \). Let \( V_1 Y_1 \) denote the set of all \( i \)-vertices of \( Y_1 \), \( i = 1, 2 \). Define \( \bar{r}_i(Y_1) := \sum_{u \in VV_1} (\deg u - 2) \). Observe that
\[
\bar{r}(Y_1) = \bar{r}_1(Y_1) + \bar{r}_2(Y_1)
\]
and that
\[
N_i(A_1, \ldots, A_m) := \sum_{[e]_{\sim_i}} (|[e]_{\sim_i}| - 2) \leq \sum_{u \in VV_1} (\deg u - 2) = 2\bar{r}_i(Y_1) \leq 2\bar{r}(Y_1). \] (3.3)

For every nonempty set \( B \subseteq E_{a_j}Y_1 \), we consider a variable \( x_{j,B} \). We also introduce a special variable \( x_s \). Note that, for given \( j \), the set of variables \( x_{j,B} \) is finite and its cardinality is less than \( 2^{\sum E_{a_j}Y_1} \).

Now we will define a system of inequalities in these variables \( x_{j,B} \), \( x_s \) so that each inequality is determined by means of an \( i \)-admissible tuple \((A_1, \ldots, A_m)\).

For an \( i \)-admissible tuple \((A_1, \ldots, A_m)\), let \( A_{j_1}, \ldots, A_{j_k} \) denote all nonempty sets in \((A_1, \ldots, A_m)\).

If \( i = 1 \), then the inequality, corresponding to the 1-admissible tuple \((A_1, \ldots, A_m)\), is defined as follows.
\[
-x_{1,A_{j_1}} - x_{2,A_{j_2}} - \ldots - x_{k,A_{j_k}} - x_s \leq -N_1(A_1, \ldots, A_m). \] (3.4)

For \( i = 2 \), then the inequality, corresponding to the 2-admissible tuple \((A_1, \ldots, A_m)\), is written in the form
\[
x_{1,A_{j_1}} + x_{2,A_{j_2}} + \ldots + x_{k,A_{j_k}} - x_s \leq -N_2(A_1, \ldots, A_m). \] (3.5)

Let
\[
\text{SLI}[Y_1]
\]

denote the system of all linear inequalities \( (3.4) - (3.5) \) constructed for all \( i \)-admissible tuples \((A_1, \ldots, A_m), i = 1, 2 \). Clearly, \( \text{SLI}[Y_1] \) is finite.

Assume that the map \( \alpha_2 : \text{core}(Y_1 \times Y_2) \to Y_2 \) is surjective. For every \( i \)-vertex \( u \in VV_2 \), consider all the edges \( e_1, \ldots, e_k \in EY_2 \) such that \( (e_1)_- = \ldots = (e_k)_- = u \) if \( i = 1 \) and \( (e_1)_+ = \ldots = (e_k)_+ = u \) if \( i = 2 \), i.e., \( \varphi(e_1), \ldots, \varphi(e_k) \in A \) and one of end vertices of \( e_1, \ldots, e_k \) is \( u \). Denote \( \varphi(e_\ell) = a_{j_\ell}, \ell = 1, \ldots, k \). If \( j \not\in \{j_1, \ldots, j_k\} \), we set \( A_j(u) := \emptyset \). Otherwise, we set
\[
A_j(u) := \alpha_1^{-1}(e_{j_\ell}) \subseteq E_{a_{j_\ell}}Y_1,
\]
where \( \alpha_2^{-1}(e_{j_\ell}) \) is the full preimage of the edge \( e_{j_\ell} \) in \( \text{core}(Y_1 \times Y_2) \). It is immediate from the definitions that the tuple \((A_1(u), \ldots, A_m(u))\) is \( i \)-admissible.
Since every \( i \)-admissible tuple \((A_1, \ldots, A_m)\) gives rise to an inequality (3.3) if \( i = 1 \) or to an inequality (3.5) if \( i = 2 \) and every \( i \)-vertex \( u \in VY_2 \) defines, as indicated above, an \( i \)-admissible tuple \((A_1(u), \ldots, A_m(u))\), it follows that every vertex \( u \in VY_2 \) is mapped to a certain inequality of the system \( SLI[Y_1] \), denoted \( inq_v(u) \). Thus we obtain a function

\[
inq_V : VY_2 \to SLI[Y_1]
\]

from the set \( VY_2 \) of vertices of a graph \( Y_2 \), with the property that the map \( \alpha_2 : \text{core}(Y_1 \times Y_2) \to Y_2 \) is surjective, to the set of inequalities of the system \( SLI[Y_1] \).

If \( q \) is an inequality of the system \( SLI[Y_1] \), written \( q \in SLI[Y_1] \), we let \( q^L \) denote the left-hand side of \( q \) and \( q^R \) denote the number of the right-hand side of the inequality \( q \).

**Lemma 3.1.** Suppose \( Y_2 \) is a finite reduced \( U_m \)-graph with the property that the map \( \alpha_2 : \text{core}(Y_1 \times Y_2) \to Y_2 \) is surjective. Then

\[
\sum_{u \in VY_2} inq_v(u)^L = -2\bar{r}(Y_2)x_s \quad \text{and} \quad \sum_{u \in VY_2} inq_v(u)^R = -2\bar{r}(\text{core}(Y_1 \times Y_2)).
\]

**Proof.** Suppose \( e \in EY_2, \varphi(e) = a_j, \) and \( e_- = u_1, e_+ = u_2 \). Clearly, \( u_i \) is an \( i \)-vertex of \( Y_2, \) \( i = 1, 2 \). Denote \( B := \alpha_1 \alpha_2^{-1}(e) \subseteq E_a Y_1 \). Then the variables \(-x_{j,B} \) and \( x_{j,B} \) of \( inq_v(u_1)^L, \) \( inq_v(u_2)^L \), resp., will cancel out in the sum \( \sum_{u \in VY_2} inq_v(u)^L \).

It is also clear that all occurrences of the variable \( \pm x_{j,B} \), where \( j = 1, \ldots, m \), \( B \subseteq E_a Y_1, |B| > 0 \), in the sum \( \sum_{u \in VY_2} inq_v(u)^L \) can be paired down by using edges of \( Y_2 \) as indicated above.

Now we observe that every vertex \( u \in VY_2 \) of degree \( d \geq 2 \) contributes \(-(d - 2)\) to the coefficient of \( x_s \) in the sum \( \sum_{u \in VY_2} inq_v(u)^L \) and that

\[
-\chi(Y_2) = \bar{r}(Y_2) = \frac{1}{2} \sum_{u \in VY_2} (\deg u - 2).
\]

Therefore, we may conclude that

\[
\sum_{u \in VY_2} inq_v(u)^L = -2\bar{r}(Y_2)x_s,
\]

as required.

The second inequality of Lemma’s statement follows from the analogous equality

\[
-\chi(W) = \bar{r}(W) = \frac{1}{2} \sum_{u \in VW} (\deg u - 2)
\]

for \( W = \text{core}(Y_1 \times Y_2) \) and from the definition of numbers \( N_i(A_1, \ldots, A_m) \) that are used in the right-hand sides of inequalities (3.4) - (3.6). \( \square \)

Let \( A = \{a_1, \ldots, a_n\} \) be a finite set. A **combination with repetitions** \( B \), denoted \( B = \{[b_1, \ldots, b_d]\} \subseteq A \), of \( A \) is a finite unordered collection of multiple copies of elements of \( A \). Hence, \( b_i \in A \) and \( b_i = b_j \) is possible for \( i \neq j \).

Observe that the graph \( Y_2 \) of Lemma 3.1 can be used to construct a combination with repetitions, denoted \( inq(VY_2) \), of the system \( SLI[Y_1] \), whose elements are individual inequalities so that every inequality \( q = inq_v(u) \) of \( SLI[Y_1] \) occurs in \( inq(VY_2) \) the number of times equal to the number of preimages of \( q \) in \( VY_2 \) under
\(\text{inq}_Y\). It follows from Lemma 3.1 that if \(\text{inq}(V Y_2) = [q_1, \ldots, q_\ell] \subseteq \text{SLI}\{Y_1\}\) then

\[
\sum_{q \in \text{inq}(V Y_2)} q^L := \sum_{i=1}^\ell q^L_i = -C x_s,
\]

where \(C \geq 0\) is an integer, \(C = 2\bar{r}(Y_2)\).

For convenience of references, we consider the following property of a (not necessarily connected) graph \(Y_2\).

\[\text{(B)} \ Y_2\ \text{is a finite reduced } U_m\text{-graph such that the map } \alpha_2 : \text{core}(Y_1 \times Y_2) \to Y_2\ \text{is surjective, } \text{core}(Y_2) = Y_2, \ \text{and } \bar{r}(Y_2) = -\chi(Y_2) > 0.\]

Note that the equality \(\text{core}(Y_2) = Y_2\) could be dropped as it follows from the surjectivity of the map \(\alpha_2 : \text{core}(Y_1 \times Y_2) \to Y_2\).

**Lemma 3.2.** Suppose \(Q\) is a nonempty combination with repetitions of \(\text{SLI}\{Y_1\}\) and

\[
\sum_{q \in Q} q^L = -C x_s, \quad (3.6)
\]

where \(C > 0\) is an integer. Then there exists a finite reduced \(U_m\text{-graph } Y_Q\) with property (B) such that \(\text{inq}(V Y_Q) = Q\). Furthermore,

\[
\sum_{q \in \text{inq}(V Y_Q)} q^L = -2\bar{r}(Y_Q)x_s \quad \text{and} \quad \sum_{q \in \text{inq}(V Y_Q)} q^R = -2\bar{r}(\text{core}(Y_1 \times Y_Q)). \quad (3.7)
\]

**Proof.** Consider a graph \(Y_Q\) whose vertices \(u_1, \ldots, u_\ell\) are in bijective correspondence \(u_i \to q_i, i = 1, \ldots, \ell, \) with elements of \(Q = [q_1, \ldots, q_\ell] \subseteq \text{SLI}\{Y_1\}\). Recall that every inequality \(q_i\) in \(Q\) has one of the form \((3.4)\)–\((3.5)\). It follows from the equality \((3.6)\) that there exists an involution map \(\tau\) on the set of all terms \(\pm x_{j,B}\) of the sum \(\sum_{j=1}^\ell q^L_{i_j}\) such that \(\tau\) takes a term \(\pm x_{j,B}\) of \(q_{i_1}\) to a term \(\mp x_{j,B}\) of \(q^L_{i_2}\), \(i_2 \neq i_1\), and \(\tau^2 = \text{id}\).

If \(\tau\) takes the term \(-x_{j,B}\) of \(q^L_{i_1}\) to the term \(x_{j,B}\) of \(q^L_{i_2}\), then we connect the vertex \(u_{i_1}\) to \(u_{i_2}\) by an oriented edge in \(Y_Q\) whose label is \(a_j \in A\). This definition means that if \(q_i \in Q\) has type \((3.4)\), then \(v_i\) is a 1-vertex. On the other hand, if \(q_i \in Q\) has type \((3.5)\), then \(v_i\) is a 2-vertex. Furthermore, it is not difficult to derive from the definitions that \(Y_Q\) is a finite reduced \(U_m\text{-graph}, \) the map \(\alpha_Q : \text{core}(Y_1 \times Y_Q) \to Y_Q\) is surjective, and \(\text{inq}(V Y_Q) = Q\). Thus \(Y_Q\) has property (B). The equalities \((3.7)\) follow from Lemma 3.1.

We combine Lemmas 3.1, 3.2 as follows.

**Lemma 3.3.** The function \(\text{inq} : Y_2 \to \text{inq}(V Y_2) = Q\) from the set of finite reduced \(U_m\text{-graphs } Y_2\) with the property (B) to the set of combinations with repetitions \(Q\) of \(\text{SLI}\{Y_1\}\) with the property that \(\sum_{q \in Q} q^L = -C x_s, \) where \(C \geq 0\) is an integer, is surjective. In addition, if \(\text{inq}(V Y_2) = Q\) then

\[
\sum_{q \in Q} q^L = -2\bar{r}(Y_2)x_s \quad \text{and} \quad \sum_{q \in Q} q^R = -2\bar{r}(\text{core}(Y_1 \times Y_2)).
\]

**Proof.** This is straightforward from Lemmas 3.1, 3.2.
4. Utilizing the method of linear programming

Let us briefly review relevant results from the theory of linear programming (LP) over the field \( \mathbb{Q} \) of rational numbers. Following the notation of Schrijver’s monograph \[21\], let \( A \in \mathbb{Q}^{m \times n'} \) be an \( m' \times n' \)-matrix, let \( b \in \mathbb{Q}^{n'} \) be a column vector, let \( c \in \mathbb{Q}^{1 \times n'} \) be a row vector, \( c = (c_1, \ldots, c_n) \), and let \( x \) be a column vector consisting of variables \( x_1, \ldots, x_{n'} \), so \( x = (x_1, \ldots, x_{n'})^\top \), where \( M^\top \) means the transpose of a matrix \( M \). The inequality \( x \geq 0 \) means that \( x_i \geq 0 \) for every \( i \).

A typical LP-problem asks about the maximal value of the objective linear function \( cx = c_1x_1 + \cdots + c_{n'}x_{n'} \) over all \( x \in \mathbb{Q}^{n'} \) subject to the system of linear inequalities \( Ax \leq b \). This value (and often the LP-problem itself) is denoted

\[
\max \{ cx \mid Ax \leq b \}.
\]

We write \( \max \{ cx \mid Ax \leq b \} = -\infty \) if the set \( \{ cx \mid Ax \leq b \} \) is empty. We write \( \max \{ cx \mid Ax \leq b \} = +\infty \) if the set \( \{ cx \mid Ax \leq b \} \) is unbounded from above and say that \( \max \{ cx \mid Ax \leq b \} \) is finite if the set \( \{ cx \mid Ax \leq b \} \) is nonempty and bounded from above. The notation and terminology for an LP-problem

\[
\min \{ cx \mid Ax \leq b \} = -\max \{ -cx \mid Ax \leq b \}
\]

is analogous with \( -\infty \) and \( +\infty \) interchanged.

If \( \max \{ cx \mid Ax \leq b \} \) is an LP-problem defined as above, then the problem

\[
\min \{ b^\top y \mid A^\top y = c^\top, y \geq 0 \},
\]

where \( y = (y_1, \ldots, y_m)^\top \), is called the dual problem of \( \max \{ cx \mid Ax \leq b \} \).

The (weak) duality theorem of linear programming can now be stated as follows, see \[21\,\text{Sect. 7.4}\].

**Theorem A.** Let \( \max \{ cx \mid Ax \leq b \} \) be an LP-problem and let \( \min \{ b^\top y \mid A^\top y = c^\top, y \geq 0 \} \) be its dual LP-problem. Then for every \( x \in \mathbb{Q}^{n} \) such that \( Ax \leq b \) and every \( y \in \mathbb{Q}^{m} \) such that \( A^\top y = c^\top, y \geq 0 \), one has that \( cx \leq y^\top Ax \leq b^\top y \) and

\[
\max \{ cx \mid Ax \leq b \} = \min \{ b^\top y \mid A^\top y = c^\top, y \geq 0 \} \tag{4.1}
\]

provided both polyhedra \( \{ x \mid Ax \leq b \} \) and \( \{ y \mid A^\top y = c^\top, y \geq 0 \} \) are not empty. In addition, the minimum, whenever it is finite, is attained at a vector \( \hat{y} \) which is a vertex of the polyhedron \( \{ y \mid A^\top y = c^\top, y \geq 0 \} \).

We will also need a corollary of the complementary slackness and Carathéodory theorem, see Corollary 7.11 \[21\].

**Theorem B.** If both optima in \( \text{(4.1)} \) are finite, then the minimum is attained at a vector \( \hat{y}, \hat{y} \geq 0 \), whose positive components correspond to linear independent columns of \( A^\top \) (or rows of \( A \)).

We now consider the problem of maximizing the objective linear function \( cx := -x_0 \), over all rational vectors \( x, x \in \mathbb{Q}^{n'} \), for a suitable \( n' \), subject to the system of linear inequalities \( \text{SL}[Y_1] \), as an LP-problem \( \max \{ cx \mid Ax \leq b \} \). Note that, in this context, \( m' = \text{in}_q \) and \( n' = \text{in}_q \), where \( \text{in}_q \) is the number of inequalities in \( \text{SL}[Y_1] \) and \( \text{in}_q \) is the number of all variables \( x_{j,A}, x_s \) in \( \text{SL}[Y_1] \).

It is straightforward to verify that the dual problem

\[
\min \{ b^\top y \mid A^\top y = c^\top, y \geq 0 \}
\]
of this LP-problem \( \max \{cx \mid Ax \leq b\} \) can be equivalently stated as follows

\[
\sum_{i=1}^{m_{\text{eq}}} y_i q_i^R \to \min \quad \text{subject to} \quad y \geq 0, \quad \sum_{i=1}^{m_{\text{eq}}} y_i q_i^L = -x_s. \tag{4.2}
\]

Hence, we can write (4.2) in the form

\[
\min \big\{ \sum_{i=1}^{m_{\text{eq}}} y_i q_i^R \bigmid \sum_{i=1}^{m_{\text{eq}}} y_i q_i^L = -x_s, y \geq 0 \big\}. \tag{4.3}
\]

In Lemma 3.3, we established the existence of a surjective function

\[\text{inq} : Y_2 \to \text{inq}(Y_2)\]

from the set of finite reduced \( U_m \)-graphs \( Y_2 \) with the property (B) to a certain set of combinations with repetitions of \( \text{SLI}[Y_1] \). Now we will relate these combinations with repetitions of \( \text{SLI}[Y_1] \) to solutions of the dual LP-problem (4.3).

Consider a combination \( Q \) with repetitions of \( \text{SLI}[Y_1] \) that has the property

\[
\sum_{q \in Q} q^L = -C(Q)x_s, \tag{4.4}
\]

where \( C(Q) \geq 0 \) is an integer. As before, let all inequalities of \( \text{SLI}[Y_1] \) be \( q_1, \ldots, q_{m_{\text{eq}}} \), let \( \ell_i(Q) \geq 0 \) denote the number of times that \( q_i \) occurs in \( Q \), and let \( \delta_i \) be the coefficient of \( x_s \) in \( q_i \). Then it follows from the definitions and (4.4) that

\[
\sum_{q \in Q} q^L = \sum_{i=1}^{m_{\text{eq}}} \delta_i \ell_i(Q) x_s = -C(Q)x_s. \tag{4.5}
\]

Consider the map

\[\text{sol} : Q \to y_Q = (y_Q, \ldots, y_Q, m_{\text{eq}})^T, \tag{4.6}\]

where \( y_{Q,i} := \frac{\ell_i(Q)}{C(Q)} \) for \( i = 1, \ldots, m_{\text{eq}} \). It follows from the definitions that \( y_Q \) is a rational vector, \( y_Q \geq 0 \) and, by (4.5), \( y_Q \) satisfies the condition that

\[
\sum_{i=1}^{m_{\text{eq}}} y_{Q,i} q_i^L = -x_s.
\]

Hence, \( y_Q \) is a vector in the polyhedron \( \{ y \mid y \geq 0, \sum_{i=1}^{m_{\text{eq}}} y_{Q,i} q_i^L = -x_s \} \) of the dual LP-problem (4.3).

Conversely, let \( \hat{y} = (\hat{y}_1, \ldots, \hat{y}_{m_{\text{eq}}})^T \) be a vector of the feasible polyhedron

\[
\{ y \mid y \geq 0, \sum_{i=1}^{m_{\text{eq}}} y_{Q,i} q_i^L = -x_s \}
\]

of the dual LP-problem (4.3). Let \( C > 0 \) be a common multiple of positive denominators of the rational numbers \( \hat{y}_1, \ldots, \hat{y}_{m_{\text{eq}}} \). Consider a combination with repetitions \( Q(\hat{y}) \) of \( \text{SLI}[Y_1] \) such that every \( q_i \) of \( \text{SLI}[Y_1] \) occurs in \( Q(\hat{y}) \) \( C\hat{y}_i \ell_i \) many times. Then it follows from the definitions that

\[
\sum_{q \in Q} q^L = \sum_{i=1}^{m_{\text{eq}}} \ell_i \hat{y}_i q_i^L = \sum_{i=1}^{m_{\text{eq}}} C\hat{y}_i q_i^L = C \sum_{i=1}^{m_{\text{eq}}} \hat{y}_i q_i^L = -C x_s. \tag{4.7}
\]

Now we can see that the vector \( y_Q = \text{sol}(Q(\hat{y})) \), defined by (4.6) for \( Q(\hat{y}) \), is equal to \( \hat{y} \).

We summarize these findings in the following.
Lemma 4.1. The map \( \text{sol} : Q \to y_Q \) defined by (1.6) is a surjective function from the set of combinations \( Q \) with repetitions of \( SL[1] \) that satisfy an equation \( \sum_{q \in Q} q^L = -C x_s \), where \( C > 0 \) is an integer, to the set of vectors of the polyhedron \( \{ y \mid y \geq 0, \sum_{i=1}^{m_{\text{inq}}} y_i q_i^L = -x_s \} \) of the dual LP-problem (1.3). Furthermore, the composition of the maps \( \text{inq} \) \& \( \text{sol} \),
\[
\text{sol} \circ \text{inq} : Y_2 \to \hat{y} = \hat{y}(Y_2),
\]
provides a surjective function from the set of graphs with property (B) to the set of points of the rational polyhedron \( \{ y \mid y \geq 0, \sum_{i=1}^{m_{\text{inq}}} y_i q_i^L = -x_s \} \) of the dual LP-problem (1.3). Under this map, the value of the objective function \( \sum_{i=1}^{m_{\text{inq}}} y_i q_i^L \) of the dual LP-problem (1.3) at \( \hat{y} = \hat{y}(Y_2) \) satisfies the equality
\[
\sum_{i=1}^{m_{\text{inq}}} y_i q_i^R = \frac{-\hat{r}(\text{core}(Y_1 \times Y_2))}{\hat{r}(Y_2)}. \tag{4.8}
\]

Proof. As was established above, see computations (4.7), \( \text{sol} \) is a surjective function, hence, by Lemma 3.3, the composition \( \text{sol} \circ \text{inq} \) is also surjective. Consider a finite irreducible \( U_m \)-graph \( Y_2 \) with property (B) and define \( Q := \text{inq}(Y_2) \), \( \hat{y} := \text{sol}(Q) \). It follows from Lemma 3.3 that
\[
\sum_{q \in Q} q^L = -2\hat{r}(Y_2)x_s \quad \text{and} \quad \sum_{q \in Q} q^R = -2\hat{r}(\text{core}(Y_1 \times Y_2)). \tag{4.9}
\]

In view of the definition (4.3) and Lemma 3.3 we have \( C(Q) = 2\hat{r}(Y_2) \). Hence, it follows from the definition (4.6) and equalities (4.9) that
\[
\sum_{j=1}^{m_{\text{inq}}} \hat{y}_j q_j^R = (\sum_{q \in Q} q^R)C(Q)^{-1} = \frac{-\hat{r}(\text{core}(Y_1 \times Y_2))}{\hat{r}(Y_2)},
\]
as required. \( \square \)

We will say that a real number \( \sigma(Y_1) \geq 0 \) is the Walter Neumann coefficient, or briefly WN-coefficient, for the graph \( Y_1 \) if
\[
\hat{r}(\text{core}(Y_1 \times Y_2)) \leq \sigma(Y_1)\hat{r}(Y_1)\hat{r}(Y_2)
\]
for every finite reduced \( U_m \)-graph \( Y_2 \) and \( \sigma(Y_1) \) is minimal with this property.

Lemma 4.2. The WN-coefficient \( \sigma(Y_1) \) for the graph \( Y_1 \) is equal to
\[
\sup \frac{-\hat{r}(\text{core}(Y_1 \times Y_2))}{\hat{r}(Y_1)\hat{r}(Y_2)} \tag{4.10}
\]
over all finite reduced \( U_m \)-graphs \( Y_2 \) with property (B).

Proof. Since \( \hat{r}(Y_1) > 0 \) and \( \text{core}(Y_1) = Y_1 \), we may use \( Y_2 = Y_1 \) to see that \( \sigma(Y_1) > 0 \) and that \( \sigma(Y_1) = \sup \frac{-\hat{r}(\text{core}(Y_1 \times Y_2))}{\hat{r}(Y_1)\hat{r}(Y_2)} > 0 \) over all finite reduced \( U_m \)-graph \( Y_2 \) such that \( \hat{r}(Y_2) > 0 \).

Suppose that \( Y_2 \) with \( \hat{r}(Y_2) > 0 \) has no property (B), i.e., the projection \( \alpha_2 : \text{core}(Y_1 \times Y_2) \to Y_2 \) is not surjective. We delete those edges and vertices in \( Y_2 \) that have no preimages in \( \text{core}(Y_1 \times Y_2) \) under \( \alpha_2 \). As a result, we obtain a subgraph \( Y_2' \) of \( Y_2 \) such that \( \text{core}(Y_1 \times Y_2') = \text{core}(Y_1 \times Y_2) \) and \( \hat{r}(Y_2') < \hat{r}(Y_2) \). This proves that graphs \( Y_2 \) with no property (B) can be disregarded when taking the supremum (4.10). \( \square \)
We remark that it is easy to show that we may assume, in addition, that graphs \( Y_2 \) in Lemma 4.2 are connected (the connectedness of \( Y_2 \) also follows from its “vertex” choice in the proof of Lemma 4.4).

**Lemma 4.3.** Both optima \( \max \{-x_s \mid \text{SLI}(Y_1)\} \) and
\[
\min \{ \sum_{j=1}^{\text{m}_{\text{inq}}} y_j q_j^R \mid y \geq 0, \sum_{j=1}^{\text{m}_{\text{inq}}} y_j q_j^L = -x_s \}
\]
are finite and satisfy the following equalities
\[
\max \{-x_s \mid \text{SLI}(Y_1)\} = \min \{ \sum_{i=1}^{\text{m}_{\text{inq}}} y_i q_i^R \mid y \geq 0, \sum_{i=1}^{\text{m}_{\text{inq}}} y_i q_i^L = -x_s \} = -\sigma(Y_1)\bar{r}(Y_1).
\]

Furthermore, the minimum is attained at a vertex \( y_\nu, y_\nu \geq 0 \), which is a vertex of the polyhedron \( \{ y \mid y \geq 0, \sum_{i=1}^{\text{m}_{\text{inq}}} y_\nu q_i^L = -x_s \} \) of the dual LP-problem (4.3), and \( \frac{1}{m-2} \leq \sigma(Y_1) \leq 1 \).

**Proof.** Setting \( Y_2 := Y_1 \), we obtain a graph \( Y_2 \) with property (B). Hence, by Lemma 4.1, \( \hat{y} = \text{sol}(\text{inq}(Y_2)) \) is a solution to the system \( y \geq 0, \sum_{j=1}^{\infty} y_j q_j^L = -x_s \) and the polyhedron \( \{ y \mid y \geq 0, \sum_{i=1}^{\text{m}_{\text{inq}}} y_i q_i^L = -x_s \} \) is not empty.

To establish that the polyhedron \( \{ x \mid \text{SLI}(Y_1)\} \) is not empty either, we will show that \( \hat{x} \), whose components are \( \hat{x}_j, A = 0 \), for all \( j = 1, \ldots, m \), \( A \subseteq E_{\nu}Y_1 \), and \( \hat{x}_s = 2\bar{r}(Y_1) \), is a solution to \( \text{SLI}(Y_1) \). To do this, we need to check that every inequality (3.3)–(3.5) of \( \text{SLI}(Y_1) \) is satisfied with these values of variables, that is,
\[
-(k-2) \cdot 2\bar{r}(Y_1) \leq -N_i(A_1, \ldots, A_m)
\]
for every \( i \)-admissible tuple \((A_1, \ldots, A_m)\) in which exactly \( k \) sets are nonempty.

It follows from the definition \( N_i(A_1, \ldots, A_m) \) of \( N_i(A_1, \ldots, A_m) = 0 \) if \( k = 2 \). Hence, if \( k = 2 \) then the inequality (4.12) is true. Since \( k \geq 2 \), we may assume that \( k > 2 \). Then, according to (3.3), \( N_i(A_1, \ldots, A_m) \leq 2\bar{r}(Y_1) \) and the inequality (4.12) is true again.

Hence, both polyhedra \( \{ x \mid \text{SLI}(Y_1)\} \), \( \{ y \mid y \geq 0, \sum_{i=1}^{\text{m}_{\text{inq}}} y_i q_i^L = -x_s \} \) are not empty, as desired.

According to Theorem A, the maximum and minimum in (4.11) are finite and equal. Referring to Theorem A again, we obtain that the minimum
\[
\min \{ \sum_{i=1}^{\text{m}_{\text{inq}}} y_i q_i^R \mid y \geq 0, \sum_{i=1}^{\text{m}_{\text{inq}}} y_i q_i^L = -x_s \}
\]
is attained at a vertex \( y_\nu \) of the polyhedron \( \{ y \mid y \geq 0, \sum_{i=1}^{\text{m}_{\text{inq}}} y_\nu q_i^L = -x_s \} \).

By Lemma 4.1, for every graph \( Y_2 \) with property (B), the ratio \( \frac{-\bar{r}(\text{core}(Y_1 \times Y_2))}{\bar{r}(Y_2)} \) is \( \sum_{i=1}^{\text{m}_{\text{inq}}} \hat{y}_i q_i^R \), where \( \hat{y} = \text{sol}(\text{inq}(Y_2)) \), and the map
\[
\text{sol} \circ \text{inq} : Y_2 \rightarrow \hat{y}
\]
covers the feasible polyhedron \( \{ y \mid y \geq 0, \sum_{i=1}^{\text{m}_{\text{inq}}} y_i q_i^L = -x_s \} \).

On the other hand, it follows from Lemma 4.2 that
\[
\sigma(Y_1)\bar{r}(Y_1) = \sup \frac{-\bar{r}(\text{core}(Y_1 \times Y_2))}{\bar{r}(Y_2)} = -\inf \frac{-\bar{r}(\text{core}(Y_1 \times Y_2))}{\bar{r}(Y_2)}
\]
over all graphs \( Y_2 \) with property (B).
Therefore, putting together these two facts, we obtain
\[
-\sigma(Y_1)\bar{r}(Y_1) = \inf \frac{-\bar{r}(\text{core}(Y_1 \times Y_2))}{\bar{r}(Y_2)} = \inf \left\{ \sum_{i=1}^{m_{\text{in}}} y_iq_i^R \mid y \geq 0, \sum_{i=1}^{m_{\text{in}}} y_iq_i^L = -x_s \right\}
\]
\[
= \min \left\{ \sum_{i=1}^{m_{\text{in}}} y_iq_i^R \mid y \geq 0, \sum_{i=1}^{m_{\text{in}}} y_iq_i^L = -x_s \right\}
\]
\[
= \sum_{i=1}^{m_{\text{in}}} y_{v,i}q_i^R = \max \{-x_s \mid SLI[Y_1]\},
\]
where \(y_v\) is a vertex of the polyhedron \(\{y \mid y \geq 0, \sum_{i=1}^{m_{\text{in}}} y_iq_i^L = -x_s\}\) at which the minimum is attained.

It remains to show that \(\frac{1}{m-2} \leq \sigma(Y_1) \leq 1\). The inequality \(\sigma(Y_1) \leq 1\) follows from the fact that the strengthened Hanna Neumann conjecture is true, see [7, L7], [3].

Let \(Y_2 := U_m\). Then \(\bar{r}(Y_2) = m - 2\) and \(\bar{r}(\text{core}(Y_1 \times Y_2)) = \bar{r}(Y_1)\) for \(Y_1 \times Y_2 = Y_1\).
Hence,
\[
\sigma(Y_1) \geq \frac{\bar{r}(\text{core}(Y_1 \times Y_2))}{\bar{r}(Y_1)\bar{r}(Y_2)} = \frac{1}{m - 2},
\]
as required.

\[\square\]

**Lemma 4.4.** There exists a finite reduced \(U_m\)-graph \(Y_2\) with property (B) such that \(\bar{r}(\text{core}(Y_1 \times Y_2)) = \sigma(Y_1)\bar{r}(Y_1)\bar{r}(Y_2), Y_2\) is connected, \(\text{sol}(\text{inq}(Y_2))\) is a vertex of the polyhedron \(\{y \mid y \geq 0, \sum_{i=1}^{m_{\text{in}}} y_iq_i^L = -x_s\}\) of the dual LP-problem (4.13), and
\[
|EY_2| < 2^{\frac{1}{2}EY_1/4 + 2 \log_2 m}.
\]

**Proof.** According to Lemma 4.3 and Theorem B, we may assume that the minimum of the dual LP-problem (4.13) is attained at a vertex \(y_v\), \(y_v \geq 0\), of the feasible polyhedron \(\{y \mid y \geq 0, \sum_{i=1}^{m_{\text{in}}} y_iq_i^L = -x_s\}\) of (4.13), whose positive components \(y_{v,i} > 0\), correspond to linearly independent left-hand sides \(q_i^L\) of the inequalities \(q_i, i = 1, \ldots, m_{\text{in}}, \text{of SLI}[Y_1]\). Reordering the inequalities of \(\text{SLI}[Y_1]\) if necessary, we may assume that \(y_{v,1}, \ldots, y_{v,r}\) are positive, \(q_1^L, \ldots, q_r^L\) are linearly independent and that \(y_{v,r+1} = \ldots = y_{v,\text{in}} = 0\).

Recall that \(Ax \leq b\) is the matrix form of \(\text{SLI}[Y_1]\). Let \(A_{r,\text{in}}\) denote the submatrix of \(A\) that consists of the first \(r\) rows of \(A\), hence, \(y_{v,1}, \ldots, y_{v,r}\) correspond to linearly independent rows of \(A_{r,\text{in}}\). Let \(A_{r \times r}\) denote a submatrix of \(A_{r,\text{in}}\) of size \(r \times r\) with \(\det A_{r \times r} \neq 0\) and let \(\tilde{y}_V = (y_{v,1}, \ldots, y_{v,r})^T\) be a truncated version of \(y_v\). Then
\[
A_{r \times r}^T \tilde{y}_V = \tilde{e}^T = (c_1, \ldots, c_r)^T.
\]
Since \(\sum_{i=1}^{m_{\text{in}}} y_{v,i}q_i^L = -x_s\), it follows that \(c_j = 0\) if \(c_j\) corresponds to a variable \(x_{\ell,B}\) and \(c_j = -1\) if \(c_j\) corresponds to the variable \(x_s\). Since \(y_v \neq 0\), we conclude that \(\tilde{e}^T \neq 0\), i.e., one of \(c_j\) is \(-1\) and all other entries in \(\tilde{e}^T\) are equal to \(0\). Since every entry of \(A_{r \times r}\) is \(0\) or \(\pm 1\) or \(-k - 2\), where \(2 \leq k \leq m\), and every row of \(A_{r \times r}\) contains at most \(m + 1\) nonzero entries, at most one of which is different from \(\pm 1\), see definitions (3.3)–(3.4), it follows that the standard Euclidean norm of any row of \(A_{r \times r}\) is at most \((m + (m - 2)^2)^{1/2} < m\) as \(m \geq 3\). Hence, by the Hadamard’s inequality, we have that
\[
|\det A_{r \times r}| < m^r.
\]
Invoking the Cramer’s rule, we further obtain that

\[
y_{V,i} = \frac{\det A^T_{rxr,i}(\tilde{c}^T)}{\det A_{rxr}}, \tag{4.14}
\]

where \( A^T_{rxr,i}(\tilde{c}^T) \) is the matrix obtained from \( A_{rxr} \) by replacing the \( i \)th column with \( \tilde{c}^T \), \( i = 1, \ldots, r \). Similarly to \( A_{rxr} \), see (4.13), we obtain that

\[
|\det A_{rxr,i}(\tilde{c}^T)| < m^r. \tag{4.15}
\]

In view of (4.13)–(4.15), we can see that a common denominator \( L > 0 \) of positive rational numbers \( y_{V,1}, \ldots, y_{V,r} \) satisfies \( L < m^r \) and that positive integers \( Ly_{V,1}, \ldots, Ly_{V,r} \) are less than \( m^r \).

It follows from the definition of the function \( \text{sol} \), see also Lemma 4.1, that if \( y_V = \text{sol}(Q) \) then

\[
|Q| < rm^r, \tag{4.16}
\]

where \( |Q| \) is the cardinality of a combination with repetitions \( Q \subseteq \text{SLI}[Y_1] \) defined so that every element \( q \in Q \) is counted as many times as it occurs in \( Q \).

It follows from the definition of the function \( \text{inq} \), see also Lemma 3.3, that if \( Y_2 \) is a graph with property (B) such that \( \text{inq}(Y_2) = Q \), then \( |VY_2| = |Q| \) and, therefore, by (4.16), we get

\[
|EY_2| \leq m|VY_2| = m|Q| < rm^r + 1. \tag{4.17}
\]

Note that \( r \) does not exceed the number \( n_{\text{inq}} \) of variables \( x_{j,B}, x_s \) of \( \text{SLI}[Y_1] \). Since \( Y_1 \) is reduced, we have \( |E_{j,Y_1}| \leq |EY_1|/4 \) for every \( j = 1, \ldots, m \). Hence, the number of variables \( x_{j,B} \) for a fixed \( j \) does not exceed \( 2|EY_1|/4 - 1 \) and, therefore,

\[
r \leq n_{\text{inq}} \leq m(2^{|EY_1|/4} - 1) + 1 < m2^{|EY_1|/4} - 2, \tag{4.18}
\]

as \( m \geq 3 \).

Finally, we obtain from (4.17)–(4.18) that

\[
|EY_2| \leq rm^r + 1 < (m \cdot 2^{|EY_1|/4} - 2)m^m \cdot 2^{|EY_1|/4} - 1
\]

\[
< 2^{|EY_1|/4} \cdot m \cdot 2^{|EY_1|/4} = 2^{|EY_1|/4 + \log_2 m \cdot m \cdot 2^{|EY_1|/4}}
\]

\[
< 2^{(1 + \log_2 m \cdot m) \cdot 2^{|EY_1|/4}} < 2^{m^2 \cdot 2^{|EY_1|/4}}
\]

\[
\leq 2^{q_2(|EY_1|/4 + 2 \log_2 m)}.
\]

It remains to show that \( Y_2 \) is connected. Note that it is easy to see that a connected component \( Y'_2 \) of \( Y_2 \) would satisfy all of Lemma’s claims except for \( \text{sol}(\text{inq}(Y'_2)) \) being a vertex of the polyhedron of the dual LP-problem (4.3).

Arguing on the contrary, assume that \( Y_2 \) is not connected and that \( Y_3, Y_4 \) are (nonempty) subgraphs of \( Y_2 \) so that there is no path in \( Y_2 \) from any vertex of \( Y_3 \) to a vertex of \( Y_4 \). Clearly, \( Y_3, Y_4 \) are graphs with property (B). Invoking Lemma 4.1 denote \( y_{V,j} := \text{sol}(\text{inq}(Y'_j)), j = 3, 4 \). By Lemma 4.1 \( y_{V,3}, y_{V,4} \) are points of the polyhedron of (4.3). It is easy to see from the definition of \( \text{sol} \) and Lemma 4.1 that \( y_{V} = \lambda_3 y_{V,3} + \lambda_4 y_{V,4} \) with some positive rational numbers \( \lambda_3, \lambda_4 \) that satisfy \( \lambda_3 + \lambda_4 = 1 \). This, however, is impossible when \( y_V \) is a vertex solution to (4.3). A contradiction proves that \( Y_2 \) is connected, as required. \( \square \)
Proof of Theorem 1.1. (a) Suppose that \( H_1 \) is a finitely generated noncyclic subgroup of the free group \( F = \pi_1(U_m, o_1) \) of rank \( m - 1 \geq 2 \). Conjugating \( H_1 \) if necessary, we may assume that a reduced \( U_m \)-graph of \( H_1 \), denoted as above by \( Y_1 \), coincides with its core, \( \text{core}(Y_1) = Y_1 \).

As in Sect. 3, consider a system of linear inequalities \( \text{SLI}[Y_1] \) with integer coefficients associated with the graph \( Y_1 = Y_1(H_1) \) and an LP-problem
\[
\text{max}\{-x_s \mid \text{SLI}[Y_1]\}.
\]
(5.1)
According to Theorem A and Lemma 4.3, the maximum of the LP-problem (5.1) is equal to \(-\sigma(Y_1)\bar{r}(Y_1)\), as required.

(b) This part follows from Lemmas 4.3–4.4.

(c) It follows from the definitions of Sect. 3 that we can effectively write down the system \( \text{SLI}[Y_1] \) and this can be done in exponential time of size of \( Y_1 \).

Next, we recall again that the size of the dual LP-problem (4.3) is equal to \(|\hat{1}|\) \( \text{SLI}[Y_1]\). Hence, the size of the primal LP-problem \text{max}\{-x_s \mid \text{SLI}[Y_1]\} as well as the size of the dual problem (4.3) are at most exponential in size of input. By Lemma 4.3, an optimal solution to the dual problem (4.3) is equal to
\[-\sigma(Y_1)\bar{r}(Y_1) = -\sigma(H_1)\bar{r}(H_1)\].

Since an LP-problem \text{max}\{\langle c, x \rangle \mid Ax \leq b\} can be solved in deterministic polynomial time of size of the problem, see [21], and since the reduced rank \( r(Y_1) = r(H_1) \) can be computed in polynomial time of size of input, it follows that the \( WN \)-coefficient \( \sigma(H_1) \) of \( H_1 \) can be computed in deterministic exponential time of size of input.

Next, we recall again that the size of the dual LP-problem (4.3), similarly to the size of the primal LP-problem \text{max}\{-x_s \mid \text{SLI}[Y_1]\}, is at most exponential in size of input and that a vertex solution \( \hat{y} \) to (4.3) can be computed in polynomial time of size of the dual LP-problem (4.3), see [21]. Note that here and below we use the notation of the proof of Lemma 4.4. Hence, both a vertex solution \( \hat{y} \) and a combination with repetitions \( \hat{Q} \) such that \( \text{sol}(\hat{Q}) = \hat{y} \) can be computed in deterministic exponential time of size of \( Y_1 \). In view of inequalities (4.16) and (4.18), we obtain that
\[|\hat{Q}| < rm^r < (m2^{EY_1}/4 - 2)m^{m2^{EY_1}/4 - 2} < 2^{O(|EY_1|)}.\]
Therefore, a desired graph \( \hat{Y}_2 = \Gamma(H_2) \) such that \( \text{inq}(\hat{Y}_2) = \hat{Q} \) can be constructed from \( \hat{Q} \) in deterministic double exponential time in size of \( Y_1 = \Psi(H_1) \) as well as in size of input. Recall that an explicit construction of a graph \( \hat{Y}_2 \) from \( \hat{Q} \) such that \( \text{inq}(\hat{Y}_2) = \hat{Q} \) is described in the proof of Lemma 3.2. The proof of Theorem 1.1 is complete.

Proof of Proposition 1.2. (a) This is immediate from part (c) of Theorem 1.1.

(b) Suppose that a finitely generated subgroup \( H \) of \( F \) is not compressed and \( K \) is a subgroup of \( F \) such that \( K \) contains \( H \) and \( K \) has a minimal reduced rank \( \bar{r}(K) \) with \( \bar{r}(K) < \bar{r}(H) \). Let \( Y, Z \) be reduced \( U_m \)-graphs of \( H, K \), resp. Since \( K \) contains \( H \), it follows that there is a locally injective map \( \eta : Y \rightarrow Z \). If \( \eta \) were not surjective then there would be a subgroup \( K' \) of \( F \), whose \( U \)-graph is \( \eta(Y) \),
such that $K'$ contains $H$, $K'$ is a free factor of $K$ and $\bar{r}(K') < \bar{r}(K)$. Hence, it follows from the minimality of $\bar{r}(K)$ that $\eta$ is surjective. This means that the map $\eta : Y \to Z$ can be factored out through a finite sequence of edge foldings which are identifications of edges and vertices and which turn the graph $Y$ into $Z$ such that $\bar{r}(Z) < \bar{r}(Y)$. Hence, our algorithm, that verifies whether $H$ is not compressed in $F$, can nondeterministically perform edge foldings over the reduced $U_m$-graph $Y$ of $H$ to check whether $Y$ can be turned into a reduced $U_m$-graph $Z$ such that $\bar{r}(Z) < \bar{r}(Y)$. Since each single edge folding decreases the number of edges by one, it follows that our algorithm runs in nondeterministic linear time of size of $Y$, as required. □

References

[1] S. Arora and B. Barak, Computational complexity – a modern approach, Cambridge Univ. Press, 2009.
[2] W. Dicks, Equivalence of the strengthened Hanna Neumann conjecture and the amalgamated graph conjecture, Invent. Math. 117 (1994), 373–389.
[3] W. Dicks, Simplified Mineyev, preprint, http://mat.uab.cat/~dicks/SimplifiedMineyev.pdf
[4] W. Dicks and E. Ventura, The group fixed by a family of injective endomorphisms of a free group, Contemp. Math. 195 (1996), 1–81.
[5] W. Dicks and S. V. Ivanov, On the intersection of free subgroups in free products of groups, Math. Proc. Cambridge Phil. Soc. 144 (2008), 511–534.
[6] W. Dicks and S. V. Ivanov, On the intersection of free subgroups in free products of groups with no 2-torsion, Illinois J. Math. 54 (2010), 223–248.
[7] J. Friedman, Sheaves on graphs, their homological invariants, and a proof of the Hanna Neumann conjecture: with an appendix by Warren Dicks, Mem. Amer. Math. Soc. 233 (2014), no. 1100. xii+106 pp.
[8] W. Haken, Theorie der Normalflächen, Acta Math. 105 (1961), 245–375.
[9] J. Hass, J. C. Lagarias and N. Pippenger, The computational complexity of knot and link problems, J. Assoc. Comput. Mach. 46 (1999), 185–211.
[10] G. Hemion, The classification of knots and 3-dimensional spaces, Oxford Univ. Press, 1993.
[11] S. V. Ivanov, On the Kurosh rank of the intersection of subgroups in free products of groups, Adv. Math. 218 (2008), 465–484.
[12] S. V. Ivanov, The computational complexity of basic decision problems in 3-dimensional topology, Geom. Dedicata 131 (2008), 1–26.
[13] S. V. Ivanov, Intersecting free subgroups in free products of left ordered groups, preprint, http://arxiv.org/abs/1607.03010
[14] S. V. Ivanov, Linear programming and the intersection of free subgroups in free products of groups, preprint, http://arxiv.org/abs/1607.03052
[15] W. H. Jaco and J. L. Tollefson, Algorithms for the complete decomposition of a closed 3-manifold, Illinois J. Math. 39 (1995), 358–406.
[16] I. Kapovich and A. G. Myasnikov, Stallings foldings and subgroups of free groups, J. Algebra 248 (2002), 608–668.
[17] I. Mineyev, Submultiplicativity and the Hanna Neumann conjecture, Ann. Math. 175 (2012), 393–414.
[18] H. Neumann, On the intersection of finitely generated free groups, Publ. Math. 4 (1956), 186–189; Addendum, Publ. Math. 5 (1957), 128.
[19] W. D. Neumann, On the intersection of finitely generated subgroups of free groups, Lecture Notes in Math. (Groups-Cambridge 1989) 1456 (1990), 161–170.
[20] C. H. Papadimitriou, Computational complexity, Addison-Wesley Publ., 1994.
[21] A. Schrijver, Theory of linear and integer programming, John Wiley & Sons, 1986.
[22] J. R. Stallings, Topology of finite graphs, Invent. Math. 71 (1983), 551–565.

Department of Mathematics, University of Illinois, Urbana, IL 61801, U.S.A.
E-mail address: ivanov@illinois.edu