Unification of bosonic and fermionic theories of spin liquids on the kagome lattice

Yuan-Ming Lu,1,2 Gil Young Cho,1,3 and Ashvin Vishwanath1,2

1Department of Physics, University of California, Berkeley, CA 94720
2Materials Sciences Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720
3Department of Physics, Institute for Condensed Matter Theory, University of Illinois at Urbana-Champaign, IL 61801

(Dated: March 5, 2014)

Recent numerical studies have provided strong evidence for a gapped $\mathbb{Z}_2$ quantum spin liquid in the kagome lattice Heisenberg model. A special feature of spin liquids is that symmetries can be fractionalized, and different patterns of fractionalization imply distinct phases. The symmetry fractionalization pattern for the kagome spin liquid remains to be determined. A popular approach to studying spin liquids is to decompose the physical spin into partons which are either bosonic (Schwinger bosons) or fermionic (Abrikosov fermions), which are then treated within mean field theory. A longstanding question has been whether these two approaches are truly distinct or describe the same phase in complementary ways. Here we show that at least four spin liquid phases on the kagome lattice can be described both in terms of bosonic and fermionic partons, unifying pairs of theories that seem rather distinct. The key idea is that for kagome lattice states that admit a Schwinger boson mean field (SBMF) description, the symmetry action on vortices is uniquely fixed. The set of four states includes the two most promising candidate states - Sachdev’s $Q_1 = Q_2$ SBMF state and Lu-Ran-Lee’s $Z_2[0,\pi]\beta$ fermionic parton state - which interestingly are found to be distinct phases. We expect these results to aid in a complete specification of the numerically observed spin liquid phase. We also discuss a set of fermionic parton phases that do not admit a SBMF representation, where spin rotation and kagome lattice symmetries lead to protected edge states.

I. INTRODUCTION

$\mathbb{Z}_2$ spin liquids (SLs) are a class of disordered many-spin states which have a finite energy gap for all bulk excitations. They differ fundamentally from symmetry breaking groundstates such as magnetically ordered phases and valence bond solids, since, in their simplest form, they preserve all the symmetries including spin rotation, time reversal and crystal symmetries. More importantly they possess bulk quasiparticles carrying fractional statistics. For example in the most common $\mathbb{Z}_2$ SL, there are three distinct types of fractionalized bulk excitations: bosonic spinon $b$ with half-integer spin, fermionic spinon $f$ with half-integer spin and bosonic vison $v$ (a vortex excitation of $\mathbb{Z}_2$ gauge theory) with integer spin. They all obey mutual semion statistics: i.e. a bosonic spinon acquires a $-1$ Berry phase when it adiabatically encircles a fermionic spinon or a vison. These statistical properties are identical to those of excitations in $\mathbb{Z}_2$ gauge theory, hence the name “$\mathbb{Z}_2$ spin liquid”.

Recently, interest in $\mathbb{Z}_2$ SLs has been rekindled by numerical studies on the spin-1/2 Heisenberg model on kagome9–11 lattice, where this state is strongly indicated. Lattices in two spatial dimensions. In particular a topological entanglement entropy12,13 of $\gamma = \log 2$ is observed in the ground state. Just like local order parameters are used to describe symmetry breaking phases, here fractional statistics and topological entanglement entropy serve as fingerprints of topological order in $\mathbb{Z}_2$ spin liquids. Analogous results have been reported for the frustrated square lattice, although the correlation lengths in that case are not as small as in the kagome lattice16,17.

Intriguingly, the experimentally studied spin-1/2 kagome materials - such as herbertsmithite18 - also remain quantum disordered down to the lowest temperature scales studied, well below the exchange energy scales. However, in contrast to the numerical studies, it does not appear to be gapped. It is currently unclear if the gaplessness is an intrinsic feature or a consequence of impurities that are known to be present in these materials. Furthermore the magnetic Hamiltonian of the material may depart from the pure Heisenberg limit. Relating the numerical results to experiments remains an important open question.

Since it preserves all symmetries of the system, is a $\mathbb{Z}_2$ SL fully characterized by its topological order? The answer is no. In fact, the interplay of symmetry and topological order leads to a very rich structure. There are many different $\mathbb{Z}_2$ spin liquids with the same $\mathbb{Z}_2$ topological order and the same symmetry group, but they cannot be continuously connected to each other without breaking the symmetry: they are dubbed “symmetry enriched topological (SET)” phases19–25. In a SET phase the quasiparticles not only have fractional statistics, but can also carry fractional symmetry quantum numbers: different SET phases are characterized by different patterns of symmetry fractionalization22.

In the literature $\mathbb{Z}_2$ SLs have been constructed in various slave-particle frameworks: the most predominant two beings to fractionalize physical spins into bosonic spinons2,26–28 and fermionic spinons3,5,19,29–32. Both approaches yield variational wavefunctions with good energetics33–35 for the kagome lattice model. It was proposed that symmetric $\mathbb{Z}_2$ SLs are classified by the projective symmetry groups19 (PSGs) of bosonic/fermionic
spinons. However it has been a long-time puzzle to understand the relation between different PSGs in bosonic-spinon representation (bSR) and fermionic-spinon representations (fSR). To be specific in the kagome lattice Heisenberg model, in bSR (Schwinger-boson approach) there are 8 different $Z_2$ SLs$^{28}$ among which the so-called $Q_1 = Q_2$ state$^{27}$ is the most promising candidate$^{33}$. Meanwhile among 20 distinct $Z_2$ SLs$^{37}$ in fSR (Abrikosov-fermion approach), the most promising one is so-called $Z_2[0, \pi, \beta]$ state$^{37}$ in the neighborhood of energetically favorable $U(1)$ Dirac SL$^{34}$. Are these two best candidates actually two different aspects of the same gapped phase? If not, what are their counterparts in the other representation?

In this paper we establish the general connection between different $Z_2$ SLs in bSR and fSR. We show that $Z_2$ SLs constructed by projecting bilinear mean-field ansatz in bSR (Schwinger-boson) will never have symmetry-protected gapless edge states. This important observation allows us to determine how visons transform under symmetry in Schwinger-boson $Z_2$ SLs, and to further relate a Schwinger-boson state to one in fSR. We demonstrate that the spinon PSG is not enough to fully characterize a $Z_2$ SL: two distinct $Z_2$ SLs in fSR can have the same PSG for fermionic spinons while only one of them has symmetry-protected gapless edge modes. However, they differ in the transformation properties of visons under symmetry (vison PSG), which provides an interesting link between symmetry implementation and topological edge states. Applying these general principles to $Z_2$ SLs on kagome lattice, we show that among the 8 different Schwinger-boson (bSR) states, only 4 have their partners in the Abrikosov-fermion (fSR) representation. This correspondence allows us to identify the possible symmetry-breaking phases in proximity to $Z_2$ SLs on kagome lattice. Moreover these results serve as a useful guide in future studies of $Z_2$ SLs.

II. SYMMETRY FRACTIONALIZATION IN $Z_2$ SPIN LIQUIDS: GENERAL CONSIDERATIONS

Spinons and visons in a $Z_2$ SL satisfy the following Abelian fusion rules$^7$:

\[
\begin{align*}
    b \times f &= v, & b \times v &= f, & f \times v &= b, \\
    b \times b &= f \times f = v \times v = 1.
\end{align*}
\]

(1)

Here 1 stands for local excitations carrying integer spins, and the fractionalized excitations will not pick up any nontrivial Berry phase when they encircle these local excitations. In other words the bound state of a bosonic spinon and a fermionic spinon is a vison, and the bound state of two bosonic (fermionic) spinons or two visons is a local excitation.

The fusion rules (1) have important implications on the symmetry properties of spinons and visons. More precisely, the symmetry quantum numbers of spinons and visons must be compatible with the fusion rules$^{22}$. Take $SU(2)$ spin rotational symmetry$^{38}$ for a simple example. Both bosonic ($b$) and fermionic ($f$) spinons carry spin-1/2 each, and vison $v$ is a spin-singlet excitation. According to rules of angular momentum addition, one can immediately see fusion rules (1) are consistent with spin quantum numbers.

Now let us apply this principle to other symmetries, i.e. time reversal $T$ and crystal symmetries of kagome lattice (see FIG. 1). Note that in a topologically ordered phase, the fractional excitations (anyons) always couple to emergent gauge fields$^{6,39}$. A direct consequence is that symmetry transformations on these anyons are not well-defined, in the sense that they are not invariant under gauge transformations. The gauge-invariant “good” symmetry quantum numbers are the Berry phases that anyons pick up through a series of symmetry operations which add up to identity operation. They correspond to different algebraic identities built up by the generators of symmetry group, as summarized in TABLE I for kagome lattice.

In a gapped $Z_2$ SL phase, these gauge-invariant Berry phases are quantized to be $\pm 1$. This is simply because any local excitation, as a bound state of an even number of spinons/visons, can only pick up a trivial ($+1$) Berry phase after these symmetry operations. These symmetry quantum numbers are termed “projective symmetry group” (PSG)$^{19}$, and two symmetric $Z_2$ SLs with different PSGs cannot be smoothly connected without breaking symmetry. TABLE I shows the spinon and vison PSGs for $Z_2$ SLs on kagome lattice constructed in Schwinger-boson (bSR) and Abrikosov-fermion (fSR) approach, which will be discussed in detail later.

The vison PSGs are fully determined by PSGs of bosonic/fermionic spinons due to fusion rule (1). To be
specific, in most cases the Berry phase $e^{i \phi_v}$ picked up by a vison equals the product of Berry phase $e^{i \phi_b}$ acquired by a bosonic and $e^{i \phi_f}$ by a fermionic spinon in the same process. The only exception in TABLE I is $(R_{\pi/3})^6$, during which all particles rotate counterclockwise around a hexagon center by a full circle. Being a bound state of a bosonic and fermionic spinon, a vison would collect an extra $-1$ Berry phase because the bosonic spinon encircles the fermionic spinon once in this process. In the case of time reversal symmetry $T$, it is easy to see that $T^2 = -1$ for spinons and $T^2 = +1$ for visons according to their spin quantum numbers.

### III. Symmetry Protected Edge States and Vison Quantum Numbers

Previously we discussed why the “good” symmetry quantum numbers for spinons/visons are their PSGs, and how the vison PSGs can be determined once we know PSGs of both bosonic and fermionic spinons. But one important question still remains to be answered: what are the physical manifestations of the vison PSG?

An important measurable property of topological phases are their edge states. Although $Z_2$ SLs in the absence of symmetries are expected to have gapped edges, the edge may be gapless in the presence of symmetries$^{24,41,42}$, or, in the case of discrete symmetries, spontaneously break symmetry at the edge. In particular, the criterion for nontrivial edges of $Z_2$ SLs with $SU(2)$ spin rotation symmetry will be established below and shown to be particularly simple. The edge modes of a $Z_2$ SL can always be fermionized$^{43}$ with the same number of right ($\psi_{R,a}$) and left ($\psi_{L,a}$) movers (velocity is set to unity):

$$L_0 = \sum_a i \psi^\dagger_{R,a} (\partial_t - \partial_x) \psi_{R,a} - i \psi^\dagger_{L,a} (\partial_t + \partial_x) \psi_{L,a} \tag{2}$$

where $a$ denotes different branches of left/right movers. One can always add backscattering terms to gap out the edge modes $\{\psi_{R/L,a}\}$

$$L_1 = \sum_{a,b} \psi^\dagger_{R,a} M_{a,b} \psi_{L,b} + \psi_{R,a} \Delta_{a,b} \psi_{L,b} + h.c. \tag{3}$$

if they are not forbidden by symmetry. In a different language, the above “mass” terms correspond to condensing either bosonic spinons $b$ or visons $v$ on the edge$^{44}$ of a $Z_2$ SL. Since the spinons carry spin-1/2 each, condensing them will necessarily break spin rotational symmetry on the edge. Therefore the only way to obtain a gapped edge without breaking the symmetry is to condense visons, unless their crystal symmetry quantum numbers (PSGs) will not allow it. The relevant symmetries here are the ones that leave the physical edge unchanged, e.g. at least one translation symmetry among $T_{1,2}$ will be broken by the edge. Therefore the existence or absence of symmetry protected edge states is a probe of the vison PSGs.

Take kagome lattice for instance, on a cylinder with open boundaries parallel to $T_1$ direction (X-edge in FIG. 1), the remaining symmetries are translation $T_1$, time reversal $T$ and mirror reflection $R_1 \equiv (R_{\pi/3})^2 R_y (R_{\pi/3})^{-1}$. If there are no symmetry protected edge states, then the remaining symmetries must act trivially on visons:

$$T_1^{-1} T^{-1} T_1 T = R_1^{-1} T^{-1} R_1 T = 1, \tag{4}$$

$$R_y^2 = T_1 R_{y,1}^{-1} T_1 R_1 = 1.$$

so no backscattering term is forbidden by symmetry.

Another inequivalent edge is perpendicular to $T_1$ direction (Y-edge in FIG. 1), which preserves translation $T_1^{-1} T_2^{-1}$, time reversal $T$ and mirror reflection $R_y$. Similarly, absence of protected edge modes necessarily implies

$$T_2^{-1} T_1^{-1} T_2^{-1} T_1 T = R_y^{-1} T^{-1} R_y T = 1, \tag{5}$$

$$R_y^2 = T_1^{-1} T_2^{-1} R_y^{-1} T_1^{-1} T_2^2 R_y = 1.$$

$i.e.$ these symmetries also have trivial actions on visons.

In addition to edge states, symmetry-protected in-gap bound state in crystal defects$^{45,46}$ (such as dislocations and disclinations) is another signature to probe crystal symmetry quantum numbers. Just like gapless edge modes, the existence of defect bound states must require non-trivial vison PSGs of the associated crystal symmetry. In our case of kagome lattice, the crystal defect associated with $R_{\pi/3}$ rotation is a disclination, centered on the hexagon with Frank angle $\Omega = \pi/3, n \in \mathbb{Z}$. Now let us imagine a vison circles around an elementary disclination with $\Omega = \pi/3$ for six times counterclockwise, and

| Algebraic Identities | Bosonic $b_i$ | Fermionic $f_i$ | Vison $v = b \times f$ |
|----------------------|--------------|----------------|-------------------|
| $T_1^{-1} T_1$ | $-1$ | $\eta_{12}$ | $-1$ |
| $T_1^{-1} R_{\pi/3} T_2 R_{\pi/3}$ | $1$ | $1$ | $1$ |
| $T_1^{-1} R_{\pi/3} T_1 R_{\pi/3}$ | $(-1)^{p_1}$ | $\eta_{12}$ | $-1$ |
| $(R_{\pi/3})^6$ | $(-1)^{p_1 + p_3}$ | $\eta f_{\pi/3} c_6$ | $1$ |
| $T_1^{-1} T^{-1} T_1$ | $1$ | $1$ | $1$ |
| $T_1^{-1} R_{\pi/3} T_1$ | $1$ | $1$ | $1$ |
| $R_y T^{-1} R_y T$ | $(-1)^{p_2}$ | $\eta \sigma T^y c_6 T$ | $1$ |
| $R_{\pi/3}^{-1} T^{-1} R_{\pi/3} T$ | $(-1)^{p_3}$ | $\eta C_6 T$ | $1$ |
| $T^2$ | $-1$ | $-1$ | $1$ |
the Berry phase it picks up in this process equals to the vison PSG $R_{\pi/3}^6$. Note that a vison only acquires a trivial $(+1)$ Berry phase when it encircles any number of spinons 6 times. Therefore if $(R_{\pi/3})^6 = -1$ for visons, there must be a nontrivial in-gap bound state localized at the $\Omega = \pi/3$ disclination. And the absence of in-gap bound state inside the disclination dictates

$$(R_{\pi/3})^6 = R_{\pi/3}^{-1}T^{-1}R_{\pi/3}T = 1. \quad (6)$$

To summarize, if a $Z_2$ SL does not host any gapless edge states or in-gap defect bound state protected by symmetry, vison PSGs must satisfy conditions (4)-(6).

Furthermore, for a Mott insulator (with an odd number of spin $1/2$ moments per unit cell) like the kagome $S=1/2$ model, a confined phase with a spin gap must double the unit cell. Therefore the visions PSG under translations must satisfy:

$$T_1T_2 = -T_2T_1 \quad (7)$$

This may be thought of as visons acquiring an extra Berry phase on going around a unit cell with an odd number of spinons. All together this leads to the vision PSGs shown in TABLE I, assuming the absence of gapless edge states or in-gap disclination bound states which are protected by symmetries. The detailed derivations are summarized in Appendix A.

### IV. UNIFICATION OF SLAVE-PARTICLE MEAN-FIELD THEORIES ON THE KAGOME LATTICE

In $b$SR or Schwinger-boson approach$^{47}$, a spin-$1/2$ on lattice site $\mathbf{r}$ is decomposed into two species of bosonic spinons $\{b_{\mathbf{r},\alpha} | \alpha = \uparrow/\downarrow\}$:

$$\hat{S}_{\mathbf{r}} = \frac{1}{2} \sum_{\alpha,\beta = \uparrow/\downarrow} \bar{b}_{\mathbf{r},\alpha} \mathbf{\sigma}_{\alpha,\beta} b_{\mathbf{r},\beta} \quad (8)$$

where $\mathbf{\sigma}$ are Pauli matrices. Meanwhile in $f$SR or Abrikosov-fermion approach$^{29}$ spin-$1/2$ is represented by two flavors of fermionic spinons$^{38}$ $\{f_{\mathbf{r},\alpha} | \alpha = \uparrow/\downarrow\}$

$$\hat{\bar{S}}_{\mathbf{r}} = \frac{1}{2} \sum_{\alpha,\beta = \uparrow/\downarrow} f_{\mathbf{r},\alpha} \mathbf{\sigma}_{\alpha,\beta} f_{\mathbf{r},\beta} \quad (9)$$

To faithfully reproduce the 2-dimensional Hilbert space for spin-$1/2$, there is a single-occupancy constraint: $\sum_{\mathbf{r},\alpha} b_{\mathbf{r},\alpha} b_{\mathbf{r},\alpha} = \sum_{\mathbf{r},\alpha} f_{\mathbf{r},\alpha} f_{\mathbf{r},\alpha} = 1$ on every lattice site $\forall \mathbf{r}$. The variational wavefunctions are obtained by implementing Gutzwiller projections$^{48}$ on spinon mean-field ground state $|MF\rangle$, in order to enforce the single-occupancy constraint. Here $|MF\rangle$ is the ground state of (quadratic) mean-field ansatz for bosonic spinons$^{27,28}$:

$$\hat{H}_{MF}^b = \sum_{x,y} \sum_{\alpha,\beta} A_{x,y} \bar{b}_{x,\alpha} b_{y,\beta} + B_{x,y} \bar{b}_{x,\alpha} c_{\alpha,\beta} b_{y,\beta} + h.c.$$ and similarly

$$\hat{H}_{MF}^f = \sum_{x,y} \begin{pmatrix} f_{x,\uparrow} \\ f_{x,\downarrow} \end{pmatrix}^T \begin{pmatrix} t_{x,y} & \Delta_{x,y} \\ \Delta_{x,y}^* & -t_{x,y}^* \end{pmatrix} \begin{pmatrix} f_{x,\uparrow} \\ f_{x,\downarrow} \end{pmatrix} + h.c.$$ for fermionic spinons$^{3,19}$. Proper on-site chemical potentials guarantee single-occupancy in $|MF\rangle$ on average.

The physical properties of a gapped $Z_2$ SL described by a projected wavefunction can be understood in terms of its mean-field ansatz. Specifically in $b$SR and $f$SR of $Z_2$ SLs, different PSGs for bosonic and fermionic spinons lead to distinct hopping/pairing patterns in mean-field ansatz $H_{MF}^b$ and $H_{MF}^f$. As pointed out in Ref. 28 there are 8 different Schwinger-boson ($b$SR) mean-field ansatz of $Z_2$ SLs on kagome lattice, while 20 distinct mean-field ansatz exists in $f$SR as shown in Ref. 37. A natural question is: what is the relation between the $Z_2$ SLs in $b$SR and those in $f$SR? Can they describe the same $Z_2$ SL phase or not?

To answer this question, we use their vison symmetry quantum numbers (PSGs) to determine the
(in)equivalence of the two representations. To be precise, a Schwinger-boson ansatz corresponds to the same phase as an Abrikosov-fermion ansatz if and only if they share the same vison PSG. As mentioned earlier, vison PSGs of a $Z_2$ SL can be probed by checking whether symmetry-protected edge states or in-gap defect (e.g. disclination) bound states exist or not. One important observation is that none of $Z_2$ SLs constructed in Schwinger-boson approach supports any gapless edge state or in-gap defect bound state. This can be verified by computing the edge spectrum or defect spectrum in a Schwinger-boson mean-field ansatz. Any gapped Schwinger-boson $Z_2$ SL ansatz can be tuned continuously to a limit that on-site chemical potential dominates over pairing/hopping terms, where it is clear no in-gap modes exist in edge/defect spectra. Therefore the vison PSGs in any Schwinger-boson ansatz is fully fixed as in TABLE I. Amazingly this result (last column in TABLE I) agrees with vison PSGs derived microscopically from Schwinger-boson ansatz.

As discussed previously fermionic/bosonic spinon and vison PSGs are restricted by fusion rules $(1)$. More concretely vison PSGs always equal the product of bosonic and fermionic spinon PSGs, except for the special case $(R_{π/3})^6$ on kagome lattice (with an extra $-1$ phase). Therefore from TABLE I, we can decide the relation between a Schwinger-boson ansatz and an Abrikosov-fermion ansatz, if they describes the same gapped $Z_2$ SL. This leads to the following conditions:

$$
\eta_{12} = (-1)^{p_1+1}, \quad \eta_{σ} = (-1)^{p_2}, \quad \eta_{σT} = (-1)^{p_2+p_3},
$$
$$
\eta_{σCN} = \eta_{C_N} = (-1)^{p_3}, \quad \eta_{C_0} = (-1)^{p_1+p_3+1}.
$$

(10)

It turns out there are only 4 distinct gaped $Z_2$ SLs, which can be described by both fSR (Abrikosov-fermion approach) and bSR(Schwinger-boson approach), as summarized in TABLE II. Moreover, the most promising Schwinger-boson state for kagome Heisenberg model, $Q_1 = Q_2$ state, describes a different phase from the most promising $Z_2[0, π]β$ state in Abrikosov-fermion approach.

V. TWO PROMISING CANDIDATES FOR KAGOME HEISENBERG MODEL

We have noted that there are 20 different Abrikosov-fermion ($f$SR) mean-field states and 8 different Schwinger-boson ($b$SR) mean-field states of $Z_2$ spin liquids on the kagome lattice. Here we will argue that we can narrow the number of candidates down to two using the following two criteria (i) energetics, as elaborated below and (ii) the requirement that the phase to be connected to a $q = 0$ magnetically ordered state via a continuous transition. Finally we will elaborate on why these two must be distinct states.

A. Indentification of two promising candidate states.

Numerical studies on the kagome lattice Heisenberg antiferromagnet supplemented by a 2nd neighbor antiferromagnetic coupling $(J_2)$ reveal that on increasing $J_2$, the quantum spin liquid phase is initially further stabilized, before undergoing a transition into a $q = 0$ magnetic order around $J_2 \approx 0.2J_1$. Since the correlation length increases as the transition is approached, it is likely to be second order. The $q = 0$ magnetic order is a coplanar magnetic ordered state in which the three sublattices have spins aligned along three directions at 120 degrees to one another. The Schwinger-boson state that naturally accounts for this is the $Q_1 = Q_2$ state in Ref. 27, where under condensation of bosonic spinons the $q = 0$ magnetic order develops via a continuous transition in the O$(4)$ universality class. Furthermore, variational permanent wavefunctions based on the $Q_1 = Q_2$ state were shown to give very competitive energies in particular for small and positive $J_2$, establishing it as a contending state.

The $Z_2[0, π]β$ mean-field state of Abrikosov fermions (fSR), also satisfies the desirable properties above. It can be thought as the $s$-wave paired superconductor of fermionic spinons near the energetically favorable $U(1)$ Dirac SL. The $s$-wave pairing opens up a gap at the Dirac point of the underlying $U(1)$ Dirac SL and this implies that the $Z_2[0, π]β$ state could be energetically competing with the $U(1)$ Dirac SL. Although variational wavefunctions that include pairing are often found to have higher energy than the underlying $U(1)$ Dirac SL, we note that this is a restricted class of states accessible via the parton construction, and a more complete search may land in the superconducting phase. For our purposes we will be content that it is proximate to the energetically favorable $U(1)$ spin liquid state. One can also describe a continuous transition from this $Z_2$ SL to the coplanar $q = 0$ magnetically ordered state, although the argument here is more involved than in the case of the Schwinger boson states. We make this argument in two parts - first by ignoring the effects of the gauge field and recalling a seemingly unrelated transition between an $s$-wave superconductor and a quantum spin Hall phase of the fermionic partons. The latter spontaneously breaks the $SU(2)$ spin rotation symmetry down to $U(1)$ that defines the direction of the conserved spin component. On including gauge fluctuations we will argue that the quantum spin Hall phase is to be identified with the $q = 0$ magnetic order.

Starting with the $U(1)$ Dirac spin liquid, the $s$-wave superconductor representing the $Z_2$ SL is obtained by including a superconducting ‘mass’ term that gaps the Dirac dispersion. Similarly, the quantum spin Hall phase is obtained by introducing a distinct ‘mass’ term that also gaps out the Dirac nodes. There are three such mass terms indexed by the direction of the conserved spin in the quantum spin Hall state. All
of these anti-commute with the superconducting mass term, which implies that a continuous transition is possible between these phases\(^{31}\). On integrating out the fermions, the coefficients of the mass terms form an \(O(5)\) vector of order parameters (real and imaginary parts of the pairing and the three quantum spin Hall mass terms), described by \(O(5)\) non-linear sigma model with a Wess-Zumino-Witten (WZW) term\(^{36,51–54}\). Then the presence of the WZW term implies a continuous phase transition from the superconductor to the quantum spin hall state with a spontaneously chosen orientation. This is most readily seen by noting that skyrmions of the quantum spin Hall director carry charge 2, which when condensed lead to a superconductor with spin rotation symmetry\(^{31}\). Now, on including gauge couplings the superconductor is converted into the \(Z_2\[0, \pi]\) SL state. The quantum spin Hall state is a gapped insulator.

In contrast, on either edge we have different PSGs for bosonic and fermion spinons. In particular, certain fermionic-spinon PSGs allow for a nontrivial band topology: this is manifested by the symmetry protected edge states in a reflection-symmetric topological superconductor of Abrikosov-fermions.

### A. Reflection protected X-edge states

There are two types of open edges in a cylinder geometry, \textit{i.e.} X-edge and Y-edge in FIG. 1. Other translational-symmetric open edges can be obtained by multiples of \(R_{\pi/3}\) rotation acting on these two prototype edges. As mentioned earlier, a cylinder with open X-edge preserves a symmetry group generated by \(\{T, T_1, R_1 \equiv (R_{\pi/3})^2 R_y(R_{\pi/3})^{-1}\}\) and \(SU(2)\) spin rotations. It’s straightforward to see that

\[
T_{1}^{-1}T^{-1}T_1 T = T_1 R_{1}^{-1} T_1 R_1 = 1.
\]

for all Abrikosov-fermion states. Since translational symmetry \(T_1\) also commutes with spin rotations, it can be disentangled from other symmetries. As discussed in Appendix C, one can further show that translational symmetry \(T_1\) won’t give rise to any nontrivial topological index. Focusing on time reversal \(T\), mirror reflection \(R_1\) (and \(SU(2)\) spin rotations which commute with both \(T\) and \(R_1\)), we have

\[
(R_1)^2 = \eta_\sigma, \quad R_{1}^{-1}T_{1}^{-1} R_{1}T = \eta_\sigma T.
\]

for Abrikosov fermions \(f\). As proven in Appendix C, only when

\[
\eta_\sigma = -1, \quad \eta_\sigma T = +1.
\]

will there be a nontrivial integer index \(Z\) for topological superconductors. As summarized in TABLE III, there are 6 fermionic-spinon PSGs (#7 \sim #12) that can support such a topological superconductor.

### B. Reflection protected Y-edge states

On a cylinder with open Y-edge the symmetry group is generated by \(\{T, T_y \equiv T_1^{-1}T_2^2, R_y\}\) and \(SU(2)\) spin rotations. Again one can easily show that

\[
T_{y}^{-1}T_{y}^{-1}T_{1} T = T_y R_{y}^{-1} T_y R_y = 1.
\]
for all Abrikosov-fermion states and we can disentangle translation $T_y$ from other symmetries. With reflection $R_y$ and time reversal $T$ on $Y$-edge we have

$$R_y^2 = \eta_\sigma \eta_\sigma c_\alpha, \quad R_y^{-1} T^{-1} R_y T = \eta_\sigma T^\sigma c_\alpha T. \quad (17)$$

Similarly a nontrivial integer index $Z$ can only happen when

$$\eta_\sigma \eta_\sigma c_\alpha = -1, \quad \eta_\sigma T^\sigma c_\alpha T = +1. \quad (18)$$

It turns out that only 6 fermionic-spinon PSGs (\#3, \#4, \#9, \#10, \#19, \#20) in TABLE III can support topological superconductors with protected gapless modes on Y-edge.

### TABLE III. 20 different Abrikosov-fermion $Z_2$ SLs on a kagome lattice in the notation of Ref. 37. Among them state \#16 corresponds to the same phase as $Q_1 = Q_2$ state in Schwinger-boson representation with 28 ($p_1, p_2, p_3$) = (0, 1, 0). Notice that among the 8 different Schwinger-boson $Z_2$ SLs with $p_i = 0$ or 1, only 4 (with $p_3 = 0$) have their counterparts in Abrikosov-fermion representation. The other 4 Schwinger-boson states (with $p_3 = 1$) cannot be realized by any Abrikosov-fermion mean-field ansatz. “Perturbatively gapped” means that fermion spinons can reach a fully-gapped superconducting ground state by perturbing the nearest neighbor (NN) hopping ansatz. “0” is a trivial topological index, indicating the absence of symmetry protected gapless modes on the edge. Meanwhile Z is a nontrivial integer topological index for possible symmetry protected gapless edge states. Among the 20 Abrikosov-fermion states, 6 can host protected edge states on X-edge, while 6 can support protected edge modes on Y-edge.

| # | \(\eta_{12}\) | \(\eta_\sigma\) | \(T\eta_{12}T\) | \(\eta_\sigma c_\alpha\) | \(T\eta_{12}C_\alpha T\) | Label | Perturbatively gapped? | X-edge | Y-edge | \((p_1, p_2, p_3)\) |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | +1 | +1 | +1 | +1 | +1 | +1 | $Z_2[0, 0]A$ | Yes | 0 | 0 | (1,0,0) |
| 2 | -1 | +1 | +1 | +1 | -1 | -1 | $Z_2[0, \pi]B$ | Yes | 0 | 0 | (0,0,0) |
| 3 | +1 | +1 | -1 | +1 | -1 | -1 | $Z_2[0, \pi]A$ | No | 0 | Z | |
| 4 | -1 | +1 | -1 | +1 | -1 | -1 | $Z_2[0, \pi]A$ | No | 0 | Z | |
| 5 | +1 | +1 | -1 | -1 | +1 | +1 | $Z_2[0, 0]B$ | Yes | 0 | 0 | |
| 6 | -1 | +1 | -1 | -1 | +1 | -1 | $Z_2[0, \pi\alpha]$ | No | 0 | 0 | |
| 7 | +1 | -1 | +1 | -1 | +1 | -1 | $Z_2[0, 0]B$ | Yes | 0 | 0 | |
| 8 | -1 | -1 | +1 | -1 | +1 | -1 | $Z_2[0, 0]D$ | Yes | 0 | 0 | |
| 9 | +1 | -1 | -1 | +1 | -1 | -1 | $Z_2[0, 0]D$ | No | 0 | 0 | |
| 10 | -1 | -1 | +1 | +1 | -1 | -1 | $Z_2[0, 0]D$ | Yes | 0 | 0 | (1,1,0) |
| 11 | +1 | +1 | -1 | +1 | -1 | -1 | $Z_2[0, 0]D$ | No | 0 | 0 | (0,1,0) |
| 12 | -1 | -1 | -1 | +1 | -1 | -1 | $Z_2[0, 0]D$ | No | 0 | 0 | |
| 13 | +1 | +1 | -1 | -1 | +1 | -1 | $Z_2[0, 0]D$ | Yes | 0 | 0 | |
| 14 | -1 | -1 | -1 | -1 | +1 | +1 | $Z_2[0, 0]D$ | No | 0 | 0 | |
| 15 | +1 | +1 | +1 | +1 | +1 | +1 | $Z_2[0, 0]D$ | Yes | 0 | 0 | |
| 16 | -1 | -1 | +1 | +1 | +1 | +1 | $Z_2[0, 0]D$ | No | 0 | 0 | |
| 17 | +1 | +1 | -1 | +1 | +1 | +1 | $Z_2[0, 0]D$ | No | 0 | 0 | |
| 18 | -1 | -1 | -1 | +1 | +1 | +1 | $Z_2[0, 0]D$ | No | 0 | 0 | |
| 19 | +1 | +1 | -1 | -1 | +1 | +1 | $Z_2[0, 0]D$ | No | 0 | Z | |
| 20 | -1 | -1 | +1 | +1 | -1 | +1 | $Z_2[0, 0]D$ | No | 0 | Z | |

As one example, in the root state ($\nu = 1$) which generates the integer ($\nu \in \mathbb{Z}$) topological index, the low-energy edge excitations are described by 2 pairs of counter-propagating fermion modes (see Appendix C for derivations)

$$\mathcal{L}_{edge}^0 = \sum_{a=\uparrow, \downarrow} \left( -i \psi_{R,a}^\dagger (\partial_t - \partial_x) \psi_{R,a} - i \psi_{L,a}^\dagger (\partial_t + \partial_x) \psi_{L,a} \right) \quad (19)$$

where the velocity is normalized as unity. The fermion modes transform under symmetries (time reversal $T$, reflection $R$ and spin rotations) in the following way:

$$\psi_{\alpha,a} \rightarrow \frac{T_3}{2} \sum_{\beta,b} [\tau_3]_{\alpha,\beta} [i \sigma_y]_{a,b} \psi_{\beta,b} \quad (20)$$

$$\psi_{\alpha,a} \rightarrow \frac{R}{2} \sum_{\beta,b} [\tau_3]_{\alpha,\beta} [i \sigma_y]_{a,b} \psi_{\beta,b} \quad (21)$$

$$\psi_{\alpha,a} \exp{i \theta \hat{\sigma}} \sum_{b} \left[ e^{\frac{i}{2} \theta \hat{\sigma}} \right]_{a,b} \psi_{\alpha,b} \quad (22)$$

where we use index $\alpha = R/L$ and $\hat{\sigma}$ matrices for chirality (right/left movers), index $a = \uparrow / \downarrow$ and $\hat{\sigma}$ matrices for spin. It’s straightforward to see that $TR = RT$ and $R^2 = -1$. There are two kinds of backscattering terms between right and left movers, which preserve $SU(2)$ spin.
rotational symmetry:
\[ \mathcal{H}_{\text{hop}} = \sum_a \left( t \psi_{R,a}^\dagger \psi_{L,a} + t^* \psi_{L,a}^\dagger \psi_{R,a} \right), \]
\[ \mathcal{H}_{\text{pair}} = \sum_{a,b} \left( \Delta \psi_{R,a}^\dagger \sigma_y \psi_{L,b} + \Delta^* \psi_{L,b}^\dagger \sigma_y \psi_{R,a}^\dagger \right). \]

Among them, imaginary pairing is forbidden by time reversal \( T \), while hopping and real pairing are both forbidden by mirror reflection \( R \).

Such a gapped \( Z_2 \) SL with symmetry protected edge states is described by a multi-component Chern-Simons theory

\[ \mathcal{L}_{\text{bulk}} = \frac{\epsilon_{\mu\nu\rho}}{4\pi} \sum_{I,J=1}^4 a_I^\dagger K_{I,J} \partial_\mu a_J^\dagger - \sum_{I,J=1}^4 a_I^\dagger J \partial_\mu \]

with a \( 4 \times 4 \) \( K \) matrix\(^{24}\)

\[ K = \begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & -1 \\
1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0
\end{pmatrix} = 2K^{-1} \quad (24) \]

The associated edge excitations are encoded by chiral boson fields \( \phi_I |1 \leq I \leq 4 \) with effective theory:

\[ \mathcal{L}_{\text{edge}} = \frac{1}{4\pi} \sum_{I,J=1}^4 \left( \partial_\mu \phi_I K_{I,J} \partial_\mu \phi_J - \partial_\mu \phi_I V_{I,J} \partial_\mu \phi_J \right) \quad (25) \]

where \( V \) is a positive-definite real symmetric matrix depending on details of the open edge. In particular there are two species of visons \( (v) \) created by vertex operators \( e^{i\phi_1} \) and \( e^{i\phi_2} \) separately, while bosonic spinons \( (b) \) are created by \( e^{i\phi_3} \) and \( e^{i\phi_4} \) operators. Under a symmetry operation \( g \) the chiral bosons transform as\(^{24,58,59}\)

\[ \phi_I \xrightarrow{g} \phi_I = \sum_J s_g [W^g]_{I,J} \phi_J + \delta \phi_I^g. \]

where the sign \( s_g = +1 \) for a unitary symmetry (such as mirror reflection \( R \) and spin rotations) and \( s_g = -1 \) for an anti-unitary symmetry (such as time reversal \( T \)).

In our case of a \( Z_2 \) SL with symmetry protected edge states, for spin rotation \( S_\theta^z \equiv \prod_i \exp(i\frac{\theta}{2} \sigma_i^z) \) by angle \( \theta \) along \( \hat{z} \)-axis

\[ W^{S_\theta^z} = 1_{4 \times 4}, \quad \delta \phi^{S_\theta^z} = \frac{\theta}{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (26) \]

For Ising symmetry \( S_\pi^x \equiv \prod_i \sigma_i^x \) (spin rotation by angle \( \pi \) along \( \hat{x} \)-axis) we have

\[ W^{S_\pi^x} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \delta \phi^{S_\pi^x} = \frac{\pi}{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (27) \]

For anti-unitary time reversal symmetry \( T \)

\[ W^T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \delta \phi^T = \frac{\pi}{2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (28) \]

For mirror reflection symmetry \( R \) we have

\[ W^R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \delta \phi^R = \frac{\pi}{2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (29) \]

Notice that under either time reversal \( T \) or mirror reflection \( R \), the \( K \) matrix changes sign i.e.

\[ (W^g)^T K W^g = -K, \quad g = T, R \quad (30) \]

The two pairs of fermionic modes are given in terms of chiral bosons as

\[ \psi_{R,\uparrow} \sim e^{i(\phi_1+\phi_4)}, \quad \psi_{R,\downarrow} \sim e^{i(\phi_2-\phi_3)}, \quad \psi_{L,\uparrow} \sim e^{i(\phi_4-\phi_1)}, \quad \psi_{L,\downarrow} \sim e^{-i(\phi_2+\phi_3)}. \quad (31) \]

to Klein factors. The spin rotational symmetric backscattering terms between right and left movers are given by

\[ \mathcal{H}_{\text{hop}} \sim |t| \left[ \cos(2\phi_1 + C_1) + \cos(2\phi_2 + C_2) \right], \]
\[ \mathcal{H}_{\text{pair}} \sim \text{Re}[\cos(\phi_1 - \phi_2) + \text{Im}[\Delta] \cdot \sin(\phi_1 - \phi_2)]. \]

where \( C_1, C_2 \) are constants. As discussed earlier they are forbidden by time reversal and mirror reflection symmetries.

\[ \text{VII. CONCLUSION AND OUTLOOK} \]

In this work we systematically establish a connection between two different representations of \( Z_2 \) SLs, i.e. Schwing-er-boson representation (bSR) and Abrikosov-fermion representation (fSR). In the presence of a symmetry group, projective symmetry groups (PSGs) are symmetry quantum numbers of the quasiparticle excitations in a \( Z_2 \) SL. We show that the vison symmetry quantum numbers can be detected by existence/absence of gapless edge modes and in-gap bound states localized at crystal defect in \( Z_2 \) SLs. Observing that there are no symmetry-protected edge modes or defect bound states in any Schwing-er-boson state, and utilizing the relation between bosonic/fermionic spinon PSGs and vison PSGs, a correspondence between Schwing-er-boson states and Abrikosov-fermion states is fully achieved. Applying this general framework to kagome lattice \( Z_2 \) SLs, we found that out of 20 distinct Abrikosov-fermion (fSR) states, only 4 have their counterparts in Schwing-er-boson representation (bSR).
There are also 4 Schwinger-boson states which cannot be realized by a mean-field ansatz in $f$SR. We believe this is due to limitations of Abrikosov-fermion representation, in other words these 4 states may be realized in a more general representation to construct $Z_2$ SLs out of fermionic spinons. Note, we have implicitly assumed that the topological order of $Z_2$ SLs is the same as that of the deconfined $Z_2$ gauge theory, based on the measured topological entanglement entropy of $\gamma = \log 2$. However, a distinct topological order, the double semion theory, which is a twisted version of the $Z_2$ gauge theory, also yields the same $\gamma$. While there is no direct evidence to support such a twisted $Z_2$ theory (in contrast to the energetic arguments for $Z_2$ spin liquids), it would be interesting in future to investigate candidate states.

With this connection in hand, we can potentially have a full understanding of the possible proximate phases and quantum phase transitions out of a $Z_2$ SL. It is well-known that the Schwinger-boson approach allows one to identify quantum phase transitions (QPT) into neighboring magnetic-ordered phases from a $Z_2$ SL, though condensation of bosonic spinons$^{27}$. Meanwhile, knowing the vison PSGs one can study possible QPTs between paramagnetic valence-bond-solid (VBS) phases and $Z_2$ SLs. On the other hand, in $f$SR it’s straightforward to track down gapless phases connected to a $Z_2$ SL through a phase transition, as well as proximate superconducting ground states upon doping a quantum SL$^{62}$. Therefore the identification between $f$SR and $b$SR can lead to a full phase diagram around a gapped $Z_2$ SL.

The correspondence obtained here also serves as important guidance towards a complete specification of the $Z_2$ spin liquid on the kagome lattice. If one of the two promising states we identified is indeed the ground state, then that provides a clear target for future studies to look for “smoking gun” signatures of these two states. Finally, we point out that similar studies can be applied to $Z_2$ SLs on the square lattice$^{16,17}$, which we leave for future work.

**ACKNOWLEDGMENTS**

We thank M. Hermele, Y. Ran, C. Xu, Y.B Kim, T. Grover, M. Lawler, P. Hosur, F. Wang, S. White, T. Senthil, P.A. Lee, D.N Sheng and S. Sachdev for helpful discussions. GYC thanks Joel E. Moore for support and encouragement during this work. The authors acknowledge support from Office of BES, Materials Sciences Division of the U.S. DOE under contract No. DE-AC02-05CH11231 (YML,AV), NSF DMR-1206515 and ICMT postdoctoral fellowship at UIUC (GYC) and in part from the National Science Foundation under Grant No. PHYS-1066293(YML). YML thanks Aspen Center for Physics for hospitality where part of the work is finished. GYC is especially thankful to M. Punk, Y. Huh and S. Sachdev for bringing Ref. 40 to our attention and for the helpful discussions and suggestions.

**Appendix A: Deriving vison PSGs in TABLE I from edge states**

In this section we explicitly show how to determine the vison PSGs in the last column of TABLE I. First of all we can always choose a proper gauge by multiplying a proper $\pm 1$ sign to symmetry actions $T_1, T_2$, so that

$$T_2 R_{\pi/3} = R_{\pi/3} T_1, \quad T_1 R_{\pi/3} = R_{\pi/3} T_2^{-1} T_1.$$

(A1)

In other words both the 2nd and 3rd rows of TABLE I are $+1$. Meanwhile as discussed in the end of section III, we have

$$T_1 T_2 = - T_2 T_1.$$  \hspace{1cm} (A2)

for visons in a spin-1/2 $Z_2$ SL on kagome lattice.

The absence of symmetry protected edge states along X-edge leads to conditions (4). In particular we have

$$T_1^{-1} T_1 = 1, \quad R_1^2 = (R_{\pi/3} R_y)^2 = 1.$$  \hspace{1cm} (A3)

and

$$R_1^{-1} T_1 R_1 T = (R_{\pi/3}^{-1} R_{\pi/3} T_1^{-1} R_y) \cdot (R_y^{-1} T_1^{-1} R_y T) = 1.$$  \hspace{1cm} (A4)

These conditions fix all the vison PSGs except for $(R_{\pi/3})^6$ and $T_2^{-1} T_1^{-1} T_2 T$. The latter one is easily determined as

$$T_2^{-1} T_1^{-1} T_2 T = 1.$$

by the absence of protected edge states in a cylinder whose edges are parallel to the direction of translation $T_2$. As discussed in section III, $(R_{\pi/3})^6 = 1$ is determined by the absence of protected mid-gap states in a disclination.

**Appendix B: Vison PSGs obtained by explicit calculations$^{40}$**

In this section we deduce the vison PSGs from the dual frustrated Ising model obtained in Ref. 40, which describes vison fluctuations of Schwinger-boson $Z_2$ SLs on
kagome lattice. The 4-component vison modes \( \{v_n|1 \leq n \leq 4\} \) in section III A of Ref. 40 transform under symmetry \( g \) as

\[
v_m \rightarrow \sum_{n=1}^{4} v_n \cdot [O_\phi(g)]_{n,m} \quad (B1)
\]

where the matrices \( \{O_\phi(g)\} \) are given by

\[
O_\phi(T_1) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (B2)
\]

\[
O_\phi(T_2) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (B3)
\]

\[
O_\phi(R_y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad (B4)
\]

\[
O_\phi(R_{\pi/3}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (B5)
\]

Note that Ref. 40 considers mirror reflection \( I_x = R_{\pi/3}R_y \) with

\[
O_\phi(I_x) = O_\phi(R_{\pi/3})O_\phi(R_y) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}
\]

It’s straightforward to check the vison PSGS

\[
O_\phi(T_1)O_\phi(T_2)O_\phi(T_1)^{-1}O_\phi(T_2)^{-1} = -1; \\
O_\phi(R_{\pi/3})^{-1}O_\phi(T_1)O_\phi(R_{\pi/3})O_\phi(T_2) = 1; \\
O_\phi(R_{\pi/3})^{-1}O_\phi(T_2)O_\phi(R_{\pi/3})O_\phi(T_2)^{-1}O_\phi(T_1) = 1; \\
O_\phi(T_1)O_\phi(T_2)^{-1}O_\phi(R_y)O_\phi(T_2)^{-1}O_\phi(R_y)^{-1} = -1; \\
O_\phi(T_1)O_\phi(R_y)O_\phi(T_1)^{-1}O_\phi(R_y)^{-1} = -1; \\
O_\phi(R_{\pi/3})^6 = 1 \\
O_\phi(R_y)^2 = 1 \\
O_\phi(R_{\pi/3})O_\phi(R_y)O_\phi(R_{\pi/3})O_\phi^{-1}(R_y) = 1 \quad (B6)
\]

which agree with the last column of TABLE I.

Appendix C: Two-dimensional TRI singlet superconductors (Class CI) with mirror reflection symmetry

In this section we discuss possible symmetry-protected edge states of a time-reversal-invariant (TRI) singlet superconductor with mirror reflection symmetry in two dimensions. In a cylinder geometry, the symmetry group is generated by translation \( T \), along the open edge (or cylinder circumference), mirror reflection \( R \), time reversal \( T \) and \( SU(2) \) spin rotations. Without loss of generality, let’s assume that \( SU(2) \) spin rotations commute with all other symmetries \( \{T,T_e,R\} \). We further assume that translation \( T_e \) acts as

\[
T_e^{-1}T_eT = T_eR^{-1}T_3R = +1. \quad (C1)
\]

for fermions.

1. Classification

Since translation \( T_e \) has trivial commutation relation with other symmetry operations, it can be disentangled from the full symmetry group. Are there any gapless edge states protected by translation symmetry? This corresponds to “weak index” of 2d topological superconductors in class CI (with time reversal and \( SU(2) \) spin rotations), which is nothing but 1d topological index of the same symmetry class. Class CI has trivial classification (0) in 1d, therefore we don’t have translation-protected edge states. Due to absence of 2d topological index in class CI, any protected edge states must come from mirror reflection symmetry \( R \). The classification of mirror reflection protected topological insulators/superconductors is resolved in Ref. 64 and 65 in the framework of K-theory. The classification of non-interacting topological phases of fermions in class CI with mirror reflection \( R \) depends on the commutation relation

\[
R^2 = s_1, \quad R^{-1}T^{-1}RT = s_2; \quad s_i = \pm 1. \quad (C2)
\]

with time reversal \( T \). When \( s_1 = s_2 = +1 \), the K-theory classification is given by \( \pi_0(R_6) = 0 \), i.e. no topological superconductors with protected edge states. When \( s_1 = s_2 = -1 \), the classification is \( \pi_0(C_5) = 0 \) i.e. no topological superconductors. When \( s_1 = +1,s_2 = -1 \) the classification is \( [\pi_0(R_4)]^2 = 0^2 = 0 \) i.e. no topological superconductors. Only when

\[
s_1 = -1, \quad s_2 = +1. \quad (C3)
\]

the classification is given by \( \pi_0(R_4) = \mathbb{Z} \) and there are topological superconductors with an integer index (\( \mathbb{Z} \)).

2. Minimal Dirac model and protected edge states

Here we construct a Dirac model for the root state \( (\nu = 1) \) of topological superconductors with integer index \( \nu \in \mathbb{Z} \) and mirror reflection \( R \) satisfying

\[
R^2 = -1, \quad RT = TR. \quad (C4)
\]

Writing spin-1/2 electrons in the Nambu basis \( \psi_k \equiv (c_{k\uparrow},c_{-k\downarrow})^T \), we use Pauli matrices \( \vec{\tau} \) for Nambu index...
and $\mu, \beta$ for orbital index. The 8-band massless Dirac Hamiltonian is given by

$$H_{\text{Dirac}} = \sum_k (k_x \mu_x + k_y \mu_y) \rho_y \tau_x$$  \hspace{1cm} (C5)

The Dirac fermion transforms as

$$\psi_k \rightarrow i \tau_y \psi_k^*, \quad \psi(k_x, k_y) \rightarrow i \mu_y \rho_y \psi(k_x, k_y)$$  \hspace{1cm} (C6)

The only symmetry-allowed mass term is

$$M = \tau_z$$  \hspace{1cm} (C7)

Now let’s create a mass domain wall across an open edge at $x = 0$ along $y$-axis. The gapless edge states is captured by zero-energy solution of differential equation

$$-i \partial_x \mu_x \rho_y \tau_x + m(x) \tau_z = 0, \quad m(x) = |m(x)| \cdot \text{Sgn}(x)$$

which is $|\text{edge}\rangle \sim e^{-\int_0^x m(\lambda) d\lambda} \mu_x \rho_y \tau_y = +1$. Therefore $\rho_y = \mu_x \tau_y$ for the protected gapless edge modes, localized on the edge between the root topological superconductor and the vacuum. The Hamiltonian for the protected edge states is given by

$$H_k = k_y \mu_y \rho_y \tau_x \rightarrow -k_y \mu_y \tau_z$$  \hspace{1cm} (C8)

Apparently there are 4 gapless modes: 2 right movers and 2 left movers. It’s straightforward to simplify such edge states to the form in section VI C.

**Appendix D: $Z_2[0, \pi] \beta$ mean-field state and its proximate phases**

We begin by revisiting $U(1)$ Dirac spin liquid \cite{34,67,68} and $Z_2[0, \pi] \beta$ state \cite{37} on a kagome lattice. The spin liquid states are proposed to be the ground state of nearest-neighbor spin-1/2 Heisenberg model on kagome lattice:

$$H = J \sum_{\langle ij \rangle} \hat{S}_i \cdot \hat{S}_j$$  \hspace{1cm} (D1)

The low-energy theory of the $U(1)$ Dirac spin liquid \cite{68} is described by a 8-component spinor $\psi$ of fermionic spinons in Dirac spectrum and a strongly fluctuating $U(1)$ gauge field $\alpha$:

$$H = \sum_k \psi_k^* \sigma^0 \mu^0 (\tau_x k_x + \tau_y k_y) \psi_k.$$  \hspace{1cm} (D2)

where $\tau^\nu$, $\sigma^\nu$ and $\mu^\nu$ are the Pauli matrices acting on Dirac, spin and valley indices of $\psi_k$ (there are two nodes or two “valleys” : hence $\psi_k$ is a 8-component spinor). Here we temporarily ignore the compact $U(1)$ gauge theory for clarity of discussion.

To obtain a $Z_2$ spin liquid from this $U(1)$ Dirac spin liquid, a BCS-type pairing term is introduced for the fermion $\psi$. We require the pairing term to be invariant under spin rotational symmetry, lattice symmetry operations (see Fig.1 for the symmetries of kagome lattice) and time-reversal symmetry, because the spin liquid found in DMRG study does not break any of the symmetries. Furthermore, the pairing should gap out the Dirac spectrum as the $Z_2$ spin liquid found numerically is fully gapped.

$$\delta H = \Delta \psi^\dagger (\sigma^y \mu^y \tau^y) \psi^* + \text{h.c.},$$  \hspace{1cm} (D3)

which is a singlet of spin, valley and Dirac spinor indices. Being a singlet under spin and valley indices guarantees that the pairing is invariant under spin rotational symmetry and lattice symmetry operations. The low-energy physics of $Z_2[0, \pi] \beta$ state is described by $H + \delta H$ in Eq.(D2) and Eq.(D3), i.e. a gapped singlet superconductor of fermionic spinons coupled to a dynamical $Z_2$ gauge field.

It is known that Dirac fermions are unstable (with sufficiently large interactions) to open up gap in various channels. Each channel is called a “mass” and is represented by a constant matrix in the Dirac spinor representation. For example in Dirac Hamiltonian (D2), $\tau^z \mu^0 \sigma^\nu$ are all mass terms, with $\alpha, \beta = 0, x, y, z$. For a 2+1-D Dirac fermion, when we can find five such mass matrices anti-commuting with each other, \cite{52,54,69} we obtain a non-linear sigma model supplemented with a topological WZW term \cite{60,71} after integrating out massive fermions. This non-linear sigma model describes fluctuating order parameters of the Dirac fermions. Most importantly, the theory can describe a Landau-forbidden second-order transition between the two phases \cite{36,51,53,54}, where the transition is driven by condensing the topological defects \cite{38,51,53,54}. The mass terms associated to the $Z_2$ spin liquid are real and imaginary pairings in (D3).

Because we seek for the nearby phases of the $Z_2$ spin liquid, the relevant mass terms should anti-commute with the pairing term (D3) and the kinetic term (D2). We immediately find two $O(3)$ vector mass terms among 26 mass terms of the $U(1)$ Dirac spin liquid \cite{68}, anti-commuting with the pairing term (D3).

Among the two $O(3)$ vector mass terms, we consider only the $O(3)$ vector chirality operator $\hat{V} \sim \psi^\dagger \tau^z \psi$ to examine the magnetically ordered phase of the $Z_2[0, \pi] \beta$ state. The operator $\hat{V}$ is spin-triplet and time-reversal symmetric. This order parameter represents the sum of the vector chirality around honeycomb plaquette $H$ on Kagome lattice.

$$\hat{V}^a \sim \sum_{\langle ij \rangle \in H} (\vec{S}_i \times \vec{S}_j)^a,$$  \hspace{1cm} (D4)

Because the vector chirality $\hat{V}$ is spin-triplet, we expect the non-linear sigma model for the unit $O(5)$ vector equal $(\Delta_x, \Delta_y, \hat{V})$ to describe the transition between the spin liquid and a magnetically ordered phase with the non-zero $\langle \hat{V} \rangle$.

To see this, we approach the critical point between the spin liquid and the magnetically ordered phase from the ordered phase. The low-energy effective theory for
the symmetry-broken phase, including the compact $U(1)$ gauge field $a_\mu$, is

$$\mathcal{L} = \psi^\dagger \sigma^\mu \partial_\mu \psi + m \hat{V} \cdot \psi^\dagger \tau^3 \bar{\psi} + \frac{1}{g^2} (\partial_\mu \hat{V})^2 + \frac{1}{2e^2} (e^{\mu\nu\lambda} \partial_\nu a_\lambda)^2 + \cdots \ (D5)$$

In the symmetry-broken phase, the $O(3)$ vector order parameter $\hat{V}$ develops a finite expectation value, and we assume $\langle \hat{V} \rangle = (0, 0, 1)$ without losing generality (other direction of $\langle \hat{V} \rangle$ can be generated by the spin rotations). With the expectation value $\hat{V}$, it is not difficult to see that the spin-up fermions and the spin-down fermions have an energy gap $|m|$ at the Dirac points with the opposite sign, and the mass gap consequently generates the “spin Hall effect” for the fermions. The quantum spin Hall effect has an important implication for the fermions. The quantum spin Hall effect generates the Goldstone modes in the magnetically ordered phase, one photon mode from the non-compact gauge field $a_\mu$, and two Goldstone modes from the ordering of the $O(3)$ vector $\hat{V}$. Meanwhile accompanying the proliferation of $a_\mu$ photons, fermionic spinons will be confined due to instanton effect of 2+1-D $U(1)$ gauge theory. Therefore indeed it is a non-collinear magnetic ordered phase with three Goldstone modes, which does not support fractionalized excitations.

Upon integrating out the massive Dirac fermion, we obtain the effective theory\textsuperscript{36,52} for the fluctuating $\hat{V}$ in the presence of the gauge field $a_\mu$

$$\mathcal{L} = \frac{1}{g^2} (\partial_\mu \hat{V})^2 + 2a_\mu J^\mu_{\text{skyr}} + \frac{1}{2e^2} (e^{\mu\nu\lambda} \partial_\nu a_\lambda)^2 + \cdots \ (D6)$$

where $J^\mu_{\text{skyr}}$ is the skyrmion current of $\hat{V}$, e.g. $J^0_{\text{skyr}} \propto \hat{V} \cdot (\partial_\tau \hat{V} \times \partial_\rho \hat{V})$ is the skyrmion density of $\hat{V}$. From the coupling between $J^\mu_{\text{skyr}}$ and $a_\mu$, it is clear that the skyrmion carries the charge-2 of the gauge field $a_\mu$. Hence, condensing the skyrmion of $\hat{V}$ breaks $U(1)$ gauge group down to $Z_2$ and the skyrmion can be thought as the pairing $\psi^\dagger \sigma^3 \psi$ of the fermionic spinons $\psi$ in (D2). As the condensation of the skyrmion would destroy the ordering in $\hat{V}$ and induce the pairing between the fermionic spinons, we will enter the $Z_2$ spin liquid phase next to the symmetry-broken phase, i.e. $Z_2[\rho, \pi/\beta]$ state. Thus we have established that the magnetically ordered phase next to the $Z_2[\rho, \pi/\beta]$ state is a non-collinear magnetically ordered phase with the non-zero vector chirality at $q = 0$. Given that the $q = 0$ magnetically ordered state is also a non-collinear magnetically ordered phase with the non-zero vector chirality at $q = 0$, the $Z_2[\rho, \pi/\beta]$ state is a natural candidate for the $Z_2$ fSL proximity to the $q = 0$ magnetically ordered state.

---

1. F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific Pub Co Inc, 1990).
2. N. Read and S. Sachdev, Phys. Rev. Lett. 66, 1773 (1991).
3. X. G. Wen, Phys. Rev. B 44, 2664 (1991).
4. R. A. Jalabert and S. Sachdev, Phys. Rev. B 44, 686 (1991).
5. C. Mudry and E. Fradkin, Phys. Rev. B 49, 5200 (1994).
6. T. Senthil and M. P. A. Fisher, Phys. Rev. B 62, 7850 (2000).
7. A. Y. Kitaev, Annals of Physics 303, 2 (2003), ISSN 0003-4916.
8. J. B. Kogut, Rev. Mod. Phys. 51, 659 (1979).
9. S. Yan, D. A. Huse, and S. R. White, Science 332, 1173 (2011).
10. H.-C. Jiang, Z. Wang, and L. Balents, Nat Phys 8, 902 (2012), ISSN 1747-2473.
11. S. Depenbrock, I. P. McCulloch, and U. Schollwck, Phys. Rev. Lett. 109, 067201 (2012).
12. A. Kitaev and J. Preskill, Phys. Rev. Lett. 96, 110404 (2006).
13. M. Levin and X.-G. Wen, Phys. Rev. Lett. 96, 110405 (2006).
14. L. D. Landau, Phys. Z. Sowjetunion 11, 26 (1937).
15. X.-G. Wen, *Quantum Field Theory Of Many-body Systems: From The Origin Of Sound To An Origin Of Light And Electrons* (Oxford University Press, New York, 2004).
I. Affleck and J. B. Marston, Phys. Rev. B 37, 3774 (1988).

I. Affleck, Z. Zou, T. Hsu, and P. W. Anderson, Phys. Rev. B 38, 745 (1988).

T. Tay and O. I. Motrunich, Phys. Rev. B 84, 020404 (2011).

Y. Ran, M. Hermele, P. A. Lee, and X.-G. Wen, Phys. Rev. Lett. 98, 117205 (2007).

Y. Iqbal, F. Becca, and D. Poilblanc, Phys. Rev. B 84, 120407 (2011).

Y.-M. Lu, Y. Ran, and P. A. Lee, Phys. Rev. B 83, 224413 (2011).

R. Moessner, S. L. Sondhi, and E. Fradkin, Phys. Rev. B 65, 024504 (2001).

Y. Huh, M. Punk, and S. Sachdev, Phys. Rev. B 84, 094419 (2011).

S.-P. Kou and X.-G. Wen, Phys. Rev. B 80, 224406 (2009).

G. Y. Cho, Y.-M. Lu, and J. E. Moore, Phys. Rev. B 86, 125101 (2012).

S.-P. Kou, M. Levin, and X.-G. Wen, Phys. Rev. B 78, 155134 (2008).

B. Bravyi and A. Y. Kitaev, arXiv:quant-ph/9811052 (1998), arXiv:quant-ph/9811052.

E. Kroner and K. H. Anthony, Annu. Rev. Mater. Sci. 5, 43 (1975), ISSN 0003-4916.

M. Kleman and J. Friedel, Rev. Mod. Phys. 80, 61 (2008).

A. Auerbach, Interacting electrons and quantum magnetism, Graduate Texts in Contemporary Physics (Springer, New York, 1994).

C. Gros, Annals of Physics 189, 53 (1989), ISSN 0003-4916.

S. Yan, D. A. Huse, and S. R. White, Bulletin of the American Physical Society 2012.

A. V. Chubukov, S. Sachdev, and T. Senthil, Nuclear Physics B 426, 601 (1994), ISSN 0550-3213.

T. Grover and T. Senthil, Phys. Rev. Lett. 100, 156804 (2008).

A. G. Abanov and P. B. Wiegmann, Nuclear Physics B 570, 685 (2000), ISSN 0550-3213.

A. Tanaka and X. Hu, Phys. Rev. Lett. 95, 036402 (2005).

T. Senthil and M. P. A. Fisher, Phys. Rev. B 74, 064405 (2006).

M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).

M. Z. Hasan and J. E. Moore, Annu. Rev. Condens. Matter Phys. 2, 55 (2011), ISSN 1947-5454.

X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).

M. Levin and A. Stern, Phys. Rev. B 86, 115131 (2012).

Y.-M. Lu and A. Vishwanath, Phys. Rev. B 86, 125119 (2012).

M. Freedman, C. Nayak, K. Shtengel, K. Walker, and Z. Wang, Annals of Physics 310, 428 (2004), ISSN 0003-4916.

M. A. Levin and X.-G. Wen, Phys. Rev. B 71, 045110 (2005).

P. A. Lee, N. Nagaosa, and X.-G. Wen, Rev. Mod. Phys. 78, 17 (2006).

Y. Ran, arXiv e-prints 1006.5454 (2010), 1006.5454.

C.-K. Chiu, H. Yao, and S. Ryu, Phys. Rev. B 88, 075142 (2013).

T. Morimoto and A. Furusaki, Phys. Rev. B 88, 125129 (2013).

A. Kitaev, AIP Conf. Proc. 1134, 22 (2009).

M. B. Hastings, Phys. Rev. B 63, 014413 (2000).

M. Hermele, Y. Ran, P. A. Lee, and X.-G. Wen, Phys. Rev. B 77, 224413 (2008).

S. Ryu, C. Mudry, C.-Y. Hou, and C. Chamon, Phys. Rev. B 80, 205319 (2009).

J. Wess and B. Zumino, Physics Letters B 37, 95 (1971), ISSN 0370-2693.

E. Witten, Nuclear Physics B 223, 422 (1983), ISSN 0550-3213.

T. Senthil, L. Balents, S. Sachdev, and T. Senthil, Phys. Rev. B 70, 144407 (2004).

Y. Ran, A. Vishwanath, and D.-H. Lee, Phys. Rev. Lett. 101, 086801 (2008).

A. M. Polyakov, Nuclear Physics B 120, 429 (1977), ISSN 0550-3213.