K-THEORY TOOLS FOR LOCAL AND ASYMPTOTIC CYCLIC COHOMOLOGY

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Abstract. A generalization of Connes-Thom isomorphism is given for stable, homotopy invariant, and split exact functors on separable $C^*$-algebras. As examples of these functors, we concentrate on asymptotic and local cyclic cohomology and the result is applied to improve some formulas in asymptotic and local cyclic cohomology of $C^*$-algebras. As an other application, it is shown that these cyclic theories are rigid after Rieffel’s deformation quantizations.

Introduction

Motivation for this paper comes from two sources. The first is Rieffel’s and Abadie’s papers [R6, A2], where it was proved that $K$-theory is rigid under Rieffel’s deformation quantizations. The second is Rosenberg’s works, [Ro3, Ro2]. In [Ro3], Rosenberg discussed the relations between $K$-theory and various quantization theories, and he suggested similar study on Connes’ cyclic homology. In [Ro2], he studied behavior of algebraic $K$-theory under formal deformation quantization. Whereas, cyclic homology of formal deformation quantization has been studied by Nest and Tsygan in [NT1,NT2], it is natural to ask how cyclic (co)homology behaves under other quantization theories. Because of the algebraic nature of Connes’ cyclic theory, we do not hope to get any satisfactory answer to this question on $C^*$-algebraic quantizations. Therefore, in such situations, we have to use other cyclic theories which do all duties of Connes’ cyclic theory and are appropriate to deal with topological algebras.

Bivariant local cyclic cohomology of admissible Fréchet algebras with compact supports and bivariant asymptotic cyclic cohomology of admissible Fréchet algebras are such theories, see [Pu1,Pu2,Pu3]. On commutative $C^*$-algebras, they are comparable with the cohomology of the character spaces as locally compact topological spaces. Moreover, they are stable and homotopy invariant bifunctors which appropriate excisions are hold in their both variables and there are bivariant Chern-Connes characters from $KK$-theory into them.

Among $C^*$-algebraic quantization theories, Rieffel’s deformation quantizations, [R3, R5], are the best theories for our purpose. Because they are associated easily to crossed product algebras (up to strong Morita equivalence), [R6, A2], and the set of their examples contains several important noncommutative spaces.

It is clear from [R6,A2] that in order to repeat their proofs for other functors, we need Morita invariance, Connes-Thom isomorphism, Pimsner-Voiculescu exact
sequence, and Bott periodicity for them. So, we provide these tools in section 1. Section 2 is devoted to the application of these tools in local and specially asymptotic cyclic cohomology. In section 3 we show local and asymptotic cyclic (co)homology groups of noncommutative and commutative Heisenberg manifolds are isomorphic. As another example of strict deformation quantization, we study behavior of local and asymptotic cyclic theories under deformation quantization by action of $\mathbb{R}^n$.

In this paper, we restrict ourselves to separable $C^*$-algebras and their dense $^*$-subalgebras. $\mathcal{K}$ denotes the $C^*$-algebra of compact operators on an infinite dimensional separable Hilbert space. The functor $S(\cdot)$ is suspension, i.e. for any $C^*$-algebra $A$, $S(A) = A \otimes C_0(\mathbb{R})$. A covariant functor is said simply functor and what will be proved for functors are hold for contravariant functors, cofunctors, by similar proofs. Let $F$ be a functor, it is called stable if for every $C^*$-algebra $A$ the natural embedding $A \to A \otimes \mathcal{K}$ induces a natural isomorphism $F(A) \to F(A \otimes \mathcal{K})$, $F$ is called homotopy invariant if $F(f) = F(g) : F(A) \to F(B)$ whenever $f, g \in \text{Hom}(A, B)$ are homotopic and $F$ is called split exact when for every split exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ with splitting $^*$-homomorphism $g : C \to B$, the map $F(f) \oplus F(g) : F(A) \oplus F(C) \to F(B)$ is an isomorphism.

By ”bivariant local cyclic cohomology" ("bivariant asymptotic cyclic cohomology"), we mean bivariant local cyclic cohomology of admissible Fréchet algebras with compact supports (bivariant asymptotic cyclic cohomology of admissible Fréchet algebras), and for a pair of $C^*$-algebras $(A, B)$ we denote it by $HC^*_{lc}(A, B)$ ($HC^*_{a}(A, B)$). We do similarly for local (asymptotic) cyclic homology and cohomology groups. Bi-

variant Chern-Connes character from $KK$-theory to bivariant local and asymptotic cyclic cohomology, as defined in [Pu1, Pu3], are denoted respectively as follows

$$ch_{btv} : KK^*(-, -) \to HC^*_{lc}(-, -)$$

and

$$ch_{btv}^* : KK^*(-, -) \to HC^*_a(S-, S-).$$

1. CONNES-THOM ISOMORPHISM

We use some techniques of $KK$-theory to prove Connes-Thom isomorphism. Our approach is known as Cuntz’s picture of $KK$-theory, [C3]. We note Bott periodicity and Pimsner-Voiculescu exact sequence were previously studied by the same way in [C2] and our proof for theorem 1.2 is an application of Fack and Skandalis result, [FS].

A quasihomomorphism between two $C^*$-algebras $A$ and $B$ is a diagram as follows:

$$A \xrightarrow{\alpha} E \xrightarrow{\beta} B,$$

where $E$ and $J$ are $C^*$-algebras, $J \triangleleft E$, and $\alpha, \tilde{\alpha}, \mu$ are $^*$-homomorphisms such that

(i) $\mu$ is an inclusion,

(ii) $E$ is the $C^*$-algebra generated by $\alpha(A)$ and $\tilde{\alpha}(A)$,

(iii) $J$ is the closed two-sided ideal generated by $\alpha(x) - \tilde{\alpha}(x), x \in A$ in $E$,

(iv) the composition of $\alpha$ and the quotient map $E \to E/J$ is injective, thus, an isomorphism.

(A diagram as above with only the property $\alpha(x) - \tilde{\alpha}(x) \in J$, for $x \in A$ is called prequasihomomorphism.) Now, let $A$ be a $C^*$-algebras, by definition, $QA$ is the
universal $^*$-algebra generated by symbols $x, q(x), x \in A$ satisfying in the relation $q(xy) = xq(y) + q(x)y - q(x)q(y)$ and let $qA$ is defined as the ideal generated by $q(x), x \in A$ in $QA$. We equip $QA$ with the largest $C^*$-norm,
\[
\|x\|_{\infty} = \sup\{\|\pi(x)\|; \pi \text{ is a } ^* - \text{representation}\}
\]
and still write $QA, qA$ for completions of $QA, qA$ with respect to this norm.

For two $C^*$-algebras $A$ and $B$, let $[qA, B \otimes K]$ be the set of homotopy classes of $^*$-homomorphisms from $qA$ to $B \otimes K$, on this set, addition is defined by $[\varphi] + [\psi] = \begin{bmatrix} \varphi & 0 \\ 0 & \psi \end{bmatrix}$. By this addition it becomes an Abelian group equal to $KK(A, B)$, (or more exactly, $KK_0(A, B)$), for details see [C1, C3].

We have two homomorphisms $\iota, \bar{\iota}$ from $A$ into $QA$ defined by $\iota(x) = x, \bar{\iota} = x - q(x)$. The quasihomomorphism given by the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{\iota} & QA \\
\pi \downarrow & & \pi \downarrow \\
& QA/\ker \varphi & qA/\ker \varphi \xrightarrow{\psi} B,
\end{array}
\]
where $\pi : QA \to QA/\ker \varphi$, $\bar{\varphi} : qA/\ker \varphi \to B$ are respectively, the quotient map, and the inclusion map defined by $\varphi$. (and vice versa, every quasihomomorphism from $A$ to $B$ give rise to a homomorphism from $qA$ to $B$, for details see proposition 1.1 of [C3].)

Following result is the main tool in our study. It was stated for functors from the category of $C^*$-algebras to the category of $Z$-modules in [C3], and can be restated for any commutative ring $R$, instated of $Z$.

**Proposition 1.1.** (Cuntz, [C3, 2.2.a]) Let $F$ be a stable, homotopy invariant, split exact functor from the category of $C^*$-algebras to the one of $R$-modules, every $\varphi \in KK(A, B)$ induces a morphism $F(\varphi) : F(A) \to F(B)$ compatible with the Kasparov product, i.e. $F(\psi \varphi) = F(\psi)F(\varphi)$ for any $\psi \in KK(B, C)$.

**Proof.** We sketch only the definition of $F(\varphi)$. Consider $\varphi$ as a $^*$-homomorphism from $qA$ to $B$, as we saw this $^*$-homomorphism is associated to the quasihomomorphism
\[
\begin{array}{ccc}
A & \xrightarrow{\iota} & QA/\ker \varphi \\
\pi \downarrow & & \pi \downarrow \\
& QA/\ker \varphi & qA/\ker \varphi \xrightarrow{\bar{\varphi}} B,
\end{array}
\]
and this quasihomomorphism induces the desired morphism as $F(\varphi) = F(\bar{\varphi})(F(\pi) - F(\bar{\pi}))$. According to the definition, it is obvious that $F(\varphi)(r-)$ for $r \in R$.

Above result shows that in order to obtain a desired morphism between $F(A)$ and $F(B)$ it is enough to find an appropriate element of $KK(A, B)$. An element of $KK(A, B)$ inducing isomorphism is called $KK$-equivalence. Equivalently, $x \in KK(A, B)$ is a $KK$-equivalence, if there is $y \in KK(B, A)$ such that $xy = 1_B, yx = 1_A$. If there is a $KK$-equivalence in $KK(A, B)$, $A$ and $B$ are called $KK$-equivalent.

In [C2] Bott periodicity and Pimsner-Voiculescu exact sequence were made for functors described in 1.1. Now Connes-Thom isomorphism is accessible by applying $KK$-equivalence $t_\alpha \in KK(A \times_\alpha R, SA)$ made in section 19.3 of [B].
Theorem 1.2. Let $F$ be a functor as 1.1, and $\alpha$ be a (strongly) continuous action of $\mathbb{R}$ on a $C^*$-algebra $A$, then $F(t_\alpha)$ is an isomorphism between $F(A \times_\alpha \mathbb{R})$ and $F(SA)$.

Remarks 1.3. (a) Indeed, theorem 1.2 is not exactly the generalization of Connes’ isomorphism. However, if we define $F_{-i}(A) = F(S^iA)$, $i \in \mathbb{Z}$, then we have Connes’ isomorphism, $F_*(A \times_\alpha \mathbb{R}) \cong F_{-1}(A)$. In $K$-theory, $K$-homology and other general $K$-functors like $KK(B_1, \mathbb{R} \otimes B_2)$, the assumption $F_{-i} = F(S^iA)$ is automatically hold, but, we still do not know whether it is true about local and asymptotic cyclic theories. So, we have to content ourselves with the theorem 1.2 at present, see next section for full Connes-Thom isomorphism for asymptotic and local cyclic cohomology.

(b) $KK$-equivalence elements of $KK(A,B)$ preserve both torsion and torsion-free parts of $K$-groups, while if the ring $R$ contains $\mathbb{Q}$, one can consider invertible elements (with respect to the Kasparov product) of the module $KK(A,B) \otimes_\mathbb{Z} R$ which preserve only torsion-free parts of $K$-groups, thus, we have other choices to construct new equivalences.

(c) $K$-amenability; locally compact group $G$ is called $K$-amenable if $\bar{p}$, the element of $KK(C^*(G), C^*_r(G))$ induced by the projection $p : C^*(G) \to C^*_r(G)$, is a $KK$-equivalence. So $K$-theory of group $C^*$-algebra of a $K$-amenable group equals $K$-theory of its reduced group $C^*$-algebra. Similarly, we can define $K$-amenability for any arbitrary functor $F$. Also, for any commutative ring $R$, we can define $K(R)$-amenability, i.e. $G$ is $K(R)$-amenable, whenever $\bar{p} \in KK(C^*(G), C^*_r(G)) \otimes_\mathbb{Z} R$ be an isomorphism.

(d) Let $\alpha$ be an automorphism of a $C^*$-algebra $A$. As it was done in [C2], Pimsner-Voiculescu exact sequence can be constructed using a $KK$-equivalence in $KK(A, T_\alpha)$, where $T_\alpha$ is Toeplitz algebra associated with the automorphism $\alpha$. This $KK$-equivalence allows us to replace $K$-groups of Toeplitz algebra with $K$-groups of $A$ in the six term exact sequence associated to the following short exact sequence, known as Toeplitz extension:

$$0 \to K \otimes A \to T_\alpha \to A \times_\alpha \mathbb{Z} \to 0$$

Therefore, in order to obtain P-V exact sequence for a functor, only conditions assumed in 1.1 is required.

Since Pimsner’s and Voiculescu’s paper, [PV], has appeared, their work has found several generalizations, [S,E2,KhS,P,AEE]. In all of them $A \times_\alpha \mathbb{Z}$ is replaced by a new $C^*$-algebra, e.g. $A \times_\mathbb{Z} \mathbb{Z}$, which generalizes the crossed product of $C^*$-algebra $A$ by $\mathbb{Z}$ and we have a generalized Toeplitz extension of $A$ by a generalized Toeplitz algebra, e.g. $T_E$, which is $KK$-equivalent to $A$. The six term exact sequences of such extensions are considered as generalizations of P-V exact sequence. One can repeat our discussion for these generalizations.

Let $\alpha$ be an automorphism of a $C^*$-algebra $A$, the mapping torus of $\alpha$ is

$$M_\alpha = \{ f : [0,1] \to A; f(1) = \alpha(f(0)) \}.$$ 

Consider the action of $\mathbb{Z}$ on $A$ defined by $\alpha$, its dual $\hat{\alpha}$ is an action of the dual of $\mathbb{Z}$, $\hat{T} = \hat{\mathbb{Z}}$, on $A \times_\alpha \mathbb{Z}$. Let $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map, then $\alpha' = \hat{\alpha} \circ \pi$ is an action of $\mathbb{R}$ on $A \times_\alpha \mathbb{Z}$, which is trivial on $\mathbb{Z}$ and $A \times_\alpha \mathbb{Z} \times_{\alpha'} \mathbb{R}$ is isomorphic to the mapping torus of $\hat{\alpha}$ on $A \times_\alpha \mathbb{Z} \times_{\hat{\alpha}} \hat{T}$, (see [B], proposition 10.3.2). By Takai duality,
it means $A \times_Z \alpha Z \times_{\alpha'} \mathbb{R}$ is isomorphic to the mapping torus of $\alpha \otimes \text{Ad} \rho$ on $A \otimes K$, where $\rho$ is the right regular representation of $Z$ and $K$ is thought of as the $C^*$-algebra of compact operators on $l^2(Z)$. Now, suppose $\alpha, \beta$ be two homotopic automorphism of $A$, then $\alpha \otimes \text{Ad} \rho$ and $\beta \otimes \text{Ad} \rho$ are homotopic too. Applying proposition 10.5.1 of [B] we deduce $A \times_Z \alpha \mathbb{R}$ is isomorphic to $A \times_{\beta} \mathbb{R}$. Thus, as a generalization of corollary 10.5.2 of [B], we have the following result:

**Corollary 1.4.** Let $F$ be a functor as 1.1 and $\alpha, \beta$ be two homotopic automorphism of a $C^*$-algebra $A$, then $F(A \times \alpha Z) \cong F(A \times \beta Z)$.

**Morita invariance.** Algebraic Morita equivalence provides the opportunity to replace an algebra $A$ with another algebra $B$ to simplify computations of (co)homology groups. For example, cyclic and Hochschild (co)homology and algebraic $K$-theory have Morita invariance property, [L, Ro1]. Since strong Morita equivalence for operator algebras was defined by Rieffel in [R1, R2], it has found several examples and applications. Now, it is an usual approximation for noncommutative spaces. For instance, $K$-theory of two strongly Morita equivalent $C^*$-algebras are isomorphic, [E1].

It is well known that two strongly Morita equivalent $\sigma$-unital $C^*$-algebras are stably isomorphic, [BGR]. Since every separable $C^*$-algebra is $\sigma$-unital and because of our assumptions, we have

**Remark 1.5.** In our discussion, all functors have strong Morita invariance property on separable $C^*$-algebras.

2. **Local and Asymptotic Cyclic Cohomology**

Some applications of $K$-theory of operator algebras are related to the structural problems on $C^*$-algebras, e.g. classification theories, while others are generalizations of well known problems on topological $K$-theory for noncommutative spaces, for instance, index theorems. Since $K$-theory is a powerful functor with a number of theorems and techniques, and also it is defined for all $C^*$- and Von Neumann algebras and their spectral invariant dense subalgebras, it seems other functors can not do any more about the first part of applications. In the second part of applications, there were some questions on existence of stable homology and cohomology theories on noncommutative spaces, $C^*$-algebras, similar to homology and cohomology theories on locally compact topological spaces, as commutative spaces. Appearance of asymptotic and local cyclic cohomology was an answer for these questions. As we have developed some tools of $K$-theory for a class of functors containing local and asymptotic cyclic theories, it is the time we study some of their applications.

Asymptotic and local cyclic cohomology are stable in both variables, (see theorem 8.18 of [Pu1] and corollary 4.10 of [Pu2]). Also, composition products and continuous homotopy theorems prove their homotopy invariances, (see theorem 6.5 and 6.15 of [Pu1] and theorem 3.5 and 3.18 of [Pu2]). But, there is a difference between local cyclic theory and asymptotic cyclic theory about split exactness. First we study asymptotic cyclic cohomology.

**Asymptotic cyclic cohomology.** At the moment, we have only stable excisions for asymptotic cyclic homology and cohomology, i.e. for any short exact sequence of separable $C^*$-algebras

$$0 \to I \xrightarrow{i} A \xrightarrow{s} B \xrightarrow{f} 0$$
with a bounded linear section \( s : B \to A \), and any \( C^* \)-algebra \( C \), there are six term exact sequences

\[
\begin{align*}
HC_0^0(C, SI) & \xrightarrow{\partial} HC_0^0(C, SA) \xleftarrow{SF_*} HC_0^0(C, SB) \\
HC_1^1(C, SB) & \xleftarrow{SF_*} HC_1^1(C, SA) \leftarrow HC_1^1(C, SI)
\end{align*}
\]

and

\[
\begin{align*}
HC_0^0(SI, C) & \leftarrow HC_0^0(SA, C) \xrightarrow{SF_*} HC_0^0(SB, C) \\
HC_1^1(SB, C) & \xleftarrow{SF_*} HC_1^1(SA, C) \rightarrow HC_1^1(SI, C)
\end{align*}
\]

If the given short exact sequence of \( C^* \)-algebras is split, these six term exact sequences give rise to four split short exact sequences of complex vector spaces as follows:

\[
\begin{align*}
0 & \to HC_0^* (C, SI) \to HC_0^* (C, SA) \to HC_0^* (C, SB) \to 0, \quad * = 0, 1, \\
0 & \to HC_1^* (SB, C) \to HC_1^* (SA, C) \to HC_1^* (SI, C) \to 0, \quad * = 0, 1.
\end{align*}
\]

Therefore, for any \( C^* \)-algebra \( C \), functor \( HC_0^* (C, S - ) \) and cofunctor \( HC_1^* (S -, C) \) are split exact, thus, we have the following results:

**Theorem 2.1.** Let \( A, B \) be two separable \( C^* \)-algebras and \( \alpha \) be a (strongly) continuous action of \( \mathbb{R} \) on \( A \), then

1. \( HC_0^* (A, B) \cong HC_0^{*+1} (SA, B) \cong HC_0^{*+1} (A, SB) \),
2. \( HC_0^* (A, B) \cong HC_0^* (SA, SB) \),
3. \( HC_1^* (A \times_\alpha \mathbb{R}, B) \cong HC_1^{*+1} (A, B) \),
4. \( HC_1^* (B, A \times_\alpha \mathbb{R}) \cong HC_1^{*+1} (B, A) \).

**Proof.** For a given \( C^* \)-algebra \( D \), suppose \( \alpha_{SD} \in HC_0^1 (S^2 D, SD) \) and \( \beta_{SD} \in HC_0^1 (SD, S^2 D) \) be respectively Dirac and Bott elements defined in definition 9.3 of [Pu1]. Also let \( \hat{i} \) be the dual action of the trivial action of \( \mathbb{R} \) on \( D \), then

\[
HC_0^* (A, B) \cong HC_0^* (S(A \times; \mathbb{R}), B) \\
\cong HC_0^* (S^2 A, B) \\
\cong HC_0^{*+1} (SA, B),
\]

where isomorphisms respectively come from Takai duality, theorem 1.2 and composition by \( \beta_{SA} \). Similarly, the second isomorphism of (1) is proved. From (1), (2)
is obvious. For (3) we observe
\[ HC^\alpha_\alpha(A \times_\alpha \mathbb{R}, B) \cong HC^\alpha_\alpha(S(A \times_\alpha \mathbb{R}), B) \]
\[ \cong HC^\alpha_\alpha(S^2 A, B) \]
\[ \cong HC^\alpha_\alpha(A, B), \]
and similarly for (4).

\[ \square \]

Parts (3) and (4) are full Connes-Thom isomorphism for asymptotic cyclic cohomology and homology, which we promised in previous section. Using above theorem we can state unstable (third!) excision theorem for asymptotic cyclic cohomology, see also [Pu1].

**Theorem 2.2.** For any \( C^\ast \)-algebra \( C \) and any short exact sequence of \( C^\ast \)-algebras
\[
0 \to I \xrightarrow{i} A \xrightarrow{s} B \to 0
\]
with a bounded linear section \( s : B \to A \), there exist following six term exact sequences:
\[
\begin{align*}
HC^0_\alpha(C, I) & \xrightarrow{\partial'} HC^0_\alpha(C, A) \xrightarrow{j^*} HC^0_\alpha(C, B) \\
HC^1_\alpha(C, I) & \xleftarrow{j^*} HC^1_\alpha(C, A) \xleftarrow{c} HC^1_\alpha(C, B)
\end{align*}
\]
\[
\begin{align*}
HC^0_\alpha(I, C) & \xleftarrow{\partial'} HC^0_\alpha(A, C) \xleftarrow{j^*} HC^0_\alpha(B, C) \\
HC^1_\alpha(B, C) & \xrightarrow{j^*} HC^1_\alpha(A, C) \xrightarrow{c} HC^1_\alpha(I, C)
\end{align*}
\]

**Notes:** Maps \( \partial', j^*, j^* \) are appropriate compositions of maps \( \partial, Sf_\ast, Sf^* \) and isomorphisms employed in the proof of theorem 2.1. Also, one can consider another Chern character form \( KK \)-theory into unstable bivariant asymptotic cyclic cohomology. Of course, we do not know whether the new Chern character is natural.

As another consequence of theorem 2.1 we have the following corollary:

**Corollary 2.3.**

1. Let \( X, Y \) be two finite CW-complexes, then
\[
HC^\alpha_\alpha(C(X), C(Y)) \cong \bigoplus_{n+m \equiv \ast \mod 2} H^n(X, \mathbb{C}), \bigoplus_{n=0}^\infty H^m(Y, \mathbb{C}),
\]
where \( H^\ast(X, \mathbb{C}) \) denotes singular cohomology of \( X \) with coefficients in \( \mathbb{C} \).

2. Let \( X \) be a locally compact metrisable topological space (or equivalently, \( C(X) \) is a separable \( C^\ast \)-algebra), then
\[
HC^\alpha_\alpha(C(X)) = HC^\alpha_\alpha(C, C(X)) \cong \bigoplus_{n=-\infty}^{\infty} H^{\ast+2n}(X, \mathbb{C}),
\]
where \( H^\ast_c(X, \mathbb{C}) \) denotes sheaf cohomology with compact supports and coefficients in \( \mathbb{C} \).
Proof. See theorems 11.2 and 11.7 of \[Pu1\].

Local cyclic cohomology. All of aforementioned results on asymptotic cyclic cohomology can be proved (even easier) for local cyclic cohomology too. Therefore, we bring only some remarks here.

Remarks 2.4. (a) Since we have a natural transformation compatible with composition products from bivariant asymptotic cyclic cohomology to bivariant local cyclic cohomology, all asymptotic cyclic equivalences like \(\alpha_S\) and \(\beta_S\), induce similar local cyclic equivalences, see 11.9.b of \[Pu1\] and 3.23 of \[Pu2\].

(b) Already there is an unstable excision for local cyclic (co)homology, see theorem 5.12 of \[Pu3\].

Since local cyclic cohomology behave reasonably under inductive limits, corollary 2.3 is adapted as follows for local cyclic theory:

**Corollary 2.5.** Let \(A, B\) be separable, commutative \(C^*\)-algebras with corresponding locally compact spaces \(X, Y\), then

\[
HC^{\ast}\alpha(A, B) \cong \lim_{\text{ind}} \left( \bigoplus_{n+m \equiv \ast (\text{mod } 2)} H^n(X, \mathbb{C}), \bigoplus_{m=0} H^m(Y, \mathbb{C}) \right).
\]

Remarks 2.6. (a) Local cyclic homology groups of commutative separable \(C^*\)-algebras are isomorphic to their local cyclic cohomology groups. This is true more generally on the class of separable \(C^*\)-algebras which are strong Morita equivalent to commutative separable \(C^*\)-algebras.

(b) Let \(C\) be the class of \(C^*\)-algebras described in the theorem 10.7 of \[Pu1\], then \(ch\) and \(ch^\ast\) yield following isomorphisms for any \(A, B\) in \(C\):

\[
ch : KK^{\ast}(A, B) \otimes_{\mathbb{Z}} C \xrightarrow{\cong} HC^{\ast}\alpha(A, B),
\]

\[
ch : KK^{\ast}(A, B) \otimes_{\mathbb{Z}} C \xrightarrow{\cong} HC^{\ast}\alpha(A, B).
\]

By proposition 6.2 of \[C2\] all commutative separable \(C^*\)-algebras belong to \(C\), thus, torsion free part of \(K\)-theory groups and \(K\)-homology groups of commutative \(C^*\)-algebras are isomorphic.

(c) As an example we have: \(HC^{\ast}\alpha(C(T^n)) \cong HC^{\ast}\alpha(C(T^n)) \cong \mathbb{C}^{2^n-1}, \ast = 0, 1\), \(HC^{\ast}\alpha(C(T^n)) \cong HC^{\ast}\alpha(C(T^n)) \cong \mathbb{C}^{2^n-1}, \ast = 0, 1\), see \[R4\].

3. Strict Deformation Quantization

Motivated by formal deformation quantization, Rieffel in \[R3\] introduced a \(C^*\)-algebraic framework for deformation quantization known as strict deformation quantization. Suppose \(A\) be a \(C^*\)-algebra with a dense \(^*\)-subalgebra \(A\) equipped with a Poisson bracket \(\{,\}\), a strict deformation quantization of \(A\) in the direction of \(\{,\}\) consists of an open interval \(I\) containing 0, and a family of pre-\(C^*\) algebra structures \(\{x_{\hbar}^*, y_{\hbar}^*| \parallel_{\hbar_i}\}\) on \(A\) which for \(\hbar = 0\) it coincide with the pre-\(C^*\) structure inherited from \(A\) such that if we denote by \(A_{\hbar}\) the completion of \(A\) under
$C^*$-structure $(\times_{\hbar^*}, \| \cdot \|_{\hbar})$, then the family $\{ A_{\hbar} \}_{\hbar \in I}$ with constant sections from $I$ into $\mathcal{A}$ constitutes a continuous field of $C^*$-algebras and for every $a, b \in \mathcal{A}$ we have

$$\lim_{\hbar \to 0} \| (a \times_{\hbar} b - ab) - \{a, b\} \| = 0.$$ 

The first examples of this definition are noncommutative tori and noncommutative Heisenberg manifolds. Deformation quantization of $C^*$-algebras by actions of finite dimensional real vector spaces, formulated in [R5], provides other examples for strict deformation quantization. In these constructions, the actions of $\mathbb{Z}$ and $\mathbb{R}$ play important roles. As a consequence, the quantized algebras naturally are associated to appropriate crossed product $C^*$-algebras by actions of $\mathbb{Z}$ and $\mathbb{R}$. Thus, Connes-Thom isomorphism and Pimsner-Voiculescu exact sequence can be used to compare $K$-theory of the quantized $C^*$-algebra with the one of the original $C^*$-algebra, [R6, A2]. Now, these tools have been generalized for stable, split exact, and homotopy invariant functors. So, we can repeat Rieffel’s and Abadie’s works to obtain the same results for these functors, in particular for local and asymptotic cyclic (co)homology.

3.1. Noncommutative Heisenberg Manifolds. For each positive integer $c$, the Heisenberg manifold $M_c$ is defined by the quotient $G/D_c$, where $G$ is the Heisenberg group,

$$G = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

and $D_c$ is the subgroup of $G$ with $x, y, cz \in \mathbb{Z}$. Every non-zero Poisson bracket on $M_c$ is determined by two real numbers $\mu, \nu$, where $\mu^2 + \nu^2 \neq 0$. According to [A2], deformation quantization of $M_c$ in the direction of non-zero Poisson bracket $(\mu, \nu)$, which is invariant under the action of $G$ by left translation, is denoted by $\{ D^{c, \hbar}_{\mu, \nu} \}_{\hbar \in \mathbb{R}}$, where $\hbar$ is the parameter of deformation. We know from [R3, A2] that for $\hbar \neq 0$, the algebra $D^{c, \hbar}_{\mu, \nu}$ is the generalized fixed-point algebra of $C_0(\mathbb{R} \times T) \times_{\lambda^\hbar} \mathbb{Z}$ under the action $\rho$, where

$$\lambda^\hbar_k(x, y) = (x + 2k\hbar\mu, y + 2k\hbar\nu), \quad k \in \mathbb{Z},$$

and if $e(x) = \exp(2\pi i x)$, then

$$(\rho_k \Phi)(x, y, p) = e(ckp(y - h_p\nu))\Phi(x + k, y, p), k \in \mathbb{Z}.$$ 

It follows from theorem 2.11 of [A2] that the algebra $D^{c, \hbar}_{\mu, \nu}$ is strongly Morita equivalent to another fixed-point algebra obtained by the action $\gamma^{\hbar}$ of $\mathbb{Z}$ on $C_0(\mathbb{R} \times T) \times_{\sigma} \mathbb{Z}$, where

$$\sigma_k(x, y) = (x - k, y), \quad k \in \mathbb{Z},$$

and

$$(\gamma^p^{\hbar} \Phi)(x, y, k) = e(-ckp(y - h_p\nu))\Phi(x - 2ph\mu, y - 2ph\nu, k), \quad p \in \mathbb{Z}.$$ 

Now, corollary 1.4 and homotopy $\hbar \to \lambda^\hbar$ show local and asymptotic cyclic (co)homology groups of $D^{c, \hbar}_{\mu, \nu}$ are independent of parameter $\hbar$. On the other hand for any real number $\hbar$, $D^{c, \hbar}_{\mu, \nu}$ and $D^{c, 1}_{h\mu, h\nu}$ are isomorphic. So, one can drop parameter $\hbar$ from the notation and simply write $D^{c, \nu}_{\mu, \nu}$ instead of $D^{c, 1}_{h\mu, h\nu}$. Also, for any pair of integers $k, l,$
$D_{\mu}^{c,\hbar}$ and $D_{\mu+k,\nu+l}^{c,\hbar}$ are isomorphic, see proposition 1 of [A1]. Thus, the assumption $\hbar \neq 0$ can be ignored. This shows every stable, homotopy invariant, and split exact (co)functor $F$ is rigid under deformation quantization of Heisenberg manifolds. As a consequence, we have

**Theorem 3.1.** For any real numbers $\mu, \nu, \hbar$ and integer $c$, even and odd asymptotic and local cyclic homology and cohomology groups of Heisenberg manifold $D_{\mu}^{c,\hbar}$ are isomorphic to $\mathbb{C}^3$.

**Proof.** It is enough to consider only the commutative, which is an easy application of remark 2.6(b) and theorem 3.4 of [A2].

3.2. **Deformation Quantization of $C^*$-Algebras by Actions of $\mathbb{R}^n$.** Let $\alpha$ be a strongly continuous action of $\mathbb{R}^n$ on $C^*$-algebra $A$, and $A^\infty$ be the dense $*$-subalgebra of its smooth vectors. Also, let $J$ be a skew-symmetric matrix on $\mathbb{R}^n$. On $A^\infty$ deformed product $\times_J$ is defined by

$$a \times_J b = \int \alpha_{Ju}(a)\alpha_{Jv}(b)e(u \cdot v) \quad a, b \in A^\infty,$$

where $e(t) = \exp(2\pi it)$. Also, a $C^*$-norm $\| \|$ and an involution $*_J$ compatible with the product $\times_J$ are defined on $A^\infty$ (for details see [R5]). The completion of this pre-$C^*$-structure is denoted by $A_J$ and called quantization of $A$ by the action of $\alpha$ in the direction of $J$. Let $A$ be separable, then it is $\sigma$-unital, and consequently $A_J$ is $\sigma$-unital too, see [R5]. It was shown in [R6] that $A_J$ is strong Morita equivalent to a stable crossed product of $A$ as follows:

$$A_J \overset{M}{\cong} A \times_\rho \mathbb{R}^n \otimes C_0(\mathbb{R}^m) \otimes \mathcal{K},$$

where $m$ is the dimension of kernel of $J$. The right side of above equivalence is separable, so $A_J$ is separable too. Thus we have

**Theorem 3.2.** Local and asymptotic cyclic homology and cohomology of $C^*$-algebra $A$ are rigid under deformation quantization by actions of $\mathbb{R}^n$.

**Proof.** We consider only asymptotic cyclic homology, others are similar. From above discussion, remark 1.5 and theorem 2.1 we have

$$HC_n^\alpha(A_J) \cong HC_n^{\alpha+m+n}(A).$$

Since $J$ is a skew-symmetric matrix, $m$ is odd if and only $n$ is odd, so always $m + n$ is even.

**Remarks 3.3.** (a) As an example, we consider $n$-dimensional noncommutative trous $\mathbb{T}_\theta^n$. Even and odd local and asymptotic cyclic homology and cohomology groups of $\mathbb{T}_\theta^n$ are isomorphic to $\mathbb{C}^{2n-1}$, see 10.2 of [R5] and remark 2.6 (b).

(b) Theorems 7.5 of [Pu1] and 3.19 of [Pu2] show the inclusion $A^\infty \to A$ induces isomorphisms between asymptotic and local cyclic homology and cohomology groups.

(c) Let $\alpha$ also denotes the action induced by $\alpha$ on $A_J$, subalgebra of smooth elements of this action is $A^\infty$ too, theorem 7.1 of [R5]. By part (b), one yields asymptotic and local cyclic (co)homology of smooth algebras are rigid after deformation quantization too.
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