HOMOTOPY TYPES OF GAUGE GROUPS
OF PU(p)-BUNDLES OVER SPHERES

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ABSTRACT. We examine the relation between the gauge groups of SU(n)- and PU(n)-bundles over S^2i, with 2 \leq i \leq n, particularly when n is a prime. As special cases, for PU(5)-bundles over S^4, we show that there is a rational or p-local equivalence \mathcal{G}_k \simeq (p) \mathcal{G}_l for any prime p if, and only if, (120, k) = (120, l), while for PU(3)-bundles over S^6 there is an integral equivalence \mathcal{G}_k \simeq (l) \mathcal{G}_l if, and only if, (120, k) = (120, l).

KEYWORDS: Gauge groups; Homotopy types; Samelson products.
2020 Mathematics Subject Classification: Primary 55P15; Secondary 55Q05.

1. Introduction

Let G be a topological group and X a space. The gauge group \mathcal{G}(P) of a principal G-bundle P over X is defined as the group of G-equivariant bundle automorphisms of P which cover the identity on X. A detailed introduction to gauge groups can be found in [8, 16]. Provided P is understood, we will use the notation \mathcal{G}(G) if we need to emphasize the structure group G.

The following problem is of interest: having fixed a topological group G and a space X, classify the possible homotopy types of the gauge groups \mathcal{G}(P) of principal G-bundles over X.

Crabb and Sutherland showed in [3] that if G is a compact, connected, Lie group and X is a connected, finite complex, then the number of distinct homotopy types of \mathcal{G}(P), as P ranges over all principal G-bundles over X, is finite. This is often in contrast with the fact that the number of isomorphism classes of principal G-bundles over X may be infinite. However, their methods did not lead to an enumeration of the classes of gauge groups. Classification results require a different kind of analysis.

The first result of this kind was obtained by Kono [12] in 1991. Using the fact that principal SU(2)-bundles over S^4 are classified by k \in \mathbb{Z} \cong \pi_3(SU(2)) and denoting by \mathcal{G}_k the gauge group of the principal bundle P_k \to S^4 corresponding to the integer k, Kono showed that there is a homotopy equivalence \mathcal{G}_k \simeq \mathcal{G}_l if, and only if, (12, k) = (12, l), where (m, n) denotes the greatest common divisor of m and n. It thus follows that there are precisely six homotopy types of SU(2)-gauge groups over S^4.
In this letter, we generalise certain results relating the classification of $\text{PU}(n)$-gauge groups to that of $\text{SU}(n)$-gauge groups from \cite{10} for the case $n = 2$, and from \cite{7} for the case $n = 3$.

Samelson products play a crucial role in the homotopy classification of gauge groups. In Section \cite{3} we show that the Samelson products on $\text{SU}(p)$ and $\text{PU}(p)$, with $p \geq 3$ a prime, are related as follows.

**Theorem 1.1.** The orders of the Samelson products $\langle \epsilon_i, 1 \rangle : S^{2i-1} \wedge \text{PU}(p) \to \text{PU}(p)$ and $\langle \delta_i, 1 \rangle : S^{2i-1} \wedge \text{SU}(p) \to \text{SU}(p)$ coincide.

In Section \cite{3} we prove the following result, which gives a sufficient condition for certain homotopy invariants of $\text{SU}(n)$- and $\text{PU}(n)$-gauge groups to coincide.

**Theorem 1.2.** Let $X$ be a connected space such that $[\Sigma X, \text{SU}(n)] \cong 0$. Then there is an isomorphism of groups $[\Sigma X, G_k(\text{PU}(n))] \cong [\Sigma X, G_k(\text{SU}(n))]$.

As special cases of our results, we obtain the following complete classifications.

**Theorem 1.3.** For $\text{PU}(5)$-bundles over $S^4$, it is the case that
(a) if $G_k(\text{PU}(5)) = G_l(\text{PU}(5))$, then $(120, k) = (120, l)$;
(b) if $(120, k) = (120, l)$, then $G_k(\text{PU}(5)) \cong G_l(\text{PU}(5))$ when localised rationally or at any prime.

**Theorem 1.4.** For $\text{PU}(3)$-bundles over $S^6$, we have $G_k(\text{PU}(3)) \cong G_l(\text{PU}(3))$ if, and only if, $(120, k) = (120, l)$.

2. Homotopy types of $\text{PU}(n)$-gauge groups

**Definition 2.1.** The projective unitary and special unitary groups are defined by quotienting out the centres. That is, one defines

$$\text{PU}(n) := \text{U}(n)/\text{U}(1) \quad \text{and} \quad \text{PSU}(n) := \text{SU}(n)/(\mathbb{Z}/n\mathbb{Z}).$$

It is well known that there is a homotopy equivalence $\text{PU}(n) \cong \text{PSU}(n)$ for each $n$, and henceforth we shall not distinguish between the two.

For each $2 \leq i \leq n$, the set of isomorphism classes of principal $\text{PU}(n)$-bundles over $S^{2i}$ is in bijection with the set

$$[S^{2i}, B\text{PU}(n)] \cong \pi_{2i}(B\text{PU}(n)) \cong \pi_{2i-1}(\text{PU}(n)) \cong \mathbb{Z}.$$ 

Let $\epsilon_i : S^{2i-1} \to \text{PU}(n)$ denote a generator of $\pi_{2i-1}(\text{PU}(n))$. Then, each isomorphism class is represented by the bundle $P_k \to S^{2i}$ induced by pulling back the universal $\text{PU}(n)$-bundle along the classifying map $k\overline{\epsilon}_i : S^{2i} \to B\text{PU}(n)$, where $\overline{\epsilon}_i$ denotes the adjoint of $\epsilon_i$ and generates $\pi_{2i}(B\text{PU}(n))$.

Let $G_k$ denote the gauge group of $P_k \to S^{2i}$. By \cite{14}, there is a homotopy equivalence $B(G_k) \cong \text{Map}(S^{2i}, B\text{PU}(n))$, the latter space being the connected component of $\text{Map}(S^{2i}, B\text{PU}(n))$ containing the classifying map $k\overline{\epsilon}_i$. 
The evaluation fibration
\[ \text{Map}^*_k(S^{2i}, BPU(n)) \to \text{Map}^*_k(S^{2i}, BPU(n)) \xrightarrow{ev} BPU(n) \]
extends to a homotopy fibration sequence
\[ \cdots \to \mathcal{G}_k \to PU(n) \to \text{Map}^*_k(S^{2i}, BPU(n)) \to B\mathcal{G}_k \to BPU(n). \]

By [17], there is a homotopy equivalence
\[ \text{Map}^*_k(S^{2i}, BPU(n)) \simeq \text{Map}^*_k(S^{2i}, BPU(n)) = \Omega_0^{2i} BPU(n) = \Omega_0^{2i-1} PU(n), \]
leading to the following homotopy fibration sequence
\[ \mathcal{G}_k \to PU(n) \xrightarrow{\partial_k} \Omega_0^{2i-1} PU(n) \to B\mathcal{G}_k \to BPU(n), \]
which exhibits the gauge group \( \mathcal{G}_k \) as the homotopy fibre of the map \( \partial_k \).

By the pointed exponential law, there is a bijection
\[ [PU(n), \Omega_0^{2i-1} PU(n)] \cong [S^{2i-1} \wedge PU(n), PU(n)] \]
Let \( \overline{\partial}_k : S^{2i-1} \wedge PU(n) \to PU(n) \) denote the adjoint of \( \partial_k : PU(n) \to \Omega_0^{2i-1} PU(n) \). By [9 Theorem 2.6], there is a homotopy \( \overline{\partial}_k \simeq \langle k \epsilon_1, 1 \rangle \), where 1 denotes the identity map on PU(n). By the bilinearity of the Samelson product, we find
\[ \overline{\partial}_k \simeq \langle k \epsilon_1, 1 \rangle \simeq k \langle \epsilon_1, 1 \rangle \simeq k \overline{\epsilon}_1, \]
and hence, taking adjoints once more, \( \partial_k \simeq k \partial_1 \). Thus, every one of the gauge groups is the homotopy fibre of the single map \( \partial_1 \) post-composed with the appropriate power map on \( \Omega_0^{2i-1} PU(n) \). Hence, if \( \partial_1 \) can be determined to have finite order in \([PU(n), \Omega_0^{2i-1} PU(n)]\), then the following lemma applies.

**Lemma 2.2 (Hamenaka, Kono [5])**. Let \( X \) be a connected CW-complex and let \( Y \) be an H-space with a homotopy inverse. Suppose that \( f \in [X, Y] \) has finite order \( n \). Then, for any \( k, l \in \mathbb{Z} \) such that \( (n, k) = (n, l) \), the homotopy fibres of \( kf \) and \( lf \) are homotopy equivalent when localised rationally or at any prime. \( \Box \)

Additionally, in the special case of principal PU(n)-bundles over \( S^{2n} \), as the homotopy groups of \( \Omega_0^{2n-1} PU(n) \) are all finite, the following stronger lemma applies.

**Lemma 2.3 (Hamenaka, Kono [5])**. Let \( X \) be a connected CW-complex and let \( Y \) be an H-space such that \( \pi_j(Y) \) is finite for all \( j \). Let \( f \in [X, Y] \) be such that \( nf \simeq * \) for some finite \( n \) and let \( k, l \in \mathbb{Z} \) satisfy \( (n, k) = (n, l) \). Then, there exists a homotopy equivalence \( h : Y \to Y \) satisfying \( hkf \simeq lf \). \( \Box \)

Note that the order of \( \partial_1 \) coincides with the order of \( \langle \epsilon_1, 1 \rangle \).
3. Samelson products on PU($p$)

Let $\delta_1: S^{2i-1} \to SU(n)$ denote the generator of $\pi_{2i-1}(SU(n)) \cong \mathbb{Z}$ corresponding to the generator $\epsilon_i$ of $\pi_{2i-1}(PU(n))$. That is, such that $q_*(\delta_1) = \epsilon_i$, where $q$ denotes the quotient map $q: SU(n) \to PU(n)$.

In this section we wish to compare the orders of the Samelson products $\langle \delta_1, 1 \rangle$ and $\langle \epsilon_1, 1 \rangle$ on $SU(n)$ and $PU(n)$, respectively. First, observe that there is a commutative diagram

\[
\begin{array}{ccc}
S^{2i-1} \wedge SU(n) & \xrightarrow{\langle \delta_1, 1 \rangle} & SU(n) \\
\downarrow{1 \wedge q} & & \downarrow{q} \\
S^{2i-1} \wedge PU(n) & \xrightarrow{\langle \epsilon_1, 1 \rangle} & PU(n)
\end{array}
\]

and recall the following well known property of the map $q$.

**Lemma 3.1.** The quotient map $q: SU(n) \to PU(n)$ induces a $p$-local homotopy equivalence $SU(n) \simeq_{(p)} PU(n)$ for any prime $p$ which does not divide $n$. \hfill $\Box$

**Lemma 3.2.** If $p$ does not divide $n$, then the $p$-primary components of the orders of the Samelson products $\langle \delta_1, 1 \rangle$ and $\langle \epsilon_1, 1 \rangle$ coincide.

**Proof.** Let $p$ be a prime which does not divide $n$. Then $q$ is a $p$-local homotopy equivalence by Lemma 3.1 and hence the commutativity of (*) yields

\[
\langle \delta_1, 1 \rangle_{(p)} = q_{(p)}^{-1} \circ \langle \epsilon_1, 1 \rangle_{(p)} \circ (1 \wedge q_{(p)}),
\]

so the $p$-primary components of the orders of $\langle \delta_1, 1 \rangle$ and $\langle \epsilon_1, 1 \rangle$ coincide. \hfill $\Box$

Hence, when $n$ is prime, the orders of $\langle \delta_1, 1 \rangle$ and $\langle \epsilon_1, 1 \rangle$ coincide up to at most their $n$-primary component.

**Lemma 3.3.** The quotient map $q: SU(n) \to PU(n)$ induces an isomorphism

\[
q_*: [S^{2i-1} \wedge SU(n), SU(n)] \to [S^{2i-1} \wedge SU(n), PU(n)].
\]

**Proof.** Recall that $q: SU(n) \to PU(n)$ fits into a homotopy fibration sequence

\[
\cdots \to \mathbb{Z}/n\mathbb{Z} \to SU(n) \xrightarrow{q} PU(n) \to B(\mathbb{Z}/n\mathbb{Z}).
\]

Since $\mathbb{Z}/n\mathbb{Z}$ is discrete, applying the functor $[S^{2i-1} \wedge SU(n), -]$ yields

\[
\cdots \to 0 \to [S^{2i-1} \wedge SU(n), SU(n)] \xrightarrow{q_*} [S^{2i-1} \wedge SU(n), PU(n)] \to 0. \hfill \Box
\]

**Corollary 3.4.** Let $p$ be a prime. If $p^k$ divides the order of $\langle \delta_1, 1 \rangle$ for some $k \geq 1$, then the order of $\langle \epsilon_1, 1 \rangle$ is at least $p^k$.

**Proof.** If $p^k$ divides the order of $\langle \delta_1, 1 \rangle$, then $p^k$ also divides the order of the composite $q \circ \langle \delta_1, 1 \rangle_{(p)}$ by Lemma 3.3. It then follows, by the commutativity of (*), that the order of $\langle \epsilon_1, 1 \rangle_{(p)}$ is at least $p^k$. \hfill $\Box$
Hence, the order of $\langle \delta, 1 \rangle$ certainly divides that of $\langle \epsilon_i, 1 \rangle$. For the remainder of this section, we shall restrict to considering $\text{PU}(n)$ when $n$ is an odd prime $p$.

Since the universal cover of $\text{PU}(p)$ is $\text{SU}(p)$ and $H_2(\text{SU}(p); \mathbb{Z})$ is torsion-free, by [11] we have the following decomposition of $\text{PU}(p)$.

**Lemma 3.5.** There is a $p$-local homotopy equivalence

$$\text{PU}(p) \simeq_{(p)} L \times \prod_{j=2}^{p-1} S^{2j-1}$$

where $L$ is the lens space $S^{2p-1}/(\mathbb{Z}/p\mathbb{Z})$. \[\square\]

Let $\alpha : L_{(p)} \to \text{PU}(p)_{(p)}$ be the inclusion. Then we can write the equivalence of Lemma [3.5] as

$$L_{(p)} \times \prod_{j=2}^{p-1} S^{2j-1}_{(p)} \xrightarrow{\alpha \times \prod_j s_{(p)}} (\text{PU}(p)_{(p)})^{p-1} \xrightarrow{\mu} \text{PU}(p)_{(p)},$$

where $\mu$ is the group multiplication in $\text{PU}(p)_{(p)}$. We note that this composite is equal to the product

$$(\alpha \circ \text{pr}_j) \cdot \prod_{j=2}^{p-1} (\epsilon_j_{(p)} \circ \text{pr}_j)$$

in the group $[L_{(p)} \times \prod_{j=2}^{p-1} S^{2j-1}_{(p)}, \text{PU}(p)_{(p)}]$, where $\text{pr}_j$ denotes the projection onto the $j$th factor.

**Lemma 3.6.** With the above notation, the localised Samelson product

$$\langle \epsilon_i, 1 \rangle_{(p)} : S^{2i-1}_{(p)} \wedge \text{PU}(p)_{(p)} \to \text{PU}(p)_{(p)}$$

is trivial if, and only if, each of $\langle \epsilon_{i(j)}(p), \alpha \rangle$ and $\langle \epsilon_i, \epsilon_j \rangle_{(p)}$, for $2 \leq j \leq p-1$, are trivial.

**Proof.** By Lemmas 3.3 and 3.4 in [7], $\langle \epsilon_i, 1 \rangle_{(p)}$ is trivial if, and only if, both $\langle \epsilon_{i(p)}, \alpha \rangle$ and $\langle \epsilon_i, \prod_j \epsilon_j \rangle_{(p)}$ are trivial. Applying the same lemmas to the second factor a further $p - 3$ times gives the statement. \[\square\]

We therefore calculate the groups $[S^{2i-1} \wedge L, \text{PU}(p)]_{(p)}$ and, for $2 \leq j \leq p-1$, the homotopy groups $\pi_{2i+2j-2}(\text{PU}(p))_{(p)}$ in order to get an upper bound on the order of $\langle \epsilon_i, 1 \rangle_{(p)}$.

**Lemma 3.7.** For $2 \leq i \leq p$ and $2 \leq j \leq p-1$, the elements of $\pi_{2i+2j-2}(\text{PU}(p))_{(p)}$ have order at most $p$.

**Proof.** Decomposing $\text{PU}(p)$ as in Lemma [3.5] and noting that $\pi_n(L) \cong \pi_n(S^{2p-1})$ for $n \geq 2$, we have

$$\pi_{2i+2j-2}(\text{PU}(p))_{(p)} \cong \pi_{2i+2j-2} \left( \prod_{k=2}^{p-1} S^{2k-1}_{(p)} \right) \cong \bigoplus_{k=2}^{p-1} \pi_{2i+2j-2}(S^{2k-1})_{(p)},$$
By Toda [21] Theorem 7.1, if \( k \geq 2 \) and \( r < 2p(p-1) - 2 \), the \( p \)-primary component of \( \pi_{(2k-1) + r}(S^{2k-1}) \) is either 0 or \( \mathbb{Z}/p\mathbb{Z} \). Since \( 2i + 2j - 2 \leq 4p - 4 \) and 

\[
4p - 4 < 2p(p-1) - 2 + (2k-1)
\]

for all \( k \geq 2 \), the statement follows. \( \square \)

Furthermore, we find:

**Lemma 3.8.** The elements of \([S^{2i-1} \wedge L, PU(p)]_p\) have order at most \( p \).

**Proof.** By [13], there is a \( p \)-local equivalence

\[
\Sigma L \simeq_p S^{2p} \vee \bigvee_{k=1}^{p-1} p^{2k+1}(p).
\]

where \( p^{2k+1}(p) \) is the mod-\( p \) Moore space given by the cofibre

\[
S^{2k} \overset{p}{\rightarrow} S^{2k} \rightarrow p^{2k+1}(p)
\]

of the degree \( p \) map on the sphere \( S^{2k} \). Note that, by extending the cofibre sequence to the right, we see that \( S^{2i-2} \wedge p^{2k+1}(p) \simeq p^{2k+2i-1}(p) \). Hence, we have

\[
[S^{2i-1} \wedge L, PU(p)]_p \cong \pi_{2i+2p-2}(PU(p))_{(p)} \oplus \bigoplus_{k=1}^{p-1} [p^{2k+2i-1}(p), PU(p)]_{(p)}.
\]

Since \( 2i + 2p - 2 \leq 4p - 2 < 2p(p-1) + 1 \) for \( p \geq 3 \), the group \( \pi_{2i+2p-2}(PU(p))_{(p)} \) consists of elements of order at most \( p \) by the same argument as in Lemma 3.7.

On the other hand, by [15] Theorem 7.1, the groups \([p^{2k+2i-1}(p), PU(p)]\) are annihilated by multiplication by \( p \) (since, for any \( m \geq 3 \), the identity on \( P^m(p) \) has order \( p \)), whence the statement. \( \square \)

Lemmas 3.6 to 3.8 combine to give:

**Corollary 3.9.** The order of the Samelson product

\[
\langle \epsilon_i, 1 \rangle_{(p)} : S^{2i-1}(p) \wedge PU(p)_{(p)} \rightarrow PU(p)_{(p)}
\]

is at most \( p \). \( \square \)

We now have all the ingredients necessary to prove Theorem 1.1.

**Proof of Theorem 1.1.** Consider the following commutative diagram

\[
\begin{array}{ccc}
S^{2i-1} \wedge S^{2(p-i)+1} & \xrightarrow{\langle \eta_i, \eta_{p-i-1} \rangle} & U(p) \\
\downarrow{1 \wedge \delta_{p-i-1}} & & \uparrow{} \\
S^{2i-1} \wedge SU(p) & \xrightarrow{\langle \delta_i, 1 \rangle} & SU(p)
\end{array}
\]

where \( \iota : SU(p) \rightarrow U(p) \) is the inclusion and \( \eta_i := L_i(\delta_i) \). By Bott [2], the map \( \langle \eta_i, \eta_{p-i-1} \rangle \) is non-trivial and \( p \) divides its order. Hence, the order of \( \langle \delta_i, 1 \rangle_{(p)} \) is at least \( p \). The result now follows from Lemma 3.2 and Corollaries 3.4 and 3.9. \( \square \)
4. Homotopy invariants of PU(n)-gauge groups

The content of Theorem 1.2 is a straightforward observation about how certain homotopy invariants of SU(n)-gauge groups relate to the corresponding invariants of PU(n)-gauge groups.

Proof of Theorem 1.2 Since $[\Sigma X, SU(n)] \equiv 0$, applying the functor $[\Sigma X, -]$ to the homotopy fibration sequence

$$
\cdots \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow SU(n) \overset{q}{\longrightarrow} PU(n) \longrightarrow B(\mathbb{Z}/n\mathbb{Z})
$$

shows that $[\Sigma X, PU(n)] \equiv 0$ also.

Applying now the functor $[\Sigma^2 X, -]$ to the homotopy fibration sequence

$$
PU(n) \overset{\partial_k}{\longrightarrow} \Omega_0^{2i-1}PU(n) \longrightarrow B\mathcal{G}_k( PU(n)) \longrightarrow BPU(n)
$$

described in Section 2 as well as to its SU(n) analogue, yields the following commutative diagram

$$
\begin{align*}
[\Sigma^2 X, SU(n)] &\overset{(\partial_k)_*}{\longrightarrow} [\Sigma^2 +1X, SU(n)] &\longrightarrow [\Sigma X, \mathcal{G}_k(SU(n))] &\longrightarrow 0 \\
\downarrow q_* & &\downarrow q_* & \\
[\Sigma^2 X, PU(n)] &\overset{(\partial_k)_*}{\longrightarrow} [\Sigma^2 +1X, PU(n)] &\longrightarrow [\Sigma X, \mathcal{G}_k( PU(n))] &\longrightarrow 0
\end{align*}
$$

where the two leftmost vertical maps are isomorphisms. The statement now follows from the five lemma.

Hamanaka and Kono showed in [5, Theorem 1.2] that, for principal SU(n)-bundles over $S^4$, the homotopy equivalence $\mathcal{G}_k(SU(n)) \simeq \mathcal{G}_l(SU(n))$ implies that $(n(n^2 - 1), k) = (n(n^2 - 1), l)$. As an application of Theorem 1.2, let us show that the analogue of this result holds for PU(n)-gauge groups.

Corollary 4.1. Let $n > 3$. For principal PU(n)-bundles over $S^4$, the homotopy equivalence $\mathcal{G}_k( PU(n)) \simeq \mathcal{G}_l( PU(n))$ implies that $(n(n^2 - 1), k) = (n(n^2 - 1), l)$.

Proof. First, suppose that $n$ is even. Note that we have

$$
\pi_{2n-4}(SU(n)) \equiv \pi_{2n-2}(SU(n)) \equiv 0.
$$

Hence, applying Theorem 1.2 with $X = S^{2n-5}$ and $X = S^{2n-3}$, we find

$$
\pi_{2n-4}(\mathcal{G}_k( PU(n))) \equiv \pi_{2n-4}(\mathcal{G}_k( SU(n)))
$$

and

$$
\pi_{2n-2}(\mathcal{G}_k( PU(n))) \equiv \pi_{2n-2}(\mathcal{G}_k( SU(n))).
$$

So the result follows for $n$ even by [3].

When $n$ is odd, we have from [5] that $[\Sigma^{2n-6}CP^2, SU(n)] \equiv 0$. Hence, applying Theorem 1.2 with $X = \Sigma^{2n-7}CP^2$, we find

$$
[\Sigma^{2n-6}CP^2, \mathcal{G}_k( PU(n))] \equiv [\Sigma^{2n-6}CP^2, \mathcal{G}_k( SU(n))].
$$
So the result follows for $n$ odd by [5 Corollary 2.6].\hfill\square

Following the work of [6], Mohammadi and Asadi-Golmankhaneh [14] recently showed that, for $SU(n)$-bundles over $S^6$, the equivalence $G_k(SU(n)) \simeq G_l(SU(n))$ implies that

\[(n - 1)n(n + 1)(n + 2), k) = ((n - 1)n(n + 1)(n + 2), l).\]

Hence, we also have:

**Corollary 4.2.** Let $n \geq 3$. For principal $PU(n)$-bundles over $S^6$, the homotopy equivalence $G_k(NU(n)) \simeq G_l(NU(n))$ implies that

\[(n - 1)n(n + 1)(n + 2), k) = ((n - 1)n(n + 1)(n + 2), l).\]

*Proof.* For $n > 3$, apply Theorem [1.2] with $X = \Sigma^{2n-3}CP^2$ and the result of [14].

For $n = 3$, apply the functor $[-, PU(3)]$ to the cofibration $S^2 \to CP^2 \to S^4$ to show that $[CP^2, PU(3)] \simeq 0$. Proceeding in the same way as in the proof of Theorem [1.2] one finds that $[CP^2, G_k(NU(3))] \simeq [CP^2, G_k(SU(3))]$. The result now follows from Hamanaka and Kono’s calculation in Section 3 of [6] or from the more general result of [14].\hfill\square

5. Special cases

5.1. $PU(p)$-bundles over $S^4$. Theriault showed in [20] that, after localisation at an odd prime $p$ and provided $n < (p - 1)^2 + 1$, the order of the Samelson product $\langle \delta_2, 1 \rangle : S^3 \wedge SU(n) \to SU(n)$ is the $p$-primary component of the integer $n(n^2 - 1)$. It then follows immediately from Theorem [1.1] that:

**Corollary 5.1.** After localisation at an odd prime, the order of the Samelson product $\langle \epsilon_2, 1 \rangle : S^3 \wedge PU(p) \to PU(p)$ is $p(p^2 - 1)$.

5.2. $PU(5)$-bundles over $S^4$. In [19], Theriault showed that the order of the Samelson product $\langle \delta_2, 1 \rangle : S^3 \wedge SU(5) \to SU(5)$ is 120. Hence, by Theorem [1.1] the order of $\langle \epsilon_2, 1 \rangle : S^3 \wedge PU(5) \to PU(5)$ is also 120.

*Proof of Theorem 1.3.* Part (i) follows from Corollary 4.1 while part (ii) follows from Lemma [2.2].\hfill\square

5.3. $PU(3)$-bundles over $S^6$. Hamanaka and Kono showed in [6] that the order of the Samelson product $\langle \delta_3, 1 \rangle : S^5 \wedge SU(3) \to SU(3)$ is 120. It follows immediately from Theorem [1.1] that the order of $\langle \epsilon_3, 1 \rangle : S^5 \wedge PU(3) \to PU(3)$ is also 120.

*Proof of Theorem 1.3.* As the homotopy groups $\pi_n(\Omega^2_3 PU(3)) \simeq \pi_{n+5}(PU(3))$ are all finite, the “if” direction follows from Lemma [2.3], while the “only if” direction follows from Corollary 4.2.\hfill\square
6. Concluding remarks

We have shown, particularly with Theorems 1.1 and 1.2, how the close relationship between the groups $SU(p)$ and $PU(p)$ is reflected in the homotopy properties of the corresponding gauge groups. Indeed, it is worth noting that, should further classifications of gauge groups of $SU(p)$-bundles over even-dimensional spheres be obtained, the aforementioned theorems would provide the corresponding results for $PU(p)$-gauge groups as immediate corollaries, provided the original results were arrived at via the standard methods.

We also note that in [7], the $PU(3)$-gauge group $G_k$ is shown to be homotopy equivalent to $\tilde{G}_k \times S^1$, where $\tilde{G}_k$ is a space whose homotopy groups are all finite. This allows the authors to apply Lemma 2.3 and obtain a classification result for $G_k$ that holds integrally. We expect the same result to apply more generally to gauge groups of $PU(n)$-bundles over $S^{2n-2}$. However, there are currently no other cases, beside that of [7], in which such a result would be applicable.

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