Casimir Physics: Geometry, Shape and Material

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The properties of fluctuation induced interactions like van der Waals and Casimir-Lifshitz forces are of interest in a plethora of fields ranging from biophysics to nanotechnology. Here we describe a general approach to compute these interactions. It is based on a combination of methods from statistical physics and scattering theory. We showcase how it is exquisitely suited to analyze a variety of previously unexplored phenomena. Examples are given to show how the interplay of geometry and material properties helps to understand and control these forces.

1. Introduction

All material objects, even if charge neutral, support instantaneous current fluctuations due to quantum and thermal fluctuations of their charge distribution. The interaction that results from the electromagnetic coupling of these currents on different objects is usually called the Casimir force. Originally, this force has been derived for two parallel perfect metal plates\(^1\) and atoms,\(^2\) and generalized later to two infinite dielectric half-spaces with planar and parallel surfaces.\(^3\)-\(^6\) The non-additivity of the Casimir force limits these results in their applicability to objects at very short separation via the so-called proximity force approximation which provides only an uncontrolled approximation of surface curvature to lowest order at vanishingly small separations and ignores the global geometrical arrangement of the objects. Generically, one encounters in practice geometries and shapes that are rather distinct from infinite, parallel and planar surfaces. Hence one faces the problem to compute the Casimir force between objects of general shape, arrangement and material decomposition.

This article summarizes recent progress that has been proofed useful in
solving this problem for a variety of geometries. (For an overview of the development of related approaches, see Ref. 7.) In order to study Casimir forces in more general geometries, it turns out to be advantageous to describe how fluctuating currents are induced on the objects by the scattering of electromagnetic waves. This representation of the Casimir interaction was developed in Refs. 7–9. Each object is characterized by its on-shell electromagnetic scattering amplitude. The separations and orientations of the objects are encoded in universal translation matrices, which describe how a solution to the source-free Maxwell’s equations in the basis appropriate to one object looks when expanded in the basis appropriate to another. These matrices hence describe the electrodynamic interaction of the multipole moments associated with the currents and depend on the displacement and orientation of coordinate systems, but not on the shape and material of the objects themselves. The scattering amplitudes and translation matrices are then combined in a simple formula that allows efficient numerical and, in some cases, analytical calculations of Casimir forces and torques for a wide variety of geometries, materials, and external conditions. The approach applies to any finite number of arbitrarily shaped objects with arbitrary linear electromagnetic response at zero or finite temperature.

To illustrate this general formulation, we provide some sample applications, including results for the interaction between metallic objects for two spheres and for a sphere and a plane, taking into account the combined effect of shape and material properties at large distances. In addition, we provide examples for the non-additivity of the interaction by considering three objects (two spheres and a plane) and for the orientation dependence in the case of spheroids. The results are presented in form of analytical expressions at large distances and as numerical results at smaller separations.

2. Fluctuating currents and T-operators

We consider the Casimir energy for neutral objects with electric and magnetic susceptibilities. The partition function $Z$ is defined through the path integral, which sums all configurations of the electromagnetic field (outside and inside the objects) with periodic boundary conditions in time between 0 and $T$. The free energy $F$ of the field at inverse temperature $\beta$ is

$$F(\beta) = -\frac{1}{\beta} \log Z(\beta).$$

(1)

The unrenormalized free energy generally depends on the ultraviolet cutoff, but cutoff-dependent contributions arise from the objects individually and
do not depend on their separations or orientations. Since we are only interested in energy differences, we can remove these divergences by subtracting the energy of the system when the objects are in some reference configuration, see below. By replacing the time $T$ by $-i\hbar\beta$, we obtain the partition function $Z(\beta)$ in 4D Euclidean space. In $A^0 = 0$ gauge, the result is simply to replace the Matsubara frequencies $\omega_n = \frac{2\pi n}{T}$ by $i\frac{2\pi n}{\hbar\beta} = ic\kappa_n$, where $\kappa_n$ is the $n^{th}$ Matsubara frequency divided by $c$. The action is quadratic, so the modes with different $\kappa_n$ decouple and the partition function decomposes into a product of partition functions for each mode. In the limit $\beta \to \infty$, the sum $\sum_{n \geq 0}$ turns into an integral $\frac{\hbar c}{2\pi} \int_0^\infty d\kappa \log Z(\kappa)$, and we have the ground state energy

$$E_0 = -\frac{\hbar c}{2\pi} \int_0^\infty d\kappa \log Z(\kappa),$$

with

$$Z(\kappa) = \int \mathcal{D}A \mathcal{D}A^* \exp \left[ -\beta \int dx E^* (n, x) \left( H_0 + \frac{1}{\kappa^2} V(n, x) \right) E(n, x) \right],$$

where we have used $\nabla \times E = i\frac{c}{\epsilon} B$ to eliminate $B$ in the action, and it is assumed that $E$ is expressed by $E = -c^{-1} \partial_t A$ in terms of the vector potential $A$. This functional integral sums over configurations of $A$. This sum must be restricted by a choice of gauge, so that it does not include the infinitely redundant gauge orbits. We will choose to work in the gauge $A^0 = 0$, although of course no physical results depend on this choice. Here we defined the Helmholtz operator

$$H_0(\kappa) = \mathbb{I} + \frac{1}{\kappa^2} \nabla \times \nabla \times,$$

which is inverted by the Green’s function that is defined by

$$\kappa^2 H_0(\kappa) G_0(\kappa, x, x') = \mathbb{I} \delta^{(3)}(x - x').$$

The potential operator is

$$V(\kappa, x) = \mathbb{I} \kappa^2 (\epsilon (ic\kappa, x) - 1) + \nabla \times \left( \frac{1}{\mu(ic\kappa, x)} - 1 \right) \nabla \times .$$

It is nonzero only at those points in space where the objects are located ($\epsilon \neq 1$ or $\mu \neq 1$). At small frequencies, typical materials have $\epsilon > 1$ and $\mu \approx 1$, and $V$ can be regarded as an attractive potential.
Next, we transform to a free field (with kernel $H_0$) by introducing fluctuating currents $J$ that are confined to the objects. To perform this Hubbard-Stratonovich-like transformation we multiply and divide the partition function of Eq. (3) by

$$W = \int D\mathbf{J}D\mathbf{J}^* |_{\text{obj}} \exp \left[-\beta \int d\mathbf{x} \mathbf{J}^* (\mathbf{x}) \cdot \nabla^{-1} (\kappa, \mathbf{x}) \mathbf{J} (\mathbf{x}) \right] = \det \nabla,$$

where $|_{\text{obj}}$ indicates that the currents are defined only over the objects, i.e., the domain where $\nabla$ is nonzero (and therefore $\nabla^{-1}$ exists), and we have represented the local potential as a matrix in position space, $\nabla (\kappa, \mathbf{x}, \mathbf{x}') = \nabla (\kappa, \mathbf{x}) \delta^3 (\mathbf{x} - \mathbf{x}')$. We then change variables in the integration, $\mathbf{J} (\mathbf{x}) = \mathbf{J}' (\mathbf{x}) + \frac{i}{\kappa} \nabla (\kappa, \mathbf{x}) \mathbf{E} (\mathbf{x})$ and $\mathbf{J}^* (\mathbf{x}) = \mathbf{J}^{**} (\mathbf{x}) + \frac{1}{\kappa} \nabla (\kappa, \mathbf{x}) \mathbf{E}^* (\mathbf{x})$, to obtain

$$Z (\kappa) = \frac{1}{W} \int D\mathbf{A}D\mathbf{A}^* D\mathbf{J}'D\mathbf{J}'^* |_{\text{obj}} \times \exp \left[-\beta \int d\mathbf{x} \mathbf{E}^* (\kappa, \mathbf{x}) \left( H_0 (\kappa) + \frac{1}{\kappa^2} \nabla (\kappa, \mathbf{x}) \right) \mathbf{E} (\kappa, \mathbf{x}) \right. \right.$$

$$+ \left. \left( \mathbf{J}^{**} (\mathbf{x}) + \frac{i}{\kappa} \nabla (\kappa, \mathbf{x}) \mathbf{E}^* (\kappa, \mathbf{x}) \right) \nabla^{-1} (\kappa, \mathbf{x}) \left( \mathbf{J}' (\mathbf{x}) + \frac{i}{\kappa} \nabla (\kappa, \mathbf{x}) \mathbf{E} (\kappa, \mathbf{x}) \right) \right] \times$$

$$= \frac{1}{W} \int D\mathbf{A}D\mathbf{A}^* D\mathbf{J}'D\mathbf{J}'^* |_{\text{obj}} \times \exp \left[-\beta \int d\mathbf{x} \mathbf{E}^* H_0 \mathbf{E} + \mathbf{J}^{**} \nabla^{-1} \mathbf{J}' + \frac{i}{\kappa} \left( \mathbf{J}^{**} \mathbf{E} + \mathbf{J}' \mathbf{E}^* \right) \right].$$

Now the free electromagnetic field can be integrated out using $H_0^{-1} = \kappa^2 \mathbf{G}_0$, yielding

$$Z (\kappa) = \frac{Z_0}{W} \int D\mathbf{J}'D\mathbf{J}'^* |_{\text{obj}} \exp \left[-\beta \int d\mathbf{x} d\mathbf{x}' \mathbf{J}'^* (\mathbf{x}) \left( \mathbf{G}_0 (\kappa, \mathbf{x}, \mathbf{x}') + \nabla^{-1} (\kappa, \mathbf{x}) \delta^3 (\mathbf{x} - \mathbf{x}') \right) \mathbf{J}' (\mathbf{x}') \right],$$

with $Z_0 = \int D\mathbf{A}D\mathbf{A}^* \exp [ -\beta \int d\mathbf{x} \mathbf{E}^* H_0 (\kappa) \mathbf{E} ]$. Both factors $W$ and $Z_0$ contain cutoff-dependent contributions but are independent of the separation of the objects. Hence these factors cancel and can be ignored when we consider a change in the energy due to a change of the object’s separations with the shape and the material composition of the objects fixed. The kernel of the action in Eq. (9) is the inverse of the T-operator, i.e., $T^{-1} = \mathbf{G}_0 + \nabla^{-1}$ which is equivalent to

$$T = \nabla (I + \mathbf{G}_0 \nabla)^{-1}.$$
The Casimir energy at zero temperature (without the cutoff-dependent parts) is hence
\[ E = -\frac{\hbar c}{2\pi} \int_0^\infty d\kappa \log \det T. \quad (11) \]

The determinant is here taken over the spatial indices \( x \) and \( x' \), which are restricted to the objects since \( T \) vanishes if \( x \) or \( x' \) are not on an object.

To compute the determinant we start from the expression for \( T^{-1} \) which yields the reciprocal of the determinant. We decompose \( T^{-1} \) by introducing separate position space basis functions for each object. The projection of the currents onto this basis defines the object’s multipole moments. This yields a division of \( T^{-1} \) into blocks where each block is labeled by an object.

The off-diagonal blocks are given by \( G_0 \) only and describe the interaction of the multipoles on different objects. To see this we choose for each object individually an eigenfunction basis to expand the free Green’s function,
\[ G_0(\kappa, x, x') = \sum_\alpha E^\text{out}_\alpha(\kappa, x_>) \otimes E^\text{reg*}_\alpha(\kappa, x'_<) \quad (12) \]

with regular solutions \( E^\text{reg}_\alpha \) and outgoing solutions \( E^\text{out}_\alpha \) of the free vector Helmholtz equation, where \( x_< \) and \( x_> \) denote the position with smaller and greater value of the “radial” variable of the separable coordinates. The multipole moments of object \( j \) are then \( Q_{j,\alpha}(\kappa) = \int d\mathbf{x} J_j(\kappa, \mathbf{x}) E^\text{reg*}_\alpha(\kappa, \mathbf{x}). \)

Regular solutions form a complete set and hence outgoing solutions can be expanded in terms of regular solutions except in a region (enclosed by a surface of constant radial variable) that contains the origin of the coordinate system of object \( i \). This expansion defines the translation matrices \( U_{ji}^{\beta\alpha} \) via
\[ E^\text{out}_\alpha(\kappa, x_i) = \sum_\beta U_{ji}^{\beta\alpha}(\kappa, \mathbf{X}_{ji}) E^\text{reg}_\beta(\kappa, x_j), \quad (13) \]

where the definition of the coordinates is shown in Fig. 1. The free Green’s function then becomes
\[ G_0(\kappa, x, x') = \sum_{\alpha,\beta} E^\text{reg}_\alpha(\kappa, x_i) \otimes U_{ji}^{\beta\alpha}(\kappa, \mathbf{X}_{ji}) E^\text{reg*}_\beta(\kappa, x'_j) \quad (14) \]

so that the off-diagonal blocks of \( T^{-1} \) are given by the translation matrices. Equivalent translation matrices can be defined between two sets of regular solutions as is necessary for one object inside another, see Ref. 7.

The diagonal blocks of \( T^{-1} \) are given by the matrix elements of the \( T \)-operators \( T_j \) of the individual objects. By multiplying \( T^{-1} \) by the \( T \)-operator \( T_\infty \) without the off-diagonal blocks which can interpreted as describing a reference configuration with infinite separations between the ob-
Fig. 1. Geometry of the configuration. The dotted lines show surfaces separating the objects on which the radial variable is constant. The translation vector $X_{ij} = x_i - x_j = -X_{ji}$ describes the relative positions of the two origins.

jects, one finds that (for objects outside each other) the diagonal blocks are given by the inverse of the matrix representing $T_j$ in the basis $E_{\text{reg}}^\alpha$. The physical meaning of this matrix follows from the Lippmann-Schwinger equation for the full scattering solution $E_{\alpha}(\kappa, x)$,

$$E_{\alpha}(\kappa, x) = E_{\text{reg}}^\alpha(\kappa, x) - G_0 \nabla_j E_{\alpha}(\kappa, x) = E_{\text{reg}}^\alpha(\kappa, x) - G_0 T_j E_{\text{reg}}^\alpha(\kappa, x). \quad (15)$$

Using the expansion of Eq. (12), the solution sufficiently far away from the object (i.e., for positions that have a radial variable larger than any point on the object) can be expressed as

$$E_{\alpha}(\kappa, x) = E_{\text{reg}}^\alpha(\kappa, x) - \sum_{\beta} E_{\text{out}}^\beta(\kappa, x) \int E_{\text{reg}}^{\alpha^*}(\kappa, x') T_j(\kappa) E_{\text{reg}}^\beta(\kappa, x') dx', \quad (16)$$

where the integral defines the scattering amplitude $F_{j,\beta\alpha}(\kappa)$ of object $j$. It can be obtained, e.g., from matching boundary conditions at the surface of a dielectric object.

The Casimir energy (without cutoff-dependent contributions from $W$ and $Z_0$) can now be expressed as

$$\mathcal{E} = \frac{\hbar c}{2\pi} \int_0^{\infty} d\kappa \log \det(M M^{-1}_\infty), \quad (17)$$

where

$$M = \begin{pmatrix} F_1^{-1} & U_{12} & U_{13} & \cdots \\ U_{21} & F_2^{-1} & U_{23} & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix} \quad (18)$$
and $M_\infty^{-1}$ is the block diagonal matrix $\text{diag}(F_1, F_2, \ldots)$. For the case of two objects this expression simplifies to

$$E = \frac{\hbar c}{2\pi} \int_0^\infty d\kappa \log \det \left( I - F_1 U_{12} F_2 U_{21} \right).$$

(19)

In order to obtain the free energy at nonzero temperature instead of the ground state energy, we do not take the limit $\beta \to \infty$ in Eq. (1). Instead, the integral $\frac{\hbar c}{2\pi} \int_0^\infty d\kappa$ is replaced everywhere by $\frac{1}{\beta} \sum'_{n}$, where $c\kappa_n = \frac{2\pi n}{\hbar \beta}$ with $n = 0, 1, 2, 3, \ldots$ is the $n$th Matsubara frequency. A careful analysis of the derivation shows that the zero frequency mode is weighted by 1/2 compared to the rest of the terms in the sum; this modification of the sum is denoted by a prime on the summation symbol.

3. Applications

In this section we demonstrate the applicability of the method through some examples. Due to the lack of space, we only present the final analytical and numerical results that all follow from Eq. (17) or Eq. (19) by truncation of the matrices at some order of partial waves, i.e., by considering only a finite set of basis functions. At asymptotically large distances, the interaction only depends on the dipole contribution while with decreasing distance the number of partial waves has to be increased. Below we will provide results both in form of a asymptotic series in the inverse separation and numerical results for a wide range of distances.

3.1. Sphere-plane

First, we consider the sphere-plane geometry that has been employed in the majority of recent experiments. At large distances, the energy can be expanded in an asymptotic series in the inverse separation. For a dielectric sphere in front of a perfectly reflecting mirror with sphere-center to mirror separation $L$ the Casimir energy is

$$E = -\frac{\hbar c}{\pi} \left\{ \frac{3}{8} (\alpha_1^E - \alpha_1^M) \frac{1}{L^4} + \frac{15}{32} (\alpha_2^E - \alpha_2^M + 2\gamma_{13}^E - 2\gamma_{13}^M) \frac{1}{L^6} ight. \right. 
+ \frac{1}{1024} \left\{ 23(\alpha_3^M)^2 - 140\alpha_1^E \alpha_1^E + 23(\alpha_1^E)^2 + 2160(\gamma_{14}^E - \gamma_{14}^M) \right\} \frac{1}{L^8} 
\left. + \frac{7}{7200} \left[ 572(\alpha_3^E - \alpha_3^M) + 675 \left( 9(\gamma_{15}^E - \gamma_{15}^M) - 55(\gamma_{23}^E - \gamma_{23}^M) \right) \right] \frac{1}{L^8} + \ldots \right\},$$

(20)

where $\alpha_l^E$, $\alpha_l^M$ are the static electric and magnetic multipole polarizabilities of the sphere of order $l$ ($l = 2$ for dipoles), and the coefficients $\gamma_{ln}^E$, $\gamma_{ln}^M$
describe finite-frequency corrections to these polarizabilities, i.e., terms \( \sim \kappa^{2l+n} \) in the low-\( \kappa \) expansion of the T-matrix element for the \( l \)th partial wave. Notice that the first three terms of the contribution at order \( L^{-7} \) have precisely the structure of the Casimir-Polder interaction between two atoms with static dipole polarizabilities \( \alpha_m \) and \( \alpha_e \) but it is reduced by a factor of \( 1/2^8 \). This factor and the distance dependence \( \sim L^{-7} \) of this term suggests that it arises from the interaction of the dipole fluctuations inside the sphere with those inside its image at a distance \( 2L \). The additional coefficient of \( 1/2 \) in the reduction factor \((1/2)(1/2^7)\) can be traced back to the fact that the forces involved in bringing the dipole in from infinity act only on the dipole and not on its image. If the sphere is also assumed to be a perfect reflector, the energy becomes

\[
E = \frac{hc}{\pi \ell} \sum_{j=4}^{\infty} b_j \left( \frac{R}{\ell} \right)^{j-1},
\]

where the coefficients up to order \( 1/L^{11} \) are

\[
\begin{align*}
b_4 &= -\frac{9}{16}, \quad b_5 = 0, \quad b_6 = -\frac{25}{32}, \quad b_7 = -\frac{3023}{4096}, \\
b_8 &= -\frac{12551}{9600}, \quad b_9 = \frac{1282293}{163840}, \\
b_{10} &= -\frac{32027856257}{722534400}, \quad b_{11} = \frac{39492614653}{412876800}.
\end{align*}
\]

Our method can be also employed to study the material dependence of the interaction. When the sphere and the mirror are described by a simple plasma model, we can obtain the interaction energy again from Eq. (19) by substituting the dielectric function on the imaginary frequency axis,

\[
\epsilon_p(\imath \kappa) = 1 + \left( \frac{2\pi}{\lambda_p \kappa} \right)^2,
\]

into the T-matrices of sphere and mirror. From this we get at large separations

\[
E = -\frac{hc}{\pi} \left[ f_4(\lambda_p/R) \frac{R^3}{\ell^4} + f_5(\lambda_p/R) \frac{R^4}{\ell^5} + O(L^{-6}) \right],
\]

with the functions

\[
\begin{align*}
f_4(z) &= \frac{9}{16} + \frac{9}{64\pi^2} z^2 - \frac{9}{32}\pi z \coth \frac{2\pi}{z} \\
f_5(z) &= -\frac{13}{20\pi} z - \frac{21}{80\pi^3} z^3 + \frac{21}{40\pi^2} z^2 \coth \frac{2\pi}{z}.
\end{align*}
\]
It is interesting that the amplitude $f_4$ of the leading term is not universal but depends on the plasma wavelength $\lambda_p$. Only in the two limits $\lambda_p/R \to 0$ and $\lambda_p/R \to \infty$ the amplitude assumes material independent values, $9/16$ and $3/8$, respectively. The first limit describes perfect reflection of electric and magnetic fields at arbitrarily low frequencies and hence agrees with the result of Eq. (21). The change to the second amplitude for large $\lambda_p$ can be understood when one considers a London superconductor that is described at zero temperature by the plasma dielectric function. If one associates $\lambda_p$ with the penetration depth, the perfect reflector limit results from the absence of any field penetration while the second limit corresponds to a large penetration depth and hence the suppression of the magnetic mode contribution to the Casimir energy, explaining the reduced amplitude of $3/8$. The latter result follows also when the objects are considered to be normal metals, described by the Drude model dielectric function

$$\epsilon_p(\omega) = 1 + \frac{(2\pi)^2}{(\lambda_p\omega)^2 + \pi\sigma/\omega}.$$  

(26)

From this function we get for a sphere and a mirror made of a Drude metal the asymptotic energy

$$\mathcal{E} = -\frac{h\omega}{\pi} \left[ \frac{3 R^3}{8 L^4} - \frac{77}{384} \frac{R^3}{\sqrt{2\pi}} \frac{1}{c} L^{9/2} - \left( \frac{c}{8\pi\sigma} - \frac{\pi}{20} \frac{\sigma R^2}{c} \right) \frac{R^3}{L^5} + O(L^{-4}) \right].$$  

(27)

In fact, one observes that the leading term is universal and agrees with the $\lambda_p \to \infty$ limit of the plasma model. Note that the result of Eq. (27) does not apply to arbitrarily large dc conductivity $\sigma$. The conditions for the validity of Eq. (27) can be written as $L \gg R$, $L \gg c/\sigma$ and $L \gg \sigma R^2/c$. The above results demonstrate strong correlations between shape and material since for two parallel, infinite plates, both the plasma and the Drude model yield at large separations the same (universal) result as a perfect mirror description.

In order to study short separations, Eq. (19) has to be evaluated numerically by including sufficiently many partial waves. The result of an extrapolation from $l = 29$ partial waves is shown in Fig. 2 in the perfect reflection limit. At small separations the result can be fitted to a power law of the form

$$\mathcal{E} = \mathcal{E}_{PFA} \left[ 1 + \theta_1 \frac{d}{R} + \theta_2 \left( \frac{d}{R} \right)^2 + \ldots \right].$$  

(28)

with $\mathcal{E}_{PFA}$ and $d$ defined in Fig. 2. The coefficients $\theta_j$ measure corrections to the proximity force approximation and are obtained from a fit of the
function of Eq. (28) to the data points for the four smallest studied separations. We find $\theta_1 = -1.42 \pm 0.02$ and $\theta_2 = 2.39 \pm 0.14$. This result is in agreement with numerical findings in Ref. 12 but is in disagreement with an asymptotic expansion for small distances.\textsuperscript{13} The latter yields $\theta_1 = -5.2$ and very small logarithmic corrections that however can be ignored at the distances considered here. The origin of this discrepancy is currently unclear but might be related to the applicability of the asymptotic expansion to only much smaller distances than accessible by current numerics.

Fig. 2. Electromagnetic Casimir energy for the sphere-plate geometry. The energy is scaled by the proximity force approximation (PFA) energy $\mathcal{E}_{\text{PFA}} = -\frac{\pi^2}{720} \hbar c R^2$. The asymptotic expansion of Eq. (21) is shown as dashed line. Inset: Corrections to the PFA at small distances as function of $d = L - R$. 
Fig. 3. Left: Geometry of the two-sphere/atom and sidewall system. Shown are also the mirror images (grey) and two- and three-body contributions (solid and dashed curly lines, respectively). Right: Typical orientations of electric (E) and magnetic (M) dipoles and image dipoles for $H/L \to 0$ and $H/L \to \infty$.

### 3.2. Three-body effects

Casimir interactions are not pair-wise additive. To study the consequences of this property, we consider the case of two identical, general polarizable objects near a perfectly reflecting wall in the dipole approximation, see Fig. 3. This situation applies to ground state atoms and also to general objects at large separations. The separation between the objects is $L$ and the separation of each of them from the wall is $H$. In dipole approximation, the retarded limit of the interaction is described by the static electric ($\alpha_z$, $\alpha_\parallel$) and magnetic ($\beta_z$, $\beta_\parallel$) dipole polarizabilities of the objects which can be different in the directions perpendicular (z) and parallel (∥) to the wall.

In the absence of the wall the potential for the two polarizable objects is given by the well-known Casimir-Polder (CP) potential

$$E_{2,\parallel}(L) = -\frac{\hbar c}{8\pi L^7}\left[33\alpha_\parallel^2 + 13\alpha_z^2 - 14\alpha_\parallel\beta_z + (\alpha \leftrightarrow \beta)\right],$$

(29)

The $L$-dependent part of the interaction energy in the presence of the wall is

$$E_{2\perp}(L, H) = E_{2,\parallel}(L) + E_{2\perp}(D, L) + E_3(D, L)$$

(30)

with $D = \sqrt{L^2 + 4H^2}$. The change in the relative orientation of the objects with $\ell = L/D$ leads to the modified 2-body CP potential

$$E_{2\perp}(D, L) = -\frac{\hbar c}{8\pi D^7}\left[26\alpha_\parallel^2 + 20\alpha_z^2 - 14\ell^2(4\alpha_\parallel^2 - 9\alpha_\parallel\alpha_z + 5\alpha_z^2)\right.\left. + 63\ell^4(\alpha_\parallel - \alpha_z)^2 - 14(\alpha_\parallel\beta_\parallel(1-\ell^2) + \ell^2\alpha_\parallel\beta_z + (\alpha \leftrightarrow \beta))\right].$$

(31)
The 3-body energy $E_3(D, L)$ describes the collective interaction between the two objects and one image object. It is given by

$$E_3(D, L) = \frac{4 \hbar c}{\pi} \frac{1}{L^3 D^4 (\ell + 1)^2} \left[ \left( 3 \ell^6 + 15 \ell^5 + 28 \ell^4 + 20 \ell^3 + 6 \ell^2 - 5 \ell - 1 \right) \times \left( \alpha_z^2 - \beta_z^2 \right) - (3 \ell^6 + 15 \ell^5 + 24 \ell^4 - 10 \ell^2 - 5 \ell - 1) \left( \alpha_z^2 - \beta_z^2 \right) + 4 \left( \ell^4 + 5 \ell^3 + \ell^2 \right) \left( \alpha_z \beta_z - \alpha_\parallel \beta_\parallel \right) \right].$$

(32)

It is instructive to consider the two limits $H \ll L$ and $H \gg L$. For $H \ll L$ $E_{2,0}$ turns out to be the CP potential of Eq. (29) with the replacements $\alpha_z \to 2 \alpha_z$, $\alpha_\parallel \to 0$, $\beta_z \to 0$, $\beta_\parallel \to 2 \beta_\parallel$. The 2-body and 3-body contributions add constructively or destructively, depending on the relative orientation of a dipole and its image which together form a dipole of zero or twice the original strength (see Fig. 3).

For $H \gg L$ the leading correction to the CP potential of Eq. (29) comes from the 3-body energy. The energy then becomes (up to order $H^{-6}$)

$$E_{2,0}(L, H) = E_{2,1}(L) + \frac{\hbar c}{2 \pi} \left[ \alpha_z^2 - \frac{\alpha_\parallel^2}{4 L^3 H^4} + \frac{9 \alpha_z^2 - \alpha_\parallel^2 - 2 \alpha_\parallel \beta_z}{8 L H^6} - (\alpha \leftrightarrow \beta) \right].$$

(33)

The signs of the polarizabilities in the leading term $\sim H^{-4}$ can be understood from the relative orientation of the dipole of one atom and the image dipole of the other atom, see Fig. 3. If these two electric (magnetic) dipoles are almost perpendicular to their distance vector they contribute attractively (repulsively) to the potential between the two original objects. If these electric (magnetic) dipoles are almost parallel to their distance vector they yield a repulsive (attractive) contribution. For isotropic polarizabilities the leading term of Eq. (33) vanishes and the electric (magnetic) part $\sim H^{-6}$ of the 3-body energy is always repulsive (attractive).

Next, we study the same geometry as before but with the objects assumed to be two perfectly reflecting spheres of radius $R$. The lengths $L$ and $H$ are measured now from the centers of the spheres, see Fig. 3. Here we do not limit the analysis to large separations but consider arbitrary distances and include higher order multipole moments than just dipole polarizability. For $R \ll L$, $H$ and arbitrary $H/L$ the result for the force can be written as

$$F = \frac{\hbar c}{\pi R^2} \sum_{j=6}^{\infty} f_j(H/L) \left( \frac{R}{L} \right)^{j+2}.$$

(34)

The functions $f_j$ can be computed exactly. We have obtained them up to
$j = 11$ and the first three are (with $s \equiv \sqrt{1 + 4h^2}$)

$$f_6(h) = -\frac{1}{16h^8} \left[ s^{-9} (18 + 312h^2 + 2052h^4 + 6048h^6 \\
+ 5719h^8) + 18 - 12h^2 + 1001h^8 \right], \quad f_7(h) = 0,$$

$$f_8(h) = -\frac{1}{160h^{12}} \left[ s^{-11} (6210 + 140554h^2 + 1315364h^4 \\
+ 6500242h^6 + 17831168h^8 + 15000675h^{12}) \\
- 6210 - 3934h^2 + 764h^4 - 78h^6 + 71523h^{12} \right].$$

(35)

(36)

For $H \gg L$ one has $f_6(h) = -1001/16 + 3/(4h^6) + O(h^{-8})$, $f_8(h) = -71523/160 + 39/(80h^6) + O(h^{-8})$ so that the wall induces weak repulsive corrections. For $H \ll L$, $f_6(h) = -791/8 + 6741h^2/8 + O(h^4)$, $f_8(h) = -60939/80 + 582879h^2/80 + O(h^4)$ so that the force amplitude decreases when the spheres are moved a small distance away from the wall. This proves the existence of a minimum in the force amplitude as a function of $H/R$ for fixed, sufficiently small $R/L$. We note that all $f_j(h)$ are finite for $h \to \infty$ but some diverge for $h \to 0$, e.g., $f_9 \sim f_{11} \sim h^{-3}$, making them important for small $H$.

To obtain the interaction at smaller separations or larger radius, we have computed the energy $E_{\infty}$ and force $F = -\partial E_{\infty}/\partial L$ between the spheres numerically.\textsuperscript{14} In order to show the effect of the wall, we plot the energy and force normalized to the results for two spheres without a wall. Fig. 4 shows the force between the two spheres as a function of the wall distance for fixed $L$. When the spheres approach the wall, the force first decreases slightly if $R/L \lesssim 0.3$ and then increases strongly under a further reduction of $H$. For $R/L \gtrsim 0.3$ the force increases monotonically as the spheres approach the wall. This agrees with the prediction of the large distance expansion. The expansion of Eq. (34) with $j = 10$ terms is also shown in Fig. 4 for $R/L \leq 0.2$. Its validity is limited to large $L/R$ and not too small $H/R$; it fails completely for $R/L > 0.2$ and hence is not shown in this range.

### 3.3. Orientation dependence

In this section we investigate the shape and orientation dependence of the Casimir force using Eq. (19). As examples we focus on ellipsoids, computing the orientation dependent force between two spheroids, and between a spheroid and a plane.\textsuperscript{15} For two anisotropic objects, the CP potential of Eq. (29) must be generalized. In terms of the Cartesian components of the standard electric (magnetic) polarizability matrix $\alpha (\beta)$, the asymptotic
large distance potential of two objects (with the $\hat{z}$ axis pointing from one object to the other), can be written as

$$E = \frac{hc}{d^2} \frac{1}{8\pi} \left\{ 13 \left( \alpha_x^1 \alpha_x^2 + \alpha_y^1 \alpha_y^2 + 2\alpha_x^1 \alpha_y^2 \right) + 20 \alpha_{zz}^1 \left[ \left( \alpha_x^1 \alpha_x^2 \alpha_y^1 \alpha_y^2 \right) + (\alpha \rightarrow \beta) \right] - 7 \left( \alpha_{xx}^1 \beta_{yy}^2 + \alpha_{yy}^1 \beta_{xx}^2 - 2\alpha_x^1 \beta_{xy}^2 \right) + (1 \leftrightarrow 2) \right\}. \tag{37}$$

For the case of an ellipsoidal object with static electric permittivity $\epsilon$ and magnetic permeability $\mu$, the polarizability tensors are diagonal in a basis oriented to its principal axes, with elements (for $i \in \{1, 2, 3\}$)

$$\alpha_{ii}^0 = \frac{V}{4\pi} \frac{1}{1 + (\epsilon - 1)n_i}, \quad \beta_{ii}^0 = \frac{V}{4\pi} \frac{1}{1 + (\mu - 1)n_i}, \tag{38}$$

where $V = 4\pi r_1 r_2 r_3 / 3$ is the ellipsoid’s volume. In the case of spheroids, for which $r_1 = r_2 = R$ and $r_3 = L / 2$, the so-called depolarizing factors can be expressed in terms of elementary functions,

$$n_1 = n_2 = \frac{1 - n_3}{2}, \quad n_3 = \frac{1 - e^2}{2e^3} \left( \log \frac{1 + e}{1 - e} - 2e \right), \tag{39}$$
where the eccentricity $e = \sqrt{1 - \frac{4R^2}{L^2}}$ is real for a prolate spheroid ($L > 2R$) and imaginary for an oblate spheroid ($L < 2R$). The polarizability tensors for an arbitrary orientation are then obtained as $\alpha = R^{-1} \alpha^0 R$, where $R$ is the matrix that rotates the principal axis of the spheroid to the Cartesian basis, i.e. $R(1, 2, 3) \rightarrow (x, y, z)$. Note that for rarefied media with $\epsilon \simeq 1, \mu \simeq 1$ the polarizabilities are isotropic and proportional to the volume. Hence, to leading order in $\epsilon - 1$ the interaction is orientation independent at asymptotically large separations, as we would expect, since pairwise summation is valid for $\epsilon - 1 \ll 1$. In the following we focus on the interesting opposite limit of two identical perfectly reflecting spheroids. We first consider prolate spheroids with $L \gg R$. The orientation of each “needle” relative to the line joining them (the initial $z$-axis) is parameterized by the two angles $(\theta, \psi)$, as depicted in Fig. 5. Then the energy is

$$E(\theta_1, \theta_2, \psi) = -\frac{\hbar c}{d^2} \left\{ \frac{5L^6}{1152\pi \left( \ln \frac{L}{R} - 1 \right)^2} \left[ \cos^2 \theta_1 \cos^2 \theta_2 \right. \right. \right.$$

$$\left. + \frac{13}{20} \cos^2 \psi \sin^2 \theta_1 \sin^2 \theta_2 - \frac{3}{8} \cos \psi \sin 2\theta_1 \sin 2\theta_2 \left. \right] + \mathcal{O} \left( \frac{L^4R^2}{\ln \frac{L}{R}} \right) \right\},$$

where $\psi \equiv \psi_1 - \psi_2$. It is minimized for two needles aligned parallel to their separation vector. At almost all orientations the energy scales as $L^6$, and vanishes logarithmically slowly as $R \rightarrow 0$. The latter scaling changes when one needle is orthogonal to $\hat{z}$ (i.e. $\theta_1 = \pi/2$), while the other is either parallel to $\hat{z}$ ($\theta_2 = 0$) or has an arbitrary $\theta_2$ but differs by an angle $\pi/2$ in its rotation about the $z$-axis (i.e. $\psi_1 - \psi_2 = \pi/2$). In these cases the energy comes from the next order term in Eq. (40), and takes the form

$$E \left( \frac{\pi}{2}, \theta_2, \frac{\pi}{2} \right) = -\frac{\hbar c}{1152\pi d^2} \frac{L^4R^2}{\ln \frac{L}{R} - 1} (73 + 7 \cos 2\theta_2),$$

which shows that the least favorable configuration corresponds to two needles orthogonal to each other and to the line joining them.

For perfectly reflecting oblate spheroids with $R \gg L/2$, the orientation of each “pancake” is again described by a pair of angles $(\theta, \psi)$, as depicted in Fig. 6. To leading order at large separations, the energy is given by

$$E = -\frac{\hbar c}{d^2} \left\{ \frac{R^6}{144\pi^3} \left[ 765 - 5(\cos 2\theta_1 + \cos 2\theta_2) + 237 \cos 2\theta_1 \cos 2\theta_2 \right. \right.$$

$$\left. + 372 \cos 2\psi \sin^2 \theta_1 \sin^2 \theta_2 - 300 \cos \psi \sin 2\theta_1 \sin 2\theta_2 \right] + \mathcal{O}(R^5L) \right\}.$$
The leading dependence is proportional to $R^6$, and does not disappear for any choice of orientations. Furthermore, this dependence remains even as the thickness of the pancake is taken to zero ($L \rightarrow 0$). This is very different from the case of the needles, where the interaction energy vanishes with thickness as $\ln^{-1}(L/R)$. The lack of $L$ dependence is due to the assumed perfectly reflectivity. The energy is minimal for two pancakes lying on the same plane ($\theta_1 = \theta_2 = \pi/2$, $\psi = 0$) and has energy $-\hbar c (173/18\pi^3) R^6/d^7$. When the two pancakes are stacked on top of each other, the energy is increased to $-\hbar c (62/9\pi^3) R^6/d^7$. The least favorable configuration is when the pancakes lie in perpendicular planes, i.e., $\theta_1 = \pi/2$, $\theta_2 = 0$, with an energy $-\hbar c (11/3\pi^3) R^6/d^7$.

Fig. 5. (Color online) Orientation of a prolate (cigar-shaped) spheroid: The symmetry axis (initially the $z$-axis) is rotated by $\theta$ about the $x$-axis and then by $\psi$ about the $z$-axis. For two such spheroids, the energy at large distances is given by Eq. (40). The latter is depicted at fixed distance $d$, and for $\psi_1 = \psi_2$, by a contour plot as function of the angles $\theta_1, \theta_2$ for the $x$-axis rotations. Minima (maxima) are marked by filled (open) dots.

For an anisotropic object interacting with a perfectly reflecting mirror, at leading order the CP potential generalizes to

$$E = -\frac{\hbar c}{d^4} \frac{1}{8\pi} \text{tr} (\alpha - \beta) + O(d^{-5}) ,$$

which is clearly independent of orientation. Orientation dependence in this system thus comes from higher multipoles. The next order also vanishes, so the leading term is the contribution from the partial waves with $l = 3$. 


for which the scattering matrix is not known analytically. However, we can obtain the preferred orientation by considering a distorted sphere in which the radius $R$ is deformed to $R + \delta f(\theta, \varphi)$. The function $f$ can be expanded into spherical harmonics $Y_{lm}(\theta, \varphi)$, and spheroidal symmetry can be mimicked by choosing $f = Y_{20}(\theta, \varphi)$. The leading orientation dependent part of the energy is then obtained as

$$E_f = -\frac{\hbar c}{640\sqrt{5\pi^3/2}} \frac{\delta R^4}{d^6} \cos(2\theta).$$

A prolate spheroid ($\delta > 0$) thus minimizes its energy by pointing towards the mirror, while an oblate spheroid ($\delta < 0$) prefers to lie in a plane perpendicular to the mirror. (We assume that the perturbative results are not changed for large distortions.) These configurations are also preferred at small distances $d$, since (at fixed distance to the center) the object reorients to minimize the closest separation. Interestingly, the latter conclusion is not generally true. In Ref. 15 it has been shown that there can be a transition in preferred orientation as a function of $d$ in the simpler case of a scalar field with Neumann boundary conditions. The separation at which this transition occurs varies with the spheroid’s eccentricity.

### 3.4. Material dependence

In this section we shall discuss some characteristic effects of the Casimir interaction between metallic nano-particles by studying two spheres with
finite conductivity in the limit where their radius $R$ is much smaller than their separation $d$. We assume further that $R$ is large compared to the inverse Fermi wave vector $\pi/k_F$ of the metal. Since typically $\pi/k_F$ is of the order of a few Angstrom, this assumption is reasonable even for nanoparticles. Theories for the optical properties of small metallic particles\textsuperscript{16} suggest a Drude dielectric function
\[ \epsilon(ik) = 1 + 4\pi \frac{\sigma(ik)}{ck}, \] (45)
where $\sigma(ik)$ is the conductivity which approaches for $k \to 0$ the dc conductivity $\sigma_{dc}$. For bulk metals $\sigma_{dc} = \omega_p^2 \tau/4\pi$ where $\omega_p = \sqrt{e^2 k_F^2/3\pi m_e}$ is the plasma frequency with electron charge $e$ and electron mass $m_e$, and $\tau$ is the relaxation time. With decreasing dimension of the particle, $\sigma_{dc}(R)$ is reduced compared to its bulk value due to finite size effects and hence becomes a function of $R$.\textsuperscript{16} In analogy to the result for a sphere and a plate that are described by the Drude model, we obtain for the large distance expansion of the energy the result
\[ E = -\hbar c \frac{23}{4\pi} \frac{R^6}{L^7} - \left( \frac{R\sigma_{dc}(R)}{c} - \frac{45}{4\pi^2} \frac{c}{R\sigma_{dc}(R)} \right) \frac{R^7}{L^8} + \ldots . \] (46)
As in the sphere-plate case, the leading term is material independent but different from that of the perfect metal limit (where the amplitude is $143/16\pi$) since only the electric polarization contributes. At next order, the first and second terms in the parentheses come from magnetic and electric dipole fluctuations, respectively. The term $\sim 1/L^8$ is absent in the interaction between perfectly conducting spheres. The limit of perfect conductivity, $\sigma_{dc} \to \infty$ cannot be taken in Eq. (46) since this limit does not commute with the large $L$ expansion.

In order to estimate the effect of finite conductivity and its dependence on the size of the nano-particle, we have to employ a theory that can describe the evolution of $\sigma_{dc}(R)$ with the particle size. A theory for the dielectric function of a cubical metallic particle of dimensions $R \gg \pi/k_F$ has been developed within the random phase approximation in the limit of low frequencies $\ll c/R$.\textsuperscript{16} In this theory it is further assumed that the discreteness of the electronic energy levels, and not the inhomogeneity of the charge distribution, is important. This implies that the particle responds only at the wave vector of the incident field which is a rather common approximation for small particles. From an electron number-conserving relaxation time approximation the complex dielectric function is obtained which yields the size-dependent dc conductivity for a cubic particle of volume $a^3$.\textsuperscript{16} It has been shown that the detailed shape of the particle does
Fig. 7. Dimensionless dc conductivity $\hat{\sigma}_{dc}(R)$ in units of $e^2/2\hbar a_0$ (with Bohr radius $a_0$) for a Aluminum sphere with $\epsilon_F = 11.63eV$, $\pi/k_F = 1.8\AA$ and $\tau = 0.8 \cdot 10^{-14}$sec as function of the radius $R$, measured in units of $\pi/k_F$. Also shown is the corresponding ratio $R\hat{\sigma}_{dc}(R)/c$ that determines the Casimir interaction of Eq. (46). The bulk dc conductivity $\hat{\sigma}_{dc}(\infty) = 17.66$ is indicated by the dashed line.

not matter much, and we can set $a = (4\pi/3)^{1/3}R$ which defines the volume equivalent sphere radius $R$. For $\pi/k_F \approx a$ the nano particle ceases to be conducting, corresponding to a metal-insulator transition due to the localisation of electrons for particles with a size of the order of the mean free path. It is instructive to consider the size dependence of $\sigma_{dc}(R)$ and of the Casimir interaction for a particular choice of material. Following Ref. 16, we focus on small Aluminum spheres with Fermi energy $\epsilon_F = 11.63eV$ and $\tau = 0.8 \cdot 10^{-14}$sec. These parameters correspond to $\pi/k_F = 1.8\AA$ and a plasma wavelength $\lambda_p = 79$nm. It is useful to introduce the dimensionless conductivity $\hat{\sigma}_{dc}(R)$, which is measured in units of $e^2/2\hbar a_0$ with Bohr radius $a_0$, so that the important quantity of Eq. (46) can be written as $R\hat{\sigma}_{dc}(R)/c = (\alpha/2)(R/a_0)\hat{\sigma}_{dc}(R)$ where $\alpha$ is the fine-structure
constant. The result is shown in Fig. 7. For example, for a sphere of radius $R = 10\text{nm}$, the dc conductivity is reduced by a factor $\approx 0.15$ compared to the bulk Drude value. If the radius of the sphere is equal to the plasma wavelength $\lambda_p$, the reduction factor $\approx 0.8$. These results show that shape and material properties are important for the Casimir interaction between nano-particles. Potential applications include the interaction between dilute suspensions of metallic nano-particles.

3.5. Further extensions

The general result of Eq. (17) and its extensions described in Ref. 7 have been recently applied to a number of new geometries and further applications are under way. Examples include so-called interior configurations with an object contained within an otherwise empty, perfectly conducting spherical shell. For this geometry the forces and torques on a dielectric or conducting object, well separated from the cavity walls, have been determined. Corrections to the proximity force approximation for this interior problem have been obtained by computing the interaction energy of a finite-size metal sphere with the cavity walls when the separation between their surfaces tends to zero. Eq. (17), evaluated in parabolic cylinder coordinates, has been used to obtain the interaction energy of a parabolic cylinder and an infinite plate (both perfect mirrors), as a function of their separation and inclination, and the cylinder’s parabolic radius. By taking the limit of vanishing radius, corresponding to a semi-infinite plate, the effect of edge and inclination could be studied.

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