ALMOST SURE INVARIANCE PRINCIPLE FOR TIME DEPENDENT NON-UNIFORMLY EXPANDING DYNAMICAL SYSTEMS

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Abstract. We prove an almost sure invariance principle for some classes of time dependent non-uniformly distance expanding dynamical systems. The main model we have in mind is a sequential version of the non-uniformly distance expanding maps considered in [3] and [24], though our results also hold true when there are some singular points. Our results rely on the theory of complex projective metrics which was developed in [23], together with the spectral methods of Gouëzel [12]. The latter was used so far only for uniformly distance expanding maps, and here we show how to apply it for non-uniformly expanding maps. A big advantage in applying this theory here is that it also yields a variety of probabilistic limit theorems for random dynamical systems, as described at the last section of this paper.

1. Introduction

Probabilistic limit theorems for deterministic dynamical systems is a well studied topic. One important generalization of such results (see, for instance [18] and [19]) is to random dynamical systems in which the system evolves according to iterates of random transformations of the form $T_{\theta_{n-1}} \circ \cdots \circ T_{\theta_{0}} \circ T_{\omega}$, $\omega \in \Omega$, where $(\Omega, F, P, \theta)$ is some ergodic invertible measure preserving system, which can be viewed as a "driving process". The central limit theorem (CLT) for partial sums generated by random dynamical systems has been studied by many authors. Recently, finer results such as the local central limit theorem (LCLT) and the Berry-Esseen theorem (optimal convergence rate in the CLT) have been obtained for several classes of random uniformly distance expanding and hyperbolic dynamical systems (see [8], [9] for an LCLT and Ch.7 of [13] for an LCLT and a Berry-Esseen theorem). These results rely on certain types of analysis of complex transfer operators, and they did not cover, for instance random non-uniformly distance expanding maps.

A related, but more general, setup is the case when the underlying sequence of random variables has the form $X_n = T_0^n x_0$, where $x_0$ is some random variable and $T_0^n = T_{n-1} \circ T_{n-2} \circ \cdots \circ T_0$ for some given sequence of maps $T_0, T_1, T_2, \ldots$.
Results in this direction where obtained, for instance, in [1], [4], [21] and references therein. Note that the setup of random dynamics is a special case of this setup, where $T_j = T_{\theta_j, \omega}$ are stationary random maps. In [7] and [17] the authors proved an almost sure central limit theorem for random and sequential dynamical systems, which means that the underlying partial Birkhoff sums can be approximated by a sum of independent Gaussian random variables with an error term which is smaller than the square root of the variance of the partial sum (such an estimate yields the law of iterated logarithm). Both papers invoked a recent result on almost sure central limit theorem for reverse martingales due to C. Cuny and F. Merlevède [5], and assumed that the underlying transfer operators preserve the same probability measure (“the conformal case”), which essentially means the exponents of the underlying potential function is the inverse Jacobian.

In this paper we will prove an almost sure central limit theorem, often referred to as an almost sure invariance principle (ASIP), for partial sums of the form $S_n = \sum_{j=0}^{n-1} u_j \circ T_0^j$, where $u_j$ is sequence of Hölder continuous or differentiable functions, and each $T_j$ is a non-uniformly distance expanding map satisfying the conditions in [3] and [24]. We will require a certain type of linear growth of the variance of $S_n$, where the main applications we have in mind is the case when the $T_j$’s belong to some $C^1$-neighborhood of a single map of this kind. Our results will also hold true for certain maps with singularities (in which the maps are not differential) such as interval maps which are only piecewise expanding on some pieces of the unit interval.

We will not obtain the ASIP using approximation by reverse martingales, and instead we will apply the spectral method of Gouëzel [12]. This method requires the underlying sequences of complex transfer operators to have certain “spectral” properties, which are obtained using the theory of complex projective metrics developed in [23], which was applied in Ch. 5 of [13] with uniformly distance expanding maps. Once the appropriate contraction properties of the projective diameter of the image of the underlying (real) transfer operator are established, in principle, it is possible in the conformal case (which do not assume) to obtain an ASIP under slower growth rates of the variances (see Remark 2.5), but this results in very strong restrictions on the Jacobian of $T_j$, and it is unclear which maps $T_j$ satisfy the required conditions (it is also unclear under which conditions the variance grow sub-linearly, but sufficiently fast). Moreover, using complex projective metrics, a Berry-Esseen theorem, a local central limit theorem and moderate and (local) large deviations principles for random non-uniformly distance expanding dynamical systems follow, which is another advantage of this method.

This paper is organized as follows. In the Section 2 we describe our results concerning the ASIP. Section 3 contains our main tool: we obtain there a sequential Ruelle-Perron-Frobenius (RPF) theorem for an appropriate sequence of parametrized complex transfer operators (by applying contraction properties of complex cones). In Section 4 we show how Gouëzel’s method yields the desired ASIP. Once a complex RPF theorem is established, the arguments in [13] also yields additional limit theorems, which are described in this section 4.3.
2. Preliminaries and main results

Let \((X_j, \rho_j), j \in \mathbb{Z}\) be a two sided sequence of bounded metric spaces, normalized in size so that \(\text{diam}(X_j) \leq 1\), and let \(T_j : X_j \to X_{j+1}\) be a sequence of maps satisfying the following

2.1. Assumption. There exist two sided sequences \((L_j), (\sigma_j), (q_j)\) and \((d_j)\) so that \((L_j)\) is bounded, for each \(j\) we have \(\sigma_j > 1\), \(q_j, d_j \in \mathbb{N}\), \(q_j < d_j\) and for any \(x, x' \in X_{j+1}\) we can write

\[
T_j^{-1}\{x\} = \{x_1, \ldots, x_{d_j}\} \quad \text{and} \quad T_j^{-1}\{x'\} = \{x'_1, \ldots, x'_{d_j}\}
\]

where for any \(i = 1, 2, \ldots, q_j\),

\[
\rho_j(x_i, x'_i) \leq L_j \rho_{j+1}(x, x')
\]

while for any \(i = q_j + 1, \ldots, d_j\),

\[
\rho_j(x_i, x'_i) \leq \sigma_j^{-1} \rho_{j+1}(x, x').
\]

In most of the applications we have mind the sequence \(d_j\) will be bounded, and the sequence \(\sigma_j - 1\) will be bounded from below by some positive constant, but our results can be formulated without these additional boundedness assumptions. The main example we have in mind is the case where \(X_j = M\) are the same compact and connected Riemannian manifold and each \(T_j = f_j : M \to M\) is a diffeomorphism satisfying the conditions in \([3]\) and \([24]\) (with constant which are uniform in \(j\)). These conditions are satisfied when all the maps \(T_j\) lie in some \(C^1\)-neighborhood of some map \(T\) satisfying the above conditions. Another model we have in mind is the case when each \(T_j\) is an interval map, which is piecewise expanding on some pieces of the unit interval, while on the other pieces \(T_j\) only has an inverse branch which is Lipschitz continuous (similar multidimensional examples with singularities can be given).

Next, let \(\alpha \in (0, 1]\) and let \(\phi_j\) be a sequence of bounded real valued Hölder functions on \(X_j\) with exponent \(\alpha\). Denote by \(H_j\) the space of such functions equipped with the norm

\[
\|g\| = \|g\|_\infty + v(g)
\]

where \(\|g\|_\infty = \sup |g|\) and \(v(g)\) is the smallest number so that \(|g(x) - g(y)| \leq v(g)(\rho_j(x, y))^{\alpha}\). In the case when \(\alpha = 1\) and each \(X_j\) is a Riemannian manifold we will also consider the norms \(\|\cdot\|\) on the space of \(C^1\) functions, where \(v(g)\) above is replaced by the supremum norm of the deferential of \(g\) (so in this case \(v(g)\) could either be the Lipschitz constant or the supremum norm of the deferential). Similarly to \([3]\), our additional requirements from the function \(\phi_j\) is summarized in the following

2.2. Assumption. We have \(\sup_j \|\phi_j\| < \infty\),

\[
\sup \phi_j - \inf \phi_j \leq \varepsilon_j \quad \text{and} \quad \sup \left( \sum_{y \in T_j^{-1}\{x\}} e^{\phi_j(y)} \right) < \infty,
\]

where \(\varepsilon_j\) is a sequence of positive constants satisfying

\[
s := \sup_j e^{\varepsilon_j q_j L_j^\alpha} (d_j - q_j) \sigma_j^{-\alpha} < 1.
\]
The inequality (2.4) is a quantitative estimate on the amount of contraction is allowed, given the amount of expansion we have in the system.

Next, let \( u_i \) be a sequence of functions so that \( u_j \in \mathcal{H}_j \) for each \( j \) and the sequence of norms \( \|u_j\| \) is bounded in \( j \). For each \( n \) and \( j \) write \( S_j, n u = \sum_{i=0}^{n-1} u_{i+j} \circ T_j^i \),

where \( T_j^i = T_j^{n+i} \circ T_j^{n-1} \circ \cdots \circ T_j \). Let \( \mu_j \) be the sequence of equivariant Gibbs measures (i.e. \( (T_j)_\# \mu_j = \mu_{j+1} \)) constructed in Theorem 3.1 so that (3.3) and (3.4) hold true. Henceforth we will refer to \( \sup_j \|u_j\| \) and the constants and least upper bounds in Assumptions 2.1 and 2.2 as the “initial parameters”.

Our approach is based on the following Ruelle-Perron-Frobenius theorem:

For each \( j \in \mathbb{Z} \) and \( z \in \mathbb{C} \), consider the transfer operators \( L_z^{(j)} \) which maps functions on \( X_j \) to functions on \( X_{j+1} \) by the formula

\[
L_z^{(j)}(g)(x) = \sum_{y \in T_j^{-1}(x)} e^{\phi(y) + z u_j(y)} g(y).
\]

Our main result here is the following:

2.3. Theorem. Suppose that Assumptions 2.1 and 2.2 hold true and that there exists a constant \( \delta_0 > 0 \) so that \( \inf_k \var_{\mu_k}(S_{k,n} u) \geq \delta_0 n \) for any sufficiently large \( n \).

Then for any \( \delta > 0 \) there exists a sequence of centered Gaussian random variables \( Z_1, Z_2, \ldots \) so that

\[
\left| \sum_{j=0}^{n-1} u_j \circ T_0^j(x) - \mu_0 \left( \sum_{j=0}^{n-1} u_j \circ T_0^j \right) - \sum_{j=1}^{n} Z_j \right| = o(n^{1/4 + \delta})
\]

where \( x \) is a \( X_0 \) valued random variable whose distribution is \( \mu_0 \) (or with respect to \( \nu_0 \) from Theorem 2.1).

The uniform lower bound on variances can be obtained in the following circumstances:

2.4. Theorem. Suppose that \( X_k = X \) for each \( k \). Let \( T : X \to X \) be a map so that Assumption 2.1 is satisfied with \( T_j = T \), and let \( \phi, u \in \mathcal{H}_j = \mathcal{H} \) be so that \( u \) does not admit a co-boundary representation with respect to \( T \) and that Assumption 2.2 holds true with \( T_j = T \) and \( \phi_j = \phi \). Then there exists \( \varepsilon_0 > 0 \), which depends only on the initial parameters, so that the following holds true: if

\[
\sup_{k \in \mathbb{Z}} \|L_z^{(k)} - L_z\| \leq \varepsilon_0
\]

for any \( z \) in some neighborhood of 0, where \( L_z \) is the transfer operator generated by \( T \) and \( f + zu \), then

\[
\inf_k \var_{\mu_k}(S_{k,n} u) \geq \delta_0 n
\]

for some \( \delta_0 > 0 \) and all sufficiently large \( n \). The constant \( \delta_0 \) depends only on the initial parameters and on the limit \( \sigma_u^2 = \lim_{n \to \infty} \var_{\mu_u} \left( \sum_{j=0}^{n-1} u \circ T_j \right) \), where \( \mu_u \) is the Gibbs measure corresponding to the map \( T \) and the potential \( \phi \).

This theorem is a particular case of Theorem 2.9 (ii) in [15]. When \( X_j = M \) are all the same Riemannian manifold, the maps \( T_j \) satisfy the conditions from [8] and they lie in some \( C^1 \)-neighborhood of a map satisfying these conditions, and the functions \( \phi_j \) and \( u_j \) all lie in some balls (in the \( C^1 \)-norm) around \( \phi \) and \( u \), respectively, then (2.3) is satisfied with the norm \( \|g\| = \sup |g| + \sup |Dg| \) (i.e. we take \( \alpha = 1 \), see Proposition 5.3 in [3]). Another example is intervals maps with finite
number of monotonicity intervals which do not depend on $j$, where on each one of them each $T_j$ and $T$ are either expanding or contracting. If each $T_j$ is obtained from $T$ by perturbing each inverse branch of $T$ in some Hölder norm, and $\phi_j$ and $u_j$ are small perturbations of $\phi$ and $u$ in this norm, then (2.3) will hold true in the appropriate Hölder norm. Similar examples can be given for maps whose inverse branches are defined on certain rectangular regions in $\mathbb{R}^d$ for $d > 1$.

2.5. Remark. When $X_j = M$ and the functions $\phi_j = -\ln J(T_j)$ satisfy Assumption 2.2, where $J(T_j)$ is the Jacobian of the map $T_j$, then the normalized volume measure $m$ is preserves under each $L_j^{(j)}$. In this case we can apply Theorem 3.1 from [17] and derive and ASIP under the weaker assumption that $\text{var}_\nu(S_{0,n},u) \geq n^{\frac{d}{2} + \delta}$ for some $\delta > 0$ and any sufficiently large $n$ (this theorem uses reverse martingale approximation). Still, it is unclear under which conditions such a growth rate can be obtained in our context (expect in the circumstances of Theorem 2.4), and these conditions on the Jacobian are quite restrictive, and it is not clear which maps $T_j$ satisfy them (since now $\phi_j$ and the constants $d_j, L_j, q_j$ and $\sigma_j$ depend on the map $T_j$). Moreover, martingale approximations do not yield the additional results obtained in Section 3.3 (while the tools developed in order to prove Theorem 2.3 yield many additional statistical properties).

3. A sequential RPF theorem via cones contractions

Let $L_j^{(j)}$ be the transfer operators defined by (2.2), where $j \in \mathbb{Z}$ and $z \in \mathbb{C}$. We also set $L_0^{(j)} = L^{(j)}$. For each $j, n$ and $z$ write

$$L_z^{j,n} = L_z^{(j+n-1)} \circ \ldots \circ L_z^{(j+1)} \circ L_z^{(j)}.$$ 

It is clear that $L_z^{(j)} \mathcal{H}_j \subset \mathcal{H}_{j+1}$. We will denote by $(L_z^{(j)})^*$ the appropriate dual operator.

3.1. Theorem. Suppose that Assumptions 2.1 and 2.2 hold true. Then there exists a neighborhood $U$ of 0, which depends only on the initial parameters, so that for any $z \in U$ there exist families $\{\lambda_j(z) : j \in \mathbb{Z}\}$, $\{h_j^{(z)} : j \in \mathbb{Z}\}$ and $\{\nu_j^{(z)} : j \in \mathbb{Z}\}$ consisting of a nonzero complex number $\lambda_j(z)$, a complex function $h_j^{(z)} \in \mathcal{H}_j$ and a complex continuous linear functional $\nu_j^{(z)} \in \mathcal{H}_j^*$ such that:

(i) For any $j \in \mathbb{Z}$,

$$(3.1) \quad L_j^{(z)}h_j^{(z)} = \lambda_j(z)h_{j+1}^{(z)}, \quad (L_j^{(z)})^*\nu_j^{(z)} = \lambda_j(z)\nu_{j+1}^{(z)} \quad \text{and} \quad \nu_j^{(z)}(h_j^{(z)}) = \nu_j^{(z)}(1) = 1$$

where 1 is the function which takes the constant value 1. When $z = t \in \mathbb{R}$ then $\lambda_j(t) > a$ and the function $h_j(t)$ takes values at some interval $[c, d]$, where $a > 0$ and $0 < c < d < \infty$ depend only on the initial parameters. Moreover, $\nu_j^{(t)}$ is a probability measure which assigns positive mass to open subsets of $\mathcal{E}_j$ and the equality $\nu_{j+1}(t)(L_j^{(z)}g) = \lambda_j(t)\nu_j^{(z)}(g)$ holds true for any bounded Borel function $g : \mathcal{E}_j \to \mathbb{C}$.

(ii) The maps

$$\lambda_j(\cdot) : U \to \mathbb{C}, \quad h_j^{(\cdot)} : U \to \mathcal{H}_j \quad \text{and} \quad \nu_j^{(\cdot)} : U \to \mathcal{H}_j^*$$
are analytic and there exists a constant \( C > 0 \), which depends only on the initial parameters such that

\[
\max \left( \sup_{z \in U} |\lambda_j(z)|, \sup_{z \in U} \| h_j^{(z)} \|, \sup_{z \in U} \| \nu_j^{(z)} \| \right) \leq C,
\]

where \( \| \nu \| \) is the operator norm of a linear functional \( \nu : \mathcal{H}_j \to \mathbb{C} \). Moreover, there exist a constant \( c > 0 \), which depends only on the initial parameters, so that \( |\lambda_j(z)| \geq c \) and \( \min_{x \in \mathcal{E}_j} |h_j^{(z)}(x)| \geq c \) for any integer \( j \) and \( z \in U \).

(ii) There exist constants \( A > 0 \) and \( \delta \in (0,1) \), which depend only on the initial parameters, so that for any \( j \in \mathbb{Z} \), \( g \in \mathcal{H}_j \) and \( n \geq 1 \),

\[
\left\| \frac{\mathcal{L}^n_{j,n} g}{\lambda_{j,n}(z)} - \nu_j^{(z)}(g) h_j^{(z)} + n \right\| \leq A \|g\| \delta^n
\]

where \( \lambda_{j,n}(z) = \lambda_j(z) \cdot \lambda_{j+1}(z) \cdots \lambda_{j+n-1}(z) \). Moreover, the probability measures \( \mu_j \), \( j \in \mathbb{Z} \) given by \( d\mu_j = h_j^{(0)} d\nu^{(0)} \) satisfy that \( (T_j)_* \mu_j = \mu_{j+1} \) and that for any \( n \geq 1 \) and \( f \in \mathcal{H}_{j+n} \),

\[
|\mu_j(g \cdot f \circ T_j^n) - \mu_j(g) \mu_{j+n}(f)| \leq A \|\mu_{j+n}\| \delta^n.
\]

We note that, as in [14], when the \( \{T_j\} \) are “sequentially non-singular” the measures \( \mu_j \) are absolutely continuous, then we have the following

3.2. Proposition. Let \( m_j \), \( j \in \mathbb{Z} \) be a family of probability measures on \( \mathcal{E}_j \), which assign positive mass to open sets, so that for each \( j \) we have \( (T_j)_* m_j \ll m_{j+1} \) and that \( e^{-f_j} = \frac{d(T_j)_* m_j}{dm_{j+1}} \). Then for any \( j \) we have \( \lambda_j(0) = 1 \) and \( \nu_j^{(0)} = m_j \).

The proof of Theorem 3.1 relies on the theory of real and complex cones. We will give a reminder of the appropriate results concerning this theory in the body of the proof of Theorem 3.3 below, and the readers are referred to Appendix A of [13] for a summary of the main definitions and results concerning contraction properties of real and complex cones.

Theorem 3.1 follows from the following

3.3. Theorem. There exist \( r, d_0 > 0 \) and a sequences \( C_j \) of complex cones so that:

(i) The cones \( C_j \) and their duals \( C_j^* := \{ \nu \in \mathcal{H}_j^* : \nu(c) \neq 0 \ \forall \nu \in \mathcal{C}_j \setminus \{0\} \} \) have bounded aperture: there exists a constant \( A > 0 \) and complex continuous linear functionals \( a_j \in \mathcal{H}_j^* \) and \( b_j \in (\mathcal{H}_j^*)^* \) so that for any \( g \in C_j \) and \( \lambda \in C_j^* \) we have

\[
\|g\| \leq A|a_j(g)| \quad \text{and} \quad \|\lambda\| \leq C|b_j(\lambda)|.
\]

(ii) The cones \( C_j \) are linearly convex, namely for any \( g \notin C_j \) there exists \( \mu \in C_j^* \) such that \( \mu(g) = 0 \).

(iii) The cones \( C_j \) are reproducing: there exist constants \( k_0 \in \mathbb{N} \) and \( r_0 > 0 \) so that for any \( j \) and \( g \in \mathcal{H}_j \) there are \( g_1, \ldots, g_{k_0} \in C_j \) so that \( g = g_1 + \ldots + g_{k_0} \) and

\[
\|g_1\| + \ldots + \|g_{k_0}\| \leq r_0\|g\|.
\]

(iv) For any \( j \in \mathbb{Z} \), and \( z \in \mathbb{C} \) so that \( |z| < r \) we have

\[
\mathcal{L}_j C_j \subset C_{j+1}
\]

and the Hilbert diameter of the image with respect to the complex projective metric corresponding to the cone \( C_{j+1} \) does not exceed \( d_0 \).

Relying on this theorem, Theorem 3.1 follows exactly as in Chapters 4 and 5 [13].
Proof of Theorem 3.3. Let $\delta > 0$ be so that $(1 + \delta)s < 1$, where $s$ is defined in (2.1), and let $\kappa > 0$ be so that $\sup_j v(\phi_j) < \kappa \delta$. Consider the real cone

$$C_{j,\mathbb{R}} = \{ g \in \mathcal{H}_j : g > 0 \text{ and } v(g) \leq \kappa \inf g \}$$

and let $C_j$ be its canonical complexification which (see Appendix A in [13]) is given by

$$C_j = \{ g \in \mathcal{H}_j : \text{Re}(\mu(g)\nu(g)) \geq 0 \quad \forall \mu, \nu \in C_{j,\mathbb{R}}^* \}$$

where $C_{j,\mathbb{R}}^* = \{ \mu \in \mathcal{H}_j^* : \mu(c) \geq 0 \quad \forall c \in C_{j,\mathbb{R}} \}$.

We begin with showing that the complex cones $C_j$ and their duals have bounded aperture. First, for any point $a \in \mathcal{X}_j$ and $g \in C_{j,\mathbb{R}}$ we have

$$\|g\| = \sup g + v(g) \leq \inf g + 2v(g) \leq (1 + 2\kappa)\inf g \leq (1 + 2\kappa)g(a)$$

where we used that $g(x) - g(y) \leq (\text{diam}(\mathcal{X}_j))^a v(g) \leq v(g)$ for any real valued function on $\mathcal{X}_j$. We conclude from Lemma 5.2 in [23] that for any $g \in C_j$ we have

$$\|g\| \leq 2\sqrt{2}(1 + 2\kappa)g(a)$$

and therefore we can take $a_j(g) = g(a)$ for an arbitrary point $a \in \mathcal{X}_j$. Next, in order to show that the cone $C_j$ has bounded aperture we will apply Lemma A.2.7 from [13] which states that

$$\|\nu\| \leq M\nu(1), \quad \forall \nu \in C_j^*$$

if the complex cone $C_j$ contains the ball of radius $1/M$ around the constant function $1$. The first step in showing that such a ball exists is the following representation of the cone:

$$C_{j,\mathbb{R}} = C_{j,\mathbb{R},\kappa} = \{ g \in \mathcal{H}_j : s_{x,\mu,\kappa}(g) \geq 0, \quad \forall (x, y, \mu, \nu) \in \Delta_j \}$$

where $\Delta_j$ is the set of triplets $(x, y, \mu, \nu) \in \mathcal{X}_j \times \mathcal{X}_j \times \mathcal{X}_j$ so that $x \neq y$ and

$$s_{x,\mu,\nu}(g) = \kappa \mu(t) - \frac{g(x) - g(y)}{\rho_j^*(x, y)}.$$ 

Then (see Appendix A in [13]), we can write

$$C_j = \{ x \in \mathcal{H}_j : \text{Re}(\mu(x)v(x)) \geq 0 \quad \forall \mu, \nu \in \Delta_j \}$$

since $\Delta_j$ generates the dual cone $C_{j,\mathbb{R}}^*$. Note that by Lemma 4.1 in [11], a canonical complexification $C_J$ of a real cone $C_{\mathbb{R}}$ is linearly convex if there exists a continuous linear functional which is strictly positive on $C_{\mathbb{R}}^* = C_{\mathbb{R}} \setminus \{0\}$.

Using (3.6), it is enough to find $\varepsilon > 0$ which does not depend on $j$ so that for any $g$ of the form $g = 1 + h$ with $\|h\| < \varepsilon$, and any $(x_i, y_i, t_i) \in \Delta_j$ for $i = 1, 2$,

$$\text{Re}(s_1(g) \cdot s_2(g)) \geq 0$$

where with $s_i = s_{x_i, y_i, t_i, \kappa}$. This is indeed enough since then we can take $M = 1/\varepsilon$. Existence of such $\varepsilon$ is clear since $s_i(g) = \kappa - s_i(h)$ and $|s_i(h)| \leq (\kappa + 1)\|h\|$. The cone $C_j$ is linearly convex since the real cone $C_{j,\mathbb{R}}$ has bounded aperture (so (ii) holds true).

Now we will prove (iii). If $g \in \mathcal{H}_j$ is real valued then $g + c_g \in C_{j,\mathbb{R}} \subset C_j$, where $c_g = \max(\sup|g|, \nu(g)/\kappa)$. Then $g = (g + c_g) - c_g$ is a sum of two members of $C_j$ so that

$$\|g + c_g\| + \| - c_g\| \leq 3(1 + \kappa^{-1})\|g\|.$$
In order to obtain the right estimate for complex valued functions we just decompose them as \( g = g_1 + ig_2 \) where \( g_1, g_2 \in \mathcal{H}_j \) are real valued.

Now we will prove (iv). We will first show that for any \( j \),

\[
\mathcal{L}^{(j)} \mathcal{C}_{j, \mathbb{R}} \subset \mathcal{C}_{j+1, \mathbb{R}, \zeta, \kappa}
\]

where \( \zeta = (1 + \delta)s < 1 \), where \( \delta \) was specified at the beginning of the proof of Theorem 4.3. This is done almost exactly as in the proof of Theorem 5.1 in [3], and the details are given below for readers' convenience. Let \( g \in \mathcal{C}_j \) and let \( x, y \in \mathcal{X}_{j+1} \). Denote by \((x_i), (y_i)\) the inverse images of \( x \) and \( y \) under \( T_j \), respectively. We have

\[
\frac{|\mathcal{L}^{(j)} g(x) - \mathcal{L}^{(j)} g(y)|}{\inf \mathcal{L}_0^{(j)}} \leq \frac{1}{d_j} \sum_{i=1}^{d_j} e^{\inf \phi_j(x_i)}|g(x_i) - g(y_i)| \leq d_j^{-1} \sum_{i=1}^{d_j} e^{\inf \phi_j(x_i)}|g(x_i) - g(y_i)| (\inf g)^{-1}
\]

\[
+d_j^{-1} \sum_{i=1}^{d_j} |(g(y_i)/\inf g) e^{-\inf \phi_j(x_i)}|e^{\inf \phi_j(x_i)} - e^{\phi_j(x_i)}| := I_1 + I_2
\]

Where \( \mathcal{L}^{(j)} = \mathcal{L}_0^{(j)} \). Since \( \rho_j(x_i, y_i) \leq L_j \rho_j(x, y) \) for any \( 1 \leq i \leq q_j \) and \( \rho_j(x_i, y_i) \leq \sigma_j^{-1}\rho_j(x, y) \) for all other preimages,

\[
I_1 \leq \rho_j^{q_j+1}(x, y) e^{\frac{\sigma_j^{-1}d_j}{2}}(L_j^\alpha q_j + (d_j - q_j)\sigma_j^{-\alpha}) = \rho_j^{q_j+1}(x, y)s\kappa
\]

where \( s \) is defined in (2.1), and we used that \( |g(x_i) - g(y_i)| \leq v(g)\rho_j^q(x_i, y_i) \leq \kappa \inf g \cdot \rho_j^q(x_i, y_i) \).

In order to bound \( I_2 \), we first observe that \( \sup g \leq \inf g + v(g) \leq (1 + \kappa)\inf g \) and that

\[
|e^{\phi_j(x_i)} - e^{\phi_j(y_i)}| \leq e^{\max(\phi_j(x_i), \phi_j(y_i))}|\phi_j(x_i) - \phi_j(y_i)| \leq e^{\inf \phi_j + \varepsilon_j} v(\phi_j)\rho_j^q(x_i, y_i).
\]

Using these estimates we obtain that

\[
I_2 \leq \rho_j^{q_j+1}(1 + \kappa)s \sup_j v(\phi_j).
\]

We conclude that

\[
v(\mathcal{L}^{(j)} g) \leq s(\kappa + \sup_j v(\phi_j)) \inf \mathcal{L}_0^{(j)} \leq s\kappa(1 + \delta) \inf \mathcal{L}_0^{(j)} = \zeta \inf \mathcal{L}_0^{(j)}.
\]

and therefore

\[
(3.7) \quad \mathcal{L}^{(j)} \mathcal{C}_{j, \mathbb{R}, \kappa} \subset \mathcal{C}_{j, \mathbb{R}, \zeta, \kappa} \subset \mathcal{C}_{j, \mathbb{R}, \kappa}.
\]

By Proposition 5.2 in [3] (see the proof of Proposition 4.3 from there), there exists \( d_0 \) which depends only on \( \kappa \) and \( \zeta \) so that the real projective diameter of \( \mathcal{C}_{j, \mathbb{R}, \zeta, \kappa} \) as a subset of \( \mathcal{C}_{j, \mathbb{R}, \kappa} \) does not exceed \( d_0 \).

We will next prove that for any \( j, \kappa, (x, y, t) \in \Delta_j \), \( g \in \mathcal{C}_{j, \mathbb{R}} \) and a complex \( z \) so that \( |z| \leq 1 \) we have

\[
(3.8) \quad |s^{x,y,t,\kappa}(\mathcal{L}_z^{(j)}) g - \mathcal{L}_0^{(j)} g| \leq c|z| s^{x,y,t,\kappa}(\mathcal{L}_0^{(j)} g)
\]

where \( c \) is some constant which does not depend on \( j \). After this is established we can apply Theorem A.2.4 from Appendix A in [13] and obtain item (iv).

We begin with the following simple result/observation: let \( A \) and \( A' \) be complex numbers, \( B \) and \( B' \) be real numbers, and let \( \varepsilon_1 > 0 \) and \( \zeta \in (0, 1) \) so that
The proof of this results is elementary, just write $B > B'$

Then $|A - A'| < 2\varepsilon_1(1 - \zeta)^{-1}$.

The proof of this results is elementary, just write

$$\frac{|A - A'|}{B - B'} - 1 \leq \frac{|A - B|}{B - B'} + \frac{|A' - B'|}{B - B'} \leq \frac{2\varepsilon_1}{B - B'} = \frac{2\varepsilon_1}{1 - B'/B}.$$

Fix some nonzero $g \in C_{j,r}$ and $(x, y, t) \in \Delta_{j+1}$. We want to apply the above results with $A = \kappa \mathcal{L}_z^{(j)}g(t)$,

$$B = \kappa \mathcal{L}_z^{(j)}g(t), \quad A' = \frac{\mathcal{L}_z^{(j)}g(x) - \mathcal{L}_z^{(j)}g(y)}{\rho_j^a(x, y)} \quad \text{and} \quad B' = \frac{\mathcal{L}_z^{(j)}g(x) - \mathcal{L}_z^{(j)}g(y)}{\rho_j^b(x, y)}.$$

We begin with noting that $B > B'$ since the function $\mathcal{L}_z^{(j)}g$ is a nonzero member of the cone $C_{j,r,c_k}$. Notice that when $|z| \leq 1$ we have

$$|A - B| = \kappa|\mathcal{L}_z^{(j)}g(t) - \mathcal{L}_z^{(j)}g(t)| = \kappa|\mathcal{L}_z^{(j)}(g(e^{z\phi_j} - 1))|$$

$$\leq \kappa\|e^{z\phi_j} - 1\|_\infty \mathcal{L}_zg(t) \leq |z|\|\phi_j\|_\infty B \leq C|z|B$$

for some constant $C > 0$, where we used that $\sup \|\phi_j\|_\infty < \infty$. Next, we have

$$|B'/B| \leq \zeta \inf \mathcal{L}_z^{(j)}g/B \leq \zeta < 1$$

where we used that $\mathcal{L}_z^{(j)}g$ is a nonzero member of the cone $C_{j,r,c_k}$. Finally, we will estimate the difference $|A' - B'|$. For each $a, b \in \mathcal{X}_j$ we define

$$\Delta_{a,b}(z) = e^{\phi_j(a)}(e^{z\phi_j(a)} - 1)g(a) - e^{\phi_j(b)}(e^{z\phi_j(b)} - 1)g(b).$$

Denote by $x_i$ and $y_i$ the preimages of $x$ and $y$ under $T_j$, respectively, where $1 \leq i \leq d_j$. Then

$$\rho_j^a(x, y)(A' - B') = \sum_{i=1}^{d_j} \Delta_{x_i, y_i}(z).$$

We first have

$$|\Delta_{a,b}(z)| = |\Delta_{a,b}(z) - \Delta_{a,b}(0)| \leq |z| \sup_{|q| \leq |z|} |\Delta_{a,b}'(q)|$$

where $\Delta_{a,b}'(\cdot)$ is the derivative of $\Delta_{a,b}(\cdot)$. Next, since the function $e^{\phi_j}$ satisfies

$$|e^{\phi_j(a)} - e^{\phi_j(b)}| \leq (e^{\phi_j(a)} + e^{\phi_j(b)})v(\phi_j)\rho_j^a(a, b)$$

using that the sequence $(L_j)$ appearing in Assumption 2.1 is bounded, we obtain that for any $|q| \leq 1$ and $1 \leq i \leq d_j$,

$$|\Delta'_{x_i, y_i}(q)| \leq C L_j(e^{\phi_j(x_i)} + e^{\phi_j(y_i)})\|g\|_\infty \rho_j^a(x, y)$$

where $C > 0$ is some constant. We conclude that there exists a constant $C > 0$ so that for any $j \in Z$ and $z \in \mathbb{C}$ with $|z| \leq 1$,

$$|A' - B'| \leq C|z|\|g\|_\infty (\mathcal{L}_z^{(j)}1(x) + \mathcal{L}_z^{(j)}1(y)) \leq C_1|z| \inf g$$
where we used that $\sup_j \|L_j^{(j)}1\|_{\infty} < \infty$. Since $\|\phi_j\|$ is bounded in $j$ there exists a constant $C_2 > 0$ so that $\inf g \leq C_2L_j^{(j)}g(t) = C_2B$ for any $j$. This completes the proof of (3.8). Applying Theorem A.2.4 in Appendix A of [13] we complete the proof of (iv). □

4. Almost sure invariance principle

4.1. Gouëzel’s method. Let $A_1, A_2, \ldots$ be a sequence of random variables which is bounded in $L^p$ for some $p > 2$. We recall assumption (H) from [12]: there exists $\varepsilon_0 > 0$ and $C, c > 0$ such that for any $n, m > 0$, $b_1 < b_2 < \ldots < b_{n+m+k}$, $k > 0$ and $t_1, \ldots, t_{n+m} \in \mathbb{R}$ with $|t_j| \leq \varepsilon_0$, we have

\[
\begin{align*}
&\left| E\left( e^{i\sum_{j=1}^{n+m} t_j(\sum_{\ell=b_j}^{b_{j+1}-1} A_\ell) + i\sum_{j=n+1}^{n+m} t_j(\sum_{\ell=b_j+k}^{b_{j+1+k}-1} A_\ell)} \right) \\
&- E\left( e^{i\sum_{j=1}^{n} t_j(\sum_{\ell=b_j}^{b_{j+1}-1} A_\ell)} \cdot E\left( e^{i\sum_{j=n+1}^{n+m} t_j(\sum_{\ell=b_j+k}^{b_{j+1+k}-1} A_\ell)} \right) \right) \right| \\
&\leq C(1 + \max |b_{j+1} - b_j|)C(n+m)e^{-ck}.
\end{align*}
\]

We will use here the following slight modification of the results of Gouëzel:

4.1. Theorem. Suppose that $A_1, A_2, \ldots$ is a sequence of centered random variables satisfying Assumption (H) which is bounded in $L^p$ for some $p > 2$. Assume, in addition, that there exist constants $0 < a$ and $b \in \mathbb{N}$ so that for any $n$ and $m \geq b$,

\[
\inf_n \text{Var}\left( \sum_{j=n+1}^{n+m} A_j \right) \geq am.
\]

Then for any $\varepsilon > 0$, by possibly enlarging the probability space there exists a sequence of independent centered normal random variable $(B_j)$ so that $P$-a.s as $n \to \infty$,

\[
\left| \sum_{j=1}^{n} (A_j - B_j) \right| = o(n^{a_p+\varepsilon})
\]

where $a_p = \frac{P}{4(p-1)}$ (which converges to $1/4$ when $p \to \infty$).

In [12] there is an assumption concerning the rate of convergence towards the asymptotic variance, which in our situation we do not have (the asymptotic variance may not exist). Still, this assumption was only used in the third step in Section 5 of [12], as well as in the proof of Lemma 5.6 from there. In our situation we can just skip the third step, whose purpose is to prescribe the variance of the approximating Gaussian random variable. Moreover, taking a careful look at the proof of Lemma 5.6 from there we see that it only relied on a lower bound of the form (4.1).

4.2. A sequential ASIP. We will show that the conditions of Theorem 4.1 hold true. Condition (4.1) is just (2.4), so we only need to show that condition (H) is satisfied. We first note that, exactly at the end of the proof of Lemma 7.4.1 in [13], there exists $\varepsilon > 0$ and $d > 0$ so that for any $j$, $n$ and $t \in [-\varepsilon, \varepsilon_0]$, we have

\[
|\lambda_{j,n}(it)| \leq |\lambda_{j,n}(0)|e^{-dn^2}
\]

where $\sigma_{j,n}^2$ we used that $\sigma_{j,n}^2 \geq \delta_0 n$. Hence, by (3.3) we have

\[
\sup_{j,n} \|L_j^{(j)}\|/\lambda_{j,n}(0) \leq C_0.
\]

\[
(4.2)
\]
Note that in our circumstances we do not have an apriori upper bound on $\|L_{it}^{jn}\|$, and so we need to use (3.3) to derive such estimates. Next, we define the transfer operators $\tilde{L}_{it}^{jn}$ by the formula

$$\tilde{L}_{it}^{jn} g = L_{it}^{jn} (gh_j^{(0)}) / \lambda_j^{(0)} h_j^{(0)}.$$ 

Then $\tilde{L}_0 1 = 1$, where $1$ is the function which takes the constant value $1$ (we will use this notation regardless of the space this function is defined on). We will also write $\tilde{L}_{it}^{jn} = L_{it}^{j+n-1} \circ \cdots \circ L_{it}^{j+1} \circ L_{it}^{j}$. Then by (3.3),

$$(4.3) \quad \sup_{j,n} \|L_{it}^{jn}\| / \lambda_j^{(0)} \leq C_1$$

where $C_1$ is some constant, and by (3.3) we have

$$(4.4) \quad \|\tilde{L}_{it}^{jn}(g) - \mu_j(b)\| \leq C_2 \|g\| \delta^n$$

where $C_2$ is some constant and $\delta \in (0, 1)$. Moreover, for any $j$ and $n$,

$$\mu_j(e^{it \sum_{k=0}^{n-1} u_{j+k} \sigma_j^k}) = \mu_j(\tilde{L}_{it}^{jn} 1).$$

We will also denote by $M_j$ the one dimensional projection given by $M_j(g) = \mu_j(g)$. Next, we assume that $\mu_j(u_j) = 0$ for any $j$. We will show next that condition (H) of Gouëzel [12] holds true with $A_t = u_t \circ T_0$. Indeed, for any $t_i \in [-\varepsilon_0, \varepsilon_0]$, $(b_i)$ and $k > 0$ we have

$$\mu_0(e^{it \sum_{j=1}^{n} t_j \sum_{\ell=1}^{n-1} A_{\ell}} + \sum_{j=0}^{n+m} t_j \sum_{\ell=1}^{n-1} A_{\ell} \circ (\prod_{j=1}^{n} \tilde{L}_{it}^{b_j+1-k}) 1) = \mu_{b_1} \left( \left( \prod_{j=n+1}^{n+m} \tilde{L}_{it}^{b_j+1-k} \right) \circ \tilde{L}_{it}^{b_{n+1-k}} \circ \left( \prod_{j=1}^{n} \tilde{L}_{it}^{b_j+1-k} \right) 1 \right)$$

$$+ \mu_{b_1} \left( \left( \prod_{j=-n-1}^{-m} \tilde{L}_{it}^{b_j+1-k} \right) \circ M_{b_{n+1-k}} \circ \left( \prod_{j=1}^{n} \tilde{L}_{it}^{b_j+1-k} \right) 1 \right) = I_1 + I_2.$$ 

Applying (4.3) and (4.4) we derive that with some constant $C > 0$ we have $|I_1| \leq C^{n+m} \delta^k$. Moreover, since $M_j = \mu_j 1$ and $\tilde{L}_{it}^{jn} 1 = 1$ we have

$$I_2 = \mu_0(e^{it \sum_{j=-n-1}^{n+m} t_j \sum_{\ell=1}^{n-1} A_{\ell}}) \cdot \mu_0(e^{it \sum_{j=n+1}^{n+m} t_j \sum_{\ell=1}^{n-1} A_{\ell}})$$

which completes the proof that condition (H) holds true with the sequence $A_t$.

4.3. Other limit theorems for random dynamical systems. Let $(\Omega, \mathcal{F}, P, \theta)$ be an ergodic and invertible measure preserving system. For the sake of simplicity we assume here that $X_j = X$ does not depend on $j$, but the case when $X_j = X_{\theta_j \omega}$, $\omega \in \Omega$, where $X_{\theta_j \omega}$ is some random compact space, can also be considered (see Chapter 5 of [13]). We will consider here the case when $T_j = T_{\theta_j \omega}$ where $T_{\omega} : X \to X$ is a random map so that the skew product $T(\omega, x) = (\theta \omega, T_{\omega} x)$ is measurable with respect to the product $\sigma$-algebra $\mathcal{F} \times \mathcal{B}$, where $\mathcal{B}$ is the Borel $\sigma$-algebra on $X$. Let $\phi(\omega, x) = \phi(\omega, \cdot)$ and $u(\omega, x)$ be two measurable functions so that $\phi_{\omega}(\cdot) = \phi(\omega, \cdot)$ and $u_{\omega}(\cdot)$ belong to $\mathcal{H} = \mathcal{H}_j$ and the norms $\|\phi_{\omega}\|$ and $\|u_{\omega}\|$ are bounded. In this case, for any $\omega$ the map $T_{\omega}$ satisfies Assumption (2.1) with constants.
Assume, in addition, that \(\int u_\omega \, d\mu = 0\), \(P\)-a.s. Then \(P\)-a.s. for any real continuous function \(g\) on \(\mathbb{R}\) with compact support,

\[
\lim_{n \to \infty} \sup_{s \in \text{supp } \nu_h} \left| \sigma \sqrt{2\pi n} \mathbb{E}_{\mu_h} g(S_n^\omega u - s) - e^{-s^2/2} \int g \, d\nu_h \right| = 0
\]
where in the lattice case \( \nu_h \) is the measure assigning mass \( h \) to each point of the lattice \( h\mathbb{Z} = \{hk : k \in \mathbb{N}\} \) while in the non-lattice case we set \( h = 0 \) and then \( \nu_0 \) is the Lebesgue measure.

Next, exactly as in Chapter 4 pf [13], we can develop a branch of the logarithm of the function \( \lambda_\omega(z) \) which will be denoted by \( \Pi_\omega(z) \). Since \( \mu_\omega(e^{zS_n u}) = \mu_\omega(\mathcal{L}_z^{u,n}(h_\omega)/h_\omega = \lambda_\omega n) \) we have that

\[
\lim_{n \to \infty} \frac{1}{n} \mu_\omega(e^{zS_n u}) = \Pi(z) := \int \ln \lambda_\omega(z) dP(\omega) - \int \ln \lambda_\omega(0) dP(\omega).
\]

Using that \( \Pi_\omega(z) \) is analytic in \( z \), a standard application of the Gärnder-Ellis theorem (see [6]) yields the following

4.5. **Theorem.** Suppose that all the above conditions hold true and that \( \sigma^2 > 0 \).

(i) Then the following (optimal) moderate deviations principle holds true: for any strictly increasing sequence \( (b_n)_{n=1}^\infty \) of real numbers so that \( \lim_{n \to \infty} b_n = 0 \) and \( \lim_{n \to \infty} \frac{b_n}{n} = \infty \) and a Borel set \( \Gamma \subset \mathbb{R} \) we have

\[
\inf_{x \in \Gamma^o} I(x) \leq \liminf_{n \to \infty} \frac{1}{a_n^2} \mu_\omega \{ x : W_n^\omega(x) \in \Gamma \} \quad \text{and} \quad \limsup_{n \to \infty} \frac{1}{a_n^2} \mu_\omega \{ x : W_n^\omega(x) \in \Gamma \} \leq \inf_{x \in \Gamma^c} I(x)
\]

where \( W_n^\omega = \frac{S_n^\omega - n \mu_\omega(S_n^\omega)}{b_n} \), \( I(x) = -\frac{x^2}{2} \), \( \Gamma^o \) is the interior of \( \Gamma \) and \( \bar{\Gamma} \) is its closer.

(ii) Let \( L(t) \) be the Legendre transform of \( \Pi(t) \). Then, (4.8) holds true for any Borel set \( \Gamma \subset [\Pi(-\delta), \Pi(\delta)] \) with \( W_n^\omega = \frac{S_n^\omega - \mu_\omega(S_n^\omega)}{n} \) and \( I(t) = L(t) \) (this is a local large deviations principle).

Observe that \( \Pi(-\delta) < \Pi(\delta) \) when \( \sigma^2 > 0 \) since then the function \( t \to \Pi(t) \) is strictly convex in some real neighbourhood of the origin.

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