Exact solutions of time-fractional generalised Burgers–Fisher equation using generalised Kudryashov method

RAMYA SELVARAJ1, SWAMINATHAN VENKATRAMAN2,∗, DURGA DEVI ASHOK3 and KRISHNAKUMAR KRISHNARAJA1

1Department of Mathematics, Srinivasa Ramanujan Centre, SASTRA Deemed to be University, Kumbakonam 612 001, India
2Discrete Mathematics Laboratory, Department of Mathematics, Srinivasa Ramanujan Centre, SASTRA Deemed to be University, Kumbakonam 612 001, India
3Department of Physics, Srinivasa Ramanujan Centre, SASTRA Deemed to be University, Kumbakonam 612 001, India
∗Corresponding author. E-mail: swaminathan@src.sastra.edu

MS received 18 March 2020; revised 14 May 2020; accepted 1 July 2020; published online 5 September 2020

Abstract. This study deals with the generalised Kudryashov method (GKM) for the time-fractional generalised Burgers–Fisher equation (TF-GBF). Using the transformation of travelling wave, the TF-GBF is transformed into a non-linear ordinary differential equation (NLODE). Later, GKM has been applied in the resultant equation which is a novel technique to obtain exact solutions. These exact solutions are plotted and the power series solution is also derived.

Keywords. Time-fractional differential equation; nonlinear differential equation; generalised Burgers–Fisher equation; Kudryashov method; power series.

PACS Nos 12.60.Jv; 12.10.Dm; 98.80.Cq; 11.30.Hv

1. Introduction

Due to its applicability in various branches of science and technology, the fractional differential equation (FDEs) plays an eminent role [1–5]. FDE has many applications in the field of magnetism, cardiac tissue–electrode interface, fluid mechanics, theory of viscoelasticity, wave propagation in viscoelastic horn, heat transfer, RLC electric circuit and so on. In recent years, for solving time-fractional differential equations (TFDEs), many researchers have proposed powerful techniques [6,7] to get an exact solution. Besides, the sine–cosine method [8,9], G′/G expansion method [10], the exp-function method [11], the tanh method [12], the subequation method [13,14], homotopy perturbation technique [15,16], the improved G′/G expansion method [17], the invariant subspace method, the generalised Riccati equation method [18], the modified Kudryashov method and some more methods are also applied [19–29].

Kudryashov method was introduced by Kudryashov [30] for reliable treatment of nonlinear (NL) wave equations. For both integer and fractional order, this method is widely used by many researchers, to find exact solutions of high order NL evolution equations (NLEEs), the Klein–Gordon equation, time-fractional KdV equations and so on [31–36]. The generalised Kudryashov method (GKM) is used to construct travelling wave solutions of several NLEEs. While comparing with other methods, the GKM is more effective and direct to construct exact solutions of high order NLEEs [37]. In this work, the GKM is applied to find exact solutions of the time-fractional generalised Burgers–Fisher equation (TF-GBFE).

A nonlinear equation which is a combination of reaction, convection and diffusion mechanism is called Burgers–Fisher equation. In the nonlinear equation, the properties of convective phenomenon from Burgers and diffusion transport as well as reaction kind of characteristics from Fisher are used. The GBFE is used in the field of fluid dynamics. It has also been found in some applications such as gas dynamics, heat conduction, elasticity and so on. The exact travelling wave and solitary wave solutions are also used in these applications [38–49].
The paper is organised as follows: In §2, the algorithmic procedure of GKM is proposed. In §3, application of GKM is presented to find the exact solutions of TF-GBFE. In §4, to find the explicit solution of the TF-GBFE, power series has been applied. Results and discussion are given in §5. The paper ends with conclusion in §6.

2. Generalised Kudryashov method

The general form of the NL partial differential equation (NLPDE) with fractional order is given as

\[ P(u, D_t^\alpha u, u_x, u_{xx}, \ldots) = 0. \]  

(1)

In the first step of GKM, we obtain the following transformation of travelling wave with arbitrary \( k \) and \( \lambda \):

\[ u(x, t) = u(\xi), \quad \xi = kx - \frac{\lambda t^\alpha}{\Gamma[1 + \alpha]}. \]  

(2)

As a result, we obtain a NLODE in the following form:

\[ N(u, u', u'', u''', \ldots) = 0, \]  

(3)

where \( \tau \) is the derivative with respect to \( \xi \).

In the second step, let the solutions of the NLODE take the form

\[ u(\xi) = \sum_{i=0}^{K} p_i R^i(\xi) = \frac{P[R(\xi)]}{Q[R(\xi)]}, \]  

(4)

where \( R = 1/(1 + \varepsilon^\xi) \). Then

\[ R_\xi = R^2 - R. \]  

(5)

Then we obtain

\[ u'(\xi) = \frac{R'P'Q - P'Q' R'}{Q^2} = \frac{P'[R(\xi)]Q - \alpha \sum_{i=0}^{K} p_{i+1} R^{i+1}(\xi) Q[R(\xi)]}{(R^2 - R)Q^2}, \]  

(6)

\[ u''(\xi) = \left[ \frac{(R^2 - R)}{Q^2} \right] \left[ (2R - 1)(P'Q - P Q') + \left( \frac{R^2 - R}{Q} \right) [Q(P''Q - P Q'')] \right] + 2Q'P'P + 2P(Q')^2, \]  

(7)

\[ u'''(\xi) = (R^2 - R)^3 \left[ ((P'''Q - P Q^{''}) - 3P''Q' - 3Q''P'Q) \right]. \]

and so on.

In the third step, the solution of eq. (3) can be expressed as

\[ u(\xi) = p_0 + p_1 R + p_2 R^2 + \ldots + p_K R^K + \ldots \]  

\[ q_0 + q_1 R + q_2 R^2 + \ldots + q_N R^N + \ldots \]  

\[ \text{where } p_i \text{ and } q_j \text{ are constants to be calculated. Balancing the term of higher order nonlinear and derivative of } u(\xi) \text{ in eq. (3), we get the values of } N \text{ and } K. \]

In the fourth step, substituting eq. (4) into eq. (3) provides a polynomial in \( R(\xi) \). Then equating the coefficients to zero, we can find the constants \( p_i \) and \( q_j \). By this way, the exact solution of eq. (1) can be found.

3. Application of GKM

Considering the TF-GBF equation

\[ u_\xi^\alpha + \beta u^\delta u_x - u_{xx} = \gamma u(1 - u^\delta), \]  

(10)

where \( \alpha \in (0, 1), \beta \) is the order of the time-fractional derivative and \( \gamma, \alpha, \delta \) are arbitrary constants. Now substituting eq. (2) in eq. (10), we obtain

\[ k^2 u'' + (\lambda - k\beta u^\delta)u' + \gamma u(1 - u^\delta) = 0. \]  

(11)

Applying the folding transformation

\[ u(\xi) = v^{1/\delta}(\xi), \]  

(12)

we obtain eq. (13) which is similar to the most general form of second-order nonlinear oscillator equation which has been already analysed by Lakshmanan et al [50,51] and Tiwari et al [52]. Moreover, they found new integrable equations and it has been further discussed by Tamizhmani et al [53].

\[ k^2 v v'' + k^2 (1 - \delta) v^2 + (\lambda - k\beta v) \delta v v' + \gamma \delta^2 (1 - v) v^2 = 0. \]  

(13)

Balancing \( v'' \) and \( v^3 \) respectively, then

\[ 2K - 2N + 2 = 3K - 3M \Rightarrow K = N + 2. \]  

(14)

Let us choose \( N = 1 \), then \( K = 3 \) and so
\[ v(\xi) = \frac{p_0 + p_1 R + p_2 R^2 + p_3 R^3}{q_0 + q_1 R}, \]  
\[ v'(\xi) = \frac{(R^2 - R)}{(q_0 + q_1 R)^2} \times [(p_1 + 2 p_2 R + 3 p_3 R^2)(q_0 + q_1 R) - q_1 (p_0 + p_1 R + p_2 R^2 + p_3 R^3)], \]  
\[ v''(\xi) = \frac{(R^2 - R)}{(q_0 + q_1 R)} \times [(2 R - 1)((p_1 + 2 p_2 R + 3 p_3 R^2) \times (q_0 + q_1 R^2) - q_1 (p_0 + p_1 R + p_2 R^2 + p_3 R^3)) + \frac{(R^2 - R)}{(q_0 + q_1 R)} ((q_0 + q_1 R) (2 p_2 + 6 p_3 R)(q_0 + q_1 R) - 2 q_1 (p_0 + p_1 R + 3 p_3 R^2) \times (q_0 + q_1 R) + 2(p_0 + p_1 R) + p_2 R^2 + p_3 R^3(q_1 R^2)]. \]  

Now we attain the exact solutions of eq. (11) and two cases are to be considered.

**Case 1**
For the choices of

\[ p_2 = p_3 = 0, \quad q_0 = p_0, \quad q_1 = 0, \quad p_1 = \frac{k(1 + \delta)p_0}{\beta \delta}, \]
\[ \lambda = k^2 + k\beta - \gamma \delta, \quad k = \frac{-\beta \delta}{(1 + \delta)}, \]

eq. (13) becomes

\[ -\beta^2 \delta v u'' + \beta^2 (\delta - 1) v^2 + (\beta^2 + \gamma (1 + \delta)^2) u v' \]
\[-\gamma (1 + \delta)^2 v^3 - (1 + \delta)(\gamma + \gamma \delta + \beta^2 v') v^2 = 0. \]

The solutions are

\[ u_1(x, t) = 1 - \frac{1}{1 + e^{\frac{k \Gamma(1 + \omega) - k^2 \beta \gamma}{\beta(1 + \omega)}}}, \]
\[ u_2(x, t) = 1 + \frac{1}{1 - e^{\frac{k \Gamma(1 + \omega) - k^2 \beta \gamma}{\beta(1 + \omega)}}}. \]

Then the exact solutions are

**4. Exact power series solutions**

Based on the power series method [54–56] and symbolic computations [57–61], we create the exact power series
solutions of eq. (10) which is differentiable. First we use a transformation

$$u(x, t) = u(\eta), \quad \eta = kx - \frac{\lambda t^\alpha}{\Gamma[1 + \alpha]},$$  \hspace{1cm} (28)

where \( k \) and \( \lambda \) are arbitrary constants with \( k, \lambda \neq 0 \).

Substituting eq. (28) into eq. (10), then we can get the nonlinear ODE

$$k^2 \delta vv'' + k^2 (1 - \delta)v^2 + (\lambda - k\beta v)\delta vv' + \gamma \delta^2 (1 - v)v^2 = 0.$$  \hspace{1cm} (29)

We suppose that the solution of eq. (13) takes the form

$$v(\xi) = \sum_{r=0}^{\infty} P_r \xi^r,$$  \hspace{1cm} (30)

where \( P_r (r = 0, 1, 2, \ldots) \) are constants. Then, we have

$$v'(\xi) = \sum_{r=0}^{\infty} (r + 1) P_{r+1} \xi^r,$$  \hspace{1cm} (31)

$$v''(\xi) = \sum_{r=0}^{\infty} (r + 1)(r + 2) P_{r+2} \xi^r.$$  \hspace{1cm} (32)

Substituting eqs. (30)–(32) into eq. (29) and solving, we get

$$P_2 = \frac{1}{2} \left[ \left( \frac{\delta - 1}{\delta} \right) \frac{P_1^2}{k^2} - \lambda \frac{P_1 P_0}{k} - \frac{\gamma}{k} \delta P_0 + \frac{\gamma}{k} \delta^2 P_0^2 \right],$$  \hspace{1cm} (33)

when \( r = 0 \). When \( r \geq 1 \), we obtain

$$P_{r+2} = \frac{1}{(r + 2)(r + 1)} \left[ \left( \frac{\delta - 1}{\delta} \right) \frac{(r + 1) P_{r+1}^2}{P_r} - \lambda \frac{P_{r+1}^2}{k^2} + \beta \frac{(r + 1) P_r P_{r+1}}{k} - \frac{\gamma}{k} \delta P_r + \frac{\gamma}{k} \delta P_r^2 + \frac{\gamma}{k} \delta^2 P_r^2 \right].$$  \hspace{1cm} (34)

We can easily prove the convergence of the power series equation (30) with the coefficients given in eqs (33) and (34). Therefore, this power series solution of eq. (30) is an exact analytic solution. Hence, the power series solution of eq. (30) can be written as follows:

$$v(\xi) = P_0 + P_1 \xi + P_2 \xi^2 + \sum_{r=1}^{\infty} P_{r+2} \xi^{r+2}$$

$$= P_0 + P_1 \xi + \frac{1}{2} \left[ \left( \frac{\delta - 1}{\delta} \right) \frac{P_1^2}{k^2} - \lambda \frac{P_1 P_0}{k} + \frac{\gamma}{k} \delta P_0 \right] \xi^2 + \sum_{r=1}^{\infty} \frac{1}{(r + 2)(r + 1)} \sum_{i=0}^{\infty} \left[ \left( \frac{\delta - 1}{\delta} \right) \frac{(r + 1) P_{r+1}^2}{P_r} - \lambda \frac{(r + 1) P_{r+1}}{k} + \beta \frac{(r + 1) P_r P_{r+1}}{k} - \frac{\gamma}{k} \delta P_r + \frac{\gamma}{k} \delta P_r^2 + \frac{\gamma}{k} \delta^2 P_r^2 \right] \xi^{r+2}.$$  \hspace{1cm} (35)
5. Results and discussion

Using the GKM, we have found the exact solutions of the TF-GBF. Figures 1 and 2 show 2D plots of eq. (21) when $\alpha = 0.5$ for different values of $x$. Figure 3 shows 2D plot of eq. (26) when $\alpha = 0.5$ for different values of $x$. Figure 4 shows 2D plot of eq. (26) when $\alpha = 0.75$ and $k = 2$ for different values of $x$. Figure 5 shows 2D plot of eq. (26) when $\alpha = 0.5$ and $k = 2$ for different values of $x$. Figure 6 shows 2D plot of eq. (26) when $\alpha = 0.85$ for different values of $x$. Figure 7 shows the
Figure 10. 3D plot of eq. (21) for $\alpha = 0.5, \beta = 2, \gamma = 3, \delta = 3, \lambda = -39/4, k = -3/2$.

Figure 11. 3D plot of eq. (21) for $\alpha = 0.75, \beta = 2, \gamma = 3, \delta = 2, \lambda = -62/9, k = -4/3$.

Figure 12. 3D plot of eq. (21) for $\alpha = 0.75, \beta = 2, \gamma = 3, \delta = 3, \lambda = -39/4, k = -3/2$.

Figure 13. 3D plot of eq. (26) for $\alpha = 0.5, \beta = 2, k = 2$.

Figure 14. 3D plot of eq. (26) for $\alpha = 0.75, \beta = 2, k = 2$.

Figure 15. 3D plot of eq. (26) for $\alpha = 0.75, \beta = 2, k = 3$. 
3D plot of eq. (21) with $\alpha = 0.25$ for different values of $x$ and time $t$. When $x$ and $t$ increase or decrease, the solution of eq. (21) also increases or decreases.

Figures 8 and 9 show the 3D plots of eq. (21) when $\alpha = 0.25$ and 0.5, respectively, for different values of $\lambda$ and $k$ for a particular range of $x$ and $t$. Figures 10 and 11 show the 3D plots of eq. (21) when $\alpha = 0.5$ and 0.75, respectively, for different values of $\lambda$ and $k$ for a particular range of $x$ and $t$. Figure 12 shows the 3D plot of eq. (21) when $\alpha = 0.75$, for different values of $t$ and $t$. From these figures, we find that when both $x$ and $t$ decrease, the solution $u$ increases. Similarly, figures 13 and 14 show the 3D plots of eq. (26) when $\alpha = 0.5$, 0.75 respectively for a particular range of $x$ and $t$. Figure 15 shows the 3D plot of eq. (26) when $\alpha = 0.75$ and $k = 3$ for a particular range of $x$ and $t$. Figure 16 shows the 3D plot of eq. (26) when $\alpha = 0.85$ and $k = 2$ for a particular range of $x$ and $t$. From these figures, we find that when both $x$ and $t$ decrease, $u$ increases.

6. Conclusion

In this article, the TF-GBFE has been transformed into a NLODE by using folding transformation. The resultant equation is similar to the most familiar general form of second-order nonlinear oscillator equation with some restrictions on the parameters. Thus, exact solutions using GKM are successfully constructed and also power series solution has been derived. Plots are given for the exact solutions with suitable parameters.

Acknowledgements

The authors thank the anonymous referees for their valuable time, effort and extensive comments which help to improve the quality of this paper. The authors also thank the Department of Science and Technology-Fund Improvement of S&T Infrastructure in Universities and Higher Educational Institutions, Government of India (SR/FST/MSI-107/2015) for carrying out this research work. The authors thank Prof. K Kannan, Tata Realty IT city-SASTRA Srinivasa Ramanujan Research Cell, SASTRA Deemed to be University, Thanjavur for his support in carrying out this research.

References

[1] A Kilbas, H Srivatsava and J Trujillo, Theory and applications of fractional differential equations (North Holland, NY, 2006)
[2] G Jumarie, Appl. Math. Lett. 23, 1444 (2010)
[3] M Eslami, Appl. Math. Comput. 285, 141 (2016)
[4] I Podlubny, Fractional differential equations (Academic Press, NY, 1999)
[5] Q Huang and R Zhdanov, Physica A 409, 110 (2014)
[6] D Kumar, J Singh, K Tanwar and D Baleanu, Int. J. Heat Mass Transf. 138, 1222 (2019)
[7] A Goswami, Sushila, J Singh and D Kumar, AIMS Math. 5(3), 2346 (2019)
[8] A Bekir, Commun. Nonlinear Sci. 14(4), 1069 (2009)
[9] M Mirzazadeh et al, Nonlinear Dynam. 81, 1933 (2015)
[10] O A Ilhan et al, Results Phys. 12, 1712 (2019)
[11] J H He and M A Abdou, Chaos Solitons Fractals 34, 1421 (2007)
[12] A Wazwaz, Appl. Math. Comput. 167, 210 (2005)
[13] J F Alzaidy, Br. J. Math. Comput. Sci. 2, 152 (2013)
[14] S Guo, Y Mei, Y Li and Y Sun, Phys. Lett. A 76, 407 (2012)
[15] A Goswami, J Singh, D Kumar and Sushila, Physica A 524, 563 (2019)
[16] D Kumar, J Singh and D Baleanu, Math. Meth. Appl. Sci. 43(1), 443 (2020)
[17] H Naher, F A Abdullah and M A Akbar, PLOS One 8(5), e64618 (2013)
[18] H Naher and F A Abdullah, J. Appl. Math. 2012(3), Article ID 486458 (2012)
[19] S Bibi et al, Results Phys. 9, 648 (2018)
[20] S Bhatter, A Mathur, D Kumar, K Sooppy Nisar and J Singh, Chaos Solitons Fractals 131, 109508 (2020)
[21] S T Demiray, Y Pandir and H Bulut, Abstr. Appl. Anal. 2014, Article ID 901540 (2014)
[22] K A Gepreel, T A Nofal and A A Alasmari, J. Egypt. Math. Soc. 25, 438 (2017)
[23] K Hosseini, P Mayeli and R Ansari, Optik 130, 737 (2017)
[24] K Hosseini et al, Opt. Quant. Electron. 49, Article No. 241 (2017)
[25] K Hosseini, P Mayeli and D Kumar, *J. Mod. Opt.* **65**, 361 (2018)
[26] M Kaplan, A Bekir and A Akbulut, *Nonlinear Dynam.* **85**, 2843 (2016)
[27] M Koparan et al., *AIP Conf. Proc.* **1798**, 020082 (2017)
[28] A Korkmaz and K Hosseini, *Opt. Quant. Electron.* **49**, Article No. 278 (2017)
[29] F Mahmud, M Samsuzzoha and M A Akbar, *Results Phys.* **7**, 4296 (2017)
[30] M Kaplan, A Bekir and A Akbulut, *Nonlinear Dynam.* **85**, 2843 (2016)
[31] M Koparan et al., *AIP Conf. Proc.* **1798**, 020082 (2017)
[32] A Korkmaz and K Hosseini, *Opt. Quant. Electron.* **49**, Article No. 278 (2017)
[33] F Mahmud, M Samsuzzoha and M A Akbar, *Results Phys.* **7**, 4296 (2017)
[34] N A Kudryashov, *Commun. Nonlinear Sci. Numer. Simul.* **17**, 2248 (2012)
[35] H Bulut, H M Baskonus and Y Pandir, *Abst. Appl. Anal.* **2013**, Article ID 636802 (2013)
[36] S M Ege and E Misirli, *Adv. Differ. Equ.* **135**, 1 (2014)
[37] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[38] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[39] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[40] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[41] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[42] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[43] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[44] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[45] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[46] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[47] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[48] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[49] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[50] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[51] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[52] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[53] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[54] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[55] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[56] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[57] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[58] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[59] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[60] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)
[61] S M Ege and E Misirli, *Int. J. Res. Adv. Technol.* **2321**, 384 (2014)