Absolute algebra and Segal’s Gamma sets

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Abstract
We show that the basic categorical concept of an $\mathcal{S}$-algebra as derived from the theory of Segal’s $\Gamma$-sets provides a unified description of several constructions attempting to model an algebraic geometry over the absolute point. It merges, in particular, the approaches using monoids, semirings and hyperrings as well as the development by means of monads and generalized rings in Arakelov geometry. The assembly map determines a functorial way to associate an $\mathcal{S}$-algebra to a monad on pointed sets. The notion of an $\mathcal{S}$-algebra is very familiar in algebraic topology where it also provides a suitable groundwork to the definition of topological cyclic homology. The main contribution of this paper is to point out its relevance and unifying role in arithmetic, in relation with the development of an algebraic geometry over symmetric closed monoidal categories.

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1. Introduction

The notion of an $\mathfrak{s}$-algebra (i.e. algebra over the sphere spectrum) is well-known in homotopy theory (cf. e.g. [10]) and in the categorical form of discrete $\Gamma$-rings used in this paper was formalized in the late 90’s. It is also intimately related to the concept of brave new rings that was first introduced in the 80’s and of functors with smash products (FSP) in the theory of spectra in algebraic topology. The goal of this short paper is to explain how the implementation of this notion in arithmetic, in terms of Segal’s $\Gamma$-sets, succeeds to unify several attempts pursued in recent times in order to define the meaning of “absolute algebra”. In particular, we refer to the development, for applications in number theory and algebraic geometry, of a suitable framework in which one would be able to make a rigorous sense of the process of taking the limit of finite fields $\mathbb{F}_q$ as $q \to 1$. In our previous work we have met and used at least three possible categories apt to handle this unification: namely the category $\mathcal{M}$ of monoids as in [2,4], the category $\mathcal{H}$ of hyperrings of [3,5,6] and finally the category $\mathcal{S}$ of semirings as in [1,9].

In [11], N. Durov developed a geometry over $\mathbb{F}_1$ suitable for Arakelov theory applications by using monads as generalizations of classical rings. In that work, a number of constructions are performed which lead to certain combinatorial structures playing, at the archimedean place(s), the role of the geometric constructions applied, at a finite prime ideal, in the process of reduction modulo that ideal. The main result of this article states that all the above structures and constructions can be naturally subsumed by a theory which is well-known in homotopy theory, namely the theory of $\mathfrak{s}$-algebras and Segal’s $\Gamma$-sets$^1$ which is taken as the natural groundwork in the recent book of B. Dundas T. Goodwillie, R. McCarthy, [10]. In particular, in Proposition 6.6 we prove that the assembly map of [16] provides a functorial way to associate an $\mathfrak{s}$-algebra to a monad on pointed sets. While in the context of [11] the tensor product $\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}$ produces an uninteresting output isomorphic to $\mathbb{Z}$, in Proposition 6.7 we show that the same tensor square, re-understood in the theory of $\mathfrak{s}$-algebras, provides a highly non-trivial object. As explained in [16], the category of pointed $\Gamma$-sets is a symmetric closed monoidal category and the theory of generalized schemes as developed by B. Töen and M. Vaquié in [19] applies directly to this category while it would not apply to the category of endofunctors under composition of [11] which is not symmetric. In fact, the paper [19] considers already the more advanced example of the model category of very special $\Gamma$-spaces which is intimately connected to the category of simplicial $\Gamma$-sets (cf. [10]).

The fundamental advantage of having unified the various attempts done in recent times to develop “absolute algebra” by the well established concept of $\mathfrak{s}$-algebra is that this latter notion is at the root of the theory of topological cyclic homology which can be understood as cyclic homology over the absolute base $\mathfrak{s}$ provided one uses the appropriate Quillen model category [10]. Our original motivation for using cyclic homology in the arithmetic context arises from the following two results:

(i) In [7] we showed that cyclic homology (a fundamental tool in noncommutative geometry) determines the correct infinite dimensional (co)homological theory for arithmetic varieties to recast the archimedean local factors of Serre as regularized determinants. The key operator in this context is the generator of the $\lambda$-operations in cyclic theory.

(ii) L. Hesselholt and I. Madsen have proven that the de Rham-Witt complex, an essential ingredient of crystalline cohomology, arises naturally when one studies the topological cyclic homology of smooth algebras over a perfect field of finite characteristic (cf. e.g. [13,14]).

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$^1$The more general notion of $\Gamma$-space is needed in homotopy theory and is simply that of a simplicial $\Gamma$-set.
Our long term objective is to use cyclic homology over the absolute base $s$ and the arithmetic site defined in [9] to obtain a global interpretation of $L$-functions of arithmetic varieties. The topos theoretic meaning of the construction of topological cyclic homology and its development at the combinatorial level will be addressed in forthcoming papers, the present article is an introduction to the algebraic framework that we plan to use.

2. $s$-algebras and Segal’s $\Gamma$-sets

Before recalling the definition of a $\Gamma$-set we comment on its conceptual yet simple meaning since one may fail to notice it at the first reading. A $\Gamma$-set is the most embracing notion of the datum provided on a set by a commutative addition with a zero element. The commutativity of the addition implies that when one performs a (finite) sum, the result only depends upon the corresponding finite set (with multiplicity). A $\Gamma$-set is best described by a covariant functor $F$ from the category $\text{Fin}$ of finite sets to the category $\text{Sets}$ of sets, mapping a finite set $X$ to the set $F(X)$ of all possible sums indexed by $X$. The covariance of $F$ embodies the associativity of the addition. To encode the presence of the 0-element (neutral for the sum) one introduces a base point $\ast \in X$ in the labelling of the sums, and a base point $\ast \in F(X)$. Thus $F$ is best described by a covariant functor from the category $\text{Fin}_{\ast}$ of finite pointed sets to the category $\text{Sets}_{\ast}$ of pointed sets. Note that while the category of covariant functors $\text{Fin} \rightarrow \text{Sets}$ is a topos the category of pointed covariant functors $\text{Fin}_{\ast} \rightarrow \text{Sets}_{\ast}$ is no longer such and admits an initial object which is also final, in direct analogy with the category of abelian groups. The theory reviewed in Chapter II of [10] is that of Segal’s $\Gamma$-spaces, i.e. of simplicial $\Gamma$-sets. The category of $\Gamma$-spaces admits a natural structure of model category (cf. [10] Definition 2.2.1.5) allowing one to do homotopical algebra. The transition from $\Gamma$-sets to $\Gamma$-spaces is parallel to the transition from abelian groups to chain complexes of abelian groups in positive degrees and the construction of topological Hochschild and cyclic homology then becomes parallel to the construction of their algebraic ancestors.

2.1 Segal’s $\Gamma$-sets

For each integer $k \geq 0$, one introduces the pointed finite set $k_{\ast} := \{0, \ldots, k\}$, where $0$ is the base point. Let $\Gamma_{\ast}$ be the small, full subcategory of $\text{Fin}_{\ast}$ whose objects are the sets $k_{\ast}$’s, for $k \geq 0$. The notion of a discrete $\Gamma$-space, i.e. of a $\Gamma$-set suffices for our applications and at the same time it is intimately related to the topos $\hat{\Gamma}$ of covariant functors $\Gamma_{\ast} \rightarrow \text{Sets}_{\ast}$. The small category $\Gamma_{\ast}$ is a pointed category i.e. it admits a (unique) initial and final object, namely the object $0_{\ast}$ formed by the base point alone. In general, if $\mathcal{C}$ is a pointed category one defines a $\Gamma$-object of $\mathcal{C}$ to be a covariant functor $\Gamma_{\ast} \rightarrow \mathcal{C}$ preserving the base point.

Definition 2.1 A $\Gamma$-set $F$ is a functor between pointed categories: $F : \Gamma_{\ast} \rightarrow \text{Sets}_{\ast}$.

The morphisms $\text{Hom}_{\Gamma_{\ast}}(M, N)$ between two $\Gamma$-sets are natural transformations of functors. The category $\Gamma\text{Sets}_{\ast}$ of $\Gamma$-sets is a symmetric closed monoidal category (cf. [10], Chapter II). The monoidal structure is given by the smash product of $\Gamma$-sets which is a Day product: we shall review this product in §2.3. The closed structure property is shown in [16] (cf. also [10] Theorem 2.1.2.4: replace the category $\mathcal{S}_{\ast}$ with $\text{Sets}_{\ast}$). The internal Hom structure is given by

$$\text{Hom}_{\Gamma_{\ast}}(M, N) := \{n_{\ast} \mapsto \text{Hom}_{\Gamma_{\ast}}(M, N_{n_{\ast}})\}$$

(1)
where \( N_{n_+}(k_+) := N(n_+ k_+) \) and for two pointed sets \( X, Y \) their smash product is obtained by collapsing the set \( X \times \{ * \} \cup \{ * \} \times Y \) to \( \{ * \} \times \{ * \} \) in the product \( X \times Y \).

Our next task is to explain how the closed monoidal category \( \Gamma \text{Sets}_* \) encompasses several attempts to model “absolute algebra” or at the geometric level an (algebraic) geometry over the “field with one element” \( \mathbb{F}_1 \).

### 2.2 A basic construction of \( \Gamma \)-sets

Let \( M \) be a commutative monoïd denoted additively and with a 0 element. One defines a functor \( HM : \Gamma^{\text{op}} \to \text{Sets}_* \) by setting

\[
HM(k_+) := M^k, \quad HM(f) : HM(k_+) \to HM(n_+), \quad HM(f)(\phi)(y) := \sum_{x|f(x)=y} \phi(x) \tag{2}
\]

where in the last formula \( y \in \{1, \ldots, n\} \) and where the sum over the empty set is the 0 element. Moreover the base point of \( HM(k_+) := M^k \) is \( \phi(x) = 0, \forall x \in \{1, \ldots, k\} \). One easily checks that the maps \( f : k_+ \to n_+ \) with \( f^{-1}(\{0\}) \neq \{0\} \) do not create any problem for the functoriality. The functor \( HM \) is in fact a particular case of the covariant functor \( \mathfrak{fin} \to \text{Sets}_* \), \( X \mapsto M^X \) defined by the formula \( \phi \ast \phi(y) = \sum_{x|f(x)=y} \phi(x) \). More precisely \( HM \) is obtained by restricting to \( \mathfrak{fin}_* \) and by subsequently dividing using the equivalence relation

\[ \phi \sim \phi' \iff \phi(x) = \phi'(x), \forall x \neq \ast. \]

By assuming that \( f : X \to Y \) preserves the base point and if \( \phi \sim \phi' \), one gets

\[
\sum_{x|f(x)=y} \phi(x) = \sum_{x|f(x)=y} \phi'(x), \forall y \neq \ast.
\]

### 2.3 \( s \)-algebras

Next, we recall the notion of an \( s \)-algebra as given in [10] (Definition 2.1.4.1). This requires to define first the smash product of two \( \Gamma \)-sets, i.e. of two pointed functors \( F_j : \Gamma^{\text{op}} \to \text{Sets}_* \), \( j = 1,2 \). The definition of the smash product \( F_1 \land F_2 \) is dictated by the internal Hom structure of (1) and the adjunction formula

\[
\text{Hom}_{\Gamma^{\text{op}}}(F_1, F_2, G) \simeq \text{Hom}_{\Gamma^{\text{op}}}(F_1, \text{Hom}_{\Gamma^{\text{op}}}(F_2, G))
\]

Thus the smash product \( F_1 \land F_2 \) is such that for any \( \Gamma \)-set \( G \), a morphism \( F_1 \land F_2 \to G \) is simply described by a map of sets, natural in both objects \( X, Y \) of \( \Gamma^{\text{op}} \)

\[
F_1(X) \land F_2(Y) \to G(X \land Y).
\]

The evaluation of the \( \Gamma \)-set \( F_1 \land F_2 \) on an object \( Z \) of \( \mathfrak{fin}_* \) is given by the following colimit

\[
(F_1 \land F_2)(Z) = \lim_{\substack{\text{v:X\land Y\to Z}\\ \text{F_1(v)\land F_2(v):F_1(X)\land F_2(Y)\to F_1(X')\land F_2(Y')}}} (F_1(X) \land F_2(Y)), \tag{3}
\]

where for any morphisms \( f : X \to X' \) and \( g : Y \to Y' \) in \( \mathfrak{fin}_* \) one uses the morphism

\[ F_1(f) \land F_2(g) : F_1(X) \land F_2(Y) \to F_1(X') \land F_2(Y') \]

provided that \( v' \circ (f \land g) = v \), with \( v' : X' \land Y' \to Z \). Thus, with the exception of the base point, a point of \( (F_1 \land F_2)(Z) \) is represented by the data \( (X, Y, v, x, y) \) given by a pair of objects \( X, Y \) of \( \mathfrak{fin}_* \), a map \( v : X \land Y \to Z \) and a pair of non-base points \( x \in F_1(X), y \in F_2(Y) \). Moreover, notice the following implication

\[ v' \circ (f \land g) = v \implies (X, Y, v, x, y) \sim (X', Y', v', F_1(f)(x), F_2(g)(y)). \]
The specialization of Definition 2.1.4.1. of [10] to the case of Γ-sets yields the following

**Definition 2.2** An $s$-algebra is a Γ-set $A: \Gamma^{\text{op}} \to \mathcal{G}ets_s$ endowed with an associative multiplication $\mu: A \wedge A \to A$ and a unit $1: s \to A$, where $s: \Gamma^{\text{op}} \to \mathcal{G}ets_s$, $s(k) := k_+$, $\forall k \geq 0$.

is the identity functor.

Since $s$ is the identity functor, the obvious identity map $X \wedge Y \to X \wedge Y$ defines the product $s \wedge s \to s$ in $s$. It is clearly associative. By construction, any $s$-algebra is an algebra over $s$. This elementary categorical object can be naturally associated to $F_1$, i.e. to the most basic algebraic structure underlying the absolute (geometric) point.

It follows from op.cit. (cf. §2.1.4.1.6) that, given an $s$-algebra $A$ and an integer $n \geq 1$, one obtains an $s$-algebra of $n \times n$ matrices by endowing the Γ-set $M_n(A) := \text{Hom}_{\mathcal{G}ets_s}(n_+, n_+ \wedge A(X))$ with the natural multiplication of matrices having only one non-zero entry in each column. There is moreover a straightforward notion of module over an $s$-algebra (cf.op.cit. Definition 2.1.5.1): an $s$-module being just a Γ-set.

### 3. Basic constructions of $s$-algebras

#### 3.1 $s$-algebras and monoids

We begin this section with an easy construction of an $s$-algebra derived from a functor from the category of (not necessarily commutative) multiplicative monoids with a unit and a zero element, to the category of $s$-algebras. This construction is given in Example 2.1.4.3, 2. of [10], where the monoid $M$ is not assumed to have a 0 element and the obtained $s$-algebra is called spherical monoid algebra.

**Definition 3.1** Let $M$ be a multiplicative monoid with a multiplicative unit and a zero element 0. We define the covariant functor $sM: \mathfrak{Fin}_s \to \mathcal{G}ets_s$, $sM(X) = M \wedge X$ with $0 \in M$ viewed as the base point and with maps $\text{Id}_M \times f$, for $f: X \to Y$.

**Proposition 3.2** Let $M$ be a multiplicative monoid with a unit and a zero element. Then the product in $M$ endows $sM$ with a structure of an $s$-algebra.

*Proof.* The product in $M$, viewed as a map $M \wedge M \to M$ determines the following map, natural in both objects $X, Y$ of $\mathfrak{Fin}_s$

$$sM(X) \wedge sM(Y) = (M \wedge X) \wedge (M \wedge Y) \to M \wedge X \wedge Y = sM(X \wedge Y).$$

The multiplicative unit $1 \in M$ determines a natural transformation $s \to sM$. This construction endows $sM$ with a structure of $s$-algebra. \hfill \Box

The multiplicative monoid $\{0, 1\}$ (frequently denoted by $F_1$) determines, in this framework, the identity functor $s$: i.e. $s\{0, 1\} = s$.

Conversely, given an $s$-algebra $A: \Gamma^{\text{op}} \to \mathcal{G}ets_s$ one obtains, using the product $A(1_+) \wedge A(1_+) \to A(1_+)$ and the unit $1_A: s \to A$, a canonical structure of multiplicative monoid (with a base point 0 and a multiplicative identity 1) on the set $M = A(1_+)$. 

5
Proposition 3.3 Let $A$ be an $s$-algebra, and $M = A(1_+)$ the associated monoid, then the following map defines a morphism of $s$-algebras

$$\rho : sM \rightarrow A, \quad \rho(j \wedge m) = \mu_A(1_A(j) \wedge m) \in A(k_+ \wedge 1_+) = A(k_+), \quad \forall j \in k_+, m \in M = A(1_+).$$

Proof. The unit $1_A : s \rightarrow A$ for $A$ defines a natural transformation of functors compatible with the product. Since the product $\mu_A : A(X) \wedge A(Y) \rightarrow A(X \wedge Y)$ is natural in $X$ and $Y$, this shows that by taking $Y = 1_+$, the morphism $\rho$ defines a natural transformation $\rho : sM \rightarrow A$. The associativity of $\mu_A$ shows that $\rho$ is multiplicative. \hfill \qed

3.2 From semirings to $s$-algebras

An important construction of an $s$-algebra is provided by the following result (cf. [10])

Lemma 3.4 Let $R$ be a semiring, then $HR$ is naturally endowed with a structure of $s$-algebra.

Proof. We first describe the product and the morphism $1 : s \rightarrow HR$. To define the product we introduce the following map, natural in both objects $X,Y$ of $\Gamma^{op}$

$$HR(X) \wedge HR(Y) \rightarrow HR(X \wedge Y), \quad (\phi, \psi) \mapsto \phi \psi, \quad \phi \psi(x,y) = \phi(x) \psi(y), \quad \forall x \in X \smallsetminus \{\ast\}, y \in Y \smallsetminus \{\ast\}.$$  

The naturality of the above operation follows from the bilinearity of the product in $R$. The morphism $1 : s \rightarrow HR$ is defined by the map

$$1_X : X \rightarrow HR(X), \quad 1_X(x) = \delta_x, \quad \delta_x(y) := \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y. \end{cases}$$

One obtains in this way a natural transformation since there is at most one non-zero value in the sum $\sum_{x \in f(\ast)} \delta_a(x)$ which computes $HR(f)(\delta_a)$. Moreover, a non-zero value occurs exactly when $y = f(a)$, which shows that $HR(f)(\delta_a) = \delta_{f(a)}$. Notice that we have defined both the product and the transformation $1 : s \rightarrow HR$ without using the additive group structure of a ring: in fact the semiring structure suffices. Finally, the axioms of $s$-algebras are checked in the same way as for rings (cf. [10] Example 2.1.4.3 for details). \hfill \qed

Proposition 3.5 The functor $H$ from semirings to $s$-algebras is fully faithful.

Proof. One needs to check that, for two semirings $A,B$ the natural map

$$\text{Hom}(A,B) \rightarrow \text{Hom}_s(HA,HB), \quad \alpha \mapsto H\alpha$$

is bijective. As a set, one obtains that $HA(1_+) \cong A$ by using the bijection defined by $\epsilon(\phi) = \phi(1) \in A$, $\forall \phi \in HA(1_+)$, which maps the base point to $0 \in A$. The product in $A$ is recovered by the map $HA(1_+) \wedge HA(1_+) \rightarrow HA(1_+ \wedge 1_+)$, using the fact that $1_+ \wedge 1_+ = 1_+$. Notice that for a pointed functor $A : \Gamma^{op} \rightarrow \text{Sets}$ of the form $A = HA$, for $A$ a semiring, one derives the special property that given two elements $x,y \in HA(1_+)$ there exists a unique element $z \in HA(2_+)$ whose images by the maps $\alpha, \beta : 2_+ \rightarrow 1_+$ of the form

$$\alpha : 0 \mapsto 0, \ 1 \mapsto 1, \ 2 \mapsto 0, \quad \beta : 0 \mapsto 0, \ 1 \mapsto 0, \ 2 \mapsto 1$$

are given by $HA(\alpha)z = x$, $HA(\beta)z = y$. One then obtains

$$x + y = \epsilon(HA(\gamma)z), \quad \text{for} \quad \gamma : 0 \mapsto 0, \ 1 \mapsto 1, \ 2 \mapsto 1.$$  \hfill (4)

This shows that a morphism of functors $\rho \in \text{Hom}_s(HA,HB)$ determines, by restriction to $1_+$, a homomorphism $\rho_1 \in \text{Hom}(A,B)$ of semirings. The uniqueness of this homomorphism is clear
using the bijection \( \epsilon : HA(1_{\ast}) \to A \). The equality \( \rho = H(\rho_1) \) follows from the naturality of \( \rho \) which implies that the projections \( \epsilon_j^A : HA(k_j) \to HA(1_{\ast}) = A \), \( \epsilon_j^A(\phi) = \phi(j) \), fulfill

\[
\rho(\phi)(j) = \epsilon_j^B(\rho(\phi)) = \rho_1(\epsilon_j^A(\phi)) = \rho_1(\phi(j)).
\]

The formula (4) for the addition in \( HA(1_{\ast}) \) retains a meaning even in the case where the above special property, for a pointed functor \( A : \Gamma^{\text{op}} \to \text{sets}_* \) is relaxed by dropping the condition that the solution to \( A(\alpha)z = x, A(\beta)z = y \) is unique. In this case, one can still define the following generalized addition \textit{i.e.} the hyper-operation

\[
x \oplus y := \{ A(\gamma)_z \mid z \in A(2_{\ast}), A(\alpha)_z = x, A(\beta)_z = y \}, \quad \forall x, y \in A(1_{\ast}).
\]

This fact suggests that one can associate an \( s \)-algebra to an hyperring. This construction will be described in more details in the next §5.

4. Smash products

The simplicity of Day’s product definition of the smash product of two \( \Gamma \)-sets hides in fact the inherent difficulty of concretely computing the colimit defined in (3). To better understand this issue, we consider the specific example of the \( s \)-algebra \( HB \), where \( B := \{0, 1\}, 1 + 1 = 1 \) is the smallest semiring of characteristic one, and compute explicitly \( HB \land HB \). To start with, we first produce an explicit description of the \( s \)-algebra \( HB \).

4.1 The \( s \)-algebra \( HB \)

**Lemma 4.1** The \( s \)-algebra \( HB \) is described by the functor \( P : \Gamma^{\text{op}} \to \text{sets}_* \) which associates to an object \( X \) of \( \Gamma^{\text{op}} \) the set \( P(X) \) of subsets of \( X \) containing the base point. The functoriality is given by the direct image \( X \ni A \mapsto f(A) \subset Y \). The product is provided by the map \( P(X) \land P(Y) \to P(X \land Y) \) which associates to the pair \( (A, B) \) the smash product \( A \land B \subset X \land Y \). The unit morphism \( 1 : s \to HB \) is given by the natural transformation \( X \to P(X) \) defined by the map \( X \ni x \mapsto \{\ast, x\} \subset X \).

**Proof.** Notice that a map \( \phi : X \to B \) is specified by the subset \( \phi^{-1}(\{1\}) = \{x \in X \mid \phi(x) = 1\} \). One associates to \( \phi \in HB(X) \) the subset \( A = \{\ast\} \cup \phi^{-1}(\{1\}) \subset X \). The formula \( \sum_{x \mid f(x) = y} \phi(x) \) which defines the functoriality shows that it corresponds to the direct image \( X \triangleright A \mapsto f(A) \subset Y \). The last two statements are straightforward to check using the product in \( B \). 

**Remark 4.2** It is interesting to compare the structures of the \( s \)-algebras \( HB \) and \( HF_2 \) because, when evaluated on an object \( X \) of \( \Gamma^{\text{op}} \), they both yield the same set. \( HF_2(X) \) is indeed equal to the set \( P(X) \) of subsets of \( X \) containing the base point: this equality is provided by associating to \( \phi \in HF_2(X) \) the subset \( A = \{\ast\} \cup \phi^{-1}(\{1\}) \subset X \). The products and the unit maps are the same in both constructions since the product in \( F_2 \) is the same as that in \( B \). The difference between the two constructions becomes finally visible by analyzing the functoriality property of the maps. The simple rule \( X \ni A \mapsto f(A) \subset Y \) holding in \( HB \) is replaced, in the case of \( HF_2 \) by adding the further condition that on the set \( f(A) \subset Y \) one only retains the points \( y \in f(A) \) such that the cardinality of the preimage \( f^{-1}(\{y\}) \) is an odd number.
4.2 k-relations and the set \((H\mathcal{B} \land H\mathcal{B})(k_+)\)

To give an explicit description of the (pointed) set \((H\mathcal{B} \land H\mathcal{B})(k_+)\) we introduce the following terminology

**Definition 4.3** (i) A k-relation is a triple \(C = (F,G,v)\) where \(F\) and \(G\) are finite sets and \(v : F \times G \to k_+\) is a map of sets such that no line or column of the corresponding matrix is identically 0.

(ii) A morphism \(\alpha : C \to C'\) between two k-relations \(C = (F,G,v)\) and \(C' = (F',G',v')\) is a pair of surjective maps \(f : F \to F'\), \(g : G \to G'\) such that \(v' \circ (f,g) = v\).

(iii) A k-relation \(C = (F,G,v)\) is said to be reduced when the lines (resp. the columns) of the matrix corresponding to \(v\) are pairwise distinct.

It follows from the colimit definition (3) that \((P \land P)(k_+)\) is the set \(\pi_0(\mathcal{C}_k)\) of connected components of the category \(\mathcal{C}_k\) of 4-tuples \(\alpha = (X,Y,v,E)\), where \(X,Y\) are objects of \(\mathfrak{Sin}_+, v : X \land Y \to k_+\) is a morphism in that category and the set \(E \in P(X) \land P(Y)\) is selected as required by the formula (3). Notice that that the set \(P(X)\) of subsets of \(X\) containing the base point can be equivalently viewed as the set of all subsets of \(X \setminus \{\ast\}\). Thus \(\alpha\) is eventually determined by the 5-tuple \((U,V,w,A,B)\) where \(U = X \setminus \{\ast\}, V = Y \setminus \{\ast\}, A \subset U, B \subset V, w : U_+ \land V_+ \to k_+\) and \(E = (A_+,B_+)\). Note also that the map \(w\) is determined uniquely by its restriction to \(U_+ \times V_+\). A morphism \(\alpha \to \alpha'\) of 5-tuples is then given by a pair of pointed maps \(f : U_+ \to U'_+, g : V_+ \to V'_+\) such that

\[
f(A_+) = A'_+, \quad g(B_+) = B'_+, \quad w' \circ (f,g) = w.
\]

**Lemma 4.4** (i) For a 5-tuple \(\alpha = (U,V,w,A,B)\) as above, let

\[
W = w^{-1}(k_+ \setminus \{0\}) \subset U \times V, \quad F = p_A((A \times B) \cap W), \quad G = p_B((A \times B) \cap W)
\]

where \(p_A\) and \(p_B\) are the projections. Then the triple \(C(\alpha) = (F,G,w|_{F \times G})\) defines a k-relation.

(ii) A morphism \(\alpha \to \alpha'\) of 5-tuples induces by restriction a morphism \(C(\alpha) \to C(\alpha')\) of k-relations.

(iii) For \(\gamma = (F,G,v)\) a k-relation, let \(\tilde{\gamma} = (F,G,v,F,G)\). Then \(\gamma \mapsto \tilde{\gamma}\) extends to a functor and one has \(C(\tilde{\gamma}) = \gamma\).

(iv) The 5-tuples \(\alpha\) and \(\tilde{\mathcal{C}}(\alpha)\) belong to the same connected component of \(\mathcal{C}_k\).

**Proof.** (i) Notice that \((A \times B) \cap W\) is a subset of \(A \times B\) whose projections are \((p_A((A \times B) \cap W)) = F\) and \(p_B((A \times B) \cap W)) = G\), thus the restriction of \(w\) to \(F \times G\) defines a k-relation.

(ii) Let \((f,g) : \alpha \to \alpha'\) be a morphism as in (6). We show that

\[
f(p_A((A \times B) \cap W)) = p_A((A' \times B') \cap W')
\]

For \((x,y) \in (A \times B) \cap W\), one has \(w'(f(x),g(y)) = w(x,y) \neq 0\), hence \((f(x),g(y)) \in X' \setminus \{\ast\}, g(y) \in Y' \setminus \{\ast\}\) and \((f(x),g(y)) \in (A' \times B') \cap W'). Then \(f(x) \in p_A'((A' \times B') \cap W')\). Conversely, let \((x',y') \in (A' \times B') \cap W'), then there exists \(x \in A, f(x) = x', y \in B, g(y) = y'\). One then has \((x,y) \in (A \times B) \cap W\) since \(w(x,y) = w'(f(x),g(y)) = w'(x',y') \neq 0\). This shows that \(f(p_A((A \times B) \cap W)) = p_A((A' \times B') \cap W')\) and similarly that \(g(p_B((A \times B) \cap W)) = p_B((A' \times B') \cap W')\). Thus the restrictions of \(f\) and \(g\) define surjections \(f : F \to F'\) and \(g : G \to G'\), where \(F = p_A((A \times B) \cap W), G = p_B((A \times B) \cap W)\). The restrictions \(w|_{F \times G} : F \times G \to k_+\), \(w'|_{F' \times G'} : F' \times G' \to k_+\) fulfill the equality \(w = w' \circ (f,g)\) and one thus obtain a morphism \(C(\alpha) \to C(\alpha')\) of k-relations.

(iii) Let \((f,g) : \gamma = (F,G,v) \to (F',G',v')\) be a morphism of k-relations, we extend both \(f,g\) to maps of pointed sets \(f_+ : F_+ \to F'_+, g_+ : G_+ \to G'_+\). Then one easily checks that one obtains
a morphism $\tilde{\gamma} \to \tilde{\gamma}'$ (cf. (6)). One has by construction $C(\tilde{\gamma}) = \gamma$.

(iv) The inclusions $A \subset U$, $B \subset V$ determine maps

$$f : A_+ \to U_+, \ g : B_+ \to V_+ \mid f(A_+) = A_+, \ g(B_+) = B_+$$

showing that in general a 5-tuple $\alpha = (U, V, w, A, B)$ is equivalent to $(A, B, w|_{(A \times B)}, A, B)$. We now prove that $(A, B, w|_{(A \times B)}, A, B)$ is equivalent to $\tilde{C}(\alpha) = (F, G, v, F, G)$, where $F = p_A((A \times B) \cap W)$, and $v = w|_{(F \times G)}$. Let $f : A_+ \to F_+$ act as the identity on $F \subset A$ and map the elements of $A \setminus F$ to the base point. We define $g : B_+ \to G_+$ in a similar manner.

Then by construction one has $f(A_+) = F_+$, $g(B_+) = G_+$. Moreover for $(a, b) \in A \times B$ one has $w(a, b) = v(f(a), g(b))$, since both sides vanish unless $(a, b) \in F \times G$ and they agree on $F \times G$. This shows that $\alpha$ and $\tilde{C}(\alpha)$ belong to the same connected component.

Every $k$-relation $C = (F, G, v)$ admits a canonical reduction $r(C) = (F/\sim, G/\sim, v|_{(F/\sim \times G/\sim)})$. This is obtained by dividing $F$ and $G$ by the following equivalence relations

$$i \sim i' \iff v(i, j) = v(i', j) \forall j \in G, \ j \sim j' \iff v(i, j) = v(i, j') \forall i \in F.$$ 

By construction, the value $v(i, j)$ only depends upon the classes $(x, y)$ of $(i, j)$ and thus the canonical reduction map $C \to r(C)$ defines a morphism of $k$-relations.

**Theorem 4.5**

(i) For $k \in \mathbb{N}$, the set $(\mathbb{K} \times \mathbb{K})(k_+) \cong \pi_0(\mathcal{G}_k)$ of connected components of the category of $k$-relations.

(ii) Two $k$-relations are in the same connected component of $\mathcal{G}_k$ if and only if their reductions are isomorphic.

(iii) The isomorphism $(\mathbb{K} \times \mathbb{K})(k_+) \cong \pi_0(\mathcal{G}_k)$ induces an equivalence of functors between $\mathbb{K} \times \mathbb{K} : \Gamma^{op} \to \text{Sets}_*$ and $\pi_0(\mathcal{G}_*), \Gamma^{op} \to \text{Sets}_*$, where a morphism $\phi \in \text{Hom}_{\Gamma^{op}}(k_+, \ell_+)$ acts on elements of $\mathcal{G}_k$ by the rule $(F, G, v) \mapsto r(F, G, \phi \circ v) \in \mathcal{G}_\ell$.

**Proof.** 

(i) It follows from Lemma 4.4 that the category of $k$-relations is a retraction of the category of 5-tuples representing elements of $(\mathbb{K} \times \mathbb{K})(k_+)$. 

(ii) By construction a $k$-relation $C$ is in the same component as its reduction since one has a reduction morphism $C \to r(C)$ in the category. Conversely, let us consider a fixed, reduced $k$-relation $\alpha = (\{1, \ldots, n\}, \{1, \ldots, m\}, w)$. It is enough to show that the set of $k$-relations whose reduction is isomorphic to $\alpha$ forms a connected component of $\mathcal{G}_k$. Let $C = (F, G, v)$ be a $k$-relation such that $r(C) \sim \alpha$. Then one has partitions $F = \bigsqcup_{1 \leq i \leq n} F_i, \ G = \bigsqcup_{1 \leq \ell \leq m} G_\ell$, whose summands are non empty sets, while $v$ is constantly equal to $w(i, \ell)$ on $F_i \times G_\ell$. Consider a morphism $(f, g) : C \to C'$ where $C'$ is an arbitrary $k$-relation. We show that $C'$ is necessarily of the same form as $C$. Let $F_i' := f(F_i), \ G_\ell' := g(G_\ell)$. We first prove that $F_i' \cap F_j' = \emptyset$ for $i \neq j$. Let $x_i \in F_i$ and $x_j \in F_j$. Since $\alpha$ is reduced, let $\ell \in \{1, \ldots, m\}$ be such that $w(i, \ell) \neq w(j, \ell)$ and let $y \in G_\ell$. It follows from

$$v'(f(x_i), g(y)) = v(x_i, y) = w(i, \ell) \neq w(j, \ell) = v(x_j, y) = v'(f(x_j), g(y))$$

that $f(x_i) \neq f(x_j)$. This shows that the $F_i' = f(F_i)$ form a partition of $F'$ and similarly one shows that $G' = \bigsqcup_{1 \leq \ell \leq m} G_\ell'$. Moreover since $v' \circ (f, g) = v$, on gets that $v'$ is constantly equal to $w(i, \ell)$ on $F_i' \times G_\ell'$. This proves that if $r(C) \sim \alpha$ and if one has a morphism $(f, g) : C \to C'$, then $r(C') \sim \alpha$. Moreover, the existence of a morphism $C' \to C$ immediately implies that $r(C') \sim \alpha$. Thus the $k$-relations such that $r(C) \sim \alpha$ form a connected component of the category $\mathcal{G}_k$ of $k$-relations.

(iii) One checks, using the naturality of the reduction map, that the isomorphism $(\mathbb{K} \times \mathbb{K}) \to \text{Sets}_*$ induces an equivalence of categories.\hfill \qed
$H \mathbb{B}) (k_+) \simeq \pi_0(\mathcal{G}_k)$ is an equivalence of functors $H \mathbb{B} \land H \mathbb{B} \to \pi_0(\mathcal{G}_k)$. More precisely, let $C = (F, G, v)$ be a $k$-relation and $\phi \in \text{Hom}_{\text{op}}(k_+, \ell_+)$ be a morphism. Let $\phi \circ C$ be the triple obtained from $(F, G, \phi \circ v)$ by deleting the lines and columns which are identically 0. One obtains a functor $\Gamma^{\text{op}} \to \mathcal{S}ets_+$, $k_+ \mapsto \pi_0(\mathcal{G}_k)$, which is by construction the same as $H \mathbb{B} \land H \mathbb{B}$.

Corollary 4.6 (i) For $n \in \mathbb{N}$, let $\text{Id}_n$ be the graph of the identity map on the set with $n$-elements. Then the 1-relations $\text{Id}_n$ belong to distinct connected components of the category $\mathcal{G}_1$ and they define distinct elements of $(H \mathbb{B} \land H \mathbb{B})(1_+)$. (ii) The action of the cyclic group $C_2$ on $(H \mathbb{B} \land H \mathbb{B})(1_+)$ is the transposition acting on isomorphism classes of reduced 1-relations. Its set of orbits of the two types (fixed and free) are both infinite and countable.

Proof. (i) The statement follows from the fact that the 1-relations $\text{Id}_n$ are reduced and pairwise non-isomorphic.

(ii) The action of the transposition $\sigma \in C_2$ replaces the element of $(H \mathbb{B} \land H \mathbb{B})(1_+)$ associated to $(X, Y, v, a, b)$, where $v : X \land Y \to 1_+$, $a \in P(X)$, $b \in P(Y)$, with the element of $(H \mathbb{B} \land H \mathbb{B})(1_+)$ associated to $(Y, X, \tilde{v}, b, a)$ where $\tilde{v} : Y \land X \to 1_+$ is obtained by composing $v$ with the transposition. This action has the effect to replace the associated reduced 1-relation $\alpha$ by its transpose $\alpha^t$. The fixed points are given by the 1-relations $\alpha \sim \alpha^t$; all the $\text{Id}_n$ as in (i) are thus fixed points. The relations between finite sets of different cardinality determine infinitely many non fixed points.

The statements proven so far show that in general the natural map $M(1_+) \land N(1_+) \to (M \land N)(1_+)$ is not surjective. For example, notice that the following reduced 1-relation

$$
\begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
$$

has equal number (3) of lines and columns but is not isomorphic to the transposed 1-relation

$$
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
$$

since the number of (non-zero) elements in the lines are respectively (3, 1, 1) and (2, 2, 1) which cannot be matched by any pair of permutations.

5. Hyperrings and $s$-algebras

The definition (5) given in §3.2 shows that the natural operation which defines a classical addition on the $s$-algebra $HR$ associated to a semiring is multivalued for a general $s$-algebra $A : \Gamma^{\text{op}} \to \mathcal{S}ets_+$. The extension of ring theory to the case of a multivalued addition has been investigated by M. Krasner: this is the theory of hyperrings and hyperfields. In our recent research we have encountered these types of generalized algebraic structures in the following two important cases (cf. [5, 6]):

- The adele class space of a global field $F$ is naturally a hyperring and it contains the Krasner hyperfield $\mathbb{K} = \{0, 1\}$ as a sub-hyperalgebra precisely because one divides the ring of the adèles of $F$ by the non-zero elements $F^\times$ of the field $F$.
The dequantization process described by the “universal perfection” at the archimedean place of a number field yields a natural hyperfield structure $\mathbb{R}^\flat$ on the set of real numbers and a natural hyperfield structure $\mathbb{C}^\flat$ in the complex case.

In particular, we notice that the notion of an hyperfield is far more demanding than that of a semifield since the operation $x \mapsto -x$ is present in the hyper-context and missing for semirings. This fact entails the existence of many “unnatural” semifields. An important common feature of the above two cases is given by the presence of an hyperring structure obtained by division of an ordinary (commutative) ring $A$ by a subgroup $G \subset A^\times$ of the group of invertible elements of $A$.

More generally, in Corollary 3.10 of [5] we proved that any (commutative) hyperring extension of the Krasner hyperfield $\mathbb{K} = \{0, 1\}$ ($1 + 1 = \{0, 1\}$), without zero divisors and of dimension > 3 is again of the form $A/G$, where $A$ is a commutative ring and $G \subset A$ is the group of non-zero elements of a subfield $F \subset A$. Moreover, Theorem 3.13 of op.cit. shows that morphisms of hyperrings of the above form lift (under mild non-degeneracy conditions) to morphisms of the pairs $(A, F)$.

5.1 From hyperrings $A/G$ to $s$-algebras

In this paragraph we provide the construction of the $s$-algebra associated to an hyperring of the form $A/G$, where $A$ is a commutative ring and $G \subset A^\times$. Notice that the structure needed to define uniquely the $s$-algebra $HA/G$ is provided by the pair $(A, G)$ and that this datum is more precise than simply assigning the hyperring $A/G$.

**Proposition 5.1** Let $A$ be a commutative ring and $G \subset A^\times$ be a subgroup of the group of invertible elements of $A$. For each object $X$ of $\mathfrak{Fin}_*$, let $(HA/G)(X)$ be the quotient of $HA(X)$ by the following equivalence relation

$$\phi \sim \psi \iff \exists g \in G \text{ s.t. } \psi(x) = g\phi(x), \ \forall x \in X, x \neq \ast.$$  

Then, the functor $HA/G : \Gamma^{op} \rightarrow \mathfrak{Sets}$, defines an $s$-algebra and the quotient map $HA \rightarrow \ldots$
HA/G is a morphism of $s$-algebras.

Proof. We first check the functoriality of the construction, i.e. that for $f : X \to Y$, the map $HA(f)$ respects the equivalence relation $\phi \sim \psi$. One has, for $y \in Y$, $y \neq *$

$$HA(f)(\psi)(y) := \sum_{x \mid f(x) = y} \psi(x) = \sum_{x \mid f(x) = y} g\phi(x) = g\left(\sum_{x \mid f(x) = y} \phi(x)\right) = gHA(f)(\phi)(y)$$

and since these equations hold for the same $g$ and for all $y \in Y$, one derives that $HA(f)(\phi) \sim HA(f)(\psi)$. The above equivalence relation is compatible with the product since, using the commutativity of $A$, one has

$$(g\phi)(h\psi)(x, y) = g\phi(x)h\psi(y) = gh\phi(x)\psi(y), \ \forall x \in X \setminus \{\ast\}, y \in Y \setminus \{\ast\}.$$

The unit map $1 : s \to HA$ defines by composition a map $1 : s \to HA/G$: this defines the natural transformation

$$1_X : X \to (HA/G)(X), \quad 1_X(x) = G\delta_1 \in (HA/G)(X) = HA(X)/G.$$ 

It follows easily from the construction that the quotient map $HA \to HA/G$ is a morphism of $s$-algebras.

Proposition 5.2 Let $A$ be a commutative ring, let $G \subset A^\times$ be a subgroup of the group of invertible elements of $A$ and $HA/G$ the $s$-algebra defined in Proposition 5.1. Then, the set $HA/G(1_+)$ endowed with the hyper-addition defined in (5) and the product $HA/G(1_+) \wedge HA/G(1_+) \to HA/G(1_+)$ is canonically isomorphic to the hyperring $A/G$.

Proof. By using the bijection $\epsilon(\phi) := \phi(1) \in A/G$, $\forall \phi \in HA/G(1_+)$ which maps the base point to $0 \in A$ one derives the bijection of sets $HA/G(1_+) \cong A/G$. The product in $A/G$ is recovered by the map $HA/G(1_+) \wedge HA/G(1_+) \to HA/G(1_+ \wedge 1_+)$ using the relation $1_+ \wedge 1_+ = 1_+$. The hyper-addition (5) gives, for $x, y \in HA/G(1_+) \cong A/G$

$$x \oplus y := \{HA/G(\gamma)z \mid z \in HA/G(2_+), \ HA/G(\alpha)z = x, \ HA/G(\beta)z = y\}.$$ 

For $x = aG$ and $y = bG$ let $\phi \in HA(2_+)$ with $\phi(1) = a$ and $\phi(2) = b$. Then the class of $a + b$ modulo $G$ belongs to $x \oplus y$ and all elements of $x \oplus y$ are of this form. In this way one recovers the hyper-addition on $A/G$. The product $HA/G(1_+) \wedge HA/G(1_+) \to HA/G(1_+)$ is the same as that in $A/G$.

Given an $s$-algebra $A$ one can guess if this structure arises from the above construction by checking whether the equations $A(\alpha)z = x, A(\beta)z = y$ have always a solution, not always unique.

5.2 W-models and associated $s$-algebras

The construction of the $s$-algebras given in Proposition 5.1 applies in particular when one uses the notion of $W$-model introduced in [6]. We recall that given a hyperfield $K$, a $W$-model of $K$ is by definition a triple $(W, \rho, \tau)$ where

$W$ is a field $\rho : W \to K$ is a homomorphism of hyperfields $\tau : K \to W$ is a multiplicative section of $\rho$.

Because $\tau : K \to W$ is a multiplicative section of $\rho$, the map $\rho$ is surjective and identifies $K$ (as a multiplicative monoid) with the quotient of $W$ by the multiplicative subgroup $G = \{x \in W \mid \rho(x) = 1\}$. Since $\rho : W \to K$ is a homomorphism of hyperfields, one has $\rho(x + y) = \rho(x) + \rho(y)$ for
any $x, y \in W$. Thus $\rho$ defines a morphism of hyperfields $W/G \to K$.

The notion of morphism of $W$-models is straightforward to define. We recall that a $W$-model of $K$ is said to be universal if it is an initial object in the category of $W$-models of $K$. When such universal model exists one easily sees that it is unique up to canonical isomorphism: we denote it by $W(K)$. In this case it is thus natural to associate to $K$ the $s$-algebra $HW/G$, $G = \{ x \in W \mid \rho(x) = 1 \}$ of its universal $W$-model $(W, \rho, \tau)$.

**Example 5.3** Let $S := \{0, \pm 1\}$ ($1 + 1 = 1, 1 - 1 = 0, \pm 1$) be the hyperfield of signs. The associated $W$-model is $(\mathbb{Q}, \rho, \tau)$ where the morphism $\rho: \mathbb{Q} \to S$ is given by the sign of rational numbers (cf. [6]). The corresponding $s$-algebra is then $A = H\mathbb{Q}/\mathbb{Q}_+^\times$. One has $A(1_+) = S$ and $A(2_+)$ is the set of half lines $L$ through the origin in the rational plane $\mathbb{Q}^2$ (including the degenerate case $L = \{0\}$). The maps $A(\alpha): A(2_+) \to A(1_+)$ and $A(\beta): A(2_+) \to A(1_+)$ are the projections onto the two axes. The map $A(\gamma): A(2_+) \to A(1_+)$ is the projection on the main diagonal. It follows that the hyper-operation (5) gives back the hyperfield structure on $S$.

One may wonder how to relate the $s$-algebra $A = H\mathbb{Q}/\mathbb{Q}_+^\times$ with $H\mathbb{B}$. To this end, one first considers the $s$-subalgebra $A_+ = H\mathbb{Q}_+/\mathbb{Q}_+^\times$ which is defined using the sub-semiring $\mathbb{Q}_+ \subset \mathbb{Q}$. The subset $A_+(k_+) \subset A(k_+)$ corresponds to the collection of half lines $L$ through the origin in $\mathbb{Q}^k$ which belong to the first quadrant $\mathbb{Q}^k_+$. One then defines a morphism of $s$-algebras $H\rho: H\mathbb{Q}_+ \to H\mathbb{B}$ by using the morphism of semirings $\rho: \mathbb{Q}_+ \to \mathbb{B}$ and one also notes that $H\rho$ induces a morphism of $s$-algebras $H\rho: A_+ = H\mathbb{Q}_+/\mathbb{Q}_+^\times \to H\mathbb{B}$. One can thus describe the relation between $A$ and $H\mathbb{B}$ by the following map

$$A \ni A_+ \xrightarrow{H\rho} H\mathbb{B}.$$ 

One should also expect a similar diagram holding more generally when passing from a semifield of characteristic 1 to an hyperfield and its $W$-model.
5.3 The levels of an $s$-algebra

In this section we shall briefly discuss how, by means of the introduction of levels for $s$-algebras, one may describe an explicit approximation to the notion of $s$-algebra. The first two levels of the approximation are obtained by restricting the functor $A : \Gamma^{op} \rightarrow \mathfrak{Sets}_s$ defining an $s$-algebra to resp. sets with $\leq 2$ elements to define the level 1, and to sets with $\leq 3$ elements for the level 2.

Level 1 The theory of multiplicative monoids

Level 2 The theory of hyperrings with partially defined addition.

In §3.1 we have seen that at level 1 an $s$-algebra $A$ is well approximated by the $s$-subalgebra corresponding to the monoid $A(1+)$. Next we discuss the compatibility of the additive structure given in (5) with the morphism of $s$-algebras $\rho : sM \rightarrow A$. By implementing (5) to $sM$ with the same notations used there, one gets

$$x \oplus y := \{sM(\gamma)z \mid z \in sM(2+) , \ sM(\alpha)z = x , \ sM(\beta)z = y\} , \ \forall x,y \in sM(1+) .$$

One has: $sM(1+) = M$, while the elements of the set $sM(2+) = 2, \wedge M$ are the base point 0 and the pairs $(j,m)$ for $j = 1,2 , m \in M , m \neq 0$. One also has

$$sM(\alpha)(1,m) = 0 , \ sM(\beta)(1,m) = m , \ sM(\alpha)(2,m) = m , \ sM(\beta)(2,m) = 0$$

and $sM(\gamma)(1,m) = sM(\gamma)(2,m) = m$. Thus the only rule related to the addition that is retained at level 1 simply states that the base point 0 plays the role of the neutral element:

$$0 + x = x + 0 = x , \ \forall x \in sM(1+) . \quad (7)$$

This rule is of course preserved by the morphism of $s$-algebras $\rho : sM \rightarrow A$. In §5.1 we have seen that the notion of an $s$-algebra is compatible with the operation of quotient by a subgroup of the multiplicative monoid associated to level 1. Moreover in §6 we shall describe how to associate a sub-$s$-algebra to a sub-multiplicative seminorm on a semiring. By combining these two constructions one provides a good approximation to the description of level 2 of an $s$-algebra. This viewpoint shows that, in general, the operation defined by (5) yields a partially defined hyper-addition so that the description of level 2 is as hyperrings with partially defined addition. The theory of $s$-algebras contains however all (non-negative) levels, even though only for the first level we have an easy explicit description.

6. $s$-algebras and Arakelov geometry

This section is motivated by the theory developed in [11] and aiming to determine an “absolute”, algebraic foundation underlying Arakelov geometry: we refer to the theory of monads which defines the groundwork of op.cit. The main fact that we want to highlight in the following paragraphs is that the assembly map of [16] determines a functorial way to associate to a monad on $\mathfrak{Sets}_s$ an $s$-algebra. Thus several main objects introduced in [11] can be naturally incorporated in the context of $s$-algebras. We start this section by reviewing some basic structures defined in op.cit., in particular, we explain how the basic construction using seminorms can be adapted in the framework of $s$-algebras. In §6.3 we shall review the assembly map which plays in Proposition 6.6 a fundamental role to define the functor that associates an $s$-algebra to a monad. Finally, Proposition 6.7 of §6.4 shows that the wedge product $s$-algebra $HZ \wedge HZ$ is (non-trivial and) not isomorphic to $HZ$. This is in sharp contrast with the result of op.cit. stating that $\mathbb{Z} \otimes_{F_1} \mathbb{Z} \cong \mathbb{Z}$ (cf. 5.1.22 p. 226).
6.1 Monads and $\text{Spec} \mathbb{Z}$

We first review briefly some of the constructions of [11]. A monad on $\mathcal{Sets}$ is given by an endofunctor $\Sigma : \mathcal{Sets} \to \mathcal{Sets}$ together with natural transformations $\mu : \Sigma \circ \Sigma \to \Sigma$ and $\epsilon : \text{Id} \to \Sigma$. The associativity of the product $\mu : \Sigma \circ \Sigma \to \Sigma$ is encoded by the commutativity of the following diagram, for any object $X$ of $\mathcal{Sets}$

$$
\begin{array}{ccc}
\Sigma^3(X) & \xrightarrow{\Sigma(\mu(X))} & \Sigma^2(X) \\
\downarrow{\mu_{\Sigma(X)}} & & \downarrow{\mu_X} \\
\Sigma^2(X) & \xrightarrow{\mu_X} & \Sigma(X)
\end{array}
$$

(8)

and one has similarly a compatibility requirement for the unit $\epsilon$. This corresponds to the notion of monoid in the monoidal category $(\text{End} \mathcal{Sets}, \circ, \text{Id})$ of endofunctors under composition, where $\text{Id}$ is the identity endofunctor. Let $R$ be a semiring, then in [11] one encodes $R$ as the endofunctor of $R$-linearization $\Sigma_R : \mathcal{Sets} \to \mathcal{Sets}$. It associates to a set $X$ the set $\Sigma_R(X)$ of finite, formal linear combinations of points of $X$, namely of formal sums $\sum_j \lambda_j \cdot x_j$, where $\lambda_j \in R$ and $x_j \in X$. One uses the addition in $R$ to simplify $\lambda \cdot x + \lambda' \cdot x = (\lambda + \lambda') \cdot x$. To a map of sets $f : X \to Y$ one associates the transformation $\Sigma_R(f) : \Sigma_R(X) \to \Sigma_R(Y)$, $\Sigma_R(f)(\sum \lambda_j \cdot x_j) := \sum \lambda_j \cdot f(x_j)$.

The multiplication in $R$ is encoded by using the simple fact that a linear combination of linear combinations is still a linear combination, thus

$$
\mu \left( \sum_i \lambda_i \cdot \left( \sum_j \mu_{ij} \cdot x_{ij} \right) \right) = \sum_{i,j} \lambda_{ij} \mu_{ij} \cdot x_{ij}.
$$

(9)

This construction defines a natural transformation $\mu : \Sigma_R \circ \Sigma_R \to \Sigma_R$ which fulfills the algebraic rules of a monad. In fact, the rules describing a monad allow one to impose restrictions on the possible linear combinations which one wants to consider. This fits well in particular with the structure of the convex sets so that one may define, for example, the monad $\mathbb{Z}_\infty$ by considering as ring $R$ the field $\mathbb{R}$ of real numbers and by restricting only to the linear combinations $\sum \lambda_j x_j$ satisfying the convexity condition $\sum |\lambda_j| \leq 1$. One can eventually replace $\mathbb{R}$ by $\mathbb{Q}$ and in this case one obtains the monad $\mathbb{Z}_{(\infty)}$. One may combine the monad $\mathbb{Z}_{(\infty)}$ with the ring $\mathbb{Z}$ as follows. The first approximation process is made by taking an integer $N$ and by considering the localized ring $B_N := \mathbb{Z}[\frac{1}{N}]$ and its intersection $A_N = B_N \cap \mathbb{Z}_{\infty}$ with $\mathbb{Z}_{\infty}$. This is simply the monad $A_N$ of linear combinations $\sum \lambda_j x_j$ with $\lambda_j \in B_N$ and $\sum |\lambda_j| \leq 1$. By computing its prime spectrum one finds ([11]) $\text{Spec} A_N = \{\infty\} \cup \text{Spec} \mathbb{Z} \setminus \{p \mid p|N\}$. One then reinstalls the missing primes $i.e.$ the set $\{p \in \text{Spec} \mathbb{Z} \mid p|N\}$, by gluing $\text{Spec} A_N$ with $\text{Spec} \mathbb{Z}$ on the large open set which they have in common. One obtains in this way the set

$$
\text{Spec} \mathbb{Z}^{(N)} := \text{Spec} A_N \cup_{\text{Spec} B_N} \text{Spec} \mathbb{Z}.
$$

Finally, one eliminates the integer $N$ by taking a suitable projective limit under divisibility. The resulting space is the projective limit space

$$
\text{Spec} \mathbb{Z} := \varprojlim \text{Spec} \mathbb{Z}^{(N)}.
$$

As a topological space it has the same topology (Zariski) as that of $\text{Spec} \mathbb{Z}$ with simply one further point (i.e. $\infty$) added. At the other non-archimedean primes this space fulfills the same
properties as \( \text{Spec } \mathbb{Z} \). One nice feature of this construction is that the local algebraic structure at \( \infty \) is given by \( \mathbb{Z}_{(\infty)} \).

### 6.2 \( \mathfrak{s} \)-algebras and seminorms

As explained in §2.2.1.1 of [10], a \( \Gamma \)-set \( M : \Gamma^\text{op} \to \mathcal{S}ets \) automatically extends, using filtered colimits over the finite subsets \( Y \subset X \), to a pointed endofunctor

\[
\tilde{M} : \mathcal{S}ets \to \mathcal{S}ets, \quad \tilde{M}(X) := \lim_{Y \subset X} M(Y). \tag{10}
\]

Let \( R \) be a (commutative) ring and \( HR \) be the associated \( \mathfrak{s} \)-algebra as in Lemma 3.4. Then the extension \( \tilde{HR} : \mathcal{S}ets \to \mathcal{S}ets \) is, except for the nuance due to the presence of the base point, the same as the functor \( \Sigma_R : \mathcal{S}ets \to \mathcal{S}ets \) just reviewed in §6.1. Then, the basic construction of [11] adapts directly as follows

**Proposition 6.1** (i) Let \( R \) be a semiring, and \( \| \| \) a sub-multiplicative seminorm on \( R \). Then \( HR \) is naturally endowed with a structure of \( \mathfrak{s} \)-subalgebra \( \| HR \|_1 \subset HR \) defined as follows

\[
\| HR \|_1 : \Gamma^\text{op} \to \mathcal{S}ets, \quad \| HR \|_1(X) := \{ \phi \in HR(X) | \sum_{X \setminus \{ \ast \}} \| \phi(x) \| \leq 1 \}. \tag{11}
\]

(ii) Let \( E \) be an \( R \)-semi-module and \( \| \| \) a seminorm on \( E \) such that \( \| a\xi \| \leq \| a \| \| \xi \| \), \( \forall a \in R \), \( \forall \xi \in E \), then for any \( \lambda > 0 \) the following defines a module \( \| HE \|_1^E \) over \( \| HR \|_1 \)

\[
\| HE \|_1^E : \Gamma^\text{op} \to \mathcal{S}ets, \quad \| HE \|_1^E(X) := \{ \phi \in HE(X) | \sum_{X \setminus \{ \ast \}} \| \phi(x) \|_E \leq \lambda \}, \tag{12}
\]

where \( HE \) is as in (2).

**Proof.** (i) The sub-multiplicative seminorm on \( R \) fulfills the rules

\[
\| x + y \| \leq \| x \| + \| y \|, \quad \| xy \| \leq \| x \| \| y \|, \quad \forall x, y \in R.
\]

In particular, the triangle inequality shows that, using the formula \( HR(f)(\phi)(y) = \sum_{x \mid f(x) = y} \phi(x) \), one derives

\[
\sum_{y \in Y, y \neq \ast} \| HR(f)(\phi)(y) \| \leq \sum_{y \in Y, y \neq \ast} \left( \sum_{x \mid f(x) = y} \| \phi(x) \| \right) \leq \sum_{X \setminus \{ \ast \}} \| \phi(x) \| \leq 1.
\]

Thus, for any morphism \( f \in \text{Hom}_\mathfrak{s}(X, Y) \), the map \( HR(f) \) restricts to a map \( \| HR \|_1(X) \to \| HR \|_1(Y) \). Finally, the stability under product derives from the sub-multiplicativity of the seminorm as follows

\[
\sum_{(X \times Y) \setminus \{ \ast \}} \left( \sum_{X \setminus \{ \ast \}} \left( \sum_{Y \setminus \{ \ast \}} \| \phi(x) \| \psi(y) \| \right) \leq \left( \sum_{X \setminus \{ \ast \}} \| \phi(x) \| \right) \left( \sum_{Y \setminus \{ \ast \}} \| \psi(y) \| \right) \leq 1.
\]

(ii) The above proof shows that \( \| HE \|_1^E \) is a \( \Gamma \)-set. It also shows that one obtains a map of sets, natural in both objects \( X, Y \) of \( \Gamma^\text{op} \)

\[
\| HR \|_1(X) \wedge \| HE \|_1^E(Y) \to \| HE \|_1^E(X \wedge Y), \quad (a \wedge \xi)(x, y) := a(x)\xi(y), \quad \forall x \in X, \ y \in Y.
\]

This gives the required \( \| HR \|_1 \)-module structure \( \| HR \|_1 \wedge \| HE \|_1^E \to \| HE \|_1^E \) on \( \| HE \|_1^E \).

**Remark 6.2** The only multiplicative seminorm on the semifield \( \mathbb{B} \) : \( \| 0 \| = 0, \| 1 \| = 1 \). In this case one obtains \( \| H \mathbb{B} \|_1 = \mathfrak{s} \) since the condition (11) restricts the functor \( P : \Gamma^\text{op} \to \mathcal{S}ets \) of Lemma 4.1 to the range of the unit map \( 1 : \mathfrak{s} \to H \mathbb{B} \).
Remark 6.3 Proposition 6.1, combined with the general theory of schemes for a symmetric closed monoidal category of B. Töen and M. Vaquié in [19], provides the tools to perform the same constructions as those of [11] reviewed above in the framework of \( s \)-algebras. We shall not pursue this point here but will now explain, using the assembly map of [16], a conceptual manner to pass from the framework of monads to that of \( s \)-algebras.

6.3 The assembly map

Next, we shall review the definition of the assembly map in the simple case of \( \Gamma \)-sets, we refer to [16, 17] for more details. Let \( N : \Gamma^{\mathrm{op}} \to \mathcal{G} \mathbf{sets} \) be a \( \Gamma \)-set, and \( \tilde{N} : \mathcal{G} \mathbf{sets} \to \mathcal{G} \mathbf{ets}_s \) its extension to a pointed endofunctor as in (10). One derives a map natural in the two pointed sets \( X,Y \) (cf. §2.2.1.2 of [10])

\[
\iota_{X,Y}^N : X \wedge \tilde{N}(Y) \to \tilde{N}(X \wedge Y)
\]

which is obtained as the family, indexed by the elements \( x \in X \), of maps \( \tilde{N}(\delta_x) : \tilde{N}(Y) \to \tilde{N}(X \wedge Y) \) with

\[
\delta_x \in \mathrm{Hom}_{\mathcal{G} \mathbf{sets}}(Y, X \wedge Y), \quad \delta_x(y) := (x, y) \in X \wedge Y, \quad \forall y \in Y.
\]

Similarly, we let \( M : \Gamma^{\mathrm{op}} \to \mathcal{G} \mathbf{ets}_s \) be a second \( \Gamma \)-set, and \( \tilde{M} \) its extension to a pointed endofunctor of \( \mathcal{G} \mathbf{ets}_s \). One then constructs a map natural in \( X \) and \( Y \) as follows

\[
\alpha_{X,Y} : \tilde{M}(X) \wedge \tilde{N}(Y) \to \tilde{M}(\tilde{N}(X \wedge Y)), \quad \alpha_{X,Y} := \tilde{M}(\iota_{X,Y}^N) \circ \iota_{\tilde{N}(Y),X}^M
\]

where \( \iota_{\tilde{N}(Y),X}^M : \tilde{M}(X) \wedge \tilde{N}(Y) \to \tilde{M}(X \wedge \tilde{N}(Y)) \), \( \tilde{M}(\iota_{X,Y}^N) : \tilde{M}(X \wedge \tilde{N}(Y)) \to \tilde{M}(\tilde{N}(X \wedge Y)) \). By restricting to finite pointed sets \( X,Y \), \( i.e. \) objects of \( \Gamma^{\mathrm{op}} \in \mathfrak{Fin}_s \subset \mathcal{G} \mathbf{ets}_s \), one obtains a natural transformation of bi-functors and hence a morphism of \( \Gamma \)-sets

\[
\alpha_{M,N} : M \wedge N \to \tilde{M} \circ \tilde{N}.
\]

Definition 6.4 Let \( M, N : \Gamma^{\mathrm{op}} \to \mathcal{G} \mathbf{ets}_s \) be \( \Gamma \)-sets. The assembly map \( \alpha_{M,N} : M \wedge N \to \tilde{M} \circ \tilde{N} \) is the morphism of \( \Gamma \)-sets as in (13).

Proposition 6.5 (i) Let \( R \) be a semiring, then the assembly map \( \alpha_{HR,HR} \) is surjective.

(ii) For \( R = \mathbb{B} \), the assembly map \( \alpha_{HR,HB} \) associates to a \( k \)-relation \( C = (F,G,v) \) the set of subsets of \( k_* \) defined by \( \alpha_{HR,HB}(C) = \{ \{v(x,y) \mid y \in G \} \mid x \in F \} \).

Proof. (i) The functor \( \overline{HR} \) coincides with the endofunctor of \( R \)-linearization \( \Sigma_R : \mathcal{G} \mathbf{ets}_s \to \mathcal{G} \mathbf{ets}_s \). For a pointed set \( Z \), we denote the elements of the set \( \overline{HR}(Z) \) as formal sums \( \sum r_z \cdot z \), where only finitely many non-zero terms occur in the sum, and the coefficient of the base point is irrelevant, \( i.e. \) the term \( r \cdot \ast \) drops out. Let \( \xi = \sum r_x \cdot x \in \overline{HR}(X) \), \( \eta = \sum s_y \cdot y \in \overline{HR}(Y) \), then one has

\[
\iota_{X,Y}^HR(\xi \wedge \eta) = \sum r_x \cdot (x \wedge \eta) \in \overline{HR}(X \wedge \overline{HR}(Y)).
\]

The image of \( x \wedge \eta \) by \( \iota_{X,Y}^HR \) is described by \( \iota_{X,Y}^HR(x \wedge \eta) = \sum s_y \cdot (x \wedge y) \in \overline{HR}(X \wedge Y) \) and one finally obtains

\[
\alpha_{HR,HR}(\xi \wedge \eta) = \sum r_x \cdot \left( \sum s_y \cdot (x \wedge y) \right) \in \overline{HR}(\overline{HR}(X \wedge Y)).
\]

By applying (3) one has

\[
(HR \wedge HR)(Z) = \lim_{\nu : X \wedge Y \to Z} (HR(X) \wedge HR(Y)).
\]
Given a 5-tuple \((X,Y,v,\xi,\eta)\) as above, the image by \(\alpha_{HR,HR}\) of the corresponding element of \((HR \wedge HR)(Z)\) is described, using (14), by

\[
\alpha_{HR,HR}(X,Y,v,\xi,\eta) = \sum r_x \cdot (\sum s_y \cdot v(x \wedge y)) \in \overline{HR}(\overline{HR}(Z)).
\]  

(16)

For \(k_+\) an object of \(\Gamma^\text{op}\), an element of \((\overline{HR} \circ HR)(k_+)\) is of the form

\[
\tau = \sum_{i \in I} \lambda_i \cdot \left( \sum_{j \in k_+ \times \ast} \mu_{ij} \cdot j \right), \quad \lambda_i \in R, \ \mu_{ij} \in R,
\]

where \(I\) is a finite set of indices. Let \(X = I_+\) and \(\xi = \sum_i \lambda_i \cdot i \in HR(X)\). Let \(Y = I_+ \wedge k_+\) and \(\eta = \sum_{i \in I, j \in k_+ \times \ast} \mu_{ij} \cdot (i,j) \in HR(Y)\). Let \(v : X \wedge Y \to k_+\) be given by \(v(i,(i',j)) = \star\) if \(i \neq i'\) and \(v(i,(i,j)) = j\), \(\forall i \in I, j \in k_+ \times \ast\). With this choice for the 5-tuple \((X,Y,v,\xi,\eta)\), one then obtains

\[
\alpha_{HR,HR}(X,Y,v,\xi,\eta) = \sum \lambda_i \cdot \left( \sum \mu_{ij} \cdot v(i \wedge (i',j)) \right) = \tau.
\]

(ii) For \(R = \mathbb{B}\), the element of \((HR \wedge HR)(k_+)\) associated to the \(k\)-relation \(C = (F,G,v)\) corresponds to the 5-tuple \((X,Y,v,\xi,\eta)\) where \(X = F_+, Y = G_+, \xi(x) = 1 \ \forall x \in F, \ \eta(y) = 1 \ \forall y \in G.\) Thus (16) becomes \(\alpha_{HR,HR}(C) = \{\{v(x,y) \mid y \in G\} \mid x \in F\}\).

Notice that (ii) of the above proposition implies that the assembly map is not injective in the case \(R = \mathbb{B}\), since by Corollary 4.6 one knows that \((HR \wedge HR)(k_+)\) is infinite countable for any \(k > 0\), while \((HR \circ HR)(k_+)\) remains finite for any \(k\).

### 6.4 \(s\)-algebras and monads

The constructions of [11] for monads which were reviewed in §6.1 are easily reproduced in the context of \(\Gamma\)-sets and yield the \(s\)-algebras: \(Z_\infty := \|HR\|_1\) and \(Z_{(\infty)} := \|HQ\|_1\), where the multiplicative seminorms are resp. the usual absolute value on \(\mathbb{R}\) and its restriction to \(\mathbb{Q} \subset \mathbb{R}\). This fact is suggestive of the existence of a functor that associates to a monad on \(\mathsf{Sets}_s\) an \(s\)-algebra. We first briefly address the use of the category of pointed sets in place of the category of sets used in [11].

One has a pair of adjoint functors \((L,F)\) where \(L : \mathsf{Sets}_s \to \mathsf{Sets}_s\), the functor \(L(X) := X_+\) of adjunction of a base point, is left adjoint to the forgetful functor \(F : \mathsf{Sets}_s \to \mathsf{Sets}\). The associated natural transformations are \(\xi : \text{Id}_{\mathsf{Sets}_s} \to F \circ L\) which maps the set \(X\) to its copy in the disjoint union \(F \circ L(X) = X \sqcup \{\star\}\) and \(\eta : L \circ F \to \text{Id}_{\mathsf{Sets}_s}\), which is the identity on \(X \subset L \circ F(X) = X \sqcup \{\star\}\) and maps the extra base point to the base point of \(X\).

Let then \(H\) be a pointed endofunctor of \(\mathsf{Sets}_s\), the composition \(H^\# := F \circ H \circ L\) is then an endofunctor of \(\mathsf{Sets}_s\). Moreover given a natural transformation \(\mu : H_1 \circ H_2 \to H_3\) of endofunctors of \(\mathsf{Sets}_s\), one obtains a natural transformation of endofunctors of \(\mathsf{Sets}_s\), \(H'^\#_1 \circ H'^\#_2 \to H'^\#_3\).

This gives a natural correspondence between monads on \(\mathsf{Sets}_s\) and monads on \(\mathsf{Sets}\). Note in particular that the monad on \(\mathsf{Sets}\) given by \(\text{Id}^\#\) where \(\text{Id} : \mathsf{Sets}_s \to \mathsf{Sets}_s\) is the identity endofunctor is the monad called \(F_1\) of [11]. Moreover for any semiring \(R,\) \(\overline{HR}\) the extension of \(HR\) to an endofunctor of \(\mathsf{Sets}_s\) as in (10), one checks that the natural monad structure on \(\overline{HR}\) corresponds to the monad \(\Sigma_R,\) i.e. that \(\overline{HR}^\# = \Sigma_R.\)

As in \textit{op.cit.}, we say that an endofunctor of \(\mathsf{Sets}_s\) is \textit{algebraic} if it preserves filtering colimits: it is the extension, preserving filtering colimits, of its restriction to \(\Gamma^\text{op} \subset \mathfrak{Fin} \subset \mathsf{Sets}_s.\)
Proposition 6.6 Let $\Sigma$ be a pointed algebraic monad on $\text{Sets}_*$, then the restriction $M$ of $\Sigma$ to $\Gamma^{op} \subset \text{Fin}_* \subset \text{Sets}_*$ defines an $s$-algebra with product $m = \mu \circ \alpha_{\Sigma,M}$ defined by the composite of the assembly map and the product $\mu : \Sigma \circ \Sigma \rightarrow \Sigma$ of the monad $\Sigma$.

Proof. One uses the fact stated in Remark 2.19 of [16] that the assembly map makes the identity functor on $\Gamma$-sets a lax monoidal functor from the monoidal category $(\Gamma \text{Sets}, \circ, s)$ to the monoidal category $(\Gamma \text{Sets}_*, \wedge, s)$. More explicitly, the composition $m = \mu \circ \alpha_{\Sigma,M}$ becomes a morphism of $\Gamma$-sets $M \wedge M \rightarrow M$. The compatibility of the assembly map with the associativity shows that the commutative diagram (8) gives the commutativity of the following diagram of maps of $\Gamma$-sets

\[
\begin{array}{ccc}
M \wedge M \wedge M & \xrightarrow{\text{Id}_{\wedge M}} & M \wedge M \\
\downarrow m \wedge \text{Id} & & \downarrow \mu \circ \alpha_{\Sigma,M} \\
M \wedge M & \xrightarrow{\mu \circ \alpha_{\Sigma,M}} & M \\
\end{array}
\]

(17)

The compatibility with the unit ($s$ in both monoidal categories) is handled in the same way. □

Using (14) one checks that, given a semiring $R$ the $s$-algebra associated to the monad $\overline{HR} \sim \Sigma_R$ using Proposition 6.6 is $HR$.

Proposition 6.5 gives the construction of an $HZ_{\infty}$-module associated to a $Z_{\infty}$-lattice in the sense of [11], indeed one is given a norm on a vector space and one can apply directly (ii) of Proposition 6.5 to obtain a structure of $HZ_{\infty}$-module on $\|HE\|_E^\Lambda$.

Since the assembly map makes the identity functor on $\Gamma$-sets a lax monoidal functor from the monoidal category $(\Gamma \text{Sets}_*, \circ, s)$ to the monoidal category $(\Gamma \text{Sets}_*, \wedge, s)$ it follows that, given an algebraic monad $\Sigma$ on $\text{Sets}_*$ one can associate to a (left or right) algebraic module $E$ on $\Sigma$ in the sense of the monoidal category $(\Gamma \text{Sets}_*, \circ, s)$ a (left or right) module over the $s$-algebra $M$ of Proposition 6.6. The underlying $\Gamma$-set is unchanged and the action of $M$ is obtained using the assembly map as in Proposition 6.6. But besides the notion of (left or right) algebraic module $E$ on $\Sigma$ in the sense of the monoidal category $(\Gamma \text{Sets}_*, \circ, s)$ discussed in [11] §4.7, a simpler category $\Sigma \rightarrow \text{Mod}$ of modules over a monad is introduced and used in [11], 3.3.5. A module $E$ on a monad $\Sigma$ on $\text{Sets}_*$ in this sense is simply a pointed set $E$ together with a map of pointed sets $\alpha : \Sigma(E) \rightarrow E$ fulfilling the two conditions: $\alpha \circ \mu_E = \alpha \circ \Sigma(\alpha)\,\alpha \circ \epsilon_E = \text{Id}_E$. This notion of a module over a monad defined in [11] implies that modules over $\Sigma_R$ correspond exactly to ordinary $R$-modules (cf. op. cit. or [12] just before Proposition 2, p. 11). In particular, by taking the monad $\Sigma = \text{Id}$ (which corresponds to $\mathbb{F}_2$ in the notations of [11]), one finds that any set $M$ is a module in the unique manner provided by the map $\alpha = \text{Id}_M : M \rightarrow M$. This fact continues to hold if one considers the monad which corresponds to $\mathbb{F}_1$ (cf. [12] §4.4 p. 24). Then one finds that the modules over $\mathbb{F}_1$ are just pointed sets.

In the framework of $s$-algebras, an $R$-module $E$ over a ring $R$ gives rise to an $HR$ module $HE$ but the modules over $HR$ are not necessarily of this form (cf. [10] Remark 2.1.5.2, 4). However, fortunately, this nuance between $R$-modules and $HR$-modules disappears at the homotopy level. Since this is a crucial fact we explain it in some details below, referring to [18] and [10] for the full treatment. Let $k$ be a commutative $s$-algebra, then one defines the smash product $M \wedge_k N$ of two $k$-modules as the coequalizer (cf. [10], Definition 2.1.5.3)

$$M \wedge_k N := \lim_{\longrightarrow} \{M \wedge k \wedge N \rightrightarrows M \wedge N\}$$
using the two maps given by the actions of $k$ on $M$ and $N$. One obtains in this way the symmetric closed monoidal category of $k$-modules and a corresponding notion of $k$-algebra. In order to perform homotopy theory, one passes to the associated categories of simplicial objects. The fundamental fact (cf. [18], Theorem 4.1) is that the homotopy category of $\mathbb{H}Z$-algebras, obtained using the simplicial version of $\mathbb{H}Z$-algebras, is equivalent to the homotopy category of ordinary simplicial rings. More generally this continues to hold if one replaces $\mathbb{Z}$ by any commutative ring $R$ and simplicial rings by simplicial $R$-algebras. This shows that, in the framework of $\mathfrak{s}$-algebras, nothing is lost in using the more relaxed notion of $HR$-module in place of the restricted notion based on monads.

### 6.5 $\mathbb{H}Z \wedge \mathbb{H}Z$ is not isomorphic to $\mathbb{H}Z$

In the framework of [11] one gets the equality $\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z} \simeq \mathbb{Z}$ (cf. [11] 5.1.22 p. 226) which is disappointing for the development of the analogy with the geometric theory over functions fields. The following result shows the different behavior of the corresponding statement over $\mathfrak{s}$.

**Proposition 6.7** The smash product $\mathfrak{s}$-algebra $\mathbb{H}Z \wedge \mathbb{H}Z$ is not isomorphic to $\mathbb{H}Z$.

**Proof.** Assume that there exists an isomorphism of $\mathfrak{s}$-algebra $\sigma : \mathbb{H}Z \wedge \mathbb{H}Z \xrightarrow{\sim} \mathbb{H}Z$. It is then unique since, by Proposition 4.1, the $\mathfrak{s}$-algebra $\mathbb{H}Z$ has no non-trivial automorphism. Furthermore, by Proposition 6.5 the assembly map $\alpha_{\mathbb{H}Z,\mathbb{H}Z}$ is surjective and this fact implies that the composition $\psi = \alpha_{\mathbb{H}Z,\mathbb{H}Z} \circ \sigma : \mathbb{H}Z \rightarrow \text{Hom}_{\mathcal{F}_{\mathbb{H}Z}}(\mathbb{H}Z, \mathbb{H}Z)$ is also surjective. Let $\mu : \text{Hom}_{\mathcal{F}_{\mathbb{H}Z}}(\mathbb{H}Z, \mathbb{H}Z) \rightarrow \mathbb{H}Z$ be the natural transformation as in (9), then the composite $\mu \circ \psi$ is a morphism of $\Gamma$-sets: $\mu \circ \psi \in \text{Hom}_{\mathcal{F}_{\mathbb{H}Z}}(\mathbb{H}Z, \mathbb{H}Z)$. Since $\mu : \text{Hom}_{\mathcal{F}_{\mathbb{H}Z}}(\mathbb{H}Z, \mathbb{H}Z) \rightarrow \mathbb{H}Z$ is surjective one then derives that the composite $\mu \circ \psi \in \text{Hom}_{\mathcal{F}_{\mathbb{H}Z}}(\mathbb{H}Z, \mathbb{H}Z)$ is also surjective, thus coming from a surjective endomorphism (i.e. an automorphism) of the abelian group $\mathbb{Z}$ (here we refer to the proof of Proposition 4.1). These facts would imply that $\psi$ is also injective, hence an isomorphism $\psi : \mathbb{H}Z \xrightarrow{\sim} \text{Hom}_{\mathcal{F}_{\mathbb{H}Z}}(\mathbb{H}Z, \mathbb{H}Z)$. The set $(\mathbb{H}Z \circ \mathbb{H}Z)(1_+)$ = $\{\sum \lambda_n \cdot n \mid \lambda_j \in \mathbb{Z}\}$ is the set of finite linear combinations, while similarly $(\text{Hom}_{\mathcal{F}_{\mathbb{H}Z}}(\mathbb{H}Z, \mathbb{H}Z))(2_+) = \{\sum \lambda_{n,m} \cdot (n,m) \mid \lambda_{i,j} \in \mathbb{Z}\}$. Moreover the three maps $\alpha, \beta, \gamma : 2_+ \rightarrow 1_+$ of (4) act as follows

$$(\mathbb{H}Z \circ \mathbb{H}Z)(\alpha)(\sum_{Z} \lambda_{n,m} \cdot (n,m)) = \sum_{Z} \lambda_{n,m} \cdot \mathbb{H}Z(\alpha)(n,m) = \sum_{Z} \lambda_{n,m} \cdot m$$

$$(\mathbb{H}Z \circ \mathbb{H}Z)(\beta)(\sum_{Z} \lambda_{n,m} \cdot (n,m)) = \sum_{Z} \lambda_{n,m} \cdot \mathbb{H}Z(\beta)(n,m) = \sum_{Z} \lambda_{n,m} \cdot n$$

$$(\mathbb{H}Z \circ \mathbb{H}Z)(\gamma)(\sum_{Z} \lambda_{n,m} \cdot (n,m)) = \sum_{Z} \lambda_{n,m} \cdot \mathbb{H}Z(\gamma)(n,m) = \sum_{Z} \lambda_{n,m} \cdot (n+m).$$

In particular, we see that given two elements $a, b \in (\mathbb{H}Z \circ \mathbb{H}Z)(1_+)$, the existence of an element $c \in (\mathbb{H}Z \circ \mathbb{H}Z)(2_+)$ such that $(\mathbb{H}Z \circ \mathbb{H}Z)(\alpha)(c) = a$ and $(\mathbb{H}Z \circ \mathbb{H}Z)(\beta)(c) = b$ implies that $\sum a_n = \sum b_n$. However this condition is not always fulfilled thus one necessarily derives that the $\Gamma$-set $(\mathbb{H}Z \circ \mathbb{H}Z)$ cannot be isomorphic to $\mathbb{H}Z$ and this determines a contradiction. \qed

**Remark 6.8** When working at the secondary level of topological spectra, it is well known that the smash product of the Eilenberg MacLane spectra $\mathbb{H}Z^\wedge \mathbb{H}Z$ is not isomorphic to $\mathbb{H}Z$. In fact the corresponding homotopy groups are not the same. Indeed, for any spectrum $X$ one has the equality $\pi_*(\mathbb{H}Z \wedge X) = H_*(X, \mathbb{Z})$ where $H_*(X, \mathbb{Z})$ denotes the spectrum homology with integral coefficients. Applying this fact to $X = \mathbb{H}Z$, one sees $\pi_*(\mathbb{H}Z \wedge \mathbb{H}Z)$ as the integral homology of the Eilenberg-Mac Lane spectrum $\mathbb{H}Z$. This homology is known to be finite in positive degrees but not trivial (cf. Theorem 3.5 of [15]). This argument however, cannot be applied directly to
provide an alternative proof of Proposition 6.7 since then one would need to compare the spectra $H \mathbb{Z} \wedge H \mathbb{Z}$ and $H \mathbb{Z} \wedge H \mathbb{Z}$ and this comparison is usually done using a cofibrant replacement $H \mathbb{Z}^c$ of the $\Gamma$-set $H \mathbb{Z}$.

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