One-particle irreducible Wilson action in the gradient flow exact renormalization group formalism

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We define a one-particle irreducible (1PI) Wilson action in the gradient flow exact renormalization group (GFERG) formalism as the Legendre transform of a Wilson action. We consider quantum electrodynamics in particular, and show that the GFERG flow equation preserves the invariance of the 1PI Wilson action (excluding the gauge-fixing term) under the conventional \(U(1)\) gauge transformation. This is in contrast to the invariance of the original Wilson action under a modified \(U(1)\) gauge transformation. The global chiral transformation also takes the conventional form for the 1PI Wilson action. Despite the complexity of the GFERG flow equation, the conventional form of the gauge and global chiral transformations may allow us to introduce a non-perturbative Ansatz for gauge and chiral invariant 1PI Wilson actions.

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1 Introduction

The exact renormalization group (ERG) \[1–8\] is a fundamental framework to investigate possible quantum field theories even at a non-perturbative level. In a series of papers \[9–11\], we and a collaborator have been developing a new formulation of ERG in gauge theory, which we call the gradient flow exact renormalization group (GFERG). The basic idea of GFERG comes from a formulation of ERG through the linear diffusion of fields \[12\]. (See also Refs. \[13–22\] for related investigations.) Using the non-linear diffusion equations introduced in Refs. \[23–27\] that preserve the gauge invariance, it might be possible to generalize ERG to have manifest gauge invariance. This reasoning has led to GFERG \[9–11\].

Though it is possible to formulate gauge theory in the standard formulation of ERG \[28–33\], the gauge transformation must be modified with dependence on the Wilson action. Accordingly the gauge invariance is difficult to implement as an Ansatz in any non-perturbative but practical studies of Wilson actions in gauge theory. (See Refs. \[34, 35\] for detailed studies of this issue in perturbation theory.) Thus, if one aims at non-perturbative applications of ERG in gauge theory, an ERG formulation with manifest gauge invariance is highly desirable if not essential\[1\].

Now, in non-perturbative applications of ERG, it has become conventional to consider the one-particle irreducible (1PI) Wilson action \[4, 43–46\] instead of the original Wilson action \[47\]. One reason for this is that the 1PI Wilson action tends to have a simpler structure and is thus more “economical”; see, for instance, Ref. \[48\] for an illustration on this point.

In this paper, with this simplicity in mind, we develop a formulation of the 1PI Wilson action as the Legendre transform of the Wilson action in GFERG. We consider only the Abelian gauge theory, quantum electrodynamics (QED), with the gauge fixing as developed in Ref. \[11\]. It turns out that a simple Legendre transformation well adopted in ERG works perfectly here. (Our prescription corresponds to the choice \(K_A(p) = e p^2 / \Lambda^2\) and \(k_A(p) = p^2 / \Lambda^2\) in Eq. (23) of Ref. \[49\], up to simple redefinitions of field variables.) We find it remarkable that the GFERG equation preserves the invariance of the 1PI Wilson action (excluding the gauge-fixing term) under the conventional form of the \(U(1)\) gauge transformation. The global chiral transformation also takes the conventional form. In return, however, we must endure the complexity of the GFERG equation satisfied by the 1PI Wilson action; up to the third-order functional derivatives of the Legendre transformed variables

\[1\] We refer the reader to Refs. \[36–42\] for alternative manifestly gauge invariant ERG formulations of gauge theory.
with respect to the original field variables are required. This is to be compared with the first-order derivatives required in the standard ERG 1PI formalism. Nevertheless, we believe that the simple form taken by the gauge transformations of the 1PI Wilson action is a great advantage when we come to introduce an Ansatz for the gauge invariant 1PI Wilson action in non-perturbative studies of the renormalization group flows.

This paper is organized as follows: In Sect. 2 we recapitulate the main results of Ref. [11] in terms of the original Wilson action. We then generalize the condition of the chiral invariance to accommodate the chiral anomaly and the associated topological effect. In Sect. 3 we introduce the Legendre transform that defines the 1PI Wilson action. It is observed that the gauge and global chiral transformations take the conventional form. The gauge invariance of the 1PI action, an a priori known gauge-fixing term excluded, is preserved under our GFERG flow. The GFERG equation for the 1PI action is derived in Sect. 4. Its complexity is the price to pay for the manifest gauge invariance. In Sect. 5, we obtain the 1PI Wilson action explicitly up to the second order in the gauge coupling $e$ by the Legendre transformation of the Wilson action, which has been calculated in Ref. [11]. We confirm that the 1PI Wilson action is simpler than the original Wilson action, which contains products of 1PI parts connected by short-distance propagators. We conclude the paper in Sect. 5.

Throughout the paper we work in the $D$-dimensional Euclidean space and set $D = 4 - \epsilon$. For the momentum integration, we use the abbreviation

$$\int_p \equiv \int \frac{d^D p}{(2\pi)^D}. \tag{1.1}$$

### 2 Wilson action in GFERG

#### 2.1 GFERG equation

We recapitulate the essence of GFERG for QED formulated in Ref. [11]. We write all the expressions in dimensionless variables, a convention suitable for the investigation of potential fixed points.

The GFERG flow equation for the Wilson action $S$ in QED [11] can be expressed as

$$\partial_\tau S = - \int d^D x \, e^{-S} \frac{\delta}{\delta A_\mu(x)} \hat{s} \left( 2\partial_{x'}^2 + \frac{D - 2}{2} + \gamma + x' \cdot \partial_{x'} \right) A_\mu(x') \hat{s}^{-1} e^S$$

\[\text{\footnotesize Here, we omit the Faddeev–Popov ghost sector, because it decouples completely under the GFERG flow [11]. In Appendix A we consider the 1PI Wilson action containing the ghost sector, and show its BRST invariance. We also omit the suffix } \tau \text{ from } S_\tau \text{ for simplicity.}\]
where $\tau$ parametrizes the logarithmic scale of the ERG transformation. The differential operator $\hat{s}$, for which we coin the name “scrambler”, is defined by

$$\hat{s} \equiv \exp \left[ \frac{1}{2} \int d^D x \frac{\delta^2}{\delta A_\mu(x) \delta A_\mu(x)} \right] \exp \left[ -i \int d^D x \frac{\delta}{\delta \psi(x)} \frac{\delta}{\delta \bar{\psi}(x)} \right].$$

In Eq. (2.1) and expressions below, it is understood that primed coordinates such as $x'$, $x''$, $x'''$, etc., are taken to $x$ only after functional differentiation. As elucidated in Ref. [11], this “point-splitting prescription” follows from a careful derivation of the GFERG equation. In Eq. (2.1), $\gamma$ and $\gamma_F$ are $\tau$-dependent anomalous dimensions associated with the photon and electron fields, respectively. These depend on the normalization condition adopted.

The GFERG equation (2.1) differs from the conventional ERG equation for QED (see, for instance, Appendix C of Ref. [11]) simply by the presence of terms containing the gauge coupling $e$; it is those terms that bring forth the fundamental property that the GFERG flow preserves manifest gauge invariance. In Ref. [11] we have introduced

$$S_{\text{inv}} \equiv S + \frac{1}{2} \int d^D x \frac{1}{\xi E(-e^{-2\tau} \partial^2) e^{2\partial^2} - \partial^2} \partial_\nu A_\nu(x),$$

where $\xi$ is the gauge-fixing parameter, $E(x)$ is an arbitrary function analytic at $x = 0$, and the gauge-fixing term is excluded from $S$. The GFERG equation (2.1) preserves the invariance of $S_{\text{inv}}$ (2.3) under the modified gauge transformation

$$\delta A_\mu(x) = \frac{\xi E(-e^{-2\tau} \partial^2) e^{2\partial^2} - \partial^2}{\xi E(-e^{-2\tau} \partial^2) e^{2\partial^2} - \partial^2} \partial_\mu \chi(x),$$

$$\delta \psi(x) = i e \chi(x) \psi(x),$$

We have generalized Eq. (5.16) of Ref. [11] by replacing $\xi$ by $\xi E(k^2 e^{-2\tau})$ given in the momentum space. This is to be explained below.
\[ \delta \bar{\psi}(x) = -ie\chi(x)\bar{\psi}(x), \] (2.4c)

where \( \chi(x) \) is an arbitrary infinitesimal function.

In Ref. [11], we have taken \( E = 1 \) since we have restricted our interest only to those \( S \) parametrized by the gauge coupling \( e \), the electron mass \( m \), and the gauge-fixing parameter \( \xi \). More generally, the Faddeev–Popov ghost action must be given by

\[ S_{\text{ghost}} = \int d^Dx \bar{c}(x) \frac{\partial^2}{E(-e^{-2\tau}\partial^2)e^{2\varphi^2} - \partial^2} c(x) = \int k \bar{c}(-k) \frac{-k^2}{E(k^2 e^{-2\tau}) e^{-2k^2} + k^2} c(k). \] (2.5)

Then, repeating the argument in Ref. [11] we obtain Eqs. (2.3) and (2.4).

We note in passing that the invariance of \( S_{\text{inv}} \) under Eq. (2.4) is equivalent to

\[ \delta S = \int d^Dx A_\mu(x) \frac{\partial^2}{\xi E(-e^{-2\tau}\partial^2)e^{2\varphi^2}} \partial_\mu \chi(x). \] (2.6)

This is to be used later.

Finally, the \( \tau \)-dependence of the gauge parameter in the above is given by the anomalous dimension \( \gamma \) as [11]

\[ \partial_\tau e = \left( \frac{\xi}{2} - \gamma \right) e, \] (2.7)

and that of the gauge-fixing parameter by

\[ \partial_\tau \xi = 2\gamma \xi. \] (2.8)

The relation (2.7) that corresponds to the Ward identity \( Z_1 = Z_3 \) in the conventional formulation follows naturally from the definition of the gauge coupling \( e \) in this formulation. Equation (2.8) follows from the underlying BRST symmetry preserved by the GFERG equation.

### 2.2 Chiral symmetry

In Ref. [10], it was shown that the GFERG equation (2.1) is consistent with a modified form of the chiral symmetry à la Ginsparg–Wilson [50]. Writing the differential operator that generates the conventional global chiral transformation as

\[ \hat{\gamma}_5 \equiv -\int d^Dx \left[ \frac{\delta}{\delta \bar{\psi}(x)} \gamma_5 \psi(x) + \frac{\delta}{\delta \bar{\psi}(x)} \bar{\psi}(x) \gamma_5 \right], \] (2.9)

we define the modified global chiral transformation by

\[ \hat{\Gamma}_5 \equiv \hat{s}\gamma_5 \hat{s}^{-1}, \] (2.10)

where \( \hat{s} \) is the scrambler defined by Eq. (2.2). Then the modified chiral invariance of the Wilson action \( S \) is given as the vanishing of \( \hat{\Gamma}_5 \) acting on \( S \):

\[ e^{-S} \hat{\Gamma}_5 e^S = \int d^Dx \left[ \frac{\delta}{\delta \bar{\psi}(x)} \gamma_5 \psi(x) + \bar{\psi}(x) \gamma_5 \frac{\delta}{\delta \bar{\psi}(x)} \psi(x) \gamma_5 S + 2iS \frac{\delta}{\delta \bar{\psi}(x)} \gamma_5 \frac{\delta}{\delta \bar{\psi}(x)} S \right]. \] (2.11)
\[ + \int d^D x \text{ tr} \left[ -2i \gamma_5 \frac{\delta}{\delta \bar{\psi}(x')} S \frac{\delta}{\delta \psi(x)} \right] \]
\[ = 0. \]  
(2.11)

This constrains the Wilson action. If we assume that \( S \) is bilinear in fermion fields, Eq. (2.11) is nothing but the Ginsparg–Wilson relation \[50\].

Equation (2.11) is an “operator equation” preserved under the GF ERG equation (2.1). To explain what we mean, we express the global chiral transformation in terms of the so-called composite operators \[6, 28\]. We first introduce composite operators corresponding to the elementary fields \( A_\mu(x), \psi(x), \) and \( \bar{\psi}(x) \) by

\[ A_\mu(x) \equiv A_\mu(x) + \frac{\delta S}{\delta A_\mu(x)} = e^{-S} \hat{s} A_\mu(x) \hat{s}^{-1} e^S, \]  
(2.12a)

\[ \Psi(x) \equiv \psi(x) + i \frac{\delta}{\delta \bar{\psi}(x)} S = e^{-S} \hat{s} \psi(x) \hat{s}^{-1} e^S, \]  
(2.12b)

\[ \bar{\Psi}(x) \equiv \bar{\psi}(x) + i S \frac{\delta}{\delta \psi(x)} = e^{-S} \hat{s} \bar{\psi}(x) \hat{s}^{-1} e^S, \]  
(2.12c)

where \( \hat{s} \) is the scrambler \[22\]. When inserted into the functional integral, each of the above plays the role of an elementary field in the modified correlation function \[51\]. For instance, for Eq. (2.12a), we obtain

\[ \int d\mu e^{S} A_\mu(x) \hat{s}^{-1} \left[ \psi(x_1) \cdots \psi(x_n) \bar{\psi}(y_1) \cdots \bar{\psi}(y_n) \right] \]
\[ = \int d\mu \hat{s} A_\mu(x) \hat{s}^{-1} e^{S} \hat{s}^{-1} \left[ \psi(x_1) \cdots \psi(x_n) \bar{\psi}(y_1) \cdots \bar{\psi}(y_n) \right] \]
\[ = \int d\mu e^{S} \hat{s} \left[ A_\mu(x) \hat{s}^{-1} \left[ \psi(x_1) \cdots \psi(x_n) \bar{\psi}(y_1) \cdots \bar{\psi}(y_n) \right] \right] \]
\[ = \int d\mu e^{S} \hat{s} \left[ A_\mu(x) \psi(x_1) \cdots \psi(x_n) \bar{\psi}(y_1) \cdots \bar{\psi}(y_n) \right] \]
\[ = \langle \langle A_\mu(x) \psi(x_1) \cdots \psi(x_n) \bar{\psi}(y_1) \cdots \bar{\psi}(y_n) \rangle \rangle. \]  
(2.14)

\[ ^{4} \text{The modified correlation function is defined by} \]
\[ \langle \langle \varphi(x_1) \cdots \varphi(x_n) \rangle \rangle \equiv \int d\mu e^{S} \hat{s}^{-1} \varphi(x_1) \cdots \varphi(x_n), \]  
(2.13)

where \( d\mu \) is the measure of functional integration over the elementary field \( \varphi \), and \( \hat{s}^{-1} \) is the inverse of the scrambler \[22\]. The modified correlation function first introduced in Ref. \[51\] differs from Eq. (2.13) by the application of inverse diffusion, and exhibits a simple scaling behavior absent in the ordinary correlation function.
Using the composite operators in Eq. (2.12), we can introduce a composite operator
\[ Q_5 \equiv -e^{-S} \int d^D x \, \text{tr} \left\{ e^{-S} \gamma_5 \Psi(x') \left[ \frac{\delta}{\delta \psi(x)} + \frac{\delta}{\delta \bar{\psi}(x)} \left[ \bar{\Psi}(x') \gamma_5 e^S \right] \right] \right\}. \] (2.15)

Using Eqs. (2.12b) and (2.12c), we find
\[ Q_5 = e^{-S} \hat{\Gamma}_5 e^S. \] (2.16)

\( Q_5 \) is a particular example of the equation-of-motion composite operator [6, 28]. The equation-of-motion composite operator, when inserted into the functional integral, induces a transformation (such as the field shift) one by one on each field in the modified correlation function. The composite operator \( Q_5 \) defined by Eq. (2.15) generates the global chiral transformation:
\[
\int d\mu e^S Q_5 \hat{s}^{-1} [\psi(x_1) \cdots \psi(x_n) \bar{\psi}(y_1) \cdots \bar{\psi}(y_n)]
= -\int d\mu e^S \hat{s}^{-1} \int d^D x \left[ \gamma_5 \psi(x') \frac{\delta}{\delta \psi(x)} + \bar{\psi}(x') \gamma_5 \frac{\delta}{\delta \bar{\psi}(x)} \right] [\psi(x_1) \cdots \psi(x_n) \bar{\psi}(y_1) \cdots \bar{\psi}(y_n)]
= -\sum_{i=1}^n \left\{ \langle \psi(x_1) \cdots \gamma_5 \psi(x_i) \cdots \psi(x_n) \bar{\psi}(y_1) \cdots \bar{\psi}(y_n) \rangle \right. \\
- \sum_{i=1}^n \left. \langle \psi(x_1) \cdots \psi(x_n) \bar{\psi}(y_1) \cdots \bar{\psi}(y_i) \gamma_5 \cdots \bar{\psi}(y_n) \rangle \right\}. \] (2.17)

Now, \( Q_5 \) being a composite operator of scale dimension zero, \( S + \eta Q_5 \) satisfies the same GFERG equation (2.1) as \( S \) up to the first order in the constant \( \eta \). In other words the \( \tau \)-dependence of \( Q_5 \) is linear in \( Q_5 \):
\[
\partial_\tau Q_5 \\
= \left[ -\int d^D x \frac{\delta}{\delta A_\mu(x)} \hat{s} \left( 2\partial_\mu^2 + \frac{D-2}{2} + \gamma + x' \cdot \partial_{x'} \right) A_\mu(x') \hat{s}^{-1}, Q_5 \right] \\
+ \left[ \int d^D x \, e^{-S} \text{tr} \left[ \frac{\delta}{\delta \psi(x)} \right] \hat{s} \left[ 2\partial_\mu^2 + 4ieA_\mu(x'') \partial_{x'} - 2e^2 A_\mu(x'') A_\mu(x''') + \frac{D-1}{2} + \gamma_F + x' \cdot \partial_{x'} \right] \psi(x') \hat{s}^{-1}, Q_5 \right] \\
+ \left[ Q_5, \int d^D x \hat{s} \text{tr} \left[ 2\partial_\mu^2 - 4ieA_\mu(x'') \partial_{x'} - 2e^2 A_\mu(x'') A_\mu(x''') + \frac{D-1}{2} + \gamma_F + x' \cdot \partial_{x'} \right] \right].
\]
where the square brackets denote commutators. If
\[ Q_5 = 0 \] (2.19)
at some \( \tau \), this holds for any \( \tau \). This is what we meant by the “operator equation” at the beginning of the paragraph below Eq. (2.11).

3 1PI action

3.1 Legendre transformation

We now define a 1PI Wilson action \( \Gamma \) from the original Wilson action \( S \) by the following Legendre transformation:
\[
\Gamma[A_\mu, \Psi, \bar{\Psi}] - \frac{1}{2} \int d^Dx A_\mu(x)A_\mu(x) + i \int d^Dx \bar{\Psi}(x)\Psi(x)
\equiv S[A_\mu, \psi, \bar{\psi}] + \frac{1}{2} \int d^Dx A_\mu(x)A_\mu(x) - i \int d^Dx \bar{\psi}(x)\psi(x)
\]
\[
- \int d^Dx A_\mu(x)A_\mu(x) + i \int d^Dx \left[ \bar{\Psi}(x)\psi(x) + \bar{\psi}(x)\Psi(x) \right],
\] (3.1)
where the field variables \( A_\mu, \Psi, \) and \( \bar{\Psi} \) are defined by Eq. (2.12). We can regard \( \Gamma \) as a functional of these new variables. As is usual with the Legendre transformation, we have relations “dual” to Eq. (2.12):
\[
\frac{\delta \Gamma}{\delta A_\mu(x)} = -A_\mu(x),
\] (3.2a)
\[
i \frac{\delta}{\delta \bar{\psi}(x)} \Gamma - \Psi(x) = -\bar{\psi}(x),
\] (3.2b)
\[
i \frac{\delta}{\delta \psi(x)} \bar{\Psi}(x) = -\bar{\psi}(x).
\] (3.2c)

These dual relations can also be summarized as
\[
\frac{\delta S}{\delta A_\mu(x)} = A_\mu(x) - A_\mu(x) = \frac{\delta \Gamma}{\delta A_\mu(x)},
\] (3.3a)
\[
i \frac{\delta}{\delta \psi(x)} S = \Psi(x) - \psi(x) = i \frac{\delta}{\delta \bar{\psi}(x)} \Gamma,
\] (3.3b)
\[
i S \frac{\delta}{\delta \psi(x)} = \bar{\Psi}(x) - \bar{\psi}(x) = i \frac{\delta \Gamma}{\delta \bar{\psi}(x)}.
\] (3.3c)
3.2 Gauge transformation and invariance

We wish to find how $\Gamma$ transforms under the gauge transformation \[2.4\]. Using Eq. \[2.6\], we obtain
\[
\delta A_\mu(x) = \delta A_\mu(x) + \frac{\delta}{\delta A_\mu(x)} \delta S = \frac{\xi E(-e^{-2\tau \partial^2})e^{2\partial^2}}{\xi E(-e^{-2\tau \partial^2})e^{2\partial^2}} \partial_\mu \chi(x) + \frac{\partial^2}{\xi E(-e^{-2\tau \partial^2})e^{2\partial^2}} \partial_\mu \chi(x) = \partial_\mu \chi(x).
\]
\[3.4\]

The transformation of $\Psi$ and $\bar{\Psi}$ remains the same as $\psi$ and $\bar{\psi}$:
\[
\delta \Psi(x) = i e \chi(x) \Psi(x),
\]
\[3.4b\]
\[
\delta \bar{\Psi}(x) = -i e \chi(x) \bar{\Psi}(x).
\]
\[3.4c\]
Thus, the U(1) gauge transformation of $A_\mu(x)$, $\Psi(x)$, and $\bar{\Psi}(x)$ takes the conventional form.

Moreover, from Eq. \[3.1\] we obtain
\[
\delta \Gamma = \int d^D x \ A_\mu(x) \delta A_\mu(x) + \delta S + \int d^D x \ A_\mu(x) \delta A_\mu(x) - \int d^D x \ [A_\mu(x) A_\mu(x)] \\
= - \int d^D x \ \partial^2 \chi(x) \frac{1}{\xi E(-e^{-2\tau \partial^2})e^{2\partial^2}} \partial_\mu A_\mu(x),
\]
\[3.5\]
where we have used Eqs. \[2.4\], \[3.4\], and \[2.6\]. This shows that the combination
\[
\Gamma_{\text{inv}} \equiv \Gamma + \frac{1}{2} \int d^D x \ e^{-\partial^2} \partial_\mu A_\mu(x) \cdot \frac{1}{\xi E(-e^{-2\tau \partial^2})} \cdot e^{-\partial^2} \partial_\nu A_\nu(x)
\]
\[3.6\]
is invariant under Eq. \[3.4\], and it is preserved by the GFERG equation. We believe that the conventional gauge invariance of $\Gamma_{\text{inv}}$ is a remarkable property of our 1PI formulation. We expect this to facilitate the search for non-trivial fixed-point Wilson actions, where some kind of truncation is inevitable in practice. It is essential that the truncation keeps a gauge invariance that is under our control.

3.3 Global chiral transformation and symmetry

Using Eq. \[3.3\], we can rewrite $Q_5$ \[2.15\] in terms of the 1PI action $\Gamma$:
\[
Q_5 = \int d^D x \ \left[ \Gamma \frac{\delta}{\delta \psi(x) \gamma_5 \psi(x')} + \Psi(x') \gamma_5 \bar{\psi}(x) \right] \\
- \int d^D x \ \text{tr} \left[ \gamma_5 \psi(x') \frac{\delta}{\delta \psi(x)} + \frac{\delta}{\delta \psi(x)} \bar{\psi}(x') \gamma_5 \right].
\]
\[3.7\]
Note that the first integral on the right-hand side is the variation of the 1PI action under the conventional global chiral transformation

\[ \delta \Psi(x) = \alpha \gamma_5 \Psi(x), \quad \bar{\delta} \bar{\Psi}(x) = \alpha \bar{\Psi}(x) \gamma_5, \quad (3.8) \]

where \( \alpha \) is an infinitesimal parameter. The second integral, which has the form of a Jacobian under the change of integration variables

\[ \delta \psi(x) = \alpha \gamma_5 \Psi(x), \quad \bar{\delta} \bar{\psi}(x) = \alpha \bar{\Psi}(x) \gamma_5, \quad (3.9) \]

is equal to the “anomalous term” in Eq. (2.11):

\[ - \int d^D x \; \text{tr} \left[ \gamma_5 \Psi(x') \frac{\delta}{\delta \psi(x)} + \frac{\bar{\delta}}{\delta \bar{\psi}(x)} \bar{\Psi}(x') \gamma_5 \right] = \int d^D x \; \text{tr}(-2i) \gamma_5 \frac{\bar{\delta}}{\delta \bar{\Psi}(x')} \bar{\Psi}(x') \gamma_5 \frac{\delta}{\delta \psi(x)}. \quad (3.10) \]

As we explained in Sect. 2.2, the vanishing of \( Q_5 \) is the condition for global chiral invariance in the present formulation. If the anomalous term (3.10) vanishes, this becomes the invariance of \( \Gamma \) under the conventional global chiral transformation:

\[ Q_5 = \int d^D x \left[ \Gamma \frac{\delta}{\delta \Psi(x)} \gamma_5 \Psi(x') + \bar{\Psi}(x') \gamma_5 \frac{\delta}{\delta \bar{\psi}(x)} \Gamma \right] = 0. \quad (3.11) \]

In \( D = 4 \) the axial vector current suffers from the axial anomaly, but if we restrict ourselves only to the field configurations satisfying

\[ \int d^4 x \; \epsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta}(x) F_{\gamma\delta}(x) = 0, \quad (3.12) \]

where \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \), the anomalous term (3.10) still vanishes, and we can impose the global axial invariance by Eq. (3.11).

Please note that if the Wilson action is bilinear in fermion fields

\[ S = \int d^D x \int d^D y \bar{\psi}(x) iD(x,y)\psi(y), \quad (3.13) \]

the anomalous term becomes

\[ \int d^D x \; \text{tr} \left[ -2i\gamma_5 \frac{\delta}{\delta \psi(x')} \bar{\Psi}(x') \gamma_5 \frac{\delta}{\delta \bar{\psi}(x)} \right] = 2 \int d^D x \; \text{tr} \gamma_5 D(x,x). \quad (3.14) \]

This is an expression of the topological charge well known in lattice gauge theory [52–57]. For computation of the chiral anomaly in GFERG, see Refs. [10, 58]. Our discussion of chiral invariance given in this subsection should be considered preliminary; we would like to come back to the subject in a later publication with a more detailed analysis.
4 GFERG equation

In this section, we derive the GFERG flow equation for the 1PI action $\Gamma$. For this, we first note that the 1PI action $\Gamma$ at each $\tau$ is given by the Legendre transformation (3.1) of the Wilson action $S$ at $\tau$. Since the Legendre transformation does not have any explicit $\tau$-dependence, we have

$$\partial_\tau \Gamma = \partial_\tau S,$$  \hfill (4.1)

where the right-hand side is given by Eq. (2.1).

Next, we introduce

$$\left[ A_\mu(x') \Psi(x) \right] \equiv e^{-S} \hat{s} A_\mu(x') \psi(x) \hat{s}^{-1} e^S,$$  \hfill (4.2a)

$$\left[ A_\mu(x') A_\mu(x'') \Psi(x) \right] \equiv e^{-S} \hat{s} A_\mu(x') A_\mu(x'') \psi(x) \hat{s}^{-1} e^S,$$  \hfill (4.2b)

$$\left[ A_\mu(x') \bar{\Psi}(x) \right] \equiv e^{-S} \hat{s} A_\mu(x') \bar{\psi}(x) \hat{s}^{-1} e^S,$$  \hfill (4.2c)

$$\left[ A_\mu(x') A_\mu(x'') \bar{\Psi}(x) \right] \equiv e^{-S} \hat{s} A_\mu(x') A_\mu(x'') \bar{\psi}(x) \hat{s}^{-1} e^S.$$  \hfill (4.2d)

These define composite operators corresponding to products of fields. Note that these appear in the GFERG equation (2.1). With this in mind, we introduce the following composite operators by combining scaling and diffusion:

$$O_\mu(x) \equiv \left( \frac{D-2}{2} + \gamma + x \cdot \partial + 2 \partial^2 \right) A_\mu(x),$$  \hfill (4.3a)

$$O_F(x) \equiv \left( \frac{D-1}{2} + \gamma_F + x \cdot \partial + 2 \partial^2 \right) \Psi(x) - 4ie \left[ A_\mu(x') \partial_\mu \Psi(x) \right] - 2e^2 \left[ A_\mu(x') A_\mu(x'') \Psi(x) \right],$$  \hfill (4.3b)

$$\bar{O}_F(x) \equiv \left( \frac{D-1}{2} + \gamma_F + x \cdot \partial + 2 \partial^2 \right) \bar{\Psi}(x) + 4ie \left[ A_\mu(x') \partial_\mu \bar{\Psi}(x) \right] - 2e^2 \left[ A_\mu(x') A_\mu(x'') \bar{\Psi}(x) \right].$$  \hfill (4.3c)

We then introduce the equation-of-motion composite operators by

$$\mathcal{E}_A(x) \equiv -e^{-S} \frac{\delta}{\delta A_\mu(x)} \left[ e^S O_\mu(x') \right],$$  \hfill (4.4a)

$$\mathcal{E}_F(x) \equiv \text{tr} \left[ e^S O_F(x') \right] \frac{\delta}{\delta \bar{\psi}(x)} e^{-S},$$  \hfill (4.4b)

$$\mathcal{E}_F(x) \equiv e^{-S} \text{tr} \frac{\delta}{\delta \bar{\psi}(x)} \left[ e^S O_F(x') \right],$$  \hfill (4.4c)
so that the GFERG equation (2.1) can be expressed concisely as

\[ \partial_\tau \Gamma = \int d^Dx \left[ \mathcal{E}_A(x) + \mathcal{E}_F(x) + \bar{\mathcal{E}}_F(x) \right]. \tag{4.5} \]

Since \( \Gamma \) is a functional of \( A_\mu, \Psi, \) and \( \bar{\Psi} \), we wish to express the right-hand side above in terms of those field variables. We do this in steps.

First, it follows from definition (4.4) that

\[ \mathcal{E}_A(x) = -\frac{\delta \Gamma}{\delta A_\mu(x)} O_\mu(x) - \frac{\delta}{\delta A_\mu(x)} O_\mu(x'), \tag{4.6a} \]

\[ \mathcal{E}_F(x) = \text{tr} \left[ O_F(x') \cdot \Gamma \frac{\delta}{\delta \Psi(x)} + O_F(x') \frac{\delta \bar{\psi}}{\delta \bar{\psi}(x)} \right], \tag{4.6b} \]

\[ \bar{\mathcal{E}}_F(x) = \text{tr} \left[ \bar{\Gamma} \cdot O_F(x') + \frac{\delta \bar{\psi}}{\delta \bar{\psi}(x)} O_F(x') \right], \tag{4.6c} \]

where we have noted Eq. (3.3). \( O_\mu(x), O_F(x), \) and \( \bar{O}_F(x') \) are given by Eq. (4.3) and they contain composite operators in Eq. (4.2). The products are explicitly given by, from Eq. (2.12),

\[ \left[ A_\mu(x') \Psi(x) \right] = A_\mu(x') \Psi(x) + \frac{\delta}{\delta A_\mu(x')} \Psi(x) \]

\[ = A_\mu(x') \Psi(x) + i \frac{\delta}{\delta \bar{\psi}(x)} A_\mu(x'), \tag{4.7a} \]

\[ \left[ A_\mu(x') \bar{\Psi}(x) \right] = A_\mu(x') \bar{\Psi}(x) + \frac{\delta}{\delta A_\mu(x')} \bar{\Psi}(x) \]

\[ = A_\mu(x') \bar{\Psi}(x) + A_\mu(x') \bar{\Psi}(x) i \frac{\delta}{\delta \bar{\psi}(x)}. \tag{4.7b} \]

The equality of the two expressions on the right-hand sides follows from the definition (4.2), because \( A_\mu(x') \psi(x) = \psi(x) A_\mu(x') \) etc. Similarly, we have

\[ \left[ A_\mu(x') A_\mu(x'') \Psi(x) \right] = \left[ A_\mu(x') + \frac{\delta}{\delta A_\mu(x')} \right] \left[ A_\mu(x'') \Psi(x) \right] \]

\[ = \left[ A_\mu(x'') + \frac{\delta}{\delta A_\mu(x'')} \right] \left[ A_\mu(x') \Psi(x) \right] \]

\[ = \left[ \Psi(x) + i \frac{\delta}{\delta \bar{\psi}(x)} \right] \left[ A_\mu(x') A_\mu(x'') \right], \tag{4.8a} \]

\[ \left[ A_\mu(x') A_\mu(x'') \bar{\Psi}(x) \right] = \left[ A_\mu(x') + \frac{\delta}{\delta A_\mu(x')} \right] \left[ A_\mu(x'') \bar{\Psi}(x) \right] \]
\[ \begin{align*}
\mathcal{A}_\mu(x''') + \frac{\delta}{\delta A_\mu(x''')} \left[ \mathcal{A}_\mu(x') \bar{\Psi}(x) \right] \\
= \left[ \mathcal{A}_\mu(x') \mathcal{A}_\mu(x'') \right] \left[ \bar{\Psi}(x) + i \frac{\delta}{\delta \bar{\psi}(x)} \right].
\end{align*} \tag{4.8b} \]

From these expressions, as the explicit forms of the equation-of-motion composite operators in Eq. (4.6), we have
\[ \begin{align*}
\mathcal{E}_A(x) &= -\frac{\delta \Gamma}{\delta A_\mu(x)} \left( 2 \partial^2 + \frac{D - 2}{2} + \gamma + x \cdot \partial \right) A_\mu(x) \\
&\quad \left( 2 \partial^2 + \frac{D - 2}{2} + \gamma + x' \cdot \partial_x' \right) \frac{\delta A_\mu(x')}{\delta A_\mu(x)}, \tag{4.9} \\
\mathcal{E}_F(x) &= -\frac{\delta \bar{\Gamma}}{\delta \bar{\Psi}(x)} \left( 2 \partial^2 + \frac{D - 1}{2} + \gamma_F + x \cdot \partial \right) \bar{\Psi}(x) \\
&\quad + \text{tr} \left( 2 \partial^2 + \frac{D - 1}{2} + \gamma_F + x' \cdot \partial_x' \right) \bar{\Psi}(x') \frac{\delta}{\delta \bar{\psi}(x)} \\
&\quad + 4ie\frac{\delta \bar{\Gamma}}{\delta \bar{\Psi}(x)} \left[ A_\mu(x) \partial_\mu \bar{\Psi}(x) + \partial_\mu \frac{\delta}{\delta A_\mu(x')} \bar{\Psi}(x) \right] \\
&\quad + 4ie \text{tr} \left[ A_\mu(x') \partial_\mu \bar{\Psi}(x') + \partial_\mu \frac{\delta}{\delta A_\mu(x''')} \bar{\Psi}(x') \right] \\
&\quad + 2e^2 \frac{\delta}{\delta \bar{\Psi}(x)} \left\{ A_\mu(x) \mathcal{A}_\mu(x) \bar{\Psi}(x) + A_\mu(x) \frac{\delta}{\delta A_\mu(x')} \bar{\Psi}(x) \\
&\quad + \frac{\delta}{\delta A_\mu(x')} \left[ A_\mu(x''') \bar{\Psi}(x) \right] + \frac{\delta^2}{\delta A_\mu(x') \delta A_\mu(x''')} \bar{\Psi}(x) \right\} \\
&\quad - 2e^2 \text{tr} \left\{ A_\mu(x''') \mathcal{A}_\mu(x''') \bar{\Psi}(x') + A_\mu(x''') \frac{\delta}{\delta A_\mu(x''')} \bar{\Psi}(x') \\
&\quad + \frac{\delta}{\delta A_\mu(x''')} \left[ A_\mu(x''') \bar{\Psi}(x') \right] + \frac{\delta^2}{\delta A_\mu(x') \delta A_\mu(x''')} \bar{\Psi}(x') \right\} \frac{\delta}{\delta \bar{\psi}(x)}, \tag{4.10} \\
\bar{\mathcal{E}}_F(x) &= -\left( 2 \partial^2 + \frac{D - 1}{2} + \gamma_F + x \cdot \partial \right) \bar{\Psi}(x) \cdot \frac{\delta}{\delta \bar{\psi}(x)} \bar{\Gamma} \\
&\quad + \text{tr} \left( 2 \partial^2 + \frac{D - 1}{2} + \gamma_F + x' \cdot \partial_x' \right) \frac{\delta}{\delta \bar{\psi}(x)} \bar{\Psi}(x')
\end{align*} \]
Thus, the first derivatives are given by the inverse of differentiation, the first derivatives satisfy

\[
-4ie \left[ \mathcal{A}_\mu(x) \partial_\mu \bar{\Psi}(x) + \partial_\mu \frac{\delta}{\delta A_\mu(x')} \bar{\Psi}(x) \right] \frac{\delta}{\delta \bar{\Psi}(x)} \Gamma \\
+ 4ie \text{tr} \frac{\delta}{\delta \bar{\psi}(x)} \left[ \mathcal{A}_\mu(x') \partial_\mu \bar{\Psi}(x') + \partial_\mu \frac{\delta}{\delta A_\mu(x'')} \bar{\Psi}(x') \right] \\
+ 2e^2 \left\{ \mathcal{A}_\mu(x) \mathcal{A}_\mu(x) \bar{\Psi}(x) + \mathcal{A}_\mu(x) \frac{\delta}{\delta A_\mu(x')} \bar{\Psi}(x) \right. \\
\quad + \left. \frac{\delta}{\delta A_\mu(x')} \left[ \mathcal{A}_\mu(x'') \bar{\Psi}(x') \right] + \frac{\delta^2}{\delta A_\mu(x') \delta A_\mu(x'')} \bar{\Psi}(x') \right\} \frac{\delta}{\delta \bar{\Psi}(x)} \Gamma \\
- 2e^2 \text{tr} \frac{\delta}{\delta \bar{\psi}(x)} \left\{ \mathcal{A}_\mu(x'') \mathcal{A}_\mu(x'') \bar{\Psi}(x) + \mathcal{A}_\mu(x'') \frac{\delta}{\delta A_\mu(x') \delta A_\mu(x'')} \bar{\Psi}(x') \right. \\
\quad + \left. \frac{\delta}{\delta A_\mu(x'')} \left[ \mathcal{A}_\mu(x''') \bar{\Psi}(x''') \right] + \frac{\delta^2}{\delta A_\mu(x'') \delta A_\mu(x'''')} \bar{\Psi}(x''') \right\}.
\]

(4.11)

These expressions contain the functional derivatives of the Legendre transformed variables \((\mathcal{A}_\mu, \Psi, \bar{\Psi})\) with respect to the original field variables \((A_\mu, \psi, \bar{\psi})\). By the chain rule of differentiation, the first derivatives satisfy

\[
\begin{pmatrix}
\delta_{\mu\nu} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\delta(x - y) =
\begin{pmatrix}
\frac{\delta A_\nu(y)}{\delta A_\mu(x)} & \frac{\delta \bar{\psi}(y)}{\delta A_\mu(x)} & \frac{\delta \psi(y)}{\delta A_\mu(x)} \\
\frac{\delta \bar{A}_\mu(x)}{\delta \psi(y)} & \frac{\delta \psi(y)}{\delta \psi(y)} & \frac{\delta \bar{\psi}(y)}{\delta \psi(y)} \\
\frac{\delta \bar{A}_\mu(x)}{\delta \bar{\psi}(y)} & \frac{\delta \bar{\psi}(y)}{\delta \bar{\psi}(y)} & \frac{\delta \psi(y)}{\delta \bar{\psi}(y)}
\end{pmatrix}
\]

\[
= \int d^D z
\begin{pmatrix}
\frac{\delta \bar{\psi}(z)}{\delta \psi(z)} \mathcal{A}_\rho(z) & \frac{\delta \bar{\psi}(z)}{\delta \psi(z)} \bar{\Psi}(z) & \frac{\delta \psi(y)}{\delta \psi(z)} \mathcal{A}_\rho(z) \\
\frac{\delta \bar{\psi}(z)}{\delta \psi(z)} \bar{\Psi}(z) & \frac{\delta \psi(y)}{\delta \psi(z)} \Psi(z) & \frac{\delta \bar{\psi}(y)}{\delta \psi(z)} \bar{\Psi}(z) \\
\frac{\delta \bar{\psi}(z)}{\delta \psi(z)} \bar{\Psi}(z) & \frac{\delta \psi(y)}{\delta \psi(z)} \psi(z) & \frac{\delta \bar{\psi}(y)}{\delta \psi(z)} \psi(z)
\end{pmatrix}
\begin{pmatrix}
\frac{\delta A_\nu(y)}{\delta A_\mu(x)} & \frac{\delta \bar{\psi}(y)}{\delta A_\mu(x)} & \frac{\delta \psi(y)}{\delta A_\mu(x)} \\
\frac{\delta \bar{A}_\mu(x)}{\delta \psi(y)} & \frac{\delta \psi(y)}{\delta \psi(y)} & \frac{\delta \bar{\psi}(y)}{\delta \psi(y)} \\
\frac{\delta \bar{A}_\mu(x)}{\delta \bar{\psi}(y)} & \frac{\delta \bar{\psi}(y)}{\delta \bar{\psi}(y)} & \frac{\delta \psi(y)}{\delta \bar{\psi}(y)}
\end{pmatrix}
\]

(4.12)

Thus, the first derivatives are given by the inverse of
where we have used Eq. (3.2). These first-order functional derivatives are common in the ordinary ERG formulation for the 1PI action. What is peculiar to the GFERG formulation is the presence of the second- and third-order functional derivatives appearing in $\mathcal{E}_F(x)$ (4.10) and $\tilde{\mathcal{E}}_F(x)$ (4.11). These higher-order derivatives are necessary for the manifest gauge invariance of $\Gamma$, and they can be obtained by differentiating further the elements of the above inverse matrix.

This extra labor is required for the sake of manifest gauge invariance for the 1PI Wilson action.

5 Perturbative solution

If we restrict ourselves only to the Wilson action parametrized by the gauge coupling $e$, the electron mass $m$, and the gauge-fixing parameter $\xi$, we can solve the GFERG equation (4.5) as a power series in $e$, where the lowest-order term is the Gaussian fixed point. We may directly solve Eq. (4.5) or apply the Legendre transformation (3.1) to the Wilson action $S$ that has been obtained perturbatively to order $e^2$ in Ref. [11]. Following the latter approach, a straightforward calculation gives

$$
\Gamma = \frac{1}{2} \int_k e^{2k^2} A_{\mu}(k) e^{2k^2} A_{\nu}(-k) \left\{ \left( \frac{\delta_{\mu\nu} k^2 - k_{\mu} k_{\nu}}{k^2} \right) \left[ 1 - e^{2} \frac{\bar{V}_{T}(k)}{k^2} \right] + \frac{1}{\xi} k_{\mu} k_{\nu} \right\}
- \int_p \bar{\Psi}(-p) e^{p^2} \left[ (\hat{p} + im) - e^{2} \bar{V}_F(p) \right] e^{p^2} \Psi(p)
+ e \int_{p,k} \bar{\Psi}(-p-k) e^{(p+k)^2} \bar{V}_{\mu}(p,k) e^{p^2} \Psi(p) e^{k^2} A_{\mu}(k)
+ e^2 \int_{p,k} \bar{\Psi}(-p-k-l) e^{(p+k+l)^2} \nabla_{\mu\nu}(p,k,l) e^{p^2} \Psi(p) e^{k^2} A_{\mu}(k) e^{l^2} A_{\nu}(l)
+ O(e^3),
$$

(5.1)
where $\bar{V}_{\mu
u}(p, k, l) = \bar{V}_{\nu\mu}(p, l, k)$ is symmetric. The functions $\bar{V}_\mu(p, k)$ and $\bar{V}_{\mu\nu}(p, k, l)$ are given in Ref. [11] as follows:

$$\bar{V}_\mu(p, k) = \gamma_\mu + 2(\not{p} + \not{k} + \text{i}m)p_\mu F((p + k)^2 - p^2 - k^2)$$
$$+ 2(\not{p} + \text{i}m)(p + k)_\mu F(p^2 - (p + k)^2 - k^2),$$  (5.2)

where

$$F(x) \equiv \frac{e^x - 1}{x}$$  (5.3)

and

$$\bar{V}_{\mu\nu}(p, k, l) = -\delta_{\mu\nu}[(\not{p} + \not{k} + \not{\ell} + \text{i}m)F((p + k + l)^2 - p^2 - k^2 - l^2)$$
$$+ (\not{\ell} + \text{i}m)F(p^2 - (p + k + l)^2 - k^2 - l^2)]$$
$$- 4X_{\mu\nu}(p, k, l),$$  (5.4)

where

$$X_{\mu\nu}(p, k, l) = X_{\nu\mu}(p, l, k)$$
$$= \frac{1}{4} \gamma_\mu p_\alpha F((p + l)^2 - p^2 - l^2) + \frac{1}{4}(p + k + l)_\mu \gamma_\nu F((p + l)^2 - (p + k + l)^2 - k^2)$$
$$+ \frac{1}{2}(\not{\ell} + \text{i}m)(p + k + l)_\mu p_\nu F((p + l)^2 - (p + k + l)^2 - k^2)F((p + l)^2 - p^2 - l^2)$$
$$+ \frac{1}{2} \frac{(\not{\ell} + \text{i}m)(p + k + l)_\mu (p + l)_\nu}{(p + k + l)^2 - (p + l)^2 - k^2}$$
$$\times [F((p + k + l)^2 - p^2 - k^2 - l^2) - F((p + l)^2 - p^2 - l^2)]$$
$$+ \frac{1}{2} \frac{(\not{\ell} + \text{i}m)(p + k + l)_\mu (p + l)_\nu}{p^2 - (p + l)^2 - l^2}$$
$$\times [F(p^2 - (p + k + l)^2 - k^2 - l^2) - F((p + l)^2 - (p + k + l)^2 - k^2)]$$
$$+ (\mu \leftrightarrow \nu, k \leftrightarrow l).$$  (5.5)

As for $\bar{V}_T(k)$ and $\bar{V}_F(p)$ in Eq. (5.1) (one-loop corrections to the kinetic terms), only the leading terms in the momentum expansions have been explicitly computed in Ref. [11].

The above results, especially the explicit form of the function $X_{\mu\nu}(p, k, l)$ (5.5), might appear complicated, but the basic structure of the 1PI action (5.1) is much simpler than that of the corresponding Wilson action $S$. For example, the Wilson action $S$ contains a
obtained from expanding the manifestly gauge invariant expression excluding the gauge-fixing term, the right-hand side of Eq. (5.1) can be obtained by possibly to verify Eq. (5.9). Thus, the 1PI Wilson action is gauge invariant. Actually, with it is easy to check that
\[ \bar{\Psi}(p, k) = e^{(p-k)^2/2} (\hat{p} + k + i m) - e^{(p-k)^2/2} \hat{p} + i m, \]
\[ (5.8) \]
and
\[ 2k \bar{\nabla}_{\mu \nu} (p, k, l) = e^{(p+k)^2 - (p+k+l)^2} \bar{V}_\mu (p, l) - e^{(p+k)^2 - k^2} \bar{V}_\sigma (p + k, l). \]
\[ (5.9) \]
It is easy to check that \( \bar{V}_\mu \) given by Eq. (5.2) satisfies Eq. (5.8). It is tedious but also possible to verify Eq. (5.9). Thus, the 1PI Wilson action is gauge invariant. Actually, with the exception of the gauge-fixing term, the right-hand side of Eq. (5.1) can be obtained by expanding the manifestly gauge invariant expression\(^5\)
\[ -\frac{1}{4} \int d^4 x \left[ \partial_\mu A_{-1\nu} (x) - \partial_\nu A_{-1\mu} (x) \right]^2 + i \int d^4 x \bar{\Psi}_{-1} (x) [\hat{\theta} - i e A_{-1} (x) - m] \Psi_{-1} (x), \]
\[ (5.10) \]
up to order \( \epsilon^2 \). Here, the fields with subscript \( t = -1 \) are the solutions of the diffusion equations
\[ \partial_t A_{t\mu} (x) = \partial_\nu \left[ \partial_\nu A_{t\mu} (x) - \partial_\nu A_{t\nu} (x) \right] + \alpha_0 \partial_\mu \partial_\nu A_{t\nu} (x), \quad A_{0\mu} (x) \equiv A_\mu (x) \]
\[ (5.11a) \]
\[ \partial_t \Psi_t (x) = \left\{ \left[ \partial_\mu - i e A_{t\mu} (x) \right]^2 + i e \alpha_0 \left[ \partial_\mu A_{t\mu} (x) \right] \right\} \Psi_t (x), \quad \Psi_0 (x) \equiv \Psi (x) \]
\[ (5.11b) \]
\[ \partial_t \bar{\Psi}_t (x) = \left\{ \left[ \partial_\mu + i e A_{t\mu} (x) \right]^2 - i e \alpha_0 \left[ \partial_\mu A_{t\mu} (x) \right] \right\} \bar{\Psi}_t (x), \quad \bar{\Psi}_0 (x) \equiv \bar{\Psi} (x) \]
\[ (5.11c) \]
solved backward from \( t = 0 \) to \( t = -1 \). Our GFERG equation (2.1) is based on the above diffusion equations with \( \alpha_0 = 1 \) (see, for instance, Eq. (2.16) of Ref. [11]). These are the flow equations introduced in Refs. [25, 27], although the solution \( (A_{t\mu} (x), \Psi_t (x), \bar{\Psi}_t (x)) \) does not

\(^5\) Here, we omit terms containing \( \bar{V}_T (k) \) and \( \bar{V}_F (p) \) for simplicity.
transform gauge covariantly under Eq. (3.4) for $\alpha_0 \neq 0$, any gauge invariant combination such as Eq. (5.10) has been shown independent of $\alpha_0$ \cite{25}, and thus is invariant under Eq. (3.4) \cite{6}.

6 Conclusion

In this paper we have introduced a one-particle irreducible Wilson action $\Gamma$ for QED using the GFERG (gradient flow exact renormalization group) formalism. This is a straightforward extension of the previous work \cite{11} where the GFERG formalism was applied to the construction of a Wilson action for QED. Realization of gauge invariance within the ERG formalism has been of interest for its potential applications to non-perturbative aspects of gauge theory, such as the search for non-trivial fixed points of the renormalization group flow. But the gauge invariance is modified within the ERG formalism in a non-trivial way, and it has been difficult to ensure gauge invariance in practical calculations. We have succeeded in constructing $\Gamma$ with manifest invariance under the conventional gauge transformation, even though the GFERG flow equation that preserves the gauge invariance is admittedly somewhat complicated.

There are three lines of development that we can pursue after this work. The first is to find the relation, perhaps equivalence, of the GFERG formalism to the ERG formalism. Do they share the same, i.e., physically equivalent, fixed points? Do they give the same universality classes? We believe that the two formalisms are physically equivalent, but the equivalence must be proven.

The second is more ambitious. We have referred to non-perturbative applications a couple of times in this paper. There are already quite a few preceding works that investigate non-perturbatively fixed points and critical exponents of Abelian gauge theory \cite{59,62}. It should be quite interesting to reexamine the results using our manifestly gauge invariant formalism.

Finally, we would like to generalize the present formulation to gauge-fixed non-Abelian gauge theory. We suspect that it would be important to figure out how to deal with Faddeev–Popov ghost fields and the Nakanishi–Lautrup auxiliary field. It would be nice to have $\Gamma$ manifestly invariant under the conventional gauge (or BRST) transformation.

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\footnote{Under the gauge transformation \cite{94}, $(A_{\mu}(x), \Psi_t(x), \bar{\Psi}_t(x))$ transforms as Eq. (5.4), where $\chi(x)$ is replaced by $\chi_t(x) = e^{i\alpha_0 \partial^2} \chi(x)$. This also explains the gauge invariance of Eq. (5.10).}
A BRST invariant 1PI action

We have shown that $\Gamma[A_\mu, \Psi, \bar{\Psi}]$ defined by Eq. (3.1) transforms as Eq. (3.5) under the gauge transformation (3.4). Now, the ghost action is given by Eq. (2.5). With

$$C(x) \equiv c(x) + \frac{\delta}{\delta c(x)} S_{\text{ghost}} = \frac{E(-e^{-2\tau \partial^2})e^{2\partial^2}}{E(-e^{-2\tau \partial^2})e^{2\partial^2} - \partial^2} c(x), \tag{A1}$$

$$\bar{C}(x) \equiv \bar{c}(x) + S_{\text{ghost}} \frac{\xi}{\delta \bar{c}(x)} = \frac{E(-e^{-2\tau \partial^2})e^{2\partial^2}}{E(-e^{-2\tau \partial^2})e^{2\partial^2} - \partial^2} \bar{c}(x), \tag{A2}$$

we define $\Gamma_{\text{ghost}}[C, \bar{C}]$ by

$$\Gamma_{\text{ghost}} - \int d^D x \bar{C}(x)C(x) = S_{\text{ghost}} + \int d^D x \bar{c}(x)c(x) - \int d^D x \left[ \bar{C}(x)c(x) + \bar{c}(x)C(x) \right]. \tag{A3}$$

This gives

$$\Gamma_{\text{ghost}}[C, \bar{C}] = \int d^D x \bar{C}(x) \frac{\partial^2}{E(-e^{-2\tau \partial^2})e^{2\partial^2}} C(x). \tag{A4}$$

To obtain a BRST transformation we choose

$$\chi(x) = \eta c(x), \tag{A5}$$

where $\eta$ is an anticommuting constant, and transform the ghost fields by

$$\delta C(x) = 0, \quad \delta \bar{C}(x) = \frac{1}{\xi} \eta \partial \cdot A(x). \tag{A6}$$

We then obtain

$$\delta \Gamma = -\int d^D x \frac{1}{\xi} \partial^2 \eta C(x) \frac{1}{E(-e^{-2\tau \partial^2})e^{2\partial^2}} \partial \cdot A(x), \tag{A7}$$

$$\delta \Gamma_{\text{ghost}} = \int d^D x \frac{1}{\xi} \eta \partial \cdot A(x) \frac{\partial^2}{E(-e^{-2\tau \partial^2})e^{2\partial^2}} C(x). \tag{A8}$$

Hence, the total 1PI Wilson action is BRST invariant:

$$\delta (\Gamma + \Gamma_{\text{ghost}}) = 0. \tag{A9}$$

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