Dualities as symmetries of the Supermembrane Theory

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ABSTRACT: In this note I review the role played by dualities in the Supermembrane Theory compactified on a torus. Supermembrane theory realize S, T, so U-duality, as exact symmetries of the theory. There are two well defined sectors: with and without central charges. Both sectors have the $SL(2,\mathbb{Z})_T \times \mathbb{Z}_2$ U-duality group in 9D as a symmetry of the theory, but the supermembrane with central charges, also exhibits an extra $SL(2,\mathbb{Z})_Σ$ symmetry of the theory associated to the diffeomorphisms not connected with the identity changing the homology basis. The $\mathbb{Z}_2$ symmetry is T-duality transformation that acts on the supermembrane and generalize that of the string. This T-duality transformation acts locally and also globally on the supermembrane theory. It has a natural description in terms of the cohomology of the base manifold and the homology of the target torus. When only string-like states are preserved, one recovers both type II string theories in 9D. If one also consider properly limit of the T-duality transformation characterizing the supermembrane, the usual closed string T-duality transformation between the type IIA and type IIB mass operators is recovered.

KEYWORDS: Supermembrane, U-duality, T-duality.

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1. Introduction

In this note we review recent, and not so recent results that together with my collaborators, we have found in different papers [5], [1], [3]. The purpose of this note is to give a unified picture of the dualities of the 2-torus compactified supermembrane. There are two well-defined sectors in the theory that have to be considered separately since they exhibit important physical differences. We have found that, however, both sectors realize S-duality, and T-duality, so U-duality, as exact symmetries of the theory although in a different way.

T-duality transformation at the worldsheet level were studied in [35]. The relation of duality and M-theory was also analyzed in [36]. A topological analysis of T-duality in terms of toroidal bundles with monodromy form a mathematical perspective has been done in [47]. In [9] it was argued that a fundamental formulation of string/M-theory should exist in which the T- and U-duality symmetries are manifest from the start. In particular, it was argued that many massive, gauged supergravities cannot be naturally embedded in string theory without such a framework [10], [11], [25], [12].

Double field theory has become an interesting arena to try to realize in a bottom-up approach, some of the properties of string theory. It is a global approach that
describe sigma models with double coordinates on a $T^{2d}$ torus fibrations such that the transition functions will be evaluated in the T-duality group $O(d, d, \mathbb{Z})$. There is evidence that string theory can be consistently defined in non-geometric backgrounds in which the transition functions between coordinate patches involve not only diffeomorphisms and gauge transformations but also duality transformations [1]. To this end it is relevant to understand the global description of dualities in terms of bundles. Some global aspects of T-duality in String theory formerly analyzed in [18], were more recently realized in this context by [19]. The type II realization has been done recently in [15, 14]. The proposed actions are such that they are invariant under duality transformations.

In this type of compactifications, T-folds are constructed by using strings formulated on a doubled torus $T^{2n}$ with n-coordinates conjugate to the momenta and the other n-coordinates conjugate to the winding modes [3], plus a constraint to guarantee the correct number of propagating degrees of freedom. Such backgrounds can arise from compactifications with duality twists [20] or from acting on geometric backgrounds with fluxes with T-duality [4, 11]. In special cases, the compactifications with duality twists are equivalent to asymmetric orbifolds which can give consistent string backgrounds [21, 22, 23, 24]. Examples of generalized T-folds can be obtained by constructing torus fibrations over base manifolds with non-contractible cycles. The same type of approach can be also implemented in M-theory. There has been works done that realize effective duality invariant approach to M-theory by a U-duality group valued Scherk-Schwarz twist [23, 26] and are related with generalized geometry [27, 20].

Our approach however will be different: we perform a top-down approach departing from a sector of M-theory: the supermembrane compactified on a torus. In the case of interest (nine noncompact dimensions) the conjectured group of U-dualities for type II string theories is the product: $SL(2, \mathbb{Z}) \times Z_2$ [17]. Previous works on the role of dualities on the supermembrane were done by [28, 29, 30]. The compactified supermembrane on a target space $M_9 \times T^2$ may be formulated in terms of sections on symplectic torus bundles [3]. There are two well-defined sectors: one in which a topological condition due to an irreducible wrapping is imposed, that corresponds to the so-called supermembrane with central charges [3], and a second one on which the previous condition vanishes. While the first one can be globally formulated in terms of sections on symplectic torus bundles with $SL(2, \mathbb{Z})$ monodromy [3] and at low energies corresponds to the type II gauged supergravities in 9D [1] through a gauging mechanism that acts on the structure of the bundles [43], the second one corresponds to the formulation on trivial symplectic torus bundles [4] associated to the maximal supergravity in 9D [14]. Physically the two sectors have very different properties among which, the most relevant one, is that the regularized supermembrane with central charges has discrete spectrum [40] - [43] - so it is a well-
defined quantum object— in distinction with the compactified supermembrane with vanishing central charge condition [8]. Due to this very relevant property it has become sensible to study the supermembrane as a quantum object in lower dimensions [34], [33], in G2 compactifications [31] or for example its non-abelian extension [32].

In this note we show that the supermembrane compactified on a torus is invariant under a S-duality and T-duality transformation of the supermembrane [1], [5]. More concretely $SL(2, \mathbb{Z})$ as a symmetry associated to the 2-torus target space appears in both sector of the theory. However there is an extra $SL(2, \mathbb{Z})$ that only acts on the supermembrane with central charges sector [5].

In [1] we showed the existence of a new $Z_2$ symmetry that plays the role of T-duality in the supermembrane interchanging the winding and KK charges but leaving the Hamiltonian invariant. T-duality becomes an exact symmetry of the symplectic torus bundle description of the supermembrane by fixing its energy tension. We will review how in the String Theory limit, the T-duality transformation for the supermembrane becomes the standard T-duality transformation of the closed superstrings compactified on a circle [3].

The outline of this note is the following: In section 2, we review the compactified supermembrane on a torus and we establish the two well differentiated topological sectors, with and without central charges, or equivalently with non-trivial or trivial second cohomology class. In section 3, we show the two types of $SL(2, \mathbb{Z})$ symmetries that the compactified supermembrane can have. However only the supermembrane with central charges exhibits both. The compactified supermembrane with vanishing second cohomology class only has the $SL(2, \mathbb{Z})$ symmetry associated to the 2-torus target space. In section 3, the T-duality transformation is introduced from a local and global approach. The differences in both sectors are pointed out. It is shown that it realizes as a symmetry of the theory when a relation between the scale of energy and the moduli is established. In section 4, the string theory limits are discussed. When only string-like configurations of the supermembrane mass operator are preserved the perturbative type IIA and type IIB $SL(2, \mathbb{Z})$ mass spectrum are recovered. If in distinction, a T-dual transformation is performed on the M2 mass operator and then the string-like configurations are the only ones preserved then the nonperturbative $SL(2, \mathbb{Z})$ mass spectrum is obtained. If also a circle limit is imposed on the T-duality transformation then the usual T-duality for type II closed string is recovered. In section 5, conclusions and comments are presented.

2. The Compactified Supermembrane in $M_9 \times T^2$

We consider now the compactified supermembrane embedded on a target space $M_9 \times T^2$ where $T^2$ is a flat torus. We consider maps $X^m, X^r$ from $M_9 \times T^2$ to the target space, where $X^m$ with $m = 3, \ldots, 9$ are single valued maps onto the Minkowski
sector of the target space while \( X^r \), with \( r = 1, 2 \) maps onto the \( T^2 \) compact sector of the target. The winding condition corresponds to

\[
\oint_{C_s} dX^1 = 2\pi R(l_s + m_s R\epsilon \tau); \quad \oint_{C_s} dX^2 = 2\pi R m_s I m \tau; \quad \oint_{C_s} dX^m = 0 \quad (2.1)
\]

where \( R, \tau \) are respectively the radius and the Teichmüller parameters of the 2-torus target-space, and \( l_s, m_s, s = 1, 2 \), are integers.

The physical Hamiltonian in the LCG is given by

\[
\mathcal{H} = \int_{\Sigma} T_{11}^{-2/3} \sqrt{W} \left[ \frac{1}{2} \left( \frac{P_m}{\sqrt{W}} \right)^2 + \frac{1}{2} \left( \frac{P_r}{\sqrt{W}} \right)^2 + \frac{T_{11}^2}{2} \{ X^r, X^m \}^2 + \frac{T_{11}^2}{4} \{ X^r, X^s \}^2 \right] \\
+ \int_{\Sigma} T_{11}^{-2/3} \sqrt{W} \left[ \frac{T_{11}^2}{4} \{ X^m, X^n \}^2 - \Psi \Gamma_m \{ X^m, \Psi \} - \Psi \Gamma_r \{ X^r, \Psi \} \right]
\]

subject to the constraints

\[
\phi_1 := d\left( \frac{P_m}{\sqrt{W}} dX^m + \frac{P_r}{\sqrt{W}} dX^r - \Psi \Gamma_m d\Psi \right) = 0, \\
\phi_2 := \oint_{C_s} \left( \frac{P_M}{\sqrt{W}} dX^M + \frac{P_r}{\sqrt{W}} dX^r - \Psi \Gamma_m d\Psi \right) = 0, \quad (2.2)
\]

associated with a residual symmetry of the theory: the infinite group of diffeomorphisms preserving the Riemann basis \( \Sigma \). So far, we have described the compactified supermembrane with no distinction between sectors. We now impose an extra topological restriction on the winding maps \([13]\): the irreducible winding constraint,

\[
\int_{\Sigma} dX^r \wedge dX^s = n e^{r s} \text{Area}(T^2) \quad n \neq 0, r, s = 1, 2. \quad (2.3)
\]

where \( \text{Area}(T^2) = (2\pi R)^2 I m \tau \).

There are two well-defined topological sectors classified according to this condition:

- When this condition holds, \( n \neq 0 \) we refer to it as the supermembrane with central charge theory \([1]\) and it implies that the winding matrix \( \Psi = \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \) has \( \det \Psi = n \neq 0 \). Globally it corresponds to a sector, with nontrivial symplectic torus bundle with monodromy \( \rho \) in \( SL(2, \mathbb{Z}) \), characterized by having \( H^2(\Sigma, \mathbb{Z}_\rho) \neq 0 \) \([4]\), \([4]\).

- When this condition vanishes \( n = 0 \), this corresponds to a sector that we will call from now on compactified M2, \( n = 0 \). Globally it corresponds to a trivial symplectic torus bundle over the base, lets choose for symplicity the flat 2-torus \( \Sigma_1 \). It is characterized by having a trivial class of \( H^2(\Sigma_1, \mathbb{Z}) \).
The Mass operator of the compactified supermembrane with winding and KK contribution \[5, 4\], is

\[
\text{Mass}^2 = T_{11}^2 ((2\pi R)^2 n \text{Im} \tau)^2 + \frac{1}{R^2} ((m_1^2 + (\frac{m|q - p|}{RI m \tau}) + T_{11}^{2/3} H
\]

where the \(H\) is defined in terms of the above hamiltonian \(H\) once the winding contribution has been extracted \(H = H - T_{11}^{-2/3} \int_\Sigma \sqrt{W} \frac{d}{dX} (X^r, X^s)^2 \) \[5\]. In the case of the sector of the compactified M2, sector \(n = 0\), the winding contribution vanishes.

3. The \(SL(2, \mathbb{Z})\) Symmetry of the supermembrane

The supermembrane is invariant under are preserving diffeomorphisms on the base manifold homotopic to the identity. This symmetry is realized by the first class constraints on the theory. Besides this standard symmetry of the supermembrane, the theory because of being compactified on a torus target is going to have a discrete symmetry \(SL(2, \mathbb{Z})\).

In order to analyze the \(SL(2, \mathbb{Z})\) symmetry of the supermembrane in detail \[4\], we first perform a Hodge decomposition of the closed one-forms. We may decompose the closed one-forms \(dX^r\) into

\[
dX^r = M^s_r d\hat{X}^s + dA^r \quad r = 1, 2
\]

where \(d\hat{X}^s, s = 1, 2\) is the basis of harmonic one-forms we have already introduced, \(dA^r\) are exact one-forms and \(M^s_r\) are constant coefficients. This condition is satisfied provided

\[
M^1_1 + iM^2_2 = 2\pi R (l_s + m_s \tau)
\]

Consequently, the most general expression for the maps \(X^r\). The general expression for the \(dX\) maps is then

\[
dX = dX_h + dA
\]

The harmonic part of \(dX\),

\[
dX_h = 2\pi R [(m_1 \tau + l_1) d\hat{X}^1 + (m_2 \tau + l_2) d\hat{X}^2].
\]

with \(l_s, m_s, s = 1, 2\), in principle of arbitrary integers.

The two different sectors of the theory although formally have the same expansion they differ and consequence of it, is the fact that they exhibit different properties of the harmonic sector depending whether \(W\) is equal or not to zero:
• For the trivial symplectic torus bundle sector associated to the compactified M2, \( n = 0 \), the harmonic sector contributions \( X_r \) are not independent. This fact classically is responsible for the existence of the string-like spikes with zero energy that render the theory unstable, and together with supersymmetry at quantum level they are also responsible for the continuity of the spectrum \([38]\).

for this reason, the supermembrane with winding has continuous spectrum as said in \([7]\). The Hodge decomposition can still be made but no gauge connection can be consistently defined in the theory since the coefficients behind the harmonic forms do not form a group as its determinant can be equal to zero. Although formally one can extract an exact 1-form this is not a gauge connection of the fibre since it is not possible to define properly the covariant derivative \([46]_1\). \([6]_1\), \([37]_1\).

• For the nontrivial symplectic torus bundle associated to the compactified M2 \( n \neq 0 \) the harmonic sector \( X_r \) are independent.

Classically There are not string-like spikes \([37]_1\) and the spectrum consists in purely isolated eigenvalues of finite multiplicity \([40]-[43]\) and then allows to define a covariant derivative, and \( A \) is a connection over this bundle with structure group the symplectomorphisms group. \( A \) is the gauge connection defined on the worldvolume associated to the monopole contribution \([13]\).

There is a compatible election for \( W \) on the geometrical picture we have defined. We define

\[
\sqrt{W} = \frac{1}{2} \epsilon_{rs} \partial_a \hat{X}^r \partial_b \hat{X}^s \epsilon^{ab},
\]

it is a regular density globally defined over \( \Sigma \). It is invariant under a change of the canonical basis of homology. \( X_h \) is a minimal immersion from \( \Sigma \) to \( T^2 \) on the target, moreover it is directly related to a holomorphic immersion of \( \Sigma \) onto \( T^2 \). The extension of the theory of supermembranes restricted by the topological constraint to more general compact sectors in the target space is directly related to the existence of those holomorphic immersions \([13]\).

3.1 The U-duality invariance

The compactified supermembrane theory on a torus, -both sectors (with and without central charge)- is also invariant under the following transformation on the target torus \( T^2 \):

\[
\begin{align*}
\tau &\rightarrow \frac{a\tau + b}{c\tau + d} \\
R &\rightarrow R|c\tau + d| \\
A &\rightarrow Ae^{i\phi} \\
W &\rightarrow \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} W
\end{align*}
\]
where \( c\tau + d = |c\tau + d| e^{-i\varphi} \) and \( \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2, \mathbb{Z}) \). The hamiltonian density \( \mathcal{H} \) of the compactified supermembrane is then invariant under (3.6). The \( SL(2, \mathbb{Z}) \) matrix acts from the left of the matrix \( \mathbb{W} \).

### 3.2 \( SL(2, \mathbb{Z}) \) of the Riemann surface

When the theory is restricted by the central charge condition (the irreducible winding condition), the theory is also invariant under an extra \( SL(2, \mathbb{Z}) \) symmetry acting on the homology basis of the base manifold \( \Sigma \), a genus one Riemann surface. This \( SL(2, \mathbb{Z}) \) realizes the modular transformations on the upper-half plane.

This extra \( SL(2, \mathbb{Z}) \) symmetry is associated to the diffeomorphisms changing the homology basis, and consequently the normalized harmonic one-forms, by a modular transformation on the Teichmüller space of the base torus \( \Sigma \). In fact, if

\[
d\hat{X}^r(\sigma) = S^r_s d\hat{X}^s(\sigma)
\]

provided

\[
\epsilon_{rs} S^r_t S^u_s = \epsilon_{tu}
\]

that is \( S \in Sp(2, \mathbb{Z}) \equiv SL(2, \mathbb{Z}) \). All conformal transformations on \( \Sigma \) are symmetries of the supermembrane with central charges [5]. We notice that under (3.7)

\[
dX \to 2\pi R(l'_s + m'_s \tau) d\hat{X}^s + dA'
\]

where \( A'(\sigma') = A(\sigma) \) is the transformation law of a scalar. and the winding matrix transforms as,

\[
\mathbb{W} \to \mathbb{W} S^{-1}
\]

In the case of the compactified M2 \( (n = 0) \) this symmetry is not present in the theory since \( W \) strongly depends on the central charge condition that restricts the values of the winding to those associated to the irreducible wrapping, to define the non trivial second Cohomolgy class associated to the nontrivial fibration.

### 4. T-duality in the Supermembrane Theory

In this section we introduce the T-duality transformations for the supermembrane theory [6]. This goes beyond the T-duality of superstring theory. The T-duality transformation we consider, is a nonlinear map which interchange the winding modes \( \mathbb{W} \), associated to the cohomology of the base manifold with the KK charges, \( Q = (p, q) \) associated to the homology of the target torus together with a transformation

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1In particular the supermembrane with central charges is invariant under the conformal maps homotopic to the identity.
of the real moduli $R \to \frac{1}{R}$ and complex moduli $\tau \to \tilde{\tau}$, both in a nontrivial way. In
the following all transformed quantities under T-duality are denoted by a tilde.
In order to define the T-duality transformation we introduce the following \(47\) dimensionless variables
\[ Z := T_{11}A\tilde{Y} \quad \tilde{Z} := T_{11}\tilde{A}Y \] (4.1)

where \(T_{11}\) is the supermembrane tension, \(A = (2\pi R)^2 Im\tau\) is the area of the target
torus and \(Y = \frac{R_{\text{Im} \tau}}{q\tau-p}\). The tilde variables \(\tilde{A}, \tilde{Y}\) are the transformed quantities under
the T-duality. See (4.3) for the explicit value of \(Z\). The T-duality transformation
we introduce is given by \[1\]:

The moduli : \[ Z\tilde{Z} = 1, \quad \tilde{\tau} = \frac{\alpha \tau + \beta}{\gamma \tau + \alpha}; \] (4.2)

The charges : \[(\tilde{p} \tilde{q}) = \Lambda_0 (\frac{p}{q}), \quad (\tilde{m}_1 \tilde{m}_2) = \Lambda_0^{-1} (\frac{l_1}{m_1}, \frac{l_2}{m_2}). \]

With \(\Lambda_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \alpha \end{pmatrix} \in SL(2, Z)\). In the above definition the T-dual supermembrane
corresponds to a new supermembrane where the role of winding and KK charges
interchanged, i.e. the KK modes are mapped onto the winding modes \((\tilde{p} \tilde{q}) = (\frac{l_1}{m_1})\)
and viceversa. The above property together with the condition \(Z\tilde{Z} = 1\) ensure that
\((\text{T-duality})^2 = \mathbb{I}\), the main property of T-duality. The explicit transformations of
the real modulus, obtained from the above T-duality transformation is
\[ \tilde{R} = \frac{|\gamma \tau + \alpha||q\tau - p|^{2/3}}{T_{11}^{2/3}(Im\tau)^{4/3}(2\pi)^{4/3}R}, \quad \tilde{\tau} = \frac{\alpha \tau + \beta}{\gamma \tau + \alpha} \quad \text{and} \quad \tilde{Z}^3 = \frac{T_{11}R^3(Im\tau)^2}{|q\tau - p|} \] (4.3)

The winding modes and KK charge contribution in the mass squared formula transform in the following way:
\[ T_{11}n^2A^2 = \frac{n^2}{Y^2}Z^2, \quad \frac{m^2}{Y^2} = T_{11}^2m^2\tilde{A}^2\tilde{Z}^2. \] (4.4)

To see how the \(H_1\) \[2,4\] transforms under T-duality it is important to realize the transformation rules for the fields,
\[ dX^m = ud\tilde{X}^m, \quad d\tilde{X} = ud^{i\varphi}dX, \quad A = u^{i\varphi}\tilde{A}, \quad \Psi = u^{3/2}\tilde{\Psi}, \quad \overline{\Psi} = u^{3/2}\overline{\tilde{\Psi}} \] (4.5)

Where \(u = Z^2 = \frac{R_{\text{Im} \tau + a}}{R}\), a phase defined in (3.22) of \[1\] and \(dX = dX^1 + idX^2\)
and respectively, its dual \(d\tilde{X}\) is
\[ d\tilde{X} = 2\pi\tilde{R}[\tilde{m}_1\tilde{\tau} + \tilde{l}_1)d\tilde{X}^1 + (\tilde{m}_2\tilde{\tau} + \tilde{l}_2)d\tilde{X}^2] \] (4.6)
The phase $e^{i\varphi}$ cancels with the h.c. the transformation of the Hamiltonian. The relation between the hamiltonians through a T-dual transformation is

$$H = \frac{1}{Z^8} \tilde{H}, \quad \tilde{H} = \frac{1}{Z^8} H. \quad (4.7)$$

We thus obtain for the mass squared formula the following identity,

$$M^2 = T_{11}^2 n^2 A^2 + \frac{m^2}{Y^2} + T_{11}^{2/3} H = \frac{1}{Z^2} \left( \frac{n^2}{Y^2} + T_{11}^2 m^2 \tilde{A}^2 \right) + \frac{T_{11}^{2/3}}{Z^8} \tilde{H}. \quad (4.8)$$

**4.1 T-Duality on Symplectic Bundles**

T-duality does not only acts at local level but also globally. As shown in [1] one can define an equivalence class with the elements of the coinvariant group associated to the monodromy group $G$, \( \{ Q + g \hat{Q} - \hat{Q} \} \), such that for any $g \in G$ and $\hat{Q} \in H_1(T^2)$, it characterizes one symplectic torus bundle. In the formulation of the supermembrane on that geometrical structure, $Q$ are identified with the KK charges. The action of $G$, the monodromy group, leaves the equivalence class invariant. We now consider the duality transformation introduced previously. Under the duality transformation the equivalence class transform as

$$\{ Q + g \hat{Q} - \hat{Q} \} \rightarrow \{ \Lambda_0 Q + (\Lambda_0 g \Lambda_0^{-1}) \hat{Q} - \Lambda_0 \hat{Q} \}, \quad (4.9)$$

hence for the dual bundle it holds, $\{ \Lambda_0 \left( \frac{l_1}{m_1} \right) + (\Lambda_0 g \Lambda_0^{-1}) \left( \frac{\tilde{l}_1}{\tilde{m}_1} \right) - \left( \frac{\tilde{l}_1}{\tilde{m}_1} \right) \}$. That is, as an element of the coinvariant group of $\Lambda_0 G \Lambda_0^{-1}$. We then conclude that the duality transformation, in addition to the transformation on the moduli $R, \tau$, also maps the geometrical structure onto an equivalent symplectic torus bundle with monodromy $\Lambda_0 G \Lambda_0^{-1}$. We notice that the transformation depends crucially on the original equivalence class of the coinvariant group. So for a nonequivalent symplectic torus bundle the dual transformations is realized with a different $SL(2, \mathbb{Z})$ matrix $\Lambda_0$. Now we are in position to determine the T-duality as a natural symmetry for the family of supermembranes with central charges. We take:

$$\tilde{Z} = Z = 1 \Rightarrow T_0 = \frac{|q\tau - p|}{R^3(Im\tau)^2}. \quad (4.10)$$

It imposes a relation between the energy scale of the tension of the supermembrane and the moduli of the torus fiber and that of its dual. Indeed we can think in two different ways: given the values of the moduli it fixes the allowed tension $T_0$ or on the other way around, for a fixed tension $T_0$, the radius, the Teichmuller parameter of the 2-torus, and the KK charges satisfy (4.10).
5. String Theory Limit

We then consider within the physical configurations of the supermembrane with central charges, the string-like configurations

\[ X^m = X^m(\tau, q_1 \widehat{X}^1 + q_2 \widehat{X}^2), \quad A^r = A^r(\tau, q_1 \widehat{X}^1 + q_2 \widehat{X}^2), \]  

(5.1)

where \( q_1, q_2 \) are relative prime integral numbers. \( X^m, A^r \) are scalar fields on the torus \( \Sigma \), a compact Riemann surface, hence they may always be expanded on a Fourier basis in term of a double periodic variable of that form. The restriction of \( q_1, q_2 \) to be relatively prime integral numbers arises from the global periodicity condition. On that configurations all the brackets

\[ \{X^m, X^n\} = \{X^m, A^r\} = \{A^r, A^s\} = 0 \]  

(5.2)

vanish. We then obtain the final expression for the mass contribution of the string states \[5\]:

\[ M^2_{11|SC} = (nT_{11}A)^2 + \left( \frac{m}{Y} \right)^2 + 8\pi^2 R_{11}T_{11}|q\tau - p|(N_L + N_R) \]  

(5.3)

where \((p, q)\) are relatively prime. We notice that \((p, q)\) may be interpreted as the wrapping of the membrane around the two cycles of the target torus. The corresponding change in the harmonic sector is \[5\]:

\[ dX_h = (qmd\widehat{X}^1 + pd\widehat{X}^2) + \tilde{\tau}(-Qnd\widehat{X}^1 + Pd\widehat{X}^2), \]  

(5.4)

the hamiltonian is invariant under that change. \( p, q \) and \( Q, P \) are now the winding numbers of the supermembrane. Given \( p, q \) there always exist \( Q \) and \( P \) with the above property, although the correspondence is not unique. The \((p, q)\) type IIB strings may indeed be interpreted as different wrappings of the supermembrane with central charges. This nice interpretation was first given in \[4\].

The \((p, q)\) IIB string compactified on a circle of radius \( R_B \) has tension \[4\]:

\[ T^2_{(p,q)} = \frac{|q\lambda_0 - p|^2}{\Im \lambda_0} T^2 \]  

(5.5)

where \( T = T_{11}^{2/3} \) is the string tension and \( \lambda_0 = \xi_0 + ie^{-i\phi_0} \) with \( \xi \) and \( \phi \) identified with the scalar fields of the type IIB theory, \( \phi \) corresponds to the dilaton fields. \( \lambda_0 \) is the asymptotic value of \( \lambda \)-the axion-dilaton of the type IIB theory- specifying the vacuum of the theory. The perturbative spectrum of the \((p, q)\) IIB string is \[4\]:

\[ M_B^2 = \left( \frac{n}{R_B} \right)^2 + (2\pi R_B m T_{(p,q)})^2 + 4\pi T_{(p,q)}(N_L + N_R). \]  

(5.6)

If we use following \[4\] a factor \( \beta^2 \) to identify term by term both mass formulas \((M_{11} = \beta M_B)\), since there were obtained using different metrics, one gets

\[ \tau = \lambda_0, \quad \beta^2 = \frac{T_{11}A_{11}^{1/2}}{T}, \quad R_B^{-2} = TT_{11}A_{11}^{3/2}. \]  

(5.7)
They were obtained by counting modes under some assumptions on the supermembrane wrapping modes, as mentioned on one of the footnotes [4]. Here we have derived the expressions from a consistent definition of the supermembrane with central charges.

The identification of (5.3) to the mass formula of IIA string compactified on a circle of radius $R_A$ and tension $T_A$ may also be performed. In order to have a consistent identification one has to take $Re\tau = 0$, $p = 1$ and hence $q = 0$ in (5.3). The mass formula for the perturbative spectrum of type IIA is

$$M^2_A = \left(\frac{m}{R_A}\right)^2 + (2\pi R_A n T_A)^2 + 4\pi T_A (N_L + N_R) \quad (5.8)$$

Identification after the limit process of the winding contributions and KK ones using a factor $(\beta \gamma)$ to compare the mass formulas, since they are obtained using different metrics, yields $R_A = \beta \gamma R_{11}$, $T_A = \gamma^{-2} (Im \tau)^{1/2} T$ which imply

$$(2\pi R_A R_B) = \left(\frac{1}{T_A T_{(p,q)}}\right)^{1/2} \quad (5.9)$$

We have thus obtained the $(p, q)$ IIB and IIA perturbative spectrum, when compactified on circles $R_B$ and $R_A$ respectively, from the string states on the supermembrane with central charges.

In this limit, by restricting the worldvolume configurations of the M2 to those of the string [5], we exactly recover the mass operator of the IIB theory as was formerly conjectured by Schwarz. The gain is that from the supermembrane with central charges the pure membrane excitations are known. If now a T-duality is performed on the supermembrane M2 mass operator restricted to string-like configurations, then an SL(2,Z) non-perturbative multiplet of IIA is obtained [5].

**String T-duality transformation limit.** Now we can take directly the limit of the T-duality transformations of the supermembrane to recover the standard T-duality transformations for the closed string operator. Before taking the limit it is convenient to consider a redefinition of $X^1$ and $X^2$. The coordinates that wrap on the $T^2$. We take

$$X^1 \rightarrow \frac{X^1}{T_{11}^{1/6} R_{11}^{2/3}}, \quad X^2 \rightarrow T_{11}^{1/6} R_{11}^{1/2} X^2, \quad (5.10)$$

such that the Lie brackets $\{X^1, X^2\}^2$ will remain invariant under the redefinition. Now we consider the following limit for the torus collapsing into a circle, by imposing $R_{11} \rightarrow 0$. Since we want to ensure $A_{11} Y$ remains finite but $A_{11} \rightarrow 0$, necessarily $q = 0$, for arbitrary $p$. Indeed this is equivalent to consider the KK charges $(p, q) = m(1, 0)$ for $m = p$. By using the M2 T-duality transformation it can also be seen that $A_{11} \rightarrow 0$, so the dual also corresponds to a string. Moreover substituting in the
previous definitions it can be seen $R_A$ is finite implies $R_B$ finite. We will define the following for the torus degenerating into a circle $S^1$

$$R_1 = \frac{R_{11}^{1/2}}{T_{11}^{1/6}}; \quad R_2 = R_{11}^{3/2}T_{11}^{1/6} Im\tau \quad (5.11)$$

Since $R_{11} \to 0$, it implies $R_1 \to 0$ but $R_2$ is finite, so it corresponds to a closed curve that topologically is a circle. Now we re-express the winding condition in terms of the new variables. In terms of the new variables we get

$$\oint C_s dX^1 = 2\pi R_1 l_s; \quad \oint C_s dX^2 = 2\pi R_2 m_s, \quad (5.12)$$

Since $R_1 \to 0$, although $l_s$ is taking finite, the first winding condition vanishes and the only residual winding condition is associated to the $S^1$ modulus is $R_2$. The former T-duality relations of the moduli in this limit become reduced to:

$$Z\bar{Z} = 1|_{\text{string}}, \quad \Rightarrow \quad T_{M2}^{A/3} \tilde{R}_2^{\tilde{A}/2} = 1 \to \tilde{R}_2 = \frac{\alpha'}{R_2}. \quad (5.13)$$

where $\tilde{R}_2 = T_{11}^{1/2} \tilde{A}^{1/2} \tilde{Y}^{1/2}$. This defines for $T_{(p,q)} = T$ on the IIB string side precisely the duality relation of the strings. The transformation on the charges and windings are given by (4.2) and we finally obtain:

$$\{R; (l_1, m)\} \overset{T-\text{duality}}{\longrightarrow} \{\tilde{R} = \frac{\alpha'}{R_1}; (m, l_1)\} \quad (5.14)$$

where $m$ is the common factor between the charges.

6. Discussion and Conclusions

We show that the supermembrane compactified on a torus, in both sectors have an $SL(2, Z)_{T^2}$ symmetry associated to the torus target space. The supermembrane with central charges has an extra symmetry associated to the diffeomorphisms not connected to the identity, changing the homology basis $SL(2, Z)_{\Sigma}$. We showed the existence of a new $Z_2$ symmetry that plays the role of T-duality in M-theory interchanging the winding and KK charges but leaving the hamiltonian invariant. The supermembrane compactified on a torus realizes this duality as an exact symmetry of the theory in both sectors ($n = 0, n \neq 0$). This is a relevant property expected for a sector of M-theory. When only string-like states are considered but it is performed this generalized T-duality of the supermembrane we obtain the mass operator IIB and the corresponding dual realizes the type IIA with all nonperturbative $SL(2, Z)$ multiplet. If we take the limit of the T-duality transformation of the supermembrane into the standard one, then only the standard $(p, q) = (1, 0)$ mass type IIA
operator is allowed meanwhile IIB mass operator is unchanged. The Supermembrane Theory compactified on a torus. Supermembrane theory realize S, T so U-duality as exact symmetries of the theory. There are two well defined sectors: without and with central charges. While the first one has as U-duality group in 9D as symmetry the group $SL(2, Z)_{T^2} \times Z_2$, the second one exhibits a larger duality group, $SL(2, Z)_\Sigma \times SL(2, Z)_{T^2} \times Z_2$ symmetry. We have been able to prove that U-dualities are symmetries of a toroidally compactified sector of M-theory, the supermembrane on a torus, as was already conjectured in [18].

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