On Two Diophantine Inequalities Over Primes

Min Zhang∗ & Jinjiang Li†
Department of Mathematics, China University of Mining and Technology∗†
Beijing 100083, P. R. China

Abstract: Let \( 1 < c < 37/18 \), \( c \neq 2 \) and \( N \) be a sufficiently large real number. In this paper, we prove that, for almost all \( R \in (N, 2N) \), the Diophantine inequality \( |p_1^c + p_2^c + p_3^c - R| < \log^{-1} N \) is solvable in primes \( p_1, p_2, p_3 \). Moreover, we also investigate the problem of six primes and prove that the Diophantine inequality \( |p_1^c + p_2^c + p_3^c + p_4^c + p_5^c + p_6^c - N| < \log^{-1} N \) is solvable in primes \( p_1, p_2, p_3, p_4, p_5, p_6 \) for sufficiently large real number \( N \).

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1 Introduction and main result

In 1952, Piatetski-Shapiro [13] considered the following analogue of the Waring-Goldbach problem. Assume that \( c > 1 \) is not an integer and let \( \varepsilon > 0 \). If \( r \) is a sufficiently large integer (depending only on \( c \)), then the inequality

\[
|p_1^c + p_2^c + \cdots + p_r^c - N| < \varepsilon
\]  \hspace{1cm} (1.1)

has a solution in prime numbers \( p_1, p_2, \cdots, p_r \) for sufficiently large \( N \). More precisely, if the least \( r \) such that (1.1) has a solution in prime numbers for every \( \varepsilon > 0 \) and \( N > N_0(c, \varepsilon) \) is denoted by \( H(c) \), then it is proved in [13] that

\[
\limsup_{c \to \infty} \frac{H(c)}{c \log c} \leq 4.
\]

In [13], Piatetski-Shapiro also proved that if \( 1 < c < 3/2 \), then \( H(c) \leq 5 \). The upper bound \( 3/2 \) for \( c \) was improved successively to

\[
\frac{14142}{8923} = 1.5848 \cdots, \quad \frac{1 + \sqrt{5}}{2} = 1.6180 \cdots, \quad \frac{81}{40} = 2.025, \quad \frac{108}{53} = 2.0377 \cdots, \quad 2.041
\]

†Corresponding author.

E-mail addresses: min.zhang.math@gmail.com (M. Zhang), jinjiang.li.math@gmail.com (J. Li).
by Zhai and Cao [20], Garaev [7], Zhai and Cao [22], Shi and Liu [16], Baker and Weingartner [1], respectively.

On the other hand, the Vinogradov-Goldbach theorem [19] suggests that at least for $c$ close to 1, one should expect $H(c) \leq 3$. The first result in this direction was obtained by D. I. Tolev [18], who showed that the inequality

$$|p_1^c + p_2^c + p_3^c - N| < \varepsilon$$

(1.2)

with $\varepsilon = N^{-(1/c)(15/14 - c)} \log^9 N$ is solvable in primes $p_1, p_2, p_3$, provided that $1 < c < 15/14$ and $N$ is sufficiently large. Later, Tolev’s range was enlarged to $1 < c < 13/12$ in Cai [3], $1 < c < 11/10$ in Cai [4] and Kumchev-Nedeva [11] independently, $1 < c < 237/214$ in Cao and Zhai [5], $1 < c < 61/55$ in Kumchev [10], $1 < c < 10/9$ in Baker and Weingartner [2].

Laporta [12] studied the corresponding binary problem, which can be viewed as an inequality analogue of the Goldbach’s conjecture for even numbers. Suppose $1 < c < 15/14$ fixed, $N$ a large real number and $\varepsilon = N^{1-15/(14c)} \log^8 N$. Then Laporta proved that the inequality

$$|p_1^c + p_2^c - R| < \varepsilon$$

(1.3)

is solvable for all $R \in (N, 2N] \setminus \mathcal{A}$ with $|\mathcal{A}| \ll N \exp \left( -\frac{1}{3} \left( \frac{\log N}{c} \right)^{1/5} \right)$. Zhai and Cao [21] improved Laporta’s [12] result and proved for $1 < c < 43/36$ fixed and for all $R \in (N, 2N] \setminus \mathcal{A}$ with $|\mathcal{A}| \ll N \exp \left( -\frac{1}{3} \left( \frac{\log N}{c} \right)^{1/5} \right)$, the inequality (1.3) is solvable with primes $p_1, p_2 \leq N^{1/c}$ and $\varepsilon = N^{1-43/(36c)}$.

In this paper we shall prove the following two Theorems.

**Theorem 1.1** Let $1 < c < 37/18$, $c \neq 2$ and $N$ be a sufficiently large real number. Then for all $R \in (N, 2N] \setminus \mathcal{A}$ with

$$|\mathcal{A}| \ll N \exp \left( -\frac{2}{15} \left( \frac{1}{c} \log \frac{2N}{3} \right)^{1/5} \right),$$

the inequality

$$|p_1^c + p_2^c + p_3^c - R| < \log^{-1} N$$

(1.4)

is solvable in three prime variables $p_1, p_2, p_3$, where $\eta$ is sufficiently small positive number.

**Remark.** The best result up to date for $H(c) \leq 3$ was obtained by Baker and Weingartner [2], who prove that $1 < c < 10/9$. From Theorem 1.1, one can expect that the range of $c$ for $H(c) \leq 3$ should be improved to $1 < c < 37/18$, $c \neq 2$. Moreover, it is
conjectured that the range of $c$, which holds for $H(c) \leq 3$, is $1 < c < 3$, $c \neq 2$. Therefore, the range of $c$ for $H(c) \leq 3$ has huge space to improve, though such a strong conjecture is out of reach at present.

**Theorem 1.2** Suppose that $1 < c < 37/18$, $c \neq 2$, then there exists a number $N_0(c)$ such that for each real number $N > N_0(c)$ the inequality

$$|p_1^c + p_2^c + p_3^c + p_4^c + p_5^c + p_6^c - N| < \log^{-1} N$$

is solvable in six prime variables $p_1, p_2, p_3, p_4, p_5, p_6$.

**Notation.** Throughout this paper, $N$ always denotes a sufficiently large real number; $\eta$ always denotes an arbitrary small positive constant, which may not be the same at different occurrences; $p$ always denotes a prime number; $n \sim N$ means $N < n \leq 2N$; $X \asymp N^{1/c}$, which is determined during each proof of the Theorems; $\tau = X^{1-c-\eta}, \varepsilon = \log^{-2} X, K = \log^5 X, \Lambda(n)$ denotes von Mangold’s function; $\mu(n)$ denotes Möbius function; $e(x) = e^{2\pi ix}$; $\mathcal{L} = \log X, E = \exp(-\mathcal{L}^{1/5})$,

$$P = \left(\frac{2}{E^2}\right)^{1/3} \mathcal{L}, \quad S(x) = \sum_{X/2 < p \leq X} \log p \cdot e(p^c x), \quad I(x) = \int_X^X e(t^c x) dt.$$

**2 Preliminary Lemmas**

**Lemma 2.1** Let $a, b$ be real numbers, $0 < b < a/4$, and let $k$ be a positive integer. There exists a function $\varphi(y)$ which is $k$ times continuously differentiable and such that

$$\begin{cases} 
\varphi(y) = 1, & \text{for } |y| \leq a - b, \\
0 < \varphi(y) < 1, & \text{for } a - b < |y| < a + b, \\
\varphi(y) = 0, & \text{for } |y| \geq a + b,
\end{cases}$$

and its Fourier transform

$$\Phi(x) = \int_{-\infty}^{+\infty} e(-xy)\varphi(y)dy$$

satisfies the inequality

$$|\Phi(x)| \leq \min \left(2a, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left(\frac{k}{2\pi|x|b}\right)^k\right). \quad (2.1)$$

**Proof.** See Piatetski-Shapiro [13] or Segal [15].
Lemma 2.2 Let \( G, F \) be twice differentiable on \([A, B]\), \( |G(x)| \leq H, G/F' \) monotonic. If \( F' \geq K > 0 \) on \([A, B]\), then
\[
\int_A^B \frac{G(x)e(F(x))}{x} \, dx \ll HK^{-1}.
\]

Proof. See Titchmarsh [17], Lemma 4.3. ■

Lemma 2.3 Suppose \( M > 1, c > 1, \gamma > 0 \). Let \( \mathcal{A}(M; c, \gamma) \) denote the number of solutions of the inequality
\[
|n_1^c + n_2^c - n_3^c - n_4^c| < \gamma, \quad M \leq n_1, n_2, n_3, n_4 \leq 2M,
\]
then
\[
\mathcal{A}(M; c, \gamma) \ll (\gamma M^{4-c} + M^2) M^n.
\]

Proof. See Robert and Sargos [14], Theorem 2. ■

Lemma 2.4 For \( 1 < c < 3, c \neq 2 \), we have
\[
\int_{-\infty}^{+\infty} I^6(x)e(-xN)\Phi(x)dx \gg \varepsilon X^{6-c}.
\]

Proof. Denote the above integral by \( \mathcal{H} \). We have
\[
\mathcal{H} := \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e\left((t_1^c + t_2^c + \cdots + t_6^c - N)x\right)\Phi(x)dxdt_1 \cdots dt_6.
\]
The change of the order of integration is legitimate because of the absolute convergence of the integral. From Lemma 2.1 with \( a = 9\varepsilon/10, b = \varepsilon/10 \), by using the Fourier inversion formula we get
\[
\mathcal{H} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \varphi(t_1^c + t_2^c + \cdots + t_6^c - N)dt_1 \cdots dt_6.
\]
By the definition of \( \varphi(y) \) we get
\[
\mathcal{H} \gg \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{dt_1 \cdots dt_6}{|t_1^c + \cdots + t_6^c - N|^{4/\varepsilon}} \geq \int_{\lambda X}^{\mu X} \cdots \int_{\lambda X}^{\mu X} \left( \int_{\frac{a}{2\varepsilon}}^{\frac{b}{2\varepsilon}} dt_6 \right) dt_1 \cdots dt_5,
\]
where \( \lambda \) and \( \mu \) are real numbers such that
\[
\frac{1}{2} < \left(\frac{4}{5}\right)^{1/c} \leq \lambda < \mu < \left(1 - \frac{1}{5} \cdot \frac{1}{2^c}\right)^{1/c} < 1
\]
\[ \mathcal{N} = \left[ \frac{X}{2}, X \right] \cap \left[ \left( N + \frac{4c}{5} - t^c_1 - \cdots - t^c_5 \right)^{1/c}, \left( N - \frac{4c}{5} - t^c_1 - \cdots - t^c_5 \right)^{1/c} \right] \]

Thus by the mean-value theorem we have
\[ H \gg \varepsilon \int_{\mathcal{M}} \cdots \int_{\mathcal{N}} (\xi t_1,t_2,t_3,t_4,t_5)^{1/c-1} dt_1 \cdots dt_5, \]
where \( \xi t_1,t_2,t_3,t_4,t_5 \approx X^c \). Therefore, \( H \gg \varepsilon X^{6-c} \), which proves the lemma. \( \square \)

**Lemma 2.5** We have
\[ A = \max_{R' \in (N,2N)} \int_N^{2N} \left| \int_{|x|<K} e((R-R')x) \, dx \right| \, dR \ll \mathcal{L}. \]

**Proof.** See Laporta \cite{12}, Lemma 1. \( \square \)

Let \( \Omega_1 \) and \( \Omega_2 \) be measurable subsets of \( \mathbb{R}^n \). Let
\[ \|f\|_j = \left( \int_{\Omega_j} |f(y)|^2 \, dy \right)^{1/2}, \quad \langle f, g \rangle_j = \int_{\Omega_j} f(y) \overline{g(y)} \, dy \quad (j = 1, 2), \]
be the usual norm and inner product in \( L^2(\Omega_j, \mathbb{C}) \), respectively.

**Lemma 2.6** Let \( c \in L^2(\Omega_1, \mathbb{C}) \), \( \xi \in L^2(\Omega_2, \mathbb{C}) \), and let \( \omega \) be a measurable complex valued function on \( \Omega_1 \times \Omega_2 \) such that
\[ \sup_{x \in \Omega_1} \int_{\Omega_2} |\omega(x,y)| \, dy < +\infty, \quad \sup_{y \in \Omega_2} \int_{\Omega_1} |\omega(x,y)| \, dx < +\infty. \]
Then we have
\[ \int_{\Omega_1} c(x) \langle \xi, \omega(x,\cdot) \rangle_2 \, dx \leq \|\xi\|_2 \|c\|_1 \left( \sup_{x' \in \Omega_1} \int_{\Omega_1} |\langle \omega(x,\cdot), \omega(x',\cdot) \rangle_2| \, dx \right)^{1/2}. \]

**Proof.** See Laporta \cite{12}, Lemma 2. \( \square \)

**Lemma 2.7** For \( 1 < c < 37/18 \), \( c \not= 2 \), we have
\[ \int_{-\tau}^{\tau} |S(x)|^2 \, dx \ll X^{2-c} \log^3 X, \quad (2.2) \]
\[ \int_{-\tau}^{\tau} |I(x)|^2 \, dx \ll X^{2-c} \log X. \quad (2.3) \]
Proof. See Tolev [18], Lemma 7. Although in Tolev’s paper, $c$ is in the range $(1, 15/14)$, it can be easily seen that his lemma is true for $c \in (1, 2) \cup (2, 3)$, and so do his Lemmas 11 – 14. In fact, the proofs of Lemma 7 and Lemmas 11 – 14 in [18] have nothing to do with the range of $c$.

**Lemma 2.8** For $1 < c < 37/18$, $c \neq 2$, $|x| \leq \tau$, then

$$S(x) = I(x) + O \left( X e^{- (\log X)^{1/5}} \right).$$

Proof. See Tolev [18], Lemma 14.

**Lemma 2.9** For $1 < c < 37/18$, $c \neq 2$, we have

$$\int_{-\tau}^{\tau} |S(x)|^4 \, dx \ll X^{4-c} \log^5 X,$$  
(2.4)  

$$\int_{-\tau}^{\tau} |I(x)|^4 \, dx \ll X^{4-c} \log^5 X.$$  
(2.5)

Proof. We only prove (2.4). Inequality (2.5) can be proved likewise.

We have

$$\int_{-\tau}^{\tau} |S(x)|^4 \, dx = \sum_{\frac{\tau}{X} < p_1, p_2, p_3, p_4 \leq X} (\log p_1) \cdots (\log p_4) \int_{-\tau}^{\tau} e \left( (p_1^c + p_2^c - p_3^c - p_4^c)x \right) \, dx$$

$$\ll \sum_{\frac{\tau}{X} < p_1, p_2, p_3, p_4 \leq X} (\log p_1) \cdots (\log p_4) \cdot \min \left( \tau, \frac{1}{|p_1^c + p_2^c - p_3^c - p_4^c|} \right)$$

$$\ll U \tau \log^4 X + V \log^4 X,$$  
(2.6)

where

$$U = \sum_{\frac{\tau}{X} < n_1, n_2, n_3, n_4 \leq X} 1, \quad V = \sum_{\frac{\tau}{X} < n_1, n_2, n_3, n_4 \leq X} \frac{1}{|n_1^c + n_2^c - n_3^c + n_4^c|}.$$  

We have

$$U \ll \sum_{\frac{\tau}{X} < n_1 \leq X} \sum_{\frac{\tau}{X} < n_2 \leq X} \sum_{\frac{\tau}{X} < n_3 \leq X} \sum_{\frac{\tau}{X} < n_4 \leq X} 1$$

$$\ll \sum_{\frac{\tau}{X} < n_1, n_2, n_3, n_4 \leq X} \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|} \ll 1$$

$$\ll \sum_{\frac{\tau}{X} < n_1, n_2, n_3 \leq X} \frac{1}{n_1^c + n_2^c - n_3^c + X^c}$$

and by the mean-value theorem

$$U \ll X^3 + \frac{1}{\tau} X^{4-c}.$$  
(2.7)
Obviously, $V \leq \sum_{\ell} V_{\ell}$, where

$$V_{\ell} = \sum_{\substack{\ell < n_1, n_2, n_3, n_4 \leq X \\ \ell < n_1 + n_2 - n_3 - n_4 \leq 2\ell}} \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|} \quad (2.8)$$

and $\ell$ takes the values $\frac{2k}{\tau}$, $k = 0, 1, 2, \cdots$, with $\ell \ll X^c$. Then, we have

$$V_{\ell} \ll \frac{1}{\ell} \sum_{\substack{\ell < n_1, n_2, n_3 \leq X \\ n_1^c + n_2^c - n_3^c \leq X^c}} \left( \frac{1}{(n_1^c + n_2^c + 2\ell)^{1/c}} - \frac{1}{(n_1^c + n_2^c - n_3^c + \ell)^{1/c}} \right) \ll X^{1-c} \quad (2.9)$$

by the mean-value theorem.

The conclusion follows from formulas (2.6)-(2.9).

**Lemma 2.10** If $1 < c < 2$, $\tau \leq |x| \leq K$, then we have

$$S(x) \ll X^{\frac{3}{4}+\eta} + X^{\frac{1}{4}+\eta}.$$  

**Proof.** See Zhai and Cao [20], Lemma 7.

**Lemma 2.11** Let $N, Q \geq 1$ and $z_n \in \mathbb{C}$. Then

$$\left| \sum_{n=1}^{N} z_n \right|^2 \leq \left(2 + \frac{N}{Q}\right) \sum_{|q| < Q} \left(1 - \frac{|q|}{Q}\right) \sum_{N < n+q, n-q \leq 2N} \overline{z_{n+q}} z_{n-q}.$$  

**Proof.** See Fouvry and Iwaniec [6], Lemma 2.

**Lemma 2.12** Suppose that

$$L(H) = \sum_{i=1}^{m} A_i H^{a_i} + \sum_{j=1}^{n} B_j H^{-b_j},$$

where $A_i, B_j, a_i$, and $b_j$ are positive. Assume that $H_1 \leq H_2$. Then there is some $H'$ with $H_1 \leq H' \leq H_2$ and

$$L(H') \ll \sum_{i=1}^{m} A_i H_1^{a_i} + \sum_{j=1}^{n} B_j H_2^{-b_j} + \sum_{i=1}^{m} \sum_{j=1}^{n} (A_i^{b_j} B_j^{a_i})^{1/(a_i+b_j)}.$$  

The implied constant depends only on $m$ and $n$.  

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Proof. See Graham and Kolesnik [8], Lemma 2.4. ■

For the sum of the form

$$\sum_{M < m \leq M_1} \sum_{N < n \leq N_1} a_m b_n e(xm^n c)$$

with

$$MN \sim X, M < M_1 \leq 2M, N < N_1 \leq 2N, a_m \ll X^{\eta}, b_n \ll X^{\eta}$$

for every fixed $\eta$, it is usually called a “Type I” sum, denoted by $S_I(M,N)$, if $b_n = 1$ or $b_n = \log n$; otherwise it is called a “Type II” sum, denoted by $S_{II}(M,N)$.

**Lemma 2.13** Let $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0, 1, 2$, $\beta \neq 0, 1, 2, 3$. For $F \gg MN^2$ and $N \geq M \geq 1$, we have

$$S_{II}(M, N) = \sum_{m \sim M} \sum_{n \sim N} a_m b_n e\left(F x^m n^\beta\right) \ll \frac{X^{1-\delta}}{F^{1/16}} + \frac{X^{725/72}}{F^{1/128}} + X^{1/2} F^{\kappa}.$$ 

Proof. See Baker and Weingartner [1], Theorem 1. ■

In the rest of this section, we always suppose $2 < c < 33/16$, $\delta = c/2 - 1 + \eta$, $F = |x| X^c$, $\tau \leq |x| \leq K$. Obviously, we have $X^{1-\eta} \ll F \ll KX^c$.

**Lemma 2.14** Suppose $2 < c < 37/18$, $b_n \ll 1$. If there holds $M \gg X^{1-725/7}$, then we have

$$S_I(M, N) = \sum_{m \sim M} \sum_{n \sim N} b_n e(xm^n c) \ll X^{1-\delta}.$$ 

Proof. Let $f(m) = xm^n c$. Then we have $|f^{(j)}(m)| \asymp (FM^{-1})^{1-j}$ for $j = 1, \ldots, 6$. By the method of exponent pairs, we get

$$S_I \ll \sum_{n \sim N} \left| \sum_{m \sim M} e(xm^n c) \right| \ll N(MF^{-1} + (FM^{-1})^\kappa M^\lambda) \ll XF^{-1} + X^{\eta} + X^{\kappa} X^{\lambda-\kappa} N^{1+\kappa-\lambda} \ll (\log X)^{3\kappa} X^{\kappa+\lambda-\kappa+1+\kappa-\lambda/3}.$$ 

The last step is due to the fact that $N \asymp XM^{-1} \ll X^{725/7} \ll X^{1/3}$. Taking the exponent pair $(\kappa, \lambda) = A^3(1/2, 1/2) = (1/30, 26/30)$, then we obtain

$$S_I(M, N) \ll X^{1-\delta}$$

by noting that $2 < c < 37/18$. ■
Lemma 2.15 Suppose \(2 < c < 37/18\), \(a_m \ll 1\), \(b_n \ll 1\). If there holds \(X^{725/7} \ll M \ll X^{1/2}\), then we have

\[ S_{II}(M, N) = \sum_{m \sim M} \sum_{n \sim N} a_m b_n e(xm^cn^c) \ll X^{-\delta}. \]

Proof. Take a suitable \(F_0 \geq MN^2\), whose value will be determined later during the following discussion. If \(F \geq F_0\), according to Theorem 1 of Baker and Weingartner [1], we obtain

\[ X^{-\eta} \cdot S_{II}(M, N) \ll M^{7/8}N^{13/16}F^{1/16} + M^{93/104}N^{23/26}F^{1/26} + M^{467/512}N^{65/64}F^{-1/128} + M^{65/72}N \]

\[ =: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4. \]

Noting that if there holds \(X^{725/7} \ll M \ll X^{1/2}\), we obtain

\[ \mathcal{I}_1 \ll X^{1-\delta}, \quad \mathcal{I}_2 \ll X^{1-\delta}, \quad \mathcal{I}_4 \ll X^{1-\delta}. \]

Therefore, for the case \(F \gg F_0 \gg MN^2\), we get

\[ S_{II}(M, N) \ll X^{1-\delta} + M^{467/512}N^{65/64}F_0^{-1/128}. \quad (2.10) \]

Next, we consider the case \(X^{1-\eta} \ll F \ll F_0\).

Take \(Q\) satisfying \(1 \ll Q \ll M\). By Cauchy’s inequality and Lemma 2.11, we have

\[ |S_{II}|^2 \ll \left( \sum_{n \sim N} |b_n|^2 \right) \left( \sum_{n \sim N} \left| \sum_{m \sim M} a_m e(xm^cn^c) \right|^2 \right). \]

\[ \ll N \sum_{n \sim N} \frac{M}{Q} \sum_{|q| < Q} \left( 1 - \frac{|q|}{Q} \right) \sum_{M < m + q, m - q \ll 2M} a_m a_{m+q} e(xm^c\Delta_c(m, q)) \]

\[ \ll \frac{M^2N^2}{Q} + \frac{MN}{Q} \sum_{1 \leq q < Q} \sum_{m \sim M} \left| \sum_{n \sim N} e(xn^c\Delta_c(m, q)) \right|, \]

where \(\Delta_c(m, q) = (m+q)^c - (m-q)^c\). Thus, it is sufficient to estimate the following sum

\[ S_0 := \sum_{n \sim N} e(xn^c\Delta_c(m, q)). \]

By the method of exponent pairs, we get

\[ S_0 \ll \frac{MN}{Fq} + \left( \frac{Fq}{MN} \right)^\kappa N^\lambda, \]

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where \((\kappa, \lambda)\) is an arbitrary exponent pair. Therefore, we have

\[
|S_{II}|^2 \ll \frac{M^2 N^2}{Q} + \frac{M N}{Q} \sum_{1 \leq q < Q \atop m \sim M} \left( \frac{M N}{F q} + \left( \frac{F q}{M N} \right)^\kappa N^\lambda \right) \\
\ll \frac{M^2 N^2}{Q} + \frac{M^3 N^2}{Q F} \log Q + Q^\kappa F^\kappa M^{2-\kappa} N^{1+\lambda-\kappa} \\
\ll \frac{M^2 N^2}{Q} + Q^\kappa F^\kappa M^{2-\kappa} N^{1+\lambda-\kappa} \\
\ll \frac{M^2 N^2}{Q} + Q^\kappa F_0^\kappa M^{2-\kappa} N^{1+\lambda-\kappa}.
\]

Set

\[ Q_0 = F_0^{-\kappa/(1+\kappa)} M^{\kappa/(1+\kappa)} N^{(1+\kappa-\lambda)/(1+\kappa)}. \]

Next, we will discuss three cases of the selection of \(Q\).

**Case 1** If \(Q_0 < 5\), then we take \(Q = 5\) and obtain

\[ |S_{II}|^2 \ll F_0^{\kappa} M^{2-\kappa} N^{1+\lambda-\kappa}. \]

**Case 2** If \(5 \leq Q_0 \leq M/2\), then we take \(Q = Q_0\), and obtain

\[ |S_{II}|^2 \ll F_0^{\kappa/(1+\kappa)} M^{2-\kappa/(1+\kappa)} N^{2-(1+\kappa-\lambda)/(1+\kappa)}. \]

**Case 3** If \(Q_0 > M/2\), then we take \(Q = M/2\), and obtain

\[ |S_{II}|^2 \ll MN^2. \]

Based on the above three cases, we have

\[
S_{II} \ll M^{1/2} N + F_0^{\kappa/2} M^{1-\kappa/2} N^{(1+\lambda-\kappa)/2} + F_0^{\kappa/(2+2\kappa)} M^{1-\kappa/(2+2\kappa)} N^{1-(1+\kappa-\lambda)/(2+2\kappa)}. 
\]

(2.11)

According to (2.10) and (2.11) and noting that \(M^{1/2} N \asymp XM^{-1/2} \ll X^{1-365/7} \ll X^{1-\delta}\), we get

\[
S_{II} \ll X^{1-\delta} + F_0^{\kappa/(2+2\kappa)} M^{1-\kappa/(2+2\kappa)} N^{1-(1+\kappa-\lambda)/(2+2\kappa)} + F_0^{\kappa/2} M^{1-\kappa/2} N^{(1+\lambda-\kappa)/2} + M^{467/512} N^{65/64} F_0^{-1/128}.
\]

According to Lemma 2.12, there exists an \(F_0\) satisfying \(MN^2 \ll F_0 \ll KN^c\) such that

\[
S_{II} \ll X^{1-\delta} + M^{467/512} N^{65/64} X^{-c/128} + \left( MN^2 \right)^{\kappa/2} M^{1-\kappa/2} N^{(1+\lambda-\kappa)/2} + \left( (M^{1-\kappa/2} N^{1+\lambda-\kappa}/2) \right)^{1/128} \left( M^{467/512} N^{65/64} \right)^{\delta/2} \ll X^{1-\delta}.
\]
\[
+ \left( (M^{1-\kappa/(2+2\kappa)} N^{1-(1+\kappa-\lambda)/(2+2\kappa)})^{1/128}
\times (M^{467/512} N^{65/64})^{\kappa/(2+2\kappa)})^{1/(1/128+\kappa/(2+2\kappa))} \right)
\]
=: \[ X^{1-\delta} + J_1 + J_2 + J_3 + J_4 + J_5. \]

Taking \((\kappa, \lambda) = ABABA^2 B(0, 1) = (1/11, 3/4),\) then under the condition \(X^{726/7} \ll M \ll X^{1/2},\) we obtain
\[ J_i \ll X^{1-\delta}, \quad i = 1, 2, 3, 4, 5. \]

Therefore, we have
\[ S_{II} \ll X^{1-\delta}. \]

This completes the proof of Lemma 2.15. 

Lemma 2.16 Suppose \(2 < c < 37/18,\) then for \(\tau \leq |x| \leq K\) we have
\[ S(x) \ll X^{1-\delta}. \]

Proof. First, we have
\[ S(x) = U(x) + O(x^{1/2}), \]
where
\[ U(x) = \sum_{X/2 < n \leq X} \Lambda(n) e(xn^c). \]

By Heath-Brown identity [9] with \(k = 3,\) it is easy to see that \(U(x)\) can be written as \(O(\log^6 X)\) sums of the form
\[ U^*(x) = \sum_{n_1 \sim N_1} \cdots \sum_{n_6 \sim N_6} \log n_1 \cdot \mu(n_1) \mu(n_3) \mu(n_6) e(x(n_1 \cdots n_6)^c), \]

where \(N_1, \cdots, N_6 \geq 1, N_1 \cdots N_6 \asymp X, n_4, n_5, n_6 \leq (2X)^{1/3}\) and some \(n_i\) may only take value 1.

Let \(F = |x| X^c.\) For \(2 < c < 37/18,\) we shall prove that for each \(U^*(x)\) one has
\[ U^*(x) \ll X^{1-\delta}. \]

Case 1 If there exists an \(N_j\) such that \(N_j \geq X^{1-726/7} > X^{1/2},\) then we must have \(j \leq 3.\) Take \(m = n_j, n = \prod_{i \neq j} n_i, M = N_j, N = \prod_{i \neq j} N_i.\) In this case, we can see that \(U^*(x)\) can be written as
\[ U^*(x) = \sum_{m \sim M} \sum_{n \sim N} a_m b_n e(xm^c n^c), \]

where \(|a_m| \leq \log m, |b_n| \leq d_5(n).\) Then \(U^*(x)\) is a sum of Type I. By Lemma 2.14, the result follows.
Case 2 If there exists an $N_j$ such that $X^{726/7} \leq N_j \leq X^{1-726/7}$, then we take $m = n_j$, $n = \prod_{i \neq j} n_i$, $M^* = N_j$, $N^* = \prod_{i \neq j} N_i$. In this case, we can see that $U^*(x)$ can be written as

$$U^*(x) = \sum_{m \sim M^*} \sum_{n \sim N^*} a_m b_n e(xm^n/n),$$

where $|a_m| \leq \log m$, $|b_n| \leq d_5(n) \log n$. If $X^{726/7} \leq M^* \leq X^{1/2}$, then $N^* \gg X^{1/2}$ and we take $(M, N) = (M^*, N^*)$. If $X^{726/7} \leq N^* \leq X^{1/2}$, then $M^* \gg X^{1/2}$ and we take $(M, N) = (N^*, M^*)$. Then $U^*(x)$ is a sum of Type II. By Lemma 2.15, the result follows.

Case 3 If $N_j < X^{726/7}$ ($j = 1, 2, 3, 4, 5, 6$), without loss of generality, we assume that $N_1 \geq N_2 \geq \cdots \geq N_6$. Let $\ell$ denote the smallest natural number $j$ such that

$$N_1 N_2 \cdots N_{\ell-1} < X^{726/7}, \quad N_1 \cdots N_\ell \geq X^{726/7},$$

then $2 \leq \ell \leq 5$. Noting that $\delta < 1/36 < 7/216$, we obtain

$$X^{726/7} \leq N_1 \cdots N_{\ell-1} \cdot N_\ell < X^{266/7}. \quad X^{726/7} < X^{1-726/7}.$$

Let $m = \prod_{i=1}^{\ell} n_i$, $n = \prod_{i=\ell+1}^{6} n_i$, $M^* = \prod_{i=1}^{\ell} N_i$, $N^* = \prod_{i=\ell+1}^{6} N_i$. At this time, we can follow the discussion of Case 2 exactly and get the result by Lemma 2.15. This completes the proof of Lemma 2.16. \[\qed\]

3 Proof of Theorem 1.1

Let us denote

$$H(R) = \int_{-\infty}^{\infty} I^3(x)e(-Rx)\Phi(x)dx, \quad H_1(R) = \int_{-\tau}^{\tau} I^3(x)e(-Rx)\Phi(x)dx,$$

$$B_1(R) = \int_{-\infty}^{\infty} S^3(x)e(-Rx)\Phi(x)dx, \quad D_1(R) = \int_{-\tau}^{\tau} S^3(x)e(-Rx)\Phi(x)dx,$$

$$D_2(R) = \int_{\tau < |x| < K} S^3(x)e(-Rx)\Phi(x)dx, \quad D_3(R) = \int_{|x| \geq K} S^3(x)e(-Rx)\Phi(x)dx.$$ 

In order to prove Theorem 1.1, it is sufficient to prove the following proposition.

Proposition 3.1 Let $1 < c < 37/18$, $c \neq 2$. Then for any sufficiently large real number $N$, we have

$$\int_N^{2N} |B_1(R) - H(R)|^2 dR \ll \varepsilon^2 N^{6/c-1} \exp \left( -\frac{1}{3} \left( \frac{1}{c \log \frac{2N}{3}} \right)^{1/5} \right). \quad (3.1)$$

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3.1 Proof of Proposition 3.1

Throughout the proof of Proposition 3.1, we always set \( X = (2N/3)^{1/\epsilon} \) and denote the function \( \Phi(x) \) which is from Lemma 2.1 with parameters 

\[
a = \frac{9\epsilon}{10}, \quad b = \frac{\epsilon}{10}, \quad k = \lfloor \log X \rfloor.
\]

We have

\[
\int_N^{2N} |B_1(R) - H(R)|^2 dR \\
= \int_N^{2N} |(D_1 - H_1) + D_2 + D_3 - (H - H_1)|^2 dR \\
\ll \int_N^{2N} |D_1 - H_1|^2 dR + \int_N^{2N} |D_2|^2 dR + \int_N^{2N} |D_3|^2 dR + \int_N^{2N} |H - H_1|^2 dR. \quad (3.2)
\]

By Lemma 2.2, we get \( I(x) \ll X^{1-c}|x|^{-1} \). By Lemma 2.1, we have

\[
\int_N^{2N} |H - H_1|^2 dR \\
\ll \int_N^{2N} \left( \int_{|x| > \tau} |I(x)|^3 |\Phi(x)| dx \right)^2 dR \\
\ll \varepsilon^2 \left( N \int_{|x| > \tau} |I(x)|^3 dx \right)^2 \\
\ll \varepsilon^2 N X^{6-6\epsilon} \left( \int_{\tau}^{\infty} \frac{dx}{x^2} \right)^2 \ll \varepsilon^2 N \frac{X^{6-6\epsilon}}{\tau^2}. \quad (3.3)
\]

For the third term on the right hand in (3.2), we have

\[
\int_N^{2N} |D_3|^2 dR \\
\ll \int_N^{2N} \left( \int_K^{+\infty} |S(x)|^3 |\Phi(x)| dx \right)^2 dR \\
\ll \left( \int_K^{+\infty} |S(x)|^3 |\Phi(x)| dx \right)^2 \\
\ll N X^6 \left( \frac{5k}{\pi K \varepsilon} \right)^{2k} \ll N X^{6+2\log(5/\pi)} X^{-4 \log A} \ll N. \quad (3.4)
\]

Take \( \Omega_1 = \{ R : N < R \leq 2N \} \), \( \Omega_2 = \{ x : \tau < |x| < K \} \), \( \xi = S^3(x) \Phi(x) \), \( \omega(x, R) = e(Rx) \), \( c(R) = \frac{D_2(R)}{D_2(R)} \). Then from Lemma 2.5 and Lemma 2.6, we obtain

\[
\int_N^{2N} |D_2(R)|^2 dR \ll 2A \int_{\tau}^{K} |S(x)|^6 |\Phi(x)|^2 dx \\
\ll \mathcal{L} \cdot \max_{\tau \leq x \leq K} |S(x)|^2 \times \int_{\tau}^{K} |S(x)|^4 |\Phi(x)|^2 dx. \quad (3.5)
\]
By the first derivative test, we have

\[ \int_\tau |S(x)|^4 |\Phi(x)|^2 \ dx \ll \varepsilon^2 \int_\tau |S(x)|^4 \ dx \]

\[ = \varepsilon^2 \sum_{\frac{X}{2} < p_1, p_2, p_3, p_4 \leq X} \prod (\log p_1) \cdot \cdots \cdot (\log p_4) \int_\tau e \left( (p_1^c + p_2^c - p_3^c - p_4^c) x \right) \ dx \]

\[ \ll \varepsilon^2 \log^4 X \sum_{\frac{X}{2} < p_1, p_2, p_3, p_4 \leq X} \min \left( K, \frac{1}{|p_1^c + p_2^c - p_3^c - p_4^c|} \right) \]

\[ \ll \sum_{\frac{X}{2} < n_1, n_2, n_3, n_4 \leq X} \min \left( K, \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|} \right). \quad (3.6) \]

Let \( u = n_1^c + n_2^c - n_3^c - n_4^c \). By Lemma 2.3, the contribution of \( K \) is (notice \( |u| \leq K^{-1} \))

\[ \ll K \cdot \mathcal{A}(X/2; c, K^{-1}) \ll (X^{4-c} + X^2)^X. \quad (3.7) \]

By a dyadic argument, the contribution from \( n_1, n_2, n_3, n_4 \) with \( |u| > K^{-1} \) is bounded by

\[ \ll \log X \times \max_{K^{-1} \leq U \leq X^c} \sum_{\frac{X}{2} < n_1, n_2, n_3, n_4 \leq X} \frac{1}{|u|} \]

\[ \ll \log X \times \max_{K^{-1} \leq U \leq X^c} U^{-1} \cdot \mathcal{A}(X/2; c, 2U) \]

\[ \ll \log X \times \max_{K^{-1} \leq U \leq X^c} (X^{4-c} + X^2 U^{-1})^X \]

\[ \ll (X^{4-c} + X^2)^X. \quad (3.8) \]

Combining (3.7) and (3.8), we have

\[ \int_\tau |S(x)|^4 |\Phi(x)|^2 \ dx \ll (X^{4-c} + X^2)^X. \quad (3.9) \]

If \( 1 < c < 2 \), then from Lemma 2.10 we get

\[ \int_{N}^{2N} |D_2(R)|^2 \ dx \ll \mathcal{L} \cdot \left( X^{(6+c)/4+\eta} + X^{28/15+\eta} \right) (X^{4-c} + X^2)^X \]

\[ \ll \left( X^{(6+c)/4+\eta} + X^{28/15+\eta} \right) X^{4-c+\eta} \]

\[ \ll X^{11/2-3c/4+\eta} + X^{88/15-c+\eta} \]

\[ \ll N^{11/(2c)-3/4+\eta} + N^{88/(15c)-1+\eta} \]

\[ \ll \varepsilon^2 N^{6/c-1-\eta} \ll \varepsilon^2 N^{6/c-1} \mathcal{L}^6 E^{2/3}. \quad (3.10) \]

If \( 2 < c < 37/18 \), then from Lemma 2.16 we get

\[ \int_{N}^{2N} |D_2(R)|^2 \ dx \ll \mathcal{L} \cdot X^{4-c-2\eta} (X^{4-c} + X^2)^X \]

\[ \ll \mathcal{L} \cdot X^{4-c-2\eta} \cdot X^{2+\eta} \ll X^{6-c-\eta} \]

\[ \ll N^{6/c-1-\eta} \ll \varepsilon^2 N^{6/c-1} \mathcal{L}^6 E^{2/3}. \quad (3.11) \]
Combining (3.10) and (3.11), for $1 < c < 37/18$, $c \neq 2$, we obtain that
\[
\int_{N}^{2N} |D_2(R)|^2 dR \ll \varepsilon^2 N^{6/c-1} \mathcal{L}^6 E^{2/3}.
\]
(3.12)

Next, we consider the first term on the right hand in (3.2). First of all, one has
\[
|D_1(R) - H_1(R)|^2 \leq \int_{-	au}^{\tau} \left( S(x)^3 - I(x)^3 \right) e(Rx) \Phi(x) dx \times \int_{-	au}^{\tau} \left( S^3(y) - I^3(y) \right) e(-Ry) \Phi(y) dy
\]
\[
= \int_{-	au}^{\tau} \left( S(x)^3 - I(x)^3 \right) \Phi(x) \left( \int_{-	au}^{\tau} (S^3(y) - I^3(y)) e(R(x-y)) \Phi(y) dy \right) dx.
\]

Therefore, we have
\[
\int_{N}^{2N} |D_1(R) - H_1(R)|^2 dR
\]
\[
= \int_{N}^{2N} \left[ \int_{-	au}^{\tau} \left( S(x)^3 - I(x)^3 \right) \Phi(x) \right. \\
\times \left( \int_{-	au}^{\tau} (S^3(y) - I^3(y)) e(R(x-y)) \Phi(y) dy \right) dx \] dR
\]
\[
= \int_{-	au}^{\tau} \left( S(x)^3 - I(x)^3 \right) \Phi(x) \left[ \int_{-	au}^{\tau} (S^3(y) - I^3(y)) \left( \int_{N}^{2N} e(R(x-y)) dR \right) \Phi(y) dy \right] dx
\]
\[
\ll \int_{-	au}^{\tau} |S^3(x) - I^3(x)||\Phi(x)|
\]
\[
\times \left( \int_{-	au}^{\tau} |S^3(y) - I^3(y)||\Phi(y)| \min \left(N, \frac{1}{|x-y|}\right) dy \right) dx.
\]
(3.13)

Applying Cauchy’s inequality to the inner integral and combining Lemma 2.9, one has
\[
\int_{-	au}^{\tau} |S^3(y) - I^3(y)||\Phi(y)| \min \left(N, \frac{1}{|x-y|}\right) dy
\]
\[
\ll \varepsilon \int_{-	au}^{\tau} |S(y) - I(y)||S^2(y) + S(y)I(y) + I^2(y)| \min \left(N, \frac{1}{|x-y|}\right) dy
\]
\[
\ll \varepsilon \left( \int_{-	au}^{\tau} |S^2(y) + S(y)I(y) + I^2(y)|^2 dy \right)^{1/2}
\]
\[
\times \left( \int_{-	au}^{\tau} |S(y) - I(y)|^2 \min \left(N, \frac{1}{|x-y|}\right)^2 dy \right)^{1/2}
\]
\[
\ll \varepsilon \left( \int_{-	au}^{\tau} |S(y)|^4 dy + \int_{-	au}^{\tau} |I(y)|^4 dy \right)^{1/2}
\]
\[
\times \left( \int_{-	au}^{\tau} |S(y) - I(y)|^2 \min \left(N, \frac{1}{|x-y|}\right)^2 dy \right)^{1/2}
\]
\[
\ll \varepsilon X^{2-c/2} \mathcal{L}^{5/2} \left( \int_{-	au}^{\tau} |S(y) - I(y)|^2 \min \left(N, \frac{1}{|x-y|}\right)^2 dy \right)^{1/2}.
\]
(3.14)
Put (3.14) into (3.13) and we get
\[
\int_{N}^{2N} |D_1(R) - H_1(R)|^2 dR 
\leq \varepsilon^{3/2} X^{2-c/2} L^{5/2} \int \left| S^3(x) - I^3(x) \right| \Phi(x)^{1/2} dx
\]
\[
\times \left( \int \left| S(y) - I(y) \right|^2 \min \left( N, \frac{1}{|x-y|} \right)^2 d\frac{1}{|x-y|} \right)^{1/2} dx
\]
\[
\leq \varepsilon^{3/2} X^{2-c/2} L^{5/2} \sup_{|x| \leq \tau} \left( \int \left| S(y) - I(y) \right|^2 \min \left( N, \frac{1}{|x-y|} \right)^2 \Phi(x) dy \right)^{1/2}
\]
\[
\times \int_{-\tau}^{\tau} \left| S^3(x) - I^3(x) \right| dx. \tag{3.15}
\]

On one hand, by Lemma 2.7, we have
\[
\int_{-\tau}^{\tau} \left| S^3(x) - I^3(x) \right| dx \leq \int_{-\tau}^{\tau} \left| S(x) \right|^3 dx + \int_{-\tau}^{\tau} \left| I(x) \right|^3 dx
\]
\[
\leq X \int_{-\tau}^{\tau} \left| S(x) \right|^2 dx + X \int_{-\tau}^{\tau} \left| I(x) \right|^2 dx
\]
\[
\leq X^{3-c/2} L^3. \tag{3.16}
\]

On the other hand, by Lemma 2.8, we have
\[
\int_{-\tau}^{\tau} \left| S(y) - I(y) \right|^2 \min \left( N, \frac{1}{|x-y|} \right)^2 \Phi(x) dy
\]
\[
\leq \varepsilon^2 \int_{y \in (x-\frac{X}{2}, x+\frac{X}{2}) \cap [-\tau, \tau]} \left| S(y) - I(y) \right|^2 dy + \varepsilon^2 \frac{N^2}{p^2} \int_{-\tau}^{\tau} \left| S(y) - I(y) \right|^2 dy
\]
\[
\leq \varepsilon^2 X^{2-c} P E^2 + \varepsilon^2 \frac{N^2}{p^2} X^{2-c} L^3
\]
\[
\leq \varepsilon^2 X^{2-c} E^{1/3} L^3. \tag{3.17}
\]

Combining (3.15), (3.16) and (3.17) we obtain
\[
\int_{N}^{2N} |D_1(R) - H_1(R)|^2 dR \ll \varepsilon^2 N^{6/c-1} L^7 E^{2/3}. \tag{3.18}
\]

From (3.2), (3.3), (3.4), (3.12) and (3.18), we know that the conclusion of Proposition 3.1 follows.

3.2 Proof of Theorem 1.1

For \( R \in (N, 2N) \), we set
\[
B := B(R) = \sum_{\frac{X}{2} < p_1, p_2, p_3 \leq X \atop \frac{1}{|p_1 p_2 p_3|} \leq \epsilon} (\log p_1)(\log p_2)(\log p_3).
\]
From Proposition 3.1, we can claim that if $1 < c < 37/18$, $c \neq 2$, there exists a set $\mathcal{P} \subset (N, 2N]$ satisfying

$$|\mathcal{P}| \ll N \exp \left( -\frac{2}{15} \left( \frac{1}{c} \log \frac{2N}{3} \right)^{1/5} \right),$$

(3.19)

such that

$$B_1(R) = H(R) + O\left( \varepsilon N^{\frac{2}{c} - 1} \exp \left( -\frac{1}{10} \left( \frac{1}{c} \log \frac{2N}{3} \right)^{1/5} \right) \right)$$

for all $R \in (N, 2N] \setminus \mathcal{P}$.

Actually, from Proposition 3.1, for $R \in \mathcal{P}$, we have

$$B_1(R) - H(R) \gg \varepsilon N^{\frac{2}{c} - 1} \exp \left( -\frac{1}{10} \left( \frac{1}{c} \log \frac{2N}{3} \right)^{1/5} \right).$$

(3.20)

Therefore, we get

$$\varepsilon^2 N^{6/c - 1} \exp \left( -\frac{1}{3} \left( \frac{1}{c} \log \frac{2N}{3} \right)^{1/5} \right)$$

$$\gg \int_N^{2N} |B_1(R) - H(R)|^2 dR$$

$$\gg \int_{\mathcal{P}} |B_1(R) - H(R)|^2 dR$$

$$\gg |\mathcal{P}| \cdot \varepsilon^2 N^{6/c - 2} \exp \left( -\frac{1}{5} \left( \frac{1}{c} \log \frac{2N}{3} \right)^{1/5} \right),$$

and (3.19) follows.

As in [18], by the Fourier transformation formula, we have

$$B_1(R) = \sum_{\frac{1}{c} < p_1, p_2, p_3 \leq X} \log p_1 \cdot \log p_2 \cdot \log p_3 \cdot \int_{-\infty}^{\infty} e((p_1^c + p_2^c + p_3^c - R)x) \Phi(x) dx$$

$$= \sum_{\frac{1}{c} < p_1, p_2, p_3 \leq X} \log p_1 \cdot \log p_2 \cdot \log p_3 \cdot \varphi (p_1^c + p_2^c + p_3^c - R) \leq B(R).$$

Hence Theorem 1.1 follows from the inequality

$$H(R) \gg \varepsilon R^{4/c - 1},$$

which can be proved proceeding as in [18], Lemma 6. This completes the proof of Theorem 1.1.

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4 Proof of Theorem 1.2

Throughout the proof of Theorem 1.2, we always set $X = (N/5)^{1/c}$ and denote the function $\varphi(y)$ which is from Lemma 2.1 with parameters

\[ a = \frac{9\varepsilon}{10}, \quad b = \frac{\varepsilon}{10}, \quad k = \lfloor \log X \rfloor. \]

Let

\[ \mathcal{B} = \sum_{\frac{N}{5} < p_1, \ldots, p_6 \leq X} (\log p_1)(\log p_2) \cdots (\log p_6). \]

Set

\[ \mathcal{B}_1 = \sum_{\frac{N}{5} < p_1, p_2, \ldots, p_6 \leq X} (\log p_1)(\log p_2) \cdots (\log p_6) \cdot \varphi(p_1^e + \cdots + p_6^e - N). \]

By the definition of $\varphi$, we have

\[ \mathcal{B} \geq \mathcal{B}_1. \tag{4.1} \]

The Fourier transformation formula gives

\[ \mathcal{B}_1 = \int_{-\infty}^{+\infty} S_6(x) e(-Nx) \Phi(x) \, dx =: D_1 + D_2 + D_3, \tag{4.2} \]

where

\[ D_1 = \int_{-\tau}^{\tau} S_6(x) e(-Nx) \Phi(x) \, dx, \]

\[ D_2 = \int_{\tau < |x| < K} S_6(x) e(-Nx) \Phi(x) \, dx, \]

\[ D_3 = \int_{|x| \geq K} S_6(x) e(-Nx) \Phi(x) \, dx. \]

By Lemma 2.1, we have

\[ D_3 \ll \int_{K}^{+\infty} |S(x)|^6 |\Phi(x)| \, dx \ll X^6 \int_{K}^{+\infty} \frac{1}{x} \left( \frac{5k}{\pi x \varepsilon} \right)^k \, dx \ll 1. \tag{4.3} \]

Let

\[ \mathcal{H}_1 = \int_{-\tau}^{\tau} I_6(x) e(-Nx) \Phi(x) \, dx \]

and

\[ \mathcal{H} = \int_{-\infty}^{+\infty} I_6(x) e(-Nx) \Phi(x) \, dx, \]

then

\[ D_1 = \mathcal{H} + (\mathcal{H}_1 - \mathcal{H}) + (D_1 - \mathcal{H}_1). \tag{4.4} \]
By Lemma 2.2, we get $I(x) \ll X^{1-c}|x|^{-1}$. By Lemma 2.1, we have

$$
\mathcal{H}_1 - \mathcal{H} \ll \int_{\tau}^{+\infty} |I(x)|^6 |\Phi(x)| dx \\
\ll X^{6-6c} \int_{\tau}^{+\infty} \frac{dx}{x^7} \\
\ll X^{6-6c}r^{-6} \ll X^{6\eta}.
$$

(4.5)

According to Lemma 2.1, Lemma 2.7 and Lemma 2.8, we have

$$
D_1 - H_1 \ll \int_{-\tau}^{\tau} \left| S^6(x) - I^6(x) \right| |\Phi(x)| dx \\
\ll \varepsilon \max_{|x| \leq \tau} \left| S(x) - I(x) \right| \times \int_{-\tau}^{\tau} \left( |S(x)|^5 + |I(x)|^5 \right) dx \\
\ll \varepsilon X^3 \max_{|x| \leq \tau} \left| S(x) - I(x) \right| \times \int_{-\tau}^{\tau} \left( |S(x)|^2 + |I(x)|^2 \right) dx \\
\ll \varepsilon X^{5-c} \log^3 X \times \max_{|x| \leq \tau} |S(x) - I(x)| \\
\ll \varepsilon X^{6-c} e^{-\frac{1}{2} \log X^{1/5}}.
$$

(4.6)

So Lemma 2.4 combining (4.4), (4.5) and (4.6) yields

$$
D_1 \gg \varepsilon X^{6-c}.
$$

(4.7)

For $D_2$, we have

$$
D_2 \ll \max_{\tau < |x| < K} |S(x)|^2 \times \int_{\tau < |x| < K} |S(x)|^4 |\Phi(x)| dx.
$$

(4.8)

For the integral on the right hand in (4.8), we can exactly follow the process of (3.9) and obtain

$$
\int_{\tau < |x| < K} |S(x)|^4 |\Phi(x)| dx \ll (X^{4-c} + X^2) X^{3\eta}.
$$

(4.9)

If $1 < c < 2$, then from (4.8), Lemma 2.10 and (4.9) we get

$$
D_2 \ll (X^{6+c}/4 + X^{28/15 + \eta})(X^{4-c} + X^2) X^{\eta} \ll X^{6-c-\eta}.
$$

(4.10)

If $2 < c < 37/18$, then from (4.8), Lemma 2.16 and (4.9) we get

$$
D_2 \ll X^{2-2\delta}(X^{4-c} + X^2) X^{\eta} \ll X^{2-2\delta} \cdot X^{2+\eta} \ll X^{6-c-\eta}.
$$

(4.11)

From (4.1), (4.2), (4.3), (4.7), (4.10) and (4.11) we get

$$
\mathcal{B} \geq \mathcal{B}_1 = D_1 + D_2 + D_3 \gg \varepsilon X^{6-c},
$$

which completes the proof of Theorem 1.2.

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