Degenerate nonlinear parabolic equations with discontinuous diffusion coefficients

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(based on joint works with Dohyun Kwon, UCLA)

Geometric and Functional Inequalities and Recent Topics in Nonlinear PDEs
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Mathematical motivation

→ Study the well-posedness and structure of solutions to diffusion equations with discontinuous nonlinearities.

→ Model problem:

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\begin{cases}
\partial_t \rho - \Delta \varphi(\rho) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0, T) \times \Omega, \\
(\nabla \varphi(\rho) + \nabla \Phi \rho) \cdot \mathbf{n} = 0, & \text{on } (0, T) \times \partial \Omega \quad \text{(NDE)}, \\
\rho(0, \cdot) = \rho_0, & \text{in } \Omega,
\end{cases}
\]

where \( T > 0, \Omega \subset \mathbb{R}^d \) smooth, bounded convex domain, \( \rho_0 \in \mathcal{P}^{ac}(\Omega) \) and \( \Phi : \Omega \to \mathbb{R} \) is a given Lipschitz continuous potential.
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→ Example of a nonlinearity

\[
\varphi : [0, +\infty) \to \mathbb{R}, \quad \varphi(s) = \begin{cases} 
\rho, & \rho \in [0, 1), \\
[\rho, 2\rho], & \rho = 1, \\
2\rho, & \rho > 1,
\end{cases}
\]
Motivation: Starvation driven diffusion in mathematical biology

→ A competition between a linear diffusion and a starvation driven diffusion:

\[ \partial_t u = d \Delta u + u(m - u - v), \quad \partial_t v = \Delta \varphi(v; m) + v(m - u - v). \]

where \( u, v \) represent two population densities and \( m \) stands for the resource density.

→ For \( 0 < l < h \), \( \varphi(v; m) := \begin{cases} lv, & \text{if } v < m, \\ hv, & \text{if } v > m. \end{cases} \)

→ Cho-Kim [2013, Bull. Math. Biol.] (“Starvation driven diffusion as a survival strategy of biological organisms”) (Ex: \( \Omega = (0, 1) \), \( m \) discontinuous with two constant values and \( u(0, \cdot) = v(0, \cdot) = m/2; l = 0.002, h = 0.004 \)
Motivation: self-organized criticality in physics

→ Bántay-Jánosi [1992, Phys. Rew. Let.] (“Avalanche dynamics from anomalous diffusion” - self organized criticality in sandpile models).

→ Same problem as (NDE), with $\Phi = 0$, $\varphi(\rho) = f(\rho)H(\rho - \rho_c)$, where $f$ is some given function (either identity, or a constant), $H$ is the Heaviside function and $\rho_c$ stands for the critical density value.

Figure: Avalanches in the Himalayas

Figure: Time evolution of $\rho$, $\rho_c = 1$ [Bántay-Jánosi, 1992]
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For $\varphi$ discontinuous:

- Blanchard-Röckner-Russo [2010, Ann. Probab.] $\rightarrow |\varphi(\rho)| \leq C\rho$; probabilistic approach in 1D; non-degenerate case.

- Barbu-Röckner-Russo, [2011, PTRF] $\rightarrow$ same model, probabilistic approach in 1D; degenerate case.
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→ Barbu-Röckner [2018, SIMA] $\rightarrow$ higher dimensions; probabilistic approach; nonlinear semigroup theory $\rightarrow$ maximal monotone operators, parabolic approximation, i.e. $\varphi_\epsilon \rightarrow \varphi$.

→ Notion of solution: generalized entropic solutions à la Kruzkov.

→ This heuristically can be written as pairs $(\rho, \eta_\rho)$ belonging to well-chosen function spaces, such that

$$\partial_t \rho - \Delta (\eta_\rho) - \nabla \cdot (\nabla \Phi \rho) = 0$$

is fulfilled and $\rho(t, x) \in \eta_\rho(t, x)$ a.e.
Our main objectives

(1) Find a unified way to treat general discontinuous nonlinearities.

(2) Give a fine characterization of the emerging critical regions \( \{\rho = 1\} \) observed in numerical experiments.
Our approach: optimal transport and gradient flows

OT toolbox

→ for \( \mu, \nu \in \mathcal{P}(\Omega) \) we define the 2-Wasserstein distance \( W_2 \) as

\[
W_2^2(\mu, \nu) := \inf \left\{ \int_{\Omega \times \Omega} |x - y|^2 \, d\gamma : \gamma \in \mathcal{P}(\Omega \times \Omega), (\pi^x)\#\gamma = \mu, (\pi^y)\#\gamma = \nu \right\}
\]

where for \( T : X \to Y \) Borel function \( T\#\mu = \nu \) means that \( \nu(A) = \mu(T^{-1}(A)) \) for any \( A \subseteq Y \) Borel set.

→ we have the dual formulation

\[
W_2^2(\mu, \nu) := \sup \left\{ \int_{\Omega} \phi \, d\mu + \int_{\Omega} \psi \, d\nu : \phi, \psi \in C_b(\Omega), \phi(x) + \psi(y) \leq |x - y|^2 \right\}.
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→ for any finite measure $\chi$ s.t. $\chi(\Omega) = 0$ we have the first variation formula

$$\left. \frac{d}{dt} \right|_{t=0} \frac{1}{2} W_2^2(\mu + t\chi, \nu) = \int_{\Omega} \phi \, d\chi.$$

→ Brenier [1991, CPAM]: if $\mu \in \mathcal{P}^{ac}(\Omega)$, then $\gamma_{\text{opt}} = (\text{id}, T)\#\mu$, with $T = \text{id} - \nabla \phi_{\text{opt}}$. 
Gradient flows in \((\mathcal{P}(\Omega), W_2)\)

→ noticed by Otto (see [2001, CPDE]), and Ambrosio-Gigli-Savaré (see [2005, Birkhäuser, Springer]) \((\mathcal{P}(\Omega), W_2)\) has a differential geometric structure
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→ for example \(\partial_t \rho - \Delta \rho = 0\) can be seen as the GF of the Boltzmann entropy \(\mathcal{I}(\rho) = \int_{\Omega} \rho \log(\rho) \, dx\).
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De Giorgi’s minimizing movements scheme (cf. Jordan-Kinderlehrer-Otto [1998, SIMA])

→ let \(\rho_0\) be given and \(N \in \mathbb{N}\) and \(\tau > 0\) be such that \(T = N\tau\). Construct the recursive sequence for all \(k \in \{1, \ldots, N\}\)

\[
\rho_k \in \operatorname{argmin}_{\rho \in \mathcal{P}(\Omega)} \left\{ \mathcal{J}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_{k-1}) \right\} 
\]  

(MM)

→ optimality condition \(\log(\rho_k) + 1 + \frac{\phi_k}{\tau} = \text{const on spt}(\rho_k)\).

→ approximate velocity \(v^T_k := \frac{x - T_k(x)}{\tau} = \frac{\nabla \phi_k}{\tau} = -\frac{\nabla \rho_k}{\rho_k}\).

→ after interpolations, the limit curve, as \(\tau \downarrow 0\) solves \(\partial_t \rho + \nabla \cdot (\rho v) = 0\).
Back to our problems

We define the energy associated to our models as

\[ \mathcal{I}(\rho) := S(\rho) + \mathcal{F}(\rho) := \int_{\Omega} S(\rho(x)) \, dx + \int_{\Omega} \Phi \, d\rho(x). \]
We define the energy associated to our models as

$$J(\rho) := S(\rho) + F(\rho) := \int_{\Omega} S(\rho(x)) \, dx + \int_{\Omega} \Phi \, d\rho(x).$$

If $S : (0, +\infty) \to \mathbb{R}$ is differentiable (or $\varphi$ is continuous), then we have the correspondence

$$\varphi(\rho) = \rho S'(\rho) - S(\rho) + S(1).$$
We consider the GF of the functional $J$ in $(\mathcal{P}(\Omega), W_2^2)$. Thus, $J$ fails to be differentiable. Therefore the classical theory does not imply directly; one needs to work with subdifferential calculus. We need to rely on the scheme (MM). To write optimality conditions, we characterize the Wasserstein subdifferential of $J$.

Back to our problems

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In our setting, we will consider $S$ convex, twice continuously differentiable, except at $\rho = 1$, where is not differentiable. Thus, some care is needed when writing the previous identity.
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$\mathcal{J}$ fails to be differentiable. Therefore the classical theory does not imply directly; one needs to work with subdifferential calculus.

We need to rely on the scheme (MM). To write optimality conditions, we characterize the Wasserstein subdifferential of $\mathcal{J}$. 
Estimates

\[ \rightarrow \text{We need to choose carefully the function spaces: we work in } L^p(\Omega), \quad 1 < p \leq +\infty. \]

**Lemma (\( L^\infty \) estimates)**

*Let \( \rho_0 \in L^\infty(\Omega) \). Let \( (\rho_k)_{k=1}^{N} \) be constructed via the scheme (MM). Then we have*

\[ \|\rho_k\|_{L^\infty} \leq C(T, \Phi)\|\rho_0\|_{L^\infty}, \quad \forall k \in \{1, \ldots, N\}. \]
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Proof: easy argument combining [Santambrogio, 2015, Springer] and [Carrillo-Santambrogio, QAM, 2018].
The model problem via GF in \((\mathcal{P}(\Omega), W_2)\)

## Estimates

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### Lemma \((L^\beta\) estimates)

Let \(\rho_0 \in \mathcal{P}(\Omega)\) such that \(J(\rho_0) < +\infty\). Let \(S''(\rho) \geq C\rho^{r-2}\), if \(\rho \in (1, +\infty)\) for some \(r \geq 1\). Let \((\rho_k)_{k=1}^N\) be constructed via the scheme (MM). Then we have

\[
\|\rho_k\|_{L^\beta} \leq C(T, \Phi, 1/\tau), \quad \forall k \in \{1, \ldots, N\},
\]

where \(\beta := \begin{cases} (2r - 1)\frac{d}{d-2}, & d \geq 3, \\ +\infty, & d = 2, \\ +\infty, & d = 1. \end{cases}\)
Estimates and optimality conditions

As a consequence, we have uniform $L^\beta([0, T] \times \Omega)$ estimates on the piecewise constant interpolations $(\rho^\tau)_{\tau > 0}$. 
Theorem

For all $k \in \{1, \ldots, N\}$, there exists $C = C(k) \in \mathbb{R}$ and $\phi_k$ such that

\[
\begin{aligned}
C - \frac{\phi_k}{\tau} - \Phi &\leq S'(0+) & \quad & \text{in } \{\rho_k = 0\}, \\
C - \frac{\phi_k}{\tau} - \Phi &\in [S'(1-), S'(1+)] & \quad & \text{in } \{\rho_k = 1\}, \\
C - \frac{\phi_k}{\tau} - \Phi &\in S' \circ \rho_k & \quad & \text{otherwise}.
\end{aligned}
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→ As a consequence, we have uniform $L^\beta([0, T] \times \Omega)$ estimates on the piecewise constant interpolations $(\rho^\tau)_{\tau>0}$.

→ We compute subdifferentials in $L^p(\Omega)^*$ (including $p = +\infty$). We have

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\end{cases}
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Theorem

For $\rho_k$ is given in (MM), if $\xi \in \partial S(\rho_k) \cap L^1(\Omega)$, then it holds that

$$
\xi \in \begin{cases}
[\infty, S'(0+)] & \text{in } \{\rho_k = 0\}, \\
[S'(1-), S'(1+)] & \text{in } \{\rho_k = 1\}, \\
S' \circ \rho_k & \text{in } \{\rho_k \neq 1\},
\end{cases} \quad (1)
$$
More on optimality conditions and a new variable

→ proof uses a theorem of Rockafellar [1971, PJM]; we can also show that $\xi^s = 0$. 

The model problem via GF in $(P(\Omega), W_2)$
More on optimality conditions and a new variable

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→ Question: how do we identify the approximate velocity, i.e. $\frac{\nabla \phi_k}{\tau}$?
More on optimality conditions and a new variable

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→ Question: how do we identify the approximate velocity, i.e. $\frac{\nabla \phi_k}{\tau}$?
→ Answer: inspired by the analysis of Maury-Roudneff-Chupin-Santambrogio [2010, M3AM] (also [M.-Santambrogio, 2016, APDE]), we introduce a new variable:

→ For $k \in \{1, \ldots, N\}$, we define $p_k : \Omega \to \mathbb{R}$ as

$$p_k := \begin{cases} 
\max \left\{ C - \frac{\phi_k}{\tau} - \Phi, S'(1-) \right\} & \text{in } \{\rho_k < 1\}, \\
C - \frac{\phi_k}{\tau} - \Phi & \text{in } \{\rho_k = 1\}, \\
\min \left\{ C - \frac{\phi_k}{\tau} - \Phi, S'(1+) \right\} & \text{in } \{\rho_k > 1\}.
\end{cases}$$

→ Or, equivalently

$$p_k = \min \left\{ \max \left\{ C - \frac{\phi_k}{\tau} - \Phi, S'(1-) \right\}, S'(1+) \right\}.$$
Now, the optimality condition can be written in a unified way!

Let us illustrate this in the example of

\[ S(\rho) := \begin{cases} \rho \log \rho, & \text{for } \rho \in [0, 1] \\ 2\rho \log \rho, & \text{for } \rho \in (1, +\infty) \end{cases} \]

We have

\[ p_k := \begin{cases} 1 & \text{in } \{\rho_k < 1\} \\ C - \phi_k \tau - \Phi & \text{in } \{\rho_k = 1\} \\ 2 & \text{in } \{\rho_k > 1\} \end{cases} \]

and the optimality conditions read as

**Lemma**

For all \( k \in \{1, \ldots, N\} \), there exists \( C \in \mathbb{R} \) such that

\[ p_k(1 + \log \rho_k) + \phi_k \tau + \Phi = C a.e. \]

In particular, both \( p_k \) and \( \rho_k \) are Lipschitz continuous and \( \rho_k > 0 \) a.e.

As a consequence,

\[ r\phi_k \tau = -r\Phi - r p_k - p_k r\rho_k \rho_k \] (since \( r p_k \log(\rho_k) = 0 \)).

---

**A model problem**

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*In particular, both* $p_k$ *and* $\rho_k$ *are Lipschitz continuous and* $\rho_k > 0$ *a.e.*

→ As a consequence, $\frac{\nabla \phi_k}{\tau} = -\nabla \Phi - \nabla p_k - p_k \frac{\nabla \rho_k}{\rho_k}$ (since $\nabla p_k \log(\rho_k) = 0$).
Uniform estimates and passing to the limit at $\tau \downarrow 0$

→ Let us notice that

$$\frac{1}{\tau} \sum_{k=1}^{N} W_{2}^{2}(\rho_{k}, \rho_{k-1}) = \frac{1}{\tau} \sum_{k=1}^{N} \int_{\Omega} |\nabla \phi_{k}|^{2} \leq J(\rho_{0}) - \inf J.$$

The model problem via GF in $(P(\Omega), W_{2})$
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From here, the piecewise constant interpolations satisfy: $(\sqrt{\rho^\tau})_{\tau > 0}$ and $(p^\tau)_{\tau > 0}$ are uniformly bounded in $L^2([0, T]; H^1(\Omega))$. 

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Theorem

Suppose that \( \rho_0 \in L^\infty(\Omega) \) and \( r \cdot \Phi \cdot n > 0 \) on \( \partial \Omega \). Then, there exists \( \rho, p \in L^\infty([0,T] \times \Omega) \cap L^2([0,T]; H^1(\Omega)) \) such that \((\rho, p)\) is a unique solution to

\[
\begin{align*}
\partial_t \rho - \Delta(p \rho) - r \cdot (r \cdot \Phi \cdot \rho) &= 0, \\
(r \cdot (p \rho) + r \cdot \Phi \cdot \rho) \cdot n &= 0, \\
\rho(\cdot, 0) &= \rho_0, \quad \text{in } \Omega,
\end{align*}
\]

in the sense of distribution.

\[\square\]

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\[
\frac{1}{\tau} \sum_{k=1}^{N} W_2^2(\rho_k, \rho_{k-1}) = \frac{1}{\tau} \sum_{k=1}^{N} \int_{\Omega} |\nabla \phi_k|^2 \leq \mathcal{J}(\rho_0) - \inf \mathcal{J}.
\]

From here, the piecewise constant interpolations satisfy: \((\sqrt{\rho^\tau})_{\tau > 0}\) and \((p^\tau)_{\tau > 0}\) are uniformly bounded in \( L^2([0,T]; H^1(\Omega)) \).

If in addition, \( \rho_0 \in L^\infty(\Omega) \), then \((\rho^\tau)_{\tau > 0}\) is uniformly bounded in \( L^2([0,T]; H^1(\Omega)) \).
Uniform estimates and passing to the limit at $\tau \downarrow 0$

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Theorem

Suppose that \(\rho_0 \in L^\infty(\Omega)\) and \(\nabla \Phi \cdot n > 0\) on \(\partial \Omega\). Then, there exists \(\rho, p \in L^\infty([0, T] \times \Omega) \cap L^2([0, T]; H^1(\Omega))\) such that \((\rho, p)\) is a unique solution to

\[
\begin{aligned}
\partial_t \rho - \Delta (p \rho) - \nabla \cdot (\nabla \Phi \rho) &= 0, & \text{in } (0, T) \times \Omega, \\
(\nabla (p \rho) + \nabla \Phi \rho) \cdot n &= 0, & \text{on } (0, T) \times \partial \Omega, \\
\rho(\cdot, 0) &= \rho_0, & \text{in } \Omega,
\end{aligned}
\]

in the sense of distribution.
Some remarks

Remark

$(\rho, p)$ satisfies

\[
\begin{cases}
    p = 1 & \text{a.e. in } \{0 < \rho < 1\}, \\
    p \in [1, 2] & \text{a.e. in } \{\rho = 1\}, \\
    p = 2 & \text{a.e. in } \{\rho > 1\}.
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Remark

If we consider more general initial, i.e. $\rho_0 \in P(\Omega)$ such that $E(\rho_0) < +\infty$, we find a solution

\[
\rho \in L^\beta([0, T] \times \Omega) \text{ and } p \in L^2([0, T]; H^1(\Omega)) \cap L^\infty([0, T] \times \Omega)
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with $\sqrt{\rho} \in L^2([0, T]; H^1(\Omega))$. 
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→ In the proof, to gain compactness we use an Aubin-Lions type argument for $\rho^\tau$.
What about more general problems?

The corresponding ‘porous medium example’ follows similar arguments with some additional care, since the sequence $(\rho_k)_k$ in general fails to be fully supported on $\Omega$. 
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→ Let

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S(\rho) := \begin{cases} 
\frac{\rho^m}{m-1}, & \text{for } \rho \in [0, 1], \\
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\]

where \(m > 1\).

→ Our main theorem for the associated entropy can be formulated as follows.
Main theorem for the PM type model problem

Theorem (Kwon-M., 2021)

For $\rho_0 \in \mathcal{P}(\Omega)$ such that $\mathcal{J}(\rho_0) < +\infty$, there exists $\rho \in L^\beta([0, T] \times \Omega)$ and $p \in L^2([0, T]; H^1(\Omega)) \cap L^\infty([0, T] \times \Omega)$ with $\rho^{m-\frac{1}{2}} \in L^2([0, T]; H^1(\Omega))$ such that $(\rho, p)$ is a weak solution of

\[
\begin{cases}
\partial_t \rho - \Delta(\left( (m-1)\rho^m + 1 \right) \frac{p}{m}) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0, T) \times \Omega, \\
\rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\
(\nabla(\left( (m-1)\rho^m + 1 \right) \frac{p}{m}) + \nabla \Phi \rho) \cdot n = 0, & \text{in } [0, T] \times \partial \Omega,
\end{cases}
\]

in the sense of distribution. Furthermore, $(\rho, p)$ satisfies

\[
\begin{cases}
p(t, x) = \frac{m}{m-1} & \text{a.e. in } \{0 < \rho < 1\}, \\
p(t, x) \in \left[ \frac{m}{m-1}, \frac{2m}{m-1} \right] & \text{a.e. in } \{\rho = 1\}, \\
p(t, x) = \frac{2m}{m-1} & \text{a.e. in } \{\rho > 1\}.
\end{cases}
\]

In addition, if $\rho_0 \in L^\infty(\Omega)$ and $\nabla \Phi \cdot n > 0$ on $\partial \Omega$, then $\rho \in L^\infty([0, T] \times \Omega)$ and $\rho^m \in L^2([0, T]; H^1(\Omega))$. 
The ‘fully’ general problem

→ Recall that if $S$ is differentiable, then we have

$$\varphi(\rho) = \rho S'(\rho) - S(\rho) + S(1)$$

→ Based on the observation and the derivation of $p$, we define the operator $L_S$ pointwisely for functions $(\rho, p) : [0, T] \times \Omega \to \mathbb{R}$ by

$$L_S(\rho, p)(t, x) := [\rho(t, x)S'(\rho(t, x)) - S(\rho(t, x)) + S(1)] \mathbb{1}_{\{\rho \neq 1\}}(t, x) + p(t, x) \mathbb{1}_{\{\rho = 1\}}(t, x)$$

→ Recall that for a.e. $(t, x) \in [0, T] \times \Omega$ the pressure variable $p : [0, T] \times \Omega \to \mathbb{R}$ satisfies a.e.

$$p(t, x) = S'(1-) \quad \text{if } 0 \leq \rho(t, x) < 1,$$

$$p(t, x) \in [S'(1-), S'(1+)] \quad \text{if } \rho(t, x) = 1,$$

$$p(t, x) = S'(1+) \quad \text{if } \rho(t, x) > 1.$$  \quad (P)

→ We aim to find a solution to the PDE

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\rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\
(\nabla (L_S(\rho, p)) + \nabla \Phi \rho) \cdot n = 0, & \text{in } [0, T] \times \partial \Omega.
\end{cases} \quad (G)$$
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\end{cases} \tag{G}
\]
Assume that $S \sim \rho^m$ in $(0, 1)$ and $S \sim \rho^r$ in $(1, +\infty)$, for some $m \geq 1$, $r \geq 1$. Set $\beta \geq 1$ as before, i.e.

$$\beta := \begin{cases} 
(2r - 1) \frac{d}{d-2} & \text{if } d \geq 3, \\
[1, \infty) & \text{if } d = 2, \\
+\infty & \text{if } d = 1.
\end{cases}$$
Our main theorem reads as:

**Theorem (Kwon-M., 2021)**

Suppose that the above growth conditions are fulfilled and $m < r + \beta^2$ holds true. For $\rho_0 \in \mathcal{P}(\Omega)$ such that $E(\rho_0) < +\infty$, there exists $\rho \in L^\beta([0, T] \times \Omega)$, $\rho_{m-\frac{1}{2}} \in L^2([0, T]; H^1(\Omega))$ and $p \in L^2([0, T]; H^1(\Omega)) \cap L^\infty([0, T] \times \Omega)$ such that $(\rho, p)$ is a solution of (G)–(P) in the sense of distributions.

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→ Based on the ‘regularization of $S$’ and the Sobolev embedding theorem, we obtain the uniform bound $L^\beta([0, T] \times \Omega)$ for $(\rho^\tau)_{\tau>0}$.
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The main idea of the proof \((m = 1, \text{the less involved case})\)

\[ \rightarrow \text{The hypothesis } m < r + \frac{\beta}{2} \text{ is always true.} \]
The main idea of the proof ($m = 1$, the less involved case)

→ The hypothesis $m < r + \frac{\beta}{2}$ is always true.

→ We define the auxiliary functions $S_a$ and $S_b : [0, +\infty) \to \mathbb{R}$ by

$$S_a(\rho) := \begin{cases} S'(1-)\rho \log \rho, & \text{for } \rho \in [0, 1], \\ S'(1+)\rho \log \rho, & \text{for } \rho \in (1, +\infty), \end{cases}$$

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\]

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\[ \rightarrow \text{We obtain that } \rho_k > 0 \text{ a.e. and the optimality condition,} \]

\[
p_k(1 + \log \rho_k) + S_b'(\rho_k) + \frac{\dot{\rho}_k}{\tau} + \Phi = C \text{ a.e.}
\]
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\end{cases} \]

Lemma

For all \(k \in \{1, \ldots, N\}\), there exists \(C \in \mathbb{R}\) such that

\[ \rho_k^{m-1} p_k = \left( C - \frac{\phi_k}{\tau} - \Phi \right)_+ \text{ a.e.} \]

In particular, \(p_k\) and \(\rho_k^{m-1}\) are Lipschitz continuous.
The main idea of the proof ($m > 1$) (Continued)

→ In order to have the strong convergence, we need spacial Sobolev estimates.
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**Lemma**

(1) If \(r \geq m\), then \((\rho \tau)^{m - \frac{1}{2}}\) is uniformly bounded in \(L^2([0, T]; H^1(\Omega))\).

(2) If \(r < m < r + \frac{\beta}{2}\), then \((\rho \tau)^{m - \frac{1}{2}}\) is uniformly bounded in \(L^q([0, T]; W^{1,q}(\Omega))\) for some \(q \in (1, 2)\).
The model problem via GF in \((\mathcal{P}(\Omega), W_2)\)

The main idea of the proof \((m > 1)\) (Continued)

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→ Together with the previous estimates, these are enough to pass to the limit, using again a refined version of the Aubin-Lions lemma.
Representation as continuity equations

Under suitable additional assumptions, our main equation (G) also reads as

\[
\begin{aligned}
\partial_t \rho - \nabla \cdot \left( \rho \nabla \left( S'(\rho) \mathbb{1}_{\rho \neq 1} + p \mathbb{1}_{\rho = 1} \right) \right) - \nabla \cdot (\rho \nabla \Phi) &= 0, & & \text{in} \ (0, T) \times \Omega, \\
\rho(0, \cdot) &= \rho_0, & & \text{in} \ \Omega, \\
\rho \left[ \nabla \left( S'(\rho) \mathbb{1}_{\rho \neq 1} + p \mathbb{1}_{\rho = 1} \right) + \nabla \Phi \right] \cdot n &= 0, & & \text{in} \ [0, T] \times \partial \Omega.
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(4)
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\end{cases}
\]

(4)

Corollary

If

\[ m < r + \frac{1}{2}, \quad \beta > 2 \quad \text{and} \quad m < \frac{\beta}{2} + \frac{1}{2}, \]

then \((\rho, p)\) is a weak solution of \((4)\) in the sense of distribution.

We underline that additional assumptions are needed to guarantee Sobolev estimates on \(S'(\rho)\).
The emergence of the region \( \{ \rho = 1 \} \)

The phenomenon observed in [Bántay-Jánosi, 1992] (they use Dirichlet boundary conditions):

\[ \rho(r) \]

\[ \text{time} \]

**Figure:** Time evolution of \( \rho \)

**Figure:** The growth of the critical region on a log-log scale
Confirming such a phenomenon

Our results support such phenomena by the simple reasoning below.

**Lemma**

If \( t \in (0, T) \) is a Lebesgue point both for \( t \mapsto \rho_t \) and \( t \mapsto p_t \) with

\[
\mathcal{L}^1(\{\rho_t < 1\}) > 0 \quad \text{and} \quad \mathcal{L}^1(\{\rho_t > 1\}) > 0
\]

then

\[
\mathcal{L}^1(\{\rho_t = 1\}) > 0.
\]

The proof is based on

\[
p(t, \cdot) \in C^{0, \frac{1}{2}}(\Omega)
\]

(coming from the \( H^1 \) spacial regularity in 1D) for all Lebesgue point \( t \) for \( t \mapsto \rho_t \) and \( t \mapsto p_t \).
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→ The proof is based on $p(t, \cdot) \in C^{0, \frac{1}{2}}(\Omega)$ (coming from the $H^1$ spacial regularity in 1D) for all Lebesgue point $t$ for $t \mapsto \rho_t$ and $t \mapsto p_t$.

→ $p = S'(1-) \text{ a.e. on } \{\rho < 1\}, p = S'(1+) \text{ a.e. on } \{\rho > 1\}$ and $S'(1-) < S'(1+)$. 
The fact that $\mathcal{L}^d(\{\rho_k = 1\}) > 0$, is supported by our numerical experiments as well.
The model problem via GF in \((\mathcal{P}(\Omega), W_2)\)

The fact that \(\mathcal{L}^d(\{\rho_k = 1\}) > 0\), is supported by our numerical experiments as well.

We computed one minimizing movement step in 1D, for \(\Phi(x) = 2x\), \(\Omega = [0, 1]\) and \(S\) in the logarithmic entropy.

\[
\rho_k := \arg\min_{\rho \in \mathcal{P}(\Omega)} \left\{ \int_\Omega S(\rho(x)) \, dx + \int_\Omega 2x \, d\rho(x) + \frac{1}{2\tau} W_2^2(\rho, \rho_{k-1}) \right\},
\]

\[
\begin{cases}
p_k(x) = 1 & \text{a.e. in } \{0 < \rho_k(x) < 1\}, \\
p_k(x) \in [1, 2] & \text{a.e. in } \{\rho_k(x) = 1\}, \\
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\]

Figure: \(\rho_0\)

Figure: \(\rho_1\)
Uniqueness of solutions

→ By an involved analysis, carefully combining ideas from [Vázquez, OSP, 2007] and [Di Marino-M., M3AS, 2016] we obtain an $L^1$ contraction result.
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Theorem

Let $(\rho^1, p^1), (\rho^2, p^2)$ be solutions to (G)-(P) with initial conditions $\rho^1_0, \rho^2_0 \in \mathcal{P}(\Omega)$ such that $\mathcal{J}(\rho^i_0) < +\infty$, $i = 1, 2$. Suppose that $L_S(\rho^i, p^i) \in L^2([0, T] \times \Omega)$, $i = 1, 2$. Then we have

$$\|\rho^1_t - \rho^2_t\|_{L^1(\Omega)} \leq \|\rho^1_0 - \rho^2_0\|_{L^1(\Omega)}, \mathcal{L}^1 - \text{a.e. } t \in [0, T].$$

→ The assumption $L_S(\rho, p) \in L^2([0, T] \times \Omega)$ seems natural in the context of the PME equation.

→ This is not needed if $\rho^i_0 \in L^\infty(\Omega)$.

→ Because of the $L^\beta([0, T] \times \Omega)$ estimates on $\rho^i$, this assumption is fulfilled already if $\beta \geq 2r$. 
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Open question: can one obtain $W_2(\rho^1_t, \rho^2_t) \leq C(t)W_2(\rho^1_0, \rho^2_0)$? (cf. [Bolley-Carrillo, CPDE, 2014]).
Singular limits

For $\varepsilon_1, \varepsilon_2 > 0$, consider $E_{\varepsilon_1, \varepsilon_2} : \mathcal{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$, defined as

$$E_{\varepsilon_1, \varepsilon_2}(\rho) := \begin{cases} \int_{\Omega} S_{\varepsilon_1, \varepsilon_2}(\rho(x)) \, dx, & \text{if } S_{\varepsilon_1, \varepsilon_2}(\rho) \in L^1(\Omega), \\ +\infty, & \text{otherwise}, \end{cases}$$

where $S_{\varepsilon_1, \varepsilon_2} : \mathbb{R} \to \mathbb{R}$ is convex and has the form

$$S_{\varepsilon_1, \varepsilon_2}(s) = \begin{cases} \varepsilon_1 S_1(s), & \text{if } s \in (0, 1), \\ \varepsilon_2 S_2(s), & \text{if } s \geq 1, \\ +\infty, & \text{otherwise}. \end{cases}$$
The model problem via GF in \((\mathcal{P}(\Omega), W_2)\)

**Singular limits**

→ For \(\varepsilon_1, \varepsilon_2 > 0\), consider \(\mathcal{E}_{\varepsilon_1, \varepsilon_2} : \mathcal{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}\), defined as

\[
\mathcal{E}_{\varepsilon_1, \varepsilon_2}(\rho) := \begin{cases} 
    \int_{\Omega} S_{\varepsilon_1, \varepsilon_2}(\rho(x)) \, dx, & \text{if } S_{\varepsilon_1, \varepsilon_2}(\rho) \in L^1(\Omega), \\
    +\infty, & \text{otherwise,}
\end{cases}
\]

where \(S_{\varepsilon_1, \varepsilon_2} : \mathbb{R} \to \mathbb{R}\) is convex and has the form

\[
S_{\varepsilon_1, \varepsilon_2}(s) = \begin{cases} 
    \varepsilon_1 S_1(s), & \text{if } s \in (0, 1), \\
    \varepsilon_2 S_2(s), & \text{if } s \geq 1, \\
    +\infty, & \text{otherwise.}
\end{cases}
\]

→ It turns out that we have uniform estimates w.r.t \(\varepsilon_1, \varepsilon_2 > 0\).

→ One can take \(\varepsilon_1 \downarrow 0\) (and \(\varepsilon_2\) fixed) to obtain the well-posedness of the original sandpile model.

→ One can take \(\varepsilon_2 \to +\infty\) (and \(\varepsilon_1\) fixed) to obtain well-posedness results for (parabolic) problems under density constraints \(\rho \leq 1\).
Open question #1

→ Can we obtain the higher regularity of $\rho$ and $p$?

→ More properties of the critical region $\{\rho = 1\}$?

→ Can we obtain the regularity of the interface $\partial\{\rho = 1\}$?
Free boundary approach

Figure: Two phases

Figure: Three phases
Free boundary approach

→ Formally, we can write the \textit{three phase free boundary problem}
\[
\Delta p = -\Delta \Phi, \quad \text{in} \{\rho = 1\}, \quad p = S'(1-) \quad \text{in} \{\rho < 1\} \quad \text{and} \quad p = S'(1+) \quad \text{in} \{\rho > 1\},
\]
Free boundary approach

Formally, we can write the three phase free boundary problem

\[ \Delta p = -\Delta \Phi, \text{ in } \{\rho = 1\}, \quad p = S'(1-) \text{ in } \{\rho < 1\} \text{ and } p = S'(1+) \text{ in } \{\rho > 1\}, \]

or more in details for our first example as

\[
\begin{align*}
\partial_t \rho &= \Delta \rho + \nabla \cdot (\nabla \Phi \rho) & \text{in } \{p\rho < 1\}, \\
-\Delta p &= \Delta \Phi, & \text{in } \{1 < p\rho < 2\}, \\
\partial_t \rho &= 2\Delta \rho + \nabla \cdot (\nabla \Phi \rho) & \text{in } \{p\rho > 2\},
\end{align*}
\]

with boundary conditions

\[
\begin{align*}
|D(p\rho)^{1+}| - |D(p\rho)^{1-}| &= 0 \text{ on } \{p\rho = 1\}.
\end{align*}
\]

and

\[
\begin{align*}
p &= 1 & \text{in } \{p\rho < 1\}, \\
\rho &= 1, & \text{in } \{1 < p\rho < 2\}, \\
p &= 2 & \text{in } \{p\rho > 2\},
\end{align*}
\]
Open questions #2

Recall the growth of $S$: $S \sim \rho^m$ in $(0, 1)$ and $S \sim \rho^r$ in $(1, +\infty)$.

→ What happens if $m \gg r$?
→ Can we obtain Sobolev estimates?
→ If not, can we observe some singular phenomena as below?

Figure: $t = 0$

Figure: $t = t^* > 0$
Thank you for your attention!