A SYMMETRY RESULT FOR THE ORNSTEIN-UHLENBECK OPERATOR

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Abstract. In 1978 E. De Giorgi formulated a conjecture concerning the one-dimensional symmetry of bounded solutions to the elliptic equation $\Delta u = F'(u)$, which are monotone in some direction. In this paper we prove the analogous statement for the equation $\Delta - \langle x, \nabla \rangle u = F'(u)$, where the Laplacian is replaced by the Ornstein-Uhlenbeck operator. Our theorem holds without any restriction on the dimension of the ambient space, and this allows us to obtain an similar result in infinite dimensions by a limit procedure.

1. Introduction

A celebrated conjecture by De Giorgi [6] asks if bounded entire solutions to the equation

\begin{equation}
\Delta u = u^3 - u
\end{equation}

which are strictly increasing in some direction are one-dimensional, in the sense that the level sets $\{u = \lambda\}$ are hyperplanes, at least if $n \leq 8$. This conjecture has been proved by Ghoussoub and Gui [14] in dimension $n = 2$, and by Ambrosio and Cabré [2] in dimension $n = 3$, and a counterexample has been given by del Pino, Kowalczyk and Wei in [7] for $n \geq 9$. While the conjecture is still open for $4 \leq n \leq 8$, a very nice proof has been presented by O. Savin [17] under the additional assumption that $u$ connects $-1$ to $1$ along the direction where it increases. See also [4] for another proof in dimension $n = 2$ and [12] for a review on the subject.

In this paper, we are interested in a variant of (1.1) where the Laplacian $\Delta$ is substituted by the Ornstein-Uhlenbeck operator $\Delta - \langle x, \nabla \rangle$. Namely, we consider the semilinear elliptic equation

\begin{equation}
\Delta u - \langle x, \nabla u \rangle + f(u) = 0
\end{equation}

and show the one-dimensional symmetry of bounded entire solutions which are monotone in some direction.

Let us state our main result.

Theorem 1.1. Let $n \in \mathbb{N}$, $\alpha \in (0, 1)$. Let $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a solution of

\begin{equation}
\Delta u - \langle x, \nabla u \rangle + f(u) = 0 \quad \text{in } \mathbb{R}^n,
\end{equation}

where $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function. Assume that

\begin{equation}
\langle \nabla u(x), w \rangle > 0 \quad \text{for any } x \in \mathbb{R}^n
\end{equation}

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for some \( w \in \mathbb{R}^n \). Then, \( u \) is one-dimensional, i.e. there exist \( U : \mathbb{R} \to \mathbb{R} \) and \( \omega \in \mathbb{R}^n \) such that
\[
u(x) = U(\langle \omega, x \rangle)
\]
for any \( x \in \mathbb{R}^n \).

Notice that (1.2) can be regarded as the analog of (1.1) in the so-called Gauss space, that is, in \( \mathbb{R}^n \) endowed with the Gaussian instead of the Lebesgue measure. Indeed, while

the Pde in (1.1) is the Euler-Lagrange equation of the Allen-Cahn Energy
\[
\int_{\mathbb{R}^n} \left( \frac{|\nabla u|^2}{2} + \frac{(u^2 - 1)^2}{4} \right) dx,
\]
the Pde in (1.2) is the Euler-Lagrange equation of the functional
\[
\int_{\mathbb{R}^n} \left( \frac{|\nabla u|^2}{2} + F(u) \right) d\gamma(x),
\]
where \( F' = -f \) and
\[
d\gamma(x) = \gamma(x)dx = \frac{e^{-|x|^2/2}}{(2\pi)^{n/2}} dx
\]
is the standard Gaussian probability measure. It is interesting to remark that Theorem 1.1 holds for general type of nonlinearities, as it happens for the conjecture of De Giorgi when \( n \leq 3 \) (see [1], and this is a major difference with respect to the techniques in [17]).

As in the case of the Laplacian, Theorem 1.1 is closely related to the Bernstein problem in the Gauss space, which asks for flatness of entire minimal surfaces which are graphs in some direction. We point out that minimal surfaces in the Gauss space are interesting geometric objects, since they correspond to self-similar shrinkers of the mean curvature flow (see for instance [8]), and satisfy the equation
\[
\kappa = \langle x, \nu \rangle
\]
where \( \kappa \) is the mean curvature at \( x \) and \( \nu \) is the normal vector. In this context, the analog of the Bernstein Theorem has been proved by Ecker and Huisken [8], under a polynomial growth assumption on the volume of the minimal surface, and more recently by Wang in [20] without any further assumption. We point out that, differently from the Euclidean case, the result holds without any restriction on the dimension of the ambient space, and in fact there is no such restriction also in Theorem 1.1. This is due to the exponential decay of the Gaussian measure associated to the Ornstein-Uhlenbeck operator which allows for better estimates than the corresponding Euclidean ones.

Since Theorem 1.1 holds in any dimension and the Gauss space \( (\mathbb{R}^n, \gamma) \) formally converges to a Wiener space \( (X, H, \gamma) \) (see Section 2.1 for a precise definition) as \( n \to \infty \), one may expect that an analogous result holds in such infinite dimensional setting. Indeed, in this paper we confirm this expectation and show the infinite dimensional extension of Theorem 1.1:

**Theorem 1.2.** Let \( u \in C^1(X) \cap L^\infty(X) \) satisfy
\[
\Delta_\gamma u = f(u)
\]
where \( f : \mathbb{R} \to \mathbb{R} \) is a locally Lipschitz function. Assume that
\[
\partial_i \partial_j u \in C(X) \quad \text{for all } i, j \in \mathbb{N}
\]
and
\[
\inf_{x \in B_R} [\nabla u(x), w] > 0
\]
for all \( x \in X \), for all \( R > 0 \) and for some \( w \in H \). Then, \( u \) is one-dimensional, in the sense that there exist \( U : \mathbb{R} \to \mathbb{R} \) and \( \omega \in X^* \) such that
\[
u(x) = U(\langle \omega, x \rangle)
\]
for all \( x \in X \).

Notice that Theorem 1.1 can be recovered as a corollary of Theorem 1.2, when the function \( u \) depends only on finitely many variables. As far as we know, Theorem 1.1 is the first result of De Giorgi conjecture type in an infinite dimensional setting. The proof that we perform exploits and generalizes some geometric ideas of [18, 19, 10, 11].

2. Notation

We denote by \((\mathbb{R}^n, \gamma)\) the \( n \)-dimensional Gauss space, where \( \gamma \) is the standard Gaussian measure on \( \mathbb{R}^n \) defined in (1.6).

2.1. The Wiener space. An abstract Wiener space is defined as a triple \((X, \gamma, H)\) where \( X \) is a separable Banach space, endowed with the norm \( \| \cdot \|_X \), \( \gamma \) is a nondegenerate centered Gaussian measure, and \( H \) is the Cameron–Martin space associated to the measure \( \gamma \), that is, \( H \) is a separable Hilbert space densely embedded in \( X \), endowed with the inner product \( [\cdot, \cdot]_H \) and with the norm \( |\cdot|_H \). The requirement that \( \gamma \) is a centered Gaussian measure means that for any \( x^* \in X^* \), the measure \( x^* \gamma \) is a centered Gaussian measure on the real line \( \mathbb{R} \), that is, the Fourier transform of \( \gamma \) is given by
\[
\hat{\gamma}(x^*) = \int_X e^{-i \langle x, x^* \rangle} d\gamma(x) = \exp \left( -\frac{\langle Q x^*, x^* \rangle}{2} \right), \quad \forall x^* \in X^*;
\]
here the operator \( Q \in \mathcal{L}(X^*, X) \) is the covariance operator and it is uniquely determined by formula
\[
\langle Q x^*, y^* \rangle = \int_X \langle x, x^* \rangle \langle x, y^* \rangle d\gamma(x), \quad \forall x^*, y^* \in X^*.
\]
The nondegeneracy of \( \gamma \) implies that \( Q \) is positive definite: the boundedness of \( Q \) follows by Fernique’s Theorem (see for instance [5, Theorem 2.8.5]), asserting that there exists a positive number \( \beta > 0 \) such that
\[
\int_X e^{\beta \| x \|^2} d\gamma(x) < +\infty.
\]
This implies also that the maps \( x \mapsto \langle x, x^* \rangle \) belong to \( L^p_\gamma(X) \) for any \( x^* \in X^* \) and \( p \in [1, +\infty) \), where \( L^p_\gamma(X) \) denotes the space of all functions \( f : X \to \mathbb{R} \) such that
\[
\int_X |f(x)|^p d\gamma(x) < +\infty.
\]
In particular, any element \( x^* \in X^* \) can be seen as a map \( x^* \in L^2_\gamma(X) \), and we denote by \( R^* : X^* \to \mathcal{H} \) the identification map \( R^* x^*(x) := \langle x, x^* \rangle \). The space \( \mathcal{H} \) given by the closure
of $R^*X^*$ in $L^2_\gamma(X)$ is called reproducing kernel. By considering the map $R : \mathcal{H} \to X$ defined as

$$R\hat{h} := \int_X \hat{h}(x)xd\gamma(x),$$

we obtain that $R$ is an injective $\gamma$–Radonifying operator, which is Hilbert–Schmidt when $X$ is Hilbert. We also have $Q = RR^* : X^* \to X$. The space $H := RH$, equipped with the inner product $\langle \cdot, \cdot \rangle_H$ and norm $| \cdot |_H$ induced by $\mathcal{H}$ via $R$, is the Cameron–Martin space and is a dense subspace of $X$. The continuity of $R$ implies that the embedding of $H$ in $X$ is continuous, that is, there exists $c > 0$ such that

$$||h||_X \leq c|h|_H, \quad \forall h \in H.$$

We have also that the measure $\gamma$ is absolutely continuous with respect to translation along Cameron–Martin directions; in fact, for $h \in H$, $h = Qx^*$, the measure $\gamma_h(B) = \gamma(B – h)$ is absolutely continuous with respect to $\gamma$ with density given by

$$d\gamma_h(x) = \exp \left( \langle x, x^* \rangle – \frac{1}{2}|h|^2_H \right) d\gamma(x). \quad (2.1)$$

**2.2. Cylindrical functions and differential operators.** For $j \in \mathbb{N}$ we choose $x_j^* \in X^*$ in such a way that $\hat{h}_j := R^*x_j^*$, or equivalently $h_j := R\hat{h}_j = Qx_j^*$, form an orthonormal basis of $H$. We order the vectors $x_j^*$ in such a way that the numbers $\lambda_j := \|x_j^*\|_{X^*}^2$ form a decreasing sequence. Given $m \in \mathbb{N}$, we also let $H_m := \langle h_1, \ldots, h_m \rangle \subseteq H$, and $\Pi_m : X \to H_m$ be the closure of the orthogonal projection from $H$ to $H_m$

$$\Pi_m(x) := \sum_{j=1}^m \langle x, x_j^* \rangle h_j \quad x \in X.$$ 

The map $\Pi_m$ induces the decomposition $X \simeq H_m \oplus X_m^\perp$, with $X_m^\perp := \ker(\Pi_m)$, and $\gamma = \gamma_m \otimes \gamma_m^\perp$, with $\gamma_m$ and $\gamma_m^\perp$ Gaussian measures on $H_m$ and $X_m^\perp$ respectively, having $H_m$ and $H_m^\perp$ as Cameron–Martin spaces. When no confusion is possible we identify $H_m$ with $\mathbb{R}^m$; with this identification the measure $\gamma_m = \Pi_m \# \gamma$ is the standard Gaussian measure on $\mathbb{R}^m$ (see [3]). Given $x \in X$, we denote by $x_m \in H_m$ the projection $\Pi_m(x)$, and by $x_m \in X_m^\perp$ the infinite dimensional component of $x$, so that $x = (x_m, x_m^\perp)$. When we identify $H_m$ with $\mathbb{R}^m$ we shall rather write $x = (x_m, x_m^\perp) \in \mathbb{R}^m \oplus X_m^\perp$.

We say that $u : X \to \mathbb{R}$ is a cylindrical function if $u(x) = v(\Pi_m(x))$ for some $m \in \mathbb{N}$ and $v : \mathbb{R}^m \to \mathbb{R}$. We denote by $\mathcal{FC}^k_b(X)$, $k \in \mathbb{N}$, the space of all $C_b^k$ cylindrical functions, that is, functions of the form $v(\Pi_m(x))$ with $v \in C^k(\mathbb{R}^n)$, with continuous and bounded derivatives up to the order $k$. We denote by $\mathcal{FC}^k_b(X, H)$ the space generated by all functions of the form $uh$, with $u \in \mathcal{FC}^k_b(X)$ and $h \in H$.

We let

$$\nabla_\gamma u := \sum_{j \in \mathbb{N}} \partial_j u h_j \quad \text{for } u \in \mathcal{FC}^1_b(X)$$

$$\text{div}_\gamma \varphi := \sum_{j \geq 1} \partial_j^* [\varphi, h_j]_H \quad \text{for } \varphi \in \mathcal{FC}^1_b(X, H)$$

$$\Delta_\gamma u := \text{div}_\gamma \nabla_\gamma u \quad \text{for } u \in \mathcal{FC}^2_b(X)$$
where \( \partial_j := \partial h_j \) and \( \partial_j^\ast := \partial_j - \hat{h}_j \) is the adjoint operator of \( \partial_j \). With this notation, the integration by parts formula holds:

\[
\int_X u \text{div}_\gamma \varphi \, d\gamma = - \int_X [\nabla_\gamma u, \varphi]_H \, d\gamma \quad \forall \varphi \in FC^1_b(X, H).
\]

In particular, thanks to \((2.2)\), the operator \( \nabla_\gamma \) is closable in \( L^p_\gamma(X) \), and we denote by \( W^{1,p}_\gamma(X) \) the domain of its closure. The Sobolev spaces \( W^{k,p}_\gamma(X) \), with \( k \in \mathbb{N} \) and \( p \in [1, +\infty] \), can be defined analogously \([5]\), and \( FC^k_b(X) \) is dense in \( W^{j,p}_\gamma(X) \), for all \( p < +\infty \) and \( k, j \in \mathbb{N} \) with \( k \geq j \).

Given a vector field \( \varphi \in L^p_\gamma(X, H) \), \( p \in (1, \infty) \), using \((2.2)\) we can define \( \text{div}_\gamma \varphi \) in the distributional sense, taking test functions \( u \) in \( W^{1,q}_\gamma(X) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). We say that \( \text{div}_\gamma \varphi \in L^p_\gamma(X) \) if this linear functional can be extended to all test functions \( u \in L^p_\gamma(X) \). This is true in particular if \( \varphi \in W^{1,p}_\gamma(X, H) \).

Let \( u \in W^{2,2}_\gamma(X) \), \( \psi \in FC^1_b(X, H) \) and \( i, j \in \mathbb{N} \). From \((2.2)\), with \( u = \partial_j u \) and \( \varphi = \psi h_i \), we get

\[
\int_X \partial_j u \partial_i \psi \, d\gamma = \int_X -\partial_j (\partial_i u) \psi + \partial_j u \psi(x_i^*, x) \, d\gamma
\]

Let now \( \varphi \in FC^1_b(X, H) \). If we apply \((2.3)\) with \( \psi = [\varphi, h_j] =: \varphi^j \), we obtain

\[
\int_X \partial_j u \partial_i \varphi^j \, d\gamma = \int_X -\partial_j (\partial_i u) \varphi^j + \partial_j u \varphi^j(x_i^*, x) \, d\gamma
\]

which, summing up in \( j \), gives

\[
\int_X [\nabla_\gamma u, \partial_i \varphi] \, d\gamma = \int_X -[\nabla_\gamma (\partial_i u), \varphi] + [\nabla_\gamma u, \varphi](x_i^*, x) \, d\gamma \quad \forall \varphi \in FC^1_b(X, H).
\]

The operator \( \Delta_\gamma : W^{2,p}_\gamma(X) \to L^p_\gamma(X) \) is usually called the Ornstein-Uhlenbeck operator. Notice that, if \( u \) is a cylindrical function, that is \( u(x) = v(y) \) with \( y = \Pi_m(x) \in \mathbb{R}^m \) and \( m \in \mathbb{N} \), then

\[
\Delta_\gamma u = \sum_{j=1}^{m} \partial_j^2 u - \langle x_j^*, x \rangle \partial_j u = \Delta v - \langle y, \nabla v \rangle_{\mathbb{R}^m}.
\]

We write \( u \in C(X) \) if \( u : X \to \mathbb{R} \) is continuous and \( u \in C^1(X) \) if both \( u : X \to \mathbb{R} \) and \( \nabla_\gamma u : X \to H \) are continuous.

For simplicity of notation, from now on we will omit the explicit dependence on \( \gamma \) of operators and spaces. We also indicate by \([\cdot, \cdot]\) and \(|\cdot|\) respectively the scalar product and the norm in \( H \). When no confusion is possible, we shall also write \( u_i \) to indicate the derivative \( \partial_i u \).

3. Proof of Theorem 1.2

Recalling the integration by parts formula \((2.2)\), equation \((1.8)\) can be written in a weak form as

\[
\int_X [\nabla u, \nabla \varphi] - f(u) \varphi \, d\gamma = 0 \quad \text{for any} \ \varphi \in W^{1,2}(X)
\]
which is meaningful for \( u \in W^{1,2}(X) \). Notice that, as \( FC^1_b(X) \) is dense in \( W^{1,2}(X) \), it is enough to require \( (3.1) \) for all \( \varphi \in FC^1_b(X) \).

**Remark 3.1.** Since \( L^{\infty}(X) \subset L^2(X) \), by \[10\] Th. 4.1 we have that a bounded weak solution of \((1.8)\) belongs to \( W^{2,2}(X) \).

### 3.1. The linearized equation

We now consider the equation solved by the derivatives of the solution \( u \).

**Lemma 3.2.** Let \( u \in W^{2,2}(X) \) satisfy \((1.8)\). For any \( i \in \mathbb{N} \) let \( u_i = \partial_i u \in W^{1,2}(X) \), then

\[
(3.2) \quad \int_X [\nabla u_i, \nabla \varphi] - f'(u)u_i \varphi + u_i \varphi \, d\gamma = 0 \quad \text{for any } \varphi \in W^{1,2}(X).
\]

**Proof.** Notice first that it is enough to prove \((3.2)\) for all \( \varphi \in FC^2_b(X) \). Letting \( \varphi \in FC^2_b(X) \), we multiply \((1.8)\) by \( \varphi_i \) and recall \((2.4)\), to get

\[
0 = \int_X [\nabla u, \nabla \varphi_i] - f(u)\varphi_i \, d\gamma
\]

\[
= \int_X -[\nabla u_i, \nabla \varphi] + (x_i^*, x)[\nabla u, \nabla \varphi] - f(u)\varphi_i \, d\gamma
\]

\[
= \int_X -[\nabla u_i, \nabla \varphi] + (x_i^*, x)\nabla u, \nabla \varphi] + f'(u)u_i \varphi - (x_i^*, x) f(u) \varphi_i \, d\gamma
\]

\[
= \int_X -[\nabla u_i, \nabla \varphi] - \varphi(x_i^*, x) + f'(u)u_i \varphi - (x_i^*, x) f(u) \varphi_i \, d\gamma
\]

where the last inequality follows from \((3.1)\), with \( \varphi \) replaced by \( (x_i^*, x) \varphi \).

### 3.2. A variational inequality implied by the monotonicity

The next result shows that monotone solutions of \((1.8)\) satisfy a variational inequality. In the Euclidean case, this fact boils down to the classical stability condition (namely, the second derivative of the energy functional being nonnegative). Differently from this, in our case, a negative eigenvalue appears in the inequality.

**Lemma 3.3.** Let \( u \in W^{2,2}(X) \) satisfy \((1.8)\) and \((1.10)\). Then, for any \( \varphi \in W^{1,2}(X) \) it holds

\[
(3.3) \quad \int_X |\nabla \varphi|^2 - f'(u)\varphi^2 \, d\gamma \geq -\int_X \varphi^2 \, d\gamma.
\]

**Proof.** The proof is a variation of a classical technique (see, e.g., \[11\]). Without loss of generality we may assume \( w = h_1 \), and we let \( \varphi \in W^{1,2}(X) \) be such that \( \varphi^2/u_1 \in W^{1,2}(X) \). Notice that, thanks to \((1.10)\), the space of such functions is dense in
We use (3.2), with $\varphi^2/u_1$, and we obtain
\[
\int_X f'(u)\varphi^2 - \varphi^2 \, d\gamma \\
= \int_X [\nabla u_1, \nabla (\varphi^2/u_1)] \, d\gamma \\
= \int_X 2(\varphi/u_1)[\nabla u_1, \nabla \varphi] - (\varphi/u_1)^2|\nabla u_1|^2 \, d\gamma \\
= \int_X |\nabla \varphi|^2 - \left((\varphi/u_1)\nabla u_1 - \nabla \varphi\right)^2 \, d\gamma \\
\leq \int_X |\nabla \varphi|^2 \, d\gamma. \quad \Box
\]

3.3. A geometric Poincaré inequality. We show that a sort of geometric Poincaré inequality stems from solutions of (1.8) satisfying (3.3). In the Euclidean case, it boils down to the inequality discovered in [18, 19].

**Lemma 3.4.** Let $u \in W^{2,2}(X)$ satisfy (1.8) and (3.3). For any $\varphi \in W^{1,\infty}(X)$ we have
\[
(3.4) \quad \int_X \left(|\nabla^2 u|^2 - |\nabla \nabla u|^2\right) \varphi^2 \, d\gamma \leq \int_X |\nabla u|^2 |\nabla \varphi|^2 \, d\gamma
\]
where
\[
|\nabla^2 u|^2 := \sum_{i,j} u^2_{ij}.
\]

**Proof.** We use (3.3) with test function $|\nabla u| \varphi$, and we see that
\[
\int_X (f'(u) - 1)|\nabla u|^2 \varphi^2 \, d\gamma \\
\leq \int_X |\nabla (|\nabla u| \varphi)|^2 \, d\gamma \\
(3.5) \quad = \int_X \varphi^2 |\nabla u|^2 + |\nabla u|^2 |\nabla \varphi|^2 + 2|\nabla (|\nabla u| \varphi)| |\nabla u| \varphi \, d\gamma \\
= \int_X \varphi^2 |\nabla u|^2 + |\nabla u|^2 |\nabla \varphi|^2 + \frac{1}{2}[|\nabla u|^2, \nabla \varphi^2] \, d\gamma.
\]

We now exploit (3.2) with test function $u_i \varphi^2$ and we get
\[
\int_X (f'(u) - 1)u_i^2 \varphi^2 \, d\gamma \\
= \int_X [\nabla u_i, \nabla (u_i \varphi^2)] \, d\gamma \\
= \int_X |\nabla u_i|^2 \varphi^2 + u_i [\nabla u_i, \nabla \varphi^2] \, d\gamma \\
= \int_X |\nabla u_i|^2 \varphi^2 + \frac{1}{2}[\nabla u_i^2, \nabla \varphi^2] \, d\gamma.
\]
Summing over \( i \in \mathbb{N} \), we conclude that
\[
\int_X \left( f'(u) - 1 \right) |\nabla u|^2 \varphi^2 \, d\gamma
= \int_X |\nabla^2 u|^2 \varphi^2 + \frac{1}{2}[|\nabla \varphi|, |\nabla u|] \, d\gamma.
\] (3.6)

From (3.5) and (3.6), we conclude that
\[
\int_X |\nabla^2 u|^2 \varphi^2 + \frac{1}{2}[|\nabla \varphi|, |\nabla u|] \, d\gamma
\leq \int_X |\nabla \varphi|^2 + |\nabla u|^2 |\nabla \varphi|^2 + \frac{1}{2}[|\nabla \varphi|, |\nabla u|] \, d\gamma
\]
which gives (3.4).

Let \( u \in C^1(X) \cap L^\infty(X) \) satisfying (1.9), let \( N \in \mathbb{N} \) and \( \overline{x}_N \in X_N^+ \). We consider the map \( \psi_{N,\overline{x}_N} : \mathbb{R}^N \to \mathbb{R} \) defined as \( \psi_{N,\overline{x}_N}(x) := u(x, \overline{x}_N) \), and let
\[
\mathcal{N}_N(\overline{x}_N) := \{ x_N \in \mathbb{R}^N : \nabla \psi_{N,\overline{x}_N}(x_N) \neq 0 \}
= \{ x_N \in \mathbb{R}^N : \exists i \in \{1, \ldots, N\} \text{ such that } u_i(x_N, \overline{x}_N) \neq 0 \}
\]
be its noncritical set. By the Implicit Function Theorem, the level set of \( \psi_{N,\overline{x}_N} \) in \( \mathcal{N}_N(\overline{x}_N) \) are \((N-1)\)-dimensional hypersurfaces of class \( C^2 \). Thus we can consider the principal curvatures of these hypersurfaces, that we denote by \( \kappa_{1,N}, \ldots, \kappa_{N-1,N} \), and the tangential gradient of \( \psi_{N,\overline{x}_N} \), that we denote by \( \nabla_{T,N} \).

With this notation, we have the following
\[\text{Lemma 3.5.} \quad \text{Let } u \in C^1(X) \cap L^\infty(X) \text{ satisfy (1.3), (1.9) and (3.3), and fix } N \in \mathbb{N}. \text{ For any } \varphi \in W^{1,\infty}(X) \text{ we have}
\int_{\mathcal{N}_N} \left( |\nabla_N u|^2 \kappa_N^2 + |\nabla_{T,N} |\nabla_N u||^2 \right) \varphi^2 \, d\gamma
\leq \int_X \left( |\nabla^2 u|^2 + |\nabla \varphi|^2 |\nabla u|^2 \right) \varphi^2 \, d\gamma
\leq \int_X |\nabla u|^2 |\nabla \varphi|^2 \, d\gamma.\] (3.7)

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1\text{The tangential gradient of a function } g \text{ along a hypersurface with normal } \nu \text{ is } \nabla g - (\nabla g \cdot \nu) \nu, \text{ that is, the tangential component of the full gradient.}
Proof. Let
\[
\mathcal{D}_N := |\nabla^2 u|^2 - |\nabla_N|\nabla_N u|^2
= \sum_{1 \leq i, j \leq N} u_{ij}^2 - \sum_{1 \leq i \leq N} \left(\frac{\nabla_N u}{|\nabla_N u|}, \nabla_N u_i\right)^2
= \sum_{1 \leq i, j \leq N} \left(u_{ij}^2 - \left(\frac{u_j u_{ij}}{|\nabla_N u|}\right)^2\right).
\]
Since \( |\nabla_{N-1} u| \leq |\nabla_N u| \) and
\[
\left|\frac{u_j u_{ij}}{|\nabla_N u|}\right| \leq \left|\frac{|u_j|}{|\nabla_N u|}\right| u_{ij} \leq |u_{ij}|
\]
for any \( i, j \leq N \), it follows that
\[
\mathcal{D}_N - \mathcal{D}_{N-1} \geq \sum_{N-1 \leq i, j \leq N} \left(u_{ij}^2 - \left(\frac{u_j u_{ij}}{|\nabla_N u|}\right)^2\right) \geq 0
\]
so that \( \mathcal{D}_N \) is nondecreasing in \( N \). Accordingly,
\[
(3.8) \quad |\nabla^2 u|^2 - |\nabla|\nabla u|^2 = \lim_{M \to +\infty} \mathcal{D}_M \geq \mathcal{D}_N
\]
for any \( N \in \mathbb{N} \). Moreover, by Stampacchia’s Theorem we have that \( \nabla_N|\nabla_N u| = 0 \) for almost any \( \mathfrak{z}_N \in \mathbb{R}^N \setminus \mathcal{N}_N(\mathfrak{p}_N) \), and similarly \( u_{ij} = 0 \) for almost any \( \mathfrak{z}_N \in \mathbb{R}^N \setminus \mathcal{N}_N(\mathfrak{p}_N) \).
Therefore
\[
(3.9) \quad \mathcal{D}_N = |\nabla_N u|^2 k_N^2 + |\nabla_{T,N}|\nabla_N u|^2 = 0 \text{ for almost any } \mathfrak{z}_N \in \mathbb{R}^N \setminus \mathcal{N}_N(\mathfrak{p}_N).
\]
On the other hand, by [19, Formula (2.1)],
\[
\mathcal{D}_N = |\nabla_N u|^2 k_N^2 + |\nabla_{T,N}|\nabla_N u|^2 \quad \text{when } \mathfrak{z}_N \in \mathcal{N}_N(\mathfrak{p}_N).
\]
From this, (3.8) and (3.9), we obtain
\[
\int_X \left(|\nabla^2 u|^2 - |\nabla|\nabla u|^2\right) \varphi^2 d\gamma \\
\geq \int_X \mathcal{D}_N \varphi^2 d\gamma \\
= \int_{\mathcal{N}_N} \mathcal{D}_N \varphi^2 d\gamma \\
= \int_{\mathcal{N}_N} \left(|\nabla_N u|^2 k_N^2 + |\nabla_{T,N}|\nabla_N u|^2\right) \varphi^2 d\gamma,
\]
which, recalling (3.4), implies (3.7). \( \square \)
3.4. A symmetry result. We now use the previous material to obtain a one-dimensional symmetry result for the $N$-dimensional projection of the solution. The idea of using geometric Poincaré inequalities as the ones in [18, 19] in order to obtain symmetry properties goes back to [10] and it was widely used in [11] in the finite dimensional Euclidean setting.

The result we present here is the following:

**Proposition 3.6.** Fix $N \in \mathbb{N}$ and $\pi_N \in X_N$. Let $u \in C^1(X) \cap L^\infty(X)$ satisfy (1.8), (1.9) and (3.3). Then, the map $\psi_N \pi_N$ is one-dimensional, i.e. there exists $U \in \mathbb{R} \rightarrow \mathbb{R}$ and $\omega \in \pi_N^* \subset X^*$ such that

$$u(x) = U(\langle \omega, x \rangle)$$

for any $x \in X_N$.

**Proof.** We fix $R > 1$, to be taken arbitrarily large in what follows, and let $\Lambda = \max_i \lambda_i$.

Let $\Phi \in C^\infty(\mathbb{R})$ be such that $\Phi(t) = 1$ if $t \leq R$, $\Phi(t) = 0$ if $t \geq R + 1$ and $|\Phi'(t)| \leq 3$ for any $t \in [R, R + 1]$. We take $\varphi(x) := \Phi(|x|)$. Then $|\nabla \varphi(x)| \leq \sqrt{\Lambda} |\Phi'(|x|)| \leq 3\sqrt{\Lambda}$, and (3.7) yields

$$\int_{N \cap \{|x| \leq R\}} |\nabla u|^2 \xi_N^2 + |\nabla T_N| |\nabla u| |^2 \, d\gamma \leq 9 \Lambda \int_{\{R \leq |x| \leq R + 1\}} |\nabla u|^2 \, d\gamma.$$ 

Also, due to our assumptions on $u$,

$$\int_X |\nabla u|^2 \, d\gamma < +\infty.$$ 

Therefore, by sending $R \rightarrow +\infty$ in (3.11), we conclude that

$$|\nabla u|^2 \xi_N^2 + |\nabla T_N| |\nabla u|^2 = 0$$

for any $x \in N_N$. From this and [11, Lemma 2.11] we get (3.10). \qed

From the finite dimensional symmetry result of Proposition 3.6 one can take the limit as $N \rightarrow +\infty$ and obtain:

**Corollary 3.7.** Let $u \in C^1(X) \cap L^\infty(X)$ satisfy (1.8), (1.9) and (3.3). Then, $u$ is necessarily one-dimensional, i.e. there exists $U : \mathbb{R} \rightarrow \mathbb{R}$ and $\omega \in X^*$ such that $u(x) = U(\langle \omega, x \rangle)$ for any $x \in X$.

**Proof.** We first show that there exists $h \in H$ such that

$$\frac{\nabla u}{|\nabla u|} = h \quad \text{in } N := \{x \in X : \nabla u(x) \neq 0\} = \bigcup_{N \in \mathbb{N}} N_N.$$ 

Let $V \subset X$ be defined as $V = \bigcup_N H_N$. Since $V$ is a dense subset of $X$, it is enough to show that (3.12) holds in $N \cap V = \bigcup_N V_N$, where $V_N := N_N \cap H_N$.

However, from Proposition 3.6 we know that

$$\frac{\nabla u}{|\nabla u|} = \omega_{N,0} \quad \text{in } V_N,$$
which implies that  

$$\frac{\nabla u}{|\nabla u|} = \lim_{N \to \infty} \frac{\nabla_N u}{|\nabla_N u|} = \lim_{N \to \infty} \omega_{N,0} = h$$  \text{ in } V.

From (3.12) it follows that there exists a function $U : \mathbb{R} \to \mathbb{R}$ such that $U(t) = u(th)$ for all $t \in \mathbb{R}$, and  

(3.14)  

$$u(x) = U(h(x)) \quad x \in X.$$  

Moreover, $U$ is a bounded nondecreasing solution to the ODE  

$$U'' - tU' + f(U) = 0 \quad t \in \mathbb{R}.$$  

Being $u$ continuous, if $U$ is nonconstant (otherwise the thesis follows immediately) then the function $\hat{h}$ is also continuous, so that $h \in QX^*$ and $\hat{h}(x) = \langle \omega, x \rangle$ for some $\omega \in X^*$, which implies the thesis. □

3.5. **Proof of Theorem 1.2.** The proof of Theorem 1.2 follows directly from Lemma 3.3 and Corollary 3.7. □

**Remark 3.8.** We observe that, in the infinite dimensional case, there may exist weak solutions to (1.8), satisfying (1.10), which are not continuous. Indeed, given $U : \mathbb{R} \to \mathbb{R}$ satisfying (4.1) and (4.2) below, the function $u(x) = U(\hat{h}(x))$ in (3.14) is a solution to (1.8), monotone in the direction given by $h$, for any $h \in H$. However, such a solution is continuous only if $h \in QX^*$. As a possible generalization of Theorem 1.2, one could ask if any bounded weak solution to (1.8), satisfying $[\nabla u, w] > 0$ for some $w \in H$, is of this form.

4. **Heteroclinic solutions**  

The results in Theorems 1.1 and 1.2 may be seen either as classification results (when one knows explicitly the solutions of the associated one-dimensional problem) or as nonexistence result (when the associated one-dimensional problem does not admit any solution). For this, we now give some simple conditions on the nonlinearity $f$ ensuring existence or nonexistence of bounded solutions to the ODE  

(4.1)  

$$U'' - tU' + f(U) = 0 \quad t \in \mathbb{R}$$  

satisfying  

(4.2)  

$$U' > 0 \quad t \in \mathbb{R}.$$  

Notice that, from (4.2) it follows that there exist $U^\pm \in \mathbb{R}$, with $U^- < U^+$, such that  

(4.3)  

$$\lim_{t \to \pm \infty} U(t) = U^\pm.$$  

Moreover, passing to the limit in (4.1) we also get  

(4.4)  

$$f(U^-) = f(U^+) = 0.$$  

We start with a nonexistence result.
**Proposition 4.1.** Assume that there exists $U_0 \in (U^-, U^+)$ such that
\[ f \geq 0 \text{ in } [U^-, U_0] \quad \text{or} \quad f \leq 0 \text{ in } [U_0, U^+]. \]
Then, there are no solutions to (4.1) satisfying (4.2).

**Proof.** Let us assume that $f \leq 0$ in $[U_0, U^+]$, since the argument is analogous in the other case, and assume by contradiction that we are given a solution $U$ of (4.1), (4.2).

Letting $t_0 > 0$ be such that $u(t_0) \in [U_0, U^+]$, we have that $U$ satisfies the differential inequality
\[ U'' \geq tU' \quad \text{for all } t \in [t_0, +\infty), \]
which implies, by direct integration,
\[ U'(t) \geq U'(t_0) e^{(t^2-t_0^2)/2} \geq U'(t_0) > 0 \quad \text{for all } t \in [t_0, +\infty), \]
contradicting (4.3). □

We consider the potential $F: \mathbb{R} \to \mathbb{R}$, defined as
\[ F(t) = -\int_0^t f(s) \, ds + k. \]
where $k \in \mathbb{R}$. Notice that, if $F$ is convex or concave, from (4.4) it follows that $f \equiv 0$ in $[U^-, U^+]$, so that by Proposition 4.1 there are no solutions to (4.1) satisfying (4.2).

Given $U: (0, +\infty) \to \mathbb{R}$, we let
\[
G(U) := \int_0^{+\infty} \left( \frac{(U'(t))^2}{2} + F(U(t)) \right) d\gamma(t)
\]
where $d\gamma(t) = e^{-t^2/2} dt$. Notice that (4.4) is the Euler-Lagrange equation of $G$.

As a counterpart of the nonexistence result in Proposition 4.1 we now give an existence result for monotone solutions to (4.1).

**Proposition 4.2.** Assume that $F$ satisfies the following properties:
\[
\begin{align*}
F(c) &= F(-c) = 0 \quad \text{for some } c > 0 \\
F(r) &> 0 \quad \text{for any } r \notin \{c, -c\} \\
F(r) &= F(-r) \quad \text{for any } r \in [0, +\infty). \\
f(r) &= 0 \quad \text{iff } r \in \{c, -c, 0\}.
\end{align*}
\]
Assume also that there exists $U \in W^{1,2}_{\gamma}((0, +\infty))$ such that $U(0) = 0$ and
\[
G(U) < G(0) = \sqrt{\frac{\pi}{2}} F(0).
\]
Then, there exists a monotone solution to (4.1), connecting $-c$ to $c$.

**Proof.** Let $\overline{U}$ be a solution to the minimum problem
\[
\min \{ G(U) : U \in W^{1,2}_{\gamma}((0, +\infty)), \; U(0) = 0 \}.
\]
Note that (4.6) implies
\[ G(\min \{ \overline{U}, c \}) \leq G(\overline{U}), \]
so that we may assume $\overline{U}(t) \in [0, c]$ for all $t \in (0, +\infty)$. 


Let now $U^*$ be the Ehrhard rearrangement of $U$ [9], which is defined in such a way that $U^*$ is nondecreasing on $(0, +\infty)$, and

$$\gamma\left(\left\{ t : U^*(t) > r \right\}\right) = \gamma\left(\left\{ t : U(t) > r \right\}\right) \quad \text{for all } r \in (0, c).$$

Notice that $U^*(0) = 0$ and $U^*(t) \in [0, c]$ for all $t \in (0, +\infty)$. By [9] (see also [13, Prop. 3.12]), we have $U^* \in W^{1,2}_\gamma((0, +\infty))$ and

$$\int_0^{+\infty} \frac{(U^*(t))^2}{2} d\gamma(t) \leq \int_0^{+\infty} \frac{(U'(t))^2}{2} d\gamma(t)$$

$$\int_0^{+\infty} F(U^*(t))d\gamma(t) = \int_0^{+\infty} F(U(t))d\gamma(t),$$

so that

$$G(U^*) \leq G(U).$$

In particular, we may assume that $U = U^*$, i.e. that $U$ is nondecreasing on $(0, +\infty)$.

As $U = c$ and $U = 0$ are solutions to (4.1), which is the Euler-Lagrange equation of $G$, we get that either $U = 0$ or $U = c$ or

$$(4.9) \quad \gamma(t) \in (0, c) \quad \text{for all } t \in (0, +\infty).$$

On the other hand, thanks to (4.7) and the fact that $U(0) = 0$, we can exclude the first two possibilities, so that (4.9) holds. Moreover, since $U$ is nondecreasing and $f(r) \neq 0$ for all $r \in (0, c)$, it follows that $U'(t) > 0$ for all $t \in (0, +\infty)$ and

$$\lim_{t \to +\infty} U(t) = c.$$  

Since by (4.6) the function $t \to -U(-t)$ is a monotone solution to (4.1) on $(-\infty, 0)$, we get that the odd extension of $U$ on $\mathbb{R}$ is a solution to (4.1) on the whole of $\mathbb{R}$ which satisfies (4.2) and connects $-c$ to $c$. \hfill \Box

**Remark 4.3.** Notice that for all $U \in W^{1,2}_\gamma((0, +\infty))$ we have

$$G(U) < \tilde{G}(U) := \int_0^{+\infty} \left( \frac{(U'(t))^2}{2} + F(U(t)) \right) dt.$$

If we let $\overline{U}$ be the unique solution to

$$U''(t) + f(U(t)) = 0 \quad t \in (0, +\infty)$$

$$U(0) = 0$$

$$\lim_{t \to +\infty} U(t) = c,$$

we have

$$G(\overline{U}) < \tilde{G}(\overline{U}) = \int_0^c \sqrt{2F(r)} dr.$$

In particular, condition (4.7) is verified whenever

$$\int_0^c \sqrt{2F(r)} dr \leq \sqrt{\frac{\pi}{2}} F(0)$$

which is satisfied, for instance, by the standard double-well potential $F(t) = (1 - t^2)^2/4.$
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