A Radó theorem for complex spaces

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Abstract. We generalize Radó’s extension theorem from the complex plane to reduced complex spaces.

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1 Introduction

A theorem due to Radó asserts that a continuous complex valued function on an open subset of the complex plane is holomorphic provided that it is holomorphic off its zero set.

Essentially this theorem was proved in [10]. Since then many other proofs have been proposed, e.g. [2], [3], [6], and [7]. The articles [1], [11] and [14] give some generalizations.

Radó’s statement remains true for complex manifolds (or, more generally, for normal complex spaces) as well as in the complex plane.

In this short note we investigate a natural extension of Radó’s theorem when the ambient space has (non normal) singularities.

Complex spaces, unless explicitly stated, are assumed to be reduced and countable at infinity. Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers;

Here we state our main results.

Proposition 1 There is an irreducible Stein curve $X$ and a continuous function $f : X \to \mathbb{C}$ that is holomorphic off its zero set but no power $f^\nu$, $\nu \in \mathbb{N}$, is globally holomorphic.

Theorem 1 Let $X$ be a complex space and $\Omega \subset X$ be a relatively compact open set. Then, there is $\nu_\Omega \in \mathbb{N}$ such that, for every continuous function $f : X \to \mathbb{C}$ that is holomorphic off its zero set, and for every integer $\nu \geq \nu_\Omega$, the power function $f^\nu$ is holomorphic on $\Omega$. 
Recall the following definition. Let \( X \) be a complex space. A continuous, complex-valued function \( f \) defined on an open set \( U \subset X \) is \( c \)-holomorphic if its restriction of to \( \text{Reg}(X) \cap U \) is holomorphic, where \( \text{Reg}(X) \) is the open set of those points of \( X \) where it is locally a manifold. The sheaf of germs of \( c \)-holomorphic functions in \( X \) is denoted by \( \mathcal{O}_X^c \); it is a coherent \( \mathcal{O}_X \)-module.

Henceforth the following remark will be used tacitly. For a complex space \( X \), any continuous function \( f : X \to \mathbb{C} \) that is holomorphic off its zero set \( f^{-1}(0) \) is \( c \)-holomorphic. (This results by the classical Radó theorem on complex manifolds.)

## 2 Proof of Proposition 1

The example of a Stein curve \( X \) is obtained by implanting generalized cusp singularities at the points \( 2, 3, \ldots \), of \( \mathbb{C} \), and then the existence of the function \( f \) is obtained via Cartan’s vanishing theorem on Stein spaces.

In order to proceed, let \( p \) and \( q \) be coprime integers \( \geq 2 \). Consider the cusp like irreducible and locally irreducible complex curve

\[
\Gamma = \{(z_1, z_2) \in \mathbb{C}^2 : z_1^p = z_2^q \} \subset \mathbb{C}^2.
\]

Its normalization is \( \mathbb{C} \) and \( \pi : \mathbb{C} \to \Gamma, t \mapsto (t^p, t^q) \), is the normalization map. Note that \( \pi \) is a homeomorphism.

A continuous function \( h : \Gamma \to \mathbb{C} \) that is holomorphic off its zero set, but fails to be globally holomorphic is produced as follows.

Select natural numbers \( m \) and \( n \) with \( mq - np = 1 \), and define \( h : \Gamma \to \mathbb{C} \) by setting for \( (z_1, z_2) \in \Gamma \),

\[
h(z_1, z_2) := \begin{cases} 
z_1^m/z_2^n & \text{if } z_2 \neq 0, \\
0 & \text{if } z_2 = 0.
\end{cases}
\]

It is easily seen that \( h \) is continuous (as \( \pi \) is a homeomorphism, the continuity of \( h \) follows from that of \( h \circ \pi \), which is equal to the identity mapping on \( \mathbb{C} \)), \( h \) is holomorphic off its zero set (incidentally, here, the regular part \( \text{Reg}(\Gamma) \) is the complement of this zero set), and \( h \) is not holomorphic about \( (0, 0) \) (use a Taylor series expansion about \( (0, 0) \in \mathbb{C}^2 \) of a presumably holomorphic extension).

Furthermore, \( h^k \) is globally holomorphic provided that \( k \geq (p-1)(q-1) \).

(Because every integer \( \geq (p-1)(q-1) \) can be written in the form \( \alpha p + \beta q \), where \( \alpha, \beta \) are integers.)
with \( \alpha, \beta \in \{0, 1, 2, \ldots\} \), and since \( h^p \) and \( h^q \) are holomorphic being the restrictions of \( z_2 \) and \( z_1 \) to \( \Gamma \) respectively.)

Also, \( z_1^p z_2^q h \) is holomorphic on \( \Gamma \) provided that \( q \frac{(m + a)/p}{+ b} \geq n \), where \([ \cdot ]\) is the floor function.

It is interesting to note that the stalk of germs of c-holomorphic functions \( \mathcal{O}_0^c \) at 0 is generated as an \( \mathcal{O}_0 \)-module by the germs at 0 of \( 1, h, \ldots, h^r \), where \( r = \min\{p, q\} \).

Now, for each integer \( k \geq 2 \), let \( \Gamma_k := \{(z_1, z_2) \in \mathbb{C}^2 ; z_1^k = z_2^{k+1}\} \).

As previously noted, \( \Gamma_k \) is an irreducible curve whose normalization map is \( \pi_k : \mathbb{C} \to \Gamma_k, t \mapsto (t^{k+1}, t^k) \), and the function \( h_k : \Gamma_k \to \mathbb{C} \) defined for \( (z_1, z_2) \in \Gamma_k \) by

\[
h_k(z_1, z_2) := \begin{cases} z_1/z_2 & \text{if } z_2 \neq 0, \\ 0 & \text{if } z_2 = 0, \end{cases}
\]

has the following properties:

a) The function \( h_k \) is c-holomorphic.

b) The power \( h_k^{k-1} \) is not holomorphic.

c) The function \( z_1^{k-1} h_k \) is holomorphic because it is the restriction of \( z_2^k \) to \( \Gamma_k \).

Here, with these examples of singularities at hand, we change the standard complex structure of \( \mathbb{C} \) at the discrete analytic set \( \{2, 3, \ldots\} \) by complex surgery, in order to obtain an irreducible Stein complex curve \( X \) and a discrete subset \( \Lambda = \{x_k : k = 2, 3, \ldots\} \) such that, at the level of germs \( (X, x_k) \) is biholomorphic to \( (\Gamma_k, 0) \).

The surgery, that we recall for the commodity of the reader (because in some monographs like [8] the subsequent condition (\( \star \)) is missing), goes as follows.

Let \( Y \) and \( U' \) be complex spaces together with analytic subsets \( A \) and \( A' \) of \( Y \) and \( U' \) respectively, such that there is an open neighborhood \( U \) of \( A \) in \( Y \) and \( \varphi : U \setminus A \to U' \setminus A' \) that is biholomorphic.

Then define

\[
X := (Y \setminus A) \sqcup_{\varphi} U' := (Y \setminus A) \sqcup U'/\sim
\]

by means of the equivalence relation \( U \setminus A \ni y \sim \varphi(y) \in U' \setminus A' \).

Then there exists exactly one complex structure on \( X \) such that \( U' \) and \( Y \setminus A \) can be viewed as open subsets of \( X \) in a canonical way provided that the following condition is satisfied:
(⋆) For every \( y \in \partial U \) and \( a' \in A' \) there are open neighborhoods \( D \) of \( y \) in \( Y \), \( D \cap A = \emptyset \), and \( B \) of \( a' \) in \( U' \) such that \( \varphi(D \cap U) \cap B \subseteq A' \).

Thus \( X \) is formed from \( Y \) by "replacing" \( A \) with \( A' \).

In practice, the condition (⋆) is fulfilled if \( \varphi^{-1} : U' \setminus A' \to U \setminus A \) extends to a continuous function \( \psi : U' \to U \) such that \( \psi(A') = A \). In this case, if \( D \) and \( V \) are disjoint open neighborhoods of \( \partial U \) and \( A \) in \( Y \) respectively, then \( B = A' \cup \varphi(V \setminus A) \) is open in \( U' \) because it equals \( \psi^{-1}(V) \) and (⋆) follows immediately. (This process is employed, for instance, in the construction of the blow-up of a point in a complex manifold!)

Coming back to the construction of the example proving Proposition 1, consider \( Y = \mathbb{C} \), \( A = \{2, 3, \ldots \} \) and for each \( k = 2, 3, \ldots \), let \( \Delta(k, 1/3) \) be the disk in \( \mathbb{C} \) centered at \( k \) of radius \( 1/3 \) that is mapped holomorphically onto an open neighborhood \( U_k \) of \( (0, 0) \in \Gamma_k \) through the holomorphic map \( t \mapsto \pi_k(z - k) \). Applying surgery, we get an irreducible Stein curve \( X \) and the discrete subset \( \Lambda \) with the aforementioned properties.

It remains to produce the function \( f \) as stated. For this we let \( \mathcal{I} \subset \mathcal{O}_X \) be the coherent ideal sheaf with support \( \Lambda \) and such that \( \mathcal{I}_{x_k} = \mathfrak{m}_{x_k}^{k-1} \) for \( k = 2, 3, \ldots \), where \( \mathfrak{m}_{x_k} \) is the maximal ideal of the analytic algebra of the stalk of \( \mathcal{O}_X \) at \( x_k \).

From the exact sequence

\[
0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0,
\]

we obtain a \( c \)-holomorphic function \( f \) on \( X \) such that, for each \( k = 2, 3, \ldots \), at germs level \( f \) equals \( h_k (\text{mod } \mathcal{I}_{x_k}) \).

By properties \( a_k \), \( b_k \) and \( a_k \) from above, it follows that there does exist \( \nu \in \mathbb{N} \) such that \( f^\nu \) becomes holomorphic on \( X \). (For instance, if \( f = h_k + g_k^{k-1} \), for certain \( g_k \in \mathfrak{m}_{x_k} \), then \( f^{k-1} \) is not holomorphic about \( x_k \).)

3 Proof of Theorem 1

This is divided into four steps. In the first step we recall, following [4], the multiplicity of an analytic set at a point. Then in Step 2, we estimate the vanishing order of a \( c \)-holomorphic function germ at a point of its zero set in terms of the multiplicity of the analytic germ where it is defined. In Step 3 we collect some useful facts about \( O^N \)-approximability due to Spallek [13] and Siu [12]. Eventually, the proof of theorem is achieved in the fourth step.
Step 1. Let \( A \) be a pure \( k \)-dimensional locally analytic subset of \( \mathbb{C}^n \). Let \( a \in A \) and select a \((n-k)\)-dimensional complex subspace \( L \subset \mathbb{C}^n \) such that \( a \) is an isolated point of the set \( A \cap \{a\} + L \). Then, as we know, there is a domain \( U \ni a \in \mathbb{C}^n \) such that \( A \cap U \cap \{a\} + L = \{a\} \) and such that the projection \( \pi_L : A \cap U \to U'_L \subset L^\perp \) along \( L \) is a \( d \)-sheeted analytic cover, for some \( d \in \mathbb{N} \), where \( L^\perp \) is the orthogonal of \( L \) with respect to the canonical scalar product in \( \mathbb{C}^n \).

The critical analytic set \( \Sigma \) of this cover does not partition the domain \( U'_L \) and is nowhere dense in it, therefore the number of sheets of this cover does not change when shrinking \( U \). Furthermore, if \( z' \) is the projection of \( z \) in \( L^\perp \) and \( z' \in U'_L \setminus \Sigma \), then
\[
\# \ A \cap U \cap \{z\} = d
\]
and all \( d \) points of the fiber above \( z' \) tend to \( a \) as \( z' \to a'. \) This number is called the multiplicity of the projection \( \pi_L|_A \) at \( a \), and is denoted by \( \mu_a(\pi_L|_A) \).

For any point \( x \in A \) in the above indicated small neighborhood \( U \ni a \) the number of sheets of the cover \( A \cap U \to U'_L \) does not exceed \( d \) in a neighborhood of \( x \) (it may be less), hence the function \( \mu_x(\pi_L|_A) \) is upper semicontinuous on \( A \cap U \). See [4], p. 102.

Thus, for every \((n-p)\)-dimensional complex plane \( L \subset \mathbb{C}^n \) such that \( a \) is an isolated point in \( A \cap \{a\} + L \), the multiplicity of the projection \( \mu_a(\pi_L|_A) \) is finite. The minimum of these numbers over all \( L \in \text{Gr}(n-p,n) \) as above is denoted \( \mu_a(A) \) and is called the multiplicity of \( A \) at \( a \).

Furthermore, it can be shown that the multiplicity \( \mu_a(A) \) does not depend on how \( A \) is locally embedded at \( a \) into a complex euclidean space.

Altogether we get a function \( A \ni x \mapsto \mu_x(A) \in \mathbb{N} \) that is upper semicontinuous. See [4], p. 120.

Step 2. For the sake of simplicity, let \( a = 0 \) and for the complex subspace \( L = \{0\} \times \mathbb{C}^{n-k} \) the projection \( \pi_L|_A \) realizes \( \mu_0(A) \), namely \( \mu_0(\pi_L|_A) = \mu_0(A) \).

With the necessary changes, by Step 1 we arrive at the following set-up.

The set \( A \) is (locally) analytic in \( D \times \mathbb{C}^{n-k} \) with \( D \) a domain of \( \mathbb{C}^k \), the map \( \pi : A \to D \) is induced by the first projection from \( \mathbb{C}^k \times \mathbb{C}^{n-k} \) onto \( \mathbb{C}^k \), \( A \ni (z,w) \mapsto \pi(x) = z \), such that \( \pi \) is a (finite) branched covering with image \( D \), covering number \( d := \mu_0(A) \), critical set \( \Sigma \), which is a nowhere dense analytic subset of \( D \), and \( \pi^{-1}(0) = \{0\} \).

Now, let \( h : A \to \mathbb{C} \) be any \( c \)-holomorphic function. For every point
\[ x = (z, w) \in (D \setminus \Sigma) \times \mathbb{C}^{n-k}, \] we define the polynomial
\[
\omega(x, t) = \prod_{\pi(x') = z} (t - h(x')) = t^d + a_1(x)t^{d-1} + \cdots + a_d(x).
\]

Since \( h \) is holomorphic on the regular part \( \text{Reg}(A) \) of \( A \) and \( h \) is continuous on \( A \), \textit{a fortiori} \( h \) is bounded on any compact subset of \( A \) (in particular, on \( \pi^{-1}(K) \), for every compact set \( K \) of \( D \)), the coefficients \( a_j \) are naturally holomorphic on \( (D \setminus \Sigma) \times \mathbb{C}^{n-k} \) and locally bounded on \( D \times \mathbb{C}^k \). Thus, granting Riemann’s extension theorem, they extend holomorphically to \( D \times \mathbb{C}^{n-k} \) (we keep the same notations for the extensions). If, furthermore, \( h(0) = 0 \), then all coefficients \( a_j(0) = 0 \) because \( \pi \) is proper and \( \pi^{-1}(0) = \{0\} \).

Therefore we obtain a distinguished Weierstrass polynomial of degree \( d \),
\[ W(x, t) = t^d + a_1(x)t^{d-1} + \cdots + a_d(x), \]
that is the unique extension of \( \omega \) to \( D \times \mathbb{C}^{n-k} \) and such that
\[ W(x, h(x)) = 0 \]
for all \( x \in A \).

Note that, if \( W(x, t) = 0 \), then the identity \( |t|^d = O(\|x\|) \) holds true as \( (x, t) \to 0 \) since \( |a_j(x)| = O(\|x\|) \), or equivalently
\[ |t| = O(\|x\|^{1/d}) \]
as \( (x, t) \to 0 \), meaning that there are positive constants \( M \) and \( \epsilon \) such that, if \( W(x, t) = 0 \) and \( \max\{|t|, \|x\|\} < \epsilon \), then \( |t| \leq M\|x\|^{1/d} \).

To sum up, coming back to the general setting, and using that for two real numbers \( \alpha \) and \( \beta \), one has \( s^\alpha = O(s^\beta) \) as \( (0, \infty) \ni s \to 0 \) if and only if \( \alpha \geq \beta \), by routine arguments, from Step 1 and the above discussion we get the following fact.

1. Let \( A \) be a locally analytic subset of \( \mathbb{C}^n \) of pure dimension. Then the multiplicity function \( \mu_x(A) \) on \( x \in A \) is upper semicontinuous. Furthermore, any point \( a \in A \) admits an open neighborhood \( U \) in \( A \) such that, for every point \( x_0 \in U \) and every non-constant, \( c \)-holomorphic germ \( h : (A, x_0) \to (\mathbb{C}, 0) \), one has
\[ |h(x)| = O(\|x - x_0\|^\alpha) \]
as \( A \ni x \to x_0 \),
where \( \alpha = 1/\mu_a(A) \).

In general, if \( (A, x) = \bigcup_j (A_j, x) \) is the decomposition of the germ \( (A, x) \) into its finitely many irreducible components, whose number might depend on \( x \in A \), then we set \( \mu_x(A) = \max_j \mu_x(A_j) \). The multiplicity function thus
defined is upper semicontinuous on \( A \) and the above “identity” in (\( \dagger \)) holds for the exponent \( \alpha \) given by \( 1/\alpha = \max_j \mu_a(A_j) \).

For the commodity of the reader, we mention that, for any complex space \( X \) we get a natural multiplicity function \( X \ni x \mapsto \mu_x(X) \in \mathbb{N} \) that is upper semicontinuous, although this information is not used hereafter.

**Step 3.** From Spallek [13] we recall the following notion. Let \( A \subset \mathbb{C}^n \) be a set and \( a \) a point of \( A \). We say that a germ function \( \varphi : (A,a) \to (\mathbb{C},\varphi(a)) \) is \( O_N \)-approximable at \( a \) if there exists a polynomial \( P(z,\overline{z}) \) of degree at most \( N - 1 \) in the variables \( z_j - a_j, \overline{z}_j - \overline{a}_j, j = 1, \ldots, n \), such that

\[
|\varphi(z) - P(z, \overline{z})| = O(\|z - a\|^N) \quad \text{as} \quad A \ni z \to a.
\]

**Example 1.** If \( \varphi \) is the restriction of a \( C^\infty \)-smooth, complex valued function defined on a neighborhood of \( a \) in \( \mathbb{C}^n \), then using Taylor formula, one has that \( \varphi \) is \( O_N \)-approximable at \( a \) for all positive integers \( N \).

**Example 2.** Let \( A \) be locally analytic at the point \( a \), and \( \nu, N \in \mathbb{N} \) that satisfy \( \nu > \mu_a(A)N \). Then, by (\( \dagger \)) it follows that for any germ of a \( c \)-holomorphic map \( h : (A,a) \to (\mathbb{C},0) \), \( \Re h^\nu \) and \( \Im h^\nu \) are \( O_N \)-approximable at \( a \).

The following result due to Siu [12] improves onto Spallek’s similar one from [13].

**Proposition 2** For every compact set \( K \) of a complex space \( X \) there exists a positive integer \( N = N(K) \) depending on \( K \) such that, if \( f \) is a \( c \)-holomorphic function germ at \( x \in K \) and \( \Re f \) is \( O_N \)-approximable at any point in some neighborhood of \( x \), then \( f \) is a holomorphic germ at \( x \).

**Step 4.** To conclude the theorem, since the assertion to be proven is local, without any loss in generality, we may assume that \( X \) is an analytic subset of some open set of \( \mathbb{C}^n \).

Now let \( K \) be a compact set of \( X \). We claim that there is \( \nu_K \in \mathbb{N} \) such that, for any \( c \)-holomorphic function \( f \) on \( X \) that is holomorphic off its zero set \( f^{-1}(0) \), the power \( f^\nu \) is holomorphic about \( K \) for all integers \( \nu \geq \nu_K \).

For this consider a compact neighborhood \( K^* \) of \( K \) in \( X \). Since the function \( X \ni x \mapsto \mu_x(X) \in \mathbb{N} \) is upper semicontinuous, there exists a natural number \( d \) such that \( \mu_x(X) < d \) for all \( x \in K^* \).

We show that \( \nu_K = dN \) is as desired, where \( N \) is selected according to Proposition 2 corresponding to the compact \( K \) of \( X \).
Indeed, in order to show that $f^\nu$ is holomorphic about $K$ for $\nu \in \mathbb{N}$ that satisfies $\nu \geq \nu_K$, we apply Proposition 2, and for this we need to check that the function $\text{Re} f^\nu$ is $O^N$-approximable at any point $x \in K^*$.

This follows by case analysis.

If $f(x) \neq 0$, since $f$ is holomorphic on the open set $X \setminus f^{-1}(0)$ of $X$ so that $\text{Re} f$ and $\text{Im} f$ are $C^\infty$-smooth there, by Example 1 it follows that $\text{Re} f^\nu$ is $O^N$-approximable at $x$.

If $f(x) = 0$, then by Example 2 the function $\text{Re} f^\nu$ is $O^N$-approximable at $x$ because $\nu \geq dN = \nu_K$.

This completes the proof of the theorem.

4 A final remark

Below we answer a question raised by Th. Peternell at the XXIV Conference on Complex Analysis and Geometry, held in Levico-Terme, June 10–14, 2019. He asked whether or not a similar statement like Theorem 1 does hold for non reduced complex spaces.

More specifically, let $(X, \mathcal{O}_X)$ be a not necessarily reduced complex space and $f : X \to \mathbb{C}$ be continuous such that, if $A$ denotes the zero set of $f$, then $X \setminus A$ is dense in $X$ and there is a section $\sigma \in \Gamma(X \setminus A, \mathcal{O}_X)$ whose reduction $\text{Red}(\sigma)$ equals $f|_{X \setminus A}$.

Is it true that, for every relatively compact open subset $D$ of $X$, there is a positive integer $n$ such that $\sigma^n$ extends to a section in $\Gamma(D, \mathcal{O}_X)$?

We show that the answer is "No".

In order to do this, recall that, if $R$ is a commutative ring with unit and $M$ is an $R$-module, we can endow the direct sum $R \oplus M$ with a ring structure with the obvious addition, and multiplication defined by

$$(r, m) \cdot (r', m') = (rr', rm' + r'm).$$

This is the Nagata ring structure from algebra [9].

Now, if $(X, \mathcal{O}_X)$ is a complex space, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, then $\mathcal{H} := \mathcal{O}_X \oplus \mathcal{F}$ becomes a coherent sheaf of analytic algebras and $(X, \mathcal{H})$ a complex space ([5], Satz 2.3).

The example is as follows. Let $^{\nu} \mathcal{O}$ denotes the structural sheaf of $\mathbb{C}^n$. The above discussion produces a complex space $(\mathbb{C}, \mathcal{H})$ such that $\mathcal{H} = \mathcal{O} \oplus ^{\nu} \mathcal{O}$, that can be written in a suggestive way $\mathcal{H} = \mathcal{O} + \epsilon \cdot \mathcal{O}$, where $\epsilon$ is a symbol.
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with $\epsilon^2 = 0$. As a matter of fact, if we consider $\mathbb{C}^2$ with complex coordinates $(z, w)$ and the coherent ideal $\mathcal{I}$ generated by $w^2$, then $\mathcal{H}$ is the analytic restriction of the quotient $\mathcal{O}/\mathcal{I}$ to $\mathbb{C}$.

The reduction of $(\mathbb{C}, \mathcal{H})$ is $(\mathbb{C}, 1_\mathcal{O})$. A holomorphic section of $\mathcal{H}$ over an open set $U \subset \mathbb{C}$ consists of couple of ordinary holomorphic functions on $U$.

Now take $f$ the identity function $id$ on $\mathbb{C}$, and the holomorphic section $\sigma \in \Gamma(\mathbb{C}^*, \mathcal{H})$ given by $\sigma = id + \epsilon g$, where $g$ is holomorphic on $\mathbb{C}^*$ having a singularity at 0, for instance $g(z) = 1/z$.

Obviously, the reduction of $\sigma$ is the restriction of $id$ on $\mathbb{C}^*$, and no power $\sigma^k$ of $\sigma$ extends across 0 to a section in $\Gamma(\mathbb{C}, \mathcal{H})$, since $\sigma^k = id + \epsilon kg$ and $g$ does not extend holomorphically across $0 \in \mathbb{C}$.

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References

[1] B. Aupetit: Une généralisation du théorème d’extension de Radó, Manuscripta Math. 23 (1977/78), 319–323.
[2] H. Behnke and K. Stein: Modifikation komplexer Mannigfaltigkeiten und Riemannscher Gebiete, Math. Ann. 124 (1951), 1–16.
[3] H. Cartan: Sur une extension d’un théorème de Radó, Math. Ann. 125 (1952), 49–50.
[4] E. M. Chirka: Complex analytic sets. Kluwer Academic Publishers Group, Dordrecht, 1989.
[5] O. Forster: Zur Theorie der Steinschen Algebren und Moduln, Math. Z. 97 (1967), 376–405.
[6] E. Heinz: Ein elementarer Beweis des Satzes von Radó-Behnke-Stein-Cartan über analytische Funktionen, Math. Ann. 131 (1956), 258–259.
[7] R. Kaufman: A theorem of Radó, Math. Ann. 169 (1967), 282.
[8] L. Kaup and B. Kaup: Holomorphic functions of several variables, de Gruyter Studies in Mathematics 3, Berlin, 1983.
[9] M. Nagata: Local rings. New York: Interscience 1962.
[10] T. Radó: Über eine nicht fortsetzbare Riemannsche Mannigfaltigkeit, Math. Z. 20 (1924), 1–6.
[11] J. Riihentaus: A note concerning Radó’s theorem, Math. Z. 182 (1983), 159–165.
[12] Y.T. Siu: $O^N$-approximable and holomorphic functions on complex spaces, *Duke Math. J.* **36** (1969), 451–454.

[13] K. Spallek: Differenzierbare und holomorphe Funktionen auf analytischen Mengen, *Math Ann.* **161** (1965), 143–162.

[14] E. L. Stout: A generalization of a theorem of Radó, *Math. Ann.* **177** (1968), 339–340.

[15] H. Whitney: Complex analytic varieties, Addison–Wesley, 1972.

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