Monotonicity of the number of positive entries in nonnegative matrix powers

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Abstract
Let $A$ be a nonnegative matrix of order $n$ and $f(A)$ denote the number of positive entries in $A$. We prove that if $f(A) \leq 3$ or $f(A) \geq n^2 - 2n + 2$, then the sequence \{$f(A^k)$\}$_{k=1}^{\infty}$ is monotonic for positive integers $k$.

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1 Introduction
A matrix is nonnegative (positive) if all of its entries are nonnegative (positive) real numbers. Nonnegative matrices have many attractive properties and are important in a variety of applications [1, 2]. For two nonnegative matrices $A$ and $B$ of the same size, the notation $A \geq B$ or $B \leq A$ means that $A - B$ is nonnegative.

A sign pattern is a matrix whose entries are from the set \{+, −, 0\}. In a talk at the 12th ILAS conference (Regina, Canada, June 26–29, 2005), Professor Xingzhi Zhan posed the following problem.

Problem ([4], p. 233) Characterize those sign patterns of square nonnegative matrices $A$ such that the sequence \{$f(A^k)$\}$_{k=1}^{\infty}$ is nondecreasing.

A nonnegative square matrix $A$ is said to be primitive if there exists a positive integer $k$ such that $A^k$ is positive. If we denote by $f(A)$ the number of positive entries in $A$, it seems that the sequence \{$f(A^k)$\}$_{k=1}^{\infty}$ is increasing for any primitive matrix $A$. However, Šidák [3] observed that there is a primitive matrix $A$ of order 9 satisfying $f(A) = 18 > f(A^2) = 16$. This is the motivation for us to investigate the nonnegative matrix $A$ such that \{$f(A^k)$\}$_{k=1}^{\infty}$ is monotonic. It is reasonable to expect that the sequence will be monotonic when $f(A)$ is too small or too large.

Since the value of each positive entry in $A$ does not affect $f(A^k)$ for all positive integers $k$, it suffices to consider the 0–1 matrix, i.e., the matrix whose entries are either 0 or 1. Denote by $E_{ij}$ the matrix with its entry in the $i$th row and $j$th column being 1 and with all other entries being 0. For simplicity we use 0 to denote the zero matrix whose size will be clear from the context.
2 Main results

Let $A$ be a nonnegative square matrix. We will use the fact that if $A^2 \geq A$ ($A^2 \leq A$), then $A^{k+1} \geq A^k$ ($A^{k+1} \leq A^k$) for all positive integers $k$ and thus $\{f(A^k)\}_{k=1}^{\infty}$ is increasing (decreasing).

**Theorem 1** Let $A$ be a 0–1 matrix of order $n$. If $f(A) \leq 2$, then the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is decreasing.

**Proof** The case $f(A) = 0$ is trivial.

If $f(A) = 1$, then $A = E_{ij}$, $1 \leq i, j \leq n$. Thus, for $k = 2, 3, \ldots$,

$$A^k = E_{ij}^k = \begin{cases} E_{ii}, & i = j; \\ 0, & i \neq j, \end{cases}$$

which implies that $\{f(A^k)\}_{k=1}^{\infty}$ is decreasing. Next suppose $f(A) = 2$.

Since $\{f(A^k)\}_{k=1}^{\infty}$ is invariant under permutation similarity or transpose of $A$, it suffices to consider the following cases.

1. $A = E_{11} + E_{22}$. Then $A^2 = A$.
2. $A = E_{11} + E_{12}$. Then $A^2 = A$.
3. $A = E_{11} + E_{23}$. Then $A^2 = E_{11} \leq A$.
4. $A = E_{12} + E_{13}$. Then $A^2 = 0$.
5. $A = E_{12} + E_{21}$. Then $A^k = E_{11} + E_{22}$ for all even $k$, $A^k = A$ for all odd $k$.
6. $A = E_{12} + E_{23}$. Then $A^2 = E_{13}, A^3 = 0$.
7. $A = E_{12} + E_{34}$. Then $A^2 = 0$.

It can be seen that in each case $\{f(A^k)\}_{k=1}^{\infty}$ is decreasing. This completes the proof. \[\square\]

**Theorem 2** Let $A$ be a 0–1 matrix of order $n$. If $f(A) = 3$, then the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is monotonic.

**Proof** Under permutation similarity and transpose, it suffices to consider the following cases.

1. $A = E_{11} + E_{22} + E_{33}$. Then $A^2 = A$.
2. $A = E_{11} + E_{22} + E_{12}$. Then $A^2 = A + E_{12} \geq A$.
3. $A = E_{11} + E_{22} + E_{13}$. Then $A^2 = A$.
4. $A = E_{11} + E_{22} + E_{34}$. Then $A^2 = E_{11} + E_{22} \leq A$.
5. $A = E_{11} + E_{12} + E_{13}$. Then $A^2 = A$.
6. $A = E_{11} + E_{12} + E_{21}$. Then $A^2 = A + E_{11} + E_{22} \geq A$.
7. $A = E_{11} + E_{12} + E_{31}$. Then $A^2 = A + E_{32} \geq A$.
8. $A = E_{11} + E_{12} + E_{23}$. Then $A^k = E_{11} + E_{12} + E_{13}$ for all $k \geq 2$.
9. $A = E_{11} + E_{12} + E_{32}$. Then $A^2 = E_{11} + E_{12} \leq A$.
10. $A = E_{11} + E_{12} + E_{34}$. Then $A^2 = E_{11} + E_{12} \leq A$.
11. $A = E_{11} + E_{23} + E_{24}$. Then $A^2 = E_{11} \leq A$.
12. $A = E_{11} + E_{23} + E_{32}$. Then $A^k = E_{11} + E_{22} + E_{33}$ for all even $k$, $A^k = A$ for all odd $k$.
13. $A = E_{11} + E_{23} + E_{34}$. Then $A^2 = E_{11} + E_{24}, A^k = E_{11}$ for all $k \geq 3$.
14. $A = E_{11} + E_{23} + E_{45}$. Then $A^2 = E_{11} \leq A$.
15. $A = E_{12} + E_{13} + E_{14}$. Then $A^2 = 0$. 

\( A = E_{12} + E_{13} + E_{21}. \) Then \( A^k = E_{11} + E_{22} + E_{23} \) for all even \( k \), \( A^k = A \) for all odd \( k \).

(17) \( A = E_{12} + E_{13} + E_{41}. \) Then \( A^2 = E_{42} + E_{43} \), \( A^3 = 0 \).

(18) \( A = E_{12} + E_{13} + E_{23}. \) Then \( A^2 = E_{13} \leq A \).

(19) \( A = E_{12} + E_{13} + E_{24}. \) Then \( A^2 = E_{14} \), \( A^3 = 0 \).

(20) \( A = E_{12} + E_{13} + E_{42}. \) Then \( A^2 = 0 \).

(21) \( A = E_{12} + E_{13} + E_{45}. \) Then \( A^2 = 0 \).

(22) \( A = E_{12} + E_{21} + E_{34}. \) Then \( A^k = E_{11} + E_{22} \) for all even \( k \), \( A^k = E_{12} + E_{21} \) for all odd \( k \geq 3 \).

(23) \( A = E_{12} + E_{23} + E_{31}. \) Then

\[
A^k = \begin{cases} 
E_{11} + E_{22} + E_{33}, & k \equiv 0 \text{ (mod3)}; \\
A, & k \equiv 1 \text{ (mod3)}; \\
E_{13} + E_{21} + E_{32}, & k \equiv 2 \text{ (mod3)}. 
\end{cases}
\]

(24) \( A = E_{12} + E_{23} + E_{34}. \) Then \( A^2 = E_{13} + E_{24} \), \( A^3 = E_{14}, A^4 = 0 \).

(25) \( A = E_{12} + E_{23} + E_{45}. \) Then \( A^2 = E_{13}, A^3 = 0 \).

(26) \( A = E_{12} + E_{34} + E_{56}. \) Then \( A^2 = 0 \).

Since in each case \( \{f(A^k)\}_{k=1}^\infty \) is either increasing or decreasing, this completes the proof. \( \square \)

**Corollary 3** Let \( A \) be a 0–1 matrix of order 2. Then the sequence \( \{f(A^k)\}_{k=1}^\infty \) is monotonic.

**Remark** When \( A \) is of order \( n \geq 3 \) with \( f(A) = 4 \), the following example shows that \( \{f(A^k)\}_{k=1}^\infty \) may not be monotonic. Consider

\[
A = E_{12} + E_{13} + E_{21} + E_{31}.
\]

Direct computation shows that

\[
A^2 = 2E_{11} + E_{22} + E_{23} + E_{32} + E_{33}, \quad A^3 = 2A.
\]

Thus \( f(A) = 4 < f(A^2) = 5 > f(A^3) = 4 \).

On the one hand, Theorems 1 and 2 show that \( \{f(A^k)\}_{k=1}^\infty \) is monotonic when \( f(A) \leq 3 \). On the other hand, \( \{f(A^k)\}_{k=1}^\infty \) is expected to be also monotonic when \( f(A) \) is large enough. Next we discuss the number of positive entries that \( A \) has to guarantee the sequence increasing.

The *permanent* of a matrix \( A = (a_{ij})_{n \times n} \) is defined as

\[
\text{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{\sigma(i),i},
\]

where \( S_n \) is the set of permutations of the integers 1, 2, \ldots, \( n \). First we have the following important fact.

**Lemma 4** Let \( A \) be a 0–1 matrix of order \( n \). If \( \text{per} A > 0 \), then the sequence \( \{f(A^k)\}_{k=1}^\infty \) is increasing.
Since $A$ is a 0–1 matrix with per $A > 0$, there exists a permutation matrix $P$ such that $A \geq P$. Now let $A = P + B$, where $B$ is also a 0–1 matrix. Then $A^{k+1} = A \cdot A^k = (P + B)A^k = P \cdot A^k + B \cdot A^k \geq P \cdot A^k$ for all positive integers $k$. Thus $f(A^{k+1}) \geq f(P \cdot A^k) = f(A^k)$, which implies that $\{f(A^k)\}_{k=1}^{\infty}$ is increasing.

\[ \square \]

**Theorem 5** Let $A$ be a 0–1 matrix of order $n$. If $f(A) \geq n^2 - 2n + 2$, then the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is increasing.

**Proof** First if per $A > 0$, by Lemma 4, $\{f(A^k)\}_{k=1}^{\infty}$ is increasing.

Next suppose per $A = 0$. Then by the Frobenius–König theorem [4, p. 46], $A$ has an $r \times s$ zero submatrix with $r + s = n + 1$. Since $f(A) \geq n^2 - 2n + 2$, $A$ has at most $2n - 2$ zero entries. Thus $rs \leq 2n - 2$. It can be seen that $r$ and $s$ must be one of the following solutions.

1. $r = 1, s = n$;
2. $r = n, s = 1$;
3. $r = 2, s = n - 1$;
4. $r = n - 1, s = 2$.

If $r = 1, s = n$ or $r = n, s = 1$, i.e., $A$ has a zero row or a zero column, then $A$ is permutation similar to a matrix of the form

\[
\begin{bmatrix}
B & C \\
0 & 0
\end{bmatrix}
\]

or its transpose, where $B$ is of order $n - 1$ and $C$ is a column vector. Since $A$ has at most $2n - 2$ zero entries, $B$ has at most $n - 2$ zero entries. Then there exists a permutation matrix $Q$ of order $n - 1$ such that $B \geq Q$. Note that

\[
\begin{bmatrix}
B & C \\
0 & 0
\end{bmatrix}^{k+1} = \begin{bmatrix}
B & C \\
0 & 0
\end{bmatrix}^k \begin{bmatrix}
B & C \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
B^k & B^{k-1}C \\
0 & 0
\end{bmatrix} \begin{bmatrix}
B & C \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
QB^k & QB^{k-1}C \\
0 & 0
\end{bmatrix}.
\]

Thus

\[
f(A^{k+1}) = f\left(\begin{bmatrix}
B & C \\
0 & 0
\end{bmatrix}^{k+1}\right) \geq f\left(\begin{bmatrix}
QB^k & QB^{k-1}C \\
0 & 0
\end{bmatrix}\right)
\]

\[
= f(QB^k) + f(QB^{k-1}C) = f(B^k) + f(B^{k-1}C)
\]

\[
= f\left(\begin{bmatrix}
B^k & B^{k-1}C \\
0 & 0
\end{bmatrix}\right) = f\left(\begin{bmatrix}
B & C \\
0 & 0
\end{bmatrix}^k\right) = f(A^k)
\]

for all positive integers $k$, which implies that $\{f(A^k)\}_{k=1}^{\infty}$ is increasing.
If \( r = 2, s = n - 1 \) or \( r = n - 1, s = 2 \), then \( A \) is permutation similar to one of the matrices \( A_1, A_2, A_1^T, A_2^T \), where

\[
A_1 = \begin{bmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1 \\
1 & \cdots & 1 & 1 \\
\vdots & \ddots & \vdots & \vdots \\
1 & \cdots & 1 & 1
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 1 \\
\vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1
\end{bmatrix}.
\]

Direct computation shows that \( A_1^2 \geq A_1, A_2^2 \geq A_2 \). Thus \( \{f(A^k)\}_{k=1}^\infty \) is increasing. This completes the proof. \( \square \)

Remark When \( f(A) = n^2 - 2n + 1 \), the following example shows that \( \{f(A^k)\}_{k=1}^\infty \) may not be increasing. Consider

\[
A = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{bmatrix}.
\]

Direct computation shows that \( f(A) = n^2 - 2n + 1 > f(A^2) = n^2 - 2n \).

3 Conclusion

This paper considers the number of positive entries \( f(A) \) in a nonnegative matrix \( A \) and deals with the question of whether the sequence \( \{f(A^k)\}_{k=1}^\infty \) is monotonic. We prove that if \( f(A) \leq 3 \) or \( f(A) \geq n^2 - 2n + 2 \), then the sequence must be monotonic. Some examples show that if \( 4 \leq f(A) \leq n^2 - 2n + 1 \) when \( n \geq 3 \), then the sequence may not be monotonic.

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