CHANGE ACTIONS: MODELS OF GENERALISED DIFFERENTIATION

MARIO ALVAREZ-PICALLO AND C.-H. LUKE ONG

University of Oxford

e-mail address: mario.alvarez-picallo@cs.ox.ac.uk

University of Oxford

e-mail address: luke.ong@cs.ox.ac.uk

ABSTRACT. Change structures, introduced by Cai et al., have recently been proposed as a semantic framework for incremental computation. We generalise change actions, an alternative to change structures, to arbitrary cartesian categories and propose the notion of change action model as a categorical model for (higher-order) generalised differentiation. Change action models naturally arise from many geometric and computational settings, such as (generalised) cartesian differential categories, group models of discrete calculus, and Kleene algebra of regular expressions. We show how to build canonical change action models on arbitrary cartesian categories, reminiscent of the Faa di Bruno construction.

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1. Introduction

Incremental computation is the process of incrementally updating the output of some given function as the input is gradually changed, without recomputing the entire function from scratch. Recently, Cai et al. [Cai et al., 2014] introduced the notion of change structure to give a semantic account of incremental computation. Change structures have subsequently been generalised to change actions [Alvarez-Picallo et al., 2019], and proposed as a model for automatic differentiation [Kelly et al., 2016]. These developments raise a number of questions about the structure of change actions themselves and how they relate to more traditional notions of differentiation.

A change action $A = (|A|, \Delta A, \oplus_A, +, 0_A)$ is a set $|A|$ equipped with a monoid $(\Delta A, +, 0_A)$ acting on it, via action $\oplus_A : |A| \times \Delta A \rightarrow |A|$. For example, every monoid $(S, +, 0)$ gives rise to a (so-called monoidal) change action $(S, S, +, +, 0)$. Given change actions $A$ and $B$, consider functions $f : |A| \rightarrow |B|$. A derivative of $f$ is a function $\partial f : |A| \times \Delta A \rightarrow \Delta B$ such that for all $a \in |A|, \delta a \in \Delta A$, $f(a \oplus_A \delta a) = f(a) \oplus_B \partial f(a, \delta a)$. Change actions and differentiable functions (i.e. functions that have a regular derivative) organise themselves into categories (and indeed 2-categories) with finite (co)products, whereby morphisms are composed via the chain rule.

The definition of change actions (and derivatives of functions) makes no use of properties of $\textbf{Set}$ beyond the existence of products. We develop the theory of change actions on arbitrary cartesian categories and study their properties. A first contribution is the notion of a change action model, which is defined to be a coalgebra for a certain (copointed) endofunctor $\text{CAct}$ on the category $\textbf{Cat}_A$ of (small) cartesian categories. The functor $\text{CAct}$ sends a category $\mathbf{C}$ to the category $\text{CAct}(\mathbf{C})$ of (internal) change actions and differential maps on $\mathbf{C}$.
There is a natural, extrinsic, notion of higher-order derivative in change action models. In such a model $\alpha : C \rightarrow \text{CAct}(C)$, a $C$-object $A$ is associated (via $\alpha$) with a change action, the carrier object of whose monoid is in turn associated with a change action, and so on ad infinitum. We construct a “canonical” change action model, $\text{CAct}_\omega(C)$, that internalises such $\omega$-sequences that exhibit higher-order differentiation. Objects of $\text{CAct}_\omega(C)$ are $\omega$-sequences of “contiguously compatible” change actions; and morphisms are corresponding $\omega$-sequences of differential maps, each map being the canonical (via $\alpha$) derivative of the preceding in the $\omega$-sequence. We show that $\text{CAct}_\omega(C)$ is the final $\text{CAct}$-coalgebra (relativised to change action models on $C$). The category $\text{CAct}_\omega(C)$ may be viewed as a kind of Faà di Bruno construction [Cruttwell, 2017; Cockett and Seely, 2011] in the more general setting of change action models.

Change action models capture many versions of differentiation that arise in mathematics and computer science. We illustrate their generality via three examples. The first, (generalised) cartesian differential categories (GCDC) [Blute et al., 2009; Cruttwell, 2017], are themselves an axiomatisation of the essential properties of the derivative. We show that a GCDC $C$—which by definition associates every object $A$ with a monoid $L(A) = (L_0(A), +_A, 0_A)$—gives rise to change action models in various non-trivial ways. First there is a canonical change action model mapping each object $A$ to the trivial action of $L(A)$ on itself. A second arises from interpreting the identity $\text{Id} : A \times L_0(A) \rightarrow A \times L_0(A)$ as defining an action of $L(A)$ on $A$ in the Kleisli category of the tangent bundle monad.

Secondly we show how discrete differentiation in both the calculus of finite differences [Jordan, 1965] and Boolean differential calculus [Steinbach and Posthoff, 2017; Thayse, 1981] can be modelled using the full subcategory $\text{Grp}_{\text{Set}}$ of $\text{Set}$ whose objects are groups. Our unifying formulation generalises these discrete calculi to arbitrary groups, and gives an account of the chain rule in these settings.

Our third example is differentiation of regular expressions. Recall that Kleene algebra $\mathbb{K}$ is the algebra of regular expressions. Thanks to Taylor’s Theorem [Hopkins and Kozen, 1999], every polynomial over a commutative $\mathbb{K}$, viewed as an endofunction on $\mathbb{K}$ qua (monoidal) change action, has a regular derivative. We show that the algebra of polynomials over a commutative Kleene algebra is a change action model. Interestingly the derivatives are not additive in the second (i.e. vectorial) argument, thus violating GCDC axiom [CD.2].

Outline. In Section 2 we present the basic definitions of change actions and differential maps, and show how they can be organised into categories. The theory of change action is extended to arbitrary cartesian categories $C$ in Section 3: we introduce the category $\text{CAct}(C)$ of internal change actions on $C$. In Section 4 we present change action models, and properties of the tangent bundle functors. In Section 5 we illustrate the unifying power of change action models via three examples. In Section 6, we study the category $\text{CAct}_\omega(C)$ of $\omega$-change actions and $\omega$-differential maps. Missing proofs are provided in the Appendix.

2. Change actions

A change action is a tuple $A = (|A|, \Delta A, \oplus_A, +_A, 0_A)$ where $|A|$ and $\Delta A$ are sets, $(\Delta A, +_A, 0_A)$ is a monoid, and $\oplus_A : |A| \times \Delta A \rightarrow |A|$ is an action of the monoid on $|A|$. We omit the subscript from $\oplus_A, +_A$ and $0_A$ whenever we can.
Remark 2.1. Change actions are closely related to the notion of change structures introduced in [Cai et al., 2014] but differ from the latter in not being dependently typed or assuming the existence of an \( \oplus \) operator. On the other hand, change actions require the change set \( \Delta A \) to have a monoid structure compatible with the map \( \oplus \), hence neither notion is strictly a generalisation of the other. Whenever one has a change structure, however, one can easily obtain a change action by considering the free monoid generated by its change set.

**Definition 2.2** (Derivative condition). Let \( A \) and \( B \) be change actions. A function \( f : |A| \to |B| \) is differentiable if there is a function \( \partial f : |A| \times \Delta A \to \Delta B \) satisfying \( f(a \oplus_A \delta a) = f(a) \oplus_B \partial f(a, \delta a) \), for all \( a \in |A|, \delta a \in \Delta A \). We call \( \partial f \) a derivative for \( f \), and write \( f : A \to B \) whenever \( f \) is differentiable.

**Lemma 2.3** (Chain rule). Given \( f : A \to B \) and \( g : B \to C \) with derivatives \( \partial f \) and \( \partial g \) respectively, the function \( \partial(g \circ f) : |A| \times \Delta A \to \Delta C \) defined by \( \partial(g \circ f)(a, \delta a) := \partial g(\partial f(a), \delta a) \) is a derivative for \( g \circ f : |A| \to |C| \).

**Proof.** Unpacking the definition, we have \( (g \circ f)(a) \oplus_C \partial(g \circ f)(a, \delta a) = g(f(a)) \oplus_C \partial g(f(a), \partial f(a, \delta a)) = g(f(a) \oplus_B \partial f(a, \delta a)) = g(f(a) \oplus_A \delta a)), \) as desired. \( \square \)

**Example 2.4** (Some useful change actions). (1) If \( (A, +, 0) \) is a monoid, \( (A, A, +, +, 0) \) is a change action (called *monoidal*).

(2) For any set \( A, A_* := (A, \{\bullet\}, \pi_1, \pi_1, \bullet) \) is a (trivial) change action.

(3) Let \( A \Rightarrow B \) be the set of functions from \( A \) to \( B \), and \( ev_{A,B} : A \times (A \Rightarrow B) \to B \) be the usual evaluation map. Then \( (A, A \Rightarrow A, ev_{A,A}, \circ, Id_A) \) is a change action. If \( U \subseteq (A \Rightarrow A) \) contains the identity map and is closed under composition, \( (A, U, ev_{A,A} |_{A \times U}, \circ |_{U \times U}, Id_U) \) is a change action.

### 2.1. Regular derivatives.

The preceding definitions neither assume nor guarantee a derivative to be additive (i.e. they may not satisfy \( \partial f(x, \Delta a + \Delta b) = \partial f(x, \Delta a) + \partial f(x, \Delta b) \)), as they are in standard differential calculus. A strictly weaker condition that we will now require is regularity: if a derivative is additive in its second argument then it is regular, but not vice versa. Under some condition, the converse is also true.

**Definition 2.5.** Given a differentiable map \( f : A \to B \), a derivative \( \partial f \) for \( f \) is regular if, for all \( a \in |A| \) and \( \delta a, \delta b \in \Delta A \), we have \( f(a, 0_A) = 0_B \) and \( \partial f(a, \delta a +_A \delta b) = \partial f(a, \delta a) +_B \partial f(a \oplus_A \delta a, \delta b) \).

**Proposition 2.6.** Whenever \( f : A \to B \) is differentiable and has a unique derivative \( \partial f \), this derivative is regular.

**Proof.** See Appendix A \( \square \)

**Proposition 2.7.** Given \( f : A \to B \) and \( g : B \to C \) with regular derivatives \( \partial f \) and \( \partial g \) respectively, the derivative \( \partial(g \circ f) = \partial g \circ (f \circ \pi_1, \partial f) \) is regular.

**Proof.** See Appendix A \( \square \)
2.2. Two categories of change actions. The study of change actions can be undertaken in two ways: one can consider functions that are differentiable (without choosing a derivative); alternatively, the derivative itself can be considered part of the morphism. The former leads to the category $\mathbf{CAct}^-$, whose objects are change actions and morphisms are the differentiable maps.

The category $\mathbf{CAct}^-$ was the category we originally proposed [Alvarez-Picallo et al., 2019]. It is well-behaved, possessing limits, colimits, and exponentials, which is a trivial corollary of the following result:

**Theorem 2.8.** The category $\mathbf{CAct}^-$ of change actions and differentiable morphisms is equivalent to $\mathbf{PreOrd}$, the category of preorders and monotone maps.

**Proof.** See Appendix A

The actual structure of the limits and colimits in $\mathbf{CAct}^-$ is, however, not so satisfactory. One can, for example, obtain the product of two change actions $A$ and $B$ by taking their product in $\mathbf{PreOrd}$ and turning it into a change action, but the corresponding monoid action map $\oplus$ is not, in general, easily expressible, even if those for $A$ and $B$ are. Derivatives of morphisms in $\mathbf{CAct}^-$ can also be hard to obtain, as exhibiting $f$ as a morphism in $\mathbf{CAct}^-$ merely proves it is differentiable but gives no clue as to how a derivative might be constructed.

A more constructive approach is to consider morphism as a function together with a choice of a derivative for it.

**Definition 2.9.** Given change actions $A$ and $B$, a differential map $f : A \to B$ is a pair $([f], \partial f)$ where $[f] : |A| \to |B|$ is a function, and $\partial f : |A| \times \Delta A \to \Delta B$ is a regular derivative for $[f]$.

The category $\mathbf{CAct}$ has change actions as objects and differential maps as morphisms. The identity morphisms are $(\text{Id}_A, \pi_1)$; given morphisms $f : A \to B$ and $g : B \to C$, define the composite $g \circ f := ([g] \circ [f], \partial g \circ ([f] \circ \pi_1, \partial f)) : A \to C$.

Finite products and coproducts exist in $\mathbf{CAct}$ (see Theorems 3.1 and 3.4 for a more general statement). Whether limits and colimits exist in $\mathbf{CAct}$ beyond products and coproducts is open.

**Remark 2.10.** If one thinks of changes (i.e. elements of $\Delta A$) as morphisms between elements of $|A|$, then regularity resembles functoriality. This intuition is explored in Appendix F, where we show that categories of change actions organise themselves into 2-categories.

2.3. Adjunctions with Set. There is an obvious forgetful functor $\mathcal{F} : \mathbf{CAct} \to \mathbf{Set}$ that maps every change action $A$ to its underlying set $|A|$ and every differential map $f : A \to B$ to the function on the underlying sets $[f] : |A| \to |B|$.

Given a change action $A$ on a set $|A|$, the structure of the change action defines a preorder $\leq$ on $|A|$ where $a \leq b$ whenever there exists some $\delta a$ such that $a \oplus \delta a = b$ (indeed, one can think of change actions as particular presentations of preorders). Then we can define a quotient functor $\mathcal{Q} : \mathbf{CAct} \to \mathbf{Set}$ that maps the change action $A$ to the set $|A|/\sim$, where $\sim$ is the transitive and symmetric closure of $\leq$; and the action on morphisms $f : A \to B$ is defined as $\mathcal{Q}(f)([a]) = [f(a)]$, where $[a]$ is the $\sim$-equivalence class of $a$. 
Finally, there is a functor $D : \text{Set} \to \text{CAct}$ that maps every set $A$ to the discrete change action $(A, 1, \text{Id}, !, !)$ (where 1 denotes the terminal object in $\text{Set}$ and $!$ the universal morphism). This functor $D$ sends every function $f$ to the differential map $(f, !)$.

In what follows we will make use of the fact that $\mathcal{F} \circ D = \mathcal{Q} \circ D = \text{Id}_{\text{Set}}$.

**Lemma 2.11.** The forgetful functor $\mathcal{F}$ is right-adjoint to the functor $D$, with the unit and counit given by:

$$
\varepsilon : D \circ \mathcal{F} \to \text{Id}_{\text{CAct}} \\
\varepsilon_A = (\text{Id}_A, 0) \\
\eta : \text{Id}_{\text{Set}} \to \mathcal{F} \circ D = \text{Id}_{\text{Set}} \\
\eta_A = \text{Id}_A
$$

**Proof.** See Appendix A

**Lemma 2.12.** The functor $D$ is right adjoint to the quotient functor $\mathcal{Q}$, with unit and counit given by:

$$
\varepsilon : \text{Id}_{\text{Set}} \cong \mathcal{Q} \circ D \to \text{Id}_{\text{Set}} \\
\varepsilon_A = \text{Id}_A \\
\eta : \text{Id}_{\text{CAct}} \to D \circ \mathcal{Q} \\
\eta_A = ([\text{Id}_A], !)
$$

where $[\text{Id}_A]$ is the map that sends an element $a$ in $A$ to the equivalence class $[a]$ of $a$ modulo $\sim_\mathcal{Q}$. Note that $\eta$ is well-defined since whenever $a \oplus \delta a = b$ it is the case that $[a] = [b]$.

In what follows we assume the Axiom of Choice. This is equivalent to the assumption that every set is the underlying set of some group. We will suppose a map $\mathcal{G}$ that sends each set $A$ to a group $\mathcal{G}_A$ whose underlying set is $A$.

**Definition 2.13.** The functor $\mathcal{G} : \text{Set} \to \text{CAct}$ maps every object to the monoidal change action $(\mathcal{G}_A, \mathcal{G}_A, +, +, 0)$, and every function $f : A \to B$ to the differential map $(f, \partial^1 f)$, with $\partial^1 f(x, \delta x) = -f(x) + f(x + \delta x)$. A straightforward consequence of this definition is that $\mathcal{G}$ is full and faithful.

**Lemma 2.14.** The functor $\mathcal{G}$ is a right adjoint to the forgetful functor $\mathcal{F}$.

**Proof.** If one uses the hom-set isomorphism definition of adjunction, it follows trivially from the fact that every function into a change action of the form $\mathcal{G}(A)$ has one and only one derivative.

**Remark 2.15.** In a nutshell, this means there is a sequence of four adjunctions

$$
\mathcal{Q} \vdash \mathcal{D} \vdash \mathcal{F} \vdash \mathcal{G}
$$

where $\mathcal{Q}$ preserves finite products and $\mathcal{D}, \mathcal{G}$ are full and faithful. This is precisely the setting of Lawvere’s notion of differential cohesion [nLab authors, 2018] (with the exception that $\text{CAct}$ is not a topos), which has been proposed to unify many settings for higher differential geometry.

It is a topic of ongoing research to put this fact in the context of the recent advances in differential cohesive type theory [Gross et al., 2018; Shulman, 2018] (the internal language...
of a topos with differential cohesion), which have recently been used to give a constructive formalization of Brouwer’s fixed point theorem [Shulman, 2018].

3. Change actions on arbitrary categories

The definition of change actions makes no use of any properties of Set beyond the existence of products. Indeed, change actions can be characterised as just a kind of multi-sorted algebra, which is definable in any category with products.

3.1. The category \( \text{CAct}(\mathbf{C}) \). Consider the category \( \text{Cat}_x \) of (small) cartesian categories (i.e. categories with chosen finite products) and product-preserving functors. We can define an endofunctor \( \text{CAct} : \text{Cat}_x \to \text{Cat}_x \) sending a category \( \mathbf{C} \) to the category of (internal) change actions on \( \mathbf{C} \).

The objects of \( \text{CAct}(\mathbf{C}) \) are tuples \( A = (|A|, \Delta A, \oplus_A, +_A, 0_A) \) where \( |A| \) and \( \Delta A \) are (arbitrary) objects in \( \mathbf{C} \), \( (\Delta A, +_A, 0_A) \) is a monoid object in \( \mathbf{C} \), and \( \oplus_A : |A| \times \Delta A \to |A| \) is a \( \mathbf{C} \)-morphism such that the following diagrams—specifying monoid action—commute (omitting the obvious structural morphisms):

\[
\begin{array}{ccc}
|A| & \xrightarrow{\text{Id}} & |A| \\
|A| \times \Delta A & \xrightarrow{\oplus_A} & |A| \\
\end{array}
\]

Given objects \( A, B \) in \( \text{CAct}(\mathbf{C}) \), the morphisms of \( \text{CAct}(A, B) \) are pairs \( f = (|f|, \partial f) \) where \( |f| : |A| \to |B| \) and \( \partial f : |A| \times \Delta A \to \Delta B \) are morphisms in \( \mathbf{C} \), such that the following diagrams commute:

\[
\begin{array}{ccc}
|A| \times \Delta A & \xrightarrow{(|f|_{\pi_1}, \partial f)} & |B| \times \Delta B \\
|A| & \xrightarrow{\partial f} & 1 \\
\end{array}
\]

The first diagram states the derivative condition: \( |f|(x \oplus_A \delta x) = |f|(x) \oplus_B \partial f(x, \delta x) \). The other two assert a diagrammatic version of the regularity of \( \partial f \).

The chain rule can then be expressed naturally by pasting two instances of the previous diagram:
The change action

**Theorem 3.1.** The following change action is the product of \( \top \) where

\[
\begin{align*}
\text{The projections are} & : \oplus & \text{where} \\
\text{See Appendix B} & \\
\text{Proof.} & \\
\end{align*}
\]

Hence \( f \circ g = \langle \langle g \circ f \rangle \circ \pi_1, \partial g \circ \langle f \circ \pi_1, \partial f \rangle \rangle \).

Now, given a product-preserving functor \( F : C \to D \), there is a corresponding functor \( \text{CA} \text{C}(F) : \text{CA} \text{C}(C) \to \text{CA} \text{C}(D) \) by the following:

\[
\begin{align*}
\text{CA} \text{C}(F)(|A|, \Delta A, \oplus_A, +_A, 0_A) & := (F(|A|), F(\Delta A), F(\oplus_A), F(+_A), F(0_A)) \\
\text{CA} \text{C}(F)(|\|f|\|, \partial f) & := (F(|f|), F(\partial f))
\end{align*}
\]

We can embed \( C \) fully and faithfully into \( \text{CA} \text{C}(C) \) via the functor \( \eta_C \) which sends an object \( A \) of \( C \) to the “trivial” change action \( \Delta_A = (A, \top, \pi_1, !) \) and every morphism \( f : A \to B \) of \( C \) to the morphism \( (f, !) \). As before, this functor extends to a natural transformation from the identity functor to \( \text{CA} \text{C} \).

Additionally, there is an obvious forgetful functor \( \varepsilon_C : \text{CA} \text{C}(C) \to C \), which defines the components of a natural transformation \( \varepsilon \) from the functor \( \text{CA} \text{C} \) to the identity endofunctor \( \text{Id} \).

Given \( C \), we write \( \xi_C \) for the functor \( \text{CA} \text{C}(\varepsilon_C) : \text{CA} \text{C}(\text{CA} \text{C}(C)) \to \text{CA} \text{C}(C) \). ¹

Explicitly, this functor maps an object \((A, B, \oplus, +, 0)\) in \( \text{CA} \text{C}(\text{CA} \text{C}(C)) \) to the object \((|A|, |B|, \|\|, |+|, |0|)\). Intuitively, \( \varepsilon_{\text{CA} \text{C}(C)} \) prefers the “original” structure on objects, whereas \( \xi_C \) prefers the “higher” structure. The equaliser of these two functors is precisely the category of change actions whose higher structure is the original structure.

### 3.2. Products and coproducts in \( \text{CA} \text{C}(C) \)

We have defined \( \text{CA} \text{C} \) as an endofunctor on cartesian categories. This is well-defined: if \( C \) has all finite (co)products, so does \( \text{CA} \text{C}(C) \).

Let \( A = (|A|, \Delta A, \oplus_A, +_A, 0_A) \) and \( B = (|B|, \Delta B, \oplus_B, +_B, 0_B) \) be change actions on \( C \). We present their product and coproducts as follows.

**Theorem 3.1.** The following change action is the product of \( A \) and \( B \) in \( \text{CA} \text{C}(C) \)

\[
A \times B := (|A| \times |B|, \Delta A \times \Delta B, \oplus_{A \times B}, +_{A \times B}, 0_{A \times B})
\]

where \( \oplus_{A \times B} := \oplus_A \circ (\pi_1 \times \pi_1), \oplus_B \circ (\pi_2 \times \pi_2) \) and \( +_{A \times B} := (+_A \circ (\pi_1 \times \pi_1), +_B \circ (\pi_2 \times \pi_2)) \).

The projections are \( \pi_1 = (\pi_1, \pi_1 \circ \pi_2) \) and \( \pi_2 = (\pi_2, \pi_2 \circ \pi_2) \), writing \( \overrightarrow{f} \) for maps \( f \) in \( \text{CA} \text{C} \) to distinguish them from \( C \)-maps.

**Proof.** See Appendix B \( \Box \)

**Theorem 3.2.** The change action \( \top = (\top, \top, \pi_1, \pi_1, \text{Id}_\top) \) is the terminal object in \( \text{CA} \text{C}(C) \), where \( \top \) is the terminal object of \( C \). Furthermore, if \( A \) is a change action every point \( |f| : \top \to |A| \) in \( C \) is differentiable, with (unique) derivative \( 0_A \).

¹One might expect \( \text{CA} \text{C} \) to be a monad with \( \varepsilon \) as a counit. But if this were the case, we would have \( \xi_C = \varepsilon_{\text{CA} \text{C}(C)} \), which is, in general, not true.
Proof. See Appendix B

Whenever we have a differential map \( f : A \times B \to C \) between change actions, we can compute its derivative \( \partial f \) by adding together its “partial” derivatives:

**Lemma 3.3.** Let \( f : A \times B \to C \) be a differential map. Then

\[
\partial f((a, b), (\delta a, \delta b)) = +C \circ \partial f((a, b), (\delta a, 0_B)), \partial f((\oplus A \circ \langle a, \delta a \rangle, b), (0_A, \delta b))
\]

(The notational abuse is justified by the internal logic of a cartesian category.)

Proof. See Appendix B

**Theorem 3.4.** If \( C \) is distributive, with law \( \delta_{A,B,C} : (A \sqcup B) \times C \to (A \times C) \sqcup (B \times C) \), the following change action is the coproduct of \( A \) and \( B \) in \( \text{CAct}(C) \)

\[
A \sqcup B := (|A| \sqcup |B|, \Delta A \times \Delta B, \oplus_{A \sqcup B}, +_{A \sqcup B}, \langle 0_A, 0_B \rangle)
\]

where \( \oplus_{A \sqcup B} := [\oplus_A \circ \langle \text{Id}_A \times \pi_1 \rangle, \oplus_B \circ \langle \text{Id}_B \times \pi_2 \rangle] \circ \delta_{A,B,C}, \text{ and } +_{A \sqcup B} := \langle +_A \circ (\pi_1 \times \pi_1), +_B \circ (\pi_2 \times \pi_2) \rangle \). The injections are \( \iota_1 = (\langle 1_1, \langle 0_1, 0_2 \rangle \rangle) \) and \( \iota_2 = (\langle 1_2, \langle 0, \pi_2 \rangle \rangle) \).

Proof. See Appendix B

### 3.3. Change actions as Lawvere theories.

According to their definition, change actions seem nothing more than multi-sorted algebras. This is perhaps misleading in that it suggests that differential maps should correspond to algebra homomorphisms, which is in fact false: a homomorphism of change actions \( A, B \) would be a pair \( (u, v) \) where \( |u| : |A| \to |B| \) is a function and \( v : \Delta A \to \Delta B \) is a monoid homomorphism such that \( u(a \oplus \delta a) = u(a) \oplus v(\delta a) \). That is to say, a homomorphism of change actions as algebras is precisely a homomorphism of monoid actions.

There is a sense, however, in which differential maps are exactly algebra homomorphisms. To make this precise, we require a few new definitions.

**Definition 3.5.** The \( \Delta \)-theory \( \Sigma_\Delta \) is the free Cartesian category generated by objects \( X, \Delta X \) and morphisms

\[
\oplus : X \times \Delta X \to X
\]

\[
+ : \Delta X \times \Delta X \to \Delta X
\]

\[
0 : 1 \to \Delta X
\]

subject to the equations:

\[
\oplus \circ \langle \text{Id}, 0! \rangle = \text{Id}
\]

\[
\oplus \circ (\text{Id} \times +) = \oplus \circ (\oplus \times \text{Id}) \circ \alpha^{-1}
\]

A \( \Delta \)-algebra on a Cartesian category \( C \) is a product-preserving functor from \( \Sigma_\Delta \) into \( C \).

**Remark 3.6.** Every \( \Delta \)-algebra \( F : \Sigma_\Delta \to C \) corresponds to a change action on \( C \) given by:

\[
F = (F(X), F(\Delta X), F(\oplus), F(+), F(0))
\]

Conversely, every change action on \( C \) induces a \( \Delta \)-algebra. However, \( \Delta \)-algebra homomorphisms do not correspond to differentiable morphisms, hence the category of \( \Delta \)-algebras and \( \Delta \)-algebra homomorphisms is not equivalent to the category \( \text{CAct} \) of change actions.

\(^2\)Alternatively, one can define the (first) partial derivative of \( f(x, y) \) as a map \( \delta_1 f \) such that \( f(x \oplus \delta x, y) = f(x, y) \oplus \delta_1(x, y, \delta x) \). It can be shown that a map is differentiable iff its first and second derivatives exist.
**Definition 3.7.** The $T$-theory $\mathcal{T}_T$ is the free Cartesian category generated by objects $X, TX$ and morphisms
\[
\begin{align*}
\oplus &: TX \to X \\
\Pi &: TX \to X
\end{align*}
\]
A $T$-algebra on a Cartesian category $C$ is a product-preserving functor from $\mathcal{T}_T$ into $C$. A homomorphism of $T$-algebras $F, G$ is a natural transformation $\phi : F \to G$.

**Lemma 3.8.** Consider the product-preserving functor $T : \mathcal{T}_T \to \mathcal{T}_{\Delta}$ defined by:
\[
\begin{align*}
T(X) &= X \\
T(TX) &= X \times \Delta X \\
T(\oplus) &= \oplus \\
T(\Pi) &= \pi_1
\end{align*}
\]
Every $\Delta$-algebra $F$ corresponds then to a $T$-algebra $F \circ T$. Furthermore, given $\Delta$-algebras $F, G$, there is a one-to-one correspondence between $T$-algebra homomorphisms $\phi : F \circ T \to G \circ T$ and pairs $(f, f')$ of a differentiable function $f$ and its derivative $f'$ between the underlying change actions $F, G$.

**Proof.** See Appendix B.

These definitions exhibit $\Delta$-algebras and $T$-algebras as multi-sorted Lawvere algebras. Differential morphisms between $\Delta$-algebras are precisely Lawvere homomorphisms when the corresponding $\Delta$-algebras are regarded as $T$-algebras.

The parallel between $\Delta$-algebras and $T$-algebras is strikingly similar to the connection between (generalized) Cartesian differential categories and categories with tangent structure that was outlined in [Cockett and Cruttwell, 2014].

### 3.4. Stable derivatives and additivity.

We do not require derivatives to be additive in their second argument; indeed in many cases they are not. Under some simple conditions, however, (regular) derivatives can be shown to be additive.

**Definition 3.9.** Given a (internal) change action $A$ and arbitrary objects $|B|, |C|$ in a cartesian category $C$, a morphism $u : |A| \times |B| \to |C|$ is **stable** whenever the following diagram commutes:
\[
\begin{array}{ccc}
(|A| \times \Delta A) \times |B| & \xrightarrow{(\oplus \times \text{Id})} & |A| \times |B| \\
\downarrow{\pi_1 \times \text{Id}} & & \downarrow{u} \\
|A| \times |B| & \xrightarrow{u} & |C|
\end{array}
\]

If one thinks of $\Delta A$ as the object of "infinitesimal" transformations on $|A|$, then the preceding definition says that a morphism $u : |A| \times |B| \to |C|$ is stable whenever infinitesimal changes on the input $A$ do not affect its output.

**Lemma 3.10.** Let $f = (|f|, \partial f)$ be a differential map in $\text{CAct}(C)$. If $\partial f$ is stable, then it is additive in its second argument\(^3\), i.e. the following diagram commutes:

\(^3\)Note that the converse is not the case, i.e. a derivative can be additive but not stable.
\[ |A| \times (\Delta A \times \Delta A) \xrightarrow{(\pi_1, \pi_1 \circ \pi_2, (\pi_1, \pi_2 \circ \pi_2))} (|A| \times \Delta A) \times (|A| \times \Delta A) \]

\[ \partial f \circ (\text{Id}_{|A|} \times +, A) \]

\[ \Delta B \quad \xrightarrow{+} \quad \Delta B \times \Delta B \]

Proof. See Appendix B.

Lemma 3.11. Let \( f = (|f|, \partial f) \) and \( g = (|g|, \partial g) \) be differential maps, with \( \partial g \) stable. Then \( \partial (g \circ f) \) is stable.

Proof. See Appendix B.

It is straightforward to see that the category \( \text{Stab}(C) \) of change actions and differential maps with stable derivatives is a subcategory of \( C\text{Act}(C) \).

4. Higher-order derivatives: the extrinsic view

In this section we study categories in which every object is equipped with a change action, and every morphism specifies a corresponding differential map. This provides a simple way of characterising categories which are models of higher-order differentiation purely in terms of change actions.

4.1. Change action models. Recall that a copointed endofunctor is a pair \((F, \sigma)\) where the endofunctor \( F : C \to C \) is equipped with a natural transformation \( \sigma : F \to \text{Id} \). A coalgebra of a copointed endofunctor \((F, \sigma)\) is an object \( A \) of \( C \) together with a morphism \( \alpha : A \to FA \) such that \( \sigma_A \circ \alpha = \text{Id}_A \).

Definition 4.1. We call a coalgebra \( \alpha : C \to C\text{Act}(C) \) of the copointed endofunctor \((C\text{Act}, \varepsilon)\) a change action model (on \( C \)).

Assumption. Throughout Sec. 4, we fix a change action model \( \alpha : C \to C\text{Act}(C) \).

Given an object \( A \) of \( C \), the coalgebra \( \alpha \) specifies a (internal) change action \( \alpha(A) = (A, \Delta A, \oplus_A, +_A, 0_A) \) in \( C\text{Act}(C) \). (We abuse notation and write \( \Delta A \) for the carrier object of the monoid specified in \( \alpha(A) \); similarly for \( +_A, \oplus_A \), and \( 0_A \).) Given a morphism \( f : A \to B \) in \( C \), there is an associated differential map \( \alpha(f) = (f, \partial f) : \alpha(A) \to \alpha(B) \). Since \( \partial f : A \times \Delta A \to \Delta B \) is also a \( C \)-morphism, there is a corresponding differential map \( \alpha(\partial f) = (\partial f, \partial^2 f) \) in \( C\text{Act}(C) \), where \( \partial^2 f : (A \times \Delta A) \times (\Delta A \times \Delta^2 A) \to \Delta^2 B \) is a second derivative for \( f \). Iterating this process, we obtain an \( n \)-th derivative \( \partial^n f \) for every \( C \)-morphism \( f \). Thus change action models offer a setting for reasoning about higher-order differentiation.

4.2. Tangent bundles in change action models. In differential geometry the tangent bundle functor, which maps every manifold to its tangent bundle, is an important construction. There is an endofunctor on change action models reminiscent of the tangent bundle functor, with analogous properties.

Definition 4.2. The tangent bundle functor \( T : C \to C \) is defined as \( TA := A \times \Delta A \) and \( Tf := (f \circ \pi_1, \partial f) \).
Notation. We use shorthand $\pi_{ij} := \pi_i \circ \pi_j$.

The tangent bundle functor $T$ preserves products up to isomorphism, i.e. for all objects $A, B$ of $\mathbf{C}$, we have $T(A \times B) \cong TA \times TB$ and $T1 \cong 1$. In particular, $\phi_{A,B} := \langle\langle \pi_{11}, \pi_{12} \rangle, \langle \pi_{21}, \pi_{22} \rangle \rangle : TA \times TB \to T(A \times B)$ is an isomorphism. Consequently, given maps $f : A \to B$ and $g : A \to C$, then, up to the previous isomorphism, $T(f, g) = (Tf, Tg)$.

A consequence of the structure of products in $\mathbf{CAct}(\mathbf{C})$ is that the map $\oplus_{A \times B}$ inherits the pointwise structure in the following sense:

**Lemma 4.3.** Let $\phi_{A,B} : TA \times TB \to T(A \times B)$ be the canonical isomorphism described above. Then $\oplus_{A \times B} \circ \phi_{A,B} = \oplus_A \times \oplus_B$.

It will often be convenient to operate directly on the functor $T$, rather than on the underlying derivatives. For these, the following results are useful:

**Lemma 4.4.** The following families of morphisms are natural transformations: $\pi_1, \oplus_A : T(A) \to A$, $z := (\text{Id}, 0) : A \to T(A)$, $1 := \langle\langle \pi_1, 0 \rangle, \langle \pi_2, 0 \rangle \rangle : T(A) \to T^2(A)$. Additionally, the triple $(T, z, \oplus)$ defines a monad on $\mathbf{C}$.

**Proof.** See Appendix C

A particularly interesting class of change action models are those that are also cartesian closed. Surprisingly, this has as an immediate consequence that differentiation is itself internal to the category.

**Lemma 4.5** (Internalisation of derivatives). Whenever $\mathbf{C}$ is cartesian closed, there is a morphism $d_{A,B} : (A \Rightarrow B) \to (A \times \Delta A) \Rightarrow \Delta B$ such that, for any morphism $f : 1 \times A \to B$, $d_{A,B} \circ \Lambda f = \Lambda(\partial f \circ \langle\langle \pi_1, \pi_{12} \rangle, \langle \pi_1, \pi_{22} \rangle \rangle)$.

**Proof.** See Appendix C

Under some conditions, we can classify the structure of the exponentials in $(\mathbf{CAct}, \varepsilon)$-coalgebras. This requires the existence of an infinitesimal object.

**Definition 4.6.** If $\mathbf{C}$ is cartesian closed, an infinitesimal object $D$ is an object of $\mathbf{C}$ such that the tangent bundle functor $T$ is represented by the covariant Hom-functor $D \Rightarrow (\cdot)$, i.e. there is a natural isomorphism $\phi : (D \Rightarrow (\cdot)) \cong T$.

**Lemma 4.7.** Whenever there is an infinitesimal object in $\mathbf{C}$, the tangent bundle $T(A \Rightarrow B)$ is naturally isomorphic to $A \Rightarrow TB$.

We would like the tangent bundle functor to preserve the exponential structure; in particular we would expect a result of the form $\frac{\partial(\lambda y t)}{\partial x} = \lambda y \frac{\partial t}{\partial x}$, which is true in differential $\lambda$-calculus [Ehrhard and Regnier, 2003]. Unfortunately it seems impossible to prove in general that this equation holds, although weaker results are available. If the tangent bundle functor is representable, however, additional structure is preserved.

**Theorem 4.8.** The isomorphism between the functors $T(A \Rightarrow (\cdot))$ and $A \Rightarrow T(\cdot)$ respects the structure of $T$, in the sense that the following diagram commutes:

\[
\begin{array}{ccc}
T(A \Rightarrow B) & \xrightarrow{\varepsilon} & A \Rightarrow T(B) \\
\oplus_{A \Rightarrow B} \downarrow & & \downarrow \text{Id}_{A \Rightarrow \oplus B} \\
A \Rightarrow B & & \end{array}
\]

\[\text{[The concept “infinitesimal object” is borrowed from synthetic differential geometry [Kock, 2006]. However, there is nothing intrinsically “infinitesimal” about these objects here.]}\]
Proof. See Appendix C

5. Examples of change action models

5.1. Generalised cartesian differential categories. Generalised cartesian differential categories (GCDC) [Cruttwell, 2017]—a recent generalisation of cartesian differential categories [Blute et al., 2009]—are models of differential calculi. We show that change action models generalise GCDC in that GCDCs give rise to change action models in three different (non-trivial) ways. In this subsection let \( C \) be a GCDC (we assume familiarity with the definitions and notations in [Cruttwell, 2017]).

1. The Flat Model. Define the functor \( \alpha : C \rightarrow \text{CAct}(C) \) as follows. Let \( f : A \rightarrow B \) be a \( C \)-morphism. Then \( \alpha(A) := (A, L_0(A), \pi_1, +_A, 0_A) \) and \( \alpha(f) := (f, D[f]) \).

Theorem 5.1. The functor \( \alpha \) is a change action model.

Proof. See Appendix D

2. The Kleisli Model. GCDCs admit a tangent bundle functor, defined analogously to the standard notion in differential geometry. Let \( f : A \rightarrow B \) be a \( C \)-morphism. Define the tangent bundle functor \( T : C \rightarrow C \) as: \( T(A) := A \times L_0(A) \), and \( Tf := (f \circ \pi_1, D[f]) \). The functor \( T \) is in fact a monad, with unit \( \eta = (\text{Id}_A, 0_A) : A \rightarrow A \times L_0(A) \) and multiplication \( \mu : (A \times L_0(A)) \times L_0(A) \rightarrow A \times L_0(A) \) defined by the composite:

\[
(A \times L_0(A)) \times L_0(A) \xrightarrow{(\pi_1 \circ \pi_1, (\pi_2 \circ \pi_1, \pi_1 \circ \pi_2)} A \times L_0(A) \times L_0(A) \xrightarrow{\text{Id} \times +_A} A \times L_0(A)
\]

Thus we can define the Kleisli category of this functor by \( C_T \) which has geometric significance as a category of generalised vector fields.

We define the functor \( \alpha_T : C_T \rightarrow \text{CAct}(C_T) \): given a \( C_T \)-morphism \( f : A \rightarrow B \), set \( \alpha_T(A) := (A, L_0(A), \text{Id}_A \times \text{Id}_{L_0(A)}, \eta \circ +_A, \eta \circ 0_A) \) and \( \alpha_T(f) := (f, D[f]) \).

Lemma 5.2. \( \alpha_T \) is a change action model.

Proof. See Appendix D

Remark 5.3. The converse is not true: in general the existence of a change action model on \( C \) does not imply that \( C \) satisfies the GCDC axioms. However, if one requires, additionally, \( (\Delta A, +_A, 0_A) \) to be commutative, with \( \Delta(\Delta A) = \Delta A \) and \( \oplus_{\Delta A} = +_A \) for all objects \( A \), and some technical conditions (stability and uniqueness of derivatives), then it can be shown that \( C \) is indeed a GCDC.

\[\text{The third, the Eilenberg-Moore model, is presented in Appendix D.0.1.}\]
5.2. Difference calculus and Boolean differential calculus. Consider the full subcategory \( \text{Grp}_{\text{Set}} \) of \( \text{Set} \) whose objects are all the groups\(^6\). This is a cartesian closed category which can be endowed with the structure of a \((\text{CAct}, \varepsilon)\)-coalgebra \( \alpha \) in a straightforward way.

Given a group \( A = (A, +, 0, -) \), define change action \( \alpha(A) := (A, A, +, +, 0) \). Given a function \( f : A \to B \), define differential map \( \alpha(f) := (f, \partial f) \) where \( \partial f(x, \delta x) := -f(x) + f(x + \delta x) \). Notice \( f(x) + \partial f(x, \delta x) = f(x) + (-f(x) + f(x + \delta x)) = f(x + \delta x) = f(x \oplus \delta x) \); hence \( \partial f \) is a (regular) derivative\(^7\) for \( f \), and \( \alpha(f) \) a map in \( \text{CAct}(\text{Grp}_{\text{Set}}) \). The following result is then immediate.

**Lemma 5.4.** \( \alpha : \text{Grp}_{\text{Set}} \to \text{CAct}(\text{Grp}_{\text{Set}}) \) defines a change action model.

This result is interesting. In the calculus of finite differences [Jordan, 1965], the discrete derivative (or discrete difference operator) of a function \( f : \mathbb{Z} \to \mathbb{Z} \) is defined as \( \delta f(x) := f(x + 1) - f(x) \). In fact the discrete derivative \( \delta f \) is (an instance of) the derivative of \( f \) qua morphism in \( \text{Grp}_{\text{Set}} \), i.e. \( \delta f(x) = \partial f(x, 1) \).

Finite difference calculus [Gleich, 2005; Jordan, 1965] has found applications in combinatorics and numerical computation. Our formulation via change action model over \( \text{Grp}_{\text{Set}} \) has several advantages. First it justifies the chain rule, which seems new. Secondly, it generalises the calculus to arbitrary groups. To illustrate this, consider the Boolean differential calculus [Steinbach and Posthoff, 2017; Thayse, 1981], a theory that applies methods from calculus to the space \( \mathbb{B}^n \) of vectors of elements of some Boolean algebra \( \mathbb{B} \).

**Definition 5.5.** Given a Boolean algebra \( \mathbb{B} \) and function \( f : \mathbb{B}^n \to \mathbb{B}^m \), the \( i \)-th Boolean derivative of \( f \) at \((u_1, \ldots, u_n) \in \mathbb{B}^n \) is the value \( \partial f_{\delta x_i}(u_1, \ldots, u_n) := f(u_1, \ldots, -u_i, \ldots, u_n) \). Writing \( u \leftrightarrow v := (u \land \neg v) \lor (\neg u \land v) \) for exclusive-or.

Now \( \mathbb{B}^n \) is a \( \text{Grp}_{\text{Set}} \)-object. Set \( \top_i := (\bot, \bot, \bot, \bot, \bot, \ldots, \bot, \bot, \bot, \bot, \top, \bot, \bot, \bot, \bot) \in \mathbb{B}^n \).

**Lemma 5.6.** The Boolean derivative of \( f : \mathbb{B}^n \to \mathbb{B}^m \) coincides with its derivative qua morphism in \( \text{Grp}_{\text{Set}} \): \( \frac{\partial f}{\delta x_i}(u_1, \ldots, u_n) = \partial f((u_1, \ldots, u_n), \top_i) \).

**Proof.** See Appendix D.0.1 \( \square \)

5.3. Polynomials over commutative Kleene algebras. The algebra of polynomials over a commutative Kleene algebra [Hopkins and Kozen, 1999; Kleene, 1956] (see [Lombardy and Sakarovitch, 2004; Esparza et al., 2010] for work of a similar vein) is a change action model. Recall that Kleene algebra is the algebra of regular expressions [Brzozowski, 1964; Conway, 1971]. Formally a Kleene algebra \( \mathbb{K} \) is a tuple \((K, +, \cdot, \cdot^*, 0, 1)\) such that \((K, +, \cdot, 0, 1)\) is an idempotent semiring under + satisfying, for all \( a, b, c \in K \):

\[
1 + a a^* = a^* \quad 1 + a^* a = a^* \quad b + a c \leq c \to a^* b \leq c \quad b + c a \leq c \to b a^* \leq c
\]

where \( a \leq b := a + b = b \). A Kleene algebra is commutative whenever \( \cdot \) is.

Henceforth fix a commutative Kleene algebra \( \mathbb{K} \). Define the algebra of polynomials \( \mathbb{K}[\bar{x}] \) as the free extension of the algebra \( \mathbb{K} \) with elements \( \bar{x} = x_1, \ldots, x_n \). We write \( p(\bar{x}) \)

\(^6\)We consider arbitrary functions, rather than group homomorphisms, since, according to this change action structure, every function between groups is differentiable.

\(^7\)Note that \( \partial f \) need not be additive in its second argument, and so derivatives in \( \text{Grp}_{\text{Set}} \) do not satisfy all the axioms of a cartesian differential category.
for the value of $p(\bar{x})$ evaluated at $\bar{x} \mapsto \bar{\alpha}$. Polynomials, viewed as functions, are closed under composition: when $p \in \mathbb{K}[\bar{x}], q_1, \ldots, q_n \in \mathbb{K}[\bar{y}]$ are polynomials, so is the composite $p(q_1(\bar{y}), \ldots, q_n(\bar{y}))$.

Given a polynomial $p = p(\bar{x})$, we define its $i$-th derivative $\frac{\partial p}{\partial x_i}(\bar{x}) \in \mathbb{K}[\bar{x}]$:

$$\frac{\partial a}{\partial x_i}(\bar{x}) = 0 \quad \frac{\partial p^*}{\partial x_i}(\bar{x}) = p^*(\bar{x}) \quad \frac{\partial x^*}{\partial x_i}(\bar{x}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial (p + q)}{\partial x_i}(\bar{x}) = \frac{\partial p}{\partial x_i}(\bar{x}) + \frac{\partial q}{\partial x_i}(\bar{x}) \quad \frac{\partial (pq)}{\partial x_i}(\bar{x}) = p(\bar{x}) \frac{\partial q}{\partial x_i}(\bar{x}) + q(\bar{x}) \frac{\partial p}{\partial x_i}(\bar{x})$$

Write $\frac{\partial p}{\partial x_i}(\bar{x})$ to mean the result of evaluating the polynomial $\frac{\partial p}{\partial x_i}(\bar{x})$ at $\bar{x} \mapsto \bar{\alpha}$.

**Theorem 5.7** (Taylor’s formula [Hopkins and Kozen, 1999]). Let $p(x) \in \mathbb{K}[x]$. For all $a, b \in \mathbb{K}[x]$, we have $p(a + b) = p(a) + b \cdot \frac{\partial p}{\partial x}(a + b)$.

The category of finite powers of $\mathbb{K}$, $\mathbb{K}_x$, has all natural numbers $n$ as objects. The morphisms $\mathbb{K}_x[m, n]$ are $n$-tuples of polynomials $(p_1, \ldots, p_n)$ where $p_1, \ldots, p_n \in \mathbb{K}[x_1, \ldots, x_m]$. Composition of morphisms is the usual composition of polynomials.

**Lemma 5.8.** The category $\mathbb{K}_x$ is a cartesian category, endowed with a change action model $\alpha : \mathbb{K}_x \to \text{CAct} K_x$ whereby $\alpha(\bar{x}) := (K, K, +, +, 0)$, $\alpha(\mathbb{K}^i) := \alpha(K)^i$; for $\bar{p} = (p_1(\bar{x}), \ldots, p_n(\bar{x})) : \mathbb{K}^m \to \mathbb{K}^n$, $\alpha(\bar{p}) := (\bar{p}, p_1, \ldots, p_n)$, where $p_i = p_i(x_1, \ldots, x_m, y_1, \ldots, y_m) := \sum_{j=1}^n y_j \cdot \frac{\partial p_i}{\partial x_j}(x_1 + y_1, \ldots, x_m + y_m)$.

*Proof.* See Appendix D.0.1

**Remark 5.9.** Interestingly derivatives are not additive in the second argument. Take $p(x) = x^2$. Then $\partial p(a + b + c) \neq \partial p(a, b) + \partial p(a, c)$. It follows that $\mathbb{K}[\bar{x}]$ cannot be modelled by GCDC (because of axiom [CD.2]).

### 6. \(\omega\)-change actions and \(\omega\)-differential maps

A change action model $\alpha : C \to \text{CAct} K(C)$ is a category that supports higher-order differentials: each $C$-object $A$ is associated with an $\omega$-sequence of change actions—$\alpha(A)$, $\alpha(\Delta A)$, $\alpha(\Delta^2 A), \ldots$—in which every change action is compatible with the neighbouring change actions. We introduce $\omega$-change actions as a means of constructing change action models “freely”: given a cartesian category $C$, the objects of the category $\text{CAct}_{\omega}(C)$ are all $\omega$-sequences of “contiguously compatible” change actions.

We work with $\omega$-sequences $[A_i]_{i \in \omega}$ and $[f_i]_{i \in \omega}$ of objects and morphisms in $C$. We write $p_k([A_i]_{i \in \omega}) := A_k$ for the $k$-th element of the $\omega$-sequence (similarly for $p_k([f_i]_{i \in \omega})$), and omit the subscript ‘$i \in \omega$’ from $[A_i]_{i \in \omega}$ to reduce clutter. Given $\omega$-sequences $[A_i]$ and $[B_i]$ of objects of a cartesian category $C$, define $\omega$-sequences, product $[A_i] \times [B_i]$, left shift $\Pi[A_i]$ and derivative space $D[A_i]$, by:

$$p_j([A_i] \times [B_i]) := A_j \times B_j \quad p_j(\Pi[A_i]) := A_{j+1}$$

$$p_0(D[A_i]) := A_0 \quad p_{j+1}(D[A_i]) := p_j D[A_i] \times p_j D(\Pi[A_i])$$
Example 6.1. Given an $\omega$-sequence $[A_i]$, the first few terms of $D[A_i]$ are:

$$
p_0 D[A_i] = A_0 \quad p_1 D[A_i] = A_0 \times A_1 \quad p_2 D[A_i] = (A_0 \times A_1) \times (A_1 \times A_2)
$$

$$
p_3 D[A_i] = ((A_0 \times A_1) \times (A_1 \times A_2)) \times ((A_1 \times A_2) \times (A_2 \times A_3))
$$

Definition 6.2. Given $\omega$-sequences $[A_i]$ and $[B_i]$, a pre-$\omega$-differential map between them, written $[f_i]: [A_i] \to [B_i]$, is an $\omega$-sequence $[f_i]$ such that for each $j$, $f_j: p_j D[A_i] \to B_j$ is a C-morphism.

We explain the intuition behind the derivative space $D[A_i]$. Take a morphism $f: A \to B$, and set $A_i = \Delta^i A$ (where $\Delta^0 := A$ and $\Delta^{n+1} := \Delta(\Delta^n A)$). Since $\Delta$ distributes over product, the domain of the $n$-th derivative of $f$ is $p_n D[A_i]$.

Notation. Define $\pi_1^{(0)} := \pi_1$ and $\pi_1^{(j+1)} := \pi_1^{(j)} \times \pi_1^{(j)}$, and define $\pi_2^{(0)} := \text{Id}$ and $\pi_2^{(j+1)} := \pi_2 \circ \pi_2^{(j)}$.

Definition 6.3. Let $[f_i]: [A_i] \to [B_i]$ and $[g_i]: [B_i] \to [C_i]$ be pre-$\omega$-differential maps. The derivative sequence $D[f_i]$ is the $\omega$-sequence defined by:

$$
p_j D[f_i] := (f_j \circ \pi_1^{(j)}, f_{j+1}) : p_{j+1} D[A_i] \to B_j \times B_{j+1}
$$

Using the shorthand $D^n[f_i] := (\Delta^n(\Delta f_i))$, the composite $[g_i] \circ [f_i]: [A_i] \to [C_i]$ is the pre-$\omega$-differential map given by $p_j ([g_i] \circ [f_i]) = g_j \circ p_0 D^j[f_i])$. The identity pre-$\omega$-differential map $\text{Id}: [A_i] \to [A_i]$ is defined as: $p_j \text{Id} := \pi_2^{(j)} : p_j D[A_i] \to A_j$.

Example 6.4. Consider $\omega$-sequences $[f_i]$ and $[g_i]$ as above. Then:

$$
p_0 D[f_i] = (f_0 \circ \pi_1^{(0)}, f_1) \quad p_1 D[f_i] = (f_1 \circ \pi_1^{(1)}, f_2)
$$

$$
p_0 D^2[f_i] = ((f_0 \circ \pi_1^{(0)}, f_1) \circ \pi_1, (f_1 \circ \pi_1^{(0)}, f_2))
$$

$$
p_1 D^2[f_i] = ((f_1 \circ \pi_1^{(1)}, f_2) \circ \pi_1^{(1)}, (f_2 \circ \pi_1^{(2)}, f_3))
$$

$$
p_0 D^3[f_i] = (p_0 D^2[f_i] \circ \pi_1^{(0)}, (f_0 \circ \pi_1^{(0)}, f_1) \circ \pi_1^{(1)}, (f_1 \circ \pi_1^{(1)}, f_2) \circ \pi_1^{(1)}, (f_2 \circ \pi_1^{(2)}, f_3))
$$

It follows that the first few terms of the composite $[g_i] \circ [f_i]$ are:

$$
p_0 ([g_i] \circ [f_i]) = g_0 \circ f_0 \quad p_1 ([g_i] \circ [f_i]) = g_1 \circ (f_0 \circ \pi_1^{(0)}, f_1)
$$

$$
p_2 ([g_i] \circ [f_i]) = g_2 \circ ((f_0 \circ \pi_1, f_1) \circ \pi_1^{(0)}, (f_1 \circ \pi_1^{(1)}, f_2))
$$

Notice that these correspond to iterations of the chain rule, assuming $f_{i+1} = \partial f_i$ and $g_{i+1} = \partial g_i$.

Proposition 6.5. For any pre-$\omega$-differential map $[f_i]$, $\text{Id} \circ [f_i] = [f_i] \circ \text{Id} = [f_i]$.

Proof. See Appendix E

Proposition 6.6. Composition of pre-$\omega$-differential maps is associative: given pre-$\omega$-differential maps $[f_i]: [A_i] \to [B_i]$, $[g_i]: [B_i] \to [C_i]$ and $[h_i]: [C_i] \to [D_i]$, then for all $n \geq 0$, $h_n \circ p_0 D^n([g_i] \circ [f_i]) = (h_n \circ p_0 D^n[g_i]) \circ p_0 D^n[f_i]$.

Proof. See Appendix E
Definition 6.7. Given pre-$\omega$-differential maps $[f_i] : [A_i] \to [B_i], [g_i] : [A_i] \to [C_i]$, the pairing $\langle [f_i], [g_i] \rangle : [A_i] \to [B_i] \times [C_i]$ is the pre-$\omega$-differential map defined by: $p_j([f_i], [g_i]) = \langle f_j, g_j \rangle$. Define pre-$\omega$-differential maps $\pi_1 := [\pi_{1i}] : [A_i] \times [B_i] \to [A_i]$ by $p_j[\pi_{1i}] := \pi_1 \circ \pi_2^{(j)}$, and $\pi_2 := [\pi_{2i}] : [A_i] \times [B_i] \to [B_i]$ by $p_j[\pi_{2i}] := \pi_2 \circ \pi_2^{(j)}$.

Definition 6.8. A pre-$\omega$-change action on a cartesian category C is a quadruple $\hat{A} = ([A_i], [\hat{+}_i A_i], [\hat{+}_i A_i], [0^i_i])$ where $[A_i]$ is an $\omega$-sequence of C-objects, and for each $j \geq 0$, $\hat{+}_i A_j$ and $+\hat{A}_j$ are $\omega$-sequences, satisfying

1. $\hat{+}_i A_j : \Pi[A_i] \times \Pi^{j+1}[A_i] \to \Pi^{j}[A_i]$ is a pre-$\omega$-differential map.
2. $\hat{+}_i A_j : \Pi^{j+1}[A_i] \times \Pi^{j+1}[A_i] \to \Pi^{j+1}[A_i]$ is a pre-$\omega$-differential map.
3. $0^i_j : \top \to A_{j+1}$ is a C-morphism.
4. $\Delta(\hat{A}, j) := (A_j, A_{j+1}, p_0 \hat{+}_i A_j, p_0 + \hat{A}_j, 0^i_j)$ is a change action in C.

We extend the left-shift operation to pre-$\omega$-change actions by defining $\Pi \hat{A} := ([A_i], [\hat{+}_i A_i], [\hat{+}_i A_i], [0^i_i])$. Then we define the change actions $D(\hat{A}, j)$ inductively by: $D(\hat{A}, 0) := \Delta(\hat{A}, 0)$ and $D(\hat{A}, j + 1) := \Delta(\hat{A}, j) \times \Delta(\Pi \hat{A}, j)$. Notice that the carrier object of $D(\hat{A}, j)$ is the $j$-th element of the $\omega$-sequence $D[A_i]$.

Definition 6.9. Given pre-$\omega$-change actions $\hat{A}$ and $\hat{B}$ (using the preceding notation), a pre-$\omega$-differential map $[f_i] : [A_i] \to [B_i]$ is $\omega$-differential if, for each $j \geq 0$, $(f_j, f_{j+1})$ is a differential map from the change action $D(\hat{A}, j)$ to $\Delta(\hat{B}, j)$. Whenever $[f_i]$ is an $\omega$-differential map, we write $\hat{f} : \hat{A} \to \hat{B}$.

We say that a pre-$\omega$-change action $\hat{A}$ is an $\omega$-change action if, for each $i \geq 0$, $\hat{+}_i A_i$ and $+\hat{A_i}$ are $\omega$-differential maps.

Remark 6.10. It is important to sequence the definitions appropriately. Notice that we only define $\omega$-differential maps once there is a notion of pre-$\omega$-change action, but pre-$\omega$-change actions need $\omega$-differential maps to make sense of the monoidal sum $+\hat{A_i}$ and action $\hat{+}_i$.

The reason for requiring each $\hat{+}_i A_i$ and $+\hat{A_i}$ in an $\omega$-change object $\hat{A} = ([A_i], [\hat{+}_i A_i], [\hat{+}_i A_i], [0^i_i])$ to be $\omega$-differential is so that $\hat{A}$ is internally a change action in CAct$_{\omega}(C)$ (see Def. 6.15).

Lemma 6.11. Let $\hat{f} : \hat{A} \to \hat{B}$ and $\hat{g} : \hat{B} \to \hat{C}$ be $\omega$-differential maps. Qua pre-$\omega$-differential maps, their composite $[g_i] \circ [f_i]$ is $\omega$-differential. Setting $\hat{g} \circ \hat{f} := [g_i] \circ [f_i] : \hat{A} \to \hat{C}$, it follows that composition of $\omega$-differential maps is associative.

Proof. See Appendix E □

Lemma 6.12. For any $\omega$-change action $\hat{A}$, the pre-$\omega$-differential map $\text{Id} : [A_i] \to [A_i]$ is $\omega$-differential. Hence $\hat{\text{Id}} := \text{Id} : \hat{A} \to \hat{A}$ satisfies the identity laws.

Proof. See Appendix E □

Definition 6.13. Given $\omega$-change actions $\hat{A}$ and $\hat{B}$, we define the product $\omega$-change action $\hat{A} \times \hat{B} := ([A_i \times B_i], \hat{[+}_i A_i], [\hat{+}_i B_i], [0^i_i])$ where

1. $\hat{[+}_i A_j] := ([\hat{+}_i A_j], \hat{[+}_i B_j]) \circ ([\hat{+}_i A_{j+1}], ([\hat{+}_i A_{j+1}], [\hat{+}_i B_{j+1}]))$
2. $\hat{[+}_i B_j] := ([\hat{+}_i A_j], [\hat{+}_i B_j]) \circ ([\hat{+}_i A_{j+1}], ([\hat{+}_i A_{j+1}], [\hat{+}_i B_{j+1}]))$
(3) \(0^j_0 \coloneqq (0^0_j, 0^1_j)\)

Notice that \(\Delta(\hat{A} \times \hat{B}, j) \coloneqq (A_j \times B_j, A_{j+1} \times B_{j+1}, p_0 \hat{\omega}_j, p_0 \hat{\omega}_j, 0^j_0)\) is a change action in \(\mathbf{C}\) by construction.

**Lemma 6.14.** The pre-\(\omega\)-differential maps \(\pi_1, \pi_2\) are \(\omega\)-differential. Moreover, for any \(\omega\)-differential maps \(\hat{f} : \hat{A} \rightarrow \hat{B}\) and \(\hat{g} : \hat{A} \rightarrow \hat{C}\), the map \(\langle \hat{f}, \hat{g} \rangle \coloneqq (\hat{f}, \hat{g})\) is \(\omega\)-differential, satisfying \(\pi_1 \circ \langle \hat{f}, \hat{g} \rangle = \hat{f}\) and \(\pi_2 \circ \langle \hat{f}, \hat{g} \rangle = \hat{g}\).

**Proof.** See Appendix E

**Definition 6.15.** Define the functor \(\mathrm{CAct}_\omega : \mathbf{Cat}_\omega \rightarrow \mathbf{Cat}_\omega\) as follows.

- \(\mathrm{CAct}_\omega(\mathbf{C})\) is the category whose objects are the \(\omega\)-change actions over \(\mathbf{C}\) and whose morphisms are the \(\omega\)-differential maps.
- If \(F : \mathbf{C} \rightarrow \mathbf{D}\) is a (product-preserving) functor, then \(\mathrm{CAct}_\omega(F) : \mathrm{CAct}_\omega(\mathbf{C}) \rightarrow \mathrm{CAct}_\omega(\mathbf{D})\) is the functor mapping the \(\omega\)-change action \((\{A_i\}, \{\hat{\omega}_i\}, [\hat{\omega}_i], [0])\) to \((\{FA_i\}, \{F\hat{\omega}_i\}, [F\hat{\omega}_i], [F0])\); and the \(\omega\)-differential map \([f_i]\) to \([Ff_i]\).

**Theorem 6.16.** The category \(\mathrm{CAct}_\omega(\mathbf{C})\) is cartesian, with product given in Def. 6.13. Moreover if \(\mathbf{C}\) is closed and has countable limits, \(\mathrm{CAct}_\omega(\mathbf{C})\) is cartesian closed.

**Proof.** See Appendix E

**Theorem 6.17.** The category \(\mathrm{CAct}_\omega(\mathbf{C})\) is equipped with a canonical change action model: \(\gamma : \mathrm{CAct}_\omega(\mathbf{C}) \rightarrow \mathrm{CAct}(\mathrm{CAct}_\omega(\mathbf{C}))\).

**Theorem 6.18** (Relativised final coalgebra). Let \(\mathbf{C}\) be a change action model. The canonical change action model \(\gamma : \mathrm{CAct}_\omega(\mathbf{C}) \rightarrow \mathrm{CAct}(\mathrm{CAct}_\omega(\mathbf{C}))\) is a relativised\(^8\) final coalgebra of \((\mathrm{CAct}, \varepsilon)\).

I.e. for all change action models on \(\mathbf{C}\), \(\alpha : \mathbf{C} \rightarrow \mathrm{CAct}(\mathbf{C})\), there is a unique coalgebra homomorphism \(\alpha_\omega : \mathbf{C} \rightarrow \mathrm{CAct}_\omega(\mathbf{C})\), as witnessed by the commuting diagram:

\[
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{\alpha} & \mathrm{CAct}(\mathbf{C}) \\
\downarrow & & \downarrow \mathrm{CAct}(\alpha_\omega) \\
\mathrm{CAct}_\omega(\mathbf{C}) & \xrightarrow{\gamma} & \mathrm{CAct}(\mathrm{CAct}_\omega(\mathbf{C}))
\end{array}
\]

**Proof.** We first exhibit the functor \(\alpha_\omega : \mathbf{C} \rightarrow \mathrm{CAct}_\omega(\mathbf{C})\).

Take a \(\mathbf{C}\)-morphism \(f : A \rightarrow B\). We define the \(\omega\)-differential map \(\alpha_\omega(f) \coloneqq \hat{f} : \hat{A} \rightarrow \hat{B}\), where \(\hat{A} \coloneqq ([A_i], [\hat{\omega}_i], [\hat{\omega}_i], [0])\) is the \(\omega\)-change action determined by \(A\) under iterative actions of \(\alpha\). I.e. for each \(i \geq 0\): \(A_i \coloneqq \Delta^i A\) (by abuse of notation, we write \(\Delta^0 A\) to mean the carrier object of the monoid of the internal change action \(\alpha(A)\), for any \(\mathbf{C}\)-object \(A\)); \(\hat{\omega}_i : \Pi^i[A_i] \times \Pi^{i+1}[A_i] \rightarrow \Pi^i[A_i]\) is specified by: \(p_k \hat{\omega}_i\) is the monoid action morphism of \(\alpha(A_{i+k})\); \(\hat{\omega}_i : \Pi^i[A_i] \times \Pi^{i+1}[A_i] \rightarrow \Pi^{i+1}[A_i]\) is specified by: \(p_k \hat{\omega}_i\) is the monoid sum morphism of \(\alpha(A_{i+k})\); \(0_i\) is the zero object of \(\alpha(A_i)\).

The \(\omega\)-sequence \(\hat{f} \coloneqq [f_i]\) is defined by induction: \(f_0 \coloneqq f\); assume \(f_n : (\mathrm{D}\hat{A})_n \rightarrow B_n\) is defined and suppose \(\alpha(f_n) = (f_n, \partial f_n)\) then define \(f_{n+1} \coloneqq \partial f_n\).

To see that the diagram commutes, notice that \(\gamma(\hat{f}) = (\hat{f}, \Pi \hat{f})\) and \(\mathrm{CAct}(\alpha_\omega)\) maps \(\alpha(f) = (f, \partial f)\) to \((\hat{f}, \Pi \hat{f})\); then observe that \(\Pi \hat{f} = \partial \hat{f}\) follows from the construction of \(\hat{f}\).

\(^8\)Here \(\mathrm{CAct}\) is restricted to the full subcategory of \(\mathbf{Cat}_\omega\) with \(\mathbf{C}\) as the only object.
Finally to see that the functor \( \alpha_{\omega} \) is unique, consider the \( C \)-morphisms \( \partial^n f \) \((n = 0, 1, 2, \ldots)\) where \( \alpha(\partial^n f) = (\partial^n f, \partial^{n+1} f) \). Suppose \( \beta : C \to C\text{Act}_{\omega}(C) \) is another homomorphism. Thanks to the commuting diagram, we must have \( \Pi^n \beta(f) = \beta(\partial^n f) \), and so, in particular \( (\beta(f))_n = (\Pi^n \beta(f))_0 = (\beta(\partial^n f))_0 = \partial^n f \), for each \( n \geq 0 \). Thus \( \hat{f} = \beta(f) \) as desired.

Intuitively any change action model on \( C \) is always a “subset” of the change action model on \( C\text{Act}_{\omega}(C) \).

**Theorem 6.19.** The category \( C\text{Act}_{\omega}(C) \) is the limit in \( \text{Cat} \times \) of the diagram

\[
\begin{array}{ccc}
D & \xrightarrow{\varepsilon} & C\text{Act}(C) \\
\downarrow & & \downarrow \\
\text{CAct}(C) & \xleftarrow{\varepsilon} & \text{CAct}(\text{CAct}(C)) \\
\downarrow & & \downarrow \\
\text{CAct}(\text{CAct}(\text{CAct}(C))) & \xleftarrow{\varepsilon} & \ldots
\end{array}
\]

**Proof.** See Appendix E

### 7. Related Work, Future Directions and Conclusions

Firstly, the present work directly expands upon work by the authors and others in [Alvarez-Picallo et al., 2019], where the notion of change action was developed in the context of the incremental evaluation of Datalog programs. This work generalizes some results in [Alvarez-Picallo et al., 2019] and addresses two significant questions that had been left open, namely: how to construct cartesian closed categories of change actions and how to formalize higher-order derivatives.

Our work is also closely related to Cockett, Seely and Cruttwell’s work on cartesian differential categories [Blute et al., 2009, 2010; Cockett and Cruttwell, 2014] and Cruttwell’s more recent work on generalised cartesian differential categories [Cruttwell, 2017]. Both cartesian differential categories and change action models aim to provide a setting for differentiation, and the construction of \( \omega \)-change actions resembles the Faà di Bruno construction [Cruttwell, 2017; Cockett and Seely, 2011] (especially its recent reformulation by Lemay [Lemay, 2018]) which, given an arbitrary category \( C \), builds a cofree cartesian differential category for it). The main differences between these two settings lie in the specific axioms required (change action models are significantly weaker: see Remark 5.3) and the approach taken to define derivatives.

This last point is of particular interest: cartesian differential categories assume there is a notion of differentiation in place that satisfies certain coherence conditions, whereas change actions give an algebraic definition of what it means for a function to have a derivative in terms of the change action structure. In this sense, the derivative condition is close to the Kock-Lawvere axiom from synthetic differential geometry [Kock, 2006; Lavendhomme, 2013], which has provided much of the driving intuition behind this work, and making this connection precise is the subject of ongoing research.

In a different direction, the nice interplay between derivatives, products and exponentials in closed change action models (see Theorem 4.8) suggests that there should be a

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9In a notable case of parallel evolution, we developed our results independently of Lemay.
reasonable calculus for change action models. It would be interesting to formulate such a calculus and explore its connections to the differential λ-calculus [Ehrhard and Regnier, 2003]. This could also lead to practical applications to languages for incremental computation, or even higher-order automatic differentiation [Kelly et al., 2016].

In conclusion, change actions and change action models constitute a new setting for reasoning about differentiation that is able to unify “discrete” and “continuous” models, as well as higher-order functions. Change actions are remarkably well-behaved and show tantalising connections with geometry and 2-categories. We believe that most ad hoc notions of derivatives found in disparate subjects can be elegantly integrated into the framework of change action models. We therefore expect any further work in this area to have the potential of benefitting these notions of derivatives.

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Appendix A. Supplementary materials for Section 2

Proposition A.1. Whenever \( f : A \rightarrow B \) is differentiable and has a unique derivative \( \partial f \), this derivative is regular.

Proof. Suppose \( a \in |A| \) (if \( |A| \) is empty the property follows trivially), and \( \delta a, \delta b \in \Delta A \). Then

\[
\begin{align*}
f(a \oplus (\delta a + \delta b)) &= f(a \oplus \delta a \oplus \delta b) \\
&= f(a \oplus \delta a) \oplus \partial f(a \oplus \delta a, \delta b) \\
&= (f(a) \oplus \partial f(a, \delta a)) \oplus \partial f(a \oplus \delta a, \delta b) \\
&= f(a) \oplus (\partial f(a, \delta a) + \partial f(a \oplus \delta a, \delta b))
\end{align*}
\]

Thus we can define the following derivative for \( f \)

\[
\partial f_a(x, \delta x) := \begin{cases} 
\partial f(a, \delta a) + \partial f(a \oplus \delta a, \delta b) & \text{when } x = a, \delta x = \delta a + \delta b \\
\partial f(x, \delta x) & \text{otherwise}
\end{cases}
\]

Since the derivative is unique, it must be the case that \( \partial f = \partial f_a \), and therefore \( \partial f(a, \delta a + \delta b) = \partial f(a, \delta a) + \partial f(a \oplus \delta a, \delta b) \). By a similar argument, \( \partial f(a, 0) = 0 \) and thus \( \partial f \) is regular.

Remark A.2. One may wonder whether every differentiable function admits a regular derivative: the answer is no. Consider the change actions:

\[
A_1 = (\mathbb{Z}_2, \mathbb{Z}_2, +, +, 0) \quad A_2 = (\mathbb{Z}_2, \mathbb{N}, [+], +, 0)
\]

where \([m][+]n = [m + n]\). The identity function \( \text{Id} : A_1 \rightarrow A_2 \) admits infinitely many derivatives, none of which are regular. The condition under which a (differentiable) function admits a regular derivative is an open question.

Proposition A.3. Given \( f : A \rightarrow B \) and \( g : B \rightarrow C \) with regular derivatives \( \partial f \) and \( \partial g \) respectively, the derivative \( \partial (g \circ f) = \partial g \circ (f \circ \pi_1, \partial f) \) is regular.

Proof.

\[
\begin{align*}
\partial g \circ (f \circ \pi_1, \partial f)(a, \delta a + \delta b) &= \partial g(f(a), \partial f(a, \delta a + \delta b)) \\
&= \partial g(f(a), \partial f(a, \delta a) + \partial f(a \oplus \delta a, \delta b)) \\
&= \partial g(f(a), \partial f(a, \delta a)) + \partial g(f(a) \oplus \partial f(a, \delta a), \partial f(a \oplus \delta a, \delta b)) \\
&= (\partial g \circ (f \circ \pi_1, \partial f))(a, \delta a) + (\partial f \circ (f \circ \pi_1, \partial f))(a \oplus \delta a, \delta b) \\
\end{align*}
\]

Thus \( \partial g \circ (f \circ \pi_1, \partial f)(a, 0) = \partial g(f(a), \partial f(a, 0)) = \partial g(f(a), 0) = 0 \).

Theorem 2.8. The category \( \mathbf{CAct}^- \) of change actions and differentiable morphisms is equivalent to \( \mathbf{PreOrd} \), the category of preorders and monotone maps.

Proof. Consider an arbitrary change action \( A = (|A|, \Delta A, \oplus, +, 0) \). Its structure as a change action induces a natural preorder on the base set \( |A| \).
Definition A.5 (Reachability preorder). For \( a, b \in |A| \), we define \( a \sqsubseteq b \) iff there is a \( \delta a \in \Delta A \) such that \( a \oplus \delta a = b \). Then \( \sqsubseteq \) defines a preorder on \( |A| \).

The intuitive significance of the reachability preorder induced by \( A \) is that it contains all the information about differentiability of functions from or into \( A \). This is made precise in the following result:

Lemma A.6. A function \( f : |A| \to |B| \) is differentiable as a function from \( A \) to \( B \) iff it is monotone with respect to the reachability preorders on \( A, B \).

Proof. Let \( f \) be a differentiable function, with \( \partial f \) an arbitrary derivative, and suppose \( a \sqsubseteq_A a' \). Hence there is some \( \delta a \) such that \( a \oplus \delta a = a' \). Then, by the derivative property, we have \( f(a') = f(a) \oplus \partial f(a, \delta a) \), hence \( f(a) \sqsubseteq_B f(a') \).

Conversely, suppose \( f \) is monotone, and pick arbitrary \( a, \delta a \in \Delta A \). Since \( a \sqsubseteq_A a \oplus \delta a \) and \( f \) is monotone, we have \( f(a) \sqsubseteq_B f(a \oplus \delta a) \) and therefore there exists a \( \delta b \in \Delta B \) such that \( f(a) \oplus \delta b = f(a \oplus \delta a) \). We define \( \partial f(a, \Delta a) \) to be precisely such a change \( \delta b \) (note that the process of arbitrarily picking a \( \delta b \) for every pair \( a, \delta a \) makes use, in general, of the Axiom of Choice).

The correspondence between a change action and its reachability preorder gives rise to a (full and faithful) functor \( \text{Reach} : \text{CAct} \to \text{PreOrd} \) that acts as the identity on morphisms.

Conversely, any preorder \( \leq \) on some set \( |A| \) induces a change action

\[
A_{\leq} := (|A|, |A|_{\perp}, \sqcup, \sqcap, \perp)
\]

where \( |A|_{\perp} \) is the set \( |A| \) extended with a bottom element \( \perp \), and \( \sqcup \) denotes the least upper bound according to the preorder \( \leq \). Note that the reachability preorder of the change action \( A_{\leq} \) is precisely \( \leq \).

This defines another full and faithful functor \( \text{Act} : \text{PreOrd} \to \text{CAct} \) that is the identity on morphisms.

It remains to check that there are natural isomorphisms \( \mathcal{U} : \text{Act} \circ \text{Reach} \to \text{Id}_{\text{CAct}} \) and \( \mathcal{V} : \text{Reach} \circ \text{Act} \to \text{Id}_{\text{PreOrd}} \). But these are trivial: it suffices to set \( \mathcal{U}_A = \text{Id}_A \) and \( \mathcal{V}(|A|_{\leq}) = \text{Id}_{|A|_{\leq}} \). Hence \( \text{Act}, \text{Reach} \) establish an equivalence of categories between \( \text{PreOrd} \) and \( \text{CAct} \).

\[
\square
\]

Adjunctions with Set.

Lemma A.7. The forgetful functor \( \mathcal{F} \) is right-adjoint to the functor \( \mathcal{D} \), with the unit and counit given by:

\[
\varepsilon : \mathcal{D} \circ \mathcal{F} \to \text{Id}_{\text{CAct}}
\]
\[
\varepsilon_A = (\text{Id}_A, 0)
\]
\[
\eta : \text{Id}_{\text{Set}} \to \mathcal{F} \circ \mathcal{D} = \text{Id}_{\text{Set}}
\]
\[
\eta_A = \text{Id}_A
\]

\[
\square
\]
Proof.
\[ \varepsilon D \circ D \eta = (\text{Id}, 0) \circ (\eta, !) \]
\[ = (\text{Id} \circ \eta, 0) \]
\[ = (\text{Id} \circ \text{Id}, 0) \]
\[ = (\text{Id}, 0) \]
\[ = \text{Id} \]

Furthermore:
\[ F(\varepsilon) \circ \eta F = F(\text{Id}, 0) \circ \text{Id} \]
\[ = \text{Id} \circ \text{Id} \]
\[ = \text{Id} \]

Appendix B. Supplementary materials for Section 3

The category $\text{CAct}(\mathcal{C})$.

Theorem B.1. The functor $\text{CAct}$ preserves all products.

Proof. Consider an $I$-indexed product of categories $\prod_{i \in I} \mathcal{C}_i$. An object $A$ of $\text{CAct}(\prod_{i \in I} \mathcal{C}_i)$ is a change action $(|A|, \Delta A, \oplus_A, +_A, 0_A)$ where:

- $|A|, \Delta A$ are $I$-indexed families of objects $|A|_i, \Delta A_i$ of $\mathcal{C}_i$
- $\oplus_A$ is an $I$-indexed family of $\mathcal{C}_i$-morphisms $\oplus_i : |A|_i \times \Delta A_i \to |A|_i$
- $+_A$ is an $I$-indexed family of $\mathcal{C}_i$-morphisms $+_i : \Delta A_i \times \Delta A_i \to |A|_i$
- $0_A$ is an $I$-indexed family of $\mathcal{C}_i$-morphisms $0_i : \top_i \to \Delta A_i$

satisfying the relevant conditions ($(\Delta A, +_A, 0_A)$ is a monoid, $\oplus_A$ is an action). But this entails that, for every $i$, the triple $(\Delta A_i, +_i, 0_i)$ defines a monoid in $\mathcal{C}_i$, and $\oplus_i$ is an action of this monoid on $\Delta A_i$. Hence we obtain an $I$-indexed family $A_i$ of change actions in $\text{CAct}(\mathcal{C}_i)$ respectively. Conversely, given any such family, we can always construct the corresponding change action in $\text{CAct}(\prod_{i \in I} \mathcal{C}_i)$.

A similar argument applies to differential maps: every differential map $f : A \to B$ in $\text{CAct}(\prod_{i \in I} \mathcal{C}_i)$ corresponds to a family of differential maps $f_i : A_i \to B_i$ in $\text{CAct}(\mathcal{C}_i)$ and vice versa. Hence the functor $\text{CAct}$ preserves all products. \hfill \Box

Products and coproducts in $\text{CAct}(\mathcal{C})$.

Theorem 3.1. The following change action is the product of $A$ and $B$ in $\text{CAct}(\mathcal{C})$

\[ A \times B := (|A| \times |B|, \Delta A \times \Delta B, \oplus_{A \times B}, +_{A \times B}, (0_A, 0_B)) \]

where $\oplus_{A \times B} := (\oplus_A \circ (\pi_1 \times \pi_1), \oplus_B \circ (\pi_2 \times \pi_2))$ and $+_A := (+_A \circ (\pi_1 \times \pi_1))$. The projections are $\pi_1 = (\pi_1, \pi_1 \circ \pi_2)$ and $\pi_2 = (\pi_2, \pi_2 \circ \pi_2)$, writing $\overline{f}$ for maps $f$ in $\text{CAct}$ to distinguish them from $\mathcal{C}$-maps.
Theorem 3.4. If $C$ is distributive, with law $\delta_{A,B,C} : (A \sqcup B) \times C \to (A \times C) \sqcup (B \times C)$, the following change action is the coproduct of $A$ and $B$ in $\text{CAct}(C)$

$$A \sqcup B := ([A] \sqcup [B], \Delta A \times \Delta B, \oplus_{A \sqcup B}, A \sqcup B, \langle 0_A, 0_B \rangle)$$

where $\oplus_{A \sqcup B} := [\oplus_A \circ (\text{Id}_A \times \pi_1), \oplus_B \circ (\text{Id}_B \times \pi_2)] \circ \delta_{A,B,C}$, and $+_{A \sqcup B} := \langle +_A \circ (\pi_1 \times \pi_1), +_B \circ (\pi_2 \times \pi_2) \rangle$. The injections are $\pi_1 = (\iota_1, \langle \pi_2, 0_B \rangle)$ and $\pi_2 = (\iota_2, \langle 0_A, \pi_2 \rangle)$.

Proof. Given any pair of differential maps $f_1 : A \to C, f_2 : B \to C$ define

$$[f_1, f_2] := ([|f_1|, |f_2|], \partial_h)$$

$$\partial_h := [\delta_{f_1} \circ (\text{Id}_A \times \pi_1), \delta_{f_2} \circ (\text{Id}_B \times \pi_2)] \circ \delta_{A,B,C}$$

where $\delta_{A,B,C} : (A \sqcup B) \times C \to (A \times C) \sqcup (B \times C)$ is the distributive law of $C$. 

Proof. Given any pair of differential maps $f_1 : C \to A, f_2 : C \to B$, define

$$(f_1, f_2) := ([|f_1|, |f_2|], \partial f_1 \circ (\pi_1 \circ \pi_1, \pi_1 \circ \pi_2), \partial f_2 \circ (\pi_2 \circ \pi_1, \pi_2 \circ \pi_2))$$

Then $\pi_1 \circ (f_1, f_2) = f_1$. Furthermore, given any map $h : C \to A \times B$ whose projections coincide with $(f_1, f_2)$, by applying the universal property of the product in $C$, we obtain $h = (f_1, f_2)$.

Proof. It is straightforward to check that, given a change action $A$, there is exactly one differential map $! : A \to \top$. Now given a differential map $(|f|, \partial f) : \top \to A$, applying regularity we obtain:

$$\partial f = \partial f \circ (\text{Id}_\top, \text{Id}_\top)$$

$$= \partial f \circ (\text{Id}_\top, 0_\top)$$

$$= 0_A$$

Lemma B.4. Let $f : A \times B \to C$ be a differential map. Then

$$\partial f((a, b), (\delta a, \delta b)) = +_C \circ \partial f((a, b), (\delta a, 0_B)), \partial f((\oplus_A \circ (a, \delta a), b), (0_A, \delta b))$$

(The notational abuse is justified by the internal logic of a cartesian category.)

Proof. Abusing the notation again, the lemma is a direct consequence of regularity:

$$\partial f((a, b), (\delta a, \delta b)) = \partial f((a, b), (\delta a, 0_B)) +_{A \times B} (0_A, \delta b)$$

$$= \partial f((a, b), (\delta a, 0_B)) +_C \partial f((a, b) \oplus_{A \times B} (\delta a, 0_B), (0_A, \delta b))$$

$$= \partial f((a, b), (\delta a, 0_B)) +_C \partial f((a \oplus_A \delta a, b), (0_A, \delta b))$$

Proof. Given any pair of differential maps $f_1 : C \to A, f_2 : C \to B$, define

$$[f_1, f_2] := ([|f_1|, |f_2|], \partial h)$$

$$\partial h := [\delta_{f_1} \circ (\text{Id}_A \times \pi_1), \delta_{f_2} \circ (\text{Id}_B \times \pi_2)] \circ \delta_{A,B,C}$$

where $\delta_{A,B,C} : (A \sqcup B) \times C \to (A \times C) \sqcup (B \times C)$ is the distributive law of $C$. 

Proof. Given any pair of differential maps $f_1 : C \to A, f_2 : C \to B$, define

$$(f_1, f_2) := ([|f_1|, |f_2|], \partial f_1 \circ (\pi_1 \circ \pi_1, \pi_1 \circ \pi_2), \partial f_2 \circ (\pi_2 \circ \pi_1, \pi_2 \circ \pi_2))$$

Then $\pi_1 \circ (f_1, f_2) = f_1$. Furthermore, given any map $h : C \to A \times B$ whose projections coincide with $(f_1, f_2)$, by applying the universal property of the product in $C$, we obtain $h = (f_1, f_2)$.
We check that, indeed, the relevant diagram commutes since:

\[
[f_1, f_2] \circ \pi_1 \\
= ([|f_1|, |f_2|, \partial h] \circ (\iota_1, \langle \pi_2, 0_B \rangle)) \\
= ([|f_1|, |f_2|] \circ \iota_1, \partial h \circ \langle \iota_1 \circ \pi_1, \langle \pi_2, 0_B \rangle \rangle) \\
= (|f_1|, \partial f_1 \circ (\iota_1 \circ \pi_1) \circ \langle \pi_2, 0_B \rangle) \\
= (|f_1|, \partial f_1) \\
= f_1
\]

The universal property of the coproduct in \( C \) entails that if \( h = (|h|, \partial h) \) is such that \( h \circ \pi_1 = f_i \), then \( h = [f_1, f_2] \). Furthermore, since

\[
(|A| \sqcup |B|) \times \Delta A \times \Delta B \cong (|A| \times \Delta A \times \Delta B) \sqcup (|B| \times \Delta A \times \Delta B),
\]

the universal property of the coproduct also shows that necessarily \( \partial h = \partial f \).

Change actions as Lawvere theories.

**Lemma B.6.** Consider the product-preserving functor \( T : \mathcal{X}_T \to \mathcal{X}_\Delta \) defined by:

\[
T(X) = X \\
T(TX) = X \times \Delta X \\
T(\oplus) = \oplus \\
T(\Pi) = \pi_1
\]

Every \( \Delta \)-algebra \( F \) corresponds then to a \( T \)-algebra \( F \circ T \). Furthermore, given \( \Delta \)-algebras \( F, G \), there is a one-to-one correspondence between \( T \)-algebra homomorphisms \( \phi : F \circ T \to G \circ T \) and pairs \((f, f')\) of a differentiable function \( f \) and its derivative \( f' \) between the underlying change actions \( F, G \).

**Proof.** Consider a natural transformation \( \phi : F \circ T \to G \circ T \), and define

\[
f = \phi_X : F(X) \to G(X)
\]

\[
f' = \pi_2 \circ \phi_{TX} : F(X) \times F(\Delta X) \to G(\Delta X)
\]

Then, by naturality of \( \phi \), it follows that:

\[
\pi_1 \circ \phi_{TX} = (G \circ T)(\Pi) \circ \phi_{TX}
= \phi_X \circ (F \circ T)(\Pi)
= f \circ \pi_1
\]

Hence \( \phi_{TX} = \langle f \circ \pi_1, f' \rangle \). Additionally, we also have:

\[
G(\oplus) \circ \langle f \circ \pi_1, f' \rangle = G(\oplus) \circ \phi_{TX}
= \phi_X \circ F(\oplus)
= f \circ F(\oplus)
\]

which states precisely that \( f' \) is a derivative for \( f \). \( \square \)
Stable derivatives and linearity.

Lemma B.7. Let $f = (|f|, \partial f)$ be a differential map in $\text{CA}(\mathbf{C})$. If $\partial f$ is stable, then it is additive in its second argument\(^{10}\), i.e. the following diagram commutes:

\[
\begin{array}{ccc}
|A| \times (\Delta A \times \Delta A) & \xrightarrow{\langle (\pi_1, \pi_1 \otimes \pi_2), (\pi_1, \pi_2 \otimes \pi_2) \rangle} & (|A| \times \Delta A) \times (|A| \times \Delta A) \\
\partial f \circ (\text{Id} \times |A| \times \Delta A) & + B & \partial f \times \partial f \\
\Delta B & \xleftarrow{\partial f \circ (\text{Id} \times \Delta A)} & \Delta B \times \Delta B \\
\end{array}
\]

Proof. Since $\partial f$ is regular, the following diagram commutes:

\[
\begin{array}{ccc}
|A| \times (\Delta A \times \Delta A) & \xrightarrow{\langle (\pi_1, \pi_1 \otimes \pi_2), a \rangle} & (|A| \times \Delta A) \times ((|A| \times \Delta A) \times \Delta A) \\
\partial f \circ (\text{Id} \times \Delta A) & + B & \partial f \times (\partial f \circ (\oplus \times \text{Id})) \\
\Delta B & \xleftarrow{\partial f \circ (\text{Id} \times \Delta A)} & \Delta B \times \Delta B \\
\end{array}
\]

But since $\partial f$ is stable, we have $\partial f \circ (\oplus \times \text{Id}) = \partial f \circ (\pi_1 \times \text{Id})$, and substituting in the previous diagram gives the desired result.

Lemma B.8. Let $f = (|f|, \partial f)$ and $g = (|g|, \partial g)$ be differential maps, with $\partial g$ stable. Then $\partial (g \circ f)$ is stable.

Proof.

\[
\begin{align*}
\partial (g \circ f) \circ (\oplus \times \text{Id}) &= \partial g \circ \langle |f| \circ \pi_1, \partial f \rangle \circ (\oplus \times \text{Id}) \\
&= \partial g \circ \langle |f| \circ \pi_1 \circ (\oplus \times \text{Id}), \partial f \circ (\oplus \times \text{Id}) \rangle \\
&= \partial g \circ \langle |f| \circ \oplus \pi_1, \partial f \circ (\oplus \times \text{Id}) \rangle \\
&= \partial g \circ (\oplus \circ (\partial f \circ (\pi_1 \circ \pi_1 \times \text{Id})) \\
&= \partial g \circ \langle |f| \circ \pi_1, \partial f \circ (\pi_1 \times \text{Id}) \rangle \\
&= \partial g \circ \langle |f| \circ \pi_1, \partial f \circ (\pi_1 \times \text{Id}) \rangle \\
&= \partial g \circ \langle |f| \circ \pi_1 \circ \pi_1, \partial f \circ (\pi_1 \times \text{Id}) \rangle \\
&= \partial g \circ \langle |f| \circ \pi_1, \partial f \circ (\pi_1 \times \text{Id}) \rangle \\
&= \partial g \circ \langle |f| \circ \pi_1 \circ \pi_1, \partial f \circ (\pi_1 \times \text{Id}) \rangle \\
&= \partial g \circ \langle |f| \circ \pi_1, \partial f \circ (\pi_1 \times \text{Id}) \rangle \\
&= \partial (g \circ f) \circ (\pi_1 \times \text{Id})
\end{align*}
\]

\(^{10}\)Note that the converse is not the case, i.e. a derivative can be additive but not stable.
**Appendix C. Supplementary materials for Section 4**

**Tangent bundles in change action models.**

**Lemma C.1.** The following families of morphisms are natural transformations: $π_1, ⊕_A : T(A) → A$, $z := (Id, 0) : A → T(A)$, $1 := ⟨⟨(π_1, 0), (π_2, 0)⟩⟩ : T(A) → T^2(A)$. Additionally, the triple $(T, z, T⊕)$ defines a monad on $C$.

*Proof.* First, we verify that $⊕ ⊙ Tz = ⊕ ⊙ z$. This is easy to do since (omitting some obvious isomorphisms):

$$T(z) = T(Id, 0) = ⟨TId, T0⟩ = ⟨Id, 0⟩ = z$$

The equation $⊕ ⊙ T⊕ = ⊕ ⊙ ⊕$ is merely an instance of the naturality of $⊕$ and thus it is satisfied trivially. 

**Lemma C.2** (Internalisation of derivatives). Whenever $C$ is cartesian closed, there is a morphism $d_{A,B} : (A ⇒ B) → (A × ΔA) ⇒ ΔB$ such that, for any morphism $f : 1 × A → B$, $d_{A,B} ◦ Δf = Δ(∂f ◦ ⟨⟨(π_1, π_1), (π_1, π_2)⟩⟩)$.

*Proof.* Whenever $C$ is cartesian closed, there is a morphism $d_{A,B} : (A ⇒ B) → (A × ΔA) ⇒ ΔB$ such that, for any morphism $f : 1 × A → B$, $d_{A,B} ◦ Δf = Δ(∂f ◦ ⟨⟨(π_1, π_1), (π_1, π_2)⟩⟩)$.

Consider the evaluation map $ev_{A,B} : (A ⇒ B) × A → B$ in $C$. Its derivative $∂ev_{A,B}$ has type

$$∂ev_{A,B} : ((A ⇒ B) × A) × (Δ(A ⇒ B) × ΔA) → ΔB$$

(note that we make no assumptions about the structure of $Δ(A ⇒ B)$).

Then consider the following composite:

$$∂ev_{A,B} ◦ ⟨⟨(π_1, π_12), (0_{A⇒B}, π_22)⟩⟩ : (A ⇒ B) × (A × ΔA) → ΔB$$

By the universal property of the exponential, we have $ev_{A,B} ◦ (Δf × Id_A) = f$ and therefore $α(∂ev_{A,B} ◦ (Δf × Id_A)) = α(f)$. Thus:

$$∂f = α(∂ev_{A,B} ◦ (Δf × Id_A)) = α(∂ev_{A,B} ◦ (Δf, Id_A)) = α(∂ev_{A,B} ◦ (Δf, Id_A) ◦ π_1, (0_{A⇒B}, π_2)) = ∂ev_{A,B} ◦ ⟨⟨(π_1, π_12), (0_{A⇒B}, π_22)⟩⟩ ◦ (Δf, (π_1, π_2)) = ∂ev_{A,B} ◦ ⟨⟨(π_1, π_12), (0_{A⇒B}, π_22)⟩⟩ ◦ (Δf, Id_{A×ΔA})$$

from which it follows trivially that

$$d_{A,B} = Δ(∂ev_{A,B} ◦ ⟨⟨(π_1, π_12), (0_{A⇒B}, π_22)⟩⟩)$$

is the desired morphism.

**Theorem 4.8.** The isomorphism between the functors $T(A ⇒ (·))$ and $A ⇒ T(·)$ respects the structure of $T$, in the sense that the following diagram commutes:
Proof. By the Yoneda lemma, the natural transformation
\[ \Gamma \circ \phi^{-1} : U \Rightarrow A \Rightarrow A \]
is precisely evaluation at some fixed element \( 1 : 1 \Rightarrow U \) (and, conversely, evaluating \( \phi(t) \) at \( 1 \) is precisely \( \oplus(t) \)).

Commutativity of the above diagram then can be shown by equational reasoning in the internal logic of the CCC \( C \):
\[
f : T(A \Rightarrow B) \vdash \lambda a. \oplus (\phi^{-1}(\lambda u. \phi(f)(u)(a))) = \lambda a. \phi(f)(1)(a) \\
= \phi(f)(1) \\
= \oplus(f)
\]

Appendix D. Supplementary materials for Section 5

Generalised cartesian differential categories.

Theorem 5.1. The functor \( \alpha \) is a change action model.

Proof. We need to check that \( \alpha \) is well-defined and a right-inverse to the forgetful functor.

First, note that \( \alpha(f) \) trivially satisfies the derivative property:
\[
f \circ \pi_1 = \pi_1 \circ (f \circ \pi_1, D[f])
\]

Furthermore, by the axiom [CD.2] of generalised cartesian differential categories, we have:
\[
D[f] \circ \langle \text{Id}, 0_A \circ ! \rangle = 0_B \\
D[f] \circ \langle a, + \circ \langle u, v \rangle \rangle = + \circ \langle D[f] \circ \langle a, u \rangle, D[f] \circ \langle a, v \rangle \rangle
\]

This entails that the map \( \alpha(f) = (f, D[f]) \) is indeed a differential map. Functoriality of \( \alpha \) is a direct consequence of axioms [CD.3] and [CD.5].

Furthermore, \( \alpha \) preserves products (up to isomorphism) since, by definition, \( L(A \times B) = L(A) \times L(B) \) and by axioms [CD.3] and [CD.4], and is trivially a right-inverse to the forgetful functor. Therefore \( \alpha \) is a change action model.

If \( f : A \Rightarrow TB, g : B \Rightarrow TC \) are \( C \)-morphisms, we denote their Kleisli composite \( \mu \circ Tg \circ f \) by \( g \diamond f \).

Lemma D.2. \( \alpha_T \) is a change action model.

Proof. First, since \( T \) preserves products (up to isomorphism), it follows that \( C_T \) is cartesian, with the product of objects \( A, B \) in \( C_T \) being precisely the product \( A \times B \) in \( C \). For brevity, we write \( \pi_i \circ \pi_j \) as \( \pi_{ij} \).

Given \( C \)-morphisms \( f : A \Rightarrow TB \) and \( g : A \Rightarrow TC \) in \( C \), write
\[
\phi := \langle \langle \pi_{11}, \pi_{12} \rangle, \langle \pi_{21}, \pi_{22} \rangle \rangle : TA \times TB \Rightarrow T(A \times B) \\
\langle f, g \rangle := \phi \circ \langle f, g \rangle \equiv \langle \langle \pi_1 \circ f, \pi_1 \circ g \rangle, \langle \pi_2 \circ f, \pi_2 \circ g \rangle \rangle
\]
with \( \phi \) being the isomorphism between \( TA \times TB \) and \( T(A \times B) \) and \( \langle f, g \rangle \) the universal morphism for the product \( A \times B \) in \( C_T \).
It is immediate that \( \alpha_T \) is functorial and preserves products, and it is trivially a section of the forgetful functor.

We need to prove that \( \alpha_T(A) \) is a change action internal to the category \( C_T \). First note that since \( (L_0(A), +_A, 0_A) \) is a commutative monoid in \( C \), the triple \( (L_0(A), \eta \circ +_A, \eta \circ 0_A) \) is a commutative monoid in \( C_T \).

We need to check that \( \text{Id}_{A \times L_0(A)} \) is a monoid action. First, note that \( (\eta \circ g) \star f = Tg \circ f \).

Then:

\[
\text{Id}_{A \times L_0(A)} \star \langle f, \eta \circ 0_A \rangle = \mu \circ T(\text{Id}) \circ \langle f, \eta \circ 0_A \rangle \\
= \mu \circ \langle f, \eta \circ 0_A \rangle \\
= (\text{Id} \times +_A) \circ \langle \pi_{11}, (\pi_{21}, \pi_{12}) \rangle \circ \phi \circ \langle f, \eta \circ 0_A \rangle \\
= \langle \pi_{11}, +_A \circ (\pi_{21}, \pi_{12}) \rangle \circ \phi \circ \langle f, \eta \circ 0_A \rangle \\
= \langle \pi_{11}, +_A \circ (\pi_{21}, \pi_{12}) \rangle \circ \langle f, \eta \circ 0_A \rangle \\
= (\pi_1 \circ f, +_A \circ (\pi_2 \circ f, \pi_1 \circ 0_A)) \\
= \langle \pi_1 \circ f, +_A \circ (\pi_2 \circ f, \pi_1 \circ (0_A, 0_A)) \rangle \\
= \langle \pi_1 \circ f, +_A \circ (\pi_2 \circ f, 0_A) \rangle \\
= \langle \pi_1 \circ f, \pi_2 \circ f \rangle \\
= f
\]

That \( \text{Id}_{A \times L_0(A)} \) respects the associativity of \( \eta \circ +_A \) follows by a similar argument.

We write in detail the proof that the derivative condition holds. In particular, what we seek to prove is

\[
f \star \oplus = \oplus \star \langle f \star (\eta \circ \pi_1), D[f] \rangle.
\]

Then, given that \( \oplus = \text{Id} \), and noting that \( \mu \circ Tf \circ \eta = f \), we obtain:

\[
\text{Id} \star \langle f \star (\eta \circ \pi_1), D[f] \rangle \\
= \mu \circ \langle \mu \circ Tf \circ \eta \circ \pi_1, D[f] \rangle \\
= \mu \circ \langle f \circ \pi_1, D[f] \rangle \\
= (\text{Id} \times +) \circ \langle \pi_{11}, (\pi_{12}, \pi_{21}) \rangle \circ \langle \langle \pi_1 \circ f \circ \pi_1, \pi_1 \circ D[f] \rangle, \langle \pi_2 \circ f \circ \pi_1, \pi_2 \circ D[f] \rangle \rangle \\
= (\text{Id} \times +) \circ \langle \pi_1 \circ f \circ \pi_1, (\pi_2 \circ f \circ \pi_1, \pi_1 \circ D[f]) \rangle \\
= \langle \pi_1 \circ f \circ \pi_1, + \circ \langle \pi_2 \circ f \circ \pi_1, \pi_1 \circ D[f] \rangle \rangle
\]

Conversely:

\[
f \star \oplus = \mu \circ Tf \circ \text{Id} \\
= (\text{Id} \times +) \circ \langle \pi_{11}, (\pi_{12}, \pi_{21}) \rangle \circ \langle f \circ \pi_1, D[f] \rangle \\
= (\text{Id} \times +) \circ \langle \pi_1 \circ f \circ \pi_1, (\pi_1 \circ D[f], \pi_2 \circ f \circ \pi_1) \rangle \\
= \langle \pi_1 \circ f \circ \pi_1, + \circ \langle \pi_1 \circ D[f], \pi_2 \circ f \circ \pi_1 \rangle \rangle
\]

Since \( + \) is commutative (by the definition of a generalised cartesian differential category), both expressions are equal and so the derivative condition holds.

Regularity is a straightforward consequence of the additivity of derivatives in generalised cartesian differential categories.

\[\square\]
D.0.1. The Eilenberg-Moore model.

**Definition D.3.** Given a category \( C \) and a monad \((T, \eta, \mu)\), a \( T \)-algebra is a pair \((A, \nu)\) where \( A \) is an object in \( C \) and \( \nu : TA \to A \) is a \( C \)-morphism such that:

- \( \nu \circ T\nu = \nu \circ \mu \)
- \( \nu \circ \eta = \text{Id} \)

A morphism of \( T \)-algebras between \( T \)-algebras \((A, \nu_A), (B, \nu_B)\) is a \( C \)-morphism \( f : A \to B \) such that \( f \circ \nu_A = \nu_B \circ Tf \).

Both of the previous change action models are in fact categories of algebras for the tangent bundle monad \( T \) on a generalised cartesian differential category \( C \) (the flat model considers algebras of the form \((A, \pi_1)\), whereas the Kleisli category for \( T \) can be understood as the category of freely generated \( T \)-algebras). This is a consequence of the following result:

**Theorem D.4.** Let \((A, \nu_A)\) be a \( T \)-algebra such that \( D[\nu_A] \circ \langle f, \langle g, 0_A \rangle \rangle = g \). Then the tuple \((A, L_0(A), \nu_A, +_A, 0_A)\) is a change action. Furthermore, given a \( C \)-morphism \( f : A \to B \), it is a \( T \)-algebra morphism between \((A, \nu_A)\) and \((B, \nu_B)\) if and only if \( Tf \) is a differential morphism between the corresponding change actions.

**Proof.** For the first part, it suffices to check that \( \nu_A \) is a monoid action.

First, note that the monad unit \( \varepsilon_A \) is precisely the map \((\text{Id}_A, 0_A)\), hence since \( \nu_A \) is a \( T \)-algebra morphism we have \( \nu_A \circ (\text{Id}_A, 0_A) = \text{Id}_A \).

For the second part, note that the monoid addition can be written in terms of the monad multiplication as follows:

\[
\mu \circ \langle \langle \pi_1, \pi_{12} \rangle, \langle \pi_{22}, 0_A \rangle \rangle = (\text{Id} \times +_A) \circ \langle \pi_1, \langle \pi_{11}, \langle \pi_{21}, \pi_{12} \rangle \rangle \rangle \circ \langle \pi_{12}, \pi_{22} \rangle \\
= \langle \text{Id} \times +_A \rangle \circ \langle \pi_1, \langle \pi_{12}, \pi_{22} \rangle \rangle \\
= \langle \text{Id} \times +_A \rangle \circ \text{Id} \\
= \langle \text{Id} \times +_A \rangle
\]

Then because \( \nu_A \) is a \( T \)-algebra homomorphism, we have:

\[
\nu_A \circ (\text{Id} \times +_A) = \nu_A \circ (\langle \langle \pi_1, \pi_{12} \rangle, \langle \pi_{22}, 0_A \rangle \rangle) \\
= \nu_A \circ T\nu_A \circ \langle \langle \pi_1, \pi_{12} \rangle, \langle \pi_{22}, 0_A \rangle \rangle \\
= \nu_A \circ (\nu_A \circ \pi_1, D[\nu_A]) \circ \langle \langle \pi_1, \pi_{12} \rangle, \langle \pi_{22}, 0_A \rangle \rangle \\
= \nu_A \circ (\nu_A \circ (\langle \langle \pi_1, \pi_{12} \rangle, D[\nu_A]) \circ \langle \langle \pi_1, \pi_{12} \rangle, \langle \pi_{22}, 0_A \rangle \rangle) \\
= \nu_A \circ (\nu_A \circ (\langle \langle \pi_1, \pi_{12} \rangle, \langle \pi_1, \pi_{12} \rangle, \langle \pi_{22}, 0_A \rangle) \rangle) \\
= \nu_A \circ \langle \pi_A \circ (\langle \langle \pi_1, \pi_{12} \rangle, \pi_{22} \rangle) \\
= \nu_A \circ (\nu_A \times \text{Id}) \circ \langle \langle \pi_1, \pi_{12} \rangle, \pi_{22} \rangle
\]

Now consider a \( C \)-morphism \( f : A \to B \). Its derivative \( D[f] \) satisfies regularity trivially, since in every generalised cartesian differential category derivatives are additive in their second argument. Then the property that \( f \) is a \( T \)-algebra morphism between \((A, \nu_A)\) and \((B, \nu_B)\) states that \( \nu_B \circ Tf = f \circ \nu_A \), which is equivalent to stating that \( (f, D[f]) \) is a differential morphism between the corresponding change actions.

\(\square\)

One might be tempted to generalise this result in the “obvious” direction and try to construct a change action model directly on the Eilenberg-Moore category \( C^T \). This is,
however, not possible: while an algebra on some object \( A \) does define a change action, it does not give any information on what should be the change action structure on \( \Delta A \).

Instead, consider the obvious projection functor \( \pi : \mathcal{C}^T \to \mathcal{C} \). Now let \( \sigma : \mathcal{C} \to \mathcal{C}^T \) be a section of \( \pi \), i.e. \( \pi \circ \sigma = \text{Id} \). For every object \( A \) of \( \mathcal{C} \), the section \( \sigma \) picks a particular \( T \)-algebra \( \sigma(A) = (A, \nu_A) \). Similarly, \( \sigma \) maps every morphism \( f : A \to B \) in \( \mathcal{C} \) onto a \( T \)-algebra homomorphism.

An immediate corollary of Theorem D.4 is that any section of \( \pi \) that maps objects of \( \mathcal{C} \) to “well-behaved” \( T \)-algebras defines a model structure on \( \mathcal{C} \).

**Lemma D.5.** Let \( \sigma : \mathcal{C} \to \mathcal{C}^T \) be a section of \( \pi : \mathcal{C}^T \to \mathcal{C} \) such that for every object \( A \) of \( \mathcal{C} \), the corresponding algebra \( (A, \nu_A) \) satisfies \( D [\nu_A] \circ \langle f, \langle g, 0_A \rangle \rangle = g \). Then there is a change action model \( \alpha : \mathcal{C} \to \text{CAct}(\mathcal{C}) \) defined by:

- \( \alpha(A) = (A, L_0(A), \nu_A, +A, 0_A) \)
- \( \alpha(f) = T f = \langle f \circ \pi_1, D[f] \rangle \)

The flat model described in Section 5.1 is an immediate corollary of this, as it is the change action model obtained from picking the section that maps every object \( A \) to the \( T \)-algebra \( \pi_2 : A \times L_0(A) \to A \).

**Groups, calculus of differences, and Boolean differential calculus.**

**Lemma D.6.** The Boolean derivative of \( f : \mathbb{B}^n \to \mathbb{B}^m \) coincides with its derivative qua morphism in \( \text{GrpSet} \):

\[
\frac{\partial f}{\partial x_i}(u_1, \ldots, u_n) = \partial f((u_1, \ldots, u_n), \top_i).
\]

**Proof.** Note first that in any Boolean algebra \( \mathbb{B} \), we have \( \neg u = u \Leftrightarrow \top \). Moreover

\[
(u_1, \ldots, \neg u_i, \ldots, u_n) = (u_1, \ldots, u_n) \oplus (\bot, \ldots, \top, \ldots, \bot)
\]

Furthermore:

\[
f(u_1, \ldots, u_n) \oplus \frac{\partial f}{\partial x_i}(u_1, \ldots, u_n) = f(u_1, \ldots, u_n) \oplus f(u_1, \ldots, \neg u_i, \ldots, u_n)
\]

\[
= f(u_1, \ldots, u_n) \Leftrightarrow f(u_1, \ldots, \neg u_i, \ldots, u_n)
\]

\[
= (f(u_1, \ldots, u_n) \Leftrightarrow f(u_1, \ldots, \neg u_i, \ldots, u_n)) \Leftrightarrow f(u_1, \ldots, \neg u_i, \ldots, u_n)
\]

\[
= \bot \Leftrightarrow f(u_1, \ldots, \neg u_i, \ldots, u_n)
\]

\[
f(u_1, \ldots, u_n) = f((u_1, \ldots, u_n) \Leftrightarrow \top_i)
\]

\[
f((u_1, \ldots, u_n) \oplus \top_i)
\]

Thus, since derivatives in \( \text{GrpSet} \) are unique, the Boolean derivative

\[
\frac{\partial f}{\partial x_i}(u_1, \ldots, u_n)
\]

is precisely the derivative \( \partial f((u_1, \ldots, u_n), \top_i) \).

\( \Box \)
Polynomials over commutative Kleene algebras.

Lemma D.7. The category $\mathbb{K}_x$ is a cartesian category, endowed with a change action model $\alpha : \mathbb{K}_x \to \text{CAct}(\mathbb{K}_x)$ whereby $\alpha(\mathbb{K}) := (\mathbb{K}, +, +, 0)$, $\alpha(\mathbb{K})^i := \alpha(\mathbb{K})^i$ for $\mathbb{K} = (p_1(\mathbb{F}), \ldots, p_n(\mathbb{F})) : \mathbb{K}^m \to \mathbb{K}^n$, where $\mathbb{F} := (\mathbb{F}_1, \ldots, \mathbb{F}_n)$, $\mathbb{F}_i(x_1, \ldots, x_m, y_1, \ldots, y_m) := \sum_{j=1}^n y_j \cdot \frac{\partial p_i}{\partial x_j}(x_1 + y_1, \ldots, x_m + y_m)$.

Proof. We consider the essential case of $m = n = 1$; the proof of the lemma is then a straightforward generalisation.

We shall make use of the following properties of commutative Kleene algebras.

1. $(a_1 + \cdots + a_m)^n = \sum\{a_1^{i_1} \cdots a_m^{i_m} \mid i_1 + \cdots + i_m = n; i_1, \ldots, i_m \geq 0\}$.
   Since $(a + b)^* = a^* b^*$, we have
   $$(a_1 + \cdots + a_m)^* = \prod\{(a_1^{i_1} \cdots a_m^{i_m})^* \mid i_1 + \cdots + i_m = n; i_1, \ldots, i_m \geq 0\}.$$ For example $((a + b + c)^2)^* = (a a)^* (a b)^* (a c)^* (b b)^* (b c)^* (c c)^*$.

2. Pilling’s Normal Form Theorem [Pilling, 1973; Hopkins and Kozen, 1999]: every (regular) expression is equivalent to a sum $y_1 + \cdots + y_n$ where each $y_i$ is a product of atomic symbols and expressions of the form $(a_1 \cdots a_k)^*$, where the $a_i$ are atomic symbols. For example $(((a b) c)^* + d)^* = d^* + (a b)^* c^* d^*$.

Take $p(x) \in \mathbb{K}[x]$, viewed as a function from change action $(\mathbb{K}, \mathbb{K}, +, +, 0)$ to itself. For $a, b \in \mathbb{K}$, we have
$$\frac{\partial p(a, b)}{\partial x} := \frac{\partial p}{\partial x}(a + b) \cdot b.$$ That this defines a derivative of $p(x)$ is an immediate consequence of Theorem 5.7.

We need to prove that the derivative is regular. Trivially $\partial p(a, 0) = 0$. It remains to prove: for $a, b, c \in \mathbb{K}$
$$\frac{\partial p}{\partial x}(u + a + b) \cdot (a + b) = \frac{\partial p}{\partial x}(u + a) \cdot a + \frac{\partial p}{\partial x}(u + a + b) \cdot b \quad (D.1)$$
which we argue by structural induction, presenting the cases of $p = q^*$ and $p = qr$ explicitly.

Let $p = q^*$. Thanks to Pilling’s Normal Form Theorem, WLOG we assume $q = x^{n+1}c$. Now $\frac{\partial x^{n+1}c}{\partial x}(x) = x^n c$. Then $\frac{\partial p}{\partial x}(x) = q^*(x) \frac{\partial q}{\partial x}(x) = (x^{n+1}c)^*(x^n c)$. Clearly RHS(D.1) $\leq$ LHS(D.1). For the opposite containment, it suffices to show
$$\frac{\partial p}{\partial x}(u + a + b) \cdot a \leq \frac{\partial p}{\partial x}(u + a) \cdot a + \frac{\partial p}{\partial x}(u + a + b) \cdot b$$
I.e.
$$(\theta^{n+1}c)^*(\theta^n c) a \leq ((u + a)^{n+1}c)^* ((u + a)^n c) a + (\theta^{n+1}c)^*(\theta^n c) b \quad (D.2)$$
using the shorthand $\theta = u + a + b$.

A typical element that matches LHS(D.2) has shape
$$\Xi := (u^{i'} a^{j'} b^{k'} c)^j (u^i a^j b^k c) a$$
satisfying
$$l \geq 0, \quad i' + j' + k' = n + 1, \quad i + j + k = n.$$ It suffices to consider two cases: $l = 0$ and $l = 1$, for if $l > 1$ and $(u^{i'} a^{j'} b^{k'} c) (u^i a^j b^k c) a$ matches RHS(D.2) then so does $(u^{i'} a^{j'} b^{k'} c)^j (u^i a^j b^k c) a$. 
• Now suppose \( l = 0 \). If \( k = 0 \) then \( \Xi \) matches the first summand of \( \text{RHS}(D.2) \); otherwise note that \( \Xi = (u^i a^{j+1} b^{k-1} c) b \) matches the second summand of \( \text{RHS}(D.2) \).

• Next suppose \( l = 1 \). If \( k = k' = 0 \) then \( \Xi \) matches the first summand of \( \text{RHS}(D.2) \); otherwise suppose \( k' > 0 \) then \( \Xi = (u^i a^{j+1} b^{k-1} c)(u^i a^{j1} b^{k'} c) b \) matches the second summand of \( \text{RHS}(D.2) \).

Let \( p(x) = q(x) r(x) \). Applying the product rule of partial derivatives, equation (D.1) is equivalent to \( L = R \) where

\[
L := \left[ r(\theta) \frac{\partial q}{\partial x}(\theta) + q(\theta) \frac{\partial r}{\partial x}(\theta) \right] \cdot (a + b)
\]

\[
R := \left[ r(u + a) \frac{\partial q}{\partial x}(u + a) + q(u + a) \frac{\partial r}{\partial x}(u + a) \right] \cdot a
+ \left[ r(\theta) \frac{\partial q}{\partial x}(\theta) + q(\theta) \frac{\partial r}{\partial x}(\theta) \right] \cdot b
\]

using the shorthand \( \theta = u + a + b \) as before. Similar to the preceding case, clearly \( R \leq L \). To show \( L \leq R \), it suffices to show:

\[
r(\theta) \frac{\partial q}{\partial x}(\theta) a \leq R
\]

\[
q(\theta) \frac{\partial r}{\partial x}(\theta) a \leq R
\]

We consider the first; the same reasoning applies to the second. As before, thanks to Pilling’s Normal Form Theorem, we may assume that \( r(x) = (x^{m+1} c)^* \) and \( q(x) = (x^{n+1} d)^* \). Then \( r(\theta) \frac{\partial q}{\partial x}(\theta) a = (\theta^{m+1} c)^* (\theta^{n+1} d)^* (\theta^n d) a \). By considering a typical element \( \Xi \) that matches the preceding expression, and using the same reasoning as the preceding case, we can then show that \( \Xi \) matches \( R \), as desired.

\[
\square
\]

Appendix E. Supplementary materials for Section 6

Notation. Let \( \alpha \) be an \( \omega \)-sequence. We use shorthand \((\alpha)_j = p_j \alpha\).

Proposition E.1. For any pre-\( \omega \)-differential map \([f_i]_1 \), \( \text{Id} \circ [f_i] = [f_i] \circ \text{Id} = [f_i] \).

Proof. To see that \( \text{Id} \circ [f_i] = [f_i] \), we show

\[
([\text{Id} \circ [f_i]])_n = \pi_2^n \circ \text{p}_0 \text{D}^n [f_i] = f_n = ([f_i])_n
\]

by a straightforward induction on \( n \).

For the other direction of the identity law, \([g_i] \circ \text{Id} = [g_i]\), it suffices to prove: for all \( n \geq 0 \)

\[
\text{p}_0 \text{D}^n \text{Id} = \text{Id} : \text{p}_n \text{D}[A_i] \rightarrow \text{p}_n \text{D}[A_i]
\]

(E.1)

For instance we have

\[
\text{p}_0 \text{D} \text{Id} = \langle \pi_1, \pi_2 \rangle : \text{p}_1 \text{D}[A_i] \rightarrow \text{p}_1 \text{D}[A_i]
\]

\[
\text{p}_0 \text{D}^2 \text{Id} = \langle \langle \pi_1, \pi_2 \rangle \circ \pi_1, \pi_2 \circ \pi_1^{(1)}, \pi_2^{(2)} \rangle \rangle : \text{p}_2 \text{D}[A_i] \rightarrow \text{p}_2 \text{D}[A_i]
\]

\[
\text{p}_0 \text{D}^3 \text{Id} = \langle \pi_1, \langle \langle \pi_2 \circ \pi_1^{(1)}, \pi_2^{(2)} \rangle \circ \pi_1^{(1)}, \pi_2 \circ \pi_1^{(2)}, \pi_2^{(3)} \rangle \rangle : \text{p}_3 \text{D}[A_i] \rightarrow \text{p}_3 \text{D}[A_i]
\]

To establish (E.1), we need to prove a stronger result:
Lemma E.2. For all $n, j \geq 0$, $p_jD^n\text{Id} = \pi_2^{(j)} : p_{n+j}D[A_i] \to p_nD\Pi^j[A_i]$.

Proof. We use lexicographical induction on $(n, j)$. The base case is straightforward. Our induction hypothesis is

$$\forall j \geq 0. p_jD^n\text{Id} = \pi_2^{(j)} : p_{n+j}D[A_i] \to p_nD\Pi^j[A_i]$$

Then

$$p_jD^{n+1}\text{Id} = (p_jD^n\text{Id} \circ \pi_1^{(j)}, p_{j+1}D^n\text{Id})$$
$$= (\pi_2^{(j)} \circ \pi_1^{(j)}, \pi_2^{(j+1)}) : p_{n+1+j}D[A_i] \to p_nD\Pi^j[A_i] \times p_nD\Pi^{j+1}[A_i]$$
$$= \pi_2^{(j+1)} : p_{n+1+j}D[A_i] \to p_nD\Pi^{j+1}[A_i]$$

as desired. The third equality uses the fact: for $j \geq 0$, $\pi_2^{(j)} \circ \pi_1^{(j)} = \pi_1 \circ \pi_2^{(j)}$, which is proved by a straightforward induction on $j$. The base case is trivial. For the inductive case:

$$\pi_2^{(j+1)} \circ \pi_1^{(j+1)} = \pi_2^{(j+1)} \circ (\pi_1^{(j)}, \pi_1^{(j)}) = \pi_2^{(j)} \circ \pi_1 \circ \pi_2 = \pi_1 \circ \pi_2^{(j+1)}.$$ 

\[\square\]

Proposition E.3. Composition of pre-$\omega$-differential maps is associative: given pre-$\omega$-differential maps $[f_i] : [A_i] \to [B_i]$, $[g_i] : [B_i] \to [C_i]$ and $[h_i] : [C_i] \to [D_i]$, then for all $n \geq 0$, $h_n \circ p_0D^n([g_i] \circ [f_i]) = (h_n \circ p_0D^n[g_i]) \circ p_0D^n[f_i]$.

Proof. For convenience and to save space, we write $p_n[f_i]$ as $f_n$. Next we first prove a couple of useful technical lemmas.

First observe that for each $n \geq 0$, and for each $0 \leq j \leq n$, there exists a morphism:

$$\pi_1^{(j)} : p_{n+1}D[A_i] \to p_nD[A_i]$$

This leads to the following.

Lemma E.4. For all $n \geq 0$ and $1 \leq j \leq n$,

$$(D^n[f_i])_0 \circ \pi_1^{(j)} = \pi_1^{(j)} \circ (D^{n+1}[f_i])_0 : p_{n+1}D[A_i] \to p_nD[B_i]$$

Notation: We use the shorthand $f^{(n,i)} = p_i(D^n[f_i])$.

Proof. A stronger induction principle is needed. We claim: for all $n, i \geq 0$ and $j \leq n$

$$f^{(n,i)} \circ \pi_1^{(i+j)} = \pi_1 \circ f^{(n+1,i)}$$

We prove by lexicographical induction on $(n, j, i)$. The base case, which is $\forall i \geq 0. f_i \circ \pi_1^{(i)} = \pi_1 \circ f^{(1,i)}$, holds trivially. For the inductive case:

$$f^{(n+1,i)} \circ \pi_1^{(i+j+1)} = (f^{(n,i)} \circ \pi_1^{(i+j)} \circ \pi_1^{(i+1)})$$
$$= (f^{(n,i)} \circ \pi_1^{(i+j)} \circ \pi_1^{(i+1)})$$
$$= (f^{(n,i)} \circ \pi_1^{(i+j)} \circ \pi_1^{(i+1)})$$
$$= (f^{(n,i)} \circ \pi_1^{(i+j)} \circ \pi_1^{(i+1)})$$
$$= (f^{(n+1,i)} \circ \pi_1^{(i+j)} \circ \pi_1^{(i+1)})$$
$$= \pi_1^{(i+j)} \circ \pi_1^{(i)} \circ \pi_1^{(i)} \circ f^{(n+1,i)}$$
$$= \pi_1^{(i+j)} \circ f^{(n+1,i)}$$

\[\square\]
Then, by applying properties of the product of change actions (e.g. that the derivative of $\pi_i$ is a special case of Lemma E.5.

The second equality appeals to the fact: for $i, j \geq 0$, $n > i + j + 1$

$$\pi_1^{(i)} \circ \pi_1^{(i+j+1)} = \pi_1^{(i+j)} \circ \pi_1^{(i)} : p_n D[A_i] \to p_{n-2} D[A_i]$$

which is easily proved by induction.

\[ \square \]

**Lemma E.5.** For all $n \geq 0$ and $0 \leq j \leq n$,

$$\left(D^n([g_1] \circ [f_1])\right)_j = (D^n[g_1])_j \circ (D^{n+j}[f_i])_0 : p_{n+j} D[A_i] \to p_n D \Pi^j[C_i].$$

**Proof.** We prove by lexicographical induction on $(n, j)$. The base case is straightforward. For the inductive case:

$$
\begin{align*}
\left(D^{n+1}([g_1] \circ [f_1])\right)_j &= \langle (D^n([g_1] \circ [f_1]) \circ \pi_1^{(j)}, (D^n([g_1] \circ [f_1]) \circ \pi_1^{(j)})\rangle_{j+1} \\
&= \langle (D^n([g_1] \circ [f_1]) \circ (D^{n+j}[f_i])_0 \circ \pi_1^{(j)}, (D^n([g_1] \circ [f_1]) \circ \pi_1^{(j)})\rangle_{j+1} \\
&= \langle (D^n[g_1]_j \circ \pi_1^{(j)} \circ (D^{n+j}[f_i])_0, (D^n[g_1] \circ \pi_1^{(j)}) \circ (D^{n+j}[f_i])_0 \\
&= \langle (D^n[g_1]_j \circ \pi_1^{(j)}, (D^n[g_1])_{j+1} \circ (D^{n+j+1}[f_i])_0 \\
&= (D^{n+1}[g_1])_j \circ (D^{n+j+1}[f_i])_0
\end{align*}
$$

The second equality appeals to the induction hypothesis. The third equality uses Lemma E.4 for the first component, and the induction hypothesis for the second.

\[ \square \]

We are now ready to prove the associativity of composition, which boils down to: for all $n \geq 0$,

$$\left(D^n([g_1] \circ [f_1])\right)_0 = (D^n[g_1])_0 \circ (D^n[f_i])_0,$$

which is a special case of Lemma E.5.

\[ \square \]

**Lemma E.6.** Whenever $\tilde{f} : \tilde{A} \to \tilde{B}$ is an $\omega$-differential map, then so is $D^n[f_i]$ for all $n \geq 0$.

**Proof.** Since $f_{i+1}$ is always a regular derivative for $f_i$, it suffices to show that $f_{i+1} \circ \pi_1^{(i+1)}$ is a regular derivative for $f_i \circ \pi_1^{(i)}$.

We abuse the notation and write $\partial f$ to denote some arbitrary derivative of $f$. Then we show by induction on $i$ that $\pi_1^{(i)} \circ \pi_2$ is a regular derivative of $\pi_1^{(i)}$. The base case is trivial. Then, by applying properties of the product of change actions (e.g. that the derivative of $\pi_i$ is $\pi_i \circ \pi_2$), we obtain:

$$\partial(\pi_1^{(i+1)}) = \partial(\pi_1^{(i)} \times \pi_1^{(i)})$$

$$= \partial(\langle \pi_1^{(i)} \circ \pi_1, \pi_1^{(i)} \circ \pi_2 \rangle)$$

$$= \langle \partial(\pi_1^{(i)} \circ \pi_1), \partial(\pi_1^{(i)} \circ \pi_2) \rangle$$

$$= \langle \pi_1^{(i)} \circ \pi_2 \circ \langle \pi_1 \circ \pi_1, \pi_1 \circ \pi_2 \rangle, \pi_2 \circ \pi_1 \circ \langle \pi_2 \circ \pi_1, \pi_2 \circ \pi_2 \rangle \rangle$$

$$= \langle \pi_1^{(i)} \circ \pi_2, \pi_1^{(i)} \circ \pi_2 \rangle$$

$$= \langle \pi_1^{(i)} \circ \pi_2 \rangle$$

$$= \pi_1^{(i+1)} \circ \pi_2$$

\[ \square \]
Hence \( \pi_1^{(i+1)} \circ \pi_2 \) is a derivative of \( \pi_1^{(i+1)} \). By the induction hypothesis, \( \pi_1^{(i)} \circ \pi_2 \) is a regular derivative for \( \pi_1^{(i)} \), and since we have obtained a derivative for \( \pi_1^{(i+1)} \) by applying the chain rule to a composition of functions with regular derivatives, the resulting derivative is also regular.

The desired result then follows from applying the chain rule again:

\[
\partial(f_i \circ \pi_1^{(i)}) = \partial f_i \circ \langle \pi_1^{(i)} \circ \pi_1, \partial \pi_1^{(i)} \rangle \\
= f_{i+1} \circ \langle \pi_1^{(i)} \circ \pi_1, \pi_1^{(i)} \circ \pi_2 \rangle \\
= f_{i+1} \circ (\pi_1^{(i)} \times \pi_1^{(i)}) \\
= f_{i+1} \circ \pi_1^{(i+1)}
\]

Hence \( f_{i+1} \circ \pi_1^{(i+1)} \) is a derivative for \( f_i \circ \pi_1^{(i)} \), and since regularity is preserved under applications of the chain rule, it is also a regular derivative.

\[\square\]

**Lemma E.7.** Let \( \hat{f} : \hat{A} \to \hat{B} \) and \( \hat{g} : \hat{B} \to \hat{C} \) be \( \omega \)-differential maps. Qua \( \omega \)-differential maps, their composite \([g_i] \circ [f_i] : \hat{A} \to \hat{C}\) is \( \omega \)-differential. Setting \( \hat{g} \circ \hat{f} := [g_i] \circ [f_i] : \hat{A} \to \hat{C} \), it follows that composition of \( \omega \)-differential maps is associative.

**Proof.** As in the previous proof, we write \( \partial f \) for an arbitrary derivative of \( f \). By the previous lemma, the map \( D^j[f_i] \) is \( \omega \)-differential and, in particular, \( \partial p_0 D^j[f_i] = p_1 D^j[f_i] \). Then, by applying the chain rule in the definition of composition, we obtain:

\[
\partial p_j([g_i] \circ [f_i]) = \partial (g_j \circ p_0 D^j[f_i]) \\
= g_{j+1} \circ \langle p_0 D^j[f_i] \circ p_1, \partial p_0 D^j[f_i] \rangle \\
= g_{j+1} \circ \langle p_0 D^j[f_i] \circ p_1, p_1 D^j[f_i] \rangle \\
= g_{j+1} \circ p_0 D^{j+1}[f_i] \\
= p_{j+1}([g_i] \circ [f_i])
\]

\[\square\]

**Lemma E.8.** For any \( \omega \)-change action \( \hat{A} \), the pre-\( \omega \)-differential map \( \text{Id} : [A_i] \to [A_i] \) is \( \omega \)-differential. Hence \( \hat{\text{Id}} := \text{Id} : \hat{A} \to \hat{A} \) satisfies the identity laws.

**Proof.** We show: for any \( \omega \)-differential maps \( \hat{f} : \hat{A} \to \hat{B} \) and \( \hat{g} : \hat{B} \to \hat{A} \), \( \hat{f} \circ \hat{\text{Id}} = \hat{f} \) and \( \hat{\text{Id}} \circ \hat{g} = \hat{g} \).

\( p_1 \hat{\text{Id}} \) is trivially a derivative for \( p_0 \hat{\text{Id}} \). Furthermore, since \( \hat{\pi}_2 \) is an \( \omega \)-differential map, we have \( p_{i+2} \hat{\text{Id}} \) is a derivative for \( p_{i+1} \hat{\text{Id}} \), therefore \( \text{Id} \) is \( \omega \)-differential.

We prove identity by simultaneous induction:

\[
p_0(\hat{f} \circ \hat{\text{Id}}) = p_0 \hat{f} \circ p_0 \hat{\text{Id}} \\
= p_0 \hat{f} \circ \text{Id} \\
= p_0 \hat{f} \\
p_0(\hat{\text{Id}} \circ \hat{g}) = p_0 \hat{\text{Id}} \circ p_0 \hat{g} \\
= \text{Id} \circ p_0 \hat{g} \\
= p_0 \hat{g}
\]
Lemma E.9. The pre-ω-differential maps π₁, π₂ are ω-differential. Moreover, for any ω-differential maps \( \hat{f} : \hat{A} \to \hat{B} \) and \( \hat{g} : \hat{A} \to \hat{C} \), the map \( \langle \hat{f}, \hat{g} \rangle := \langle [f_i], [g_i] \rangle \) is ω-differential, satisfying \( \pi_1 \circ \langle \hat{f}, \hat{g} \rangle = \hat{f} \) and \( \pi_2 \circ \langle \hat{f}, \hat{g} \rangle = \hat{g} \).

Proof. First we prove that the map \( \pi_1 : \hat{A} \times \hat{B} \to \hat{A} \) is ω-differential. For this we need to check that \( p_{i+1} \pi_1 \) is a regular derivative for \( p_i \pi_1 \) as a map from the change action \( D(\hat{A} \times \hat{B}, i) \) to the change action \( \Delta(\hat{A}, i) \).

The key insight is that \( D(\hat{A} \times \hat{B}, i) = D(\hat{A}, i) \times D(\hat{B}, i) \). Verifying the derivative property then boils down to applying the structure of products of change actions.

Consider the map \( \pi_1 \circ \pi_2^{(i)} \) from the carrier \( |D(\hat{A}, i) \times D(\hat{B}, i)| \) into \( |\Delta(\hat{A}, i)| \). This is in fact the composition of \( i + 1 \) differentiable maps, and hence we can apply the chain rule to compute a regular derivative for it in terms of the (regular) derivatives for \( π₁, π₂ \), which we know are \( π_{21}, π_{22} \) respectively. We abuse the notation and write \( \partial f \) to denote some arbitrary derivative of \( f \). Then we show by induction on \( i \) that \( π_1 \circ π_2^{(i+1)} \) is a derivative for \( π_1 \circ π_2^{(i)} \). The base case is trivial. For the inductive case:

\[
\partial(π_1 \circ π_2^{(i+1)}) = \partial(π_1 \circ π_2^{(i)} \circ π_2)
\]

\[
= \partial(π_1 \circ π_2^{(i)}) \circ (π_2 \circ π_1, \partial π_2)
\]

\[
= π_1 \circ π_2^{(i+1)} \circ (π_2 \circ π_1, π_2 \circ π_2)
\]

\[
= π_1 \circ π_2^{(i)} \circ π_2 \circ π_2 \circ π_2
\]

\[
= π_1 \circ π_2^{(i)} \circ π_2 \circ π_2
\]

Hence \( \hat{π}_1 \) is an ω-differential map; similarly for \( \hat{π}_2 \). A similar argument also shows that \( \langle \hat{f}, \hat{g} \rangle \) is ω-differential whenever \( \hat{f} \) and \( \hat{g} \) are.

We then check that \( \hat{π}_1 \circ \langle \hat{f}, \hat{g} \rangle = \hat{f} \). For this, the following auxiliary lemma will be of use:

Lemma E.10. For any pre-ω-differential map \([f_i] \), for all \( j, k \geq 0 \),

\[
π_2^{(j)} \circ p_k D^j[f_i] = f_{j+k}.
\]

Proof. We proceed by induction on \( j \). For the case \( j = 0 \) we have \( π_2^{(0)} \circ p_k D^0[f_i] = p_k[f_i] \)

\[
p_i(\Pi \hat{f} \circ (\mathrm{Id} \circ π_1, \Pi \hat{f}))) = p_i(\Pi \hat{f} \circ (π_1, π_2)) = p_i(\Pi \hat{f}) = p_i+1 \hat{f}.
\]

In fact the composition of \( i \) differentiable maps, and hence we can apply the chain rule to check that \( π_1 \circ \hat{f}, \hat{g} \) are. Hence \( \pi_{21}, \pi_{22} \) respectively. We abuse the notation and write \( \partial f \) to denote some arbitrary derivative of \( f \). Then we show by induction on \( i \) that \( π_1 \circ π_2^{(i)} \) is a derivative for \( π_1 \circ π_2^{(i)} \). The base case is trivial. For the inductive case:

\[
\partial(π_1 \circ π_2^{(i+1)}) = \partial(π_1 \circ π_2^{(i)} \circ π_2)
\]

\[
= \partial(π_1 \circ π_2^{(i)}) \circ (π_2 \circ π_1, \partial π_2)
\]

\[
= π_1 \circ π_2^{(i+1)} \circ (π_2 \circ π_1, π_2 \circ π_2)
\]

\[
= π_1 \circ π_2^{(i)} \circ π_2 \circ π_2 \circ π_2
\]

\[
= π_1 \circ π_2^{(i)} \circ π_2 \circ π_2
\]

Hence \( \hat{π}_1 \) is an ω-differential map; similarly for \( \hat{π}_2 \). A similar argument also shows that \( \langle \hat{f}, \hat{g} \rangle \) is ω-differential whenever \( \hat{f} \) and \( \hat{g} \) are.

We then check that \( \hat{π}_1 \circ \langle \hat{f}, \hat{g} \rangle = \hat{f} \). For this, the following auxiliary lemma will be of use:

Lemma E.10. For any pre-ω-differential map \([f_i] \), for all \( j, k \geq 0 \),

\[
π_2^{(j)} \circ p_k D^j[f_i] = f_{j+k}.
\]

Proof. We proceed by induction on \( j \). For the case \( j = 0 \) we have \( π_2^{(0)} \circ p_k D^0[f_i] = p_k[f_i] \)
For the inductive case:
\[ \pi_2^{(j+1)} \circ p_k D^{j+1}[f_i] = \pi_2^{(j)} \circ p_2 \circ \langle \ldots, p_{k+1} D^j[f_i] \rangle \]
\[ = \pi_2^{(j)} \circ p_{k+1} D^j[f_i] \]
\[ = f_{(j+1)+k} \]

The desired equation follows as a trivial corollary. Indeed, for any \( i \)
\[ p_i(\hat{\pi}_1 \circ \langle \hat{f}, \hat{g} \rangle) = p_i \hat{\pi}_1 \circ p_0 D^i \langle \hat{f}, \hat{g} \rangle \]
\[ = \pi_1 \circ \pi_2^{(i)} \circ p_0 D^i \langle \hat{f}, \hat{g} \rangle \]
\[ = \pi_1 \circ p_i \langle \hat{f}, \hat{g} \rangle \]
\[ = p_i \hat{f} \]

Hence \( \hat{\pi}_1 \circ \langle \hat{f}, \hat{g} \rangle = \hat{f} \).

**Theorem 6.16.** The category \( \text{CAct}_\omega(\mathbf{C}) \) is cartesian, with product given in Def. 6.13. Moreover if \( \mathbf{C} \) is closed and has countable limits, \( \text{CAct}_\omega(\mathbf{C}) \) is cartesian closed.

**Proof.** Consider \( \omega \)-change actions \( \hat{A}, \hat{B}, \hat{C} \) and let \( \hat{f} : \hat{A} \to \hat{B}, \hat{g} : \hat{A} \to \hat{C} \) be \( \omega \)-differential maps. Lemma 6.14 already shows that \( \hat{\pi}_1 \circ \langle \hat{f}, \hat{g} \rangle = \hat{f} \), and similarly for \( \hat{\pi}_2 \). It remains to establish uniqueness.

Suppose there is an \( \omega \)-differential map \( \hat{h} : \hat{A} \to \hat{B} \times \hat{C} \) satisfying
\[ \hat{\pi}_1 \circ \hat{h} = \hat{f} \quad \hat{\pi}_2 \circ \hat{h} = \hat{g} \]

Then, for every \( i \), applying Lemma E.10 we obtain
\[ p_i(\hat{\pi}_1 \circ \hat{h}) = p_i \hat{\pi}_1 \circ p_0 D^i \hat{h} \]
\[ = \pi_1 \circ \pi_2^{(i)} \circ p_0 D^i \hat{h} \]
\[ = \pi_1 \circ h_i \]
\[ = f_i \]

Applying a similar reasoning to \( g_i \), and by the universal property of \( \langle f_i, g_i \rangle \) in \( \mathbf{C} \), we obtain that \( h_i = \langle f_i, g_i \rangle = p_i \langle \hat{f}, \hat{g} \rangle \) and hence \( \hat{h} = \langle \hat{f}, \hat{g} \rangle \). Therefore, \( \hat{A} \times \hat{B} \) is the categorical product in \( \text{CAct}_\omega(\mathbf{C}) \).

We can construct a terminal \( \omega \)-change action \( \hat{T} \) by picking the terminal object of \( \mathbf{C} \) at every level. This uniquely determines the entire structure of \( \hat{T} \), since the only possible choice for every morphism is the universal morphism \( ! \) in \( \mathbf{C} \).

\[ \hat{T} := ([\top], [!], [?!], [!!!]) \]

Note that the \( \omega \)-sequences \([!]\) are all \( \omega \)-differential maps, and hence \( \hat{T} \) is an \( \omega \)-change action.

Now take an arbitrary \( \omega \)-change action \( \hat{A} \). It is straightforward to check that there is exactly one morphism \( \hat{!} : \hat{A} \to \hat{T} \), namely the morphism given by \( p_i \hat{!} = ! \). Therefore \( \hat{T} \) is the terminal object in \( \text{CAct}_\omega(\mathbf{C}) \).
We now sketch a proof that \( \text{CAct}_\omega(\mathbf{C}) \) has exponentials, provided that \( \mathbf{C} \) is cartesian closed and has all countable limits. First, consider \( \omega \)-sequences of \( \mathbf{C} \)-objects \([A_i]\) and \([B_i]\). Since \( \mathbf{C} \) has all countable products, one can construct the infinite product
\[
p_j([A_i] \Rightarrow [B_i]) := p_j \mathbf{D}[A_i] \Rightarrow p_j[B_i]
\]
Intuitively, this object of \( \mathbf{C} \) represents the pre-\( \omega \)-differential maps between \([A_i]\) and \([B_i]\).

If \( \hat{A}, \hat{B} \) are \( \omega \)-change actions on the \( \omega \)-sequences \([A_i], [B_i]\), we can consider the subobject of \( p_j([A_i] \Rightarrow [B_i]) \times p_{j+1}([A_i] \Rightarrow [B_i]) \) where the second element is the derivative of the first (i.e. of differential maps) by taking the limit of the following diagram:

\[
\begin{array}{ccc}
p_j([A_i] \Rightarrow [B_i]) \times p_{j+1}([A_i] \Rightarrow [B_i]) & \rightarrow & (p_j \mathbf{D}[A_i] \times p_{j+1} \mathbf{D}[A_i]) \Rightarrow (p_j[B_i] \times p_{j+1}[B_i]) \\
p_j \mathbf{D}[A_i] \Rightarrow p_j[B_i] & \downarrow & (p_j \mathbf{D}[A_i] \times p_{j+1} \mathbf{D}[A_i]) \Rightarrow p_j[B_i] \\
\pi_1 & \nearrow \gamma & \text{Id} \Rightarrow p_0 \widehat{\otimes} j
\end{array}
\]

We can further restrict the space to only regular derivatives by taking the limit of a similar diagram, requiring that
\[
\partial f(a, 0) = 0 \quad \text{and} \quad \partial f(a, \delta a + \delta b) = \partial f(a, \delta a) + \partial f(a + \delta a, \delta b).
\]
Pasting all these diagrams together, we can define the space of \( \omega \)-differential maps between \( \hat{A} \) and \( \hat{B} \) as a limit object \([\hat{A} \Rightarrow \hat{B}]\) internal to \( \mathbf{C} \).

The \( \omega \)-sequence \([\hat{A} \Rightarrow \Pi_\omega \hat{B}]\) is then a pre-\( \omega \)-change action that forms the basis for the exponential \( \hat{A} \Rightarrow \hat{B} \) in \( \text{CAct}_\omega(\mathbf{C}) \). The structure morphisms, \( \widehat{\otimes}, \widehat{\ominus} \) and \( \widehat{0} \), are obtained by lifting the structure morphisms in \( \Pi_\omega \hat{B} \) pointwise.

\[\square\]

**Theorem 6.17.** The category \( \text{CAct}_\omega(\mathbf{C}) \) is equipped with a canonical change action model: \( \gamma : \text{CAct}_\omega(\mathbf{C}) \rightarrow \text{CAct}(\text{CAct}_\omega(\mathbf{C})) \).

**Proof.** Given an object \( \hat{A} = ([A_i], [\widehat{\otimes}_i], [\widehat{\ominus}_i], [0_i]) \) of \( \text{CAct}_\omega(\mathbf{C}) \), the canonical coalgebra \( \gamma : \text{CAct}_\omega(\mathbf{C}) \rightarrow \text{CAct}(\text{CAct}_\omega(\mathbf{C})) \) maps the \( \omega \)-change action to itself. That \( \hat{A} \) is a internal change action of \( \text{CAct}_\omega(\mathbf{C}) \) follows at once from the definition of \( \omega \)-change action: \( \Delta \hat{A} = \Pi_\omega \hat{A} \) and \( p_n(\mathbf{D}(\hat{A} \times \Pi_\omega \hat{A})) = p_{n+1}(\mathbf{D} \hat{A}) \); and \( \widehat{\ominus}_0 : \hat{A} \times \Delta \hat{A} \rightarrow \hat{A} \) and \( \widehat{\otimes}_0 : \Delta \hat{A} \times \Delta \hat{A} \rightarrow \Delta \hat{A} \) are \( \omega \)-differential. The functor \( \gamma \) maps an \( \omega \)-differential map \( \hat{f} : \hat{A} \rightarrow \hat{B} \) to the differential map \( \gamma(\hat{f}) := (\hat{f}, \partial \hat{f}) \) where \( \partial \hat{f} : \Delta \hat{A} \rightarrow \Delta \hat{B} \) is just the \( \omega \)-differential map \( \Pi_\omega \hat{f} \).

\[\square\]

**Theorem 6.19.** The category \( \text{CAct}_\omega(\mathbf{C}) \) is the limit in \( \text{Cat}_\omega \) of the diagram
Proof. First we construct by induction on \( i \geq 1 \) a family of forgetful functors \( \varepsilon_i : \text{CAct}_\omega(C) \to \text{CAct}^i(C) \) that make the diagram commute.

When \( i = 1 \), we use the operator \( \Delta(\hat{A},i) \) from Definition 6.8 to define \( \varepsilon_1 \):

- \( \varepsilon_1(\hat{A}) := \Delta(\hat{A},1) \)
- \( \varepsilon_1(\hat{f}) := (p_0\hat{f},p_1\hat{f}) \)

For \( i \geq 0 \), the functor \( \varepsilon_{i+1} \) is defined inductively by:

- \( \varepsilon_{i+1}(\hat{A}) := (\varepsilon_i(\hat{A}),\varepsilon_i(\Pi\hat{A}),\varepsilon_{i+1}(\oplus_0^\hat{A}),\varepsilon_{i+1}(\oplus_0^\hat{A}),p_0\hat{A}) \)
- \( \varepsilon_{i+1}(\hat{f}) := (\varepsilon_i(\hat{f}),\varepsilon_i(\Pi\hat{f})) \)

It is straightforward to check that the required diagram does commute. For example, for \( i = 1 \), \( \varepsilon_2(\hat{A}) \) is the change action

\[
\varepsilon_2(\hat{A}) = (A_{01},A_{12},\oplus_{012},\ldots)
\]

\[
A_{01} = (A_0,A_1,(p_0\oplus_0,p_1\oplus_0),\ldots)
\]

\[
A_{12} = (A_1,A_2,(p_0\oplus_1,p_1\oplus_1),\ldots)
\]

\[
\oplus_{012} = ((p_0\oplus_0,p_1\oplus_1),(p_1\oplus_1,p_2\oplus_2))
\]

hence the “lower” structure extracted by \( \varepsilon \) and the “higher” one extracted by \( \xi \) coincide.

To prove the universal property, consider a category \( D \) and functors \( \varepsilon'_i : D \to \text{CAct}^i(C) \) making the diagram commute. Then there is a unique functor \( [\varepsilon'_i] : D \to \text{CAct}_\omega(C) \) satisfying \( \varepsilon'_i = \varepsilon_i \circ [\varepsilon'_i] \). To construct it, first consider an object \( U \) of \( D \). We define the \( \omega \)-sequence \( [U] \) by:

\[
U_0 := [\varepsilon'_1(U)]
\]

\[
U_{j+1} := \Delta^{j+1}\varepsilon'_{j+1}(U)
\]

Note that, for every \( j \), \( \varepsilon'_{j+1}(U) \) is a change action on \( \text{CAct}^{j+1}(C) \) and, therefore, \( \Delta^j\varepsilon'_{j+1}(U) \) is a change action in \( \text{CAct}(C) \). In particular, \( \oplus\Delta^j\varepsilon'_{j+1} \) is an action of \( U_{j+1} \) on \( U_j \). Hence we define the pre-\( \omega \)-differential map \( \widehat{\oplus^j_U} \) as follows:

\[
p_k\widehat{\oplus^j_U} = \partial^k \oplus \Delta^j\varepsilon'_{j+k+1}
\]

and similarly for \( \hat{\oplus^j_U},\hat{\oplus^j_U} \).

Then the action of \( [\varepsilon'_i] \) on an object \( U \) of \( D \) can be defined as:

\[
[\varepsilon'_i](U) = ([U_j],\hat{\oplus^j_U},\hat{\oplus^j_U},[0^U])
\]

Note that this is indeed an \( \omega \)-change action, since the maps \( \hat{\oplus^j_U},\hat{\oplus^j_U} \) are \( \omega \)-differential and satisfy the required equations by construction.
Whenever \( f : U \rightarrow V \) is a morphism in \( D \), the morphism \( \varepsilon'_{j+1}(f) \) is a differential map in \( \text{CAct}^{j+1} (C) \), hence its \( j \)-th derivative \( \partial^{j+1}f \) is a morphism in \( C \) of the appropriate type, so we can express the action of \( [\varepsilon'_i] \) on morphisms of \( U \) by:
\[
p_0[\varepsilon'_i](f) = |\varepsilon'_1(f)|p_{j+1}[\varepsilon'_i](f) = \partial^{j+1}\varepsilon'_{j+1}(f)
\]
which is an \( \omega \)-differential morphism by construction.

**Remark E.14.** Note that the previous statement depends on \( \varepsilon'_{j+1} \) equalising \( \varepsilon \) and \( \xi \). If this were not the case, then we could have \( \varepsilon'_2(f) = ((f_0, f_1), (f'_1, f_2)) \) with \( f_1 \neq f'_1 \). Then, according to the above definition:
\[
p_0[\varepsilon'_i](f) = f_0
p_1[\varepsilon'_i](f) = f_1
p_2[\varepsilon'_i](f) = f_2
\]
However, since \( f_1 \neq f'_1 \), there is no guarantee that \( f_2 \) is a derivative for \( f_1 \), hence \([\varepsilon'_i](f)\) is not \( \omega \)-differential.

Thus defined, \([\varepsilon'_i]\) is a functor from \( D \) into \( \text{CAct}_\omega(C) \) such that \( \varepsilon'_i = \varepsilon_i \circ [\varepsilon'_i] \), and it is clear from the construction that it is unique. Therefore, \( \text{CAct}_\omega(C) \) is precisely the desired limit.

\[
\square
\]

**Appendix F. Change actions as 2-categories**

Consider a change action \( A \) in \( \textbf{Set} \). The change action induces the structure of a category \( \text{Cat}(A) \) on \(|A|\) as follows:
- The objects of \( \text{Cat}(A) \) are the elements of \(|A|\).
- The morphisms \( \text{Cat}(A)(a_1, a_2) \) are the changes \( \delta a : \Delta A \) such that \( a_1 \oplus \delta a = a_2 \).
- The identity morphism \( \text{Id}_A \) is the object \( 0 : \Delta A \).
- Composition \( \delta a_2 \circ \delta a_1 \) is the sum \( \delta a_1 + \delta a_2 \).

Since \( \oplus \) is a monoid action, the composition of morphisms is well-typed. Associativity and identity follow from the fact that \( \Delta A \) is a monoid.

Now let \( f = (|f|, \partial f) \) be a differential map between change actions \( A \) and \( B \). Clearly \( f \) can be seen as a functor \( \text{Cat}(f) \) between the corresponding categories \( \text{Cat}(A), \text{Cat}(B) \) in the following way:
- If \( a \) is an object of \( \text{Cat}(A) \), then \( \text{Cat}(f)(a) \) is the element \(|f|(a)\) considered as an object of \( \text{Cat}(B) \).
- If \( \delta a \) is a morphism from \( a_1 \) to \( a_2 \), then \( \text{Cat}(f)(\delta a) \) is \( \partial f(a_1, \delta a) \).

This definition is well-typed since \( \partial f(a_1, \delta a) \) is a change mapping \(|f|(a_1)\) into \(|f|(a_1 \oplus \delta a) = |f|(a_2)\), and hence a morphism from \(|f|(a_1)\) into \(|f|(a_2)\) in the category \( \text{Cat}(B) \). Functoriality follows from (and is equivalent to) regularity of \( \partial f \).

Conversely, let \( F \) be a functor from \( \text{Cat}(A) \) into \( \text{Cat}(B) \). This induces a differential map \( F = (F, \partial F) \) from \( A \) into \( B \) defined by:
\[
F(a) := F(a)
\partial F(a, \delta a) := F(\delta a)
\]
Regularity follows from functoriality of $F$, and the derivative property is a direct consequence of the fact that $F$ is well-typed.

**Lemma F.1.** The category $\mathbf{CAct}$ embeds fully and faithfully into the 2-category $\mathbf{Cat}$ of (small) categories and functors.

Given differential maps $f, g : A \to B$, a natural transformation $U : f \to g$ maps every object $a : |A|$ to a change $U(a) : \Delta B$ such that the following diagram commutes:

$$
\begin{array}{c}
|f|(a) \xrightarrow{\partial f(a, \delta a)} |f|(a \oplus \delta a) \\
\text{U(a)} \downarrow \hspace{1cm} \downarrow \text{U(a\oplus\delta a)} \\
|g|(a) \xrightarrow{\partial g(a, \delta a)} |g|(a \oplus \delta a)
\end{array}
$$

In particular, this means natural transformations are a subset of the set $|A| \to \Delta B$, which can be read as generalized vector fields (mapping the space $A$ to $\Delta B$ rather than $\Delta A$).

**Remark F.2.** Consider a natural transformation from functor $\text{Cat}(f)$ into $\text{Cat}(g)$. This is, first and foremost, a map that assigns to every element $a \in |A|$ a change $\delta a \in \Delta A$. This is precisely the space of functional changes $\Delta(A \to B)$ in $\mathbf{CAct}_\omega$ (see Sec. 6) and, in general, in any change action model (see Sec. 4) equipped with an infinitesimal object.

More generally, the category $\mathbf{CAct}(C)$ of change actions on an arbitrary base cartesian category $C$ can be regarded as a 2-category. Indeed, given change actions $A, B$, we define the category of differential maps $\text{Diff}(A, B)$ as follows:

- The objects of $\text{Diff}(A, B)$ are differential maps $f : A \to B$.
- The morphisms between $f, g$ are $C$-morphisms $U : |A| \to \Delta B$ such that the following diagrams (in $C$) commute:

$$
\begin{array}{c}
A \xrightarrow{\langle f, U \rangle} B \times \Delta B \\
\circ \downarrow \hspace{1cm} \downarrow U \circ \pi_1, \partial g \\
B \xrightarrow{\Delta B \times \Delta B} \Delta B \\
\end{array}
$$

Intuitively, the first diagram asserts that $U$ has the “type” of a natural transformation $f$ to $g$, whereas the second diagram states naturality of $U$.

The identity objects in $\text{Diff}(A, B)$ are the constant zero maps $\text{Id}_f := (0_B)\circ!$. Given $\text{Diff}(A, B)$-morphisms $U : f \to g, V : g \to h$, their composition is defined by:

$$
V \bullet U := + \circ \langle U, V \rangle
$$

which is a $\text{Diff}$-map between $f$ and $h$ - indeed:

$$
\begin{align*}
    h &= \circ \circ (g, V) \\
    &= \circ \circ (\circ \circ (f, U), V) \\
    &= \circ \circ (f, + \circ \langle U, V \rangle) \\
    + \circ \langle \partial f, + \circ \langle U, V \rangle \circ \oplus \rangle &= + \circ \langle + \circ \langle U \circ \pi_1, \partial g \rangle, V \circ \oplus \rangle \\
    &= + \circ \langle + \circ \langle U \circ \pi_1, + \circ \langle U \circ \pi_1, \partial g \rangle, V \circ \oplus \rangle \\
    &= + \circ \langle + \circ \langle U \circ \pi_1, + \circ \langle V \circ \pi_1, \partial g \rangle \rangle \\
    &= + \circ \langle + \circ \langle U, V \rangle \circ \oplus, \partial g \rangle
\end{align*}
$$
which entails the required diagram commutes. Associativity and identity follow from the
definition of change action.

Furthermore, composition of differential maps can be lifted to a functor on the cor-
responding categories. More precisely, let \( A, B, C \) be change actions. Define the functor
\( \text{Comp} : \text{Diff}(A, B) \times \text{Diff}(B, C) \rightarrow \text{Diff}(A, C) \) as follows:

- If \( (f, g) \) is an object in \( \text{Diff}(A, B) \times \text{Diff}(B, C) \), then \( \text{Comp}(f, g) \) is just the composition
  of differential maps, i.e.
  \[
  \text{Comp}(f, g) = ([g] \circ [f], \partial g \circ ([f] \circ \pi_1, \partial f))
  \]
- If \( (U, V) \) is a morphism from \((f_1, g_1)\) into \((f_2, g_2)\), then \( \text{Comp}(U, V) \) is defined as:
  \[
  + \circ (\partial g_1 \circ (f_1, U), V \circ |f_2|)
  \]

**Proof.** We need to show that, as defined above, the morphism \( \text{Comp}(U, V) \) is indeed a 2-cell
(i.e. the required diagrams commute).

It’s easy to verify the first. Indeed:

\[
\begin{align*}
g_2 \circ f_2 &= \oplus \circ (g_1, U) \circ f_2 \\
&= \oplus \circ (g_1 \circ f_2, V \circ f_2) \\
&= \oplus \circ (g_1 \circ g_2 \circ f_2, V \circ f_2) \\
&= \oplus \circ (\partial g_1 \circ (f_1, U), V \circ f_2)
\end{align*}
\]

For the second:

\[
+ \circ (\text{Comp}(U, V) \circ \pi_1, \partial (g_2 \circ f_2))
\]

\[
= + \circ (+ \circ (\partial g_1 \circ (f_1, U), V \circ f_2) \circ \pi_1, \partial g_2 \circ (f_2 \circ \pi_1, \partial f_1))
\]

\[
= + \circ (\partial g_1 \circ (f_1, U) \circ \pi_1, + \circ (V \circ f_2 \circ \pi_1, \partial g_2 \circ (f_2 \circ \pi_1, \partial f_2))
\]

\[
= + \circ (\partial g_1 \circ (f_1, U) \circ \pi_1, + \circ (\partial g_1 \circ (f_1, U), V \circ f_2 \circ \pi_1, \partial g_2 \circ (f_2 \circ \pi_1, \partial f_2))
\]

\[
= + \circ (\partial g_1 \circ (f_1, U) \circ \pi_1, + \circ (\partial g_1 \circ (f_1, U), V \circ \oplus \circ (f_2 \circ \pi_1, \partial f_2))
\]

\[
= + \circ (+ \circ (\partial g_1 \circ (f_1, U) \circ \pi_1, \partial g_1 \circ (f_2 \circ \pi_1, \partial f_2)), V \circ \oplus \circ ((f_2 \circ \pi_1, \partial f_2))
\]

\[
= + \circ (+ \circ (\partial g_1 \circ (f_1, U) \circ \pi_1, \partial g_1 \circ (+(\circ (f_1, U), V \circ \pi_1, \partial f_2)), V \circ \oplus \circ ((f_2 \circ \pi_1, \partial f_2))
\]

(by regularity of \( \partial g_1 \))

\[
= + \circ (\partial g_1 \circ (f_1 \circ \pi_1, + \circ (U \circ \pi_1, \partial f_2)), V \circ \oplus \circ ((f_2 \circ \pi_1, \partial f_2))
\]

(by regularity of \( \partial g_1 \) and reassociating)

\[
= + \circ (\partial g_1 \circ (f_1 \circ \pi_1, \partial f_1), + \circ (\partial g_1 \circ (\partial f_1, U \circ \oplus)), V \circ \oplus \circ ((f_2 \circ \pi_1, \partial f_2))
\]

\[
= + \circ (\partial g_1 \circ (f_1 \circ \pi_1, \partial f_1), + \circ (\partial g_1 \circ (\partial f_1, U \circ \oplus)), V \circ f_2 \circ \oplus)
\]

\[
= + \circ (\partial g_1 \circ (f_1 \circ \pi_1, \partial f_1), \text{Comp}(U, V) \circ \oplus)
\]

That \( \text{Comp} \) is a functor follows straightforwardly from monoidality. 
\( \square \)
Theorem F.3. The category $\mathbf{CAct}(\mathbf{C})$ of change actions on a base category $\mathbf{C}$ is a 2-category with the structure described above.