BOUNDS BETWEEN LAPLACE AND STEKLOV EIGENVALUES
ON NONNEGATIVELY CURVED MANIFOLDS

MIKHAIL KARPUKHIN

(Communicated by Tobias Colding)

Abstract. Consider a compact Riemannian manifold with boundary. In this short note we prove that under certain positive curvature assumptions on the manifold and its boundary the Steklov eigenvalues of the manifold are controlled by the Laplace eigenvalues of the boundary. Additionally, in two dimensions we obtain an upper bound for Steklov eigenvalues in terms of topology of the surface without any curvature restrictions.

1. Introduction and main results

Let \((M, g)\) be a smooth compact Riemannian manifold with smooth boundary. We consider two elliptic self-adjoint operators defined on \(\partial M\). The first is the usual Laplace-Beltrami operator \(\Delta = \Delta_{\partial M}\) acting on \(C^\infty(\partial M)\) with respect to the induced metric \(g|_{\partial M}\). Since \(\partial M\) is compact, the spectrum of \(\Delta\) is discrete and consists of eigenvalues which we denote by

\[0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots,\]

where eigenvalues are counted with multiplicities. Note that \(\partial M\) is not necessarily connected, so the eigenvalue 0 might not be simple. That is the reason why we start our numeration with \(\lambda_1\) as opposed to \(\lambda_0\).

The second operator is the Dirichlet-to-Neumann operator \(D\), which is defined in the following way. For any \(u \in C^\infty(\partial M)\), there is a unique harmonic extension \(\hat{u} \in C^\infty(M)\), i.e., there is a unique \(\hat{u}\) such that \(\Delta_M \hat{u} = 0\) and \(\hat{u}|_{\partial M} = u\). Then one defines \(D(u) = \partial_n \hat{u}\), where \(n\) is an outward unit normal vector to \(\partial M\). Similar to the Laplacian, the operator \(D\) is an elliptic self-adjoint operator with discrete spectrum. We denote the corresponding sequence of eigenvalues by

\[0 = \sigma_1 \leq \sigma_2 \leq \sigma_3 \leq \cdots,\]

where eigenvalues are counted with multiplicities. Note again that the numeration starts with \(\sigma_1\) as opposed to \(\sigma_0\). The numbers \(\sigma_i\) and the sequence \(\{\sigma_i\}\) are }

©2016 American Institute of Mathematical Sciences
sometimes referred to as Steklov eigenvalues of $M$ and Steklov spectrum of $M$
respectively. The study of Steklov spectrum has become rather popular in the recent
years, see, e.g., survey paper [6] and references therein.

The aim of this paper is to show that under certain curvature restrictions on $M$ and $\partial M$ the sequence $\{\sigma_i\}$ is controlled from above by the sequence $\{\lambda_i\}$. The
precise statement is the following.

**Theorem 1.1.** Let $(M,g)$ be a compact Riemannian manifold of dimension $n \geq 3$
with boundary. Suppose that the Weitzenböck curvature on 2-forms $W^{[2]}$ is
non-negative and the lowest $(n-2)$-curvature $c_{n-2}$ of $\partial M$ is bounded from below by a
positive constant $c$. Then, if $n \geq 4$, for all $m \geq 1$ one has the inequality

$$\sigma_m \leq \frac{n-2}{(n-1)c} \lambda_m. \quad (1)$$

Moreover, if $n=3$ then for all $m > b_0(M)$

$$\sigma_m < \frac{2}{3c} \lambda_m,$$

where $b_0(M)$ is the number of connected components of $M$.

In Section 2 we define the Weitzenböck curvature on $p$-forms and the lowest
$p$-curvatures of the boundary for all $p=1,\ldots,n$. For now, let us mention that
convexity of $\partial M$ is sufficient to guarantee $\lambda_p \geq pc > 0$ for all $p$, where $c > 0$
is a lower bound for principal curvatures of the boundary. At the same time, non-
negativity of the curvature operator suffices for non-negativity of $W^{[p]}$ for all $p$.

**Remark 1.2.** Bound (1) is sharp for $m=2$ on a Euclidean ball $B^n(\frac{1}{r})$ of radius $\frac{1}{r}$.
Indeed, on the standard sphere $S^{n-1}(\frac{1}{r})$ of radius $\frac{1}{r}$ all principal curvatures are
constant and equal to $c$, therefore, $c_{n-2} \equiv (n-2)r$ (see definition in Section 2).
Additionally, it is well-known that $\sigma_2(S^{n-1}(\frac{1}{r})) = r$, while $\lambda_2(S(\frac{1}{r})) = (n-1)r^2$.
Combining these observations, one obtains an equality in (1) with $c = (n-2)r$.
Moreover, both these eigenvalues have multiplicity $n$, therefore, we have that
inequality (1) is sharp for all $m \leq n+1$. However, as it follows from the remark after
Theorem 3.2, the equality in inequality (1) can only be achieved on a ball, i.e., for
$m > n+1$ inequality (1) is strict.

**Remark 1.3.** For a fixed manifold $M$ bound of the type (1), i.e., inequality bounding $\sigma_m$ in terms of a linear function of $\lambda_m$, can not possibly be sharp for all $m$.
Indeed, according to Weyl’s law (see, e.g., [6]) $\sigma_m \sim m^{\frac{1}{n-1}}$, whereas $\lambda_m \sim m^{\frac{n-2}{n-1}}$.

A curious feature of the proof is that it uses the extension of the Dirichlet-to-
Neumann map to the space of differential forms of higher degree defined by Raulot
and Savo [14], whereas the final statement does not refer to differential forms.

In case $M$ is a surface we have the following theorem.

**Theorem 1.4.** Let $(M,g)$ be a connected oriented Riemannian surface. Then one
has

$$\sigma_{p+1} \sigma_{q+1} L^2(\partial M) \leq \begin{cases} \pi^2(p+q+2\gamma+2k-3)^2 & \text{if } p+q \equiv 1 \pmod{2}, \\ \pi^2(p+q+2\gamma+2k-2)^2 & \text{if } p+q \equiv 0 \pmod{2}, \end{cases} \quad (2)$$

where $L(\partial M)$ is the length of $\partial M$, $k$ is the number of boundary components, and
$\gamma$ is the genus of $M$. In particular, when $p=q$ one obtains

$$\sigma_{p+1} L(\partial M) \leq 2\pi(p+\gamma+k-1). \quad (3)$$
Remark 1.5. Theorem 1.4 is a consequence of Theorem 3.1 below. As it is explained in [19], Theorem 3.1 can be seen as a generalisation of Hersch-Payne-Schiffer inequality [9], which is inequality (2) for simply connected planar domains, i.e., $k = 1, \gamma = 0$. Theorem 1.4 makes the connection between Theorem 3.1 and Hersch-Payne-Schiffer inequality more evident. A particular case $k = 1$ of inequality (2) have already been pointed out in [19]. A similar inequality was obtained by Girouard and Polterovich in [7]:

$$
\sigma_{p+1}L^2(\partial M) \leq \begin{cases} 
\pi^2(p + q - 1)^2(\gamma + k)^2 & \text{if } p + q \equiv 1 \\
\pi^2(p + q)^2(\gamma + k)^2 & \text{if } p + q \equiv 0 \pmod{2}.
\end{cases}
$$

However, one can easily see that inequality (2) yields a better upper bound unless either $\gamma = 0$, $k = 1$, or $(p, q) = (1, 1), (2, 1)$, when both inequalities yield the same bound. Let us also mention a remarkable series of papers by Fraser and Schoen [3, 4], where the authors investigate the optimal upper bound of the form (3) for $p = 1$. Note that in the mentioned papers the enumeration of Steklov eigenvalues starts with $\sigma_0$, which is compensated by the fact that we have $p + 1$ and $q + 1$ in the left hand side of (2).

Remark 1.6. Bound (3) has an advantage of being linear in all the parameters involved. In fact, Hassanezhad [8] proved that there exists a bound of the form

$$
\sigma_{p+1}L(\partial M) \leq A\gamma + Bp
$$

with implicit universal constants $A$ and $B$. Thus, by introducing linear dependence on $k$ we are able to make the constants in the above bound explicit. Let us also mention that for $p = 1$, Kokarev proved in [12] an upper bound

$$
\sigma_2L(\partial M) \leq 8\pi \left[ \frac{\gamma + 3}{2} \right].
$$

The paper is organized as follows. Section 2 contains necessary background from differential geometry. In Section 3 we describe the extension of the operator $D$ to differential forms due to Raulot and Savo. Theorem 1.4 is proved in Section 4. We prove Theorem 1.1 for orientable manifolds in Section 5 and in the subsequent section we show how to generalize this proof to non-orientable manifolds. Finally, in the last section, we compare our results to a similar result due to Wang and Xia [17].

2. Background in differential geometry

Let $\nabla$ denote the Levi-Civita connection on $M$ associated to the metric $g$. The operator $\nabla$ and the metric $g$ have natural extensions to the bundle $\bigwedge^p(T^*M)$ of differential $p$-forms on $M$ for all $p = 1, \ldots, n$, which will be denoted by the same letters. Let $\nabla^*$ be the formal adjoint of $\nabla$. Then the Weitzenböck curvature $W[p]$ on $p$-forms is defined via the Bochner formula

$$
\Delta_M\omega = \nabla^*\nabla\omega + W[p]\omega,
$$

where $\omega \in \Omega^p(M) = \Gamma(\bigwedge^p(T^*M))$ and $\Delta_M$ is the Hodge Laplacian on $\Omega^p(M)$.

According to [13] the condition $W^{[2]} \geq 0$ can be expressed in terms of Riemann curvature tensor $R(\cdot, \cdot, \cdot, \cdot)$. In fact, non-negativity of $W^{[2]}$ is equivalent to requiring non-negativity of the following expressions:
By summing inequality (5) over tangent space of $p$ all that the non-negativity of the curvature operator implies non-negativity of $p$ defined as eigenvalues of the second fundamental form of $\omega$, e.g., $[1, 10, 11, 14, 16]$. Here we discuss the one due to Raulot and Savo. For $\partial M$ equivalent to the convexity of $\{D_i\}_{i=1,\ldots,n-1}$ where eigenvalues are written with multiplicities. Since $\lambda_1 > 0$ is equivalent to the convexity of $\partial M$ and $c_{n-1}$ is proportional to the mean curvature of $\partial M$.

### 3. Dirichlet-to-Neumann operator on differential forms

There are several definitions of Dirichlet-to-Neumann operator on forms, see, e.g., $[1, 10, 11, 14, 16]$. Here we discuss the one due to Raulot and Savo. For $\omega \in \Omega^p(\partial M)$ there exists a unique differential form $\hat{\omega} \in \Omega^p(M)$, see, e.g., $[15]$, such that

$$\Delta \hat{\omega} = 0, \quad \hat{\omega}|_{\partial M} = \omega, \quad i_n \hat{\omega} = 0,$$

where $i_n \hat{\omega}$ stands for contraction of $\hat{\omega}$ with the unit outer normal vector field $n$, i.e., $\hat{\omega}(\cdot, \cdot, \ldots, \cdot) = \hat{\omega}(n, \cdot, \ldots, \cdot)$, which (since $n$ is only defined on the boundary) yields a well-defined element of $\Omega^{p-1}(\partial M)$. Then the Dirichlet-to-Neumann operator $D^{(p)}$ on $\Omega^p(\partial M)$ is defined as $D^{(p)}\omega = i_n d\hat{\omega}$. Raulot and Savo proved that $D^{(p)}$ is a positive elliptic self-adjoint operator and therefore its spectrum consists of a sequence of eigenvalues

$$0 \leq \sigma_1^{(p)} \leq \sigma_2^{(p)} \leq \sigma_3^{(p)} \leq \cdots,$$

where eigenvalues are written with multiplicities. Since $D^{(0)} = D$, we sometimes interchange notations $\sigma_i^{(0)}$ and $\sigma_i$ for the $i$th eigenvalue of the Dirichlet-to-Neumann operator on $C^\infty(\partial M)$. As in the case of $D$, the numbers $\sigma_i^{(p)}$ are sometimes referred to as Steklov eigenvalues of $M$ on differential $p$-forms.

This particular definition of $D^{(p)}$ is of interest to us due to the following two theorems.

**Theorem 3.1** (Yang, Yu [19]). Let $(M, g)$ be a compact oriented $n$-dimensional Riemannian manifold with nonempty boundary. Let $\sigma_m^{(p)}$ be the $m$th Steklov eigenvalue on differential $p$-forms of $M$ and $\lambda_m$ be the $m$th eigenvalue for the Laplacian operator on $\partial M$. Then for any two positive integers $m$ and $r$, one has

$$\sigma_m^{(0)} \sigma_m^{(n-2)} \lambda_m \leq \sigma_m^{(p)} \leq \lambda_m + r b_{n-2} f_{n-1} - 1,$$

where $b_k$ is the $k$th absolute Betti number of $M$. 

\[\Box\]
Thus, one can rewrite inequality (9) in terms of 
\[ \sigma_1^{(p)} \geq \frac{n - p + 1}{n - p} c. \] (7)

If \( p \geq \frac{n}{2} \) one has
\[ \sigma_1^{(p)} \geq \frac{p + 1}{p} c. \] (8)

\textbf{Remark 3.3.} As it was pointed out in [14], \( \dim \ker D^{(p)} = b_p \). Since \( \sigma_1^{(p)} > 0 \) iff \( \dim \ker D^{(p)} = 0 \), it implies that under assumptions of the theorem one has \( b_p = 0 \).

\textbf{Remark 3.4.} In the same paper [14] it is also proved that equality in (8) implies that \( M \) is a Euclidean ball of radius \( \frac{2}{c} \).

4. Proof of Theorem 1.4

For a connected oriented surface \( n = 2 \) inequality (6) with \( m = p + 1 \) and \( r = q \) reads
\[ \sigma_{p+1}\sigma_{q+1} \leq \lambda_{p+q+(k+2\gamma-1)}, \] (9)

where \( k \) is the number of boundary components of \( M \) and \( \gamma \) is the genus of \( M \). In this section we obtain an upper bound for the left hand side independent of the spectrum of the boundary.

Let us introduce some notation. For each vector \( l = (l_1, \ldots, l_k) \in (\mathbb{R}_{>0})^k \) the sequence \( S_l \) is defined in the following way. For each \( i = 1, \ldots, k \) consider an arithmetic progression \( T_i = \left\{ \frac{d}{r_i} \right\}_{d=1}^{\infty} \). Then \( S_l \) is a nondecreasing sequence obtained by taking the union of sequences \( T_i \) and reordering the entries. The sequence \( S_l \) is closely related to the spectrum of \( \partial M \), see, e.g., [5]. Eigenfunctions of Laplacian of a single boundary component of length \( l_i \) are given by \( \{1, \sin(\frac{2\pi d}{l_i}), \cos(\frac{2\pi d}{l_i})\} \) with \( d \geq 1 \), i.e., the spectrum of that component is given by \( \{0\} \cup 4\pi^2 T_i^2 \cup 4\pi^2 T_i^2 \) (an arithmetic operation applied to a sequence denotes an application of that operation to each entry individually). Spectrum of a manifold is a union of spectra of its connected components, i.e., the spectrum of \( \partial M \) is given by
\[ \{0, \ldots, 0\} \cup 4\pi^2 S_l^2 \cup 4\pi^2 S_l^2. \]

Thus, one can rewrite inequality (9) in terms of \( S_l \)
\[ \sigma_{p+1}\sigma_{q+1} \leq \begin{cases} 4\pi^2 S_l \left[ \frac{p+q}{2} + \gamma \right]^2 & \text{if } p + q \equiv 1 \pmod{2}, \\ 4\pi^2 S_l \left[ \frac{p+q}{2} + \gamma \right]^2 & \text{if } p + q \equiv 0 \pmod{2}. \end{cases} \] (10)

where \( A[n] \) denotes the \( n^{th} \) entry of a sequence \( A \). Denoting by \( |l| \) the \( L^1 \) norm of \( l \), i.e., \( |l| = l_1 + \ldots + l_k \), inequality (10) implies
\[ \sigma_{p+1}\sigma_{q+1}|l|^2 \leq \begin{cases} 4\pi^2 \sup_l \left( |l| S_l \left[ \frac{p+q}{2} + \gamma \right] \right)^2 & \text{if } p + q \equiv 1 \pmod{2}, \\ 4\pi^2 \sup_l \left( |l| S_l \left[ \frac{p+q}{2} + \gamma \right] \right)^2 & \text{if } p + q \equiv 0 \pmod{2}. \end{cases} \] (11)

Thus, in order to prove Theorem 1.4 it is sufficient to prove the following proposition.
Proposition 4.1. For any $d \in \mathbb{N}$ one has
\[
\sup_l |S_l[d]| = d + k - 1.
\]

Proof. Let us first assume that there exists a vector $l = (l_1, \ldots, l_k)$ maximising the function $|l| S_l[d]$ and let $M$ denote the corresponding maximal value.

Claim 1. For each $i = 1, \ldots, k$ there is $d_i \in \mathbb{N}$ such that $\frac{d_i}{l_i} = \frac{M}{|l|}$. Indeed, let $d_i$ be the smallest integer, such that $\frac{d_i}{l_i} > \frac{M}{|l|}$. Assume that $\frac{d_i}{l_i}$ is strictly greater than $\frac{M}{|l|}$. We will show that in that case $M$ is not a maximum. Indeed, by definition of $d_i$ it is easy to see that
\[
S_{(l_1, \ldots, l_i, \ldots, l_k)}[d] = S_{(l_1, \ldots, \frac{d_i}{l_i} \ldots, l_k)}[d] = \frac{M}{|l|}.
\]
At the same time, $\frac{d_i|l|}{M} > l_i$, therefore the replacement $l_i \mapsto \frac{d_i|l|}{M}$ increases the value of the function $|l| S_l[d]$. We arrive at a contradiction.

Claim 2. $S_{(l_1, \ldots, l_k)}[d] > S_{(l_1, \ldots, l_k)}[d - 1]$. The proof of this claim is similar to the previous one. Indeed, by previous claim $l_1 = \frac{d_1|l|}{M}$. Then, argue by contradiction, one has $S_{l'}[d] = S_{l'[d - 1]} = S_{(l'[d], \ldots, l_k)}[d]$, where $l'_1 = \frac{(d_1 + 1)|l|}{M}$. Once again $l'_1 > l_1$ and we arrive at a contradiction. The component $l_1$ was chosen only for the notation convenience, the same argument follows through for any $l_i$.

According to the first claim, $l_i = \frac{d_i|l|}{M}$. Hence, for each $i$ the sequence $T_i$ has exactly $d_i - 1$ entries less than $\frac{M}{l_i}$ below. Therefore, the sequence $S_t$ contains $\sum(d_i - 1)$ entries less than $\frac{M}{l_i}$. Since $S_t[d] = \frac{M}{l_i}$, Claim 2 implies
\[
d = \sum_{i=1}^k (d_i - 1) + 1.
\]
At the same time, $\sum l_i = |l|$, which implies $\sum d_i = M$. Substituting this into equality (12) one obtains $M = d + k - 1$. Additionally, note that this argument provides an explicit formula for all maximizers. Namely, any maximizer $l$ should have the form
\[
l = \frac{|l|}{d + k - 1}(d_1, \ldots, d_k),
\]
where $d_i \in \mathbb{N}$ are such that $\sum d_i = d + k - 1$. It is easy to see that any such vector satisfies $|l| S_l[d] = d + k - 1$. Regardless of the existence of maximizer the arguments above prove that
\[
\sup_l |l| S_l[d] \geq d + k - 1,
\]
where one has equality provided the existence of a maximizer.

Let us turn to proving that the supremum is achieved. First, note that for any positive constant $\lambda > 0$, one has $S_M = \frac{1}{\lambda} S_l$, therefore the function $|l| S_l[d]$ is scale-invariant for any $d$. Therefore, without loss of generality we may assume that $|l| = l_1 + \cdots + l_k = 1$. Let $L_k$ denote the space of vectors $l \in \mathbb{R}^k$ with strictly positive components such that $|l| = 1$. The function $S_l[d]$ is obviously continuous on $L_k$, but $L_k$ is not compact. To circumvent this difficulty one needs to control the behaviour of the functional $S_l[d]$ near the boundary $\partial L_k$. To that end it is useful to note that $S_l[d]$ is bounded as it follows from the inequality $S_l[d] \leq T_l[d] = \frac{d}{l}$ which holds for all $i$. Since $|l| = 1$ there exists an index $i$ with $l_i \geq \frac{1}{k}$ and, therefore,
\[
S_l[d] \leq kd.
\]
The space $L_k$ is an interior of a $(k-1)$-simplex whose boundary is a union of faces $L_k^{(i)} = \{ l \mid l_i = 0 \}$. Consider an open neighbourhood $U^{(i)} \subset L_k$ of $L_k^{(i)}$ defined by $U^{(i)} = \{ l \mid 0 < l_i < \frac{1}{kd} \}$. Then one has $T_i[l] > kd \geq S_l[d]$ by inequality (14). Thus, the presence of that $l_i$ does not affect $S_l[d]$ and setting $l^{(i)} = (l_1, \ldots, \hat{l}_i, \ldots, l_k) \in \mathbb{R}^{k-1}$ (here we use the hat symbol to denote the omission) one has the following inequality for all $l \in U^{(i)}$:

$$
S_l[d] = S_{l^{(i)}}[d] = \frac{1}{1-l_i}(|l^{(i)}|S_{l^{(i)}}[d]) < \frac{kd}{kd - 1}(|l^{(i)}|S_{l^{(i)}}[d]).
$$

(15)

Now we are ready to complete the proof. We proceed by induction on the number of boundary components $k$. The base $k = 1$ is obvious since in that case the space $L_1$ is a single point. The step of induction guarantees that for $l' \in (\mathbb{R}_{>0})^{k-1}$ one has $\sup |l'|S_{l'}[d] = d + k - 2$. Combining it with the right hand side of inequality (15) one obtains that for $l \in U^{(i)}$ the following inequality holds:

$$
S_l[d] < \frac{kd}{kd - 1}(k + d - 2) \leq k + d - 1 \leq \sup_{l \in (\mathbb{R}_{>0})^k} |l| S_l[d] = \sup_{l \in L_k} S_l[d],
$$

where we used inequality (13) and obvious observations $d \geq 1$ and $k \geq 2$. Therefore, the supremum of $S_l[d]$ over $L_k$ coincides with the supremum of $S_l[d]$ over compact subset $L_k \cup \bigcup_{i=1}^{k} U^{(i)}$, i.e., supremum is achieved and is equal to $d + k - 1$ by inequality (13).

Note that the above proposition is purely algebraic. Nevertheless, as a corollary we obtain Theorem 1.4.

5. Proof of Theorem 1.1 in the orientable case

The proof is obtained by an easy combination of Theorem 3.1 and Theorem 3.2. First of all, let us make a couple of observations.

**Observation 1.** Applying Hodge $*$-operator to both sides of formula (4) implies that $W^{[n-p]} = *W^{[p]}$. Indeed, both $\Delta$ and $\nabla^*\nabla$ are formally self-adjoint, which implies that they commute with $*$, therefore so does $W$. Thus, non-negativity of $W^{[p]}$ is equivalent to non-negativity of $W^{[n-p]}$.

**Observation 2.** Application of Theorem 3.2 for $p = n - 2$ now yields that under the conditions of Theorem 1.1 one has the inequality

$$
\sigma_1^{(n-2)} \geq \frac{n - 1}{n - 2} c
$$

if $n \geq 4$ and

$$
\sigma_1^{(1)} \geq \frac{3}{2} c
$$

if $n = 3$. In particular, by the remark after Theorem 3.2, it yields $b_{n-2} = 0$. Similarly, since $W^{[2]} \geq 0$ implies $Ric = W^{[1]} \geq 0$ and $c_{n-2} > 0$ implies $c_{n-1} > 0$, we can apply Theorem 3.2 for $p = n - 1$ and conclude that $b_{n-1} = 0$.

Taking into account $b_{n-2} = b_{n-1} = 0$, an application of Theorem 3.1 for $q = 1$ yields the inequality

$$
\sigma_m \sigma_1^{(n-2)} \leq \lambda_m.
$$

Combining this inequality with inequalities from Observation 2, one completes the proof of Theorem 1.1 for orientable manifolds.
6. Non-orientable manifolds

It is possible to generalize Theorem 3.1 to include the case of non-orientable manifolds in a way that allows us to apply the arguments of the previous section. Below we give the statement and the short outline of the argument.

Let us pass to an orientable double cover \( \tilde{M} : \tilde{M} \to M \) and endow it with the metric \( \pi^*g \) such that the involution \( \tau \) interchanging the leaves of \( \pi \) is an isometry. Then \( \tau \) induces the decomposition of differential forms on \( \tilde{M} \) into even and odd with respect to \( \tau \) which is compatible with the Hodge-Morrey decomposition (for details on the Hodge-Morrey decomposition see [19]). Similarly, the eigenvalues of Dirichlet-to-Neumann and Laplace operators are divided into those corresponding to odd and even eigenforms respectively. If we denote by \( \lambda_{1,\text{even}} \) the \( i \)th even eigenvalue of \( \Delta \) on \( C^\infty(\partial M) \) and similarly by \( \lambda^{(p)}_{j,\text{odd}} \) the \( j \)th odd Steklov eigenvalue on \( \Omega^p(\partial \tilde{M}) \), then inequality (6) for \( \tilde{M} \) becomes

\[
\sigma^{(0)}_{m,\text{even}} \lambda^{(n-2)}_{b_n-2(M) - b_{n-2}(M) + r,\text{odd}} \leq \lambda_{m+r+b_{n-1}(M)-1,\text{even}}.
\] (16)

In order to prove (16) one follows the proof of Theorem 3.1 accounting for the presence of \( \tau \). The key points are the following:

- differential \( d \) is compatible with the decomposition into odd and even forms;
- Hodge star operator sends odd forms to even and vice versa;
- Since the kernel of \( D^{(p)} \) consists of Neumann harmonic \( p \)-fields and even Neumann harmonic fields on \( \tilde{M} \) are the pullbacks of Neumann harmonic fields on \( M \), then \( \dim(\ker D^{(p)} \cap \Omega^n_{\text{even}}) = b_p(M) \), \( \dim(\ker D^{(p)} \cap \Omega^n_{\text{odd}}) = b_p(\tilde{M}) - b_p(M) \).

To complete the proof of Theorem 1.1 one makes the following two observations. First, local curvature conditions on \( M \) pass to \( \tilde{M} \), therefore Betti numbers disappear from the inequality (16). Second, there is an obvious inequality \( \sigma^{(n-2)}_1 \geq \sigma^{(n-2)}_1 \). Since even eigenvalues of \( \tilde{M} \) coincide with the eigenvalues of \( M \), the same arguments as in Section 5 conclude the proof.

7. Comparison with earlier results

In this section we compare results of Theorem 1.1 with the following theorem.

**Theorem 7.1** (Wang, Xia [17]). Let \( (M, g) \) be an \( n \)-dimensional compact connected Riemannian manifold with non-negative Ricci curvature and boundary \( \partial M \). Assume that the principal curvatures of \( \partial M \) are bounded from below by a positive constant \( c \). Then one has the inequality

\[
\sigma_2(M) \leq \frac{\sqrt{\lambda_2(\partial M)}}{(n-1)c} \left( \sqrt{\lambda_2(\partial M)} + \sqrt{\lambda_2(\partial M) - (n-1)c^2} \right).
\] (17)

We start our comparison with the case \( n = 3 \), where conditions of our theorem coincide with conditions of Theorem 7.1 (since \( W^{[1]} = \ast W^{[2]} \ast = Ric \)). Theorem 1.1 for \( m = 2 \) and \( n = 3 \) yields

\[
\sigma_2(M) < \frac{2}{3c} \lambda_2(\partial M),
\] (18)

while inequality (17) for \( n = 3 \) becomes

\[
\sigma_2(M) \leq \frac{\sqrt{\lambda_2(\partial M)}}{2c} \left( \sqrt{\lambda_2(\partial M)} + \sqrt{\lambda_1(\partial M) - 2c^2} \right).
\] (19)
It is easy to see now that the inequality (18) yields a better bound once $\lambda_2(\partial M) \geq \frac{9}{4} c^2$.

**Example 7.2.** Let $M = \{(x, y, z) \in \mathbb{R}^3 | \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\}$. Then $\text{Ric}(M) = 0$ and $M$ is convex, i.e., it satisfies the conditions of the theorem. If $a > b > c$ the lower bound for principal curvatures is $\frac{c}{a^2}$, while the lower bound $K_0$ for Gaussian curvature of $\partial M$ is $\frac{c^2}{a^2 b^2}$. Using the classical bound $\lambda_2(\partial M) \geq 2K_0$, see, e.g., \cite[2, p. 186]{2}, one sees that the bound given by Theorem 1.1 is better once $b^2 \leq \frac{8}{9} a^2$, i.e., once the ellipsoid is oblong enough.

In the case $n \geq 4$ condition $W^{[2]} \geq 0$ implies condition $\text{Ric} \geq 0$ of Theorem 7.1. At the same time, the assumption "principle curvatures are bounded from below by $c > 0$" of Theorem 7.1 implies the assumption "the lowest $(n-2)$-curvature is bounded from below by $(n-2)c$". Thus, while the condition of Theorem 1.1 on the interior curvature is stronger, the condition on the curvature of $\partial M$ is weaker. Suppose that $n \geq 4$ and $(M, g)$ satisfies the conditions of both Theorem 1.1 and Theorem 7.1, i.e., $W^{[2]} \geq 0$ and all principle curvatures of $\partial M$ are bounded from below by $c > 0$, then the lowest $(n-2)$-curvature is bounded from below by $(n-2)c$. An application of Theorem 7.1 yields inequality (17), which can be rewritten as

$$\sigma_2 \leq \frac{\lambda_2}{(n-1)c} + \frac{\sqrt{\lambda_2 (\lambda_2 - (n-1)c^2)}}{(n-1)c},$$

while Theorem 1.1 for $m = 1$ gives

$$\sigma_2 \leq \frac{\lambda_2}{(n-1)c},$$

which is clearly stronger. The right hand sides of those inequalities differ by an expression, which is equal to zero iff $\lambda_2 = (n-1)c^2$. According to the results of Xia \cite{18} the latter happens only if $M$ is a Euclidean ball of radius $\frac{1}{c}$. Therefore, our estimate yields a better bound for any convex bounded domain of the Euclidean space other than a ball. Moreover, our estimate can also be applied to non-convex domains, since our condition on the curvature of $\partial M$ is only $c_{n-2} > 0$.

**References**

\cite{1} M. Belishev and V. Sharafutdinov, Dirichlet to Neumann operator on differential forms, *Bull. Sci. Math.*, 132 (2008), 128–145. MR 2387822

\cite{2} I. Chavel, *Riemannian Geometry, A Modern Introduction*, 2nd edition, Cambridge University Press, New York, 2006. MR 2229062

\cite{3} A. Fraser and R. Schoen, The first Steklov eigenvalue, conformal geometry, and minimal surfaces, *Adv. Math.*, 226 (2011), 4011–4030. MR 2770439

\cite{4} A. Fraser and R. Schoen, Sharp eigenvalue bounds and minimal surfaces in the ball, *Inventiones mathematicae*, 203 (2016), 823–890. MR 3461367

\cite{5} A. Girouard, L. Parnovski, I. Polterovich and D. Sher, The Steklov spectrum of surfaces: Asymptotics and invariants, *Math. Proc. Camb. Phil. Soc.*, 157 (2014), 379–389. MR 3286514

\cite{6} A. Girouard and I. Polterovich, Spectral geometry of the Steklov problem, *J. of Spectral Theory*, 7 (2017), 321–359. MR 3662010

\cite{7} A. Girouard and I. Polterovich, Upper bounds for Steklov eigenvalues on surfaces, *Electron. Res. Announc. Math. Sci.*, 19 (2012), 77–85. MR 2970718

\cite{8} A. Hassannezhad, Conformal upper bounds for the eigenvalues of the Laplacian and Steklov problem, *J. of Functional Analysis*, 261 (2011), 3419–3436. MR 2838029

\cite{9} J. Hersch, L. Payne and M. Schiffer, Some inequalities for Stekloff eigenvalues, *Arch. Rational Mech. Anal.*, 57 (1975), 99–114. MR 0387837
[10] M. S. Joshi and W. R. B. Lionheart, An inverse boundary value problem for harmonic differential forms, *Asymptot. Anal.*, 41 (2005), 93–106. MR 2129227

[11] M. Karpukhin, Steklov problem on differential forms, Preprint, arXiv:1705.08951.

[12] G. Kokarev, Variational aspects of Laplace eigenvalues on Riemannian surfaces, *Adv. Math.*, 258 (2014), 191–239. MR 3190427

[13] P. Petersen, *Demystifying the Weitzenböck curvature operator*, Preprint available from: http://www.math.ucla.edu/~petersen/.

[14] S. Raulot and A. Savo, On the first eigenvalue of the Dirichlet-to-Neumann operator on forms, *J. of Functional Analysis*, 262 (2012), 889–914. MR 2863852

[15] G. Schwarz, *Hodge Decomposition – A Method for Solving Boundary Value Problems*, Lecture Notes in Math., Springer, 1995. MR 1367287

[16] V. Sharafutdinov and C. Shonkwiler, The complete Dirichlet-to-Neumann map for differential forms, *J. Geom. Anal.*, 23 (2013), 2063–2080. MR 3107691

[17] Q. Wang and C. Xia, Sharp bounds for the first non-zero Stekloff eigenvalues, *J. of Functional Analysis*, 257 (2009), 2635–2644. MR 2555007

[18] C. Xia, Rigidity for compact manifolds with boundary and non-negative Ricci curvature, *Proc. Amer. Math. Soc.*, 125 (1997), 1801–1806. MR 1415343

[19] L. Yang and C. Yu, A higher dimensional generalization of Hersch-Payne-Schiffer inequality for Steklov eigenvalues, *J. of Functional Analysis*, 272 (2017), 4122–4130. MR 3626035