On \( n \)-connected minors of the \( es \)-splitting binary matroids

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Abstract

The \( es \)-splitting operation on an \( n \)-connected binary matroid may not yield an \( n \)-connected matroid for \((n \geq 3)\). In this paper, we show that given an \( n \)-connected binary matroid \( M \) of rank \( r \), the resulting \( es \)-splitting binary matroid has an \( n \)-connected minor of rank-\((r+1)\) having \(|E(M)|+1\) elements.

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1. Introduction

Slater [13] specified the \( n \)-line splitting operation on graphs as follows. Let \( G \) be a graph and \( e = uv \) be an edge of \( G \) with \( \deg u \geq 2n - 3 \) with \( u \) adjacent to \( v, x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_h \), where \( k \) and \( h \geq n - 2 \). Let \( H \) be the graph obtained from \( G \) by replacing \( u \) by two adjacent vertices \( u_1 \) and \( u_2 \), with \( v \) adj \( u_1, v \) adj \( u_2, u_1 \) adj \( x_i \) (\( 1 \leq i \leq k \)), and \( u_2 \) adj \( y_j \) (\( 1 \leq j \leq h \)), where \( \deg u_1 \geq n \) and \( \deg u_2 \geq n \). The transition from \( G \) to \( H \) is called an \( n \)-line splitting operation. We also say that \( H \) is obtained from \( G \) by an \( n \)-line splitting operation. This construction is explicitly illustrated with the help of Figure 1.

Slater [13] proved that if \( G \) is \( n \)-connected and \( H \) is obtained from \( G \) by \( n \)-line-splitting operation, then \( H \) is \( n \)-connected. In fact, he characterized 4-connected graphs, in terms of the 4-line
splitting operation along with some other operations. The notion of connectivity of graphs also has been studied in [6, 13] and connectivity of binary matroids has been studied in [3, 14].

Suppose $G$ is a graph with $n$ vertices and $m$ edges. Let $X = \{e, x_1, x_2, \ldots, x_k\}$ be a subset of $E(G)$. The incident matrix $A$ of $G$ is a matrix of size $n \times m$. The row corresponding to the vertex $u$ has 1 in the columns of $e, x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_h$ and 0 in the other columns. The graph $H$ has $(n + 1)$ vertices and $(m + 2)$ edges. The incidence matrix $A'$ of $H$ is a matrix of size $(n + 1) \times (m + 2)$. The row corresponding to $u_2$ has 1 in the columns of $y_1, y_2, \ldots, y_h, \gamma$ and 0 in the other columns, where as the row corresponding to the vertex $u_1$ has 1 in the columns of $e, x_1, x_2, \ldots, x_k$ and 0 in other columns. One can check that the matrix $A'$ can be obtained from $A$ by adjoining an extra row corresponding to the vertex $u_1$ to $A$ with entries zero every where except in the columns corresponding to $e, x_1, x_2, \ldots, x_k$ where it takes the value 1. The row vector obtained by addition (mod 2) of row vectors corresponding to vertices $u$ and $u_1$ will corresponds to the row vector of the vertex $u_2$ in $A'$.

Noticing the above s Azanchiler [1] extended the notion of $n$-line-splitting operation from graphs to binary matroids in the following way:

**Definition 1.** Let $M$ be a binary matroid on a set $E$ and let $X$ be a subset of $E$ with $e \in X$. Suppose $A$ is a matrix representation of $M$ over GF(2). Let $A_X^e$ be a matrix obtained from $A$ by adjoining an extra row $\delta_X$ to $A$ with entries zero every where except in the columns corresponding to the elements of $X$, where it takes the value 1 and then adjoining two columns labelled $a$ and $\gamma$ to the resulting matrix such that the column labelled $a$ is zero everywhere except in the last row where it takes the value 1, and $\gamma$ is sum of the two column vectors corresponding to the elements $a$ and $e$. The vector matroid of the matrix $A_X^e$ is denoted by $M_X^e$. The transition from $M$ to $M_X^e$ is called an $e$-$s$-splitting operation. We call the matroid $M_X^e$ as $e$-$s$-splitting matroid.

The following proposition characterizes the circuits of the matroid $M_X^e$ in terms of the circuits of the matroid $M$.

**Proposition 1.1.** [1] Let $M(E, C)$ be a binary matroid together with the collection of circuits $C$. Suppose $X \subseteq E$, $e \in X$ and $a, \gamma \notin E$. Then $M_X^e = (E \cup \{a, \gamma\}, C')$, where $C' = C_0 \cup C_1 \cup C_2 \cup C_3 \cup \{\Delta\}$ with $\Delta = \{e, a, \gamma\}$ and
\[ C_0 = \{C \in \mathcal{C} \mid \text{C contains an even number of elements of } X \}; \]
\[ C_1 = \text{The set of minimal members of } \{C_1 \cup C_2 \mid C_1, C_2 \in \mathcal{C}, C_1 \cap C_2 = \phi \text{ and each of } C_1 \text{ and } C_2 \text{ contains an odd number of elements of } X \text{ such that } C_1 \cup C_2 \text{ contains no member of } C_0 \}; \]
\[ C_2 = \{C \cup \{a\} \mid C \in \mathcal{C} \text{ and } C \text{ contains an odd number of elements of } X\}; \]
\[ C_3 = \{C \cup \{e, \gamma\} \mid C \in \mathcal{C}, e \notin C \text{ and } C \text{ contains an odd number of elements of } X \} \cup \{(C \setminus e) \cup \{\gamma\} \mid C \in \mathcal{C}, e \in C \text{ and } C \text{ contains an odd number of elements of } X \} \cup \{(C \setminus e) \cup \{a, \gamma\} \mid C \in \mathcal{C}, e \in C \text{ and } C \setminus e \text{ contains an odd number of elements of } X \}. \]

Throughout this paper we assume that \( M \) is a loopless and coloopless binary matroid, \( X \subseteq E(M) \) and \( M^e_X \) is the \( es \)-splitting matroid of \( M \). We denote by \( C_{OX} \) the set of all circuits of a matroid \( M \) each of which contains an odd number of elements of the set \( X \). The members of the set \( C_{OX} \) are called \( OX \)-circuits. On the other hand, \( C_{EX} \) denotes the set of all circuits of a matroid \( M \) each of which contains an even number of elements of the set \( X \). The members of the set \( C_{EX} \) are called \( EX \)-circuits.

It is interesting to observe that \( M^e_X \setminus \gamma \) and \( M^e_X \setminus \{a, \gamma\} \) are isomorphic with element splitting matroid and splitting matroid of \( M \), respectively. The main theorems of this paper, Theorem 3.1 and Theorem 3.2 are motivated by a series of earlier work on splitting operation, element splitting operation and \( es \)-splitting operation [1, 2, 4, 7, 8, 10, 11, 12, 15, 17].

The following result characterizes the rank function of the matroid \( M^e_X \) in terms of the rank function of the matroid \( M \) [4].

**Lemma 1.1.** Let \( r \) and \( r' \) be the rank functions of the matroids \( M \) and \( M^e_X \), respectively. Suppose that \( A \subseteq M(E) \). Then

1. \( r'(A) = r(A) + 1 \), if \( A \) contains an \( OX \)-circuit of the matroids \( M \);
   \[ = r(A); \text{ otherwise}. \]

2. \( r'(A \cup a) = r(A) + 1 \).

3. \( r'(A \cup \{\gamma\}) = r(A) \), if not \( A \) but \( A \cup \{e\} \) contains an \( OX \)-circuit of \( M \);
   \[ = r(A) + 2; \text{ if } A \text{ contains an } OX \text{-circuit of } M \text{ and } e \notin cl(A); \]
   \[ = r(A) + 1; \text{ otherwise}. \]

4. \( r'(A \cup \{a, \gamma\}) = r(A) + 1 \), if \( e \in cl(A) \);
   \[ = r(A) + 2; \text{ if } e \notin cl(A). \]

Using Lemma 1.1, one can obtain the following corollary.

**Corollary 1.1.** Let \( r \) and \( r' \) be the rank functions of the matroids \( M \) and \( M^e_X \), respectively. Then
\[ r'(M^e_X) = r(M) + 1. \]

**Remark 1.1.** As \( \Delta = \{a, e, \gamma\} \subseteq M^e_X \), we have \( r'(M^e_X) = r'(M^e_X \setminus \{a\}) = r'(M^e_X \setminus \{e\}) = r'((M^e_X \setminus \{\gamma\})). \)
We recall that matroid \( M \) is connected if and only if for every pair of distinct elements of \( E(M) \), there is a circuit containing both. The concept of \( n \)-connected matroids was introduced by W. T. Tutte [14]. If \( k \) is positive integer, the matroid \( M \) is \( k \)-separated if there is a subset \( X \subset E(M) \) such that \( |X| \geq k, |E\setminus X| \geq k \) and \( r(X)+r(E\setminus X)-r(M) = k-1 \). Connectivity \( \lambda(M) \) of \( M \) is the list positive integer \( j \) such that \( M \) is \( j \)-separated. If there is no such integer we say \( \lambda(M) = \infty \). Note that \( \lambda(U_{2,4}) = \infty \). The following result from [9] provides a necessary condition for a matroid to be \( n \)-connected.

**Lemma 1.2.** If \( M \) is a \( n \)-connected matroid and \( |E(M)| \geq 2(n-1) \) then all circuits and all cocircuits of \( M \) have at least \( n \) elements.

Let \( M \) be an \( n \)-connected binary matroid and \( X \subset E(M) \). Note that if \( |X| < n \) then \( X \cup \{a\} \) will be a cocircuit of \( M^e_X \). Further, if \( |X \cup \{a\}| < n \) then, by Lemma 1.2, \( M^e_X \) is not \( n \)-connected. Azanchiler [1] proved that \( es \)-splitting operation on a connected binary matroid yields a connected binary matroid. In fact, he proved the following theorem.

**Theorem 1.1.** Let \( M \) be a connected binary matroid and \( X \subset E(M) \) with \( |X| \geq 2 \). Then \( M^e_X \) is connected binary matroid.

In the following result Dhotre, Malavadkar and Shikare [4], provided a sufficient condition for the \( es \)-splitting operation to yield a 3-connected binary matroid from a 3-connected binary matroid.

**Theorem 1.2.** Let \( M \) be a 3-connected binary matroid, \( X \subset E(M) \) and \( e \in X \). Suppose that \( M \) has an \( OX \)-circuit not containing \( e \). Then \( M^e_X \) is a 3-connected binary matroid.

In particular, when \( X = \{x, y\} \) the \( es \)-splitting maroid is denoted by \( M^e_{x,y} \). As a consequence of the above result, Dhotre, Malavadkar and Shikare [4] obtained a splitting lemma for \( es \)-splitting matroid \( M^e_{x,y} \).

**Corollary 1.2.** (Splitting Lemma). If \( M \) is a 3-connected binary matroid then, \( M^e_{x,y} \) is a 3-connected binary matroid for any pair \( \{x, y\} \) of elements of \( E(M) \).

2. 3-Connected Minors of the \( es \)-splitting Matroids.

In this section, we provide a sufficient condition for a 3-connected binary matroid \( M \) of rank \( r \), where \( M^e_X \setminus e \) and \( M^e_X \setminus \gamma \) are 3-connected minors of rank \( r+1 \) of the matroid \( M^e_X \).

Let \( M \) be a cycle matroid of a wheel \( W_5 \) as shown in the Figure 2, \( X = \{8, 9, 10\} \) and \( e = 10 \). Then \( M^e_X \) is the \( es \)-splitting matroid of \( M \). Observe that \( M^e_X \setminus e \) and \( M^e_X \setminus \gamma \) are 3-connected minors of \( M^e_X \). But \( M^e_X \setminus a \) is not a 3-connected minor of \( M^e_X \).

In the following result, we provide a sufficient condition for a 3-connected binary matroid \( M \) where \( M^e_X \setminus e \) is a 3-connected minor of \( M^e_X \).

**Lemma 2.1.** Let \( M \) be a 3-connected binary matroid, \( |E(M)| \geq 4 \) and let \( X \subset E(M) \), where \( |X| \geq 3 \). Suppose for \( x \in E(M) \) there is an \( OX \)-circuit of \( M \) not containing \( x \). Then \( M^e_X \setminus e \) is a 3-connected binary matroid.
Case 1. Let $A \subseteq \{a, \gamma \} \subset A$. Then there is an $OX$-circuit $C$ of $M$ not containing $x$ and $C \subset B'$. Thus, by Lemma 1.1 (2) and (3), $r'(A) = r(A) + 1$ and $r'(B) \geq r(B') + 1$. By inequality (1), we have $r(A') + 1 + r(B') - r(M) - 1 \leq 1$. That is, $r(A') + r(B') - r(M) \leq 0$ where $|A'|, |B'| \geq 1$. This implies $(A', B')$ is a $1$-separation of $M$, a contradiction. If $B = 2$, and suppose $B = \{\gamma, x\}$. Then by the argument similar to one as given above we get a contradiction to the $3$-connectedness of $M$.

Subcase 1.2. $|A|, |B| > 2$.

Then, by Lemma 1.1 (2) and (3), $r'(A) = r(A') + 1$ and $r'(B) \geq r(B')$. By inequality (1), we have $r(A') + r(B') - r(M) - 1 \leq 1$. That is, $r(A') + r(B') - r(M) \leq 1$ where $|A'|, |B'| \geq 2$. This implies $(A', B')$ is a $2$-separation of $M$, a contradiction.

Case 2. $\{a, \gamma\} \subset A$.

Let $A = A \setminus \{a, \gamma\}$ and $B' = B$. We have the following three subcases.

Subcase 2.1. $|A| = 2$.

Then $A = \{a, \gamma\}$, $r'(A) = 2$ and $A' = \{\phi\}$. By hypothesis, there is an $OX$-circuit $C$ of $M$ not containing $e$ and thus, $C \subseteq B$. Then, by Lemma 1.1 (1), $r'(B) = r(B') + 1$. So, by inequality (1),
2 + r(B') + 1 − r(M) − 1 ≤ 1. That is, r(B') − r(M) ≤ −1 or r(B') ≤ r(M) − 1. This is a contradiction.

**Subcase 2.2.** |A| = 3. Suppose A = \{a, γ, x\}. Then e ∉ Cl(A') since otherwise, \{x, e\} forms a 2-circuit, which is not possible in a 3-connected matroid M. We conclude that e ∉ Cl(A'). Consequently, by Lemma 1.1 (4), r'(A) = r(A') + 2. Also, by Lemma 1.1 (1), r'(B) ≥ r(B'). Thus, by inequality (1), r(A') + 2 + r(B') − r(M) − 1 ≤ 1. That is, r(A') + r(B') − r(M) ≤ 0 and |A'|, |B'| ≥ 1. This gives a 1-separation of M, a contradiction.

**Subcase 2.3.** |A| > 3.

Applying Lemma 1.1 to A and B, we get r'(A) ≥ r(A) + 1 and r'(B) ≥ r(B). Then, by inequality (1), we get r(A') + 1 + r(B') − r(M) − 1 ≤ 1. That is, r(A') + r(B') − r(M) ≤ 1 and |A'|, |B'| ≥ 2. This leads to a 2-separation of M; a contradiction. The above facts imply that M_X has no 2-separation. We conclude that M_X \ e is 3-connected matroid.

In the following lemma, we provide a sufficient condition for a 3-connected binary matroid M so that M_X \ γ is a 3-connected minor of the es-splitting matroid M_X.

**Lemma 2.2.** Let M be a 3-connected binary matroid, |E(M)| ≥ 4. Let X ⊂ E(M) with |X| ≥ 3. Suppose for x ∈ E(M) there is an OX-circuit of M not containing x. Then, M_X \ γ is a 3-connected binary matroid.

**Proof.** If x = e then, by hypothesis, there is an OX-circuit of M not containing x. So, by Theorem 1.2, M_X is 3-connected and M_X \ γ is connected. Suppose M_X \ γ is not 3-connected and let (A, B) be a 2-separation of E(M_X \ γ). Then min \{|A|, |B|\} ≥ 2 and

\[ r'(A) + r'(B) - r'(M_X \setminus \gamma) ≤ 1. \]   (2)

Assume that \{a\} ⊂ A. Let A' = A \ a and B' = B. Then, by Lemma 1.1, r'(A) = r(A') + 1 and r'(B) ≥ r(B'). Now one of the following two cases occurs.

**Case 1.** |A| = 2.

Suppose A = \{z, a\} and A' = \{z\} where z ∈ E(M). Then, by Lemma 1.1 (2), r'(A) = r(A') + 1. Now M contains an odd circuit C of M and \{z\} ∩ C = φ, implies C ⊆ B. Then, by Lemma 1.1 (1), r'(B) = r(B') + 1. Thus, by inequality (2), r(A') + 1 + r(B') + 1 - r(M) - 1 ≤ 1. That is, r(A') + r(B') − r(M) ≤ 0, and |A'|, |B'| ≥ 1. This implies that (A', B') is a 1-separation of M, a contradiction.

**Case 2.** |A| > 2.

By (1) and (2) of Lemma 1.1, r'(A) ≥ r(A') + 1 and r'(B) ≥ r(B'). Then, by inequality (2), r(A') + 1 + r(B') − r(M) − 1 ≤ 1. That is, r(A') + r(B') − r(M) ≤ 1 and |A'|, |B'| ≥ 2. This leads to a 2-separation of M, a contradiction. Thus, M_X \ γ has no 2-separation. We conclude that M_X \ γ is a 3-connected binary matroid.

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3. \(n\)-Connected Minors of the es-splitting Matroids.

In this section, we provide a sufficient condition for an \(n\)-connected binary matroid \(M (n ≥ 4)\) of rank \(r\), where \(M_X \setminus e\) and \(M_X \setminus \gamma\) are \(n\)-connected minors of rank \(r + 1\) of the es-splitting
matroid $M_X^\epsilon$.  

Let $M$ be an $n$-connected binary matroid $(n \geq 4)$, $X \subseteq E(M)$ and $e \in X$. Suppose that $M$ has an $OX$-circuit not containing $e$. Then, by Theorem 1.2, the binary matroid $M_X^\epsilon$ is 3-connected. Note that the matroid $M_X^\epsilon$ contains a triangle $\triangle = \{a,e,\gamma\}$. Hence, by Proposition 1.2, $M_X^\epsilon$ is not 4-connected. We observe that for any $x \in E(M_X^\epsilon)$, $M_X^\epsilon/x$ contains a 2-circuit or a triangle and therefore it is not 4-connected. Further, for any $x \in (E(M_X^\epsilon) - \triangle)$, the minor $M_X^\epsilon \setminus x$ contains the triangle $\triangle$ and therefore, it is not 4-connected. Thus, the possible 4-connected minors of $M_X^\epsilon$ are $M_X^\epsilon \setminus e$ and $M_X^\epsilon \setminus \gamma$.

In the following theorem, we give a sufficient condition for an $n$-connected binary matroid $M$ where $M_X^\epsilon \setminus e$ is an $n$-connected minor of $M_X^\epsilon$.

**Theorem 3.1.** Let $M$ be an $n$-connected binary matroid where $n \geq 4$, $|E(M)| \geq 2(n-1)$ and let $X \subset E(M)$ with $|X| \geq n$. Suppose that for any $(n-2)$-element subset $S$ of $E(M)$ there is an $OX$-circuit $C$ of $M$ such that $S \cap C = \emptyset$. Then $M_X^\epsilon \setminus e$ is $n$-connected.

**Proof.** The proof is by induction on $n$. First we prove the case $n = 4$. The matroid $M_X^\epsilon \setminus e$ is 3-connected by Lemma 2.1. To prove that $M_X^\epsilon \setminus e$ is 4-connected, it is enough to show that it has no 3-separation. On the contrary, suppose $(A, B)$ forms a 3-separation of $M_X^\epsilon \setminus e$. Then

$$\min\{|A|, |B|\} \geq 3 \quad \text{and} \quad r'(A) + r'(B) - r'(M_X^\epsilon \setminus e) \leq 2. \quad (3)$$

Now one of the following two cases occurs.

**Case 1.** $a \in A$ and $\gamma \in B$

**Subcase 1.1.** $|A| = 3$ Let $A = \{a, x, y\}$. Then there is an $OX$-circuit $C$ of $M$ not containing $x$, $y$ and $C \subset B'$. Thus, by Lemma 1.1 (2) and (3), $r'(A) = r(A') + 1$ and $r'(B') \geq r(B') + 1$. By inequality (1), we have $r(A') + 1 + r(B') + 1 - r(M) - 1 \leq 2$. That is, $r(A') + r(B') - r(M) \leq 1$ where $|A'|, |B'| \geq 2$. This implies $(A', B')$ is a 2- separation of $M$, a contradiction.

**Subcase 1.2.** $|A|, |B| > 3$

Then, by Lemma 1.1 (2) and (3), $r'(A) = r(A') + 1$ and $r'(B') \geq r(B')$. By inequality (1), we have $r(A') + 1 + r(B') - r(M) - 1 \leq 2$. That is, $r(A') + r(B') - r(M) \leq 2$ where $|A'|, |B'| \geq 3$. We conclude that $(A', B')$ is a 3- separation of $M$, a contradiction.

**Case 2.** $\{a, \gamma\} \subset A$

Let $A' = A \setminus \{a, \gamma\}$ and $B' = B$. We have the following three subcases.

**Subcase 2.1.** $|A| = 3$ and $A = \{a, \gamma, x\}$, where $x \in E(M) \setminus e$

If $e \in Cl(A')$, then $\{x, e\}$ forms a 2-circuit of $M$. This is not possible, since $M$ is 4-connected. Thus, $e \notin Cl(A')$ and by Lemma 1.1 (4), $r'(A) = r(A') + 2$. Also, there is an $OX$-circuit $C$ of $M$ not containing $x$ and $C \subseteq B'$. Therefore, $r'(B) = r(B') + 1$. Consequently, by inequality (3),

$$r(A') + 2 + r(B') + 1 - r(M) - 1 \leq 2.$$

That is, $(A') + r(B') - r(M) \leq 0$ and $|A'|, |B'| \geq 1$. So $M$ has a 1-separation; a contradiction.

**Subcase 2.2.** $|A| = 4$ and $A = \{a, \gamma, x, y\}$ where $x, y \in E(M) \setminus e$

If $e \in Cl(A')$, then the set $\{x, y, e\}$ itself is a 3-circuit or contains a 2-circuit of $M$. This is not possible, since $M$ is 4-connected. Thus, $e \notin Cl(A')$ and, by Lemma 1.1 (4), $r'(A) = r(A') + 2$.  

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Now there is an $OX$-circuit $C$ of $M$ not containing $x$ and $y$ and $C \subseteq B'$. So, $r'(B) = r(B') + 1$. Therefore, by inequality (3),

$$r(A') + 2 + r(B') + 1 - r(M) - 1 \leq 2.$$ 

That is, $r(A') + r(B') - r(M) \leq 0$ and $|A'|, |B'| \geq 1$. We conclude that $M$ has a 1-separation, a contradiction.

**Subcase 2.3.** $|A| > 4$

Now by (1) and (4) of Lemma 1.1, $r'(B) \geq r(B)$ and $r'(A) \geq r(A) + 1$. By inequality (3), we get

$$r(A') + 1 + r(B') - r(M) - 1 \leq 2.$$ 

That is, $r(A') + r(B') - r(M) \leq 2$ and $|A'|, |B'| \geq 3$. This leads to a 3-separation of $M$, a contradiction.

Thus, $M_X^e$ has no 3-separation. We conclude that $M_X^e \setminus e$ is 4-connected.

Now we assume that the result is true for $k \geq 4$ and prove that the result is true for $k + 1$.

Let $M$ be a $(k + 1)$-connected binary matroid and $M_X^e$ be the es-splitting matroid of $M$ and any $(k - 1)$-element subset $S$ of $E(M)$ there is an $OX$-circuit $C$ of $M$ such that $S \cap C = \phi$. Note that $M_X^e \setminus e$ is a $k$-connected minor by induction hypothesis. Thus, it is enough to show that $M_X^e \setminus e$ has no $k$-separation.

On the contrary, suppose $M_X^e \setminus e$ is not $(k+1)$-connected. Let $(A, B)$ be a $k$-separation of $E(M_X^e \setminus e)$. Then, $\min \{|A|, |B|\} \geq k$, and

$$r'(A) + r'(B) - r'(M_X^e \setminus a) \leq k - 1. \quad (4)$$

Now one of the following two cases occurs.

**Case 1.** $a \in A$ and $\gamma \in B$

Let $A' = A \setminus a$ and $B' = B \setminus \gamma$. Then, by (2) and (3) of Lemma 1.1, $r'(A) = r(A') + 1$ and $r'(B) \geq r(B') + 1$. By inequality (4), we have $r(A') + 1 + r(B') + 1 - r(M) - 1 \leq k - 1$. That is, $r(A') + r(B') - r(M) \leq k - 2$, where $|A'|, |B'| \geq k$. Thus, $(A', B')$ is a $k$-separation of $M$ and this is a contradiction.

**Case 2.** $\{a, \gamma\} \subset A$

Let $A' = A \setminus \{a, \gamma\}$ and $B' = B$. We have the following two subcases.

**Subcase 2.1.** $|A| = 4$

If $e \in Cl(A')$ then the set $\{x, y, e\}$ itself is a 3-circuit or contains a 2-circuit of $M$. This is not possible, since $M$ is 4-connected. If $e \notin Cl(A')$ then, by Lemma 1.1 (4), $r'(A) = r(A') + 2$. Since there is an $OX$-circuit $C$ of $M$ not containing $x$ and $y$, $C \subseteq B'$. So $r'(B) = r(B') + 1$. Consequently, by inequality (4),

$$r(A') + 2 + r(B') + 1 - r(M) - 1 \leq k - 1.$$ 

That is, $r(A') + r(B') - r(M) \leq k - 2$ and $|A'|, |B'| \geq k$. This implies that $M$ has a $k$-separation, a contradiction.

**Subcase 2.2.** $|A| > 4$

Now by (1) and (4) of Lemma 1.1, $r'(B) \geq r(B)$ and $r'(A) \geq r(A) + 1$. By inequality (4), we get
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$r(A') + 1 + r(B') - r(M) = 1 \leq k - 1$. That is, $r(A') + r(B') - r(M) \leq k - 2$ and $|A'|, |B'| \geq k$.

This leads to a $k$-separation of $M$, a contradiction.

Thus, $M_X^e$ has no $k$-separation. We conclude that $M_X^e \setminus e$ is $k + 1$-connected. We conclude that, by principle of mathematical induction, the result is true for all $n \geq 4$.

In the following theorem, we give a sufficient condition for an $n$-connected binary matroid $M$ so that $M_X^e \setminus e$ is an $n$-connected minor of $M_X^e$.

**Theorem 3.2.** Let $M$ be an $n$-connected binary matroid with $n \geq 4$, $|E(M)| \geq 2(n - 1)$ and let $X \subset E(M)$, where $|X| \geq n$. Suppose that for any $(n - 2)$-element subset $S$ of $E(M)$ there is an $OX$-circuit $C$ of $M$ such that $S \cap C = \phi$. Then $M_X^e \setminus \gamma$ is $n$-connected.

The proof follows by the arguments similar to one as given for the proof of Theorem 3.1.

Thus, we proved that given an $n$-connected binary matroid $M$ of rank $r$, $M_X^e \setminus e$ and $M_X^e \setminus \gamma$ are the $n$-connected minors of rank $(r + 1)$ of the $es$-splitting matroid $M_X^e$. In other words, we provide a procedure to obtain $n$-connected matroids of rank $(r + 1)$ from an $n$-connected matroid of rank $r$. The matroids also have the property that each of them has exactly one additional element than $M$. We illustrate Theorems 3.1 and 3.2 with the help of the following example.

**Example 1.** Let matrix $M$ be a cycle matroid of a complete bipartite graph $K_{4,4}$ shown in Figure 3. $M$ is 4-connected matroid. Let $X = \{1, 2, 5, 6\}$. Observe that there is an $OX$-circuit in $M$ avoiding every pair of elements $\{x, y\}$. Let $A$ be the matrix representation of the cycle matroid $M$ over $GF(2)$ where

$$A = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

Let $X = \{1, 2, 5, 6\}$ and $10 = e$. Then representation of $es$-splitting matroid $M_X^e$ over the field $GF(2)$ is given by the matrix
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Note that, by Theorem 1.2, the $es$-splitting matroid $M^e_X$ is 3-connected. But if $A = \{a, e, \gamma\}$ and $B = E(M^e_X) \setminus A$, then $r'(A) + r'(B) - r'(M^e_X) = 2 + r'(B) - 8 \leq 2$. Thus $(A, B)$ is a 3-separation of $M^e_X$ and hence $M^e_X$ is not 4-connected. Further, it is easy to verify that $M^e_X \setminus e$ and $M^e_X \setminus \gamma$ are 4-connected minors of the $es$-splitting matroid $M^e_X$.

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