Nonlinear wave damping due to multi-plasmon resonances

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Abstract

For short wavelengths, it is well known that the linearized Wigner–Moyal equation predicts wave damping due to wave-particle interaction, where the resonant velocity shifted from the phase velocity by a velocity \( v_\parallel = \frac{\hbar k}{2m} \). Here \( h \) is the reduced Planck constant, \( k \) is the wavenumber and \( m \) is the electron mass. Going beyond linear theory, we find additional resonances with velocity shifts \( v_n \), \( n = 2, 3, \ldots \), giving rise to a new wave-damping mechanism that we term multi-plasmon damping, as it can be seen as the simultaneous absorption (or emission) of multiple plasmon quanta. Naturally this wave damping is not present in classical plasmas. For a temperature well below the Fermi temperature, if the linear (\( n = 1 \)) resonant velocity is outside the Fermi sphere, the number of linearly resonant particles is exponentially small, while the multi-plasmon resonances can be located in the bulk of the distribution. We derive sets of evolution equations for the case of two-plasmon and three-plasmon resonances for Langmuir waves in the simplest case of a fully degenerate plasma. By solving these equations numerically for a range of wave-numbers we find the corresponding damping rates, and we compare them to results from linear theory to estimate the applicability. Finally, we discuss the effects due to a finite temperature.

Keywords: wave-particle interaction, Wigner equation, Langmuir waves, nonlinear wave damping

(Some figures may appear in colour only in the online journal)

1. Introduction

Wave-particle interaction has been of long-standing interest in plasma physics, where Landau damping of Langmuir waves has been a prominent example. In a classical context Landau damping has been studied extensively, both linearly as well as in the nonlinear regime when effects such as bounce oscillations enters the picture, see e.g. [1–6]. Including quantum effects, new mechanisms come into play, for example the effect of the electron spin [7, 8], nonlinear modifications of the Fermi surface [9], quantum modifications of the radiation pressure [10], exchange effects [11, 12] and bounce-like oscillations in the absence of trapped particles [13], even in a ‘weak’ quantum regime. It can also be noted that the classical-quantum transition of plasmons recently has been probed experimentally, using metallic nanoparticles [14].

In the present paper we will instead be interested in a ‘strong’ quantum regime [15–17]. By weak (strong) we mean that the Langmuir wavelength is much less than (comparable to) the characteristic de Broglie wavelength. The main difference between the weak and strong quantum regimes is that in the former, the resonant particles still have a velocity close to the phase velocity of the wave, just as in classical theory. Nevertheless it should be stressed that significant effects, e.g. quantum suppression of nonlinear bounce oscillations [13, 18, 19], may occur already in the weak quantum regime. In the strong quantum regime the resonant particles instead fulfill \( \omega - kv_z \pm \hbar k^2/2m = 0 \) in linearized theory [16, 17, 20, 21]. Here \( \omega \) and \( k = k_\parallel + k_\perp \) are the wave frequency and wavevector, \( v_z \) is the electron velocity along \( \parallel \), \( h = 2\pi\hbar \) is Planck’s constant, and \( m \) is the electron mass.

We will demonstrate that if we study the nonlinear evolution, there appear multi-plasmon resonances, that is, resonant wave-particle interaction with a resonant velocity

\[
v_{\text{res,}+\hbar} = \frac{\omega}{k} \pm \frac{n\hbar k}{2m},
\]

(1)
where \( n \) is a positive integer. The case with \( n = 1 \) corresponds to one-plasmon processes (linear Landau damping), \( n = 2 \) corresponds to two-plasmon processes, etc. The physical meaning of the resonance condition (1) and the integer \( n \) can be understood as follows. When a particle absorbs or emits a wave quantum its momentum can increase or decrease according to

\[
h\mathbf{k}_1 \pm n h\mathbf{k} = h\mathbf{k}_2
\]

and at the same time the energy changes according to

\[
h\omega_1 \pm n h\omega = h\omega_2.
\]

Next we identify \( h\mathbf{k}_1/m \) (or equally well \( h\mathbf{k}_2/m \)) with the resonant velocity, and note that for small amplitude waves the particle frequencies and wavenumbers \( (\omega_{1,2}, \mathbf{k}_{1,2}) \) obey the free particle dispersion relation \( \omega_{1,2} = h\mathbf{k}_{1,2}^2/2m \), that we get directly from the single particle Schrödinger equation by letting the wave field go to zero. Using these relations we see that the energy momentum relations (2) and (3) imply the quantum modification of the resonant velocity as seen in equation (10), from linearized theory. The linear quantum modifications of the resonant denominators is what is obtained by putting \( n = 1 \) in equation (1).

A possibility that requires nonlinear theory is the simultaneous absorption of multiple wave quanta, rather than a single wave quanta at a time. In that case equations (2) and (3) are replaced by

\[
h\mathbf{k}_1 \pm n h\mathbf{k} = h\mathbf{k}_2
\]

and

\[
h\omega_1 \pm n h\omega = h\omega_2,
\]

where \( n = 1, 2, 3 \ldots \) is an integer. Using equations (4) and (5) together with the free particle dispersion relations we recover equation (1). Thus it is clear that the integer \( n \) in this equation represents the number of wave quanta that is simultaneously absorbed or emitted. When we pick the minus sign in (1) the resonant velocity for absorbing multiple wave quanta can be considerably smaller, provided the wavelengths are short.

For pedagogical reasons, and as the well-known already from linear theory, we will treat a fully degenerate plasma and assume that the wavenumber \( k \) is such that one-plasmon processes are forbidden due to a lack of resonant particles. This is the case if \( k < k_{\alpha} \) where \( k_{\alpha} \) is the critical wavenumber computed in [16]. Specifically we will focus on two-plasmon and three-plasmon processes, that can be studied if we include up to cubic terms in an amplitude expansion. The damping rates of Langmuir waves, which decrease with the decaying amplitude, are found both for \( n = 2 \) and \( n = 3 \), and the evolution of the Wigner function is computed numerically. We stress here that wave damping due to multi-plasmon resonances (i.e. \( n \geq 2 \)) is a fundamentally new process. Contrary to ordinary Landau damping it does not exist in the limit \( h \to 0 \). However, as indicated by the above discussion, the new resonances are present also for completely non-degenerate systems. The effects of a finite temperature will be discussed in the final section.

2. Basic equations and preliminaries

We take as our starting point the Wigner–Moyal equation for electrons

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = -\frac{i q \mathbf{v}}{\hbar} \int \frac{d^4 \mathbf{r}' d^4 \mathbf{v}'}{(2\pi)^3} e^{i \mathbf{r}'(\mathbf{v}' - \mathbf{v})/\hbar} \times \left[ \Phi \left( \mathbf{r} + \frac{\mathbf{r}'}{2} \right) - \Phi \left( \mathbf{r} - \frac{\mathbf{r}'}{2} \right) \right] f(\mathbf{r}, \mathbf{v}', t) = 0.
\]

Here \( f \) is the Wigner function, \( q = -|e| \) is the electron charge, and we have restricted ourselves to the electrostatic case, where \( \Phi \) is the scalar potential. The system is closed with Poisson’s equation,

\[
-\nabla^2 \Phi = \frac{q}{\epsilon_0} \int d^3 v f - \frac{q}{\epsilon_0} n_i,
\]

where \( n_i \) is a constant neutralizing ion background.

2.1. Short review of linearized theory

While the theory of multi-plasmon resonances requires a study of the nonlinear Wigner equation, there is much information that follows directly from the linearized theory. In particular the parameters encountered in the nonlinear theory are functions of \( \omega \) and \( k \), whose relation is determined by the linear dispersion relation to leading order in an amplitude expansion. A highly useful study of linear theory is provided by [16]. Our presentation here will briefly review some of these results, but also point out a few aspects of the linear theory that are of special relevance for the (nonlinear) multi-plasmon processes.

Dividing the Wigner function as \( f = F_0 + f_1(\mathbf{v}) \exp[i(kz - \omega t)] \) where \( F_0 \) is the background distribution and linearizing equation (6), the solution for the Wigner function is

\[
f_1 = -\frac{q \Phi[F_0(\mathbf{v} + \mathbf{v}_k) - F_0(\mathbf{v} - \mathbf{v}_k)]}{h(\omega - kv_k)},
\]

where we have introduced \( \mathbf{v}_k = v_k \mathbf{z} = (h k/2m) \mathbf{z} \). See [16] for further details. Inserting this expression into Poisson’s equation (7), the linear dispersion relation becomes

\[
1 = -\frac{q^2}{\hbar k^2 \epsilon_0} \int \frac{F_0(\mathbf{v} + \mathbf{v}_k) - F_0(\mathbf{v} - \mathbf{v}_k)}{(\omega - kv_k)} d^3 \mathbf{v}.
\]

Changing integration variables the dispersion relation can be written as

\[
1 + \frac{q^2}{\hbar k^2 \epsilon_0} \int \left( \frac{F_0(\mathbf{v})}{\omega - kv_k + \hbar k^2/2m} - \frac{F_0(\mathbf{v})}{\omega - kv_k - \hbar k^2/2m} \right) d^3 \mathbf{v} = 0.
\]

Taking \( F_0 \) as a Fermi–Dirac distribution with \( T = 0 \) we have \( F_0 = 0 \) for \( |\mathbf{v}| > v_F \) and \( F_0 = 2m^3/(2\pi \hbar)^3 \) for \( |\mathbf{v}| \leq v_F \), where

\[
v_F = (2k_B T_F/m)^{1/2}
\]

is the Fermi velocity. In accordance with the assumption of no linear poles, we will limit ourselves to values of \( k \) such that \( F_0 = 0 \) whenever the denominator in the integral is zero. Some intermediate steps to get the solutions for the integrals are given in [16]. Here
we just write out the final form for the dispersion relation that applies in the absence of linear poles. The dispersion relation is then given as

$$1 + \chi_e = 0,$$

where the linear susceptibility $\chi_e$ is given by

$$\chi_e = -\frac{3\omega_e^2}{4k^2v_F^2} \left\{ 2 - \frac{m_e}{hkv_F} \left[ v_F^2 - \left( \frac{\omega}{k} + \frac{h\omega}{2m} \right)^2 \right] \times \ln \left[ \frac{\omega - v_F - \frac{h\omega}{2m}}{\omega + \frac{h\omega}{2m}} \right] + \frac{m_e}{hkv_F} \left[ v_F^2 - \left( \frac{\omega}{k} + \frac{h\omega}{2m} \right)^2 \right] \ln \left[ \frac{\omega - v_F - \frac{h\omega}{2m}}{\omega + \frac{h\omega}{2m}} \right] \right\}. \tag{11}$$

2.2. Numerical solutions—location of multi-plasmon resonances

Now we want to separate the wavenumber spectrum into different regimes, depending on what type of resonances that occur. For this purpose we focus on the negative sign in equation (1), as this gives the lowest resonant velocity. Moreover, we introduce the normalized wavenumber $\tilde{k} = k^2/(mv_F)$, and the normalized wavenumber $\tilde{\omega} = \omega/\omega_p$, in which case $\chi_e$ obtains the form

$$\chi_e = \frac{3H^2}{4k^2} \left\{ 2 - \frac{1}{k} \left[ 1 - \left( \frac{H\tilde{k}}{k} + \frac{\tilde{k}}{2} \right)^2 \right] \times \ln \left[ \frac{H\tilde{k} - 1 + \tilde{k}/2}{H\tilde{k} + 1 + \tilde{k}/2} \right] + \frac{1}{k} \left[ 1 - \left( \frac{H\tilde{k}}{k} + \frac{\tilde{k}}{2} \right)^2 \right] \ln \left[ \frac{H\tilde{k} - 1 - \tilde{k}/2}{H\tilde{k} + 1 - \tilde{k}/2} \right] \right\}, \tag{12}$$

where $H = h\omega_p/(mv_F^2)$. Since $H$ scales with density only as $n^{-1/6}$, and is of order unity for a wide range of densities including metallic densities, we will here limit ourselves to the case $H = 1$. A few numerical tests in the range $0.5 < H < 2$ have been made, and the conclusions remain qualitatively the same. A numerical solution $\omega = \omega(k)$ for $H = 1$ is given in figure 1. The solution only applies provided the condition of no linear poles is met, and the part which violates this condition is indicated. Still it should be noted that the dashed part of the curve gives a good description of the electron-acoustic branch of the dispersion relation provided that wave-particle interaction is not too strong (see e.g. [22] for a discussion of the electron-acoustic branch). While the electron-acoustic branch also is subject to multi-plasmon resonances, we will be concerned with the Langmuir branch from now on. Our purpose is to avoid the presence of linear resonances altogether in order to focus our attention on physics induced by the multi-plasmon resonances. Since we are solely focused on the case without linear poles, this means that we assume $\nu_{res,-1} > \nu_F$, where we apply the negative sign and $n = 1$ in equation (1). Assuming that both $\omega$ and $k$ are positive and keeping $H = 1$ the inequality $\nu_{res,-1} > \nu_F$ implies

$$\tilde{\omega} - \tilde{k} - \tilde{k}^2/2 > 0. \tag{13}$$

First focusing on the case of two-plasmon resonance we want to combine (13) with the presence of two-plasmon resonances, that is $\nu_{res,-2} < \nu_F$. In terms of normalized frequencies this condition reads

$$\tilde{\omega} - \tilde{k} - \tilde{k}^2 < 0. \tag{14}$$

Simultaneously requiring conditions (13) and (14) to hold give us the regime where two-plasmon processes are present, but the poles of linear theory are absent. Next we would like to study three-plasmon processes. In that case we are particularly interested in cases where two-plasmon processes are forbidden but three-plasmon processes are allowed. In this case we demand $\nu_{res,-3} < \nu_F < \nu_{res,-2}$ which is written as

$$\tilde{\omega} - \tilde{k} - \tilde{k}^2 > 0 \tag{15}$$

and

$$\tilde{\omega} - \tilde{k} - 3\tilde{k}^2/2 < 0. \tag{16}$$

when $H = 1$.

We are now interested in separating the dispersion curves into the two-plasmon part (points simultaneously satisfying (13) and (14)), the three-plasmon part (points simultaneously satisfying (15) and (16)), and the other parts. How such a division turns out for $H = 1$ is illustrated in figure 2. Picking specific points $(\tilde{k}, \tilde{\omega})$ of these curves determine the numerical values of the coefficients in the two-plasmon and three-plasmon system of equations that will be derived in the next section.

2.3. The amplitude expansion

If the nonlinearity is weak, we can use the slowly varying amplitude approximation. In this approximation, all quantities have the form of linear combinations of plane waves where the amplitude vary slowly as a function of time compared to the natural frequency $\omega$. We will work to cubic order in the amplitude as this is the lowest order that exhibits multi-plasmon resonances. To cubic order, we can find resonances according to equation (1) with $n = 2, 3$ (again, $n = 1$ corresponds to linear theory). In principle, we are interested in resonances with arbitrary $n$, but to explicitly compute them for an arbitrary $n$ means carrying out an amplitude expansion to all orders.

To obtain evolution equations up to cubic order, we need an Ansatz for the electrostatic potential that contains second harmonic terms and low-frequency terms. For homogenous initial
conditions, particle conservation excludes a low-frequency contribution to the potential. Thus, we will take the electric potential to be of the form

$$\Phi(t) = \Phi_1(t) \exp[i(kz - \omega t)] + \Phi_2(t) \exp[2i(kz - \omega t)] + \text{c.c.}$$

(17)

Here c.c. denotes the complex conjugate, which is added to ensure that the potential is real.

Similarly, the Wigner function is taken to be a periodic function

$$f = f_0(v, t) + \sum_{n=1}^{\infty} f_n(v, t) \exp[in(kz - \omega t)] + \text{c.c.}$$

(18)

Here $f_0$ is the initial (constant) background distribution and $f_n(v, t)$ is the nonlinearly induced background modification. Keeping only up to cubic terms in the amplitude expansion will allow us to truncate the summation, but we keep it general for now to show the structure of the equations.

We stress again that the time dependence of the amplitudes $\Phi_1$, $\Phi_2$, and $f_n$ is assumed to be slow compared to $\omega$.

Inserting these Ansätze into equation (6) we will obtain a hierarchy of equations for $\Phi_1$, $\Phi_2$, and the $f_n$. With a potential of the form equation (17) the integral over $v'$ in equation (6) will produce a linear combination of Dirac delta functions. Viz., the terms in $\Phi(r + r'/2) - \Phi(r - r'/2)$ proportional to $e^{ikz - \omega t}$ are $\Phi_1 e^{ikz - \omega t}(e^{ikz'/2} - e^{-ikz'/2})$ and

$$\int \frac{d^3r'd^3v'}{(2\pi\hbar)^3} e^{i(r-r')\cdot\mathbf{v'}} \frac{m(v-v')}{h} \Phi_1 e^{ikz - \omega t} f$$

$$\int \frac{d^3r'd^3v'}{(2\pi\hbar)^3} e^{i(r-r')\cdot\mathbf{v'}} \frac{m(v-v')}{h} \Phi_1 e^{ikz - \omega t} f$$

$$\int d^3v'[\delta(v - v' + v_0) - \delta(v - v' - v_0)] \Phi_1 e^{ikz - \omega t} f$$

$$= \Phi_2 e^{i(kz - \omega t)} D_n f.$$  

(19)

Here we have introduced a quantum velocity shift $v_q$ and the velocity shift operator $D_n$, defined by

$$v_q = \frac{\hbar k}{2m^2}$$

and

$$(D_n f)(v) = f(v + v_0) - f(v - v_0).$$

(20)

Similarly, the terms in the potential proportional to $e^{-ikz - \omega t}$, $e^{2ikz - \omega t}$, and $e^{-2ikz - \omega t}$ will give $-\Phi_2 e^{-ikz - \omega t} D_1 f$, $\Phi_2 e^{2ikz - \omega t} D_2 f$, and $\Phi_2 e^{-2ikz - \omega t} D_2 f$, respectively, where the star denotes complex conjugate.

Since the amplitudes vary on timescales much longer than $\omega$, we can multiply any expression by $e^{-mi(kz - \omega t)}$ and average over one wave period to pick out the terms proportional to $e^{mi(kz - \omega t)}$, where $m$ is an integer. (Essentially, we are equating Fourier components and using the convolution theorem.) Applying this to the Wigner equation equation (6) and using the previous results, we find the hierarchy for the $f_n$.

$$\partial_t f_0 = \frac{iq}{\hbar} (\Phi_2 D_1 f_1 - \Phi_2 D_1 f_1 + \Phi_2 D_2 f_2 - \Phi_2 D_2 f_2)$$

(21a)
The curve has been adjusted to the resonant velocity \( v = \omega/k - n\nu/k/2m \), where \( n = 2 \) for two-plasmon damping and \( n = 3 \) for three-plasmon damping. For comparison the red region shows the background distribution.

These equations can be summarized as

\[
\begin{align*}
\partial_t f_n &= i(\omega - kv_n)f_n = \frac{iq}{\hbar}\left(\Phi_1^* D_1 f_{n+1} + f_{n-1}^* \Phi_1^* D_1 f_{n-1} - \Phi_1^* D_1 f_{n+1} - \Phi_1^* D_1 f_{n-1}\right) \\
\partial_t f_{n+1} &= i(\omega - kv_{n+1})f_{n+1} = \frac{iq}{\hbar}\left(\Phi_1^* D_1 f_{n+2} + f_{n-2}^* \Phi_1^* D_1 f_{n-2} - \Phi_1^* D_1 f_{n+2} - \Phi_1^* D_1 f_{n-2}\right)
\end{align*}
\]  

\[(21b)\]

\[
\begin{align*}
\partial_t g_n &= \frac{iq}{\hbar}\left(\Phi_2 D_2 f_{n+2}^* - \Phi_2 D_2 f_{n+2}^*\right) n > 1.
\end{align*}
\]  

\[(21c)\]

These equations can be summarized as \( f_n \) coupling to \( D_1 f_{n+1} \) through \( \Phi_1 (\Phi_1^*) \) and to \( D_2 f_{n+2} \) through \( \Phi_2 (\Phi_2^*) \), if one takes \( f_{n-1} = f_n^* \) (as is the case for the Fourier series of a real function).

So far, we have kept terms in the Wigner function to all orders. For \( n \geq 3 \), \( f_n \) is at least cubic in the amplitude according to equation \((21c)\); \( f_0 \) and \( f_2 \) are quadratic. We thus only need equations for the first three Fourier components of the Wigner function, and can discard some terms on the right hand side that are higher order in the amplitude expansion. Furthermore, since the problem is 1-dimensional in velocity space, we can integrate over \( v_x \) and \( v_y \) and work with the reduced Wigner function and background distribution,

\[
g_n(v_z) = \int dv_x dv_y f_n \quad \text{and} \quad G_0(v_z) = \int dv_x dv_y F_0.
\]  

\[(22)\]

\[\]  

Our final evolution equations for the Wigner function, valid to cubic order in the amplitude, then read

\[
\begin{align*}
\partial_t g_0 &= \frac{iq}{\hbar}\left(\Phi_2 D_2 (g_0 + G_0) - \Phi_2 D_2 g_0 + \Phi_2 D_2 g_0\right) \\
\partial_t \omega g_1 &= \frac{iq}{\hbar}\left(\Phi_1 D_1 (g_0 + G_0) - \Phi_1 D_1 g_1 + \Phi_1 D_1 g_1\right) \\
\partial_t \omega g_2 &= \frac{iq}{\hbar}\left(\Phi_1 D_1 g_1 + \Phi_2 D_2 G_0\right).
\end{align*}
\]  

\[(23a)\]

\[(23b)\]

\[(23c)\]

where \( \delta \omega = \omega - kv_n \). We note that so far we have made no assumptions regarding the background distribution.

### 3. Multi-plasmon damping processes

Now, the idea is to divide velocity space into a resonant region and a non-resonant region. Unless we are close to the resonant velocity, we can treat \( \partial_t g_0 \) and \( \partial_t g_n \) as small corrections in equation \((23)\). The part of velocity space where this approximation is applicable is the non-resonant region, and the part where the time derivative is essential for the solution is the resonant region. The size of the resonant region is not sharply defined. It should be taken large enough, i.e., such that \( g_n \) is negligible near the edge of the resonant region as compared to the center, but still small compared to \( \nu_p \), the size of the whole velocity distribution. It is possible to fulfill both conditions, according to our numerical simulations, and our results are not sensitive to the exact size of the resonant region.

In the non-resonant region, we solve equations \((23b)\) and \((23c)\) for \( g_1 \) and \( g_2 \) to lowest order by neglecting the time derivative and dividing by \( -i\delta \omega \). Recursively inserting these solutions, the repeated action of the velocity shift operators \( D_1 \) and \( D_2 \) will induce resonances for velocities fulfilling equation \((1)\), that is, multi-plasmon resonances.

We will treat the two-plasmon and three-plasmon processes separately, starting with the two-plasmon case. As can be realized, when two-plasmon processes are allowed three-plasmon processes will also take place (see figure 2), and the damping rates are of the same magnitude. Thus in principle two- and three-plasmon processes should be studied simultaneously. However, the physics is easier to understand when either is studied in isolation. In practice the wave damping due to the two processes can be added together afterwards, which is the approach we will chose. We note that since there is a regime that forbids two-plasmon processes but allows
three-plasmon processes, the latter can be studied in isolation, in a cubic order amplitude expansion.

3.1. The two-plasmon case

We first concentrate on the two-plasmon case, where \( \omega/k - \nu_F > v_F \) (no linear resonances) but \( \omega/k - 2\nu_F < v_F \). In this case only \( g_2 \) couples directly to the background distribution, and it will be resonant when \( \delta \omega \approx 0 \) in equation (23c). This means that \( g_2 \) couples resonantly to \( G_0(\omega/k \pm 2\nu_F) \) through the velocity shift operator of the second harmonic term \( \propto \Phi_2 \). Since \( g_2(\nu_F) \) couples to \( g_1(\nu_F \pm \nu_E) \), \( g_1 \) will need to be treated as resonant in two regions. Figure 3 represents the two-plasmon process schematically, where the arcs illustrates the couplings induced by the velocity shift operators.

To close the system, we use Poisson’s equation to derive evolution equations for the potential. Using the division into resonant and non-resonant regions, the component proportional to \( e^{ikz\omega-t} \) is

\[
4k^2\Phi_2 = \frac{q}{\epsilon_0} \int_{\mathbb{R}} g_2 \, dv_z + \frac{q}{\epsilon_0} \int_{\text{res}} g_2 \, dv_z. \tag{24}
\]

In the non-resonant region, where \( \delta \omega \) is large, the time derivative on \( g_2 \) is negligible, and we can substitute the linear expression for \( g_1 \) in the evolution equation (23c) to get

\[
4k^2\Phi_2 = \frac{q^2}{\epsilon_0 \hbar} \int_{\mathbb{R}} \left(D_2G_0\right) dv_z = \frac{q}{\epsilon_0} \int_{\text{res}} g_2 \, dv_z \]

\[
+ \frac{q^3\Phi_2^2}{2\hbar^2\epsilon_0} \int_{\mathbb{R}} \left(G_0(\nu_F + 2\nu_F) - G_0(\nu_F) \right) \frac{G_0(\nu_F) - G_0(\nu_F - 2\nu_F)}{\omega - k\nu_F} \, dv_z.
\]

We identify the LHS of equation (25) as \( 4k^2D(2\omega, 2k)\Phi_2 \), where \( D(\omega, k) \) is the linear dispersion function including only the non-resonant regions.

\[
\Phi_2 = \frac{q^2}{\epsilon_0 \hbar} \int_{\mathbb{R}} \left(D_2G_0\right) dv_z + \frac{q^3\Phi_2^2}{2\hbar^2\epsilon_0} \int_{\mathbb{R}} \left(G_0(\nu_F + 2\nu_F) - G_0(\nu_F) \right) \frac{G_0(\nu_F) - G_0(\nu_F - 2\nu_F)}{\omega - k\nu_F} \, dv_z. \tag{25}
\]

The coefficient for \( \Phi_1^2 \) can be approximated in terms of the linear susceptibility, under the assumption that the resonant region is small. For details, see the appendix. This approximation should be no cruder than the ones already made, and since we have studied the system for a range of modes, our qualitative conclusions seem robust to changes in the \( \omega, k \)-dependent coefficients.

For convenience we introduce the notation

\[
\chi_n = \chi(n\omega, nk), \tag{26}
\]

where \( \chi \) is the linear susceptibility. Note that the expression equation (12) is only an approximation when computing \( \chi_2 \) and \( \chi_3 \), as these expressions have pole contributions which have been dropped when deducing equation (12). Unless the pole contributions are small \( \chi_2 \) and \( \chi_3 \) should be computed from the susceptibility given in equation (10) rather than that given in equation (12). Since \( \omega, k \) obey the linear dispersion relation, \( \chi_n = 1 \), equation (25) can be written

\[
(1 + \chi_2)\Phi_2 = -\frac{q^2\chi_2(1 + \chi_4)}{\hbar kv_F} \Phi_1^2 + \frac{q}{\epsilon_0 k^2} \int_{\text{res}} g_2 \, dv_z. \tag{27}
\]

This equation determines \( \Phi_2 \) in terms of the other variables.

Now we turn to the nonlinear back-reaction on \( \Phi_1 \). Poisson’s equation gives

\[
\epsilon_0 k^2\Phi_1 = q \int g_1 \, dv_z = q \int_{\text{res}} g_1 \, dv_z + q \int_{\text{nr}} g_1 \, dv_z. \tag{28}
\]

Here by resonant region, we mean the region of velocity space where \( \omega \approx \pm kv_F \). This region is where the source term \( D_1g_1 \) has a contribution from \( g_2 \) evaluated near \( \delta \omega = 0 \).

In the non-resonant region we write

\[
(\omega - k\nu_F)g_1 = -i \frac{\partial g_1}{\partial t} - \frac{q}{\hbar} \Phi_1 \int g_1 \, dv_z - \frac{q}{\hbar} \Phi_2 \int g_1 \, dv_z + \frac{q}{\hbar} \int g_1 \, dv_z. \tag{29}
\]

Since the linear result \( g_1^L \propto \Phi_1(\omega - k\nu_F) \) holds to quadratic order, and time derivatives are assumed slow, we can use this in the term in the small (but important) term \( -i \partial_t g_1 \) and still be correct to cubic order. Inserting this into Poisson’s
equation, we obtain
\[
\left(1 + \frac{q^2}{\varepsilon_0 \hbar k^2} \int_{\omega} \frac{D_0 G_0}{\omega - k v_z} \, dv_z\right) \Phi_1
- i \frac{q^2}{\varepsilon_0 \hbar k^2} \int_{\omega - k v_z} \frac{D_0 G_0}{\omega - k v_z} \, dv_z \frac{\partial \Phi_1}{\partial t}
= \frac{q^2}{\varepsilon_0 \hbar k^2} \int_{\omega - k v_z} \frac{1}{\omega - k v_z} (\Phi_1^* D_1 g_2 - \Phi_2 D_2 g_2^*) \, dv_z
+ \frac{q^2}{\varepsilon_0 \hbar k^2} \int_{\omega - k v_z} g_1 \, dv_z. \tag{30}
\]

As \( \omega, k \) satisfy the linear dispersion relation, the first two terms on the left-hand side sum to zero, and the last is \( \frac{i \partial \Phi_1}{\partial \omega} \).

Thus the evolution equation for \( \Phi_1 \) is
\[
i \frac{\partial \Phi_1}{\partial \omega} \frac{\partial \Phi_1}{\partial t} = \frac{q^2}{\varepsilon_0 \hbar k^2} \int_{\omega - k v_z} \frac{1}{\omega - k v_z} (\Phi_1^* D_1 g_2 - \Phi_2 D_2 g_2^*) \, dv_z
+ \frac{q^2}{\varepsilon_0 \hbar k^2} \int_{\omega - k v_z} g_1 \, dv_z. \tag{31}
\]

By substituting for \( g_2, g_2^* \) according to the evolution equations and the linear result for \( g_1 \), the first integral is similar to that in equation (25) and can also be approximated in terms of the linear susceptibility. This is again detailed in an appendix. In the resonant region, where \( \delta \omega \approx k v_q \), we can solve for \( g_1 \) using equation (29) but dropping the time derivative. Since only \( g_2 \) is resonant, the other contributions simply modify the coefficients of the terms from the first integral.

After evaluating the coefficients, we have
\[
i \frac{\partial \Phi_1}{\partial \omega} \frac{\partial \Phi_1}{\partial t} = \frac{q^2}{\varepsilon_0 \hbar k^2} \left(27 \chi_3 - 32 \chi_2 - 6\right)|\Phi_1|^2 \Phi_1
+ \frac{q^2}{\varepsilon_0 \hbar k^2} \int_{\omega - k v_z} g_2 (v_z + v_q) - g_2 (v_z - v_q) \, dv_z. \tag{32}
\]

In each component of the integration region only one of the the terms in \( D_0 g_2 \) contributes, viz.
\[
\int_{\omega - k v_z} g_2 (v_z + v_q) - g_2 (v_z - v_q) \, dv_z
\approx \int_{\omega - k v_z} g_2 (v_z + v_q) \, dv_z - \int_{\omega - k v_z} g_2 (v_z - v_q) \, dv_z, \tag{33}
\]

\[
\approx \int_{\omega - k v_z} \left(\frac{1}{\omega - k (v_z + v_q)} - \frac{1}{\omega - k (v_z - v_q)}\right) g_2 (v_z) \, dv_z \tag{34}
+ \int_{\omega - k v_z} \frac{2 v_q g_2}{k (v_q - \tilde{v})^2} \, dv \tag{35}
\]

where \( \tilde{v} = v_z - \frac{\omega}{k} \) measures velocity relative to the resonance. This gives us the final evolution equation for \( \Phi_1 \),

\[
i \frac{\partial \Phi_1}{\partial \omega} \frac{\partial \Phi_1}{\partial t} = \frac{q^2 A}{\hbar^2 k^2 v_q^2} |\Phi_1|^2 \Phi_1 + \frac{q B}{\hbar k v_q} \Phi_1^* \Phi_2
+ \frac{2 q^2 \Phi_1^* \Phi_2}{\hbar k v_q} \int_{\omega - k v_z} \frac{g_2}{v_q - \tilde{v}^2} \, dv \tag{36}
\]

where the coefficients are
\[
A = \frac{1}{4} \left(27 \chi_3 - 32 \chi_2 - 6\right) \quad \text{and} \quad B = 8 \chi_2. \tag{37}
\]

The first term represents a nonlinear frequency shift. Since \( \Phi_2 \sim \Phi_1^* + \int g_2 \), the second term also contains a nonlinear frequency shift, as well as an additional coupling to \( g_2 \). It is the wave-particle interaction of the coupling to \( g_2 \) that is responsible for the damping; the nonlinear frequency shift turns out to be relatively unimportant.

Next we introduce normalized dimensionless variables by \( t \rightarrow t_\gamma, v_z \rightarrow (v - \omega/k)/v_F, \Phi \rightarrow q \Phi/(\hbar v_\gamma) \) and \( g \rightarrow q^2 g/y_\gamma^3/(\varepsilon_0 \hbar k^2 v_\gamma^2) \). As the evolution of the system occurs on a timescale much longer than the natural scales \( \omega^{-1}, v_F^{-1} \) we take an arbitrary value of \( \gamma \), giving an arbitrary parameter \( \alpha = k v_F / \gamma \) in the equations. Then equations (27) and (36) take the form

\[
\Phi_2 = \frac{v_F}{v_q} \left(1 + \frac{\chi_3}{4} \alpha^2 \Phi_1^2 \right) \tag{38a}
\]

\[
i \frac{\partial \Phi_1}{\partial t} = \frac{A}{\alpha v_q} \Phi_1^2 + \frac{B}{v_q} \Phi_1 \Phi_2^* + \Phi_2^* \int_{v_q - \tilde{v}^2} g_2 \, dv \tag{38b}
\]

and equation (23c) becomes

\[
i \frac{\partial g_2}{\partial t} - 2i \alpha v g_2 = i \phi_1^2 G_0 (v - 2v_q) - i \phi_2 G_0 (v - 2v_q)
+ \frac{i \phi_1^2 g_2}{\alpha (v - v_q) \Phi_1^2}. \tag{38c}
\]

where the integrals are over the resonant region. The dimensionless background distribution is given by \( G_0 (v) = (3 \hbar^2 \alpha / 32 v_q^2) \max(1 - v_q^2, 0) \).

In equation (38c), we have discarded terms containing the background distribution outside the Fermi sphere, that are non-zero for finite temperatures. However, at finite but low temperature, just outside the Fermi sphere \( v_F \), \( G_0 \approx \exp[(T_F/T) (1 - v_q^2/v_F^2)] \), and the discarded terms are therefore exponentially small. Thus the importance of multi-plasmon damping is not restricted to the extreme case of full degeneracy.

### 3.2. The three-plasmon case

Next we consider three-plasmon resonances, which become dominant for slightly longer wavelengths, such that the lowest resonance \( \omega - k v_z - m v_q = 0 \) occurs for \( n = 3 \), for some velocity \( v_z < v_F \). As indicated by the previous case of two-plasmon resonances, the nonlinear frequency shift only plays a modest role for the damping rate, and thus these terms are dropped henceforth. Overall the calculations use the same methods as the two-plasmon case but in this case the
important component of $g$ is $g_1$, near $\omega - kv \approx 0$. The process is presented schematically in figure 4.

With the same type of derivation as for the 2-plasmon damping, we find the set of normalized equations

$$
\begin{align*}
(1 + \chi_2 + \int_{v_{\text{res}}} G_0(v - 3v_q) \, dv) \Phi_2 &= \frac{-\chi_2 + 1/4}{\omega v_q} \Phi_1^2 + \Phi_1 \frac{1}{\omega} \int \frac{g_1}{v^2 - v^2} \, dv, \\
\frac{1}{\omega} \frac{\partial D}{\partial \Phi_1} &= \frac{-\chi_1}{\omega} \int g_1 \, dv \\
\frac{\partial g_1}{\partial t} &= -i\alpha v g_1 = \frac{\chi_2 - 1/4}{1 + \chi_2} \frac{\Phi_1^2}{\omega} (a\Phi_1 G_0(v) + b v g_1),
\end{align*}
$$

where the coefficients are given by $a = \frac{3}{2} + O(v/v_q)$ and $b = 1 + O(v^2/v_q^2)$. We omit the higher order terms as we have already made approximations at least this crude, and we confirm numerically that this omission shifts the shape of $g_1$ somewhat, but does not affect the damping rate significantly.

To obtain proper values for all parameters in the two-plasmon and three-plasmon system respectively, we need to solve the linear dispersion relation $\omega(k)$, which serves as the input for $\partial D/\partial \Phi_1$, $\chi_2$, etc., as shown in section 2.1.

4. Numerical results and validity

Next we study the two-plasmon system equations (38a)–(38c) and the three-plasmon system equations (40) and (41) numerically. We have implemented a Runge–Kutta scheme [23], fourth order in time, intended for such problems and obtained results stable against changing both the width of the resonant region and the resolution.

The damping of the wave amplitude scales the same in the two-plasmon and the three-plasmon case, and in both cases the following analytical fit

$$
|\Phi(t)| = |\Phi(0)|/(1 + t/t_0)^{1/2}
$$

is a good approximation. Here $t_0$ is a characteristic damping time that scales as

$$
t_0 = C(v_q, \omega/k) \left( \frac{\hbar \omega}{q\Phi(0)} \right)^2 \frac{1}{\omega}.
$$

The amplitude evolution for some typical cases is shown in figure 5. The factor $C$ varies between 0.03 and 0.5, depending on the precise location of the resonance, as shown in figure 6. The damping time does not decrease monotonously as the resonance is moved further into the bulk of the background distribution as one would perhaps expect prima facie; this is explained by the dependence of the coefficients in equations (38a)–(38c) on $k$.

Figure 5 also shows amplitude oscillations. These are due to a frequency shift that is an artefact of taking a finite resonant region. Taking a larger resonant region decreases the oscillations, but if the resonant region has to be made very large, the approximations we have used cannot be justified anymore. For the three-plasmon case, these artefacts are much smaller. The term $\propto \Phi_1^2$ in equation (38a) gives a term $\propto \Phi_1 |\Phi_1|^2$ when the expression for $\Phi_2$ is substituted into equation (38b), i.e., a nonlinear frequency shift term. However, this term is relatively unimportant and solving equations (38a)–(38c) with or without this term, does not alter the damping rate significantly. Thus the omission of the nonlinear frequency shift term when studying three-plasmon processes is justified—at least as far as we are only concerned with the evolution of the wave amplitude. If we shift our interest to the evolution of the resonant particles, see figure 7, the situation is somewhat changed. Here a frequency shift changes the exact location of the resonance. Since the approximation of a finite resonant region also gives rise to a frequency shift, this is still seen in figure 7. This effect is much less pronounced in the
three-plasmon case, as the size of the resonant region is less relevant in this case.

5. Conclusion and discussion

In the present paper we have studied wave-particle interaction due to multi-plasmon resonances. We note that the appearance of these new resonances is independent of the background distribution, and hence our results can be immediately generalized to cases with a finite temperature. At finite temperature, there are, however, always linearly resonant particles in the tail of the Fermi–Dirac distribution. Since the tail outside the Fermi sphere is exponentially small in $T/T_F$, we can expect the physics to be the same for low temperatures as for zero temperatures. In effect, by taking a completely degenerate plasma we have only ignored exponentially few particles, compared to the many particles near the multi-plasmon resonances.

A more quantitative estimate comes from comparing the linear dispersion relation based on equation (12) with that computed for an arbitrary degeneracy, see [17]. For low
temperatures, $T \lesssim 0.2 T_F$, the linear damping rate $\gamma$ is of the order $10^{-3} \omega_p^2$ or even smaller for the range of wave numbers considered in this paper. In figure 5, we can see that multi-plasmon damping happens on a similar timescale. For higher initial amplitudes nonlinear multi-plasmon damping is faster, while the linear damping time is the same, as seen from equation (43). This shifts the relative importance toward multi-plasmon damping when the amplitude is larger the allowed limit, resulting from the assumption that the resonant region is small.

The separation of the resonant velocity between the linear resonance and the multi-plasmon resonances with $n=1, 2, 3, \ldots$ is directly proportional to the standard quantum parameter $H = \hbar \omega_p / E_k$. Here $E_k$ is the characteristic kinetic energy of the particles, given by $E_k = k_B T$ for a degenerate plasma and $E_k = k_B T$ in the non-degenerate case. For plasma densities of the order of metallic systems, $H$ is of the order of unity when $T \ll T_F$, and hence the resonances for $n=1$ and $n=3$ are well separated. Increasing the temperature, $H$ is kept of the order unity at least up to $T \sim T_F$. However, increasing the temperature further to temperatures $T \gg T_F$, the relative importance of the new resonances gradually diminishes, since the separation from the classical resonances become small.

For $T \ll T_F$, two plasmon processes occur for wave-numbers $k_3 < k < k_{cr}$, where $k_3 \simeq 0.69 \omega_p / v_F$ and $k_{cr} = 0.98 \omega_p / v_F$. Whenever two-plasmon processes occur, we also have three-plasmon processes, which give a damping of comparable magnitude. A more precise comparison of two- and three-plasmon damping is made in figure 6. For $k_3 < k < k_2$, where $k_2 = 0.60 \omega_p / v_F$, three-plasmon damping dominates and two-plasmon processes are absent. This holds because $n$-plasmon processes with $n \geq 4$ are of higher order in an amplitude expansion. For multi-plasmon processes with $n = 2$ or $n = 3$, the damping rate decreases with the amplitude, as given by equation (43). Thus these processes are of limited importance for very small amplitudes, in which case collisional damping will be the dominant damping mechanism when $k < k_{cr}$.

In the present paper we have been concerned with damping mechanisms, but just like standard Landau damping can turn into an instability by modifying the initial conditions.
distribution, the same is true for multi-plasmon resonances. In this context it should be noted that generally for resonant velocities in equation (1), the negative signs in the expression are associated with resonant particles absorbing wave quanta from the wave (leading to wave damping), whereas the positive signs are associated with particles emitting quanta to the wave (leading to wave growth), if the wave is propagating in the positive direction. In our calculation we have started from a background distribution in thermodynamic equilibrium (more specifically a fully degenerate Fermi–Dirac distribution), and have chosen a wavenumber with resonant velocities such that no resonant particles with the positive sign in equation (1) are present initially. Actually, during the wave damping process particles with a velocity equal to \( v_{res,-2} \) (for two-plasmon damping) or \( v_{res,-3} \) (for three-plasmon damping) are transferred to velocities \( v_{res,2} \) and \( v_{res,3} \) respectively, when absorbing multiple wave quanta. However, for the modest initial amplitude considered in our paper this is a small effect affecting only a limited number of particles, that does not deplete the background distribution significantly. As a consequence the nonlinear modification of the background distribution that takes place is not sufficient to reverse the damping process. However, for a different background distribution where the initial number of particles is larger at the positive resonant velocity (i.e. a background distribution where \( f_{0}(v_{res,+2}) > f_{0}(v_{res,-3}) \)) we should expect the process to run in the opposite direction, i.e. we should have a wave instability rather than wave damping. In the limit where \( \hbar k/2m \to 0 \) this turns into the standard bump-on-tail instability that follows from a negative slope of the dispersion function. The generality of the mechanism studied here follows from the fact that wave-particle damping of any wave mode, provided the wavelength is not much longer than the de Broglie wavelength. The generality for broad classes of systems. In particular multi-plasmon resonances can play a role in dense plasmas such as metallic plasmas, white dwarf stars and inertially confined plasmas. More generally, the methods used here can be used to study resonant interaction involving multiple wave quanta of other types of waves, which can be of significance for wave-particle damping of any wave mode, provided the wavelength is not much longer than the de Broglie wavelength. The generality of the mechanism studied here follows from the fact that equation (1) can be derived from energy-momentum conservation only. As a consequence, resonant wave-particle interaction in high-density plasmas involving intense short-wavelength electromagnetic radiation can be possible for \( n \geq 1 \), potentially leading to new opportunities for particle acceleration. However, this remains a project for further study.

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Appendix. Detailed calculation of coefficients

We detail how to relate the coefficient for \( \Phi_{1}^{2} \) in (25) to the dispersion function. The coefficient is

\[
q^{3} \frac{\hbar^{2}}{8} \epsilon_{0} \int_{nr} \frac{G_{0}(v_{c} + 2v_{q}) - G_{0}(v_{c})}{(\omega - k_{v}c)[\omega - k(v_{c} + v_{q})]} dv_{v} = - \frac{G_{0}(v_{c}) - G_{0}(v_{c} - 2v_{q})}{(\omega - k_{v}c)[\omega - k(v_{c} - v_{q})]} \]

where by partial fractions decomposition is a sum of terms of the form \( \int_{nr} G_{0}(v_{c})/(\omega - k_{v}c + mv_{q}) \) where \( m \) is an integer. These integrands are of the same form as in the linear dispersion function (9), but the integral is over the non-resonant region only. Since by assumption the resonant region is small, it is valid to approximate \( \int_{nr} \approx \int_{nr} + \int_{res} \) and use the expression (12), in case the pole contribution is small. Otherwise the susceptibility should be computed from (11).

Concretely, the two terms in the integrand as

\[
\frac{G_{0}(v_{c} + 2v_{q}) - G_{0}(v_{c})}{(\omega - k_{v}c)[\omega - k(v_{c} + v_{q})]} = \frac{G_{0}(v_{c}) - G_{0}(v_{c} - 2v_{q})}{(\omega - k_{v}c)[\omega - k(v_{c} - v_{q})]} \]

\[
\frac{1}{\omega - k(v_{c} + v_{q})} - \frac{1}{\omega - k_{v}c} \]

(A1)

\[
\frac{G_{0}(v_{c}) - G_{0}(v_{c} - 2v_{q})}{(\omega - k_{v}c)[\omega - k(v_{c} - v_{q})]} = \frac{1}{\omega - k_{v}c} - \frac{1}{\omega - k(v_{c} - v_{q})} \]

(A2)

Now, by shifting the integration variable it is seen that

\[
q^{3} \frac{\hbar^{2}}{8} \epsilon_{0} v_{q} \int_{nr} \frac{G_{0}(v_{c} + 2v_{q}) - G_{0}(v_{c})}{\omega - k(v_{c} + v_{q})} dv_{v} = \frac{q}{8\hbar k_{v}c} \chi_{1} = \frac{q}{8\hbar k_{v}c} \]

(A3)

where the last step is that \( \chi_{1} = -1 \), as \( \omega, k \) verify the linear dispersion relation. Furthermore, the \( G_{0}(v_{c})/(\omega - k_{v}c) \) terms enter with opposite signs and cancel. Therefore, the coefficient is

\[
\frac{-q}{4\hbar k_{v}c} - \frac{q}{4\hbar k_{v}c} \chi_{2} = \frac{-q}{4\hbar k_{v}c} \left( \frac{1}{4} + \chi_{2} \right) \]

(A4)
We have
\[ q^2 \Phi^*_2 = \int_{\Omega} \frac{1}{\omega - k (v_i + v_j)} \left[ g_1(v_i + 2v_j) - g_1(v_i) \right] - [G_0(v_i + v_j) - G_0(v_i - v_j)] \Phi_1 + [G_0(v_i + v_j) - G_0(v_i - v_j)] \Phi_2 \]
and
\[ q^2 \Phi^*_1 = \int_{\Omega} \frac{1}{\omega - k (v_i + v_j)} \left[ g_1(v_i + 2v_j) - g_1(v_i) \right] - [G_0(v_i + v_j) - G_0(v_i - v_j)] \Phi_2 \]
respectively. Adding these together and taking into account the integration variable, we can write these as
\[ \int_{\Omega} \frac{1}{\omega - k (v_i + v_j)} \left[ g_1(v_i + 2v_j) - g_1(v_i) \right] - [G_0(v_i + v_j) - G_0(v_i - v_j)] \Phi_2 \]
Thus we need to evaluate for the \( \Phi_1 \) part,
\[ \int_{\Omega} \frac{1}{\omega - k (v_i + v_j)} \left[ g_1(v_i + 2v_j) - g_1(v_i) \right] - [G_0(v_i + v_j) - G_0(v_i - v_j)] \Phi_2 \]
Disregarding the prefactor, after a shift of integration variable and partial fractions, the first fraction is
\[ \int_{\Omega} \frac{1}{k^2 v_q} \left[ 2(\omega - k (v_i + v_j)) - 1 \right] \]
while the last fraction is
\[ \int_{\Omega} \frac{1}{k^2 v_q} \left[ 2(\omega - k (v_i + v_j)) - 1 \right] \]
These enter with the same sign. With suitable shifts of integration variable, we can write these as
\[ \int_{\Omega} \frac{1}{k^2 v_q} \left[ 2(\omega - k (v_i + v_j)) - 1 \right] \]
respectively. Adding these together and taking into account the prefactor again yields
\[ \frac{q^2}{2h^2 k^2 v_q} \left( \frac{27}{2} \chi_3 - 8 \chi_2 - 1 \right) \]
Therefore, the integral is

\[ \frac{q^2 \Phi_2^* G_0}{\hbar} \int_{\mathcal{M}} \frac{G_0(v_c + 3v_q) - G_0(v_c + v_q)}{\omega - k v_c} \frac{G_0(v_c - v_q) - G_0(v_c - 3v_q)}{\omega - k (v_c - 2v_q)} \frac{1}{2kv_q} - \frac{1}{\omega - k (v_c + 2v_q)}] \]

With partial fractions, the two terms are

\[ \frac{q^2 \Phi_2^* G_0}{\hbar} \int_{\mathcal{M}} \frac{G_0(v_c + 3v_q) - G_0(v_c + v_q)}{\omega - k v_c} \frac{G_0(v_c - v_q) - G_0(v_c - 3v_q)}{\omega - k (v_c - 2v_q)} \frac{1}{2kv_q} - \frac{1}{\omega - k (v_c + 2v_q)}] \]

Therefore, the integral is

\[ \frac{q^2 \Phi_2^* G_0}{\hbar} \int_{\mathcal{M}} \frac{G_0(v_c + 3v_q) - G_0(v_c + v_q)}{\omega - k (v_c + 2v_q)} \frac{G_0(v_c - v_q) - G_0(v_c - 3v_q)}{\omega - k (v_c - 2v_q)} \frac{1}{2kv_q} - \frac{1}{\omega - k (v_c + 2v_q)}] \]

Our conclusion is that

\[ \frac{q^2}{\epsilon_0 \hbar k^3} \int_{\mathcal{M}} \frac{1}{\omega - k v_c} (\Phi_1^* D_1 g_2^* - \Phi_2^* D_2 g_2^*) \text{dv}_z \]

\[ = \frac{q^2}{4 \hbar^2 k^3 v_q^2} (27 \chi_3 - 32 \chi_2 - 6) |\Phi_1|^2 \Phi_1 \]

\[ + \frac{q}{2 \hbar k v_q} (27 \chi_3 + 16 \chi_2 + 1) \Phi_1^* \Phi_2 \]

\[ = \frac{q^2}{4 \hbar^2 k^3 v_q^2} (27 \chi_3 - 32 \chi_2 - 6) |\Phi_1|^2 \Phi_1 + \frac{q}{\hbar k v_q} 8 \chi_2 \Phi_1^* \Phi_2. \]

\[ \text{(A21)} \]

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