Abstract. In this paper we present a complete computation of the cosmic microwave background (CMB) anisotropies up to third order from gravitational perturbations accounting for scalar, vector and tensor perturbations. We then specify our results to the large-scale limit, providing the evolution of the gravitational potentials in a flat universe filled with matter and cosmological constant which characterizes the integrated Sachs–Wolfe effect. As a by-product in the large scale approximation we are able to give non-perturbative solutions for the photon geodesic equations. Our results are the first step towards providing a complete theoretical prediction for cubic nonlinearities which are particularly relevant for characterizing the level of non-Gaussianity in the CMB through the detection of the four-point angular connected correlation function (trispectrum). For this purpose we also allow for generic initial conditions due to primordial non-Gaussianity.

Keywords: CMBR theory, cosmological perturbation theory, inflation

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1. Introduction

The three-year dataset of the Wilkinson Microwave Anisotropy Probe (WMAP) on cosmic microwave background (CMB) temperature anisotropies has offered a wealth of information about the evolution of the universe of unprecedented accuracy [1] and prepared the way for even more ambitious future missions such as those of the Planck satellite [2] and CMB polarization observations. Such precise measurements have been accompanied by increasing theoretical efforts in predicting the signatures on the CMB from various cosmological scenarios. In particular a lot of attention in the last few years has been dedicated to the statistical properties of the CMB beyond the power spectrum, in search for possible non-Gaussian signatures [3]. Non-Gaussianity of the CMB owes its importance to the possibility of unveiling crucial aspects of the physics of both the early and the late universe, which would be unreachable using only the CMB power spectra information. Different mechanisms for the generation of cosmological perturbations predict different amplitudes and shapes of primordial non-Gaussianity; thus a positive detection of NG or an upper limit on its amplitude is powerful in discriminating among the various competing scenarios which would be indistinguishable otherwise [3]. An illuminating example is given by the standard models of inflation which typically predict a very low content of non-Gaussianity [4, 5] and, as such, they could be completely ruled out if a positive detection could be achieved. CMB non-Gaussianity can also have a non-primordial origin due to secondary anisotropies which arise when the CMB photons leave the last scattering surface and cross the large-scale structure of the universe. These include both secondary scatterings (such as the thermal and kinetic Sunyaev–Zel’dovich effects, produced by the thermal and bulk motions of electrons in clusters, and the Ostriker–Vishniac effect, due to bulk motions modulated by linear density perturbations) and gravitational secondaries such as gravitational lensing and the nonlinear integrated Sachs–Wolfe effect or, on smaller scales, the Rees–Sciama effect due to the nonlinear evolution of the potentials. Such nonlinearities can give important and new information about the dark matter and the dark energy...
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Statistics like the bispectrum (the Fourier transform of the three-point correlation function) and the trispectrum (four-point correlation) can then be used to assess the level of primordial non-Gaussianity on various cosmological scales and to discriminate it from the ones induced by secondary anisotropies and systematic effects.

Up to now most of the attention has focused on the three-point statistics both for the primordial non-Gaussianity and for the analysis of the angular bispectrum of the CMB temperature and polarization anisotropies [3,10], [14]–[17]. On the other hand, the four-point correlation function could display interesting features. From the observational point of view it has been argued that it could be even more sensitive to primordial non-Gaussianity than the bispectrum for very small angular scales in the next CMB experiments [18,19]. From the theoretical point of view the four-point connected correlation function for the large-scale CMB anisotropies has been computed in [20] giving the theoretical predictions for the quadratic and cubic nonlinearities (which include generic non-Gaussian initial conditions), while [21,22] have computed the trispectrum from single-field and uncoupled multiple fields in slow-roll inflation showing that it is of the order of the slow-roll parameters. Interestingly, in some cases, such as in some configurations of the curvaton scenarios [23,24], the main source of a non-Gaussian signal can come from the four-point correlation functions, the three-point correlation function being suppressed (see also the phenomenological model discussed in [1,25]).

However, most of these computations deal with the trispectrum of the uniform density curvature perturbation \( \zeta \) on large scales within a given inflationary model, but this is not the physical quantity that is observed. The same is true of course for the bispectrum of the curvature perturbation. But, if for the case of the curvature perturbation bispectrum one is sure that the information about the primordial non-Gaussianity in the final observable quantity (the CMB anisotropies) is just the one obtained by evolving linearly the curvature bispectrum, the same is not true any more for the trispectrum. Let us consider the standard way to characterize the level of non-Gaussianity in the gravitational potential by expanding it as

\[
\Phi = \Phi_L + f_{NL}(\Phi_L^2 - \langle \Phi_L^2 \rangle) + g_{NL}\Phi_L^3,
\]

where \( \Phi_L \) is the linear first-order Gaussian part, and \( f_{NL} \) and \( g_{NL} \) are the parameters which measure the quadratic and cubic nonlinearities. In fact, the nonlinearity parameters might have, in general, a non-trivial scale dependence. At linear order the curvature perturbation is \( \zeta^{(1)} = 5\Phi_L/3 \) during the matter-dominated epoch, and, for example, the Sachs–Wolfe effect tells us that \( \Delta T/T = -\Phi_L/3 \). The standard approach followed in the literature is to extend both of these two relations also at higher order, so that, for example, for quadratic non-Gaussianity one writes \( \zeta = 5\Phi/3 \) using the expansion (1.1) up to second order. Evolving linearly the perturbations in such a way just accounts for the primordial content of non-Gaussianity and is justified only when one assumes that the primordial level of non-Gaussianity is much larger of the second-order corrections (in the form of first-order squared perturbations) which arise both in the relation between the curvature perturbation and the gravitational potential and when computing the CMB anisotropies. One is guaranteed that such primordial contributions in the final CMB anisotropies will appear as computed in this way because it is already an intrinsically second-order nonlinearity and all the transfer functions will therefore be the same as at linear order. For the CMB bispectrum this is also all that is necessary to account for

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the primordial non-Gaussianity. For the trispectrum the situation is different, however. Evolving linearly the primordial cubic non-Gaussianity in equation (1.1) would just lead to \( \zeta^{(3)}/6 = 5g_{NL}/3\Phi_L^3 \) (splitting \( \zeta = \zeta^{(1)} + \zeta^{(2)}/2 + \cdots \) into a first- and higher-order parts). Since the curvature perturbation remains constant on superhorizon scales (for adiabatic perturbations) it is a useful quantity to keep track of the primordial non-Gaussianity. Therefore, suppose we parameterize the primordial non-Gaussianity as

\[
\zeta = \zeta^{(1)} + (a_{NL} - 1) \left( \zeta^{(1)} \right)^2 + (b_{NL} - 1) \left( \zeta^{(1)} \right)^3,
\]

where the two nonlinearity parameters depend on the physics of a given scenario for the generation of the perturbations (for example, for standard single-field models of inflation \( a_{NL} = 1 \) and \( b_{NL} = 1 \), plus tiny corrections proportional to the slow-roll parameters). Thus the relation \( \zeta^{(3)}/6 = 5g_{NL}/3\Phi_L^3 \) is equivalent to

\[
g_{NL} = \frac{25}{9}(b_{NL} - 1).
\]

However, this does not catch at all the whole information about the primordial non-Gaussianity that is contained in the CMB anisotropies up to third order. This is due to two reasons: first, as a source for the evolution of the gravitational potentials at third order now there are also the gravitational potentials at second order which inevitably contain primordial nonlinearities proportional to the \( f_{NL} \) (or \( a_{NL} \)) parameter; and second, when computing the CMB anisotropies at third order, one still gets additional terms proportional to \( f_{NL}\Phi_L^3 \). This point is clear in the results we present in sections 3 and 4. See, for example, the first line of equation (3.5), or equations (4.2) and (4.7). Another example can be found in [20], where a fully nonlinear expression for the Sachs–Wolfe effect has been obtained for generic non-Gaussian initial conditions. The corresponding parameter \( g_{NL} \) for CMB anisotropies has been computed (see equation (67) of that reference) by accounting for all possible dependences on the primordial non-Gaussianity finding an expression of the type

\[
g_{NL} = \frac{25}{9}(b_{NL} - 1) + A(k_1, k_2, k_3; a_{NL}),
\]

which, contrary to equation (1.3), depends also on the quadratic nonlinearity parameter (here \( k_i \) are the perturbation wavenumbers in Fourier space). A similar expression to equation (1.4) is given for the gravitational potentials at third order in equation (4.23). These examples show that, even in the case of a ‘local’ model for quadratic non-Gaussianity (i.e. a constant \( f_{NL} \) parameter), at higher orders such nonlinearities are modulated by first-order perturbations generating a scale-dependent non-Gaussianity.

The best limits to date on the \( f_{NL} \) parameter come from measurements of the CMB bispectrum on the WMAP data giving \(-36 \leq f_{NL} \leq 100 \) [1,14,26], while there is at present no real bound on \( g_{NL} \). Given the increasing precision of future measurements of CMB anisotropies, it is clear that it is of fundamental importance to provide accurate predictions for all the cubic nonlinearities that enter in the evolution of the cosmological perturbations. This is not only mandatory to be able to evaluate the trispectrum of CMB anisotropies, but becomes crucial if one wants to account for the precise dependence of the trispectrum on the primordial non-Gaussianity. Spurred on by these considerations, in this paper we will focus on the CMB anisotropies up to third order from gravitational perturbations due to the redshift the photons suffer when they travel from the last scattering surface to the observer. The computation will be performed following [27]–[29] by perturbing, at
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In the last part of the paper we focus on the large-scale approximation and on scalar perturbations for a universe filled with non-relativistic matter and a non-vanishing cosmological constant. In fact, one of the goals is to characterize the evolution of the gravitational potentials on large scales which is responsible for the (late) integrated Sachs–Wolfe effect, thus completing the analysis for the large-scale CMB anisotropies at third order already started in [20]. Of course it is outside the goals of this paper to deal with the anisotropies generated from the dynamics at recombination and due to the scattering terms.

2. Temperature anisotropies

We are interested in the pattern of fluctuations of the CMB temperature as measured by an observer in a perturbed flat Friedmann–Robertson–Walker spacetime.

The line element can be written as
\[ \text{d}s^2 = a^2(\eta) g_{\mu\nu} \text{d}x^\mu \text{d}x^\nu = a^2(\eta) \left[ g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(1)} + g_{\mu\nu}^{(2)} + g_{\mu\nu}^{(3)} + \cdots \right] \text{d}x^\mu \text{d}x^\nu, \]
where \( a(\eta) \) is the scale factor in conformal time \( \text{d}\eta = \text{d}t/a(t) \), \( g_{\mu\nu}^{(0)} \) is the background Minkowski metric and \( g_{\mu\nu}^{(r)} \) is the \( r \)th-order metric perturbation.

Photons travel along null geodesics \( x^\mu(\lambda) \), where \( \lambda \) is an affine parameter in the conformal metric \( g_{\mu\nu} \). The photon path connects the point of observation, with coordinates \( x^\mu_O = (\eta_O, x_O) \), to the hypersurface of emission, defined as the spacelike hypersurface of constant conformal time \( \eta_E \). The actual last scattering surface is the intersection of the observer’s past light cone with this hypersurface. We assume that on the constant \( \eta_E \) hypersurface every point \( p \) emits thermal radiation, characterized by a temperature \( T(p, \hat{d}) \) which depends on the point of emission \( p \) and direction \( \hat{d} \), described by the vector \( \hat{d} \) normalized to unity in the conformal background metric. The different photon paths are specified by the direction from which they arrive at \( O \), described by a vector \( \hat{e} \), normalized to unity in the conformal background metric; this unit vector can be thought of as the direction toward which the observer is pointing an antenna. The initial conditions \( x^\mu_O, \hat{e} \) determine the point \( p \) and direction \( \hat{d} \) of emission.

During their travel from the last scattering surface to the observer the CMB photons suffer a redshift determined by the ratio of the emitted frequency \( \omega_E \) and the observed one \( \omega_O \). For a blackbody spectrum, the ratio \( \omega/T \) is constant along the photon path, and the temperature measured by an observer is given by
\[ T_O(x_O, \hat{e}) = \frac{\omega_O}{\omega_E} T_E(p, \hat{d}). \]

The expression for the frequency is
\[ \omega = -g_{\mu\nu} u^\mu k^\nu, \]
where \( u^\mu \) is the 4-velocity of the observer or emitter, normalized to \( a^2 g_{\mu\nu} u^\mu u^\nu = -1 \), and \( k^\nu(\lambda) = \text{d}x^\nu/\text{d}\lambda \) is the photon wavevector, tangent to the null geodesic \( x^\nu(\lambda) \).

A complete computation of the CMB anisotropies due to the nonlinear dynamics taking place at the last scattering epoch has been performed in [30, 31], where the full system of Boltzmann equations up to second order for photons, baryons and cold dark matter have been presented together with analytical solutions in the tight coupling approximation.
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Given the initial conditions \( \mathbf{v}_i, \omega, \omega \), we need to compute the quantities \( \mathbf{p}, \mathbf{d}, \omega \), propagating the photons back from the observation to the emission surface. These will depend on the photon path and the associated wavevector, which we expand in series of the metric perturbations \( g^{(r)}_{\mu \nu} \) and their derivatives:

\[
x^\mu (\lambda) = x^{(0)} \mu (\lambda) + x^{(1)} \mu (\lambda) + x^{(2)} \mu (\lambda) + x^{(3)} \mu (\lambda) + \cdots ,
\]

(2.4)

\[
k^{\mu} (\lambda) = k^{(0)} \mu (\lambda) + k^{(1)} \mu (\lambda) + k^{(2)} \mu (\lambda) + k^{(3)} \mu (\lambda) + \cdots .
\]

(2.5)

We find it useful to write the perturbed (conformal) metric as

\[
g_{\mu \nu} \frac{d x^\mu}{d \tau} \frac{d x^\nu}{d \tau} = -e^{2 \Phi} d\eta^2 + 2 \omega_i \partial_i d\eta + (e^{-2 \Psi} \delta_{ij} + \chi_{ij}) dx^i dx^j,
\]

(2.6)

which is valid at any order in perturbation theory. The two gravitational potentials \( \Phi \) and \( \Psi \) correspond to scalar metric perturbations, \( \omega_i \) includes a scalar and a vector perturbation, and \( \chi_{ij} \) contains another scalar, as well as vector and tensor perturbations. To make contact with the usual perturbative calculations one expands every quantity as \( \Phi = \Phi^{(1)} + \frac{1}{2} \Phi^{(2)} + \frac{1}{6} \Phi^{(3)} + \cdots \) and analogously for the others. Notice that, unlike \( \omega_i^{(r)} \) and \( \chi_{ij}^{(r)} \), the quantities \( \Phi^{(r)} \) and \( \Psi^{(r)} \) are not the usual \( r \)-th order scalars \( \phi^{(r)}, \psi^{(r)} \) appearing in [3, 32]. However, it is immediate to find the relation between these quantities. Up to third order \( \Phi^{(1)} = \phi^{(1)}, \Psi^{(1)} = \psi^{(1)}, \Phi^{(2)} = \phi^{(2)} - 2 \phi_0^2, \Psi^{(2)} = \psi^{(2)} + 2 \psi_0^2, \Phi^{(3)} = \phi^{(3)} - 6 \phi_0 \phi^{(2)} + 8 \phi_0^3, \Psi^{(3)} = \psi^{(3)} + 6 \psi_0 \psi^{(2)} + 8 \psi_0^3 \). The form of the metric (2.6) greatly helps in the intermediate computations: however, some results will be expressed in the variables \( \phi^{(r)} \) and \( \psi^{(r)} \) because they appear in a more compact form. Notice that in this section we will not choose a particular gauge so all the following expressions for the CMB anisotropies are valid in any gauge.

The 4-velocity will be expanded as

\[
u^\mu = \frac{1}{a} \left( \delta^\mu_0 + v^{(1) \mu} + \frac{1}{2} v^{(2) \mu} + \frac{1}{6} v^{(3) \mu} + \cdots \right).
\]

(2.7)

The zero component of the velocity is fixed from the normalization condition; we find

\[
v_0^{(1)} = - \phi^{(1)}; \]

(2.8)

\[
v_0^{(2)} = - \phi^{(2)} + 3 \left( \phi^{(1)} \right)^2 + 2 \omega^{(1) i} v_i^{(1)} + v_i^{(1)} v_i^{(1)}; \]

(2.9)

\[
v_0^{(3)} = - \phi^{(3)} + 3 \omega^{(2) i} v_i^{(1)} + 3 \omega^{(1) i} v_i^{(2)} + 9 \phi^{(2)} \phi^{(1)} + 3 \left( \chi^{(1) i} - 2 \psi^{(1) i} \delta_{ij} \right) v_i^{(1)} v_j^{(1)}
- 12 \phi^{(1)} \omega^{(1) i} v_i^{(1)} - 15 \left( \phi^{(1)} \right)^3 - 3 \phi^{(1)} v_i^{(1)} v_i^{(1)} + 3 \nu_i^{(2) i} v_i^{(1)}. \]

(2.10)

In order to compute the observed temperature up to third order in perturbations, we expand the frequency as

\[
\omega = \omega^{(0)} \left( 1 + \omega^{(1)} + \omega^{(2)} + \omega^{(3)} + \cdots \right),
\]

(2.11)

and the temperature at emission as

\[
T_e (\mathbf{p}, \mathbf{d}) = T_e^{(0)} (1 + \tau (\mathbf{p}, \mathbf{d})).
\]

(2.12)
where $\tau = \tau^{(1)} + \tau^{(2)} + \tau^{(3)} + \cdots$ is the intrinsic temperature fluctuation at emission. A calculation of this quantity up to third order is beyond the goal of this paper, since we are interested in the additional effects of gravity along the photon path. However, on large scales (bigger than the horizon at recombination) it has been computed in a non-perturbative way in [20]. Its fully nonlinear expression is very simple: $\tau = e^{-2\Phi/3} - 1$. Moreover, on smaller scales, a complete treatment of the CMB anisotropies including the acoustic oscillations on the surface of last scattering up to second order have been studied analytically in [30,31] and a full numerical evaluation can be performed [33] using the set of Boltzmann equations provided in [30].

Now, we have to take into account that we need to expand the point $p$ and direction $\hat{d}$ at emission as $p = p^{(0)} + p^{(1)} + \cdots$ and $\hat{d} = \hat{d}^{(0)} + \hat{d}^{(1)} + \cdots$; then we Taylor expand $\tau(p, \hat{d})$ around the background values, obtaining

$$\tau(p, \hat{d}) = \tau^{(1)} + \left[ \tau^{(2)} + p^{(1)}_i \frac{\partial \tau^{(1)}}{\partial x^i} + d^{(1)}_i \frac{\partial \tau^{(1)}}{\partial d^i} \right]$$

$$+ \left[ \tau^{(3)} + p^{(1)}_i \frac{\partial \tau^{(2)}}{\partial x^i} + d^{(1)}_i \frac{\partial \tau^{(2)}}{\partial d^i} + \frac{1}{2} p^{(1)}_i p^{(1)}_j \frac{\partial^2 \tau^{(1)}}{\partial x^i \partial x^j} + p^{(1)}_i d^{(1)}_j \frac{\partial^2 \tau^{(1)}}{\partial x^i \partial d^j} \right] \frac{\partial d^{(2)}_i}{\partial d^i} + \frac{1}{2} d^{(1)}_i d^{(1)}_j \frac{\partial^2 \tau^{(1)}}{\partial x^i \partial d^j} + \frac{1}{2} d^{(2)}_i \frac{\partial \tau^{(1)}}{\partial d^i} + \frac{1}{2} d^{(2)}_i \frac{\partial \tau^{(1)}}{\partial d^i} \right],$$

(2.13)

where it is understood that all quantities are evaluated at $p^{(0)}$, $\hat{d}^{(0)}$. We find $p^{(1)i} = x^{(1)i} - k^{(0)i} x^{(0)i}$, $p^{(2)i} = x^{(2)i} - k^{(0)i} x^{(0)i} - k^{(1)i} x^{(1)i}$, $d^{(1)i} = (k^{(0)i} + k^{(1)i})/(|k^{(0)i}| + |k^{(1)i}) - k^{(0)i}/|k^{(0)i}|$, $d^{(2)i} = (k^{(0)i} + k^{(1)i} + k^{(2)i})/(|k^{(0)i}| + k^{(1)i} + k^{(2)i}) - (k^{(0)i} + k^{(1)i})/(|k^{(0)i}| + k^{(1)i})$. Notice that therefore the quantities $d^{(0)}$ are just the difference between the photon normalized wavevector up to the th $i$th order and that at the $(i-1)$th order.

Performing all the expansions in equation (2.2) and dividing by the background temperature $a_\infty/a_O T^0_\infty$, we find

$$\frac{\delta T^{(1)}}{T} = \tilde{\omega}^{(1)}_O - \tilde{\omega}^{(1)}_E + \tau^{(1)};$$

(2.14)

$$\frac{\delta T^{(2)}}{T} = \tilde{\omega}^{(2)}_O - \tilde{\omega}^{(2)}_E + \tau^{(2)} + p^{(1)}_i \frac{\partial \tau^{(1)}}{\partial x^i} + d^{(1)}_i \frac{\partial \tau^{(1)}}{\partial d^i} + \left( \tau^{(1)} - \tilde{\omega}^{(1)}_E \right) \left( \tilde{\omega}^{(1)}_O - \tilde{\omega}^{(1)}_E \right);$$

(2.15)

$$\frac{\delta T^{(3)}}{T} = \tilde{\omega}^{(3)}_O - \tilde{\omega}^{(3)}_E + \tau^{(3)} + p^{(1)}_i \frac{\partial \tau^{(2)}}{\partial x^i} + d^{(1)}_i \frac{\partial \tau^{(2)}}{\partial d^i} + \frac{1}{2} p^{(1)}_i p^{(1)}_j \frac{\partial^2 \tau^{(1)}}{\partial x^i \partial x^j}$$

$$+ p^{(1)}_i d^{(1)}_j \frac{\partial \tau^{(1)}}{\partial x^i \partial d^j} + \frac{1}{2} d^{(1)}_i d^{(1)}_j \frac{\partial^2 \tau^{(1)}}{\partial x^i \partial d^j} + p^{(1)}_i \frac{\partial \tau^{(1)}}{\partial x^i} + d^{(1)}_i \frac{\partial \tau^{(1)}}{\partial d^i}$$

$$+ \left( \tau^{(2)} + p^{(1)}_i \frac{\partial \tau^{(1)}}{\partial x^i} + d^{(1)}_i \frac{\partial \tau^{(1)}}{\partial d^i} - \tilde{\omega}^{(2)}_E \right) \left( \tilde{\omega}^{(1)}_O - \tilde{\omega}^{(1)}_E \right)$$

$$+ \left( \tau^{(1)} - \tilde{\omega}^{(1)}_E \right) \left( \tilde{\omega}^{(2)}_O - \tilde{\omega}^{(2)}_E + (\tilde{\omega}^{(1)}_E)^2 - \tilde{\omega}^{(1)}_E \tilde{\omega}^{(1)}_O \right).$$

(2.16)

The expansion of the frequency using equation (2.3) yields

$$\tilde{\omega}^{(1)}_E = k^{(0)}_i + \phi^{(1)}_i + \omega^{(1)}_i v^{(1)}_i + v^{(1)}_i e^i;$$

(2.17)
\[
\tilde{\omega}_c^{(2)} = k_0^{(2)} + \frac{1}{2} \omega^{(2)}_i e^i + \frac{1}{2} \psi^{(1)}_i v^{(1)}_i - \frac{1}{2} (\phi^{(1)})^2 + \phi^{(1)} k_0^{(1)} - \omega^{(1)}_i k_0^{(1)} - v^{(1)}_i k_0^{(1)} - \phi^{(1)} \omega^{(1)}_i e^i - 2 \psi^{(1)}_i v^{(1)}_i e^i + \chi_{ij}^{(1)} v^{(1)}_i e^i + \frac{1}{2} v^{(2)}_i e^i \\
+ \Delta \lambda^{(1)} \frac{dk^{(0)}}{d\lambda} + p^{(1)}_i \left( \partial_i \phi^{(1)} + \partial_i \omega^{(1)} e^i + \partial_i v^{(1)}_j e^j \right);
\]

(2.18)

\[
\tilde{\omega}_c^{(3)} = k_0^{(3)} + \Delta \lambda^{(1)} \frac{dk^{(0)}}{d\lambda} + \Delta \lambda^{(2)} \frac{dk^{(1)}}{d\lambda} + \frac{1}{2} \left( \Delta \lambda^{(1)} \right)^2 \frac{d^2 k_0^{(0)}}{d\lambda^2} \\
+ \frac{1}{6} \phi^{(3)} + \frac{1}{2} p^{(1)}_i \partial_i \phi^{(2)} + p^{(2)}_i \partial_i \phi^{(1)} + \frac{1}{2} p^{(1)}_i p^{(1)}_j \partial_i \partial_j \phi^{(1)} \\
+ e^i \left( \frac{1}{6} v^{(3)}_i + \frac{1}{2} p^{(1)}_i \partial_i v^{(2)}_i + p^{(2)}_i \partial_j v^{(1)}_i + \frac{1}{2} p^{(1)}_j p^{(1)}_k \partial_j \partial_k v^{(1)}_i \right) \\
- \frac{1}{2} \phi^{(2)} \phi^{(1)} + \frac{1}{2} \left( \chi_{ij}^{(1)} - 2 \psi^{(1)} \delta_{ij} \right) v^{(1)}_i e^j \\
+ \frac{1}{2} \left( \phi^{(1)} \right)^3 + \frac{1}{2} v^{(1)}_i v^{(2)}_i + \frac{1}{2} \phi^{(1)} v^{(1)}_i v^{(1)}_i - \phi^{(1)} i p^{(1)} \partial_i \phi^{(1)} \\
+ \frac{1}{2} k_0^{(1)} \left[ \phi^{(2)} + 2 p^{(1)}_i \partial_i \phi^{(1)} - \left( \phi^{(1)} \right)^2 + v^{(1)}_i v^{(1)}_i \right] + \phi^{(1)} \left( k_0^{(2)} + \Delta \lambda^{(1)} \frac{dk^{(1)}}{d\lambda} \right) \\
+ \frac{1}{6} \omega^{(3)}_i e^i + \frac{1}{2} p^{(1)}_i \partial_i \omega^{(2)} e^i + p^{(2)}_i \partial_i \omega^{(1)} e^i + \frac{1}{2} p^{(1)}_i p^{(1)}_j \partial_i \partial_j \omega^{(1)} e^k \\
+ \omega^{(1)}_i \left\{ \phi^{(1)} k_0^{(1)} + \frac{1}{2} e^{(1)} \left[ - \phi^{(2)} - 2 \phi^{(1)} \partial_j \omega^{(1)} + 3 \phi^{(1)} - 2 \omega^{(1)} v^{(1)} + v^{(1)} v^{(1)}_j \right] \right\} \\
- \delta_{ij} k_0^{(1)} \left( \frac{1}{2} v^{(2)}_j + p^{(1)}_k \partial_k v^{(1)}_j \right) - \delta_{ij} v^{(1)}_j \left( k_0^{(2)} + \Delta \lambda^{(1)} \frac{dk^{(1)}}{d\lambda} \right) \\
+ \left( \chi_{ij}^{(1)} - 2 \psi^{(1)} \delta_{ij} \right) \left( \frac{1}{2} v^{(2)}_j e^j + p^{(1)}_k \partial_k v^{(1)}_j e^j - v^{(1)}_j k_0^{(2)} \right) \\
+ \left[ \left( \frac{1}{2} \chi_{ij}^{(2)} - \psi^{(2)} \delta_{ij} \right) + p^{(1)}_k \partial_k \left( \chi_{ij}^{(1)} - 2 \psi^{(1)} \delta_{ij} \right) \right] v^{(1)}_i e^i.
\]

(2.19)

In these expressions, \( \Delta \lambda^{(1)} \) and \( \Delta \lambda^{(1)} + \Delta \lambda^{(2)} \) are the differences in affine parameter between the points where the first order (respectively, the second order) and the background geodesics intersect the \( \eta = \eta_e \) hypersurface. They are given by \( \Delta \lambda^{(1)} = -x^{(1)0} \) and \( \Delta \lambda^{(2)} = -x^{(2)0} + k_0^{(1)} x^{(1)0} \).

The next step is to obtain the null geodesics up to third order, using the formalism set up in [28, 29]. This will allow us to compute at the desired order the photon path \( x^\mu(\lambda) \) and the associated wavevector \( k^\mu(\lambda) \) appearing in the previous expressions for the perturbations of the photon frequency. The \( a \)-th order geodesic equation can be recast as the forced Jacobi equation [28, 29]

\[
\frac{d^2 x^\mu(a)}{d\lambda^2} + 2 \Gamma^{(0)\mu}_{\alpha\beta} k^{(0)\alpha} k^{(a)\beta} + \partial_\sigma \Gamma^{(0)\mu}_{\alpha\beta} k^{(0)\alpha} k^{(0)\beta} x^{(a)\sigma} = f^{(a)\mu}.
\]

(2.20)

In a flat background, the solutions are

\[
x^{(a)\mu} = (\lambda - \lambda_0) k^{(a)\mu} (\lambda_0) + \int^{\lambda}_\lambda (\lambda - \tilde{\lambda}) f^{(a)\mu} (\tilde{\lambda}) d\tilde{\lambda},
\]

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\[ k^{(a)\mu} = k^{(a)\mu} (\lambda) + \int_{\lambda_0}^{\lambda} f^{(a)\mu} (\tilde{\lambda}) \, d\tilde{\lambda}. \]  

(2.22)

The forcing terms, in a flat background, are given by

\[ f_{(1)}^{\mu} = -\Gamma^{(1)\mu}_{\alpha\beta} k^{(0)\alpha} k^{(0)\beta}, \]  

(2.23)

\[ f_{(2)}^{\mu} = -2\Gamma^{(1)\mu}_{\alpha\beta} k^{(0)\alpha} k^{(1)\beta} - x^{(1)\sigma} \partial_\sigma \Gamma^{(1)\mu}_{\alpha\beta} k^{(0)\alpha} k^{(0)\beta} - \Gamma^{(2)\mu}_{\alpha\beta} k^{(0)\alpha} k^{(0)\beta}, \]  

(2.24)

\[ f_{(3)}^{\mu} = -\Gamma^{(1)\mu}_{\alpha\beta} \left( 2k^{(0)\beta}_0 k^{(1)}_{(0)} - 2k^{(0)}_{(1)} k^{(0)}_0 \right) - 2\kappa^{(1)}_{(1)} k^{(0)}_0 \left( \Gamma^{(2)\mu}_{\alpha\beta} + \partial_\sigma \Gamma^{(1)\mu}_{\alpha\beta} x^{(1)}_\sigma \right) - k^{(0)}_{(0)} k^{(0)}_0 \left( \Gamma^{(3)\mu}_{\alpha\beta} + \partial_\sigma \Gamma^{(1)\mu}_{\alpha\beta} x^{(1)}_\sigma + \frac{1}{3} \partial_\sigma \partial_\tau \Gamma^{(1)\mu}_{\alpha\beta} x^{(1)}_\sigma x^{(1)}_{\tau(1)} \right). \]  

(2.25)

The Christoffel symbols are calculated in appendix A, where one can find their expressions for the conformal metric (2.6). Notice that we have recast them in a compact form without specifying the order of the perturbations, even though they can be used just up to third order (at least for the terms coming from the vector and tensor perturbations). The method that uses equations (2.23)–(2.25) is an iterative method: once one knows the geodesic equations of rth order, he or she is able to determine those at the next order through equation (2.21). Thus, in order to have all the means to compute the photon geodesics up to order in the following, we report the expressions that we have computed for the wavevectors \( k^{(\mu)} \) up to second order:

\[ k^{(0)}_{(1)} = -2\Phi^{(1)} - \omega^{ij}_1 e^i + \int_{\lambda_0}^{\lambda} d\tilde{\lambda} A^{(1)}(\tilde{\lambda}) = -2\Phi^{(1)} - \omega^{ij}_1 e^i + I_1(\lambda), \]  

(2.26)

\[ k^{(i)}_{(1)} = -2\Psi^{(1)} e^i - \omega^{ij}_1 + \chi^{ij}_{(1)} e^j - \int_{\lambda_0}^{\lambda} d\tilde{\lambda} \partial^i A^{(1)}(\tilde{\lambda}) = -2\Psi^{(1)} e^i - \omega^{ij}_1 + \chi^{ij}_{(1)} e^j - I_1^i(\lambda), \]  

(2.27)

for the first-order wavevectors and

\[ k^{(0)}_{(2)} = -\Phi^{(2)} - 2\Phi^{(1)^2} - \frac{1}{2} \omega^{ij}_2 e^i - 2k^{(0)}_1 \Phi^{(1)} - 2x^{(1)}_j \partial_\mu \Phi^{(1)} + \omega^{(1)}_i k^{(1)}_j - x^{(1)}_j \partial_\mu \omega^{(1)}_i e^i + \int_{\lambda_0}^{\lambda} d\tilde{\lambda} \left[ \frac{1}{2} A^{(2)}(\tilde{\lambda}) + 2 \left( \Phi^{(1)}(\Phi^{(1)})' - \Psi^{(1)}(\Psi^{(1)})' \right) \right] + \left( \chi^{ij}_{(1)} e^i - \omega^{ij}_1 \right) \left( e^i k^{(0)}_1 + k^{(i)}_1 \right) + 2k^{(0)}_1 A^{(1)}(\tilde{\lambda}) + 2\Psi^{(1)} A^{(1)}(\tilde{\lambda} + x^{(1)}_j \partial_\mu A^{(1)}(\tilde{\lambda})), \]  

(2.28)

\[ k^{(i)}_{(2)} = -e^i \Psi^{(2)} + 2e^i \Psi^{(1)^2} - 2e^i x^{(1)} \partial_\mu \Psi^{(1)} + 2k^{(1)} \Psi^{(1)} + \frac{1}{2} \chi^{ij}_{(2)} e^i + x^{(1)} \partial_\mu \chi^{ij}_{(1)} e^j - \chi^{ij}_{(1)} k^{(1)}_j - \frac{1}{2} \omega^{ij}_2 - x^{(1)}_j \partial_\mu \omega^{(1)}_i - k^{(0)}_1 \omega^{(1)}_i + \int_{\lambda_0}^{\lambda} d\tilde{\lambda} \left[ -\frac{1}{2} \partial_\tau A^{(2)}(\tilde{\lambda}) - x^{(1)}_j \partial_\mu \partial^j A^{(1)}(\tilde{\lambda}) - 2k^{(1)} \partial^i \Phi^{(1)} + \partial^i \left( \Psi^{(1)^2} - \Phi^{(1)^2} \right) \right] + 2e_j k^{(1)} \partial^j \Psi^{(1)} - e^j k^{(1)} \partial^j \chi^{(1)}_{jk} + k^{(1)} \partial^j \omega^{(1)}_j - k^{(0)}_j \partial^j \omega^{(1)}_j e^j], \]  

(2.29)
at second order, where
\[ A^{(r)} \equiv \Phi^{(r)} + \Psi^{(r)} + \omega_i^{(r)} e^i - \frac{1}{2} \lambda_{ij}^{(r)} e^i e^j. \]

As a non-trivial test for the correctness of the above expressions, we have explicitly verified that the first-and second-order wavevectors do satisfy the null vector condition \( k^\mu k_\mu = 0 \). When expanded perturbatively, this gives at linear order
\[ k^0_{(1)} + e_i k^i_{(1)} = -A^{(1)}, \]
and at second order
\[ k^0_{(2)} + e_i k^i_{(2)} + \frac{1}{2} (k^0_{(1)})^2 - \frac{1}{2} k^i_{(1)} k^i_{(1)} + 2 \Phi_{(1)} k^0_{(1)} - \omega_i^{(1)} k^i_{(1)} + \omega_i^{(1)} e^i k^0_{(1)} - 2 \Psi_{(1)} e_i k^i_{(1)} + x^\mu_{(1)} \partial_\mu A^{(1)} = 0. \]

Both equations are satisfied when one substitutes the expressions (2.26)–(2.29).

3. CMB anisotropies on large scales from scalar perturbations

In this section, we focus on scalar perturbations and we give the expressions for the CMB anisotropies on large scales. Two main results are achieved. First we are able to give, using a non-perturbative method, an integral solution for the photon geodesic equations which holds at any order in perturbation theory. Second, we solve for the first time the evolution of the third-order gravitational potentials in a universe filled with non-relativistic matter and a cosmological constant. The presence of a non-vanishing cosmological constant makes the gravitational potentials vary at late times, producing on large scales an integrated–Sachs–Wolfe effect which is the counterpart of what happens at linear order. Of course, due to the nonlinear evolution of the perturbations, other integrated terms will appear in the form of cubic corrections. The ISW constitutes the main effect for large-scale CMB anisotropies together with the Sachs–Wolfe effect originating at the last scattering surface. In fact, both of them keep a memory of the initial primordial non-Gaussianity of the cosmological perturbations (see details in section 4.2), and as such our expressions can be of particular relevance when trying to pin down the primordial non-Gaussian content from the higher-order statistics of the CMB anisotropies as the bispectrum or the trispectrum. Let us see in detail these two results.

In the limit of large scales, it is convenient to adopt a non-perturbative formalism to get the temperature anisotropies and the Einstein equations. We take the metric to be
\[ ds^2 = a^2(\eta) [-e^{2\Phi} dr^2 + e^{-2\Phi} \delta_{ij} dx^i dx^j] \]
and for simplicity we take the points of emission and observation to be comoving (in such a way we lose the Doppler effects: however, they are important on scales smaller than those we are interested here). In writing this metric we have neglected vector and tensor perturbation modes. For the vector perturbations the reason is that we are interested in long-wavelength perturbations, while vector modes will contain gradient terms being produced as nonlinear combination of scalar modes and thus they will be more important on small scales (linear vector modes are not generated in the standard mechanism for cosmological perturbations, as inflation). For example the results of [34] show clearly this for second-order perturbations. In order to study the CMB anisotropies from scalar perturbations in the large-scale limit, the tensor contribution can be neglected, since on large scales it has been proven to remain constant and to
give a negligible effect, being of the order of (powers of) the slow-roll parameters during inflation \[5,35\].

We start from equation (2.2) and we write the frequency as
\[
\omega = -g_{\mu\nu}u^\mu k^\nu = e^{2\Phi} u^0 k^0 = e^{\Phi} k^0,
\]
where \( u^0 = e^{-\Phi} \) is given by the normalization condition. The observed temperature is given by
\[
T_O(x, \hat{e}) = \frac{\alpha \tilde{E}}{a_O} e^{\Phi - \Phi_{\epsilon} k^0} T_{\epsilon}(p, d).
\]

We make another simplification: namely, we reabsorb the terms computed at the point of observation into a redefinition of \( T_O \). This amounts to losing the monopole term, which, however, is unobservable.

At this point, we can perform the double expansion in perturbation orders and around the background geodesic (i.e. the line of sight). We find
\[
\frac{\delta^{(1)} T}{T} = \tau^{(1)} - \Phi^{(1)} - k^{(1)} \tau^{(1)};
\]
\[
\frac{\delta^{(2)} T}{T} = \tau^{(2)} + \frac{p_1^I}{2} \frac{\partial \tau^{(1)}}{p^I} + d^{(1)}_1 \frac{\partial \tau^{(1)}}{p^I} - \frac{1}{2} \Phi^{(2)} - p_1^I \frac{\partial \Phi^{(1)}}{p^I} + \frac{1}{2} (\Phi^{(1)})^2
\]
\[
- k^{(2)} - \Delta \lambda^{(1)} \frac{d k^{(1)}}{d \lambda} + \left( k^{(1)} \right)^2 - \left( \tau^{(1)} \Phi^{(1)} - \tau^{(1)} k^{(1)} - \Phi^{(1)} k^{(1)} \right) ;
\]
\[
\frac{\delta^{(3)} T}{T} = \tau^{(3)} + \frac{1}{6} \Phi^{(3)} + \frac{1}{2} \left( \Phi^{(1)} k^{(2)} - \frac{1}{2} \left( \Phi^{(1)} \right)^3 + 2 k^{(1)} \right) - \left( \Phi^{(1)} \right)^3 - \tau^{(2)} \Phi^{(1)}
\]
\[
+ \left[ k^{(1)} - \left( k^{(1)} \right)^2 \right] \left( \Phi^{(1)} - \tau^{(1)} \right) + \frac{1}{2} \left( \Phi^{(1)} \right)^2 \right[ - \Phi^{(2)} + \left( \Phi^{(1)} \right)^2 \right]
\]
\[
+ k^{(1)} - \left( k^{(1)} \right)^2 \left( \Phi^{(1)} - \tau^{(1)} \right) \left( \Phi^{(1)} - \Phi^{(1)} \right) + \Phi^{(1)} \partial_\Phi^{(1)}
\]
\[
+ \partial_\Phi^{(1)} \left( \partial_\Phi^{(1)} - \partial_\Phi^{(1)} \right) + \Phi^{(1)} \partial_\Phi^{(1)}
\]
\[
+ \partial_\Phi^{(1)} \left( \partial_\Phi^{(1)} - \partial_\Phi^{(1)} \right) + \Phi^{(1)} \partial_\Phi^{(1)}
\]
\[
+ \frac{1}{2} \partial_\Phi^{(1)} \left( \partial_\Phi^{(1)} - \partial_\Phi^{(1)} \right) + \Phi^{(1)} \partial_\Phi^{(1)}
\]
\[
+ \frac{1}{2} \partial_\Phi^{(1)} \left( \partial_\Phi^{(1)} - \partial_\Phi^{(1)} \right) + \Phi^{(1)} \partial_\Phi^{(1)}
\]
\[
+ \frac{1}{2} \partial_\Phi^{(1)} \left( \partial_\Phi^{(1)} - \partial_\Phi^{(1)} \right) + \Phi^{(1)} \partial_\Phi^{(1)}
\]
\[
+ \Delta \lambda^{(1)} \left( \frac{d k^{(2)}}{d \lambda} - \frac{1}{2} \Delta \lambda^{(1)} \frac{d k^{(1)}}{d \lambda} + \frac{1}{2} \left( \Phi^{(1)} - \tau^{(1)} \right) \frac{d k^{(1)}}{d \lambda} \right)
\]
\[
- \Delta \lambda^{(2)} \frac{d k^{(1)}}{d \lambda}.
\]

In order to compute the photon wavevectors and the null geodesics, we will proceed in a different way than the usual standard perturbative method introduced in \[28,29\] which makes use of equations (2.23)–(2.25). Let us instead start from a fully nonlinear geodesic.
equation (time component) which can be written as

$$\frac{\text{d}k^0}{\text{d}\lambda} = -2\frac{\text{d}\Phi}{\text{d}\lambda} k^0 + (\Phi' + \Psi') (k^0)^2,$$

where we have used the null vector condition $g_{\mu\nu}k^\mu k^\nu = 0$. We can write the formal solution (formal since the potentials depend on the true geodesic) as

$$k^0(\lambda) = e^{-2\Phi(\lambda)} \left[ 1 - \int_{\lambda_0}^{\lambda} e^{-2\Phi(x(\tilde{\lambda}))} \left( \Phi'(x(\tilde{\lambda})) + \Psi'(x(\tilde{\lambda})) \right) \text{d}\tilde{\lambda} \right]^{-1}. \quad (3.7)$$

The advantage of equation (3.7) is that it can be straightforwardly expanded to find the perturbations to the background wavevector (and, by integration, to the background geodesic), up to any desired order. To make contact with the standard perturbative results, in addition to the expansion in perturbative orders we need to perform a Taylor expansion of the true geodesic around the background geodesic for the potentials, in order to express all the quantities on the background geodesic. For example, if we have the nonlinear quantity $\alpha(x)$ computed on the true geodesic, its expansion up to second order will be $\alpha(x) = \alpha^{(1)}(x(0)) + \partial_\mu \alpha^{(1)}(x(0)) x^{(1)}\mu + \alpha^{(2)}(x(0))$, and similarly at higher orders.

We find

$$k^{(1)0}(\lambda) = -2\Phi^{(1)} + I_1(\lambda), \quad (3.8)$$

$$k^{(2)0}(\lambda) = -\Phi^{(2)} + 2(\Phi^{(1)})^2 - 2x^{(1)\nu} \partial_\nu \Phi^{(1)} + I_2(\lambda) - 2\Phi^{(1)} I_1(\lambda) + I_1(\lambda)^2, \quad (3.9)$$

$$k^{(3)0}(\lambda) = -\frac{1}{3}\Phi^{(3)} - x^{(1)\mu} \partial_\mu \Phi^{(2)} - x^{(1)\mu} x^{(1)\nu} \partial_\mu \partial_\nu \Phi^{(1)} - 2x^{(2)\nu} \partial_\mu \Phi^{(1)} + \frac{8}{3}(\Phi^{(1)})^3$$

$$+ (2\Phi^{(1)} - I_1(\lambda)) \left[ \Phi^{(2)} - 2(\Phi^{(1)})^2 + 2x^{(1)\nu} \partial_\nu \Phi^{(1)} \right]$$

$$+ I_3(\lambda) + 2I_1(\lambda) I_2(\lambda) + I_1(\lambda)^3 - 2\Phi^{(1)} (I_2(\lambda) + I_1(\lambda)^2), \quad (3.10)$$

where

$$I_1(\lambda) = \int_{\lambda_0}^{\lambda} \text{d}\tilde{\lambda} A^{(1)\nu}(\tilde{\lambda}), \quad (3.11)$$

$$I_2(\lambda) = \int_{\lambda_0}^{\lambda} \text{d}\tilde{\lambda} \left[ \frac{1}{2} A^{(2)\nu}(\tilde{\lambda}) + x^{(1)\nu} \partial_\nu A^{(1)\nu}(\tilde{\lambda}) - 2\Phi^{(1)} A^{(1)\nu}(\tilde{\lambda}) \right], \quad (3.12)$$

$$I_3(\lambda) = \int_{\lambda_0}^{\lambda} \text{d}\tilde{\lambda} \left[ \frac{1}{6} A^{(3)\nu} + \frac{1}{2} x^{(1)\mu} \partial_\mu A^{(2)\nu}(\tilde{\lambda}) + x^{(1)\mu} x^{(1)\nu} \partial_\mu \partial_\nu A^{(1)\nu}(\tilde{\lambda}) + x^{(2)\nu} \partial_\mu A^{(1)\nu}(\tilde{\lambda}) + x^{(1)\nu} \partial_\mu A^{(1)\nu}(\tilde{\lambda}) \right]$$

$$- 2\Phi^{(1)} \left( \frac{1}{2} A^{(1)\nu} + x^{(1)\nu} \partial_\nu A^{(1)\nu} \right) + A^{(1)\nu} \left( 2\Phi^{(1)} - \Phi^{(2)} - 2x^{(1)\nu} \partial_\mu \Phi^{(1)} \right)], \quad (3.13)$$

$$A^{(n)} = \Phi^{(n)} + \Psi^{(n)}. \quad (3.14)$$

The above expressions are completely known once we find the geodesic $x^\mu(\lambda)$ (it is sufficient to compute them up to second order, as is clear by looking at equation (3.13)). For the
time component the integration of $k^{(0)}(\lambda)$ yields

$$x^{(1)0}(\lambda) = \int_{\lambda_0}^{\lambda} d\tilde{\lambda} \left[ -2\Phi^{(1)} + \left( \lambda - \tilde{\lambda} \right) A^{(1)′} \right],$$

(3.15)

$$x^{(2)0}(\lambda) = \int_{\lambda_0}^{\lambda} d\tilde{\lambda} \left[ -\Phi^{(2)} + 2\left( \Phi^{(1)} \right)^2 - 2x^{(1)μ} \partial_μ \Phi^{(1)} - 2\Phi^{(1)} I_1(\tilde{\lambda}) + I_1(\tilde{\lambda})^2 \right. $$

$$+ \left. \left( \lambda - \tilde{\lambda} \right) \left( \frac{1}{2} A^{(2)′} + x^{(1)μ} \partial_μ A^{(1)′} - 2\Phi^{(1)} A^{(1)′} \right) \right].$$

(3.16)

Notice that from equations (3.6) to (3.13) we have never used the large-scale approximation, so these expressions are general and we recover exactly the corresponding expressions for the second-order quantities already found in [27, 28]. On the other hand, to compute the spatial components $x^i(\lambda)$ we do make use of the large-scale approximation in the following way. We start from the non-perturbative spatial geodesic equation

$$\frac{dk^i}{d\lambda} = 2\frac{d\Psi}{d\lambda} k^i - \left( \partial^i \Psi + \partial^i \Phi \right) \delta_j k^j k^i.$$

(3.17)

To find the solution of this equation we first decompose $k^i = k_∥ e^i + k_⊥$ where $k_∥ = e_j k^j$ and $k_⊥ = (δ^i_j - e^i e_j) k^j$ are the parallel and transverse parts of the wavevector with respect to the background geodesic. Then, in the geodesic equation we approximate $δ_{ij} k^i k^k = k_∥^2 + δ_{jk} k_⊥^2 k^k \approx k_∥^2$, since the second term will be negligible on large angular scales. Then, projecting equation (3.17), we can split it into the two equations

$$\frac{dk_∥}{d\lambda} = 2\frac{d\Psi}{d\lambda} k_∥ - e_i \left( \partial^i \Psi + \partial^i \Phi \right) k_∥^2,$$

(3.18)

$$\frac{dk_⊥}{d\lambda} = 2\frac{d\Psi}{d\lambda} k_⊥ - (δ^i_j - e^i e_j) \left( \partial^i \Psi + \partial^i \Phi \right) k_∥^2,$$

(3.19)

which can be solved. The solution for $k_∥$ is analogous to the one for $k^0$:

$$k_∥ = e^{2\Psi} \left[ -1 + \int_{\lambda_0}^{\lambda} d\tilde{\lambda} e^{2\Psi} e^i \partial_i \left( \Psi + \Phi \right) \right]^{-1},$$

(3.20)

from which the solution for $k_⊥$ is easily obtained:

$$k_⊥ = e^{2\Psi} \int_{\lambda_0}^{\lambda} d\tilde{\lambda} e^{2\Psi} (δ^i_j - e^i e_j) \partial_j \left( \Psi + \Phi \right) \left[ -1 + \int_{\lambda_0}^{\tilde{\lambda}} d\tilde{\lambda} e^i \partial_i \left( \Psi + \Phi \right) \right]^{-2},$$

(3.21)

since as initial condition we have $k_⊥(\lambda_0) = 0$. Perturbing equations (3.20) and (3.21) up to second order we find

$$k^{(1)}_i = -2\Psi^{(1)} e^i - \int_{\lambda_0}^{\lambda} d\tilde{\lambda} \partial^i A^{(1)}(\lambda),$$

(3.22)

$$k^{(2)}_i = -\Psi^{(2)} e^i - 2x^{(1)μ} \partial_μ \Psi^{(1)} e^i - 2\Psi^{(1)} e^i - 2\Psi^{(1)} \int_{\lambda_0}^{\lambda} d\tilde{\lambda} \partial^i A^{(1)}(\lambda)$$

$$+ \int_{\lambda_0}^{\lambda} d\tilde{\lambda} \left[ -\frac{1}{2} \partial^i A^{(2)}(\lambda) - x^{(1)μ} \partial_μ A^{(1)}(\lambda) - 2\Psi^{(1)} \partial^i A^{(1)}(\lambda) - 2\partial^i A^{(1)} \int_{\lambda_0}^{\lambda} d\lambda' e^j \partial_j A^{(1)}(\lambda') \right].$$

(3.23)
CMB temperature anisotropies from third-order gravitational perturbations

Notice that, in fact, the above expressions for $k^4$ are exact, as we have checked explicitly by using the perturbative method of equations (2.23)–(2.25). The reason is that up to this order the large-scale approximation we used also corresponds to neglecting higher-order terms in the perturbations. Moreover, as a further consistency check, we have explicitly verified that the expressions (3.8)–(3.10) and (3.22), (3.23) are the same one gets with the perturbative method employed in section 2.

The integrals (3.11)–(3.13) along the line of sight are at the origin of the late (and early) integrated Sachs–Wolfe effect, since they express the redshift the photons suffer from the last scattering surface to the observer while traveling through a time varying gravitational potential. In particular the first integrand terms, $A^{(2)^{r}}$ and $A^{(3)^{r}}$, in each of the integrals at second and third order, $I_{2}(\lambda)$ and $I_{3}(\lambda)$, are just the straightforward extension of the well-known first-order ISW effect. However, one must consider all the additional contributions which are integrated terms and which depend on the time variation of the gravitational potentials. So terms of this type will appear also from the first two lines of equation (3.10). Notice in particular that terms containing spatial gradients like $x^{(r)}\partial_{i}A^{(m)^{r}}$ (and similar) coming from the combination $x^{(r)}\partial_{i}A^{(m)^{r}} = x^{(r)}\partial_{i}A^{(m)^{r}} + x^{(r)}\partial_{b}A^{(m)^{r}}$ cannot be neglected a priori even in the large-scale approximation because they are integrated along the line of sight.

Finally, plugging the expressions (3.8)–(3.10) for $k^{(0)}$ up to third order into equation (3.5) the expression for the third-order temperature fluctuations on large scales is

$$\frac{\delta(3)T}{T} = \tau^{(3)} + \frac{1}{6}\Phi_{\xi}^{(3)} + \tau(2)\Phi_{\xi}^{(1)} + \tau^{(1)}\frac{1}{2}\Phi_{\xi}^{(2)} + \tau^{(1)}\frac{1}{2}\Phi_{\xi}^{(1)^{2}} + \frac{1}{2}\Phi_{\xi}^{(1)}\Phi_{\xi}^{(2)} + \frac{1}{6}(\Phi_{\xi}^{(1)})^{3}$$

$$+ x^{(1)0}\Phi_{\xi}^{(2)^{r}} + (x^{(1)0})^{2}\Phi_{\xi}^{(1)^{r}} + 2x^{(1)0}\Phi_{\xi}^{(1)^{r}} + 2x^{(1)0}\Phi_{\xi}^{(1)^{r}}(\Phi_{\xi}^{(1)} + \tau^{(1)})$$

$$- I_{3\xi} - I_{2\xi} [\tau^{(1)} + \Phi_{\xi}^{(1)}]$$

$$- I_{1\xi} [\tau^{(2)} + \tau^{(1)}\Phi_{\xi}^{(1)} + \frac{1}{2}\Phi_{\xi}^{(2)} + \frac{1}{2}\Phi_{\xi}^{(1)^{2}} + 2x^{(1)0}\Phi_{\xi}^{(1)}]$$

$$+ d_{(1)}^{i} \left[ \frac{\partial\tau^{(1)}}{\partial d^{i}} \right] + (\Phi_{\xi}^{(1)} - I_{1\xi}) \frac{\partial\tau^{(1)}}{\partial d^{i}} + \frac{1}{2}d_{(1)}^{i}d_{(1)}^{j} \frac{\partial^{2}\tau^{(1)}}{\partial d^{i}\partial d^{j}} + d_{(2)}^{i} \frac{\partial\tau^{(1)}}{\partial d^{i}}$$

$$+ x^{(1)0}(\partial_{b}(\Phi_{\xi}^{(2)} + 2\Phi_{\xi}^{(1)^{2}} - 2x^{(1)0}\Phi_{\xi}^{(1)^{r}}) + \frac{1}{2}A_{\xi}^{(2)^{r}} + x^{(1)0}A_{\xi}^{(1)^{r}}$$

$$- 4\Phi_{\xi}^{(1)} A_{\xi}^{(1)^{r}} + I_{1\xi} A_{\xi}^{(1)^{r}} + I_{1\xi}(-2\Phi_{\xi}^{(1)^{r}} + A_{\xi}^{(1)^{r}})$$

$$- \frac{1}{2}x^{(1)0}(A_{\xi}^{(1)^{r}} - 2\Phi_{\xi}^{(1)^{r}})$$

$$- (2I_{1\xi} - 3\Phi_{\xi}^{(1)} - \tau^{(1)})\left(A_{\xi}^{(1)^{r}} - 2\Phi_{\xi}^{(1)^{r}}\right) + 2\Delta \lambda^{(2)}\Phi_{\xi}^{(1)^{r}} - \Delta \lambda^{(2)} A_{\xi}^{(1)^{r}},$$

where $A^{(r)}$ is given by equation (3.14) and we recall that $\Delta \lambda^{(2)} = -x^{(2)0} + k^{(1)0}x^{(1)0}$. We recognize various contributions to the temperature anisotropies in equation (3.24).

The first line includes the Sachs–Wolfe effect, which combines the intrinsic temperature.

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6 In fact, one can check that, for a term like, for example, $x^{(r)}\partial_{i}A^{(m)^{r}}$, only the parts of such gradients transverse to the background geodesic can be neglected on large scales, while the longitudinal part combines with $x^{(r)}\partial_{b}A^{(m)^{r}}$ to give a term which does not contain spatial gradients.
fluctuations $\tau$ with the photon redshift on the last scattering surface due to the gravitational potential perturbations $\Phi_\delta$. The terms in the third and fourth lines correspond to the integrated Sachs–Wolfe effect, while in the fifth line there are contributions due to a lensing effect at the last scattering surface, being dependent on the vector $\hat{d}$, which specifies the direction of emission of the photon. Finally all the remaining terms are due to a possible time dependence of the gravitational potentials at the last scattering epoch, due to the fact that, by that time, the universe is still not completely matter-dominated (similarly to the early integrated Sachs–Wolfe effect). At second order, in equation (3.4), the lensing-like contributions appear as well. Notice that, in order to make a comparison with the bispectrum computed using the different technique of [36], one has to take into account also these lensing terms along with the Sachs–Wolfe effect.

4. Third-order scalar perturbations of a flat $\Lambda$CDM universe

4.1. Evolution of the gravitational potentials on large scales

We now consider a spatially flat universe filled with a cosmological constant $\Lambda$ and a non-relativistic perfect pressureless fluid and we solve the third-order Einstein equations for the scalar perturbations, in the large-scale limit. The energy–momentum tensor of the matter component is $T^\mu_\nu = \rho u^\mu u_\nu$, with energy density $\rho$ and 4-velocity $u^\mu$. The derivation of the relevant equations is sketched in appendix B.

The evolution equation for the third-order gravitational potential $\psi^{(3)}$ is

$$\psi^{(3)}'' + 3H \psi^{(3)} + a^2 \Lambda \psi^{(3)} = S(\eta, x), \quad (4.1)$$

where the source term is given by

$$S(\eta, x) = \frac{4}{H^2 \Omega_m} \nabla^{-2} \left\{ \left[ (H^2 + a^2 \Lambda) E + HE' \right] \left[ 3 \nabla^{-2} \partial_i \partial_j (\varphi_0 \partial^i \varphi_0 \partial_j \varphi_0) - \varphi_0 (\partial^i \varphi_0) (\partial_i \varphi_0) \right] + \left[ (H^2 + a^2 \Lambda) F + HF' \right] \left[ 3 \nabla^{-2} \partial_i \partial_j (\partial^i \varphi_0 \partial_j \varphi_0 + \partial^i \varphi_j \partial_j \varphi_0) - 2 (\partial^i \varphi_0) (\partial_i \varphi_0) \right] - (g' + 2Hg) \left[ 3H (g')^2 + a^2 \Lambda g^2 (2f - 1) \right] \varphi_0 \partial_0 \right\} - 6 (g')^2 \varphi_0^3 + 3 \left( g' A \varphi_0^3 + g' B \varphi_0 \partial_0 \right) + \text{gradients}, \quad (4.2)$$

where $\nabla^{-2}$ stands for the inverse of the Laplacian operator. Here $g(\eta)$ is the linear growth function, defined by $\varphi(x, \eta) = g(\eta) \varphi_0(x)$, where $\varphi_0(x)$ is the peculiar gravitational potential linearly extrapolated to the present time and $f(\eta) = 1 + g'(\eta)/Hg(\eta)$. The growth-suppression factor is given by $g(\eta) = D_+(\eta)/a(\eta)$, where $D_+(\eta)$ is the linear growing mode of density fluctuations in the Newtonian limit. The exact form of $g$ can be found in [37]–[39]. A very good approximation for $g$ as a function of redshift $z$ is given in [37, 38]

$$g \propto \Omega_m \left[ \Omega_m^{4/7} - \Omega_\Lambda + \left( 1 + \Omega_m/2 \right) \left( 1 + \Omega_\Lambda/70 \right) \right]^{-1}, \quad (4.3)$$

with $\Omega_m = \Omega_m(1 + z)^3/E^2(z)$, $\Omega_\Lambda = \Omega_\Lambda/E^2(z)$, $E(z) = (1 + z)H(z)/H_0 = [\Omega_0m(1 + z)^3 + \Omega_0\Lambda]^{1/2}$ and $\Omega_m, \Omega_\Lambda = 1 - \Omega_m, \Omega_\Lambda$ the present-day density parameters of non-relativistic matter and cosmological constant, respectively. We will normalize the growth-suppression factor so that $g(z = 0) = 1$. The function $f(\eta)$ can be written as a function of $\Omega_m$ as $f(\Omega_m) \approx \Omega_m(z)^{4/7}$ [37, 38]. In the $\Lambda = 0$ case $g = 1$ and $f(\eta) = 1$. 

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In equation (4.2) $A(\eta)$ and $B(\eta)$ enter in the solutions for the second-order gravitational potentials on large scales [13]:

$$\Psi^{(2)}(x, \eta) = A(\eta)\varphi_0^2(x) + B(\eta)\alpha_0(x),$$  \hspace{1cm} (4.4)

$$\Phi^{(2)}(x, \eta) = A(\eta)\varphi_0^2(x) + C(\eta)\alpha_0(x),$$  \hspace{1cm} (4.5)

and their explicit expressions are

$$\alpha_0(x) = \nabla^2 \left[ \partial_\eta \varphi_0 \partial_\eta \varphi_0 - 3 \nabla^2 \partial_i \partial_j (\partial_i \varphi_0 \partial_j \varphi_0) \right],$$  \hspace{1cm} (4.6)

$$A(\eta) = B_1(\eta) + 2g^2(\eta) - 2g_m g(\eta) - \frac{10}{3} (a_{NL} - 1) g_m g(\eta),$$  \hspace{1cm} (4.7)

$$B(\eta) = B_2(\eta) - \frac{4}{3} g_m g(\eta),$$  \hspace{1cm} (4.8)

$$C(\eta) = B(\eta) + \frac{4}{3} g_m g(\eta) \left( \frac{f^2(\eta)}{\Omega_m(\eta)} + \frac{3}{2} \right),$$  \hspace{1cm} (4.9)

where the functions $B_1(\eta)$ and $B_2(\eta)$ are given in [13] while $g(\eta_m) = g$ is the value of the growth-suppression factor during matter domination, when the cosmological constant was still negligible. A good approximation is $g_m \approx \frac{5}{7} \Omega_{0m} \left( \Omega_{0m}^{1/3} + \frac{2}{5} \Omega_{0m} \right)$.

Notice that in $A(\eta)$ appears the parameter $a_{NL}$ specifying the level of quadratic primordial non-Gaussianity which depends on the particular scenario for the generation of cosmological perturbations, as we will discuss later in more detail. For example, for standard single-field models of slow-roll inflation $a_{NL} = 1 + \mathcal{O}(\epsilon, \eta)$ [3]–[5], [40] where $\epsilon$ and $\eta$ are the standard slow-roll parameters [41]. The reason it appears in equation (4.2) is due to the fact that at third order some of the source terms contain products of linear and second-order gravitational potentials, as detailed in appendix B.

We have also introduced the functions

$$E(\eta) = 2A'g' + \left( 5H^2 - a^2\Lambda \right)Ag - 2Hg'g^2 + 2H(A'g + g'A) - 2g(g')^2$$

$$- 2g(g' + Hg)^2 + 2 \frac{gf}{\Omega_m} (g' + Hg)^2,$$  \hspace{1cm} (4.10)

$$F(\eta) = \frac{1}{2} B'g' + \frac{1}{4} \left( 5H^2 - a^2\Lambda \right) Cg + \frac{1}{2} H(Cg' + B'g).$$  \hspace{1cm} (4.11)

Notice that when $\Lambda = 0$ then $B_1(\eta)$ and $B_2(\eta) \to 0$, $B(\eta) \to -4/3$ and $A(\eta) \to 2 - (10/3)a_{NL}$, and therefore $E(\eta)$ simplifies to $E(\eta) = -(50/3)a_{NL}H^2$ while $F(\eta) = (5/2)H^2$. Thus, looking at equation (4.2), it is simple to see that all the terms in the last two lines are specific only to the case of a non-vanishing cosmological constant.

The solution of equation (4.1) is given by Green’s formula

$$\psi^{(3)}(\eta) = \frac{g}{g_m} \psi^{(3)}_m + \psi_+(\eta) \int_{\eta_m}^{\eta} d\eta' \frac{\psi_-(\eta')}{W(\eta')} S(\eta') + \psi_-(\eta) \int_{\eta_m}^{\eta} d\eta' \frac{\psi_+(\eta')}{W(\eta')} S(\eta'),$$  \hspace{1cm} (4.12)

where

$$\psi_+(\eta) = g(\eta), \hspace{1cm} \psi_-(\eta) = \frac{H(\eta)}{a^2(\eta)}, \hspace{1cm} W(\eta) = \frac{H_0^2}{a^3(\eta)} \left( f_0 + \frac{3}{2} \Omega_{0m} \right),$$  \hspace{1cm} (4.13)

are, respectively, the growing and decaying mode solutions and the Wronskian of the homogeneous equation. Here the suffix '0' stands for the value of the corresponding quantities at the present time, while $\psi^{(3)}_m \equiv \psi^{(3)}(\eta_m)$ represents the initial condition taken deep
in the matter-dominated era on superhorizon scales, $\eta_m$ being the epoch when full matter domination starts. It is such an initial value that must be properly determined in order to account for the primordial cubic non-Gaussianity in the cosmological perturbations.

The solution for $\phi^{(3)}$ is then obtained from the relation between $\psi^{(3)}$ and $\phi^{(3)}$ obtained in appendix B (equation (B.7)):

$$
\phi^{(3)} = \psi^{(3)} + 12g(\eta)A(\eta)\varphi_0^3 + 6g(\eta) (B(\eta) + C(\eta)) \alpha_0 \varphi_0 - \frac{4}{H^2\Omega_m} \left[ E(\eta)\mu_0 + F(\eta)\nu_0 \right],
$$

where we have defined the following kernels:

$$
\mu_0(x) = 3\nabla^{-4}\partial_i\partial^j \left( \varphi_0 \partial^i \varphi_0 \partial_j \varphi_0 \right) - \nabla^{-2} (\varphi_0 \partial^k \varphi_0 \partial_k \varphi_0),
$$

$$
\nu_0(x) = 3\nabla^{-4}\partial_i\partial^j \left( \partial^i \alpha_0 \partial_j \varphi_0 + \partial^i \varphi_0 \partial_j \alpha_0 \right) - 2\nabla^{-2} (\partial^k \alpha_0 \partial_k \varphi_0).
$$

### 4.2. Initial conditions from primordial non-Gaussianity

We now discuss the key issue of the initial conditions, which are conveniently fixed at the time when the relevant modes of the perturbations are well outside the Hubble radius. In order to follow the superhorizon evolution of the density perturbations produced during inflation, we use the curvature perturbations on uniform-density hypersurfaces $\zeta$, which will be expanded as

$$
\zeta = \zeta^{(1)} + \frac{1}{3}(1 + w) \ln \frac{\bar{\rho}}{\rho}.
$$

where just for simplicity in this expression we have assumed a constant equation of state parameter $w = \bar{p}/\bar{\rho}$, $\bar{\rho}$ and $\bar{p}$ being the background energy density and pressure, respectively. This variable has two crucial features: it is gauge invariant and, most important, it is constant in time for adiabatic, superhorizon perturbations. This allows us to set the initial conditions at the time when $\zeta$ becomes constant and follow them until the perturbation mode reenters the horizon. The primordial non-Gaussianity can be parameterized in terms of the curvature perturbation as

$$
\zeta^{(2)} = 2(a_{NL} - 1) \left( \zeta^{(1)} \right)^2, \quad \zeta^{(3)} = 6(b_{NL} - 1) \left( \zeta^{(1)} \right)^3,
$$

where the two nonlinearity parameters depend on the physics of a given scenario of generation of the perturbations. For example, for standard single-field models of inflation $a_{NL} = 1$ and $b_{NL} = 1$ (plus tiny corrections proportional to the slow-roll parameters), while for other scenarios they might well be non-negligible.

On the other hand, the physical observable quantity is given by the CMB anisotropies, which are one of the best tools to detect or constrain the primordial non-Gaussianity.

---

7 As a consistency test, we have used equation (4.17), together with the Einstein equations contained in appendix B, to verify that $\zeta'$ turns out to be proportional to the LHS of equation (4.1) minus the very same source term appearing in equation (4.2). This provides an independent check for the expression of the source term which turns out to be consistent with $\zeta' = 0$ on large scales.
generated on large scales [3]. The standard procedure is to introduce the quadratic and cubic nonlinearity parameters, \( f_{\text{NL}} \) and \( g_{\text{NL}} \) (which can be, in fact, also non-trivial kernels in Fourier and harmonic space) characterizing the non-Gaussianity in the large-scale temperature anisotropies [1, 10, 14, 18, 44]. In the limit of large non-Gaussianity, \( |a_{\text{NL}}| \gg 1 \) and \( |b_{\text{NL}}| \gg 1 \) [3, 20, 45], the resulting size of the non-Gaussianity in the CMB anisotropies can be estimated as \( f_{\text{NL}} \sim 5a_{\text{NL}}/3 \) and \( g_{\text{NL}} \sim 25b_{\text{NL}}/9 \) if accounting only for the contribution to the CMB anisotropies from the Sachs–Wolfe effect. In fact, for the cubic nonlinearities entering in the Sachs–Wolfe effect such an estimate is too rough, and the correct expression relating the observable quantity \( g_{\text{NL}} \) to the primordial nonlinearity parameters \( a_{\text{NL}} \) and \( b_{\text{NL}} \) is given by equation (67) of [20].

Our results allow us to take into account also the contribution from the (late) integrated Sachs–Wolfe effect. The memory of the initial non-Gaussianity is kept in the Sachs–Wolfe effect such an estimate is too rough, and the correct expression relating the observable quantity \( g_{\text{NL}} \) to the primordial nonlinearity parameters \( a_{\text{NL}} \) and \( b_{\text{NL}} \) is given by equation (67) of [20].

\[
\zeta_m = -\Psi_m - \frac{4}{3} \Phi_m, \tag{4.19}
\]

where we have employed the energy constraint (B.1) to replace the energy density.

We can expand both sides: at first order we find the usual result

\[
\zeta_m^{(1)} = -\frac{5}{3} g_m \varphi_0; \tag{4.20}
\]

at second order

\[
\phi_m^{(2)} = -\frac{3}{5} \zeta_m^{(2)} + 2 g_m^2 \left( \varphi_0^2 + \alpha_0 \right) = 2 g_m^2 \left[ -\frac{5}{3} (a_{\text{NL}} - 1) \varphi_0^2 + \varphi_0^2 + \alpha_0 \right]; \tag{4.21}
\]

and at third order

\[
\phi_m^{(3)} = -\frac{3}{5} \zeta_m^{(3)} + 6 \varphi_m \phi_m^{(2)} - 8 \varphi_m^3 + \frac{12}{5 \mathcal{H}^2 \Omega_m} \left[ E(\eta_m) \mu_0 + F(\eta_m) \nu_0 \right]. \tag{4.22}
\]

With the use of equation (4.14) evaluated in the matter-dominated period, we get as initial conditions

\[
\begin{align*}
\phi_m^{(3)} &= 6 g_m^3 \left[ \left( \frac{25}{9} (b_{\text{NL}} - 1) - \frac{10}{3} (a_{\text{NL}} - 1) + \frac{3}{2} \right) \varphi_0^3 + 2 \varphi_0 \alpha_0 - \frac{20}{3} (a_{\text{NL}} - 1) \mu_0 + \nu_0 \right], \tag{4.23} \\
\psi_m^{(3)} &= 6 g_m^3 \left[ \left( \frac{25}{9} (b_{\text{NL}} - 1) + \frac{10}{3} (a_{\text{NL}} - 1) + \frac{3}{2} \right) \varphi_0^3 - \frac{4}{3} \varphi_0 \alpha_0 - \frac{160}{9} (a_{\text{NL}} - 1) \mu_0 + 8 \nu_0 \right]. \tag{4.24}
\end{align*}
\]

Finally, from equations (4.12) and (4.14), the large-scale solutions for \( \psi^{(3)} \), \( \phi^{(3)} \) are given by

\[
\begin{align*}
\psi^{(3)}(\eta, \mathbf{x}) &= 6 g(\eta) g_m^2 \left[ \left( \frac{25}{9} (b_{\text{NL}} - 1) + \frac{10}{3} (a_{\text{NL}} - 1) + \frac{3}{2} \right) \varphi_0^3(\mathbf{x}) \\
&\quad - \frac{4}{3} \varphi_0(\mathbf{x}) \alpha_0(\mathbf{x}) - \frac{160}{9} (a_{\text{NL}} - 1) \mu_0(\mathbf{x}) + 8 \nu_0(\mathbf{x}) \right] \\
&\quad + \mathcal{H}_0^{-2} \left( f_0 + \frac{3}{2} \Omega_m \right)^{-1} \left[ g(\eta) \int_{\eta_m}^{\eta} d\tilde{\eta} \alpha(\tilde{\eta}) \mathcal{H}(\tilde{\eta}) S(\tilde{\eta}, \mathbf{x}) \\
&\quad - \frac{\mathcal{H}(\eta)}{a^2(\eta)} \int_{\eta_m}^{\eta} d\tilde{\eta} \alpha^3(\tilde{\eta}) g(\tilde{\eta}) S(\tilde{\eta}, \mathbf{x}) \right]; \tag{4.25}
\end{align*}
\]

\[
\phi^{(3)}(\eta, \mathbf{x}) = \psi^{(3)}(\eta, \mathbf{x}) + 12 g(\eta) A(\eta) \varphi_0^3(\mathbf{x}) + 6 g(\eta) \left( B(\eta) + C(\eta) \right) \alpha_0(\mathbf{x}) \varphi_0(\mathbf{x}) \\
&\quad - \frac{4}{\mathcal{H}^2 \Omega_m} \left[ E(\eta) \mu_0(\mathbf{x}) + F(\eta) \nu_0(\mathbf{x}) \right]. \tag{4.26}
\]
5. Conclusions

In this paper we have computed the expression for the CMB anisotropies due to the redshift the photons suffer when they travel from the last scattering surface to the observer up to third order in the gravitational perturbations. We thus have completed the findings of [20], where a fully nonlinear expression for the Sachs–Wolfe effect has been obtained, by including also the integrated Sachs–Wolfe effect and lensing effects at the last scattering surface. To achieve this goal we have proposed an alternative method to the standard perturbative one in order to solve for the geodesic photon equation which provides a fully nonlinear integral solution. Moreover, we have studied the evolution of the gravitational potentials on large scales, allowing for generic non-Gaussian initial conditions. Our results, together with those of [20], are of particular relevance when facing the trispectrum of the CMB anisotropies, in that, as pointed out in section 1, they include all the relevant cubic nonlinearities for a coherent prediction of such statistics from various cosmological scenarios. In particular they include those contributions related to primordial non-Gaussianity, some of which would inevitably be lost if one would stick to the linear evolution approximation that is often used in the literature.

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Appendix A. Connection coefficients

We start from the line element of the conformal metric

\[ ds^2 = -e^{2\Phi} \, d\eta^2 + 2\omega_i \, d\eta \, dx^i + \left( e^{-2\Psi} \delta_{ij} + \chi_{ij} \right) \, dx^i \, dx^j. \]  

(A.1)

It is useful to consider as ‘background’ metric the one with the exponentiated scalars:

\[ \bar{g}_{00} = -e^{2\Phi}, \quad \bar{g}_{0i} = 0, \quad \bar{g}_{ij} = e^{-2\Psi} \delta_{ij}, \]  

(A.2)

while treating vectors and tensors perturbatively (so here \( n \) refers to the order in vectors and tensors):

\[ \delta g^{(n)}_{00} = 0, \quad \delta g^{(n)}_{0i} = \frac{1}{n!} \varpi^{(n)}_i, \quad \delta g^{(n)}_{ij} = \frac{1}{n!} \chi^{(n)}_{ij}. \]  

(A.3)

The inverse metric will be computed perturbatively up to third order. At zeroth order

\[ \bar{g}^{00} = -e^{-2\Phi}, \quad \bar{g}^{0i} = 0, \quad \bar{g}^{ij} = e^{2\Psi} \delta^{ij}. \]  

(A.4)

At first order, \( \delta g^{\mu\nu}_{(1)} = -\bar{g}^{\mu\lambda} \bar{g}^{\nu\rho} \delta g^{(1)}_{\lambda\rho} \) and we find

\[ \delta g^{00}_{(1)} = 0, \quad \delta g^{0i}_{(1)} = e^{2(\Psi - \Phi)} \omega_{(1)}^i, \quad \delta g^{ij}_{(1)} = -e^{4\Psi} \chi_{(1)}^{ij}. \]  

(A.5)
At second order, \( \delta g_{(2)}^{\mu\nu} = -\bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \delta g_{(2)}^{\rho\sigma} - \delta g_{(1)}^{\mu\nu} \bar{g}^{\rho\sigma} \delta g_{(1)}^{\rho\sigma} \) and we find
\[
\delta g_{(2)}^{ji} = e^{2 \Phi - 4 \Phi} \left[ \frac{1}{2} \omega_{i}^{(1)} - e^{2 \Phi} \omega_{i}^{(1)} \chi_{ii}^{(1)} \right].
\]
(A.6)

At third order, \( \delta g_{(3)}^{\mu\nu} = -\bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \delta g_{(3)}^{\rho\sigma} - \delta g_{(1)}^{\mu\nu} \bar{g}^{\rho\sigma} \delta g_{(2)}^{\rho\sigma} - \delta g_{(2)}^{\mu\nu} \bar{g}^{\rho\sigma} \delta g_{(1)}^{\rho\sigma} \).
\[
\delta g_{(3)}^{0i} = e^{2 \Phi - 4 \Phi} \left[ \omega_{i}^{(1)} \omega_{j}^{(2)} - e^{2 \Phi} \omega_{i}^{(1)} \omega_{j}^{(1)} \chi_{ii}^{(1)} \right],
\]
\[
\delta g_{(3)}^{ij} = e^{2 \Phi - 4 \Phi} \left[ \frac{1}{6} \omega_{i}^{(1)} - \frac{1}{2} e^{2 \Phi} \omega_{k}^{(1)} \chi_{ki}^{(1)} \right] - \frac{1}{2} \chi_{ij}^{(1)} e^{2 \Phi} \left[ \frac{1}{2} \omega_{k} - e^{2 \Phi} \omega_{k}^{(1)} \chi_{ki}^{(1)} \right] \right].
\]
(A.7)

It is now convenient to resum the inverse metric, retaining terms up to third order; we find
\[
g^{0i} = - e^{2 \Phi} + e^{2 \Phi - 4 \Phi} \left( \omega_{k} \right)^{2} - \omega_{i} \omega_{j} \chi_{ij},
\]
\[
g^{0i} = e^{2 \Phi - \psi} \omega_{i} - e^{2 \Phi - 2 \Phi} \omega_{i} \chi_{ij} \omega_{k},
\]
\[
g^{ij} = e^{2 \Phi} \omega_{i} \omega_{j} - e^{2 \Phi - 2 \Phi} \omega_{i} \omega_{j} + e^{4 \Phi} \chi_{ik} \omega_{j} + \omega_{k} \left( \omega_{i} \omega_{j} + \omega_{j} \omega_{k} + \omega_{k} \omega_{i} \right) - \chi_{i} \chi_{j} \chi_{k}.
\]
(A.8)

It is now immediate to compute the connection coefficients with the usual formula
\[
\Gamma_{\alpha i j}^{k} = \frac{1}{2} g^{k \lambda} \left( \partial_{\alpha} g_{j k} + \partial_{j} g_{\alpha i} - \partial_{i} g_{\alpha j} \right),
\]
where \( g^{\mu\nu} \) is the resummed inverse metric in equation (A.8). We find, keeping terms up to third order:
\[
\Gamma_{00}^{0} = \Phi' \left[ 1 - (\omega_{k})^{2} \right] + e^{2 \Phi} \partial_{\alpha} \Phi + \omega_{j} \left[ e^{2 \Phi} \partial_{\alpha} \Phi + \omega_{j} \right] ;
\]
\[
\Gamma_{0i}^{0} = \partial_{i} \Phi \left[ 1 - (\omega_{k})^{2} \right] + \Psi' \left[ - e^{2 \Phi} \omega_{i} + \chi_{ij} \omega_{j} \right],
\]
\[
\Gamma_{ij}^{0} = \Psi' \delta_{ij} \left[ - \left( \omega_{k} \right)^{2} + \frac{1}{2} \left( \chi_{ij} - \partial_{i} \partial_{j} \right) \left[ e^{2 \Phi} \left( \partial_{\alpha} \Phi + \omega_{j} \right) \right] \right] ;
\]
(A.10)
\[
\Gamma_{00}^{0} = \Phi' \left[ - e^{2 \Phi} \omega_{i} + \omega_{j} \chi_{ij} \right] + \left( \omega_{j} + e^{2 \Phi} \partial_{j} \Phi \right) \left[ e^{2 \Phi} \delta_{ij} - e^{4 \Phi} \chi_{ij} - \omega_{i} \omega_{j} + \chi_{ik} \chi_{kj} \right] ;
\]
\[
\Gamma_{ij}^{0} = \partial_{j} \Phi \left[ - e^{2 \Phi} \omega_{i} + \omega_{k} \chi_{ik} \right] + \left( \chi_{jk} - 2 \Psi \omega_{i} \omega_{j} + \partial_{j} \omega_{k} - \partial_{k} \omega_{i} \right) \times \left. e^{2 \Phi} \delta_{ik} - e^{4 \Phi} \chi_{ik} - \omega_{i} \omega_{k} + \chi_{ik} \chi_{kj} \right] ;
\]
\[
\Gamma_{jk}^{i} = \frac{1}{2} \left[ \left( \partial_{i} \omega_{k} + \partial_{k} \omega_{i} \right) + 2 \Psi \left( e^{2 \Phi} \delta_{jk} - \omega_{j} \chi_{ik} \right) \right] \left[ e^{2 \Phi} \omega_{i} - \omega_{i} \chi_{ik} \right] ;
\]
\[
\times \left[ e^{2 \Phi} \delta_{ik} - e^{4 \Phi} \chi_{ik} + \omega_{i} \omega_{k} - \chi_{ik} \chi_{kj} \right] ;
\]
(A.11)

At second order, \( \delta g_{(2)}^{ij} = -\bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \delta g_{(2)}^{\rho\sigma} \) and we find
\[
\delta g_{(2)}^{ij} = e^{2 \Phi - 4 \Phi} \left[ \frac{1}{2} \omega_{i}^{(1)} - e^{2 \Phi} \omega_{i}^{(1)} \chi_{ii}^{(1)} \right],
\]
\[
\delta g_{(2)}^{ij} = e^{4 \Phi} \left[ \frac{1}{2} \omega_{i}^{(1)} - e^{2 \Phi} \omega_{i}^{(1)} \chi_{ii}^{(1)} \right].
\]
(A.12)

At third order, \( \delta g_{(3)}^{\mu\nu} = -\bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \delta g_{(3)}^{\rho\sigma} - \delta g_{(1)}^{\mu\nu} \bar{g}^{\rho\sigma} \delta g_{(2)}^{\rho\sigma} - \delta g_{(2)}^{\mu\nu} \bar{g}^{\rho\sigma} \delta g_{(1)}^{\rho\sigma} \).
\[
\delta g_{(3)}^{0i} = e^{2 \Phi - 4 \Phi} \left[ \omega_{i}^{(1)} \omega_{j}^{(2)} - e^{2 \Phi} \omega_{i}^{(1)} \omega_{j}^{(1)} \chi_{ii}^{(1)} \right],
\]
\[
\delta g_{(3)}^{ij} = e^{2 \Phi - 4 \Phi} \left[ \frac{1}{6} \omega_{i}^{(1)} - \frac{1}{2} e^{2 \Phi} \omega_{k}^{(1)} \chi_{ki}^{(1)} \right] - \frac{1}{2} \chi_{ij}^{(1)} e^{2 \Phi} \left[ \frac{1}{2} \omega_{k} - e^{2 \Phi} \omega_{k}^{(1)} \chi_{ki}^{(1)} \right] \right].
\]
(A.13)

At second order, \( \delta g_{(2)}^{ij} = -\bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \delta g_{(2)}^{\rho\sigma} \) and we find
\[
\delta g_{(2)}^{ij} = e^{2 \Phi - 4 \Phi} \left[ \frac{1}{2} \omega_{i}^{(1)} - e^{2 \Phi} \omega_{i}^{(1)} \chi_{ii}^{(1)} \right],
\]
\[
\delta g_{(2)}^{ij} = e^{4 \Phi} \left[ \frac{1}{2} \omega_{i}^{(1)} - e^{2 \Phi} \omega_{i}^{(1)} \chi_{ii}^{(1)} \right].
\]
(A.14)
Appendix B. Einstein equations for the gravitational potentials

In this appendix we derive the evolution equations for the third-order scalar perturbations $\psi^{(3)}$, $\phi^{(3)}$.

We start by writing the fully nonlinear Einstein equations for the metric $ds^2 = a^2(\eta)[-e^{2\Phi} d\eta^2 + e^{-2\Psi} \delta_{ij} dx^i dx^j]$, for a $\Lambda$CDM model:

- $0-0$
\[ e^{-2\Phi} (\mathcal{H} - \Psi')^2 = \frac{a^2}{3} (8\pi G \rho + \Lambda) \tag{B.1} \]

- $0-i$
\[ e^{-\Phi} [\partial_i \Psi' + (\mathcal{H} - \Psi') \partial_i \Phi] = -4\pi G a \rho u_i; \tag{B.2} \]

- $i-j$ traceless
\[ \partial^i \partial_j \Psi - \frac{1}{3} \nabla^2 \Psi \delta^i_j - \partial^i \partial_j \Phi + \frac{1}{3} \nabla^2 \Phi + \partial^i \Psi \partial_j \Psi - \frac{1}{3} \partial^i \Psi \partial_j \Psi + \frac{1}{3} \partial^i \Phi \partial_j \Phi \]
\[ = 8\pi G a^2 e^{-2\Psi} (T^i_j - \frac{1}{3} \delta^i_j T) \] (B.3)

- $i-j$ trace
\[ e^{-2\Phi} [(\mathcal{H} - \Psi') (-\mathcal{H} + 3\Psi' + 2\Phi') - 2\mathcal{H}' + 2\Psi''] \]
\[ + \frac{e^{2\Phi}}{3} \left[ 2 \nabla^2 (\Phi - \Psi) + \partial^i \Psi \partial_i \Phi + 2 \partial^i \Phi \partial_i \Phi - 2 \partial^i \Phi \partial_i \Psi \right] \]
\[ = 8\pi G a^2 e^{-2\Psi} T^i_j. \tag{B.4} \]

In writing down the first two equations, we made use of the normalization condition for the velocity, $g^{\mu\nu} u_\mu u_\nu = -1$, which gives
\[ u_0 = -e^\Phi \left( a^2 + e^{2\Psi} u^l u_l \right)^{1/2} \simeq -ae^\Phi; \tag{B.5} \]
the last equality is valid at any order on large scales, and up to third order on smaller scales.

We start from the traceless equation, which we project onto the scalar modes by applying the operator $\partial_i \partial^i$:
\[ \nabla^2 \nabla^2 (\Psi - \Phi) = -\frac{1}{3} \nabla^2 \left[ (\partial^i \Phi)(\partial_i \Phi) - (\partial^i \Phi)(\partial_i \Psi) + 2(\partial^i \Phi)(\partial_i \Psi) + 8\pi G a^2 e^{-2\Psi} T \right] \]
\[ + \frac{1}{2} \partial_i \partial^i (\partial^i \Phi)(\partial_i \Phi) - (\partial^i \Phi)(\partial_i \Psi) \]
\[ + (\partial^i \Phi)(\partial_i \Psi) + (\partial^i \Psi)(\partial_i \Phi) + 8\pi G a^2 e^{-2\Psi} T^i_j \]. (B.6)

It is convenient to rewrite it as
\[ \Psi - \Phi = \mathcal{Q}, \tag{B.7} \]
where we define
\[ \nabla^2 \mathcal{Q} \equiv -P + 3N \equiv -P^l_l + 3\nabla^2 \partial_i \partial^i P^j_j \] (B.8)
and

\[ P_j^i = \frac{1}{2} [ (\partial^i \Phi)(\partial_j \Phi) - (\partial^i \Psi)(\partial_j \Psi) ] + \frac{1}{2} [ (\partial^i \Phi)(\partial_j \Psi) + (\partial^i \Psi)(\partial_j \Phi) ] + 4\pi Ga^2 e^{-2\Psi}T_j^i. \]  

(3)

For a ΛCDM model, the spatial part of the stress–energy tensor is

\[ T_j^i = \frac{e^{2\Psi}}{a^2} \rho u_j u_i - \frac{\Lambda}{8\pi G} \delta_j^i \]

\[ = \frac{1}{16\pi^2 G^2} \frac{e^{2(\Psi - \Phi)}}{a^2 \rho} \left[ \partial^i \Psi' + (\mathcal{H} - \Psi') \partial^i \Phi \right] [ \partial_j \Psi' + (\mathcal{H} - \Psi') \partial_j \Phi ] - \frac{\Lambda}{8\pi G} \delta_j^i, \]

(4)

where in the last equality we substitute for the spatial velocities using the 0 – i equation:

\[ u_i = -\frac{1}{4\pi Ga \rho} e^{-\Phi} \partial_i \Psi' + (\mathcal{H} - \Psi') \partial_i \Phi. \]  

(5)

Substituting this expression into \( P_j^i \) we find

\[ P_j^i = \frac{1}{2} [ \partial^i \Phi \partial_j \Phi - \partial^i \Psi \partial_j \Psi ] + \frac{1}{2} [ \partial^i \Phi \partial_j \Psi + \partial^i \Psi \partial_j \Phi ] + 4\pi Ga^2 e^{-2\Psi} \]

\[ \quad + \frac{1}{4\pi G a^2} \frac{e^{-2\Psi}}{\rho} \left[ \partial^i \Psi' + (\mathcal{H} - \Psi') \partial^i \Phi \right] [ \partial_j \Psi' + (\mathcal{H} - \Psi') \partial_j \Phi ] - \frac{1}{2} e^{-2\Psi} a^2 \Lambda \delta_j^i. \]

(6)

Take now the trace equation, and using the background equation \( \mathcal{H}^2 + 2\mathcal{H}' = a^2 \Lambda \) and equations (B.7) and (B.8), we find

\[ e^{-2\Psi} \left[ -a^2 \Lambda - 2\Psi' \Psi' - 3(\Psi')^2 + 2\mathcal{H}(3\Psi' - \mathcal{Q}') + 2\Psi'' \right] \]

\[ + \frac{e^{2\Psi}}{3} \left[ 2\partial^i \Phi \partial_i \Phi + \partial^i \Psi \partial_i \Psi - 2\partial^i \Phi \partial_i \Psi + 2(\mathcal{P} - 3N) \right] = \frac{8\pi G}{3} a^2 T. \]  

(7)

At this point, we can expand equation (B.13) at third order. Using the first-order solution \( \Phi^{(1)} = \Psi^{(1)} = \varphi \), we find

\[ \Psi^{(3)}'' + 3\mathcal{H} \Psi^{(3)}' + a^2 \Lambda \Psi^{(3)} = 6a^2 \Lambda Q^{(3)} + 6a^2 \Lambda \varphi \Phi^{(2)} - 4a^2 \Lambda \varphi^3 \]

\[ - 30\varphi(\varphi')^2 + 3\varphi' (4\Psi^{(2)'} + \Phi^{(2)}) + 18\mathcal{H} \Phi^{(2)} \varphi' - 36\mathcal{H} \varphi^2 \varphi' + 18\mathcal{H} \Phi^{(2)'} - 12\mathcal{H} \varphi Q^{(2)'} + 6\mathcal{H} Q^{(3)'} \]

\[ + 6\Phi^{(2)} \varphi'' - 12\varphi \varphi'' + 6\varphi \Psi^{(2)'''} - 2\varphi (\partial_k \varphi)^2 - \partial^i \Phi^{(2)} \partial_i \varphi \]

\[ + 4\varphi \nabla^2 Q^{(2)} + 2 \nabla^2 Q^{(3)} + 8\pi Ga^2 T^{(3)}, \]

(8)

where \( Q = Q^{(1)} + Q^{(2)} + Q^{(3)} + \ldots \).

Since we are interested in the large-scale solution, it is more convenient to use the lower-case variables \( \psi^{(r)}, \varphi^{(r)} \), related to the upper-case ones as explained after equation (2.6). Finally, using the first-order equation \( \varphi'' + 3\mathcal{H} \varphi' + a^2 \Lambda \varphi = 0 \), we get

\[ \psi^{(3)''} + 3\mathcal{H} \psi^{(3)'} + a^2 \Lambda \psi^{(3)} \]

\[ = 6a^2 \Lambda Q^{(3)} + 6\mathcal{H} Q^{(3)'} - 6\varphi(\varphi')^2 + 3\varphi' \Psi^{(2)'} - 6\varphi' Q^{(2)'} \]

\[ - 12\mathcal{H} \varphi Q^{(2)'} - 2\varphi (\partial_k \varphi)^2 - (\partial^i \Phi^{(2)} \partial_i \varphi) \]

\[ + 4\varphi \nabla^2 Q^{(2)} + 2 \nabla^2 Q^{(3)} + 8\pi Ga^2 T^{(3)}. \]  

(9)
At this point we need to explicit the terms $Q^{(3)}$, $Q^{(3)'}$ and $T^{(3)}$. It is straightforward to show that

$$T^{(3)} = \frac{1}{(4\pi G a)^2} \rho_0 \left( \partial_i \varphi' + \mathcal{H} \partial_i \varphi \right) \left[ \partial_j \Psi^{(2)'} + \mathcal{H} \partial_j \Phi^{(2)} - 2 \varphi' \partial_i \varphi - \delta^{(1)} \left( \partial_i \varphi' + \mathcal{H} \partial_i \varphi \right) \right] \tag{B.16}$$

and

$$P_j^i = \frac{1}{(4\pi G a)^2} \rho_0 \left\{ \frac{1}{2} \left[ \partial_j \Psi^{(2)'} \partial_j \varphi' + \partial_k \varphi' \partial_k \Phi^{(2)} \right] + \frac{1}{4} \left( 5 \mathcal{H}^2 - a^2 \Lambda \right) \left[ \partial^i \Phi^{(2)} \partial_j \varphi + \partial^i \varphi \partial_j \Phi^{(2)} \right] - 2 \mathcal{H} \partial^i \varphi \partial_j \varphi + \frac{1}{2} \mathcal{H} \left[ \partial^i \Phi^{(2)} \partial_j \varphi' + \partial^i \varphi' \partial_j \Phi^{(2)} + \partial^i \Psi^{(2)'} \partial_j \varphi + \partial^i \varphi \partial_j \Psi^{(2)'} \right] \right. \right.$$

$$- \left. \left. \varphi' \left[ \partial^i \varphi' \partial_j \varphi + \partial^i \varphi \partial_j \varphi' \right] - \left( \delta^{(1)} + 2 \varphi \right) \left( \partial^i \varphi' + \mathcal{H} \partial^i \varphi \right) \partial_j \varphi \right\} \tag{B.17}$$

We now use the solutions at first and second order, and the first-order (0–0) equation

$$\delta^{(1)} = -2(\eta f/\Omega_m) \varphi_0, \text{ to obtain}$$

$$\nabla^2 Q^{(3)} = \frac{1}{4\pi G a^2} \rho_0 \left\{ E(\eta) \left[ 3 \nabla^2 - \partial_j \partial^j \left( \varphi_0 \partial^i \varphi_0 \partial_j \varphi_0 \right) - \left( \varphi_0 \partial^k \varphi_0 \partial_k \varphi_0 \right) \right] \right.$$\n
$$+ F(\eta) \left[ 3 \nabla^2 - \partial_j \partial^j \left( \partial^i \varphi_0 \partial_j \varphi_0 + \partial_j \varphi_0 \partial_j \alpha_0 \right) - 2 (\partial^k \varphi_0 \partial_k \varphi_0) \right] \right\}, \tag{B.18}$$

where $E(\eta), F(\eta)$ and $\alpha_0(\chi)$ are defined in equations (4.10), (4.11) and (4.6), respectively.

By differentiating the last equation we find also

$$\nabla^2 \dot{Q}^{(3)} = \mathcal{H} \nabla^2 \dot{Q}^{(3)} + \frac{1}{4\pi G a^2} \rho_0 \left\{ E' \left[ 3 \nabla^2 - \partial_j \partial^j \left( \varphi_0 \partial^i \varphi_0 \partial_j \varphi_0 \right) - \left( \varphi_0 \partial^k \varphi_0 \partial_k \varphi_0 \right) \right] \right.$$\n
$$+ F' \left[ 3 \nabla^2 - \partial_j \partial^j \left( \partial^i \varphi_0 \partial_j \varphi_0 + \partial_j \varphi_0 \partial_j \alpha_0 \right) - 2 (\partial^k \varphi_0 \partial_k \varphi_0) \right] \right\}, \tag{B.19}$$

using the background equation $\rho_0 = -3\mathcal{H} \rho_0$; and

$$T^{(3)} = \frac{1}{(4\pi G a^2)^2} \rho_0 \left[ (g' + \mathcal{H} g) \left( 2 A' + \mathcal{H} A + \mathcal{H} C - 2gg' + 2\mathcal{H} \frac{q^2 f^2}{\Omega_m} \right) \left( \varphi_0 \partial^k \varphi_0 \partial_k \varphi_0 \right) \right.$$\n
$$+ (g' + \mathcal{H} g) B' \left( \partial^k \varphi_0 \partial_k \varphi_0 \right) \right]. \tag{B.20}$$

where $A(\eta), B(\eta), C(\eta)$ are defined in equations (4.7)–(4.9), respectively.

Finally, putting all these expressions together, we arrive at equation (4.1), while equation (4.14) is simply given by expanding equation (B.7).

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