ON THE CASIMIR $\mathcal{WA}_N$ ALGEBRAS AS THE TRUNCATED $\mathcal{W}_\infty$ ALGEBRA

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Abstract: The complete structure of the Casimir $\mathcal{WA}_N$ algebras are shown to exist in such a way that the Casimir $\mathcal{WA}_N$ algebra is a kind of truncated type of $\mathcal{W}_\infty$ algebra both in the primary and in the quadratic basis, first using the associativity conditions in the basis of primary fields and second using the Miura basis coming from the free field realization as a different basis. Finally one can say that the Casimir $\mathcal{WA}_N$ algebra is a kind of truncated type of $\mathcal{W}_\infty$ algebra, so it is clear from any construction of $\mathcal{W}_\infty$ algebra that by putting infinite number of fields $W_s$ with $s > N$ to zero we arrive at the Casimir $\mathcal{WA}_N$ algebra.
1 Introduction

Although a wide variety of $W_N$ algebras was discovered since the last quarter century, nowadays we still do not have an explicit form for the Operator Product Expansions (OPEs) between primary fields and the structure constants for sufficiently large values of $N$, therefore the higher spin extensions of the conformal symmetries is still an open problem.

Conformal symmetry play an important role in the study of two-dimensional conformal field theories and thus it founds striking applications in string theory [1] and in the study of critical phenomena in statistical physics [2], as well as in mathematics [3]. Underlying symmetry algebra of this symmetry is Virasoro algebra which appears naturally in two-dimensional conformal field theories. The idea to extend Virasoro algebra with the introduction of higher conformal spin generators is also seem to be relevant in these theories. A seminal type of these extensions is the so-called $W_N$ algebras and Virasoro algebra is a $W_2$ algebra within this framework. First example of the extended symmetry algebra, $W_3$, was constructed by Zamolodchikov in [4, 5]. This algebra was obtained by extending the Virasoro algebra by addition of spin-3 conformal field. This algebras have been studied dealing with the classification and also construction from a variety of view points [6–10], as well as from the Casimir $WA_N$ algebras [11–13] point of view.

Another intriguing $W$ algebra very closely related to $W_N$ algebra is $W_\infty$ family, that is, $W$ algebra can be interpreted in the context of a $W_\infty$ type algebra [14–17]. Such an algebra is based on an infinite number of higher-spin operators with spin $s \geq 1$, or 2 and multiplicity 1. One emphasize here that the authors were studying $W_\infty$ for special value of parameters where the algebra linearizes. The first signs of connection between $W_\infty$ and $W_N$ is in the seminal Ref. [17]. From today’s point of view one can say that $W_\infty$ is two-parametric family algebra [18–20]. The connection between one-parametric algebra $W_N$ and two-parametric family $W_\infty$ is similar to the construction of the higher spin algebra $hs(\lambda)$ in Vasiliev theory [21]. From today’s point of view, the main idea of related studyings is first to understand $W_\infty$ and truncate it to $W_N$ for any $N$. There are some explicit results for quantum case as considered in [22] where the author focuses on the construction of $W_\infty$ in the primary basis and also in the Miura basis [23–28] which is different from the primary bases.
We also emphasize here that a wide variety of $\mathcal{W}_N$ algebras was discovered as expressed above, but nowadays we still do not have an explicit form for the Operator Product Expansions between primary fields and the structure constants for sufficiently large values of $N$. Therefore the main purpose of our work is to try to fill in this gap by finding the complete structure of the Casimir $\mathcal{W}_N\mathcal{A}_N$ algebra for sufficiently large values of $N$ and to establish a method in this direction in the Casimir $\mathcal{W}_N\mathcal{A}_N$ algebra point of view as was done in [13].

The paper is organized as follows: In section 2 we consider that $\mathcal{W}_N$ algebra is a kind of truncated type of $\mathcal{W}_\infty$ algebra, then, we construct the most general $\mathcal{W}_N$ algebra in the primary basis, and explain how the different structure constants can be determined recursively from the Jacobi identity. For this, we first count primary fields in the field content of the $\mathcal{W}_N$ just as in the field content of the $\mathcal{W}_\infty$, that is, using the character formula for $\mathcal{W}_\infty$, one can get a counting function which counts the Virasoro primaries for $\mathcal{W}_\infty$, so one can get truncated form of this character formula which counts the Virasoro primaries for $\mathcal{W}_N$. The method is rather straightforward: one begins with the stress-energy tensor and extends the algebra by addition of the higher-spin primary fields $W_s(z)$ with spin $s = 3, 4, ..., N$. These primary fields generate all other fields by using the conditions of the associativity of the OPEs.

In the next part, we made use of construction known as Miura transformation with Feigin-Fuchs type of free massless scalar fields, as a different starting point. It is seen that this gives us the possibility to exploit a relation between $\mathcal{W}_N$-algebras and $\mathcal{A}_{N-1}$-Lie algebras. This brings us to the fact that one can define the primary fields of $\mathcal{W}_N\mathcal{A}_5$ Casimir algebra as was done in [13]. So we proceed similarly to what we did in the first part and use the associativity conditions to compute the OPEs. In this way we can make comparisons between results of the different parts of the article. In the Appendix we give the explicit OPEs that overlap with either $\mathcal{W}_N$-algebras having structure constants (2.13) in section 2 and the Casimir $\mathcal{W}_N\mathcal{A}_N$ algebras having structure constants (3.17) in the section 3. All these have been possible with a dense application of Mathematica Package program [29].

2 $\mathcal{W}_N$ algebra in the primary basis

In this section we will study the $\mathcal{W}_N$ algebra in the basis of the Virasoro primary fields. To get the $\mathcal{W}_N$ algebra, we will extend the Virasoro algebra generated by stress-energy tensor $T(z)$

$$T(z_1)T(z_2) \sim \frac{s}{z_{12}^4} + \frac{2T}{z_{12}^3} + \frac{\partial T}{z_{12}}$$  \hspace{1cm} (2.1)

by higher spin primary generators $W_s(z)$ of spin $s = 3, 4, ..., N$:

$$T(z_1)W_s(z_2) \sim \frac{sW_s}{z_{12}^4} + \frac{\partial W_s}{z_{12}}.$$  \hspace{1cm} (2.2)

We will usually denote the Virasoro generator by $T(z) \equiv W_2(z)$ and $z_{12} = z_1 - z_2$. The algebraic structure will be fixed by imposing the conditions of associativity of the operator product algebra. We do not start by reviewing the construction and algebraic properties of the normal ordering in the radial quantization since this formalism is standard enough. A large part of this article was carried out with this formalism by using the Mathematica package OPEdefs by Kris Thielemans [30] to find all structure constants of related algebra. We also emphasize here that the power of the Thielemans’ package cannot be denied.

2.1 Conformal Bootstrap and $\mathcal{W}_N$ algebra in the primary basis

We can first define the field content of $\mathcal{W}_N$ and apply the basic properties of operator product expansions (OPE) to compute the OPE of $\mathcal{W}_N$.  

- 2 -
2.1.1 Field content of $W_N$ from $W_{\infty}$ algebra and counting $W_s(z)$ primaries

One can say that $W_N$ algebra is a kind of truncated type of $W_{\infty}$ algebra, so it is clear from any construction of $W_{\infty}$ algebra that by putting infinite number of fields $W_s$ with $s \geq N$ to zero we arrive at $W_N$ algebra. The definition of field content of $W_N$ will be our starting point in this paper. As field content, this algebra consist of two part; the first part have simple primary fields $W_s(z)$ of spin $s = 3, 4, \ldots, N$, and all other fields of $W_N$ will be obtained by taking derivatives an normal ordered products of these fields as composite primary fields.

Using the character formula

$$\chi[q] = q^{-s} \prod_{s=2}^{\infty} \prod_{j=s}^{\infty} \frac{1}{1-q^j}$$

(2.3)

for the vacuum representation of $W_{\infty}$ algebra, we can write down the function $P[q]$ and $P_N$ is the integer counting the number of Virasoro primaries of conformal dimension $h$

$$P[q] = \sum_{h=0}^{\infty} P_h q^h$$

(2.4)

that counts the Virasoro primaries for $W_{\infty}$ algebra. Then we find a formula for $P[q]$

$$P(q) = q + (1 - q) \prod_{s=2}^{\infty} \prod_{j=s}^{\infty} \frac{1}{1- \frac{1}{w_s} q^j} \approx 1 + q^3 + q^4 + q^5 + 2q^6 + 2q^7 + 5q^8 + 6q^9 + 11q^{10} + 14q^{11} + 26q^{12} + \cdots$$

(2.5)

To end this, one can use the more detailed counting function

$$q + (1 - q) \prod_{s=2}^{\infty} \prod_{j=s}^{\infty} \frac{1}{1- \frac{1}{w_s} q^j}$$

(2.6)

shows what $W_s$ primaries create the composite one (with $w_2 = 1$). For example, at order 10 in $q$ we find the coefficient

$$w_{10} + w_3 w_7 + w_4 w_6 + w_5 w_5 + w_3 w_3 w_4 + w_4 w_5 + w_3 w_6 + w_3 w_4 + w_3 w_5 + w_4 w_4 + w_3 w_3$$

(2.7)

which is simply reflected by the last two lines of the table below.

| spin | count | even composite primary field | odd composite primary field |
|------|-------|----------------------------|---------------------------|
| 0    | 1     | $1$                        | $-$                       |
| 3    | 1     | $-1$                       | $W_3$                     |
| 4    | 1     | $W_4$                      | $-$                       |
| 5    | 1     | $W_5$                      | $-$                       |
| 6    | 2     | $W_6, [W_3 W_3]$           | $-$                       |
| 7    | 2     | $-1$                       | $W_7, [W_3 W_4]$          |
| 8    | 5     | $W_8, [W_3 W_5], \partial^2[W_3 W_3], [W_4 W_4]$ | $\partial[W_3 W_4]$ |
| 9    | 6     | $\partial[W_3 W_5]$        | $W_9, [W_3 W_6], [W_2 W_5], [W_3 W_3]$ |
| 10   | 11    | $W_{10}, [W_3 W_7], [W_4 W_6], [W_5 W_5]$ | $\partial^2[W_3 W_4], \partial[W_4 W_5], \partial[W_3 W_6]$ |
|      |       | $[W_3 W_3 W_3], \partial^2[W_3 W_5], \partial^4[W_3 W_3]$ | $\partial^3[W_3 W_4]$ |
| 11   | 14    | $\partial[W_3 W_4 W_3], \partial[W_4 W_6], \partial^3[W_3 W_5]$ | $W_{11}, [W_3 W_3], [W_4 W_7]$ |
|      |       | $\partial^2[W_3 W_3 W_3], \partial[W_3 W_7]$ | $[W_3 W_6], [W_3 W_3 W_3]$ |
|      |       |                              | $[W_3 W_4 W_4], \partial^2[W_3 W_5]$ |
|      |       |                              | $\partial[W_4 W_5], \partial^4[W_3 W_4]$ |
Here these brackets are used symbolically to denote the composite primary fields. We see that from the table schematically all the composite fields are obtained by taking derivatives and normal ordered products of single primary fields.

### 2.1.2 Conformal Bootstrap and an ansatz for OPEs of \( \mathcal{W}_5 \) algebra with associativity

We emphasize here that \( \mathcal{W}_5 \) algebra is a kind of truncated type of \( \mathcal{W}_{\infty} \) algebra, so it is clear from any construction of \( \mathcal{W}_{\infty} \) algebra that by putting infinite number of fields \( W_s \) with \( s > 5 \) to zero we arrive at \( \mathcal{W}_5 \) algebra. We can summarize all field content of \( \mathcal{W}_5 \) algebra by the following table:

| spin | count | even composite primary field | odd composite primary field |
|------|-------|-----------------------------|-----------------------------|
| 0    | 1     | \( \mathbb{1} \)            | -                           |
| 3    | 1     | -                           | \( W_3 \)                   |
| 4    | 1     | \( W_4 \)                   | -                           |
| 5    | 1     | -                           | \( W_5 \)                   |
| 6    | 1     | \([W_3W_3]\)                 | -                           |
| 7    | 1     | -                           | \([W_3W_4]\)                |
| 8    | 4     | \([W_3W_5], \partial^2[W_3W_3], [W_4W_4]\) | \( \partial[W_3W_4]\) |
| 9    | 4     | \([W_3W_5], [W_4W_5], \partial^2[W_3W_4]\) | -                           |

To get this table we use the following truncated form of equation (2.6)

\[
q + (1 - q) \sum_{\alpha=3}^{5} \prod_{j=\alpha}^{\infty} \frac{1}{1 - w_j q^j}
\]

which shows what \( W_s \) primaries create the composite one for \( \mathcal{W}_5 \) algebra. For example, at order 8 in \( q \) we find the coefficient

\[
w_3w_5 + w_3w_3 + w_4w_4 + w_3w_4
\]

(2.9)

Let me start by using **OPEdefs** package to find all structure constants \( C_{ij} \) of the \( \mathcal{W}_5 \) algebra following from associativity of the OPE algebra. Technically, we will use **OPEconf** package as an extension of **OPEdefs**. We can write an ansatz symbolically to represent OPE algebra of the primary fields. So it can be written as:

\[
W_3W_3 \sim \frac{c}{3} \mathbb{1} + C_{33}^4 W_4
\]

\[
W_3W_4 \sim C_{34}^4 W_3 + C_{44}^4 W_5
\]

\[
W_4W_4 \sim \frac{c}{4} \mathbb{1} + C_{44}^4 W_4 + C_{44}^{[33]} [W_3W_3]
\]

\[
W_3W_5 \sim C_{34}^4 W_4 + C_{35}^{[33]} [W_3W_3]
\]

\[
W_4W_5 \sim C_{45}^{[33]} W_3 + C_{45}^{[34]} [W_3W_4] + C_{45}^{[34]} \partial [W_3W_4]
\]

\[
W_5W_5 \sim \frac{c}{5} \mathbb{1} + C_{55}^4 W_4 + C_{55}^{[33]} [W_3W_3] + C_{55}^{[35]} [W_3W_5] + C_{55}^{[44]} [W_4W_4] + C_{55}^{[33]''} \partial^2 [W_3W_3]
\]

(2.10)

Where all composite fields can be calculated in the presence of stress-energy tensor \( T(z) \) by thinking all multiple normal ordered products of single primary \( W_s \) fields must be primary as in the eq (2.2). The first two lines of ansatz (2.10) let us expression for composite primary operators \([W_3W_3]\) , \([W_3W_4]\) and we find

\[
[W_3W_3] = W_3 W_3 - \frac{22 C_{33}^{[4]} T W_4}{3(c+24)} - \frac{(5c+76) C_{33}^{[4]} \partial^2 W_4}{36(c+24)}
\]

\[
- \frac{16(191 c + 22)}{3(2c-1)(5c+22)(7c+68)} T T T - \frac{225 c^2 + 1978 c + 776}{2(2c-1)(5c+22)(7c+68)} \partial T \partial T
\]

(2.11)
The resulting relations are:

\[ \frac{94 C_{34}^5}{11c + 350} \] T – \frac{4(257c + 83) C_{34}^5}{(c + 23)(5c - 4)(7c + 114)} T T W_3 \]

\[ + \frac{(355c^3 - 329c^2 - 52214c - 12072) C_{34}^5 \partial^2 W_3}{18(c + 2)(c + 23)(5c - 4)(7c + 114)} \]

\[ - \frac{(437c^2 + 908c + 222454c - 76152) C_{34}^5 \partial^2 W_3}{12(c + 2)(c + 23)(5c - 4)(7c + 114)} \]

\[ + \frac{(25c^4 - 93c^3 - 17157c^2 + 115358c + 26904) C_{34}^5 \partial^4 W_3}{432(c + 2)(c + 23)(5c - 4)(7c + 114)} \]

Where RHS of these equations contains all the descendants at given level. In spite of the fact that the coefficients look very complicated, they are all established by the Virasoro algebra. Using function \textsc{OPEP Pole} in the \textsc{OPEConf} package one can find these results, as well as all the fields appearing on the RHS of eq. (2.10). The first significant Jacobi identity is \(W_3 W_4 W_5\) which gives us the first nontrivial relations for the structure constants appearing on the RHS of eq. (2.10). Then remainder Jacobi identities \(W_3 W_4 W_5\), \(W_4 W_5 W_5\), \(W_3 W_5 W_5\), \(W_4 W_5 W_5\) and \(W_5 W_5 W_5\) give us respectively all the structure constants appearing on the RHS of eq. (2.10). For completeness, the resulting relations are:

\[ C_{33}^3 = \sqrt{\frac{1024(c + 2)(c + 23)}{3(5c + 22)(7c + 68)}} \]

\[ C_{34}^5 = \frac{960(3c + 116)}{(7c + 68)(7c + 114) C_{33}^3 C_{45}^4} \]

\[ C_{44}^4 = \frac{96 (c^2 + 70c - 128)}{(5c + 22)(7c + 68) C_{33}^3} \]

\[ C_{44}^{[34]} = \frac{9(5c + 22)}{2(c + 2)(c + 23)} \]

\[ C_{45}^{[33]} = \frac{2(2c - 1) C_{44}^{[34]}}{3c + 116} \]

\[ C_{45}^{[34]} = \frac{-80 (11c^3 + 204c^2 + 9340c + 70272)}{(5c + 22)(7c + 68)(7c + 114) C_{33}^3} \]

\[ C_{45}^{[34]} = \frac{(37c + 334) S_{45}^{[34]}}{(3c + 116) C_{33}^3} \]

\[ C_{55}^{[33]} = \frac{3 (181c^2 + 14880c^2 + 248948c + 1507824)}{2(c + 2)(c + 23)(3c + 116)(7c + 114)} \]

\[ C_{55}^{[44]} = \frac{-64 (7c + 114)(3c + 116)(5c + 22)}{(3c + 116)(5c + 22)} \]

\[ C_{55}^{[33]} = \frac{4 (11c^2 - 306c - 13656)}{(2 + c)(116 + 3c)(114 + 7c)} \]
3 The Quantum Miura transformation and the Casimir $\mathcal{WA}_5$ algebra

The $\mathcal{W}_N$ algebra is generated by a set of chiral currents \{\(U_k(z)\)\}, of conformal dimension \(k (k = 1, \cdots, N)\). Let us define a differential operator \(R_N(z)\) of degree \(N\) \[31\]

\[
R_N(z) = - \sum_{k=0}^{N} U_k(z)(\alpha_0 \partial)^{N-k} =: N \prod_{j=1}^{N} (\alpha_0 \partial_z - h_j(z)) ;
\]

where the double dots denote the normal ordering of the free fields \(\varphi(z)\). Here \(\varphi(z)\) is an \(N-1\) component Feigin Fuchs-type of free massless scalar fields. This transformation is called the quantum Miura transformation and it determines completely the fields \{\(U_k(z)\)\} with

\[
h_j(z) = i\mu_j \partial \varphi(z)
\]

Here, \(\mu_i\)'s, \((i = 1, \cdots, N)\) are the weights of the fundamental representation of \(SU(N)\), satisfying \(\sum_{i=1}^{N} \mu_i = 0\) and \(\mu_i + \mu_j = \delta_{ij} - \frac{1}{N}\). The simple roots of \(SU(N)\) are given by \(\alpha_i = \mu_i - \mu_{i+1}, \quad (i = 1, \cdots, N-1)\). The Weyl vector of \(SU(N)\) is denoted as \(\rho = \frac{1}{2} \sum_{\alpha>0} \alpha^+\) where \(\alpha^+\) are the positive roots of \(SU(N)\). An OPE of \(h_j(z)\) with itself is given by

\[
h_{j_1}(z_1) h_{j_2}(z_2) \sim \frac{\delta_{j_1} - \frac{1}{N}}{z_{12}}
\]

The fields \{\(U_k(z)\)\} can be obtained by expanding \(R_N(z)\). We present a first five for \(\mathcal{W}_N\) algebra explicitly as in the following \[13\]

\[
\begin{align*}
U_0(z) &= -1 \\
U_1(z) &= \sum_i h_i(z) = 0 \\
U_2(z) &= - \sum_{i<j} (h_i h_j)(z) + \alpha_0 \sum_i (i-1) \partial h_i(z) \\
U_3(z) &= \sum_{i<j<k} (h_i h_j h_k)(z) - \alpha_0 \sum_{i<j} (i-1) \partial \left( (h_i h_j)(z) \right) \\
&\quad - \alpha_0 \sum_{i<j} (j-i-1) (h_i \partial h_j)(z) + \frac{1}{2} \alpha_0^2 \sum_i (i-1)(i-2) \partial^2 h_i(z) \\
U_4(z) &= - \sum_{i<j<k<l} (h_i h_j h_k h_l)(z) + \alpha_0 \sum_{i<j<k} (i-1) (\partial h_i h_j h_k)(z) + \alpha_0 \sum_{i<j<k} (k-3) (h_i h_j \partial h_k)(z) \\
&\quad + \alpha_0^2 \sum_{i<j} (j-2) (h_i \partial h_j h_k)(z) + \alpha_0^2 \sum_{i<j} (k-3) (h_i \partial^2 h_j)(z) \\
&\quad + \frac{\alpha_0^2}{2} \sum_{i<j} (i-1)(j-3) (\partial h_i \partial h_j)(z) - \frac{\alpha_0^2}{2} \sum_{i<j} (j-2)(j-3)(h_i \partial^2 h_j)(z) \\
&\quad - \frac{\alpha_0^2}{2} \sum_{i<j} (i-1)(i-2)(\partial^2 h_i h_j)(z) + \frac{\alpha_0^3}{3!} \sum_i (i-1)(i-2)(i-3) \partial^3 h_i(z)
\end{align*}
\]
\[U_s(z) = \sum_{i<j<k<l<m} (h_i h_j h_k h_m)(z) - \alpha_6 \sum_{i<j<k<l} (i-1)(\partial h_i h_j h_k h_l)(z) - \alpha_6 \sum_{i<j<k<l} (j-2)(h_i \partial h_j h_k h_l)(z) - \alpha_6 \sum_{i<j<k<l} (k-3)(h_i h_j \partial h_k h_l)(z) - \alpha_6 \sum_{i<j<k<l} (l-4)(h_i h_j h_k \partial h_l)(z) + \frac{\alpha_6^2}{2} \sum_{i<j<k} (i-1)(i-2)(\partial^2 h_i h_j h_k)(z) + \frac{\alpha_6^2}{2} \sum_{i<j<k} (j-2)(j-3)(\partial h_i \partial h_j h_k)(z) + \frac{\alpha_6^2}{2} \sum_{i<j<k} (k-3)(k-4)(h_i h_j \partial^2 h_k)(z) + \frac{\alpha_6^2}{2} \sum_{i<j<k} (i-1)(i-3)(\partial \partial h_i h_j h_k)(z) + \frac{\alpha_6^2}{2} \sum_{i<j<k} (j-2)(i-4)(h_i \partial h_j \partial h_k)(z) - \frac{\alpha_6^3}{6} \sum_{i<j}(i-1)(i-2)(i-3)(\partial^3 h_i h_j h_k)(z) - \frac{\alpha_6^3}{6} \sum_{i<j}(i-1)(i-2)(j-4)(\partial^2 h_i h_j h_k)(z) - \frac{\alpha_6^3}{6} \sum_{i<j}(i-1)(j-4)(\partial h_i \partial^2 h_j h_k)(z) - \frac{\alpha_6^3}{6} \sum_{i<j}(j-2)(j-3)(j-4)(h_i \partial^3 h_j h_k)(z) + \frac{\alpha_6^4}{4!} \sum_{i<j}(i-1)(i-2)(i-3)(i-4)(\partial^4 h_i h_j h_k)(z)\]

One can see that \(U_s(z) \equiv T(z)\) has spin 2, which is called the stress-energy tensor, \(U_s(z)\) has spin s. The standard OPE of \(T(z)\) with itself is

\[T(z_1)T(z_2) \sim \frac{c}{z_{12}^2} + \frac{2T}{z_{12}} + \frac{\partial T}{z_{12}}\]  

(3.7)

where the central charge, for \(SU(N)\), is given by

\[c = (N-1)(1-N(N+1)\alpha_6^2).\]  

(3.8)

### 3.1 Primary Field content of the Casimir \(\mathcal{W}_A_5\) algebra in the Quantum Miura basis

If one compare the fields \(U_s(z)\) and the fields \(W_s(z)\) as in the section-1, the fields \(U_s(z)\) are not primary since [13]

\[T(z_1)U_s(z_2) \sim \sum_{k=1}^{N} \frac{(N-k+s)!}{(N-k)!} a_s^{s-2} ((s-1)(N-1) + 2(k-1)) \alpha_6^{s-1} \frac{U_{s-1}}{z_{12}^{s-1}} + \frac{k \alpha_6^2}{z_{12}^2} + \frac{\partial U_s}{z_{12}}\]  

(3.9)

Using above OPE, we want to construct the Casimir \(\mathcal{W}_3\) algebra. Therefore we must obtain spin 3 , spin 4 and spin 5 primary fields. Here we first write down the spin 3 primary field for \(SU(N)\) as [13]

\[U_s(z) = U_3(z) - \frac{(N-2)}{2} \alpha_6 \partial T(z)\]  

(3.10)
the spin 4 primary field as [13]

\[ \mathcal{U}_4(z) = \frac{(N - 3)}{2} a_0 \partial \mathcal{U}_3(z) + \frac{(N - 2)(N - 3)}{4N(22 + 5c)} [-3 + N(13 + 3N + 2c) a_0^2] \partial^2 T(z) \\
+ \frac{(N - 2)(N - 3)}{2N(22 + 5c)} [5 - N(5N + 7) a_0^2] (TT)(z) \]  

(3.11)

and finally the spin 5 primary field as [13]

\[ \mathcal{U}_5(z) = \frac{(N - 4)}{2} a_0 \partial \mathcal{U}_4(z) + \frac{3}{4} \frac{(N - 3)(N - 4)}{N(114 + 7c)} [-2 + N(20 + C + 2N) a_0^2] \partial^2 \mathcal{U}_4(z) \\
+ \frac{(N - 2)(N - 3)(N - 4)}{12N(114 + 7c)} [9 - N(33 + C + 9N) a_0^2] \partial^3 \mathcal{U}_4(z) \\
+ \frac{(N - 3)(N - 4)}{N(114 + 7c)} [7 - N(13 + 7N) a_0^2] (U_2 U_3)(z) \\
+ \frac{(N - 1)(N - 3)(N - 4)}{2N(114 + 7c)} [-7 + N(13 + 7N) a_0^2] (U_2 \partial U_2)(z) \]  

(3.12)

3.2 OPEs of higher spin primary fields

To obtain OPE of the two primary fields \( \{ \mathcal{U}_k(z) \} \) and \( \{ \overline{\mathcal{U}}_k(z) \} \) which gives the central terms in the known form as

\[ W_s(z_1) W_s(z_2) \sim \frac{c}{2z_1^2} + \cdots \]  

(3.13)

so, we must take care of the normalized forms of all the primary fields \( \{ \overline{\mathcal{U}}_k(z) \} \). Therefore the normalized form of the Casimir \( \mathcal{W}_5 \) algebra generators are given by the following expressions:

\[ \overline{\mathcal{U}}_k(z) = \sqrt{\theta_3} W_k(z), \overline{\mathcal{U}}_4(z) = \sqrt{\theta_4} W_4(z), \overline{\mathcal{U}}_5(z) = \sqrt{\theta_5} W_5(z) \]  

(3.14)

where \( \theta_3 \), \( \theta_4 \) and \( \theta_5 \) are the normalization factors for \( SU(5) \) and they can be written explicitly:

\[ \theta_3 = \frac{7c + 68}{80}, \theta_4 = \frac{(c + 2)(c + 23)(7c + 68)}{300(5c + 22)}, \theta_5 = \frac{(c + 2)(c + 23)(3c + 116)(7c + 68)}{24000(7c + 114)} \]  

(3.15)

A straightforward calculation gives us the non-trivial OPEs of the Casimir \( \mathcal{W}_3A_5 \) algebra as in the Appendix, with structure constants which can be calculated explicitly:

\[ C_{33}^{4} = \frac{320}{(7c + 68)\theta_4}, C_{34}^{4} = \frac{\sqrt{1024(c + 2)(c + 23)}}{3(5c + 22)(7c + 68)}, \]

\[ C_{34}^{4} = \frac{3}{4} C_{33}^{3}, C_{34}^{5} = \frac{5\theta_3 \theta_4}{\theta_5}, C_{44}^{4} = \sqrt{\frac{25(3c + 116)(5c + 22)}{(7c + 68)(7c + 114)}}, \]

\[ C_{44}^{4} = \frac{16}{9} (c^2 + 70c - 128) \frac{(c + 2)(c + 23)(7c + 68)\theta_4}{(c + 2)(c + 23)(7c + 68)^2} = \sqrt{\frac{27 (c^2 + 70c - 128)^2}{(c + 2)(c + 23)(5c + 22)(7c + 68)}}, \]

\[ C_{44}^{33} = \frac{9(5c + 22)}{2(c + 2)(c + 23)}, C_{45}^{4} = \frac{(3c + 116)(5c + 22) \theta_3 \theta_5}{20(7c + 114)\theta_4} = \frac{16(3c + 116)(5c + 22)}{(7c + 68)(7c + 114)}, \]
$$C^{(33)}_{45} = \frac{6(2c - 1)\theta_3}{5(7c + 114)\theta_3} = \sqrt{\frac{432(2c - 1)^2}{(c + 2)(c + 23)(3c + 116)(7c + 114)^2}}, \quad C^{3}_{45} = \frac{3}{4}C^{4}_{35},$$

$$C^{2}_{45} = -\frac{(11c^3 + 204c^2 + 9340c + 70272)}{4(5c + 22)(7c + 114)}\theta_4 = \sqrt{\frac{75(11c^3 + 204c^2 + 9340c + 70272)^2}{4(c + 2)(c + 23)(5c + 22)(7c + 68)(7c + 114)^2}},$$

$$C^{(34)}_{45} = \frac{(37c + 334)\theta_3}{5(7c + 114)\theta_3} = \sqrt{\frac{12(37c + 334)^2}{(c + 2)(c + 23)(3c + 116)(7c + 114)^2}}, \quad C^{4}_{55} = \frac{4}{5}C^{5}_{45},$$

$$C^{(33)}_{55} = \frac{120\left(181c^3 + 14880c^2 + 248948c + 1507824\right)}{(c + 2)(c + 23)(3c + 116)(7c + 68)(7c + 114)\theta_4^2} = \frac{3\left(181c^3 + 14880c^2 + 248948c + 1507824\right)}{2(c + 2)(c + 23)(3c + 116)(7c + 114)}\theta_3^2,$$

$$C^{(35)}_{45} = -\frac{2(3c + 116)\theta_3}{(7c + 114)\theta_3} = -\frac{10}{3}C^{(34)}_{45}, \quad \left(C^{(34)}_{45}\right)^2 = \frac{108(3c + 116)}{(c + 2)(c + 23)(7c + 114)},$$

$$C^{(44)}_{55} = \frac{19200(7c + 114)}{(c + 2)(c + 23)(3c + 116)(7c + 68)\theta_4^2} = \frac{64(7c + 114)}{(3c + 116)(5c + 22)}.$$

These structure constants overlap with the structure constants (2.13) in the Casimir $\mathcal{W}_A_{3\lambda}$ algebras in the section 2.
A Appendix

A.1 Explicit OPEs of the Casimir $W_{A_5}$ algebras

Here we give all the non-trivial OPEs of the Casimir $W_{A_5}$ algebra, as follows:

$$W_3(z_1)W_3(z_2) \sim \frac{5}{z_{12}^3} + \frac{2 T}{z_{12}^2} + \frac{\partial T}{z_{12}^2} + \frac{1}{z_{12}^2} \left( C_{43}^4 W_4 + \frac{32}{5c + 22} T T + \frac{3(c - 2)}{2(5c + 22)} \partial^2 T \right) + \frac{1}{z_{12}^2} \left( C_{43}^4 \frac{2}{5c + 22} \partial W_4 + \frac{32}{5c + 22} \partial T T + \frac{c - 2}{3(5c + 22)} \partial^3 T \right)$$

(A.1)

$$W_3(z_1)W_4(z_2) \sim + \frac{1}{z_{12}^2} \left( \frac{3 C_{43}^4}{4} W_3 + \frac{1}{z_{12}^2} \frac{C_{43}^4}{4} \partial W_3 + \frac{1}{z_{12}^2} \left( C_{53}^5 W_5 + \frac{39 C_{43}^4}{7c + 114} T W_3 + \frac{3(c - 6) C_{43}^4}{8(7c + 114)} \partial^2 W_3 \right) \right)$$

$$\quad + \frac{1}{z_{12}^2} \left( \frac{2}{5} \frac{C_{34}^3}{z_{34}} \partial W_5 + \frac{3(9c - 2) C_{34}^3}{2(c + 2)(7c + 114)} T \partial W_3 + \frac{15(5c + 22) C_{34}^3}{4(c + 2)(7c + 114)} \partial T W_3 + \frac{(c^2 - 3c - 6) C_{43}^4}{16(c + 2)(7c + 114)} \partial^3 W_3 \right)$$

(A.2)

$$W_4(z_1)W_4(z_2) \sim \frac{5}{z_{12}^3} + \frac{2 T}{z_{12}^2} + \frac{\partial T}{z_{12}^2} + \frac{1}{z_{12}^2} \left( C_{44}^4 W_4 + \frac{42}{5c + 22} T T + \frac{3(c - 4)}{2(5c + 22)} \partial^2 T \right) + \frac{1}{z_{12}^2} \left( C_{44}^4 \frac{2}{5c + 22} \partial W_4 + \frac{42}{5c + 22} \partial T T + \frac{c - 4}{3(5c + 22)} \partial^3 T \right)$$

$$\quad + \frac{1}{z_{12}^2} \left( \frac{128(7c - 118)}{(5c + 22)(7c + 68) C_{43}^4} T W_4 + \frac{72(3c + 38)(4c - 1)}{(c + 2)(c + 23)(5c + 22)(7c + 68)} \partial^2 T T \right)$$

$$\quad + \frac{295c^3 + 5052c^2 + 16164c - 34768}{8(5c - 106c - 1456)(7c + 68) C_{34}^3} \partial^2 W_4 + \frac{5c^4 + 20c^3 - 1344c^2 - 10928c + 4336}{12(c + 2)(c + 23)(5c + 22)(7c + 68)} \partial^4 T \right)$$

$$\quad + \frac{1}{z_{12}^2} \left( C_{34}^{33} \partial W_3 W_3 + \frac{64}{(5c + 22)(7c + 68) C_{35}^5} T \partial W_4 + \frac{64}{(5c + 22)(7c + 68) C_{35}^5} \partial T W_4 \right)$$

$$\quad + \frac{108(3c + 38)(4c - 1)}{(c + 2)(c + 23)(5c + 22)(7c + 68)} \partial^4 T \right) + \frac{3(59c^3 + 492c^2 - 3204c - 5312)}{(c + 2)(c + 23)(5c + 22)(7c + 68)} \partial^2 T T T \right)$$

$$\quad + \frac{3(c - 8)(13c^2 + 226c + 832)}{8(c^2 - 38c - 8)} \partial^3 T T + \frac{2(c + 2)(c + 23)(5c + 22)(7c + 68)}{3(5c + 22)(7c + 68) C_{34}^3} \partial^3 W_4$$

$$\quad + \frac{5c^4 + 20c^3 - 1344c^2 - 10928c + 4336}{80(c + 2)(c + 23)(5c + 22)(7c + 68)} \partial^5 T \right)$$

(A.3)
\[
\begin{align*}
W_4(z_1)W_5(z_2) & \sim \frac{1}{z_{12}^3}C_{45}^3 W_3 + \frac{1}{z_{12}^3}3C_{45}^3 \partial W_3 \\
+ \frac{1}{z_{12}^3} \left( \frac{2c - 1}{3(c + 116)} \right) C_{45}^{[34]'} T\partial W_4 & + \frac{7c + 114}{6(c + 116)} \partial T W_4 - \frac{32(191c + 22)C_{45}^{[34]'}}{3(c + 116)(5c + 22)(7c + 68)} \partial T T T T \\
+ \frac{4(2c - 1)C_{45}^{[34]'}}{3(5c + 116)} \partial W_3 W_3 & - \frac{30(3c^2 + 6c + 4) C_{45}^{[34]'}}{(3c + 116)(5c + 22)(7c + 68)} \partial^2 T T T T - \frac{4(89c^2 + 266c - 1624) C_{45}^{[34]'}}{9(3c + 116)(5c + 22)(7c + 68)} \partial^3 T T \\
+ \frac{(c^2 - 70c - 24) C_{45}^{[34]'}}{576(3c + 116)} \partial^3 W_4 & - \frac{(c - 8)(5c^2 + 60c + 4) C_{45}^{[34]'}}{30(3c + 116)(5c + 22)(7c + 68)} \partial^5 T \\
\end{align*}
\]
\[
W_5(z_1)W_5(z_2) \sim \frac{\xi}{z_{12}^6} + \frac{2 T}{z_{12}^3} + \frac{\partial T}{z_{12}} + \frac{1}{2} \left( \frac{C_{33}^{[33]} W_4 + \frac{52}{5c+22} TT + \frac{3(c-6)}{2(5c+22)} \partial^2 T}{z_{12}} \right) + \frac{1}{z_{12}} \left( \frac{C_{33}^{[3]} \partial W_4}{z_{12}} + \frac{52}{5c+22} \partial TT + \frac{c-6}{3(5c+22)} \partial^3 T \right) \\
+ \frac{1}{z_{12}} \left( C_{33}^{[3]} \partial W_3 W_3 - \frac{3 (187c^3 + 11520c^2 + 452876c + 3767568) C_{33}^{[3]} \partial W_4}{16(c+2)(c+23)(3c+116)(7c+114)} \right) \\
\frac{1}{z_{12}} \left( \frac{C_{33}^{[3]} \partial W_3 W_3 - \frac{3 (187c^3 + 11520c^2 + 452876c + 3767568) C_{33}^{[3]} \partial W_4}{16(c+2)(c+23)(3c+116)(7c+114)} \right) \\
\frac{1}{z_{12}} \left( \frac{C_{33}^{[3]} \partial W_3 W_3 - \frac{3 (187c^3 + 11520c^2 + 452876c + 3767568) C_{33}^{[3]} \partial W_4}{16(c+2)(c+23)(3c+116)(7c+114)} \right) \\
\frac{1}{z_{12}} \left( \frac{C_{33}^{[3]} \partial W_3 W_3 - \frac{3 (187c^3 + 11520c^2 + 452876c + 3767568) C_{33}^{[3]} \partial W_4}{16(c+2)(c+23)(3c+116)(7c+114)} \right)
\]
\[
\begin{align*}
&- 128 \left(235c^4 + 16593c^3 + 444596c^2 + 2945348c - 2562704\right) \frac{\partial T \partial W_4}{(3c + 116)(5c + 22)(7c + 68)(7c + 114)c_{33}^4} \\
&+ 245c^3 + 8536c^2 + 31524c - 259536 \partial W_3 \partial W_3 + 3 \left(111c^2 + 2704c^2 - 44676c - 924080\right) \partial^2 W_3 W_3 \\
&- 32 \left(557c^4 + 23830c^3 + 223380c^2 - 10662136c - 105450464\right) \partial^2 T W_4 \\
&+ 2 \left(30884c^4 + 1049739c^3 + 4568478c^2 - 161344916c^2 - 1352580792c + 345598368\right) \partial^2 T T T \\
&+ 3 \left(2303c^6 + 79492c^5 - 477096c^4 - 30376928c^3 + 111133680c^2 + 1333008960c + 277883904\right) \partial^2 T \partial T^2 T \\
&+ \frac{30625c^6 + 406272c^5 + 4388712c^4 - 432405472c^3 - 4946700400c^2 - 8004630912c + 12394720512}{(c + 2)(c + 23)(3c + 116)(5c + 22)(7c + 68)(7c + 114)} \partial^3 T \partial T \\
&+ 5425c^6 + 160666c^5 - 3951116c^4 - 170672872c^3 - 1654958176c^2 - 2594536192c + 5584883712 \partial^4 T T \\
&+ 1 \left(3c + 116\right) \left(5c + 22\right) \left(7c + 68\right) \left(7c + 114\right) c_{33}^4 \left(3c + 116\right) \left(5c + 22\right) \left(7c + 68\right) \left(7c + 114\right) c_{33}^4 \left(3c + 116\right) \left(5c + 22\right) \left(7c + 68\right) \left(7c + 114\right) c_{33}^4 \\
&+ 1225c^7 + 30940c^6 - 2227980c^5 - 77243512c^4 - 744781216c^3 - 1705759584c^2 + 7306651392c - 2606286336 \partial^6 T \\
&- \frac{1}{12} \left(c_{55}^4 \partial W_4 W_4 + c_{33}^3 \partial (W_3 W_3) - 8192 \left(43c^4 + 2393c^3 + 23131c - 5262\right) \frac{\partial T W_3 W_3}{(3c + 116)(5c + 22)(7c + 68)(7c + 114)c_{33}^4} \right) \partial T \partial W_4 \\
&- \frac{48(2c - 1)(31c + 572)}{(c + 2)(c + 23)(3c + 116)(7c + 114)} \frac{\partial T W_3 W_3}{(3c + 116)(5c + 22)(7c + 68)(7c + 114)c_{33}^4} \partial T W_4 + \frac{64 \left(155c^4 + 9936c^3 + 158844c^2 + 1871664c - 2291840\right)}{3(c + 116)(5c + 22)(7c + 68)(7c + 114)c_{33}^4} T \partial^3 W_4 \\
&- 16384 \left(43c^3 + 2393c^2 + 23131c - 5262\right) \frac{(3c + 116)(5c + 22)(7c + 68)(7c + 114)c_{33}^4}{(c + 2)(c + 23)(3c + 116)(7c + 114)c_{33}^4} \partial T T W_4 \\
&- 1536 \left(504c^4 + 17652c^3 + 17193c^2 - 84704c + 10972\right) \frac{\partial T T T T}{(c + 2)(c + 23)(3c + 116)(5c + 22)(7c + 68)(7c + 114)c_{33}^4} \\
&+ 4 \left(866c^4 + 25699c^3 + 186946c^2 - 1053888c + 262944\right) \frac{\partial T \partial T \partial T}{(c + 2)(c + 23)(3c + 116)(5c + 22)(7c + 68)(7c + 114)c_{33}^4} \\
&+ 8 \left(7756c^5 + 254217c^4 + 1074504c^3 - 41564620c^2 + 248593104c + 41783424\right) \frac{\partial^2 T \partial T T}{(c + 2)(c + 23)(3c + 116)(5c + 22)(7c + 68)(7c + 114)c_{33}^4} \\
&+ 32 \left(157c^6 + 10430c^5 + 67296c - 250272\right) \frac{\partial^2 T \partial^2 W_3}{(c + 2)(c + 23)(3c + 116)(5c + 22)(7c + 68)(7c + 114)c_{33}^4} \\
&+ 8 \left(7756c^5 + 254217c^4 + 1074504c^3 - 41564620c^2 + 248593104c + 41783424\right) \frac{\partial^2 T \partial^2 W_4}{(c + 2)(c + 23)(3c + 116)(5c + 22)(7c + 68)(7c + 114)c_{33}^4} \\
&+ 35c^3 + 1096c^2 + 12684c - 41136 \frac{\partial^2 W_3 \partial W_3}{(c + 2)(c + 23)(3c + 116)(7c + 114)c_{33}^4} \\
&+ 32 \left(281c^4 + 3268c^3 - 430508c^2 - 10725536c + 50220992\right) \frac{\partial^2 T \partial^2 W_3}{(c + 2)(c + 23)(3c + 116)(5c + 22)(7c + 68)(7c + 114)c_{33}^4} \\
&+ 256 \left(46c^4 + 73c^3 - 98752c^2 - 3039700c - 18902512\right) \frac{\partial^2 T W_4}{(c + 2)(c + 23)(3c + 116)(5c + 22)(7c + 68)(7c + 114)c_{33}^4} \\
&+ 8 \left(1708c^5 + 60145c^4 + 77830c^3 - 16938364c^2 - 128486776c + 25371680\right) \frac{\partial^3 T T T}{(c + 2)(c + 23)(3c + 116)(5c + 22)(7c + 68)(7c + 114)c_{33}^4} \\
&+ 2303c^6 + 79492c^5 - 3882096c^4 - 48628928c^3 - 215981680c^2 + 1148688960c + 94139904 \frac{\partial^3 T \partial^2 T}{(c + 2)(c + 23)(3c + 116)(5c + 22)(7c + 68)(7c + 114)c_{33}^4} \\
&+ \cdots
\end{align*}
\]
\[\begin{align*}
&+ \frac{1365c^6 + 37234c^5 - 1037316c^4 - 44429640c^3 - 386767936c^2 - 483555456c + 91093296}{6(c + 2)(c + 23)(3c + 116)(5c + 22)^2(7c + 68)(7c + 114)} \partial^4 T \partial T \\
&+ \frac{1015c^6 + 30858c^5 - 929600c^4 - 40783408c^3 - 373238160c^2 - 472204384c + 1790162304}{15(c + 2)(c + 23)(3c + 116)(5c + 22)^2(7c + 68)(7c + 114)} \partial^5 TT \\
&- \frac{8 (35c^5 + 1590c^4 + 48252c^3 + 422632c^2 - 25026336c + 13821696)}{45(3c + 116)(5c + 22)^2(7c + 68)(7c + 114)c^4} \partial^4 W_4 \\
&+ \frac{245c^7 + 6188c^6 - 592564c^5 - 21763712c^4 - 204399584c^3 - 340348544c^2 + 2548955904c - 556655616}{2520(c + 2)(c + 23)(3c + 116)(5c + 22)^2(7c + 68)(7c + 114)c^4} \partial^7 T
\end{align*}\]

(A.6)

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