Spectral Super Elements for Beams with Arbitrary Cross Section

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Spectral Super Elements are semi-analytical elements for the calculation of the wave propagation in structures with a constant cross section. They base on a finite element discretization in the cross-sectional directions and wave functions as ansatz for the longitudinal direction. No discretization in the latter direction has to be carried out and therefore the number of degrees of freedom is independent of the length the structure.

The wave functions used as ansatz are gained as eigenvalues (wave numbers) and –vectors (wave forms) from a quadratic eigenvalue problem of the infinite structure (in longitudinal direction). The infinite structure is reduced to a 2D problem with the help of a Fourier transformation in longitudinal direction. This method is called Waveguide Finite Element Method or 2.5D Finite Element Method in literature.

In this study we compare the performance of Spectral Super Elements for beams with arbitrary cross sections with analytical solutions for the Euler-Bernoulli beam at low frequencies and with results form a 3D Finite Element analysis with commercial software for higher frequencies.

1 Introduction

The origin of the Waveguide Finite Element Method (WFEM) dates back to the early seventies (Aalami [1]). Gravic [2] and Finnveden [3] used this technique later for the calculation of dispersion curves. Birgersson et al. [4] have developed Spectral Super Elements (SSEs) based on two-noded (8 DOF) WFEM plate elements. Ryue et al. [5] have used semi-infinite SSEs for beams with arbitrary cross sections based on four-noded (12 DOF) WFEM volume elements. In this study we use finite SSEs similar to [4] which are based on the same WFEM elements as [5].

The number of degrees of freedom (DOFs) of a SSE is independent of its length with same accuracy. The coupling (also with conventional FEs) and boundary condition application at the start and end of the SSE is carried out analogously to conventional FE. The only requirement is a constant (arbitrary) cross section.

2 Theory

The mathematical-mechanical theory of SSEs in general is described e.g. by Birgersson et al. [4]. The theory consist of two major steps. In the first step wave numbers and shapes of waves occurring in an infinite waveguide with the considered cross section and material at the defined frequency are determined (Waveguide FEM). The results are used in the second step to formulate an ansatz for the displacement of an finite beam element. A linear system of equations for determining the DOFs in the FE mesh at the start and end (comp. Fig. 3) of the SSE is found.

2.1 Waveguide FEM

The cross section of an infinite waveguide is discretised with quadrilateral four-noded elements as depicted in Fig 1. The DOFs (ui(x), vi(x), wi(x)) are considered to be (unknown) analytical functions of x. Bi-linear shape functions are used for the displacement ansatz in the cross-sectional directions (yi, zi). No external loading is considered. With the help of Hamilton’s Principle a system of differential equations is found:

\[
K_{2} v'' + K_{1} v' + \left( K_{0} - \omega^{2} M \right) v = 0
\]

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where \( \vec{v} \) is the vector of unknown DOF functions, \( \omega \) is the angular excitation frequency, \( K \) and \( M \) are stiffness and mass matrices and the apostrophe ' denotes the partial derivative w.r.t. the coordinate \( x \). This system of differential equations is transferred to a system of algebraic equations with the help of Fourier transformation form the spatial domain \( x \) to the wave number domain \( k_x \):

\[
\left( -k_x^2 K_{k_x} + i k_x K_{k_x}^1 + K_{k_x}^0 - \omega^2 M \right) \tilde{v}(k_x) = 0
\]

This represents a quadratic eigenvalue problem for the wave number \( k_x \), if we prescribe a frequency \( \omega \). Twice the number of DOFs in the cross section of eigenvalues and corresponding eigenvectors (visualized exemplarily in Fig. 2) are found. This eigenvalue problem causes already approx. half of the computational effort of the whole SSE calculation.

### 2.2 Spectral Super Element

The eigenvalues \( k_x \) and eigenvectors (wave shapes, concatenated in the matrix \( \Phi \)) are used to formulate the displacement ansatz in (1) for a finite element like in Fig. 3. \( E(x) \) is a diagonal matrix of wave functions of the form \( e^{ik_x x} \) which have to be shifted in \( x \)-direction for numerical stability according to [4] and [6]. The unknown wave contribution factors \( a \) are mapped with (2) onto the FE DOFs at the start and end of the SSE (\( W_1 \) and \( W_2 \)) by inserting the length of the SSE into (1):

\[
\begin{align*}
\vec{v}(-l_x) &= \Phi E(-l_x)a = W_1 \\
\vec{v}(l_x) &= \Phi E(l_x)a = W_2 \\
\Rightarrow a &= \left( \frac{\Phi E(-l_x)}{\Phi E(l_x)^{-1}} \right)^{-1} \left\{ \frac{W_1}{W_2} \right\} = A W \\
\Rightarrow \vec{v}(x) &= \Phi E(x) A W
\end{align*}
\]

Inserting the ansatz of (2) into the Lagrangian obtained with Hamilton’s principle in the Waveguide FEM and carrying out the integral in \( x \)-direction from \(-l_x\) to \( l_x \) analytically leads through Euler’s differential equation to an system of algebraic equations of the form \( K W - F = 0 \). \( F \) is the load vector of nodal forces at the FE DOFs at the start and end of the SSE.

### 3 Numerical Example

The numerical example is a single span concrete beam, discretised with one SSE, loaded with a moment \( M_y \) at one bearing (Fig. 4a). Cross section: \( b = 0.4m \), \( h = 0.6m \); span: \( l = 10m \); material: Young’s modulus \( E = 28300000 \frac{MPa}{m^2} \), Poisson’s ratio \( \nu = 0.2 \), density \( \rho = 2.5 \frac{kg}{m^3} \). Fig. 4b) shows exemplarily the deformation at 200Hz. Figs. 4c)+d) show the frequency response at \( x = 2.6m \). Euler-Bernoulli assumptions are sufficient only for the first eigenfrequency. The ANSYS model (8-node solid185 elements with “simplified enhanced strain formulation”) is at higher frequencies to stiff if the same element size is used as for the discretisation of the SSE cross section.

![Fig. 4: Numerical example. a) Undef. system with loading; b) Def. system at 200Hz; c)+d) Vertical deflection at \( x = 2.6m \) (mean of all cross-sectional nodes), green: SSE, orange: Spectral Element based on Euler-Bernoulli assumptions, blue: ANSYS with ESIZE= 0.1m (same as for cross sectional mesh used for the SSE), red: ANSYS with ESIZE= 0.05m. Green and blue are almost identical in c).](image)

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