Unstability of pseudoharmonic maps between pseudo-Hermitian manifolds

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Abstract

In this paper, we derive the second variation formula of pseudoharmonic maps into any pseudo-Hermitian manifolds. When the target manifold is an isometric embedded CR manifold in complex Euclidean space or a pseudo-Hermitian immersed submanifold in Heisenberg group, we give some conditions on Weingarten maps to obtain some unstability of pseudoharmonic maps between these pseudo-Hermitian manifolds.

1 Introduction

As known to all, a harmonic map is a critical point of the energy integral. A harmonic map is called stable if it has nonnegative second variation, that is, the index of the map is 0. The stability problem is an important problem in the theory of harmonic maps. In [7], R.T. Smith estimated the index of the identity map of a Riemannian manifold, in particular, he showed that the identity map on $S^m$ is $m + 1$. In [8], Y.L. Xin proved that for $m \geq 3$, any nonconstant harmonic map $f : S^m \to N^n$ is unstable. A result of Leung [5] states that any nonconstant map from a compact Riemannian manifold to the sphere is unstable too. In [4], the authors extended the Leung’s result to the case that the target manifold is a compact immersed submanifold of Euclidean space.

In recent years, some generalized harmonic maps have been introduced and investigated in various geometric backgrounds. For example, some pseudoharmonic maps were introduced in the field of pseudo-Hermitian geometry (cf. [1],[2],[6]). For a map $f : (M^{2m+1}, H(M), J, \theta) \to (N, \tilde{H}(N), \tilde{J}, \tilde{\theta})$ between two pseudo-Hermitian maps, Petit [6] introduced a new horizontal energy functional $E_{H,\tilde{H}}(f)$. He derived the first variation formula of $E_{H,\tilde{H}}(f)$ and called a critical point of the energy a pseudoharmonic map. Note that there is an extra condition on the pull-back of the torsion on the target manifold. The second author [1] modified Petit’s variational problem slightly by restricting the variational vector field to be horizontal. In [1], the critical point of the restricted variational problem about is refered to as a pseudoharmonic too. Among other results, the second author derived the second variation formula of pseudoharmonic maps.

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into Sasakian manifold and proved that any nonconstant horizontal pseudoharmonic map from a closed pseudo-Hermitian manifold into the odd dimensional sphere is unstable.

In this paper, we will extend their results to the case that the target manifold is an isometric embedded CR manifold or a pseudo-Hermitian immersed submanifold of Heisenberg group and give a low bound of the index of identity map \( I : S^{2n+1} \to S^{2n+1} \) with \( n \geq 1 \). Firstly, we derive the second variation formula of pseudoharmonic maps into any pseudo-Hermitian manifolds. Then we give a condition on Weingarten map which implies that there is no nonconstant horizontal pseudoharmonic map from a closed pseudo-Hermitian manifold into an isometric embedded CR manifold. Next we consider the identity map \( I : S^{2n+1} \to S^{2n+1} \). From the above result we know that it is unstable. Following the result of [7] we discuss the degree of the instability and derive that \( \text{index}(I) \geq 2n+2 \). In the end, we give a condition on CR Weingarten map which also implies there is no nonconstant pseudoharmonic map from a closed pseudo-Hermitian manifold into a pseudo-Hermitian immersed submanifold of Heisenberg group.

2 Basic Notions

We will follow mainly the notations and terminologies in [2], but a somewhat different wedge multiplication for forms.

Let \( M \) be a real \((2m+1)\)-dimensional \( C^\infty \) differentiable manifold. Let \( TM \otimes \mathbb{C} \) be the complexified tangent bundle over \( M \). Let \( T_{1,0}M \subseteq TM \otimes \mathbb{C} \) be a complex subbundle of complex rank \( m \). \( T_{1,0}M \) is called an CR structure on \( M \), if \( T_{1,0}M \cap T_{0,1}M = 0 \) and \([T_{1,0}M, T_{1,0}M] \subseteq (T_{1,0}M)\). Here \( T_{0,1}M = \overline{T_{1,0}M} \).

A pair \((M, T_{1,0}M)\) consisting of a \( C^\infty \) manifold and a CR structure is a CR manifold. The integer \( m \) is the CR dimension.

Its Levi distribution is the real rank \( 2m \) subbundle \( H(M) \subseteq TM \) given by \( H(M) = \text{Re}\{T_{1,0}M \oplus T_{0,1}M\} \). It carries the complex structure \( J_b : H(M) \to H(M) \) given by \( J_b(V + \overline{V}) = \sqrt{-1}(V - \overline{V}) \) for any \( V \in \Gamma(T_{1,0}M) \).

**Definition 2.1.** Let \((M, T_{1,0}M)\) and \((N, T_{1,0}N)\) be two CR manifolds. A \( C^\infty \) map \( f : M \to N \) is a CR map if

\[ (dxf)(T_{1,0}M)_{x} \subseteq (T_{1,0}N)_{f(x)}, \]

for any \( x \in M \), where \( dxf \) or \((f_*)_x\) is the differential of \( f \) at \( x \).

There exists a global nonvanishing 1-form \( \theta \) such that \( \ker(\theta_x) \supseteq H(M)_x \) for any \( x \in M \). Any such section \( \theta \) is referred to as a pseudo-Hermitian structure on \( M \). Given a pseudo-Hermitian structure \( \theta \) on \( M \), the Levi-form \( L_\theta \) is defined by \( L_\theta(Z, W) = -\frac{1}{2\sqrt{-1}}(d\theta)(Z, \overline{W}) \) for any \( Z, W \in \Gamma(T_{1,0}M) \).

We can define the bilinear form \( G_\theta \) by setting \( G_\theta(X, Y) = \frac{1}{2}(d\theta)(X, J_bY) \) for any \( X, Y \in \Gamma(H(M)) \). Since \( L_\theta \) and (the \( \mathbb{C} \) linear extension to \( H(M) \otimes \mathbb{C} \)) of \( G_\theta \) coincide on \( H(M) \otimes H(M) \), then \( G_\theta(J_bX, J_bY) = G_\theta(X, Y) \) for any \( X, Y \in \Gamma(H(M)) \). In particular, \( G_\theta \) is symmetric.
Definition 2.2. Let \((M, T_{1,0}M)\) be an orientable CR manifold and \(\theta\) a fixed pseudo-Hermitian structure on \(M\). We call that \((M, T_{1,0}M)\) is a strictly pseudoconvex CR manifold if its Levi form \(L_0\) is positive definite.

Remark 2.3. In this paper, we only consider the strictly pseudoconvex CR manifolds. \((M, H(M), J_b, \theta)\) is called a pseudo-Hermitian manifold, if \((M, T_{1,0}M)\) is a strictly pseudoconvex CR manifold and \(\theta\) is the pseudo-Hermitian structure such that \(L_0\) is positive definite.

If \((M, H(M), J_b, \theta)\) is a pseudo-Hermitian manifold, there exists a unique globally defined nowhere zero tangent vector field \(T\) on \(M\), which transverse to \(H(M)\), satisfying

\[
\theta(T) = 1, \quad T \cdot d\theta = 0. \tag{2.1}
\]

The vector \(T\) is referred to as the characteristic direction. Then we have the decomposition: \(TM = H(M) \oplus R_T\). Using the decomposition we may extend \(G_\theta\) to a Riemannian metric \(g_\theta\) on \(M\). Let \(g_\theta\) be the Riemannian metric given by \(g_\theta(X,Y) = G_\theta(\pi_H X, \pi_H Y) + \theta(X)\theta(Y)\) for any \(X, Y \in \Gamma(H(M))\), where \(\pi_H : TM \to H(M)\) is the projection associated with the direct sum decomposition. \(g_\theta\) is called the Webster metric. In this paper we always write \(g_\theta\) or \(G_\theta\) as \(\langle \cdot, \cdot \rangle\) for simplicity.

If \(\nabla\) is a linear connection on \(M\), we use \(T_\nabla\) to denote its torsion field. A vector-valued 1-form \(\tau : TM \to TM\) is defined by \(\tau X = T_\nabla(T, X)\) for any \(X \in \Gamma(TM)\).

Let us extend \(J_b\) to a \((1,1)\) tensor field \(J\) on \(M\) by requiring that \(JT = 0\). Then \((M, H(M), J, \theta)\) becomes a pseudo-Hermitian manifold.

On a pseudo-Hermitian manifold, there is a canonical linear connection preserving the complex structure and the webster metric. Precisely, we have the following theorem:

Theorem 2.4. ([2]) Let \((M, H(M), J, \theta)\) be a pseudo-Hermitian manifold. Let \(T\) be the characteristic direction and \(J\) the complex structure in \(H(M)\). Let \(g_\theta\) be the webster metric. There is a unique linear connection on \(M\) (called the Tanaka-Webster connection) such that

(i) The Levi distribution \(H(M)\) is parallel with respect to \(\nabla\).

(ii) \(\nabla J = 0, \nabla g_\theta = 0\).

(iii) The torsion \(T_\nabla\) of \(\nabla\) satisfies the following conditions:

\[
T_\nabla(Z,W) = 0, \tag{2.2}
\]

\[
T_\nabla(Z,W) = 2\sqrt{-1}L_\theta(Z,\overline{W})T, \tag{2.3}
\]

\[
\tau J + J\tau = 0 \tag{2.4}
\]

for any \(Z, W \in \Gamma(T_{1,0}M)\).
By the Theorem 2.4 we may conclude that $\nabla T = 0$ and $\nabla \theta = 0$. The vector-valued 1-form $\tau$ on $M$ is called the pseudo-Hermitian torsion of $\nabla$.

**Remark 2.5.** The pseudo-Hermitian torsion $\tau$ is $H(M)$-valued. It is self-adjoint with respect to $g_\theta$ and trace $\tau = 0$ (cf. [2], page 37).

Let us set $A(X, Y) = g_\theta(\tau X, Y)$. Then $A(X, Y) = A(Y, X)$. In particular, $(M, H(M), J, \theta)$ is called a Sasakian manifold, if the pseudo-Hermitian torsion $\tau$ is zero.

Since $g_\theta$ is a Riemannian metric on $M$, then there exists the Levi-Civita connection of $(M, g_\theta)$ denoted by $\nabla_\theta$. We have the following lemma:

**Lemma 2.6.** ([2]) Let $(M, H(M), J, \theta)$ be a pseudo-Hermitian manifold. Let $\nabla$ be the Tanaka-Webster connection. Then the torsion tensor field $T_\nabla$ of $\nabla$ is given by

$$T_\nabla = \theta \wedge \tau + d\theta \otimes T. \quad (2.5)$$

Moreover, the Levi-Civita connection $\nabla^0$ of $(M, g_\theta)$ is related $\nabla$ by

$$\nabla^0 = \nabla - \left( \frac{1}{2} d\theta + A \right) \otimes \tau + \theta \otimes \tau + 2 \theta \otimes J. \quad (2.6)$$

Here $\otimes$ denotes the symmetric tensor product. For instance, $2(\theta \otimes J)(X, Y) = \theta(X)JY + \theta(Y)JX$ for $\forall X, Y \in \Gamma(TM)$.

**Example 2.7.** (Heisenberg group). The Heisenberg group $H_n$ is obtained by $\mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, t) \cdot (w, s) = (z + w, t + s + 2 \text{Im}(z, w)).$$

Here $(z, t) = (z^1 = x^1 + y^1, \cdots, z^n = x^n + y^n, t)$ is the natural coordinates.

Let us consider the complex vector fields on $H_n$,

$$T_\alpha = \frac{\partial}{\partial z_\alpha} + \sqrt{-1} \frac{\partial}{\partial t}, \quad \alpha = 1, \cdots, n.$$  

Here $\frac{\partial}{\partial z_\alpha} = \frac{1}{2}(\frac{\partial}{\partial x_\alpha} - \sqrt{-1} \frac{\partial}{\partial y_\alpha})$ and $z_\alpha = x_\alpha + y_\alpha$. The CR structure $T_{1,\theta}H_n$ is spanned by $\{T_1, \cdots, T_n\}$. There is a pseudo-Hermitian structure $\theta$ on $H_n$ defined by

$$\theta = dt + 2 \sum_{i=1}^{n} (x_i dy_i - y_i dx_i)$$

such that $(H_n, H(H), J, \theta)$ becomes a pseudo-Hermitian manifold. Moreover, it is a Sasakian manifold. See [2] for details.
Now let us discuss the divergence of a vector field on a pseudo-Hermitian manifold.

For a vector field $X \in \Gamma(TM)$, the divergence $\text{div}(X)$ can be locally computed as:

$$\text{div}(X) = \text{trace}_{g_{\theta}}(\nabla^\theta X) = \sum_{\lambda=1}^{2m} \langle \nabla_{e^\lambda} X, e^\lambda \rangle + \langle \nabla^\theta_T X, T \rangle,$$

(2.7)

where $\{e^\lambda\}_{\lambda=1}^{2m}$ is a local orthonormal frame of $H(M)$.

Using (2.6) and $\nabla^\theta g_{\theta} = 0$, we have

$$\text{div}X = \sum_{\lambda=1}^{2m} e^\lambda \langle X, e^\lambda \rangle - \sum_{\lambda=1}^{2m} \langle X, \nabla_{e^\lambda} e^\lambda \rangle + \frac{1}{2} \langle d\theta + A \rangle (e^\lambda, e^\lambda) \theta(X) + T(\theta(X)).$$

(2.8)

Since $\text{trace} \tau = 0$, then

$$\text{div}X = \sum_{\lambda=1}^{2m} \langle \nabla_{e^\lambda} X, e^\lambda \rangle + T(\theta(X)).$$

In particular, for $X \in \Gamma(H(M))$ the identity (2.8) becomes

$$\text{div}X = \sum_{\lambda=1}^{2m} \langle \nabla_{e^\lambda} X, e^\lambda \rangle$$

(2.9)

### 3 Pseudoharmonic map

Assume that $(M, H(M), J, \theta)$ and $(N, \tilde{H}(N), \tilde{J}, \tilde{\theta})$ are two pseudo-Hermitian manifolds and $M$ is closed, $\dim_H M = 2m + 1$ and $\dim_H N = 2n + 1$.

For any smooth map $f : M \to N$ Petit([6]) introduced the following horizontal energy $E_{H, \tilde{H}}(f) = \frac{1}{2} \int_M |df|_{H, \tilde{H}}^2 dv_{g_{\theta}}$. Here $df_{H, \tilde{H}} = \pi_{\tilde{H}} \circ df \circ i_H$, $\pi_{\tilde{H}} : TN \to \tilde{H}(N)$ is the natural projection and $i_H : H(M) \to TM$ is the natural inclusion.

Let $\nabla$ and $\nabla_{\tilde{\theta}}$ be the Tanaka-Webster connections on $M$ and $N$ respectively. According to [6], we may define the second fundamental form with respect to the data $(\nabla, \nabla_{\tilde{\theta}})$

$$\beta(X, Y) = \langle \nabla_{\tilde{X}} df(Y), \nabla_X df(Y) - df(\nabla_X Y) \rangle.$$

Here we also use $\nabla_{\tilde{\theta}}$ to denote the pull-back connection in $f^{-1}TN$.

In [6], Petit derived the first variation formula of the energy $E_{H, \tilde{H}}$ and call a critical point of $E_{H, \tilde{H}}$ a pseudoharmonic map. Note that there is an extra condition on the pull-back of the pseudo-Hermitian torsion. Dong([1]) modified the variational problem slightly and considered the restricted variational problem by requiring the variational vector fields to be horizontal. Then we have
Proposition 3.1. ([1], cf also [6]) For any horizontal vector field $\nu \in \Gamma(f^{-1}\tilde{H}(N))$, let $\{f_t\}_{|t| < \varepsilon}$ with $f_0 = f$ and $\nu = \frac{\partial f}{\partial t}|_{t=0}$ be a one parameter variation. Then we have

$$\frac{dE_{H,\tilde{H}}(f_t)}{dt}|_{t=0} = -\int_M \langle \nu, \tau_{H,\tilde{H}}(f) \rangle d\nu,$$

where

$$\tau_{H,\tilde{H}}(f) = \text{tr}_{G_{\theta}}(\beta_{H,\tilde{H}} + (f^*\tilde{\theta} \otimes f^*\overline{\tau})_{H,\tilde{H}});$$

$$\beta_{H,\tilde{H}} = \pi_{\tilde{H}} \circ \beta_{|H(M) \times H(M)};$$

$$(f^*\tilde{\theta} \otimes f^*\overline{\tau})_{H,\tilde{H}} = \pi_{\tilde{H}}[f^*\tilde{\theta} \otimes f^*\overline{\tau}]_{H(M) \times H(M)}.$$

We call $\tau_{H,\tilde{H}}$ the pseudo-tension field of $f$.

Definition 3.2. ([1]) A $C^\infty$ map $f : (M, H(M), J, \theta) \rightarrow (N, \tilde{H}(N), \tilde{J}, \tilde{\theta})$ is called a pseudoharmonic map if it is a critical point of $E_{H,\tilde{H}}$ for any horizontal vector field $\nu \in \Gamma(f^{-1}\tilde{H}(N))$.

Corollary 3.3. ([1]) Let $f : (M, H(M), J, \theta) \rightarrow (N, \tilde{H}(N), \tilde{J}, \tilde{\theta})$ be a $C^\infty$ map. Then $f$ is pseudoharmonic if and only if $\tau_{H,\tilde{H}} = 0$, that is,

$$\text{tr}_{G_{\theta}}(\beta_{H,\tilde{H}}(f) + [(f^*\tilde{\theta} \otimes (f^*\overline{\tau})]_{H,\tilde{H}} = 0.$$

Remark 3.4. As we have mentioned, the notion of pseudoharmonic maps is slightly different from that in [6]. In [1], the author has shown that Petit’s pseudoharmonic maps coincide with Dong’s if the target manifold is Sasakian. Henceforth we will investigate pseudoharmonic maps in the sense of Definition 3.2.

We end this section by proving a lemma which will be used later.

Lemma 3.5. Let $f : (M, H(M), J, \theta) \rightarrow (N, \tilde{H}(N), \tilde{J}, \tilde{\theta})$ be a $C^\infty$ map between two pseudo-Hermitian manifolds. Then

$$\nabla_X df(Y) - \nabla_Y df(X) = df([X, Y]) + \theta(df(X))\overline{\tau}(df(Y)) - \overline{\theta}(df(Y))\overline{\tau}(df(X)) + d\tilde{\theta}(df(X), df(Y))\tilde{T} \quad (3.1)$$

for any $X, Y \in \Gamma(TM)$.

Proof. Let $(U, x^1, \cdots, x^{2m+1})$ and $(V, y^1, \cdots, y^{2n+1})$ be two local coordinate systems on $M$ and $N$ respectively $(f(U) \subseteq V)$. It is clear that both sides are $C^\infty$ linear. Since $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$, by (2.3) we only need to prove that

$$\nabla_{\frac{\partial}{\partial x^i}} df(\frac{\partial}{\partial x^j}) - \nabla_{\frac{\partial}{\partial x^j}} df(\frac{\partial}{\partial x^i}) = T_{\nabla}(df(\frac{\partial}{\partial x^i}), df(\frac{\partial}{\partial x^j})). \quad (3.2)$$
Then we get

\[ \nabla_{\partial_t} df(\frac{\partial}{\partial x^j}) - \nabla_{\partial x^j} df(\frac{\partial}{\partial t}) \]

\[ = \sum_{a=1}^{2n+1} \left[ \nabla_{\partial x^j} \partial f^a_{\partial x^j} \cdot \left( \frac{\partial}{\partial y^a} \right) f - \nabla_{\partial y^a} \partial f^a_{\partial x^j} \cdot \left( \frac{\partial}{\partial y^a} \right) \right] \]

\[ = \sum_{a,b=1}^{2n+1} \left[ \partial f^a_{\partial x^j} \partial f^b_{\partial x^j} \nabla_{\partial x^j} \partial f^a_{\partial y^a} - \partial f^a_{\partial x^j} \partial f^b_{\partial y^a} \right] \]

\[ = \sum_{a,b=1}^{2n+1} \partial f^a_{\partial x^j} \partial f^b_{\partial x^j} T_{\tilde{\psi}} \left( \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b} \right) \]

\[ = T_{\tilde{\psi}}(df(\frac{\partial}{\partial x^j}), df(\frac{\partial}{\partial x^j})). \quad (3.3) \]

\[ \square \]

4 The second variation formula

In [1], the author derived the second variation formula for pseudoharmonic maps into Sasakian manifolds. In following, we will derive the second variation formula for pseudoharmonic maps into any pseudo-Hermitian manifolds.

Firstly, let \( f : (M, H(M), J, \theta) \to (N, H(N), J, \theta) \) be a pseudoharmonic map between two pseudo-Hermitian manifolds. Assume that \( M \) is closed.

Given any one-parameter variation \( \{ f_t \} \) \( |t| < \varepsilon \) with \( f_0 = f, V = \frac{\partial f_t}{\partial t} |_{t=0} \), we set \( \Phi(\cdot, t) = f_t \) and \( V_t = \frac{\partial f_t}{\partial t} \). Moreover, we require \( V_t = \frac{\partial f_t}{\partial t} \) to be horizontal for all \( t \), i.e. \( \frac{\partial f_t}{\partial t} \in \Gamma(J^{-1}(\tilde{H}(N))). \)

From the first variation formula in [1], we have

\[ \frac{dE_{H, \tilde{H}}(f_t)}{dt} = \int_M \nabla_{\frac{\partial}{\partial t}} d\Phi_{H, \tilde{H}}(e_\lambda), d\Phi_{H, \tilde{H}}(e_\lambda)) dV_\theta \]

\[ = \int_M \left[ \left\langle \nabla_{\frac{\partial}{\partial t}} d\Phi(e_\lambda), d\Phi_{H, \tilde{H}}(e_\lambda) \right\rangle + \bar{\theta}(d\Phi(e_\lambda)) \langle \bar{\tau}(d\Phi(e_\lambda)), d\Phi_{H, \tilde{H}}(e_\lambda) \rangle \right. \]

\[ - \bar{\theta}(d\Phi(e_\lambda)) \langle \bar{\tau}(d\Phi(e_\lambda)), d\Phi_{H, \tilde{H}}(e_\lambda) \rangle dV_\theta \]

\[ = \int_M \left[ -\langle d\Phi(e_\lambda), (\nabla_{\frac{\partial}{\partial t}} d\Phi_{H, \tilde{H}})(e_\lambda) \rangle + \bar{\theta}(d\Phi(e_\lambda)) \langle \bar{\tau}(d\Phi(e_\lambda)), d\Phi_{H, \tilde{H}}(e_\lambda) \rangle \right. \]

\[ - \bar{\theta}(d\Phi(e_\lambda)) \langle \bar{\tau}(d\Phi(e_\lambda)), d\Phi_{H, \tilde{H}}(e_\lambda) \rangle \right] dV_\theta. \]

Then we get

\[ \frac{dE_{H, \tilde{H}}(f_t)}{dt} = \int_M \left[ -\langle V_t, \tau_{H, \tilde{H}}(f_t) \rangle + \bar{\theta}(d\Phi(e_\lambda)) \langle \bar{\tau}(d\Phi(e_\lambda)), d\Phi_{H, \tilde{H}}(e_\lambda) \rangle \right] dV_\theta. \]
Note that $f$ is pseudoharmonic and $\frac{\partial f}{\partial t} \in \Gamma(f_{t}^{-1}(\tilde{H}(N)))$ for all $t$, one gets

$$\frac{d^2 E_{H,\tilde{H}}(f_t)}{dt^2}|_{t=0} = - \frac{d}{dt}\{ \int_{M} \langle V, \tau_{H,\tilde{H}}(f_t) \rangle dV_\theta \}|_{t=0}$$

$$= - \int_{M} \langle V, \tilde{\nabla}_{\partial f} \tau_{H,\tilde{H}}(f_t) \rangle |_{t=0} dV_\theta .$$

Let $\tilde{R}$ be the curvature tensor fields of $\tilde{\nabla}$. Using (3.11) and $\tilde{\nabla} \theta = 0$ we derive

$$\tilde{\nabla}_{\partial f} \tau_{H,\tilde{H}}(f_t)|_{t=0}$$

$$= \sum_{\lambda=1}^{2m} \tilde{\nabla}_{\partial f} [\langle \tilde{\nabla}_{e_\lambda} d f_{H,\tilde{H}}(e_\lambda) + (f_0^t \tilde{\theta})(e_\lambda) \tilde{\tau}(d f_{H,\tilde{H}}(e_\lambda)) \rangle |_{t=0}$$

$$= \sum_{\lambda=1}^{2m} \langle \tilde{\nabla}_{\partial f} \tilde{\nabla}_{e_\lambda} d \Phi_{H,\tilde{H}}(e_\lambda) |_{t=0} + \sum_{\lambda=1}^{2m} \tilde{\nabla}_{\partial f} \tilde{\theta}(d \Phi(e_\lambda)) \tilde{\tau}(d \Phi_{H,\tilde{H}}(e_\lambda)) |_{t=0}$$

$$= \sum_{\lambda=1}^{2m} [\tilde{R}(\frac{\partial}{\partial t}, e_\lambda) \tilde{\Phi}_{H,\tilde{H}}](e_\lambda) |_{t=0} + \sum_{\lambda=1}^{2m} \tilde{\nabla}_{e_\lambda} \tilde{\nabla}_{\partial f} \tilde{\Phi}_{H,\tilde{H}}(e_\lambda) |_{t=0}$$

$$+ \sum_{\lambda=1}^{2m} \tilde{\theta}(d f(e_\lambda)) \tilde{\tau}(\tilde{\nabla}_{e_\lambda} V - \tilde{\theta}(d f(e_\lambda)) \tilde{\tau}(V))$$

Note that $G_{\tilde{\theta}}(\cdot, \cdot) = \frac{1}{2} \tilde{d} \tilde{\theta}(\cdot, \cdot)$. Then

$$\tilde{\nabla}_{\partial f} \tau_{H,\tilde{H}}(f_t)|_{t=0}$$

$$= \sum_{\lambda=1}^{2m} [\tilde{R}(\frac{\partial}{\partial t}, e_\lambda) \tilde{\Phi}_{H,\tilde{H}}](e_\lambda) |_{t=0} + \sum_{\lambda=1}^{2m} \tilde{\nabla}_{e_\lambda} \tilde{\nabla}_{\partial f} \tilde{\Phi}_{H,\tilde{H}}(e_\lambda) |_{t=0}$$

$$+ \sum_{\lambda=1}^{2m} [2\langle \tilde{J} V, d \tilde{f}_{H,\tilde{H}}(e_\lambda) \rangle \tilde{\tau}(d \tilde{f}_{H,\tilde{H}}(e_\lambda)) + \tilde{\theta}(d f(e_\lambda)) \tilde{\tau}(V) \tilde{\tau}(d \tilde{f}_{H,\tilde{H}}(e_\lambda))]$$

$$+ \sum_{\lambda=1}^{2m} \tilde{\theta}(d f(e_\lambda)) \tilde{\tau}(\tilde{\nabla}_{e_\lambda} V - \frac{\sum_{\lambda=1}^{2m} \tilde{\theta}(d f(e_\lambda))}{\sum_{\lambda=1}^{2m} \tilde{\theta}(d f(e_\lambda))} \tilde{\tau}^2(V))$$

Taking into account the identity

$$d f(e_\lambda) = \tilde{\theta}(d f(e_\lambda)) \tilde{T} + d \tilde{f}_{H,\tilde{H}}(e_\lambda),$$

we have

$$\langle \tilde{R}(V, d f(e_\lambda)) d \tilde{f}_{H,\tilde{H}}(e_\lambda), V \rangle = \tilde{\theta}(d f(e_\lambda)) \langle \tilde{R}(V, \tilde{T}) d \tilde{f}_{H,\tilde{H}}(e_\lambda), V \rangle + \langle \tilde{R}(V, d \tilde{f}_{H,\tilde{H}}(e_\lambda)) d \tilde{f}_{H,\tilde{H}}(e_\lambda), V \rangle.$$
By (1.77) in [2], we deduce that
\[
\langle \tilde{R}(V, df(e_\lambda))df_{H, \tilde{H}}(e_\lambda), V \rangle = -\tilde{\theta}(df(e_\lambda))\langle \widetilde{\mathcal{S}}(df_{H, \tilde{H}}(e_\lambda), V), V \rangle + \langle \tilde{R}(V, df_{H, \tilde{H}}(e_\lambda))df_{H, \tilde{H}}(e_\lambda), V \rangle
\]
\[
= -\tilde{\theta}(df(e_\lambda))\langle \nabla_{df_{H, \tilde{H}}(e_\lambda)}\tilde{\tau}V - \nabla_V\tilde{\tau}(df_{H, \tilde{H}}(e_\lambda)), V \rangle + \langle \tilde{R}(V, df_{H, \tilde{H}}(e_\lambda))df_{H, \tilde{H}}(e_\lambda), V \rangle.
\]
(4.2)

Here \( \widetilde{\mathcal{S}} \) is given by \( \widetilde{\mathcal{S}}(X, Y) = (\tilde{\nabla}_X\tilde{\tau})(Y) - (\tilde{\nabla}_Y\tilde{\tau})(X) \) for any \( X, Y \in \Gamma(TN) \).

Let \( X \in \Gamma(H(M)) \) be (locally) defined by
\[
X = \sum_{\lambda=1}^{2m} \langle \nabla_{\tilde{e}_\lambda} df_{H, \tilde{H}}(e_\lambda), V \rangle e_\lambda \big|_{t=0}.
\]

Then we compute the divergence of \( X \). By the divergence theorem, one deduces from (5.1) that
\[
\int_M \sum_{\lambda=1}^{2m} \langle \nabla_{\tilde{e}_\lambda} \nabla_{\tilde{e}_\lambda} df_{H, \tilde{H}}(e_\lambda) \rangle_{t=0} V d\theta
\]
\[
= -\int_M \sum_{\lambda=1}^{2m} \langle \nabla_{\tilde{e}_\lambda} df_{H, \tilde{H}}(e_\lambda), \nabla_{\tilde{e}_\lambda} V \rangle_{t=0} dV_\theta
\]
\[
= -\int_M \sum_{\lambda=1}^{2m} \langle \nabla_{\tilde{e}_\lambda} V, \nabla_{\tilde{e}_\lambda} V \rangle - \tilde{\theta}(df(e_\lambda))\langle \tilde{\tau}(V), \nabla_{\tilde{e}_\lambda} V \rangle dV_\theta
\]
(4.3)

It follows from (4.1), (4.2) and (4.3) that

**Theorem 4.1.** Let \( f : (M, H(M), J, \theta) \to (N, \tilde{H}(N), \tilde{J}, \tilde{\theta}) \) be a pseudoharmonic map between two pseudo-Hermitian manifolds. Assume that \( M \) is closed. Let \( \{f_t\}(|t| < \varepsilon) \) be a family of maps with \( f_0 = f, \frac{\partial f_t}{\partial t} \in \Gamma(f_t^{-1}(\tilde{H}(N))) \) for all \( t \).

Set \( V = \frac{\partial f_t}{\partial t} \big|_{t=0} \). Then the second variation formula of the energy functional \( E_{H, \tilde{H}} \) is given by
\[
\frac{d^2 E_{H, \tilde{H}}(f_t)}{dt^2} \big|_{t=0} = \int_M \sum_{\lambda=1}^{2m} \langle \nabla_{\tilde{e}_\lambda} V, \nabla_{\tilde{e}_\lambda} V \rangle - \tilde{R}(V, df_{H, \tilde{H}}(e_\lambda), V, df_{H, \tilde{H}}(e_\lambda))
\]
\[
+ \tilde{\theta}(df(e_\lambda))\langle \nabla_{df_{H, \tilde{H}}(e_\lambda)}\tilde{\tau}V, V \rangle + [\tilde{\theta}(df(e_\lambda))]^2 \langle \tilde{\tau}(V), \tilde{\tau}(V) \rangle
\]
\[
- 2\tilde{\theta}(df(e_\lambda))\langle \nabla_{df_{H, \tilde{H}}(e_\lambda)}\tilde{\tau}(V), df_{H, \tilde{H}}(e_\lambda) \rangle + \langle \tilde{\tau}(V), \nabla_{\tilde{e}_\lambda} V \rangle
\]
\[
- 2\langle df_{H, \tilde{H}}(e_\lambda), \tilde{\tau}V \rangle \langle df_{H, \tilde{H}}(e_\lambda), \tilde{J}V \rangle \rangle dV_\theta,
\]
(4.4)

where \( \{e_\lambda\}_{\lambda=1}^{2m} \) is a local orthonormal frame of \( H(M) \).

**Definition 4.2.** ([6]) Let \( f : (M, H(M), J, \theta) \to (N, \tilde{H}(N), \tilde{J}, \tilde{\theta}) \) be a \( C^\infty \) map between two pseudo-Hermitian manifolds. We say \( f \) is horizontal if
\[
\langle d_x f \rangle(H_x M) \subseteq \tilde{H}_{f(x)}(N),
\]
(4.5)

for any \( x \in M \).
Corollary 4.3. Let \( f : (M, H(M), J, \theta) \to (N, \tilde{H}(N), \tilde{J}, \tilde{\theta}) \) be a horizontal pseudoharmonic map (i.e. \( f \) is horizontal and pseudoharmonic) between two pseudo-Hermitian manifolds. Assume that \( M \) is closed. Let \( \{f_t\} \) be a family of maps with \( f_0 = f, \nu = \frac{\partial f}{\partial t}|_{t=0} \). Moreover, we require \( \frac{\partial f}{\partial t} \) to be horizontal for all \( t \), i.e. \( \frac{\partial f}{\partial t} \in \Gamma(f_t^{-1}(\tilde{H}(N))) \). Then the second variation formula of the energy functional \( E_{H, \tilde{H}} \) is given by

\[
\frac{d^2 E_{H, \tilde{H}}(f_t)}{dt^2}|_{t=0} = \int_M \sum_{\lambda=1}^{2m} \left( \langle \tilde{\nabla}_{e_\lambda} V, \tilde{\nabla}_{e_\lambda} V \rangle - \tilde{R}(V, df(e_\lambda), V, df(e_\lambda)) - 2\langle df(e_\lambda), \tilde{V} \rangle \langle df(e_\lambda), \tilde{J}V \rangle \right) dV_{\theta}.
\]

(4.6)

Similar to the case of harmonic maps between Riemannian manifolds, we may introduce the notion of stability for pseudo-Hermitian harmonic maps as follows.

Definition 4.4. Let \( f : (M, H(M), J, \theta) \to (N, \tilde{H}(N), \tilde{J}, \tilde{\theta}) \) be a pseudo-Harmonic map. We say \( f \) is stable, if \( \frac{d^2 E_{H, \tilde{H}}(f_t)}{dt^2}|_{t=0} \geq 0 \) for any variation \( \{f_t\} \) with \( f_0 = f, \nu = \frac{\partial f}{\partial t}|_{t=0} \in \Gamma(f_t^{-1}(\tilde{H}(N))) \). If \( \frac{d^2 E_{H, \tilde{H}}(f_t)}{dt^2}|_{t=0} < 0 \) for some variation \( f_t \) with \( f_0 = f, \nu = \frac{\partial f}{\partial t}|_{t=0} \in \Gamma(f_t^{-1}(\tilde{H}(N))) \), \( f \) is called unstable.

Note that if we require \( \frac{\partial f}{\partial t} \in \Gamma(f_t^{-1}(\tilde{H}(N))) \) to be horizontal for all \( t \), by Theorem 4.1 we have

\[
\frac{d^2 E_{H, \tilde{H}}(f_t)}{dt^2}|_{t=0} = \int_M \sum_{\lambda=1}^{2m} \left( \langle \tilde{\nabla}_{e_\lambda} V, \tilde{\nabla}_{e_\lambda} V \rangle - \tilde{R}(V, df(H, \tilde{H})(e_\lambda), V, df(H, \tilde{H})(e_\lambda)) + \tilde{\theta}(df(e_\lambda))(\langle \tilde{\nabla}_{df(H, \tilde{H})(e_\lambda)} \tilde{\nabla}%_{e_\lambda} V, V \rangle + \langle \tilde{\theta}(df(e_\lambda)), \tilde{\nabla}_{e_\lambda} V \rangle) - 2\tilde{\theta}(df(e_\lambda))\langle \langle \tilde{\nabla}_{\tilde{V}} \tilde{\nabla}_{e_\lambda} \tilde{V}, \tilde{\nabla}_{e_\lambda} V \rangle \rangle \right) dV_{\theta}.
\]

(4.7)

In following, we will mainly use (4.7) to verify the instability of a pseudo-Harmonic map. For convenience, we denote the right hand of (4.7) by \( H(f, V, V) \).

Remark 4.5. When the target manifold \( N \) is Sasakian, the second author derived the second variation formula of \( E_{H, \tilde{H}} \) which is given by (cf. [1])

\[
\frac{d^2 E_{H, \tilde{H}}(f_t)}{dt^2}|_{t=0} = \int_M \sum_{\lambda=1}^{2m} \left( \langle \tilde{\nabla}_{e_\lambda} V, \tilde{\nabla}_{e_\lambda} V \rangle - \tilde{R}(V, df(H, \tilde{H})(e_\lambda), V, df(H, \tilde{H})(e_\lambda)) \right) dV_{\theta}.
\]

(4.8)

where \( \{f_t\} \) is the variation of \( f \) corresponding to \( V \) and \( V_{\tilde{H}} \) denotes the horizontal part of \( V \). He used this formula to establish some stability and unstability results for pseudoharmonic maps.
5 Pseudoharmonic maps into isometric embedded CR manifolds

In [1], the author has shown that any nonconstant horizontal pseudoharmonic map from a closed pseudo-Hermitian manifold to the odd dimensional sphere is unstable. In this section we generalize his result to the case that the target manifold is an isometric embedded CR manifold.

Let $\tilde{i} : N \hookrightarrow \mathbb{C}^{n+k}(\cong \mathbb{R}^{2n+2k})$ be a real $(2n+1)$-dimensional submanifold($k \geq 1$). If $N$ is a CR manifold whose CR structure is induced from $\mathbb{C}^{n+k}$, i.e.

$$T_{1,0}(N) = T^{1,0}(\mathbb{C}^{n+k}) \cap (TN \otimes \mathbb{C}),$$

then $N$ is referred to as an embedded CR manifold.

For some pseudo-Hermitian structure $\tilde{\theta}$ on $N$, let $g_{\tilde{\theta}}$ be the Webster metric of $(N, \tilde{\theta})$ and $g_{\text{can}}$ be the canonical metric on $\mathbb{C}^{n+k}(\cong \mathbb{R}^{2n+2k})$. We call $(N, g_{\tilde{\theta}})$ an isometric embedded CR manifold if $N$ is an embedded CR manifold and $g_{\tilde{\theta}} = \tilde{i}^\ast g_{\text{can}}$. Then $(N, \tilde{H}(N), \tilde{J}, \tilde{\theta})$ is a pseudo-Hermitian manifold, where $\tilde{H} = \ker \tilde{\theta}$ and $\tilde{J}$ is induced from the standard complex structure $\hat{J}$ of $\mathbb{C}^{n+k}(\cong \mathbb{R}^{2n+2k})$. More precisely, if $X \in \Gamma(\tilde{H}(N))$, we have

$$\tilde{J}X = \hat{J}X.$$ (5.1)

Let $\tilde{T}$ be the characteristic direction of $(N, \tilde{\theta})$. If $T^\perp N$ is the normal bundle of $N$ in $\mathbb{C}^{n+k}(\cong \mathbb{R}^{2n+2k})$, there exists a vector field $\xi \in \Gamma(T^\perp N)$ such that

$$\tilde{T} = \hat{J}\xi|_N.$$ (5.2)

**Example 5.1.** ([2], cf also [1]) The standard odd-dimensional sphere $i : S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ is a isometric embedded CR manifold. Moreover, it is a Sasakian manifold.

In this section we always assume that $(N, \tilde{H}(N), \tilde{J}, \tilde{\theta})$ is an isometric embedded CR manifold. Let $\tilde{\nabla}$ be the standard flat connection on $\mathbb{C}^{n+k}(\cong \mathbb{R}^{2n+2k})$, $\tilde{\nabla}^\theta$ the Levi-Civita connection of $(N, g_\theta)$ and $h$ the second fundamental form of $N$ in $\mathbb{C}^{n+k}(\cong \mathbb{R}^{2n+2k})$. These are related by

$$\tilde{\nabla}_X Y = \tilde{\nabla}^\theta_X Y + h(X, Y),$$ (5.3)

where $X, Y \in \Gamma(TN)$.

For $\eta \in \Gamma(T^\perp N)$ and $X \in \Gamma(TN)$, we can define the Weingarten map $A_\eta X$ and the connection $\nabla^\perp_X \eta$ in the normal bundle by

$$\tilde{\nabla}_X \eta = -A_\eta X + \nabla^\perp_X \eta.$$ (5.4)

The tensors $h$ and $A$ are related by

$$\langle A_\eta X, Y \rangle = \langle h(X, Y), \eta \rangle,$$ (5.5)
where $X$ and $Y$ are tangent to $N$ and $\eta$ is normal to $N$. Obviously, $h(X,Y)$ is symmetric in $X$ and $Y$ and for each $\eta$ the linear map $A_\eta$ is self-adjoint.

Let $\{v_{2n+2}, \cdots, v_{2n+2k}\}$ be an orthonormal basis for the normal space $T^\perp_y N$ to $N$ at $y$. Define a linear map $Q^N_y: T_y N \to T_y N$ by

$$Q^N_y = \sum_{\alpha=2n+2}^{2n+2k} \left\{ 2(\pi_{\tilde{H}} A_{v_\alpha})^2 - \text{trace}_{\tilde{G}}(A_{v_\alpha}) \cdot A_{v_\alpha} + 2A_G^2 - 4Id \right\}, \quad (5.6)$$

where $\text{trace}_{\tilde{G}}(A_{v_\alpha}) = \sum_{j=1}^{2n} \langle A_{v_\alpha}(X_j), X_j \rangle$ for some (local)orthonormal frame $\{X_j : 1 \leq j \leq 2n\}$ of $\tilde{H}(N)$. The definition of $Q^N_y$ does not depend on the choice of orthonormal basis at $y$. Note that for any $X, Y \in \tilde{H}_y(N)$ we have

$$\langle Q^N_y X, Y \rangle = \langle X, Q^N_y Y \rangle.$$

By the Gauss equation for submanifolds, the curvature tensor of $\tilde{\nabla}$ is given by

$$\tilde{R}(X, Y, X, Y) = \langle h(Y, W), h(X, Z) \rangle - \langle h(X, W), h(Y, Z) \rangle, \quad (5.7)$$

where $X, Y, Z, W \in \Gamma(TN)$. Let $\tilde{\nabla}$ be the Tanaka-Webster connection of $(N, \tilde{H}(N), \tilde{J}, \tilde{\theta})$ and $\tilde{\tau}$ be pseudo-Hermitian torsion. The sectional curvature of $\tilde{\nabla}$ is given by (cf. [2], page 49)

$$\tilde{R}(X, Y, X, Y) = \langle h(X, X), h(Y, Y) \rangle - |h(X, Y)|^2 + 3\langle \tilde{J}X, Y \rangle^2 - \langle \tilde{\tau}X, Y \rangle^2 + \langle \tilde{\tau}X, X \rangle \langle \tilde{\tau}Y, Y \rangle, \quad (5.8)$$

where $X, Y \in \Gamma(\tilde{H}(N))$. By (5.7) we obtain

$$\tilde{R}(X, Y, X, Y) = \langle h(X, X), h(Y, Y) \rangle - |h(X, Y)|^2 + 3\langle \tilde{J}X, Y \rangle^2 - \langle \tilde{\tau}X, Y \rangle^2 + \langle \tilde{\tau}X, X \rangle \langle \tilde{\tau}Y, Y \rangle,$$

where $X, Y \in \Gamma(\tilde{H}(N))$.

Next we give the following lemma which will be useful to us later.

**Lemma 5.2.** Let $\xi \in \Gamma(T^\perp N)$ be the vector field in (5.2). For any $X \in \Gamma(\tilde{H}(N))$ we have

(i) $\langle A_\xi \tilde{T}, X \rangle = 0$

(ii) $A_\xi X = \tilde{\tau}X - X$.

**Proof.** Firstly, let us set $X, Y = \tilde{T}$ in (5.2) to obtain (since $\tilde{\nabla}^\theta_{\tilde{T}} \tilde{T} = 0$)

$$\tilde{\nabla}_{\tilde{T}} \tilde{T} = \tilde{\nabla}^\theta_{\tilde{T}} \tilde{T} + h(\tilde{T}, \tilde{T}) = h(\tilde{T}, \tilde{T}). \quad (5.9)$$

Using (5.2), for any $X \in \Gamma(\tilde{H}(N))$ we have

$$\langle A_\xi \tilde{T}, X \rangle = -\langle (\tilde{\nabla}_{\tilde{T}} \xi)^\top, X \rangle = \langle \tilde{\nabla}^\theta_{\tilde{T}} \tilde{T}, X \rangle. \quad (5.10)$$
Note that \( \hat{\nabla} \hat{J} = 0 \) and \( \hat{J}^2 = -1 \). By (5.1) and (5.9), (5.10) may be written as
\[
(A_\xi \hat{T}, X) = -\langle \hat{\nabla}_T \hat{J} X, \hat{J} X \rangle = -\langle h(\hat{T}, \hat{T}), \hat{J} X \rangle.
\] (5.11)

Since \( h(\hat{T}, \hat{T}) \in \Gamma(T^\perp N) \) and \( \hat{J} X \in \Gamma(TN) \), (i) is proved.

Next we may use (2.6) and \( \hat{\nabla}_T = 0 \) to perform the following calculations:
\[
\hat{\nabla}_T \hat{J} X = \hat{\nabla}_T \theta \hat{J} X + h(\hat{T}, X) = \hat{\nabla}_T \hat{J} X + \hat{J} X + h(\hat{T}, X).
\] (5.12)

Using the fact that \( \hat{\nabla} \) is torsion free, the above identity becomes
\[
A_\xi X \hat{T} = \hat{\nabla}_T \hat{J} X = \hat{J} \hat{T} X - X + \hat{J} h(\hat{T}, X);
\] hence
\[
A_\xi X = \hat{J} \hat{T} X - X + \hat{J} h(\hat{T}, X)^\top.
\] (5.14)

For any \( Y \in \Gamma(H(N)) \). Since \( h(\hat{T}, X) \in \Gamma(T^\perp N) \) and \( \hat{J}^2 = -1 \), by (5.1) we have
\[
\langle \hat{J} h(\hat{T}, X), Y \rangle = -\langle h(\hat{T}, X), \hat{J} Y \rangle = 0.
\] (5.15)

On the other hand, Using \( \hat{J}^2 = -1 \) again, by (5.1), (5.2) and (5.5) a computation shows that
\[
\langle \hat{J} h(\hat{T}, X), \hat{T} \rangle = (A_\xi \hat{T}, X).
\] (5.16)

We may conclude that
\[
\langle \hat{J} h(\hat{T}, X), \hat{T} \rangle = 0
\] (5.17)

due to (i). It follows from (5.15) and (5.17) that \( [\hat{J} h(\hat{T}, X)]^\top = 0 \). Substitution into (5.14) shows that \( A_\xi X = \hat{J} \hat{T} X - X \).

Let \( a \) be a vector field in \( \mathbb{C}^{n+k}(\cong \mathbb{R}^{2n+2k}) \). We define a vector field \( a^\top \) tangent to \( N \) and a vector field \( a^\perp \) normal to \( N \).

According to the decomposition \( TN = H(N) \oplus R \hat{T} \) we may write \( a^\top \) as
\[
a^\top = \pi_H a^\top + \langle a, \hat{T} \rangle \hat{T}.
\]
Let us set \( \tilde{a}_\tilde{H} = \pi_\tilde{H}a^\top \). Then we have

\[
a_{\tilde{H}} = a - a^\perp - \langle a, \tilde{T} \rangle \tilde{T},
\]

where \( a_{\tilde{H}} \in \Gamma(\tilde{H}(N)) \).

To prove the main result in this section we start with the following lemma.

**Lemma 5.3.** Assume that \( a \) is a constant vector field. For the vector field \( a_{\tilde{H}} \) defined above, we have

\[
\tilde{\nabla}_X a_{\tilde{H}} = A_{a^\perp} X - (A_{a^\perp} X, \tilde{T}) \tilde{T} - \langle a, \tilde{T} \rangle \tilde{T} X - \langle a, \tilde{T} \rangle JX
\]

for any \( X \in \Gamma(\tilde{H}(N)) \).

**Proof.** Note that \( G_{\tilde{\theta}} = \frac{1}{2} d\tilde{\theta}(\cdot, \tilde{\cdot}) \) and \( \tilde{\nabla} a = 0 \). Then using (2.6) and (5.13) we perform the following calculations:

\[
\tilde{\nabla}_X a_{\tilde{H}} = \tilde{\nabla}_X^h a_{\tilde{H}} + \frac{1}{2} d\tilde{\theta}(X, a_{\tilde{H}}) \tilde{T} + \tilde{A}(X, a_{\tilde{H}}) \tilde{T}
\]

\[
= \left[ \tilde{\nabla}_X (a - a^\perp - \langle a, \tilde{T} \rangle \tilde{T}) \right] + \langle \tilde{J}X, a \rangle \tilde{T} + \langle \tilde{T}X, a \rangle \tilde{T}
\]

\[
= \left( - \tilde{\nabla}_X a^\perp \right) - \langle a, \tilde{T} \rangle \tilde{T} - \langle a, \tilde{T} \rangle (\tilde{\nabla}_X \tilde{T})^\top + \langle \tilde{T}X, a \rangle \tilde{T} + \langle \tilde{T}X, a \rangle \tilde{T}
\]

\[
= A_{a^\perp} X - \langle a, \tilde{\nabla}_X \tilde{T} - \langle a, \tilde{T} \rangle (\tilde{\nabla}_X \tilde{T})^\top \rangle \tilde{T} + \langle \tilde{T}X, a \rangle \tilde{T} + \langle \tilde{T}X, a \rangle \tilde{T}
\]

\[
= A_{a^\perp} X - \langle a, \tilde{\nabla}_X \tilde{T} \rangle \tilde{T} - \langle a, \tilde{T} \rangle \tilde{T} X - \langle a, \tilde{T} \rangle JX.
\]

Now we investigate the unstability of horizontal pseudoharmonic maps from a closed pseudo-Hermitian manifold \((M^{2m+1}, H(M), J, \theta)\) into an isometric embedded CR manifold \((N^{2n+1}, \tilde{H}(N), \tilde{J}, \tilde{\theta})\).

Let \( f: (M, H(M), J, \theta) \rightarrow (N, \tilde{H}(N), \tilde{J}, \tilde{\theta}) \) be a horizontal pseudoharmonic map between \( M \) and \( N \). Let \( a \) be a constant vector field in \( \mathbb{C}^{n+k} (\cong \mathbb{R}^{2n+2k}) \). We use \( \varphi_t ([t] < \epsilon) \) to denote the flow or one-parameter group of diffeomorphisms generated by \( a_{\tilde{H}} \). For the horizontal variational vector field \( a_{\tilde{H}} \) along \( f \), the one-parameter variation is \( f_t = \varphi_t \circ f \) with \( f_0 = f \), \( \frac{\partial f_t}{\partial t}|_{t=0} = a_{\tilde{H}} \). Since \( a_{\tilde{H}} \) is a horizontal vector field, then \( \frac{\partial f_t}{\partial t} = \frac{\partial a_{\tilde{H}}}{\partial t} \circ f \) is horizontal for all \( t \).

By Corollary 4.3 the second variation formula can be written in the following form:

\[
\frac{d^2}{dt^2} \bigg|_{t=0} E_{H, \tilde{H}}(f_t) = \int_M \sum_{\lambda=1}^{2m} \left[ (\nabla_d f(e_\lambda)) a_{\tilde{H}} \right]^2 - \tilde{R}(a_{\tilde{H}}, df(e_\lambda), a_{\tilde{H}}, df(e_\lambda))
\]

\[
- 2\langle df(e_\lambda), \tilde{T}a_{\tilde{H}} \rangle \langle df(e_\lambda), \tilde{J}a_{\tilde{H}} \rangle dv_{\tilde{\theta}}
\]

(5.19)

Here \( \{e_\lambda\}_{\lambda=1}^{2m} \) is a local orthonormal frame of \( H(M) \).
Theorem 5.4. Let \( f : (M, H(M), J, \theta) \to (N, \bar{H}(N), \bar{J}, \bar{\theta}) \) be a nonconstant horizontal pseudoharmonic map from a closed pseudo-Hermitian manifold into an isometric embedded CR manifold. Assume that \( Q_N \) is negative definite on \( \bar{H}(N) \) at each point \( y \) of \( N \) (i.e. \( \langle Q_N X, X \rangle < 0 \) for all \( X \neq 0 \) and \( X \in \Gamma(\bar{H}(N)) \)). Then \( f \) is unstable.

Proof. Let \( f : M \to N \) be a nonconstant horizontal pseudoharmonic map. We consider the horizontal vector field \( a_{\bar{H}} \) on \( N \) as above. By Definition 4.4 and using (5.19), we have

\[
H_f(a_{\bar{H}}, a_{\bar{H}}) = \left. \frac{d^2 E_{H, \bar{H}}(f_t)}{dt^2} \right|_{t=0}
= \int_M \sum_{k=1}^{2m} (\nabla_{df(e_k)} a_{\bar{H}})^2 - \bar{R}(a_{\bar{H}}, df(e_k), a_{\bar{H}}, df(e_k))
- 2\langle df(e_k), \bar{\tau} a_{\bar{H}} \rangle \langle df(e_k), \bar{J} a_{\bar{H}} \rangle \rangle \ dv_0,
\]

where \( \{e_k\}_{k=1}^{2m} \) is a local orthonormal frame of \( H(M) \). Since \( f \) is horizontal and by Lemma 5.3 we get

\[
\nabla_{df(e_k)} a_{\bar{H}} = A_a df(e_k) - \langle A_a df(e_k), \bar{T} \rangle \bar{T} - \langle a, \bar{T} \rangle \bar{J} df(e_k) - \langle a, \bar{T} \rangle \bar{J} df(e_k)
\]

and thus

\[
(|\nabla_{df(e_k)} a_{\bar{H}}|^2 = \langle A_a df(e_k), A_a df(e_k) \rangle - \langle A_a df(e_k), \bar{T} \rangle^2
+ \langle a, \bar{T} \rangle^2 \langle \bar{T}, df(e_k) \rangle \rangle \langle df(e_k), df(e_k) \rangle
\]

\[
- 2\langle a, \bar{T} \rangle \langle A_a df(e_k), \bar{J} df(e_k) \rangle - 2\langle A_a df(e_k), \bar{J} df(e_k) \rangle \langle a, \bar{T} \rangle
+ 2\langle a, \bar{T} \rangle^2 \langle \bar{T}, \bar{J} df(e_k) \rangle \langle df(e_k), df(e_k) \rangle
\]

Next using (5.8), we derive that

\[
\bar{R}(a_{\bar{H}}, df(e_k), a_{\bar{H}}, df(e_k))
= \langle h(a_{\bar{H}}, a_{\bar{H}}), h(df(e_k), df(e_k)) \rangle - \langle h(a_{\bar{H}}, df(e_k)), df(e_k) \rangle^2
+ 3\langle \bar{J} a_{\bar{H}}, df(e_k) \rangle \langle \bar{J} df(e_k), df(e_k) \rangle
\]

\[
- \langle a_{\bar{H}}, \bar{\tau} df(e_k) \rangle^2 + \langle \bar{\tau} a_{\bar{H}}, a_{\bar{H}} \rangle \langle \bar{\tau} df(e_k), df(e_k) \rangle
= \langle A_h(a_{\bar{H}}, a_{\bar{H}}) df(e_k), df(e_k) \rangle - \langle h(a_{\bar{H}}, df(e_k)), df(e_k) \rangle^2
+ 3\langle a_{\bar{H}}, \bar{J} df(e_k) \rangle^2
\]

For any fixed point \( y \), we choose a real orthonormal basis \( \{a_1, \ldots, a_{2n+2k}\} \) of \( \mathbb{C}^{n+k} \cong \mathbb{R}^{2n+2k} \) such that \( \{a_1, \ldots, a_{2n}\} \rangle_y \) is a basis of \( \bar{H}_y(N), \ a_{2n+1} \rangle_y = \bar{T}_y \) and \( \{a_{2n+2}, \ldots, a_{2n+2k}\} \rangle_y \) is a basis of \( T_{\bar{y}} N \). Then we use \( a_i \ (i = 1, \ldots, 2n+2k) \) to construct the vector field \( (a_i)_{\bar{H}} \) as above. By a direct computation at \( y \) and
Using (5.21), (5.22) and \(\text{trace} \tilde{\tau} = 0\), we may get
\[
\sum_{i=1}^{2n+2k} \langle \nabla df(e_\lambda)(a_i) , \nabla df(e_\lambda)(a_i) \rangle = \sum_{i=2n+2}^{2n+2k} \langle [A_{a_i} df(e_\lambda), A_{a_i} df(e_\lambda)] - \langle A_{a_i} df(e_\lambda), \tilde{T} \rangle \rangle + \langle \tilde{T}^2 df(e_\lambda), df(e_\lambda) \rangle + |df(e_\lambda)|^2 + 2\langle \tilde{T} J df(e_\lambda), df(e_\lambda) \rangle
\]
and
\[
\sum_{i=1}^{2n+2k} \langle h(a_i) , df(e_\lambda) \rangle = \sum_{i=2n+2}^{2n+2k} \langle [\pi_{\tilde{H}} A_{a_i} df(e_\lambda), \pi_{\tilde{H}} A_{a_i} df(e_\lambda)] - \langle A_{h(a_i), a_j} df(e_\lambda), df(e_\lambda) \rangle + \langle h(a_j, df(e_\lambda)), h(a_j, df(e_\lambda)) \rangle + 2\langle \tilde{T}^2 df(e_\lambda), df(e_\lambda) \rangle - 2|df(e_\lambda)|^2 + 4\langle \tilde{T} J df(e_\lambda), df(e_\lambda) \rangle \rangle \rangle dv_\theta
\]

Using (5.23), an easy calculation shows
\[
\sum_{i=1}^{2n+2k} \langle h(a_i) , df(e_\lambda) \rangle = 2\langle df(e_\lambda), \tilde{T} J df(e_\lambda) \rangle
\]

It follows from (5.27), (5.24) and (5.25) that
\[
\sum_{i=1}^{2n+2k} H_f((a_i)_{\tilde{H}}, (a_i)_{\tilde{H}})
= \int_M \sum_{\lambda=1}^{2n} \{ \sum_{i=2n+2}^{2n+2k} \langle [\pi_{\tilde{H}} A_{a_i} df(e_\lambda), \pi_{\tilde{H}} A_{a_i} df(e_\lambda)] - \langle A_{h(a_i), a_j} df(e_\lambda), df(e_\lambda) \rangle + \langle h(a_j, df(e_\lambda)), h(a_j, df(e_\lambda)) \rangle + 2\langle \tilde{T}^2 df(e_\lambda), df(e_\lambda) \rangle - 2|df(e_\lambda)|^2 + 4\langle \tilde{T} J df(e_\lambda), df(e_\lambda) \rangle \rangle \rangle dv_\theta
\]

By (5.28) and the choice of the basis, we obtain
\[
\sum_{j=1}^{2n} \langle h(a_j, df(e_\lambda)), h(a_j, df(e_\lambda)) \rangle = \sum_{j=1}^{2n} \sum_{i=2n+2}^{2n+2k} \langle A_{a_i} df(e_\lambda), a_j \rangle = \sum_{i=2n+2}^{2n+2k} \langle \pi_{\tilde{H}} A_{a_i} df(e_\lambda), \pi_{\tilde{H}} A_{a_i} df(e_\lambda) \rangle
\]

Using (5.28) again, we perform the following calculations:
\[
\sum_{j=1}^{2n} \langle h(a_j, df(e_\lambda)), h(a_j, df(e_\lambda)) \rangle = \sum_{j=1}^{2n} \sum_{i=2n+2}^{2n+2k} \langle A_{a_i} df(e_\lambda), a_j \rangle \langle A_{a_i} df(e_\lambda), df(e_\lambda) \rangle
= \sum_{i=2n+2}^{2n+2k} \langle \langle \text{trace}_{G_a} A_{a_i} \rangle A_{a_i} df(e_\lambda), df(e_\lambda) \rangle
\]
By (ii) in Lemma 5.2 and (2.4), we have
\[
\tau^2 X = (\bar{\tau}J) \cdot (\bar{\tau}J) X = (-A_\xi - Id) \cdot (-A_\xi - Id) X
= (A_\xi^2 + 2A_\xi + Id) X
\]
(5.29)
for any \( X \in \Gamma(\bar{H}(N)) \). It follows from (5.29) that
\[
2(\tau^2 df(e_\lambda), df(e_\lambda)) - 2|df(e_\lambda)|^2 + 4(\tau \bar{J} df(e_\lambda), df(e_\lambda))
= 2(A_\xi^2 df(e_\lambda) + 2A_\xi df(e_\lambda) + df(e_\lambda), df(e_\lambda)) - 2|df(e_\lambda)|^2
- 4\langle A_\xi df(e_\lambda), df(e_\lambda) \rangle - 4|df(e_\lambda)|^2
= \langle (2A_\xi^2 - 4Id) df(e_\lambda), df(e_\lambda) \rangle
\]
(5.30)
Finally substituting (5.27), (5.28), (5.30) into (5.26), we get
\[
\sum_{i=1}^{2n+2k} H_f((a_i)_\bar{H}, (a_i)_\bar{H}) = \int_M \sum_{\lambda=1}^{2m} \langle Q^N(df(e_\lambda)), df(e_\lambda) \rangle dv_\theta
\]
(5.31)
Under the assumption, \( \int_M \sum_{\lambda=1}^{2m} \langle Q^N(df(e_\lambda)), df(e_\lambda) \rangle dv_\theta \) is negative. We see that at least one \( H_f((a_i)_\bar{H}, (a_i)_\bar{H}) \) must be negative. Then by Definition 4.4, \( f \) is unstable.

**Remark 5.5.** By (5.31), we observe that the condition \( \langle Q^N X, X \rangle < 0 \) can be relaxed in such a way that
\[
\sum_{\lambda=1}^{2m} \langle Q^N(df(e_\lambda)), df(e_\lambda) \rangle < 0.
\]
Here \( \{e_\lambda\}_{\lambda=1}^{2m} \) is a local orthonormal frame of \( H(M) \).

Using Theorem 5.4 we may recapture the result in [1] as follows:

**Corollary 5.6.** (cf.[1]) Suppose \( f : (M, H(M), J, \theta) \to (S^{2n+1}, \bar{H}, \bar{J}, \bar{\theta}) \) is a nonconstant horizontal pseudoharmonic map from a closed pseudo-Hermitian manifold to the odd dimensional sphere. Then \( f \) is unstable.

Before we proof the corollary, we need the following lemma.

**Lemma 5.7.** Suppose \( f : (M, H(M), J, \theta) \to (N, \bar{H}(N), \bar{J}, \bar{\theta}) \) is a horizontal map. If \( f \) is nonconstant, then it is horizontally nonconstant.

**Proof.** we only have to show that if \( df|_{H(M)} = 0 \) then \( df|_{\mathcal{T}M} = 0 \). For any \( X, Y \in \Gamma(H(M)) \), since \( df|_{H(M)} = 0 \), by (3.1) we have
\[
(\tilde{\nabla}_X df)(Y) - (\tilde{\nabla}_Y df)(X) = -d\theta(X, Y) df(T).
\]
As the levi distribution \( H(M) \) is parallel with respect to \( \nabla \), we get \( df(T) = 0 \). 

\[\square\]
Now we are ready to proof the Corollary 5.6. Let $N = S^{2n+1}$. By Example 5.1 we know that the standard odd-dimensional sphere $i : S^{2n+1} \hookrightarrow C^{n+1}$ is an isometric embedded CR manifold. Therefore we have $T = J\nu$. Here $\nu$ be the exterior unit normal to $S^{2n+1}$. It is known that $A_n X = X$ for all $X \in \Gamma(TS^{2n+1})$. Then $Q^{S^{2n+1}} = 2n\tilde{H} - (2n+2)Id$. Since $f$ is nonconstant, by Lemma 5.7 we can get $\sum_{\lambda=1}^{2n} |df(e_\lambda)|^2 > 0$. It follows that $\sum_{\lambda=1}^{2n} \langle Q^N(df(e_\lambda)), df(e_\mu) \rangle < 0$. By Remark 5.5 we will obtain the result.

Next, we will consider the simplest map, i.e. the identity map $I : S^{2n+1} \to S^{2n+1}$. Obviously $I$ is a horizontal pseudoharmonic map. By Corallary 5.6 we know that $I$ is unstable. Following the method in [7], we want to investigate the unstable degree of $I$ as a pseudoharmonic map.

Let $V \in \Gamma(TS^{2n+1})$. According to the decomposition $TS^{2n+1} = span\{\tilde{T}\} \oplus \tilde{H}(S^{2n+1})$, we may write $V$ as

$$V = V_\tilde{T} + V_\tilde{H},$$

where $V_\tilde{T} = \langle V, \tilde{T}\rangle \tilde{T}$ and $V_\tilde{H} = V - \langle V, \tilde{T}\rangle \tilde{T}$. Obviously $V_\tilde{H}$ is a section of $\tilde{H}(S^{2n+1})$. It is also a section of $I^{-1}(\tilde{H}(S^{2n+1}))$.

Let $L_V g_\tilde{g}$ be the Lie derivative of the webster metric $g_\tilde{g}$ in the direction of $V$. It is the symmetric 2-tensor on $S^{2n+1}$. Firstly, we give the following lemma:

**Lemma 5.8.** For any $V \in \Gamma(TS^{2n+1})$,

$$\frac{1}{2} \int_{S^{2n+1}} |L_V g_\tilde{g}|^2_{\tilde{H}} dV_\tilde{g} = \int_{S^{2n+1}} \left\{ \sum_{\lambda=1}^{2n} \left[ \langle \nabla_{e_\lambda} V_\tilde{H}, \nabla_{e_\lambda} V_\tilde{H} \rangle - \langle \tilde{R}(V, e_\lambda, V, e_\lambda) \rangle \right] 
+ (\text{div} V_\tilde{H})^2 - 2 \langle \nabla_{\tilde{T}} V_\tilde{H}, J V_\tilde{H} \rangle \right\} dV_\tilde{g}.$$

Here $|L_V g_\tilde{g}|^2_{\tilde{H}} = \sum_{\lambda=1}^{2n} |(L_V g_\tilde{g})(e_\lambda, e_\mu)|^2$ and $\{e_\lambda\}_{\lambda=1}^{2n}$ is a local orthonormal frame of $\tilde{H}(S^{2n+1})$.

**Proof.** By (5.32), we have

$$(L_V g_\tilde{g})(e_\lambda, e_\mu) = (L_V g_\tilde{g})(e_\lambda, e_\mu) + (L_V g_\tilde{g})(e_\lambda, e_\mu),$$

where $1 \leq \lambda, \mu \leq 2n$. Since $S^{2n+1}$ is a Sasakian manifold and $\nabla_{\tilde{T}} = 0$, we may perform the following calculations:

$$(L_V g_\tilde{g})(e_\lambda, e_\mu) = -\langle [V_\tilde{T}, e_\lambda], e_\mu \rangle - \langle e_\lambda, [V_\tilde{T}, e_\mu] \rangle
= -\tilde{g}(V)[\langle \nabla_{\tilde{T}} e_\lambda, e_\mu \rangle + \langle e_\lambda, \nabla_{\tilde{T}} e_\mu \rangle]
= 0;$$

hence

$$(L_V g_\tilde{g})(e_\lambda, e_\mu) = (L_V g_\tilde{g})(e_\lambda, e_\mu).$$
\[
\sum_{\lambda, \mu = 1}^{2n} \left[ (L_{V^\xi g^\theta})_\lambda (e_\lambda, e_\mu) \right] = \sum_{\lambda = 1}^{2n} \langle \tilde{\nabla}_e V^\phi_{\tilde{H}}, \tilde{\nabla}_e V^\phi_{\tilde{H}} \rangle + \sum_{\lambda, \mu = 1}^{2n} \langle \tilde{\nabla}_e V^\phi_{\tilde{H}}, e_\mu \rangle \langle e_\lambda, \tilde{\nabla}_e e_\lambda, V^\phi_{\tilde{H}} \rangle \tag{5.34}\]

Let \( X, Y \in \Gamma(\tilde{H}(S^{2n+1})) \) be (locally) defined by \( X = \tilde{\nabla}^\phi_{\tilde{H}} V^\phi_{\tilde{H}} \) and \( Y = \sum_{i=1}^{2n} \langle \tilde{\nabla}_e V^\phi_{\tilde{H}}, e_\lambda \rangle V^\phi_{\tilde{H}} \). Let us compute the divergence of \( X \) and \( Y \) respectively. A calculation shows that

\[
\text{div} X - \text{div} Y = \sum_{\lambda = 1}^{2n} \tilde{R}(V^\phi_{\tilde{H}}, e_\lambda, V^\phi_{\tilde{H}}, e_\lambda) + \sum_{\lambda, \mu = 1}^{2n} \langle [\tilde{\nabla}_e V^\phi_{\tilde{H}}, e_\mu] V^\phi_{\tilde{H}}, e_\lambda \rangle - \langle \tilde{\nabla}_e V^\phi_{\tilde{H}}, \tilde{\nabla}_e e_\lambda V^\phi_{\tilde{H}}, e_\mu \rangle
\]

Taking into account (2.5) and \( \bar{\theta} = 0 \) we may actually express \( [e_\lambda, e_\mu] \) as

\[
[e_\lambda, e_\mu] = \tilde{\nabla}_e e_\lambda - \tilde{\nabla}_e e_\mu - d\bar{\theta}(e_\lambda, e_\mu) T.
\]

Then

\[
\text{div} X - \text{div} Y = \sum_{\lambda = 1}^{2n} \tilde{R}(V^\phi_{\tilde{H}}, e_\lambda, V^\phi_{\tilde{H}}, e_\lambda) + \sum_{\lambda, \mu = 1}^{2n} \langle \tilde{\nabla}_e V^\phi_{\tilde{H}}, e_\mu \rangle \langle \tilde{\nabla}_e e_\lambda V^\phi_{\tilde{H}}, e_\lambda \rangle - \langle \tilde{\nabla}_e V^\phi_{\tilde{H}}, \tilde{\nabla}_e e_\lambda V^\phi_{\tilde{H}}, e_\mu \rangle
\]

On the other hand, we have

\[
\sum_{\lambda, \mu = 1}^{2n} \langle \tilde{\nabla}_e V^\phi_{\tilde{H}}, e_\lambda \rangle \langle V^\phi_{\tilde{H}}, e_\mu \rangle = \sum_{\lambda, \mu, \nu = 1}^{2n} \langle \tilde{\nabla}_e e_\mu, e_\nu \rangle \langle \tilde{\nabla}_e e_\lambda V^\phi_{\tilde{H}}, e_\lambda \rangle \langle V^\phi_{\tilde{H}}, e_\mu \rangle - \langle \tilde{\nabla}_e V^\phi_{\tilde{H}}, \tilde{\nabla}_e e_\lambda V^\phi_{\tilde{H}}, e_\mu \rangle - (\text{div} V^\phi_{\tilde{H}})^2 \tag{5.35}\]
Similarly, by a easy calculation we get

\[
\sum_{\lambda,\mu=1}^{2n} \langle \overline{\nabla}_{e_{\mu}} e_{\lambda} V_{\tilde{H}}, e_{\lambda} \rangle \langle V_{\tilde{H}}, e_{\mu} \rangle = - \sum_{\lambda,\mu=1}^{2n} \langle \overline{\nabla}_{e_{\lambda}} V_{\tilde{H}}, \overline{\nabla}_{e_{\mu}} e_{\lambda} \rangle \langle V_{\tilde{H}}, e_{\mu} \rangle.
\]

Therefore (5.35) becomes (by \(G_{\tilde{\theta}}(\cdot, \cdot) = \frac{1}{2} \tilde{d}\tilde{\theta}(\cdot, \tilde{J} \cdot)\))

\[
div X - div Y = \sum_{\lambda=1}^{2n} \tilde{R}(V_{\tilde{H}}, e_{\lambda}, V_{\tilde{H}}, e_{\lambda}) + \sum_{\lambda,\mu=1}^{2n} \langle \overline{\nabla}_{e_{\lambda}} V_{\tilde{H}}, \overline{\nabla}_{e_{\mu}} V_{\tilde{H}} \rangle \langle \overline{\nabla}_{e_{\mu}} V_{\tilde{H}}, e_{\lambda} \rangle
\]

\[
- (div V_{\tilde{H}})^2 + 2 \langle \tilde{\nabla}_{T} V_{\tilde{H}}, e_{\lambda} \rangle \langle \tilde{J} V_{\tilde{H}}, e_{\lambda} \rangle.
\]

Then

\[
\sum_{\lambda,\mu=1}^{2n} \langle \overline{\nabla}_{e_{\lambda}} V_{\tilde{H}}, e_{\mu} \rangle \langle \overline{\nabla}_{e_{\mu}} V_{\tilde{H}}, e_{\lambda} \rangle
\]

\[
= div X - div Y - \sum_{\lambda=1}^{2n} \tilde{R}(V_{\tilde{H}}, e_{\lambda}, V_{\tilde{H}}, e_{\lambda}) + (div V_{\tilde{H}})^2
\]

\[
- 2 \langle \tilde{\nabla}_{T} V_{\tilde{H}}, e_{\lambda} \rangle \langle \tilde{J} V_{\tilde{H}}, e_{\lambda} \rangle.
\]

(5.36)

Since \(S^{2n+1}\) is Sasakian, we have(cf.[2])

\[
\langle \tilde{R}(T), Y \rangle Z, W \rangle = \langle \tilde{S}(Z, W), Y \rangle = 0.
\]

Thus

\[
\tilde{R}(V_{\tilde{H}}, e_{\lambda}, V_{\tilde{H}}, e_{\lambda}) = \tilde{R}(V, e_{\lambda}, V, e_{\lambda}).
\]

Then (5.36) becomes

\[
\sum_{\lambda,\mu=1}^{2n} \langle \overline{\nabla}_{e_{\lambda}} V_{\tilde{H}}, e_{\mu} \rangle \langle \overline{\nabla}_{e_{\mu}} V_{\tilde{H}}, e_{\lambda} \rangle
\]

\[
= div X - div Y - \sum_{\lambda=1}^{2n} \tilde{R}(V, e_{\lambda}, V, e_{\lambda}) + (div V_{\tilde{H}})^2
\]

\[
- 2 \langle \tilde{\nabla}_{T} V_{\tilde{H}}, e_{\lambda} \rangle \langle \tilde{J} V_{\tilde{H}}, e_{\lambda} \rangle.
\]

We may substitute into (5.34) to obtain:

\[
\frac{1}{2} |L_{V_{\tilde{H}}} g_{\tilde{H}}|^2 = \sum_{\lambda=1}^{2n} |\langle \overline{\nabla}_{e_{\lambda}} V_{\tilde{H}}, \overline{\nabla}_{e_{\lambda}} V_{\tilde{H}} \rangle - \tilde{R}(e_{\lambda}, V, e_{\lambda}, V) \rangle
\]

\[
- 2 \langle \tilde{\nabla}_{T} V_{\tilde{H}}, e_{\lambda} \rangle \langle \tilde{J} V_{\tilde{H}}, e_{\lambda} \rangle + (div V_{\tilde{H}})^2 + div X - div Y.
\]
hence

\[
\frac{1}{2} \int_{S^{2n+1}} |L_V g_\tilde{g}|^2_H dV_\tilde{g} = \int_{S^{2n+1}} \left\{ \sum_{\lambda=1}^{2n} \left[ \langle \tilde{\nabla}_{e_\lambda} V_\tilde{H}, \tilde{\nabla}_{e_\lambda} V_\tilde{H} \rangle - \langle \tilde{R}(V, e_\lambda, V, e_\lambda) \rangle \right] + (\text{div} V_\tilde{H})^2 - 2\langle \tilde{\nabla}_{\tilde{H}} V_\tilde{H}, \tilde{J} V_\tilde{H} \rangle \right\} dV_\tilde{g}.
\]

By Remark 4.5 and Lemma 5.8, we have the following result.

**Proposition 5.9.** For any \( V \in \Gamma(TS^{2n+1}) \), we have

\[
H_I(V, V) = \int_{S^{2n+1}} \left\{ \sum_{\lambda=1}^{2n} \left[ \langle \tilde{\nabla}_{e_\lambda} V_\tilde{H}, \tilde{\nabla}_{e_\lambda} V_\tilde{H} \rangle - \langle \tilde{R}(V, e_\lambda, V, e_\lambda) \rangle \right] + (\text{div} V_\tilde{H})^2 - 2\langle \tilde{\nabla}_{\tilde{H}} V_\tilde{H}, \tilde{J} V_\tilde{H} \rangle \right\} dV_\tilde{g},
\]

where \( V_\tilde{H} \) denotes the horizontal part of \( V \).

Let \( i \) denote the algebra of infinitesimal isometries, i.e. vector fields \( V \) satisfying \( L_V g_\tilde{g} = 0 \) and let \( c \) denote the algebra of conformal fields (here we follow the notations of [3]). We have the following conclusion:

**Proposition 5.10.** A vector field \( V \) on \( S^{2n+1} \) is conformal iff \( L_V g_\tilde{g} = \frac{\text{div} V_\tilde{H}}{n} g_\tilde{g} \).

**Proof.** A vector field is conformal iff \( L_V g_\tilde{g} = \sigma g_\tilde{g} \), where \( \sigma \) is a function. In an orthonormal frame of \( H(S^{2n+1}) \), we see that

\[
(L_V g_\tilde{g})(e_\lambda, e_\mu) = \sigma g_\tilde{g}(e_\lambda, e_\mu) = \sigma \delta_{\lambda\mu}.
\]

By (5.33), we may conclude that

\[
(L_V g_\tilde{g})(e_\lambda, e_\mu) = (L_{V_\tilde{H}} g_\tilde{g})(e_\lambda, e_\mu) = \langle \tilde{\nabla}_{e_\lambda} V_\tilde{H}, e_\mu \rangle + \langle e_\lambda, \tilde{\nabla}_{e_\mu} V_\tilde{H} \rangle.
\]

Therefore

\[
\langle \tilde{\nabla}_{e_\lambda} V_\tilde{H}, e_\mu \rangle + \langle e_\lambda, \tilde{\nabla}_{e_\mu} V_\tilde{H} \rangle = \sigma \delta_{\lambda\mu}.
\]

Then contract this identity to obtain

\[
\sigma = \frac{\text{div} V_\tilde{H}}{n}
\]

We are now ready to prove the following result:

**Proposition 5.11.** If \( n \geq 1 \), then index(\( I \)) \( \geq 2n+2 \). Here the index of \( I \) is the dimension of the largest subspace of \( \Gamma(TS^{2n+1}) \) on which \( H_I \) is negative.
Proof. For $V \in \mathcal{L}$, we have $|L_V g_{\tilde{\theta}}|^2 = 2 \left( \text{div} V \right)^2$ so that (5.37) yields

$$H_I(V, V) = \int_{S^{2n+1}} \frac{1-n}{n} (\text{div} V_{\tilde{\theta}})^2 + 2(\tilde{\nabla}_V V_{\tilde{\theta}}, \tilde{\nabla}_V V_{\tilde{\theta}}) dV_{\tilde{\theta}}$$

(5.39)

Moreover, if $V$ is in the orthogonal complement of $\mathcal{L}$, it may be seen as restrictions of the constant vector fields on $R^{2n+2}$ by the standard embedding $S^{2n+1} \to R^{2n+2}$. It has been shown that(cf. Lemma 6.9 in [1]) $\tilde{\nabla}_V V_{\tilde{\theta}} = -\tilde{J} V_{\tilde{\theta}}$. We may substitute this identity into (5.39) to get

$$H_I(V, V) = \int_{S^{2n+1}} \left[ 1-n \frac{1}{n} (\text{div} V_{\tilde{\theta}})^2 - 2|V_{\tilde{\theta}}|^2 \right] dV_{\tilde{\theta}}.$$

If $V$ is in the orthogonal complement of $\mathcal{L}$ in $\mathcal{C}$, it may be seen as restrictions of the constant vector fields on $R^{2n+2}$ by the standard embedding $S^{2n+1} \to R^{2n+2}$. It has been shown that(cf. Lemma 6.9 in [1]) $\tilde{\nabla}_V V_{\tilde{\theta}} = -\tilde{J} V_{\tilde{\theta}}$. We may substitute this identity into (5.39) to get

$$H_I(V, V) = \int_{S^{2n+1}} \left[ 1-n \frac{1}{n} (\text{div} V_{\tilde{\theta}})^2 - 2|V_{\tilde{\theta}}|^2 \right] dV_{\tilde{\theta}}.$$ 

(5.39)

Moreover, if $V$ is in the orthogonal complement of $\mathcal{L}$ in $\mathcal{C}$, by Proposition 5.10 we have $\nabla V_{\tilde{\theta}} \neq 0$; hence $H_I(V, V) < 0$. Then we obtain $\text{index}(I) \geq \dim(\mathcal{C}/\mathcal{L})$, i.e. $\text{index}(I) \geq 2n+2$.

\section{Pseudoharmonic maps into pseudo-Hermitian submanifolds in Heisenberg groups}

In this section, we want to give a condition on the CR weigarten map of a pseudo-Hermitian immersed submanifold $N$ of Heisenberg group which implies that any nonconstant pseudoharmonic map $f$ from a closed pseudo-Hermitian manifold to $N$ is unstable. Firstly we introduce some notions(see [2] for details).

Let $(M, H(M), J, \theta)$ and $(K, H(K), J_K, \Theta)$ be two pseudo-Hermitian manifolds of real dimensions $m$ and $m+k$ respectively. We say that a map $f : M \to K$ is a CR immersion if $f$ is a CR map and $\text{rank}(d_x f) = \dim M$ at any $x \in M$.

Definition 6.1. ([2]) Let $f : M \to K$ be a CR immersion. Then $f$ is called an isopseudo-Hermitian immersion if $f^* \Theta = \theta$.

Remark 6.2. Using the fact that $f$ is a CR map, we have $J_K \circ f_* = f_* \circ J_M$ and thus $f^* G_{\Theta} = G_{\theta}$. In general, the immersion $f$ is not isometric(with respect to the Riemannian metrics $g_{\Theta}$ and $g_{\theta}$).

For simplicity, let’s identify $M$ with $f(M)$ and denote the immersion by $i : M \hookrightarrow K$. Since $(K, H(K), J_K, \Theta)$ is a pseudo-Hermitian manifold, $(K, g_{\Theta})$ is a Riemannian manifold. We can define a vector field $W^\perp$ normal to $N$ by $W^\perp(x) = \text{tan}_x N$ and $W^\perp(x) = \text{nor}_x W$ for any $x \in M$.

We denote $\nabla^K$ to be the Tanaka-Webster connection on $K$. Then we set

$$\nabla_X Y = (\nabla^K_X Y)^\perp,$$

(6.1)

$$\alpha(X, Y) = (\nabla_X^\Theta Y)^\perp$$

(6.2)

for any $X \in \Gamma(TM)$ and $\eta \in \Gamma(T^\perp M)$. It is easy to prove that $\nabla$ is a linear connection on $M$, while $\alpha$ is $C^\infty(M)$-bilinear and has values in $T^\perp M$. Then we
obtain the following CR Gauss formula ([2]):
\[ \nabla^K_X Y = \nabla_X Y + \alpha(X, Y) \]  
(6.3)

We can also set
\[ a_\eta X = - (\nabla^K_X \eta) \]  
(6.4)
\[ \nabla^\perp_X \eta = (\nabla^K_X \eta) \]  
(6.5)
for any \( X \in \Gamma(TM) \) and \( \eta \in \Gamma(T^\perp M) \). Then \( a \) is \( C^\infty(M) \)-bilinear, while \( \nabla^\perp \) is a connection in \( T^\perp M \). So we have the following CR Wegarten formula ([2]):
\[ \nabla^K_X \eta = -a_\eta X + \nabla^\perp_X \eta. \]  
(6.6)

The connection \( \nabla \) in (6.1) does not coincide with the Tanaka-Webster connection of \((M, \theta)\) in general, nor is \( \alpha(f) \) symmetric.

Let \( i : M \rightarrow K \) be an isopseudo-Hermitian CR immersion between two pseudo-Hermitian manifolds. In [2], it has been proved that \( i^* g_\theta = g_\Theta \) if and only if \( T_K = 0 \).

According to the above statement, we have the following definition.

**Definition 6.3.** ([2]) A pseudo-Hermitian immersion is an isopseudo-Hermitian CR immersion with the additional property \( T_K = 0 \).

**Remark 6.4.** If \( i : M \hookrightarrow K \) is a pseudo-Hermitian immersion, we have \( T_M = T_K|_M \) (see details for [2]).

Then we have the following theorem. (cf. [2], page 354)

**Theorem 6.5.** ([2]) Let \((M, H(M), J, \theta)\) and \((K, H(K), J_K, \Theta)\) be two pseudo-Hermitian manifolds and \( i : M \rightarrow K \) a pseudo-Hermitian immersion. Then

(i) \( \nabla \) is the Tanaka-Webster connection of \((M, \theta)\).

(ii) \( \pi_H \alpha \) is symmetric. Here \( \pi_H \alpha \) be a vector-valued form defined by
\[ (\pi_H \alpha)(X, Y) = \alpha(\pi_H X, \pi_H Y) \]
for any \( X, Y \in TM \).

(iii) \( a_\eta \) is \( H(M) \)-valued and for any \( x \in M \), \( (a_\eta)_x : H(M)_x \rightarrow H(M)_x \) is self-adjoint (with respect to \( G_{\theta, x} \)).

In fact, the tensors \( \alpha \) and \( a \) can also related by
\[ g_\Theta(\alpha(X, Y), \eta) = g_\theta(a_\eta X, Y) \]  
(6.7)
for any \( X, Y \in \Gamma(TM) \), \( \eta \in \Gamma(T^\perp M) \).

Next we will consider a special pseudo-Hermitian immersion. Let \( K = \)
$H_{n+k}$ be the Heisenberg group (with the standard strictly pseudoconvex pseudo-Hermitian structure). By Example 2.7 we know it is a Sasakian manifold. Let $(N, \tilde{H}(N), \tilde{J}, \tilde{\theta})$ be a pseudo-Hermitian manifold of dimension $2n+1$.

From now on, we always assume that $i : (N, \tilde{H}(N), \tilde{J}, \tilde{\theta}) \to H_{n+k}$ is a pseudo-Hermitian immersion.

Let $\nabla, \tilde{\nabla}$ be the Tanaka-Webster connections on $H_{n+k}$ and $N$ respectively. Since $i$ is a pseudo-Hermitian immersion, by Theorem 6.5 we have

\[
\nabla_X Y = \tilde{\nabla}_X Y + \alpha(X, Y) \tag{6.8}
\]

and

\[
\nabla_X \eta = -a_\eta X + \nabla_{\tilde{X}} \eta \tag{6.9}
\]

for any $X, Y \in \Gamma(T N)$, $\eta \in \Gamma(T^\perp N)$.

We have the following lemma:

**Lemma 6.6.** For any $\eta \in \Gamma(T^\perp N)$, $X \in \Gamma(T N)$, $a_\eta\tilde{T} = 0$ and $\alpha(\tilde{T}, X) = 0$. Here we still use $\tilde{T}$ to denote the characteristic direction on $N$.

**Proof.** For $X \in \Gamma(T N)$, by (6.7) we have

\[
\langle a_\eta\tilde{T}, X \rangle = \langle \alpha(\tilde{T}, X), \eta \rangle. \tag{6.10}
\]

Since $H_{n+k}$ is Sasakian, using $T_{H_{n+k} \mid N} = \tilde{T}, \nabla T_{H_{n+k}} = 0$, and $\tilde{\nabla}\tilde{T} = 0$ we get

\[
\alpha(\tilde{T}, X) = (\nabla_{\tilde{T}} X)^\perp = (\nabla_X \tilde{T} + [\tilde{T}, X])^\perp = \nabla_X T_{H_{n+k}} - \tilde{\nabla}_X \tilde{T} = 0.
\]

\[\square\]

Since $H_{n+k}$ with the standard strictly pseudoconvex pseudo-Hermitian structure is Sasakian, then $(H, \tilde{H}(N), \tilde{J}, \tilde{\theta})$ is also Sasakian. In fact, by Definition 6.3 Theorem 6.5 and $\tilde{\nabla}\tilde{T} = 0$ again, we can perform the following calculations:

\[
\tilde{\tau}(X) = T_{\tilde{\nabla}}(\tilde{T}, X) = (\nabla_{\tilde{T}} X)^T - [\tilde{T}, X] = (\nabla_{T_{H_{n+k} \mid N}} X - [T_{H_{n+k} \mid N}, X])^T = (\tau_{H_{n+k} \mid N}(X))^T = 0 \tag{6.11}
\]

for any $X \in \Gamma(T N)$. Here $\tilde{\tau}$ is the pseudo-Hermitian torsion on $N$.

For each $x \in N$, let $\{\eta_{2n+2}, \ldots, \eta_{2n+2k+1}\}$ be an orthonormal basis for the normal space $T_x^\perp N$. According to $(iii)$ in Theorem 6.5 we can define a self-adjoint linear map $P_x^N : H(N)_x \to H(N)_x$ by

\[
P_x^N = \sum_{i=2n+2}^{2n+2k+1} [2a_{\eta_i} - tr_{g_x}(a_{\eta_i} \cdot a_{\eta_i})]. \tag{6.12}
\]
It is easy to see that $P^N_x$ does not depend on the choice of $\{\eta_i\}_{i=2n+2}^{2n+2k+1}$.

Let $V$ be a vector in $H_{n+k}$. $V$ can be identified with a parallel vector field on $H_{n+k}(\nabla V = 0)$. For any $W \in \Gamma(TN)$, we define a tensor $A^W$ in $\text{Hom}(TN,TN)$ corresponding to $W$ by $A^W(X) = \tilde{\nabla}_X W$, for any $X \in \Gamma(TN)$.

If we let $W$ be $V^\top$, using (6.8), (6.9) and $\tilde{\nabla}V = 0$ by a direct computation we obtain

$$A^{V^\top}(X) = \tilde{\nabla}_X V^\top = a_{V^\top}(X)$$

and

$$\nabla^\perp_X V^\top = (\tilde{\nabla}_X V^\top)^\perp = -\alpha(X, V^\top).$$

These imply

$$(\nabla^\perp_X A^{V^\top}) = (\nabla^\perp V^\top a)_{V^\top} - a_{(V^\top, V^\top)}$$

(6.13)

where $(\nabla_X a)_{V^\top} = (\tilde{\nabla}_X a_{V^\top}) - a_{\nabla^\perp X},$ for any $X \in \Gamma(TN)$.

**Lemma 6.7.** For any $X \in \Gamma(TN)$,

$$\tilde{R}(V^\top, X, V^\top, X) = -\langle(\nabla^\perp V^\top A^{V^\top})(X), X \rangle - \langle A^{V^\top} A^{V^\top}(X), X \rangle$$

$$+ \langle \nabla_X (A^{V^\top}(V^\top)), X \rangle$$

(6.14)

**Proof.** By the definition of the curvature tensor we have

$$\langle \tilde{R}(V^\top, X)V^\top, X \rangle = \langle \nabla^\perp X V^\top - \tilde{\nabla}_X \tilde{V}^\top - \tilde{\nabla}_{[V^\top, X]} V^\top, X \rangle$$

(6.15)

Since $N$ is sasakian, by (2.5) we can get

$$[V^\top, X] = \tilde{\nabla}^\perp_{V^\top} X - \tilde{\nabla}_X V^\top - T_\Phi(V^\top, X)$$

$$= \tilde{\nabla}^\perp_{V^\top} X - \tilde{\nabla}_X V^\top - d\tilde{\theta}(V^\top, X)$$

Putting this into (6.15), Using Lemma 6.6 we obtain

$$\langle \tilde{R}(V^\top, X)V^\top, X \rangle = \langle \nabla^\perp X V^\top, X \rangle - \langle \tilde{\nabla}_X \tilde{V}^\top, X \rangle$$

$$- \langle \nabla^\perp_{\tilde{\nabla}^\perp V^\top} X, V^\top \rangle + \langle \tilde{\nabla}_X \tilde{V}^\top, X \rangle + d\tilde{\theta}(V^\top, X) \langle \nabla_{\tilde{\nabla}^\perp X} V^\top, X \rangle$$

$$= \langle (\tilde{\nabla}^\perp V^\top A^{V^\top})(X), X \rangle + \langle A^{V^\top} A^{V^\top}(X), X \rangle - \langle \tilde{\nabla}_X (A^{V^\top}(V^\top)), X \rangle.$$ 

Then we can get

$$\tilde{R}(V^\top, X, V^\top, X) = -\langle (\tilde{\nabla}^\perp V^\top A^{V^\top})(X), X \rangle - \langle A^{V^\top} A^{V^\top}(X), X \rangle$$

$$+ \langle \tilde{\nabla}_X (A^{V^\top}(V^\top)), X \rangle,$$

since $\tilde{R}(V^\top, X, V^\top, X) = -\langle \tilde{R}(V^\top, X)V^\top, X \rangle$. 

□

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Theorem 6.8. Let \( i : (N, \tilde{H}(N), \tilde{J}, \tilde{\theta}) \rightarrow H_{n+k} \) be a pseudo-Hermitian immersion and \((M, H(M), J, \theta)\) be a closed pseudo-Hermitian manifold. Let \( f : (M, H(M), J, \theta) \rightarrow (N, \tilde{H}(N), \tilde{J}, \tilde{\theta}) \) be a nonconstant pseudoharmonic map. Assume that \( P^N_y \) is negative definite on \( \tilde{H}(N) \) at each point \( y \) of \( N \) (i.e. for all \( X \neq 0 \) and \( X \in \Gamma(\tilde{H}(N)), \langle P^N X, X \rangle < 0 \)). Then \( f \) is unstable.

Proof. Firstly we consider the second variation of \( f \) corresponding to \( V^T \) (as above). Since \( N \) is sasakian, by Remark 4.5 we have

\[
H_f(V^T, V^T) = \int_M \sum_{\lambda=1}^{2m} (\tilde{\nabla}_{df(e_\lambda)}(V^T)_H, \tilde{\nabla}_{df(e_\lambda)}(V^T)_H) - \langle \tilde{R}(V^T, \pi_{\tilde{H}} df(e_\lambda), V^T, \pi_{\tilde{H}} df(e_\lambda)) \rangle d\theta,
\]

(6.16)

where \( \{e_\lambda\}_{\lambda=1}^{2m} \) is a local orthonormal frame of \( H(M) \).

Using (iii) in Theorem 6.5 and Lemma 6.6 the first term in (6.16) may be written as

\[
\int_M \sum_{\lambda=1}^{2m} (\tilde{\nabla}_{df(e_\lambda)}(V^T)_H, \tilde{\nabla}_{df(e_\lambda)}(V^T)_H)
= \int_M \sum_{\lambda=1}^{2m} (a_{V^T}(\pi_{\tilde{H}} df(e_\lambda)), a_{V^T}(\pi_{\tilde{H}} df(e_\lambda)))
= \int_M \sum_{\lambda=1}^{2m} (a_{V^T}(\pi_{\tilde{H}} df(e_\lambda)), a_{V^T}(\pi_{\tilde{H}} df(e_\lambda)))
= \int_M \sum_{\lambda=1}^{2m} (a_{V^T}(\pi_{\tilde{H}} df(e_\lambda)), a_{V^T}(\pi_{\tilde{H}} df(e_\lambda)))
\]

(6.17)

By (6.13) and Lemma 6.7 the second term in (6.16) may be written as

\[
\int_M \sum_{\lambda=1}^{2m} \tilde{R}(V^T, \pi_{\tilde{H}} df(e_\lambda), V^T, \pi_{\tilde{H}} df(e_\lambda))
= \int_M \sum_{\lambda=1}^{2m} [-((\bar{\nabla}_{V^T} A^{V^T})(\pi_{\tilde{H}} df(e_\lambda)), \pi_{\tilde{H}} df(e_\lambda)) - (A^{V^T} A^{V^T}(\pi_{\tilde{H}} df(e_\lambda)), \pi_{\tilde{H}} df(e_\lambda))]
+ \langle \tilde{\nabla}_{\pi_{\tilde{H}} df(e_\lambda)}(A^{V^T}(V^T)), \pi_{\tilde{H}} df(e_\lambda)) \rangle d\theta
= \int_M \sum_{\lambda=1}^{2m} [-((\bar{\nabla}_{V^T a^T}(\pi_{\tilde{H}} df(e_\lambda)), \pi_{\tilde{H}} df(e_\lambda)) + \langle a_{(V^T a^T)}(\pi_{\tilde{H}} df(e_\lambda)), \pi_{\tilde{H}} df(e_\lambda) \rangle)
- \langle a_{V^T}(\pi_{\tilde{H}} df(e_\lambda)), \pi_{\tilde{H}} df(e_\lambda) \rangle + \langle \tilde{\nabla}_{\pi_{\tilde{H}} df(e_\lambda)}(V^T), \pi_{\tilde{H}} df(e_\lambda) \rangle] d\theta
\]

(6.18)
The last term in (6.18) may be computer as

\[
\int_M \sum_{\lambda=1}^{2m} (\nabla \pi_{\hat{\beta}} df(e_\lambda) (\nabla_{V^T} V^T), \pi_{\hat{\beta}} df(e_\lambda))
\]

\[
= \int_M \sum_{\lambda=1}^{2m} [(\nabla df(e_\lambda) (\nabla_{V^T} V^T), \pi_{\hat{\beta}} df(e_\lambda)) - \tilde{\theta}(df(e_\lambda))(\nabla_{\hat{\beta}} (\nabla_{V^T} V^T), \pi_{\hat{\beta}} df(e_\lambda))]d\theta
\]

\[
= \int_M \sum_{\lambda=1}^{2m} [e_\lambda (\nabla_{V^T} V^T, \pi_{\hat{\beta}} df(e_\lambda)) - (\nabla_{V^T} V^T, \nabla_{e_\lambda} (\pi_{\hat{\beta}} df(e_\lambda))]

- \tilde{\theta}(df(e_\lambda))(\nabla_{\hat{\beta}} (\nabla_{V^T} V^T), \pi_{\hat{\beta}} df(e_\lambda))]d\theta.
\]

Let \( Y \in \Gamma(H(M)) \) be locally defined by \( Y = (\nabla_{V^T} V^T, \pi_{\hat{\beta}} df(e_\lambda))e_\lambda \). Let us computer the divergence of \( Y \). We have

\[
div(Y) = \sum_{\lambda=1}^{2m} [e_\lambda (\nabla_{V^T} V^T, \pi_{\hat{\beta}} df(e_\lambda)) - (\nabla_{V^T} V^T, \pi_{\hat{\beta}} df(e_\lambda))]
\]

Then

\[
\int_M \sum_{\lambda=1}^{2m} (\nabla \pi_{\hat{\beta}} df(e_\lambda) (\nabla_{V^T} V^T), \pi_{\hat{\beta}} df(e_\lambda))
\]

\[
= \int_M [-(\nabla_{V^T} V^T, trG_{\theta} \beta_{H, \hat{\beta}}) - \sum_{\lambda=1}^{2m} \tilde{\theta}(df(e_\lambda))(\nabla_{\hat{\beta}} (\nabla_{V^T} V^T), \pi_{\hat{\beta}} df(e_\lambda))]d\theta
\]

Since \( f \) is pseudoharmonic, by (6.13) and Lemma 6.6 we obtain

\[
\int_M \sum_{\lambda=1}^{2m} (\nabla \pi_{\hat{\beta}} df(e_\lambda) (\nabla_{V^T} V^T), \pi_{\hat{\beta}} df(e_\lambda))
\]

\[
= - \int_M \sum_{\lambda=1}^{2m} \tilde{\theta}(df(e_\lambda))(\nabla_{\hat{\beta}} (\nabla_{V^T} V^T), \pi_{\hat{\beta}} df(e_\lambda))d\theta
\]

\[
= - \int_M \sum_{\lambda=1}^{2m} \tilde{\theta}(df(e_\lambda))(\nabla_{\hat{\beta}} A_{V^T})(V^T), \pi_{\hat{\beta}} df(e_\lambda))d\theta
\]

\[
= - \int_M \sum_{\lambda=1}^{2m} \tilde{\theta}(df(e_\lambda))(\nabla_{\hat{\beta}} a_{V^T})(V^T) - a_{\alpha(\hat{\beta}, V^T)}(V^T), \pi_{\hat{\beta}} df(e_\lambda))d\theta
\]

\[
= - \int_M \sum_{\lambda=1}^{2m} \tilde{\theta}(df(e_\lambda))(\nabla_{\hat{\beta}} a_{V^T})(V^T), \pi_{\hat{\beta}} df(e_\lambda))d\theta
\]
Then we obtain
\[
\int_M \sum_{\lambda=1}^{2m} \tilde{R}(V^\top, \pi_{\tilde{H}} df(e_\lambda), V^\top, \pi_{\tilde{H}} df(e_\lambda))
\]
\[
= \int_M \sum_{\lambda=1}^{2m} \left[ -\langle a^2_{V^\top} (\pi_{\tilde{H}}(df(e_\lambda))), \pi_{\tilde{H}}(df(e_\lambda)) \rangle - \langle (\nabla_{V^\top} a)_{V^\top} (\pi_{\tilde{H}}df(e_\lambda)), \pi_{\tilde{H}}df(e_\lambda) \rangle \right]
\]
\[
+ \langle a_\alpha(V^\top, V^\top) (\pi_{\tilde{H}}df(e_\lambda)), \pi_{\tilde{H}}df(e_\lambda) \rangle - \bar{\theta}(df(e_\lambda)) \langle (\nabla_{\pi^\top} a)_{V^\top} (V^\top), \pi_{\tilde{H}}df(e_\lambda) \rangle \rangle dV_\theta
\]
(6.19)

It follows from (6.16), (6.17) and (6.19) that
\[
H_f(V^\top, V^\top) = \left. \frac{d^2}{dt^2} \right|_{t=0} E_{H, \tilde{H}}(f_t)
\]
\[
= \int_M \sum_{\lambda=1}^{2m} \left[ 2\langle a^2_{V^\top} (\pi_{\tilde{H}}(df(e_\lambda))), \pi_{\tilde{H}}(df(e_\lambda)) \rangle + \langle (\nabla_{V^\top} a)_{V^\top} (\pi_{\tilde{H}}df(e_\lambda)), \pi_{\tilde{H}}df(e_\lambda) \rangle \right]
\]
\[
- \langle a_\alpha(V^\top, V^\top) (\pi_{\tilde{H}}df(e_\lambda)), \pi_{\tilde{H}}df(e_\lambda) \rangle \rangle dV_\theta
\]
(6.20)

We choose an orthonormal basis \( \{V_1, \cdots, V_{2n+2k+1}\} \) of \( H_{\alpha+k} \) such that \( \{V_1, \cdots, V_{2n+1}\} \) is a basis of \( T_x N \) and \( \{V_{2n+2}, \cdots, V_{2n+2k+1}\} \) is a basis of \( T^\perp_x N \). Then \( (\nabla_{V_j^\top} a)_{V_j^\top} = 0 \) and \( (\nabla_{\pi^\top} a)_{V_j^\top} (V^\top) = 0 \) as for each \( j \) one of \( V_j^\top|_x \) or \( V_j^\perp|_x \) is zero.

Then we obtain
\[
\sum_{j=1}^{2n+2k+1} H_f(V_j^\top, V_j^\top) = \int_M \sum_{\lambda=1}^{2m} \sum_{i=2n+2}^{2n+2k+1} 2\langle a^2_{V^\top} (\pi_{\tilde{H}}(df(e_\lambda))), \pi_{\tilde{H}}(df(e_\lambda)) \rangle
\]
\[
- \sum_{j=1}^{2n+1} \langle a_\alpha(V_j, V_j) (\pi_{\tilde{H}}df(e_\lambda)), \pi_{\tilde{H}}df(e_\lambda) \rangle \rangle dV_\theta
\]
(6.21)

By (6.7) a calculation shows that
\[
\langle a_\alpha(V_j, V_j) (\pi_{\tilde{H}}df(e_\lambda)), \pi_{\tilde{H}}df(e_\lambda) \rangle = \langle \alpha(\pi_{\tilde{H}}df(e_\lambda), \pi_{\tilde{H}}df(e_\lambda)), \alpha(V_j, V_j) \rangle
\]
\[
= \sum_{i=2n+2}^{2n+2k+1} \langle \alpha(\pi_{\tilde{H}}df(e_\lambda), \pi_{\tilde{H}}df(e_\lambda)), V_i \rangle \langle \alpha(V_j, V_j), V_i \rangle
g \]
\[
= \sum_{i=2n+2}^{2n+2k+1} \langle a_{V_i} (\pi_{\tilde{H}}df(e_\lambda)), \pi_{\tilde{H}}df(e_\lambda) \rangle \langle a_{V_i} V_j, V_j \rangle
g \]
\[
= \sum_{i=2n+2}^{2n+2k+1} \langle tr_{g_\xi} a_{V_i} \cdot a_{V_i} (\pi_{\tilde{H}}df(e_\lambda)), \pi_{\tilde{H}}df(e_\lambda) \rangle
g
\]
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Substitute into (6.21), then
\[\sum_{j=1}^{2n+2k+1} H_f(V_j^T, V_j^T) = \int_M \sum_{\lambda=1}^{2m} \sum_{i=2n+2}^{2n+2k+1} [2\langle a^2_i (\pi_{H_i}(df(e_{\lambda}))), \pi_{H_i}(df(e_{\lambda})) \rangle - \langle tr g_i a_{V_i} \cdot a_{V_i}(\pi_{H_i}df(e_{\lambda})), \pi_{H_i}df(e_{\lambda}) \rangle]dV_{\theta}
\]
\[= \int_M \sum_{\lambda=1}^{2m} \langle PN(\pi_{H_i}df(e_{\lambda})), \pi_{H_i}df(e_{\lambda}) \rangle dV_{\theta}.
\]
Under the assumption, \(P^N\) is negative. We see that at least one \(H_f(V_j^T, V_j^T)\) must be negative. Then \(f\) is unstable.

References

[1] Dong Y X. Pseudoharmonic maps between pseudo-Hermitian manifolds, to appear.

[2] Dragomir S, Tomassini G. Differential Geometry and Analysis on CR manifolds. Progress in Mathematics, Birkhauser, 2006, 246.

[3] Eells J, Lemaire L. Selected topics in harmonic maps. Conf Board Math Sci, 1983, 50.

[4] Howard R, Wei S W. Nonexistence of stable harmonic maps to and from certain homogeneous spaces and submanifolds of Euclidean space. Trans Amer Math Soc, 1986, 294: 319-331.

[5] Leung P F. On the stability of harmonic maps. Lecture Notes in Mathematics, 1982, 949: 122-129.

[6] Petit R. Mok-Siu-Yeung type formulas on contact locally sub-symmetric spaces. Ann Glob Anal Geom, 2009, 35: 1-37.

[7] Smith R T. The second variation formula for harmonic mappings. Pro Amer Math Soc, 1975, 47: 229-236.

[8] Xin Y L. Some results on stable harmonic maps. Duke Math J, 1980, 47: 609-613.

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