ROSENBERG–ZELINSKY SEQUENCES FOR TENSORS AND NON-ASSOCIATIVE ALGEBRAS

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Abstract. We produce a long exact sequence of unit groups of associative algebras that behave as automorphisms of tensors in a manner similar to inner automorphisms for associative algebras. Analogues for Lie algebras of derivations of a tensor are also derived. These sequences, which are basis invariants of the tensor, generalize similar ones used for associative and non-associative algebras; they similarly facilitate inductive reasoning about, and calculation of the groups of symmetries of a tensor. The sequences can be used for problems as diverse as understanding algebraic structures to distinguishing entangled states in particle physics.

In memory of C.C. Sims.

1. Introduction

The purpose of this work is to provide tools to expose the symmetries of a tensor. By a tensor we mean a vector, \( t \), that can be interpreted as a multilinear map \( \langle t \rangle : U_1 \times \cdots \times U_1 \rightarrow U_0 \). (Throughout \( \rightarrow \) will distinguish multilinear from linear maps.) For instance, a \((d_2 \times d_1 \times d_0)\)-grid, \( t = [t_{ij}^k] \), of scalars may be interpreted as a multilinear map \( \langle t \rangle : \mathbb{K}^{d_2} \times \mathbb{K}^{d_1} \rightarrow \mathbb{K}^{d_0} \) evaluated on \( u_2, u_1 \) by:

\[
\langle t \rangle |_{u_2, u_1} = \left( \sum_{i=1}^{d_2} \sum_{j=1}^{d_1} u_{2i} t_{ij}^1 u_{1j}, \ldots, \sum_{i=1}^{d_2} \sum_{j=1}^{d_1} u_{2i} t_{ij}^{d_0} u_{1j} \right) \in \mathbb{K}^{d_0}.
\]

Tensors describe diverse structures, including distributive products in algebra, affine connections in differential geometry, quantum entanglement in particle physics, and measurements and meta-data in statistical models.

A natural objective in the study of tensors is to discover properties invariant under basis change. We are particularly concerned with one such invariant, namely the group of symmetries of a tensor:

\[
\text{Aut}(t) = \left\{ \varphi \in \prod_{a=0}^1 \text{GL}_2(U_a) \mid \varphi_0 \langle t \rangle |_{u_0, \ldots, u_1} = \langle t \rangle |_{\varphi_0 u_0, \ldots, \varphi_1 u_1} \right\}.
\]

We introduce to the study of tensors an analog of a powerful tool—Rosenberg–Zelinsky exact sequences—that is well known in the study of automorphisms of associative algebras. In that context, given an associative algebra \( A \) one studies...
Aut(A) by placing it in an exact sequence
\[(1.2) \quad 1 \to \text{Inn}(A) \to \text{Aut}(A) \to J(A),\]
where \(\text{Inn}(A)\) is the group of inner automorphisms and \(J(A)\) is a group into which outer automorphisms of \(A\) are naturally represented; one in which we can explore \(J(A)\) without knowing \(\text{Aut}(A)\). For instance, Skolem-Noether type theorems use Galois automorphisms for \(J(A)\), whereas general Rosenberg–Zelinsky sequences relate \(J(A)\) to module theory of \(A\) via Picard groups; see [AH,GM,BFRS].

A fundamental problem with tensors is that there is no a priori notion of inner automorphisms, and creating them forces us to extend the Rosenberg–Zelinsky sequences to the left of \(\text{Aut}(A)\). To guide us, however, we have variations on Skolem-Noether theorems, first by Jacobson and others for Lie and Jordan algebras \([\mathcal{J}]\), and secondly for bilinear maps by the third author \([W]\). (It was not recognized at the time that this was part of a general strategy for all tensors.)

1.1. **Notation.** Throughout, the Hebrew letter \(\gamma\) (\(\text{vav}\) to evoke *valence*) is a non-negative integer set \(\{1\} = \{0, \ldots, \gamma\}\) and \(\binom{[\gamma]}{i} = \{A \subset \gamma \mid i = |A|\}\). As context permits we let the letters \(a, b, c\) range over \(\gamma\). For \(A \subset \gamma\), write \(\bar{A} = \gamma - A\) and \(\bar{a} = \gamma - \{a\}\). Throughout, \(\mathbb{K}\) will be a commutative unital ring and \(U_1, \ldots, U_0\) will denote finitely generated \(\mathbb{K}\)-modules. Define
\[
U_1 \otimes U_0 = \text{hom}(U_1, U_0) \quad U_1 \otimes \cdots \otimes U_0 = U_1 \otimes (U_{i-1} \otimes \cdots \otimes U_0).
\]

Then \(U_a \otimes (-)\) is a functor on modules with (left) adjoint functor \((-) \otimes U_a\) giving rise to the following natural isomorphisms of \(\mathbb{K}\)-modules:
\[
U_1 \otimes \cdots \otimes U_1 \otimes U_0 \cong (U_1 \otimes \cdots \otimes U_a) \otimes U_{a-1} \otimes \cdots \otimes U_0 \\
\cong (U_1 \otimes \cdots \otimes U_{a+1}) \otimes U_{a} \otimes \cdots \otimes U_0 \\
\cong U_1 \otimes \cdots \otimes U_0.
\]

A *tensor space* is a \(\mathbb{K}\)-module \(T\) equipped with a \(\mathbb{K}\)-module monomorphism
\[(1.3) \quad \langle | \rangle : T \hookrightarrow U_1 \otimes \cdots \otimes U_0.
\]

An element \(t \in T\) is a *tensor*, and \(\langle t \rangle : U_1 \times \cdots \times U_1 \to U_0\) is its associated multilinear map. For \(|u| = |u_1, \ldots, u_\gamma|\) \(\in \prod_{a \in \gamma} U_a\), write \(\langle t | u \rangle \in U_0\) to mean the evaluation of \(\langle t \rangle\) at \(|u|\). The set \(\{U_0, \ldots, U_\gamma\}\) of modules is the *frame* of \(T\), and \(\gamma\) *valence*. For brevity, we often write \(S \subset U_1 \otimes \cdots \otimes U_0\) to denote a set of tensors and its frame.

For \(f_a \in \text{End}(U_a)\) (the ring of \(\mathbb{K}\)-linear endomorphisms) write \(f_a u_a, u_a\) to apply \(f_a\) to \(u_a\) while leaving the other coordinates fixed. If, for each \(a \in \gamma\), \(\langle t | u_a, U_a \rangle = 0\) implies \(u_a = 0\), then \(t\) is *nondegenerate*; if \(U_0 = \langle t | U_0 \rangle\) then \(t\) is *full*. We say \(t\) is *fully nondegenerate* if it is full and nondegenerate, and we lose no essential information by assuming all our tensors are of this type.

1.2. **Main results.** We adopt Albert’s *autotopisms* and Leger–Luks’ *generalized derivations* \([LL]\) as the principal invariants to study. Write \(\text{Aut}(U_a)\) for \(\text{End}(U_a)^\times\), the group of units of \(\text{End}(U_a)\). For \(S \subset U_1 \otimes \cdots \otimes U_0\),

\[
\text{Der}(S) = \left\{ \delta \in \prod_{a \in \gamma} \text{End}(U_a) \mid \forall t \in S, \quad \delta t \langle t | u \rangle = \sum_{\varepsilon \in \gamma, \varepsilon \neq 0} \langle t | \delta \varepsilon u_\varepsilon, u_\varepsilon \rangle \right\}
\]

\(\delta \in \prod_{a \in \gamma} \text{End}(U_a)\)
is the (Lie) algebra of derivations of $S$, and

$$\text{Aut}(S) = \left\{ \alpha \in \prod_{a \in [1]} \text{Aut}(U_a) \mid \forall t \in S, \alpha_0(t \mid u) = \langle t \mid \alpha_0 u \rangle \right\}$$  

(1.5)

is the group of automorphisms (also called autotopisms) of $S$. For $0 < a < b \leq 1$,

$$\text{Nuc}_{ab}(S) = \{ \omega \in \text{End}(U_a) \otimes \text{End}(U_b) \mid \langle t \mid \omega_a v_a, v_b \rangle = \langle t \mid \omega_b v_b, v_b \rangle \}$$  

(1.6)

and power of our methods. These examples include detailed examinations of sym-

alar we demonstrate how our theorems constitute part of a general strategy to attach

classifications of nuclei are associative rings.) For $A \subseteq [1] - 0$, the associative rings

$$\text{Cen}_A(S) = \left\{ \omega \in \prod_{a \in A} \text{End}(V_a) \mid \forall a, b \in A, \langle t \mid \omega_a v_a, v_b \rangle = \langle t \mid \omega_b v_b, v_b \rangle \right\}$$  

(1.7)

are called centroids of $S$. The assumption that $S$ is fully nondegenerate ensures that all centroids are commutative. For $2 < k \leq 1$, put

$$\text{Nuc}(S) := \bigoplus_{\lambda \in \mathbb{C}} \text{Nuc}_\lambda(S) \quad \text{Cen}_k(S) := \bigoplus_{\lambda \in \mathbb{C}} \text{Cen}_\lambda(S).$$  

(1.8)

**Theorem A.** For each fully nondegenerate $S \subseteq U_\nu \otimes \cdots \otimes U_0$, there is an exact sequence of $\mathbb{K}$-Lie algebras

$$0 \rightarrow \text{Cen}_1(S) \rightarrow \cdots \rightarrow \text{Cen}_3(S) \rightarrow \text{Nuc}(S) \rightarrow \text{Der}(S).$$

We also establish the following analogue of Theorem A for groups.

**Theorem B.** For each fully nondegenerate $S \subseteq U_\nu \otimes \cdots \otimes U_0$, there is an exact sequence of groups

$$1 \rightarrow \text{Cen}_1(S) \rightarrow \cdots \rightarrow \text{Cen}_3(S) \rightarrow \text{Nuc}(S) \rightarrow \text{Aut}(S).$$

1.3. Outline. The paper is organized as follows. In Section 2, we describe more general operators than those used in the sequences of Theorems A and B. In particular we demonstrate how our theorems constitute part of a general strategy to attach polynomial-based invariants to tensors, which are then analyzed by restricting to subsets of variables. Geometrically this equates to taking sections and inspecting fibers. Section 3 constructs the exact sequences in Theorems A and B, and identifies a certain combinatorial property needed to prove exactness. This property is examined in isolation in Section 4, followed by the proofs our main theorems in Section 5. Section 6 provides several examples that help to illustrate the breadth and power of our methods. These examples include detailed examinations of symmetries of tensors as well as a brief look at quantum particle entanglement, where two states are distinguished using our exact sequences.

2. Creating the sequences

Observe that the sequences in Theorems A and B concern either a group or a Lie algebra but involve lists of associative algebras. Although this is a convenient conversion from a pragmatic viewpoint—associative algebras are better understood,
structurally, than groups and Lie algebras—switching between categories is at a minimum unnatural, and at worst a sign of confusion. However, the conversion has a natural explanation when viewed through a broader geometric lens.

In order to introduce our sequences and expose their purpose we make use of a device to record operators on a tensor space by polynomial identities as introduced in [FMW]. This allows for a convenient algebro-geometric vocabulary.

2.1. Operator sets. Our sequences originate by inducing different representations for the operators within multiple endomorphism rings. We describe the set of operators, and then demonstrate how restrictions generate our sequence.

Throughout, fix a frame of $K$-modules $\{U_0, \ldots, U_t\}$, and let $t$ be a tensor of $U_1 \otimes \cdots \otimes U_t$. Set $K[X] = K[x_1, \ldots, x_0]$. For $p(X) = \sum e \lambda_p X^e \in K[X]$ and $\omega \in \prod_{a \in [t]} \text{End}(U_a)$, define
\[
\langle t| p(\omega)| u \rangle = \sum e \lambda_p \omega_0^e \langle t| \omega_1^e u_1, \ldots, \omega_t^e u_t \rangle.
\]
For each set $S$ of tensors, and each $P \subset K[X]$, define $\langle S| P(\omega)| U \rangle$ to be the subspace generated by $\langle t| p(\omega)| u \rangle$ as $t$ ranges over $S$, $p$ over $P$, and $|u|$ over $\prod_{a \geq 0} U_a$. Then,
\[
(2.1) \quad \mathcal{Z}(S, P) = \left\{ \omega \in \prod_{a \in [t]} \text{End}(U_a) \mid \langle S| P(\omega)| U \rangle = 0 \right\}.
\]
is the operator sets for the pair $S, P$. Although our focus on groups and algebras may encourage us to regard $\mathcal{Z}(S, P)$ simply as sets, it helps our intuition to consider them as geometries. Over an algebraically closed field these are algebraic zero-sets. For other rings (such as finite fields as required of several problems in algebra) one still has the notion of $\mathcal{Z}(S, P)$ as an affine $K$-scheme [FMW].

Remark 2.2. Note that the polynomials that define $\mathcal{Z}(S, P)$ as an affine $K$-scheme are derived from the formula for $P$, but they are in general quite different from $P$. For instance, the number of variables in $P$ is $r + 1$, whereas in general the polynomials describing $\mathcal{Z}(S, P)$ involve $\sum (\dim U_a)^2$ variables. Thus, when defining functions on the sets $\mathcal{Z}(S, P)$, one should not expect their images or fibers to again be sets of the form $\mathcal{Z}(S, P)$.

2.2. Fibers of restriction. We describe a general approach to study the zero-sets $\mathcal{Z}(S, P)$ for an arbitrary set of polynomials; eventually, we return to the cases with which we are concerned.

Our sequence starts by restricting the operator sets $\mathcal{Z}(S, P)$ to subsets of the frame. Fix $A \subset [t]$, and define the projection
\[
\Lambda_A : \mathcal{Z}(S, P) \to \prod_{a \in A} \text{End}(U_a), \quad \Lambda_A(\omega_b : b \in [t]) = (\omega_a : a \in A).
\]
Write $\mathcal{Z}(S, P)|_A$ for the image of $\Lambda_A$. As noted in Remark 2.2, $\mathcal{Z}(S, P)|_A$ need not be an operator set. However, the fibers of $\Lambda_A$ are still comprised of operators that satisfy the polynomials $P$ on tensors $S$. These we might describe as sets $\mathcal{Z}(S, Q)$ for various $Q$ related to $P$. Extending our notation slightly, for $\omega \in \mathcal{Z}(S, P)$, write
\[
\mathcal{Z}(S, P(\omega, X_A)) := \Lambda_A^{-1}(\omega) = \{ (\tau_A, \omega_A) \mid \langle S| P(\tau_A, \omega_A)| U \rangle = 0 \}.
\]
There are two problems. First, the formula depends on \( \omega_A \). This we can largely ignore, since fibers over generic points—those not lying on a proper subvariety—are invariant for a fixed irreducible component. Thus, we have a notion of generic fibers over the components of \( \mathcal{Z}(S, P) \) that is independent of \( \omega \). Secondly, \( P(\omega_A, X_A) \) is partially evaluated at linear operators and thus is no longer a polynomial in \( \mathbb{K}[X] \).

To get to an honest polynomial, it suffices to evaluate \( P \) at \( \omega_A = (\lambda_a I_{u_a} : a \in A) \) with \( \lambda_a \in \mathbb{K} \). If \( \omega_A \) is generic, its fibers are operator sets:

\[
P_A(X_A) = P(\lambda_A, X_A) \in \mathbb{K}[X_A].
\]

In this way, generic fibers are isomorphic to \( \mathcal{Z}(S, P_A) \) and can be regarded as an operator set like in (2.1). Indeed, we will confine ourselves to a case when \( \mathcal{Z}(S, P) \) is a group and \( \Lambda_A \) is a group homomorphism and so the choice of \( \lambda_a \in \{0,1\} \) will be appropriate. This offers a better opportunity to actually compute the sequences.

2.3. Polynomials defining groups or algebras. The next fact considers two specific polynomials, related to the derivation algebra and the automorphism group of \( S \). The proof follows from the definitions (1.4), (1.5), and (2.1).

Fact 2.3. Letting \( D(X) = x_1 + \cdots + x_1 - x_0 \), and \( G(X) = x_1 \cdots x_1 - x_0 \),

\[
\text{Der}(S) = \mathcal{Z}(S, D) \quad \text{Aut}(S) = \mathcal{Z}(S, G).
\]

Hereafter, we assume \( P \) is chosen so that \( \mathcal{Z}(S, P) \) is closed under one of two group operations: addition of endomorphisms or composition of automorphisms. In the latter case we still write \( \mathcal{Z}(S, P) \) but consider only invertible endomorphisms. Let \( \epsilon \) denote the appropriate identity (0 or 1) for \( \mathcal{Z}(S, P) \), which we represent naturally as constant in \( \mathbb{K} \). In [FMW] a characterization of such \( P \) is given; however, it follows easily from Fact 2.3 that both \( D(X) \) and \( G(X) \) have this property. Observe that, as a projection map, \( \Lambda_A \) is also a group homomorphism. Furthermore, the fibration we created from \( \Lambda_A \) has a generic fiber, since every fiber is a coset of the kernel. In particular, for each \( A \subseteq [1] \), we have an exact sequence

\[
\{\epsilon\} \longrightarrow \mathcal{Z}(S, P_A)|_A \longrightarrow \mathcal{Z}(S, P) \xrightarrow{\Lambda_A} \mathcal{Z}(S, P)|_A \longrightarrow \{\epsilon\}.
\]

Translated into the language of our introduction, we observe the origins of our replacements for inner derivations and inner automorphisms.

Fact 2.5. For \( \emptyset \neq A \subseteq [1] \), we have exact sequences:

\[
0 \longrightarrow \text{Der}_A(S) \longrightarrow \text{Der}(A) \xrightarrow{\Delta_A} \text{Der}(S)|_A \longrightarrow 0
\]

\[
0 \longrightarrow \text{Aut}_A(S) \longrightarrow \text{Aut}(A) \xrightarrow{\Gamma_A} \text{Aut}(S)|_A \longrightarrow 0.
\]

where \( \text{Der}_A(S) = \mathcal{Z}(S, D_A)|_A \) and \( \text{Aut}_A(S) = \mathcal{Z}(S, G_A)|_A \). (For emphasis, when specializing to derivations and autotopisms, shall replace the restriction maps \( \Lambda \) with \( \Delta \) and \( \Gamma \), respectively.)

2.4. Chains of derivations and automorphisms. We obtain a global outlook by summing over all restrictions to sets of a common cardinality. In this way we have one parameter to consider instead of exponentially many subsets.

Fact 2.6. For each \( k \in [1] \), there exists group homomorphisms \( \Lambda^k \) \((i = 1, 2)\) that make the following diagram commute, and ensure that \( \ker(\Lambda_1^k) = \text{im}(\Lambda_2^k) \).
A is a similar twisting to turn

\[ \prod_{A \in \{1\}^k} \mathcal{Z}(S, P_A) \xrightarrow{\Lambda^k} \mathcal{Z}(S, P) \xrightarrow{\Lambda^k} \prod_{A \in \{1\}^k} \mathcal{Z}(S, P) \]

Fact 2.6 follows from the exact sequences in (2.4). Using \( P = D \) we arrive at the following as a corollary.

**Fact 2.7.** There is an exact sequence

\[ \bigoplus_{A \in \{1\}^k} \text{Der}_A(S) \xrightarrow{\Delta^k} \text{Der}(S) \xrightarrow{\Delta^k} \prod_{A \in \{1\}^k} \text{Der}(S) \] \( \text{der}_A \)

In formulating a group analogue we face the problem that finite coproducts of groups are not isomorphic to products. This will prevent us from extending our sequence to the left in the case of \( P = G \). The solution to the problem is implied by Theorem B, where we swapped from groups to units in a ring.

**Lemma 2.8.** For \( 0 \leq a < b \leq 1 \) and \( C = [1] - \{a, b\} \), there is a natural isomorphism

\[ \mathcal{Z}(S, G_{ab})]_{ab} \to \mathcal{Nuc}_{ab}(S)^\times. \]

**Proof.** If \( 0 = a \) then \( G_{ab} = x_b - x_0 \) and the identity map provides the isomorphism. Otherwise \( 0 < a \) so \( G_{ab} = x_b x_a - 1 \). So if \( (\omega_a, \omega_b) \in \mathcal{Z}(S, G_{ab}) \) then \( (\omega_a^{-1}, \omega_b) \in \mathcal{Z}(S, x_b - x_a) \). This embedding is invertible.

By composing the inclusion \( \mathcal{Nuc}_{ab}(S)^\times \to \mathcal{Z}(S, G_{ab}) \) with \( \mathcal{Z}(S, G_{ab}) \to \mathcal{Z}(S, G) \) in the diagram of Fact 2.6 we can replace \( \prod \) with \( \bigoplus \) in the category of rings. Restricting to the units, we obtain the following.

**Fact 2.9.** There is an exact sequence

\[ \mathcal{Nuc}(S)^\times \xrightarrow{\Gamma_2} \text{Aut}(S) \xrightarrow{\Gamma_1} \prod_{A \in \{1\}^k} \text{Aut}(S) \] \( \text{der}_A \)

**Remark 2.10.** Note that for \( 0 < a < b \leq 1 \), \( \mathcal{Z}(S, x_a + x_b) = \text{Der}_{ab}(S) \) and is naturally isomorphic to \( \mathcal{Z}(S, x_a - x_b) = \mathcal{Nuc}_{ab}(S) \) as vector spaces. This implicit isomorphism explains how nuclei appear in Theorem A instead of derivations.

Hereafter, we treat all products as coproducts, and use superscripts \( \omega^A \), for \( A \subseteq [1] \), to record the factor from which an operator is taken. In this way we obtain explicit (additive) notation for the functions \( \Lambda^k \):

\[ \Lambda^k_2((\omega^A : a \in A) : A \in \binom{[1]}{k}) = \sum_{A \in \{1\}^k} (\epsilon_A, \omega^A). \]

Now our goal is to extend the sequence to an exact sequence ending in \( \{\epsilon\} \).

**2.5. Summary.** It is possible to state the exact sequences between the algebras and \( \text{Cen}_a(S) \) and \( \mathcal{Nuc}(S) \) by explicit formulas, and these will be given below. As noted, however, these maps are made for convenience of working with associative algebras when in fact that change in categories is unnatural. For instance, Fact 2.9 depends on a twisting of \( \mathcal{Nuc}(S) \), and Remark 2.10 is a similar twisting to turn
an associative ring into a Lie algebra. Indeed, Nuc(S) and Cen(S) only become the rings we seek after we restrict the operator sets \( \mathcal{Z}(S, P_\lambda) \) to \( \prod_{a \in A} \text{End}(U_a) \). Later we reveal an exponential number of seemingly arbitrary choices in signs that make the sequences. Even so, as we hope we have demonstrated in this section, there a clear, canonical picture being interpreted through these choices. As a final demonstration, before leaving this section we study a toy example.

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Figure 2.1. Geometric picture of the operator sets for the exact sequences when \( \gamma = 2 \).

![Figure 2.1](image-url)

Defining \( \langle t \rangle : \mathbb{K} \times \mathbb{K} \rightarrow \mathbb{K} \) with \( \langle t | u_2, u_1 \rangle = u_2 u_1 \), we have

\[
\begin{align*}
\text{Der}(t) &= \mathcal{Z}(t, x_a - x_b) = \{(\lambda_2, \lambda_1, \lambda_2 + \lambda_1)\} \cong \mathbb{K}^2 \\
\mathcal{Z}(t, x_a - x_b) &= \{(\lambda_2, \lambda_1, \lambda_0) \mid \lambda_a = \lambda_b\} \cong \mathbb{K}^2 \\
\text{Nuc}_{ab}(t) &= \mathcal{Z}(t, x_a - x_b)|_{ab} = \{(\lambda, \lambda) \in \mathbb{K} \times \mathbb{K}\} \cong \mathbb{K} \\
\text{Cen}(t) &= \{(\lambda, \lambda, \lambda)\} \cong \mathbb{K}.
\end{align*}
\]

(2.12)

Figure 2.1 illustrates the sets in the projective space \( \mathbb{P}^2 \). One can see from this figure and the data in (2.12) that the nuclei as rings are not the same as the operator sets, because of the restriction. Furthermore, the centroid does not exist inside the nuclei. But it does exist inside the full operator sets \( \mathcal{Z}(t, x_a - x_b) \). Similarly, the nuclei are not in general subsets of derivations (though in our picture because \( |\mathbb{K}| = 2 \) and \(-1 = +1\) we do not see the distinction). However, the full operator sets \( \mathcal{Z}(t, x_a - x_b) \) can be adjusted naturally to \( \mathcal{Z}(t, x_a + x_b) \) and those intersect nontrivially with Der(t) from the embedding we described.

We come to understand that for \( \gamma = 2 \) the sequence is prescribed as follows. Fix a projective line \( h \) in \( \mathbb{P}^2 \). Also fix points \( n_{ab} \), \( 0 \leq a < b \leq \gamma \) in general position on \( h \), and a point \( c \) not on \( h \). For example, if we use \( n_{10}, n_{21} \), and \( c \) to coordinatize the (affine) plane then \( h \) is given by a formula of the form \( \lambda_2 x_2 + \lambda_1 x_1 + \lambda_0 x_0 = 0 \) with each \( \lambda_a \neq 0 \). This parameterizes an operator set \( \mathcal{Z}(S, \lambda_2 x_2 + \lambda_1 x_1 + \lambda_0 x_0) \) on which we can attach a Lie algebra structure (though now the natural Lie brackets are weighted by the coefficients \( \lambda_a \)). We likewise use the lines through each \( n_{ab} \) and \( c \) to define the sets of nucleus type \( \mathcal{Z}(S, \mu^{ab}_a x_a - \mu^{ab}_b x_b) \), and the centroid lies in their intersection. After possibly deforming the scalars of the nuclei, their operator sets intersect the derivation set at the points \( n_{ab} \). We obtain the desired embedding of Cen(S) into Nuc(S) with cokernel embedded into Der(S).

All this occurs in generic terms and can be reasoned for nonlinear structures such as the operator sets of groups. Our specific interest in derivations and automorphisms are just two natural demonstrations of an otherwise general technique.
3. Exact sequences of groups and algebras

We specialize our discussion from Section 2 to the sequences in Theorems A and B. With our notation, we consider the case where \( P \in \{D, G\} \) and \( k = 2 \):

\[
\Lambda^2_2 : \bigoplus_{A \in \binom{\{\hat{1}\}}{2}} \mathfrak{Z}(S, PA)|_A \to \mathfrak{Z}(S, P).
\]

The function should also preserve further relevant structure such as being a Lie algebra homomorphism in the case of derivations and being a group homomorphism in the case of automorphisms. Recall, for the automorphisms we moved out of the category of groups into the category of rings. We return to groups through the units of these rings.

Recall from Fact 2.9 and Remark 2.10, we must adjust \( \Lambda^2_2 \) by twisting. To accomplish this, we define an auxiliary function \( \sigma \), whose input is a pair of subsets of \([1]\) differing by one element and whose output is \( \pm 1 \). For \( A = \{a, b\} \), we put \( C_A := x_a - x_b \), and remind the reader that subsets of size 1 are written without \( \{\} \).

Next, take \( \sigma \) and \( \Upsilon \):

Define \( \Upsilon^2 : \bigoplus_{A \in \binom{\{\hat{1}\}}{2}} \mathfrak{Z}(S, C_A) \to \mathfrak{Z}(S, P) \) by

\[
\bigoplus_{A \in \binom{\{\hat{1}\}}{2}} (\omega^A_a : a \in A) \mapsto \left( \sum_{\in [1]-a} \sigma(a, a \cup b) \cdot \omega^A_{a \cup b} : a \in [1] \right).
\]

For \( A \subseteq [1] \) of size at least two, let \( C_A = \left\{ C_B : B \in \binom{A}{2} \right\} \). Note, for \( 2 \leq k \leq \hat{1} + 1 \), if \( A \in \binom{[1]}{k} \), from (1.7) we have \( \mathfrak{Z}(S, C_A) = \text{Cen}_k(S) \). Hence,

\[
\bigoplus_{A \in \binom{[1]}{k}} \mathfrak{Z}(S, C_A) = \text{Cen}_k(S).
\]

For \( 2 \leq k \leq \hat{1} \), we define \( \Upsilon^{k+1} : \bigoplus_{A \in \binom{[1]}{k+1}} \mathfrak{Z}(S, C_A) \to \bigoplus_{B \in \binom{[k]}{k}} \mathfrak{Z}(S, C_B) \) by

\[
\bigoplus_{A \in \binom{[1]}{k+1}} (\omega^A_a : a \in A) \mapsto \bigoplus_{B \in \binom{[k]}{k}} \left( \sum_{a \notin B} \sigma(B, B \cup a) \cdot \omega^B_{b \cup a} : b \in B \right).
\]

It remains to determine the conditions on \( \sigma \) that ensure the functions \( \Upsilon^k \) are well-defined and exact. We first deal with well-definedness.

**Lemma 3.1.** For \( 3 \leq k \leq \hat{1} + 1 \), the maps \( \Upsilon^k \) are well-defined. The map \( \Upsilon^2 \) is well-defined if, and only if,

\[
\sigma(a, a \cup b) = \begin{cases} 
\sigma(b, a \cup b) & 0 = a < b \leq \hat{1}, \\
-\sigma(b, a \cup b) & 0 < a < b \leq \hat{1}.
\end{cases}
\]

**Proof.** Suppose that \( P \in \{D, G\} \). First, let \( A = \{a, b\} \). Then

\[
(\epsilon_A, \sigma(a, A) \cdot \omega^A_a, \sigma(b, A) \cdot \omega^B_b) \in \mathfrak{Z}(S, P),
\]

if, and only if, (3.2) holds (Fact 2.6 and Lemma 2.8).

Next, take \( A \subseteq [1] \) of size \( k \geq 3 \). If \( \omega = (\omega^A_a : a \in A) \in \mathfrak{Z}(S, C_A) \), then

\[
\Upsilon^k(\omega) = \bigoplus_{a \in A} (\sigma(A - a, A) \cdot \omega^B_b : b \in A - a)
\]

Thus, each summand is contained in \( \mathfrak{Z}(S, C_{A-a}) \). \( \square \)
For the next two lemmas, we assume that \( \sigma \) satisfies (3.2) so that the maps \( \Upsilon^k \) are well-defined for \( 2 \leq k \leq \gamma + 1 \).

**Lemma 3.3.** Let \( S \) be a fully nondegenerate tensor space. For all \( 3 \leq k \leq \gamma + 1 \), \( \Upsilon^k \) is a homomorphism, and \( \Upsilon^2 \) is a homomorphism if, and only if,

\[
(\forall 0 \leq a < b \leq \gamma) \quad \sigma(a, a \cup b) = \begin{cases} 1 & a = 0, \\ -1 & a > 0. \end{cases}
\]

**Proof.** If \( S \) is fully nondegenerate, then for \( 3 \leq k \leq \gamma + 1 \) and \( A \in \binom{[k]}{1} \), \( \mathfrak{Z}(S, P_A) \) is a commutative ring, and so has trivial Lie bracket and abelian unit group.

We may therefore assume \( k = 2 \). We give a proof for derivations (where \( P = D \)); by Lemma 2.8 the proof for autopisms is similar. First, consider \( A = \{0, b\} \). By Lemma 3.1, \( \sigma(0, A) = \sigma(b, A) \). Recalling that \( \text{Nuc}_{ab}(S) \subseteq \text{End}(U_b) \times \text{End}(U_0) \), \( \Upsilon^k_A \) is a homomorphism if, and only if, \( \sigma(0, A) = \sigma(0, A)^2 \). Next, if \( 0 < a < b \), then \( \text{Nuc}_{ab}(S) \subseteq \text{End}(U_a)^{op} \times \text{End}(U_b) \). As the \( a \)-coordinate is contained in the opposite ring, \( \Upsilon^k_A \) is a homomorphism if, and only if, \( \sigma(a, A) = -\sigma(a, A)^2 \).

The notation for the calculations required to prove the exactness of these maps is simplified if, for \( C \subset \{1\} \) of order at most \( \gamma - 1 \) and \( a, b \in C \), we set

\[
(3.4) \quad \tau(C, a, b) = \sigma(C, C \cup a)\sigma(C \cup a, C \cup \{a, b\}) + \sigma(C, C \cup b)\sigma(C \cup b, C \cup \{a, b\}).
\]

We determine the properties of \( \sigma \) that guarantees the maps \( \Upsilon^k \) are a chain complex. This turns out to be enough to also ensure exactness.

**Lemma 3.5.** Fix \( 2 \leq k \leq \gamma \). Then \( \Upsilon^k \circ \Upsilon^{k+1} = \epsilon \) if, and only if, for all \( C \in \binom{[k]}{k-1} \) and distinct \( a, b \in C \), \( \tau(C, a, b) = 0 \).

**Proof.** For \( 2 \leq k \leq \gamma \),

\[
\Upsilon^k \circ \Upsilon^{k+1} \left( \bigoplus_{A \in \binom{[k]}{k+1}} (\omega^A_a : a \in A) \right)
= \bigoplus_{C \in \binom{[k]}{k-1}} \left( \sum_{a \notin C} \sigma(C, C \cup b) \sum_{a \notin C \cup b} \sigma(C \cup b, C \cup \{a, b\}) \omega^{C \cup \{a, b\}}_c : c \in C \right)
= \bigoplus_{C \in \binom{[k]}{k-1}} \left( \sum_{(a, b) \in \binom{[1]}{2}} \tau(C, a, b) \omega^{C \cup \{a, b\}}_c : c \in C \right). \quad \square
\]

To prove exactness, we express \( \Upsilon^k \) as a matrix with entries in \( \{-1, 0, 1\} \). The nonzero entries are determined by the function \( \sigma \). Our approach is to derive a suitable transition matrix \( M_k \) and study instead \( \Upsilon^k M_k \).

**Lemma 3.6.** Fix \( 2 \leq k \leq \gamma \). If \( \Upsilon^k \circ \Upsilon^{k+1} = \epsilon \), then \( \ker(\Upsilon^k) = \im(\Upsilon^{k+1}) \).

**Proof.** First, we arrange the components of \( \bigoplus_{A \in \binom{[k]}{k+1}} \mathfrak{Z}(S, C_A) \) to arrange for the first summand to be over \([1]\) and thus organize together those operators that act on the same term in the frame.

\[
\bigoplus_{A \in \binom{[k]}{k+1}} (\omega^A_a : a \in A) \mapsto \bigoplus_{a \in [1]} \left( \omega^{B \cup a}_a : B \in \binom{[1] - a}{k} \right).
\]
Reordering $\mathcal{Y}^2$ into these coordinates we define $\mathcal{Y}^{k+1}$ in terms of its restrictions $\mathcal{Y}^{k+1}_a$ to each $\text{End}(U_a)$, as follows:

\[
\mathcal{Y}^{k+1} \left( \bigoplus_{A \in \{0,1\}} (\omega^A_a : a \in A) \right) = \bigoplus_{a \in [1]} \mathcal{Y}^{k+1}_a \left( \omega^B_{aB} : B \in \binom{[1]}{k} - a \right).
\]

Secondly, we wish to use a matrix to describe each $\mathcal{Y}^{k+1}_a$ as it leads in a natural way to the concept of echelonizing, which will identify the image of each map. Rather than work with a block diagonal matrix $\mathcal{Y}^{k+1} = \bigoplus_a \mathcal{Y}^{k+1}_a$, we instead fix $a \in [1]$ and focus on the matrix for just that coordinate. Thus, for each $k$ we define a $\{-1, 0, 1\}$-valued matrix $M_{k+1}$ whose rows range over $\binom{[1]}{k} - a$, and whose columns range over $\binom{[1]}{k-1}$. Fixing an order (e.g. lexicographic) on subsets of $[1] - a$, define

\[
\begin{bmatrix}
M_{k+1}^{(a)}
\end{bmatrix}_{AB} = \begin{cases}
\sigma(B, A) & B \subseteq A, \\
0 & B \nsubseteq A.
\end{cases}
\]

For $b = \min[1] - a$, we observe the following partition of $M_{k+1}^{(a)}$:

\[
M_{k+1}^{(a)} = \begin{cases}
b \in A & B \subseteq A \\
b \not\in A & B \nsubseteq A
\end{cases}
\begin{bmatrix}
Y_{k+1} & Z_{k+1} \\
0 & X_{k+1}
\end{bmatrix}, \quad Z_{k+1} = \bigoplus \sigma(B, B \cup a).
\]

The block of $0$’s in the lower left follows from the fact that those rows and columns are indexed by subsets $A$ and $B$ for which $B \nsubseteq A$. Similarly in the upper right block $B \subseteq A$ at exactly the row $A = B \cup a$ and column $B$, for our fixed $a$. As $\sigma(B, B \cup a) \in \{-1, 1\}$, it follows that $Z_{k+1}$ is diagonal, invertible, and of order at most 2. Letting $Z_1 = 1$ for the base case, we now claim that for each $k$,

\[
Y_{k+1} = -Z_{k+1}X_k Z_k.
\]

Indeed, as $Z_{k+1}$ is an involution we can rewrite this identity equivalently as

\[
Z_{k+1}X_k + Y_{k+1}Z_k = 0.
\]

Focusing just on non-zero coordinates, condition (3.8) translates precisely to the condition $\tau(C, c, d) = 0$, as in (3.4). Thus, by Lemma 3.5, equation (3.7) holds.

Having identified the structure of the matrices, for each $a \in [1]$ we can echelonize the coordinates of the transform $\mathcal{Y}^{k+1}_a$ to identify the symbolic rank of each matrix. This will ensure that the image equals the kernel. To simplify notation for inverses, we put $N_k = M_{k}^{(a)}$, and compute

\[
N^{-1}_a \mathcal{Y}_a^{k+1} = \begin{bmatrix}
Z_2 \\
-X_2 Z_2 \\
0 \end{bmatrix} \begin{bmatrix}
Z_2 \\
X_2 \\
0
\end{bmatrix} = \begin{bmatrix} 1 \\
0 \\
0 \end{bmatrix}
\]

\[
\vdots
\]

\[
N^{-1}_a \mathcal{Y}_a^{k+1} N_k = \begin{bmatrix}
Z_{k+1} \\
-X_{k+1} Z_{k+1} \\
0 \end{bmatrix} \begin{bmatrix}
Z_{k+1} \\
X_{k+1} \\
0
\end{bmatrix} = \begin{bmatrix} 1 \\
0 \\
0 \end{bmatrix}
\]

\[
\vdots
\]

\[
\mathcal{Y}_a^{k+1} N_1 = \begin{bmatrix}
Z_1 \\
Z_1 \left( X_1 - 1 \right) Z_1 - 1 \\
Z_1 \\
Z_1 \left( X_1 - 1 \right) Z_1 - 1 \\
0
\end{bmatrix} = \begin{bmatrix} 1 \\
0 \end{bmatrix}
\]
Evidently the image (column span) of each matrix on the right is the kernel (right null space) of the one below it. Since each row-column operation is carried out by unimodular transform (a permutation or transvection with \( \pm 1 \)-values) it is possible to carry out these operations symbolically on our coordinates. Thus, elements in the kernel of \( \Upsilon_a^k \) can be used to write elements in the image of \( \Upsilon_a^{k+1} \).

To summarize, we require three properties of \( \sigma \) that ensure the homomorphisms \( \Upsilon^* \) form an exact sequence. First, by Lemma 3.1, we require

\[
\sigma(a, a \cup b) = \begin{cases} 
\sigma(b, a \cup b) & 0 = a < b \leq 1, \\
-\sigma(b, a \cup b) & 0 < a < b \leq 1,
\end{cases}
\]

so that the maps are well-defined. Secondly, from Lemma 3.3 we require

\[
\sigma(a, a \cup b) = \begin{cases} 
1 & 0 = a < b \leq 1, \\
-1 & 0 < a < b \leq 1,
\end{cases}
\]

to ensure the homomorphism property. Finally, by Lemmas 3.5 and 3.6 we require that for all \( C \in [\ell] \) with order at most \( \ell - 1 \), and for all distinct \( a, b \notin C \),

\[
\sigma(C, C \cup a)\sigma(C \cup a, C \cup \{a, b\}) + \sigma(C, C \cup b)\sigma(C \cup b, C \cup \{a, b\}) = 0.
\]

4. Directed graphs

The goal of this section is to prove the existence of a function \( \sigma \), as in Section 3, which we accomplish using a directed graph.

Let \( G \) be a graph whose vertices are subsets of \([\ell]\). Two vertices \( A, B \subseteq [\ell] \) are adjacent if there exists \( b \notin A \) such that \( A \cup b = B \). Our objective is to define an orientation on \( G \), namely to assign a direction to each edge in \( G \) that encodes the nonzero values of \( \sigma \). A directed edge \( C \cup a \to C \) can be thought of as “down” in the underlying poset and carries a value of 1, while \( C \to C \cup a \) is “up” and carries the value \(-1\).

To state the key result, we introduce some convenient terminology. The edges from \( A \subseteq [\ell - 1] \) to \( A \cup 1 \) are called controlling edges. For distinct \( a, b \in [\ell] \) and \( C \subseteq [\ell] - \{a, b\} \), denote by \( D(C, a, b) \) the subgraph of \( G \) induced on the four vertices labeled by \( C, C \cup a, C \cup b, \) and \( C \cup \{a, b\} \). We refer to subgraphs \( D(C, a, b) \) as diamonds of \( G \). We say an orientation on \( G \) is oddly acyclic if every diamond of \( G \) is acyclic with a path of length 3. For consistency, we say that every orientation on \( G_0 \) is (vacuously) oddly acyclic.

Our first result collects the diamonds of \( G \) into three buckets. For \( A \subseteq [\ell] \), we say a diamond \( D \) is contained in \( 2^A \) if the vertex labels of \( D \) are contained in \( A \).

**Lemma 4.1.** For \( \ell \geq 1 \), every diamond of \( G \) is either contained in \( 2^{[\ell - 1]} \), contained in \( 2^{[\ell]} - 2^{[\ell - 1]} \), or contains exactly two controlling edges. This partitions the set of diamonds of \( G \).

**Proof.** No diamond in \( 2^{[\ell - 1]} \) or \( 2^{[\ell]} - 2^{[\ell - 1]} \) contains a controlling edge, so the three sets are disjoint. Suppose \( D \) is a diamond not contained in \( 2^{[\ell - 1]} \) or \( 2^{[\ell]} - 2^{[\ell - 1]} \). Because \( 2^{[\ell]} \) and \( 2^{[\ell]} - 2^{[\ell - 1]} \) are subset-closed, it follows that \( D \) has two vertices with labels in \( 2^{[\ell]} \) and two vertices with labels in \( 2^{[\ell]} - 2^{[\ell - 1]} \). \( \square \)

The following is the key result in this section.

**Lemma 4.2** (Rihanna’s Lemma). Suppose \( \ell \geq 1 \). For every oddly acyclic orientation on the subgraph \( G_{\ell - 1} \) of \( G \), and for every orientation on the controlling edges of \( G_{\ell} \), there exists a unique induced oddly acyclic orientation for \( G \).
Proof. Diamonds contained in $2^{[i]}$ are green diamonds, and diamonds contained in $2^{[i]} - 2^{[i-1]}$ are yellow diamonds. Call a diamond controlling if it contains two controlling edges. Observe that every noncontrolling edge of $G$ lies in a unique controlling diamond.

First, we will construct an orientation induced from the orientation $s$ on $G$ and the controlling edges of $G$. For every $a \in [i-1]$ and $C \subseteq [i-1] - a$, the (yellow) edge incident to $C \cup \{a, 1\}$ and $C \cup \{1\}$ lies in a unique controlling diamond $D(C, a, 1)$, call it $y$. From the proof of Lemma 4.1, the orientation of exactly three edges of $D(C, a, 1)$ is determined by the oddly acyclic orientation on $G$ and the orientation on the controlling edges of $G$. Therefore, there is a unique choice of orientation of $y$ such that $D(C, a, 1)$ is oddly acyclic. Do this for all yellow edges, and suppose $G$ has this orientation.

By Lemma 4.1 we need to show that every yellow diamond is oddly acyclic. Such a diamond has the form $D = D(C, a, b)$ where $\{1\} \subseteq C \subseteq [i]$ and $a, b \in [i-1]$ with $a \neq b$. Let $D' = D(C - 1, a, b)$, the corresponding green diamond. There are exactly four controlling edges incident to both $D$ and $D'$. Two controlling edges incident to the same edges, $y$ in $D$ and $g$ in $D'$, point in the same direction if, and only if, the edges $y$ and $g$ point in opposite directions. This implies that $D$ is oddly acyclic if, and only if, $D'$ is oddly acyclic. Therefore, $G$ is oddly acyclic. □

We illustrate Rihanna’s Lemma in Figure 4.1 for $i = 2$. The orientations for the green diamond and the controlling edges have been given, and by Rihanna’s Lemma there is a unique choice of orientation for the yellow diamond so that $G_2$ is oddly acyclic.

We now assert the existence of a suitable function $\sigma$ to use in our exact sequences. An orientation on $G$, is positively swapped if the subgraph induced on the vertices labeled $\{a, b\}$, $\{a\}$, and $\{b\}$ is a path of length 2 if, and only if, $0 < a < b \leq i$.

Lemma 4.3. For every $i \geq 1$, there exists an oddly acyclic and positively swapped orientation for $G_i$ such that for every $A \in \binom{[i]}{2}$, there is a directed edge from $A$ to $\max(A)$.

Proof. We prove this by induction. From Rihanna’s Lemma, there are 8 oddly acyclic orientations for $G_1$, exactly 4 of those positively swapped. Of the 4 orientations, exactly 2 have a directed edge from $\{0, 1\}$ to $\{1\}$.
Now suppose the subgraph $\mathcal{G}_{\mathcal{G}}$ is oddly acyclic and positively swapped such that for every $A \in \{0, 1\}$, there is a directed edge from $A$ to $\max(A)$. The only choice we impose to get positively swapped is that the controlling edge between $\{1\}$ and $\emptyset$ points in a direction such that the subgraph on $\{0\}, \{1\}$, and $\emptyset$ is a path of length 2. By induction, the subgraph on $\{0\}, \{a\}$, and $\emptyset$ is a path of length 2, for all $0 < a < 1$. Therefore, regardless of the choice of orientation for the remaining controlling edges, the resulting orientation from Rihanna’s Lemma will be positively swapped. To ensure the last property, we choose the orientation for the controlling edges so that the edge points from $\{a, 1\}$ to $\{1\}$, for all $a \in [1 - 1]$.

5. Proof of Theorems A and B

From our work in Section 3, both theorems follow easily from the following.

**Lemma 5.1.** For $\gamma \geq 1$, there exists a function $\sigma$ such that

(i) for all $A \subset [\gamma]$ and $b \notin A$,

$$\sigma(A, A \cup b) = \pm 1,$$

(ii) for all $0 \leq a < b \leq \gamma$,

$$\sigma(a, a \cup b) = \begin{cases} \sigma(b, a \cup b) & 0 = a, \\ -\sigma(b, a \cup b) & 0 < a, \end{cases} \quad \sigma(a, a \cup b) = \begin{cases} 1 & 0 = a, \\ -1 & 0 < a, \end{cases}$$

(iii) for all $C \in ([\gamma])$ and distinct $a, b \notin C$,

$$\sigma(C, C \cup a)\sigma(C \cup a, C \cup \{a, b\}) + \sigma(C, C \cup b)\sigma(C \cup b, C \cup \{a, b\}) = 0.$$

**Proof.** Translating the directions on edges in an oriented graph $\mathcal{G}$ to values $\pm 1$—as described at the start of Section 4—each such graph encodes some function $\sigma$ with the correct domain and range in property (i). By Lemma 4.3, the orientation on $\mathcal{G}$ can be chosen so that properties (ii) and (iii) hold.

**Proof of Theorems A and B.** By Lemmas 3.1, 3.3, 3.6, 5.1, and full nondegeneracy the homomorphisms $\Upsilon^k$ form an exact sequence. From Fact 2.3, if $P \in \{D, G\}$, then the zero set $\mathcal{S}(S, P)$ is either $\text{Der}(S)$ or $\text{Aut}(S)$. In the case $P = G$, apply Lemma 2.8 to obtain the sequence in Theorem B from the zero-sets. □

6. Applications & Examples

The techniques developed in Sections 2 and 3 to prove Theorems A and B are new, and it is likely not clear to the reader how they can be used to study tensors, nor how they might be applied in more general settings. We therefore devote this final section to a range of examples and applications that illustrate the scope and potential of these new tools.

In a forthcoming paper [BMW], the authors use the exact sequences established in this paper as one half of a two-pronged attack on the group $\text{Aut}(t_G)$ for the tensor $t_G$ of a $p$-group $G$. The sequences provide a systematic way to build up the group $\text{Aut}(t_G)$ iteratively from successively easier-to-construct pieces. The second (complementary) part of the attack chisels down towards $\text{Aut}(t_G)$ by first building the normalizer of the derivation algebra $\text{Der}(t_G)$. This requires new tools to compute with representations of Lie algebras. By simultaneously building up and cutting down toward the target group we often achieve vastly improved results.
6.1. **Entangled quantum states.** We demonstrate that our sequences in Theorem A give a basis-free description of quantum entangled states. In particular we recreate a discovery of Dür-Vidal-Cirac [D] that creates two maximally entangled states on 3 qubits that are inequivalent. Our derivations based sequences provide a basis free characterization. Here are the principal details.

First, suppose $\mathbb{H}$ is an 8-dimensional Hilbert space with $|\cdot|: \mathbb{H} \to \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ (more commonly represented as $(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) \otimes \mathbb{C}$ - the dual space to $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$). In the usual convention $\mathbb{C}^2 \cong \mathbb{C} \times \mathbb{C}$, and so a basis for $\mathbb{H}$ is $|a,b,c\rangle$ with $a, b, c \in \{0, 1\}$. Note that $\tau = 2$, so there is only one sequence to construct.

Our first state is the Greenberger-Horne-Zeilinger state:

$$\langle GHZ \rangle = \frac{\sqrt{2}}{2} (|000\rangle + |111\rangle).$$

In this case, $\text{Cent}(GHZ) \cong \mathbb{C}^2$, and the sequence is

$$0 \longrightarrow \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow 0.$$

Next we have the $W$ state from [D]:

$$\langle W \rangle = \frac{\sqrt{3}}{3} (|000\rangle + |010\rangle + |001\rangle).$$

It follows that $\text{Cent}(W) \cong \mathbb{C}[x]/(x^2)$. Even though the centroid is also 2-dimensional like the previous tensor, the algebra is fundamentally different—in particular, it has a nontrivial Jacobson radical. The corresponding sequence for $W$ contains nontrivial outer derivations, and with $A = \mathbb{C}[x]/(x^2)$, the sequence is

$$0 \longrightarrow A \longrightarrow A \oplus A \oplus A \longrightarrow C \oplus A \oplus A \longrightarrow C \longrightarrow 0.$$

We note that all 2-dimensional algebras take the form $\mathbb{C}[x]/(ax^2 + bx + c)$ and the two separate examples demonstrate the two generic possibilities: either $ax^2 + bx + c$ has a repeated root (the $W$ case), or it has distinct roots (the $GHZ$ case).

6.2. **Composition and Matrix products.** This example uses the familiar concept of tensor contraction (also known as hyper-matrix multiplication), which for convenience we model as a special case of composition in a module category.

Recall in our notation $U \otimes V = \text{hom}(U, V)$, which is again a module. Hence, composition of functions in the $K$-module category is a bilinear map

$$\circ : C \otimes B \times B \otimes A \to C \otimes A.$$

If $A = K^a$, $B = K^b$, and $C = K^c$, we can, after fixing bases, identify composition with the matrix multiplication tensor

$$\text{M}_{c \times b}(K) \times \text{M}_{b \times a}(K) \to \text{M}_{c \times a}(K).$$

The matrix multiplication tensor has been studied extensively and its derivations and automorphisms are known. Thus, it is a good example to illustrate our methods, as they provide another means to see the structure.

Within composition we have three self-evident contributions to the nuclei:

- $\text{Der}^{10}(\circ) = \text{End}(A)$
- $\text{Aut}^{10}(\circ) = \Gamma L_K(A)$
- $\text{Der}^{21}(\circ) = \text{End}(B)$
- $\text{Aut}^{21}(\circ) = \Gamma L_K(B)$
- $\text{Der}^{20}(\circ) = \text{End}(C)$
- $\text{Aut}^{20}(\circ) = \Gamma L_K(C)$
Therefore, while $B$ does not occur in the codomain, its influence in the middle of the domain can be identified by local derivations. This may not seem surprising given how we introduced the product, but such a tensor could be given as a black-box. Then the product would take the form $K^{ab} \times K^{bc} \to K^{ac}$, and that is a completely ambiguous decomposition. We consider a specific example.

**Example 6.1.** If $t : \mathbb{M}_{2 \times 3}(\mathbb{C}) \times \mathbb{M}_{3 \times 4}(\mathbb{C}) \to \mathbb{M}_{2 \times 4}(\mathbb{C})$, then $\text{Der}_t^2(t) = \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_3(\mathbb{C}) \oplus \mathbb{M}_4(\mathbb{C})$, and $\text{Der}_t^2(t) = \mathbb{C}$. Therefore, the sequences in Theorem A and Theorem B have the form

$$0 \to \mathbb{C} \to \mathbb{M}_2 \oplus \mathbb{M}_3 \oplus \mathbb{M}_4 \to \mathbb{C}^2 \times (\mathfrak{sl}_2 \times \mathfrak{sl}_3 \times \mathfrak{sl}_4) \to 0,$$

$$1 \to \mathbb{C}^\times \to \text{GL}_2 \times \text{GL}_3 \times \text{GL}_4 \to \text{SL}_3 \times (\text{GL}_2 \times \text{GL}_4) \to 1.$$

In other words, all automorphisms of tensors given by matrix multiplication are “inner,” in the sense that they are realized as the groups of units of the various nuclei.

### 6.3. Strictly non-associative tensor decompositions.

Tensor contraction is expressible in other ways partly because it relates to our familiar associative matrix multiplication. However, tensors can be constructed in substantially non-associative ways; one need only consider products of Lie algebras to witness such cases. Our next examples demonstrate how we detect nonassociative components of a tensor.

**Example 6.2.** Let $K$ be a field such that $2K = K$. Let $\langle t \rangle : K^4 \times K^4 \times K^4 \to \Lambda^3 K^4$ be given by the exterior cube of $K^4$. Also, let $\langle s \rangle : K \times \mathfrak{sl}_2 \times \mathfrak{sl}_2 \to \mathfrak{sl}_2$ be the tensor that maps $(k, X, Y)$ to $k[X, Y]$. Define the tensor product (over $K$) of $K$-tensors $t$ and $s$ as $\langle t \otimes s \rangle : (K^4 \otimes K) \times (K^4 \otimes \mathfrak{sl}_2) \times (K^4 \otimes \mathfrak{sl}_2) \to K^4 \otimes \mathfrak{sl}_2$, where

$$\langle t \otimes s | u \otimes v \rangle = \langle t | u \rangle \otimes \langle s | v \rangle.$$

A calculation shows that $\text{Der}(t) \cong K^3 \otimes \mathfrak{sl}_4(K)$. For all other $A \subset [3]$, with $|A| \geq 2$, $\text{Der}_A(t) = K^{|A| - 1}$. For $s$, on the other hand, $\text{Der}_{\{0,1,2\}}(s) \cong K^2 \otimes \mathfrak{sl}_3(K)$ and $\text{Der}(s) = K \times \text{Der}_{\{0,1,2\}}(s)$. For all other $A \subset [3]$, $\text{Der}_A(s) = K^{|A| - 1}$. These derivation algebras are detected in $t \otimes s$, and in fact, there are no larger algebras:

$$0 \to K \to K^4 \to K^6 \to K \oplus \mathfrak{gl}_3 \oplus \mathfrak{gl}_4 \to \mathfrak{sl}_3 \oplus \mathfrak{sl}_4 \to 0.$$

**Example 6.3.** Let $K$ be a field with degree 2 and 3 extensions denoted by $E$ and $F$ respectively. Let $t$ and $s$ be the dot products on $E^2$ and $F^2$, respectively. We concatenate a 1-dimensional coordinate (over $K$) to both $t$ and $s$, making them $K$-trilinear: $\langle t' \rangle : K \times K^4 \times K^4 \to K^2$ and $\langle s' \rangle : K^6 \times K^6 \times K \to K^3$. Let $r = t' \otimes s'$. Instead of writing out the sequence like in the previous examples, we display the dimension of every algebra over $K$ in Figure 6.1. While $r$ is only $K$-trilinear (the centroid of $r$ is isomorphic to $K$), the local centroids detect the fact that $r$ was built from tensors that are bi-linear over extensions, namely degree 2 and 3 extensions.

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Figure 6.1. A graphical description of the sequence in Theorem A. Here, we have separated the direct summands of the terms of the sequence, and we are only displaying their dimensions over $K$. The sequence starts at the bottom and goes to the top, with the last nontrivial term being the 26-dimensional derivation algebra. The vertical sequence on the left aligns with the dimensions of the direct summands—in lex-least order.

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