\( \hat{Z} \)-Invariant for \( SO(3) \) and \( OSp(1|2) \) Groups

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Abstract. Three-manifold invariants \( \hat{Z} \) ("Z-hat"), also known as homological blocks, are \( q \)-series with integer coefficients. Explicit \( q \)-series form for \( \hat{Z} \) is known for \( SU(2) \) group, supergroup \( SU(2|1) \) and orthosymplectic supergroup \( OSp(2|2) \). We focus on \( \hat{Z} \) for \( SO(3) \) group and orthosymplectic supergroup \( OSp(1|2) \) in this paper. Particularly, the change of variable relating \( SU(2) \) link invariants to the \( SO(3) \) and \( OSp(1|2) \) link invariants plays a crucial role in explicitly writing the \( q \)-series.

1. Introduction

Knot theory has attracted attention from both mathematicians and physicists during the last 40 years. The seminal work of Witten [25] giving a three-dimensional definition for Jones polynomials of knots and links, using \( SU(2) \) Chern–Simons theory on \( S^3 \), triggered a tower of new colored link invariants. Such new invariants are given by expectation value of Wilson loops carrying higher dimensional representation \( R \in \mathcal{G} \) in Chern–Simons theory where \( \mathcal{G} \) denotes gauge group. These link invariants are in variable \( q \) which depends on the rank of the gauge group \( \mathcal{G} \) and the Chern–Simons coupling constant \( k \in \mathbb{Z} \) (For eg: when \( \mathcal{G} = SU(N) \) then \( q = \exp \left( \frac{2\pi i}{k+N} \right) \). Witten’s approach also gives three-manifold invariant \( Z_k^G[M; q] \), called Chern–Simons partition function for manifold \( M \), obtained from surgery of framed links on \( S^3 \) (Lickorish–Wallace theorem [17, 23]). Witten–Reshetikhin–Turaev (WRT) invariants \( \tau_k^G[M; q] \) known in the mathematics literature are proportional to the Chern–Simons partition function:
These WRT invariants can be written in terms of the colored invariants of framed links [14,17,22,23].

It was puzzling observation that the colored knot polynomials appear as Laurent series with integer coefficients. There must be an underlying topological interpretation of such integer coefficients. This question was answered both from mathematics and physics perspective. Initial work of Khovanov [15] titled ‘categorification’ followed by other papers on bi-graded homology theory including Khovanov–Rozansky homology led to new homological invariants. Thus, the integer coefficients of the colored knot polynomials are interpreted as the dimensions of vector space of homological theory. From topological strings and intersecting branes [8,9,20], the integers of HOMFLY-PT polynomials are interpretable as counting of BPS states. Further, the connections to knot homologies within topological string context were initiated in [13] resulting in concrete predictions of homological invariants for some knots (see review [19] and references therein). Such a physics approach involving brane set up in $M$-theory [6,11,12,18] suggests the plausibility of categorification of WRT invariants $\tau^G_k[M; q]$ for three-manifolds. However, the WRT invariants for simple three-manifolds are not a Laurent series with integer coefficients.

The detailed discussion on $U(N)$ Chern–Simons partition function on Lens space $L(p, 1) \equiv S^3/\mathbb{Z}_p$ (see section 6 of [12]) shows a basis transformation $Z^G_k[M; q] \rightarrow \hat{Z}^G_k[M; q]$ so that $\hat{Z}$ are $q$-series(where variable $q$ is an arbitrary complex number inside a unit disk) with integer coefficients (GPPV conjecture [11]). These $\hat{Z}$ are called the homological blocks of WRT invariants of three-manifolds $M$. Physically, the new three-manifold invariants $\hat{Z}^G_k[M; q]$ are the partition function $Z^T_G[M][D^2 \times S^1]$ for simple Lie groups. Here, $T^G_k[M]$ denote the effective 3d $\mathcal{N} = 2$ theory on $D^2 \times S^1$ obtained by reducing 6d $(2, 0)$ theory (describing dynamics of coincident $M5$ branes) on $M$.

For a class of negative definite plumbed three-manifolds as well as link complements [2,10,11,21], $\hat{Z}^{SU(N)}$ has been calculated. Further, $\hat{Z}$ invariants for super unitary group $SU(n|m)$ supergroup with explicit $q$-series for $SU(2|1)$ are presented in [6]. Generalization to orthosymplectic supergroup $OSp(2|2n)$ with explicit $q$-series for $Osp(2|2)$ [1] motivates us to look at $\hat{Z}$ for other gauge groups.

Our goal in this paper is to extract $\hat{Z}$ for the simplest orthogonal group $SO(3)$ and the simplest odd orthosymplectic supergroup $OSp(1|2)$. We take the route of relating $SU(2)$ colored link invariants to the link invariants for these two groups to obtain $\hat{Z}$ invariants.

The plan of the paper is organized as follows. In Sect. 2, we will review the developments on the invariants of knots, links and three-manifolds. We will first briefly present Chern–Simons theory and colored link invariants with explicit results for $SU(2)$ gauge group and indicate how colored $SO(3)$ and $OSp(1|2)$ link invariants can be obtained from the colored $SU(2)$ polynomials. Then, we will summarize the developments of the homological invariants. In
Sect. 3, we briefly review Z-series invariant for SU(2) group for the negative definite plumbed three-manifolds. This will serve as a warmup to extend to SO(3) and OSP(1|2) group which we will present in Sect. 4. We summarize the results and also indicate future directions to pursue in the concluding Sect. 5.

2. Knots, Links and Three-Manifold Invariants

In this section, we will briefly summarize new invariants in knot theory from the physics approach as well as from the mathematics approach.

2.1. Chern–Simons Field Theory Invariants

Chern–Simons theory based on gauge group G is a Schwarz type topological field theory which provides a natural framework for study of knots, links and three-manifolds M. Chern–Simons action $S_{CS}^G(A)$ is explicitly metric independent:

$$S_{CS}^G(A) = \frac{k}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

(2)

Here, A is the matrix valued gauge connection based on gauge group G and $k \in \mathbb{Z}$ is the coupling constant. In fact, the expectation value of Wilson loop operators associated with any m-component link $L_m$ is the link invariants:

$$V_{R_1, R_2, \ldots, R_m}^G[L_m; \mathcal{L}_m] = \langle W_{R_1, R_2, \ldots, R_m}[L_m] \rangle$$

$$= \int D A \exp(iS_{CS}) P \left( \prod_i \text{Tr}_{R_i} \exp \oint_{K_i} A \right) \frac{\int D A \exp(iS_{CS})}{Z^G_{[M; q]}},$$

(3)

where $K_i$’s denote the component knots of link $L_m$ carrying representations $R_i$’s of gauge group G and $Z^G_{[M; q]}$ defines the Chern–Simons partition function encoding the topology of the three-manifold M.

Exploiting the connection between Chern–Simons theory, based on group G, and the corresponding Wess–Zumino–Witten (WZW) conformal field theory with the affine Lie algebra symmetry $\mathfrak{g}_k$, the invariants of these links embedded in a three-sphere $M = S^3$ can be explicitly written in variable q:

$$q = \exp \left( \frac{2\pi i}{k + C_v} \right),$$

(4)

which depends on the coupling constant $k$ and the dual Coxeter number $C_v$ of the group G. These link invariants include the well-known polynomials in the knot theory literature.
2.1.1. Link Invariants. As indicated in the above table, Jones’ polynomial corresponds to the fundamental representation $R = \boxdot \equiv 1 \in SU(2)$ placed on all the component knots:

$$V_{1,1,1,...,1}^{SU(2)}[\mathcal{L}_m; q] = J_{2,2,...,2} \left[ \mathcal{L}_m; q = \exp \left( \frac{2\pi i}{k + 2} \right) \right],$$

(5)

where the subscript ‘2’ in Jones polynomial $J_{2,2,...,2}[\mathcal{L}_m; q]$ denotes the dimension of $R = \boxdot$. Higher dimensional representations placed on the component knots $R_i = \boxdot_{n_i-1}$ are the colored Jones invariants:

$$V_{n_1-1,n_2-1,...,n_m-1}^{SU(2)}[\mathcal{L}_m; q] \equiv J_{n_1,n_2,...,n_m} \left[ \mathcal{L}_m; q = \exp \left( \frac{2\pi i}{k + 2} \right) \right],$$

(6)

and the invariants with these representations belonging to $SU(N)$ ($SO(N)$) are known as colored HOMFLY-PT (colored Kauffman) invariants. For clarity, we will restrict to $SU(2)$ group to write the invariants explicitly in terms of $q$ variable.

We work with the following unknot $(\bigcirc)$ normalization:

$$J_{n+1}[\bigcirc; q] = \dim_q \bigotimes_{n}^{n} = \frac{q^{\frac{(n+1)}{2}} - q^{-\frac{(n+1)}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = \frac{\sin(\frac{\pi (n+1)}{k+2})}{\sin(\frac{\pi}{k+2})} = \frac{S_{0n}}{S_{00}},$$

(7)

where $\dim_q \bigotimes_{n}^{n}$ denotes quantum dimension of the representation $\bigotimes_{n}^{n}$ and $S_{n_1,n_2}$ are the modular transformation matrix elements of the $su(2)_k$ WZW conformal field theory whose action on the characters is $\chi_{n_1} (\tau) \rightarrow \chi_{n_2} (-\frac{1}{\tau})$, where $\tau$ denotes the modular parameter. These knot and link polynomials with the above unknot normalization are referred as unnormalized colored Jones invariant.

For framed unknots with framing number $f$, the invariant will be

$$J_{n+1}[\bigcirc; q] = q^f \left( \frac{(n+1)^2 - 1}{4} \right) \left( \frac{q^{\frac{(n+1)}{2}} - q^{-\frac{(n+1)}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \right) \propto (T_{nn})^f \frac{S_{0n}}{S_{00}},$$

(8)

where the action of the modular transformation matrix $T$ on characters is $\chi_n (\tau) \rightarrow \chi_n (\tau + 1)$. The colored Jones invariant for the Hopf link can also be written in terms of $S$ matrix:

$$J_{n_1+1,n_2+1}[H; q] = \left( \frac{q^{\frac{(n_1+1)(n_2+1)}{2}} - q^{-\frac{(n_1+1)(n_2+1)}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \right) = \frac{S_{n_1,n_2}}{S_{00}}.$$  

(9)
The invariant for a framed Hopf link $H(f_1, f_2)$, with framing numbers $f_1$ and $f_2$ on the two component knots, in terms of $T$ and $S$ matrices is

$$J_{n_1+1, n_2+1}[H(f_1, f_2); q] \propto (T_{n_1 n_1})^{f_1} (T_{n_2 n_2})^{f_2} \frac{S_{n_1 n_2}}{S_{00}}.$$  \hspace{1cm} (10)

We will look at a class of links obtained as a connected sum of framed Hopf links. For instance, the invariant for the connected sum of two framed Hopf links $H(f_1, f_2) \# H(f_2, f_3)$ will be

$$J_{n_1+1, n_2+1, n_3+1}[H(f_1, f_2) \# H(f_2, f_3); q] \propto \prod_{i=1}^{3} T_{n_i n_i} \frac{S_{n_1 n_2} S_{n_2 n_3}}{S_{00} S_{n_2 0}}$$

$$= \prod_{i=1}^{2} J_{n_i+1, n_{i+1}+1}[H(f_i, f_{i+1}); q] \frac{1}{\prod_{(v_1, v_2) \in Edges} S_{n_{v_1} n_{v_2}}}.$$ \hspace{1cm} (11)

Such a connected sum of two framed Hopf links, which is a 3-component link, can be denoted as a linear graph

$$\begin{align*}
\ & f_1 \quad \quad \quad \quad f_2 \quad \quad \quad \quad f_3
\end{align*}$$

with three vertices labeled by the framing numbers and the edges connecting the adjacent vertices. These are known as ‘plumbing graphs’. Another plumbing graph $\Gamma$ with 8 vertices denoting the link $L(\Gamma)$ (the connected sum of many framed Hopf links) is illustrated in Fig. 1. The colored invariant for these links $L(\Gamma)$ can be written in terms of $S$ and $T$ matrices.

For a general $m$ vertex plumbing graph with vertices $v_1, v_2, \ldots, v_m \in V$ labelled by framing numbers $f_1, f_2, \ldots, f_m$, there can be one or more edges connecting a vertex $v$ with the other vertices. The degree of any vertex $v$ ($\deg(v)$) is equal to the total number of edges intersecting $v$. For the graph in Fig. 1, $\deg(2) = \deg(4) = \deg(6) = 3$. The colored Jones’ invariant for any plumbing graph $\Gamma$ is

$$J_{n_1+1, n_2+1, \ldots, n_m+1}[L; q]$$

$$\propto \frac{1}{S_{00}} \prod_{i=1}^{m} \{(T_{n_i n_i})^{f_i} (S_{0 n_i})^{1-\deg(v_i)}\} \prod_{(v_1, v_2) \in Edges} (S_{n_{v_1} n_{v_2}}).$$ \hspace{1cm} (12)
Even though we have presented the colored Jones invariants \((10, 11, 12)\), the formal expression of these link invariants in terms of \(S\) and \(T\) matrices is applicable for any arbitrary gauge group \(\mathcal{G}\).

**\(SO(3)\) and \(OSp(1|2)\) Link invariants**

Using group theory arguments, it is possible to relate colored link invariants between different groups. For instance, the representations of the \(SO(3)\) can be identified with a subset of \(SU(2)\) representations. As a consequence, the \(SO(3)\) link invariants can be related to the colored Jones invariants as follows:

\[
V_{n_1,n_2,n_3,...,n_m}^{SO(3)} [\mathcal{L}_m; Q] = J_{2n_1+1,2n_2+1,...2n_m+1}[\mathcal{L}_m; q] |_{q^2=Q},
\]

where the level \(K\) of the affine \(\mathfrak{so}(3)_K\) Lie algebra must be an even integer \((K \in 2\mathbb{Z})\).

Similarly, the representations of the orthosymplectic supergroup \(OSp(1|2)\) can be related to the representations of the \(SU(2)\) group from the study of \(\mathfrak{osp}(1|2)_K\) WZW conformal field theory and the link invariants \(V_{n_1,n_2,n_3,...,n_m}^{OSp(1|2)} [\mathcal{L}_m; \hat{Q}] = \exp \left( \frac{2\pi i}{2K+3} \right)\) \([5]\). Particularly, there is a precise identification of the polynomial variable \(\hat{Q}\) to \(SU(2)\) variable \(q\). Further, the fusion rules of the primary fields of \(\mathfrak{osp}(1|2)_K\) WZW conformal field theory can be compared to integer spin primary fields of the \(\mathfrak{su}(2)_k\). Particularly, the \(\hat{S}\) and \(\hat{T}\)-matrices of \(\mathfrak{osp}(1|2)_K\):

\[
\hat{S}_{n_1n_2} = \sqrt{\frac{4}{2K+3}} (-1)^{n_1+n_2} \cos \left( \frac{(2n_1+1)(2n_2+1)}{2(2K+3)} \pi \right),
\]

\[
\hat{T}_{n_1,n_2} \propto \delta_{n_1,n_2} \hat{Q}^{(2n_1+1)^2-1},
\]

are related to the \(S\) and \(T\) matrices of \(\mathfrak{su}(2)_k\) in the following way:

\[
\hat{S}_{n_1,n_2} = S_{2n_1,2n_2} |_{q=\hat{Q}}; \quad \hat{T}_{n_1,n_1} = T_{2n_1,2n_1} |_{q=\hat{Q}}
\]

(16)

Using these relations, we can show that the \(OSp(1|2)\) colored invariant match the colored Jones invariant for any arbitrary link \(\mathcal{L}_m\) in the following way:

\[
V_{n_1,n_2,n_3,...,n_m}^{OSp(1|2)} [\mathcal{L}_m; \hat{Q}] = \epsilon J_{2n_1+1,2n_2+1,...2n_m+1}[\mathcal{L}_m; q] |_{q=\hat{Q}},
\]

where \(\epsilon\) could be \(\pm 1\) depending on the link \(\mathcal{L}\) and the representations \(n_i\)'s. For example, the colored \(OSp(1|2)\) invariant for framed Hopf link is

\[
V_{n_1,n_2}^{OSp(1|2)} [H(f_1, f_2); \hat{Q}] = \hat{Q}^{\frac{(2n_1+1)^2}{4}} \hat{Q}^{\frac{(2n_2+1)^2}{4}} (-1)^{(n_1+n_2)} \times \left( \frac{\hat{Q}^{(2n_1+1)(2n_2+1)}}{\hat{Q}^{\frac{1}{2}} + \hat{Q}^{-\frac{1}{2}}} \right)
\]

\[
= J_{2n_1+1,2n_2+1}[H(f_1, f_2), -\hat{Q}].
\]

(17)
In fact, for any link $\mathcal{L}(\Gamma)$ denoted by the graph $\Gamma$, the invariants will be

$$V_{n_1, n_2, \ldots, n_m}^{OSp(1|2)} [\mathcal{L}(\Gamma); \hat{Q}] = \frac{1}{Q^{2} + Q^{-2}} \prod_{i=1}^{m} (-1)^{n_i} \frac{f_i((2n_i+1)^2-1)}{4} \left( \hat{Q}^{\frac{2n_i+1}{2}} + \hat{Q}^{-\frac{2n_i+1}{2}} \right) \deg(v_i) \cdot \prod_{(v_1, v_2) \in \text{Edges}} \left( \hat{Q}^{(2n_{v_1}+1)(2n_{v_2}+1)} + \hat{Q}^{-(2n_{v_1}+1)(2n_{v_2}+1)} \right).$$

(19)

As three-manifolds can be constructed by a surgery procedure on any framed link, the Chern–Simons partition function/WRT invariant (1) can be written in terms of link invariants [14, 17, 22, 23]. We will now present the salient features of such WRT invariants.

2.1.2. Three-Manifold Invariants. Let us confine to the three-manifold $M[\Gamma]$ obtained from surgery of framed link associated with $L$-vertex graph (an example illustrated in Fig. 1). These kind of manifolds is known in the literature as plumbed three-manifolds. The linking matrix $B$ is defined as

$$B_{v_1, v_2} = \begin{cases} 1, & v_1, v_2 \text{ connected}, \\ f_{v}, v_1 = v_2 = v, & v_i \in \text{Vertices of $\Gamma \cong \{1, \ldots, L\}$}, \\ 0, & \text{otherwise}. \end{cases}$$

(20)

The algebraic expression for the WRT invariant $\tau^G_k[M(\Gamma); q]$ is

$$\tau^G_k[M(\Gamma); q] = \frac{F^G[\mathcal{L}(\Gamma); q]}{F^G[\mathcal{L}(+\bullet); q]^{b_+} F^G[\mathcal{L}(-\bullet); q]^{b_-}}$$

(21)

where $b_\pm$ are the number of positive and negative eigenvalues of a linking matrix $B$, respectively, and $F^G[\mathcal{L}(\Gamma); q]$ is defined as

$$F^G[\mathcal{L}(\Gamma); q] = \sum_{R_1, R_2, \ldots, R_L} \left( \prod_{i=1}^{L} V_{R_i}^G[\emptyset; q] \right) V_{R_1, R_2, \ldots, R_L}^G[\mathcal{L}(\Gamma); q],$$

(22)

where the summation indicates all the allowed integrable representations of affine $\mathfrak{g}_k$ Lie algebra. By construction, any two homeomorphic manifolds must share the same three-manifold invariant. There is a prescribed set of moves called Kirby moves on links which gives the same three-manifold. For framed links depicted as plumbing graphs, these moves are known as Kirby–Neumann moves as shown in Fig. 2. Hence, the three-manifold invariant must obey

$$\tau^G_k[M(\Gamma); q] = \tau^G_k[M(\Gamma'); q],$$

(23)

where the plumbing graphs $\Gamma, \Gamma'$ are related by the Kirby–Neumann moves.

Toward the end of twentieth century, attempts to give a topological interpretation for the integer coefficients in the Laurent series expression for Jones
polynomial (HOMFLY-PT) as well as the corresponding colored invariants for any knot $\mathcal{K}$

$$J_n[\mathcal{K}; q] = \sum_s a_s q^s, \quad \{a_s\} \in \mathbb{Z}$$

have resulted in developments on homology theories as well as physics explanation. We will discuss these ‘homological invariants’ and their appearance in string/M-theory in the following section.

### 2.2. Knot, Link and Three-Manifold Homologies

We will first review the developments on homological invariants of knots accounting for these integers $a_s$ (24) as dimension of the vector space $H_{\mathcal{K}}$ of a homological theory. Then, we will present the topological string/M-theory approach where these integers count number of BPS states.

#### 2.2.1. Homological Invariants of Knots

The pioneering work of Khovanov [15] on bi-graded homology theory led to categorification of the Jones polynomial. This was extended to colored $\mathfrak{sl}_2$ knot homology $H^{\mathfrak{sl}_2; n}_{i,j}$ [3, 7, 24] leading to new homological invariants $P_n^{\mathfrak{sl}_2}[\mathcal{K}, q, t]$ which categorifies the colored Jones polynomial:

$$P_n^{\mathfrak{sl}_2}[\mathcal{K}, q, t] = \sum_{i,j} t^i q^j \dim H^{\mathfrak{sl}_2; n}_{i,j}.$$  \hspace{1cm} (25)

The subscripts $i$ and $j$ on the colored $\mathfrak{sl}_2$ homology $H^{\mathfrak{sl}_2; n}_{i,j}$ are called the polynomial grading and the homological grading, respectively. In fact, the $q$-graded Euler characteristic of the colored $\mathfrak{sl}_2$ knot homology gives the colored Jones invariant:

$$J_n[\mathcal{K}; q] = \sum_{i,j} (-1)^j q^i \dim H^{\mathfrak{sl}_2; n}_{i,j},$$

explaining the reasons behind the integers $a_s(24)$. Khovanov and Rozansky [16] constructed $\mathfrak{sl}_N$ homology using matrix factorizations. This led to the categorification of colored HOMFLY-PT polynomials of knots. There has been interesting insight on these homological invariants within topological strings context and $M$-theory. We will now discuss the essential features from physics approach.
2.2.2. Topological Strings and M-Theory. The parallel developments from topological strings and intersecting branes in M-theory [9,20] interpreted the integers of unnormalized HOMFLY-PT (24) as counting of BPS states. Invoking topological string duality in the presence of any knot $\mathcal{K}$, Ooguri-Vafa conjectured

$$V^{SU(N)}[\mathcal{K}; q, \lambda = q^N] = \frac{1}{(q^{1/2} - q^{-1/2})} \sum_{Q,s} N_{\square}^{Q,s} \lambda^Q q^s,$$

(27)

where the integers $N_{\square}^{Q,s}$ count $D4 - D2$ bound states in string theory. Further, the relation between the BPS spectrum and $sl_N$/Khovanov–Rozansky knot homology was conjectured within the topological string context [13]:

$$N_{\square}^{Q,s} = \sum_j (-1)^j D_{Q,s,j},$$

(28)

where the integers $D_{Q,s,j}$ are referred to as refined BPS invariants. The extra charge/homological grading $j$ are explainable by the appearance of extra $U(1)$ symmetry in $M$-theory compactified on Calabi–Yau three-folds $\text{CY}_3$. The topological string duality and the dualities of physical string theories compactified on $\text{CY}_3$ imply that the vector space of knot homologies is the Hilbert space of BPS states (see review [19] and references therein):

$$\mathcal{H}_{\mathcal{K}} \equiv \mathcal{H}_{\text{BPS}}.$$

The impact of knot homology on the categorification of the WRT invariants has been studied in the last six years. We now present a concise summary of the recent developments in this direction.

2.2.3. Three-Manifold Homology. As WRT invariants (21) of three-manifolds involves invariants of links, logically we would expect the homology of three-manifold $\mathcal{H}^G_M$ such that

$$\tau^G_k[M; q] = \sum_{i,j} (-1)^j q^i \dim \mathcal{H}^G_{i,j}.$$

(29)

However, the WRT invariants known for many three-manifolds are not seen as $q$-series (29). We will now review the necessary steps [11] of obtaining a new three-manifold invariant $\hat{Z}$, as $q$-series, from $U(N)$ Chern–Simons partition function for Lens space $M = L(p,1) \equiv S^3/\mathbb{Z}_p$. The space of flat connections $\{a\}$ denoted by $\pi_1[S^3/\mathbb{Z}_p] \equiv \mathbb{Z}_p$. Hence, $Z^{U(N)}_k[L[p,1]; q]$ can be decomposed as a sum of perturbative Chern–Simons $Z^{U(N)}_a[L[p,1]; q]$ around these abelian flat connections $a$ [11]:

$$Z^{U(N)}_k[L[p,1]; q] = \sum_a \exp[S_{CS}^{(a)}] Z^{U(N)}_a[L[p,1]; q],$$

(30)

where $S_{CS}^{(a)}$ is the corresponding classical Chern–Simons action. The following change of basis by $S$ matrix of $u(1)^N_p$ affine algebra:

$$Z^{U(N)}_a[L(p,1); q] = \sum_b S_{ab} \hat{Z}^{U(N)}_b[L[p,1]; q] \bigg|_{q \to q_i},$$

(31)
is required so that
\[ \hat{Z}^U(N)_{[L[p, 1]; q]} = q^{\Delta_b} \mathbb{Z}[[q]], \ \Delta_b \in \mathbb{Q}. \] (32)

Physically, the \( \hat{Z}^U(N)_{[L[p, 1]; q]} \) is also the vortex partition function \( \hat{Z}^U(T[L[p, 1]])[D^2 \times_q S^1] \) obtained from reducing 6d \((2,0)\) theory (describing dynamics of \( N \)-coincident \( M5 \) branes on \( L[p, 1] \times D^2 \times_q S^1 \)) on \( L[p, 1] \). The effective 3-d \( \mathcal{N} = 2 \) theory on \( D^2 \times_q S^1 \) (cigar geometry) is denoted as \( T^{U(N)}[L[p, 1]] \).

As discussed in Sect. 2.2.3 [12], the expression for Lens space partition function
\[ Z^{(35)}_{[L(p, 1); q]} = \sum_{a, b \in \mathbb{Z}_p} S_{ab} \exp[i S^{(a)}_{CS}] \hat{Z}^U(N)_{[L[p, 1]; q]} \\bigg|_{q \to q}, \] (34)
where \( \hat{Z}^U_{[L[p, 1]; q]} = q^{\Delta_b} \mathbb{Z}[[q]], \ \Delta_b \in \mathbb{Q} \). (35)

led to the following conjecture [10, 11] for any closed oriented three manifold \( M \) known as GPPV conjecture:
\[ Z^{SU(2)}_{k}[M; q] = (i \sqrt{2(k + 2)})^{b_1(M) - 1} \sum_{a, b \in \mathbb{Z}_2} S_{ab} \exp[2\pi i (k + 2)\ell k(a, a)] \times |W_b|^{-1} S_{ab} \hat{Z}^{SU(2)}_{b}[M; q] \bigg|_{q \to q = \exp\left(\frac{2\pi i}{k + 2}\right)} \] (36)
where
\[ \hat{Z}^{SU(2)}_{b}[M; q] = 2^{-c} q^{\Delta_b} \mathbb{Z}[[q]] \ \Delta_b \in \mathbb{Q}, \ \ c \in \mathbb{Z}_+ \] (37)
is convergent for \( |q| < 1 \) and
\[ S_{ab} = \frac{e^{2\pi i \ell k(a, b)} + e^{-2\pi i \ell k(a, b)}}{|W_a| \sqrt{|H_1(M, \mathbb{Z})|}}. \] (38)

3. Review of \( SU(2) \) İ̻̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇̇
Here, $\mathcal{W}_a$ is the stabilizer subgroup defined as
\[
\mathcal{W}_a \equiv \text{Stab}_\mathbb{Z}_2(a) = \begin{cases} \mathbb{Z}_2, & a = -a, \\ 1, & \text{otherwise}. \end{cases}
\] (39)
and $\ell k$ denotes the linking pairing on $H_1(M, \mathbb{Z})$:
\[
\ell k : H_1(M, \mathbb{Z}) \otimes H_1(M, \mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z} \\
[a] \otimes [b] \mapsto \#(a \cap \hat{b})/n
\] (40)
where $\hat{b}$ is a two-chain complex such that $\partial \hat{b} = nb$ with $n \in \mathbb{Z}$. Such a $\hat{b}$ and $n$ exists because $[b] \in H_1(M, \mathbb{Z})$. The number $\#(a \cap \hat{b})$ counts the intersection points with signs determined by the orientation. The set of orbits is the set of $\text{Spin}^c$ structures on $M$, with the action of $\mathbb{Z}_2$ by conjugation.

Although the relation (36) is true for any closed oriented three-manifold $M$, the explicit $q$ series expression for $\hat{Z}$ is waiting to be discovered for a general three-manifold.

In the following subsection, we will review the $\hat{Z}^{SU(2)}$ for the plumbed manifolds. We begin with the WRT invariant for a plumbing graph, of the type shown in Fig. 1, discussed in Sect. (2.1.2). Then, analytically continue $q \rightarrow \tilde{q}$ to get the $\hat{Z}^{SU(2)}$-invariant. We will see that the analytic continuation procedure is doable only for negative definite plumbed manifolds (i.e., the signature of linking matrix $B$, $\sigma = b_+ - b_- = -L$).\(^1\) Moreover, as explained in [10], the $\text{Spin}^c$-structure in case of plumbed three-manifold with $b_1(M) = 0$ is given by $H_1(M, \mathbb{Z}) \cong \text{Coker}\, B = \mathbb{Z}^L / B \mathbb{Z}^L$.

3.1. $\hat{Z}^{SU(2)}_b(a)$

The WRT invariant $\tau^{SU(2)}_k[M(\Gamma); q]$,\(^2\) for plumbed three-manifold $M(\Gamma)(21)$, obtained from surgery of framed link $L(\Gamma)$ in $S^3$, is
\[
\tau^{SU(2)}_k[M(\Gamma); q] = \frac{F^{SU(2)}[\mathcal{L}(\Gamma); q]}{F^{SU(2)}[\mathcal{L}(+1\bullet); q]^b_v \cdot F^{SU(2)}[\mathcal{L}(-1\bullet); q]^b_v}.
\]
where $F^{SU(2)}[\mathcal{L}(\Gamma); q] = \sum_{n \in \{1, \ldots, k+1\}^L} J[\mathcal{L}(\Gamma)]_{n_1, \ldots, n_L} \prod_{v=1}^L \frac{q^{n_v/2} - q^{-n_v/2}}{q^{1/2} - q^{-1/2}}$.

(41)

Note $b_\pm$ are the number of positive and negative eigenvalues of a linking matrix $B$, respectively, and the colored Jones polynomial of link $\mathcal{L}(\Gamma)$ (12) in variable $q = \exp(2i\pi/(k + 2))$ is
\[
J[\mathcal{L}(\Gamma)]_{n_1, \ldots, n_L} = \frac{2i}{q^{1/2} - q^{-1/2}} \prod_{v \in \text{Vertices} \cong \{1, \ldots, L\}} q^{f_v(n_v^2 - 1)/(4)}.
\]

\(^1\)In principle, this procedure is also doable when $B$ is negative on a certain subspace of $\mathbb{Z}^L$.

\(^2\)Normalized such that $\tau^0_k[S^3; q] = 1$ and $k$ is the bare level for $SU(2)$ Chern–Simons theory.
Using the following Gauss sum reciprocity formula
\[
\sum_{n \in \mathbb{Z}^{L/2k} \mathbb{Z}} \exp \left( \frac{\pi i}{2k} (n, Bn) + \frac{\pi i}{k} (\ell, n) \right) = e \frac{\pi i}{|\det B|^{1/2}} \sum_{a \in \mathbb{Z}^{L/2k} \mathbb{Z}} \exp \left( -2\pi i k \left( a + \frac{\ell}{2k} B^{-1} \left( a + \frac{\ell}{2k} \right) \right) \right)
\]
where \( \ell \in \mathbb{Z}^{L} \), \((\cdot, \cdot)\) is the standard pairing on \( \mathbb{Z}^{L} \) and \( \sigma = b_{+} - b_{-} \) is the signature of the linking matrix \( B \), we can sum
\[
F_{k}^{\text{SU}(2)}[\mathcal{L}(\pm 1 \bullet); q] = \sum_{n} q^{\pm n^{2}} \left( q^{n/2} - q^{-n/2} \right)^{2} = \frac{[2(k + 2)]^{1/2} e^{\pm 2\pi i / 4} q^{1/2}}{q^{1/2} - q^{-1/2}}
\]
for the unknot with framing \( \pm 1 \). Incorporating the above equation and the fact that \( L - |\text{Edges}| = 1 \) for the framed link \( \mathcal{L}(\Gamma) \), the WRT invariant simplifies to
\[
\tau_{k}^{\text{SU}(2)}[M(\Gamma); q] = e^{\frac{-\pi i}{4} q^{3/4}} \frac{2 (2(k + 2))^{L/2} (q^{1/2} - q^{-1/2})}{q^{n_{v}/2} / q^{n_{v}/2}}
\]
\[
\times \sum_{n \in \mathbb{Z}^{L/2k+2} \mathbb{Z}} \prod_{v \in \text{Vertices}} q^{f_{v}(n_{v}^{2} - 1)} \frac{1}{q^{n_{v}/2} - q^{-n_{v}/2}} \prod_{(v', v'') \in \text{Edges}} \frac{q^{n_{v'}/2} - q^{n_{v''}/2}}{2}
\]
where we used invariance of the summand under \( n_{v} \rightarrow -n_{v} \). The prime \('\) in the sum means that the singular values \( n_{v} = 0 \), \( k + 2 \) are omitted. Let us focus on the following factor for general plumbed graph:
\[
\prod_{(v', v'') \in \text{Edges}} \left( q^{n_{v'}/2} - q^{n_{v''}/2} \right) = \sum_{p \in \{\pm 1\}^{\text{Edges}}} \prod_{(v', v'') \in \text{Edges}} p^{n_{v'}/2} n_{v''}/2.
\]
Note that, under \( n_{v} \rightarrow -n_{v} \) on any vertex \( v \) of degree \( \text{deg}(v) \), the factor with a given configuration of signs associated with edges (i.e., \( p \in \{\pm 1\}^{\text{Edges}} \)) will transform into a term with a different configuration times \((-1)^{\text{deg}(v)}\). For the class of graphs \( \Gamma \) (like Fig. 1), the sequence of such transforms can be finally brought to the configuration with all signs \( +1 \). Hence, the WRT invariant (45) for these plumbed three-manifolds can be reduced to this form:
\[ 
\tau_{k}^{SU(2)}[M(\Gamma)] = \frac{e^{-\frac{\pi i \alpha}{4}} q_{1}^{\frac{3\sigma - \sum_{v} f_{v}}{4}}}{2 (2(k + 2))^{L/2} (q_{1}^{1/2} - q_{-1}^{-1/2})} \times \sum_{n \in \mathbb{Z}^{L}/2(k+2)\mathbb{Z}} q_{1}^{\frac{(n,Bn)}{4}} \prod_{v \in \text{Vertices}} \left( \frac{1}{q_{1}^{n_{v}/2} - q_{-n_{v}/2}} \right)^{\text{deg}(v)-2}.
\] (47)

In the above expression, the points 0 and \(k + 2\) are excluded in the summation but in the reciprocity formula (43) no point is excluded. So, to apply the reciprocity formula we have to first regularize the sum. This is achieved by introducing the following regularizing parameters:

\[ \Delta_{v} \in \mathbb{Z}_{+} : \Delta_{v} = \text{deg}(v) - 1 \pmod{2}, \quad \forall v \in \text{Vertices}, \]

so that the sum in eqn.(47) is rewritable as \(\omega \to 1\):

\[ \sum_{n \in \mathbb{Z}^{L}/2(k+2)\mathbb{Z}} q_{1}^{\frac{(n,Bn)}{4}} \prod_{v \in \text{Vertices}} \left( \frac{1}{q_{1}^{n_{v}/2} - q_{-n_{v}/2}} \right)^{\text{deg}(v)-2} = \lim_{\omega \to 1} \frac{1}{2L} \sum_{n \in \mathbb{Z}^{L}/2(k+2)\mathbb{Z}} q_{1}^{\frac{(n,Bn)}{4}} F_{\omega}(x_{1}, \ldots, x_{L})|_{x_{v} = q_{1}^{n_{v}/2}}, \]

where

\[ F_{\omega}(x_{1}, \ldots, x_{L}) = \prod_{v \in \text{Vertices}} (x_{v} - 1/x_{v})^{\Delta_{v}} \times \left\{ \left( \frac{1}{x_{v} - \omega/x_{v}} \right)^{\text{deg}(v)-2+\Delta_{v}} + \left( \frac{1}{\omega x_{v} - 1/x_{v}} \right)^{\text{deg}(v)-2+\Delta_{v}} \right\} \]

(50)

Note that, we can perform a binomial expansion taking \((\omega/x_{v}^{2})\) small in the first term and \((\omega x_{v}^{2})\) small in the second term to rewrite \(F_{\omega}(x_{1}, \ldots, x_{L})\) as a formal power series:

\[ F_{\omega}(x_{1}, \ldots, x_{L}) = \sum_{\ell \in \mathbb{Z}^{L}+\delta} F_{\omega}^{\ell} \prod_{v} x_{v}^{\ell_{v}} \in \mathbb{Z}[\omega][[x_{1}^{1}, \ldots, x_{L}^{1}]], \]

(51)

where \(\delta \in \mathbb{Z}^{L}/2\mathbb{Z}^{L}, \delta_{v} \equiv \text{deg}(v) \pmod{2}\) and

\[ F_{\omega}^{\ell} = \sum_{m \in \mathcal{I}_{m}} N_{m,\ell} \omega^{m} \in \mathbb{Z}[\omega] \]

(52)

with \(\mathcal{I}_{m}\) being a finite set of elements from \(\mathbb{Z}^{L}\). By definition, \(\lim_{\omega \to 1} F_{\omega}^{\ell}\) is not dependent on \(\Delta \in \mathbb{Z}^{L}\) (48). However, this \(\omega \to 1\) limit in eqn. (50) will restrict the binomial expansion range of the first term to be \(x \to \infty\) and that of the second term to \(x \to 0\):

\[ F_{\omega \to 1}(x_{1}, \ldots, x_{L}) = \sum_{\ell \in \mathbb{Z}^{L}+\delta} F_{\omega \to 1}^{\ell} \prod_{v} x_{v}^{\ell_{v}} \]

\[ = \lim_{\omega \to 1} \prod_{v \in \text{Vertices}} \left\{ \text{Expansion as } x \to \infty \left( \frac{1}{x_{v} - \omega/x_{v}} \right)^{\text{deg}(v)-2} \right\} \]

\[ = \lim_{\omega \to 1} \prod_{v \in \text{Vertices}} \left\{ 1 \right\} \]
Now, let us assume that the quadratic form $B : \mathbb{Z}^L \times \mathbb{Z}^L \to \mathbb{Z}$ is negative definite, i.e., $\sigma = -L$. Then, we can define the following series in $q$ which is convergent for $|q| < 1$:

$$
\hat{Z}_b^{SU(2)}[M(\Gamma); q] \overset{\text{Def}}{=} 2^{-L}q^{-\frac{3L+\sum_v f_v}{4}} \sum_{\ell \in 2B\mathbb{Z}^L+b} F^\ell_{\omega \rightarrow 1} q^{-\frac{(\ell, B^{-1}\ell)}{4}} \in 2^{-c}q^{\Delta_v} \mathbb{Z}[q]
$$

(54)

where $c \in \mathbb{Z}_+, c \leq L$ and

$$
b \in (2\mathbb{Z}^L + \delta)/2B\mathbb{Z}^L/\mathbb{Z}_2 \cong (2\text{Coker } B + \delta)/\mathbb{Z}_2, \quad \Delta_b = -3L + \sum_v f_v - \max_{\ell \in 2M\mathbb{Z}^L+b} \frac{(\ell, B^{-1}\ell)}{4} \in \mathbb{Q}
$$

(56)

where $\mathbb{Z}_2$ action takes $b \rightarrow -b$ and is the symmetry of (54).

Using relation (49) and applying Gauss reciprocity formula (43), we arrive at the following expression for the WRT invariant:

$$
\tau_k^{SU(2)}[M(\Gamma); q]
= e^{-\frac{\pi L}{4}} q^{-\frac{3L+\sum_v f_v}{4}} \lim_{\omega \rightarrow 1} \sum_{n \in \mathbb{Z}^L/(2k+2)\mathbb{Z}^L} q^{-\frac{(n, Bn)}{4}} F_\omega(x_1, \ldots, x_L) |_{x_v = q^{n_v/2}}
$$

$$
= \frac{2^{-L}q^{-\frac{3L+\sum_v f_v}{4}}}{2(q^{1/2} - q^{-1/2}) | \det B|^{1/2}} \sum_{a \in \text{Coker } B} e^{-2\pi i(a, B^{-1}a)} e^{-2\pi i(k+2)(a, B^{-1}a)}
$$

$$
\times \lim_{\omega \rightarrow 1} \sum_{\ell \in 2B\mathbb{Z}^L+b} F^\ell_{\omega} q^{-\frac{(\ell, B^{-1}\ell)}{4}}.
$$

(57)

Assuming that the limit $\lim_{q \rightarrow q_0} \hat{Z}_b^{SU(2)}(q)$ exists, where $q$ approaches $(k+2)$-th primitive root of unity from inside of the unit disc $|q| < 1$, we expect

$$
\lim_{\omega \rightarrow 1} \sum_{\ell \in 2B\mathbb{Z}^L+b} F^\ell_{\omega} q^{-\frac{(\ell, B^{-1}\ell)}{4}} = \lim_{q \rightarrow q_0} \sum_{\ell \in 2B\mathbb{Z}^L+b} F^\ell_{\omega} q^{-\frac{(\ell, B^{-1}\ell)}{4}}.
$$

(58)

Thus, we obtain GPPV conjecture form:

$$
\tau_k^{SU(2)}[M(\Gamma), q] = \frac{1}{2(q^{1/2} - q^{-1/2}) | \det B|^{1/2}} \sum_{a \in \text{Coker } B} e^{-2\pi i(k+2)(a, B^{-1}a)} \sum_{b \in \text{Coker } B + \delta} e^{-2\pi i(a, B^{-1}b)} \lim_{q \rightarrow q_0} \hat{Z}_b^{SU(2)}[M(\Gamma); q].
$$

(59)

There is also an equivalent contour integral form for the homological blocks (54):
\[ \hat{Z}_{b}^{SU(2)}[M(\Gamma); q] = q^{-\frac{3L + \sum_v f_v}{4}} \cdot \text{v.p.} \int_{|z_v|=1} \prod_{v \in \text{Vertices}} \frac{dz_v}{2\pi i z_v} (z_v - 1/z_v)^{2 - \deg(v)} \cdot \Theta_{b}^{-B}(z), \]

(60)

where \(\Theta_{b}^{-B}(x)\) is the theta function of the lattice corresponding to minus the linking form \(B\):

\[ \Theta_{b}^{-B}(x) = \sum_{\ell \in 2\mathbb{Z}^L + b} q^{-\frac{(\ell, B^{-1} \ell)}{4}} \prod_{i=1}^{L} x_i^{\ell_i}, \]

(61)

and “v.p.” refers to principle value integral (i.e. take half-sum of contours \(|z_v| = 1 \pm \epsilon\)). This prescription corresponds to the regularization by \(\omega\) made in eqn.(50).

Thus, we can obtain explicit \(SU(2)\) \(q\)-series for any negative definite plumbed three-manifolds. For completeness, we present the \(q\)-series for some examples.

### 3.2. Examples

- Poincare homology sphere is a well-studied three-manifold corresponding to the graph:

\[ \text{graph} \]

As \(H_1(M, \mathbb{Z}) = 0\), we obtain only single homological block \(\hat{Z}_{b_1}\). Solving eqns.(53,54), we get

\[ \hat{Z}_{b_1}^{SU(2)} = q^{-3/2}(1 - q - q^3 - q^7 + q^8 + q^{14} + q^{20} + q^{29} - q^{31} - q^{42} + \cdots). \]

(63)

- The next familiar example with \(H_1(M, \mathbb{Z}) = 0\) is Brieskorn homology sphere. A particular example of this class is \(\Sigma(2, 3, 7)\) with the following equivalent graphs:
The homological block turns out to be
\[ \hat{Z}_{b_1}^{SU(2)} = q^{1/2}(1 - q - q^5 + q^{10} - q^{11} + q^{18} + q^{30} - q^{41} + q^{43} - q^{56} - q^{76} \cdots). \]

• For a three-manifold with non-trivial \( H_1(M, \mathbb{Z}) = \mathbb{Z}_3 \) as drawn below,

\[
\begin{align*}
\hat{Z}^{SU(2)} = \begin{pmatrix}
1 - q + q^6 - q^{11} + q^{13} - q^{20} + q^{35} + O(q^{41}) \\
q^{5/3} (-1 + q^3 - q^{21} + q^{30} + O(q^{41})) \\
q^{5/3} (-1 + q^3 - q^{21} + q^{30} + O(q^{41}))
\end{pmatrix},
\end{align*}
\]

where two of them are equal.

Our focus is to obtain explicit \( q \)-series for \( SO(3) \) and \( OSp(1|2) \) groups. Using the relation between \( SU(2) \) and \( SO(3) \), \( SU(2) \) and \( OSp(1|2) \) link invariants (2.1.1), we will investigate the necessary steps starting from the WRT invariant for \( SO(3) \) and \( OSp(1|2) \) eventually leading to the \( \hat{Z} \)-invariant. This will be the theme of the following section.

4. \( \hat{Z} \) for \( SO(3) \) and \( OSp(1|2) \)

Our aim is to derive the \( \hat{Z} \)-invariant for \( SO(3) \) and \( OSp(1|2) \) groups. We will first look at the WRT invariants \( \tau_K^{SO(3)}[M(\Gamma); Q] \) for plumbed three-manifolds written in terms of colored Jones invariants of framed links \( \mathcal{L}[\Gamma] \) in the following subsection and then, discuss \( OSp(1|2) \) \( \hat{Z} \) in the subsequent section.
4.1. \( SO(3) \) WRT Invariant and \( \hat{Z}^{SO(3)} \) Invariant

Recall that the framed link invariants are written in variable \( q \) which is dependent on Chern–Simons coupling and the rank of the gauge group \( G \). For \( SO(3) \) Chern–Simons with coupling \( K \in 2\mathbb{Z} \), the variable \( Q = e^{\frac{2\pi i}{K+1}} \). Hence, \( F^{SO(3)}[\mathcal{L}(\Gamma); Q] \) in WRT \( \tau^{SO(3)}_K[M(\Gamma); Q] \) is

\[
F^{SO(3)}[\mathcal{L}(\Gamma); Q] = \sum_{n_1, n_2, \ldots, n_L \in \{0, 1, \ldots, K\}} V^{SO(3)}_{n_1, n_2, \ldots, n_L}(\mathcal{L}(\Gamma); Q) \prod_{v=1}^L V^{SO(3)}_{n_1, n_2, \ldots, n_L}(\bigcirc; Q)
\]

\[
= \sum_{n_1, n_2, \ldots, n_L \in \{1, 3, \ldots, 2K+1\}} J^{SU(2)}_{n_1, n_2, \ldots, n_L} \left( \mathcal{L}(\Gamma); q = e^{\frac{2\pi i}{2K+2}} \right)
\]

\[
\times \prod_{v=1}^L \frac{q^{n_v/2} - q^{-n_v/2}}{q^{1/2} - q^{-1/2}} \bigg|_{q^2 \to Q},
\]

(68)

where we have used the relation (13) to write \( SO(3) \) link invariants in terms of the colored Jones invariants. Notice that the summation is over only odd integers and hence, WRT invariant for \( SO(3) \) is different from the WRT for \( SU(2) \) group. Further, the highest integrable representation in the summation indicates that the Chern–Simons coupling for \( SU(2) \) group is \( 2K + 2 \). After performing the summation, we can convert the \( q = Q^{1/2}(13) \) to obtain \( SO(3) \) WRT invariant. We need to modify the Gauss sum reciprocity formula to incorporate the summation over odd integers in \( F^{SO(3)}[\mathcal{L}(\Gamma); Q] \).

Using the following Gauss sum reciprocity formula

\[
\sum_{n \in \mathbb{Z}^L/k\mathbb{Z}^L} \exp \left( \frac{2\pi i}{k} (n, Bn) + \frac{2\pi i}{k} (\ell, n) \right) = \frac{e^{\frac{2\pi i}{k}} (k/2)^{L/2}}{|\det B|^{1/2}} \sum_{a \in \mathbb{Z}^L/2B\mathbb{Z}^L} \exp \left( -\frac{\pi ik}{2} (a + \frac{\ell}{k}, B^{-1} (a + \frac{\ell}{k})) \right),
\]

(69)

for \( k = 2K + 2 \), we can obtain the summation over odd integers by replacing \( n \to \frac{n-1}{2} \);

\[
\sum_{n_1, n_2, \ldots, n_L \in \{1, 3, \ldots, 4K+3\}} q^{\frac{(n, Bn)}{4} + \frac{(n, d)}{4}} = \frac{e^{\frac{\pi i}{4}} (K+1)^{L/2}}{|\det B|^{1/2}} q^{\frac{(d, B^{-1}d)}{4}} \sum_{a \in \mathbb{Z}^L/2B\mathbb{Z}^L} \exp \left[ -\pi i (K+1)(a, B^{-1}a) \right] \exp \left[ -\pi i (a, B^{-1}(d + BI)) \right],
\]

(70)

where \( d = \ell - BI \) with \( I \) denoting \( L \) component vector with entry 1 on all the components. That is, the transpose of the vector \( I \) is

\[
I^T = [1, 1, \ldots, 1].
\]

(71)
For unknot with framing ±1, the $F^{SO(3)}[\mathcal{L}(-1\bullet); Q = q^2]$ involving summation over odd integers simplifies to

$$F^{SO(3)}[\mathcal{L}(\pm \bullet); Q = q^2] = \frac{\sqrt{K + 1}}{q^{1/2} - q^{-1/2}} e^{\pm \pi i 3/4} q^{3/4} (1 + e^{\pi i K}),$$  \hspace{1cm} (72)$$
as the coupling $K \in 2\mathbb{Z}$ for the $SO(3)$ Chern–Simons theory. Hence, the WRT invariant takes the following form:

$$\tau_K^{SO(3)} [M(\Gamma); Q = q^2] = \frac{e^{-\pi i 4}}{q^{4}} e^{\pm \pi i 4} q^{3/4} q^{2} \bigg( \sum_{n \in \{1,3,\ldots,2K+1\}^L} \prod_{v \in \text{Vertices}} \frac{1}{q_{v}^{n_v/2} - q^{-n_v/2}} \bigg)^{\deg(v) - 2} \prod_{(v',v'') \in \text{Edges}} \left( q^{n_{v'} n_{v''}/2} - q^{-n_{v'} n_{v''}/2} \right).$$ \hspace{1cm} (73)$$

In above equation, the terms involving edges of the graph $\Gamma$

$$\prod_{(v',v'') \in \text{Edges}} \left( q^{n_{v'} n_{v''}/2} - q^{-n_{v'} n_{v''}/2} \right) = 2^{L-1} \prod_{(v',v'') \in \text{Edges}} \frac{q^{n_{v'} n_{v''}/2} - q^{-n_{v'} n_{v''}/2}}{2},$$
can also be rewritten as

$$\prod_{(v',v'')} (q^{n_{v'} n_{v''}/2} - q^{-n_{v'} n_{v''}/2}) = \sum_{p \in \{\pm 1\}^\text{Edges}} \prod_{(v',v'') \in \text{Edges}} p_{(v',v'')} q^{p_{(v',v'')} n_{v'} n_{v''}/2}.$$  

Here again, if we make a change of variable as $n_v \longrightarrow -n_v$ at any vertex, a term in the sum with a given configuration of signs associated with edges (that is $p \in \{\pm 1\}^\text{Edges}$) will transform into a term with a different configuration times $(-1)^{\deg(v)}$. However, for these plumbing graphs $\Gamma$, the signs of such configuration can be brought to the configuration with all signs +1. Incorporating this fact, the WRT invariant(73) simplifies to

$$\tau_K^{SO(3)} [M(\Gamma); Q = q^2] = e^{-\pi i 4} q^{3\pi - \sum_v f_v} q^{3\pi - \sum_v f_v} 2^{(K + 1)/2} (q^{1/2} - q^{-1/2}) \bigg( \sum_{n \in \{1,3,\ldots,2K+1\}^L} \prod_{v \in \text{Vertices}} \frac{1}{q^{n_v/2} - q^{-n_v/2}} \bigg)^{\deg(v) - 2}.$$  \hspace{1cm} (74)$$

Further, we double the range of summation so as to use the reciprocity formula(70)

$$\tau_K^{SO(3)} [M(\Gamma); Q = q^2] = \frac{e^{-\pi i 4} q^{\sum_v f_v}}{4 (K + 1)/2 (q^{1/2} - q^{-1/2})} \bigg( \sum_{n \in \{1,3,\ldots,2K+1\}^L} \prod_{v \in \text{Vertices}} \frac{1}{q^{n_v/2} - q^{-n_v/2}} \bigg)^{\deg(v) - 2}.$$
\[
\times \sum_{n \in \{1, 3, \ldots, 4K+3\}} q^{(n, B_n)/4} \prod_{v \in \text{Vertices}} \left( \frac{1}{q^{n_v/2} - q^{-n_v/2}} \right)^{\deg(v)-2}.
\]

(75)

The steps discussed in the SU(2) context to extract \( \hat{Z} \) can be similarly followed for SO(3). This procedure leads to

\[
\tau_{SO(3)} K [M(\Gamma); Q = q^2] = \frac{1}{2(q^{1/2} - q^{-1/2}) |\det B|^{1/2}} \sum_{a \in \text{Coker} B} e^{-\pi i (K+1)(a, B^{-1}a)} \sum_{b \in 2\text{Coker} B + \delta} e^{-\pi i (a, B^{-1}(b + Bf))} \lim_{q \to \hat{Q}} \hat{Z}^{SO(3)}_b [M(\Gamma); q].
\]

(76)

We observe that the SO(3) WRT invariant is different from the SU(2) invariant due to the factor highlighted in blue color in the summand whereas the \( \hat{Z}^{SO(3)}_b [M(\Gamma); \hat{Q}] \) is exactly same as the SU(2) \( q \)-series. Even though SO(3) \( \cong SU(2)/\mathbb{Z}_2 \), it is surprising to see that the factor group SO(3) shares the same \( \hat{Z} \) as that of the parent group SU(2). The case of \( \hat{Z}^{SO(3)} \) was also considered in [4], but they took a different route by considering the refined WRT invariant which is consistent with our result.

In the following subsection, we will extract \( \hat{Z} \) from the WRT invariant \( \tau_{OSp(1\mid2)} \) for OSp(1\mid2) supergroup. We will see that the OSp(1\mid2) \( q \)-series are related to \( \hat{Z}^{SU(2)} \).

### 4.2. OSp(1\mid2) WRT and \( \hat{Z}^{OSp(1\mid2)} \) Invariant

Using the relation between OSp(1\mid2) and SU(2) link invariants (17), the WRT invariant can be written for plumbed manifolds \( M(\Gamma) \) as

\[
\tau_{OSp(1\mid2)} K [M(\Gamma); \hat{Q} = q^2] = \frac{e^{-\pi i \sigma} q^{3\sigma}}{(2\hat{K} + 3)^{L/2}} \left( q^{1/2} + q^{-1/2} \right) \sum_{n_1, n_2, \ldots, n_L \in \{1, 3, \ldots, 2\hat{K}+1\}} \prod_{v \in \text{Vertices}} q^{f_v(n_v^2 - 1)/2} \left( \frac{1}{q^{n_v/2} + q^{-n_v/2}} \right)^{\deg(v)-2} \prod_{(v', v'') \in \text{Edges}} \left( q^{n_v n_{v''}/2} + q^{-n_v n_{v''}/2} \right)\bigg|_{q = \hat{Q}}.
\]

(77)

Here again, we use the Gauss reciprocity(70) as the summation is over odd integers to work out the steps leading to \( \hat{Z}^{OSp(1\mid2)} [M(\Gamma); \hat{Q}] \). Note that, the highest integrable representation \( 2\hat{K} + 1 \) which fixes the \( q \) as \( (2\hat{K}+2) \)-th root of unity. However, to compare the result with OSp(1\mid2) WRT, we have to replace \( \hat{K} + 1 \rightarrow 2\hat{K} + 3 \) which is equivalent to \( q = \hat{Q} \).

Following similar steps performed for SU(2), we find the following expression for OSp(1\mid2) WRT invariant:
\[
\frac{1}{2 (q^{1/2} + q^{-1/2}) | \det B|^{1/2}} \sum_{a \in \text{Coker } B} e^{-\pi i(2\hat{K}+3)(a,B^{-1}a)} \times \sum_{b \in 2\text{Coker } B + \delta} e^{-\pi i(a,B^{-1}(b+\delta))} \lim_{q \to \hat{Q}} \hat{Z}_b^{OSp(1|2)} [M(\Gamma); q],
\] (78)

where \( I \) is again the column vector \((71)\) and \( \hat{Z}_b^{OSp(1|2)} [M(\Gamma); q] \) is given by the following algebraic expression:

\[
\hat{Z}_b^{OSp(1|2)} [M(\Gamma); q] = 2^{-L} q^{-\frac{3L+\sum_v f_v}{4}} \sum_{d \in 2BZ^L + b} F^d_1 q^{-\frac{(d,B^{-1}d)}{4}},
\] (79)

with coefficient \( F^d_1 \) is obtained by following relation

\[
\sum_{d \in 2Z^L + \delta} F^d_1 \prod_v x_v^{d_v} = \prod_{v \in \text{Vertices}} \left\{ \text{Expansion at } x \to 0 \frac{1}{(x_v + 1/x_v)^{\deg v - 2}} + \text{Expansion at } x \to \infty \frac{1}{(x_v + 1/x_v)^{\deg v - 2}} \right\}.
\] (80)

Equivalently, \( \hat{Z}^{OSp(1|2)} [M(\Gamma); q] \) can also represented as the following contour integral:

\[
\hat{Z}_b^{OSp(1|2)} [M(\Gamma); q] = q^{-\frac{3L+\sum_v f_v}{4}} \text{ v.p.} \int_{|z_v| = 1} \prod_{v \in \text{Vertices}} \frac{dz_v}{2\pi i z_v} (z_v + 1/z_v)^{2-\deg(v)} \cdot \Theta^{-B}_b(z).
\] (81)

Here, \( \Theta^{-B}_b(x) \) is the theta function of the lattice corresponding to minus the linking form \( B \):

\[
\Theta^{-B}_b(x) = \sum_{d \in 2BZ^L + b} q^{-\frac{(d,B^{-1}d)}{4}} \prod_{i=1}^L x_i^{d_i}
\] (82)

and “v.p.” again means that we take principle value integral (\( i.e. \) take half-sum of contours \( |z_v| = 1 \pm \epsilon \)). Comparing Eqns.\( (79,80) \) with the \( SU(2) \) expressions\( (53,54) \), we can see that the \( \hat{Z} \) for \( OSp(1|2) \) are different from \( SU(2) \) q-series. We will now present explicit q-series for some examples.

4.2.1. Examples.

- For the Poincare homology sphere \((62)\), we find the following \( OSp(1|2) \) q-series

\[
\hat{Z}_b^{OSp(1|2)} = q^{-3/2} (1 + q + q^3 + q^7 + q^8 + q^{14} + q^{20} - q^{29} + q^{31} - q^{42} - q^{52} + \cdots).
\] (83)

- In the case of Brieskorn homology sphere \((64)\), the \( OSp(1|2) \) q-series is
\[ \hat{Z}_{b_i}^{OSp(1|2)} = q^{1/2} (1 + q + q^5 + q^{10} + q^{11} + q^{18} + q^{30} + q^{41} - q^{43} - q^{56} - q^{76} \cdots ). \]  

(84)

- For the case of plumbing graph (66), the three homological blocks are

\[
\hat{Z}^{OSp(1|2)} = \begin{pmatrix}
1 + q + q^6 + q^{11} - q^{13} - q^{20} - q^{35} + O(q^{41}) \\
q^{5/3} (1 + q^3 - q^{21} - q^{30} + O(q^{41})) \\
q^{5/3} (1 + q^3 - q^{21} - q^{30} + O(q^{41}))
\end{pmatrix}.
\]

(85)

After comparing the \(q\)-series for \(SU(2)\) and \(OSp(1|2)\), we noticed that these two \(q\)-series are related by a simple change of variable which is \(q \rightarrow -q\). This change of variable applies only to the series not to the overall coefficient outside the series.

- Lens space \(L(p,q)\) is a well studied three-manifold. For \(L(-5,11) \sim L(-13,29)\) whose plumbing graph is shown below, we obtain the five homological blocks

\[ \hat{Z}^{OSp(1|2)} = \begin{pmatrix}
q^{1/10} \\
q^{-1/10} \\
0 \\
q^{-1/10} \\
q^{1/10}
\end{pmatrix} \text{ as } H_1(M,\mathbb{Z}) = \mathbb{Z}_5. \]

(86)

- For the following plumbing graph, \(H_1(M,\mathbb{Z}) = \mathbb{Z}_{13},\)
5. Conclusions and Future Directions

Our goal was to investigate \( \hat{Z} \) for \( SO(3) \) and \( OSp(1|2) \) groups for negative definite plumbed three-manifolds. The change of variable and color indeed relates invariants of framed links \( \mathcal{L}[\Gamma](13,17) \) of \( SO(3) \) and \( OSp(1|2) \) to colored Jones. Such a relation allowed us to go through the steps of GPPV conjecture to extract \( \hat{Z} \) from WRT invariants. Interestingly, we observe that the \( \hat{Z}^{SO(3)} \) is same as \( \hat{Z}^{SU(2)} \) even though the WRT invariants are different. We know that \( SU(2)/\mathbb{Z}_2 \equiv SO(3) \) and it is not at all obvious that the homological blocks are same for both the groups. It is important to explore other factor groups and the corresponding \( \hat{Z} \) invariants.

We have checked for many examples that under \( q \to -q \) in the \( OSp(1|2) \) \( q \)-series (not affecting the overall coefficient), we obtain the \( SU(2) \) \( q \)-series.
For the odd orthosymplectic supergroup $OSp(1|2)$, we observe from our computations for many negative definite plumbing graph $\Gamma$:

$$\hat{Z}^{OSp(1|2)}_b(\Gamma; q) = 2^{-c} q^{\Delta_b} \left( \sum_n a_n q^n \right)$$

whereas their $SU(2)$ q-series is

$$\hat{Z}^{SU(2)}_b(\Gamma; q) = 2^{-c} q^{\Delta_b} \left( \sum_n a_n (-q)^n \right)$$

where $c \in \mathbb{Z}_+$, $\Delta_b \in \mathbb{Q}$.

The brane setup in string theory for $U(N)$ gauge group gives a natural interpretation for these q-series as partition function of the theory $T^G[M]$. In principle, there should be a natural generalization to orthogonal $SO(N)$ and symplectic group $Sp(2n)$ involving orientifolds. It will be worth investigating such a construction to obtain $\hat{Z}$ for $SO(N)$ group and compare with our $SO(N = 3)$ results. Extension of $\hat{Z}$ to the two variable series for link complements [10] is another direction to pursue. We hope to report on these aspects in future.

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