Worldsheet Instantons, Torsion Curves and Non-Perturbative Superpotentials

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Abstract

As a first step towards computing instanton-generated superpotentials in heterotic standard model vacua, we determine the Gromov-Witten invariants for a Calabi-Yau threefold with fundamental group $\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$. We find that the curves fall into homology classes in $H_2(X, \mathbb{Z}) = \mathbb{Z}^3 \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_3)$. The unexpected appearance of the finite torsion subgroup in the homology group complicates our analysis. However, we succeed in computing the complete genus-0 prepotential. Expanding it as a power series, the number of instantons in any integral homology class can be read off. This is the first explicit calculation of the Gromov-Witten invariants of homology classes with torsion. We find that some curve classes contain only a single instanton. This ensures that the contribution to the superpotential from each such instanton cannot cancel.

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1 Introduction

One of the challenges of string theory is to stabilize all moduli. There has been considerable progress in this direction in recent years, particularly for closed string moduli in type II compactifications [1, 2, 3, 4, 5]. Comparably less progress has been made in stabilizing the moduli of $N = 1$ supersymmetric $E_8 \times E_8$ heterotic string vacua. Non-vanishing flux can remove some moduli, most notably those associated with deformations of the complex structure. However, stabilizing the remaining moduli is made difficult by the fact that it is not possible to generate a perturbative superpotential for them. This follows from $N = 1$ supersymmetry and the fact that, classically, deformations of the Calabi-Yau threefold and holomorphic vector bundle are unobstructed. If there were only perturbative corrections, this would rule out $N = 1$ supersymmetric heterotic string compactifications, even those with non-zero flux. However, the complete string theory admits non-perturbative corrections as well. The best understood of these is the superpotential generated by gaugino condensation in the strongly coupled hidden sector [6]. Gaugino condensation can help to fix at least some of the remaining moduli. However, this can never stabilize them all. In particular, the moduli associated with the deformations of the holomorphic vector bundle do not enter either the flux or the gaugino condensate superpotentials. Hence, the potential energy remains flat in those directions.

To fix the remaining moduli in $N = 1$ supersymmetric heterotic vacua, one must consider the non-perturbative superpotential generated by worldsheet instantons. It was shown in [7, 8, 9, 10, 11, 12, 13] that a string wrapped on a single genus-0 holomorphic curve, $C$, in the background Calabi-Yau threefold generates a non-vanishing contribution to the superpotential. This contribution is of the form $\text{const} \cdot P(\phi) \cdot \exp \int_\gamma \omega$, where $P(\phi)$ is the “Pfaffian”, typically a homogeneous polynomial of the vector bundle moduli $\phi$, and $\omega$ is the Kähler form. Note that the vector bundle moduli enter the superpotential through these worldsheet contributions and, at least in principle, can now be stabilized. This was demonstrated within the context of a simplified theory in [12, 14, 15]. Clearly, computing the complete instanton superpotential requires one to sum over all such contributions and, hence, to count and classify all holomorphic genus-0 curves. It is well-known that these are specified by the “instanton numbers”, or Gromov-Witten invariants, of the Calabi-Yau threefold.

There are two approaches towards computing these invariants. For some manifolds, one can directly compute the instanton corrections to the prepotential of the A-model topological string theory. For Calabi-Yau manifolds with a “mirror” threefold, a second method is much easier [16]. First, the classical prepotential of the B-model topological theory on the mirror manifold is computed. Mirror symmetry identifies this with the prepotential of the A-model on the original Calabi-Yau space. One finally expands this in a power series to find the instanton numbers. Using these methods, the instanton numbers of a variety of Calabi-Yau manifolds have been computed, such as toric complete intersection threefolds.
Once the Gromov-Witten invariants of the Calabi-Yau threefold of a heterotic vacuum are known, one can attempt to compute the complete instanton superpotential. To do this, the holomorphic vector bundle must be specified. For generic heterotic vacua, the calculation of the superpotential is difficult. Hence, it has only been fully carried out under very restrictive conditions, such as on a toric complete intersection Calabi-Yau manifold with the vector bundle induced from the ambient space. Rather remarkably, in all of these simple cases, the instanton-generated superpotential vanishes \[17, 18, 19, 20, 21\]. This cancellation occurs in a very distinctive way. The contribution to the superpotential of each curve in a fixed homology class contains the same factor $\exp \int_C \omega$. Therefore, when one sums over the curves in this class the exponential factors out leaving a sum over the Pfaffian components. In the restrictive cases analyzed in \[17, 18, 19, 20, 21\], this sum of Pfaffians cancels. Hence, the complete instanton induced superpotential vanishes.

In this paper, we begin the calculation of the non-perturbative superpotential induced by the worldsheet instantons of a very different class of heterotic string vacua, namely, the “heterotic standard model” compactifications presented in \[22, 23, 24, 25, 26, 27, 28\]. These theories have the exact matter spectrum of the minimal supersymmetric standard model in the visible sector, and a relatively small number of both geometric and vector bundle moduli. The associated Calabi-Yau manifold, $X$, is the quotient of an elliptically fibered threefold on which the finite group $\mathbb{Z}_3 \times \mathbb{Z}_3$ acts freely. Hence, $\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$. This does not fall into the category of manifolds, such as toric complete intersection spaces, considered in the previous paragraph. Furthermore, the holomorphic vector bundle is constructed by extension and is not related to the tangent bundle of the Calabi-Yau space nor is it coming from the ambient space. It follows that the vanishing theorems do not apply. Indeed, various arguments lead to the conclusion that the complete instanton superpotential of heterotic standard model vacua is non-zero. Nevertheless, there remains a possibility of cancellations between the different instantons in each given curve class. Therefore, the complete superpotential in this context must be computed with great care.

To evaluate the instanton superpotential of heterotic standard model vacua, one must first compute the degree-2 integer homology of $X$. We find that

$$H_2(X, \mathbb{Z}) = H_2(X, \mathbb{Z})_{\text{free}} \oplus H_2(X, \mathbb{Z})_{\text{tors}} = \mathbb{Z}^3 \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_3).$$

(1)

That is, the homology classes of curves are not just a free rank 3 lattice, but contain a torsion subgroup $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. Non-vanishing torsion subgroups in the homology of Calabi-Yau spaces have been calculated in a small number of differing contexts \[29, 30, 31, 32\]. As discussed below, it is essential when computing the complete instanton superpotential that the torsion subgroup of the degree-2 integral homology be explicitly computed.

Second, one must compute the number of holomorphic rational curves\(^1\) on the

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\(^1\)That is, $\mathbb{P}^1 \subset X$. Due to holomorphy, higher genus curves do not generate a superpotential.
Calabi-Yau threefold $X$. As discussed previously, this is accomplished by calculating the genus-0 prepotential in the associated topological string theories. However, for the non-simply connected, elliptic Calabi-Yau threefolds of heterotic standard model vacua, this calculation is complicated by two facts. First, it is usually assumed that mirror symmetry will allow one to completely solve for the instanton numbers. However, trying to do the computation in the present context one quickly realizes that this is not so. Since $X$ is not a complete intersection, one cannot simply apply the toric algorithms. Second, no one has previously computed the topological string prepotential for torsion curves. We have solved both of these problems. First consider the A-model.

- Quotienting the prepotential on the covering space $\tilde{X}$ given in [33], we can compute much of the prepotential on $X = \tilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. This prepotential includes torsion information, but is only valid to linear order in one of the Kähler moduli.

- The same part of the prepotential on $X$ can be computed directly by counting curves on $X$, with complete agreement.

These results can be extended using the B-model and mirror symmetry.

- Mirror symmetry for the toric complete intersection $\tilde{X}$ is an algorithm to compute the prepotential. However, there are many non-toric divisors. Therefore, after descending to $X$ the prepotential is valid to all orders in the Kähler moduli but the torsion information is lost.

- We show that $\tilde{X}$ is self-mirror. It turns out that in the mirror representation all moduli are toric, so that we can give a numerical algorithm for calculating the instanton numbers on $X$ to any degree in the Kähler moduli and including all torsion information.

We will present the calculation of the integer homology, including torsion, in [34]. In that paper, the prepotential and the associated Gromov-Witten invariants will be computed purely within the context of the A-model. The B-model calculations will be given in a second paper [35]. Here, we simply summarize the arguments and present the final results.

Having computed the instanton numbers, the final step is to introduce the holomorphic vector bundles associated with heterotic standard model vacua. Specifying this, one calculates the non-perturbative superpotential induced by each holomorphic curve and then sums over all curves in a given homology class and over all homology classes. This will be attempted in future publications. However, we can already state one important result. Namely, we will find that in some curve homology classes on $X$ there is a single instanton, which locally looks like $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore, its contribution cannot cancel against other instanton contributions.

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2On the quintic, quite distinctly, each effective curve class contains thousands of instantons.
2 The Calabi-Yau threefold

To obtain phenomenological low energy gauge groups within the context of the $E_8 \times E_8$ heterotic string, it is necessary to spontaneously break the visible sector $E_8$. One way to do this, which was used in the context of heterotic standard models \[23\,24\,25\,22\], is to first embed an $SU(4)$ gauge instanton in the $E_8$, breaking it initially to its commutant $Spin(10)$. One then adds a $Z_3 \times Z_3$ Wilson line, breaking the gauge group further to

$$E_8 \longrightarrow Spin(10) \longrightarrow SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}.$$  \(2\)

In order to turn on discrete Wilson lines, the manifold must have sufficiently many non-contractible curves. Hence, we are interested in a Calabi-Yau threefold $X$ with $\pi_1(X) = Z_3 \times Z_3$.

In this paper, we consider a slightly different quotient than the one in \[36\]. This is done so as to simplify the application of toric mirror symmetry. However, in practice there is little difference between the two spaces and it is a simple exercise to extend the results of this paper to the heterotic standard model manifold. The threefold we consider here is the free $Z_3 \times Z_3$ quotient

$$X = \tilde{X} / (Z_3 \times Z_3)$$  \(3\)

of the complete intersection

$$\tilde{X} = \left\{ \begin{array}{l}
  t_0 \left( x_0^3 + x_1^3 + x_2^3 \right) + t_1 \left( x_0 x_1 x_2 \right) = 0 \\
  \left( \lambda_1 t_0 + t_1 \right) \left( y_0^3 + y_1^3 + y_2^3 \right) + \left( \lambda_2 t_0 + \lambda_3 t_1 \right) \left( y_0 y_1 y_2 \right) = 0 \\
  \end{array} \right\}$$  \(4\)

The defining equations allow for three complex parameters $\lambda_1$, $\lambda_2$, $\lambda_3$ respecting the free $Z_3 \times Z_3$ group action on $\tilde{X}$,

$$g_1 : \begin{cases}
  [x_0 : x_1 : x_2] \mapsto [x_0 : \zeta x_1 : \zeta^2 x_2] \\
  [t_0 : t_1] \mapsto [t_0 : t_1] \ (\text{no action}) \\
  [y_0 : y_1 : y_2] \mapsto [y_0 : \zeta y_1 : \zeta^2 y_2] 
\end{cases}$$  \(5\)

$$g_2 : \begin{cases}
  [x_0 : x_1 : x_2] \mapsto [x_1 : x_2 : x_0] \\
  [t_0 : t_1] \mapsto [t_0 : t_1] \ (\text{no action}) \\
  [y_0 : y_1 : y_2] \mapsto [y_1 : y_2 : y_0],
\end{cases}$$

These 3 parameters are the $h^{21}(X) = 3$ complex structure moduli.
where \( \zeta = e^{2\pi i/3} \) is a third root of unity. The Hodge numbers and homology groups of the covering space \( \tilde{X} \) have been worked out previously [37, 38]. They are

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 19 & 0 \\
0 & 0 & 19 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
h^{p,q}(\tilde{X}) = 1 \quad 19 \quad 19 \quad 1, \quad H_i(\tilde{X}, \mathbb{Z}) = \begin{cases}
\mathbb{Z} & i = 6 \\
0 & i = 5 \\
\mathbb{Z}^{19} & i = 4 \\
\mathbb{Z}^{40} & i = 3 \\
\mathbb{Z}^{19} & i = 2 \\
0 & i = 1 \\
\mathbb{Z} & i = 0
\end{cases}
\]  

(6)

The homology of the quotient \( X = \tilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3) \) is more complicated to determine. We will investigate it in the following section. But before doing so, note that there are nine particularly simple genus-0 curves in \( X \), which turn out to be important. Inspecting the defining equations, we see that they are identically satisfied if

\[
x_0^3 + x_1^3 + x_2^3 = 0 = x_0x_1x_2, \quad y_0^3 + y_1^3 + y_2^3 = 0 = y_0y_1y_2.
\]

(7)

Since two cubics intersect in nine points, there are nine such solutions \([x_0^{(i)} : x_1^{(i)} : x_2^{(i)}]\) for the first equation and nine independent solutions \([y_0^{(j)} : y_1^{(j)} : y_2^{(j)}]\) for the second equation. This yields \( 9 \cdot 9 = 81 \) genus-0 curves

\[
\tilde{C}_{ij} : \mathbb{P}^1 \to \tilde{X}, \quad [t_0 : t_1] \mapsto (x_0^{(i)} : x_1^{(i)} : x_2^{(i)}), [t_0 : t_1], [y_0^{(j)} : y_1^{(j)} : y_2^{(j)}])
\]

(8)

on \( \tilde{X} \), each wrapping the \( \mathbb{P}^1_{[t_0 : t_1]} \). The \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) group action identifies them in 9-tuples and, hence, they define 9 genus-0 curves

\[
\{s_0, \ldots, s_8\} = \{\tilde{C}_{ij}/(\mathbb{Z}_3 \times \mathbb{Z}_3)\}
\]

(9)

on the quotient threefold \( X \).

### 3 Quotient Homology

How can we determine the homology of the quotient \( X \) in terms of the homology of the covering space \( \tilde{X} \) and, in particular, where does the torsion part \( H_2(X, \mathbb{Z})_{\text{tors}} = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \) come from? The most complete answer to this question would be the Cartan-Leray spectral sequence. However, one can understand \( H_2(X, \mathbb{Z}) \) much more simply as follows.

Each curve on the covering space yields a curve on the quotient by taking its image under the quotient map \( q : \tilde{X} \to X \). Now for each rational curve \( \tilde{C} \) on \( \tilde{X} \), there are
8 other images \( g \tilde{C}, \ g \in \mathbb{Z}_3 \times \mathbb{Z}_3 \) under the group action. Clearly, their projection to the quotient is the same,

\[
q(\tilde{C}) = q(g \tilde{C}).
\] (10)

Hence, one should impose the relations \( \tilde{C} - g \tilde{C} = 0 \) on the homology of \( \tilde{X} \) to learn about the homology of \( X \). This is called the **coinvariant homology** of \( \tilde{X} \). We find that

\[
H_2(\tilde{X}, \mathbb{Z})_{\mathbb{Z}_3 \times \mathbb{Z}_3} = H_2(\tilde{X}, \mathbb{Z}) / \text{span} \{ \tilde{C} - g \tilde{C} \} = \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3.
\] (11)

It turns out that these relations contain \( 3(s_i - s_j) \), but do not include \( s_i - s_j \). This is the origin of torsion in the quotient. The 8 non-trivial torsion homology classes can be represented by the differences \( s_i - s_0, \ i = 1, \ldots, 8 \).

In general, the coinvariant homology is only one ingredient in the Cartan-Leray spectral sequence and need not coincide with the homology of the quotient. However, in our case this turns out to be sufficient for the curve classes [34]. Hence, we obtain

\[
H_2(X, \mathbb{Z}) = H_2(\tilde{X}, \mathbb{Z})_{\mathbb{Z}_3 \times \mathbb{Z}_3} = \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3.
\] (12)

In summary, the Hodge numbers and homology groups of the quotient \( X = \tilde{X} / \mathbb{Z}_3 \times \mathbb{Z}_3 \) are

\[
h^{p,q}(X) = 1 \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{array}, \quad H_i(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 6 \\ 0 & i = 5 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 & i = 4 \\ \mathbb{Z}^8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 & i = 3 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 & i = 2 \\ \mathbb{Z}_3 \oplus \mathbb{Z}_3 & i = 1 \\ \mathbb{Z} & i = 0 \end{cases} \] (13)

Note that these results are in accord with the self-mirror property [35] and the conjecture of [30] that the torsion parts \( H_1(X, \mathbb{Z})_{\text{tors}} \) and \( H_2(X, \mathbb{Z})_{\text{tors}} \) are exchanged under mirror symmetry.

### 4 The Prepotential

Having found the homology classes of curves, we are ready to count the number of genus-0 curves in each homology class. This is achieved by calculating the worldsheet

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4There is no analytic free map \( \mathbb{P}^1 \to \mathbb{P}^1 \) without fixed points, so the free group action must map genus-0 curves to different genus-0 curves.
instanton corrections to the genus-0 prepotential in the A-model topological string on $X$. Formally, the non-perturbative prepotential is given by

$$F_{np}^{X,0} = \sum_{C \in H_2(X,\mathbb{Z})} n_C \, \text{Li}_3 \left( e^{2\pi i f_C \omega} \right),$$

where $\omega$ is the Kähler class. The integer $n_C$ is the instanton number in the homology class $C$ which we will compute in the following.

First, note that the prepotential seemingly does not distinguish to rsion homology classes, since $C$ only appears in the integral $\int_C \omega$ and the integral of a closed form over a torsion homology class vanishes. However, this neglects the fact that $\omega$ is the complexified Kähler class

$$\omega = t^a e_a = B + iJ.$$

One can always expand the Kähler form in a basis of harmonic $(1,1)$-forms $\{e_a\}$, but that is not entirely true for the $B$-field. To define the topological string on a Kähler manifold, the $B$-field must be locally closed, $dB = 0$. However, this only requires that its characteristic class $H \in H^3(X,\mathbb{Z})$ vanishes in $H^3(X,\mathbb{R})$. In other words, $B$ need not be globally defined and, in that case, $\int_C B$ makes no sense.

The correct way to define the instanton contribution to the path integral, $e^{iS(C)} = e^{2\pi i f_C \omega}$, is as an abstract map

$$e^{iS} : H_2(X,\mathbb{Z}) \to \mathbb{C}^\times.$$ (16)

Such a map is determined by the image of the generators. In the case at hand, we need 3 generators for $H_2(X,\mathbb{Z})_{\text{free}} = \mathbb{Z}^3$ and 2 more for $H_2(X,\mathbb{Z})_{\text{tors}} = \mathbb{Z}_3 \oplus \mathbb{Z}_3$. We denote the image of the free generators as

$$e^{iS}(s_0) = p = e^{2\pi i t^1}, \quad q = e^{2\pi i t^2}, \quad r = e^{2\pi i t^3},$$

where $s_0$ is the curve defined in eq. (9). Finally, we denote the image of the two torsion generators as $b_1$ and $b_2$. Since three times any torsion curve class is zero, the torsion generators satisfy

$$b_1^3 = b_2^3 = 1.$$ (18)

The instanton factor $e^{iS}(C)$ of any curve class

$$C = (n_1, n_2, n_3, m_1, m_2) \in \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 = H_2(X,\mathbb{Z})$$

is then a monomial in these generators, that is,

$$e^{iS}(C) = p^{n_1} q^{n_2} r^{n_3} b_1^{m_1} b_2^{m_2}.$$ (20)

It follows that the genus-0 non-perturbative superpotential eq. (14) is a power-series expansion in $p, q, r, b_1, b_2$ given by

$$F_{np}^{X,0} = \sum_{C \in H_2(X,\mathbb{Z})} n_C \, \text{Li}_3 \left( p^{n_1} q^{n_2} r^{n_3} b_1^{m_1} b_2^{m_2} \right).$$ (21)
5 Counting Curves

To explicitly compute the instanton number for each curve class, one must evaluate the non-perturbative superpotential in the A-model topological field theory on $X$. After expanding as a power series in the generators, the instanton numbers can be read off by comparing with eq. (21). Although the general form of the expansion for non-simply connected Calabi-Yau threefolds with torsion has been known for a long time [39], no such example has ever been explicitly computed. The obvious guess as to how to do this would be to use mirror symmetry. However, it is not known how to deal with torsion curves in this context.

To compute the non-perturbative prepotential on $X$, we use a combination of different techniques. To begin with, we evaluate the prepotential directly in the A-model as follows.

- Part of the prepotential on $\tilde{X}$ is known analytically [33]. Quotienting this result, one can calculate the prepotential on $X$ to all orders in $q, r, b_1, b_2$, but only to $O(p)$. We also compute the same part of the prepotential by a direct A-model computation on $X$.

The detailing calculations will be given in [34]. Here, we simply present the results. We find that

$$F^{\text{np}}_{X,0}(p, q, r, b_1, b_2) = \left( \sum_{i,j=0}^{2} pb_i b_j^2 \right) P(q)^4 P(r)^4 + O(p^2)$$

(22)

Expanding this as a power series and comparing it to eq. (21), one can read off the instanton numbers. The constant part in $p$ vanishes, so

$$n_{(0,n_2,n_3,m_1,m_2)} = 0 \quad \forall n_2, n_3 \in \mathbb{Z}, m_1, m_2 \in \mathbb{Z}.$$

(23)

At linear order in $p$, that is, $n_1 = 1$, the instanton numbers are non-vanishing. However, they do not depend on the torsion part of the homology class. That is,

$$n_{(1,n_2,n_3,m_1,m_2)} = n_{(1,n_2,n_3,0,0)} \quad \forall m_1, m_2 \in \{0, 1, 2\}.$$

(24)

We list the instanton numbers for $n_2, n_3 \leq 5$ in Table 1. An important conclusion can be drawn from these results. The 9 curves $s_0, \ldots, s_8$ of eq. (9) contribute at degree $pb_i b_j^2$. We see from Table 1 that

$$n_{(1,0,0,m_1,m_2)} = 1, \quad \forall m_1, m_2 \in \{0, 1, 2\}.$$

(25)
| $n_2$ | $n_3$ 0 1 2 3 4 5          |
|-------|-------------------------|
| 0     | 1 14 40 105 252         |
| 1     | 4 16 56 160 420 1008    |
| 2     | 14 56 196 560 1470 3528 |
| 3     | 40 160 560 1600 4200 10080 |
| 4     | 105 420 1470 4200 11025 26460 |
| 5     | 252 1008 3528 10080 26460 63504 |

**Table 1:** Instanton numbers $n_{(1,n_2,n_3,*,*)}$ computable in the A-model. In this case (for $n_1 = 1$), the instanton number is independent of the torsion part of the homology class.

Hence, each of the curves $s_0, \ldots, s_8$ is the lone genus-0 curve in their respective homology class. It follows from this that the worldsheet instanton induced non-perturbative superpotential can *not* vanish in heterotic standard model vacua. The explicit superpotential in these theories will be computed elsewhere. Note that had we ignored the torsion subgroup in the degree-2 integer homology, the number of instantons in the $(1,0,0)$ homology class would have been found to be $n_{(1,0,0)} = 9$. One would then be unable to reach any conclusion about whether or not the superpotential vanished. Hence, it is essential that the torsion information be explicitly included when computing the instanton numbers.

Although these results are already significant, one would like to extend them to any order in the modulus $p$, not just linear order. This can be accomplished in two ways.

- While a direct application of mirror symmetry to the covering space $\tilde{X}$ fails (there are not enough toric divisors), one can compute the prepotential on the Batyrev-Borisov mirror $\tilde{X}^*$. This can be used to determine any desired term in the A-model prepotential of $X$, but is computationally very intensive.

- The toric mirror of the partial quotient $\tilde{X}/\mathbb{Z}_3$ is manageable, and can be used to compute the expansion of the A-model prepotential on $X$ to any desired order.

Using these techniques, one can obtain the expansion of the non-perturbative prepotential to any order in each of the generators. The explicit calculations will be given in [35]. Here, we simply present some of the results. We find that up to total degree
4 in \( p, q, r \), the genus-0 non-perturbative prepotential is

\[
\mathcal{F}_{X,0}^{np}(p, q, r, b_1, b_2) =
\]

\[
+ \sum_{i,j=0}^2 \left( \text{Li}_3(p b_i b_j) + 4 \text{Li}_3(pq b_i b_j) + 4 \text{Li}_3(pr b_i b_j) + 14 \text{Li}_3(pq^2 b_i b_j) + 16 \text{Li}_3(pqr b_i b_j) + 14 \text{Li}_3(pr^2 b_i b_j) + 40 \text{Li}_3(pq^3 b_i b_j) + 56 \text{Li}_3(pq^2 r b_i b_j) + 56 \text{Li}_3(pqr^2 b_i b_j) + 40 \text{Li}_3(pq^3 b_i b_j) - 2 \text{Li}_3(p^2 q b_i b_j) - 2 \text{Li}_3(p^2 r b_i b_j) - 28 \text{Li}_3(p^2 q^2 b_i b_j) + 32 \text{Li}_3(p^2 q r b_i b_j) - 28 \text{Li}_3(p^2 r^2 b_i b_j) \right)
\]

\[
+ 3 \text{Li}_3(p^3 q) + 3 \text{Li}_3(p^3 r) + \text{(degree} \geq 5)\right) .
\]

(26)

The instanton numbers can be read off by comparing this expansion to eq. (21). First, we find that the order \( p \) instanton numbers obtained from this expansion are identical to those given in Table 1, as they must be. Second, the order \( p^2 \) instanton numbers do not depend on the torsion part of the homology class, as was the case at order \( p \).

Some illustrative results at order \( p^3 \) are given in Table 2. Interestingly, we observe that at order \( p^3 \) the number of instantons does depend on the torsion part of their homology class. Note, for example, that there are 3 instantons in the homology class \( (3,1,0,0,0) \in H_2(X, \mathbb{Z}) \), but there are no instantons in the other 8 curve classes \( (3,1,0,i,j) \in H_2(X, \mathbb{Z}) \) that differ in their torsion part.

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