High frequency band sensitivity of the gravitational wave interferometer

N I Kolosnitzyn\textsuperscript{1} and V N Rudenko\textsuperscript{2}

\textsuperscript{1}Shmidt Earth Physics Institute of RAS, Moscow, B. Gruzinskaya 10
nikkols_emto@mail.ru
\textsuperscript{2}Sternberg Astronomical Institute, MSU, Moscow, Universitetskii prospect 13
rvn@sai.msu.ru

Abstract. It is shown that a reception frequency bandwidth of the large interferometer gravitational wave detectors in principle can be expanded in the high frequency side up to 100 kHz due to special “windows of reconstructed sensitivity” which have to appear in regions of EM-gravity resonances.

PACS: 04.80.Nn

1. Introduction

Recently the two long base laser interferometer gravitational wave detectors LIGO [1] and VIRGO [2] performed the first relatively short joint scientific run (S5 in LIGO, VSR-1 in VIRGO) with a total duration of 4.5 months [3]. After some new upgrade they have started (July 2009) with a second joint scientific run (S6, VSR-2).

General principles of such detectors were developed during the last quarter of the past century and published in many papers, monographs and textbooks (see for example [4]).

A conventional theory of GW interferometers usually was considered with two main physical restrictions.

First, it was accepted that a frequency of cosmic gravitational waves has to be much smaller the optical (a laser pump) frequency.

This restriction introduces a small parameter \( \frac{\lambda_e}{\lambda_g} \ll 1 \) that allows using of a linear theory of gravitational wave interaction with the interferometer. Indeed for typical astrophysical GW sources this parameter does not exceed \( \sim 10^{-10} \) and even for the high frequency tail of relic gravitational wave background it remains to be very small \( \sim 10^{-4} \) or less [5].

Second, it was considered the length of the interferometer base \( L \) was much less the gravitational wave length, or more exactly, the interferometer light circulation frequency (or free spectral range frequency for FP cavity) \( \nu_0 = c/2L \) was much larger the gravitational wave frequency i.e. \( \left( \nu_g / \nu_0 \right) \ll 1 \). For Michelson interferometers with FP cavities into its arms [1,2] the physical length has to be replaced by the effective value \( L_{\text{eff}} = FL \) (where \( F \) is the finesse of FP cavity) with the limitation \( L_{\text{eff}} \leq \frac{\lambda_g}{2} \). The last condition also means that such interferometer as a GW detector operates in a regime of “slow motions”: the period of mirror’s oscillation \( \tau_g \) produced by the GW action has to be smaller the optical relaxation time of the FP cavity \( \tau^* = F / \nu_0 \).

Under this “slow motion” (or “large wave length”) approximation one can define the interferometer antenna pattern (as a GW detector) independently on a GW frequency.

At frequencies \( \nu_g \geq \nu_0 / F \) a detector transfer function begins to decrease \( \sim \nu_0 / F \nu_g \). It is just the reason of sensitivity degradation at the high frequency wing for the large GW interferometer detectors despite the optical shot noise spectrum density is kept at the constant level.

However it was noted in [6] that the interferometer transfer function at high frequencies might be reconstructed inside special narrow windows around frequencies of so called “gravitational - electromagnetic resonance” when GW frequency \( \nu_g \) equals to an integer number of the interferometer free spectral range \( \nu_0 \).

In this paper we continue this investigation more in detail. To simplify calculations we consider below the Michelson structure with multi-pass delay lines instead of FP cavities in the interferometer’s arms.
estimate the reconstructed sensitivity inside the resonance windows and also calculate a correspondent structure of the detector antenna pattern.

2. Optical delay line.

Electrical field of the laser beam going along the $x$ direction through the optical delay line (ODL) in the presence of a gravitational wave can be read as [7, 8]

$$
E^{(-)} = \frac{1}{2} r_1^{N-1} r_2^N \exp(i\omega_0 t) \left( h\hat{D}^N A \right) + c.c.,
$$  \hspace{1cm} (1)

here $\omega_0 = 2\pi\nu_e$, $\omega_g = 2\pi\nu_g$ - optical and gravitational frequencies; $r_1$, $r_2$ - amplitude reflection factors of the front and far mirrors separated by the distance $L$; $N$ – the number of round trips of the laser beam; the vectors $h$, $A$ are defined as:

$$
h = \left( \frac{1}{2} h_{11} \exp(i\omega_0 t), \frac{1}{2} h_{11} \exp(-i\omega_0 t) \right), \quad A = (E_0, 0, 0).
$$  \hspace{1cm} (2)

If the gravitational wave falls down inclined in respect of the interferometer plane the matrix $\hat{D}$ looks like [7, 8]

$$
\hat{D} = \exp(-i\mu) \begin{pmatrix}
1 & 0 & 0 \\
(k_0 / 2k_g)K & \exp(-i\varepsilon) & 0 \\
-(k_0 / 2k_g)K^* & 0 & \exp(i\varepsilon)
\end{pmatrix}
$$  \hspace{1cm} (3)

where the factor $K$ is defined as

$$
K(i\varepsilon) = \frac{1}{1 + \alpha} + \frac{2\alpha}{1 - \alpha^2} \cdot \exp(-i(1 + \alpha)\varepsilon / 2) - \frac{1}{1 - \alpha} \cdot \exp(-i\varepsilon),
$$  \hspace{1cm} (4)

$$
\mu = 2k_0L, \quad \varepsilon = 2k_gL = \omega_g / \nu_0, \quad \nu_0 = c / 2L, \quad \alpha = \sin \theta \cos \varphi.
$$  \hspace{1cm} (5)

A substitution (2), (3) into (1) results in

$$
E^{(-)} = -r_1^{N-1} r_2^N E_0 e^{-iN\mu} \left\{ \frac{1}{2} e^{i\omega_0 t} 1 + h_{11} \mu \frac{\sin(N\varepsilon / 2)}{4\varepsilon} \left[ e^{i(N-1)\varepsilon / 2} K_x - e^{-i(N-1)\varepsilon / 2} K_y^* \right] \right\} + c.c.
$$  \hspace{1cm} (6)

3. Michelson interferometer with delay lines in arms

The formula (6) presents the electrical field in the output light going along the $x$ direction. At the output of a second delay line placed in $y$ direction the electrical field will be described by a similar formula with following substitutions:

$$
h_{11} \rightarrow h_{22}, \quad \mu_1 = k_0 2L_1 \rightarrow \mu_2 = k_g 2L_2, \quad \alpha = \sin \theta \cos \varphi \rightarrow \beta = \sin \theta \sin \varphi,
$$

$$
K_x \rightarrow K_y, \quad \varepsilon_1 = k_g 2L_1 \rightarrow \varepsilon_2 = k_g 2L_2, \quad \varepsilon_1 \neq \varepsilon_2.
$$  \hspace{1cm} (7)

The light beams from the both delay lines pass through a beam-splitter and interfere in two perpendicular outputs, so called the differential and common mode outputs. At an optical resonance a luminos-
The upper bar reflects a time averaging performed by photodiodes filtering high frequency components. Low indexes in (8) mark ODL orientation; signs \(\pm\) correspond to the common (symmetrical) and differential (anti symmetrical) modes.

The substitution in (8) electrical fields of the arm’s ODL (6), (7) results in the light intensity at output photo diodes

\[
I_x = I_0 \left( T_x^{(\Delta \mu)} + T_x^{(\pm)} h H_{ODL}(i \omega_g) \exp(i \omega_g t) + c.c. \right)
\]

\[
T_x^{(\Delta \mu)} = 1 \pm \cos(N \Delta \mu), \quad T_x^{(\pm)} = \frac{d T_x}{d(\Delta \mu)} = \mp \sin(N \Delta \mu),
\]

\[
H_{ODL}(\omega_g) = -\frac{\mu}{4 \epsilon} \frac{\sin(N\epsilon / 2)}{\sin(\epsilon / 2)} \exp(-i(N-1)\epsilon / 2)[h_{11} K_x(\epsilon) - h_{22} K_y(\epsilon)].
\]

here \(K_{x(y)}\) is defined by (4); a transition from \(K_x\) to \(K_y\) corresponds to substitutions \(\alpha \rightarrow \beta, \quad l_1 \rightarrow l_2\). Other parameters are: \(h = (h_1^2 + h_2^2)^{1/2}\), \(\Delta \mu = \mu_2 - \mu_1 = k_0 (L_2 - L_1)\), \(I_0 = r_1^{2N-2} r_2^{2N} I^*\), \(I^*\) - the input intensity of laser beam; \(T_x^{(\Delta \mu)}\) characterizes a background luminosity at photo diodes. The second term in the right part of (9) describes variations of luminosity induced by gravitational wave. Its maximum can be achieved by the arm length tuning so that \(L = 2(L_2 - L_1) = (\epsilon / (4N))(1 + 2k)\), \(k\) is an integer number.

Formula (11) presents a transfer function of the interferometer as a GW detector. It depends on the frequency and angular variables included into the factor

\[
F(\epsilon, \theta, \varphi) = K_x(\epsilon, \alpha) \frac{h_{11}}{h} - K_y(\epsilon, \beta) \frac{h_{22}}{h},
\]

This factor \(F(\epsilon, \theta, \varphi)\) can be considered as the detector antenna pattern. However in a general case this pattern will be frequency dependent.

For the particular case of orthogonal fall down \(\varphi = 0, \quad \epsilon_1 = -h_{22}, \quad \epsilon_2 = \epsilon\), \(K_x = K_y = 1 - e^{-i \epsilon} = -i 2 e^{-i \epsilon / 2} \sin(\epsilon / 2)\) the frequency transfer function reduced to the well known form [7]

\[
H_{ODL}(i \omega_g) = -\frac{i \mu}{2} \frac{\sin(N\epsilon / 2)}{\epsilon / 2} \frac{h_{11}}{h} \exp(-N\epsilon / 2).
\]

4. Long wave approximation

Conventional theory treats interferometer GW detectors as effective devices in the “long wave approximation” i.e. when the gravitational wave length is much larger the length of interferometer arms: \(\lambda_g >> L\).
In this situation $\varepsilon << 1$. In the first order on $\varepsilon$ the functions $K_x(\varepsilon, \alpha)$, $K_y(\varepsilon, \beta)$ do not depend on direction angles, and reduced to the value $K_x = K_y = i\varepsilon$.

Under transferring to the real form the expression (11) is factorized separating a frequency transfer function and angular antenna pattern.

$$H_{lw} = \frac{\mu \sin(N\varepsilon / 2)}{4 \sin(\varepsilon / 2)} F_0(\theta, \varphi), \quad \varepsilon << 1. \quad (14)$$

$F_0(\theta, \varphi)$ is the angular antenna pattern:

$$F_0(\theta, \varphi) = (h_{11} - h_{22}) / h = (h_1 / h)(1 + \cos^2 \theta) \cos 2\varphi - (h_2 / h)2 \cos \theta \sin 2\varphi. \quad (15)$$

Maximum of the transfer function (14) in the long wave limit $\varepsilon \to 0$ results in:

$$\max H_{lw} = \frac{\mu}{4} NF_0(\theta, \varphi) = \frac{\mu}{4} N \max \left(\frac{h_{11} - h_{22}}{h}\right). \quad (16)$$

5. Gravity-optical resonance (GOR)

Let’s consider the transfer function (11) under an inclined gravitational wave falling down. We select frequencies $\nu_g$ equal an integer number of the circulation frequency $\nu_0 = c / 2L$ so that $\nu_g = n\nu_0$. Under integer value of $n$ (1, 2, 3….) the factor $\sin(N\varepsilon / 2)/\sin(\varepsilon / 2)$ in (11) has an uncertainty which can be easy solved. Taking into account that $\varepsilon / 2 = \pi(\nu_g / \nu_0) = \pi n$, one comes to the following:

$$\frac{\sin(N\varepsilon / 2)}{(\varepsilon / 2) \sin(\varepsilon / 2)} = \frac{\sin \pi Nn}{\pi n \sin \pi n} = \lim_{x \to 0} \frac{\sin \pi N(n + x)}{\pi n \sin \pi(n + x)} = \lim_{x \to 0} \frac{\cos \pi Nn}{\pi n \cos \pi n} \frac{\sin \pi x}{\sin \pi x} = (-1)^{(-n-1)r} \frac{N}{\pi n} \quad (17)$$

Then coming back to the formula (11) one gets in the maximum transfer function

$$H_{GOR}(n, \theta, \varphi) = -i \frac{\mu}{8\pi} \frac{N}{n} \left(K_x \frac{h_{11}}{h} - K_y \frac{h_{22}}{h}\right). \quad (18)$$

In the theory of spectral analysis the factor $\sin N\chi / \sin \chi$, is known as the Bui-Ballot filter [9]: It filters with the same weights harmonics at frequencies equal integer number of $\pi$ (i.e. $\nu = n\pi$).

But in our case according to formula (18) the weight of harmonic is inverse proportional of $n$.

In a close around of $\nu_g = n\nu_0$ ($n << N$) the transfer function symmetrically grows up to the value (18); one can call it as a “gravity-optical resonance” and expects an increasing sensitivity of GW-detection in these regions on a “white type” noise background.

Character of the transfer function (11) at resonance is defined mainly by the same Bui-Ballot factor with (other factors $F(\varepsilon, \theta, \varphi)$ and $\varepsilon$ are changed slowly). Calculating the real part of (11) and omitting factors with module on the order of unit one can find the following estimate of the resonance transfer function

$$H_{GOR}(n, x, \theta, \varphi) = \frac{\mu}{8\pi} \frac{1}{n} \frac{\sin \pi N(x - x_0)}{\sin \pi(x - x_0)} F(n, \theta, \varphi), \quad |x| < x_0 = n \quad (19)$$
In the long wave approximation the transfer function (14) also contains the Bui-Ballot factor but in the low frequency region \( v_g \ll v_0 \) which one can call a "region of zero resonance".

It is worth to note a similar form of each resonance. Resonance number affects only the amplitude of transfer function (\( \sim 1/n \)).

6. Sensitivity enhancement in resonance zones

Appearance of gravity-optical resonances at frequencies \( v_g = n v_0 \) allows in principle to modify the sensitivity curve of large gravitational wave interferometers in its high frequency tail. For LIGO interferometers with the arm’s length 4 km the circulation frequency of light is \( v_0 = 37.5720 \, \text{kHz} \); the second and third zone corresponds to frequencies 75.040 kHz and 112.560 kHz. Widths of these zones are on the order of double zero zone width. It is interesting to estimate a factor of sensitivity increasing, using formulae for the frequency transfer function (14), (18). Conventional description of the GW-interferometer sensitivity is carried out in the form of the noise spectral density reduced to the instrument input. At the high frequency part the main noise source is the white photon shot noise depending only on the light intensity. Recalculation of this noise to the interferometer input is performed in fact through a simple division its spectral density on the instrument transfer function. Let’s consider consequently the “long wave “ and “gravity-optical resonance” regimes.

A) Long-wave regime.

In the frequency zone \( v \leq v_g = \tilde{n}/2NL \) signal intensity is estimated \( \Delta I_s = I_0 NH_{LW} h \), where the frequency characteristic \( H_{LW} \) is defined by the formulae (14), (16).

In zone of the “effective reception” i.e. \( v_g \leq v_h, \, \epsilon \ll 1 \), one can get from (14)

\[
| H_{LW} |^{-1} = \frac{-\mu N}{4} \sin \frac{x}{x} \sim \frac{1}{\pi \nu e} \sim \text{const}, \quad x = N \epsilon / 2 = \pi v_g / \nu c \leq 1. \quad (20)
\]

At more high frequencies \( v_g > v_c \) the transfer function (14) is falling down. Its inverse value is growing up proportional \( v_g \) :

\[
| H_{LW} |^{-1} = \frac{v_g}{v_c} \quad (21)
\]

At frequencies equaled integer numbers of circulation frequency this formula looks like

\[
| H_{LW} |^{-1} = \frac{v_c}{v_e} n \cdot N \quad (22)
\]

There is a low frequency plateau \( v \leq v_h = \tilde{n}/2NL \) with maximal sensitivity which has a linear decreasing for \( v_g > v_c \).

B) Gravity-optical resonance regime.

At high frequencies \( v_g > v_c \) under the condition \( v_g = n v_0 \) the transfer function (18) inverted to the interferometer input is estimated as

\[
| H_{GOR} |^{-1} = \frac{\mu N}{8\pi n} F(n, \theta, \varphi) \mid^{-1} \approx \frac{4 \nu c}{v_c n} \quad (23)
\]
Ratio of the inverted transfer functions (22) to (23) presents a relative value of the noise spectral densities in regions of gravity-optical resonance and long wave regime

\[
\frac{|H_{\text{GOR}}|^{-1}}{|H_{\text{LW}}|^{-1}} = \frac{4}{N}
\]  

(24)

Thus inside zones of gravity-optical resonance with \(n=1, n=2\), a depressed sensitivity of LW regime is reconstructed partly reaching the level (23) i.e. the factor of increasing equals \(N/4\).

7. Antenna pattern at gravity optical resonances.

Formulae of antenna patterns for the first gravity-optical resonances in a general case are too complex and we will consider below a particular case of direction cosines equality \(\alpha = \beta = \pm(1/\sqrt{2} \cdot \sin \theta)\). Using the polarization formulae

\[
h_{11} = h_\varphi \frac{1}{2} (\cos^2 \theta - 1) - h_x \cos \theta, \quad h_{22} = h_\varphi \frac{1}{2} (\cos^2 \theta - 1) + h_x \cos \theta.
\]  

(25)

supposing \(h_\varphi = h_\varphi; \varepsilon/2 = \pi \cdot n\) and looking at (4) one has for \(K_\varphi(\varepsilon, \alpha)\)

\[
K_\varphi(n, \alpha) = \frac{1}{1 + \alpha^2} \cdot \frac{2\alpha}{1 - \alpha^2} \cdot e^{-i(1+\alpha)\pi n} - \frac{1}{1 - \alpha} \cdot e^{-i2\pi n}.
\]

Taking into account that \(e^{-i2\pi n} = 1, e^{-i(1+\alpha)\pi n} = (-1)^n e^{-i\alpha n}\) one can rewrite finally

\[
K_\varphi(n, \alpha) = -\frac{4\alpha}{1 - \alpha^2} \cdot e^{-i\alpha \pi n/2} \begin{cases} 
\cos\left(\frac{\alpha \pi n}{2}\right) & n - \text{odd}, \\
\sin\left(\frac{\alpha \pi n}{2}\right) & n - \text{even}.
\end{cases}
\]

A similar expression takes place for the second arm \(K_\varphi(n, \beta) = K_\varphi(n, \alpha)\). At last coming from the complexity form to the real expression we find the antenna pattern as

\[
F(n, \theta, \varphi) = \frac{4\alpha}{1 - \alpha^2} \cdot \frac{h_{11} - h_{22}}{h} \begin{cases} 
\cos\left(\frac{\alpha \pi n}{2}\right) & n - \text{odd}, \\
\sin\left(\frac{\alpha \pi n}{2}\right) & n - \text{even}.
\end{cases}
\]  

(26)

Here the antenna pattern specifics is defined only by the number of resonance \(n\) and does not depend on the number of round trip \(N\) in each arm.

Structures of antenna pattern (26) in the cross section \(\varphi = \pi / 4\), including the “zero resonance” (14) are given at figure 1a. The antenna pattern of “zero resonance” in the same cross section is reduced to \(F_0(\theta, \varphi)_{\alpha = \beta} = \sqrt{2} \cos \theta\). Its maximum corresponds to the gravitational wave with the normal falling down at the interferometer plane \(\theta = 0\).
Figure 1a, $n = 0$, $\text{maxF} = 2,221$

Figure 1b, $n = 1$, $\text{maxF} = 1,889$, 
Figure 1c, $n = 2$, $\max(1/2)F = 1.385$

Figure 1d, $n = 3$, $\max(1/3)F = 0.984$
8. Discussion

Antenna pattern configurations presented above demonstrate specific features of the gravity optical resonance – an angular shift of the maximum sensitivity depending on the number of resonance (or the receiver frequency) as well as a directivity (sharpness) enhancement with increasing a resonance number. It means that angular position of a source would be defined more accurate (compare with the antenna directivity in the low frequency limit) if it will be detected through the gravity-optical resonance with a large number \( n \).

It is interesting to estimate the value of maximal “reconstructed sensitivity” inside the resonance zones. One can easily get such estimation for the case of dominant optical shot noise background (other noises are considered as much smaller). Then according to the simple “detection rule” \( \Delta P_s \geq \Delta P_{f_1} \) using (18) one estimates the minimal detectable metric variation as

\[
P_0 H_{\text{GSR}} h \geq \left( 2 \hbar \omega P_0 \Delta f \right)^{1/2} \quad \text{or} \quad h_{\text{min}} \geq \frac{\lambda \cdot n}{N} \sqrt{\frac{\hbar \omega \cdot \Delta f}{P_0}}
\]

(27)

References

1. http://www.ligo.caltech.edu/
2. http://www.virgo.infn.it/
3. Abbott B P et al. 2009 In upper limit on stochastic gravitational-wave background of cosmological origin. Nature 460 990-994
4. Saulson P R 1997 How an interferometer extracts and amplifies power from a gravitational wave. Class. Quantum Grav. 19 2435-2454
5. Bisnovatyi-Kogan G S and Rudenko V N 2004 Very high frequency gravitational wave background in the Universe. Class. Quantum Grav. 21 1-13
6. Rudenko V N and Sazhin M V 1980 Laser interferometer as a detector of gravitational waves Kvantovaya electronica 7 2344-2355 (in Russian) translated in English 1980 as "Sov. Quantum Electron." 10 1366-1373
7. Vinet J-Y 1986 Recycling interferometric antennas for periodic gravitational waves. Physique. 47 639-643
8. Kolosnitin N I 1996 Maxwell equations and gravitational wave detection. Gravitation & Cosmology. 2 262-266
9. Serebryanikov G and Pervozvanskii oscow 1965 Detection of covered periodicity