On one example and one counterexample in counting rational points on graph hypersurfaces.

Dzmitry Doryn

Abstract

In this paper we present a concrete counterexample to the conjecture of Kontsevich about the polynomial countability of graph hypersurfaces. In contrast to this, we show that the "wheel with spokes" graphs $WS_n$ are polynomially countable.

Introduction

Let $\Gamma$ be a connected graph with a set of edges $E$ and vertexes $V$. We can define the graph polynomial of $G$ by

$$\Psi_\Gamma := \sum_T \prod_{e \notin T} A_e \in \mathbb{Z}[A_e | e \in E],$$

where the sum goes over all spanning trees and $A_e$'s are variables. The vanishing locus of such a polynomial in the affine space $\mathbb{A}^{|E|}_\mathbb{Z}$ (or in $\mathbb{P}^{|E|}_\mathbb{Z}$) defines the graph hypersurface $X_\Gamma$. It is convenient for us to use this affine notation since the big part of computations was done with PC.

We are interested in the number of $\mathbb{F}_q$-rational points of graph hypersurface. Consider a function $F_\Gamma : q \mapsto \#X_\Gamma(\mathbb{F}_q)$ defined on the set of prime powers. We known that all know periods of Feynman graphs are elements of a $\mathbb{Q}$-subalgebra of $\mathbb{R}$ generated by multiple zeta values (see [BrK] and [Sch]). There was a hope that the arithmetic of graph hypersurfaces is very simple. In '97 M.Kontsevich made the following conjecture on the number of points of graph hypersurfaces: For any graph $\Gamma$, $F_\Gamma \in \mathbb{Z}[q]$. The conjecture was wrong. That was proved in [BB] in '00. The proof is very technical and gives no concrete example of a graph for which the function $F_{X_\Gamma}$ is not a polynomial.

In this paper we give a concrete counterexample with a strong proof. There are parallel computations done by Oliver Schnetz independently, where he proposed 6 counterexamples in $\phi^4$ theory with 14 edges. Focusing on finding of all counterexamples in $\phi^4$ theory, he computes $F_\Gamma(q)$
for $q$ up to 7 and finds the graphs, for which the coefficients after (partial) interpolation in these points become very large; he speculates on the good behaviour of values $F_\Gamma(q)$ at prime powers $q = 2^k$ or away from this set (5 examples), and with respect to the residue modulo 3 (one other example).

Actually, if you are sure about a graph being a counterexample, you can use cluster computer system to proof this. My aim is to show how it was done for our example with a home computer using the stratification of graph hypersurface studied in [BEK] and [DD]. My graph was chosen by some thoughts of symmetry and after some tries, and one of the Oliver’s graphs is isomorphic to my counterexample.

You can find the implementation of the algorithm on \url{http://doryn.org/progs/conterexample.html}.

The consequence of the result of ([BB]) is that the $F_\Gamma$ is not a polynomial for almost all graphs. On the other hand, Spencer Bloch proves in [Bl] that the (finite) sum of all $F_\Gamma$ for connected graphs with $n$ edges (counted with some multiplicities) is a polynomial for any $n \geq 3$. Can the big enough graph be polynomially countable? The example is the cycle $O_n$ of length $n$, since $F_{O_n}(q) = q^n - 1$. We avoid such trivial cases. Recall that primitively log divergent graphs are the graphs $\Gamma$ such that $|E(\Gamma)| = 2h_1(\Gamma)$ and for all subgraphs $\Gamma' \subset \Gamma$ the inequality $E(\Gamma') > 2h_1(\Gamma')$ holds. We restrict our attention to the primitively log divergent graphs since for such graphs the periods are defined. So, the natural question is whether we have a primitively log divergent graph $\Gamma_n$ with the Betti number $h_1(\Gamma) = n$ such that $F_X$ is a polynomial for each $n \geq 3$. The answer is yes. The function $F_\Gamma$ is a polynomial for each graph from the $WS_n$ series. In the last part of the paper we prove this statement by computing the class of the graph hypersurface of $WS_n$ in the Grothendick ring of varieties.

1 Counterexample

We start with presenting our graph $\Gamma = XStrip$. It looks like a strip of 3 squares and with $X$’s inside both pairs of consequent squares (see drawing). For a graph polynomial we use the presentation as a determinant of a matrix. We choose some orientation of edges and the direction of loop tracing, and we build an $h_1(\Gamma) \times N - \text{table } Tab(\Gamma)$ with $N = |E(\Gamma)|$. The $Tab(\Gamma)_{ij} = 1$ if the edge $e_j$ in the $i$’s loop is in the tracing direction of the loop and $Tab(\Gamma)_{ij} = -1$ if this edge is in the opposite direction; otherwise $Tab(\Gamma)_{ij} = 0$. Then the desired matrix is $M_\Gamma(T) := \sum_{k=1}^{N} T_k M^k$ in some variables $T_0, \ldots, T_N$, where $M^k_{ij} = Tab(\Gamma)_{ik} \cdot Tab(\Gamma)_{jk}$. For more details see ([DD], Example 1.2.5).
To have a better chances in fighting with polynomials, we change the variables to have as much independent entries of the matrix as possible. Then we come to (denoting by the same letter) matrix $\mathcal{M}$ in variables $A_0, \ldots, A_6, B_0, \ldots, B_6$.

$$\mathcal{M}_\Gamma(A, B) = \begin{pmatrix} B_0 & A_0 & A_2 & A_1 & 0 & 0 & A_1 \\ A_0 & B_1 & A_2+ A_3 & A_3 & 0 & A_3 & A_3 \\ A_2 & A_2+ A_3 & B_2 & A_3-A_4 & A_4 & A_3 & A_3-A_4 \\ A_1 & A_3 & A_3-A_4 & B_3 & A_4 & A_3 & A_3-A_4 \\ 0 & 0 & A_4 & A_4 & B_4 & A_5 & A_6 \\ 0 & A_3 & A_3 & A_3 & A_5 & B_5 & A_4+A_6 \\ A_1 & A_3 & A_3 & A_3-A_1 & A_6 & A_3+A_6 & B_6 \end{pmatrix} \tag{2}$$

In this section we use the affine notion of graph hypersurface. So, $X := X_\Gamma \subset \mathbb{A}^{14}(A, B)$ defined by $\det(\mathcal{M}) = 0$ in the affine space with coordinates all of $A$'s and $B$'s, where $\mathcal{M} := \mathcal{M}_\Gamma(A, B)$. We write $X = V(\det(\mathcal{M}))$ in this situation. More generally, we denote by $V(\mathcal{I})$ or $V(f_1, \ldots, f_n)$ the variety in $\mathbb{A}^N(T_1, \ldots, T_N)$ defined by the vanishing locus of the ideal generated by polynomials $f_1, \ldots, f_n \in \mathbb{Z}[T_1, \ldots, T_N]$. Sometimes we write $V(\mathcal{I})^{(N)}$ indicating the dimension $N$ of the ambient affine space.

Consider the function $F_\Gamma : q \mapsto \#X_\Gamma(\mathbb{F}_q)$. The core of this article is the following

**Theorem 1.1**

If $\Gamma = X_{\text{Strip}}$, then $F_\Gamma$ is not a polynomial.

Assume that $F_\Gamma(q)$ is a polynomial. The proof is based on the computer program, but there are several steps of optimization needed to get the answer in a reasonable time. We use a shape of the matrix and make a stratification of graph hypersurface.

**Step 1.** First we explain the simple projection techniques used ([BEK]) and ([DD]). The polynomial $I_7 = M := \det(\mathcal{M})$ is linear in $B_0$: $I_7 = B_0 I_6 - G_6$ with $G_6 := -I_7|_{B_0=0}$. If $I_6 = 0$, then the equation $I_7 = 0$ implies $G_6 = 0$ and we forget the variable $B_0$. So, the good idea is to consider the image of
$X \cap I_6$ under the ”forgetting $B_0$” projection from $\mathbb{A}^{14}$ to the affine space of one less dimension $\mathbb{A}^{13}(A, B; no B_0)$. In the other case — when $I_{n-1} \neq 0$ — we can express $B_0$ from the equation $I_n = 0$ and also project to $\mathbb{A}^{13}$ getting an isomorphism $X \setminus X \cap V(I_6) \cong \mathbb{A}^{13} \setminus V(I_6)^{(13)}$.

Consider the the class $[X]$ of the graph hypersurface in the Grothendieck ring $K_0(\text{Var}_K)$ of varieties over a field $K$ of characteristic 0. By definition,

$$[X] = [V(I_7)] = [X \cap V(I_6)] + [X \setminus X \cap V(I_6)].$$  (3)

Using the explained projections, one gets

$$[X \cap V(I_6)] = [V(I_6, I_7)^{(14)}] = [L][V(I_6, G_6)]$$  (4)

with $V(I_6, G_6)$ living in $\mathbb{A}^{13}(A, B, no B_0)$ and $L = [\mathbb{A}^1]$. Also

$$[X \setminus X \cap V(I_6)] = [\mathbb{A}^{13} \setminus V(I_6)^{(13)}] = [L][\mathbb{A}^{12} \setminus V(I_6)] = L^{13} - L[V(I_6)]$$  (5)

with $V(I_6) \subset \mathbb{A}^{12}(A, B; no B_0, A_0)$ since $I_6$ is independent of $A_0$. The polynomials $I_5$ and $G_5$ are independent of $B_1$, while $I_5$ is also independent of $A_2$. Thus, repeating the procedure for $V(I_6)$, we obtain

$$[V(I_6)] = [V(I_6, I_5)] + [V(I_6) \setminus V(I_6, I_5)] = [V(I_5, G_5)^{(12)}] + [L^2 \setminus V(I_5)] = [L^2 \setminus V(I_5, G_5)]$$  (6)

where $V(I_5, G_5) \subset \mathbb{A}^{11}(A, B, no B_0, B_1, A_0)$ and $\mathbb{A}^{10} \setminus V(I_5) \subset \mathbb{A}^{10}(A, B; no B_0, B_1, A_0, A_2)$.

Now we consider $G_6$ as a quadratic polynomial of variables $A_0$, $A_1$ and $A_2$ (sitting in the first row and column). The coefficient of $A_0$ is then $I_5$. Now we use two tricks from the paper [DD]. First, by Corollary 1.5, the product polynomial $G_6I_5$ is a square of a linear polynomial in $A_0$, $A_1$, $A_2$ on the locus where $I_6 = 0$ and $I_5 \neq 0$. Using this one can express $A_0$ from that linear polynomial on $V(I_6) \setminus V(I_6, I_5)$ and get rid of $G_6$. Second is Theorem 1.6: if $I_5 = 0$, then $G_6$ has not just the zero coefficient of $A_0^2$, but is independent of $A_0$ on $V(I_6)$ at all. It follows that

$$[V(I_6, G_6)] = [V(I_5, I_6, G_6)] + [V(I_6, G_6) \setminus V(I_5, I_6, G_6)] =$$

$$L[L[V(I_5, I_5, \tilde{G}_6)] + [A^{11} \setminus V(I_5)^{(11)}] = L[V(I_5, I_6, \tilde{G}_6)] + L[A^{10} \setminus V(I_5)]$$  (7)

where $\tilde{G}_6 := G_6|_{A_0 = 0}$. Collecting everything together, we obtain

$$[X] = L[L[V(I_5, G_6)] + L^{13} - L[V(I_6)] = L \left( L[L[V(I_5, I_6, \tilde{G}_6)] + L[A^{10} \setminus V(I_5)] \right)$$

$$+ L^{13} - L \left( L[L[V(I_5, G_5)] + L[A^{10} \setminus V(I_5)] \right) =$$

$$L^{13} - L^2[V(I_5, G_5)] + L^2[V(I_5, I_6, \tilde{G}_6)].$$  (8)
**Step 2.** The formula above is interesting as itself (holds in general for all primitively divergent graph) and we will use similar technique to optimize the algorithm of computation further, but the one direct consequence is the following.

**Proposition 1.2**

The polynomial $F_\Gamma(q)$ is divisible by $q^2$.

This is implied by the fact that the functor of counting rational points factors through Grothendick ring of varieties.

Let us look closely at $F_\Gamma$ assumed being a polynomial. Since $X$ is a hypersurface in $\mathbb{A}^{14}$, the degree of $F_X(q)$ is at most 13. Recall that the hypersurface associated to a primitively log divergent graph is always irreducible. This can be proved easily by induction. As a consequence, the leading term of $F_X(q)$ is $q^{13}$. By Proposition 1.2 we can rewrite $F_X(q)$ like

$$F_X(q) = q^{13} + q^2 \tilde{F}(q),$$

(9)

where $\tilde{F}$ is polynomial of degree at most 10. Such polynomial can be uniquely defined by its 11 values. So we need to compute $\#X(F_q)$ for at least 12 prime powers. Of course, we take the first prime powers starting with 2 and up to 19.

Our algorithm computes $\tilde{F}(19)$ in three days on the home computer, but for this reason the formula (8) is not enough, we need to stratify further using the shape of the matrix. It becomes complicated, we do this in steps.

**Step 3.** We return to the formula (8) The polynomial $\tilde{G}_6$ is of degree 1 as a polynomial of $B_1$. Write

$$\tilde{G}_6 = \tilde{G}_6^1 B_1 + \tilde{G}_6^2,$$

(10)

For $\mathcal{V}(I_5, I_6, \tilde{G}_6) = \mathcal{V}(I_5, G_5, \tilde{G}_6) \subset \mathbb{A}^{12}(A,B \text{ no } A_0, B_0)$, we separate into two cases according to whether $\tilde{G}_6^1$ equals zero or not, and we get

$$[\mathcal{V}(I_5, G_5, \tilde{G}_6)] = [\mathcal{V}(I_5, G_5, \tilde{G}_6, \tilde{G}_6^1)] + [\mathcal{V}(I_5, G_5, \tilde{G}_6) \setminus \mathcal{V}(I_5, G_5, \tilde{G}_6, \tilde{G}_6^1)].$$

(11)

On the first variety on the right we forget $B_1$, while on the last open scheme we can express $B_1$ from the equation $\tilde{G}_6 = 0$, projecting down to $\mathbb{A}^{11}$. So, we obtain

$$[\mathcal{V}(I_5, G_5, \tilde{G}_6)] = L[\mathcal{V}(I_5, G_5, \tilde{G}_6^1, \tilde{G}_6^2)] + [\mathcal{V}(I_5, G_5) \setminus \mathcal{V}(I_5, G_5, \tilde{G}_6^1)].$$

(12)
By (3) and (12), one gets

\[
[X] = \mathbb{L}^{13} - \mathbb{L}^{2}[\mathcal{V}(I_{5}, G_{5})] + \mathbb{L}^{2}[\mathcal{V}(I_{5}, I_{6}, \tilde{G}_{6})] = \mathbb{L}^{13} - \mathbb{L}^{2}[\mathcal{V}(I_{5}, G_{5})] \\
+ \mathbb{L}^{3}[\mathcal{V}(I_{5}, G_{5}, \tilde{G}_{6}^{1}, \tilde{G}_{6}^{2})] + \mathbb{L}^{2}(\mathcal{V}(I_{5}, G_{5}) - [\mathcal{V}(I_{5}, G_{5}, \tilde{G}_{6}^{1})]) \\
= \mathbb{L}^{13} - \mathbb{L}^{2}[\mathcal{V}(I_{5}, G_{5}, \tilde{G}_{6}^{1})] + \mathbb{L}^{3}[\mathcal{V}(I_{5}, G_{5}, \tilde{G}_{6}^{1}, \tilde{G}_{6}^{2})] \quad (13)
\]

**Step 4.** This step is more closely to the implementation of the algorithm. The situation is the following. To compute the number of rational points \( \#X(\mathbb{F}_{q}) \) we need to count the number of solutions \((a_{0}, \ldots, a_{7}, b_{1}, \ldots, b_{7})\) \(\in\mathbb{A}^{14}\) of the equation \(I_{7} = 0\). The brute force strategy is to put each of \(q^{14}\) 14-tuples into the equation and check if it is a solution. Then the complexity is \(O(14)\) times the complexity of one such check. If we apply the Gauss algorithm for this, the total complexity will be \(O(14)O(3) = O(17)\). This is impossible to compute \(\#X(\mathbb{F}_{q})\) with this strategy for \(q = 19\) in 1 year with a home PC.

Formula (13) helps to restrict the complexity to \(O(11)O(3) = O(14)\) since we deal with 11-tuples \(a_{1}, \ldots, b_{7}\) (no \(b_{0}, a_{0}\) or \(b_{1}\)) approximately. The complexity depends on the branch of the algorithm where some polynomials vanish or not. The last useful trick is the following. The polynomial \(I_{5} = B_{2}I_{4} - G_{4}\) is zero on both varieties \(Y = \mathcal{V}(I_{5}, G_{5}, \tilde{G}_{6}^{1})\) and \(Z = \mathcal{V}(I_{5}, G_{5}, \tilde{G}_{6}^{1}, \tilde{G}_{6}^{2})\) appearing in the formula. We can decrease the number of tuples further. For each 10-tuple \(a_{1}, \ldots, b_{7}\) (no \(b_{0}, a_{0}, b_{1}\) or \(b_{2}\)) such that \(I_{4} \neq 0\) we have unique \(b_{2}\) such that \(I_{5}\) is zero. In the opposite case, when \(I_{4} = 0\), we have no vanishing of \(I_{5}\) if \(G_{4} \neq 0\), and the vanishing for all \(b_{2}\) if \(G_{4} = 0\). Then on the branch where \(I_{4} \neq 0\) and \(I_{5} = 0\) we can compute \(a_{2}\) in a way to have \(G_{5} = 0\), so we do not need to run over all values of \(a_{2}\). Such tricks restrict the running time. For more explanation see the program.

### 2 Results

The program gives us 13 points \(F(2), F(3), \ldots, F(23)\). The biggest one, \(F(23)\) was computed in 2 weeks on my office computer in Universität Duisburg-Essen on the Christmas vacation 2008-2009. Then we use Lagrange interpolation formula for first 11 of them to interpolate \(\tilde{F}\) in (9). Then we take \(F(19)\) and check whether the point \((19, \tilde{F}(19))\) is lying on that graph or not. It is not! After multiplying with determinants the problem becomes integral and can be done even by hand.

The interesting thing is that for the 10 odd prime powers \(q = 3, 5, 7, 9, 11, 13, 17, 19, 23\) the values \(F(k)\) are exactly the values of the following
polynomial

\[ F_1(q) = q^{13} + q^{11} + 23q^{10} - 78q^9 + 90q^8 - 35q^7 + (q - 2)q^6 
- 34q^5 + 66q^4 - 32q^3 + (q - 1)q^2 \]  (14)

But the values in the even prime powers 2, 4, 8, 16 are the values of the polynomial \( F_1 - (q - 1)q^2 \). We believe that for \( q \) prime to 2 the number of rational points \( F_{X_1}(q) = F_0(f) \), and that for even prime powers \( F_{X_1}(2^k) = F_0(2^k) - (2^k - 1)2^{2k} \). This will be not very surprising since the first counterexample must be ”not very bad”, and the simple picture for this is that we have a polynomial in all but one prime. At this prime and all its powers we have another polynomial and the difference must be divisible by \( q^2 \) by Lemma [1.2]. The results are compatible with that of Schnetz, [Sch], our graph is isomorphic to the graph on Fig 1.a without vertex 1. If one goes from the affine case to the projective complement of graph hypersurface, he gets the polynomial (2.30),[Sch].

3 Example

Here we study the well-known series \( WS_n \). These are the simplest examples of primitive log divergent graphs. The graphs \( WS_n, n \geq 3 \) looks like \( n \) points on the circle and one point inside, the first \( n \) edges connect the center to the points on the circle and the other \( n \) edges are the arcs that the circle is divided on. To make the formulas a bit more readable later, we consider \( WS_{n+1} \).

In this section we work in projective setting and the graph hypersurface \( X \) lives in \( \mathbb{P}^{n+1} \). Now the \( \mathcal{V}(I) \) or \( \mathcal{V}(f_1, \ldots, f_n) \) means the vanishing locus of the ideal generated by homogenous polynomials \( f_1, \ldots, f_n \in \mathbb{Z}[T_1 : \ldots : T_N] \) in projective space. If \( Y_p = \mathcal{V}(f_1, \ldots, f_n) \subset \mathbb{P}^{N-1} \) in this setting and \( Y_a = \mathcal{V}(f_1, \ldots, f_n) \subset \mathbb{A}^N \) in the setting of Section 1, then the natural projection \( \mathbb{A}^N \setminus \{0\} \longrightarrow \mathbb{P}^{N-1} \) induces an equality \( [Y_a] - 1 = (\mathbb{L} - 1)[Y_p] \). Thus \#\( Y_a(\mathbb{F}_q) = 1 + q \cdot \#Y_p \), and for proving that \( F_1 \) is a polynomial it does not matter, whether we work in the affine setting or in the projective one.

The graph polynomial of \( WS_{n+1} \) is the determinant of the following
Using the same technique as in the first section, we stratify the matrix.

\[ \mathcal{M}_{n+1} = \mathcal{M}_W s_{n+1} = \begin{pmatrix}
B_0 & A_0 & 0 & \vdots & 0 & 0 & A_n \\
A_0 & B_1 & A_1 & \vdots & 0 & 0 & 0 \\
0 & A_1 & B_2 & \vdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \vdots & B_{n-2} & A_{n-2} & 0 \\
0 & 0 & 0 & \vdots & A_{n-2} & B_{n-1} & A_{n-1} \\
A_n & 0 & 0 & \vdots & 0 & A_{n-1} & B_n
\end{pmatrix}. \quad (15) \]

We again work the class of \( X = \det \mathcal{M}_{n+1} \) in the Grothendieck ring. Using the same technique as in the first section, we stratify

\[ [X] = \mathbb{L}^2[\mathbb{P}^{2n-2}\backslash \mathcal{V}(I_n)] + 1 + \mathbb{L}[\mathcal{V}(I_n, G_n)]. \quad (16) \]

Here \( I_n \) is independent of \( B_0, A_0 \) and \( A_n \); and \( G_n = -I_{n+1}|_{B_0=0} \). Since \( I_{n+1}(0; n) = A_0A_1 \ldots A_{n-1} \), we can write

\[ G_n = A_0^2 I_{n-1} - 2A_0 A_n \cdot A_0 A_1 \ldots A_{n-1} + A_n^2 I_{n-1} \quad (17) \]

Knowing \( G_n \), we go on

\[ [X] = \mathbb{L}^2([\mathbb{P}^{2n-2}] - 1) - \mathbb{L}^3[\mathcal{V}(I_{n-1}, G_{n-1})] - \mathbb{L}^3[\mathbb{P}^{2n-4}\backslash \mathcal{V}(I_{n-1})] + \\
1 + \mathbb{L} + \mathbb{L}^2[\mathcal{V}(I_{n-1}, I_n, \tilde{G}_n)] + \mathbb{L}^2[\mathbb{P}^{2n-4}\backslash \mathcal{V}(I_{n-1})] = 1 + \mathbb{L} + \\
\mathbb{L}^2([\mathbb{P}^{2n-2}] - 1) - \mathbb{L}^3[\mathcal{V}(I_{n-1}, G_{n-1})] + \mathbb{L}^2[\mathcal{V}(I_{n-1}, I_n, \tilde{G}_n)] = \\
1 + \mathbb{L} + \mathbb{L}^2([\mathbb{P}^{2n-2}] - 1) - \mathbb{L}^3[\mathcal{V}'(I_{n-1}, G_{n-1})] + \mathbb{L}^2[\mathcal{V}(I_{n-1}, I_n, \tilde{G}_n)] = \quad (18) \]

where \( G_{n-1} = -I_{n+1}|_{B_0=0} = A_0^2 I_{n-2} \) and \( \tilde{G}_n = -I_{n+1}|_{B_0=0=0} = A_0^2 I_{n-1} \).

First consider \( Y' = \mathcal{V}(I_{n-1}, A_1 I_{n-2}) \). For each \( i, 1 \leq i \leq n-1 \), we define \( Y'_i := \mathcal{V}(I_i, A_{n-i-1} I_{i-1}) \subset \mathbb{P}^{2i-1} \) in the projective space with coordinates all \( A \)'s and \( B \)'s appearing in the the submatrix of \( I_i \) and \( A_{n-i-1} \). Similarly \( Y'_i := \mathcal{V}(I_i) \) is defined in \( \mathbb{P}^{2i-2} \) with the same coordinates but no \( A_{n-i-1} \). In this notation \( Y' = Y'_{n-1} \). Set \( y_i := [Y_i] \) and \( y'_i = [Y'_i] \) in \( K_0(Var_K) \).

**Lemma 3.1**

For \( y'_i \) and \( y_i \), as above, we have \( y'_i, y_i \in \mathbb{Z}[\mathbb{L}] \).

**Proof.** We prove the statement by induction on \( i \). For \( i = 1 \), \( Y_1 = \mathcal{V}(A_{n-2}; B_{n-1}) \subset \mathbb{P}^1 \) and \( Y'_1 = B_{n-1} \subset \mathbb{P}^0 \), so \( y'_1 = y_1 = 0 \in \mathbb{Z}[\mathbb{L}] \). As-
sume now that for $i = k < n - 2$ the statement is true. Then for $i = k + 1$:

$$[Y_{k+1}] = [\mathcal{V}(I_{k+1}, A_{n-k-2}I_k)] = [\mathcal{V}(I_{k+1}, A_{n-k-2})] + [\mathcal{V}(I_{k+1}, I_k)(2k)] - [\mathcal{V}(I_{k+1}, A_{n-k-2}, I_k)] = [Y_{k+1}]_{2k} + [Y_{k+1}]_{2k-1} + [Y_{k+1}]_{2k-2} + \cdots + [Y_{k+1}]_{1} + [Y_{k+1}]_{0},$$

$$[Y_{k+1}] = [\mathcal{V}(I_{n-k-1}I_k - A^2_{n-k-1}I_{k-1}, I_k)] = [Y_{k+1}]_{n-k-1} + [Y_{k+1}]_{n-k-2} + \cdots + [Y_{k+1}]_{1} + [Y_{k+1}]_{0},$$

(19)

We get recurrence formulas

$$y_{k+1} = 1 + \mathbb{L}y_k + \mathbb{L}[\mathbb{P}^{2k-2}] - \mathbb{L}y_k,$$

with $y_1 = 0$. If one likes, the substitution $v_k := [Y_k]$ and $u_k := [Y_k] - [Y_k]$ brings us to a one recurrence formula for $u_k$, which is easy to compute

$$v_{k+1} = 1 + \mathbb{L}[\mathbb{P}^{2k-1}] + \mathbb{L}u_k,$$

$$u_{k+1} = \mathbb{L} + \mathbb{L}(\mathbb{L} - 1)(u_k + 1) + \mathbb{L}u_{k-1} + \mathbb{L}[\mathbb{P}^{2k-3}).$$

(22)

Now elements $y_i$ and $y'_i$ are now evidently polynomials of $\mathbb{L}$. The first terms are listed in the table below.

| $i$ | 1  | 2   | 3   | 4   |
|-----|----|-----|-----|-----|
| $y_i$ | 0  | $\mathbb{L} + 1$ | $\mathbb{L}^2 + 2\mathbb{L}^2 + \mathbb{L} + 1$ | $\mathbb{L}^3 + 3\mathbb{L}^4 + \mathbb{L}^2 + \mathbb{L} + 1$ |
| $y'_i$ | 0  | $2\mathbb{L} + 1$ | $3\mathbb{L}^3 + \mathbb{L}^2 + \mathbb{L} + 1$ | $4\mathbb{L}^5 + \mathbb{L}^4 + \mathbb{L}^2 + \mathbb{L} + 1$ |

Now we return to the last summand in (13):

$$[Z'] = [\mathcal{V}(I_{n-1}, I_n, \tilde{C}_n)] = [\mathcal{V}(I_{n-1}, I_n, A^2_{n-1}I_{n-1})] + [\mathcal{V}(I_{n-1}, I_n, I_{n-1})] - [\mathcal{V}(I_{n-1}, I_n, I_{n-1})] = [\mathcal{V}(I_{n-1}, I_n, A_{n})] + 1 + (\mathbb{L} - 1)[\mathcal{V}(I_{n-1}, I_n, A_{n})] + [\mathcal{V}(I_{n-1}, I_n, I_{n-1})] = 2 + \mathbb{L}y'_{n-1} + (\mathbb{L} - 1)[Z].$$

(23)

We use the same trick as appeared in [BEK]. Define $Z_1 := \mathcal{V}(I_n, I_{n-1})$ and $Z_2 := \mathcal{V}(I_n, I_{n-1})$. Then $Z = Z_1 \cap Z_2$. Recall that

$$I_{n-1}I_{n-2} = S_{n-1}^2 = I_nI_{n-2},$$

(24)
where \( S_{n-1} := I_n(0; n - 1) \), the determinant of the matrix when we throw away 0-th column and \((n-1)\)-th row. Define

\[
T := Z_1 \cup Z_2 = \mathcal{V}(I_n, S_{n-1}) = \mathcal{V}(I_n, \prod_{i=1}^{n-1} A_i). \tag{25}
\]

Note that \( Z_1 \cong Z_2 \) is a cone over \( Y'_i \) in the setting of Lemma 3.1 and

\[
[Z] = 2(1 + \mathbb{L} y'_{n-1}) - [T]. \tag{26}
\]

Define \( T_I = T \cup \bigcap_{i \in I} \mathcal{V}(A_i) \) for each \( I \subset \{1, \ldots, n-1\} \). Then

\[
[T] = \sum_i [T_i] - \sum_{i,j} [T_{ij}] + \sum_{i,j,k} [T_{ijk}] - \ldots + (-1)^{n-1} [T_{1,2,\ldots,n-1}]. \tag{27}
\]

For \( 1 \leq i_1 < i_2 < \ldots < i_p \leq n - 2 \),

\[
T_{i_1,\ldots,i_p} = \mathcal{V}(I_i^{-i_1} I_{i_2-i_1}^{n-i_2}, \ldots, I_{i_p-i_{p-1}}^{n-i_p} I_{n-i_p}). \tag{28}
\]

One can decompose \( T_{i_1,\ldots,i_p} := T' \) similar to (27)

\[
T_{i_1,\ldots,i_p} = T' \cap \mathcal{V}(I_i^{-i_1} I_{i_2-i_1}^{n-i_2}) + T' \cap \mathcal{V}(I_i^{-i_2} I_{i_3-i_2}^{n-i_3}) + \ldots + T' \cap \mathcal{V}(I_n^{-i_p} I_{n-i_p}). \tag{29}
\]

Note that each of the first \( p + 1 \) summands in the last sum is a cone over some \( I_j^k \), that is a determinant of a 3-diagonal matrix of the right dimension similar to \( Y_j \) in Lemma 3.1. Next, \( T' \cap \mathcal{V}(I_j^k, I_j') \) is a cone over \( Y_j \times Y_j' \). And so on. Substituting all sums like (28) into (27), we get an expression for \([T]\) as a big sum in terms of \( y_i \).

One likes to get a convenient recursive formula to compute the number of points with a computer quickly. The good way to do this is the following. Define \( T^j = \mathcal{V}(I_j, S_{j-1}) \) for \( 2 \leq j \leq n \), so that \( T^n = T \). Let \( S_{i(i)} \) be the sum consisting of the summands of a big sum where \( i_1 = i \) in (28). First, \([\mathcal{V}(B_1 I_{n-1})] \) \( \in S_{i(i)} \). We write

\[
[\mathcal{V}(B_1 I_{n-1})] = (1 + \mathbb{L} [\mathcal{V}(I_{n-1})]) + ([\mathbb{P}^{2n-4}] + \mathbb{L}^{2n-3} [\mathcal{V}(B_1)]) -
[\mathcal{V}(B_1, I_{n-1})] = 1 + (\mathbb{L} - 1) y_{n-1} + ([\mathbb{P}^{2n-4}] + \mathbb{L}^{2n-3} y_1). \tag{30}
\]

Next, consider a summand with exactly one \( A_i \) vanishing (except \( A_1 \)). We get

\[
[\mathcal{V}(B_1 I_{n-1}^{n-i} I_{n-i})] = 1 + (\mathbb{L} - 1) [\mathcal{V}(I_{n-1}^{n-i} I_{n-i})] + ([\mathbb{P}^{2n-5}] + \mathbb{L}^{2n-4} y_1). \tag{31}
\]
In general, for \( P = I_{i_1}^{n-i_1} I_{i_2}^{n-i_2} \cdots I_{i_p}^{n-i_p}, I_{n-i_p} \) we compute

\[
[\mathcal{V}(B_1 P)] = 1 + (\mathbb{L} - 1)[\mathcal{V}(P)] + [\mathbb{P}^{2n-4-p}] + \mathbb{L}^{2n-3-p} y_1. \tag{32}
\]

For a fixed \( p \) we have exactly \( \binom{n-2}{p} \) different summands \( P \). Taking a sum over all such \( P \) for all \( p \) with right signs, we come to the sum

\[
S_{(1)} = \sum_{p=0}^{n-2} (-1)^p \left( \frac{n-2}{p} \right) + (\mathbb{L} - 1)y_{n-1} - (\mathbb{L} - 1)[T^{n-1}] +
\]

\[
\sum_{p=0}^{n-2} (-1)^p \left( \frac{n-2}{p} \right) [\mathbb{P}^{2n-4-p}] + y_1 \sum_{p=0}^{n-2} (-1)^p \left( \frac{n-2}{p} \right) \mathbb{L}^{2n-3-p} =
\]

\[
(\mathbb{L} - 1)(y_{n-1} - [T^{n-1}]) + \sum_{p=0}^{n-2} (-1)^p \left( \frac{n-2}{p} \right) [\mathbb{P}^{2n-4-p}] + \mathbb{L}^{n-1}(\mathbb{L} - 1)^{n-2} y_1. \tag{33}
\]

Consider now a summand \([\mathcal{V}(I_i P)]\) of \( S_{(i)} \) with \( P = I_{i_1}^{n-i_1} I_{i_2}^{n-i_2} \cdots I_{n-i_p}, 1 < i < n - 1 \). Similar to \((31)\), one obtains

\[
[\mathcal{V}(I_i^{n-i} P)] = \left( [\mathbb{P}^{2i-2}] + \mathbb{L}^{2i-1} [\mathcal{V}(P)] \right) + \left( [\mathbb{P}^{2n-2i-p-2}] + \mathbb{L}^{2n-2i-p-1} [\mathcal{V}(I_i^{n-i})] \right) - [\mathcal{V}(I_i^{n-i}, P)] = \left( [\mathbb{P}^{2i-2}] - y_i \right) + \left( \mathbb{L}^{2i-1} - y_i (\mathbb{L} - 1) - 1 \right) [\mathcal{V}(P)] + [\mathbb{P}^{2n-2i-p-2}] + \mathbb{L}^{2n-2i-p-1} y_i. \tag{34}
\]

We used here that for two polynomials \( P \) and \( Q \) of different non-intersected sets of variables (both of cardinality at least 2), one has \([\mathcal{V}(P, Q)] = (\mathbb{L} - 1) [\mathcal{V}(P)] [\mathcal{V}(Q)] + [\mathcal{V}(P)] + [\mathcal{V}(Q)] \) in the corresponding projective spaces. Taking the sum over all \( p \) and \( P \) with the right signs and grouping like in \((33)\), we obtain

\[
S_{(i)} = (\mathbb{L}^{2i-1} - y_i (\mathbb{L} - 1) - 1)(y_{n-i} - [T^{n-i}]) +
\]

\[
\sum_{p=0}^{n-1-i} (-1)^p \left( \frac{n-1-i}{p} \right) [\mathbb{P}^{2n-2i-2-p}] + \mathbb{L}^{n-i}(\mathbb{L} - 1)^{n-1-i} y_i. \tag{35}
\]

for \( i < n - 1 \) and

\[
S_{(n-1)} = \mathbb{L}^{2n-3} y_1 + 1 + (\mathbb{L} - 1)y_{n-1} + [\mathbb{P}^{2n-4}]. \tag{36}
\]

The big sum for \([T] = [T^n]\) is a sum of all \( S_{(i)} \). Considering \( n \) to be
non-fixed from the beginning, we get a recurrence formula

\[
[T^n] = \sum_{i=1}^{n-1} S_{i(i)} = \left[\mathbb{P}^{2n-4}\right] + \sum_{i=1}^{n-2} (\mathbb{L}^{2i-1} - y_i(\mathbb{L} - 1) - 1)
\]

\((y_{n-i} - [T^{n-1}]) + \sum_{i=1}^{n-2} \sum_{p=0}^{n-2-i} (-1)^p \binom{n-1-i}{p} \left[\mathbb{P}^{2n-2i-2-p}\right] + \sum_{i=1}^{n-2} \mathbb{L}^{n-i}(\mathbb{L} - 1)^{n-1-i}y_i + 1 + (\mathbb{L} - 1)y_{n-1}\]  

(37)

with \([T^2] = 2\). We can simplify the sum of projective spaces

\[
\sum_{p=0}^{n-2-i} (-1)^p \binom{n-1-i}{p} \left[\mathbb{P}^{2n-2i-2-p}\right] = \left[\mathbb{P}^{2n-2i-2}\right] + \sum_{p=0}^{n-2-i} (-1)^p \binom{n-2-i}{p} \left[\mathbb{P}^{2n-2i-2-p}\right]
\]

(38)

for \(1 \leq i \leq n - 2\). The formula for \([T]\) simplifies to

\[
[T^n] = \left[\mathbb{P}^{2n-4}\right] + \sum_{i=1}^{n-2} (\mathbb{L}^{2i-1} - y_i(\mathbb{L} - 1) - 1)(y_{n-i} - [T^{n-1}]) + \sum_{i=1}^{n-2} \mathbb{L}^{n-i}(\mathbb{L} - 1)^{n-2-i}(1 + (\mathbb{L} - 1)y_i) + 1 + (\mathbb{L} - 1)y_{n-1}.
\]  

(39)

First terms are

\[
[T^2] = 2,
\]

\[
[T^3] = 4\mathbb{L}^2 - \mathbb{L} + 2,
\]

\[
[T^4] = 6\mathbb{L}^4 - 4\mathbb{L}^3 + 5\mathbb{L}^2 - 2\mathbb{L} + 2,
\]

\[
[T^5] = 8\mathbb{L}^6 - 8\mathbb{L}^5 + 9\mathbb{L}^4 - 7\mathbb{L}^3 + 8\mathbb{L}^2 - 3\mathbb{L} + 2.
\]  

(40)

By (18), (23) and (26), we obtain

\[
[X] = 1 + \mathbb{L} + \mathbb{L}^2\left[\mathbb{P}^{2n-2}\right] - \mathbb{L}^2 - \mathbb{L}^3y_{n-1}' + \mathbb{L}^2(2 + \mathbb{L}y_{n-1}' + (\mathbb{L} - 1)(2(1 + \mathbb{L}y_{n-1}')) - [T^n])) = \]

\[
1 + \mathbb{L} + \mathbb{L}^2\left[\mathbb{P}^{2n-2}\right] + \mathbb{L}^2 + 2\mathbb{L}^2(\mathbb{L} - 1)(1 + \mathbb{L}y_{n-1}') - \mathbb{L}^2(\mathbb{L} - 1)[T^n].
\]  

(41)
For example,

\[
[X_3] = \mathbb{L}^4 + \mathbb{L}^3 + 2\mathbb{L}^2 + \mathbb{L} + 1,
\]

\[
[X_4] = \mathbb{L}^6 + \mathbb{L}^5 + 4\mathbb{L}^4 - 2\mathbb{L}^3 + 2\mathbb{L}^2 + \mathbb{L} + 1,
\]

\[
[X_5] = \mathbb{L}^8 + \mathbb{L}^7 + 7\mathbb{L}^6 - 8\mathbb{L}^5 + 8\mathbb{L}^4 - 3\mathbb{L}^3 + 2\mathbb{L}^2 + \mathbb{L} + 1.
\]

(42)

for \(X_k := X_{W S_k}\).

References

[BB] Belkale,P. and Brosnan,P. Matroids, motives and a conjecture of Kontsevich Duke Math. Journal, Vol.116 (2003) 147-188.

[BEK] Bloch,S., Esnault,H., and Kreimer,D. On motives associated to graph polynomials. Commun. Math. Phys. 267, 181-225 (2006).

[Bl] Bloch,S. Motives associated to sums of graphs. arXiv.org:0810.1313v1

[BrKr] Broadhurst,D., and Kreimer,D. Knots and numbers in \(\Phi^4\) theory to 7 loops and beyond. Int. J. Mod. Phys. C 6, 519 (1995).

[BrK] Broadhurst,D., and Kreimer,D. Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops. Phys. Lett. B 393 (1997) 403.

[DD] Doryn,D. Cohomology of graph hypersurface associated to certain Feynman graphs. arXiv.org:0811.0402v1

[Sch] Schnetz,O. Quantum field theory over \(\mathbb{F}_q\). arXiv.org:0909.0905v1