Coulomb blockade of anyons

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Coulomb interaction turns anyonic quasiparticles of a primary quantum Hall liquid with filling factor \( \nu = 1/(2m + 1) \) into hard-core anyons. We have developed a model of coherent transport of such quasiparticles in systems of multiple antidots by extending the Wigner-Jordan description of 1D abelian anyons to tunneling problems. We show that the anyonic exchange statistics manifests itself in tunneling conductance even in the absence of quasiparticle exchanges. In particular, it can be seen as a non-vanishing resonant peak associated with quasiparticle tunneling through a line of three antidots.

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Quasiparticles of two-dimensional (2D) electron liquids in the regime of the Fractional Quantum Hall effect (FQHE) have unusual properties of fractional charge \([1]\) and fractional exchange statistics \([2, 3]\). The fractional quasiparticle charge was observed in experiments on antidot tunneling \([4]\) and shot-noise measurements \([5, 6]\). The situation with fractional statistics is so far less certain even in the case of the abelian statistics, which is the subject of this work. Although the recent experiments \([7]\) demonstrating unusual flux periodicity of conductance of a quasiparticle interferometer can be interpreted as a manifestation of the fractional statistics \([8, 9]\), this interpretation is not universally accepted \([10, 11]\). There is a number of theoretical proposals (see, e.g., \([12, 13]\)) suggesting tunnel structures where the statistics manifests itself through noise properties. Partly due to complexity of noise measurements, such experiments have not been performed successfully up to now. In this work, we show that coherent quasiparticle dynamics in multi-antidot structures should provide clear signatures of the exchange statistics in dc transport. Most notably, in tunneling through a line of three antidots, fractional statistics leads to a non-vanishing peak of the tunnel conductance which would vanish for integer statistics.

These effects rely on the ability of quantum antidots to localize individual quasiparticles of the QH liquids \([4, 14, 15]\). The resulting transport phenomena in antidots are very similar to those associated with the Coulomb blockade \([16]\) in tunneling of individual electrons in dots. For instance, similarly to a quantum dot \([19]\), the linear conductance of one antidot shows periodic oscillations with each period corresponding to the addition of one quasiparticle \([14, 15, 17, 18]\). Recently, we have developed a theory of such Coulomb-blockade-type tunneling for a double-antidot system \([20]\), where quasiparticle exchange statistics does not affect the transport. The goal of this work is to extend this theory to antidot structures where the statistics does affect the conductance. The two simplest structures with this property consist of three antidots and have quasi-1D geometries with either periodic or open boundary conditions (Fig. 1). A technical issue that needed to be resolved to calculate the tunnel conductance is that the anyonic field operators defined through the Wigner-Jordan transformation \([21, 22, 23, 24]\), are not fully sufficient in the situations of tunneling. As we show below, to obtain correct matrix elements for anyon tunneling, one needs to keep track of the appropriate boundary conditions of the wavefunctions which are not accounted for in the field operators.

Specifically, we consider the antidots coupled by tunneling among themselves and to two opposite edges of the quantum Hall liquid (Fig. 1). The edges play the role of the quasiparticle reservoirs with the transport voltage \( V \) applied between them. We assume that the antidot-edge coupling is weak and can be treated as a perturbation. Quasiparticle transport through the antidots is governed then by the kinetic equation similar to that for Coulomb-blockade transport through quantum dots with a discrete energy spectrum \([25]\). Coherent quasiparticle dynamics requires that the relaxation rate \( \Gamma_d \) created by direct Coulomb antidot-edge coupling is weak. This condition should be satisfied if the edge-state confinement is sufficiently strong \([20]\). The requirement on the confinement is less stringent in the case of the antidot line (Fig. 1b), in which antidot quasiparticles move along the edge, suppressing the antidot-edge coupling at low frequencies. We also assume that all quasiparticle energies

![FIG. 1: Tunneling of anyonic quasiparticles between opposite edges of an FQHE liquid through quasi-1D triple-antidot systems: (a) loop, (b) open interval. Quasiparticles tunnel between the edges and the antidots with rates \( \Gamma_{1,2} \). The antidots are coupled coherently by tunnel amplitudes \( \Delta \).](image)
on the antidots, tunnel amplitudes $\Delta$, temperature $T$, Coulomb interaction energies $U$ between quasiparticles on different antidots, are much smaller than the energy gap $\Delta^*$ for excitations on each antidot. This condition ensures that the state of each antidot is characterized completely by the occupation number $n$ of its relevant quantized energy level. In any given range of the background voltage or magnetic field (which produces the overall shift of the antidot energies - see, e.g., [4, 14, 15]), there can be at most one quasiparticle on each antidot, $n = 0, 1$. This "hard-core" property of the quasiparticles means that they behave as fermions in terms of their occupation factors, despite the anyonic exchange statistics. All these assumptions can be summarized as: $\Gamma_d, \Gamma_j \ll \Delta, U, T \ll \Delta^*$.

Under these conditions, the antidot tunneling is dominated by the antidot energies. The quasi-1D geometry of the antidot systems we consider makes it possible to introduce the quasiparticle "coordinate" $x$ numbering successive antidots; e.g., $x = -1, 0, 1$ for systems in Fig. 1. The quasiparticle Hamiltonian can be written as

$$H = \sum_x [\epsilon_x n_x - (\Delta x \xi^\dagger_{x+1} \xi_x + \text{h.c.})] + \sum_{x \neq y} U_{x,y} n_x n_y ,$$

where $\epsilon_x$ are the energies of the relevant localized states on the antidots (taken relative to the common chemical potential of the edges at $V = 0$). $\Delta x$ is the tunnel coupling between them, $U_{x,y}$ is the quasiparticle Coulomb repulsion, and $n_x = \xi^\dagger_x \xi_x$. The quasiparticle operators $\xi^\dagger_x, \xi_x$ in (1) can be viewed as the Klein factors left in the standard operators for the edge-state quasiparticles when all the edge magneto-plasmon modes are suppressed by the gap $\Delta^*$. Characteristics of such Klein factors depend on the geometry of a specific tunneling problem; non-trivial examples can be found in [12, 13, 26, 27]. In the Hamiltonian (1), $\xi_x$ describe the hard-core anyons with exchange statistics $\pi \nu$. Wigner-Jordan transformation expresses them through the Fermi operators $\epsilon_x$ [21]:

$$\xi_x = e^{i\pi(\nu - 1)\sum_{x < x} n_x} \epsilon_x , \quad \xi^\dagger_x = \xi_x \epsilon^\dagger x e^{i\pi \nu \text{sgn}(x-y)} ,$$

with similar relations for $\xi^\dagger_x$. Anyonic statistics creates an effective interaction between the quasiparticles which can be understood as the Aharonov-Bohm (AB) interaction between a flux tube "attached" to one of the particles and the charge carried by another. In general, this interaction can be masked by the direct Coulomb interaction $U_{x,y}$. In the antidot loop (Fig. 1a), however, $U_{x,y}$ is constant, $U_{x,y} = U$, and the interaction term in (1) reduces to $Un(n-1)/2$, with $n = \sum_x n_x$ - the total number of the quasiparticles on the antidots. In this case, the Coulomb interaction contributes to the energy separation between the group of states with different $n$, but does not affect the level structure for given $n$. The hard-core property of quasiparticlesc limits $n$ to the interval $[0, 3]$. For $n = 0$ and $n = 3$, the system has the "empty" and "completely filled" state with respective energies $E_0 = 0, E_3 = \sum \epsilon_x + 3U$. The spectrum $E_{1k}$ of the three $n = 1$ states $|1k\rangle = \sum \phi_k(x) \xi_{x}^\dagger |0\rangle$, is obtained as usual from (1). In the uniform case $\epsilon_x = \epsilon, \Delta x = \Delta$, with an external AB phase $\phi$, one has $\phi_k(x) = e^{ikx}/L^{1/2}$ and

$$E_{1k} = \epsilon - \Delta \cos k , \quad k = (2\pi m + \varphi)/L ,$$

where $m = 0, 1, 2$, and the loop length is $L = 3$.

Anyonic statistics can be seen in the $n = 2$ states, $|2l\rangle = (1/\sqrt{2}) \sum_{xy} \psi_l(x,y) \xi_{x}^\dagger \xi_{y}^\dagger |0\rangle$. The fermion-anyon relation (2) suggests that the stationary two-quasiparticle wavefunctions should coincide up to the exchange phase with that for free fermions:

$$\psi_l(x,y) = \frac{e^{i\pi(1-\nu)\text{sgn}(x-y)/2}}{\sqrt{2}} \det \left( \begin{array}{cc} \phi_q(x) & \phi_q(y) \\ \phi_p(x) & \phi_p(y) \end{array} \right) .$$

Here $\phi_q$ are the single-particle eigenstates of the Hamiltonian (1). (The states $\phi_q$ are numbered with the index $l$ of the third "unoccupied" state with respective to the two occupied ones $q, p$. The boundary conditions for the $\phi$ are affected by the exchange phase in Eq. (4). To find them, we temporarily assume for clarity that coordinates $x, y$ are continuous and lie in the interval $[0, L]$. Subsequent discretization does not change anything substantive in this discussion. The 1D hard-core particles are impenetrable and can be exchanged only by moving one of them, say $x$, around the loop from $x = y + 0$ to $x = y - 0$ (Fig. 2b). Since the loop is imbedded in the underlying 2D system, such an exchange means that the wavefunction acquires the phase factor $e^{i\pi \nu}$, in which the sign of $\nu$ is fixed by the properties of the 2D system, e.g. the direction of magnetic field in the case of FQHE liquid. Next, if the second particle is moved similarly, from $y = x + 0$ to $y = x - 0$, the wavefunction changes in the same way, for a total factor $e^{i2\pi \nu}$. Equation (4) shows that only one of these changes can agree with the 1D form of the exchange phase. As a result, the wavefunction (4) satisfies different boundary conditions in $x$ and $y$:

$$\psi_l(L,y) = \psi_l(0,y) e^{i\varphi} , \quad \psi_l(x,L) = \psi_l(x,0) e^{i(\varphi + 2\pi \nu)} .$$

Conditions (5) on the wavefunction (4) mean that the single-particle functions $\phi$ in (1) satisfy the boundary condition that correspond to the effective AB phase $\varphi' = \varphi + \pi - \pi \nu$, i.e. the addition of an extra quasiparticle to the loop changed the AB phase by $\pi - \pi \nu$, where $-\pi \nu$ comes from the exchange statistics and $\pi$ from the hard-core condition. This gives the energies of the two-quasiparticle states (1) as $U + E_{1q} + E_{1p}$, where, if the loop is uniform, the single-particle energies are given by Eq. (3) with $\varphi \rightarrow \varphi'$. In this case, $\sum_k E_{1k} = 0$, and the energies $E_{2l}$ of the two-quasiparticle states can be written as:

$$E_{2l} = 2\epsilon + U - \Delta \cos l , \quad l = (2\pi n' + \varphi - \pi \nu)/3 ,$$

where $n' = l / 2\pi$.
Specific anyonic interaction between quasiparticles can formally coordinates \( x \) and \( y \) the character of the boundary conditions (5) between \( \nu \) and \( \gamma \), where \( \gamma \) is the overall magnitude of the tunneling rate, and 

\[
f_i(E) = (2\pi T/\omega_c)^{\nu-1} |\Gamma(\nu/2+iE/2\pi T)|^2 e^{-E^2/2T}/2\pi \Gamma(\nu)
\]

is its energy dependence associated with the Luttinger-liquid correlations in the edges [28]. Here \( \Gamma(z) \) is the gamma-function and \( \omega_c \) is the cut-off energy of the edge excitations. The rates \( \Gamma_j(E) \) can be used in the standard kinetic equation to calculate the conductance of the antidot system [20]. Anyonic statistics of quasiparticles affects the position and amplitude of the conductance peaks through the shift of the energy levels by quasiparticle tunneling (described, e.g., by Eq. (9)) and through the kinetic effects caused by the anyonic features in the matrix elements (8). In the case of the antidot loop (Fig. 1b), however, effects of statistics are masked by the external AB flux \( \varphi \) through the loop. Since the area of practical antidots is much larger than the internal area of the loop, \( \varphi \) is essentially random and can not be controlled by external magnetic field on the relevant scale of one period of conductance oscillations. Below, we present the results for conductance for the similar case of a line of antidots (Fig. 1b), the conductance of which is insensitive to the AB flux, and shows effects of fractional statistics in the tunneling matrix elements.

As before, the quasiparticle Hamiltonian is given by Eq. (1). In this geometry, the interaction energy \( U_1 = U_{\nu=0} = U_{\nu=-1} \) between the nearest-neighbor antidots is in general different from the interaction \( U_2 = U_{\nu=1} \) between the quasiparticles at the ends. The localization energies on the antidots can be written as \( \epsilon_j = \epsilon + x\delta + 2\lambda|x| \). We consider first the unbiased line, \( \delta = 0 \). At low temperatures, \( T \ll \Delta, U \), only the ground states of \( n \) quasiparticles with energies \( E_n \) participate in transport: \( E_0 = 0 \), \( E_1 = \epsilon + \lambda - \omega \), \( E_2 = 2\epsilon + 3\lambda - \omega + (U_a + U_b)/2 \), and \( E_3 = 3\epsilon + 2U_a + U_b + 4\lambda \), where \( \omega = (\Delta_1^2 + \Delta_2^2 + \lambda^2)^{1/2} \) and \( \omega \) is given by the same expression with \( \lambda \) replaced by \( \lambda - (U_1 - U_2)/2 \). In this regime, the linear conductance \( G \) consists of three peaks, with each peak associated with addition of one more quasiparticle to the antidots,

\[
G = \frac{(e\nu)^2}{T} \frac{\gamma_1 \gamma_2 a_n f_n(E_{n+1} - E_n)}{\gamma_1 + \gamma_2 + 1 + \exp[-(E_{n+1} - E_n)/T]},
\]

where \( a_n \equiv |\langle n | \xi_0 | n \rangle|^2 \). The amplitudes \( a_0, a_2 \) are effectively single-particle, and thus, independent of the

![FIG. 2: Exchanges of hard-core anyons on a 1D loop: (a) real exchanges by transfer along the loop embedded in a 2D system; (b) formal exchanges describing the assumed boundary conditions (6) of the wavefunction.](image)
tunnel matrix element. Fractional statistics of quasiparticles makes this destructive interference incomplete. Finite bias $\delta \neq 0$ along the line suppresses this interference making the effect of the statistics smaller. One can still distinguish the fractional statistics by looking at the dependence of the amplitude of the middle peak of conductance $G_0$ on the bias $\delta$ shown in the right inset in Fig. 3.

In conclusion, we have developed a model of coherent transport of anyonic quasiparticles in systems of multiple antidots. In antidot loops, addition of individual quasiparticles shifts the quasiparticle energy spectrum by adding statistical flux to the loop. In the case without loops, energy levels are insensitive to quasiparticle statistics, but the statistics still manifests itself in the quasiparticle tunneling rates and hence dc tunnel conductance of the antidot system.

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