Ultrametric Cantor sets and Growth of Measure

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Abstract

A class of ultrametric Cantor sets \((C, d_u)\) introduced recently (Raut, S and Datta, D P (2009), Fractals, 17, 45-52) is shown to enjoy some novel properties. The ultrametric \(d_u\) is defined using the concept of relative infinitesimals and an inversion rule. The associated (infinitesimal) valuation which turns out to be both scale and reparametrisation invariant, is identified with the Cantor function associated with a Cantor set \(\tilde{C}\) where the relative infinitesimals are supposed to live in. These ultrametrics are both metrically as well as topologically inequivalent compared to the topology induced by the usual metric. Every point of the original Cantor set \(C\) is identified with the closure of the set of gaps of \(\tilde{C}\). The increments on such an ultrametric space is accomplished by following the inversion rule. As a consequence, Cantor functions are reinterpreted as locally constant functions on these extended ultrametric spaces. An interesting phenomenon, called growth of measure, is studied on such an ultrametric space. Using the reparametrisation invariance of the valuation it is shown how the scale factors of a Lebesgue measure zero Cantor set might get deformed leading to a deformed Cantor set with a positive measure. The definition of a new valuated exponent is introduced which is shown to yield the fatness exponent in the case of a positive measure (fat) Cantor set. However, the valuated exponent can also be used to distinguish Cantor sets with identical Hausdorff dimension and thickness. A class of Cantor sets with Hausdorff dimension \(\log_3 2\) and thickness 1 are constructed explicitly.

Key Words: Ultrametric, Cantor set, Cantor function, Scale invariance, Measure

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1 Introduction

A Cantor set is a compact, perfect, totally disconnected, metrisable topological space. In this work we consider a Cantor set that is realized as a (proper) subset of the real line. It is of measure zero if the Lebesgue measure of the set is zero. Otherwise this has a positive measure. Such a set is also said to be a fat Cantor set. Hausdorff measure and Hausdorff dimension are generally considered to reveal the characteristic features of a Cantor set. A set \( C \) is said to be an \( s \)-set if the corresponding Hausdorff measure has a finite non-zero value viz; \( 0 < H^s(C) < \infty \) \([1]\). Cantor sets are known to carry a natural ultrametric structure. Here we study a family of Cantor sets with same Hausdorff dimension. We endow a Cantor set with a scale invariant ultrametric structure which coincides with the usual ultrametric only for a special choice of the valuation. For a general valuation this offers one with an insight how a measure zero set might grow into a fat Cantor set because of a distortion (deformation) of scales in an infinitesimal element of the original thin Cantor set. Recently, there has been some interest in ultrametric Cantor sets \([2, 3, 4, 5, 6]\). Let us recall that a metric on a Cantor set is regular if the topology induced by the metric is equivalent to the usual topology. The relationship of the present class of ultrametrics with those considered in the so-called Michon’s correspondence with rooted trees \([2]\) will be considered separately. In the present work, we show that the valued measure induced by our non-archimedean valuation retrieves the nontrivial measure of \([2]\) in a natural way. This valuation also gives a new handle to distinguish two sets with identical Hausdorff dimension. Various other measures are available in literature to characterise different aspects of a Cantor set. Thickness is used to study arithmetic sum and difference of Cantor sets that arise in number theory and dynamical systems \([7, 8]\). However, thickness of a positive measure set is infinite and fails to reveal the intrinsic geometric structures of such a set. A commonly used measure is the fatness exponent \([9, 10]\). An analogous but, nevertheless, a slightly different parameter is called the exterior dimension or the uncertainty exponent \([11]\). We compare our renormalised valuated parameter with these parameters.

1.1 Main Results

In Sec. 2, we collect, for completeness, the basic definitions and properties that are necessary for this paper.

In Sec. 3, we present, in some detail, the definition(s) of a class of inequivalent ultrametrics \([4, 5, 6]\) and discuss the salient properties, mostly, in the context of the classical middle third Cantor set. The definition of ultrametrics depends on the concept of relative infinitesimals which are new elements in the neighbourhood of 0 and satisfy an inversion rule. That is to say, the increments among infinitesimals as well as between an infinitesi-
mal and a (real) point of the Cantor set are accomplished by *inversions*, rather than by *translations* that is generally considered in standard real analysis. Some of the results presented here are new (Theorem 1 and Proposition 2): (i) The relation of the nontrivial valuation with the Cantor function, considered in Refs. [4, 5], is made precise by proving that the valuation is indeed given by an appropriate Cantor function. (ii) The multiplicative representation that exists because of the nontrivial infinitesimals and the scale invariant ultrametric for every element of the Cantor set is verified explicitly, leading to a proof that the non-archimedean absolute value $||x|| = 3^{-ns}$, $x \in C$, $s = \log_3 2$ precisely corresponds to the ultrametric of [2] in the context of noncommutative geometry and (iii) the analysis of convergence of sequences of the form $\varepsilon^{n-l}$, $0 < \varepsilon, l < 1$. The usual limit 0 is replaced by the constant $l$ in the present ultrametrics. This establishes the metric as well as the topological inequivalence of these scale invariant ultrametrics. The ultrametrics considered in [2] are only metrically distinct.

In Sec. 4, we discuss the relationship of the scale invariant differential equation (DE)

$$x \frac{dX}{dx} = X \tag{1}$$

with a Cantor set $C$. First, we reinterpret the so-called non-smooth solutions of Ref. [13, 14] in the context of Cantor sets, when Cantor set elements are replaced by infinitesimal copies of an inverted ultrametric Cantor set. Infinitesimals are supposed to live in such a set which is defined as the closure of the set of gaps of a Cantor set. Eq(1) is thus well defined in the neighbourhood of each $x \in C$. The non-smooth solutions of [14] are realised as locally constant functions (LCF) $\phi(x)$ on $C$. The nontrivial ultrametric valuation that is shown to correspond to the locally constant (LC) Cantor function associated with $C$ is, therefore, raised to the class of smooth (LC) solutions of the scale free equation (1) in the double logarithmic (log log $x^{-1}$) scale (c.f. eq.(26)) (Theorem 2).

In Sec. 5, we study the interesting phenomenon of *growth of measure* that becomes meaningful in the context of the present ultrametric. (i) Starting from the observation of the reparametrisation invariance of the definition of a LC valuation in an ultrametric space, we demonstrate explicitly how a possible infinitesimal scale variation in the reparametrisation invariant infinitesimal can lead to a deformation of scale factors of a (Lebesgue) measure zero Cantor set, so that the associated deformed Cantor set may have a positive measure (Theorem 3). Next, (ii) we define a higher (double logarithmic) order valuation and an *valuated exponent* that is shown to equal the fatness exponent that is used in literature [9, 10] to characterise Cantor sets with positive measure. However, (iii) the valuated exponent is identified here with the inverse of the Hausdorff dimension of a residual (measure zero) Cantor set that would arise as local fine structures in the neighbourhoods of the points of the original (positive measure) Cantor set. (iv) The valuated exponent is also shown to characterise a family of measure zero sets all having identical
Hausdorff dimension and thickness. This is demonstrated by constructing a family of such sets with Hausdorff dimension $\log_3 2$.

In the concluding Sec.6, we remark on the implications of the inequivalent ultrametrics and the associated multiplicative structure, called the generalized Euler’s factorisation on the real number system. The nature of the asymptotic limit $x \to 0^+$ is altered significantly in the presence of relative infinitesimals.

### 2 Basic Definitions

A Cantor set $C$ is defined as a countable intersection of finite unions of closed (and bounded) subsets of $R$. For definiteness, let $C \subset I = [0,1]$. Then, by definition, $C = \bigcap_1^\infty F_n = \bigcap_{n=1}^\infty \bigcup_{m=1}^p F_{nm}$ where $F_{nm} \subset I$ are closed with $F_{00} = I$. Equivalently, $C$ is also defined as $C = I - \bigcup_{i=1}^\infty O_i$ where $O_i$ are open intervals which are deleted recursively from $I$. Consequently, a Cantor set is often defined as the limit set of an iterated function system (IFS) $f = \{ f_i : I \to I, i = 1, 2, ..., p \}$ so that $C = f (C)$. For definiteness, we consider binary Cantor sets in which each application of the IFS removes an open interval from a closed subinterval splitting it into two disjoint closed subintervals of the form

$$F = F_0 \cup O \cup F_1.$$  

(2)

The deleted interval $O$ is called the gap and the two closed components are the bridges. As an example let us consider a middle $\alpha$ Cantor set $C_\alpha$ which arises as the limit set under the IFS

$$f_i(x) = \beta x + i(1 - \beta), \quad i = 0, 1$$  

(3)

where the scale factor $\beta$ is defined by $\alpha + 2\beta = 1$. Each iteration of the IFS removes an open interval (i.e. a gap) of length proportional to $\alpha$ from a closed subinterval of $I$, leaving out two bridges of size proportional to $\beta$ each. The IFS (2) satisfies the open set condition (OSC) if there exists a non-empty bounded open set $S$ such that $\bigcup_i f_i (S) \subseteq S$. It follows accordingly that $\beta \in (0, \frac{1}{2})$. Since the total length of the disjoint open intervals viz., $\Sigma_{i=1}^{\infty} |O_i| = \Sigma_{i=1}^{\infty} \alpha(2\beta)^{i-1} = 1$, the middle $\alpha$ Cantor set is of measure zero with the Hausdorff dimension $s = \frac{\log 2}{\log \frac{1}{\beta}}$. More generally, when $q$ open intervals each of size $\alpha$ are deleted leaving out $p$ equal closed intervals of size $\beta$ so that $q\alpha + p\beta = 1$, then the OSC gives $\beta \in (0, \frac{1}{p})$. The length of the deleted open intervals add up to 1 viz., $\Sigma (q\alpha)(p\beta)^{n-1} = 1$. The corresponding measure zero set $C_{\alpha,p}$ has the Hausdorff dimension $\frac{\log p}{\log \frac{1}{p}}$.

Returning to the discussion of the binary Cantor set we recall that the set $C_\alpha$ is also a homogeneous and uniform Cantor set. It is homogeneous since the scale factors in each
component of the IFS are same. The set is uniform because each deleted open interval also is of constant proportion \( \alpha \) of the length of the previous (defining) closed interval.

A positive 1-set \( \tilde{C} \), on the other hand, is obtained if the deletion process removes open intervals of variable sizes.

**Example 1:** Let at each step we remove \( \alpha_n \) portion of the length of each component of the previous closed set \( F_{n-1} \) so that \( F_{n-1} = F_0 \cup O_n \cup F_{n1} \) and \( |O_n| = \alpha_n |F_{n-1}|, |F_{n0}| = |F_{n1}| = \frac{1}{2} (1 - \alpha_n) |F_{n-1}| \). By induction, each of \( 2^n \) components of \( F_n \) has length \( |F_{ni}| = \frac{1}{2^n} \prod_{0}^{n}(1 - \alpha_j), i = 1, 2, \ldots 2^n \). Consequently, \( m(\tilde{C}) = \lim_{n \to \infty} |F_{n-1}| = \prod_{0}^{\infty} (1 - \alpha_i) > 0 \) when \( \sum \alpha_n < \infty \).

**Example 2:** Suppose at the \( n \)th step an open interval of length \( \frac{\delta}{3^n}, (0 < \delta < 1) \) is removed from each of the \( 2^n \) components of \( F_n \). The length of each component of \( F_n \) is \( \frac{1}{2^n} \left( 1 - \frac{\delta}{3} - \cdots - \frac{\delta^{n+1}}{3^n} \right) \). The sum of the lengths of all the open intervals removed is \( \sum \frac{2^n \delta}{3^{n+1}} = \delta \), so that \( m(\tilde{C}) = 1 - \delta \).

In both the above examples, \( 0 < H^s(\tilde{C}) = m(\tilde{C}) = l < \infty \) when \( s = 1 \). Even as the Cantor set \( \tilde{C} \) is nowhere dense in \( I \), it has a non-zero measure because of the fact that the defining iteration process now removes the open interval at a slower rate in comparison to the middle \( \alpha \) set \( C_\alpha \). In fact, the relative difference of the lengths of the deleted open middle \( \alpha \) and \( \alpha_n \) intervals \( O_{\alpha_n} \) and \( \tilde{O}_{\alpha_n} \) respectively is given by

\[
\left| \alpha \beta^n - \frac{1}{2^n} \alpha_n \prod_{1}^{n}(1 - \alpha_i) \right| = \left| 1 - \frac{\alpha_n}{\alpha} \prod_{1}^{n} \frac{1 - \alpha_i}{1 - \alpha} \right| |O_{\alpha_n}| \geq |1 - \gamma| |O_{\alpha_n}|
\]

where \( \frac{\alpha_n}{\alpha} \prod_{1}^{n} \frac{1 - \alpha_i}{1 - \alpha} \to \gamma \) for \( n \to \infty \). Note that the above lower bound exists, otherwise \( \tilde{C} \) would have been a set of measure zero. For the modified middle \( \frac{\delta}{3} \)rd set (Example 2) one has the exact equality

\[
\left| \alpha \beta^n - \frac{1}{2^n} \alpha_n \prod_{1}^{n}(1 - \alpha_i) \right| = (1 - \delta) |O_{\frac{\delta}{3^n}}|
\]

The emergence of the positive measure of \( \tilde{C} \) can, therefore, be explained in a dynamical sense provided the said set \( \tilde{C} \) is seen as being evolved from a given zero measure set because of a possible **principle** allowing for a deformation of scales in the deletion process. As is evident the relative scaling of the lengths of infinitesimal elements of the deleted open intervals \( \tilde{O}_{\alpha_n} \) over the corresponding \( \alpha \) interval \( O_{\alpha_n} \), indeed captures the origin of the positive measure even in the totally disconnected perfect set \( \tilde{C} \). In the following we offer a new ultrametric explanation of the growth of the positive measure of \( \tilde{C} \) over \( C_\alpha \). The class of ultrametrics that we consider is not only **scale invariant** (in the sense of a power law), but also **reparametrisation invariant** (that is, the invariance under a reparametrisation of the form \( X(t) \to \tilde{X}(t) = X(f(t)) \) where the otherwise arbitrary function \( f \) satisfies
the conditions $f' > 0$, along with the boundary condition $f(0) = 0$, $f(1) = 1$. As will be explained later, the scale variation can, therefore, be interpreted as a reflection of the underlying reparametrisation invariance of the valuation. Incidentally, we note that for a measure 1 set the functions defining an IFS may not have a closed form [12].

2.1 Thickness

The measure of thickness of a Cantor set has various applications in number theory and dynamical systems. To recall the definition, let $F_i = I - \bigcup_{i=0}^{i} O_i$ form a defining sequence of the Cantor set $C$. The $2^i$ components of $F_i$ are the closed intervals $F_{ij}$, $j = 1, 2, \cdots 2^i$, which are the bridges. The deleted intervals $O_i$ are the gaps of $C$. Let $O_{ik}$ denote the open deleted subinterval of a bridge $F_{ij}$ of $F_i$ which divides $F_{ij}$ into two smaller bridges $F_{ij}^L$ and $F_{ij}^R$ of $F_i$. Let

$$\tau(F_i) = \inf_j \left\{ \frac{|F_{ij}^L|}{|O_{ik}|}, \frac{|F_{ij}^R|}{|O_{ik}|} \right\}$$

The thickness $\tau(C)$ is defined by $\tau(C) = \sup_i \tau(F_i)$, where sup is evaluated over the defining sequences of $C$. For a set $A$ containing an interval, $\tau(A) = \infty$, by definition.

For the middle $\alpha$- Cantor set $C_\alpha$, it follows that

$$\frac{|F_{ij}^L|}{|O_{ik}|} = \frac{|F_{ij}^R|}{|O_{ik}|} = \frac{\beta}{\alpha} |F_{ij}|$$

so that $\tau(C) = \frac{\beta}{\alpha}$.

For a positive measure set $\tilde{C}$, on the other hand, one has $\frac{|F_{ij}^L|}{|O_{ik}|} = \frac{|F_{ij}^R|}{|O_{ik}|} = \frac{1-\alpha}{2\alpha}$, leading to $\tau(\tilde{C}) = \infty$, as expected.

2.2 Cantor Function

A Cantor function is a nonconstant and non-decreasing continuous function $\phi : [0, 1] \to [0, 1]$ such that $\phi'(x) = 0$ a.e. with points of nondifferentiability $x$ lying, for instance, in the Cantor set $C_\alpha$.

To construct $\phi$ explicitly, let $\phi(0) = 0$, $\phi(1) = 1$. Assign $\phi(x)$ a constant value $\phi(x) = i2^{-n}$, $i = 1, 2, \cdots, 2^n - 1$ on each of the deleted open intervals (including the end points of the deleted interval) of $C_\alpha$. Next, let $x \in C_\alpha$. Then, at the $n$th iteration, $x$ belongs to the interior of exactly one of the $2^n$ remaining closed intervals each of length $\beta^n$. Let $[a_n, b_n]$ be one such intervals. Then $b_n - a_n = \beta^n$. Moreover, $\phi(b_n) - \phi(a_n) = 2^{-n}$. At the next iteration, assuming $x \in [a_{n+1}, b_{n+1}]$, ($a_n = a_{n+1}$), say, we have $\phi(a_n) \leq \phi(a_{n+1}) < \phi(b_{n+1}) \leq \phi(b_n)$. Define $\phi(x) = \lim_{n \to \infty} \phi(a_n) = \lim_{n \to \infty} \phi(b_n)$. Then $\phi : [0, 1] \to [0, 1]$
is a continuous, non-decreasing function. Also $\phi'(x) = 0$ for $x \in I \setminus C_\alpha$ when it is not differentiable at any $x \in C_\alpha$. (c.f., [4 5]).

2.3 Ultrametric

The topology of a Cantor set is equivalent to an ultrametric topology. A point $x \in C_\alpha$ has the unique infinite word representation

$$x = (1 - \beta)^\infty \sum_0^\infty x_i \beta^i = x_0 x_1 x_2 \cdots, \ x_i \in \{0, 1\}$$

Let $L(x, y) = n$ such that $x_0 = y_0, \ldots, x_{n-1} = y_{n-1}, x_n \neq y_n, x, y \in C_\alpha$. The ultrametric $\tilde{d}_u$ is defined by $\tilde{d}_u(x, y) = p^{-L(x,y)}$ for any $p > 1$. This ultrametric is equivalent to the usual metric

$$C_1 \tilde{d}_u(x, y) \leq d(x, y) \leq C_2 \tilde{d}_u(x, y)$$

for two positive constants $C_1$ and $C_2$, where $d(x, y)$ is the usual metric.

The Cantor set thus consists of towers of closed balls (intervals) with countable intersection property. Further the fundamental neighbourhood system of any point consists of clopen balls.

We now define an inequivalent class of ultrametrics, called valuated ultrametrics, on a Cantor set. The definition of the ultrametric makes use of a concept of relative infinitesimals introduced recently [4 5 6].

3 Inequivalent Ultrametrics

Let $C$ be a (measure zero) Cantor set. We consider an infinitesimal gap $O_{\inf}$ of $C$ in $I = [0, 1]$. Given an arbitrarily small $x \in C - \{0\}$ (in the sense that $x \to 0^+$ on $C - \{0\}$), $\exists \epsilon \in I$ and $\epsilon < x$ and an open interval $\tilde{I} \subset (0, \epsilon)$ such that $\tilde{I} \cap C = \Phi$, the null set. This follows from the total disconnectedness of $C$. An element $\tilde{x}$ in $\tilde{I}$ satisfying $0 < \tilde{x} < \epsilon < x$ and the inversion rule

$$\frac{\tilde{x}}{\epsilon} = \lambda^{-1}(\epsilon) \frac{\epsilon}{x}$$

for a real constant $\lambda$ ($0 < \lambda \leq 1$) is called a relative infinitesimal relative to the scale $\epsilon$. For each choice of $x$ and $\epsilon$, we have a unique $\tilde{x}$ for a given $\lambda \in (0, 1)$. Consequently, by varying $\lambda$ in an open subinterval of $(0,1)$, we get an open interval of relative infinitesimals in the interval $(0, \epsilon)$, all of which are related to $x$ by the inversion formula. The infinitesimal gap $O_{\inf} \subset \tilde{I}$ is, by definition, the set of these relative infinitesimals satisfying the inversion rule (4), as $\epsilon \to 0$, in an asymptotic sense (c.f. Remarks 1 and 2). In the limit $\epsilon \to 0$, $O_{\inf} = \Phi$, in the usual topology. However, the corresponding set of scale invariant infinitesimals
\( \tilde{O}_{\text{inf}} = \lim_{\epsilon \rightarrow 0} \{ \tilde{X} | \tilde{X} = \tilde{x} \approx \mu e^\alpha \pm o(e^\beta) \} \) where \( \mu \) is a constant and \( 1 > \beta > \alpha \geq 0 \), may be a non-null subset of \((0,1)\) (for instance, when \( \alpha = 0 \), in particular) (for an explanation of the asymptotic expansion of \( \tilde{X} \) see Remark 2.1). Notice that constants \( \alpha, \beta, \) and \( \mu \) may, however, depend on \( \lambda \). Notice also that the infinitesimal gap \( O_{\text{inf}} = O(x, \epsilon, \lambda) \) depends on \( \epsilon \), but apparently also on the arbitrarily small element \( x \) of the Cantor set along with the parameter \( \lambda \) appearing in the inversion law. But \( x \) and \( \epsilon \) are very closely related, so that \( O_{\text{inf}} \) essentially depends only on \( \epsilon \) and \( \lambda(\epsilon) \).

For a point \( x \) from a Cantor set \( C \), it is natural to assume that the scale \( \epsilon \) is determined by the privileged scale of the Cantor set. Two relative infinitesimals \( \tilde{x} \) and \( \tilde{y} \) must necessarily satisfy the condition \( 0 < \tilde{x} \leq \tilde{y} < \tilde{x} + \tilde{y} < \epsilon \). As indicated already, the inversion rule maps an open interval of (relative) infinitesimals of size determined by the parameter \( \lambda \) to an arbitrarily small element \( x \) of \( C \). These relative infinitesimals are endowed with a scale free (non-archimedean) absolute value \( | \cdot |_u : O_{\text{inf}} \rightarrow [0,1] \) defined by

\[
| \tilde{x} |_u = \lim_{\epsilon \rightarrow 0} \log_{e^{-1}} \frac{\epsilon}{| \tilde{x} |}, \quad \tilde{x} \neq 0
\]

and \( | \tilde{x} |_u = 0 \), for \( \tilde{x} = 0 \).

The proof follows in two steps (see, for instance, [6]). One first verifies that \( | \cdot |_u \) defined by (5) is a semi-norm viz: (i) \( | \tilde{x} |_u > 0, \tilde{x} \neq 0 \) (ii) \( | -\tilde{x} |_u = | \tilde{x} |_u \) (iii) \( | \tilde{x} + \tilde{y} |_u \leq \max \{ | \tilde{x} |_u, | \tilde{y} |_u \} \), known as the stronger triangle inequality. (For the property (ii), \( | \cdot |_u \) should be defined on an infinitesimal neighbourhood of \( \epsilon \) in \( [-1,1] \)).

**Remark 1:** As remarked already, the set of infinitesimals \( O_{\text{inf}} = \Phi \) when \( \epsilon \rightarrow 0 \). However, the corresponding asymptotic expression for the scale free (invariant) infinitesimals is nontrivial, in the sense that the associated valuations [5] can be shown to exist as finite real numbers.

Choosing \( \epsilon = \beta^r \), the Cantor set scale factor, the scale free infinitesimal gaps can be identified as \( \tilde{O}_{\text{inf}}^m = (0, \beta^m) \) when \( \epsilon \rightarrow 0 \) is realised as \( n \rightarrow \infty, r = n + m, \ m = 1, 2, \ldots \). Assign nonzero constant valuation \( | \tilde{x}|_m = \alpha_m \forall \tilde{x} = \tilde{x}/\epsilon \in \tilde{O}_{\text{inf}}^m \). The set of all possible scale free infinitesimals \( \cup \tilde{O}_{\text{inf}}^m \subset (0,1) \) is now realised as nested clopen circles \( S_m : \{ \tilde{x} : | \tilde{x}|_m = \alpha_m \} \). The ordinary \( 0 \) of \( C \) is replaced by this set of scale free infinitesimals \( 0 \rightarrow 0 = O_{\text{inf}} / \sim = \{ 0, \cup S_m \}, \ 0 \) being the equivalence class under the equivalence relation \( \sim \), where \( x \sim y \) means \( |x|_u = |y|_u \). The existence of \( \tilde{x} \) could also be conceived dynamically as a computational model [4, 5, 6], in which a number, for instance, \( 0 \) is identified as an interval \( [-\epsilon, \epsilon] \) at an accuracy level determined by \( \epsilon = \beta^m \).

**Remark 2:** 1. The concept of infinitesimals and the associated absolute value considered here become significant only in a limiting problem (or process), which is reflected in the explicit presence of “lim” in the relevant definitions. Recall that for a continuous real valued function \( f(x) \), the statement \( \lim_{x \rightarrow 0} f(x) = f(0) \), means that \( x \rightarrow 0 \) essentially is \( x = 0 \). This may be considered to be a passive evaluation (interpretation) of limit. The present approach is dynamic, in the sense that it offers not only a more refined
evaluation of the limit, but also provides a clue how one may induce new (nonlinear) structures (ingredients) in the limiting (asymptotic) process. The inversion rule (4) is one such nonlinear structure which may act nontrivially as one investigates more carefully the motion of a real variable \( x \) (and hence of the associated scale \( \epsilon < x \)) as it goes to 0 more and more accurately. Notice that at any “instant”, elements defined by inequalities \( 0 < \tilde{x} < \epsilon < x \) in the limiting process, are well defined; relative infinitesimals are meaningful only in that dynamic sense (classically, these are all zero, as \( x \) itself is zero). Scale invariant infinitesimals \( \tilde{X} \), however, may or may not be zero classically. \( \tilde{X} = \mu \neq 0 \), a constant, for instance, is nonzero even when \( x \) and \( \epsilon \) go to zero. On the other hand, \( \tilde{X} = \epsilon^\alpha, \ 0 < \alpha < 1 \), of course, vanish classically, but as shown below, are nontrivial in the present formalism. As a consequence, relative (and scale invariant) infinitesimals may be said to exist even as real numbers in this dynamic sense. The accompanying metric \( |.|_u \), however, is an ultrametric.

2. However, a genuine (nontrivial) scale free infinitesimal \( \tilde{X} \) can not be a constant. Let \( \tilde{x}_0 = \mu \epsilon, \ 0 < \mu < 1, \mu \) being a constant. Then \( |\tilde{x}_0|_u = \lim_{\epsilon \rightarrow 0} \log \mu = 0 \), so that \( \tilde{x}_0 \) is essentially the trivial infinitesimal 0 (more precisely, such a relative infinitesimal belongs to the equivalence class of 0).

3. The scale free infinitesimals of the form \( \tilde{X}_m \approx \epsilon^{\alpha_m} \) go to 0 at a slower rate compared to the linear motion of the scale \( \epsilon \). The associated nontrivial absolute value \( |\tilde{x}_m|_u = \alpha_m \) essentially quantifies this decelerated motion.

Next, one notes that \( O_{inf} \) is the union of countable family of disjoint clopen balls (this is, in fact, true even for the natural ultrametric). Recall that an (a) open (closed) ball is defined, as usual, using the seminorm \( | \cdot |_u \) viz:\  a set \( B_r(a) = \{ \tilde{x} : |\tilde{x} - a|_u < r \} \) is open while \( \bar{B}_r(a) = \{ \tilde{x} : |\tilde{x} - a|_u \leq r \} \) is a closed ball. We now assume that the mapping \( | \cdot |_u : O_{inf} \rightarrow [0,1] \) is constant on each of the component balls \( B(a_i) \subset O_{inf} \subseteq \cup B(a_i) \). Again for a given \( \epsilon \) the closure of \( O_{inf} \) is compact and so is covered by a finite number of clopen balls \( B(a_i), i = 0, 1, \cdots , n \). Consequently, \( | \cdot |_u \) is discretely (finitely) valued and so there exists a constant \( 0 < \sigma_{\epsilon} < 1 \) such that

\[
|\tilde{x}|_u = \sigma_{\epsilon} s, \ \tilde{x} \in B(a_i), \ i = 0, 1, \cdots , n
\]

for an ascending sequence of valuations \( 0 < s_0 < s_1 < \cdots < s_n \). For an ascending sequence \( \alpha_i > 0 \), the above can be written alternatively as

\[
|\tilde{x}|_u = \alpha_i s^\tilde{s}, \ \tilde{x} \in B(a_i), i = 0, 1, \cdots , n \quad (6)
\]

where \( \tilde{s} = s_n \). It follows from (5) that \( \tilde{x}_i \approx \epsilon^{\alpha_i} s^\tilde{s} + \mu_{\epsilon} \), where \( \mu_{\epsilon} \) goes to zero faster than \( \sigma_{\epsilon} \) as \( \epsilon \rightarrow 0^+ \).
Now, as in Remark 1, the limit $\epsilon \to 0^+$ on infinitesimal scales could be evaluated via a countable number of different sequences of the form $\epsilon^{m+n}$ as $n \to \infty$. For each such sequence, there exists a special value of $\sigma = \sigma_m$, all of which may be assumed to arrange in a descending order. As a consequence, $\sigma$ assumes values from a decreasing sequence of scales (of the form $\sigma_m$), called the secondary scales. The sequence $\epsilon^n$ defines a set of primary scales.

Next, the (infinitesimal) valuation $|\tilde{x}|_u = v(\tilde{x})$ (say), as a mapping from $O_{\text{inf}}$ to $I \subset \mathbb{R}$ is continuous. The equation (6), however, defines $v(.)$ only for points in the clopen balls $B(a_i)$, $i = 1, 2, \cdots$. The definition can be extended continuously over the entire set $O_{\text{inf}}$ for points outside the clopen balls. Indeed, let for a given primary scale $\epsilon$, $\sigma_i$ be the secondary scale. Let also that $y \in O_{\text{inf}} \cup B(a_i)$. Then there exist $y_i \in B(a_i)$, $y_{i+1} \in B(a_{i+1})$ such that $y_i < y < y_{i+1}$, and $v(y_{i+1}) - v(y_i) = (\alpha_{i+1} - \alpha_i)\sigma_i$. Clearly, the sequence $v(y_{i+1})$ is increasing and $v(y_i)$ is decreasing. Thus, $v(y) := \lim v(y_i)$, as $i \to \infty$. We have thus proved that the scale invariant valuation $v(\tilde{x})$ is indeed given by a Cantor function. Conversely, given a Cantor function $\phi(x)$, $x \in [0, 1]$, one can define a set of infinitesimals by the asymptotic formula $\tilde{x} \approx \epsilon\phi(\tilde{x}/\epsilon)$ as $\epsilon \to 0$ that is assumed to live in a nontrivial neighbourhood of 0.

With this class of valuations, the seminorm now extends to a non-archimedean absolute value, satisfying also the product rule (iv) $|\tilde{x} \tilde{y}|_u = |\tilde{x}|_u |\tilde{y}|_u$.

We have thus proved the

**Theorem 1.** The infinitesimal valuation (5) is a non-archimedean absolute value, and is given by a Cantor function associated with the Cantor set containing the relative infinitesimals. Conversely, given a Cantor function, there exists a class of infinitesimals that live in an extended ultrametric neighbourhood of 0.

Notice that an infinitesimal gap of the form $(0, \epsilon)$, $\epsilon \to 0^+$ is a segment of a line (i.e. an open interval) in the usual topology inherited from the real line. In the topology induced from the ultrametric $| \cdot |_u$, the interval $(0, \epsilon)$ instead is realized as a totally disconnected set and so itself is equivalent to a Cantor set $\tilde{C}$. The structure of $\tilde{C}$ can, in general, be very arbitrary and even be independent of the original Cantor set $C$. In ref.[4, 5], we have, however, shown how this new ultrametric valuation for relative infinitesimals can lead to a topology identical to that of the original Cantor set $C$. In the following we give a brief outline of the procedure and also comment on the origin of the inequivalence of this ultrametric.

### 3.1 Middle Third Cantor Set: An Example

Let $C$ be the standard middle $1/3$rd Cantor set. As will become clear our discussion will apply generically to any measure zero Cantor set considered in this paper. The Cantor
set $C$ offers us with a privileged set of scales $\epsilon_n = 3^{-n}$. For a sufficiently large $n$ viz : as $n \to \infty$, suppose an infinitesimal gap of the form $(0, \epsilon_{n+m})$ is decomposed into a finite number $M$ of open subintervals $\tilde{I}_i$ of relative infinitesimals with constant valuations defined by

$$| \tilde{x}_i |_u = i \cdot 2^{-n}, \quad i = 1, 2, \ldots, M, \quad \tilde{x}_i \in I_i.$$  \hspace{1cm} (7)

The valuation assigned by (7) is the triadic Cantor function $\phi : I \to I$ so that $M = 2^m - 1$ corresponding to the scale $\epsilon_m = 3^{-m}$. We call infinitesimals $\tilde{x}$ leaving in the island $\tilde{I}_i \subset (0, \epsilon_{n+m})$ valued infinitesimals having the valuation (7) induced by the Cantor function. Any element $x$ of the original Cantor set $C$ is now endowed with a set of valued neighbours

$$X^i_\pm = x \cdot x^{\mp |\tilde{x}_i|_u}.$$  \hspace{1cm} (8)

Finally, the element $x$ is assigned the ultrametric valuation

$$\| x \| = \inf \log_{\lambda^{-1}} \frac{X^i_+}{x} = \inf \log_{\lambda^{-1}} \frac{x}{X^i_-},$$  \hspace{1cm} (9)

so that $\| x \| = 2^{-n} = 3^{-ns}$ where $s = \frac{\log 2}{\log 3}$, the Hausdorff dimension of the triadic Cantor set $C$ and $n \to \infty$. As it turns out, this valuation exactly reproduces the nontrivial measure of [2] derived in the context of noncommutative geometry (c.f., definition of valued measure in Sec.3.2.)

Now, to make contact with the absolute value (5) and the inversion rule (4), let for a sufficiently large but fixed $n$, $\tilde{x}_i \in \tilde{I}_i$ has the form (we set for definiteness $m = 0$)

$$\tilde{x}_i = 3^{-n} \cdot 3^{-n \cdot 2^{r-i}} \times a_i$$  \hspace{1cm} (10)

where $ni = 2^r \cdot k_i$, $k_i$ being, in general, a sufficiently large real number and $a_i = \sum a_{ij} 3^{-j} \in O_i$, a gap of size $3^{-r}$ of the Cantor set $C$ and $a_{ij} \in \{0, 1, 2\}$. Then $0 < \tilde{x}_i < 3^{-n}$ and $|\tilde{x}_i| = i \cdot 2^{-r}$. One also verifies that for an $x_i = 1 + a_{0i}$, $|x_i|_u = \lim_{n \to \infty} \log_{3^n} (a_i/3^{-n}) = 1$. By the inversion rule (4) the elements $\tilde{x}_i$ of the ball $\tilde{I}_i$ now connect to an $x_i \in C$ given by

$$x_i = 3^{-n} \cdot 3^{-n \cdot 2^{r-i}} \times b_i, \quad b_i = \sum b_{ij} 3^{-j}, \quad b_{ij} \in \{0, 2\}$$  \hspace{1cm} (11)

where $\lambda = a_i \times b_i \in (0, 1)$. (Infinitesimal) Scales $\epsilon_n = 3^{-n}$, are the primary scales when the scales $3^{-k_i}$ (or equivalently $2^{-r}$) are the secondary scales.

Finally, to verify the new multiplicative representation one notes that there exists $y_i \in C$ in a neighbourhood of $x_i$ so that

$$y_i = 3^{-n} c_i, \quad c_i = \sum c_{ij} 3^{-j}, \quad c_{ij} \in \{0, 2\}$$
and

\[ x_i = y_i \cdot y_i^{i.2^{-r}} \]  

(12)

so as to satisfy the identity

\[ c_i^{n-k_i} = b_i^n. \]  

(13)

To verify that (13) is not empty we note that for the end points \( \frac{1}{3} \) and \( \frac{2}{3} \), both belonging to \( C \), (13) means \( \left( \frac{2}{3} \right)^n = \left( \frac{1}{3} \right)^{n-k_1} \) yielding \( k_1 = ns, s = \frac{\log 2}{\log 3} \). For this value of \( k_1 \), (13) now tells that \( c_i^{1-s} = b_i \), so that \( c_i = \left( \frac{2}{3} \right)^r \) and \( b_i = \left( \frac{2}{3} \right)^r \) for a suitable \( r \). Similar estimates for \( k_i \) are available for other (consecutive) end points of (higher order) gaps. It thus follows that the representation (8) is realised at the level of the finite Hausdorff measure of the set, when the value of the constant \( k \) is real (rather than a natural number).

Accordingly, it follows that a gap \( O \) in \( I/C \) (which is a connected interval in the usual topology) containing a point \( x \) of the Cantor set \( C \) is indeed realised as an “infinitesimal” Cantor set in the valuation defined by the Cantor function associated with the original Cantor set \( C \) itself. One thus concludes that

**Proposition 2.** Any element \( x \) of an ultrametric Cantor set \( C \) has the multiplicative representation (8) and the non-archimedean absolute value \( ||x|| = \inf_i v(\tilde{x}_i) \).

### 3.2 Valued Measure

Next it is easy to verify that the valued (metric) measure defined by the ultrametric induced by the valued norm (9) naturally yields the finite Hausdorff measure of \( C \).

To recall, the valued measure \( \mu_v : C \to \mathbb{R}_+ \) is defined by

(a) \( \mu_v(\Phi) = 0 \), \( \Phi \) the null set

(b) \( \mu_v[(0, x)] = ||x|| \), when \( x \in C \),

(c) For any \( E \subset C \), we have

\[ \mu_v(E) = \lim_{\delta \to 0} \inf \Sigma \{ d_u(I_i) \} \]  

(14)

where \( I_i \in \tilde{I}_\delta \) and the infimum is over all countable \( \delta \)-covers of \( E \) by clopen balls \( I_i \) and \( d_u(I_i) = \) the non-archimedean “diameter” of \( I_i = \sup \{ ||x-y|| : x, y \in I_i \} \).

It follows that \( \mu_v \) is a metric (Lebesgue) outer measure on \( C \) realised as an ultrametric space. Now for an infinitesimal (primary) scale \( \epsilon_n = \beta^n \), we can choose a sufficiently small secondary scale \( \tilde{\epsilon}_m = 2^{-m} \), so that for any two \( x \) and \( y \in C \) with \( || x - y || = d \), we have \( d_u(x, y) = || x - y || = \tilde{\epsilon}_m^s \leq d^s \leq \{d(x, y)\}^s \), where \( s = \frac{\log 2}{\log 1/\beta} \). Accordingly, it follows that \( d_u(I_i) \leq \{d(I_i)\}^s \). Using this fact and also using the monotonicity of measures one can
now deduce that \( \mu_v(E) = H^s(E) \). Choosing the full set \( C \) for \( E \) and noting that \( s \) is the Hausdorff dimension it thus follows that \( \mu_v(C) = 1 \). We remark that the metric properties of the present ultrametric are indeed distinct from the natural ultrametric (c.f., [4, 5]), since the Lebesgue measure of \( C \) in the natural ultrametric is zero, but in the present case, the corresponding valued measure equals the Hausdorff measure. More importantly, topologies induced by the two ultrametrics are also different, as seen in the following example.

**Example 3:** The sequence \( \epsilon_n = \epsilon^{n-nl}, 0 < \epsilon < 1, 0 < l < 1 \) converges to 0 in the usual metric (ultrametric), but converges to \( l \) in the present ultrametric. For a sufficiently large \( n \), choose \( \epsilon^n \) as scale factor and then relative infinitesimals are \( \tilde{\epsilon}_n = \lambda^{-1} \epsilon^{n+nl}, 0 \ll \lambda < 1 \). Then, letting the secondary scale \( \epsilon \to 0 \), we have \( |\tilde{\epsilon}_n|_u = \lim \log_{\epsilon-n} (\epsilon^n/\tilde{\epsilon}_n) = l \) and hence \( ||\epsilon_n|| = l \), by eq(9), for a sufficiently large \( n \). Thus, \( \{\epsilon_n\} \to l \) in the ultrametric \(||.||\).

Letting \( \epsilon = \tilde{\epsilon}^m \), the sequence \( \epsilon^{n-nl} \) is replaced by \( \epsilon^{N-Nl}, N = nm \), so that the limit \( \epsilon \to 0 \) of the secondary scale is well defined, since it is realised as \( m \to \infty \). Note, however, that the sequence \( \{\epsilon^n\} \) converges to 0, even in \(||.||\). For a sufficiently large but fixed \( n \), we choose \( \epsilon^{n+1} \) as the scale factor, so that \( \tilde{\epsilon}_n = \lambda^{-1} \epsilon^{n+2} \), are relative infinitesimals and \( |\tilde{\epsilon}_n|_u = \frac{1}{n+1} \). More generally, for scales \( \epsilon^{n+r}, r \) being a nonnegative real, we have \( |\tilde{\epsilon}_n|_u = \frac{r}{n+r} \). Thus, \( ||\epsilon^n|| = \inf_{r \in \epsilon^{n+r}} = 0 \). Further details of the convergence of sequences and series (in a scale invariant real analysis) will be considered elsewhere.

This example also gives an alternative proof that the metric \(||.||\) is really an ultrametric.

### 4 Differential Equation

We now study the relationship of the scale free DE of the form

\[
x \frac{dX}{dx} = X
\]

with a Cantor set \( C \). As a preparation, let us recall how the simplest Cauchy problem

\[
\frac{dX}{dx} = 1, \ X (1) = 1
\]

is solved on the interval \( I = [0, 1] \). One considers a partition \( 0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1 \). The desired result \( X (x) = x \) is obtained as a limit of a sum:

\[
\lim_{\Delta x_j \to 0} \sum_{j=1}^{i} \Delta x_j \text{ where } x_{i-1} < x < x_i. \text{ The scale free Cauchy problem}
\]

\[
x \frac{dX}{dx} = X, \ X (1) = 1
\]

is also solved exactly in an analogous fashion.
Let us now solve (17) in a slightly unconventional “multiplicative” approach [14, 15].

First we note that the neighbourhood of a point \( x_0 \) is mapped to that of \( x = 1 \) by a rescaling \( x \to \frac{x}{x_0} \). So we concentrate only in the neighbour of \( x = 1 \). Let \( x_{\pm} = 1 \pm \eta \), \( X_{\pm} = X(x_{\pm}) \). Then the DE in (17) splits into two branches

\[
x_{\pm} \frac{dX_{\pm}}{dx_{\pm}} = X_{\pm}.
\]

Let us solve the left hand branch \( X_- \). The standard solution is \( X_- = 1 - \eta \). We now write

\[
X_- = \frac{1}{1 + \eta} X_{1-}
\]

so that the correction factor \( X_{1-} \) solves the renormalised (self-similar) equation

\[
x_{1-} \frac{dX_{1-}}{dx_{1-}} = X_{1-}
\]

where \( x_{1-} = 1 - \eta^2 \). Iterating ad infinitum, the desired solution is retrieved (in a non-trivial form) thus

\[
X_- = \prod_{i=0}^{\infty} \frac{1}{1 + \eta^{2^i}}.
\]

The right hand branch, however, has the form

\[
X_+ = \frac{1}{1 - \eta} \prod_{i=1}^{\infty} \frac{1}{1 + \eta^{2^i}}.
\]

The infinite product representation of \( X_- \), for instance, is interpreted as follows. The first iterated value exceeds the exact value by an amount \( \frac{\eta^2}{1 + \eta} \) which is canceled progressively in a self-similar manner over smaller and smaller inverted scales \( \log \left( 1 - \eta^{2^i} \right)^{-1} \), \( i = 1, 2, 3, \ldots \).

We note that the higher order correction factors \( X_c = \prod_{i=1}^{\infty} \frac{1}{1 + \eta^{2^i}} \) may therefore be reinterpreted as a deletion process: viz; a portion of a line segment is deleted progressively and self-similarly, analogous to the formation of a Cantor set. Alternately, a product of the form \( (1 - \eta) (1 + \eta) = 1 - \eta^2 \) could be considered to represent a deletion: a length of size \( \eta \) in the neighbourhood of \( 1^- \), is deleted progressively as \((1 - \eta) (1 + \eta) (1 + \eta^2) \cdots (1 - \eta^{2^{n-1}}) = (1 - \eta^{2^n})\), \( n \to \infty \).

A second possibility is to interpret the multiplicative iteration process defined above as a dynamical process in which the dynamic (independent) variable undergoes increments not by the usual linear translations but by inversions (hoppings) over smaller and smaller sizes. This then provides one with a mechanism of deletion process stated above. Notice
that the renormalised corrections $X_i$ satisfy the self-similar scale free equations of the
form (20) over smaller (logarithmic) scales (hopping sizes) \( \log \left( 1 - \eta^i \right)^{-1} \). Actually, these
logarithmic scales inhabits concomitant smaller scales of the from \( \eta^j \), that ordinarily arise
in the context of an IFS. To justify this in a greater detail, let us assume that the scale
free problem (17) is now defined on a closed subset \( \tilde{C} \subseteq I \), called an inverted Cantor set,
where \( \tilde{C} = \bigcup \tilde{I}_i \) is a countable union of disjoint closed intervals \( \tilde{I}_i \) of varying sizes. \( \tilde{I}_i \) in
fact, is the closure of a corresponding gap \( O_i \) (inclusive of the end points ) of the original
Cantor set \( C \). Suppose, a dynamic variable (say, a particle) in motion on this set \( \tilde{C} \), hops
between the end points of each of such disjoint closed intervals \( \tilde{I}_i \), following the scale free
DE (17). Let \( |\tilde{I}_0| = \eta \) be the maximum hopping size. Then the smaller hopping sizes are \( \eta \) proportion of the remaining sizes of the set \( \tilde{C} \) viz., \( |\tilde{I}_i| = \eta (1 - \eta)^{i-1} \equiv \eta_i \) (say), when
we assume \( |\tilde{C}| = 1 \). Because of the rescaling symmetry (scale invariance) of the DE (17)
each of the component intervals \( \tilde{I}_i \) could be imagined to have been symmetrically placed
at 1 with end points at \( x_{i \pm} = \frac{1}{2} (1 \pm \eta_i) \). Now, the particle at left end point \( x_{0-} \) of \( \tilde{I}_0 \)
say) hops to the right end point \( x_{0+} \) following the rule (c.f., (19))
\[
x_{0-} \rightarrow x_{0-}^{-1} = x_{0+} X_1
\]
so that we have, using (22)
\[
(1 - \eta)^{-1} = (1 + \eta) X_c, \quad X_c = \prod_{i=1}^{\infty} \left( 1 + \eta^2 \right)
\]
because of the scale invariance. Eqn(23) tells that hopping motion of the type considered
above, of any given size \( \eta \) is accomplished by an infinite cascade of self similar smaller
scale inverted motions of sizes \( \eta^2, i = 1, 2, \cdots \). The total length covered by all these self-
similar jumps , viz., 1, is reached multiplicatively i.e. as \( 1 = \lim_{n \to \infty} \left( 1 - \eta^n \right) \), reminiscent
of an ultrametric limiting process. Notice that in the ordinary sense, the total jump size
is determined additively as an infinite series viz., \( \sum_{i=1}^{\infty} \eta (1 - \eta)^{i-1} = \sum \eta_i = 1 \).

The above multiplicative model of the Cauchy problem (17) should have a natural
relevance in the context of a Cantor set (or, equivalently to an ultrametric space). Let
us now assume that the problem (17) is defined on a Cantor set \( C \), i.e. \( x \in C \). This
Cantor set is now realised as an inequivalent ultrametric space. Accordingly, each \( x \in C \) is
replaced by an infinitesimal copy of the inverted Cantor set \( \tilde{C} \). Because of scale invariance,
the DE (17) at an \( x \in C \), which is actually not defined in the usual (even in the natural
ultrametric) sense, is now raised to an equation which is well defined on a closed set of
the form \( \tilde{C} \) and treated as a multiplicative model.

We note that the solution (21) and (22) is the standard solution derived in an uncon-
ventional way and interpreted non-trivially. On a Cantor set, however, the equation (17)
can accommodate a host of new solutions in consonance with the multiplicative model
interpretation. The origin of these new solutions could be explained in the context of locally constant functions (LCF). To justify, in a most natural way, the existence of locally constant functions, let us write a solution of (17) in the form

\[ X = x \cdot x^{\tau(x^{-1})}. \]  

The function \( \tau(x) \) here represents a LCF and is defined by the scale free equation on logarithmic variables, viz:

\[ \log x^{-1} \frac{d\tau}{d\log x^{-1}} = \tau. \]  

(26)

Clearly \( \tau(x) \) corresponds to our non-trivial valuation \( v(\tilde{x}) = |\tilde{x}|_u \). To verify \( v(\tilde{x}) \), indeed is a LCF, we note that

\[ \frac{d}{dx} v(\tilde{x}) = \lim_{\epsilon \to 0} \frac{d}{dx} \left( \frac{\log x}{\log \epsilon} + 1 \right) = 0. \]  

(27)

Equation (26), on the other hand, reveals the variability of a LCF over smaller logarithmic scales. Of course, the valuation also passes this test

\[ \log v(\tilde{x}) = \log \log \tilde{x} + \log \log \epsilon \lambda - \log \log \epsilon^{-1} \]

leading to equation (26) in the inverted rescaled real variable \( \tilde{x} \) (in the log log scale). We have already seen that \( v(\tilde{x}) \) relates to an appropriate Cantor function. Consequently, a Cantor function \( \phi(x) \) is shown to be a LCF with variability over log log scales. Equation (25) constitutes an ultrametric extension not only of a Cantor set, but of any connected interval of \( R \). This is already verified explicitly in the \( \frac{1}{3} \) rd Cantor set, in Sec.3.1.

The main results derived in this section are summarised in

**Theorem 3.** An element of an ultrametric Cantor set \( C \) is replaced by the set of gaps of the Cantor set \( \tilde{C} \) where relative infinitesimals are supposed to live in. Increments on such an extended Cantor set \( C \) is accomplished by following an inversion rule of the form (4). A scale free differential equation of the form eq(17) is well defined on such an ultrametric space and accommodates Cantor functions as locally constant functions. The associated infinitesimal valuation \( v(\tilde{x}) \) is a locally constant function with variability over double logarithmic scales.

5 Growth of Measure

In Sec 3 we studied the valued ultrametric structure of a measure zero Cantor set. Here we study a few more general properties of the valued ultrametricity. We note, at first,
that the valued structures of \( \tilde{x} \) and \( x \) of the middle third Cantor set \( C_{1/3} \), viz., equation \((10)\) and \((11)\) actually correspond to the solution \((24)\) when \( a_i \), \( b_i \) and \( \lambda \) are identified with \((1 - \eta), (1 + \eta)\) and \( \tilde{X} \) respectively. The function \( \tilde{X} \) is a LCF. The valuation \( v(\tilde{x}) \) is identified with the LC Cantor function corresponding to \( C_{1/3} \). However, as verified in equation \((27)\), \( v(\tilde{x}) \) is indeed a LCF satisfying
\[
\frac{d}{dx} v(\tilde{x}(x)) = 0.
\]
Consequently, \( v \) is, not only a LCF, but more importantly is a reparametrisation invariant object. As a result, \( v \) does not require to be an explicit function of the original variable \( x \) but may be a function instead of any monotonic, continuously first differentiable function of \( x \). By the same token, \( v \) does not depend explicitly on the scale \( \epsilon \) inherited from the original (mother) Cantor set, as we did in the example of \( C_{1/3} \). In the following example, we show that relative infinitesimals may instead live in a positive measure Cantor set. Notice that in the general representation of the valued ultrametric in \((5)\), the parameter may be a constant independent of an explicit \( \epsilon \).

**Example 4:** Suppose that eq\((10)\) and \((11)\) are replaced by
\[
\tilde{x} = \beta^n \left( \beta^n \right)^{n\beta_n(1 + \gamma_m)} \times a
\]
where \( a = (1 - \beta) \left( 1 + \sum_1^\infty a_i \beta^i \right) \), \( a_i \in \{0, 1\} \), \( \beta = 1/2(1 - \alpha) \), and \( \beta_n \) and \( \gamma_n \) are two non-increasing sequence of positive numbers such that \( \beta_n \to 0 \) as \( n \to \infty \) and \( m \) may be independent of \( n \) or may vary with \( n \) more slowly, and \( \delta > 1 \) is a constant.

Although \( \beta_n \to 0 \) as \( n \to \infty \), the valuation \( v(\tilde{x}) \) could be non-trivial, since
\[
v(\tilde{x}) = \lim_{n \to \infty} \log_{\beta^{-n}} \frac{\beta^n}{\tilde{x}}
= \lim_{n \to \infty} \left[ n^\delta \beta_n (1 + \gamma_m) + \log a \right]_{\beta^{-n}}
= l + \tilde{\gamma}_{m_n}(\delta)
\]
when we assume \( n^\delta \beta_n \to l \) as \( n \to \infty \) and \( n^\delta \beta_n \gamma_m \to \tilde{\gamma}_{m_n}(\delta) \) is a sub-dominant slowly varying non-increasing sequence, for a real \( m_n > 0 \). The representation \((28)\) tells that a scale free infinitesimal \( \frac{\tilde{x}}{\beta^m} \) may live in a Cantor set \( \tilde{C}_p \) of Example 1 (Sec 2), so that \( m(\tilde{C}_p) = l \). Let the original Cantor set be a middle \( \alpha \) set \( C_\alpha \) with the uniform scale factor \( \beta = 1/2(1 - \alpha) \). For the positive measure set \( \tilde{C}_p \) the scale factor at the \( n \)th iteration is \( \tilde{\beta}_n = 2^{-n} \sum_{i=1}^{\infty} (1 - \alpha_i) \) and \( l = m \left( \tilde{C}_p \right) = \prod_{i=1}^{\infty} (1 - \alpha_i) = \lim_{n \to \infty} 2^n \tilde{\beta}_n \). Let us choose \( \delta > 1 \) such that \( \tilde{\beta}_n = \beta^{n^\delta} \). Then \( n^\delta \beta_n \to l \) tells that \( \beta_n \approx \frac{\log \beta}{\log \beta_n} \) as \( n \to \infty \). Thus the dominant term \( l \) of the valuation \( v(\tilde{x}) \) is a constant while the subdominant asymptotic
\( \tilde{\gamma}_m(\delta) \) could be a genuine LCF (i.e. a Cantor function for a sub dominant Cantor like set \( C_s \) (say)), precise determination of which depends on the explicit model of the Cantor set \( \tilde{C}_p \). It follows, therefore, from (38) and (39) the ultrametric valuation of \( x \in C_\alpha \) now has the form

\[
\| x \| = l + \tilde{\gamma}_m(\delta).
\]

(30)

For larger and larger values of \( n \) (\( \to \infty \)), we can disregard the sub-dominant term (since \( \tilde{\gamma}_m \to 0 \) as \( m \to \infty \)) so that

\[
\| x \| = l \quad \forall x \in C_\alpha, \ x \neq 0.
\]

(31)

Clearly the trivial ultrametric (31) reveals that the mother set \( C_\alpha \) must get deformed to a positive measure set \( C_p \) so that \( \mu_v(C_p) = m(C_p) = l \), when the reparametrisation invariance of LC correction factors is invoked. Indeed, we have \( \| x - y \| = l \) for any two \( x, y \in C_p \). Thus, any single clopen ball \( B(x_0), x_0 \in C_p \) (say) covers the compact \( C_p \) and hence \( \mu_v(C_p) = d_u(B(x_0)) = l \).

To summarise, we have shown that any element \( x \in C_\alpha \) when deformed by the non-trivial, reparametrisation invariant valuation of relative infinitesimals, is identified with an element of a 1-set \( C_p \). Because of this invariance, the relative infinitesimals may be assumed to live in a positive measure set \( C_p \), which, in turn, determines the measure (size) of the deformed set \( C_p \). Since each element \( x \in C \subset [0, 1] \) is written as the arithmetic sum of two elements \( x_0 \in C_\alpha \) and \( x_1 \in C_{\alpha'} \) (\( C_{\alpha'} \) being the Cantor set of infinitesimal neighbours of \( x_0 \)), it follows from a theorem of Solomyak [16] that for \( \beta = \frac{1}{2} (1 - \alpha) \in (0, \frac{1}{2}) \), there exists \( C_{\alpha'} \) for a.e. \( \beta' = \frac{1}{2} (1 - \alpha') \in (0, \frac{1}{2}) \) so that \( C_{\alpha} + C_{\alpha'} \), has positive measure and \( \frac{1}{\log \frac{1}{2}} + \frac{1}{\log \frac{1}{2}} \geq \frac{1}{\log 2} \). This, therefore, constitutes an alternative proof for the said assertion. Indeed, in the above construction, the set of infinitesimals \( C_{\alpha'} \) itself is a 1-set \( \tilde{C}_p \).

It follows, accordingly, that a slower rate of removal of middle open sets compared to a measure zero Cantor set hides a positive measure in an infinitesimal scaling factor which is exposed under the present scale invariant valuation. The uniform rate of deletion in the case of a measure zero set is violated because of the underlying reparametrisation invariance. Further, in a dynamical process leading to a Cantor set, a positive measure Cantor set \( C_p \) is favoured a.s (almost surely) compared to a measure zero set \( C_\alpha \) since relative infinitesimal neighbours a.s. lie in a Cantor set \( C_{\alpha'} \) satisfying the above constraints.

The generic result that follows from this example is stated thus

**Theorem 4.** Because of the reparametrisation invariance of the infinitesimal valuation, a measure zero Cantor set \( C_\alpha \) is a.s. deformed to a positive measure Cantor set \( C_p \), the measure of which is determined by the Cantor set \( \tilde{C}_p \) in which the relative infinitesimals are supposed to live.
Next to expose the significance of the sub-dominant term, let us first define a renormalized valuation $v_R(\tilde{x})$:

$$v_R(\tilde{x}) = \log \beta_n \log \beta_n \left[ \frac{\tilde{x}}{\left(\beta_n^{1+v_0(\tilde{x})}\right)} \right], \quad n \to \infty \quad (32)$$

where $v_0(\tilde{x}) = l < 1$ is the dominant valuation of the infinitesimal $\tilde{x}$. The LCF $\tilde{\gamma}_m(\delta)$ is now given by (c.f.(6))

$$\tilde{\gamma}_m(\delta) = \alpha_i \beta^m \rho(\delta) \quad (33)$$

where the $\delta$–dependent constant $\rho$ is called a renormalised valued exponent and the non-zero constant $\alpha_i$ assumes values from a finite set for a secondary scale $\beta^m$. As will become clear the valued exponent $\rho$ is useful to distinguish two sets with identical Hausdorff dimensions.

5.1 Applications

1. Middle third Cantor sets: As an application of the renormalised valued exponent, let us first consider a class of $s$-sets where $s = \log_3 2$, constructed as a slight variation of the process of Example 2, Sec.2. Let $I = [0,1]$. Also let $0 < \delta_n = 3^{-(n+1)\alpha_n} \lesssim 1$, $n = 1,2,\cdots$ (so that $\delta_n^{-1} \gtrsim 1$), be a non-increasing sequence. For definiteness, one may choose $\alpha_n = q^{-n}$, for a sufficiently large positive integer $n$ and $q > 1$. In that case $\alpha_n$ may be considered to belong to the range set of an appropriate Cantor function. Delete the middle open interval of length $1/3$. Next, delete a length $3^{-2(1+\alpha_1)}$ from each of the two closed subintervals. Then, delete the length $3^{-(3(1-\alpha_2))}$ from each of $2^2$ closed subintervals. Call these two operations together $O_1$. $O_n$ consists of two steps: deletion of $2^{n+1}$ open intervals of length $3^{-(n+1)(1+\alpha_n)}$, which is succeeded by the next deletion of lengths $3^{-(n+2)(1-\alpha_{n+1})}$ from $2^{n+2}$ remaining closed subintervals. Notice that we are considering a set of fluctuating scale factors, i.e., in the $(n+1)$th step open intervals of slightly smaller sizes compared to the middle third set are removed. In the next step, however, open intervals of slightly bigger sizes are removed. As a consequence, we get a family of limit sets which are indistinguishable and equivalent to the middle third Cantor set at the level of the Hausdorff dimension, but nevertheless, distinguishable at the level of renormalised valued exponents. Indeed, the total length of deleted open intervals viz.,

$$\frac{1}{3} + \frac{2}{3^2} \delta_1 + \frac{2^2}{3^3} \delta_2^{-1} + \cdots = 1 + \sum u_n$$

equals 1, when the series of real numbers $\sum u_n$ vanishes. The sequence $u_n$ is determined by the sequence $\alpha_n$, i.e., $\alpha_n = \log_3(\alpha_{n+1} + 1 - \frac{2^{n+1}}{2^{n+1}} \bar{u}_n)^{-1}$ and $\alpha_{n+1} = \log_3(\alpha_{n+2} + 1 + \frac{2^{n+2}}{2^{n+2}} \bar{u}_{n+1})$ so that $u_n = -\bar{u}_n$, $u_{n+1} = \bar{u}_{n+1}$. Clearly, such a series exists. Hence, all such sets are of measure zero.

Now, to determine the Hausdorff dimension, we first note that the scaling of closed intervals (bridges) follows the recurrence $2l_n = l_{n-1} - \delta_n^{\pm 1} 3^{-(n+1)}$, where + sign goes with
an odd $n$ and the - sign with $n$ even and $l_n$ denotes the length of each closed interval at level $n$. Accordingly, $l_n = \frac{1}{3^{2n}}[1 - \frac{\delta_1}{3} - \frac{2^m-1}{3^2} - \cdots - \frac{2^{n-1} \delta_1}{3^{n+1}}] \approx \frac{\delta_1}{3^{n+2}}$, for a sufficiently large $n$. As a consequence, the scale factors behave as either $\beta_{n+1} = 3^{-(n+1)(1+\alpha_n)}$ or $\beta_{n+2} = 3^{-(n+2)(1-\alpha_n+1)}$ respectively, and hence, the lower and upper box dimensions and the Hausdorff dimension are all equal and equal to $\lim_{n \to \infty} \frac{\log 2}{\log 3^{(1+\alpha_n)}} = \log_3 2$.

One may also estimate the thickness of these sets easily. Because of the above scaling, the limiting length of the closed intervals (bridges) coincides with that of the corresponding gap (viz., $\delta_{n+1} 3^{-(n+2)}$) at the $n$th level. It follows therefore that the ratio of sizes of bridges and gaps (c.f., Sec.2.1) has the limiting value 1. Hence, thickness of all these sets coincides with that of the classical middle third Cantor set as well.

However, a higher order (renormalised) valuated exponent can indeed reveal the local dissimilarities of such an $s$-set. Extending the representations (11) and (23) a little further to suit the present problem, we would now have for an element $x$ of the $s$-set,

$$x_{i \pm} = 3^{-n} \cdot 3^{-n(i-2m+1)} \times b, \ ||b|| = 1$$  \hspace{1cm} (34)

where $i$ assumes values from a finite set and $m_n \to \infty$ at a slower rate as $n \to \infty$, so that a renormalised valuation is defined as

$$v_R(x) = \inf_{i} \log_{3^{2m_n}} \log_{3^{-n}} (x_{i \pm}/x_0) = \alpha_{m_n}, \ x_0 = 3^{-n} \cdot 3^{-n(i-2m_n)}.$$  \hspace{1cm} (35)

It now follows from the definition of $\alpha_{m_n}$, that one can find a sufficiently large natural number $q >> 1$ such that $\alpha_{m_n} = q^{-m_n}$. Consequently, we obtain $v_R(x) = \alpha_{m_n} = 3^{-\rho m_n}$, where $\rho = \log_3 q > 0$ is the valuated exponent, for suitable positive integers $r$ and $m_n$.

Now, to justify the existence of such a $q$, let us first assume $\tilde{u}_{2m} = u_m^1$ and $\tilde{u}_{2m+1} = u_m^2$ such that $\sum u_m^i = l$. Then $\sum_{2}^{\infty} u_n = \sum \tilde{u}_{2m+1} - \sum \tilde{u}_{2m} = l - l = 0$. Consequently,

$$\alpha_{2m} = \log_{3^{(2m+1)}}(1 - \frac{3^{(2m+1)}}{2^{m+1}} u_m^2) \quad \alpha_{2m+1} = \log_{3^{(2m+2)}}(1 + \frac{3^{(2m+2)}}{2^{m+1}} u_m^2).$$

Let $\eta_{2m} = \frac{3^{(2m+1)}}{2^{m+1}} u_m^1$ and $\tilde{\eta}_{2m+1} = \frac{3^{(2m+2)}}{2^{m+1}} u_m^2$. Then the functions $(1 - \eta_{2m})^{-1}$ and $1 + \tilde{\eta}_{2m+1}$ are identified as LCF of the form eq(23) and eq(24) (Sec.4), in the neighbourhood of 1. Using scale invariance, we can choose for $x$ in eq(23) as $x = 3^{-n}(1 - \eta_n)$ (or $x = 3^{-n}(1 + \tilde{\eta}_n)$), and the scale factor $\epsilon = 3^{-n}$. Thus, there exists a Cantor function $\tau(\tilde{x})$, $\tilde{x} = x/\epsilon$ such that $\log_{\epsilon^{-1}} \tilde{x}^{-1} = \tau(\tilde{x})$ (or $\log_{\epsilon^{-1}} \tilde{x} = \tau(\tilde{x})$). As a result, there exists positive integers $q$ and $m_n$ so that the sequence $\{q^{-m_n}\} \subset \text{Range}(\tau(\tilde{x}))$. More generally, because of the local constancy, the limiting form $\alpha_n$ could be $\alpha_n = \tilde{l} + q^{-m_n}$, where $\tilde{l}$ is a non-negative constant, $0 \leq \tilde{l} < 1$.

We remark that the exponent $\rho$ may be considered to be the inverse of the Hausdorff dimension of a residual Cantor set that would remain attached with infinitesimal scales in a neighbourhood of a point (of the original Cantor set). For the classical middle third
Cantor set $\alpha_n = 0 \forall n$ and so $\rho = \infty$, which is consistent with the fact that the residual set is null. Since, sets with infinite Hausdorff dimension $s = \infty$ are excluded, by definition, $\rho$ indeed is positive $\rho > 0$.

2. 1-sets: Irregular 1-sets \[1\] are positive measure Cantor sets and are generally classified on the basis of fatness and/or uncertainty exponents. The LC renormalised valuation (32) and (33) now tells that $v_R(\tilde{x})$ is a Cantor function corresponding to a subdominant residual Cantor set $C_s$, and so has the form $v_R(\tilde{x}) = \alpha_i \beta^m n^p$. As for the $s$-sets, the valued exponent $\rho > 0$ equals the inverse of the Hausdorff dimension of the residual set $C_s$. For $\rho = \infty$, the double exponential factor in (28) drops out (i.e., reduces to the trivial factor $\beta^n$), and hence the 1-set is a regular set \[1\] having connected components (actually corresponds to a nonfractal set). Consequently, $0 < \rho \leq \infty$.

Now, to compare with the fatness exponent \[9, 10\], we first recall the relationship between the uncertainty exponent $\alpha$, $0 < \alpha \leq 1$ \[11\] and the fatness exponent $\tilde{\beta}$, $0 < \tilde{\beta} \leq \infty$. It is shown \[10\] that $\tilde{\beta} = \alpha$ in $[0,1]$, so that there is essentially the fatness exponent that has to be considered. We claim that $\rho = \tilde{\beta}$. The parameter $\tilde{\beta}$ is defined as

$$\tilde{\beta} = \lim_{\epsilon \to 0} \frac{\log[\mu(\epsilon) - \mu(0)]}{\log \epsilon} \quad (36)$$

where $\mu(\epsilon)$ is a LC measure which tells the scaling of smaller gap sizes when the smaller gaps are coarse grained by fattening by the amount $\epsilon$ and $\mu(0)$ equals the positive (Lebesgue) measure of the set. In our multiplicative representation (c.f. (28) and (33)), the fattening size is $\epsilon = \beta^n$ and

$$x = \beta^n (\beta^n)^{-l + k \beta^n} \times b \quad (37)$$

where $k$ is a constant independent of $\beta$, so that the exponent $\rho$ is defined by (36) when we identify $\mu(\beta^n) = \log_{\beta^n}(x/\beta^n)$. Notice that $\mu(0) = \lim_{n \to \infty} \log_{\beta^n}(x/\beta^n) = l$. Notice also that the measure $\mu$ here is nothing but the valuation of relative infinitesimals at the fattened scale $\epsilon$, which equals the full measure of the Cantor set $\tilde{C}_p$ at the scale $\epsilon$ (c.f. Example 4) where the infinitesimals live. Because of the reparametrisation invariance, we may suppose that $\tilde{C}_p$ is determined by the original 1-set and vice versa. At the scale $\epsilon$, the gaps of $\tilde{C}_p$ are fattened by the amount $\epsilon$, and in the presence of a positive measure, the said valuation is determined by the sum of the fattened gap sizes. For a zero measure set, this valuation, on the other hand, is determined instead by the finite Hausdorff measure, upto a finer (double logarithmic) scale correction that arises from the possible presence of local fine structures (c.f., above application). This observation proves the claim.
6 Final Remarks

The general representation of the fine structure induced by relative infinitesimals to any point \(x\) of a Cantor set \(C\), with a nonnegative measure \(m(C) \geq 0\), that has emerged from the above analysis, has the form

\[
X_\pm(x) = x \left( x^{\pm v(\tilde{x})(1-H^s(\tilde{C}))} \cdot x^{\pm x^{\pm v_R(\tilde{x})(1-H^{s_1}(\tilde{C}_r))}} \cdots \right) \times a. \tag{38}
\]

Here, relative infinitesimals \(\tilde{x}\) are supposed to live in a Cantor set \(\tilde{C}\). The actual nature of \(\tilde{C}\) is determined by \(C\) and vice versa. The first (logarithmic) order valuation \(v(\tilde{x}) = m(\tilde{C})\) when \(m(\tilde{C}) > 0\) (and \(H^s(\tilde{C}) = 1\) for \(s = 1\), thus eliminating the first order corrected factor). When \(m(\tilde{C}) = 0\), \(v(\tilde{x}) > 0\) and \(H^s(\tilde{C})\) is the finite \(s\) dimensional Hausdorff measure of the (Lebesgue) measure zero set \(\tilde{C}\). The higher order (double logarithmic) valuation has two interpretations: for a positive Lebesgue measure set, \(v_R\) relates to the fatness exponent, which is interpreted as the inverse of the Hausdorff dimension of a residual Cantor set \(\tilde{C}_r\). On the other hand, when the Lebesgue measure is zero, \(V_R\) gives rise to a valuated exponent \(\rho\) which can be used to distinguish between sets having identical Hausdorff dimension \(s\) and thickness. \(\tilde{s} = 1/\rho\) is the Hausdorff dimension of \(\tilde{C}_r\) and \(H^{\tilde{s}}(\tilde{C}_r)\) is the corresponding \(\tilde{s}\) dimensional Hausdorff measure. The nontrivial factors involving Hausdorff measures may be seen to arise analogous to (29), in view of the reparametrisation invariance of the higher order valuations \(v_R\). Finally, the ellipses indicate the possible existence of higher order exponential factors. The ordinary (real analysis) representation is reproduced for \(\tilde{C} = \Phi\).

A detail study of the generalized Euler’s factorisation (38) will be taken up separately. Let us only remark here that the nontrivial measure dependent exponents tell that the asymptotic limit \(x \to 0^+\) actually depends on the size and nature of the underlying set on which the variable \(x\) is supposed to live in. Exploiting the reparametrisation invariance, the structure of the underlying set \(C\) could also be altered significantly, that is, a measure zero set may acquire a positive measure.

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