UNIFORM ASYMPTOTIC FORMULAS FOR THE FOURIER COEFFICIENTS OF CERTAIN INVERSE THETA FUNCTIONS

ZHI-GUO LIU AND NIAN HONG ZHOU

Abstract. In this paper we investigate the asymptotics of the Fourier coefficients of certain inverse theta functions, which play a key role in theory of partitions. We establish some uniform asymptotic formulas for those coefficients, which will follows from a more general result about the asymptotic expansion of alternating sum of a class of functions over certain quadratic polynomials. As corollaries, we derived the asymptotics monotonicity properties for rank and crank statistics for integer partitions.

1. Introduction and statement of results

The theory of Jacobi forms was first systematically studied by Eichler and Zagier [1], which play an important role in combinatorics, algebraic geometry and physics. One example of Jacobi forms is the following theta function:

\[ \vartheta(z, \tau) = q^{1/8}(\zeta^{1/2} - \zeta^{-1/2}) \prod_{n \geq 1} (1 - q^n)(1 - \zeta q^n)(1 - \zeta^{-1} q^n), \]

where \( q = e^{2\pi i \tau}, \zeta = e^{2\pi iz} \) with \( \tau, z \in \mathbb{C} \) and \( \Im(\tau) > 0 \). The integers \( M_k(m, n) \) for positive integer \( k \) are defined by the generating function

\[ C_k(\zeta; q) := \sum_{n \geq 0} \sum_{m \in \mathbb{Z}} M_k(m, n) \zeta^m q^n := \frac{(\zeta^{1/2} - \zeta^{-1/2})q^{k/24} \eta(\tau)^{3-k}}{\vartheta(z, \tau)}, \]

where \( \eta(\tau) \) is the Dedekind eta function defined as

\[ \eta(\tau) = q^{1/24} \prod_{j \geq 1} (1 - q^j). \]

For the case \( k = 1 \), \( M_1(m, n) \) is the number of partitions of \( n \) with crank \( m \), which was introduced conjecturally to explain Ramanujan’s famous partition congruences modulo 11 by Dyson [2], and found by Andrews and Garvan [3, 4]. For the case \( k = 2 \), \( M_2(m, n) \) corresponding the birank of partitions, which introduced by Hammond and Lewis [5]. For the cases \( k \geq 3 \), the functions \( C_k(\zeta; q) \) are well-known to be generating functions of Betti numbers of moduli spaces of Hilbert schemes on \((k - 3)\)-point blow-ups of the projective plane [6], see [7] and references therein for detail.

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In [8], Dyson gave the following asymptotic formulae conjecture for the crank statistic for integer partitions:

\begin{equation}
M_1(m, n) \sim \frac{\pi}{4 \sqrt{6n}} \operatorname{sech}^2 \left( \frac{\pi m}{2 \sqrt{6n}} \right) p(n), n \to +\infty.
\end{equation}

He then asked the question about the precise range of \( m \) in (1.1), which holds and about the error term.

Let \( p_k(n) \) be defined by

\begin{equation}
\sum_{n \geq 0} p_k(n) q^n := q^{k/24} \eta(\tau)^{-k},
\end{equation}

the number of partitions of integer \( n \) allowing \( k \) colors. In [9], Bringmann and Dousse has been proved above Dyson conjecture by

**Theorem 1.1.** Let \( k \) be a positive integer. If \( |m| \leq \frac{1}{2} \sqrt{26k} \log n \), we have

\begin{equation}
M_k(m, n) = \frac{\pi}{4} \frac{1}{\sqrt{6n}} \operatorname{sech}^2 \left( \frac{\pi m}{2 \sqrt{6n}} \right) p_k(n) \left( 1 + O \left( \frac{|m|^{1/3} + 1}{n^{1/4}} \right) \right)
\end{equation}

holds for \( n \) tends to infinity.\(^a\)

It is well known that \( 1/\vartheta(z; \tau) \) can be represented as a so-called Lerch sum:

\[
\frac{1}{\vartheta(z; \tau)} = -\frac{\zeta^{1/2}}{\eta(\tau)^3} \sum_{n \in \mathbb{Z}} (-1)^n \frac{q^{(n+1)}_{m(n+1)}}{1 - \zeta q^n},
\]

which means that

\begin{equation}
\sum_{n \geq 0} M_k(m, n) q^n = \frac{q^{k/24}}{\eta(\tau)^k} \sum_{n \geq 1} (-1)^{n-1} q^{\frac{n(n+1)}{2} + n|m|} (1 - q^n),
\end{equation}

see [4, 7] for detail. Let integer \( k \geq 2 \), the Garvan \( k \)-rank \( N_k(m, n) \) is the number of partitions of \( n \) into at least \( (k - 1) \) successive Duree squares with \( k \)-rank equal to \( m \in \mathbb{Z} \). Garvan [10] proved

\begin{equation}
\sum_{n \geq 0} N_k(m, n) q^n = \frac{q^{1/24}}{\eta(\tau)} \sum_{n \geq 1} (-1)^{n-1} q^{\frac{n(2k-1)(n-1)}{2} + |m|n(1 - q^n)}.
\end{equation}

For \( k = 2 \), \( N(m, n) = N_2(m, n) \) is the rank statistic for integer partitions; it was introduced by Dyson [2] to explain Ramanujan’s famous partition congruences with modulus 5 and 7. It is clear that equation (1.5) has similar structure as equation (1.4), and \( M_1(m, n) = N_1(m, n) \).

In [11], Dousse and Mertens proved that (1.3) also holds for \( N(m, n) \). For the case of fixed \( m \), the complete asymptotic expansion of \( M_k(m, n) \) as \( n \) tends to infinity has been established in [7]. For more results on asymptotics for rank and crank statistics for integer partitions, see for example [12, 13, 14, 15].

The aim of this paper is give a uniform asymptotic expansion for \( M_k(m, n) \) and \( N_k(m, n) \). We shall define \( S_k(a, b, n) \) for \( k \in \mathbb{Z}_+ \) by

\begin{equation}
\sum_{n \in \mathbb{Z}} S_k(a, b, n) q^n = \frac{q^{k/24}}{\eta(\tau)^k} \sum_{n \geq 1} (-1)^{n-1} q^{an^2 + bn},
\end{equation}

\(^a\)In [9], the O-term of (1.3) is \( O(|m|^{1/3}n^{-1/4}) \), the case of \( m = 0 \) was missed.
for some $a > 0, b \geq 0$ such that $ax^2 + bx$ be an integer-valued polynomial for $x \in \mathbb{Z}$. We also introduce the forward difference operator $\Delta$ defined as

$$\Delta^0_{n} f(n) = f(n),$$

$$\Delta_{n} f(n) = f(n+1) - f(n)$$

and

$$\Delta^{k+1}_{n} f(n) = \Delta_{n} (\Delta^{k}_{n} f(n)) \text{ for } k \in \mathbb{Z}_{+}.$$ 

Then, it is easy to deduce the following relations by (1.4), (1.5) and (1.6).

**Proposition 1.2.** For each positive integer $k$, nonnegative integers $n$ and $m$, we have

$$M_{k}(m, n) = -\Delta_{m} S_{k} \left( \frac{1}{2}, m - \frac{1}{2}, n \right)$$

and

$$N_{k}(m, n) = -\Delta_{m} S_{1} \left( k - \frac{1}{2}, m - \frac{1}{2}, n \right).$$

We shall investigate the symptoms for $S_{k}(a, b, n)$ defined by (1.6) with $n$ tends to infinity and arbitrary $b \geq 0$. From Proposition 1.2, the asymptotics for $M_{k}(m, n)$ and $N_{k}(m, n)$ follows from the symptoms for $S_{p}(a, b, n)$. The complete asymptotic expansion for $S_{k}(a, b, n)$ allow us to determine the asymptotic monotonicity properties of $M_{k}(m, n)$ and $N_{k}(m, n)$ in $m$ as $n$ tends to infinity.

We begin with the obvious fact that, for $m \in \mathbb{Z}$ and $N \in \mathbb{Z}_{+},$

$$S_{k}(a, b, N) = \sum_{n \geq 1} \frac{(-1)^{n-1} p_{k} \left( N - (a n^2 + b n) \right)}{a n^2 + b n \leq N},$$

which follows from (1.2) and (1.6). We now consider a function $f$ which have similar asymptotic expansion to partition functions $p(n)$. More precisely, we shall let $f : \mathbb{R} \to \mathbb{C}$ be a real function and has asymptotic expansion of form

$$f(X) \sim X^{a_{f}} e^{\beta_{f} \sqrt{X}} \sum_{n \geq 0} \frac{\gamma_{n}(f)}{X^{n/2}}, X \to +\infty,$$

with $\alpha_{f}, \beta_{f}, \gamma_{n}(f) \in \mathbb{C}$ for all integer $n \geq 0$ and $\gamma_{0}(f) \neq 0$. In this paper, we focus on the following sum

$$(1.8) \quad S_{f}(a, b, X) := \sum_{n \geq 1} \frac{(-1)^{n-1} f \left( X - a(n^2 + 2bn) \right)}{a(n^2 + 2bn) \leq X},$$

with $a \in \mathbb{R}_{+}$ and $b \in \mathbb{R}_{\geq 0}$. It is clear that $S_{k}(a, b, N) = S_{p_{k}}(a, b/(2a); N)$.

Our main result of this paper is stated in the following theorem.

**Theorem 1.3.** Let $\mu \in \mathbb{R}$ be given. We have for $0 \leq b = a(X^{3/4})$, as $X \to +\infty$

$$\frac{S_{f}(a, b + \mu; X)}{f(X)} \sim \sum_{g \geq 0} X^{-g/2} L_{g}(\mu, b, \partial_{\alpha}) \left|_{a=a_{f}} \left\{ \frac{1}{1 + e^{\alpha}} \right\} \right. ,$$

where $L_{g}(\mu, b, \partial_{\alpha})$ is a differential operator defined as

$$L_{g}(\mu, b, \partial_{\alpha}) = \sum_{r, \ell, s \geq 0, 3r + 2\ell + 2s \leq 2g} C_{r, \ell, s}(2g; f, a)(2b)^{r} \mu^{\ell} \alpha^{r+\ell+2s},$$
\[ \partial_a := \frac{d}{da}, \text{ and the coefficients } C_{r, \ell, s}(g; f, a) \text{ is given by } (3.3) \text{ depending only on } r, \ell, s, f \text{ and } f \text{ and } a. \]

Remark 1.1. A parameter \( \mu \) was introduced in above theorem is for conveniently compute finite difference of \( S_f(a, b; X) \) with respect to \( b \).

To completely our main result we shall give the following more easy theorem.

**Theorem 1.4.** For \( X^{1/2} \leq b \leq X/(2a) \), as \( X \to \infty \),

\[ \frac{S_f(a, b; X)}{f(X - (1 + 2b)a)} = 1 + O \left( b^{-1} X \exp \left( -\frac{ab\beta_f}{\sqrt{X}} \right) \right). \]

Remark 1.2. Such type result actually has been proved in [15]. From which we can find finite difference of \( S_f(a, b + \mu, X) \) with respect to \( \mu \) for \( b > C\sqrt{X} \log X \) with some large enough \( C \in \mathbb{R}_+ \).

From Theorem 1.3, we prove

**Corollary 1.5.** Let \( \mu \in \mathbb{R} \setminus \{0\} \) be given and \( X \) be sufficiently large. We have for \( 0 \leq b = o(X^{3/4}) \),

\[ \frac{S_f(a, b; X)}{f(X)} = \frac{1}{2} \left( 1 - \tanh \left( \frac{ab\beta_f}{2\sqrt{X}} \right) \right) \left( 1 + O \left( \frac{X + b^2}{X^{3/2}} \right) \right); \]

\[ \Delta_u \big|_{u=0} S_f(a, b + u\mu; X) \big| \frac{f(X)}{f(X)} = \frac{u\beta_f}{4\sqrt{X}} \left( a(2b - \mu)\beta_f \right) \left( 1 + O \left( \frac{X + b^2}{X^{3/2}} \right) \right); \]

and for \( 1 \ll b - \mu = o(X^{3/4}) \),

\[ \Delta_u^2 \big|_{u=0} S_f(a, b + u\mu; X) \big| \frac{f(X)}{f(X)} = \frac{(u\beta_f)^2}{4X} \left( \frac{a(b - \mu)\beta_f}{2\sqrt{X}} \right) \left( 1 + O \left( \frac{X + b^2}{X^{3/2}} \right) \right). \]

As applications of above Corollary 1.5, we have the following uniform asymptotic formulas for \( M_k(m, n) \) and \( N_k(m, n) \), which improve the result of Bringmann and Dousse [9] for \( M_k(m, n) \), and the result of Dousse and Mertens [11] for \( N(m, n) \).

**Corollary 1.6.** We have for \( |m| = o(n^{3/4}) \) and each \( k \in \mathbb{Z}_+ \), as \( n \to +\infty \)

\[ \frac{M_k(m, n)}{p_k(n)} = \frac{1}{4} \sqrt{\frac{k\pi^2}{6n}} \operatorname{sech}^2 \left( \frac{m}{2} \sqrt{\frac{k\pi^2}{6n}} \right) \left( 1 + O \left( \frac{n + m^2}{n^{3/2}} \right) \right), \]

and

\[ \frac{N_k(m, n)}{p(n)} = \frac{1}{4} \sqrt{\frac{\pi^2}{6n}} \operatorname{sech}^2 \left( \frac{m}{2} \sqrt{\frac{\pi^2}{6n}} \right) \left( 1 + O \left( \frac{n + m^2}{n^{3/2}} \right) \right). \]

Also, by Corollary 1.5 we have the following asymptotic monotonicity properties for \( M_k(m, n) \) and \( N_k(m, n) \).

**Corollary 1.7.** We have for \( 0 \leq m = o(n^{3/4}) \) and each \( k \in \mathbb{Z}_+ \), as \( n \to +\infty \)

\[ \frac{M_k(m, n) - M_k(m + 1, n)}{p_k(n)} = \frac{k\pi^2}{24n} \frac{\sinh \left( \frac{2m+1}{4} \sqrt{\frac{k\pi^2}{6n}} \right)}{\cosh^3 \left( \frac{2m+1}{4} \sqrt{\frac{k\pi^2}{6n}} \right)} \left( 1 + O \left( \frac{n + m^2}{n^{3/2}} \right) \right), \]
and
\[
\frac{N_k(m, n) - N_k(m + 1, n)}{p(n)} = \frac{\pi^2}{24n} \sinh^3 \left( \frac{2m+1}{6n} \alpha \right) \left( 1 + O \left( \frac{n}{m^{3/2}} \right) \right).
\]

**Remark 1.3.** We remark that the monotonicity properties of \( N(m, n) \) has been investigated by S.H. Chan and R. Mao \[16\].

This paper is organized as follows. In Section 2, we prove some fundamental results, which play a crucial role in this paper. In Section 3, we prove Theorem 1.3 and Theorem 1.4. In Section 4, we finish the proof of remaining results of this paper.

**Notation.** Notation is standard or otherwise introduced when appropriate. The symbols \( \mathbb{Z}_{\geq 0}, \mathbb{Z}_+, \mathbb{R}_{\geq 0} \) and \( \mathbb{R}_+ \) denote the set of nonnegative integer, positive integer, nonnegative real and positive real numbers, respectively. We use \( Y \ll_{a_1, \ldots, a_k} X \) to denote \( Y \leq C_{a_1, \ldots, a_k} X \) for some finite positive quantity \( C_{a_1, \ldots, a_k} \) depending only on \( a_1, \ldots, a_k \) and \( Y \gg X \) to denote \( Y \geq CX \) for some absolute real number \( C > 0 \).

## 2. Fundamental results

### 2.1. Shift of the asymptotic expansion for \( f(X) \)

We begin with the following asymptotic result for a shift of an asymptotic expansion, which will be used to deduce the approximation for \( f(X + r) \) with \( r = o(X^{3/4}) \).

**Proposition 2.1.** We have if \( r = o(X^{3/4}) \) then \( f(X + r) \) has asymptotic expansion of form
\[
f(X + r) \sim f(X) \sum_{j \geq 0} \left\{ \Lambda_j(f, X) \partial_y^j \right\} \bigg|_{y = \frac{r}{2\sqrt{X}}} e^{yr}, X \to +\infty,
\]
with
\[
\left\{ \Lambda_j(f, X) \partial_y^j \right\} \bigg|_{y = \frac{r}{2\sqrt{X}}} e^{yr} \ll_{j, f} e^{\frac{3r}{2\sqrt{X}} (X^{-3/4}|r|)^j},
\]
\( \Lambda_j(k, X) \) has asymptotic expansion of form
\[
\Lambda_j(f, X) \sim X^{-3j/4} \sum_{n \geq 0} \lambda_{n,j}(f) X^{-n/4}
\]
for some constants \( \lambda_{n,j}(f) \in \mathbb{C} \) depending only on \( n, j \) and \( f \) are defined as (2.5). In particular, \( \lambda_{n,j} = 0 \) for \( n \neq j \) mod 2, \( \lambda_{n,0} = 1, \lambda_{n,0} = 0 \) for all \( n \geq 1, \lambda_{1,1} = \alpha_f, \lambda_{0,2} = -\frac{\beta_f}{8}, \) and \( \lambda_{1,3} = -\frac{\alpha_f \beta_f}{8} + \frac{\beta_f}{16} \).

**Proof.** First of all, since \( r = o(X^{3/4}) \) we have
\[
(1 + \frac{r}{X})^{\alpha_f} = \sum_{\ell \geq 0} \binom{\alpha_f}{\ell} r^{\ell} X^{-\ell},
\]
by generalized binomial theorem. Also, by generalized binomial theorem again,
\[
\sqrt{X} \left( \sqrt{1 + \frac{r}{X}} - 1 - \frac{r}{2X} \right) = \sqrt{X} \sum_{h \geq 2} \binom{1/2}{h} \left( \frac{r}{X} \right)^h = O \left( \frac{r^2}{X^{3/2}} \right) = o(1),
\]
we have
\[
e^{\beta_f \sqrt{X}(\sqrt{1 + \frac{r}{X}} - \frac{r}{X})} = \sum_{\ell \geq 0} \frac{(\beta_f X^{1/2})^{\ell}}{\ell!} \left( \sum_{h \geq 2} \left( \frac{1}{2} \right) \left( \frac{r}{h} X \right)^h \right)^{\ell}
\]
(2.2)
\[
= \sum_{\ell \geq 0} r^{2\ell} \left( X^{-3\ell/2} \sum_{k \geq 0} d_{k,\ell}(f) \left( \frac{r}{X} \right)^k \right).
\]

If we denote \( g(X) = X^{-\alpha_f} e^{-\beta_f f(X)} \) then \( g(X) \) has asymptotic expansion of form
\[
g(X) \sim \sum_{n \geq 0} \frac{\gamma_n(f)}{X^{n/2}} X \to +\infty.
\]
By the basic results of asymptotic analysis, see for example [17, 18], since \( r/X = o(X^{-1/4}) \), we have
\[
g(X + r) \sim \sum_{n \geq 0} \frac{\gamma_n(f)}{(X + r)^{n/2}}
\]
\[
\sim \sum_{n \geq 0} \frac{\gamma_n(f)}{X^{n/2}} \sum_{g \geq 0} \left( \frac{-n/2}{g} \right) r^g X^{-g}
\]
\[
\sim \sum_{g \geq 0} r^g X^{-g} \sum_{n \geq 0} \left( \frac{-n/2}{g} \right) \gamma_n(f) X^{-n/2}.
\]
Thus if we define \( c_{g,h}(f) \) for each nonnegative integers \( g \) and \( h \) that
\[
\sum_{h \geq 0} c_{g,h}(f) X^{-h/2} := \left( \sum_{n \geq 0} \frac{\gamma_n(f)}{X^{n/2}} \right)^{-1} \left( \sum_{n \geq 0} \left( \frac{-n/2}{g} \right) \gamma_n(f) X^{-n/2} \right),
\]
formally, then
\[
\frac{g(X + r)}{g(X)} \sim \sum_{g \geq 0} r^g X^{-g} \sum_{h \geq 0} c_{g,h}(f) X^{-h/2}.
\]

From (2.1), (2.2) and (2.4) we have
\[
\frac{f(X + r)}{f(X)} = \exp \left( \frac{\beta_{fr}}{2\sqrt{X}} \left( 1 + \frac{r}{X} \right)^{\alpha_f} e^{\beta_f \sqrt{X}(\sqrt{1 + \frac{r}{X}} - \frac{r}{X})} \frac{g(X + r)}{g(X)} \right)
\]
\[
\sim \exp \left( \frac{\beta_{fr}}{2\sqrt{X}} \sum_{s \geq 0} \left( \frac{\alpha_f}{s} \right) X^s \sum_{k,\ell \geq 0} d_{k,\ell}(f) \frac{r^{k+2\ell}}{X^{3\ell/2}} \sum_{h \geq 0} c_{g,h}(f) X^{-h/2} \right)
\]
\[
\sim \exp \left( \frac{\beta_{fr}}{2\sqrt{X}} \sum_{j \geq 0} r^j X^{-j} \sum_{s, k, \ell, g, h \geq 0} X^{-(s+3\ell/2-k-g-h/2)} \left( \frac{\alpha_f}{s} \right) c_{g,h}(f) d_{k,\ell}(f) \right)
\]
\[
\sim \exp \left( \frac{\beta_{fr}}{2\sqrt{X}} \sum_{j \geq 0} r^j X^{-3j/4} \sum_{n \geq 0} X^{-n/4} \sum_{s, k, g, h, \ell \geq 0} X^{-(s+3\ell/2+g-h/2)} \left( \frac{\alpha_f}{s} \right) c_{g,h}(f) d_{k,\ell}(f) \right).
\]
Namely we have the following asymptotic expansion
\[
\frac{f(X + r)}{f(X)} \sim \exp \left( \frac{\beta f r}{2\sqrt{X}} \right) \sum_{j \geq 0} \Lambda_j(f, X)r^j
\]
with
\[
\Lambda_j(f, X) \sim X^{-3j/4} \sum_{n \geq 0} \lambda_{n,j}(f) X^{-n/4}
\]
for some constants \( \lambda_{n,j}(f) \in \mathbb{C} \) depending only on \( f, n \) and \( j \). In particular,
\[
(2.5) \quad \lambda_{n,j}(f) = \sum_{s,k,g,h \geq 0} \frac{2(\ell - h) + n}{s + k + 2\ell + g = j} \alpha_s f \beta_{g,h}(f) \gamma_{k,\ell}(f)
\]
with \( \beta_{g,h}(f) \) and \( \gamma_{k,\ell}(f) \) are determined by (2.2) and (2.3), respectively.

2.2. Uniformly asymptotics of a false theta function.

In this section we consider the asymptotics of the following false theta function:
\[
(2.6) \quad T_b(z) = \sum_{n \geq 1} (-1)^{n-1} e^{-(n^2 + 2bn)z},
\]
where \( b \in \mathbb{R} \) and \( z \in \mathbb{C} \) with \( \Re(z) > 0 \).

The above partial theta function (2.6) have recently appeared in several areas of the theory of \( q \)-series, integer partitions and quantum topology, see for example [19] for detail. In all of the these aspects, it is important to understand the asymptotic properties of (2.6). The asymptotic expansion of partial theta function (2.6) with \( b \) fixed at \( z = 0 \) has been investigated in many literature, see for example [20] and [21], which play an important role in the studies of the asymptotic expansion of rank and crank statistic for integer partitions, see for example [7], [13] and [12]. However, there is no such results which holds uniformly for \( b \in \mathbb{R} \).

We first deduce the following proposition.

**Proposition 2.2.** Let \( \alpha, z \in \mathbb{C} \) with \( \Re(\alpha), \Re(z) > 0 \), \( \ell \in \mathbb{Z}_{\geq 0} \) and \( N \in \mathbb{Z}_+ \). We have
\[
\sum_{n \geq 1} (-1)^{n-1} n^{\ell} e^{-n^2 z - n\alpha} = \sum_{k=0}^{N-1} \frac{(-z)^k}{k!} \partial^{2\ell} \left( \frac{1}{1 + e^\alpha} \right) - z^N R_N(\ell, \alpha, z),
\]
where,
\[
R_N(\ell, \alpha, z) = \frac{(-1)^{N+\ell}}{(N-1)!} \int_0^1 (1 - t)^{N-1} \partial^{2N+\ell} \left( \sum_{n \geq 1} (-1)^n e^{-n^2 zt - n\alpha} \right) dt.
\]

**Proof.** We first denote by
\[
F(z) = \sum_{n \geq 1} (-1)^n n^{\ell} e^{-n^2 z - n\alpha}.
\]
It is clear that for each \( k \in \mathbb{N} \),

\[
F^{(k)}(0^+) := \lim_{z \to 0^+} F^{(k)}(z)
= \lim_{z \to 0^+} \sum_{n \geq 1} (-1)^{n+k}r^{2k+\ell}e^{-n^2z-\alpha n}
= (-1)^k \sum_{n \geq 1} (-1)^n n^{2k+\ell}e^{-\alpha n}
\]

which means that

\[
(2.7) \quad F^{(k)}(0^+) = -(-1)^{k+\ell} \partial_\alpha^{2k+\ell} \left( \frac{e^{-\alpha}}{1+e^{-\alpha}} \right).
\]

By Taylor theorem we have for each \( N \in \mathbb{Z}_+ \),

\[
F(z) = \sum_{k=0}^{N-1} \frac{1}{k!} F^{(k)}(0^+)(z-0^+)^k + \int_0^z \frac{F^{(N)}(t)}{(N-1)!} (z-t)^{N-1} dt
\]

\[
= -\sum_{k=0}^{N-1} \frac{(-z)^k}{k!} \partial_\alpha^{2k+\ell} \left( \frac{1}{1+e^{-\alpha}} \right) + z^N R_N(\ell, \alpha, z),
\]

with

\[
R_N(\ell, \alpha, z) = \frac{1}{(N-1)!} \int_0^1 \partial_u^N \big|_{u=zt} (F(u)) (1-t)^{N-1} dt
\]

\[
= \frac{(-1)^N}{(N-1)!} \int_0^1 (1-t)^{N-1} \sum_{n \geq 1} (-1)^n n^{2N+\ell}e^{-n^2zt-\alpha n} dt
\]

\[
= \frac{(-1)^{N+\ell}}{(N-1)!} \int_0^1 (1-t)^{N-1} \partial_\alpha^{2N+\ell} \left( \sum_{n \geq 1} (-1)^n e^{-n^2zt-\alpha n} \right) dt,
\]

which completes the proof.

To estimate \( R_{N+1}(\alpha, z) \) of above we need the following lemma.

**Lemma 2.3.** Let \( x \in \mathbb{C} \) with \(|x| < 1\). Also let the function \( f : \mathbb{Z} \to \mathbb{C} \) satisfy \( \lim_{n \to \infty} f(n) = 0 \). Then, we have

\[
\sum_{n \geq 0} x^n f(n) = \sum_{r=0}^{K-1} \frac{x^r \Delta^r f(0)}{(1-x)^{r+1}} + \frac{x^K}{(1-x)^{K}} \sum_{n \geq 0} x^n \Delta^K f(n), K \in \mathbb{Z}_+.
\]

**Proof.** This lemma is called Euler transform and very easy to prove by mathematical induction, and hence we omit its detail.

**Proposition 2.4.** Let \( R_N(\ell, \alpha, z) \) be defined as 2.2 and \( N, \ell \in \mathbb{Z}_{\geq 0} \). We have for \( z = x + iy \) with \( x, y \in \mathbb{R} \), \( x > 0 \) and \( |y| \leq x \),

\[
e^{2bz} R_N(\ell, 2bz, z) \ll_{N, \ell} 1,
\]

uniformly for \( b \geq 0 \).
Proof. First of all, from Proposition 2.2 we have

(2.8) \[ R_N(\ell, 2bz, z) \ll N \int_0^1 dt \left| \partial_\alpha^{2N+\ell} \left( \sum_{n \geq 1} (-1)^n e^{-n^2zt-n\alpha} \right) \right|. \]

If we let \( E_u(n) = e^{-n^2u} \) and let \( K \in \mathbb{Z}_{\geq 0} \) be given, then by the Taylor expansion for \( e^z \), we have for all \( |zt| \leq 1 \),

\[
e^{n^2zt} \Delta^K E_{zt}(n) = e^{n^2zt}(-1)^K \sum_{r=0}^K \binom{K}{r} (-1)^r e^{-2rzt} \Delta^K E_{zt}(n) \]

\[
\ll \sum_{r=0}^K \binom{K}{r} (-1)^r e^{-2rzt} \left( \sum_{0 \leq \ell < K/2} \frac{(-r^2zt)^\ell}{\ell!} + O_K(|zt|^{K/2}) \right). \]

If \( \Re(nzt) \geq 0 \) then we have further,

\[
e^{n^2zt} \Delta^K E_{zt}(n) \ll_K \sum_{0 \leq \ell < K/2} \frac{(-zt)^\ell}{\ell!} \sum_{r=0}^K \binom{K}{r} (-1)^r e^{-2rzt} \Delta^K E_{zt}(n) + |zt|^{K/2} \]

\[
= \sum_{0 \leq \ell < K/2} \frac{(-zt)^\ell}{\ell!} \partial_\alpha^{2\ell} |_{u=2nzt} (1 - e^{-u})^K + |zt|^{K/2}. \]

Thus if \( 0 \leq n \leq 1/x, t \geq 0 \) and \( |zt| \leq 1 \) then

\[
e^{n^2zt} \Delta^K E_{zt}(n) \ll_K \sum_{0 \leq \ell < K/2} n^{K-2\ell} |zt|^{K-\ell} + |zt|^{K/2} \ll_K |nzt|^K + |(xt)|^{K/2}. \]

Combining the trivial estimate for \( \Delta^K E_{zt}(n) \) with \( n \gg 1/x \), we get further,

(2.9) \[ \Delta^K E_{zt}(n) \ll_K \min \left( 1, (xt)^{K/2} + |nzt|^K \right) e^{-n^2zt}, n \geq 0. \]

On the other hand, by Lemma 2.3 we have for any integer \( K \geq 2 \) be given,

\[
\sum_{n \geq 1} (-1)^n e^{-n^2u-n\alpha} = -1 + \sum_{r=0}^{K-1} \frac{e^{-ra} \Delta^r E_u(0)}{(1+e^{-\alpha})^{r+1}} + \frac{e^{-Ka}}{(1+e^{-\alpha})^K} \sum_{n \geq 1} (-1)^n e^{-n\alpha} \Delta^K E_u(n) \]

(2.10) \[ = \sum_{r=1}^{K-1} \frac{e^{-ra} \Delta^r E_u(0)}{(1+e^{-\alpha})^{r+1}} + \frac{e^{-Ka}}{(1+e^{-\alpha})^K} \sum_{n \geq 0} (-1)^n e^{-n\alpha} \Delta^K E_u(n). \]

Hence by inserting (2.10) to (2.8), combining (2.9), and the fact that \( |y| \leq x \) then

\[
e^{2bz} \partial_\alpha^M |_{\alpha=2bz} \left( \frac{e^{-ra}}{(1+e^{-\alpha})^{r+1}} \right) \ll_r,M 1 + \frac{1}{|1+e^{-2bz}|^{r+M+1}} \ll_r,M 1, \]

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holds for all $r \geq 1$ and $M \in \mathbb{Z}_{\geq 0}$, directly calculation yields
\[
e^{2b_z} R_N(\ell, 2b_z, z) \ll N, \ell, K + \int_0^1 dt \left| \sum_{0 \leq j \leq 2N+\ell} \sum_{n \geq 0} (-1)^n n^j e^{-2bnz} \Delta^K E_u(n) \right| \ll N, \ell, K + \int_0^1 dt \left| \sum_{n \geq 0} n^{2N+\ell} \min \left( 1, (xt)^{K/2} + (nxt)^K \right) e^{-n^2 xt} \right|.
\]

We now estimate the sum lie on above integral. For $0 < u \leq 1$, since
\[
\sum_{n \geq 0} n^{2N+\ell} \min \left( 1, u^{K/2} + (nu)^K \right) e^{-n^2 u} \ll \sum_{0 \leq n < u^{-2/3}} (u^{K/2} + (nu)^K) + \sum_{n \geq u^{-2/3}} n^{2N+\ell} e^{-n^2 u}
\]
\[
\ll N, \ell u^{K/2 - \frac{1}{2}(2N+\ell+1)} + 1,
\]
we have if one set $K = 4N + 2\ell + 2$ in above then
\[
e^{2b_z} R_N(\ell, 2b_z, z) \ll N, \ell, 1,
\]
which completes the proof.

We prove the following estimate which will be used in the following and the proof of Corollary 1.5.

**Lemma 2.5.** Let $J \in \mathbb{Z}_+$ be given and let $\alpha = x + iy$ with $x, y \in \mathbb{R}$ and $|y| \leq x$. we have uniformly for $x \geq 0$,
\[
e^{\alpha} \partial^J \alpha \left\{ \frac{1}{1 + e^{\alpha}} \right\} \ll 1,
\]
\[
\partial^{2J-1} \alpha \left\{ \frac{1}{1 + e^{\alpha}} \right\} \ll \partial^J \alpha \left\{ \frac{1}{1 + e^{\alpha}} \right\} \text{ and } \partial^{2J} \alpha \left\{ \frac{1}{1 + e^{\alpha}} \right\} \ll \partial^2 \alpha \left\{ \frac{1}{1 + e^{\alpha}} \right\}.
\]

**Proof.** Since
\[
\partial^J \alpha \left( \frac{1}{1 + e^{\alpha}} \right) = \partial^J \alpha \left( \sum_{n \geq 0} (-1)^n e^{-(n+1)\alpha} \right)
\]
\[
= (-1)^J e^{-\alpha} \sum_{n \geq 0} (n+1)^J (-e^{-\alpha})^n = \frac{(-1)^J e^{-\alpha} A_J(-e^{-\alpha})}{(1 - (-e^{-\alpha}))^{J+1}},
\]
where $A_J(t)$ are the Eulerian polynomials, we have
\[
\partial^J \alpha \left( \frac{1}{1 + e^{\alpha}} \right) = (-1)^J \frac{A_J(-e^{-\alpha})}{(1 + e^{-\alpha})^{J+1}} \ll e^{-\alpha}.
\]
Which completes the proof of the first estimate of the lemma. To prove the next two estimate, we note that for the cases $x \gg 1$, the lemma is clear, thus we just need consider the cases $x = o(1)$. If $x = o(1)$ then $\alpha = o(1)$, by the definition of
Bernoulli number $B_J$ we have

$$\partial^J_\alpha \left( \frac{1}{1 + e^\alpha} \right) = \partial^J_\alpha \left( \frac{1}{e^\alpha - 1} - \frac{2}{e^{2\alpha} - 1} \right)$$

$$= \partial^J_\alpha \left( \sum_{j \geq 0} \frac{(1 - 2j+1)B_{j+1} \alpha^j}{(j + 1)!} \right) = \sum_{j \geq 0} \frac{(1 - 2j+j+1)B_{j+j+1} \alpha^j}{j!(j + J + 1)}$$

$$= \frac{1 - 2J+1}{J + 1} B_{J+1} + \frac{1 - 2J+2}{J + 2} B_{J+2} \alpha + O(|\alpha|^2).$$

Thus by note the fact that $B_{J+1}^2 + B_{J+2}^2 > 0$ for all $J \in \mathbb{Z}_{\geq 0},$

$$\partial^J_\alpha \left( \frac{1}{1 + e^\alpha} \right) \sim \frac{1 - 2J}{J} B_{J} \alpha, \text{ for } J \geq 1;$$

and

$$\partial^{J-1}_\alpha \left( \frac{1}{1 + e^\alpha} \right) \sim \frac{1 - 2J}{J} B_{J} \alpha \text{ for } J > 1.$$
Take derivative of both sides above,
\[
\frac{\partial^J}{\partial z^J} T_{b+\mu}(z) = \frac{\partial^J}{\partial z^J} \left( \sum_{\ell \geq 0} \frac{(-2\mu)^\ell}{\ell!} z^\ell \sum_{n \geq 1} (-1)^{n-1} n^\ell e^{-n^2z-2bnz} \right)
\]
(2.12) \[= \sum_{\ell \geq 0} \frac{(-2\mu)^\ell}{\ell!} \sum_{v=0}^{J} \left( \frac{J}{v} \right) (-1)^v (-\ell)^v z^{-v} \frac{\partial^J}{\partial z^J} \sum_{n \geq 1} (-1)^{n-1} n^\ell e^{-n^2z-2bnz}.
\]
On the other hand, we have for each nonnegative integer \(N\),
\[
\frac{\partial^J}{\partial z^J} \sum_{n \geq 1} (-1)^n n^\ell e^{-n^2z-2bnz} = \sum_{n \geq 1} (-1)^{n+N} n^\ell (n^2 + 2bn)^N e^{-(n^2+2bn)z}
\]
(2.13) \[= \sum_{r=0}^{N} \left( \frac{N}{r} \right) (2b)^r \sum_{n \geq 1} (-1)^{n+N} n^\ell + 2N - r e^{-(n^2+2bn)z}.
\]
Inserting (2.13) to (2.12) implies that
\[
\frac{\partial^J}{\partial z^J} T_{b+\mu}(z) = \sum_{\ell \geq 0} (-2\mu)^\ell \sum_{v=0}^{J} \left( \frac{J}{v} \right) (-1)^v (-\ell)^v z^{-v} \frac{\partial^J}{\partial z^J} \sum_{n \geq 1} (-1)^{n-1} n^\ell e^{-n^2z-2bnz}
\]
\[= (-1)^J \sum_{v=0}^{J} \sum_{r=0}^{J-v} \left( \frac{J}{v, r} \right) (2b)^r \sum_{\ell \geq 0} \left( \frac{2\mu)^v+\ell(-z)\ell}{\ell!} \right) \sum_{n \geq 1} (-1)^{n-1} n^\ell + 2J - v e^{-(n^2+2bn)z}.
\]
Here and throughout, \( \left( \frac{J}{v, r} \right) \) is multinomial coefficients. Using Theorem 2.6 in above yields:
\[
\frac{\partial^J}{\partial z^J} T_{b+\mu}(z) \sim (-1)^J \sum_{v, r \geq 0 \atop v + r \leq J} \left( \frac{J}{v, r} \right) (2b)^r \sum_{\ell \geq 0} \left( \frac{2\mu)^v+\ell(-z)\ell}{\ell!} \right) \times \sum_{n \geq 0} \left( \frac{(-z)^n}{n!} \right)^{2n + \ell + 2J - v - r} \left|_{\alpha = 2bn} \left\{ \frac{1}{1 + e^\alpha} \right\} \right.
\]
\[\sim \sum_{k \geq 0} \left( \frac{(-z)^k}{k!} \right) \sum_{r, s \geq 0 \atop s + r \leq k + J} C_{s, r}(k, J) \mu^{k+j-s-r} (2b)^r \frac{\partial^{s+k+j}}{\partial z^{s+k+j}} \bigg|_{\alpha = 2bn} \right\} \frac{1}{1 + e^\alpha}
\]
with
\[
C_{s, r}(k, J) = (-1)^{k+j} k! \mu^{-(k+j-s-r)} \sum_{v, n, \ell \geq 0 \atop n + \ell + 2J - v = s + k + J} \left( \frac{J}{v, r} \right) \frac{(2\mu)^v+\ell(-1)^{\ell+n}}{\ell!n!}
\]
\[= 2^{k+j-s-r} (-1)^J \sum_{\ell, v \geq 0 \atop k + j - s - r = \ell + v} \left( \frac{k}{\ell} \right) \left( \frac{J}{v, r} \right),
\]
which completes the proof. \(\square\)
3. ASYMPTOTIC EXPANSION FOR THE SUM OF $f(X)$ OVER VALUE OF CERTAIN QUADRATIC POLYNOMIAL

In this section we prove the main result of this paper. We shall first prove the main theorem of this paper.

3.1. The proof of Theorem 1.3.

We first prove the following uniform asymptotic expansion for $S_f(a, b; X)$ in terms of false theta function $T_b(z)$ defined as Subsection 2.2.

**Proposition 3.1.** If $0 \leq b = o(X^{3/4})$ then we have the asymptotic expansion of form,

$$
\frac{S_f(a, b; X)}{f(X)} \sim \sum_{\ell \geq 0} \left\{ \Lambda(\ell, f, X) \partial_x^\ell \right\} \bigg|_{z = \frac{a}{X^{1/2}}} T_b(az), X \to +\infty,
$$

where for each $\ell \geq 0$,

$$
\left\{ \Lambda(\ell, f, X) \partial_x^\ell \right\} \bigg|_{z = \frac{a}{X^{1/2}}} T_b(az) \ll (1 + b^\ell) X^{-3/4}.
$$

**Proof.** Let $\varepsilon \in (0, 1/10)$. We first split the sum $S_f(a, b; X)$ into two part as follows:

$$
S_f(a, b; X) = \left( \sum_{n \geq 1} \frac{a}{X^{1/2} \varepsilon < a(n^2 + 2bn) \leq X} \right) \left( -1 \right)^{n-1} f(X - a(n^2 + 2bn))
\quad := \ M + E.
$$

For the part $E$, we estimate that

$$
R \ll \sum_{n \geq 1} \frac{f(X - a(n^2 + 2bn))}{X^{1/2 + \varepsilon} < a(n^2 + 2bn) \leq X}
\quad \ll \sqrt{X} f(X - X^{1/2+\varepsilon}) \ll f(X) \exp \left( -X^{\varepsilon/2} \right).
$$

(3.1)

For the part $M$, since $0 \leq a(n^2 + 2bn) \leq X^{1/2+\varepsilon} = o(X^{3/4})$, hence application Proposition 2.1 implies that,

$$
\frac{M}{f(X)} \sim \sum_{n \geq 1} \frac{(-1)^{n-1} \sum_{\ell \geq 0} \Lambda(\ell, f, X) \partial_x^\ell \bigg|_{y = \frac{a}{2X}}} {a(n^2 + 2bn) \leq X^{1/2+\varepsilon} \leq X^{1/2+\varepsilon}}
\quad \sim \sum_{n \geq 1} \frac{(-1)^{n-1} \Lambda(\ell, f, X) \partial_x^\ell \bigg|_{y = \frac{a}{2X}}} {a(n^2 + 2bn) \leq X^{1/2+\varepsilon} \leq X^{1/2+\varepsilon}}.
$$
Since for each $\ell \in \mathbb{Z}_+$, 
\[
\partial_y \bigg|_{y = \frac{\lambda}{2\sqrt{n}}} \left( \sum_{n \geq 1} (-1)^{n-1} e^{-y(n^2 + 2bn)} \right) 
\ll \sum_{n \geq 1} |n^2 + 2bn| e^{-\frac{a\beta}{2\sqrt{n}}(n^2 + 2bn)} \ll e^{-X^{1/2+\epsilon}} \sum_{n \in \mathbb{Z}} |n^2 + 2bn| e^{-\frac{a\beta}{2\sqrt{n}}(n^2 + 2bn)} 
\ll e^{-X^{2s/3}} \sum_{n \in \mathbb{Z}} (n^2 + b^2) e^{-\frac{a\beta (s^2 + 1)}{2\sqrt{n}}} \ll \exp \left(-X^{s/2}\right),
\]
we have the following asymptotic expansion:
\[(3.2) \quad M \sim f(X) \sum_{\ell \geq 0} \left\{ \Lambda_\ell (f, X) \partial_x^\ell \right\} \bigg|_{x = \frac{\lambda}{2\sqrt{n}}} T_\ell (az).\]

Thus combining (3.1) and (3.2) completes the proof. 

We now give the proof of Theorem 1.3. From the Proposition 2.7 and Proposition 3.1 we have 
\[
\frac{S_f(a, b + \mu; X)}{f(X)} \sim \sum_{\ell \geq 0} \left\{ \Lambda_\ell (f, X) \partial_x^\ell \right\} \bigg|_{x = \frac{\lambda}{2\sqrt{n}}} T_{b+\mu}(az) 
\sim \sum_{k, n, j \geq 0} \lambda_{n,j}(f) X^{-(n+3j)/4} (a\beta f(2X^{1/2})^{-1}) k \left(-\frac{1}{k}\right) 
\times \sum_{s, r, u, v \geq 0} \sum_{s+r \leq j+k} C_{s,r}(k,j) \mu^{k+j-s-r} (2b)^r \partial_x^{s+k+j} \bigg|_{\alpha = \frac{a\beta}{2\sqrt{n}}} \left\{ \frac{1}{1 + e^{\alpha}} \right\} 
\sim \sum_{g \geq 0} \frac{1}{X^{g/4}} \sum_{r,\ell,s \geq 0} C_{r,\ell,s}(g; f, a) (2b)^r \mu^{\ell} \partial_x^{r+\ell+2s} \bigg|_{\alpha = \frac{a\beta}{2\sqrt{n}}} \left\{ \frac{1}{1 + e^{\alpha}} \right\},
\]
where 
\[
C_{r,\ell,s}(g; f, a) = \sum_{\substack{n+2k+3j=g \\ k+j-s-r=\ell \\ n,k,j \geq 0}} \lambda_{n,j}(f) \left(\frac{a\beta f}{2}\right)^k C_{s,r}(k,j) \left(-\frac{1}{k}\right) 
\times \sum_{\substack{n+2k+3j=g \\ k+j-s-r=\ell \\ k+j-s-r=u+v \\ n,k,j \geq 0}} \lambda_{n,j}(f) g^{k+j-s-r} \left(-\frac{a\beta f}{2}\right)^k \left(k\right) \left(\binom{j}{u}\right) \left(\binom{j}{v}\right).
\]

From above and notice that $\lambda_{n,j} = 0$ if $n \neq j$ mod 2 (by Proposition 2.1) it is not difficult to prove that 
\[
C_{r,\ell,s}(g; f, a) = 1_{g \equiv 0 \mod 2} 1_{3r+2\ell+2s \leq g} C_{r,\ell,s}(g; f, a),
\]
where $\mathbf{1}_{\text{condition}} = 1$ if the 'condition' is true, and equals to 0 if the 'condition' is false. Hence we obtain that

\[
C_{r,\ell,s}(g; f, a) = \sum_{n,k \geq 0, j \geq r, n+2k+3j = g} \frac{(-a\beta_f)^k}{2^{k-1}k!} \binom{k}{\ell-v} \binom{j}{v, r} \lambda_{n,j}(f),
\]

with $\lambda_{n,j}(f)$ depending only on $n$, $j$ and $f$ be defined by Proposition 2.1, which completes the proof of Theorem 1.3.

We now give some special value of $C_{r,\ell,s}(g; f, a)$, which will be used in Subsection 4.1. From Theorem 1.3 and Proposition 2.1, we have for $J \in \mathbb{Z}_{\geq 0}$,

\[
C_{0,0,0}(2J; f, a) = \frac{(-a\beta_f)^J}{J!} \lambda_{0,0}(f) = \frac{(-a\beta_f)^J}{J!},
\]

\[
C_{0,0,0}(2J + 2; f, a) = \frac{(-a\beta_f)^J}{J!} \lambda_{2,0}(f) + 2 \frac{(-a\beta_f)^{J-1}}{\Gamma(J)} \alpha_f - \frac{(-a\beta_f)^{J-2}}{2\Gamma(J-1)} \beta_f,
\]

\[
C_{0,0,1}(2J + 2; f, a) = \frac{(-a\beta_f)^{J+1}}{2(J+1)!} \binom{J+1}{J} \lambda_{0,0}(f) = \frac{(-a\beta_f)^{J+1}}{2J!}.
\]

\[
C_{1,0,0}(2J + 4; f, a) = 2 \frac{(-a\beta_f)^{J-1}}{\Gamma(J)} \binom{2}{1,1} \lambda_{0,2}(f) = \frac{(-a\beta_f)^J}{J!} \lambda_{1,1}(f)
\]

\[
= -\frac{(-a\beta_f)^{J-1}}{2\Gamma(J)} \beta_f + \frac{(-a\beta_f)^J}{J!} \alpha_f,
\]

and

\[
C_{2,0,0}(2J + 6; f, a) = \frac{(-a\beta_f)^J}{J!} \lambda_{0,2}(f) = -\frac{(-a\beta_f)^J}{8J!} \beta_f.
\]

### 3.2. The proof of Theorem 1.4.

In this subsection we shall prove Theorem 1.4. First of all, if $X^{1/2} \leq b \leq X/(2a)$, then from the definition:

\[
S_f(a, b, X) = \sum_{n \geq 1} \sum_{\alpha(n^2 + 2bn) \leq X} (-1)^{n-1} f \left( X - a(n^2 + 2bn) \right),
\]

we have for $X/(2a) \geq b \geq X/(4a)$,

\[
S_f(a, b, X) = f(X - a(1 + 2b));
\]
For $X^{1/2} \leq b < 3X/(8a)$,
\[
\frac{S_f(a, b, X)}{f(X - a(1 + 2b))} - 1 = \sum_{n \geq 2} (-1)^{n-1} \frac{f(X - a(n^2 + 2bn))}{f(X - a(1 + 2b))} \lesssim \sum_{n \geq 2} \frac{f(X - a(4 + 4b))}{f(X - a(1 + 2b))} \lesssim X^{-1} f\left(\frac{X - 4a - 4ab - \sqrt{X - a - 2ab}}{\sqrt{X}}\right),
\]
which complete the proof of the Theorem 1.4.

4. The proof of Corollaries

In this section we prove the corollaries of this paper. We shall first consider Corollary 1.5.

4.1. Leading asymptotic for Theorem 1.3 and the proof of Corollary 1.5.

We begin with the following lemma.

Lemma 4.1. Let $\ell, J \in \mathbb{Z}_0$. We have:
\[
\Delta^J u \big|_{u = 0} u^\ell = \begin{cases} 
0 & 0 \leq \ell < J, \\
\ell! & \ell = J, \\
J(J + 1)!/2 & \ell = J.
\end{cases}
\]

Proof. For $J, \ell \in \mathbb{Z}_0$, by note that
\[
\Delta^J u \big|_{u = 0} u^\ell = (-1)^J \sum_{j = 0}^J (-1)^j \binom{J}{j} j^\ell \\
= (-1)^\ell \partial_x^\ell \big|_{x = 0} \left((-1)^J \sum_{j = 0}^J (-1)^j \binom{J}{j} e^{-jx}\right) \\
= (-1)^\ell J^\ell \partial_x^\ell \big|_{x = 0} (1 - e^{-x})^J = (-1)^\ell J^\ell \partial_x^\ell \big|_{x = 0} \left(x^J - \frac{J}{2} x^{J+1} + \ldots\right)
\]
we finish the proof of this lemma. □

We now compute some special value of $\Delta^J u \big|_{u = 0} \mathcal{L}_g(\mu u, b, \partial_\alpha)$. From Theorem 1.3 and above Lemma 4.1, we have for $g < J$,
\[
\Delta^J u \big|_{u = 0} \mathcal{L}_g(\mu u, b, \partial_\alpha) = \sum_{r, s \geq 0, \ell \geq J, 3r + 2\ell + 3s \leq 2g} \mathcal{C}_{r, \ell, s}(2g; f, a)(2b)^r \Delta^J u \big|_{u = 0} (u^\ell) \mu^r \partial_\alpha^{r+\ell+2s} = 0;
\]
for \( g = J \),

\[
\Delta^J_u|_{u=0} = \sum_{\ell,s \geq 0 \atop 3r+2\ell+2s \leq 2J} \mathcal{C}_{r,s}(2J; f, a)(2b)^r \Delta^J_u|_{u=0}(u^\ell) \mu^\ell \partial^{r+\ell+2s}_a
\]

(4.2)

\[
= J! \mu^J \mathcal{C}_{0,J}(2J; f, a) \partial^J_a;
\]

for \( g = J + 1 \),

\[
\Delta^J_u|_{u=0} = \sum_{\ell,s \geq 0 \atop 3r+2\ell+2s \leq 2J+2} \mathcal{C}_{r,s}(2J+2; f, a)(2b)^r \Delta^J_u|_{u=0}(u^\ell) \mu^\ell \partial^{r+\ell+2s}_a
\]

(4.3)

\[
+ \frac{J(J+1)!}{2} \mu^{J+1} \mathcal{C}_{0,J+1}(2J+2; f, a) \partial^{J+1}_a;
\]

for \( g = J + 2 \),

\[
\Delta^J_u|_{u=0} = \sum_{\ell,s \geq 0 \atop \ell+s \leq 1} \mathcal{C}_{0,J+1,s}(2J+2; f, a) \Delta^J_u|_{u=0}(u^{J+\ell}) \mu^{J+\ell} \partial^{J+\ell+2s}_a
\]

(4.4)

\[
= J! \mu^J \mathcal{C}_{0,J}(2J+4; f, a) J! \mu^J \partial^{J+1}_a + b^0 (\ldots) ;
\]

for \( g = J + 3 \),

\[
\Delta^J_u|_{u=0} = \sum_{\ell,s \geq 0 \atop \ell+s \leq 1} \mathcal{C}_{0,J+2,s}(2J+6; f, a) J! \mu^J \partial^{J+2}_a + b^1 (\ldots) + b^0 (\ldots).
\]

For \( g \geq J + 4 \), we estimate that

\[
\Delta^J_u|_{u=0} = \sum_{\ell,s \geq 0 \atop 3r+2\ell+2s \leq 2g} \mathcal{C}_{r,s}(2g; f, a)(2b)^r \Delta^J_u|_{u=0}(u^\ell) \mu^\ell \partial^{r+\ell+2s}_a|_{\alpha = a^{b/\sqrt{X}} \frac{1}{1 + e^\alpha}} \leq (1 + b^{(2r-2J-1)} \frac{1}{1 + e^\alpha}).
\]

(4.6)

by use Theorem 1.3, Lemma 4.1 and Lemma 2.5.

From Theorem 1.3, above (4.1)–(4.5) and estimate (4.6), we shall denote by

\[
\mathcal{M}_{f,J}(a, b, \mu, X, \partial_\alpha)
\]

\[
= J! \mu^J \mathcal{C}_{0,J}(2J; f, a) \partial^J_a + \frac{J! \mu^J}{\sqrt{X}} \left( \mathcal{C}_{0,J}(2J+2; f, a) \partial^J_a
\]

\[
+ \mathcal{C}_{0,J+1}(2J+2; f, a) \partial^{J+2}_a + \frac{J(J+1)}{2} \mu \mathcal{C}_{0,J+1,0}(2J+2; f, a) \partial^{J+1}_a
\]

\[
+ J! \mu^J \mathcal{C}_{1,J}(2J+4; f, a) \partial^{J+1}_a + J! \mu^J \mathcal{C}_{2,J}(2J+6; f, a) \partial^{J+2}_a
\]

\[
+ \frac{J(J+1)!}{2} \mu^{J+1} \mathcal{C}_{0,J+1,0}(2J+2; f, a) \partial^{J+1}_a
\]

\[
+ J! \mu^J \mathcal{C}_{1,J}(2J+4; f, a) \partial^{J+1}_a + J! \mu^J \mathcal{C}_{2,J}(2J+6; f, a) \partial^{J+2}_a.
\]

\[
\]
Further, from (3.4)–(3.8) we find that
\[
\mathcal{M}_{f,J}(a,b,\mu,X,\partial_\alpha) = \left(1 - \frac{J(4a_\alpha + J - 1)}{2a^2\beta f \sqrt{X}}\right) \partial_\alpha^j - \frac{a\beta f + \beta f X^{-1/2}b^2}{2\sqrt{X}} \partial_\alpha^{j+2}
\]
\[(4.7)\]
By above argument and Theorem 1.3 one has the following estimates:
\[
\frac{\Delta^j}{u=0} S_f(a, b + \mu u; X) f(X) = X^{-j/2} \mathcal{M}_{f,J}(a,b,\mu,X,\partial_\alpha) \bigg|_{\alpha = \frac{ab\beta f}{\sqrt{X}}} \left\{ \frac{1}{1 + e^\alpha} \right\}
\]
\[
\ll X^{-(J+2)/2} e^{-\frac{ab\beta f}{\sqrt{X}}} + bX^{-J/2} e^{-\frac{ab\beta f}{\sqrt{X}}}
\]
\[
+ \sum_{g > J+3} X^{-g/2}(1 + b^{2x+2j}) e^{-\frac{ab\beta f}{\sqrt{X}}}
\]
\[
\ll X^{-J/2-1} \left(1 + (X^{-1/2}b)^4 \right) e^{-\frac{ab\beta f}{\sqrt{X}}}.
\]
We conclude above as the following proposition.

**Proposition 4.2.** Let \(J \in \mathbb{Z}_{\geq 0}\) and let the deferential operator \(\mathcal{M}_{f,J}(a,b,\mu,X,\partial_\alpha)\) defined as (4.7). We have if \(0 \leq b = o(X^{3/4})\) and \(\mu \neq 0\) then as \(X \to +\infty\),
\[
X^{j/2} \frac{\Delta^j}{u=0} S_f(a, b + \mu u; X) = \mathcal{M}_{f,J}(a,b,\mu,X,\partial_\alpha) \bigg|_{\alpha = \frac{ab\beta f}{\sqrt{X}}} \left\{ \frac{1}{1 + e^\alpha} \right\} + O \left( X^{-1} e^{-\frac{ab\beta f}{\sqrt{X}}} (1 + O(X^{-1/2}b)^2) \right).
\]
To complete the proof of Corollary 1.5 we need the following estimates.

**Lemma 4.3.** Let \(z > -1\) and \(\epsilon = o(1)\), we have
\[
(\partial_\alpha^j + \epsilon \partial_\alpha^{j+1}) \bigg|_{\alpha = z} \left\{ \frac{1}{1 + e^\alpha} \right\} = (1 + O(\epsilon^2)) \bigg|_{\alpha = z + \epsilon} \left\{ \frac{1}{1 + e^\alpha} \right\} + O(\epsilon^3 e^{-z}).
\]

**Proof.** This lemma is easy that follows from Taylor mean value theorem, hence we omit its proof. \(\square\)

Now, for \(J \in \mathbb{Z}_{\geq 0}\), using Lemma 4.3 and Lemma 2.5 in Proposition 4.2 yields
\[
\frac{\Delta^j}{u=0} S_f(a, b + \mu u; X) f(X) = \\
= \left(-\frac{mu\beta f}{\sqrt{X}}\right)^j \left(1 + O \left( \frac{1}{X} \right) \right) \bigg|_{\alpha = \frac{a(2b - \mu)\beta f}{2\sqrt{X}}} \left\{ \frac{1}{1 + e^\alpha} \right\} + \\
O \left( \frac{1}{X^{j/2 + j/2} e^{\frac{a(2b - \mu)\beta f}{2\sqrt{X}}} + \frac{X^{-1/2}b^2}{X^{j+1/2}} \bigg|_{\alpha = \frac{2b\beta f}{\sqrt{X}}} \left\{ \frac{1}{1 + e^\alpha} \right\} \right) \right)
\]
\[
+ O \left( \frac{b + 1 + (X^{-1/2}b)^4}{X^{j+1/2}} e^{-\frac{a\beta f}{\sqrt{X}}} \right).
\]
By using Lemma 4.3 and Lemma 2.5 again in above,
\[
\Delta'_u|_{u=0} S_f(a, b + \mu u; X) / f(X) \\
= \left( \frac{-\mu \alpha \beta_f}{\sqrt{X}} \right)^J \left( 1 + O \left( \frac{1 + X^{-1} b^2}{\sqrt{X}} \right) \right) \partial^J_{\alpha} \Big|_{\alpha = \frac{a(2b - \mu J) \beta_f}{2 \sqrt{X}}} \left\{ \frac{1}{1 + e^\alpha} \right\} \\
+ O \left( \frac{1 + b + (X^{-1/2} b)^J}{X^{1+J/2}} \right) e^{-\frac{ab \beta_f}{2 \sqrt{X}}}.
\]  
(4.8)

If 0 ≤ b = o(X^{1/2}), using (2.11) in (4.8) we have
\[
X^{J/2} \Delta'_u|_{u=0} S_f(a, b + \mu u; X) / (-\mu \alpha \beta_f)^J f(X) \\
= \left( 1 + O \left( \frac{1}{\sqrt{X}} \right) \right) \partial^J_{\alpha} \Big|_{\alpha = \frac{a(2b - \mu J) \beta_f}{2 \sqrt{X}}} \left\{ \frac{1}{1 + e^\alpha} \right\} + O \left( \frac{1 + b}{X} \right) \\
= O \left( \frac{1 + b}{X} \right) + \left( 1 + O \left( \frac{1 + |2b - \mu J|}{\sqrt{X}} \right) \right) \\
\times \left( \frac{1 - \frac{2}{J + 1} B_{J+1} + \frac{1 - \frac{2}{J + 2} \frac{a(2b - \mu J) \beta_f}{2 \sqrt{X}}}{J + 2 \frac{a(2b - \mu J) \beta_f}{2 \sqrt{X}}} \right).
\]

We conclude above as the following corollary, which gives the leading asymptotics for finite difference of S(a, b; X) with respect to b when b = o(X^{1/2}).

**Corollary 4.4.** If (J + 1)/2 ∈ ℤ_+ ∪ {1/2} and 0 ≤ b = o(X^{1/2}) then
\[
X^{J/2} \Delta'_u|_{u=0} S_f(a, b + \mu u; X) / (-\mu \alpha \beta_f)^J f(X) = \frac{1 - \frac{2}{J + 1} B_{J+1} + O \left( \frac{1 + |2b - \mu J|}{\sqrt{X}} \right)}{J + 1}.
\]

If J/2 ∈ ℤ_+ and 1 ≤ |2b - \mu J| = o(X^{1/2}) then
\[
X^{J/2} \Delta'_u|_{u=0} S_f(a, b + \mu u; X) / (-\mu \alpha \beta_f)^J f(X) = \frac{1 - \frac{2}{J + 2} \frac{a(2b - \mu J) \beta_f}{2 \sqrt{X}}}{J + 2} + O \left( \frac{1 + b + |2b - \mu J|^2}{X} \right).
\]

We note that if for some J ∈ ℤ_+ and μ, one has
\[
e^\alpha \partial^J_{\alpha} \left\{ \frac{1}{1 + e^\alpha} \right\} \neq 0,
\]  
for all α ≥ -o(1), then by (4.8), it is clear that
\[
\Delta'_u|_{u=0} S_f(a, b + \mu u; X) / f(X) \\
= \left( \frac{-\mu \alpha \beta_f}{\sqrt{X}} \right)^J \left( 1 + O \left( \frac{1 + X^{-1} b^2}{\sqrt{X}} \right) \right) \partial^J_{\alpha} \Big|_{\alpha = \frac{a(2b - \mu J) \beta_f}{2 \sqrt{X}}} \left\{ \frac{1}{1 + e^\alpha} \right\} \\
+ O \left( \frac{1 + X^{-1} b^3}{X^{(1+J)/2}} \right) e^{-\frac{a(2b - \mu J) \beta_f}{2 \sqrt{X}}}.
\]  
(4.9)
holds for all \( b \geq 0 \) such that \( 1 \ll 2b - \mu J = o(X^{3/4}) \). Further, since (4.9) holds for all \( J \) such that \( (J + 1)/2 \in \mathbb{Z}_+ \cup \{1/2\} \) and \( \alpha = o(1) \), thus by (4.8),

\[
\frac{\Delta'_u}{u=0} S_f(a, b + \mu u; X) f(X) = \left( -\frac{\mu a \beta f}{\sqrt{X}} \right)^J \left( 1 + O \left( \frac{1 + X^{-1}b^2}{\sqrt{X}} \right) \right) \frac{\partial^J}{\partial a} \left|_{a = \frac{n^{(2b - \mu J)/\beta}}{n^{1/2}}} \right\{ \frac{1}{1 + e^\alpha} \right\},
\]

holds for all \( b = o(X^{1/2}) \) with \( (J + 1)/2 \in \mathbb{Z}_+ \cup \{1/2\} \).

Now, if \( J \in \{0, 1\} \) then it is clear that

\[
e^\alpha \frac{\partial^J}{\partial a} \left|_{a = \frac{n^{(2b - \mu J)/\beta}}{n^{1/2}}} \right\{ \frac{1}{1 + e^\alpha} \right\} \neq 0
\]

for all \( b \geq 0 \), and for \( J = 2 \),

\[
e^\alpha \frac{\partial^J}{\partial a} \left|_{a = \frac{n^{(2b - \mu J)/\beta}}{n^{1/2}}} \right\{ \frac{1}{1 + e^\alpha} \right\} \neq 0
\]

for all \( b \geq 0 \) such that \( 2b - \mu J = 2(b - \mu) \gg 1 \). In fact,

\[
\frac{\partial^0}{\partial a} \left\{ \frac{1}{1 + e^\alpha} \right\} = \frac{1}{1 + e^\alpha} = \frac{1}{2} \left( 1 - \tanh \left( \frac{\alpha}{2} \right) \right),
\]

\[
\frac{\partial}{\partial a} \left\{ \frac{1}{1 + e^\alpha} \right\} = -\frac{e^\alpha}{(1 + e^\alpha)^2} = -\frac{1}{4} \text{sech}^2 \left( \frac{\alpha}{2} \right)
\]

and

\[
\frac{\partial^2}{\partial a^2} \left\{ \frac{1}{1 + e^\alpha} \right\} = \frac{e^\alpha(e^\alpha - 1)}{(1 + e^\alpha)^3} = \frac{1}{4} \frac{\sinh \left( \frac{\alpha}{2} \right)}{\cosh^3 \left( \frac{\alpha}{2} \right)}.
\]

Which completes the proof of Corollary 1.5.

4.2. Uniform asymptotic formulas for \( M_k(m, n) \) and \( N_k(m, n) \).

To prove Corollary 1.6 and Corollary 1.7, we need the following asymptotic result for \( p_k(n) \).

**Proposition 4.5.** Let \( k \in \mathbb{Z}_+ \). We have asymptotic expansion of form

\[
p_k(n) \sim n^{-\frac{k}{4}} e^{\frac{bk}{\sqrt{n}}} \sum_{\ell \geq 0} \gamma_\ell(k)n^{-\ell/2}, \quad n \to \infty
\]

for some \( \gamma_\ell(k) \in \mathbb{R} \) with \( \gamma_0(k) > 0 \).

**Proof.** Let \( n \) be sufficiently large. By Hardy and Ramanujan [22] (see also Rademacher and Zuckerman [23]) we have

\[
p_k(n) = \frac{2\pi}{(n - k/24)^{1/2} \sqrt{2\pi k}} \left( \frac{k}{24} \right)^{1/2} I_{1+k/2} \left( b_k \sqrt{n - k/24} \right) + O \left( e^{b_k(1-1/(2k)) \sqrt{n}} \right),
\]

where \( b_k = 2\sqrt{k/6} \) and \( I_{1+k/2} \) is the \((1 + k/2)\)-th Bessel’s function. Thus by the asymptotic expansion of Bessel function \( I_\nu(x) \) (see for example [24]):

\[
I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \sum_{\ell \geq 0} (-1)\ell \frac{\partial^\ell (\nu)}{\ell!}, \quad x \to \infty,
\]
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with \( a_0(\nu) = 1 \) and

\[
a_\ell(\nu) = \frac{1}{8\ell!} \prod_{1 \leq r \leq \ell} (4\nu^2 - (2r - 1)^2)
\]

for \( \ell \geq 1 \), we obtain the proof of Proposition 4.5. \( \square \)

Now, by Proposition 1.2 and the fact that \( S_k(a, b, n) = S_{p_k}(a, b/(2a), n) \), it is easy to check that

\[
M_k(m, n) = \Delta_u \big|_{u=0} S_{p_k} \left( \frac{1}{2}, m + \frac{1}{2} - u, n \right),
\]

\[
N_k(m, n) = \Delta_u \big|_{u=0} S_{p_1} \left( k - \frac{1}{2}, \frac{m + \frac{3}{2}}{2k - 1} - \frac{u}{2k - 1}, n \right),
\]

\[
M_k(m, n) - M_k(m + 1, n) = \Delta_u \big|_{u=0} S_{p_k} \left( \frac{1}{2}, m + \frac{3}{2} - u, n \right),
\]

and

\[
N_k(m, n) - N_k(m + 1, n) = \Delta_u \big|_{u=0} S_{p_1} \left( k - \frac{1}{2}, \frac{m + \frac{3}{2}}{2k - 1} - \frac{u}{2k - 1}, n \right).
\]

Then, it is clear that Corollary 1.6 and Corollary 1.7 for nonnegative \( m \) follows from above relations, Proposition 4.5 and Corollary 1.5. The cases of \( m \) is a negative integer for Corollary 1.6 follows from the fact that \( M_k(m, n) = M_k(|m|, n) \) and \( N_k(m, n) = N_k(|m, n|) \).

REFERENCES

[1] Martin Eichler and Don Zagier. The theory of Jacobi forms, volume 55 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1985.

[2] F. J. Dyson. Some guesses in the theory of partitions. Eureka, (8):10–15, 1944.

[3] George E. Andrews and F. G. Garvan. Dyson’s crank of a partition. Bull. Amer. Math. Soc. (N.S.), 18(2):167–171, 1988.

[4] F. G. Garvan. New combinatorial interpretations of Ramanujan’s partition congruences mod 5, 7 and 11. Trans. Amer. Math. Soc., 305(1):47–77, 1988.

[5] Paul Hammond and Richard Lewis. Congruences in ordered pairs of partitions. Int. J. Math. Math. Sci., (45-48):2509–2512, 2004.

[6] Lothar Götsche. The Betti numbers of the Hilbert scheme of points on a smooth projective surface. Math. Ann., 286(1-3):193–207, 1990.

[7] Kathrin Bringmann and Jan Manschot. Asymptotic formulas for coefficients of inverse theta functions. Commun. Number Theory Phys., 7(3):497–513, 2013.

[8] Freeman J. Dyson. Mappings and symmetries of partitions. J. Combin. Theory Ser. A, 51(2):169–180, 1989.

[9] Kathrin Bringmann and Jehanne Dousse. On Dyson’s crank conjecture and the uniform asymptotic behavior of certain inverse theta functions. Trans. Amer. Math. Soc., 368(5):3141–3155, 2016.

[10] Frank G. Garvan. Generalizations of Dyson’s rank and non-Rogers-Ramanujan partitions. Manuscripta Math., 84(3-4):343–359, 1994.

[11] Jehanne Dousse and Michael H. Mertens. Asymptotic formulae for partition ranks. Acta Arith., 168(1):83–100, 2015.

[12] Byungchan Kim, Eunmi Kim, and Jeehyeon Seo. Asymptotics for q-expansions involving partial theta functions. Discrete Math., 338(2):180–189, 2015.

[13] Renrong Mao. Asymptotic inequalities for k-ranks and their cumulation functions. J. Math. Anal. Appl., 409(2):729–741, 2014.

[14] Daniel Parry and Robert C. Rhoades. On Dyson’s crank distribution conjecture and its generalizations. Proc. Amer. Math. Soc., 145(1):101–108, 2017.
[15] Nian Hong Zhou. On the distribution of rank and crank statistic for integer partitions. 
arXiv:1803.11081.
[16] Song Heng Chan and Renrong Mao. Inequalities for ranks of partitions and the first moment 
of ranks and cranks of partitions. Adv. Math., 258:414–437, 2014.
[17] J. G. van der Corput. Asymptotic expansions. I. Fundamental theorems of asymptotics. 
Department of mathematics, University of California, Berkeley, Calif., 1954.
[18] J. G. van der Corput. Asymptotic developments. I. Fundamental theorems of asymptotics. 
J. Analyse Math., 4:341–418, 1955/56.
[19] Kathrin Bringmann, Amanda Folsom, and Antun Milas. Asymptotic behavior of partial and 
false theta functions arising from Jacobi forms and regularized characters. J. Math. Phys., 
58(1):011702, 19, 2017.
[20] Bruce C. Berndt and Byungchan Kim. Asymptotic expansions of certain partial theta func-
tions. Proc. Amer. Math. Soc., 139(11):3779–3788, 2011.
[21] Renrong Mao. Some new asymptotic expansions of certain partial theta functions. Ramanujan 
J., 34(3):443–448, 2014.
[22] G. H. Hardy and S. Ramanujan. Asymptotic Formulae in Combinatory Analysis. Proc. 
London Math. Soc. (2), 17:75–115, 1918.
[23] Hans Rademacher and Herbert S. Zuckerman. On the Fourier coefficients of certain modular 
forms of positive dimension. Ann. of Math. (2), 39(2):433–462, 1938.
[24] E. T. Whittaker and G. N. Watson. A course of modern analysis. An introduction to the 
general theory of infinite processes and of analytic functions: with an account of the principal 
transcendental functions. Fourth edition. Reprinted. Cambridge University Press, New York, 
1962.