Oriented cobicircular matroids are $GSP^*$

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Abstract

Colouring and flows are well-known dual notions in Graph Theory. In turn, the definition of flows in graphs naturally extends to flows in oriented matroids. So, the colour-flow duality gives a generalization of Hadwiger’s conjecture about graph colourings, to a conjecture about cflows of oriented matroids. The first non-trivial case of Hadwiger’s conjecture for oriented matroids reads as follows. If $O$ is an $M(K_4)$-minor free oriented matroid, then $O$ has a nowhere 3-cflow, i.e., it is 3-colourable in the sense of Hochstättler-Nešetřil. The class of generalized series parallel ($GSP$) oriented matroids is a class of 3-colourable oriented matroids with no $M(K_4)$-minor. So far, the only technique towards proving that all orientations of a class $C$ of $M(K_4)$-minor free matroids are $GSP$ (and thus 3-colourable), has been to show that every matroid in $C$ has a positive coline. Towards proving Hadwiger’s conjecture for the class of gammoids, Goddyn, Hochstättler, and Neudauer conjectured that every gammoid has a positive coline. In this work we disprove this conjecture by exhibiting an infinite class of strict gammoids that do not have positive colines. We conclude by proposing a simpler technique for showing that certain oriented matroids are $GSP$. In particular, we recover that oriented lattice path matroids are $GSP$, and we show that oriented cobicircular matroids are $GSP$.

Keywords: Flows, Colourings, Matroids, Oriented matroids, Bicircular matroids

1 Introduction

Hadwiger’s Conjecture is a well-known and long open conjecture regarding proper graph colourings. It states that for every positive integer $k$ if a graph $G$ contains no $K_{k+1}$-minor, then $G$ is $k$-colourable. This conjecture has been proven true for $k \leq 5$ \cite{13}, and remains open for larger integers.

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The notion of proper graph colourings is the dual concept of nowhere-zero flows (NZ flows). The latter, has a natural generalization to oriented matroids, which Hochstädtler and Nešetřil \cite{8} use to propose a definition of the chromatic number of an oriented matroid. It turns out that Hadwiger’s conjecture can be generalized to this context; but in this scenario, the first non-trivial case remains open and it reads as follows.

**Conjecture 1.** Every (loopless) $M(K_4)$-minor free oriented matroid has a nowhere-zero 3-coflow.

Goddyn and Hochstädtler \cite{6} observed that Hadwiger’s conjecture for regular oriented matroids, includes the cases $k = 4$ and $k = 5$ of Tutte’s $k$-flow conjecture \cite{15, 16, 17}, which remain open.

Towards proving Conjecture \cite{11} Goddyn, Hochstädtler and Neudauer, introduce the class of Generalized Series Parallel ($GSP$) oriented matroids and show that every $GSP$ oriented matroid has a NZ 3-coflow \cite{7}. It might be too much to hope for, but if every $M(K_4)$-free oriented matroid is $GSP$, then Conjecture \cite{11} follows directly. In any case, this raises the fundamental problem of determining when a class $\mathcal{C}$ of oriented matroids is a subclass of $GSP$ oriented matroids. To this end and in the same work, the previously mentioned authors show that if $\mathcal{C}'$ is a class of orientable matroids closed under minors such that every member of $\mathcal{C}'$ has a positive coline, then the class $\mathcal{C}$ of all orientations of matroids in $\mathcal{C}'$ is a class of $GSP$ oriented matroids. Finally, they show that every bicircular matroid has a positive coline, so, if $\mathcal{O}$ is an oriented bicircular matroid, then $\mathcal{O}$ is $GSP$ and thus it has a NZ 3-coflow.

Bicircular matroids are transversal matroids, and the smallest class closed under minors that contains transversal matroids is the class of gammoids. In turn, the class gammoids is a class of $M(K_4)$-free orientable matroids, so Goddyn, Hochstädtler and Neudauer pose the following conjecture.

**Conjecture 2.** \cite{7} Every simple gammoid of rank at least two has a positive coline.

In this work, we disprove this conjecture by exhibiting a large class of cobicircular matroids that do not have positive colines; but we show that nonetheless, every orientation of a cobicircular matroid is $GSP$.

The rest of this work is organized as follows. In Section \ref{section2} we introduce all concepts needed to state Conjecture \ref{conjecture2}. In Section \ref{section3} we introduce bicircular matroids and prove the necessary results to exhibit a class of counterexamples to the previously mentioned conjecture. In Section \ref{section4} we prove that all orientations of cobicircular matroids are $GSP$. Finally, in Section \ref{section5} we conclude this work by posing some further questions and problems that arose from this work.

### 2 Preliminaries

We assume basic familiarity with matroids and with oriented matroids, standard references are \cite{2} and \cite{12}.
2.1 GSP oriented matroids

Consider an oriented matroid $O$ with ground set $E$ and collection of signed circuits $C$. The signed vector of a signed circuit $C = (C^+, C^-)$ is the characteristic vector of $C$ in $\{0, 1, -1\}^E$. In other words, $C(e) = 1$ if $e \in C^+$; $C(e) = -1$ if $e \in C^-$; and $C(e) = 0$ if $e \notin C$. The flow lattice of $O$, denote by $\mathcal{F}_O$ is the integer lattice generated by the signed vectors of the circuits of $O$. In symbols,

$$\mathcal{F}_O = \left\{ \sum_{C \in C} \lambda_C C \mid \lambda_C \in \mathbb{Z} \right\}.$$

We call any element $x \in \mathcal{F}_O$ a flow of $O$. The flow lattice of uniform matroids is characterized in [9]. In particular, the following statements hold.

**Lemma 3.** Let $O$ be an orientation of a rank $r$ uniform matroid of at least $r + 2$ elements. For any pair of elements $e, f \in E$, the following statements hold:

1. if $r$ is even, then there is a flow $x \in \mathcal{F}_O$ such that $x(e) = 1$ and $x(e') = 0$ for every $e' \neq e$, and
2. if $r$ is odd, then there is a flow $x \in \mathcal{F}_O$ such that $|x(e)| = |x(f)| = 1$ and $x(e') = 0$ for every $e' \notin \{e, f\}$.

**Proof.** The descriptions of flow lattice used in this proof are taken from [9]. With out loss of generality suppose that $E = [n]$. The first statement holds because the flow lattice of a uniform matroid of even rank $r$, $r \leq n - 2$, is $\mathbb{Z}^n$. Regarding the flow lattice of oriented uniform matroids of odd rank, there are two possibilities. The first option, is that there is a vector $v \in \{1, -1\}^n$ such that $\mathcal{F}_O$ is the orthogonal space $\left\{v\right\}^\perp$ intersected with $\mathbb{Z}^n$. In this case, for any $i, j \in [n]$ with $i \neq j$, consider the vector $x$ defined as follows: $x(k) = 0$ for any $k \notin \{i, j\}$; $x(i) = v(i)$ and $x(j) = -v(j)$. Clearly, $x \in \left\{v\right\}^\perp \cap \mathbb{Z}^n$, and so, this case is settled.

The other option, is that the flow lattice of an oriented uniform matroid of odd rank is characterized as follows. Let $v_1$ be the all 1 vector in $\{0, 1, -1\}^n$. The flow lattice $\mathcal{F}_O$ is the set of points $x \in \{0, 1, -1\}^n$ such that the product $x^Tv_1$ is even. In this case, for any $i, j \in [n]$ with $i \neq j$, consider the vector $x$ where $x(i) = x(j) = 1$, and $x(k) = 0$ for every $k \notin \{i, j\}$. Clearly, $x \in \mathcal{F}_O$ and thus, the claim holds in both cases. \qed

Analogously, the coflow lattice of an oriented matroid $O$ with signed cocircuits $D$, is the integer lattice generated by the signed vectors of the cocircuits of $O$. Clearly, the coflow lattice of $O$ is the flow lattice of the dual $O^*$, and so, we denote by $\mathcal{F}_{O^*}$ the coflow lattice of $O$. An element $x \in \mathcal{F}_{O^*}$ is a coflow of $O$. Such an element is a nowhere-zero-$k$ coflow if $0 < |x(e)| < k$ for every $e \in E$.

Having introduced all this nomenclature, we restate the first non-trivial open case of Hadwiger’s conjecture for oriented matroids (and its dual statement in terms of flows, that follows because $M(K_4)$ is self-dual).

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1 This case occurs if and only if $O$ is a reorientation of a so-called neighbourly oriented matroid
**Conjecture 1.** Every (loopless) $M(K_4)$-minor free oriented matroid has a nowhere-zero 3-coflow. Equivalently, every (coloop free) $M(K_4)$-minor free oriented matroid has a nowhere-zero 3-flow.

This conjecture restricted to graphic matroids, is equivalent to the statement: every graph with no $K_4$ minor is 3-colourable. This equivalent statement can be easily proven, even in the case of regular oriented matroids, since the class of $M(K_4)$-minor free regular oriented matroids correspond to series parallel graphs. Thus, with a simple inductive argument one can show that every graph with no $K_4$-minor is 3-colourable.

In an attempt to mimic the previous technique for oriented matroid and NZ-3 coflows, Goddyn, Hochstättler, and Neudauer introduce the class of generalized series parallel (GSP) oriented matroids. An oriented matroid $O$ is GSP if every simple minor of $O$ has a \{0, 1, -1\}-coflow with at most two non-zero entries. In the same work, the authors show that every GSP oriented matroid has a NZ 3-coflow.

**Proposition 4.** [7] Let $O$ be a GSP oriented matroid. If $O$ has no loops, then $O$ has a NZ-3-coflow.

Not much is known about GSP oriented matroids. In particular, it is not known whether GSP oriented matroids are closed under duality. For this reason, we introduce the dual class of generalized series parallel oriented matroids. An oriented matroid $O$ is coGSP if every cosimple minor of $O$ has a \{0, 1, -1\}-flow with at most two non-zero entries.

### 2.2 Positive colines

Let $M$ be a matroid. A **copoint** of $M$ is a hyperplane, that is, a flat of codimension 1. A **coline** of $M$ is a flat of codimension 2. If a coline $L$ is contained in a copoint $H$, we say that $H$ is a copoint on $L$. It is not hard to notice that if $L$ is a coline of a matroid $M$ then there is a partition $(H_1, \ldots, H_k)$ of $E(M) \setminus L$ such that the copoints on $L$ are $L \cup H_i$ for $i \in \{1, \ldots, k\}$. Furthermore, this partition is unique (up to permutations) so we call it the **copoint partition** of $L$, and define the degree of $L$ to be $k$. A class $H_i$ is **singular** if $|H_i| = 1$; otherwise it is a **multiple** class. A coline $L$ is **positive** if there are more singular than multiple classes in its copoint partition. It turns out that positive colines and GSP oriented matroids are related as follows.

**Proposition [7].** Let $C$ be minor closed class of orientable matroids. If every simple matroid in $C$ has a positive coline, then every orientation of a matroid in $C$ is GSP.

Using this proposition, Goddyn, Hochstättler, and Neudauer [7] show that every orientation of a bicircular matroid is GSP. Every bicircular matroid is a transversal matroid, and the class of gammoids is the smallest dually and minor closed class that contains transversal matroids [10]. These facts motivate Goddyn, Hochstättler, and Neudauer to conjecture that every orientation of a gammoid is GSP. Furthermore, they conjecture that the following statement is true.

**Conjecture 2.** [7] Every simple gammoid of rank at least two has a positive coline.
2.3 Double circuits

Circuits are the dual complements of hyperplanes. That is, if $H$ is a hyperplane of a matroid $M$ then $E(M) \setminus H$ is a circuit of $M^*$. A double circuit of a matroid $M$ is a set $D$ such that $r(D) = |D| - 2$ and for every element $d \in D$ the rank of $D - d$ does not decrease, i.e. $r(D - d) = |D| - 2 = r(D)$. Dress and Lovász [5] show that if $D$ is a double circuit then $D$ has a partition $(D_1, \ldots, D_k)$ such that the circuits of $D$ are $D \setminus D_i$ for $i \in \{1, \ldots, k\}$. We call this partition the circuit partition of $D$ and say that the degree of $D$ is $k$.

**Observation 5.** Let $M$ be a matroid and $D \subseteq E(M)$ a double circuit of $M$. If $D$ is a degree $k$ double circuit, then $M[D]$ is a series extension of $U_{k-2,k}$.

*Proof.* Without loss of generality suppose that $D = E(M)$, and let $(D_1, \ldots, D_k)$ be the circuit partition of $D$. If $|D_i| = 1$ for every $i \in \{1, \ldots, k\}$, then $M \cong U_{k-2,k}$. Now, the claim follows by a straightforward induction over the difference $|D| - k$. 

Similar to how circuits are the dual complements of copoints, double circuits are the complements of colines. Moreover, the copoint partitions and circuit partitions relate as follows.

**Observation 6.** Let $M$ be a matroid, $L \subseteq E(M)$ and $(H_1, \ldots, H_k)$ a partition of $E(M) \setminus L$. Then, $L$ is a coline of $M$ with copoint partition $(H_1, \ldots, H_k)$ if and only if $E(M) \setminus L$ is a double circuit of $M^*$ with circuit partition $(H_1, \ldots, H_k)$.

A positive double circuit $D$ is a double circuit with more singular than multiple classes in its circuit partition. By Observation 5, a matroid $M$ has a positive double circuit if and only if $M^*$ has a positive coline. Since gammoids are closed under duality, the following conjecture is equivalent to Conjecture 2.

**Conjecture 7.** [7] Every cosimple gammoid of corank at least two has a positive double circuit.

3 Bicircular matroids and double circuits

Every bicircular matroid is a transversal matroid [11], and so, every bicircular matroid is a gammoid. In this section, we disprove Conjecture 2 by showing that its dual statement, Conjecture 7, does not hold for bicircular matroids. To do so, we begin by briefly introducing the class of bicircular matroids.

A standard reference for graph theory is [3]. In particular, given a graph $G$ and a subset of edges $I$ we denote by $G[I]$ the subgraph of $G$ induced by $I$. That is, $G[I]$ is the subgraph of $G$ with edge set $I$ and no isolated vertices.

Let $G$ be a (not necessarily simple) graph with vertex set $V$ and edge set $E$. The bicircular matroid of $G$ is the matroid $B(G)$ with base set $E$ whose independent sets are the edge sets $I \subseteq E$ such that $G[I]$ contains at most one cycle in every connected component. Equivalently, the circuits of $B(G)$ are the edge sets of subgraphs which are subdivisions of one of the graphs: two loops on the same vertex, two loops joined by an edge, or three parallel edges joining a pair of vertices.

Mathews [11] noticed that there are only a few uniform bicircular matroids.
Theorem 8. [14] The uniform bicircular matroids are precisely the following:

- $U_{1,n}$, $U_{2,n}$, $U_{n,n}$, $(n \geq 0)$;
- $U_{n-1,n}$ $(n \geq 1)$;
- $U_{3,5}$, $U_{3,6}$ and $U_{4,6}$.

Recall that if a matroid $M$ has a double circuit of degree $k$ then $M$ contains a $U_{k-2,k}$ minor (Observation [5]).

Corollary 9. The degree of a double circuit in a bicircular matroid is at most 6.

Suppose that a graph $G$ is obtained by subdividing edges of a graph $H_G$ with minimum degree 3. It is not hard to notice that $H_G$ is unique up to isomorphism. The subdivision classes of $G$ are the sets of edges that correspond to a series of subdivisions of an edge in $H_G$. An unsubdivided edge of $G$ is an edge of $G$ that is an edge of $H_G$. Clearly, if $G$ has no leaves an edge $xy$ is an unsubdivided edge of $G$ if and only if $d_G(x), d_G(y) \geq 3$.

Lemma 10. Let $G$ be a graph and $D \subseteq E$ a double circuit of $B(G)$. Then $G[D]$ has no leaves and contains at most 4 vertices $x_1, x_2, x_3$ and $x_4$ of degree greater than or equal to 3. Moreover, every subdivision class of $G[D]$ belongs to the same class of the circuit partition of $D$.

Proof. Since every element of $D$ belongs to a circuit of $D$, then $D$ has no coloops so $G[D]$ has no leaves. Let $V'$ be the vertex set of $G[D]$ and $r'$ the rank of $D$, so $r' = |V'|$. Since $D$ has no leaves then $d(e) \geq 2$ for every $e \in V'$. Let $t$ be the number of vertices in $V'$ with degree at least 3 in $G[D]$. By the handshaking lemma, $2|D| \geq 3t + 2(r' - t)$. Since $D$ is a double circuit then $r' = |D| - 2$, and thus $2(r' + 2) \geq 3t + 2(r' - t)$, so $t \leq 4$. Finally, let $S$ be a subdivision class of $G[D]$. Then $S$ is the edge set of a path $P$ path such that the internal vertices of $P$ have degree 2 in $G[D]$. Thus, every cycle in $G[D]$ that contains an edge of $P$ contains all edges of $P$. Hence, every circuit of $B(G)$ in $D$ that contains an edge in $P$ contains all of them, so $E(P)$ is contained in some circuit class of $D$.

Given a double circuit $D$ of a bicircular matroid $B(G)$, a distinguished vertex of $D$ is a vertex of degree at least 3 in $G[D]$. Since $G[D]$ has no leaves, the subdivision classes of $G[D]$ correspond to paths that contain distinguished vertices (only) as endpoints. In particular, unsubdivided edges of $G[D]$ are edges incident in distinguished vertices.

Theorem 11. Let $G$ be a graph. If $\text{girth}(G) \geq 5$ then $B(G)$ has no positive double circuits.

Proof. We proceed by contrapositive. Suppose that $D$ is a positive double circuit in $B(G)$ of degree $k$ with circuit partition $(D_1, \ldots, D_k)$ where $k \leq 6$ by Corollary 9. If $k \in \{1, 2, 3, 4\}$ with out loss of generality assume that $(D_1, \ldots, D_{k-1})$ are singular circuit classes of $D$. Since the union $D'$ of these classes is $D \setminus D_k$, then this union is a circuit of $D$. Thus, $G[D']$ contains two cycles of $G$, so $\text{girth}(G) \leq |D'| \leq 3$. 

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Now suppose that $k \in \{5, 6\}$ and $(D_1, \ldots, D_k)$ has at least 4 simple classes $\{e_1\}, \ldots, \{e_4\}$. By the moreover statement of Lemma 10 the edges $e_1, e_2, e_3$ and $e_4$ must be unsubdivided edges of $G[D]$. Thus, the endpoints of these edges are distinguished vertices of $G[D]$, which by the same lemma there are at most 4 of these vertices. Putting all of this together we conclude that $D'$ is a set of four edges such that $G[D']$ has at most 4 vertices. Therefore, $G[D']$ contains at least one cycle of $G$, and so $\text{girth}(G) \leq |D'| \leq 4$.

The only remaining case is when the degree of $D$ is 5 and it has 3 singular classes. In this case there are 3 unsubdivided edges $e_1, e_2$, and $e_3$ of $G[D]$. We claim that $\{e_1, e_2, e_3\}$ contains a cycle of $G$, and thus $\text{girth}(G) \leq 3$. Anticipating a contradiction, suppose that $\{e_1, e_2, e_3\}$ does not contain a cycle of $G$. This implies that $D$ has four distinguished $x_1, x_2, x_3$, and $x_4$. Notice that if we contract a subdivided edge $e$ of $G[D]$ we obtain a double circuit $D'$ of $B(G)/e$ with the same degree as $D$. Inductively, we end up with a double circuit $D_0$ and a graph $H$ with edge set $D_0$ such that $\{e_1, e_2, e_3\} \subseteq D_0$ and $V(H) = \{x_1, x_2, x_3, x_4\}$. Moreover, $\{e_1, e_2, e_3\}$ spans a tree of $H$. On the other hand, each edge $e_i$ for $i \in \{1, 2, 3\}$ belongs to a singular class of the circuit partition of $D_0$. Since the rank of $D_0$ is 4 then $D_0$ contains at most six edges. Also, the circuit partition of $D_0$ has 5 classes, so there must be an edge $e_4 \in D_0 \setminus \{e_1, e_2, e_3\}$ that belongs to a singular class. Notice that that $\{e_1, e_2, e_3, e_4\}$ contains at most one cycle of $H$ since $\{e_1, e_2, e_3\}$ spans a tree of $H$. On the other hand, $D_0 \setminus \{e_1, e_2, e_3, e_4\}$ is a class of $D_0$, so $\{e_1, e_2, e_3, e_4\}$ is a circuit of $D_0$, i.e. $\{e_1, e_2, e_3, e_4\}$ contains two cycles of $H$. We arrive at this contradiction by assuming that $\{e_1, e_2, e_3\}$ does not contain a cycle of $G$, thus $\text{girth}(G) \leq 3$, and the theorem follows.

This statement yields a large class $C$ of bicircular matroids with no positive double circuits. For instance, if $G$ is the dodecahedron graph and $P$ the Petersen graph (Figure 1) then $B(G)$ and $B(P)$ are cosimple bicircular matroids of corank at least two that do not have positive double circuits. Dually, $B(G)^*$ and $B(P)^*$ are simple matroids of rank at least two that do not have positive colines.

![Figure 1: The Petersen graph and the dodecahedron.](image)

Recall that every transversal matroid is a gammoid, and since bicircular matroids are transversal matroids [11], Theorem [11] yields a class of cosimple gammoids of corank at least

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two that do not have positive double circuits.

**Corollary 12.** Not every cosimple gammoid of corank at least two has a positive double circuit. Equivalently, not every simple gammoid of rank at least two has a positive coline.

## 4 Subclasses of GSP oriented matroids

In this section, we show that if a matroid $M$ has a double circuit with two singular classes, then every orientation of $M$ has a $\{0, 1, -1\}$-flow with at most two non-zero entries. In particular, we will use this observation to show that oriented cobicircular matroids are GSP.

The flow lattice or oriented uniform matroids is characterized in [9]. The properties of these lattices that are interesting for this work are stated in Lemma 3. The following observation is an implication of the first part of Lemma 3.

**Observation 13.** Let $O$ be an oriented matroid with underlying matroid $M$. If $M$ is a series extension of a uniform matroid of even rank, then for any series class $C \subseteq E(M)$ there is a $\{0, 1, -1\}$-flow $F$ of $O$, such that $|F(e)| = 1$ if and only if $e \in C$.

*Proof.* The claim follows from the fact that for each pair $e$ and $f$ of coparallel elements of $M$, the sign of $e$ and $f$ are either the same in every circuit of $O$, or are opposite in every circuit of $O$. This, part 1 of Lemma 3 proves the claim.

With similar arguments to the ones in the previous proof, and using part 2 of Lemma 3 we conclude that the following statement holds.

**Observation 14.** Let $O$ be an oriented matroid with underlying matroid $M$. If $M$ is a series extension of a uniform matroid of odd rank, then for any pair $C_1$ and $C_2$ of series classes of $M$, there is a $\{0, 1, -1\}$-flow $F$ of $O$, such that $|F(e)| = 1$ if and only if $e \in C_1 \cup C_2$.

In Observation 3 we argued that if $D$ is a double circuit in a matroid $M$, then $M[D]$ is a series extension of some uniform matroid. This statement together with Observations 13 and 14 imply the following statement.

**Lemma 15.** Let $O$ be an oriented matroid with underlying matroid $M$. If $D$ is a double circuit of $M$ such that:

- $D$ has even degree and at least one singular class, or
- $D$ has odd degree and at least two singular classes,

then $O$ has a $\{0, 1, -1\}$-flow with at most two non-zero entries.

As previously mentioned, Goddyn, Hochstädtler and Neudauer, show that finding positive double circuits in certain matroids imply that the orientations of these matroids are coGSP. Lemma 15 yields a simple tool to improve the previous sufficient condition as follows.
Proposition 16. Let $C$ be a minor closed class of orientable matroids. If every cosimple matroid in $C$ has a double circuit $D$ such that:

- $D$ has even degree and at least one singular class, or
- $D$ has odd degree and at least two singular classes,

then every orientation of a matroid in $C$ is coGSP.

For the sake of completeness, we state the dual version of Proposition 16.

Proposition (Dual version of Proposition 16). Let $C$ be a minor closed class of orientable matroids. If every simple matroid in $C$ has a coline $L$ such that:

- there is a simple copoint on $L$ and $L$ has even degree, or
- there is a pair of simple copoints on $L$ and $L$ has odd degree,

then every orientation of a matroid in $C$ is GSP.

In the subsections below, we will use these propositions to show that certain classes of oriented matroids are GSP (coGSP). To do so, we begin by proving the following lemma which guarantees the existence of double circuits with at least two singular classes.

Lemma 17. For any matroid $M$ the following statements are equivalent:

- there is a double circuit $D \subseteq M$ with at least two singular classes, and
- there is a pair $C_1$ and $C_2$ of circuits such that $|C_1 \triangle C_2| = 2$.

Proof. If $D = (D_1, \ldots, D_k)$ is a double circuit such that $|D_1| = |D_2| = 1$, then $D - D_1$ and $D - D_2$ are a pair of circuits such that $|(D - D_1) \triangle (D - D_2)| = |D_1 \cup D_2| = 2$. Now suppose that $C_1$ and $C_2$ satisfy the second statement. Let $D = C_1 \cup C_2$. It is clear that $|D| = k + 2 = r(D) + 2$. Moreover, since any $e \in D$ belongs to $C_i$ for some $i \in \{1, 2\}$, then $r(D - e) \geq r(C_i - e) = k$. So, for any $e \in D$ the rank of $D - e$ does not decrease, which shows that $D$ is a double circuit. Finally, notice that $D \setminus C_1$ and $D \setminus C_2$ form two singular classes of the circuit partition of $D$. The claim follows.

Lemma (Dual version of Lemma 17). For any matroid $M$ the following statements are equivalent:

- there is a coline $L \subseteq M$ with at least two singular classes, and
- there is a pair $H_1$ and $H_2$ of hyperplanes of $M$ such that $|H_1 \triangle H_2| = 2$. 
4.1 Bicircular matroids

In this subsection, we show that oriented bicircular matroids are coGSP. To do so, we will show that for every bicircular matroid $B(G)$ there is a pair of circuits $C_1$ and $C_2$ such that $|C_1 \triangle C_2| = 2$. We begin by proving the following lemma.

Lemma 18. Consider a graph $G$. If $B(G)$ is a cosimple matroid, then $B(G)$ contains a pair of circuits $C_1$ and $C_2$ such that $|C_1 \triangle C_2| = 2$.

Proof. Since $G$ is cosimple, every vertex of $G$ is incident with at least 3 edges. Consider a maximum path $P$ of $G$, where $P = v_1e_1v_2 \ldots v_{k-1}e_{k-1}v_k$. Let $e_k \neq e_{k-1}$ be an edge incident in $v_k$. By the choice of $P$, $e_k$ has both endpoints in $P$. Denote by $E'$ the set of edges $\{e_1, \ldots, e_k\}$. Let $e$ and $f$ be a pair of edges incident in $v_1$ different from each other and different to $e_1$. By the choice of $P$, both endpoints of $e$ and both endpoints of $f$ belong to $P$. It is straightforward to notice that $E' \cup \{e\}$ and $E' \cup \{f\}$ are both circuits of $B(G)$. These circuits also satisfy that $|E' \cup \{e\} \triangle (E' \cup \{f\})| = 2$. Therefore, $B(G)$ contains a pair of circuits $C_1$ and $C_2$ such that $|C_1 \triangle C_2| = 2$. □

Lemmas 17 and 18 imply that every cosimple bicircular matroid has a double circuit with at least two singular classes. So, using Proposition 16 we conclude that any oriented bicircular matroid is coGSP.

Proposition 19. Every oriented bicircular matroid is coGSP.

An equivalent reformulation of Proposition 19 states that every oriented cobicircular matroid is GSP. Since every GSP matroid has a NZ-3 coflow, we conclude the following statement.

Corollary 20. Every oriented cobicircular matroid has a NZ-3-coflow.

4.2 Clone reducible matroids

We conclude this section by defining the class of clone reducible matroids. In particular, the graphic matroids in this class correspond to graphic matroids of series parallel graphs, also any orientation of a clone reducible matroid is a GSP oriented matroid.

Consider a matroid $M$ and a pair of elements $e$ and $f$ of $M$. We say that $e$ and $f$ are clones (in $M$) if permuting $e$ and $f$ is an automorphism of $M$. We say that a matroid $M$ is clone reducible if every minor of $M$ (with at least two elements) has a pair of clones. For instance, uniform matroids, matroids of rank at most 2 and matroids of corank at most 2 are examples of clone reducible matroids.

It is not hard to notice that if $e$ and $f$ are clones in $M$, and $g \in E(M) - \{e, f\}$, then $e$ and $f$ are clones in $M - g$. Also notice that if $e$ and $f$ are clones in $M$, then $e$ and $f$ are clones in $M^*$. Putting these two observations together, we conclude that if $e$ and $f$ are clones of $M$, and $N$ is a minor of $M$ such that $e, f \in E(N)$, then $e$ and $f$ are clones in $N$. This implies that $M$ is a clone reducible matroid, if and only if there is a linear ordering $e_1 \leq e_2 \leq \cdots \leq e_n$ of $E(M)$, such that $e_i$ has a clone in the restriction $M[e_1, \ldots, e_i]$, for all $i \in \{2, \ldots, n\}$.
Notice that if \( e \) and \( f \) are a pair of parallel edges in a graph \( G \), then \( e \) and \( f \) are clones in \( M(G) \). Similarly, if \( G \) has a vertex \( v \) of degree two, and \( e \) and \( f \) are incident with \( v \), then \( e \) and \( f \) are also clones in \( M(G) \). So, graphic matroids of series parallel graphs (\( M(K_4) \)-free graphic matroids) are clone reducible matroids.

**Proposition 21.** For a binary matroid \( M \) the following statements are equivalent:

1. \( M \) does not contain an \( M(K_4) \)-minor, and
2. \( M \) is a clone reducible matroid.

**Proof.** On the one hand, \( M(K_4) \)-free binary matroids correspond to graphic matroids of series parallel graphs. Thus, by the arguments preceding this statement, the first item implies the second one. On the other hand, \( M(K_4) \) does not contain a pair of clones, thus the second statement implies the first one. \( \square \)

We have previously observed that the class of clone reducible matroids is closed under duality. Now we show that every orientation of a clone reducible matroid is \( GSP \), and thus \( coGSP \).

**Observation 22.** Every orientation of a clone reducible matroid is a \( GSP \) and \( coGSP \) oriented matroid.

**Proof.** It suffices to show that any orientation of a clone reducible matroid is \( coGSP \). Suppose that \( e \) and \( f \) are pair of clones in a cosimple clone reducible matroid \( M \). In particular, \( e \) and \( f \) are not coparallel elements, so there is a circuit \( C_1 \) such that \( e \in C_1 \) but \( f \not\in C_1 \). Since \( e \) and \( f \) are clones, then the set \( C_2 \) defined by \((C_1 - e) \cup \{f\}\) is a circuit of \( M \). Thus, \( C_1 \) and \( C_2 \) are a pair of circuits of \( M \) such that \(|C_1 \triangle C_2| = 2\). Thus, by Lemma 17 \( M \) has a double circuit with two singular classes. So, by Proposition 16 we conclude that every orientation of a clone reducible matroid is a \( coGSP \) oriented matroid. \( \square \)

In [1] the author shows that orientations of lattice path matroids are \( GSP \). We propose a simple proof of this fact by showing that every lattice path matroid is a clone reducible matroid.

**Lemma 23.** Every lattice path matroid on at least two elements has a pair of clones.

**Proof.** Consider a lattice path matroid \( L \) on \([n]\), and suppose that \( L \) has no coloops. It is not hard to notice that if \( I \) is an independent set such that \( 1 \in I \), then \((I - 1) \cup \{2\}\) is also an independent set. With out loss of generality suppose that \( I \) is a base. If \( 2 \in I \), then \((I - 1) \cup \{2\} = I - 1 \) and the claim is trivial. On the other hand, if \( 2 \not\in I \), by the maximality of \( I \), there is a lattice path \( P \) whose north steps include 1 and its east steps include 2, i.e., \( P = n_1e_2Q \), for some lattice path \( Q \) starting in \((1,1)\). Thus, the lattice path \( e_1n_2Q \) is a lattice path which certifies that \((I - 1) \cup \{2\}\) is an independent set in \( I \). With similar arguments we observe that if \( I \) is an independent set such that \( 2 \in I \), then \((I - 2) \cup \{1\}\) is also an independent set. Putting these observations together, we conclude that if \( C \) is a circuit such that \( 1 \in C \) and \( 2 \not\in C \), then \( C - 1 \cup \{2\}\) is a circuit (and viceversa). Therefore, the function \( f: [n] \to [n] \) that fixes \([n] \setminus \{1,2\}\) and permutes 1 with 2, maps circuits \( M \) to circuits of \( M \), i.e., \( f \) is an automorphism of \( M \). Thus, 1 and 2 are clones in \( M \). \( \square \)
Proposition 24. Every orientation of a lattice path matroid is a GSP oriented matroid.

Corollary 25. Every orientation of a lattice path matroid has a NZ 3-coflow.

4.3 Rank 3 matroids

Consider a rank 3 simple and cosimple non-uniform oriented matroid $O$ on the set $E$. Hochstättler and Nickel [9], show that if $O$ is not an orientation of $M(K_4)$, then the flow lattice $F_O$ is $\mathbb{Z}^E$. In particular, this implies that in this case, $O$ has a $\{0, 1, -1\}$-flow with at most one non-zero entry.

Proposition 26. Every $M(K_4)$-free oriented matroid of rank at most 3 is coGSP. Dually, every $M(K_4)$-free oriented matroid of corank at most 3 is GSP

Proof. Recall that we want to show that every such cosimple oriented matroid $O$ has a $\{0, 1, -1\}$-flow with at most two non-zero entries. The arguments above this proposition takes care of the case when $O$ is a simple rank 3 non-uniform oriented matroid. Now, if $O$ has a circuit $C$ of size at most two, then $C$ is a $\{0, 1, -1\}$-flow with at most two non-zero entries. Finally, if $O$ is uniform or has rank at most 2, then every minor of the underlying matroid of $O$ has a pair of clones. So, the claim follows by Observation [22].

5 Conclusions

A simple observation used to prove Theorem 11 shows that the degree of double circuits in bicircular matroids is bounded above by 6. This raises the natural problem of describing the classes of matroids obtained by considering bicircular matroids whose double circuits have degree at most $k$ where $k \in \{3, 4, 5\}$. This question has been answered from another perspective for $k = 3$: the positive double circuits in a bicircular matroid $B$ have degree at most 3 if and only if $B$ is a binary bicircular matroid [11]. If we also restrict the degree of colines, for the case $k = 4$ we recover ternary bicircular matroids [14]. For the case $k = 5$, considering bicircular matroids with double circuits and colines of degree at most $k$, we do not recover representation over $GF(4)$ since $P_6$ has no positive double circuits nor colines of degree greater that or equal to 5, but is a bicircular matroid not representable over $GF(4)$ [4]. We are interested in knowing if there is a meaningful description of bicircular matroids whose double circuits have bounded degree.

Problem. Provide a meaningful description of bicircular matroids where every double circuit has degree at most $k$ for $k \in \{4, 5\}$.

The motivation of this work was to settle Conjecture 2; we showed that it does not hold even for cobicircular matroids. Nonetheless, we prove that all oriented cobicircular matroids are GSP. We did so by using Proposition 16, which yields a weaker condition (to that in Conjecture 2) for proving that certain classes oriented matroids are GSP. We propose the following question.

Question. Is it true that every cosimple gammoid $M$ contains a pair of circuits $C_1$ and $C_2$ such that $|C_1 \Delta C_2| = 2$?
At the end of Section 4, we introduce the class of clone reducible matroids. In particular, we showed that lattice path matroids and graphic matroids of series parallel are examples of clone reducible matroids. Since lattice path matroid and graphic matroids form a pair of incomparable matroid classes, then the class of clone reducible matroids properly contains both classes. We also noticed that $M(K_4)$ does not have a pair of clones, and it is straightforward to observe that the three whirl does not have a pair of clones either. What are the excluded minors to the class of clone reducible matroids?

**Problem.** Characterize the class of clone reducible matroids.

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**References**

References

[1] I. Albrecht, Contributions to the Problems of Recognizing and Coloring Gammoids, Doctoral Dissertation, FernUniversität in Hagen, 2018. https://doi.org/10.18445/20180820-090543-4

[2] A. Björner, M. L. Vergnas, B. Sturmfels, N. White, G. M. Ziegler, Oriented Matroids, Cambridge University Press, 1993.

[3] J.A. Bondy and U.S.R Murty, Graph Theory, Springer, Berlin, 2008.

[4] D. Chun, T. Moss, D. Slilaty, X. Zhou, Bicircular Matroids representable over $GF(4)$ and $GF(5)$, Discrete Mathematics 339(9) (2016) 2239–2248.

[5] A. Dress and L. Lovász, On some combinatorial properties of algebraic matroids, Combinatorica 7(1) (1987) 39–48.

[6] L. Goddyn and W. Hochstättler, Nowhere-zero flows in regular matroids and Hadwiger’s conjecture, Seminarberichte der Mathematik – FernUniversität in Hagen 87 (2015) 97–102.

[7] L. Goddyn, W. Hochstättler, N. Neudauer, Bicircular matroids are 3-colourable, Discrete Mathematics 339(5) (2016) 1425–1429.

[8] W. Hochstättler and J. Nešetřil, Antisymmetric flows in matroids, European Journal of Combinatorics 27(7) (2006) 1129–1134.

[9] W. Hochstättler and R. Nickel, The flow lattice of oriented matroids, Contributions to Discrete Mathematics 2(1) (2007) 68–86.
[10] A.W. Ingleton, Gammoids and Transversal Matroids, Journal of Combinatorial Theory (B) 15 (1973) 51–68.

[11] L. R. Matthews, Bicircular Matroids, Quarterly Journal of Mathematics (2) 28(110) (1997) 213–227.

[12] J.G. Oxley, Matroid Theory, The Clarendon Press Oxford University Press, New York (1992).

[13] N. Robertson, P. Seymour, R. Thomas, Hadwiger’s conjecture for $K_6$-free graphs, Combinatorica 13 (1993) 279–361.

[14] V. Sivaraman, Bicircular signed-graphic matroids, Discrete Mathematics 328 (2014) 1–4.

[15] W. Tutte, A contribution to the theory of chromatic polynomials, Canadian Journal of Mathematics 6 (1954) 80–91.

[16] W. Tutte, A geometrical version of the four color problem, R.C. Bose, T.A. Dowling (Eds.), Combinatorial Mathematics and its Applications, University of North Carolina Press, Chapel Hill, NC (1967) 553–560.

[17] W. Tutte, On the algebraic theory of graph colorings, Journal of Combinatorial Theory 1(1) (1966) 15–50.