Cosmological Solutions in Bimetric Gravity and their Observational Tests

Mikael von Strauss*, Angnis Schmidt-May†, Jonas Enander‡, Edvard Mörtsell§, S. F. Hassan¶

Department of Physics & The Oskar Klein Centre for Cosmoparticle Physics,
Stockholm University, AlbaNova University Centre, SE-106 91 Stockholm, Sweden

ABSTRACT: We obtain the general cosmological evolution equations for a classically consistent theory of bimetric gravity. Their analytic solutions are demonstrated to generically allow for a cosmic evolution starting out from a matter dominated FLRW universe and relaxing towards a de Sitter (anti-de Sitter) phase at late cosmic time. In particular, we examine a subclass of models which contain solutions that are able to reproduce the expansion history of the cosmic concordance model inspite of the nonlinear couplings of the two metrics. This is demonstrated explicitly by fitting these models to observational data from Type Ia supernovae, Cosmic Microwave Background and Baryon Acoustic Oscillations. In the appendix we comment on the relation to massive gravity.

KEYWORDS: modified gravity, dark energy theory.

*mvs@fysik.su.se
†angnis.schmidt-may@fysik.su.se
‡enander@fysik.su.se
§edvard@fysik.su.se
¶fawad@fysik.su.se
1. Introduction and summary

Cosmological observations confirm to a high degree of precision that our universe is homogeneous and isotropic and can be described by a single metric using Einstein’s theory of general relativity with the addition of a cosmological constant $\Lambda_{\text{obs}}$. At the same time,
one of the corner stones of this theory is the equivalence principle which, taken together
with quantum field theory, provides one of the most serious unresolved problems of modern
theoretical physics. Quantum field theory suggests the reality of vacuum energy and
naturally provides a cosmological constant \( \Lambda_{\text{vac}} \), while the equivalence principle guarantees
that this energy gravitates with the same strength as any other source. The main problem
is to reconcile the expected large value of \( \Lambda_{\text{vac}} \) with the observed small value of \( \Lambda_{\text{obs}} \), without
recourse to extreme fine tuning. This is one of the main motivations for considering
theories of modified gravity.

An intuitive modification of general relativity is massive gravity which generically also
suffers from a ghost instability \([1]\). Such theories must contain at least two metrics, say,
\( g_{\mu \nu} \) and \( f_{\mu \nu} \), where \( f_{\mu \nu} \) is treated as fixed and non-dynamical. This extra metric is needed
to construct non-derivative, non-linear mass terms in the action; since \( \sqrt{-g} \) and \( g^{\mu \nu} = 4 \)
alone are not adequate for the purpose. Massive gravity has received increased attention
since a class of massive actions, formulated with a flat \( f_{\mu \nu} \), was proposed in \([4, 5]\) and were
shown to be ghost free at the completely non-linear level \([6]\). These were soon generalized
to arbitrary \( f_{\mu \nu} \) and proved to remain ghost free \([6, 7]\). The potential relevance of massive
gravity to the cosmological constant problem can be argued on a heuristic level, based
on the Yukawa suppression of the massive amplitudes over large distances. This may be
expected to weaken gravity over such distances and mimic cosmic acceleration or possibly
screen out a large cosmological constant. “Self-accelerated” solutions of massive gravity
with a small cosmological constant have been considered, for example, in \([7, 8]\). However,
the screening mechanism does not seem to work without fine tuning \([9]\).

Here, we will consider a generalization of massive gravity, the bimetric gravity, where
\( f_{\mu \nu} \) is promoted to a dynamical variable with its own kinetic term. The idea of studying
bimetric gravity for cosmological purposes has a long history \([10, 11]\) and such theories
are known to admit cosmological vacuum solutions \([12]\). For more recent work on bimetric
theories see \([13, 14, 15, 16, 17, 18, 19]\) and references therein. In these studies the nonlinear
bimetric interaction potential is often chosen with the only restriction that it reduces to
the Fierz-Pauli form \([20, 21]\) at the linearized level. However, these theories also contain
the Boulware-Deser ghost \([1]\), similar to massive gravity, and hence are not consistent. It
has been known that internal consistency imposes restrictions on the kinetic structure of
bi-metric theories \([22]\). Recently, a class of ghost free bimetric theories were constructed
in \([23]\). Here we will focus on the cosmological implications of these theories. Spherically
symmetric vacuum solutions of these models were considered in \([24]\).

Bimetric theories describe a pair of interacting massless and massive spin-2 fields
\([12, 25]\) which are combinations of the two metrics. The presence of the massless mode
distinguishes bimetric gravity from pure massive gravity. A priori it is not obvious what
combination of the spin-2 fields should couple to the matter, except that the coupling
should not reintroduce the ghost. In this paper we consider the bimetric theory of \([23]\) and
assume that only one of the metrics, \( g_{\mu \nu} \), is coupled to matter. Then, although not a mass
eigenstate, this metric alone will determine the geodesics and the causal structure of the
spacetime.

Looking for cosmological solutions we take both of the metrics to be homogeneous

\[ \sqrt{-g} = \text{const}, \quad \frac{\partial g_{\mu \nu}}{\partial x^\lambda} = 0, \]

\[ \frac{\partial f_{\mu \nu}}{\partial x^\lambda} = 0. \]

However, these theories also contain
the Boulware-Deser ghost \([1]\), similar to massive gravity, and hence are not consistent. It
has been known that internal consistency imposes restrictions on the kinetic structure of
bi-metric theories \([22]\). Recently, a class of ghost free bimetric theories were constructed
in \([23]\). Here we will focus on the cosmological implications of these theories. Spherically
symmetric vacuum solutions of these models were considered in \([24]\).

Bimetric theories describe a pair of interacting massless and massive spin-2 fields
\([12, 25]\) which are combinations of the two metrics. The presence of the massless mode
distinguishes bimetric gravity from pure massive gravity. A priori it is not obvious what
combination of the spin-2 fields should couple to the matter, except that the coupling
should not reintroduce the ghost. In this paper we consider the bimetric theory of \([23]\) and
assume that only one of the metrics, \( g_{\mu \nu} \), is coupled to matter. Then, although not a mass
eigenstate, this metric alone will determine the geodesics and the causal structure of the
spacetime.

Looking for cosmological solutions we take both of the metrics to be homogeneous
and isotropic and obtain the general cosmological equations for the metric $g_{\mu\nu}$. Solving these in their most general form involves finding the zero locus of a quartic polynomial. This guarantees that solutions of the general equations exist and they are straightforward to derive. From the general equation we conclude some generic properties of the solution inherent to the model. First, they generically asymptote towards a de Sitter or anti-de Sitter (AdS) spacetime and hence will in general be able to mimic late-time cosmic acceleration. Secondly, one branch of the solutions always allows for an expansion history which starts out from an ordinary FLRW like universe at high densities, and hence the usual early universe considerations still apply for this solution.

Displaying the full solutions of the quartic in their analytic form is not very illuminating for general values of the parameters of the theory. We study instead the analytic details of two simpler classes of solutions, where the quartic is reduced to a quadratic. These solutions are shown to exhibit the general behaviour expected of the full solution. In particular, we identify and study a “minimal” bimetric model in more detail. In this model, the Friedman equation is shown to be completely degenerate with general relativity, up to a possible rescaling of $G_N$, the Newton’s constant. Hence, observationally it is indistinguishable from the usual concordance model on cosmological scales. We demonstrate this explicitly by fitting the model to observations, using data from Type Ia supernovae (SNe Ia), Cosmic Microwave Background (CMB) and Baryon Acoustic Oscillations (BAO). We further study neighbouring models of the minimal model and find that cosmological data favours models close the minimal one.

The paper is organized as follows: In section 2 we review ghost free bi-metric gravity and present the equations of motion for a homogeneous and isotropic ansatz for both metrics. In section 3 we discuss the general properties of cosmological solutions and then focuses on two more constrained models that allow for solutions close to those of general relativity. We specify the model used for the comparison to observations in section 4 and summarize the cosmological data considered in section 5. Section 6 contains the results of the model fit, which are discussed in section 7. Appendix A contains the details for the equations of motion in our metric ansatz. In Appendix B we discuss the Bianchi constraints and comment on the relation to some cosmological solutions recently obtained in the massive gravity literature.

2. The bimetric gravity equations in the cosmological ansatz

In this section we review the structure of ghost free bimetric gravity and its equations of motion. We then write the equations of motion using a cosmological ansatz for the metrics.

2.1 Review of bimetric gravity action and equations of motion

The action for the metric $g_{\mu\nu}$ interacting with another spin-2 field $f_{\mu\nu}$ through a non-derivative potential is determined by the requirement of the absence of the Boulware-Deser ghost. The most general action of this type, modulo choice of coupling to matter, has the
form \[ S = -\frac{M_g^2}{2} \int d^4 x \sqrt{-\det g} \, R(g) - \frac{M_f^2}{2} \int d^4 x \sqrt{-\det f} \, R(f) + m^2 M_g^2 \int d^4 x \sqrt{-\det g} \sum_{n=0}^4 \beta_n e_n(\sqrt{g^{-1}f}) + \int d^4 x \sqrt{-\det g} \, \mathcal{L}_m(g, \Phi). \] (2.1)

Here, $\beta_n$ are free parameters and $e_n(\mathcal{X})$ are elementary symmetric polynomials of the eigenvalues of the matrix $\mathcal{X}$ given explicitly by

$$e_0(\mathcal{X}) = 1, \quad e_1(\mathcal{X}) = [\mathcal{X}], \quad e_2(\mathcal{X}) = \frac{1}{2}([\mathcal{X}]^2 - [\mathcal{X}^2]), \quad e_3(\mathcal{X}) = \frac{1}{6}([\mathcal{X}]^3 - 3[\mathcal{X}][\mathcal{X}^2] + 2[\mathcal{X}^3]), \quad e_4(\mathcal{X}) = \det(\mathcal{X}), \tag{2.2}$$

where the square brackets denote the matrix trace. The non-trivial point here is the appearance of the square root matrix $\mathcal{X} = \sqrt{g^{-1}f}$ which is necessary to avoid the ghost at the nonlinear level.

As demonstrated through a nonlinear ADM analysis in [6, 23], this action is ghost free and contains 7 propagating degrees of freedom; 2 corresponding to a massless spin-2 graviton and 5 corresponding to a massive spin-2 field. Both $g_{\mu\nu}$ and $f_{\mu\nu}$ are combinations of these massless and massive degrees of freedom.

Since $e_n \sim (\sqrt{g^{-1}f})^n$, the $\beta_n$ parameterize the order of interactions at the nonlinear level. As such they are more convenient to use than other parameterizations. Besides the two Planck masses, the action (2.1) contains five free parameters $\beta_n$. Of these, $\beta_0$ and $\beta_4$ parameterize the cosmological constants of $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively. One combination of the remaining $\beta$’s gives the mass of the massive mode, leaving us with two extra free parameters to characterize the nonlinear interactions.

For convenience, and also not to violate the equivalence principle in any drastic way, we only couple $g_{\mu\nu}$ to matter. This metric determines the geodesics and the causal structure of spacetime. Apart from this coupling, the action is invariant under the exchange,

$$g \leftrightarrow f, \quad \beta_n \rightarrow \beta_{4-n}, \quad M_g \leftrightarrow M_f, \quad m^2 \rightarrow m^2 M_g^2/M_f^2 \tag{2.3}$$

where the last replacement is needed due to our asymmetric parameterization of the mass scale in terms of $m^2$.

Setting $\beta_3 = 0$ in the action (2.1) eliminates the highest order interaction term in $\sqrt{g^{-1}f}$. However, in view of (2.3) we still have a cubic order interaction term in $\sqrt{f^{-1}g}$ which can in turn be eliminated by setting $\beta_1 = 0$. In this sense, the choice $\beta_1 = \beta_3 = 0$ leads to the “minimal” bimetric action, which is the simplest in the class.

Let us now consider the equations of motion and the Bianchi constraints that follow from (2.1). Varying the action with respect to $g_{\mu\nu}$ gives the equations of motion \[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{m^2}{2} \sum_{n=0}^{3} (-1)^n \beta_n \left[ g_{\mu\lambda} Y_{(n)\nu}^\lambda (\sqrt{g^{-1}f}) + g_{\nu\lambda} Y_{(n)\mu}^\lambda (\sqrt{g^{-1}f}) \right] = \frac{1}{M_g^2} T_{\mu\nu}, \tag{2.4} \]
where $Y^\lambda_{(n)\mu}(\sqrt{g^{-1}f})$ are defined below. Similarly, varying with respect to $f_{\mu\nu}$ gives,

$$R_{\mu\nu} - f_{\mu\nu} R + \frac{m^2}{2M_2^4} \sum_{n=0}^3 (-1)^n \beta_4 - n \left[ f_{\mu\lambda} Y^\lambda_{(n)\nu}(\sqrt{f^{-1}g}) + f_{\nu\lambda} Y^\lambda_{(n)\mu}(\sqrt{f^{-1}g}) \right] = 0,$$

(2.5)

where the overbar indicate $f_{\mu\nu}$ curvatures and we have introduced the dimensionless ratio of Planck masses

$$M_2^2 \equiv \frac{M_2^2}{M_5^2}.$$  

(2.6)

Note that (2.5) is essentially obtainable from (2.4) through the replacements (2.3). Finally, the matrices $Y^\lambda_{(n)\mu}(\mathcal{X})$ in (2.4) and (2.5) are given by (with square brackets denoting the trace)

$$Y^0(\mathcal{X}) = 1, \quad Y^1(\mathcal{X}) = \mathcal{X} - 1[\mathcal{X}],$$

$$Y^2(\mathcal{X}) = \mathcal{X}^2 - \mathcal{X}[\mathcal{X}] + \frac{1}{2}2(\mathcal{X}^2 - [\mathcal{X}^2]),$$

$$Y^3(\mathcal{X}) = \mathcal{X}^3 - \mathcal{X}^2[\mathcal{X}] + \frac{1}{2}\mathcal{X}([\mathcal{X}^2 - [\mathcal{X}^2]) - \frac{1}{6}2(\mathcal{X}^3 - 3[\mathcal{X}][\mathcal{X}^2] + 2[\mathcal{X}^3]).$$

(2.7)

As a consequence of the Bianchi identity and the covariant conservation of $T_{\mu\nu}$, the $g_{\mu\nu}$ equation of motion (2.4) leads to the Bianchi constraint,

$$\nabla^\mu \sum_{n=0}^3 (-1)^n \beta_n \left[ g_{\mu\lambda} Y^\lambda_{(n)\nu}(\sqrt{g^{-1}f}) + g_{\nu\lambda} Y^\lambda_{(n)\mu}(\sqrt{g^{-1}f}) \right] = 0.$$  

(2.8)

Similarly, the $f_{\mu\nu}$ equation of motion (2.5) leads to the Bianchi constraint,

$$\nabla^\mu \sum_{n=0}^3 (-1)^n \beta_4 - n \left[ f_{\mu\lambda} Y^\lambda_{(n)\nu}(\sqrt{f^{-1}g}) + f_{\nu\lambda} Y^\lambda_{(n)\mu}(\sqrt{f^{-1}g}) \right] = 0.$$  

(2.9)

where the overbar indicates covariant derivatives with respect to the $f_{\mu\nu}$ metric. Both of these Bianchi constraints follow from the invariance of the interaction term under the diagonal subgroup of the general coordinate transformations of the two metrics and hence are equivalent. From now on we explicitly concentrate on (2.8).

Before proceeding further, let us briefly remark on the relation to massive gravity. This corresponds to freezing the dynamics of $f_{\mu\nu}$ and hence loosing the corresponding equation of motion (2.7). Hence in the bimetric theory $f_{\mu\nu}$ is much more constrained than in massive gravity.

### 2.2 Equations in the cosmological ansatz

We look for solutions where both of the metrics exhibit spatial isotropy and homogeneity. For simplicity, we also assume both metrics to have the same spatial curvature $k = 0, \pm 1$. Then, modulo time reparameterizations, the most general form for the metrics is,

\[ g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\vec{x}^2 \]

\[ f_{\mu\nu} dx^\mu dx^\nu = -X^2(t) dt^2 + Y^2(t) d\vec{x}^2 \]

(2.10)

\[ ^1\text{Modulo non-perturbative solutions that can exist for certain values of the } \beta_n \text{ parameters [24] (see also [25]).} \]
where
\[ \text{d}x^2 = \frac{\text{d}r^2}{1 - kr^2} + r^2 \left( \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2 \right). \] 

(2.11)

Obviously, we have used time reparameterizations to set \( g_{00} = -1 \) so that the \( g_{\mu\nu} \) metric is in the usual FLRW form. No other transformations are available to further reduce the number of functions in the metric ansatz. Thus we need to keep three arbitrary functions to describe the metrics with the specified symmetries. Below we will see that the Bianchi constraint (implied by general covariance) enforces
\[ X = \frac{\dot{Y}}{a} = \frac{dY}{da}, \]
thus leaving only two free functions to work with.

Now, we write down the Bianchi constraint and the equations of motion for the above parameterization of the metrics. For the ansatz (2.10) the Bianchi constraint (2.8), or (2.9), gives,
\[ 3m^2 \left[ \beta_1 + 2\frac{Y}{a} \beta_2 + \frac{Y^2}{a^2} \beta_3 \right] \left( \dot{Y} - \ddot{a}X \right) = 0. \]

(2.12)

One way this can be satisfied is by setting to zero the expression within the square brackets. This implies solutions where \( Y \propto a \), with special values for the constant of proportionality.

As will be evident from the Friedmann equation (2.14) below, this leads to the ordinary general relativistic equations with a cosmological constant of order \( m^2 \). Further, the special values imply a vanishing mass for the massive spin-2 field (as is discussed further in the Appendix [3]). One can also check that linear metric perturbations around these backgrounds are indistinguishable from general relativity. Hence one concludes that this is effectively a decoupled class of solutions that do not modify general relativity on any scale.

The true dynamical constraint is enforced by the vanishing of the expression within the round brackets,
\[ X = \frac{\dot{Y}}{a} = \frac{dY}{da}. \]

(2.13)

Using this result together with the ansatz (2.10), the \( g_{\mu\nu} \) equations of motion (2.4) lead to the modified Friedmann equation (for details see Appendix [4]),
\[ -3 \left( \frac{\dot{a}}{a} \right)^2 - 3 \frac{k}{a^2} + m^2 \left[ \beta_0 + 3\beta_1 \frac{Y}{a} + 3\beta_2 \frac{Y^2}{a^2} + \beta_3 \frac{Y^3}{a^3} \right] = \frac{1}{M_5^2} T_0^0, \]

(2.14)

and the acceleration equation,
\[ -2 \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 - \frac{k}{a^2} + m^2 \left[ \beta_0 + 2 \beta_1 \left( \frac{Y}{a} + \frac{\dot{Y}}{a} \right) \right] + \beta_2 \left( \frac{Y^2}{a^2} + 2 \frac{YY}{aa} \right) + \beta_3 \frac{Y^2 \dot{Y}}{a^2 \ddot{a}} \right] = \frac{1}{M_5^2} T_1^1. \]

(2.15)

Here we have used \( T_1^1 = T_2^2 = T_3^3 \), consistent with the symmetries of the spacetime. These obviously reduce to the ordinary Friedmann and acceleration equations of cosmology in the limit \( m^2 \to 0 \), although the solutions will not always be well defined in this limit. Similarly,
the $f_{\mu\nu}$ equations of motion (2.5) lead to,
\[-3 \left( \frac{\dot{a}}{Y} \right)^2 - 3 \frac{k}{Y^2} + \frac{m^2}{M^2} \left( \beta_4 + 3 \beta_3 \frac{a}{Y} + 3 \beta_2 \frac{a^2}{Y^2} + \beta_1 \frac{a^3}{Y^3} \right) = 0, \quad (2.16)\]
and
\[-2 \frac{\ddot{a}}{YY} \left( \frac{\dot{a}}{Y} \right)^2 - \frac{k}{Y^2} + \frac{m^2}{M^2} \left[ \beta_3 + \beta_3 \left( 2 \frac{a}{Y} + \frac{\dot{a}}{Y} \right) + \beta_2 \left( \frac{a^2}{Y^2} + 2 \frac{a\dot{a}}{YY} \right) + \beta_1 \frac{a^2\ddot{a}}{Y^2} \right] = 0. \quad (2.17)\]
The first of these, the $f$-Friedmann equation (2.16), is in general a cubic equation for $Y$ so the system can be solved exactly. Further, as a consequence of already having imposed the Bianchi constraint the two equations are not independent. Indeed, the $f$-acceleration equation (2.17) is obtained by acting with $(3 + (Y/\dot{Y})\partial_t)/3$ on the $f$-Friedmann equation. Thus we only need to consider (2.16).

Let us briefly comment on the source structure. In what follows we will assume a perfect fluid source $T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}$ so that in the rest frame,
\[T^0_0 = -\rho, \quad T^i_i = P \quad \text{(no sum implied)}. \quad (2.18)\]
Assuming also an equation of state of the usual form, $P(t) = w\rho(t)$, the continuity equation,
\[\frac{3}{M_g^2} \frac{\dot{a}}{a} \left( P + \rho + \frac{1}{3} \frac{a}{\dot{a}} \dot{\rho} \right) = 0, \quad (2.19)\]
tells us that, for $w \neq -1$,
\[\rho = \rho_0 \left( \frac{a}{a_0} \right)^{-3(1+w)}, \quad \text{and} \quad H = -\frac{\dot{\rho}/\rho}{3(1+w)}. \quad (2.20)\]
Here $\rho_0$ is the present day energy density. Now, in an expanding universe any source with $w > -1$ will get diluted as the scale factor grows. Hence, $\dot{\rho} < 0$ and $H > 0$.

### 3. Viable cosmological solutions

In this section we will consider cosmological solutions in bimetric gravity. In particular we concentrate on parameter values for which the solutions are close to cosmological solutions in general relativity and hence are not observationally ruled out.

#### 3.1 General features of solutions

In order to make the analysis more transparent, we define the dimensionless combinations
\[\Upsilon \equiv \frac{Y}{a}, \quad \rho_* \equiv \rho/3M_g^2m^2. \quad (3.1)\]
In terms of these, the two Friedmann equations (2.14) and (2.16) can be written as
\[\frac{\beta_3}{3} \Upsilon^3 + \beta_2 \Upsilon^2 + \beta_1 \Upsilon + \frac{\beta_0}{3} + \rho_* - \frac{H^2}{m^2} - \frac{k}{m^2a^2} = 0. \quad (3.2)\]
Subtracting these two equations to eliminate $H^2$ gives in general a quartic equation for $\Upsilon$:

$$\frac{\beta_3}{3 \Lambda^2} \Upsilon^4 + \left(\frac{\beta_2 + \beta_3}{3 \Lambda^2}\right) \Upsilon^3 + \left(\frac{\beta_1 - \beta_3}{\Lambda^2}\right) \Upsilon^2 + \left(\rho_* + \frac{\beta_0}{3} \right) \Upsilon - \frac{\beta_1}{3 \Lambda^2} = 0. \quad (3.4)$$

This determines $\Upsilon$ as a function of $\rho_*$. The analytic solutions are straightforward to derive but lengthy to display. For generic values of parameters there is the further complexity that in some cases one has to match different branches of solutions in order to keep $\Upsilon$ real for all $\rho$, although a real solution always exists. In order to keep the discussion transparent we simply state some generic properties here and then consider a few special cases in more detail.

First, at late times in an expanding universe, $\rho$ generically approaches a constant value $\rho_{\text{vac}}$, the vacuum energy density$^2$. Then from (3.4) it is obvious that at late times $\Upsilon$ approaches a constant value and this holds for generic values of parameters. The Friedmann equations (3.2) and (3.3) then imply that at late times the universe always asymptotes towards a de Sitter or anti-de Sitter geometry.

On the other hand, in the early time limit of $\rho_* \to \infty$ (in practice, $\rho_* \gg \beta_n/\Lambda^2$, but not necessarily including $\beta_0$) the linear term in (3.4) dominates and forces a solution $\Upsilon \to 0$. The Friedmann equation (3.2) then implies an evolution for $H^2$ dominated by $\rho_*$ and a cosmological constant contribution $\beta_0$, as in general relativity. Of course, even in this limit, the full quartic equation has three more solutions (two being complex), but these diverge in the limit $\rho_* \to \infty$, as can be inferred from the explicit form of the solution. Thus, the solution that vanishes is physically preferable since we then get back ordinary general relativity very early in the expansion history. While a priori this may not be necessary, it guarantees that ordinary early universe considerations remain valid.

As a caveat, note that while the discussion above is formulated in terms of $\Upsilon$ and holds in general, at the end, it is only the behaviour of $H^2$ that concerns us. Later we will display an example of an analytic solution where $\Upsilon$ diverges at early times while the Friedmann equation for $H^2$ is completely equivalent to the usual general relativistic equation.

To summarize, the equations admit generic solutions that start out from a universe described by the ordinary general relativistic Friedmann equation with the cosmological constant $\sim \beta_0 \Lambda^2$, and evolve toward a de Sitter (or AdS) universe, with a cosmological constant depending on the parameters of the theory.

### 3.2 Models with $\beta_3 = 0$

In the following we consider primarily theories with $\beta_3 = 0$. Although our main intent is to simplify the analysis of the solutions we note that this choice is of interest as it corresponds to neglecting the highest (cubic) order nonlinear interactions in $g^{-1}f$ in (2.1). Also, in the massive gravity limit, which corresponds to freezing the dynamics of $f_{\mu\nu}$, only for this

---

$^2$Of course, we can always choose $\rho_{\text{vac}} = 0$ and include the vacuum energy contribution entirely in the $\beta_0$ parameter, or vice versa.
choice the authors in [28] find spherically symmetric cosmological solutions that exhibit the Vainshtein mechanism in a manner consistent with observations.

For the choice $\beta_3 = 0$, (3.4) reduces to a cubic equation for $\Upsilon$ and one can integrate the $g_{\mu\nu}$ - Friedmann equation (3.2) for $H$. Writing down the explicit solution is not very illuminating, and we simply note that it exists for generic $\beta_1$. This solution is fitted to data in the following sections.

Here, however, to illustrate the general discussion of section 3.1, we look for simpler analytic solutions. Note that the more restrictive choice $\beta_3 = 0$ and $\beta_4 = 3\beta_2 M_2^{*2}$ converts (3.4) into a quadratic equation (the other possibility $\beta_3 = 0$, $\beta_1 = 0$ is discussed below).

Now, solutions exist only for $\beta_1 \neq 0$,

$$\Upsilon = -\frac{3\rho_0 + 3\beta_0 M_2^* - 6\beta_1}{6\beta_1} \pm \frac{\Upsilon(\rho_0)}{6\beta_1},$$

where for convenience we have defined

$$\Psi = \sqrt{(3\rho_0 + 3\beta_0 - 3\beta_2 M_2^*)^2 + 12\beta_1^2 M_2^2}.$$  

(3.5)

Choosing the solution with the positive sign ensures that $\Upsilon \to 0$ as $\rho \to \infty$, recovering standard early cosmology as discussed above. For this choice of sign, the Friedmann equation (3.2) becomes,

$$H^2 + \frac{k}{a^2} = C_1 \frac{\rho}{3M_g^2} + m^2 C_2 - \frac{m^2}{12} (2C_1 - 3) \Psi(\rho_0) + \frac{m^2 \beta_2}{2\beta_1} \rho_0^2 - \frac{m^2 \beta_2}{6\beta_1^2} \rho_0 \Psi(\rho_0),$$

(3.7)

where $C_{1,2}$ are constants given by

$$C_1 = \frac{1}{6\beta_1^2} \left( 3\beta_1^2 + 2\beta_0 \beta_2 - 6\beta_1^2 M_2^* \right),$$

$$C_2 = \frac{1}{18\beta_1^2} \left( 3\beta_0^2 \beta_2 + 3\beta_0 \beta_2 + 15\beta_1^2 \beta_2 - 6\beta_0 \beta_2^2 M_2^* + 9\beta_2^2 M_2^* \right).$$

(3.8)

Although this appears to be a highly nontrivial modification of the general relativistic Hubble parameter, in the high energy limit of large $\rho$ we have,

$$H^2 + \frac{k}{a^2} \sim \frac{\rho}{3M_g^2} + \frac{m^2 \beta_0}{3},$$

(3.9)

while as the energy density dilutes towards a constant value $\rho_{vac}$,

$$H^2 + \frac{k}{a^2} \sim C_1 \frac{\rho_{vac}}{3M_g^2} + m^2 C_3,$$

(3.10)

where the constant $C_3$ can be read off from the above expressions. This explicitly demonstrates the general arguments of section 3.1.

---

[3] Explicitly, [28] find that for $\beta_3 \neq 0$, the Vainshtein mechanism screens not only the scalar mode but also the tensor modes and hence is ruled out.
3.3 Models with $\beta_1 = \beta_3 = 0$

Another particularly interesting class of solutions arise when both $\beta_1 = \beta_3 = 0$, and $\beta_2 < \beta_4/3M_2^2$. As discussed in section 2.1, this corresponds to the minimal bimetric model where the interactions are of the lowest order simultaneously for both $\sqrt{g^{-1}f}$ and $\sqrt{f^{-1}g}$. This keeps only the quadratic order nonlinear interactions in both sectors such that the action (2.1) looks like a nonlinear action with a mass potential.

For these values we find the solution

$$\Upsilon^2 = \frac{3\rho^* + \beta_0 - 3\beta_2 M_2^2}{\beta_4 M_2^2 - 3\beta_2}$$, (3.11)

and the Friedmann equation (3.2) is given by

$$H^2 + \frac{k}{a^2} = \frac{\beta_4}{\beta_4 - 3\beta_2 M_2^2} \frac{\rho}{3 M_2^2} + \frac{m^2(\beta_0 \beta_4 - 9 \beta_2^2)}{3(\beta_4 - 3\beta_2 M_2^2)}$$. (3.12)

In general, we can also split the energy density into matter and vacuum contributions, $\rho = \rho_m + \rho_{\text{vac}}$. This equation is completely degenerate with the ordinary general relativistic equation with a rescaled Planck mass and a shifted cosmological constant. Thus it can be easily fitted to cosmological data. From the observational perspective any discrepancy with data must then be looked for by examining the corresponding solutions at smaller scales where the homogeneous solutions are not appropriate, e.g. cluster scales. To make this point more explicit and also consider neighbouring models with $\beta_1 \neq 0$ we proceed to fit these models against observational data from supernovae, cosmic microwave background, and baryon acoustic oscillations in the upcoming sections.

Note that this solution corresponds to a case where even though $\Upsilon$ diverges for large $\rho$, the equation for $H$ has the general relativistic form. This case evades the general arguments of section 3.1.

4. Parameterization of the solution

The bimetric action (2.1) has a number of degenerate parameters. In this section we discuss the choice of parameters to facilitate comparison to data.

4.1 Parameterization used for model fitting

In order to make the comparison with standard cosmology more transparent, we define

$$M^2 \equiv \frac{m^2}{H_0^2}, \quad E^2 \equiv \frac{H^2}{H_0^2}$$, (4.1)

where $H_0$ is the present day Hubble parameter. We further define the density parameters,

$$\Omega \equiv \frac{\rho}{3 M_2^2 H_0^2}, \quad \Omega_k \equiv -\frac{k}{a_0^2 H_0^2}, \quad \Omega_\Upsilon \equiv M^2 \left(\frac{\beta_3}{3} \Upsilon^3 + \beta_2 \Upsilon^2 + \beta_1 \Upsilon + \frac{\beta_0}{3}\right)$$, (4.2)

where as usual

$$\Omega = \Omega_\gamma (1 + z)^4 + \Omega_m (1 + z)^3 + \Omega_\Lambda + \ldots$$ (4.3)


is given in terms of the fluid components, respectively, for radiation, matter and vacuum etc., at redshift $z = 0$. Dividing through by $H_0^2$, the Friedmann equation then assumes the form

$$E^2 = \Omega + \Omega_k (1 + z)^2 + \Omega_{\gamma},$$

where $E^2 = 1$ at redshift $z = 0$ by definition.

In the action (2.1), the parameter $\beta_0$ is a cosmological constant for the metric $g_{\mu\nu}$ and hence degenerate with the vacuum energy $\rho_{\text{vac}}$. This can be used to set

$$\beta_0 = -3\beta_1 - 3\beta_2 - \beta_3.$$  (4.5)

The reason for this choice is simply to have similar conventions to those sometimes used in massive gravity (where this choice eliminates the contribution of the mass potential to the cosmological constant but only for backgrounds where $g = f$). This parameter choice factorizes $\Omega_{\gamma}$ as

$$\Omega_{\gamma} = M^2 Y - \frac{1}{3} \left[ \beta_3 (Y^2 + Y + 1) + 3\beta_2 (Y + 1) + 3\beta_1 \right].$$

Note that $Y_0 = 1$ is a consistent normalization that can be achieved on rescaling $f_{\mu\nu}$ by adjusting $M_f^2$. This fixes, $\Omega + \Omega_k = 1$ at the present epoch. The Friedmann equation for $f_{\mu\nu}$ (3.3) then allows us to determine, e.g., $\beta_4$ as,

$$\beta_4 = -(\beta_1 + 3\beta_3) - 3 \left( \beta_2 - \frac{k}{a_0 m^2} - \frac{M_f^2}{M^2} \right).$$

Finally, note that the parameters $\beta_1, \beta_2, \beta_3$ are degenerate with the mass scale $m^2$. In particular, we can fix,

$$\beta_1 = -1 - 2\beta_2 - \beta_3,$$

(4.8)
to render the Fierz-Pauli mass of the massive fluctuation independent of the $\beta$'s. Then for this choice,

$$m_{\text{FP}}^2 = \frac{M_f^2 + 1}{M_f^2} m^2$$

(4.9)
is the Fierz-Pauli mass of the massive fluctuation when expanding the metrics around a common background for canonically normalized fluctuations. For $M_f >> M_g$ the $f_{\mu\nu}$ sector decouples and $m^2$ becomes the mass for the massive fluctuation of $g_{\mu\nu}$.

**4.2 Model specific considerations**

We consider a model with zero spatial curvature, $k = 0$, and $\beta_3 = 0$. With the above choices for $\beta_0, \beta_1, \beta_4$, these can now be parameterized as,

$$\beta_0 = 3 + 3\beta_2 = 6 - 3\alpha$$

$$\beta_1 = -1 - 2\beta_2 = -3 + 2\alpha$$

$$\beta_4 = 1 - \beta_2 + 3 \frac{M_f^2}{M^2} = \alpha + 3 \frac{M_f^2}{M^2}$$

(4.10)
where we have defined
\[ \alpha \equiv 1 - \beta_2. \quad (4.11) \]
Note that this is the \( \bar{\alpha} \) parameter of \( \text{(3)} \), discussed further in Appendix \( \text{[3.1]} \). We can now write the two Friedmann equations \( \text{(3.2)} \) and \( \text{(3.3)} \) in terms of \( \alpha \) as
\[ E^2 = \Omega + (2 - \alpha)M^2 + (2\alpha - 3)M^2\Upsilon + (1 - \alpha)M^2\Upsilon^2, \quad (4.12) \]
and
\[ \left[ \frac{M_*^2}{M^2} + \frac{\alpha}{3} \right] \Upsilon^3 + \left[ 1 - \alpha - \frac{M_*^2}{M^2}E^2 \right] \Upsilon + \frac{2\alpha}{3} - 1 = 0. \quad (4.13) \]
Since our goal is to obtain an analytical expression for \( E(z) \), we first substitute the expression for \( E^2 \) given in \( \text{(4.12)} \) into \( \text{(4.13)} \) to obtain
\[ \left[ \frac{\alpha}{3} + \frac{M_*^2}{M^2} + (\alpha - 1)M_*^2 \right] \Upsilon^3 + (3 - 2\alpha)M_*^2\Upsilon^2 \]
\[ + \left[ 1 - \alpha - \Omega \frac{M_*^2}{M^2} + (\alpha - 2)M_*^2 \right] \Upsilon + \frac{2\alpha}{3} - 1 = 0. \quad (4.14) \]
We then solve this cubic equation for \( \Upsilon \) (being careful to pick the solution corresponding to \( \Upsilon_0 = 1 \)) and put this back into the Friedmann equation \( \text{(4.12)} \). The resulting solution is fitted to data in the next sections.

In the case that \( \alpha = 3/2 \) (corresponding to \( \beta_1 = 0 \) discussed earlier), we obtain a particularly simple form for the expansion history,
\[ E^2 = \Omega \left[ 1 - \Omega_{\Lambda}^{\text{eff}} \right] + \Omega_{\Lambda}^{\text{eff}}. \quad (4.15) \]
where,
\[ \Omega_{\Lambda}^{\text{eff}} = \frac{M_*^2M^2}{M^2 + M_*^2(2 + M^2)}. \quad (4.16) \]
appears as an effective cosmological constant and also contributes to an effective Planck mass. To compare with data, we do not include any vacuum contribution in \( \Omega \) which, at late times, is then entirely given by \( \rho_m \). The actual outcome of this evolution equation depends on the relation between the scale \( M_g \) and the physical Planck mass \( M_P \) (or equivalently, the Newton constant \( G_N \)). In general this will be of the form,
\[ M_P = Q M_g, \quad (4.17) \]
where \( Q \) is given by the parameters of the theory. To explicitly determine \( Q \), we need localized bimetric solutions that are not well understood at present, so we treat it as a parameter \(^4\). Then, in view of \( \text{(4.2)} \) \( \Omega \) is related to the physical density parameter by
\[ \Omega = \Omega_{\text{phys}} Q^2 \quad (4.18) \]

\(^4\)It is easy to determine \( Q \) in the linearized theory which will also exhibit a milder version of vDVZ discontinuity. However, the real \( Q \) for non-linear solution will be different due to the Vainshtein effect.
and the evolution equation becomes,

\[ E^2 = \Omega_{\text{phys}} Q^2 (1 - \Omega^\text{eff}_\Lambda) + \Omega^\text{eff}_\Lambda. \]  \hspace{1cm} (4.19)

Comparison to data, gives \( \Omega^\text{eff}_\Lambda = 0.7 \) so that one always has \( \Omega_{\text{phys},0} Q^2 = 1 \). The limiting values of \( Q \) are \( Q = 1 \) \( (M_g = M_P) \) and \( Q^2 (1 - \Omega^\text{eff}_\Lambda) = 1 \) which is equivalent to the concordance model.

As some limits, let us consider,

\[ M \to \infty \Rightarrow \Omega^\text{eff}_\Lambda \to \frac{M^2_s}{1 + M^2_s}, \]  \hspace{1cm} (4.20)

and

\[ M_s \to \infty \Rightarrow \Omega^\text{eff}_\Lambda \to \frac{M^2}{2 + M^2}. \]  \hspace{1cm} (4.21)

This means that we always have a cosmological constant universe, even as either \( M \) or \( M_s \) are extremely large, as long as the other mass ratio is of the appropriate size. As an example, let us set \( Q = 1 \). Then the first of the above limits, eq. (4.20), corresponds to

\[ M >> 1 \iff m >> H_0. \]  \hspace{1cm} (4.22)

This, as can be seen from eq. (4.13), implies a very large mass for the spin-2 field measured in Hubble units. As is straightforward to verify, given that \( m/H_0 >> 1 \) we need \( M_f \sim 1.5 M_g \) in order for \( \Omega^\text{eff}_\Lambda \) to mimic \( \Omega_\Lambda \sim 0.7 \). It is indeed interesting that this bimetric model can mimic the concordance model even for a very large mass for the massive spin-2 mode. It is not clear, however, how this will effect smaller scale physics.

The second limit, eq. (4.21), corresponds to

\[ M_s >> 1 \iff M_f >> M_g. \]  \hspace{1cm} (4.23)

This will effectively decouple the \( f_{\mu\nu} \) field, making it a free field determined by the vacuum Einstein equations. This in turn makes the fluctuations of \( g_{\mu\nu} \) massive with mass \( m \), which again can be seen from (4.9). Now, in order for \( \Omega^\text{eff}_\Lambda \) to mimic \( \Omega_\Lambda \sim 0.7 \), we need \( m \sim 2.2 H_0 \).

We also note that the limit

\[ M << 1 \iff m << H_0, \]  \hspace{1cm} (4.24)

can be seen from eq. (4.16) to imply \( \Omega^\text{eff}_\Lambda \sim 0 \), so that the limit of vanishing mass for the massive spin-2 field (for non-zero \( M_f \)) gives no cosmological contribution.

5. Data

In this study, we limit ourselves to purely geometrical tests of the expansion history of the universe. That is, tests only involving cosmological distances. We defer possible constraints involving smaller scale gravity and structure formation to upcoming work.
5.1 Type Ia supernova data

As being standardizable candles and thus effective distance indicators, Type Ia supernovae (SNe Ia) are one of the most direct probes we have of the expansion history of the universe. In this paper, we use the Union2 compilation of SNe Ia. This data set contains SNe Ia from, e.g., the Supernova Legacy Survey, ESSENCE survey and HST observations. After selection cuts, the data set amounts to 557 SNe Ia, spanning a redshift range of \(0 \lesssim z \lesssim 1.4\), analyzed in a homogeneous fashion using the spectral-template based fit method SALT2.

5.2 Cosmic Microwave Background and Baryon Acoustic Oscillations

The position of the first Cosmic Microwave Background (CMB) power-spectrum peak, representing the angular scale of the sound horizon at the era of recombination, is given by

\[
\ell_A = \pi \frac{d_A(z_*)}{r_s(z_*)},
\]

where \(d_A(z_*)\) is the comoving angular-diameter distance to recombination while the comoving sound horizon at photon decoupling, \(r_s\), is given by

\[
r_s = \int_{z_*}^{\infty} \frac{c_s}{H(z)} \, dz,
\]

which depends upon the speed of sound before recombination, \(c_s\). Here we use CMB measurements from the seven-year Wilkinson Microwave Anisotropy Probe (WMAP) observations \([30]\), in this case the WMAP7.2 results reported at lambda.gsfc.nasa.gov, adopting the value \(\ell_A = 302.56 \pm 0.78\). We further assume \(z_* = 1091.12\) exactly (variations within the uncertainties about this value do not give significant differences to the results).

Baryon Acoustic Oscillations (BAO) observations are often compared to theoretical models using measurements of the ratio of the sound horizon scale at the drag epoch, \(r_s(z_d)\), to the dilation scale, \(D_V(z)\). The drag epoch, \(z_d \approx 1020\), is the epoch at which the acoustic oscillations are frozen in. A more model-independent constraint can be achieved by multiplying the BAO measurement of \(r_s(z_d)/D_V(z)\) with the CMB measurement \(\ell_A = \pi d_A(z_*)/r_s(z_*),\) thus cancelling some of the dependence on the physical size of the sound horizon scale \([31]\). In doing this, we are effectively only left with the assumption that the observed inhomogeneities in the large scale distribution of galaxies and in the CMB temperature reflects the same (redshifted) physical scale.

In \([32]\), measurements of the ratio \(r_s(z_d)/D_V(z)\) at two redshifts, \(z = 0.2\) and \(z = 0.35\), are reported as \(r_s(z_d)/D_V(0.2) = 0.1905 \pm 0.0061\) and \(r_s(z_d)/D_V(0.35) = 0.1097 \pm 0.0036\). Before matching to cosmological models, we need to implement a correction for the difference between the sound horizon at the end of the drag epoch, \(z_d \approx 1020\), and the sound horizon at last-scattering, \(z_* \approx 1091\), the first being relevant for the BAO and the second for the CMB. Here, we use \(r_s(z_d)/r_s(z_*) = 1.0451 \pm 0.0158\), again using WMAP7.2 results. A possible caveat is that this ratio was calculated using standard cosmology for the evolution between the two redshifts. However, we expect this to be a good approximation since the redshift difference is relatively small, and the sound horizon at decoupling and
Figure 1: Cosmological constraints for $\alpha = 3/2$ using supernova distances (SN) and the ratio of the observed scales of the baryon acoustic oscillations as imprinted in the cosmic microwave background and the large scale galaxy distribution (CMB/BAO). In the left panel, a value of $M_* = 1$ is assumed, in the right panel, $M_* = 3$.

drag is mostly governed by the fractional difference between the number of photons and baryons. Combining this with $\ell_A$ gives the numbers we employ in our cosmology fits,

$$\frac{d_A(z_*)}{D_V(0.2)} = 17.55 \pm 0.62,$$

$$\frac{d_A(z_*)}{D_V(0.35)} = 10.11 \pm 0.34. \tag{5.3}$$

We take into account the correlation between these measurements using a correlation coefficient of 0.337 calculated in [32].

6. Results of observational tests

6.1 $\alpha = 3/2$ ($\beta_1 = 0$)

If we fix the value of $M_* = M_f/M_\gamma$ and allow for a cosmological constant, we can (assuming a flat universe) fit for the spin-2 mass parameter $m$ and the matter density $\Omega_m$. We expect to get a good fit for $M = 0$ and $\Omega_m \sim 0.3$ since this corresponds to the concordance cosmology with $\Omega_\Lambda = 0.7$. If we are able to get a good fit also when $\Omega_m \to 1$, that is with no cosmological constant, depends on the value of $M_*$. From Figure 1, we see that in the case of $M_* = 1$ (left panel), this is not possible, while in the case of $M_* = 3$ (right panel), it is, as expected from Eq. (4.13). In this and all figures hereafter, shaded contours shows constraints for SN and CMB/BAO data, respectively, corresponding to 95% confidence interval for two parameters. Combined constraints are shown with solid lines corresponding to 95% and 99.9% confidence intervals for two parameters.

In what follows, we will set $\Omega_m = 1$ and $\Omega_\Lambda = 0$ in order to investigate whether bi-metric gravity models can explain the apparent accelerated expansion seen in cosmological...
6.2 General $\alpha$ ($\beta_1 \neq 0$)

We now proceed to fit also the value of $\alpha$ (assuming $\Omega_{\Lambda} = 0$). For $M_* = 3$, we obtain the result depicted in the left panel of Figure 3 showing a good fit to the data for $\alpha \sim 1.4$ and $M \sim 3.0$.

Even more generally, we want to fit $\alpha$, $M$ and $M_*$ simultaneously. This requires care when projecting results on two dimensional surfaces. Since we are able to obtain good fits to the data for both $M \to \infty$ and $M_* \to \infty$, we will not be able to cover the entire non-negligible probability function in our grid of tested parameter values which in principle is required to perform a proper marginalization. This will mostly affect the projected constraints for $M$ and $M_*$ where generally a larger parameter space in one of the parameters will give more weight to the likelihood function of the other parameter at lower values (see Figure 3). In the following, we present results for $M_{\text{max}} = M_{\text{max}}^* = 10$, the equivalent of putting a flat prior on the values of $M$ and $M_*$ to be in the interval $[0, 10]$.

Results in the $[\alpha, M]$-plane after marginalizing over $M_*$ are shown in the right panel of Figure 3. Comparing to the left panel, we can see that allowing $M_*$ to vary slightly shifts and widens the allowed values of $\alpha$ and $M$ as compared to the case of a fixed value of $M_*$. Marginalizing down to one parameter surfaces (assuming flat prior probabilities), our data constrains the parameter values of the bimetric gravity model to be (at $95\%$ confidence level for one parameter)

$$1.1 \lesssim \alpha \lesssim 1.5, \quad 2 \lesssim M \lesssim 3.5, \quad 1.5 \lesssim M_* \lesssim 3.$$  \hfill (6.1)
Figure 3: Left panel: Cosmological constraints in the $[\alpha, M]$-plane assuming $\Omega_m = 1$ and $M_\star = 3$. Right panel: Constraints in the $[\alpha, M]$-plane for $\Omega_m = 1$ after marginalizing over $M_\star$ with a flat prior in the interval $[0, 10]$.

6.3 Interpretation of results

As expected from the form of the cosmological solutions, the bimetric gravity model is perfectly capable of matching geometric cosmological data. For the case of $\alpha = 3/2$ ($\beta_1 = 0$), this was obviously going to be possible for some values, but we find that data clearly favours the regions $M \sim 2.5, M_\star \gtrsim 1.5$ or $M \sim 2.5, M_\star \sim 1.5$ (assuming $M_P = M_g$ for simplicity, the more general case having been discussed in section 4.2). Allowing for an arbitrary $\alpha$ is equivalent to switching on the $\beta_1$ interactions, with the only restriction that we focus here on bigravity models for which $\beta_1 = -1 - 2\beta_2$, which include massive gravity limits of bigravity. In this case, we find that data favours a rather narrow region of parameter values. In particular, large deviations from $\alpha = 3/2$ as well as $m = 3H_0$ and $M_f = 2M_g$ are being disfavoured by data.

7. Conclusions

We have investigated cosmological solutions in the unique classically consistent theory of bimetric gravity. Under the assumption that both metrics respect the symmetries of spatial isotropy and homogeneity, we derived the most general cosmological evolution equations for this theory. The generic solution was demonstrated to always allow for a cosmic evolution starting out from an ordinary FLRW matter dominated universe while evolving towards a de Sitter (or AdS) geometry at late times in the expansion history. We explored further a particular class of solutions, corresponding to neglecting cubic nonlinear interactions in the potential for the two metrics. A subclass of these solutions, the minimal model, was shown to be completely degenerate with the evolution of a universe described by the usual cosmic concordance model in general relativity. Using recent data from SNe Ia, CMB and BAO we demonstrated that data favoured regions in parameter space close to this
minimal model. Interestingly, assuming $M_P \sim M_g$, we found that even for a very large mass for the massive spin-2 mode can this theory match observations as good as general relativity, although under the assumption of order unity values for the relative couplings, data favoured a mass of the order of the Hubble scale $m \sim H_0$.

Since the bimetric gravity theory is a highly nontrivial but consistent modification of general relativity, it is important to explore its consequences further. In particular, exploring whether more general classes of solutions than the subclass studied in this paper are able to match observations. Since we have demonstrated the existence of bimetric solutions that match general relativity on cosmic scales it is important to study perturbations on these solutions and also obtain the corresponding small scale solutions in order to see what this theory predicts for observations on e.g. cluster and galaxy scale. Another important issue is the determination of the observed Planck scale in terms of the parameters of the theory.

Acknowledgments

We would like to thank Bo Sundborg, Rachel Rosen, Stefan Sjörs, for useful discussions and comments. We are indebted to M. Crisostomi for pointing out an erroneous statement in our first draft. EM acknowledges support for this study by the Swedish Research Council.

Note Added: When the writing of this paper was being finalized, a paper with similar intent appeared [27] with some overlap with section 3 of the present paper. After this paper was submitted to the arXiv, [33] appeared which also considers cosmological implications of bimetric gravity

A. Derivation of the equations of motion

In this appendix we give some details on the derivation of the equations of motion, equations (2.14), (2.15), (2.16), and (2.17). We insert our ansatz for the two metrics (2.11), into the $g$- and $f$-equations of motion (2.4) and (2.5), respectively. With our ansatz the 00-component of the $g_{\mu\nu}$ equation of motion (2.4) becomes

$$-3 \dot{a}^2 - 3 k a^2 + m^2 \left( \beta_0 + 3 \beta_1 \frac{Y}{a} + 3 \beta_2 \frac{Y^2}{a^2} + \beta_3 \frac{Y^3}{a^3} \right) = \frac{T^0_0}{M^2_g}, \quad (A.1)$$

and the 11-component reads

$$-2 \ddot{a} - \dot{a}^2 - \frac{k}{a} a^2 + m^2 \left( \beta_0 + \left[ \frac{2 Y}{a} + X \right] \beta_1 + \left[ \frac{Y^2}{a^2} + 2 \frac{XY}{a} \right] \beta_2 + \frac{XY^2}{a^2} \beta_3 \right) = \frac{T^1_1}{M^2_g}. \quad (A.2)$$

Now we insert the solution of the Bianchi constraint,

$$X = \frac{\dot{Y}}{\dot{a}} = \frac{dY}{da}, \quad (A.3)$$
to rewrite (A.2) as
\[-2\ddot{a} - \frac{a^2}{a^2} - \frac{k}{a^2} + m^2 \left( \beta_0 + \left[ \frac{2Y}{a} + \dot{Y} \right] \beta_1 + \left[ \frac{Y^2}{a^2} + 2 \frac{YY}{aa} \right] \beta_2 + \frac{YY^2}{aa^2} \beta_3 \right) = \frac{T_1}{M_g}.
\] (A.4)

In order to compute the contribution from the kinetic terms to the equations of motion (2.5), we need the expression for the Ricci tensor $R_{\mu\nu}(f)$ in terms of the two scale factors. The non-vanishing Christoffel symbols for $f_{\mu\nu}$ are
\[
\Gamma^0_{00} = \frac{\dot{X}}{X}, \quad \Gamma^0_{11} = \frac{\dot{Y}Y}{X^2(1 - kr^2)}, \quad \Gamma^0_{22} = \frac{\dot{Y}r^2}{X^2}, \quad \Gamma^0_{33} = \frac{\dot{Y}r^2 \sin^2 \theta}{X^2},
\]
\[
\Gamma^1_{0i} = \frac{\dot{Y}}{Y}, \quad \Gamma^1_{11} = \frac{kr}{1 - kr^2}, \quad \Gamma^1_{22} = -r(1 - kr^2), \quad \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r},
\]
\[
\Gamma^3_{33} = -r \sin^2 \theta (1 - kr^2), \quad \Gamma^3_{23} = -\sin \theta \cos \theta, \quad \Gamma^3_{23} = \cot \theta.
\] (A.5)

From these we compute the non-vanishing components of the Ricci tensor,
\[
R_{00} = \frac{3}{Y^2} \left( -Y\ddot{Y} + \frac{Y\dot{Y}\dot{X}}{X} \right),
\]
\[
R_{11} = \frac{1}{X^2(1 - kr^2)} \left( Y\ddot{Y} - Y\dot{Y}\dot{X} + 2\dot{Y}^2 + 2kX^2 \right),
\]
\[
R_{22} = \frac{r^2}{X^2} \left( Y\ddot{Y} - \frac{Y\dot{Y}\dot{X}}{X} + 2\dot{Y}^2 + 2kX^2 \right),
\]
\[
R_{33} = \frac{r^2 \sin^2 \theta}{X^2} \left( Y\ddot{Y} - \frac{Y\dot{Y}\dot{X}}{X} + 2\dot{Y}^2 + 2kX^2 \right).
\] (A.6)

Thus, the curvature scalar is given by
\[
R = \frac{6}{X^2} \left( \frac{\ddot{Y}}{Y} - \frac{\dot{Y}\dot{X}}{YX} + \frac{\dot{Y}^2}{Y^2} + \frac{kX^2}{Y^2} \right),
\] (A.7)

The Einstein tensor then has the nonvanishing components
\[
R_{00} - \frac{1}{2} f_{00} R = \frac{3}{Y^2} \ddot{Y}^2 + 3 \frac{kX^2}{Y^2},
\]
\[
R_{11} - \frac{1}{2} f_{11} R = -\frac{1}{X^2(1 - kr^2)} \left( 2\ddot{Y} - 2Y\dot{Y}\dot{X} + \dot{Y}^2 + kX^2 \right),
\]
\[
R_{22} - \frac{1}{2} f_{22} R = -\frac{r^2}{X^2} \left( 2\ddot{Y} - 2Y\dot{Y}\dot{X} + \dot{Y}^2 + kX^2 \right),
\]
\[
R_{33} - \frac{1}{2} f_{33} R = -\frac{r^2 \sin^2 \theta}{X^2} \left( 2\ddot{Y} - 2Y\dot{Y}\dot{X} + \dot{Y}^2 + kX^2 \right).
\] (A.8)
For the equations of motion, we raise one index with $f^{\mu\nu}$ and consider the 00- as well as the 11-component. The 00-component of the equations of motion is then obtained as

$$-3\frac{\dot{Y}^2}{X^2Y^2} - 3\frac{k}{Y^2} + \frac{m^2}{M_*^2} \left[ \frac{a^2}{Y^2} \beta_1 + \frac{3a^2}{Y^2} \beta_2 + \frac{3a}{Y} \beta_3 + \beta_4 \right] = 0,$$

(A.9)

whereas the 11-component reads

$$0 = -\frac{1}{X^2} \left( 2\frac{\ddot{Y}}{Y} - 2\frac{\dot{Y} \dot{X}}{YX} + \frac{\dot{Y}^2}{Y^2} \right) - \frac{k}{Y^2} + \frac{m^2}{M_*^2} \left[ \frac{a^2}{XY^2} \beta_1 + \left( \frac{a^2}{Y^2} + \frac{2a}{XY} \right) \beta_2 + \left( \frac{2a}{Y} + \frac{1}{X} \right) \beta_3 + \beta_4 \right].$$

(A.10)

Inserting $X = \dot{Y} / \ddot{a}$ into (A.9) and (A.10) we arrive at

$$0 = -3\frac{\ddot{a}^2}{Y^2} - 3\frac{k}{Y^2} + \frac{m^2}{M_*^2} \left[ \frac{a^2}{Y^2} \beta_1 + \frac{3a^2}{Y^2} \beta_2 + \frac{3a}{Y} \beta_3 + \beta_4 \right],$$

(A.11)

$$0 = -2\frac{\dddot{a} \ddot{a}}{YY} - \ddot{a}^2 \frac{\ddot{Y}}{Y^2} - \frac{k}{Y^2} + \frac{m^2}{M_*^2} \left[ \frac{a^2 \ddot{a}}{YY^2} \beta_1 + \left( \frac{a^2 \ddot{a}}{Y^2} + \frac{2a \dddot{a}}{YY} \right) \beta_2 + \left( \frac{2a}{Y} + \frac{\ddot{a}}{Y} \right) \beta_3 + \beta_4 \right].$$

(A.12)

We now observe that acting with $(3 + (Y / \dot{Y}) \partial_t) / 3$ on (A.11) gives (A.12). Thus, the two equations are equivalent.

**B. Comments related to massive gravity**

In order to facilitate a comparison with other works on both bimetric and massive gravity, using the consistent interaction term of the action (2.1), we note here the relation between the different parameters of the action that has been established in the recent literature on massive gravity, in particular [3, 9]. We also study the nature of the Bianchi constraint and clarify some points related to cosmological solutions recently obtained in the massive gravity literature.

**B.1 Parameters of massive gravity**

In massive gravity, the metric that does not couple to matter is regarded as a fixed background, usually taken to be a flat reference metric (although this is not necessary). This implies removing the kinetic strength $M_f$. One must also fix $\beta_0 = -3\beta_1 - 3\beta_2 - \beta_3$ in order to cancel terms linear in perturbations around this background. In order for $m^2$ to correspond to the mass of the massive spin-2 mode one has to fix also $\beta_1 = -1 - 2\beta_2 - \beta_3$. This effectively eliminates three parameters. Since $\beta_4$ does not enter the equations of motion for $g_{\mu\nu}$ one can consistently eliminate this parameter as well. Thus, out of the five $\beta_n$ parameters of the general bimetric theory only two are important for massive gravity. One is left with four free parameters (including the Planck mass and the spin-2 mass). The remaining two parameters in the interaction can be chosen arbitrarily and (at least) two
conventions have occurred in the literature. First, we note the relations between $\beta_n$ and the $\bar{\alpha}_n$ used in \[9\]

$$
\begin{align*}
\beta_0 &= 6 - 4\bar{\alpha}_3 + \bar{\alpha}_4 , \\
\beta_1 &= -3 + 3\bar{\alpha}_3 - \bar{\alpha}_4 , \\
\beta_2 &= 1 - 2\bar{\alpha}_3 + \bar{\alpha}_4 , \\
\beta_3 &= \bar{\alpha}_3 - \bar{\alpha}_4 .
\end{align*}
$$

(B.1)

The relation between these and the parameters of \[3\] is

$$
\begin{align*}
\bar{\alpha}_3 &= -3\alpha_3 = 6c_3 , \\
\bar{\alpha}_4 &= 12\alpha_4 = -48d_5 .
\end{align*}
$$

(B.2)

B.2 The Bianchi constraint

We recall the Bianchi constraint (2.12)

$$
\frac{3m^2}{a} \left[ \beta_1 + 2\frac{Y}{a} \beta_2 + \frac{Y^2}{a^2} \beta_3 \right] \left( \dot{Y} - \dot{a}X \right) = 0 .
$$

(B.3)

Clearly we can enforce this by looking for solutions for $Y$ which force the left bracket to vanish. This will however only result in the ordinary general relativistic equations for $g_{\mu\nu}$ with the addition of a cosmological constant proportional to $m^2$ as is evident from (2.14), independent of any dynamics for $f_{\mu\nu}$. As such they represent a particular class of screening solutions with special values for the parameters of the action.

The reason these values are special can be quantified further. If we consider perturbations around solutions where the metrics are proportional, i.e.

$$
g = \bar{g} + \delta g , \quad f = C\bar{g} + \delta f ,
$$

we have that

$$
g^{-1} f \approx C + \bar{g}^{-1} (\delta f - C\delta g) \equiv C + \delta M ,
$$

(B.5)

where $\bar{g}\delta M$ defines the massive fluctuations up to a constant of proportionality. Using this and expanding the interaction term in the equations of motion (2.4) to linear order we obtain (excluding linear contributions from the cosmological term)

$$
\sum_{n=0}^{3} (-1)^n \beta_n g\lambda (\mu Y_{(n)\nu}) (\sqrt{g^{-1}} f) \approx (\beta_1 + 2C\beta_2 + C^2\beta_3) [\bar{g} (\text{Tr}(\delta M) - \delta M)]_{(\mu\nu)} .
$$

(B.6)

This is just the Fierz-Pauli mass contribution to the linearized equations of motion. Comparing this to the Bianchi constraint (B.3) we see that forcing the left bracket of the Bianchi constraint to vanish for proportional metrics is equivalent to choosing a constant of proportionality such that the Fierz-Pauli mass term for the fluctuations vanish. A similar conclusion holds also for the cosmological solutions when the spatial metrics are proportional but this requires more work to show than the simple example discussed here.
For our ansatz (2.10), the Bianchi constraint (B.3) allows to be less strict and only demand that the spatial part of the metrics are proportional, i.e. \( Y(t) = Ca(t) \). Then by forcing the left bracket of the Bianchi constraint to vanish we can find bimetric solutions that read

\[
Y^2(t) = C^2 a^2(t), \quad X^2(t) = \frac{-3C^2M^2_2 \dot{a}^2}{-3kM^2 + m^2(\beta_2 + C(2\beta_3 + C\beta_4))a^2},
\]

where \( X \) is determined from the \( f_{\mu\nu} \) equations of motion (2.5) and the constant \( C \) is determined from the Bianchi constraint (B.3) to be given by

\[
C_\pm = -\frac{\beta_2 \pm \sqrt{\beta_2^2 - 3\beta_1\beta_3}}{\beta_3}.
\]

Although the existence of these solutions might appear interesting, in the bimetric theory where we include dynamics for \( f_{\mu\nu} \), they do not modify general relativity in any respect apart from the contribution of a cosmological source given explicitly by the addition

\[
m^2 \left[ \beta_0 + \frac{1}{\beta_3} \left( 2\beta_2^2 - 3\beta_1\beta_2 \pm 2\beta_2 \sqrt{\beta_2^2 - \beta_1\beta_3} + 2\beta_1 \sqrt{\beta_2^2 - \beta_1\beta_3} \right) \right],
\]

to the usual Friedmann and acceleration equations of general relativity without a cosmological constant. From our previous reasoning this can be understood as a consequence of using the parameters of the general theory to impose the vanishing of the massive fluctuations. Hence, the theory contains only massless spin-2 fluctuations and must be equivalent to ordinary general relativity. Since we keep all parameters arbitrary such solutions could be expected, and a very similar conclusion was reached when examining spherically symmetric solutions in the consistent bimetric theory of gravity in [24].

If we do not include a kinetic term for \( f_{\mu\nu} \), as in the massive gravity setups, these solutions do allow for regular cosmological evolution for \( g_{\mu\nu} \), but again only contribute with a constant source addition (c.f. (2.3)). For such a scenario \( X \) is not given as in (B.7), since we have used the \( f_{\mu\nu} \) equations of motion to derive that solution. Indeed, without the dynamical term for \( f_{\mu\nu} \), \( X \) can be an arbitrary function of time and in particular it is possible to choose \( X \) such that \( f_{\mu\nu} \) represents an open chart of the Minkowski metric, as was recently demonstrated in [8]. From this perspective it is also clear why it is not possible to find solutions with positive or zero spatial curvature for a non-dynamical \( f_{\mu\nu} \), there is simply no representation of Minkowski space as a homogeneous and isotropic space with nontrivial scale factor for these cases.

**B.3 \( X = \text{constant} \)**

More generally, we recall the true dynamical Bianchi constraint (2.13),

\[
X = \frac{\dot{Y}}{a} = \frac{dY}{da},
\]

obtained from the vanishing of the right bracket of (B.3). Note that this encodes also the \( Y \propto a \) solutions obtainable from enforcing the vanishing of the left bracket, but in this case
will enforce $f_{\mu\nu} \propto g_{\mu\nu}$. Thus, for arbitrary parameters of the theory (B.10) contains all non-trivial information about the nature of the constraint. This allow for the simple class of solutions

$$Y = C_1 a + C_2 \Rightarrow \Upsilon = C_1 + \frac{C_2}{a}. \quad \text{(B.11)}$$

where $C_{1,2}$ are constants. The Friedmann equation for $g_{\mu\nu}$ (2.4) is then given by

$$H^2 \frac{k}{m^2} + \frac{k}{a^2 m^2} = \rho_\star + \frac{\beta_0}{3} + C_1 \left( \beta_1 + \beta_2 C_1 + \frac{\beta_3 C_1^2}{3} \right) + \left( \beta_1 + 2\beta_2 C_1 + \beta_3 C_1^2 \right) \frac{C_2}{a}$$

$$+ \left( \beta_2 + \beta_3 C_1 \right) \frac{C_2^2}{a^2} + \frac{\beta_3}{3} \frac{C_2^3}{a^3}. \quad \text{(B.12)}$$

For $C_2 = 0$ this clearly only amounts to a cosmological constant contribution to the usual general relativistic equations. Moreover, if $C_1 = 1$ we see that the choice $\beta_0 = -3\beta_1 - 3\beta_2 - \beta_3$ completely eliminates this contribution (c.f. discussion in Appendix B.1).

More interestingly, if $C_2 \neq 0$ the addition to the general relativistic Friedmann equation is precisely of the form to add extra fluid components of the conventional type. From the perspective of the usual formulation of massive gravity then, where there is no dynamical equation for $f_{\mu\nu}$, these solutions can be interesting since they can exist for an arbitrary source and give a contribution to all the fluid components except for radiation. Solutions of this type were recently discussed in that context in [34]. The added degeneracy in parameter space is however a rather unpleasant feature.

In the context of bimetric theory that we are considering here, the equations of motion for $f_{\mu\nu}$ (3.3) imply that these solutions will constrain the source, as is evident from the quartic equation (3.4). In fact, they will tell us that the source will contain terms of the form

$$\frac{a}{C_1 a + C_2}, \quad \frac{C_2}{C_1 a + C_2}, \quad \frac{C_2}{a(C_1 a + C_2)}, \quad \frac{C_2}{a^2(C_1 a + C_2)}. \quad \text{(B.13)}$$

up to various multiplicative constants. For $C_2 = 0$, they are of the standard matter fluid type but then only the constant contribution can remain, as we have also remarked on earlier. For $C_1 = 0$ (such that $f_{\mu\nu}$ is flat) the Bianchi constraint (B.10) tell us that either $X = 0$ or $a$ is a constant and that there can be no cosmological evolution and hence this case is not interesting for our purposes. This conclusion was reached also in the massive gravity context in [35]. In that case however no dynamics where considered for $f_{\mu\nu}$ and the possibility of finding solutions by enforcing the non-dynamical Bianchi constraint was missed due to a too restrictive parameterization of Minkowski space.

If we do proceed and eliminate the source in the above prescribed manner we end up with an evolution equation where the Hubble expansion is driven by terms of exactly the same form as in (B.13). Although it is intriguing that these terms actually do scale in accordance with the usual matter fluid components for large values of $a$, for the purposes of this paper we do not want to impose any such restrictions on the matter sector.

**References**

[1] D. G. Boulware and S. Deser, Phys. Rev. D 6 (1972) 3368.
[2] C. de Rham and G. Gabadadze, Phys. Rev. D 82 (2010) 044020 [arXiv:1007.0443 [hep-th]].
[3] C. de Rham, G. Gabadadze and A. J. Tolley, Phys. Rev. Lett. 106 (2011) 231101 [arXiv:1011.1232 [hep-th]].
[4] S. F. Hassan and R. A. Rosen, arXiv:1106.3344 [hep-th].
[5] S. F. Hassan, R. A. Rosen and A. Schmidt-May, arXiv:1109.3230 [hep-th].
[6] S. F. Hassan and R. A. Rosen, arXiv:1111.2070 [hep-th].
[7] C. de Rham, G. Gabadadze, L. Heisenberg and D. Pirtskhalava, Phys. Rev. D 83 (2011) 103516 [arXiv:1010.1780 [hep-th]].
[8] A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, JCAP 1111 (2011) 030 [arXiv:1109.3845 [hep-th]].
[9] S. F. Hassan and R. A. Rosen, JHEP 1107 (2011) 009 [arXiv:1103.6055 [hep-th]].
[10] N. Rosen, Phys. Rev. 57 (1940) 150.
[11] N. Rosen, In *Erice 1975, Proceedings, Topics In Theoretical and Experimental Gravitation Physics*, New York 1977, 273-294.
[12] A. Salam and J. A. Strathdee, Phys. Rev. D 16 (1977) 2668.
[13] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, Annals Phys. 305 (2003) 96 [hep-th/0210184].
[14] M. Banados, A. Gomberoff, D. C. Rodrigues and C. Skordis, Phys. Rev. D 79 (2009) 063515 [arXiv:0811.1270 [gr-qc]].
[15] Z. Berezhiani, D. Comelli, F. Nesti and L. Pilo, Phys. Rev. Lett. 99 (2007) 131101 [hep-th/0703264 [HEP-TH]].
[16] D. Blas, D. Comelli, F. Nesti and L. Pilo, Phys. Rev. D 80 (2009) 044025 [arXiv:0905.1699 [hep-th]].
[17] N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, Nucl. Phys. B 597 (2001) 127 [hep-th/0007220].
[18] T. Damour and I. I. Kogan, Phys. Rev. D 66 (2002) 104024 [hep-th/0206042].
[19] M. Milgrom, Phys. Rev. D 80 (2009) 123536 [arXiv:0912.0790 [gr-qc]].
[20] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A 173 (1939) 211.
[21] W. Pauli and M. Fierz, Helv. Phys. Acta 12 (1939) 297.
[22] C. Aragone and S. Deser, Nuovo Cim. A 3 (1971) 709.
[23] S. F. Hassan and R. A. Rosen, arXiv:1109.3515 [hep-th].
[24] D. Comelli, M. Crisostomi, F. Nesti and L. Pilo, arXiv:1110.4967 [hep-th].
[25] C. J. Isham, A. Salam and J. A. Strathdee, Phys. Rev. D 3 (1971) 867.
[26] T. Damour, I. I. Kogan and A. Papazoglou, Phys. Rev. D 66 (2002) 104025 [hep-th/0206044].
[27] M. S. Volkov, arXiv:1110.6153 [hep-th].
[28] K. Koyama, G. Niz and G. Tasinato, Phys. Rev. D 84 (2011) 064033 [arXiv:1104.2143 [hep-th]].

[29] R. Amanullah, C. Lidman, D. Rubin, G. Aldering, P. Astier, K. Barbary, M. S. Burns and A. Conley et al., Astrophys. J. 716 (2010) 712 [arXiv:1004.1711 [astro-ph.CO]].

[30] E. Komatsu et al. [WMAP Collaboration], Astrophys. J. Suppl. 192 (2011) 18 [arXiv:1001.4538 [astro-ph.CO]].

[31] J. Sollerman, E. Mortsell, T. M. Davis, M. Blomqvist, B. Bassett, A. C. Becker, D. Cinabro and A. V. Filippenko et al., Astrophys. J. 703 (2009) 1374 [arXiv:0908.4276 [astro-ph.CO]].

[32] B. A. Reid et al. [SDSS Collaboration], Mon. Not. Roy. Astron. Soc. 401 (2010) 2148 [arXiv:0907.1660 [astro-ph.CO]].

[33] D. Comelli, M. Crisostomi, F. Nesti and L. Pilo, arXiv:1111.1983 [hep-th].

[34] A. H. Chamseddine and M. S. Volkov, Phys. Lett. B 704 (2011) 652 [arXiv:1107.5504 [hep-th]].

[35] G. D’Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava and A. J. Tolley, arXiv:1108.5231 [hep-th].