Lepton flavor violating decays $l_j \to l_i \gamma$ in the $U(1)_X$ SSM model within the mass insertion approximation

Tong-Tong Wang$^{1,2,a}$, Shu-Min Zhao$^{1,2,b}$, Jian-Fei Zhang$^{1,2,c}$, Xing-Xing Dong$^{1,2,d}$, Tai-Fu Feng$^{1,2,3,e}$

1 Department of Physics, Hebei University, Baoding 071002, China
2 Key Laboratory of High-precision Computation and Application of Quantum Field Theory of Hebei Province, Baoding 071002, China
3 Department of Physics, Chongqing University, Chongqing 401331, China

Abstract Three singlet new Higgs superfields and right-handed neutrinos are added to MSSM to obtain $U(1)_X$ SSM model. Its local gauge group is $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$. In the framework of $U(1)_X$ SSM, we study muon anomalous magnetic moment and lepton flavor violating decays $l_j \to l_i \gamma (j = 2, 3; i = 1, 2)$ within the mass insertion approximation (MIA). Through the MIA method, we can find the parameters that directly affect the analytical result of the lepton flavor violating decays $l_j \to l_i \gamma$, which make our work more convenient. We want to provide a set of simple analytic formulas for the form factors and the associated effective vertices, that may be very useful for future phenomenological studies of the lepton flavor violating decays. According to the accuracy of the numerical results which the influence of different sensitive parameters, we come to the conclusion that the non-diagonal elements which correspond to the generations of the initial lepton and final lepton are main sensitive parameters and lepton flavor violation (LFV) sources. This work can provide a clear signal of new physics (NP).

1 Introduction

Lepton has unitary matrix similar to Cabibbo–Kobayashi–Maskawa (CKM) mixed matrix. The breaking theory of electric weak symmetry and neutrino oscillation experiment show that lepton flavor violation (LFV) exists both theoretically and experimentally [1–3]. The standard model (SM) is already a mature theory. However, the lepton number is conserved in the SM, so there is no LFV process in the SM [4]. Through research, it is necessary to expand the SM. Any sign of LFV can be regarded as evidence of the existence of new physics (NP) [5].

Physicists have extended SM and obtained a large number of extended models, among which the minimum supersymmetric standard model (MSSM) is the most concerned model. However, it is gradually found that MSSM also has problems: $\mu$ problem [6] and zero mass neutrinos [7]. To solve these problems, we pay attention to the $U(1)$ expansions of MSSM. We extend the MSSM with $U(1)_X$ gauge group, whose symmetry group is $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$ [8–10]. It adds three Higgs singlet superfields and right-handed neutrino superfields beyond MSSM [11]. There are five neutral CP-even Higgs component fields in the model, which come from two Higgs doublets and three Higgs singlets respectively. Therefore, the mass mixing matrix is $5 \times 5$, and the 125.1 GeV Higgs particle [12,13] corresponds to the lightest mass eigenstate.

To improve the corrections to LFV processes of $l_j \to l_i \gamma$, people discuss different SM extended models [14], for example minimal R-symmetric supersymmetric standard model [15], MSSM extension with gauged baryon and lepton numbers [16], SM extension with a hidden $U(1)_X$ gauged symmetry [17] and lepton numbers and supersymmetric low-scale seesaw models [18]. It is worth noting that in our previous work, we have studied lepton flavor violating decays $l_j \to l_i \gamma$ in the $U(1)_X$ SSM model [19]. The above works and most of the research on LFV are studied with the mass eigenstate method. Using this method to find sensitive parameters is often not intuitive and clear enough, which depends on the mass eigenstates of the particles and rotation matrices. It will lead us to pay too much attention to many unimportant parameters. Now we use a novel calculation method called as mass insertion approximation (MIA) [20–23], which uses the electroweak interaction eigenstate and treats perturba-
tively the mass insertions changing slepton flavor. By means of mass insertions inside the propagators of the electroweak interaction sleptons eigenstates, at the analytical level, we can find many parameters that have direct impact on LFV. It is worth noting that these parameters are considered between all possible flavor blends among SUSY partner of sleptons, in which their particular origin has no assumption and is independent of the model [21]. In addition, the MIA method has been applied to other works related to LFV, including the $h$, $H$, $A \to \tau \mu$ decays induced from SUSY loops [21], effective lepton flavor violating $H \ell_i \ell_j$ vertex from right-handed neutrinos [22], one-loop effective LFV $Zl_\ell l_\mu$ vertex from heavy neutrinos [23] and so on. This method provides very simple and intuitive analytical formula, and is also clear about the changes of the main parameters affecting lepton taste destruction, which provides a new idea for other work of LFV in the future.

In the process of LFV, because the mass of $\tau$ lepton is much greater than $\mu$ and $e$, there are more LFV decay channels [24]. The decay processes of $l_j \to l_i \gamma$ are the most interesting. This work is to study the LFV of the $l_j \to l_i \gamma$ processes under the $U(1)_X$SSM model. The effects of different reasonable parameter spaces on the branching ratio $Br(l_j \to l_i \gamma)$ are compared. The latest upper limits on the LFV branching ratio of $\mu \to e\gamma$, $\tau \to \mu\gamma$ and $\tau \to e\gamma$ at 90% confidence level (CL) [25] are

$$Br(\mu \to e\gamma) < 4.2 \times 10^{-13}, \quad Br(\tau \to \mu\gamma) < 4.4 \times 10^{-8}, \quad Br(\tau \to e\gamma) < 3.3 \times 10^{-8}.$$  

(1)

The paper is organized as follows. In Sect. 2, we mainly introduce the $U(1)_X$SSM including its superpotential and the general soft breaking terms. In Sect. 3, we give analytic expressions for muon anomalous magnetic moment and the branching ratios of $l_j \to l_i \gamma$ decays in the $U(1)_X$SSM. In Sect. 4, we give the numerical analysis, and the summary is given in Sect. 5.

2 The $U(1)_X$SSM

$U(1)_X$SSM is the $U(1)_X$ extension of MSSM, whose local gauge group is $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \otimes U(1)_X$ [7,20,26]. On the basis of MSSM, $U(1)_X$SSM has new superfields such as three Higgs singlets $\tilde{\eta}$, $\bar{\eta}$, $\tilde{S}$ and right-handed neutrinos $\tilde{\nu}$. Through the seesaw mechanism, light neutrinos obtain tiny masses at the tree level. The neutral CP-even parts of $H_u$, $H_d$, $\eta$, $\bar{\eta}$ and $S$ mix together and form a $5 \times 5$ mass squared matrix, whose lightest mass eigenvalue corresponds to the lightest CP-even Higgs. The particle content and charge assignments for $U(1)_X$SSM can be found in our previous work [7]. To get 125.1 GeV Higgs mass [27,28], the loop corrections should be taken into account.

The sneutrinos are dispursed into CP-even sneutrinos and CP-odd sneutrinos, and their mass squared matrix are both extended to $6 \times 6$.

In $U(1)_X$SSM, the concrete form of the superpotential is:

$$W = l_W \tilde{S} + \mu \tilde{H}_u \tilde{H}_d + M \tilde{S} \tilde{S} - Y_d d \tilde{Q} \tilde{H}_d - Y_c c \tilde{L} \tilde{H}_d + \lambda_\eta \tilde{S} \tilde{H}_u \tilde{H}_d + \lambda_c \tilde{S} \tilde{S} \tilde{S} + \kappa \tilde{S} \tilde{S} \tilde{S} + Y_\eta \tilde{Q} \tilde{H}_u + Y_X \tilde{\nu} \tilde{\nu} + Y_\nu \tilde{\nu} \tilde{L} \tilde{H}_u.$$  

(2)

We collect the explicit forms of two Higgs doublets and three Higgs singlets here

$$H_u = \left( \frac{1}{\sqrt{2}} \left( v_u + H_u^0 + i P_u^0 \right) \right),$$

$$H_d = \left( \frac{1}{\sqrt{2}} \left( v_d + H_d^0 + i P_d^0 \right) \right),$$

$$\eta = \frac{1}{\sqrt{2}} \left( v_\eta + \phi_{\eta}^0 + i P_{\eta}^0 \right),$$

$$\bar{\eta} = \frac{1}{\sqrt{2}} \left( v_{\bar{\eta}} + \phi_{\bar{\eta}}^0 + i P_{\bar{\eta}}^0 \right),$$

$$S = \frac{1}{\sqrt{2}} \left( v_S + \phi_{S}^0 + i P_{S}^0 \right).$$  

(3)

The vacuum expectation values(VEVs) of the Higgs superfields $H_u$, $H_d$, $\eta$, $\bar{\eta}$ and $S$ are denoted by $v_u$, $v_d$, $v_\eta$, $v_{\bar{\eta}}$ and $v_S$ respectively. Two angles are defined as $\tan \beta = v_u/v_d$ and $\tan \beta_\eta = v_\eta/v_{\bar{\eta}}$.

The soft SUSY breaking terms of this model are shown as

$$\mathcal{L}_{soft} = \mathcal{L}_{soft}^{MSSM} + B_S S^2 - L_S$$

$$- T^\alpha \bar{S}^\alpha - T_{\lambda_1} \lambda_1 \tilde{S}^H \tilde{H}_d^H$$

$$- T_X \bar{\eta} \tilde{\nu}_R^\dagger \tilde{\nu}_R + \epsilon_1 T_{\lambda_2} \tilde{S} \tilde{H}_d^H \tilde{H}_d^H$$

$$- (m^2_{\nu} \bar{\nu}_R^\dagger \tilde{\nu}_R + \frac{1}{2} (M_X \lambda_1^2 + 2 M_B X \lambda_1 \tilde{X}) + h.c..$$  

(4)

We have proven that $U(1)_X$SSM is anomaly free in our previous work [26]. Two Abelian groups $U(1)_Y$ and $U(1)_X$ produce a new effect called as the gauge kinetic mixing in the $U(1)_X$SSM, which is MSSM never before.

In general, the covariant derivatives of $U(1)_X$SSM can be written as [29–32]

$$D_\mu = \partial_\mu - i \left( Y, X \right) \left( g_Y, g_X \right) \left( A^Y_\mu, A^X_\mu \right),$$  

(5)

with $A^Y_\mu$ and $A^X_\mu$ representing the gauge fields of $U(1)_Y$ and $U(1)_X$ respectively.

Under the condition that the two Abelian gauge groups are unbroken, we use the rotation matrix $R$ [29,31,32] to perform a change of the basis

\[ Springer \]
Here, we show some needed couplings in this model. We use two mixing angles \( \theta_1 \) and \( \theta_2 \) to get mass eigenvalues of the matrix in Eq. (9). \( \theta_1 \) is the Weinberg angle and the new mixing angle \( \theta'_W \) is defined from the following formula

\[
\sin^2 \theta'_W = \frac{1}{2} - \frac{[(g_{yx} + g_{x})^2 - g_1^2 - g_2^2]v^2 + 4g_X^2\xi^2}{2\sqrt{[g_{yx} + g_{x}]^2 + g_1^2 + g_2^2]v^4 + 8g_X^2[(g_{yx} + g_{x})^2 - g_1^2 - g_2^2]v^2\xi^2 + 16g_X^4\xi^4}}.
\]

It appears in the couplings involving \( Z \) and \( Z' \). The exact eigenvalues of Eq. (9) are deduced [20]

\[
m_Y^2 = 0,
\]
\[
m_{Z,Z'}^2 = \frac{1}{8} \left[ (g_1^2 + g_2^2 + (g_{yx} + g_{x})^2)v^2 + 4g_X^2\xi^2 \right.
\]
\[
\left. \mp \sqrt{[g_1^2 + g_2^2 + (g_{yx} + g_{x})^2]v^4 + 8[(g_{yx} + g_{x})^2 - g_1^2 - g_2^2]v^2\xi^2 + 16g_X^4\xi^4}. \right]
\]

The used mass matrices can be found in the work [7,19]. Here, we show some needed couplings in this model. We deduce the vertexes of \( \bar{\ell}_i - X_j - \tilde{\nu}_R \). \( \tilde{\nu}_R \) is the \( \tilde{\nu}_R \).

\[
\mathcal{L}_{\bar{\ell}_i X_j \tilde{\nu}_R} = \frac{1}{\sqrt{2}} \bar{\ell}_i \left\{ \tilde{\nu}_L^i Y_{ji}^* P_L \tilde{H}_1 - g_2 \tilde{\nu}_R^i P_R \tilde{W}^- \right\},
\]
\[
\mathcal{L}_{\bar{\ell}_i X_j \tilde{\nu}^t} = \frac{i}{\sqrt{2}} \bar{\ell}_i \left\{ \tilde{\nu}_L^i Y_{ji}^* P_L \tilde{H}_1 - g_2 \tilde{\nu}_R^i P_R \tilde{W}^- \right\}.
\]
disparted into CP-even sneutrinos $\tilde{\nu}^R$ and CP-odd sneutrinos $\tilde{\nu}^I$. After our analysis, in Fig. 1f since right-handed sneutrinos are strongly depressed by $Y$, the situation of right-handed sneutrinos here is neglected. In other words, there are only two cases of left-handed CP-even sneutrinos $\nu$ here, which implies that there are only two cases of left-handed CP-even sneutrinos $\tilde{\nu}_L^R$ and left-handed CP-odd sneutrinos $\tilde{\nu}_L^I$ in Fig. 1f. In order to more directly express the influencing factors of LFV of $l_j \to l_i \gamma$, we use $C_{f}^{L} = C_{f}^{R} (j = 1 \cdots 6)$ to express the one-loop corrections by MIA.

1. The one-loop contributions from $\tilde{B}(\lambda_{\tilde{\nu}}) \cdot \bar{L}_j^R \cdot \tilde{L}_j^R$.

$$C_{2}^{L} (\bar{L}_j^R, \tilde{L}_j^R, \tilde{B}) = \frac{-1}{2m_{jl} \Delta_{ij}^{LR}} \frac{g_{YX}^2}{\sqrt{\lambda_{\tilde{\nu}}}} [I_{1}(x_{\tilde{\nu}_{j}^R}, x_{\tilde{\nu}_{j}^R}) + I_{2}(x_{\tilde{\nu}_{j}^R}, x_{\tilde{\nu}_{j}^R}) - 2I_{2}(x_{\tilde{\nu}_{j}^R}, x_{\tilde{\nu}_{j}^R})], \quad (15)$$

$$C_{2}^{L} (\bar{L}_j^R, \tilde{L}_j^R, \lambda_{\tilde{\nu}}) = \frac{-1}{2m_{jl} \Delta_{ij}^{LR}} \frac{g_{YX}^2}{\sqrt{\lambda_{\tilde{\nu}}}} [I_{1}(x_{\tilde{\nu}_{j}^R}, x_{\tilde{\nu}_{j}^R}) + I_{2}(x_{\tilde{\nu}_{j}^R}, x_{\tilde{\nu}_{j}^R}) - 2I_{2}(x_{\tilde{\nu}_{j}^R}, x_{\tilde{\nu}_{j}^R})], \quad (16)$$

here, $m$ is the particle mass, with $x = \frac{m^2_{jl}}{\lambda_{\tilde{\nu}}}$. The functions $I_{1}(x, y)$ and $I_{2}(x, y)$ are

$$I_{1}(x, y) = \frac{1}{32\pi^2} \left[ \frac{1}{x(x-y)} - \frac{2 + 2 \log x}{(x-y)^2} + \frac{2 \log x - 2y \log y}{(x-y)^2} \right], \quad (17)$$

$$I_{2}(x, y) = \frac{1}{96\pi^2} \left[ \frac{2}{x(x-y)} - \frac{9 + 6 \log x}{(x-y)^2} + \frac{6x + 12 \log y}{(x-y)^2} \right]. \quad (18)$$

2. The one-loop contributions from $\tilde{B}(\lambda_{\tilde{\nu}}) \cdot \bar{H}_j^0 \cdot \tilde{L}_j^R \cdot \tilde{L}_j^R$.

$$C_{2}^{L} (\bar{H}_j^0, \tilde{L}_j^R, \tilde{B}, \tilde{H}_j^0) = \frac{1}{2A^4} \frac{g_{YX}}{\sqrt{\lambda_{\tilde{\nu}}}} \frac{\Delta_{ij}^{LR}}{\Delta_{ij}^{RR}} [2I_{4}(x_{\tilde{\nu}_{j}^R}, x_{\tilde{\nu}_{j}^R}) + 3I_{3}(x_{\tilde{\nu}_{j}^R}, x_{\tilde{\nu}_{j}^R})], \quad (19)$$

$$C_{2}^{L} (\bar{H}_j^0, \tilde{L}_j^R, \lambda_{\tilde{\nu}}) \cdot \tilde{H}_j^0 = \frac{1}{2A^4} \frac{1}{2} \left[ 2g_{YX} + g_{X} \right] (2g_{YX} + g_{X}) \frac{\Delta_{ij}^{LR}}{\Delta_{ij}^{RR}} [2I_{4}(x_{\tilde{\nu}_{j}^R}, x_{\tilde{\nu}_{j}^R}) + 3I_{3}(x_{\tilde{\nu}_{j}^R}, x_{\tilde{\nu}_{j}^R})], \quad (20)$$

here $\mu_{\tilde{H}} = \frac{\lambda_{\tilde{H}} \mu_{\tilde{H}}}{\sqrt{2}}$ and $x_{\tilde{\nu}_{j}^R} = \frac{\mu_{\tilde{\nu}_{j}^R}}{\lambda_{\tilde{\nu}}}$. The specific forms of $I_{3}(x, y, z)$ and $I_{4}(x, y, z)$ are...
\textbf{3. The one-loop contributions from }\bar{\tilde{B}} - \tilde{\chi} - \bar{L}_j \bar{L}_i^L + \bar{L}_j^R - \bar{L}_i^L.

\begin{equation}
I_3(x, y, z) = \frac{1}{32\pi^2} \left\{ \frac{6x + 12x \log x}{(y - x)(y - z)^3} + \frac{6x + 12x \log x}{(y - z)^2(x - z)^2} + \frac{6x \log x}{(y - z)^4(x - z)} \right. \\
\left. - \frac{6x \log x}{(y - z)(x - z)^2} + \frac{6x^2 \log x}{(y - z)^3(x - z)^3} + \frac{6x^2 \log x}{(y - z)^3(x - z)^2} \right. \\
\left. - \frac{6x^2 \log x}{(y - z)^4(x - z)} \right\}
\end{equation}

\begin{align}
I_4(x, y, z) &= \frac{1}{32\pi^2} \left\{ \frac{2 + 2 \log x}{(x - y)(x - z)^2} + \frac{2 + 2 \log x}{(x - y)(x - z)} + \frac{1}{x(x - y)(x - z)} + \frac{2x \log x}{(x - y)(x - z)^2} + \frac{2x \log x}{(x - y)(x - z)} + \frac{1}{x(x - y)(x - z)} \right. \\
\left. - \frac{2y \log y}{(x - y)^2(x - z)} + \frac{2y \log y}{(x - y)(x - z)^2} + \frac{2y \log y}{(x - y)^3(x - z)} + \frac{2y \log y}{(x - y)^2(x - z)} \right. \\
\left. + \frac{2y \log y}{(x - y)(x - z)} + \frac{2y \log y}{(x - y)^2(x - z)^2} + \frac{2y \log y}{(x - y)(x - z)^3} + \frac{2y \log y}{(x - y)^3(x - z)^2} \right. \\
\left. - \frac{2y \log y}{(x - y)^3(x - z)} \right\}
\end{align}

\textbf{5. The one-loop contributions from }\bar{\tilde{B}} - \tilde{\chi} - \bar{L}_j^L - \bar{L}_i^R.

\begin{equation}
C_2^L(\bar{L}_j^L, \bar{L}_i^R, \bar{L}_j^L, \lambda_{\tilde{\chi}})
= \frac{-1}{2m_{ij} A^4} A_{ij}^{LL} |g_1 g_{YY} | \tilde{B} \left\{ I_4(x_{\bar{L}_j^L}, x_{\bar{L}_i^R}, x_{\bar{L}_j^L}, x_{\bar{L}_i^L}) + \left[ I_4(x_{\bar{L}_j^L}, x_{\bar{L}_i^L}, x_{\bar{L}_j^L}, x_{\bar{L}_i^L}) + 2I_5(x_{\bar{L}_j^L}, x_{\bar{L}_i^L}, x_{\bar{L}_j^L}, x_{\bar{L}_i^L}) \right] \right. \\
\left. + \left[ I_4(x_{\bar{L}_j^L}, x_{\bar{L}_i^L}, x_{\bar{L}_j^L}, x_{\bar{L}_i^L}) + 2I_5(x_{\bar{L}_j^L}, x_{\bar{L}_i^L}, x_{\bar{L}_j^L}, x_{\bar{L}_i^L}) \right] \right\}.
\end{equation}

\textbf{6. The one-loop contributions from chargino and left-handed CP-even/odd sneutrino.}

\begin{equation}
C_2^6(\tilde{\chi}, \tilde{\chi}, \tilde{\chi}, \tilde{\chi}, \tilde{\chi}, \tilde{\chi})
= \frac{1}{2A^4} \beta \left\{ |g_{YY} | \tilde{B} \left\{ I_4(x_{\bar{L}_j^L}, x_{\bar{L}_i^L}, x_{\bar{L}_j^L}, x_{\bar{L}_i^L}) + \left[ I_4(x_{\bar{L}_j^L}, x_{\bar{L}_i^L}, x_{\bar{L}_j^L}, x_{\bar{L}_i^L}) + 2I_5(x_{\bar{L}_j^L}, x_{\bar{L}_i^L}, x_{\bar{L}_j^L}, x_{\bar{L}_i^L}) \right] \right. \\
\left. + \left[ I_4(x_{\bar{L}_j^L}, x_{\bar{L}_i^L}, x_{\bar{L}_j^L}, x_{\bar{L}_i^L}) + 2I_5(x_{\bar{L}_j^L}, x_{\bar{L}_i^L}, x_{\bar{L}_j^L}, x_{\bar{L}_i^L}) \right] \right\}.
\end{equation}

The one-loop functions \(I_7(x, y, z)\), \(I_8(x, y, z)\) and \(I_9(x, y, z)\) read as

\begin{equation}
I_7(x, y, z) = \frac{-1}{32\pi^2} \left\{ \frac{8x \log x - 4x}{(x - y)(x - z)^3} + \frac{2x + 4x \log x}{(x - y)^2(x - z)^2} + \frac{3 + 2 \log x}{(x - y)(x - z)^2} \right. \\
\left. + \frac{2 + \log x}{(x - y)(x - z)} + \frac{1}{x(x - y)(x - z)} \right\}.
\end{equation}
\[
C_2 = \sum_{i=1}^{6} C^i_2, \quad \Gamma(l_j \to l_i \gamma) = \frac{e^2}{8\pi m^2_{l_j}} |C_2|^2.
\]  

(34)

The branching ratio of \(l_j \to l_i \gamma\) is

\[
Br(l_j \to l_i \gamma) = \frac{\Gamma(l_j \to l_i \gamma)}{\Gamma_{l_j}}.
\]  

(35)

3.2 Degenerate result

In order to more intuitively analyze the factors affecting lepton flavor violating processes \(l_j \to l_i \gamma\), we suppose that all the masses of the superparticles are almost degenerate. In other words, we give the one-loop results (chargino-neutralino, neutralino-slepton) in the extreme case, where the masses for superparticles \((M_1, M_2, \mu'_H, m_{\tilde{L}_L}, m_{\tilde{L}_R}, M_{\tilde{\chi}_+}, M_{\tilde{M}_{BB}})\) are equal to \(M_{SUSY}\) [20]:

\[
M_1 = |M_2| = \mu'_H = m_{\tilde{L}_L} = m_{\tilde{L}_R} = M_{\tilde{\chi}_+} = |M_{BB}| = M_{SUSY}.
\]

The functions \(I_i (i = 1 \cdots 9)\) and \(\Delta^{AB}_{ij} (A, B = L, R)\) are much simplified as

\[
I_1(1, 1) = -\frac{1}{96\pi^2}, \quad I_2(1, 1) = -\frac{1}{192\pi^2}, \quad I_3(1, 1, 1) = -\frac{1}{480\pi^2},
\]

\[
l_i(1, 1) = \frac{1}{192\pi^2}, \quad l_i(1, 1, 1) = \frac{1}{480\pi^2}, \quad l_i(1, 1) = -\frac{1}{320\pi^2},
\]

\[
l_i(1, 1, 1) = \frac{1}{384\pi^2}, \quad \Delta^{LR}_{ij} = m_{l_j} m_{\tilde{L}_j} \delta^{LR}_{ij}, \quad \Delta^{LL}_{ij} = m_{l_j}^2 \delta^{LL}_{ij},
\]

\[
\Delta^{RR}_{ij} = m_{l_j}^2 \delta^{RR}_{ij}.
\]

(36)

Then, we obtain the much simplified one-loop results of \(C_2\)

\[
C_2 = \begin{array}{c}
\frac{2g_1^2 \text{sign}[M_1 \mu'_H] + (2g_T^2 + 3g_{YX} + g_X^2) \text{sign}[M_{\tilde{\chi}_+} \mu'_H] \tan \beta \delta^{RR}_{ij}}{960\pi^2 M^2_{SUSY}}
\end{array}
\]

\[
+ \frac{4g_2^2 \text{sign}[M_1 \mu'_H] + (2g_T^2 + 3g_{YX} + g_X^2) \text{sign}[M_{\tilde{\chi}_+} \mu'_H] \tan \beta \delta^{LL}_{ij}}{960\pi^2 M^2_{SUSY}}
\]

\[
+ \frac{-4g_2^2 \text{sign}[M_2^2] - 4g^2_2 \text{sign}[\mu'_H^2] - 12g_2^2 \text{sign}[\mu'_H M_2] + 5g_2^2 \tan \beta \delta^{LL}_{ij}}{3840\pi^2 M^2_{SUSY}}
\]

\[
+ \frac{1}{1920\pi^2 M^2_{SUSY}} \times \{5g_1^2 \text{sign}[M_1] + 5g_1^2 g_{YX} \text{sign}[M_{\tilde{\chi}_+}] - 4g_1 g_{YX} \text{sign}[M_{BB} M_{\tilde{\chi}_+}] + g_1 g_{YX} \text{sign}[M_{BB'}] \delta^{LR}_{ij}\}.
\]

(38)
In the above formula, after simple approximation, we can find that in the formula of the second line, due to the existence of \( \frac{m_{ij}^2}{m_{ij}} \), the result is 2–3 orders of magnitude smaller than other terms. Therefore, we will not consider the term with \( \frac{m_{ij}^2}{m_{ij}} \) here. It can be found that 
\[ \text{sign}[M_1], \text{sign}[M_2], \text{sign}[M_{\mu H}], \text{sign}[\mu'_{H}] \] and 
\[ \text{sign}[M_{BB'}] \] have a certain impact on the correction of \( C_2 \).

According to \( 1 > g X > g X > 0 \), we assume 
\[ \text{sign}[M_1] = \text{sign}[M_{\mu H}] = 1 \] and 
\[ \text{sign}[M_2] = \text{sign}[M_{BB'}] = -1, \] and get the larger value of \( C_2 \)

\[
C_2 = \frac{(5g_1^2 + 5(g_2^2 + 1/2g_2g_3g_5) + 3g_1g_3)\delta_{ij}^{LR}}{1920\pi^2 M_{\text{SUSY}}^2} \\
+ \frac{3g_2^2 \tan \beta \delta_{ij}^{LL}}{1280\pi^2 M_{\text{SUSY}}^2} \\
+ \frac{2(2g_1^2 + (3g_1g_3 + g_2^2)) \tan \beta \delta_{ij}^{RR}}{960\pi^2 M_{\text{SUSY}}^2}. \tag{39}
\]

Due to the different orders of magnitude of branching ratios, we set \( \tan \beta = 9, M_{\text{SUSY}} = 1000 \text{ GeV} \) and discuss it in two cases:

1. \( \mu \to e \gamma \)

We take \( g_{XY}, g_X, \delta_{ij}^{LR}, \delta_{ij}^{LL}, \) and \( \delta_{ij}^{RR} \) as variables to study the effect on \( Br(\mu \to e \gamma) \). In Fig. 2, it can be found that both \( \delta_{ij}^{LR} \) and \( \delta_{ij}^{RR} \) have great influence on \( Br(\mu \to e \gamma) \) and they are all increasing trend. Larger the values of \( \delta_{ij}^{LR} \) and \( \delta_{ij}^{RR} \), the easier it is to approach the upper limit of the experiment.

Similar to the above, in Fig. 3, with the increase of \( \delta_{ij}^{LL} \), the value of \( Br(\mu \to e \gamma) \) gradually increases, and when \( g_X \) increases, \( Br(\mu \to e \gamma) \) also increases. When \( \delta_{ij}^{LL} = 9 \times 10^{-4} \) and \( g_X = 0.65 \), \( Br(\mu \to e \gamma) \) reaches the experimental upper limit. In numerical terms, the effect of \( \delta_{ij}^{LL} \) is greater than \( g_X \).

2. \( \tau \to \mu \gamma \)

Since the numerical results of \( Br(\tau \to \mu \gamma) \) and \( Br(\tau \to e \gamma) \) are close and have similar characteristic, we take \( \tau \to \mu \gamma \) as an example. In Fig. 4, when the values of \( \delta_{ij}^{RR} \) and \( g_X \) enlarger, the value of \( Br(\tau \to \mu \gamma) \) also increases, which can well reach the experimental measured value.

In Fig. 5, we analyze \( \delta_{ij}^{LL} \) and \( g_X \) on the \( Br(\tau \to \mu \gamma) \) .

The value of \( Br(\tau \to \mu \gamma) \) also increases with the increasing \( \delta_{ij}^{LL} \) and \( g_X \), but the effect from \( g_X \) is greater than \( \delta_{ij}^{LL} \). So the correction of \( g_X \) to \( Br(\tau \to \mu \gamma) \) is greater than that of \( \delta_{ij}^{LL} \).

All in all, we can find that \( g_X, \delta_{ij}^{LR}, \delta_{ij}^{LL}, \) and \( \delta_{ij}^{RR} \) all have direct impact on the correction to \( Br(\mu \to e \gamma) \), \( Br(\tau \to \mu \gamma) \) and \( Br(\tau \to e \gamma) \).

3.3 Muon anomalous magnetic moment

The one-loop corrections to muon anomalous magnetic moment are obtained with MIA. Here, we show the one-loop contributions from chargino and CP-even/odd sneutrino as [20]

\[
a_i(\tilde{\chi}^L_{ij}, \tilde{H}^\pm, \tilde{W}^\pm) = \frac{m_{\chi}^2}{g_2^2} \frac{g_2^2}{2} \left\{ \frac{2}{g_2^2} \frac{g_2^2}{2} \tan \beta \left[ -2 \mathcal{I}(x_{\mu_{ij}}, x_{\tau_{ij}^L}, x_{\tau_{ij}^R}) + \mathcal{J}(x_{\mu_{ij}}, x_{\tau_{ij}^L}, x_{\tau_{ij}^R}) \right] \right.
\]

\[
+ \frac{1}{4} \left[ -2 \mathcal{I}(x_{\mu_{ij}}, x_{\tau_{ij}^L}, x_{\tau_{ij}^R}) + \mathcal{J}(x_{\mu_{ij}}, x_{\tau_{ij}^L}, x_{\tau_{ij}^R}) \right] \right\}.
\tag{40}
\]

The concrete forms of the one-loop functions \( \mathcal{I}(x, y, z) \) and \( \mathcal{J}(x, y, z) \) are

\[
\mathcal{I}(x, y, z) = \frac{1}{16\pi^2} \left[ \frac{x(y - 2z + yz)}{(x - y)(x - z)(y - z)^2} \right].
\]

\[
\mathcal{J}(x, y, z) = \frac{1}{16\pi^2} \left[ \frac{y(z - x y)}{(x - z)(x - y)(y - z)^2} \right]. \tag{41}
\]

The other one-loop contributions are obtained from \( B(\lambda_{\tilde{\chi}}) - L_{\tilde{\chi}} - L_{\tilde{H}} \), \( B(\lambda_{\tilde{\chi}}) - H^0_{\tilde{L}} \), \( B(\tilde{H}^0_{\tilde{L}}) \), \( L_{\tilde{H}} \) and \( B_{\tilde{L}} - \lambda_{\tilde{\chi}} - \tilde{L}_{\tilde{L}} \).

4 Numerical results

In this section, we study the numerical results and consider the constraints from lepton flavor violating processes \( l_j \to l_i \gamma \). In addition, we have considered the following conditions: 1. the lightest CP-even Higgs mass \( m_{H^0} = 125.1 \text{ GeV} \) [34, 35], 2. the latest experimental results of the mass of the heavy
Fig. 2 Under the condition that $g_{YX} = 0.2$, $g_X = 0.3$ and $\delta_{ij}^{LR} = 1 \times 10^{-3}$, the effect of $\delta_{ij}^{LR}$ and $\delta_{ij}^{RR}$ on $\text{Br}(\mu \to e\gamma)$. The x-axis representing the range of $\delta_{ij}^{LR}$ is from $1 \times 10^{-5}$ to $1 \times 10^{-2}$, and the y-axis represents $1 \times 10^{-5} < \delta_{ij}^{RR} < 1 \times 10^{-3}$. The rightmost icon is the color corresponding to the value of $\text{Br}(\mu \to e\gamma)$.

Fig. 3 Under the condition that $g_{YX} = 0.2$, $\delta_{ij}^{RR} = 1 \times 10^{-6}$ and $\delta_{ij}^{LR} = 1 \times 10^{-2}$, $\delta_{ij}^{LL}$ versus $g_X$ about $\text{Br}(\mu \to e\gamma)$. The abscissa is $1 \times 10^{-5} < \delta_{ij}^{LL} < 1 \times 10^{-3}$ and the ordinate represents $0.2 < g_X < 0.7$. The icon on the right shows the value of $\text{Br}(\mu \to e\gamma)$.

vector boson $Z'$ is $M_{Z'} > 5.1$ TeV [36]. 3. The limits for the masses of other particles beyond SM. 4. The bound on the ratio between $M_{Z'}$ and its gauge coupling $g_X$ is $M_{Z'}/g_X \geq 6$ TeV at 99% CL [37,38]. 5. The constraint from LHC data, $\tan \beta \eta < 1.5$ [39]. 6. The scalar lepton masses larger than 700 GeV and chargino masses larger than 1100 GeV [40].

Considering the above constraints in the front paragraph, we use the following parameters

\begin{align*}
\xi &= 17 \text{ TeV}, \quad Y_{X11} = Y_{X22} = Y_{X33} = 1, \quad g_X = 0.3, \\
\kappa &= 1, \quad \lambda_c = -0.08, \quad \nu_S = 4.3 \text{ TeV}, \\
M_{BB'} &= 0.4 \text{ TeV}, \quad T_{\lambda H} = 0.3 \text{ TeV}, \quad M_{L_{11}}^2 = M_{L_{22}}^2 = M_{L_{33}}^2 = M_L^2 = 0.5 \text{ TeV}^2, \quad l_W = 4 \text{ TeV}^2, \\
\lambda_H &= 0.1, \quad T_{e_{11}} = T_{e_{22}} = T_{e_{33}} = 5 \text{ TeV}, \quad \tan \beta_\eta = 0.8, \\
B_\mu &= B_S = 1 \text{ TeV}^2, \quad \mu = 0.5 \text{ TeV}, \\
T_{\lambda C} &= -0.1 \text{ TeV}, \quad M_{E_{11}}^2 = M_{E_{22}}^2 = M_E^2 = 3.6 \text{ TeV}^2.
\end{align*}

(45)
Fig. 4 Under the condition that $\delta_{LR} = 0.1$, $g_YX = 0.2$ and $\delta_{LL} = 0.1$, the effect of $\delta_{RR}^{ij}$ and $g_X$ on $Br(\tau \to \mu\gamma)$. The x-axis representing the range of $\delta_{RR}^{ij}$ is from $10^{-3}$ to 1, and the y-axis represents $0.2 < g_X < 0.7$. The rightmost icon is the color corresponding to the value of $Br(\tau \to \mu\gamma)$.

Fig. 5 Under the condition that $\delta_{RR}^{ij} = 0.2$, $g_YX = 0.3$ and $\delta_{LL}^{ij} = 0.5$, $\delta_{LL}^{ij}$ versus $g_YX$ about $Br(\tau \to \mu\gamma)$. The abscissa is $1 \times 10^{-5} < \delta_{LL}^{ij} < 0.5$ and the ordinate represents $0.1 < g_YX < 0.5$. The icon on the right shows the value of $Br(\tau \to \mu\gamma)$.

To simplify the numerical research, we use the relations for the parameters and they vary in the following numerical analysis:

$$
M_{L12}^2 = M_{L21}^2, \quad M_{L13}^2 = M_{L31}^2, \quad M_{L23}^2 = M_{L32}^2,
M_{E12}^2 = M_{E21}^2, \quad M_{E13}^2 = M_{E31}^2, \quad M_{E23}^2 = M_{E32}^2, \quad Te_{12} = Te_{21}, \quad Te_{13} = Te_{31}, \quad Te_{23} = Te_{32}, \quad \tan \beta.
$$

Without special statement, the non-diagonal elements of the parameters are supposed as zero.

4.1 Muon anomalous magnetic moment

In this subsection, we study the one-loop $g - 2$ in $U(1)_X$SSM model by MIA and expect to get some inspiration about using MIA to find LFV. The new experiment data of muon $g - 2$ is reported by the workers at Fermilab National Accelerator Laboratory (FNAL) [41–44]. Combined with the previous Brookhaven National Laboratory (BNL) E821 result [45], we get the new averaged experiment value of muon anomaly is $a_{\mu}^{\exp} = 116592061(41) \times 10^{-11}(0.35 \text{ ppm})$. The departure
from the SM prediction is $\Delta a_\mu = a^\exp_\mu - a^SM_\mu = 251(59) \times 10^{-11}$, which is about 4.2$\sigma$. We set the particle masses as: $M_1 = 800$ GeV, $\mu = 350$ GeV, $m_{\tilde{\nu}_\tau} = 150$ GeV, $m_{\tilde{\tau}_1} = 140$ GeV, $m_{\tilde{\nu}_e} = 800$ GeV, $m_{\tilde{\nu}_\tau} = 850$ GeV and $m_{\tilde{\nu}_e} = 450$ GeV to get Fig. 6. When $g_{YX} = 0.2$ in Fig. 6a, we can see that from bottom to top are solid line (tan $\beta = 30$), dashed line (tan $\beta = 40$) and dotted line (tan $\beta = 50$) and the overall trend of the three lines is downward. That is to say, tan $\beta$ is a sensitive parameter and larger tan $\beta$ leads to larger $a_\mu$.

We set tan $\beta = 50$ in Fig. 6b. It is obvious that three lines with the same tendency to decrease and then increase. The dotted line ($g_X = 0.5$) is below the dashed line ($g_X = 0.4$), and the dashed line is below the solid line ($g_X = 0.3$). When $g_X$ increases, $a_\mu$ decreases. The influence of $g_{X}$ on $a_\mu$ mainly depends on its own value: when $g_{YX}$ is less than 0.3, $g_{YX}$ increases and $a_\mu$ decreases, but when $g_{YX} > 0.3$, the situation is just the opposite. The value of $a_\mu$ can reach $2 \times 10^{-9}$ at most, it can reach about 80% of the departure($\Delta a_\mu$), which better meets the experimental limitations. The above conclusion is the same as Eq. (44), so we can find other sensitive parameters more intuitively through formula Eq. (44).

4.2 The processes of $\mu \rightarrow e\gamma$

In order to study the parameters affecting LFV, we need to study some sensitive parameters. To show the numerical results clearly, we draw the relation diagrams and scatter diagrams of $Br(\mu \rightarrow e\gamma)$ with different parameters.

The gray area is current limit on LFV decay $\mu \rightarrow e\gamma$ in Fig. 7. With the parameters $M_{L12}^2 = 0$, $M_{L12}^2 = 0$ and $\tan \beta = 0$, we plot $Br(\mu \rightarrow e\gamma)$ versus $m_{\tilde{\nu}_\tau}$ in the Fig. 7a. The dashed curve corresponds to $M_{L12}^2 = 500$ GeV$^2$ and the solid line corresponds to $M_{L12}^2 = 200$ GeV$^2$. On the whole, both lines show a downward trend. $m_{\tilde{\nu}_\tau}$ and $Br(\mu \rightarrow e\gamma)$ are inversely proportional. The smaller $m_{\tilde{\nu}_\tau}$ is, the greater the value of $Br(\mu \rightarrow e\gamma)$ is. Separately, the dashed line is larger than the solid line, and the ranges consistent with the experimental value are 1400 GeV–4000 GeV and 900 GeV–1400 GeV respectively. $m_{L12}^2$ and $Br(\mu \rightarrow e\gamma)$ are positively correlated. If the value of $M_{L12}^2$ gets smaller, the value of $m_{\tilde{\nu}_\tau}$ can be less than 1000 GeV.

We show $Br(\mu \rightarrow e\gamma)$ varying with $M_{L12}^2$ by the solid curve ($T_{e12} = 100$ GeV) and dashed curve ($T_{e12} = 50$ GeV) in the Fig. 7b. We can see that the overall values meet the limit, and the trend is a subtractive function, and the solid line is greater than the dotted line. So we can conclude that as $T_{e12}$ increases, $Br(\mu \rightarrow e\gamma)$ also increases. When $M_{L12}^2$ increases, $Br(\mu \rightarrow e\gamma)$ decreases. The numerical results are tiny and at the order of $10^{-19}$.

Finally, we analyze the effects of the parameter $T_{e12}$ on branching ratio of $\mu \rightarrow e\gamma$. The numerical results are shown in the Fig. 7c by the dashed curve (tan $\beta = 9$) and solid curve (tan $\beta = 20$). The value of the solid line is greater than that of the dashed line, and both show an upward trend. Therefore, The relationship between tan $\beta$ and $Br(\mu \rightarrow e\gamma)$, $T_{e12}$ and $Br(\mu \rightarrow e\gamma)$ is the similar, and they are all positively correlated.

For more multidimensional analysis of sensitive parameters, we scatter points according to Table 1 (part of $\mu \rightarrow e\gamma$) to get Fig. 8. We set the range of $\Delta$ ($0 < Br(\mu \rightarrow e\gamma) < 1.5 \times 10^{-13}$), $\Delta$ ($1.5 \times 10^{-13} \leq Br(\mu \rightarrow e\gamma) < 3.5 \times 10^{-13}$) and $\Delta$ ($3.5 \times 10^{-13} \leq Br(\mu \rightarrow e\gamma) < 4.2 \times 10^{-13}$) to represent the results in different parameter spaces for the process of $\mu \rightarrow e\gamma$.

The relationship between $M_{L12}^2$ and $m_{\tilde{\nu}_\tau}$ is shown in Fig. 8a. $\Delta$ are mainly concentrated in the upper left corner, then the outer layer are $\Delta$ and finally $\bullet$. When $M_{L12}^2$ is near 0 and $m_{\tilde{\nu}_\tau}$ is near 2500 GeV, $Br(\mu \rightarrow e\gamma)$ gets the minimum value. Figure 8b is plotted in the plane of $m_{\tilde{\nu}_\tau}$ versus $m_{\tilde{\nu}_\tau}$. We can clearly find that the points are mainly concentrated in the right, and the color gradually deepens from lower left to upper right. The effects of $M_{L12}^2$ and $m_{\tilde{\nu}_\tau}$ on $Br(\mu \rightarrow e\gamma)$ are shown in the Fig. 8c. All points are mainly concentrated in the upper left corner and on both sides of axis $M_{L12}^2 = 0$. 

![Fig. 6](image_url)
and axis $m_{\tilde{u}_L} = 3000$ GeV. From the inside to the outside, they are $\diamondsuit$, $\Delta$ and $\bullet$. When $M^2_{L12}$ becomes larger and $m_{\tilde{u}_L}$ becomes smaller, the value of $\text{Br}(\mu \rightarrow e\gamma)$ increases to reach the experimental measurement. Figure 8d shows the effect of $\tan \beta$ and $M^2_{L12}$ on $\text{Br}(\mu \rightarrow e\gamma)$. All points are mainly concentrated near the x-axis, and the value increases from bottom to top.

4.3 The processes of $\tau \rightarrow \mu \gamma$

To study the influence of parameters $M^2_{L23}$ and $M^2_{E23}$ on $\text{Br}(\tau \rightarrow \mu \gamma)$ in Fig. 9, we suppose the parameters $M^2_{L12} = 0$, $M^2_{E12} = 0$ and $T_{e12} = 0$ and plot the solid line (tan $\beta = 9$) and dashed line (tan $\beta = 20$). In Fig. 9a, we can see that $M^2_{L23}$ corresponds to $\text{Br}(\tau \rightarrow \mu \gamma)$. We plot $M^2_{E23}$ varying with $\text{Br}(\tau \rightarrow \mu \gamma)$ in the Fig. 9b. Both figures show an upward trend within the experimental limit, and the dashed line is larger than the solid line, so we can draw a conclusion: when $M^2_{L23}$ or $M^2_{E23}$ increases, $\text{Br}(\tau \rightarrow \mu \gamma)$ also increases. In the whole, the numerical results in Fig. 9 are very tiny.

We scatter points according to the parameters given in Table 1 (part of $\tau \rightarrow \mu \gamma$) to obtain Fig. 10. Where $\diamondsuit$, $\Delta$ and $\bullet$ represent $(0 < \text{Br}(\tau \rightarrow \mu \gamma) < 1.0 \times 10^{-10})$, $(1.0 \times 10^{-10} \leq \text{Br}(\tau \rightarrow \mu \gamma) < 9.0 \times 10^{-10})$ and $(9.0 \times 10^{-10} \leq \text{Br}(\tau \rightarrow \mu \gamma) < 4.4 \times 10^{-8})$ respectively.

$M^2_{L23}$ corresponds to $m_{\tilde{L}}$ in Fig. 10a. Horizontally, $\diamondsuit$ are mainly concentrated in $0 < M^2_{L23} < 1500$ GeV$^2$, $\Delta$ are in $1500$ GeV$^2 < M^2_{L23} < 3200$ GeV$^2$, and $\bullet$ are distributed in $3200$ GeV$^2 < M^2_{L23} < 5000$ GeV$^2$. Vertically, $\Delta$ and $\bullet$ are concentrated near $m_{\tilde{L}} = 500$ GeV, and there are obvious stratification, from bottom to top are $\bullet$, $\Delta$ and $\diamondsuit$. So we can know that as $M^2_{L23}$ increases, $\text{Br}(\tau \rightarrow \mu \gamma)$ increases, and when $m_{\tilde{L}}$ increases, $\text{Br}(\tau \rightarrow \mu \gamma)$ decreases. We plot $m_{\tilde{L}}$ varying with $T_{e23}$ in the Fig. 10b. The three types of points are almost symmetrical about $T_{e23} = 0$. The smaller the $m_{\tilde{L}}$ is, the greater the value of $\text{Br}(\tau \rightarrow \mu \gamma)$. The farther away the value of $T_{e23}$ from the 0 axis, the greater the value of $\text{Br}(\tau \rightarrow \mu \gamma)$.

Figure 10c is shown in the plane of $T_{e23}$ versus $m_{\tilde{u}_L}$, where the centralized distributions of the three types of points are distributed in a “U” shape on both sides of the $T_{e23} = 0$ axis. $\diamondsuit$ distribute on the innermost side, followed by $\Delta$ and $\bullet$ on the outermost side. So as $m_{\tilde{u}_L}$ increases, $\text{Br}(\tau \rightarrow \mu \gamma)$ decreases. Finally, we analyze the effects from parameters $\tan \beta$ and $m_{\tilde{u}_L}$ in Fig. 10d. All points are $\diamondsuit$, $\Delta$ and $\bullet$ from top to bottom. The smaller the value of $m_{\tilde{u}_L}$, the larger the value of $\text{Br}(\tau \rightarrow \mu \gamma)$.
Table 1  Scanning parameters for Figs. 8, 10 and 11. Without special statement, the non-zero values of non-diagonal elements $m_{Li j}^2$, $m_{Eij}^2$, $T_{ei j}$ corresponding to $l_j \rightarrow l_i \gamma$ are shown in the column

| Parameters/range/processes | $\mu \rightarrow \gamma (j = 2, i = 1)$ | $\tau \rightarrow \gamma (j = 3, i = 2)$ | $\tau \rightarrow \gamma (j = 3, i = 1)$ |
|---------------------------|--------------------------------------|--------------------------------------|--------------------------------------|
| $\tan \beta$              | $0.5 \sim 50$                        | $0.5 \sim 50$                        | $0.5 \sim 50$                        |
| $M_{Li j}^2/\text{GeV}^2$ | $0 \sim 5000$                        | $0 \sim 5000$                        | $0 \sim 5000$                        |
| $M_{Eij}^2/\text{GeV}^2$  | $0 \sim 10^4$                        | $0 \sim 10^4$                        | $0 \sim 10^4$                        |
| $T_{ei j}/\text{GeV}$      | $-1 \sim 1$                         | $-50 \sim 50$                        | $-50 \sim 50$                        |
| $m_{\nu Li}/\text{GeV}$   | $100 \sim 3000$                     | $100 \sim 3000$                     | $100 \sim 3000$                     |
| $m_{\nu Li}/\text{GeV}$   | $400 \sim 2500$                     | $400 \sim 2500$                     | $400 \sim 2500$                     |

Fig. 8  Under the premise of lower current limit on lepton flavor violating decay $\mu \rightarrow \gamma$, reasonable parameter space is selected to scatter points, where ♦ mean the value of $Br(\mu \rightarrow \gamma)$ less than $1.5 \times 10^{-13}$, ▲ mean $Br(\mu \rightarrow \gamma)$ in the range of $1.5 \times 10^{-13}$ to $3.5 \times 10^{-13}$, • show $3.5 \times 10^{-13} \leq Br(\mu \rightarrow \gamma) < 4.2 \times 10^{-13}$

Fig. 9  Below the experimental limit, the line diagram of parameters and $Br(\tau \rightarrow \mu \gamma)$. In a, b solid lines and dotted lines represent $\tan \beta = 9$ and $\tan \beta = 20$
4.4 The processes of $\tau \to e\gamma$

Based on the Table 1 (part of $\tau \to e\gamma$), we analyze $\tau \to e\gamma$ to study the possibility of LFV in Fig. 11. The branching ratio of $\tau \to e\gamma$ process is denoted by: $\bullet$ ($0 < Br(\tau \to e\gamma) < 1 \times 10^{-10}$), $\blacktriangle$ ($1 \times 10^{-10} \leq Br(\tau \to e\gamma) < 9 \times 10^{-10}$) and $\blacklozenge$ ($Br(\tau \to e\gamma)$ from $9 \times 10^{-10}$ to $4.4 \times 10^{-8}$).

The Fig. 11a shows the effects from $m_{\tilde{\nu}_L}$ and $M_{L_{13}}^2$. Most points are concentrated in lower right quarter. $\blacklozenge$ are on the innermost side of the whole region. $\blacktriangle$ are in the middle and $\bullet$ are on the outermost side. The numerical performance is that the larger the $M_{L_{13}}^2$ and the smaller the $m_{\tilde{\nu}_L}$, the larger the $Br(\tau \to e\gamma)$. In Fig. 11b, we analyze the effects of $m_{\tilde{\nu}_L}$ and $m_{\tilde{\nu}_L}$ on $Br(\tau \to e\gamma)$. In the whole figure, $\blacktriangle$ are mainly close to both sides of the x-axis and y-axis, and then $\bullet$ with the same trend, and the rest is $\blacklozenge$. $Br(\tau \to e\gamma)$ decreases with the increase of $m_{\tilde{\nu}_L}$ and $m_{\tilde{\nu}_L}$. Figure 11c has two axes $M_{L_{13}}^2$ versus $T_{e13}$. All three points show “>” shaped distribution, from left to right are $\blacklozenge$, $\blacktriangle$, $\bullet$. So $Br(\tau \to e\gamma)$ increases with the increase of $T_{e13}$.

5 Discussion and conclusion

The $U(1)_X$ SSM has new superfields including right-handed neutrinos and three Higgs superfields $\hat{\eta}$, $\tilde{\eta}$, $\tilde{S}$, and its local gauge group is $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$. We use MIA to study muon anomalous magnetic moment. Combined with the latest experimental data, our numerical results can reach about $2 \times 10^{-9}$, which can better fit the measurement result, and play a certain role in promoting the study of LFV.

We use the method of MIA to study lepton flavor violating decays $l_j \to l_i\gamma$ in the $U(1)_X$ SSM model. From the order of magnitude of branching ratio and data analysis, we can find that the restriction on lepton flavor violation in the process of $\mu \to e\gamma$ is stronger. This provides a reference for other lepton flavor violation work in the future.

We take into account the constraints from the upper limits on LFV branching ratios of $l_j \to l_i\gamma$. In the numerical calculation, we take many parameters as variables including $\tan\beta$, $g_X$, $g_{YX}$, $M_{L_i}^2$, $M_{\tilde{L}_i}^2$, $M_{\tilde{E}_{ij}}^2$, $\delta_{ij}^{AB}$, $m_{\tilde{L}_i}$, $m_{\tilde{\nu}_L}$ and $T_{eij}$. Through the analysis of the numerical results, we find that $M_{L_i}^2$, $M_{\tilde{L}_i}^2$, $g_X$, $g_{YX}$, $\delta_{ij}^{AB}$, $m_{\tilde{L}_i}$, $m_{\tilde{\nu}_L}$ and $T_{eij}$ are sensitive parameters. $Br(l_j \to l_i\gamma)$ is an increasing function of $M_{L_i}^2$, $M_{\tilde{L}_i}^2$, $T_{eij}$, $g_X$, $\delta_{ij}^{AB}$, and decreasing function of $m_{\tilde{L}_i}$ and $m_{\tilde{\nu}_L}$. $g_X$ can also give influence on the numerical results but not very large. That is to say they give mild influences on the numerical results. Finally, we come to the conclusion that the non-diagonal elements which correspond to the generations of the initial lepton and final lepton are main sensitive parameters and LFV sources.
Fig. 11  For the scatter diagrams of the parameters below the experimental limit $Br(\tau \to e\gamma)$, different points represent the different ranges of $Br(\tau \to e\gamma)$. ♦ represent less than $1.0 \times 10^{-10}$, ▲ represent the range of $1.0 \times 10^{-10}$ to $8.0 \times 10^{-10}$, and • represent the range of $8.0 \times 10^{-10}$ to $3.3 \times 10^{-8}$.

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