An Alternative Proof of the $H$-Factor Theorem *

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Abstract

Let $H : V(G) \rightarrow 2^\mathbb{N}$ be a set mapping for a graph $G$. Given a spanning subgraph $F$ of $G$, $F$ is called a general factor or an $H$-factor of $G$ if $d_F(x) \in H(x)$ for every vertex $x \in V(G)$. $H$-factor problems are, in general, $NP$-complete problems and imply many well-known factor problems (e.g., perfect matchings, $f$-factor problems and $(g, f)$-factor problems) as special cases. Lovász [The factorization of graphs (II), Acta Math. Hungar., 23 (1972), 223–246] gave a structure description and obtained a deficiency formula for $H$-optimal subgraphs. In this note, we use a generalized alternating path method to give a structural characterization and provide an alternative and shorter proof of Lovász’s deficiency formula.

1 Introduction

In this paper, we consider finite undirected graphs without loops and multiple edges. For a graph $G = (V, E)$, the degree of $x$ in $G$ is denoted by $d_G(x)$, and the set of vertices adjacent to $x$ in $G$ is denoted by $N_G(x)$. For $S \subseteq V(G)$, the subgraph of $G$ induced by $S$ is denoted by $G[S]$ and $G - S = G[V(G) - S]$. For vertex subsets $S$ and $T$, $E_G(S, T)$ is the set of edges between $S$ and $T$ in $G$. We use $\omega(G)$ for the number of connected components in $G$. Notations and terminologies not defined here may be found in [4].

For a given graph $G$, we associate an integer set $H(x)$ with each vertex $x \in V(G)$ (i.e., $H$ is a set mapping from $V(G)$ to $2^\mathbb{N}$). Given a spanning subgraph $F$ of $G$, $F$ is a general factor or an $H$-factor of $G$ if $d_F(x) \in H(x)$ for every vertex $x \in V(G)$. By specifying $H(x)$

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to be an interval or a special set, an $H$-factor becomes an $f$-factor, an $[a, b]$-factor or a $(g, f)$-factor, respectively. For a general mapping $H$, the decision problem of determining whether a graph has an $H$-factor is known to be $\text{NP}$-complete. In fact, when $H(x)$ contains a “gap” with more than one element, $H$-factor problem is an $\text{NP}$-complete problem. Interestingly, Lovász \[3\] showed that the 3-edge-colorability problem is reducible to $H$-factor problem with $H(x) = \{1\}$ or $\{0, 3\}$ for each $x \in V$. Furthermore, he proved that Four-Colors Problem is equivalent to the existence of such a special $H$-factor. So it is reasonable to conclude that finding a characterization for $H$-factors in general is a challenging problem and so it is natural to turn our attention to $H$-factor problems in which $H(x)$ contains only one-element gaps. Furthermore, Lovász also conjectured that the general factor problem with one-element gaps could be solved in polynomial time and Cornuéjols \[1\] proved the conjecture.

Assume that $H$ satisfies the property:
\[(*) \quad \text{if } i \notin H(x), \text{ then } i + 1 \in H(x), \text{ for } mH(x) \leq i \leq MH(x),\]
where $mH(x) = \min\{r \mid r \in H(x)\}$ and $MH(x) = \max\{r \mid r \in H(x)\}$. Let $MH(S) = \sum_{u \in S} MH(u)$, $mH(S) = \sum_{v \in S} mH(v)$ and $H \pm c = \{i \pm c \mid i \in H\}$. Lovász \[2\] obtained a sufficient and necessary condition for the existence of $H$-factors with the properties $(\ast)$ and a deficiency formula for $H$-optimal subgraphs. In this paper, we use the traditional technique – alternating path – which has dealt effectively with other factor problems to prove Lovász’s deficiency formula. However, we need to modify the usual alternating paths to changeable trails to handle the more complicated structures in this case.

Let $H$ be a set function satisfying the property $(\ast)$ and $F$ any spanning subgraph of $G$. For a vertex $x \in V(G)$, if
\[d_F(x) \in H(x),\]
then vertex $x$ is called feasible. So a subgraph $F$ is an $H$-factor if and only if every vertex is feasible. Given a spanning subgraph $F$ and a subset $S \subseteq V(G)$, the deficiency of subgraph $G[S]$ in $F$ is defined as
\[\text{def}_H(F; S) = \sum_{x \in S} \text{dist}(d_F(x), H(x)),\]
where $\text{dist}(d_F(x), H(x))$ is the distance of $d_F(x)$ from the set $H(x)$. For convenience, we use $\text{def}(F; S)$ in short. So $\text{def}(F; x) = \text{dist}(d_F(x), H(x))$ is the deficiency of vertex $x$ in $F$. We can measure $F$’s “deviation” from condition $(\ast)$ by defining the deficiency of $F$ with respect to $H$ as
\[\text{def}_H[F] = \sum_{x \in V(F)} \text{dist}(d_F(x), H(x)).\]
The total deficiency of $G$ with respect to $H$ is

$$
def_H(G) = \min \{ \def_H[F] \mid F \text{ is a spanning subgraph of } G \}. $$

Clearly, $\def_H(G) = 0$ if and only if there exists an $H$-factor. A subgraph $F$ is called $H$-optimal, if $\def_H[F] = \def_H(G)$. Of course, any $H$-factor is $H$-optimal.

Let $I_H(x) = \{ d_F(x) \mid F \text{ is any } H\text{-optimal subgraph} \}$. Lovász [2] studied the structure of $H$-factors in graphs by introducing a Gallai-Edmonds type of partition for $V(G)$ as follows:

$$
C_H(G) = \{ x \mid I_H(x) \subseteq H(x) \}, \\
A_H(G) = \{ x \mid \min I_H(x) \geq MH(x) \}, \\
B_H(G) = \{ x \mid \max I_H(x) \leq mH(x) \}, \\
D_H(G) = V(G) - A_H(G) - B_H(G) - C_H(G).
$$

Based on this canonical partition, Lovász obtained a sufficient and necessary conditions of $H$-factors with property (*) and as well as the deficiency formula for $H$-optimal subgraphs. In this paper, we give an alternative description of the partition $(A, B, C, D)$ by deploying changeable trails and therefore provide a new proof of the deficiency formula for $H$-optimal subgraphs. Our approach is as follows:

Suppose that $G$ does not have $H$-factors. Choose a spanning subgraph $F$ of $G$ such that for all $v \in V(G), d_F(v) \leq MH(v)$ and the deficiency is minimized over all such choices. Moreover, we choose $F$ such that the $E(F)$ is minimal. Necessarily, there is a vertex $v \in V$ such that $d_F(v) \not\in H(v)$, so the deficiency of $F$ is positive. Set

$$
B_0 = \{ x \in V(G) \mid d_F(x) \not\in H(x) \}.
$$

Since $E(F)$ is minimal and $H$ satisfies (*), we have

$$
B_0 = \{ x \in V(G) \mid d_F(x) < mH(x) \}.
$$

A trail $P = v_0v_1 \ldots v_k$ is called a changeable trail if it satisfies the following condition:

(a) $v_0 \in B_0$, and $v_0v_1 \not\in E(F)$;

(b) $\def(F; x) = \def(F \triangle P; x) = 0$, for every $x \in V(P) - v_0 - v_k$;

(c) if $v_0 = v_i \neq v_k$, then $\def(F; v_0) > \def(F \triangle P; v_0)$;

(d) for all $l \leq k$, sub-trail $P' = v_0v_1 \ldots v_l$ satisfies conditions (a)-(c) as well.

A changeable trail $P$ is odd if the last edge doesn’t belong to $F$; otherwise, $P$ is even. Moreover, the trails of length zero are considered as even changeable trails.
For a given graph $G$, we define $D(G)$ to be a vertex set consisting of three types of vertices as follows:

(i) $\{v \mid \exists$ both of an even changeable trail and an odd changeable trail from $B_0$ to $v\}$;

(ii) $\{v \mid mH(v) < d_F(v) \leq MH(v)$ and $\exists$ an even changeable trail from $B_0$ to $v\}$;

(iii) $\{v \mid mH(v) \leq d_F(v) < MH(v)$ and $\exists$ an odd changeable trail from $B_0$ to $v\}$.

The sets $A(G)$ and $B(G)$ are defined as follows:

$$B(G) = \{v \mid \exists \text{ an even changeable trail ending at } v\} - D,$$

$$A(G) = \{v \mid \exists \text{ an odd changeable trail ending at } v\} - D,$$

and $C(G) = V(G) - A(G) - B(G) - D(G)$. We abbreviate $D(G), A(G), B(G)$ and $C(G)$ by $D, A, B$ and $C$, respectively.

If $v \in B$, then $d_F(v) \leq mH(v)$. Otherwise, as $v \not\in D$, we can swap edges in $F$ along an even changeable trail ending at $v$ and thus decrease the deficiency. Similarly, if $v \in A$, then $d_F(v) = MH(v)$. Otherwise, as $v \not\in D$, we can likewise decrease the deficiency by swapping edges in $F$ along an odd changeable trail ending at $v$. By the definitions, clearly $A, B, C$ and $D$ are a partition of $V(G)$. We call a changeable trail $P$ an augmenting changeable trail if $\text{def}(F \triangle P; G) < \text{def}(F; G)$. Following the above discussion, when $H(v)$ is an integer interval with more than an element, then $v \not\in D$.

## 2 Main Theorem

In the following lemmas, we always assume that $G$ has no $H$-factors and $F$ is an $H$-optimal subgraph with minimal $E(F)$. Let $\tau = \omega(G[D])$ and $D_1, \ldots, D_\tau$ be the components of the subgraph of $G$ induced by $D$.

**Lemma 2.1** An $H$-optimal subgraph $F$ does not contain an augmenting changeable trail.

**Lemma 2.2** $\text{def}(F; D_j) \leq 1$ for $j = 1, \ldots, \tau$.

**Proof.** Suppose, to the contrary, that $\text{def}(F; D_i) > 1$. Let $v_0 \in D_i$ and $\text{def}(F; v_0) \geq 1$. Since $E(F)$ is minimal, so $d_F(v_0) < mH(v_0)$. Hence $v_0$ is of type (i) and there exists an odd changeable trail $P$ from a vertex $x$ of $B_0$ to $v_0$. Then $x = v_0$. Otherwise, $\text{def}(F; G) > \text{def}(F \Delta P; G)$, a contradiction since $F$ is $H$-optimal. Furthermore, if $\text{def}(F; v_0) \geq 2$, then $\text{def}(F; G) > \text{def}(F \Delta P; G)$, a contradiction again. So we have
$def(F; v_0) = 1$ and $def(F; u) \leq 1$ for any $u \in D_i - v_0$. Moreover, $d_F(v_0) + 1 \in H(v_0)$ and $d_F(v_0) + 2 \notin H(v_0)$.

We define $D_i^1$ to be a vertex set consisting of three types of vertices as follows:

1. $\{w \in D_i \mid \exists$ an even changeable trail and an odd changeable trail from $v_0$ to $w\}$;

2. $\{w \in D_i \mid mH(w) < d_F(w) \leq MH(w)$ and $\exists$ an even changeable trail from $v_0$ to $w\}$;

3. $\{w \in D_i \mid mH(w) \leq d_F(w) < MH(w)$ and $\exists$ an odd changeable trail from $v_0$ to $w\}$.

Now we choose a maximal subset $D_i^2$ of $D_i^1$ such that $P \subseteq D_i^2$ and the trails, which are of type (1), (2) or (3), belongs to $D_i^2$.

Claim. $D_i^2 = D_i$.

Otherwise, since $D_i$ is connected, there exists an edge $xy \in E(G)$ such that $x \in D_i - V(D_i^2)$ and $y \in V(D_i^2)$. We consider $xy \in E(F)$ (or $xy \notin E(F)$).

Then there exists an even (resp. odd) changeable trail $P_1$ from $v_0$ to $x$, where $xy \in P_1$ and $V(P_1) - x \subseteq V(D_i^2)$. So $x$ is type (i) or type (ii) (resp. type (i) or type (iii)). Since $x \notin D_i^2$, $x$ can only be of type (i). So there exists an odd (resp. even) changeable trail $P_2$ from a vertex $t$ of $B_0$ to $x$. Thus $t \neq v_0$; otherwise, we have $V(P_1 \cup P_2) \subseteq D_i^2$, a contradiction to the maximality of $D_i^2$. If $E(P_1) \cap E(P_2) = \emptyset$, then $def(F; G) > def(F \triangle (P_1 \cup P_2); G)$, a contradiction since $F$ is $H$-optimal. Let $z \in P_2$ be the first vertex which also belongs to $D_i^2$ and denote the subtrail of $P_2$ from $t$ to $z$ by $P_3$. If $z$ is of type (1), by the definition, there exist both an odd changeable trail $P_4$ from $v$ to $z$ and an even changeable trail $P_5$ from $v$ to $z$ such that $V(P_4 \cup P_5) \subseteq V(D_i^2)$. Thus either $P_4 \cup P_3$ or $P_5 \cup P_3$ is an augmenting trail, a contradiction to Lemma 2.1. If $z$ is type (2) or type (3), the argument is similar. We complete the claim.

Let $u \in V(D_i) - v_0$ and $def(F; u) = 1$. Since $u$ is not type (2), there exists an odd changeable trail $P_6$ from $v_0$ to $u$. We have $def(F; G) > def(F \triangle P_6; G)$, a contradiction since $F$ is $H$-optimal. \hfill $\square$

Using the above lemma, we have the following result.

**Lemma 2.3** For $i = 1, \ldots, \tau$, if $def(F; D_i) = 1$, then

(a) $E_G(D_i, B) \subseteq E(F)$;

(b) $E_G(D_i, A) \cap E(F) = \emptyset$. 

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Proof. Let \(\text{def}(F; r) = 1\), where \(r \in V(D_i)\). Suppose the lemma does not hold.

To show (a), let \(uv \notin E(F)\), where \(u \in V(D_i)\) and \(v \in V(B)\). If \(u\) is of type (i) or type (ii), from the proof of Lemma 2.2, then there exists an even changeable trail \(P \subseteq G[D_i]\) from \(r \) to \(u\). Hence \(P \cup uv\) be an odd changeable trail from \(r \) to \(v\), a contradiction to \(v \in B\). If \(u\) is of type (iii), then there exists an odd changeable trail \(P \subseteq G[D_i]\) from \(r \) to \(u\). Since \(F\) is \(H\)-optimal and \(H\) has the property (*), so \(d_F(u) \in H(u), d_F(u) + 1 \notin H(u)\) and \(d_F(u) + 2 \in H(u)\). Hence \(P \cup uv\) is an odd changeable trail from \(r \) to \(v\), a contradiction to \(v \in B\) again.

Next we consider (b). Let \(uv \in F\), where \(u \in D_i\) and \(v \in A\). If \(u\) is of type (i) or type (iii), from the proof of Lemma 2.2, then there exists an odd changeable trail \(P \subseteq G[D_i]\) from \(r \) to \(u\). Then \(P \cup uv\) be an even changeable trail from \(r \) to \(v\), a contradiction to \(v \in A\). If \(u\) is of type (ii), then there exists an even changeable trail \(P \subseteq G[D_i]\) from \(r \) to \(u\). Since \(F\) is \(H\)-optimal and \(H\) has the property (*), so \(d_F(u) \in H(u), d_F(u) - 1 \notin H(u)\) and \(d_F(u) - 2 \in H(u)\). Hence \(P \cup uv\) is an even changeable trail from \(r \) to \(v\), a contradiction to \(v \in A\) again. \(\Box\)

From the definition of partition \((A, B, C, D)\), it is not hard to see the next lemma.

**Lemma 2.4** \(E_G(B, C \cup B) \subseteq E(F)\), \(E_G(A, A \cup C) \cap E(F) = \emptyset\) and \(E_G(D, C) = \emptyset\).

**Lemma 2.5** (a) \(F\) misses at most one edge of \(E_G(D_i, B)\). Moreover, if \(F\) misses one edge of \(E_G(D_i, B)\), then \(E_G(D_i, A) \cap E(F) = \emptyset\);

(b) \(F\) contains at most one edge of \(E_G(D_i, A)\). Moreover, if \(F\) contains one edge of \(E_G(D_i, A)\), then \(E_G(D_i, B) \subseteq E(F)\).

**Proof.** By Lemma 2.3 we may assume \(\text{def}(F; D_i) = 0\). Let \(u \in V(D_i)\), by the definition of \(D\), there exists a changeable trail \(P\) from a vertex \(x\) of \(B_0\) to \(u\). Denote the first vertex in \(P\) belonging to \(D_i\) by \(y\), and the sub-trail of \(P\) from \(x\) to \(y\) by \(P_1\). Let \(y_1y \in E(P_1)\), where \(y_1 \notin D_i\). Without loss of generality, assume that \(P_1\) is an odd changeable trail (when \(P_1\) is an even changeable trail, the proof is similar). Since \(P_1\) is a changeable trail, so \(y_1 \in B\) and \(y_1y \notin F\). Because \(y \in D\), \(y\) is of type (i) or type (iii). We define the subset \(D_i^1 \subseteq D_i\) which consists of the following vertices:

1. \(\{w \in D_i \mid \exists \text{ an even changeable trail and an odd changeable trail along } P_1 \text{ from } x \text{ to } w\}\);
2. \(\{w \in D_i \mid mH(w) < d_F(w) \leq MH(w) \text{ and } \exists \text{ an even changeable trail along } P_1 \text{ from } x \text{ to } w\}\);
3. \(\{w \in D_i \mid mH(w) \leq d_F(w) < MH(w) \text{ and } \exists \text{ an odd changeable trail along } P_1 \text{ from } x \text{ to } w\}\).

Now we choose a maximal subset \(D_i^2\) of \(D_i^1\) such that the trails, which are of type (1), (2) or (3), except \(V(P_1) - y\), belongs to \(D_i^2\).
Claim 1. \( D_1^1 = D_i = D_i^2 \).

Suppose that \( D_i \neq D_i^2 \). Let \( v_1v_2 \in E(G) \), where \( v_1 \in D_i^1 \) and \( v_2 \in D_i - D_i^2 \). Firstly, we show that \( D_i^2 \neq \emptyset \). If \( y \) is type (iii), then \( y \in D_i^2 \). If \( y \) is type (i) or type (ii), then there exists an even changeable trail \( R_1 \) from a vertex \( w \) of \( B_0 \) to \( y \). We have \( yy_1 \in E(R_1) \); otherwise \( R_1 \cup yy_1 \) is an odd changeable trail from \( w \) to \( y_1 \), contradicting to \( y_1 \in B \). Hence, we may assume \( w = x \) and \( P_1 \) is a subtrail of \( R_1 \). So \( V(R_1) - (V(P_1) - y) \subseteq D_i^2 \) and \( D_i^2 \neq \emptyset \).

We consider \( v_1v_2 \in E(F) \). Then there exists an even changeable trail \( R_2 \) from \( x \) to \( v_2 \) such that \( V(R_2) - (V(P_1) - y) - v_2 \subseteq D_i^2 \). If \( v_2 \) is type (ii), by the definition of \( D_i^2 \), then we have \( v_2 \in D_i^2 \), contradicting to the maximality of \( D_i^2 \). If \( v_2 \) is type (i) or type (iii), then there exists an odd changeable trail \( R_3 \) from a vertex \( w_2 \) of \( B_0 \) to \( v_2 \). Next we show that \( yy_1 \in R_3 \). If \( V(R_3) \cap D_i^2 \neq \emptyset \), let \( z \) be first vertex in \( R_3 \) belonging \( D_i^2 \); else let \( z = v_2 \). Without loss of generality, we suppose that the subtrail \( R_4 \) from \( w_2 \) to \( z \) along \( R_3 \) is an odd changeable trail and \( z \in D_i^2 \). If \( z \in D_i^2 \) is type (1) or (2), then there is an even changeable trail, say \( R_5 \), from \( x \) to \( z \) along \( P_1 \) such that \( V(R_5) - (V(P_1) - y) \subseteq D_i^2 \). Let \( R_6 \) is a subtrail from \( y_1 \) to \( z \) along \( R_5 \). Then \( R_4 \cup R_6 \) is an odd changeable trail from \( w_2 \) to \( y_1 \), contradicting to \( y_1 \in B \). If \( z \in D_i^2 \) is type (3), then \( d_F(z) \in H(z) \), \( d_F(z) + 1 \notin H(z) \), and \( d_F(z) + 2 \in H(z) \). Moreover, there is an odd changeable trail \( R_7 \) along \( P_1 \) from \( x \) to \( z \). Let \( R_8 \) is a subtrail from \( y_1 \) to \( z \) along \( R_7 \). Then \( R_4 \cup R_8 \) is an odd changeable trail from \( w_2 \) to \( y_1 \), contradicting to \( y_1 \in B \) again. So \( yy_1 \in R_3 \). Let \( R_9 \) be the subtrail from \( y \) to \( v_2 \) along \( R_3 \). Then we have \( V(R_9) \subseteq D_i^2 \), contradicting to the maximality of \( D_i^2 \). By the symmetry of definition of \( D_i \) and \( D_i^2 \), for \( v_1v_2 \notin E(F) \), the proof is similar. We complete the proof of the claim.

Let \( x_3y_3 \in E(G) - yy_1 \). We have the following two claims.

Claim 2. If \( x_3 \in D_i \) and \( y_3 \in B \), then \( x_3y_3 \in E(F) \).

Otherwise, \( x_3y_3 \notin E(F) \). If \( x_3 \) is of type (1) or (2), by the definition of set \( D_i^2 \) and \( D_i = D_i^2 \), then there exists an even changeable trail \( P_{10} \) from \( x \) to \( x_3 \) such that \( V(P_{10}) - (V(P_1) - y) \subseteq D_i \). Then \( P_{10} \cup x_3y_3 \) is an odd changeable trail from \( x \) to \( y_3 \), contradicting to \( y_3 \in B \). If \( x_3 \) is of type (3), then there exists an odd changeable trail \( P_{11} \) from \( x \) to \( x_3 \) such that \( V(P_{11}) - (V(P_1) - y) \subseteq D_i \). Note that \( d_F(x_3) \in H(x_3) \). Since \( F \) is \( H \)-optimal, so \( d_F(x_3) + 1 \notin H(x_3) \) and \( d_F(x_3) + 2 \in H(x_3) \). Then \( P_{11} \cup x_3y_3 \) is an odd changeable trail from \( x \) to \( y_3 \), contradicting to \( y_3 \in B \). We complete Claim 2.

Claim 3. If \( x_3 \in D_i \) and \( y_3 \in A \), then \( x_3y_3 \notin E(F) \).

Otherwise, \( x_3y_3 \in E(F) \). If \( x_3 \) is of type (1) or (3), then there exists an odd changeable trail \( P_{12} \) from \( x \) to \( x_3 \) such that \( V(P_{12}) - (V(P_1) - y) \subseteq D_i \). Then \( P_{12} \cup x_3y_3 \) is an even changeable trail from \( x \) to \( y_3 \), contradicting to \( y_3 \in A \). If \( x_3 \) is of type (2), then there exists
an even changeable trail $P_{13}$ from $x$ to $x_3$ such that $V(P_{13}) - (V(P_i) - y) \subseteq D_i$. Note that $d_F(x_3) \in H(x_3)$. Since $F$ is $H$-optimal and $H$ has the property (*), so $d_F(x_3) - 1 \notin H(x_3)$ and $d_F(x_3) - 2 \in H(x_3)$. Then $P_{13} \cup x_3y_3$ is an even changeable trail from $x$ to $y_3$, contradicting to $y_3 \in A$ again.

We complete the proof. \hfill \Box

Now we present and prove deficiency formula for $H$-optimal subgraphs. Recall that $\tau$ is the number of components in $G[D]$.

**Theorem 2.6** $\text{def}_H(G) = \tau + \sum_{v \in B}(mH(v) - d_{G-A}(v)) - \sum_{v \in A} M_H(v)$.

**Proof.** Let $\tau_1$ denote the number of components $D_i$ of $G[D]$ which satisfies $\text{def}(F; D_i) = 1$. Let $\tau_B$ (or $\tau_A$) be the number of components $T$ of $G[D]$ such that $F$ misses (or contains) one edge from $T$ to $B$ (or $A$). By Lemmas 2.2 and 2.5 we have $\tau = \tau_1 + \tau_A + \tau_B$. Note that $d_F(v) \leq m_H(v)$ for all $v \in B$ and $d_F(v) = M_H(v)$ for all $v \in A$. So

$$
\text{def}_H(G) = \tau_1 + m_H(B) - \sum_{v \in B}d_F(v)
= \tau_1 + m_H(B) - \left(\sum_{v \in B}d_{G-A}(v) - \tau_B\right) - (M_H(A) - \tau_A)
= \tau + m_H(B) - \sum_{v \in B}d_{G-A}(v) - M_H(A).
$$

\hfill \Box

Let $X, Y$ be two disjoint subsets of $V(G)$. Define the modified prescription of $H$ to be

$$
H_{(X,Y)}(u) = H(u) - |E_{G}(u,Y)| \quad \text{for} \quad u \in V(G) - X - Y.
$$

Let $K$ be a component of $G - X - Y$. We defined $H_{(X,Y)|K}$ as follows:

$$
H_{(X,Y)|K}(u) = H_{(X,Y)}(u) \quad \text{for} \quad u \in V(K).
$$

**Theorem 2.7** Let $F_i = F[V(D_i)]$ for $i = 1, \ldots, \tau$. Then $\text{def}_{H(A,B)|D_i}(F_i) = 1$ and $F_i$ is $H_{(A,B)|D_i}$-optimal for $i = 1, \ldots, \tau$.

**Proof.** By Lemmas 2.2 and 2.5 we have $\text{def}_{H(A,B)|D_i}(F_i) \leq 1$. Suppose that the theorem doesn’t hold. Let $F_i^*$ be $H_{(A,B)|D_i}$-optimal. Then we have $\text{def}_{F_i^*}(D_i) = 0$. We consider two cases.

**Case 1.** $\text{def}_H(F; D_i) = 1$.

Then we have $\text{def}_H((F - F_i) \cup F_i^*) < \text{def}_H(F)$, but $F$ is $H$-optimal, a contradiction.

**Case 2.** $F$ contains (or misses) an edge of $E(D_i, A)$ (resp. $E(D_i, B)$).

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Let \( yy_1 \in E(D_i, A) \cap E(F) \), where \( y \in D_i \) and \( y_1 \in A \) (resp. \( yy_1 \in E(D_i, B) \) and \( yy_1 \notin E(F) \), where \( y \in D_i \) and \( y_1 \in B \)). Since \( y \in D_i \), there is a changeable trail \( P \) from a vertex \( x \) of \( B_0 \) to \( y \). By Lemma 2.3, we have \( yy_1 \in E(P) \). Let \( P_1 \) be a subtrail of \( P \) such that \( V(P_1) \cap V(D_i) = \{ y \} \). Then we have \( \text{def}_H(((F - F_i) \Delta P_1) \cup F_1^*) < \text{def}_H(F) \), a contradiction again. \( \square \)

Now we prove Lovász's classic deficiency formula.

**Theorem 2.8 (Lovász [2])** The total deficiency is

\[
\text{def}_H(G) = \max_{S,T} \tau_H(S,T) - \sum_{x \in T} d_{G-S}(x) - MH(S) + mH(T),
\]

where \( \tau_H(S,T) \) denotes the number of components \( K \) of \( G - S - T \) such that \( K \) contains no \( H_{(S,T)|K} \)-factors. Moreover, a graph \( G \) has an \( H \)-factor if and only if for any pair of disjoint sets \( S, T \subseteq V(G) \),

\[
\tau_H(S,T) - \sum_{x \in T} d_{G-S}(x) - MH(S) + mH(T) \leq 0.
\]

**Proof.** Let \( M \) be an arbitrary \( H \)-optimal graph of \( G \). Firstly, we show that

\[
\text{def}_H(G) \geq \max_{S,T} \tau_H(S,T) - \sum_{x \in T} d_{G-S}(x) - MH(S) + mH(T).
\]

Let \( S \) and \( T \) be arbitrary disjoint subsets of \( V(G) \). Let \( \tau_H(S,T) \) be defined as in the above. For \( i = 1, \ldots, \tau_H(S,T) \), let \( C_i \) denote the component of \( G - S - T \) containing no \( H_{(S,T)|C_i} \)-factors. Let \( W = C_1 \cup \cdots \cup C_{\tau_H(S,T)} \). Since \( C_i \) contains no \( H_{(S,T)|C_i} \)-factors, so if \( \text{def}_M(C_i) = 0 \), then \( M \) either misses at least an edge of \( E(C_i, T) \) or contains at least an edges of \( E(C_i, S) \). Let \( \tau_1 \) denote the number of components of \( W \) such that \( M \) misses at least an edge of \( E(C_i, T) \) and \( \tau_2 \) denote the number of the components of \( W \) such that \( M \) contains at least an edge of \( E(C_i, S) \). Then we have

\[
\text{def}_H(G) = \text{def}_H(M) \geq \tau_H(S,T) - \tau_1 - \tau_2 + \sum_{x \in S \cup T} \text{dist}(d_M(x), H(x))
\]

\[
\geq \tau_H(S,T) - \tau_1 - \tau_2 + \sum_{x \in S} (d_M(x) - MH(x)) + \sum_{x \in T} (MH(x) - d_M(x))
\]

\[
\geq \tau_H(S,T) - \tau_1 - \tau_2 + (e_M(S,T) + \tau_2 - MH(S)) + \sum_{x \in T} (MH(x) - d_M(x))
\]

\[
= \tau_H(S,T) - \tau_1 + (e_M(S,T) - MH(S)) + \sum_{x \in T} (MH(x) - d_M(x))
\]

\[
\geq \tau_H(S,T) - \tau_1 + (e_M(S,T) - MH(S)) + (MH(T) - (e_M(S,T) + \sum_{x \in T} d_{G-S}(x) - \tau_1))
\]

\[
= \tau_H(S,T) + MH(T) - MH(S) - \sum_{x \in T} d_{G-S}(x).
\]
By Theorems 2.6 we have

\[ \text{def}_H(G) = \tau + \sum_{v \in B} (mH(v) - d_{G-A}(v)) - \sum_{v \in A} MH(v). \]

By Theorem 2.7, \( D_i \) contains no \( H_{(A,B)|D_i} \)-factors for \( i = 1, \ldots, \tau \). So we have

\[ \text{def}_H(G) = \tau_H(A,B) + \sum_{v \in B} (mH(v) - d_{G-A}(v)) - \sum_{v \in A} MH(v) \]

\[ = \max_{S,T: S \cap T = \emptyset} \tau_H(S,T) - \sum_{x \in T} d_{G-S}(x) - MH(S) + mH(T). \]

We complete the proof. \( \square \)

The proof of Theorem 2.8 also imply the following result.

**Theorem 2.9** Let \( R \) be an arbitrary \( H \)-optimal graph of \( G \). Then

1. \( d_R(v) \in H(v) \) for all \( v \in C \);
2. \( d_R(v) \geq MH(v) \) for all \( v \in A \);
3. \( d_R(v) \leq mH(v) \) for all \( v \in B \).

From Theorem 2.9 we can see that the partition \((A(G), B(G), C(G), D(G))\) defined in this paper is equivalent to the original partition \((A_H, B_H, C_H, D_H)\) introduced by Lovász in [2].

**Theorem 2.10** \( C_H = C, A_H = A, B_H = B \) and \( D_H = D \).

**Proof.** By Theorem 2.9, we have \( C \subseteq C_H, A \subseteq A_H, B \subseteq B_H \). However, the definition of \( D \) implies \( v \notin C_H \cup A_H \cup B_H \) for every \( v \in D \). So we have \( D \subseteq D_H \). This completes the proof. \( \square \)

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