A Soluble Model of Four-Fermion Interactions in de Sitter Space

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Abstract

We consider the theory of four-fermion interactions with N-component fermions in de Sitter space. It is found that the effective potential for a composite operator in the theory is calculable in the leading order of the 1/N expansion. The resulting effective potential is analyzed by varying both the four-fermion coupling constant and the curvature of the space-time. The critical curvature at which the dynamically generated fermion mass disappears is found to exist and is calculated analytically. The dynamical fermion mass is expressed as a function of the space-time curvature.

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According to the well-accepted scenario of early universe it is believed that the GUT phase is broken down to the QCD and electroweak phase through the symmetry breaking due to the Higgs mechanism. At this era the quantum gravity plays a minor role while the space-time curvature due to the external strong gravity is important in triggering the phase transition. Thus it is of interest to consider quantum field theory in curved space-time in connection with physics in the very early universe.\(^1\) On the other hand the Higgs mechanism is often explained as a dynamical effect due to the emergence of the composite Higgs field. A typical example of such a model is the technicolor model.\(^2\) In this regard it is interesting to deal with a quantum field theory with the composite Higgs field. The four-fermion interaction theory is one of the prototype models of the composite Higgs theory.\(^3\)

Under these circumstances we find it useful to discuss the phase structure of the four-fermion theory in curved space-time. Since the phase transition is a nonperturbative phenomenon, we need to use the method free from the perturbative approach and also to avoid any approximation in dealing with the space-time curvature, e. g. a weak curvature approximation in which we rely on an expansion in powers of the curvature.

In the present paper we adopt the \(1/N\) expansion method as a nonperturbative approach and try to find the effective potential without making any approximation in the space-time curvature. For this purpose we restrict ourselves to the specific space-time, i. e. the de Sitter space-time, and calculate the effective potential for the composite operator made of a fermion-antifermion pair. By the use of the effective potential we shall argue the symmetry breaking caused by the curvature effect.

We consider the theory in the curved space-time defined by the action,

\[
S = \int d^Dx \sqrt{-g(x)} \left[ \sum_{k=1}^{N} \bar{\psi}_k \gamma^\mu \nabla_\mu \psi_k + \frac{\lambda_0}{2N} \left( \sum_{k=1}^{N} \bar{\psi}_k \psi_k \right)^2 \right],
\]  

(1)

where index \(k\) represents the flavor of the fermion field \(\psi\), \(N\) is the number of fermion
species, $g$ the determinant of the space-time metric $g_{\mu\nu}$, $\gamma^\mu$ the Dirac matrix in the curved space and $\nabla_\mu \psi$ the covariant derivative of the fermion field $\psi$. Throughout the paper we work in arbitrary space-time dimension $D$. For simplicity we neglect the flavor index and the summation on it. Our notation is the $(+,+,+)$ convention as defined in the book by Misner, Thorne and Wheeler.\footnote{In the following calculations it is more convenient to introduce auxiliary field $\sigma$ and to consider the action,}

$$S' = \int d^Dx \sqrt{-g(x)} \left[ \bar{\psi} \gamma^\mu \nabla_\mu \psi + \bar{\psi} \sigma \psi - \frac{N}{2\lambda_0} \sigma^2 \right]. \quad (2)$$

It is well-known that the physics described by action (2) is equivalent to that described by action (1). We are interested in estimating the effective potential for the composite field $\bar{\psi} \psi$ which is essentially the same as the auxiliary field $\sigma$. We calculate the effective potential $V(\sigma)$ for field $\sigma$ in the theory defined by action $S'$ in Eq.(2).

We start with the generating functional given by

$$Z = \int [d\psi d\bar{\psi} d\sigma] \exp(iS). \quad (3)$$

Performing the integration over the fermion fields $\psi$ and $\bar{\psi}$ we obtain

$$Z = \int [d\sigma] \exp(iNS_{eff}), \quad (4)$$

where $S_{eff}$ is given by

$$S_{eff} = -\int d^Dx \sqrt{-g(x)} \frac{1}{2\lambda_0} \sigma^2 - i \text{Tr} \ln(\gamma^\mu \nabla_\mu + \sigma). \quad (5)$$

In the following argument we restrict ourselves to the space-time where the path integrals (3) and (4) are well-defined. As it may be seen that quantum corrections relevant to the auxiliary field $\sigma$ are only of higher order in the $1/N$ expansion, we realize that the field $\sigma$ is regarded as a classical field in the leading order of the $1/N$ expansion and the effective potential $V(\sigma)$ reads

$$V(\sigma) = \frac{1}{2\lambda_0} \sigma^2 + i \text{tr} \langle x | \ln \frac{\gamma^\mu \nabla_\mu + \sigma}{\gamma^\mu \nabla_\mu} | x \rangle + O(1/N), \quad (6)$$

3
where the potential is normalized so that $V(0) = 0$ and variable $\sigma$ is independent of the space-time coordinate.

To estimate the second term on the right-hand side of Eq.(6) we adopt the Schwinger proper time method, i.e.

$$V(\sigma) = \frac{1}{2\lambda_0} \sigma^2 + \text{itr} \int_0^\sigma ds \, G(x,x; s),$$

(7)

where $G(x,y; s)$ is defined by

$$G(x,y; s) = \langle x| (\gamma^\mu \nabla_\mu + s)^{-1} |y\rangle.$$  

(8)

The function $G(x,y; s)$ satisfies the differential equation

$$(\gamma^\mu \nabla_\mu + s)G(x,y; s) = \delta^D(x,y),$$

(9)

where $\delta^D(x,y)$ is the Dirac delta function in the curved space. With this equation we recognize that $G(x,y; s)$ is essentially equal to the Green function for the massive free fermion with mass $s$ in the curved space-time.

Our problem of calculating the effective potential for the composite field in the leading order of the $1/N$ expansion for the four-fermion theory in the curved space-time is now reduced to the problem of finding the Green function for the free massive fermion in the curved space-time. Fortunately it is known that the Green function for the free massive fermion is exactly calculable in de Sitter space.\footnote{We then restrict ourselves to de Sitter space and closely follow the method developed by Candelas and Raine.\footnote{The de Sitter space is a maximally symmetric curved space-time with constant curvature. The de Sitter space of $D$ dimensions is represented as a hyperboloid}}

\footnote{The de Sitter space of $D$ dimensions is represented as a hyperboloid}

$$r^2 = -\xi_0^2 + \xi_1^2 + \cdots + \xi_D^2,$$

(10)

embedded in the $D + 1$-dimensional Minkowski space. The metric in de Sitter space with
variable $\xi_\mu$ ($\mu = 0, 1, 2, \ldots, D - 1$) is given by

$$g_{\mu \nu} = \eta_{\mu \nu} + \frac{\xi_\mu \xi_\nu}{r^2 - \xi^2}. \quad (11)$$

The space-time curvature $R$ for de Sitter space reads

$$R = D(D - 1)r^{-2}. \quad (12)$$

We concentrate ourselves on solving Eq.(9) in de Sitter space with the Dirac delta function,

$$\delta(x, x') = ie^{-iD\pi/4} \frac{2\pi}{(4\pi\epsilon)^{D/2}} e^{i\frac{\pi}{2} \sqrt{-\gamma}}, \quad (13)$$

where $\epsilon$ is a parameter which is set equal to zero after calculations. We first note that the Green function $G(x, y; s)$ in de Sitter space depends on two variables $x$ and $y$ through

$$\sigma(x, x') = \frac{1}{2}(\xi - \xi')^2, \quad (14)$$

according to the maximal symmetry of de Sitter space. As is pointed out by Candelas and Raine, the Green function $G(x, y; s)$ may be decomposed into invariant amplitudes $A(\sigma)$ and $B(\sigma)$ in the following way:

$$G(x, y; s) = H(x, y)\Phi(x, y), \quad (15)$$

with $\Phi(x, x) = \text{unit matrix}$, and

$$H(x, y) = A(\sigma) + B(\sigma)\sigma_\alpha \gamma^\alpha. \quad (16)$$

We substitute the expression (13) with Eq.(16) into Eq.(9) and take the trace on both side of the equation. After some algebra we obtain for the invariant amplitude $A(\sigma)$

$$z(z - 1)\frac{d^2 A}{dz^2} + D(z - \frac{1}{2}) \frac{dA}{dz} + r^2(\frac{i}{2r}(2 - D) + s)(\frac{i}{2r}D + s)A = r^2(\frac{i}{2r}(2 - D) + s)ie^{-iD\pi/4} \frac{2\pi}{(4\pi\epsilon)^{D/2}} e^{i\sigma/2\epsilon}, \quad (17)$$
where we made the change of variable: \( z = \frac{\sigma^2}{2r^2} \). Eq. (17) is the hypergeometric differential equation whose solution is given by

\[
A = \frac{ar^{1-D}}{(4\pi)^{D/2}} \frac{\Gamma(a)\Gamma(b)}{\Gamma(D/2)} F(a, b, D/2; 1 - \frac{\sigma^2}{2r^2}),
\]

where \( F(a, b, c; z) \) is the hypergeometric function of variable \( z \) with parameters \( a, b, c \)

\[
\begin{align*}
  a &= \frac{D - 2}{2} + isr, \\
  b &= \frac{D}{2} - isr.
\end{align*}
\]

From the solution (18) \( \text{tr}G(x, x; s) \) may be easily obtained with recourse to Eq. (15) with

\[
\text{tr}G(x, x; s) = \frac{ar^{1-D}}{(4\pi)^{D/2}} \frac{\Gamma(a)\Gamma(b)(-a - b + D/2)}{\Gamma(-a + D/2)\Gamma(-b + D/2)} \text{tr}1.
\]

Here by \( \text{tr}1 \) we mean the trace of the unit Dirac matrix. Inserting Eq. (20) into Eq. (7) our final expression of the effective potential is obtained.

\[
V(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - \int_0^\sigma ds \frac{sr^{2-D}}{(4\pi)^{D/2}} \frac{\Gamma\left(D + isr\right)}{\Gamma\left(1 + isr\right)\Gamma\left(1 - isr\right)} \Gamma\left(1 - \frac{D}{2}\right) \text{tr}1.
\]

Equation (21) is an exact expression of the effective potential for the model of four-fermion interactions in de Sitter space in the leading order of the \( 1/N \) expansion.

The effective potential (21) is divergent in two and four dimensions. In four dimensions the theory is nonrenormalizable and so we restrict ourselves to the case \( D < 4 \). In two dimensions the theory is renormalizable so that the effective potential is given a finite expression by the renormalization procedure. All the divergent terms in the effective potential are essentially of the same form as those in the flat space-time. (Note that according to our normalization \( V(0) = 0 \) divergences associated with the vacuum energy are absent.) Thus the necessary renormalization can be performed in the vanishing curvature limit, i.e. in the Minkowski space. Taking the limit \( R \rightarrow 0 \) in Eq. (21) we obtain
the effective potential \( V_0(\sigma) \) in the \( D \)-dimensional Minkowski space:

\[
V_0(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - \frac{1}{(4\pi)^{D/2}D} \Gamma\left(1 - \frac{D}{2}\right) \sigma^D \text{tr} \mathbf{1}.
\] (22)

Here the following formula has been utilized:

\[
\frac{\Gamma\left(\frac{D}{2} + isr\right)\Gamma\left(\frac{D}{2} - isr\right)}{\Gamma\left(1 + isr\right)\Gamma\left(1 - isr\right)} = \frac{|\Gamma\left(\frac{D}{2} + isr\right)|^2}{|\Gamma\left(1 + isr\right)|^2} \sim (|s|r)^{D-2} \quad (r \to \infty).
\] (23)

We impose the renormalization condition

\[
\left. \frac{\partial^2 V_0(\sigma)}{\partial \sigma^2} \right|_{\sigma = \sigma_0} = \frac{\sigma_0^{D-2}}{\lambda_r},
\] (24)

with \( \sigma_0 \) the renormalization scale and \( \lambda_r \) the renormalized dimensionless coupling constant. From Eq.(24) we find

\[
\frac{1}{\lambda_0} = \frac{\sigma_0^{D-2}}{\lambda_r} + \frac{D - 1}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) \sigma_0^{D-2} \text{tr} \mathbf{1}.
\] (25)

Replacing \( \lambda_0 \) by \( \lambda_r \) in Eq.(22) we obtain the renormalized effective potential in the Minkowski space,

\[
\frac{V_0(\sigma)}{\sigma_0^D} = \frac{1}{2\lambda_r} \left(\frac{\sigma}{\sigma_0}\right)^2 + \frac{D - 1}{2(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) \left(\frac{\sigma}{\sigma_0}\right)^2 \text{tr} \mathbf{1}
\[
- \frac{1}{(4\pi)^{D/2}D} \Gamma\left(1 - \frac{D}{2}\right) \left(\frac{\sigma}{\sigma_0}\right)^D \text{tr} \mathbf{1}.
\] (26)

Through the gap equation \( \left. \frac{\partial V_0(\sigma)}{\partial \sigma} \right|_{\sigma = m_0} = 0 \) we obtain the dynamical fermion mass

\[
m_0 = \sigma_0 \left[ \frac{(4\pi)^{D/2}}{\Gamma\left(1 - \frac{D}{2}\right) \text{tr} \mathbf{1}} \left(\frac{1}{\lambda_r} - \frac{1}{\lambda_{cr}}\right) \right]^{1/(D-2)},
\] (27)

if \( \lambda_r > \lambda_{cr} \) where

\[
\frac{1}{\lambda_{cr}} \equiv \frac{1 - D}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) \text{tr} \mathbf{1}.
\] (28)
Substituting the renormalized coupling constant $\lambda_r$ into the Eq.(21) we find the renormalized expression of the effective potential $V(\sigma)$ in de Sitter space,

$$V(\sigma) = \left[ \frac{1}{2\lambda_r} + \frac{D - 1}{2(4\pi)^{D/2}} \Gamma \left( 1 - \frac{D}{2} \right) \right] \sigma_0^{D-2} \sigma^2$$

$$- \int_0^\sigma ds \frac{s^{2-D}}{(4\pi)^{D/2}} \frac{\Gamma \left( \frac{D}{2} + isr \right) \Gamma \left( \frac{D}{2} - isr \right)}{\Gamma (1+isr) \Gamma (1-isr)} \Gamma \left( 1 - \frac{D}{2} \right) \tr 1.$$  \hspace{1cm} (29)

Note that Eq.(29) reduces to

$$\frac{V_{D=2}(\sigma)}{\sigma_0^2} = \left[ \frac{1}{2\lambda_r} + \frac{1}{2\pi} \left( \ln \frac{1}{r_0} - 1 \right) \right] \left( \frac{\sigma}{\sigma_0} \right)^2$$

$$+ \frac{1}{\sigma_0^2} \int_0^\sigma ds \left[ \psi(1 + isr) + \psi(1 - isr) \right], \hspace{1cm} (30)$$

in two dimensions which is different from the expression obtained in Ref.11 and is consistent with the one in Ref.12. In Fig.1 the behavior of the effective potential given by Eq.(29) is presented in the case of $D = 2.5$ for several typical values of the curvature. It is observed in Fig.1 that, if $\lambda_r < \lambda_{cr}$, the theory is always in the symmetric phase as the curvature changes while, if $\lambda_r > \lambda_{cr}$, the symmetry restoration takes place as the curvature exceeds its critical value $1/r_{cr} = \sigma_0/2.74$. This observation remains true if the space-time dimension is arbitrarily changed.

To discuss the dynamical mass of the fermion we study the minimum of this effective potential more precisely. A necessary condition for the minimum is given by

$$\left. \frac{\partial V(\sigma)}{\partial \sigma} \right|_{\sigma=m} = 0,$$  \hspace{1cm} (31)

where the non-trivial solution $m$ of this equation corresponds to the dynamical fermion mass. Equation (31) reads

$$\frac{1}{\lambda_r} \sigma_0^{D-2} + \frac{D - 1}{(4\pi)^{D/2}} \Gamma \left( 1 - \frac{D}{2} \right) \sigma_0^{D-2} \tr 1$$

$$- \frac{r^{2-D}}{(4\pi)^{D/2}} \Gamma \left( \frac{D}{2} + imr \right) \Gamma \left( \frac{D}{2} - imr \right) \Gamma (1 + imr) \Gamma (1 - imr) \Gamma \left( 1 - \frac{D}{2} \right) \tr 1$$

$$= 0.$$  \hspace{1cm} (32)

8
Fig. 1 Behavior of the effective potential is shown at $D = 2.5$ for (a) $\lambda_r < \lambda_{cr}(\lambda_r = 0.9\lambda_{cr})$ and (b) $\lambda_r > \lambda_{cr}(\lambda_r = 1.25\lambda_{cr})$ where $\lambda_{cr} = 3.2/\text{tr}$. The critical curvature is given by $1/r_{cr} = \sigma_0/2.74$. 

(a) $\lambda_r = 0.9\lambda_{cr}$

(b) $\lambda_r = 1.25\lambda_{cr}$
In Fig. 2 we present the solution of the gap equation (32).

Taking the two-dimensional limit Eq. (32) reduces to

\[
\frac{1}{\lambda} r - \frac{1}{2\pi} \left[ \ln(r^2\sigma_0^2) - \psi(1 + imr) - \psi(1 - imr) \right] = 0. \quad (33)
\]

For three-dimensions it reads\textsuperscript{13}

\[
\frac{1}{\lambda} \sigma_0 - \frac{2}{\pi} \sigma_0 + \frac{1}{\pi r} \frac{\Gamma \left( \frac{3}{2} + imr \right)}{\Gamma (1 + imr)} \frac{\Gamma \left( \frac{3}{2} - imr \right)}{\Gamma (1 - imr)} = 0, \quad (34)
\]

while it reduces to

\[
\frac{1}{\lambda} r - \frac{3}{(2\pi)^2} \left( C_{uv} - \frac{2}{3} \right) - \frac{2r^{-2}}{(2\pi)^2\sigma_0^2} \\
+ \frac{r^{-2} + m^2}{(2\pi)^2\sigma_0^2} \left( C_{uv} + \ln(r^2\sigma_0^2) - \psi(1 + imr) - \psi(1 - imr) \right) = 0, \quad (35)
\]

in the limit of \( D \to 4 \), where

\[
C_{uv} = \frac{2}{4 - D} - \gamma + \ln 2\pi + 1. \quad (36)
\]

In the weak curvature limit Eq. (34) reproduces the result obtained in Ref. 13. For the weak curvature limit \( r \to \infty \) Eq. (33) tends to

\[
\frac{1}{\lambda} - \frac{1}{(2\pi)^2} \left[ 3 \left( C_{uv} - \frac{2}{3} \right) - \left( C_{uv} - \ln \frac{m^2}{\sigma_0^2} \right) \frac{m^2}{\sigma_0^2} - \left( C_{uv} - \frac{13}{6} - \ln \frac{m^2}{\sigma_0^2} \right) \frac{1}{r^2\sigma_0^2} \right] = 0. \quad (37)
\]

We find that there is a simple correspondence between this result (37) and the result given in Ref. 14 if we make a replacement

\[
C_{uv} + \frac{1}{6} \frac{1}{(m^2 - 3\sigma_0^2)r^2} \leftrightarrow \ln \frac{\Lambda^2}{\sigma_0^2}, \quad (38)
\]

where \( \Lambda \) is a cut-off of the divergent integral appearing in Ref. 14. Note here that the direct comparison of our result with the result in Ref. 14 is possible only after renormalizing the coupling constant \( \lambda \) in Ref. 14 under the renormalization condition (24).
As is seen in Fig.1 and Fig.2 the phase transition is of the second order. Accordingly by setting $m = 0$ in the gap equation (32) we may derive the equation which determines the critical radius $r_{cr}$,

$$
\frac{1}{\lambda_r} \sigma_0^{D-2} + \frac{D - 1}{(4\pi)^{D/2}} \Gamma \left(1 - \frac{D}{2}\right) \sigma_0^{D-2} \text{tr} \mathbf{1} - \frac{r_{cr}^{2-D}}{(4\pi)^{D/2}} \Gamma^2 \left(\frac{D}{2}\right) \Gamma \left(1 - \frac{D}{2}\right) \text{tr} \mathbf{1} = 0. \quad (39)
$$

Hence the critical radius is given by

$$
r_{cr} = \frac{1}{\sigma_0} \left[ \frac{(4\pi)^{D/2}}{\Gamma^2 \left(\frac{D}{2}\right) \Gamma \left(1 - \frac{D}{2}\right) \text{tr} \mathbf{1}} \frac{1}{\lambda_r} + \frac{D - 1}{\Gamma^2 \left(\frac{D}{2}\right)} \right]^{1/(2-D)}. \quad (40)
$$

For some special values of $D$ Eq.(40) simplifies to:

$$
\begin{align*}
    r_{cr} &= \frac{1}{\sigma_0} \exp \left( \frac{\pi}{\lambda_r} - 1 - \gamma \right) ; D = 2, \\
    r_{cr} &= \frac{1}{\sigma_0} \left( \frac{8}{\pi} - 4 \frac{1}{\lambda_r} \right)^{-1} \, ; D = 3, \\
    r_{cr} &= \sqrt{3} \frac{1}{3} \sigma_0 \, ; D = 4.
\end{align*}
$$

(41)

In Fig.3 we show the critical radius $r_{cr}$ as a function of the coupling constant $\lambda_r$. 

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Fig. 2 Behavior of the Dynamical fermion mass as a function of the radius $r$ at $D = 2.0, 2.5, 3.0, 3.5, 3.99$ where $m_0$ is the dynamical fermion mass in flat space-time.
Fig. 3 Critical radius $r_{cr}$ as a function of four-fermion coupling $\lambda_r$
We found that the phase structure of the four-fermion interaction theory in de Sitter space is analyzable in the sense of the $1/N$ expansion and discovered an existance of the critical curvature at which the symmetry is restored. Although our model is too primitive to be adopted to the symmetry restoration of the unified theories in early universe, we hope that our analysis will help building a more realistic composite Higgs model in early stage of the universe.

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