Quotients for sheets of conjugacy classes

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Abstract

We provide a description of the orbit space of a sheet $S$ for the conjugation action of a complex simple simply connected algebraic group $G$. This is obtained by means of a bijection between $S/G$ and the quotient of a shifted torus modulo the action of a subgroup of the Weyl group and it is the group analogue of a result due to Borho and Kraft. We also describe the normalisation of the categorical quotient $S//G$ for arbitrary simple $G$ and give a necessary and sufficient condition for $S//G$ to be normal in analogy to results of Borho, Kraft and Richardson. The example of $G_2$ is worked out in detail.

1 Introduction

Sheets for the action of a connected algebraic group $G$ on a variety $X$ have their origin in the work of Kostant [16], who studied the union of regular orbits for the adjoint action on a semisimple Lie algebra, and in the work of Dixmier [10]. Sheets are the irreducible components of the level sets of $X$ consisting of points whose orbits have the same dimension. In a sense they provide a natural way to collect orbits in families in order to study properties of one orbit by looking at others in its family. For the adjoint action of a complex semisimple algebraic group $G$ on its Lie algebra they were deeply and systematically studied in [2, 4]. They were described as sets, their closure was well-understood, they were classified in terms of pairs consisting of a Levi subalgebra and suitable nilpotent orbit therein, and they were used to answer affirmatively to a question posed by Dixmier on the multiplicities in the module decomposition of the ring of regular
functions of an adjoint orbit in $\mathfrak{sl}(n, \mathbb{C})$. If $G$ is classical then all sheets are smooth [14, 24]. The study of sheets in positive characteristic has appeared more recently in [26].

In analogy to this construction, sheets of primitive ideals were introduced and studied by W. Borho and A. Joseph in [3], in order to describe the set of primitive ideals in a universal enveloping algebra as a countable union of maximal varieties. More recently, Losev in [18] has introduced the notion of birational sheet in a semisimple Lie algebra, he has shown that birational sheets form a partition of the Lie algebra and has applied this result in order to establish a version of the orbit method for semisimple Lie algebras. Sheets were also used in [25] in order to parametrise the set of 1-dimensional representations of finite $W$-algebras, with some applications also to the theory of primitive ideals. Closures of sheets appear as associated varieties of affine vertex algebras, [1].

In characteristic zero, several results on quotients $S/G$ and $\overline{S}/G$, for a sheet $S$ were addressed: Katsylo has shown in [15] that $S/G$ has the structure of a quotient and is isomorphic to the quotient of an affine variety by the action of a finite group [15]; Borho has explicitly described the normalisation of $\overline{S}/G$ and Richardson, Broer, Douglass-Röhrle in [27, 6, 11] have provided the list of the quotients $\overline{S}/G$ that are normal.

Sheets for the conjugation action of $G$ on itself were studied in [8] in the spirit of [4]. If $G$ is semisimple, they are parametrized in terms of pairs consisting of a Levi subgroup of parabolic subgroups and a suitable isolated conjugacy class therein. Here isolated means that the connected centraliser of the semisimple part of a representative is semisimple. An alternative parametrisation can be given in terms of triples consisting of a pseudo-Levi subgroup $M$ of $G$, a coset in $Z(M)/Z(M)^\circ$ and a suitable unipotent class in $M$. Pseudo-Levi subgroups are, in good characteristic, centralisers of semisimple elements and up to conjugation they are subroot subgroups whose root system has a base in the extended Dynkin diagram of $G$ [22]. It is also shown in [7] that sheets in $G$ are the irreducible components of the parts in Lusztig’s partition introduced in [19], whose construction is given in terms of Springer’s correspondence.

Also in the group case one wants to reach a good understanding of quotients of sheets. An analogue of Katsylo’s theorem was obtained for sheets containing spherical conjugacy classes and all such sheets are shown to be smooth [9]. The proof in this case relies on specific properties of the intersection of spherical conjugacy classes with Bruhat double cosets, which do not hold for general classes. Therefore, a straightforward generalization to arbitrary sheets is not immediate. Even in absence of a Katsylo type theorem, it is of interest to understand the orbit
In this paper we address the question for $G$ simple provided $G$ is simply connected if the root system is of type $C$ or $D$. We give a bijection between the orbit space $S/G$ and a quotient of a shifted torus of the form $Z(M)^0s$ by the action of a subgroup $W(S)$ of the Weyl group, giving a group analogue of \cite[Theorem 3.6]{17},[2] Satz 5.6. In most cases the group $W(S)$ does not depend on the unipotent part of the triple corresponding to the given sheet although it may depend on the isogeny type of $G$. This is one of the difficulties when passing from the Lie algebra case to the group case. The restriction on $G$ needed for the bijection depends on the symmetry of the extended Dynkin diagram in this case: type $C$ and $D$ are the only two situations in which two distinct subsets of the extended Dynkin diagram can be equivalent even if they are not of type $A$. We illustrate by an example in $\text{HSpin}_{10}(\mathbb{C})$ that the restriction we put is necessary in order to have injectivity so our theorem is somehow optimal.

We also address some questions related to the categorial quotient $\overline{S}///G$, for a sheet in $G$. We obtain group analogues of the description of the normalisation of $\overline{S}///G$ from \cite{2} and of a necessary and sufficient condition on $\overline{S}///G$ to be normal from \cite{27}. Finally we apply our results to compute the quotients $S/G$ of all sheets in $G$ of type $G_2$ and verify which of the quotients $\overline{S}///G$ are normal. This example will serve as a toy example for a forthcoming paper in which we will list all normal quotients for $G$ simple.

## 2 Basic notions

In this paper $G$ is a complex simple algebraic group with maximal torus $T$, root system $\Phi$, weight lattice $\Lambda$, set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$, Weyl group $W = N(T)/T$ and corresponding Borel subgroup $B$. The numbering of simple roots is as in \cite{5}. Root subgroups are denoted by $X_\alpha$ for $\alpha \in \Phi$ and their elements have the form $x_\alpha(\xi)$ for $\xi \in \mathbb{C}$. Let $-\alpha_0$ be the highest root and let $\tilde{\Delta} = \Delta \cup \{\alpha_0\}$. The centraliser of an element $h$ in a closed group $H \leq G$ will be denoted by $H^h$ and the identity component of $H$ will be indicated by $H^o$. If $\Pi \subset \tilde{\Delta}$ we set

$$G_\Pi := \langle T, X_{\pm\alpha} \mid \alpha \in \Pi \rangle.$$

Conjugates of such groups are called pseudo-Levi subgroups. We recall from \cite[§6]{22} that if $s \in T$ then its connected centraliser $G^{s^o}$ is conjugated to $G_\Pi$ for some $\Pi$ by means of an element in $N(T)$. By \cite[2.2]{13} we have $G^s = \langle G^{s^o}, N(T)^s \rangle$. $W_\Pi$ indicates the subgroup of $W$ generated by the simple reflections with respect to roots in $\Pi$ and it is the Weyl group of $G_\Pi$. 

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We realize the groups \( \text{Sp}_{2\ell}(\mathbb{C}) \), \( \text{SO}_{2\ell}(\mathbb{C}) \) and \( \text{SO}_{2\ell+1}(\mathbb{C}) \), respectively, as the groups of matrices of determinant 1 preserving the bilinear forms: \( \left( \begin{array}{cc} 0 & I_{\ell} \\ -I_{\ell} & 0 \end{array} \right), \left( \begin{array}{cc} 0 & I_{\ell} \\ I_{\ell} & 0 \end{array} \right) \) and \( \left( \begin{array}{cc} 1 & I_{\ell} \\ I_{\ell} & 0 \end{array} \right) \), respectively.

If \( G \) acts on a variety \( X \), the action of \( g \in G \) on \( x \in X \) will be indicated by \( (g, x) \mapsto g \cdot x \). If \( X = G \) with adjoint action we thus have \( g \cdot h = ghg^{-1} \).

For \( n \geq 0 \) we shall denote by \( X_{(n)} \) the union of orbits of dimension \( n \). The nonempty sets \( X_{(n)} \) are locally closed and a sheet \( S \) for the action of \( G \) on \( X \) is an irreducible component of any of these. For any \( Y \subset X \) we set \( Y^{\text{reg}} \) to be the set of points of \( Y \) whose orbit has maximal dimension. We recall the parametrisation and description of sheets for the action of \( G \) on itself by conjugation and provide the necessary background material.

A Jordan class in \( G \) is an equivalence class with respect to the equivalence relation: \( g, h \in G \) with Jordan decomposition \( g = su, h = rv \) are equivalent if up to conjugation \( G^{s} = G^{r}, r \in Z(G^{s})^{0} s \) and \( G^{s} \cdot u = G^{s} \cdot v \). As a set, the Jordan class of \( g = su \) is thus \( J(su) = G \cdot ((Z(G^{s})^{0}s)^{\text{reg}}u) \) and it is contained in some \( G_{(n)} \). Jordan classes are parametrised by \( G \)-conjugacy classes of triples \((M, Z(M)^{0}s, M \cdot u)\) where \( M \) is a pseudo-Levi subgroup, \( Z(M)^{0}s \) is a coset in \( Z(M)/Z(M)^{0} \) such that \((Z(M)^{0}s)^{\text{reg}} \subset Z(M)^{\text{reg}}\) and \( M \cdot u \) is a unipotent conjugacy class in \( M \). They are finitely many, locally closed, \( G \)-stable, smooth, see [20, 3.1] and [8, §4] for further details.

Every sheet \( S \subset G \) contains a unique dense Jordan class, so sheets are parametrised by conjugacy classes of a subset of the triples above mentioned. More precisely, a Jordan class \( J = J(su) \) is dense in a sheet if and only if it is not contained in \((J')^{\text{reg}}\) for any Jordan class \( J' \) different from \( J \). We recall from [8, Proposition 4.8] that

\[
J(su)^{\text{reg}} = \bigcup_{z \in Z(G^{s})} G \cdot (s \text{Ind}_{G^{s}}^{G^{s0}} (G^{s0} \cdot u)),
\]

where \( \text{Ind}_{G^{s0}}^{G^{s}} (G^{s0} \cdot u) \) is Lusztig-Spaltenstein’s induction from the Levi subgroup \( G^{s0} \) of a parabolic subgroup of \( G^{s0} \) of the class of \( u \) in \( G^{s0} \), see [21]. So, Jordan classes that are dense in a sheet correspond to triples where \( u \) is a rigid orbit in \( G^{s} \), i.e., such that its class in \( G^{s} \) is not induced from a conjugacy class in a proper Levi subgroup of a parabolic subgroup of \( G^{s0} \).

A sheet consists of a single conjugacy class if and only if \( \overline{S} = J(su) = G \cdot su \) where \( u \) is rigid in \( G^{s} \) and \( G^{s} \) is semisimple, i.e., if and only if \( s \) is isolated and \( u \) is rigid in \( G^{s0} \). Any sheet \( S \) in \( G \) is the image through the isogeny map \( \pi \) of a sheet \( S' \) in the simply-connected cover \( G_{\text{sc}} \) of \( G \), where \( S' \) is defined up
to multiplication by an element in Ker(\(\pi\)). Also, \(Z(G^{(s)\circ}) = \pi(Z(G^{s\circ}))\) and \(Z(G^{(s)\circ})^\circ = \pi(Z(G^{s\circ})^\circ) = Z(G^{s\circ})^\circ\) Ker(\(\pi\)).

3 A parametrization of orbits in a sheet

In this section we parametrize the set \(S/G\) of conjugacy classes in a given sheet. Let \(S = J(su)\) with \(s \in T\) and \(u \in U\cap G^{s\circ}\). Let \(Z = Z(G^{s\circ})\) and \(L = C_G(Z^\circ)\). The latter is always a Levi subgroup of a parabolic subgroup of \(G\), [29, Proposition 8.4.5, Theorem 13.4.2] and if \(\Psi_s\) is the root system of \(G^{s\circ}\) with respect to \(T\), then \(L\) has root system \(\Psi := Q^{\Psi_s} \cap \Phi\).

Let \((3.2)\)

\[ W(S) = \{ w \in W \mid w(Z^\circ s) = Z^\circ s \}. \]

We recall that \(C_G(Z(G^{s\circ})^\circ) = G^{s\circ}\). Thus, for any lift \(\hat{w}\) of \(w \in W(S)\) we have \(\hat{w} \cdot G^{s\circ} = G^{s\circ}\), so \(\hat{w} \cdot Z^\circ = Z^\circ\) and therefore \(\hat{w} \cdot L = L\). Thus, any \(w \in W(S)\) determines an automorphism of \(\Psi_s\) and \(\Psi\). Let \(O = G^{s\circ} \cdot u\). We set:

\[ (3.3) \quad W(S)^u = \{ w \in W(S) \mid \hat{w} \cdot O = O \}. \]

The definition is independent of the choice of the representative of each \(w\) because \(T \subset L\).

**Lemma 3.1** Let \(\Psi_s\) be the root system of \(G^{s\circ}\) with respect to \(T\), with basis \(\Pi \subset \Delta \cup \{-\alpha_0\}\). Let \(W_{\Pi}\) be the Weyl group of \(G^{s\circ}\) and let \(W^{\Pi} = \{ w \in W \mid w \Pi = \Pi \}\). Then

\[ W(S) = W_\Pi \rtimes (W^{\Pi})_{Z^\circ s} = \{ w \in W_\Pi W^{\Pi} \mid w Z^\circ s = Z^\circ s \}. \]

In particular, if \(G^{s\circ}\) is a Levi subgroup of a parabolic subgroup of \(G\), then \(W(S) = W_\Pi \rtimes W^{\Pi} = N_W(W_\Pi)\) and it is independent of the isogeny class of \(G\).

**Proof.** Let \(W_X\) denote the stabilizer of \(X\) in \(W\) for \(X = Z^\circ s, G^{s\circ}, Z, Z^\circ\). We have the following chain of inclusions:

\[ W(S) = W_{Z^\circ s} \leq W_{G^{s\circ}} \leq W_Z \leq W_{Z^\circ}. \]

We claim that \(W_{G^{s\circ}} = W_\Pi \rtimes W^{\Pi}\). Indeed, \(W_\Pi W^{\Pi} \leq W_{G^{s\circ}}\) is immediate and if \(w \in W_{G^{s\circ}}\) then \(w \Psi_s = \Psi_s\) and \(w \Pi\) is a basis for \(\Psi_s\). Hence, there is some \(\sigma \in W_\Pi\) such that \(\sigma w \in W^{\Pi}\). By construction \(W^{\Pi}\) normalises \(W_\Pi\). The elements
of $W_{G^s}$ permute the connected components of $Z = Z(G^s)$ and $W_{Z^s}$ is precisely the stabilizer of $Z^s$ in there. Since the elements of $W_{\Pi}$ fix the elements of $Z(G^s)$ pointwise, they stabilize $Z^s$, whence the statement. The last statement follows from the equality $W_{\Pi} \times W_{\Pi} = N_{W}(W_{\Pi})$ in [12] Corollary3 and [22] Lemma 33] because in this case $Z^s = zZ^s$ for some $z \in Z(G)$, so $W_{Z^s} = W_{Z^s}$.

Remark 3.2 If $G^s$ is not a Levi subgroup of a parabolic subgroup of $G$, then $W(S)$ might depend on the isogeny type of $G$. For instance, if $\Phi$ is of type $C_5$ and $s = \operatorname{diag}(-I_2, x, I_2, -I_2, x^{-1}, I_2) \in \operatorname{SP}_{10}(\mathbb{C})$ for $x^2 \neq 1$, then:

$$\Pi = \{\alpha_0, \alpha_1, \alpha_4, \alpha_5\}$$

$$Z = Z(G^s) = \{\operatorname{diag}(\epsilon I_2, y, \eta I_2, \epsilon I_2, y^{-1}, \eta I_2), y \in \mathbb{C}^*, \epsilon^2 = \eta^2 = 1\},$$

$$Z^s = \{\operatorname{diag}(-I_2, I_2, y, -I_2, I_2, y^{-1}), y \in \mathbb{C}^*\},$$

and $W_{\Pi} = \langle s_{a_1+a_2+a_3+a_4}s_{a_2+a_3}, s_{a_3+a_4}\rangle$. Since $s_{a_1+a_2+a_3+a_4}s_{a_2+a_3}(Z^s) = -Z^s$ we have $W(S) = W_{\Pi}$. However, if $\pi : \operatorname{Sp}_{10}(\mathbb{C}) \to \operatorname{PSp}_{10}(\mathbb{C})$ is the isogeny map, then $W_{\Pi}$ preserves $\pi(Z^s)$ so $W(\pi(S)) = W_{\Pi} \times W_{\Pi}$. Taking $u = 1$ have an example in which also $W(S)^u$ depends on the isogeny type.

Table 1: Kernel of the isogeny map; $\Phi$ of type $B_\ell$, $C_\ell$ or $D_\ell$

| type | parity of $\ell$ | group | \(\ker \pi\) |
|------|-----------------|-------|----------------|
| $B_\ell$ | any | $SO_{2\ell+1}(\mathbb{C})$ | $\langle \alpha_0^\ell(-1)\rangle$ |
| $C_\ell$ | any | $\operatorname{PSp}_{2\ell}(\mathbb{C})$ | $\bigg\langle \prod_{j \text{ odd}} \alpha_j^\ell(-1)\bigg\rangle = \langle -I_{2\ell}\rangle$ |
| $D_\ell$ | even | $\operatorname{PSO}_{2\ell}(\mathbb{C})$ | $\bigg\langle \prod_{j \text{ odd}} \alpha_j^\ell(-1), \alpha_{\ell-1}^\ell(-1)\alpha_\ell^\ell(-1)\bigg\rangle$ |
| $D_\ell$ | odd | $\operatorname{PSO}_{2\ell}(\mathbb{C})$ | $\bigg\langle \prod_{j \text{ odd} \leq \ell-2} \alpha_j^\ell(-1)\alpha_{\ell-1}^\ell(i)\alpha_\ell^\ell(i^3)\bigg\rangle$ |
| $D_\ell$ | any | $SO_{2\ell}(\mathbb{C})$ | $\langle \alpha_{\ell-1}^\ell(-1)\alpha_\ell^\ell(-1)\rangle$ |
| $D_\ell$ | even | $\operatorname{HSpin}_{2\ell}(\mathbb{C})$ | $\bigg\langle \prod_{j \text{ odd}} \alpha_j^\ell(-1)\bigg\rangle$ |

Next Lemma shows that in most cases $W(S)^u$ can be determined without any knowledge of $u.$
Lemma 3.3  Suppose \( G \) and \( S = J(su)^{reg} \) are not in the following situation:

"\( G \) is either \( \mathrm{PSp}_{2\ell}(\mathbb{C}) \), \( \mathrm{HSpin}_{2\ell}(\mathbb{C}) \), or \( \mathrm{PSO}_{2\ell}(\mathbb{C}) \);
\[ [G^{s0}, G^{s0}] \) has two isomorphic simple factors \( G_1 \) and \( G_2 \) that are not of type \( A \);
the components of \( u \) in \( G_1 \) and \( G_2 \) do not correspond to the same partition."

Then, \( W(S) = W(S)^u \).

Proof. The element \( u \) is rigid in \( [G^{s0}, G^{s0}] \leq G^{s0} \) and this happens if and only if each of its components in the corresponding simple factor of \( [G^{s0}, G^{s0}] \) is rigid. Rigid unipotent elements in type \( A \) are trivial \([28, \text{Proposition 5.14}]\), therefore what matters are only the components of \( u \) in the simple factors of type different from \( A \). In addition, rigid unipotent classes are characteristic in simple groups, \([2, \text{Lemma 3.9, Korollar 3.10}]\). For all \( \Phi \) different from \( C \) and \( D \), simple factors that are not of type \( A \) are never isomorphic. Therefore the statement certainly holds in all cases with a possible exception when: \( \Phi \) is of type \( C_\ell \) or \( D_\ell \); \( [G^{s0}, G^{s0}] \) has two isomorphic factors of type different from \( A \); and the components of \( u \) in those two factors, that are of type \( C_m \) or \( D_m \), respectively, correspond to different partitions.

Let us assume that we are in this situation. Then, \( W(S) = W(S)^u \) if and only if the elements of \( W(S) \), acting as automorphisms of \( \Psi_s \), do not interchange the two isomorphic factors in question. We have 2 isogeny classes in type \( C_\ell \), 3 in type \( D_\ell \) for \( \ell \) odd, and 4 (up to isomorphism) in type \( D_\ell \) for \( \ell \) even.

If \( \Phi \) is of type \( C_\ell \) and \( G = \mathrm{Sp}_{2\ell}(\mathbb{C}) \) up to a central factor \( s \) can be chosen to be of the form:

\[
(3.4) \quad s = \text{diag}(I_m, t, -I_m, I_m, t^{-1}, -I_m)
\]

where \( t \) is a diagonal matrix in \( \text{GL}_{\ell-2m}(\mathbb{C}) \) with eigenvalues different from \( \pm 1 \). Then \( \Pi \) is the union of \( \{\alpha_0, \ldots, \alpha_{m-1}\}, \{\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_{\ell-m+1}\} \) and possibly other subsets of simple roots orthogonal to these. Then \( W^{\Pi} \) is the direct product of terms permuting isomorphic components of type \( A \) with the subgroup generated by \( \sigma = \prod_{j=1}^{m} s_{\alpha_j + \ldots + \alpha_{\ell-j}} \). In this case the elements of \( Z^s s \) are of the form \( \text{diag}(I_m, r, -I_m, I_m, r^{-1}, -I_m) \), where \( r \) has the same shape as \( t \) and \( \sigma(Z^s s) = -Z^s s \). Thus, \( W^{\Pi} \) does not permute the two factors of type \( C_m \) and \( W(S) = W(S)^u \).

If, instead, \( G = \text{PSp}_{2\ell}(\mathbb{C}) \) and the sheet is \( \pi(S) \), we may take \( J = J(\pi(su)) \) where \( s \) as is in \( (3.4) \). Then, \( \sigma \) preserves \( \pi(Z^s s) \) and therefore \( W(\pi(S)) \neq W(\pi(S))^{\pi(u)} \).
Let now \( \Phi \) be of type \( D_\ell \) and \( G = Spin_{2\ell}(\mathbb{C}) \). With notation as in [29], we may take

\[
(3.5) \quad s = \left( \prod_{j=1}^{m} \alpha_j^\vee (e^j) \right) \left( \prod_{i=m+1}^{l-m-1} \alpha_i^\vee (c_i) \right) \left( \prod_{b=2}^{m} \alpha_{\ell-b}^\vee (d^b \eta^b) \right) \alpha_{\ell-1}^\vee (\eta d) \alpha_\ell (d)
\]

with \( \epsilon^2 = \eta^2 = 1 \), \( \epsilon \neq \eta \), and \( d, c_i \in \mathbb{C}^* \).

Here \( \Pi \) is the union of \( \{ \alpha_0, \ldots, \alpha_{m-1} \} \), \( \{ \alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_{\ell-m+1} \} \) and possibly other subsets of simple roots orthogonal to these. Then \( W^\Pi \) is the direct product of terms permuting isomorphic components of type \( A \) and \( \langle \sigma \rangle \) where \( \sigma = \prod_{j=1}^{m} s_{\alpha_j} \ldots s_{\alpha_{\ell-1}} s_{\alpha_\ell} \).

The coset \( Z^\circ s = Z_{\epsilon, \eta} \) consists of elements of the same form as \( (3.5) \) with constant value of \( \epsilon \) and \( \eta \), and \( Z^\circ = Z_{1,1} \) consists of the elements of similar shape with \( \eta = \epsilon = 1 \). Then \( \sigma(Z_{\epsilon, \eta}) = Z_{\eta, \epsilon} \), hence \( \sigma \not\in W(S) \), so \( W(S) \) preserves the components of \( \Psi_s \) of type \( D \) and \( W(S) = W(S)^u \).

If \( \ell = 2q \) and \( G = HSpin_{2\ell}(\mathbb{C}) \) and \( \pi : Spin_{2\ell}(\mathbb{C}) \to HSpin_{2\ell}(\mathbb{C}) \) is the isogeny map we see from Table I that \( Ker(\pi) \) is generated by an element \( k \) such that \( kZ_{\epsilon, \eta} = Z_{-\epsilon, \eta} \), so \( \sigma \) as above preserves \( \pi(Z^\circ s) \) whereas it does not preserve the conjugacy class of \( \pi(u) \). Therefore \( \sigma \in W(\pi(S)) \neq W(\pi(S))^u. \)

If \( G = SO_{2\ell}(\mathbb{C}) \) and \( \pi : Spin_{2\ell}(\mathbb{C}) \to SO_{2\ell}(\mathbb{C}) \) is the isogeny map, then \( Ker(\pi) \) is generated by an element \( k \) such that \( kZ_{\epsilon, \eta} = Z_{\epsilon, \eta} \). In this case \( \sigma \) does not preserve \( \pi(Z^\circ s) \), whence \( \sigma \not\in W(\pi(S)) = W(\pi(S))^u. \)

If \( G = PSO_{2\ell}(\mathbb{C}) \) and \( \pi : Spin_{2\ell}(\mathbb{C}) \to PSO_{2\ell}(\mathbb{C}) \), then by the discussion of the previous isogeny types we see that \( \sigma(Z_{\epsilon, \eta}) \subset Ker(\pi)Z_{\epsilon, \eta} \), so \( \sigma \) preserves \( \pi(Z^\circ s) \) whence \( \sigma \in W(\pi(S)) \neq W(\pi(S))^u. \)

Following [2] §5 and according to [8] Proposition 4.7 we define the map

\[
\theta : Z^\circ s \to S/G,
zs \mapsto Ind_L^G(L \cdot szu)
\]

where \( L = C_G(Z(G^{\circ o})^o) \).

**Lemma 3.4** With the above notation, \( \theta(zs) = \theta(w \cdot (zs)) \) for every \( w \in W(S)^u. \)

**Proof.** Let us observe that, since \( z \in Z(L) \) and \( G^{\circ o} \subset L \) there holds \( L^{z_{G^{\circ o}}} = G^{z_{G^{\circ o}}}. \) In particular, \( G^{\circ o} \) is a Levi subgroup of a parabolic subgroup of \( G^{z_{G^{\circ o}}} \). Let \( U_P \) be the unipotent radical of a parabolic subgroup of \( G \) with Levi factor \( L \) and let \( \tilde{w} \) be
a representative of $w$ in $N_G(T)$. By [8, Proposition 4.6] we have

$$\text{Ind}_L^G(L \cdot (w \cdot zs)u) = G \cdot (w \cdot (zs)u U)_{\text{reg}}$$

$$= G \cdot (zs(\dot{w}^{-1} \cdot u) U_{\dot{w}^{-1}, p})_{\text{reg}}$$

$$= \text{Ind}_L^G(L \cdot (zs(\dot{w}^{-1} \cdot u)))$$

$$= G \cdot (zs \text{ Ind}_{G^{s_o}}^G(\dot{w}^{-1} \cdot (G^{s_o} \cdot u)))$$

$$= G \cdot (zs \text{ Ind}_{G^{s_o}}^G(G^{s_o} \cdot u))$$

$$= \text{Ind}_L^G(L \cdot (zsu))$$

where we have used that $L = \dot{w} \cdot L$ for every $w \in W(S)^u \leq W(S)$ and independence of the choice of the parabolic subgroup with Levi factor $L$, [8, Proposition 4.5].

\[\square\]

**Remark 3.5** The requirement that $w$ lies in $W(S)^u$ rather than in $W(S)$ is necessary. For instance, we consider $G = \text{PSp}_{2l}(\mathbb{C})$ with $l = 2m + 1$ and $s$ the class of $\text{diag}(I_m, \lambda, -I_m, I_m, \lambda^{-1}, -I_m)$ with $\lambda^4 \neq 1$ and $u$ rigid with non-trivial component only in the subgroup $H = \langle X_{\pm \alpha_j}, j = 0, \ldots m - 1 \rangle$ of $G^{s_o}$. The element $\sigma = \prod_{j=1}^m s_{\alpha_j + \cdots + \alpha_{l-j}}$ lies in $W(S) \setminus W(S)^u$. Taking $\theta(s)$ we have

$$\text{Ind}_L^G(L \cdot su) = G \cdot su$$

whereas

$$\text{Ind}_L^G(L \cdot w(s)u) = \text{Ind}_L^G(L \cdot s(\dot{w} \cdot u)) = G \cdot (s(\dot{w} \cdot u)),$$

where $\dot{w}$ is any representative of $w$ in $N_G(T)$. These classes would coincide only if $u$ and $\dot{w} \cdot u$ were conjugate in $G^s$. They are not conjugate in $G^{s_o}$ because they lie in different simple components. Moreover, $G^s$ is generated by $G^{s_o}$ and the lifts of elements in the centraliser $W^s$ of $s$ in $W$ [13, 2.2], which is contained in $W(S)$. Since $\lambda^4 \neq 1$ we see that the elements of $W^s$ cannot interchange the two components of type $C_m$ in $G^{s_o}$. Hence,

$$\theta(s) = \text{Ind}_L^G(L \cdot su) \neq \text{Ind}_L^G(L \cdot w(s)u) = \theta(w(s)).$$

In analogy with the Lie algebra case we formulate the following theorem. The proof follows the lines of [2, Satz 5.6] but a more detailed analysis is necessary because the naive generalization of statement [2, Lemma 5.4] from Levi subalgebras in a Levi subalgebra to Levi subgroups in a pseudo-Levi subgroup does not hold.
Theorem 3.6  Assume \( G \) is simple and different from \( \text{PSO}_{2\ell}(\mathbb{C}) \), \( \text{HSpin}_{2\ell}(\mathbb{C}) \) and \( \text{PSp}_{2\ell}(\mathbb{C}) \), \( \ell \geq 5 \). Let \( S = \overline{J(su)} \), with \( s \in T \), \( Z = Z(G^{s_0}) \) and let \( W(S) \) be as in (3.2). The map \( \theta \) induces a bijection \( \overline{\theta} \) between \( Z^0 s/W(S) \) and \( S/G \).

Proof.  Recall that under our assumptions Lemma \( \text{3.3} \) gives \( W(S) = W(S)^u \). By Lemma \( \text{3.4} \) \( \theta \) induces a well-defined map \( \overline{\theta} : Z^0 s/W(S) \to S/G \). It is surjective by \( \text{[8, Proposition 4.7]} \). We prove injectivity.

Let us assume that \( \theta(zs) = \theta(z's) \) for some \( z,z' \in Z^0 \). By construction, \( Z^0 \subset T \). By \( \text{[8, Proposition 4.5]} \) we have

\[
G \cdot (zs \left( \text{Ind}_{G^{s_0}}^{G^{s_0}} (G^{s_0} \cdot u) \right)) = G \cdot (z's \left( \text{Ind}_{G^{s_0}}^{G^{s_0}} (G^{s_0} \cdot u) \right)).
\]

This implies that \( z's = \sigma \cdot (zs) \) for some \( \sigma \in W \). Let \( \dot{\sigma} \in N(T) \) be a representative of \( \sigma \). Then

\[
\theta(zs) = \theta(z's) = G \cdot (\sigma \cdot zs \left( \text{Ind}_{G^{s_0}}^{G^{s_0}} (G^{s_0} \cdot u) \right)) = G \cdot (zs \left( \text{Ind}_{G^{s_0}}^{G^{s_0}} (\dot{\sigma}^{-1} \cdot (G^{s_0} \cdot u)) \right)) = G \cdot (zs \left( \text{Ind}_{G^{s_0}}^{G^{s_0}} (\dot{\sigma}^{-1} \cdot (G^{s_0} \cdot u)) \right)).
\]

Since the unipotent parts of \( \theta(zs) \) and \( \theta(z's) \) coincide, for some \( x \in G^{s_0} \) we have

\[
x \cdot \left( \text{Ind}_{G^{s_0}}^{G^{s_0}} (G^{s_0} \cdot u) \right) = \text{Ind}_{\dot{\sigma}^{-1} \cdot (G^{s_0})}^{G^{s_0}} (\dot{\sigma}^{-1} \cdot (G^{s_0} \cdot u)).
\]

The element \( x \) may be written as \( \dot{w}g \) for some \( \dot{w} \in N(T) \cap G^{s_0} \) and some \( g \in G^{s_0} \) \( \text{[13, \S 2.2]} \). Hence,

\[
\text{Ind}_{G^{s_0}}^{G^{s_0}} (G^{s_0} \cdot u) = \dot{w}^{-1} \cdot \left( \text{Ind}_{\dot{\sigma}^{-1} \cdot (G^{s_0})}^{G^{s_0}} (\dot{\sigma}^{-1} \cdot (G^{s_0} \cdot u)) \right) = \text{Ind}_{\dot{w}^{-1} \dot{\sigma}^{-1} \cdot (G^{s_0})}^{G^{s_0}} ((\dot{w}^{-1} \dot{\sigma}^{-1}) \cdot (G^{s_0} \cdot u)).
\]

Let us put

\[
M := G^{s_0} = \langle T, X_\alpha, \alpha \in \Phi_M \rangle, \quad L_1 := G^{s_0} = \langle T, X_\alpha, \alpha \in \Psi \rangle
\]

with \( \Phi_M = \bigcup_{j=1}^J \Phi_j \) and \( \Psi = \bigcup_{i=1}^m \Psi_i \) the decompositions in irreducible root subsystems. We recall that \( L_1 \) and \( L_2 := (\dot{w}^{-1} \dot{\sigma}^{-1}) \cdot L_1 \) are Levi subgroups of some parabolic subgroups of \( M \). We claim that if \( L_1 \) and \( L_2 \) are conjugate in \( M \), then \( zs \) and \( z's \) are \( W(S) \)-conjugate. Indeed, under this assumption, since \( L_1 \) and
$L_2$ contain $T$, there is $\hat{\tau} \in N_M(T)$ such that $L_1 = \hat{\tau} \cdot L_2 = \hat{\tau}w^{-1}\sigma^{-1} \cdot L_1$, so
\[\tau w^{-1}\sigma^{-1}(Z^o) = Z^o.\] Then, $\tau w^{-1}\sigma^{-1}(Z^o s) = zs$ and therefore
\[\tau w^{-1}\sigma^{-1}(Z^o s) = \tau w^{-1}\sigma^{-1}(Z^o z' s) = Z^o zs = Z^o s.\]

Hence $zs$ and $z's$ are $W(S)$-conjugate. By Lemma 3.3 we have the claim. We show that if $\Phi_M$ has at most one component different from type $A$, then $L_1$ is always conjugate to $L_2$ in $M$. We analyse two possibilities.

$\Phi_j$ is of type $A$ for every $j$. In this case the same holds for $\Psi_1$ and $u = 1$. We recall that in type $A$ induction from the trivial orbit in a Levi subgroup corresponding to a partition $\lambda$ yields the unipotent class corresponding to the dual partition [28, 7.1]. Hence, equivalence of the induced orbits in each simple factor $M_i$ of $M$ forces $\Phi_j \cap \Psi \cong \Phi_j \cap w^{-1}\sigma^{-1}\Psi$ for every $j$. Invoking [2, Lemma 5.5], in each component $M_i$ we deduce that $L_1$ and $L_2$ are $M$-conjugate.

There is exactly one component in $\Phi_M$ which is not of type $A$. We set it to be $\Phi_1$. Then, there is at most one $\Psi_j$, say $\Psi_1$, which is not of type $A$, and $\Psi_1 \subset \Phi_1$. In this case, $w^{-1}\sigma^{-1}\Phi_1 \subset \Psi_1$. Equivalence of the induced orbits in each simple factor $M_j$ of $M$ forces $\Phi_j \cap \Psi \cong \Phi_j \cap w^{-1}\sigma^{-1}\Psi$ for every $j > 1$. By exclusion, the same isomorphism holds for $j = 1$. Invoking once more [2, Lemma 5.5] for each simple component, we deduce that $L_1$ and $L_2$ are $M$-conjugate.

Assume now that there are exactly two components of $\Phi_M$ which are not of type $A$. This situation can only occur if $\Phi$ is of type $B_\ell$ for $\ell \geq 6$, $C_\ell$ for $\ell \geq 4$ or $D_\ell$ for $\ell \geq 8$ (we recall that $D_2 = A_1 \times A_1$ and $D_3 = A_3$). By a case-by-case analysis we directly show that $\sigma$ can be taken in $W(S)$.

If $G = \text{Sp}_{2\ell}(\mathbb{C})$ we may assume that
\[s = \text{diag}(I_m, t, -I_p, I_m, t^{-1}, -I_p)\]
with $p, m \geq 2$ and $t$ a diagonal matrix with eigenvalues different from 0 and $\pm 1$. Then $Z^o s$ consists of matrices in this form, so $zs$ and $z's$ are of the form $zs = \text{diag}(I_m, h, -I_p, I_m, h^{-1}, -I_p)$ and $z's = \text{diag}(I_m, g, -I_p, I_m, g^{-1}, -I_p)$, where $h$ and $g$ are invertible diagonal matrices. The elements $zs$ and $z's$ are conjugate in $G$ if and only if $\text{diag}(h, h^{-1})$ and $\text{diag}(g, g^{-1})$ are conjugate in $G' = \text{Sp}_{2(\ell - p - m)}(\mathbb{C})$. This is the case if and only if they are conjugate in the normaliser
of the torus $T' = G' \cap T$. The natural embedding $G' \rightarrow G$ given by

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \mapsto
\begin{pmatrix}
I_m & B \\
C & D
\end{pmatrix}
$$

gives an embedding of $N_{G'}(T') \leq N_G(T)$ whose image lies in $W(S)$. Hence, $zs$ and $z's$ are necessarily $W(S)$-conjugate. This concludes the proof of injectivity for $G = \text{Sp}_{2\ell}(\mathbb{C})$.

If $G = \text{Spin}_{2\ell+1}(\mathbb{C})$, then we may assume that

$$
s = \left( \prod_{j=1}^m \alpha_j^\vee((1)^2) \right) \left( \prod_{b=m+1}^{\ell-p-1} \alpha_b^\vee(c_b) \right) \left( \prod_{q=1}^p \alpha_{\ell-q}^\vee(c^2) \right) \alpha_1^\vee(c)
$$

where $m \geq 4$, $p \geq 2$, $c, c_b \in \mathbb{C}^*$ are generic. Here $Z^o s$ consists of elements of the form

$$
\left( \prod_{j=1}^m \alpha_j^\vee((1)^2) \right) \left( \prod_{b=m+1}^{\ell-p-1} \alpha_b^\vee(d_b) \right) \left( \prod_{q=1}^p \alpha_{\ell-q}^\vee(c^2) \right) \alpha_1^\vee(d)
$$

with $d_b, d \in \mathbb{C}^*$. The reflection $s_{a_1+\ldots+a_\ell} = s_{\varepsilon_1}$ maps any $y \in Z^o s$ to $y\alpha_1^\vee((1)^2) \in Z(G)Z^o s = Z^o s$.

Let us consider the natural isogeny $\pi : G \rightarrow G_{ad} = \text{SO}_{2\ell+1}(\mathbb{C})$. Then

$$
\pi(s) = \text{diag}(1, -I_m, t, I_p, -I_m, t^{-1}, I_p)
$$

where $t$ is a diagonal matrix with eigenvalues different from $0$ and $\pm 1$. A similar calculation as in the case of $\text{Sp}_{2\ell}(\mathbb{C})$ shows that $\pi(zs)$ is conjugate to $\pi(z's)$ by an element $\sigma_1 \in W(\pi(S)) = W(\pi(S))^u$. Then, $\sigma_1(zs) = k z's$, where $k \in Z(G)$. If $k = 1$, then we set $\sigma = \sigma_1$ whereas if $k = \alpha_1^\vee((1)^2)$ we set $\sigma = s_{a_1+\ldots+a_\ell}$. Then $\sigma(zs) = z's$ and $\sigma(Z^o s) = Z(G)Z^o s = Z^o s$. This concludes the proof for $\text{Spin}_{2\ell+1}(\mathbb{C})$ and $\text{SO}_{2\ell+1}(\mathbb{C})$.

If $G = \text{Spin}_{2\ell}(\mathbb{C})$, up to multiplication by a central element we may assume that

$$
s = \left( \prod_{j=m+1}^{\ell-p-1} \alpha_j^\vee(c_j) \right) \left( \prod_{q=2}^p \alpha_{\ell-q}^\vee((-1)^q c^2) \right) \alpha_{\ell-1}^\vee(-c)\alpha_1^\vee(c)
$$

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where \( m, p \geq 4, c, c_j \in \mathbb{C}^* \) are generic. The elements in \( Z^0s \) are of the form

\[
\left( \prod_{j=m+1}^{\ell-p-1} \alpha_j^\vee (d_j) \right) \left( \prod_{q=2}^{p} \alpha_{\ell-q}^\vee (\alpha_{\ell}^\vee -1) \right) \alpha_{\ell+1}^\vee (-d) \alpha_{\ell}^\vee (d)
\]

with \( d_j, d \in \mathbb{C}^* \). We argue as we did for type \( B_\ell \), considering the isogeny \( \pi : G \to \text{SO}_{2\ell}(\mathbb{C}) \). The Weyl group element \( s_{\alpha_1} s_{\alpha_{\ell-1}} \) maps any \( y \in Z^0s \) to \( y \alpha_{\ell-1}^\vee (-1) \alpha_{\ell}^\vee (-1) \in \text{Ker}(\pi)Z^0s = Z^0s \). The group \( \pi(Z^0s) \) consists of elements of the form

\[
\text{diag}(I_m, t, -I_p, I_m, t^{-1}, -I_p)
\]

where \( t \) is a diagonal matrix in \( GL_{2(\ell-m-p)}(\mathbb{C}) \). Two elements

\[
\begin{align*}
\pi(zs) &= \text{diag}(I_m, h, -I_p, I_m, h^{-1}, -I_p), \\
\pi(z's) &= \text{diag}(I_m, g, -I_p, I_m, g^{-1}, -I_p)
\end{align*}
\]

therein are \( W \)-conjugate if and only if \( \text{diag}(1, h, 1, h^{-1}) \) and \( (1, g, 1, g^{-1}) \) are con-
jugate by an element \( \sigma_1 \) of the Weyl group \( W' \) of \( G' = \text{SO}_{2(\ell-m-p+1)}(\mathbb{C}) \). More precisely, even if \( h \) and \( g \) may have eigenvalues equal to 1, we may choose \( \sigma_1 \) in the subgroup of \( W' \) that either fixes the first and the \( (\ell - m - p + 2) \)-th eigenval-
ues or interchanges them. Considering the natural embedding of \( G' \) into \( \text{SO}_{2\ell}(\mathbb{C}) \) in a similar fashion as we did for \( \text{SO}_{2\ell}(\mathbb{C}) \), we show that \( \sigma_1 \in W(\pi(S)) \). This proves injectivity for \( \text{SO}_{2\ell}(\mathbb{C}) \). Arguing as we did for \( \text{Spin}_{2\ell+1}(\mathbb{C}) \) using \( s_{\alpha_1} s_{\alpha_{\ell-1}} \) concludes the proof of injectivity for \( \text{Spin}_{2\ell}(\mathbb{C}) \). \( \square \)

The translation isomorphism \( Z^0s \to Z^0 \) determines a \( W(S) \)-equivariant map

where \( Z^0 \) is endowed with the action \( w \bullet z = (w \cdot zs)^{-1} \), which is in general not
an action by automorphisms on \( Z^0 \). Hence, \( S/G \) is in bijection with the quotient
\( Z^0/W(S) \) of the torus \( Z^0 \) where the quotient is with respect to the \( \bullet \) action.

**Remark 3.7** Injectivity of \( \overline{\theta} \) does not necessarily hold for the adjoint groups \( G = \text{PSp}_{2\ell}(\mathbb{C}), \text{PSO}_{2\ell}(\mathbb{C}) \) and for \( G = \text{HSpin}_{2\ell}(\mathbb{C}) \). We give an example for \( G = \text{HSpin}_{20}(\mathbb{C}) \), in which \( W(S) = W(S)^u \) and \( G^s_{\varnothing} \) is a Levi subgroup of a parabolic
subgroup of \( G \). Let \( \pi : \text{Spin}_{20}(\mathbb{C}) \to G \) be the central isogeny with kernel \( K \) as in Table[II Let \( u = 1 \) and

\[
s = \alpha_1^\vee (a) \alpha_2^\vee (a^2) \alpha_3^\vee (a^3) \alpha_4^\vee (b) \alpha_5^\vee (c) \alpha_6^\vee (d^{-2}e^2) \alpha_7^\vee (e) \alpha_8^\vee (d^2) \alpha_9^\vee (d) \alpha_{10}^\vee (-d)K
\]

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with \( a, b, c, d, e \in \mathbb{C}^* \) sufficiently generic. Then, \( G^s \) is generated by \( T \) and the root subgroups of the subsystem with basis indexed by the following subset of the extended Dynkin diagram:

\[
\begin{array}{cccccccc}
& \bullet & \bullet & \circ & \circ & \circ & \circ & \bullet & \bullet \\
\end{array}
\]

Here \( Z^o \) is given by elements of shape:

\[
\alpha_1^\vee (a_1) \alpha_2^\vee (a_2^2) \alpha_3^\vee (a_3^3) \alpha_5^\vee (c_1) \alpha_6^\vee (d_1^{-2} e_1^2) \alpha_7^\vee (e_1) \alpha_8^\vee (d_1^2) \alpha_9^\vee (d_1) \alpha_{10}^\vee (-d_1) \ K
\]

with \( a_1, b_1, c_1, d_1, e_1 \in \mathbb{C}^* \). Let

\[
z s = \alpha_5^\vee (c) \alpha_6^\vee (d_1^2) \alpha_7^\vee (-d_1^2) \alpha_8^\vee (d_1^2) \alpha_9^\vee (d_1) \alpha_{10}^\vee (-d_1) K
\]

obtained by setting \( a_1 = b_1 = 1, c_1 = c, d_1 = d \) and \( e_1 = -d_1^2 \), and

\[
z' s = \alpha_5^\vee (-c) \alpha_6^\vee (d_1^2) \alpha_7^\vee (-d_1^2) \alpha_8^\vee (d_1^2) \alpha_9^\vee (d_1) \alpha_{10}^\vee (-d_1) K
\]

obtained by setting \( a_1 = b_1 = 1, c_1 = -c, d_1 = d \) and \( e_1 = -d_1^2 \). The subgroup \( M := G^{z s} = G^{z' s} \) is generated by \( T \) and the root subgroups of the subsystem with basis indexed by the following subset of the extended Dynkin diagram:

\[
\begin{array}{cccccccc}
& \bullet & \bullet & \circ & \circ & \circ & \circ & \bullet & \bullet \\
\end{array}
\]

For \( \sigma = \prod_{j=1}^4 s_{a_j + \cdots + a_{10-j}} \) we have \( \sigma \cdot z s = z' s \). We claim that \( z s \) and \( z' s \) are not \( W(S) \)-conjugate. Equivalently, we show that \( \sigma W^{z s K} \cap W(S) = \emptyset \), where \( W^{z s K} \) is the stabiliser of \( z s \) in \( W \). Let \( \sigma w \) be an element lying in such an intersection. We observe that if \( \sigma w \in W(S) \), then \( \sigma w(G^s) = G^s \) hence \( \sigma w \) cannot interchange the component of type \( 3A_1 \) with the component of type \( A_2 \) therein. Thus, it cannot interchange the two components of type \( D_4 \) in \( M \). However, by looking at the projection \( \pi' \) onto \( G/Z(G) = \text{PSO}_{10}(\mathbb{C}) \), we see that \( z s Z(G) \) is the class of the matrix

\[
\text{diag}(I_4, c, c^{-1}d^2, -I_4, I_4, d^{-2}c, c^{-1}, -I_4)
\]

which cannot be centralized by a Weyl group element interchanging these two factors if \( c \) and \( d \) are sufficiently generic. A fortiori, this cannot happen for the class \( z s K \). Hence, \( z s \) and \( z' s \) are not \( W(S) \)-conjugate.
Let now $M_1$ and $M_2$ be the simple factors of $M$ corresponding respectively to the roots $\{\alpha_j, 0 \leq j \leq 3\}$, and $\{\alpha_k, 7 \leq k \leq 10\}$, let $L_1 = M_1 \cap G^{s_0}$ and $L_2 = M_2 \cap G^{s_0}$. Then,

$$\theta(zs) = \text{Ind}^G_L(L \cdot zs) = G \cdot (zs(\text{Ind}^M_{G^{s_0}}(1))) = G \cdot (zs(\text{Ind}^{M_1}_{L_1}(1))(\text{Ind}^{M_2}_{L_2}(1)))$$

and

$$\theta(z's) = \text{Ind}^G_L(L \cdot z's) = G \cdot (z's(\text{Ind}^M_{G^{s_0}}(1))) = G \cdot (z's(\text{Ind}^{M_1}_{L_1}(1))(\text{Ind}^{M_2}_{L_2}(1))).$$

Since $\sigma(zs) = z's$ we have, for some representative $\dot{\sigma} \in N(T)$:

$$\theta(z's) = G \cdot \left(zs(\text{Ind}^{M_{\dot{\sigma}^{-1}, L_1}}_{\dot{\sigma}^{-1}, L_1}(1))(\text{Ind}^{M_{\dot{\sigma}^{-1}, L_2}}_{\dot{\sigma}^{-1}, L_2}(1)))\right) = G \cdot \left(zs(\text{Ind}^{M_{\dot{\sigma}, L_1}}_{\dot{\sigma}, L_1}(1))(\text{Ind}^{M_{\dot{\sigma}, L_2}}_{\dot{\sigma}, L_2}(1)))\right).$$

By [23, Example 3.1] we have $\text{Ind}^{M_{\dot{\sigma}, L_1}}_{\dot{\sigma}, L_1}(1) = \text{Ind}^{M_{\dot{\sigma}^{-1}, L_1}}_{\dot{\sigma}^{-1}, L_1}(1)$ and $\text{Ind}^{M_{\dot{\sigma}, L_2}}_{\dot{\sigma}, L_2}(1) = \text{Ind}^{M_{\dot{\sigma}^{-1}, L_2}}_{\dot{\sigma}^{-1}, L_2}(1)$ so $\theta(zs) = \theta(z's)$.

**Remark 3.8** The parametrisation in Theorem 3.6 cannot be directly generalised to arbitrary Jordan classes. Indeed, if $u \in L$ is not rigid, then $L \cdot u$ is not necessarily characteristic and it may happen that for some external automorphism $\tau$ of $L$, the class $\tau(L \cdot u)$ differs from $L \cdot u$ even if they induce the same $G$-orbit. Then the map $\overline{\theta}$ is not necessarily injective.

### 4 The quotient $\overline{S} // G$

In this section we discuss some properties of the categorical quotient $\overline{S} // G = \text{Spec}(C[\overline{S}])^G$ for $G$ simple in any isogeny class. Since $\overline{S} // G$ parametrises only semisimple conjugacy classes it is enough to look at the so-called Dixmier sheets, i.e., the sheets containing a dense Jordan class consisting of semisimple elements.

In addition, since every such Jordan class is dense in some sheet, studying the collection of $\overline{S} // G$ for $S$ a sheet in $G$ is the same as studying the collection of $\overline{J(s)} // G$ for $J(s)$ a semisimple Jordan class in $G$.

The following Theorem is a group version of [2, Satz 6.3], [17, Theorem 3.6(c)] and [27, Theorem A].

**Theorem 4.1** Let $S = \overline{J(s)}^{reg} \subset G$. 

---

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1. The normalisation of \( S/G \) is 
\[ Z(G^{s_0})^s/W(S). \]

2. The variety \( S//G \) is normal if and only if the natural map

\[ \rho: C[T]^W \to C[Z(G^{s_0})^s]^W(S) \]

induced from the restriction map \( C[T] \to C[Z(G^{s_0})^s] \) is surjective.

**Proof.** 1. The variety \( Z(G^{s_0})^s/W(S) \) is the quotient of a smooth variety (a shifted torus) by the action of a finite group, hence it is normal. Every class in \( J(s) \) meets \( T \) and \( T \cap J(s) = W \cdot (Z(G^{s_0})^s) \). Also, two elements in \( T \) are \( G \)-conjugate if and only if they are \( W \)-conjugate, hence we have an isomorphism \( J(s)//G \cong W \cdot (Z(G^{s_0})^s)//W \) induced from the isomorphism \( G//G \cong T/W. \)

We consider the morphism \( \gamma: Z(G^{s_0})^s/W(S) \to W \cdot (Z(G^{s_0})^s)/W \) induced by \( zs \mapsto W \cdot (zs) \). It is surjective by construction, bijective on the dense subset \( (Z(G^{s_0})^s)^{\text{reg}}/W(S) \) and finite, since the intersection of \( W \cdot (zs) \) with \( Z(G^{s_0})^s \) is finite. Hence \( \gamma \) is a normalisation morphism.

2. The variety \( S//G \) is normal if and only if the normalisation morphism is an isomorphism. This happens if and only if the composition

\[ Z(G^{s_0})^s/W(S) \cong S//G \subseteq G//G \cong T/W \]

is a closed embedding, i.e., if and only if the corresponding algebra map between the rings of regular functions is surjective. \( \square \)

5  **An example: sheets and their quotients in type \( G_2 \)**

We list here the sheets in \( G \) of type \( G_2 \) and all the conjugacy classes they contain. We shall denote by \( \alpha \) and \( \beta \), respectively, the short and the long simple roots. Since \( G \) is adjoint, by [7, Theorem 4.1] the sheets in \( G \) are in bijection with \( G \)-conjugacy classes of pairs \( (M, u) \) where \( M \) is a pseudo-Levi subgroup of \( G \) and \( u \) is a rigid unipotent element in \( M \). The corresponding sheet is \( J(su)^{\text{reg}} \) where \( s \) is a semisimple element whose connected centralizer is \( M \). The conjugacy classes of pseudo-Levi subgroups of \( G \) are those corresponding to the following subsets \( \Pi \) of the extended Dynkin diagram:

1. \( \Pi = \emptyset \), so \( M = T, u = 1, s \) is a regular semisimple element and \( S \) consists of all regular conjugacy classes;
2. \( \Pi = \{\alpha\} \). Here \([M, M]\) is of type \( \tilde{A}_1 \), so \( u = 1 \) and \( s = \alpha^\vee(\zeta)\beta^\vee(t^2) = (3\alpha + 2\beta)^\vee(\zeta^{-1}) \) for \( \zeta \neq 0, \pm 1 \);

3. \( \Pi = \{\beta\} \). Here \([M, M]\) is of type \( A_1 \) so \( u = 1 \) and \( s = \alpha^\vee(\zeta^2)\beta^\vee(\zeta^3) = (2\alpha + \beta)^\vee(\zeta) \) for \( \zeta \neq 0, 1, e^{2\pi i/3}, e^{-2\pi i/3} \),

4. \( \Pi = \{\alpha_0, \beta\} \). Here \([M, M]\) is of type \( A_2 \) so \( u = 1 \); the corresponding \( s = (2\alpha + \beta)^\vee(e^{2\pi i/3}) \) is isolated and \( S = G \cdot s \);

5. \( \Pi = \{\alpha_0, \alpha\} \). Here \([M, M]\) is of type \( \tilde{A}_1 \times A_1 \) so \( u = 1 \), the corresponding \( s = (3\alpha + 2\beta)^\vee(-1) \) is isolated and \( S = G \cdot s \);

6. \( \Pi = \{\alpha, \beta\} \) so \( L = G \). In this case we have three possible choices for \( u \)
rigid unipotent, namely \( 1, x_\alpha(1) \) or \( x_\beta(1) \) (cfr. [28]). Each of these classes is a sheet on its own.

The only sheets containing more than one conjugacy classes are the regular one \( S_0 = G^{\text{reg}} \) corresponding to \( \Pi = \emptyset \) and the two subregular ones, corresponding to \( \Pi_1 = \{\alpha\} \) and \( \Pi_2 = \{\beta\} \). For \( S_0 \) we have \( Z^0S = T, W(S) = W \) so \( S_0/G \) is in bijection with \( T/W \) and \( S_0//G \simeq G//G \) which is normal. For \( S_1 \) and \( S_2 \) we have:

\[
S_1 = \overline{J((3\alpha + 2\beta)^\vee(\zeta_0))}^{\text{reg}} = \left( \bigcup_{\zeta \neq 0, 1} G \cdot (3\alpha + 2\beta)^\vee(\zeta) \right) \cup \text{Ind}_{A_1}^{G}(1) \cup G \cdot \left( (3\alpha + 2\beta)^\vee(-1) \text{Ind}_{\tilde{A}_1}^{A_1}(1) \right)
\]

for \( \zeta_0 \neq 0, 1 \) and

\[
S_2 = \overline{J((2\alpha + \beta)^\vee(\xi_0))}^{\text{reg}} = \left( \bigcup_{\epsilon \neq 0, 1} G \cdot (2\alpha + \beta)^\vee(\xi) \right) \cup \text{Ind}_{A_1}^{G}(1) \cup G \cdot \left( (2\alpha + \beta)^\vee(e^{2\pi i/3}) \text{Ind}_{\tilde{A}_1}^{A_2}(1) \right)
\]

for some \( \xi_0 \neq 0, 1, e^{\pm 2\pi i/3} \).

In both cases \( M \) is a Levi subgroup of a parabolic subgroup of \( G \). By Lemmata 3.1 and 3.3 we have \( W(S_1) = W(S_1)^u = \langle s_\alpha, s_{3\alpha + 2\beta} \rangle \) and \( W(S_2) = W(S_1)^u = \langle s_\beta, s_{2\alpha + \beta} \rangle \). Also \( Z(M)^0 = Z(M)^0S \) in both cases, so

\[
S_1/G \simeq (3\alpha + 2\beta)^\vee(\mathbb{C}^\times)/\langle s_\alpha, s_{3\alpha + 2\beta} \rangle \simeq (3\alpha + 2\beta)^\vee(\mathbb{C}^\times)/\langle s_{3\alpha + 2\beta} \rangle
\]

\[
S_2/G \simeq (2\alpha + \beta)^\vee(\mathbb{C}^\times)/\langle s_\beta, s_{2\alpha + \beta} \rangle \simeq (2\alpha + \beta)^\vee(\mathbb{C}^\times)/\langle s_{2\alpha + \beta} \rangle
\]

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where the $\simeq$ symbols stand for the bijection $\bar{\theta}$.

Let us analyze normality of $\overline{S_2} // G$. Here, $Z(M) = (3\alpha + 2\beta)^\vee(\mathbb{C}^*) \simeq \mathbb{C}^*$, so $\mathbb{C}[Z(M)]^W = \mathbb{C}[\zeta + \zeta^{-1}]$. On the other hand, since $G$ is simply connected, $\mathbb{C}[T]^W = (\mathbb{C} \Lambda)^W$ is the polynomial algebra generated by $f_1 = \sum_{\gamma \in \Phi_{\text{short}}} e^{\gamma}$ and $f_2 = \sum_{\gamma \in \Phi_{\text{long}}} e^{\gamma}$, [5, Ch.VI, §4, Théorème 1] Then,

$$\rho(f_1)(3\alpha + 2\beta)^\vee(\zeta) = f_1((3\alpha + 2\beta)^\vee(\zeta)) = \sum_{\gamma \in \Phi_{\text{short}}} \zeta^{(\gamma, (3\alpha + 2\beta)^\vee)} = 2 + 2\zeta + 2\zeta^{-1}$$

so the restriction map is surjective and $\overline{S_1} // G$ is normal.

Let us consider normality of $\overline{S_2} // G$. Here, $Z(M) = (2\alpha + \beta)^\vee(\mathbb{C}^*) \simeq \mathbb{C}^*$, so $\mathbb{C}[Z]^\Gamma = \mathbb{C}[\zeta + \zeta^{-1}]$. Then,

$$\rho(f_1)(2\alpha + \beta)^\vee(\zeta) = f_1((2\alpha + \beta)^\vee(\zeta)) = \sum_{\gamma \in \Phi_{\text{short}}} \zeta^{(\gamma, (2\alpha + \beta)^\vee)} = \zeta^2 + \zeta^{-2} + 2(\zeta + \zeta^{-1})$$

whereas

$$\rho(f_2)(2\alpha + \beta)^\vee(\zeta) = f_2((2\alpha + \beta)^\vee(\zeta)) = \sum_{\gamma \in \Phi_{\text{long}}} \zeta^{(\gamma, (2\alpha + \beta)^\vee)} = 2 + 2\zeta^3 + 2\zeta^{-3}.$$

Let us write $y = \zeta + \zeta^{-1}$. Then, $(\zeta^2 + \zeta^{-2}) = y^2 - 2$ and $\zeta^3 + \zeta^{-3} = y^3 - 3y$ so $\text{Im}(\rho) = \mathbb{C}[y^2 + 2y, y^3 - 3y] = \mathbb{C}[(y + 1)^2, y^3 + 3y^2 + 6y + 3 - 3y] = \mathbb{C}[(y + 1)^2, (y + 1)^3]$. Hence, $\rho$ is not surjective and $\overline{S_2} // G$ is not normal.

We observe that $\text{Im}(\rho)$ is precisely the identification of the coordinate ring of $\overline{S_2} // G$ in $\mathbb{C}[T]^W$. We may thus see where this variety is not normal. We have: $\text{Im}(\rho) = \mathbb{C}[(y + 1)^2, (y + 1)^3] \cong \mathbb{C}[Y, Z]/(Y^3 - Z^2)$ so this variety is not normal at $y + 1 = 0$, that is, for $\zeta + \zeta^{-1} + 1 = 0$. This corresponds precisely to the closed, isolated orbit $G \cdot ((2\alpha + \beta)^\vee(e^{2\pi i/3}))x_{\alpha_0}(1) = G \cdot ((2\alpha + \beta)^\vee(e^{-2\pi i/3}))x_{\alpha_0}(1)$. This example shows two phenomena: the first is that even if the sheet corresponding to the set $\Pi_2$ in $\text{Lie}(G)$ has a normal quotient [6] Theorem 3.1, the same does not hold in the group counterpart. The second phenomenon is that the non-normality locus corresponds to an isolated class in $\overline{S_2}$. In a forthcoming paper we will address the general problem of normality of $\overline{S} // G$ and we will prove and make use of the fact that if the categorical quotient of the closure a sheet in $G$ is not normal, then it is certainly not normal at some isolated class.
References

[1] T. ARAKAWA, A. MOREAU, Sheets and associated varieties of affine vertex algebras, Adv. Math. 320, 157–209, (2017).

[2] W. BORHO, Über Schichten halbeinfacher Lie-Algebren, Invent. Math., 65, 283–317 (1981/82).

[3] W. BORHO, A. JOSEPH, Sheets and topology of primitive spectra for semisimple Lie algebras, J. Algebra 244, 76–167, (2001). Corrigendum 259, 310–311 (2003).

[4] W. BORHO, H. KRAFT, Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen, Comment. Math. Helvetici, 54, 61–104 (1979).

[5] N. BOURBAKI, Éléments de Mathématique. Groupes et Algèbres de Lie, Chapitres 4,5, et 6, Masson, Paris (1981).

[6] A. BROER, Decomposition varieties in semisimple Lie algebras, Can. J. Math. 50(5), 929–971 (1998).

[7] G. CARNOVALE, Lusztig’s partition and sheets, with an appendix by M. Bulois, Mathematical Research Letters, 22(3), 645-664, (2015).

[8] G. CARNOVALE, F. ESPOSITO, On sheets of conjugacy classes in good characteristic, IMRN 2012(4), 810–828, (2012).

[9] G. CARNOVALE, F. ESPOSITO, A Katsylo theorem for sheets of spherical conjugacy classes, Representation Theory, 19, 263-280 (2015).

[10] J. DIXMIER, Polarisations dans les algèbres de Lie semi-simples complexes Bull. Sci. Math. 99, 45-63, (1975).

[11] J.M. DOUGLASS, G. RÖHRLE, Invariants of reflection groups, arrangements, and normality of decomposition classes in Lie algebras, Compos. Math. 148, 921–930, (2012).

[12] R. B. HOWLETT, Normalizers of parabolic subgroups of reflection groups, J. London Math. Soc. 21, 62–80 (1980).

[13] J. HUMPHREYS, Conjugacy Classes in Semisimple Algebraic Groups, AMS, Providence, Rhode Island (1995).
[14] A. E. Hof, *The sheets in a classical Lie algebra*, PhD thesis, Basel, [http://edoc.unibas.ch/257/](http://edoc.unibas.ch/257/)(2005).

[15] P.I. Katsylo, *Sections of sheets in a reductive algebraic Lie algebra*, Math. USSR Izvestia 20(3), 449–458 (1983).

[16] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. 85, 327-404 (1963).

[17] H. Kraft, *Parametrisierung von Konjugationsklassen in sln*, Math. Ann. 234, 209-220 (1978).

[18] I. Losev, *Deformation of symplectic singularities and orbit method for semisimple Lie algebras*, [arXiv:1605.00592v1](http://arxiv.org/abs/1605.00592v1), (2016).

[19] G. Lusztig, *On conjugacy classes in a reductive group*. In: Representations of reductive groups, 333–363, Progr. Math., 312, Birkhäuser/Springer (2015).

[20] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. 75, 205–272 (1984).

[21] G. Lusztig, N. Spaltenstein, *Induced unipotent classes*, J. London Math. Soc. (2), 19, 41–52 (1979).

[22] G. McNinch, E. Sommers, *Component groups of unipotent centralizers in good characteristic*, J. Algebra 270(1), 288–306 (2003).

[23] A. Moreau, *Corrigendum to: On the dimension of the sheets of a reductive Lie algebra*, Journal of Lie Theory 23(4), 1075–1083 (2013).

[24] D. Peterson, *Geometry of the Adjoint Representation of a Complex Semisimple Lie Algebra* PhD Thesis, Harvard University, Cambridge, Massachusetts, (1978).

[25] A. Premet, L. Topley, *Derived subalgebras of centralisers and finite W-algebras*, Compos. Math. 150(9), 1485–1548, (2014).

[26] A. Premet, D. Stewart, *Rigid orbits and sheets in reductive Lie algebras over fields of prime characteristic*, Journal of the Institute of Mathematics of Jussieu, 1–31, (2016).
[27] R. W. Richardson,  *Normality of G-stable subvarieties of a semisimple Lie algebra*, In: Cohen et al., Algebraic Groups, Utrecht 1986, Lecture Notes in Math. 1271, Springer-Verlag, New York, (1987).

[28] N. Spaltenstein,  *Classes Unipotentes et Sous-Groupes de Borel*, Springer-Verlag, Berlin (1982).

[29] T.A. Springer,  *Linear Algebraic Groups, Second Edition*  Progress in Mathematics 9, Birkhäuser (1998).