NON-ABELIAN MONOPOLES ON FOUR-MANIFOLDS

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ABSTRACT

We present a non-abelian generalization of Witten monopole equations and we analyze the associated moduli problem, which can be regarded as a generalization of Donaldson theory. The moduli space of solutions for SU(2) monopoles on Kähler manifolds is discussed. We also construct, using the Mathai-Quillen formalism, the topological quantum field theory corresponding to the new moduli problem. This theory involves the coupling of topological Yang-Mills theory to topological matter in four dimensions.

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1. Introduction.

Topological quantum field theory has shown to be a very fruitful arena in both, physics and mathematics. Since the formulation of Donaldson theory [1,2,3] in the language of quantum field theory [4] many other interesting aspects of topology and algebraic geometry have been reached. In recent years, the physical and mathematical aspects of topological quantum field theory have benefited from each other. On the one hand, using insight from topological aspects related to Donaldson-Witten theory [5], theories with $N = 4$ [6] and $N = 2$ [7] supersymmetry in four dimensions have been solved in the infrared; on the other hand, these developments in physical theories have been used to show that Donaldson-Witten theory itself can be mapped to a simpler theory involving abelian monopoles[8].

Donaldson theory involves the study of certain differential forms on the moduli space of self-dual gauge connections. This theory, initially formulated for $SU(2)$, can be reformulated in terms of quantum field theory [4] and generalized to other gauge groups. The theory is intrinsically non-abelian. In [8] Witten showed that Donaldson theory with gauge group $SU(2)$ is equivalent to a new moduli problem which involves an abelian connection coupled to matter in a pair of monopole equations. The topological quantum field theory associated to this new moduli space has been recently constructed in [9] using the Mathai-Quillen formalism. The resulting theory turned out to be an abelian Donaldson-Witten theory coupled to a twisted version of the $N = 2$ supersymmetric hypermultiplet [10,11,12]. A non-abelian version of this model, associated though to a simpler moduli problem, was presented in [13,14]. Related topological quantum field theories have been analyzed in [15], and their connection to [8] has been indicated in [16].

In this paper we present the non-abelian version of the monopole equations proposed in [8], we study the moduli problem and the associated moduli space for the simple case of $SU(2)$ and matter fields in the fundamental representation, and we construct the corresponding topological quantum field theory action using the Mathai-Quillen formalism. The generalization of the monopole equations involve
the following data: a four-dimensional spin manifold $X$ endowed with a metric $g$, a gauge group $G$ and a representation $R$. The data are therefore enlarged with respect to the ordinary Donaldson-Witten theory by the presence of matter in a representation $R$. This is an indication that the set of topological quantities associated to this new theory might be richer than in the ordinary case. It is interesting to notice that the step given in [8] to map Donaldson theory to a simpler one is now utilized to enlarge the original theory. In fact we will show that the conditions to have a well-defined moduli problem are the same than in the Donaldson case, and the non-abelian monopole theory appears thus as a natural generalization of the Donaldson theory. We will argue that the new moduli space contains the moduli space of anti self-dual connections and in addition new branches of solutions which in the $SU(2)$ case are similar to the ones that appear in the abelian theory.

The analysis carried out in this work although often is particularized for simple cases can in principle be extended to the general situation. We have concentrated in the specific cases working along the lines of [2] and [8] but presumably similar arguments can be used in general.

The paper is organized as follows. In sect. 2 we present the monopole equations and we analyze the corresponding moduli problem. In sect. 3 the previous analysis is extended for the case of $SU(2)$ monopoles on Kähler manifolds, with the matter fields in the fundamental representation. In sect. 4 the non-abelian topological action is constructed using the Mathai-Quillen formalism. Finally, in sect. 5 we state our conclusions. An appendix contains the spinor conventions used in the paper.
2. Non-abelian monopole equations.

Let $X$ be an oriented, closed four-manifold endowed with a Riemannian structure given by a metric $g$. We will restrict ourselves to spin manifolds, although the generalization to arbitrary manifolds can be done using a $\text{Spin}_c$ structure. We will denote the positive and negative chirality spin bundles on $X$ by $S^+$ and $S^−$, respectively. In [8], Witten introduced a new moduli problem involving an abelian Yang-Mills field, associated to a $U(1)$ complex line bundle $L$, coupled to a spinor field of positive chirality in a pair of “monopole equations”:

$$F^{+}_{\alpha\beta} + \frac{i}{2} \tau_{\alpha \beta} M = 0,$$
$$D_{\alpha \dot{\alpha}} M^{\alpha} = 0,$$

where $D_{\alpha \dot{\alpha}}$ is the Dirac operator and $F^{+}_{\alpha\beta}$ the self-dual part of the gauge field-strength (see eq. (A31,A40) in the appendix for the conventions used). This moduli problem turns out to be equivalent to the $SU(2)$ Donaldson theory on $X$. In principle a non-abelian generalization of these monopole equations would give rise to a generalization of Donaldson theory which from the physical point of view corresponds to a coupling of a topological Yang-Mills theory to topological matter in four dimensions [13, 14]. As we will see, the non-abelian monopole equations share many properties of Donaldson theory as well as of the abelian theory proposed in [8].

Before going on with the non-abelian generalization of (2.1), it is important to recall the topological framework for the field-theory approach to moduli problems in the context of the Mathai-Quillen formalism [17,19]: given a (infinite-dimensional) field space $\mathcal{M}$ and a vector bundle over $\mathcal{M}$, $\mathcal{V}$, the basic equations of the problem are defined as sections of this vector bundle. Let us denote generically these sections as $s : \mathcal{M} \rightarrow \mathcal{V}$. In the situations in which there is a gauge symmetry, as will be the case under consideration, one has to take into account the action of a group $\mathcal{G}$ on both, the manifold $\mathcal{M}$ and the vector bundle. This is done “dividing by $\mathcal{G}$”
which implies that the section \( s \) must be taken to be gauge-equivariant and hence one must consider the associated section \( \hat{s} : M/\mathcal{G} \to V/\mathcal{G} \).

The basic topological invariant associated to the moduli problem is the Euler characteristic of the bundle \( V \), which can be obtained, as in the finite-dimensional case, integrating the pullback under \( s \) of its Thom class on \( M \). In the situations with gauge symmetries the interest resides in the computation of this class for the corresponding quotient bundle. The computation of the Thom class involves the construction of the Mathai-Quillen form [20] which leads to the topological quantum field theory associated to the moduli problem. The Mathai-Quillen form for the non-abelian monopole theory will be obtained in sect. 4. We will be interested in the special situation in which the vector bundle \( V \) is trivial and can be written in the form \( V = M \times F \), where \( F \) is the fibre on which a \( \mathcal{G} \)-invariant metric is defined. Considering the moduli space \( M \) as a principal bundle with group \( \mathcal{G} \) the quotient bundle is the associated vector bundle \( E = M \times_{\mathcal{G}} F \). This is the situation common to Donaldson-Witten theory and monopoles on four-manifolds as described in [17] and [9] respectively. A discussion for the case in which there are non-trivial vector bundles can be found in [18, 19].

In Witten’s monopole theory, as it is discussed in [9], the geometrical data are a four-manifold \( X \) and a complex line bundle \( L \) over \( X \), and the field space is \( \mathcal{M} = A \times \Gamma(X, S^+ \otimes L) \), where \( A \) is the moduli space of \( U(1) \) abelian connections on \( L \), and \( \Gamma(X, S^+ \otimes L) \) are the sections of the product bundle \( S^+ \otimes L \), i.e., positive chirality spinors taking values in \( L \). The vector bundle over \( M \) is a trivial one with fibre \( \mathcal{F} = \Omega^{2+}(X) \oplus \Gamma(X, S^- \otimes L) \), where the first factor denotes the self-dual differential forms of degree 2 on \( X \). The equations (2.1) define a moduli space which is the zero locus of a section of this bundle. The group \( \mathcal{G} \) is the group of gauge transformations of the principal \( U(1) \)-bundle associated to the connection \( A \).

The obvious way to construct the non-abelian moduli problem consists of considering, instead of a complex line bundle, a principal fibre bundle \( P \) with some
compact, connected, simple Lie group $G$. The Lie algebra of $G$ will be denoted by $\mathfrak{g}$. For the matter part we need an associated vector bundle $E$ to the principal bundle $P$ by means of a representation $R$ of the Lie group $G$. The field space is $\mathcal{M} = \mathcal{A} \times \Gamma(X, S^+ \otimes E)$, where $\mathcal{A}$ is now the moduli space of $G$-connections on $E$, and the spinors take values now in this representation space. The vector bundle over $\mathcal{M}$ is again a trivial one with fibre $\mathbb{F} = \Omega^2(X, \mathfrak{g}_E) \oplus \Gamma(X, S^- \otimes E)$, and the self-dual differential forms take values in the representation of the Lie algebra of $G$ associated to $R, \mathfrak{g}_E$. The group $\mathcal{G}$ is the group of gauge transformations of the bundle $E$, and its action on the moduli space is given locally by:

$$
g^*(A_\mu) = -igd_\mu g^{-1} + gA_\mu g^{-1} ,$$
$$
g^*(M_\alpha) = gM_\alpha ,$$

(2.2)

where $M \in \Gamma(X, S^+ \otimes E)$ and $g$ takes values in the group $G$ in the representation $R$. Notice that in terms of the covariant derivative $d_A = d + i[A, ]$ the infinitesimal form of the transformations (2.2) becomes, after considering $g = \exp(i\phi)$:

$$
\delta A = -d_A \phi ,$$
$$
\delta M_\alpha = i\phi M .
$$

(2.3)

The group of gauge transformations also acts on the fibre $\mathcal{F}$, but we must use $g^{-1}$, as the construction of an associated vector bundle imposes. The Lie algebra of the group $\mathcal{G}$ is $\text{Lie}(\mathcal{G}) = \Omega^0(X, \mathfrak{g}_E)$. The tangent space to the moduli space at the point $(A, M)$ is just $T_{(A,M)}\mathcal{M} = T_A\mathcal{A} \oplus T_M\Gamma(X, S^+ \otimes E) = \Omega^1(X, \mathfrak{g}_E) \oplus \Gamma(X, S^+ \otimes E)$, for $\Gamma(X, S^+ \otimes E)$ is a vector space. We can define a gauge-invariant Riemannian metric on $\mathcal{M}$ given by:

$$
\langle (\psi, \mu), (\theta, \nu) \rangle = \int_X \text{Tr}(\psi \wedge *\theta) + \frac{1}{2} \int_X e(\bar{\mu}^\alpha \nu^\alpha + \mu^\alpha \bar{\nu}^\alpha) ,
$$

(2.4)

where $e = \sqrt{g}$. The spinor notation used in this paper is conveniently compiled in the appendix. An analogous expression gives the inner product on the fibre $\mathcal{F}$. 

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The Lie algebra of the gauge group of transformations \( \text{Lie}(G) \) is also endowed with a metric given, as in (2.4), by the trace and the inner product on the space of zero-forms.

Within this general framework it is easy to write out explicitly the non-abelian monopole equations for the cases in which Donaldson theory has been proved more useful: the \( SU(N) \) and \( SO(N) \) cases. The non-abelian monopole equations in this case are simply:

\[
F_{\alpha\beta}^{+ij} + \frac{i}{2} \overline{M}_{(\alpha} M_{\beta)}^{ij} - \frac{\delta_{ij}^{k}}{d_{R}} \overline{M}_{(\alpha} M_{\beta)}^{k} = 0, \quad \text{for } SU(N),
\]

\[
F_{\alpha\beta}^{+ij} + \frac{i}{2} \overline{M}_{(\alpha} M_{\beta)}^{ij} = 0, \quad \text{for } SO(N),
\]

where \( F_{\alpha\beta}^{+ij} \) are in a representation \( R \), i.e., \( F_{\alpha\beta}^{+ij} = F_{\alpha\beta}^{+a} (T^{a})^{ij} \) being \( T^{a} \) the generators of the Lie algebra taken in the representation \( R \). In the first equation of (2.5) (and similar ones in this paper), a sum in the repeated index \( k \) is understood, and \( d_{R} \) denotes the dimension of the representation \( R \). The other monopole equation is simply the Dirac equation with the Dirac operator coupled to the gauge connection in the corresponding representation.

The structure of equations (2.5) for other groups possesses a similar structure. It can be written in a compact form as:

\[
F_{\alpha\beta}^{+a} + \frac{i}{2} \overline{M}_{(\alpha} (T^{a}) M_{\beta)} = 0, \quad \text{(2.6)}
\]

where \( \overline{M}_{(\alpha} (T^{a}) M_{\beta)} \) is shortened form for \( \overline{M}_{(\alpha} (T^{a})^{ij} M_{\beta)}^{j} \) (this convention will be used throughout the paper). The expressions in (2.5) are obtained from this equation after contracting it with \( T^{a} \) and using the fact that the normalization of the generators can be chosen such that for the representation \( R \) one has \( (T^{a})^{ij} (T^{a})^{kl} = \delta^{il} \delta^{jk} - \frac{1}{d_{R}} \delta^{ij} \delta^{kl} \) for \( SU(N) \) and \( (T^{a})^{ij} (T^{a})^{kl} = \delta^{il} \delta^{kj} \) for \( SO(N) \). In the rest of this paper we will mainly focus on the \( SU(N) \) case. The generalization for \( SO(N) \) and for other groups is straightforward.
When writing the section of the bundle $V$ from (2.5) it will be useful to rescale the first monopole equation by the factor $1/\sqrt{2}$, as in the abelian theory. The section reads therefore, in the $SU(N)$ case:

$$s(A, M) = \left( \frac{1}{\sqrt{2}} (F_{\alpha\beta}^{+ij} + \frac{i}{2} (M_{i\alpha}^j M_{\beta}^i - \delta^{ij} M_{i\alpha}^k \frac{d}{dR} (\bar{M}_{i\alpha}^k M_{\beta}^k))), (D_{\alpha\beta} M_{i\alpha}^i) \right). \quad (2.7)$$

A first step to understand the structure of the moduli space of solutions to the non-abelian monopole equations modulo gauge transformations is to construct the associated instanton deformation complex, which allows one to compute (under certain assumptions) the dimension of the tangent space to this moduli space (the virtual dimension of the moduli space). For this we need an explicit construction of the gauge orbits, which are given by the vertical tangent space on the principal bundle with group $G$. This space is the image of a map from the Lie algebra of the group $G$ to the tangent space to $M$,

$$C : \text{Lie}(G) \longrightarrow T \mathcal{M}, \quad (2.8)$$

which can be obtained from (2.3) and reads:

$$C(\phi) = (-d_A \phi, i \phi^{ij} M^j) \in \Omega^1(X, \mathfrak{g}_E) \oplus \Gamma(X, S^+ \otimes E), \quad \phi \in \Omega^0(X, \mathfrak{g}_E). \quad (2.9)$$

Using the metrics in (2.4) and the analogous one on the fibre, we can compute the adjoint operator $C^\dagger$ which will be needed later to obtain the topological lagrangian of the theory. Let us consider $(\psi, \mu) \in T_{(A, M)} \mathcal{M} = \Omega^1(X, \mathfrak{g}_E) \oplus \Gamma(X, S^+ \otimes E)$. One finds,

$$C^\dagger(\psi, \mu)^{ij} = -(d_A^* \psi)^{ij} + \frac{i}{2} (\bar{\mu}^{ij} M_{i\alpha}^j \bar{M}_{i\alpha}^i - \delta^{ij} \frac{d}{dR} (\bar{\mu}^{ik} M_{i\alpha}^k \bar{M}_{i\alpha}^k)) \in \Omega^0(X, \mathfrak{g}_E). \quad (2.10)$$

We also need the linearization of the non-abelian monopole equations, which
can be understood as a map \( ds : T_{(A,M)}M \to \mathcal{F} \). The result is:

\[
\begin{align*}
ds(\psi, \mu) &= \left( \frac{1}{\sqrt{2}} \left( (p^+(dA\psi))^{ij}_{\alpha\beta} + \frac{i}{2} (M^i_{(\alpha\mu_\beta)} + \bar{\mu}^i_{(\alpha} M^i_{\beta)}) - \frac{\delta^{ij}}{dR} (M^k_{(\alpha\mu_\beta)} + \bar{\mu}^k_{(\alpha} M^k_{\beta)}) \right) \\
&\quad \left( D^{\alpha}_{\alpha} \mu^i + i \psi^{ij}_{\alpha\bar{\alpha}} M^{\alpha j} \right),
\end{align*}
\]

(2.11)

where \( p^+ \) is the projector defined in the appendix (eq. (A40)). The maps \( ds \) and \( C \) fit into the instanton deformation complex:

\[
0 \to \Omega^0(X, g_E) \xrightarrow{C} \Omega^1(X, g_E) \oplus \Gamma(X, S^+ \otimes E) \xrightarrow{ds} \Omega^2(X, g_E) \oplus \Gamma(X, S^- \otimes E) \to 0.
\]

(2.12)

The index of this complex can be computed dropping terms of order zero in the operators \( C \) and \( ds \) (as their leading symbol is not changed). In this way, the complex (2.12) splits into the complex associated to the anti-self-dual (ASD) connections of Donaldson theory:

\[
0 \to \Omega^0(X, g_E) \xrightarrow{dA} \Omega^1(X, g_E) \xrightarrow{p^+dA} \Omega^2(X, g_E) \to 0.
\]

(2.13)

and the complex of the twisted Dirac operator. The index will be simply the sum of the virtual dimension of the moduli space of ASD instantons, \( \mathcal{M}_{\text{ASD}} \), and of twice the index of the twisted Dirac complex (for we are considering \( S^+ \otimes E, S^- \otimes E \) as real vector bundles, in order to obtain the real dimension of the moduli space).

This is easily computed and gives:

\[
\text{index } D = \int_X \text{ch}(E) \hat{A}(X) = -\frac{dR}{8} \sigma - c_2(E),
\]

(2.14)

where \( \sigma \) is the signature of the four-manifold \( X \), which is given, according to the Hirzebruch signature formula, by \( \sigma = p_1(X)/3 \), and \( c_2(E) \) is the second Chern class of the representation bundle. The virtual dimension of the moduli space for
the non-abelian theory, $\mathcal{M}_{\text{NA}}$, in the $SU(N)$ case, is thus

$$\dim \mathcal{M}_{\text{NA}} = \dim \mathcal{M}_{\text{ASD}} + 2\text{index} \ D = (4N - 2)c_2(E) - \frac{N^2 - 1}{2}(\chi + \sigma) - \frac{d_R}{4}\sigma,$$

(2.15)

where $\chi$ is the Euler characteristic of $X$. The generalization of this expression to other gauge groups is straightforward.

There are two points that are important in order to understand the non-abelian monopole equations and the possibility of extracting some new topological information from them. The basic topological invariant associated to the non-abelian equations (2.5) is the Euler characteristic of the bundle $\mathcal{E}$ and can be interpreted as the partition function of the corresponding topological theory (which we will construct in sect. 4). This partition function is defined for $\dim \mathcal{M}_{\text{NA}} = 0$, i.e., when the moduli space of solutions of the monopole equations consists of a finite set of points. On the tangent space to each of these points we can consider the elliptic operator associated to the complex (2.12), $T = C^\dagger \oplus ds$, and the sign of the determinant of this operator. Using standard arguments [4] one can show that the partition function of the theory is given by the sum over the set of solutions of the monopole equations of the signs of $\det T$ (this is an infinite dimensional version of the Poincaré-Hopf theorem [19, 6]). But to have a well-defined topological quantity we need the determinant bundle of $T$ to be a trivial one. From a field-theory point of view, this is equivalent to require that the corresponding topological field theory does not have global anomalies. Now, recall that the operator $T$ can be deformed to a direct sum of the Dirac operator and the elliptic operator for the complex of ASD connections (2.13). The determinant line bundle of the Dirac operator, when this is regarded as a real operator, has a natural trivialization coming from its underlying complex structure. Therefore one must prove the triviality of the determinant bundle of the operator $\delta_A = d_A^* \oplus p^+ d_A$ coming from (2.13). But this is in fact guaranteed by Donaldson theory [21,3] and it is equivalent to the orientability of the moduli space of irreducible connections with $H^2_A = \text{coker} \ p^+ d_A = 0$ (for such connections, coker $\delta_A = 0$ and therefore the determinant line bundle of $\delta_A$
coincides with $\Lambda^{\text{max}} \text{Ker} \delta_A = \Lambda^{\text{max}} T_A \mathcal{M}_{\text{ASD}}$).

The second important question which arises in order to have a well-defined moduli problem is whether or not the group of gauge transformations has a free action on the space of solutions to the non-abelian monopole equations. This is rather obvious when we look at the partition function as the Euler class of the bundle $\mathcal{E}$: if the action of the gauge group has fixed points, we will have singularities when making the quotient by $\mathcal{G}$ and we will not be able to get rid of the gauge degrees of freedom. From the field-theory point of view [22] a non-free action of the gauge group gives a moduli space larger than the space to which we want to localize the path integral of the corresponding topological field theory. Notice that, for the non-abelian case, the only way to have a fixed point in the space of solutions to the monopole equations is to have, as in the abelian case, a solution with $M = 0$. In this case, the monopole equations reduce to the equation which defines an ASD connection. In fact, the non-abelian monopole equations have always the solution $M = 0, F^+ = 0$, and therefore $\mathcal{M}_{\text{ASD}} \subset \mathcal{M}_{\text{NA}}$. A non-free action of the gauge group can only be possible in the subset of ASD connections, i.e., when we have reducible ASD connections. The conditions for a free action are thus the same than in Donaldson theory. This analysis and the one we did for establishing the triviality of the determinant line bundle show that the non-abelian monopole theory appears as a rather natural generalization of Donaldson theory: the moduli space of solutions contains $\mathcal{M}_{\text{ASD}}$ as a subset, and the conditions for having a well defined moduli problem are essentially the same. In this way it seems that the coupling of non-abelian topological Yang-Mills theory to topological matter in four dimensions given by the non-abelian monopole equations could provide an adequate extension of the Donaldson framework. Indeed, we will try to argue in the next section that the moduli space of solutions to the non-abelian equations has in principle a richer structure than $\mathcal{M}_{\text{ASD}}$.

Another aspect of the relation of the non-abelian monopole theory to Donaldson theory is the following. In the abelian case, Witten showed [8], making use of vanishing theorems, that there are only a finite number of isomorphism classes of
line bundles for which the moduli space of solutions has a positive or zero virtual dimension. In the $SU(N)$ case, as for the abelian monopoles, vanishing theorems are obtained by computing the squared norm of the section (2.7) using the natural Riemannian metric on the fibre. Let us carry out the corresponding analysis for the non-abelian case.

Taking into account the Weitzenböck formula (see eq. (A44) in the appendix):

$$D_{\alpha \dot{\alpha}} D_{\beta \dot{\beta}} M^{\dot{\beta} i} = (g^{\mu \nu} D_{\mu} D_{\nu} - \frac{1}{4} R) M^{i}_{\alpha} + i F^{+ij}_{\alpha \beta} M^{j \beta}, \quad (2.16)$$

being $R$ the scalar curvature on $X$, one finds:

$$|s(A, M)|^2 = \int_{X} e D_{\alpha \dot{\alpha}} \overline{M}^{\alpha} D_{\beta \dot{\beta}} M^{\dot{\beta}} + \frac{1}{2} \int_{X} e (F^{+\alpha \beta ji} + \frac{i}{2} (\overline{M}^{i (\alpha M^{\beta) j} - \frac{1}{d_R} (\overline{M}^{i M^{j (\alpha M^{k \beta)})} - \frac{1}{d_R} (\overline{M}^{i M^{j (\alpha M^{k \beta)})})))

= \int_{X} e \left[ g^{\mu \nu} D_{\mu} M^{\alpha} D_{\nu} M_{\alpha} + \frac{1}{4} R \overline{M}^{\alpha} M_{\alpha} + \frac{1}{2} \text{Tr}(F^{+\alpha \beta F^{+\alpha \beta})}

- \frac{1}{8} (\overline{M}^{i (\alpha M^{\beta)})} (\overline{M}^{j (\alpha M^{i \beta)})} - \frac{1}{d_R} (\overline{M}^{i (\alpha M^{j \beta)})} (\overline{M}^{j (\alpha M^{i \beta)})}). \right]. \quad (2.17)$$

After using the fact that for $SU(N)$ the normalization of the generators $T^{a}$ can be chosen in such a way that for the representation $R (T^{a})^{ij} (T^{a})^{kl} = \delta^{il} \delta^{jk} - \frac{1}{d_R} \delta^{ij} \delta^{kl}$, the last term in (2.17) can be written as:

$$- \frac{1}{8} (\overline{M}^{i (\alpha T^{a} M^{\beta)})} (\overline{M}^{j (\alpha T^{a} M^{\beta)})}, \quad (2.18)$$

where a sum over $a$ is must be understood. Notice that if one denotes the components of $M^{a}_{\alpha}$ by $M^{i}_{\alpha} = (a^{i}, b^{i})$, this term is in fact,

$$\frac{1}{4} \left( (1 - \frac{1}{d_R}) \left( \sum_{i} (|a^{i}|^2 + |b^{i}|^2) \right)^2 + (1 + \frac{2}{d_R}) \sum_{ij} |a^{i} b^{j}|^2 \right) \quad (2.19)$$

and therefore it is positive definite. The factor $i \overline{M}^{a} F^{+}_{\alpha \beta} M^{\beta}$ has cancelled in the sum, and then each term in the second expression for $|s(A, M)|^2$ in (2.17) is positive.
definite except the one involving the scalar curvature. This was the reason of choosing the factor $1/\sqrt{2}$ in (2.7).

The advantage of the form (2.17), which will become the bosonic sector of the topological action is that, as discussed in sect. 3, one can apply vanishing theorems which improve the analysis of the space of solutions of the monopole equations as in [8,6]. As in the abelian case, we can get from this expression an upper bound for the squared norm of the self-dual part of the curvature on solutions of the monopole equations:

$$I^+ = \int_X eF^+_{a+\alpha\beta}F^+_{a,-\alpha\beta} \leq \frac{1}{8(1-1/d_R)} \int_X eR^2$$

But, in contrast to the abelian case, we cannot find an upper bound for the anti self-dual part $I^-$ when we impose that dimension of the moduli space (2.15) to be greater than or equal to zero. This is so because a theory of $SU(N)$ connections involves the instanton number $c_2(E)$, which equals $I^- - I^+$. Therefore the non-abelian theory is in this respect like Donaldson theory.

3. $SU(2)$ monopoles on Kähler manifolds.

In this section we will analyze in more detail the non-abelian monopole equations on a compact Kähler manifold $X$ and for the case in which the gauge group is $SU(2)$, following the procedure of [8]. We will show that many of the characteristics of the abelian case are shared by the non-abelian equations, and this will allow us to propose a more concrete picture of $\mathcal{M}_{\text{NA}}$ in this case. In particular we want to argue that this moduli space is in fact “larger” than $\mathcal{M}_{\text{ASD}}$, and therefore that the non-abelian theory can give a different kind of topological information which may be useful for studying the geometry of four-manifolds.

On a Kähler manifold the spinor bundle $S^+$ splits into $K^{1/2} \oplus K^{-1/2}$, where $K^{1/2}$ is a square root of the canonical bundle $K$. Let $E$ be the vector bundle
associated to the fundamental representation of $SU(2)$, and denote by $\alpha = (\alpha^1, \alpha^2)$ and $-i\beta = (-i\beta^1, -i\beta^2)$ the components of $M_\alpha^i$ in $K^{1/2} \otimes E$ and $K^{-1/2} \otimes E$, respectively. If we denote by $\omega$ the Kähler form on $X$, we have the decomposition of self-dual forms $\Omega^+ = \Omega^{2,0} \oplus \Omega^{0,2} \oplus \Omega^0 \omega$. According to this decomposition we can write the first $SU(2)$ monopole equation as:

$$F^{ij}_{2,0} = \alpha^i \beta^j - \frac{1}{2} \delta^{ij} \alpha^k \beta^k,$$

$$F^{ij}_\omega = -\frac{\omega}{2} \left( \alpha^i \overline{\alpha}^j - \beta^i \beta^j - \frac{\delta^{ij}}{2} (|\alpha|^2 - |\beta|^2) \right),$$

$$F^{ij}_{0,2} = \overline{\alpha}^i \beta^j - \frac{1}{2} \delta^{ij} \overline{\alpha}^k \beta^k.$$  (3.1)

Now we can use expression (2.17) to obtain vanishing results for the solutions of (3.1), as in [8,6]. Suppose $(A, \alpha, \beta)$ is a solution to (3.1), and hence (2.17) vanishes. Then $(A, \alpha, -\beta)$ makes (2.17) vanish too, and we obtain another solution to (3.1). Therefore, any solution of these equations verifies:

$$F^{ij}_{2,0} = F^{ij}_{0,2} = 0.$$  (3.2)

This tells us that the connection $A$ endows $E$ with the structure of a holomorphic bundle, as it happens in Donaldson theory for ASD connections and in the abelian theory.

The most general solution to the equation

$$\alpha^i \beta^j - \frac{1}{2} \delta^{ij} \alpha^k \beta^k = 0,$$  (3.3)

is $\alpha \neq 0$, $\beta = 0$ or $\alpha = 0$, $\beta \neq 0$ (with $\alpha$, $\beta$ understood as vectors). Of course we also have the solution $\alpha = \beta = 0$, which corresponds to an ASD instanton. We want to consider the first kind of solutions. Suppose $\alpha \neq 0$, $\beta = 0$. The Dirac equation for this kind of solution is simply $\overline{\partial}_A \alpha = 0$, with $\overline{\partial}_A$ the twisted Dolbeault operator on $E$. As $A$ defines a holomorphic structure on $E$, according to (3.2),
the Dirac equation simply tells us that $\alpha$ is a holomorphic section of the bundle $K^{1/2} \otimes E$. For $\alpha = 0$, $\beta \neq 0$ we have the symmetric situation, with $\beta$ a holomorphic section of the bundle $K^{1/2} \otimes \overline{E}$. Now we want to consider the second equation of (3.1). For this we will use some techniques of symplectic geometry which have been proved to be useful both in Donaldson theory [3] and in the abelian case [8]. Suppose again we are in the case $\alpha \neq 0$, $\beta = 0$. We define a symplectic structure on $\mathcal{M}_{\beta=0} = A \times \Gamma(X, K^{1/2} \otimes E)$ according to:

$$\Omega((\psi, \mu), (\theta, \nu)) = \int_X \text{Tr}(\psi \wedge \theta) \wedge \omega - \frac{i}{2} \int_X \omega \wedge \omega (\bar{\mu}^i \nu^i - \mu^i \bar{\nu}^i), \quad (3.4)$$

where $\psi, \theta$ are in $\Omega^1(X, g_E)$ and $\mu, \nu$ in $\Gamma(X, K^{1/2} \otimes E)$. This symplectic form is obviously preserved by the action of the group of gauge transformations. We consider $\Omega^4(X, g_E)$ as the dual of $\text{Lie}(\mathcal{G}) = \Omega^0(X, g_E)$, and the pairing is given by the integration over $X$ of the trace of the wedge product. A moment map for the action of the group of gauge transformations is a map,

$$m : \mathcal{M}_{\beta=0} \longrightarrow \Omega^4(X, g_E), \quad (3.5)$$

verifying:

$$\langle (dm)_{(A, \alpha)}(\psi, \mu), \phi \rangle = \Omega((\psi, \mu), C(\phi)), \quad (3.6)$$

for all $\phi \in \Omega^0(X, g_E)$, and $C$ is the map given in (2.9). The brackets denote the dual pairing. The explicit expression of this map is given by:

$$m(A, \alpha) = F^{ij} \wedge \omega + \frac{\omega \wedge \omega}{2} (\alpha^i \alpha^j - \delta^{ij} |\alpha|^2). \quad (3.7)$$

The first piece of this map is just the corresponding map for Donaldson theory, and the second piece contains the dependence on the monopole part. The property
(3.6) is easily verified from the expression for the differential of (3.7):

$$(dm)_{(A,\alpha)}(\psi, \mu) = (d_A\psi)^{ij} \wedge \omega + \frac{\omega \wedge \omega}{2} \left( \mu^i \overline{\alpha}^j + \alpha^i \overline{\mu}^j - \frac{\delta^i_j}{2}(\mu^k \overline{\alpha}^{k} + \alpha^k \overline{\mu}^k) \right). \quad (3.8)$$

The solutions of the second equation in (3.1) are precisely the zeroes of the moment map (3.7), as it happens in the abelian case. This indicates that the moduli space of solutions of the $SU(2)$ monopole equations with $\beta = 0$ can be identified with the symplectic quotient $m^{-1}(0)/\mathcal{G}$. Furthermore, under certain stability conditions this symplectic quotient can be identified with what is called the complex quotient of $\mathcal{M}_{\beta=0}$, i.e., the quotient by the complexification of the group of gauge transformations $\mathcal{G}^c$, which in this case is simply the group of $Sl(2, C)$ gauge transformations. More precisely, in order to identify the symplectic quotient with the complex quotient one must get rid in the former of the points of the moduli space in which the group $\mathcal{G}$ has a non-free action. As we have discussed in the preceding section, as far as $M \neq 0$ the group of gauge transformations acts freely, and therefore we don’t need any additional restriction when considering $\mathcal{M}_{\beta=0}$. Now recall that, because of (3.2), the connections in $A$ define holomorphic structures on the bundle $E$, and it can be seen that two connections define isomorphic holomorphic structures if and only if they are related by a complex gauge transformation. Then we can identify equivalence classes of connections under the complexified gauge group with equivalence classes of holomorphic $Sl(2, C)$ bundles.

Concerning the stability conditions for the complex quotient, although an accurate treatment requires the analysis of the gradient flow lines associated to the moment map, it seems that, when $M \neq 0$, there are no topological restrictions on the holomorphic structures of the bundles. This is already the case in the abelian theory [8] and is in contrast with Donaldson theory [23, 3], where a $\mathcal{G}^c$ orbit contains an irreducible ASD connection if and only if the holomorphic bundle $E$ verifies a certain algebro-geometric condition. Therefore, on a compact Kähler, spin manifold, the moduli space of solutions to the $SU(2)$ monopole equations has three branches: the first one corresponds to the irreducible ASD connections.
with $M = 0$, and can be identified with the equivalence classes of stable holomorphic $Sl(2, C)$ bundles $E$. The second branch corresponds to pairs consisting of an equivalence class of holomorphic $Sl(2, C)$ bundles $E$ together with a holomorphic section of $K^{1/2} \otimes E$ modulo $Sl(2, C)$ gauge transformations (the case $\alpha \neq 0$, $\beta = 0$ discussed before). The third branch is similar to the second branch, but now $\alpha = 0$, $\beta \neq 0$, and consequently we must consider instead holomorphic sections of $K^{1/2} \otimes \overline{E}$. The structure of this moduli space has obvious similarities with the abelian case and could be refined along the same lines, but it strongly suggests that the non-abelian monopole theory has a richer content than Donaldson theory, as one would expect from a highly non-trivial coupling of the topological Yang-Mills multiplet to topological matter.

4. The Topological Action.

In this section we will build the non-abelian generalization of the topological action presented in [9]. As in that case, we will use the Mathai-Quillen formalism [20]. This formalism is very well suited for our purposes since it provides a procedure to construct the action of a topological quantum field theory starting from a moduli problem formulated in purely geometrical terms. Indeed, we will apply it to the moduli problem discussed in sect. 2.

The Mathai-Quillen form is essentially an adequate representative of the Thom class of the bundle $\mathcal{E}$. As we discussed in sect. 2, when we integrate over the space $\mathcal{M}/\mathcal{G}$ the pullback of this Thom class under a section $s$ of $\mathcal{E}$ we obtain the Euler characteristic of $\mathcal{E}$. In addition, because of its localization properties, we can use this pullback to compute intersection numbers in the moduli space constituted by the zeroes of $s$. From the field-theory point of view, as the pullback of the Thom class corresponds to $\exp(-S)$, the Euler characteristic can be interpreted as the partition function of the topological field theory, and the intersection numbers as topological correlation functions. The Mathai-Quillen form is constructed making use of a connection defined on $\mathcal{E}$. For the case in which the space $\mathcal{M}$ has a $\mathcal{G}$-
invariant metric defined on it there is a natural way to construct it as follows [17]: consider on the principal bundle $\mathcal{M} \rightarrow \mathcal{M}/\mathcal{G}$ the connection defined by declaring the horizontal subspaces to be the orthogonal ones to the vertical subspaces. The latter are just the gauge orbits given by the action of the group $\mathcal{G}$. This connection on the principal bundle $\mathcal{M}$ induces a connection on the associated bundle $\mathcal{E}$ in the standard way, and this is just the connection that one needs to construct the Mathai-Quillen form.

With the help of the connection which has been introduced we are now in the position to write down the Mathai-Quillen form. We will use the Cartan model for the equivariant cohomology which gives the BRST symmetry of the theory. Hence we will deal with the Cartan model of the Mathai-Quillen form. This is an equivariant differential form of the fibre $\mathcal{F}$ which can be written as:

$$U = e^{-|x|^2} \int D\chi \exp\left(\frac{1}{4}\langle \chi, \Omega \chi \rangle + i\langle dx, \chi \rangle\right).$$

(4.1)

In this expression, $x$ denotes a (commuting) vector coordinate for the fibre $\mathcal{F}$, $\chi$ a Grassman coordinate and the bracket a $\mathcal{G}$-invariant metric on $\mathcal{F}$. $\Omega$ is the universal curvature which acts on the fibre according to the action of the group $\mathcal{G}$. Now, in order to obtain a differential form on the base space $\mathcal{M}/\mathcal{G}$ we must use the Chern-Weil homomorphism which has the effect of substituting $\Omega$ by the actual curvature on $\mathcal{M}$ and thus gives a basic differential form on $\mathcal{M} \times \mathcal{F}$. However, in the Cartan model, due to the relation between the Cartan model and the Weil model for equivariant cohomology, one needs to make an horizontal projection in order to obtain a closed form on $\mathcal{E}$. In other words, the differential form on $\mathcal{M} \times \mathcal{F}$ must be evaluated on the horizontal subspace of $\mathcal{M}$. Once we do that, we have a form on $\mathcal{E}$ which descends to a form on $\mathcal{M}/\mathcal{G}$ by simply taking the pullback by the section $\hat{s}$. This has the effect of substituting the coordinate $x$ by the section $\hat{s}$.

Let us describe in detail how to construct the connection on $\mathcal{M}$ and how to enforce the horizontal projection. The gauge orbits are given by the image of the map $C$ introduced in (2.8). Consider now the operator $R = C^\dagger C$. The connection
one-form is given by [17],
\[ \Theta = R^{-1}C^\dagger. \]  
(4.2)

As the Cartan representative acts on horizontal vectors, we can write the curvature as
\[ \Omega = d\Theta = R^{-1}dC^\dagger. \]  
(4.3)

Now, to enforce the horizontal projection we should have to integrate over the vertical degrees of freedom which amounts to an integration over the Lie group. Alternatively, we can introduce a “projection form” [19] which, besides of projecting on the horizontal direction, automatically involves the Weil homomorphism which substitutes the universal curvature by the actual curvature on the bundle (4.3). The projection form also allows to write the correlation functions on the quotient moduli space \( \mathcal{M}/\mathcal{G} \) as integrals over \( \mathcal{M} \), in such a way that we can consider the original section \( s \) instead of \( \hat{s} \). Taking into account all these facts, and after some suitable manipulations, we obtain the following expression for the Thom class of the bundle \( \mathcal{E} \):
\[
\int D\eta D\chi D\phi D\lambda \exp \left( -|s|^2 + \frac{1}{4} \langle \chi, \phi \chi \rangle + i \langle ds, \chi \rangle + i \langle dC^\dagger, \lambda \rangle - i \langle \phi, R\lambda \rangle + i \langle C^\dagger \theta, \eta \rangle \right).
\]  
(4.4)

Here, \( \phi, \lambda \) are commuting Lie algebra variables and \( \eta \) is a Grassmann one. The variables \( (P, \theta) \) (the first one is commuting and present in \( s \), the second one is Grassmann) are the usual superspace coordinates for the integration of differential forms on \( \mathcal{M} \). The bracket with the subscript \( g \) is the Cartan-Killing form of \( \text{Lie}(\mathcal{G}) \). This expression is to be understood as a differential form on \( \mathcal{M} \) which when integrated out with the measure \( DPD\theta \) gives the Euler characteristic of \( \mathcal{E} \).

This ends our brief introduction of the Mathai-Quillen formalism.

We will apply the previous formalism to the moduli problem of non-abelian monopoles on four-manifolds introduced in sect. 2. We will restrict ourselves to the gauge group \( SU(N) \) but a similar construction holds in the general case. The
operators $C$ and $C^\dagger$ have been explicitly computed in (2.9) and (2.10), respectively. The operator $R = C^\dagger C$ is easily obtained:

$$R(\phi)^{ij} = (d^*_A d_A \phi)^{ij} + \frac{1}{2}(\overline{M}^{ak} \phi^{kj} M^j_a + \overline{M}^{aj} \phi^{ik} M^k_a) - \frac{\delta^{ij}}{dR} \overline{M}^{ak} \phi^{kl} M^l_a, \quad \phi \in \Omega^0(X, \mathfrak{g}).$$  

(4.5)

The other operator involved in (4.4), $ds$, has been computed in (2.11).

In order to write the topological quantum field theory associated to the moduli problem we must indicate the field content and the topological symmetry. These are determined by the geometrical structure that we have been developing. For the moduli space we have commuting fields $P = (A, M) \in \mathcal{M} = \mathcal{A} \times \Gamma(X, S^+ \otimes E)$, with ghost number 0 and their superpartners, representing a basis of differential forms on $\mathcal{M}$, $\theta = (\psi, \mu)$, with ghost number 1. Now, we must introduce fields for the fibre which we denote by $(\chi_{\mu\nu}, v_{\dot{a}}) \in \Omega^{2, +}(X, \mathfrak{g}_E) \oplus \Gamma(X, S^- \otimes E)$, with ghost number 1. It is also useful in the construction of the action from gauge fermions to introduce auxiliary commuting fields with the same geometrical content, $(H_{\mu\nu}, h_{\dot{a}})$. The gauge symmetry makes necessary to introduce three fields in Lie($\mathcal{G}$), as we have remarked in writing (4.4). The field $\phi \in \Omega^0(X, \mathfrak{g}_E)$, with ghost number 2, is a commuting one. It roughly corresponds to the universal curvature and enters in the equivariant cohomology of $\mathcal{M}$. The fields $\lambda$ and $\eta$, also in $\Omega^0(X, \mathfrak{g}_E)$ but anticommuting and with ghost number $-2$ and $-1$, respectively, come from the projection form, as explained in [19]. The BRST cohomology of the model is:

\[
\begin{align*}
[Q, A] &= \psi, \\
\{Q, \psi\} &= d_A \phi, \\
[Q, \phi] &= 0, \\
\{Q, \chi_{\mu\nu}\} &= H_{\mu\nu}, \\
[Q, H_{\mu\nu}] &= i[\chi_{\mu\nu}, \phi], \\
[Q, \lambda] &= \eta, \\
[Q, M^i_\alpha] &= \mu^i_\alpha, \\
\{Q, \mu^i_\alpha\} &= -i\phi^{ij} M^j_\alpha, \\
\{Q, v^i_{\dot{a}}\} &= h^i_{\dot{a}}, \\
\{Q, h^i_{\dot{a}}\} &= -i\phi^{ij} v^j_{\dot{a}}, \\
\{Q, \eta\} &= i[\lambda, \phi].
\end{align*}
\]

(4.6)

This BRST gauge algebra closes up to a gauge transformation generated by $-\phi$ (recall that the group acts on the fibre with $g^{-1}$).
We are now in the position to write down the action of the theory. Let us consider first the last five terms in the exponential of the Thom class (4.4),

\[ -i\langle \phi, R\lambda \rangle_g = -i \int_X \text{Tr}(\lambda \wedge *d_A^*d_A \phi) - \frac{i}{2} \int_X e\overline{M}^\alpha\{\lambda, \phi\} M_\alpha, \]

\[ i\langle \chi, v \rangle, ds = \frac{i}{\sqrt{2}} \int_X \text{Tr}(\chi \wedge *p^+d_A \psi) - \frac{1}{\sqrt{2}} \int_X e(M_\alpha \chi^\alpha_\beta \mu_\beta - \bar{\mu}_\alpha \chi^\alpha_\beta M_\beta) \]

\[ + \frac{i}{2} \int_X e(\bar{v}^\alpha D_\alpha \mu_\alpha - \bar{\mu}_\alpha D_\alpha \psi^\alpha) + \frac{1}{2} \int_X e(M_\alpha \psi_\alpha \mu_\alpha v^\alpha - \bar{v}^\alpha \psi_\alpha \bar{\mu}_\alpha M_\alpha), \]

\[ i\langle C^\dagger(\psi, \mu), \eta \rangle_g = - \int_X \text{Tr}(\eta \wedge *d_A^* \psi) + \frac{1}{2} \int_X e(\bar{\mu}_\alpha \eta M_\alpha + \overline{M}^\alpha \eta \mu_\alpha), \]

\[ \frac{1}{4} \langle (\chi, v), (\phi, \psi) \rangle = -\frac{i}{4} \int_X \text{Tr}(\chi \wedge *[\phi, \chi]) - \frac{i}{4} \int_X e\bar{v}^\alpha \phi v_\alpha, \]

\[ i\langle dC^\dagger, \lambda \rangle_g = \int_X \text{Tr}(\lambda \wedge *[\psi, *\psi]) + \int_X e\bar{\mu}_\alpha \lambda \mu_\alpha. \]

(4.7)

The section term in (4.4) has been computed in (2.17), and the action resulting after adding to it all the terms in (4.7) constitutes the field theoretical representation of the Thom class of the bundle \( \mathcal{E} \). This action is invariant under the transformations (4.6) once the auxiliary field \( H_{\alpha\beta} \) and \( h_\alpha \) are introduced. It can be obtained in its off-shell form using the nilpotent transformations (4.6) (up to a gauge transformation) and an appropriate gauge invariant gauge fermion. This approach was first used in [24] (for a review on subsequent developments see [25]) and reformulated in the context of the Mathai-Quillen formalism in [19]. We will now construct the topological action using this last point of view. In a topological field theory with gauge symmetry there exists a localization gauge fermion which comes directly from the Cartan model representative of the Thom class (4.4) with additional auxiliary fields \( H_{\alpha\beta} \) and \( h_\alpha \). In our case, the appropriate gauge fermion turns out to be:

\[ \Psi_{loc} = -i\langle (\chi, v), s(A, M) \rangle - \frac{1}{4}\langle (\chi, v), (H, h) \rangle, \]

(4.8)
while the projection gauge fermion, which implements the horizontal projection, is,

\[ \Psi_{\text{proj}} = i \langle \lambda, C^\dagger(\psi, \mu) \rangle_g. \] (4.9)

Making use of the \( Q \)-transformations (4.6) one easily computes the localization and the projection lagrangians:

\[
\{ Q, \Psi_{\text{loc}} \} = \left\{ Q, \int_X e \left[ -i \chi_{\alpha\beta} \delta_{ji} \left( \frac{1}{\sqrt{2}} \left( F_{\alpha\beta}^{+ij} + \frac{i}{2} (M_\alpha^j M_\beta^i - \delta_{ij} M_\alpha^k M_\beta^k) \right) \right) - \frac{i}{4} H_{\alpha\beta} \right] 
\]

\[
= \int_X e \left[ -i \frac{1}{\sqrt{2}} \left( \chi_{\alpha\beta} \delta_{ji} \left( \frac{1}{\sqrt{2}} \left( F_{\alpha\beta}^{+ij} + \frac{i}{2} (M_\alpha^j M_\beta^i - \delta_{ij} M_\alpha^k M_\beta^k) \right) \right) \right] 
\]

\[
= \int_X e \left[ -i \frac{1}{\sqrt{2}} \left( \chi_{\alpha\beta} \delta_{ji} \left( \frac{1}{\sqrt{2}} \left( F_{\alpha\beta}^{+ij} + \frac{i}{2} (M_\alpha^j M_\beta^i - \delta_{ij} M_\alpha^k M_\beta^k) \right) \right) \right] 
\]

\[
= \int_X e \left[ -i \frac{1}{\sqrt{2}} \left( \chi_{\alpha\beta} \delta_{ji} \left( \frac{1}{\sqrt{2}} \left( F_{\alpha\beta}^{+ij} + \frac{i}{2} (M_\alpha^j M_\beta^i - \delta_{ij} M_\alpha^k M_\beta^k) \right) \right) \right] 
\]

\[
= \int_X e \left[ -i \frac{1}{\sqrt{2}} \left( \chi_{\alpha\beta} \delta_{ji} \left( \frac{1}{\sqrt{2}} \left( F_{\alpha\beta}^{+ij} + \frac{i}{2} (M_\alpha^j M_\beta^i - \delta_{ij} M_\alpha^k M_\beta^k) \right) \right) \right] 
\]

\[
= \int_X e \left[ -i \frac{1}{\sqrt{2}} \left( \chi_{\alpha\beta} \delta_{ji} \left( \frac{1}{\sqrt{2}} \left( F_{\alpha\beta}^{+ij} + \frac{i}{2} (M_\alpha^j M_\beta^i - \delta_{ij} M_\alpha^k M_\beta^k) \right) \right) \right] 
\]

\[
= \int_X e \left[ -i \frac{1}{\sqrt{2}} \left( \chi_{\alpha\beta} \delta_{ji} \left( \frac{1}{\sqrt{2}} \left( F_{\alpha\beta}^{+ij} + \frac{i}{2} (M_\alpha^j M_\beta^i - \delta_{ij} M_\alpha^k M_\beta^k) \right) \right) \right] 
\]

\[
= \int_X e \left[ -i \frac{1}{\sqrt{2}} \left( \chi_{\alpha\beta} \delta_{ji} \left( \frac{1}{\sqrt{2}} \left( F_{\alpha\beta}^{+ij} + \frac{i}{2} (M_\alpha^j M_\beta^i - \delta_{ij} M_\alpha^k M_\beta^k) \right) \right) \right] 
\]

\[
= \int_X e \left[ -i \frac{1}{\sqrt{2}} \left( \chi_{\alpha\beta} \delta_{ji} \left( \frac{1}{\sqrt{2}} \left( F_{\alpha\beta}^{+ij} + \frac{i}{2} (M_\alpha^j M_\beta^i - \delta_{ij} M_\alpha^k M_\beta^k) \right) \right) \right] 
\]

The sum of (4.10) and (4.11) is just the same as the sum of the terms in (4.7) plus \(-|s(A, M)|^2\) as given in (2.17) once the auxiliary fields \( H_{\alpha\beta} \) and \( h_{\dot{a}} \) have been integrated out. This is indeed the exponent appearing in the Thom class (4.4) which must be identified as minus the action, \(-S\), of the topological quantum field.
theory. After carrying out the integration of the auxiliary fields the resulting action turns out to be:

\[ S = \int X e^{\int g^{\mu\nu} D_\mu M^\alpha D_\nu M_\alpha + \frac{1}{4} R M^\alpha M_\alpha + \frac{1}{2} \text{Tr}(F^{+\alpha\beta} F_{\alpha\beta}^+) - \frac{1}{8} (M^{(\alpha T^a M^\beta)}(M_\alpha T^a M_\beta))} + \int \text{Tr}(\eta \wedge *d^*_A \psi - \frac{i}{\sqrt{2}} \chi^{\alpha\beta} (p^+(d_A \psi))_{\alpha\beta} - \frac{i}{4} \chi^{\alpha\beta} [\chi_{\alpha\beta}, \phi] + i \lambda \wedge *d^*_A d_A \phi + \lambda \wedge *[\psi, \psi]) + \int e^{(-i M^\alpha \{\phi, \lambda\} M_\alpha + \frac{1}{\sqrt{2}} (M^\alpha \chi^{\alpha\beta} \mu_\beta - \bar{\mu}_\alpha \chi^{\alpha\beta} M_\beta) - \frac{i}{2} (\bar{v}^\alpha D_{\alpha\dot{a}} \mu^\alpha - \bar{\mu}^\alpha D_{\alpha\dot{a}} \bar{v}^{\dot{a}}) - \frac{1}{2} (M^\alpha \psi_{\alpha\dot{a}} \bar{v}^{\dot{a}} - \bar{v}^{\dot{a}} \psi_{\alpha\dot{a}} M^\alpha) - \frac{1}{2} (\bar{\mu}^\alpha \eta M_\alpha + \overline{M^\alpha \eta} \mu_\alpha) + \frac{i}{4} \bar{v}^\alpha \phi v^{\dot{a}} - \bar{\mu}^\alpha \lambda \mu_\alpha)}, \]

(4.12)

where we have used (2.18) to write the term quartic in the fields \( M_\alpha \). Although this action has been computed considering an arbitrary representation of the gauge group \( SU(N) \), its form is also valid for any other gauge group. This action is invariant under the modified BRST transformations which are obtained from (4.6) after taking into account the modifications which appear once the auxiliary fields have been integrated out. It contains the standard gauge fields of a twisted \( N = 2 \) vector multiplet, or Donaldson-Witten fields, coupled to the matter fields of the twisted \( N = 2 \) hypermultiplet.

One important question is the analysis of the observables of the theory. Certainly one has the observables corresponding to ordinary Donaldson theory. These have been written down explicitly for the abelian case in [9]. For the non-abelian case these are the ones in [4] and their generalization for an arbitrary gauge group. The issue now is to study if there are some observables involving matter fields, i.e., BRST invariant operators which are not \( Q \)-exact. We have not found any.
5. Conclusions.

In this paper we have presented the non-abelian generalization of Witten’s monopole equations. These equations lead to the study of a new moduli problem which is a generalization of Donaldson theory. We have performed a first analysis of the space of solutions and it has been argued that they constitute an enlarged moduli space which might be richer than the ordinary one. In addition, the paper contains the formulation of the topological quantum field theory associated to the moduli problem corresponding to the non-abelian monopole equations. This has been carried out using the Mathai-Quillen formalism and the resulting theory contains twisted gauge and matter $N = 2$ supersymmetric multiplets.

This work opens a variety of investigations. Certainly, the moduli space of solutions should be further analyzed. Our study can be consider as a preliminary one which seems to suggests that there exist an interesting structure, but a refined analysis should be performed. Another important study which should be carried out is the analysis of the moduli problem presented in this paper from the point of view of the underlying untwisted $N = 2$ theory. In particular, it would be interesting to know if the topological quantum field theory can be analyzed along the lines of [5] for the Kähler case, or using the techniques of [7] in the general case. One could ask then if there exist a theory with a moduli problem equivalent to the one presented in this work similarly as it happens between ordinary Donaldson theory and the abelian monopole equations.

Another important aspect of the topological quantum field theory presented in this work is the study of its possible relation to string theory following the type of analysis done in [26] for the case of ordinary Donaldson theory.

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**Note added:** After this work was completed we became aware of [27] where a topological quantum field theory similar to ours is considered.
APPENDIX

In this appendix we summarize the conventions used in this paper. Basically we will describe the elements of the positive and negative chirality spin bundles $S^+$ and $S^-$ on a four-dimensional spin manifold $X$ endowed with a vierbein $e^m{\mu}$ and a spin connection $\omega_{\mu}^{\rho\nu}$. Let us begin recalling that a Dirac spinor in Euclidean four-dimensional space corresponds to $S^+ \oplus S^-$ and it is associated to a representation of dimension four of the group of rotations $SO(4)$. This representation is reducible in terms of the simplest irreducible representation of $SO(4)$: the one associated to two-component Weyl spinors. These describe locally the elements of $\Gamma(X, S^+)$ and $\Gamma(X, S^-)$ on the spin manifold $X$. The four-dimensional representation of $SO(4)$ associated to $S^+ \oplus S^-$ is built out of gamma matrices satisfying:

$$\{\gamma_m, \gamma_n\} = 2\delta_{mn}. \quad (A.1)$$

These can be chosen to be hermitian:

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_a = \begin{pmatrix} 0 & i\tau_a \\ -i\tau_a & 0 \end{pmatrix}, \quad a = 1, 2, 3, \quad (A.2)$$

where $1$ is the $2 \times 2$ unit matrix and $\tau_a, a = 1, 2, 3$ are the Pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (A.3)$$

Throughout this appendix latin indices at the beginning of the alphabet $a, b, \cdots$ will run from 1 to 3 (unless otherwise indicated), while the ones at the middle $m, n, \cdots$ will run from 0 to 3. The Pauli matrices satisfy:

$$\tau_a \tau_a = i\epsilon_{abc}\tau_c + \delta_{ab}1, \quad (A.4)$$

where $\epsilon_{abc}$ is the totally antisymmetric tensor with $\epsilon_{123} = 1$. The projection from a Dirac four-component spinor to a Weyl two-component one is carried out with
the help of the matrix $\gamma_5$ which verifies $\{\gamma_5, \gamma_m\} = 0$ and is chosen to be:

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hfill (A.5)

The projection into the positive and negative chirality spin bundles $S^+$ and $S^-$ is performed using $\frac{1}{2}(1 + \gamma_5)$ and $\frac{1}{2}(1 - \gamma_5)$ respectively.

The four-dimensional representation of $SO(4)$ associated to $S^+ \oplus S^-$ has the following matrices:

$$\gamma_{mn} = \frac{i}{4} [\gamma_m, \gamma_n],$$ \hfill (A.6)

which have been chosen to be hermitian. They satisfy the $SO(4)$ algebra:

$$[\gamma_{mn}, \gamma_{pq}] = i \left( \delta_{np} \gamma_{mq} - \delta_{nq} \gamma_{mp} + \delta_{mp} \gamma_{qn} - \delta_{mq} \gamma_{pn} \right).$$ \hfill (A.7)

Using the representation (A.2) and (A.6) one immediately obtains the two two-dimensional representations of $SO(4)$ which correspond to $S^+$ and $S^-$. Indeed, the matrices $\gamma_{mn}$ have the form:

$$\gamma_{mn} = \begin{pmatrix} \sigma_{mn} & 0 \\ 0 & \tilde{\sigma}_{mn} \end{pmatrix},$$ \hfill (A.8)

where the matrices $\sigma_{mn}$ and $\tilde{\sigma}_{mn}$ are antisymmetric in $m$ and $n$ and have the form,

$$\sigma_{0a} = \frac{1}{2} \tau_a, \quad \sigma_{ab} = -\frac{1}{2} \epsilon_{abc} \tau_c, \quad \tilde{\sigma}_{0a} = -\frac{1}{2} \tau_a, \quad \tilde{\sigma}_{ab} = -\frac{1}{2} \epsilon_{abc} \tau_c.$$ \hfill (A.9)

Certainly, from (A.7) and (A.8) follows that the matrices $\sigma_{mn}$ and $\tilde{\sigma}_{mn}$ satisfy the $SO(4)$ algebra. Furthermore, the matrices of the two sets are hermitian.
Under an infinitesimal $SO(4)$ rotation a Weyl spinor $M^\alpha$, $\alpha = 1, 2$, associated to $S^+$, transforms as:

$$\delta M^\alpha = \frac{1}{2} \epsilon_{mn} (\sigma_{mn})^\alpha_\beta M^\beta,$$

(A.10)

where $\epsilon_{mn} = -\epsilon_{nm}$ are the infinitesimal parameters of the transformation. On the other hand, a Weyl spinor $N_{\dot{\alpha}}$, $\dot{\alpha} = 1, 2$, associated to $S^-$, transforms as,

$$\delta N_{\dot{\alpha}} = \frac{1}{2} \epsilon_{mn} (\bar{\sigma}_{mn})_{\dot{\alpha}}^{\dot{\beta}} M_{\dot{\beta}}.$$

(A.11)

The Pauli matrix $(\tau_2)_{\alpha\beta}$ is an invariant tensor as one can easily verify:

$$(\sigma_{mn})_{\alpha}^{\gamma} (\tau_2)_{\gamma\beta} + (\sigma_{mn})_{\beta}^{\gamma} (\tau_2)_{\alpha\gamma} = (\sigma_{mn} \tau_2 + \tau_2 \sigma_{mn}^\top)_{\alpha\beta} = 0,$$

(A.12)

after using the fact that since

$$\tau_2 \tau_a^\top \tau_2 = -\tau_a, \quad a = 1, 2, 3,$$

(A.13)

one has,

$$\tau_2 \sigma_{mn}^\top \tau_2 = -\sigma_{mn}, \quad \tau_2 \bar{\sigma}_{mn}^\top \tau_2 = -\bar{\sigma}_{mn}.$$

(A.14)

The invariant matrix $\tau_2$ can be used to raise and lower spinor indices. Following the conventions in [28] we define:

$$C_{\alpha\beta} = (\tau_2)_{\alpha\beta}, \quad C_{\dot{\alpha}\dot{\beta}} = (\tau_2)_{\dot{\alpha}\dot{\beta}},$$

(A.15)

and their inverse tensors:

$$C^{\alpha\beta} = -(\tau_2)_{\alpha\beta}, \quad C^{\dot{\alpha}\dot{\beta}} = -(\tau_2)_{\dot{\alpha}\dot{\beta}},$$

(A.16)

so that,

$$C^{\alpha\beta} C_{\gamma\beta} = \delta_{\gamma}^\alpha, \quad C^{\dot{\alpha}\dot{\beta}} C_{\dot{\gamma}\dot{\beta}} = \delta_{\dot{\gamma}}^{\dot{\alpha}},$$

(A.17)
and therefore can be utilized to raise and lower spinor indices:

\[
M_\alpha = M^\beta C^{\beta\alpha}, \quad N^\dot{\alpha} = C^{\dot{\alpha}\dot{\beta}} N_\beta, \quad (A.18)
\]

\[
M^\alpha = C^{\alpha\beta} M_\beta, \quad N_\dot{\alpha} = N^{\dot{\beta}} C^{\dot{\beta}\dot{\alpha}}.
\]

It is useful to study how \( M^\alpha \) and \( N_\dot{\alpha} \) transform under infinitesimal \( SO(4) \) rotations. One finds in this way another two realizations of the two-dimensional representation. Using (A.10) and (A.18) one finds:

\[
\delta M_\alpha = \frac{1}{2} \epsilon_{mn}(\sigma_{mn})^{\beta\alpha} M_\beta, \quad \delta N^\dot{\alpha} = \frac{1}{2} \epsilon_{mn}(\hat{\sigma}_{mn})^{\dot{\beta}\dot{\alpha}} N_\beta, \quad (A.19)
\]

where,

\[
\sigma'_{mn} = \tau_2 \sigma_{mn} \tau_2 = -\sigma_{mn}^\top, \quad \hat{\sigma}_{mn} = \tau_2 \hat{\sigma}_{mn} \tau_2 = -\hat{\sigma}_{mn}^\top. \quad (A.20)
\]

In order to write down the Dirac equation for a Weyl spinor we need to introduce new matrices. Let us define the set of four matrices \( \sigma_m, \ m = 0, 1, 2, 3, \) as:

\[
\sigma_0 = 1, \quad \sigma_a = i \tau_2 \tau_a \tau_2 = -i \tau_a^\top, \quad a = 1, 2, 3. \quad (A.21)
\]

This is a convenient choice because on the one hand the determinant of \( P_m \sigma_n \) is,

\[
\det[P_m \sigma_n] = P_0^2 + P_1^2 + P_2^2 + P_3^2, \quad (A.22)
\]

and, on the other hand, it transforms as a vector under \( SO(4) \) rotations. Indeed, one has,

\[
\frac{i}{2} \epsilon_{mn}(\sigma_{mn})^{\alpha\beta} (\sigma_p)_{\beta\dot{\alpha}} + \frac{i}{2} \epsilon_{mn}(\hat{\sigma}_{mn})^{\dot{\alpha}\dot{\beta}} (\sigma_p)_{\alpha\dot{\beta}} = \frac{1}{2} \epsilon_{mn} \delta_{p[m}(\sigma_{n])_{\alpha\dot{\alpha}}. \quad (A.23)
\]

where \( \sigma_{mn} \) and \( \hat{\sigma}_{mn} \) are given in (A.9) and (A.20) respectively.
Let us consider the covariant derivative $D_\mu$ on the manifold $X$. Acting on an element of $\Gamma(X, S^+)$ it has the form:

$$D_\mu M_\alpha = \partial_\mu M_\alpha - \frac{i}{2} \omega_{\mu mn}^\alpha (\sigma_{mn})^\beta_\alpha M_\beta, \quad (A.24)$$

where $\omega_{\mu mn}^\alpha$ is the spin connection. Defining $D_{\alpha \dot{a}}$ as,

$$D_{\alpha \dot{a}} = (\sigma_n)_{\alpha a} e^{n\mu} D_\mu, \quad (A.25)$$

where $e^{n\mu}$ is the vierbein on $X$, the Dirac equation for $M \in \Gamma(X, S^+)$ and $N \in \Gamma(X, S^-)$ can be simply written as,

$$D_{\alpha \dot{a}} M^\alpha = 0, \quad D_{\alpha \dot{a}} N^\dot{a} = 0. \quad (A.26)$$

Explicit computations show that these equations are equivalent to

$$\frac{1}{2} e^{mn} \gamma_\mu (1 \pm \gamma_5) \Psi = 0, \quad (A.27)$$

where $\Psi \in \Gamma(X, S^+ \oplus S^-)$ is the Dirac spinor,

$$\Psi = \begin{pmatrix} M^\alpha \\ N^\dot{a} \end{pmatrix}. \quad (A.28)$$

Let us now introduce a $G$ gauge connection $A$ and let us consider that the Weyl spinors $M^i_\alpha$ realize locally an element of $\Gamma(S^+ \otimes E)$, i.e., they transform under an $G$ gauge transformation in a representation $R$:

$$\delta M^i_\alpha = i \phi^{ij} M^j_\alpha = i \phi^a (T^a)^{ij} M^j_\alpha, \quad (A.29)$$

where $T^a$, $a = 1, \cdots, N^2 - 1$ are the generators of $G$ in the representation $R$, which are traceless and chosen to be hermitian. In (A.29) $\phi^a$, $a = 1, \cdots, N^2 - 1$, denote
the infinitesimal parameters of the gauge transformation. We use the same type of indices as in (A.4) to label the group generators but there is not risk to be mistaken because their meaning will be always clear from the context. Using the gauge connection $A$ and taking (A.24) we define now the full covariant derivative,

$$D_\mu M^i_\alpha = \partial_\mu M^i_\alpha - \frac{i}{2} \omega^{mn}_\mu (\sigma_{mn})_\alpha^\beta M^i_\beta + i A^i_\mu M^i_\alpha,$$

(A.30)

and its analogue in (A.25):

$$D_\alpha \dot{\alpha} = (\sigma_n)_{\alpha\dot{\alpha}} e^{n\mu} D_\mu.$$

(A.31)

In the context under consideration the Dirac equations (A.26) become:

$$D_\alpha \dot{\alpha} M^{\alpha i} = 0, \quad D_\alpha \dot{\alpha} N^{\dot{\alpha} i} = 0.$$

(A.32)

Given an element of $\Gamma(X, S^+ \otimes E)$, $M^i_\alpha = (a^i, b^i)$ we define $\overline{M}^{\alpha i} = (a^{i*}, b^{i*})$ to be the corresponding one of $\Gamma(S^+ \otimes \overline{E})$ where $\overline{E}$ denotes the bundle associated to the representation conjugate to $R$. In this way, given $M, N \in \Gamma(X, S^+ \otimes E)$, the gauge-invariant quantity entering the metric (2.4),

$$\overline{M}^{\alpha i} N^{\alpha i} + \overline{N}^{\dot{\alpha} i} M^{\dot{\alpha} i},$$

(A.33)

is positive definite.

Acting on an element of $\Gamma(X, S^+ \otimes E)$ the covariant derivatives satisfy:

$$[D_\mu, D_\nu] M^i_\alpha = i F^{ij}_{\mu\nu} M^j_\alpha + i R^{mn}_{\mu\nu} (\sigma_{mn})^\beta_\alpha M^i_\beta,$$

(A.34)

where $F^{ij}_{\mu\nu}$ are the components of the two-form field strength:

$$F = dA + A \wedge A,$$

(A.35)
and $R_{\mu\nu}^{\ mn}$ the components of the curvature two-form,

$$R^{mn} = d\omega^{mn} + \omega^{mp} \wedge \omega^{pn},$$  \hspace{1cm} (A.36)

being $\omega^{mn}$ the spin connection one-form. The scalar curvature is defined as:

$$R = \epsilon^\mu e^\nu_{\ m} R_{\mu\nu}^{\ mn}. \hspace{1cm} (A.37)$$

The two-form $F$ can be decomposed into its self-dual and anti-self-dual parts:

$$F^\pm = \frac{1}{2} (F \pm \ast F), \hspace{1cm} (A.38)$$

or, in components, after defining,

$$F^{ij}_{\ \alpha\dot{\alpha},\beta\dot{\beta}} = (\sigma_m)_{\alpha\dot{\alpha}} (\sigma_n)_{\beta\dot{\beta}} e^{m\mu} e^{n\nu} F^{ij}_{\mu\nu} = C_{\alpha\dot{\alpha}} F^{\ -ij}_{\beta\dot{\beta}} + C_{\beta\dot{\beta}} F^{+ij}_{\alpha\dot{\alpha}}, \hspace{1cm} (A.39)$$

as,

$$F^{+ij}_{\alpha\dot{\alpha}} = (p^+(F))^{ij}_{\alpha\dot{\alpha}} = e^{m\mu} e^{n\nu} C^{\dot{\alpha}\beta} (\sigma_m)_{\alpha\dot{\alpha}} (\sigma_n)_{\beta\dot{\beta}} \frac{1}{2} (F^{ij}_{\mu\nu} + \frac{1}{2} \epsilon_{\mu\nu}^{\rho\sigma} F^{ij}_{\rho\sigma}), \hspace{1cm} (A.40)$$

$$F^{-ij}_{\dot{\alpha}\dot{\beta}} = (p^-(F))^{ij}_{\dot{\alpha}\dot{\beta}} = e^{m\mu} e^{n\nu} C^{\alpha\beta} (\sigma_m)_{\alpha\dot{\alpha}} (\sigma_n)_{\beta\dot{\beta}} \frac{1}{2} (F^{ij}_{\mu\nu} - \frac{1}{2} \epsilon_{\mu\nu}^{\rho\sigma} F^{ij}_{\rho\sigma}), \hspace{1cm} (A.40)$$

where the projector $p^\pm$ has been introduced. From their definition, the components $F^{+ij}_{\alpha\dot{\alpha}}$ and $F^{-ij}_{\dot{\alpha}\dot{\beta}}$ of the self-dual and anti-self-dual parts of $F$ are symmetric in their tangent-space indices. From (A.40) one can verify that the quantity:

$$F^{+\alpha\beta ij} F^{+ji}_{\alpha\beta} = \text{Tr}(F^{+\alpha\beta} F^{+}_{\alpha\beta}), \hspace{1cm} (A.41)$$

is positive definite.
Equation (A.34) can be rewritten in terms of the covariant derivatives (A.31) in the following way:

\[
[D_{\alpha \dot{\alpha}}, D_{\beta \dot{\beta}}] M^i_\gamma = i R^{ij}_{\alpha \dot{\alpha}, \beta \dot{\beta}} M^j_\gamma + i R^{mn}_{\alpha \dot{\alpha}, \beta \dot{\beta}} (\sigma_{mn})^\gamma_\delta M^i_\delta, \quad (A.42)
\]

where \( R^{ij}_{\alpha \dot{\alpha}, \beta \dot{\beta}} \) is given in (A.39) and,

\[
R^{mn}_{\alpha \dot{\alpha}, \beta \dot{\beta}} = (\sigma_p)_{\alpha \dot{\alpha}} (\sigma_q)_{\beta \dot{\beta}} \epsilon^{pq} \epsilon^{\mu \nu} R^{\mu \nu, mn}_{\nu \nu}. \quad (A.43)
\]

Using (A.42) and the fact that for arbitrary spinors \( M_\alpha \) and \( N_\alpha \) one has \( M_{[\alpha} N_{\beta]} = C_{\alpha \beta} M_\gamma N^\gamma \), one finds,

\[
D_{\alpha \dot{\alpha}} D_{\beta \dot{\beta}} M^{\beta i} = (g^{\mu \nu} D_\mu D_\nu - \frac{1}{4} R) M^{\beta i}_\alpha + i F^{+ij}_{\alpha \beta} M^{\beta j}_\alpha, \quad (A.44)
\]

where \( R \) is the scalar curvature (A.37).
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