Completeness of Wilson loop functionals
on the moduli space of $SL(2,C)$ and $SU(1,1)$-connections

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Abstract.
The structure of the moduli spaces $\mathcal{M} := \mathcal{A}/\mathcal{G}$ of (all, not just flat) $SL(2,C)$ and $SU(1,1)$ connections on a n-manifold is analysed. For any topology on the corresponding spaces $\mathcal{A}$ of all connections which satisfies the weak requirement of compatibility with the affine structure of $\mathcal{A}$, the moduli space $\mathcal{M}$ is shown to be non-Hausdorff. It is then shown that the Wilson loop functionals --i.e., the traces of holonomies of connections around closed loops-- are complete in the sense that they suffice to separate all separable points of $\mathcal{M}$. The methods are general enough to allow the underlying n-manifold to be topologically non-trivial and for connections to be defined on non-trivial bundles. The results have implications for canonical quantum general relativity in 4 and 3 dimensions.

1. Introduction

The structure of the moduli spaces $\mathcal{M} := \mathcal{A}/\mathcal{G}$ of connections has been studied in detail in the case when the gauge group $\mathcal{G}$ is compact and has been shown to admit the structure of an infinite dimensional manifold except for “conical singularities” at those points where the connections admit symmetries (so that the holonomy group is a proper sub-group of the full gauge group). $^1$ (See, e.g. [1].) In the non-compact case, on the other hand, relatively little seems to be known. From a physical standpoint, this was not considered to be handicap because one can restrict oneself to the compact case in realistic gauge theories. In recent years, however, general relativity in 3 and 4 dimensions has been recast as a theory of connections (see, e.g., [2,3]), and the relevant gauge groups -- $SU(1,1)$ and $SL(2,C)$ respectively-- are non-compact. It is therefore of considerable physical interest to extend the previous work and analyse the structure of the moduli spaces of corresponding connections.

The issue of completeness of the Wilson loop functionals was analysed in detail recently [4]. While for $SU(2)$-connections, these functionals separate all points of $\mathcal{M}$, for $SL(2,C)$ and $SU(1,1)$-connections, this is not the case; the Wilson loop functionals now fail to capture the full gauge invariant information in the connections. This failure can occur when the connection is reducible, i.e. only on “sets of measure zero” in $\mathcal{M}$. Nonetheless, this limitation is significant in quantization of the theory since the “missing information” can lead to physically irrelevant superselection rules [3,5].

In this Letter, we will show that the failure occurs simply because the points in question are not separable in any reasonable topology. Thus, the Wilson loop functionals

$^1$ This structure is analogous to that of Wheeler’s superspace of 3-geometries, which had been analysed by Fisher, Marsden and others already in the seventies.
are in fact “as complete as they can be.” The implications of this result to the quantization procedure are not yet fully understood because we have very little experience in quantizing systems whose configuration spaces fail to be Hausdorff. On the mathematical side, on the other hand, the ramifications of these results seem more transparent. Since non-Hausdorffness occurs at certain reducible connections, it is tempting to conjecture that in the passage from compact gauge groups to non-compact, extra care would be needed only at such connections. While in the compact case $\mathcal{M}$ fails to have a nice differential structure at these points, in the non-compact case, problems may arise already at the topological level. In the compact case, the failure occurs because the orbits in $\mathcal{A}$ of the gauge group through these connections are “thinner” than generic orbits. In the non-compact case, not only are they thinner but they may even be contained in the closure of other orbits.

In application to 4 (and 3)-dimensional general relativity, the $SL(2, \mathbb{C})$ (respectively $SU(1,1)$) connections are defined on 3 (respectively 2)-dimensional manifolds, the Cauchy surfaces. In this Letter, however, we will consider the general case and consider connections on any principal $SL(2, \mathbb{C})$ or $SU(1,1)$ bundle over an $n$-dimensional real manifold $\Sigma$. We will begin with some preliminaries, then explain the origin of the non-Hausdorff character using a trivial bundle and finally establish the main theorem in full generality.

2. Preliminaries

Standard definitions and statements about bundles and connections are available from Kobayashi and Nomizu [6] and Steenrod [7]. We denote by $\mathcal{A}$ the set of connections defined on a principal fibre bundle $P(\Sigma, G)$ with the structure group $G$ which is either $SL(2, \mathbb{C})$ or $SU(1,1)$. Following the notation introduced in [8], which has become standard in quantum general relativity, we will denote the Wilson loop functional associated with a closed loop $\alpha$ by $T_{\alpha}$. Thus, associated with a piecewise $C^1$ loop

$$\alpha : [0,1] \to \Sigma,$$

with $\alpha(0) = \alpha(1)$, we have a function on $\mathcal{A}$:

$$T_{\alpha}(A) = \frac{1}{2} Tr H(\alpha, A)$$

where $H(\alpha, A)$ is an element of $G$ assigned to $\alpha$ and $A$ by the holonomy map. (Although $H(\alpha, A)$ depends on the choice of a point in the fiber of $P(\Sigma, G)$ over $\alpha(0)$, the $Tr H(\alpha, A)$ is uniquely defined). Since $T_{\alpha}$ is invariant with respect to the group $\mathcal{G}$ of gauge transformations acting on $\mathcal{A}$, we can consider it as a function on the quotient $\mathcal{M} := A/\mathcal{G}$.

We can now specify our topological assumption. We assume that $\mathcal{A}$ is equipped with a topology compatible with the affine structure defined on the space of connections; i.e. that every line in $A$,

$$A(t) = tA_1 + (1-t)A_2, \quad A_1, A_2 \in A, \quad (1)$$

is continuous. This is a very weak assumption. In practice, one normally equips $\mathcal{A}$ with the structure of a suitable Sobolev space [1] and then our assumption is trivially satisfied. The topology on $\mathcal{M}$ is induced by this topology on $\mathcal{A}$ via the quotient construction.

The origin of the non-Hausdorff character of $\mathcal{M}$ can be seen rather easily in the case when the bundle is trivial. Let $(\tau_1, \tau_2, \tau_3)$ be a basis in $su(2)$ which is orthonormal with
respect to the scalar product given by $-\frac{1}{2} \text{Tr}$. (Thus, the $\tau_i$ are $i$ times the Pauli matrices). Next, define null basis:

$$\tau_+ := \tau_1 + i\tau_2, \quad \tau_- := \tau_1 - i\tau_2.$$  

We consider hereafter $sl(2,C)$ as a complexification of $su(2)$ and $su(1,1)$ as a real sub-algebra of $sl(2,C)$ generated by $(\tau_+, \tau_-, i\tau_3)$, and extend this identification to the level of groups. Consider a connection which (when pulled down by some global section) is given by the following (Lie algebra)-valued 1-form

$$A = A^+\tau_+ + A^3\tau_3$$  

$A^+$ and $A^3$ being arbitrary complex 1-forms on $\Sigma$. The gauge orbit passing through $A$ includes a line

$$A(\lambda) = e^{-2\lambda}A^+\tau_+ + A^3\tau_3,$$

which is the image of $A$ under the action of the 1-dimensional subgroup of $SU(1,1)$, represented in this gauge by the constant $SU(1,1)$-valued functions

$$g_\lambda := e^{i\lambda\tau_3},$$  

where the real $\lambda$ is a parameter in the subgroup. But in the limit, we have:

$$\lambda \to \infty, \quad A(\lambda) \to A^3\tau_3.$$  

It therefore follows that for every continuous and gauge invariant function $f$ on $A$, we must have:

$$f(A^+\tau_+ + A^3\tau_3) = f(A^3\tau_3).$$  

Note that the connections $A^+\tau_+ + A^3\tau_3$ and $A^3\tau_3$ have distinct holonomy groups and therefore define distinct points of $\mathcal{M}$. Eq.(4) implies that these points can not be separated; $\mathcal{M}$ is not Hausdorff. This, incidentally, can be regarded as a “topological explanation” of the fact that the loop variables $T_\alpha$ in quantum general relativity are insensitive to the term proportional to $\tau_+$ if the connection has the form (2).

### 3. Main Result

Our aim now is to show that the set of all the functions $T_\alpha$ separates all the separable points of $\mathcal{M}$. Let us begin by fixing the notation. Denote by $L$ the set of piecewise $C^1$ loops in $M$. Next, given a connection $A \in \mathcal{A}$ we will denote its holonomy group by $G_H(A)$ and define its degeneracy, $\text{Deg}(A)$, as follows:

$$\text{Deg}(A) := \{A' \in \mathcal{A} \mid \text{for every } \alpha \in \mathcal{L}, \quad T_\alpha(A') = T_\alpha(A)\}.$$  

We will let $A\mathcal{G}$ stand for the orbit in $\mathcal{A}$ of the (local) gauge group $\mathcal{G}$ which contains $A$. Note that, since every $T_\alpha$ is a gauge invariant function on $\mathcal{A}$, $\text{Deg}(A)$ contains the entire orbit $A\mathcal{G}$. Finally, two sub-groups of $SL(2,C)$ (respectively $SU(1,1)$) will play an important role in what follows. First is the group of null rotations to be denoted by $G(\pm, 3)$. This is the group generated by the Lie algebra of complex (respectively, real) linear combinations of $(\tau_+, \tau_3)$. Similarly, we will denote by $G'(\pm)$ the group generated by the Lie algebra of complex (real) multiples of $\tau_+$ and by $G(3)$ the group generated by the Lie algebra of complex (real) multiples of $\tau_3$. 

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The main result can be stated as follows:

**Theorem** Suppose that $A_1, A_2 \in \mathcal{A}$ and

$$T_\alpha(A_1) = T_\alpha(A_2)$$

for every loop $\alpha \in L$. Then, for every continuous and gauge invariant function $f$ defined on $\mathcal{A}$, we have:

$$f(A_1) = f(A_2).$$

**Proof:** The proof consists of three steps which we extract in the form of lemmas stated below. The key issue is: i) whether there exist connections $A$ for which $AG$ is smaller than $\text{Deg}(A)$; and, if this happens, ii) whether the point of $\mathcal{M}$ defined by $A$ is non-Hausdorff, i.e., whether the closure $\overline{AG}$ of $AG$ contains other gauge orbits $A_0G$.

**Lemma 1** The property $AG < \text{Deg}(A)$ holds if and only if the holonomy group $G_H(A)$ of $A$ is a subgroup of the group of null rotations $G(+, 3)$.

**Lemma 2** If the holonomy group $G_H(A)$ of $A$ is a subgroup of $G(+, 3)$, then there exists a unique gauge orbit $A_0G \subset \text{Deg}(A)$ such that $G_H(A_0) \subset G(3)$.

**Lemma 3** Suppose that the holonomy group $G_H(A)$ of a connection $A \in \mathcal{A}$ is a subgroup of $G(+, 3)$. Then, in the closure $\overline{AG}$ of the orbit $AG$, there is a connection $A_0$ such that $G_H(A_0) \subset G(3)$ and $A_0 \in \text{Deg}(A)$.

It follows from the above lemmas that if $T_\alpha$ fail to separate a point of $\mathcal{M}$, i.e., if there exists $A \in \mathcal{A}$ such that $AG < \text{Deg}(A)$, then there is a unique gauge orbit $A_0G$ in $\text{Deg}(A)$ which is contained in the closure $\overline{AG}$ of $AG$. Therefore, for any $A_1, A_2 \in \text{Deg}(A)$, we have:

$$\overline{A_1G} \cap \overline{A_2G} \neq \emptyset,$$

because the intersection contains the connection $A_0$. Since a gauge invariant and continuous function $f$ on $\mathcal{A}$ is constant on the closed of orbits, it is necessarily true that $f(A_1) = f(A_2)$. 

**proof of Lemma 1:** The analysis of the invertibility of the mapping

$$H(\cdot, A) \to T_\alpha(A)$$

for a connection $A$ which has a connected holonomy group has been performed in [4]. It was shown there that, unless $G_H(A) \subset G(+, 3)$, we can reconstruct the element $H(\alpha, A)$ of $G$ provided that we know the value $T_\beta(A)$ for every loop $\beta \in L$. Thus, to establish the Lemma, we need only consider the disconnected subgroups of $SL(2, C)$ that can arise as holonomy groups. These were classified by Jacobson and Romano [9]. The only subgroup of $SL(2, C)$ which is not contained in $G(+, 3)$ is that denoted in [9] by $G(3, Z_2)$. This is the union of two connected components: $G(3)$ and $G(3) \circ \tau_2$ where $\tau_2$ is now regarded as an element of $SL(2, C)$. But if the holonomy mapping takes values in this group then there exists a loop $\alpha_1$ such that

$$T_{\alpha_1} = 0, \quad T_{\alpha_1 \circ \alpha_1} = -1$$
(Actually, the first equality above implies the second one.) Thus we can identify the holonomy group. But then we know that in some gauge, every $H(\alpha, A)$ is either diagonal or antidiagonal. Moreover, modulo $G(3)$ gauge transformations, $H(\alpha_1, A)$ is just $\tau_2$. Finally, from values of $T(A)$ taken on suitable products of loops, we can easily recover $H(\alpha, A)$, whence, for such a connection $A$, $\text{Deg}(A) = A\mathbb{G}$.

**Proof of Lemma 2:** Suppose that the holonomy group of a connection $A'$ is a subgroup of $G(+, 3)$. Then we can find another $A$ gauge equivalent to $A'$ such that the holonomy map of $A$ takes values in $G(+, 3)$ and has the form

$$H_\alpha(A) = \cos\theta_\alpha(A) + \tau_3\sin\theta_\alpha(A) + \tau_+\phi_\alpha(A)$$

where $\theta_\alpha$ and $\phi_\alpha$ are complex-valued functions of $A$ ($\theta_\alpha$ not necessarily continuous). We define a map

$$\tilde{H} : L \ni \alpha \to H_\alpha(A) - \tau_+\phi_\alpha(A) \in G(3).$$

It not difficult to check that $\tilde{H}$ satisfies all the conditions [10] sufficient for the existence of a connection $A_0$ such that $\tilde{H}(\alpha)$ coincides with the holonomy mapping $H(\alpha, A_0)$. Furthermore, $A_0 \in \text{Deg}(A)$, since by (5) for every loop $\alpha$

$$T_\alpha(A_0) = T_\alpha(A).$$

This establishes the existence. The uniqueness of a $G(3)$ connection satisfying (6) follows from the fact that, up to gauge transformations, $A_0$ can be completely reconstructed from $T_\alpha$'s.

**Proof of Lemma 3:** The idea of the proof is to find a one parameter subgroup of gauge transformations analogous to (3), allowing, however, for the bundle to be non-trivial. Now, there exists an open covering $\{V_I\}$ on $\Sigma$ and local sections

$$s_I : V_I \to P,$$

such that

$$A_I := s_I \ast A = A^3_I\tau_3 + A^+_I\tau_+,$$

which means that locally defined $A_I$’s take values in the Lie algebra of $G(+, 3)$. Moreover, every $G(+, 3)$ principal bundle over $\Sigma$ is reducible to a $U(1)$ principal bundle because, topologically, $G(+, 3)/U(1) \equiv \mathbb{R}^3$ (Rendall [11], Steenrod [7]). Therefore, we can choose the sections $s_I$ in such a way that the transition functions $a_{IJ}$, given by $s_I a_{IJ} = s_J$ take values in $U(1)$. Therefore, the part of $A$ in (7) proportional to $\tau_3$ itself defines a connection $A_0$ on $P$, s.t.

$$s_I^*A_0 := A^3_I\tau_3.$$ (8)

We can now find a 1-parameter family of automorphisms on the bundle $P$ which, in the limit as the parameter tends to infinity, squeezes $A$ to $A_0$. Let

$$\psi_\lambda(x) := e^{i\lambda\tau_3}$$

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where $\lambda$ is a real constant. By using the sections $s_I$ we lift $\psi_\lambda$ to a well defined constant function on the holonomy bundle of $A$. Next, we determine $\psi_\lambda$ at any point of $P$ by the condition that $\psi_\lambda(pg) = g^{-1}\psi(p)g$. Hence, $\psi_\lambda$ defines an automorphism of $P$. In addition, applying $\psi_\lambda$ to $A$ we obtain

$$
\psi_\lambda^*A = A_0 + e^{-2\lambda}(A - A_0).
$$

By taking the limit $\lambda \to \infty$ we see that

$$
A_0 \in \overline{A_G}.
$$

On the other hand, we see from (7) and (8) and from the fact that the transition functions are $U(1)$ valued that $T_\alpha(A_0) = T_\alpha(A)$ for any loop $\alpha$. Thus, we have:

$$
A_0 \in \text{Deg}(A) \quad \text{and} \quad G_H(A_0) \subset G(3),
$$

(whence $A_0G$ is the unique gauge orbit of Lemma 2). This completes the proof of Lemma 3 and hence of the Theorem.

Remarks:

1. Note that, in the above analysis, we have not assumed that the Wilson loop functionals $T_\alpha$ are continuous on $\mathcal{M}$. If they are as is the case if one uses a standard topology [1] on $\mathcal{A}$ the Theorem has a stronger implication: $\mathcal{M}$ is non-Hausdorff only at those points which can not be separated by the $T_\alpha$. Furthermore, the arguments used in the proof provide a classification of these points. We have a natural projection $G(+,3) \to G(3)$ and the $G(3)$ part of $H(A,.)$ coincides with $H(A_0,.)$ which in turn characterizes $\text{Deg}(A)$.

2. It is important to note the sense in which the Wilson loop functionals have been shown to be complete: they suffice to separate all separable points of $\mathcal{M}$. In the physics literature, one often assumes completeness in a different sense, namely that “all (relevant) gauge invariant functionals of connections can be expressed as a limit of polynomials of the Wilson loop functionals.” While for finite dimensional manifolds, the two senses of completeness are essentially equivalent, in the case of $\mathcal{M}$, we do not have a corresponding result.

3. In 4-dimensional general relativity, one can associate 4-metrics with points of $\mathcal{M}$. Remarkably, the time evolution given by Einstein’s equations preserves the holonomy group and hence, in particular, the degeneracy of a point in $\mathcal{M}$. (See [4] for the treatment of the vacuum case and [9] for the case with a non-vanishing cosmological constant and topological nontrivialities.) The non-separable points of $\mathcal{M}$ correspond to metrics which admit a covariantly constant spinor direction [4]. The Einstein Equations in this class of metrics has been solved completely (see [12] for the vacuum case and [13] for the case with a cosmological constant). These metrics are, in a certain sense, the $(-+++)$ analogs of the Kähler metrics with Euclidean signature [14].
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