Critical Dimension for Stable Self-gravitating Stars in $AdS$

Zhong-Hua Li
Department of Physics, China West Normal University, Nanchong 637002, China

Rong-Gen Cai
Key Laboratory of Frontiers in Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, P.O.Box 2735, Beijing 100190, China

Abstract

We study the self-gravitating stars with a linear equation of state, $P = a\rho$, in AdS space, where $a$ is a constant parameter. There exists a critical dimension, beyond which the stars are always stable with any central energy density; below which there exists a maximal mass configuration for a certain central energy density and when the central energy density continues to increase, the configuration becomes unstable. We find that the critical dimension depends on the parameter $a$, it runs from $d = 11.1429$ to $10.1291$ as $a$ varies from $a = 0$ to $1$. The lowest integer dimension for a dynamically stable self-gravitating configuration should be $d = 12$ for any $a \in [0, 1]$ rather than $d = 11$, the latter is the case of self-gravitating radiation configurations in AdS space.
I. INTRODUCTION

Self-gravitating configuration is a subject of long-standing interest in general relativity since its thermodynamic behavior is quite different from that of a usual thermal system without gravity, due to the attractive long-range, unshielded nature of gravitational potential. For example, it is well known that the canonical ensemble is not defined in asymptotically flat space. This is because having thermal radiation at constant temperature at infinity is not compatible with asymptotic flatness. One can avoid this problem by enclosing the system in a box, which is unphysical, or by working in anti-de Sitter (AdS) space which needs not any unphysical perfectly reflecting walls at finite radius, the rising gravitational potential in AdS space, plus natural boundary conditions at infinity, acts to confine whatever is inside. On the other hand, due to the conjecture of AdS/CFT correspondence [1], which says that string theory/M theory on an AdS space (times a compact manifold) is dual to a strong coupling conformal field theory (CFT) residing on the boundary of the AdS space, over the past decade, a lot of attention has been focused on AdS space and relevant physics. Further, Witten [2] argued that thermodynamics of black holes in AdS space can be identified to that of dual strong coupling CFTs. Therefore one could study thermodynamics and phase structure of strong coupling CFTs by investigating thermodynamics and phase structure of AdS black holes. It is well-known that thermodynamics of AdS black holes is quite different from that of their counterparts in asymptotically flat space and that there is a Hawking-Page phase transition between large black hole and thermal radiation in AdS space [3]. Therefore it is also of great interest to study self-gravitating configurations in AdS space and their thermodynamics.

Self-gravitating radiation gas has been investigated thoroughly. Sorkin, Wald and Zhang [4] have studied the equilibrium configurations of self-gravitating radiation in a spherical box of radius \( R \) in asymptotically flat space. It was found that for locally stable configuration, the total gravitational mass of radiation obeys the inequality \( M < \mu_{\text{max}} R \), where \( \mu_{\text{max}} = 0.246 \). In AdS space, Page and Philips [5] examined the self-gravitating configuration of radiation in four dimensional space-time. The configuration can be labeled as its mass, entropy and temperature versus the central energy density. They found that there exist locally stable radiation configurations all the way up to a maximum red-shifted temperature, above which there are no solutions; there is also a maximum mass and maximum entropy configuration occurring at a higher central density than the maximal temperature configuration. Beyond their peaks the temperature, mass and entropy undergo an infinite series of damped oscillations, which indicates the configurations in this regime are unstable. The self-gravitating radiation in five dimensional AdS space has been studied in [6] (see also [7]) with similar
conclusions. Recently, Vaganov and Hammersley independently discussed the self-gravitating radiation configurations in higher dimensional AdS spaces. They found that in the case of $4 \leq d \leq 10$, the situation is qualitatively similar to the case in four dimensions, while $d \geq 11$, the oscillation behavior disappears. Namely, there is a critical dimension, $d_{c,ads} = 11$ (very close, but not exact), beyond which, the temperature, mass and entropy of the self-gravitating configuration are monotonic functions of the central energy density, asymptoting to their maxima as the central density goes to infinity. The equilibrium configurations of self-gravitating radiation gas in AdS space are quite different from those of their counterparts in asymptotically flat space.

Recently Chavanis investigated spherically symmetric equilibrium configurations of relativistic stars with a linear equation of state $P = a \rho$ in flat space. The equation of state $P = a \rho$ says the pressure $P$ is proportional to the energy density $\rho$, where $a$ is a ratio coefficient. Theoretically, different coefficients correspond to different stars, although it is impossible to define names for all the stars corresponding to each coefficient. The speed of sound is given by $(dP/d\rho)^2 c = a^2 c$, where $c$ is the speed of light. Thus causality requires $a^2 \leq 1$. As a result, $0 \leq a \leq 1$ is considered only. One could identify relativistic stars by an equation of state with their different ratio coefficients. In the Newtonian limit $a \to 0$, it returns the classical isothermal equation of state. On the other hand, the case with $a = 1/(d-1)$, where $d$ is the dimension of space-time, corresponds to a gas of self-gravitating radiation; the core of neutron stars or so called “photon stars” where the pressure is entirely due to radiation. A gas of baryons interacting through a vector meson field for the case of $a = 1$ is called the “stiffest” star. Chavanis found that the structure of the system is highly dependent on the dimensionality of space-time and that the oscillations in the mass-central density profile disappear above a critical dimension $d_{c,flat}(a)$ depending on the coefficient $a$. Above this dimension, the equilibrium configurations are stable for any central density, contrary to the case $d < d_{c,flat}$. For Newtonian isothermal stars ($a \to 0$), the critical dimension $d_{c,flat}(0) = 11$. For the stiffest stars ($a = 1$), $d_{c,flat}(1) = 10$ and for a self-gravitating radiation($a = 1/(d-1)$), $d_{c,flat} = 1 + 9.96404372...$ very close to 11. The oscillations exist for any $a \in [0,1]$ when $d \leq 10$ and they cease to exist for any $a \in [0,1]$ when $d \geq 11$.

In a previous paper, we have studied self-gravitating radiation configurations with plane symmetry in AdS space and found that the situation is quite different from the one in spherically symmetric AdS space. In the present paper we continue this study and consider the self-gravitating configurations with a linear equation of state $P = a \rho$ in higher $(d \geq 4)$-dimensional spherically symmetric AdS space, in order to see the dependence of the critical dimension on the parameter $a$ and to see whether there is any essential difference
between the configurations in flat space and the configurations in AdS space.

The organization of the paper is as follows. In the next section, we give a general formulism to describe the relativistic stars with a linear equation of state \( P = a\rho \) in AdS space. The numerical results are given in Sec. III. The Sec. IV is devoted to the conclusions.

II. RELATIVISTIC STARS WITH LINEAR EQUATION OF STATE IN ADS SPACE

Consider a \( d \)-dimensional asymptotically AdS space with metric

\[
ds^2 = -e^{2\delta(r)}h(r)dt^2 + h^{-1}(r)dr^2 + r^2\gamma_{ij}dx^i dx^j,
\]

where \( \delta \) and \( h \) are two functions of the radial coordinate \( r \), and \( \gamma_{ij} \) is the metric of a \((d-2)\)-dimensional Einstein manifold with constant scalar curvature \((d-2)(d-3)\). We take the gauge \( \lim_{r\to\infty}\delta(r) = 0 \), and rewrite the metric function \( h(r) \) as

\[
h(r) = 1 + \frac{r^2}{l^2} - \frac{16\pi Gm(r)}{(d-2)\Omega r^{d-3}},
\]

where \( l \) denotes the radius of the AdS space with cosmological constant \( \Lambda = -(d-1)(d-2)/2l^2 \), \( \Omega \) is the volume of the Einstein manifold, and \( m(r) \) is the mass function of the solution. In this gauge, the total gravitational mass of the solution is just

\[
M = \lim_{r\to\infty} m(r).
\]

The Einstein field equations with the cosmological constant and energy-momentum tensor \( T_{\mu\nu} \) are

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{(d-1)(d-2)}{2l^2}g_{\mu\nu} = 8\pi G T_{\mu\nu}.
\]

The metric function \( \delta(r) \) and the mass function \( m(r) \) satisfy the following equations

\[
\delta'(r) = -\frac{8\pi Gr}{(d-2)h(r)}(T^t_t - T^r_r),
\]

\[
m'(r) = -\Omega r^{d-2}T^t_t.
\]

The stress-energy tensor of perfect fluid is \( T_{\mu\nu} = (\rho + P)U_\mu U_\nu + Pg_{\mu\nu} \), where \( U_\mu \) is the four-velocity of the fluid. For self-gravitating radiation, the equation of state obeys \( P = \rho/(d-1) \). For linear relativistic stars, \( P = a\rho \) with \( a \in [0,1] \). Thus the equations in (5) reduce to

\[
\delta'(r) = -\frac{8\pi Gr}{(d-2)h(r)}(1 + a)\rho,
\]

\[
m'(r) = \Omega r^{d-2}\rho.
\]
In this paper we are particularly interested in the relation between the total mass of configuration and the central energy density. To integrate (6) and (7), we make a scaling transformation as follows,

\[ r \rightarrow l r, \quad \rho \rightarrow l^{-2} \rho, \quad m(r) \rightarrow l^{d-3} m(r), \]  

so that \( r, \rho \) and \( m \) become dimensionless. In the numerical integration, we will adopt the units \( 8\pi G = 1 \) and \( l = 1 \), and rescale the mass function as

\[ 16\pi G m(r)/(d-2)\Omega \rightarrow m(r). \]

In that case, the gravitational “mass” \( M \) in the plots in the next section in fact is the gravitational mass density, \( 16\pi G M/(d-2)\Omega \), of corresponding relativistic star configurations. From the conservation of the energy-momentum tensor, one can derive the energy density \( \rho(r) \) satisfies the following equation

\[
\frac{d\rho}{dr} = \frac{(a+1)\rho r}{2a} \left( \frac{d-3}{r^2} - \frac{d-1}{h(r)} - \frac{d-3}{r^2 h(r)} - \frac{2\rho}{(d-2)h(r)} \right).
\]

Further, note that the equations (7) and (9) are singular at \( r = 0 \). To avoid this, in the numerical calculations, we will start the integration from \( r = \epsilon = 10^{-5} \) to \( r = L = 100 \). Obviously, the accuracy of the numerical calculations depends on the values of \( \epsilon \) and \( L \).

III. NUMERICAL RESULTS

To be more clear to discuss all kinds of critical dimensions of relativistic stars with a linear equation of state in AdS space, let us first revisit the self-gravitating radiation case. In \cite{9}, Hammersley found that in the case of \( 4 \leq d \leq 10 \), there exist locally stable radiation configurations all the way up to a maximum mass, beyond
FIG. 2: The case of $a = 0.5$ in AdS space: The mass of relativistic star configurations versus the central energy density from dimension $d = 4$ (bottom curve) to $d = 12$ (top curve).

their peaks the curves undergo an infinite series of damped oscillations, which indicates the configurations in this regime are unstable. However, with $d \geq 11$, the oscillation behavior disappears. The configurations turn to be monotonic functions of the central energy density, asymptoting to their maxima as the central density goes to infinity. Namely, there is a critical dimension. By numerical analysis, Hammersley [9] found a semi-empirical model

$$\log \rho_c \approx 0.50d + \frac{5.75}{\sqrt{11.0 - d}} - 2.20,$$  \hspace{1cm} (10)

which gives a critical dimension $d_{c,\text{ads}} = 11$, and a similar conclusion was also reached by Vaganov [8] independently. We reproduce the result in Fig. 1.

Now we turn to the case of relativistic stars with a linear equation of state $P = a\rho$. Due to the requirement of causality, we consider the cases with $0 \leq a \leq 1$. In Fig. 2 we plot the mass of relativistic stars configurations versus the central energy density from dimension $d = 4$ to $d = 12$ for the case of $a = 0.5$. Note that in Fig. 2 we rescale the mass $M$ with scale $10^{8-2d(d-3)^2}$ in each dimension for better displaying all curves in one figure. We notice that for a certain value of $a$, the configurations of different dimensions from 4 to 12 are similar to the case of self-gravitating radiation stars. For the case of lower dimensions, there exist obvious oscillations which can be seen from Fig 3. As for the higher dimensions, the oscillations become weaker and weaker, and finally beyond some critical dimensions, configurations become monotonic functions of the central energy density. Fig. 4 is the case of $d = 11$. Note that in Fig. 3 and Fig. 4 each curve corresponds to each relativistic star with coefficient $a$ which undergoes a continuous variation from 0.01 to 1.0 with step 0.1, and also we rescale the mass density $M$ to $M/M_{\max}$ in each case of coefficient $a$ for better comparing those different cases.

As a matter of fact, it is not really accurate to determine the critical dimensions only by the naked eyes to
FIG. 3: The case of \( d = 4 \) in AdS space: The mass of relativistic star configurations versus the central energy density from the coefficient \( a = 1 \) (left curve) to \( a = 0.01 \) (right curve).

FIG. 4: The case of \( d = 11 \) in AdS space: The mass of relativistic star configurations versus the central energy density from the coefficient \( a = 1 \) (left curve) to \( a = 0.01 \) (right curve).

Watch these curves. To be more accurate to determine the critical dimensions, we take the numerical analysis approach. In Fig. 3 one can see clearly that for each curve there exists a saturation point, \( \log \rho_c \) (the red dot in the figure), which is the location of the first local maximum. The saturation point moves towards the right side as the dimension \( d \) increases. Beyond the saturation points, the curves undergo an infinite series of damped oscillations. When the dimension increases, the oscillations become weaker and weaker, while the saturation point \( \log \rho_c \) becomes larger and larger, and finally, beyond a critical dimension \( d_{c,\text{ads}} \), the saturation point \( \log \rho_c \) goes to infinity. So we can determine the critical dimension by analyzing the variation of the saturation points.

Some data of saturation points \( \log \rho_c \) are listed in Table I. Obviously, \( \log \rho_c \) depends on the coefficient \( a \).
FIG. 5: The self-gravitating radiation in AdS space: The mass of the self-gravitating radiation configuration versus the central energy density from dimension \( d = 4 \) (top curve) to \( d = 10 \) (bottom curve). The red dots are the saturation points \( \log \rho_c \).

and dimension \( d \), so namely \( \log \rho_c(a, d) \). We use a formula like \([10]\) and obtain the critical dimensions \( d_c(a) \) for different coefficient \( a \), which are listed in Table [II]. As can be seen from Table [II] in the case of \( d = 11 \) some data singularity appears. So we only use data from \( d = 4 \) to \( d = 10 \) to fit.

| TABLE I: Saturation Points \( \log \rho_c(a, d) \) |
|---|---|---|---|---|---|---|---|---|
| \( a \) | \( d=4 \) | \( d=5 \) | \( d=6 \) | \( d=7 \) | \( d=8 \) | \( d=9 \) | \( d=10 \) | \( d=11 \) |
| 0.01 | 3.90 | 3.60 | 3.90 | 4.20 | 5.10 | 6.00 | 8.10 | 21.00 |
| 0.1 | 3.00 | 3.00 | 3.30 | 3.90 | 4.80 | 6.00 | 8.40 | 21.90 |
| 0.2 | 2.40 | 2.40 | 3.00 | 3.90 | 4.80 | 6.30 | 8.70 | \( \infty \) |
| 0.3 | 1.80 | 2.10 | 3.00 | 3.90 | 4.80 | 6.30 | 9.00 | \( \infty \) |
| 0.4 | 1.50 | 2.10 | 3.00 | 3.90 | 5.10 | 6.60 | 9.60 | \( \infty \) |
| 0.5 | 1.20 | 2.10 | 3.00 | 3.90 | 5.10 | 6.60 | 10.20 | \( \infty \) |
| 0.6 | 1.20 | 2.10 | 3.00 | 4.20 | 5.10 | 6.90 | 11.10 | \( \infty \) |
| 0.7 | 1.20 | 2.10 | 3.00 | 4.20 | 5.40 | 7.20 | 12.30 | \( \infty \) |
| 0.8 | 1.20 | 2.10 | 3.30 | 4.20 | 5.40 | 7.50 | 14.10 | \( \infty \) |
| 0.9 | 1.20 | 2.10 | 3.30 | 4.20 | 5.70 | 7.50 | 15.60 | \( \infty \) |
| 1.0 | 1.20 | 2.10 | 3.30 | 4.20 | 5.70 | 7.80 | 16.50 | \( \infty \) |

| TABLE II: Critical Dimensions \( d_{c.\text{ads}}(a) \) |
|---|---|---|---|---|---|---|---|
| \( a \) | 0.01 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| \( d_{c.\text{ads}}(a) \) | 11.1061 | 11.0952 | 10.9989 | 10.9254 | 10.7465 | 10.6333 |
| \( a \) | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| \( d_{c.\text{ads}}(a) \) | 10.4928 | 10.3707 | 10.2602 | 10.2023 | 10.1763 |

| TABLE III: Numerical Fitting Results |
|---|---|---|---|---|---|
| \( d_{c.\text{ads}}(a) \) | linear | second-order | third-order | average |
| low bound | 10.1063 | 10.1048 | 10.1763 | 10.1291 |
| up bound | 11.1601 | 11.1591 | 11.1097 | 11.1429 |
FIG. 6: The critical dimensions $d_{c,ads}(a)$ versus the coefficient $a$. The blue hollow dots are the scattered data of the critical dimensions $d_{c,ads}(a)$. The red dashed curve is the case of linear fitting. The blue dashed curve is the case of the third-order fitting. The black thick curve is the case of average fitting.

In Fig. 6 we plot the scattered diagram of the critical dimensions $d_{c,ads}(a)$ versus the coefficient $a$. It can be seen that the value of $d_{c,ads}(a)$ becomes smaller and smaller as the value of the coefficient $a$ goes from 0 to 1. So $d_{c,ads}(a)$ can be considered to be a monotonic function of the coefficient $a$. The red dashed line in Fig. 6 is the result of a linear numerical fitting for these scattered data. It could be seen that for all kinds of relativistic stars with a linear equation of state $P = a \rho$ ($a \in [0, 1]$), the critical dimensions $d_{c,ads}(a) \in [10.1063, 11.1601]$. Obviously, the lower and upper bounds of $d_{c,ads}(a)$ for this kind of fitting are a little wider because these scattered data are not really linear related. To obtain more accurate results, we need to adopt non-linear fitting which could better fit the scattered data. The results of the second and third-order fitting are listed in Table III. The blue dashed curve in Fig. 6 is the case of the third-order fitting. Note that the curve of the second-order fitting is not plotted in Fig. 6 because it is overlapped with the most of the linear fitting curve. It can be seen from Fig. 6 that the curve of the third-order fitting can better fit the scattered data. Namely, in AdS space for all kinds of relativistic stars with a linear equation of state $P = a \rho$ ($a \in [0, 1]$) the critical dimensions $d_{c,ads}(a) \in [10.1763, 11.1097]$. The minimum for the stiffest star in AdS space is $d_{c,ads}(1) = 10.1763$ and the maximum for the Newtonian isothermal star is $d_{c,ads}(0) = 11.1097$. In Fig. 7 we plot the saturation points $\log \rho_c$ versus the dimension $d$ for each relativistic star (corresponding to each coefficient $a$ ) from $a = 0.1$ (right blue curve) to $a = 1.0$ (left red curve) in the range $9.4 \leq d \leq 11.4$. The low and up bounds of the critical dimensions $d_{c,ads}(a) \in [10.1763, 11.1097]$ are also plotted in this figure (the blue
FIG. 7: The saturation points \( \rho_c \) versus the dimension \( d \) from \( a = 0.1 \) (right blue curve) to \( a = 1.0 \) (left red curve). The blue dashed curves are the low and up bounds of the critical dimensions \( d_{c,\text{ads}}(a) \in [10.1763, 11.1097] \), respectively. The black thick curves are the low and up bounds of the critical dimensions \( d_{c,\text{ads.aver}}(a) \in [10.1291, 11.1429] \), respectively.

dashed curves). One can find that for the stiffest stars \( (a = 1) \), the left red curve increases steeply and finally up to a critical dimension \( d = 10.1763 \). However, for Newtonian isothermal stars \( a \to 0 \), the right blue curve changes more slowly, up to the critical dimension \( d = 11.1097 \).

By an analysis of asymptotic behavior, Chavanis [10] found that the critical dimension runs from \( d = 10 \) to 11 when the coefficient \( a \) decreases from \( a = 1 \) to 0 in flat space. Namely the difference of dimension \( \Delta d_{\text{flat}} = d_{c,\text{flat}}(0) - d_{c,\text{flat}}(1) = 1 \). In our case, namely in AdS space, the critical dimension runs from \( d = 11.1097 \) to 10.1763 with \( \Delta d_{\text{ads}} = d_{c,\text{ads}}(0) - d_{c,\text{ads}}(1) = 0.9334 \) when \( a \) varies from 0 to 1. If taking the averaged values, then one has \( \overline{\Delta d}_{\text{ads}} = d_{c,\text{ads.aver}}(0) - d_{c,\text{ads.aver}}(1) = 11.1429 - 10.1291 = 1.0138 \). There exists some difference between the flat and AdS cases. Therefore we guess that for all kinds of relativistic stars with a linear equation of state regardless of in AdS and flat spaces, the critical dimensions \( d_{c,\text{ads}}(a) \) and \( d_{c,\text{flat}}(a) \) vary just within 1 dimension when \( a \) changes from 0 to 1, and the inaccuracy between the two cases is very small, within 1.38%.

IV. CONCLUSION

There exists a critical dimension for self-gravitating configurations in general relativity, beyond which the configurations with any central energy density are always stable; below which there exists a maximal mass
configuration for a certain central energy density, when the central energy density increases, the configuration becomes unstable. In this paper we studied the self-gravitating configurations (relativistic stars) with a linear equation of state $P = a\rho$ in AdS space, where $a$ is a constant parameter within $a \in [0, 1]$. We found that the critical dimension depends on the parameter $a$, it runs from $d = 11.1097$ to 10.1763 in the third order fitting as $a$ varies from $a = 0$ to 1. The result is a little different from the case in flat space. It runs from $d = 11.1429$ to 10.1291 if one takes the averaged fitting. In that case, it is very close to the case in flat space within the inaccuracy 1.38%. Therefore it is of interest to study the self-gravitating configurations in de Sitter space and to see whether there exist any differences among the three cases.

In [8] and [9], it was found that the critical dimension of self-gravitating radiation configurations is $d = 11$. Combing the results obtained in [11] and in the present paper, we see that in fact, the dimension $d = 11$ might not have any other special physical meaning, except for the stability divide. Note that the fact that no special happens for thermodynamics of AdS Schwarzschild black holes in $d \geq 4$, it is of interest to understand the meaning of the existence of the critical dimension in the AdS/CFT correspondence.

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