ON THE CAUCHY PROBLEMS ASSOCIATED TO A ZK-KP-TYPE FAMILY EQUATIONS WITH A TRANSVERSAL FRACTIONAL DISPERSION

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Abstract. In this paper we examine the well-posedness and ill-posedness of the Cauchy problems associated with a family of equations of ZK-KP-type
\[
\begin{align*}
    u_t &= u_{xxx} - \mathcal{H} D_x^\alpha u_{yy} + uu_x, \\
    u(0) &= \psi \in Z
\end{align*}
\]
in anisotropic Sobolev spaces, where $1 \leq \alpha \leq 1$, $\mathcal{H}$ is the Hilbert transform and $D_x^\alpha$ is the fractional derivative, both with respect to $x$.

INTRODUCTION

Nonlinear evolution equations play an important role in different areas of science and engineering. Some of them are worth mentioning: fluid mechanics, plasma physics, fiber optics, solid state physics, chemical kinetics, chemical physics and geochemistry, among others. From the study of their solutions, an attempt is made to understand the effects of dispersion, diffusion, reaction and convection associated with

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the models described by them. For example, the Korteweg-de Vries (KdV) equation
\[ u_t = u_{xxx} + uu_x \quad (x, t) \in \mathbb{R}^2, \] (1)
which models the behavior of water waves in shallow channels, has solitary waves as solutions that behave like particles, which is why Kruskal and Zabusky called them solitons in their 1965 work (37). These solitons are stable, in the sense that if a solution of the KdV equation (equation (1)) that differs very little in shape from soliton-type solutions, at the beginning, its shape will maintain an aspect that will differ very little from the shape of a soliton-type solution through time (see [4] and [6]); in fact, these solutions eventually take the form of solitons (see [31]). From a practical point of view, the notion of soliton stability guarantees that, with meticulous care, in the laboratory we will be able to reproduce these phenomena, first observed by J. Scott Russell in 1834.

The Benjamin-Bona-Mahony (BBM) equation
\[ u_t + uu_x - u_{xxx} = 0, \] (2)
was introduced in [5] with the intention to modeling the propagation of long waves of small amplitude, where the dispersion effect is purely nonlinear. The way in which this was obtained, it was pursued to arrive at an equation equivalent to the equation KdV (1). It is interesting to note that despite this intention, from a purely mathematical point of view, these equations present significant and interesting differences.

Other one-dimensional equations are, one, introduced independently by Benjamin in [3] and Ono in [30],
\[ u_t + \mathcal{H} u_{xx} + uu_x = 0. \] (3)
which models the internal waves into deep stratified fluids, where \( \mathcal{H} \) is the Hilbert transform. The another, is the regularized Benjamin-Ono (rBO)
\[ u_t + u_x + uu_x + \mathcal{H} u_{xt} = 0 \] (4)
where \( u = u(x, t) \) is a real function, with \( x, t \in \mathbb{R} \). This equation is a model for wave time evolution with large ridges at the interface between two immissible fluids.

There are two-dimensional versions that extend the equations previously mentioned. In the case of the KdV equation we have the Kadomtsev-Petviashvilli (KP) equation, see [10],
\[ (u_t + auu_x + u_{xxx})_x \pm u_{yy} = 0, \] (5)
that describes waves in thin films of high surface tension. Another one is the Zakharov-Kuznetsov (ZK) equation, see [38],

$$u_t = (u_{xx} + u_{yy})_x + uu_x,$$

(6)

which arises in the study of the dynamics of geophysical fluids in isotropic sets (means in which the characteristics of the bodies do not depend on direction) and ionic acoustic waves in magnetic plasmas.

As a two-dimensional extension of the Benjamin-Ono equation we consider the following family of equations

$$u_t + u^p u_x + \mathcal{H}(u_{xx} + \alpha u_{yy})_x - \gamma u_{yy} = 0 \quad p \in \mathbb{N}.$$  

(7)

This is the model of weakly nonlinear dispersive long wave motion in a two-fluid system, where the interface is capillarized and the bottom fluid is infinitely deep (see [1], [2] and [18]). Another one version that has received some attention in recent literature is the ZK-BO equation,

$$u_t = (\mathcal{H} u_x + u_{yy})_x + uu_x.$$  

(8)

To finish this introduction we will mention the equations that present a more general type of dispersion that include as particular cases those mentioned above. A case of these is the integro differential equation of Whitham

$$u_t + \alpha u u_x + k * u_x = 0.$$  

(9)

This was introduced by Whitman in [36] to model the breakage of dispersive waves in water. It is clear that if $k = \delta + \delta''$ or $k = \text{v.p.} \frac{1}{x}$ we have the aforementioned KdV and BO equations. There is also what Lannes and Saut called the fractional dispersion equation KdV, fKdV (see [22])

$$u_t + \alpha uu_x + \partial_x D^\alpha u = 0,$$  

(10)

where $D = \sqrt{-\partial_x^2}$. In the papers [8], [19], [24] and [25] it is established the relationship between the parameter $\alpha$ and the arising of singularities or the global existence of solutions in their energy space, as well as how this behavior is related to the existence and stability of the solitary waves associated with these equations. It should be said that in this direction there are interesting open problems, although they can really be very difficult.

At this time we also think it is important to mention the work of Kenig, Martel and Robbiano ([16]) that demonstrates the blow-up of solutions in the energy space of the generalized scattering equation BO (a slightly different version of the fKdV)

$$u_t + |u|^{2\alpha} u_x + \partial_x D^\alpha u = 0,$$  

(11)
when the initial condition is “larger” than the “shape” of the solitary wave associated with this equation.

In the two-dimensional case we have the equation introduced by Lannes in [21] to model long waves of small amplitude in weakly transverse regimes

\[
\frac{\partial u}{\partial t} + \mu \frac{3}{2} u \frac{\partial u}{\partial x} + c_{ww}(\sqrt{\mu} |D^\alpha|) \left(1 + \frac{D_x^2}{D_y^2}\right)^{\frac{1}{2}} u = 0, \tag{12}
\]

where

\[
c_{ww}(k) = \left(1 + \beta k^2 \frac{\tanh(k)}{k}\right)^{\frac{1}{2}} \quad \text{and} \quad |D^\alpha| = \sqrt{D_x^2 + \mu D_y^2}.
\]

In [22] the relation between the equations KP and FDKP is established. In fact, they highlight the difference in the dispersive character with respect to the parameter \(\beta\). Furthermore, by limiting the parameter \(\mu\), the equations KPII or KPI are obtained when \(\beta = 0\) or \(\beta > 0\) respectively. An interesting conjecture here is to regard the existence of solitary waves for values \(\beta\) greater than \(1/3\).

In this paper we consider the ZK-KP type Cauchy problem

\[
\begin{cases}
\frac{\partial u}{\partial t} = u_{xxx} - \mathcal{H} D_x^\alpha u_{yy} + uu_x, \\
u(0) = \psi \in Z
\end{cases} \tag{13}
\]

for \(-1 \leq \alpha \leq 1\), where \(\mathcal{H}\) denotes the Hilbert transform in the \(x\) variable defined by

\[
\mathcal{H}(f) = \text{p.v.} \frac{1}{\pi} \int \frac{f(y)}{x-y} \, dy \quad f \in H^\alpha(\mathbb{R}),
\]

for each \(f \in \mathcal{S}\), \(D_x^\alpha\) is the homogeneous fractional derivative in \(x\) variable defined by

\[
\widehat{D_x^\alpha f}(\xi, \eta) = |\xi|^{\alpha} \hat{f}(\xi, \eta),
\]

and \(Z\) is one of the Sobolev spaces \(X^{s_1, s_2}, \hat{X}^{s_1, s_2}, Y^{s_1, s_2}\) and \(\hat{Y}^{s_1, s_2}\), that we will specify in the notations.

The cases \(\alpha = 1\) and \(\alpha = -1\) are the Cauchy problems corresponding to the very popular ZK and KPI equations, respectively, that were mentioned earlier. If \(u(x, y, t)\) is a solution to (13), then \(u_{\lambda}\) given by

\[
u_{\lambda}(x, y, t) = \lambda^2 u(\lambda x, \lambda^\frac{3-\alpha}{2} y, \lambda^3 t)
\]

is also a solution to (13) and

\[
\|u_{\lambda}(t)\|_{\dot{H}^{s_1, s_2}(\mathbb{R}^2)} = \lambda^{s_1 + \frac{3-\alpha}{4}} \|u(t)\|_{\dot{H}^{s_1, s_2}(\mathbb{R}^2)}
\]
with $s_2 = \frac{2\alpha}{2-\alpha}$. This suggests that the local well-posedness could be guaranteed in $H^{s_1,\frac{2s_1}{3-\alpha}}(\mathbb{R}^2)$ for $s_1 \geq -\frac{3+\alpha}{4}$.

In this work we propose to show the local well-posedness of the Cauchy problem (13) in the Sobolev spaces $Z$ mentioned above. For this purpose, we use Kato’s theory for quasilinear equations and the ideas introduced by Kenig [15] for the KP-I equation and developed by Linares, Pilod and Saut in [26] for the f-KPI and f-KPII equations. Specifically, it is making the use of the Strichartz estimate provided by the group generated by the homogeneous linear equation associated with (13), making use of energy estimates. We will also make some ill-posedness observations of this equation for $\alpha < 0$. For this we will use the ideas developed by Molinet, Saut and Tzvetkov in [29]. More precisely, we will show that the flow associated with the solutions of (13) is not of class $C^2$. This, in particular, implies that it cannot be applied the Picard iteration method to get a solution to the integral equation obtained by the Duhamel principle applied to (13).

**Notation**

1. $S(\mathbb{R}^2) = S$ denotes the Schwartz space and $S'(\mathbb{R}^2) = S'$ denotes its topological dual vector space, the tempered distributions.
2. $H^s(\mathbb{R}^2) = H^s$ is the $s$th-order Sobolev space.
3. For a variable or an operator $u$, we denote by $\langle u \rangle$ the expression $(1 + u^2)^{\frac{1}{4}}$.
4. For $s_1, s_2 \in \mathbb{R}$, the anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2)$ is defined by

$$H^{s_1,s_2}(\mathbb{R}^2) = \left\{ f \in S' \left| \int_{\mathbb{R}^2} (\langle \xi \rangle^{2s_1} + \langle \eta \rangle^{2s_2}) |\hat{f}(\xi, \eta)|^2 d\xi d\eta < \infty \right. \right\}.$$ 

The norm in this space is given by

$$\|f\|_{H^{s_1,s_2}(\mathbb{R}^2)} = \sqrt{\int_{\mathbb{R}^2} (\langle \xi \rangle^{2s_1} + \langle \eta \rangle^{2s_2}) |\hat{f}(\xi, \eta)|^2 d\xi d\eta},$$

for all $f$ in this space. When there is no risk of confusion, we denote this space by $H^{s_1,s_2}$. Observe that $H^{s,s} = H^s$, and the immediately above norm is equivalent to the usually given in the literature.
(5) \( D_x^s, D_y^s, J_x^s, J_y^s \) and \( J^s \) denotes the operators defined, via Fourier transform, by
\[
\hat{D_x^s f} = |\xi|^s \hat{f}, \\
\hat{D_y^s f} = |\eta|^s \hat{f}, \\
\hat{J_x^s f} = (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}, \\
\hat{J_y^s f} = (1 + |\eta|^2)^{\frac{s}{2}} \hat{f}
\]
for any \( f \in S'({\mathbb{R}}^2) \).
(6) For \( s_1, s_2 \geq 0 \), we denote by \( X^{s_1, s_2}({\mathbb{R}}^2) = X^{s_1, s_2} \) the space
\[
X^{s_1, s_2}({\mathbb{R}}^2) = \{ f \in H^{s_1, s_2} \mid \partial_x^{-1} f \in H^{s_1, s_2} \}.
\]
The norm in this space is given by
\[
\| f \|_{X^{s_1, s_2}}^2 = \| f \|_{H^{s_1, s_2}}^2 + \| \partial_x^{-1} f \|_{H^{s_1, s_2}}^2.
\]
(7) For \( s_1, s_2 \geq 0 \), we denote by \( \hat{X}^{s_1, s_2}({\mathbb{R}}^2) = \hat{X}^{s_1, s_2} \) the space
\[
\hat{X}^{s_1, s_2}({\mathbb{R}}^2) = \{ f \in H^{s_1, s_2} \mid \partial_x^{-1} f \in L^2 \}.
\]
The norm in this space is
\[
\| f \|_{\hat{X}^{s_1, s_2}}^2 = \| f \|_{H^{s_1, s_2}}^2 + \| \partial_x^{-1} f \|_{L^2}^2.
\]
(8) For \( s_1, s_2 \geq 0 \), we denote by \( X_\alpha^{s_1, s_2}({\mathbb{R}}^2) = X_\alpha^{s_1, s_2} \) the space
\[
X_\alpha^{s_1, s_2}({\mathbb{R}}^2) = \{ f \in H^{s_1, s_2} \mid \partial_x^{-\alpha} f \in L^2 \}.
\]
The norm in this space is given by
\[
\| f \|_{X_\alpha^{s_1, s_2}}^2 = \| f \|_{H^{s_1, s_2}}^2 + \| \partial_x^{-\alpha} f \|_{H^{s_1, s_2}}^2.
\]
(9) For \( s_1, s_2 \geq 0 \), we denote by \( Y^{s_1, s_2}({\mathbb{R}}^2) = Y^{s_1, s_2} \) the space
\[
Y^{s_1, s_2}({\mathbb{R}}^2) = \{ f \in H^{s_1, s_2} \mid \partial_x^{-1} \partial_y f \in H^{s_1, s_2} \}.
\]
The norm in this space is
\[
\| f \|_{Y^{s_1, s_2}}^2 = \| f \|_{H^{s_1, s_2}}^2 + \| \partial_x^{-1} \partial_y f \|_{H^{s_1, s_2}}^2.
\]
(10) For \( s_1, s_2 \geq 0 \), we denote by \( \hat{Y}^{s_1, s_2}({\mathbb{R}}^2) = \hat{Y}^{s_1, s_2} \) the space
\[
\hat{Y}^{s_1, s_2}({\mathbb{R}}^2) = \{ f \in H^{s_1, s_2} \mid \partial_x^{-1} \partial_y f \in L^2 \}.
\]
The norm in this space is given by
\[
\| f \|_{\hat{Y}^{s_1, s_2}}^2 = \| f \|_{H^{s_1, s_2}}^2 + \| \partial_x^{-1} \partial_y f \|_{L^2}^2.
\]
(11) We define \( H^\infty({\mathbb{R}}^2) = \bigcap_{s_1, s_2 \geq 0} H^{s_1, s_2}({\mathbb{R}}^2) \). Analogously \( X^\infty, \hat{X}^\infty, Y^\infty \) and \( \hat{Y}^\infty \).
1. Preliminaries

We start this section of preliminaries by observing that the spaces $X^{s_1,s_2}$, $\hat{X}^{s_1,s_2}$, $Y^{s_1,s_2}$, and $\hat{Y}^{s_1,s_2}$ are Hilbert spaces. Thanks to the next lemma, each of these spaces is dense in $L^2$.

**Lemma 1.1.** The space $\partial_x S$ is dense in $L^2$. In general, it is dense also in $H^{s_1,s_2}$, $X^{s_1,s_2}$, $\hat{X}^{s_1,s_2}$, $Y^{s_1,s_2}$, and $\hat{Y}^{s_1,s_2}$.

*Proof.* Take a nonnegative function $\phi \in C^\infty$ defined on the real numbers, identically zero on the interval $[-1/2,1/2]$ and identically 1 out of $[-1,1]$. For any $\psi \in S$ we define $\psi_\lambda$ using the equation

$$
\hat{\psi}_\lambda(\xi,\eta) = \phi(\lambda \xi) \hat{\psi}(\xi,\eta),
$$

for all $(\xi,\eta) \in \mathbb{R}^2$. The Plancherel theorem allows us show that $\psi_\lambda$ converges to $\psi$ in $L^2$, as $\lambda \to \infty$. The same argument allows us to show that $\partial_x S$ is dense in any of the spaces mentioned in the lemma statement. □

By a duality argument, we can conclude that $L^2$ is densely contained in the dual spaces of any of the spaces mentioned in the lemma.

As a consequence of the previously discussed, we can extend the operator $\partial_x^3 - \mathcal{K} D_x^\alpha \partial_y^2$ to the entire $L^2$, with image in the $X^3$ dual, $(X^3)^*$. Indeed, $-\partial_x^3 + \mathcal{K} D_x^\alpha \partial_y^2$ is bounded from $X^3$ to $L^2$. So its adjoint operator is bounded from $L^2$ to $(X^3)^*$. Thanks to the Fourier transform, it can be seen that this adjoint operator is an extension of $\partial_x^3 - \mathcal{K} D_x^\alpha \partial_y^2$ to all $L^2$.

As a corollary of above, we have the following lemma.

**Lemma 1.2.** Let $W_\alpha$ be the unitary group of operators generated by the operator $\partial_x^3 - \mathcal{K} D_x^\alpha \partial_y^2$ and let $f$ and $u$ be continuous functions from an open interval $I$ to $L^2$. Then

$$
u = W_\alpha(t) \psi + \int_0^t W_\alpha(t-t') f(t') \, dt'$$

if, and only if $u$ has continuous derivative on $I$ with values in $(X^3)^*$, the dual space $X^3$, and

$$
\partial_t u = \partial_x^3 u - \mathcal{K} D_x^\alpha \partial_y^2 u + f. \quad (14)
$$

This lemma gives sense to the well-posedness results in $H^{s_1,s_2}$, that we shall enunciate later, when $\alpha$ is negative.

The next result is a technical lemma that we use later.
Lemma 1.3. Let us assume that \(u\) and \(f\) are continuous functions in the \(L^2\) space and satisfy (14) in the sense described there. Then,

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 = (u, f).
\] (15)

Proof. Assume that \(u_\lambda = e^{\lambda(\Delta - D_x^{-1})} u\) and \(f_\lambda\) is defined in the same way. \(u_\lambda\) and \(f_\lambda\) are in \(X^\infty\) and they are uniformly convergent, as \(\lambda\) tends to 0, on closed intervals to \(u\) and \(f\) in \(L^2\). Also, they satisfy the equation (14). From the antisimmetry of the operator \(\mathcal{H} D_x^\alpha \mathcal{D}_y^\beta\), it is easy to see that \(u_\lambda\) and \(f_\lambda\) satisfy (15). When we make \(\lambda\) tend to 0, we get the lemma. □

Next we shall state a set of results about the properties of the spaces with we work in this paper. Maybe one the most known of these results is the Sobolev lemma. Here we present a version for the \(H^{s_1, s_2}(\mathbb{R}^2)\) spaces.

Lemma 1.4 (Sobolev). Let \(s_1\) and \(s_2\) be positive real numbers such that \(\frac{1}{s_1} + \frac{1}{s_2} < 2\). Then, \(H^{s_1, s_2}(\mathbb{R}^2) \subset C_\mathcal{C}(\mathbb{R}^2)\) (the set of continuous functions on \(\mathbb{R}^2\) vanishing at infinity), with continuous embedding.

Proof. See [33]. □

Lemma 1.5. Let \(1 \leq p < q \leq \infty\) and \(f \in L^p \cap L^q\). Then \(f \in L^r\) for \(r = \theta p + (1 - \theta)q\), where \(\theta \in (0, 1)\), and we have

\[
\|f\|_{L^r} \leq \|f\|_{L^p}^{\theta p} \|f\|_{L^q}^{(1 - \theta)q}.
\]

Proof. The proof is immediate consequence of the Hölder inequality. □

Lemma 1.6. If \(s \in (0, n/2)\), then \(H^s(\mathbb{R}^n)\) is a continuous embedding in \(L^p(\mathbb{R}^n)\), with \(p = \frac{2n}{n - 2s}\), i.e. \(s = n\left(\frac{1}{2} - \frac{1}{p}\right)\). Furthermore, for \(f \in H^s(\mathbb{R}^n)\), \(s \in (0, n/2)\)

\[
\|f\|_{L^p(\mathbb{R}^n)} \leq c_{n, s} \|D^s f\|_{L^2(\mathbb{R}^n)} \leq c \|f\|_{H^s(\mathbb{R}^n)}
\]

where

\[
D^s f = (-\Delta)^{s/2} = (|\xi|^s \hat{f})^\vee.
\]

Proof. See the Linares and Ponce book [27], page 48. □

Lemma 1.7. Let \(s_1, s_2 \in \mathbb{R}\) and assume that \(D^{s_1} f \in L^p(\mathbb{R})\) and \(D^{s_2} f \in L^q(\mathbb{R})\). Then, for all \(\theta \in [0, 1]\), \(D^s f \in L^r\), and

\[
\|D^s f\|_{L^r(\mathbb{R})} \leq C_\theta \|D^{s_1} f\|_{L^p(\mathbb{R})} \|D^{s_2} f\|_{L^q(\mathbb{R})}^{1 - \theta}
\]

where \(\theta = \frac{s_2 - s}{s_2 - s_1}\) and \(\frac{1}{r} = \frac{\theta}{p} + \frac{1 - \theta}{q}\).
Lemma 1.12. Let

$$\| [J^s, f] g \|_{L^p(\mathbb{R}^n)} \lesssim \| \partial^s f \|_{L^\infty(\mathbb{R}^n)} \| J^{s-1} g \|_{L^p(\mathbb{R}^n)} + \| J^s f \|_{L^p(\mathbb{R}^n)} \| g \|_{L^\infty(\mathbb{R}^n)},$$

(16)

for all $f$ and $g \in \mathcal{S}(\mathbb{R}^n)$.

The next estimate was proved by Kato and Ponce in [14] and it will be very useful later in this work.

Lemma 1.18. For $s > 0$ and $1 < p < \infty$, we have

$$\| [J^s, f] g \|_{L^p(\mathbb{R}^n)} \lesssim \| \partial^s f \|_{L^\infty(\mathbb{R}^n)} \| J^{s-1} g \|_{L^p(\mathbb{R}^n)} + \| J^s f \|_{L^p(\mathbb{R}^n)} \| g \|_{L^{\infty(\mathbb{R}^n)}},$$

(16)

for all $f$ and $g \in \mathcal{S}(\mathbb{R}^n)$.

\[\| \partial_x^s g \|_{L^2(\mathbb{R})} \lesssim \| \partial_x g \|_{L^\infty(\mathbb{R})} \| \partial_x^{s-1} f \|_{L^2(\mathbb{R})} + \| \partial_x^s g \|_{L^2(\mathbb{R})} \| f \|_{L^\infty(\mathbb{R})},\]

(18)

The following estimate for the commutator operator can be seen in [27] page 51.

Lemma 1.10. For $s > 0$, we have

$$\| [\partial_x^s, g] f \|_{L^2(\mathbb{R})} \lesssim \| \partial_x g \|_{L^\infty(\mathbb{R})} \| \partial_x^{s-1} f \|_{L^2(\mathbb{R})} + \| \partial_x^s g \|_{L^2(\mathbb{R})} \| f \|_{L^\infty(\mathbb{R})},$$

(18)

The next result is the Leibniz rule for fractional derivatives and it was proved by Kenig, Ponce and Vega in [17].

Lemma 1.11. For $\alpha \in (0, 1)$, we have

$$\| D_x^\alpha (f g) \|_{L^p(\mathbb{R})} \lesssim \| D_x^\alpha (f) \|_{L^p(\mathbb{R})} \| g \|_{L^q(\mathbb{R})} + \| D_x^\alpha g \|_{L^p(\mathbb{R})} \| f \|_{L^q(\mathbb{R})},$$

(19)

where $1 < p_1, p_2, q_1, q_2 \leq \infty$ and satisfy $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{p}$.\]

Lemma 1.12. Let $-1 \leq \alpha \leq 1$ and $0 < p \leq \frac{8}{1-\alpha}$.

$$\| f \|_{L^{p+2}(\mathbb{R}^2)} \lesssim \| f \|_{L^{p+2}(\mathbb{R}^2)} \| \partial_x^\alpha f \|_{L^{p+2}(\mathbb{R}^2)} \| D_x^{1-\alpha} \|_{L^{p+2}(\mathbb{R}^2)} \| D_y^\alpha \|_{L^{p+2}(\mathbb{R}^2)} \| f \|_{L^{p+2}(\mathbb{R}^2)}.$$\]

Proof. First, let us prove the lemma for $p = p^* = \frac{8}{1-\alpha}$. From Lemmas 1.6 and 1.7 we have

$$\| f \|_{L^{p^*+2}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |f(x, y)|^{p^*+2} dx dy \]

$$= \int_{\mathbb{R}} \| f(\cdot, y) \|_{L^{p^*+2}(\mathbb{R}^2)} dx dy \]

$$\leq C \int_{\mathbb{R}} \| D_y^{1/2} f(\cdot, y) \|_{L^{p^*+2}(\mathbb{R}^2)}^{p^*+2} dy \]

$$\leq C \int_{\mathbb{R}} \| \partial_x f(\cdot, y) \|_{L^{p^*+2}(\mathbb{R}^2)}^{p^*+2} dy \]

$$\leq C \| \partial_x f \|_{L^{p^*+2}(\mathbb{R}^2)}^{p^*+2} \sup_{y \in \mathbb{R}} \| D_x^{p^*+2} f(x, y) \|_{L^{p^*+2}(\mathbb{R}^2)}^{p^*+2}.$$

(20)
On the other hand, for all \( y \in \mathbb{R} \),
\[
\left\| D_x^{\frac{n}{2}-1} f(\cdot, y) \right\|_{L_x^2}^2 = \int_{\mathbb{R}} \left| D_x^{\frac{n}{2}-1} f(x, y) \right|^2 \, dx \\
= 2 \int_{\mathbb{R}} \int_{-\infty}^{\infty} D_x^{\frac{n}{2}} f(x, \eta) D_x^{\frac{1}{2}} \partial_y f(x, \eta) \, d\eta \, dx \\
= 2 \int_{-\infty}^{\infty} \int_{\mathbb{R}} D_x f(x, \eta) D_x^{\frac{1}{2}} \partial_y f(x, \eta) \, dx \, d\eta \\
\leq 2 \left\| \partial_x f(\cdot, \eta) \right\|_{L_x^2} \left\| D_x^{\frac{1}{2}} \partial_y f(\cdot, \eta) \right\|_{L_x^2} \, d\eta \\
\leq 2 \left\| \partial_x f \right\|_{L_x^{2}(\mathbb{R}^2)} \left\| D_x^{\frac{1}{2}} \partial_y f \right\|_{L_x^{2}(\mathbb{R}^2)}.
\]
Therefore,
\[
\left\| f \right\|_{L_x^{p+2}(\mathbb{R}^2)} \leq c \left\| \partial_x f \right\|_{L_x^{\frac{p^n(\alpha+\rho)}{1+\rho}}(\mathbb{R}^2)} \left\| D_x^{\frac{1}{2}} \partial_y f \right\|_{L_x^{2}(\mathbb{R}^2)}.
\]

The inequality for \( 0 < p < p^* \) follows immediately from Lemma 1.5 and the last inequality. \( \square \)

1.1. **Kato’s theory.** We will make a brief presentation of Kato’s theory described in [11]. With this it can be showed the well-posedness of the Cauchy problems associated to linear and quasilinear evolution equations.

1.1.1. **Linear case.** Suppose that \( X \) and \( Y \) are reflexives Banach spaces with \( Y \subseteq X \) in a dense and continuous way, and let \( \{A(t)\}_{t \in [0,T]} \) be a operators family such that

(1) \( A(t) \in G(X, 1, \beta) \). In other words, \( -A(t) \) generates a \( C_0 \)-semigroup such that
\[
\left\| e^{-sA(t)} \right\| \leq e^{\beta s},
\]
for all \( s \in [0, \infty) \).

(2) There exists an isomorphism \( S : Y \to X \) such that \( SA(t)S^{-1} = A(t) + B(t) \), where \( B(t) \in B(X) \), for \( 0 \leq t \leq T \), \( t \to B(t)x \) is strongly measurable, for each \( x \in X \), and \( t \to \left\| B(t) \right\|_X \) is integrable in \( [0, T] \).

(3) \( Y \subseteq D(A(t)) \), for \( 0 \leq t \leq T \), and \( t \to A(t) \) is strongly continuous from \( [0, T] \) to \( B(Y, X) \).

**Theorem 1.13.** Under the above conditions, there exists a operators family \( \{U(t, s)\}_{0 \leq s \leq t \leq T} \) such that:

(1) \( U \) is strongly continuous from \( \Delta \to B(X) \), where \( \Delta = \{(t, s) : 0 \leq s \leq t \leq T \} \).
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(2) \( U(t, s)U(s, r) = U(t, r) \) for \( (t, s) \in \Delta \), and \( U(s, s) = I \).

(3) \( U(t, s)Y \subset Y \) and \( U \) is strongly continuous from \( \Delta \to B(Y) \).

(4) \( \frac{dU(t, s)}{dt} = -A(t)U(t, s), \frac{dU(t, s)}{ds} = U(t, s)A(s) \), in the strong sense in \( B(X, Y) \) space and are strongly continuous from \( \Delta \to B(X, Y) \).

The operators family \( \{U(t, s)\}_{0 \leq s \leq t \leq T} \) in the previous theorem is called the evolution operators associated to \( A(t) \). An immediate consequence from the last theorem is, for \( \varphi \in Y \), \( u(t) = U(t, s)\varphi \) is solution to the Cauchy problem

\[
\frac{du}{dt} + A(t)u = 0 \quad \text{for} \quad s \leq t \leq T, \\
\quad u(s) = \varphi.
\]

Moreover, if \( f \in C([0, T]; X) \cap L^1([0, T]; Y) \), then

\[
u(t) = U(t, 0)\varphi + \int_0^t U(t, s)f(s)ds
\]

if and only if \( u \in C([0, T]; Y) \cap C^1((0, T); X) \) and

\[
\frac{du}{dt} + A(t)u = f(t) \quad \text{for} \quad 0 \leq t \leq T,
\]

\[u(0) = \varphi.\]

1.1.2. Quasilinear Case. Let \( X \) and \( Y \) be reflexives Banach spaces, \( Y \subset X \), with dense and continuous embedding. Let us consider the following problem

\[
\partial_t u + A(t, u)u = f(t, u) \in X, \quad 0 < t,
\]

\[u(0) = u_0 \in Y, \quad (23)\]

where, for each \( t \), \( A(t, u) \) is a linear operator from \( Y \) to \( X \) and \( f(t, u) \) is a function from \( \mathbb{R} \times Y \) in \( X \). Let us also consider the next conditions:

(X) There exists an isometric isomorphism \( S \) from \( Y \) to \( X \).

There exist \( T_0 > 0 \) and \( W \) an open ball with \( w_0 \) as center such that:

(A1) For each \( (t, y) \in [0, T_0] \times W \), the linear operator \( A(t, y) \) belongs to \( G(X, 1, \beta) \), where \( \beta \) is a positive real number. In the other words, \(-A(t, y)\) generate a \( C_0 \) semigroup such that

\[
\|e^{-sA(t,y)}\|_{B(X)} \leq e^{\beta s}, \quad \text{for} \quad s \in [0, \infty).
\]

Note that if \( X \) is a Hilbert space, \( A \in G(X, 1, \beta) \) if, and only if,

a) \( \langle Ay, y \rangle_X \geq -\beta \|y\|_X^2 \) for all \( y \in D(A) \),

b) \( (A + \lambda) \) is onto for all \( \lambda > \beta \).
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(See [13] or [32])

(A2) For each \((t, y) \in [0, T_0] \times W\) the operator \(B(t, y) = [S, A(t, y)]S^{-1} \in \mathcal{B}(X)\) and is uniformly bounded, i.e., there exists \(\lambda_1 > 0\) such that

\[
\|B(t, y)\|_{\mathcal{B}(X)} \leq \lambda_1 \quad \text{for all } (t, y) \in [0, T_0] \times W,
\]

Furthermore, for some \(\mu_1 > 0\), we have that, for all \(y, z \in W\),

\[
\|B(t, y) - B(t, z)\|_{\mathcal{B}(X)} \leq \mu_1 \|y - z\|_Y.
\]

(A3) \(Y \subseteq D(A(t, y))\), for each \((t, y) \in [0, T_0] \times W\), (the restriction of \(A(t, y)\) to \(Y\) belongs to \(\mathcal{B}(Y, X)\)) and, for each \(y \in W\) fix, \(t \mapsto A(t, y)\) is strongly continuous. Besides, for all \(t \in [0, T_0]\) fix, it is satisfied the following Lipschitz condition,

\[
\|A(t, y) - A(t, z)\|_{\mathcal{B}(Y, X)} \leq \mu_2 \|y - z\|_X,
\]

where \(\mu_2 \geq 0\) is constant.

(A4) \(A(t, y)w_0 \in Y\) for all \((t, y) \in [0, T] \times W\). Also, there exists a constant \(\lambda_2\) such that

\[
\|A(t, y)\|_Y \leq \lambda_2, \quad \text{for all } (t, y) \in [0, T_0] \times W.
\]

(f1) \(f\) is a bounded function in \([0, T_0] \times W\) to \(Y\), i.e., there exists \(\lambda_3\) such that

\[
\|f(t, y)\|_Y \leq \lambda_3, \quad \text{for all } (t, y) \in [0, T_0] \times W,
\]

Also, the function \(t \in [0, T_0] \mapsto f(t, y) \in Y\) is continuous with respect to the topology of \(X\) and for all \(y, z \in Y\) we have that

\[
\|f(t, y) - f(t, z)\|_X \leq \mu_3 \|y - z\|_X,
\]

where \(\mu_3 \geq 0\) is a constant.

**Theorem 1.14** (Kato). Assume that the conditions \((X), (A_1) - (A_4)\) and \((f_1)\) are satisfied. Given \(u_0 \in Y\), there exist \(0 < T < T_0\) and a unique \(u \in C([0, T]; Y) \cap C^1((0, T); X)\) solution to \((23)\). Furthermore, the map \(u_0 \mapsto u\) is continuous in the following sense: consider the sequence of Cauchy problems,

\[
\begin{align*}
\partial_t u_n + A_n(t, u_n)u_n &= f_n(t, u_n), \quad t > 0 \\
u_n(0) &= u_{n_0}, \quad n \in \mathbb{N}.
\end{align*}
\]

Suppose that the conditions \((X), (A_1) - (A_4)\) and \((f_1)\) are also satisfied for all \(n \geq 0\) in \((24)\), with the same \(X, Y, S\), and the corresponding \(\beta, \lambda_1 - \lambda_3, \mu_2 - \mu_3\) can be chosen independent of \(n\). Also, let us
suppose that
\[
\lim_{n \to \infty} A_n(t, w) = A(t, w) \quad \text{in} \quad B(X, Y),
\]
\[
\lim_{n \to \infty} B_n(t, w) = B(t, w) \quad \text{in} \quad B(X),
\]
\[
\lim_{n \to \infty} f_n(t, w) = f(t, w) \quad \text{in} \quad Y,
\]
\[
\lim_{n \to \infty} u_n(t) = u_0 \quad \text{in} \quad Y,
\]
where \(s\)-\(\lim\) denotes the strong limit. Then, \(T\) can be taken in such a
way that \(u_n \in C([0, T], Y) \cap C^1((0, T), X)\) and
\[
\lim_{n \to \infty} \sup_{[0, T]} \|u_n(t) - u(t)\|_Y = 0.
\]
A proof of this theorem can be found in [11] and [20].

1.2. Other results.

Proposition 1.15 (Kato’s inequality). Let \(f \in H^s\), \(s > 2\), \(\Lambda = (1 - \Delta)^{1/2}\) and \(M_f\) be the multiplication operator by \(f\). Then, for \(|\bar{t}|, |\bar{s}| \leq s - 1\), \(\Lambda^{-\bar{s}}[\Lambda^{\bar{s} + i + 1}, M_f]\Lambda^{-\bar{t}} \in B(L^2(\mathbb{R}^2))\) and
\[
\left\| \Lambda^{-\bar{s}}[\Lambda^{\bar{s} + i + 1}, M_f]\Lambda^{-\bar{t}} \right\|_{B(L^2(\mathbb{R}^2))} \leq c \|\nabla f\|_{H^{s+1}}.
\] (25)

Proposition 1.16. Let \(f : \mathbb{R}^2 \to \mathbb{R}\) be a bounded continuous function
such that \(\partial_x f\) exists and is continuous and bounded. Then, if \(A = f\partial_x\),
\[
\langle A(u), u \rangle_{L^2} \geq -\frac{1}{2} \|\partial_x f\|_{L^\infty} \|u\|^2_{L^2},
\] (26)
for all \(u \in D(A)\), \(A + \lambda\) is onto, for all \(\lambda > \frac{1}{2} \|f\|_{L^\infty}\). In particular,
\(A \in G \left(L^2(\mathbb{R}^2), 1, \frac{1}{2} \|f\|_{L^\infty}\right).

Proof. The inequality (26) is obtained immediately after using the integration
by parts. Let us see that \(A + \lambda\) is onto, if \(\lambda > \frac{1}{2} \|f\|_{L^\infty}\). Suppose that \(\psi\) is such that \(\langle (A + \lambda)(u), u \rangle_{L^2} = 0\), for all \(u \in D(A)\).
Then \(\psi \in D(A^*) \subseteq D(A)\). From (26), it follows that
\[
0 \geq \langle (a + \lambda)(u), u \rangle_{L^2} \geq (\lambda - \frac{1}{2} \|f\|_{L^\infty}) \|\psi\|^2_{L^2}.
\]
Hence, \(\psi = 0\) and, therefore, \(A + \lambda\) is onto. \(\square\)

2. Local well-posedness in Sobolev spaces of \(s\)th order
with \(s > 2\)

In this section we examine the local well-posedness of the problem
(13) in the Sobolev spaces \(H^s, X^s, \tilde{X}_\alpha^s, Y^s\) and \(\tilde{Y}^s\), for \(s > 2\).
2.1. **Local well-posedness in** $H^s(\mathbb{R}^2)$. In this section, we will make use of Kato’s theory to show the local well-posedness of (13) in the $H^s$ spaces. More precisely we have the following theorem.

**Theorem 2.1.** Let $s$ and $\alpha$ be real numbers such that $s > 2$ and $-1 \leq \alpha \leq 1$. For $\psi \in H^s(\mathbb{R}^2)$, there exist $T > 0$, that depends only on $\|\psi\|_{H^s}$, and a unique $u \in C([0, T], H^s(\mathbb{R}^2)) \cap C^1([0, T], H^{s-3}(\mathbb{R}^2) \cap (X^3)\ast)$ solution to the Cauchy problem (13). Moreover, the map $\psi \rightarrow u$ from $H^s$ to $C([0, T], H^s(\mathbb{R}^2))$ is continuous.

**Proof.** Let $W_\alpha(t)$ be the operators group defined by

$$W_\alpha(t)\psi = e^{it(\partial_x^3 - \partial_y^3)}\psi = \left(e^{-it\xi^3 + sgn(\xi)|\xi|^{\alpha}y^2}\psi\right)^\vee,$$

for all $\psi \in H^s$. $u$ is solution to the problem (13) if and only if $v = W_\alpha(t)u$ is solution to the problem

$$\begin{cases}
v_t + A(t, v)v = 0 \\
v(0) = \psi,
\end{cases} \quad (27)$$

where $A(t, v) = W_\alpha(t)(W_\alpha(-t)v)\partial_x W_\alpha(-t)$. Let us see that this last problem satisfies each condition of Kato’s theorem (Theorem 1.14).

Let $X = L^2(\mathbb{R}^2)$, $Y = H^s(\mathbb{R}^2)$ and $S = \Lambda_s = J^s$. From Plancherel’s theorem, it is evident that $S$ is an isomorphism between $X$ and $Y$.

With the following lemmas we show that the conditions (A1)-(A4) are satisfied.

**Lemma 2.2.** $A(t, v) \in G(X, 1, \beta(v))$, where $\beta(v) = \frac{1}{2} \sup_t \|\partial_x W_\alpha(t)v\|_{L^\infty}$

**Proof.** Since $\{W_\alpha(-t)\}$ is an strongly continuous unitary operators group and $u \in H^s(\mathbb{R}^2)$, from Proposition 1.16 we obtain the result. \qed

**Lemma 2.3.** For $S$ given as above,

$$SA(t, v)S^{-1} = A(t, v) + B(t, v),$$

where $B(t, v)$ is a bounded operator in $L^2$, for all $t \in \mathbb{R}$ and all $v \in H^s$, and satisfies the inequalities

$$\|B(t, v)\|_{B(L^2)} \leq \lambda(v) \quad (28)$$

$$\|B(t, v) - B(t, v')\|_{B(L^2)} \leq \mu\|v - v'\|_{H^s} \quad (29)$$

for $t \in \mathbb{R}$ and all $v$ and $v' \in H^s(\mathbb{R}^2)$. Where $\mu$ is a positive real number and

$$\lambda(v) = \sup_t C_s\|v\|_{H^s}.$$
Lemma 2.4. From Lemma 1.15 it follows that \( [S, W_\alpha(-t)v] S^{-1} \in \mathcal{B}(L^2) \) y 
\[
\| [S, W_\alpha(-t)v] S^{-1} \|_{\mathcal{B}(L^2)} \leq C_s \| v \|_{H^s}.
\]

Therefore \( B(t, v) \in \mathcal{B}(L^2) \) and satisfies (28).

Proof. Inasmuch as \( \mu \) where \( \psi \) conditions of Theorem 1.14. Therefore, for each \( \psi \) the preceding lemmas show that the Cauchy problem (13) satisfies the □ conditions. Furthermore, the function \( \mu(\alpha) \) is as the lemma above. 

Now, for all \( f \in H^s \), we have
\[
\| A(t, v)f - A(t', v)f \|_{L^2} \leq \| (W_\alpha(t) - W_\alpha(t')) W_\alpha(-t)v \|_2 + \| \hat{\alpha}(t) \|_{L^2} + \| \hat{\alpha}(t') \|_{L^2}.
\]

Since the group \( \{W_\alpha(t)\}_{t \in \mathbb{R}} \) is strongly continuous, \( t \mapsto A(t, v) \) is strongly continuous from \( \mathbb{R} \) to \( \mathcal{B}(H^s, L^2) \). Finally, for any \( t \in \mathbb{R} \), we have
\[
\| A(t, v)f - A(t, v')f \|_{L^2} \leq \mu(\| \hat{\alpha}(t) \|_{L^2} + \| \hat{\alpha}(t') \|_{L^2} + \| \hat{\alpha}(t') \|_{L^2} + \| \hat{\alpha}(t') \|_{L^2}).
\]

This ends the lemma proof. □

If we take the open ball \( W \) of \( v \in H^s \) such that \( \| v \|_{H^s(\mathbb{R}^2)} < R \), the preceding lemmas show that the Cauchy problem (13) satisfies the conditions of Theorem 1.14. Therefore, for each \( \psi \in H^s(\mathbb{R}^2) \), with \( s > 2 \), there exists \( T > 0 \), that depends on \( \| \psi \|_{H^s} \), and a unique \( v \in C([0, T], H^s(\mathbb{R}^2)) \cap C^1([0, T], H^{s-1}(\mathbb{R}^2)) \) solution to the problem (27). Moreover, the map \( \psi \mapsto v \) is continuous from \( H^s(\mathbb{R}^2) \) to \( C([0, T], H^s(\mathbb{R}^2)) \).

Now, from the group \( W_\alpha(t) \) properties it can be verified that \( u(t) = \).
where

\[ w_\alpha(-t)v \]

is solution to Cauchy problem \[13\] and satisfies all properties stated in the theorem.

**Theorem 2.5.** The existence time of the solution of the Cauchy problem \[13\] can be chosen independently of \( s \) in the following sense: if \( u \in C([0, T], H^r(\mathbb{R}^2)) \) is the solution to \[13\] with \( \psi \in H^r(\mathbb{R}^2) \), for some \( r > s \), then \( u \in C([0, T], H^r(\mathbb{R}^2)) \).

In particular, if \( \psi \in H^{\infty}(\mathbb{R}^2) \), \( u \in C([0, T], H^{\infty}(\mathbb{R}^2)) \)

Proof. Let \( r > s \), \( u \in C([0, T], H^r(\mathbb{R}^2)) \) solution to \[13\] and \( v = W_\alpha(-t)u \). Suppose that \( r \leq s + 1 \). If we apply \( \partial_x^2 \) on both sides of the differential equation \[27\], we arrive to the following linear evolution equation for \( w(t) = \partial_x^2 v(t) \)

\[
\frac{dw}{dt} + A(t)w + B(t)w = 0. \tag{30}
\]

where

\[
A(t) = \partial_x W_\alpha(t)u(t)W_\alpha(-t) \quad \tag{31}
\]

and

\[
B(t) = 2W_\alpha(t)u_x(t)W_\alpha(-t). \quad \tag{32}
\]

Since \( v \in C([0, T]; H^s(\mathbb{R}^2)) \), then \( w \in C([0, T]; H^{s-2}(\mathbb{R}^2)) \). Besides \( w(0) = \psi_{xx} \in H^{r-2}(\mathbb{R}^2) \), because \( \psi \in H^r(\mathbb{R}^2) \). It is needed to see that \( w \in C([0, T]; H^{r-2}(\mathbb{R}^2)) \). For this we shall prove that the Cauchy problem associated to the linear equation \[30\] is locally well-posed for \( 1 - s \leq k \leq s - 1 \), for which we have the following lemma whose proof is similar to that of Lemma 3.1 in \[12\].

**Lemma 2.6.** The family \( \{A(t)\}_{t \in [0, T]} \) has a unique family of evolution operators associated, \( \{U(t, \tau)\}_{t \in [0, T]} \), in the spaces \( X = H^h \), \( Y = H^h \), where

\[
- s \leq h \leq s - 2 \quad 1 - s \leq k \leq s - 1 \quad k + 1 \leq h. \quad \tag{33}
\]

In particular, \( U(t, \tau) : H^r \to H^r \) for \( -s \leq s \leq s - 1 \).

Then, \( w \) satisfies the equation

\[
w(t) = U(t, 0)\psi_{xx} + \int_0^t U(t, \tau)[-B(\tau)w(\tau) + f(\tau)]d\tau. \tag{34}
\]

Since \( \psi_{xx} \in H^{r-2} \), \( B(t) \), given by \[32\], is an operators family in \( H^{r-2} \) that is strongly continuous for \( t \) in the interval \([0, T]\). From Lemma 2.6, the solution to \[34\] belongs to \( C([0, T]; H^{r-2}(\mathbb{R}^2)) \). In other words, \( \partial_x^2 u \in C([0, T]; H^{r-2}(\mathbb{R}^2)) \).
If \( w_1(t) = \partial_x \partial_y v(t) \), we have
\[
\frac{dw_1}{dt} + A(t)w_1 + B_1(t)w_1 = f_1(t),
\]
where
\[
B_1(t) = \mathcal{W}(t)u_x(t)\mathcal{W}(-t) = \frac{1}{2}B(t),
\]
and
\[
f_1(t) = -\mathcal{W}(t)(u_{xx}(t)u_y(t)).
\]
As before, we have
\[
w_1(t) = U(t, 0)\psi_{xy} + \int_0^t U(t, \tau)(-B_1(\tau)w_1(\tau) + f_1(\tau)) \, d\tau.
\]
Since \( u_{xx} \in C([0, T]; H^{r-2}(\mathbb{R}^2)) \), \( f_1 \in C([0, T]; H^{r-2}(\mathbb{R}^2)) \). Inasmuch as, also, \( B_1(t) \in (H^{r-2}(\mathbb{R}^2)) \) is strongly continuous on the interval \([0, T] \). Arguing as before we have that \( w_1 \in C([0, T]; H^{r-2}(\mathbb{R}^2)) \), or equivalently \( u_{xy} \in C([0, T]; H^{r-2}(\mathbb{R}^2)) \).

Analogously, if \( w_2(t) = \partial_y^2 v(t) \), we have that
\[
\frac{dw_2}{dt} + A(t)w_2 = f_2(t),
\]
where
\[
f_2(t) = -2\mathcal{W}(t)(u_{xy}u_y(t)).
\]
Therefore,
\[
w_2(t) = U(t, 0)\psi_{yy} + \int_0^t U(t, \tau)f_2(\tau) \, d\tau.
\]
Since \( u_{xy} \in C([0, T]; H^{r-2}(\mathbb{R}^2)) \), \( f_2 \in C([0, T]; H^{r-2}(\mathbb{R}^2)) \). Repeating the argument above, we can conclude that \( w_2 \in C([0, T]; H^{r-2}(\mathbb{R}^2)) \), or equivalently, \( \partial_y^2 u \in C([0, T]; H^{r-2}(\mathbb{R}^2)) \).

Hence, we have showed that if \( s < r \leq s + 1 \) and \( \psi \in H^r, \ u \in C([0, T]; H^r(\mathbb{R}^2)) \). To see the case \( r > s + 1 \), since \( \psi \in H^s \), for \( s' < r \), we can use over and over again what we have proved so far, to obtain that \( u \in C([0, T]; H^r(\mathbb{R}^2)) \).

2.2. Local well-posedness in \( X^s, \hat{X}^s_\alpha \) and \( Y^s \). Let us finish this section showing the local well-posedness of (13) in the spaces \( X^s, \hat{X}^s_\alpha \) and \( Y^s \).
Theorem 2.7. Let $s$ and $\alpha$ be as in Theorem 2.1. Let $Z$ also be any of the spaces $X^s$, $\hat{X}^s$, $\tilde{X}^s$, $Y^s$ and $\tilde{Y}^s$. Then, if $\psi \in Z$ and $u \in C([0, T], H^s)$ is solution to (13) with $u(0) = \psi$, $u \in C([0, T], Z)$. Moreover, $\psi \mapsto u$ is continuous from $Z$ to $C([0, T], Z)$.

Proof. Suppose that $\psi$ and $u$ are as in the hypothesis of the theorem. Then, from the fundamental calculus theorem, we have that

$$u(t) = W_\alpha(-t)\psi + \int_0^t W_\alpha(-t + \tau)\partial_x \left( \frac{u^2(\tau)}{2} \right) d\tau.$$

So that,

$$\partial_x^{-1}u(t) = W_\alpha(-t)\partial_x^{-1}\psi + \int_0^t W_\alpha(-t + \tau) \left( \frac{u^2(\tau)}{2} \right) d\tau.$$

From here it follows that (13) is locally well-posed in the spaces $\hat{X}^s$ and $X^s$.

On the other hand,

$$\partial_x^{-1}\partial_y u(t) = W_\alpha(-t)\partial_x^{-1}\partial_y \psi + \int_0^t W_\alpha(-t + \tau)(u\partial_y u)(\tau) d\tau.$$

Then, (13) is locally well-posed in the space $\hat{Y}^s$.

To see the local well-posedness in $Y^s$ is slightly more complicated. Let $v$ be as in (27). Hence, $w = \partial_x^{-1}\partial_y v$ satisfies

$$\begin{cases}
w_t + A(t, v)w = 0 \\
w(0) = \partial_x^{-1}\partial_y \psi,
\end{cases}$$

(42)

where $A(t, v)$ is as in (27). From linear case of Kato’s theory, more specifically from Theorem 1.14 when taking $X = L^2$ and $Y = H^s$, we have that, if $\partial_x^{-1}\partial_y \psi \in H^s$, $w$ is continuous on $t$ and depends continuously on this data. This shows the local well-posedness of the problem (13) in $Y^s$.

Analogously we can show that (13) is local well-posedness in space $X^s_\alpha$. □

3. Local well-posedness in low regularity spaces

In this section we examine the local well-posedness of equation (13) in $H^{s_1, s_2}(\mathbb{R}^2)$, for $-1 \leq \alpha \leq 1$, $s_1 > \frac{17}{12} - \frac{\alpha}{4}$ and $1 < s_2 \leq s_1$. We will use the dispersive properties of the group generated by the linear equation associated to (13). This is the same strategy used by Kenig in [14] for the KP-I equation and by Linares, Pilod and Saut in [26] for the f-KPI and f-KPII equations.
3.1. Linear estimates. The linear Cauchy problem associated to the problem (13) is
\[
\begin{align*}
    \begin{cases}
        u_t = u_{xxx} - \mathcal{H} D_x^2 u_{yy}, \\
        u(0) = \psi,
    \end{cases}
\end{align*}
\]  
where $-1 \leq \alpha \leq 1$. The unitary group generated by this problem is
\[
W_\alpha(t) \psi(x, y) = \left( e^{-it(\xi^3 + sgn(\xi)|\xi|^{\alpha} \eta^2)} \psi \right)^\vee = S_t^\alpha \ast \psi(x, y),
\]  
where
\[
S_t^{\alpha}(x, y) = \int_{\mathbb{R}^2} e^{-it(\xi^3 + sgn(\xi)|\xi|^{\alpha} \eta^2) + i\eta x + iy \eta} d\xi d\eta
\]  
We shall examine this group properties.

**Lemma 3.1.** Let $-1 \leq \alpha \leq 1$ and $\frac{\alpha}{2} - 1 < \text{Re} \beta < \frac{\alpha}{2}$. Then,
\[
\| D_x^\beta W_\alpha(t) \psi \|_{L^\infty(\mathbb{R}^2)} \lesssim |t|^{-\frac{5 + 2\beta - \alpha}{6}} \| \psi \|_{L^1(\mathbb{R}^2)}.
\]

**Proof.** Effectively,
\[
D_x^\beta S_t^{\alpha}(x, y) = \int_{\mathbb{R}^2} |\xi|^\beta e^{-it(\xi^3 + sgn(\xi)|\xi|^{\alpha} \eta^2) + i\eta x + iy \eta} d\xi d\eta
\]  
\[= \int_{\mathbb{R}_\xi} |\xi|^\beta e^{-it\xi^3 + i\xi \eta} \int_{\mathbb{R}_\eta} e^{-it sgn(\xi)|\xi|^{\alpha} \eta^2 + iy \eta} d\eta d\xi
\]  
\[= \int_{\mathbb{R}_\xi} |\xi|^\beta e^{-it\xi^3 + i\xi \eta} \int_{\mathbb{R}_\eta} e^{-it sgn(\xi)|\xi|^{\alpha} \eta^2} \left[ y - \frac{y}{2 sgn(\xi)|\xi|^{\alpha}} \right]^2 d\eta d\xi
\]  
\[= \pi^{\frac{1}{2}} \frac{|t|^{\frac{1}{2}}}{|t|^{\frac{5 + 2\beta - \alpha}{6}}} \int_{\mathbb{R}} e^{\left( \theta^3 - x^3 \theta - \frac{1}{2} - \frac{sgn(\theta)|\theta|^{2}}{4|\theta|^4} |\theta|^{\frac{3}{2}} - 1 \right) |\theta|^{\beta - \frac{\alpha}{2}}} d\theta.
\]  
Let us see that the last integral is bounded. For this let us take a function $\chi$ defined on all $\mathbb{R}$, infinitely differentiable, with support in the interval $[-2, 2]$ such that $\chi \equiv 1$ in the interval $[-1, 1]$. So,
\[
\int_{\mathbb{R}} e^{\left( \theta^3 - x^3 \theta - \frac{1}{2} - \frac{sgn(\theta)|\theta|^{2}}{4|\theta|^4} |\theta|^{\frac{3}{2}} - 1 \right) |\theta|^{\beta - \frac{\alpha}{2}}} d\theta
\]  
\[= \int_{\mathbb{R}} e^{\left( \theta^3 - x^3 \theta - \frac{1}{2} - \frac{sgn(\theta)|\theta|^{2}}{4|\theta|^4} |\theta|^{\frac{3}{2}} - 1 \right) |\theta|^{\beta - \frac{\alpha}{2}}} \chi(\theta) d\theta +
\]  
\[+ \int_{\mathbb{R}} e^{\left( \theta^3 - x^3 \theta - \frac{1}{2} - \frac{sgn(\theta)|\theta|^{2}}{4|\theta|^4} |\theta|^{\frac{3}{2}} - 1 \right) |\theta|^{\beta - \frac{\alpha}{2}}} (1 - \chi(\theta)) d\theta.
\]
Clearly the first integral on the right hand side is bounded. To see that the second integral on the right hand side is bounded, we will make use of the Van der Corput lemma (see [27], Corollary 1.1). The second and third derivatives of phase function $\varphi_\alpha(x, y) = \theta^2 - x\theta t^{\frac{1}{5}} - \frac{\text{sgn}(\theta)}{4|\theta|^\alpha} |t|^{-\frac{1}{3}}$ in the integral are

$$\dot{\varphi}_\alpha''(\theta) = 6\theta + \text{sgn}(\theta)\alpha(\alpha + 1) \frac{y^2}{4} |t|^{\frac{\alpha}{2} - 1} |\theta|^{-\alpha - 2}$$

and

$$\dot{\varphi}_\alpha'''(\theta) = 6 + \alpha(\alpha + 1)(\alpha + 2) \frac{y^2}{4} |t|^{\frac{\alpha}{2} - 1} |\theta|^{-\alpha - 3}.$$ 

It is easily verified that, for $|\theta| \geq 1$, $|\dot{\varphi}_\alpha''(\theta)| \geq 6$ when $-1 \leq \alpha \leq 0$ and that $|\dot{\varphi}_\alpha''(\theta)| \geq 6$ when $0 \leq \alpha \leq 1$. Since the function $\theta \mapsto |\theta|^{\frac{1}{2} - \frac{\alpha}{4}} (1 - \chi(\theta))$ is uniformly bounded and integrable on the set $|\theta| \geq 1$, it is verified that the second integral on the right side of (48) is bounded. Which verifies that the last integral in (47) is uniformly bounded. Therefore,

$$\|D_\alpha^\beta S_\alpha^\gamma(x, y)\|_{L^\infty(\mathbb{R}^2)} \leq |t|^\frac{5 + 2\beta - \alpha}{6}.$$ 

The theorem follows immediately from Young’s inequality for convolution.

**Remark 1.** The last result coincides with that proved by Linares and Pastor in [23] for the ZK equation (case $\alpha = 1$). The results of Saut in [34], for the KPI equation (case $\alpha = -1$), and of Lizarazo in his PhD thesis (see [28]) (case $\alpha = 0$) are better estimates.

**Corollary 3.2.** Let $-1 \leq \alpha \leq 1$, $0 < \epsilon < 1$ and $0 \leq \theta \leq 1$. Then,

$$\left\|D_\alpha^\theta S_\alpha^\gamma(x, y)\right\|_{L^\infty(\mathbb{R}^2)} \leq |t|^\frac{\theta(\alpha - 2\epsilon)}{p} \|\psi\|_{L^p(\mathbb{R}^2)}$$

(49)

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $p = \frac{2}{1 - \epsilon}$.

**Proof.** Let $\psi \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. The corollary follows from Stein’s interpolation theorem for analytical operator families (see [35]). Let us set $z = \theta + i\gamma \in \mathbb{C}$. For each $z$ we define the operators $T_z$ by

$$T_z \psi = D_\alpha^z S_\alpha^\gamma(x, y) \psi.$$ 

This family $\{T_z\}$ is an admissible family of operators. By Lemma 3.1 we have

$$\|T_{1 + i\gamma} \psi\|_{L^\infty(\mathbb{R}^2)} = \left\|D_\alpha^z S_\alpha^\gamma(x, y) \psi\right\|_{L^\infty(\mathbb{R}^2)} \leq c |t|^{-\frac{\alpha(\alpha - 2\epsilon)}{2}} \|\psi\|_{L^1(\mathbb{R}^2)}.$$ 

(50)
Also, since \( \{W_\alpha(t)\}_{t \in \mathbb{R}} \) is an strongly continuous group on the parameter \( t \) in \( L^2(\mathbb{R}^2) \), we have
\[
\|T_\gamma \psi\|_{L^2(\mathbb{R}^2)} = \left\| D_x^{\left(\frac{4}{p} - \epsilon\right)} (i \gamma) W_\alpha(t) \psi \right\|_{L^2(\mathbb{R}^2)} = \|\psi\|_{L^2(\mathbb{R}^2)}.
\] (51)

From Stein’s interpolation theorem we obtain
\[
\|T_{\tilde{\theta}} \psi\|_{L^p(\mathbb{R}^2)} \leq c |t|^{-\theta(\frac{12}{2}\mathcal{E})} \|\psi\|_{L^p(\mathbb{R}^2)},
\] (52)
what was we wanted to prove.

**Corollary 3.3.** Let \(-1 \leq \alpha \leq 1, 0 < \epsilon < 1\) and \(0 < \theta \leq 1\). Then,
\[
\left\| D_x^{\left(\frac{4}{p} - \epsilon\right)} \int_{-\infty}^{\infty} W_\alpha(t - t') F(t') dt' \right\|_{L^p_x L^2_y} \lesssim \|F\|_{L^p_x L^2_y},
\] (53)
where \(\frac{1}{q} + \frac{1}{q'} = 1\), \(p = \frac{2}{1 - \mathcal{E}}, \frac{2}{q} = \frac{\theta(5 - 2\mathcal{E})}{6}\) and \(\frac{1}{p} + \frac{1}{q} = \frac{1}{2} - \frac{\theta(1 + 2\mathcal{E})}{12}\).

**Proof.** From Minkowski’s inequality, Corollary 3.2 and from the Hardy-Littlewood-Sobolev theorem, we have
\[
\left\| D_x^{\theta(\frac{4}{p} - \epsilon)} \int_{-\infty}^{\infty} W_\alpha(t - t') F(t') dt' \right\|_{L^p_x L^2_y}
\leq \left\| \int_{-\infty}^{\infty} D_x^{\theta(\frac{4}{p} - \epsilon)} W_\alpha(t - t') F(\cdot, \cdot, t') dt' \right\|_{L^p_x L^2_y}
\lesssim \left\| \int_{-\infty}^{\infty} D_x^{\left(\frac{4}{p} - \epsilon\right)} W_\alpha(t - t') F(\cdot, \cdot, t') dt' \right\|_{L^p_x L^2_y}
\lesssim \left\| \int_{-\infty}^{\infty} |t - t'| \frac{\theta(5 - 2\mathcal{E})}{6} \|F(\cdot, \cdot, t')\|_{L^p_{x'y}} dt' \right\|_{L^2_t}
\lesssim \left\| \|F(\cdot, \cdot, t')\|_{L^p_{x'y}} \right\|_{L^2_t} = \|F\|_{L^p_{x'y}},
\]

Using the Stein-Thomas argument (see [9]) we have.

**Proposition 3.4.** Let \(\alpha, \epsilon, \theta, p\) and \(q\) be as in the previous corollary. Then,
\[
\left\| D_x^{\frac{12}{5(2\mathcal{E})}} (\frac{4}{p} - \epsilon) W_\alpha(t) \psi \right\|_{L^p_x L^2_y} \lesssim \|\psi\|_{L^2_x}.
\] (54)

From the above we have the following two very useful corollaries in the proof of the local well-posedness in this section.
Corollary 3.5. For each \( T > 0 \) and \( 0 < \epsilon < 1 \), we have that

\[
\| W_\alpha(t) \psi \|_{L^2_T L^\infty_{xy}} \lesssim T^{\frac{14+2\alpha}{12}} \| D_x^{\frac{1}{2}(-\frac{\alpha}{2})} \psi \|_{L^2_y}.
\]  

(55)

Proof. From Hölder’s inequality and Proposition 3.4 we have that

\[
\| W_\alpha(t) \psi \|_{L^2_T L^\infty_{xy}} \lesssim T^{\frac{14+2\alpha}{12}} \| W_\alpha(t) \psi \|_{L^2_T L^\infty_{xy}} \lesssim T^{\frac{14+2\alpha}{12}} \| D_x^{\frac{1}{2}(-\frac{\alpha}{2})} \psi \|_{L^2_y}.
\]

□

Just like in [15] and [26], we prove a refined Strichartz estimate for the solution to non-homogeneous linear problem

\[
\partial_t w = w_{xxx} - \mathcal{H} D_x^\alpha w_{yy} + F.
\]

(56)

So, we have the next lemma.

Lemma 3.6. Let \(-1 \leq \alpha \leq 1\), \(0 < \epsilon < 1\) and \(T > 0\). If \(w\) is solution to (56), then there exists \(c_\epsilon > 0\) such that

\[
\| \partial_x w \|_{L^1_T L^\infty_{xy}} \leq c_\epsilon T^{\frac{14+2\alpha}{12}} \left( \sup_{t \in [0,T]} \left\| \int_0^t \mathcal{H} D_x^{\alpha-\frac{\alpha}{2} + \frac{2\alpha}{3}} w \right\|_{L^2_y} + \int_0^T \left\| D_x^{\alpha-\frac{\alpha}{2} + \frac{2\alpha}{3}} F(t) \right\|_{L^2_y} dt \right).
\]

(57)

Proof. We will make use of a Littlewood-Paley type decomposition of \(w\) on \(x\) variable. For this, let \(\varphi_0, \varphi \in C_0^\infty\) with \(\text{supp}(\varphi_0) = \{ |\xi| < 2 \}\) and \(\text{supp}(\varphi) = \{ \frac{1}{2} < |\xi| < 2 \}\) such that \(\varphi_0(\xi) + \sum_{k=1}^\infty \varphi(2^{-k}\xi) = 1\) for all \(\xi \in \mathbb{R}\). Let us define \(P_k w\) via Fourier transform by \(\hat{P_0} w(\xi, \eta) = \varphi_0(\xi) \hat{w}(\xi, \eta)\) and \(\hat{P_k} w(\xi, \eta) = \varphi(2^{-k}\xi) \hat{w}(\xi, \eta)\) for \(k \geq 1\), in such a way that

\[
w = \sum_{k=0}^\infty P_k w.
\]

Let us first make an estimate for \(\| \partial_x P_0 w \|_{L^1_T L^\infty_{xy}}\). Since \(w\) satisfies (56), then \(P_0 w\) satisfies the integral equation when apply \(P_0\) on both sides of the equation. So, from the Cauchy-Schwarz inequality and (55), it
follows that

$$\| \partial_x P_0 w \|_{L^1_t L^\infty_{x,y}} \leq$$

$$\leq \| W_\alpha(t) \partial_x P_0 w(0) \|_{L^1_t L^\infty_{x,y}} + \int_0^t \| W_\alpha(t-t') \partial_x P_0 F(t') \|_{L^1_t L^\infty_{x,y}} \, dt'$$

$$\leq T^{\frac{1}{2}} \left( \| W_\alpha(t) \partial_x P_0 w(0) \|_{L^2_t L^\infty_{x,y}} + \int_0^t \| W_\alpha(t-t') \partial_x P_0 F(t') \|_{L^2_t L^\infty_{x,y}} \, dt' \right)$$

$$\leq T^{\frac{1}{2} + \frac{14k}{12}} \left( \left\| \partial_x P_0 w(0) \right\|_{L^2_t L^\infty_{x,y}} + \int_0^t \left\| \partial_x P_0 F(t') \right\|_{L^2_t L^\infty_{x,y}} \, dt' \right)$$

$$\leq cT^{\frac{7 + 2k}{12}} \left( \left\| \partial_x P_0 w(0) \right\|_{L^2_t L^\infty_{x,y}} + \int_0^t \left\| \partial_x P_0 F(t') \right\|_{L^2_t L^\infty_{x,y}} \, dt \right).$$

(58)

Now, let us estimate $\| \partial_x P_k w \|_{L^1_t L^\infty_{x,y}}$, when $k \geq 1$. For this, we will make a suitable partition in time in such a way that allow us to control the localized frequencies associated to $x$ variable. So, let $\mathcal{P} = \{ a_0, a_1, \ldots, a_{2^k} \}$ be a partition from interval $[0, T]$ with $a_j = jT2^{-k}$, $j = 0, 1, \cdots, 2^k$. We denote by $I_j$ the interval $[a_{j-1}, a_j]$.

Then, thanks to Young’s inequality, Cauchy-Schwarz’s inequality and the inequality $\left\| (\chi_{(2^{k-1} < |\xi| < 2^k)} \xi) \right\|_{L^1(\mathbb{R})} \leq 2^k$, we have

$$\| \partial_x P_k w \|_{L^1_t L^\infty_{x,y}} \leq \left\| (\chi_{(2^{k-1} < |\xi| < 2^k)} \xi) \right\|_{L^1(\mathbb{R})} \| P_k w \|_{L^1_t L^\infty_{x,y}}$$

$$\leq 2^k \sum_{j=1}^{2^k} \| P_k w \|_{L^1_{I_j} L^\infty_{x,y}}$$

$$\leq (2^k) \sum_{j=1}^{2^k} \left( a_j - a_{j-1} \right)^{\frac{1}{2}} \| P_k w \|_{L^2_{I_j} L^\infty_{x,y}}$$

$$= (2^k T)^{\frac{1}{2}} \sum_{j=1}^{2^k} \| P_k w \|_{L^2_{I_j} L^\infty_{x,y}}. \quad (59)$$

From Duhamel’s principle on each interval $[a_{j-1}, a_j]$, we have that, for each $t \in [a_{j-1}, a_j]$,

$$P_k w(t) = W_\alpha(t - a_j) P_k w(a_j) + \int_{a_{j-1}}^t W_\alpha(t - t') P_k F(t') \, dt'.$$
By (59) and (55), we get
\[
\|c_k P_k w\|_{L^1_t L^\infty_{x,y}} \leq (2^k T)^{\frac{1}{2}} \sum_{j=1}^{2^k} \|P_k w\|_{L^2_{t[a_j, a_{j+1}]} L^\infty_{x,y}}
\]
\[
\leq (2^k T)^{\frac{1}{2}} \sum_{j=1}^{2^k} \left( \|W_\alpha(t - a_j) P_k w(\cdot, a_j)\|_{L^2_{t} L^\infty_{x,y}} + \int_{a_{j-1}}^t \|W_\alpha(t - t') P_k F(t')\|_{L^2_{t} L^\infty_{x,y}} dt' \right)
\]
\[
\leq 2^{-\frac{k(5+2k)}{12}} T^{\frac{7+2k}{12}} \sum_{j=1}^{2^k} \left( \|D_x^{\frac{1}{2}} (\cdot - \frac{\alpha}{2}) P_k w(\cdot, a_j)\|_{L^2_{x,y}} + \int_{a_{j-1}}^t \left( \|D_x^{\frac{1}{2}} (\cdot - \frac{\alpha}{2}) P_k F(t')\|_{L^2_{x,y}} \right) dt' \right)
\]
\[
\leq T^{\frac{7+2k}{12}} \left( \sup_{t \in [0, T]} \left\|D_x^{\frac{17}{2} - \frac{\alpha}{4} + \frac{2k}{3}} P_k w(t)\right\|_{L^2_{x,y}} + \int_0^T \left\|D_x^{\frac{17}{2} - \frac{\alpha}{4} + \frac{2k}{3}} P_k F(t')\right\|_{L^2_{x,y}} dt' \right).
\]
(60)

For the sake of the completeness, let us explain the inequality
\[
2^{-\frac{k(5+2k)}{12}} \sum_{j=1}^{2^k} \left\|D_x^{\frac{1}{2}} (\cdot - \frac{\alpha}{2}) P_k w(\cdot, a_j)\right\|_{L^2_{x,y}} \leq \sup_{t \in [0, T]} \left\|D_x^{\frac{17}{2} - \frac{\alpha}{4} + \frac{2k}{3}} P_k w(t)\right\|_{L^2_{x,y}} ,
\]
what was used to prove the inequality above. Observe that, for each integer \( j \leq 2^k \), we have
\[
\left\|D_x^{\frac{1}{2}} (\cdot - \frac{\alpha}{2}) P_k w(\cdot, a_j)\right\|_{L^2_{x,y}} \leq J^{-\frac{k(17+2k)}{12}} \left\|D_x^{\frac{17}{2} - \frac{\alpha}{4} + \frac{2k}{3}} P_k w(\cdot, a_j)\right\|_{L^2_{x,y}} .
\]
Summing on \( j \) from 1 to \( 2^k \) and considering
\[
\sum_{j=1}^{2^k} J^{-\frac{k(17+2k)}{12}} \sim 2^{-\frac{k(5+2k)}{12}}
\]
it follows the inequality (61).

(60) together with (58) show the present lemma. \( \square \)

3.2. Energy Estimate.

**Lemma 3.7.** Let \(-1 \leq \alpha \leq 1\), \( T > 0 \) and assume that \( u \in C([0, T]; H^2(\mathbb{R}^2)) \) is a solution to the Cauchy problem (13). Then, there exists a positive
constant $C$ such that, for $1 \leq s_2 \leq s_1$,

$$
\|u\|_{L^\infty_T H^{s_1,s_2}_y} \leq \|\psi\|_{H^{s_1,s_2}_y} e^{C(|\partial_x u|_{L^1_T L^2_y} + |\partial_y u|_{L^1_T L^2_y})}. \tag{62}
$$

**Proof.** Let $u$ as in the statement in lemma. Let us estimate first $\|J_x^{s_1} u\|_{L^2_y}$. Operating with $J_x^{s_1}$ and then multiplying by $J_x^{s_1} u$ on both sides of the equation (13), and integrating with respect to $x,y$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (J_x^{s_1} u)^2 dx dy = \int_{\mathbb{R}^2} J_x^{s_1} (u \partial_x u) J_x^{s_1} u dx dy
\begin{equation}
= \int_{\mathbb{R}^2} [J_x^{s_1}, u] \partial_x u J_x^{s_1} u dx dy + \int_{\mathbb{R}^2} u J_x^{s_1} \partial_x u J_x^{s_1} u dx dy. \tag{63}
\end{equation}
$$

From inequalities of Cauchy-Schwarz and Kato-Ponce (Lemma 1.8), applied in the $x$ variable, we have

$$
\int_{\mathbb{R}^2} [J_x^{s_1}, u] \partial_x u J_x^{s_1} u dx dy \leq \int_{\mathbb{R}_y} \|J_x^{s_1}, u\|_{L^2_x} \|\partial_x u\|_{L^2_x} \|J_x^{s_1} u\|_{L^2_y} dy
\begin{equation}
\leq \int_{\mathbb{R}_y} \|\partial_x u\|_{L^\infty_x} \|J_x^{s_1} u\|_{L^2_x}^2 dy
\leq \|\partial_x u\|_{L^\infty_x} \|J_x^{s_1} u\|_{L^2_y}^2 \leq \|\partial_x u\|_{L^\infty_x} \|u\|_{H^{s_1,s_2}_y}^2. \tag{64}
\end{equation}
$$

On the other hand,

$$
\int_{\mathbb{R}^2} u J_x^{s_1} \partial_x u J_x^{s_1} u dx dy = -\frac{1}{2} \int_{\mathbb{R}^2} \partial_x u (J_x^{s_1} u)^2 dx dy \leq \|\partial_x u\|_{L^\infty_y} \|J_x^{s_1} u\|_{L^2_y}^2. \tag{65}
$$

By (63), (64) and (65), we have

$$
\frac{d}{dt} \|J_x^{s_1} u\|_{L^2_x}^2 \leq \|\partial_x u\|_{L^\infty_y} \|J_x^{s_1} u\|_{L^2_y}^2 \leq \|\partial_x u\|_{L^\infty_y} \|u\|_{H^{s_1,s_2}_y}^2. \tag{66}
$$

In a totally analogous way we shall estimate $\|J_y^{s_2} u\|_{L^2_y}$. So,

$$
\frac{1}{2} \frac{d}{dt} \|J_y^{s_2} u\|_{L^2_y}^2 = \int_{\mathbb{R}^2} [J_y^{s_2}, u] \partial_x u J_y^{s_2} u dx dy + \int_{\mathbb{R}^2} u J_y^{s_2} \partial_x u J_y^{s_2} u dx dy. \tag{67}
$$
Proceeding as before, we have

\[ \int_{\mathbb{R}^2} \left[ J_y^{s_2} u \right] \partial_x u J_y^{s_2} u \, dx \, dy \leq \int_{\mathbb{R}^2} \left\| \left[ J_y^{s_2} u \right] \partial_x u \right\|_{L_y^2} \left\| J_y^{s_2} u \right\|_{L_y^2} \, dx \lesssim \]

\[ \lesssim \int_{\mathbb{R}^2} \left( \left\| \partial_y u \right\|_{L_y^\infty} \left\| J_y^{s_2-1} \partial_x u \right\|_{L_y^2} + \left\| \partial_x u \right\|_{L_y^\infty} \left\| J_y^{s_2} \partial_x u \right\|_{L_y^2} \right) \left\| J_y^{s_2} u \right\|_{L_y^2} \, dx \]

\[ \leq \left\| \partial_y u \right\|_{L_y^\infty} \left\| J_x^1 u \right\|_{L_y^2} \left\| J_y^{s_2} \partial_x u \right\|_{L_y^2} + \left\| \partial_x u \right\|_{L_y^\infty} \left\| J_y^{s_2} u \right\|_{L_y^2} \]

\[ \lesssim \left( \left\| \partial_x u \right\|_{L_y^2} + \left\| \partial_y u \right\|_{L_y^2} \right) \left\| u \right\|^2_{H_x^{s_1+s_2}} . \tag{68} \]

So, integrating by parts in the second term on the right side of the inequality (67) and from last inequality, we have

\[ \frac{d}{dt} \left\| J_y^{s_2} u \right\|^2_{L_y^2} \lesssim \left( \left\| \partial_x u \right\|_{L_y^2} + \left\| \partial_y u \right\|_{L_y^2} \right) \left\| u \right\|^2_{H_x^{s_1+s_2}} . \tag{69} \]

Gathering (66) and (69), we obtain

\[ \frac{d}{dt} \left\| u \right\|^2_{H_x^{s_1+s_2}} \lesssim \left( \left\| \partial_x u \right\|_{L_y^2} + \left\| \partial_y u \right\|_{L_y^2} \right) \left\| u \right\|^2_{H_x^{s_1+s_2}} . \tag{70} \]

From Gronwall inequality we get the lemma. \[ \square \]

### 3.3. A Strichartz Estimate Type

The energy estimate suggests that we have to control the norms \( \left\| \partial_x u \right\|_{L_t^1 L_y^{s_2}} \) and \( \left\| \partial_y u \right\|_{L_t^1 L_y^{s_2}} \) in order to prove the local well-posedness of the Cauchy problem (13). The next lemma shows how we can control these norms.

**Lemma 3.8.** Let \(-1 \leq \alpha \leq 1, T > 0\) and assume that \( u \in C([0, T]; H^\infty_x(\mathbb{R}^2)) \) is a solution to initial value problem (13) with initial condition \( \psi \).

Then, for any \( s_1 > \frac{17}{12} - \frac{\alpha}{4} \) such that

\[ \begin{cases} 
    s_2 > 1, & \text{if } 0 \leq \alpha \leq 1, \\
    \frac{1}{s_2} - \frac{\alpha}{4s_1} < 1, & \text{if } -1 \leq \alpha \leq 0
\end{cases} \]

and \( s_1 \geq s_2 \), there exist constants \( C_{s_1,s_2} \) and \( k_{s_1,s_2} \in (7/12, 1) \) such that

\[ f(T) = \left\| u \right\|_{L_t^1 L_y^{s_2}} + \left\| \partial_x u \right\|_{L_t^1 L_y^{s_2}} + \left\| \partial_y u \right\|_{L_t^1 L_y^{s_2}} \leq C_{s_1,s_2} \]

\[ (1 + f(T)) \left\| u \right\|_{H_x^{s_1+s_2}} . \tag{71} \]

satisfies

\[ f(T) \leq C_{s_1,s_2} T^{k_{s_1,s_2}} (1 + f(T)) \left\| u \right\|_{H_x^{s_1+s_2}} . \tag{72} \]
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Proof. First let us examine an estimate for \( \| \partial_x u \|_{L^1_t L^{\alpha}_{xy}} \). From the refined Strichartz estimate (57), we have

\[
\| \partial_x u \|_{L^1_t L^{\alpha}_{xy}} \leq c_\epsilon T^{\frac{7+2\alpha}{12}} \left( \sup_{t \in [0,T]} \left\| J^\frac{5}{12} \partial_x^x + \frac{2\alpha}{3} u \right\|_{L^2_{xy}} + \int_0^T \left\| D_x^\frac{5}{12} \partial_x^x + \frac{2\alpha}{3} (u \partial_x u) \right\|_{L^2_{xy}} dt \right).
\]

(73)

Let us choose \( 0 < \epsilon_1 < 1 \) in such a way that \( \frac{5}{12} - \frac{\alpha}{4} + \frac{2\alpha}{3} < 1 \) and \( \frac{17}{12} - \frac{\alpha}{4} + \frac{2\alpha}{3} < s_1 \). So, the first term from the right side of (73) we have

\[
\sup_{t \in [0,T]} \left\| J^\frac{5}{12} \partial_x^x + \frac{2\alpha}{3} u \right\|_{L^2_{xy}} \leq \sup_{t \in [0,T]} \left\| J^s_{x} u \right\|_{L^2_{xy}} \leq \| u \|_{L^\infty_x H^s_{xy}} .
\]

(74)

For the second term on the right side of (74), from the Leibniz rule (equation (19)) in \( x \), we have

\[
\int_0^T \left\| D_x^\frac{5}{12} \partial_x^x + \frac{2\alpha}{3} (u \partial_x u) \right\|_{L^2_{xy}} dt \leq \\
\int_0^T \left\| J^\frac{5}{12} \partial_x^x + \frac{2\alpha}{3} u \right\|_{L^2_{xy}} \| \partial_x u \|_{L^\infty_x} + \left\| J^s_{x} u \right\|_{L^2_{xy}} \| u \|_{L^\infty_x} dt \\
\leq \int_0^T \left( \| u \|_{L^\infty_x} + \| \partial_x u \|_{L^\infty_x} \right) \| J^s_{x} u \|_{L^2_{xy}} dt \\
\leq \| u \|_{L^\infty_x H^s_{xy}} f(T)
\]

(75)

Then, from (75), (74) and (73)

\[
\| \partial_x u \|_{L^1_t L^{\alpha}_{xy}} \leq C_1(s_1) T^{\frac{7+2\alpha}{12}} (1 + f(T)) \| u \|_{L^\infty_x H^s_{xy}} .
\]

(76)

Let us estimate now \( \| u \|_{L^1_t L^{\alpha}_{xy}} \). From Duhamel’s principle, Cauchy-Schwarz inequality in \( t \) and Corollary 3.5, we have that

\[
\| u \|_{L^1_t L^{\alpha}_{xy}} \leq \left\| W_\alpha(t) \psi \right\|_{L^1_t L^{\alpha}_{xy}} + \int_0^T \left\| W_\alpha(t-t')(u \partial_x u)(t') \right\|_{L^1_t L^{\alpha}_{xy}} dt' \\
\leq T^{\frac{5}{4}} \left( \left\| W_\alpha(t) \psi \right\|_{L^1_t L^{\alpha}_{xy}} + \int_0^T \left\| W_\alpha(t-t')(u \partial_x u)(t') \right\|_{L^1_t L^{\alpha}_{xy}} dt' \right) \\
\leq C_2 T^{\frac{7+2\alpha}{12}} \left( \left\| D_x^\frac{5}{8} \psi \right\|_{L^2_{xy}} + \int_0^T \left\| D_x^\frac{5}{8} (u \partial_x u)(t') \right\|_{L^2_{xy}} dt' \right),
\]

(77)

for some \( \epsilon_2 = \epsilon_2(s_1) \) such that \( 0 < \frac{1}{2}(\epsilon_2 - \frac{\alpha}{2}) \) and \( \frac{1}{2}(\epsilon_2 - \frac{\alpha}{2}) + 1 \leq s_1 \). The first term in the last line of equation above, thanks to the choice
of $\epsilon_2$, is less than $\|J_x^s \psi\|_{L_y^{2s}}$. For the term within the integral, from the Leibniz rule in the $x$ variable, we have

$$\left\| D_x^{\frac{1}{2}(\epsilon_2 - \frac{\alpha}{2})} (u \partial_x u) \right\|_{L_y^{2s}} \leq \left\| D_x^{\frac{1}{2}(\epsilon_2 - \frac{\alpha}{2})} u \right\|_{L_y^{2s}} \| \partial_x u \|_{L_y^{2s}} + \left\| D_x^{\frac{1}{2}(\epsilon_2 - \frac{\alpha}{2})} \partial_x u \right\|_{L_y^{2s}} \left\| u \right\|_{L_y^{2s}}$$

So that

$$\left\| u \right\|_{L_y^{1} L_y^{2s}} \leq C_2 T \int_0^T \left( \left\| J_x^s \psi \right\|_{L_y^{2s}} + \int_0^T \left( \left\| u \right\|_{L_y^{2s}} + \| \partial_x u \|_{L_y^{2s}} \right) \left\| J_x^s u \right\|_{L_y^{2s}} \, dt \right) \leq C_2 T \int_0^T \left( \left\| J_x^s \psi \right\|_{L_y^{2s}} + \int_0^T \left( \left\| u \right\|_{L_y^{2s}} + \| \partial_x u \|_{L_y^{2s}} \right) \left\| J_x^s u \right\|_{L_y^{2s}} \, dt \right) \leq C_2 T \int_0^T \left( 1 + f(T) \right) \left\| u \right\|_{L_y^{2s}}$$

where $C_2 = C_2(s_1)$. Lastly, let us estimate $\| \partial_y u \|_{L_y^{1} L_y^{2s}}$. In the same way as we estimate the other two terms, we have that

$$\left\| \partial_y u \right\|_{L_y^{1} L_y^{2s}} \leq C_3 T \int_0^T \left( \left\| D_x^{\frac{1}{2}(\epsilon_3 - \frac{\alpha}{2})} \partial_y \psi \right\|_{L_y^{2s}} + \int_0^T \left\| D_x^{\frac{1}{2}(\epsilon_3 - \frac{\alpha}{2})} \partial_y(u \partial_x u) \right\|_{L_y^{2s}} \, dt \right)$$

where we will choose $\epsilon_3$ later. For the term within the integral, from the commutator estimate (Lemma 1.10) in the $x$ variable, we obtain

$$\left\| D_x^{\frac{1}{2}(\epsilon_3 - \frac{\alpha}{2})+1} (u \partial_y u) \right\|_{L_y^{2s}} \leq \left\| D_x^{\frac{1}{2}(\epsilon_3 - \frac{\alpha}{2})+1} u \right\|_{L_y^{2s}} \| \partial_y u \|_{L_y^{2s}} + \left\| u D_x^{\frac{1}{2}(\epsilon_3 - \frac{\alpha}{2})} \partial_y u \right\|_{L_y^{2s}}$$

Let us, then, examine $\left\| D_x^{\frac{1}{2}(\epsilon_3 - \frac{\alpha}{2})+1} u \right\|_{L_y^{2s}}$ and $\left\| D_x^{\frac{1}{2}(\epsilon_3 - \frac{\alpha}{2})} \partial_y u \right\|_{L_y^{2s}}$. We will do this for two cases in $\alpha$. The first case is when $-1 \leq \alpha \leq 0$. In this case $\frac{1}{2}(\epsilon_3 - \frac{\alpha}{2}) > 0$ for $0 < \epsilon_3 < 1$. Choose $\epsilon_3$ in such a way that $\frac{1}{2}(\epsilon_3 - \frac{\alpha}{2}) + 1 < s_1$ and $\frac{1}{2s_1} - \frac{\alpha}{4s_1} + \frac{1}{s_2} < 1$, or equivalently $\frac{1}{2}(\epsilon_3 - \frac{\alpha}{2}) \frac{s_2}{s_2-1} < s_1$. So,

$$\left\| D_x^{\frac{1}{2}(\epsilon_3 - \frac{\alpha}{2})+1} u \right\|_{L_y^{2s}} \leq \left\| J_x^s u \right\|_{L_y^{2s}} \leq \left\| u \right\|_{H_y^{1+s_2}}$$
and

\[
\left\| D_x^{(\epsilon_3 - \frac{\alpha}{2})} \partial_y u \right\|_{L^2_{xy}} \leq \left\| J_x^{(\epsilon_3 - \frac{\alpha}{2})} J_y u \right\|_{L^2_{xy}} \leq \left\| J_x^{(\epsilon_3 - \frac{\alpha}{2})} J_x^{s_2} u \right\|_{L^2_{xy}}^\frac{s_2-1}{s_2} \left\| J_y^{s_2} u \right\|_{L^2_{xy}} \leq \left\| J_x^{s_1} u \right\|_{L^2_{xy}}^\frac{s_2-1}{s_2} \left\| J_y^{s_2} u \right\|_{L^2_{xy}} \leq \left\| u \right\|_{H^{s_1+s_2}} \left( \psi \right) \] (83)

The second case is when \( 0 \leq \alpha \leq 1 \). Choose, then, \( \epsilon_3 > \frac{\alpha}{2} \) such that also \( \frac{1}{2}(\epsilon_3 - \frac{\alpha}{2}) + \frac{\epsilon_3}{2s_1} - \frac{\alpha}{4s_1} + \frac{1}{s_2} < 1 \). So we are in the same situation of the first case, what leads us to the same two inequalities (82) and (83). This inequalities together to (81) and (80) allow us show that

\[
\left\| \partial_y u \right\|_{L^1_t L^\infty_{xy}} \leq C_3 T^{\frac{7+4\epsilon}{12}} \left( \left\| \psi \right\|_{H^{s_1,s_2}} + (1 + f(T)) \left\| u \right\|_{L^\infty_t H^{s_2}} \right). \] (84)

The theorem follows from (76), (79) and (84), taking \( C_{s_1,s_2} = C_1 + C_2 + C_3 \) and \( k_{s_1,s_2} = \max_{i=1,2,3} \frac{7+4\epsilon}{12} \).

3.4. Local well-posedness. We already have all the necessary ingredients to prove the following theorem.

**Theorem 3.9.** Let \(-1 < \alpha < 1\) and \(0 < \epsilon < 1\) be such that

\[
s_1 > \frac{17 + 4\epsilon - 3\alpha}{12}, \quad s_2 \leq s_1 \text{ and } \begin{cases} s_2 > 1, & \text{if } \alpha > 0, \\ \frac{5+4\epsilon-3\alpha}{12s_1} + \frac{1}{s_2} < 1, & \text{if } \alpha \leq 0. \end{cases}
\]

Then, for any \( \psi \in H^{s_1,s_2} (\mathbb{R}^2) \), there exist a time \( T = T(\left\| \psi \right\|_{H^{s_1,s_2}}) \) and a unique solution \( u \in C([0,T] : H^{s_1,s_2}) \) to the Cauchy problem (13) such that \( u, \partial_x u, \partial_y u \in L^\infty_t L^\infty_{xy} \). Furthermore, if \( 0 < T' < T \), there exists a neighborhood \( V \) of \( \psi \) in \( H^{s_1,s_2} (\mathbb{R}^2) \) such that \( \psi \mapsto u(t) \) is continuous.

From the local well-posedness of (13) in \( H^\sigma \) for \( \sigma > 2 \) (\( \sigma \) fix), for \( \psi \in H^\sigma (\mathbb{R}^2) \), there exists a unique solution \( u \in C([0,T^*] : H^\sigma (\mathbb{R}^2)) \), where \( T^* \) is the maximal existence time of the solution, such that if \( T^* < \infty \), then

\[
\lim_{t \uparrow T^*} \left\| u(t) \right\|_{H^\sigma} = \infty. \] (85)

Thanks to the above we have the following a priori estimate that will be useful in the proof of the existence of the solution to problem (13).

**Lemma 3.10.** Let \( 2 < \sigma, \ -1 \leq \alpha \leq 1 \) and \( s_1, s_2 \) be as in Theorem 3.9 such that \( s_i \leq \sigma, \ i = 1,2 \). Let, also, \( \psi \in H^\sigma \) and \( u \in C([0,T^*]; H^\sigma (\mathbb{R}^2)) \) such that \( u(0) = \psi \), where \( T^* \) is the maximal time.
of existence of $u$. Then, there exist $K_0 = K_0(s_1, s_2) > 0$ and $L_{s_1, s_2} > 0$ such that $T^* > T$, where $(L_{s_1, s_2})^\psi_{H^{s_1, s_2}} + 1)^{\frac{1}{2}} T = 1$, and
\[
\|u\|_{L^\infty_T H^{s_1, s_2}} \leq 2 \|\psi\|_{H^{s_1, s_2}}
\]
\[
f(T) = \|u\|_{L^1_T L^\infty_y} + \|\partial_x u\|_{L^1_T L^\infty_y} + \|\partial_y u\|_{L^1_T L^\infty_y} \leq K_0
\]
(86)

Proof. For $s_1$ and $s_2$ as in the statement of lemma, let
\[
T_0 = \sup_{T \in (0, T^*)} \left\{ \bar{T} \mid \|u\|_{L^\infty_T H^{s_1, s_2}} \leq 2 \|\psi\|_{H^{s_1, s_2}} \right\}
\]
(87)

From the local well-posedness, this set is not empty. Let $C_{s_1, s_2}$ be as in Lemma 3.7, $L_{s_1, s_2} = 2(2C - 1)C_{s_1, s_2}$ and
\[
T = \frac{1}{(L_{s_1, s_2})^\psi_{H^{s_1, s_2}} + 1)^{\frac{1}{2}}}
\]
Let us see that $T < T_0$. Suppose not. Thanks to Lemma 3.8
\[
f(T_0) \leq C_{s_1, s_2} T_0^{k_{s_1, s_2}} (1 + f(T_0)) \|u\|_{L^\infty_T H^{s_1, s_2}} \leq C_{s_1, s_2} T_0^{k_{s_1, s_2}} (1 + f(T_0)) \|u\|_{L^\infty_T H^{s_1, s_2}}
\]
(88)

Since $\|u\|_{L^\infty_T H^{s_1, s_2}} \leq 2 \|\psi\|_{H^{s_1, s_2}}$ and $T_0 < T$,
\[
f(T_0) \leq 2C_{s_1, s_2} \left( 1 + f(T_0) \right) \|u\|_{L^\infty_T H^{s_1, s_2}} + 1 \|u\|_{H^{s_1, s_2}}
\]
or equivalently
\[
(4C_{s_1, s_2} \|\psi\|_{H^{s_1, s_2}} + 1) f(T_0) \leq 2C_{s_1, s_2} \|\psi\|_{H^{s_1, s_2}}
\]

Therefore,
\[
f(T_0) \leq \frac{1}{2C}
\]

By Lemma 3.7 we would have
\[
\|u(T_0)\|_{H^{s_1, s_2}} \leq \epsilon \|\psi\|_{H^{s_1, s_2}} < 2 \|\psi\|_{H^{s_1, s_2}}
\]

From the continuity of $u$, it follows that there exists $\bar{T} > T_0$ such that
\[
\|u\|_{L^\infty_{\bar{T}} H^{s_1, s_2}} \leq 2 \|\psi\|_{H^{s_1, s_2}}
\]
which contradicts the choice of $T_0$. Then, $T < T_0$. In particular,
\[
\|u\|_{L^\infty_{T} H^{s_1, s_2}} \leq 2 \|\psi\|_{H^{s_1, s_2}}
\]

and, repeating the above reasoning, from inequality (88), taking $T$ instead of $T_0$ we have that
\[
f(T) \leq \frac{1}{2C}
\]

This proves the lemma. \qed
**Corollary 3.11.** Let \( \psi \) and \( T \) be as in the last lemma. If \( \psi \in H^\infty \), then the solution \( u \) to the Cauchy problem (13), with \( u(0) = \psi \), belongs to the set \( C([0, T]; H^\infty) \).

**Proof.** Let \( T \) be as in the lemma above. From that lemma, for any number \( \sigma \) that satisfies the condition established there, we have \( u \in C([0, T]; H^\infty) \). From here it follows the corollary.

**Corollary 3.12.** Let \( R > 0 \) and \( \psi \in H^\infty \) be such that \( \|\psi\|_{H^{s_1, s_2}} \leq R \). Then, there exists \( T_0 \) that depends on \( R \) and \( M \), constant that depends only on \( s_1, s_2 \), such that \( u \in C([0, T_0]; H^\infty) \) and

\[
f(T_0) = \|u\|_{L^1_{t_0} L^2_{xy}} + \|\partial_x u\|_{L^1_{t_0} L^2_{xy}} + \|\partial_y u\|_{L^1_{t_0} L^2_{xy}} \leq M.
\]

**Proof.** If we set \( T_0 = 1/(L_{s_1, s_2} R + 1)^{\frac{1}{4}} \), we have that \( T_0 \leq T \). From the proof of that lemma it follows that \( f(T_0) \leq 1/2C \), that does not depend on initial data, only on \( s_1 \) and \( s_2 \). Making \( M = 1/2C \) shows the corollary.

3.4.1. **Proof of Theorem 3.9**

**Lemma 3.13.** Suppose that \( \psi \) and \( \phi \in H^\infty \) and that \( u \) and \( v \in C([0, T]; H^\infty) \) are the solutions to the problem (13) with initial conditions \( \psi \) and \( \phi \), respectively. Then,

\[
\|u - v\|_{L^2(T_0(R))} \leq \|\psi - \phi\|_{L^2} e^{CM},
\]

where \( T_0(R) \) and \( M \) are as in Corollary 3.12 \( C \) is a constant that depends only on \( s_1 \) and \( s_2 \), and \( R \) is the maximum between the norm of \( \phi \) and \( \psi \) in the space \( H^{s_1, s_2} \).

**Proof.** The proof is analogous to obtaining the energy estimate. Let us see. Let \( u \) and \( v \) be as in statement of lemma. Then,

\[
\partial_t(u-v) = \partial_x^3(u-v) - HD^\alpha \partial_y^2(u-v) + \frac{1}{2}(u+v)\partial_x(u-v) + \frac{1}{2}(u-v)\partial_x(u+v).
\]

Multiplying by \( u - v \) on both sides the equation and integrating by parts we have that

\[
\frac{1}{2} \frac{d}{dt} \|u - v\|_2^2 \leq \frac{1}{4} (\|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}) \|u - v\|_2^2.
\]

From the Gronwall Lemma and Corollary 3.12 it follows the lemma.

Now, let \( \psi \in H^{s_1, s_2} \) and suppose that \( \psi_n \) is a sequence of functions in \( H^\infty \) that converges to \( \psi \) in \( H^{s_1, s_2} \). Taking \( R = \sup_n \|\psi_n\| \), from lemma above, the solutions to (13) \( u_n \in C([0, T_0]; H^\infty(\mathbb{R}^2)) \), with initial condition \( \psi_n \), converge uniformly to a function \( u \) in \( C([0, T_0]; L^2(\mathbb{R}^2)) \). Moreover, from Corollary 3.12 the functions \( u_n \) are uniformly bounded in
$H^{s_1,s_2}$. Therefore, from Banach-Alaoglu theorem, $u_n(t)$ have a subsequence that converges weakly in $H^{s_1,s_2}$. From the uniform convergence of $u_n(t)$ to $u(t)$ in $L^2$, it follows that $u(t) \in H^{s_1,s_2}$, for each $t \in [0,T_0]$. From the continuity of $u$ from $[0,T_0]$ to $L^2$ follows the weak continuity of $u$ from $[0,T_0]$ to $H^{s_1,s_2}$. In particular, from uniform boundedness of the sequence $u_n(t)$ and Lemma 3.14, it follows that $u_n$ converges strongly and uniformly in $H^{s_1,s_2}$ to $u$, for any pair of non negative real numbers $s_1', s_2'$ strictly less than $s_1$, $s_2$, respectively. Since each one of the functions $u_{n,i}$ satisfies the integral equation associated to (13),

$$w = W_\alpha(t)w(0) + \frac{1}{2} \int_0^t W_\alpha(t-t')\partial_x(w^2(t')) dt',$$

in $H^{s_1,-1,s_2-1}$, it follows that function $u$ satisfies this same integral equation in $H^{s_1,-1,s_2-1}$, $1 < s'_2 < s_2$. From Lemma 3.15 considering $17/12 - \alpha/4 < s'_1 < s_1$, follows that $\partial_x u \in L^1_T L^\infty_{x,y}$. In the same way, thanks to the inequalities (77) and (80), we obtain that $u$ and $\partial_y u \in L^1_T L^\infty_{x,y}$. So,

**Lemma 3.14.** Let $\psi \in H^{s_1,s_2}(\mathbb{R}^2)$. Then, there exists $T > 0$ and a unique $u \in C([0,T]; H^1(\mathbb{R}^2)) \cap C([0,T]; H^2(\mathbb{R}^2)) \cap (X^1) \cap (X^3)^*$ solution to problem (13). Furthermore, $u$, $\partial_x u$, $\partial_y u \in L^1_T L^\infty_{x,y}$ and the map $\psi \mapsto u$ is Lipschitz continuous from $L^2(\mathbb{R}^2)$ to $C([0,T]; L^2(\mathbb{R}^2))$.

**Proof.** It only remains to prove that $u$ is unique and that the map $\psi \mapsto u$ is Lipschitz continuous from $L^2(\mathbb{R}^2)$ to $C([0,T]; L^2(\mathbb{R}^2))$. Let $u$ and $v$ be as in the statement of lemma. Since $u$ and $v$ are solutions to the integral equation, from Lemma 3.15 $u_x$ and $v_x \in L^1_T L^\infty_{x,y}$. Now proceed as in the lemma above, but we need Lemma (13), to obtain that

$$\frac{1}{2} \frac{d}{dt}\|u-v\|_{L^2}^2 = \frac{1}{4} \|\partial_x u + \partial_x v, (u-v)^2\|_{L^2} \leq (\|u_x\|_{L^\infty_{x,y}} + \|v_x\|_{L^\infty_{x,y}})\|u-v\|_{L^2}^2.$$ 

Since $\|u_x\|_{L^\infty_{x,y}} + \|v_x\|_{L^\infty_{x,y}}$ is integrable and $\|u-v\|_{L^2}^2$ is continuous, from Gronwall lemma, it follows the theorem.

**Lemma 3.15.** For $\psi \in H^{s_1,s_2}$ as in the lemma above, the solution $u$ to (13) described there is strongly continuous in $H^{s_1,s_2}$

**Proof.** Let $\psi$, $\psi_n$, $u_n$, $n = 1, 2, 3, \ldots$, and $u$ be as before. Let, also, $M$, $T_0$ be as in Corollary 3.12 and $T \leq T_0$. It is clear that $f_n(T) = \|u_n\|_{L^\infty_{x,y}} + \|\partial_x u_n\|_{L^\infty_{x,y}} + \|\partial_y u_n\|_{L^\infty_{x,y}}$ satisfies that

$$f_n(T) \leq M,$$

for all $n$. From Lemma 3.8 and Corollary 3.12 it follows that

$$f_n(T) \leq 2C_{s_1,s_2}T^{k_{s_1,s_2}} + (1 + M)R.$$
From Lemma 3.7

\[ \|u_n(T)\|_{H^{s_1,s_2}} \leq \|\psi_n\|_{H^{s_1,s_2}} e^{2C_{s_1,s_2} T^{k s_1 + s_2} (1+M) R}, \]

for all \(n\). From the weak convergence of \(u_n(T)\) to \(u(T)\) in \(H^{s_1,s_2}\) and since \(\psi_n\) converges to \(\psi\) in \(H^{s_1,s_2}\), it follows that

\[ \|u(T)\|_{H^{s_1,s_2}} \leq \|\psi\|_{H^{s_1,s_2}} e^{2C_{s_1,s_2} T^{k s_1 + s_2} (1+M) R}. \]

From this last inequality and the weak continuity of \(u\) in \(H^{s_1,s_2}\), it follows that

\[ \|\psi\|_{H^{s_1,s_2}} \leq \liminf_{T \to 0^+} \|u(T)\|_{H^{s_1,s_2}} \leq \limsup_{T \to 0^+} \|u(T)\|_{H^{s_1,s_2}} \leq \|\psi\|_{H^{s_1,s_2}}. \]

Then, \(u\) is right strongly continuous at 0 in the space \(H^{s_1,s_2}\). Since \(u(-t,-x,-y)\) is also solution to the equation (13), we have that \(u\) is also left strongly continuous at 0 in the space \(H^{s_1,s_2}\). Now, for any \(t^* \in [0,T]\), \(u(t + t^*)\) is also solution to the equation (13) with initial condition \(u(t^*)\). Then from the unicity of the solution, \(u\) also is strongly continuous at \(t^*\) in the space \(H^{s_1,s_2}\). This ends the proof. \(\Box\)

Now examine the continuity of the solutions to (13) with respect to initial data. For this purpose we will use a technique very useful and recurrently used in the literature related to the well-posedness of evolution equations. This technique is the Bona-Smith method of approximation introduced in [7]. Let us see.

**Lemma 3.16.** Let \(\phi \in H^{s_1,s_2}\), \(s_1\) and \(s_2\) be positive real numbers. For each \(\tau > 0\) define \(\phi^\tau\) by

\[ \phi^\tau(x,y) = \left( \hat{\phi}(\xi,\eta) \exp \left( -\tau \left( (1 + |\xi|^2)^{\frac{s_1}{2}} + (1 + |\eta|^2)^{\frac{s_2}{2}} \right) \right) \right) \sim (x,y). \]  

Then

\[ \lim_{\tau \to 0^+} \|\phi^\tau - \phi\|_{H^{s_1,s_2}} = 0 \]

and there exists a constant \(C = C(s)\) such that

\[ \|\phi^\tau\|_{H^{s_1+1,s_2}} \leq C \left( \frac{1}{\tau s_1} \right)^{\frac{1}{s_1}} \|\phi\|_{H^{s_1,s_2}}, \]

\[ \|\phi^\tau\|_{H^{s_1,s_2+1}} \leq C \left( \frac{1}{\tau s_2} \right)^{\frac{1}{s_2}} \|\phi\|_{H^{s_1,s_2}} \]

and

\[ \|\phi^\tau - \phi^\theta\|_{L^2} \leq C |\tau - \theta| \|\phi\|_{H^{s_1,s_2}} \]
Proposition 3.17. Let \( R > 0 \), and assume that \( \Lambda \) is a set and \( \psi_{\lambda \in \Lambda} \) is a collection of functions in \( H^\infty \) such that \( \|\psi_\lambda\|_{H^{s_1,s_2}} \leq R \), for all \( \lambda \in \Lambda \). Also, let \( \psi_\lambda^\tau \) be the approximations defined from \( \psi_\lambda \) as in (90), and assume that \( u_\lambda^\tau \) is the solution to (13) with initial condition \( \psi_\lambda^\tau \), for all \( \lambda \in \Lambda \). Suppose that \( 0 \leq \theta \leq \tau \). Then, for \( \nu > 0 \)

\[
\|u_\lambda^\tau - u_\lambda^\theta\|_{H^{s_1,s_2}}^2 \leq C \left( \|\psi_\lambda^\tau - \psi_\lambda^\theta\|_{H^{s_1,s_2}}^2 + \tau^\nu \right).
\]

for all \( \lambda \in \Lambda \).

Proof. It is evident that \( u_\lambda^\tau(t) \) also is defined on \([0,T_0]\), for all \( n \) and \( \tau \). We will proceed as in the proof of Lemma 3.7. Then,

\[
\frac{1}{2} \frac{d}{dt} \|J_x^{s_1}(u_\lambda^\tau - u_\lambda^\theta)\|_{L^2}^2 = \int_{\mathbb{R}^2} J_x^{s_1}(u_\lambda^\tau \partial_x u_\lambda^\tau - u_\lambda^\theta \partial_x u_\lambda^\theta) J_x^{s_1}(u_\lambda^\tau - u_\lambda^\theta) \, dx \, dy + \frac{1}{2} \frac{\partial}{\partial \tau} \int_{\mathbb{R}^2} J_x^{s_1}(u_\lambda^\tau \partial_x u_\lambda^\tau - u_\lambda^\theta \partial_x u_\lambda^\theta) J_x^{s_1}(u_\lambda^\tau - u_\lambda^\theta) \, dx \, dy + \frac{1}{2} \frac{\partial}{\partial \nu} \int_{\mathbb{R}^2} J_x^{s_1}(u_\lambda^\tau \partial_x u_\lambda^\tau - u_\lambda^\theta \partial_x u_\lambda^\theta) J_x^{s_1}(u_\lambda^\tau - u_\lambda^\theta) \, dx \, dy.
\]

We estimate the last two terms that appear in the inequality above. From Lemma 1.3 for the first term, we have

\[
\frac{1}{2} \int_{\mathbb{R}^2} J_x^{s_1}(u_\lambda^\tau \partial_x u_\lambda^\tau - u_\lambda^\theta \partial_x u_\lambda^\theta) J_x^{s_1}(u_\lambda^\tau - u_\lambda^\theta) \, dx \, dy \leq C(\|\partial_x(u_\lambda^\tau - u_\lambda^\theta)\|_{L^2} \|J_x^{s_1-1}(u_\lambda^\tau - u_\lambda^\theta)\|_{L^2} \|J_x^{s_1}(u_\lambda^\tau - u_\lambda^\theta)\|_{L^2} + \|J_x^{s_1}(u_\lambda^\tau - u_\lambda^\theta)\|_{L^2} \|J_x^{s_1}(u_\lambda^\tau - u_\lambda^\theta)\|_{L^2}) - \frac{1}{2} \int_{\mathbb{R}^2} \partial_x(u_\lambda^\tau - u_\lambda^\theta) J_x^{s_1}(u_\lambda^\tau - u_\lambda^\theta) \, dx \, dy.
\]

\[
\leq C(\|\partial_x(u_\lambda^\tau + u_\lambda^\theta)\|_{L^\infty} \|J_x^{s_1-1}(u_\lambda^\tau + u_\lambda^\theta)\|_{L^2} \|J_x^{s_1}(u_\lambda^\tau + u_\lambda^\theta)\|_{L^\infty} \|J_x^{s_1}(u_\lambda^\tau - u_\lambda^\theta)\|_{L^2}) - \frac{1}{2} \int_{\mathbb{R}^2} \partial_x(u_\lambda^\tau + u_\lambda^\theta) (J_x^{s_1}(u_\lambda^\tau - u_\lambda^\theta))^2 \, dx \, dy.
\]

\[
(95)
\]

\[
(94)
\]
With the second term we proceed in the same way to obtain the following inequality

\[
\frac{1}{2} \int_{\mathbb{R}^2} J_x^{s_2} \left( \partial_x (u_\lambda^\tau + u_\lambda^\theta) (u_\lambda^\tau - u_\lambda^\theta) \right) \partial_x (u_\lambda^\tau - u_\lambda^\theta) \, dx \, dy \leq \]

\[
\leq C \left( \left\| \partial_x u_\lambda^\tau \right\|_{L^\infty} + \left\| \partial_x u_\lambda^\theta \right\|_{L^\infty} \right) \left\| J_x^{s_2} (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^2}^2 + \]

\[
+ \left\| J_x^{s_2} (u_\lambda^\tau + u_\lambda^\theta) \right\|_{L^2} \left\| \partial_x (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^\infty} \left\| J_x^{s_2} (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^2} \right) \quad (96)
\]

So, \( \| \psi^\lambda_\lambda \|_{H^{s_1, s_2}} \leq R \), for all \( \lambda \), from (94), (95), (96) and Lemma 3.10 we have

\[
\frac{1}{2} \frac{d}{dt} \left\| J_x^{s_1} (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^2}^2 \leq C \left( \left\| \partial_x u_\lambda^\tau \right\|_{L^\infty} + \left\| \partial_x u_\lambda^\theta \right\|_{L^\infty} \right) \left\| J_x^{s_2} (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^2}^2 + \]

\[
+ \left\| \partial_x (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^\infty} \left\| J_x^{s_2} (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^2} \quad (97)
\]

On the other hand,

\[
\frac{1}{2} \frac{d}{dt} \left\| J_y^{s_2} (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^2}^2 = \int_{\mathbb{R}^2} J_y^{s_2} \left( (u_\lambda^\tau + u_\lambda^\theta) (u_\lambda^\tau - u_\lambda^\theta) \right) \partial_x (u_\lambda^\tau - u_\lambda^\theta) \, dx \, dy
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^2} J_y^{s_2} \partial_x \left( (u_\lambda^\tau + u_\lambda^\theta) (u_\lambda^\tau - u_\lambda^\theta) \right) J_y^{s_2} \partial_x (u_\lambda^\tau - u_\lambda^\theta) \, dx \, dy
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^2} J_y^{s_2} \left( (u_\lambda^\tau + u_\lambda^\theta) \partial_x (u_\lambda^\tau - u_\lambda^\theta) \right) J_y^{s_2} (u_\lambda^\tau - u_\lambda^\theta) \, dx \, dy + \]

\[
+ \frac{1}{2} \int_{\mathbb{R}^2} J_y^{s_2} \partial_x (u_\lambda^\tau + u_\lambda^\theta) (u_\lambda^\tau - u_\lambda^\theta) \, J_y^{s_2} (u_\lambda^\tau - u_\lambda^\theta) \, dx \, dy. \quad (98)
\]

Proceeding in the same way that allows us to obtain (95) and (96), it follows that

\[
\frac{1}{2} \int_{\mathbb{R}^2} J_y^{s_2} \left( (u_\lambda^\tau + u_\lambda^\theta) \partial_x (u_\lambda^\tau - u_\lambda^\theta) \right) J_y^{s_2} \partial_x (u_\lambda^\tau - u_\lambda^\theta) \, dx \, dy \leq \]

\[
\leq C \left( \left\| \partial_y u_\lambda^\tau \right\|_{L^\infty} + \left\| \partial_y u_\lambda^\theta \right\|_{L^\infty} \right) \times
\]

\[
\times \left( \left\| J_y^{s_2-1} \partial_x (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^2} \left\| J_y^{s_2} (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^2}^2 + \left\| J_y^{s_2} (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^2} \left\| J_y^{s_2} (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^2} \right) + \]

\[
+ \left\| J_y^{s_2} (u_\lambda^\tau + u_\lambda^\theta) \right\|_{L^2} \left\| \partial_x (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^\infty} \left\| J_y^{s_2} (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^2} \right)
\]

\[
\leq C \left( \left\| \partial_y u_\lambda^\tau \right\|_{L^\infty} + \left\| \partial_y u_\lambda^\theta \right\|_{L^\infty} \right) \left\| J_x^{s_2} (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^2}^2 + \left\| J_y^{s_2} (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^2} \right) + \]

\[
+ \left\| \partial_x (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^\infty} \left\| J_y^{s_2} (u_\lambda^\tau - u_\lambda^\theta) \right\|_{L^2} \right) \quad (99)
\]
and

\[
\frac{1}{2} \int_{\mathbb{R}^2} J_y^{s_2}(\partial_x(u^r_\lambda + u^\vartheta_\lambda)(u^r_\lambda - u^\vartheta_\lambda)) J_y^{s_2}(\partial_x(u^r_\lambda - u^\vartheta_\lambda)) \, dx \, dy \leq \\
\leq C(\|\partial_x u^r_\lambda \|_{L^\infty} + \|\partial_x u^\vartheta_\lambda \|_{L^\infty} + \|\partial_y u^r_\lambda \|_{L^\infty} + \|\partial_y u^\vartheta_\lambda \|_{L^\infty}) \times \\
\times \left( \|J_x^{s_1}(u^r_\lambda - u^\vartheta_\lambda)\|_{L^2}^2 + \|J_y^{s_2}(u^r_\lambda - u^\vartheta_\lambda)\|_{L^2}^2 \right) + \\
+ \|\partial_x(u^r_\lambda - u^\vartheta_\lambda)\|_{L^2} + \|\partial_y(u^r_\lambda - u^\vartheta_\lambda)\|_{L^2} \right) (100)
\]

Therefore, from (98), (99) and (100), we have

\[
\frac{1}{2} \frac{d}{dt} \|J_y^{s_2}(u^r_\lambda - u^\vartheta_\lambda)\|_{L^2}^2 \leq \\
\leq C(\|\partial_x u^r_\lambda \|_{L^\infty} + \|\partial_x u^\vartheta_\lambda \|_{L^\infty} + \|\partial_y u^r_\lambda \|_{L^\infty} + \|\partial_y u^\vartheta_\lambda \|_{L^\infty}) \times \\
\times \left( \|J_x^{s_1}(u^r_\lambda - u^\vartheta_\lambda)\|_{L^2}^2 + \|J_y^{s_2}(u^r_\lambda - u^\vartheta_\lambda)\|_{L^2}^2 \right) + \\
+ \|\partial_x(u^r_\lambda - u^\vartheta_\lambda)\|_{L^\infty} + \|\partial_y(u^r_\lambda - u^\vartheta_\lambda)\|_{L^\infty}) (101)
\]

Gathering (97) and (101), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|u^r_\lambda - u^\vartheta_\lambda\|_{H^{s_1, r_2}}^2 \leq \\
\leq C((\|\partial_x u^r_\lambda \|_{L^\infty} + \|\partial_x u^\vartheta_\lambda \|_{L^\infty} + \|\partial_y u^r_\lambda \|_{L^\infty} + \|\partial_y u^\vartheta_\lambda \|_{L^\infty})\|u^r_\lambda - u^\vartheta_\lambda\|_{H^{s_1, r_2}}^2 + \\
+ \|\partial_x(u^r_\lambda - u^\vartheta_\lambda)\|_{L^\infty} + \|\partial_y(u^r_\lambda - u^\vartheta_\lambda)\|_{L^\infty}) (102)
\]

Integrating, it follows that

\[
\|u^r_\lambda - u^\vartheta_\lambda\|_{H^{s_1, r_2}}^2 \leq \|\psi^r_\lambda - \psi^\vartheta_\lambda\|_{H^{s_1, r_2}}^2 + \\
+ C \int_0^t ((\|\partial_x u^r_\lambda \|_{L^\infty} + \|\partial_x u^\vartheta_\lambda \|_{L^\infty})\|u^r_\lambda - u^\vartheta_\lambda\|_{H^{s_1, r_2}}^2 dt' + \\
+ \|\partial_x(u^r_\lambda - u^\vartheta_\lambda)\|_{L^\infty} + \|\partial_y(u^r_\lambda - u^\vartheta_\lambda)\|_{L^2} \right) (103)
\]

Let us estimate the last two terms of the last inequality. Observe that

\[
u^r_\lambda - u^\vartheta_\lambda = W_\alpha(t)(\psi^r_\lambda - \psi^\vartheta_\lambda) + \int_0^t W_\alpha(t - t')\partial_x((u^r_\lambda + u^\vartheta_\lambda)(u^r_\lambda - u^\vartheta_\lambda)) (t') \, dt'.
\]
Therefore, proceeding as in the proof of Lemma 3.8 from refined the Strichartz estimate (57) and Lemma 3.13, we have

\[
\| \partial_x (u_\lambda^\tau - u^\theta_\lambda) \|_{L_t^1 L_x^p} \leq C (\| u_{\lambda}^\tau - u^\theta_\lambda \|_{L_t^\infty H_x^{s_1, s_2}} + \| (u_{\lambda}^\tau + u^\theta_\lambda) (u_{\lambda}^\tau - u^\theta_\lambda) \|_{L_t^2 H_x^{s_1, s_2}})
\]

\[
\leq C (\| u_{\lambda}^\tau - u^\theta_\lambda \|_{L_t^\infty H_x^{s_1, s_2}} + \| u_{\lambda}^\tau + u^\theta_\lambda \|_{L_t^2 H_x^{s_1, s_2}} | u_{\lambda}^\tau - u^\theta_\lambda |_{L_t^2 H_x^{s_1, s_2}})
\]

\[
\leq C \| u_{\lambda}^\tau - u^\theta_\lambda \|_{L_t^\infty H_x^{s_1, s_2}}
\]

\[
\leq C \| \psi_{\lambda}^\tau - \psi^\theta_\lambda \|_{L_x^\nu} \| u_{\lambda}^\tau - u^\theta_\lambda \|_{L_t^\nu H_x^{s_1, s_2}}^{1-\nu} \leq C \tau^{\nu},
\]

where \(s_i', s_i''\) and \(\nu\) are such that \(1 < s_i' = \nu s_i < s_i, i = 1, 2\). In the same way, again, if we proceed as in the proof of Lemma 3.8, more precisely, in the way that we got (84), we also have

\[
\| \partial_y (u_{\lambda}^\tau - u^\theta_\lambda) \|_{L_t^1 L_x^p} \leq C \tau^{\nu}.
\]

From these last two inequalities with (103) and the Gronwall Lemma follows the proposition.

\[\square\]

**Corollary 3.18.** Let \(\psi_{\lambda}^\tau\) and \(u_{\lambda}^\tau\) be as in the result above. Then, \(\{u_{\lambda}^\tau\}_{\tau > 0}\) converges uniformly in \(t\) to \(u_{\lambda}\) when \(\tau \to 0^+\). In other words,

\[
\lim_{\tau \to 0^+} \sup_{t \in [0, T]} \| u_{\lambda}^\tau(t) - u_{\lambda}(t) \|_{H^{s_1, s_2}} = 0.
\]

**Proof.** Take \(\theta = 0\) in Proposition 3.17. \[\square\]

**Theorem 3.19.** In \(H^{s_1, s_2}\) the map \(\psi \mapsto u\), where \(u\) is solution to (13) with initial condition \(\psi\), is continuous. More precisely, if \((\psi_n)_{n \in \mathbb{N}}\) is a sequence such that \(\psi_n \to \psi\) in \(H^{s_1, s_2}\) and if \(u_n \in C([0, T_0]; H^{s_1, s_2})\) are the corresponding solutions to (13) with initial condition \(\psi_n\), then

\[
\lim_{n \to \infty} \sup_{t \in [0, T_0]} \| u_n(t) - u(t) \| = 0.
\]

**Proof.** Let \(\psi \in H^{s_1, s_2}\) and \((\psi_n)\) be a sequence in \(H^{s_1, s_2}\) that converges strongly to \(\psi\) in this space. Let, also, \(R = \max (\sup_n \| \psi_n \|_{H^{s_1, s_2}}, \| \psi \|_{H^{s_1, s_2}})\). Now, let us take \(\psi_{m,n} = e^{-\frac{t}{T_0} \Delta} \psi_n\) and \(\psi_m = e^{-\frac{t}{T_0} \Delta} \psi\). Let \(u_n, u, u_{m,n}, u_m, u_n^\tau, u_m^\tau, u_{m,n}^\tau, u_m^\tau\) be the corresponding solutions to (13) in \([0, T_0]\) with conditions \(\psi_n, \psi, \psi_{m,n}, \psi_m, \psi_n^\tau, \psi_m^\tau, \psi_{m,n}^\tau, \psi_m^\tau\), respectively. Also, observe that \(u_{m,n}, u_m, u_{m,n}^\tau\) and \(u_m^\tau\) converge uniformly to \(u_n, u, u_n^\tau\) and
exists \tau in the weak sense, as \( m \to \infty \). Therefore,

\[
\langle u_n - u, \varphi \rangle_{H^{s_1, s_2}} = \lim_{m \to \infty} \langle u_{m,n} - u_{m,n}, \varphi \rangle_{H^{s_1, s_2}} + \langle u_{m,n} - u_{m}, \varphi \rangle_{H^{s_1, s_2}} + \langle u_{m,n} - u_{m}, \varphi \rangle_{H^{s_1, s_2}} + \langle u_{m} - u_{m}, \varphi \rangle_{H^{s_1, s_2}}
\]

\[
= \lim_{m \to \infty} \left[ \langle u_{m,n} - u_{m,n}, \varphi \rangle_{H^{s_1, s_2}} + \langle u_{m,n} - u_{m}, \varphi \rangle_{H^{s_1, s_2}} \right] + \langle u_{m} - u_{m}, \varphi \rangle_{H^{s_1, s_2}}.
\]

On the other hand, Corollary 3.18 implies that, given \( \epsilon > 0 \), there exists \( \tau_0 \) such that for \( 0 < \tau \leq \tau_0 \)

\[
|\langle u_{m,n} - u_{m,n}, \varphi \rangle_{H^{s_1, s_2}} + \langle u_{m,n} - u_{m}, \varphi \rangle_{H^{s_1, s_2}}| \leq \epsilon \|\varphi\|_{H^{s_1, s_2}},
\]

for all \( m > 0 \). Then,

\[
|\langle u_n - u, \varphi \rangle_{H^{s_1, s_2}}| \leq \epsilon \|\varphi\|_{H^{s_1, s_2}} + \|u_n - u\|_{H^{s_1, s_2}} \|\varphi\|_{H^{s_1, s_2}},
\]

for all \( \varphi \in H^{s_1, s_2} \). Therefore,

\[
\|u_n - u\|_{H^{s_1, s_2}} \leq \epsilon + \|u_n - u\|_{H^{s_1, s_2}}. \tag{104}
\]

Arguments similar to those used in Proposition 3.17 allow us show that, for \( \tau \) small enough,

\[
\|u_n - u\|_{H^{s_1, s_2}} \leq C \|\psi_n - \psi\|_{H^{s_1, s_2}} \tau^{-\frac{1}{2}} \leq \|\psi_n - \psi\|_{H^{s_1, s_2}} \tau^{-\frac{1}{2}}.
\]

Then, fixing \( \tau \) small enough, we can conclude from (104) that

\[
\|u_n - u\|_{H^{s_1, s_2}} \leq 2\epsilon,
\]

for \( n \) large enough. \( \square \)

Let us see now that the problem is locally well-posed in the spaces \( X^{s_1, s_2}, \tilde{X}^{s_1, s_2} \) and \( Y^{s_1, s_2} \). Since the solution to (13) satisfy the integral equation

\[
u = W_\alpha(t)\psi + \int_0^t W_\alpha(t - t')(u\partial_x u)(t') \, dt',
\]

we have that \( \partial_x^{-1}u \) and \( \partial_x^{-1}\partial_y u \) satisfy the equations

\[
\partial_x^{-1}u = W_\alpha(t)\partial_x^{-1}\psi + \int_0^t W_\alpha(t - t') \left( \frac{u^2}{2} \right) (t') \, dt'
\]

and

\[
\partial_x^{-1}\partial_y u = W_\alpha(t)\partial_x^{-1}\partial_y \psi + \int_0^t W_\alpha(t - t')(u\partial_y u)(t') \, dt',
\]

respectively. From here and Theorem 3.9, for \( s_1 \) and \( s_2 \) as in that theorem, it follows that (13) is locally well-posed in the spaces \( \tilde{X}^{s_1, s_2}, X^{s_1, s_2} \) and \( Y^{s_1, s_2} \). The case \( Y^{s_1, s_2} \) requires some additional effort. For \( \psi \in Y^{s_2} \), the solution \( u \) to (13), with initial condition \( \psi \), belongs to
satisfies the Duhamel equation associated to the equation (13), i.e.,

Proof. Let’s see first what we should show. For this we consider

and that the flow

C([0, T_0]; Y^\infty). We can proceed as in the proof Lemma 3.7 to show that

\[
\frac{1}{2} \frac{d}{dt} \| \partial_x^{-1} \partial_y u \|^2 \leq C(\|u_x\| + \|u_y\|) \|u\|_{Y^{s_1,s_2}}^2.
\]

This inequality with (70) allow us prove that

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2_{Y^{s_1,s_2}} \leq C(\|u_x\| + \|u_y\|) \|u\|_{Y^{s_1,s_2}}^2.
\]

(105)

Thanks to the Gronwall lemma we have the following generalization of Lemma 3.7

\[
\|u\|_{Y^s} \leq \|\psi\|_{Y^{s_1,s_2}} e^{C(\|u_x\|_{L^1_t L^\infty_x} + \|u_y\|_{L^1_t L^\infty_y})}.
\]

Now, if \(\psi\) is arbitrary, in the same way that we prove for the space \(H^{s_1,s_2}\), the solution \(u \in C([0, T]; Y^{s_1,s_2})\). To see that the map \(\psi \mapsto u\) is continuous from \(Y^{s_1,s_2}\) to \(C([0, T]; Y^{s_1,s_2})\) we repeat the same argument of Bona-Smith that we use before. So, summarizing, we paraphrase Theorem (2.7) for the current situation.

Theorem 3.20. Let \(s_1, s_2\) and \(\alpha\) be as in Theorem 3.7. Let, also, \(Z\) any of the spaces \(X^{s_1,s_2}, \hat{X}^{s_1,s_2}, Y^{s_1,s_2}\) and \(\hat{Y}^{s_1,s_2}\). Then, if \(\psi \in Z\) and \(u \in C([0, T]; H^{s_1,s_2})\) is solution to (13) with \(u(0) = \psi\), then \(u \in C([0, T]; Z)\). Moreover, \(\psi \mapsto u\) is continuous from \(Z\) to \(C([0, T]; Z)\)

4. Remarks on ill-posedness of the equation (13)

In this section we prove that the flow associated to the equation (13) is not of class \(C^2\) for \(-1 \leq \alpha < 0\). In particular, we have that we cannot apply Picard iterative process to solve the Duhamel equation associated to this equation.

For this we will use the ideas given in (29) to prove that the flow associated with the KP-I equation is not of class \(C^2\).

4.1. The flow associated to the problem (13) is not \(C^2\).

Theorem 4.1. Let \((s_1, s_2) \in \mathbb{R}^2\) and \(-1 \leq \alpha < 0\). Then, there does not exist \(T > 0\) such that (13) has a unique solution \(u\) for all \(\phi \in H^{s_1,s_2}\) and that the flow \(S_t : \phi \mapsto u\) is not of class \(C^2\) at 0 from \(H^{s_1,s_2}\) to \(H^{s_1,s_2}\)

Proof. Let’s see first what we should show. For this we consider \(u(\lambda, t) = S_t(\lambda \phi), S_t\) the flow associated to the problem (13). So this solution satisfies the Duhamel equation associated to the equation (13), i.e.,

\[
u(\lambda, t) = \lambda W_\alpha(t) \phi - \int_0^t W_\alpha(t-t')u(t')u_x(t') dt'.
\]

(106)
If the flow is twice differentiable around 0 in $H^{s_1,s_2}$ then, thanks to the chain rule,

$$\partial_\lambda u(0,t) = W_\alpha(t)\phi,$$

and

$$\partial^2_\lambda u(0,t) = -2 \int_0^t W_\alpha(t-t')W_\alpha(t')\phi W_\alpha(t')\phi_x dt'.$$

Which would imply that the map

$$\phi \mapsto \int_0^t W_\alpha(t-t')W_\alpha(t')\phi W_\alpha(t')\phi_x dt'$$

would be a quadratic form coming from a continuous symmetric bilinear transformation in $H^{s_1,s_2}$, and that, in particular, for some fixed $C$

$$\left\| \int_0^t W_\alpha(t-t')W_\alpha(t')\phi W_\alpha(t')\phi_x dt' \right\|_{H^{s_1,s_2}} \leq C\|\phi\|_{H^{s_1,s_2}}^2$$

for all $\phi \in H^{s_1,s_2}$. So let’s show that this is not the case. To do this, suppose that this inequality is valid and consider the function $\phi$ defined via the Fourier transform by

$$\hat{\phi} = \gamma^{-3/2}1_{D_1} + \gamma^{-3/2}N^{-\alpha_1-\frac{3-\alpha}{2}}1_{D_2},$$

where $\gamma$ and $N$ are positive numbers such that $\gamma \ll 1$ and $N \gg 1$, $D_1$ and $D_2$ are the sets

$$D_1 = \left[\frac{\gamma}{2}, \gamma\right] \times \left[-\frac{\gamma^2}{6}, \frac{\gamma^2}{6}\right] \text{ and } D_2 = [N, N+\gamma] \times \left[\sqrt{-\frac{3}{\alpha}}N^{-\frac{3-\alpha}{2}}, \sqrt{-\frac{3}{\alpha}}N^{-\frac{3-\alpha}{2}} + \gamma^2\right]$$

and $1_{D_i}$ are the characteristic functions of the sets $D_i, i = 1, 2$. $\|\phi\|_{H^{s_1,s_2}} \sim 1$ for whatever values that we take for $\gamma$ and $N$. Let us see that for a convenient choice in parameters of $\gamma$ and $N$,

$$\left\| \int_0^t W_\alpha(t-t')W_\alpha(t')\phi W_\alpha(t')\phi_x dt' \right\|_{H^{s_1,s_2}} \leq C\|\phi\|_{H^{s_1,s_2}}^2 \quad (107)$$

can be as large as we want. Calculating its Fourier transform, it follows that

$$\int_0^t W_\alpha(t-t')W_\alpha(t')\phi W_\alpha(t')\phi_x dt' \quad (108)$$

is $f_1 + f_2 + f_3$ where

$$\hat{f}_1(t, \xi, \eta) = \frac{i\xi e^{it(\xi^3 + \text{sgn}(\xi)|\xi|^\alpha\eta^2)}}{2\gamma^3} \int_{(\xi, \eta_1) \in D_1} e^{-it \chi(\xi, \xi_1, \eta_1)} \frac{1}{\chi(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1,$$

$$\hat{f}_2(t, \xi, \eta) = \frac{i\xi e^{it(\xi^3 + \text{sgn}(\xi)|\xi|^\alpha\eta^2)}}{2\gamma^3 N^{2(s_1 + \frac{3-\alpha}{2})}} \int_{(\xi, \eta_1) \in D_2} e^{-it \chi(\xi, \xi_1, \eta_1)} \frac{1}{\chi(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1,$$

$$\hat{f}_3(t, \xi, \eta) = \frac{i\xi e^{it(\xi^3 + \text{sgn}(\xi)|\xi|^\alpha\eta^2)}}{2\gamma^3} \int_{(\xi, \eta_1) \in D_1} e^{-it \chi(\xi, \xi_1, \eta_1)} \frac{1}{\chi(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1.$$
and
\[
\hat{f}_3(t, \xi, \eta) = \frac{i\xi e^{it(\xi^3 + \text{sgn}(\xi)|\xi|^\theta)^2}}{2\gamma^3 N^{s_1 + \frac{3}{2} s_2}} \int_{(\xi_1, \eta_1) \in D_2 \atop (\xi - \xi_1, \eta - \eta_1) \in D_1} \frac{e^{-it\chi(\xi, \xi_1, \eta, \eta_1)} - 1}{\chi(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1 + 
\]
\[
+ \frac{i\xi e^{it(\xi^3 + \text{sgn}(\xi)|\xi|^\theta)^2}}{2\gamma^3 N^{s_1 + \frac{3}{2} s_2}} \int_{(\xi, \eta_1) \in D_1 \atop (\xi - \xi_1, \eta - \eta_1) \in D_2} \frac{e^{-it\chi(\xi, \xi_1, \eta, \eta_1)} - 1}{\chi(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1,
\]
where \(\chi\) is the resonant function
\[
\chi = \chi(\xi, \xi_1, \eta, \eta_1) = \vartheta(\xi, \eta) - \vartheta(\xi_1, \eta_1) - \vartheta(\xi - \xi_1, \eta - \eta_1)
= 3\xi_1(\xi - \xi_1) + \text{sgn}(\xi) \frac{\eta^2}{|\xi|^\theta} - \text{sgn}(\xi_1) \frac{\eta_1^2}{|\xi_1|^\theta} - \text{sgn}(\xi - \xi_1) \frac{(\eta - \eta_1)^2}{|\xi - \xi_1|^\theta},
\]
where
\[
(110)
\]
is the phase function and \(\theta = -\alpha\), which is between 0 and 1. Since in our case \(\xi, \xi_1\) and \(\xi - \xi_1\) are positive, we have
\[
\chi = 3\xi_1(\xi - \xi_1) + \frac{\eta^2}{\xi^\theta} - \frac{\eta_1^2}{\xi_1^\theta} - \frac{(\eta - \eta_1)^2}{(\xi - \xi_1)^\theta},
\]
(111)

Observe that to estimate (107) it is enough estimate \(\|f_3(t)\|_{H^{s_1+s_2}}\), in fact,
\[
\left\| \int_0^t W_\alpha(t - t') W_\alpha(t') \phi W_\alpha(t') \phi_x \, dt' \right\|_{H^{s_1+s_2}} \geq \|f_3(t)\|_{H^{s_1+s_2}}.
\]

To continue we need the following lemma.

**Lemma 4.2.** Suppose that
\[
(\xi_1, \eta_1) \in D_1 \quad \text{and} \quad (\xi - \xi_1, \eta - \eta_1) \in D_2
\]
or
\[
(\xi_1, \eta_1) \in D_2 \quad \text{and} \quad (\xi - \xi_1, \eta - \eta_1) \in D_1.
\]
Then,
\[
|\chi(\xi, \xi_1, \eta, \eta_1)| \lesssim \gamma^2 N.
\]

**Proof.** First let us calculate the \(\eta\) such that \(\chi(\xi, \xi_1, \eta, \eta_1) = 0\). So, we have
\[
[\xi^\theta - (\xi - \xi_1)^\theta]\xi_1^\theta \eta^2 - 2\xi^\theta \xi_1 \eta \eta_1 + [\xi_1^\theta + (\xi - \xi_1)^\theta]\xi^\theta \eta_1^2 - 3\xi^1+\theta \xi_1^{1+\theta} (\xi - \xi_1)^{1+\theta} = 0
\]
(112)
Then, we get
\[
\eta = \frac{\xi^\theta \eta_1}{[\xi^\theta - (\xi - \xi_1)^\theta]} \pm \sqrt{3\xi^{1+\theta}\xi_1(\xi - \xi_1)^{1+\theta} + \frac{\xi^\theta(\xi - \xi_1)^\theta[\xi^\theta + (\xi - \xi_1)^\theta - \xi^\theta]}{[\xi^\theta - (\xi - \xi_1)^\theta]^2} \eta_1^2},
\]
(113)

Let \(\eta^*\) be the smallest zero between the two calculated above. So,
\[
\eta^* - \eta_1 = \frac{(\xi - \xi_1)^\theta \eta_1}{[\xi^\theta - (\xi - \xi_1)^\theta]} - \sqrt{3\xi^{1+\theta}\xi_1(\xi - \xi_1)^{1+\theta} + \frac{\xi^\theta(\xi - \xi_1)^\theta[\xi^\theta + (\xi - \xi_1)^\theta - \xi^\theta]}{[\xi^\theta - (\xi - \xi_1)^\theta]^2} \eta_1^2},
\]
(114)

Let
\[
R = \frac{(\xi - \xi_1)^\theta \eta_1}{[\xi^\theta - (\xi - \xi_1)^\theta]} + \sqrt{3\xi^{1+\theta}\xi_1(\xi - \xi_1)^{1+\theta} + \frac{\xi^\theta(\xi - \xi_1)^\theta[\xi^\theta + (\xi - \xi_1)^\theta - \xi^\theta]}{[\xi^\theta - (\xi - \xi_1)^\theta]^2} \eta_1^2},
\]

Then
\[
|\eta^* - \eta_1| = \frac{\frac{(\xi^\theta - \xi_1^\theta)(\xi - \xi_1)^\theta}{(\xi^\theta - (\xi - \xi_1)^\theta)} |\eta_1^2 - 3\xi^{1+\theta}\xi_1^{1+\theta} g(\xi, \xi_1)|}{R},
\]
(115)

where
\[
g(\xi, \xi_1) = \begin{cases} 
\frac{\xi - \xi_1}{\xi^\theta - \xi_1^\theta} & \text{if } \xi \neq \xi_1 \\
\frac{1}{\theta} & \text{in another case.}
\end{cases}
\]

Now, since
\[
\frac{1}{\theta} \xi_1^{-\theta} \leq g(\xi, \xi_1) \leq \frac{1}{\theta} \xi_1^{-\theta}
\]
(116)

and
\[
R \geq \frac{(\xi - \xi_1)^\theta \eta_1}{(\xi^\theta - (\xi - \xi_1)^\theta)};
\]
we have
\[
|\eta^* - \eta_1| = \frac{\xi^\theta - \xi_1^\theta}{\xi_1^\theta} \frac{|\eta_1^2 - 3\xi^{1+\theta}\xi_1^{1+\theta} g(\xi, \xi_1)|}{\eta_1} \]
\[
\leq 3 \frac{\xi^\theta - \xi_1^\theta}{\xi_1^\theta} \left| \eta_1 - \sqrt{3 \xi^{1+\theta}\xi_1^{1+\theta} \sqrt{g(\xi, \xi_1)}} \right| \]
\[
\leq 3 \theta \frac{\xi - \xi_1}{\xi_1} \left| \eta_1 - \frac{3}{\theta} \xi_1^{-\theta} \xi_1^{\frac{3}{2}+\theta} \left( \xi_1^{-\frac{1}{2}} \sqrt{g(\xi, \xi_1)} - \frac{1}{\sqrt{\theta}} \xi_1^{\frac{1}{2}} \right) \right|,
\]

Remember that \(\eta_1\) take values in \([\sqrt{\frac{3}{\theta}} N^{\frac{3}{2}+\theta}, \sqrt{\frac{3}{\theta}} N^{\frac{3}{2}+\theta} + \gamma^2]\) and that \(\xi_1\) in \([N, N + \gamma]\). Whence, thanks to the Taylor formula with remainder
applied to the function $x \mapsto x^{\frac{3+\theta}{2}}$,
\[
\sqrt[3]{3} \xi^{\frac{3+\theta}{2}} \in \left[ \sqrt[3]{3} \frac{3+\theta}{2} N^{\frac{3+\theta}{2}} \gamma + \sqrt[3]{3} \frac{3+\theta}{2} N^{\frac{3+\theta}{2}} \gamma + \sqrt[3]{3} (3+\theta)(1+\theta) \gamma^2 \right]
\]
and, moreover
\[
|\eta_1 - \sqrt[3]{3} \xi^{\frac{3+\theta}{2}}| \leq 2 \sqrt[3]{3} N^{\frac{3+\theta}{2}} \gamma + \sqrt[3]{3} \gamma^2.
\]
On the other hand from (116)
\[
\xi^{\frac{1+\theta}{2}} \sqrt{g(\xi, \xi_1)} - \frac{1}{\sqrt{\theta}} \xi_1 \leq \frac{1}{\sqrt{\theta}} (\xi - \xi_1) \leq \frac{1}{\sqrt{\theta}} \gamma.
\]
Therefore,
\[
|\eta^*-\eta_1| \leq 3 \theta \frac{\gamma}{N} \left( 2 \sqrt[3]{3} \frac{1+\theta}{2} N^{\frac{1+\theta}{2}} \gamma + \sqrt[3]{3} \frac{1+\theta}{2} N^{\frac{1+\theta}{2}} \gamma + \gamma \right) \leq 18 \sqrt[3]{3} \frac{\gamma^2}{N^{\frac{1+\theta}{2}}}
\]
By the mean value theorem, there exists $\bar{\eta} \in [\eta, \eta^*]$ such that
\[
\chi(\xi, \xi_1, \eta, \eta_1) = \chi(\xi, \xi_1, \eta^*, \eta_1) + (\eta - \eta^*) \partial_\eta \chi(\xi, \xi_1, \bar{\eta}, \eta_1)
\]
\[
= - (\eta - \eta^*) \frac{2(\bar{\eta} - \eta_1 \xi^\theta - (\xi - \xi_1)^\theta - \eta_1 \xi^\theta)}{\xi^\theta (\xi - \xi_1)^\theta}
\]
\[
= - (\eta - \eta^*) \frac{2(\bar{\eta} - \eta_1 \xi^\theta - (\xi - \xi_1)^\theta - \eta_1 \xi^\theta)}{\xi^\theta (\xi - \xi_1)^\theta}
\]
\[
= - (\eta - \eta^*) \frac{2(\bar{\eta} - \eta_1 \xi^\theta - (\xi - \xi_1)^\theta - \eta_1 \xi^\theta)}{\xi^\theta (\xi - \xi_1)^\theta}
\]
So,
\[
|\chi(\xi, \xi_1, \eta, \eta_1)| \leq \frac{\gamma^2}{N^{\frac{1+\theta}{2}}} \left( \frac{\gamma^2}{N^{\frac{1+\theta}{2}} \gamma^\theta} + \frac{N^{\frac{3+\theta}{2}}}{N^\theta} \right)
\]
\[
\leq \gamma^2 N
\]
The lemma follows immediately observing that
\[
\chi(\xi, \xi_1, \eta, \eta_1) = \chi(\xi, \xi - \xi_1, \eta, \eta - \eta_1).
\]
Let us finish the proof of the theorem. Let us choose $\gamma$ and $N$ in such a way that $\gamma^2 N = N^{-\varepsilon}$ for $\varepsilon \ll 1$. Thanks to the previous lemma we have that
\[
\left| \frac{e^{it\xi} - 1}{\xi} \right| = |t| + O(N^{-\varepsilon})
\]
for \((\xi_1, \eta_1) \in D_1\) and \((\xi - \xi_1, \eta - \eta_1) \in D_2\) or \((\xi_1, \eta_1) \in D_2\) and \((\xi - \xi_1, \eta - \eta_1) \in D_1\). So

\[
\|f_3(t, \cdot, \cdot)\|_{H^s} \gtrsim \frac{NN^{\frac{3+6}{4}N^3\gamma^2}}{N^{\frac{3+6}{4}N^3\gamma^2}} = \gamma^2N = N^{(1-3\varepsilon)/4}.
\]

This leads to a contradiction since

\[
1 \sim \|\phi\|_{H^s}^2 \gtrsim \|f_3(t, \cdot, \cdot)\|_{H^s}.
\]

\(\square\)

An immediate consequence of the previous theorem is the following theorem.

**Theorem 4.3.** For \((s_1, s_2) \in \mathbb{R}^2\), \(-1 \leq \alpha < 0\) and a positive real number \(T\), there does not exist a space \(X_T\) continuously embedded in \(C([-T, T]; H^{s_1, s_2})\) such that, for a fix constant \(C\),

\[
\|W_\alpha(\cdot)\phi\|_{X_T} \leq C\|\phi\|_{H^{s_1, s_2}},
\]

for all \(\phi \in H^{s_1, s_2}\), and

\[
\left\| \int_0^t W_\alpha(t - t')(u(t')u_x(t')) dt' \right\|_{X_T} \leq C\|u\|_{X_T}^2,
\]

for all \(u \in X_T\).

Note that the estimates given in the theorem statement are necessary to prove the contraction properties of the operator \(\Phi\) defined by

\[
\Phi(u) = W_\alpha(t)\phi + \int_0^t W_\alpha(t - t')(u(t')u_x(t')) dt'.
\]

**Proof.** Suppose that we have (117) and (118) for all \(\phi \in H^{s_1, s_2}\) and for all \(u \in X_T\). Let \(\phi \in H^{s_1, s_2}\) and set \(u(t) = W_\alpha(t)\phi\). Then,

\[
\left\| \int_0^t W_\alpha(t - t')(W_\alpha(t')\phi W_\alpha(t')\phi_x) dt' \right\|_{X_T} \leq C\|W_\alpha(t)\phi\|_{X_T}^2 \leq \|\phi\|_{H^{s_1, s_2}}^2.
\]

Since \(X_T\) is continuously embedded in \(C([-T, T]; H^{s_1, s_2})\)

\[
\left\| \int_0^t W_\alpha(t - t')(W_\alpha(t')\phi W_\alpha(t')\phi_x) dt' \right\|_{H^{s_1, s_2}} \leq C\|\phi\|_{H^{s_1, s_2}}^2,
\]

which is contradictory with the previous theorem. This ends the proof. \(\square\)

Another immediate corollary is the following theorem.

**Theorem 4.4.** The flow associated to the problem (13), for \(-1 \leq \alpha < 0\), whose well-posedness was proved in the previous section, is not of class \(C^2\).
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