Exact solutions of initial boundary value problems for Sobolev type equations of elastic oscillations

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Abstract. The article is devoted to the problem of the unique solvability of initial boundary value problems for partial differential equations. These initial boundary value problems are dissipative dynamic mathematical models from the field of linear elasticity. The considered equations are not resolved with respect to the highest time derivative. Therefore, these equations are referred to the so-called Sobolev type partial differential equations. The research of initial boundary value problems is carried out from general position in the form of the complete second order differential equation in abstract spaces. The theory of the distributions with values in Banach spaces and the concept of a fundamental solution of differential operator are used. Theorems of existence and uniqueness of the solutions of considered initial boundary value problems in the class of functions of finite smoothness with respect to time are proved. These solutions are obtained as an functional series of the Kummer confluent hypergeometric function.

1. Introduction
The paper is devoted to the study of the unique solvability of initial boundary value problems for Sobolev type differential equations. The name of these equations goes back to the pioneering work of S. L. Sobolev [1], where the problem of small oscillations of a rotating liquid was considered. This term means a partial differential equation that is not resolved with respect to the highest time derivative [2]. Partial differential equations of the Sobolev type, represented in an abstract differential operator form, are also called degenerate or singular. The results of research in the field of Sobolev type equations and a wide bibliography are given in monographs of Demidenko and Uspenskii [2], Carrol and Showalter [3], Favini and Yagi [4], Sidorov, Loginov, Sinitysyn and Falaleev [5], Sviridyuk and Fedorov [6], Al’shin, Korpusov, Šveshnikov [7] and others. Many partial differential equations of continuum mechanics can be represented as an abstract evolution equation of the second order with respect to time variable. Complete second order evolution equations are of particular interest in applied problems, because the terms with the first time derivative of an unknown function allow to formalize internal friction, energy dissipation, influence of gyroscopic forces, and other effects. Among the works devoted to the complete second order abstract differential equations and their applications in the form of the Sobolev type partial differential equations, let us cite the research papers, which are closest to our studies. They are [8] by Batty, Chill and Srivastava, [9] by Andronova and Kopachevskii, [10] by Bulatov and Lee, [11] by Keyantuo and Lizama, [12] by Falaleev and Grazhdantseva, [13] by Melnikova and Filinkov, [14] by Oka, [15] by Tijun and Jin.
2. Problem Statement

In this paper we consider an initial boundary value problem

\[
\lambda' u_{tt} - u_{xxtt} - a(u_{xxt} - \lambda'' u_t) - b(u_{xx} - \lambda''' u) = f(x, t), \quad x \in [0; h], \quad t > 0; \quad (1)
\]

\[
u(x, t)|_{t=0} = u_0(x), \quad u_t(x, t)|_{t=0} = u_1(x); \quad u(x, t)|_{x=0} = u(x, t)|_{x=h} = 0, \quad t \geq 0; \quad (2)
\]

which describes small longitudinal oscillations of an elastic rod by taking inertia into account. The equation (1) is called the Boussinesq–Love equation [16]. The unknown function \( u = u(x, t) \) is the displacement of points of the principal axis of the rod of length \( h > 0 \), and the given function \( f = f(x, t) \) is the distribution of the external mass load per unit length of the rod. Real parameters \( a, b, \lambda', \lambda'', \lambda''' \) depend on properties of rod material such that density, Young modulus, Poisson ratio and Lamé coefficients. The initial boundary value problem (1), (2) has been studied thoroughly by Zamyslyeva, Bychkov and Tsyplenkova in [16]. Our research deals with construction of new exact solutions of this problem in terms of special functions.

Another object of the study is an initial boundary value problem

\[
\lambda v_{tt} - \Delta v_{tt} - \Delta v + \Delta^2 v = f(r, t), \quad r \in K, \quad t > 0; \quad (3)
\]

\[
 v(r, t)|_{t=0} = v_0(r), \quad v_t(r, t)|_{t=0} = v_1(r); \quad v(r, t)|_{r \in \partial K} = 0, \quad \Delta v(r, t)|_{r \in \partial K} = 0, \quad t \geq 0. \quad (4)
\]

This is a linearized dissipative model of small transversal oscillations of a thin homogeneous isotropic thermoelastic plate [17]. Here \( \Delta \) is the Laplace operator with respect to the spatial variables \( r = xi + yj \) defined on the square \( K = [0; h] \times [0; h] \), and \( \Delta^2 \) is the biharmonic operator. The unknown function \( v : K \times [0; +\infty) \to \mathbb{R} \) describes the transversal displacements of a two-dimensional plate occupying \( K \). The parameter \( \lambda > 0 \) is inversely proportional to the plate thickness, and the term \( -\Delta u_{tt} \) describes the rotational inertia of the plate. The given function \( f : K \times [0; +\infty) \to \mathbb{R} \) is the distribution of the external mass load per unit area of the plate. The boundary conditions (4) defined on \( \partial K \) are called the Navier boundary conditions [18]. The boundary value problem with Dirichlet boundary conditions for equation (3) has been studied by Bisognin, Bisognin, Perla Menzala and Zuazua in [17]. The uniform exponential decay of the energy of the solutions of this problem has been proved.

3. Methods and Materials

Let \( E_1, E_2 \) be real Banach spaces. Consider the following second order linear differential operator

\[
D(u(t)) = Bu''(t) - A_1 u'(t) - A_0 u(t),
\]

where \( B, A_1, A_0 \) are closed linear operators from \( E_1 \) into \( E_2 \) such that \( D(B) \subseteq D(A_1) \cap D(A_0) \).

In this section we consider an initial value problem

\[
D(u(t)) = f(t), \quad t > 0; \quad u(0) = u_0, \quad u'(0) = u_1, \quad (5)
\]

where \( u : [0; +\infty) \to E_1 \) is unknown function, and \( f : [0; +\infty) \to E_2 \) is given function.

Definition 1. A function \( u = u(t) \) that is strongly continuously differentiable two times over semi-axis \([0; +\infty)\) and satisfies the differential equation (5) with initial conditions is called a classical solution of this initial value problem.

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\[ D(B) = D(A_1) = D(A_0) = H_{\mathcal{K}}^{t+4} = \{ v(r) \in W_{2K}^{t+4} : v(r)|_{r \in \partial K} = 0, \Delta v(r)|_{r \in \partial K} = 0 \}; \]

then the initial value problem (5) becomes the initial boundary value problem (3), (4). By \( W_{2K}^t \) or \( H_{\mathcal{K}}^t \) denote the Sobolev space, where \( t \in \{0\} \cup \mathbb{N} \), and by \( L_{2K}^t \) denote the Lebesgue space of square-integrable on \( K \) functions.

To study the unique solvability of the initial value problem (5) we used the Sobolev–Schwartz theory of distributions with values in Banach spaces [5]. Denote by \( \delta \) the Dirac delta function. In the space \( K_1^t(E_1) \) of distributions with left-bounded support and values in \( E_1 \) the initial value problem (5) has the convolutional form

\[ D(\delta(t)) * \tilde{u}(t) = \tilde{g}(t), \tag{6} \]

with given distribution \( \tilde{g}(t) \in K_1^t(E_2) \) such that

\[ \tilde{g}(t) = f(t)\theta(t) + (Bu_1 - A_1u_0)\delta(t) + Bu_0\delta'(t), \]

and \( \tilde{u}(t) = u(t)\theta(t) \), where \( u = u(t) \) is the classical solution of the initial value problem (5).

**Definition 2.** A solution \( \tilde{u}(t) \in K_1^t(E_1) \) of the equation (6) is called a generalized solution of the initial value problem (5).

**Definition 3.** A generalized operator valued function \( E(t) \) is called a fundamental solution or fundamental operator-function of the second order differential operator \( D(\delta(t)) \) if the following convolutional equalities \( D(\delta(t)) * E(t) * v(t) = v(t) \) and \( E(t) * D(\delta(t)) * w(t) = w(t) \) are true for any distributions \( v(t) \in K_1^t(E_2) \) and \( w(t) \in K_1^t(E_1) \).

If fundamental operator-function \( E(t) \) is known, then generalized solution of the initial value problem (5) is given by formula

\[ \tilde{u}(t) = E(t) * \tilde{g}(t). \tag{7} \]

By the first equality of the Definition 3, it follows that the distribution (7) is the solution of the equation (6). The existence of this solution explains by the existence of convolution in the class of distributions with left-bounded support. The uniqueness of the solution (7) of the equation (6) is easy to prove. Let \( \tilde{v}(t) \in K_1^t(E_1) \) such that \( \tilde{v}(t) \neq \tilde{u}(t) \), and \( D(\delta(t)) * \tilde{v}(t) = \tilde{g}(t) \), i.e. there exists another solution of the equation (6). By the second equality of Definition 3, it follows that \( \tilde{v}(t) = E(t) * D(\delta(t)) * \tilde{v}(t) = E(t) * \tilde{g}(t) = \tilde{u}(t) \), which contradicts our previous assumptions that \( \tilde{v}(t) \neq \tilde{u}(t) \).

Suppose that linear operator \( B \) is continuously invertible and \( D(B) \subseteq D(A_1) \cap D(A_0) \). Let us denote \( A_1 = A_1B^{-1} \) and \( A_0 = A_0B^{-1} \). It follows from the closed graph theorem that mappings \( A_1 \) and \( A_0 \) are bounded linear operators from \( E_2 \) to itself. By \( \exp(A_1t) \) denote a uniformly continuous semigroup of bounded linear operators with an infinitesimal generator \( A_1 \). Consider an operator valued function \( V(t) = \exp(-A_1t)A_0 \exp(A_1t) \). By the Baker–Campbell–Hausdorff formula [19], it follows that

\[ V(t) = A_0 + [A_0,A_1] \frac{t}{1!} + [[A_0,A_1],A_1] \frac{t^2}{2!} + [[[[A_0,A_1],A_1],A_1],A_1] \frac{t^3}{3!} + \ldots, \]

where \([A_0,A_1] = A_0A_1 - A_1A_0 \) is the commutator of linear operators \( A_0 \) and \( A_1 \). Let us remark that operator valued function \( V(t) \) and linear operator \( A_1 \) satisfy the Lax equation \( V'(t) = [V(t),A_1] \). In the theory of integrable systems, such operators are called a Lax pair [20]. The initial condition \( V(0) = A_0 \) associates \( V(t) \) and \( A_0 \).

**Theorem 1.** Suppose \( B \) is continuously invertible linear operator; then a fundamental operator-function of differential operator \( D(\delta(t)) \) is given by

\[ E(t) = B^{-1} \sum_{k=1}^{+\infty} \int_0^t (t-s)^{k-1} \frac{1}{(k-1)!} \exp(A_1s)U_{k-1}(s)ds \theta(t), \]
where operator valued functions \( U_{k-1}(t) \) are such that

\[
U_k(t) = \int_0^t V(s)U_{k-1}(s)ds, \quad U_0(t) = I_2;
\]

\( I_1 \) and \( I_2 \) are identity operators in \( E \).

To prove this theorem, it is necessary to check two convolutional equalities of the Definition 3 using recurrence formula for \( \{U_k(t)\}_{k \in \mathbb{N}} \) and the following asymptotic and differential relations

\[
U_{k-1}(t) \sim A_0^{k-1} \frac{t^{k-1}}{(k-1)!}, \quad t \to 0; \quad U_k'(t) - [U_k(t), A_1] = U_{k-1}(t)A_0.
\]

**Corollary 1.** Let the conditions of the Theorem 1 be satisfied and the composition of linear operators \( A_0 \) and \( A_1 \) is commutative; then a fundamental operator-function of differential operator \( D(\delta(t)) \) is given by

\[
\mathcal{E}(t) = B^{-1} \sum_{k=1}^{+\infty} \frac{t^{2k-1}}{(2k-1)!} A_0^{k-1} _1 F_1(k; 2k; A_1t) \theta(t),
\]

where \(_1 F_1(a; b; t)\) is the *Kummer confluent hypergeometric function* [21] in an integral form

\[
_{1}F_{1}(a; b; t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_{0}^{1} \exp(t\tau)\tau^{a-1}(1-\tau)^{b-a-1}d\tau, \quad b > a > 0.
\]

**Corollary 2.** Let the conditions of the Theorem 1 be satisfied and \( f(t) \in C([0; +\infty); E_2) \); then the initial value problem (5) has a unique generalized solution

\[
\tilde{u}(t) = u(t)\theta(t) = B^{-1}\left[\exp(A_1t)Bu_0 + \sum_{k=1}^{+\infty} \int_0^t \int_0^{t-s} \frac{(t-s)^{k-1}}{(k-1)!} \exp(A_1\tau)U_{k-1}(\tau)f(s)d\tau ds +
\right.
\]
\[
\left. + \sum_{k=1}^{+\infty} \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} \exp(A_1s)U_k(s)ds Bu_0 + \sum_{k=1}^{+\infty} \int_0^t \frac{(t-s)^k}{k!} \exp(A_1s)U_k(s)ds (Bu_1 - A_1u_0) \right] \theta(t).
\]

As has been mentioned above, a unique generalized solution of the initial value problem (5) is given by (7). The form of this solution is obtained in the Corollary 2 under the assumption that \( f = f(t) \) is strongly continuous. Generalized function \( \tilde{u}(t) \in K_+^1(E_1) \) is regular and coincides with function \( u = u(t) \) that is strongly continuously differentiable two times and satisfies the differential equation (5) with initial conditions.

**Corollary 3.** Let the conditions of the Theorem 1 be satisfied and \( f(t) \in C([0; +\infty); E_2) \); then the initial value problem (5) has a unique classical solution \( u = u(t) \) from the Corollary 2.

4. Results and Discussion

Let us consider a homogeneous boundary value problem

\[
\varphi''(x) - \lambda \varphi(x) = 0, \quad x \in [0; h]; \quad \varphi(0) = \varphi(h) = 0.
\]
The eigenvalues and eigenfunctions of this problem are given by

\[ \lambda_m = -\frac{\pi^2 m^2}{h^2}, \quad \varphi_m(x) = \left(\frac{2}{h}\right)^{1/2} \sin \frac{\pi m x}{h}, \quad m \in \mathbb{N}. \] (9)

The system of eigenfunctions is orthonormal in the sense of the scalar product \( \langle \cdot, \cdot \rangle \) of the Lebesgue space \( L^2_{[0, h]} \). Assume that \( \lambda' \notin \sigma(\partial_{xx}) \), i.e., \( \lambda' \neq \lambda_m \) for any \( m \in \mathbb{N} \), then the operator \( B = \lambda' - \partial_{xx} \) is continuously invertible, and the following theorem is due to the Corollary 3.

**Theorem 2.** Suppose \( \lambda' \notin \sigma(\partial_{xx}) \), \( f(x, t) \in C(t \geq 0; L^2_{[0, h]}) \), \( u_0(x), u_1(x) \in H^{t+2}_{0; [0, h]} \); then the initial boundary value problem (1), (2) has a unique solution of class \( C^2(t \geq 0; H^{t+2}_{0; [0, h]}) \), and this solution is given by

\[
u(x, t) = \sum_{m=1}^{+\infty} \sum_{k=1}^{+\infty} \left[ \langle u_0(x), \varphi_m(x) \rangle \exp(\alpha_m t) + \int_0^t \left( \frac{(t-s)^{2k-1}}{(2k-1)!} F_1(k; 2k; \alpha_m t) + \frac{\beta_{m}^k}{\lambda' - \lambda_m} \int_0^t (t-s)^{2k-1} F_1(k; 2k; \alpha_m t) (f(x, s), \varphi_m(x)) \right) \varphi_m(x) \right],
\]

where \( \alpha_m = \frac{\lambda_m - \lambda''}{\lambda' - \lambda_m} \), \( \beta_m = \frac{\lambda_m - \lambda''}{\lambda' - \lambda_m} \), and \( F_1(a; b; t) \) is a special function defined by (8).

Let us consider a homogeneous boundary value problem

\[ \Delta \varphi(x) = \lambda \varphi(x), \quad r \in \mathbb{K}; \quad \varphi(x)|_{r \in \partial \mathbb{K}} = 0, \quad \Delta \varphi(x)|_{r \in \partial \mathbb{K}} = 0. \]

The spectrum \( \sigma(\Delta) \) consists of the eigenvalues \( \lambda_{m,n} = \lambda_m + \lambda_n \), where \( \lambda_m, \lambda_n \in \sigma(\partial_{xx}) \) are given by (9). It necessary that \( \lambda_{m,n} h^2 = -\pi^2 s \), where \( s \in \mathbb{N} \). The multiplicity \( d(\lambda_{m,n}) \) of eigenvalue \( \lambda_{m,n} \) is equal to the number of different solutions \((m, n) \in \mathbb{N}^2 \) of the equation \( m^2 + n^2 = s \) with given \( s \in \mathbb{N} \). By the theorem on sums of two squares from [22], it follows that

\[ d(\lambda_{m,n}) = \sum_{d \in \mathbb{N} ; s | d} \chi_4(d) - \sum_{a \in \mathbb{N}} \delta_{s, a^2}, \]

where \( \chi_4(d) \) is Dirichlet character modulo 4 of \( d \in \mathbb{N} \), and \( \delta_{a^2} \) is Kronecker delta. The systems of eigenfunctions \( \varphi_{m,n}(r) = \varphi_m(x) \varphi_n(y) \) is orthonormal in the sense of the scalar product \( \langle \cdot, \cdot \rangle \) of the Lebesgue space \( L^2_{\mathbb{K}} \), where \( \varphi_m(x), \varphi_n(y) \) from (9). Assume that \( \lambda \notin \sigma(\Delta) \), then the operator \( B = \lambda - \Delta \) is continuously invertible, and the following theorem is due to the Corollary 3.

**Theorem 3.** Suppose \( \lambda \notin \sigma(\Delta) \), \( f(r, t) \in C(t \geq 0; L^2_{\mathbb{K}}) \), \( v_0(r), v_1(r) \in H^{t+4}_{0; \mathbb{K}} \); then the initial boundary value problem (3), (4) has a unique solution of class \( C^2(t \geq 0; H^{t+4}_{0; \mathbb{K}}) \), and this solution is given by

\[
u(r, t) = \sum_{m,n=1}^{+\infty} \sum_{k=1}^{+\infty} \left[ \exp(\mu_{m,n} t) \langle v_0(r), \varphi_{m,n}(r) \rangle + \langle v_1(r), \varphi_{m,n}(r) \rangle \right] F_1(k; 2k; \mu_{m,n} t) + \frac{(-\lambda_m \lambda_n)^{k-1}}{(2k-1)!} \left[ F_1(k; 2k; \mu_{m,n} t) + \right.
\]

\[ + \left. \left( -\lambda_m \mu_{m,n} \right)^{k-1} \langle v_0(r), \varphi_{m,n}(r) \rangle - \mu_{m,n} \langle v_0(r), \varphi_{m,n}(r) \rangle \right] \right]. \]
\[ \frac{(\lambda_{m,n} \mu_{m,n})^{k-1}}{\lambda - \lambda_{m,n}} \int_0^t \frac{(t-s)^{2k-1}}{(2k-1)!} \, \text{I}_1(k; 2k; \mu_{m,n}(t-s)) \langle f(r,s), \varphi_{m,n}(r) \rangle \, ds \varphi_{m,n}(r) \],

where \( \mu_{m,n} = \frac{\lambda_{m,n}}{\lambda - \lambda_{m,n}} \), and \( \text{I}_1(a; b; t) \) is a special function defined by (8).

5. Conclusion
Thus, the present paper deals with nonclassical equations of mathematical physics. Initial boundary value problems of the theory of elasticity are considered. The unique solvability of initial boundary value problems in the class of the functions of finite smoothness with respect to time are proved. Explicit formulas of the solutions are obtained. The object of direct study in the article is complete second order differential equation in Banach spaces. Original initial boundary value problems are considered as concrete examples of this abstract equation. The applied approach develops the theory and methods for solving not only these initial boundary value problems, but also any others, which are a particular case of the abstract differential equation under consideration.

Acknowledgments
The reported study was funded by Russian Foundation for Basic Research according to the research projects No. 18-01-00643 A and No. 18-51-54001 Viet_a.

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