Inverse Boundary Value Problem for the Helmholtz Equation: Multi-Level Approach and Iterative Reconstruction

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Abstract. We study the inverse boundary value problem for the Helmholtz equation using the Dirichlet-to-Neumann map at selected frequency as the data. We develop an explicit reconstruction of the wavespeed using a multi-level nonlinear projected steepest descent iterative scheme in Banach spaces. We consider wavespeeds containing discontinuities. A conditional Lipschitz stability estimate for the inverse problem holds for wavespeeds of the form of a linear combination of piecewise constant functions with an underlying domain partitioning, and gives a framework in which the scheme converges. The stability constant grows exponentially as the number of subdomains in the domain partitioning increases. To mitigate this growth of the stability constant, we introduce hierarchical compressive approximations of the solution to the inverse problem with piecewise constant functions. We establish an optimal bound of the stability constant, which leads to a condition on the compression rate pertaining to these approximations.

1. Introduction. In this paper, we study the inverse boundary value problem for the Helmholtz equation using the Dirichlet-to-Neumann map at selected frequency as the data. We focus on developing an explicit iterative reconstruction of the wavespeed. This inverse problem arises, for example, in reflection seismology and inverse obstacle scattering problems for electromagnetic waves [4, 24, 5]. We consider wavespeeds containing discontinuities.

Uniqueness of the mentioned inverse boundary value problem was established by Sylvester & Uhlmann [23] assuming that the wavespeed is a bounded measurable function. This inverse problem has been extensively studied from an optimization point of view. We mention, in particular, the work of [6]. Using multi-frequency data, to intuitively stabilize the iterative schemes, so-called frequency progression has been introduced [9, 22, 3, 10]. Frequency progression also appears naturally in our approach.

It is well known that the logarithmic character of stability of the inverse boundary value problem for the Helmholtz equation [11, 19] cannot be avoided. In fact, in [17] Mandache proved that despite of regularity or a-priori assumptions of any order on the unknown wavespeed, logarithmic stability is optimal. However, conditional Lipschitz stability estimates can be obtained: Accounting for discontinuities, such an estimate holds if the unknown wavespeed is a finite linear combination of piecewise constant functions with an underlying known domain partitioning [7]. We can extend this result to a framework of discrete approximations of the unique solution. It was obtained following an approach introduced by Alessandrini and Vessella [2] and further developed by Beretta and Francini [8] for Electrical Impedance Tomography (EIT). Here, we revisit the Lipschitz stability estimate for the full Dirichlet-to-Neumann map using complex geometrical optics solutions which gives rise to an optimal bound of the Lipschitz constant in terms of the number of subdomains in the domain partitioning. We develop the estimate in $L^2(\Omega)$. This result aids in the design of our multi-level iterative scheme for reconstruction through increasing the number of subdomains.

To be more precise, we introduce a multi-scale hierarchy of compressive approximations, while refining the domain partitioning, of the unique solution to the inverse problem with piecewise constant functions. Assuming a bound on the compression, we mitigate the mentioned growth of the stability constant in a multi-level projected steepest descent method yielding a condition which couples the approximation errors and stability constants between neighboring levels [12]. Tracking the domain partitioning dependencies in the approximation errors we then arrive at a procedure to refine the domain partitioning, which are related to the scales, guaranteeing convergence from level to level, that is, to progressively more accurate approximations following the multi-scale hierarchy.

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As a part of the analysis, we study the Fréchet differentiability of the direct problem and obtain the frequency and domain partitioning dependencies of the relevant constants away from the Dirichlet spectrum. Our results hold for finite-frequency data including frequencies arbitrarily close to zero while avoiding Dirichlet eigenfrequencies; in view of the estimates, inherently, there is a finest scale which can be reached.

**Frequency data.** In many applications such as reflection seismology, frequency data are obtained from solutions to the corresponding boundary value problem for the wave equation by applying a Fourier transform. Let $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^3$ and $c = c(x)$ be a strictly positive bounded measurable function. We consider the boundary value problem for the wave equation

\[
\begin{cases}
\partial_t^2 u(x, t) - c^2(x) \Delta u(x, t) = 0, & (x, t) \in \Omega \times (0, \infty), \\
u(x, t) = f(x, t), & (x, t) \in \partial\Omega \times (0, \infty), \\
u(x, 0) = 0, & x \in \Omega.
\end{cases}
\]

The hyperbolic Dirichlet-to-Neumann map, $\Lambda_{c^{-2}}$, is given by

\[
\Lambda_{c^{-2}} : H \rightarrow L^2(\partial\Omega \times (0, \infty)),
\]

\[
f \mapsto \partial_n u^f |_{\partial\Omega \times (0, \infty)},
\]

where $\partial_n$ denotes the normal derivative at $\partial\Omega$ and $H = \{ f \in H^1(\partial\Omega \times (0, \infty)) \mid f(x, 0) = 0 \}$. One, indeed, may take the Fourier transform of $\partial_n u^f$, since it is a tempered distribution \([14]\), and thus obtain multi-frequency data.

**2. Direct problem.** We describe the direct problem and some properties of the data, that is, the Dirichlet-to-Neumann map. We formulate the direct problem as a nonlinear operator mapping $L^2(\Omega)$ to $L(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$. We invoke

**Assumption 2.1.** There exist two positive constants $B_1, B_2$ such that

\[
B_1 \leq c^{-2} \leq B_2 \quad \text{in } \Omega.
\]

Following the above mentioned Fourier transform, we consider the boundary value problem,

\[
\begin{cases}
(-\Delta - \omega^2 c^{-2}(x))u = 0, & \text{in } \Omega, \\
u = g & \text{on } \partial\Omega.
\end{cases}
\]

We summarize some results concerning the well-posedness of problem (2.2) which we will use in the proofs of the properties of the Dirichlet-to-Neumann map.

**Proposition 2.2.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^3$, $f \in L^2(\Omega)$, $g \in H^{1/2}(\partial\Omega)$ and $c^{-2} \in L^\infty(\Omega)$ satisfying (2.1). Then, there exists a discrete set $\Sigma_{c^{-2}} := \{ \lambda_n \mid \lambda_n > 0, \forall n \in \mathbb{N} \}$ such that, for any $\omega^2 \in C \setminus \Sigma_{c^{-2}}$, there exists a unique solution $u \in H^1(\Omega)$ of

\[
\begin{cases}
(-\Delta - \omega^2 c^{-2}(x))u = f, & \text{in } \Omega, \\
u = g & \text{on } \partial\Omega.
\end{cases}
\]

Furthermore, there exists a positive constant $C$ such that

\[
\|u\|_{H^1(\Omega)} \leq C \left( \|g\|_{H^{1/2}(\partial\Omega)} + \|f\|_{L^2(\Omega)} \right),
\]

where $C = C(B_2, \omega^2, \Omega, d)$ where $d := \text{dist}(\omega^2, \Sigma_{c^{-2}})$ and $C$ blows up as $d \to 0$. 


Proof. We first prove the result for $g = 0$. Consider the linear operator $K = (-\Delta)^{-1}M_{c-2} : L^2 \rightarrow H^1_0(\Omega)$ where $M_{c-2} : L^2(\Omega) \rightarrow L^2(\Omega)$ is the multiplication operator $h \mapsto c^{-2}h$. Then, problem (2.3) is equivalent to

$$(I - \omega^2 K)u = (-\Delta)^{-1}f =: h.$$  

We observe that $K$ is a compact operator from $L^2(\Omega)$ to $L^2(\Omega)$. In fact,

$$\| Ku \|_{H^1_0(\Omega)} \leq C \| u \|_{L^2(\Omega)},$$

and by the Rellich-Kondrachev compactness theorem $H^1_0(\Omega) \subset\subset L^2(\Omega)$. Furthermore, by Assumption 2.1 and the properties of $(-\Delta)^{-1}$ and of $M_{c-2}$ it follows that $K$ is self-adjoint and positive. Hence, $K$ has a discrete set of positive eigenvalues $\{\alpha_n\}_{n \in \mathbb{N}}$ such that $\alpha_n \to 0$ as $n \to \infty$. Let $\tilde{\lambda}_n := \frac{1}{\alpha_n}$, $n \in \mathbb{N}$ and define $\Sigma_{c-2} := \{\lambda_n : n \in \mathbb{N}\}$ and let $\omega^2 \in \mathbb{C}\setminus\Sigma_{c-2}$. Then, by the Fredholm alternative, there exists a unique solution $u \in H^1_0(\Omega)$ of (2.5).

To prove estimate (2.4) we observe that

$$u = \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n, \quad Ku = \sum_{n=1}^{\infty} \alpha_n \langle u, e_n \rangle e_n$$

where $\{e_n\}_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$. Hence we can rewrite (2.5) in the form

$$\sum_{n=1}^{\infty} (1 - \omega^2 \alpha_n) \langle u, e_n \rangle e_n = \sum_{n=1}^{\infty} \langle h_n, e_n \rangle e_n$$

Hence,

$$\langle u, e_n \rangle = \frac{1}{1 - \frac{\omega^2}{\lambda_n}} \langle h, e_n \rangle, \quad \forall n \in \mathbb{N}$$

and

$$u = \sum_{n=1}^{\infty} \frac{1}{1 - \frac{\omega^2}{\lambda_n}} \langle h, e_n \rangle e_n$$

so that

$$(2.6) \quad \| u \|_{L^2(\Omega)} \leq \left(1 + \frac{\omega^2}{d(\omega^2, \Sigma_{c-2})}\right) \| h \|_{L^2(\Omega)} \leq \left(1 + \frac{\omega^2}{d(\omega^2, \Sigma_{c-2})}\right) \| f \|_{L^2(\Omega)}$$

Now by multiplying equation (2.3) by $u$, integrating by parts, using Schwartz’ inequality, Assumptions (2.1) and (2.6) we derive

$$(2.7) \quad \| \nabla u \|_{L^2(\Omega)} \leq C \left(1 + \frac{\omega^2}{d(\omega^2, \Sigma_{c-2})}\right) \| f \|_{L^2(\Omega)}$$

Hence, by (2.6) and (2.7) we finally get

$$\| u \|_{H^1(\Omega)} \leq C \left(1 + \frac{\omega^2}{d(\omega^2, \Sigma_{c-2})}\right) \| f \|_{L^2(\Omega)}.$$  

If $g$ is not identically zero then we reduce the problem to the previous case by considering $v = u - \tilde{g}$ where $\tilde{g} \in H^1(\Omega)$ is such that $\tilde{g} = g$ on $\partial \Omega$ and $\| \tilde{g} \|_{H^1(\Omega)} \leq \| g \|_{H^{1/2}(\partial \Omega)}$. □
If \( \omega^2 \in \mathbb{C} \setminus \Sigma_{c-2} \), then, by \( \text{Lemma 2.2} \), for any \( g \) is in \( H^{1/2}(\partial \Omega) \), there exists a unique solution to \( \text{Equation 2.2} \) which belongs to \( H^1(\Omega) \). Therefore, \( \nabla u \) is in \( L^2(\Omega) \) and as a consequence \( \nabla u |_{\partial \Omega} \) belongs to \( H^{-1/2}(\partial \Omega) \). One can then introduce the Dirichlet-to-Neumann map, \( \Lambda_{\omega^2 \in \Sigma_{c-2}} \), according to

\[
\Lambda_{\omega^2 \in \Sigma_{c-2}} g = \nu \cdot \nabla u |_{\partial \Omega} = \frac{\partial u}{\partial \nu} \in H^{-1/2}(\partial \Omega).
\]

Proceeding similarly as in the proof of Proposition \( \text{Lemma 2.2} \) and using standard regularity results we can prove also the following

**Proposition 2.3.** Let \( \Omega \) be a bounded \( C^{1,1} \) domain in \( \mathbb{R}^3 \), \( c^{-2} \in L^\infty(\Omega) \) satisfying \( \text{Equation 2.1} \), \( f \in L^p(\Omega) \), \( g \in W^{2-\frac{1}{p},p}(\partial \Omega) \) with \( 1 < p < \infty \). Then, if \( \omega^2 \in \mathbb{C} \setminus \Sigma_{c-2} \), there exists a unique solution \( u \in W^{2,p}(\Omega) \) to the problem

\[
\begin{cases}
(\Delta - \omega^2 c^{-2}(x))u = f & \text{in } \Omega, \\
u u = g & \text{on } \partial \Omega.
\end{cases}
\]

Moreover,

\[
\|u\|_{W^{2,p}(\Omega)} \leq C(\|g\|_{W^{2-\frac{1}{p},p}(\partial \Omega)} + \|f\|_{L^p(\Omega)})
\]

where \( C = C(B_2,\omega^2,\Omega,d) \) where \( d := \text{dist}(\omega^2,\Sigma_{c-2}) \) and \( C \) blows up as \( d \to 0 \).

The constants appearing in the estimates of \( \text{Equation 2.2} \) and \( \text{Lemma 2.3} \) depend on \( c^{-2} \) and \( \Sigma_{c-2} \) which are unknown. To our purposes it would be convenient to have constants depending only on a priori parameters \( B_1,B_2 \) and known parameters. In fact, we can prove the following

**Proposition 2.4.** Suppose that the assumptions of Proposition \( \text{Lemma 2.2} \) (Proposition \( \text{Lemma 2.3} \)) are satisfied. Let \( \{\lambda_n\}_{n \in \mathbb{N}} \) denote the Dirichlet eigenvalues of \( -\Delta \). Then, for any \( n \in \mathbb{N} \),

\[
\frac{\lambda_n}{B_2} \leq \lambda_n \leq \frac{\lambda_n}{B_1}.
\]

If \( \omega^2 \) is such that,

\[
0 < \omega^2 \leq \omega_0^2 < \frac{\lambda_1}{B_2},
\]

or, for some \( n \geq 1 \),

\[
\frac{\lambda_n}{B_1} < \omega_1^2 \leq \omega^2 \leq \omega_n^2 < \frac{\lambda_{n+1}}{B_2},
\]

then there exists a unique solution \( u \in H^1(\Omega) \) (\( u \in W^{2,p}(\Omega) \)) of \( \text{Equation 2.2} \) while estimate \( \text{Equation 2.4} \) (\( \text{Equation 2.10} \)) is satisfied and \( C = C(B_1,B_2,\omega_0^2,\Sigma) \) \((C = C(B_1,B_2,\omega_1^2,\omega_n^2,\Sigma))\), where \( \Sigma := \{\lambda_n\}_{n \in \mathbb{N}} \) and \( C \) blows up as \( |\omega_0^2 - \frac{d^2}{B_2}| \to 0 \) \((\min(|\omega_1^2 - \frac{d^2}{B_1}|, |\omega_n^2 - \frac{d^2}{B_2}|) \to 0)\).

**Proof.** To derive estimate \( \text{Equation 2.11} \) we consider the Rayleigh quotient related to equation \( \text{Equation 2.2} \)

\[
\frac{\int_\Omega |\nabla v|^2}{\int_\Omega c^{-2}v^2}.
\]

By assumption \( \text{Equation 2.1} \), for any non trivial \( v \in H^1_0(\Omega) \),

\[
\frac{1}{B_2} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2} \leq \frac{\int_\Omega |\nabla v|^2}{\int_\Omega c^{-2}v^2} \leq \frac{1}{B_1} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2}.
\]
which, by the Courant-Rayleigh minimax principle, immediately gives

\[
\frac{\lambda_n}{\lambda_2} \leq \tilde{\lambda}_n \leq \frac{\lambda_n}{\lambda_1}, \quad \forall n \in \mathbb{N}.
\]

Hence, we have well-posedness of problem (2.2) if select an \( \omega^2 \) satisfying (2.12) or (2.13), and since \( d(\omega^2, \Sigma_{c^{-2}}) \geq |\omega_0^2 - \frac{\Lambda_{\Sigma_{c^{-2}}}}{\lambda_1}| \) or \( d(\omega^2, \sigma_{c^{-2}}) \geq \min(\{\omega_1^2 - \frac{\Lambda_{\Sigma_{c^{-2}}}}{\lambda_1}, |\omega_2^2 - \frac{\Lambda_{\Sigma_{c^{-2}}}}{\lambda_1}|\}) \) the claim follows. \( \square \)

We observe that in order to derive the uniform estimates in Proposition 2.4 below, we need to assume either that the frequency is sufficiently small or that the oscillation of \( c^{-2} \) is sufficiently small. This clearly depends on the fact that we compare the eigenvalues of our equation with those of the Laplacian. In the applications we have in mind we aim to recover also wavespeeds with high contrasts at not too small frequencies. For these purposes, the following local result will be relevant.

**Proposition 2.5.** Let \( \Omega \) and \( c_0^{-2} \) satisfy the assumptions of Proposition 2.2, 2.3 and let \( \omega^2 \in \mathbb{C}\backslash \Sigma_{c^{-2}} \) where \( \Sigma_{c^{-2}} \) is the Dirichlet spectrum of equation (2.2) for \( c^{-2} = c_0^{-2} \). There exists \( \delta > 0 \) such that, if

\[
\|c^{-2} - c_0^{-2}\|_{L^\infty(\Omega)} \leq \delta,
\]

then \( \omega^2 \in \mathbb{C}\backslash \Sigma_{c^{-2}} \) and the estimates (2.14) (2.11) hold, the constant \( C = C(B_2, \omega^2, \Omega, d_0) \) where \( d_0 := \text{dist}(\omega^2, \Sigma_{c_0^{-2}}) \) and \( C \) blows up as \( d_0 \to 0 \).

**Proof.** We limit ourselves to prove the well-posedness in \( H^1(\Omega) \). Let \( \delta c := c^{-2} - c_0^{-2} \) and consider \( u_0 \in H^1(\Omega) \) the unique solution of (2.3) for \( c_0^{-2} \) and consider the problem

\[
\begin{cases}
-\Delta v - \omega^2 c_0^{-2} v - \omega^2 \delta c v = \omega^2 u_0, & x \in \Omega, \\
v = 0, & x \in \partial \Omega.
\end{cases}
\]

Let \( L := -\Delta - \omega^2 c_0^{-2} \). Then, by assumption, \( L \) is invertible from \( H^1(\Omega) \) to \( L^2(\Omega) \) and we can rewrite problem (2.14) in the form

\[
(I - K)v = h,
\]

where \( K = L^{-1} M_{\delta c} \) where \( M_{\delta c} \) is the multiplication operator and \( h = L^{-1}(\omega^2 u_0 \delta c) \). Observe now that

\[
\|K\| \leq \|L^{-1}\| \|M_{\delta c}\| \leq \|L^{-1}\| \delta \leq \frac{\omega^2}{d_0} \delta.
\]

Hence, choosing \( \delta = \frac{1}{2}(B_2 \omega^2 (1 + \frac{\omega^2}{d_0}))^{-1} \), we have that there exists a unique solution \( v \) of (2.15) in \( H^1_0 \) satisfying (2.2) with \( C = C(B_2, \omega^2, \Omega, d_0) \) and since \( u = u_0 + v \) the statement follows. \( \square \)

We define the direct operator, \( F_\omega \), as

\[
F_\omega : L^2(\Omega) \to \mathcal{L}(H^{1/2}(\partial \Omega), H^{-1/2}(\partial \Omega)),
\]

\[
c^{-2}(x) \mapsto \Lambda_{\omega^2 c^{-2}},
\]

and examine its properties in the following lemmas.

**Lemma 2.6** (Fréchet differentiability). Assume that \( \omega^2 \) satisfies (2.12) or (2.13). Then the direct operator \( F_\omega \) is Fréchet differentiable at \( c^{-2} \).

**Proof.** We start from Alessandrini’s identity,

\[
\int_\Omega \omega^2(c_1^{-2} - c_2^{-2})u_1 u_2 \, dx = (\Lambda_{\omega^2 c_1^{-2}} - \Lambda_{\omega^2 c_2^{-2}})u_1|_{\partial \Omega}, u_2|_{\partial \Omega},
\]
where \( \langle \cdot, \cdot \rangle \) is the dual pairing with respect to \( H^{-1/2}(\partial \Omega) \) and \( H^{1/2}(\partial \Omega) \) and \( u_1 \) and \( u_2 \) are the solutions of the Helmholtz equation with Dirichlet boundary condition and coefficient \( c_1 \) and \( c_2 \), respectively. Let \( \delta c^{-2} \in L^\infty(\Omega) \). We observe, while substituting \( c^{-2} \) and \( c^{-2} + \delta c^{-2} \) for \( c_1^{-2} \) and \( c_2^{-2} \), that

\[
(2.17) \quad \langle (\Lambda_{\omega^2(c^{-2}+\delta c^{-2})} - \Lambda_{\omega^2 c^{-2}})g, h \rangle = \omega^2 \int_{\Omega} \delta c^{-2} uv \, dx,
\]

where \( u \) and \( v \) solve the boundary value problems,

\[
\begin{align*}
(\Delta - \omega^2(c^{-2} + \delta c^{-2}))u &= 0, \quad x \in \Omega, \\
u &= g, \quad x \in \partial \Omega,
\end{align*}
\]

and

\[
\begin{align*}
(\Delta - \omega^2 c^{-2})v &= 0, \quad x \in \Omega, \\
v &= h, \quad x \in \partial \Omega,
\end{align*}
\]

respectively. We show that

\[
(2.18) \quad \langle DF_{\omega}(c^{-2})(\delta c^{-2})g, h \rangle = \omega^2 \int_{\Omega} \delta c^{-2} \tilde{u} v \, dx,
\]

where \( \tilde{u} \) solves the equation

\[
\begin{align*}
\langle (\Delta - \omega^2 c^{-2})\tilde{u}, \tilde{u} \rangle &= 0, \quad x \in \Omega, \\
\tilde{u} &= g, \quad x \in \partial \Omega.
\end{align*}
\]

In fact, by (2.17), we have that

\[
(2.19) \quad \langle (\Lambda_{\omega^2(c^{-2}+\delta c^{-2})} - \Lambda_{\omega^2 c^{-2}})g, h \rangle - \omega^2 \int_{\Omega} \delta c^{-2} \tilde{u} v \, dx = \omega^2 \int_{\Omega} \delta c^{-2} (u - \tilde{u}) v \, dx.
\]

By using the H"older inequality twice and the Sobolev embedding theorem, we obtain that

\[
(2.20) \quad \left| \omega^2 \int_{\Omega} \delta c^{-2} (u - \tilde{u}) v \, dx \right| \leq \omega^2 \|\delta c^{-2}\|_{L^2(\Omega)} \|u - \tilde{u}\|_{L^2(\Omega)} \|v\|_{L^4(\Omega)}.
\]

We note that \( u - \tilde{u} \) solves the equations

\[
\begin{align*}
(\Delta - \omega^2 c^{-2})(u - \tilde{u}) &= -\omega^2 \delta c^{-2} u, \quad x \in \Omega, \\
0 &= u - \tilde{u}, \quad x \in \partial \Omega.
\end{align*}
\]

Therefore, by the Sobolev embedding theorem and Proposition 2.4, we find that

\[
(2.21) \quad \|u - \tilde{u}\|_{L^4(\Omega)} \leq C\|u - \tilde{u}\|_{W^{1,\frac{4}{3}}(\Omega)} \leq C\omega^2 \|\delta c^{-2} u\|_{L^2(\Omega)} \leq C\omega^2 \|\delta c^{-2}\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}.
\]

The right-most inequality is obtained by applying the H"older inequality to \( \int_{\Omega} |\delta c^{-2} u|^{12/11} \, dx \) with indexes 11/6 and 11/5. By using the interpolation of \( L^p \) spaces,

\[
\|u\|_{L^{p\theta}(\Omega)} \leq \|u\|^{1-\theta}_{L^2(\Omega)} \|u\|^\theta_{L^6(\Omega)}, \quad \forall \theta \in [0,1];
\]

with \( p_\theta \) defined by \( \frac{1}{p_\theta} = \frac{1}{2}(1-\theta) + \frac{1}{6} \theta \), we conclude that

\[
(2.22) \quad \|u - \tilde{u}\|_{L^4(\Omega)} \leq C\omega^2 \|\delta c^{-2}\|_{L^2(\Omega)} \|u\|^{3/4}_{L^6(\Omega)} \|u\|^{1/4}_{L^6(\Omega)}.
\]
Then we estimate the left-hand side of (2.19) using inequalities (2.20), (2.21) and the interpolation of $L^4$ space,

\[
\left| \left( \Lambda \omega^2 c_1^{-2} - \Lambda \omega^2 c_2^{-2} \right) g, h \right| - \omega^2 \int_{\Omega} \delta c_2^{-2} \Delta \omega \, dx = \left| \omega^2 \int_{\Omega} \delta c_2^{-2} (u - \tilde{u}) v \, dx \right|
\]

for some constant $C$. Next, by applying the Sobolev embedding, $H^1(\Omega) \subset L^6(\Omega)$, and Proposition 2.4 we arrive at the following inequality,

\[
\|u\|^{3/4}_{L^2(\Omega)} \|u\|^{1/4}_{H^n(\Omega)} \leq C \|u\|^{3/4}_{L^2(\Omega)} \|u\|^{1/4}_{H^{1/2}(\partial \Omega)} \leq C \|u\|^{1/2}_{H^1(\Omega)}.
\]

This, with (2.23) and the same procedure for $v$, leads to the Fréchet differentiability of $F_\omega$ at $c^{-2}$. \(\blacksquare\)

**Lemma 2.7.** Let $c^{-2} \in L^2(\Omega)$ satisfy Assumption 2.1. Then, if (2.12) (2.13) holds, there exists a constant $\delta_0$, which depends on $(\Omega, B_2, \lambda_1, \omega^2_0)$ $(\Omega, B_1, B_2, \omega_1^2, \omega_2^2, \Sigma)$ such that

\[
\|DF_\omega(c^{-2})\|_{L(\Omega)} \leq \delta_0 \omega^2;
\]

$\delta_0$ blows up as $|\omega^2_0 - \frac{\lambda_1}{B_2}| \to 0$ (min$(|\omega^2_1 - \frac{\lambda_1}{B_1}|, |\omega^2_2 - \frac{\lambda_{i+1}}{B_2}|) \to 0$).

**Proof.** We start from Alessandrini’s identity (2.10). By applying the Hölder inequality twice, we find that

\[
\left| \left( \Lambda \omega^2 c_1^{-2} - \Lambda \omega^2 c_2^{-2} \right) u_1 |_{\partial \Omega}, u_2 |_{\partial \Omega} \right| = \left| \int_{\Omega} \omega^2 (c_1^{-2} - c_2^{-2}) u_1 u_2 \, dx \right|
\]

\[
\leq \omega^2 \|c_1^{-2} - c_2^{-2}\|_{L^2(\Omega)} \|u_1\|_{H^n(\Omega)} \|u_2\|_{L^n(\Omega)}.
\]

Similarly as in the proof of Lemma 2.8 by the interpolation of $L^4$ space, the Sobolev embedding theorem and Proposition 2.4 we obtain

\[
\|u_i\|_{L^n(\Omega)} \leq C \|u_i\|_{H^1(\Omega)} \leq C \|u_i\|_{H^{1/2}(\partial \Omega)}, \quad i = 1, 2,
\]

where $C$ is a generic constant. Hence,

\[
\|\Lambda \omega c_1^{-2} - \Lambda \omega c_2^{-2}\|_{L(H^{1/2}(\partial \Omega), H^{-1/2}(\partial \Omega))} \leq \delta_0 \omega^2 \|c_1^{-2} - c_2^{-2}\|_{L^2(\Omega)},
\]

with $\delta_0 = C \|u_i\|_{L^n(\Omega)}$, from which (2.24) follows. \(\blacksquare\)

**Lemma 2.8.** Let $c_1^{-2}, c_2^{-2} \in L^2(\Omega)$ satisfy Assumption 2.1. Then, if (2.12) (2.13) holds there exists a positive constant $\delta_0$, such that

\[
\|DF_\omega(c_1^{-2}) - DF_\omega(c_2^{-2})\|_{L(\Omega)} \leq \delta_0 \omega^4 \|c_1^{-2} - c_2^{-2}\|_{L^2(\Omega)};
\]

where $\delta_0 = \delta_0(\Omega, B_1, B_2, \omega_1^2, \omega_2^2, \Sigma)$ and $\delta_0$ blows up as $|\omega^2_0 - \frac{\lambda_1}{B_2}| \to 0$ (min$(|\omega^2_1 - \frac{\lambda_1}{B_1}|, |\omega^2_2 - \frac{\lambda_{i+1}}{B_2}|) \to 0$).

**Proof.** Let $g, h \in H^{1/2}(\Omega)$ and $u_i, v_i, \ i = 1, 2$, solve the boundary value problems,

\[
\begin{cases}
(-\Delta - \omega^2 c_1^{-2}) u_i = 0, \quad x \in \Omega, \\
u_i = h, \quad x \in \partial \Omega,
\end{cases}
\]

\[
\begin{cases}
(-\Delta - \omega^2 c_2^{-2}) v_i = 0, \quad x \in \Omega, \\
v_i = g, \quad x \in \partial \Omega,
\end{cases}
\]
respectively. By using identity (2.18) and applying the Hölder inequality twice, we have

\[ ||(DF_\omega(c_1^{-2})(\delta c^{-2}) - DF_\omega(c_2^{-2})(\delta c^{-2}))g, h|| \]

\[ = |\omega^2 \int_\Omega \delta c^{-2} (u_1 v_1 - u_2 v_2) dx| \]

\[ \leq \omega^2 \|\delta c^{-2}\|_{L^2(\Omega)} (||u_1 - u_2||_{L^4(\Omega)} ||v_1||_{L^4(\Omega)} + ||u_2||_{L^4(\Omega)} ||v_1 - v_2||_{L^4(\Omega)}). \]

We note that

\[ u_1 - u_2 \]

solves the equations

\[ \begin{cases} (-\Delta - \omega^2 c_1^{-2}) (u_1 - u_2) = \omega^2 (c_1^{-2} - c_2^{-2}) u_2, & x \in \Omega, \\ u_1 - u_2 = 0, & x \in \partial\Omega. \end{cases} \]

Using an argument similar to the one in the proof of Lemma 2.6, we derive that

\[ ||u_1 - u_2||_{L^4(\Omega)} \leq C \omega^2 \|c_1^{-2} - c_2^{-2}\|_{L^2(\Omega)} \|u_2||_{L^4(\Omega)} \leq C \omega^2 \|c_1^{-2} - c_2^{-2}\|_{L^2(\Omega)} ||g||_{H^{1/2}(\partial\Omega)} \]

and, analogously,

\[ ||v_1 - v_2||_{L^4(\Omega)} \leq C \omega^2 \|c_1^{-2} - c_2^{-2}\|_{L^2(\Omega)} \|v_2||_{L^4(\Omega)} \leq C \omega^2 \|c_1^{-2} - c_2^{-2}\|_{L^2(\Omega)} ||h||_{H^{1/2}(\partial\Omega)}. \]

Hence

\[ ||(DF_\omega(c_1^{-2})(\delta c^{-2}) - DF_\omega(c_2^{-2})(\delta c^{-2}))g, h|| \]

\[ \leq C \omega^4 \|\delta c^{-2}\|_{L^2(\Omega)} \|c_1^{-2} - c_2^{-2}\|_{L^2(\Omega)} ||g||_{H^{1/2}(\partial\Omega)} ||h||_{H^{1/2}(\partial\Omega)}, \]

which gives that

\[ ||DF_\omega(c_1^{-2}) - DF_\omega(c_2^{-2})||_{L(L^2(\Omega), L(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)))} \leq C \omega^4 \|c_1^{-2} - c_2^{-2}\|_{L^2(\Omega)}. \]

\[ \square \]

**Remark 2.9.** In the above lemmas, we analyze the properties of the Fréchet derivative. For simplicity, we use \( L^2 \). With Assumption 2.4, we enforce that \( c^{-2} \) belongs to \( L^\infty(\Omega) \). For \( L^p \) optimization, \( p > 3/2 \), Lemma 2.6 and 2.7 can be generalized with the \( L^2 \) norm replaced by the \( L^p \) norm; the constants \( \Sigma_0 \) and \( \Sigma_0 \) will also depend on \( p \).

**3. Stability of the inverse problem.** Let \( B_2, r_0, r_1, A, L, N \) be positive with \( N \in \mathbb{N}, N \geq 2, r_0 < 1 \). In the sequel we will refer to these numbers as to the a-priori data. To prove the results of this section we invoke the following common assumptions

**Assumption 3.1.** \( \Omega \subset \mathbb{R}^n \) is a bounded domain such that

\[ |x| \leq Ar_1, \quad \forall x \in \Omega. \]

Assume

\[ \partial \Omega \] of Lipschitz class with constants \( r_1 \) and \( L. \)

Consider a partitioning \( D_N \) of \( \Omega \) given by

\[ D_N \equiv \{D_1, D_2, \ldots, D_N\} \mid \bigcup_{j=1}^N \overline{D_j} = \Omega, (D_j \cap D_{j'})^0 = \emptyset, j \neq j', \]

with

\[ \{\partial D_j\}_{j=1}^N \] of Lipschitz class with constants \( r_0 \) and \( L. \)
Assumption 3.2. The function $c^{-2} \in W_N$, that is, satisfies

$$B_1 \leq c^{-2} \leq B_2, \text{ in } \Omega$$

and is of the form

$$c^{-2}(x) = \sum_{j=1}^{N} c_j \chi_{D_j}(x),$$

where $c_j, j = 1, \ldots, N$ are unknown numbers and $D_j$ are known open sets in $\mathbb{R}^3$.

Assumption 3.3.

$$0 < \omega^2 \notin \Sigma_{c^{-2}}, \forall c^{-2} \in W_N.$$ 

Under the above assumptions we can state the following result

Lemma 3.4. Let assumptions 3.1 and 3.2 hold and let $c^{-2} \in W_N$. Then, for every $s' \in (0, 1/2)$, there exists a positive constant $C$ with $C = C(L, s')$ such that

$$\|c^{-2}\|_{H^{s'}(\Omega)} \leq C(L, s') \frac{1}{r_0^s} \|c^{-2}\|_{L^2(\Omega)}.$$ 

Proof. The proof is based on the extension of a result of Magnanini and Papi in [16] to the three dimensional setting. In fact, following the argument in [MP], one has that

$$\|\chi_{D_j}\|_{H^{s'}(\Omega)}^2 \leq \frac{16\pi}{(1-2s')(2s')^{1+2s'}}|D_j|^{1-2s'}|\partial D_j|^{2s'}.$$ 

By Assumption 3.2, recalling that the $\{D_j\}_{j=1}^{N}$ is a partition of disjoint sets of $\Omega$, we have that

$$\|c^{-2}\|_{H^{s'}(\Omega)}^2 = \sum_{j=1}^{N} c_j^2 \|\chi_{D_j}\|_{H^{s'}(\Omega)}^2$$ 

so that from (3.3)

$$\|c^{-2}\|_{H^{s'}(\Omega)}^2 \leq C(s') \sum_{j=1}^{N} c_j^2 |D_j| \left( \frac{|\partial D_j|}{|D_j|} \right)^{2s'} \leq \frac{C(L, s')}{r_0^{2s'}} \|c^{-2}\|_{L^2(\Omega)}^2.$$ 

We have the following stability result

Proposition 3.5. Assume (3.1) and let $c_1^{-2}, c_2^{-2} \in \tilde{W}_N$ and $\omega^2$ satisfies (3.3). Then, there exists a positive constant $K$ depending on $A, r_1, L$ such that,

$$\|c_1^{-2} - c_2^{-2}\|_{L^2(\Omega)} \leq \frac{1}{\omega^2} e^{K(1+\omega^2B_2)(|\Omega|/r_0^3)} \|\Lambda_1 - \Lambda_2\|_{L^2(H^{1/2}(\partial \Omega), H^{-1/2}(\partial \Omega))},$$

where $\Lambda_k = \Lambda_{\omega^2 c_k^{-2}}$ for $k = 1, 2$.

Proof. To prove our stability estimate we follow the original idea of Alessandrini of using CGO solutions but we use slightly different ones than those introduced in [23] and in [1] to obtain better
constants in the stability estimates as proposed by [20]. We also use the estimates proposed in [20] (see Theorem 3.8) and due to [13] concerning the case of bounded potentials.

In fact, by Theorem 3.9 of [20], since \( c^{-2} \in L^\infty(\Omega) \), \( \| c^{-2} \|_{L^\infty(\Omega)} \leq B_2 \), then there exists a positive constant \( C \) such that for any \( \zeta \in \mathbb{C}^3 \) satisfying \( \zeta \cdot \zeta = 0 \) and \( |\zeta| \geq C \) the equation

\[-\Delta u - \omega^2 c^{-2} u = 0\]

has a solution of the form

\[ u(x) = e^{ix \cdot \zeta}(1 + R(x)) \]

where \( r \in H^1(\Omega) \) satisfies

\[ \| R \|_{L^2(\Omega)} \leq \frac{C}{|\zeta|}, \]

\[ \| \nabla R \|_{L^2(\Omega)} \leq C. \]

Let \( \xi \in \mathbb{R}^3 \) and let \( \tilde{\omega}_1 \) and \( \tilde{\omega}_2 \) be unit vectors of \( \mathbb{R}^3 \) such that \( \{ \tilde{\omega}_1, \tilde{\omega}_2, \xi \} \) is an orthogonal set of vectors of \( \mathbb{R}^3 \). Let \( s \) be a positive parameter to be chosen later and set for \( k = 1, 2 \),

\[ \zeta_k = \begin{cases} 
(-1)^{k-1} \frac{s}{\sqrt{2}} \left( (1 - \frac{|\xi|^2}{2s^2}) \tilde{\omega}_1 + (-1)^{k-1} \frac{1}{\sqrt{2s}} \xi + i \tilde{\omega}_2 \right) & \text{for } \frac{|\xi|}{\sqrt{2s}} < 1, \\
(-1)^{k-1} \frac{1}{\sqrt{2s}} \left( (-1)^{k-1} \frac{1}{\sqrt{2s}} \xi + i \sqrt{(\frac{|\xi|^2}{2s^2} - 1)} \tilde{\omega}_1 + \tilde{\omega}_2 \right) & \text{for } \frac{|\xi|}{\sqrt{2s}} \geq 1.
\end{cases} \]

Then a straightforward computation gives

\[ \zeta_k \cdot \zeta_k = 0 \]

for \( k = 1, 2 \) and

\[ \zeta_1 + \zeta_2 = \xi. \]

Furthermore, for \( k = 1, 2 \),

\[ |\zeta_k| = \begin{cases} 
s & \text{for } \frac{|\xi|}{\sqrt{2s}} < 1, \\
|\xi| \sqrt{2s} & \text{for } \frac{|\xi|}{\sqrt{2s}} \geq 1.
\end{cases} \]

Hence,

\[ |\zeta_k| = \max\{s, \frac{|\xi|}{\sqrt{2s}}\}. \]

Then, by Theorem 3.9 of [20], for \( |\zeta_1|, |\zeta_2| \geq C_1 = \max\{C_0 \omega^2 B_2, 1\} \), with \( C_0 = C_0(A, r_1) \), there exist \( u_1, u_2 \), solutions to \(-\Delta u_k - \omega^2 c_k^{-2} u_k = 0 \) for \( k = 1, 2 \) respectively, of the form

\[ u_1(x) = e^{ix \cdot \zeta_1}(1 + R_1(x)), \quad u_2(x) = e^{ix \cdot \zeta_2}(1 + R_2(x)) \]

with

\[ \| R_k \|_{L^2(\Omega)} \leq \frac{C_0 \sqrt{|\Omega|}}{s} \omega^2 B_2 \]

and

\[ \| \nabla R_k \|_{L^2(\Omega)} \leq C_0 \sqrt{|\Omega|} \omega^2 B_2 \]
for \( k = 1, 2 \).

Consider Alessandrini’s identity
\[
\int_{\Omega} \omega^2 (c_1^{-2} - c_2^{-2}) u_1 u_2 \, dx = \langle (\Lambda_1 - \Lambda_2) u_1 |_{\partial \Omega}, u_2 |_{\partial \Omega} \rangle,
\]
where \( u_k \in H^1(\Omega) \) is any solution of \(-\Delta u_k - \omega^2 c_k^2 u_k = 0\) for \( k = 1, 2 \). Inserting the solutions \(3.9\) in Alessandrini’s identity we derive
\[
(3.12) \quad \left| \int_{\Omega} \omega^2 (c_1^{-2} - c_2^{-2}) e^{i\xi \cdot x} \, dx \right| \leq \| \Lambda_1 - \Lambda_2 \| \| u_1 \|_{H^{1/2}(\partial \Omega)} \| u_2 \|_{H^{1/2}(\partial \Omega)}
\]
\[
+ \left| \int_{\Omega} \omega^2 (c_1^{-2} - c_2^{-2}) e^{i\xi \cdot x} (R_1 + R_2 + R_1 R_2) \, dx \right|
\]
\[
\leq \| \Lambda_1 - \Lambda_2 \| \| u_1 \|_{H^1(\Omega)} \| u_2 \|_{H^1(\Omega)} + E(\| R_1 \|_{L^2(\Omega)} + \| R_2 \|_{L^2(\Omega)} + \| R_1 \|_{L^2(\Omega)} \| R_2 \|_{L^2(\Omega)}).
\]

where \( E := \| \omega^2 (c_1^{-2} - c_2^{-2}) \|_{L^2(\Omega)} \) By \(3.10\), \(3.11\), \(3.8\) and since \( \Omega \subset B_{2R}(0) \) we have
\[
\| u_k \|_{H^1(\Omega)} \leq C \sqrt{|\Omega|} (s + |\xi|) e^{A_{r_1}(s+|\xi|)}, \quad k = 1, 2.
\]

Let \( s \geq C_2 \) so that \( s + |\xi| \leq e^{A_{r_1}(s+|\xi|)} \). Then, for \( s \geq C_3 = \max(C_1, C_2) \), using \(3.10\) and \(3.11\) we get
\[
(3.13) \quad \| \omega^2 (c_1^{-2} - c_2^{-2}) \|_{L^2(\mathbb{R}^3)} \leq C \sqrt{|\Omega|} \left( e^{A_{r_1}(s+|\xi|)} \| \Lambda_1 - \Lambda_2 \| + \frac{\omega^2 B^2 E}{s} \right)
\]
where the \( \omega^2 c_k^{-2} \)’s have been extended to all \( \mathbb{R}^3 \) by zero. Hence, we get
\[
(3.14) \quad \| \omega^2 (c_1^{-2} - c_2^{-2}) \|_{L^2(\mathbb{R}^3)} \leq C |\Omega| \rho^3 \left( e^{A_{r_1}(s+\rho)} \| \Lambda_1 - \Lambda_2 \| \rho^2 + \frac{\omega^4 B^2 E^2}{s^2} \right) + \int_{|\xi| \geq \rho} \| \omega^2 (c_1^{-2} - c_2^{-2}) \|_{L^2(\mathbb{R}^3)} \, d\xi
\]
where \( C = C(A, r_1) \). By \(3.2\) and \(3.4\) we have that
\[
\| \omega^2 (c_1^{-2} - c_2^{-2}) \|_{H^{s'}(\Omega)} \leq \frac{C}{r_0^{2s'}} E^2,
\]
where \( C \) depends on \( L, s' \) and hence
\[
\rho^{2s'} \int_{|\xi| \geq \rho} \| \omega^2 (c_1^{-2} - c_2^{-2}) \|_{L^2(\mathbb{R}^3)} \, d\xi \leq \int_{|\xi| \geq \rho} \| \xi |^{2s'} \| \omega^2 (c_1^{-2} - c_2^{-2}) \|_{L^2(\mathbb{R}^3)} \, d\xi
\]
\[
\leq \int_{\mathbb{R}^3} (1 + |\xi|^2)^{s'} \| \omega^2 (c_1^{-2} - c_2^{-2}) \|_{L^2(\mathbb{R}^3)} \, d\xi \leq \frac{C E^2}{r_0^{2s'} \rho^{2s'}}.
\]

Hence, we get
\[
\int_{|\xi| \geq \rho} \| \omega^2 (c_1^{-2} - c_2^{-2}) \|_{L^2(\mathbb{R}^3)} \, d\xi \leq \frac{C E^2}{r_0^{2s'} \rho^{2s'}}
\]
for any \( s' \in (0, 1/2) \). Inserting last bound in \(3.14\) we derive
\[
\| \omega^2 (c_1^{-2} - c_2^{-2}) \|_{L^2(\mathbb{R}^3)} \leq C \left( \rho^3 |\Omega| \left( e^{A_{r_1}(s+\rho)} \| \Lambda_1 - \Lambda_2 \| \rho^2 + \frac{\omega^4 B^2 E^2}{s^2} \right) + \frac{E^2}{r_0^{2s'} \rho^{2s'}} \right).
\]
where \( C = C(L, s') \). Pick up
\[
\sqrt{|\Omega|} \rho = \left( \frac{|\Omega|}{r_0^3} \right)^{\frac{2s'}{3s+2s'}} \left( \frac{1}{r_0} \right)^{\frac{1}{s+2s'}} \frac{1}{r_0^2} \frac{s}{s+2s'},
\]
with \( \alpha = \max\{1, \omega^d B_2^2\} \). Then, by Assumption 3.1 and observing that we might assume without loss of generality that \( \frac{\Omega}{r_0^3} > 1 \), we get

\[
\|\omega^2 (c_1^{-2} - c_2^{-2})\|_{L^2(\Omega)}^2 \leq CE^2 \left( \frac{\Omega}{r_0^3} \right)^{\frac{2\gamma'}{\gamma + 2\gamma'}} \left( c_4 \left( \frac{\Omega}{r_0^3} \right)^{\frac{2\gamma'}{\gamma + 2\gamma'}} \right) \left( \frac{\|A_1 - A_2\|}{E} \right)^2 + \left( \frac{\alpha}{8^2} \right)^{\frac{2\gamma'}{\gamma + 2\gamma'}}
\]

for \( s \geq C_3 \) and where \( C \) depends on \( s', L, A, r_1 \) and \( C_4 \) depends on \( L, A, r_1 \). Let us choose

\[
s = \frac{1}{C_4 \left( \frac{\Omega}{r_0^3} \right)^{\frac{2\gamma'}{\gamma + 2\gamma'}}} \log \frac{\|A_1 - A_2\|}{E}
\]

where we have assumed that

\[
\frac{\|A_1 - A_2\|}{E} < c := e^{-C \max\{1, \omega^d B_2\} \left( \frac{\Omega}{r_0^3} \right)^{\frac{2\gamma'}{\gamma + 2\gamma'}}}
\]

with \( \bar{C} = \tilde{C}(R) \) so that \( s \geq C_3 \). Under this assumption,

\[
\|\omega^2 (c_1^{-2} - c_2^{-2})\|_{L^2(\Omega)} \leq C(\sqrt{\alpha})^{\frac{2\gamma'}{\gamma + 2\gamma'}} \left( \frac{\Omega}{r_0^3} \right)^{\frac{2\gamma'}{\gamma + 2\gamma'} + \frac{9 + 10\gamma'}{3(3 + 2\gamma')}} E \left( \log \frac{\|A_1 - A_2\|}{E} \right)^{\frac{2\gamma'}{\gamma + 2\gamma'}}
\]

where \( C = C(L, s', A, r_1) \) and we can rewrite last inequality in the form

\[
E \leq C(1 + \omega^2 B_2)^{\frac{2\gamma'}{\gamma + 2\gamma'}} \left( \frac{\Omega}{r_0^3} \right)^{\frac{2\gamma'}{\gamma + 2\gamma'} + \frac{9 + 10\gamma'}{3(3 + 2\gamma')}} E \left( \log \frac{\|A_1 - A_2\|}{E} \right)^{\frac{2\gamma'}{\gamma + 2\gamma'}}
\]

which gives

\[
E \leq e^{C(1 + \omega^2 B_2) \left( \frac{\Omega}{r_0^3} \right)^{\frac{9 + 10\gamma'}{3(3 + 2\gamma')}}} \|A_1 - A_2\|
\]

where \( C = C(L, s', A, r_1) \). On the other hand if

\[
\frac{\|A_1 - A_2\|}{E} \geq c,
\]

then

\[
\|\omega^2 (c_1^{-2} - c_2^{-2})\|_{L^2(\Omega)} \leq e^{-1} \|A_1 - A_2\| \leq e^{C(1 + \omega^2 B_2) \left( \frac{\Omega}{r_0^3} \right)^{\frac{9 + 10\gamma'}{3(3 + 2\gamma')}}} \|A_1 - A_2\|
\]

Hence, from (3.17) and (3.18) and recalling that \( s \in (0, \frac{1}{2}) \), we have that

\[
E \leq e^{C(1 + \omega^2 B_2) \left( \frac{\Omega}{r_0^3} \right)^{\frac{9 + 10\gamma'}{3(3 + 2\gamma')}}} \|A_1 - A_2\|
\]

Choosing \( s' = \frac{1}{7} \), we derive

\[
\|c_1^{−2} - c_2^{−2}\|_{L^2(\Omega)} \leq \frac{1}{\omega^2} e^{K(1 + \omega^2 B_2) \left( \frac{\Omega}{r_0^3} \right)^{\frac{1}{7}}} \|A_1 - A_2\|
\]

where \( K = K(L, A, r_1, s') \) and the claim follows. \( \Box \)
Remark 3.6. In [7] the following lower bound of the stability constant has been obtained in the case of a uniform polyhedral partition $\mathcal{D}_N$

$$C_N \geq \frac{1}{4} e^{K_1 N^{\frac{1}{4}}}$$

Choose a uniform cubical partition $\mathcal{D}_N$ of $\Omega$ of mesh size $r_0$. Then,

$$|\Omega| = N r_0^3$$

and estimate (3.4) of Proposition 3.5 gives

$$C_N = \frac{1}{\omega^2} e^{K(1+\omega^2 B_2)N^{\frac{1}{4}}}$$

which proves optimality of the bound on the Lipschitz constant with respect to $N$ when the global DtN map is known. In [7] a Lipschitz stability estimate has been derived in terms of the local DtN map using singular solutions. This type of solutions allows to recover the unknown piecewise constant wavespeeds by determining it on the outer boundary of the domain and then, by propagating the singularity inside the domain, to recover step by step the wavespeed on the interface of all subdomains of the partition. This iterative procedure does not lead to sharp bounds of the Lipschitz constant appearing in the stability estimate. It would be interesting if one can get a better bound of the Lipschitz constant using oscillating solutions.

Remark 3.7. Observe that the result obtained in Proposition 3.5 extends to the case

$$c^{-2}(x) = \sum_{j=1}^{N} c_j \psi_j(x)$$

with $\psi_j$ with support in $D_j$ for each $j$ and satisfying

$$r_0^s \| \psi_j \|_{H^{s'}(D_j)} \leq M, \quad j = 1, \cdots, N$$

for $s' \in (0, \frac{1}{2})$. Then

$$(3.20) \quad \| c^{-2} \|_{H^{-s'}(\Omega)}^2 = \sum_{j=1}^{N} c_j^2 \| \psi_j \|_{H^{s'}(D_j)}^2 \leq M \frac{1}{r_0^{2s}} \| c^{-2} \|_{L^2(\Omega)}^2$$

and we might apply the same arguments of Proposition 3.5 to derive estimate (3.5).

Remark 3.8. Observing that

$$\| c_1^{-2} - c_2^{-2} \|_{L^\infty(\Omega)} \leq C \frac{r_0^{s/2}}{r_0^{s/2}} \| c_1^{-2} - c_2^{-2} \|_{L^2(\Omega)},$$

we immediately get the following stability estimate in the $L^\infty$ norm

$$\| c_1^{-2} - c_2^{-2} \|_{L^\infty(\Omega)} \leq C \frac{1}{\omega^2} e^{K(1+\omega^2 B_2)\left(\frac{1}{r_0^2}\right)^{\frac{1}{4}}} \| \Lambda_1 - \Lambda_2 \|$$

with $C = C(L, r_0)$. 

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4. Multi-level projected steepest descent iteration in $L^2(\Omega)$. Thus far, we have analyzed the data operator, which is contained in the Banach space, $Y$ say, of bilinear form on $H^{1/2}(\partial \Omega) \times H^{1/2}(\partial \Omega)$, and the stability of the inverse problem, and quantitatively estimated the relevant constants. Here, we consider the iterative reconstruction in $X := L^2(\Omega)$. We apply a projected steepest descent iteration \[12\] in a multi-level setting.

In the following lemma, we summarize some basic notions associated with iterative methods in $L^2(\Omega)$. For a detailed introduction to the Bregman distance, the duality mapping and the geometry of Banach spaces, we refer to \[11, 21\].

**Lemma 4.1.**
(a) The normalized duality mapping, $J_2$, is the identity mapping.
(b) The Bregman distance, $\Delta_2(f, \hat{f}) = \frac{1}{2} \| f - \hat{f} \|_{L^2(\Omega)}^2$, is non-expansive.
(c) Given a closed convex set $Z \subset L^2(\Omega)$, the projection of an element $f \in L^2(\Omega)$ into $Z$ given by
\[(4.1) \quad P_Z(f) = \arg \min_{\hat{f} \in L^2(\Omega)} \Delta_2(f, \hat{f}),\]
is non-expansive.
(d) Assume that $\{D_j\}_{j=1}^N$ is a domain partitioning of $\Omega$ as in Section 3. Let $Z$ be defined by $Z = \text{span}\{\chi_{D_1}, \ldots, \chi_{D_N}\}$. We have that
\[P_Z(f) = \sum_{j=1}^N g_j \chi_{D_j} \quad \text{with} \quad g_j = \frac{1}{|D_j|} \int_{D_j} f(x) \, dx.\]

The assumptions concerning the Fréchet derivative, $DF$, of the direct operator, $F : X \to Y$, associated with the projected steepest descent iteration can be summarized as \[12\]
(a) (uniformly bounded)
\[\|DF\| \leq \hat{L}\]
(b) (Lipschitz continuous)
\[\|DF(c_1^{-2}) - DF(c_2^{-2})\| \leq \mathcal{L}\|c_1^{-2} - c_2^{-2}\|, \quad \forall c_1^{-2}, c_2^{-2} \in X\]
(c) (conditionally stable)
\[\|c_1^{-2} - c_2^{-2}\| \leq \mathcal{C}\|F(c_1^{-2}) - F(c_2^{-2})\|, \quad \forall c_1^{-2}, c_2^{-2} \in Z \subset X.\]

In the previous sections, we examined these conditions and estimated the behaviors of the constants with respect to the frequency, $\omega$, and the number, $N$, of subdomains in the domain partitioning. With Lemma 2.7, Lemma 2.8 and Proposition 3.5, we have
\[\hat{L} = \hat{L}_0 \omega^2, \quad \mathcal{L} = L_0 \omega^4, \quad \mathcal{C} = \omega^{-2} e^{K(1+\omega^2 \eta_3)(|\Omega|/r_0^2)^\Delta}\]
while assuming that $\omega^2$ satisfies the conditions in Proposition 2.4. We elucidated in Remark 3.6 that $|\Omega|/r_0^2$ is effectively equal to $N$ in the case of a uniform domain partitioning.

We now introduce the framework of discrete approximations of the unique solution. Let $c_\dagger$ denote the unique solution of the inverse boundary value problem and $y_\omega$ the data, that is, $y_\omega = F_\omega(c_\dagger^{-2})$.  

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**Definition 4.2.** For a given domain partitioning $\mathcal{D}_N$, the approximation error, $\eta_{\omega, \mathcal{D}_N}$, is given by

$$
\eta_{\omega, \mathcal{D}_N} : L(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)) \to [0, \infty)
$$

We consider the error of the discrete approximation and assume that it is bounded by some positive function $\varphi$ which is monotonically decreasing in $N$:

$$
(4.3) \quad \text{dist}(c_{-2}^0, W_N) \leq \varphi(N).
$$

Lemma 2.7 implies that

$$
(4.4) \quad \eta_{\omega, \mathcal{D}_N}(y) \leq \hat{L}_{0,\omega}^2 \varphi(N).
$$

Indeed the error decreases as the frequency decreases.

The following projected steepest descent iteration is taken from [12]. We let $Z = W_N$.

**Algorithm 4.3.** For $c_{-2}^k$, $k = 0, 1, 2, \ldots$, we let

$$
(4.5) \quad R_k = F_{\omega, \mathcal{D}_N}(c_{-2}^k) - y_\omega, \quad T_k = DF_{\omega, \mathcal{D}_N}(c_{-2}^k)^* j_2(R_k), \quad r_k = \|R_k\|, \quad t_k = \|T_k\|,
$$

where $j_2$ stands for a single-valued selection of the normalized duality mapping from the data space $Y$ to its dual. Moreover, we define

$$
(4.6) \quad \tilde{\mathcal{C}} := 2\mathcal{C}^2,
$$

$$
(4.7) \quad \rho := \frac{1}{2}(2\tilde{\mathcal{C}}^{-2})^{-1} \left(1 + \sqrt{1 - 8\tilde{\mathcal{C}}^{-2}4\mathcal{C}^{-2}}\right)^2.
$$

For $k = 0, 1, \ldots$ we set

$$
(4.8) \quad \mu_k := t_k^{-2}u_k r_k.
$$

The iteration is given by

**S0** Choose a starting point $c_{-2}^0 \in Z$ such that

$$
(4.9) \quad \Delta_2(c_{-2}^0, z^\dagger) < \rho,
$$

where $z^\dagger \in Z$ satisfies

$$
\text{dist}(y_\omega, F_{\omega, \mathcal{D}_N}(z^\dagger)) = \eta_{\omega, \mathcal{D}_N}(y_\omega).
$$

**S1** Compute the new iterate via

$$
(4.10) \quad c_{-2}^{k+1} = c_{-2}^k - \mu_k T_k,
$$

$$
(4.11) \quad \tilde{c}_{-2}^{k+1} = \mathcal{P}_Z(c_{-2}^{k+1}).
$$

Set $k \leftarrow k + 1$ and repeat this step.
Due to the projection $P_z$ applied, all iterates belong to the ‘stable subset’ $Z$, which in general can only offer an approximation to the unique solution. The dimension, $N$, of $Z$ should be low to ensure a large radius of convergence, $\rho$ (cf. (1.0)). In [12] we introduced a multi-level approach to enable a gradual refinement of the domain partitioning. Let $n$ denote the level index. As $n$ increases, the number of subdomains, $N_n$, grows while the approximation error decreases. We introduce a hierarchy of domain partitionings, $D_{N_n}$, and corresponding

$$Z_n = W_{N_n}. \tag{4.11}$$

Given a frequency $\omega$, to each level, we assign a domain partitioning. We identify $D_{N_n}$ with $D_n$, $F_{\omega,D_{N_n}}$ with $F_n$ and $\eta_{\omega,D_{N_n}}$ with $\eta_n$. For the given frequency $\omega$, the (attainable) data $y_{\omega}$ is denoted by $y$. We identify $\hat{\mathcal{L}}_n$, $\hat{\mathcal{L}}_n$ and $\mathcal{C}_n$ with the expressions in (4.2) replacing $N$ by $N_n$. Similarly, we identify the radius $\rho_n$ with the expression for $\rho$ in (1.0) replacing $\hat{\mathcal{L}}$, $\mathcal{L}$ and $\mathcal{C}_n$ and $\eta_n$, respectively. For simplicity of notation, we omit the subscript in the operator norm. In the following algorithm, $c_{n,k}^{-2}$ denotes the $k$th iterate at level $n$.

**Algorithm 4.4.**
1. **(S0)** Set $n = 0$. Use $c_{0,0}^{-2}$ as the starting point, with $\Delta_2(c_{0,0}^{-2},c_{1,0}^{-2}) < \rho_0$ where $c_{1,0}^{-2}$ is the best $D_0$ approximation to $c_{1}^{-2}$.

2. **(S1) Iteration.** Use $F_n$ and $Z_n$ as the modelling operator and the convex subset to run Algorithm 4.3 with the discrepancy criterion given by

$$K_n = \min\{k \in \mathbb{N} \mid \|F_n(c_{n,k}^{-2}) - y_n\| \leq (3 + \varepsilon)\eta_n\}, \tag{4.12}$$

where $\varepsilon > 0$ is a given small tolerance constant. That is, this algorithm stops at $k = K_n$.

3. **(S2)** Set $c_{n+1,0} = c_{n,K_n}^{-2}$, refine the domain partitioning to $D_{n+1}$ such that the corresponding constants $\hat{\mathcal{L}}_{n+1}$, $\hat{\mathcal{L}}_{n+1}$, $\mathcal{C}_{n+1}$ and the approximation error $\eta_{n+1}$ satisfy the inequalities

$$\left\{ \begin{array}{l}
8\hat{\mathcal{C}}_{n+1,\eta_{n+1}} < 1, \\
(3 + \varepsilon)\eta_n + \eta_{n+1} \leq 2^{-5/2} (\hat{\mathcal{L}}_{n+1}\mathcal{C}_{n+1}\hat{\mathcal{C}}_{n+1})^{-1}
\end{array} \right. \tag{4.13}$$

Set $n = n + 1$ and go to step (S1).

**Remark 4.5.** Because

$$1 - 8\hat{\mathcal{C}}_{n+1,\eta_{n+1}} > 0$$

implies that

$$1 + \sqrt{1 - 8\hat{\mathcal{C}}_{n+1,\eta_{n+1}} - 4\hat{\mathcal{C}}_{n+1,\eta_{n+1}}} > 1/2,$$

the inequalities in (4.13) imply the original multi-level criterion (13)

$$3 + \varepsilon)\eta_n < 2^{-1/2} (\hat{\mathcal{L}}_{n+1}\mathcal{C}_{n+1})^{-1} \left( \frac{1 + \sqrt{1 - 8\hat{\mathcal{C}}_{n+1,\eta_{n+1}}}}{2\mathcal{C}_{n+1}} - 2\eta_{n+1} \right) - \eta_{n+1}. \tag{4.14}$$

This algorithm is adapted to the inverse boundary value problem for the Helmholtz equation as follows. The multi-level conditions in (4.13) are replaced by

$$\varphi(N_{n+1}) - 8^{-1}\omega^{-2}(\hat{\mathcal{L}}_0\hat{\mathcal{C}}_0)^{-1}e^{-2K(1+\omega^2B_2)N_{n+1}} < 0, \tag{4.15}$$
and

\[(3 + \varepsilon) \varphi(N_n) + \varphi(N_{n+1}) - 2^{-5/2} \omega^{-2}(\tilde{\Delta}_{0}^{2} \Omega_{0})^{-1} e^{-3K(1+\omega^{2}B_2)N_{n+1}^{4}} \leq 0.\]

We now show that (4.15)-(4.16) are the proper multi-level conditions. These provide an upper bound for the increment in the number of subdomains from level to level. One can no longer increase \(N\) when \(N_{\max}\) is reached, which follows from setting \(N_{n+1} = N_n = N_{\max}\) in (4.16):

\[(4 + \varepsilon) \varphi(N_{\max}) - 2^{-5/2} \omega^{-2}(\tilde{\Delta}_{0}^{2} \Omega_{0})^{-1} e^{-3K(1+\omega^{2}B_2)N_{\max}^{4}} = 0.\]

We note that \(\varphi\) signifies a compression rate while approximating the unique solution with piecewise constant functions corresponding with the given hierarchy of domain partitionings. If \(\varphi\) decays fast enough, such a (finite) \(N_{\max}\) does not exist.

Indeed, let \(c_{\dagger,n}\) be the \(D_n\)-best approximation to \(c_{\dagger}\) which gives the approximation error \(\eta_n = \eta_{\omega,D_n}(y)\) in Definition 3.6. We have

\[\|c_{n,K_n} - c_{\dagger,n}\| \leq \|c_{n,K_n} - c_{\dagger,n}\| + \|c_{\dagger,n} - c_{\dagger}\|.
\]

For the first term, noting that both \(c_{n,K_n}\) and \(c_{\dagger,n}\) belong to the ‘stable subset’, we can apply the stability estimate, Proposition 3.5, to arrive at

\[\|c_{n,K_n} - c_{\dagger,n}\| \leq \omega^{-2} e^{-2K(1+\omega^{2}B_2)N_n^{4}} \|F_n(c_{n,K_n}) - F_n(c_{\dagger,n})\|.
\]

Here, we have replaced \(|\Omega|/r_0^3\) by the number, \(N\), of subdomains in the partitioning following Remark 3.6.

By the discrepancy criterion (4.12), we know that

\[\|F_n(c_{n,K_n}) - y_n\| \leq (3 + \varepsilon) \eta_n.
\]

Hence,

\[\|F_n(c_{n,K_n}) - F_n(c_{\dagger,n})\| \leq \|F_n(c_{n,K_n}) - y_n\| + \|F_n(c_{\dagger,n}) - y\| \leq (4 + \varepsilon) \eta_n.
\]

Substituting the above inequality into (4.18), we obtain

\[\|c_{n,K_n} - c_{\dagger,n}\| \leq (4 + \varepsilon) \omega^{-2} e^{-2K(1+\omega^{2}B_2)N_n^{4}} \eta_n
\]

so that

\[(4.19) \|c_{n,K_n} - c_{\dagger,n}\| \leq (4 + \varepsilon) \omega^{-2} e^{-2K(1+\omega^{2}B_2)N_n^{4}} \eta_n + \|c_{\dagger,n} - c_{\dagger}\|.
\]

We use the estimates of \(\eta_{n+1}\) in (4.13), and of \(\mathcal{L}_{n+1}\) in Lemma 2.8 to obtain,

\[\mathbf{\check{\epsilon}}_{n+1} \eta_{n+1} \leq \mathcal{L}_0 \mathbf{\check{\epsilon}}_0 \mathbf{\check{c}}_{n+1} \omega^6 \varphi(N_{n+1}).
\]

Substituting the estimate of the stability constant in Proposition 3.5 into the above inequality, we get

\[(4.20) \mathbf{\check{\epsilon}}_{n+1} \eta_{n+1} \leq \mathcal{L}_0 \mathbf{\check{\epsilon}}_0 \omega^2 e^{2K(1+\omega^{2}B_2)N_{n+1}^{4}} \varphi(N_{n+1}).
\]

We also find that

\[(4.21) \hat{\mathbf{\check{\epsilon}}}_{n+1} \mathbf{\check{c}}_{n+1} ((3 + \varepsilon) \eta_n + \eta_{n+1}) \leq \hat{\mathbf{\check{\epsilon}}}_0 \mathbf{\check{\epsilon}}_0 \omega^2 e^{3K(1+\omega^{2}B_2)N_{n+1}^{4}} [(3 + \varepsilon) \varphi(N_n) + \varphi(N_{n+1})].
\]
Using (4.20), we conclude that (4.15) implies that the first inequality in (4.13) is satisfied; using (4.21), we find that (4.16) implies that the second inequality in (4.13) is satisfied.

In our inverse problem, we fix the frequency $\omega$ and specify the level by the domain partitioning. Our multi-level scheme starts at a coarse domain partitioning. We let $D_{N_0}$ be an initial domain partitioning of $\Omega$ with the number of subdomains being equal to $N_0$. This initial domain partitioning needs to allow a large-scale approximation of the unique solution in $\Omega$. Algorithm 4.4 is designed such that the starting point $c_{-2}^{n,K_n}$ at level $n+1$, which equals $c_{-2}^{n,K_n}$, is within the $(n+1)$-level radius of convergence, $\rho_{n+1}$ [12]. Therefore, the iterations can continue until the accuracy limitation is reached. In the above, (4.13) yields a sufficient multi-level condition balancing the competition between the approximation errors and the convergence radii of neighboring levels [12].

**Lowering the frequency.** In the next theorem, we prove that by letting the frequency go to zero, the overall convergence radius tends to infinity. We note that the statement in the theorem does not give a lower bound of the frequency that is uniform for all unique solutions and starting points. The frequency needs to satisfy the conditions in Propositions 2.4 or 2.5 in the absence of prior knowledge of $\Sigma_{e^{-2}}$.

**Theorem 4.6.** Let $D_{N_0}$ be an arbitrary initial domain partitioning and $y$ be the given data. Then, for any positive number $M_0$, there exists a frequency $\omega_0$ low enough such that the convergence radius $\rho \geq M_0$.

**Proof.** The proof follows by examining the behavior of the convergence radius with respect to the frequency. All the constant are associated with the initial level. For simplicity of notation, we omit the subscripts. We note that $\rho$ in (4.6) can be written as

$$2\rho^{1/2} = \frac{1 + \sqrt{1 - 8\tilde{\mathcal{C}}\eta - 4\tilde{\mathcal{C}}\eta}}{2\mathcal{L}e^2\hat{\mathcal{L}}}.$$  \hfill (4.22)

The constant $\tilde{\mathcal{C}}$ does not depend on frequency. First, we provide a uniform lower bound for the numerator

$$1 + \sqrt{1 - 8\tilde{\mathcal{C}}\eta - 4\tilde{\mathcal{C}}\eta}.$$  

From (4.2) and (4.4), we conclude that

$$\tilde{\mathcal{C}}\eta = \mathcal{L}e^2\eta \leq \hat{\mathcal{L}}_0 e^{2K(1+\omega^2B_2)(|\Omega|/r_0^3)^{4} \hat{\mathcal{L}}_0^2 \omega^2 \varphi(N)}.$$  

We choose $\omega$ sufficiently small such that $8\tilde{\mathcal{C}}\eta < 1$. Then,

$$1 + \sqrt{1 - 8\tilde{\mathcal{C}}\eta - 4\tilde{\mathcal{C}}\eta} = \frac{1}{2}(\sqrt{1 - 8\tilde{\mathcal{C}}\eta + 1})^2 \geq \frac{1}{2}.$$  

Furthermore, the denominator, $2\mathcal{L}e^2\hat{\mathcal{L}}$, in (4.22) satisfies the estimate,

$$2\mathcal{L}e^2\hat{\mathcal{L}} \leq \hat{\mathcal{L}}_0 e^{2K(1+\omega^2B_2)(|\Omega|/r_0^3)^{4} \omega^2},$$  

following from (4.2). Hence, it tends to zero as the frequency goes to zero. This completes the proof. \[\square\]

**5. Discussion.** As an example of forming a multi-scale hierarchy, $W_{N_n}$, we mention the use of Haar wavelets and ‘local’ scale refinement. We refer to convergence of our scheme from the following point of view. Within one level, $n$, the iterates converge in the sense that the distance between the iterate and the ball centered at the $Z_n$-best approximation with a certain tolerance radius tends to zero. This tolerance radius is determined by the approximation error. We use a ball, instead of a
point, to describe the convergence. This is because if the iterates get sufficiently close to the $Z_n$-best approximation, the descent direction will mainly drive the iterate out of the ‘stable set’ while, still, the projection will pull it back.

In the multi-level scheme, convergence means that the result $c_{n,K_n}$ converges to the true solution, $c^1$, as the approximation error $\eta_n$ tends to 0. Unless $\varphi$ decays sufficiently fast, such a convergence is not guaranteed. We mention that the high-frequency regime yields different estimates\footnote{Recently, increasing stability-type estimates for the inverse boundary value problem for the Helmholtz equation with increasing frequency have been obtained \cite{18,14}; however, the stability constants in these estimates do not enable the application of our scheme.} and we expect that the multi-level condition will change accordingly.

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