A Study of Ultraviolet Renormalon Ambiguities
in the Determination of $\alpha_s$ from $\tau$ Decay

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Abstract

The divergent large-order behaviour of the perturbative series relevant for the determination of $\alpha_s$ from $\tau$ decay is controlled by the leading ultraviolet (UV) renormalon. Even in the absence of the first infrared (IR) renormalon, an ambiguity of order $\Lambda^2/m_\tau^2$ is introduced. We make a quantitative study of the practical implications of this ambiguity. We discuss the magnitude of UV renormalon corrections obtained in the large-$N_f$ limit, which, although unrealistic, is nevertheless interesting to some extent. We then study a number of improved approximants for the perturbative series, based on a change of variable in the Borel representation, such as to displace the leading UV renormalon singularity at a larger distance from the origin than the first IR renormalon. The spread of the resulting values of $\alpha_s(m_\tau^2)$ obtained by different approximants, at different renormalization scales, is exhibited as a measure of the underlying ambiguities. Finally, on the basis of mathematical models, we discuss the prospects of an actual improvement, given the signs and magnitudes of the computed coefficients, the size of $\alpha_s(m_\tau^2)$ and what is known of the asymptotic properties of the series. Our conclusion is that a realistic estimate of the theoretical error cannot go below $\delta\alpha_s(m_\tau^2) \sim \pm 0.060$, or $\delta\alpha_s(m_Z^2) \sim \pm 0.006$. 
1. Introduction

The possibility of measuring $\alpha_s$ from $\tau$-decay has been extensively studied in a series of interesting papers, in particular by Braaten, Narison and Pich [1-4]. The relevant quantity is $R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons})}{\Gamma(\tau \to \nu_\tau + l\nu)}$, with $l = e, \mu$. At present [5] the ALEPH collaboration finds

$$ (R_\tau)_{\text{exp}} = 3.645 \pm 0.024. \quad (1.1) $$

Based on this result, it is argued, if the formalism of QCD sum rules is assumed, according to SVZ [6], that

$$ \alpha_s(m^2_\tau) = 0.355 \pm 0.021 \quad (1.2) $$

and finally

$$ \alpha_s(m^2_Z) = 0.121 \pm 0.0016(\text{exp}) \pm 0.0018(\text{th}) = 0.121 \pm 0.0024. \quad (1.3) $$

Given that $m_\tau$ is so small, this determination of $\alpha_s(m^2_Z)$ appears [7-10] a bit too precise!

In defence of this method [11] one can certainly point out that $R_\tau$ has several combined advantages. Dropping some inessential complications, $R_\tau$ is an integral in $s$ of a spectral function $R(s)$ which is the analogue of $R_{e^+e^-}(s)$ but for the case of charged weak currents. Thus, first, $R_\tau$ is even more inclusive than $R_{e^+e^-}(s)$ and one expects that the asymptotic regime is more precocious for more inclusive quantities. Second, one can use analyticity in order to transform the relevant integral into an integral over the circle $|s| = m^2_\tau$ [4]. This not only gives some confidence that the appropriate scale of energy for the evaluation of $R_\tau$ is of order $m_\tau$, but also shows that the integration over the low-energy domain helps very much in smearing out the complicated behaviour in the resonance region. Also important is the presence of a phase-space factor that kills the sensitivity of the spectral function near Re $s = m^2_\tau$, where there is a gap of validity of the asymptotic approximations due to the vicinity of the cut singularities and also to the nearby charm threshold. On the circle $|s| = m^2_\tau$, asymptotic formulae should be approximately valid for the correlator. The perturbative component of $R(s)$ is known up to terms of order $\alpha_s(m^2_Z)^3$ [12]. One can hope to get some control of the non-perturbative corrections by using the operator
product expansion and some estimate (either experimental or by some model) of the dominant condensates, in the spirit of the QCD sum rules [3].

This series of virtues of $R_\tau$ is indeed real but would not be sufficient in itself to justify the precision on $\alpha_s(m^2_\tau)$ which is claimed. The real point is that no corrections of order $1/m^2_\tau$ are assumed to exist. The fact that there is no operator with the corresponding dimension in the short distance expansion is not sufficient, because there could be non-leading corrections in the coefficient function of the leading operator. We think it is a fair statement that there is no theorem that guarantees the absence of $\Lambda^2/m^2_\tau$ terms in $R_\tau$ in the massless limit; no theorem that proves that terms of order $\Lambda^2/m^2_\tau$ cannot arise from the mechanism that generates confinement. But even if in principle the above theorem would exist, still, in practice, there would be ambiguities on the leading-term perturbative expansion of order $\Lambda^2/m^2_\tau$ from the ultraviolet renormalon sequence associated to the divergence of the perturbative series for the spectral function [13-21]. The present note is mainly devoted to a quantitative discussion of the impact of UV renormalon ambiguities on the determination of $\alpha_s(m^2_\tau)$. On the basis of the accumulated knowledge on renormalon behaviour, we address the question of what is the theoretical error on $\alpha_s(m^2_\tau)$ and examine possible ways to decrease it. We discuss the magnitude of UV renormalon corrections obtained from explicit calculations [15-21], which although based on unrealistic simplified schemes, are nevertheless interesting to some extent. We then study a number of improved approximants for the perturbative series, based on a change of variable in the Borel representation [4,17], such as to displace the leading UV renormalon singularity at a larger distance from the origin than the first infrared (IR) renormalon. The spread of the resulting values of $\alpha_s(m^2_\tau)$ obtained by different approximants, at different renormalisation scales is exhibited as a measure of the underlying ambiguities. Finally, on the basis of mathematical models, we discuss the prospects of an actual improvement, given the signs and the magnitudes of the computed coefficients, the size of $\alpha_s(m^2_\tau)$ and what is known of the asymptotic properties of the series. Our conclusion is that a realistic estimate of the theoretical error cannot go below $\delta\alpha_s(m^2_\tau) \sim \pm 0.060$, or $\delta\alpha_s(m^2_Z) \sim \pm 0.006$.

The organisation of this article is as follows. In sect.2 we summarise the basic formulae and discuss different procedures to do the integration over the circle that differ by resumming or not an infinite series of “large $\pi^2$ terms”. We discuss the
relative merits of the various procedures and their scale dependence. In sect. 3 we introduce the problems related to the divergence of the perturbative series, we review the Borel transform method and the renormalon singularities. In sect. 4 we derive some useful formulae obtained in the Borel space after integration on the circle. In sect. 5 we consider the explicit form for the leading UV renormalon singularity derived in perturbation theory in the large $N_f$ limit, $N_f$ being the number of flavours. This limit is not meant to be realistic, but, for orientation, we evaluate the quantitative impact that such an UV renormalon would have on the determination of $\alpha_s(m^2_\tau)$. We find that this effect is rather small. In sects. 6, 7, which contain the main original results of this work, we introduce and study a number of improved approximants that could in principle suppress the ambiguity from the leading UV renormalon. We study the combined effects of different, a priori equivalent, procedures, different accelerators of convergence and different choices of the renormalisation scale. We also study in a simple mathematical model under which conditions for the known coefficients of the series the accelerator method leads to a better approximation of the true result. Finally, in sect. 8 we present our conclusion.

2. Basic Formulae and Truncation Ambiguities

The quantity of interest is the integral over the hadronic squared mass $s$ in $\tau$ decay of a function $R(s)$ analogous to $R_{e^+e^-}(s)$, weighted by a phase-space factor. In the limit of massless $u, d, s$ quarks we have [1-3]:

$$R_\tau = \int_0^{m^2_\tau} \frac{ds}{s^2} \left[ \left(1 - \frac{s}{m^2_\tau}\right)^2 \left(1 + \frac{2s}{m^2_\tau}\right) R(s). \right]$$

(2.1)

$R(s)$ is proportional to the imaginary part of a current-current correlator:

$$R(s) = \frac{N}{\pi} \text{Im} \Pi(s) = \frac{N}{2\pi i} \left[ \Pi(s + ie) - \Pi(s - ie) \right].$$

(2.2)

The normalization factor $N$ is defined in such a way that, in zeroth order in perturbation theory, $R(s) = 3$. In turn, the correlator $\Pi(s)$ is related to the Adler function $D_\tau(s)$, defined in such a way as to remove a constant:

$$D_\tau(s) = -s \frac{d}{ds} N \Pi(s).$$

(2.3)
By first integrating by parts and then using the Cauchy theorem one obtains for \( R_\tau \) the result
\[
R_\tau = \frac{1}{2\pi i} \oint_{|s|=m_\tau^2} ds \left( 1 - \frac{s}{m_\tau^2} \right)^3 \left( 1 + \frac{s}{m_\tau^2} \right) D_\tau(s).
\] (2.4)

The Adler function \( D_\tau \) has a perturbative expansion of the form:
\[
D_\tau(s) = D_\tau^0 \sum_{n=0}^\infty D_n a(-s)^n \simeq D_\tau^0 \left[ 1 + D_1 a(-s) + D_2 a^2(-s) + D_3 a^3(-s) + \ldots \right],
\] (2.5)

where \( a = \alpha s / \pi \), \( D_\tau^0 = 3(1+\delta) \) where \( \delta \) is a known small electroweak correction, and, for \( N_f = 3 \) in the \( \overline{\text{MS}} \) scheme,
\[
D_1 = 1
\]
\[
D_2 = \left[ \frac{11}{2} - 4\zeta(3) \right] \beta + \frac{C_A}{12} - \frac{C_F}{8} = 1.640
\]
\[
D_3 = \left[ \frac{151}{18} - \frac{19}{3} \zeta(3) \right] 4\beta^2 + 2C_A \left[ \frac{31}{6} - \frac{5}{3} (\zeta(3) + \zeta(5)) \right] \beta
\]
\[
+ 2C_F \left[ \frac{29}{32} - \frac{19}{2} \zeta(3) + 10\zeta(5) \right] \beta + C_A^2 \left[ -\frac{799}{288} - \zeta(3) \right]
\]
\[
+ C_A C_F \left[ -\frac{827}{192} + \frac{11}{2} \zeta(3) \right] + C_F^2 \left( -\frac{23}{32} \right) = 6.371,
\] (2.6)

where \( \zeta(3) = 1.20206 \) and \( \zeta(5) = 1.03693 \), \( C_A = N_C = 3 \) and \( C_F = (N_C^2 - 1)/(2N_C) = 4/3 \). The quantity \( \beta = (11C_A - 2N_f)/12 \) is the first beta function coefficient \( [22] \):
\[
\mu^2 \frac{da(a)}{d\mu^2} = \beta \left(a(a)\right); \quad \beta(a) = -\beta a^2(1 + \beta'a + \ldots)
\] (2.7)

\( (\beta = 9/4, \beta' = 16/9 \) for \( N_f = 3 \)).

The expansion in eq. (2.5) defines the Adler function at all complex \( s \) with a cut for \( s > 0 \). In the spacelike region, where \( s < 0 \), \( a(-s) \) is real and given asymptotically by \( (\mu^2 > 0) \):
\[
\frac{1}{a(-s)} = \frac{1}{a(a^2)} + \beta \log \frac{-s}{\mu^2} = \beta \log \frac{-s}{\Lambda^2}.
\] (2.8)

If we want \( a(-s) \) at some complex value of the argument, e.g. \( s = -|s| \exp(i\theta) \), we can use the formula
\[
a(-s) = a(|s|) \frac{1}{1 + \beta a(|s|) i\theta}
\] (2.9)
where the angle $\theta$ is $-\pi$ on the upper tip of the cut for $s$ real and positive, $+\pi$ on the lower tip and zero on the negative real axis. The more accurate two-loop expression is given by

$$a(-s) = \frac{a(|s|)}{1 + \beta a(|s|)i\theta + \beta' a(|s|) \log(1 + \beta a(|s|)i\theta)}. \quad (2.10)$$

The expansion for $R(s)$ (for $s$ real and positive) can be obtained from that of $D_\tau$ by the relation

$$R(s) = \frac{1}{2\pi i} \oint_{|s'|=s} ds' D_\tau(s'). \quad (2.11)$$

Performing the integration by using the expansion for $D_\tau$ in eq. (2.5) and the expression in eq. (2.10) for a complex argument, one obtains

$$R(s) = D^0_\tau \left[ 1 + F_1 a(s) + F_2 a^2(s) + F_3 a^3(s) + \ldots \right], \quad (2.12)$$

and

$$F_1 = D_1; \quad F_2 = D_2; \quad F_3 = D_3 - \frac{\beta^2 \pi^2}{3}. \quad (2.13)$$

The origin of the $\beta^2 \pi^2/3$ term is easily understood. By using the one-loop expansion for $a(s)$, eq. (2.9), one gets

$$R(s) = D^0_\tau \left[ 1 + \frac{1}{2\pi i} \log \frac{1 + i\pi \beta a(s)}{1 - i\pi \beta a(s)} + \ldots \right] = D^0_\tau \left[ 1 + a(s) - \frac{\beta^2 \pi^2 a^3(s)}{3} + \ldots \right] \quad (2.14)$$

We have the following observations on this result. First, for $N_f = 3$, the coefficients $F_{1,2,3}$ in the expansion of $R(s)$ for $\tau$ decay coincide with those of $R_{e^+e^-}$ because the potentially different terms proportional to $(\sum Q_i)^2$, with $Q_i$ being the quark charges, vanish in this case. Second, we observe that eq. (2.13) is obtained by a truncation of higher-order terms in the quantity $\beta^2 a^2 \pi^2 \approx 0.7$ with $a = a(m_\tau^2)$. Note that a similar problem of truncation arises when the integration over the circle in eq. (2.4) is performed. In the early treatments of this problem (e.g. in ref. [4]) the expression of $a(-s)$, which appears on the circle, is taken from eq. (2.10) and expanded in a consistently to the order $a^3$. With this procedure, one obtains

$$R^{(BNP)}_\tau = D^0_\tau \left[ 1 + H_1 a(m_\tau^2) + H_2 a^2(m_\tau^2) + H_3 a^3(m_\tau^2) + \ldots \right], \quad (2.15)$$

with

$$H_1 = 1; \quad H_2 = 5.2023; \quad H_3 = 26.3666. \quad (2.16)$$
We will refer to this result as BNP formula (for the authors of ref. [2]). More recently in ref. [4] it was advocated that a better procedure for performing the integration on the circle is to keep the full three-loop expression for \( a(-|s|e^{i\theta}) \), according to the formula

\[
R^{(LP)}_\tau = \frac{D^0_\tau}{2\pi i} \oint_{|z|=1} \frac{dz}{z} (1-z)^3 (1+z) \left[ 1 + D_1 a(-zm^2_\tau) + D_2 a^2(-zm^2_\tau) + D_3 a^3(-zm^2_\tau) + \ldots \right].
\]

(2.17)

It is this procedure which is currently adopted (LeDiberder-Pich, or LP method). At fixed experimental value of \( R_\tau \), the two procedures lead to values of \( \alpha_s(m^2_\tau) \) that differ by terms of order \( \delta\alpha_s(m^2_\tau) \sim (\beta^2 a^2 \pi^2 a^2) \sim 0.01 \) or \( (\beta^2 a^2 \pi^2 a^2) \sim 0.05 \). These “large \( \pi^2 \) terms” always arise when one goes from the spacelike to the timelike region (e.g. similar terms arise [22] when one relates Drell-Yan processes to electroproduction or fragmentation functions to structure functions). There have been many discussions in the past on the opportunity of resumming these terms [23]. If there was a good argument to consider the expansion for \( D_\tau(s) \) in some respect superior to that for \( R(s) \) it could be worthwhile at low energies to keep the expression in eq. (2.14) in its resummed form rather than to expand in \( \beta^2 a^2 \pi^2 \). A glance at eqs. (2.6) shows that, for \( n \leq 3 \), there are no explicit “large \( \pi^2 \) terms” in the coefficients \( D_n \) of the expansion for \( D_\tau \). Similarly when the integration on the circle is performed with the complete formula for \( a(-|s|e^{i\theta}) \) all terms are kept up to order \( a^3(\beta^2 a^2 \pi^2)^n \), i.e. up to order \( a^3 \) one expands in \( a \) but keeps \( \beta^2 a^2 \pi^2 \) unexpanded. Sure enough this sequel of terms exists in reality, so why not take them into account? However, the counter-argument is that there are in perturbation theory terms involving \( \pi^2 \) that arise from origins other than the spacelike-timelike connection and, in any case, there are many terms of the same general magnitude (for example, the term proportional to \( \beta^2 \) in \( D_3 \), eq. (2.6)), so that the advantage of keeping this particular class of terms is likely to be completely illusory.

One can further consider the scale dependence of the different procedures. One can write

\[
R^{(BNP)}_\tau = D^0_\tau \left( 1 + \tilde{H}_1 a(\mu^2) + \tilde{H}_2 a^2(\mu^2) + \tilde{H}_3 a^3(\mu^2) + \ldots \right),
\]

(2.18)

where

\[
\tilde{H}_1(\mu^2) = H_1
\]
$$\tilde{H}_2(\mu^2) = H_2 - H_1 \beta \log \frac{m_\tau^2}{\mu^2}$$

$$\tilde{H}_3(\mu^2) = H_3 - 2H_2 \beta \log \frac{m_\tau^2}{\mu^2} + H_1 \left[ \beta^2 \log^2 \frac{m_\tau^2}{\mu^2} - \beta \beta' \log \frac{m_\tau^2}{\mu^2} \right]. \quad (2.19)$$

Analogously, we can study the scale dependence in the LP method. In this case we have

$$R_{\tau}^{(LP)} = \frac{D_0^{\tau}}{2\pi i} \oint_{|z|=1} \frac{dz}{z} (1-z)^3(1+z) \left[ 1 + \tilde{D}_1 a(-z\mu^2) + \tilde{D}_2 a^2(-z\mu^2) + \tilde{D}_3 a^3(-z\mu^2) + \ldots \right], \quad (2.20)$$

where the $\tilde{D}_n$ coefficients are related to the $D_n$ as the $\tilde{H}_n$ to the $H_n$ in eqs. (2.19).

The results of the LP and BNP methods are shown in fig. 1, where, assuming a measured value of 3.6 for $R_{\tau}$, we show the corresponding determination of $\alpha_s(m_\tau^2)$ as a function of the renormalization scale $\mu$. For comparison, we also show (dashed curve) the determination obtained with a simplified LP method, in which we use the one-loop expression eq. (2.9) for $a$. For $\mu > 1$ GeV there is less $\mu$ dependence if the resummed expressions are used. This stability is often taken as a possible indication that resumming is better.

In conclusion, we agree that the resummed formulae provide a less ambiguous result with respect to a change of scale than the unresummed expression. However, it is true that a priori it is not possible to guarantee that a more accurate result is obtained in one way or the other. As a consequence the spread shown in fig. 1 for different choices of $\mu$ and of procedure is to be taken as a real ambiguity. In particular the large discrepancy at $\mu \sim 1$ GeV is a genuine signal of trouble, especially in view of the fact that several proposed scale-fixing procedures lead to small values of $\mu$, e.g. minimal sensitivity, BLM scheme etc. [24]. A small value of $\mu$ is also suggested by physical considerations, because the average hadronic mass is well below $m_\tau$. So, on the one hand, one cannot sensibly reject the option of small values of $\mu$. On the other hand, the corresponding value of $a(m_\tau^2)$ becomes very ambiguous at small $\mu$.

3. Renormalons and Borel Transformation

In the following discussion we first study the properties of $D_\tau(s)$ in itself, and only later we consider the integration over the circle. The problem that we now consider
Figure 1: The value of $\alpha_s(m_T^2)$ obtained from $R_T = 3.6$, as a function of the renormalization scale $\mu$. 

- Solid: LP method (exact)
- Dashed: LP method (approximate)
- Dotted: BNP method
is the well known fact that the series for $D_\tau$ is divergent. Indeed one can identify sequences of diagrams, depicted in fig. 2, called “renormalon” terms \[13\-21\], that provide the leading behaviour at large $n$ for the $n$-th coefficient of the expansion for $D_\tau$. The renormalon contribution is of the form:

$$D_n \sim C_k n! n^{\gamma_k} \left( \frac{\beta}{\bar{k}} \right)^n [1 + O(1/n)] \quad (n \text{ large}). \quad (3.1)$$

Note the $n!$ behaviour which implies that the series is divergent. Here, $C_k$ and $\gamma_k \geq 0$ are not known for the real theory but only, to some extent, in the large-$N_f$ expansion \[18\-21\], and the index $k$ runs over a discrete set of values:

$$k = -1, -2, -3, \ldots \quad \text{Ultraviolet (UV) Renormalons}$$

$$k = +1(?), +2, +3, \ldots \quad \text{Infrared (IR) Renormalons} \quad (3.2)$$

In the above list we omit the contribution from instantons, which appear at rather large values of $k$. They have been computed in ref. \[21\] and shown to be small. The UV or IR renormalons arise from the limits of large or small virtuality, respectively, for the exchanged gluon(s). As hinted by the question mark, the $k = +1$ IR renormalon is probably absent in perturbation theory, but the issue is not really settled \[16\-21\]. The absence of this term is necessary for the consistency of the assumption that all non-perturbative effects can be absorbed in the condensates. In the following we will assume that the $k = 1$ IR renormalon is indeed absent.
In view of the divergence of the perturbative expansion, one can possibly give a meaning to the quantity

\[ d(a) = \frac{D_2}{D_1} - 1 = D_1 a + D_2 a^2 + D_3 a^3 + \ldots \]  

(3.3)

by the Borel transform method \[27\]. One defines the perturbative expansion of the Borel transform \( B(b) \) of \( d(a) \) by removing the \( n! \) factors:

\[ B(b) = \sum_{n=0}^{\infty} D_{n+1} \frac{n^n}{n!} = D_1 + D_2 b + D_3 \frac{b^2}{2} + \ldots \]  

(3.4)

Then, formally

\[ d(a) = \int_0^{\infty} db \ e^{-b/a} B(b) \]  

(3.5)

in the sense that the expansion for \( B(b) \) reproduces the expansion for \( d(a) \) term by term. What is needed for \( d(a) \) to be well defined is that the integral converges (this cannot be true at all \( s \) \[13,17\] because of the singularities of \( d(a) \) in the \( s \) plane, but this problem can be neglected in our context) and that \( B(b) \) has no singularities in the integration range. But, as already mentioned, the large-\( n \) expansion of \( B(b) \) leads to singularities on the real axis. In fact, at large \( n \), \( B(b) \) is essentially given by a geometric series:

\[ B(b) \sim C_k \sum_n n^{\gamma_k} \left( \frac{\beta b}{k} \right)^n \sim C_k \Gamma(\gamma_k + 1) \left( 1 - \frac{\beta b}{k} \right)^{-\gamma_k - 1} + \text{less singular terms} \]  

(3.6)

so that it is singular at \( b = k/\beta \). Thus the UV renormalons correspond to singularities at \( b = -1/\beta, -2/\beta, \ldots \) and the IR renormalons at \( +2/\beta, +3/\beta, \ldots \). As a consequence, the convergence radius of the expansion for \( B(b) \) near the \( b \)-origin is determined by the UV renormalon at \( b = -1/\beta \), independent of the existence of the IR renormalon at \( b = +1/\beta \).

Thus, the perturbative expansion for \( B(b) \) can be directly used only to perform the integration up to \( b = +1/\beta \). The contribution from \( b = +1/\beta \) up to \( b = \infty \), where the expansion is not valid, could typically lead to terms of order \( \Lambda^2/s \) (or even worse). For example, if \( B(b) \) is sufficiently well behaved at \( b = +1/\beta \) and at \( b = \infty \),

\[ \Delta d(a) = \int_{1/\beta}^{\infty} db e^{-b/a} B(b) \sim aB(1/\beta) \exp(-1/\beta a) \sim aB(1/\beta) \Lambda^2/s, \]  

(3.7)
where we used $a^{-1} \simeq \beta \log(s/\Lambda^2)$ and the fact that the exponential cuts away all large-$b$ contributions so that $B(b)$ was approximated by its value near $b = +1/\beta$. From a different point of view, at large $n$, the series for $d(a)$ is dominated by the UV renormalon behaviour with $D_n \sim n^5 n! (-\beta)^n$. At fixed small $a$, the individual terms $|D_n|a^n$ first decrease with $n$, then flatten out and eventually increase because of the $n!$ factor. The best estimate of the sum is obtained by stopping at the minimum, for $n \sim n_{opt}$, given by $|D_n|a^n \sim |D_{n-1}|a^{n-1}$, or $n_{opt} \sim 1/\beta a$. From the theory of asymptotic series [27], the corresponding uncertainty $\Delta d(a)$ is of order $|D_{n_{opt}}|a^{n_{opt}}$:

$$|D_{n_{opt}}|a^{n_{opt}} \sim (1/\beta a)^\gamma (1/\beta a)! (\beta a)^{(1/\beta a)}$$

$$\sim (1/\beta a)^\gamma (1/\beta a)^{(1/\beta a)} e^{(1/\beta a)} \sqrt{2\pi/\beta a (\beta a)^{(1/\beta a)}}$$

$$\sim (1/\beta a)^\gamma e^{(1/\beta a)} \sqrt{2\pi/\beta a} \sim \Lambda^2/s \times \text{logarithms},$$  

(3.8)

(where the Stirling approximation was used: $n! \simeq n^n e^{-n} \sqrt{2\pi n}$). Thus one could improve the accuracy of the perturbative expansion by computing more subleading terms until $n \sim n_{opt}$ is reached and then add a residual term of order $\Lambda^2/s$. Note that the estimate $n_{opt} \sim 1/\beta a \sim 4$ indicates a rather small value. However this estimate is obtained from the behaviour of the leading UV renormalon series, while there is no alternation of signs and in general no evidence of renormalon behaviour in the few known terms of the series.

While an accuracy of order $\Lambda^2/s$ is what one gets in practice from the three-loop expression of $d(a)$, it is true that, in principle, if there is no IR renormalon at $b = +1/\beta$, $d(a)$ can be better defined. In fact, as the location of the leading UV renormalon at $b = -1/\beta$ is not in the integration range, there is the possibility of defining $B(b)$ by analytic continuation up to $b = +2/\beta$. If this is realised then the remaining ambiguity, of order $(\Lambda^2/s)^2$, is unavoidable because the corresponding singularity at $b = +2/\beta$ is on the real axis, so that an arbitrary procedure to go around it must be defined and the difference between two such procedures would be of that order. However, since operators of dimension 4 do exist in the operator expansion, this ambiguity can be reabsorbed in the non-perturbative condensate terms [13-15]. We also understand that the absence of the IR renormalon at $b = +1/\beta$ is necessary for the consistency of the SVZ [8] approach because operators of dimension 2 are absent in this channel. If there would be a singularity at $b = +1/\beta$ the corresponding ambiguity could not be absorbed in a condensate. There are indeed indications in
perturbation theory that the first IR renormalon does not appear \[19,21\]. However there could be non-perturbative sources of breaking of the operator expansion at non-leading level. After all no theory of confinement could be built up from perturbation theory and renormalons. But, in practice, independent of the existence of the IR renormalon at \(b = +1/\beta\), the accuracy to be expected from the first three terms in the expansion for \(d(a)\), as they have been used so far in the actual determination of \(\alpha_s\), is of order \(\Lambda^2/s\).

4. The Integration over the Circle in the Borel Transform Formalism

In this section we show that the integration over the circle in eq. (2.4) for \(R_\tau\) is particularly simple in the Borel representation. Starting from eqs. (2.9) and (3.5) we have

\[
R_\tau = \frac{1}{D_\tau^0} - 1 = \frac{1}{2\pi i} \int |s| = m_\tau^2 \frac{ds}{s} \left(1 - \frac{s}{m_\tau^2}\right)^3 \left(1 + \frac{s}{m_\tau^2}\right) \int_0^\infty db e^{-b/a(-s)} B(b). \tag{4.1}
\]

We work in the approximation where the two-loop coefficient \(\beta'\) in the beta function is neglected. Then, according to eq. (2.9), we can replace \(1/a(-s)\) by \(1/a+i\beta\theta\), where //a(|s|) = a(m_\tau^2), invert the integration order and write \(s/m_\tau^2 = -\exp(i\theta)\):

\[
r = \int_0^\infty db e^{-b/a} B(b) \frac{1}{2\pi i} \int_{-\pi}^{\pi} i\theta (1 + e^{i\theta})^3(1 - e^{i\theta})e^{-ib\beta\theta}. \tag{4.2}
\]

The integration is easily performed, with the result

\[
r = \int_0^\infty db e^{-b/a} B(b) \frac{-12 \sin(\beta b\pi)}{\beta b(\beta b - 1)(\beta b - 3)(\beta b - 4)\pi} = \int_0^\infty db e^{-b/a} B(b) F(\beta b). \tag{4.3}
\]

We see that, in first approximation, the effect of going from \(a(-s)\) to \(a(|s|)\) by integrating over the circle is to multiply the Borel transform \(B(b)\) by the factor \(F(\beta b)\). For real \(x\), the function \(F(x)\) is shown in fig. 3. It is an entire function in the whole complex plane with a good behaviour at infinity on the real axis. The factor \(F(x)\) has simple zeros at the location of all UV renormalons and also of all IR renormalons with
Figure 3: Plot of the function $F(x)$. 
the exception of those at $\beta b = 1$ (if any), 3 and 4. Since, in general, the corresponding singularities are not simple poles, they are not eliminated, but their strength is attenuated (this point will be discussed in more detail in sect. 5.). Equation (4.3) obviously coincides with the LP approach in the limit $\beta' = \beta'' = 0$. One can also repeat the procedure by expanding in $a(\mu^2)$ instead of $a(m_\tau^2)$, according to eq. (2.20). The resulting $\mu$ dependence is shown in fig. together with the analogous results for the BNP and the LP formulae.

5. Large-$N_f$ Evaluation of Renormalons and their Resummation

As well known, the typical renormalon diagrams of QED and QCD can be evaluated in the large-$N_f$ limit and their structure is simple in this limit. In the abelian case the large-$N_f$ limit corresponds to the large-$\beta$ limit, and this is believed to be true also in the non-abelian gauge theory in spite of the fact that the beta function in this case cannot be evaluated only in terms of vacuum polarisation diagrams. The sequence of dominant terms in $\beta^n$ generalises the terms in $\beta$ and $\beta^2$ that appear in $\tilde{D}_2$ and $\tilde{D}_3$ respectively. As argued in a recent paper, ref. [20], the determination of the exact behaviour of the UV renormalon series may be very different from the one indicated in the large-$\beta$ limit. While we do not know the complete form of the leading UV renormalon at $b = -1/\beta$, we can nevertheless compute, for orientation, the quantitative impact of its approximate form at large $\beta$ on the determination of $a(m_\tau^2)$. From eq. (42) of ref. [21] one obtains the large-$\beta$ expression of the contribution of the leading UV renormalon at $b = -1/\beta$ to the Borel transform $B(b)$:

$$B(b) = \frac{2}{9} e^{-5/3} \sum_n [7 + 2n] (-x)^n = \frac{2}{9} e^{-5/3} \left[ 7 - \frac{2x}{1+x} \right] \frac{1}{1+x},$$  \hspace{1cm} (5.1)$$

where $x = \beta b$ and the factor $e^{-5/3}$ transforms the result from the MOM into the $\overline{MS}$ scheme. We observe that the leading UV renormalon is a double pole. Since in the large-$\beta$ limit $\beta'$ can be neglected, the corresponding expression for $R_\tau$ in the approximation of eq. (4.3) is appropriate, and it turns the double pole into a simple pole.

We now study the numerical effect of including the whole renormalon series with
respect to a truncated result up to the order $b^2$. We first consider the impact on $d(a)$, i.e. before the integration on the circle. In order to get the correction to $d(a)$ from the higher-order terms in the UV renormalon we must subtract from $B(b)$ its expansion up to $O(b^2)$ and perform the inverse Borel transform, eq. (3.5):

$$\Delta d(a) = \frac{1}{\beta} \int_0^\infty dx \, e^{-x/\beta a} \left[ B(x) - \frac{2}{9} e^{-5/3} (7 - 9x + 11x^2) \right].$$  \hspace{1cm} (5.2)

For $\beta = 27/12 = 2.25$ and $a = 0.12$ (or $\beta a = 0.27$) one finds $\Delta d \sim -3.7 \times 10^{-3}$ which corresponds to a $\sim 2\%$ increase in the value of $\alpha_s(m_T^2)$ at fixed $d(0.12) = 0.155$ ($\delta \alpha_s(m_T^2) \sim 0.007$).

We now repeat the same exercise for the function $r$, given in eq. (4.3), obtained after integration over the circle. We compute the variation

$$\Delta r(a) = \frac{1}{\beta} \int_0^\infty dx \, e^{-x/\beta a} \left[ B(x) - \frac{2}{9} e^{-5/3} (7 - 9x + 11x^2) \right] F(x).$$  \hspace{1cm} (5.3)

Numerically we find $\Delta r(0.12) \sim -6.0 \times 10^{-3}$ which, at fixed $r(0.12) = 0.220$, again corresponds to $\delta \alpha_s(m_T^2) \sim 0.007$. Thus the extra factor $F(\beta b)$ has practically no influence on the effect on $\alpha_s(m_T^2)$ of the nearest UV singularity.

In conclusion the overall effect of the UV renormalon singularity in this model is small and not much changed by the integration over the circle.

### 6. Search for More Convergent Approximants

Assuming that indeed there is no IR renormalon at $b = +1/\beta$ one can in principle try to obtain by analytic continuation a definition of the Borel transform, valid on the positive real $b$ axis up to $b = +2/\beta$, outside the radius of convergence of its expansion. We now discuss how the analytic continuation could be implemented in practice.

Starting from eq. (3.4) we can make a change of variable $z = z(b)$ with inverse $b = b(z)$, $z(0) = 0$ and $z(\infty) = 1$ (so that the interval from 0 to $\infty$ in $b$ is mapped into the 0 to 1 range in $z$), such that the IR singularities are mapped onto the interval between $z_0 = z(2/\beta)$ and 1 and the UV singularities are pushed away at
\[ |z| \geq z_0. \text{ Changing variable one obtains:} \]

\[ d(a) = \int_0^\infty db \, e^{-b/a} B(b) \int_0^1 dz \left| \frac{db}{dz} \right| e^{-b(z)/a} B(b(z)). \] \hfill (6.1)

Using the expansion

\[ b(z) = c_1 z + c_2 z^2 + \ldots \] \hfill (6.2)

the series

\[ B(b) = D_1 + D_2 b + D_3 \frac{b^2}{2} + \ldots \] \hfill (6.3)

goes into

\[ B(b(z)) = D_1 + D_2 c_1 z + (D_2 c_2 + D_3 \frac{c_2^2}{2}) z^2 + \ldots \] \hfill (6.4)

which is convergent up to \( z = z_0 \), while the original \( b \) expansion was convergent only up to \( b = 1/\beta \), corresponding to \( z(1/\beta) < z_0 \). The improved approximation for \( d(a) \) is therefore given by

\[
\begin{align*}
    d(a) &\approx \int_z^{z_0} dz \left| \frac{db}{dz} \right| e^{-b(z)/a} \left[ D_1 + D_2 c_1 z + (D_2 c_2 + D_3 \frac{c_2^2}{2}) z^2 + \ldots \right] \\
    &= \int_0^{2/\beta} db \, e^{-b/a} \left[ D_1 + D_2 c_1 b + (D_2 c_2 + D_3 \frac{c_2^2}{2}) b^2 + \ldots \right], \hfill (6.5)
\end{align*}
\]

where the full expression of \( z \) as function of \( b \) is inserted in the integral. In this way, an infinite sequence of terms is added to the \( b \) expansion. For \( a \) small, the upper limit of integration can be replaced with infinity without significant effect.

One possible example is given by [17]:

\[ z(b) = \frac{\sqrt{1 + \beta b} - 1}{\sqrt{1 + \beta b} + 1} \rightarrow b(z) = \frac{4z}{\beta(1-z)^2}. \] \hfill (6.6)

In this case the first UV singularity is at \( z = -1 \), and all higher UV renormalons are on the unit circle \( |z| = 1 \). IR renormalons are between \( z_0 = (\sqrt{3} - 1)/(\sqrt{3} + 1) \) and \( z = 1 \). In this example, \( c_1 = 4/\beta, c_2 = 8/\beta \). Other examples are

\[ z(b) = \frac{\beta b}{k + \beta b} \rightarrow b(z) = \frac{kz}{\beta(1-z)}, \] \hfill (6.7)

with \( k = 1, 2 \) or 3. Also in these cases the first IR renormalon at \( b = 2/\beta \) becomes the closest singularity to \( z = 0 \), while the UV are pushed further away. Here we have \( c_1 = c_2 = k/\beta \).
Before discussing numerical applications, we observe that the present method relies simply on the position of the IR and UV renormalon singularities in the Borel plane and not on the nature and the strength of the singularities. We have seen that the integration over the circle in eq. (4.3) does not change the position of the singularities in the $b$ plane, but simply affects their strength. Thus, we can as well consider the effect of the accelerators on the expansions for $R^{\text{BNP}}_\tau$ given in eq. (2.18) or on the LP expression of eq. (2.20). For example, the improved version of eq. (4.3) simply becomes

$$r \simeq \int_0^\infty db \, e^{-b/a} \left[ D_1 + D_2 c_1 z(b) + \left( D_2 c_2 + D_3 \frac{c_2^2}{2} \right) z(b)^2 + \ldots \right] F(\beta b). \quad (6.8)$$

We now consider the following numerical exercise. We assume that experiments have measured $R_\tau = 3.6$. We then compute $a(m_\tau^2)$ with the LP formula as a function of the scale $\mu$, and we perform the same calculation applying our acceleration procedures to the LP method [1]. The results are shown in fig. [4]. We see that relatively large differences in the fitted value of $\alpha_s(m_\tau^2)$ are obtained, especially at large $\mu$ for different accelerators and in comparison to the non-accelerated formulae. We do not see a priori compelling reasons to prefer one or the other procedure. The fact that a priori equivalent methods lead to results with a sizeable spread must be considered as an indication of a real ambiguity. Even if we only consider the method of ref. [1] for the integration over the circle, it is impossible to go below an uncertainty of the order $\delta \alpha_s(m_\tau^2) \sim \pm 0.050$ for $\mu$ in the range from 1 to 3 GeV. The ambiguity becomes even larger if we extend the comparison to the formulae with truncation in $\beta^2 a^2 \pi^2$ a–la BNP (fig. [3]).

7. Study of More Convergent Approximants in a Model

The method for accelerating the convergence discussed in the previous section only relies on the position of the singularities of the Borel transform and not on their nature and strength. It looks rather surprising that one can compensate for the effect of renormalons without actually knowing their form in detail. In this section we study a simple mathematical model to clarify under which conditions the method can be successful, in the sense that it provides a better approximation to the true result.
Figure 4: Effect of the accelerators on the determination of $\alpha_s(m_T^2)$ with the LP method, for $R_T = 3.6$. The curves b, c and d refer to the change of variable of eq. (6.7), while e refers to eq. (6.6).
Figure 5: As in Fig. 4 for the BNP method.
We consider as a model the case where the Borel transform is exactly specified by

\[ B_{\text{true}}(b) = 1 + D_2 b + D_3 b^2/2 + \rho \sum_{n=3}^{\infty} \left( \frac{n + \gamma - 1}{n!} \right) (-\beta b)^n \]

\[ = 1 + D_2 b + D_3 b^2/2 + \rho \left[ \frac{1}{(1 + \beta b)^\gamma} - 1 + \gamma \beta b - \frac{\gamma (\gamma + 1)}{2} (\beta b)^2 \right]. \]  

(7.1)

The added sum stands for the higher-order contribution that could arise from a leading UV renormalon at \( b = -1/\beta \) with a degree of singularity specified by \( \gamma \) and a fixed overall strength given by \( \rho \). We will take \( \rho = 1 \) in the following discussion. It is convenient to re-express eq. (7.1) in terms of \( x = \beta b \):

\[ B_{\text{true}}(x) = 1 + D_2 x + D_3 x^2/2 + \rho \left[ \frac{1}{(1 + x)^\gamma} - 1 + \gamma x - \frac{\gamma (\gamma + 1)}{2} x^2 \right]. \]

(7.2)

where \( D_2 = D_2/\beta \) and \( D_3 = D_3/\beta^2 \). Similarly we can introduce \( B_{\text{pert}}(x) \) and \( B_{\text{accel}}(x) \), the perturbative Borel functions without and with acceleration, respectively:

\[ B_{\text{pert}}(x) = 1 + D_2 x + D_3 x^2/2 \]

(7.3)

\[ B_{\text{accel}}(x) = 1 + D_2 \overline{c}_1 z(x) + (D_2 \overline{c}_2 + D_3 \frac{\tau^2_2}{2}) z(x)^2, \]

(7.4)

where \( \overline{c}_{1,2} = \beta c_{1,2} \). In all cases the corresponding \( d \) function, \( d_{\text{true}}, d_{\text{pert}} \) and \( d_{\text{accel}} \) is given by

\[ \beta d(a) = \int_0^\infty dx \ e^{-x/\beta a} B(x). \]

(7.5)

We consider the ratio

\[ H = \frac{d_{\text{true}} - d_{\text{accel}}}{d_{\text{true}} - d_{\text{pert}}} = 1 - \overline{D}_2 I_2(\beta a) - \overline{D}_3 I_3(\beta a), \]

(7.6)

where the quantities \( I_{2,3} \) are given by

\[ I_2(\beta a) = \frac{1}{I_0(\beta a)} \int_0^\infty dx \ e^{-x/\beta a} \left( \overline{c}_1 z(x) + \overline{c}_2 z(x)^2 - x \right) \]

(7.7)

\[ I_3(\beta a) = \frac{1}{2} \frac{1}{I_0(\beta a)} \int_0^\infty dx \ e^{-x/\beta a} \left( \overline{c}_1^2 z(x)^2 - x^2 \right) \]

(7.8)
and

\[ I_0(\beta a) = \int_0^\infty dx \, e^{-x/\beta a} \left[ \frac{1}{(1+x)^\gamma} - 1 + \gamma x - \frac{\gamma(\gamma+1)}{2} x^2 \right]. \]  

(7.9)

Clearly, \(|H| < 1\) is the condition for the acceleration method to be successful. In particular for \(H = 0\), \(d_{\text{true}} = d_{\text{accel}}\) coincide. For each value of \(\beta a\) and \(\gamma\), in a given model specified by \(z(x)\) and the corresponding coefficients \(\varphi_{1,2}\), the condition \(H = 0\) is satisfied on a straight line in the plane \(\overline{D}_2, \overline{D}_3\), while the inequality \(|H| < 1\) is satisfied in a band defined by two straight lines parallel to the \(H = 0\) line. In figs. 6-8 we plot the lines \(H = 0\) for \(\beta a = 0.27\) for fixed \(\gamma\) and \(z(x)\) given by eq. (6.6) (case labelled by 0), or by eq. (6.7) with \(k = 1\) or 2 (cases 1 and 2). The values of \(\gamma\) in figs. 6-8 are \(\gamma = 0.5, 1, 2.\) We see that for each choice of \(z(x)\) the lines have different negative slopes. The lines tend to cross each other in a region of the plane not far from the point \(\overline{D}_2 = -\gamma, \overline{D}_3 = \gamma(\gamma+1)\), i.e. the values that correspond to the first few terms of the expansion of the asymptotic function \(1/(1+x)^\gamma\). The region where

\[ \begin{align*}
\text{Figure 6: Lines corresponding to } H = 0 \text{ for the method “0” (solid), “1” (dashed) and “2” (dotted) for } \gamma = 0.5. \text{ The circle corresponds to expansion of } 1/(1+x)^\gamma. 
\end{align*} \]
Figure 7: As in Fig. 6, for $\gamma = 1$. 
Figure 8: As in Fig. 6 for $\gamma = 2$. 
the lines cross is more sharply defined if \( \gamma \) is small, i.e. if the asymptotic function is not too singular. In figs. 9 \& 10 we show the bands \(|H| < 1\) for \( \gamma = 0.5, 1\) in cases 0 and 1.

![Graph](image)

Figure 9: Bands corresponding to \(|H| < 1\) for the method “0” (solid), “1” (dashed) for \( \gamma = 0.5\).

The conclusion is that the method for accelerating the convergence works well only if the coefficients \(D_2, D_3\) resemble those of the asymptotic series, in other words if the known terms in the expansion are sufficiently representative of the asymptotic series. In particular we see that it is very unlikely to get an improvement if the coefficients \(D_2\) and \(D_3\) are of the same sign, as is unfortunately the case for the series of interest for us (see eq.6).

From a different point of view we now consider the simple function \(B(\beta b)\) given by

\[
B(\beta b) = \frac{1}{(1 + \beta b)^\gamma}
\]  

(7.10)
Figure 10: As in Fig. 9, for $\gamma = 1$. 
and we plot the relation of the exact result

\[ \beta d(a) = \int_0^\infty dx \, e^{-x/\beta a} B(x) \]  

(7.11)

with its accelerated or non-accelerated series approximants, as a function of \( \beta a \) and of the order of the expansion. The results obtained for the accelerating function \( z(b) \) given in eq. (6.6) (the case labeled by 0) and \( \gamma = 1, 2 \) are shown in figs. We see that when, as in this case, the coefficients of the expansion coincide with their asymptotic form the accelerated formulae provide a much better approximation to the true result, more so if the singularity is weaker (i.e. \( \gamma \) is smaller). The non-accelerated formulae are only good for small enough \( \beta a \). The physically interesting case of \( \beta a \sim 0.27 \) appears to be at the limit of the range where the non-accelerated formulae are acceptable.

Figure 11: Effect of the resummation technique described in the text, for a function with Borel transform \( B(b) = 1/(1 + \beta b)^\gamma, \gamma = 1 \)
Figure 12: Effect of the resummation technique described in the text, for a function with Borel transform $B(b) = 1/(1 + \beta b)^\gamma$, $\gamma = 2$
8. Conclusion

The determination of \( \alpha_s(m_\tau^2) \) or \( \alpha_s(m_Z^2) \) from \( \tau \) decay is nominally very precise and the experimental errors are extremely small at LEP. Certainly the dominant ambiguity is at present the theoretical error. The nominal precision is large because in the massless limit no explicit \( 1/m_\tau^2 \) corrective terms are present in the operator expansion. But it has become clear by now that one cannot sensibly talk of power-suppressed corrections if the ambiguities in the leading term are not under control [28]. We think that there is no real theorem that prevents non-perturbative corrections in the coefficient function of the leading term in the operator expansion at the level of \( \Lambda^2/m_\tau^2 \). Equivalently, there could be an IR renormalon singularity at \( b = +1/\beta \), which would create an irreducible (being located on the integration path) ambiguity of order \( \Lambda^2/m_\tau^2 \). In all-order perturbative evaluations of the singularity pattern in the Borel plane the IR singularity at \( b = +1/\beta \) is probably absent, a result consistent with the idea that all irreducible ambiguities can be reabsorbed in condensates. Even if the IR renormalon singularity at \( b = +1/\beta \) is indeed absent, the radius of convergence of the expansion is limited by the leading UV renormalon singularity at \( b = -1/\beta \). If this disease is not cured or cannot be cured the resulting ambiguity is still of order \( \Lambda^2/m_\tau^2 \). In principle the problem could be solved if the exact nature and strength of the singularity was known, by simply taking its effect into account in the evaluation of \( \alpha_s(m_\tau^2) \), along the way indicated in section 5 in the case of the estimate of the singularity in the unrealistic limit of large \( \beta \). But the exact determination of the singularity appears to be beyond the scope of presently known methods. In the actual case where the UV renormalon singularity at \( b = -1/\beta \) is not specified, one can still try, in principle, to bypass the problem by a transformation of variables that pushes the leading UV singularity to a larger distance from the origin than the first IR renormalon singularity at \( b = +2/\beta \). Expanding in the new variable is equivalent to add a specified infinite sequel of terms to the original expansion. The convergence of the series should be improved by these accelerators of convergence and the ambiguity decreased. We have studied the quantitative effect of implementing a number of such accelerators with different choices of the renormalisation scale \( \mu \). An indication of the size of the ambiguities on \( \alpha_s(m_\tau^2) \) is obtained from the spread of the results for different starting formulae (e.g. with \( a(-s) \) taken in the integration over the circle in its renormalisation group improved form or in a fixed order truncated.
expansion), different accelerator functions and different choices of the renormalisation scale. The difference between the resummed or truncated expression for $a(-s)$ on the circle are especially large at small values of $\mu$, while the variations induced by the different accelerators are especially pronounced at large values of $\mu$. The relative stability of the unaccelerated result of ref. [4] versus changes of $\mu$ appears as largely accidental in that the accelerated formulae based on it are much less stable at large $\mu$. It was argued in ref. [19] that if one expands in $\alpha_s(\mu^2)$ instead of expanding in $\alpha_s(m_r^2)$ the scale dependence of the UV renormalon correction becomes of order $(\Lambda^2/m_r^2)(m_r^2/\mu^2)^2$. Can then one be safe if $\mu$ is chosen sufficiently large? Clearly in the true result the sum of the perturbative terms plus the remainder must be scale independent. When $\mu$ is changed, the number of terms to be added before the series becomes asymptotic changes and must compensate for the difference. The increased sensitivity of the accelerated formulae at large $\mu$ is not encouraging for invoking that large $\mu$ is safer. All together, from fig.4 we find it difficult to imagine that the theoretical error on the strong coupling can be taken smaller then, say, $\delta\alpha_s(m_r^2) \sim 0.050$ (which approximately corresponds to $\delta\alpha_s(m_2^2) \sim 0.005$).

The accelerator method is based on the mere knowledge of the position of the singularity and not on its precise form. Clearly such a method can only work if the known terms of the expansion carry enough information on the asymptotic form of the series. We have quantitatively confirmed this statement by studying the problem on a simple mathematical model where the true result is known. The performance of different accelerators is studied as a function of the coefficients of the first few terms. These results indicate that there is little hope of improving the ambiguity from the leading UV renormalon because the first few coefficients of the actual expansion show no evidence for the asymptotic behaviour, in particular no sign alternance. This last argument (as well as the one on the $\mu$ dependence of the UV renormalon) can be interpreted in different ways. If one is a great optimist, he can argue that the series does not resemble at all to the renormalon asymptotics, hence the normalisation of the renormalon term is very small (as is the case for the explicit form of the singularity obtained in the large $N_f$ limit). Or, if one is more cautious, as one should be in estimating errors, he can say that since the known terms do not show sign of asymptotia, they are dominated by subasymptotic effects and cannot be used to estimate the remainder. In this spirit we do not propose the accelerators as a better way to determine the true result but simply as a criterium to evaluate the theoretical
error. In fact, while all accelerators tend to increase the resulting value of $\alpha_s$, the amount of the upward shift is sizeably different for different accelerators.

In order to bypass all possible objections one should be able to fit at the same time $\alpha_s(m^2)$ and $C_2$, the coefficient of $1/m^2$ corrections to $R_\tau$. Note that in the ALEPH moment analysis $C_2$ is fixed to zero while the coefficients of some higher-dimension operators are fitted. This is not very relevant to the main issue. If $C_2$ is not fixed it is found that the sensitivity to $\alpha_s(m^2)$ is much reduced. In an interesting paper Narison attempted to put an upper bound on $C_2$ from the data on $e^+e^- \rightarrow$ hadrons. This is an important issue that would deserve further study. Our interpretation of the analysis of ref. is that values of $C_2$ of order $(500 \text{ MeV})^2$ are not at all excluded. Narison derives a more stringent limit $|C_2| < (374 \text{ MeV})^2$ but we feel he relies too much on the so called optimisation procedure. Indeed, something that should be a constant in a dummy variable turns out to be a steep parabola. The value at the tip is taken, with a small error, as the best estimate because of the vanishing of the derivative at that point, instead of considering the span of the results in a priori reasonable range for the irrelevant parameter. In a recent paper an estimate of $C_2$ from Argus data on hadronic $\tau$ decay was obtained and the results are compatible with $|C_2| < (500 \text{ MeV})^2$.

We ignored here other possible sources of error beyond those arising from higher orders in perturbation theory. These include errors from the freezing mechanism for $\alpha_s$, errors from the translation of $\alpha_s(m^2)$ in terms of $\alpha_s(m_Z^2)$, from the region of the circle integration near the positive real axis and so on. These errors are presumably smaller than our current estimate of the error from higher order terms in the perturbative expansion. Taking all the other uncertainties into account we end up with a total theoretical error around $\delta\alpha_s(m^2_Z) \sim 0.006$. As a result, in spite of the fact that our estimate of the error is larger than usually quoted, the determination of $\alpha_s(m^2_Z)$ from $\tau$ remains one of the best determinations of the strong coupling constant.

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