Benedicks–Amrein–Berthier theorem for the Heisenberg motion group

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Abstract
It is well known that \((\mathbb{H}_n \rtimes U(n), U(n))\) is a Gelfand pair, an exact analogue of the Heisenberg group result due to Narayanan and Ratnakumar is not possible for the Heisenberg motion group. We prove that if an integrable function on the Heisenberg motion group is supported on a set of finite measure, and its Weyl transform is non-zero only for finitely many Fourier-Wigner pieces and have finite rank, then the function must be zero. In the end, a quantitative interpretation of this result is described through strong annihilating pair for the Weyl transform.

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1 | INTRODUCTION

In an interesting article [3], Benedicks proved that if \(f \in L^1(\mathbb{R}^n)\), then both the sets \(\{x \in \mathbb{R}^n : f(x) \neq 0\}\) and \(\{\xi \in \mathbb{R}^n : \hat{f}(\xi) \neq 0\}\) cannot have finite Lebesgue measure, unless \(f = 0\). Concurrently, in the article [1], Amrein–Berthier reached to the same conclusion via the Hilbert space theory. The aforesaid fundamental result got further attention in general Lie groups.

Let \(G\) be a locally compact group and \(\hat{m}\) denotes the Plancherel measure on the unitary dual group \(\hat{G}\). Then \(G\) is said to satisfy qualitative uncertainty principle (QUP) if for each \(f \in L^2(G)\) with \(m\{x \in G : f(x) \neq 0\} < m(G)\) and

\[
\int_{\hat{G}} \text{rank} \hat{f}(\lambda) \, d\hat{m}(\lambda) < \infty
\]  \hspace{1cm} (1.1)
implies \( f = 0 \). In [2], QUP was proved for certain unimodular groups of type I. A brief survey of QUP is presented in [7]. Although condition (1.1) works for too general class of groups, however, for specific groups uncertainty may arise with some weaker conditions than (1.1). In the case of the Heisenberg group \( \mathbb{H}^n \), the condition (1.1) of QUP implies \( \hat{f} \) should be supported on a set of finite Plancherel measure together with rank \( \hat{f}(\lambda) \) is finite for almost all \( \lambda \).

In [10], Narayanan and Ratnakumar proved that if \( f \in L^1(\mathbb{H}^n) \) is supported on \( B \times \mathbb{R} \), where \( B \) is a compact subset of \( \mathbb{C}^n \), and \( \hat{f}(\lambda) \) has finite rank for each \( \lambda \), then \( f = 0 \). Thereafter, Vemuri [14] replaced the compactness condition on \( B \) by finite measure. In [5], authors consider \( B \) as a rectangle in \( \mathbb{R}^{2n} \) while proving an analogous result on step two nilpotent Lie groups and a version of this result, with a strong assumption on rank, derived therein for the Heisenberg motion group. Later in the article [8], this result is extended to arbitrary set \( B \) of finite measure for step two nilpotent Lie groups. In this article, we prove the following analogue result on the Heisenberg motion group.

First, consider the Heisenberg motion group \( G \), which is the semidirect product of \( \mathbb{H}^n \) and \( U(n) \), the unitary group on \( \mathbb{C}^n \). Denote \( K = U(n) \). Then, due to the fact that \((G, K)\) is a Gelfand pair [4], the Fourier transform of a \( K \)-bi-invariant integrable function has rank one, irrespective of support of the function. Thus, an exact analogue of the Heisenberg group result due to Narayanan and Ratnakumar [10] is not inevitable for the Heisenberg motion group.

However, we prove that if an integrable function is supported on a set of finite measure, and its Weyl transform is non-zero only for finitely many Fourier-Wigner pieces and have finite rank, then the function is zero. Consequently, we obtain that if each Fourier–Wigner piece of a non-trivial function is supported on a set of finite measure, then all of its Fourier transform can not have finite rank. Further, we explain the naturality of this new version of the Benedicks–Amrein–Berthier theorem by comparing it with the qualitative uncertainty principle.

The proof of our result follows in a similar spirit as described by Amrein-Berthier in [1]. However, specifying the appropriate set of projections in the setup of the Heisenberg motion group was a major bottleneck. This result, as of now, is the most general analogue of Benedicks–Amrein–Berthier theorem in this setup.

## 2 | HEISENBERG MOTION GROUP

The Heisenberg group \( \mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \) is a step two nilpotent Lie group having center \( \mathbb{R} \) that equipped with the group law

\[
(z, t) \cdot (w, s) = (z + w, t + s + \frac{1}{2} \text{Im}(z \cdot \bar{w})).
\]

By the Stone-von Neumann theorem, the infinite-dimensional irreducible unitary representations of \( \mathbb{H}^n \) can be parameterized by \( \mathbb{R}^* = \mathbb{R} \setminus \{0\} \). That is, each \( \lambda \in \mathbb{R}^* \) defines a Schrödinger representation \( \pi_\lambda \) of \( \mathbb{H}^n \) via

\[
\pi_\lambda(z, t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda(x \cdot \xi + \frac{1}{2} x \cdot y)} \varphi(\xi + y),
\]

where \( z = x + iy \) and \( \varphi \in L^2(\mathbb{R}^n) \).

Having chosen sub-Laplacian \( \mathcal{L} \) of the Heisenberg group \( \mathbb{H}^n \) and its geometry, there is a larger group of isometries that commute with \( \mathcal{L} \), known as Heisenberg motion group. The Heisenberg...
motion group $G$ is the semidirect product of $\mathbb{H}^n$ with the unitary group $K = U(n)$. Since $K$ defines a group of automorphisms on $\mathbb{H}^n$, via $k \cdot (z, t) = (kz, t)$, the group law on $G$ can be expressed as

$$(z, t, k_1) \cdot (w, s, k_2) = \left( z + k_1 w, t + s - \frac{1}{2} \text{Im}(k_1 w \cdot z), k_1 k_2 \right).$$

Since a right $K$-invariant function on $G$ can be thought of as a function on $\mathbb{H}^n$, the Haar measure on $G$ is given by $dg = dz dt dk$, where $dz dt$ and $dk$ are the normalized Haar measure on $\mathbb{H}^n$ and $K$, respectively.

For $k \in K$, define another set of representations of the Heisenberg group $\mathbb{H}^n$ by $\pi_{\lambda, k}(z, t) = \pi_{\lambda}(kz, t)$. Since $\pi_{\lambda, k}$ agrees with $\pi_{\lambda}$ on the center of $\mathbb{H}^n$, it follows by the Stone-Von Neumann theorem for the Schrödinger representation that $\pi_{\lambda, k}$ is equivalent to $\pi_{\lambda}$. Hence there exists an intertwining operator $\mu_{\lambda}(k)$ satisfying

$$\pi_{\lambda}(kz, t) = \mu_{\lambda}(k)\pi_{\lambda}(z, t)\mu_{\lambda}(k)^*.$$ 

By an appropriate selection of $\mu_{\lambda}$, it becomes a unitary representation of $K$ on $L^2(\mathbb{R}^n)$, called metaplectic representation. For details, we refer to [6], chapter 4. Let $(\sigma, H_\sigma)$ be an irreducible unitary representation of $K$ and $H_\sigma = \text{span}\{e_j^\sigma : 1 \leq j \leq d_\sigma\}$. For $k \in K$, the matrix coefficients of the representation $\sigma \in \hat{K}$ are given by

$$\varphi^\sigma_{ij}(k) = \langle \sigma(k)e_i^\sigma, e_j^\sigma \rangle,$$

where $i, j = 1, \ldots, d_\sigma$. By the Peter–Weyl theorem for compact groups [13], it follows that the set $\{\sqrt{d_\sigma}\varphi^\sigma_{ij} : 1 \leq i, j \leq d_\sigma, \sigma \in \hat{K}\}$ is an orthonormal basis for $L^2(K)$. Then for each $\lambda \in \mathbb{R}^n$, the set $\{\varphi^\lambda_{ij} : \alpha \in \mathbb{Z}^n_+\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$. By letting $P^\lambda_m = \text{span}\{\varphi^\lambda_{ij} : |\alpha| = m\}$, $\mu_{\lambda}$ becomes an irreducible unitary representation of $K$ on $P^\lambda_m$. Hence, the action of $\mu_{\lambda}$ can be realized on $P^\lambda_m$ by

$$\mu_{\lambda}(k)\varphi^\lambda_{ij} = \sum_{|\alpha| = |\gamma|} \eta_{\gamma\alpha}(k)\varphi^\lambda_{\gamma j},$$

(2.1)

where functions $\eta_{\gamma\alpha}$ are the matrix coefficients of $\mu_{\lambda}$. Define a bilinear form $\varphi^\lambda_{ij} \otimes e_j^\sigma$ on $L^2(\mathbb{R}^n) \times H_\sigma$ by $\varphi^\lambda_{ij} \otimes e_j^\sigma = \varphi^\lambda_{ij} e_j^\sigma$. Then $\{\varphi^\lambda_{ij} \otimes e_j^\sigma : \alpha \in \mathbb{Z}^n_+, 1 \leq j \leq d_\sigma\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n) \otimes H_\sigma$. Denote $H_\sigma^2 = L^2(\mathbb{R}^n) \otimes H_\sigma$. Define a representation $\rho^\lambda_o$ of $G$ on the space $H_\sigma^2$ by

$$\rho^\lambda_o(z, t, k) = \pi_{\lambda}(z, t)\mu_{\lambda}(k) \otimes \sigma(k).$$

Then $\rho^\lambda_o$ are all possible irreducible unitary representations of $G$ those participate in the Plancherel formula [12]. Thus, in view of the above discussion, we shall denote the partial dual of the group $G$ by $G' \cong \mathbb{R}^n \times \hat{K}$. For $(\lambda, \sigma) \in G'$, the Fourier transform of $f \in L^1(G)$, defined by

$$\hat{f}(\lambda, \sigma) = \int_k \int_{\mathbb{C}^n} f(z, t, k)\rho^\lambda_o(z, t, k)dz dt dk,$$
is a bounded linear operator on $\mathcal{H}_\sigma^2$. As the Plancherel formula [12]

$$\int_K \int_{\mathbb{H}^n} |f(z, t, k)|^2 dz dt dk = (2\pi)^{-n} \sum_{\sigma \in K} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \|\hat{f}(\lambda, \sigma)\|^2_{HS} |\lambda|^n d\lambda$$

holds for $f \in L^2(G)$, it follows that $\hat{f}(\lambda, \sigma)$ is a Hilbert–Schmidt operator on $\mathcal{H}_\sigma^2$.

In order to prove our main result on the Heisenberg motion group $G$, it is enough to consider a similar proposition for the Weyl transform on $G^\times = \mathbb{C}^n \times K$. For that, we require to set some preliminaries about the Weyl transform on $G^\times$.

Let $f^\lambda$ be the inverse Fourier transform of the function $f$ in the $t$ variable defined by

$$f^\lambda(z, k) = \int_{\mathbb{R}} f(z, t, k)e^{i\lambda t} dt. \quad (2.2)$$

Then

$$\hat{f}(\lambda, \sigma) = \int_K \int_{\mathbb{C}^n} f^\lambda(z, k) \rho^\lambda_\sigma(z, k) dz dk,$$

where $\rho^\lambda_\sigma(z, k) = \rho^\lambda_\sigma(z, 0, k)$.

For $(\lambda, \sigma) \in G'$, define the Weyl transform $W^\lambda_\sigma$ on $L^1(G^\times)$ by

$$W^\lambda_\sigma(g) = \int_K \int_{\mathbb{C}^n} g(z, k) \rho^\lambda_\sigma(z, k) dz dk.$$

Then $\hat{f}(\lambda, \sigma) = W^\lambda_\sigma(f^\lambda)$, and hence $W^\lambda_\sigma(g)$ is a bounded operator if $g \in L^1(G^\times)$. On the other hand, if $g \in L^2(G^\times)$, then $W^\lambda_\sigma(g)$ becomes a Hilbert-Schmidt operator satisfying the Plancherel formula

$$\int_K \int_{\mathbb{C}^n} |g(z, k)|^2 dz dk = (2\pi)^{-n} |\lambda|^n \sum_{\sigma \in K} d_\sigma \||\hat{W}^\lambda_\sigma(g)\|^2_{HS}. \quad (2.3)$$

**Fourier–Wigner representation:** Define the Fourier–Wigner transform $V^\eta_\zeta$ of the functions $\zeta, \eta \in \mathcal{H}_\sigma^2$ by

$$V^\eta_\zeta(z, k) = (2\pi)^{-\frac{n}{2}} |\lambda|^{-\frac{n}{2}} \langle \rho^\lambda_\sigma(z, k) \zeta, \eta \rangle,$$

where $(z, k) \in G^\times$. The following orthogonality relation is derived in [5]. A version of this result is also appeared in [12].

**Lemma 2.1.** For $\zeta_l, \eta_l \in \mathcal{H}_\sigma^2$, $l = 1, 2$, the corresponding Fourier–Wigner transforms satisfy

$$\int_K \int_{\mathbb{C}^n} V^\eta_{\zeta_1}(z, k) \overline{V^\eta_{\zeta_2}(z, k)} dz dk = \langle \zeta_1, \zeta_2 \rangle \langle \eta_1, \eta_2 \rangle.$$
In particular, $V_\eta^\zeta \in L^2(G^\times)$. Let $\mathcal{V}_\sigma^V = \overline{\text{span}\{V_\eta^\zeta : \zeta, \eta \in H_\sigma^2\}}$ and set $\Psi^\sigma_{\alpha,j} = \phi^\lambda_\alpha \otimes e^\sigma_j$. Since $B_\sigma^j = \{\psi^\sigma_{\alpha,j} : \alpha \in \mathbb{Z}_+^n, 1 \leq j \leq d_\sigma\}$ forms an orthonormal basis for $H_\sigma^2$, by Lemma 2.1, we infer that

$$V_{B_\sigma^j} = \left\{ V_{\psi^\sigma_{\alpha_1,j_1}}^{\psi^\sigma_{\alpha_2,j_2}} : \psi^\sigma_{\alpha_1,j_1}, \psi^\sigma_{\alpha_2,j_2} \in B_\sigma^j \right\}$$

is an orthonormal basis for $V_\sigma^V$. The next result, which is followed as a corollary of the Peter–Weyl theorem [13], will be the desired decomposition of $L^2(G^\times)$.

**Proposition 2.2.** The set $\bigcup_{\sigma \in \hat{K}} V_{B_\sigma^j}$ is an orthonormal basis for $L^2(G^\times)$.

Since $V_{B_\sigma^j}$ is an orthonormal basis for $V_\sigma^V$, by Proposition 2.2, we infer that $L^2(G^\times) = \bigoplus_{\sigma \in \hat{K}} V_\sigma^V$. We shall call this as the Fourier–Wigner decomposition and $V_\sigma^V$ as Fourier–Wigner representation of $G^\times$.

**Remark 2.3.** Consider the Fourier–Wigner transform $V_\eta^\zeta$ of $\zeta, \eta \in H_\sigma^2$. Then Lemma 2.1 ensures that the range of $W_\sigma^{\lambda}(V_\eta^\zeta)$ is spanned by $\eta$ (see [6, p. 32], for the analogous result on the Heisenberg group). Further, the set of all integrable functions on $G$, which are $K$-bi-invariant, form a commutative convolution algebra. Hence, the Fourier transform of a $K$-bi-invariant integrable function has rank one. However, the later set of functions differ from the Fourier–Wigner transform in terms of the Fourier–Wigner decomposition.

Although the decomposition in Proposition 2.2 is being followed by the Peter–Weyl theorem, it is quite finer than the usual Peter–Weyl decomposition of function on $K$, due to the presence of the metaplectic representation. And as an effect, even if $g \in L^2(G^\times)$ is $K$-bi-invariant on $G^\times$, it need not fall into the trivial Fourier–Wigner representation. This fact can be explained more explicitly via the following example of right $K$-invariant functions, which can be made $K$-bi-invariant by some particular choice.

Consider the one-dimensional Heisenberg motion group $\mathbb{H}^1 \rtimes U(1)$. Realize $U(1) \cong S^1$. Let $(z, t, e^{i\theta}) \in \mathbb{H}^1 \rtimes U(1)$. Then for each $(\lambda, \alpha) \in \mathbb{R}^+ \times \mathbb{Z}$, the unitary irreducible representations $(\rho^\lambda_{\alpha}, L^2(\mathbb{R}))$ of $\mathbb{H}^1 \rtimes U(1)$ can be defined by

$$\rho^\lambda_{\alpha}(z, t, e^{i\theta}) = e^{-i\alpha \theta} \pi^\lambda_{\alpha}(z, t) \mu^\lambda_{\alpha}(e^{i\theta}).$$

In fact, the action of $\rho^\lambda_{\alpha}$ on Hermite function $\phi^\lambda_{\beta}$, where $\beta \in \mathbb{Z}_+$, will be given by

$$\rho^\lambda_{\alpha}(z, t, e^{i\theta}) \phi^\lambda_{\beta} = e^{-i(\alpha - \beta) \theta} \pi^\lambda_{\alpha}(z, t) \phi^\lambda_{\beta}. \quad (2.4)$$

For more details, see [11]. By Proposition 2.2, we have $L^2(C \times S^1) = \bigoplus_{\alpha \in \mathbb{Z}} V_\alpha^V$, where

$$V_\alpha^V = \{ \varphi \in L^2(C \times S^1) : W^\lambda_{\alpha}(\varphi) = 0 \text{ for all } \alpha \neq \alpha' \}.$$
From (2.4), it follows that
\[ \langle \rho^\lambda_\alpha(z, e^{i\theta}) \phi^\lambda_\beta, \phi^\lambda_\gamma \rangle = e^{-i(\alpha - \beta)\theta} \langle \pi^\lambda_\alpha(z) \phi^\lambda_\beta, \phi^\lambda_\gamma \rangle = e^{-i(\alpha - \beta)\theta} \Phi^\lambda_\beta \gamma(z), \]
where \( \Phi^\lambda_\beta \gamma(z) \) is the special Hermite function. Denote \( \tilde{\Phi}^\lambda_\alpha \beta \gamma(z, e^{i\theta}) = e^{-i(\alpha - \beta)\theta} \Phi^\lambda_\beta \gamma(z) \). Then \( \{ \tilde{\Phi}^\lambda_\alpha \beta \gamma : \beta, \gamma \in \mathbb{Z}_+ \} \) will be an orthonormal basis for \( \mathcal{V}^\lambda_\alpha \). In particular, corresponding to the trivial representation, \( \Phi^\lambda_0 \beta \gamma(z) \) are basis elements of \( \mathcal{V}^\lambda_0 \). Thus, the presence of \( e^{i\beta\theta} \) in the basis, concludes that an arbitrary \( h \in L^2(\mathbb{C}) \) need not be contained in \( \mathcal{V}^\lambda_0 \). To be explicit, consider a function \( h \in L^2(\mathbb{C}) \) supported on a set of finite measure. Further, define a function \( g \) on \( \mathbb{C} \times S^1 \) by \( g(z, e^{i\theta}) = h(z) \). The Weyl transform of \( g \) is defined by
\[ W_\lambda^\alpha(g) = \int_{\mathbb{C} \times [0, 2\pi]} g(z, e^{i\theta}) \rho_\alpha^\lambda(z, e^{i\theta}) dz d\theta. \]
From (2.4), we have
\[ W_\lambda^\alpha(g) \phi^\lambda_\beta = W^\lambda_\beta(h) \phi^\lambda_\beta \int_0^{2\pi} e^{-i(\alpha - \beta)\theta} d\theta. \]

Hence, for each \( \alpha \in \mathbb{Z} \), \( W^\lambda_\alpha(g) \) can have rank at most one. Further \( W^\lambda_\beta(g) \phi^\lambda_\beta = W^\lambda_\beta(h) \phi^\lambda_\beta \) for \( \beta \in \mathbb{Z}_+ \). Since \( h \) is a non-zero function supported on a finite measure set, \( W^\lambda_\beta(h) \) cannot be a finite rank operator, see \([8, 10]\). Therefore, there exist infinitely many \( \beta \) such that \( W^\lambda_\beta(g) \phi^\lambda_\beta \neq 0 \). Hence, \( g \) is not contained in any \( \mathcal{V}^\lambda_\alpha \). In fact, \( g \) fails to be a member of any finite union of subspaces \( \mathcal{V}^\lambda_\alpha \).

The above discussion brings forward the following question. Does there exist a non-trivial function \( g \in L^1(G^\times) \) which is supported on a set of finite measure, and whose Weyl transform \( W^\lambda_\sigma(g) \) has finite rank only for finitely many \( \sigma \) and \( W^\lambda_\sigma(g) = 0 \) otherwise? The non-existence of such a function is guaranteed by Proposition 2.9.

Next, we prove the inversion formula for the Weyl transform \( W^\lambda_\sigma \) which is a key ingredient while proving our main result. For this, we need the fact that
\[ \rho^\lambda_\sigma(z, k_1) \rho^\lambda_\sigma(w, k_2) = e^{-\frac{i}{2} \text{Im}(k_1 w \bar{z})} \rho^\lambda_\sigma(z + k_1 w, k_1 k_2), \]
where \( (z, k_1), (w, k_2) \in G^\times \).

**Theorem 2.4** (Inversion formula). Let \( g \in L^1 \cap L^2(G^\times) \). Then
\[ g(z, k) = (2\pi)^{-n} |\lambda|^n \sum_{\sigma \in \hat{K}} d_\sigma \text{tr}(W^\lambda_\sigma(g)(\rho^\lambda_\sigma)^<)(z, k)), \]
where the series converges in \( L^2(G^\times) \).

**Proof.** First consider the element \((0, e) \in G^\times \), where \( e \) is the identity element in \( K \). Then
\[ \text{tr}(W^\lambda_\sigma(g)) = \sum_{\gamma \in \mathbb{Z}_+} \int_{G^\times} g(w, k_2) \langle \rho^\lambda_\sigma(w, k_2)(\phi_\gamma \otimes e^j), \phi_\gamma \otimes e^j \rangle dw dk_2. \]
By (2.1), the above expression takes the form
\[
\text{tr}(W^j_\sigma(g)) = \sum_{\gamma \in \mathbb{Z}_+^n} \sum_{|\alpha| = |\gamma|} \int_{\mathcal{K}} \eta^j_\gamma(\kappa_2) \int_{\mathbb{C}^n} g(w, \kappa_2) \Phi^j_{\alpha\gamma}(w) q^j(\kappa_2) dw d\kappa_2,
\]
where \( \Phi^j_{\alpha\gamma}(w) = \langle \pi_\lambda(w) \phi^j_\alpha, \phi^j_\gamma \rangle \). Therefore by the Peter–Weyl theorem (inversion) for the compact groups, we derive that
\[
\sum_{\sigma \in \hat{K}} d_\sigma \text{tr}(W^j_\sigma(g)) = \sum_{\gamma \in \mathbb{Z}_+^n} \sum_{|\alpha| = |\gamma|} \int_{\mathcal{K}} \eta^j_\gamma(e) \int_{\mathbb{C}^n} g(w, e) \Phi^j_{\alpha\gamma}(w) dw = \sum_{\gamma \in \mathbb{Z}_+^n} \int_{\mathbb{C}^n} g(w, e) \Phi^j_{\gamma\gamma}(w) dw.
\]
Further, in view of the inversion formula for the Weyl transform on the Heisenberg group, we infer that
\[
\sum_{\sigma \in \hat{K}} d_\sigma \text{tr}(W^j_\sigma(g)) = (2\pi)^n |\lambda|^{-n} g(0, e). \tag{2.7}
\]

The proof will complete by the following translation argument. For \((z, k_1) \in G^\times\), define a twisted translation operator \(\tau(z, k_1)\) on \(L^1 \cap L^2(G^\times)\) by
\[
(\tau(z, k_1)g)(w, k_2) = e^{i\frac{\lambda}{2} \text{Im}(w \cdot k_1^{-1}z)} g(z + k_1 w, k_1 k_2).
\]
Then
\[
W^j_\sigma(\tau(z, k_1)g) = \int_{G^\times} e^{i\frac{\lambda}{2} \text{Im}(w \cdot k_1^{-1}z)} g(z + k_1 w, k_1 k_2) \rho^j_\sigma(w, k_2) dw d\kappa_2
\]
\[
= \int_{G^\times} e^{i\frac{\lambda}{2} \text{Im}(k_1^{-1}w \cdot k_1^{-1}z)} g(w, k_2) \rho^j_\sigma(k_1^{-1}w - k_1^{-1}z, k_1^{-1}k_2) dw d\kappa_2.
\]
Hence, from the relation (2.5) and using \((\rho^j_\sigma)^*(z, k_1) = \rho^j_\sigma(-k_1^{-1}z, k_1^{-1})\), we get
\[
W^j_\sigma(\tau(z, k_1)g) = (\rho^j_\sigma)^*(z, k_1) W^j_\sigma(g).
\]
Thus, from (2.7), we have
\[
g(z, k_1) = (\tau(z, k_1)g)(0, e) = (2\pi)^{-n} |\lambda|^n \sum_{\sigma \in \hat{K}} d_\sigma \text{tr}(W^j_\sigma(\tau(z, k_1)g))
\]
\[
= (2\pi)^{-n} |\lambda|^n \sum_{\sigma \in \hat{K}} d_\sigma \text{tr}(W^j_\sigma(g)(\rho^j_\sigma)^*(z, k_1)). \qed
\]

For simplicity, we assume \(\lambda = 1\) and denote \(\rho_\sigma(z, k) = \rho^1_\sigma(z, k)\), \(W_\sigma = W^1_\sigma\). Further, throughout this section, we shall assume \(A\) is a Lebesgue measurable subset of \(\mathbb{C}^n\) with finite measure. Next, we define a set of orthogonal projection operators which is core in formulating a problem analogous to Benedicks–Amrein–Berthier type theorem.

Let \(\sigma \in \hat{K}\) and \(B_{N_\sigma}\) be an \(N_\sigma\) dimensional subspace of \(\mathcal{H}^2_\sigma\). Then, there exists an orthonormal basis \(\{\psi^l_\sigma : l \in \mathbb{N}\}\) of \(\mathcal{H}^2_\sigma\) such that \(B_{N_\sigma} = \text{span}\{\psi^l_\sigma : 1 \leq l \leq N_\sigma\}\). Define an orthogonal projection
$P_{N_{\sigma}}$ of $H_0^2$ onto $R(P_{N_{\sigma}}) = B_{N_{\sigma}}$. Consider a finite subset $J$ of $\hat{K}$ and let $N = \max_{\sigma \in J} N_{\sigma}$. Now, we define a pair of orthogonal projections $E_A$ and $F_N$ of $L^2(G^\times)$ by

$$E_A g = \chi_{A \times K} g \quad \text{and} \quad W_{\sigma}(F_N g) = \begin{cases} P_{N_{\sigma}} W_{\sigma}(g) & \text{if } \sigma \in J, \\ 0 & \text{otherwise,} \end{cases}$$

where $\chi_{A \times K}$ denotes the characteristic function of $A \times K$. Then, it is easy to see that $R(E_A) = \{ g \in L^2(G^\times) : g = g \chi_{A \times K} \}$ and

$$R(F_N) = \{ g \in L^2(G^\times) : R(W_{\sigma}(g)) \subseteq B_{N_{\sigma}} \text{ for } \sigma \in J \text{ and } R(W_{\sigma}(g)) = 0 \text{ for } \sigma \notin J \}.$$

Now, we derive a key lemma that enables us to recognize $E_A F_N$ as an integral operator.

**Lemma 2.5.** The operator $E_A F_N$ is an integral operator on $L^2(G^\times)$.

**Proof.** Let $g \in L^2(G^\times)$. By inversion formula (2.6), we get

$$(F_N g)(z, k_1) = \sum_{\sigma \in K} a_{\sigma} \text{tr}(W_{\sigma}(F_N g) \rho_{\sigma}^*(z, k_1)) = \sum_{\sigma \in J} a_{\sigma} \text{tr}(P_{N_{\sigma}} W_{\sigma}(g) \rho_{\sigma}^*(z, k_1)) = \sum_{\sigma \in J} a_{\sigma} \int_K \int_{C^n} g(w, k_2) \text{tr}(P_{N_{\sigma}} \rho_{\sigma}(w, k_2) \rho_{\sigma}^*(z, k_1)) \, dw \, dk_2,$$

where $a_{\sigma} = (2\pi)^{-n} d_{\sigma}$. Hence,

$$(E_A F_N g)(z, k_1) = \chi_{A \times K}(z, k_1)(F_N g)(z, k_1) = \int_K \int_{C^n} g(w, k_2) \mathcal{K}((z, k_1), (w, k_2)) \, dw \, dk_2,$$

where $\mathcal{K}((z, k_1), (w, k_2)) = \sum_{\sigma \in J} a_{\sigma} \chi_{A \times K}(z, k_1) \text{tr}(P_{N_{\sigma}} \rho_{\sigma}(w, k_2) \rho_{\sigma}^*(z, k_1))$. \hfill \square

Further, the integral operator $E_A F_N$ is a Hilbert–Schmidt operator and satisfies the following dimension condition.

**Lemma 2.6.** $E_A F_N$ is a Hilbert–Schmidt operator with $\|E_A F_N\|_{HS}^2 \leq c_J m(A) N$, where $c_J = (2\pi)^n m(K)|J| \sum_{\sigma \in J} a_{\sigma}^2 < \infty$.

**Proof.** From Lemma 2.5, it follows that

$$\|E_A F_N\|_{HS}^2 = \int_{G^\times} \int_{G^\times} |\mathcal{K}((z, k_1), (w, k_2))|^2 \, dw \, dk_2 \, dz \, dk_1 = \int_{G^\times} \int_{G^\times} \left| \sum_{\sigma \in J} a_{\sigma} \chi_{A \times K}(z, k_1) \text{tr}(P_{N_{\sigma}} \rho_{\sigma}(w, k_2) \rho_{\sigma}^*(z, k_1)) \right|^2 \, dw \, dk_2 \, dz \, dk_1.$$
If the cardinality of $J$ is denoted by $|J|$, from Hölder’s inequality, we get

$$
\|E_{A,F_N}\|_{HS}^2 \leq |J| \sum_{\sigma \in J} a_\sigma^2 \int_{G^x} |\chi_{A \times K}(z,k_1)|^2 \int_{G^x} |\text{tr}(P_{N_\sigma} \rho_\sigma(w,k_2) \rho_\sigma^*(z,k_1))|^2 dwdk_2 dzdk_1.
$$

(2.8)

Now, we shall simplify the inner integral

$$
\int_{G^x} |\text{tr}(P_{N_\sigma} \rho_\sigma(w,k_2) \rho_\sigma^*(z,k_1))|^2 dwdk_2 = \int_{G^x} \left| \sum_{1 \leq l \leq N_\sigma} \langle \rho_\sigma(w,k_2) \rho_\sigma^*(z,k_1) \psi_l^\sigma, \psi_l^\sigma \rangle \right|^2 dwdk_2
$$

$$
= \int_{G^x} \left| \sum_{1 \leq l \leq N_\sigma} \langle \rho_\sigma(w,k_2) \eta_l^\sigma, \psi_l^\sigma \rangle \right|^2 dwdk_2,
$$

where $\eta_l^\sigma = \rho_\sigma^*(z,k_1) \psi_l^\sigma \in H_\sigma^2$. The above integral can be written in terms of Fourier–Wigner transform by

$$
\int_{G^x} \left| \sum_{1 \leq l \leq N_\sigma} \langle \rho_\sigma(w,k_2) \eta_l^\sigma, \psi_l^\sigma \rangle \right|^2 dwdk_2 = (2\pi)^n \int_{G^x} \left| \sum_{1 \leq l \leq N_\sigma} V_{\eta_l^\sigma}^l(w,k_2) \right|^2 dwdk_2
$$

$$
= (2\pi)^n \sum_{1 \leq l_1, l_2 \leq N_\sigma} \int_{G^x} V_{\eta_{l_1}^\sigma}^l(w,k_2) \overline{V_{\eta_{l_2}^\sigma}^l(w,k_2)} dwdk_2.
$$

Since,

$$
\langle \eta_{l_1}^\sigma, \eta_{l_2}^\sigma \rangle = \langle \rho_\sigma^*(z,k_1) \psi_{l_1}^\sigma, \rho_\sigma^*(z,k_1) \psi_{l_2}^\sigma \rangle = \langle \psi_{l_1}^\sigma, \psi_{l_2}^\sigma \rangle = \delta_{l_1,l_2},
$$

by Lemma 2.1, we have

$$
\int_{G^x} |\text{tr}(P_{N_\sigma} \rho_\sigma(w,k_2) \rho_\sigma^*(z,k_1))|^2 dwdk_2 = (2\pi)^n \sum_{1 \leq l_1, l_2 \leq N_\sigma} \langle \eta_{l_1}^\sigma, \eta_{l_2}^\sigma \rangle \langle \psi_{l_1}^\sigma, \psi_{l_2}^\sigma \rangle = (2\pi)^n N_\sigma.
$$

(2.9)

Thus, from (2.8) and (2.9), we get

$$
\|E_{A,F_N}\|_{HS}^2 \leq \frac{(2\pi)^n m(A)m(K)N |J| \sum_{\sigma \in J} a_\sigma^2}{\sum_{\sigma \in J} a_\sigma^2} < \infty,
$$

where $N = \max_{\sigma \in J} N_\sigma$ as defined above.
We need the following result that describes an interesting property of Lebesgue measurable sets \([1]\). Denote \(wA = \{ z \in \mathbb{C}^n : z - w \in A \}\).

**Lemma 2.7** \([1]\). Let \(B\) be a measurable set in \(\mathbb{C}^n\) with \(0 < m(B) < \infty\). If \(B_0\) is a measurable subset of \(B\) with \(m(B_0) > 0\), then for each \(\varepsilon > 0\) there exists \(w \in \mathbb{C}^n\) such that

\[
m(B) < m(B \cup wB_0) < m(B) + \varepsilon.
\]

We also need the following basic fact about the orthogonal projection, which help in deciding the disjointness of the projections \(E_A\) and \(F_N\) while \(m(A) < \infty\).

For given orthogonal projections \(E\) and \(F\) of a Hilbert space \(\mathcal{H}\), let \(E \cap F\) denote the orthogonal projection of \(\mathcal{H}\) onto \(R(E) \cap R(F)\). Then

\[
\|E \cap F\|_{HS}^2 = \dim R(E \cap F) \leq \|EF\|_{HS}^2. \quad (2.10)
\]

Let \(F_N^\perp = I - F_N\), and \(A^c\) be the complement of \(A\).

**Proposition 2.8.** Let \(A\) be a measurable subset of \(\mathbb{C}^n\) of finite Lebesgue measure. Then the projection \(E_A \cap F_N = 0\).

**Proof.** Assume toward a contradiction that there exists a non-zero function \(g\) in \(R(E_A \cap F_N)\). Then \(R(W_\sigma(g)) \subseteq B_{N_\sigma}\) for \(\sigma \in J\) and \(R(W_\sigma(g)) = 0\) for \(\sigma \in \hat{K} \setminus J\). Consider \(A_0 = \{ z \in A : \exists\) a positive measure set \(K_z \subseteq K\) with \(g(z,k) \neq 0, \forall k \in K_z\}. Then \(0 < m(A_0) < \infty\). Let \(g_0(z,k) = \chi_{A_0}(z)g(z,k)\). Thus \(g = g_0\) a.e. and hence \(g_0 \in R(E_A \cap F_N)\). Choose \(s \in \mathbb{N}\) such that \(s > 2c_J m(A_0)N\). Now, we construct an increasing sequence of sets \(\{A_l : l = 1, \ldots, s\}\). Using Lemma 2.7 with \(\varepsilon = \frac{1}{2c_J N}, B_0 = A_0\) and \(B = A_{l-1}\), there exists \(w_l \in \mathbb{C}^n\) such that

\[
m(A_{l-1}) < m(A_{l-1} \cup w_l A_0) < m(A_{l-1}) + \frac{1}{2c_J N}.
\]

Denote \(A_l = A_{l-1} \cup w_l A_0\). Then from (2.10), we get

\[
\dim R(E_A \cap F_N) \leq c_J m(A_s)N \leq \left\{ m(A_0) + \frac{s}{2c_J N} \right\} c_J N < s. \quad (2.11)
\]

On the other hand, we construct \(s + 1\) linearly independent functions in the space \(R(E_A \cap F_N)\), after verifying \(R(F_N)\) is a twisted translation invariant space.

Let \(g_l(z,k) = e^{\frac{i}{2} \text{Im}(z,\bar{w}_l)} g_0(z - w_l, k)\). Then for \(\eta^\sigma \in \mathcal{H}_\sigma^2\) and \(p > N_\sigma\), where \(\sigma \in J\), we have

\[
\langle W_\sigma(g_l)\eta^\sigma, \psi_p^\sigma \rangle = \int_{G^\times} g_l(z,k)(\rho_\sigma(z,k)\eta^\sigma, \psi_p^\sigma)dzdk
\]

\[
= \int_{G^\times} e^{\frac{i}{2} \text{Im}(z,\bar{w}_l)} g_0(z - w_l,k)(\rho_\sigma(z,k)\eta^\sigma, \psi_p^\sigma)dzdk
\]

\[
= \int_{G^\times} e^{\frac{i}{2} \text{Im}(z,\bar{w}_l)} g_0(z,k)(\rho_\sigma(z + w_l,k)\eta^\sigma, \psi_p^\sigma)dzdk.
\]
Since $\rho_\sigma(z, k)\rho_\sigma(k^{-1}w, e) = e^{i\text{Im}(z\bar{w})}\rho_\sigma(z + w, k)$, where $e$ is the identity element in $K$, we get

$$\langle W_\sigma(g_l)\eta_\sigma, \psi_\sigma^p \rangle = \int_{G^x} g_0(z, k)\langle \rho_\sigma(z, k)\rho_\sigma(k^{-1}w_l, e)\eta_\sigma, \psi_\sigma^p \rangle dzdk$$

$$= \int_{G^x} g_0(z, k)\langle \rho_\sigma(z, k)\xi_\sigma, \psi_\sigma^p \rangle dzdk$$

$$= \langle W(g_0)\xi_\sigma, \psi_\sigma^p \rangle = 0.$$

Thus, $R(W_\sigma(g_l)) \subseteq B_{N_\sigma}$ for $\sigma \in J$. Similarly, for $\sigma \notin J$, it can be shown that $R(W_\sigma(g_l)) = 0$. Since $A_m = A_0 \cup w_1A_0 \cup \ldots \cup w_mA_0$ and $g_l = 0$ on $w_lA_0 \times K$, we have $E_{A_m}g_l = g_l$ for $l = 0, 1, \ldots, m$. Furthermore, $E_{A_m \setminus A_{m-1}}g_l = 0$ for $l = 0, \ldots, m-1$ and by the definition of $A_0$, it follows that $E_{A_m \setminus A_{m-1}}g_m \neq 0$ on a set of positive measure. Therefore, it shows that $g_m$ is not a linear combination of $g_0, \ldots, g_{m-1}$. Hence, $g_0, \ldots, g_s$ are $s + 1$ linearly independent functions in $R(E_{A_s} \cap F_N)$ that contradicts (2.11). This completes the proof.

This leads to the following version of the Benedicks–Amrein–Berthier theorem for the Weyl transform.

**Proposition 2.9.** Let $g \in L^1(G^x)$ and $\{(z, k) \in G^x : g(z, k) \neq 0\} \subseteq A \times K$, where $m(A) < \infty$. Suppose $J$ be a finite subset of $\hat{K}$. If $W_\sigma(g)$ is a finite rank operator for each $\sigma \in J$ and $W_\sigma(g) = 0$ for $\sigma \in \hat{K} \setminus J$, then $g = 0$.

If $g \in L^1(G^x)$, by the Plancherel formula (2.3), assumed rank condition implies $g \in L^2(G^x)$. Further, for $g \in L^2(G^x)$, proof of Proposition 2.9 follows from Proposition 2.8.

In the Heisenberg motion group, in terms of Fourier transform, the above result takes the following form.

**Theorem 2.10.** Let $f \in L^1(G)$ and $\{(z, t, k) \in G : g(z, t, k) \neq 0\} \subseteq A \times \mathbb{R} \times K$, where $m(A) < \infty$. For each $\lambda \in \mathbb{R}^\ast$, consider a finite subset $J_\lambda$ of $\hat{K}$. If for each $\lambda \in \mathbb{R}^\ast$, $\hat{f}(\lambda, \sigma)$ has finite rank for $\sigma \in J_\lambda$ and $\hat{f}(\lambda, \sigma) = 0$ for $\sigma \in \hat{K} \setminus J_\lambda$, then $f = 0$.

**Remark 2.11.** Note that, for the Heisenberg motion group, the QUP condition (1.1) is equivalent to

$$\int_{\mathbb{R} \setminus \{0\}} \left( \sum_{\sigma \in \hat{K}} d_\sigma \text{rank}\hat{f}(\lambda, \sigma) \right)|\lambda|^n d\lambda < \infty. \quad (2.12)$$

Therefore, the rank condition in Theorem 2.10 will not satisfy (2.12), and hence Theorem 2.10 improves the QUP in that perspective. Further, the assumption in Theorem 2.10 that, for each $\lambda \in \mathbb{R}^\ast$, $\hat{f}(\lambda, \sigma) = 0$ except finitely many $\sigma$, looks natural in view of (2.12).

As a consequence of Proposition 2.9, we obtain the following analogue result in terms of the Fourier–Wigner decomposition. For this, we recall the Fourier–Wigner decomposition. Let $g \in L^2(G^x)$. By Proposition 2.2, we get $g = \bigoplus_{\sigma \in \hat{K}} g_\sigma$. 


Proposition 2.12. Let \( g \in L^2(G^\times) \) and \( \{ (z, k) \in G^\times : g_\sigma(z, k) \neq 0 \} \subseteq A_\sigma \times K \), where \( m(A_\sigma) < \infty \), whenever \( \sigma \in \hat{K} \). If \( W_\sigma(g) \) is a finite rank operator for each \( \sigma \), then \( g = 0 \).

Proof. For \( \varphi, \psi \in \mathcal{H}_\sigma^2 \), we have

\[
\langle W_\sigma(g) \varphi, \psi \rangle = \int_K \int_{\mathbb{C}^n} g(z, k) \langle \rho_\sigma(z, k) \varphi, \rho_\sigma(z, k) \psi \rangle \, dz \, dk
\]

\[
= \int_K \int_{\mathbb{C}^n} g_\sigma(z, k) \langle \rho_\sigma(z, k) \varphi, \rho_\sigma(z, k) \psi \rangle \, dz \, dk
\]

\[
= \langle W_\sigma(g_\sigma) \varphi, \psi \rangle.
\]

Hence, for \( \sigma_o \in \hat{K} \), \( R(W_{\sigma_o}(g_{\sigma_o})) = R(W_{\sigma_o}(g)) \) be a finite-dimensional subspace of \( \mathcal{H}_{\sigma_o}^2 \) and \( R(W_{\sigma}(g_{\sigma_o})) = 0 \) for \( \sigma(\neq \sigma_o) \in \hat{K} \). Thus by Proposition 2.9, we get \( g_{\sigma_o} = 0 \). Since \( \sigma_o \in \hat{K} \) is arbitrary, we infer that \( g = 0 \).

Remark 2.13.

(a) For \( U(n) \)-bi-invariant function, the rank condition in Proposition 2.12 is obviously true. Thus support condition is enough for the conclusion. In dimension one, it can argue by the fact that each Fourier–Wigner piece will be of the form \( \tilde{\tilde{g}}_{\alpha} = \tilde{\alpha} \rtimes \Phi_{\alpha \alpha} \), where \( \tilde{\alpha} \rtimes (z) = \tilde{\alpha}(-z) \), which is real analytic. Hence it can not have finite support.

(b) After a close examination of the utility of \( U(n) \) to obtain the decomposition of \( L^2(G^\times) \) as in Proposition 2.2, we observed that \( U(n) \)-invariance is nevermore used except while realizing the irreducible action of metaplectic repression \( \mu_\lambda \) on \( P_m^\lambda \). If we consider a compact subgroup \( K \) of \( U(n) \) which makes \( (\mathbb{H}^n \rtimes K, K) \) a Gelfand pair, then \( P_m^\lambda \) will be decomposed into finitely many irreducible pieces according to the metaplectic representation \( \mu_\lambda \) of \( K \). To avoid further complexity in the calculation, we have preferred to prove the results for the Gelfand pair \( (\mathbb{H}^n \rtimes U(n), U(n)) \) instead of \( (\mathbb{H}^n \rtimes K, K) \).

Strong annihilating pair: Let \( A \subseteq \mathbb{R} \) and \( \Sigma \subseteq \mathbb{R} \) be measurable sets. Then the pair \((A, \Sigma)\) is called weak annihilating pair if \( \text{supp} f \subseteq A \) and \( \text{supp} \hat{f} \subseteq \Sigma \), implies \( f = 0 \). The pair \((A, \Sigma)\) is called strong annihilating pair if there exists a positive number \( C = C(A, \Sigma) \) such that

\[
\|f\|_2^2 \leq C\left( \|f\|_{L^2(A^c)}^2 + \|\hat{f}\|_{L^2(\Sigma^c)}^2 \right) \tag{2.13}
\]

for every \( f \in L^2(\mathbb{R}) \). It is obvious that every strong annihilating pair is a weak annihilating pair. In [3], Benedicks had proved that \((A, \Sigma)\) is a weak annihilating pair when \( A \) and \( \Sigma \) both have finite measure. In [1], Amrein-Berthier had proved that \((A, \Sigma)\) is a strong annihilating under the identical assumption as in [3].

Since, Fourier transform on the Heisenberg motion group is an operator valued function, we could not expect a similar conclusion as (2.13). However, we can adequately describe a strong annihilating pair.

Definition 2.14. For each \( \sigma \in \hat{K} \), let \( A_\sigma \) be a measurable subset of \( \mathbb{C}^n \) and \( S_\sigma \) be a closed subspace of \( \mathcal{H}_\sigma^2 \). By abuse of notations, denote \( A = \{ A_\sigma \}_{\sigma \in \hat{K}} \) and \( S = \{ S_\sigma \}_{\sigma \in \hat{K}} \). We say that the pair \((A, S)\) is
a strong annihilating pair for the Weyl transform, if there exist positive numbers $C_{\sigma} = C_\sigma(A_\sigma, S_\sigma)$ such that for every $g \in L^2(C^n \times K)$,

$$
\| g \|_2^2 \leq \sum_{\sigma \in \hat{K}} C_{\sigma} \left( \| g_\sigma \|_{L^2(A_\sigma \times K)}^2 + \| P_{S_\sigma}^\perp W_\sigma^\lambda (g) \|_{HS}^2 \right),
$$

(2.14)

where functions $g_\sigma$ are the orthogonal pieces of $g$, according to Proposition 2.2, and $P_{S_\sigma}$ is the projection of $H^2_\sigma$ onto $S_\sigma$.

**Proposition 2.15.** If $A_\sigma$ has finite measure and dimension of $S_\sigma$ is finite for each $\sigma \in \hat{K}$, then $(A, S)$ is a strong annihilating pair.

To prove Proposition 2.15, we need the following basic result from [9, p. 88].

**Lemma 2.16** [9]. Let $P$ and $Q$ be two orthogonal projections on a complex Hilbert space $H$. Then $\|PQ\| < 1$ if and only if there exists a positive number $C$ such that for each $x \in H$

$$
\| x \|_2^2 \leq C \left( \| P^\perp x \|_2^2 + \| Q^\perp x \|_2^2 \right).
$$

Proof of Proposition 2.15. For $\sigma_o \in \hat{K}$, let $S_{\sigma_o}$ be a finite-dimensional subspace of $H^2_{\sigma_o}$ and $A_{\sigma_o}$ be any subset of $C^n$ with finite measure. Then, recall the set of projections for $J = \{ \sigma_0 \}$, $E_{A_{\sigma_o}} g(z, k) = \chi_{A_{\sigma_o}}(z) g(z, k)$ and $W_\sigma^\lambda (F_{S_\sigma} g) = P_{S_\sigma}^\perp W_\sigma^\lambda (g)$, $W_\sigma^\lambda (F_{S_\sigma} g) = 0$ for $\sigma \neq \sigma_o$. Now, by Lemma 2.6 $E_{A_{\sigma_o}} F_{S_\sigma}$ is a compact operator and from Proposition 2.8, we have $E_{A_{\sigma_o}} \cap F_{S_\sigma} = 0$. Therefore, we must have $\|E_{A_{\sigma_o}} F_{S_\sigma}\| < 1$. Since $W_\sigma^\lambda (F_{S_\sigma}^\perp g) = P_{S_\sigma}^\perp W_\sigma^\lambda (g)$ and $W_\sigma^\lambda (F_{S_\sigma}^\perp g) = W_\sigma^\lambda (g)$ for $\sigma \neq \sigma_o$, by Lemma 2.16, there exists $C_{\sigma_o} = C_\sigma(A_\sigma, S_\sigma) > 0$ such that

$$
\| g \|_2^2 \leq C_{\sigma_o} \left( \| g \|_{L^2(A_\sigma \times K)}^2 + d_{\sigma_o} \| P_{S_\sigma}^\perp W_\sigma^\lambda (g) \|_{HS}^2 + \sum_{\sigma \neq \sigma_o} d_{\sigma} \| W_\sigma^\lambda (g) \|_{HS}^2 \right),
$$

for all $g \in L^2(C^n \times K)$. In particular, for any $g_{\sigma_o} \in \gamma_{\sigma_o}^2$ we have

$$
\| g_{\sigma_o} \|_2^2 \leq C_{\sigma_o} \left( \| g_{\sigma_o} \|_{L^2(A_{\sigma_o} \times K)}^2 + \| P_{S_{\sigma_o}}^\perp W_{\sigma_o}^\lambda (g_{\sigma_o}) \|_{HS}^2 \right).
$$

(2.15)

For any $g \in L^2(C^n \times K)$, by Proposition 2.2, $g = \bigoplus_{\sigma \in \hat{K}} g_\sigma$. Since $\sigma_o$ is arbitrary, from (2.15) we can conclude that $(A, S)$ is a strong annihilating pair, whenever $A_\sigma$ has finite measure and dimension of $S_\sigma$ is finite for each $\sigma \in \hat{K}$. \qed

**Remark 2.17.** Consider the hypothesis of Proposition 2.9. There exist two large classes of functions of which one satisfies the support condition, and the other satisfies the rank condition. However, in Proposition 2.12, it is not clear which functions will fulfill such a support condition. In other words, whether the assumption of finite support condition in each piece is strong enough for the conclusion of Proposition 2.12. We know this is true for the $U(n)$-bi-invariant functions. However, we reached out to a quantitative estimate (2.14, 2.15) of Proposition 2.12, which is true for all square integrable functions, irrespective of their support.
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