Special functions, KZ type equations
and Representation theory

Notes of a course given at MIT during the spring of 2002 by

Alexander Varchenko

Notes taken by:
Josh Scott: Lectures 1,2,3
Matías Graña: Lectures 4,5,6
Igor Mencattini: Lectures 7,8,9,10,11,12
This paper is a set of lecture notes of my course “Special functions, KZ type equations, and representation theory” given at MIT during the spring semester of 2002. The notes were prepared by J.Scott (Lectures 1-3), M.Graña (Lectures 4-6), and I.Mencattini (Lectures 7-12).

The notes do not contain new results, and are an exposition (mostly without proofs) of various published results in this area, illustrated by the simplest nontrivial examples. Some references are given at the end of each lecture, but their list is not complete; the reader is referred to the original articles for proofs and for complete references.
Contents

Lecture 1.
1. Setting of Knizhnik–Zamolodchikov equation 5
2. Solutions 8

Lecture 2.
1. Hyperplane Arrangements 12
2. Classical Hypergeometric Series 14
3. Properties of the Hypergeometric series (Gauss 1812) 14

Lecture 3.
1. Quantum Groups 19
2. Braid Groups 20
3. Quantum Singular Vectors 21

Lecture 4.
1. Monodromy of KZ equations 23
2. Topological monodromy 23
3. $R$-matrix 23
4. Relationship 24
5. Remark on integration cycles 25
6. Example: the Selberg integral 25
7. Connection with finite reflection groups 26

Lecture 5.
1. Determinant formulas 29
2. Resonances 32
3. Dynamical equations 33

Lecture 6.
1. Dynamical equations 35
2. Trigonometric $r$-matrix 36
3. Hypergeometric solutions of trigonometric equations 36
4. Quasiclassical asymptotics of hypergeometric functions and Bethe ansatz 38
5. Asymptotic solutions of KZ and eigenvectors of $H_i(z) = \sum_{j \neq i} \frac{\Omega^{(ij)}}{z_i - z_j}$ 39

Lecture 7.

Lecture 8.
1. Differential Equations with Regular Singular Points 49
2. Solutions at a neighborhood of an ordinary point  
3. Solutions at a neighborhood of a regular singular point  
4. Fuchsian Equations  
5. Enumerative Algebraic Geometry

Lecture 9.  
1. Asymptotic Solutions  
2. Special Solutions and Conformal Blocks  
3. Theta functions of level $\kappa$.  
4. Integral Formulas

Lecture 10.  
1. Modular Transformations of Conformal Blocks  
2. Macdonald Polynomials  
3. Trace of Intertwining Operators  
4. $q$-KZ Equations  
5. Yangian $Y(gl_2)$ and Rational R-matrices

Lecture 11.  
1. Classical KZ-system  
2. Geometric Construction of Quantized KZ Equation  
3. $q$-KZ Equation: $sl_2$ case

Lecture 12.  
1. Trigonometric case  
2. Hypergeometric Pairing  
3. Quantization of the Drinfeld-Kohno theorem

Bibliography
1. Setting of Knizhnik–Zamolodchikov equation

Given $m = (m_1, \ldots, m_n)$ and $z = (z_1, \ldots, z_n)$ fixed vectors in $\mathbb{C}^n$ and given fixed $\kappa \in \mathbb{C}$ we define the following multi-valued function and associated set of 1-forms:

$$\Phi = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{m_i m_j} \prod_{l=1}^n (t - z_l)^{-m_l}$$

$$\eta_j = \Phi \frac{dt}{t - z_j} \quad j = 1, \ldots, n$$

The 1-forms $\eta_j$ are closed and, moreover, are cohomologically dependent as they satisfy the relation

$$m_1 \eta_1 + \cdots + m_n \eta_n = -\kappa d\phi.$$

Let $G = \pi_1(\mathbb{C} - \{z_1, \ldots, z_n\}, \ast)$ and choose any contour $\gamma$ emanating from the base point $\ast$ with the property that $[\gamma]$ resides in the commutator subgroup $[G, G]$. Since $[\gamma] \in [G, G]$ one can pull back the multi-valued 1-form $\eta_j$ to a single valued 1-form on $S^1$ - after choosing a branch of $\eta_j$ about the base point $\ast$. In this manner the contour integral $\int_\gamma \eta_j$ becomes well defined and its value will only depend on $\gamma$’s homotopy class. Define $I^\gamma$ to be the following holomorphic vector valued function:

$$I^\gamma = (I_1, \ldots, I_n) = \left( \int_\gamma \eta_1, \ldots, \int_\gamma \eta_n \right)$$

**Theorem 1.1.** $I^\gamma$ satisfies the following system of differential equations:

$$\frac{\partial I}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} I \quad \text{for} \quad i = 1, \ldots, n \quad \text{where}$$

$$\Omega_{ij} = \frac{m_i m_j}{z_i - z_j}.$$
\[
\Omega_{ij} = \begin{pmatrix}
i \\
\vdots \\
(m_i - 2) \frac{m_i}{2} \\
\vdots \\
\vdots \\
m_i \\
\vdots \\
m_j \\
\vdots \\
m_j \\
\vdots \\
\vdots \\
(\frac{m_j(m_j - 2)}{2}) \\
\vdots \\
\vdots
\end{pmatrix}
\]

All other diagonal entries are \( \frac{m_i m_i}{2} \) and the remaining off-diagonal entries are all zero.

**Proof.** For example, in the case of \( i = 1 \) and the component \( \frac{\partial I_{ij}^\gamma}{\partial z_1} \) of \( \frac{\partial I_{ij}^\gamma}{\partial z_1} \), we have

\[
\frac{\partial I_{ij}^\gamma}{\partial z_1} = \int \gamma \frac{\partial \eta}{\partial z_1} dt = \int \gamma \left( \sum_{j \neq 1}^{n} \frac{m_i m_j}{2\kappa} \frac{1}{z_1 - z_j} + \frac{m_1}{\kappa} \frac{1}{t - z_1} \right) \frac{\Phi}{t - z_2} dt.
\]

Using the identity \( \frac{1}{t - z_1} + \frac{1}{t - z_2} = \frac{1}{z_1 - z_2} \left( \frac{1}{t - z_1} - \frac{1}{t - z_2} \right) \) we obtain the desired result

\[
\frac{\partial I_{ij}^\gamma}{\partial z_1} = \int \gamma \left( \frac{m_1(m_2 - 2)}{2\kappa} \frac{1}{z_1 - z_2} \eta_2 + \sum_{j \notin \{1, 2\}} \frac{m_1 m_j}{2\kappa} \frac{1}{z_1 - z_j} \eta_2 + \frac{m_1}{\kappa} \frac{1}{z_1 - z_2} \eta_1 \right).
\]

\[\square\]

We will now describe a generalization of the above system of differential equations which arises in the context \( \mathfrak{sl}_2(\mathbb{C}) \)-representations.

Recall that \( \mathfrak{sl}_2(\mathbb{C}) \) is the Lie algebra with generators

\[
e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]

satisfying the familiar commutation relations

\[
[h, e] = 2e, \quad [e, f] = h, \quad [h, f] = -2f.
\]

The element \( \Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h \in \mathfrak{g} \otimes \mathfrak{g} \) is called the Casimir element and for all \( x \in \mathfrak{sl}_2(\mathbb{C}) \) it satisfies the commutation relation \([x \otimes 1 + 1 \otimes x, \Omega] = 0\) inside \( \mathcal{U}(\mathfrak{sl}_2(\mathbb{C})) \otimes \mathcal{U}(\mathfrak{sl}_2(\mathbb{C})) \).

Let \( V_1, \ldots, V_n \) be \( \mathfrak{sl}_2(\mathbb{C}) \)-modules and set \( V = V_1 \otimes \cdots \otimes V_n \). An element \( x \in \mathfrak{sl}_2(\mathbb{C}) \) acts on \( V \) by \( x \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes x \). For indices \( 1 \leq i < j \leq n \) let \( \Omega_{ij} : V \rightarrow V \) be the operator which acts by \( \Omega \) on the \( i \)th and \( j \)th positions and as the identity on all others. For example:
\[ \Omega^{12}(v_1 \otimes \cdots \otimes v_n) = \Omega(v_1 \otimes v_2) \otimes v_3 \cdots \otimes v_n \]

**Definition 1.2.** The KZ equation for a \( V \)-valued function \( \phi(z_1, \ldots, z_n) \) is

\[ \frac{\partial \phi}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega^{ij}}{z_i - z_j} \phi \quad \text{for } i = 1 \ldots n. \]

This equation is defined on \( U = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \} \).

Mention connections with CFT and decorated Riemann surfaces.

A solution \( \phi \) of the KZ equation one can be translated by an element \( x \in \mathfrak{sl}_2(\mathbb{C}) \) to obtain a new \( V \)-valued function \( x \cdot \phi \). It follows from the commutation relation \( [\Omega, x \otimes 1 + 1 \otimes x] = 0 \) that \( x \cdot \phi \) is also a solution of the KZ equation. From this we can conclude that

**Corollary 1.3.** For any \( x \in \mathfrak{sl}_2(\mathbb{C}) \) the KZ equation preserves the eigenspaces of the action of \( x \) on \( V \). In other words, if \( \phi \) is a KZ solution such that \( x \cdot \phi(z_0) = \lambda \phi(z_0) \) for some \( z_0 \in \mathbb{C}^n \) then \( x \cdot \phi(z) = \lambda \phi(z) \) for all \( z \in \mathbb{C}^n \).

For \( m \in \mathbb{C} \), the Verma module \( M_m \) is the infinite dimensional \( \mathfrak{sl}_2(\mathbb{C}) \)-representation with highest weight \( m \) generated by a single vector \( v_m \) where \( hv_m = mv_m \) and \( ev_m = 0 \). The vectors \( f^k v_m \) for \( k = 0, 1, \ldots \) form a basis for \( M_m \). The generators \( e, f, \) and \( h \) act on this basis as indicated below:

\[ f \cdot f^k v_m = f^{k+1} v_m \quad h \cdot f^k v_m = (m - 2k)f^k v_m \quad e \cdot f^k v_m = k(m - k + 1)f^{k-1} v_m \]

If \( m \notin \mathbb{Z}_{\geq 0} \) then \( M_m \) is irreducible, otherwise \( < f^{m+1} v_m > = \mathfrak{sl}_2(\mathbb{C}) \cdot f^{m+1} v_m \) will be an invariant subspace. Let \( L_m = M_m / < f^{m+1} v_m > \). This quotient is irreducible with basis \( v, fv, \ldots, f^m v \).

It is a familiar exercise in theory of \( \mathfrak{sl}_2(\mathbb{C}) \)-representations to check the following tensor decomposition:

\[ L_m \otimes L_l = L_{m+l} \oplus L_{m+l-2} \oplus \cdots \oplus L_{m-l} \]

Let \( m = (m_1, \ldots, m_n) \in \mathbb{C}^n \) and set \( |m| = m_1 + \cdots + m_n \). Let \( M^\otimes m = M_{m_1} \otimes \cdots \otimes M_{m_n} \). For \( J = (j_1, \ldots, j_n) \in \mathbb{Z}_{\geq 0} \) let \( f_J v = f_1^{j_1} v_m_1 \otimes \cdots \otimes f_n^{j_n} v_m_n \) where \( v_{m_i} \) is the primitive vector in \( M_{m_i} \). The vectors \( f_J v \) form a basis for the module \( M^\otimes m \).

Now \( h \cdot f_J v = \left( |m| - 2|J| \right) f_J v \). Define \( M^\otimes m[\lambda] = \{ v \in M^{\otimes m} \mid h \cdot v = \lambda v \} \) and \( \text{Sing} M^\otimes m[\lambda] = \{ v \in M^{\otimes m}[\lambda] \mid e \cdot v = 0 \} \). The singular eigenspaces \( \text{Sing} M^\otimes m[\lambda] \) generate the entire module \( M^{\otimes m} \) and, in view of Corollary 1, it follows that it is enough to produce KZ solutions for the singular eigenspaces only.
2. Solutions

Case 1: Sing $M^{\otimes \mathbb{m}}[|m|]$

Sing $M^{\otimes \mathbb{m}}[|m|]$ is spanned by the single vector $v_{m_1} \otimes \cdots \otimes v_{m_n}$. If $\phi$ is a KZ solution residing in the $|m|$-eigenspace then it is of the form $\phi = I_0(z_1, \ldots, z_n)v_{m_1} \otimes \cdots \otimes v_{m_n}$ where $I_0$ is a scalar valued function. In an effort to compute the right hand side of the KZ equation we obtain:

$$\Omega^{ij}(v_{m_1} \otimes \cdots \otimes v_{m_n}) = \frac{m_i m_j}{2} v_{m_1} \otimes \cdots \otimes v_{m_n}$$

In this case the KZ equation reads as:

$$\frac{\partial I_0}{\partial z_i} = \sum_{j \neq i} \frac{m_i m_j}{2\kappa} \frac{I_0}{z_i - z_j} \quad \text{for } i = 1 \ldots n$$

with solution given by

$$I_0 = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{\frac{m_i m_j}{2\kappa}}$$

In view of Corollary 1 we can conclude that $f^k \cdot I_0 v_{m_1} \otimes \cdots \otimes v_{m_n} = I_0 f^k \cdot v_{m_1} \otimes \cdots \otimes v_{m_n}$ is also a KZ solution for any $k \in \mathbb{Z}_{\geq 0}$.

Case 2: Sing $M^{\otimes \mathbb{m}}[|m| - 2]$

This eigenspace is

$$\text{Sing } M^{\otimes \mathbb{m}}[|m| - 2] = \left\{ w = \sum_{l=1}^{m} t_l v_{m_1} \otimes \cdots \otimes f \cdot v_{m_l} \otimes \cdots \otimes v_{m_n} \mid e \cdot w = 0 \right\}$$

where the $t_l$'s are scalars. In this case the condition $e \cdot w = 0$ can be reformulated as $m_1 t_1 + \cdots + m_n t_n = 0$. In order to compute the right hand side of the KZ equation we compute the action of $\Omega^{ij}$ on the basis vectors $v_{m_1} \otimes \cdots \otimes f \cdot v_{m_l} \otimes \cdots \otimes v_{m_n}$. Assume $i < j$. If the basis vector is of the form $v = \cdots \otimes v_{m_i} \otimes \cdots \otimes v_{m_j} \otimes \cdots$ then $\Omega^{ij}(v) = \frac{m_i m_j}{2} v$. If $v$ is of the form $\cdots \otimes f \cdot v_{m_i} \otimes \cdots \otimes v_{m_j} \otimes \cdots$ then

$$\Omega^{ij} v = m_i \left( \cdots \otimes v_{m_i} \otimes \cdots \otimes f \cdot v_{m_j} \otimes \cdots \right) + \frac{(m_i - 2)m_j}{2} \left( \cdots \otimes f \cdot v_{m_i} \otimes \cdots \otimes v_{m_j} \otimes \cdots \right).$$

If the basis vector $v$ is of the form $\cdots \otimes v_{m_i} \otimes \cdots \otimes f \cdot v_{m_j} \otimes \cdots$ then

$$\Omega^{ij} v = m_j \left( \cdots \otimes f \cdot v_{m_i} \otimes \cdots \otimes v_{m_j} \otimes \cdots \right) + \frac{(m_j - 2)m_i}{2} \left( \cdots \otimes v_{m_i} \otimes \cdots \otimes f \cdot v_{m_j} \otimes \cdots \right).$$

From these computations we may surmise that the matrix representing $\Omega^{ij}$ is of the form
\[
\Omega_{ij} = \begin{pmatrix}
i & j \\
\vdots & \vdots \\
(i-(m-2)m_i)/2 & m_j \\
\vdots & \vdots \\
j & m_i \\
\vdots & \vdots \\
(j-(m-2)m_j)/2 & m_i \\
\vdots & \vdots \\
\end{pmatrix}
\]

where all other diagonal entries are \(\frac{m_im_j}{2}\) and the remaining off-diagonal entries are all zero. Of course this is the same matrix we encountered in the beginning of the lecture. We know explicit solutions namely,

\[
\phi^\gamma(z_1, \ldots, z_n) = \sum_{j=1}^{n} \int_{t} \Phi \frac{dt}{t - z_j} v_{m_1} \otimes \cdots \otimes f \cdot v_{m_j} \otimes \cdots \otimes v_{m_n} \quad \text{where}
\]

\[
\Phi = \prod_{i<j} (z_i - z_j)^{\frac{m_im_j}{2k}} \prod_{l=1}^{n} (t - z_l)^{-\frac{m}{k}}.
\]

Applying our translation argument again we may conclude that \(f^k \cdot \phi^\gamma\) is a KZ solution with values in \(M^{\otimes m}\left[|m| - 2 - 2k\right].\)

Case 3: Sing \(M^{\otimes m}\left[|m| - 2k\right]\) where \(k \in \mathbb{Z}_{\geq 0}\)

Define \(\Phi_{k,n}(t, z, m) = \Phi_{k,n}(t_1, \ldots, t_k, z_1, \ldots, z_n, m_1, \ldots, m_n)\) to be the following expression:

\[
\prod_{i<j} (z_i - z_j)^{\frac{m_im_j}{2k}} \prod_{l=1}^{k} \prod_{i=1}^{n} (t - z_l)^{-m_i}.
\]

Comments about discriminantal arrangement of hyperplanes associated to the singularities \(t_i = z_l\) and vanishings \(t_i = t_j\).

Recall that \(M^{\otimes m}\left[|m| - 2k\right]\) has a basis of vectors \(f_J v\) where \(J = (j_1, \ldots, j_n)\) and \(|J| = k\). Associate to each basis vector \(f_J v\) the following rational function:

\[
A_J(t, z) = \frac{1}{j_1! \cdots j_n!} \text{Sym}_t \left[ \prod_{l=1}^{n} \prod_{i=1}^{j_l} \frac{1}{t_{j_1+\cdots+j_{l-1}+i-z_l}} \right] \quad \text{where}
\]

\[
\text{Sym}_t f(t_1, \ldots, t_k) := \sum_{\sigma \in S_k} f(t_{\sigma(1)}, \ldots, t_{\sigma(k)}).
\]
For example: $A(1, 0, \ldots, 0) = \frac{1}{t_1 - z_1}$ and $A(2, 0, \ldots, 0) = \frac{1}{t_1 - z_1} \frac{1}{t_2 - z_2}$ and
\[
A(1, 1, 0, \ldots, 0) = \frac{1}{t_1 - z_1} \frac{1}{t_2 - z_2} + \frac{1}{t_2 - z_1} \frac{1}{t_1 - z_2}.
\]

We define the following $M^\otimes m[|m| - 2k]$-valued functions:
\[
\phi^\gamma(z_1, \ldots, z_n) = \sum_{|J|=k} \left( \int_{\gamma} \Phi_{k,n}(t, z, m) \frac{1}{k!} A_J(t, z) \, dt_1 \wedge \cdots \wedge dt_k \right) f_J.
\]

In the next lecture we shall prove the following theorem:

**Theorem 1.4.** $\phi^\gamma(z) \in \text{Sing } M^\otimes m[|m| - 2k]$ and it is a KZ solution.

**Definition 1.5.** The functions $I_J^\gamma := \int_{\gamma} \Phi_{k,n} \frac{1}{k!} A_J \, dt_1 \wedge \cdots \wedge dt_k$ are called the hypergeometric functions associated with $\text{Sing } M^\otimes m[|m| - 2k]$.

**Claim 1.6.** Let $J + 1_l = (j_1, \ldots, j_l + 1, \ldots, j_n)$. If $|J| = k - 1$ then
\[
\sum_{l=1}^n I_{J+1_l}(z)(j_l + 1)(m_l - j_l) = 0.
\]

This statement is equivalent to the claim that $\phi^\gamma(z) \in \text{Sing } M^\otimes m[|m| - 2k]$.

**References for this chapter:** [V2], [SV].
Lecture 2

We begin by sketching the proof of Theorem 2 from Lecture 1. To do so, we modify the expressions for \( \phi^\gamma \) slightly. For any differential form \( f(t_1, \ldots, t_k) \) let

\[
\text{Ant}_t f(t_1, \ldots, t_k) = \sum_{\sigma \in S_k} (-1)^l(\sigma) f(t_{\sigma(1)}, \ldots, t_{\sigma(k)})
\]

where \( l(\sigma) \) is the length of the permutation \( \sigma \). For \( J = (j_1, \ldots, j_n) \in \mathbb{Z}_{\geq 0}^n \) with \( |J| = j_1 + \cdots + j_n = k \) set

\[
\eta_J = \frac{1}{j_1! \cdots j_n!} \text{Ant}_t \left[ \prod_{l=1}^n \prod_{i=1}^{j_l} \frac{d(t_{j_l+\cdots+j_l-1+i}-z_l)}{t_{j_l+\cdots+j_l-1+i}-z_l} \right].
\]

When all \( z_i \) specialized to a fixed constant then \( \eta_J \) is \( A_J \, dt_1 \wedge \cdots \wedge dt_k \) as defined in Lecture 1. For \( \kappa \in \mathbb{C}^* \) let

\[
\phi^\gamma = \sum_{|J|=k} \int_\gamma \Phi_{k,n}^J \eta_J f_J v.
\]

Claim 2.1. The fact that \( \phi^\gamma(z) \in \text{Sing} M^{\otimes m} \left[ |m| - 2k \right] \) follows from the identities

\[
\left[ \kappa d\left( \Phi_{k,n}^J \eta_J \right) + \sum_{l=1}^n \Phi_{k,n}^J \eta_{J+1_l}(j_l+1)(m_l-j_l) \right]_{\text{all } z_i = \text{constant}} = 0
\]

where \( J = (j_1, \ldots, j_n) \) is any multi-index with \( |J| = k - 1 \).

The validity of this identity is easily checked by expanding \( d(\Phi_{k,n}^J \eta_J) \). To see that this identity implies \( \phi^\gamma \in \text{Sing} M^{\otimes m} \left[ m - 2k \right] \) integrate the above expression over \( \gamma \). Since \( \gamma \) is a cycle the integral \( \int_\gamma \kappa d(\Phi_{k,n}^J \eta_J) \) vanishes. The remaining part of the integral is exactly the coefficient of \( f_J v \) in \( e \cdot \phi^\gamma \).
Claim 2.2. The fact that $\phi^i$ is a KZ solution follows from the identity

$$d\left(\sum_{|J|=k} \Phi^i_{k,n} \eta_J f_J v\right) = \frac{1}{\kappa} \sum_{i<j} \Omega^{ij} \left(\frac{d(z_i - z_j)}{z_i - z_j} \wedge \sum_{|J|=k} \Phi^i_{k,n} \eta_J f_J v\right).$$

The identity is easily checked by expanding the left hand side with $d = dz + dt$. Integrate the above expression over a cycle $\gamma$ in the coordinate $t$. From the identity we obtain

$$\int_\gamma dt \left[\sum_{|J|=k} \Phi^i_{k,n} \eta_J f_J v\right] = -\int_\gamma dz \left[\sum_{|J|=k} \Phi^i_{k,n} \eta_J f_J v\right] + \int_\gamma \frac{1}{\kappa} \sum_{i<j} \Omega^{ij} \left(\frac{d(z_i - z_j)}{z_i - z_j} \wedge \sum_{|J|=k} \Phi^i_{k,n} \eta_J f_J v\right).$$

By Stokes’ Theorem we know that $\int_\gamma dt \left[\sum_{|J|=k} \Phi^i_{k,n} \eta_J f_J v\right]$ vanishes so we may conclude that

$$\int_\gamma dz \left[\sum_{|J|=k} \Phi^i_{k,n} \eta_J f_J v\right] = \int_\gamma \frac{1}{\kappa} \sum_{i<j} \Omega^{ij} \left(\frac{d(z_i - z_j)}{z_i - z_j} \wedge \sum_{|J|=k} \Phi^i_{k,n} \eta_J f_J v\right)$$

or

$$d_z \left[\int_\gamma \sum_{|J|=k} \Phi^i_{k,n} \eta_J f_J v\right] = \frac{1}{\kappa} \sum_{i<j} \Omega^{ij} \left(\frac{d(z_i - z_j)}{z_i - z_j}\right).$$

By isolating the $\frac{\partial}{\partial z_i}$ component in the right hand side of the above expression we obtain the KZ equation as required.

1. Hyperplane Arrangements

The modern theory of arrangements has some origins in Hilbert’s 13th problem. An algebraic function $z = z(x_1, \ldots, x_k)$ is a multi-valued function defined by an equation of the form

$$z^n + P_1(x_1, \ldots, x_k)z^{k-1} + \cdots + P_n(x_1, \ldots, x_k) = 0$$

where the $P_i$’s are polynomials.

**Hilbert’s 13th Problem:** Show that an algebraic function $z$ defined by

$$z^7 + az^2 + bz + c = 0$$

can not be represented as a composition of continuous functions in two variables.

Kolmogorov and Arnold showed that in fact one can find such a two variable decomposition. Despite the Kolmogorov/Arnold result it is believed that the composition is impossible if one restricts to algebraic functions. Arnold wanted to understand why algebraic functions of many variables become more complicated. The idea was to invent invariants of algebraic functions which
would detect when compositions are possible. The simplest characteristic of an algebraic function 
\( z(x_1, \ldots, x_k) \) is its discriminant

\[
\Delta_P = \left\{ x \in \mathbb{C}^k \mid z^n + P_1(x) + \cdots + P_n(x) = 0 \text{ has multiple roots} \right\}.
\]

Any algebraic function is induced from the universal algebraic function 
\( z^n + a_1 z^{n-1} + \cdots + a_n = 0 \) by the map 
\( P : \mathbb{C}^k \to \mathbb{C}^n \) given by 
\( x = (x_1, \ldots, x_k) \mapsto (P_1(x), \ldots, P_n(x)) \). The discriminant can then be expressed as 
\( \Delta_P = P^{-1}(\Delta) \) where

\[
\Delta = \left\{ (a_1, \ldots, a_n) \in \mathbb{C}^n \mid z^n + a_1 z^{n-1} + \cdots + a_n = 0 \text{ has multiple roots} \right\}.
\]

By passing to cohomology we obtain the map \( P^* : H^*(\mathbb{C}^n - \Delta) \to H^*(\mathbb{C}^k - \Delta_P) \). For any \( \alpha \in H^*(\mathbb{C}^n - \Delta) \) one has a characteristic class \( P^* \alpha \) in \( H^*(\mathbb{C}^k - \Delta_P) \) for the algebraic function \( z(x_1, \ldots, x_k) \). One might hope that the cohomology of the compliment of the discriminant might give obstructions to the representability of an algebraic function as a composition. As a first step one might try to describe \( H^*(\mathbb{C}^n - \Delta) \). There is, however, a simplification. The Vieta map \( V : \mathbb{C}^n \to \mathbb{C}^n \) is defined by \( z = (z_1, \ldots, z_n) \mapsto (a_1(z), \ldots, a_n(z)) \) where \( a_k \) is \((-1)^k\) times the \( k \)th elementary symmetric function in the variables \( z_1, \ldots, z_n \). Under the Vieta map the set \( D = \{ (z_1, \ldots, z_n) \mid z_i = z_j \ \exists \ i, j \} \) maps to \( \Delta \). The problem of computing \( H^*(\mathbb{C}^n - \Delta) \) thus reduces to the study of \( H^*(\mathbb{C}^n - D) \) since \( H^*(\mathbb{C}^n - D) \) is the symmetric part of the \( S_n \) action on \( H^*(\mathbb{C}^n - \Delta) \).

Arnold described \( H^*(\mathbb{C}^n - D) \). Consider the 1-forms \( \omega_{ij} = \frac{1}{2\pi i} \frac{d(z_i - z_j)}{z_i - z_j} \). Clearly \( \omega_{ij} = \omega_{ji} \). An easy exercise of Arnold reveals that for distinct \( i, j, \) and \( k \)

\[
\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ki} + \omega_{ki} \wedge \omega_{ij} = 0.
\]

**Theorem 2.3** (Arnold). Let \( A = \mathbb{C}[\omega_{ij}] \) be the exterior algebra generated by the differential forms \( \omega_{ij} \). The map \( \alpha \mapsto [\alpha] \) of \( A \) to \( H^*(\mathbb{C}^n - D) \) is an isomorphism.

**Theorem 2.4** (Arnold). Let \( P_\Delta(t) = \sum_{k=0}^{n} \dim H^k(\mathbb{C}^n - D)t^k \) be the Poincaré polynomial, then

\[
P_\Delta(t) = \left(1 + t\right)\left(1 + 2t\right) \cdots \left(1 + (n-1)t\right).
\]

Brieskorn proved Theorem 1 for any hyperplane arrangement and Orlik and Solomon combinatorially described the algebra of logarithmic differential forms. Realize each hyperplane \( H \) in the arrangement \( \mathcal{H} \) as the vanishing of an equation \( f_H \). Associate to \( f_H \) the closed 1-form \( \omega_H = \frac{1}{2\pi i} \frac{dH}{Tu} \); note that \( \omega_H \) does not depend on the choice of \( f_H \). Consider the exterior algebra \( \mathcal{A} = \mathbb{C}[\omega_H]_{H \in \mathcal{H}} \). The relations in \( \mathcal{A} \) are given as follows. A collection of hyperplanes \( H_1, \ldots, H_j \) is said to intersect transversally if \( \operatorname{codim} (H_1 \cap \cdots \cap H_j) = j \). To obtain the relations for \( \mathcal{A} \), take any collection of hyperplanes \( H_1, \ldots, H_{j+1} \) such that \( H_1 \cap \cdots \cap H_{j+1} \neq \emptyset \) and such that \( H_1, \ldots, H_{j+1} \) do not intersect transversally. Associate to this collection the relation
\[
\sum_{i=1}^{j+1} (-1)^j \omega_{H_1} \wedge \cdots \wedge \omega_{H_i} \wedge \cdots \wedge \omega_{H_{j+1}} = 0.
\]

**Theorem 2.5 (Orlik & Solomon).** These are the defining relations for \( \mathcal{A} = \mathbb{C}[\omega_H] \).

Theorem 2 is generalized in the following manner. Let \( G \) be a Coxeter group and let \( \mathcal{H} \) be the collection of reflecting hyperplanes for \( G \). Set \( \mathcal{H}_\mathbb{C} \) to be \( \mathcal{H} \)'s complexification.

**Theorem 2.6 (Terao).** Let \( P(t) \) be the Poincaré polynomial of \( \mathbb{C}^k - \bigcup_{H \in \mathcal{H}} H \) then,

\[
P(t) = \prod_j \left( 1 + (d_j - 1)t \right)
\]

where the \( d_j \)'s are the degrees of the basis set of homogeneous polynomials invariant under the action of \( G \).

### 2. Classical Hypergeometric Series

For a complex number \( z \) define the \( n \)th rising power \((z)_n = z(z+1)\cdots(z+n-1)\) with the provision that \((z)_0 = 1\). The classical hypergeometric series is defined as:

\[
F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}
\]

The rising power can be expressed via the Gamma function as \((z)_n = \frac{\Gamma(a+n)}{\Gamma(a)}\) where

\[
\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt.
\]

Note that the Gamma function is defined and holomorphic for \( z \in \mathbb{C} - \mathbb{Z}_{<0} \).

### 3. Properties of the Hypergeometric series (Gauss 1812)

1. The series \( F(a, b, c; z) \) is convergent for \(|z| < 1\) and it converges for \(|z| = 1\) provided \( \Re(c - a - b) > 0 \).

2. Contiguous Recurrence Relations:

\[
(c - a - b)F + a(1 - z)F(a+) - (c - b)F(b-) = 0
\]
where \( F(a\pm) = F(a\pm 1, b, c; z) \), \( F(b\pm 1) = F(a, b\pm 1, c; z) \), and \( F(c\pm 1) = F(a, b, c\pm 1; z) \) are the associated continuous functions. Gauss showed there is a linear relation between \( F \) and any two continuous functions with coefficients which are linear in \( z \).

3. \( F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \).

**Theorem 2.7 (Euler).** \( F \) satisfies the Gauss Hypergeometric Equation

\[
z(1-z)\frac{d^2u}{dz^2} + (c-(a+b+1)z)\frac{du}{dz} - abu = 0.
\]

**Theorem 2.8 (Euler).**

\[
F(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt.
\]

If the specialization \( z = 1 \) is made in the above expression formula #3 is recovered.

Set \( J = \int_a^b (s-a)^\alpha (b-s)^\beta (z-s)^\gamma ds \).

If we make the change of variables \( t = \frac{s-a}{b-a} \) in the integral \( J \) we obtain

\[
J = \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} (b-a)^{\alpha+\beta+1}(z-a)^\gamma F\left(-\gamma, \alpha + 1, \alpha + \beta + 2, \frac{b-a}{z-a}\right).
\]

References for this chapter: [V2], [SV], [R], [OS]
In this lecture we address which cycles $\gamma$ can be chosen in order to make the expressions

$$\phi^\gamma(z) = \sum_{|J|=k} \int_{\gamma} \Phi^\frac{1}{2}_{k,n}(t, z, m) A_J(t, z) dt_1 \wedge \cdots \wedge dt_k$$

single valued non trivial KZ solutions, where $\kappa \in \mathbb{C}^*$ and $m = (m_1, \ldots, m_n) \in \mathbb{C}^n$ are fixed and

$$\Phi_{k,n}(t, z, m) = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{-m_im_j} \prod_{1 \leq i \leq k} (t_i - t_j)^2 \prod_{l=1}^n \prod_{i=1}^k (t_i - z_l)^{-m_l}$$

$$A_J(t, z) = \frac{1}{j_1! \cdots j_n!} \text{Sym}_t \left[ \prod_{l=1}^n \prod_{i=1}^{j_l} \frac{1}{t_{j_1 + \cdots + j_l - i + z_l}} \right].$$

When $k = 1$ the cycle $\gamma$ should be chosen so that its homotopy class $[\gamma]$ resides in the commutator subgroup of $\pi_1(\mathbb{C} - \{z_1, \ldots, z_n\})$. For example, a Pochhammer cycle about any points $z_i$ and $z_j$ will serve for $\gamma$. In general the appropriate $\gamma$ will reside in homology groups twisted by coefficients associated with $\Phi^\frac{1}{2}_{1,n}$. We start by defining and computing this homology for $k = 1$.

**Definition 3.1.** A twisted $k$-cell is a pair $(\Delta^k, s)$ where $\Delta^k \subset \mathbb{C} - \{z_1, \ldots, z_n\}$ is a singular $k$-cell and $s$ is a univalent branch of $\Phi^\frac{1}{2}_{1,n}$ over $\Delta^k$. A twisted $k$-chain is a formal linear (over $\mathbb{C}$) combination of $k$-cells.

Define the boundary operator $d_k$ on any $k$-cell by the formula $d_k(\Delta^k, s) = (\partial \Delta^k, s|_{\Delta^k})$. Extend it by linearity to all $k$-chains.

The $k$th $\Phi^\frac{1}{2}_{1,n}$-twisted homology is defined as

$$H_k(\mathbb{C} - \{z_1, \ldots, z_n\}, \Phi^\frac{1}{2}_{1,n}) := \frac{\ker d_k}{\text{Im} d_{k+1}}.$$

Note that $\phi^\gamma$ is a well-defined non-trivial KZ solution provided $\gamma \in H_1$.

For what follows let $q = e^{\frac{2\pi i}{\kappa}}$ and for any $a \in \mathbb{C}$ set
where \( q^n = e^{\frac{2\pi i}{n}} \). Note that if \( q \to 1 \) (or \( \kappa \to \infty \)) then \([a] \to a\). We will begin by computing the twisted homology in a toy example; namely \( H_*(\mathbb{C} - \{0\}, t^{-m}) \). Take \( p \in \mathbb{C} - \{0\} \) and choose a loop \( l_1 \) about 0 passing through \( p \). Fix a value \( s_0 \) of \( t^{-m} \) at \( p \) and a branch \( s \) of \( t^{-m} \) over \( l \) whose value at \( l \)'s endpoint is \( s_0 \). Computing the boundary map \( d_1 \) we obtain:

\[
d_1(l, s) = (p, s|_{\text{end point}}) - (p, s|_{\text{start point}}) = (p, s_0) - (p, q^m s_0)
\]

\[
= (1 - q^m)(p, s_0)
\]

**Lemma 3.2.** The chain complex \( 0 \to \mathbb{C} \cdot (l, s) \xrightarrow{d_1} \mathbb{C} \cdot (p, s_0) \to 0 \) computes the homology groups \( H_*(\mathbb{C} - \{0\}, t^{-m}) \).

**Corollary 3.3.** \( H_0 \) and \( H_1 \) are both 0 if \( [m] \neq 0 \) and they are both 1 dimensional if \( [m] = 0 \).

Consider \( H_*\left(\mathbb{C} - \{z_1, \ldots, z_n\}, \Phi_{1,n}^\perp\right) \) and assume that \( z_1 < \cdots < z_n \) are all real. Fix a point \( p_0 \) in the upper-half plane and choose loops \( l_i \) emanating from \( p_0 \) and circling only around \( z_i \) for \( i = 1, \ldots, n \). Choose branches \( s_0, s_1, \ldots, s_n \) of \( \Phi_{1,n}^\perp \) over \( p_0, l_1, \ldots, l_n \) respectively. Note that if we analytically continue \( s_i \) along the loop \( l_j \) (for \( 1 \leq i, j \leq n \)) then \( s_i \) at \( l_j \)'s endpoint is \( q^{-m_j} \) times its value at the start point.

**Lemma 3.4.** The complex \( 0 \to \bigoplus_{j=1}^n \mathbb{C} \cdot (l_j, s_j) \xrightarrow{d_1} \mathbb{C} \cdot (p_0, s_0) \to 0 \) computes \( H_*\left(\mathbb{C} - \{z_1, \ldots, z_n\}, \Phi_{k,n}^\perp\right) \).

**Lemma 3.5.** One can choose \( c_j \in \mathbb{C} \) for \( j = 0, \ldots, n \) such that

\[
d_1(\omega_j) = [m_j] \left( q^{-m_1-\cdots-m_{j-1}-m_{j+1}+\cdots+m_n} \right) \omega_0 \quad j = 1, \ldots, n
\]

where \( \omega_0 = c_0(p_0, s_0), \ldots, \omega_n = c_n(l_n, s_n) \).

**Corollary 3.6.**

\[
H_1 = \left\{ I_1 \omega_1 + \cdots + I_n \omega_n \mid \sum_{j=1}^n [m_j] q^4 \left( -\sum_{i<j} m_i + \sum_{i>j} m_i \right) I_j = 0 \right\}
\]

Note that as \( q \to 1 \) the formula above becomes \( \sum_{j=1}^n m_j I_j = 0 \) which is precisely the singular vector condition; i.e. the condition to be in \( \text{Sing} M^{\otimes m} [m] - 2 \).
Corollary 3.7. If \([m_1] = \cdots = [m_n] = 0\) then \(\dim H_0 = 1\) and \(\dim H_1 = n\). Otherwise \(\dim H_0 = 0\) and \(\dim H_1 = n - 1\).

1. Quantum Groups

In what follows set \(q^x = e^{\frac{2\pi i}{\kappa}x}\) for any linear operator \(x\).

Definition 3.8. The algebra \(U_q(\mathfrak{sl}_2)\) is the algebra, with unit 1, generated by \(e, f,\) and \(q^x h\) (for \(x \in \mathbb{C}\)) subject to the following relations:

\[
q^{x h} e = eq^{x(h+2)} \quad q^{x h} f = fq^{x(h-2)}
\]

\[
[e, f] = \frac{q^k - q^{-k}}{q^\frac{1}{4} - q^{-\frac{1}{4}}} \quad q^{x h} q^{x' h} = q^{(x + x') h}
\]

We also require \(q^{0 h} = 1\). The algebra \(U_q(\mathfrak{sl}_2)\) is called the quantum group of \(\mathfrak{sl}_2(\mathbb{C})\). It is equipped with a Hopf algebra structure with comultiplication \(\Delta\) given by

\[
\Delta(e) = e \otimes q^{\frac{h}{4}} + q^{-\frac{h}{4}} \otimes e
\]

\[
\Delta(f) = f \otimes q^{\frac{h}{4}} + q^{-\frac{h}{4}} \otimes f
\]

\[
\Delta(q^{x h}) = q^{x h} \otimes q^{x h}.
\]

Note that when \(q \to 1\) these relations degenerate into the defining relations for the classical enveloping algebra \(U(\mathfrak{sl}_2)\). By analogy with \(\mathfrak{sl}_2(\mathbb{C})\), we define the quantum Verma modules. For \(\lambda \in \mathbb{C}\), the Verma module \(M_q\lambda\) is the infinite dimensional \(U_q(\mathfrak{sl}_2)\)-module generated by one vector \(v_\lambda\) satisfying \(e \cdot v_\lambda = 0\) and \(q^{x h} \cdot v_\lambda = q^{x\lambda} v_\lambda\). It has a basis given by \(v_\lambda, f v_\lambda, f^2 v_\lambda, \ldots\) subject to

\[
f \cdot f^k v_\lambda = f^{k+1} v_\lambda \quad q^{x h} \cdot f^k v_\lambda = q^{x(\lambda - 2k)} f^k v_\lambda \quad e \cdot f^k v_\lambda = [k][\lambda - k + 1] f^{k-1} v_\lambda.
\]

If \(\lambda \in \mathbb{Z}_{\geq 0}\) then \(M_q\lambda\) possesses a non-trivial submodule spanned by the vectors \(f^{\lambda+1} v_\lambda, f^{\lambda+2} v_\lambda, \ldots\). The quotient \(L_q\lambda\) is irreducible. Again, as \(q \to 1\) the representations \(L_q\lambda\) degenerate to the classical finite dimensional irreducible representations \(L_\lambda\) of \(\mathfrak{sl}_2(\mathbb{C})\). If no confusion arises we shall omit the superscript \(q\) in the notation and write only \(M_\lambda\) and \(L_\lambda\).

Unlike the classical theory, the linear isomorphism \(v \otimes w \mapsto w \otimes v\) between \(V \otimes W\) and \(W \otimes V\) of two \(U_q(\mathfrak{sl}_2)\)-modules \(V\) and \(W\) is not an isomorphism of representations. Indeed, the \(U_q(\mathfrak{sl}_2)\)-module isomorphism is constructed using the \(R\) matrix:

\[
R = q^{\frac{h \otimes h}{4}} \sum_{k \geq 0} q^{-k(k+1)} \left( \frac{q^\frac{1}{2} - q^{-\frac{1}{2}}}{[k]!} \right)^k q^{\frac{kk}{2}} c^k \otimes q^{\frac{-hh}{4}} f^k \in U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)
\]
The above expression is in fact a finite sum when applied to any particular vector in $V \otimes W$ since $e$ is always locally nilpotent. The $U_q(\mathfrak{sl}_2)$-isomorphism between $V \otimes W$ and $W \otimes V$ is $P \cdot R$ where $P(v \otimes w) = w \otimes v$.

As an example, consider the case of $L_m \otimes L_l$. Its basis is given by vectors of the form $v_m \otimes v_l, f v_m \otimes v_l, v_m \otimes f v_l, \ldots$. Assume $q$ is not a root of unity (i.e. $\kappa$ is not rational). We will evaluate $P \cdot R$ on the first three basis vectors. Since the highest power of $f$ occurring in any of these vectors is 1 it is enough to truncate $R$ as

$$q^\frac{h \otimes h}{4} + q^\frac{h \otimes h - 1}{4} e \otimes q^\frac{h}{4} f.$$

The isomorphism $P \cdot R : L_m \otimes L_l \rightarrow L_l \otimes L_m$ evaluates on $v_m \otimes v_l, f v_m \otimes v_l, v_m \otimes f v_l$ as

\[
\begin{align*}
    v_m \otimes v_l & \mapsto q \frac{4}{m l} v_l \otimes v_m \\
    v_m \otimes f v_l & \mapsto q \frac{4}{m(l-2)} v_l \otimes f v_m \\
    f v_m \otimes v_l & \mapsto q \frac{4}{4} v_l \otimes f v_m + [m] q \frac{4}{4} m - m - 1 f v_l \otimes v_m
\end{align*}
\]

**Theorem 3.9.** Let $V_1, V_2, and V_3$ be $U_q(\mathfrak{sl}_2)$-modules then the following diagram is commutative

\[
\begin{array}{ccc}
    V_1 \otimes V_2 \otimes V_3 & \xrightarrow{(23)} & V_1 \otimes V_3 \otimes V_2 \\
    (12) \downarrow & & (12) \downarrow \\
    V_2 \otimes V_1 \otimes V_3 & \xrightarrow{(23)} & V_2 \otimes V_3 \otimes V_1 \\
    & & (12) \downarrow \\
    & & V_3 \otimes V_2 \otimes V_2
\end{array}
\]

where $(12) = PR \otimes 1$ and $(23) = 1 \otimes PR$.

Commutativity of the above diagram implies that the $R$-matrix satisfies the Yang-Baxter equation.

**2. Braid Groups**

**Definition 3.10.** The Braid group $B_n$ on $n$-strings is the group with generators $b_1, \ldots, b_{n-1}$ with defining relations

- $b_i b_j = b_j b_i$ if $|i - j| > 1$
- $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ for $1 \leq i \leq n - 2$.

The Pure Braid group $PB_n$ is defined as the kernel of the homomorphism $\sigma : B_n \rightarrow S_n$ given by $b_i \mapsto (i, i + 1)$. 
An element of the Braid group is visualized as a family of \( n \) non-intersecting paths in space joining \( n \) points in a plane to \( n \) point in a parallel plane. Multiplication in \( B_n \) is interpreted as concatenation of paths. The Pure Braid group \( PB_n \) is visualized as families of paths which start and end at the same point - where the second plane and the points are identified with the first.

Recall that \( \Delta = \{ (a_1, \ldots, a_n) \in \mathbb{C}^n \mid z^n + a_1 z^{n-1} + \cdots + a_n \text{ has multiple roots} \} \). Every element in the compliment \( \mathbb{C}^n - \Delta \) can be identified with a unique monic \( n \)th order polynomial with distinct roots; namely associate the point \( (a_1, \ldots, a_n) \) with the polynomial \( (z - a_1) \cdots (z - a_n) \).

Given a braid - which we interpret as a family of non-intersecting paths joining the distinct points \( (a_1, \ldots, a_n) \) we may associate a loop in \( \mathbb{C}^n - \Delta \) emanating from \( (a_1, \ldots, a_n) \). The position of the loop at time \( t \in [0, 1] \) is the point \( (b_1, \ldots, b_n) \) where each \( (b_i, t) \) is the intersection point of a strand of the braid with the plane \( \mathbb{C} \times \{t\} \). Using this construction one obtains the following theorem.

**Theorem 3.11.** The classifying space for the Braid group \( B_n \) is \( \mathbb{C}^n - \Delta \). In other words, \( \pi_1(\mathbb{C}^n - \Delta) = B_n \) and \( \pi_k(\mathbb{C}^n - \Delta) = 0 \) for \( k > 1 \).

Let \( D = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = z_j \ \exists \ i \neq j \} \). By a similar construction using pure braids one obtains the following theorem.

**Theorem 3.12.** The classifying space for the Pure Braid group \( PB_n \) is \( \mathbb{C}^n - D \).

Let \( V_1, \ldots, V_n \) be \( U_q(\mathfrak{sl}_2) \)-modules and let \( R_i^\vee : V_1 \otimes \cdots \otimes V_i \otimes V_{i+1} \otimes \cdots \otimes V_n \to V_1 \otimes \cdots \otimes V_{i+1} \otimes V_i \otimes \cdots \otimes V_n \) be the map \( P \cdot R \) in the \( i \), \( i + 1 \) positions and identity in all others.

**Theorem 3.13.** For \( 1 \leq i \leq n-2 \) we have the Yang-Baxter identity \( R_i^\vee R_{i+1}^\vee R_i^\vee = R_{i+1}^\vee R_i^\vee R_{i+1}^\vee \).

Let \( V \) be a \( U_q(\mathfrak{sl}_2) \)-module. In view of Theorem 4 we can surmise that the map \( B_n \to \text{GL}(V^\otimes n) \) given by \( b_i \mapsto R_i^\vee \) is a representation of \( B_n \). Moreover, since \( (R_i^\vee)^2 \) maps \( V_1 \otimes \cdots \otimes V_n \) to itself it follows that the map \( PB_n \to \text{GL}(V_1 \otimes \cdots \otimes V_n) \) given by \( b_i \mapsto R_i^\vee \) is a representation of the Pure Braid group \( PB_n \).

### 3. Quantum Singular Vectors

For \( m = (m_1, \ldots, m_n) \in \mathbb{C}^n \) let \( M^\otimes m \) be the \( U_q(\mathfrak{sl}_2) \) Verma module \( M_{m_1} \otimes \cdots \otimes M_{m_n} \). Let \( \text{Sing} M^\otimes m \left[ \lfloor m \rfloor - 2k \right] = \{ w \in M^\otimes m \mid e \cdot w = 0 \text{ and } q^{|\lambda|} w = q^{\lambda(\lfloor m \rfloor - 2k)} w \} \). Note that \( e \) acts as \( e \otimes q^{\frac{h_1}{4}} \otimes \cdots \otimes q^{\frac{h_n}{4}} + \cdots + q^{\frac{h_1}{4}} \otimes \cdots \otimes q^{\frac{h_n}{4}} \otimes e \). As in the classical case, \( \text{Sing} M^\otimes m \left[ \lfloor m \rfloor \right] \) is spanned by the vector \( v_{m_1} \otimes \cdots \otimes v_{m_n} \). When \( k = 1 \) we see that \( \text{Sing} M^\otimes m \left[ \lfloor m \rfloor - 2 \right] \) is given by

\[
\{ w = \sum_{j=1}^{n} f_j v_{m_1} \otimes \cdots \otimes f v_{m_j} \otimes \cdots \otimes v_{m_n} \mid \text{eigenspace condition and} \}
\]
\[
\sum_{j=1}^{n} I_j[m_j]q^{\frac{1}{2}}\left(\sum_{i>j} m_i - \sum_{i<j} m_i\right) = 0 \}
\]

**Corollary 3.14.** The map \( v^{m_1} \otimes \cdots \otimes f v^{m_j} \otimes \cdots \otimes v^{m_n} \xrightarrow{\psi} \omega_j = c_j(l_j, s_j) \) and \( v^{m_1} \otimes \cdots \otimes v^{m_n} \xrightarrow{\psi} \omega_0 = c_0(p_0, s_0) \) makes the following diagram commute:

\[
\begin{array}{ccc}
0 & \xrightarrow{\psi} & M^{\otimes m}[|m| - 2] & \xrightarrow{e} & M^{\otimes m}[|m|] & \xrightarrow{\psi} & 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \xrightarrow{\bigoplus_{j=1}^{n} \mathbb{C} \cdot \omega_j} & d_1 & \xrightarrow{d_1} & \mathbb{C} \cdot \omega_0 & \xrightarrow{} & 0 \\
\end{array}
\]

As an immediate consequence of this corollary we obtain

\[
\text{Sing } M^{\otimes m}[|m| - 2] \cong H_1\left(\mathbb{C} - \{z_1, \ldots, z_n\}, \phi_{1,n}^z\right)
\]

for \( z_1 < \cdots < z_n \) real.

**References for this chapter:** [V2].
1. Monodromy of KZ equations

KZ equations are defined over $U = \mathbb{C}^n - D$, where

$$D = \{ z \in \mathbb{C}^n \mid \exists i, j \text{ s.t. } z_i = z_j \}.$$ 

For any $A \in U$, denote $\text{Sol}_A$ the space of all solutions of the equations in a neighborhood of $A$. For any path $p$ from $A$ to $B$ in $U$ the continuation of solutions along $p$ gives an isomorphism

$$\alpha_p : \text{Sol}_A \to \text{Sol}_B.$$ 

If $A, B$ are fixed, then $\alpha_p$ does not depend on continuous deformations of the path. If $p$ is a loop, then the isomorphism becomes an automorphism

$$M(p) : \text{Sol}_A \to \text{Sol}_A$$

and gives the monodromy representation

$$(4.1) \quad \pi_1(\mathbb{C}^n - D) \to \text{GL(Sol}_A).$$

2. Topological monodromy

Consider on $\mathbb{C}^n - D$ the vector bundle whose fiber on $(z_1, \ldots, z_n)$ is $H_1(\mathbb{C} - \{z_1, \ldots, z_n\}, \Phi_{1,n}^{1/\kappa})$. There is a flat connection on it: we deform the cycles for close points $(z_1, \ldots, z_n) \sim (z'_1, \ldots, z'_n)$ (this is known as the Gauss–Manin connection). In particular, if we have a loop in $\mathbb{C}^n - D$ we get a map

$$H_1(\mathbb{C} - \{z_1, \ldots, z_n\}, \Phi_{1,n}^{1/\kappa}) \to H_1(\mathbb{C} - \{z_1, \ldots, z_n\}, \Phi_{1,n}^{1/\kappa}).$$

We get thus another action of the group

$$(4.2) \quad \pi_1(\mathbb{C}^n - D) \to \text{Aut}(H_1).$$

3. $R$-matrix

Recall from the previous lecture that we have an action of the pure braid group

$$(4.3) \quad PB_n \to \text{Aut(Sing } M^{\otimes m}[|m| - 2])$$

for $M^{\otimes m} = M_{m_1} \otimes \cdots \otimes M_{m_n}$. Recall also that $PB_n = \pi_1(\mathbb{C}^n - D).$
4. Relationship

**Claim 4.4.** (a) Via the correspondence of the space of solutions of KZ with \( H_1 \), we get an isomorphism between the two first representations (namely, (4.1) and (4.2)).

(b) Via the correspondence of \( \text{Sing} M^\otimes m \otimes [m] - 2 \) with \( H_1(C^n - \{z_1, \ldots, z_n\}, \Phi_{1|n}^{1/\kappa}) \), we get an isomorphism between the second and third representations (namely, (4.2) and (4.3)).

The first part of the claim is clear; the second one is remarkable. It was proved first by Kohno without hypergeometric functions and then by Drinfeld, who developed for this purpose the theory of quasi Hopf algebras.

We give a sketch of a third proof. Consider the KZ equation with values in \( \text{Sing} M^\otimes m \otimes [m] - 2k \); solutions of it are given, as we know, by

\[
\varphi(\gamma)(z_1, \ldots, z_n) = \sum_{|J|=k} \int \Phi_{k,n}^{1/\kappa}(t, z, m) A_J(t, z) \, dt_1 \wedge \cdots \wedge dt_k
\]

\[
\Phi_{k,n} = \prod_{i,j}(z_i - z_j)^{m_i m_j/2} \prod_i (t_i - t_j)^2 \prod_i \prod_l (t_i - z_l)^{-m_l}
\]

For \( z \in C^n \), let \( C_{k,n}(z) = \{ t \in C^k \mid t_i \neq z_j, \ t_i \neq t_j \ \forall i, j \} \). The \( k \)-cycles \( \gamma \) are elements of \( H_k(C_{k,n}(z), \Phi_{1|n}^{1/\kappa}) \) and each \( \gamma \) of this type defines a solution of KZ. However, not all \( \gamma \) give a nonzero solution. Notice that the differential forms \( \eta_J = \Phi_{k,n}^{1/\kappa} A_J dt_1 \wedge \cdots \wedge dt_k \) are skew symmetric with respect to permutations of \( t_1, \ldots, t_k \):

\[
\eta(t_{\sigma(1)}, \ldots, t_{\sigma(k)}) = (-1)^{\sigma(\gamma)} \eta(t_1, \ldots, t_k) \ \forall \sigma \in S_k.
\]

The symmetric group \( S_k \) acts also on the \( k \)-cells. This action induces an action of \( S_k \) on the homology spaces. Let \( H_k(C_{k,n}(z), \Phi_{1|n}^{1/\kappa})^- \) be the subspace of skew symmetric elements \( \gamma \) (i.e., \( \sigma \gamma = (-1)^{\sigma(\gamma)} \gamma \)). Let \( H_k^- \) be the sum of all other isotypical components, \( H_k = H_k^- \oplus H_k^\ast \). Then, for any \( \gamma \in H_k^- \) we have \( \varphi(\gamma) = 0 \). Thus, solutions of KZ are labeled by \( H_k(C_{k,n}(z), \Phi_{1|n})^- \).

**Theorem 4.5.** (a) Let \( z \in C^n \) be such that \( z_1 < \cdots < z_n \). Then there is a natural isomorphism

\[
H_k(C_{k,n}(z), \Phi_{1|n})^- \simeq \text{Sing} M^\otimes m, q \otimes [m] - 2k].
\]

Thus, the solutions of KZ are labeled by this space.

(b) Under this identification, the monodromy of KZ is identified with the \( R \)-matrix representation of \( PB_n \) in \( \text{Sing} M^\otimes m, q \otimes [m] - 2k] \).

The construction of the isomorphism is analogous to the construction of the isomorphism for \( k = 1 \). The vector \( f_{J,v} = f_{j_1,v_m} \otimes \cdots \otimes f_{j_n,v_m} \) is identified with the following cell: take \( j_i \) loops around each \( z_i \) (see Figure 1). This is a product of \( k \) 1-dimensional cells, and then it is a \( k \)-dimensional cell. Then, take the antisymmetrization of it.

This proof was developed later than that of Drinfeld. In it, one realizes KZ equations geometrically, as equations for hypergeometric functions associated with the master functions \( \Phi_{k,n} \). In this geometric approach the isomorphism of the KZ monodromy with the \( R \)-matrix representation is given explicitly, through the hypergeometric pairing

\[
\gamma \otimes \Phi_{1/\kappa} A_J dt \mapsto \int_{\gamma} \Phi_{1/\kappa} A_J dt.
\]
We address now the following

**Problem:** Show that the construction we made gives many linearly independent solutions. This will be solved in the next lecture; we develop here some tools.

### 5. Remark on integration cycles

If \( \kappa, m_1, \ldots, m_n < 0, \kappa > 0 \) and \( \kappa \gg 1 \), then all exponents of \( \Phi_{k,n} \) are big positive integers. Assume \( z \in \mathbb{R}^n \). We can consider in \( \mathbb{R}^k \) the hyperplanes \( \{ t_i = z_j \} \) and \( \{ t_i = t_j \} \). Call \( C_{k,n}(z)(z) = \mathbb{R}^k - \{ t \in \mathbb{R}^k \mid t_i = t_j, \ t_i = z_j \} \) complement to these hyperplanes. For \( k = 2 \) we can picture it as in Figure 2. We can take now \( \gamma \) as any bounded domain in \( C_{k,n}(z)(z) \) (we shaded two of such domains in figure 2).

**Claim 4.6.** Under these assumptions, \( \varphi^{(\gamma)}(z_1, \ldots, z_n) \) is a solution of the KZ equation with values in \( \text{Sing} \ M^\otimes m [[m] - 2k] \).

### 6. Example: the Selberg integral

Let \( n = 2 \); consider KZ with values in

\[
\text{Sing} \ M_{m_1} \otimes M_{m_2}[m_1 + m_2 - 2k] = \{ \varphi = \sum_{j_1 + j_2 = k} I_{(j_1,j_2)} f^{j_1} v_{m_1} \otimes f^{j_2} v_{m_2} \mid e \varphi = 0 \}.
\]
Calculating explicitly the condition $e\varphi = 0$ we can see that the space is one dimensional and is generated by

$$\omega = \sum_{j_1+j_2=k} \frac{(-1)^j_1}{j_1!j_2!} \prod_{i=0}^{j_1-1} \frac{1}{m_1-i} \prod_{i=0}^{j_2-1} \frac{1}{m_2-i} f^{j_1}v_{m_1} \otimes f^{j_2}v_{m_2}.$$ 

The hypergeometric solutions have the form

$$\varphi^{(\gamma)}(z_1, z_2) = \sum_{j_1+j_2=k} \int_{\Delta(z_1, z_2)} \Phi_{k,2}^A(j_1, j_2) \ dt_1 \wedge \cdots \wedge dt_k \ f^{j_1}v_{m_1} \otimes f^{j_2}v_{m_2},$$

where $\Delta(z_1, z_2) = \{ t \in \mathbb{R}^k \mid z_1 \leq t_k \leq \cdots \leq t_1 \leq z_2 \}$. Then $\varphi(z_1, z_2)$ is proportional to $\omega$. To see that $\varphi \neq 0$, we consider the $(k,0)$ coordinate of the solution (i.e., $(j_1, j_2) = (k,0)$):

$$\int_{\Delta(z_1, z_2)} \prod_{i=1}^k (t_i - z_1)^{-m_1} - (t_i - z_2)^{-m_2} \ dt_1 \wedge \cdots \wedge dt_k = 0.$$

Using new variables $t_i = (z_2 - z_1)s_i + z_1$, we can write this as

$$(z_1 - z_2)^{-\Delta(m_1, m_2, -2k)} \int_k \Gamma\left(\frac{m_1}{\kappa}, -\frac{m_2}{\kappa} + 1, \frac{1}{\kappa}\right),$$

where $\Delta(a) = \frac{a(a+2)}{4}$. Here $\int_k(a, b, c)$ is the Selberg integral:

$$\int_{k} (a, b, c) = \sum_{0 \leq s_1 < \cdots < s_k \leq 1} \prod_{i=1}^k s_i^{a-1} (1-s_i)^{b-1} \prod_{1 \leq i < j \leq k} (s_i - s_j)^{2c} \ ds_1 \wedge \cdots \wedge ds_k.$$ 

In 1944 Selberg showed that

$$\int_{k} (a, b, c) = \prod_{j=0}^{k-1} \frac{\Gamma(1+c+jc)}{\Gamma(1+c)} \frac{\Gamma(a+jc)\Gamma(b+jc)}{\Gamma(a+b+(k+j-1)c)}.$$ 

For $n = 1$, we get the beta integral $\int_{1} (a, b, c) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. The Selberg integral is one of the most remarkable hypergeometric functions. In particular, we see that our hypergeometric solution is not zero for generic $m_1, m_2, \kappa$. Taking a suitable limit of the Selberg integral we get

$$\int_{\mathbb{R}^k} e^{-a \sum_{i=1}^k s_i^2} \prod_{i<j} (s_i - s_j)^{2c} \ ds_1 \wedge \cdots \wedge ds_k = (2\pi)^{\frac{k}{2}} (2a)^{\frac{(k-1)+1}{2}} \prod_{j=1}^k \frac{\Gamma(1+jc)}{\Gamma(1+c)}.$$ 

7. Connection with finite reflection groups

Macdonald (1982) observed that $s_i - s_j = 0$, $1 \leq i < j \leq k$ are equations of the reflecting hyperplanes of the finite group $A_{k-1}$. There are other finite groups generated by reflections. In the $k$-dimensional Euclidean space $\mathbb{R}^k$ consider a certain number of hyperplanes, all passing through the origin. If the angles between the hyperplanes are properly chosen, then the group generated by reflections corresponding to them is finite. These groups were enumerated and classified by Coxeter as

$$A_k \ (k \geq 1), \ B_k \ (k \geq 2), \ D_k \ (k \geq 4) \ E_6, \ E_7, \ E_8, \ F_4, \ G_2, \ H_3, \ H_4, \ I_2(p) \ (p > 5).$$
Let $G$ be one of these reflecting groups. Let $P(s)$ be the product of distances of the point $(s_1, \ldots, s_k) = s$ from all reflecting hyperplanes belonging to $G$. Let $N$ be the number of hyperplanes. Not knowing Selberg integrals, Macdonald conjectured that

$$\int_{\mathbb{R}^k} e^{-\sum s_i^2/2} |P(s)|^{2c} ds_1 \wedge \cdots \wedge ds_k = 2^{-Nc} (2\pi)^{k/2} \prod_{j=1}^k \frac{\Gamma(1 + cd_j)}{\Gamma(1 + c)},$$

where $d_j$ are the degrees of a basis set of the space of homogeneous polynomials which are invariant with respect to $G$.

The conjecture was proved by Opdam. The $q$-analogs were proved by Cherednik.

References for this chapter: [V1, V2, V4]
A. Varchenko. Special functions, KZ type equations and Representation theory
The goal of this lecture is to see how many independent solutions one can get by taking cycles on the bounded domains in the case $\kappa > 0, m_i < 0 \forall i$.

1. Determinant formulas

Consider the KZ equation with values in $\text{Sing} M^\otimes m [[m] - 2]$:

$$
\Phi_{1,n} = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{m_i m_j} \prod_{l=1}^{n} (t - z_l)^{-m_l},
$$

$$
\varphi^{(\gamma)}(z_1, \ldots, z_n) = \sum_{j=1}^{n} \int_{\gamma} \Phi_{1,n}^{1/\kappa} \frac{dt}{t - z_j} v_{m_1} \otimes \cdots \otimes f v_{m_j} \otimes \cdots \otimes v_{m_n}
$$

and call $I_{\gamma}^j(z) = \int_{\gamma} \Phi_{1,n}^{1/\kappa} \frac{dt}{t - z_j}$. We have $I_{\gamma}^j(z) = -\frac{1}{m_1} \sum_{j=2}^{n} m_j I_{\gamma}^j(z)$, which is the condition for the vector $\sum_j I_{\gamma}^j v_{m_1} \otimes \cdots \otimes f v_{m_j} \otimes \cdots \otimes v_{m_n}$ to be a singular vector.

**Corollary 5.1.** If $m_1 \neq 0$, then $\dim \text{Sing} M^\otimes m [[m] - 2] = n - 1$.

Let $z_1 < \ldots < z_n$ be real. Let $\Delta_i(z) = [z_{i-1}, z_i], i = 2, \ldots, n$. Let $\kappa, m_1, \ldots, m_n \in \mathbb{R}, \kappa > 0, m_1, \ldots, m_n < 0$. Then $\varphi^{(\Delta_i)}(z_1, \ldots, z_n), i = 2, \ldots, n$ are solutions of KZ.

**Theorem 5.2.** These solutions are linearly independent for generic $m_1, \ldots, m_n, \kappa$.

The theorem follows from the following lemma

**Lemma 5.3 (Determinant Formula).** Let $\varphi^1, \ldots, \varphi^{n-1}$ be a set of solutions. Let $w^1, \ldots, w^{n-1}$ be a basis of $\text{Sing} M^\otimes m [[m] - 2]$, and write $\varphi^i = \sum_j \varphi^j w^j$. Then

$$
\det(\varphi^i_j) = (\text{const}) \prod_{i<j}(z_i - z_j)^{(n-1)\frac{m_i m_j}{2m} - \frac{m_i}{m} - \frac{m_j}{m} - \frac{m_i m_j}{m^2}}
$$

**Proof.** Notice that if one has a differential equation $\frac{dy}{dx} = A(x)y$, where $A, y$ are $n \times n$ matrices, then $\det' y = \text{tr} A(x) \det y$ (since $\det(A + \epsilon I) = 1 + \epsilon \text{tr} A + o(\epsilon^2)$). Our system is

$$
\frac{\partial y}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega^{(ij)}}{z_i - z_j} y,
$$

and then $\frac{\partial \det y}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\text{tr} \Omega^{(ij)}}{z_i - z_j} \det y$, whence $\det y = (\text{const}) \prod_{i<j}(z_i - z_j)\frac{\Omega^{(ij)}}{\kappa}$. 

29
We show now that $\text{tr } \Omega^{(ij)} = \text{tr } \Omega^{(ij)}|_{\text{Sing } M^\otimes m|_{[m]-2}} = (n - 1) \frac{m_i m_j}{2\kappa} - \frac{m_i}{\kappa} - \frac{m_j}{\kappa}$. We prove it for $\Omega^{(12)}$, the general case is analogous. We can use the basis

$$w^1 = \frac{1}{m_1} fv \otimes v \otimes \cdots \otimes v - \frac{1}{m_2} v \otimes fv \otimes \cdots \otimes v,$$

$$w^i = \frac{1}{m_1 + m_2} (fv \otimes \cdots \otimes v + v \otimes fv \otimes \cdots \otimes v) - \frac{1}{m_i} v \otimes \cdots \otimes fv \otimes \cdots \otimes v$$

and then

$$\Omega^{(12)} w^1 = \left(\frac{1}{2} h \otimes h + e \otimes f + f \otimes e\right) \otimes 1 \otimes \cdots \otimes 1 (w^1)$$

$$= \left(\frac{1}{2} m_1 m_2 - m_1 - m_2\right) w^1$$

$$\Omega^{(12)} w^i = \frac{1}{2} m_1 m_2 w^i.$$

\[\square\]

We compute now the constant in the determinant. Let $\bar{\Phi} = \prod_{i=1}^n (t - z_i)^{\alpha_i}$, and the differential forms $\eta_j = \frac{dt - z_j}{t - z_j}$, $j = 1, \ldots, n$. Let $z_1 < \ldots < z_n$ as before. We fix then the arguments in each interval $\Delta_i(z)$, and then the integrals $\int_{z_i}^{z_{i+1}} \bar{\Phi} \eta_j$ are well defined.

**Theorem 5.4.**

$$\det_{2 \leq i,j \leq n} (\alpha_j \int_{z_i}^{z_j} \bar{\Phi} \eta_j) = \frac{\Gamma(\alpha_1 + 1) \cdots \Gamma(\alpha_n + 1)}{\Gamma(\alpha_1 + \cdots + \alpha_n + 1)} \prod_{i<j} (z_i - z_j)^{\alpha_i + \alpha_j}.$$---

Before giving the proof we notice that for $n = 2$, $z_1 = 0$ and $z_2 = 1$ the theorem says

$$\alpha_2 \int_0^1 t^{\alpha_1} (1 - t)^{\alpha_2 - 1} dt = \frac{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)}{\Gamma(\alpha_1 + \alpha_2 + 1)},$$

and this is Euler’s formula.

**Proof.** By induction. For $n = 2$, we saw that this is Euler’s formula. Consider then $z_1, \ldots, z_{n-1}$ fixed and let $z_n \to \infty$; we will see that the ratio RHS/LHS tends to 1.

We remark that

1. For $i, j \leq n - 1$,

$$\int_{z_i}^{z_j} \alpha_j \bar{\Phi} \frac{dt}{t - z_j} = (-z_n)^{\alpha_n} \left[ \int_{z_i}^{z_j} \alpha_j \prod_{l=1}^{n-1} (t - z_l)^{\alpha_l} \frac{dt}{t - z_j} + O\left(\frac{1}{z_n}\right) \right]$$

2. $\int_{z_i}^{z_j} \alpha_n \bar{\Phi} \frac{dt}{t - z_n} = O\left((-z_n)^{\alpha_n - 1}\right)$

3. $\int_{z_{n-1}}^{z_n} \alpha_n \bar{\Phi} \frac{dt}{t - z_{n-1}} = O\left((-z_n)^{\alpha_1 + \cdots + \alpha_n}\right)$

4. $\int_{z_{n-1}}^{z_n} \alpha_n \bar{\Phi} \frac{dt}{t - z_{n-1}} = (-z_n)^{\alpha_1 + \cdots + \alpha_n} \int_0^1 \alpha_n t^{\alpha_1 + \cdots + \alpha_{n-1}} (1 - t)^{\alpha_n - 1} dt + O\left(z_n^\alpha\right)$.

We have then a matrix
Thus, dividing the \( i \)-th row \((i < n)\) by \((-z_n)^\alpha\) and the \( n \)-th row by \((-z_n)^{\alpha_1+\cdots+\alpha_n}\), we get

\[
\det_{\Delta}(z_1, \ldots, z_n) = (-z_n)^{(n-1)\alpha} \det_{\Delta}(z_1, \ldots, z_{n-1}) \frac{\Gamma(\alpha_1 + \cdots + \alpha_{n-1} + 1)\Gamma(\alpha_n + 1)}{\Gamma(\alpha_1 + \cdots + \alpha_n + 1)}.
\]

Now, it is easy to prove that \( \text{RHS}/\text{LHS} \) is holomorphic with respect to \( z_n \). Since its limit at \( \infty \) is 1, we finished.

**Remark 5.5.** We proved that we constructed all solutions for generic \( m_1, \ldots, m_n, \kappa \). The case of degenerate parameters is very explicit: it happens in the poles of \( \Gamma(\alpha_1 + \cdots + \alpha_n + 1) \).

**Remark 5.6.** Considering solutions with values in \( \text{Sing} M^{\otimes m}[|m| - 2k] \), one can similarly show that the set of hypergeometric solutions generate all solutions for generic \( m_1, \ldots, m_n, \kappa \). For \( n = 2 \), one has to replace Euler’s formula by the Selberg integral.

There are remarkable general determinant formulas by Terao and Douai. Let \( A \) be a finite collection of hyperplanes in \( \mathbb{R}^k \) and chose for each \( H \in A \) a number \( \lambda_H \) and a linear function \( f_H \). Then

\[
\det \left( \int_{\Delta} \prod_{H \in A} f_H^{\lambda_H} \omega \right) = (\text{Alternating product of } \Gamma \text{ functions}) \prod \left( \text{values of } f_H^{\lambda_H} \text{ at vertices} \right),
\]

where \( \Delta \) runs through bounded domains and \( \omega \) through a suitable basis of logarithmic differential forms. In our case there is an action of the symmetric group and one must consider isotypical components.

Another determinant formula: let \( A \) be again an arrangement of hyperplanes in \( \mathbb{R}^k \), now in generic position. It can be proved that the number of bounded domains is \((n-1)^{k-1}\). On the other hand, the number of monomials \( t^a = t_1^{a_1} \cdots t_k^{a_k} \) such that \( a_1 + \cdots + a_k \leq n - k - 1 \) is also \((n-1)^{k-1}\). One can consider \( \det(\int_{\Delta} t^a dt_1 \cdots dt_k) \), where \( \Delta \) runs on the bounded domains and \( t^a \) runs on the preceding monomials. We have

\[
\det = \pm[(n-1)!]^{\frac{n-2}{k-1}} \prod_{S \subset \mathbb{R}^k} (k! \text{ vol}(S)),
\]

where \( S \) runs on the \( k \)-simplices with faces in \( A \).

Yet another example: let \( z_1 < \cdots < z_n \in \mathbb{R} \). Consider \( \det(\int_{z_i}^{z_{i+1}} t^j dt) \). This is a Vandermonde determinant, and we have

\[
\det(\int_{z_i}^{z_{i+1}} t^j dt) = \text{const} \prod_{j < i} (z_i - z_j).
\]
2. Resonances

Consider the case $k = 1$. Then

$$\det(\text{KZ Solutions}) = \frac{\prod_j \Gamma(-\frac{m_j}{\kappa} + 1)}\Gamma(-\frac{\sum m_i}{n} + 1) \prod (z_i - z_j)^v$$

When $-\frac{\sum m_i}{\kappa} + 1 = 0$ then we have a pole in the denominator and then we get linearly dependent solutions. In this case, the bundle $\text{Sing} L^\otimes m[|m| - 2]$ has a nontrivial subbundle which is invariant by the KZ equation. We want to describe this subbundle. It turns out that in CFT the appearance of such subspace was predicted, and it was described in terms of representation theory.

For a general $k$, let $m_1, \ldots, m_n \in \mathbb{Z}_{>0}$ and consider $L^\otimes m = L_{m_1} \otimes \cdots \otimes L_{m_n}$ the tensor product of irreducible f.d. representations. Assume that $\kappa \in \mathbb{Z}$, $\kappa > 2$, $0 \leq m_1, \ldots, m_n$ and $|m| - 2 \leq \kappa - 2$. Consider the KZ equation with values in

$$\text{Sing} L^\otimes m[|m| - 2k] = \{v \in L^\otimes m \mid hv = (|m| - 2k)v, \ ev = 0\}.$$ 

Set $p = \kappa - 1 - |m| + 2k$, this is a positive integer. Take $W(z_1, \ldots, z_n) \subset \text{Sing} L^\otimes m[|m| - 2k]$ given by

$$W(z) = \{w \in L^\otimes m \mid hw = (|m| - 2k)w, \ ev = 0, \ (ze)^p w = 0\},$$

where $ze : L^\otimes m \rightarrow L^\otimes m$ is the operator $w_1 \otimes \cdots \otimes w_n \mapsto \sum_i z_i w_1 \otimes \cdots \otimes ew_i \otimes \cdots \otimes w_n$.

**Theorem 5.7.** The subspaces $W(z) \subset \text{Sing} L^\otimes m[|m| - 2k]$ are preserved by KZ.

**Definition 5.8.** These subspaces are called spaces of conformal blocks.

**Theorem 5.9.** All the hypergeometric solutions $\varphi^{(\gamma)}(z)$ belong to the bundle of conformal blocks.

This means that hypergeometric functions “know” about CFT and its conformal blocks. Consider the case $k = 1$. We have

$$\text{Sing} L^\otimes m[|m| - 2] = \{w = \sum_{j=1}^n I_j v_{m_1} \otimes \cdots \otimes f v_{m_j} \otimes \cdots \otimes v_{m_n} \mid \sum_{j=1}^n I_j m_j = 0\}.$$ 

Here $p = \kappa + 1 - |m| \in \mathbb{Z}_{>0}$. If $p > 1$ then $W(z) = \text{Sing} L^\otimes m[|m| - 2]$ (by a degree argument $(ze)^2 = 0$). If $p = 1$ then

$$W(z) = \{w = \sum I_j v_{m_1} \otimes \cdots \otimes f v_{m_j} \otimes \cdots \otimes v_{m_n} \mid \sum m_j I_j = 0, \ \sum z_j m_j I_j = 0\}.$$ 

Notice that $p = 1 \iff \kappa = \sum m_j$. Then

**Theorem 5.10.** For any hypergeometric solution $\varphi^{(\gamma)}(z)$ we have $\sum z_j m_j I_j = 0$.

**Proof.** It is similar to the proof that $\sum m_j I_j = 0$. In that case, we have

$$-\kappa d_t(\Phi^{1/\kappa}) = \sum m_j \Phi^{1/\kappa} \frac{dt}{t - z_j}.$$
and integrating over a cycle $\gamma$ we get the identity. Now, we have $\kappa d_t(t\Phi^{1/\kappa}) = \sum z_j m_j \Phi^{1/\kappa} \frac{dt}{t-z_j}$, i.e.,

$$d_t(t \prod (z_i - z_j)^m \prod (t-z_l)^{-m_l/\kappa}) = \Phi^{1/\kappa} dt + \Phi^{1/\kappa} \sum - \frac{m_l (t - z_l) + z_l}{t - z_l}$$

$$= (1 - \sum_{l=1}^n \frac{m_l}{\kappa}) \Phi^{1/\kappa} dt - \sum_{l=1}^n \frac{m_l \Phi^{1/\kappa} dt}{t - z_l},$$

and, again, integrating over a cycle $\gamma$ we get the identity. \qed

### 3. Dynamical equations

Let $\Phi^{1/\kappa}_{k,n} = \prod_{i,j} (z_i - z_j)^{m_{ij}} \prod_{i,j} (t_i - t_j)^{\frac{1}{\kappa}} \prod_i \prod_j (t_i - z_j)^{-m_{ij}}$. Let

$$\varphi(\gamma)(z_1, \ldots, z_n) = \sum_{|J|=k} I_J \Phi^{1/\kappa},$$

$$I_J(z_1, \ldots, z_n) = \int_{\gamma} \Phi^{1/\kappa} A_J dt_1 \wedge \cdots \wedge dt_k.$$

be solutions of KZ with values in $\text{Sing } M^{\otimes m}[m-2k]$, $\gamma \in H_k(C_{k,n}(z), \Phi^{1/\kappa}_{k,n})$. Shift $m_1 \mapsto m_1 + \kappa$. Then the integrand is multiplied by $\frac{\text{const}(z)}{(t_1-z_1)\cdots(t_k-z_1)}$. This is a rational univalued function; the same cycle $\gamma$ is good for the new integral. This defines a map

$$\{I_J(z, m_1, \ldots, m_n)\} \to \{I_J(z, m_1 + \kappa, \ldots, m_n)\}$$

in $H^1(C_{k,n}(z), \Phi^{1/\kappa}_{k,n})$.

**Problem:** Describe this map.

Again, the answer will be given in terms of $\mathfrak{sl}_2$-representation theory. Relations of this type will be called *dynamical equations*. Before going over the problem, we make a

**Remark 5.11.** Since $\varphi(\gamma) \in \text{Sing } M^{\otimes m}[m-2k]$, there are many linear relations between $\{I_J\}$.

**Lemma 5.12.** If $m_1 \notin \{0, 1, \ldots, k-1\}$, then for any $J_0$, $|J_0| = k$, the integral $I_{J_0}$ is a linear combination of integrals $\{I_J\}$ with $J$ having the form $(0, j_2, \ldots, j_n)$, $j_1 + \cdots + j_n = k$. The coefficients of this linear combination do not depend on $\gamma$.

For example, for $k = 1$ we have $I_1 = \frac{1}{m_1} \sum_{j=2}^n m_j I_j$. We will study the transformation $m_1 \mapsto m_1 + \kappa$ for “restricted vector valued functions” $\psi(\gamma)(z_1, \ldots, z_n) = \sum_{J=(0, j_2, \ldots, j_n)} I_J f_J v$.

**Remark 5.13.** Having the restricted function $\psi(\gamma)$ one can reconstruct the initial function $\varphi(\gamma)$. Therefore one can rewrite the KZ differential equation for $\varphi(\gamma)(z)$ as a suitable differential equation for the restricted function. This new differential equation is called *trigonometric KZ equation*.
References for this chapter: [V1, V2, V3, V4], [MTV], [DT], [FeSV]
The goal of this lecture is to describe the trigonometric KZ equation and dynamical equation and see their interaction.

1. Dynamical equations

Let

$$\Phi_{k,n}^{1/\kappa} = \prod_{i,j} (z_i - z_j)^{m_{ij}/2\kappa} \prod_{i,j} (t_i - t_j)^{2} \prod_{i} \prod_{l} (t_i - z_l)^{-m_l/\kappa}$$

$$\Gamma_j(z_1, \ldots, z_n) = \int_{\gamma} \Phi_{k,n}^{1/\kappa} A_j \, dt_1 \wedge \cdots \wedge dt_k,$$

$$\varphi(\gamma)(z_1, \ldots, z_n) = \sum_{|J| = k} \Gamma_J f_J v.$$

be solutions of KZ with values in $\text{Sing} \, M^\otimes m[|m| - 2k]$, $\gamma \in H_k(C_{k,n}(z), \Phi_{k,n}^{1/\kappa})$. Shift $m_1 \mapsto m_1 + \kappa$.

Then the integrand is multiplied by $\frac{\text{const}(z)}{(t_1-z_1) \cdots (t_k-z_1)}$. This is a rational univalued function; the same cycle $\gamma$ is good for the new integral. This defines a map in cohomology, given by

$$\{\Gamma_J(z, m_1, \ldots, m_n)\} \mapsto \{\Gamma_J(z, m_1 + \kappa, \ldots, m_n)\}.$$

**Problem:** Describe this map.

The answer will be given in terms of $\mathfrak{sl}_2$-representation theory. Relations of this type will be called dynamical equations. Before going over the problem, we make a

**Remark 6.1.** Since $\varphi(\gamma) \in \text{Sing} \, M^\otimes m[|m| - 2k]$, there are many linear relations between $\{\Gamma_J\}$.

**Lemma 6.2.** If $m_1 \notin \{0, 1, \ldots, k - 1\}$, then for any $J_0$, $|J_0| = k$, the integral $\Gamma_{J_0}$ is a linear combination of integrals $\{\Gamma_J\}$ with $J$ having the form $(0, j_2, \ldots, j_n)$, $j_1 + \cdots + j_n = k$. The coefficients of this linear combination do not depend on $\gamma$.

For example, for $k = 1$ we have $I_1 = \frac{1}{m_1} \sum_{j=2}^{n} m_j I_j$. We will study the transformation $m_1 \mapsto m_1 + \kappa$ for “restricted vector valued functions”

$$\psi(\gamma)(z_1, \ldots, z_n) = \sum_{J = (0, j_2, \ldots, j_n), |J| = k} \Gamma_J f_J v.$$

**Remark 6.3.** Having the restricted function $\psi(\gamma)$ one can reconstruct the initial function $\varphi(\gamma)$. Therefore one can rewrite the KZ differential equation for $\varphi(\gamma)(z)$ as a suitable differential equation for the restricted function. This new differential equation is called trigonometric KZ equation.
2. Trigonometric $r$-matrix

Set
\[
\begin{align*}
\Omega^+ &= \frac{1}{4}h \otimes h + e \otimes f \\
\Omega^- &= \frac{1}{4}h \otimes h + f \otimes e
\end{align*}
\] \in \mathfrak{sl}_2^2
\]
We have $\Omega^+ + \Omega^- = \Omega$, the Casimir element. Define the trigonometric $r$-matrix $\gamma(z) = \frac{\Omega^+ z + \Omega^-}{z-1}$. A more symmetric form is got when one considers $\gamma(z_i/z_j) = \frac{\Omega^+ z_i + \Omega^- z_j}{z_i-z_j}$. Let $\mathfrak{h} = \mathbb{C} h \subset \mathfrak{sl}_2$ be the Cartan subalgebra. Let $V = V_2 \otimes \cdots \otimes V_n$ be a tensor product of $\mathfrak{sl}_2$ modules, $\nu \in \mathbb{C}$ and $\bar{\lambda} \in \mathfrak{h}$.

**Definition 6.4.** The **trigonometric KZ equations** on a $V$-valued function $u(z_2, \ldots, z_n, \bar{\lambda}) \in V$ are the equations
\[
\kappa z_i \frac{\partial}{\partial z_i} u = \sum_{j \neq i} \frac{(\Omega^+ z_i + \Omega^- z_j)^{(i,j)}}{z_i - z_j} u + \bar{\lambda}^{(i)} u, \quad i = 2, \ldots, n,
\]
where $\bar{\lambda}^{(i)} = 1 \otimes \cdots \otimes 1 \otimes \bar{\lambda} \otimes 1 \otimes \cdots \otimes 1$ in the $i$-th factor. $\bar{\lambda}$ and $\kappa$ are parameters of the equation.

We introduce the **trigonometric KZ operators**
\[
\nabla_i(\kappa, \bar{\lambda}) = \kappa z_i \frac{\partial}{\partial z_i} - \sum_{j \neq i} r^{(i,j)}(z_i/z_j) - \bar{\lambda}^{(i)}, \quad i = 2, \ldots, n.
\]
Then the trigonometric KZ equations have the form
\[
\nabla_i(\kappa, \bar{\lambda}) u = 0, \quad i = 2, \ldots, n.
\]

3. Hypergeometric solutions of trigonometric equations

We consider solutions with values in $V = M_{m_2} \otimes \cdots \otimes M_{m_n}[\sum_{j=2}^n m_j - 2k]$. This is so because the trigonometric KZ equation does not preserve singular spaces, though it does preserve weight spaces (there is a correspondence between singular spaces in $V_{m_1} \otimes \cdots \otimes V_{m_n}$ and weight spaces of $V_{m_2} \otimes \cdots \otimes V_{m_n}$). Take $\bar{\lambda} = \lambda \frac{h}{2}$, $\lambda \in \mathbb{C}$. Set $\nu = \sum_{j=2}^n m_j - 2k$ and put
\[
\Phi(t_1, \ldots, t_k, z_2, \ldots, z_n) = \prod_{2 \leq i < j \leq n} (z_i - z_j)^{-m_i m_j} \prod_{i=2}^n \frac{1}{z_i^{m_i(\lambda+(m_i-\nu)/2)}}
\]
\[
\times \prod_{i=1}^k \prod_{j=2}^n (t_i - z_j)^{-m_j} \prod_{1 \leq i < j \leq k} (t_i - t_j)^2 \prod_{i=1}^k t_i^{-(\lambda-1-\nu)/2}
\]
For any $J = (j_2, \ldots, j_n)$, $|J| = k$, introduce the function $A_J$ by the same formulas as before. Set $f_J v = f_{j_2} v_{m_2} \otimes \cdots \otimes f_{j_n} v_{m_n}$.

**Remark 6.5.** The complement to the singularities of $\Phi$ with respect to $t_1, \ldots, t_k$ is the space $\mathcal{C}_{k,n} \subset \mathbb{C}^k$, which is also the complement to the singularities of $\Phi_{k,n}$ with $z = (0, z_2, \ldots, z_n)$.

**Theorem 6.6.** For any $\gamma(z) \in H_k(\mathcal{C}_{k,n}(z), \Phi^{1/\kappa})$, the function
\[
\varphi^{(\gamma)}(z_1, \ldots, z_n) = \sum_{|J| = k} \int_{\gamma(z)} \Phi^{1/\kappa} A_J dt_J v
\]
is a solution of the trigonometric KZ equation with parameter $\bar{\lambda} = \lambda \frac{h}{2}$. 
Let us describe then the Dynamical Equations $\vec{\lambda} \mapsto \vec{\lambda} + \kappa \frac{h}{2}$ (this corresponds to $\lambda \mapsto \lambda + \kappa$). For $\lambda \in \mathbb{C}$, set

$$P(\lambda) = \sum_{i=0}^{\infty} f^i e^{\frac{i}{2} \sum_{j=1}^{i} \frac{1}{\lambda - \frac{h}{2} - j}}.$$  

**Remark 6.7.** If $V$ is a f.d. $\mathfrak{sl}_2$ module or a Verma module, then $P(\lambda) : V \to V$ is well defined. Similarly, $P(\lambda)$ is well defined on a tensor product of such modules.

**Definition 6.8.** The Dynamical Equation for $u(z_2, \ldots, z_n) \in V_2 \otimes \cdots \otimes V_n$ is

$$u(z_2, \ldots, z_n, \lambda + \kappa) = \prod_{i=2}^{n} \left( z_i^{\frac{h}{2}} \right)^{(i)} P(\lambda) u(z_2, \ldots, z_n, \lambda).$$

Denote $K(z_2, \ldots, z_n, \lambda) = \prod_{i=2}^{n} \left( z_i^{\frac{h}{2}} \right)^{(i)} P(\lambda)$. It is clear that $K(z, \lambda)$ preserves the weight decomposition of $V$.

**Theorem 6.9.** The Dynamical Equation is compatible with the trigonometric KZ equation. Explicitly,

$$\nabla_i (\kappa, \vec{\lambda} + \kappa \frac{h}{2}) K(z, \lambda) = K(z, \lambda) \nabla_i (\kappa, \vec{\lambda}).$$

**Corollary 6.10.** If $u(z_2, \ldots, z_n, \lambda)$ is a solution of the trigonometric KZ equation with parameter $\vec{\lambda} = \lambda \frac{h}{2}$, then $K(z, \lambda) u(z, \lambda)$ is a solution of the trigonometric KZ equation with parameter $\vec{\lambda} + \kappa \frac{h}{2}$.

For hypergeometric solutions we can say more: consider a hypergeometric solution of the trigonometric KZ equation with values in $M_{m_2} \otimes \cdots \otimes M_{m_n} [\sum_{i=2}^{n} m_i - 2k]$,

$$\varphi^{(\gamma)}(z_2, \ldots, z_n, \lambda) = \sum_{J = \{j_2, \ldots, j_n\}}^{J = \{2, \ldots, n\}} \int_\gamma \Phi_{1/\kappa}^{A_{J}} dt_1 \wedge \cdots \wedge dt_k.$$

**Theorem 6.11.** The function $\varphi^{(\gamma)}$ satisfies also the Dynamic Equation

$$\varphi^{(\gamma)}(z_2, \ldots, z_n, \lambda + \kappa) = K(z, \lambda) \varphi^{(\gamma)}(z_2, \ldots, z_n, \lambda).$$

**Remark 6.12.** The function $\varphi^{(\gamma)}(z_2, \ldots, z_n, \lambda + \kappa)$ is well defined since the corresponding integrand differs from the integrand of $\varphi^{(\gamma)}(z_2, \ldots, z_n, \lambda)$ by a univalued factor.

**Example 6.13.** Take $V = M_m$ and consider a weight subspace $V[m - 2k] = \mathbb{C} f^k v_m$. Let $\bar{u}(z, \lambda) \in V[m - 2k]$, $\bar{u}(z, \lambda) = u(z, \lambda) f^k v_m$ for a scalar $u(z, \lambda)$. We describe trigonometric KZ, Dyn. Eqn. and its hypergeometric solutions (here $\lambda, z \in \mathbb{C}$).

**Trigonometric KZ:** $\kappa z \frac{\partial}{\partial z} \bar{u} = \vec{\lambda} \bar{u}, \vec{\lambda} = \lambda \frac{h}{2}$. Then the equation reads as

$$\kappa z \frac{\partial}{\partial z} u = \lambda \frac{m - 2k}{2} u.$$

Since $K(z, \lambda) f^k v_m = z^{h/2} P(\lambda) f^k v_m$, we must compute $P(\lambda) f^k v_m$.  

---
Theorem 6.14. We have

\[ P(\lambda) f^k v_m = (\lambda + \frac{m}{2})(\lambda + \frac{m}{2} - 1) \cdots (\lambda + \frac{m}{2} - k + 1) \frac{1}{(\lambda - \frac{m}{2})}(\lambda - \frac{m}{2} + 1) \cdots (\lambda - \frac{m}{2} + k - 1) f^k v_m. \]

Proof. \[ P(\lambda) f^k v_m = \sum_{j \geq 0} \frac{k(k-1) \cdots (k-j+1)(m-k+1) \cdots (m-k+j)}{j!(\lambda - \frac{m-2k}{2} - 1) \cdots (\lambda - \frac{m-2k}{2} - j)} f^k v_m \]

= \( F(-k, m-k+1, -\lambda + \frac{m-2k}{2} + 1, z = 1) f^k v_m, \)

and by Gauss’ formula we have \( F(a, b, c, 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \) \( \Box \)

Corollary 6.15. The Dynamical Equation is

\[ u(z, \lambda + \kappa) = z^{\frac{m-2k}{2}} \prod_{j=0}^{k-1} \frac{(\lambda + \frac{m}{2} - j)}{(\lambda - \frac{m}{2} + j)} u(z, \lambda). \]

The hypergeometric solution is given by

\[ u(z, \lambda) = \int_{0 \leq t_1 \leq \cdots \leq t_k \leq z} \Phi^{1/\kappa} A dt_1 \wedge \cdots \wedge dt_k, \]

where \( \Phi^{1/\kappa} = z^{\frac{1}{\kappa}(\lambda+k)m} \prod_{i=1}^{k} t_i^{-\frac{1}{2}(\lambda-1-\frac{m-2k}{2})} \prod_{i,j} (t_i - t_j)^{2/\kappa} \)

\[ A = \frac{1}{(t_1 - z) \cdots (t_k - z)}. \]

Changing variables \( t_i = zs_i \) we get

\[ u(z, \lambda) = z^{\frac{m-2k}{2k}} \int_k \left( \frac{1}{\kappa}(\lambda-1-\frac{m-2k}{2}) + 1, -\frac{m}{\kappa}, \frac{1}{\kappa} \right), \]

where as before \( \int_k (a, b, c) = \prod_{j=0}^{k-1} \frac{\Gamma(a+j+1)c \Gamma(a+jc)\Gamma(a+jc)}{\Gamma(a+b+j+c)} \) is the Selberg integral, and we have the relation

\[ \int_k (a + 1, b, c) = \prod \frac{a+jc}{a+b+(k+j-1)c} \int_k (a, b, c), \]

from where we can see that \( u(z, \lambda) \) satisfies both equations.

4. Quasiclassical asymptotics of hypergeometric functions and Bethe ansatz

Let \( V = V_1 \otimes \cdots \otimes V_n \) be a tensor product of \( \mathfrak{sl}_2 \) modules. Set \( H_i(z) = \sum_{j \neq i} \frac{\Omega^{(ij)}}{z_i - z_j} \) as operators on \( V \).

Theorem 6.16. For any \( z \), the operators \( H_i(z) \) commute.

Proof. Follows from the identity \( [\Omega, x \otimes 1 + 1 \otimes x] = 0 \) \( \forall x \in \mathfrak{sl}_2 \). \( \Box \)
The operators \( \{H_i(z)\} \) are called the Hamiltonians of the Gaudin model of an “inhomogeneous magnetic chain”. In integrable models of quantum mechanics, usually one has a vector space 
\( V = V_1 \otimes \cdots \otimes V_n \) called the space of states. One may think about \( n \) particles at the points \( z_1, \ldots, z_n \in \mathbb{C} \), and the spaces \( V_1, \ldots, V_n \) are their spin spaces. Usually one has a family of commutative linear operators on \( V \) called Hamiltonians. The main problem is to diagonalize them and to find their eigenvalues. The next problem is to find the limit when \( n \to \infty \). The Bethe ansatz is a method for diagonalizing the Hamiltonians. One looks for eigenvectors of the form
\[
v(t_1, \ldots, t_k) \in V,
\]
where \( v(t_1, \ldots, t_k) \) is some suitable function of parameters \( \vec{t} \). One then tries to find the values of the parameters such that \( v(t_1, \ldots, t_k) \) is an eigenvector of the Hamiltonians. One proves that if \( t_1, \ldots, t_k \) satisfy some explicit system of equations
\[
F_j(t_1, \ldots, t_k) = 0, \quad j = 1, \ldots, k,
\]
then \( v(t_1, \ldots, t_k) \in V \) is an eigenvector.

The equations (6.17) are called the Bethe equations, \( v(t) \) is called the Bethe vector. The method is called the Bethe ansatz method.

The standard conjectures are
1. The number of solutions properly counted = dim of the space of states.
2. Bethe vectors from a basis

Important observation: In all known examples there exists a single (master) function
\[
\Phi(t_1, \ldots, t_k)
\]
such that \( \frac{\partial \Phi}{\partial t_j} = F_j, \ j = 1, \ldots, n \). Thus, with a given integrable model of quantum mechanics there is associated a (master) function \( \Phi(t_1, \ldots, t_k) \) such that

Conjectures (reformulated)
1. The number of critical points of \( \Phi = \text{dim of the space of states}, \)
2. the Bethe vectors \( \{v(t_0)\} \mid t_0 \in \text{crit. pts.} \) form a basis,
3. the Hessian \( \det(\frac{\partial^2 \ln \Phi}{\partial t_i \partial t_j})(t_0) \) “=” square of the length of the Bethe vector \( v(t_0) \)

Remark 6.18. For f.d. irreducible representations of \( \mathfrak{sl}_2 \) this is recently known to be true.

5. Asymptotic solutions of KZ and eigenvectors of \( H_i(z) = \sum_{j \neq i} \frac{\Omega^{(i)}}{z_i - z_j} \)

Let \( V = V_1 \otimes \cdots \otimes V_n \) be a tensor product of \( \mathfrak{sl}_2 \) modules. The KZ equation for \( u(z_1, \ldots, z_n) \in V \) is
\[
x \frac{\partial}{\partial z_i} u = H_i(z) u, \quad i = 1, \ldots, n.
\]

Definition 6.20. Let
\[
u = e^{-\frac{\Omega^{(i)}}{\kappa}} \left( f_0(z) + \kappa f_1(z) + \kappa^2 f_2(z) + \cdots \right)
\]
for certain functions $S, f_0, \ldots$. We say that $u$ is an asymptotic solution of KZ if the substitution of $u$ in (6.19) gives 0 in the expansion as a formal power series in $\kappa$.

**Theorem 6.22.** Let $u(z)$ be an asymptotic solution of KZ. Then $\forall i$ the vector $f_0(z)$ is an eigenvector of $H_i(z)$ with eigenvalue $\frac{\partial S}{\partial z_i}(z)$.

**Proof.**

$$\kappa \left[ \frac{1}{\kappa} \frac{\partial S}{\partial z_i} e^{\frac{s(z)}{\kappa}} (f_0 + \kappa f_1 + \cdots) + e^{\frac{s(z)}{\kappa}} \left( \frac{\partial f_0}{\partial z_i} + \kappa \frac{\partial f_1}{\partial z_i} + \cdots \right) \right] = H_i e^{\frac{s(z)}{\kappa}} (f_0 + \kappa f_1 + \cdots),$$

and then

$$\frac{\partial S}{\partial z_i} f_0 = H_i f_0, \quad \frac{\partial S}{\partial z_i} f_1 + \frac{\partial f_0}{\partial z_i} = H_i f_1, \ldots$$

$\square$

**Remark 6.23.** With the following powers we can in general recover $f_i$ from $f_0 \forall i > 0$.

**5.1. Shapovalov form.** Let $V = V_m$ generated by $v$. There is a unique bilinear symmetric form $S$ on $V$ such that $S(v, v) = 1$, $S(\kappa x, y) = S(x, \kappa y)$, $S(\kappa x, y) = S(x, \kappa y)$ for all $x, y \in V$. For example $S(fv, fv) = S(v, efv) = S(v, hv) = m$, and in general we get

$$S(f^k v, f^k v) = k! m (m-1) \cdots (m-k+1).$$

The form $S$ is called Shapovalov form.

Consider now $V_1 \otimes V_2$ and $S = S_1 \otimes S_2$ the tensor product of the Shapovalov forms on $V_1$ and $V_2$, i.e., $S(x_1 \otimes x_2, y_1 \otimes y_2) = S(x_1, y_1) S(x_2, y_2)$. We have

**Lemma 6.24.** $S(\Omega(x_1 \otimes x_2), y_1 \otimes y_2) = S(x_1 \otimes x_2, \Omega(y_1 \otimes y_2))$.

**Corollary 6.25.** For $V = V_1 \otimes \cdots \otimes V_n$, $S = S_1 \otimes \cdots \otimes S_n$, we have $\forall x, y \in V$

$$S(H_i(z)x, y) = S(x, H_i(z)y).$$

**Lemma 6.26.** Let $u_1(z)$ be a solution of KZ with parameter $\kappa$. Let $u_2(z)$ be a solution with parameter $-\kappa$. Then $S(u_1(z), u_2(z))$ does not depend on $z$.

**Proof.**

$$\frac{\partial}{\partial z_i} S(u_1, u_2) = S(u_1, \frac{\partial}{\partial z_i} u_2) + S(u_1, \frac{\partial}{\partial z_i} u_2) = S(\frac{1}{\kappa} H_i u_1, u_2) + S(u_1, -\frac{1}{\kappa} H_i u_2) = 0.$$

$\square$

**Lemma 6.27.** Let $u(z, \kappa) = e^{\frac{s(z)}{\kappa}} (f_0(z) + \kappa f_1(z) + \cdots)$ be an asymptotic solution of KZ with parameter $\kappa$. Then $S(f_0(z), f_0(z))$ does not depend on $z$.

**Proof.** Let $u_2(z, \kappa) = u(z, -\kappa)$. It is an asymptotic solution to KZ with parameter $-\kappa$, and then $S(u, u_2)$ does not depend on $z$. But

$$S(e^{\frac{s(z)}{\kappa}} (f_0(z) + \kappa f_1(z) + \cdots), e^{-\frac{s(z)}{\kappa}} (f_0(z) - \kappa f_1(z) + \cdots)) = S(f_0(z), f_0(z)) + O(\kappa).$$

$\square$

References for this chapter: [FMTV], [TV3], [MaV], [EV1], [F], [G], [RV], [TV4]
LECTURE 7

During the last lecture we proved the following result:
given
\[ u = e^{\frac{S(z)}{\kappa}}(f_0(z) + \kappa f_1(z) + \kappa^2 f_2(z) + \ldots) \]
asymptotic solution of the KZ equation, \( f_0(z) \) is a common eigenvector for the set of commutating operators \( H_i = H_i(z), \) \( i = 1, \ldots, n \), where:
\[ H_i(z) = \sum_{i \neq j}^{n} \Omega^{(ij)} \frac{z_i - z_j}{z_i - z_j} \]
with eigenvalues \( \frac{\partial S}{\partial z_i}(z) \). We also introduced the Shapovalov form \( S \): given \( V = V_m \), generated by \( v \), \( S \) is the unique bilinear symmetric form such that:
\[ S(f^k v, f^k v) = k! m(m - 1) \ldots (m - k + 1) \]
and 0 otherwise, and we proved that if \( V = V_1 \otimes \ldots \otimes V_n \), \( S = S_1 \otimes \ldots \otimes S_n \) and \( u = e^{\frac{S(z)}{\kappa}}(f_0(z) + \kappa f_1(z) + \kappa^2 f_2(z) + \ldots) \) asymptotic solution of the KZ equation, then \( S(f_0(z), f_0(z)) \) does not depend on \( z \). During this lecture we will discuss how to find explicit asymptotic solutions of KZ equation and hence eigenvectors for the set of hamiltonian operators \( H_i(z) \).

Let’s consider the space of \( \text{Sing} M^\otimes m[\mid m \mid - 2k] \). We know that given:
\[ \Phi_{k,n}(t, z, m) = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{m_i m_j \over 2} \prod_{1 \leq i < j \leq k} (t_i - t_j)^2 \prod_{l=1}^{n} \prod_{i=1}^{k} (t_i - z_l)^{-m_i} \]
\[ A_j(t, z) = \frac{1}{j_1! \ldots j_n!} \text{Sym}_t \left[ \prod_{l=1}^{n} \prod_{i=1}^{j_l} \frac{1}{t_{j_l} + \ldots + t_{j_{l+1}} - z_l} \right] \]
\[ u^\gamma = \Sigma \int \Phi_{k,n}^\gamma(t, z) A_j(t, z) dt_1 \wedge \ldots \wedge dt_k \]
is a solution of the KZ equation.
We will construct asymptotic solutions taking the limit of \( u^\gamma \) for \( \kappa \to 0 \).

Let’s consider the following examples where we introduce the Steepest Descend Method.

EXAMPLE 7.6. Let’s consider the following function:
\[ I(\kappa) = \int_{-\infty}^{+\infty} e^{-\frac{a(s)}{\kappa}} ds. \]
If $a(0) \neq 0$ we can conclude that:

$$I(\kappa) = \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2\kappa}} a(s) ds \sim \sqrt{\pi},$$

(7.8)

since:

$$\int_{-\infty}^{+\infty} e^{-\frac{s^2}{2\kappa}} a(s) ds = \sqrt{\pi} \kappa (a(0) + \frac{a''(0)}{4} + ...),$$

(7.9)

In the same way we could analyze the function:

$$I(\kappa) = \int_{-\infty}^{+\infty} e^{i\frac{s^2}{2\kappa}} a(s) ds$$

(7.10)

**Definition 7.11.** The function $a(s)$ is called the **amplitude** while the function $\phi(s) = -\frac{s^2}{\kappa}$ is called the **phase**.

**Example 7.12.** Given a phase function of several variables, $S = S(t_1, ..., t_n)$ whose critical points are isolated and non-degenerate, we can reduce the analysis of the behavior of

$$I(\kappa) = \int_{\mathbb{R}^n} e^{-\frac{S(t_1, ..., t_n)}{\kappa}} a(t_1, ..., t_n) dt_1 ... dt_n$$

(7.13)

to that studied in the previous example. In fact in a neighborhood of any of the critical point of $S$, we can find a new set of coordinates $s_1, ..., s_n$, such that:

$$S(s_1, ..., s_n) = A \pm s_1^2 \pm s_2^2 ... \pm s_n^2$$

(7.14)

Applying this method to:

$$\int \Phi_{k,n}^1(t, z) A_j(t, z) dt_1 ... dt_k$$

(7.15)

we can conclude that it localizes at the critical points of the function $\Phi_{k,n}^1(t, z)$.

Let’s define $S = \frac{\log \Phi_{k,n}(t, z)}{\kappa}$, let’s suppose that $z^o \in \mathbb{C}^n$ has distinct coordinates and let $t^o \in \mathbb{C}^k$ be a non-degenerate critical point of $\Phi_{k,n}$ w.r.t $t$, i.e

$$\frac{\partial \Phi_{k,n}}{\partial t_i}(t^o, z^o) = 0$$

(7.16)

and

$$\det \frac{\partial^2 \Phi_{k,n}}{\partial t_i \partial t_j}(t^o, z^o) \neq 0.$$  

(7.17)

**Theorem 7.18.** ( Implicit Function.) There exist a unique holomorphic $\mathbb{C}^k$ valued function $t = t(z)$, such that:

1. $\frac{\partial \Phi_{k,n}}{\partial t_i}(t(z), z) = 0$ if $i = 1, ..., k$;

2. $t^o = t(z^o)$.

Under these hypothesis we have the following:
Theorem 7.19. The KZ equation with values in $\text{Sing } M^{\otimes m}[|m| - 2k]$ has an asymptotic solution, $u$, in a neighborhood of $z^o$ given by the following expression:

\begin{equation}
(7.20) \quad u(z) = \frac{\Phi_{k,n}(t(z),z)}{\sqrt{\det \frac{\partial^2}{\partial t_i \partial t_j} \ln \Phi_{k,n}(t(z),z)}} \left( \sum_{|J|=k} A_J(t(z),z) f_J v + o(\kappa) \right)
\end{equation}

Corollary 7.21. If $t^o \in \mathbb{C}^k$ is a non degenerate critical point of $\Phi_{k,n} = \Phi_{k,n}(t,z^o)$ then the vector:

\begin{equation}
(7.22) \quad \omega(t^o,z^o) = \sum_{|J|=k} A_J(t(z),z) f_J v
\end{equation}

is an eigenvector of the set of commutating hamiltonians $H_i = H_i(z)$, $i = 1,\ldots, n$.

Definition 7.23. $\omega(t^o,z^o)$ is called the Bethe vector.

We also have the following:

Corollary 7.24.

\[ S(\omega(t(z),z),\omega(t(z),z)) = \det \frac{\partial^2}{\partial t_i \partial t_j} \ln \Phi_{k,n}(t(z),z) = \text{cost} \]

Question 7.25. What is this constant?

Remark 7.26. Even if this question makes sense for arbitrary Kac-Moody algebras the answer is known only for $\mathfrak{sl}_2$; in this case we have:

Theorem 7.27.

\[ S(\omega(t(z),z),\omega(t(z),z)) = \det \frac{\partial^2}{\partial t_i \partial t_j} \ln \Phi_{k,n}(t(z),z) \]

i.e. the constant is 1.

Remark 7.28. The set of critical points $t^o = (t_1,\ldots, t_k)$ of $\Phi_{k,n} = \Phi_{k,n}(t,z^o)$ is invariant w.r.t the permutation of the coordinates, moreover the Bethe vector corresponding to the point belonging to the same orbit are equal.

we will address the following:

Question 7.29. Is it true that:

1. the number of the orbits of critical points of $\Phi_{k,n}$ equals the dim. of $\text{Sing } M^{\otimes m}[|m| - 2k]$?
2. the Bethe vectors form a basis?

To answer these question we introduce the following general result; given an arrangement of hyperplanes $A = \{H\}$ in $\mathbb{R}^k$, defined by the equations $f_H = 0$, set:

\begin{equation}
(7.30) \quad \Phi_A^\lambda := \prod_{H \in A} f_H^{\lambda_H}, \lambda \in \mathbb{C}.
\end{equation}

Question 7.31. What is the number of critical points of the function $\Phi_A^\lambda$ for generic $\{\lambda_H\}$?
The equation that gives us the critical points is \( \frac{\partial \Phi}{\partial \beta} = 0 \) or in other words:

\[
\sum \lambda_H \frac{\partial f_H}{\partial t_i} = 0, \quad i = 1, \ldots, k.
\]

**Remark 7.32.** Considering a real arrangement in \( \mathbb{R}^k \), and positive \( \lambda_H \), then each bounded component has a critical point. It is also possible to show that each of these critical points is non-degenerate.

Let \( A_R \in \mathbb{R}^k \) be a real arrangement, \( A \subset \mathbb{C}^k \) its complexification and let’s define \( M = \mathbb{C}^k - \bigcup_{H \in A} H \). Then for positive \( \{ \lambda_H \} \) it is possible to prove the following:

**Theorem 7.33.**

\[ z \left( \text{critical points of } \Phi^*_{A} \text{ in } M \right) = z \left( \text{bounded components of } \mathbb{R}^k - \bigcup_{H \in A} H \right) = |\chi(M)| \]

where \( \chi(M) \) is the Euler characteristic of the open complex manifolds \( M \).

**Remark 7.34.** It was conjectured by Varchenko that for generic \( \{ \lambda_k \} \) this result holds for any complex arrangement \( A \subset \mathbb{C}^k \). This conjecture was proved by Orlik and Terao.

From the theorem follows:

**Corollary 7.35.** If the highest weights \( m_1, \ldots, m_n \) are negative, i.e. the modules \( M_{m_j} \) are all irreducible

\[
z \left( \text{orbits of critical points of } \Phi_{k,n} \right) = \dim \text{Sing } M^\otimes m[|m| - 2k].
\]

**Remark 7.36.** Varchenko and Reshetikhin checked that for generic points \( z_1, \ldots, z_n \) and negative \( m_1, \ldots, m_n \) the Bethe vectors form a basis in \( \text{Sing } M^\otimes m[|m| - 2k] \). In this case we have:

\[
\dim \text{Sing } M^\otimes m[|m| - 2k] = \binom{k + n - 2}{n - 2} = \frac{\mathcal{B}}{k!}
\]

where \( \mathcal{B} \) in the number of the bounded components of the arrangement.

Let’s now consider the following:

**Example 7.37.** Let’s consider the Selberg integrand:

\[
\tilde{\Phi}(t; \alpha, \beta, \gamma) = \prod_{j=1}^{k} t_j^\alpha (1 - t_j)^\beta \prod_{1 \leq i < j \leq k} (t_i - t_j)^{2\gamma}
\]

where \( \alpha, \beta, \gamma > 0 \). This function has \( k! \) critical points, one on each bounded domain. The symmetric group acts on the set of critical points and we have exactly one orbit.

Let’s consider the elementary symmetric functions \( \lambda_1 = \sum_{i=1}^{k} t_i, \lambda_2 = \sum_{i<j} t_i t_j, \ldots, \lambda_k = \prod_{i=1}^{k} t_i \). We have the following: The critical points of the Selberg integrand satisfy the following equations:

\[
\frac{\alpha}{t_i} + \frac{\beta}{t_i - 1} + \sum_{j \neq i} \frac{2\gamma}{t_i - t_j} = 0,
\]

for \( i = 1, \ldots, k \). If \( (t_1, \ldots, t_k) \) is a critical point then:
Theorem 7.39.  
(7.40) \[
\lambda_j = \binom{k}{j} \prod_{i=1}^{j} \frac{\alpha(k-i)\gamma}{\alpha + \beta + (2k-i-1)\gamma},
\]
for \( j = 1, ..., k \).

Proof. The equation for the critical point can be rearranged in the following way:
(\( j+1 \) \( \alpha + j\gamma \))\( \lambda_{i+1} = (k-j)(\alpha - \beta + (k+j-1)\gamma)\lambda_j \).

Now let \( t \in \mathbb{C}^k \) be a singular point of the function \( \tilde{\Phi}(t; -m_1, -m_2, 1) \) and let’s consider the Bethe vector:

\[
\omega = \sum_{p=0}^{k} f^{k-p} v_{m_1} \otimes f^{p} v_{m_2} A_{k-p,p}(t).
\]

where:

\[
A_{k-p,p}(t) = \sum_{1 \leq i_1 < \ldots < i_p \leq k} \frac{1}{(t_{i_1} - 1) \ldots (t_{i_p} - 1)} \prod_{i \notin \{i_1, \ldots, i_p\}} \frac{1}{t_i}
\]

This vector belongs to \( \text{Sing} M_{m_1} \otimes M_{m_2}[m_1 + m_2 - 2k] \) and w.r.t. the standard generator \( \omega_{m_1+m_2-2k} = \sum_{p=0}^{k} \frac{f^{k-p} v_{m_1} \otimes f^{p} v_{m_2}}{(k-p)!m_1(m_1 - 1)\ldots(m_1 - k + p + 1)m_2(m_2 - 1)\ldots(m_2 - k + p + 1)} \) of \( \text{Sing} M_{m_1} \otimes M_{m_2}[m_1 + m_2 - 2k] \) we have the following:

Theorem 7.42.  
\[ \omega_{\text{Bethe}} = k! \prod_{j=1}^{k} (m_1 + m_2 - (2k-j-1))\omega_{m_1+m_2-2k}. \]

Theorem 7.43. Let \( S = S_{m_1} \otimes S_{m_2} \) the Shapovalov form on \( M_{m_1} \otimes M_{m_2} \) and \( t^o \) a critical point of the function \( \Phi(t; -m_1, -m_2, 1) \). Then:

\[ S(\omega_{\text{Bethe}}, \omega_{\text{Bethe}}) = \det \left( \frac{\partial^2}{\partial t_i \partial t_j} \ln \Phi(t) \right) \]

The theorem follows from the following lemmas:

Lemma 7.44.

\[ S(\omega_{\text{Bethe}}, \omega_{\text{Bethe}}) = k! \prod_{l=0}^{k-1} \frac{(m_1 + m_2 - 2k + l + 2)^3}{(m_1 - l)(m_2 - l)} \]

Proof. The result is based on:

\[ F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \]
**Lemma 7.45.**
\[
\det \left( \frac{\partial^2}{\partial t_i \partial t_j} \ln \Phi(t^0, -m_1, -m_2, 1) \right) = k! \prod_{l=0}^{k-1} \frac{(m_1 + m_2 - 2k + l + 2)^3}{(m_1 - l)(m_2 - l)}.
\]

**Proof.** This follows comparing the asymptotic expansion of the Selberg integral:
\[
I(\kappa) = \int_{0 \leq t_k \leq \ldots \leq t_1 \leq 1} \Phi^\frac{1}{k} dt_1 \wedge \ldots \wedge dt_k
\]
with its expression in term of \(\Gamma\)-functions. \(\square\)

The important example for applications is the case of the tensor product of finite dimensional representation and not for generic Verma modules. Given \(m_1, \ldots, m_n \in \mathbb{Z}_{>0}\) let’s consider the function:
\[
\Phi_{k,n}(t, z, m) = \prod_{1 \leq i < j \leq k} (t_i - t_j)^2 \prod_{l=1}^n \prod_{i=1}^k (t_i - z_l)^{-m_l}
\]

**General Problem 7.46.** Study the critical set of this function.

**Remark 7.47.** It’s natural to conjecture that the number of orbits of the critical points must be related to the dimension of \(S\text{Sing} M^\otimes m[|m| - 2k]\).

**Remark 7.48.** Let \(m_1, \ldots, m_n \in \mathbb{Z}_{>0}\) and let’s consider
\[
L^\otimes m = L_{m_1} \otimes \ldots \otimes L_{m_n} = \oplus L_a
\]
where each \(a\) has the form:
1. \(a = |m| - 2k\);
2. \(a \geq 0\).

Let’s denote \(\omega(m, k)\) the multiplicity of \(L_{|m| - 2k}\) in \(L^\otimes m\).

**Example 7.49.** Let take \(m = (1, 1, 1), \ |m| = 3\), so:
\[
L_1 \otimes L_1 \otimes L_1 = L_3 \oplus L_1 \oplus L_1.
\]
In this case we have \(\omega(m, 0) = 1\) and \(\omega(m, 1) = 2\).

Note that \(\dim M^\otimes m[|m| - 2k] = \omega(m, k)\) and each direct summand \(L_a\) contains one singular vector of weight \(a\).

Now we are ready to describe the critical set of the function
\[
\Phi_{k,n}(t, z, m) = \prod_{1 \leq i < j \leq k} (t_i - t_j)^2 \prod_{l=1}^n \prod_{i=1}^k (t_i - z_l)^{-m_l}
\]
for \(m_1, \ldots, m_n \in \mathbb{Z}_{>0}\). Let \(\lambda_1 = \sum_{i=1}^k t_i, \lambda_2 = \sum_{i<j} t_it_j, \ldots, \lambda_k = \prod_{i=1}^k t_i\) the standard symmetric functions of \(t_1, \ldots, t_k\) and denote \(\mathbb{C}_\lambda^k\) the space with coordinates \(\lambda_1, \ldots, \lambda_k\).

**Theorem 7.50.** Let \(m_i, i + 1, \ldots, n, \) as before and \(k \in \mathbb{Z}_{>0}\).
1. If \(|m| + 1 - k > k\), then for generic \(z\) all critical points of \(\Phi_{k,n}(t)\) are non degenerate and the critical set consists of \(\omega(m, k)\) orbits.
2. If \(|m| + 1 - k = k\), then for any \(z\) the function \(\Phi_{k,n}(t)\) has no critical points.
3. If \(0 \leq |m| + 1 - k < k\) then for generic \(z\) the function \(\Phi_{k,n}(t)\) may have only non isolated critical points. Written in symmetric coordinates \(\lambda_1, ..., \lambda_k\), the critical set consists of \(\omega(m, |m| + 1 - k)\) straight lines in the space \(\mathbb{C}_\lambda^k\).

4. If \(|m| + 1 - k < 0\), then for any \(z\) the function \(\Phi_{k,n}(t)\) has no critical points.

**Example 7.51.** Let \(m = (1, 1, 1)\), \(|m| = 3\). In this case we have:

\[
\prod_{1 \leq i < j \leq k} (t_i - t_j)^2 \prod_{l=1}^{3} \prod_{i=1}^{k} (t_i - z_l)^{-1}.
\]

If:

1. \(k = 1\), \(|m| + 1 - k = 3\) so \(\Phi_{1,3}(t)\) has two non degenerate critical points;
2. \(k = 2\), \(\Phi_{2,3}(t)\) has no critical points;
3. \(k = 3\), \(\Phi_{3,3}(t)\) has two straight lines of critical points;
4. \(k = 4\), \(\Phi_{4,3}(t)\) has one straight line of critical points;
5. \(k \geq 5\), \(\Phi_{k,3}(t)\) has no critical points.

**References for this chapter:** [F], [G], [RV], [OT], [V1], [MV], [TV4], [ScV]
A. Varchenko. Special functions, KZ type equations and Representation theory
LECTURE 8

During the last lecture we studied the critical points of the function:

\[ \Phi_{k,n}(t, z, m) = \prod_{1 \leq i < j \leq k} (t_i - t_j)^2 \prod_{l=1}^{n} \prod_{i=1}^{k} (t_i - z_l)^{-m_l} \]

We studied the action on the set of the critical points of the symmetric group and we discussed the following:

**Theorem 8.1.**

\[ \dim \text{Sing } M \otimes^m \lfloor |m| - 2k \rfloor = \binom{k + n - 2}{n - 2} \]

and also:

**Theorem 8.2.** Let \( m_i \in \mathbb{Z}_{>0} \) for \( i = 1, ..., n \), \( k \in \mathbb{Z}_{>0} \) and let \( \omega(m = (m_1, ..., m_n), k) \) be the multiplicity of \( L_{|m|-2k} \) in \( L_{m_1} \otimes ... \otimes L_{m_n} \). Then:

1. If \( |m| + 1 - k > k \), then for generic \( z \) all critical points of \( \Phi_{k,n}(t) \) are non degenerate and the critical set consists of \( \omega(m, k) \) orbits.
2. If \( |m| + 1 - k = k \), then for any \( z \) the function \( \Phi_{k,n}(t) \) has no critical points.
3. If \( 0 \leq |m| + 1 - k < k \) then for generic \( z \) the function \( \Phi_{k,n}(t) \) may have only non isolated critical points. Written in symmetric coordinates \( \lambda_1, ..., \lambda_k \), the critical set consists of \( \omega(m, |m| + 1 - k) \) straight lines in the space \( \mathbb{C}^k \).
4. If \( |m| + 1 - k < 0 \), then for any \( z \) the function \( \Phi_{k,n}(t) \) has no critical points.

Today we will discuss the relation of the previous theorem (in particular the statements 2., 3., and 4.) with the theory of the Fuchsian differential equations.

### 1. Differential Equations with Regular Singular Points

We will focus to the case of second order equations:

\[ u'' + p(z)u' + q(z)u = 0 \]

(8.3)

Let’s start with the following:

**Definition 8.4.** A point \( z_o \in \mathbb{C} \) is called ordinary for the differential equation if the functions \( p(z) \) and \( q(z) \) are holomorphic at a neighborhood of this point.

To study the case \( z = \infty \) put \( z = \frac{1}{t} \).
\[
\frac{du}{dz} = \frac{d\xi}{dz} \frac{du}{d\xi} = -\xi^2 \frac{du}{d\xi}
\]
and
\[
\frac{d^2 u}{d\xi^2} = \xi^4 \frac{d^2 u}{d\xi^2} + 2\xi^3 \frac{du}{d\xi}.
\]

With this change of variables the equation turns into:

\[
(8.5) \quad \frac{d^2 u}{d\xi^2} + \left[\frac{2}{\xi} - \frac{1}{\xi^2} p\left(\frac{1}{\xi}\right)\right] \frac{du}{d\xi} + \frac{1}{\xi^4} q\left(\frac{1}{\xi}\right) u.
\]

So we have the following:

**Definition 8.6.** The point \(z = \infty\) is an ordinary point of the equation if \(\xi = 0\) is an ordinary point of the transformed equation, i.e if:

\[
\left[\frac{2}{\xi} - \frac{1}{\xi^2} p\left(\frac{1}{\xi}\right)\right]
\]

and

\[
\frac{1}{\xi^4} q\left(\frac{1}{\xi}\right)
\]

are holomorphic functions at \(\xi = 0\).

**Definition 8.7.** If \(z_o\) is called singular point for the equation if it is not ordinary.

**Definition 8.8.** The point \(z_o\) is called regular singular if the following conditions are verified:

1) \(z_o\) is singular;
2) at \(z_o\) the function \(p(z)\) at has a pole of order \(\leq 1\);
3) at \(z_o\) the function \(q(z)\) has a pole of order \(\leq 2\).

**Remark 8.9.** We have analogous definitions for the point \(z = \infty\) and local coordinate \(z = \frac{1}{\xi}\).

In the following two section we will discuss the solutions of the equation at an ordinary point and at a singular point.

**2. Solutions at a neighborhood of an ordinary point**

Let \(z_o\) an ordinary point, so we can write:

\[
p(z) = \sum_{n=0}^{\infty} b_n (z - z_o)^n
\]

and

\[
q(z) = \sum_{n=0}^{\infty} d_n (z - z_o)^n
\]

for \(|z - z_o| < R\). We look for solutions of the form:

\[
u(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n.
\]

Under these hypothesis we have the following:
Theorem 8.10. For any choice of $a_0, a_1$ there exist a unique series

$$u(z) = \sum_{n=0}^{\infty} a_n(z - z_0)$$

with a non-zero radius of convergence, satisfying the differential equation:

$$u'' + p(z)u' + q(z)u = 0.$$ 

Proof. Plugging $u(z) = \sum_{n=0}^{\infty} a_n(z - z_0)$ into the equation $u'' + p(z)u' + q(z)u = 0$ we get:

for $n = 0$, $0a_0 = 0$,

for $n = 1$, $0a_1 + 0a_0b_0 = 0$,

while for $n \geq 2$,

$$n(n - 1)a_n + \sum_{k=0}^{n-1} ka_kb_{n-k-1} + \sum_{k=0}^{n-2} a_kb_{n-k-2}.$$ 

3. Solutions at a neighborhood of a regular singular point

Let’s assume $z_0$ be a regular singular point so that the coefficients of the equation can be written in the following way:

$$p(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^{n-1}$$

and

$$q(z) = \sum_{n=0}^{\infty} d_n(z - z_0)^{n-2},$$

for $|z - z_0| < R$. In this case we look for solutions of the form:

$$u(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^{n+c}$$

where $a_0 \neq 0$ and $c$ is a constant that has to be determined. Plugging the function $u(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^{n+c}$ into the equation, we get the following equation:

$$\sum_{n=0}^{\infty} (n + c)(n + c - 1)a_n(z - z_0)^{n+c-2} + \sum_{n=0}^{\infty} p_n(z - z_0)^{n-1} \left[ \sum_{n=0}^{\infty} (n + c)a_n(z - z_0)^{n+c-1} \right] +$$

$$+ \left[ \sum_{n=0}^{\infty} q_n(z - z_0)^{n-2} \right] \left[ \sum_{n=0}^{\infty} a_n(z - z_0)^{n+c} \right] = 0.$$ 

From this equation we get the following recurrence relations:

for $n = 0$:

(8.11) $[c(c - 1) + cp_0 + q_0]a_0 = 0,$

while for $n \geq 1$:

(8.12) $[(n + c)(n + c - 1) + (n + c)p_0 + q_0]a_n = -\sum_{k=0}^{n-1} a_k[(k + c)p_{n-k} + q_{n_k}].$
Since $a_0 \neq 0$, we have:

\[(8.13) \quad c^2 + (p_0 - 1) + q_0 = 0.\]

**Definition 8.14.** The equation $c^2 + (p_0 - 1) + q_0 = 0$ is called indicial equation, while the roots $c_1$ and $c_2$ are called the exponents of the equation at the point $z = z_0$.

Let’s write $F(c) = c^2 + (p_0 - 1) + q_0$ so that we have:

\[F(n + c)a_n = -\sum_{k=0}^{n-1} a_k[(k + c)p_{n-k} + q_{n-k}]\]

for $n \geq 1$, and $F(c) = 0$, for the indicial equation. Let’s order the roots $c_1, c_2$ of the polynomial $F(c)$ in such a way $\Re c_1 \geq \Re c_2$ and set $s = c_1 - c_2$. Now if we start with $c = c_1$, we get:

\[F(n + c_1) = n(n + c_1 - c_2) = n(n + s)\]

from which we see that the recurrence equations are satisfied; so we can state the following:

**Theorem 8.15.** If $z_0$ is a regular singular point with exponentials $c_1$ and $c_2$ such that $\Re c_1 \geq \Re c_2$, then the series:

\[(8.16) \quad u_1(z) = (z - z_0)^{c_1} + \sum_{n=1}^{\infty} a_n(z - z_0)^{n+c_1}\]

satisfies the equation and it is convergent in a region of the form $0 < |z - z_0| < r$.

**Question 8.17.** How to find the second solution?

The answer to this question depends on $s = c_1 - c_2$. Put $c = c_2$, then the equation for $n = 0$ is satisfied; for $n \geq 1$ we get:

\[n(n-s)a_n = \sum_{k=0}^{n-1} q_k[(k + c_2)p_{n-k} + q_{n-k}] :\]

**Theorem 8.18.** If $s \notin \mathbb{Z}$ then there exists a series:

\[(8.19) \quad u_2(z) = (z - z_0)^{c_2} + \sum_{n=1}^{\infty} a_n(z - z_0)^{n+c_1}\]

that satisfies the equation and converges in a region of the form $0 < |z - z_0| < r$.

In the case $s = 0$ we don’t get any new solution, while if $s \in \mathbb{Z}_{>0}$, for $n = s$, we get:

\[0 = \sum_{k=0}^{s-1} q_k[(k + c_2)p_{s-k} + q_{s-k}] .\]

This gives a non trivial constraint: given arbitrary $a_0$ and $a_s$ we can get:

\[u_1(z) = (z - z_0)^{c_1} + \sum_{n=1}^{\infty} a_n(z - z_0)^{n+c_1}\]
and
\[ u_2(z) = (z - z_0)^{c_2} + \sum_{n=1}^{\infty} a_n(z - z_0)^{n+c_2} \]

if \(a_0\) and \(a_s\) automatically satisfy the recurrence relation or:
\[ u_2(z) = u_1 \ln(z - z_0) + \sum_{n=0}^{\infty} d_n(z - z_0)^{n+c_2} \]

if not.

### 4. Fuchsian Equations

Let’s start with the following:

**Definition 8.20.** [Lazarus Fuchs (1833-1902)] A linear differential equation is called fuchsian if all its singular points are regular singular.

**Remark 8.21.** During this section we will focus only on second order differential equations.

**Remark 8.22.** If the equation:
\[ u'' + p(z)u' + q(z)u = 0 \]
is fuchsian, then the coefficients \(q(z)\) and \(p(z)\) are rational functions.

We also have the following:

**Proposition 8.23.** A fuchsian equation has at least a singular point.

In what follows we will give a description of the second order fuchsian differential equations with one two and three singular points. Let’s start with the first case. Let’s assume that \(u'' + p(z)u' + q(z)u = 0\) has only one singular point at \(z = z_1\), then:

**Proposition 8.24.** The equation has the form:
\[
(8.25) \quad u'' + \frac{2}{z - z_1} u' = 0.
\]
and its solutions are of the form \(u(z) = a + \frac{b}{z - z_1}\) for given \(a\) and \(b\).

**Proof.** From the hypothesis we have that:
\[ p(z) = \frac{p_1(z)}{z - z_1} \]
and
\[ q(z) = \frac{q_1(z)}{(z - z_1)^2} \]
where \(p_1(z)\) and \(q_1(z)\) are polynomials. Since \(z = \infty\) is an ordinary point we have:
\[ 2z - z^2 p(z) = 2z - \frac{z^2 p_1(z)}{z - z_1} = \frac{2z^2 - z^2 p_1 - 2z_1 z}{z - z_1} \]
and
\[ z^4 q(z) = \frac{z^4 q_1(z)}{(z - z_1)^2} \]
are holomorphic at $z = \infty$. From the last equation we deduce that $q_1 \equiv 0$ (otherwise $\deg z^4 q_1(z) \geq 4$) while from the first one follows that $p_1 = 2$. Clearly $u_1(z) = 1$ and $u_2(z) = \frac{1}{z-z_1}$ are solutions.

We also have the following result:

**Theorem 8.26.** If $z = \infty$ is the only singular point of a fuchsian differential equation, then the equation is:

\[ u'' = 0. \]  

(8.27)

**Proof.** In this case the coefficients $p(z)$ and $q(z)$ are two polynomials. So we have that the function $2z - z^2 p(z)$ has poles of order $\leq 1$ while the function $z^4 q(z)$ has poles of order $\leq 2$. From this observation follows that $p(z) = q(z) = 0$. □

Now let’s discuss the case of fuchsian equations with two singular points $z_1$ and $z_2$. In this case we have two different cases:

1. $z_2 = \infty$, or:
2. $z_1$ and $z_2 \in \mathbb{C}$.

In the first case we have:

**Proposition 8.28.** The fuchsian equation can be written as:

\[ u'' + \frac{k_1}{z - z_1} u' + \frac{k_2}{(z - z_1)^2} u = 0, \]  

(8.29)

with $k_1, k_2 \in \mathbb{C}$. Moreover the solutions are:

\[ u(z) = A(z - z_1)^{c_1} + B(z - z_1)^{c_2} \]

if $c_1 \neq c_2$, or:

\[ u(z) = (z - z_1)^c (A + B \ln(z - z_1)) \]

(8.30)

if $c_1 = c_2 = c$.

For the second case, when both $z_1$ and $z_2$ belong to $\mathbb{C}$, it is possible to prove the following:

**Proposition 8.32.** The fuchsian equation has the form:

\[ u'' + \frac{2z + k_3}{(z - z_1)(z - z_2)} u' + \frac{k_4}{(z - z_1)^2(z - z_2)^2} u = 0. \]

(8.31)

Let’s now start to discuss the case of three singular point.

**Remark 8.34.** We have the following general:

**Theorem 8.35.** A second order fuchsian equation with $n + 1$ singular points $z_1, ..., z_n, \infty$ has the form:

\[ u'' + \frac{T_{n-1}}{(z - z_1)...(z - z_n)} u' + \frac{T_{2n-2}}{(z - z_1)^2...(z - z_n)^2} u = 0, \]

(8.32)

where $T_{n-1}$ and $T_{2n-2}$ are polynomials of degree less or equal to $n - 1$ and $2n - 2$.

If we call $\alpha_{1,k}$ and $\alpha_{2,k}$ the exponents of the equation at the point $z = z_k$ and $\alpha_{1,\infty}$ and $\alpha_{2,\infty}$ the exponents at the point $z = \infty$, then:
Theorem 8.37. [Fuchsian Invariants.]

\[ \alpha_{1,\infty} + \alpha_{2,\infty} + \sum_{k=1}^{n} (\alpha_{1,k} + \alpha_{2,k}) = n - 1. \]

Let's focus on the case of three singular points \( z_1, z_2 \) and \( \infty \). In this case the exponents are: \((\alpha_{1,\infty}, \alpha_{2,\infty})\) (\(\alpha_{1,1}, \alpha_{2,1}\)) (\(\alpha_{1,2}, \alpha_{2,2}\)) and we have:

\[ \alpha_{1,\infty} + \alpha_{2,\infty} + \sum_{k=1}^{2} (\alpha_{1,k} + \alpha_{2,k}) = 1. \]

Claim 8.40. the fuchsian equation can be written in the following way:

\[
\frac{1}{z - z_1} + \frac{1}{z - z_2} \frac{\alpha_{1,1} \alpha_{2,2}}{(z - z_1)^2} + \frac{\alpha_{2,1} \alpha_{2,2} - \alpha_{1,1} \alpha_{2,1} - \alpha_{1,2} \alpha_{2,2}}{(z - z_1)(z - z_2)} u' + \\
\frac{1 - \alpha_{1,1} - \alpha_{2,2}}{z - z_1} \frac{1 - \alpha_{1,2} - \alpha_{2,2}}{z - z_2} \frac{1 - \alpha_{1,1} \alpha_{2,2}}{(z - z_1)^2} + \frac{(z - z_1)(z - z_2)}{(z - z_1)(z - z_2)} u = 0
\]

Remark 8.41. The previous equation is the form of the general Fuchsian equation of the second order and with three regular singular points written in terms of the exponents.

Using the notation \( z = a, b, c \) for the singular points (all of them different by \( \infty \)) and \((a', a''), (b', b''), (c', c'')\) for the correspondents exponents, we have:

Theorem 8.42. The previous equation can be written as:

\[
u'' + \left[ \frac{1 - a' - a''}{z - a} + \frac{1 - b' - b''}{z - b} + \frac{1 - c' - c''}{z - c} \right] u' + \\
\frac{a''(a - b)(a - c)}{z - c} + \frac{b''(b - a)(b - c)}{z - b} + \frac{c''(c - a)(c - b)}{z - c} u = 0.
\]

Remark 8.43. This equation is called Riemann-Papperitz equation.

Definition 8.44. [Riemann P-symbol.]

Let's write:

\[ u = P \begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} \]

This is called the Riemann P-symbols. This denotes a solution of a Fuchsian differential equation with singular points at \( a, b, c \) (distinct but can be \( \infty \)) and exponentials \((a', a''), (b', b''), (c', c'')\).

Remark 8.46. Under these hypothesis we have:

\[ a' + a'' + b' + b'' + c' + c'' = 1. \]

Remark 8.47. The informations given in the P-symbol determines the equation uniquely.
Theorem 8.48. Let’s suppose $a, c \in \mathbb{C}$ and:

\[
(8.49) \quad u = P \begin{pmatrix}
    a & b & c \\
    a' & b' & c' \\
    a'' & b'' & c'' \\
\end{pmatrix}.
\]

If

\[
u = \left(\frac{z-a}{z-c}\right)^k w
\]

then:

\[
(8.50) \quad u = P \begin{pmatrix}
    a & b & c \\
    a' - k & b' & c' + k \\
    a'' - k & b'' & c'' + k \\
\end{pmatrix}.
\]

Theorem 8.51. If:

\[
(8.52) \quad u = P \begin{pmatrix}
    a & b & \infty \\
    a' & b' & c' \\
    a'' & b'' & c'' \\
\end{pmatrix}
\]

and $u = (z-a)^k$ then:

\[
(8.53) \quad u = P \begin{pmatrix}
    a & b & \infty \\
    a' - k & b' & c' + k \\
    a'' - k & b'' & c'' + k \\
\end{pmatrix}.
\]

Theorem 8.54. If

\[
(8.55) \quad u = P \begin{pmatrix}
    a & b & c \\
    a' & b' & c' \\
    a'' & b'' & c'' \\
\end{pmatrix}
\]

the fractional transformation:

\[
v = \frac{Az + B}{Cz + D}
\]

with $AD - CB \neq 0$ maps $z = a, b, c$ into $v = a_1, b_1, c_1$ and the solution $u$ can be written in terms of $v$ in the following way:

\[
(8.56) \quad u = P \begin{pmatrix}
    a_1 & b_1 & c_1 \\
    a'_1 & b'_1 & c'_1 \\
    a''_1 & b''_1 & c''_1 \\
\end{pmatrix} v
\]

We also have the following general:

Lemma 8.57. Any linear fractional transformation:

\[
v = \frac{Az + B}{Cz + D}
\]

with $AD - BC \neq 0$ is the composition of elementary transformations:

$v = \gamma z$, $v = \gamma + z$ and $v = \frac{1}{z}$.
We have the following important:

**Theorem 8.58.** Any P-symbols can be reduced to a P-symbols of the form:

\[
y = P \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1 - \gamma & \gamma - \alpha - \beta & \beta \end{pmatrix} x
\]

**(8.59)**

**Remark 8.60.** The corresponding differential equation is the following:

\[
\frac{d^2y}{dx^2} + \left[ \frac{1 - (1 - \gamma)}{x} + \frac{1 - (\gamma - \alpha - \beta)}{x - 1} \right] \frac{dy}{dx} + \frac{\alpha \beta}{x(x - 1)} y = 0.
\]

**(8.61)**

Multiplying both sides by \(x(x - 1)\) we get:

\[
x(x - 1)y''(x) + [\gamma - (\alpha + \beta + 1)x]y' - \alpha \beta y = 0,
\]

i.e. the Gauss differential equation.

Let’s start the discussion the connection of the theory of the Fuchsian differential equations with the critical points of the master function. Let’s consider the following:

\[
F(x)u''(x) + G(x)u'(x) + H(x)u(x) = 0
\]

where \(F, G\) and \(H\) are polynomials of degree \(n, n - 1\) and \(n - 2\) respectively.

**Claim 8.64.** If the polynomial \(F\) has no multiple roots then the previous equation is Fuchsian.

In this case let’s write:

\[
F(x) = \prod_{j=1}^{n} (x - z_j)
\]

\[
\frac{G(x)}{F(x)} = \sum_{j=1}^{n} \frac{m_j}{x - z_j}
\]

for suitable complex numbers \(m_j\) and \(z_j\).

**Claim 8.65.** It is possible to prove that 0 and \(m_j + 1\) are exponents of the above equation and if \(-k\) is one of the exponents at \(z = \infty\) then the other is \(k - l(m) - 1\).

**Problem:** Given polynomials \(F(x)\) and \(G(x)\) as above:
1. find a polynomial \(H(x)\) of degree at most \(n - 2\) such that the equation has a polynomial solution of a preassigned degree \(k\);
2. find the number of solutions of 1.

we have the following classical results (reference: Szego G., “Orthogonal Polynomials”):

**Theorem 8.66.** 1. Let \(u(x)\) a polynomial solution of degree \(k\) of the equation, with roots \(t_1^0, \ldots, t_k^0\) of multiplicity one. Then \(t^0 = (t_1^0, \ldots, t_k^0)\) is a critical point of the function \(\Phi_{k,n}(t; z, m)\), where \(z = (z_1, \ldots, z_n)\) and \(m = (m_1, \ldots, m_n)\).
2. Let \( t^0 \) be a critical point of the function \( \Phi_{k,n}(t; z, m) \), then the polynomial \( u(x) = (x - t^0_1) \cdots (x - t^0_k) \) of degree \( k \) is a solution of the equation with

\[
H(x) = \left( -\frac{F(x)u''(x) - G(x)u'(x)}{u(x)} \right)
\]

being a polynomial of degree at most \( n - 2 \).

**Remark 8.67.** Recall that the critical points of the function \( \Phi_{k,n}(t; z, m) \) are given by:

\[
\sum_{j=1}^{n} \frac{-m_j}{t_i - z_j} + \sum_{j \neq i} \frac{2}{t_i - t_j}, \quad i = 1, \ldots, k.
\]  

(8.68)

**Remark 8.69.** This can be seen as an equivalent version of the Bethe ansatz problem.

**Proof.** (of part (1)) Let \( u(x) = (x - t^0_1) \cdots (x - t^0_k) \) be a solution of the equation:

\[
Fu'' + Gu' + Hu = 0,
\]

and set \( x = t^0_i \); then:

\[
F(t^0_i)u''(t^0_i) + G(t^0_i)u'(t^0_i) + H(t^0_i)u(t^0_i) = 0.
\]

From this we can deduce:

\[
\frac{u''(t^0_i)}{u'(t^0_i)} + \frac{G(t^0_i)}{F(t^0_i)} = 0,
\]

but by definition we have:

\[
\frac{G(t^0_i)}{F(t^0_i)} = \sum_{j \neq i} \frac{-m_i}{t^0_i - t^0_j} + \sum_{j \neq i} \frac{2}{t^0_i - t^0_j}.
\]

To conclude we need the following:

**Lemma 8.70.** If \( u(x) = (x - t^0_1) \cdots (x - t^0_k) \), then:

\[
\frac{u''(t^0_i)}{u'(t^0_i)} = \sum_{j \neq i} \frac{2}{t^0_i - t^0_j}.
\]  

(8.71)

**Example 8.72.** If:

\[
u(x) = (x - t^0_1)(x - t^0_2)
\]

then:

\[
u(x) = (x - t^0_1) + (x - t^0_2),
\]

\[u'' = 2\]

so:

\[
\frac{u''(t^0_1)}{u'(t^0_1)} = \frac{2}{t^0_1 - t^0_2}
\]

This give us part i. of the previous theorem.
Remark 8.73. The critical points label solutions of the problem 2., and a critical point defines a differential equation and a polynomial solution of that equation.

Remark 8.74. In 19-th century Heine and Stieltjes showed that for fixed real $z_1, ..., z_n$ and negative $m_1, ..., m_n$ the problem 2. has:

$$\binom{k + n - 2}{n - 2}$$

solutions (reference: Szego G., “Orthogonal Polynomials”).

Let $m_1, ..., m_n \in \mathbb{Z}_{\geq 0}$ and $z \in \mathbb{Z}_{>0}$, such that $|m| - 2k \geq 0$, then $k < |m| - k + 1$. Assume that $t^0$ is a critical point of $\Phi_{k,n}(t; z, m)$, then it is possible to construct a line of critical points of the function $\Phi_{|m|-k+1,n}(t; z, m)$.

Example 8.75. Let $m = (1, 1, 1)$ and

$$\Phi_{1,3}(t; z, m) = (t - z_1)^{-1}(t - z_2)^{-1}(t - z_3)^{-1}.$$ 

it is possible to construct a line of critical points of the function: Given the two critical points of the function $\Phi_{1,3}(t; z, m)$ we get two lines of critical points of the function.

$$\Phi_{3,3}(t; z, m) = \prod_{i=1}^{3} \prod_{t=1}^{3} (t_i - z_i)^{-1} \prod_{1 \leq i < j \leq 3} (t_i - t_j)^2.$$ 

Moreover it is possible to show that for generic $z_1, z_2, z_3$ the points on those two lines are the only critical points.

5. Enumerative Algebraic Geometry

It is possible to reformulate the main theorem in terms of Enumerative Algebraic Geometry. Let $V$ a two dimensional space of polynomials of one variable with coefficients in $\mathbb{C}$. Let $k_1$ the degree of a typical polynomial and $k_2$ the degree of a special polynomial ($k_1 < k_2$)

Definition 8.76. Given two functions $f(x)$ and $g(x)$ let’s denote with:

\begin{equation}
W(f, g) = \begin{vmatrix}
f & g \\
f' & g'
\end{vmatrix}
\end{equation}

their Wronskian.

Remark 8.78. If $(f, g)$ and $(\tilde{f}, \tilde{g})$ are two basis of $V$ then there exits a constant $c$ such that:

$$W(f, g) = cW(\tilde{f}, \tilde{g}).$$

Lemma 8.79. If $deg f = k_1$ and $deg g = k_2$ then $deg W(f, g) = k_1 + k_2 - 1$.

Proof. This follows from direct calculation:

\begin{equation}
W(f, g) = \begin{vmatrix}
x^{k_1} + \ldots & x^{k_2} + \ldots \\
k_1 x^{k_1 - 1} + \ldots & k_2 x^{k_2 - 1} + \ldots
\end{vmatrix} = (k_2 - k_1)x^{k_1 + k_2 - 1} + ...
\end{equation}

Let $W_V = x^{k_1 + k_2 - 1} + ...$ the Wronskian of $V$ and let’s suppose that $W_V(x) = \prod_{l=1}^{n} (x - z_i)^{m_l}$, where $m_l \geq 1$. 
Definition 8.81. We will say that $V$ is not singular if $\forall x_0 \in \mathbb{C}$ there is $f \in V$ such that $f(x_0) \neq 0$.

Now we can state the following:

Problem: Assume that $k_1 < k_2$ and $W_V(x) = \prod_{l=1}^{n} (x - z_l)^{m_l}$ are fixed. What is the number of non singular vector spaces $V$ of polynomials with such data?

It is possible to get an answer to question in terms of representation of $\mathfrak{sl}_2$:

Theorem 8.82. For generic $z_1, \ldots, z_n$ the number of non singular vector spaces with such data is equal to:

$$\text{mult}L_{k_1-k_1-1} \subset L_{m_1} \otimes \ldots \otimes L_{m_n}.$$ 

References for this chapter: [R], [ScV]
LECTURE 9

During this lecture we will study the elliptic generalization of the hypergeometric functions. Motivations come from CFT (Conformal Field Theory) where to any Riemann surfaces $\Sigma$ with marked points decorated with finite dimensional representations of a simple Lie algebra is assigned the (finite dimensional) vector space of conformal blocks. The set of all conformal blocks forms a vector bundle over the moduli space of the marked Riemann surfaces, this vector bundle carries a projectively flat connection. We will discuss in detail the case of genus 1:

$$\Sigma = T = \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}, \quad Im \tau > 0.$$  

Notations: we will consider the Lie algebra $g = sl_2$, the Cartan subalgebra $h = \mathbb{C} h$, the module:

$$V = V_1 \otimes \ldots \otimes V_n,$$

and we define:

$$V[0] = \{ v \in V | hv = 0 \}.$$  

The conformal block will be given by the function:

$$u = u(z_1, \ldots, z_n, \lambda, \tau) \in V[0], \quad \lambda, z_i \in \mathbb{C}, \ i = 1, \ldots, n \ \tau \in \mathbb{H},$$

i.e will be a function of $\lambda$ and will depend on the parameter $z_1, \ldots, z_n, \tau$.

We will discuss the following equations:

\begin{align}
1. \quad & \kappa \frac{\partial u}{\partial z_i} = \sum_{j \neq i} r^{(i,j)}(z_i - z_j, \lambda, \tau) u - h^{(i)} \frac{\partial u}{\partial \lambda}, \ i = 1, \ldots, n; \\
2. \quad & 4\pi i \kappa \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \lambda^2} + \sum_{i,j} H^{(i,j)}(z_i - z_j, \lambda, \tau) u, \ i = 1, \ldots, n.
\end{align}

Remark 9.3. The equation 9.2 is called KZB heat equation.

Example 9.4. For $n = 1$ we have $V = V_1 = L_{2p}$ (dim $L_{2p} = 2p + 1$) and $V[0] = L_{2p}[0] = \mathbb{C} f^p v_{2p}$. In this case the function $u = u(\lambda, z, \tau)$ is a scalar valued function and we have that:

\begin{align}
1. \quad & \frac{\partial u}{\partial z_1} = 0, \\
2. \quad & 2\pi i \kappa \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \lambda^2} + p(p + 1) \rho'(\lambda, \tau) u,
\end{align}

where:

$$\rho(\lambda, \tau) = \frac{\theta'(\lambda, \tau)}{\theta(\lambda, \tau)},$$

and

$$\theta(\lambda, \tau) = - \sum_{j \in \mathbb{Z}} e^{\pi i (j + \frac{1}{2})^2 \tau + 2\pi i (j + \frac{1}{2})(\lambda + \frac{1}{2})}$$

61
Remark 9.7. The derivative in the equation 9.6 is taken w.r.t \( \lambda \), i.e \( (\frac{\partial}{\partial \lambda}) \).

Remark 9.8. The function \( \theta \) is the first Jacobi theta function.

Remark 9.9. We have that:
\[
\rho'(\lambda, \tau) = -\varphi(\lambda, \tau) + c(\tau)
\]
where:
\[
\varphi(z, \tau) = \frac{1}{z^2} + \sum_{n,m \in \mathbb{Z}^2 \setminus (0,0)} \left( \frac{1}{(z - m - n\tau)^2} - \frac{1}{(m + n\tau)^2} \right)
\]
is the Weierstrass \( \wp \)-function.

Example 9.10. For \( g = \mathfrak{sl}_N, n = 1, V = S^{\mathfrak{p}N}_\mathbb{C}^N \) and \( h = \mathbb{C}/\mathbb{C}(1, ..., 1) \), the KZB equation becomes:
\[
4\pi i \kappa \frac{\partial u}{\partial \tau} = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial \lambda^2_i} + 2p(p+1) \sum_{i<j} \rho'(\lambda_i - \lambda_j, \tau)u.
\]
The RHS of the previous equation is the hamiltonian operator of the quantum elliptic Calogero-Moser N-body system.

Remark 9.12. A remarkable fact about all form of KZ equation is that them can be realized geometrically. Their solutions can be expressed in terms of hypergeometric integrals depending on parameters. The KZ equation(s) can also be realized as Gauss-Manin connection.

Example 9.13. Let \( n = 1, p = 1 \) so \( V = L_2 \) (dim \( L_2 = 3 \)). In this case the solution of the KZB equation can be written in the following form:
\[
u(\lambda, \tau) = u_g(\lambda, \tau) = \int_0^1 \left( \frac{\theta(t, \tau)}{\theta'(0, \tau)} \right)^{-\frac{2}{\kappa}} \frac{\theta(\lambda - t, \tau)\theta'(0, \tau)}{\theta(\lambda, \tau)\theta(t, \tau)} g(\lambda - \frac{2}{\kappa}t, \tau) dt,
\]
where \( g = g(\lambda, \tau) \) is any holomorphic solution of:
\[
2\pi i \kappa \frac{\partial g}{\partial \tau} = \frac{\partial^2 u}{\partial \lambda^2}.
\]
For instance we can take the function:
\[
g = e^{\lambda \mu + \frac{\mu^2}{2\pi i \kappa} \tau} \quad \mu \in \mathbb{C}.
\]

Remark 9.14. The simplest KZ equations have the classical Gauss hypergeometric function as their solution. So it is natural to consider solutions of all KZ type equations as generalized hypergeometric functions. The Gauss hypergeometric function is:
\[
\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, b; c; z) = \int_1^\infty t^{a-1}(t-1)^{c-b-1}(t-z)^{-a} dt
\]
We can see that the function:
\[
\left( \frac{\theta(t, \tau)}{\theta'(0, \tau)} \right)^{\alpha}
\]
is the analogue of the function \( t^{\alpha} \), while:
\[
\frac{\theta(\lambda - t, \tau)\theta'(0, \tau)}{\theta(\lambda, \tau)\theta(t, \tau)}
\]
is the analogue of the function $\frac{1}{t}$.

1. Asymptotic Solutions

Let’s consider the asymptotic behavior of the function $u = u(\lambda, \tau)$, i.e let’s study the behavior of this function for $\kappa \to 0$.

**Remark 9.15.** Under this hypothesis we are dealing with the following eigenvalue problem:

$$\frac{d^2}{d\lambda^2} v(\lambda) - 2\varphi(\lambda, \tau)v(\lambda) = E v(\lambda)$$

where $E$ is a number. The equation:

$$\frac{d^2}{d\lambda^2} v(\lambda) - p(p+1)\varphi(\lambda, \tau)v(\lambda) = E v(\lambda) \quad (9.16)$$

is called Lame’ equation and its solutions are called Lame’ functions.

We are looking for solutions of the form:

$$u(\lambda, \tau) = e^{\frac{S(\tau)}{2\kappa}} (f_0(\lambda, \tau) + \kappa f_1(\lambda, \tau) + \ldots)$$

Plugging this expression into the equation:

$$2\pi i\kappa \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \lambda^2} - p(p+1)\rho'(\lambda, \tau)u \quad (9.17)$$

we get recurrence relations for the functions $f_i(\lambda, \tau)$ $n = 0, 1, \ldots$.

**Definition 9.18.** We say that the function $u = u(\lambda, \tau)$ is a formal asymptotic solution if all the recurrence relations are satisfied.

**Corollary 9.19.** If $u = u(\lambda, \tau)$ is an asymptotic solution then for every fixed $\tau$ the function $f_0 = f_0(\lambda, \tau)$ is an eigenvector of the operator:

$$\frac{\partial^2}{\partial \lambda^2} + p(p+1)\rho'$$

with eigenvalue $\frac{\partial S}{\partial \tau}$.

Applying the stationary phase method to the integral:

$$u(\lambda, \tau) = \int_0^1 \left( \frac{\theta(t, \tau)}{\theta'(0, \tau)} \right)^{\frac{\mu}{2}} \frac{\theta(\lambda - t, \tau)\theta'(0, \tau)}{\theta(\lambda, \tau)\theta(t, \tau)} e^{\lambda \mu + \frac{\mu^2}{2}\kappa \tau} dt,$$

we get the following:

**Theorem 9.20.** For any $\mu \in \mathbb{C}$, the function:

$$v(\lambda) = e^{\lambda \mu} \frac{\theta(\lambda - t_0, \tau)}{\theta(\lambda, \tau)}$$

is an eigenfunction of:

$$\frac{\partial^2}{\partial \lambda^2} - 2\varphi(\lambda, \tau)$$
if \( t_0 \) is a critical point, w.r.t \( t \), of the function:
\[
\theta(t, \tau) e^{-\mu t}.
\]

Remark 9.21. This is an example of Bethe ansatz. Hermite developed this method for the Lame’ equation (1872).

Remark 9.22. Similar formulas are available for the eigenfunctions of the Many-Body Elliptic Calogero-Moser Hamiltonian.

2. Special Solutions and Conformal Blocks

In what follows we will consider the differential equation:
\[
2\pi i \kappa \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \lambda^2} + p(p+1)\rho' (\lambda, \tau) u.
\]

Let’s fix \( \kappa \in \mathbb{Z}_{>0} \) and \( p \in \mathbb{Z}_{\geq 0} \) and assume that \( \kappa \geq 2p+2 \). We will consider holomorphic solutions \( u = u(\lambda, \tau) \) of the KZB equation with the following properties:

1. \( u(\lambda + 2, \tau) = u(\lambda, \tau) \);
2. \( u(\lambda + 2\tau, \tau) = e^{-2\pi i \kappa (\lambda + \tau)} u(\lambda, \tau) \);
3. \( u(-\lambda, \tau) = (-1)^{p+1} u(\lambda, \tau) \);
4. \( u(\lambda, \tau) = \mathcal{O}(e^{-2\pi i (\lambda - m - n\tau)^{p+1}}) \), as \( \lambda \to m + n\tau, \ m, n \in \mathbb{Z} \).

Definition 9.24. Holomorphic solutions of 9.23 satisfying the conditions 1., 2., 3., and 4. are called conformal blocks associated with the elliptic curve \( T = \mathbb{C} / \mathbb{Z} + \tau \mathbb{Z} \) and the point \( z = 0 \).

Remark 9.25. It known that under this hypothesis the space of conformal blocks has dimension equal to \( \kappa - 2p - 1 \).

3. Theta functions of level \( \kappa \).

Before describing the space of conformal blocks it is useful discuss some elementary properties of theta functions.

Definition 9.26. Let \( \kappa \in \mathbb{Z}_{>0} \). A holomorphic function \( u = u(\lambda, \tau) \) is called a theta function of level \( \kappa \) w.r.t the lattice \( 2\mathbb{Z} + 2\tau \mathbb{Z} \) if:

1. \( u(\lambda + 2, \tau) = u(\lambda, \tau) \);
2. \( u(\lambda + 2\tau, \tau) = e^{-2\pi i \kappa (\lambda + \tau)} u(\lambda, \tau) \).

Lemma 9.27. The space of theta functions of level \( \kappa \) is \( 2\kappa \) dimensional. A basis is given by:
\[
\theta_{n,\kappa}(\lambda, \tau) = \sum_{j \in \mathbb{Z}} e^{2\pi i (j + \frac{m}{\kappa})^2 + 2\pi i (j + \frac{n}{\kappa}) \lambda}
\]
\( n = 0, \ldots, 2\kappa - 1 \)

Proof. Follows from the definition and from the Fourier expansion. \( \square \)

These functions have the following remarkable:

Properties 9.28. Let \( q = e^{\frac{\pi i}{\kappa}} \):
1. \( \theta_{n,\kappa}(\lambda, \tau) \) is a solution of:
\[
2\pi i \kappa \frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial \lambda^2};
\]
2. \( \theta_{n,\kappa}(\lambda, \tau) = \theta_{n+2\kappa,\kappa}(\lambda, \tau) \);
3. \( \theta_{n,\kappa}(-\lambda, \tau) = \theta_{-n,\kappa}(\lambda, \tau) \);
4. \( \theta_{n,\kappa}(\lambda + \frac{2\kappa}{\tau}, \tau) = q^{2n} \theta_{n,\kappa} \);
5. \( \theta_{n,\kappa}(\lambda + \frac{2\kappa}{\tau}, \tau) e^{-2\pi i t \lambda^2 / \kappa} \theta_{n+2\kappa}(\lambda, \tau) \).

### 3.1. Modular Transformations.

Let’s start with:

**Definition 9.29.** The Modular Group is \( M = SL(2, \mathbb{Z})/(\pm Id) \). The generators are:
\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
and they satisfy the relations:
\[
S^2 = 1, \quad (ST)^3 = 1.
\]

**Remark 9.30.** Elliptic curves corresponding to parameters \( \tau, \tau + 1, -\frac{1}{\tau} \) are isomorphic so it natural to expect relations among conformal blocks associated to these curves.

Let’s introduce the following maps:

**Definition 9.31.**
\[
(Tu)(\lambda, \tau) = u(\lambda, \tau + 1)
\]
and
\[
(Su)(\lambda, \tau) = e^{-\pi i \kappa \lambda^2 / 2\tau} \tau^{-1/2} u(\lambda/\tau, -1/\tau).
\]

**Definition 9.34.** \( \Theta_\kappa \) will denote the space of theta functions of level \( \kappa \).

**Lemma 9.35.** The maps \( T \) and \( S \) preserve the space of \( \Theta_\kappa \)-functions:
\[
1. \quad \theta_{n,\kappa}(\lambda, \tau + 1) = q^{\frac{n^2}{2}} \theta_{n,\kappa}(\lambda, \tau);
\]
\[
2. \quad \theta_{n,\kappa}(\frac{\lambda}{\tau}, -\frac{1}{\tau}) = \sqrt{-\frac{i\tau}{2\kappa}} e^{\pi i \kappa \lambda^2 / 2\tau} \sum_{m=0}^{2\kappa-1} q^{-mn} \theta_{m,\kappa}(\lambda, \tau).
\]

**Remark 9.38.** The maps \( T \) and \( S \) define a projective representation of the Modular group in the space of \( \Theta_\kappa \)-functions.

### 3.2. Symmetric Theta Functions.

**Definition 9.39.** Let’s define:
\[
\theta_{n,\kappa}^S(\lambda, \tau) = \theta_{n,\kappa}(\lambda, \tau) + \theta_{n,\kappa}(-\lambda, \tau) = \theta_{n,\kappa}(\lambda, \tau) + \theta_{-n,\kappa}(\lambda, \tau).
\]

\( \theta_{n,\kappa}^S \) is called symmetric theta function of level \( \kappa \).

It is possible to prove the following:
Lemma 9.41. The space of symmetric theta functions of level \( \kappa \) has dimension equal to \( \kappa + 1 \) and it is generated by \( \theta_{0,\kappa}^S, \ldots, \theta_{\kappa,\kappa}^S \).

Let’s go back to the general theory of theta function and discuss the following:

Example 9.42 (First Jacobi theta function). The first Jacobi theta function is defined by the following formula:

\[
\theta(\lambda, \tau) = - \sum_{j \in \mathbb{Z}} e^{\pi i (j + \frac{1}{2})^2 \tau + 2 \pi i (j + \frac{1}{2})(\lambda + \frac{1}{2})}.
\]

(9.43)

This function satisfies the following product formula:

\[
\theta(\lambda, \tau) = i e^{\pi i \left( \frac{\tau}{4} - \lambda \right)} (\tilde{q}, \tilde{q})(\tilde{q}, \tilde{q})
\]

(9.44)

where \( \tilde{q} = e^{2 \pi i \tau}, \kappa = e^{2 \pi i \lambda} \) and \( (\kappa, \tilde{q}) = \prod_{n=0}^{\infty} (1 - \kappa q^n) \). Moreover this function has the following:

Properties 9.45.
1. \( \theta(-\lambda, \tau) = -\theta(\lambda, \tau) \);
2. \( \theta(\lambda + 1, \tau) = -\theta(\lambda, \tau) \);
3. \( \theta(\lambda + \tau, \tau) = e^{-\pi i \tau - 2 \pi i \lambda} \theta(\lambda, \tau) \);
4. \( \theta(\lambda, \tau) \) has simple zeros at \( \mathbb{Z} + \tau \mathbb{Z} \);
5. \( \theta(\lambda, \tau) \) is a theta function of level 2;
6. \( \theta(\lambda, \tau) \) is a solution of the heat equation:

\[
2 \pi i \kappa \frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial \lambda^2};
\]

7. (Modular property) The function \( \theta(\lambda, \tau) \) satisfies the following two equations:

\[
\theta(\lambda, \tau + 1) = e^{\frac{\pi i}{4}} \theta(\lambda, \tau);
\]

(9.46)

\[
\theta\left(\frac{\lambda}{\tau}, -\frac{1}{\tau}\right) \sqrt{-\frac{i \tau}{\tau}} e^{\pi i \lambda^2 / \tau} \theta(\lambda, \tau)
\]

(9.47)

From the property we see that \( \theta \) generates a one dimensional subspace, invariant w.r.t the action of the modular group.

Let’s go back to the general theory and let’s start to investigate the relation between conformal blocks and theta functions.

Lemma 9.48. For fixed \( \tau \) the space of conformal blocks has dimension \( \kappa - 2p + 1 \) and consists of linear combinations of functions of the form:

\[
\theta^{n+1}(\lambda, \tau) \theta_{n,\kappa-2p}^S(\lambda, \tau), \quad n = 0, \ldots, \kappa - 2p.
\]

Example 9.49. Let’s suppose that \( \kappa = 2p + 2 \); then the space of conformal blocks is one dimensional. This implies that the holomorphic solutions of the KZB equation must be of the form:

\[
u(\lambda, \tau) = A(\tau) \theta^{n+1}(\lambda, \tau).
\]

(9.50)

Lemma 9.51. The function \( u(\lambda, \tau) = \theta^{n+1}(\lambda, \tau) \) is a solution of the KZB heat equation for \( \kappa = 2p + 2 \).
4. Integral Formulas

In this section we will introduce the integral formulas for the conformal blocks.

**Notation 9.52.** Let’s define:

\[ \sigma_\lambda(t, \tau) = \frac{\theta(\lambda - t, \tau)\theta'(0, \tau)}{\theta(\lambda, \tau)\theta(t, \tau)}; \]

\[ E(t, \tau) = \frac{\theta(t, \tau)}{\theta'(0, \tau)}. \]

Let’s also define the Master function:

\[ \Phi(t_1, ..., t_p) = \prod_{j=1}^{p} E(t_j, \tau)^{-\frac{2\pi}{\kappa}} \prod_{1 \leq i < j \leq p} E(t_i - t_j, \tau)^{\frac{2}{\kappa}}. \]

**Remark 9.56.** 9.55 is the elliptic analogous of the Selberg integral.

We also introduce the integration cycles:

\[ \Delta_k \subset \mathbb{R}^k \subset \mathbb{C}^k \]

where

\[ \Delta_k = \{(t_1, ..., t_k) | 0 \leq t_k \leq ... \leq t_1 \leq 1\}. \]

We define \( \tilde{\Delta}_k \) the image of the standard real simplex under the map:

\[ (t_1, ..., t_k) \mapsto (\tau t_1, ..., \tau t_k). \]

**Remark 9.58.** 9.57 define a rotation of the standard simplex.

For \( 0 \leq k \leq p \) let’s define:

\[ J_{n, \kappa}^{[k]}(\lambda, \tau) = \int_{\Delta_k} \prod_{j=1}^{k} \sigma_\lambda(t_j, \tau)dt_j \int_{\tilde{\Delta}_{p-k}} \prod_{j=k+1}^{p} \sigma_\lambda(t_j, \tau)dt_j \Phi(t_1, ..., t_p, \tau)\theta_{n, \kappa}(\lambda + \frac{2}{\kappa} \sum_{j=1}^{p} t_j, \tau). \]

**Definition 9.59.**

\[ u_{n, \kappa}^{[k]}(\lambda, \tau) = J_{n, \kappa}^{[k]}(\lambda, \tau) + (-1)^{p+1} J_{n, \kappa}^{[k]}(-\lambda, \tau). \]

**Theorem 9.61.** For any \( 0 \leq k \leq p \) and any \( n \)

\[ u_{n, \kappa}^{[k]}(\lambda, \tau) \]

is a conformal block.

We have the following:

**Definition 9.62.** The

\[ u_{n, \kappa}^{[k]}(\lambda, \tau) \]

are called elliptic hypergeometric functions.

Now we state some elementary properties of the elliptic hypergeometric functions:

**Properties 9.63.**

1. \( u_{n, \kappa}^{[k]}(\lambda, \tau) = u_{n,2+\kappa}^{[k]}(\lambda, \tau); \)
2. \( w_n^{[k]}(\lambda, \tau) = -q^{2k(n+p-k)}w_n^{[k]}(\kappa, -n-2(p-k)). \)

**Theorem 9.64** (on Basis). 1. The set \( w_n^{[p]}(\lambda, \tau), \ n = p + 1, \ldots, \kappa - p - 1, \) is a basis in the space of conformal blocks.

2. \( w_n^{[p]}(\lambda, \tau) \equiv 0 \) if \( 0 \leq n \leq p \) or \( \kappa - p \leq n \leq \kappa. \)

**Example 9.65.** Let \( \kappa = 2p + 2 \) so the dimension of the space of conformal blocks is equal to 1. In this case we have that:

\[
u_p^{[p]}(\lambda, \tau) \equiv 0 \text{ if } n \neq p + 1 \mod \kappa.
\]

In this case the \( p \)-dimensional integral can be calculate explicitly. This integral can be called the elliptic Selberg integral. Recall that:

\[
P_p(\alpha, \beta, \gamma) = \int_{\Delta_p} \prod_{j=1}^{p} t_j^{\alpha-1}(1 - t_j)^{\beta-1} \prod_{1 \leq i < j \leq p} (t_j - t_k)^{2\gamma} dt_1 \wedge \ldots \wedge d_k =
\]

\[
\frac{1}{p!} \prod_{j=0}^{p-1} \frac{\Gamma(1 + \gamma(1 + j)) \Gamma(\alpha + j\gamma) \Gamma(\beta + j\gamma)}{\Gamma(1 + \gamma) \Gamma(\alpha + \beta + (p + j - 1)\gamma)}.
\]

Now we can state the following:

**Theorem 9.66.**

\[
u_{2p+2}^{[p]}(\lambda, \tau) = C_p P_p\left(\frac{1}{2} + \frac{1}{2(p+1)}, -1 + \frac{1}{p+1}, \frac{1}{2(p+1)}\right) \theta(\lambda, \tau)^{p+1},
\]

where:

\[
C_p = (2\pi)^{\frac{p}{2}} e^{-\pi i \left[\frac{(3p-1)}{4(p+1)} + \frac{p+1}{2}\right]} \prod_{j=1}^{p} (1 - e^{-2\pi i \frac{j}{2p+2}})
\]

**References for this chapter:** [FV], [FSV1, FSV2]
LECTURE 10

During the last lecture we introduced the elliptic analogous of the KZ differential equations. In particular, given \( \tau \in \mathbb{H}_+ \) and \( \lambda \in \mathbb{C} \), we studied the following equation:

\[
2\pi i \kappa \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \lambda^2} + p(p+1)\rho'({\lambda},\tau)u,
\]

where \( \kappa, p \in \mathbb{Z}_+ \) and \( \kappa \geq 2p + 2 \) and \( \rho \) is the logarithmic derivative of the first Jacobi theta function:

\[
\rho = \frac{\theta'({\lambda},\tau)}{\theta'({\lambda},\tau)}.
\]

We also introduced the space of conformal blocks, i.e the space of holomorphic solution of (10.1) that satisfy the following properties:

1. \( u(\lambda + 2, \tau) = u(\lambda, \tau) \);
2. \( u(\lambda + 2, \tau) = e^{-2\pi i \kappa (\lambda + \tau)}u(\lambda, \tau) \);
3. \( u(-\lambda, \tau) = (-1)^{p+1}u(\lambda, \tau) \);
4. \( u(\lambda, \tau) = O((\lambda - m - n\tau)^{p+1}) \) as \( \lambda \to n + m\tau \in \mathbb{Z} + \tau\mathbb{Z} \).

During this lecture we will study the action of the modular group on the space of conformal blocks. To this end we remind the integral formulas introduced at the end of the last lecture. We defined:

\[
E(t, \tau) = \frac{\theta(t, \tau)}{\theta'(0, \tau)},
\]

\[
\sigma_\lambda(t, \tau) = \frac{\theta(\lambda - t, \tau)\theta'(0, \tau)}{\theta(\lambda, \tau)\theta(t, \tau)},
\]

and:

\[
\Phi(t_1, \ldots, t_p) = \prod_{j=1}^{p} E(t_j, \tau)^{-\frac{2\kappa}{\kappa}} \prod_{1 \leq i < j \leq p} E(t_i - t_j, \tau)^{\frac{2}{\kappa}}.
\]

If \( \Delta_k = \{(t_1, \ldots, t_k) | 0 \leq t_k \leq \ldots \leq t_1 \leq 1 \} \) is the standard k-simplex and \( \tilde{\Delta}_k \) is the image of \( \Delta_k \) under the map:

\[
(t_1, \ldots, t_k) \mapsto (\tau t_1, \ldots, \tau t_k),
\]

we have that:

\[
J_{n,k}^{[k]}(\lambda, \tau) = \int_{\Delta_k} \prod_{j=1}^{k} \sigma_\lambda(t_j, \tau)dt_j \int_{\Delta_{p-k}} \prod_{j=k+1}^{p} \sigma_\lambda(t_j, \tau)dt_j \Phi(t_1, \ldots, t_p, \tau)\theta_{\kappa,n}(\lambda + \frac{2}{\kappa} \sum_{j=1}^{p} t_j, \tau),
\]

(10.2)
where \( \theta_{\kappa,n} \) is a theta function of level \( \kappa \).

We also defined the functions:

\[
 u^{[k]}_{\kappa,n}(\lambda, \tau) = J^{[k]}_{\kappa,n}(\lambda, \tau) + (-1)^{p+1} J^{[k]}_{\kappa,n}(-\lambda, \tau)
\]

we observed that:

**Properties 10.3.**
1. \( u^{[k]}_{\kappa,n}(\lambda, \tau) = u^{[k]}_{\kappa,2+n}(\lambda, \tau) \).
2. \( u^{[k]}_{\kappa,n}(\lambda, \tau) = -q^{2k(n+p-k)} u^{[k]}_{\kappa,-n-2(p-k)} \)

and we stated the following important:

**Theorem 10.4 (on Basis).**
1. The set \( u^{[p]}_{\kappa,n}(\lambda, \tau), \ n = p + 1, \ldots, \kappa - p - 1 \), is a basis in the space of conformal blocks.
2. \( u^{[p]}_{\kappa,n} \equiv 0 \) if \( 0 \leq n \leq p \) or \( \kappa - p \leq n \leq \kappa \).

In what follows it will be convenient use the following:

**Notation 10.5.**

\[
 q = e^{\frac{2i\pi}{\kappa}},
\]

\[
 [n] = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]

The following theorem describes some important relations satisfied by the functions \( u^{[k]}_{\kappa,n} \) with different \( k \).

**Remark 10.8.** We recall that \( k \) represents the number of horizontal integrations performed in 10.2.

**Theorem 10.9 (Stokes Theorem).** For \( 0 \leq k \leq p - 1 \) and any \( n \), we have:

\[
 [p-k](q^{m+p-k} - q^{-m-p+k}) u^{[k]}_{\kappa,m} = q^{-m-k-1}[k+1](q^{-2(k+1)} u^{[k+1]}_{\kappa,m+2} - u^{[k+1]}_{\kappa,m}).
\]

**Proof.** We will give the idea of the proof of the theorem. Let \( p = 1 \) and \( k = 0 \). In these hypothesis the integrals are 1 dimensional:

\[
 \int_{\text{interval}} \Phi(t, \tau) \sigma_{\lambda}(t, \tau) \theta_{\kappa,m}(\lambda + \frac{2}{\kappa} t, \tau) dt.
\]

Now will perform the integration on the fundamental domain \((1, \tau)\). Since the integrand is holomorphic we get that:

\[
 I_1 + I_2 + I_3 + I_4 = 0
\]

and from the periodicity properties of the \( \theta \)-functions, we get \( I_1 = c_1 u^{[1]}_m, I_4 = c_4 u^{[0]}_m, I_2 = c_2 u^{[0]}_m, I_3 = c_1 u^{[1]}_{m+2} \).

**Remark 10.12.** The Stokes theorem allows to express an arbitrary function \( u^{[k]}_{\kappa,m} \) in terms of basic ones.
1. Modular Transformations of Conformal Blocks

Let’s introduce the following two transformations:

\[(Tu)(\lambda, \tau) = u(\lambda, \tau + 1),\]
\[(Su)(\lambda, \tau) = e^{-\pi i \kappa \frac{\lambda^2}{\tau^2} - \frac{1}{2}} u(\frac{\lambda}{\tau}, -\frac{1}{\tau}).\]

We have the following:

**Theorem 10.15.** If \(u = u(\lambda, \tau)\) is a solution of the KZB heat equation the \(Tu\) and \(Su\) are also solutions of the same equation. Moreover the maps \(T\) and \(S\) preserve the space of conformal blocks.

We also have the following:

**Proposition 10.16.** The restrictions of the maps \(T\) and \(S\) to space of conformal blocks satisfy the following identities:

\[S^2 = (-1)^p i q^{-p(p+1)} \text{Id},\]
\[(ST)^3 = (-1)^p i q^{-p(p+1)} \text{Id},\]

i.e they define a projective representation of the modular group on the space of conformal blocks.

Now we want to discuss the following problem: find an explicit description of the operators \(S\) and \(T\).

The solution of this problem is contained in the following two lemmas:

**Lemma 10.19.**

\[Tu_{\kappa,n}^{[p]}(\lambda, \tau) = q^{\frac{n^2}{\tau}} u_{\kappa,n}^{[p]},\]

and

**Lemma 10.20.**

\[Su_{\kappa,n}^{[p]}(\lambda, \tau) = e^{-\frac{\pi i}{4} \sqrt{2\kappa}} \sum_{m=0}^{2\kappa-1} q^{-mn} u_{\kappa,m}^{[0]}(\lambda, \tau).\]

**Remark 10.22.** The proofs of the previous lemmas are straightforward. We observe that in 10.21 the RHS is given in terms of functions obtained taking p-vertical integrations while the LHS is a function obtained taking p-horizontal integrations. To get the formula 10.21 we need to apply p-times the Stokes theorem to get the form:

\[Su_{\kappa,n}^{[p]} = \sum_m S_{n,m} u_{\kappa,m}^{[p]},\]

To describe the coefficients \(S_{n,m}\) in 10.23 of the previous remark we need the following:

**Theorem 10.24.**

\[S_{n,m} = \sqrt{-i} \frac{q^{np(n-p)-p(p+1)}}{2\kappa} (q^{-m} - q^{n}) \prod_{j=1}^{p} (q^{-n+j} - q^{n-j}) P_{n-p-1}^{(p+1)}(m).\]

**Definition 10.26.** The function \(P_{n-p-1}^{(p+1)}(x)\) is the Macdonald polynomial of level \(p + 1\) and degree \(n - p - 1\) associated with the Lie algebra \(\mathfrak{sl}_2\).
2. Macdonald Polynomials

The Macdonald polynomials of level $k$ associated with $\mathfrak{sl}_2$ are $x$-even polynomials of $q^{nx}$, $m \in \mathbb{Z}$ defined by the following rules:

1. $P_{0}(k)(x) \equiv 1$ and $P_{n}(k)(x) = q^{nx} + q^{-nx} + \text{lower order terms}$;
2. $\langle P_{m}(k)(x), P_{n}(k)(x) \rangle = 0$ if $m \neq n$, where:

\[
\langle f(x), g(x) \rangle = \text{constant term } \left( f(x)g(x) \prod_{j=0}^{k-1} (1 - q^{2(j+x)})(1 - q^{2(j-x)}) \right).
\]

**Example 10.27.** For $n > 0$ we have that:

\[
P_{0}(0)(x) = q^{nx} + q^{-nx}.
\]

In the following we will define an operator that play an important role in the theory of Macdonald polynomials:

**Definition 10.28 (Shift Operator).** The shift operator $D$, in the $\mathfrak{sl}_2$-case, is defined by:

\[
Df(x) = \frac{f(x - 1) + f(x + 1)}{q^x + q^{-x}}.
\]

where $f = f(x)$ is a given function.

Given the shift operator it is possible to obtain Macdonald polynomials of higher level knowing polynomials of lower level. In fact we have the following:

**Theorem 10.30 (Askey-Ismail).**

\[
(DP_{0}(k))(x) = (q^{-n} - q^{n})P_{n-1}(k+1)(x).
\]

**Example 10.32.** For $n > 0$ we have that:

\[
P_{1}(n-1)(x) = \frac{q^{nx} + q^{-nx}}{q^x + q^{-x}}.
\]

This follows applying the definition of the shift operator and previous theorem to $P_{0}(0)(x) = q^{nx} + q^{-nx}$. From 10.33 we get that $P_{0}(2)(x) = 1$, $P_{1}(2)(x) = q^{x} + q^{-x}$ and that

\[
P_{2}(2)(x) = q^{2x} + \frac{1 + 2q^2 + q^4}{1 + q^2 + q^4} + q^{-2x}.
\]

The following theorem describes the relation between the shift operator and the modular transformations of the basic elliptic hypergeometric functions:

**Theorem 10.34.** For $0 \leq k \leq p$ we have:

\[
Su_{k,n}^{[p]}(\lambda, \tau) = \sqrt{-\frac{i}{2\kappa}} \sum_{m=-p+2k+1}^{\kappa-p-1} q^{\kappa m - \frac{k(k+1)}{2}} \left[ \begin{array}{c} p \\ k \end{array} \right]_{q}^{-1} (q^{n+p+k} - q^{m+p-k})
\]

\[
\times \prod_{j=1}^{k} (q^{n+j} - q^{n-j}) P_{n-k+1}(m + p - k) u_{m}^{[k]}(\lambda, \tau).
\]

**Remark 10.35.** Comparing this formula and the Stokes theorem we see that the Stokes formula gives the shift operator relation 10.31.
3. Trace of Intertwining Operators

In what follows $q = e^{2\pi \iota}$. The quantum group $U_q(\mathfrak{sl}_2)$ has generators $E$, $F$ and $q^{ch}$, where $c \in \mathbb{C}$, with relations:

1. $q^{ch}q^{-ch} = q^{(c+c)h}$;
2. $q^{ch}Eq^{-ch} = q^{2ch}E$;
3. $q^{ch}Fq^{-ch} = q^{-2ch}F$;
4. $EF - FE = \frac{q^h - q^{-h}}{q - q^{-1}}$. The comultiplication is defined by the following:

$\Delta(E) = E \otimes q^h + 1 \otimes E$, $\Delta(F) = F \otimes 1 + q^{-h} \otimes F$, $\Delta(q^{ch}) = q^{ch} \otimes q^{ch}$. Let’s identify weights for $U_q(\mathfrak{sl}_2)$ with complex number as follows and let’s say that a vector $v$ in a $U_q(\mathfrak{sl}_2)$-module has weight $\nu \in \mathbb{C}$ if $q^h v = q^\nu v$. Let $M_\mu$ the Verma module over $U_q(\mathfrak{sl}_2)$ with highest weight $\mu$ and let $v_\mu$ be its highest vector. Let $k$ be a non negative integer such that $\kappa \geq 2k + 2$, $U$ an irreducible finite dimensional representation of $U_q(\mathfrak{sl}_2)$ of weight $2k$ and let’s denote with $U[0]$ the zero weight subspace of $U$. Let $u \in U[0]$. For generic $\mu$ let:

\[(10.36) \quad \Phi^\nu_\mu : M_\mu \longrightarrow M_\mu \otimes U \]

be the intertwining operator defined by:

\[(10.37) \quad \Phi^\nu_\mu v_\mu = v_\mu \otimes u + \frac{Fv_\mu}{[-\mu]} \otimes Eu + ... + \frac{F^jv_\mu}{[j]!(-\mu, q)_j} \otimes E^ju + ... \]

where $(-\mu, q)_j = [n][n + 1]...[n + j - 1]$. Introduce now and $\text{End}(U[0])$-valued function $\psi^{(k)}(q, \nu, \mu)$ defined by:

\[(10.38) \quad \psi^{(k)}(q, \nu, \mu)u = Tr|_{M_\mu}(\Phi^\nu_\mu q^{j\mu}). \]

Since $U[0]$ is 1 dimensional, this function is scalar. It is possible give an explicit description of this function:

**Theorem 10.39 (Etingof-Varchenko).** The function $\psi^{(k)}(q, \nu, \mu)$ is given by the formula:

\[
\psi^{(k)}(q, \nu, \mu) = q^{\kappa \mu} \sum_{j=0}^{k} (-1)^j q^{ \frac{j(j+3)}{2} (q - q^{-1})^{-j-1} \frac{[k+j]!}{[j]![k-j]!} \prod_{l=0}^{j-1} [\mu-l] \prod_{l=0}^{j} [\nu-l] q^{-j\mu-(j-1)\nu}. 
\]

Let’s introduce now the renormalized trace functions:

**Definition 10.40.**

\[
\Psi^{(k)}(q, \nu, \mu) = \prod_{j=1}^{k} \left( \frac{q^{\mu+1-j} - q^{-\mu-1+j}}{q^{\nu+j} - q^{-\nu-j}} \right). 
\]

**Remark 10.41.** The functions $\Psi^{(k)}$ are holomorphic functions of $\mu$.

The relation between Macdonald polynomials and the trace functions is given by the following:

**Theorem 10.42 (Etingof-Styrkas).**

\[
\Psi^{(k)}(q^{-1}, -m - p + k, n - 1) - \Psi^{(k)}(q^{-1}, -m - p + k, -n - 1) = P_{n-k-1}^{(k+1)}(x) \prod_{j=1}^{k} (q^{-n+2j} - q^n). 
\]
Now we will explain how the trace functions enter in the picture. Our starting point was the formula 10.21:

\[ Su_{\kappa,n}^{[p]}(\lambda, \tau) = \frac{e^{-\frac{\pi i}{2}}}{\sqrt{2\kappa}} \sum_{m=0}^{2\kappa-1} q^{-mn} u^{[q]}_{\kappa,m}(\lambda, \tau). \]

Then using the Stokes theorem:

\[ [p - k](q^{m+p-k} - q^{-m-p+k}) u^{[k]}_{\kappa,m} = q^{-m-k-1}(k+1)(q^{-2(k+1)} u^{[k+1]}_{\kappa,m+2} - u^{[k+1]}_{\kappa,m}), \]

we obtained for any 0 ≤ k ≤ p an expression of the form:

\[ Su_{\kappa,n}^{[p]}(\lambda, \tau) = \sqrt{\frac{-i}{2\kappa}} \left[ \sum_{m=-p+2k+1}^{k-p-1} f^{(k)}_{m,n} u^{[k]}_{\kappa,m}(\lambda, \tau) + \sum_{m=\kappa-p+2k+1}^{2\kappa-p-1} f^{(k)}_{m,n} u^{[k]}_{\kappa,m}(\lambda, \tau) \right] \]

for suitable numbers \( f^{(k)}_{m,n} \) given by the following:

**Theorem 10.43.**

(10.44) \[ f^{(k)}_{m,n} = q^{pm-km-k(k+1)}(q^{m+p-k} - q^{-m-p+k}) \left[ p \right]^{-1}_q \Psi^{(k)}(q^{-1}, -m - p + k, n - 1). \]

4. \( q \)-KZ Equations

In what follows we will discuss the theory of the \( q \)-hypergeometric functions and the related KZ-equations. These are *difference equations* that can be thought as a quantized version of the previous ones. In the “classical” case we had that given \( V = V_1 \otimes \ldots \otimes V_n \), tensor product of \( \mathfrak{sl}_2 \) modules, we could define the following system of differential KZ-equations:

\[ \kappa \frac{\partial u}{\partial z_i} = \sum_{j \neq i} \Omega^{(i,j)} \frac{u}{z_i - z_j} \]

where \( u = u(z_1, \ldots, z_n) \) is a \( V \)-valued function and \( \Omega = e \otimes f + f \otimes e + \frac{1}{2}h \otimes h \) is the Casimir operator. These equations define a flat connection on the flat vector bundle \( V \times \mathbb{C}^n \longrightarrow \mathbb{C}^n \), that has regular singularities along the diagonals. The monodromy of this connection can be described in terms of the universal \( R \)-matrix associated to the quantum group \( U_q(\mathfrak{sl}_2) \), where \( q = e^{2\pi i \hbar} \), acting on the tensor product \( V^q = V_1^q \otimes \ldots \otimes V_n^q \), where the \( V_i^q \) are \( U_q(\mathfrak{sl}_2) \)-modules associated to the \( \mathfrak{sl}_2 \) modules \( V_i \) for \( i = 1, \ldots, n \). Moreover the KZ equations can be also realized geometrically:

if we consider the map \( p : \mathbb{C}^{k+n} \longrightarrow \mathbb{C}^n \) and the Master function \( \Phi_{k,n} : \mathbb{C}^{k+n} \longrightarrow \mathbb{C} \), then the KZ-equations can be identified with the Gauss-Manin connection of the vector bundle \( \mathcal{H} \) whose whose fibers are the homology groups \( H_k(p^{-1}(z), \Phi) \). Now we want to study the discrete version of all these objects. There are three versions of the classical KZ-equation:

1. *rational*;
2. *trigonometric* and;
3. *elliptic*.

In what follows we will describe the discretization of the rational KZ-system. Let’s consider the tensor product of \( \mathfrak{sl}_2 \) modules \( V = V_1 \otimes \ldots \otimes V_n \).
Definition 10.45. The rational q-KZ equation on a $V$-valued function \( \omega = \omega(z_1, ..., z_n) \) has the form:
\[
\omega(z_1, ..., z_m + p, ..., z_n) = K_m(z_1, ..., z_n, p) \omega(z_1, ..., z_n), \quad m = 1, ..., n.
\]
(10.46)

\( p \) is called the \textit{step} of the equations and \( K_m : V \to V \) are linear operators.

Now we have the following data: the step map, a trivial fibration \( \pi : V \times \mathbb{C}^n \to \mathbb{C}^n \) and the set of linear maps \( \{ K_m(z, p) \}_{m=1, ..., n} \). These operators define identification of the neighboring fibers w.r.t the lattice generated by \( p \mathbb{Z}^n \subset \mathbb{C}^n \):

Definition 10.47. We’ll call this structure a discrete connection and we will say that such connection is flat if the translation along the sides of any elementary square gives the identity operator, i.e:
\[
K_i(z_1, ..., z_j + p, ..., z_n)K_j(z_1, ..., z_i + p, ..., z_n)K_i(z_1, ..., z_n)
\]
(10.48)

Remark 10.49. The q-KZ discrete connection is flat and the q-KZ equations are the equations for flat sections of the discrete connection.

In the next section we will discuss the explicit formulas for the q-KZ operators \( K_m \).

5. Yangian \( Y(gl_2) \) and Rational R-matrices

Definition 10.50. The \textit{Yangian} \( Y(gl_2) \) is an associative algebra with unit and generators \( T_{i,j}^{(k)} \), \( i, j = 1, 2 \) \( k = 1, 2, 3, ... \).

To write the relations let’s introduce the generating series:
\[
T_{i,j}(u) = \delta_{i,j} + \sum_{k=1}^{\infty} T_{i,j}^{(k)} u^{-k}.
\]
(10.51)

The relations are given by the following:
\[
(u - v)[T_{i,j}(u), T_{k,l}(v)] = T_{j,k}(v)T_{i,l}(u) - T_{k,j}(u)T_{i,l}(v).
\]
(10.52)

The Yangian is a Hopf algebra whose coproduct \( \Delta : Y(gl_2) \to Y(gl_2) \otimes Y(gl_2) \) is given by:
\[
(\Delta T_{i,j})(u) = \sum_{k=1}^{2} T_{i,k}(u) \otimes T_{k,j}(u).
\]
(10.53)

On \( Y(gl_2) \) are also defined:

1. a family of homomorphisms \( \rho_z : Y(gl_2) \to Y(gl_2) \), where:
\[
(\rho_z T_{i,j})(u) = T_{i,j}(u - z), \quad \rho_z(\frac{1}{u^k}) = \frac{1}{u^k} \frac{1}{(1 - z u)^k},
\]
(10.54)

and:

2. the evaluation homomorphism \( \epsilon : Y(gl_2) \to U(sl_2) \), such that:
\[
T_{1,1}^k = \delta_{1,k} h \quad T_{1,2}^k = \delta_{1,k} f,
\]
(10.55)

\[
T_{2,1}^k = \delta_{1,k} e \quad T_{2,2}^k = -\delta_{1,k} h.
\]
(10.56)
If $V$ is any $\mathfrak{sl}_2$-module:

**Definition 10.57.** Let's denote $V(z)$ the Yangian module defined via the homomorphism:
\[ (10.58) \quad \epsilon \circ \rho_z : Y(g_{\mathfrak{sl}_2}) \longrightarrow U(\mathfrak{sl}_2). \]

Now let $V_1$ and $V_2$ two highest weight $\mathfrak{sl}_2$ modules with highest weight vectors $v_1 \in V_1$ and $v_2 \in V_2$. For generic complex numbers $x, y \in \mathbb{C}$ the Yangian modules $V_1(x) \otimes V_2(y)$ and $V_2(y) \otimes V_1(x)$ are isomorphic.

**Definition 10.59.** The rational R-matrix $R_{V_1,V_2}(x-y)$ can be defined as the unique element of $\text{End}(V_1 \otimes V_2)$, such that:

1. $R_{V_1,V_2} : v_1 \otimes v_2 \longrightarrow v_1 \otimes v_2$ (identity operator on the tensor product of the highest weight vectors),
2. $V_1(x) \otimes V_2(y) \xrightarrow{R_{V_1,V_2}} V_1(x) \otimes V_2(y) \xrightarrow{P} V_2(y) \otimes V_1(x)$ (the composition $P \circ R_{V_1,V_2}$ is an isomorphism of Yangian modules).

**Remark 10.60.** $P$ is the permutation operator.

**Example 10.61.** Let $V_1 = V_2 = \mathbb{C}^2$, $v_+, v_-$ a basis for $\mathbb{C}^2$ where $v_+$ is highest weight vector. In $V_1 \otimes V_2$ we have a basis given by $v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+ \text{ and } v_- \otimes v_-$. Under these hypotheses we have that the R-matrix with respect to the ordered basis written above is:
\[ (10.62) \quad R(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{x}{x+1} & \frac{1}{x+1} & 0 \\ 0 & \frac{1}{x+1} & \frac{x}{x+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

**Remark 10.63.** The R-matrices satisfy the QYBE:
\[ R_{V_1,V_2}^{(12)}(x_1-x_2) R_{V_1,V_3}^{(13)}(x_1-x_3) R_{V_2,V_3}^{(23)}(x_2-x_3) = R_{V_2,V_3}^{(23)}(x_2-x_3) R_{V_1,V_3}^{(13)}(x_1-x_3) R_{V_1,V_2}^{(12)}(x_1-x_2) \]

Now let $V = V_1 \otimes \ldots \otimes V_n$ the tensor product of highest weight $\mathfrak{sl}_2$-modules. Then:

**Definition 10.64.** The q-KZ operators $K_m$ are defined by the following:
\[ K_m = R_{V_m,V_{m-1}}(z_m - z_{m-1} + p) \ldots R_{V_m,V_1}(z_m - z_1 + p) e^{\mu h(m)} \]
\[ R_{V_m,V_n}(z_m - z_n) \ldots R_{V_m,V_{m+1}}(z_m - z_{m+1}) \]

**Exercise 10.65.** Check that the q-KZ discrete connection defined by the operators $K_m$ given in the previous definition is flat.

**Hint.** Use the fact that the R-matrices satisfy the WYE.

**Remark 10.66.** The rational q-KZ difference equations turn into the KZ differential equations under the following limiting procedure.
Let $z_m = SZ_m$, $m = 1, \ldots, n$, with $S >> 1$ a number and $Z_m$ a new variable and $\mu = \frac{\nu}{S}$, where $\nu$ is a new parameter. Introduce $u(Z) = \omega(Sz)$. Then the q-KZ equation is:
\[ (10.67) \quad u(Z_1, \ldots, Z_m + \frac{p}{S}, \ldots Z_n) = K_m(SZ, p)u(Z), \quad m = 1, \ldots, n \]
where:

\[ K_m(SZ, p) = 1 + \frac{1}{S} \sum_{j \neq m} \frac{\Omega^{(m,j)} + c_{m,j}}{Z_m - Z_j} + \frac{\nu h^{(m)}}{S}, \]

(10.68)

for suitable numbers \( c_{m,j} \). As \( S \to \infty \) the difference equation 10.67 turns into a differential equation:

\[ p \frac{\partial u}{\partial z_m} = \sum_{j \neq m} \frac{\Omega^{(m,j)} + c_{m,j}}{z_m - z_j} + \nu h^{(m)}, \quad m = 1, \ldots, n. \]

(10.69)

References for this chapter: [FV], [FSV1, FSV2], [EV2], [EFK]
A. Varchenko. Special functions, KZ type equations and Representation theory
During the last lecture we introduced the quantized version of the KZ-equations and we studied in detail the rational case. Let’s recall the main features of this construction. Let $V = V_1 \otimes \ldots \otimes V_n$ a tensor product of $\mathfrak{sl}_2$-modules, then the q-KZ equation is given by:

\begin{equation}
\omega(z_1, \ldots, z_m + p, \ldots, z_n) = K_m(z_1, \ldots, z_n, p)\omega(z_1, \ldots, z_n), \quad m = 1, \ldots, n,
\end{equation}

where $u = u(z_1, \ldots, z_n)$ is a $V$-valued function, $p$ is called the step of the equation, $K_m : V \rightarrow V, \ m = 1, \ldots, n$ are linear operators that are given in terms of R-matrices, i.e:

\[ K_m = R_{V_m, V_{m-1}}(z_m - z_{m-1} + p) \ldots R_{V_m, V_1}(z_m - z_1 + p)e^{\mu h(m)} 
\]

We also described the geometrical interpretation of this discretized version of the KZ-equation: the q-KZ equation can be interpreted as discrete connection on the trivial vector bundle $V \times \mathbb{C}^n \rightarrow \mathbb{C}^n$. Moreover this connection is flat, i.e:

\[ K_i(z_1, \ldots, z_j + p, \ldots, z_n)K_j(z_1, \ldots, z_i + p, \ldots, z_n) = K_j(z_1, \ldots, z_i + p, \ldots, z_n)K_i(z_1, \ldots, z_n). \]

Finally we proved that under a suitable limiting procedure, the q-KZ equation turns into the KZ equation.

**Lemma 11.2.** Let $\vec{z} = (z_1, \ldots, z_n)$ and let:

\begin{equation}
\omega(\vec{z}, p) = e^{\frac{s(\vec{z})}{p}} (f_0(\vec{z}) + pf_1(\vec{z}) \ldots),
\end{equation}

be an asymptotic solution of 11.1, as $p \rightarrow 0$. Then:

\[ K_i(\vec{z}, p = 0)f_0(\vec{z}) = \lambda_i(\vec{z}) f_0, \quad i = 1, \ldots, n, \]

where \( \lambda_i(\vec{z}) = e^{\frac{\partial s(\vec{z})}{\partial z_i}} \).

**Proof.** Since $\omega = \omega(\vec{z}, p)$ is a solution of 11.1 we have that:

\[ \omega(z_1, \ldots, z_i + p, \ldots, z_n, p) = K_i(\vec{z}, p)\omega(\vec{z}, p). \]

Plugging 11.3 in the previous equation we get:

\begin{equation}
\frac{e^{s(z_1, \ldots, z_i + p, \ldots, z_n)}}{p}(f_0(z_1, \ldots, z_i + p, \ldots, z_n) + pf_1(z_1, \ldots, z_i + p, \ldots, z_n) + \ldots) = K_i(\vec{z}, p)e^{\frac{s(\vec{z})}{p}} (f_0(\vec{z}) + \ldots).
\end{equation}

expanding with respect to the $z_i$ coordinate:

\[ e^{\frac{s(z_1, \ldots, z_i + p, \ldots, z_n)}{p}} = e^{\frac{s(\vec{z})}{p} + \frac{\partial s}{\partial z_i} + \ldots}, \]
Plugging this result in the LHS of 11.4 and taking the limit for \( p \to 0 \), we get:

\[
K_i(z, p = 0)f_0(z) = \lambda_i(z)f_0(z).
\]

From the previous lemma we deduce that the leading term \( f_0(z) \) is an eigenvector of commuting operators \( \{K_m(z, p = 0)\}_{m=1,...,n} \). These operators are the Hamiltonians of the XXX-quantum spin chain model. So having solutions of the q-KZ equation and computing its quasi-classical asymptotics it is possible to construct eigenvectors for the operators \( \{K_m(z, p = 0)\}_{m=1,...,n} \). These eigenvectors are the so called Bethe vectors and they are constructed by the Bethe ansatz of the XXX-model.

The main topic of this lecture will be the quantization of the KZ system of differential equation and its geometric interpretations. Let’s start recalling the main features of the classical KZ differential equation.

1. Classical KZ-system

Given \( m_1, ..., m_n, \kappa \in \mathbb{C} \) consider the following function:

\[
\Phi = \prod_{l=1}^{n} (t - z_l)^{m_l}.
\]

and let \( \gamma_m = [z_m, z_{m+1}] \) be the oriented interval. Let’s define the following function:

\[
I^\gamma(z_1, ..., z_n) = (\int_{\gamma} \Phi \frac{dt}{t - z_1}, ..., \int_{\gamma} \Phi \frac{dt}{t - z_n}),
\]

where \( \gamma = \gamma_1, ..., \gamma_{n-1} \), then:

\[
\frac{\partial I^\gamma}{\partial z_l} = \frac{1}{\kappa} \sum_{i \neq j} \frac{\Omega_{i,j}}{z_i - z_j} I^\gamma.
\]

Remark 11.7. In 11.6 \( \Omega \) depends only on \( m_1, ..., m_n \).

Remark 11.8. The system 11.6 is an example of KZ equation.

1.1. Differential Forms. For \( z_1, ..., z_n \) fixed consider the complex:

\[
0 \to \Omega^0 \xrightarrow{d} \Omega^1 \to 0,
\]

where by definition:

\[
\Omega^0 = \{f(t)\Phi(t, z)|f \text{ is a rational function on } \mathbb{C}\backslash\{z_1, ..., z_n\}\},
\]

\[
\Omega^1 = \{\omega = f(t)\Phi(t, z)dt|f \text{ is a rational function on } \mathbb{C}\backslash\{z_1, ..., z_n\}\}
\]

and \( d \) is the usual Cartan differential. Under these hypothesis we have the following:

Theorem 11.12. For generic \( m_1, ..., m_n \) and \( \kappa \), \( H^0 = 0 \) and \( \dim H^1 = n - 1 \). Moreover the differential forms \( \omega_l = \Phi \frac{dt}{t - z_l}, \ l = 1, ..., n - 1 \), form a basis in \( H^1 \).

Remark 11.13. From the previous theorem follows that \( I^\gamma \) for \( \gamma = \gamma_1, ..., \gamma_{n-1} \), consists of integrals of closed forms.
Let $H_1 = (H^1)^*$ be the dual space of the first cohomology group. Each interval $\gamma_m$, $m = 1, ..., n - 1$, defines a linear functional on $\Omega^1$:

$$[\gamma_m] : f \Phi dt \mapsto \int_{\gamma_m} f \Phi dt.$$  

By the Stokes theorem we have that $\int_{\gamma_m} d(f \Phi) = 0$, hence $[\gamma_m]$ defines an element of $H_1$. We have that:

**Theorem 11.14.** The elements $[\gamma_m]$, $m = 1, ..., n - 1$, form a basis in $H_1$.

**Proof.** The proof of the theorem is based on the following formula:

$$\det_{1 \leq l, m \leq n-1} \left( \frac{m_l}{\kappa} \int_{\gamma_m} \frac{dt}{t - z_l} \right) \frac{\Gamma(m_l/\kappa + 1) \cdots \Gamma(m_n/\kappa + 1)}{\Gamma(m_1 + \cdots + m_n/\kappa + 1)} \prod_{i \neq j} (z_i - z_j)^{m_j/\kappa}.$$ (11.15)

**Example 11.16.** Let’s suppose $n = 2$ and $(z_1, z_2) = (0, 1)$. Then the formula 11.15 becomes:

$$b \int_0^1 t^a(1-t)^{b-1}dt = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}.$$ (11.17)

**Remark 11.18.** In the following table we summarize the main relations between conformal field theory and the geometry of the classical KZ equation:

| Conformal Field Theory | Geometry of hypergeometric functions |
|------------------------|--------------------------------------|
| Space of Conformal Blocks | Cohomology space $H^1$ associated to the function $\Phi(t, z)$ |
| KZ equation | Gauss-Manin connection |
| Solutions to KZ equation | Cycles in $H_1$ |

In what follows we proceed to the quantization of the geometry associated to the KZ equation.

### 2. Geometric Construction of Quantized KZ Equation

We know that KZ can be realized as differential equation for integrals of the basic closed differential one forms over cycles. It turns out that there is a quantization of all those geometric object leading to a geometric construction of the q-KZ equation. Now we want introduce the p-analogous of the main ingredients of the classical picture. Let’s start to define the p-analogous of the function $\Phi(t, z) = \prod_{l=1}^{n} (t - z_l)^{m_l/\kappa}$.

The function $\Phi_p(t, z)$: first we observe that the monomial $T^m/\kappa$ satisfies the following differential equation:

$$\frac{dY}{dT} = \frac{m}{T} Y(T).$$ (11.19)

The analogous of 11.19 is:

$$y(t + p) = \frac{t + m}{t - m} y(t).$$ (11.20)

In fact if we pose $t = ST$ and and we define $Y(T) = y(ST)$, then:

$$Y(T + \frac{p}{S}) = (1 + \frac{2m}{S} \frac{1}{T - \frac{m}{S}})$$ (11.21)
and as $S \to \infty$ we get that (11.21) turns into the equation:

\begin{equation}
\frac{dY}{dT} = \frac{2m}{T} Y(T).
\end{equation}

**p-Solutions:** the solution of the equation (11.20) is given by the following:

\[ y(t,m) = \Gamma\left(\frac{t + m}{p}\right) \Gamma\left(1 - \frac{t - m}{p}\right) e^{-\frac{\pi i t}{p}}, \]

so the p-analogous of the Master function will be given by:

\[ \Phi_p(t) = \prod_{l=1}^{n} y(t - z_l, m_l) = \prod_{l=1}^{n} \Gamma\left(\frac{t - z_l + m_l}{p}\right) \Gamma\left(1 - \frac{t - z_l + m_l}{p}\right) e^{-\frac{\pi i t}{p}}. \]

**Singularities of the function $\Phi_p(t,z)$:** these correspond to the points:

\[ t = z_l - m_l - Np \]
\[ t = z_l + m_l + (N + 1)p \]

where \( l = 1, \ldots, n \), \( N = 0, 1, 2, \ldots \). We will also introduce the following:

**Notation 11.23.** $\text{Sing}=\{t \in \mathbb{C} \mid t = z_l - m_l + (N + 1)p, t = z_l + m_l - Np, l = 1, \ldots, n, N = 0, 1, 2, \ldots \}$

The complex: now we define the p-analogous of the complex given in 11.9. Let’s introduce the space of the 0 and 1 p-form:

\[ \Omega^0_p = \Omega^1_p = \{\Phi_p f(t) \mid f \text{ is a rational function regular on } \mathbb{C}\setminus\text{Sing an with first order poles on it}\}, \]

and let the p differential $D_p : \Omega^0_p \to \Omega^1_p$ defined by the following:

\begin{equation}
(D_p h)(t) = h(t + p) - h(t).
\end{equation}

Starting from these data we have the following:

**Definition 11.25.** the p analogous of the complex 11.9 is:

\begin{equation}
0 \to \Omega^0_p \xrightarrow{D_p} \Omega^1_p \to 0.
\end{equation}

The first cohomology group is given by:

\[ H^1_p = \Omega^1_p \backslash D_p \Omega^0_p, \]

and the first homology group will be:

\[ H_1^p = (H^1_p)^*. \]

**Theorem 11.27.** 1. For generic $z_1, \ldots, z_n, p, m_l$, $H^1_p$ is (n-1)-dimensional and the following 1 forms form a basis:

\[ \omega_l = \frac{1}{t - z_l - m_l} \cdot \frac{t - z_{l-1} + m_{l-1}}{t - z_{l-1} - m_{l-1}} \cdots \frac{t - z_1 + m_1}{t - z_1 - m_1} \, dt. \]
2. Let \( t = ST, \ z_j = SZ_j \), where \( S > > 1 \) and \( T, \ Z_j \) are new variables. As \( S \to \infty \) the \( p \)-complex turns into the de Rham complex of \( \prod_{l=1}^{n} (T - Z_l) \) and \( \{ \omega_l \}_{l=1, \ldots, n} \) turn into \( \{ \frac{dT}{T - Z_l} \}_{l=1, \ldots, n} \) which give a basis in \( H^1 \) of the de Rham complex.

### 2.1. \( p \)-Homology Theory

In this subsection we will construct linear functionals on the space \( \Omega^1_{(p)} \setminus D_p \), i.e. linear functionals:

\[
\lambda : \Omega^1_{(p)} \to \mathbb{C}
\]

such that:

\[
\lambda(g(t + p) - g(t)) = 0, \ \forall g \in \Omega^1_{(p)}.
\]

**Naive approach:** let \( \xi \in \mathbb{C} \) and \( h(t) = \Phi_{(p)}(t)f(t) \).

**Definition 11.28 (Jackson’s Integral).** The Jackson integral of the function \( h = h(t) \) over \( [\xi]_p \) is given by the following formula:

\[
\int_{[\xi]_p} h(t)dt = p \sum_{l=-\infty}^{\infty} h(\xi + lp).
\]

**Properties 11.30.**

1. If \( h(t) = g(t + p) - g(t) \) then:

\[
\int_{[\xi]_p} h(t)dt = p \sum_{l=\infty}^{\infty} (g(\xi + (l+1)p) - g(\xi + lp)) = 0;
\]

2. \( \lim_{p \to 0} \int_{[\xi]_p} h(t)dt = \int_{\xi + \text{line } \mathbb{R}_p} h(t)dt. \)

In our case this definition does not work since the integral 11.29 is divergent.

**Right approach:** let’s choose our data \( z_1, \ldots, z_n, m_1, \ldots, m_n, p \) such that the points \( z_i \), for \( i = 1, \ldots, n \), lie on the imaginary line and \( m_1, \ldots, m_n, p \in \mathbb{R} \), with \( m_1, \ldots, m_n, p < 0 \). For a given \( m \in \{1, \ldots, n-1\} \) let’s introduce the following function:

\[
G_m : \mathbb{C} \to \mathbb{C}
\]

\[
t \mapsto \exp\left(\frac{2\pi imt}{p}\right)
\]

**Remark 11.31.** The \( G_m \) are \( p \)-periodic functions.

Now for every \( m \) let’s consider the following functionals:

\[
[G_m] : \Omega^1_{(p)} \to \mathbb{C}
\]

\[
\Phi_{(p)}f \longmapsto \int_{\mathbb{R}} G_m(t)\Phi_{(p)}(t, z)dt
\]

**Remark 11.32.** The functionals \( [G_m] \) are the \( p \) analogous of the intervals \( [\gamma_m] \).

In what follows we will describe some of the properties of the functionals \( [G_m] \) just introduced.

**Properties 11.33.**

1. The functionals \( [G_m] \) are well defined if \( m \in \{1, \ldots, n-1\} \) and are not defined if \( m \notin \{1, \ldots, n-1\} \).

2. \( [G_m]|_{D_p\Omega^1_{(p)}} \equiv 0 \). From the previous property we get the following:
COROLLARY 11.34. The functionals $[\mathbb{G}_m]$, $m \in \{1, \ldots, n-1\}$, define elements in $H_1^{(p)}$.  
3. If $z_l = SZ_l$, then:
   $$\lim_{t \to +\infty} [\mathbb{G}_m] = [z_m, z_{m+1}]$$
   in the following sense:
   $$\int_{t \in \mathbb{R}} \mathbb{G}_m(t)\Phi_{(p)}(t, z)\omega_1 dt \longrightarrow C_m(S, p)\left( \int_{t \in \mathbb{R}} \prod_{j=1}^{n} (T - Z_j)^{2m_j} z_{m_j} \frac{dT}{T - Z_l} + O(S^{-1}) \right), \forall m.$$ 
   Here $C_m = C_m(S, p)$ is some explicitly known function.
4. Let $u_m = u_m(z_1, \ldots, z_n)$, for $m = 1, \ldots, n-1$, be the following function:
   $$(11.35) \quad u_m(z_1, \ldots, z_n) = \left( \int_{t \in \mathbb{R}} \mathbb{G}_m(t)\Phi_{(p)}(t, z)\omega_1 dt, \ldots, \int_{t \in \mathbb{R}} \mathbb{G}_m(t)\Phi_{(p)}(t, z)\omega_{n-1} dt \right).$$
   Then for every $m \in \{1, \ldots, n-1\}$, 11.35 is a solution of the q-KZ equation with values in $M_{S_m} \otimes \cdots \otimes M_{S_1} |m| = 2$ with $\mu = 0$. The functions $u_m = u_m(z_1, \ldots, z_n)$ define by 11.35 are called q-hypergeometric functions associated with $\Phi_{(p)}$.
5. The functionals $[\mathbb{G}_m]$, $m \in \{1, \ldots, n-1\}$, form a basis in the homology space $H_1^{(p)}$.

PROOF. 1. Follows applying the Stirling formula to the definition of the p-master function $\Phi_{(p)}$:
   $$\Phi_{(p)} = \prod_{l=1}^{n} y(t - z_l, m_l) = \prod_{l=1}^{n} \Gamma \left( \frac{t - z_l + m_l}{p} \right) \Gamma \left( 1 - \frac{t - z_l + m_l}{p} \right) e^{-\frac{z_l}{p}}.$$ 
2. Follows from the periodicity of the functions $\mathbb{G}_m$, in fact since these functions are p periodic we have:
   $$\int_{t \in \mathbb{R}} \mathbb{G}_m(t) \left[ \Phi_{(p)}(t + p, z) f(t + p) - \Phi_{(p)}(t, z) f(t) \right] dt = \int_{t \in \mathbb{R} + p} \mathbb{G}_m(t)\Phi_{(p)}(t, z) f(t) dt$$
   $$- \int_{t \in \mathbb{R} + p} \mathbb{G}_m(t)\Phi_{(p)}(t, z) f(t) dt = 0.$$ 
3. Follows from the Stirling formula.
5. This is a consequence of the following formula:
   $$\prod_{1 \leq l, m \leq n-1} \left( \frac{2m_l}{p} \int_{t \in \mathbb{R}} \mathbb{G}_m\Phi_{(p)}(t)\omega_l dt \right) = \det_{\frac{2m_l}{p}} \prod_{1 \leq l, m \leq n-1} \Gamma \left( \frac{z_l + m_l - z_l + m_l}{p} \right) \Gamma \left( 1 + \frac{z_l + m_l - z_l + m_l}{p} \right).$$

EXAMPLE 11.36. For $n = 2$ from the previous determinant we get the following classical formula:
   $$(11.37) \quad \int_{-\infty}^{\infty} \Gamma(a + t)\Gamma(a + t)\Gamma(c - t)\Gamma(d - t) dt = 2\pi i \frac{\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)}{\Gamma(a + b + c + d)}.$$ 

REMARK 11.38. 11.37 is the q-analogous of the $\beta$-function and it is called is called Barnes’ formula.
In the next section we will describe explicit solutions for the q-KZ equation in terms of q-hypergeometric functions.

3. q-KZ Equation: \( sl_2 \) case

Let’s start with some preliminary remarks about the action of the twisted symmetric group \( S^k \) on space \( \mathcal{F}_k = \{ \text{function of } k\text{-variables } t_1, \ldots, t_k \} \).

Given \( h = h(x) \) a function such that

\[
(11.39) \quad h(x)h(-x) = 1.
\]

let’s define:

\[
(11.40) \quad s_j : f(t_1, \ldots, t_k) \mapsto f(t_1, \ldots, t_{j+1}, t_j, \ldots, t_k)h(t_i - t_j).
\]

**Definition 11.41.**

\( S^k = \{ s_j : \mathcal{F}_k \rightarrow \mathcal{F}_k \text{ such that } 11.40 \text{ is satisfied} \} \).

**Lemma 11.42.** The maps \( s_j, j = 1, \ldots, k-1 \), given by 11.40, define an action of \( S^k \) on the space \( \mathcal{F}_k \).

**Proof.** It suffices check that \( s_j^2 = 1 \) and \( s_js_{j+1}s_j = s_{j+1}s_j s_{j+1} \) for every \( j \in \{1, \ldots, k-1\} \). □

**Example 11.43.** The following are examples of functions that satisfy the condition 11.39.

\[
(11.44) \quad h(x) = \frac{x-1}{x+1},
\]

\[
(11.45) \quad h(x) = \frac{\sin(\frac{\pi(x-1)}{p})}{\sin(\frac{\pi(x+1)}{p})},
\]

\[
(11.46) \quad h(x) = \frac{\theta(x-a, \tau)}{\theta(x+a, \tau)}.
\]

**Remark 11.47.** In the formula 11.46 \( \theta = \theta(x, \tau) \) is the first Jacobi theta-function.

**Remark 11.48.** The functions just described in the previous example enter in the description of the solutions of the q-KZ equation. In particular the functions 11.44 and 11.45 are used in the rational case. For the trigonometric case we need the functions 11.45 and 11.46, while for the elliptic one we need only 11.46.

We need the following:

**Definition 11.49.** Let \( F^{k}_{m_1, \ldots, m_n}(z_1, \ldots, z_n) \subset \mathcal{F}_k \) be the \( \mathbb{C} \) vector space of the functions of the form:

\[
P(t_1, \ldots, t_k) \prod_{l=1}^{n} \prod_{i=1}^{k} \frac{1}{t_i - z_l - m_l} \prod_{1 \leq i < j \leq k} \frac{t_i - t_j}{t_i - t_j + 1},
\]

where \( P = P(t_1, \ldots, t_k) \) is a symmetric polynomial (in the standard sense), of degree less than \( n \) in each of the \( k \) variables.
EXAMPLE 11.50. For $k = 1$ we have that the function in $F^k_{m_1,...,m_n}(z_1, ..., z_n)$ are of the form:

$$P(t) \over (t - z_1 - m_1) \cdots (t - z_n - m_n),$$

with $\deg P < n$.

DEFINITION 11.51. Let’s define:

$$F_{m_1,...,m_n}(z_1, ..., z_n) = \bigoplus_{k=0}^{\infty} F^k_{m_1,...,m_n}(z_1, ..., z_n),$$

where by definition, $F^0_{m_1,...,m_n}(z_1, ..., z_n) = \mathbb{C}$.

The spaces $F^k_{m_1,...,m_n}(z_1, ..., z_n)$ have the following:

PROPERTIES 11.52.  
1. $\dim F^k_{m_1,...,m_n}$ = $\dim M_{m_1} \otimes \cdots \otimes M_{m_n} [m] - 2k$;
2. $F^k_{m_1,...,m_n}$ consists of functions that are symmetric with respect to the action of $S^k$ with $h$ given by 11.44.

EXAMPLE 11.53. If

$$g = \frac{1}{t_1 - z_1 - m_1} \cdot \frac{1}{t_2 - z_1 - m_1} \cdot \frac{t_1 - t_2}{t_1 - t_2 + 1},$$

then:

$$s_1(g) = \frac{1}{t_1 - z_1 - m_1} \cdot \frac{1}{t_2 - z_1 - m_1} \cdot \frac{t_2 - t_1 - 1}{t_2 - t_1 + 1} \cdot \frac{t_1 - t_2 - 1}{t_1 - t_2 + 1} =$$

$$\frac{1}{t_1 - z_1 - m_1} \cdot \frac{1}{t_2 - z_1 - m_1} \cdot \frac{t_2 - t_1}{t_2 - t_1 + 1} = g.$$

3. $F^k_{m_1,...,m_n}(z_1, ..., z_n) = F^k_{m_1,...,m_{j+1},m_j,...,m_n}(z_1, ..., z_{j+1}, j, ..., z_n)$.

EXAMPLE 11.54. For $n = 1$ we have that the space $F^k_{m}(z)$ is one dimensional and it is spanned by:

$$(11.55) \omega_k(t_1, ..., t_k, z) = \prod_{a < b} \frac{t_a - t_b}{t_a - t_b + 1} \prod_{a=1}^{k} \frac{1}{t_a - z - m}.$$ 

Let’s introduce the following map:

$$F^{k'}_{m_1,...,m'_n}(z_1, ..., z'_{n}) \otimes F^{k''}_{m_{n'+1},...,m_{n'+n''}}(z_{n'+1}, ..., z_{n'+n''}) \rightarrow F^{k'+k''}_{m_1,...,m_{n'+n''}}(z_1, ..., z_{n'+n''})$$

$$f \otimes g \rightarrow k(t_1, ..., t_{k'+k''})$$

where

$$k(t_1, ..., t_{k'+k''}) = \frac{1}{k'! k''!} \text{Sym}_h(f(t_1, ..., t'_k) g(t_{k'+1}, ..., t_{k'+k''}) \cdot \prod_{k < j \leq k'+k''} \prod_{1 \leq l \leq n'} \frac{t_j - z_l + m_l}{t_j - z_l - m_l})$$
Example 11.56. For $k' = 0$ and $k'' = 1$ we have:

$$F_{m_1}^0(z_1) \otimes F_{m_2}^1(z_2) \longrightarrow F_{m_1, m_2}^1(z_1, z_2)$$

and

$$1 \otimes \frac{1}{t-z_2-m_2} \longmapsto \frac{t-z_1+m_1}{t-z_1-m_1} \frac{1}{t-z_2-m_2}$$

Claim 11.57. For generic values of $z_1, \ldots, z_{m}$:

$$\bigoplus_{k'+k''=k} F_{m_1, \ldots, m_n}^{k'}(z_1, \ldots, z_n') \otimes F_{m_{n'+1}, \ldots, m_{n'+n''}}^{k''}(z_{n'+1}, \ldots, z_{n'+n''}) \longrightarrow F_{m_1, \ldots, m_{n'+n''}}^{k}(z_1, \ldots, z_{n'+n''})$$

is an isomorphism.

Now we can define:

Definition 11.58.

$$\omega(z) : (V_m)^* \longrightarrow F_m(z)$$

$$(f^k v)^* \longmapsto \omega_k$$

The map $\omega_z$ induces a map:

$$(11.59) \quad \omega(z_1, \ldots, z_n) : V_{m_1}^* \otimes \ldots \otimes V_{m_n}^* \longrightarrow F_{m_1, \ldots, m_n}(z_1, \ldots, z_n).$$

Claim 11.60. For generic $z_1, \ldots, z_n$, the map $\omega = \omega(z_1, \ldots, z_n)$, 11.59 is an isomorphism.

References for this chapter: [V5], [TV1, TV2]
A. Varchenko. Special functions, KZ type equations and Representation theory
The last part of the last lecture was devoted to discuss the following problem: how to solve the q-KZ equation in terms of q-hypergeometric functions. Let’s recall the main points of this construction. We introduced the action of the twisted symmetric group $S^k$ in the space of the function $F_k = \{\text{functions of } k \text{ variables } t_1, \ldots, t_k\}$, via the maps:

\[ s_j : f(t_1, \ldots, t_k) \mapsto f(t_1, \ldots, t_{j+1}, t_j, \ldots, t_k)h(t_i - t_j), \]

where $h = h(x)$ is a function that satisfies the condition:

\[ h(x) \cdot h(-x) = 1, \]

and we also defined the space:

\[ F_{\mathbf{m}_1, \ldots, \mathbf{m}_n}(z_1, \ldots, z_n) \subset F_k \]

of the functions of the form:

\[ P(t_1, \ldots, t_k) \prod_{l=1}^n \prod_{i=1}^k \frac{1}{t_i - z_l - m_l} \prod_{1 \leq i < j \leq k} \frac{t_i - t_j}{t_i - t_j + 1}, \]

where $P = P(t_1, \ldots, t_k)$ is a symmetric polynomial (in the standard sense).

We also noticed that the space $F_{\mathbf{m}_1, \ldots, \mathbf{m}_n}(z_1, \ldots, z_n) = \bigoplus_{k=0}^\infty F_{\mathbf{k}, \mathbf{m}_1, \ldots, \mathbf{m}_n}(z_1, \ldots, z_n)$ satisfies the following:

**Properties 12.2.**

1. $\dim F_{\mathbf{m}_1, \ldots, \mathbf{m}_n} = \dim M_{\mathbf{m}_1} \otimes \ldots \otimes M_{\mathbf{m}_n}[|m| - 2k]$;
2. $F_{\mathbf{m}_1, \ldots, \mathbf{m}_n}^k$ consists of functions that are symmetric with respect to the action of $S^k$ with
   \[ h = \frac{x - 1}{x + 1}, \]
3. $F_{\mathbf{m}_1, \ldots, \mathbf{m}_n}^k(z_1, \ldots, z_n) = F_{\mathbf{m}_1, \ldots, \mathbf{m}_{j+1}, \mathbf{m}_j, \ldots, \mathbf{m}_n}^k(z_1, \ldots, z_{j+1}, z_j, \ldots, z_n)$.

Then we defined a tensor product:

\[ F_{\mathbf{m}_1, \ldots, \mathbf{m}_n}^{k'}(z_1, \ldots, z_{n'}) \otimes F_{\mathbf{m}_1', \ldots, \mathbf{m}_{n'+n''}}^{k''}(z_{n'+1}, \ldots, z_{n'+n''}) \rightarrow F_{\mathbf{m}_1, \ldots, \mathbf{m}_{n'+n''}}^{k'+k''}(z_1, \ldots, z_{n'+n''}) \]

\[ f \otimes g \mapsto k(t_1, \ldots, t_{k'+k''}) \]

where

\[ k(t_1, \ldots, t_{k'+k''}) = \frac{1}{k'n''k''!} \text{Sym}_h \left( f(t_1, \ldots, t_{k'}) g(t_{k'+1}, \ldots, t_{k'+k''}) \cdot \prod_{k < j \leq k'+k''} \prod_{1 \leq l \leq n'} \frac{t_j - z_l + m_l}{t_j - z_l - m_l} \right) \]

and we state the following:
Claim 12.3. For generic values of $z_1, \ldots, z_n$:

$$\bigoplus_{k' + k'' = k} F_{m_1, \ldots, m_n'}^{k'}(z_1, \ldots, z_{n'}) \otimes F_{m_{n'+1}, \ldots, m_{n'+n''}}^{k''}(z_{n'+1}, \ldots, z_{n'+n''}) \longrightarrow F_{m_1, \ldots, m_{n'+a''}}^{k}(z_1, \ldots, z_{n'+n''})$$

is an isomorphism.

Finally we defined:

$$\omega(z) : (V_m)^* \longrightarrow F_m(z)$$

where

$$(12.4) \quad \omega_k(t_1, \ldots, t_k, z) = \prod_{a < b} \frac{t_a - t_b}{t_a - t_b + 1} \prod_{a = 1}^k \frac{1}{t_a - z - m},$$

is a generator of the one dimensional vector space $F_m(z)$ and from 12.4 and the tensor product we also defined the map:

$$(12.5) \quad \omega(z_1, \ldots, z_n) : V_{m_1}^* \otimes \ldots \otimes V_{m_n}^* \longrightarrow F_{m_1, \ldots, m_n}(z_1, \ldots, z_n);$$

that has the following property:

Claim 12.6. For generic $z_1, \ldots, z_n$, the map $\omega = \omega(z_1, \ldots, z_n)$, defined in 12.5, is an isomorphism.

Now we will proceed introducing a suitable R-matrix to relate basis in different spaces. Let $z, u \in \mathbb{C}$ such that the map:

$$(12.7) \quad \omega(z, u) : M_a^* \otimes M_b^* \longrightarrow F_{a, b}(z, u)$$

be invertible. Then:

Definition 12.8. The rational $R$-matrix $R_{a, b}(z, u)$ is the dual map of the composition:

$$(12.9) \quad M_a^* \otimes M_b^* \xrightarrow{p} M_b^* \otimes M_a^* \xrightarrow{\omega(u, z)} F_{a, b}(z, u) \xrightarrow{\omega^{-1}(z, u)} M_a^* \otimes M_b^*,$$

Remark 12.10. Alternatively we can think the map $R_{a, b}(z, u)$ defined by 12.9 as the matrix that express the basis $\omega(u, z)((f^m v_b)^* \otimes (f^l v_a)^*)$ in terms of the basis $\omega(z, u)((f^l v_a)^* \otimes (f^m v_b)^*)$.

The rational $R$-matrix just introduced satisfies the following:

Properties 12.11. 1. $R_{a, b}(z, u)$ depends only on the difference $u - z$;
2. $R_{a, b}(z, u)$ is the standard $R$-matrix associated with the Yangian module structure of the tensor product $M_a(z) \otimes M_b(u)$.

In the next section we will describe the trigonometric analogous of the previous construction.
1. Trigonometric case

In what follows with \( p \) we will denote the step of the q-KZ equation, \( q = e^{2\pi i p} \) and \( M^q_{m_j}, j = 1, .., n \), will be a \( \mathcal{U}_q(\mathfrak{sl}_2) \)-Verma module. Let’s first introduce the trigonometric analogous of the space

\[
F^k_{m_1, .., m_n}(z_1, ..., z_n) \subset \mathcal{F}_k
\]

**Definition 12.12.** Let \( F^k_{m_1, .., m_n}(z_1, ..., z_n) \subset \mathcal{F}_k \) the space of function of the form:

\[
\mathcal{P}(\xi_1, ..., \xi_l) \prod_{l=1}^n \prod_{i=1}^k \exp\left(\frac{\pi i (z_l - t_a)}{p}\right) \frac{\sin\left(\frac{\pi (t_a - z_l - m_i)}{p}\right)}{\sin\left(\frac{\pi (t_a - z_l)}{p}\right)} \prod_{a<b} \frac{\sin\left(\frac{\pi (t_a - t_b)}{p}\right)}{\sin\left(\frac{\pi (t_a - t_b + 1)}{p}\right)},
\]

where \( \xi_a = \exp\left(\frac{2\pi i t_a}{p}\right) \) and \( \mathcal{P} = \mathcal{P}(\xi_1, ..., \xi_l) \) is a symmetric polynomial in \( \xi_1, ..., \xi_l \), in the standard sense of degree \( n \) in each of its variable.

In analogy to the rational case we have:

**Definition 12.14.**

\[
\mathcal{F}_{m_1, .., m_n}(z_1, ..., z_n) = \bigoplus_{k=0}^{\infty} \mathcal{F}^k_{m_1, .., m_n},
\]

where, by definition, \( \mathcal{F}^0_{m_1, .., m_n} = \mathbb{C} \).

The spaces \( \mathcal{F}^k_{m_1, .., m_n} \) satisfy the following:

**Properties 12.15.**

1. \( \dim \mathcal{F}^k = \binom{n+k-1}{k} = \dim (M^q_{m_1} \otimes .. \otimes M^q_{m_n}) |m| - 2k \);

2. \( \mathcal{F}^k \) consists of symmetric functions with respect to the action of the group \( S^k \) defined by the function:

\[
\tilde{h}(x) = \frac{\sin\left(\frac{\pi(x-1)}{p}\right)}{\sin\left(\frac{\pi(x+1)}{p}\right)};
\]

3. \( \mathcal{F}^k_{m_1, .., m_n}(z_1, ..., z_n) = \mathcal{F}^k_{m_1, .., m_{j+1}, m_j, .., m_n}(z_1, ..., z_{j+1}, z_j, ..., z_n) \).

**Example 12.16.** As in the rational case we have an explicit description of the generators for the case \( n = 1 \). In fact in this hypothesis we have that the space \( \mathcal{F}^k_m(z) \) is generated by:

\[
W_k(t_1, ..., t_k, z) = \prod_{a=1}^k \exp\left(\frac{\pi i (z - t_a)}{p}\right) \prod_{a<b} \frac{\sin\left(\frac{\pi (t_a - t_b)}{p}\right)}{\sin\left(\frac{\pi (t_a - t_b + 1)}{p}\right)}.
\]

Let’s introduce the tensor product:

\[
\mathcal{F}^{k'}_{m_1, .., m_{n'}}(z_1, ..., z_{n'}) \otimes \mathcal{F}^{k''}_{m_{n'+1}, .., m_{n'+n''}}(z_{n'+1}, ..., z_{n'+n''}) \rightarrow \mathcal{F}^{k' + k''}_{m_1, .., m_{n'+n''}}(z_1, ..., z_{n'+n''})
\]

where:

\[
f \otimes g \rightarrow u(t_1, ..., t_{k' + k''})
\]
\[ u(t_1, \ldots, t_{k'+k''}) = \frac{1}{k'!k''!} \text{Sym}_k(f(t_1, \ldots, t_{k'})g(t_{k'+1}, \ldots, t_{k'+k''})) \prod_{k'<j<k'+k''} \frac{\sin(\frac{t_j-z+i}{p})}{\sin(\frac{t_j-z-m_i}{p})} \]

Also in this case we have that:

**Claim 12.18.** For generic values of \( z_1, \ldots, z_n \):

\[ \bigoplus_{k'+k''=k} F_{m_1,\ldots,m_{a'}}(z_1, \ldots, z_{a'}) \otimes F_{m_{a'+1},\ldots,m_{a'+a''}}(z_{a'+1}, \ldots, z_{a'+a''}) \rightarrow F_{m_1,\ldots,m_{a'+a''}}(z_1, \ldots, z_{a'+a''}) \]

is an isomorphism.

Now let’s define:

**Definition 12.19.**

\[ W(z) : (M_m)^q \rightarrow F_m(z) \]

(12.20)

\[ (f^k v) \quad \mapsto \quad C_k W_k(t_1, \ldots, t_n, z), \]

where:

\[ C_k = \prod_{s=0}^{k-1} \frac{\sin(\frac{\pi(2m-s)}{p})}{\sin(\frac{\pi}{p})} = \prod_{s=0}^{k-1} \frac{q^{2(m-s)} - q^{-2(m-s)}}{q - q^{-1}}. \]

The map 12.20 induces a map:

\[ W(z_1, \ldots, z_n) : M_{m_1} \otimes \ldots \otimes M_{m_n} \rightarrow F_{m_1,\ldots,m_n}(z_1, \ldots, z_n). \]

We have:

**Claim 12.22.** For generic \( z_1, \ldots, z_n \) the map 12.21 is non degenerate.

Let’s now introduce the trigonometric version of the R-matrix 12.7. Let \( z, u \in \mathbb{C} \) such that

\[ W(u, z) : V_b \otimes V_a \rightarrow F_{a,b}(z, u) \]

is non degenerate. Then:

**Definition 12.23.** The trigonometric R-matrix \( R_{a,b}^{\text{trig}} \) is given by the following composition:

(12.24)

\[ M_a \otimes M_b \xrightarrow{W(z,u)} F_{a,b}(z, u) \xrightarrow{W^{-1}(u,z)} M_b \otimes M_a \xrightarrow{p} M_a \otimes M_b, \]

The trigonometric R-matrix defined by 12.24 has the following:

**Properties 12.25.**

1. \( R_{a,b}^{\text{trig}} \) is a map between \( M_a^q \otimes M_b^q \) and itself;
2. \( R_{a,b}^{\text{trig}} \) depends only on the difference \( z - u \), is \( p \)-periodic and coincides with the trigonometric R-matrix \( R_{M_a^q,M_b^q}^{\text{trig}}(\exp(\frac{2\pi i(z-u)}{p})) \).

**Remark 12.26.** For any highest weight \( \mathcal{U}_q(\mathfrak{sl}_2) \)-modules \( V_1^q \) and \( V_2^q \), there is \( R_{V_1^q,V_2^q}(z) \in \text{End}(V_1^q \otimes V_2^q) \) satisfying the QYBE:

\[ R_{V_1^q,V_2^q}(\frac{z_1}{z_2}) R_{V_1^q,V_3^q}(\frac{z_1}{z_3}) R_{V_2^q,V_3^q}(\frac{z_2}{z_3}) = R_{V_2^q,V_1^q}(\frac{z_2}{z_3}) R_{V_2^q,V_3^q}(\frac{z_1}{z_3}) R_{V_1^q,V_2^q}(\frac{z_1}{z_2}). \]
\( R_{V_1^q, V_2^q}(z) \) is called trigonometric R-matrix, it is constructed using \( \mathcal{U}_q(\mathfrak{sl}_2) \) and normalized requiring that:

\[
R_{V_1^q, V_2^q}(z)(v_1 \otimes v_2) = v_1 \otimes v_2
\]

where \( v_i \in V_i^q \) are highest weight vectors.

**Example 12.27.** Let \( V_1^q = V_2^q = \mathbb{C}^2 \), then:

\[
(12.28) \quad R^{\text{trig}}(x) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{x^{-1}}{xq^{-1}} & \frac{q^{-1}}{xq^{-1}} & 0 \\
0 & \frac{x^{-1}}{xq^{-1}} & \frac{q^{-1}}{xq^{-1}} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

**Remark 12.29.** There is an analogous construction for the elliptic q-KZ equation.

## 2. Hypergeometric Pairing

In what follows we will assume that \( p \in \mathbb{R} \ (p < 0) \). Let’s introduce the following map:

**Definition 12.30.**

\( J(z_1, ..., z_n) : \mathcal{F}_{m_1, ..., m_n}(z_1, ..., z_n) \otimes \mathcal{F}_{m_1, ..., m_n}(z_1, ..., z_n) \to \mathbb{C} \)

such that:

\[
(12.31) \quad J(z_1, ..., z_n)(f \otimes g) = \int_{\gamma} \Phi_{m_1, ..., m_n}(t_1, ..., t_k, z_1, ..., z_n) f(t_1, ..., t_k) g(t_1, ..., t_k) dt_1 \cdots dt_k,
\]

where \( \Phi_{m_1, ..., m_n}(t_1, ..., t_k, z_1, ..., z_n) \) is a q-deformation of the Master function, defined by:

\[
\Phi_{m_1, ..., m_n}(t_1, ..., t_k, z_1, ..., z_n) = \exp(\mu \sum_{a=1}^{k} t_a/p) \prod_{i=1}^{n} \prod_{a=1}^{k} \frac{\Gamma((t_a - z_i + m_i)/p)}{\Gamma((t_a - z_i - m_i)/p)} \prod_{a<b} \frac{\Gamma((t_a - t_b - 1)/p)}{\Gamma((t_a - t_b + 1)/p)},
\]

and the cycle \( \gamma \) is defined by:

\[
\gamma = \{(t_1, ..., t_k) \in \mathbb{C}^k | \Re t_i = 0, \forall i = 1, ..., k\}.
\]

**Remark 12.32.** If \( m_1, ..., m_n \in \mathbb{R} \) and \( < < 0 \) and \( z_1, ..., z_n \in i\mathbb{R} \) the integral 12.31 is convergent. If the previous conditions are not satisfied we define 12.31 by analytic continuation.

The hypergeometric pairing has the following property:

**Proposition 12.33.** 1. For generic \( z_1, ..., z_n \) and generic \( m_1, ..., m_n \) 12.31 is non degenerate.

**Proof.** The proof follows from the calculation of suitable determinants of hypergeometric integrals.

**Example 12.34** (q-Selberg integral). Let \( n = 1 \). Then:

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{k} (u^{2i} \Gamma(a + t_i) \Gamma(a - t_i)) \prod_{i<j} \frac{\Gamma(t_i - t_j + x) \Gamma(t_j - t_i + x)}{\Gamma(t_i - t_j) \Gamma(t_j - t_i)} dt_1 \cdots dt_k =
\]

\[
(2\pi \sqrt{-1})^k (u + u^{-1})^{k(2a + (k-1)x)} \prod_{i=1}^{k} \frac{\Gamma(1 + ix) \Gamma(2a + (i - 1)x)}{\Gamma(1 + x)}
\]
Remark 12.35. The q-Selberg integral defined in the previous example gives the multidimensional generalization of the Barnes’ formula.

Now let’s consider:

\[ J(z_1, \ldots, z_n) : F_{m_1, \ldots, m_n}(z_1, \ldots, z_n) \otimes F_{m_1, \ldots, m_n}(z_1, \ldots, z_n) \rightarrow \mathbb{C} \]

and let \( \nu, \tau \in S^n \), where \( S^n \) is the permutation group. Then \( J \) defines the following paring:

\[ (12.36) \quad (M_{m_1}^q \otimes \ldots \otimes M_{m_n}^q) \otimes (M_{m_1}^q \otimes \ldots \otimes M_{m_n}^q)^* \rightarrow \mathbb{C} \]

or, equivalently, the following map:

\[ (12.37) \quad I_{\tau, \nu}(z_1, \ldots, z_n) : M_{m_1}^q \otimes \ldots \otimes M_{m_n}^q \rightarrow M_{m_1} \otimes \ldots \otimes M_{m_n}. \]

Let’s now consider 12.37 for \( \nu = id \):

\[ (12.38) \quad I_{\tau, id}(z_1, \ldots, z_n) = I_{\tau}(z_1, \ldots, z_n) : M_{m_1}^q \otimes \ldots \otimes M_{m_n}^q \rightarrow M_{m_1} \otimes \ldots \otimes M_{m_n}. \]

We can now state the following:

Theorem 12.39. For any \( \tau \in S^n \), \( I_{\tau} \) defined in 12.38 satisfies the rational q-KZ equation. with values in \( M_{m_1} \otimes \ldots \otimes M_{m_n} \).

Remark 12.40. We can rephrase the content of the previous theorem saying that the solution of the rational q-KZ equation, with values in \( M_{m_1} \otimes \ldots \otimes M_{m_n} \), product of \( \mathfrak{sl}_2 \)-modules, are labeled elements of the product of \( \mathcal{U}_q(\mathfrak{sl}_2) \) corresponding modules.

We observe that for \( \tau \in S^n \), the product \( M_{m_1}^q \otimes \ldots \otimes M_{m_n}^q \) has a natural basis given by:

\[ (12.41) \quad f_{j_1} v_{j_2} = f_{j_1}^q v_{j_1}^q \otimes \ldots \otimes f_{j_n}^q v_{j_n}^q, \]

where \( J = (j_1, \ldots, j_n) \) and \( v_{j_1}^q \in M_{m_1}^q \) is the generating vector. The basis \( \{ f_{j_1} v_{j_2} \} \) given in 12.41 corresponds to a solution of the q-KZ equation and for any element \( \tau \in S^n \) we have a basis of solutions. It turns out that these basis have remarkable asymptotic properties.

3. Quantization of the Drinfeld-Kohno theorem

In this section we will discuss the discrete analogue of the Drinfeld-Kohno theorem. Let’s start describing the monodromy of a difference equation.

Example 12.42. Let’s consider the following scalar difference equation:

\[ (12.43) \quad \psi(z + p) = K(z) \psi(z), \quad z \in \mathbb{C} \text{ and } p < 0. \]

We have that if \( \psi = \psi(z) \) is a non zero solution then any other solution \( \tilde{\psi} \) of this equation has the form:

\[ \tilde{\psi}(z) = P(z) \psi(z), \]

where \( P = P(z) \) is \( p \)-periodic, i.e:

\[ P(z + p) = P(z) \forall z. \]

Let’s now let’s suppose that the function \( K = K(z) \) satisfies the following properties:

\[ (12.44) \quad K(z) = e^{az}(1 + \alpha/z + \mathcal{O}(1/z)), \quad z \rightarrow +\infty, \]

\[ (12.45) \quad K(z) = e^{bz}(1 + \beta/z + \mathcal{O}(1/z)), \quad z \rightarrow -\infty, \]
where \( a, b, \alpha \) and \( \beta \) are some numbers. Then we have that there are solutions \( f_+ \) and \( f_- \) of \( 12.43 \) such that:

\[
f_+(z) \sim e^{az/p} z^{\alpha/p} (1 + \mathcal{O}(1/z)) \quad \text{as} \quad z \to +\infty
\]

and similarly

\[
f_-(z) \sim e^{bz/p} z^{\beta/p} (1 + \mathcal{O}(1/z)) \quad \text{as} \quad z \to -\infty.
\]

The function:

\[
(12.46) \quad P_{+, -} = P_{+, -}(z) = \frac{f_+(z)}{f_-(z)}
\]

is called the scattering or transition matrix associated to the solutions \( 12.44 \) and \( 12.45 \).

**Remark 12.47.** The scattering matrix replaces the concept of monodromy in the case of difference equations.

We want to apply a procedure similar to the one described in the previous example to the q-KZ equation.

**Remark 12.48.** We observe that the q-KZ equation has \( n! \) asymptotic zones:

\[
A_{\tau}, \quad \tau \in S^n :
\]

\[
A_{\tau} : \Re z_{\tau_1} << \Re z_{\tau_2} << \ldots \Re z_{\tau_n},
\]

i.e. we can approach \( \infty \) in \( n! \) different ways.

**Definition 12.49.** We will say that a basis of solutions \( \psi_1, \psi_2, \ldots \) of the q-KZ equation, forms an asymptotic solution in a given zone \( A \), if:

\[
(12.50) \quad \psi_j = \exp \left( \sum_{m=1}^{n} a_{mj} z_m/p \right) \prod_{m<l} (z_l - z_m)^{b_{jlm}} (v_j + \mathcal{O}(1)) \forall j,
\]

where \( a_{mj}, b_{jlm} \) are numbers, \( v_1, v_2, \ldots \) are vectors forming a basis in \( V \) and \( \mathcal{O}(1) \) tends to zero as \( z \to \infty \) in \( A \).

**Remark 12.51.** We recall that \( K_m = K_m(z_1, \ldots, z_n, p) \) can be written in terms of R-matrices as:

\[
K_m = R_{V_m, V_m-1} (z_m - z_{m-1} + p) \ldots R_{V_m, V_1} (z_m - z_1 + p) e^{\mu h^{(m)}}
\]

\[
R_{V_m, V_n} (z_m - z_n) \ldots R_{V_{m+1}, V_{m+1}} (z_m - z_{m+1}).
\]

In every asymptotic zone \( A_{\tau} \) we have:

\[
K_m(z_1, \ldots, z_n) \sim e^{\mu h^{(m)}},
\]

i.e at \( \infty \) the \( K_m(z_1, \ldots, z_n) \) operators are diagonal and the vectors \( v_1, v_2, \ldots \) introduced in the previous definition are common eigenvectors. These vectors can be easily described, namely consider the vectors \( \{ f^1 v_{m_1} \otimes \ldots \otimes f^n v_{m_n} \} \): they form an eigenbasis for \( h^{(m)} \). This basis will be called the monomial basis in \( M_{m_1} \otimes \ldots \otimes M_{m_n} \).

Now we can state the main result:

**Theorem 12.52.** For every asymptotic zone \( A_{\tau} \) the basis in the space of solutions corresponding to the basis:

\[
\{ f_j \psi_{\tau} \} \subset M_{m_1} \otimes \ldots \otimes M_{m_n}
\]

is asymptotic in the zone \( A_{\tau} \).
We also have the following description of the scattering matrices:

**Corollary 12.53.** The transition functions between the asymptotic solutions corresponding to neighboring zones are given in terms of the corresponding trigonometric R-matrices. Namely \( \forall \tau \in S^n, m \in \{1, \ldots, n\} \), let

\[
T_{\tau,m}(z_1, \ldots, z_n) : M^q_{\tau_1} \otimes \ldots \otimes M^q_{\tau_{m+1}} \otimes M^q_{\tau_m} \otimes \ldots \otimes M^q_{\tau_n} \longrightarrow M^q_{\tau_1} \otimes \ldots \otimes M^q_{\tau_n}
\]

be the composition of \( R_{M^q_{\tau_{m+1}}, M^q_{\tau_m}}(e^{2\pi i(z_{\tau_{m+1}} - z_{\tau_m})/p}) \) and:

\[
P : M^q_{\tau_{m+1}} \otimes M^q_{\tau_m} \longrightarrow M^q_{\tau_m} \otimes M^q_{\tau_{m+1}}.
\]

Then the transition function between the solutions associated to the zones \( A_\tau \) and \( A_{\sigma \tau} \) is identified with \( T_{\tau,m}(z_1, \ldots, z_n) \).

**Example 12.55.** Let \( a, b, c, d, \theta \in \mathbb{C} \) with \( \Re \theta > 0 \).

Set \( \lambda = \sqrt{a^2 - bc}, h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( A(u) = \frac{1}{d + u} \begin{pmatrix} a + u & b \\ c & a - u \end{pmatrix} \).

We have that:

\[
A(u, \theta) = \theta^u A(u) \theta^{-uh}.
\]

We have the following:

**Theorem 12.56.**

\[
\lim_{s \to \infty} \left( s^{-ah} h^s \left( \prod_{r=-s}^{s} A(u + r; \theta) \right) h^s s^{ah} \right) = A^q(u),
\]

where:

\[
A^q(u) = \frac{1}{\sin(\pi(d + u))} \begin{pmatrix} \sin(\pi(a + u)) & \frac{\pi b(\theta + \theta^{-1})^{2a}}{\Gamma(1+a+\lambda)\Gamma(1-a-\lambda)} \\ \frac{\pi c(\theta + \theta^{-1})^{-2a}}{\Gamma(1-a+\lambda)\Gamma(1-a-\lambda)} & \sin(\pi(a - u)) \end{pmatrix}
\]

References for this chapter: [TV1, TV2]
Bibliography

[DT] Antoine Douai, Hiroaki Terao, The determinant of a hypergeometric period matrix, Invent. Math. 128 (1997), no. 3, 417–436.

[EFK] Pavel Etingof, Igor Frenkel, Alexander Kirillov Jr., Lectures on representation theory and Knizhnik-Zamolodchikov equations, Mathematical Surveys and Monographs, 58, American Mathematical Society, Providence, RI, 1998.

[EV1] Pavel Etingof, Alexander Varchenko, Dynamical Weyl groups and applications, math.QA/0011001.

[EV2] Pavel Etingof, Alexander Varchenko, Traces of intertwiners for quantum groups and difference equations. I, Duke Math. J. 104 (2000), no. 3, 391–432.

[F] L. Faddeev, Lectures on the Quantum Inverse Scattering Method, In: Integrable Systems, ed. by X.-G. Song, Nankai Lectures Math. Phys., World Scientific, Teaneck, N.J., 1990, 23–70.

[FMTV] G. Felder, Y. Markov, V. Tarasov, A. Varchenko, Differential Equations Compatible with KZ Equations, math.QA/0001184.

[FSV1] G. Felder, L. Stevens, A. Varchenko, Elliptic Selberg integrals, math.QA/0103227.

[FSV2] G. Felder, L. Stevens, A. Varchenko, Modular transformations of the elliptic hypergeometric functions, Macdonald polynomials, and the shift operator, math.QA/0203049.

[FV] G. Felder, A. Varchenko, Special functions, conformal blocks, Bethe ansatz, and SL(3, Z), Phil. Trans. Roy. Soc. Lond. Ser. A, 359 (2001) 1365-1374, math.QA/0101136.

[FeSV] Boris Feigin, Vadim Schechtman, Alexander Varchenko, On algebraic equations satisfied by hypergeometric correlators in WZW models, I: Comm. Math. Phys. 166 (1994), no. 1, 173–184. II: Comm. Math. Phys. 170 (1995), no. 1, 219–247.

[G] M. Gaudin, Diagonalization d’une class hamiltoniens de spin, Journ. de Physique 37, no. 10 (1976), 1087–1098.

[KZ] V. Knizhnik and A. Zamolodchikov, Current algebra and the Wess-Zumino model in two dimensions, Nucl. Phys. B 247 (1984), 83–103.

[MaV] Y. Markov, A. Varchenko, Solutions of Trigonometric KZ Equations satisfy Dynamical Difference Equations, Adv. Math. 166 (2002), no. 1, 100–147, math.QA/0103226.

[MTV] Y. Markov, V. Tarasov, A. Varchenko, The Determinant of a Hypergeometric Period Matrix, math.AG/9709017.

[MV] E. Mukhin and A. Varchenko, Remarks on Critical Points of Phase Functions and Norms of Bethe Vectors, In: Arrangements - Tokyo 1998, Advanced Studies in Pure Mathematics 27 (2000), 239–246.

[OS] Peter Orlik, Louis Solomon, Combinatorics and topology of complements of hyperplanes, Invent. Math. 56 (1980), no. 2, 167–189.

[OT] P. Orlik and H. Terao, The Number of Critical Points of a Product of Powers of Linear Functions, Invent. Math. 120 (1995), no. 1, 1–14.

[R] E. Rainville, Intermediate differential equations, The Macmillan Company, 1964.

[RV] N. Reshetikhin, A. Varchenko, Quasiclassical Asymptotics of Solutions to the KZ Equations, In: Geometry, Topology, and Physics for Raoul Bott, International Press, 1994, 293–322.

[Si] R. Silvotti, On a conjecture of Varchenko, Invent. Math. 126 (1996), no. 2, 235–248.

[Sk1] E. Sklyanin, Separation of variables in the Gaudin model, J. Soviet Math. 47 (1989), 2473–2488.

[Sk2] E. Sklyanin, The functional Bethe ansatz. In: Integrable and Superintegrable Systems, ed. by B. Kupershmidt, World Scientific, Singapore, 1990, 8–33.
I. Scherbak and A. Varchenko, Critical point of functions, $\mathfrak{sl}_2$ representations and Fuchsian differential equations with only univalued solutions, math.QA/0112269.

V. Schechtman, A. Varchenko, Arrangements of hyperplanes and Lie algebra homology, Invent. Math., 106 (1991), 139–194.

G. Szego, Orthogonal polynomials, AMS, 1939.

V. Tarasov and A. Varchenko, Geometry of $q$-Hypergeometric Functions, Quantum Affine Algebras and Elliptic Quantum Groups, Astérisque 246 (1997) 1–135, math.QA/9703044.

V. Tarasov and A. Varchenko, Geometry of $q$-Hypergeometric Functions as a Bridge between Yangians and Quantum Affine Algebras, Invent. Math., 128 (1997), 501–588, math.QA/9604011.

V. Tarasov and A. Varchenko, Difference Equations Compatible with Trigonometric KZ Differential Equations, math.QA/0002132.

V. Tarasov and A. Varchenko, Completeness of Bethe Vectors and Difference Equations with Regular Singular Points, IMRN 13 (1995), 637–669.

A. Varchenko, Critical points of the product of powers of linear functions and families of bases of singular vectors, Compositio Mathematica 97 (1995), 385–401.

A. Varchenko, Multidimensional hypergeometric functions and representation theory of Lie algebras and quantum groups, Advances in Math. Phys., 21, World Scientific, 1995.

A. Varchenko, The Euler beta function, the Vandermonde determinant, Legendre’s equation, and critical points of linear function on a configuration of hyperplanes, I. Math. USSR, Izvestia 35 (1990), 543–571; II. Math. USSR, Izvestia 36 (1991), 155–168.

A. Varchenko, Determinant formula for Selberg type integrals, Funct. Analys and Its Appl. 4 (1991), 65–66.

Alexander Varchenko, Quantization of geometry associated to the quantized Knizhnik-Zamolodchikov equations, math.QA/9606006.