Numerical simulations of a flux qubit coupled to a high quality resonator

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Abstract. Circuit-QED - the electrical analogue of cavity quantum electrodynamics in atomic physics- is an emerging field of research. By coupling superconducting qubits to an on-chip high-quality resonator it is possible to study a number of effects that are of interest for fundamental tests of quantum mechanics and also have potential applications in quantum information processing, metrology and communication. We have performed computer simulations of a system comprising a flux qubit strongly coupled to a high-quality coplanar waveguide resonator driven by an external microwave source. Using realistic values for the various parameters we have studied, among other things, the transmittance of the cavity and the spectrum of the emitted radiation.

1. Introduction
The rapid development of solid state quantum information processing (QIP) over the past few years has now reached a point where the basic building block -the single qubit- is experimentally quite well understood. But there are still many challenging problems, in particular how to improve on the short coherence time which still makes it very difficult to implement even simple gate operations in multi-qubit systems. Nevertheless, in some systems of interest only a single qubit is needed, making the problem of designing systems with long coherence times somewhat less challenging. One such example is on-demand generation of single microwave photons which has potential applications in metrology and quantum communication. Hence, even existing technologies might be useful in some practical QIP applications.

One such system arises from circuit Quantum Electrodynamics (circuit-QED) where a superconducting qubit is either capacitively or inductively coupled to a resonator. This coupling can give rise to a plethora of effects which can be studied using e.g. spectroscopy. The origin of these effects is that the qubit can be made to exchange energy with the resonator. In a high-quality resonator at low temperatures the quantization of the electromagnetic field becomes important and can be modeled as a quantum mechanical harmonic oscillator $\hbar \omega_r (a^\dagger a + 1/2)$ where the creation(annihilation) operator creates(does) one excitation (i.e. a photon) in the resonator. If the qubit parameters are chosen in such a way that the energy splitting $\omega_0$ can be tuned so that $\omega_0 = \omega_r$, it can exchange energy with the resonator, in essence by emitting a photon when flipping from its higher to lower energy state or, vice-versa by being excited to its higher state by absorbing a photon. If the lifetime of the photons in the resonator is long
and the coupling is strong enough this can happen many times during the coherence time of the qubit. This coherent exchange of energy is what makes this system so interesting. It also sets it apart from systems where a qubit interacts with low Q resonators where the lifetime of the photons in the resonator is so short that the interaction is essentially "classical".

The pioneering experiments in this field were done by Schoelkopf and co-workers [1, 2] using a charge qubit. However, other qubit systems have also been considered [3] and c-QED effects have also been reported for systems with flux qubits [4, 5]. Here we will discuss the case of a three-junction persistent current qubit (PCQ) [6] inductively coupled to a coplanar resonator.

Superconducting circuit-QED systems have been theoretically considered by several authors, see e.g. [7, 8, 9]. Note, however, that some of the ideas presented in those papers are not directly applicable to the case where a PCQ is used, the reason is that an ordinary (i.e. one loop) PCQ only has one tuning parameter (the flux threading the loop) as opposed to a split-Cooper box charge qubit where both flux and charge can be tuned independently. While this means that a PCQ is less flexible it also means that it is intrinsically less sensitive to charge noise.

The system under consideration can be seen in fig 1. A PCQ with an area $A$ is placed in the gap between the centre conductor and ground plane of a coplanar resonator. Both these structures can be fabricated using conventional lithography. The qubit can be biased with a flux $\Phi_x$ using an external coil and/or via on-chip control lines. The system can then be measured by using standard methods from microwave engineering to probe the resonator.

In order to examine the viability of this system we have performed numerical simulations using realistic values for all parameters. In section II of this paper we will briefly outline the underlying theory and in section III the results of the simulations.

2. Theory

As was shown in [10] the full Hamiltonian of the system can be written

$$H = \hbar \omega_r \left( a^{\dagger} a + \frac{1}{2} \right) + \frac{\hbar \omega_0}{2} \sigma_z - \frac{\hbar g}{2} (a^{\dagger} + a) \sin \theta + i \hbar g (a^{\dagger} - a) \sigma_z \sin \theta$$

where $\theta = \arctan \frac{\Delta}{\epsilon}$ is the mixing angle, $\delta$ is the level repulsion, $\epsilon = (2I_p/\hbar)(\Phi_x - \Phi_0/2)$ and $I_p$ the persistent current which is approximately equal to the critical current of the smallest
junction in the loop. Note that $\epsilon$ can be tuned by sweeping the external field $\Phi_x$ which is how the Larmor frequency $\omega_0(\Phi_x) = h^{-1}\sqrt{\epsilon^2 + \delta}$ of the qubit can be controlled.

The coupling factor $g$ which describes the interaction strength between the qubit and the cavity is given by

$$g \approx \frac{I_p A \mu_0}{\hbar \pi r} \sqrt{\frac{\hbar \omega_r}{2L}}.$$  \hspace{1cm} (2)

using a dipole approximation.

The Hamiltonian (1) reduces to the Jaynes-Cummings Hamiltonian, well known from quantum optics, when the qubit is operated near the degeneracy point $\theta = \pi/2$, i.e. $g \sin \theta = g$ and the last term $\hbar g (a^\dagger + a)\sigma_z \cos \theta$ is zero. In what remains we will assume that this approximation is valid.

For this system there are three different dissipation channels: the qubit relaxes at a rate $\gamma_1$; is subject to pure dephasing (due to e.g flux noise) at a rate $\gamma_\phi$; the resonator is losing energy at at rate $\kappa/2\pi = \omega_r/Q$ where $Q$ is the quality factor. Assuming the losses in the film and the substrate are negligible $\kappa$ is determined by the strength of the coupling to the transmission lines used to probe the system.

If $g \gg \kappa, \gamma_1$ the system is in the so-called strong coupling regime. This regime is of particular interest since the interaction between qubit and resonator is so strong that they can exchange energy many times before they decay. This gives rise to a Rabi splitting $\hbar g$ which manifests itself as a splitting of the single peaked spectrum to a double peaked spectrum centered around $\omega_r$ where the two peaks are separated in frequency by $2g$ when $\omega_0 = \omega_r$.

Assuming the interaction with the environment is Markovian the evolution of the reduced density matrix of the system can be described by a master equation of Lindblad form [11]

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} [H, \rho] + \sum_{k=1}^3 \left( C_k \rho C_k^\dagger - \frac{1}{2} \left( C_k^\dagger C_k \rho + \rho C_k^\dagger C_k \right) \right)$$  \hspace{1cm} (3)

In our case there are therefore three Lindblad operators that can be written $C_1 = \sqrt{\gamma_1} \sigma^- , C_2 = \sqrt{\kappa} \sigma_z$ and $C_3 = \sqrt{\epsilon} a$ where $C_1, C_2$ describes the energy relaxation and pure dephasing of the qubit, respectively, and $C_3$ gives the relaxation of the resonator. The problem has therefore been reduced to solving the master equation $\dot{\rho} = \mathcal{L} \rho$. This can be done using various numerical methods. Here we have used the routines including in the "Quantum Optics Toolbox" by Tan[12].

3. Simulations

We will show the results of some of our numerical calculations. For definiteness we will use the following parameters in the simulations (in units of frequency): $\omega_r = 6$ GHz, $\kappa = 0.1$ MHz, $\gamma_1 = 1$ Mhz, $\gamma_\phi = 10$ MHz, $g = 35$ MHz and $\epsilon = 5.8$ GHz. We will also make a further assumption that $\gamma_1, \gamma_\phi$ are approximately independent of the bias point in the range considered here.

By solving the master equation numerically a number of experimentally accessible quantities can be calculated. Experimentally the transmittance of the circuit is of particular importance since it is relatively easy to measure. This is usually described in terms of the scattering matrix element $S_{21}$.

$$S_{21}(\omega) = S_{12}(\omega) = \frac{2Z_0 n^2}{2\pi^2 Z_0 + Z}$$  \hspace{1cm} (4)

where $Z_0 = 50\Omega$ is the impedance of measurement system and $n$ is a parameter that describes the coupling strength between the resonator and the measurement lines [13]. The steady-state
Figure 2. Left: The transmittance $|S_{21}|$ of the resonator as a function of the external flux. Right: $S_{21}$ magnitude and phase on- and far from resonance.

impedance $Z$ of the system can, in the linear regime, be calculated from the susceptibility of the circuit [11, 14]

$$\chi(\omega) = \frac{i}{\hbar} \int_0^\infty e^{i\omega t} \langle [q(t), q(0)] \rangle dt$$  \hspace{1cm} (5)

where $q$ is the operator

$$q = -i \sqrt{\frac{\hbar \omega_r}{2L}} (a - a^\dagger)$$  \hspace{1cm} (6)

We then get the impedance from $Z(\omega) = i\omega/\chi(\omega)$ and finally the transmissivity from eq. 4. The results of this calculation have been plotted in fig. 2.

As can be seen from the figure the system goes through an avoided crossing when the qubit is tuned in and out of resonance with the resonator. Obviously, this type of behavior will only be seen for a system where $\omega_0(\Phi_x)$ can be tuned into resonance with the resonator but unfortunately the coherence times goes down drastically as the qubit is moved from its degeneracy point. This means that the qubit must be designed to have a level repulsion $\delta/\hbar$ just slightly smaller than $\omega_r$. Experimentally, this imposes rather strict conditions on the accuracy of fabrication.

In order to calculate the power spectrum we use the same method as in [10]. The spectrum of the intracavity field can be calculated from the 2-time correlation function $\langle a^\dagger(t + \tau)a(\tau) \rangle [15]$

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega \tau} \langle a^\dagger(\tau + t)a(t) \rangle d\tau$$  \hspace{1cm} (7)

The power spectrum is important since it gives information about the state of the system not readily seen in e.g. the transmittance. As was shown in e.g. our earlier work[10] the symmetry of the spectrum is very sensitive to pure dephasing $\gamma_\phi$. Since the dephasing time in solid state qubits tend to be quite short, this is an important effect, but at the same time it cannot readily
Figure 3. Power spectrum of the emitted radiation as a function of probe power. The spectra have been normalized to have the maximum value 1. The parameters are the same as in fig. 2. The maximum power corresponds to $<n> \approx 5$. Note also the logarithmic scale on the y-axis.

be observed in an ordinary transmittance experiment (which always records an asymmetric spectrum unless exactly on resonance).

Another important effect which is readily seen in the power spectrum is the dependence on probe power. In the linear regime this power is assumed to be small enough not to affect the dynamics of the system. However, once the average photon number $\langle n \rangle$ start to be significant the drive can have a large impact. To see this we add a classical drive of the form $\xi (e^{-i\omega t} a^\dagger + e^{i\omega t} a)$ to the Hamiltonian, $\xi$ being a measure of the drive amplitude, and then calculate the spectrum using eq. 7. In fig. 3 we show the power spectrum on resonance as a function of drive power. At strong drive power $\langle n \rangle \sim 10^{-3}$ we are no longer probing the properties of the 'bare' system since the drive has changed the spectrum significantly. Hence, in experiments it is important to keep the probe power to a minimum. The power leaking out of the resonator when $\langle n \rangle = 1$ is $\hbar \omega_K/2$ which is equal to a power of about -160 dBm in the case considered here, a very small number indeed. In order not to perturb the system significantly this power should preferably be reduced by another 20 dB. Hence, a very sensitive measurement setup is required.

4. Conclusion
Circuit-QED systems based on superconducting qubits can be of interest both because they allow us to test fundamental relations in quantum mechanics and because they can be used in applications such as single photon generation and various types of quantum enhanced metrology. The fact that the qubit-resonator system is 'simple' in the sense that it is well described by the Jaynes-Cummings Hamiltonian is important since it allows us to create realistic models of experiments that, at least in principle, allow us to understand the dynamics of real circuits in...
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