Approximate Equations of Motion for Compact Spinning Bodies in General Relativity

James L. Anderson

Stevens Institute of Technology and
Hoboken, NJ 07030

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Abstract

Approximate equations are derived for the motion of a gyroscope on the earth’s gravitational field using the Einstein, Infeld, Hoffmann surface integral method. This method does not require a knowledge the energy-momentum-stress tensor associated with the gyroscope and uses only its exterior field for its characterization. The resulting equations of motion differ from those of previous derivations.
Already by 1927 Einstein had realized that the field equations of general relativity contained information about the motion of the sources of the gravitational field. A complete working out of this possibility was achieved by Einstein, Infeld and Hoffmann (EIH) and elaborated later in two papers by Einstein and Infeld. Their method used only the source-free field equations of general relativity and avoided any consideration of the interiors of the sources, assuming only that they are compact, i.e. their sizes are small compared to the distance between them and that their velocities are small compared to the speed of light $c$. The nature of the sources is characterized by their exterior fields alone. The main drawback to the EIH procedure is that it requires a large amount of tedious calculation.

An alternate approach to deriving equations of motion was introduced by Fock and later developed further by Papapetrou. This second method takes into account the interiors of the sources and makes use of the conservation laws

$$T^{\mu\nu}$$

where $T^{\mu\nu}$ is the matter tensor associated with the source. (Here and in what follows Greek indices take the values 0, 1, 2, 3, Latin indices take the values 1, 2, 3, the Einstein summation convention is assumed, a comma denotes ordinary differentiation and a semicolon denotes covariant differentiation.) While this method requires somewhat less calculations, it has the drawback that one must specify $T^{\mu\nu}$. Papapetrou later used the Fock approach to derive equations of motion for ‘test’ bodies moving in an external gravitational field and applied it to the case of a spinning test body. However, the equations obtained by him were insufficient to determine completely the dynamical variables $X^{\mu}$ and $S^{\mu\nu}$, the position and spin angular momentum of the body, appearing in them. As a consequence it was necessary to impose restrictions from the outside on the components of $S^{\mu\nu}$ in order to close the system of equations. These restrictions all require the vanishing of its space-time components (the i0 components) but differ in which frame of reference these components vanish. Various authors have proposed different restrictions leading to different equations of motion for $S^{\mu\nu}$. In a completely different approach using the Principle of Equivalence, Weinberg has obtained yet another set of equations for this variable that differs from those based on the Papapetrou equations. It should be pointed out that neither of these derivations make full use of the Einstein field equations - Papapetrou used them only to derive the conservation laws exhibited in equation (1) and Weinberg did not use them at all. A resolution of these
different results has taken on a certain urgency with the launch of the NASA-Standard Gravity Probe B (GPB) designed to measure the change in orientation of a small gyroscope in earth orbit. This experiment was first suggested by Schiff in 1960, whose calculations were based on the spin equations of motion obtained using the supplementary conditions proposed by Corinaldesi and Papapetrou (CP).

In an attempt to resolve these issues I have undertaken to derive approximate equations of motion for compact spinning bodies using methods similar to the ones used by EIH. Today two developments have made it possible to perform the calculations needed for this purpose without undue labor—high speed personal computers and symbolic manipulation programs such as Mathematica and Maple together with the wonderful program grtensor. With these tools it is now possible to obtain equations of motion in a matter of minutes with the assurance of correctness that would have required days or even weeks to perform by hand.

Since the equations of motion obtained using the EIH approach are only approximate, it is necessary to specify the system one wishes to apply these equations to, in this case a compact gyroscope orbiting the earth, and to identify the small dimensionless parameters associated with this system that will be used in the expansions employed. The gyroscope used in GPB (there are actually four of them) consists of an almost perfect sphere of fused quartz with a radius \( r_g = 0.019 \) m and a mass \( m = 0.075 \) kg. and an initial angular velocity \( \omega = 27000 \) rad/s. The gyroscope was launched into a near perfect circular polar orbit (eccentricity = .0014) of radius \( R = 7027 \) km. These values will determine the relative importance of the terms in the approximate expression for the gravitational field to be used to evaluate the surface integral terms that arise in the EIH approach.

The form of the field equations for the gravitational field \( g_{\mu\nu} \) to be used here are due to Landau and Lifshitz. Exterior to the field sources they have the form

\[
U_{\mu\nu,\rho} = -g t_{LL}^{\mu\nu},
\]

where

\[
U_{\mu\nu} = U_{\mu\nu} = (1/16\pi)\{-g(g_{\mu\rho}g^{\rho\sigma} - g_{\mu\sigma}g^{\mu\rho})(g^{\nu\sigma})\}, \quad (3)
\]

\( g = \det(g_{\mu\nu}) \) and \( t_{LL}^{\mu\nu} \) is the Landau-Lifshitz pseudotensor. Because of the antisymmetry of \( U_{\mu\nu}^{\rho} \) in its last two indices, it follows that \( U_{\mu\nu}^{\rho} \) is a three-dimensional curl and therefore
when equation (2) is integrated over a two-surface in a $t = \text{constant}$ hypersurface, one gets

$$\oint_S (U^\mu r^0, 0 + g t_{LL}^{\mu r}) n_r dS,$$  \hspace{1cm} (4)

where $n_r$ is a unit surface normal. In a like manner one gets

$$\oint_S \{ (x^\mu U^{rr0})_0 - (x^\nu U^{\nu r0})_0 + g x^\mu t_{LL}^{\mu r} - g x^\nu t_{LL}^{\nu r}$$

$$+ (1/16\pi) \{ g (g^{rr} g^{\mu 0} - g^{\mu r} g^{r 0}) \}_0 \} n_r dS = 0.$$  \hspace{1cm} (5)

It is these two last equations that are used in the EIH procedure to obtain equations of motion.

In order to use equations (4) and (5) it is necessary to obtain solutions of the field equations corresponding to the type of system being considered. Since in the case of GPB no such exact solution exists it is necessary to use approximate ones. In the case of GPB the system consists of two bodies, the earth and the gyroscope. Since the ratio of the masses $M$ and $m$ of the earth and the gyroscope is $1.25 \times 10^{-26}$ it is clear that we can ignore completely any effect the gyroscope has on the earth’s motion. We can therefore take the earth to be at rest at the origin of an inertial frame characterized by coordinates $\{ct, x, y, z\}$. At the location of the gyroscope the earth’s field has the dimensionless value $MG/Re^2 = 6.5 \times 10^{-10}$ where $G$ is the Newtonian gravitational constant and $M$ is the mass of the earth. The gyroscope’s contribution to the gravitational field consists of three parts: its static part $mG/r_g^2 = 2.9 \times 10^{-27}$, an induction part $mGV/r_g^3 = 7.3 \times 10^{-32}$, where $V = 7.5 \times 10^3$ m/s is the orbital velocity of the gyroscope, and a spin contribution. The latter contribution depends on the gyroscope’s spin angular momentum $S = 0.29$ kg m$^2$/s and is given by $SG/r_g^2 c^3 = 2.1 \times 10^{-33}$. From these numbers we can form the three small dimensionless parameters to be used in the construction of the approximate gravitational field. The first of these is the slowness parameter $\epsilon = V/c = 2.5 \times 10^{-5}$. The second one is the ratio of the gyroscope’s monopole field to that of the earth at the surface of the gyroscope, $\epsilon 1 = mR/M r_g = 4.6 \times 10^{-18}$. Finally, the third small parameter is the ratio of the spin angular momentum of the gyroscope to its orbital angular momentum $\epsilon 2 = = S/m VR = 7.4 \times 10^{-11}$.

Taking these considerations into account, using units with $G = c = 1$ and measuring masses in units of $M$ and lengths in units of $R$, the components of the gravitational field
$g_{\mu\nu}$ have the form

$$g_{00} = 1 - 2\epsilon^2\frac{M}{r} - 2\epsilon^2\epsilon 1\frac{m}{r^1} - 2\epsilon^4\epsilon 2\gamma_i V_i$$  \hspace{1cm} (6a)$$

$$g_{ij} = \delta_{ij}(-1 - 2\epsilon^2\frac{M}{r} - 2\epsilon^2\epsilon 1\frac{m}{r^1}) - \epsilon^4\epsilon 2(\gamma_i V_j + \gamma_j V_i)$$  \hspace{1cm} (6b)$$

and

$$g_{i0} = 2\epsilon^3\epsilon 1\frac{mV_i}{r^1} + \epsilon^3\epsilon 2\gamma_i$$  \hspace{1cm} (6c)$$

where

$$\gamma_i = \epsilon_{ijk}\frac{x^1 j s_k}{r^1 3}$$  \hspace{1cm} (7)

and where $r^2 = x^2 + y^2 + z^2$ is the distance from the earth’s center with coordinates $\{0, 0, 0\}$ to the field point $\{x, y, z\}$, $r^1 2 = x^1 2 + x^1 2 + x^1 3$ is the distance from the center of the gyroscope with coordinates $\{R_1, R_2, R_3\}$ to this field point and $\epsilon_{ijk}$ is the antisymmetric density with values +1 or -1 depending on whether $ijk$ is an even or odd permutation of 123 and zero otherwise. Here only those terms that are needed to determine the lowest order equations of motion for the $s_k$ have been included in these expressions for the approximate components of $g_{\mu\nu}$. It is to be noted that the gyroscope monopole and dipole contributions to $g_{\mu\nu}$ are taken to have the same effective centers so that no supplementary conditions are needed to determine the time dependent of the $s_k$.

To obtain the above expressions for the gravitational field one makes use of the field of a stationary spinning, spherically symmetric, body given by $g_{i0}$ above. Since the GPB gyroscopes are moving in the earth’s gravitational field, it is necessary to boost the static field to the velocity of the moving gyroscope. This boost is responsible for the terms in the expressions for $g_{00}$ and $g_{ij}$ above that depend on $s_i$. In addition it is necessary that these fields satisfy the harmonic coordinate conditions to an accuracy that insures that the $g_{\mu\nu}$ are in fact approximate solutions of the Einstein field equations. In the present case this requirement is satisfied if

$$(\sqrt{-gg})_{,\nu} = \mathcal{O}(\epsilon^4\epsilon 1)$$  \hspace{1cm} (8)

This will be the case provided that

$$s_{i,0} = \mathcal{O}(\epsilon^3)$$  \hspace{1cm} (9)

Finally, it is necessary to discuss the dependence on time of the dynamical variables $R_i$ and $s_i$. In their original paper, EH introduced what they called the slow-motion approximation
by assuming that the source coordinates depended on the time \( t \) through the combination \( \epsilon t \). Their procedure is equivalent to what is known today as a multiple time formalism \((\href{#14}{14})\) and will be used in what follows. This being the case, condition \((\href{#9}{9})\) can be satisfied if one assumes that

\[ s_i = s_{0i} + \epsilon s_{1i}(\epsilon t) \]  

(10)

where \( s_{0i} \) is independent of \( \epsilon t \). (In higher orders of approximation \( s_{0i} \) will in general depend on \( \epsilon^3 t \).)

All that remains is to substitute the above expressions for \( g_{\mu\nu} \) into the surface integrals in equations \((\href{#4}{4})\) and \((\href{#5}{5})\) and evaluate the integrals over a sphere surrounding the gyroscope. Most of the terms so obtained will depend on the radius of the sphere chosen and so must cancel as a consequence of the field equations \((\href{#2}{2})\). Those terms that are independent of the sphere radius must vanish as a consequence of the motion of the sources, here the gyroscope, and hence are the desired equations of motion. Evaluating the surface integrals in equation \((\href{#4}{4})\) to \( \mathcal{O}(\epsilon^4 \epsilon 1) \) yields the Newtonian equations of motion for a particle moving in the earth’s gravitational field

\[ R' = -\frac{MR}{R^3} \]  

(11)

where a prime denotes differentiation with respect to \( \epsilon t \).

To get equations for the spin variables \( s_i \) it is necessary to evaluate the surface integrals in equation \((\href{#5}{5})\) to \( \mathcal{O}(\epsilon^6 \epsilon 2) \), a task that would have been beyond my abilities to perform without the help of Mathematica and grtensor. One finds that these equations can be written in vector form as

\[ ds_1/d(\epsilon t) = \frac{1}{10} \frac{M}{R^3} \left\{ -(s_0 \cdot \hat{R}) R + 19(s_0 \cdot R) \hat{R} + 16s_0(R \cdot \hat{R}) \right\} \]  

(12)

These equation are to be compared to the ones obtained by Cornalidesi and Papapetrou given by

\[ ds/dt = 2\frac{M}{R^3} \left\{ -(\frac{1}{2}(s \cdot \hat{R}) R + (s \cdot R) \hat{R} + 2s(R \cdot \hat{R}) - \frac{3}{2} \frac{(R \cdot \hat{R})}{R^2} (s \cdot R) R \right\} \]  

(13)

and Weinberg’s equation

\[ ds/d(t) = \frac{M}{R^3} \left\{ -2(s \cdot \hat{R}) R + (s \cdot R) \hat{R} - 2s(R \cdot \hat{R}) \right\}. \]  

(14)

where a dot over a quantity denotes differentiation with respect to \( t \). These latter two sets of equations are considered to be exact by their authors but can be solved approximately
by substituting for \( s \) the expression given for it in equation (10). It is clear that they will
give different results than those obtained from equations (12). In all of these equations the
dependence of \( R \) on \( t \) is gotten by solving the Newtonian equations of motion (11).

In the case of a circular motion in the xy-plane we can take, in equation (12),
\[
R_1 = R \cos(\epsilon \omega t) \quad \text{and} \quad R_2 = R \sin(\epsilon \omega t).
\]  
where \( \omega \) is the angular velocity of the gyroscope in its orbit. To find the secular change in
\( s \) with time we can average Equation (12) over an orbital period \( T = 2\pi/\omega \). If one takes \( s_0 \)
to lie in the plane of the orbit and the xy-axes are chosen so that \( s_{01} = s_0 \) and \( s_{02} = 0 \), the
resultant change \( \Delta s_1 \) is given by
\[
\Delta s_{11} = 0
\]
and
\[
\Delta s_{12} = 2\pi s_0 M/R
\]
so that the angular change \( \Delta \theta \) in the direction of \( s \) is given by
\[
\Delta \theta = 2\pi M/R.
\]
A similar analysis yields a value \( \Delta \theta = M/2R \) for the CP equations and \( 3M/2R \) for the
Weinberg equations. Why the difference in the three results? In the case of the Papapetrou-
Corinaldesi equations the authors made assumptions concerning the matter tensor \( T^{\mu \nu} \) that
are not justified and Weinberg relied on the Principle of Equivalence and identified the space
part of a four-vector with an axial three-vector.

It is also possible to take account of the spin-spin interaction between the gyroscope and
the earth’s rotation about its axis. To do so it is first necessary to introduce a fourth small
parameter \( \epsilon_3 = \Omega/\omega = 6.8 \times 10^{-2} \), where \( \Omega \) is the angular velocity of the earth, into our
expansions. The gravitational field of the earth’s rotation can be taken account of by adding
to the expression (6c) for \( g_{i0} \) a term
\[
g'_{i0} = \epsilon^3 \epsilon_3 \varepsilon_{ijk} \frac{R_j S_k}{R^3}
\]
where \( S_k = I \Omega_k \) is the earth’s angular momentum and \( I \) is its moment of inertia about
its axis of rotation. With this addition to the gravitational field, the surface integral (5) introduces an additional term \( \tau_S \) in the equation of motion (12) for \( s_1 \) given by
\[
\tau_S = \frac{\epsilon^3}{2R^3} \left\{ R^2 (s_0 \times S) - 3(S \cdot R) (s_0 \times R) \right\}.
\]
For a circumpolar orbit with xy-axes now chosen so that \( S = \{S, 0, 0\} \), assuming that \( s_0 = \{s_0 \cos(\varphi), s_0 \sin(\varphi), 0\} \) and after averaging over an orbital period one finds an additional change in \( s_1 \) given by

\[
\Delta s_{13} = \frac{2\pi}{4R^3} I \frac{\Omega}{\omega} s_0 \sin(\varphi) 
\]

with a corresponding angular change in the direction of \( s \) given by

\[
\Delta \theta = \frac{2\pi}{4R^3} I \frac{\Omega}{\omega} \sin(\varphi). 
\]

It is amusing to think that if this additional change in direction could be measured with enough accuracy one could use the result to determine \( I \) by assuming that general relativity was the correct theory of gravity.

* e-mail address: jlanders@stevens.edu

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