Noise-enhanced reconstruction of attractors

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Abstract

In principle, the state space of a chaotic attractor can be partially or wholly reconstructed from interspike intervals recorded from experiment. Under certain conditions, the quality of a partial reconstruction, as measured by the spike train prediction error, can be increased by adding noise to the spike creation process. This phenomenon for chaotic systems is an analogue of stochastic resonance.
The seminal articles [1] and the theorem of Takens in [2] demonstrated the theoretical and practical possibility of reconstructing the topological structure of the state space underlying an experimental system, using the measurement of a generic scalar or multivariate signal from the system. This possibility is especially welcome for nonlinear systems, where the potential exists for extremely complicated state space attractors. A great deal of subsequent research effort has gone into developing data processing techniques for the detection, analysis and exploitation of nonlinear, and in particular chaotic, processes.

Often, a delay coordinate reconstruction of a compact attractor from an evenly-sampled time series of measurements can be found that is topologically equivalent to the attractor. The function mapping the attractor to the reconstructed copy is called an embedding. According to [3], as long as the embedding dimension is greater than twice the box-counting dimension of the attractor, an embedding results for a probability-one choice of measurement functions. Recent theoretical work has attempted to widen the scope of dynamical data that can lead to an embedding. It has been shown, for example, that a similar reconstruction result holds when using spike train data (the recorded times between firings) from a model integrate-and-fire dynamical system with chaotic dynamics [4].

Attractor reconstruction can be viewed as a type of information transfer. In the case of a topological embedding, no information is lost. The set of states of the underlying experimental system is reproduced exactly in the copy that is reconstructed from measured data. In other cases, the reconstruction may be incomplete. When a chaotic signal is fed into a threshold crossing detector, the time-delay plot of time intervals between crossings reconstructs something akin to a Poincaré section of the underlying chaotic attractor. The dimension is decreased by one [5]. A recent study [6] of neuron models subjected to chaotic input points out that the two-dimensional FitzHugh-Nagumo differential equation [7] (hereafter referred to as FHN2) acts as a kind of threshold-crossing “filter” for input signals, and in particular fails to completely reconstruct the attractor which generated the input signal. One focus of the present article is to clarify the distinction, for attractor reconstruction purposes, between threshold-crossing (TC) and integrate-and-fire (IF) filters. In contrast to
the fact that FHN2 acts as a TC filter, we exhibit a variation on FHN2 which acts principally as an IF filter instead, and does fully reconstruct the input attractor. This differential equation model, which we shall denote FHN3, is a three-dimensional version of excitable FitzHugh-Nagumo-type dynamics. With appropriate parameter settings, it has the following remarkable property: When a signal from a system attractor is added to one of FHN3’s variables, another variable undergoes a deterministic spiking behavior whose interspike intervals, embedded as \( m \)-tuples, embed the original attractor in \( m \)-dimensional reconstruction space. Just as Takens’ theorem guarantees that multidimensional state space information can be condensed into a single evenly-spaced time series, the same can be accomplished with spike timings from the FHN3 filter.

Viewing attractor reconstruction as information transfer raises questions about the efficiency of the transfer process, and the possible effects of noise on this process. Surprisingly, in some instances noise can have a beneficial effect on information transmission, analogous to the stochastic resonance phenomenon [8] observed for multistable potentials and excitable media. This seemingly contradictory effect is the observed amplification of a filtered signal achieved when stochastic noise is added to the input signal.

Stochastic resonance has been shown to amplify signals generated by linear models, such as sine waves. Recently, it has been shown [9] that the same basic effect can be seen with aperiodic (stochastic) input signals if the means of measurement is appropriately modified. Since the spectrum of a random signal is not discrete, the SNR must be replaced in this case with a power norm sensitive to shape-matching and/or signal correlation.

Our present interest in stochastic resonance is somewhat different. Our goal is in maximizing the amount of state information carried by the spike train. We pose the question whether the quality of state information from a deterministic signal (as opposed to the stochastic realization used in [9]) can be improved by injecting random noise into the process. In particular, we are interested in the case where the input signal is chaotic. To determine the quality of state information contained in the spike train, we measure the ability to predict the spike train from its own history, using nonlinear prediction techniques.
We will show examples in which the interspike interval prediction error of the output signal decreases (predictability increases) with increasing noise added to the input signal. Showing that nonlinear predictability is enhanced by adding noise is analogous to the enhanced SNR shown in stochastic resonance studies.

The filters we will use to create spike trains capable of carrying low-dimensional deterministic state information are based on the well-known FitzHugh-Nagumo equation \[7\]. The two dimensional system FHN2

\[
\begin{align*}
\epsilon \dot{v} &= -v(v - 0.5)(v - 1) - w + S \\
\dot{w} &= v - w - b
\end{align*}
\]

is a simple differential equation that exhibits a fast spike followed by a refractory period. There is an equilibrium at \((v, w) = (v_0, v_0 - b)\), where \(v_0\) is a real-valued root of \(v_0(v_0 - 0.5)(v_0 - 1) = S + b - v_0\). A stability check of the equilibrium \(v_0\) shows the existence of a supercritical Hopf bifurcation for \(S_H = v_H(v_H - 0.5)(v_H - 1) + v_H - b\), where \(v_H = 0.5 - \sqrt{3 - 12\epsilon}/6\). Therefore, if \(b, \epsilon\) are fixed and the bifurcation parameter \(S\) is increased, the system undergoes a Hopf bifurcation at \(S_H\), resulting in a periodic orbit of the system encircling the formerly stable equilibrium. The periodic orbit is manifested in rhythmical spiking by the variable \(v\). For example, setting \(b = 0.15, \epsilon = 0.005\), there is a Hopf bifurcation point at \(S_H \approx 0.112331\ldots\). For \(S < S_H\), the system is quiescent; the equilibrium is stable. For \(S > S_H\), the system spikes at a rate of approximately 1 Hz.

Now consider FHN2 as a nonlinear filter by substituting for the constant \(S\) in (1) a signal \(S(t)\) from another system. Fig. 1 shows a plot of the variable \(v\) from (1) where \(S\) has been replaced by a signal from the Rössler system \([10]\)

\[
\begin{align*}
\dot{x} &= \tau(-y - z) \\
\dot{y} &= \tau(x + ay) \\
\dot{z} &= \tau(b + (x - c)z)
\end{align*}
\]

where the standard parameters are set to \(a = 0.36, b = 0.4, c = 4.5,\) and \(\tau = 0.5\) causes
the trajectory to run at half speed. The input signal, also plotted in Fig. 1, is \( S(t) = 0.09 + 0.013x(t) \), where \( x(t) \) is the \( x \) variable of (2). The bias of the signal is \( \langle S(t) \rangle = 0.093 \), and its root mean square amplitude is \( \sqrt{\langle (S(t) - \langle S(t) \rangle)^2 \rangle} = 0.035 \). Fig. 2 demonstrates the threshold-crossing detection capability of FHN2. When the peak height of \( S(t) \) is greater than \( \approx 0.15088 \), FHN2 fires a burst of spikes. Note that this threshold is significantly higher than the Hopf bifurcation value \( S_H \), which would be the threshold in the limit of an \( S(t) \) which oscillates infinitely slowly.

Although we expect the spike sequences to carry state information of the Rössler system, because of its threshold detection behavior we do not expect it to carry enough to reconstruct the entire attractor. On the other hand, the benefit of a TC filter is that noise can in some cases enhance the reconstruction quality, as measured by prediction error. As in studies of stochastic resonance, we will add white noise to the input signal of the filter (in this case, the FHN2 spike generator). The equation with noise term is

\[
\epsilon \dot{v} = -v(v - 0.5)(v - 1) - w + S(t) + \xi(t) \\
\dot{w} = v - w - b,
\]

where \( \epsilon = 0.005, b = 0.15 \), and \( \xi(t) \) is Gaussian white noise with zero mean and autocorrelation \( \langle \xi(t)\xi(s) \rangle = 2D\delta(t - s) \). For small values of the noise level \( D \) (including all those considered here), the variable \( v \) exhibits a clearly distinguishable spiking behavior, often in bursts of more than one spike, as in Fig. 1. For analysis purposes, we found it more convenient to collect series of interburst intervals, each defined to be the elapsed time between the final spike of one burst and the first spike of the next burst. After using (3) to make a series of 1024 interburst intervals, we used a standard nonlinear prediction algorithm to measure the level of determinism in the series. The fact that state information from a deterministic system is contained in a spike train, even when the spike train is chaotic, can be detected by measuring the nonlinear predictability of the interburst intervals. If it can be shown that the ISI series is predictable “beyond the power spectrum”, that is, if there is predictability beyond that which is guaranteed by linear autocorrelation, then there is
evidence of nonlinear dynamics in the series.

The prediction algorithm works as follows. Given an ISI vector \( V_0 = (t_{i_0}, \ldots, t_{i_0-m+1}) \), the 1\% of other reconstructed vectors \( V_k \) that are nearest to \( V_0 \) are collected, omitting vectors \( V_k \) close in time. The ISI for some number \( h \) of steps ahead are averaged for all \( k \) to make a prediction. That is, the average \( p_{i_0} = \langle t_{i_0+h} \rangle_k \) is used to approximate the future interval \( t_{i_0+h} \). The difference \( p - t_{i_0+h} \) is the \( h \)-step prediction error at step \( i_0 \). We could instead use the series mean \( m \) to predict at each step; this \( h \)-step prediction error is \( m - t_{i_0+h} \).

The ratio of the root mean square errors of the two possibilities (the nonlinear prediction algorithm and the constant prediction of the mean) gives the normalized prediction error

\[
\text{NPE} = \frac{\langle (p_{i_0} - t_{i_0+h})^2 \rangle^{1/2}}{\langle (m - t_{i_0+h})^2 \rangle^{1/2}}
\]

where the averages are taken over the entire series. The normalized prediction error is a measure of the (out-of-sample) predictability of the ISI series. A value of NPE less than 1 means that there is linear or nonlinear predictability in the series beyond the baseline prediction of the series mean.

The results of the predictability of the interburst interval series from (3) are shown in Fig. 3. For these parameter settings, unlike those for Fig. 1, no spikes occurs in the absence of noise. As the noise power \( D \) is increased from zero, spikes begin to occur for very small noise levels, although the interburst series show no predictability (NPE \( \approx 1 \)) until \( D \) is raised beyond \( 10^{-11} \). The prediction error then drops to a minimum and raises again when the noise becomes large enough to swamp the system. The clearly noticeable improvement in predictability due to extremely small noise input is essentially a stochastic resonance effect. These results show evidence of nonlinear determinism, since Gaussian-scaled surrogate series [12] created from all burst series considered in Fig. 3 have NPE \( \approx 1 \).

A slight alteration in the FitzHugh-Nagumo equations yields a nonlinear filter that acts as an integrate-and-fire processor. Define the system FHN3 by

\[
\begin{align*}
\dot{u} &= -au - cw + S(t) \\
\dot{v} &= -v(v - 0.5)(v - 1) + u - dw \\
\dot{w} &= v^2 - w - b.
\end{align*}
\]
This system is similar to FHN2 in that if $S(t)$ is set to be a constant parameter $S$, there is a Hopf bifurcation as $S$ is increased. Setting parameters $a = 0.1, b = 0.15, c = 0.5, d = 0.5, \epsilon = 0.005$, the bifurcation point is $S_H \approx -0.059$. Its success as an information processor is shown in Fig. 4. As with FHN2, we replace the parameter $S$ with an input signal $S(t)$ from the Rössler attractor. The signal is $S(t) = 0.0023x(t) - 0.04$, which corresponds to bias $\langle S(t) \rangle = -0.04$ and root mean square amplitude $\sqrt{\langle (S(t) - \langle S(t) \rangle)^2 \rangle} = 0.006$. Comparing with FHN2 in Fig. 1, we see a marked difference in the way FHN3 processes the input signal. Fig. 5(a) shows a three-dimensional plot of the vectors $(t_i, t_{i-1}, t_{i-2})$, where $t_i = T_i - T_{i-1}$ is the time interval between spikes of the $v$ variable of FHN3. Fig. 5(b) shows a similar reconstruction where the input signal $S(t)$ is the $x$-coordinate from the Lorenz equations

\[
\begin{align*}
\dot{x} &= \tau(\alpha(y-x)), \\
\dot{y} &= \tau(\rho x - y - xz), \\
\dot{z} &= \tau(-\beta z + xz),
\end{align*}
\]

where the parameters are set to the standard values $\alpha = 10, \rho = 28, \beta = 8/3$, and $\tau = 0.01$. Apparently, the interspike intervals recovered from (1) do an effective job of reconstructing the chaotic attractor which produced the input signal $S(t)$, for both the Rössler and Lorenz examples. Nonlinear prediction on a length 1024 series of spikes created as in Fig. 4 yields $NPE = 0.1$. This very low NPE supports the visual indication in Fig. 5(a) of a faithful reconstruction of the underlying Rössler attractor. This is similar to the mechanism that was studied in the generic integrate-and-fire model of [4], where firing times $T_i$ were generated recursively by

\[
\int_{T_i}^{T_{i+1}} S(t) dt = \Theta
\]

for a fixed threshold $\Theta$. Theoretical reconstruction results for spike trains generated by model (3) are discussed in [4].

Creating spikes using FHN2 or FHN3 means imposing a type of highly nonlinear filter on the attractor signal, a filter which edits out amplitude information (since the spike waveforms are essentially alike) and converts the information entirely to event timings. Our purpose is to gain insight into the data processing methods used in systems which communicate through spike timings, as is conjectured for certain neural systems [14]. We have shown by example that noise may be useful for this communication, in that it can amplify transmission.
of deterministic, nonlinear state information as measured by nonlinear prediction error. For the latter spike generation model (FHN3), we have the possibility of complete reconstruction of attractor states.

ACKNOWLEDGMENTS

The research of T.S. was supported in part by the National Science Foundation (Computational Mathematics and Physics programs).
REFERENCES

[1] N. Packard, J. Crutchfield, J. D. Farmer, and R. Shaw, Phys. Rev. Lett., 45, 712 (1980).
J. C. Roux, A. Rossi, S. Bachelart, C. Vidal, Phys. Lett. A77, 391 (1980). J. C. Roux,
H. Swinney, in: Nonlinear Phenomena in Chemical Dynamics, eds. C. Vidal, A. Pacault.
Springer-Verlag, Berlin (1981).

[2] F. Takens, Lecture Notes in Math. 898, Springer-Verlag (1981).

[3] T. Sauer, J. A. Yorke, M. Casdagli, J. Stat. Phys. 65, 579-616 (1991).

[4] T. Sauer, Phys. Rev. Lett. 72, 3911 (1994). T. Sauer, in Nonlinear Dynamics and
Time Series, eds. C. Cutler, D. Kaplan. Fields Institute Publications, Amer. Math. Soc.
(1996).

[5] R. Castro, T. Sauer, Phys. Rev. E 55 (1997).

[6] D. Racicot, A. Longtin, Physica D (in press).

[7] R. FitzHugh, In: Biological Engineering, Ed. by H.P. Schwann. McGraw-Hill, New York
(1962). J. Nagumo, S. Arimoto, S. Yoshizawa, Proc. IRE 50, 2061 (1962).

[8] K. Weisenfeld, F. Moss, Nature 373, 33 (1995). F. Moss, A. Bulsara, M.F. Schlesinger,
Eds. Proc. of NATO Advanced Research Workshop on Stochastic Resonance in Physics
and Biology, J. Stat. Phys. 70 (1993). A. Longtin, A. Bulsara, F. Moss, Phys. Rev. Lett.
67, 656 (1991).

[9] J.J. Collins, C.C. Chow, T.T. Imhoff, Phys. Rev. E 52, R3321 (1995).

[10] O. E. Rössler, Physics Letters 57A, 397 (1976).

[11] The method of R. Manella and V. Palleschi, Phys. Rev. A 40, 3381 (1989) was used for
integrating (3), with a step size of 0.001.

[12] J. Theiler, S. Eubank, A. Longtin, B. Galdrakian, J.D. Farmer, Physica D 58, 77 (1992).
[13] E. Lorenz, J. Atmos. Sci. 20 131 (1963).

[14] W. Bialek, F. Rieke, R. R. de Ruyter van Steveninck, D. Warland, Science 252, 1854 (1991). L. Abbott, Quart. Rev. Biophys. 27, 291 (1994). E. Vaadia, I. Haalman, M. Abeles, H. Bergman, Y. Prut, H. Slovin, A.M.H.J. Aertsen, Nature 373, 5151 (1995). Z. Mainen, T. J. Sejnowski, Science 268, 1503 (1995).
FIGURES

FIG. 1. The solid curve is the variable \( v \) of FHN2, the FitzHugh-Nagumo equation (1) with \( S \) replaced by \( S(t) = 0.09 + 0.013x(t) \), where \( x(t) \) is a solution of the Rössler system (2). The dashed curve is \( S(t) \). When a peak of \( S(t) \) is greater than \( \approx 0.15 \), a burst is triggered in FHN2.

FIG. 2. Peak heights of the signal \( S(t) \) from Fig. 1 graphed versus time. The height is plotted as an asterisk if it triggers a burst from FHN2; as an open circle if not. All of the asterisks lie above all of the open circles, signifying precise threshold detection by FHN2.

FIG. 3. Normalized prediction error of spike trains generated by (3), where \( S(t) \) is a signal formed using the Rössler \( x \)-variable of (2) with bias 0.075 and rms amplitude 0.020 (open circles) or 0.023 (asterisks). As the input noise power \( D \) increases, the NPE displays a minimum, corresponding to maximum information transfer. Each plotted point is an average over 5 noise realizations; standard error is less than 0.02 for each.

FIG. 4. The solid curve is the variable \( v \) of FHN3, equation (4), with \( S(t) = 0.0023x(t) - 0.04 \). The dashed curve is \( x(t) \), the \( x \)-variable of the Rössler attractor (2). A plot of 3-tuples of interspike intervals from this equation is shown in Fig. 5a.

FIG. 5. Interspike interval reconstructions of (a) the Rössler-FHN3 intervals from Fig. 4 (b) Lorenz-FHN3 intervals from (3) with \( S(t) = .0005x(t) - 0.04 \), where \( x(t) \) is the \( x \)-variable of the Lorenz system. In (b), fewer points are plotted, and they are connected with line segments.
