Well-Founded Iterations of Infinite Time Turing Machines

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Applications

Useful for ordinal analysis
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Iteration and hyper-iteration/feedback
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- Turing jump \(\mapsto\) hyperarithmetic sets
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- Turing jump $\leftrightarrow$ hyperarithmetic sets
- Inductive definitions $\leftrightarrow$ the $\mu$-calculus
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- Turing jump $\mapsto$ hyperarithmetic sets
- Inductive definitions $\mapsto$ the $\mu-$calculus
- ITTM $\mapsto$ ???
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First Definitions

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- the machine is in a dedicated state
- the head is on the $0^{th}$ cell
- the content of a cell is limsup of the previous contents (i.e. 0 if eventually 0, 1 if eventually 1, 1 if cofinally alternating)
Writable reals and ordinals

Definition

$R \subseteq \omega$ is writable

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**Proposition**

$R \subseteq \omega$ is writable iff $R \in L_\lambda$. 
Eventually writable reals and ordinals

Definition
$R \subseteq \omega$ is **eventually writable** if its characteristic function is on the output tape, never to change, of a computation.

An ordinal $\alpha$ is **eventually writable** if some real coding $\alpha$ (via some standard representation) is eventually writable.

$\zeta := \sup \{ \alpha \mid \alpha \text{ is eventually writable} \}$

Proposition
$R \subseteq \omega$ is eventually writable iff $R \in L_\zeta$. 
Accidentally writable reals and ordinals

Definition
\( R \subseteq \omega \) is \textbf{accidentally writable} if its characteristic function is on the output tape at any time during a computation.

An ordinal \( \alpha \) is \textbf{accidentally writable} if some real coding \( \alpha \) (via some standard representation) is accidentally writable.

\[ \Sigma := \sup \{ \alpha \mid \alpha \text{ is accidentally writable} \} \]

Proposition
\( R \subseteq \omega \) is \textbf{accidentally writable} iff \( R \in L\Sigma \).
Summary and conclusions

$\lambda$ is the supremum of the writables.
$\zeta$ is the supremum of the eventually writables.
$\Sigma$ is the supremum of the accidentally writables.
Clearly, $\lambda \leq \zeta \leq \Sigma$. 
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Clearly, \( \lambda \leq \zeta \leq \Sigma \).

**Theorem**

(Welch) \( \zeta \) is the least ordinal \( \alpha \) such that \( L_\alpha \) has a \( \Sigma_2 \)-elementary extension. (\( \zeta \) is the least \( \Sigma_2 \)-extendible ordinal.) The ordinal of that extension is \( \Sigma \). \( L_\lambda \) is the least \( \Sigma_1 \)-elementary substructure of \( L_\zeta \).
Time to iterate

Definition

\[ 0^\downarrow = \{ (e, x) | \phi_e(x) \downarrow \} \]
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$0^\nabla = \{(e, x) | \phi_e(x) \downarrow \}$

Proposition

*The definitions of $\lambda, \zeta, \text{ and } \Sigma$ relativize (to $\lambda^\nabla, \zeta^\nabla, \text{ and } \Sigma^\nabla$) to computations from $0^\nabla$. Furthermore, $\zeta^\nabla$ is the least $\Sigma_2$-extendible limit of $\Sigma_2$-extendibles, the ordinal of its $\Sigma_2$ extension is $\Sigma^\nabla$, and $\lambda^\nabla$ is the ordinal of its least $\Sigma_1$-elementary substructure.*
Time to iterate

ITTMs with arbitrary iteration:
A computation may ask a convergence question about another computation. This can be considered calling a sub-computation. That sub-computation might do the same. This can continue, generating a *tree of sub-computations*. Eventually, perhaps, a computation is run which calls no sub-computation. This either converges or diverges. That answer is returned to its calling computation, which then continues.
Good examples
Good examples
Bad example
One can naturally define the course of a computation if and only if the tree of sub-computations is well-founded. How is this to be dealt with?
When the main computation makes a sub-call, the call must be made with an ordinal. When a sub-call makes a sub-call itself, that must be done with a smaller ordinal. The definitions of $\lambda, \zeta, \text{and } \Sigma$ relativize (to $\lambda^{it\downarrow}, \zeta^{it\downarrow}, \text{and } \Sigma^{it\downarrow}$).
Definition

β is 0- (or 1-) extendible if its $\Sigma_2$-extendible.

β is $(\alpha+1)$-extendible if its a $\Sigma_2$-extendible limit of $\alpha$-extendibles.

β is $\kappa$-extendible if its a $\Sigma_2$-extendible limit of $\alpha$-extendibles for each $\alpha < \kappa$. 
Results

Definition

\( \beta \) is 0- (or 1-) extendible if its \( \Sigma_2 \)-extendible.

\( \beta \) is \((\alpha+1)\)-extendible if its a \( \Sigma_2 \)-extendible limit of \( \alpha \)-extendibles.

\( \beta \) is \( \kappa \)-extendible if its a \( \Sigma_2 \)-extendible limit of \( \alpha \)-extendibles for each \( \alpha < \kappa \).

Proposition

\( \zeta^{it\downarrow} \) is the least \( \kappa \) which is \( \kappa \)-extendible, \( \Sigma^{it\downarrow} \) is its \( \Sigma_2 \) extension, and \( \lambda^{it\downarrow} \) its least \( \Sigma_1 \) substructure.
Option II

Allow all possible sub-computation calls, even if the tree of sub-computations is ill-founded, and consider only those for which the tree of sub-computations just so happens to be well-founded.
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Allow all possible sub-computation calls, even if the tree of sub-computations is ill-founded, and consider only those for which the tree of sub-computations just so happens to be well-founded. So some legal computations have an undefined result. Still, among those with a defined result, some computations are halting, and some divergent computations have a stable output.
BIG FACT
If a real is eventually writable in this fashion, then it’s writable.
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Proof.
Given $e$, run the computation of $\phi_e$. Keep asking “if I continue running this computation until cell 0 changes, is that computation convergent or divergent?” Eventually you will get “divergent” as your answer. Then go on to cell 1, then cell 2, etc. After going through all the natural numbers, you know the real on your output tape is the eventually writable real you want. So halt.
**BIG FACT**
If a real is eventually writable in this fashion, then it’s writable.

**SECOND BIG FACT**
If a real is accidentally writable in this fashion, then it’s writable.
QUESTION
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ANSWER
You can’t run a universal machine.
As soon as a machine with code for a universal machine makes an ill-founded sub-computation call, it freezes.
Definition

$R$ is **freezingly writable** if $R$ appears anytime during such a computation, even if that computation later freezes.
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**Claim** In order to understand the writable reals in this context, one needs to understand the freezingly writable reals. One also needs to understand the tree of sub-computations for freezing computations.
**Notation** Let \( \Lambda \) be the supremum of the ordinals so writable (i.e. with well-founded oracle calls).
Prospects

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c) the total number of nodes is bounded beneath $\Lambda$. 
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b) every level has size less than $\Lambda$, but those sizes are cofinal in $\Lambda$, or

c) the total number of nodes is bounded beneath $\Lambda$.

**Proposition**

*Options a) and b) are incompatible: there cannot be one tree of sub-computations with more than $\Lambda$-much splitting beneath a node and another with the splittings beneath all the nodes cofinal in $\Lambda$.***
Option III

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computations converge?” The computation asked about has an
index $e$, parameter $x$, and free variable $n$. 
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to be continued ...
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