Discovering Relic Gravitational Waves in Cosmic Microwave Background Radiation

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(Dated: March 9, 2010)

Abstract

The authority of J. A. Wheeler in many areas of gravitational physics is immense, and there is a connection with the study of relic gravitational waves as well. I begin with a brief description of Wheeler’s influence on this study. One part of the paper is essentially a detailed justification of the very existence of relic gravitational waves, account of their properties related to the quantum-mechanical origin, derivation of the expected magnitude of their effects, and reasoning why they should be detectable in the relatively near future. This line of argument includes the comparison of relic gravitational waves with density perturbations of quantum-mechanical origin, and the severe criticism of methods and predictions of inflationary theory.

Another part of the paper is devoted to active searches for relic gravitational waves in cosmic microwave background radiation (CMB). Here, the emphasis is on the temperature-polarization $T E$ cross-correlation function of CMB. The expected numerical level of the correlation, its sign, statistics, and the most appropriate interval of angular scales are identified. Other correlation functions are also considered. The overall conclusion is such that the observational discovery of relic gravitational waves looks like the matter of a few coming years, rather than a few decades.

PACS numbers: 98.70.Vc, 98.80.Cq, 04.30.-w

* Based on the invited lecture at the 1-st J. A. Wheeler School on Astrophysical Relativity, June 2006, Italy.
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I. INTRODUCTION

It is my honor and pleasure to be a lecturer at the first course of J. A. Wheeler School on Astrophysical Relativity. Wheeler is one of the founding fathers of the field of gravitational physics, and many of us are strongly influenced by his work and his personality. In particular, we often find inspiration in the great textbook of him and his colleagues [1].

I was fortunate to speak with J. A. Wheeler right before the publication of my first papers on relic gravitational waves [2]. That conversation helped me to shape my views on the subject. Being a young researcher, I was somewhat nervous about evaluation of my work by the towering scientist, but to my relief, Wheeler quickly understood the work and agreed with it. In what follows, we will be discussing relic gravitational waves systematically and in details, but I would like to start from describing my initial doubts and worries, and how Wheeler helped me to see them in a different light.

I showed that the wave-equation for a gravitational wave \( h(\eta, x) = \mu(\eta)/a(\eta) \) \( e^{in \cdot x} \) in a homogeneous isotropic universe with the scale factor \( a(\eta) \) can be written as a Schrodinger-like equation

\[
\mu'' + \mu \left[ n^2 - \frac{a''}{a} \right] = 0. \tag{1}
\]

It follows from this equation that the fate of the wave with the wavenumber \( n \) depends on the comparative values of \( n^2 \) and the effective potential \( U(\eta) = \frac{a''}{a} \). If \( n^2 \) is much larger than \( |U(\eta)| \), the wave does not feel the potential and propagates with the adiabatically changing amplitude \( h \propto 1/a \). In the opposite limit, the interaction with the potential is strong and the wave changes dramatically. The amplitude of the initial wave gets amplified over and above the adiabatic law \( h \propto 1/a \), and at the same time a wave propagating in the opposite direction is being created. This process results effectively in the production of standing waves. I called this phenomenon the superadiabatic amplification. Over the years, other cosmological wave equations were also modeled on Eq. [1]. In terms of the variable \( h(\eta) \), where \( h(\eta) = \mu(\eta)/a(\eta) \), Eq. [1] has the form

\[
h'' + \frac{2a'}{a} h' + n^2 h = 0. \tag{2}
\]

While analyzing various scale factors \( a(\eta) \) and potentials \( U(\eta) \), I was imagining them as being ‘drawn by a hand’. My first example was the potential \( U(\eta) \) archtypical for quantum mechanics - a rectangular barrier. This means that \( U(\eta) = \text{const} \) in some interval of \( \eta \)-time between \( \eta_a \) and \( \eta_b \), while \( U(\eta) = 0 \) outside this interval. I have shown that the waves interacting with this barrier are necessarily amplified. However, the postulated \( U(\eta) \) caused some concerns.
It is easy to make $U(\eta) = 0$ before $\eta_a$ and after $\eta_b$. Indeed, if $a''/a = 0$, the scale factor $a(\eta)$ is either a constant, like in a flat Minkowski world, or is proportional to $\eta$, like in a radiation-dominated universe. It is not difficult to imagine that the Universe was radiation-dominated before and after some crucial interval of evolution. However, if it is assumed that $a''/a = \text{const}$ between $\eta_a$ and $\eta_b$, it is not easy to find a justification for this evolution, as the scale factor $a(\eta)$ should depend exponentially on $\eta$-time in this interval. In terms of $t$-time, which is related to $\eta$-time by $c \, dt = a \, d\eta$, the scale factor $a(t)$ should be proportional to $t$. The Einstein equations allow this law of expansion, but they demand that the ‘matter’ driving this evolution should have the effective equation of state $p = -(1/3)\epsilon$. Some other potentials $U(\eta)$ drawn by a hand do also require strange equations of state.

Having shown the inevitability of superadiabatic amplification, as soon as $a''/a \neq 0$, I was somewhat embarrassed by the fact that in some parts of my study I operated with scale factors drawn by a hand and driven by matter with unusual equations of state. Although equations of state with negative pressure had already been an element of cosmological research, notably in the work of A. D. Sakharov, I feared that Wheeler may dislike this idea and may say ‘forget it’. To my surprise, he accepted the approach and even suggested a wonderful name: the ‘engine-driven cosmology’. The implication was that although we may not know the nature of the ‘engine’ which drives a particular $a(\eta)$, this knowledge is not, for now, our high priority. Being inspired by Wheeler’s attitude, I hurried to include the notion of the engine-driven cosmology, with reference to Wheeler, in the very first paper on the subject.

It is interesting to note that E. Schrödinger felt uneasy about wave solutions in an expanding universe. (I became aware of his work much after the time of my first publications.) Schrödinger identified the crucial notion of the “mutual adulteration of positive and negative frequency terms in the course of time”. He was thinking about electromagnetic waves and he called the prospect of photon creation in an expanding universe an “alarming phenomenon”. From the position of our present knowledge, we can say that Schrödinger was right to be doubtful. Indeed, electromagnetic waves cannot be amplified and photons cannot be created in a nonstationary universe. Even though the wavelengths of electromagnetic and gravitational waves change in exactly the same manner, their interactions with external gravitational field are drastically different. The corresponding effective potential $U(\eta)$ in the Maxwell equations is strictly zero and the “alarming phenomenon” does not take place. All physical fields are tremendously ‘stretched’ by expansion, but only some of them are being amplified.

In his paper, Schrödinger was operating with a variant of scalar electrodynamics, wherein the
coupling of scalar fields to gravity is ambiguous and can be chosen in such a way that the wave equation becomes identical to Eq. (1), making the amplification of scalar waves possible (for more details, see introductory part to Ref. [5]). L. Parker [6] undertook a systematic study of the quantized version of test scalar fields in FLRW (Friedmann-Lemaître-Robertson-Walker) cosmologies. As for the gravitational waves, there is no ambiguity in their coupling to gravity since the coupling follows directly from the Einstein equations. As we see, the “alarming phenomenon” does indeed take place for gravitational waves [2]. (The authors of publications preceding Ref. [2] explicitly denied the possibility of graviton creation in FLRW universes. In the end, it was only Ya. B. Zeldovich who wrote to me: “Thank you for your goal in my net”.)

It is important to realize that the possibility of generation of relic gravitational waves relies only on the validity of general relativity and quantum mechanics. The governing principles are part of the well-understood and tested physics. The underlying equation (1) admits an analogy with the Schrödinger equation (outlined above), but it can also be viewed as an equation for a classical oscillator with variable frequency. The phenomenon of superadiabatic (parametric) amplification of the waves’ zero-point quantum oscillations is at the heart of the cosmological generating mechanism. In order to better appreciate this phenomenon, we shall briefly review the closely related laboratory-type problem of parametric amplification in a classical pendulum.

Let us consider an ideal pendulum hanging in a constant gravitational field characterized by the free-fall acceleration $g$ (see Fig. 1). The frequency of the oscillator is given by $\omega_0 = \sqrt{g/l}$, where $l$ is the length of the pendulum,

$$\ddot{x} + \frac{g}{l} \dot{x} = 0.$$ 

The amplitude of oscillations can be enhanced either by force acting directly on the pendulum’s mass or by an influence which changes a parameter of the oscillator – in this case, its frequency. The equation for small horizontal displacements $x(t)$ of the oscillator’s mass takes the form

$$\ddot{x} + \omega^2(t)x = 0. \tag{3}$$

The simplest parametric intervention makes the length $l$ time-dependent, as shown in Fig. 1b. Then, $\omega^2(t)$ in Eq. (3) is $\omega^2(t) = (g - \ddot{l})/l(t)$. Note that even if the gravitational acceleration $g$ remains constant, it still gets modified by the acceleration term $\ddot{l}$ arising due to the variation of length $l(t)$. So, the correct $\omega^2(t)$ differs from the naive expectation $\omega^2(t) = g/l(t)$. (For more details about parametrically excited oscillators, see [7].) In general, both $g(t)$ and $l(t)$ are functions
of time, in which case $\omega^2(t) = [g(t) - \ddot{l}]/l(t)$ and Eq. (3) reads

$$\ddot{x} + x \left[ \frac{g(t)}{l(t)} - \frac{\ddot{l}}{l} \right] = 0.$$  \hfill (4)

If the external influence on the oscillator is very slow, i. e. $\omega_0 \gg |\dot{\omega}(t)/\omega(t)|$, the ratio of the slowly changing energy $E(t)$ and frequency $\omega(t)$,

$$\frac{E}{\hbar \omega} = N,$$  \hfill (5)

will remain constant. This ratio is called an adiabatic invariant \[8\]. The quantity $N$ is also a ‘number of quanta’ in a classical oscillator (see paper by Ya. B. Zeldovich, signed by a pseudonym, on how quantum mechanics helps understand classical mechanics \[9\]).

On the other hand, if the oscillator was subject to some interval of appropriate parametric influence, the amplitude of oscillations and the number of quanta $N$ will significantly increase, as shown in Fig. 1b. To get a significant effect, the function $\omega(t)$ does not have to be periodic, but in its Fourier spectrum there should be enough power at frequencies around $\omega_0$. The final frequency does not need to differ from the initial $\omega_0$.

One can notice the striking analogy between Eq. (1) and Eq. (4). This analogy extends further if one goes over from the displacement $x(t)$ to the dimensionless angle variable $\phi(t)$ related to $x(t)$ by $\phi(t) = x(t)/l(t)$. (Compare with the cosmological relationship $h(\eta) = \mu(\eta)/a(\eta)$.) Then, Eq. (4) takes the form

$$\ddot{\phi} + 2\frac{\dot{l}}{l} \dot{\phi} + \frac{g(t)}{l(t)} \phi = 0,$$  \hfill (6)
which is strikingly similar to Eq. (2). In cosmological equations, the analog of the ratio $g(t)/l(t)$ is $n^2$.

There are two lessons to be learned from this discussion. First, the necessary condition for a significant amplification of the wave is the availability of a regime where the characteristic time $|a/a'|$ of variation of the external gravitational field (represented by the scale factor $a(\eta)$) becomes comparable and much shorter than the wave period $2\pi/n$,

$$n \ll \left| \frac{a'(\eta)}{a(\eta)} \right|. \quad (7)$$

Before and after this regime, the wave may be a high-frequency wave, that is, it may satisfy the condition $n \gg |a'(\eta)/a(\eta)|$. But if during some interval of evolution the opposite condition (7) is satisfied, and $a''/a \neq 0$, the wave will be superadiabatically amplified, regardless of whether the model universe is expanding or contracting [2].

Second, a classical pendulum should initially be in a state of oscillations – excited state – in order to have a chance to be amplified by an external ‘pump’ influence. Otherwise, if it is initially at rest, i.e. hanging stright down, the time-dependent change of $l$ or $g$ will not excite the oscillator, and the energy of oscillations will remain zero. However, in the quantum world, even if the oscillator is in the state of lowest energy (ground, or vacuum, state) it will inevitably possess the zero-point quantum oscillations. One can think of these zero-point quantum oscillations as those that are being amplified by the external pump (assuming, of course, that the oscillator is coupled to the pump). This is where the quantum mechanics enters the picture. The initial ground state of the parametrically excited oscillator evolves into a multi-quantum state, and the mean number of quanta grows, when condition (7) is satisfied [11]. There is no quantum state lower than the ground state, so the engine-driven cosmology will necessarily bring the properly coupled quantum oscillator into an excited state.

The concept of the engine-driven cosmology remains perfectly adequate for the present-day research. We do not know what governed the scale factor of the very early Universe. It could be a lucky version of the scalar field (inflation) or something much more sophisticated and dictated by the ‘theory of everything’. The lack of this knowledge is not important for the time being. By observing relic gravitational waves we may not be able to determine at once the nature of the cosmological ‘engine’, but we will be able to determine the behaviour of the early Universe’s scale factor. Therefore, we will gain unique information about the ‘birth’ of the Universe and its very early dynamical evolution.
II. COSMOLOGICAL OSCILLATORS

The physics of laboratory-type oscillators have direct relevance to cosmological oscillators. In cosmology, we normally consider a universe filled with some matter and slightly perturbed in all constituents. It is convenient to write the perturbed metric of a flat FLRW universe in the form

\[ ds^2 = -c^2 dt^2 + a^2(t) \delta_{ij} dx^i dx^j = a^2(\eta) \left[ -d\eta^2 + (\delta_{ij} + h_{ij}) dx^i dx^j \right]. \]  

(8)

The six functions \( h_{ij}(\eta, x) \) can be expanded over spatial Fourier harmonics \( e^{\pm in \cdot x} \), where \( n \) is a dimensionless time-independent wave-vector,

\[ h_{ij}(\eta, x) = C \frac{(2\pi)^{3/2}}{\sqrt{2n}} \sum_{s=1,2} \left[ p_{ij}(n) \hat{h}_n(\eta) e^{in \cdot x} \hat{c}_n + p_{ij}^*(n) \hat{h}_n^*(\eta) e^{-in \cdot x} \hat{c}_n^* \right]. \]  

(9)

This representation requires some explanations. The mode functions \( \hat{h}_n(\eta) \) obey differential equations that follow from the perturbed Einstein equations (as an example, look at Eq. (1)). The wavelength of the mode \( n \) is given by \( \lambda = 2\pi a/n \), where the wavenumber \( n \) is \( n = (\delta_{ij} n^i n^j)^{1/2} \). It is convenient to take today’s scale factor \( a(\eta_R) \) to be equal to the ‘size of the Universe’, that is, \( a(\eta_R) = 2l_H \), where \( l_H = c/H_0 \) is today’s Hubble radius and \( H_0 = H(\eta_R) \) is today’s Hubble parameter. Then, for a fixed moment of time, today, we can also write the laboratory-type expression for the same wavelength \( \lambda \): \( \lambda = 2\pi/k \), where \( k \) has the dimensionality of inverse length and is related to the dimensionless \( n \) by \( k = n/2l_H \). The wave whose length is equal to today’s ‘size of the Universe’ has \( n = 2\pi \). It is assumed that wavelengths can always be measured by unchangeable laboratory standards.

By systematically writing \( n \) and distinguishing it from \( k \) we essentially follow the original motivations of E. M. Lifshitz (see, for example, [10]). In contrast, some modern authors think that Lifshitz needs to be ‘simplified’ and ‘modernized’. They use only one letter \( k \) and write \( k \) everywhere where Lifshitz was writing \( n \). This created quite a mess in wavenumbers and wavelengths. Among them you will see physical, non-physical, coordinate, comoving, proper, etc.

Moving to complex Fourier coefficients \( \hat{c}_n, \hat{c}_n^* \) we note that they are some particular numbers, if the left-hand-side (l.h.s) of Eq.(9) is a deterministic, even if arbitrarily complicated, function. In the rigorous quantum-mechanical version of the theory, these coefficients will be promoted to the status of quantum-mechanical annihilation and creation operators \( \hat{c}_n, \hat{c}_n^\dagger \) acting on some quantum states. In the CMB applications, we will treat, for simplicity, \( \hat{c}_n, \hat{c}_n^* \) as random numbers taken from some probability distributions. The factor \( 1/\sqrt{2n} \) in Eq.(9) is a useful insertion inspired by quantum field theories. The normalization constant \( C \) will be discussed later.
The gravitational field polarization tensors \( \hat{p}_{ij}(n) \) deserve special attention. As we shall see below, the polarization properties of the CMB radiation – our final destination – are intimately connected with the structure of these metric polarization tensors. Polarization of CMB and polarization of metric perturbations is not simply a coincidence in the usage of the word polarization. The polarization tensors \( \hat{p}_{ij}(n) \) have different forms depending on whether the functions \( h_{ij}(\eta, x) \) represent gravitational waves, rotational perturbations, or density perturbations. Each class of these perturbations have two polarization states, so \( s = 1, 2 \) for each of them. In what follows we will be considering gravitational waves and density perturbations.

In the case of gravitational waves, two independent linear polarization states can be described by two real polarization tensors

\[
\begin{align*}
\hat{1}_{p_{ij}}(n) &= l_i m_j - m_i l_j, \\
\hat{2}_{p_{ij}}(n) &= l_i m_j + m_i l_j,
\end{align*}
\]

where spatial vectors \((l, m, n/n)\) are unit and mutually orthogonal vectors. The polarization tensors \(10\) satisfy the conditions

\[
\begin{align*}
\hat{p}_{ij} \delta_{ij} &= 0, \\
\hat{p}_{ij} n^i &= 0, \\
\hat{p}_{ij}^* \hat{p}_{ij} &= 2 \delta_{s's}.
\end{align*}
\]

Two circular polarization states are described by

\[
\begin{align*}
L_{p_{ij}} &= \frac{1}{\sqrt{2}} \left( \hat{1}_{p_{ij}} + i \hat{2}_{p_{ij}} \right), \\
R_{p_{ij}} &= \frac{1}{\sqrt{2}} \left( \hat{1}_{p_{ij}} - i \hat{2}_{p_{ij}} \right).
\end{align*}
\]

The left and right polarizations interchange under a coordinate reflection (altering the sign of \( l^i \) or \( m^i \)). In other words, gravitational waves can support a chirality, or ‘handedness’.

In the case of density perturbations, the polarization tensors are

\[
\begin{align*}
\hat{1}_{p_{ij}} &= \sqrt{\frac{2}{3}} \delta_{ij}, \\
\hat{2}_{p_{ij}} &= -\sqrt{\frac{3}{n^2}} n_i n_j + \frac{1}{\sqrt{3}} \delta_{ij}.
\end{align*}
\]

These polarization tensors satisfy the last of the conditions \(12\). The polarization tensors \(13\) remain unchanged under coordinate mirror reflections, so density perturbations cannot support handedness.

It is important to stress that from the position of general relativity, cosmological density perturbations represented by metric perturbations \( h_{ij} \) with polarization structure \(13\), can be viewed as ‘scalar’, or spin-0, gravitational waves. Indeed, although in the absense of matter, i.e. for \( T_{\mu\nu} = 0 \), the linearised Einstein equations admit spin-0 wave solutions, i.e. solutions with the structure \(13\), these solutions do not carry energy, do not affect the relative motion of test masses, and can be nullified by coordinate transformations. It is only spin-2 wave solutions, i.e. solutions with the
structure (10), that carry energy, affect the relative motion of test masses, and cannot be removed by coordinate transformations. These spin-2 solutions are called gravitational waves.

However, in cosmology, the spin-0 solutions survive and become non-trivial, as soon as metric perturbations with the structure (13) are supported by non-vanishing matter perturbations, that is, when $\delta T_{\mu}^{\nu} \neq 0$. These solutions, which unite gravitational field and matter perturbations, are called cosmological density perturbations. Therefore, cosmological density perturbations and cosmological gravitational waves, although separate from the point of view of algebraic classification of the tensor $h_{ij}$, are not entirely disconnected entities that should be treated by different theories. On the contrary, they should be viewed as parts of a common set of gravitational (metric) degrees of freedom. This can be regarded as a physical principle that will later guide our choice of initial conditions for density perturbations.

One more comment is in order. In our presentation, we will be consistently using the class of synchronous coordinate systems (8), that is, we assume that $h_{00} = 0$ and $h_{0i} = 0$. We are not losing anything in terms of physics but we gain significantly in terms of technical simplicity and universality of our approach to gravitational waves and density perturbations. In principle, one can work in arbitrary coordinates, assuming that all components of metric perturbations, including $h_{00}$ and $h_{0i}$, are non-zero. This will not bring you any real advantages, but will complicate calculations and can mislead you in issues of interpretation. Especially if you attempt to compare, say, gravitational waves described in synchronous coordinates with density perturbations described in the ‘Newtonian-gauge’ or other ‘gauge’ coordinates. Nevertheless, since various gauges and gauge-invariant formalisms are popular in contemporary literature, we will indicate, where necessary, how our formulas would be modified had we used arbitrary coordinates.

### III. QUANTIZATION OF GRAVITATIONAL WAVES

Cosmological gravitational waves exist in the absence of matter perturbations, i.e. for $\delta T_{\mu}^{\nu} = 0$. For each wavenumber $n$ and polarization state $s = 1, 2$ the mode functions $h_n(\eta) = \psi_n(\eta)$ satisfy the familiar equation (11). We assume that each of gravitational-wave oscillators was initially, at some $\eta = \eta_0$, in its ground state. Although certain oscillators could be somewhat excited without violating our perturbative assumptions, we do not see physical justification for such non-vacuum initial states.

There is no such thing as the ground (vacuum) state without explicitly indicating the Hamiltonian for which the state is the ground state. Shortly, we will explicitly write down the Hamiltonian
for gravitational waves (and, later, for density perturbations too). Specifying the Hamiltonian will also make more precise the concept of normalization of the initial mode functions to the zero-point quantum oscillations, or in other words the normalization to a ‘half of the quantum in each mode’.

It is important to remember, however, that, qualitatively, we already know the answer [2]. The energy of a gravitational wave with wavelength $\lambda_0$, contained in a volume $\left(\frac{\lambda_0}{2}\right)^3$, is equal to a half of the quantum, i.e. $N = 1/2$ in Eq.(5), if the amplitude of the wave is at the level

\[ h_i(n) \approx \sqrt{\frac{G\hbar}{c^3 a_0}} \approx \frac{l_{pl}}{\lambda_0}, \tag{14} \]

where $l_{pl} = \sqrt{G\hbar/c^3}$, $a_0 = a(\eta_0)$ and $\lambda_0 = 2\pi a_0/n$. Eq.(14) defines the initial vacuum spectrum of the gravitational wave (g.w.) amplitudes: $h_i(n) \propto n$.

Shifting the initial time $\eta_0$ up to the boundary between the adiabatic and superadiabatic regimes at $\eta = \eta_i$, we derive the estimate $h_i \sim l_{pl}/\lambda_i \sim l_{pl}H_i/c$. Then, we use the constancy of the metric amplitude $h$ throughout the long-wavelength (superadiabatic) regime $n \ll a'/a$, Eq.(7), that is, the regime in which the wave is ‘under the barrier $a'/a$’. The constancy of $h$ follows from the constancy of the dominant (first) term in the approximate long-wavelength solution [2] to the equations (1), (2):

\[ \frac{\mu}{a} = C_1 + C_2 \int \frac{dn}{a^2}. \tag{15} \]

This allows us to write $h_f \approx h_i$, where $h_f$ is the estimate of $h$ at the end of the superadiabatic regime. After having emerged from ‘under the barrier $a'/a$’ the wave will again behave adiabatically.

Using initial conditions (14) and evolving classical mode functions through all the barriers and intervals of adiabatic evolution, one can derive today’s metric amplitudes as a function of frequency, i.e. today’s amplitude spectrum. For a typical engine-driven expanding cosmology shown in Fig.2 and for its associated barrier $a'/a$ shown in Fig.3, one expects to arrive at today’s spectrum qualitatively shown in Fig.4 (for more details, see [12]).

The today’s amplitudes in different parts of the spectrum are mostly determined by the thickness of the barrier $a'/a$, that is, by the duration of time that a given mode $n$ spent under the barrier. All modes start with the initial $N_i = 1/2$, but if the mode $n$ enters the barrier (i.e. satisfies the condition $\lambda_i \approx c/H_i$) when the scale factor was $a_i(n)$, and leaves the barrier (i.e satisfies the condition $\lambda_f \approx c/H_f$) when the scale factor was $a_f(n)$, the final number of quanta in the mode will be $N_f(n) = (a_f/a_i)^2 = (\lambda_f/\lambda_i)^2 = (H_i/H_f)^2$. The very high-frequency modes with $n$ above the tip of the barrier, $n > n_1$, remain in the adiabatic regime throughout their evolution, and their energy is being renormalised to zero. Therefore, the resulting amplitude spectrum quickly drops to zero at the high-frequency end.
FIG. 2: A typical scale factor $a(\eta)$ as a function of time from the era of imposing the initial conditions, $i$-stage, and up to the present time.

FIG. 3: The barrier $a'/a$ built from the scale factor $a(\eta)$ of Fig.2 versus crucial wavenumbers $n$ defined by this barrier.

The power-law index of today’s spectrum $h(n)$ in an arbitrary narrow interval of frequencies, say, between $n_a$ and $n_b$, as outlined in Fig.4 is expressible in terms of the initial vacuum spectrum $h_i(n) \propto n$ and the power-law indeces $\beta$ of the scale factor $a(\eta) \propto |\eta|^{1+\beta}$

which approximates exact cosmological evolution in short intervals of time when the left and the right sides of the relevant portion of the barrier $a'/a$ were formed. Today’s spectral index of $h(n)$ in the discussed narrow interval of frequencies is given by

$$h(n) \propto n^{1+\beta_i} n^{-(1+\beta_f)} \propto n^{1+\beta_i-\beta_f}, \quad (16)$$

where $\beta_i$ and $\beta_f$ refer to the left and to the right slopes of the barrier, respectively.
FIG. 4: The characteristic gravitational-wave amplitude \( h(n) \) today as a function of frequency \( n \). The size of the amplitude is mostly determined by the thickness of the barrier \( a'/a \) in the place, where it is traversed by the wave with a given wavenumber \( n \). The barrier from Fig. 3 is shown at the top of the figure.

The vacuum spectrum processed only on the left slope of the barrier, i.e. considered at times before processing on the right slope of the barrier, is called the primordial spectrum \( h_p(n) \):

\[
h_p(n) \propto n^{2+\beta_i}.
\] (17)

For some historical and notational reasons, one and the same spectral index \( 2(2 + \beta_i) \) in the power spectrum \( h^2(n) \) of primordial metric perturbations is called \( n_t \) in the case of gravitational waves and \( n_s - 1 \) in the case of density perturbations (more details below, Eq.(65)). We will use the notation \( 2(2 + \beta_i) = n - 1 \).

If the cosmological evolution \( a(\eta) \) is such that the barrier \( a'/a \) is ‘one-sided’ and has only the right slope, the determination of the primordial spectrum requires additional considerations. The under-barrier amplification of the waves is still taking place \[2\], but the lack of the initial high-frequency regime makes the definition of the initial amplitudes somewhat ambiguous.

To summarise, for a given cosmological evolution \( a(\eta) \), one can qualitatively predict today’s
piecewise amplitudes and slopes of the spectrum $h(n)$ without doing any particularly detailed calculations. A barrier, whose shape is more complicated than the one shown in Fig.4 would have resulted in a more complicated shape of the generated spectrum. On the other hand, from the measured g.w. spectrum one can, in principle, reconstruct cosmological evolution $a(\eta)$ [13].

We will now turn to rigorous calculations based on quantum theory. The traditional, but not obligatory [14], approach to quantum theory begins with a classical Lagrangian. The Lagrangian for a gravitational-wave oscillator of frequency $n$, and for each of two polarizations $s = 1, 2$, has the form (15 and references there):

$$L_{gw} = \frac{1}{2n} \left( \frac{a}{a_0} \right)^2 \left[ (\bar{h}^s)'^2 - n^2 \bar{h}^2 \right],$$

where

$$\bar{h} = \left( \frac{\hbar}{32\pi^2} \right)^{1/2} \lambda_0 \frac{m}{l_P h}.$$  

This Lagrangian can be derived from the total Hilbert-Einstein quadratic action, where both gravity and matter parts of the action are taken into account. The classical equation of motion derivable from the Lagrangian (18) in terms of the variable $h$ is Eq. (2), and the equation of motion derivable in terms of the variable $\mu$, where $h = \mu/a$, is Eq. (1).

The canonical pair of position $q$ and momentum $p$ for the oscillator can be taken as

$$q = \bar{h}, \quad p = \frac{\partial L_{gw}}{\partial \bar{h}'} = \frac{1}{n} \left( \frac{a}{a_0} \right)^2 \bar{h}'.$$  

Then, the classical Hamiltonian $H_{gw} = pq' - L_{gw}$ reads

$$H_{gw} = \frac{n}{2} \left[ \left( \frac{a_0}{a} \right)^2 p^2 + \left( \frac{a}{a_0} \right)^2 q^2 \right].$$

We now promote $q$ and $p$ to the status of quantum-mechanical operators and denote them by bold-face letters. Since we are interested in the initial conditions imposed in the early high-frequency regime, i.e. when $n \gg a'/a$, we can write the following asymptotic expressions for the operators:

$$q = \sqrt{\frac{\hbar}{2a}} a_0 \left[ ce^{-in(\eta-\eta_0)} + c^\dagger e^{in(\eta-\eta_0)} \right],$$

$$p = i\sqrt{\frac{\hbar}{2a}} a_0 \left[ -ce^{-in(\eta-\eta_0)} + c^\dagger e^{in(\eta-\eta_0)} \right].$$
The commutation relationships for \( q, p \) operators, and for the annihilation and creation \( c, c^\dagger \) operators, are given by

\[
[q, p] = i\hbar, \quad [c, c^\dagger] = 1.
\]

The asymptotic expression for the Hamiltonian \( H_{gw} \) takes the form

\[
H_{gw} = \hbar n c^\dagger c.
\] (24)

Obviously, the quantum state \(|0\rangle\) satisfying the condition

\[
c|0\rangle = 0
\]

is the state of the lowest energy, i.e. the ground (vacuum) state of the Hamiltonian (24). At \( \eta = \eta_0 \) we get the relationships

\[
\langle 0|q^2|0\rangle = \langle 0|p^2|0\rangle = \frac{\hbar}{2}, \quad \Delta q \Delta p = \frac{\hbar}{2}.
\]

The root-mean-square value of \( q \) in the vacuum state \(|0\rangle\) is \( q_{rms} = \sqrt{\frac{\hbar}{2}} \). Combining this number with (19) we derive

\[
h_{rms} = \left( \langle 0|h^2|0\rangle \right)^{1/2} = \frac{\sqrt{2} (2\pi)^{3/2} l_{Pl}}{\lambda_0},
\] (25)

which agrees with the qualitative estimate (14). The adopted notations require \( C = \sqrt{16\pi l_{Pl}} \) in the general expression (9).

Now we will have to discuss exact Hamiltonians. The Hamiltonian built on the canonical pair \( \mu, \partial L_{gw}/\partial \mu' \) manifestly illustrates the underlying pair creation process for gravitational waves. Specifically, we introduce

\[
Q = \sqrt{\frac{\hbar}{8\pi n}} \frac{1}{\sqrt{n} l_{Pl}}, \quad P = \frac{\partial L_{gw}}{\partial Q'} = Q' - \frac{a'}{a} Q
\]

and write the classical Hamiltonian in the form

\[
H_{gw} = \frac{1}{2} \left[ P^2 + n^2 Q^2 + \frac{a'}{a} (PQ + QP) \right].
\]

The associated annihilation and creation operators are

\[
c = \sqrt{\frac{n}{2}} \left( Q + i\frac{P}{n} \right), \quad c^\dagger = \sqrt{\frac{n}{2}} \left( Q - i\frac{P}{n} \right).
\]

The full quantum Hamiltonian can be written as

\[
H(\eta) = nc^\dagger c + \sigma c^{\dagger 2} + \sigma^* c^2,
\] (26)
where coupling to the external field is given by the function \( \sigma(\eta) = (i/2)(a'/a) \).

The interaction with the external field can be neglected when \( n \gg a'/a \), and then the first term in (26) dominates and represents a free oscillator. The early-time asymptotic expressions for the Heisenberg operators,

\[
\begin{align*}
c(\eta) &= c e^{-in(\eta-\eta_0)}, \\
c^\dagger(\eta) &= c^\dagger e^{in(\eta-\eta_0)},
\end{align*}
\]

enter into formulas (22), (23).

One can note that the notion of the barrier \( a'/a \) is convenient when one thinks of the problem in terms of the interaction Hamiltonian (26) and the first-order Heisenberg equations of motion, whereas the barrier \( a''/a \) is more convenient when one thinks of the problem in terms of the second-order Schrödinger-like equation (1).

The emerging correlation between the traveling modes \( n \) and \( -n \), which leads to the production of standing waves referred to in the Introduction, is described by the 2-mode Hamiltonian (see [11] and references there):

\[
H(\eta) = nc_n^\dagger c_n + nc_{-n}^\dagger c_{-n} + 2\sigma c_n^\dagger c_{-n} + 2\sigma^* c_n c_{-n}.
\]  

(27)

This Hamiltonian can be viewed as the sum of two Hamiltonians (26). The Hamiltonian (26) is called a 1-mode Hamiltonian.

The defined quantum-mechanical operators and Hamiltonians, plus the assumption about a particular initial state of the field, fully determine dynamical evolution of the field, its statistical properties, correlation functions, and eventually the observational predictions for later times.

IV. SQUEEZING AND POWER SPECTRUM

The quantum-mechanical Schrödinger evolution transforms the initial vacuum state \( |0_n\rangle|0_{-n}\rangle \) into a 2-mode squeezed vacuum state, which is equivalent to a pair of 1-mode squeezed vacuum states (see [11] and references there). In the Heisenberg picture, the initial state of the field does not evolve, but the operators do, and their evolution is ultimately described by the mode functions \( h_n(\eta) \).

It is the variance of the oscillator’s phase that is being strongly diminished (squeezed), whereas the variance of the amplitude is being strongly increased. The Gaussian nature of the initial vacuum state is maintained in the course of the Schrödinger evolution, but the variances of phase and amplitude in the resulting squeezed vacuum quantum state are dramatically different. The
parameter of squeezing and the mean number of quanta are growing all the way up in the amplifying regime, and stop growing only at its end \[11\]. (The multi-quantum nature of the developing squeezed vacuum quantum state is behind the continuing debate over the ‘quantum-to-classical’ transition, ‘decoherence’, etc.) The phenomenon of squeezing allows us to treat the resulting quantum states as a stochastic collection of standing waves. The squeezing and the associated picture of standing waves is very important observationally \[11\]. It leads to oscillatory features in the metric power spectrum and, as a consequence, to oscillatory features in the angular power spectrum of CMB temperature and polarization anisotropies. We will discuss these oscillations later.

Having accepted the initial vacuum state \( |0\rangle \) of the gravitational (metric) field \( \Psi \), we can calculate the variance of the field:

\[
\langle 0| h_{ij}(\eta, x) h^{ij}(\eta, x) |0\rangle = \frac{C^2}{2\pi^2} \int_0^{\infty} n^2 \sum_{s=1,2} | \hat{s}_h n(\eta) |^2 \frac{dn}{n}.
\]  

(28)

The quantity

\[ h^2(n, \eta) = \frac{C^2}{2\pi^2} n^2 \sum_{s=1,2} | \hat{s}_h n(\eta) |^2 \]

(29)

gives the mean-square value of the gravitational field perturbations in a logarithmic interval of \( n \) and is called the metric power spectrum. The spectrum of the root-mean-square (rms) amplitude \( h(n, \eta) \) is determined by the square root of Eq. (29).

Having evolved the classical mode functions \( \hat{s}_h n(\eta) \) up to some arbitrary instant of time \( \eta \) one can find the spectrum \( h(n, \eta) \) at that instant of time. As mentioned before, the spectrum calculated at times when the waves of today’s interest were in their long-wavelength regime (that is, longer than the Hubble radius at that times) is called the primordial spectrum. The today’s spectrum, i.e. spectrum calculated at \( \eta = \eta_R \), is normally expressed in terms of frequency \( \nu \) measured in Hz, \( \nu = n H_0/4\pi = n\nu_H/4\pi \). The spectral rms-amplitude is denoted by \( h_{rms}(\nu) \), or simply \( h(\nu) \). The mean-square value of the gravitational-wave field in some interval of frequencies between \( \nu_1 \) and \( \nu_2 \) is given by the integral:

\[ \langle h^2 \rangle = \int_{\nu_1}^{\nu_2} h^2(\nu) \frac{d\nu}{\nu} . \]

(30)

The spectral function \( h^2(\nu) \) depends on frequency \( \nu \), but is dimensionless. The dimensionality of \( Hz^{-1} \) is carried by the function \( h^2(\nu)/\nu \).
For gravitational waves which are comfortably shorter than the Hubble radius, one can also calculate the spectral gravitational-wave energy density $\rho_{gw}(\nu)c^2$ and the total g.w. energy density in some interval of frequencies. These quantities are expressible in terms of $h^2(\nu)$:

$$\rho_{gw}(\nu) = \frac{\pi}{8G} h^2(\nu)\nu^2, \quad \rho_{gw}(\nu_1, \nu_2) = \int_{\nu_1}^{\nu_2} \rho_{gw}(\nu) \frac{d\nu}{\nu}. \quad (31)$$

For the purpose of comparing a g.w. background with other sorts of matter, it is convenient to introduce the cosmological $\Omega_{gw}$-parameter and its spectral value $\Omega_{gw}(\nu)$. As with all other sorts of radiation, this parameter is defined by

$$\Omega_{gw}(\nu) = \frac{\rho_{gw}(\nu)}{\rho_{crit}}, \quad (32)$$

where $\rho_{crit} = 3H_0^2/8\pi G$. Using Eq.(31) one can also write

$$\Omega_{gw}(\nu) = \frac{\pi^2}{3} h^2(\nu) \left( \frac{\nu}{\nu_H} \right)^2. \quad (33)$$

The total $\Omega_{gw}$ between frequencies $\nu_1, \nu_2$ is given by

$$\Omega_{gw}(\nu_1, \nu_2) = \int_{\nu_1}^{\nu_2} \Omega_{gw}(\nu) \frac{d\nu}{\nu}.$$ 

One should be weary of the confusing definition

$$\Omega_{gw}(\nu) = \frac{1}{\rho_{crit}} \frac{d\rho_{gw}(\nu)}{d\ln \nu}$$

often floating in the literature. As it stands, this relationship is incorrect. It can be made consistent with the correct definition (32) only if one assumes that what is being differentiated in this formula is not the spectral density $\rho_{gw}(\nu)$, but a logarithmic integral of this quantity in the limits between some fixed $\nu_1$ and a running $\nu$.

To make more precise the expected qualitative graph for $h_{rms}(n)$ in Fig.4 as well as previous theoretical graphs in Fig.4 of Ref.[16], we need to use some available observational data. We make the fundamental assumption that the observed CMB anisotropies are indeed caused by cosmological perturbations of quantum-mechanical origin. If so, the contribution of relic gravitational waves to the large-scale anisotropies should be of the same order of magnitude as the contribution of density perturbations (we will show this in more detail below). This allows us to determine the position and the slope of the function $h_{rms}(\nu)$ at frequencies near the Hubble frequency $\nu_H \approx 2 \times 10^{-18} Hz$ and then extrapolate the spectrum to higher frequencies.

We choose $h_{rms}(\nu_H)$ and primordial spectral index $n$ in such a way that the lower-order CMB multipoles produced by relic gravitational waves are at the level of the actually observed values.
Then, today’s spectra for $h_{\text{rms}}(\nu)$ with spectral indeces $n = 1$ ($\beta_i = -2$) and $n = 1.2$ ($\beta_i = -1.9$) are shown in Fig.5 (for more details, see [17] and [12]). At frequencies around $\nu_H$ we have $h_{\text{rms}}(\nu_H) \approx 10^{-5}$, so that the present-day mean number of quanta $N_f$ exceeds $10^{100}$.

When extrapolating the spectrum to higher frequencies we assume that the entire left slope of the barrier in Fig.3 was formed by a single power-law scale factor with one and the same $\beta_i = \beta$:

$$a(\eta) = l_0|\eta|^{1+\beta}.$$ (34)

This seems to be a reasonable assumption given the relative featurelessness of the initial stage – the overall energy density was 10 orders of magnitude lower than the Planckian density and was barely changing. Specifically, at the $i$-stage, we consider two examples: $\beta = -2$ and $\beta = -1.9$. The power-law indeces $\beta_f$ at the matter-dominated and radiation-dominated stages are well known: $\beta_f = 1$ and $\beta_f = 0$, respectively. The effective pressure $p$ and the energy density $\epsilon$ of matter driving the general power-law evolution (34) with a given constant $\beta$ are related by the effective equation of state $p = [(1 - \beta)/3(1 + \beta)]\epsilon$.

As for the high-frequency part of the spectrum, it was calculated under the assumption that the $z$-stage of evolution shown in Fig.2 was governed by matter with a stiff equation of state $p = \epsilon$ ($\beta_f = -(1/2)$) advocated long ago by Zeldovich. The issue of the back reaction of the created gravitons on the “pump” field $a(\eta)$ becomes important for this sort of values of $\beta_f$ [2, 18]. Of course, the existence of such an interval in the past evolution of the very early Universe cannot be guaranteed. In any case, the waves with frequencies above $10^{10}$ Hz have never been in superadiabatic regime, they remain in the vacuum state. The renormalization (subtraction of a “half of the quantum” from each mode) cuts off the spectrum at these high frequencies. At lower frequencies, the renormalization has practically no effect on the spectrum.

The spectra of $\Omega_{gw}(\nu)$, shown in Fig.6, are derived from $h_{\text{rms}}(\nu)$ according to Eq.(33). As was already mentioned, the substantial rise of the spectrum at very high frequencies cannot be guaranteed, and the shown graphs in this area of frequencies should be regarded as the upper allowed limits (especially for the model with the primordial spectral index $n = 1.2$).

There are two comments to be made about Fig.5 and Fig.6. First, the phenomenon of squeezing and standing waves production is reflected in the oscillations of the metric power spectrum and $\Omega$-spectrum as functions of frequency $\nu$. The first few cycles of these oscillations are shown in the graph of $h_{\text{rms}}(\nu)$. In the CMB sections of the paper, we will show how these oscillations that existed in the recombination era translate into peaks and dips of the angular power spectra for CMB temperature and polarization anisotropies observed today.
FIG. 5: The present-day spectrum for $h_{\text{rms}}(\nu)$. The solid line corresponds to the primordial spectral index $\beta = -1.9$, i.e. $n = 1.2$, while the dashed line is for $\beta = -2$, i.e. $n = 1$.

Second, since the primordial spectrum of quantum-mechanically generated perturbations has the form (see Eq. (17)):

$$h^2(n) \propto n^{2(\beta+2)},$$

(35)

theoretical considerations suggest that the preferred range of values for the primordial spectral index $n$ is $n > 1$, i.e. preferred spectra are ‘blue’. Indeed, for the opposite range $n < 1$, the primordial spectra would be ‘red’, and the integral would be power-law divergent at the lower limit. To avoid the infra-red divergency one would need to make extra assumptions about bending the spectrum down in the region of very small wavenumbers $n$. In its turn, a primordial spectral index $n > 1$ requires that the power-law index $\beta$ of the initial scale factor $a(\eta)$ should be $\beta > -2$. This is because $\beta$ and $n$ are related by $n = 2\beta + 5$. The value $n = 1$ ($\beta = -2$) describes the spectrum known as a flat, or Harrison-Zeldovich-Peebles, or scale-invariant spectrum. It was originally proposed in the context of density perturbations. The spectral index $n = 1$ marks the
FIG. 6: The present-day spectrum for $\Omega_{gw}(\nu)$. The solid line corresponds to the primordial spectral index $\beta = -1.9$, i.e. $n = 1.2$, while the dashed line is for $\beta = -2$, i.e. $n = 1$.

beginning of the trouble – the metric variance is logarithmically divergent at the lower limit.

V. DENSITY PERTURBATIONS

To discover relic gravitational waves in CMB anisotropies we have to distinguish their effects from other possible sources of CMB anisotropies, and first of all from density perturbations. In comparison with gravitational waves, the possibility of quantum-mechanical generation of density perturbations requires an extra hypothesis, namely, the appropriate parametric coupling of the matter field to external gravity. But we can assume that this hypothesis was satisfied by, for example, a lucky version of the scalar field. Can the amplitudes of quantum-mechanically generated density perturbations be many orders of magnitude larger than the amplitudes of relic gravitational waves? This would make the search for relic gravitational waves practically hopeless. We will show that this can never happen. Density perturbations can compete with relic gravitational waves
in producing the large-scale CMB anisotropies, but can never be overwhelming.

We are mostly interested in the gravitational field (metric) sector of density perturbations. Indeed, it is perturbations in the gravitational field that survived numerous transformations of the matter content of the Universe. Primordial matter (for example, scalar field) has decayed, together with its own perturbations, long ago. But it is primordial metric perturbations that have been inherited by density perturbations at radiation-dominated and matter-dominated stages.

As a driving ‘engine’ for the very early Universe, it is common to consider the so-called minimally-coupled scalar field \( \varphi(\eta, x) \), with the energy-momentum tensor

\[
T_{\mu\nu} = \varphi,_{\mu} \varphi,_{\nu} - g_{\mu\nu} \left[ \frac{1}{2} g^{\alpha\beta} \varphi,_{\alpha} \varphi,_{\beta} + V(\varphi) \right].
\]

The polarization structure (13) of density perturbations allows us to write the \( n \)-mode metric perturbation:

\[
h_{ij} = h(\eta)Q\delta_{ij} + h_l(\eta)n^{-2}Q_{,ij},
\]

where \( Q = e^{\pm in \cdot x} \). Two polarization amplitudes \( h(\eta) \) and \( h_l(\eta) \) are accompanied by the third unknown function - the amplitude \( \varphi_1(\eta) \) of the scalar field perturbation:

\[
\varphi(\eta, x) = \varphi_0(\eta) + \varphi_1(\eta)Q.
\]

The perturbed Einstein equations allow us to find all three unknown functions. The important fact is that, for any \( V(\varphi) \), there exists only one second-order differential equation, of the same structure as Eq.(1), that needs to be solved [19]:

\[
\mu'' + \mu \left[ n^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}} \right] = 0.
\]

This equation coincides with the gravitational wave equation (1) if one makes there the replacement

\[
a(\eta) \to a(\eta)\sqrt{\gamma(\eta)}.
\]

We will call function \( \mu \) satisfying Eq.(38) \( \mu_S \), and that satisfying Eq.(1) \( \mu_T \).

The function \( \gamma(\eta) \) is defined by

\[
\gamma(\eta) = 1 + \left( \frac{a}{a'} \right)' = -\frac{c H'}{a H^2}.
\]

For power-law scale factors (34), this function reduces to a set of constants,

\[
\gamma = \frac{2 + \beta}{1 + \beta}.
\]
The constant $\gamma$ degenerates to zero in the limit of the expansion law with $\beta = -2$, that is, in the limit of the gravitational pump field (an interval of deSitter evolution) which is responsible for the generation of cosmological perturbations with flat primordial spectrum $n = 1$.

In terms of $t$-time the function $\gamma$ is

$$\gamma(t) = -\frac{\dot{H}}{H^2}.$$  

It is also related to the cosmological deceleration parameter $q(t) = -\ddot{a}/\dot{a}^2$: $\gamma(t) = 1 + q(t)$. The unperturbed Einstein equations allow us to write

$$\kappa (\varphi_0')^2 = 2 \left( \frac{a'}{a} \right)^2 \gamma(\eta),$$  

or, equivalently,

$$\frac{\dot{\varphi}_0}{H} = \sqrt{\frac{2}{\kappa}} \sqrt{\gamma(t)},$$  

where $\kappa = 8\pi G/c^4$. The function $\gamma(t)$ is sometimes denoted in the literature by $\epsilon(t)$.

As soon at the appropriate solution to Eq.(38) is found, all unknown functions are easily calculable:

$$h(\eta) = \frac{1}{c} H(\eta) \left[ \int_{\eta_0}^{\eta} \mu \sqrt{\gamma} d\eta + C_i \right],$$  

$$h'_1(\eta) = \frac{a}{a'} \left[ h'' - \frac{H''}{H} h' + n^2 h \right],$$  

$$\varphi_1(\eta) = \frac{\sqrt{\gamma}}{\sqrt{2\kappa}} \left[ \frac{\mu}{a \sqrt{\gamma}} - h \right].$$  

The arbitrary constant $C_i$ in Eq.(42) reflects the remaining coordinate freedom within the class of synchronous coordinate systems (8). Indeed, a small coordinate transformation (45)

$$\overline{\eta} = \eta - \frac{C}{2a} Q$$

generates a ‘gauge transformation’

$$\overline{h}(\eta) = h(\eta) + C \frac{a'}{a^2} = h(\eta) + C \frac{H(\eta)}{c}.$$  

The term with constant $C$ is automatically present in the properly written general solution (42) for $h(\eta)$. 

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By combining the transformation (46) with its time-derivative, one can build a quantity which does not contain the constant $C$ and therefore is a ‘gauge-invariant’ quantity with respect to transformations preserving synchronous coordinates. Specifically, one can check that

$$h - h' \frac{H}{H'} = \bar{h} - \bar{h}' \frac{\bar{H}}{\bar{H'}}.$$  

Denoting this quantity by $\zeta$, and taking into account Eq.(42), we write

$$\zeta \equiv h - h' \frac{H}{H'} = \frac{\mu S}{a\sqrt{\gamma}}.$$  

(47)

In terms of the variable $\zeta(\eta)$ the fundamental equation (38) takes the form

$$\zeta'' + 2\frac{(a\sqrt{\gamma})'}{a\sqrt{\gamma}} \zeta' + n^2 \zeta = 0.$$  

(48)

Not surprisingly, this equation coincides with the gravitational-wave equation (2), if one replaces the gravitational-wave function $h$ with $\zeta$, and $a$ with $a\sqrt{\gamma}$. For density perturbations, the metric variable $\zeta$ is the physically relevant quantity that plays the same role as the metric variable $h$ for gravitational-wave perturbations.

The fact that we are working with scalar gravitational waves supported by scalar field fluctuations, instead of normal tensor gravitational waves, has boiled down to the necessity of a single modification: the substitution (39) in the gravitational-wave equations.

It is appropriate to say a few words about cosmological gauge transformations in general. Their origin is related to the notion of Lie transport performed on a manifold covered by some coordinates $x^\alpha$. Lie transport is being carried out along a given vector field $\xi^\mu(x^\alpha)$. This is a quite formal construction which respects only the transformation properties of fields defined on the manifold. These fields are not required to satisfy any physical equations. An infinitesimal Lie transport changes the field by an amount equal to its Lie derivative. This change can be viewed as a rule by which new values of the field are assigned to the same point $x^\alpha$. It is only with some reservation that we borrow the name gauge transformation from physical field theories and apply it to this, always valid, mathematical procedure.

In gravitational applications, the vector field $\xi^\mu(x^\alpha)$ is usually associated with an infinitesimal coordinate transformation $\overline{x}^\mu = x^\mu - \xi^\mu(x^\alpha)$. Then, for example, an infinitesimal increment of the metric tensor $g_{\mu\nu}(x^\alpha)$ is given by the Lie derivative of $g_{\mu\nu}$. It can be written as

$$\delta g_{\mu\nu}(x^\alpha) = \xi_{\mu;\nu} + \xi_{\nu;\mu}.$$

A particular case of these transformations is represented by our Eqs.(45, 46).
Since we operate with only four components of an arbitrary vector $\xi^\mu$ and, potentially, with many fields transforming with the same $\xi^\mu$, one can build some combinations of these transformed fields, in which the functions $\xi^\mu$ cancel out. Therefore, these combinations become gauge-invariant quantities. An example is given by our Eq.(47).

Certainly, it is an exaggeration to claim that “only gauge-invariant quantities have any inherent physical meaning”. It is as if you were told that where specifically you are going in your car has no physical meaning, because the components of velocity are coordinate-dependent, and it is only the readings of your speedometer that have an inherent physical meaning, because they are coordinate-independent. In any case, there exists an infinite number of gauge-invariant quantities. For example, a product of a gauge-invariant quantity with any ‘background’ function produces a new gauge-invariant quantity. The fact that a quantity is gauge-invariant does not answer the question of its physical interpretation. Nevertheless, it is useful to know gauge-invariant quantities.

Had we started from arbitrary coordinates, the perturbed metric (8) would have contained two more unknown functions, namely $A(\eta)$ and $B(\eta)$. The perturbed metric components $h$ and $A$ would transform as follows (any possible $x^i$-transformations do not participate in these relationships):

$$h = h - 2 \frac{a'}{a} T, \quad A = A - T' - \frac{a'}{a} T.$$  \hspace{1cm} (50)

The $\eta$-transformation generalizing our Eq.(45) would read

$$\eta = \eta + T(\eta)Q,$$  \hspace{1cm} (49)

where $T(\eta)$ is an arbitrary function of $\eta$. The perturbed metric components $h$ and $A$ would transform as follows (any possible $x^i$-transformations do not participate in these relationships):

$$\bar{h} = h - 2 \frac{a'}{a} T, \quad \bar{A} = A - T' - \frac{a'}{a} T.$$  \hspace{1cm} (50)

The functions $T$ and $T'$ cancel out in the gauge-invariant metric combination

$$\zeta_g = h - \frac{H}{H'} h' - \frac{2A}{\gamma}.$$  \hspace{1cm} (51)

Obviously, $\zeta_g$ reduces to $\zeta$ for transformations preserving synchronous conditions, that is, when $A = \bar{A} = 0$. The previously introduced quantity $\zeta_{BST}$, where $BST$ stands for Bardeen, Steinhardt, Turner, can also be reduced, after some work, to our $\zeta$, up to a coefficient $-(1/2)$.

Since we assume that in addition to the tensor metric field $g_{\mu\nu}$, a scalar field

$$\varphi(\eta, \mathbf{x}) = \varphi(\eta) + \delta\varphi(\eta)Q$$
is also defined on the manifold, we can write down its gauge transformation under the action of Eq. (49):

$$\delta \varphi = \delta \varphi + \varphi'_0 T.$$ (51)

The arbitrary function $T(\eta)$ cancels out in the gauge-invariant combination

$$V(\eta) = \delta \varphi + \frac{1}{2a} \varphi'_0 h.$$ (52)

Although the combination (52) is often quoted in the literature, this object is something like a ‘half of a horse plus half of a cow’. It combines physically separate quantities – one from metric another from matter – and its physical interpretation is obscure. However, using the unperturbed Einstein equation (40) and solution (44) for $\delta \varphi = \varphi_1$ one can show that $V$ reduces to the gauge-invariant metric perturbation $\zeta$ times a background-dependent factor:

$$V = \frac{1}{\sqrt{2\kappa}} \sqrt{\gamma} \zeta.$$ 

The quantity $\sqrt{2\kappa} a V$ is called Chibisov, Lukash, Mukhanov, Sasaki (23–25) variable $u_{CLMS}$:

$$u_{CLMS} = a \sqrt{\gamma} \zeta.$$ (53)

In our approach, this variable is simply the function $\mu_S$ satisfying Eq. (38), modeled on Eq. (1). The factor $\sqrt{\gamma}$ in Eq. (53) will be a matter of great attention when we come to the discussion of quantization procedures.

Returning to the function $\zeta(\eta)$, one can notice that this metric amplitude is practically constant in the regime when $n^2$ is much smaller than the effective potential $(a \sqrt{\gamma})''/a \sqrt{\gamma}$. This behaviour is similar to the constancy of the gravitational wave amplitude $h = \mu_T/a$ throughout the long-wavelength regime. Indeed, in full analogy with the long-wavelength solution (15), the general solution to Eq. (38) in this regime reads

$$\frac{\mu_S}{a \sqrt{\gamma}} = C_1 + C_2 \int \frac{d\eta}{(a \sqrt{\gamma})^2}.$$ 

For usually considered expanding cosmologies, the term with constant $C_2$ is decreasing, and therefore the dominant solution is $h \approx C_1$ for gravitational waves and $\zeta \approx C_1$ for density perturbations. The constancy of $\zeta$ allows one to easily estimate the value of metric perturbations at much later times, at the radiation-dominated and matter-dominated stages, as soon as one knows the initial value of $\zeta$.

The constancy of $\zeta$ (when one can neglect the term with $C_2$) is sometimes called a conservation law. This association is incorrect. Genuine conservation laws reflect symmetries of the system, and
conserved quantities are constants independently of the initial conditions. For example, the energy of a free oscillator is a constant independently of initial positions and velocities. In our problem, the genuine conservation law for $\zeta$ would look like an empty statement $0 = 0$.

VI. QUANTIZATION OF DENSITY PERTURBATIONS

A quantum system is defined by its Hamiltonian. Whether or not the Hamiltonian follows from some assumed classical Lagrangian should not be a question of major concern [14]. From this point of view, the quantization of density perturbations, similarly to the quantization of gravitational waves (26), is defined by the Hamiltonian [19]

$$H(\eta) = nc^c + \sigma c^{*^2} + \sigma^* c^2.$$  \quad (54)

The pair creation of scalar gravitational waves $\zeta$ by the external pump field is regulated by the coupling function $\sigma(\eta) = (i/2)(a\sqrt{\gamma})'/a\sqrt{\gamma}$. However, a more traditional approach begins with a Lagrangian, and here too, the Hamiltonian (54) can be derived from quadratic perturbation terms in the total Hilbert-Einstein action, where both gravity and matter (scalar field) are taken into account.

The Lagrangian for an $n$-mode of density perturbations, after some transformations of the total Hilbert-Einstein quadratic action, can be written in the form [15]

$$L_{dp} = \frac{1}{2n} \left( \frac{a\sqrt{\gamma}}{a_0\sqrt{\gamma_0}} \right)^2 \left[ (\bar{\zeta}')^2 - n^2 \bar{\zeta}^2 \right],$$ \quad (55)

where

$$\bar{\zeta} = \left( \frac{\hbar}{32\pi^3} \right)^{1/2} \frac{\lambda_0}{l_{Pl}} \zeta,$$ \quad (56)

and $a_0$, $\gamma_0$ are values of the functions $a(\eta)$, $\gamma(\eta)$ at $\eta = \eta_0$ where the initial conditions are being set. The Euler-Lagrange equations derivable from this Lagrangian in terms of $\zeta$ are Eq.(48), and in terms of $\mu_S$ – Eq.(38).

Obviously, the Lagrangian (55) for $\zeta$ coincides with the gravitational wave Lagrangian (18) for $h$ after the replacement of the factor $a/a_0$ with the factor $a\sqrt{\gamma}/a_0\sqrt{\gamma_0}$. Starting from the Lagrangian (55) and building on the canonical pair $\mu_S$, $\partial L_{dp}/\partial \mu'_S$ one can derive the Hamiltonian (54) by doing exactly the same steps that have led us from the Lagrangian (18) to the Hamiltonian (26).

It should be noted that the total classical Lagrangian (55) admits, as always, some freedom of modifications without affecting the Euler-Lagrange equations. In particular, the derived equations
of motion (48), (38) will remain exactly the same, if one changes the Lagrangian (55) to a new one by multiplying (55) with a constant, for example with a constant $\gamma_0$:

$$L_{dp(new)} = \frac{1}{2n} \left( \frac{a\sqrt{\gamma}}{a_0\sqrt{\gamma_0}} \right)^2 \gamma_0 \left[ (\bar{\zeta}')^2 - n^2 \bar{\zeta}^2 \right] = \frac{1}{2n} \left( \frac{a\sqrt{\gamma}}{a_0\sqrt{\gamma_0}} \right)^2 \gamma \left[ (\bar{\zeta}')^2 - n^2 \bar{\zeta}^2 \right]. \quad (57)$$

This new Lagrangian degenerates to zero in the limit $\gamma \to 0$. We will discuss later the subtleties in quantum theory that arise after such a modification of the Lagrangian.

Similarly to gravitational waves, we impose initial conditions in the early regime of a free oscillator, that is, when $n \gg (a\sqrt{\gamma})'=a_0\sqrt{\gamma}$. We choose the canonical pair

$$q = \bar{\zeta}, \quad p = \frac{\partial L_{dp}}{\partial \bar{\zeta}'} = \frac{1}{n} \left( \frac{a\sqrt{\gamma}}{a_0\sqrt{\gamma_0}} \right)^2 \bar{\zeta}', \quad (58)$$

and write the asymptotic expressions for the operators:

$$q = \sqrt{\frac{\hbar}{2}} \frac{a_0\sqrt{\gamma}}{a_0\sqrt{\gamma_0}} \left[ ce^{-in(\eta-\eta_0)} + c^\dagger e^{in(\eta-\eta_0)} \right], \quad (59)$$

$$p = i\sqrt{\frac{\hbar}{2}} \frac{a\sqrt{\gamma}}{a_0\sqrt{\gamma_0}} \left[ -ce^{-in(\eta-\eta_0)} + c^\dagger e^{in(\eta-\eta_0)} \right], \quad (60)$$

with

$$[q, p] = i\hbar, \quad [c, c^\dagger] = 1. \quad (61)$$

Obviously, a quantum state satisfying the condition

$$c|0\rangle = 0 \quad (62)$$

is the ground (vacuum) state of the Hamiltonian (54). Calculating the mean square values of $q$ and its canonically conjugate momentum $p$, we find

$$\langle 0|q^2|0\rangle = \langle 0|p^2|0\rangle = \frac{\hbar}{2}, \quad \Delta q\Delta p = \frac{\hbar}{2}.$$ 

Returning to $\zeta$ from $\bar{\zeta}$ according to Eq.(56), we find

$$\zeta_{rms} = \left( \langle 0|\zeta^2|0\rangle \right)^{1/2} = \sqrt{2(2\pi)^{3/2}l_{Pl}} / \lambda_0.$$ 

that is, exactly the same value as the initial amplitudes (25) for each of two polarization components of gravitational waves.

Extrapolating the initial time $\eta_0$ up to the boundary between the adiabatic and superadiabatic regimes at $\eta = \eta_i$, we derive the evaluation $\zeta_{rms} \sim l_{Pl}/\lambda_i$. This evaluation, plus the constancy of
the quantity $\zeta$ throughout the long-wavelength regime, is the foundation of the result according to which the final (at the end of the long-wavelength regime) metric amplitudes of gravitational waves and density perturbations should be roughly equal to each other [19].

The primordial $\zeta$-spectrum has the same form as the primordial gravitational-wave spectrum in Eq.(35):

$$\zeta^2(n) \propto n^{2(\beta+2)}.$$  

(64)

In this approximation, the spectral indeces are equal:

$$n_s - 1 = n_t = 2(\beta + 2) \equiv n - 1.$$  

(65)

Scalar fields (36) can support cosmological scale factors with $\beta$ in the interval $-1 \leq \beta, \beta \leq -2$. The ratio of (35) to (64) is approximately 1 for all spectral indeces near and including $\beta = -2$ ($n = 1$).

Having strictly defined the dynamical equations and the initial (quantum ground state) conditions, one can calculate from formula (29) the exact power spectrum of metric perturbations associated with density perturbations. In this formula, two polarization components $\tilde{s}_n (\eta)$ are now determined by the conventions (13), (37), and the constant $C$ is $C = \sqrt{24\pi} l_{Pl}$.

The employed approximations cannot guarantee that in the real Universe the coefficients in Eq.(35) and Eq.(64) should be exactly equal to each other. But there is no reason for them to be different by more than a numerical factor of order 1. Therefore, the lower order CMB multipoles, induced primarily by metric perturbations which are still in the long-wavelength regime, should be approximately at the equal numerical levels for, both, density perturbations and gravitational waves.

As we shall now see the inflationary theory differs from the described quantum theory of density perturbations by many orders of magnitude, let alone numerical coefficients of order 1. The discrepancy becomes infinitely large in the limit of the observationally preferred spectral index $n = 1$ ($\beta = -2, \gamma = 0$).

VII. WHAT INFLATIONARY THEORY SAYS ABOUT DENSITY PERTURBATIONS, AND WHAT SHOULD BE SAID ABOUT INFLATIONARY THEORY

For many years, inflationists keep insisting that in contrast to the generation of gravitational waves [2], which begins with the amplitude $h \approx l_{Pl}/\lambda$ (“half of the quantum in the mode”) and
finishes with the amplitude $h \approx l_{Pl}/(c/H) \approx H/H_{Pl}$ and power $h^2 \approx (H/H_{Pl})^2$ (see 2 and Secs. III IV), the generation of primordial density perturbations (i.e. scalar metric perturbations) is more efficient by many orders of magnitude. Contrary to the calculations reviewed in Sec. VI inflationary theory claims that in a cosmological model with the same $H$, the resulting amplitude and power of the quantity $\zeta$ should contain a huge extra factor, tending to infinity in the limit of the standard de Sitter inflation.

Traditionally, the claimed inflationary derivation of density perturbations goes along the following lines. One starts from the ground state quantum fluctuations taken from a different theory, namely from a theory of a free test scalar field, where metric perturbations are ignored altogether, and writes

$$\delta \varphi \approx \frac{H}{2\pi}.$$  

Then, from the gauge transformation (51), assuming that the l.h.s. is equal to zero, one finds the characteristic time interval $\delta t$ “to the end of inflation” and puts into $\delta t$ the estimate of $\delta \varphi$ from the above-mentioned ‘quantum’ evaluation:

$$\delta t \approx \frac{\delta \varphi}{\dot{\varphi}_0} \approx \frac{H}{2\pi\dot{\varphi}_0}.$$  

Then, one declares that the dimensionless ratio $\delta t/H^{-1}$ is what determines the dimensionless metric and density variation amplitudes in the post-inflationary universe. And this is being presented as the “famous result” of inflationary theory:

$$\delta H \approx \frac{\delta \rho}{\rho} \approx \frac{H^2}{2\pi\dot{\varphi}_0} = \frac{H/2\pi}{\dot{\varphi}_0/H}.$$  

This “inflationary mechanism” of generation of density perturbations is claimed to have been confirmed in numerous papers and books 58. The most dramatic feature of the claimed inflationary result is the factor $\dot{\varphi}_0/H$ in the denominator of the final expression. I call it a ‘zero in the denominator’ factor 15. According to Eq. (41) this factor is $\sqrt{\gamma}$; in the literature, this factor appears in several equivalent incarnations, including such combinations as $V_{,\varphi}/V$, $1 + p/\rho c^2 \equiv 1 + w$, $\epsilon$, where $\epsilon \equiv \gamma \equiv -\dot{H}/H^2$, and so on.

The early Universe Hubble parameter $H$ featuring in the numerator of the final expression cannot be too small because $H(t)$ in scalar field driven cosmologies is a decreasing, or at most constant, function of time. The parameter $H$ in the very early Universe cannot be smaller than, say, the $H$ in the era of primordial nucleosynthesis. So, the numerator of the “famous result” cannot be zero, but the denominator can, at $\gamma = 0$. Therefore, the “famous result” prescribes the
arbitrarily large numerical values to the amplitudes of density perturbations in the limit of the de Sitter evolution $\gamma = 0$ ($\beta = -2$). One has to be reminded that it is this gravitational pump field that generates gravitational waves and other perturbations with primordial spectrum $n = 1$, advocated long ago on theoretical grounds by Harrison, Zeldovich, Peebles, and which is in the vicinity of the primordial spectral shape currently believed to be preferred observationally.

The inflation theory claims to have predicted a “nearly” scale-invariant spectrum of density perturbations because on the strictly scale-invariant spectrum the predicted amplitudes blow up to infinity.

More recent literature operates with the ‘curvature perturbation $R$', equivalent to our $\zeta$ from Eq.(47). A typical quotation states: “The amplitude of the resulting scalar curvature perturbation is given by

$$\langle R^2 \rangle^{1/2} = \left( \frac{H}{\dot{\phi}} \right) \langle \delta \phi^2 \rangle^{1/2},$$

where $H$ is the Hubble parameter, $\dot{\phi}$ is the time derivative of the inflaton $\phi$, and $\delta \phi$ is the inflaton fluctuation on a spatially flat hypersurface. The quantum expectation value of the inflaton fluctuations on super-horizon scales in the de Sitter space-time is

$$\langle \delta \phi^2 \rangle = \left( \frac{H}{2\pi} \right)^2.$$

Here again one is invited to believe that the ground state quantum fluctuations in one theory (without metric perturbations) are responsible for the appearance of arbitrarily large (factor $\gamma$ in the denominator of the $R$ power spectrum) gravitational field fluctuations in another theory. Inflationists are keen to make statements about the resulting gauge-invariant curvature perturbations without having included metric perturbations in the initial conditions. The infinitely large curvature perturbation $R$ is supposed to occur at $\gamma = 0$, that is, exactly in the de Sitter model from which all the ‘quantum’ reasoning about scalar field fluctuations has started.

According to inflationary views on the generating process, the amount of the created scalar particles-perturbations is regulated not by the strength of the external gravitational ‘pump’ field (basically, space-time curvature in the early Universe) but by the closeness of the metric to a de Sitter one. In a sequence of space-times with very modest and approximately equal values of $H$ you are supposed to be capable of generating arbitrarily large amplitudes of the scalar metric perturbation by simply going to smaller and smaller values of $\dot{H}$.

Since the quantity $R$ (as well as the quantity $\zeta$ and the gravitational wave function $h$) oscillates with a slowly decreasing amplitude in the initial short-wavelength regime (see Eq.(48)) and remains
constant during the subsequent long-wavelength regime, the arbitrarily large amplitude of the resulting “inflation-predicted” scalar curvature perturbation $R$ must have been implanted from the very beginning, i.e. from the times in the high-frequency regime of evolution, when the initial quantum state for density perturbations was defined. It is important to remember that for many years inflationists claimed that the reason for the huge difference between the resulting scalar and tensor perturbations was the “big amplification during reheating” experienced by the long-wavelength scalar metric perturbations. These days, the explanation via the “big amplification during reheating” is not even mentioned. These days, the most sophisticated inflationary texts put forward, as the foundation for their belief in arbitrarily large resulting scalar metric perturbations $R$ and $\zeta$, the “Bunch-Davies vacuum”, i.e. a concept from the theory of a test scalar field, where the gravitational field (metric) perturbations are absent altogether. (Moreover, the Bunch-Davies vacuum was originally introduced as a de Sitter invariant state prohibiting the particle production by definition.) The absurd proposition of inflationists with regard to the density perturbations is sometimes called a “classic result”, and it is widely used for derivation of further incorrect conclusions in theory and data analysis.

The most common, and most damaging, inflationary claim is that the amplitudes of relic gravitational waves must be “suppressed”, “sub-dominant”, “negligibly small” in comparison with primordial density perturbations, especially for models with $\gamma \rightarrow 0$. Having arrived at a divergent formula for density perturbations, i.e. with a ‘zero in the denominator’ factor, inflationists compose the ratio of the gravitational-wave power spectrum $P_T$ to the derived divergent scalar metric power spectrum $P_\zeta$ (the so-called ‘tensor-to-scalar ratio’ $r \equiv P_T/P_\zeta$) and write it as

$$r = 16\epsilon = -8n_t.$$  \hspace{1cm} (66)

The inflationary theory binds the amplitude of scalar perturbations with the spectral index, and makes the absurd prediction of arbitrarily large amplitudes of density perturbations in the limit of models with $\epsilon \equiv \gamma = 0$ ($n_s = 1$, $n_t = 0$). But the inflationary “consistency relation” encourages and misleads one to believe that everything is perfect – as if it were the amount of gravitational waves that must go to zero in this limit. To make the wrong theory look acceptable, inflationary model builders keep $P_\zeta$ fixed at the observationally required level of $10^{-9}$ or so, and move $H/H_{Pl}$ down whenever $\epsilon$ goes to zero in the inflationary divergent formula

$$P_\zeta \approx \frac{1}{\epsilon} \left( \frac{H}{H_{Pl}} \right)^2 ,$$

thus making the amount of relic gravitational waves arbitrarily small.
It is instructive to see how the problem of initial conditions is dealt with by S. Weinberg [27]. The author operates with equations for \( \zeta \) (equivalent to our Eq.(48)), gravitational waves (equivalent to our Eq.(2) for \( h \)), and a test scalar field \( \sigma \) (which is known [2] to be similar to the equation for gravitational waves). When it comes to the initial conditions, the author says that they are “designed to make” \( \dot{\zeta} \phi_0/H, h, \) and \( \sigma \) behave like conventionally normalized free fields in the remote past. In other words, the initial conditions for \( h \) and \( \sigma \) are not “designed to make” a potentially vanishing factor to enter the normalization, but the initial conditions for \( \zeta \) are. The normalization of \( \zeta \) becomes now proportional to \( 1/\sqrt{\epsilon} \), and the power spectrum \( P_\zeta \) of \( \zeta \) acquires the factor \( \epsilon \) in the denominator. The author calls this divergent power spectrum of scalar metric perturbations a “classic” result.

One can imagine why this divergent formula is called “classic”. Something repeated in the literature so many times could become “classic” more or less automatically, regardless of its true value. However, Weinberg has not explicitly stated that the “classic” result is a correct result. On the contrary, the recent paper [28], which deals with gravitational waves, makes assumptions diametrically opposite to the prescriptions of the “classic” result (but the paper does not explicitly say that the “classic” result is an incorrect result). That paper chooses for the analysis a model with \( n_t = 0 \) and ‘tensor/scalar ratio’ \( r = 1 \). This choice is in agreement with conclusions of the quantum theory that I advocated and reviewed in Sec.VI but it is in conflict with what is demanded by inflationary Eq.(66) (quoted also in [29]). Indeed, the “classic” result for the case \( n_t = 0 \) implies the non-existence of the very subject of discussion, namely, relic gravitational waves. Although the choice \( n_t = 0, r = 1 \) has been made [28] purely for numerical convenience, it seems to me that the cautious formulation “designed to make” [27], together with the earlier [29] and more recent [28] treatments, testify to a certain evolution of views on the subject of initial conditions.

It appears that some attempts of technical derivation of the inflationary \( \zeta \)-normalization suffer from serious inaccuracies in dealing with quantum operators and quantum states. Certainly, it is incorrect to think that by “demanding that \( a^\dagger \) and \( a \) obey the standard creation and annihilation commutation relations we get a normalization condition for \( \zeta \)”. To put it in the context of a medical analogy: the rules for a surgical operation do not identify the patient on whom you want to operate. Let us discuss this point in more detail.

Suppose, being (mis)guided by various inflationary prejudices, you decided to write, instead of Eqs.(59), (60), the following asymptotic expressions for the operators \( q, p \):

\[
q = \sqrt{\frac{\hbar}{2}} \frac{1}{\sqrt{\gamma}} \left[ b e^{-i(n-\eta_0)} + b^\dagger e^{i(n-\eta_0)} \right],
\]

(67)

\[
p = \sqrt{\frac{\hbar}{2}} \frac{\gamma}{\sqrt{\gamma}} \left[ b e^{-i(n-\eta_0)} + b^\dagger e^{i(n-\eta_0)} \right],
\]

(68)
\[ p = i \sqrt{\frac{\hbar}{2a_0}} \sqrt{\gamma} \left[ -be^{-i(n-\eta_0)} + b^\dagger e^{i(n-\eta_0)} \right] . \] (68)

The commutation relationships \([q, p] = i\hbar\) dictate exactly the same commutation relationships for the operators \(b^\dagger, b\) as they did for the operators \(c^\dagger, c\). Namely, \([b, b^\dagger]\) = 1.

The commutation relationships for the annihilation and creation operators are exactly the same, but the quantum state \(|0_s\rangle\) annihilated by \(b\),

\[ b|0_s\rangle = 0, \]

is totally different, it is not the ground state of the Hamiltonian (54). Calculation of the mean square value of the variable \(\bar{\zeta}\) and its canonically-conjugate momentum gives at \(\eta = \eta_0\):

\[ \langle 0_s|q^2|0_s\rangle = \frac{\hbar}{2}\frac{1}{\gamma_0}, \quad \langle 0_s|p^2|0_s\rangle = \frac{\hbar}{2}\gamma_0, \]

and the factor \(\sqrt{\gamma_0}\) cancels out in the uncertainty relation:

\[ \Delta q \Delta p = \frac{\hbar}{2}. \]

The initial rms value of \(\zeta\) is proportional to \(1/\sqrt{\gamma_0}\) and therefore contains the ‘zero in the denominator’ factor, but this happens only because the quantum state \(|0_s\rangle\) is an excited (multi-quantum) squeezed vacuum state (for more details, see [15] and references there). The choice of this state as an initial state for \(\zeta\)-perturbations would make them arbitrarily large, in the limit of \(\gamma_0 \to 0\), right from the very beginning, i.e. from the time of imposing the initial conditions at \(\eta = \eta_0\). The multi-quantum state \(|0_s\rangle\) is not a choice of the initial state that is regarded physically motivated.

At the level of classical equations, inflationists have reacted to the visual analogy between equations (1), (38), rather than, say, to the visual analogy between equations (2), (48). They have modeled initial conditions for the function \(\mu_S\) on the initial conditions for the function \(\mu_T\), instead of modeling initial conditions for \(\zeta\) on the initial conditions for the g.w. function \(h\). The inflationary initial conditions are usually written in the form

\[ \mu(T,S) = \frac{1}{\sqrt{2k}} e^{-ik\eta} \quad \text{for} \quad k|\eta| \gg 1. \]

These initial conditions are correct for gravitational waves, but are incorrect for density perturbations. As is seen from Eq.(47), these initial conditions for \(\mu_S\) would require the gauge-invariant metric perturbation \(\zeta\) (as well as curvature perturbation \(R\)) to be arbitrarily large, in the limit of models with \(\gamma \to 0\), right from the very beginning, i.e. from the time of imposing initial conditions.
in the short-wavelength regime. In other words, the inflationary initial conditions are “designed to make” the quantity $\zeta$ to be divergent as $1/\sqrt{\gamma}$ from the start. These incorrect initial conditions are used in all inflationary calculations and conclusions, including the latest claims of inflationists on what the observations “really” tell them about inflation.

Finally, let us assume that the ‘correct’ Lagrangian is given by Eq. (57). (Surely, you may assume correct whichever Lagrangian you wish, because it is taxpayers [30] who will be paying the price at the end of the day.) In contrast to the Lagrangians [18], [55], this new Lagrangian vanishes in the most interesting limit of models with $\gamma \rightarrow 0$.

For this new Lagrangian, the canonical quantization would require us to write (instead of Eqs. (59), (60)):

$$q = \sqrt{\frac{\hbar}{2a\sqrt{\gamma}} \frac{1}{\sqrt{\gamma_0}}} \left[ d e^{-i(n-\eta_0)} + d^\dagger e^{i(n-\eta_0)} \right],$$

$$(69)$$

$$p = i\sqrt{\frac{\hbar}{2a\sqrt{\gamma} \sqrt{\gamma_0}}} \left[ -de^{-i(n-\eta_0)} + d^\dagger e^{i(n-\eta_0)} \right],$$

$$(70)$$

where $[q,p] = i\hbar$ and $[d,d^\dagger] = 1$.

The ground state $|0_{\text{new}}\rangle$ of the new Hamiltonian associated with this new Lagrangian obeys the condition

$$d|0_{\text{new}}\rangle = 0.$$

The mean square value of the variable $\tilde{\zeta}$ in this state at $\eta = \eta_0$ is given by

$$\langle 0_{\text{new}}|q^2|0_{\text{new}}\rangle = \hbar \frac{1}{2\gamma_0}.$$

Technically speaking, one could argue that although the initial rms value of $\zeta$ is divergent as $1/\sqrt{\gamma_0}$, the divergency takes place for the ground state (of this new Hamiltonian), not for an excited state (of this new Hamiltonian). But this technical subtlety requires a vanishing Lagrangian, and in any case it does not change much from the physical point of view.

Indeed, independently of technical arguments, it is important to realize that a proposal for a divergent scalar metric power spectrum in the limit of $n = 1$, whatever the reasons for this proposal might be, is in conflict with available observations. The currently derived ‘best fit’ value for $n$ is slightly lower than 1, but the data allow $n = 1$, even if with a smaller likelihood. This means that for the tested spectral indices in the vicinity and including $n = 1$ the data are consistent with finite and small amplitudes at the level of the best fit amplitude (as implied by the quantum
theory discussed in Sec. VI). But there is absolutely nothing in the data that would suggest the need for a catastrophic growth of the amplitude (demanded by inflationary theory) when the data are fitted against spectra with indeces approaching and crossing $n = 1$. (Surely, the same comparison of inflationary predictions with available observations is addressed by inflationists as “one of the most remarkable successes ... confirmed by observations”.) What would we conclude about the assumed Lagrangian whose associated Hamiltonian leads to predictions contradicting observations? We “would conclude that it was the wrong Lagrangian” [14].

The inflationary ‘zero in the denominator’ factor (if you decided to commit suicide and include it in the scalar metric power spectrum) should be taken at the moment of time, for a given mode $n$, when the mode begins its superadiabatic evolution. In general, this factor is $n$-dependent, and hence it affects not only the overall normalization of the spectrum, but also its shape. It enables one to ‘generate’ a flat, or even a blue, power spectrum of scalar perturbations by gravitational pump fields which in reality can never do this. It enables one to derive all sorts of wrong conclusions in various subjects of study, ranging from the formation of primordial black holes and up to perturbations in cyclic and brane-world cosmologies. In particular, recent claims stating that a given ultra-modern theory predicts an “extremely small $r \lesssim 10^{-24}$” or such an $r$ that “a tensor component...is far below the detection limit of any future experiment”, mean only that the predictions were based on the incorrect (inflationary) formula for scalar metric perturbations, with the ‘zero in the denominator’ factor. Whatever the incorrectly derived $r$ may be, small or large, the use of inflationary Eq. (66) in data analysis (what, unfortunately, is regularly being done in the CMB data analysis [31, 32, 33]) can only spoil the extraction of physical information on relic gravitational waves from the data.

The self-contradictory nature of conclusions based on the inflationary formula (66) is dramatically illustrated by the claimed derivation of limits on the amount of gravitational waves from the WMAP data [34]. The authors use Eq. (66) and explicitly quote, in the form of their equations (17) and (18), the power-spectrum amplitude $\Delta^2_R$ of inflationary scalar perturbations and the ‘tensor-to-scalar’ ratio $r$:

$$\Delta^2_R = \frac{V}{M^4} \frac{\langle l \rangle}{24\pi^2 \epsilon}, \quad r = 16\epsilon. \quad (I)$$

The conclusion of this (and many other subsequent papers) is such that according to the likelihood analysis of the data the value of $r$ must be small or zero. In other words, the maximum of the likelihood function is either at $r = 0$ or, in any case, the value of the observed $r$ is perfectly well “consistent with zero”. Comparing this conclusion with the inflationary formulas (I), quoted in
the same paper, one has to decide whether this conclusion means that the WMAP data are also perfectly well consistent with arbitrarily large scalar amplitudes $\Delta^2 R$, and hence with arbitrarily large CMB anisotropies caused by density perturbations, or that the cited inflationary formulas are wrong and have been rejected by observations (see also a discussion below, in Sec. XII). The CMB data and the inflationary theory of density perturbations are in deep conflict with each other for long time, but inflationists and their followers keep claiming that they are in “almost perfect agreement”.

It seems to me that the situation in this area of physics and cosmology remains unhealthy for more than 25+ years. (This is my mini-version of ‘an obligation to inform the public’ – from [35].) It appears that more than 2+ generations of researchers have been ‘successfully’ misled by the “standard inflationary results”. One can only hope that the present generation of young researchers will be smarter and more insightful.

VIII. WHY RELIC GRAVITATIONAL WAVES SHOULD BE DETECTABLE

The detailed analysis in previous sections is crucial for proper understanding of the very status of relic gravitational waves. Are we undertaking difficult investigations because we want to find an optional “bonus”, “smoking gun”, “limitation” on dubious theories, or we search for something fundamental that ‘must be there’, and at a measurable level?

Strictly speaking, it is still possible that the observed CMB anisotropies have nothing to do with cosmological perturbations of quantum-mechanical origin, that is, with the superadiabatic evolution of the ground state of quantized perturbations. The first worry is that, even if the superadiabatic (parametric) mechanism is correct by itself, the ‘engine’ that drove the cosmological scale factor was unfortunate and the pump field was too weak. In this case, a relic gravitational-wave signal (as well as scalar perturbations of quantum-mechanical origin) could be too small for discovery. I think this possibility is unlikely. First, it is difficult to imagine an equally unavoidable mechanism – basic laws of quantum mechanics and general relativity – for the generation of the presently existing, as well as the processed in the past, long-wavelength cosmological perturbations. Second, the inevitable quantum-mechanical squeezing and the standing-wave character of the generated perturbations, which, among other things, should have resulted in the scalar metric power spectrum oscillations (see Sec. IV), seems to have already revealed itself [36] in the observed CMB power spectrum oscillations. (These oscillations are often being associated – in my opinion, incorrectly – with baryonic acoustic waves and are called “acoustic” peaks).
A second worry is that we may be wrong in extrapolating the laws of general relativity and quantum mechanics to extreme conditions of the very early Universe. Although we apply these laws in environments that are still far away from any Planckian or ‘trans-Planckian’ ambiguities, it is nevertheless an extremely early and unfamiliar Universe (which we want to explore). In principle, it is possible that something has intervened and invalidated the equations and rules that we have used for derivation of relic gravitational waves. This would also invalidate the equations and rules used in the derivation of density perturbations, but the generation of scalar perturbations requires an extra hypothesis in any case. In other words, even if the driving cosmological ‘engine’ was right, the employed quantum theory of arising perturbations could be wrong. Hopefully, this complication also did not take place.

Assuming that the dangers did not materialize, we come to the conclusion that we have done our job properly. The theoretical calculations were adequate, the normalization of $h_{\text{rms}}$ on the CMB lowest-order multipoles was justified, and therefore the relic gravitational wave background must exist, and probably at the level somewhere between the dotted and solid lines in Fig.5. The other side of the same argument is that if we do not detect relic gravitational waves at this level, something really nasty, like the above-mentioned dangers, had indeed happened. From not seeing relic gravitational waves, we would at least learn something striking about the limits of applicability of our main theories.

It is important to stress again that what is called here the relic gravitational waves is not the same thing which is sometimes called the inflationary gravitational waves. As the name suggests, statements about the inflationary gravitational waves are based on the inflation theory, as applied to density perturbations and gravitational waves. Although, conceptually, the inflationary derivation of density perturbations is an attempt of using the mechanism of superadiabatic (parametric) amplification (see Sec.I), the actual implementation of this approach has led inflationists to the divergency in the scalar metric spectrum (see Sec.VII), which has been converted, for the purpose of “consistency”, into a statement about small $r$. In particular, for the “standard inflation” ($\gamma \equiv \epsilon = 0$), the inflationary theory predicts (see Eq.(66)) the “smoking gun of inflation” in the form of a zero amount of inflationary gravitational waves ($r = 0$). This is in sharp contrast with the superadiabatic (parametric) prediction of a finite and considerable amount of relic gravitational waves, as shown by a dotted line in Fig.5.

Some recent papers, still based on the incorrect theory with Eq.(66), start making additional artificial assumptions about the “natural” inflationary conditions, which amounts to postulating that $\epsilon$ should not be too small, but should, instead, be at the level of, say, 0.02. This kind of
argument gives some authors “more reason for optimism” with regard to the detection of primordial gravitational waves. Surely, an incorrect theory can lead to predictions not very much different from predictions of a correct theory in some narrow range of parameters that are supposed to be deduced from observations. One should remember, however, that theories are tested not only by what they predict but also by what they do not predict. Any observation consistent with the parameter value other than you postulated is against your theory.

Since the spectrum of relic gravitational-wave amplitudes is decreasing towards the higher frequencies, see Fig.5, it is the lowest frequencies, or better to say the longest wavelengths, that provide the most of opportunities for the (indirect) detection. It is known for long time [37–40] that gravitational waves affect the CMB temperature and polarization. The low-order CMB multipoles (\( \ell \lesssim 100 \)) are mostly induced by cosmological perturbations with wavelengths ranging from 10 times longer and up to 10 times shorter than the present-day Hubble radius \( l_H \). And this is the range of scales that will be in the center of our further analysis.

**IX. INTENSITY AND POLARIZATION OF THE CMB RADIATION**

A radiation field, in our case CMB, is usually characterized by four Stokes parameters \( (I, Q, U, V) \) [41, 10]. \( I \) is the total intensity of radiation (or its temperature \( T \)), \( Q \) and \( U \) describe the magnitude and direction of linear polarization, and \( V \) is the circular polarization. The Stokes parameters of the radiation field arriving from a particular direction in the sky are functions of a point \( \theta, \phi \) on a unit sphere centered on the observer. The metric tensor \( g_{ab} \) on the sphere can be written as:

\[
dσ^2 = g_{ab} dx^a dx^b = dθ^2 + \sin^2 θ dφ^2.
\]

(71)

The Stokes parameters are also functions of the frequency of radiation \( ν \), but angular dependence is more important for our present discussion.

The Stokes parameters are components of the polarization tensor \( P_{ab} \) [10], which can be written

\[
P_{ab}(θ, φ) = \frac{1}{2} \begin{pmatrix}
I + Q & -(U - iV) \sin θ \\
-(U + iV) \sin θ & (I - Q) \sin^2 θ
\end{pmatrix}.
\]

(72)

As every tensor, the polarization tensor \( P_{ab} \) transforms under arbitrary coordinate transformations on the sphere, but some quantities remain invariant.

We can build invariants, linear in \( P_{ab} \) and its derivatives, with the help of the tensor \( g_{ab} \) and
the unit antisymmetric pseudo-tensor $\epsilon^{ab}$. First two invariants are easy to build:

\[ I(\theta, \phi) = g^{ab}(\theta, \phi) P_{ab}(\theta, \phi), \quad V(\theta, \phi) = i \epsilon^{ab}(\theta, \phi) P_{ab}(\theta, \phi). \]  

(73)

Two other invariants involve second covariant derivatives of $P_{ab}$:

\[ E(\theta, \phi) = -2 (P_{ab}(\theta, \phi))^{;a;b}, \quad B(\theta, \phi) = -2 (P_{ab}(\theta, \phi))^{;b;d} \epsilon^{b}_{d}, \]

(74)

Being invariant, quantities $E, B$ do not mix with each other. To calculate $E$ and $B$ in a given point $\theta, \phi$ we do not need to know the polarization pattern all over the sky, but we do need to know the derivatives of $P_{ab}$ at that point. Whatever the numerical values of $E$ or $B$ in a given point are, $E$ and $B$ will retain these values under arbitrary coordinate transformations (smoothly reducable to an ordinary rotation). The invariant $B$ is built with the help of a pseudo-tensor $\epsilon_{ab}$, and therefore $B$ is a pseudo-scalar rather than an ordinary scalar. While $E$ does not change sign under a coordinate reflection, $B$ does. With $B$ one can associate the notion of chirality, or handedness (compare with polarization tensors (12), (13)). Clearly, if given cosmological perturbations are such that they themselves are incapable of supporting the handedness, it will not arise in the CMB polarization which these perturbations are responsible for.

The invariant quantities $I, V, E, B$ can be expanded over ordinary spherical harmonics $Y_{\ell m}(\theta, \phi), Y_{\ell m}^{*} = (-1)^{m} Y_{\ell,-m}$:

\[ I(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{T\ell m} Y_{\ell m}(\theta, \phi), \]  

(75a)

\[ V(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{V\ell m} Y_{\ell m}(\theta, \phi), \]  

(75b)

\[ E(\theta, \phi) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \left[ \frac{(\ell + 2)!}{(\ell - 2)!} \right]^{\frac{1}{2}} a_{E\ell m}^{T} Y_{\ell m}(\theta, \phi), \]  

(75c)

\[ B(\theta, \phi) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \left[ \frac{(\ell + 2)!}{(\ell - 2)!} \right]^{\frac{1}{2}} a_{B\ell m}^{T} Y_{\ell m}(\theta, \phi). \]  

(75d)

The set of (complex) multipole coefficients $(a_{T\ell m}, a_{V\ell m}, a_{E\ell m}, a_{B\ell m})$ completely characterizes the intensity and polarization of the CMB.

The same multipole coefficients participate in the expansion of the tensor $P_{ab}$ itself, not only in the expansion of its invariants, but the expansion of $P_{ab}$ requires the use of generalized spherical functions, the so-called spin-weighted or tensor spherical harmonics [42, 43]. For example, the
tensor $P_{ab}$ can be written as

\[
P_{ab} = \frac{1}{2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left( g_{ab} a_{\ell m}^T - i e_{ab} a_{\ell m} V \right) Y_{\ell m}(\theta, \phi)
\]

\[
+ \frac{1}{\sqrt{2}} \sum_{\ell=2}^{\infty} \sum_{m=-l}^{l} \left( -a_{\ell m} Y_{(\ell m)ab}^G(\theta, \phi) + a_{\ell m} Y_{(\ell m)ab}^C(\theta, \phi) \right),
\]

where $Y_{(\ell m)ab}^G(\theta, \phi)$ and $Y_{(\ell m)ab}^C(\theta, \phi)$ are the ‘gradient’ and ‘curl’ tensor spherical harmonics.

In what follows, we will not be considering the circular polarization $V$, and we will sometimes denote the multipole coefficients collectively by $a_X^{\ell m}$, where $X = I, E, B$. These coefficients are to be found from solutions to the radiative transfer equations in a slightly perturbed universe.

**X. RADIATIVE TRANSFER IN A PERTURBED UNIVERSE**

Polarization of CMB arises as a result of Thompson scattering of the initially unpolarized light on free electrons residing in a slightly perturbed universe, Eq.(8). Following [41], [44], [40], it is convenient to describe Stokes parameters in terms of a 3-component symbolic vector $\hat{n}$:

\[
\hat{n} = \begin{pmatrix}
\hat{n}_1 \\
\hat{n}_2 \\
\hat{n}_3
\end{pmatrix} = \frac{c^2}{4\pi \hbar \nu^3} \begin{pmatrix}
I + Q \\
I - Q \\
-2U
\end{pmatrix}.
\]  

(76)

In the zero-order approximation, we assume that all $h_{ij} = 0$ and the CMB radiation field is fully homogeneous, isotropic, and unpolarized. Then,

\[
\hat{n}^{(0)} = n_0(\nu a(\eta)) \hat{u},
\]

(77)

where

\[
\hat{u} = \begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix}.
\]

In the presence of metric perturbations, we write

\[
\hat{n} = \hat{n}^{(0)} + \hat{n}^{(1)},
\]

(78)

where $\hat{n}^{(1)}$ is the first order correction. The functions $\hat{n}^{(1)}$ depend on $(\eta, x^i, \tilde{\nu}, e^i)$, where $\tilde{\nu} = \nu a(\eta)$ and $e^i$ is a unit spatial vector along the photon’s path. Our final goal is to predict, with as much completeness as possible, the values of the Stokes parameters at the time of observation $\eta = \eta_R$. 

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The functions $\hat{n}^{(1)}$ satisfy the linear version of the radiation transfer equations. These equations can be written

$$
\left[ \frac{\partial}{\partial \eta} + q(\eta) + e^i \frac{\partial}{\partial x^i} \right] \hat{n}^{(1)}(\eta, x^i, \tilde{\nu}, e^j) = f(\tilde{\nu}) n_0(\tilde{\nu}) n^{(1)}(\eta, x^i, \tilde{\nu}, e^j) + q(\eta) \frac{1}{4\pi} \int d\Omega' \hat{P}(e^i; e'^j) \hat{n}^{(1)}(\eta, x^i, \tilde{\nu}, e'^j).$$

(79)

The astrophysical inputs from unpolarized radiation and free electrons are described, respectively, by $f(\tilde{\nu}) n_0(\tilde{\nu})$ and $q(\eta)$, while the scattering process is described by the Chandrasekhar matrix $\hat{P}(e^i; e'^j)$. The input from the gravitational field (metric) perturbations is given by the combination $e^i e'^j \partial h_{ij}/\partial \eta$. Certainly, when all $h_{ij} = 0$, all $\hat{n}^{(1)}$ vanish if they were not present initially, which we always assume.

The combination $e^i e'^j \partial h_{ij}/\partial \eta$ gives rise to disparate angular structures for gravitational waves and density perturbations. Let us consider a particular Fourier mode with the wavevector $\mathbf{n} = (0, 0, n)$. The polarization tensors of gravitational waves generate the structure

$$
e^i e^j s^{ij}(\mathbf{n}) = (1 - \mu^2) e^{\pm 2i\phi},$$

(80)

where $\mu = \cos \theta$ and the $\pm$ signs correspond to the left $s = 1 = L$ and right $s = 2 = R$ polarization states, respectively. Solving Eq.(79) for $\hat{n}^{(1)}$ in terms of a series over $e^{im\phi}$, one finds that the terms proportional to $e^{\pm 2i\phi}$ are retained while other components $e^{im\phi}$ are not arising. This is because the Chandrasekhar matrix does not create any new $m\phi$ dependence.

In contrast to gravitational waves, the same metric combination with polarization tensors of density perturbations produces the structure which is $\phi$-independent. Although in the case of density perturbations, Eqs.(79) contain one extra term, proportional to the electron fluid velocity, $e^i v_i = -i\mu v_b$ (expressible in terms of metric perturbations via perturbed Einstein equations), this term is also $\phi$-independent. Since the invariant $B$ depends on the derivative of $\hat{n}^{(1)}$ over $\phi$, one arrives at the conclusion that $B = 0$ for density perturbations and $B \neq 0$ for gravitational waves. These results for one special Fourier mode $\mathbf{n} = (0, 0, n)$ can then be generalized to any arbitrary $\mathbf{n}$. As we anticipated on the grounds of handedness, gravitational waves can generate the $B$-mode of CMB polarization, but density perturbations can not.
XI. STATISTICS AND ANGULAR CORRELATION FUNCTIONS

The linear character of the radiation transfer equations (79) makes the randomness of \( h_{ij} \) being inherited by \( \hat{n}^{(1)}(1) \). A consistent handling of Eq.(79) requires us to use the spatial Fourier expansion

\[
\hat{n}^{(1)}(\eta, x^i, \tilde{\nu}, e^j) = \frac{C}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \frac{d^3 n}{\sqrt{2n}} \sum_{s=1,2} \left[ \hat{n}^{(1)}_{n,s}(\eta, \tilde{\nu}, e^j) e^{i n \cdot x^s c_n} + \hat{n}^{(1)*}_{n,s}(\eta, \tilde{\nu}, e^j) e^{-i n \cdot x^s c_n^*} \right],
\]

where random coefficients \( c_n \) are the same entities which enter the metric field Eq.(9). The CMB anisotropies are random because the underlying metric perturbations are random.

In the strict quantum-mechanical version of the theory, where \( c_n \) are quantum-mechanical operators, the CMB field \( \hat{n}^{(1)}(1) \) itself becomes a quantum-mechanical operator. If the initial quantum state of the system is chosen to be the ground state \( |0\rangle \), \( c_n |0\rangle = 0 \), all statistical properties of the system are determined by this choice.

Note the extra degree of uncertainty that we will have to deal with. Usually, the signal is deterministic, even if totally unknown, and the randomness of the observed outcomes arises at the level of the measurement process, as a consequence of the uncontrollable noises. In our problem, if cosmological perturbations do indeed have quantum-mechanical origin, the signal itself is inherently random and is characterized by a quantum state, or a wave-function, or a probability distribution function. At best, we can predict only a probability distribution function for possible CMB maps (outcomes). This is true even if the dynamics and cosmological parameters are strictly fixed and the measurement process is strictly noiseless.

Each mode of cosmological perturbations has started its life in the initial vacuum state. This state is Gaussian. In course of time it evolved into a squeezed vacuum state. Squeezed vacuum states retain the Gaussianity, even though developing the strongly unequal variances in amplitudes and phases. Therefore, the actually observed coefficients \( a^{X}_{\ell m} \) (our own realisation of the CMB map belonging to the theoretical ensemble of CMB maps) are supposed to be drawn from the zero-mean Gaussian distributions for \( a^{X}_{\ell m} \). If the observed CMB map looks a bit strange to you, due to the presence, for example, of some hints on ‘axes’ and ‘voids’, this is not necessarily an indication of non-Gaussianity of the underlying cosmological perturbations. Even if the observed CMB map consists entirely of your own images, this is not a proof of non-Gaussianity. It is a legitimate procedure to postulate some sort of a non-Gaussian distribution, by introducing a new parameter \( f_{NL} \), and try to find \( f_{NL} \) from the data of a single actually observed map. But it is even more a legitimate procedure to insist on Gaussianity of perturbations, because it follows from the very
deep foundations explained above, and therefore regard \( f_{NL} \equiv 0 \) by definition. From this position, if the set of the observed coefficients \( a_{\ell m}^X \) looks a bit strange to you, this is simply because our Universe is not as dull and ‘typical’ as you might expect.

In this paper, we simplify the problem and treat cosmological fields classically rather than quantum-mechanically. We also make mild statistical assumptions. Specifically, we assume that classical random complex numbers \( \hat{c}_n^s \) satisfy a limited set of statistical requirements:

\[
\langle \hat{c}_n^s \rangle = \langle c_{n'}^{s*} \rangle = 0, \quad \langle \hat{c}_n^s \hat{c}_{n'}^{s*} \rangle = \langle c_n^{s*} c_{n'}^s \rangle = \delta_{ss'} \delta^{(3)}(n - n'), \quad \langle \hat{c}_n^s c_{n'}^s \rangle = \langle c_n^s c_{n'}^s \rangle = 0,
\]

(82)

where the averaging is performed over some probability distributions. We do not even assume outright that these distributions are Gaussian. The rules (82) are sufficient for the most of our further calculations.

We want to know the value of quantities \( a_{\ell m}^X \) at the time of observation \( \eta = \eta_R \). To find these quantities we have to integrate Eq. (79) over time, with all the astrophysical and gravitational inputs taken into account. The derived coefficients \( a_{\ell m}^X \) are random, because the participating coefficients \( \hat{c}_n^s \) are random. We can calculate various correlation functions of the CMB by calculating the quantities \( \langle a_{\ell m}^X a_{\ell' m'}^{X'} \rangle \). Using the rules (82), one can show that these quantities take the form

\[
\langle a_{\ell m}^X a_{\ell' m'}^{X'} \rangle = C_{\ell m}^{XX'} \delta_{\ell \ell'} \delta_{mm'},
\]

(83)

where \( C_{\ell}^{XX'} \) depend on the gravitational mode functions and astrophysical input. \( C_{\ell}^{XX'} \) are calculable as general theoretical expressions.

The quantities \( C_{\ell}^{XX'} \) are called the multipoles of the corresponding CMB power spectrum \( XX' \). As usual, the power spectrum of a field contains less information than the field itself, but we will mostly ignore this loss of information. Also, it is worth noting that practically any feature in the CMB power spectrum can be “predicted” and “explained” solely by the properly adjusted primordial spectrum of cosmological perturbations. But we will not go along this line and will stick to simple power-law primordial spectra.

In the case of relic gravitational waves, the general shape of today’s CMB power spectra and their features in the \( \ell \)-space are almost in one-to-one correspondence with the processed metric power spectra and their features in the wavenumber \( n \)-space. The processed metric power spectra should be taken at the time of decoupling of CMB. The \( TT \) power spectrum is determined by the power spectrum of the metric itself, \( hh \). The \( EE \) and \( BB \) power spectra are largely determined by the power spectrum of the metric’s first time-derivative, \( h' h' \). And \( TE \) power spectrum is determined by the cross power spectrum of the metric and its first time-derivative, \( hh' \). For the gravitational-wave background with parameters indicated by dotted line in Fig. 5, the corresponding metric and
FIG. 7: The left panel shows (a) the power spectrum of temperature anisotropies $\ell(\ell + 1)C_{TT}^{\ell}$ (in $\mu K^2$) generated by (b) the power spectrum of g.w. metric perturbations $hh$ (29), $\beta = -2$. The right panel shows (c) the power spectra of polarization anisotropies $\ell(\ell + 1)C_{BB}^{\ell}$ (solid line) and $\ell(\ell + 1)C_{EE}^{\ell}$ (dashed line), panel (d) shows the power spectrum of the first time derivative of the same g.w. field, $h'h'$. CMB power spectra are shown in Fig.7 and Fig.8. For the gravitational-wave background with parameters indicated by a solid line in Fig.5, the summary of CMB spectra is shown in Fig.9. The summary also includes the reionization ‘bump’ at $\ell \lesssim 12$. (More details about these figures can be found in [17]).

All CMB power spectra have been calculated as averages over a theoretical ensemble of all possible realizations of the CMB field. In its turn, this randomness of CMB anisotropies ensues from the randomness of the gravitational field coefficients $c_n, c_n^*$. The characteristic parameters of a stochastic field, such as its mean values and variances, are, by definition, averages over the ensemble of realizations. However, in CMB observations, we have access to only one realization of this ensemble, which can be thought of as a single observed set of coefficients $a_{Xlm}^X$. Is it possible to find the parameters of a stochastic process by studying only one realization of this process?

The answer is yes, if the correlations of the stochastic process decay sufficiently quickly at large separations in time or space where the process is defined [47]. The process allowing the derivation
FIG. 8: The bottom panel shows the cross-power spectrum $hh'$ of gravitational waves, whereas the top panel shows the angular power spectrum $\ell(\ell + 1)C_{\ell}^{TE}$ caused by these waves. The negative values of these functions are depicted by broken lines.

of its true parameters, with probability arbitrarily close to 1, from a single realization is called ergodic. For example, the distribution of galaxies, or a stochastic density field, in an infinite 3-space (our Universe) may be ergodic, and then by studying a single realization of this distribution we could extract the true parameters of the underlying random process. In the theory that we are discussing here, the randomness of the linear density field is also described by the random coefficients $s_c^n$, $s_c^*$ appearing in the metric perturbations.

If the process is non-ergodic, there will be an inevitable uncertainty surrounding the parameter’s estimation derived from a single realization. This uncertainty should not be mis-taken for the ‘cosmic variance’, often quoted and plotted on the observational graphs. The cosmic variance is a mathematically correct statement about the size of the variance of the $\chi^2$-distribution for $2\ell + 1$ independent Gaussian variables. There is nothing particularly cosmic in the cosmic variance. The elementary theorem, called cosmic variance, has nothing to say about the ergodicity or non-ergodicity of a given stochastic process. In particular, the universal validity of cosmic variance cannot prevent the extraction of exact parameters from the observation of a single realization of
the stochastic density field in our Universe, if the density field is ergodic, – and it might be ergodic. The problem is, however, that on a compact 2-sphere, where the random CMB is defined, ergodic processes do not exist. We will always be facing some uncertainty related to non-ergodicity. The size of this uncertainty about the derived parameter depends on the statistics and employed estimator. Under some conditions, this uncertainty is close, numerically, to the size of the usually quoted cosmic variance [48]. This discussion is important, because we are now approaching the observational predictions and the ways of discovering relic gravitational waves in the CMB data.

XII. TEMPERATURE-POLARIZATION CROSS-CORRELATION FUNCTION

The numerical levels of primordial spectra for tensor and scalar metric perturbations are approximately equal. This means that the amplitudes of those Fourier modes \( n \) which have not started yet their short-wavelength evolution are numerically comparable, and the metric mode functions are practically constant in time. In contrast, in the short-wavelength regime, the amplitudes of gravitational waves adiabatically decrease, while the amplitudes of gravitational field perturbations associated with density perturbations may grow. Specifically, at the matter-dominated stage, the function \( h_l(\eta) \) grows and overtakes \( h(\eta) \), which remains constant.

In the context of CMB anisotropies, the crucial time is the epoch of decoupling. The wavelenghts

FIG. 9: The summary of CMB temperature and polarization anisotropies due to relic gravitational waves with \( n = 1.2 \) and \( R = 1 \).
FIG. 10: The dotted line shows the contribution of density perturbations alone, while the dashed line shows the contribution of gravitational waves alone. The solid line is the sum of these contributions. It is seen from the graph that the inclusion of g.w. makes the total curve to be below the d.p. curve.

of modes with $n \lesssim 100$ were comfortably longer than the Hubble radius at the decoupling. The influence of these modes, both in gravitational waves (g.w.) and density perturbations (d.p.), have been projected into today’s $XX'$ anisotropies at $\ell \lesssim 100$. In this interval of $\ell$, the g.w. contribution is not a small effect in comparison with the d.p. contribution. The g.w. contribution to the CMB power spectra is illustrated in Fig.7, Fig.8, and Fig.9.

One way of detecting relic gravitational waves is based on measuring the $BB$ auto-correlation. This method is clean, in the sense that density perturbations do not intervene, but the expected signal is very weak, see Fig.7, Fig.9.

We propose [17] to concentrate on the $TE$ cross-correlation (without, of course, neglecting the $BB$ searches). The $TE$ signal is about two orders of magnitude stronger than the $BB$ signal, and the use of a cross-correlation is always better than an auto-correlation, in the sense of fighting the noises. The special feature allowing to distinguish the g.w. part of $TE$ from the d.p. part of $TE$ is the difference in their signs. In the interval $\ell \lesssim 100$ the $(TE)_{gw}$ must be negative (see Fig.8), while the $(TE)_{dp}$ must be positive at lowest $\ell'$s and up to, at least, $\ell \approx 50$. This difference in sign of $TE$ correlation functions is the consequence of the difference in sign of the g.w. and d.p. metric cross-power spectra $hh'$ in the interval $n \lesssim 70$ [17]. (The difference in sign of QT correlation functions is discussed in the earlier paper [49].)

An example of expected g.w. and d.p. contributions to the $TE$ correlation function is shown in Fig.10. To include reionization and enhance the lowest $\ell'$s, we plot the function $(\ell + 1)/2\pi C^{TE}$. 

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rather than the usual $\ell(\ell + 1)/2\pi C_\ell^{TE}$. The g.w. and d.p. metric power spectra are normalized in such a way that they give $R = 1$, where

$$R \equiv \frac{C_{\ell=2}^{TT}(gw)}{C_{\ell=2}^{TT}(dp)}.$$  

More specifically, they give equal contributions to the total ‘best fit’ temperature quadrupole $C_{\ell=2}^{TT}$. The dashed line in Fig.10 shows the effect of the g.w. background marked by dotted line in Fig.5. The dotted line in Fig.10 shows the effect of d.p. metric perturbations, with the same primordial spectral index, $n = 1$. The sum of the two contributions is shown by a solid line. The fact that the two contributions may almost cancel each other does not mean that the g.w. signature is weak and hard to measure. The g.w. signal is in the strong deviation of the total $TE$ spectrum from the expectation of the d.p. model.

It is important to remember that the widely publicized “tight limits” on the ‘tensor-to-scalar ratio’ $r$ which allegedly rule out $R = 1$ and any $R \neq 0$, unless $R$ is small, were derived by making use of the inflationary “consistency relation” (see Sec.VII). In these derivations, the g.w. spectral index $n_t$ is being taken from the relationship $n_t = -r/8$, which automatically sends $r$ to zero when $n_t$ approaches zero. If this relationship is regarded as an artificial extra condition on g.w. parameters, then the results of such a data analysis may be of some value to those who are interested in this ad hoc condition, but not to those who are interested in determination of the true amount of relic gravitational waves. If, on the other hand, this relationship is regarded as part of inflationary theory, then such a data analysis is deeply self-contradictory. Indeed, the invariably derived conclusion, according to which the maximum of the likelihood function for $r$ is at $r = 0$, or at least the value of the observed $r$ is “consistent with zero”, means that the most likely values of the inflationary density perturbation amplitudes, together with the CMB anisotropies induced by them, are infinitely large, or at least the data are “consistent” with such an infinity (see end of Sec.VII). It goes without saying that we are not using this relationship. But we do use the relationship $n_t = n_s - 1$ which is a consequence of the superadiabatic generating mechanism, see Eq.(65).

The WMAP community seems to be satisfied with the data analysis which states that the CMB data can be described by a small number of parameters which include density perturbations, but with no necessity for gravitational waves. The usual logic in these derivations is first to find the best-fit parameters assuming that only the density perturbations are present, and then to claim that there is no much room left for inclusion of gravitational waves. This looks like being satisfied with the statement that most of what is known about the human race can be described by a small
number of parameters which includes one leg of individuals, but with no need for another. And when you propose to the data analysts that it is better to treat the data under the assumption that humans have two legs, they reply that this would be one extra parameter, and the proposer should be penalized for that. Anyway, in our analysis, we include the (inevitable) gravitational waves from the very beginning.

The total $TE$ signal is the sum of g.w. and d.p. contributions. Even if this sum is positive in the interval $\ell \lesssim 70$, the effect of gravitational waves can (and expected to) be considerable. In this case, the amount of gravitational waves can be estimated through the analysis of all correlation functions together. However, if the total $TE$ signal is negative in this interval of $\ell$, there is little doubt that a significant g.w. component is present and is responsible for the negative signal, because the mean value of the $TE$ signal cannot be negative without gravitational waves (the issues of statistics are discussed below).

It is intriguing that the WMAP team [32] explicitly emphasizes the detection of a negative correlation (i.e. anticorrelation) at the multipoles near $\ell \approx 30$: “The detection of the $TE$ anticorrelation near $\ell \approx 30$ is a fundamental measurement of the physics of the formation of cosmological perturbations...”. The motivation for this statement is the continuing concern of CMB observers about the so-called ‘defect’ models of structure formation. These ‘causal’ models cannot produce any correlations, positive or negative, at $\ell \lesssim 100$. So, a detected correlation near $\ell \approx 30$ is an evidence against them. At the same time, the better detected $TE$ anticorrelation in the region of higher $\ell \approx 150$ could still be accommodated by the causal models, and therefore this more visible feature is not a direct argument against these models. However, the motivation for the WMAP team’s statement is not essential for the present discussion. Even if the available data are not sufficient to conclude with confidence that the excessive $TE$ anticorrelation at lower $\ell$’s has been actually detected, it seems that the WMAP’s published data (together with the published statement, quoted above) can serve at least as an indication that this is likely to be true. In the framework of the theory that we are discussing here, such an $TE$ anticorrelation is a natural and expected feature due to relic gravitational waves, but of course this needs to be thoroughly investigated.

The ensemble-averaged correlation functions do not answer all the questions. It is necessary to know what one would get with individual realizations of $a_{\ell m}^X$ caused by individual realizations of random coefficients $c_n, c_n^*$. The general unbiased estimator $D_{\ell}^{XX'}$ of $C_{\ell}^{XX'}$ is given by

$$D_{\ell}^{XX'} = \frac{1}{2(2\ell + 1)} \sum_{m=-\ell}^{\ell} \left( a_{\ell m}^X a_{\ell m}^{X'*} + a_{\ell m}^{X'} a_{\ell m}^X \right),$$

where $a_{\ell m}^X$ depend in a complicated, but calculable, manner on mode functions, astrophysical input,
and, in general, on all random coefficients $c_n, s_n$. (It was explicitly shown [48] that the estimator $D^T_{\ell}$ is not only unbiased, but also the best, i.e. the minimum-variance, estimator among quadratic estimators.) One can write, symbolically,

$$
\alpha_{\ell m}^X = \int_{-\infty}^{+\infty} d^3 n \sum_{s=1,2} \left[ f^X_{\ell m}(n, s) \hat{c}_n + f^{X*}_{\ell m}(n, s) \hat{s}_n^* \right].
$$

(85)

Some features of the averaged $XX'$ functions remain the same in any realization, i.e. for any choice of random coefficients $s_n, s_n^*$. One example is the absence of $BB$ correlations for density perturbations. In the case of density perturbations, the $BB$ correlation function vanishes not just on average, but in every realization. This happens because all the functions $f^B_{\ell m}(n, s)$ in Eq.(85) are zeros. Another example is the retention, in every realization, of positive sign of the auto-correlation functions $XX'$, with $X' = X$. Indeed, expression [85] may be arbitrarily complicated, but the estimator (84) for $X' = X$ is always positive, as it is the sum of strictly positive terms. In different realizations the estimates will be different, but they will always be positive. That is, in any realization, including the actually observed one, the quantity $D^X_{\ell}$ has the same sign as the quantity $C^XX_{\ell}$.

The important question is how often the $TE$ estimations (84) retain the same sign as the already calculated ensemble-averaged quantity $C^{TE}_{\ell}$. In other words, suppose a negative $TE$ is actually observed in the interval $\ell \lesssim 50$. Can it be a statistical fluke of density perturbations alone, rather than a signature of gravitational waves? In general, the answer is ‘yes’, but we will present arguments why in the problem under discussion the answer may be ‘no’, or ‘very likely no’.

To quantify the situation, it is instructive to start from two zero-mean Gaussian variables, say, $a^T$ and $a^E$:

$$
\langle a^T a^T \rangle = \sigma^2, \quad \langle a^E a^E \rangle = \sigma^2, \quad \langle a^T a^E \rangle = \rho \sigma^2, \quad 0 \leq \rho \leq 1.
$$

(86)

The probability density function for the product variable $a^T a^E$ involves the modified Bessel function $K_0$ and is known as an exact expression [50]. In the situation, like ours, where $\rho$ is close to 1, the probability of finding negative products $a^T a^E$ is small. A crude evaluation shows that, with probability 68% and higher, the values of $a^T a^E$ lie in the interval $(0.3 - 1.7)$ around the positive mean value $\langle a^T a^E \rangle$, whereas for $\rho = 1$ the probability of finding a negative $\langle a^T a^E \rangle$ is strictly zero. For $2\ell + 1$ degrees of freedom the scatter around the mean value is expected to be much narrower. Nevertheless, if $\rho \neq 1$, infrequent realizations with negative values of the product $a^T a^E$ are still possible.
We are probably quite fortunate in our concrete case of $TE$ cross-correlation functions. It is true that $a^{T\ell m}$ and $a^{E\ell m}$ contain a large number of independent random coefficients $s_{cn}$, multiplied by different deterministic functions $f^{T\ell m}(n, s)$, $f^{E\ell m}(n, s)$. However, in practice, these deterministic functions are such that the integrands in Eq. (85) are quite sharply peaked at $n \approx \ell$. Ideally, for every $\ell$, the random variables $a^{T\ell m}$, $a^{E\ell m}$ become proportional to one and the same linear combination of random coefficients with $|n| = n \approx \ell$. The square of this combination is always positive, so the sign of the estimator $D^{TE}_\ell$ would be the same as the sign of the mean value $C^{TE}_\ell$. This sign is determined by the known deterministic functions, not by statistics. Although this practical retention of the sign is not a strictly proven theorem, I think it is a very plausible conjecture.

If the above-mentioned conjecture is correct, the negative sign of any observed $TE$ correlation in the interval $\ell \lesssim 50$ is unlikely to be a statistical fluke of density perturbations alone, it should be a signature of presence of gravitational waves. One especially interesting interval of $TE$ searches is the interval between $\ell \approx 30$ and $\ell \approx 50$. In this interval of $\ell$, the total signal $C^{TE}_\ell$ should be negative, whereas the contribution from density perturbations is still positive (see Fig.10 for a realistic example). The total $TE$-signal is a factor 50, or so, larger than the expected $BB$-signal. All the logic described above suggests that if a negative $TE$ is detected in this interval of $\ell$’s it must have occurred due to relic gravitational waves.

**XIII. PROSPECTS OF THE CURRENT AND FORTHCOMING OBSERVATIONS**

It is difficult to predict the future, but it seems to me that the Planck mission \[51\], as well as the ground-based experiments, such as BICEP \[52, 53\], Clover \[54\], QUIET \[55\], have a very good chance of detecting relic gravitational waves through the $TE$ anticorrelation described above. A level of the expected $BB$ correlation is shown in Fig.9. This level is natural for the discussed theory, but may be a little bit optimistic, as it is shown for a relatively high primordial spectral index $n = 1.2$. Observations with the help of space and ground facilities, and the measurement of all relevant correlation functions, should bring positive results given the expected sensitivities of those observations. It seems to me that the detection of relic gravitational waves in CMB is the matter of a few coming years, rather than a few decades. The analysis of the latest WMAP data provides serious indications of the presence of relic gravitational waves in the CMB anisotropies, see \[56, 57\].
Acknowledgments

I am grateful to D. Baskaran and A. G. Polnarev for collaboration on the joint paper [17] that was extensively quoted in this presentation, and to S. Weinberg for helpful comments on this manuscript.

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[58] All inflationary quotations in the present article are taken from real publications. However, I do not see much sense in giving precise references. I am confronting here something like a culture, distributed over hundreds of publications, rather than a persistent confusion of a few authors. This situation was once qualified as “the controversy between Grishchuk and the rest of the community”.
