ADJUNCTION OF ROOTS, ALGEBRAIC $K$-THEORY AND CHROMATIC REDSHIFT

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Abstract. Given an $E_1$-ring $A$ and a class $a \in \pi_{mk}(A)$ satisfying a suitable hypothesis, we define a map of $E_1$-rings $A \to A(\sqrt[m]{a})$ realizing the adjunction of an $m$th root of $a$. We define a form of logarithmic THH for $E_1$-rings, and show that root adjunction is log-THH-étale for suitably tamely ramified extensions, which provides a formula for $\text{THH}(A(\sqrt[m]{a}))$ in terms of $\text{THH}$ and log-THH of $A$. If $A$ is connective, we prove that the induced map $\text{K}(A) \to \text{K}(A(\sqrt[m]{a}))$ in algebraic $K$-theory is the inclusion of a wedge summand. Using this, we obtain a new proof of the fact that Lubin-Tate spectra satisfy the redshift conjecture.

1. Introduction

Let $A$ be an $E_1$-algebra spectrum, and let $a \in \pi_{mk}(A)$ with $m > 0$ and even $k \geq 0$. In this paper, we define under a certain Hypothesis 4.1, an $E_1$-algebra extension $A \to A(\sqrt[m]{a})$ realizing the adjunction of an $m$th-root of $a$ in homotopy rings,

$$\pi_* A \to \pi_* (A(\sqrt[m]{a})) \cong \pi_* (A)[z]/(z^m - a),$$

and then study how the algebraic $K$-theory of $A(\sqrt[m]{a})$, or its topological Hochschild and cyclic homology, relates to that of $A$. Hypothesis 4.1 holds for example if $A$ is an $E_2$-ring for which $\pi_* A$ is concentrated in even degrees.

In general, the existence of a suitable root adjunction to a ring spectrum and its effect on such invariants is an intriguing question; for example it has been shown by Schwänzl, Vogt and Waldhausen [SVW99, Proposition 2], precisely by considering topological Hochschild homology, that it is not possible, in $E_\infty$-ring spectra, to adjoin a fourth root of unity $i$ to the sphere spectrum $S$ (in a sense made precise in loc. cit.). Nevertheless, Lawson [Law20] introduces a construction that allows, under some assumption, to adjoin roots of a homotopy degree zero unit in $E_\infty$ ring spectra.

For classes in positive homotopy degrees, examples exist and have shown to be relevant, in particular in studying redshift for algebraic $K$-theory. We have the Adams splitting of connective complex $K$-theory $ku$ completed at an odd prime $p$,

$$ku_p \simeq \bigvee_{1 \leq i < p-1} \Sigma^{2i} \ell_p,$$

and the extension $\ell_p \to ku_p$ can be interpreted, on homotopy rings, as the root adjunction

$$\pi_* \ell_p \cong \mathbb{Z}_p[v_1] \to \mathbb{Z}_p[u] \cong \pi_* ku_p,$$

and then study how the algebraic $K$-theory of $A(\sqrt[m]{a})$, or its topological Hochschild and cyclic homology, relates to that of $A$. Hypothesis 4.1 holds for example if $A$ is an $E_2$-ring for which $\pi_* A$ is concentrated in even degrees.

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were \( v_1 \mapsto u^{p-1} \), giving
\[
\pi_* ku_p \cong (\pi_* \ell_p)[u]/(u^{p-1} - v_1).
\]
Sagave showed in [Sag14, 4.15] that \( ku_p \) can be constructed as an extension of \( \ell_p \), establishing how \( \ell_p \rightarrow ku_p \) qualifies as a tamely ramified extension in \( E_{\infty} \)-rings. In [AR02, Aus10], Rognes and the first author had computed the algebraic \( K \)-theory of \( \ell_p \) and \( ku_p \) with coefficients in a Smith-Toda complex \( V(1) = S/(p, v_1) \), for \( p \geq 5 \).

Taking \( T(2) = V(1)[v_1^{p-1}] \), one has the formula
\[
T(2)_* K(ku_p) \cong (T(2)_* K(\ell_p))[b]/(b^{p-1} + v_2)
\]
relating the two computations, hinting at a chromatic shift (or redshift) of this tamely ramified root adjunction.

After our construction of root adjunction in \( E_1 \)-rings, we offer an investigation of how the obtained extension is reflected in algebraic \( K \)-theory. In particular, we have the following spectrum-level splitting of algebraic \( K \)-theory in the tamely ramified case, which applies to a wide array of examples.

**Theorem 1.1** (Theorem 5.8). Assume Hypothesis \( 4.4 \) with \( p \nmid m \) and \( |a| > 0 \). Furthermore, assume that \( A \) is \( p \)-local and connective. In this situation, the map in algebraic \( K \)-theory
\[
K(A) \rightarrow K(A(\sqrt[m]{a}))
\]
induced by the extension \( A \rightarrow A(\sqrt[m]{a}) \) is the inclusion of a wedge summand.

This is deduced from the corresponding result for topological cyclic homology, see Theorem 5.11. An analogous splitting result for algebraic \( K \)-theory in the non-connective case is provided in Corollary 5.11.

For an integer \( n > 0 \), we say that a spectrum \( E \) is **height** \( n \) if \( L_{T(n)} E \not\cong 0 \) and \( L_{T(m)} E \cong 0 \) for \( m > n \), where, \( T(n) \) denotes a height \( n \) telescope. We say an \( E_1 \)-ring \( A \) of height \( n \) **exhibits redshift** if \( K(A) \) is of height \( n + 1 \). Due to [LMMT20, Purity Theorem], \( K(A) \) is of height at most \( n + 1 \), so \( A \) will exhibit redshift if \( L_{T(n+1)} K(A) \not\cong 0 \). The following result is thus an immediate consequence of our splitting results, Theorem 5.8 and Corollary 5.11.

**Corollary 1.2.** Assume Hypothesis \( 4.4 \) with \( p \nmid m \) and \( |a| > 0 \). If \( A \) exhibits redshift, then so does \( A(\sqrt[m]{a}) \).

A key feature of the defined root adjunction is that \( A(\sqrt[m]{a}) \) is endowed with the structure of an \( E_1 \)-algebra in the symmetric monoidal category \( \text{Fun}(\mathbb{Z}/m, \text{Sp}) \) of \( m \)-graded spectra, which is reflected in an Adams’ type splitting of spectra
\[
A(\sqrt[m]{a}) \simeq \bigvee_{0 \leq k < m} \Sigma^k A.
\]
Such a grading on a spectrum, compatible with further additional algebraic structures, has already proven to be very useful: let us mention the computations of Hesselholt-Madsen [HM97] of the \( K \)-theory of truncated polynomial algebras, and of the second and third authors [BM22] of the \( K \)-theory of the free \( E_1 \)-algebras in degree 2 over finite fields. In the present paper, the grading corresponding to the root adjunction, and the induced splitting of \( \text{THH} \) as developed in [AMMN22, Appendix A], are essential ingredients in the proofs of our various splitting results.
We use the theory of logarithmic topological Hochschild homology for an in depth study of the THH of $A(\sqrt[m]{a})$. Hesselholt and Madsen $[\text{HM03}]$ introduced logarithmic topological Hochschild homology for studying the algebraic $K$-theory of complete discrete valuation rings in mixed characteristic, and proved a descent property of log THH in the case of tamely ramified extensions. Rognes $[\text{Rog09}]$ then initiated a study of logarithmic structures, logarithmic André-Quillen homology and of log THH in the context of $E_\infty$ ring spectra. With Sagave and Schlichtkrull $[\text{RSS15}, \text{RSS18}]$, they then established the existence of localization sequences for log THH and proved that it satisfies tamely ramified descent in the example of the extension $\ell_p \rightarrow ku_p$.

Here, we offer an alternative definition of log THH that applies to a more general class of ring spectra. More precisely, we define log THH for a given $E_1$-ring spectrum $A$ and a class $a \in \pi_{mk}(A)$ satisfying Hypothesis 4.4, associated to the pre-log structure given by the monoid generated, under multiplication, by $a \in \pi_*(A)$ (see Definition 6.6); it is denoted $\text{THH}(A | a)$. For instance, this applies to the Morava $K$-theory spectrum $k(n)$ for $v_n \in \pi_{k}(n)$. We prove the following form of tamely ramified descent (see also Theorem 6.33):

**Theorem 1.3** (Theorem 6.33). If $A$ is $p$-local and $p \nmid m$, there is an equivalence of spectra:

$$\text{THH}(A(\sqrt[m]{a})) \simeq \text{THH}(A) \vee \bigvee_{0<i<m} \Sigma^i \text{THH}(A | a).$$

We also prove the existence of a localization cofibre sequence

$$\text{THH}(A) \rightarrow \text{THH}(A | a) \rightarrow \Sigma \text{THH}(A/a),$$

see Theorem 6.28. This is an analogue in the present setting of the localization sequences constructed in $[\text{RSS15}]$, but note that our definition only applies to the case of a pre-log structure given by a monoid on a single generator. We would like to point out also the recent preprint of Binda, Lundemo, Park and Østvær $[\text{BLPO22}]$, where a version of logarithmic Hochschild homology for simplicial commutative rings is constructed.

We now mention examples of application of the above results.

**Topological $K$-theory.** We prove in Theorems 4.10 and 7.6 that, at an odd prime $p$, there are equivalences of $E_1$-ring spectra

$$ku_p \simeq \ell_p(\sqrt[p]{v_1}) \quad \text{and} \quad ko_p \simeq \ell_p(\sqrt[p]{v_1}).$$

These equivalences upgrade the splitting result of Adams into a $\mathbb{Z}/(p-1)$-graded, respectively $\mathbb{Z}/(\frac{p^2-1}{2})$-graded $E_1$-ring structure. We prove that when $p = 1$ in $\mathbb{Z}/m$, the splitting in Theorem 1.1 can be improved to a more refined splitting of $K(A(\sqrt[m]{a}))$. In the case of $ku_p \simeq \ell_p(\sqrt[p]{v_1})$, this more refined splitting reads as

$$(1.4) \quad K(ku_p) \simeq \bigvee_{0 \leq i \leq p-1} K(ku_p)_i.$$

Here $K(ku_p)_0 \simeq K(\ell_p)$, and for $p > 3$, and we can compute the $V(1)$-homotopy groups of each of the $i$-th graded piece $V(1), K(ku_p)_i$, using first author’s computation of $V(1), K(ku_p)$. 

Using this refined splitting, the second author makes a simplified computation of $T(2), K(ku)$ in [Bay23]. We also obtain complete descriptions of $V(1), K(ko_p)$ and $T(2), K(ko_p)$, see Theorem 4.10.

We note that the splitting (1.4) can be considered as a version of Adams’ splitting for the cohomology theory represented by $K(ku)$, with classes corresponding to 2-categorical complex vector bundles, as developed in [BDRR11].

**Johnson-Wilson spectra and Morava $E$-theory.** Let $n \geq 1$ be an integer, and let $E(n)$ and $E_n$ be the Johnson-Wilson and Morava $E$-theory spectra. Let $\widehat{E}(n)$ be the $K(n)$-localization of $E(n)$. These spectra have coefficient rings given as

$$\pi_*E(n) \cong \mathbb{Z}_p[v_1, ..., v_{n-1}, v_n^{\pm1}], \quad \pi_*E_n \cong W(\mathbb{F}_p)[[u_1, ..., u_{n-1}][u^{\pm1}]$$

and $\pi_*\widehat{E}(n) \cong \pi_*E(n)_{\widehat{\mathbb{Q}}}$, where $|u_i| = 0$, $|u| = -2$, and $I_n = (p, v_1, ..., v_n)$. The Galois group $Gal = \text{Gal}(\mathbb{F}_p^w/\mathbb{F}_p)$ acts on $E_n$, and let $E_n^{hGal}$ be the homotopy fixed point spectrum with coefficients $\pi_*E_n^{hGal} \cong \mathbb{Z}_p[[u_1, ..., u_{n-1}][u^{\pm1}]$.

We prove in Theorem 9.6 that there are equivalences of $E_1$-rings

$$E_n \cong S_{W(\mathbb{F}_p^n)} \wedge_{\mathbb{Z}_p} \widehat{E}(n)(\sqrt[n]{v^n}), \quad \text{and} \quad E_n^{hGal} \cong \widehat{E}(n)(\sqrt[n]{v^n}).$$

This promotes the $E_1$-ring structure on Morava $E$-theories to a non-trivial $\mathbb{Z}_p/\langle p \rangle$-graded $E_1$-ring structure.

Using this description of $E_n$ and the log THH étaleness of root adjunction, we show in Theorem 9.15 that the canonical maps

$$\text{THH}(\widehat{E}(n)) \wedge_{\widehat{E}(n)} E_n \xrightarrow{\approx} \text{THH}(E_n),$$

is an equivalence after $p$-completion. The relationship between such equivalences and the Galois descent question for THH are studied in [Mat17].

Applying Corollary 5.11 to these root adjunctions of non-connective spectra, we deduce the following result.

**Theorem 1.6 (Theorem 9.9).** The canonical maps:

$$K(E(n)) \to K(E_n^{hGal})$$

$$K(S_{W(\mathbb{F}_p^n)} \wedge E(n)) \to K(E_n)$$

are inclusions of wedge summands after $T(n + 1)$-localization.

**Lubin-Tate spectra.** We can also apply our results to Lubin-Tate spectra that can be constructed, in several steps, from the truncated Brown-Peterson spectra $BP_\langle n \rangle$, with coefficients $\pi_*BP_\langle n \rangle \cong \mathbb{Z}_p[v_1, ..., v_n]$. In more precise terms, we consider an $E_3 MU[\sigma_{2(p^n-1)}]$-algebra form of $BP_\langle n \rangle$ as constructed by Hahn and Wilson [HW22, Remark 2.1.2]. In this situation, we can construct $BP_\langle n \rangle(\sqrt[n]{v^n})$ as an $E_3 MU[\sigma_2]$-algebra (Remark 1.11).

Let be $k$ a perfect field of characteristic $p$, and let $S_{W(k)}$ denote the spherical Witt vectors spectrum. We prove in Proposition 8.6 that the $MU[\sigma_2]$-orientation above provides a formal group law $\Gamma$ of height $n$ over $k$, and that there is an equivalence of $E_3$-rings

$$L_{K(n)}(S_{W(k)} \wedge BP_\langle n \rangle)(\sqrt[n]{v^n}) \cong E_{(k, \Gamma)}.$$
where $E_{(k,\Gamma)}$ denotes the Lubin-Tate spectrum corresponding to $\Gamma$. Due to [LMMT20, Purity Theorem], we have an equivalence

$$L_{T(n+1)}K(S_{W(k)} \wedge BP\langle n \rangle) \simeq L_{T(n+1)}K(E_{(k,\Gamma)}).$$

By [HW22], $BP\langle n \rangle$ satisfies the redshift conjecture; following an argument suggested to us by Hahn, we show that $S_{W(k)} \wedge BP\langle n \rangle$ also satisfies the red-shift conjecture, c.f. Proposition 8.2. By Corollary 1.2, we deduce that $E_{(k,\Gamma)}$ satisfies the redshift conjecture. Indeed, we deduce from this (see Theorem 8.9) a new proof of Yuan’s result [Yua21] that all Lubin-Tate spectra satisfy the redshift conjecture. We also obtain the following from Corollary 5.11

**Theorem 1.7** (Theorem 8.8). The induced map

$$L_{T(n+1)}K(S_{W(k)} \wedge BP\langle n \rangle) \rightarrow L_{T(n+1)}K(E_{(k,\Gamma)})$$

is the inclusion of a non-trivial wedge summand.

We expect the above result to be relevant also for explicit computations. For example, in [AKAC+22], the authors compute $V(2)_* TC(BP\langle 2 \rangle)$ for $p \geq 7$ which, provides an explicit description of $T(3)_* K(BP\langle 2 \rangle)$. Through the inclusion above, we deduce that $T(3)_* K(BP\langle 2 \rangle)$ maps isomorphically to a summand of $T(3)_* K(E_{(\mathbb{F}_p,\Gamma)})$ for a height 2 formal group law $\Gamma$ over $\mathbb{F}_p$. To our knowledge, this is the first explicit, quantitative result on the algebraic $K$-theory groups of Lubin-Tate spectra for height larger than 1.

Note that it is not known if the $E_3$ $MU$-algebra forms of $BP\langle n \rangle$ constructed in [HW22] map into the Morava $E$-theories mentioned in the preceding sub-section.

**Remark 1.8.** In [BSY22, Theorem G], the authors construct an $E_{\infty}$-map $MU[\sigma_2] \rightarrow E_{(\mathbb{F}_p,\Gamma)}$ for the unique height $n$ formal group law $\Gamma$ on $\mathbb{F}_p$ solving an open question on the existence of orientations on Lubin-Tate spectra. Our constructions provide a similar orientation for a “smaller” form of Lubin-Tate spectra. Namely, we obtain that $E_{(\mathbb{F}_p,\Gamma)}$ is an $E_3$ $MU[\sigma_2]$-algebra where $\sigma_2$ acts through $u^{-1} \in \pi_* E_{(\mathbb{F}_p,\Gamma)}$; see Example 8.7. Moreover, there is a grading on $E_{(\mathbb{F}_p,\Gamma)}$ that respects this structure. To be precise, $E_{(\mathbb{F}_p,\Gamma)}$ is an $E_3$ $MU[\sigma_2]$-algebra in the $\infty$-category of $\mathbb{Z}/(p^n-1)$-graded spectra where $u^{-1} \in \pi_* E_{(\mathbb{F}_p,\Gamma)}$ is of weight 1.

**Remark 1.9.** The above construction of Lubin-Tate spectra by root adjunction is used in an essential way in the construction of a counter-example to the telescope conjecture in forthcoming work by Burklund, Hahn, Levy and Schlank.

**Morava K-theory.** In Section 9.3, we construct two-periodic Morava $K$-theories from Morava $K$-theories through root adjunction. By Corollary 1.2, we deduce that two-periodic Morava $K$-theories satisfy the redshift conjecture if the redshift conjecture holds for Morava $K$-theories (Corollary 9.11).

For $p > 3$, the $V(1)$-homotopy of $K(k(1))$ is computed by the first author and Rognes in [AR12] where it is also shown that $k(1)$ satisfies the redshift conjecture. From this, we deduce that the two-periodic first Morava $K$-theory $ku/p$ also satisfies the redshift conjecture (Corollary 9.12). Moreover, through the interpretation of $ku/p$ as $k(1)(n,\sqrt{\nu_1})$, the second author makes the first computation of $T(2)_* K(ku/p)$ in [Bay23].
Outline. We begin with a quick introduction of graded objects in Section 2. In Section 3, we construct a family of graded $E_2$ “polynomial” algebras and establish their even cell decompositions. In Section 4, we provide our central construction for root adjunctions (Construction 4.6) and prove our first splitting result on the THH of ring spectra obtained via a root adjunction. In Section 5 we prove Theorem 1.6. Section 6 is devoted to studying the variant of log THH we set forth, as well as the logarithmic THH-étaleness of root adjunctions. Section 7 contains our results on the algebraic $K$-theory of real and complex topological $K$-theories. We apply our results to Lubin-Tate spectra in Section 8. In Section 9, we study the THH and the algebraic $K$-theory of Morava $E$-theories.

Notation 1.10. (1) We work freely in the setting of $\infty$-categories and higher algebra from [Lur09, Lur17].

(2) For an $E_2$-algebra $R$ in a symmetric monoidal $\infty$-category, when we say $T$ is an $R$-algebra (or an $E_1$ $R$-algebra), we mean that it is an $E_1$-algebra in right $R$-modules. If we mean an $E_1$-algebra in left $R$-modules, we call this a left $E_1$ $R$-algebra. If $R$ is $E_\infty$, we do not need to denote the distinction.

(3) When we say $E_n$-ring, we mean an $E_n$-algebra in the $\infty$-category of spectra $Sp$.

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2. Recollections on graded objects

Let $m \geq 0$ be an integer and let $\mathbb{Z}/m$ denote the discrete $\infty$-groupoid whose objects are the elements of the set of integers modulo $m$. For a presentably symmetric monoidal $\infty$-category $\mathcal{V}$, we define the $\infty$-category of $m$-graded objects in $\mathcal{V}$ to be the functor category $\text{Fun}(\mathbb{Z}/m, \mathcal{V})$. For a functor $F$ in $\text{Fun}(\mathbb{Z}/m, \mathcal{V})$, we denote $F(i)$ by $F_i$ for every $i \in \mathbb{Z}/m$. Since $\mathbb{Z}/m$ is discrete, we have:

$$\text{Fun}(\mathbb{Z}/m, \mathcal{V}) \simeq \prod_{i \in \mathbb{Z}/m} \mathcal{V}.$$ 

For $m = 0$, this is given by $\text{Fun}(\mathbb{Z}, \mathcal{V})$ where $\mathbb{Z}$ is the corresponding discrete $\infty$-groupoid. In this case, we omit $m$ and we call $\text{Fun}(\mathbb{Z}, \mathcal{V})$ the $\infty$-category of graded objects in $\mathcal{V}$. We are mainly interested in the case $\mathcal{V} = Sp$. We call an object of
Fun(\(\mathbb{Z}/m, \text{Sp}\)) an \textit{m-graded spectrum}; for \(m = 0\), we drop \(m\) and call it a \textit{graded spectrum}.

Using the symmetric monoidal structure on \(\mathbb{Z}/m\) given by addition, we equip Fun(\(\mathbb{Z}/m, \text{Sp}\)) with the Day convolution closed symmetric monoidal structure \([\text{Gla} 16]\). Since \(\mathbb{Z}/m\) is discrete, this boils down to the following.

\[
(F \otimes_{\text{Day}} G)_k = \coprod_{i + j = k \text{ in } \mathbb{Z}/m} F_i \otimes G_j
\]

2.1. \textbf{Algebras in graded spectra.} We are interested in \(E_n\)-algebras in the \(\infty\)-category of \(m\)-graded spectra and the algebras over these \(E_n\)-algebras.

\textbf{Definition 2.1.} An \textit{m-graded} \(E_n\)-\textit{ring} \(A\) is an \(E_n\)-algebra in Fun(\(\mathbb{Z}/m, \text{Sp}\)). For \(k < n\), an \textit{m-graded} \(E_k\) \textit{A-algebra} is an \(E_k\) \textit{A-algebra} in Fun(\(\mathbb{Z}/m, \text{Sp}\)). Similarly, an \textit{m-graded} (left) right \textit{A-module} is a (left) right \textit{A-module} in Fun(\(\mathbb{Z}/m, \text{Sp}\)).

\textbf{Remark 2.2.} Note that the notion of an \textit{m-graded} \(E_k\)-algebra in \(\text{Sp}\) is in general different than the notion of an \textit{m-graded} object in the \(\infty\)-category of \(E_k\)-algebras in \(\text{Sp}\).

2.2. \textbf{Manipulations on graded objects.} For a symmetric monoidal functor \(\mathbb{Z}/m \rightarrow \mathbb{Z}/m'\), there is an induced adjunction between Fun(\(\mathbb{Z}/m, \text{Sp}\)) and Fun(\(\mathbb{Z}/m', \text{Sp}\)) where the left adjoint is symmetric monoidal and given by left Kan extension \([\text{Nik} 16, \text{Corollary 3.8}]\). The right adjoint is given by restriction. This provides the following adjunctions which allow us to move between various gradings. Let \(n > 0\) and \(s \geq 0\) be integers.

- We let \(D^n_{sn} \dashv Q\) denote the adjunction induced by the quotient map \(\mathbb{Z}/sn \rightarrow \mathbb{Z}/n\) sending 1 to 1.
- We often use \(D^n_{sn}\) for \(s = 0\) which allows us to obtain an \(n\)-graded object out of a graded object \(X\) in \(\text{Sp}\). We let \(D^n\) denote \(D^n_0: \text{Fun}(\mathbb{Z}, \text{Sp}) \rightarrow \text{Fun}(\mathbb{Z}/n, \text{Sp})\) and we have
  \[
  D^n(X)_i \simeq \bigvee_{j \in \mathbb{Z}/n | j \equiv i \mod n} X_j.
  \]
- For \(n = 1\), we denote \(D^1_s\) by \(D\). In this case,
  \[
  D: \text{Fun}(\mathbb{Z}/s, \text{Sp}) \rightarrow \text{Sp}
  \]
  is given by \(D(X) \simeq \bigvee_{j \in \mathbb{Z}/s} X_j\), i.e. left Kan extension along \(\mathbb{Z}/s \rightarrow 0\). For an \(s\)-graded (spectrum) \(E_n\)-\textit{ring} \(X\), we call \(D(X)\) the \textbf{underlying (spectrum)} \(E_n\)-\textit{ring} of \(X\). We often omit \(D\) in our notation.
- For \(s \in \mathbb{Z}\), let \(L_s \dashv R_s\) denote the adjunction on Fun(\(\mathbb{Z}, \text{Sp}\)) induced by the map \(\mathbb{Z} \xrightarrow{s} \mathbb{Z}\) given by multiplication by \(s\). For a graded spectrum \(X\), we have
  \[
  L_s(X)_i \simeq X_i
  \]
  for every \(i\) and \(L_s(X)_j \simeq 0\) whenever \(s \nmid j\).
- Let \(F \dashv G\) denote the adjunction induced by the trivial map \(0 \rightarrow \mathbb{Z}/m\). We have \(G(X) = X_0\). For an \textit{m-graded} \(E_n\)-\textit{ring} \(A\), \(F(G(A))\) is given by \(A_0\) in weight 0 and it is trivial on the other degrees. Therefore, we sometimes abuse
notation and denote the $m$-graded $E_n$-ring $F(G(A))$ by $A_0$. The counit of this adjunction provides a map

$$A_0 \to A$$

of $m$-graded $E_n$-rings. If $A_i \simeq 0$ for $i \neq 0$, then this map is an equivalence and we say that $A$ is **concentrated in weight zero**. The following lemma states that in this situation, there is an equivalence of $E_n$-rings between the underlying $E_n$-ring of $A$ and the weight zero piece $G(A) = A_0$ of $A$. Therefore, we often do not distinguish between $A$, $G(A) = A_0$ and $D(A)$ in our notation when $A$ is concentrated in weight zero.

**Lemma 2.3.** Let $A$ be an $m$-graded $E_n$-ring concentrated in weight zero. There is an equivalence of $E_n$-rings

$$A_0 \simeq D(A)$$

where $A_0$ denotes $G(A)$. In particular, we have $F(D(A)) \simeq A$ as $m$-graded $E_n$-rings.

**Proof.** Since $A$ is concentrated in weight zero, we have $D(A) \simeq DFG(A)$. As $D \circ F$ Kan extends through the composite $0 \to \mathbb{Z}/m \to 0$, it is equivalent to the identity functor. We obtain that $DFG(A) \simeq G(A) \simeq A_0$.

For the second statement, note that $F(D(A)) \simeq F(G(A))$ due to the first statement. Since $A$ is concentrated in weight 0, we have $F(G(A)) \simeq A$. \qed

### 3. A family of $E_2$ polynomial rings in graded spectra

In this section, we introduce the construction of a family of $E_2$-algebras in graded spectra. These have appeared in the work of Hahn and Wilson in [HW22] and are also studied in greater depth in [DM22]. These will be central to our constructions.

For every $r, w \in \mathbb{Z}$, one constructs a graded $E_2$-ring $\mathbb{S}[\sigma_{2r}]$ which may be thought of as a “polynomial” algebra with a generator in homotopical degree $2r$ and grading weight $w$. However, these are not polynomial algebras in the precise sense, as they are only demonstrated to admit $E_2$ structures. By mapping these $E_2$-rings into each other, we will be able to construct $E_2$-ring extensions.

#### 3.1. Shearing preliminaries

The main mechanism underlying this construction is that of shearing, which we now briefly review. It has appeared in [Rak20], and is also studied in [DM22]. In what follows, $Gr(Sp)$ denotes $Fun(\mathbb{Z}, Sp)$, i.e. the $\infty$-category of graded spectra.

**Proposition 3.1.** There exists an endofunctor on graded spectra

$$sh : Gr(Sp) \to Gr(Sp)$$

given by

$$sh(M)_i := M_i[-2i]$$

with the following properties:

- $sh$ is an equivalence, with inverse given by $sh^{-1}(M_i) = M_i[2i]$
- $sh$ admits an $E_2$-monoidal structure, with respect to the Day convolution product on $Gr(Sp)$. 
Proof. This appears in the \( \mathbb{Z} \)-linear setting in [Rak20] and is also studied in [DM22]. However, for the sake of completeness, we sketch the basic ideas underlying the construction. In [Lur15], Lurie constructs an \( E_2 \)-monoidal map of spaces
\[
\phi : \mathbb{Z} \to \text{Pic}(\text{Sp})
\]
sending \( n \mapsto \mathbb{S}^{-2n} \). We now define \( \text{sh} \) as the functor, obtained by adjunction, from the assignment
\[
\mathbb{Z} \times \text{Gr}(\text{Sp}) \to \text{Sp}
\]
given by the composition
\[
\mathbb{Z} \times \text{Gr}(\text{Sp}) \xrightarrow{\phi \times ev} \text{Pic}(\text{Sp}) \times \text{Sp} 
\]
Here, the first map sends \( (n, M) \mapsto (\phi(n), M_n) \). The fact that this latter composition is \( E_2 \) follows from the fact that \( \phi \) is itself \( E_2 \). This further implies that \( \text{sh} \) is itself \( E_2 \)-monoidal. To see that this is an equivalence, one displays, as in [Rak20], an inverse in the same way by precomposing \( \phi \) with the map \( \mathbb{Z} \leftarrow \mathbb{Z} \).

\[\square\]

Variant 3.2. One can precompose the map \( \phi : \mathbb{Z} \to \text{Pic}(\text{Sp}) \) with the map \( \cdot (k) : \mathbb{Z} \to \mathbb{Z} \). We denote the composition by
\[
\phi^k : \mathbb{Z} \to \text{Pic}(\text{Sp})
\]
As in the above, we use this to define an endofunctor
\[
\text{sh}^k : \text{Gr}(\text{Sp}) \to \text{Gr}(\text{Sp}).
\]
This acquires the same formal properties as above, e.g it will be an \( E_2 \)-monoidal autoequivalence on \( \text{Gr}(\text{Sp}) \). Furthermore, one has the description \( (\text{sh}^k M)_i \simeq M_i[-2ki] \).

3.2. Sheared polynomial algebras. Recall that there exists an \( E_\infty \) algebra \( S[t] \in \text{Gr}(\text{Sp}) \), which gives a graded enhancement of the “flat” polynomial algebra. One can obtain this, for example, by observing that the restriction map from filtered spectra to graded spectra
\[
\text{Res} : \text{Fil}(\text{Sp}) \to \text{Gr}(\text{Sp})
\]
is lax symmetric monoidal. In more detail, this will be the restriction
\[
\text{Fil}(\text{Sp}) = \text{Fun}((\mathbb{Z}, \leq), \text{Sp}) \to \text{Fun}(\mathbb{Z}^{ds}, \text{Sp}) = \text{Gr}(\text{Sp})
\]
along \( \mathbb{Z} \leftarrow (\mathbb{Z}, \leq) \) so that in particular we forget the structure maps of the filtration, cf [Lur15]. We remind the reader that this is different from the associated graded functor. One then sets \( S[t] := \text{Res}(1) \), where 1 denotes the unit of the symmetric monoidal structure on \( \text{Fil}(\text{Sp}) \). Thus, \( S[t] \) (which is given by \( S \) in nonpositive weights and 0 in positive weights) acquires the structure of an \( E_\infty \)-algebra in graded spectra.

Construction 3.3. As described in Proposition 3.1 there exists an \( E_2 \)-monoidal autoequivalence \( \text{sh} \). We set
\[
S[\sigma_2] := \text{sh}(S[t]),
\]
and more generally for \( k > 0 \),
\[
S[\sigma_{2k}] := \text{sh}^k(S[t]);
\]
that is, one applies \( \text{sh}^k \) to \( S[t] \) to obtain a family of \( E_2 \)-algebras in graded spectra. For \( k=0 \), we set \( S[\sigma_0] := S[t] \). It follows by inspection that the underlying graded \( E_1 \)-ring
of $S[\sigma_k]$ is the free graded $E_1$-ring on $S^{2k}(-1)$ where $S^{2k}(-1)$ is $S^{2k}$ concentrated in weight $-1$.

Remark 3.4. For $w \in \mathbb{Z}$ and even $k \geq 0$, we apply the functor $L_{-w}$ which left Kan extends along the multiplication map

$\cdot(-w) : \mathbb{Z} \to \mathbb{Z}$

to obtain weight sifted variants of $S[\sigma_k]$. We often omit $L_{-w}$ when the weight of $\sigma_k$ is clear from the context but when we wish to be explicit, we write $S[\sigma_{k,w}]$ for the graded $E_2$-ring $L_{-w}S[\sigma_k]$ where $\sigma_{k,w}$ is in weight $w$. As before, the underlying graded $E_1$-ring of $S[\sigma_{k,w}]$ is the free graded $E_1$-ring $F E_1(S^k(w))$.

To adjoin roots, we often start with $S[\sigma_{mk}]$ and $S[\sigma_k]$ with $\sigma_{mk}$ and $\sigma_k$ in weights $m$ and 1 respectively where $m > 0$ and $k > 0$ is even.

**Proposition 3.5.** In the situation above, there exists a map of graded $E_2$-rings

$S[\sigma_{mk}] \to S[\sigma_k]$ that carries $\sigma_{mk}$ to $\sigma_k^m$ in homotopy. This provides a map of $m$-graded $E_2$-rings

$D^m(S[\sigma_{mk}]) \to D^m(S[\sigma_k])$

where $\sigma_k \in \pi_1^*D^m(S[\sigma_k])$ is of weight 1 and $D^m(S[\sigma_{mk}])$ is concentrated in weight 0.

Furthermore, we have $D^m(S[\sigma_{mk}]) \simeq F(D(S[\sigma_{mk}]))$ as $m$-graded $E_2$-rings; here $F$ Kan extends through $0 \to \mathbb{Z}/m$.

**Proof.** We first remind the reader that there is an identification

$S[\sigma_{mk,-1}] := \text{sh}^m(S[t]) \simeq R_m(S[\sigma_{k,-1}])$

of graded $E_2$-rings. This follows from the very definition the $k$th shearing functor $\text{sh}^k$, in particular that it sends $S[t]$ to the negative weight part of the graded spectrum given by the map

$\phi^k : \mathbb{Z} \times^k \mathbb{Z} \to \text{Pic}(Sp)$.

Thus we may identify $S[\sigma_{mk,-1}]$ with $R_mS[\sigma_{k,-1}] \simeq R_m \text{sh}^k(S[t])$, negative weight part of the graded spectrum given by the map

$\phi^{mk} : \mathbb{Z} \to \text{Pic}(Sp)$.

Therefore, the counit of the adjunction $L_m \dashv R_m$ provides a map of graded $E_2$-rings

$S[\sigma_{mk,-m}] \to S[\sigma_{k,-1}]$.

Applying the functor $D^m$ to this map, we obtain the map of graded $E_2$-rings $S[\sigma_{mk}] \to S[\sigma_k]$ claimed in the proposition.

The functor $D^m$ gives the desired map

$D^m(S[\sigma_{mk}]) \to D^m(S[\sigma_k])$

of $m$-graded $E_2$-algebras. Note that $\sigma_{mk} \in \pi_1^*D^m(S[\sigma_{mk}])$ is of weight 0, which ensures that $D^m(S[\sigma_{mk}])$ is concentrated in weight 0.

To see the last statement, we have

(3.6) $F D D^m(S[\sigma_{mk}]) \simeq D^m(S[\sigma_{mk}])$
due to Lemma 2.3 since \( D^m(\mathbb{S}[\sigma_{mk}]) \) is concentrated in weight 0. Furthermore, \( DD^m \) is Kan extension through the composite \( \mathbb{Z} \rightarrow \mathbb{Z}/m \rightarrow 0 \) which is the same as the Kan extending through \( \mathbb{Z} \rightarrow 0 \). Therefore, \( DD^m(\mathbb{S}[\sigma_{mk}]) \simeq D(\mathbb{S}[\sigma_{mk}]) \). This, together with (3.6) provides the desired equivalence \( D^m(\mathbb{S}[\sigma_{mk}]) \simeq F(D(\mathbb{S}[\sigma_{mk}])) \). \( \square \)

We often omit the functor \( D^m \) in our notation and denote the map of \( m \)-graded \( E_2 \)-rings \( D^m(\mathbb{S}[\sigma_{mk}]) \rightarrow D^m(\mathbb{S}[\sigma_k]) \) as \( \mathbb{S}[\sigma_{mk}] \rightarrow \mathbb{S}[\sigma_k] \).

**Remark 3.7.** To adjoin a root to a degree 0 class, we need the \( k = 0 \) case of the proposition above. In other words, we need an analogous \( E_2 \)-map \( \mathbb{S}[\sigma_{mk}] \rightarrow \mathbb{S}[\sigma_k] \) for \( k = 0 \). For this, we start with the graded \( E_2 \)-map \( \mathbb{S}[\sigma_{m2}] \rightarrow \mathbb{S}[\sigma_2] \) and apply the functor \( \text{sh} \). This procedure provides a graded \( E_2 \)-map \( \mathbb{S}[\sigma_{0,m}] \rightarrow \mathbb{S}[\sigma_{0,1}] \) that carries \( \sigma_{0,m} \) to \( \sigma_{0,1}^m \) as desired.

3.3. **Cell structures on sheared polynomial algebras.** A key technical result for us will be the even cell decomposition of \( \mathbb{S}[\sigma_k] \) as an \( E_2 \)-algebra. As we will see in the remainder of this section, this is what will allow for us to define \( E_2 \)-algebra maps to a given \( E_2 \)-algebra \( A \), along which we will then adjoin roots.

**Remark 3.8.** In the second arxiv version of [HW22], Hahn and Wilson also construct \( E_2 \) even cell decompositions on the free \( E_2 \)-algebra \( \mathbb{S}[\sigma_k] \) using a Koszul duality argument for even \( k \geq 0 \). However, they removed this result in the later versions of their paper since they found a simpler argument for their redshift results that avoid the use of these even cell decompositions. Since this does not appear in the published version of [HW22], we give a proof of the \( E_2 \) even cell decompositions on \( \mathbb{S}[\sigma_k] \) that we use. We would like to note that our methods are different than the ones used in [HW22, Arxiv version 2].

Before doing this, we make precise what exactly we mean by even cell decomposition. The following notions are heavily inspired by Section 6.3 of [GKRW18].

**Definition 3.9.** Let \( f : S \rightarrow R \in \text{Alg}_{E_2}(\text{Sp}^Z) \) be a map of \( E_2 \)-algebras in graded spectra. We say \( f \) has a **filtered cellular decomposition** if there exists a tower in \( \text{Alg}_{E_2}(\text{Fil}(\text{Fun}(\mathbb{Z}, \text{Sp}))) \) for which

\[
S = \text{sk}_{-1}(f) \rightarrow \text{sk}_0(f) \rightarrow \text{sk}_1(f) \rightarrow \cdots \rightarrow \text{colim}_i \text{sk}_i(f) =: \text{sk}(f) \simeq R
\]

such that each \( \text{sk}_i(f) \) is obtained from \( \text{sk}_{i-1}(f) \) via the following pushout diagram:

\[
\begin{array}{ccc}
\text{Free}_{E_2}(\bigsqcup_{\alpha \in I_{n_i}} \partial D^{g_{n_i,m_i}}[i - 1]) & \longrightarrow & \text{sk}_{i-1}(f) \\
\downarrow & & \downarrow \\
\text{Free}_{E_2}(\bigsqcup_{\alpha \in I_{n_i}} D^{g_{n_i,m_i}}[i]) & \longrightarrow & \text{sk}_i(f).
\end{array}
\]

The notation \( X[n] \) in the above means that the object \( X \) is placed in filtering degree \( n \).

In particular, in each degree \( i \) of the tower we are adding cells in increasing dimension \( n_i \). Thus if \( i \leq j \), then \( n_i \leq n_j \), and \( I_{n_i} \) refers to the set of \( n_i \) cells of \( R \). It may be the case, that \( n_i = i \), but we do not require this for the sake of flexibility of the definition, which is a point of departure from the notion in [GKRW18]. If \( f : 1 \rightarrow R \) is the map from the unit, we call this a filtered cellular decomposition of \( R \).
Definition 3.10. A map $f: S \to R$ of graded $E_2$-rings admits a cell decomposition if it is the colimit of a tower

$$S = sk_d(f) \to sk_{d-1}(f) \to \cdots \to \colim sk_{-1}(f) \simeq R$$

in graded $E_2$-rings where each stage is obtained from the previous via a cell attachment in graded $E_2$-rings. In particular, if $f$ admits a filtered cellular decomposition, taking levelwise colimits provides a cellular decomposition of $f$.

Our first step is to establish the decomposition for $\mathbb{S}[t]$.

Proposition 3.11. As an $E_2$-algebra in graded spectra, $\mathbb{S}[t]$ admits a (filtered) cellular decomposition with cells in even degrees.

Proof. The two key inputs for our argument are Theorem 11.21 and Theorem 13.7 of [GKRW18]. The former result applied to the map $f : 0 \to I$ of non-unital $E_2$-algebras in graded spectra (where $I$ is the augmentation ideal of the map $\mathbb{S}[t] \to \mathbb{S}$) says that there exists a relative CW decomposition

$$0 \to \colim sk_n(f) \simeq I.$$

Moreover, the proof of this fact in loc. cit. constructs a minimal cell structure, one that has the smallest possible number of cells in a given bidegree (the extra degree here arises since we are working in graded spectra). In particular the colimit they construct, $\colim sk_n(f)$, will have cells precisely in bidegree $b^{E_2}_{g,d}(\mathbb{S}[t]) := \dim_k H^{E_2}_{g,d}(\mathbb{S}[t], S, k) \in \mathbb{N} \cup \{\infty\}$, where $H^O_{g,d}(R; k)$ is the $O$-homology of an $O$-algebra $R$, with coefficients in the ring $k$; we will in fact set $k = \mathbb{Z}$.

We sketch the argument given there applied to our particular case for the sake of completeness. One proceeds by inductively constructing a factorization

$$0 = sk_{-1} \xrightarrow{h_0} \cdots \xrightarrow{h_\epsilon} sk_\epsilon \xrightarrow{f_\epsilon} I,$$

in the $\infty$-category of (increasingly) filtered objects of Fun($\mathbb{Z}, \mathbb{S}$). Here $h_\epsilon : sk_{\epsilon-1} \to sk_\epsilon$ comes with the structure of a (filtered) CW attachment of dimension $\epsilon$, where $sk_\epsilon$ denotes the $\epsilon$th skeleton equipped with the skeletal filtration leading up to that degree. Taking the colimit along $\epsilon$ gives an induced map $f_\infty : \colim sk_\epsilon(f) \to I$, which is an equivalence.

For the inductive step in their argument, they show that the Hurewicz map

$$\pi_{*,\epsilon}(\mathbb{S}[t], sk_{\epsilon-1}) \to H^{E_2}_{*,\epsilon}(\mathbb{S}[t], sk_{\epsilon-1}; k)$$

from relative homotopy to relative $E_2$-homology with coefficients in $k$ is surjective. Using this, one is able to choose a set of maps

$$\{E_\alpha : (D', \partial D') \to (\mathbb{S}[t](g), sk_{\epsilon-1}(g))\},$$

whose images generate $H^{E_2}_{*,\epsilon}(\mathbb{S}[t], sk_{\epsilon-1}; k)$ as a $k$-module. The boundary maps are then used to attach filtered cells $(g, \epsilon)$ to $sk_{\epsilon-1}$ to form $sk_{\epsilon}$ and the corresponding $E_\alpha$ is used to extend $f_{\epsilon-1}$ to $f_\epsilon$. Putting all this together, we see that the attachment of the cells is parameterized by the dimensions of the $E_2$-algebra homology groups with coefficients in $k$. To see, in our particular setup, that this cell decomposition is concentrated in even degrees, it is therefore enough to verify that $H^{E_2}_{g,d}(\mathbb{S}[t], k)$ vanishes whenever $\epsilon \equiv 1 \mod 2$. For this we apply [GKRW18, Theorem 13.7] which states that the $k$-fold iterated bar construction of an $E_k$ algebra is equivalent to...
the $k$-suspension of the $E_k$-cotangent complex. By [Lur15, Proposition 5.4.9], there is an equivalence $\text{Bar}^{(2)}(S[t]) \simeq \text{gr}(S[C^\infty P^n]_{n \geq 0})$ in graded spectra, where the right hand side is the associated graded of the filtration on spherical chains on $C^\infty P^\infty$, with filtration induced by the skeletal filtration on infinite projective space. Tensoring this with $k = \mathbb{Z}$ in our particular situation, we obtain (a 2-fold shift) of chains on $C^\infty P^\infty$ with coefficients in $\mathbb{Z}$ which has a cell in each bidegree $(-n, 2n - 2)$. By taking into account units, we conclude that $S[t]$ may be constructed from $S$ by attaching the same cells.

**Remark 3.12.** We remark that we may take the levelwise colimit in the above filtered cellular decomposition to obtain an $E_2$ cellular decomposition for $S \rightarrow S[t]$ in the sense of Definition 3.10.

**Corollary 3.13.** The degree zero piece of the above cellular decomposition is the free algebra $\text{Free}_{E_2}(S(-1))$, i.e. the free $E_2$-algebra with generator in degree 0 and weight $-1$. Moreover, the map $f_0 : \text{Free}_{E_2}(S(-1)) \rightarrow S[t]$ itself admits a cellular decomposition with even cells of positive dimensions.

**Proof.** In degree zero, we have the following pushout square in the Alg $E_2$ (Fun($\mathbb{Z}$, Sp)):

\[
\begin{array}{ccc}
\mathbb{S} & \xrightarrow{\text{Free}_{E_2}(\emptyset)} & \text{sk}_{-1}(S[t]) \simeq S \\
\downarrow & & \downarrow \\
\text{Free}_{E_2}(S(-1)) \simeq \text{Free}_{E_2}(D^0) & \xrightarrow{\text{sk}_{0}(S[t])} & .
\end{array}
\]

Since this is a pushout square we obtain an equivalence

\[\text{sk}_{0}(S[t]) \simeq \text{Free}_{E_2}(S(-1))\]

Moreover, by starting in degree zero with the zero cells already attached, we may conclude that the map

\[f_0 : \text{Free}_{E_2}(S(-1)) \rightarrow S[t].\]

itself admits a cellular decomposition with even cells of positive dimension.

By the above corollary, we have a cellular decomposition on the map of graded $E_2$-rings $\text{Free}_{E_2}(S(-1)) \rightarrow S[t]$. We can apply shearing to this map to obtain a map

\[\text{Free}_{E_2}(S^k(-1)) \rightarrow S[\sigma_k].\]

**Proposition 3.14.** Let $k > 0$ be even. The map $f_0 : \text{Free}_{E_2} S^k(-1) \rightarrow S[\sigma_k]$ admits a cell decomposition with cells concentrated in even degrees. Left Kan extending along the multiplication map $\mathbb{Z} \xrightarrow{w} \mathbb{Z}$, we conclude that $f_0 : \text{Free}_{E_2} S^k(w) \rightarrow S[\sigma_{k,w}]$ admits a cell decomposition with cells concentrated in even degrees.

**Proof.** By construction, $S[t]$ may be written as a filtered colimit of a diagram of $E_2$ algebras,

\[\text{Free}_{E_2}(S(-1)) \rightarrow \text{sk}_1(f) \rightarrow \cdots \rightarrow \text{sk}_{i-1}(f) \rightarrow \text{sk}_i(f) \rightarrow \cdots\]

where each $\text{sk}_i(f)$ is formed as a pushout from $\text{sk}_{i-1}$ along a map $\text{Free}_{E_2}(S^{2n+1}) \rightarrow S$. We may apply $\text{sh}^{k/2}$ to this diagram, and take note of the fact that this will commute.
with colimits along the filtered diagram, together with the free $E_2$-algebra functor. Thus we conclude with an even cell presentation for the induced map:

$$sh^{k/2}(\text{Free}_{E_2}(S(-1))) \simeq \text{Free}_{E_2}(S^k(-1)) \to S[\sigma_k].$$

By left Kan extending along the multiplication by $-w$ map on $Z$ (i.e. applying $L_{-w}$), we conclude analogously for the map

$$\text{Free}_{E_2}(S^k(w)) \to S[\sigma_{k,w}].$$

\[\Box\]

**Proposition 3.15.** Let $A$ be a (graded) $E_2$-ring whose homotopy groups are concentrated in even degrees and let $a \in \pi_k A$ be a weight $w$ class for some even $k \geq 0$. Then there is a (graded) $E_2$-ring map

$$S[\sigma_{k,w}] \to A$$

which carries $\sigma_k$ to $a$.

**Proof.** By Proposition 3.14 the map

$$f : \text{Free}_{E_2}(S^k(w)) \to S[\sigma_{k,w}]$$

admits an even cell decomposition. Let $a \in \pi_k(A)$ be as in the hypothesis of the proposition. This induces an $E_2$-algebra map $\text{Free}_{E_2}(S^k(w)) \to A$, which we would like to extend inductively along the above tower. In order to do this, it is enough to note that in degree $i$, we would need to trivialize the induced map $\text{Free}_{E_2}(S^{k+2i-1}) \to A$. Using the free/forgetful adjunction between $\text{Alg}_{E_2}(\text{Fun}(Z, \text{Sp}))$ and $\text{Fun}(Z, \text{Sp})$ algebras and graded spectra, this will now follow from the fact that $\pi_{2n-1}(A) = 0$ for all $n$. \[\Box\]

**Remark 3.16.** In Remark 3.7 we mentioned that we are going to use $\text{sh}(S[\sigma_{m2,m}])$ to adjoin roots to degree 0 classes. We remark that $\text{sh}(S[\sigma_{m2,m}])$ also satisfies the lifting property in the proposition above. This follows by the fact that the even cell decomposition for $S[\sigma_{m2,m}]$ provides an even cell decomposition for $\text{sh}(S[\sigma_{m2,m}])$ since $\text{sh}$ is an $E_2$-monoidal left adjoint functor.

**Remark 3.17.** We remark that another way to construct an $E_2$-algebra map $S[t] \to A$ comes from the filtration on $S[t]$ given its filtered cell decomposition. The mapping space

$$\text{Map}_{\text{Alg}_{E_2}}(S[t], A)$$

obtains a filtration from the filtration on the source; this will have associated graded

$$\text{gr Map}_{\text{Alg}_{E_2}}(S[t], A) \simeq \text{Map}(\bigoplus_{n \geq 1} S^{2n}, A) \simeq \text{Map}(\bigoplus_{n \geq 1} S^{2n}, A).$$

Now if $X$ has even homotopy groups, then so does the associated graded, so that the resulting spectral sequence computing the homotopy groups of the limit collapses. Thus, $a \in \pi_0(A)$ gives a class in $x \in \pi_0 \text{Map}(S^{2k}, X) \subset \pi_0 \text{Map}(\bigoplus_{n \geq 1} S^{2n+2nk-2}, X)$, which will be an infinite cycle, and thus corresponds to an $E_2$-algebra map $S[t] \to A$. We remark that this approach should allow for one to define maps $S[\sigma_k] \to A$ in the general case as well. We thank Oscar Randal-Williams for suggesting this approach.
4. Adjoining roots and THH

Here, we introduce our construction for adjoining roots to ring spectra and prove our first results on the THH of ring spectra obtained through this construction.

4.1. Background on algebras over $E_n$-algebras. Here is a quick background on some of the standard facts that we often use from [Lur17].

For an $E_\infty$-algebra $R$ in a presentably symmetric monoidal $\infty$-category $\mathcal{C}$, the $\infty$-category of $E_n$ $R$-algebras is a symmetric monoidal $\infty$-category with the pointwise tensor product [Lur17, Example 3.2.4.4]. Therefore, for two $E_n$ $R$-algebras $A$ and $B$, $A \otimes_R B$ is an $E_n$ $R$-algebra.

In this work, we often consider algebras over an $E_n$-algebra $R$ and in this case, the $\infty$-category of $E_m R$-algebras (for $m \leq n - 1$) are not known to carry an appropriate $E_n - 1$-monoidal structure. To work around this problem, we use the following facts which can be extracted from [Lur09, Corollary 4.8.5.20].

The $\infty$-category of (left) right $R$-modules is an $E_n - 1$-monoidal $\infty$-category. We call an $E_m$-algebra in the $\infty$-category of right $R$-modules an $E_m R$-algebra where $m \leq n - 1$.

Furthermore, for a map $f: R \rightarrow S$ of $E_n$-algebras in $\mathcal{C}$, one obtains an $E_n - 1$-monoidal functor $- \otimes_R S$ between the respective $\infty$-categories of modules. For every $m \leq n - 1$, this induces a functor:

\[(4.1) \quad - \otimes_R S: \text{Alg}_{E_m}(\text{RMod}_R) \rightarrow \text{Alg}_{E_m}(\text{RMod}_S).\]

In particular, for an $E_m$ $R$-algebra $A$, $A \otimes_R S$ is an $E_m S$-algebra. Furthermore, the forgetful functor induced by $f$, i.e. the right adjoint of $- \otimes_R S$, is $E_n - 1$-lax monoidal and therefore it induces a functor:

\[(4.2) \quad \text{Alg}_{E_m}(\text{RMod}_S) \rightarrow \text{Alg}_{E_m}(\text{RMod}_R).\]

The unit of this adjunction provides a map of $E_m$ $R$-algebras:

\[(4.3) \quad A \rightarrow A \otimes_R S.\]

Since $S$ is the monoidal unit in $\text{RMod}_S$, $S$ is an $E_n - 1$ $S$-algebra, and forgetting through $\text{(4.2)}$, it is an $E_n - 1$ $R$-algebra. In summary, an $E_n$-algebra map $R \rightarrow S$ equips $S$ with the structure of an $E_n - 1$ $R$-algebra.

4.2. A construction for adjoining roots to ring spectra. We now introduce our construction for adjoining roots to ring spectra. For this we use the following hypothesis. Recall that we often omit the functor $D$ and let $S[\sigma_k]$ denote the underlying $E_2$-ring of the graded $E_2$-ring $S[\sigma_k]$.

Hypothesis 4.4 (Root adjunction hypothesis). Given an $E_1$-ring $A$ with an $a \in \pi_{mk} A$, there is a chosen $S[\sigma_{mk}]$-algebra structure on $A$ for which the structure map $S[\sigma_{mk}] \rightarrow A$ carries $\sigma_{mk}$ to $a \in \pi_{mk} A$. Here, $m > 0$ and $k \geq 0$ is even. See Proposition 4.5 for the cases of interest where this is satisfied.

The hypothesis above may not seem very natural but it is satisfied in the following general situations.

Proposition 4.5. Let $k \geq 0$ be even and $m > 0$, an $E_1$-ring $A$ satisfies Hypothesis 4.4 for $a \in \pi_{mk} A$ if:
(1) A is an $E_2$-ring for which $\pi_* A$ is concentrated in even degrees, or
(2) A is an $R$-algebra for an $E_2$-ring $R$ where $\pi_* R$ is concentrated in even degrees and $a$ is in the image of the map $\pi_* R \to \pi_* A$.

Proof. Assume that $A$ is as in (2), let $r \in \pi_{mk} R$ detect $a$ through the map $\pi_* R \to \pi_* A$. We choose an $E_2$-ring map $g : S[\sigma_{mk}] \to R$ that carries $\sigma_{mk}$ to $r$, see Proposition 4.13. Forgetting through $g$, see (4.2), one obtains a $S[\sigma_{mk}]$-algebra structure on $A$. Indeed, through this structure, $\sigma_{mk}$ acts through $a$ as desired.

If $A$ is as in (1), then $A$ is an $A$-algebra and $A$ satisfies the assumption in (2). Therefore, $A$ satisfies Hypothesis 4.4. $\square$

For instance, the Morava $K$-theory spectrum $K(n)$ and all $E_1 MU_{(p)}$-algebra forms of $BP(n)$ satisfy Hypothesis 4.4 with respect to their non-negative degree homotopy classes.

Notice that we are not assuming any preexisting non-trivial grading on $A$; in fact this will allow us to view it as an $m$-graded spectrum concentrated in weight zero. Given $a \in \pi_{mk} A$, the following construction adjoins an $m$-root to $a$.

**Construction 4.6.** Assume Hypothesis 4.4 We consider $S[\sigma_{mk}]$ as an $m$-graded $E_2$-ring and $A$ as an $m$-graded $S[\sigma_{mk}]$-algebra, both concentrated in weight 0, using the functor $F$ from Section 2.2; we omit $F$ in our notation.

Due to Proposition 3.5 (Remark 3.7 for $k = 0$), there is a map
\[
\phi : S[\sigma_{mk}] \to S[\sigma_k]
\]
of $m$-graded $E_2$-rings that carries $\sigma_{mk}$ to $\sigma_k^m$ in homotopy where $\sigma_k$ is of weight 1 and $\sigma_{mk}$ is of weight 0. Note that we omit the functor $D^m$ in our notation. Considering the corresponding extension of scalars functor $- \wedge_{S[\sigma_{mk}]} S[\sigma_k]$ between the $\infty$-categories of $m$-graded $S[\sigma_{mk}]$-algebras and $m$-graded $S[\sigma_k]$-algebras, (see (4.1)), we define the $m$-graded $E_1 S[\sigma_k]$-algebra $A(\sqrt{a})$ through:
\[
A(\sqrt{a}) := A \wedge_{S[\sigma_{mk}]} S[\sigma_k].
\]
This comes equipped with a map $A \to A(\sqrt{a})$ of $m$-graded $E_1 S[\sigma_{mk}]$-algebras, see (4.3).

Since $\pi_*(S[\sigma_k])$ is free as a $\pi_*(S[\sigma_{mk}])$-module, one obtains an isomorphism of rings:
\[
\pi_* A(\sqrt{a}) \cong \pi_*(A)[z]/(z^m - a).
\]
Therefore, we say $A(\sqrt{a})$ is obtained from $A$ by adjoining an $m$-root to $a$.

When $A$ is $p$-local, observe that we have and equivalence of $m$-graded $E_1 S[\sigma_k]$-algebras:
\[
A(\sqrt{a}) \simeq A \wedge_{S_{(p)}[\sigma_{mk}]} S_{(p)}[\sigma_k],
\]
where $S_{(p)}[\sigma_i]$ denotes the $p$-localization of $S[\sigma_i]$.

It follows that the weight pieces of $A(\sqrt{a})$ are given by the following
\[
A(\sqrt{a})_i \simeq \Sigma^{ik} A
\]
for each $0 \leq i < m$.

Note that $A(\sqrt{a})$ might possibly depend on the $S[\sigma_{mk}]$-algebra structure chosen on $A$. Therefore, everytime we apply Construction 4.6 we fix an $S[\sigma_{mk}]$-algebra structure on $A$. 

Hence, there is an equivalence of $E$-algebras.

*Remark 4.9.* In general, we do not expect the root adjunction $A \to A(\sqrt{a})$ to satisfy a universal property. On the other hand, if $A$ is an $E_3$-ring, $A(\sqrt{a})$ is an $E$-algebra and the map $\pi_*A \to \pi_*A(\sqrt{a})$ is étale, then it follows by [HP22, Theorem 1.10] that there is a bijection between homotopy classes of $A$-algebra maps $A(\sqrt{a}) \to B$ and $\pi_A$-algebra maps $\pi_*(A(\sqrt{a})) \to \pi_* B$ for any étale $A$-algebra $B$.

For the following, we fix an $E_2$-map $S(p)[\sigma_{2(p-1)}] \to \ell$ carrying $\sigma_{2(p-1)}$ to $v_1$.

**Theorem 4.10.** There is an equivalence

$$ku_p \simeq \ell_p(\sqrt[p]{v_1})$$

of $E_1$ $\ell_p$-algebras.

*Proof.* By Remark 4.8 above, $\ell_p(\sqrt[p]{v_1})$ is an $\ell_p$-algebra. Let $L_p$ denote the non-connective $p$-completed Adams summand. The $E_1$ $L_p$-algebra

$$\ell_p(\sqrt[p]{v_1}) \wedge \ell_p L_p$$

is an étale $E_1 L_p$-algebra in the sense of Hesselholt-Pstragowski [HP22] and there is an isomorphism of $\pi_*(L_p)$-algebras

$$\pi_*(\ell_p(\sqrt[p]{v_1}) \wedge \ell_p L_p) \cong \pi_* (KU_p).$$

It follows by [HP22, Theorem 1.10] that there is an equivalence of $L_p$-algebras

$$\ell_p(\sqrt[p]{v_1}) \wedge \ell_p L_p \simeq KU_p.$$

Through this, $\ell_p(\sqrt[p]{v_1})$ serves as the connective cover of $KU_p$ in $E_1$ $\ell_p$-algebras. Hence, there is an equivalence of $E_1$ $\ell_p$-algebras $ku_p \simeq \ell_p(\sqrt[p]{v_1})$. \hfill \Box

In Theorem 4.6 we show that the Morava $E$-theory spectrum $E_n$ is given by $S_{W(F_p^n)} \wedge_k E(n)(\sqrt[p^n]{v_n})$ as an $E_1$-ring where $E(n)$ is the $K(n)$-localized Johnson-Wilson spectrum.

*Remark 4.11.* In certain cases, it is possible to equip $A(\sqrt{a})$ with the structure of an $E_n$-algebra for $n > 1$. For this, one may use the graded Morava $E$-theory spectrum $MU[\sigma_k]$ [HW22, Construction 2.6.1] where $k > 0$ is even. Indeed, $MU[\sigma_k]$ is the free graded $E_1 MU$-algebra over $\Sigma^{k}MU$. There is a map of graded $E_\infty$-rings

$$MU[\sigma_{2(p^n-1)}] := L_{p^n-1}R_{p^n-1}MU[\sigma_2] \to MU[\sigma_2]$$

and Proposition 3.11 provides maps $S[\sigma_k] \to MU[\sigma_k]$ of graded $E_2$-rings.

It follows by [HW22, Remark 2.1.2] that a form of $BP\langle n \rangle$ admits the structure of an $E_3$ $MU[\sigma_{2(p^n-1)}]$-algebra where $\sigma_{2(p^n-1)}$ acts through $v_n \in \pi_* BP\langle n \rangle$. We obtain an equivalence of $p^n-1$-graded $E_1$ $S[\sigma_2]$-algebras

$$BP\langle n \rangle(\sqrt[p^n]{v_n}) := BP\langle n \rangle \wedge_{S[\sigma_{2(p^n-1)}]} S[\sigma_2] \simeq BP\langle n \rangle \wedge_{MU[\sigma_{2(p^n-1)}]} MU[\sigma_2].$$

This equips $BP\langle n \rangle(\sqrt[p^n]{v_n})$ with the structure of a $p^n-1$-graded $E_3$ $MU[\sigma_2]$-algebra.
4.3. **The weight zero piece of THH.** Here, we prove our first result regarding the topological Hochschild homology of the ring spectra obtained via root adjunction. Namely, we show that $\text{THH}(A(\sqrt[m]{a}))$ contains $\text{THH}(A)$ as a summand whenever $A$ is $p$-local and $p \nmid m$.

It follows by [AMMN22, Example A.10] that for an $m$-graded $E_1$-ring $Y$, the $m$-grading on $\text{THH}(Y)$ is obtained by applying the cyclic bar construction $b_s(Y)$ of $Y$ in the $\infty$-category of $m$-graded spectra. In simplicial level $s$ and weight $i$, the $m$-graded cyclic bar construction of $Y$ is given by the following.

$$b_s(Y)_i \simeq \bigvee_{k_0 + \cdots + k_s = i \in \mathbb{Z}/m} Y_{k_0} \wedge \cdots \wedge Y_{k_s}$$

Due to [AMMN22, Corollary A.15], one has the following equality

$$(4.12) \quad \text{THH}(D(Y)) \simeq D(\text{THH}(Y))$$

where the functor $D(-)$ provides the underlying spectrum as usual; we often omit $D$ in our notation. Furthermore, $\text{THH}(Y)$ is an $S^1$-equivariant $m$-graded spectrum in a canonical way and the equivalence above is an equivalence of $S^1$-equivariant spectra.

**Construction 4.13.** Let $R$ be an $E_2$-ring and let $S$ be an $E_1 R$-algebra. For us this will mean that the pair $(R, S) \in \text{RMod}^{(2)}(\text{Sp})$, where

$$\text{RMod}^{(2)}(\mathcal{C}) = \text{Alg}(\text{RMod}(\mathcal{C}))$$

for an arbitrary symmetric monoidal $\infty$-category $\mathcal{C}$. Here, $\text{RMod}(\mathcal{C})$ is the $\infty$-category of pairs $(A, M)$ where $A$ is an $E_1$-algebra and $M \in \text{RMod}_A(\mathcal{C})$. Thus, objects in $\text{RMod}^{(2)}(\mathcal{C})$ may be identified with pairs $(A, M)$ where $A$ is an $E_2$-algebra, and $M$ is an $E_1 A$-algebra in $\mathcal{C}$.

We remark that in general, $\text{RMod}^{(2)}(\mathcal{C})$ may be written as $\text{Alg}_\mathcal{O}(\mathcal{C})$ where $\mathcal{O}$ is the tensor product of operads

$$\mathcal{O} := \text{RMod} \times E_1$$

This tensor product of operads, studied in depth in [Lur17, Section 2.2.5] is symmetric and satisfies the following universal property at the level of algebra objects:

$$\text{Alg}_\mathcal{O}(\mathcal{C}) \simeq \text{Alg}_{E_1}(\text{Alg}_{\text{RMod}}(\mathcal{C})) \simeq \text{Alg}_{\text{RMod}}(\text{Alg}_{E_1}(\mathcal{C}))$$

Hence, applying the discussion to $\mathcal{C} = \text{Sp}$ and $R$ and $S$ as above, we may view $S$ as a right $R$-module in $E_1$-algebras.

Since $\text{THH}$ is a symmetric monoidal functor from $E_1$-rings to spectra [NS18, Section IV.2] we deduce that $\text{THH}(S)$ is a right $\text{THH}(R)$-module.

**Proposition 4.14.** Let $F$ be an $m$-graded $E_1 E$-algebra and $E \to F'$ be a map of $m$-graded $E_2$-rings. There is a natural equivalence of $m$-graded right $\text{THH}(F')$-modules in $S^1$-equivariant spectra:

$$\text{THH}(F \wedge_E F') \simeq \text{THH}(F) \wedge_{\text{THH}(E)} \text{THH}(F')$$

whose underlying undegraded equivalence is that of right $\text{THH}(F')$-modules in cyclotomic spectra. If $E$ and $F$ are concentrated in weight zero, then we have the following.

$$\text{THH}(F \wedge_E F')_i \simeq \text{THH}(F) \wedge_{\text{THH}(E)} (\text{THH}(F'))_i$$
Proof. Let us recall that the functor
\[ \text{THH} : \text{Alg}_{\text{Sp}} \to \text{CycSp} \]
is symmetric monoidal. Furthermore, it commutes with sifted colimits; indeed this can be seen from the fact that it can be decomposed into a composition of functors comprised of taking tensor products and realizations of simplicial objects, both of which commute with sifted colimits. Thus there will be a natural equivalence
\[
\text{THH}(F \wedge E F') \cong \text{THH}(|\text{Bar}_\bullet(F, E, F')|)
\cong |\text{Bar}_\bullet(\text{THH}(F)_\bullet, \text{THH}(E)_\bullet, \text{THH}(F')_\bullet)|
\cong \text{THH}(F) \wedge_{\text{THH}(E)} \text{THH}(F')
\]
This allows us to deduce that THH preserves the sifted colimit given by the double sided Bar construction; this can be computed at the level of underlying spectra by the bilinear pairing
\[ F \text{BMod}_E \times F \text{BMod}_{F'} \to F \text{BMod}_{F'} \]
corresponding to the relative tensor product. Furthermore, as THH preserves the sifted colimits corresponding to this relative tensor product, the above equivalence is compatible with right THH$(F')$ module structures. The analogous claims all hold when accounting for additional gradings, by recalling that THH promotes to a sifted colimit preserving symmetric monoidal functor from algebras in graded spectra to $S^1$-equivariant objects in graded spectra. In particular, if $E$ and $F$ are concentrated in weight zero, we deduce the equivalence
\[
\text{THH}(F \wedge E F')_i \cong \text{THH}(F)_i \wedge \text{THH}(E)_i \text{THH}(F')_i.
\]
of graded THH$(F)$-modules. \qed

Remark 4.15. The $m = 1$ case of the proposition above provides the non-graded case.

One may consider $S[\sigma_k]$ as an $E_1$-ring obtained by adjoining an $m$-root to $S[\sigma_{mk}]$. Proposition 4.17 identifies the weight zero piece of THH$(S(p)[\sigma_k])$. Before Proposition 4.17, we state and prove a well known fact.

Lemma 4.16. Let $\varphi : M \to N$ be a map between bounded below spectra. Then $\varphi$ is an equivalence if and only if $HZ \wedge \varphi$ is an equivalence. If furthermore $M$ and $N$ are $p$-local, then $\varphi$ is an equivalence if and only if $HZ(p) \wedge \varphi$ is an equivalence.

Proof. Let $K$ be the fiber of $\varphi$ and let $i$ be the smallest $i$ such that $\pi_i K \neq 0$. Due to the Tor spectral sequence of [EKMM97, Theorem IV.4.1], we have $\pi_i(HZ \wedge K) = \pi_i K$. Therefore, if $HZ \wedge K \cong 0$ then $K \cong 0$ and $\varphi$ is an equivalence.

If $M$ and $N$ are $p$-local, then $\varphi$ is an equivalence if and only if $S_{(p)} \wedge \varphi$ is an equivalence. It follows by the previous result that $\varphi$ is an equivalence if and only if $HZ \wedge S_{(p)} \wedge \varphi \cong HZ_{(p)} \wedge \varphi$ is an equivalence. \qed

For the following, let $k \geq 0$ be even and let $m > 1$. Furthermore, fix a map of $m$-graded $E_2$-rings $S_{(p)}[\sigma_{mk}] \to S_{(p)}[\sigma_k]$ provided by Proposition 3.5 (Remark 3.7 for $k = 0$).

Proposition 4.17. In the situation above, assume that $p \nmid m$. The induced map
\[
\text{THH}(S_{(p)}[\sigma_{mk}])_0 \cong \text{THH}(S_{(p)}[\sigma_k])_0
\]
is an equivalence of $E_1$-rings. Since $S_p[σ_{mk}]$ is concentrated in weight zero, we obtain the following chain of equivalences
\[ D(ZH(S_p[σ_{mk}])) \cong THH(S_p[σ_{mk}])_0 \rightarrow THH(S_p[σ_k])_0 \]
of $E_1$-rings using Lemma \[ \text{2.3} \]

Proof. It suffices to prove that the map
\[ HZ_p \wedge THH(S_p[σ_{mk}])_0 \rightarrow HZ_p \wedge THH(S_p[σ_k])_0 \]
is an equivalence, see Lemma \[ \text{1.16} \].

By the base change formula for $THH$, this is equivalent to the following map
\[ THH^{HZ_p}(HZ_p[σ_{mk}]) \rightarrow THH^{HZ_p}(HZ_p[σ_k]) \]
being an equivalence in weight zero. Here, $HZ_p[σ_k]$ denotes the free $HZ_p$-algebra on $S_p[σ_k]$. The map $HZ_p[σ_{mk}] \rightarrow HZ_p[σ_k]$ induces a map
\[ φ^r : E^r \rightarrow F^r \]
from the Bökstedt spectral sequence computing $THH^{HZ_p}(HZ_p[σ_{mk}])$ to the Bökstedt spectral sequence computing $THH^{HZ_p}(HZ_p[σ_k])$. Since the weight grading on the $THH$ of an $m$-graded ring spectrum comes from a weight grading on the corresponding cyclic bar construction, the Bökstedt spectral sequence computing $THH$ of an $m$-graded ring spectrum admits an $m$-grading, i.e. it splits into $m$ summands in a canonical way. Therefore, in our situation, it is sufficient to show that $φ^2$ is an isomorphism on weight zero.

We have
\[ φ^2 : \mathbb{Z}_p[σ_{mk}] \otimes \Lambda_{\mathbb{Z}_p}(d(σ_{mk})) \rightarrow \mathbb{Z}_p[σ_k] \otimes \Lambda_{\mathbb{Z}_p}(d(σ_k)) \]
where $d$ denotes the Connes operator. The degrees of the classes above are given by the following.
\[ \deg(σ_{mk}) = (0, mk) \quad \deg(d(σ_{mk})) = (1, mk) \]
\[ \deg(σ_k) = (0, k) \quad \deg(d(σ_k)) = (1, k) \]
Furthermore, $σ_{mk}$ and $d(σ_{mk})$ are in weight 0 and $σ_k$ and $d(σ_k)$ are in weight 1. In particular, all of $E^2$ is weight zero and the weight zero piece of $F^2$ is the $\mathbb{Z}_p$-module generated by the classes $σ_{ki}^m$ and $σ_{ki}^{(i+1)m-1}d(σ_k)$ over $i \geq 0$.

Since $φ^2(σ_{mk}) = σ_{km}^m$, we obtain that
\[ φ^2(d(σ_{mk})) = d(φ^2(σ_{mk})) = d(σ_{km}^m) = mσ_{km}^{m-1}d(σ_k) \]
Therefore, we have
\[ φ^2(σ_{mk}^i) = σ_{km}^{im} \quad \text{and} \quad φ^2(d(σ_{mk}^i)) = mσ_{km}^{(i+1)m-1}d(σ_k) \]

Since $p ∤ m$, we have that $m$ is a unit. Using this, one observes that $φ^2$ is an isomorphism after restricting and corestricting to weight zero as desired.

In the situation of Hypothesis \[ \text{1.23} \], $A \rightarrow A(\sqrt{a})$ is a map of $m$-graded $E_1$-rings and $A$ is concentrated in weight zero. Therefore, there is a map
\[ (4.19) \quad THH(A) \rightarrow THH(A(\sqrt{a}))_0 \]
where $A$ above denotes the underlying $E_1$-ring of $A$. 


**Theorem 4.20.** Assume Hypothesis \([4,4]\) and that \(A\) is \(p\)-local and \(p \nmid m\). Then (4.19) is an equivalence.

**Proof.** Recall that \(A(\sqrt[p]{a})\) is given by

\[ A \wedge_{S(p)[\sigma_m]} S(p)[\sigma_k] \]

where \(A\) and \(S(p)[\sigma_{mk}]\) are concentrated in weight zero. Due to Proposition 4.14, we have

\[ \text{THH}(A(\sqrt[p]{a}))_0 \simeq \text{THH}(A) \wedge_{\text{THH} (S(p)[\sigma_k])} (\text{THH}(S(p)[\sigma_k])_0) \]

and it follows by Proposition 4.17 that the map

\[ \text{THH}(S(p)[\sigma_{mk}]) \xrightarrow{\simeq} \text{THH}(S(p)[\sigma_k])_0 \]

is an equivalence. This identifies \(\text{THH}(A(\sqrt[p]{a}))_0\) with \(\text{THH}(A)\) as desired. \(\square\)

5. **ADJOINING ROOTS AND ALGEBRAIC \(K\)-THEORY**

We now prove Theorem 1.1 from the introduction. For the rest of this section, assume Hypothesis \([4,3]\). We established that \(A(\sqrt[p]{a})\) is an \(m\)-graded ring spectrum and therefore \(\text{THH}(A(\sqrt[p]{a}))\) is an \(S^1\)-equivariant \(m\)-graded spectrum, see (4.12). One might define \(\text{TC}^*(A(\sqrt[p]{a}))\) as an \(m\)-graded spectrum given by:

\[ \text{TC}^*(A(\sqrt[p]{a}))_i \simeq \text{THH}(A(\sqrt[p]{a}))^{hS^1}_i. \]

Since \(m\) is finite, the underlying spectrum of an \(m\)-graded spectrum, provided by the functor \(D\), is given by a finite coproduct which is equivalent to the corresponding finite product. In particular, \(D\) commutes with all limits and colimits. Because of this, we have

\[ D(\text{TC}^*(A(\sqrt[p]{a}))) \simeq \text{TC}^*(D(A(\sqrt[p]{a}))) \]

and therefore, we often omit \(D\) in our notation.

Similarly, \(\text{TP}(A(\sqrt[p]{a}))\) and \((\text{THH}(A(\sqrt[p]{a}))^{hS^1}_{iC_p})\) admit the structure of \(m\)-graded spectra and these constructions commute with the functor \(D\) as above. Combining this, with Theorem 4.20 we obtain the following result.

**Theorem 5.1.** Assume Hypothesis \([4,4]\) that \(A\) is \(p\)-local, and that \(p \nmid m\). The maps

\[ \text{TC}^*(A) \xrightarrow{\simeq} \text{TC}^*(A(\sqrt[p]{a}))_0 \]

\[ \text{TP}(A) \xrightarrow{\simeq} \text{TP}(A(\sqrt[p]{a}))_0 \]

\[ (\text{THH}(A)^{hS^1}_{iC_p}) \xrightarrow{\simeq} ((\text{THH}(A(\sqrt[p]{a}))^{hS^1}_{iC_p}))_0 \]

induced by \(A \to A(\sqrt[p]{a})\) are all equivalences. \(\square\)

When \(A\) is connective and \(p\)-local, \(A(\sqrt[p]{a})\) is also connective and \(p\)-local. By [NS18, Corollary 1.5] (and the discussion afterwards), the topological cyclic homology of \(A(\sqrt[p]{a})\) is defined via the following fiber sequence.

\[ \text{TC}(A(\sqrt[p]{a})) \to \text{THH}(A(\sqrt[p]{a}))^{hS^1}_{\sqrt[p]{H^p} \to \text{can}} \to (\text{THH}(A(\sqrt[p]{a}))^{hS^1}_{iC_p}) \]

As mentioned above, the middle term and the third term above admit canonical splittings into \(m\)-cofactors. Furthermore, \(\text{can}\) respects this splitting since it only depends on the \(S^1\)-equivariant structure of \(\text{THH}(A(\sqrt[p]{a}))\).
However, \(TC(A(\sqrt{a}))\) do not necessarily split into \(m\)-cofactors. This is due to the fact that the Frobenius map does not necessarily respect the grading. Indeed, the Frobenius is given by maps
\[
\varphi_p: THH(A(\sqrt{a})),_i \to THH(A(\sqrt{a}))^{IC_p},
\]
see [AMMN22, Corollary A.9]. On the other hand, we obtain the following splitting of \(TC(A(\sqrt{a}))\).

**Construction 5.3.** Assume Hypothesis 4.4 and that \(A\) is connective and \(p\)-local where \(p \nmid m\). In this situation, \(p\) is a non-zero divisor in \(\mathbb{Z}/m\). Therefore, the Frobenius map on \(THH(A(\sqrt{a}))\) carries pieces of non-zero weight to non-zero weight pieces. Moreover, \(\varphi_p\) carries weight zero to weight zero. Therefore, the map \(\varphi_p - can\) splits as a coproduct of their restriction to weight zero and their restriction to non-zero weight. In particular, the fiber sequence in (5.2) admits a splitting as follows.

\[
TC(A(\sqrt{a}))_0 \vee TC(A(\sqrt{a})))_{1} \to THH(A(\sqrt{a}))_{0}^{hS^1} \vee THH(A(\sqrt{a}))_{0}^{hS^1}
\]
\[
\xrightarrow{(\varphi_p)_0 - can_0 - can_{>0}} (THH(A(\sqrt{a}))^{IC_p})_{0}^{hS^1} \vee (THH(A(\sqrt{a}))^{IC_p})_{0}^{hS^1}
\]

Here, \((-)_{>0}\) denotes restriction to weight not equal to 0. We have

\[
(5.4) \quad TC(A(\sqrt{a})) \simeq TC(A(\sqrt{a}))_0 \vee TC(A(\sqrt{a}))_1
\]

where \(TC(A(\sqrt{a}))_0\) denotes the fiber of the map \((\varphi_p)_0 - can_0\) and \(TC(A(\sqrt{a}))_1\) denotes the fiber of the map \((\varphi_p)_{>0} - can_{>0}\).

**Remark 5.5.** There are interesting cases where one obtains further splittings of the topological cyclic homology spectrum \(TC(A(\sqrt{a}))\). For instance, if \(p = 1\) in \(\mathbb{Z}/m\), then and one obtains that \(TC(A(\sqrt{a}))\) splits into \(m\)-summands. This happens to be the case when \(m = p - 1\) or when \(p\) is odd and \(m = 2\). We exploit this in Construction 7.3 to obtain a splitting of \(TC(ku_p)\) into \(p - 1\) summands. Moreover, if \(m = p^n - 1\), then one obtains an underlying \(p - 1\)-grading of \(THH(A(\sqrt{a}))\) by Kan extending through \(\mathbb{Z}/(p^n - 1) \to \mathbb{Z}/(p - 1)\). This provides a \(p - 1\)-grading for \(TC(A(\sqrt{a}))\).

**Theorem 5.6.** Assume Hypothesis 4.4 with \(p \nmid m\) and that \(A\) is \(p\)-local and connective. Under the equivalence (5.4), the canonical map

\[
TC(A) \to TC(A(\sqrt{a}))
\]

is equivalent to the inclusion of the first wedge summand.

**Proof.** Since \(A\) is concentrated in weight zero, the map \(TC(A) \to TC(A(\sqrt{a}))\) factors through the map

\[
(5.7) \quad TC(A) \to TC(A(\sqrt{a}))_0
\]

induced by the canonical map \(THH(A) \to THH(A(\sqrt{a}))_0\). The map \(THH(A) \to THH(A(\sqrt{a}))_0\) of cyclotomic spectra is an equivalence due to Theorem 4.20. Considering the construction of \(TC(A(\sqrt{a}))_0\), one observes that this equivalence induces an equivalence between the fiber sequences defining \(TC(A)\) and \(TC(A(\sqrt{a}))_0\). In other words, (5.7) is an equivalence as desired.

Finally, we obtain the desired splitting for \(K(A(\sqrt{a}))\).
Theorem 5.8 (Theorem 1.1). Assume Hypothesis 4.4 with $p \nmid m$ and $k > 0$. Furthermore, assume that $A$ is $p$-local and connective. In this situation, the following map

$$K(A) \to K(A(\sqrt[ ]{a}))$$

is the inclusion of a wedge summand.

Proof. Since $|a| = mk$ and since $k > 0$, we have

(5.9) \[ \pi_0 A(\sqrt[ ]{a}) = \pi_0 A. \]

We start by constructing a map of $m$-graded $E_1$-algebras

(5.10) \[ A(\sqrt[ ]{a}) \to H\pi_0 A \]

that induces an isomorphism on $\pi_0$ where $H\pi_0 A$ is concentrated in weight 0. Weight 0 Postnikov truncation [HW22, Lemma B.0.6] provides a map of graded $E_2$-rings $\mathbb{S}[\sigma_k] \to \mathbb{S}$ that we consider as a map of $m$-graded $E_2$-rings by left Kan extending through $\mathbb{Z} \to \mathbb{Z}/m$.

This provides a map of $m$-graded $E_1$-rings $A(\sqrt[ ]{a}) \simeq A \wedge_{\mathbb{S}[\sigma_{mk}]} \mathbb{S}[\sigma_k] \to A \wedge_{\mathbb{S}[\sigma_{mk}]} \mathbb{S}$ (see (4.3)) where the right hand side is concentrated in weight 0. Postcomposing with the ordinary Postnikov truncation, we obtain (5.10).

Due to the Dundas-Goodwillie-McCarthy theorem [DGM13], there is a pullback square

$$
\begin{array}{ccc}
K(A(\sqrt[ ]{a})) & \longrightarrow & \text{TC}(A(\sqrt[ ]{a})) \simeq \text{TC}(A) \vee \text{TC}(A(\sqrt[ ]{a}))_1 \\
\downarrow & & \downarrow \\
K(H\pi_0 A) & \longrightarrow & \text{TC}(H\pi_0 A)
\end{array}
$$

provided by the map (5.10). The equivalence on the upper right corner follows by Construction 5.3 and Theorem 5.6.

The map $A(\sqrt[ ]{a}) \to H\pi_0 A$ induces a map of $m$-graded spectra $f: \text{THH}(A(\sqrt[ ]{a})) \to \text{THH}(H\pi_0 A)$.

Since $H\pi_0 A$ is concentrated in weight zero, $\text{THH}(H\pi_0 A)$ is also concentrated in weight zero. Therefore, the map $f$ is trivial on $\text{THH}(A(\sqrt[ ]{a}))_{>0}$. This shows that the right vertical map above induces the trivial map on $\text{TC}(A(\sqrt[ ]{a}))_1$. Using this, we obtain that the pullback square above splits as a coproduct of the pullback squares

$$
\begin{array}{ccc}
K(A) & \longrightarrow & \text{TC}(A) \\
\downarrow & & \downarrow \\
K(H\pi_0 A) & \longrightarrow & \text{TC}(H\pi_0 A)
\end{array}
$$

and

$$
\begin{array}{ccc}
\text{TC}(A(\sqrt[ ]{a}))_1 & \longrightarrow & \text{TC}(A(\sqrt[ ]{a}))_1 \\
\downarrow & & \downarrow \\
* & \longrightarrow & *
\end{array}
$$
This shows that

\[ K(A(\sqrt{a})) \simeq K(A) \vee TC(A(\sqrt{a}))_1 \]

as desired. \qed

Using the Purity theorem for algebraic \(K\)-theory, we obtain the following for non-connective \(A\).

**Corollary 5.11.** Assume Hypothesis 4.4 with \(p \nmid m\) and \(k > 0\). If \(A\) is \(p\)-local, then the map

\[ L_{T(i)}K(A) \to L_{T(0)}K(A(\sqrt{a})) \]

is the inclusion of a wedge summand for every \(i \geq 2\). In particular, if \(A\) is of height larger than 0 and \(A\) satisfies the redshift conjecture, then \(A(\sqrt{a})\) also satisfies the redshift conjecture.

**Proof.** Let \(cA\) denote the connective cover of \(A\) in \(S[\sigma_{mk}]\)-algebras. We consider the following commuting diagram of \(m\)-graded \(E_1\)-rings.

\[
\begin{array}{ccc}
\text{cA} & \longrightarrow & A \\
\downarrow & & \downarrow \\
A(\sqrt{a}) & \longrightarrow & A(\sqrt{a})
\end{array}
\]

Every spectrum with bounded above homotopy is \(T(i)\)-locally trivial for every \(i \geq 1\). Taking fibers, one obtains that the horizontal arrows above are \(T(i)\)-equivalence for every \(i \geq 1\), see (4.7).

It follows by [LMMT20, Purity Theorem] that the horizontal maps above induce \(T(i)\)-equivalences in algebraic \(K\)-theory for every \(i \geq 2\). The result follows by applying Theorem 5.8 to the left vertical map. \qed

6. A variant of logarithmic THH

Here, we introduce our definition of logarithmic THH and identify THH\((A(\sqrt{a}))\) using THH\((A)\) and logarithmic THH of \(A\) whenever \(A\) is \(p\)-local and \(p \nmid m\). Through our definition, logarithmic THH admits a canonical structure of a cyclotomic spectrum; in upcoming work, Devalapurkar and the third author develop a very general notion of logarithmic structures for \(E_2\)-algebras and a corresponding theory of log THH which subsumes the definition we use here. This will in particular recover the variant due to Rognes, which is defined by way of the replete bar construction, cf. [Rog09, RSS15].

Our definition of log THH starts with a definition of the log THH of the free algebra \(S[\sigma_k]\) where \(k \geq 0\) is even as before. We consider \(\sigma_k\) to be in weight 1.

For a graded \(E_n\)-ring spectrum \(E\), we denote the weight connective cover of \(E\) by \(E_{\geq 0}\). Indeed, the weight connective cover is obtained by restricting and then left Kan extending through the inclusion \(\mathbb{N} \to \mathbb{Z}\). The counit of this adjunction provides a map \(E_{\geq 0} \to E\) of graded \(E_n\)-algebras.

**Construction 6.1.** Analogous to Construction 3.3, let \(S[\sigma_k^{\pm 1}] := sh^k(S[t^{\pm 1}])\). The graded \(E_\infty\)-map \(S[t] \to S[t^{\pm 1}]\) provides a graded \(E_2\)-map \(S[\sigma_k] \to S[\sigma_k^{\pm 1}]\). Furthermore, by the definition of the shearing functor, \(S[\sigma_k^{\pm 1}]\) is indeed given by \(\phi^k\) of Variant
Example 6.4. Considering $\mathbb{H}Z$ by showing that its Hochschild homology of the free algebra $S\sigma$ of $S\sigma$, where $d\sigma$ is of weight 1 and degree $k + 1$ and $\sigma$ is of weight 1 and degree $k$. The isomorphism above follows by the usual Bökstedt spectral sequence considerations applied together with the HKR theorem. Taking the weight connective cover of $(6.5)$, we obtain:

$$H\mathbb{Z}_* \text{THH}(S[\sigma_k] \mid \sigma_k) \cong \mathbb{Z}[\sigma_k] \otimes \Lambda(d\sigma_k)$$

where $d\log\sigma_k$ is of weight 0 and homotopical degree 1 and it corresponds to $(d\sigma_k)/\sigma_k$. Furthermore, the map

$$H\mathbb{Z}_* \text{THH}(S[\sigma_k]) \to H\mathbb{Z}_* \text{THH}(S[\sigma_k] \mid \sigma_k)$$

Remark 6.2. One may also take weight connective covers in the $\infty$-category of graded $S^1$-equivariant spectra by using the left Kan extension/restriction adjunction induced by $\mathbb{N}^d \to \mathbb{Z}$. This provides a map

$$\text{THH}(S[\sigma_k]) \to \text{THH}(S[\sigma_k^{\pm 1}]_{\geq 0})$$

of graded $E_1$-algebras in $S^1$-equivariant spectra factoring the map $\text{THH}(S[\sigma_k]) \to \text{THH}(S[\sigma_k^{\pm 1}])$.

The following is analogous to the description of the replete bar construction of commutative $J$-space monoids generated by a single element; c.f. [Rog09, Proposition 3.21], [SS19, Section 8.5] and [RSS15, Sections 6 and 7].

Definition 6.3. Let $k \geq 0$ be even. The logarithmic THH of $S[\sigma_k]$ with respect to $\sigma_k \in \pi_k S[\sigma_k]$ is the weight connective cover of the topological Hochschild homology of $S[\sigma_k^{\pm 1}]$. In other words, it is the $S^1$-equivariant $E_1$-algebra:

$$\text{THH}(S[\sigma_k] \mid \sigma_k) := \text{THH}(S[\sigma_k^{\pm 1}]_{\geq 0}).$$

Similarly, the $p$-local counterpart is defined as follows.

$$\text{THH}(S(p)[\sigma_k] \mid \sigma_k) := \text{THH}(S(p)[\sigma_k^{\pm 1}]_{\geq 0})$$

The following example provides a justification for this definition of logarithmic THH by showing that its $H\mathbb{Z}$-homology provides what should be the logarithmic Hochschild homology of the free algebra $\mathbb{Z}[\sigma_k]$, c.f. [KN19, Example 10.3].

Example 6.4. Considering $H\mathbb{Z}$ as a graded $E_\infty$-algebra concentrated in weight 0, we deduce that

$$H\mathbb{Z} \wedge (\text{THH}(S[\sigma_k^{\pm 1}]_{\geq 0}) \simeq (H\mathbb{Z} \wedge \text{THH}(S[\sigma_k^{\pm 1}]))_{\geq 0}. $$

Therefore, $H\mathbb{Z}_* \text{THH}(S[\sigma_k] \mid \sigma_k)$ is given by the weight connective cover of

$$(6.5)$$

$$\text{THH}_{H\mathbb{Z}}(H\mathbb{Z}[\sigma_k^{\pm 1}]) \cong \mathbb{Z}[\sigma_k^{\pm 1}] \otimes \Lambda(d\sigma_k)$$

where $d\sigma_k$ is of weight 1 and degree $k + 1$ and $\sigma_k$ is of weight 1 and degree $k$. The isomorphism above follows by the usual Bökstedt spectral sequence considerations applied together with the HKR theorem. Taking the weight connective cover of $(6.5)$, we obtain:

$$H\mathbb{Z}_* \text{THH}(S[\sigma_k] \mid \sigma_k) \cong \mathbb{Z}[\sigma_k] \otimes \Lambda(d\log\sigma_k)$$

where $d\log\sigma_k$ is of weight 0 and homotopical degree 1 and it corresponds to $(d\sigma_k)/\sigma_k$. Furthermore, the map

$$H\mathbb{Z}_* \text{THH}(S[\sigma_k]) \to H\mathbb{Z}_* \text{THH}(S[\sigma_k] \mid \sigma_k)$$

3.2 In particular, $S[\sigma_k^{\pm 1}]$ is the restriction of $S[\sigma_k^{\pm 1}]$ along $\mathbb{Z} \to \mathbb{Z}$. Applying the adjunction $L_m \dashv R_m$ induced by $\cdot m$ to the map $S[\sigma_{mk}] \to S[\sigma_k]$, one observes that $S[\sigma_{mk}]$ is also the restriction of $S[\sigma_k]$ along $m$. Therefore, the counit of $L_m \dashv R_m$ provides a commutative diagram of graded $E_2$-rings:

$$S[\sigma_{mk}] \longrightarrow S[\sigma_k]$$

$$\downarrow \quad \downarrow$$

$$S[\sigma_k] \longrightarrow S[\sigma_k^{\pm 1}].$$

Remark 6.2. One may also take weight connective covers in the $\infty$-category of graded $S^1$-equivariant spectra by using the left Kan extension/restriction adjunction induced by $\mathbb{N}^d \to \mathbb{Z}$. This provides a map

$$\text{THH}(S[\sigma_k]) \to \text{THH}(S[\sigma_k^{\pm 1}]_{\geq 0})$$

of graded $E_1$-algebras in $S^1$-equivariant spectra factoring the map $\text{THH}(S[\sigma_k]) \to \text{THH}(S[\sigma_k^{\pm 1}])$.

The following is analogous to the description of the replete bar construction of commutative $J$-space monoids generated by a single element; c.f. [Rog09, Proposition 3.21], [SS19, Section 8.5] and [RSS15, Sections 6 and 7].

Definition 6.3. Let $k \geq 0$ be even. The logarithmic THH of $S[\sigma_k]$ with respect to $\sigma_k \in \pi_k S[\sigma_k]$ is the weight connective cover of the topological Hochschild homology of $S[\sigma_k^{\pm 1}]$. In other words, it is the $S^1$-equivariant $E_1$-algebra:

$$\text{THH}(S[\sigma_k] \mid \sigma_k) := \text{THH}(S[\sigma_k^{\pm 1}]_{\geq 0}).$$

Similarly, the $p$-local counterpart is defined as follows.

$$\text{THH}(S(p)[\sigma_k] \mid \sigma_k) := \text{THH}(S(p)[\sigma_k^{\pm 1}]_{\geq 0})$$

The following example provides a justification for this definition of logarithmic THH by showing that its $H\mathbb{Z}$-homology provides what should be the logarithmic Hochschild homology of the free algebra $\mathbb{Z}[\sigma_k]$, c.f. [KN19, Example 10.3].

Example 6.4. Considering $H\mathbb{Z}$ as a graded $E_\infty$-algebra concentrated in weight 0, we deduce that

$$H\mathbb{Z} \wedge (\text{THH}(S[\sigma_k^{\pm 1}]_{\geq 0}) \simeq (H\mathbb{Z} \wedge \text{THH}(S[\sigma_k^{\pm 1}]))_{\geq 0}. $$

Therefore, $H\mathbb{Z}_* \text{THH}(S[\sigma_k] \mid \sigma_k)$ is given by the weight connective cover of

$$(6.5)$$

$$\text{THH}_{H\mathbb{Z}}(H\mathbb{Z}[\sigma_k^{\pm 1}]) \cong \mathbb{Z}[\sigma_k^{\pm 1}] \otimes \Lambda(d\sigma_k)$$

where $d\sigma_k$ is of weight 1 and degree $k + 1$ and $\sigma_k$ is of weight 1 and degree $k$. The isomorphism above follows by the usual Bökstedt spectral sequence considerations applied together with the HKR theorem. Taking the weight connective cover of $(6.5)$, we obtain:

$$H\mathbb{Z}_* \text{THH}(S[\sigma_k] \mid \sigma_k) \cong \mathbb{Z}[\sigma_k] \otimes \Lambda(d\log\sigma_k)$$

where $d\log\sigma_k$ is of weight 0 and homotopical degree 1 and it corresponds to $(d\sigma_k)/\sigma_k$. Furthermore, the map

$$H\mathbb{Z}_* \text{THH}(S[\sigma_k]) \to H\mathbb{Z}_* \text{THH}(S[\sigma_k] \mid \sigma_k)$$
carries $d\sigma_k$ to $d\sigma_k = \sigma_k d\log \sigma_k$.

Recall from Construction 4.13 that when $A$ is a $S[\sigma_k]$-algebra, $\text{THH}(A)$ admits the structure of a $\text{right THH}(S[\sigma_k])$-module. We use this structure in the following definition. Recall that Proposition 4.13 provides various cases of interest where the assumptions on $A$ in the following definition are satisfied.

**Definition 6.6.** Let $A$ be an $E_1$ $S[\sigma_k]$-algebra and assume that the unit map $S[\sigma_k] \to A$ carries $\sigma_k \in \pi_k S[\sigma_k]$ to $a \in \pi_k A$ with even $k \geq 0$. We define the logarithmic $\text{THH}$ of $A$ relative to $a$ as the following $S^1$-equivariant spectrum.

$$\text{THH}(A | a) := \text{THH}(A) \wedge_{\text{THH}(S[\sigma_k])} \text{THH}(S[\sigma_k] | \sigma_k)$$

If $A$ is assumed to be $p$-local, we use the following equivalent definition

$$\text{THH}(A | a) := \text{THH}(A) \wedge_{\text{THH}(S(p)[\sigma_k])} \text{THH}(S(p)[\sigma_k] | \sigma_k).$$

The definition of logarithmic $\text{THH}$ we provide above is analogous to the definitions used in [Rog09, SS15, RSS15].

**Remark 6.7.** We remark that $\text{THH}(S[\sigma_k] | \sigma_k)$ should be a cyclotomic spectrum as the Frobenius maps of $\text{THH}$ multiply the weight by $p$ and this should provide $\text{THH}(A | a)$ above with the structure of a cyclotomic spectrum. However, since we don’t explicitly need this for our application, displaying this will take us too far afield, and so, we leave the details to the future work of Devalapurkar and the third author.

Since the definition of logarithmic $\text{THH}$ is given by the extension of scalars functor:

$$- \wedge_{\text{THH}(S[\sigma_k])} \text{THH}(S[\sigma_k] | \sigma_k) : \text{RMod}_{\text{THH}(S[\sigma_k])} \to \text{RMod}_{\text{THH}(S[\sigma_k])},$$

corresponding to the $E_1$-algebra map $\text{THH}(S[\sigma_k]) \to \text{THH}(S[\sigma_k] | \sigma_k)$, we deduce that $\text{THH}(A | a)$ is equipped with the structure of a right $\text{THH}(S[\sigma_k] | \sigma_k)$-module. Furthermore, the unit of the adjunction given by the extension of scalars functor above and the corresponding forgetful functor provides a map

$$\text{THH}(A) \to \text{THH}(A | a) \quad (6.8)$$

of right $\text{THH}(S[\sigma_k])$-modules, see [4.3].

**Remark 6.9.** Using $\text{MU}[\sigma_k]$ mentioned in Remark 4.11, it is possible to equip logarithmic $\text{THH}$ with the structure of an $E_n$-algebra for $n > 0$ in favorable cases. For instance, for the $E_2$ $\text{MU}[\sigma_{2(p^n - 1)}]$-algebra form of $BP(n)$ constructed in [HW22], $\text{THH}(BP(n) | v_n)$ admits the structure of an $E_1$-ring. Indeed, using the map of $E_2$-rings $\text{THH}(\text{MU}[\sigma_{2(p^n - 1)}]) \to \text{THH}(BP(n))$, we obtain an $E_1$-ring:

$$\text{THH}(BP(n) \wedge_{\text{THH}(\text{MU}[\sigma_{2(p^n - 1)}])} \text{THH}(\text{MU}[\sigma_{2(p^n - 1)}]) \geq 0,$$

equivalent to $\text{THH}(BP(n) | v_n)$. This equivalence follows by the following chain of equivalences

$$\text{THH}(BP(n) \wedge_{\text{THH}(\text{MU}[\sigma_{2(p^n - 1)}])} \text{THH}(\text{MU}[\sigma_{2(p^n - 1)}]) \geq 0$$

$$\simeq \text{THH}(BP(n) \wedge_{\text{THH}(\text{MU}) \wedge \text{THH}(S[\sigma_{2(p^n - 1)}])} \text{THH}(\text{MU}) \wedge \text{THH}(S[\sigma_{2(p^n - 1)}]) \geq 0$$

$$\simeq \text{THH}(BP(n) \wedge_{\text{THH}(S[\sigma_{2(p^n - 1)}])} \text{THH}(S[\sigma_{2(p^n - 1)}])$$

obtained from the equivalence of $E_2$ $\text{MU}$-algebras $\text{MU}[\sigma_{2(p^n - 1)}] \simeq \text{MU} \wedge S[\sigma_{2(p^n - 1)}]$ mentioned in Remark 4.11.
Furthermore, Hahn and Yuan [HY20, 1.11 and 1.12] show that there is an $E_\infty$-map $MU[\sigma_2] \to ku_p$ for $p = 2$ and claim that their methods provide such a map for odd primes too. In this situation, $\text{THH}(ku_p | u_2)$ is equipped with the structure of an $E_\infty$-ring (by arguing as above) where $u_2$ denotes the Bott element. Note that the logarithmic THH of $ku_p$ relative to $u_2$ is also constructed as an $E_\infty$-ring in [RSS18].

Remark 6.10. In work in progress, S. Devalapurkar and the second author show in a general context, that for every $E_2$-ring with even homotopy, logarithmic THH, as in our definition, may be equipped with a canonical $E_1$-algebra structure in cyclotomic spectra.

To study the log THH of $E_1$-rings obtained via root adjunctions, we use the following constructions.

**Construction 6.11.** Using Construction 6.1 and the weight connective cover adjunction mentioned in Remark 6.2, we obtain the following commuting diagram of graded $E_1$-algebras.

\[
\begin{array}{ccc}
S[\sigma_{mk}] & \longrightarrow & \text{THH}(S[\sigma_{mk}]) \\
\downarrow & & \downarrow \\
S[\sigma_k] & \longrightarrow & \text{THH}(S[\sigma_k])
\end{array}
\]  
\[
(6.12)
\]

**Construction 6.13.** Assume Hypothesis 1.4. By Proposition 4.14, there is an equivalence:

$$\text{THH}(A(\sqrt{a}) \simeq \text{THH}(A) \wedge_{\text{THH}(S[\sigma_{mk}])} \text{THH}(S[\sigma_k]).$$

This equips $\text{THH}(A(\sqrt{a}))$ with the structure of a right $\text{THH}(S[\sigma_k])$-module in $m$-graded spectra. Considering the map

$$\text{THH}(S[\sigma_k]) \rightarrow \text{THH}(S[\sigma_k] | S[\sigma_k])$$

as a map of $m$-graded $E_1$-ring spectra, Definition 6.6 may be employed at the level of $m$-graded spectra. This shows that $\text{THH}(A(\sqrt{a}) | \sqrt{a})$ admits a canonical structure of a right $\text{THH}(S[\sigma_k] | \sigma_k)$-module in $m$-graded spectra. Furthermore, the map $\text{THH}(A(\sqrt{a}) \rightarrow \text{THH}(A(\sqrt{a}) | \sqrt{a})$ is a map of $m$-graded $\text{THH}(S[\sigma_k])$-modules.

6.1. Logarithmic THH-étale root adjunctions. Here, our goal is to show that when $A$ is $p$-local and $p \nmid m$, root adjunction is logarithmic THH-étale. In other words, we show that there is an equivalence of $m$-graded spectra:

$$\text{THH}(A | a) \wedge_{S(p)[\sigma_{mk}]} S(p)[\sigma_k] \simeq \text{THH}(A(\sqrt{a}) | \sqrt{a}).$$

Remark 6.14. A notion of logarithmic THH-étalenes is already defined in [RSS18]. In the language of Rognes, Sagave and Schlichtkrull [RSS18], logarithmic THH-étaleness of $A \rightarrow A(\sqrt{a})$ would be expressed by an equivalence:

$$\text{THH}(A(\sqrt{a}) | \sqrt{a}) \simeq A(\sqrt{a}) \wedge_A \text{THH}(A | a).$$

(6.15)

Since we only assume $A$ to be $E_1$, $\text{THH}(A | a)$ may not admit an $A$-module structure and therefore, the right hand side above may not be defined in our generality. On the other hand, if one starts with an $E_3$-algebra $A$ with even homotopy, the logarithmic THH of $A$ may be given an $A$-module structure and we obtain that $A \rightarrow A(\sqrt{a})$ is logarithmic THH-étale in the sense of (6.15) whenever $A$ is $p$-local and $p \nmid m$. 


Proposition 6.16. For $k \geq 0$, the spectra $\text{THH}(S[\sigma_k] \mid \sigma_k)$ and $\text{THH}(S(p)[\sigma_k] \mid \sigma_k)$ are connective in homotopy.

Proof. This follows by the fact that the weight connective part of the cyclic bar construction on $S[\sigma_k^{\pm 1}]$ ($S(p)[\sigma_k^{\pm 1}]$) is connective in homotopy in each simplicial degree.

We start with proving a logarithmic THH étaleness result for the $p$-localized free $E_1$-algebra $S(p)[\sigma_m]$.

Proposition 6.17. Let $k \geq 0$ be even and let $m > 0$ with $p \nmid m$. In this situation, there is an equivalence of left $\text{THH}(S(p)[\sigma_m] \mid \sigma_m)$-modules in $m$-graded spectra:

$\text{THH}(S(p)[\sigma_m] \mid \sigma_m) \wedge_{S(p)[\sigma_m]} S(p)[\sigma_k] \simeq \text{THH}(S(p)[\sigma_k] \mid \sigma_k)$

where the left $\text{THH}(S(p)[\sigma_m] \mid \sigma_m)$-module structure on the right hand is provided by Construction 6.11.

Proof. We start by constructing the desired map. First, there is a composite map of $m$-graded $E_1$-ring spectra,

$\text{(6.18)} \quad S(p)[\sigma_k] \rightarrow \text{THH}(S(p)[\sigma_k]) \rightarrow \text{THH}(S(p)[\sigma_k] \mid \sigma_k)$

which is in particular a map of left $S(p)[\sigma_m]$-modules in $m$-graded spectra by forgetting structure through the $m$-graded $E_1$-ring map $S(p)[\sigma_m] \rightarrow S(p)[\sigma_k]$. Using the extension of scalars functor induced by the map

$\text{(6.19)} \quad S(p)[\sigma_m] \rightarrow \text{THH}(S(p)[\sigma_m] \mid \sigma_m)$

of $m$-graded $E_1$-algebras, we obtain the desired map:

$f: \text{THH}(S(p)[\sigma_m] \mid \sigma_m) \wedge_{S(p)[\sigma_m]} S(p)[\sigma_k] \rightarrow \text{THH}(S(p)[\sigma_k] \mid \sigma_k)$,

of left $\text{THH}(S(p)[\sigma_m] \mid \sigma_m)$-modules in $m$-graded spectra from (6.18). Here, we used the fact that the left $S(p)[\sigma_m]$-module structure on $\text{THH}(S(p)[\sigma_k] \mid \sigma_k)$ used in (6.18) is compatible with the one obtained by forgetting the canonical left $\text{THH}(S(p)[\sigma_m] \mid \sigma_m)$-module structure on $\text{THH}(S(p)[\sigma_k] \mid \sigma_k)$ through (6.19); this follows by the $p$-local version of Diagram (6.12).

What remains is to show that $f$ is an equivalence. Since $f$ is a map between $p$-local connective spectra (Proposition 6.10), it is sufficient to show that $HZ(p) \wedge f$ is an equivalence, see Lemma 4.16.

By inspection on the two sided bar construction defining relative smash products, one obtains that

$HZ(p) \wedge \text{THH}(S(p)[\sigma_m] \mid \sigma_m) \wedge_{S(p)[\sigma_m]} S(p)[\sigma_k] \simeq (HZ(p) \wedge \text{THH}(S(p)[\sigma_m] \mid \sigma_m)) \wedge_{HZ(p)[\sigma_m]} HZ(p)[\sigma_k]$.

Using the base change formula for THH, we obtain that $HZ(p) \wedge f$ is given by the canonical map

$HZ(p) \wedge f: \text{THH}^{HZ(p)}(HZ(p)[\sigma_m^{\pm 1}]_{\geq 0}) \wedge_{HZ(p)[\sigma_m]} HZ(p)[\sigma_k] \rightarrow \text{THH}^{HZ(p)}(HZ(p)[\sigma_k^{\pm 1}]_{\geq 0})$.

To prove that $HZ(p) \wedge f$ is an equivalence, we argue as in the proof of Proposition 4.14. The map of Bökstedt spectral sequences computing the map

$\text{(6.20)} \quad \text{THH}^{HZ(p)}(HZ(p)[\sigma_m^{\pm 1}]) \rightarrow \text{THH}^{HZ(p)}(HZ(p)[\sigma_k^{\pm 1}])$
is given on the second page, due to the HKR theorem, by the ring map

\[(6.21) \ \phi: \mathbb{Z}(p)[\sigma_{mk}^{\pm 1}] \otimes \Lambda_{\mathbb{Z}(p)}(d(\sigma_{mk})) \to \mathbb{Z}(p)[\sigma_k^{\pm 1}] \otimes \Lambda_{\mathbb{Z}(p)}(d(\sigma_k))\]

satisfying

\[\phi(\sigma_{mk}) = \sigma_k^m \text{ and } \phi(d(\sigma_{mk})) = m\sigma_k^{m-1}d(\sigma_k)\]

where \(m\) is a unit as \(p \nmid m\). In particular,

\[(6.22) \ \phi(\sigma_{mk}^{-1}d(\sigma_{mk})) = \sigma_k^{-m}m\sigma_k^{m-1}d(\sigma_k) = m\sigma_k^{-1}d(\sigma_k).\]

Here, \(\sigma_{mk}\) and \(\sigma_k\) are in degrees \((0, mk)\) and \((0, k)\) respectively and \(d(\sigma_{mk})\) and \(d(\sigma_k)\) are in degrees \((1, mk)\) and \((1, k)\) respectively. Furthermore, \(\sigma_{mk}\) and \(d(\sigma_{mk})\) are of weight \(m\) and \(\sigma_k\) and \(d(\sigma_k)\) are of weight \(1\). In particular, both Bökstedt spectral sequences degenerate on the second page and the map \(\phi\) provides the map \((6.20)\).

Taking connective covers in weight direction and identifying \(\sigma_{mk}^{-1}d(\sigma_{mk})\) as \(d\log\sigma_{mk}\) and \(\sigma_k^{-1}d(\sigma_k)\) as \(d\log\sigma_k\), we obtain that the map

\[\text{THH}^\mathcal{H}_{\mathbb{Z}(p)}(\mathcal{H}_{\mathbb{Z}(p)}[\sigma_{mk}^{\pm 1}])_{\geq 0} \to \text{THH}^\mathcal{H}_{\mathbb{Z}(p)}(\mathcal{H}_{\mathbb{Z}(p)}[\sigma_k^{\pm 1}])_{\geq 0}\]

is given by a map

\[\mathbb{Z}(p)[\sigma_{mk}] \otimes \Lambda_{\mathbb{Z}(p)}(d\log\sigma_{mk}) \to \mathbb{Z}(p)[\sigma_k] \otimes \Lambda_{\mathbb{Z}(p)}(d\log\sigma_k)\]

that carries \(\sigma_{mk}\) to \(\sigma_k^m\) and \(d\log\sigma_{mk}\) to \(d\log\sigma_k\) up to a unit due to \((6.22)\) as \(p \nmid m\).

Upon extending scalars with respect to the map \(\mathbb{Z}(p)[\sigma_{mk}] \to \mathbb{Z}(p)[\sigma_k]\), this map becomes an isomorphism. In other words, \(\pi_*(\mathcal{H}_{\mathbb{Z}(p)} \wedge f)\) is an isomorphism and therefore, \(f\) is an equivalence.

The following provides the logarithmic THH-étaleness of root adjunction in ring spectra.

**Theorem 6.23.** Assume Hypothesis 4.4 with \(p \nmid m\) and that \(A\) is \(p\)-local. In this situation, there is an equivalence of \(m\)-graded spectra

\[\text{THH}(A(\sqrt[m]{a}) \mid \sqrt[m]{a}) \simeq \text{THH}(A \mid a) \wedge_{\mathbb{S}(p)[\sigma_{mk}]} \mathbb{S}(p)[\sigma].\]

In other words, as an \(m\)-graded spectrum, \(\text{THH}(A(\sqrt[m]{a}) \mid \sqrt[m]{a})\) is given by

\[\text{THH}(A(\sqrt[m]{a}) \mid \sqrt[m]{a})_i \simeq \Sigma^{ik} \text{THH}(A \mid a)\]

for every \(0 \leq i < m\).

**Proof.** We have the following chain of equivalences

\[\text{THH}(A(\sqrt[m]{a}) \mid \sqrt[m]{a}) \simeq \text{THH}(A(\sqrt[m]{a}) \wedge_{\text{THH}(\mathbb{S}(p)[\sigma_k])} \text{THH}(\mathbb{S}(p)[\sigma_k] \mid \sigma_k)\]

\[\simeq (\text{THH}(A) \wedge_{\text{THH}(\mathbb{S}(p)[\sigma_{mk}])} \text{THH}(\mathbb{S}(p)[\sigma_k] \mid \sigma_k)) \wedge_{\text{THH}(\mathbb{S}(p)[\sigma_k])} \text{THH}(\mathbb{S}(p)[\sigma_k] \mid \sigma_k)\]

\[\simeq \text{THH}(A) \wedge_{\text{THH}(\mathbb{S}(p)[\sigma_{mk}])} \text{THH}(\mathbb{S}(p)[\sigma_k] \mid \sigma_k)\]

\[\simeq \text{THH}(A) \wedge_{\text{THH}(\mathbb{S}(p)[\sigma_{mk}])} \text{THH}(\mathbb{S}(p)[\sigma_k] \mid \sigma_{mk}) \wedge_{\text{THH}(\mathbb{S}(p)[\sigma_{mk}] \mid \sigma_{mk})} \text{THH}(\mathbb{S}(p)[\sigma_k] \mid \sigma_k)\]

\[\simeq \text{THH}(A) \wedge_{\mathbb{S}(p)[\sigma_{mk}]} \text{THH}(\mathbb{S}(p)[\sigma_k] \mid \sigma_k)\]

\[\simeq \text{THH}(A) \wedge_{\mathbb{S}(p)[\sigma_{mk}]} \text{THH}(\mathbb{S}(p)[\sigma_k] \mid \sigma_{mk}) \wedge_{\mathbb{S}(p)[\sigma_{mk}] \mid \sigma_{mk}} \text{THH}(\mathbb{S}(p)[\sigma_k] \mid \sigma_k)\]

\[\simeq \text{THH}(A) \wedge_{\mathbb{S}(p)[\sigma_{mk}] \mid \sigma_{mk}} \text{THH}(\mathbb{S}(p)[\sigma_k] \mid \sigma_k)\]
The first and the fifth equivalences follow by the definition of logarithmic THH, the second equivalence follows by our definition of root adjunction and Proposition 4.14 and the sixth equivalence follows by Proposition 6.17.

□

Remark 6.24. In [RSS18], the authors show that $\ell \to ku_{(p)}$ is logarithmic THH-étale. This compares to our result above since we show that $ku_{(p)} \simeq \ell_{(p)}(\sqrt[p]{\eta})$ in Theorem 4.10.

6.2. Relating THH and logarithmic THH. The goal of this section is to show that there is a fiber sequence

$$\text{THH}(A) \to \text{THH}(A \mid a) \to \Sigma \text{THH}(A/a)$$

under our usual assumptions. The $E_2$-ring $A/a$ above is described in the following construction which is analogous to [RSS15, Lemmas 6.14 and 6.15].

Construction 6.25. Let $A$ be an $S[\sigma_k]$-algebra where $\sigma_k$ acts through $a \in \pi_k A$ where $k \geq 0$ is even. The weight 0 Postnikov section of $S[\sigma_k]$ provides a map $S[\sigma_k] \to S$ of $E_2$-rings [HW22, B.0.6]. Considering the extension of scalars functor $- \otimes_{S[\sigma_k]} S$ from the $\infty$-category of $E_1 S[\sigma_k]$-algebras to the $\infty$-category of $E_1 S$-algebras, one equips $A/a := A \otimes_{S[\sigma_k]} S$ with the structure of an $E_1$-ring spectrum. Since $S$ is the cofiber of the map $S[\sigma_k] \to S[\sigma_k]$, $A/a$ is indeed the cofiber of the map $A \to A$.

Considering $S[\sigma_k]$ as a graded $E_2$-ring, we have

$$\text{THH}(S[\sigma_k])_0 \simeq S.$$

This can be observed by inspection on the cyclic bar construction on $S[\sigma_k]$ or by computing the $HZ$-homology of the left hand side above. This is used in the statement of the following proposition.

Remark 6.26. The following proposition is analogous to [RSS15, Proposition 6.11]. We remark that unlike in loc cit., we do not take $S^1$-equivariance into account, which leads to a simpler proof.

Proposition 6.27. The cofiber of the map

$$\text{THH}(S[\sigma_k]) \to \text{THH}(S[\sigma_k] \mid \sigma_k)$$

is given by $\Sigma S$ concentrated in weight 0 as a left $\text{THH}(S[\sigma_k])$-module in graded spectra. Here, the left $\text{THH}(S[\sigma_k])$-module structure on $S$ is given by the weight-Postnikov truncation map of graded $E_1$-rings

$$\text{THH}(S[\sigma_k]) \to \text{THH}(S[\sigma_k])_0 \simeq S.$$

Proof. Let $M$ be the cofiber of the map $f$ below in left $\text{THH}(S[\sigma_k])$-modules in graded spectra.

$$\text{THH}(S[\sigma_k]) \xrightarrow{f} \text{THH}(S[\sigma_k], \mid \sigma_k) \to M$$

We start by computing $HZ_* M$. The map

$$HZ_* f: \text{THH}_{HZ}^*(HZ[\sigma_k]) \to \text{THH}_{HZ}^*(HZ[\sigma_k^{\pm 1}])_{\geq 0}$$

is given by the ring map

\[ \mathbb{Z}[\sigma_k] \otimes \Lambda(d(\sigma_k)) \to \mathbb{Z}[\sigma_k] \otimes \Lambda(d\log_x) \]

that carries \( \sigma_k \) to \( \sigma_k \) and \( d(\sigma_k) \) to \( \sigma_k d\log_x \); this follows by the Böksedt spectral sequences in (4.18) and (6.21). This map is injective and the only class that is not in the image is \( d\log_x \). We obtain,

\[ H\mathbb{Z} \wedge M \simeq \Sigma H\mathbb{Z} \]

where the right hand side is concentrated in weight 0. Due to Proposition 6.10, \( f \) is a map between connective spectra. In particular, \( M \) is connective and we obtain an equivalence of spectra

\[ M \simeq \Sigma S. \]

We need to improve this to an equivalence of left \( \text{THH}(S[\sigma_k]) \)-modules in graded spectra. Since \( M \) is a left \( \text{THH}(S[\sigma_k]) \)-module in graded spectra, there is a map

\[ \Sigma \text{THH}(S[\sigma_k]) \to M \]

of left \( \text{THH}(S[\sigma_k]) \)-modules in graded spectra carrying 1 to 1 in homotopy. Taking weight 0 Postnikov sections [HW22, B.0.6], we obtain an equivalence of left \( \text{THH}(S[\sigma_k]) \)-modules in graded spectra

\[ \Sigma \text{THH}(S[\sigma_k])_0 \simeq M. \]

This map is an equivalence because it carries 1 to 1 in homotopy by construction and since both sides are equivalent as spectra to \( \Sigma S \).

We are ready to provide the cofiber sequence relating THH to logarithmic THH.

**Theorem 6.28.** Let \( A \) be an \( S[\sigma_k] \)-algebra where \( \sigma_k \) acts through \( a \in \pi_k A \) with even \( k \geq 0 \). In this situation, there is a cofiber sequence of spectra:

\[ \text{THH}(A) \to \text{THH}(A \mid a) \to \Sigma \text{THH}(A/a). \]

The corresponding cofiber sequence for \( \text{THH}(A(\sqrt[n]{\sigma_k}) \mid \sqrt[n]{\sigma_k}) \) is a cofiber sequence of \( m \)-graded spectra.

**Proof.** Proposition 6.27 provides the following cofiber sequence of left \( \text{THH}(S[\sigma_k]) \)-modules in graded spectra.

\[ \text{THH}(S[\sigma_k]) \to \text{THH}(S[\sigma_k] \mid \sigma_k) \to \Sigma S \]

Applying the functor \( \text{THH}(A) \wedge_{\text{THH}(S[\sigma_k])} - \) to this cofiber sequence, we obtain the following cofiber sequence

\[ (6.29) \quad \text{THH}(A) \to \text{THH}(A \mid a) \to \text{THH}(A) \wedge_{\text{THH}(S[\sigma_k])} \Sigma S. \]

What is left is to identify the cofiber above as \( \text{THH}(A/a) \). We have

\[ (6.30) \quad \text{THH}(A) \wedge_{\text{THH}(S[\sigma_k])} \Sigma S \simeq \Sigma \text{THH}(A) \wedge_{\text{THH}(S[\sigma_k])} S. \]

Here, \( S \) on the right hand side denotes the de-suspension of \( \Sigma \text{THH}(S[\sigma_k])_0 \) as a left \( \text{THH}(S[\sigma_k]) \)-module in graded spectra, see Proposition 6.27. This is \( \text{THH}(S[\sigma_k])_0 \) which admits the structure of a graded \( E_1 \)-ring spectrum equipped with a map \( \text{THH}(S[\sigma_k]) \to \text{THH}(S[\sigma_k])_0 \) of graded \( E_1 \)-ring spectra given by the relevant weight 0 Postnikov section map. Indeed, due to the universal property of Postnikov sections,
this weight 0 Postnikov section map factors the map of graded $E_1$-rings $\text{THH}(S[\sigma_k]) \to \text{THH}(S)$ induced by the weight 0 Postnikov section map $S[\sigma_k] \to S$; i.e. we have a factorization of this map of graded $E_1$-algebras as

$$\text{THH}(S[\sigma_k]) \to \text{THH}(S[\sigma_k])_0 \simeq \text{THH}(S).$$

The second map above is an equivalence as its domain and codomain are equivalent to $S$ as spectra and it carries the unit to the unit by construction. In particular, we can replace $S$ on the right hand side of (6.30) with $\text{THH}(S[\sigma_k])$. This provides the first equivalence below.

$$\Sigma \text{THH}(A) \wedge_{\text{THH}(S[\sigma_k])} S \simeq \Sigma \text{THH}(A) \wedge_{\text{THH}(S[\sigma_k])} \text{THH}(S)$$

(6.31)

The second equivalence follows by Proposition 4.14 and the third equivalence follows by our description of the $E_1$-algebra $A/a$ in Construction 6.25. Equations (6.30) and (6.31) identify the cofiber in (6.29) as $\text{THH}(A/a)$ providing the cofiber sequence claimed in the theorem.

The statement regarding the cofiber sequence in $m$-graded spectra for $A(\sqrt[m]{a})$ follows by utilizing same arguments. □

Remark 6.32. The above localization sequence is of fundamental importance in the theory of log THH. A proof of the above localization sequence for general $E_2$ log structures, using more general methods, will be supplied in [DM22].

6.3. THH after root adjunction. Here, we identify $\text{THH}(A(\sqrt[m]{a}))$ in terms of $\text{THH}(A)$ and $\text{THH}(A \mid a)$.

Theorem 6.33 (Theorem 1.3). Assume Hypothesis 4.4 with $p \nmid m$ and that $A$ is $p$-local. In this situation, the $m$-graded spectrum $\text{THH}(A(\sqrt[m]{a}))$ is given by

$$\text{THH}(A(\sqrt[m]{a}))_0 \simeq \text{THH}(A)$$

and

$$\text{THH}(A(\sqrt[m]{a}))_i \simeq \Sigma^i k \text{THH}(A \mid a) \quad \text{for } 0 < i < m.$$ 

In particular, there is an equivalence of spectra:

$$\text{THH}(A(\sqrt[m]{a})) \simeq \text{THH}(A) \vee \bigvee_{0 < i < m} \Sigma^i k \text{THH}(A \mid a).$$

Proof. The identification of $\text{THH}(A(\sqrt[m]{a}))_0$ is provided by Proposition 4.20. Therefore, it is sufficient to provide the identification of $\text{THH}(A(\sqrt[m]{a}))_i$ for $i \neq 0$.

Due to Theorem 6.23

$$\text{THH}(A(\sqrt[m]{a})_i \simeq \Sigma^i k \text{THH}(A \mid a).$$

Therefore, it is sufficient to show that

$$\text{THH}(A(\sqrt[m]{a}))_i \simeq \text{THH}(A(\sqrt[m]{a})_i \simeq \text{THH}(A(\sqrt[m]{a})_i,$$

whenever $i \neq 0$. This follows once we show that the cofiber of the the map

$$\text{THH}(A(\sqrt[m]{a})) \to \text{THH}(A(\sqrt[m]{a}) \mid \sqrt[m]{a})$$

(6.36)
of $m$-graded spectra is concentrated in weight 0. Due to Theorem 6.28 the cofiber of this map is given by

$$\text{THH}(A(\sqrt[1]{a})/\sqrt[1]{a})$$

where $A(\sqrt[1]{a})/\sqrt[1]{a}$ is defined to be $A(\sqrt[1]{a}) \wedge_{S[\sigma_k]} S$. Therefore, we have

$$A(\sqrt[1]{a})/\sqrt[1]{a} := A(\sqrt[1]{a}) \wedge_{S[\sigma_k]} S \simeq A \wedge_{S[\sigma_{mk}]} S[\sigma_k] \wedge_{S[\sigma_k]} S \simeq A \wedge_{S[\sigma_{mk}]} S.$$ 

Since $A$, $S[\sigma_{mk}]$, and $S$ are concentrated in weight 0, we obtain that $A(\sqrt[1]{a})/\sqrt[1]{a}$ and therefore $\text{THH}(A(\sqrt[1]{a})/\sqrt[1]{a})$ are also concentrated in weight 0. This proves that the cofiber of (6.36) is concentrated in weight 0 which proves (6.35) and this, together with (6.34) proves the theorem. □

7. Algebraic $K$-theory of complex and real topological $K$-theories

Here, we start by showing that $K(ku_{p})$ splits into $p - 1$ non-trivial summands. Afterwards, we show that $ku_{p}$ may be constructed from $ko_{p}$ via root adjunction. We use this to obtain an explicit description of the $V(1)$-homotopy of $K(ko_{p})$ from the first authors computation of $V(1), K(ku_{p})$ [Aus10].

7.1. Adams’ splitting result for 2-vector bundles. Recall that $\pi_{*}ku_{p} \cong \mathbb{Z}_{p}[u]$ and $\pi_{*}\ell_{p} \cong \mathbb{Z}_{p}[v_{1}]$, where $|u| = 2$ and $|v_{1}| = 2p - 2$. The $E_{\infty}$-map $\ell_{p} \to ku_{p}$ carries $v_{1}$ to $u^{p-1}$ in homotopy. For the rest of this section, we fix a map $S[\sigma_{2(p-1)}] \to \ell_{p}$ of $E_{2}$-algebras carrying $\sigma_{2(p-1)}$ to $v_{1}$ and perform root adjunction using this map. Recall from Theorem 4.10 that there is an equivalence of $E_{1}$ $\ell_{p}$-algebras

$$\ell_{p}(\sqrt[p]{v_{1}}) \cong ku_{p}.$$ 

This equips $ku_{p}$ with the structure of a $p - 1$-graded $E_{1}$ $\ell_{p}$-algebra which further equips $\text{THH}(ku_{p})$ with the structure of a $p - 1$-graded $S^{1}$-equivariant spectrum.

Let $p > 3$ be a prime and let $V(1)$ denote the type-2 finite spectrum used in [Aus05]; $V(1)$ is a homotopy ring spectrum.

There is another grading on $V(1), \text{THH}(ku_{p})$ that the first author calls the $\delta$-grading [Aus05]. The group $\Delta := \mathbb{Z}/(p - 1)$ acts on the $E_{\infty}$-ring $ku_{p}$ through Adams operations. Let $\delta \in \Delta$ be a chosen generator and let $\alpha \in \mathbb{F}_{p}^{x}$ satisfy $\pi_{*}(S/p \wedge \delta)(u) = \alpha u$ where

$$\pi_{*}(S/p \wedge \delta) : \pi_{*}(S/p \wedge ku_{p}) \to \pi_{*}(S/p \wedge ku_{p}) \cong \mathbb{F}_{p}[u].$$

We say $u^{i}$ has $\delta$-weight $i$ as $\pi_{*}(S/p \wedge \delta)(u^{i}) = \alpha^{i} u^{i}$. Similarly, one says $x \in V(1), \text{THH}(ku_{p})$ has $\delta$-weight $i$ if the self map of $V(1), \text{THH}(ku_{p})$ induced by $\delta$ carries $x$ to $\alpha^{i} x$. One defines $\delta$-weight in a similar way on other invariants of $ku_{p}$ [Aus05, Definition 8.2].

Proposition 7.1. The group $V(1), \text{THH}(ku_{p})_{i}$ is given by the classes of $\delta$-weight $i$ in $V(1), \text{THH}(ku_{p})$.

Proof. Since $HF_{p} \wedge ku_{p}$ is a $p - 1$ graded $E_{1} HF_{p}$-algebra, there is a $p - 1$-grading on $HH_{*}^{HF}(HF_{p} \wedge ku_{p})$. By inspection on the Hochschild complex, one observes that the $\delta$-weight grading on $HH_{*}^{HF}(HF_{p} \wedge ku_{p})$ agrees with the weight grading. In particular, the $\delta$-weight grading and the weight grading agree on the second page of the Bökstedt spectral sequence computing $HH_{*}^{HF}(HF_{p} \wedge ku_{p})$. Due to [Aus05, Section 9], this shows that the $\delta$-weight grading and the weight grading agree on $HH_{*}^{HF}(HF_{p} \wedge ku_{p})$. 

Furthermore, there is a basis of $\text{HH}^F_{p}(HF_{p} \wedge k\mathfrak{u}_p)$ as an $F_p$-module where $\delta$-weight is defined for each basis element. Therefore, the $HF_{p}$-module $\text{HH}^F_{p}(HF_{p} \wedge k\mathfrak{u}_p)$ splits as a coproduct of suspensions of $HF_{p}$ in a way that the map $\text{HH}^F_{p}(HF_{p} \wedge \delta)$ is given by the respective multiplication map corresponding to the $\delta$-weight on each cofactor. Using this, one observes that the $\delta$-weight and the weight grading agree on $H_{*}(V(1) \wedge \text{THH}(k\mathfrak{u}_p); F_p)$.

The Hurewicz map

$$V(1)_{*} \text{THH}(k\mathfrak{u}_p) \rightarrow H_{*}(V(1) \wedge \text{THH}(k\mathfrak{u}_p); F_p)$$

is injective and this map preserves both gradings. From this, we deduce that the weight grading and the $\delta$-weight grading agree on $V(1)_{*} \text{THH}(k\mathfrak{u}_p)$. □

In general, THH of $m$-graded ring spectra may not result in an $m$-graded cyclotomic spectrum as the Frobenius map do not preserve the grading; it multiplies the grading by $p$. On the other hand, for $k\mathfrak{u}_p$, THH($k\mathfrak{u}_p$) is $p-1$-graded and $p = 1$ in $\mathbb{Z}/(p-1)$. In particular, the Frobenius map preserves the grading and one obtains that THH($k\mathfrak{u}$) is a $p-1$-graded cyclotomic spectrum.

**Proposition 7.2.** The $S^1$-equivariant structure on THH($k\mathfrak{u}_p$)$_i$ lifts to a cyclotomic structure for which there is an equivalence

$$\text{THH}(k\mathfrak{u}) \simeq \prod_{i \in \mathbb{Z}/(p-1)} \text{THH}(k\mathfrak{u}_p)_i$$

of cyclotomic spectra.

**Proof.** The monoid $\mathbb{Z}/(p-1)$ satisfies the conditions in [AMMN22, Appendix A] needed endow THH($k\mathfrak{u}$) with an $L_p$ twisted cyclotomic structure. However, since $p \equiv 1 \mod p-1$, this ends up being the identity functor on $\mathbb{Z}/(p-1)$-graded spectra. Thus one obtains a sequence of $S^1$-equivariant maps

$$\text{THH}(k\mathfrak{u}_p)_i \rightarrow \text{THH}(k\mathfrak{u}_p)^{IC_p}$$

for each $i \in \mathbb{Z}/(p-1)$, which is precisely the relevant additional piece of structure needed to view this as a cyclotomic object. □

**Construction 7.3.** Here, we construct a splitting of $K(k\mathfrak{u}_p)$ using Proposition 7.2. Since the product mentioned in Proposition 7.2 is a finite product, it is at the same time a coproduct. In particular, it commutes with all limits and colimits. Therefore, the fiber sequence defining TC($k\mathfrak{u}_p$) splits into a product of fiber sequences

$$\text{TC}(k\mathfrak{u}_p)_i \rightarrow \text{THH}(k\mathfrak{u}_p)_{i}^{S^1} \xrightarrow{(\varphi_p)_{i}^{\text{can}_i}} (\text{THH}(k\mathfrak{u}_p)^{IC_p})_{i}^{S^1}.$$ 

Hence, there is a splitting of TC($k\mathfrak{u}_p$):

$$\text{TC}(k\mathfrak{u}_p) \simeq \prod_{i \in \mathbb{Z}/(p-1)} \text{TC}(k\mathfrak{u}_p)_i$$

where $\text{TC}(k\mathfrak{u}_p)_i := \text{TC}($THH($k\mathfrak{u}_p)$)$_i$). Arguing as in the proof of Theorem 5.8, one obtains a map $k\mathfrak{u}_p \rightarrow H\mathbb{Z}_p$ of $p-1$-graded $E_i$-rings where $H\mathbb{Z}_p$ is concentrated in weight 0. Therefore, the induced map THH($k\mathfrak{u}_p$) → THH($\mathbb{Z}_p$) of $p-1$-graded spectra is trivial in non-zero weight. By inspection on the product splitting of the fiber sequence defining TC($k\mathfrak{u}_p$), we consider TC($k\mathfrak{u}_p$) → TC($\mathbb{Z}_p$) as a map of $p-1$ graded
spectra where $TC(\mathbb{Z}_p)$ is concentrated in weight 0. Again, as in the proof of Theorem 5.8 this splits the pull-back square (from Dundas-Goodwillie-McCarthy theorem) relating $TC(ku_p)$ to $K(ku_p)$ resulting in a splitting of $K(ku_p)$ that we denote by

$$K(ku_p) \simeq \bigvee_{i \in \mathbb{Z}/(p-1)} K(ku_p)_i.$$  

Here, $K(ku_p)_0 \simeq K(\ell_p)$ due to Theorem 5.8.

To understand the resulting splitting of $K(ku_p)$, we identify the $V(1)$-homotopy of each weight piece. The computation of $V(1)_*K(ku_p)$ is due to the first author [Aus10, Theorem 8.1] and these groups are given below.

$$V(1)_*K(ku_p) \cong \mathbb{F}_p[b] \otimes \Lambda(\lambda_1, a_1) \oplus \mathbb{F}_p[b] \otimes \mathbb{F}_p(\partial \lambda_1, \partial b, \partial a_1, \partial \lambda_1 a_1)$$

$$\oplus \mathbb{F}_p[b] \otimes \Lambda(a_1) \otimes \mathbb{F}_p\{\lambda_2 b^{d-1} | 0 < d < p\}$$

$$\oplus \mathbb{F}_p[b] \otimes \Lambda(\lambda_1) \otimes \mathbb{F}_p\{\sigma_n, \lambda_2 b^{d-p} | 1 \leq n \leq p-2\}$$

$$\oplus \mathbb{F}_p[s]$$

(7.4)

Here, $|b| = 2p+2$, $|\partial| = -1$, $|\lambda_1| = 2p-1$, $|a_1| = 2p+3$, $|\sigma_n| = 2n+1$, $|t| = -2$, $|\lambda_2| = 2p^2-1$ and $|s| = 2p-3$. We assign weights to these classes in a way that turns $V(1)_*K(ku_p)$ into a $p-1$-graded abelian group. The weights of $\sigma_n$, $b$, $a_1$, $\partial$, $\lambda_1$, $t$, $\lambda_2$ and $s$ are given by $n$, 1, 1, 0, 0, 0 and 0 respectively. Classes denoted by tensor products or products above have the canonical degrees and weights. Furthermore, the isomorphism above is that of $\mathbb{F}_p[b]$-modules and $b^{d-1} = -v_2$.

**Theorem 7.5.** For the equivalence of spectra

$$K(ku_p) \simeq \bigvee_{i \in \mathbb{Z}/(p-1)} K(ku_p)_i$$

provided by Construction 7.3, there is an equivalence:

$$K(ku_p)_0 \simeq K(\ell_p)$$

and there are isomorphisms

$$V(1)_*(K(ku_p)) \cong (V(1)_*K(ku_p))_i$$

for each $i \in \mathbb{Z}/(p-1)$ where the right hand side denotes the weight $i$ piece of the $p-1$-grading on $V(1)_*K(ku_p)$ described above.

**Proof.** The identification of $K(ku_p)_0$ is given in Construction 7.3. This provides the identification of $V(1)_*K(ku_p)_0$ as $(V(1)_*K(ku_p))_0$ since this is precisely the image of the map

$$V(1)_*K(\ell_p) \to V(1)_*K(ku_p),$$

see [Aus05, Theorem 10.2]. The identification of $V(1)_*(K(ku_p))_i$ for $i \neq 0$ follows by noting from Proposition 7.4 that it is sufficient to keep track of the contribution of $\delta$-weight $i$ classes in $V(1)_*THH(ku_p)$ to $V(1)_*TC(ku_p)$. This follows by inspection on [Aus10, Section 7] and [Aus10, Section 5].
7.2. Algebraic $K$-theory of real $K$-theory. Let $p > 3$. Using Theorem 5.8, the splitting of $K(\mathcal{K}u_p)$ discussed above and our root adjunction formalism, we obtain a straightforward computation of $V(1)_*K(ko_p)$ from our knowledge of $V(1)_*K(\mathcal{K}u_p)$ from [Aus10]. Here, $ko_p$ denotes the connective cover of the $p$-completed real topological $K$-theory spectrum $KO_p$. We have $\pi_*KO_p \cong \mathbb{Z}_p[\alpha^{\pm 1}]$ with $|\alpha| = 4$.

There is a subgroup of $C_2$ of $\Delta \cong \mathbb{Z}/(p-1)$ such that $KO_p \cong KU^h\Delta_2$. Through this, the induced map $KO_p \rightarrow KU_p$ carries $\alpha$ to $\alpha^2$ up to a unit that we are going to omit. Since $L \cong (KU_p)^{h\Delta}$, we obtain a sequence of $E_\infty$-maps

$$L_p \rightarrow KO_p \rightarrow KU_p$$

where the first map carries $v_1$ to $\alpha^{p-1}$ in homotopy.

**Theorem 7.6.** For $p > 3$, there is an equivalence

$$ko_p \simeq \ell_p(\sqrt[p]{v_1})$$

of $E_1$ $\ell_p$-algebras.

**Proof.** This follows as in the proof of Theorem 4.10 by noting that $p \nmid \frac{p-1}{2}$. \qed

Furthermore, $\mathcal{K}u_p$ may also be obtained from $ko_p$ via root adjunction; for this root adjunction, we use the $\mathbb{S}_p[\sigma_4]$-algebra structure on $ko_p$ provided by Theorem 7.6. To identify the resulting 2-graded $E_1$-ring structure on $\mathcal{K}u_p$, we use the symmetric monoidal functor

$$D': \text{Fun}(\mathbb{Z}/(p-1), \text{Sp}) \rightarrow \text{Fun}(\mathbb{Z}/2, \text{Sp})$$

given by left Kan extension through the canonical map $\mathbb{Z}/(p-1) \rightarrow \mathbb{Z}/2$.

**Proposition 7.7.** For $p > 3$, there is an equivalence

$$ko_p(\sqrt[p]{\alpha}) \simeq D'(\mathcal{K}u_p)$$

of 2-graded $E_1$-algebras where $D'$ is defined above and the $p - 1$-grading on $\mathcal{K}u_p$ is given by Theorem 4.10.

**Proof.** Due to Theorem 7.6, $ko_p$ is an $\mathbb{S}[\sigma_4]$-algebra given by

$$\ell_p \wedge_{\mathbb{S}[\sigma_2(p-1)]} \mathbb{S}[\sigma_4].$$

To adjoin a root to $ko_p$ using this structure, we use the sequence of maps

$$\mathbb{S}[\sigma_2(p-1)] \rightarrow \mathbb{S}[\sigma_4] \rightarrow D'(\mathbb{S}[\sigma_2])$$

of 2-graded $E_2$-ring spectra where $\mathbb{S}[\sigma_2(p-1)]$ and $\mathbb{S}[\sigma_4]$ are concentrated in weight 0 and $\mathbb{S}[\sigma_2]$ above is given its canonical $p - 1$-grading so that $\sigma_2$ in $D'(\mathbb{S}[\sigma_2])$ lies in weight 1.

We obtain the following equivalences of 2-graded $E_1$-rings.

(7.8) $$ko_p(\sqrt[p]{\alpha}) \simeq \ell_p \wedge_{\mathbb{S}[\sigma_2(p-1)]} \mathbb{S}[\sigma_4] \wedge_{\mathbb{S}[\sigma_4]} D'(\mathbb{S}[\sigma_2]) \simeq \ell_p \wedge_{\mathbb{S}[\sigma_2(p-1)]} D'(\mathbb{S}[\sigma_2])$$

The functor $D'$ is a left adjoint and it is symmetric monoidal. Therefore, it commutes with the two sided bar construction defining relative smash products. This provides the second equivalence in the following equivalences of 2-graded $E_1$-algebras.

(7.9) $$D'(\mathcal{K}u_p) \simeq D'(\ell_p \wedge_{\mathbb{S}[\sigma_2(p-1)]} \mathbb{S}[\sigma_2]) \simeq D'(\ell_p \wedge_{D'(\mathbb{S}[\sigma_2(p-1)])} D'(\mathbb{S}[\sigma_2]))$$

The first equivalence above follows by Theorem 4.10 and the relative smash product in the middle is taken in $p - 1$-graded spectra. Since $\ell_p$ and $\mathbb{S}[\sigma_2(p-1)]$ are concentrated
in weight 0, the right hand side above is equivalent to the right hand side of (7.8); this follows by Lemma 2.3. In other words, (7.8) and (7.9) agree. □

Recall that the spectra $K(ku_p)_i$ are given in Construction 7.3 and the groups $V(1)_*K(ku_p)_i$ are identified in Theorem 7.5.

**Theorem 7.10.** For $p > 3$, there is an equivalence of spectra:

$$K(ko_p) \simeq \bigvee_{0 \leq i < (p-1)/2} K(ku_p)_{2i}.$$ 

Therefore, we have

$$V(1)_*K(ko_p) \cong \bigoplus_{0 \leq i < (p-1)/2} V(1)_*K(ku_p)_{2i}.$$ 

and $V(1)_*K(ko_p)_i$, as an abelian group, is given by:

$$V(1)_*K(ko_p) \cong \mathbb{F}_p[b^2] \otimes \Lambda(\lambda_1, ba_1) \oplus \mathbb{F}_p[b^2] \otimes \mathbb{F}_p\{\partial \lambda_1, b \partial b, b \partial a_1, b \partial \lambda_1 a_1\}$$

$$\oplus \mathbb{F}_p[b^2] \otimes \Lambda(ba_1) \oplus \mathbb{F}_p\{t^d \lambda_1 | 0 < d < p\}$$

$$\oplus \mathbb{F}_p[b^2] \otimes \Lambda(\lambda_1) \otimes \mathbb{F}_p\{ b^{e(n)} \sigma_n, \lambda_2 t^{i^2-p} \mid 1 \leq n \leq p - 2 \}$$

$$\oplus \mathbb{F}_p\{s\},$$

where $e(n) = 1$ if $n$ is odd and $e(n) = 0$ if $n$ is even. Here, the class denoted by $(b^2)^{(p-1)/2}$ is $-v_2$.

As a consequence, we have an isomorphism of abelian groups:

$$T(2)_*K(ko) \cong T(2)_*K(\ell_p)[b^2]/((b^2)^{(p-1)/2} + v_2).$$

**Proof.** We start by identifying $\text{THH}(ko_p)$ as a cyclotomic spectrum. We have the following chain of equivalences

$$\text{THH}(ko_p) \simeq \text{THH}(ko_p(\sqrt[p]{\alpha}))_0$$

$$\simeq \text{THH}(D'(ku_p))_0$$

$$\simeq (D'(\text{THH}(ku_p)))_0$$

$$\simeq \prod_{0 \leq i < (p-1)/2} \text{THH}(ku_p)_{2i}$$

The first equivalence above follows by Theorem 1.20, the second equivalence follows by Proposition 7.4 and the third equivalence is a consequence of [AMMN22, Corollary A.15]. The last equivalence above follows by the description of $D'$ as a left Kan extension, see Section 2.2. Indeed, this shows that the following composite map of cyclotomic spectra is an equivalence.

$$\text{THH}(ko_p) \rightarrow \text{THH}(ku_p) \simeq \prod_{0 \leq i < p-1} \text{THH}(ku_p)_i \rightarrow \prod_{0 \leq i < (p-1)/2} \text{THH}(ku_p)_{2i}$$

Here, the equivalence in the middle follows by Proposition 7.2. The last map above is the canonical projection.
The composite equivalence of cyclotomic spectra above shows that
\[ TC(\text{THH}(ko_p)) \simeq TC(\prod_{0 \leq i < (p-1)/2} \text{THH}(ku_p)) \simeq \prod_{0 \leq i < (p-1)/2} TC(ku_p). \]
Considering the Dundas-Goodwillie-McCarthy theorem with respect to the composite \( ko_p \to ku_p \to HZ_p \), we obtain that the splitting of the pullback square relating \( K(ku_p) \) with \( TC(ku_p) \) (mentioned in Construction 7.3) provides a splitting for \( K(ko_p) \) given by
\[ (7.11) \quad K(ko_p) \simeq \prod_{0 \leq i < (p-1)/2} K(ku_p). \]
The first and the second statements in the theorem follow from this splitting. The third statement follows by this, and by inspection on (7.2).

For the last statement, note that \( T(2)_* K(ko) \cong T(2)_* K(ko_p) \) due to the purity of algebraic \( K \)-theory and \([LMMT20, \text{Lemma 2.2 (vi)}]\). It follows by Theorem 7.5, that the map
\[ T(2)_* K(ku_p) \overset{b'}{\to} T(2)_* K(ku_p) \]
carries \( T(2)_* K(ku_p) \) to \( T(2)_* K(ku_p) \) for \( i < p-1 \) where the map above multiplies by \( b' \). Using this fact, together with \([\text{Aus10, Proposition 1.2 (b)}]\), provides isomorphisms
\[ T(2)_* K(ku_p) \cong T(2)_* K(ku_p) \overset{\cong}{\to} T(2)_* K(ku_p), \]
given by \( b' \) for \( i < p-1 \). This, together with (7.11) provides the desired identification of \( T(2)_* K(ko) \cong T(2)_* K(ko_p) \) as \( T(2)_* K(ku_p) [b^2]/((b^2)^{(p-1)/2} + v_2) \).

8. Root adjunction and Lubin-Tate spectra

Recall that in \([\text{HW22}]\), Hahn and Wilson prove that there are \( E_3 \) \( MU \)-algebra forms of \( BP(n) \). Furthermore, their constructions provide an \( E_3 \) \( MU[\sigma_2^{(p-1)}] \)-algebra form of \( BP(n) \) where \( \sigma_2^{(p-1)} \) acts through \( v_n \) \([\text{HW22, Remark 2.1.2}]\).

To relate particular forms of \( BP(n) \) to Lubin-Tate spectra, we use the spherical Witt vectors constructed by Lurie \([\text{Lur18, Example 5.2.7}]\). For a given discrete perfect \( S \)-algebra form \( B \), this provides an \( E_\infty \)-ring \( S_{W(B)} \) that is flat over \( S \) in the sense of \([\text{Lur17, Definition 7.2.2.10}]\). Therefore, it follows by \([\text{Lur17, Proposition 7.2.2.13}]\) and \([\text{Mao20, Proposition 2.7}]\) that
\[ (8.1) \quad \pi_n(S_{W(B)} \wedge F) \cong W(B) \otimes \pi_n F. \]
for every spectrum \( F \). We would like to thank Jeremy Hahn for showing us the proof of the following proposition.

**Proposition 8.2.** Fix an \( E_3 \) \( MU \)-algebra form of \( BP(n) \). Then \( S_{W(B)} \wedge BP(n) \) satisfies the redshift conjecture for every discrete perfect \( F \)-algebra \( B \).

**Proof.** There is an equivalence of \( E_2 \)-algebras in \( S^1 \)-equivariant spectra
\[ S_{W(B)} \wedge \text{THH}^*_{BP(n)} \cong \text{THH}^*_{S_{W(B)} \wedge BP(n)}. \]
Therefore, it follows by (8.1) that we have
\[ (8.3) \quad \text{THH}^*_{S_{W(B)} \wedge BP(n)} \cong W(B) \otimes \text{THH}^*_{BP(n)}. \]
and this is a polynomial algebra over $W(B_0)[v_1, ..., v_n]$ due to [HW22, Theorem 2.5.4] where one of the generators is denoted by $\sigma^2 v_{n+1}$. The rest of the argument follows as in the proofs of [HW22, Theorems 2.5.4 and 5.0.1, Corollary 5.0.2] by considering
\[
\pi_*(\text{THH}^{E_{\infty}}(B_0) \wedge MU)(S_{W(B_0)} \wedge BP(n))^{hS^1})
\]
instead of $\pi_*(\text{THH}^{MU}(BP(n))^{hS^1})$.

Namely, we obtain from [HW22, Theorem 2.5.4] that $[S,3]$ is concentrated in even degrees. Therefore, the corresponding homotopy fixed point spectral sequence degenerates on the second page providing the $W(B_0)[v_1, ..., v_n]$-algebra $[S,4]$ as
\[
W(B_0) \otimes \text{THH}^{MU}_{\ast}(BP(n))[t]
\]
since $[S,4]$ is polynomial. Using the map from $\pi_*(\text{THH}^{MU}(BP(n))^{hS^1})$ to $[S,4]$, we deduce from [HW22, Theorem 5.0.1] that $v_{n+1}$ in $[S,4]$ is represented by the class $t\sigma^2 v_{n+1}$.

Considering the action of $v_0, ..., v_{n+1}$ on $[S,4]$ described above, one observes that
\[
L_{T(n+1)} \text{THH}^{E_{\infty}}(S_{W(B_0)} \wedge BP(n))^{hS^1} \neq \ast.
\]
Using this and the $E_2$-map
\[
L_{T(n+1)}K(S_{W(B_0)} \wedge BP(n)) \to L_{T(n+1)} \text{THH}^{E_{\infty}}(S_{W(B_0)} \wedge BP(n))^{hS^1},
\]
we deduce that $L_{T(n+1)}K(S_{W(B_0)} \wedge BP(n)) \neq *$ as desired. $\square$

**Construction 8.5.** Fix an $E_3 MU[\sigma_{2(p^n-1)}]$-algebra form of $BP(n)$ where $\sigma_{2(p^n-1)}$ acts through $v_n$ and let $k$ be a perfect field of characteristic $p$. Recall that in this situation, degree $p^n - 1$-root adjunction to $v_n$ provides a $p^n - 1$-graded $E_3 MU[\sigma_2]$-algebra, see Remark 4.11. We consider the $E_3 MU$-algebra:
\[
E := (L_{K(n)}(S_{W(k)} \wedge BP(n)))(\sqrt[p^n]{v_n}).
\]
It follows by [Hov97, Theorem 1.5.4] and [Hov95, Theorem 1.9] that
\[
\pi_\ast E \cong W(k)[u_1, ..., u_{n-1}][u^\pm 1]
\]
where $|u_i| = 0$ and $|u| = -2$. Furthermore, the resulting $E_3$-map $MU \to BP(n) \to E$ provides a formal group law over $\pi_\ast E$.

For a given perfect $\mathbb{F}_p$-algebra $B_0$ and a height $n$ formal group law $\Gamma$ over $B_0$, we let $E_{(B_0, \Gamma)}$ denote the corresponding Lubin-Tate spectrum. By Lurie’s generalization [Lur18, Section 5] of the Goerss-Hopkins-Miller theorem [Rez98, GH04], $E_{(B_0, \Gamma)}$ is an $E_{\infty}$-ring.

**Proposition 8.6.** In the setting of Construction 8.5, $E$ is equivalent to $E_{(k, \Gamma)}$ as an $E_1$-ring for some height $n$ formal group law $\Gamma$ over $k$.

Proof. By construction, the map $\pi_\ast MU(p) \to \pi_\ast E$ carries $v_i$ to $u_i u^{-(p^n-1)}$ for $0 < i < n$ and $v_n$ to $u^{-(p^n-1)}$. Therefore, the corresponding formal group law on $\pi_\ast E$ is the universal deformation of the resulting height $n$ formal group law $\Gamma$ over $k$. This follows by [Lur10, Theorem 5 in Lecture 21]. Alternatively, one can directly check the conditions given in [LT66, Proposition 1.1]. It follows from Hopkins-Miller theorem that there is an equivalence of $E_1$-rings $E \cong E_{(k, \Gamma)}$ [Rez98, Theorem 7.1]. $\square$
Burklund, Hahn, Levy and Schlank are going to use the following example in their construction of a counterexample to the telescope conjecture.

**Example 8.7.** Let \( k \) be a perfect algebraic extension of \( \mathbb{F}_p \). We know from [Ram23, Corollary 4.31] that the \( E_1 \)-algebra structure on \( E(k, \Gamma) \) lifts to a unique \( E_d \)-algebra structure for every \( 1 \leq d \leq \infty \). Since \( E \) in Construction 8.5 is an \( E_3 \)-ring, it follows from Proposition 8.6 that there is an \( E_3 \)-equivalence 

\[
E \simeq E(k, \Gamma)
\]

for \( \Gamma \) as in Proposition 8.6. Through this equivalence, we may equip \( E(k, \Gamma) \) with the structure of an \( E_3 \) \( MU \)-algebra. In particular, we obtain a map of \( E_3 \) \( MU \)-algebras 

\[
BP(\langle n \rangle) \rightarrow E(k, \Gamma).
\]

Furthermore, \( E(k, \Gamma) \) is a \( \mathbb{Z}/(p^n - 1) \)-graded \( E_3 \) \( MU[\sigma_2] \)-algebra where the weight 1 class \( \sigma_2 \) acts through \( u^{-1} \).

**Theorem 8.8.** In the setting of Construction 8.5 and Proposition 8.6, the canonical map 

\[
L_{T(n+1)}K(S_W(k) \wedge BP(\langle n \rangle)) \rightarrow L_{T(n+1)}K(E(k, \Gamma))
\]

is the inclusion of a non-trivial wedge summand.

**Proof.** It follows by Corollary 5.11 and Proposition 8.6 that 

\[
L_{T(n+1)}K(L_{K(n)}(S_W(k) \wedge BP(\langle n \rangle))) \rightarrow L_{T(n+1)}K(E(k, \Gamma))
\]

is the inclusion of a wedge summand. Since 

\[
S_W(k) \wedge BP(\langle n \rangle) \rightarrow L_{K(n)}(S_W(k) \wedge BP(\langle n \rangle))
\]

is a \( T(n) \vee T(n+1) \)-equivalence, the result follows by [LMMT20, Purity Theorem].

We finally prove the following theorem of Yuan.

**Theorem 8.9 ([Yua21]).** For every perfect \( \mathbb{F}_p \)-algebra \( B_0 \) and height \( n \) formal group law \( \Gamma \) over \( B_0 \), the Lubin-Tate spectrum \( E(B_0, \Gamma) \) satisfies the redshift conjecture.

**Proof.** There is an \( E_\infty \)-map \( E(B_0, \Gamma) \rightarrow E(k, \Gamma') \) for some algebraically closed field \( k \) of characteristic \( p \) and \( \Gamma' \) is the corresponding height \( n \) formal group law on \( k \). Since there is an induced \( E_\infty \)-map \( K(E(B_0, \Gamma)) \rightarrow K(E(k, \Gamma')) \), it suffices to prove the redshift conjecture for \( E(k, \Gamma') \).

Since \( k \) is algebraically closed, there is a unique formal group law of height \( n \) over \( k \) [Laz55, Theorem IV]. Using Proposition 8.6 we deduce that \( E(k, \Gamma') \simeq E \) for \( E \) as in Construction 8.5. Combining Proposition 8.2 and Theorem 8.8 we obtain that \( E(k, \Gamma') \) satisfies the redshift conjecture as desired.

We remark that it should be possible to prove the redshift conjecture for all \( E_1 \) \( MU \)-algebra forms of \( BP(\langle n \rangle) \) by constructing maps \( BP(\langle n \rangle) \rightarrow E(k, \Gamma) \) through root adjunction.
9. Algebraic $K$-theory and THH of Morava $E$-theories

In this section, we work with a particular form of Lubin-Tate spectra. This is the Morava $E$-theory spectrum $E_n$ and $E_n$ is central in the Ausoni-Rognes program for the computation of $K(S)$. When we say Morava $E$-theory $E_n$, we mean the Lubin-Tate spectrum corresponding to the height $n$ Honda formal group. This formal group is characterized by its $p$-series
\[ [p]_n(x) = x^{p^n}, \]
and admits a canonical form over $\mathbb{F}_{p^n}$, in the sense that all of its endomorphisms are defined over this field. In this section, we prove a splitting result for the algebraic $K$-theory of the Morava $E$-theory $E_n$ and the corresponding two periodic Morava $K$-theory. Furthermore, we show that the THH of $E_n$ may be obtain from the THH of the $K(n)$-localized Johnson-Wilson spectrum through base change.

9.1. An identification of Morava $E$-theory. Here, we provide an alternate description of $E_n$ in terms of its fixed points and spherical Witt vectors. We have
\[ \pi_* E_n \cong W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}][[u^{\pm 1}]] \]
where $|u_i| = 0$ and $|u| = -2$.

**Proposition 9.1.** The map
\[ \pi_* S_{W(\mathbb{F}_{p^n})} \cong W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \pi_* S_p \to \pi_* E_n \]
obtained via the map $\pi_* S_p \to \pi_* E_n$ and the canonical $W(\mathbb{F}_{p^n})$-module structure on $\pi_* E_n$ lifts to a map of $E_\infty$ $S_p$-algebras
\[ S_{W(\mathbb{F}_{p^n})} \to E_n. \]

**Proof.** This is a consequence of Lurie’s theory of thickenings of relatively perfect morphisms [Lur18, Section 5.2]. Indeed, $S_p \to S_{W(\mathbb{F}_{p^n})}$ is an $S_p$-thickening of $H\mathbb{F}_p \to H\mathbb{F}_{p^n}$ in the sense of [Lur18, Definition 5.2.1], see [Lur18, Example 5.2.7].

In particular, this implies that the space of $E_\infty$ $S_p$-algebra maps from $S_{W(\mathbb{F}_{p^n})}$ to the connective cover $cE_n$ of $E_n$ is given by the set of $\mathbb{F}_{p^n}$-algebra maps
\[ (9.2) \quad \text{hom}_{\mathbb{F}_{p^n}\text{-Alg}}(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}[[u_1, \ldots, u_{n-1}]]) \]
where this correspondence is given by the functor $\pi_0(-)/p$.

Let $f : S_{W(\mathbb{F}_{p^n})} \to cE_n$ be the map of $E_\infty$ $S_p$-algebras corresponding to the canonical $\mathbb{F}_{p^n}$-algebra map in (9.2); in particular, $\pi_0(f)/p$ is the canonical map in (9.2). We first show that $\pi_0f$ is given by the canonical inclusion
\[ W(\mathbb{F}_{p^n}) \to W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]]. \]

Since $W(\mathbb{F}_{p^n})$ is the ring of integers of the unique unramified extension $\mathbb{Q}_p[\mu_{p^n-1}]$ of $\mathbb{Q}_p$ of degree $d$, $W(\mathbb{F}_{p^n})$ is generated as a $\mathbb{Z}_p$-algebra by a primitive $p^n - 1$ root of the unit. Since the roots of the unit of $\pi_0 cE_n$ are all in the image of the canonical inclusion $W(\mathbb{F}_{p^n}) \to W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]]$, one observes that the map $\pi_0 f$ has to factor through the canonical inclusion $W(\mathbb{F}_{p^n}) \to W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]]$. Furthermore, there is a unique ring map $W(\mathbb{F}_{p^n}) \to W(\mathbb{F}_{p^n})$ that lifts the identity map on $\mathbb{F}_{p^n}$. This shows that $\pi_0 f$ is given by the canonical inclusion $W(\mathbb{F}_{p^n}) \to W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]]$. Since $\pi_* f$ is a map of $\pi_* S_p$-modules, it follows that the composition of $f$ with the map $cE_n \to E_n$ provides the map claimed in the proposition.
Let $\text{Gal}$ denote the Galois group $\text{Gal}(\mathbb{F}_p^\infty, \mathbb{F}_p)$. Due to Goerss-Hopkins-Miller theorem, there is an action of $\text{Gal}$ on the $E_\infty$-algebra $E_n$ for which

$$\pi_* E_n^{bhGal} \cong \mathbb{Z}_p[[u_1, \ldots, u_{n-1}][[u^\pm1]]$$

where the degrees of the generators are as in $\pi_* E_n$.

**Proposition 9.3.** There is an equivalence of $E_\infty$-$\mathbb{S}_p$-algebras:

$$\mathbb{S}_W(\mathbb{F}_p^n) \wedge_{\mathbb{S}_p} E_n^{bhGal} \simeq E_n.$$

**Proof.** This equivalence is given by the following composite map of $E_\infty$-$\mathbb{S}_p$-algebras

$$\mathbb{S}_W(\mathbb{F}_p^n) \wedge_{\mathbb{S}_p} E_n^{bhGal} \to E_n \wedge_{\mathbb{S}_p} E_n \to E_n$$

where the first map is induced by the map provided by Proposition 9.1 and the second map is given by the multiplication map of $E_n$. Due to the flatness of $\mathbb{S}_W(\mathbb{F}_p^n)$, this map induces the canonical map

$$W(\mathbb{F}_p^n) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[u_1, \ldots, u_{n-1}][[u^\pm1]] \to W(\mathbb{F}_p^n)[[u_1, \ldots, u_{n-1}][[u^\pm1]].$$

at the level of homotopy groups [Lur1]. Since $W(\mathbb{F}_p^n)$ is a free $\mathbb{Z}_p$-module of finite rank, the functor $W(\mathbb{F}_p^n) \otimes_{\mathbb{Z}_p} -$ is given by taking a $n$-fold product of $-$. In particular, the functor $W(\mathbb{F}_p^n) \otimes_{\mathbb{Z}_p} -$ commutes with completions. This shows that (9.5) is an isomorphism as desired. \qed

9.2. Algebraic $K$-theory of Morava $E$-theories. There is a finite subgroup $\mathbb{F}_p^n$ of the Morava stabilizer group such that $K = \mathbb{F}_p^n \rtimes \text{Gal}$ acts on the $E_\infty$-algebra $E_n$. Furthermore,

$$E_n^{hK} \simeq \widehat{E(n)}$$

where $\widehat{E(n)}$ denotes the $K(n)$-localization of the Johnson-Wilson spectrum $E(n)$, see [Rog08, Section 5.4.7]. We have

$$\pi_* E(n) \cong \mathbb{Z}_p[[v_1, \ldots, v_{n-1}][[v^\pm1]] \text{ and } \pi_* \widehat{E(n)} \cong \mathbb{Z}_p[[v_1, \ldots, v_{n-1}][[v^\pm1]^I]$$

where $I$ denotes the ideal $(p, v_1, \ldots, v_{n-1})$. Since $\widehat{E(n)}$ is given by $E_n^{hK}$, there is a $K$-equivariant map of $E_\infty$-algebras $E(n) \to E_n$. In particular, this provides a map

$$\widehat{E(n)} \to E_n^{hGal}$$

of $E_\infty$-algebras. This map carries $v_i$ to $u^{-(p^n)^{i-1}}$ and $v_i$ to $u_i u^{-(p^n)^{i-1}}$ for $0 < i < n$.

For the following, we fix an $E_2$-map $\mathbb{S}[\sigma_2(p^n-1)] \to \widehat{E(n)}$ to adjoin roots. Recall from Remark 4.8 that in this situation, $\widehat{E(n)}(r^n \sqrt{-v_n})$ is an $\widehat{E(n)}$-algebra.

**Theorem 9.6.** There are equivalences of $E_1$-$\widehat{E(n)}$-algebras:

$$E_n^{hGal} \simeq \widehat{E(n)}(r^n \sqrt{-v_n})$$

$$E_n \simeq \mathbb{S}_W(\mathbb{F}_p) \wedge_{\mathbb{S}_p} \widehat{E(n)}(r^n \sqrt{-v_n})$$

where the class $u^{-1}$ corresponds to $r^n \sqrt{-v_n}$ at the level of homotopy groups for both of these equivalences.
In particular, $E_n^{hGal}$ and $E_n$ are $p^n - 1$-graded $E_1$-algebras with
\[(E_n^{hGal})_i \simeq \Sigma^{2i} E(n)\]
and
\[(E_n)_i \simeq \Sigma^{2i} S_{W(F_p)} \wedge S_p E(n)\]
for every $0 \leq i < p^n - 1$.

**Proof.** By inspection, one observes that
\[\pi_* (\hat{E}(n)(p^n \sqrt{v_n})) \cong \pi_* E_n^{hGal},\]
see [Rog08, 5.4.9]. Furthermore, the map of rings,
\[\pi_* \hat{E}(n) / \pi_* \hat{E}(n)(p^n \sqrt{v_n}) \cong (\pi_* \hat{E}(n))[z]/(zp^n - 1 - v_n)\]
is a map of étale rings as $v_n$ and $p^n - 1$ are invertible in $\pi_* \hat{E}(n)$. Through [HP22, Theorem 1.10], we obtain the first equivalence in the theorem. The second equivalence follows by the first equivalence and Proposition 9.3. The statement on graded ring structures follows by the fact that root adjunction results in $m$-graded ring spectra, see Construction 4.6. \hfill \square

We are ready to prove our result on the $K$-theory of Morava $E$-theories. For this, we use the following composite map
\[(9.7) \quad E(n) \to \hat{E}(n) \to \hat{E}(n)(p^n \sqrt{v_n}) \cong E_n^{hGal}\]
of $E_1$-rings where the last map above is given by Theorem 9.6. Using Proposition 9.3 we obtain the following composite:
\[(9.8) \quad S_{W(F_{pn})} \wedge E(n) \to S_{W(F_{pn})} \wedge \hat{E}(n) \to S_{W(F_{pn})} \wedge E_n^{hGal} \to S_{W(F_{pn})} \wedge S_p E_n^{hGal} \cong E_n.\]

**Theorem 9.9.** The maps
\[K(E(n)) \to K(E_n^{hGal})\]
\[K(S_{W(F_{pn})} \wedge E(n)) \to K(E_n)\]
induced by those above are inclusions of wedge summands after $T(n+1)$-localization.

**Proof.** The first map in (9.7) is a $T(n) \vee T(n+1)$-equivalence and hence induces a $T(n+1)$-equivalence in algebraic $K$-theory [LMMT20, Purity Theorem]. Therefore, the first equivalence in the theorem follows by applying Corollary 5.11 to $\hat{E}(n)(p^n \sqrt{v_n})$.

Similarly, the first and the third maps in (9.8) induce $T(n + 1)$-equivalences in algebraic $K$-theory. The second equivalence in the theorem follows by observing that there is an equivalence of $E_1$-algebras:
\[S_{W(F_{pn})} \wedge (\hat{E}(n)(p^n \sqrt{v_n})) \cong (S_{W(F_{pn})} \wedge \hat{E}(n))(p^n \sqrt{v_n})\]
and then applying Corollary 5.11. \hfill \square
9.3. Two-periodic Morava $K$-theories. We obtain analogous results for two-periodic Morava $K$-theories. Taking a quotient with respect to a regular sequence in $\pi_* S_{W(F_{p^n})} \wedge_{S_p} \widehat{E(n)}$, one obtains an $\widehat{E(n)}$-algebra $K(n)$ \cite{Laz03, Ang08, HW18}. Here, $K(n)$ is the Morava $K$-theory spectrum with coefficients in $F_{p^n}$. We have

$$\pi_* K(n) \cong F_{p^n}[u^\pm 1].$$

Using the $\widehat{E(n)}$-algebra structure on $K(n)$, we adjoin roots and define the two periodic Morava $K$-theory as follows

$$K_n := K(n)(\sqrt[n]{v_n}).$$

In this case,

$$\pi_* K_n \cong F_{p^n}[u^\pm 1]$$

where $|u| = -2$. Together with Theorem 9.6, this provides a commuting diagram of $E_1$-rings

$$\xymatrix{ S_{W(F_{p^n})} \wedge_{S_p} \widehat{E(n)} \ar[d] \ar[r] & E_n \ar[d] \\ K(n) \ar[r] & K_n }$$

which justifies our definition of $K_n$. In particular, $K_n$ is a $p^n - 1$-graded $E_1$-ring in a non-trivial way and we obtain the following from Corollary 5.11.

**Theorem 9.10.** The following map

$$K(K(n)) \to K(K_n)$$

is the inclusion of a wedge summand after $T(i)$-localization for every $i \geq 2$.

**Corollary 9.11.** If $K(n)$ satisfies the redshift conjecture, then so does $K_n$.

The $V(1)$-homotopy of $K(k(1))$ is computed by Ausoni and Rognes in \cite{AR12} for $p > 3$. In particular, their computation shows that $K(1)$ satisfies the redshift conjecture. We obtain the following.

**Corollary 9.12.** The two periodic Morava $K$-theory $K_1$ of height one satisfies the redshift conjecture for $p > 3$.

9.4. THH descent for Morava $E$-theories. Theorem 6.28 identifies THH of various periodic ring spectra with their logarithmic THH. For instance, the Morava $E$-theory spectrum $E_n$ is periodic with a unit $u$ in degree $-2$. Since $E_n/(u^{-1}) \cong 0$, the canonical map

$$\text{THH}(E_n) \xrightarrow{\cong} \text{THH}(E_n \mid u^{-1})$$

is an equivalence. Using this, together with our result on logarithmic THH-étaleness of root adjunction, we show that THH$(E_n)$ may be obtained from THH$(\widehat{E(n)})$ via base-change up to $p$-completion. Such base-change formulas and their relationship with Galois descent problems for THH were studied by Mathew in \cite{Mat17}.

**Theorem 9.13.** The canonical map:

$$\text{THH}(\widehat{E(n)}) \wedge_{\widehat{E(n)}} E_n^{h_{\text{Gal}}} \xrightarrow{\cong} \text{THH}(E_n^{h_{\text{Gal}}}),$$

is an equivalence.
Proof. Recall from Theorem 9.10 that there is an equivalence of $\hat{E}(n)$-algebras
\[ E_n^{hGal} \simeq \hat{E}(n)(\sqrt[n]{p^n-1}v_n). \]
Therefore, it follows by Construction 4.6 that
\[ THH(\hat{E}(n)) \wedge_{\hat{E}(n)} E_n^{hGal} \simeq THH(\hat{E}(n)) \wedge_{\hat{E}(n)} E_n \wedge_{\mathbb{S}(p)|\{p^n-1\}} \mathbb{S}(p)[\sigma_2] \]
\[ \simeq THH(\hat{E}(n)) \wedge_{\mathbb{S}(p)|\{p^n-1\}} \mathbb{S}(p)[\sigma_2]. \]
(9.14)
Since $v_n$ is a unit in $\hat{E}(n)$ and $u^{-1}$ is a unit in $E_n^{hGal}$, Theorem 6.28 provides the equivalences:
\[ THH(\hat{E}(n)) \simeq THH(\hat{E}(n)|v_n) \text{ and } THH(E_n^{hGal}) \simeq_{\cdot} THH(E_n^{hGal}|u^{-1}). \]
Using these equivalences together with Theorem 6.23, we obtain that the following canonical map is an equivalence.
\[ THH(\hat{E}(n)) \wedge_{\mathbb{S}(p)|\{p^n-1\}} \mathbb{S}(p)[\sigma_2] \simeq_{\cdot} THH(E_n^{hGal}) \]
This, together with (9.14), provides the desired result. \qed

Theorem 9.15. The canonical map:
\[ THH(\hat{E}(n)) \wedge_{\hat{E}(n)} E_n \xrightarrow{\simeq_p} THH(E_n), \]
is an equivalence after $p$-completion.

Proof. The first equivalence below follows by Proposition 9.3, and the second follows by Theorem 9.13.
\[ THH(\hat{E}(n)) \wedge_{\hat{E}(n)} E_n \simeq THH(\hat{E}(n)) \wedge_{\hat{E}(n)} (E_n^{hGal} \wedge_{\mathbb{S}_p} \mathbb{S}_{W(F_{p^n})}) \]
\[ \simeq THH(E_n^{hGal}) \wedge_{\mathbb{S}_p} \mathbb{S}_{W(F_{p^n})}. \]
Therefore, it is sufficient to show that the canonical map
\[ THH(E_n^{hGal}) \wedge_{\mathbb{S}_p} \mathbb{S}_{W(F_{p^n})} \to THH(E_n^{hGal}) \wedge_{\mathbb{S}_p} \mathbb{S}_{W(F_{p^n})} \]
is an equivalence after $p$-completion. This follows by the following canonical diagram of $\mathbb{S}/p$-equivalences.
\[ \begin{array}{ccc}
THH(E_n^{hGal}) \wedge_{\mathbb{S}_p} \mathbb{S}_{W(F_{p^n})} & \xrightarrow{\simeq_p} & THH(E_n^{hGal}) \wedge_{\mathbb{S}_p} \mathbb{S}_{W(F_{p^n})}) \\
\downarrow{\simeq_p} & & \downarrow{\simeq_p} \\
\overline{THH}^p(E_n^{hGal}) \wedge_{\mathbb{S}_p} \mathbb{S}_{W(F_{p^n})} & \xrightarrow{\simeq} & \overline{THH}^p(E_n^{hGal}) \wedge_{\mathbb{S}_p} \mathbb{S}_{W(F_{p^n})}) \\
\end{array} \]
The right hand vertical map and the upper left vertical map are $\mathbb{S}/p$-equivalences due to [Mao20, Lemma 5.20]. The fact that the lower left vertical map is an $\mathbb{S}/p$-equivalence follows by [Mao20, proof of Lemma 5.20] and the fact that the composite $\mathbb{S}_{W(F_{p^n})} \to THH(\mathbb{S}_{W(F_{p^n})}) \to \mathbb{S}_{W(F_{p^n})}$ is an equivalence. This shows that the upper horizontal map is an $\mathbb{S}/p$-equivalence proving the theorem. \qed
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