Gauge Invariances in Second Class Constrained Systems - a comparative look at two methods

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Abstract

We look at and compare two different methods developed earlier for inducing gauge invariances in systems with second class constraints. These two methods, the Batalin-Fradkin method and the Gauge Unfixing method, are applied to a number of systems. We find that the extra field introduced in the Batalin-Fradkin method can actually be found within the original phase space itself.

1 Introduction

In recent years there have been a number of papers on gauge invariances in systems with second class constraints. Basically this involves unravelling, using the language of constraints, gauge symmetries hidden in such systems. By doing so it has sometimes been possible to obtain a deeper and more illuminating view of these systems.

In unravelling such hidden symmetries, the basic idea is that the original system is a gauge fixed version of a certain gauge theory; the latter reverts to the former under certain gauge fixing conditions. The advantage in having a gauge theory lies in the fact that other gauges can also be considered. Further it is possible to get more than one gauge theory for the same second class constrained system.

Two methods have been suggested to make this conversion of second class theories into gauge theories. One method, based originally on the idea of Faddeev and Shatashvili, is now called the Batalin-Fradkin method and is formulated in an enlarged phase space. The other method, based on the idea of Mitra-Rajaraman is what we call gauge unfixing, and does not use any enlarged phase space. Rather it confines itself to the phase space of the original second class system.

Even though these methods look very different in their formulations, when they are applied to many systems like Chern-Simons theory, chiral Schwinger model, etc., the results are basically the same, implying that as far as these examples are concerned, the two

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methods are essentially equivalent. In what follows we illustrate this equivalence for three such examples. We compare the results of the two methods for these systems. Any conclusions arising out of such a comparison might be illuminating when the formal equivalence is considered. Such a formal equivalence will be considered separately.

In Section 2 we review the two methods, and look at specific systems in Section 3. We conclude in Section 4.

2 The Formalisms

We consider a finite dimensional system in phase space with co-ordinates $q^i$ and conjugate momenta $p_i$ ($i = 1, 2, \ldots N$). The system has two second class constraints,

$$Q_1(q, p) \approx 0, \quad Q_2(q, p) \approx 0 \quad (1)$$

defining a constraint surface $\Sigma_2$. Due to their second class nature, the $2 \times 2$ antisymmetric matrix $E$ whose elements $E_{mn}$ are Poisson brackets among the $Q$’s,

$$E_{ab}(q, p) = \{Q_a, Q_b\} \quad a, b = 1, 2 \quad (2)$$

is invertible everywhere, even on the surface $\Sigma_2$. The canonical Hamiltonian is $H_c$ and the total Hamiltonian is

$$H = H_c + \mu_1 Q_1 + \mu_2 Q_2, \quad (3)$$

where the multipliers $\mu_1, \mu_2$ are determined on the surface $\Sigma_2$ by demanding consistency of the two constraints with respect to $H$.

2.1 Batalin - Fradkin (BF) method

As mentioned earlier, the idea behind this method \[1, 2\] is to enlarge the phase space by including new variables. Since we have taken the number of second class constraints here to be two, we introduce two variables $\Phi^a$($a = 1, 2$). The enlarged phase space $(q, p, \Phi)$ has the basic Poisson brackets

$$\{q^i, p_j\} = \delta^i_j \quad \{\Phi^a, \Phi^b\} = \omega^{ab} \quad (4)$$

with all other Poisson brackets zero. The antisymmetric $2 \times 2$ matrix $\omega^{ab}$ is a constant matrix, unspecified for the present.

The first class constraints are obtained as functions in this extended phase space. Since we had initially two second class constraints, there will now be two first class constraints, given in general by

$$\tilde{Q}_a(q^i, p_i, \Phi^a) = Q_a + \sum_{m=1}^{\infty} Q^{(m)}_a, \quad Q^{(m)}_a \sim (\Phi^a)^m \quad (5)$$

$$\tilde{Q}_a(q^i, p_i, 0) = Q^{(0)}_a = Q_a$$
where the second line gives the boundary condition. The terms of various orders in the expansion for $\tilde{Q}_a$ are obtained by demanding that the $\tilde{Q}_a$ are strongly first class,

$$\{\tilde{Q}_a, \tilde{Q}_b\} = 0. \quad (6)$$

For instance for the first order this requirement gives

$$E_{ab} + X_{ac}\omega^{cd}X_{db} = 0 \quad (7)$$

which can be satisfied, using (4), if we write and substitute

$$Q^{(1)}_a = X_{ab}(q,p)\Phi^b, \quad (8)$$
in (6) and consider terms up to first order. For many systems, such as the ones we consider in the next section, the higher order terms are all zero. It must be noted that there is an inherent arbitrariness in the choice of the $\Phi^a$ and the coefficients $X_{ab}$. This choice may be exploited to advantage.

To get gauge invariant observables, we note that relevant quantities of the original second class system in general are not gauge invariant with respect to the new first class constraints. They are made gauge invariant by modifying them in the extended phase space. In particular the gauge invariant Hamiltonian \[2\] can be written in general as

$$\tilde{H} = H + \sum_{m=1}^{\infty} H^{(m)} \quad H^{(m)} \sim (\Phi^a)^m \quad (9)$$

and the terms of various orders are obtained by demanding that

$$\{\tilde{H}, \tilde{Q}_m\} = 0. \quad (10)$$

A similar procedure is followed to obtain other gauge invariant quantities also \[2\]. We finally remark that eqn. (7) can always be written so in the case of 2 constraints. For more than 2 second class constraints, this has to be taken as an assumption, which need not hold in the very general case. In a sense, the $X_{ab}$ can be called the “square root” of the matrix $E$ \[2\].

### 2.2 The Gauge Unfixing method

This method \[3, 4\], in contrast to the BF method, makes no enlargement of the phase space. Rather, since the number of second class constraints is even (we consider here only bosonic constraints), only half of these constraints are chosen to form a first class subset, and the other half as the corresponding gauge fixing subset. This latter subset is discarded, retaining only the first class subset, and so we have a gauge theory.
In a general system, getting a first class subset is a non-trivial issue; this might be possible only under certain conditions. However in the case of only two second class constraints (which we consider here) the first class constraint can always be chosen.

For instance, we can choose \( Q_1 \) as our first class constraint, and \( Q_2 \) as its gauge fixing constraint. We redefine, using (2),

\[
Q_1 \rightarrow \chi = E_{12}^{-1}Q_1 \\
Q_2 = \psi
\]

and no longer consider the \( \psi \) as a constraint. The gauge invariant Hamiltonian and other physical quantities are obtained by defining the projection operator

\[
P \cong : e^{-\psi \hat{\chi}} : \quad \hat{\chi}(A) \equiv \{ \chi, A \}
\]

and operating \( P \) on any phase space function \( A \). There is an ordering prescribed for the action of \( P \); the \( \psi \) is always outside the Poisson bracket in the expansion of the exponential. The action of \( P \) on relevant quantities gives the gauge invariant quantities.

It must be noted that even in this method, there is an inherent arbitrariness; either of the two second class constraints can be chosen to be first class. The two choices define two different projection operators, and the gauge theories so constructed will in general be different. This arbitrariness can be exploited to advantage.

### 3 Examples

#### 3.1 Chiral Schwinger Model

This well known anomalous gauge theory involves chiral fermions coupled to a \( U(1) \) gauge field in 1 + 1 dimensions. Classically the theory has gauge invariance, but this is lost upon quantisation. We look at its bosonised version, the advantage being that the corresponding classical theory itself has no gauge invariance. We have

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} (\partial_\mu \phi)^2 + e(g^{\mu\nu} - \epsilon^{\mu\nu})(\partial_\mu \phi)A_\nu + \frac{1}{2} e^2 \alpha A_\mu^2
\]

where \( g^{\mu\nu} = \text{diag}(1, -1) \), \( \epsilon^{01} = -\epsilon^{10} = 1 \) and \( \alpha \) is the regularisation parameter. The theory is gauge non-invariant for all values of \( \alpha \). We consider the case \( \alpha > 1 \).

The canonical Hamiltonian density is

\[
\mathcal{H}_c = \frac{1}{2} \pi_1^2 + \frac{1}{2} \pi_\phi^2 + \frac{1}{2} (\partial_1 \phi)^2 + e(\partial_1 \phi + \pi_\phi)A_1 + \frac{1}{2} e^2 (\alpha + 1)A_1^2
\]

\[
- A_0 \left[ -\partial_1 \pi_1 \frac{1}{2} e^2 (\alpha - 1) A_0 + e(\partial_1 \phi + \pi_\phi) + e^2 A_1 \right]
\]

(14)
where \( \pi_1 = F^{01} = \partial^0 A^1 - \partial^1 A^0 \), and \( \pi_\phi = \partial_0 \phi + e(A_0 - A_1) \) are the momenta conjugate to \( A_1 \) and \( \phi \) respectively. The constraints are

\[
Q_1 = \pi_0 = 0 \\
Q_2 = -\partial_1 \pi_1 + e^2(\alpha - 1)A_0 + e(\partial_1 \phi + \pi_\phi) + e^2A_1 = 0
\]

(15)
defining a constraint surface \( \Sigma_2 \). These are of the second class,

\[
E_{12} = \{ Q_1(x), Q_2(y) \} = -e^2(\alpha - 1)\delta(x - y).
\]

(16)

Following the BF method \( \text{[6]} \), the phase space is extended by introducing two fields \( \Phi_1, \Phi_2 \), with Poisson bracket relations of the form (4). To get the first class constraints (5) and (8), we recall that there is a natural arbitrariness in choosing the matrices \( \omega^{ab} \) and \( X^{ab} \). This allows the choice

\[
\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta(x - y) \quad X(x, y) = e\sqrt{\alpha - 1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(x - y)
\]

(17)

This choice allows the two new fields to form a canonically conjugate pair. The terms beyond the first order in the expansion (5) are all zero. We then get

\[
\tilde{Q}_m = Q_m + e\sqrt{\alpha - 1}\Phi_m, \quad m = 1, 2
\]

(18)

which, using (16) and (17), can be verified to be strongly first class.

Using similar arguments the gauge invariant Hamiltonian for the choice (17) is

\[
\tilde{H}_{BF} = H_c + \int dx \left[ -\frac{e\pi_1 + e(\alpha - 1)\partial_1 A_1}{\sqrt{\alpha - 1}} \Phi_1 + \frac{e^2}{2(\alpha - 1)} \Phi_1^2 \\
+ \frac{1}{2}(\partial_1 \Phi_1)^2 + \frac{1}{2}(\Phi_2)^2 - \frac{\tilde{Q}_2 \Phi_2}{e\sqrt{\alpha - 1}} \right]
\]

(19)

where \( H_c \) is the canonical Hamiltonian. This \( \tilde{H}_{BF} \) has zero PBs with the first class constraints (18).

Coming to the Gauge Unfixing (GU) method \( \text{[7]} \), we reiterate that no new field need be introduced. The first class constraint is taken to be just one of the two existing constraints. We choose, after a rescaling

\[
\chi = \frac{1}{e^2(\alpha - 1)}Q_2,
\]

(20)

so that the relevant constraint surface is \( \Sigma_1 \) defined by \( \chi \cong 0 \). The gauge fixing-like constraint is \( \psi = 0 \), and is discarded (that is unfixes). To get the gauge invariant Hamiltonian
we construct a projection operator $P$ of the form (12) and use it on the canonical Hamiltonian $H_c$,
\[
\tilde{H}_{GU} = H_c + \int dx \left[ \frac{\pi_1 + (\alpha - 1)\partial_1 A_1}{\alpha - 1} Q_1 + \frac{1}{2e^2(\alpha - 1)} (\partial_1 Q_1)^2 + \frac{1}{2(\alpha - 1)^2} Q_1^2 \right]
\] (21)
which is gauge invariant with respect to the $\chi$.

We see that, apart from the term $\int dx \left( \frac{\Phi_2^2 - \Phi_2}{2\sqrt{\alpha - 1}} \right)$, the $\tilde{H}_{GF}$ and the $\tilde{H}_{GU}$ in (19) and (21) are the same, if we make the identification $\Phi_1 = \frac{Q_1}{e\sqrt{\alpha - 1}}$. We however emphasise the two rather different paths used to get these Hamiltonians. One requires the introduction of an extra (canonical) pair of fields, while the other doesn’t need this. In both cases extra terms are needed to make the original Hamiltonian gauge invariant. For the $\tilde{H}_{BF}$ these terms had to be written down by introducing extra fields, whereas in the $\tilde{H}_{GU}$ these terms involve a variable *already present* in the original theory.

We look at the path integral quantisation for these two Hamiltonians. For Hamiltonian $\tilde{H}_{BF}$ we have,
\[
Z_{BF} = \int \mathcal{D}(\pi, A^\mu, \pi_0, \phi, \Phi_1, \Phi_2, \mu_1, \mu_2) e^{iS_{BF}}
\]
\[
S_{BF} = \int dx dt \left[ \pi_0 A^0 + \pi_1 A^1 + \pi_0 \phi + \Phi_2 \dot{\Phi}_1 - \tilde{H}_{BF} - \mu_1 \dot{Q}_1 - \mu_2 \dot{Q}_2 \right].
\]
Here $\mu_1, \mu_2$ are undetermined Lagrange multipliers corresponding to the first class constraints $\dot{Q}_1, \dot{Q}_2$ respectively. If we make the transformations $\mu_2 \rightarrow \mu'_2 = \mu_2 - \frac{\Phi_2}{e(\alpha - 1)}$, $A_0 \rightarrow A'_0 = A_0 - \mu'_2$, $\pi_1 \rightarrow \pi'_1 = \pi_1 + \partial_0 A_1 - \partial_1 A'_0$, $\pi_0 \rightarrow \pi'_0 = \pi_0 - \dot{\phi} - e(A'_0 - A_1)$, and then integrate over $\pi'_1, \pi'_0, \Phi_2$, we get
\[
Z_{BF} = \int \mathcal{D}(A^\mu, \phi, \Phi_1) e^{iS_{BF}}
\]
\[
S_{BF} = \int dx dt \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^2 \alpha}{2} A^\mu A^\mu + e(\eta^\mu^\nu - e^\mu^\nu)(\partial_\mu \phi) A_\nu + \frac{1}{2} (\partial_\mu \phi)^2 \right.
\]
\[
+ \frac{1}{2} (\partial_\mu \Phi_1)^2 - \frac{e}{\sqrt{\alpha - 1}} \Phi_1 (\alpha - 1) \eta^\mu_\nu + e^\mu_\nu)(\partial_\mu A_\nu) \right].
\] (23)
The action $S_{BF}$ above is just the gauge invariant version for the chiral Schwinger model. As is well known, this action was first obtained by merely adding the (Schwinger) terms in the variable $\Phi_1$ to the original bosonised action (13). It has also been obtained using other arguments. In the Batalin-Fradkin approach, these Schwinger terms and $\Phi_1$ come up due to the extension of the phase space.

Coming to the Hamiltonian $\tilde{H}_{GU}$ of the Gauge Unfixing method, we have
\[
Z_{GU} = \int \mathcal{D}(A^\mu, \pi, \phi, \pi_0, \lambda) e^{iS_{GU}}
\]
\[
S_{GU} = \int dx dt \left[ \pi_0 A^0 + \pi_1 A^1 + \pi_0 \phi - \tilde{H}_{GU} - \lambda \chi \right],
\] (24)
where $\lambda$ is the arbitrary Lagrange multiplier. We make the transformations $A_0 \to A_0' = A_0 - \frac{\lambda}{e(\alpha - 1)}$, $\pi_1 \to \pi_1' = \pi_1 + \partial_0 A_1 - \partial_1 A_0 + \frac{\pi_0}{\alpha - 1}$, $\pi_\phi \to \pi_\phi' = \pi_\phi - \dot{\phi} + eA_1 - eA_0'$ and $\lambda \to \lambda' = \lambda + \partial_0 \pi_0$. We then integrate over $\pi_1', \pi_\phi'$ and $\lambda'$ in the path integral to get

\[
Z_{GU} = \int \mathcal{D}(A^\mu, \phi, \pi_0) e^{iS_{GU}}
\]

\[
S_{GU} = \int dx dt \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{e^2 \alpha}{2} A_\mu A^\mu + e(\eta^{\mu \nu} - \epsilon^{\mu \nu})(\partial_\mu \phi)A_\nu + \frac{1}{2}(\partial_\mu \phi)^2 \right) + \frac{1}{2} e^2(\alpha - 1) + \frac{\pi_0}{\alpha - 1}[(\alpha - 1)\eta^{\mu \nu} + \epsilon^{\mu \nu}](\partial_\mu A_\nu).
\]

We see that on making the replacement $\pi_0 = -e\sqrt{\alpha - 1}\Phi_1$ in (25), we get the same result as in the Batalin-Fradkin case (23). This is achieved here by introducing no extra fields. The extra field of the BF method is found here within the original phase space. Further the Schwinger terms are the same in both cases. We also note that the extra term $\int \left( \frac{(\Phi_2)^2}{2} + \ldots \right)$ which comes upon comparing the Hamiltonians in (19) and (21) have been integrated away in the path integral (23).

### 3.2 The abelian Proca model

This $(3 + 1)$ - dimensional theory is given by the Lagrangian \[\square\]

\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{m^2}{2} A_\mu A^\mu
\]

where $m$ is the mass of the $A_\mu$ field, $g_{\mu \nu} = \text{diag} (+, -, -, -)$ and $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The canonical Hamiltonian is given by

\[
H_c = \int d^3 x \mathcal{H}_c = \int d^3 x \left( \frac{1}{2} \pi_i \pi_i + \frac{1}{4} F_{ij} F_{ij} - \frac{m^2}{2} (A_0^2 - A_i^2) + A_0 (\partial_i \pi_i) \right),
\]

with $\pi_i = -F_{0i}$ the momenta conjugate to the $A^i$. The second class constraints are

\[
Q_1 = \pi_0(x) \approx 0, \quad Q_2 = (-\partial_i \pi_i + m^2 A_0)(x) \approx 0,
\]

which together define the surface $\Sigma_2$ in the phase space. Their second class nature is due to the mass term in the Lagrangian. The matrix $E$ of eqn. (2) is here

\[
E = \begin{pmatrix}
0 & -m^2 \\
m^2 & 0
\end{pmatrix} \delta(x - y),
\]

whose determinant is non-zero everywhere.

Using the Batalin - Fradkin method \[\square\], we introduce an extra canonical pair of fields $\theta$ and $\pi_\theta$, with $\{\theta(x), \pi_\theta(y)\} = \delta(x - y)$. As earlier, transformations in this extended phase space modify the constraints (28) to

\[
\tilde{Q}_1 = Q_1 + m^2 \theta \quad \tilde{Q}_2 = Q_2 + \pi_\theta,
\]
which, using (29) can be seen to be strongly first class. The form for these constraints corresponds to the choice of $\theta$ and $\pi_\theta$ as a canonically conjugate pair.

The corresponding gauge invariant Hamiltonian is

$$\tilde{H}_{BF} = H_c + \int d^3x \left( \frac{\pi_\theta^2}{2m^2} + \frac{m^2}{2} (\partial_i \theta)^2 - m^2 \theta \partial_i A_i \right),$$

with respect to which the first class constraints satisfy

$$\{ \bar{Q}_1, \tilde{H}_{BF} \} = \bar{Q}_2 \quad \{ \bar{Q}_2, \tilde{H}_{BF} \} = 0$$

On the other hand, in the Gauge Unfixing method, there is only one first class constraint, one of the two in (28). For our purposes we choose this constraint to be

$$\chi = \frac{1}{m^2} (-\partial_0 \pi_0 + m^2 A_0) \approx 0,$$

and throw away the other $\psi = \pi_0$. The relevant constraint surface is defined by $\chi \approx 0$. The $H_c$ of (27) does not have zero PB with this $\chi$ on this new surface, and hence is not gauge invariant. Using a projection operator of the form (12) on $H_c$ we get the gauge invariant Hamiltonian

$$\tilde{H}_{GU} = H_c + \int d^3x \left[ \psi \partial_i A_i - \frac{1}{2m^2} \psi \partial_i^2 \psi \right].$$

Note the similarity between the Hamiltonians $\tilde{H}_{BF}$ and $\tilde{H}_{GU}$. Indeed, apart from the term $\int d^3x \frac{\pi_\theta^2}{2m^2}$ in (31) the two Hamiltonians are the same if we make the identification $\psi = -m^2 \theta$.

We look at path integral quantisations. For the $\tilde{H}_{BF}$, we have

$$Z_{BF} = \int \mathcal{D}(\pi_0, A_0, \pi_i, A^i, \pi_\theta, \theta, \mu_1, \mu_2) \exp i S_{BF}$$

$$S_{BF} = \int d^4x \left[ -\pi_0 \dot{A}_0 + \pi_i \dot{A}^i + \pi_\theta \dot{\theta} - H_c - \frac{\pi_\theta^2}{2m^2} - \frac{m^2}{2} (\partial_i \theta)^2 + m^2 \theta \partial_i A_i - \mu_1 \bar{Q}_1 - \mu_2 \bar{Q}_2 \right],$$

where $\mu_1$ and $\mu_2$ are undetermined Lagrange multipliers. After some redefinitions of fields and integration over momenta and the $\mu$'s, we get

$$Z_{BF} = \int \mathcal{D}(A^\mu, \theta) \exp i \int d^4x \left[ -\frac{1}{4} F^2_{\mu\nu} + \frac{m^2}{2} A^2_\mu + \frac{m^2}{2} \partial_\mu \theta \partial^\mu \theta - m^2 \theta \partial_\mu A^\mu \right].$$

The last line gives just the Stückelberg gauge invariant action [8]. The $\theta$ field is called the Stückelberg scalar, which was originally introduced by Stückelberg directly into the Proca Lagrangian to make it gauge invariant. Thus in the BF formalism, the extra field introduced is the Stückelberg scalar.

On the other hand, the path integral for the gauge unfixed Hamiltonian $\tilde{H}_{GU}$ is

$$Z_{GU} = \int \mathcal{D}(A^\mu, \pi_\mu, \lambda) \exp i \int d^4x \left( \pi_0 \dot{A}_0 + \pi_i \dot{A}^i - \tilde{H}_{GU} - \lambda \chi \right),$$
with no extra fields. After redefinition of $A_0, \pi_i$ and the $\lambda$ and integrating over $\lambda'$ and $\pi_i'$, we get
\[
Z_{GU} = \int \mathcal{D}(A^\mu, \theta) \exp \left( i \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}^2 + \frac{m^2}{2} A_\mu^2 + \frac{m^2}{2} (\partial_\mu \theta)^2 - m^2 \theta \partial_\mu A^\mu \right] \right),
\]
where $\theta = -\frac{\pi_0}{m^2}$. This is just the path integral (37) obtained earlier using the Batalin-Fradkin method. No extra fields were introduced. Rather the extra coordinate field of the BF method which was recognised earlier as the Stückelberg scalar corresponds here to $-\frac{\pi_0}{m^2}$, which was already present in the phase space of the original second class constrained theory. This suggests that the extra field need not be introduced at all.

### 3.3 Abelian Chern-Simons Theory

This 2 + 1 dimensional theory \[4, 11\] consists of a complex field interacting with an abelian Chern-Simons field. The theory is described by the Lagrangian density
\[
\mathcal{L} = (\mathcal{D}_\mu \phi)^* (\mathcal{D}^\mu \phi) + \frac{\alpha}{4\pi} \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda,
\]
with $\mathcal{D}_\mu \phi = (\partial_\mu - i A_\mu) \phi$, $g_{\mu\nu} = \text{diag} (1, -1, -1)$, and $\mu, \nu, \lambda = 0, 1, 2$. We also have
\[
H_c = \int d^2x \left[ (\vec{\nabla} + i \vec{A}) \phi^* \cdot (\vec{\nabla} - i \vec{A}) \phi + \pi_\phi^* \pi_\phi + A_0 (j_0 - \frac{\alpha}{2\pi} \epsilon_{ij} \partial_i A_j) \right]
\]
with $\pi_\phi = \dot{\phi}^* + i A_0 \phi^*$, $\pi_{\phi^*} = \dot{\phi} - i A_0 \phi$, and $j_0 = i (\phi \pi_\phi - \phi^* \pi_{\phi^*})$.

The constraints are
\[
\pi_0 \approx 0 \quad Q_3 = (j_0 - \frac{\alpha}{2\pi} \epsilon_{ij} \partial_i A_j) - \frac{\alpha}{2\pi} \partial_1 Q_1 + \partial_2 Q_2 \approx 0
\]
\[
Q_1 = -\frac{2\pi}{\alpha} (\pi_1 + \frac{\alpha}{4\pi} A_2) \approx 0 \quad Q_2 = (\pi_2 - \frac{\alpha}{4\pi} A_1) \approx 0
\]
with the first line showing the first class constraints. The second line gives the second class constraints of the theory,
\[
\{Q_1(x), Q_2(y)\} = \delta(x - y).\]

Instead of the canonical Hamiltonian (40), we will consider the total Hamiltonian which guarantees the time consistency of $Q_1$ and $Q_2$ (on the surface defined by both $Q_1$ and $Q_2$),
\[
H = \int d^2x \{H_c + u_1 Q_1 + u_2 Q_2\} \quad u_1 = i \phi^* \mathcal{D}_2 \phi - i \phi (\mathcal{D}_2 \phi)^* + \frac{\alpha}{2\pi} \partial_1 A_0
\]
\[
u_2 = \frac{2\pi}{\alpha} \left[ i \phi^* \mathcal{D}_1 \phi - i \phi (\mathcal{D}_1 \phi)^* - \frac{\alpha}{2\pi} \partial_2 A_0 \right].\]

We now get the gauge theory using the Batalin-Fradkin method \[11\]. The new variables $\Phi^1, \Phi^2$ serve to enlarge the phase space, and have the Poisson brackets (4). In this enlarged
phase space, we have the strongly first class constraints (after appropriate choice of the $\omega$ and the $X$ matrices),

$$\tilde{Q}_1 = Q_1 + \Phi^1 \quad \quad \tilde{Q}_2 = Q_2 - \Phi^2.$$  

(44)

The corresponding Batalin-Fradkin gauge invariant Hamiltonian is

$$\tilde{H}_{BF} = H + \int d^2x \left\{ \phi\phi^*(\Phi^1)^2 + \left( \frac{2\pi}{\alpha} \right)^2 \phi\phi^*(\Phi^2)^2 - 2\phi\phi^* \left[ \Phi^1\tilde{Q}_1 - \Phi^2\tilde{Q}_2 \right] \right\}$$

(45)

We apply the gauge unfixing method [4] to this theory. We redefine

$$\chi = -\frac{2\pi}{\alpha}(\pi_1 + \frac{\alpha}{4\pi}A_2) \quad \quad \psi = (\pi_2 - \frac{\alpha}{4\pi}A_1),$$

(46)

with \( \{\chi(x), \psi(y)\} = \delta(x - y) \). As usual we choose the \( \chi \) as the first class constraint and discard the \( \psi \). The gauge invariant Hamiltonian with respect to \( \chi \) is given by

$$\tilde{H}_{GU} = H + \int d^2x \left[ \left( \frac{2\pi}{\alpha} \right)^2 \phi\phi^*\psi^2 \right].$$

(47)

We see that apart from the term \( \int d^2x \phi\phi^*(\Phi^1)^2 \) (and those proportional to the \( \tilde{Q}_1 \) and \( \tilde{Q}_2 \) in (45)), the Hamiltonians \( \tilde{H}_{BF} \) and \( \tilde{H}_{GU} \) are the same. However this extra term can be easily introduced in the Hamiltonian \( \tilde{H}_{GU} \), since it is basically proportional to the gauge unfixing first class constraint \( \chi \).

By making use of these extra terms in \( \tilde{H}_{GU} \), we can see this equivalence using the path integral too. After various redefinitions and integrations, we get the final gauge invariant action to be

$$S = \int d^2x dt \left\{ (D_\mu \phi)^* D^\mu \phi + \frac{\alpha}{4\pi} \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda - \left( \frac{2\pi}{\alpha} \right)^2 \phi\phi^*(\Phi^2)^2 \right.$$

$$\left. - \left( \frac{2\pi}{\alpha} \Phi^2 \right) \left[ i\phi^* D_1 \phi - i\phi(D_1 \phi)^* + \frac{\alpha}{2\pi} F_{02} \right] \right.$$

$$\left. + \frac{1}{4\phi\phi^*} \left[ \Phi^2 + \frac{\alpha}{2\pi} F_{01} - [i\phi^* D_1 \phi - i\phi(D_1 \phi)^* + \frac{\alpha}{2\pi} F_{02}]^2 \right] \right\}.$$ 

(48)

We have deliberately omitted the subscripts BF and GU here, in order to emphasize that the same result is obtained for both the methods. The BF action is as given above, with the terms in the $\Phi^2$ as extra terms in order to give the new gauge symmetry. In the GU result the action is the same as in (48), with the $\Phi^2$ being replaced by the $\psi$. It may also be noted that the action in its final form is not manifestly Lorentz invariant.

### 4 Conclusion

In conclusion, we have seen that for the three systems above, the two vastly different methods described in Section 2 unearth essentially the same gauge theories. This is seen
both classically and using the path integral. In both methods, extra terms have to be introduced in the Hamiltonian; in the $\tilde{H}_{BF}$ case these terms involve new fields, whereas in $\tilde{H}_{GU}$ these terms are found in the original phase space. Thus, at least as far as the above three systems are concerned, one need not really introduce a totally new variable. In other words to get the hidden gauge symmetries one need not look outside the original system, they are present within the original system itself.

When their second class constraint structures are considered, the above three systems are simple ones. The $E$ matrices of (2) involve only constants, so that getting the gauge invariant Hamiltonians is quite easy; the new Hamiltonians will have a finite number of terms. This situation however need not be seen for other systems. Sometime the gauge invariant Hamiltonians may be in series form, in which case it has to be seen if closed form expressions are possible. It is to be seen if the two methods are equivalent in such more general cases also. This question is being currently looked into.

In our analysis above, the three systems involved only two second class constraints. It was mentioned earlier that in the gauge unfixing case, this does not pose any problem in choosing the first class constraint; either one of the two can be chosen, to give more than one gauge theory. However in the Batalin-Fradkin case, both constraints are to be converted into first class; but even here this conversion is always possible, the reason being the “square root” matrix $X$ of (8) can always be obtained from the $E$ matrix.

On the other hand, in the case of more than two second class constraints, we may have additional problems; in the GU method, the choice of the first class subset becomes non-trivial, and in the BF method finding the “square root” $X$ matrix becomes non-trivial. But once the first class subset (or $X$ matrix) is found, then the new gauge symmetry is defined. In this regard we mention that in the GU method, a certain assumption regarding the $E$ matrix of (2) has to be used to obtain the first class subset.
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