Quantum synchronization of a driven self-sustained oscillator

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Synchronization is a universal phenomenon that is important both in fundamental studies and in technical applications. Here we investigate synchronization in the simplest quantum-mechanical scenario possible, i.e., a quantum-mechanical self-sustained oscillator coupled to an external harmonic drive. Using the power spectrum we analyze synchronization in terms of frequency entrainment and frequency locking in close analogy to the classical case. We show that there is a step-like crossover to a synchronized state as a function of the driving strength. In contrast to the classical case, there is a finite threshold value in driving. Quantum noise reduces the synchronized region and leads to a deviation from strict frequency locking.

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$\frac{dp}{dt} = -i \left[ -\Delta \hat{b} \hat{b} + i \Omega (\hat{b} - \hat{b}^\dagger), \rho \right] + \gamma_1 \hat{D}\{\hat{b}\} \rho + \gamma_2 \hat{D}\{\hat{b}^\dagger\} \rho,

(1)

where $\Delta$ is the detuning of the drive frequency with respect to the natural frequency of the undriven oscillator, $\Omega$ determines the strength of the external drive, and we have set $\hbar = 1$. The coefficients $\gamma_1$ and $\gamma_2$ describe negative and non-linear damping, respectively, and $\hat{D}$ are Lindblad dissipators, $\hat{D}\{O\} \rho = O \rho \hat{D} - \frac{1}{2} \{ O, \hat{D} \}$.

The corresponding equation of motion for $\beta = \langle \hat{b} \rangle$,

$\frac{d}{dt} \beta = i \Delta \beta + \frac{\gamma_1}{2} \beta - \gamma_2 |\beta|^2 \beta - \Omega,

(2)

is the equation of motion for the classical van der Pol oscillator. Separating the dynamics of the complex variable $\beta = re^{i\phi}$ into the dynamics of amplitude $r$ and phase $\phi$ leads to the classical amplitude equation $dr/dt = (\gamma_1/2 - \gamma_2 r^2)r -$
Ω cos φ and dφ/dt = Δ + (Ω/γ) sin φ which is called Adler equation [4] if the amplitude dynamics are negligible (i ̇ = 0).

**Synchronization regimes.**— The synchronization regimes of the classical driven van der Pol oscillator can be determined from the phase-space trajectories of its late-time dynamics, for details on synchronization regimes and a phase diagram of (classical) synchronization see [5]. To characterize phase locking in the system we investigate the probability histogram of the phase, P(φ). In the classical case, Pcl(φ) can be obtained from the late-time phase-space trajectory. In the quantum case, we use the Wigner function, a quasi-probability distribution for the canonically conjugate operators ̂x = x_{p} = 1/2mω_{0} is the amplitude of the zero-point fluctuations of the oscillator with natural frequency ω_{0}, and similarly p_{p} = √mω_{0}/2. Here, P_{qm}(φ) can be calculated from W_{qm}(x,p) by transforming from the variables x and p to radius and phase and integrating over the radius. In Figs. 1 and 2 we show the classical phase-space trajectory, the Wigner function, and the corresponding probability histograms for the phase variable. In both figures, we decrease the detuning at fixed driving strength.

**FIG. 1:** (color online). Classical phase-space trajectory (black solid line), Wigner function, and the corresponding phase probability histograms for Ω/γ_{1} = 1 and γ_{2}/γ_{1} = 0.1. a), e) Δ/γ_{1} = 16, b), f) Δ/γ_{1} = 0.6, c), g) Δ/γ_{1} = 0.1, and d), h) Δ/γ_{1} = 0.

**FIG. 2:** (color online). Classical phase-space trajectory (black solid line), Wigner function, and the corresponding phase probability histograms for Ω/γ_{1} = 3 and γ_{2}/γ_{1} = 3. a), e) Δ/γ_{1} = 100, b), f) Δ/γ_{1} = 12, c), g) Δ/γ_{1} = 0.1, and d), h) Δ/γ_{1} = 0.

In the classical case, we observe phase locking, as indicated by the δ-function like shape of P_{cl}(φ) in g), h), whereas in the quantum case P_{qm}(φ) is smeared out even in this regime due to quantum noise. In the quantum case, a trend towards synchronization of the quantum oscillator is visible since the expectation values ̂x and ̂p become finite with smaller Δ, see Figs. 1d) and 2d) where the Wigner function concentrates in a displaced blob.

The difference between the cases γ_{2} < γ_{1} (Fig. 1) and γ_{2} > γ_{1} (Fig. 2) is most prominent for large detuning Ω. For γ_{2} < γ_{1} the Wigner function is of ring shape and the classical trajectory almost coincides with the maximum of the Wigner function, Fig. 1a). For γ_{2} > γ_{1} however, the maximum of the Wigner function and the classical trajectory no longer coincide. In this case, the quantum nature (i.e., the discrete level structure) of the self-sustained oscillator becomes more important. In the limit γ_{2} > γ_{1} the rate at which the oscillator loses two phonons dominates over the rate at which it gains one phonon, see Eq. (1). Thus, in steady state for a small and moderate driving strength only the lowest Fock states of the mode ̂b are occupied.

**Spectra and observed frequency.**— To investigate frequency entrainment and locking we use the classical and quantum-mechanical power spectrum. In the classical case, it reads

\[ S_{cl}(ω) = \int_{-∞}^{∞} dt e^{iωt} \langle \beta(0)\beta(t) \rangle , \]  

where the bar denotes time averaging. In the quantum case, we consider the spectrum

\[ S_{qm}(ω) = \int_{-∞}^{∞} dt e^{iωt} \langle ̂b^\dagger(t) ̂b(t) \rangle , \]

where \( \langle \cdot \rangle \) denotes the average with respect to the full quantum-mechanical density matrix.

**FIG. 3** shows typical spectra. The classical spectrum has a δ-shaped peak at the observed frequency ω_{obs} (in the frame rotating with the drive). In addition to this main peak, satellite peaks emerge at higher harmonics of the main frequency [1] (not shown in Fig. 3). In the quantum case, the spectrum is not an even function of frequency, and the peak has finite width. In the classical and the quantum case, the observed frequency can be obtained from the position of the maximum of \( S_{cl,qm}(ω) \) if the peaks are well-separated. The spectra are thus a convenient tool to study synchronization properties, and we will use them in the following to analyze the dependence of ω_{obs} on the detuning Δ and the driving strength Ω.

**Synchronization.**— To investigate synchronization we plot in Figs. 4a) and 4b) the observed frequency ω_{obs} as a function of detuning Δ for fixed driving strengths Ω and compare the classical case with the quantum case for γ_{2} < γ_{1}.

Figure 4a) shows that for weak driving the classical oscillator is frequency-locked to the drive over a finite range of Δ. In the quantum-mechanical model, however, there is only weak frequency entrainment (cf. inset), except at Δ = 0 where the frequency of the oscillator trivially agrees with the drive. This is in stark contrast to the classical case, where an arbitrarily weak drive leads to a small, but finite synchronized region. At
a larger driving strength, strong frequency entrainment and a tendency towards frequency locking for the quantum model can be seen in Fig. 4b). However, as the inset shows, there is no exact frequency locking at finite detuning $\Delta \neq 0$. We attribute the absence of frequency locking to quantum noise, since for the classical van der Pol oscillator, noise can reduce frequency coincides with the detuning (black line). Driving the classical model is frequency-locked to the drive, while the quantum model is not. The other parameters are $\Omega/\gamma_1 = 1$ and $\gamma_2/\gamma_1 = 0.1$.

To analyze the synchronization behavior in more detail, we now fix the detuning $\Delta$ and study the dependence of $\omega_{\text{obs}}$ on the driving strength $\Omega$ and the non-linear damping rate $\gamma_2$.

Figure 5 shows $\omega_{\text{obs}}(\Omega/\gamma_1)$ for different damping rates $\gamma_2/\gamma_1$. A weak drive leads to a small amount of frequency entrainment. However, if the drive exceeds a threshold whose position depends on the non-linear damping $\gamma_2/\gamma_1$, a step-like crossover to strong frequency entrainment occurs. Moreover, the influence of $\gamma_2$ is different below and above the threshold. For weak driving, increasing $\gamma_2/\gamma_1$ (and thereby increasing the quantum nature of the oscillator) reduces frequency entrainment, i.e., for small driving a large value of $\gamma_2/\gamma_1$ inhibits synchronization. For large driving, the situation is opposite: increasing the non-linear damping rate $\gamma_2$ supports frequency entrainment (see right inset of Fig. 5). To investigate the case of large $\gamma_2/\gamma_1$ in more detail, we solve Eq. (1) perturbatively. In the limit $\gamma_2/\gamma_1 \to \infty$ and for $\Omega/\gamma_1 = \Delta/\gamma_1 = 0$ the steady state of Eq. (1) can be found: $\rho_{ss} \to \frac{3}{2}|0\rangle \langle 0| + \frac{1}{2}|1\rangle \langle 1|$. Including a finite drive and detuning leads to non-zero off-diagonal matrix elements $\langle 1|\rho_{ss}|0\rangle$ and $\langle 0|\rho_{ss}|1\rangle$. In this case, making use of the quantum regression theorem, we obtain an analytic expression (too lengthy to be displayed here) for the spectrum $S_{\text{qm}}(\omega)$ which is independent of $\gamma_2$. The analytic result for the observed frequency is also shown in Fig. 5. Although we do not find genuine frequency-locking in the quantum case, the difference between observed frequency and drive frequency can be made arbitrarily small. Our analytical treatment predicts the observed frequency to vanish like $(\Omega/\gamma_1)^{-2}$ (see left inset of Fig. 5).

To analyze the dependence on the non-linear damping rate, Fig. 6 shows $\omega_{\text{obs}}(\gamma_2/\gamma_1)$ for different driving strengths $\Omega$. Without driving there is no entrainment and the observed frequency coincides with the detuning (black line). Driving the oscillator stronger generally enhances frequency entrainment. However, in the left part of Fig. 6, i.e., for $\gamma_2/\gamma_1 \lesssim 1$, there is an interesting difference in the dependence on $\gamma_2/\gamma_1$. For weak driving, frequency entrainment is reduced by increasing $\gamma_2/\gamma_1$. For strong driving, on the other hand, increasing $\gamma_2/\gamma_1$ enhances frequency entrainment and synchronization (lower three curves). In the limiting case $\gamma_2/\gamma_1 \to \infty$, $\omega_{\text{obs}}$ becomes independent of $\gamma_2$. The observed frequency obtained from our analytical solution is shown at the very right of Fig. 6.

Realization.— Optomechanical systems are suitable candidates to study synchronization. We therefore describe a possible scheme to realize the dynamics given by Eq. (1) in an optomechanical system. The two dissipative processes $\gamma_1 \hat{D}[\hat{b}]$ and $\gamma_1 \hat{D}[\hat{b}^2]$ can readily be engineered in a “membrane-in-the-middle” optomechanical setup [20]. Such a setup gives rise to a quadratic optomechanical coupling which can be used to
favor two-phonon absorption and emission [21]. The dissipative terms can be created using two lasers. The first laser is detuned to the blue one-phonon sideband and gives rise to the negative damping term \( \sim -\gamma_1 \). The second laser is detuned to the red two-phonon sideband leading to the non-linear damping \( \sim -\gamma_2 \). Synchronization can be probed by changing the frequency of the harmonic drive and observing the spectrum of the mechanical oscillator. A different strategy based on trapped ions to realize the dynamics of the van der Pol oscillator in a quantum system was put forward in Ref. [13].

**Conclusion.**—We have analyzed the quantum van der Pol oscillator in the presence of an external harmonic drive. Our study establishes the power spectrum, that is readily accessible in current nanomechanical experiments, as an important observable of quantum synchronization. The steady-state Wigner function can indicate a synchronized state, but it is not sufficient to obtain the observed frequency and thus does not allow for studying frequency entrainment and its locking. The information contained in the power spectrum thus neatly complements previous analyses of phase locking. We have shown the existence of weak and strong frequency entrainment in the quantum case and found a threshold in the strength of the drive below which there is only weak frequency entrainment between the oscillator and the drive. Our study contributes new insights in understanding synchronization in the quantum regime and also allows various extensions as for instance the investigation of synchronization of two (or an ensemble of) nonidentical quantum van der Pol oscillators. Another interesting question is whether engineered noise [22, 23] (classical and/or quantum) can favor synchronization.

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**Note added.**—Just before submitting this manuscript we became aware of the work of Lee and Sadeghpour [13] studying a quantum van der Pol oscillator with a resonant weak drive.

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DETAILS ON DIFFERENT SYNCHRONIZATION REGIMES

Here, we give a brief overview of the different synchronization regimes. Figure 7 shows a schematic phase diagram of synchronization. At large detuning $\Delta$ the classical van der Pol oscillator exhibits stable limit cycles with a nearly constant radius $r_A = \sqrt{\frac{\gamma_1}{2\gamma_2}}$ (region $C$ in Fig. 7). Following the direction of the dashed arrow in Fig. 7 to smaller values of $\Delta$, the frequency of the self-sustained oscillator is pulled towards the frequency of the drive, and at $\Delta_A = \Omega/r_A$ the oscillator synchronizes with the drive. This leads to a fixed phase relation between self-oscillation and drive (region $A$ in Fig. 7). The region at the tail of the dotted arrow (large $\Omega$ and $\Delta$, region $C$) is characterized by stable limit cycles encircling the origin. At smaller detunings $\Delta$ (in direction of the dotted arrow in Fig. 7) the focus of the limit cycle shifts, its radius decreases, and it no longer encircles the origin (region $D$).

FIG. 7: Schematic phase diagram for a harmonically driven van der Pol oscillator, Eq. (2) of the main text. Regions $A$ and $B$ correspond to a synchronized state of the classical dynamical system [in $A$ Eq. (2) of the main text has three steady states one of which is stable, in $B$ it has one steady state which is stable]. Region $C$ ($D$) corresponds to stable limit cycles (not encircling the origin. The open and black dots correspond to the phase-space trajectories shown in Fig. 1a) and 1b) of the main text, and the cross to a point where the oscillator is synchronized with the drive, see e.g. Fig. 1c) of the main text.