Persistence in higher dimensions: a finite size scaling study

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We show that the persistence probability $P(t, L)$, in a coarsening system of linear size $L$ at a time $t$, has the finite size scaling form $P(t, L) \sim L^{-\theta} f(\frac{t}{L^z})$ where $\theta$ is the persistence exponent and $z$ is the coarsening exponent. The scaling function $f(x) \sim x^{-\theta}$ for $x \ll 1$ and is constant for large $x$. The scaling form implies a fractal distribution of persistent sites with power-law spatial correlations.

We study the scaling numerically for Glauber-Ising model at dimension $d = 1$ to $4$ and extend the study to the diffusion problem. Our finite size scaling ansatz is satisfied in all these cases providing a good estimate of the exponent $\theta$.

Persistence decay has been the subject of considerable research activity in recent years. The basic quantity under investigation is the persistence probability $P(t)$, which is the probability that a given stochastic variable under investigation is the persistence probability for finite lattice sizes can be summarised in the following dynamical scaling form.

$$P(t, L) = L^{-\theta} f(t/L^z)$$

where the scaling function $f(x) \sim x^{-\theta}$ for $x \ll 1$ and $f(x) \to$ constant at large $x$. Similar finite size scaling ideas have been used in a previous work in the context of global persistence exponent for nonequilibrium critical phenomena.

The finite-size scaling form given by Eq 2 implies the presence of scale-invariant spatial correlations in the system, characteristic of fractals. To show this, we consider the two-point correlation function $C(r, t)$, which we define as the probability of finding a persistent spin at a distance $r$ from another persistent spin. For a $d$-dimensional system, $C(r, t)$ satisfies the normalisation condition $\int_0^L C(r, t) dr^d = L^d P(t, L)$. After substituting Eq 2, this becomes
\[ \int_0^L C(r, t) r^{d-1} dr \sim L^{d-z_0} f(t/L^z) \] (3)

Let us rewrite this equation in terms of a new function \( F(a, b) = a^\theta C(a, b) \) and dimensionless variables \( x = r/L \) and \( \tau = t/L^z \).

\[ \int_0^1 F(Lx, L^2 \tau)x^{d-1-z_0} dx \sim f(\tau) \] (4)

Since the RHS of the equation has no explicit \( L \)-dependence, LHS should also be likewise. This is possible only if \( F(a, b) = g(ab^{-\tau}) \), where \( g(\eta) \) is given by the integral relation

\[ \tau^{\frac{d}{2}-\theta} \int_0^\infty \eta^{\theta-\left(\frac{d}{2}+\frac{\tau}{\theta}\right)} g(\eta) d\eta \sim z f(\tau) \] (5)

Using the above equation, the limiting behaviour of the function \( g(\eta) \) for small and large values of the argument could be deduced from the known behaviour of the function \( f(\tau) \). Consider \( \tau \gg 1 \), where \( f(\tau) \) is constant. From Eq. 5, this implies that \( g(\eta) \) is constant for large \( \eta \). In the other extreme of \( \tau \ll 1 \), \( f(\tau) \sim \tau^{-\theta} \). We split the integral in Eq. 5 as \( \int_0^\infty = \int_0^1 + \int_1^\infty \) and note that \( g(\eta) \) is constant in the second integral for sufficiently large \( \alpha \). The second integral vanishes as \( \tau^{\frac{d}{2}-\theta} \) as \( \tau \to 0 \), whereas the RHS diverges as \( \tau^{-\theta} \). This can be consistent only if the first integral diverges as \( \tau^{-\theta} \), which would imply that \( g(\eta) \sim \eta^{-\theta} \) as \( \eta \to 0 \). This leads to the following dynamical scaling form for \( C(r, t) \).

\[ C(r, t) = r^{-z_0} g\left(\frac{t}{r^z}\right) \] (6)

For small separations \( r \ll t^{1/z} \), this scaling form implies scale-free correlations, i.e., \( C(r, t) \sim r^{-z_0} \), characteristic of a fractal with fractal dimension \( d_f = d - z_0 \). On the other hand, over large length scales, \( C(r, t) \sim t^{-d} \), which is indicative of the absence of any spatial correlations. This scaling description was introduced by us [2] in the context of \( A + A \to \emptyset \) model, and later verified numerically in 2-dimensional Ising model [3] also.

To check the finite-size scaling form given by Eq. 6, we simulate Ising spin systems of various sizes in spatial dimension \( d = 1 \) to 4. Starting from a random initial configuration, the spins are quenched to zero temperature and are updated sequentially using the Glauber updating rule by which a spin is always flipped if the resulting energy change \( \Delta E < 0 \), never flipped if \( \Delta E > 0 \), and flipped with probability \( \frac{1}{2} \) if \( \Delta E = 0 \). One MC time step was counted after every spin in the lattice was updated once.

The persistence probability at any time \( t \) was determined as the fraction of spins that did not flip even once till time \( t \) since the time evolution started. The data is averaged typically over 1000 starting random configurations for small \( L \) and low \( d \) and over 50 starting configurations for large \( L \) and high \( d \).

Figure 1. The persistence probability \( P(t/L) \) is plotted against time \( t \) (measured in MC steps) for three different lattice sizes \( L \) in \( d = 2 \) Glauber Ising model.

Figure 2. Same as Fig. 1, except that the scaling function \( f(x) = L^{2\theta} P(t, L) \) is plotted against the dimensionless scaling variable \( x = t/L^z \). The data for different \( L \) values were found to collapse well to a single curve for \( \theta = 0.21 \) and \( z = 2.12 \pm 0.05 \).

For \( T = 0 \) Glauber dynamics of Ising model, the persistence exponent \( \theta \) is exactly known to be \( 3/8 \) in \( d = 1 \) [3]. In higher dimensions, simulations predict \( \theta \simeq 0.22 (d = 2) \), \( 0.18 (d = 3) \) and \( \theta \simeq 0.16 (d = 3) \) [4]. In our finite size scaling analysis of the simulation data, we adopt the following procedure. For \( d = 1, 2 \) and 3, we fix \( \theta \) at its known value and adjust \( z \) to find the value which gives the best data collapse. In all cases, we find \( z \simeq 2 \), which is the accepted value of the coarsening exponent for non-conserved scalar models [5]. (In \( d = 3 \) Glauber dynamics, a slower \( t^{1/3} \) coarsening has been observed before [6]. This is presumably due to lattice effects, but we have not seen any signature of this effect in our simulations). In \( d = 4 \), on the other hand, we fix \( z \) at 2, and adjust \( \theta \) to collapse the data to a single curve. The results are displayed in Figs. 1 to 4.

In \( d = 4 \), we find that for \( z = 2 \), \( \theta = 0.12 \pm 0.02 \) gives reasonably good data collapse over the time scales and system sizes studied. Fig.4 shows the scaled data in \( d = 4 \). It may be mentioned that in \( d = 4 \), earlier simula-
tions had suggested that the persistence decay might be slower than a power-law, and perhaps logarithmic [10]. However, the agreement of our data with the scaling form Eq. 2 suggests that persistence follows a power-law decay in $d = 4$ also. For $d > 4$, blocking of spins has been shown to lead to a limiting value of $P(t, L)$ as $t \to \infty$, which is independent of $L$ [10]. We could simulate only small lattice sizes for $d = 5$ from which we cannot make any conclusive remark at this stage.

In the diffusion problem, we have a scalar field $\phi(x, t)$ evolving according to the diffusion equation. The initial values $\phi(x, 0)$ are taken to be independent random variables with zero mean.

$$\frac{\partial \phi(x, t)}{\partial t} = \nabla^2 \phi(x, t); \quad \langle \phi(x, 0) \phi(x', 0) \rangle = \delta^d(x - x')$$  

(7)

For this problem, it has been shown using approximate analytic theories [14,15] that $P(t) \sim t^{-\theta}$ in all dimensions. The predicted exponent values in low dimensions were in good agreement with simulation results. The exponent was found to increase with dimension, and has been suggested to have the asymptotic value $\theta(d) \approx \alpha \sqrt{d}$ as $d \to \infty$. The constant $\alpha$ has been estimated to be $\approx 0.14$ [14,15] and $\approx 0.18$ [13] by different authors. For $d = 1, 2$ and 3, the exponent values are found to be $\theta \approx 0.12, 0.18$ and 0.23 respectively.

To simulate Eq. 7 numerically, we use the finite difference Euler discretization scheme on cubic lattices of $L^d$ sites [14,15].

$$\phi(x, t + \Delta t) = \phi(x, t) + a \sum_{x'} \phi(x', t) - 2d \phi(x, t)$$  

(8)

where $x'$ runs over all the $2d$ nearest neighbour lattice sites of $x$ in the cubic lattice and $a = \frac{\Delta t}{\Delta x^2} < \frac{1}{2d}$ for stability of the discretization scheme. We have taken $a = \frac{1}{4d}$ in our simulations as this value has been observed to provide the fastest approach to the asymptotic regime [1].

For the diffusion problem, simple scaling arguments suggest that the dynamical exponent $z = 2$ in all dimensions. In all dimensions studied, we found excellent scaling collapse with $z \approx 2$ and the $\theta$ values quoted above. Upon substitution of the exponent values into Eq. 1, it can be easily seen that the condition for fractal formation is satisfied for $d = 1, 2$ and 3. For $d = 1$, this has already been confirmed by an earlier numerical study [2]. Our results for the persistence probability and the scaling function for three different lattice sizes in $d = 2$ is displayed in Fig. 5 and 6.

It is also possible to extrapolate these results to the $d \to \infty$ limit using the asymptotic form suggested for $\theta$. We see that in this limit, the LHS of Eq. 1 vanishes as $\frac{1}{\sqrt{d}}$, leading us to conjecture that fractal formation persists in all dimensions for the diffusion problem.
L = 50 ---
L = 70 ---

FIG. 6. Same as Fig. 5, except that the scaling function
\[ f(x) = L^z \theta P(t, L) \] is plotted against the dimensionless scaling variable
\[ x = t/L^z \]. The data for different \( L \) values were found to collapse well to a single curve for \( \theta = 0.186 \) and \( z = 2.05 \pm 0.04 \).

To conclude, we have proposed a finite size scaling ansatz for the persistence probability in a coarsening system. The scaling form corresponds to the fractal structure and dynamic scaling characterising the spatio-temporal evolution of the persistent set. We check the scaling form numerically for Glauber-Ising model and for the diffusion problem. Finite size scaling enables us to study persistence reliably in higher dimensions. Our results agree with the known values of \( \theta \) in the case of Ising model (from \( d = 1 \) to \( 3 \)) and in the diffusion problem (we have checked upto \( d = 3 \)). For \( d = 4 \) Ising model, we find the signature of algebraic decay of persistence with \( \theta \approx 0.12 \), in contrast with what had been reported earlier [10].

We thank G. I. Menon and D. Dhar for a critical reading of the manuscript and valuable suggestions. G. M also thanks C. Sire, S. N. Majumdar and A. J. Bray for helpful discussions and illuminating remarks.

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