CORK TWISTS AND AUTOMORPHISMS OF 3-MANIFOLDS

SELMAN AKBULUT

Abstract. Here we study two interesting smooth contractible manifolds, whose boundaries have non-trivial mapping class groups. The first one is a non-Stein contractible manifold, such that every self diffeomorphism of its boundary extends inside; implying that this manifold can not be a loose cork. The second example is a Stein contractible manifold, with an interesting involution on its boundary \( f : \partial W \to \partial W \), which makes \((W, f)\) a candidate to be a cork. Interestingly, this map \( f \) is not obtained by “zero-dot exchange” process, a technique which provides a tool for checking if maps are cork maps. By [AM] we know that any homotopy 4-sphere is obtained gluing together two contractible Stein manifolds along their common boundaries by a diffeomorphism (not necessarily cork twisting map). As an example, we study \( \Sigma = -W \sim_f W \):

We first prove that \( \Sigma \) is a Gluck twisted \( S^4 \), then from this we obtain a 3-handle free handlebody description of \( \Sigma \).

0. Introduction

A cork is a pair \((W, f)\), where \( W \) is a compact contractible Stein manifold, and \( f : \partial W \to \partial W \) is an involution, which extends to a homeomorphism of \( W \), but does not extend to a diffeomorphism of \( W \).

We say \((W, f)\) is a cork of \( M \), if there is an imbedding \( W \hookrightarrow M \) and cutting \( W \) out of \( M \) and re-gluing with \( f \) produces an exotic copy \( M' \).

\[ M \hookrightarrow M' = W \cup_f [M - W] \]

The operation \( M \hookrightarrow M' \) is called cork-twisting \( M \) along \( W \). The first example of a cork appeared in [A1], then in [M], [CFHS] it was proven that any exotic copy \( M' \) of a closed simply connected 4-manifold \( M \) is obtained by twisting along a contractible manifold by an involution as above. Furthermore in [AM] it was shown that this contractible manifold can be taken to be a Stein manifold. In particular, if the boundary of a cork is \( S^3 \) it has to be \( B^4 \) (Eliashberg’s theorem). A cork without the “Stein” condition is called a loose-cork. It is not known if loose-corks are corks (they have to contain corks by above).

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Question 1. Is there any loose-cork with irreducible boundary which can not be a cork?

The infinite corks of [G] and [A2] could provide examples to this question. Recently, Mark and Tosun proved that the contractible manifold $W(0, 2)$ cannot be a Stein manifold. In the notation of [AK] and [A3] this manifold is drawn in Figure 1. Taking a connected sum of two copies gives an answer to Question 1 without the requirement of irreducibility ([MT, Corollary 1.8]). So the only other way a contractible Stein (or non-Stein) manifold $W$ fails to be a cork (or a loose-cork) is when all the self diffeomorphisms of $\partial W$ smoothly extend inside $W$. Here we test this on two specific examples from the family $W(0, n)$ which was introduced in [AK] Figure 1.

\[ \begin{align*}
\begin{array}{c}
\text{n} \\
\mathcal{C}
\end{array} \\
\end{align*} \]

Figure 1. $W(0, n)$

From its Legendrian picture it is easy to check that when $n \leq 1$, $W(0, n)$ is a Stein manifold. Moreover, $\partial W(0, 1)$ and $\partial W(0, 2)$ can be identified as $+1$ and $-1$ surgeries of the Stevedore knot $K$, respectively. It is known that $K^{+1}$ is a hyperbolic manifold [BW], and $K^{-1}$ is the Brieskorn homology sphere $\Sigma(2, 3, 13)$ (e.g. [A3]).

\[ \begin{align*}
\begin{array}{c}
\text{+1} \\
\mathcal{C}
\end{array} \\
\mathcal{C} \\
\text{-1}
\end{align*} \]

Figure 2. $\partial W(0, 1)$ and $\partial W(0, 2)$

Theorem 1. The non-Stein manifold $W(0, 2)$ can not be a loose-cork.
We will construct a curious diffeomorphism $f : \partial W(0, 1) \to \partial W(0, 1)$ which is not obtained by the usual “zero-dot exchange” process like cork twisting maps. This leads us to the natural question whether the homotopy sphere $\Sigma = -W(0, 1) \sim_f W(0, 1)$ obtained by gluing two copies of $W(0, 1)$ by $f$ is the standard smooth copy of $S^4$? Recall that, by [AM] we know every smooth homotopy sphere can be decomposed as a union of two contractible Stein manifolds glued along their boundaries (not necessarily by a cork twisting map). Therefore, it is only natural to seek counterexamples to 4-dimensional Poincaré conjecture among such manifolds. Surprisingly, as the previous well known examples ([A3], [A5], [A6]), $\Sigma$ also decomposes as a Gluck twisted $S^4$, twisted along a knotted $S^2 \hookrightarrow S^4$ (Figure 16). By using this fact, we can cancel all the 3-handles of the handlebody of $\Sigma$. This lengthy process results a seemingly simple, 3-handle free handlebody picture of the homotopy sphere $\Sigma$, this is the content of Theorem 2.

**Theorem 2.** The Figure 3 is a 3-handle free handlebody picture of $\Sigma$

![Figure 3. $\Sigma$](image)

We hope to investigate this handlebody closely in a future paper. Reader is encouraged to check that its boundary $S^3$. Close inspection of this handlebody shows that, the 3-manifold $Y$ obtained by 0-surgery to Stevedore knot admits a self diffeomorphism induced by rotation, similar to that of $S^1 \times S^2$ obtained by $\pi_1(SO_3) = \mathbb{Z}_2$. This could be used as a generalization of the Gluck twisting operation in 4-manifolds.

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1. Proofs

1.1. Proof of Theorem 1. Let $\text{MCG}(X)$ denote the mapping class group of a topological space $X$. It is orientation-preserving diffeomorphisms of $X$ modulo isotopy.

**Lemma 3.** $\text{MCG}(K^{+1})$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and is generated by the symmetries induced by the rotations $R$ and $S$ of the knot $K$, as indicated in Figure 4 and $\text{MCG}(K^{-1}) = \mathbb{Z}_2$, which is generated by the symmetry $T$ of Figure 5. In this figure we used another identification of $\Sigma(2,3,13)$ (from Exercise 12.3 of [A3]), which is equivalent to this plumbing).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure4.png}
\caption{Self diffeomorphisms $R$ and $S$ of $K^{+1}$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure5.png}
\caption{Self diffeomorphism $T$ of $K^{-1}$}
\end{figure}

*Proof.* $\text{MCG}(K^{-1})$ is straightforward to calculate, by using the fact that it is a Seifert fibered space. From [AK], we know that $K^{-1}$ is the Brieskorn sphere $\Sigma(2,3,13)$. This is a 'small' Seifert fibered space, from which it follows that any element in $\text{MCG}(K^{-1})$ is isotopic to a fiber-preserving diffeomorphism [BO]. In particular, any orientation preserving self-diffeomorphism is isotopic to the identity, or to an involution that reverses the orientation of both base and fiber. For the identification of $\text{MCG}(K^{+1})$ we refer reader to [R]. $\square$
Remark 1. Manifolds $W(0, n)$ have an interesting feature: Blowing them up $n$ times produces absolutely exotic manifold pairs with a cork inside (as in [A7]). We will call contractible manifolds with this property “almost corks”. Figure 6 demonstrates this process when $n = 1$, iterating gives an absolutely exotic $W(0, n) \# n\mathbb{CP}^2$ containing $W(0, 0)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure6.png}
\caption{$W(0, 1) \# \mathbb{CP}^2$ and its absolutely exotic copy}
\end{figure}

Proof. of Theorem 1: We will show the generator $T$ of $\text{MCG}(\partial W(0, 2))$ extends to a self-diffeomorphism of $W(0, 2)$. To show $T$ extends we need to recall the identification $\partial W(0, 2)$ with $-K^{-1}$ as shown in Figure 7.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure7.png}
\caption{The identification $\partial W(0, 2)$ with $-K^{-1}$}
\end{figure}

It suffices to show that the diffeomorphism $T$ keeps the isotopy type of the dual loop of the 2-handle of $W(0, 2)$; then we can extend $T$ to a self-diffeomorphism of $W(0, 2)$ by carving ([A3]). To see this we need to analyze the identification of $\partial W(0, 2)$ with $-K^{-1}$ closely. This is done in Figure 8. Now Figure 9 shows that the loop $T(\gamma)$ is isotopic to $\gamma$ (just slide $T(\gamma)$ over the -2 framed 2-handle along the dotted line). \qed

We now trace the image $R(\gamma)$ of the dual circle $\gamma$ of the 2-handle of $W(0, 1)$ to $\partial K^{+1}$. It follows from the construction that $R(\gamma) = S(\gamma)$, so $S(\gamma)$ can not bound any smooth disc in $W(0, 1)$ when $R(\gamma)$ doesn’t. Figure 10 shows how to identify $\partial W(0, 1)$ with $K^{+1}$; it also identifies the action of $R$ as 180° rotation. After sliding $R(\gamma)$ over the 2-handle we get Figure 11. In Figure 12 we draw the positions of $\gamma$ and $R(\gamma)$ (in Figures 10 and 11) in the same picture of $W(0, 1)$. 
Then after sliding $\gamma$ and $R(\gamma)$ over the 2-handle (along the indicated arrows) we arrive Figure 13 which describes the action of $R$ on the loop $\gamma$. Figure 14 is the same only the 1-handle notation has been changed.

Figure 8. Tracing dual circle of $W(0,2)$ to $\Sigma(2,3,13)$
Figure 9. $T(\gamma)$ is isotopic to $\gamma$

Figure 10. Tracing dual circle of 2-handle of $W(0, 1)$ to $K^{+1}$
Figure 11. Sliding $R(\gamma)$ over the 2-handle

Figure 12. Positions of $\gamma$ and $R(\gamma)$ in $\partial W(0,1)$

Figure 13. $R : \partial W(0,1) \to \partial W(0,1)$

Figure 14. Another view of $R : \partial W(0,1) \to \partial W(0,1)$
1.2. **Proof of Theorem** 2. Clearly, up to 3-handles, the handlebody of $\Sigma = -W(0, 1) \sim_{R(\gamma)} W(0, 1)$ can be obtained by attaching 0-framed 2-handle to $W(0, 1)$ along $R(\gamma)$, which is the dual 2-handle of $W(0, 1)$. This means attaching a 2-handle to the second picture of Figure 13 along $R(\gamma)$. Careful reader will notice that the big left twist between the curves in the middle of Figure 14 can represented in a simpler way by introducing a cancelling $1/2$- handle pair (i.e. a dotted circle and small linking circle with $-1$ framing). This gives us the first picture of Figure 15. This observation will now lead us to Gluck construction: Now we slide the middle 1-handle over the 2-handle as indicated in this figure (note that 1-handles do not slide over the 2-handles, unless they are in the form of Figure 1.16 of \[A3\]). Then proceed to the last picture of Figure 15. The dotted arcs in Figure 15 are the ribbon moves reminding us the bounding disk of the ribbon 1-handle. Drawing the last picture of Figure 15 in a more symmetric way we get the first picture of Figure 16. This last picture reveals an amazing fact: It describes a Gluck twisted $S^4$ twisted along an imbedded $S^2 \subset S^4$, where this 2-sphere is obtained by putting together the two different ribbon disks $D_{\pm} \subset B^4_{\pm}$, which the Stevedore knot $K$ bounds. The two distinct ribbon moves are related to each other by $180^\circ$ rotation of $S^3$ as indicated by the picture. The second picture of Figure 16 (after ribbon move performed) is the picture of the Glucked 2-sphere $S^2 \subset S^4$. Now $R(\gamma)$ represents a zero framed 2-handle.

Figure 17 is the same as the second picture of Figure 16 drawn more symmetric way. It is easy to check that in this picture the linking circle of the small $-1$ framed circle is unknot in $S^3$ boundary. Now by using this unknot, we can attach a $2/3$ - canceling handle pair (the new 2-handle is the 0-framed small red circle in the first picture of Figure 18). Next we use the trick from (Figure 14.11 of \[A3\]): That is, we cancel the circle with dot at top with its linking $-1$ framed circle and get the second picture of Figure 18. Then after the obvious handle slide over the large 0-framed 2-handle at the top right picture of Figure 18, we obtain the third picture of Figure 18 where we can now see a cancelling $1/2$-handle pair. So this picture can be thought of a handlebody without 1-handles (i.e. it consists of two 2-handles and two 3-handles), and hence turning it upside down we will give get a handlebody without 3-handles! Having noted this, we can now turn this handlebody upside down (as the process described in \[A3\]).
For this, we ignore the cancelling 1/2 handle pair, and carry the duals of the remaining 2-handles to the boundary of \( \#2(S^1 \times B^3) \) by any diffeomorphism. Starting with the third picture of Figure 18, we proceed from Figures 19 to Figure 21 by self described steps similar to in Figure 14.11 of [A3], and end up with Figure 21 and the first picture of Figure 22. By canceling a 1/2-handle pair gives the second picture of Figure 22 (dotted Stevedore knot represents a ribbon 1-handle). \( \square \)

![Figure 15. Tracing \( R(\gamma) \subset \partial W(0, 1) \)](image)

![Figure 16. Two different ribbons describing \( S^2 \subset S^4 \)](image)
Figure 17. Another view of $R(\gamma) \subset W(0, 1)$. Attaching 2-handle to $R(\gamma)$ turns picture to Gluck twisted $S^4$

Figure 18. Using Gluck twist to cancel 1-handle

Figure 19. Starting the process of turning upside down
Figure 20. Turning upside down process

Figure 21. $\Sigma$ without 3-handles

Figure 22. $\Sigma$
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DEPT. OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, MI, 48824

E-mail address: akbulut@math.msu.edu