Stochastic Binary Opinion Model

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We study a stochastic binary opinion model where agents in a group are considered to hold an opinion of 0 or 1 at each moment. An agent in the group updates his/her opinion based on the group’s opinion configuration and his/her personality. Considering the number of agents with opinion 1 as a continuous time Markov process, we analyze the long-term probabilities for large population size in relation to the personalities of the group. In particular, we focus on the question of “balance” where both opinions are present in nearly equal numbers as opposed to “dominance” when one opinion is dominant.

I. INTRODUCTION

The study of opinion dynamics focuses on decision making process in multi-agent systems. This line of research dates back to 1950’s [1]. Since then, the study of opinion dynamics has attracted researchers from diverse areas, and hence various approaches have been proposed for the modeling of evolution of opinions. It is natural to assume that an agent’s decision making process is influenced by the information he/she receives from the society. The influence of society has been considered in the form of agent’s pairwise interactions with his/her “neighbors” [2–7]. An agent’s neighbor can be chosen based on agent’s opinion [6–9] or based on a given communication graph independent of the opinions [10–15].

The set of possible opinions of a given agent at a given time may be regarded as a finite discrete set without any additional structure, the simplest example being a binary set, or alternatively as a continuum of values in \( \mathbb{R} \) or \( \mathbb{R}^d \). The structure of the \( \mathbb{R}^d \) imposes a concept of nearby opinions and leads to the notion of confidence bounds where an agent may only interact with other agents within his/her confidence bound. See for instance [6, 8, 10]. In this paper we shall focus on the binary opinion case and as such the notion of bounded confidence does not apply.

One simple way an agent can choose to update his/her opinion when interacting with a neighbor agent would be simply adapting the opinion of the neighbor agent. This updating rule is used in what is known as voter models in opinion dynamics. One example of voter models is when a group of agents has to make a yes/no decision on a subject. These binary voter models where an agent chooses one of his/her neighbor uniformly at random and adapts his/her opinion is studied [17, 18]. It is important to note that, this opinion updating rule does not take agents’ personality into account. In this respect, [13, 14, 19] introduces personality to the group including stubborn agents who do not change their opinion, however influence other agents. The presence of leaders is considered in [11, 20]. In [20], every agent in the group is given a personality defined through parameters of persuasiveness and supportiveness. These parameters decide on the strength of pair interactions with the opposite opinion or the same opinion. General voter models and the analysis of different types of social influence such as conformity and nonconformity on the limiting decision of the group is discussed in [21].

In this study, we propose a binary opinion model (say 0 or 1) for a group in which agents are considered to have personality defined by a conformity function and a spontaneity coefficient. An agent’s personality determines the effect of social influence on the agent. Moreover, in our model, contrary to pair-wise interaction of agents that are defined to be neighbors, agents are considered to be informed on the distribution of opinions in the group at each time \( t \). We note that, our model can be thought of as the result of a situation where agents do not change their mind after one (pairwise or group) interaction, but rather after several interactions. In this case, assuming all agents can interact with all others, in a large population an agent interacts with sufficiently many others before changing their mind and the sample from the sufficiently many can be taken as a good approximation of sampling the entire population.

We also assume (as is done in other models) that there is no “natural bias” towards one of the opinions. As an example, if the opinion is about whether the earth is flat or not, one would expect that in an informed society, agents are more likely to believe the truth. We are focused on the long-term probability distribution for the number of agents with opinion 1 when the number \( N \) of total agents is very large.

It is natural to focus on two possible extreme outcomes: balance of opinions and complete dominance of one opinion. In reality, the situation is not “black or white”; for instance, one may find that in the long-term, one opinion

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is likely to be held by say 70% of the agents. We investigate various personality traits and their effect on the group’s limiting behavior. In particular, the personality traits that lead to dominance of one opinion is our main interest.

The paper is organized as follows. In Section II we introduce our binary model and focus on a homogeneous group. In section III we study the effects of personality of the (homogeneous) group to the group’s limiting behavior. We extend our model to heterogeneous groups and examine limiting decision behavior when the group is formed by two extreme personality classes in Section IV. The concluding remarks follow in Section V.

II. THE MODEL AND THE HOMOGENEOUS CASE

We consider a group of $N$ agents where each agent holds an opinion from the set $\{0, 1\}$. An agent flips his/her opinion based on the group’s current configuration and his/her personality. Here, personality of an agent $i$, $i = 1, \ldots , N$, is given by the pair $(\phi_i, \beta_i)$ where $\phi_i : [0, 1] \rightarrow [0, \infty)$ is a monotonic function that accounts for conformity, $\beta_i$ is a nonnegative quantity that accounts for spontaneity. We call a group  homogeneous if all agents in the group share the same personality, $\phi = \phi_1, \beta = \beta_i \forall i = 1, \ldots , N$. We first look at the case when the group is homogeneous.

We define $X^N(t)$ to be the number of agents holding opinion 1 at time $t \in [0, \infty)$ and assume that a given agent flips his/her opinion during a time interval $(t, t+h]$ with a probability

\[
\lambda_i^N(N) = \phi(\frac{i}{N}) + \beta_i \quad \text{and} \quad \mu_i^N(N) = \beta(N - i),
\]

\[
\lambda_i^N(N) = \phi(\frac{i}{N}) + \beta_i \quad \text{and} \quad \mu_i^N(N) = \beta_i(N - i).
\]

Since the state space $\{0, 1, 2, \ldots , N\}$ is finite, $\lambda_i^N = 0$ and $\mu_i^N = 0$. Using these transition rates one can construct a transition rate matrix $Q^N = [q_{ij}^N]$, where $q_{ij}^N(t) = \lambda_i^N(N) q_{ij}^N = -\sum_{j \neq i} q_{ij}^N$ and $q_{ij}^N = 0 \forall j \notin \{i - 1, i, i + 1\}$.

We may rewrite the birth rate $\lambda_i^N(N)$ and the death rate $\mu_i^N(N)$ as follows:

\[
\lambda_i^N(N) = N\bar{\lambda} \left( \frac{i}{N} \right), \quad \mu_i^N(N) = N\bar{\mu} \left( \frac{i}{N} \right) \quad i = 0, 1, \ldots , N,
\]

where $\bar{\lambda}, \bar{\mu} : [0, 1] \rightarrow [0, \infty)$ are given by

\[
\bar{\lambda}(x) = (1 - x)\phi(x) + \beta(1 - x),
\]

\[
\bar{\mu}(x) = x\phi(1 - x) + \beta x.
\]

We define the probability $p_i^N(t) = (p_{0}^N(t), p_{1}^N(t), \ldots , p_{N}^N(t))$, where $p_i^N(t) = P[X^N(t) = i]$ for each $i = 0, 1, \ldots , N$. The probability mass function satisfies the Kolmogorov’s forward equation

\[
p_i^N(t) = \sum_k p_k^N(t) q_{ik}^N.
\]

It should be noted that when spontaneity coefficient $\beta = 0$, the states $i = 0$ and $i = N$ are absorbing states since $\lambda_0^N = \mu_0^N = 0$. When $\beta > 0$, the birth-death process $X^N(t)$ is an irreducible Markov process with the finite state space $\{0, 1, 2, \ldots , N\}$ and thus, $X^N(t)$ is ergodic and attains a unique stationary probability distribution as $t \rightarrow \infty$. The probability vector $p_i^N(t) \rightarrow \pi^N = (\pi_0^N, \pi_1^N, \ldots , \pi_N^N)$ as $t \rightarrow \infty$ and $\pi^N$ does not depend on the initial state $X^N(0)$. Using the detailed balance condition at stationarity, one can obtain

\[
\pi_n^N = \frac{\lambda_{n-1}^N \pi_{n-1}^N \cdots \lambda_0^N \pi_0^N}{\mu_0^N \mu_1^N \cdots \mu_n^N}, \quad n = 1, 2, \ldots , N.
\]

Hence,

\[
\pi_n^N = \frac{R_n^N}{\sum_{k=0}^n R_k^N}, \quad n = 0, 1, \ldots , N,
\]

\[
\pi_0^N = \frac{1}{\sum_{k=0}^n R_k^N}
\]

where

\[
R_n^N = r\pi_{n-1}^N \cdots r_0^N, \quad n = 1, 2, \ldots , N
\]

for $r_n^N = \frac{\lambda_n^N}{\mu_0^N}$ and $P_0^N = 1$.

III. EFFECT OF THE CONFORMITY FUNCTION : THE HOMOGENEOUS CASE

In this section, we study various conformity functions $\phi(x)$ and analyze their effect on the behavior of $X^N(t)$ for large $t$ and large $N$.

In order to study $X^N(t)$ for large $N$ and $t$, we shall consider the normalized process $X_N(t) = \frac{X^N(t)}{N}$. As $N \rightarrow \infty$, in the fluid limit, one expects $X_N$ to converge to $X$ where $X$ satisfies the ODE

\[
\dot{X}(t) = F(X(t)) = \bar{\lambda}(X(t)) - \bar{\mu}(X(t)),
\]
which simplifies to
\[
X(t) = \phi(X(t))(1 - X(t)) - \phi(1 - X(t))X(t) + \beta(1 - 2X(t))
\]

(9)

Intuitively, when \(N\) and \(t\) are both large, one expects the peaks of the probability distribution of \(X_N(t)\) to occur near the stable equilibria of this ODE. This observation will motivate the rest of the analysis in this paper.

While we do not make new claims about rigorous limits as \(t \to \infty\) and \(N \to \infty\) jointly, some rigorous limits exist in literature that we mention here. A major result is that if \(F\) is \(C^1\), then given any finite time interval \([0, T]\), as \(N \to \infty\), \(X_N \to X\) uniformly on \([0, T]\) with probability one, and moreover a diffusion approximation for \(X_N\) is also available [22]. Since this result only considers the limit as \(N \to \infty\) over finite intervals of time, one needs to be cautious in interpreting the large \(N\) and large \(t\) approximation. In particular, if one fixes any large final time \(t\), and considers increasing \(N\), then one expects distributions at time \(t\) to have peaks around the stable equilibria of the ODE. When \(F\) has a unique globally attractive equilibrium \(\overline{X}\), under suitable conditions, as \(N \to \infty\) one can rigorously justify a Gaussian approximation with mean \(\overline{X}\) for the stationary probability distribution (see Theorem 2.7 in [23]).

Henceforth we shall study the system \((1)\) for its stable equilibria. We note that \(F(0) > 0, F(1) < 0\) and \(\overline{X} = \frac{1}{2}\) is always an equilibrium for the dynamics \((1)\). If \(\overline{X} = \frac{1}{2}\) is the unique equilibrium, it will be globally attractive on \([0, 1]\). Then, for very large \(N\), the stationary probability distribution will be a narrow Gaussian with mean \(\overline{X}\) and hence, the model leads to balance of opinions.

We shall see that the shape of the conformity function \(\phi\) plays an important role in deciding if dominance of an opinion is likely. Intuitively, one may expect that greater conformity leads to dominance of one opinion while greater spontaneity leads to balance of opinions via a law of large number effect. However, our examples suggest that when \(\phi\) is strictly convex, for sufficiently small \(\beta\) we see dominance. When \(\phi\) is not strictly convex or if it is concave, we do not see dominance in our examples.

Remark: For the sake of precision, we shall use the term balance to mean the situation where there is only one stable equilibrium of \((1)\) which is \(\overline{X} = 1/2\). We shall use the term dominance rather loosely to stand for lack of balance.

We will consider different examples of \(\phi\) and investigate the stable equilibria to predict dominance or balance. We shall also check the prediction from the stable equilibria of the ODE model against computational results of the stationary probability distributions.

We note that, there are two methods to compute the stationary distributions. One is to use the formulas \((6)\) and \((7)\) and the other is to use an ODE solver to compute the solution to \((1)\). For very large \(N\) values, our numerical experiments suggest that using \((6)\) and \((7)\) provide more accurate results compared to the ODE solver. Hence, throughout this study, we refer to \((6)\) and \((7)\) to verify our predictions from the stable equilibria of \((1)\).

Before we proceed with examples of \(\phi\), we note that the most natural conditions on \(\phi: [0, 1] \to [0, \infty)\) are that \(\phi(0) = 0\) and \(\phi\) is increasing for conformity, and \(\phi(1) = 0\) and decreasing for rebelliousness (opposite of conformity).

Example 1 Consider the simplest example of \(\phi(x) = x\). This is convex, but not strictly so. In this case, the rate that an agent changes his/her opinion is \(\frac{X}{2} + \beta\). Using \((9)\), we can conclude that as \(N\) gets large \(X_N(t)\) converges to \(X(t)\), where
\[
\dot{X}(t) = (1 - 2X(t))\beta.
\]

(10)

Since this ODE has a unique equilibrium at \(\overline{X} = \frac{1}{2}\) that is globally attractive, regardless of spontaneity coefficient \(\beta > 0\), it is expected that the group will reach a balance of opinions as can be observed in Fig. 4.

It is thus interesting to note that as long as \(\beta > 0\), no matter how small, one expects balance of opinions.

![FIG. 1. Exact values of the stationary probabilities calculated using (10) and (11) for \(\phi(x) = x\) and \(N = 1000\). (a) \(\beta = 5\). (b) \(\beta = 0.01\).](image)

Example 2 Consider \(\phi(x) = x^2\) which is strictly convex. The limiting ODE is
\[
\dot{X}(t) = (1 - 2X(t))(X(t)^2 - X(t) + \beta).
\]

(11)

It is easy to see that \((11)\) has three possible equilibria: \(\overline{X}_1 = \frac{1}{2}\) and \(\overline{X}_2, 3 = \frac{1}{2} \pm \sqrt{\frac{3\beta}{4}}\). Hence, based on the choice of spontaneity coefficient \(\beta\), different scenarios are expected. When \(\beta \geq \frac{1}{4}\), the stable equilibrium is \(\overline{X}_1 = \frac{1}{2}\) and the model leads to balance of opinions. On the other hand, when \(\beta < \frac{1}{4}\), the stable equilibrium are \(\overline{X}_2, 3\) and the model leads to dominance. In Fig. 4(a), \(\beta = 5\) and the model leads to balance of opinions. On the other hand, when \(\beta = 0.2\), as can be seen in Fig. 4(b), the model leads to dominance of one opinion.

Example 3 We consider \(\phi(x) = 1 - x^2\) to explore the impact of rebelliousness in our model. In this case, the rate an agent changes his/her opinion is \(\frac{X^2}{\sqrt{X^2} + \beta}\). Hence, as the number of agents with the opposite opinion increases, the
rate of opinion change decreases. The limiting ODE for this model is

$$\dot{X}(t) = (X(t)^2 - X(t) - 1 - \beta)(2X(t) - 1).$$

One can see that (12) has a unique equilibrium at $X = \frac{1}{2}$ regardless of the spontaneity coefficient $\beta \geq 0$. Thus, we can conclude that the model leads to balance of opinions as can also be observed in Fig. 3.

**Example 4** Let $\phi(x) = \sqrt{x}$ which is concave. Then the corresponding ODE is

$$\dot{X}(t) = (1 - 2X(t)) \left[ \frac{X(t)\sqrt{1 - X(t)}}{\sqrt{1 - X(t) + \sqrt{X(t)}}} + \beta \right].$$

(13)

We first observe that $F$ is not $C^1$ on $[0,1]$ and the fluid limit theorem does not apply. Nevertheless, we proceed heuristically to look for the stable equilibria. One can observe that $X = 1/2$ is the only equilibrium for (13). Thus, one may expect that the model will lead to balance of opinions regardless of the choice of the spontaneity coefficient $\beta$ as can be observed in Fig. 4.

We provide a proof in Appendix B that for the case of $\phi(x) = x^r$ with $r > 1$, the ODE model predicts dominance for sufficiently small $\beta > 0$.

**Example 5** Consider the monotone increasing, convex conformity function $\phi(x) = \frac{x}{x - 2}$, so that the conformity function is the ratio of the fraction of agents with opposite opinion to those with the same opinion. Then, the rate of change of one’s opinion is $n/(N - n) + \beta$. We note that $\phi(x)$ has a singularity at $x = 1$ and that (5) does not hold for $i = 0, N$, and hence the fluid limit theorem does not hold. Nevertheless, we proceed heuristically to consider $F$ which is given by

$$F(x) = (\beta - 1)(1 - 2x),$$

which has only one equilibrium $X = 1/2$ and it is (asymptotically) stable if and only if $\beta > 1$. Thus we expect balance for $\beta > 1$. When $\beta < 1$, we expect dominance of one opinion. This heuristic is verified both by our numerical computation of the stationary probabilities as well as the asymptotic formulas for the stationary probabilities (6) and (7) that we derive in Appendix A.

In Fig. 5 we present an example for $\beta < 1$ case. In this experiment we use $\beta = 0.2$ and $N = 100$. Fig. 5 shows the asymptotic approximation $\tilde{\pi}^N = (\tilde{\pi}_1^N, \tilde{\pi}_2^N, \ldots, \tilde{\pi}_N^N)$ calculated in (A13). As shown in Fig. 5 our model leads to dominance of one opinion. On the other hand, an example of $\beta > 1$ case is shown in Fig. 6. Fig. 6 gives the approximations for stationary probabilities, $\tilde{\pi}^N$, calculated using (A17). As seen in Fig. 6 stationary probabilities show the balance of opinions.

**IV. HETEROGENEOUS BINARY OPINION DYNAMICS**

Let us consider the case where the group is heterogeneous. Namely, suppose we have $m$ personality classes of agents such that all agents within a class $i$ (where $i = 1, \ldots, m$) have the same personality $(\phi_i, \beta_i)$, but personalities differ among
being the total number of class personalities of agents is fixed in time, thus \( \beta = 10 \). (b) agents. We note that during a time interval \((t, t + h)\) an agent from class \(i\) will flip with probability

\[
(\phi_i(n/N) + \beta_i)h + o(h) \quad h \to 0^+,
\]

where \(n\) is the total number of all agents who have the opposite opinion to that of the given agent. We shall be concerned the classes. This results in a Markov process model where the state is a vector \(x = (x_1, \ldots, x_m)\) where \(0 \leq x_i \leq N_i\) is the number of class \(i\) agents who hold opinion 1 with \(N_i\) being the total number of class \(i\) agents. We assume that the personalities of agents is fixed in time, thus \(N_i\) is a constant for each \(i\) and \(N = \sum N_i\) is the total number of all agents. We note that during a time interval \((t, t + h)\) an agent from class \(i\) will flip with probability

\[
(\phi_i(n/N) + \beta_i)h + o(h) \quad h \to 0^+,
\]

where \(\phi_i\) is the fraction of all agents who hold opinion 1. We shall be concerned with the case of large \(N\) with the fractions \(k_i = N_i/N\) within classes being held constant.

This results in a family of Markov process \(X^N(t)\) which undergo a jump \(c_i\) or \(-c_i\) for \(i = 1, \ldots, m\) (here \(c_i \in \mathbb{R}^m\) is the vector with \(i\)th component equal to one and all others equal to zero) with corresponding class \(i\) birth and death rates given by

\[
\begin{align*}
\lambda_i^N(x) &= \phi_i \left( \frac{|x|}{N} \right) (N_i - x_i) + \beta_i (N_i - x_i) \\
\mu_i^N(x) &= \phi_i \left( 1 - \frac{|x|}{N} \right) x_i + \beta_i x_i
\end{align*}
\]

(14)

where given the state \(x = (x_1, \ldots, x_m)\) we denote by \(|x|\) the total number of agents holding opinion 1:

\[
|x| = \sum_{i=1}^{m} x_i.
\]

We note that \(0 \leq x_i \leq N_i\). We shall consider the normalized process \(X^*_N(t) = X^N(t)/N\), where \(X^*_N(t)\) is the fraction of class \(i\) agents with opinion 1 where the fraction is normalized by \(N\) and not \(N_i\). We may write

\[
\begin{align*}
\lambda_i^N(x) &= N \bar{\lambda}_i \left( \frac{x_i}{N} \right), \\
\mu_i^N(x) &= N \bar{\mu}_i \left( \frac{x_i}{N} \right)
\end{align*}
\]

where

\[
\begin{align*}
\bar{\lambda}_i(x) &= \phi_i(|x|)(k_i - x_i) + \beta_i (k_i - x_i), \\
\bar{\mu}_i(x) &= \phi_i (1 - |x|) x_i + \beta_i x_i
\end{align*}
\]

(15)

In the fluid limit, as \(N \to \infty\), one expects \(X^*_N\) to converge to \(X\) where \(X\) satisfies the ODE

\[
\dot{X}(t) = F(X(t))
\]

where the \(m\) dimensional vectorfield \(F\) is given by

\[
F_i(x) = \bar{\lambda}_i(x) - \bar{\mu}_i(x), \quad i = 1, \ldots, m,
\]

which simplifies to

\[
F_i(x) = \beta_i (k_i - 2x_i) + \phi(|x|)(k_i - x_i) - \phi(1 - |x|) x_i
\]

(16)

for \(i = 1, \ldots, m\). When \(N\) and \(t\) are both large, we expect to see the peaks of the probability distribution of \(X^*_N(t)\) to

\[
\begin{align*}
\text{FIG. 4. Exact values of the stationary probabilities calculated using (6) and (7) for } \phi(x) = \sqrt{x} \text{ and } N = 1000. \quad \text{(a)} \beta = 10. \quad \text{(b)} \beta = 0.1.
\end{align*}
\]

\[
\begin{align*}
\text{FIG. 5. Asymptotic approximation for } \tilde{\pi}^N \text{ when } \phi(x) = \frac{x}{\sqrt{x}} \text{, } \beta = 0.2 \text{ and } N = 100.
\end{align*}
\]

\[
\begin{align*}
\text{FIG. 6. Asymptotic approximation for } \tilde{\pi}^N \text{ when } \phi(x) = \frac{x}{\sqrt{x}}, \quad \beta = 10 \text{ and } N = 100.
\end{align*}
\]
occur near the stable equilibria of this ODE. We note that \( \bar{x} = (k_1/2, \ldots, k_m/2) \) is always an equilibrium.

**Example with extreme two personality classes**

We consider the case of two extreme personality classes \((\phi_i, \beta_i)\) for \(i = 1, 2\) where

\[
\begin{align*}
\phi_1(\xi) &= \xi^e, \quad \beta_1 = 0, \\
\phi_2(\xi) &= 0, \quad \beta_2 = \beta > 0.
\end{align*}
\]

We note that class 1 corresponds to total conformity and class 2 corresponds to total spontaneity. Let us write \( k_2 = 2k \) (thus \( k \) is half the fraction of class 2) and thus \( k_1 = 1 - 2k \). This results in

\[
\begin{align*}
F_1(x) &= (x_1 + x_2)^r (1 - 2k - x_1) - (1 - x_1 - x_2)^r x_1, \\
F_2(x) &= \beta(2k - 2x_2).
\end{align*}
\]  
(18)

At an equilibrium, clearly \( x_2 = k \) and \( (1/2 - k, k) \) is always an equilibrium. Additional equilibria are found by solving the equation

\[
(x_1 + k)^r (1 - 2k - x_1) - (1 - k - x_1)^r x_1 = 0
\]

for \( x_1 \). Before analyzing the model for different \( r \) values, we can observe that class 2 (spontaneous class) is always expected to reach a balance since at an equilibrium \( x_2 = k \).

We first consider \( r = 1 \). The system has a unique equilibrium \((1/2, k)\) and the Jacobian at the equilibrium is

\[
J(1/2, k) = \begin{bmatrix} -2k & 1 - 2k \\ 0 & -2\beta \end{bmatrix}.
\]

Since all the eigenvalues of \( J \) have negative real part, regardless of the choice of \( \beta \) and \( k \) the equilibrium is stable. Hence, when \( r = 1 \) this model leads to balance in each class \( i \). Therefore, we reach a balance for the whole group.

Now we consider \( r = 2 \). In this case, when \( k \geq 1/4 \) there is only one equilibrium \((1/2 - k, k)\). On the other hand, when \( k < 1/4 \) the equilibria are \((1/2 - k \pm \sqrt{1 - 2k}, k)\).

Assume \( k \geq 1/4 \). At the equilibrium \((1/2 - k, k)\), eigenvalues are

\[
e_1 = \frac{1}{2} - 2k, \quad e_2 = -2\beta.
\]

Since \( k \geq 1/2 \), we conclude that \((1/2 - k, k)\) is stable.

Assume \( k < 1/4 \). In this case we can analyze the sign of \( e_1 \) and obtain that \( e_1 < 0 \) for the equilibria \((1/2 - k \pm \sqrt{1 - 2k}, k)\). Thus \((1/2 - k \pm \sqrt{1 - 2k}, k)\) is stable.

In summary, when \( r = 2 \), for our model, class 2 is expected to have balance, however class 1, based on the choice of \( k \), may have balance or dominance. When \( k \geq 1/4 \), that is when spontaneous class is more than 50% of the population, class 1 (conformists) is also expected to reach a balance since \((1/2 - k, k)\) is the only equilibrium (stable) for the corresponding ODE. Thus resulting a balance for the whole group.

When \( k < 1/4 \), that is when spontaneous class is less than 50% of the population, class 1 (conformists) reaches a dominance. And, this leads to dominance for the whole group. One Example is given in Fig. 7 where probabilities are computed using Monte Carlo simulations of 10,000 trajectories. Here, total number of agents is \( N = 120 \) with \( N_2 = 20 \) \((k = 1/12)\) being the spontaneous class population where \( \phi(x) = x^2 \) and spontaneity coefficient \( \beta = 0.02 \).

**V. CONCLUSIONS**

We have proposed a simple binary model where agents hold an opinion from the set \( \{0, 1\} \) at any time \( t \geq 0 \). An agent flips his/her opinion based on the number of agents with opinion 1 in the entire population. The influence of the group on an agent is determined by his/her personality. A personality is formed by a conformity function \( \phi \) and spontaneity coefficient \( \beta \). When all agents in the group share the same personality, we call the group homogeneous.

Initially, focusing on a homogeneous group, we analyzed the long time probabilities for a large population size, for different personality characteristics of the group. The question of what personality characteristics lead to dominance of one opinion was studied. We found that the shape of the conformity function, namely strict convexity or lack thereof, seems to be an important determining factor in whether dominance of one opinion occurs for sufficiently small \( \beta \).

We extended our model to a heterogeneous group, where the group consists of different personality classes. In particular, when the group is formed by two extreme classes, complete conformity and complete spontaneity, the dominance of group opinion is analyzed with various conformity functions. In this example, we found that the fraction of the pure conformists was a key determining factor of dominance along with the strict convexity of \( \phi \).

**Appendix A: Asymptotic approximations in Example 5**

In order to obtain an approximation for the values of \( \pi_n^N \) in (6) and (7) for large \( N \), we first approximate \( R_n^N \) given by (5) and then acquire an approximation for the sum \( S^N = \sum_{k=0}^{N} R_k^N \). For \( n = 1, 2, \ldots, N \), we may write

\[
\frac{1}{N} \ln(R_n^N) = \frac{1}{N} \sum_{j=1}^{n} \ln \left( \lambda \left( \frac{j-1}{N} \right) \right) - \frac{1}{N} \sum_{j=1}^{n} \ln \left( \mu \left( \frac{j}{N} \right) \right).
\]  
(A1)

The following lemma will help us to understand the right hand side of (A1) asymptotically.
Lemma 1. Let $h : [0,1] \to \mathbb{R}$ be a $C^2$ function on its domain. Consider the partition $0 = x_0 < x_1 < \ldots < x_N = 1$ with $x_k - x_{k-1} = \frac{1}{N}$. Consider $x = \frac{N}{2}$ be fixed. Then,

$$N \int_0^x h(t) \, dt - \sum_{k=1}^N h(x_k) = \frac{h(x) - h(0)}{2}, \quad (A2)$$

$$\sum_{k=1}^N h(x_k) - N \int_0^x h(t) \, dt \to \frac{h(x) - h(0)}{2} \quad \text{as} \quad N \to \infty. \quad (A3)$$

Proof. Using Taylor expansion of $h(t)$, $t \in [x_k-1, x_k]$ about $x_{k-1}$, we may write

$$h(t) - h(x_{k-1}) = h'(x_{k-1})(t-x_{k-1}) + \frac{1}{2} h''(\xi(t))(t-x_{k-1})^2,$$

where $\xi(t) \in (x_{k-1}, t)$. Integrating both sides of the equation above on the interval $[x_{k-1}, x_k]$,

$$e_k := \int_{x_{k-1}}^{x_k} h(t) \, dt - h(x_{k-1})(x_k - x_{k-1})$$

$$= \frac{1}{2N} h'(x_{k-1}) + T_k, \quad (A4)$$

where $T_k = \int_{x_{k-1}}^{x_k} \frac{1}{2} h''(\xi(t))(t-x_{k-1})^2 \, dt$. Now, summing over $k = 1, \ldots, n$ we reach

$$\sum_{k=1}^n e_k = \int_0^x h(t) \, dt - \frac{1}{N} \sum_{k=1}^n h(x_{k-1}) \to \frac{h(x) - h(0)}{2}, \quad (A5)$$

Moreover, since $h''(x)$ is continuous and hence bounded on $[0,1]$, for some $M < \infty$ we may write

$$\sum_{k=1}^n T_k \leq \frac{1}{2} \sum_{k=1}^n \frac{M}{N^2} = \frac{M}{N^2}, \quad (A6)$$

Hence, for $N \to \infty$,

$$N \sum_{k=1}^n T_k \to \frac{M}{N^2} \to 0. \quad (A7)$$

Considering (A4) together with (A5) and (A6), we may write

$$N \left( \frac{1}{N} \sum_{k=1}^n h(x_{k-1}) \right) \sim N \int_0^x h(t) \, dt - \frac{h(x) - h(0)}{2}, \quad N \to \infty. \quad (A8)$$

This proves (A2). Similarly, one can easily obtain (A3). \qed

Let us define

$$F(x) := \int_0^x \ln \left( \frac{\lambda(t)}{\mu(t)} \right) \, dt, \quad H(x) := \frac{1}{2} \ln \left( \frac{\lambda(x)\mu(x)}{\lambda(0)\mu(0)} \right). \quad (A8)$$

Then,

$$R_n^N \sim \exp \left( NF \left( \frac{n}{N} \right) - H \left( \frac{n}{N} \right) \right), \quad N \to \infty. \quad (A9)$$

Therefore,

$$S^N = \sum_{n=0}^N R_n^N \sim \sum_{n=0}^N \exp \left( NF \left( \frac{n}{N} \right) - H \left( \frac{n}{N} \right) \right), \quad N \to \infty. \quad (A10)$$

We note that

$$F(x) = \int_0^x \ln \left( \frac{(1-\beta)x + \beta}{(\beta-1)x + 1} \right) \, dt, \quad (A11)$$

$$H(x) = \frac{1}{2} \ln \left( 1 + \frac{(1-\beta)^2}{\beta} x (1-x) \right).$$

Based on the choice of $\beta$ one can observe that $F$ takes its maximum value either in the middle of the interval $[0,1]$ or at its end points. One can reach this conclusion by examining $F(x)$ in (A10). In fact, the integrand, $F'(x)$, is an increasing function with zero at $x = \frac{1}{\beta}$ if $\beta < 1$ and a decreasing function with zero at $x = \frac{1}{\beta}$ if $\beta > 1$. Now, using this observation on $F(x)$, and Laplace’s method, we can reach an approximation for $S^N$ (A10).

We first consider the case when $\beta < 1$. In this case $F(x)$ attains its maximum at the end points of the interval $[0,1]$. Since the largest contribution to the sum $S^N$ (A10) comes from some $\epsilon > 0$ neighborhood of the maximum and $F$ is symmetric, we have $S^N \sim 2S^N$ where

$$S^N_k = \sum_{n=0}^N R_n^N \sim \sum_{n=0}^N e^{NF \left( \frac{n}{N} \right) - H \left( \frac{n}{N} \right)}.$$
Let $I_r(N) = \sum_{m=0}^{M^2} \exp \left\{ -K \left( \frac{m}{M} + \frac{1}{2} \right)^2 \right\}$, where $K = -F''(\frac{1}{2})$ and $M = \sqrt{N}$. Then for large $M$

$$\frac{1}{M} I_r(M) = \sum_{m=0}^{M^2} \exp \left\{ -K \left( \frac{m}{M} + \frac{1}{2} \right)^2 \right\} \sim \int_0^{\infty} \exp \left\{ -K \left( \frac{t}{2} \right)^2 \right\} dt. \quad (A14)$$

Using the dominated convergence theorem [24], we can show that

$$\int_0^{\infty} \exp \left\{ -K \left( \frac{t}{2} \right)^2 \right\} dt \to \int_0^{\infty} \exp \left\{ -K \left( \frac{t}{2} \right)^2 \right\} dt \text{ as } M \to \infty.$$  

Note that the integral above is a Gaussian integral

$$\int_0^{\infty} e^{-ax^2} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \text{ with } a = \frac{1}{2}.$$  

Hence,

$$I_r(N) \sim \frac{1}{2} \sqrt{\frac{2\pi N}{F''(\frac{1}{2})}}.$$  

This gives us the approximation

$$S^N \sim S_0^N \sim e^{(NF(\frac{1}{2}) - H(\frac{1}{2}))} \sqrt{\frac{2\pi N}{F''(\frac{1}{2})}}.$$  

Using definitions of $F$ and $H$ in [410],

$$S^N \sim \sqrt{\frac{2N\pi \beta}{\beta^2 - 1}} e^{NF(\frac{1}{2})}, \quad (A15)$$

Hence,

$$\pi_0 \sim \frac{\sqrt{(\beta^2 - 1)}}{2N\pi\beta} e^{-NF(\frac{1}{2})}. \quad (A16)$$

where $F(\frac{1}{2}) = \int_0^{\infty} \frac{\sqrt{t}}{2\pi e^{\frac{t}{2}}} \ln \left( \frac{\lambda(t)}{\mu(t)} \right) dt = \frac{x^{(1+\beta)} \beta \beta}{(1+\beta)^{1+\beta} \beta N}$. Similarly, for $n = 1, 2, \ldots, N$, we obtain

$$\pi_n \sim \pi_0 \frac{\frac{n}{\beta n + (N-n)}}{\beta n + (N-n)} \frac{2^{\frac{n-1}{2}}}{2^{\frac{n-1}{2}}}. \quad (A17)$$

In order to check the accuracy of our asymptotic approximations, we calculate the relative error $E = (E_1, \ldots, E_N)$ defined as $E_i = \frac{\pi_i}{\pi_i^N}$ for $i = 0, \ldots, N$, where $\pi_i^N$ represents the exact calculations (28), (7), and $\pi^N$ is the asymptotic approximations (A13), (A17). We observed that the relative error is in the order of $10^{-10}$ and decreases as $N$ increases.

**Appendix B: Conformity function $\phi(x) = x^r$ with $r > 0$**

Here we study the homogeneous case with $\phi(x) = x^r$ for $r > 0$. Recall that the fluid limit vectorfield on $[0, 1]$ is given by

$$F(x) = \phi(x)(1 - x) - x\phi(1 - x) + \beta(1 - 2x).$$

It is convenient to introduce $G$:

$$G(x) = \phi(x)(1 - x) - x\phi(1 - x).$$

Then

$$F(x) = G(x) + \beta(1 - 2x).$$

We note that $G(0) = G(1/2) = G(1) = 0$. Thus $F(0) = \beta > 0$, $F(1) = -\beta < 0$ and $F(1/2) = 0$. So $1/2$ is always an equilibrium. If $F'(1/2) > 0$ then $1/2$ is unstable, and there must be at least two (symmetrically placed) equilibria, one in $(0, 1/2)$, and the other in $(1/2, 1)$. Moreover, generically, these equilibria will be stable, leading to dominance.

Since $F'(1/2) = G'(1/2) - 2\beta$, let us focus on computing $G'(1/2)$.

Consider $\phi(x) = x^r$ where $r > 0$. We get

$$G(x) = (1-x)x^r - x(1-x)^r.$$  

If $r = 1$, then $G(x) = 0$ for all $x$. Hence, only one equilibrium at $1/2$ which is stable, resulting in balance of opinions. If $r \neq 1$,

$$G'(x) = rx^{r-1}(1-x) - x^r - (1-x)^r + r(1-x)^{r-1}.$$  

After simplifying,

$$G'(1/2) = (1/2)^{r-1}(r-1).$$

If $r > 1$, $G'(1/2) > 0$, and since $F'(1/2) = G'(1/2) - 2\beta$, for sufficiently small $\beta$ we have $F'(1/2) > 0$ and thus there exist at least two (symmetrically placed) equilibria, one in $(0, 1/2)$, and the other in $(1/2, 1)$ which will both be generically stable.
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