Modified method of variable elasticity parameters for solving problems of dynamics of rod systems taking into account physical and geometric nonlinearities

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Abstract. The article describes an algorithm for calculating vibrations of rod systems by the finite element method, based on an iterative process that allows taking into account physical and geometric nonlinearity. The iterative process, which takes into account physical nonlinearity, is based on the introduction of the cubic dependence of stresses on deformations into the calculation and the comparison of the secant elastic modulus at each step of the iteration. When calculating the stiffness matrix, an additional term (matrix) is introduced that takes into account geometric nonlinearity. A comparative analysis of the results of the calculation with a linear dependence between stresses and deformations and the calculation with a cubic dependence of stresses on deformations is given.

1. Introduction

Building structures and elements of structures under the influence of force factors can significantly change their shapes and sizes, which is why the principle of small movements, on which the methods of linear construction mechanics are based, is not applicable. When large deformations occur, the stress values in structural elements may exceed the elastic limit, and the stress-strain relationship becomes nonlinear. Therefore, to describe the actual operation of such structures, nonlinear factors must be taken into account. In particular, the rod domes of locators, antennas, and other similar structures can receive large bending movements during seismic impacts. This requires upgrading direct methods for solving differential equations of the oscillatory motion of a mechanical system to account for nonlinear components of deformations and stresses.

This article provides formulas and describes an algorithm for calculating vibrations of rod systems by the method of Central differences, taking into account the geometric non-linearity and physical non-linearity of the material at each step in time. Accounting for physical non-linearity is implemented on the basis of power polynomials of the stress dependence on the relative deformation of elements. We will take into account geometric nonlinearity by constructing and entering a geometric matrix into the calculation.

2. Accounting for geometric non-linearity

Positive directions of nodal movements and forces of the final element and their numbering as shown in figure 1.
Figure 1. Direction of movement and deformations in the rod nodes

Denote by \( u, v, w \) the axial displacements of points along the axes \( X, Y, Z \), respectively; \( \varphi_x, \varphi_y, \varphi_z \) - the angles of turns of the rod sections around the axes \( X, Y, Z \). And then we will consider

\[
\varphi_x = \varphi, \quad \varphi_y = \frac{dv}{dx} = v', \quad \varphi_z = \frac{dv}{dx} = v''
\]

(1)

For a compressed-curved (stretched-curved) rod, the exact strain-displacement expression is determined by the dependency

\[
\varepsilon = u' - y \cdot v'' + z \cdot w'' + \frac{1}{2} \cdot (v')^2 + \frac{1}{2} \cdot (w')^2
\]

(2)

In (2), the first three terms are the components of axial and flexural deformation, and the fourth and fifth terms, which are nonlinear, take into account the change in the geometry of the rod at a large oscillation amplitude. Following R. Gallagher [1], the potential energy of deformation of a prismatic rod is written as:

\[
V_n = \frac{1}{2} \cdot E \cdot \int \int_D \left( \varepsilon_x \right)^2 \cdot dA \cdot dx
\]

(3)

When representing the components of the rod displacement vector as

\[
U = \begin{bmatrix} u_1 & v_1 & w_1 & \varphi_{x1} & \varphi_{y1} & \varphi_{z1} & u_2 & v_2 & w_2 & \varphi_{x2} & \varphi_{y2} & \varphi_{z2} \end{bmatrix}^T
\]

(4A)

or

\[
U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} & u_{17} & u_{18} & u_{19} & u_{10} & u_{11} & u_{12} \end{bmatrix}^T
\]

(4B)

differentiating expression (2) by these components, we obtain a well-known matrix of the stiffness of the spatial rod of the twelfth order and a so-called geometric matrix with dimensions \([12 \times 12]\), which takes into account the change in the geometry of the system:
For solving dynamic problems taking into account geometric nonlinearity in the differential equation of motion of a mechanical system

$$M \ddot{U} + B \dot{U} + KU = P(t)$$  \(6)$$

the stiffness matrix will be represented as:

$$K = K_L - K^G,$$  \(7)$$

where $M$ is the diagonal mass matrix;

$U$ – the displacement vector of the system;

$B$-damping matrix (motion resistance matrix);

$P(t)$ is a vector of nodal loads composed of the products of the acceleration values for a given earthquake accelerogram on the masses of the corresponding nodes.

3. Accounting for the physical non-linearity of the material

When writing equilibrium equations taking into account physical nonlinearity it is convenient to use dependencies of the form

$$\sigma = f(\varepsilon)$$  \(8)$$

In a fairly General form, this dependence (8) can be written as follows [2-5]:

$$\sigma = \sum_{i=1}^{n} A_i \cdot \varepsilon^{k_i} = A_1 \cdot \varepsilon^{k_1} + A_2 \cdot \varepsilon^{k_2} + \ldots + A_n \cdot \varepsilon^{k_n},$$  \(9)$$

where are some physical constants that have a stress dimension; dimensionless exponents that can be any positive number (integer, fractional).
\[ \sigma = \sum_{k=1,3,...}^{n} E_k \cdot \varepsilon^k = E_1 \cdot \varepsilon + E_3 \cdot \varepsilon^3 \pm \cdots \pm E_n \cdot \varepsilon^n \]  
(10)

For most practical calculations, you can limit yourself to two members of this series:

\[ \sigma = \sum_{k=1,3}^{3} E_k \cdot \varepsilon^k = E \cdot \varepsilon - E_3 \cdot \varepsilon^3, \]

that is, the cubic dependence [4], which ensures the symmetry of the diagram with respect to the stretching-compression.

Here \( E \) is the initial modulus of elasticity of the material; \( a \) is the parameter

\[ E_3 = \frac{4 \cdot E^3}{27 \cdot \sigma_u}, \]

\( \sigma_u \) – the ultimate strength of the material.

Formula (11) is equally convenient for approximating the \( \sigma-\varepsilon \) dependencies for both concrete and steel.

Substituting (12) and (11) in (2), we obtain an equation that establishes the relationship between the stresses and displacements \( u, v, w, \phi_x, \phi_y, \phi_z \).

We will distinguish between the concepts of static and dynamic rigidity of the frame elements [6-8]. "Static" refers to the stiffness determined by slow processes of structural deformation, and "dynamic " refers to the stiffness determined by fairly fast cyclic processes of structural deformation (usually by natural oscillation frequencies).

When assigning static stiffness, the stress-strain diagrams of the materials that make up the structure are used. In this case, the calculated values correspond to the upper limits of static stresses or deformations of materials. Dynamic stiffness is usually assumed to be higher than static, but the purpose of the absolute value here is due to the influence of many factors.

For most construction materials, the diagram type \( \sigma-\varepsilon \) is assumed to be linear when unloading. When loading, the shape of the diagram depends on the rate of increase in load, but the character of deformation itself is approximately preserved at any speed of loading. This makes it possible to use diagrams similar to static ones in dynamic calculations, but with adjustments to the main parameters, such as the strength limit for concrete or the time resistance limit for low-alloy steel. In addition, when structures vibrate, internal and external processes occur that cause variable resistance to movement.

4. The calculation algorithm

For the practical calculation of nonlinear stresses and deformations, we will use the method of successive approximations (the method of variable elasticity parameters) developed by I. A. Birger [8] for solving static problems, but we will adapt it for dynamic problems. To do this, at each time step, we assume that the values of the stiffness matrix at each time step are iteratively determined using the values of the vector obtained at the previous stage of the elastic solution.

The algorithm of the modified method of variable elasticity parameters is as follows:

1. Using the method of Central differences, we determine the movements of mechanical system nodes at moments of time \( U_{t+\Delta t} \):

\[ U_{t+\Delta t} = \left( \frac{M}{(\Delta t)^2} + \frac{B}{2 \cdot \Delta t} \right)^{-1} \left( P_t - M \cdot \frac{U_{t-\Delta t} - 2 \cdot U_t + U_{t+\Delta t}}{(\Delta t)^2} + B \cdot \frac{U_{t+\Delta t} - U_{t-\Delta t}}{2 \cdot \Delta t} - K(U_{t+\Delta t}) \cdot U_t \right) \]

(13)
In the first step of the iteration in the parameters of elasticity use elastic solution with the initial modulus $E$.

2. The values of the displacements defined by the strain components $\varepsilon_{ii}$:

$$ w'_i = \frac{-3 \cdot u_{i3} - 2 \cdot u_{i4} \cdot l_i + 3 \cdot u_{i9} - 3 \cdot u_{i11} \cdot l_i x + 3 \cdot 2 \cdot u_{i3} + u_{i5} \cdot l_i - 2 \cdot u_{i9} + u_{i11} \cdot l_i x}{(l_i)^2}, $$

$$ v'_i = 2 \cdot \left( \frac{-3 \cdot u_{i2} - 2 \cdot u_{i6} \cdot l_i + 3 \cdot u_{i8} - 3 \cdot u_{i12} \cdot l_i x + 3 \cdot 2 \cdot u_{i2} + u_{i6} \cdot l_i - 2 \cdot u_{i8} + u_{i12} \cdot l_i x}{(l_i)^2} \right), $$

$$ w''_i = u_{i5} + 2 \cdot \frac{-3 \cdot u_{i3} - 2 \cdot u_{i4} \cdot l_i + 3 \cdot u_{i9} - 3 \cdot u_{i11} \cdot l_i x + 3 \cdot 2 \cdot u_{i3} + u_{i5} \cdot l_i - 2 \cdot u_{i9} + u_{i11} \cdot l_i x^2}{(l_i)^3}, $$

$$ v''_i = u_{i6} + 2 \cdot \frac{-3 \cdot u_{i2} - 2 \cdot u_{i6} \cdot l_i + 3 \cdot u_{i8} - 3 \cdot u_{i12} \cdot l_i x + 3 \cdot 2 \cdot u_{i2} + u_{i6} \cdot l_i - 2 \cdot u_{i8} + u_{i12} \cdot l_i x^2}{(l_i)^3}, $$

$$ \varepsilon_{x,j} = w'_j - y' \cdot v''_j + z' \cdot w''_j + \frac{1}{2} \cdot (v'_j)^2 + \frac{1}{2} \cdot (w'_j)^2. $$

It is advisable to determine the deformations at the most remote points of the section. For example, for a rectangular section (see figure 1), $z = \pm \frac{b}{2}$, $y = \pm \frac{h}{2}$.

3. Determine the voltage:

$$ \sigma_{x,j} = E \cdot \varepsilon_{x,j} - E_3 \cdot \left( \varepsilon_{x,j} \right)^3. $$

4. Determine the secant elastic modulus for structural elements:

$$ E_i = \frac{\sigma_{x,j}}{\varepsilon_{x,j}}. $$

5. Performing verification:

$$ |E_i - E_{i-1}| \leq \delta, $$

where $\delta$ is the specified error rate.

When this ratio is fulfilled, we finish the calculation and proceed to a new step in time. Otherwise, go to point 1.

5. The results of dynamic calculations

Dynamic calculation was performed on the example of locators with radii of 4.0-4.5 m and lifting boom height of 0.8 m (figure 2). A load was applied to the locator nodes, which changed according to the law:

$$ P_x = 10 M_i \cdot \sin \left( 20 \cdot t \right); $$

$$ P_y = 10 M_i \cdot \sin \left( 20 \cdot t \right) - M_i \cdot g; $$

$$ P_z = 10 M_i \cdot \sin \left( 20 \cdot t \right). $$

Based on the calculation results, element 22 received the largest relative deformations (marked in red in Figure 2A). Figure 3 shows the stress-strain diagram for element 22 under this load (vertical axis-stress, PA; horizontal axis-relative strain). Figures 4-6 show graphs of linear movements of the node on the outer ring along the global coordinate axes (vertical axis-movements, m; horizontal axis-number of steps over time, step value 0.000125 s). Red graph of movements-without taking into account non-linearity. The blue displacement graph takes into account non-linearity.
The results of the calculations are summarized in table 1, where $x_{\text{max}}$, $y_{\text{max}}$, $z_{\text{max}}$ - maximum positive displacement along the axes $x$, $y$, $z$, respectively; $x_{\text{min}}$, $y_{\text{min}}$, $z_{\text{min}}$ – the maximum negative displacement along the axes $x$, $y$, $z$, respectively. The frequency of free vibrations of the system was determined at the time when the movement along the $x$-axis reached $x_{\text{max}}$.

**Figure 2.** a) design diagram of the locator; b) Cross-section of all elements
Figure 3. Stress-strain diagram for element 22

Figure 4. Moving along the x axis
Figure 5. Moving along the $y$ axis

Figure 6. Moving along the $z$ axis
Table 1. Comparison of results of linear and nonlinear calculations

| Radius of the locator dome, m | Linear calculation | Calculation taking into account non-linearity |
|------------------------------|-------------------|---------------------------------------------|
|                              | \(x_{\text{max}}, \text{m}\) | \(x_{\text{max}}, \text{m}\) |
|                              | \(-0.037713\)     | \(-0.037829\) |
|                              | \(y_{\text{max}}, \text{m}\) | \(y_{\text{max}}, \text{m}\) |
|                              | \(-0.019468\)     | \(-0.019516\) |
|                              | \(z_{\text{max}}, \text{m}\) | \(z_{\text{max}}, \text{m}\) |
|                              | \(-0.015311\)     | \(-0.015041\) |
| The period of oscillation, s | 0.272875           | 0.278625 |
| Natural oscillation frequency (first form), Hz | 25.885             | 24.096 |
| Percentage of increase in the amplitude of movements along the x axis, % | 13.55              | 29.02 |
| Percentage of increase in the amplitude of movements along the y axis, % | 14.44              | 34.63 |
| Percentage of increase in the amplitude of movements along the z axis, % | 14.75              | 38.30 |

6. Conclusions

The calculation results (table 1) show an increase in the amplitude of movements along the x, y and z axes to 15; 44 and 24%, respectively. The graphs of movements up to 440 steps coincide, i.e. both systems operate within the elastic limits. After the 440 steps, the stresses of some elements take values close to the limit, so there is a significant deviation of the graphs of nonlinear calculation in comparison with the linear calculation.

In the calculations, at a radius of 4 m, 6 elements are observed in which the voltage reaches the limit (marked in red and green in figure 2a), and at a radius of 4.5 m - 27 elements (all radial elements). Since the stiffness matrix of the entire system changes at each time step in accordance with the algorithm proposed above, the natural oscillation frequencies also change (decrease). As a result, the graphs (figures 4-6) the corresponding amplitude values of displacements in linear and nonlinear calculations are achieved at different times, and the period of oscillations in the calculation with an account of non-linearity is greater than in the linear one (table 1). In other words, if large deformations occur during vibrations, the effect of beats may not occur at the frequency that was expected in the linear calculation.

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