CATEGORIES ENRICHED OVER OPLAX MONOIDAL CATEGORIES

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Abstract

We define a notion of category enriched over an oplax monoidal category $V$, extending the usual definition of category enriched over a monoidal category. Even though oplax monoidal structures involve infinitely many ‘tensor product’ functors $V^n \to V$, the definition of categories enriched over $V$ only requires the lower arity maps ($n \leq 3$), similarly to the monoidal case.

The focal point of the enrichment theory shifts, in the oplax case, from the notion of $V$-category (classically given by collections of objects and hom-objects together with composition and unit maps) to the one of categories enriched over $V$ (genuine categories equipped with additional structures).

One of the merits of the notion of categories enriched over $V$ is that it becomes straightforward to define both enriched functors and enriched natural transformations. We show moreover that the resulting 2-category $\text{Cat}_V$ can be put in correspondence (via the theory of distributors) with the 2-category of modules over $V$.

We give an example of such an enriched category in the framework of operads: every cocomplete symmetric monoidal category $C$ is enriched over the category of sequences in $C$ endowed with an oplax monoidal structure stemming from the usual operadic composition product, whose monoids are still the (planar) operads.

As an application of the study of the 2-functor $V \mapsto \text{Cat}_V$, we show that when $V$ is also endowed with a compatible lax monoidal structure—thus forming a lax-oplax duoidal category—the 2-category $\text{Cat}_V$ inherits a lax 2-monoidal structure, thereby generalising the corresponding result when the enrichment base is a braided monoidal category. We illustrate this result by discussing in details the lax-oplax structure on the category of $(R^e,R^e)$-bimodules, whose bimonoids are the bialgebroids.

We conclude by commenting on the relations between the enrichment theory over oplax monoidal categories and other enrichment theories (monoidal, multicategories, skew and lax).

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In the 1960s arose the first formalisations of the notion of enriched categories [1, 2], where the set of arrows between two objects in a category could bear additional structure. Given two objects \(x, y\), one would associate a mapping object \([x, y]\) of any other external ‘base category’ \(V\), provided that this category is endowed with a tensor structure, necessary to define the composition morphisms

\[ [y, z] \otimes [x, y] \rightarrow [x, z] \]

and composition units \(1 \otimes [x, x] \rightarrow [x, x]\), which plays the rôle of the usual product structure on the category of sets. In addition, the regular set of ‘morphisms’ between two objects \(x\) and \(y\) could be recovered as the set of maps \(1 \otimes [x, y] \rightarrow [x, y]\) in \(V\).

The notion of enriched category then got generalised through the years, allowing enrichments over multicategories, double categories etc. This culminated at the end of the century with the definition by Leinster of ‘the most

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The work of Thomas Basile was supported by the National Research Foundation (Korea) through the grants NRF-2018H1D3A1A02074698, NRF-2019K1A3A1A21031365 and NRF-2019R1F1A1044065, and by the Fonds de la Recherche Scientifique — FNRS under Grant No. F.4544.21 ("HigherSpinGraWave"). The work of Damien Lejay was supported by IBS-R003-D1. The work of Kevin Morand was supported by Brain Pool Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science and ICT 2018H1D3A1A01030137 and by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education NRF-2020R1A6A1A03047877.
general structure’ one could use to enrich a category: fc-multicategories [3], which are ‘very general kinds of two-dimensional structures, encompassing bicategories, monoidal categories, double categories and ordinary multicategories’.

Here we shall go in the opposite direction and develop an enrichment theory which is only slightly more general than the classical one: enrichment over an oplax monoidal category. Being only slightly more general, it has the benefit that the theory is mostly similar to the traditional case.

In a tensor category, the symbol $x \otimes y \otimes z$ is formally undefined but tacitly assumed to mean $(x \otimes y) \otimes z$ (the other choice being possible as well), in which case one has an isomorphism $x \otimes y \otimes z \cong x \otimes (y \otimes z)$ which is simply the associator. In an oplax monoidal category, the symbol $x \otimes y \otimes z$ is supplied together with two decomposition maps

$$(x \otimes y) \otimes z \leftarrow x \otimes y \otimes z \rightarrow x \otimes (y \otimes z)$$

which are not assumed to be invertible. These non-invertible maps are all that is required to formulate the associativity axiom of enriched categories as a symmetric hexagon

$$[y, z] \otimes [x, y] \otimes [w, x]$$

$$\rightarrow$$

$$([y, z] \otimes [x, y]) \otimes [w, x] \quad [y, z] \otimes ([x, y] \otimes [w, x])$$

$$\rightarrow$$

$$[x, z] \otimes [w, x] \quad [y, z] \otimes [w, y]$$

$$\rightarrow$$

$$[w, z]$$

thus replacing the usual—asymmetric—pentagon.

Even though the definition of an oplax monoidal structure involves a 4-ary symbol $w \otimes x \otimes y \otimes z$, a 5-ary symbol et cetera up to infinity, we shall explain why the oplax nature of the structure of $\mathcal{V}$ makes the use of the $n$-ary symbols redundant for $n > 3$ when it comes to enrichment [3.14]. Thanks to this, the definition of a category enriched over an oplax monoidal category remains very akin to the usual case.

The main departure from the usual theory concerns the necessary distinction between the concept of $\mathcal{V}$-category and the more relevant concept of ‘category enriched over $\mathcal{V}$’. In the second case, one starts with a genuine category $\mathcal{C}$ which one then endows with a bifunctor

$$\mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{\{\cdot, \cdot\}} \mathcal{V}$$

thus yielding two distinct notions of morphisms: the usual morphisms of $\mathcal{C}$ (the strong morphisms) versus the maps of the form $1_{\otimes} \rightarrow [x, y]$ (the weak morphisms). In this regard, a category enriched over $\mathcal{V}$ can really be thought of as ‘a category with an extra enrichment structure’ thus making the definition of the 2-category $\mathcal{C}_{\mathcal{V}}$ of categories enriched over $\mathcal{V}$ straightforward to articulate.

Contradistinctly, working with $\mathcal{V}$-categories—created by assigning objects $[x, y] \in \mathcal{V}$ to each pair of objects $x, y$ etc—one is quickly led to the problem that
the assignment $(x,y) \mapsto [x,y]$ is not generically functorial and that as a result, there is no obvious notion of $\mathcal{V}$-natural transformation between $\mathcal{V}$-functors that one could use to build a 2-category $\mathcal{V}$-$\text{cat}$.

This chasm between $\mathcal{V}$-$\text{cat}$ and $\text{Cat}_\mathcal{V}$ was already found by Campbell in the skew context [4]. It is not of a mere technical nature but is also justified from a theoretical standpoint. Whenever $\mathcal{V}$ is endowed with a monoidal structure, $\mathcal{V}$-$\text{cat}$ can be put in correspondence—via the use of distributors—with the 2-category of $\mathcal{V}$-modules (also called $\mathcal{V}$-actegories) [5], where the denomination ‘module’ here refers to the fact that monoidal categories correspond to pseudo-monoids in the 2-category $\text{Cat}$. If $\mathcal{V}$ is instead endowed with an oplax monoidal structure, it can then be seen as an oplax monoid in $\text{Cat}$ [6] making it natural to consider its 2-category of modules. Making use of distributors again to build a correspondence with $\mathcal{V}$-modules on one side, what one obtains on the other side is precisely the 2-category $\text{Cat}_\mathcal{V}$ rather than $\mathcal{V}$-$\text{cat}$.

As an application of the general formalism, the present paper provides two examples of this enrichment theory which were motivational to the authors. The first example occurs within the theory of operads in a symmetric monoidal category $\mathcal{C}$ [§ 3.6]. When the monoidal structure on $\mathcal{C}$ is closed, a theorem by Kelly [7], central to the theory of operads, asserts that (planar) operads in $\mathcal{C}$ can be realised as the monoids in the category of sequences in $\mathcal{C}$ endowed with an operadic-composition monoidal structure. However, when the monoidal structure on $\mathcal{C}$ is not closed, the operadic-composition no longer necessarily defines a monoidal structure. Rather, the operadic-composition induces an oplax monoidal structure on $\mathcal{C}$, as shown by Ching [8], whose monoids are still the (planar) operads. The base category $\mathcal{C}$ then becomes enriched over the oplax monoidal category of sequences via a typical formula from the theory of operads. This setup is at the hearth of the theory since an algebra over an operad in $\mathcal{C}$ simply becomes a representation of a monoid in the enriching oplax monoidal category.

The second example deals with the formalisation of the theory of (non-commutative) bialgebroids [§ 5]. Given a ring $R$, let $R^e$ denote the enveloping ring $R \otimes R^{\text{op}}$. Following Takeuchi [9], a bialgebroid can be defined as a $(R^e, R^e)$-bimodule, denoted $A$, together with structure maps among which two structure maps

$$A \otimes_{R^e} A \longrightarrow A \quad \text{and} \quad A \longrightarrow A \times_{R^e} A \subset A \otimes_{R^e} A$$

enjoying compatibility conditions resembling the ones of a bialgebra. On the one hand the category $(R^e, R^e)$-$\text{bimod}$ of $(R^e, R^e)$-bimodules is endowed with a genuine tensor structure given by $\otimes_{R^e}$, so that a bialgebroid is in particular a monoid in this monoidal category. On the other hand, the restricted tensor product $\times_{R^e} \subset \otimes_{R^e}$ due to Sweedler [10] and Takeuchi [9] is not associative. We shall show that it endows $(R^e, R^e)$-$\text{bimod}$ with a lax monoidal structure, of which only the first stages were described by Takeuchi. In addition, we shall show that the two structures $\otimes_{R^e}$ and $\times_{R^e}$ possess natural compatibility structures endowing $(R^e, R^e)$-$\text{bimod}$ with a lax-oplax duoidal structure. This general structure, involving both a lax monoidal structure and an oplax monoidal structure, constitutes a natural generalisation of the notion of duoidal structure in which the notion of bimonoid still makes sense [11].

We shall then be able to say that:

‘Bialgebroids are the bimonoids in the lax-oplax duoidal category of $(R^e, R^e)$-bimodules.’
Lastly, the study of the 2-functor $\mathcal{V} \mapsto \text{Cat}_\mathcal{V}$ mapping each oplax monoidal category $\mathcal{V}$ to its corresponding 2-category of categories enriched over $\mathcal{V}$ [§ 4] will allow us to claim that, for every lax-oplax duoidal category $\mathcal{V}$, its associated 2-category $\text{Cat}_\mathcal{V}$ is canonically endowed with a lax monoidal structure, thus extending a well-known result about the 2-category of categories enriched over a braided monoidal category. Applied to $(R^e, R^e)$-bimod, the 2-category of categories enriched over $(R^e, R^e)$-bimod becomes endowed with a lax monoidal structure stemming from the restricted tensor product $\times_{R^e}$. A natural example of such a category is the category of $(R, R)$-bimodules. More generally, the category of modules over a bialgebroid is canonically enriched over the lax-oplax duoidal category of $(R^e, R^e)$-bimodules, which the authors expect should lead to new developments in the theory of bialgebroids.

NOTATIONS FOR INTEGRAL CALCULUS

INTEGRALS Following Yoneda’s original notation [12], we shall denote

$$\int_b F(b, b)$$

the integral of a functor $F: \mathcal{B}^{\text{op}} \times \mathcal{B} \to \mathcal{C}$. This convention is the opposite of the Australian convention where subscripts usually denote cointegrals (‘ends’) instead of integrals (‘coends’). We shall however denote the cointegral of $F$ using the superscript

$$\int^b F(b, b)$$

which differs from both Yoneda’s and the Australian school’s conventions, but is instead standard in the literature about bialgebroids [9, 10, 13] which we shall discuss in a dedicated section [§ 5].

SEQUENCE OF OBJECTS We shall denote a sequence of $n + 1$ objects in a category by

$$x := (x_0, \ldots, x_n)$$

and

$$x \oplus y = (x_0, \ldots, x_m, y_0, \ldots, y_n)$$

for the union of two sequences $x$ and $y$. We shall also write

$$x = (x_a, x_b, x_c)$$

whenever a sequence can be split into three subsequences $x_a, x_b$ and $x_c$, and

$$x \otimes y := (x_0 \otimes y_0, \ldots, x_n \otimes y_n)$$

the sequence of term-by-term tensor products if the category is monoidal.

HOM-SETS For any category $\mathcal{C}$, we shall denote its hom-sets vertically as

$$\mathcal{C}(^x_y) := \text{Hom}_\mathcal{C}(x, y)$$

and likewise, denote the values of a (set-valued coloured) operad $\mathcal{P}$ by $\mathcal{P}(^x_y)$. The composition of two arrows then becomes a map written $\mathcal{C}(^y_z) \times \mathcal{C}(^x_y) \to \mathcal{C}(^x_z)$ for example.
1 OPLAX MONOIDAL CATEGORIES

We shall start by reviewing the definition and basic features of the 2-category \( \text{Oplax} \) of oplax monoidal categories, a notion that has yet to receive a full treatment in the literature. Along the way, we introduce several conventions of notation that will be abundantly used throughout the paper.

In order to maximise the readability of a subject which necessarily involves an infinite amount of structural maps, we have chosen to: avoid writing indices and variables as much as possible; start all indexations from \( n = -1 \), so as to obtain an easy \( p + q \) additivity rule on the structural maps; change the set of notations depending on the case at hand. Other presentation and notation choices can be found in Day & Street [6], Leinster [14], Batanin & Weber [15] or Böhm & Vercruysse [11].

1.1 Definition and notations

Definition 1.1 (Oplax monoidal category). An oplax monoidal category \( \mathcal{V}^{\omega} \) is a category \( \mathcal{V} \) equipped with the following data:

- functorial structure maps
  \[
  \mathcal{V}^{n+1} \xrightarrow{\omega^n} \mathcal{V}
  \]
  for every integer \( n \geq -1 \);

- natural transformations
  \[
  \omega^{p+q} \xrightarrow{\alpha^{p,q}_{i,j}} \omega^p \circ (\text{Id}, \ldots, \text{Id}, \omega^q, \text{Id}, \ldots, \text{Id})
  \]
  for every \( p \geq 0 \), every \( q \geq -1 \) and every \( i \geq 0 \), \( j \geq 0 \) with \( i + j = p \), called associators, which we shall abbreviate as
  \[
  \omega^{p+q} \xrightarrow{\omega^{p+q}(\cdot, \omega^q, \cdot)}
  \]

- a counit natural transformation
  \[
  \omega^0 \xrightarrow{i} \text{Id}
  \]

satisfying the following axioms

**Counitality**

\[
\begin{align*}
\omega^p & \xrightarrow{\omega^0 \omega^p} \omega^p \\
\omega^p & \xrightarrow{\omega^p (-, \omega^0, \cdot)}
\end{align*}
\]

and commute whenever they make sense;

**Parallel decomposition**

\[
\begin{align*}
\omega^{p+q+r} & \xrightarrow{\omega^{p+q+r}(-, \omega^r, \cdot)} \omega^{p+q}(-, \omega^r, \cdot) \\
\omega^{p+q+r}(-, \omega^r, \cdot) & \xrightarrow{\omega^p (-, \omega^q, -, \omega^r, \cdot)} \omega^p(-, \omega^q, -, \omega^r, \cdot)
\end{align*}
\]

commutes whenever it makes sense;
A lax monoidal structure on a category $\mathcal{V}$ is the data of an oplax monoidal structure on the opposite category $\mathcal{V}^{op}$. A lax (or oplax) monoidal structure is called strong whenever the structural natural transformations are all invertible.

Lax monoidal categories were originally introduced by Day and Street as an example of lax monoids in $\text{Cat}$ [6]. Notice however that our presentation differs slightly from theirs, in that they require an associator of the form $\omega^n \to \omega^p(\omega^{n_0}, \ldots, \omega^{n_p})$ with $n_0 + \cdots + n_p = n - p$ and $0 \leq p < n$, i.e., a natural transformation which relates the $n$th structure map to the composition of $\omega^p$ with any collection of $\{\omega^{n_k}\}$ corresponding to a partition of $n - p$ in $p + 1$ parts. Consequently, the parallel and sequential decomposition conditions are replaced by the fact that two successive decompositions using a different sequence of partitions are identical. These two presentations can be seen as the counterpart of the presentations of operads in terms of full composition maps and partial composition maps, and hence turn out to be equivalent, as explained by Ching [8].

Notation 1.2. For $\mathcal{V}^\omega$ an oplax monoidal category, we shall let

$$1_{\omega} := \omega^{-1}, \quad x \otimes_{\omega} y := \omega^{1}(x, y) \quad \text{and} \quad x \otimes_{\omega} y \otimes_{\omega} z := \omega^{2}(x, y, z)$$

for any objects $x, y, z$ of $\mathcal{V}$. The same notations will be used whenever $\mathcal{V}^{\lambda}$ is a lax monoidal category.

To increase readability when the number of variables is unspecified, we shall write $\omega(x)$ to designate either $1_{\omega} = \omega^{-1}$ if $x$ is the empty sequence or $\omega^n(x_0, \ldots, x_n)$ if $x = (x_0, \ldots, x_n)$.

Remark 1.3. Any monoidal category $\mathcal{V}^\otimes$ can be turned into a strong oplax category by setting $\omega^{-1} = 1_{\otimes}, \omega^{0} = \text{Id}_V$ and

$$\omega^n(x_0, \ldots, x_n) := (((x_0 \otimes x_1) \otimes \ldots) \otimes x_n)$$

for $n \geq 0$. This is not a trivial result and we shall give more details about this in a dedicated section [§ 6].

Example 1.4. By definition, $\omega^0$ satisfies the axioms of a comonad. Conversely, if $W$ is a comonad on a category with finite coproducts, one can define an oplax monoidal structure by setting

$$\omega([x_i]_{i \in I}) := \coprod_{i \in I} W(x_i)$$

for $I$ finite. This construction is dual to the one given by Batanin and Weber where a lax monoidal structure was built from a monad [15, 2.3].
1.2 Lax functors

We give two definitions of lax functors, depending on whether the source category is endowed with a lax or an oplax structure.

**Definition 1.5** (Lax functor with oplax source). Let $F: U^\Psi \to V^\omega$ be a functor between two oplax monoidal categories. A lax monoidal structure on $F$ consists in natural transformations

$$\omega^n \circ (F \times \cdots \times F) \xrightarrow{l^n} F \circ \psi^n$$

which we shall abbreviate as

$$\omega^n F \to F \psi^n$$

for every $n \geq -1$. It is required that

**Counit**

$$\omega^0 F \quad \xrightarrow{\rho} \quad F \psi^0 \quad \xrightarrow{\rho} \quad F$$

commute;

**Decomposition**

$$\omega^{p+q} F \quad \xrightarrow{l^{p+q}} \quad \omega^p (-, \omega^q, -)F \xrightarrow{\rho} \omega^p (-, \psi^q, -) \xrightarrow{l^p (-, \psi^q, -)} F \psi^p (-, \psi^q, -)$$

commute whenever it makes sense.

**Definition 1.6** (Lax functor with lax source). Let $F: U^\lambda \to V^\omega$ be a functor between a lax monoidal category and an oplax monoidal category. A lax monoidal structure on $F$ is the data of natural transformations

$$\omega^n \circ (F \times \cdots \times F) \xrightarrow{l^n} F \circ \lambda^n$$

which we shall abbreviate as

$$\omega^n F \to F \lambda^n$$

for every $n \geq -1$. It is required that

**Unitality**

$$\omega^0 F \quad \xrightarrow{\rho} \quad F \lambda^0$$

commute;
ADDITIVITY

\[
\begin{array}{c}
\omega^{p+q} F \\
\omega^p (-, \omega^q, -) F \\
\omega^p (-, \lambda^q, -) \\
F \lambda^p (-, \lambda^q, -) \\
\end{array}
\]

commute whenever it makes sense.

The additivity rule just described makes it easy to guess that the higher maps \( l^n \) for \( n > 1 \) as well as \( l^0 \) can be reconstructed from \( l^1 \) and \( l^{-1} \). The proof of this result proceeds via careful inductions.

**Proposition 1.7** (Reconstruction of lax monoidal functors). Let \( F : \mathcal{U} \to \mathcal{V}^\omega \) be a functor between a lax monoidal category and an oplax monoidal category. Let

\[
F(-) \otimes_\omega F(-) \xrightarrow{l^1} F(- \otimes \lambda -)
\]

be a natural transformation and let

\[
1_\omega \xrightarrow{l^{-1}} F(1_\lambda)
\]

be a morphism. Moreover assume that

ASSOCIATIVITY the associativity diagram

\[
\begin{array}{c}
F(-) \otimes_\omega F(-) \otimes_\omega F(-) \\
(F(-) \otimes_\omega F(-)) \otimes_\omega F(-) \\
F(- \otimes_\lambda -) \otimes_\omega F(-) \\
F(-) \otimes_\omega (F(-) \otimes_\omega F(-)) \\
F(- \otimes_\lambda - \otimes_\lambda -) \\
F(- \otimes_\lambda - \otimes_\lambda -) \\
F(- \otimes_\lambda - \otimes_\lambda -)
\end{array}
\]

commutes;
then the pair \((l^{-1}, l^1)\) can be extended, in a unique way, into a sequence of natural transformations

\[ \omega^n F \xrightarrow{l^n} F \lambda^n, \quad n \geq -1 \]

endowing \( F : U^\lambda \to V^\omega \) with a structure of lax monoidal functor.

**Proof.** Let us index the additivity diagrams in the definition of a lax functor by \( A(p, q, i) \), where \( 0 \leq i \leq p \) is the position of \( \omega^q \) inserted in \( \omega^p \).

Define \( l^0 \) as the composition

\[ \omega^0 F \to F \to F \lambda^0 \]

so that the unitality diagram for lax functors commute by definition, and define \( l^{q+1} \) by induction: for any \( q \geq 1 \), let \( l^{q+1} \) be such that \( A(1, q, 0) \) commutes.

Furthermore, let \( f(p, q) \) denote the hypothesis that \( A(p, q, i) \) hold for every position \( 0 \leq i \leq p \). We shall show that \( f(p, q) \) is true for every \( p \geq 0 \) and every \( q \geq -1 \).

Let us start by the case of \( f(0, q) \) with \( q \geq -1 \). The following two squares
commute. The first by functoriality of \( \omega^0 \); the commutativity of the second one follows from the commutativity of the first one, in addition to the unitality axiom for \( \omega \). As a consequence, the top part of the diagram commutes and the bottom part commutes by unitality of \( \lambda \).

We shall now address \( H(1,q) \) for every \( q \geq -1 \). The hypothesis \( H(1,-1) \) is satisfied thanks to the two unitality assumptions. For \( H(1,0) \), the two diagrams

\[
\begin{align*}
F(-) \otimes \omega F(-) & \rightarrow F\lambda^0(-) \otimes \omega F(-) & F(-) \otimes \omega F(-) & \rightarrow F\lambda^0(-) \otimes \omega F(-) \\
\downarrow_{\mu} & \downarrow_{\mu}\left(\lambda^0(-)\right) & \downarrow_{\mu} & \downarrow_{\mu}\left(\lambda^0(-)\right) \\
F(- \otimes \lambda -) & \rightarrow F\left(\lambda^0(-) \otimes \lambda -\right) & F(- \otimes \lambda -) & \leftarrow F\left(\lambda^0(-) \otimes \lambda -\right)
\end{align*}
\]

commute: the first one by functoriality of \(- \otimes \lambda -\); the second one because of the commutativity of the first one coupled with the unitality of \( \lambda \). As a consequence, the bottom part of the diagram

\[
\begin{align*}
F(-) \otimes \omega F(-) & \rightarrow F\lambda^0(-) \otimes \omega F(-) & F(-) \otimes \omega F(-) & \rightarrow F\lambda^0(-) \otimes \omega F(-) \\
\downarrow_{\mu} & \downarrow_{\mu}\left(\lambda^0(-)\right) & \downarrow_{\mu} & \downarrow_{\mu}\left(\lambda^0(-)\right) \\
F(- \otimes \lambda -) & \rightarrow F\left(\lambda^0(-) \otimes \lambda -\right) & F(- \otimes \lambda -) & \leftarrow F\left(\lambda^0(-) \otimes \lambda -\right)
\end{align*}
\]

commutes and the top part commutes by unitality of \( \omega \). This shows that half of \( H(1,0) \) holds, the other half can be obtained symmetrically.
The hypothesis $H(1,1)$ is true by the associativity axiom, we can now show that $H(1,q)$ is true by induction on $q \geq 1$. The diagram

commutes. Indeed, the top two squares commute by sequential decomposition; the middle two squares by functoriality; the octagon commutes by the associativity axiom and the bottom two squares commute by sequential composition. Using $H(1,q)$, one can see that the left full composite map equals $l^{q+2}$ and thus, the commutativity of this diagram gives us the commutativity of $A(1,q+1,1)$, since $A(1,q+1,0)$ commutes by definition, we have shown $H(1,q) \implies H(1,q+1)$, and hence $H(1,q)$ holds for any integer $q \geq -1$. 
Finally, assume that $h(p, q)$ holds. Then the diagram

\[
\begin{array}{ccc}
\omega^{p+q+1}F & \rightarrow & \omega^{p+1}(-, \omega^q, -)F \\
F \otimes_\omega \omega^{p+q}F & \rightarrow & \omega^{p+1}F(-, \lambda^q, -) \\
F \otimes_\omega (\omega^p (-, \omega^q, -))F & \rightarrow & \omega^{p+1}F(-, \lambda^q, -) \\
- \otimes_\omega \omega^{p+q} & \rightarrow & F \otimes_\omega (\omega^p F(-, \lambda^q, -)) \\
\end{array}
\]

commutes whenever it makes sense. The top square by sequential decomposition; the left pentagon by $h(p, q)$; the right pentagon by definition of $l^{p+1}$; the left and the right squares by functoriality, and the bottom square by sequential composition. Using $h(1, p+q)$, one can see that the left total composite map equals $l^{p+q+1}$. This shows that $h(p, q)$ implies the commutativity of $A(p+1, q, i)$ for every $0 < i \leq p + 1$. The mirror diagram involving $l^{p+q} \otimes_\omega -$ instead of $- \otimes_\omega l^{p+q}$ shows that $A(p+1, q, i)$ commutes for every $0 \leq i < p + 1$. Thus we have shown that $h(p, q) \Rightarrow h(p + 1, q)$ which concludes the proof of the proposition.

**Remark 1.8.** Note that if both categories in the previous proposition are strong monoidal, then we also recover the usual notion of a lax monoidal functor between (strong) monoidal categories.

Furthermore, out of the four possible combinations of lax functors between (op)lax categories, lax functors with lax source and oplax target are privileged as they are the only ones for which the additivity diagram can be used to define the higher maps $l^{p+q}$ in terms of lower ones, as necessary for the reconstruction result to hold.
1.3 Oplax functors

The previous section provided two (non-exhaustive) definitions of lax functors. Using opposite categories, we can readily define two corresponding types of oplax functors: oplax functors with lax source and lax target as well as oplax functors with oplax source and lax target. Since none of these involve an oplax functor with oplax target, we add to the above taxonomy the following definition.

**Definition 1.9.** Let $\mathcal{U}_\psi$ and $\mathcal{V}_\omega$ be two oplax monoidal categories. An oplax structure on a functor $F: \mathcal{U} \to \mathcal{V}$ consists in natural transformations

$$F \circ \psi^n \xrightarrow{\omega^n \circ (F \times \cdots \times F)} \text{ for every } n \geq -1.$$

which we shall abbreviate as

$$F\psi^n \longrightarrow \omega^n F$$

for every $n \geq -1$. It is required that

**Counit**

$$F\psi^0 \longrightarrow \omega^0 F$$

commute;

**Decomposition**

$$F\psi^{p+q} \longrightarrow F\psi^p (-, \psi^q, -) \xrightarrow{\omega^q (-, \psi^q, -)} \omega^q F (-, \psi^q, -) \xrightarrow{\omega^p (-, -)} \omega^{p+q} (-, -) \longrightarrow F$$

commute whenever it makes sense.

**Remark 1.10.** Note that oplax monoidal structures on functors, from an oplax monoidal category to a lax monoidal one, enjoy a reconstruction theorem similar to that of lax monoidal functors with lax monoidal source and oplax monoidal target [1.7]. This is simply a consequence of the fact that, for an oplax monoidal structure on a functor $F: \mathcal{U}_\omega \to \mathcal{V}^\lambda$, with $\mathcal{U}_\omega$ oplax monoidal and $\mathcal{V}^\lambda$ lax monoidal, the decomposition condition in the previous definition can be used to define the natural transformations $l^n: F \omega^n \to \lambda^n F$ for $n \geq 2$ in terms of the pair $(l^1, l^{-1})$, while the counit condition define $l^0$ as the composition of the unit and counit of $\lambda$ and $\omega$. 
1.4 Monoidal natural transformations

**Definition 1.11** (Monoidal natural transformation). Let \( \mathcal{U}^\Psi \) and \( \mathcal{V}^\omega \) be two oplax monoidal categories. Let \( F, G: \mathcal{U} \to \mathcal{V} \) be two lax functors with respective lax monoidal structure \( \{k^n\}_{n \geq -1} \) and \( \{l^n\}_{n \geq -1} \). A natural transformation \( \alpha: F \Rightarrow G \) is said to be monoidal if the diagrams

\[
\begin{array}{c}
\omega^n F \xrightarrow{k^n} F\psi^n \\
\downarrow \alpha^n \downarrow \alpha^n \\
\omega^n G \xrightarrow{l^n} G\psi^n
\end{array}
\]

commute for any \( n \geq -1 \). One can define monoidal transformations between oplax monoidal functors in a similar way.

For monoidal natural transformations between lax monoidal functors that can be truncated, i.e. whose source and target categories are lax and oplax monoidal respectively, the infinite collection of conditions in the previous definition is actually redundant, as illustrated by the following lemma.

**Lemma 1.12.** Let \( F, G: \mathcal{U}^\lambda \to \mathcal{V}^\omega \) be two lax monoidal functors from a lax monoidal category to an oplax monoidal one, with respective lax monoidal structures \( \{k^n\} \) and \( \{l^n\} \). A natural transformation \( \alpha: F \to G \) between such functors is monoidal if and only if the following diagrams

\[
\begin{array}{c}
(F \otimes \omega F) \xrightarrow{\alpha \otimes \omega \alpha} (G \otimes \omega G) \\
\downarrow k^i \downarrow l^i \\
F(- \otimes \lambda -) \xrightarrow{\alpha \otimes \lambda \alpha} G(- \otimes \lambda -) \quad \text{and} \quad 1_{\omega} \xrightarrow{1_{\omega}} 1_{\omega} \xrightarrow{1_{\lambda}} G(1_{\lambda})
\end{array}
\]

commute.

**Proof.** Let us denote by \( M(n) \), with \( n \geq -1 \), the diagrams encoding the conditions that \( \alpha \) is monoidal. The diagrams \( M(-1) \) and \( M(1) \) correspond to those required in the above lemma. The diagram \( M(0) \) can be written as

\[
\begin{array}{c}
\omega^0 F \xrightarrow{\omega^0(\alpha)} \omega^0 G \\
\downarrow k^0 \downarrow 1^0 \\
F \xrightarrow{\alpha} G \quad \text{and} \quad \downarrow \rho^0 \\
\alpha_{\lambda 0} \xrightarrow{\alpha(\lambda 0)} G \lambda 0
\end{array}
\]

and hence commutes as a consequence of the unitality conditions of the lax monoidal structures \( k \) and \( l \), and of the naturality of the co/unit of \( \omega \) and \( \lambda \).
Now assume that $M(k)$ commutes for all integers $k < n$. Then the diagram

\[
\begin{array}{ccc}
\omega^n F & \xrightarrow{\omega^n(A,\ldots,A)} & \omega^n G \\
\downarrow & & \downarrow \\
(-\otimes_\omega, \omega^{n-1}) F & \xrightarrow{\alpha\otimes_\omega \omega^{n-1}(A,\ldots,A)} & (-\otimes_\omega, \omega^{n-1}) G \\
\downarrow & & \downarrow \\
k^n & \xrightarrow{\omega, \omega^{n-1}} & \omega^n F \\
\downarrow & & \downarrow \\
\omega^n F & \xrightarrow{\alpha\otimes_\omega \omega^{n-1}(A,\ldots,A)} & \omega^n G \\
\downarrow & & \downarrow \\
k^n & \xrightarrow{\omega, \omega^{n-1}} & \omega^n F \\
\downarrow & & \downarrow \\
F, \lambda^n & \xrightarrow{\alpha\otimes_\lambda \lambda^{n-1}(A,\ldots,A)} & G, \lambda^n \\
\downarrow & & \downarrow \\
F & \xrightarrow{\alpha\otimes_\lambda \lambda^{n-1}(A,\ldots,A)} & G \\
\downarrow & & \downarrow \\
F & \xrightarrow{\alpha\otimes_\lambda \lambda^{n-1}(A,\ldots,A)} & G \\
\end{array}
\]

commutes: the top and bottom (exterior) squares by naturality of the associators of $\omega$ and $\lambda$, the left and right (exterior) squares by additivity of the lax monoidal structures $k$ and $l$ and the interior two squares by assumption. This proves that $M(n)$ also commutes and hence proves the lemma by recursion.

Remark 1.13. Note that the same result holds for monoidal natural transformations between two oplax monoidal functors, whose source and target categories are respectively oplax and lax monoidal. In other words, the monoidality condition on natural transformations between op/lax monoidal functors can be truncated whenever the op/lax monoidal structure of these functors can be.

Remark 1.14. When both $\lambda$ and $\omega$ are strong monoidal, the previous lemma reproduces the usual definition of a monoidal natural transformation between monoidal functors.

1.5 Monoids in oplax monoidal categories

Definition 1.15 (Category of monoids). A monoid in an oplax monoidal category $\mathcal{V}^{\omega}$ is a lax monoidal functor from the punctual category $\ast \to \mathcal{V}^{\omega}$, and a morphism between two monoids is a monoidal transformations between the corresponding functors. These data define a category, that will be denoted $\text{Mon}(\mathcal{V}^{\omega})$.

The category of comonoids in $\mathcal{V}^{\omega}$ is the category of oplax functors $\ast \to \mathcal{V}^{\omega}$ and natural transformations between them.

In more details, the functor $\ast \to \mathcal{V}^{\omega}$ singles out an object $A$ in the target category $\mathcal{V}$, while the lax monoidal structure on this functor consists in a collection of morphisms

\[
\omega^n(A,\ldots,A) \xrightarrow{m^n_A} A
\]
of \( V \), such that \( m_A^0 \) identifies with the counit \( \omega^0 \to \text{Id}_V \), and which obey the additivity conditions

\[
\omega^{p+q} A \\
\downarrow m_A^{p+q} \\
\omega^p A \\
\downarrow m_A^p \\
A
\]

for all integers \( p, q \geq -1 \). A morphism between two monoids \( (A, \{m^n_A\}) \) and \( (B, \{m^n_B\}) \) is a morphism \( f: A \to B \) such that

\[
\begin{array}{ccc}
\omega^n A & \xrightarrow{\omega^n(f, \ldots, f)} & \omega^n B \\
\downarrow m_A^n & & \downarrow m_B^n \\
A & \xrightarrow{f} & B
\end{array}
\]

commutes for all integers \( n \geq -1 \).

The punctual category is (trivially) strong monoidal, and hence we can apply the reconstruction theorem for lax monoidal functors [1.7] to obtain the following result [11, Th. 3.5].

**Corollary 1.16** (Truncation of monoids). A structure of monoid on an object \( A \) in an oplax monoidal category \( V^\omega \) can be uniquely reconstructed from a pair of morphisms

\[ \mu_A: A \otimes \omega A \to A, \quad \text{and} \quad \eta_A: 1 \otimes \omega A \to A, \]

such that the associativity and unitality diagrams

\[
\begin{array}{ccc}
(A \otimes \omega A) \otimes \omega A & \xrightarrow{\rho_A \otimes \omega \rho_A} & A \otimes \omega (A \otimes \omega A) \\
\downarrow \rho_A & & \downarrow \omega \rho_A \rho_A \\
A \otimes \omega A & \xrightarrow{\rho_A} & A \otimes \omega A
\end{array}
\]

\[
\begin{array}{ccc}
\omega^0 (A) & \xrightarrow{1 \otimes \omega \mu_A} & A \otimes \omega (1 \otimes \omega A) \\
\downarrow \omega \eta_A & & \downarrow \omega \eta_A \\
A \otimes \omega A & \xrightarrow{\rho_A} & A \otimes \omega A
\end{array}
\]

commute.

**Remark 1.17.** When \( V^\otimes \) is monoidal, one recovers the usual pentagonal axiom (on the left diagram) and the two triangular axioms (on the right diagram) of the usual definition of a monoid, independently of a choice of lift of the monoidal structure into a strong monoidal structure.

When \( V^\otimes \) is strictly normal, one recovers a theorem from Ching [8, 3.4].
Remark 1.18 (Comonoids in oplax monoidal categories). Contrarily to the case of monoids, a structure of comonoid cannot be truncated to lower arity maps, as oplax monoidal functors with an oplax monoidal category as their target are not subject to the reconstruction theorem discussed previously. This can also be seen from the above diagram: it is not possible to use the latter as a definition of the higher arity maps in terms of the lower arity ones.

Remark 1.19 (Comonoids in lax monoidal categories). One can also define monoids and comonoids in a lax monoidal category $V^\lambda$ as lax and oplax monoidal functors respectively, from the punctual category. In this case, the situation is reversed: comonoid structures can be truncated, while monoid structures cannot. This is simply due to the fact that lax monoidal structures on functors with lax monoidal target categories cannot be truncated.

Lemma 1.20 ($1_\omega$ is a canonical comonoid). Let $\omega^\mu$ be an oplax monoidal category. Let us define by induction a map

$$1_\omega \xrightarrow{\omega^n} \omega^n 1_\omega := \omega^n(1_\omega, \ldots, 1_\omega)$$

for every $n \geq -1$ by first letting $w^{-1}: 1_\omega \to 1_\omega$ be the identity of $1_\omega$. Then, assuming $w^n$ to be defined for $n \geq -1$, let $w^{n+1}$ be defined by the composition

$$1_\omega \xrightarrow{\omega^n} \omega^n 1_\omega \xrightarrow{a_i^{n+1,-1}1_\omega} \omega^{n+1} 1_\omega$$

so that:

- for every $n \geq 0$ and every $0 \leq i \leq n$, one has $w^{n+1} = (a_i^{n+1,-1}1_\omega) \circ w^n$;
- the sequence of maps $\{w^n\}_{n \geq -1}$ makes $1_\omega$ a comonoid.

Proof. The definition of the maps $w^n$ can be shown to be canonical by induction using the parallel decomposition axiom for oplax monoidal categories. The counit axiom for $1_\omega$ follows directly from the counit axiom for $\omega^\mu$. Let us index the decomposition diagrams for oplax functors by $D(p,q,i)$, where $0 \leq i \leq p$ is the insertion index for $\omega^q$ inside $\omega^p$. Let $f(p,q)$ denote the hypothesis that $D(p,q,i)$ commute for every $0 \leq i \leq p$.

By construction $f(p,-1)$ holds for every $p \geq 0$. Furthermore, the diagram

$$
\begin{array}{ccc}
1_\omega & \xrightarrow{\omega^p} & \omega^p 1_\omega \\
\downarrow w^{p+1} & & \downarrow \omega^p (-, \omega^q, -) \\
\omega^{p+q} 1_\omega & \xrightarrow{\omega^{p+q}(-, \omega^q, -)} & \omega^{p+q} (-, \omega^q+1, -) 1_\omega \\
\downarrow & & \downarrow \\
\omega^{p+q+1} 1_\omega & \xrightarrow{\omega^{p+q+1}(-, \omega^q+1, -)} & \omega^{p+q+1} (-, \omega^q+1, -) 1_\omega
\end{array}
$$

commutes by virtue of $f(p,q)$ for the top square and by sequential decomposition for the bottom square. In other terms, we showed $f(p,q) \implies f(p,q+1)$ for every $p \geq 0, q \geq -1$ hence $(1_\omega, \{w^n\}_{n \geq -1})$ is a comonoid.

Remark 1.21. Notice that the two parallel structure maps $\omega^0(1_\omega) \Rightarrow 1_\omega \otimes_\omega 1_\omega$ may not be equal, but the two induced maps $1_\omega \to 1_\omega \otimes_\omega 1_\omega$ are equal.
1.6 Normal oplax monoidal categories

Definition 1.22. An oplax monoidal category $\mathcal{V}^\omega$ will be called normal whenever the structure map

$$\omega^0(x) \rightarrow x$$

is invertible for every object $x \in \text{Ob}(\mathcal{V})$.

We shall say that $\omega$ is strictly normal whenever $\omega^0(x) = x$,

$$x = \omega^0(x) \rightarrow x$$

is the identity of $x$ for every $x \in \text{Ob}(\mathcal{V})$ and the decomposition natural transformations

$$\omega^p \rightarrow \omega^0 \omega^p \quad \text{and} \quad \omega^p \rightarrow \omega^p (-, \omega^0, -)$$

are also identity natural transformations.

Remark 1.23. The terminology ‘normal’ originates from Day and Street [6]. It is also used by Ching [8] to describe what we have called ‘strictly normal’ structures.

Notice in particular that the set of axioms defining strictly normal oplax monoidal categories is simpler: there is no need to specify $\omega^0$ and the counitality axioms can be removed.

Remark 1.24 (Strictification). The distinction between ‘normal’ and ‘strictly normal’ is almost irrelevant since one can always strictify a normal structure.

Every oplax monoidal structure $\omega$ on a category $\mathcal{V}$ has an obvious underlying strictly normal oplax monoidal structure $\bar{\omega}$ obtained by replacing $\omega^0$ with the identity functor and every decomposition natural transformation involving $\omega^0$ with an identity.

In addition, the identity functor of $\mathcal{V}$ has an obvious structure of oplax monoidal functor

$$\omega \Rightarrow \bar{\omega}$$

which becomes an isomorphism of oplax monoidal structures on $\mathcal{V}$ whenever $\omega$ is normal.

1.7 Comonad twist of an oplax monoidal category

We shall expand the example of oplax monoidal category given earlier using a comonad $W$ and the coproduct [1.4]. That example is a special case of a general construction where one can twist an oplax monoidal structure with a lax monoidal comonad, which in turn provides a plethora of examples of non-normal oplax monoidal structures.

Definition 1.25. Let $\mathcal{V}^\omega$ be an oplax monoidal category. A lax monoidal comonad on $\mathcal{V}^\omega$ is the data of a comonad $(W: \mathcal{V} \rightarrow \mathcal{V}, w: W \Rightarrow W^2, t: W \Rightarrow \text{Id})$ on $\mathcal{V}$ together with a lax monoidal structure $\{l^n\}_{n \geq 1}$ on $W: \mathcal{V} \rightarrow \mathcal{V}$ such that $w$ and $t$ are monoidal natural transformations.

Example 1.26. If $\mathcal{V}$ admits finite coproducts, then any comonad admits a canonical structure of lax monoidal comonad for the coproduct strong monoidal structure $\mathcal{V}^{\text{cop}}$. 

Proposition 1.27. Given a lax monoidal comonad \((W, w, t, \{l^n\}_{n \geq -1})\) on an oplax monoidal category \(V^\omega\), one can obtain a new oplax monoidal structure on \(V\) by setting
\[
\psi^n := \omega^n W
\]
for every \(n \geq -1\). The counit natural transformation \(\psi^0 \Rightarrow \text{Id}\) is given as the composite
\[
\omega^0 W \xrightarrow{\psi^0} W \xrightarrow{t} \text{Id}
\]
and the decompositions are given by
\[
\begin{align*}
\omega^{p+q} W & \quad \omega^p (-, \omega^q W, -) \quad \omega^p (-, \omega^q W, -) W \\
\omega^p (-, \omega^q W, -) W & \quad \omega^p (-, \omega^q W, -).
\end{align*}
\]

Proof. The two counitality axioms follow straightforwardly from the counitality axioms of the comonad, the counit axioms of \(\omega\), the fact that \(t\) is a monoidal natural transformation and the naturality of the counit natural transformation associated with \(\omega\).

Parallel decomposition comes from parallel decomposition for \(\omega\) and the fact that \(w\) is a monoidal natural transformation.

Sequential decomposition follows from sequential decomposition for \(\omega\), the coassociativity of \(W\) and the fact that \(w\) is a monoidal natural transformation.

\[\square\]

1.8 Oplax monoidal categories as multicategories

Recall that usually the words ‘multicategory’ and (set valued, coloured) ‘operads’ are synonymous. In the present context, it will be useful to introduce a slightly more general definition of a multicategory so as to be able to compare it with non-normal oplax monoidal categories.

Definition 1.28 (Multicategory). A multicategory is the data of a category \(M\), a coloured operad \(M_0\) and a morphism of coloured operads \(M \rightarrow M_0\) (where \(M\) is considered as a coloured operad with only operations of arity 1), which is the identity on objects.

Given two objects \(a, b \in \text{Ob}(M)\), the set \(M(\beta)\) will be called the set of strict morphisms between \(a\) and \(b\) while the set \(M_0(\beta)\) will be referred to as the set of weak morphisms between \(a\) and \(b\).

A multicategory will be called normal whenever \(M(\beta) = M_0(\beta)\) for all objects \(a, b \in \text{Ob}(M)\), in which case \(M\) is simply the underlying category of the operad \(M_0\).

One can form a 2-category \(\text{Multi}\) of multicategories in an obvious way where 1-morphisms are given by compatible pairs of operad morphisms \((F: M \rightarrow N, F_0: M_0 \rightarrow N_0)\) and 2-morphisms are given by compatible pairs of transformations \((\alpha: F \Rightarrow G, \alpha_0: F_0 \Rightarrow G_0)\).

Remark 1.29. The above definition of multicategories [1.28] corresponds to the definition of ‘lax promonoidal categories’ of Day and Street [6, 6.3].
induces a multicategory: given an oplax monoidal category \( \mathcal{V}^\omega \), the operadic structure is given by

\[
\mathcal{V}_{0}(\omega^0(a)) := \mathcal{V}(\omega^0(a))
\]

for every object \( b \in \text{Ob}(\mathcal{V}) \) and every finite sequence \( a \) of elements of \( \mathcal{V} \). The composition maps are given by

\[
\mathcal{V}((\omega^p(-,x),\omega^q(-,y)) \times \mathcal{V}(\omega^p(x,-)) \rightarrow \mathcal{V}(\omega^{p+q}(x,y))
\]

where the first arrow is simply the composition of morphisms in \( \mathcal{V} \) and the second one is defined using the associator of \( \mathcal{V}^\omega \). Given two objects \( a, b \in \text{Ob}(\mathcal{V}) \), the maps

\[
\mathcal{V}(\omega^p(a)) \rightarrow \mathcal{V}(\omega^q(b)) := \mathcal{V}(\omega^q(a))
\]

are obtained using the counit \( \omega^0(b) \rightarrow a \).

**Remark 1.30.** It is straightforward to check that the assignment above extends to a fully faithful 2-functor

\[
\text{Oplax}_{\text{lax}} \hookrightarrow \text{Multi}
\]

thereby generalising the normal case, described by Aguiar, Hiam and López Franco [16, 1.7].

Notice in particular that, an oplax monoidal category is normal if and only if the multicategory it generates is normal.

**Remark 1.31.** In the case where a multicategory is represented by an oplax monoidal category \( \mathcal{V}^\omega \), the map sending strong morphisms with source \( \omega(a) \) and target \( b \) to weak morphisms with source \( \omega(a) \) and target \( b \)

\[
\mathcal{V}(\omega^p(a)) \overset{\cong}{\rightarrow} \mathcal{V}(\omega^q(a))
\]

is a bijection.

## 2 LAX–OPLAX DUOIDAL CATEGORIES

We review a generalisation of the notion of duoidal categories to the oplax setting, originally introduced by Böhm and Vercruysse [11].

### 2.1 Definition and notations

**Notation 2.1.** We shall denote by \( \tau^{p,q} \) the permutation natural transformation stemming from the symmetric monoidal structure of \( \text{Cat} \) and whose components are given by

\[
\tau^{p,q}: (C_{0,0} \times \cdots \times C_{0,q}) \times \cdots \times (C_{p,0} \times \cdots \times C_{p,q}) \rightarrow (C_{0,0} \times \cdots \times C_{p,0}) \times \cdots \times (C_{0,q} \times \cdots \times C_{p,q})
\]

where \( C_{i,j} \) are categories for \( 0 \leq i \leq p \) and \( 0 \leq j \leq q \). In plain words, \( \tau^{p,q} \) is the functor that sends the product of \( p+1 \) products of \( q+1 \) categories, to the product of \( q+1 \) products of \( p+1 \) categories. One has

\[
\tau^{1,1}(x_0, x_1, x_2, x_3) = (x_0, x_2, x_1, x_3)
\]

for example.
**Definition 2.2** (Lax-oplax duoidal category). A lax-oplax duoidal structure on a category $D$ is the data of a lax monoidal structure $D^\lambda$ and an oplax monoidal structure $D^\omega$ together with natural transformations

$$
\omega^p \circ (\lambda^q \times \cdots \times \lambda^q) \xrightarrow{\chi^{p,q}} \lambda^q \circ (\omega^p \times \cdots \times \omega^p) \circ \tau^{p,q}
$$

for every $p, q \geq -1$, such that

- the transformations $\chi^{p,q}$ endow

  $$(D \times \cdots \times D)^\omega \xrightarrow{\lambda^q} D^\omega,$$  
  with a lax monoidal structure, for every $q \geq -1$;

- the transformations $\chi^{p,q} \tau^{p,q}$ endow

  $$(D \times \cdots \times D)^\lambda \xrightarrow{\omega^p} D^\lambda,$$  
  with an oplax monoidal structure for every $p \geq -1$.

**Remark 2.3.** For the case $(p, q) = (-1, -1)$, we get a morphism

$$1 \rightarrow 1\lambda$$

and for the case $(p, q) = (1, 1)$, we get morphisms

$$(a \otimes \lambda b) \otimes (c \otimes \lambda d) \rightarrow (a \otimes \omega c) \otimes (b \otimes \omega d)$$

for every tuple $(a, b, c, d)$ of objects of $D$. When both $\lambda$ and $\omega$ are strong, these correspond to the structure maps of a (usual) duoidal structure on $D$ [17].

**Remark 2.4.** For the case $p = -1$, we obtain structure maps

$$1 \omega \rightarrow \lambda^q(1 \omega, \ldots, 1 \omega)$$

turning $1 \omega$ into a $\lambda$-comonoid.

Symmetrically, for $q = -1$, we get structure maps

$$\omega^p(1 \lambda, \ldots, 1 \lambda) \rightarrow 1 \lambda$$

turning $1 \lambda$ into an $\omega$-monoid.

### 2.2 Bimonoids in lax-oplax duoidal categories

**Proposition 2.5** [11, Th. 4.4]. Let $D^{\lambda,\omega}$ be a lax-oplax duoidal category. Then $\lambda$ induces a lax monoidal structure on the category of monoids $\text{Mon}(D^\omega)$ with respect to the oplax monoidal structure $\omega$. Similarly, $\omega$ induces an oplax monoidal structure on the category of comonoids $\text{Comon}(D^\lambda)$ with respect to the lax monoidal structure $\lambda$.

We can now recall the definition of a bimonoid in a lax-oplax duoidal category.
**Definition 2.6** (Bimonoid). A bimonoid in a lax-oplax duoidal category $D^\lambda,\omega$ is a comonoid in the category of monoids of $D^\omega$, or equivalently, a monoid in the category of comonoids in $D^\lambda$.

Very concretely, a bimonoid in $D^\lambda,\omega$ consists in an object $A$ in $D$, together with morphisms

$$m^n_A: \omega^n(A,\ldots,A) \to A \quad \text{and} \quad w^n_A: A \to \lambda^n(A,\ldots,A)$$

for any integer $n \geq -1$, so that $(A, m_A)$ be a monoid in $D^\omega$ and $(A, w_A)$ be a comonoid in $D^\lambda$. On top of that, these morphisms have to be such that the diagram

$$\begin{array}{ccc}
\omega^p A & \xrightarrow{\omega^p \omega^q} & \omega^p \lambda^q A \\
\downarrow{m^p_A} & & \downarrow{\chi^{p,q}} \\
\lambda^q \omega^p A & & \\
\downarrow{\lambda^q m^p_A} & & \\
A & \xrightarrow{w^q_A} & \lambda^q A
\end{array}$$

commute for any integers $p, q \geq -1$. These diagrams encode the requirement that the maps $w_A$ are monoid morphisms, or equivalently that the maps $m_A$ are comonoid morphisms.

Thanks to the reconstruction theorem for monoids in an oplax monoidal category, the data needed to describe a bimonoid in a lax-oplax duoidal category is similar to the data needed to describe a bimonoid in a usual duoidal category.

**Proposition 2.7.** The data of a bimonoid $A$ in a lax-oplax duoidal category $D^\lambda,\omega$ is equivalent to the data of:

**PRODUCT AND UNIT** a product map and a unit map

$$\mu_A: A \otimes_{\omega} A \to A \quad \text{and} \quad \eta_A: 1_{\omega} \to A,$$

such that the associativity and unitality diagrams

$$\begin{array}{ccc}
A \otimes_{\omega} A & \xrightarrow{\mu_A} & A \\
\downarrow{\mu_A} & & \\
A & & \\
\downarrow{\mu_A} & & \\
A & & \\
\end{array}$$

$$\begin{array}{ccc}
A \otimes_{\omega} (A \otimes_{\omega} A) & \xrightarrow{\mu_A \otimes_{\omega} \mu_A} & A \otimes_{\omega} A \\
\downarrow{\mu_A} & & \\
(A \otimes_{\omega} A) \otimes_{\omega} A & & \\
\downarrow{\mu_A} & & \\
A & & \\
\end{array}$$

$$\begin{array}{ccc}
\omega^0(A) & \xrightarrow{\eta_A \otimes_{\omega} \eta_A} & A \otimes_{\omega} 1_{\omega} \\
\downarrow{\eta_A} & & \\
1_{\omega} \otimes_{\omega} A & & \\
\downarrow{\eta_A} & & \\
A \otimes_{\omega} A & & \\
\end{array}$$

commute;

**COPRODUCT AND COUNIT** A coproduct and a counit

$$\delta_A: A \to A \otimes_{\lambda} A \quad \text{and} \quad \epsilon_A: A \to 1_{\lambda},$$
such that the coassociativity and counitality diagrams commute:

which are also required to satisfy the compatibility conditions

which impose that $\mu_A$ and $\eta_A$ are comonoid morphisms, or equivalently $\delta_A$ and $\epsilon_A$ are monoid morphisms.

Proof. The truncations of a monoid structure in $D^\omega$ and of a comonoid structure in $D^\lambda$ follow from the reconstruction theorem [1.7] on certain lax and oplax monoidal functors, whereas the truncations of the monoid and comonoid morphism condition follow from the truncation of the monoidality condition between such functors [1.12].

The notion of bimonoid in a lax-oplax duoidal category extends the one of bimonoid in a duoidal category [17] which itself generalises the notion of bimonoid in a braided monoidal category. The princeps of such a bimonoid in a braided monoidal category is the one of bialgebras. A particularly interesting example of bimonoid in a genuine (i.e. not braided) duoidal category is given by bialgebroids over a commutative ring $R$ defined as bimonoids in the duoidal category of $(R,R)$-bimodules [17, 6.44]. However bialgebroids over a non-commutative ring do not admit a characterisation as bimonoids in a duoidal category [17, 6.45]. The relevant framework in this case is given by lax-strong duoidal categories, that is lax-oplax duoidal categories for which the oplax monoidal structure is strong monoidal.
2.3 Lax-strong duoidal categories

The example that we shall give of a lax-oplax duoidal category [§ 5] is in fact lax-strong. In the case where the oplax structure is strong, the description of the lax-oplax structure can be simplified.

**Proposition 2.8 (Lax-strong duoidal category).** A lax-strong duoidal structure on a category $D$ can be described as the data of a lax monoidal structure $D^{\lambda}$, a strong monoidal structure $D^{\otimes}$, together with natural transformations

\[
\lambda^{n} \otimes \lambda^{n} \xrightarrow{\chi^{n}} \lambda^{n} \circ (\otimes \times \cdots \times \otimes) \circ \tau^{1,n}
\]

and morphisms

\[
1_{\otimes} \xrightarrow{w^{n}} \lambda^{n} 1_{\otimes}
\]

for all integers $n \geq -1$ such that

- each pair $(\chi^{n}, w^{n})$ endows

  \[ \lambda^{n} : (D \times \cdots \times D)^{\otimes} \to D^{\otimes} \]

  with a structure of lax monoidal functor (between strong monoidal categories);

- the natural transformations $\chi^{n} \circ \tau^{1,n}$ endow

  \[ \otimes : (D \times D)^{\lambda} \to D^{\lambda} \]

  with a structure of oplax monoidal functor (between lax monoidal categories);

- the sequence $\{w^{n}\}_{n\geq-1}$ endows the unit object $1_{\otimes}$ with a structure of comonoid in $D^{\lambda}$.

**Proof.** According to the previously given definition of a lax-oplax duoidal category $D^{\lambda,\omega}$ [2.2], the functors $\lambda^{n}$ are lax functors between oplax monoidal categories. However, in the special case where $\omega = \otimes$ is strong monoidal, the proposition on the reconstruction of the lax structure of a functor [1.7] applies and tells us that for every integer $n \geq -1$, the natural transformations

\[
\omega^{k} \lambda^{n} \xrightarrow{\lambda^{k,n}} \lambda^{n} \omega^{k} \tau^{k,n}
\]

are completely determined by $w^{n} := \chi^{-1,n}$ and $\chi^{n} := \chi^{1,n}$. The conditions listed in the above proposition follow from this identification. \qed

Let us spell out the conditions to be verified by the data of a lax-strong duoidal category $(D, \lambda, \otimes, \chi, w)$. That each pair $(\chi^{n}, w^{n})$ defines a structure of lax monoidal functor on $\lambda^{n}$ imposes that the associativity diagram

\[
\begin{array}{ccc}
(\lambda^{n}(x) \otimes \lambda^{n}(y)) \otimes \lambda^{n}(z) & \xrightarrow{\chi^{n} \circ \otimes} & \lambda^{n}(x) \otimes (\lambda^{n}(y) \otimes \lambda^{n}(z)) \\
\downarrow^{\chi^{n} \circ \otimes} & & \downarrow^{- \otimes \chi^{n}} \\
\lambda^{n}(x \otimes y) \otimes \lambda^{n}(z) & & \lambda^{n}(x \otimes (y \otimes z)) \\
\downarrow^{\chi^{n}} & & \downarrow^{\chi^{n}} \\
\lambda^{n}((x \otimes y) \otimes z) & & \lambda^{n}(x \otimes (y \otimes z))
\end{array}
\]
as well as the two unitality diagrams

\[
\begin{array}{c}
\text{Categories enriched over oplax monoidal categories} \\
\text{Oplax monoidal categories can be seen as the oplax monoids of \(\text{Cat}^\times\). Thus one is naturally led to study its modules [§ 3.1]. As one could expect, modules over}
\end{array}
\]
an oplax monoidal category consist in a category together with infinitely many functors (instead of a single bifunctor) encoding the action of the (arbitrarily many copies of the) oplax monoidal category on it. We shall then repeat this process of constructing a lax monoid and its lax modules in a bigger bicategory $\text{Cat} \subset \text{Cat}_\omega$, which is obtained from $\text{Cat}$ by replacing functors with distributors. We shall refer to these modules as $\omega$-modules, and show that a particular class of the latter, namely those which are left representable (defined below) are in fact equivalent to ordinary modules over an oplax monoidal category.

One can also identify another class of $\omega$-modules, which we shall call right representable, and relate them to categories endowed with additional structures defined in terms of an oplax monoidal structure. We shall identify the latter as categories enriched over an oplax monoidal category. Recall that given a monoidal category $\mathcal{V}^\otimes$, a $\mathcal{V}$-category $\mathcal{C}$ has a set of objects $\text{Ob}(\mathcal{C})$ and $\mathcal{V}$-hom objects $[x,y] \in \text{Ob}(\mathcal{V})$ for every pair $x,y \in \text{Ob}(\mathcal{C})$. There is also a composition map $\mu: [y,z] \otimes [x,y] \to [x,z]$. Now if $\mathcal{V}^\omega$ is an oplax monoidal category, one expects to also have composition maps $[y,z] \otimes \omega [x,y] \to [x,z]$ and also countably many higher composition maps $\mu^n$ such as $[y,z] \otimes \omega [x,y] \otimes \omega [w,x] \otimes \omega [w,z]$ which are evident when $\mathcal{V}^\omega$ is strong.

There are a few new phenomena that appear when enriching over an oplax monoidal category:

1. By the oplax nature of the tensor structure, one can always recover the higher composition maps $\mu^n$ for $n \geq 2$ from $\mu^1$ and $\mu^{-1}$. This leads to two equivalent definitions for $\mathcal{V}$-categories: an extended one including all maps $\mu^n$ and an abridged one, including only $\mu^1$ and $\mu^{-1}$.

2. By the non-normal nature of the oplax monoidal structure, even though a $\mathcal{V}$-category—by which we mean a collection of objects $x,y,\ldots$ and hom-objects $[x,y],\ldots$, together with composition maps as above—always possesses an underlying $\text{Sets}^\times$-category with set of arrows $x \to y$ given by the maps $1_\omega \to [x,y]$, the object $[x,y]$ is usually not functorial in $x$ and $y$. Instead it is $[x,y]_\omega := \omega^0([x,y])$ which is functorial. This leads us to define a category enriched over $\mathcal{V}$ as a category $\mathcal{C}$ endowed with additional structure, namely a bifunctor $[-,-]: \mathcal{C} \times \mathcal{C}^{\text{op}} \to \mathcal{V}$ and natural transformations $\mu$ between tensor products of $[-,-]$. In other words, we require the enrichment structure to be functorial instead of recovering it as a byproduct, as happens when enriching over a strong monoidal category.

We shall spell out in more details the notion of categories enriched over an oplax monoidal category $\mathcal{V}^\omega$ [§ 3.3], and show that they are equivalent to the aforementioned right representable $\mathcal{V}$-$\omega$-modules [§ 3.4]. We shall then compare categories enriched over $\mathcal{V}$ with the notion of $\mathcal{V}$-categories, which is obtained by trying to mimic as closely as possible the standard construction of enrichment over strong monoidal categories, i.e. without requiring the enrichment bifunctor or the associated composition maps to be functorial [§ 3.5]. Finally, we shall finish by presenting an example of category enriched over an oplax monoidal category, in the context of the theory of operads [§ 3.6].

To help jumping between definitions, we shall continue using the notations $1_\omega, x \otimes_\omega y, x \otimes_\omega y \otimes_\omega z$ or $\omega^p(x_0,\ldots,x_p)$ depending on when it is the most appropriate.
3.1 \( \mathcal{V} \)-modules

In this subsection, we shall fix \( \mathcal{V}^{\omega} \) to be an oplax monoidal category. As the latter is nothing but an oplax monoid in \( \text{Cat}^X \), one can consider the 2-category \( \mathcal{V} \text{-mod}_{lax} \) of modules over it with lax maps, which will be defined hereafter.

**Definition 3.1 (\( \mathcal{V} \)-module).** A \( \mathcal{V} \)-module \( M^\rho \) is the data of a category \( M \) equipped with

- structure maps
  \[
  \mathcal{V}^n \times M \xrightarrow{\rho^n} M
  \]
  for any \( n \geq 0 \);

- natural transformations
  \[
  \rho^{p+q} \to \rho^p(-, \omega^q, -)
  \]
  for every \( p \geq 1, q \geq -1 \), and \( 0 \leq i < p \) is the number of entries on the left of \( \omega^q \);

- natural transformations
  \[
  \rho^{p+q} \to \rho^p(-, \rho^q)
  \]
  for every \( p, q \geq 0 \);

- a natural transformation
  \[
  \rho^0 \to \text{Id}_M
  \]

such that

**Counitalities**

\[
\begin{align*}
\rho^p & \Rightarrow \rho^0 \rho^p \Rightarrow \rho^p \\
\rho^p & \Rightarrow \rho^p(-, \rho^0) \Rightarrow \rho^p(-, \omega^0, -)
\end{align*}
\]

and

\[
\begin{align*}
\rho^p & \Rightarrow \rho^p(-, \rho^0) \Rightarrow \rho^p(-, \omega^0, -) \\
\rho^p & \Rightarrow \rho^p(-, \omega^q, -) \Rightarrow \rho^p(-, \omega^q, -)
\end{align*}
\]

commute whenever they make sense;

**Parallel Decompositions**

\[
\begin{align*}
\rho^{p+q+r} & \Rightarrow \rho^{p+q}(-, \rho^r) \Rightarrow \rho^{p+q}(-, \omega^r, -) \\
\rho^{p+q+r} & \Rightarrow \rho^{p+q}(-, \omega^r, -) \Rightarrow \rho^{p+q}(-, \omega^r, -)
\end{align*}
\]

commute whenever they make sense;
Sequential decompositions

\[
\begin{align*}
\rho^{p+q+r} &\rightarrow \rho^{p+q}(-, \omega^q, -) & \rho^{p+q+r} &\rightarrow \rho^{p+q}(-, \rho^r) \\
\rho^p(-, \rho^{q+r}) &\rightarrow \rho^p(-, \rho^q(-, \omega^q, -)) & \rho^p(-, \rho^{q+r}) &\rightarrow \rho^p(-, \rho^q(-, \rho^r))
\end{align*}
\]

and

\[
\begin{align*}
\rho^{p+q+r} &\rightarrow \rho^{p+q}(-, \omega^q, -) \\
\rho^p(-, \omega^{q+r}, -) &\rightarrow \rho^p(-, \omega^q(-, \omega^q, -), -)
\end{align*}
\]

commute whenever they make sense.

**Remark 3.2.** Note that the counitality conditions, as well as the parallel and sequential decomposition conditions, of a \(\mathcal{V}\)-module are simply obtained from those of an oplax monoidal category by replacing the oplax monoidal structure with a module structure map whenever it is possible. In particular, any oplax monoidal category is a module over itself.

**Definition 3.3** (Lax \(\mathcal{V}\)-module morphism). A lax morphism between two modules \(\mathcal{M}^\rho\) and \(\mathcal{N}^\sigma\) is the data of a functor \(F: \mathcal{M} \rightarrow \mathcal{N}\) together with natural transformations \(\sigma_n\) for all \(n \geq 0\), such that the diagrams

\[
\begin{align*}
\sigma^0(F) &\rightarrow F^{p+q}\sigma(-, \omega^q, -) & \sigma^0(F) &\rightarrow F^{p+q}\sigma(-, \rho^r) \\
\sigma^0(F) &\rightarrow F^{p+q}\sigma(-, \omega^q, -) & \sigma^0(F) &\rightarrow F^{p+q}\sigma(-, \rho^r)
\end{align*}
\]

and

\[
\begin{align*}
\sigma^p\sigma(-, \omega^q, -) &\rightarrow \sigma^p\sigma(-, \omega^q, -) \\
\sigma^p\sigma(-, \rho^r) &\rightarrow \sigma^p\sigma(-, \rho^r)
\end{align*}
\]

commute whenever they make sense.

**Definition 3.4** (Natural transformation of \(\mathcal{V}\)-module morphisms). A natural transformation of \(\mathcal{V}\)-module morphisms between two lax morphisms \(F, G: \mathcal{M}^\rho \rightarrow \mathcal{N}^\sigma\), with structure \(f\) and \(g\) respectively, is a natural transformation \(\alpha: F \Rightarrow G\) such that the diagram

\[
\begin{align*}
\sigma^0(F) &\rightarrow F^{p+q}\sigma(-, \omega^q, -) & \sigma^0(F) &\rightarrow F^{p+q}\sigma(-, \rho^r) \\
\sigma^0(F) &\rightarrow F^{p+q}\sigma(-, \omega^q, -) & \sigma^0(F) &\rightarrow F^{p+q}\sigma(-, \rho^r)
\end{align*}
\]

commute for any \(n \geq 0\).
3.2 $\mathcal{V}$-modules

We shall now introduce a ‘distributor version’ of $\mathcal{V}$-modules, that we shall call $\mathcal{V}$-modules and we shall show that the 2-category of $\mathcal{V}$-modules is equivalent to a certain subcategory of left representable $\mathcal{V}$-modules.

**Distributors** Recall that distributors—initially introduced by Bénabou [18] and Lawvere [19] under the name ‘bimodule’—are a particular kind of $\text{Sets}$-valued functors. More precisely, a distributor between two categories $\mathcal{B}$ and $\mathcal{C}$ is a functor $T : \mathcal{B}^{\text{op}} \times \mathcal{C} \to \text{Sets}$, denoted $T : \mathcal{B} \to \mathcal{C}$. We shall write $T(b)$ for the set obtained by evaluating the distributor $T$ on an object $b$ of $\mathcal{B}$ and an object $c$ of $\mathcal{C}$. Note that distributors are also called ‘profunctors’ or ‘bimodules’ while the 2-category $\text{Cat}_\delta$ of categories, distributors and natural transformations is also denoted $\text{Prof}$ in the literature.

The composition of two distributors $T : \mathcal{A} \to \mathcal{B}$ and $U : \mathcal{B} \to \mathcal{C}$, is again a distributor $U \circ T : \mathcal{A} \to \mathcal{C}$, defined as the integral

$$ (U \circ T)(a)(c) := \int_b U(b)(c) \times T(a), $$

for any objects $a \in \text{Ob}(\mathcal{A})$ and $c \in \text{Ob}(\mathcal{C})$. Note that this composition is associative and unital (the identity distributor is simply the hom-functor) only up to an isomorphism, so that $\text{Cat}_\delta$ is a bicategory [20, Chap. 5].

Every functor $F : \mathcal{B} \to \mathcal{C}$ defines a distributor $C(F)(-)$, and this assignment induces a 2-functor $\text{Cat}_2^{\text{op}} \to \text{Cat}_\delta$. Its essential image, that we shall denote by $\text{Cat}^{1-\text{rep}}_\delta$, is comprised of distributors that are called left representable.

**Definition 3.5 ($\mathcal{V}$-module).** A lax $\mathcal{V}$-module $M^R$ is a category $M$ equipped with

- distributors $R^n : \mathcal{V}^n \times M \to M$ for all natural numbers $n \geq 0$;
- natural transformations $R^p(-, \omega^q, -) \rightarrow R^{p+q}$ for every $p \geq 1, q \geq -1$ and every $0 \leq i < p$ representing the number of entries on the left of $\omega^q$;
- natural transformations $R^p(-, R^q) \rightarrow R^{p+q}$ for every $p, q \geq 0$;
- and a natural transformation $\text{Id}_M \rightarrow R^0$ such that
and
\[
\begin{align*}
R^p(-, \text{Id}_M) & \to R^p(-, R^0) \\
R^p(-, \text{Id}, -) & \to R^p(-, \omega^0, -)
\end{align*}
\]

commute whenever they make sense;

**Parallel Compositions**
\[
\begin{align*}
R^p(-, \omega^q, -, R^r) & \to R^{p+q}(-, R^r) \\
R^p(-, \omega^q, -, \omega^r, -) & \to R^{p+q}(-, \omega^r, -)
\end{align*}
\]

\[
\begin{align*}
R^{p+r}(-, \omega^q, -) & \to R^{p+q+r} \\
R^{p+r}(-, \omega^q, -) & \to R^{p+q+r}
\end{align*}
\]

commute whenever they make sense;

**Sequential Compositions**
\[
\begin{align*}
R^p(-, \omega^q(-, \omega^r, -), -) & \to R^{p+q}(-, \omega^r, -) \\
R^p(-, R^q(-, \omega^r, -)) & \to R^{p+q}(-, \omega^r, -)
\end{align*}
\]

\[
\begin{align*}
R^p(-, \omega^q+r, -) & \to R^{p+q+r} \\
R^p(-, R^q+r) & \to R^{p+q+r}
\end{align*}
\]

and
\[
\begin{align*}
R^p \circ (-, R^q(-, R^r)) & \Rightarrow R^p \circ (-, R^q) \circ (-, R^r) \\
R^p(-, R^q+r) & \Rightarrow R^{p+q+r}
\end{align*}
\]

commute whenever they make sense.

If the structure maps $R^n$ are left representable distributors, then $M^R$ will be called a left representable lax $\mathcal{V}$-$\delta$-module.

**Remark 3.6.** Via $\text{Cat}^{2\text{-op}} \to \text{Cat}_\delta$, every oplax monoid is sent to a (representable) lax monoid. The definition of $\mathcal{V}$-$\delta$-module then corresponds to the notion of module over the lax monoid induced by $\mathcal{V}^{\omega}$ in $\text{Cat}_\delta$.

**Lemma 3.7.** Let $\mathcal{M}$ be a category and $R^n : \mathcal{V}^n \times \mathcal{M} \to \mathcal{M}$ be left representable distributors for $n \geq 0$. For each representations $\rho^n : \mathcal{V}^n \times \mathcal{M} \to \mathcal{M}$, for $n \geq 0$, of the previous collection of distributors, there is a bijection between lax $\mathcal{V}$-$\delta$-module structures extending $M^R$ and $\mathcal{V}$-$\delta$-module structures extending $M^\rho$.

**Proof.** Straightforward thus omitted. \qed

Considering $\mathcal{V}$-$\delta$-modules as modules over a lax monoid in $\text{Cat}_\delta$, one can naturally define a 2-category of $\mathcal{V}$-$\delta$-modules. We shall however restrict ourselves to a sub-bicategory of $\mathcal{V}$-$\delta$-mod$_{\text{oplax}}$, that we shall denote $\mathcal{V}$-$\delta$-mod$_{\text{fun}}$, wherein the oplax morphisms of $\mathcal{V}$-$\delta$-modules are taken to be functors (instead of distributors) equipped with some natural transformations defined precisely below.
Definition 3.8 (Oplax functor between $\mathcal{V}$-$\delta$-modules). An oplax functor between two $\mathcal{V}$-$\delta$-modules $M^R$ and $N^S$ is the data of:

- a functor $F: M \to N$;
- natural transformations

$$R^p(a_x^y) \xrightarrow{fp} S^p(a_x^{F(y)})$$

with components for any collection $a$ of objects in $\mathcal{V}$, pair of objects $x, y$ in $M$ and integers $p \geq 0$,

such that the diagrams commute whenever they make sense.

$\begin{array}{ccc}
M(y^x) & \xrightarrow{F} & R^0(x^y) \\
\downarrow^p & & \downarrow^p \\
N(F(x)^{F(y)}) & \xrightarrow{S^0(F(x)^{F(y)})} & S^0(F(x)^{F(y)}) \\
\end{array}$

$\begin{array}{ccc}
R^p(a_x^y)^{-x} & \rightarrow & R^{p+q}(-x)^y \\
\downarrow^p & & \downarrow^p \\
S^p(-a_x^{F(x)}^{F(y)}) & \rightarrow & S^{p+q}(-F(x)^{F(y)}) \\
\end{array}$

$\begin{array}{ccc}
R^p(-, q)^y_x & \rightarrow & R^{p+q}(-x)^y \\
\downarrow & & \downarrow \\
\int_m S^p(-F(m))^{F(y)} \times S^q(-F(x))^{F(m)} & \rightarrow & S^{p+q}(-F(x)^{F(y)}) \\
\end{array}$

$\begin{array}{ccc}
R^p(a_x^y)^{-x} & \rightarrow & R^{p+q}(-x)^y \\
\downarrow & & \downarrow \\
S^p(a_{G(x)}^{G(y)}) & \rightarrow & S^{p+q}(-a_{G(x)}^{G(y)}) \\
\end{array}$

Definition 3.9 (Natural transformation between oplax functors). A natural transformation $\alpha: F \Rightarrow G$ between a couple of oplax functors $F, G: M \to N$, with structure $f$ and $g$ respectively, is a natural transformation between the underlying functors such that the diagram

$\begin{array}{ccc}
R^p(a_x^y)^{-x} & \rightarrow & R^{p+q}(-x)^y \\
\downarrow & & \downarrow \\
S^p(a_x^{G(y)}) & \rightarrow & S^{p+q}(-a_x^{G(y)}) \\
\end{array}$

commutes for any $p \geq 0$.

There is an evident 2-functor

$$\mathcal{V}\text{-mod}_{\text{lax}} \longrightarrow \mathcal{V}\text{-mod}_{\text{oplax}}$$

sending every $\mathcal{V}$-module $M^p$ to the lax $\mathcal{V}$-$\delta$-module with the same category $M$ and structure maps $M^p$. Lax morphisms between $\mathcal{V}$-modules induce oplax functors between the associated $\mathcal{V}$-$\delta$-modules in the obvious way and the same goes for the natural transformations.
**Proposition 3.10.** The 2-functor above induces a 2-equivalence
\[ \mathcal{V} \text{-mod}_{\text{lax}} = \text{left rep. } \mathcal{V} \text{-mod}_{\text{oplax}} \]
between the 2-category of \( \mathcal{V} \)-modules, lax morphisms and natural transformations, and the 2-category of left representable \( \mathcal{V} \)-\( \delta \)-modules, oplax functors and natural transformations.

**Proof.** The previous lemma relating left representable \( \mathcal{V} \)-\( \delta \)-modules to \( \mathcal{V} \)-modules \([3.7]\) implies essential surjectivity on objects.

Given a functor \( F: \mathcal{M}^\rho \to \mathcal{N}^\sigma \) between two \( \mathcal{V} \)-modules, one has a bijection between natural transformations \( \mathcal{N}(\rho, F) \to \mathcal{M}(\sigma, F) \). In one direction, this is given by the composition
\[ \mathcal{M}(\rho) \longrightarrow \mathcal{N}(F^\rho) \longrightarrow \mathcal{N}(\mathcal{V}(\rho, F)) \]
and by the embedding lemma of Yoneda in the other direction. The three diagrams defining lax morphisms of \( \mathcal{V} \)-modules on one hand, and oplax functors between \( \mathcal{V} \)-\( \delta \)-modules on the other hand, are in one-to-one correspondence under this bijection. Bijectivity on natural transformations is obtained in a similar way. \( \square \)

### 3.3 Categories enriched over \( \mathcal{V} \)

In this subsection, we introduce the notion of a category enriched over an oplax monoidal category \( \mathcal{V}^{\omega} \), as well as the associated notions of enriched functors and natural transformations. An oplax monoidal category is endowed with structure maps of arbitrarily high arity which naturally leads one to consider composition maps of arbitrary arity as well when defining an enrichment structure.

There is also an abridged version of the definition of an enriched category which is closer to the familiar definition of an enriched category over a tensor category. As we shall see, in the case of enrichment over an oplax monoidal category, the two definitions are equivalent.

**Notation 3.11.** Given three categories \( \mathcal{C}, \mathcal{V}, \) and \( \mathcal{W} \) and a bifunctor \([ -, - ]: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{V} \), for every \( n \geq 0 \) and every functor \( F: \mathcal{V}^n \to \mathcal{W} \) we shall use the notation
\[ [ -, \ldots, - ]_F := F([ -, \ldots, - ]) \]
for the functor \((\mathcal{C}^{\text{op}} \times \mathcal{C})^n \to \mathcal{W}\) obtained by composing \([ -, - ]^n \) with \( F \).

**Definition 3.12 (Categories enriched over \( \mathcal{V} \), extended version).** A category enriched over \( \mathcal{V} \) is the data of a category \( \mathcal{C} \) endowed with

- a bifunctor \([ -, - ]: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{V} \)

  taking values in \( \mathcal{V} \);

- for every \( n \geq -1 \) and each \((n + 2)\)-tuple of objects \( x_0, \ldots, x_{n+1} \) of \( \mathcal{C} \), transformations
\[ \omega^n([x_{n+1}, x_0, x_1], \ldots, [x_{n+1}, x_1]) \to [x_0, x_{n+1}] \]
which are natural in \( x_0 \) and \( x_{n+1} \), extranatural in \( x_1, \ldots, x_n \), and which we shall abbreviate as

\[
[x_0, \ldots, x_{n+1}]_{\omega^m} \rightarrow [x_0, x_{n+1}]
\]

and collectively abbreviate as

\[
[-, \ldots, -]_{\omega^m} \rightarrow [-, -]
\]

It is required that:

**Unitality** the morphism \( m^0 \) be the counit natural transformation of \( \mathcal{V} \);

**Additivity** the diagram

\[
\begin{array}{ccc}
[-, \ldots, -]_{\omega^{p+q}} & \quad & [-, \ldots, -]_{\omega^p (-, \omega_q -, -)} \\
\downarrow \quad m^{p+q} & \quad \downarrow \quad \omega^p (-, m^q -, -) & \quad \downarrow \\
[-, \ldots, -]_{\omega^p} & \quad \downarrow \quad m^p & \quad [-, -]
\end{array}
\]

commute whenever it makes sense.

Now let us introduce an abridged version of the above definition, wherein one only requires the existence of a binary composition and a unit map.

**Definition 3.13** (Category enriched over \( \mathcal{V} \), abridged version). A category \( \mathcal{C} \) enriched over \( \mathcal{V} \) is a category \( \mathcal{C} \) equipped with

- a bifunctor
  \[
  [-]: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{V}
  \]
  taking values in \( \mathcal{V} \);

- an extranatural transformation with components
  \[
  1_\omega \rightarrow [x, x]
  \]
  for each \( x \in \text{Ob}(\mathcal{C}) \), called the unit map;

- a transformation called the composition map, with components
  \[
  [y, z] \otimes_\omega [x, y] \rightarrow [x, z]
  \]
  for each triplet of objects \( x, y, z \in \text{Ob}(\mathcal{C}) \) being natural in \( x \) and \( z \) while being extranatural in \( y \);

such that:
**Associativity** the associativity diagram in $\mathcal{V}$

\[
[y,z] \otimes_{\omega} [x,y] \otimes_{\omega} [w,x] \\
\mu_{y,z} \otimes_{\omega} (x,y) \otimes_{\omega} [w,x]
\]

commutes for every $w, x, y, z \in \text{Ob}(\mathcal{C})$;

**Unitality** the two unitality diagrams in $\mathcal{V}$

\[
[x,y]_{\omega} \otimes_{\omega} [x,y] \\
\eta_{x,y} \otimes_{\omega}
\]

and

\[
[x,y] \otimes_{\omega} [x,x] \\
\mu_{x,y}
\]

commute for every $x, y \in \text{Ob}(\mathcal{C})$.

This definition is close to that of a category enriched over a strong monoidal category, the main differences being that we still require the composition and unit maps to be (extra)natural, and that the associativity and unitality condition be slightly modified due to the oplax nature of $\mathcal{V}^\omega$.

**Proposition 3.14.** The two above definitions of a $\mathcal{V}$-category are canonically isomorphic.

**Proof.** The proof of the equivalence between the extended and the abridged version can be done by induction and is very similar to the one of the reconstruction of the lax structure of a functor [1.7], apart from the fact that the diagrams involved are a bit smaller. □

**Definition 3.15 (\(\mathcal{V}\)-enriched functor, extended version).** Let $(\mathcal{B}, [-,-], \{l^n\}_{n \geq -1})$ and $(\mathcal{C}, [-,-], \{m^n\}_{n \geq -1})$ be two categories enriched over $\mathcal{V}$. A $\mathcal{V}$-enriched functor between $\mathcal{B}$ and $\mathcal{C}$ is the data of a functor $F: \mathcal{B} \to \mathcal{C}$ equipped with natural transformations with components

\[
[x,y] \xrightarrow{F_{x,y}} (F(x), F(y))
\]
for each pair of objects \(x, y\) of \(B\), such that the diagrams

\[
\begin{array}{cccccc}
[x_0, \ldots, x_{n+1}]_\omega & \xrightarrow{\omega^n(F_{x_0, x_{n+1}} \cdots F_{x_{n+1}, x_0})} & \langle F(x_0), \ldots, F(x_{n+1}) \rangle_\omega \\
\downarrow F_{x_0, x_{n+1}} & & \downarrow m^n \\
[x_0, x_{n+1}] & \xrightarrow{F_{x_0, x_{n+1}}} & \langle F(x_0), F(x_{n+1}) \rangle
\end{array}
\]

commute for all integers \(n \geq -1\) and all sequences of objects \(x_0, \ldots, x_{n+1}\) of \(B\).

Similarly as for categories enriched over \(V\), the notion of \(V\)-enriched functor admits an abridged version, involving only two compatibility diagrams.

**Definition 3.16 (\(V\)-enriched functor, abridged version).** A \(V\)-enriched functor between two categories \((B, [-, -], \eta, \mu)\) and \((C, \langle - , - \rangle, \theta, \nu)\) enriched over \(V\) is a functor \(F : B \to C\) equipped with natural transformations with components

\[
[x, y] \xrightarrow{F_{x,y}} \langle F(x), F(y) \rangle
\]

for each pair of objects \(x, y\) of \(B\), such that:

- **Unit** the unit diagram in \(V\)

\[
\begin{array}{cccc}
1_\omega & \xrightarrow{\theta_{F(x)}} & \langle F(x), F(x) \rangle \\
\downarrow \eta_x & & \\
[x, x] & \xrightarrow{F_{x,x}} & \langle F(x), F(x) \rangle
\end{array}
\]

commutes for every \(x \in \text{Ob}(B)\);

- **Composition** the composition diagram in \(V\)

\[
\begin{array}{cccccc}
[y, z] \otimes \omega [x, y] & \xrightarrow{F_{x,y} \otimes \omega F_{x,y}} & \langle F(y), F(z) \rangle \otimes \omega \langle F(x), F(y) \rangle \\
\downarrow \mu_{x,y,z} & & \downarrow \nu_{F(x), F(y), F(z)} \\
[x, z] & \xrightarrow{F_{x,z}} & \langle F(x), F(z) \rangle
\end{array}
\]

commutes for every \(x, y, z \in \text{Ob}(B)\).

**Proposition 3.17.** The two above definitions of \(V\)-functors are canonically equivalent.

**Proof.** Let us denote \(f(n)\) the hypothesis that the \(n\)-th diagram commutes with \(n \geq -1\). The hypotheses \(f(-1)\) and \(f(1)\) are satisfied due to the unit and composition diagrams, respectively, while \(f(0)\) holds by naturality of the counit of \(\omega\). From the additivity of the structure maps \(m^n\), a simple diagram shows that, by writing \(m^{n+1}\) as \(m^n \otimes -\), and symmetrically for \(l^{n+1}\), one has \(f(n) \land f(1) \implies f(n + 1)\).

**Definition 3.18 (Natural \(V\)-enriched transformations, extended version).** Let \((B, [-, -], [l^m]_{m \geq -1})\) and \((C, \langle - , - \rangle, [m^n]_{n \geq -1})\) be two categories enriched over \(V\).
Let furthermore $F, G : B \to C$ be two $\mathcal{V}$-enriched functors. A natural $\mathcal{V}$-enriched transformation between $F$ and $G$ is a natural transformation $\alpha : F \Rightarrow G$ such that the diagram

![Diagram](image)

commutes in $\mathcal{V}$, for every $n \geq 0$ and every $x_0, \ldots, x_{n+1} \in \text{Ob}(B)$.

Once again, we can consider an abridged version of the previous definition allowing to reexpress the above compatibility diagram in the usual square form.

**Definition 3.19 (Natural $\mathcal{V}$-enriched transformations, abridged version).** Let $F, G : B \to C$ be two $\mathcal{V}$-enriched functors between two categories $(B, [\cdot, \cdot], \eta, \mu)$ and $(C, \langle \cdot, \cdot \rangle, \theta, \nu)$ enriched over $\mathcal{V}$. A natural $\mathcal{V}$-enriched transformation between $F$ and $G$ is a natural transformation $\alpha : F \Rightarrow G$ such that the diagram

![Diagram](image)

commutes for every pair $x, y \in \text{Ob}(B)$.

**Proposition 3.20.** The two above definitions of natural $\mathcal{V}$-enriched transformations are canonically equivalent.

**Proof.** The fact that the extended version implies the abridged version follows straightforwardly from the specialisation of the defining diagram to $n = 0$. In
order to prove the converse implication, we use the naturality of \( m^n \) and \( \alpha \) to rewrite the defining diagram \([3.18]\) as

\[
\begin{align*}
&\omega^n(F_{x_0,\ldots,x_{n+1}}) \quad \omega^n(G_{x_0,\ldots,x_{n+1}}) \\
&\langle F(x_0),\ldots,F(x_{n+1}) \rangle_\omega \quad \langle G(x_0),\ldots,G(x_{n+1}) \rangle_\omega \\
&\langle x_0,\ldots,x_{n+1} \rangle_\omega \quad \langle x_0,\ldots,x_{n+1} \rangle_\omega \\
&\langle F(x_0),F(x_{n+1}) \rangle_\omega \quad \langle G(x_0),G(x_{n+1}) \rangle_\omega \\
&\langle -,- \rangle \quad \langle -,- \rangle \\
&\langle F(x_0),G(x_{n+1}) \rangle_\omega \quad \langle F(x_0),G(x_{n+1}) \rangle_\omega
\end{align*}
\]

so that the top squares commute by virtue of the property of \( V \)-enriched functors while the bottom square is the abridged \( V \)-enriched naturality condition.

\[\square\]

3.4 Categories enriched over \( V \) vs \( V-\delta \) modules

Having previously compared \( V \)-modules with \( V-\delta \) modules, we shall now compare categories enriched over \( V \) with \( V-\delta \) modules, and shall show that the former identify with a subclass of the latter (obeying a certain representability condition). First, let us start by showing that any category enriched over \( V \) induces a \( V-\delta \) module.

**Proposition 3.21.** Any category \( M \) enriched over \( V \) defines a \( V-\delta \) module with structure maps

\[
R^{n+1}(a,x,y) := \mathcal{V}(\omega^n(a))
\]

for any sequence \( a \) of \( n+1 \) objects of \( V \), pair \( x, y \) of objects of \( M \) and integer \( n \geq -1 \).

**Proof.** First, we need to construct the associators, i.e. the natural transformations

\[
R^p(-,\omega^q,\omega^q \circ (-,R^q) \rightarrow R^{p+q} \quad \text{and} \quad \text{Id}_M \rightarrow R^0
\]

The first type of associators is induced from those of the oplax structure \( \omega \). The second type of associators is obtained from the composite

\[
\mathcal{V}(\omega^{p+1}(a)) \times \mathcal{V}(\omega^{q+1}(b)) \rightarrow \mathcal{V}(\omega^{p+1}(a) \otimes \omega^{q+1}(b)) \rightarrow \mathcal{V}(\omega^{p+q+1}(a \otimes b))
\]

where \( a \) and \( b \) are sequences of respectively \( p \) and \( q \) objects of \( V \) (with \( p,q \geq 0 \)), and \( m,x,y \) are objects of \( M \). The first map is simply given by the tensor product of morphisms in \( V \), the second arrow is induced by either of the compositions

\[
\begin{align*}
\omega^{p+q+1} \quad \omega^p(-,\omega^q) \\
\omega^q(\omega^{p+1},-) \quad \omega^p-1 \otimes \omega \omega^q-1
\end{align*}
\]
which are equivalent by parallel decomposition, and the last arrow from the composition map of $\mathcal{M}$. The total composite map is extranatural in $m$, due to the fact that $[-,-]$ is a bifunctor and that $\mu$ is extranatural in $m$. Hence the universal property of integrals ensures the existence of natural transformations

$$\int_m \mathcal{V}(\omega^{p-1}(a)_{[m,y]}) \times \mathcal{V}(\omega^{q-1}(b)_{[x,m]}) \to \mathcal{V}(\omega^{p+q-1}(a \oplus b)_{[x,y]})$$

for all $p, q \geq 0$ which play the role of associators $R^p \circ (-, R^q) \to R^{p+q}$.

The last associator, i.e. the unit of the $\mathcal{V}$-$\delta$-module structure,

$$\mathcal{M}(x) \to \mathcal{V}(1_{[x,x]}),$$

is simply given by

$$\begin{array}{ccc}
1_{\omega} & \xrightarrow{\eta_x} & [x,x] \\
\downarrow \eta_y & & \downarrow [-f] \\
[y,y] & \xrightarrow{[f,-]} & [x,y]
\end{array}$$

for any morphism $f : x \to y$, and which commutes by extranaturality of the unit $\eta$ of $\mathcal{M}$.

Now we also need to verify that these natural transformations do define a $\mathcal{V}$-$\delta$-module structure. One can check that the unitality condition containing $\omega^0$ is verified as a consequence of the counitality condition of $\omega$, while the other two unitality conditions (both involving $R^0$) are satisfied by virtue of the unitality condition for the enrichment of $\mathcal{M}$ over $\mathcal{V}$ as well as the counitality condition of $\omega$. Moreover, the parallel composition condition involving two copies of $\omega$ is satisfied as a consequence of the parallel decomposition of $\omega$ while the one involving one copy of $\omega$ holds by virtue of both the parallel and sequential decompositions of $\omega$. Finally, the two sequential composition conditions involving at least one $\omega$ are verified as a consequence of the sequential composition conditions of $\omega$, while the third one (only involving the structure maps $R^n$) is verified due to the associativity of the composition maps $\mu$, as well as the parallel and sequential decomposition conditions of $\omega$.

Notice that the structure of $\mathcal{V}$-$\delta$module induced by a category enriched over $\mathcal{V}$ is such that the diagrams

$$\begin{array}{ccc}
\mathcal{V}(\omega^n(a)_{[x,y]}) & \to & \mathcal{V}(\alpha^0 \omega^n(a)_{[x,y]}) = R^1(\omega^n(a), x) \\
\downarrow R^n+1(a,x) & & \downarrow R^n+1(a,x)
\end{array}$$

commute for any sequence of $n + 1$ objects $a$ of $\mathcal{V}$, any pair of objects $x, y$ of $\mathcal{M}$ and any integer $n \geq 0$. This is simply a consequence of the counitality condition that the oplax monoidal structure satisfies. Abstracting away the previous property leads to consider the following definition of a right representable $\mathcal{V}$-$\delta$-module.

**Definition 3.22** (Right representable $\mathcal{V}$-$\delta$-modules). A $\mathcal{V}$-$\delta$-module $\mathcal{M}^R$ will be called right representable if there exists
• a bifunctor
\[ [-, -]: \mathcal{M}^{\text{op}} \times \mathcal{M} \to \mathcal{V} \]
taking values in \( \mathcal{V} \);
• a natural isomorphism
\[ \phi: \mathcal{V}(\omega^0(-)) \cong R^1(-) \]
such that the maps
\[ \mathcal{V}(\omega^{p-1}(-)) \xrightarrow{\phi^p} R^p(-) \]
defined by the diagrams
\[
\begin{array}{ccc}
\mathcal{V}(\omega^{p-1}(a)) & \xrightarrow{\phi^p} & \mathcal{V}(\omega^p \omega^{p-1}(a)) \\
& \downarrow{\phi} & \downarrow{\phi} \\
R^p(a, x) & \xrightarrow{\phi^p} & R^p(\omega^{p-1}(a), x)
\end{array}
\]
are natural isomorphisms, for \( p \geq 0 \).

**Remark 3.23.** By counitality of \( \omega^0 \), one recovers \( \phi \) as \( \phi^1 \).

**Remark 3.24.** In a right representable \( \mathcal{V} \)-\( \delta \)-module, the composition map
\[ R^1(\omega^p, -) \to R^{p+1} \]
admits a canonical section
\[ R^{p+1} \to R^1(\omega^p, -) \]
which, using sequential composition, allows us to rewrite the associators
\[ R^p(-, R^q) \to R^{p+q} \]
as the composite
\[ R^p(-, R^q) \to R^1(\omega^{p-1}, R^1(\omega^{q-1}, -)) \to R^2(\omega^{p-1}, \omega^{q-1}, -) \to R^{p+q} \]
for \( p, q \geq 0 \).

**Lemma 3.25.** In a right representable \( \mathcal{V} \)-\( \delta \)-module, the square
\[
\begin{array}{ccc}
\mathcal{V}(\omega^{p-1}(-, -)) & \xrightarrow{\phi^p} & R^p(-, -) \\
& \downarrow{\phi^{p+q}} & \downarrow{\phi^{p+q}} \\
\mathcal{V}(\omega^{p+q-1}(-)) & \xrightarrow{\phi^{p+q}} & R^{p+q}(-)
\end{array}
\]
commutes whenever it makes sense.
Proof. For this it is enough to consider the diagram

\[
\begin{array}{cccccc}
\mathcal{V}(\omega_{0}^{-1}(-,\omega_{0}^{-1})_{[-,-]}) & \rightarrow & \mathcal{V}(\omega_{0}^{0} \omega_{0}^{-1}(-,\omega_{0}^{-1})_{[-,-]}) & \rightarrow & R^{1}(\omega_{0}^{-1}(-,\omega_{0}^{-1})_{[-,-]}) & \rightarrow & R^{p}(-,\omega_{0}^{-1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{V}(\omega_{0}^{p+1}(-,\omega_{0}^{p+1})_{[-,-]}) & \rightarrow & \mathcal{V}(\omega_{0}^{0} \omega_{0}^{p+1}(-,\omega_{0}^{p+1})_{[-,-]}) & \rightarrow & R^{1}(\omega_{0}^{p+1}(-,\omega_{0}^{p+1})_{[-,-]}) & \rightarrow & R^{p+1}(-) \\
\end{array}
\]

whose first two squares commute by naturality and whose last square commutes by sequential composition. \(\square\)

The notion of right representable \(\mathcal{V}\)-\(\delta\)-module identifies with the one of category enriched over \(\mathcal{V}\), as shown in the following proposition.

**Proposition 3.26.** Let \(\mathcal{M}\) be a category and \(R^n : \mathcal{V}^n \times \mathcal{M} \rightarrow \mathcal{M}\) be distributors for \(n \geq 0\). Given a bifunctor \([-,-] : \mathcal{M}^{op} \times \mathcal{M} \rightarrow \mathcal{V}\), and natural isomorphisms \(\psi^p : \mathcal{V}(\omega_{0}^{p-1}) \cong R^p\) for \(p \geq 0\), there is a bijection between right representable \(\mathcal{V}\)-\(\delta\)-module structures extending \((\mathcal{M}, R)\) and structures of category enriched over \(\mathcal{V}\) extending \((\mathcal{M}, [-,-])\).

**Proof.** Starting from an extension of structure \((\mathcal{M}, [-,-])\) into a category enriched over \(\mathcal{V}\), we have already explained how to build a canonical \(\mathcal{V}\)-\(\delta\)-module structure on \(\mathcal{M}\) \([3.21]\). Using this construction and the isomorphisms \(\psi^p\), one can then extend \((\mathcal{M}, R)\) into a \(\mathcal{V}\)-\(\delta\)-module structure. Finally, letting \(\phi := \psi^1\), one can show that for this \(\mathcal{V}\)-\(\delta\)-module structure \((\mathcal{M}, R)\), we have \(\phi^p = \psi^p\) as in the definition of right representable modules; it is thus right representable.

We conversely now prove that any right representable \(\mathcal{V}\)-\(\delta\)-module yields a category enriched over \(\mathcal{V}\). Let \(\mathcal{M}\) be a right representable \(\mathcal{V}\)-\(\delta\)-module with bifunctor \([-,-]\). Let us start by showing that the \(\mathcal{V}\)-\(\delta\)-module associators induce an enrichment over \(\mathcal{V}\) on the category \(\mathcal{M}\) endowed with \([-,-]\). Thanks to the associator

\[
R^1 (-, R^1) \rightarrow R^2
\]

of the right representable \(\mathcal{V}\)-\(\delta\)-module structure, one can obtain the composite map

\[
(\mathcal{V} \times \mathcal{V})(\omega_{0}^{a,b})_{[m,y] \times [x,m]} \rightarrow (\mathcal{V} \times \mathcal{V})(\omega_{0}^{0}(a) \times \omega_{0}^{0}(b))_{[m,y] \times [x,m]} \rightarrow \int_m \mathcal{V}(\omega_{0}^{0}(a))_{[m,y]} \times \mathcal{V}(\omega_{0}^{0}(b))_{[x,m]} \rightarrow \mathcal{V}(\omega_{0}^{0}(a) \times \omega_{0}^{0}(b))_{[m,y]}_{[x,m]}
\]

natural in \(a, b\). Such maps are in bijection with transformations

\[
[m, y] \otimes [x, m] \xrightarrow{\mu_{m,y}} [x, y]
\]

which are natural in \(x, y\) and extranatural in \(m\). In a simpler fashion, the unit \(\text{Id}_{\mathcal{M}} \rightarrow R^0\) yields

\[
1_{\omega_{0}} \xrightarrow{\eta_{x}} [x, x]
\]

which is extranatural in \(x\).

The associativity condition for \([-,-]\) follows from this

\[
R^1 o (-, R^1(-, R^1)) \xrightarrow{\Phi} R^1(-, R^1) o (-, R^1) \rightarrow R^2(-, R^1) \\
R^1(-, R^2) \rightarrow R^3
\]
sequential composition diagram, while the two unitality conditions follow from these two

\[
\begin{align*}
\text{Id}_M & \circ R^p \rightarrow R^0 \circ R^1 \\
& \text{and} \\
R^1 & \circ (-, \text{Id}_M) \rightarrow R^1(-, R^0)
\end{align*}
\]

unitality diagrams. This concludes the proof that any right representable \(\mathcal{V}\)-\(\delta\)-modules induces a category enriched over \(\mathcal{V}\).

The fact that these two procedures are inverse to one another follows from integral calculus, the counitality of \(\omega^0\) and from the fact that the structure of a right representable \(\mathcal{V}\)-\(\delta\)-module can be reconstructed from the two transformations \(\text{Id}_M \rightarrow R^0\) and \(R^1(-, R^1) \rightarrow R^2\) (together with the oplax structure of \(\mathcal{V}^\omega\)) as explained earlier \([3.24, 3.25]\).

We have previously described how to create a right representable \(\mathcal{V}\)-\(\delta\)-module from a category enriched over \(\mathcal{V}\). This construction can be extended into a 2-functor

\[
\text{Cat}_\mathcal{V} \rightarrow \mathcal{V}\text{-}\text{mod}_{\text{fun}}^\text{oplax}
\]

in the obvious way.

**Remark 3.27.** Let \((F, \{f_p\}) : \mathcal{M}^R \rightarrow \mathcal{N}^S\) be an oplax functor between two \(\mathcal{V}\)-\(\delta\)-modules. If \(\mathcal{M}^R\) is right representable, then each \(f_p\) can be rewritten as

\[
\begin{align*}
R^p(\omega^{-1}(-), x) & \rightarrow R^1(\omega^{-1}(-), y) \\
& \text{and} \\
S^p(\omega^{-1}(-), F(x)) & \rightarrow S^1(\omega^{-1}(-), F(x), F(y))
\end{align*}
\]

thanks to the second diagram defining oplax functors \([3.8]\) and thanks to the canonical section \(R^p \rightarrow R^1(\omega^{-1}, -)\) \([3.24]\).

**Theorem 3.28.** Let \(\mathcal{V}^\omega\) be a normal oplax monoidal category, then the above map induces a 2-equivalence

\[
\text{Cat}_\mathcal{V} = \text{right rep. } \mathcal{V}\text{-}\text{mod}_{\text{fun}}^\text{oplax}
\]

between the 2-category of categories enriched over \(\mathcal{V}\) and the one of right representable \(\mathcal{V}\)-\(\delta\)-modules.

**Proof.** The previous proposition implies essential surjectivity on objects. We now turn to 1-fully faithfulness. Let \((\mathcal{B}, [-, -])\) and \((\mathcal{C}, \langle - , - \rangle)\) be two categories enriched over \(\mathcal{V}\) and let us denote by \(\mathcal{B}^R\) and \(\mathcal{C}^S\) the two respective associated \(\mathcal{V}\)-\(\delta\)-modules. We need to build from every oplax functor \((F, \{f_p\}) : \mathcal{B}^R \rightarrow \mathcal{C}^S\) a functor \(F : \mathcal{B} \rightarrow \mathcal{C}\) enriched over \(\mathcal{V}\). The underlying functor will be the same.
For its additional structure, we start by noting that since \( V^\omega \) is normal, one gets natural transformations

\[
R^1(\tilde{x}, \tilde{y}) := \mathcal{V}(\tilde{x}, \tilde{y}) \xrightarrow{f^1} \mathcal{V}((F(x), F(y))) =: S^1(\tilde{x}, \tilde{y})
\]

from which one extracts the maps \( F_{x,y} : [x, y] \to \langle F(x), F(y) \rangle \). These maps satisfy the composition and unit axioms for \( V \)-enriched functors \([3.16]\) because the diagrams

\[
\begin{array}{ccc}
B(\tilde{x}) & \longrightarrow & R^0(\tilde{x}) \\
\downarrow & & \downarrow f^0 \\
C(\tilde{x}) & \longrightarrow & S^0(F(x))
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
R^1(\langle x, y \rangle) & \longrightarrow & R^2(\langle x, y \rangle) \\
\downarrow f^1 \times f^1 & & \downarrow f^2 \\
S^1(\langle F(x), F(y) \rangle) & \longrightarrow & S^2(\langle F(x), F(y) \rangle)
\end{array}
\]

commute and because both \( f^0 \) and \( f^2 \) can be rewritten using the components \( F_{x,y} \) \([3.27]\). From this process one receives a \( V \)-enriched functor. One can check that this process is inverse to the one sending \( V \)-enriched functors to op-lax functors using the usual arguments in addition to the remark above that all maps \( f^p \) can be recovered from \( f^1 \) \([3.27]\).

The proof of the 2-fully faithfulness can be obtained similarly. \( \square \)

### 3.5 Categories enriched over \( V \) vs \( V \)-categories

As mentioned at the beginning of this section, the definition of a category \( C \) enriched over an op-lax monoidal category \( V^\omega \) differs from the standard definition of a \( V \)-category in that the former consists in a category together with additional structure, whereas the latter is a collection of objects and hom-objects together with composition maps, from which an underlying category is recovered.

**Definition 3.29 (\( V \)-category).** A \( V \)-category \( C \) is the data of

- a set of objects \( \text{Ob}(C) \);
- an object \( [x, y] \) of \( V \) for each pair of objects \( x, y \in \text{Ob}(C) \);
- for each \( x \in \text{Ob}(C) \), \( 1_x \xrightarrow{\eta_x} [x, x] \) a unit map;
- for all triplets of objects \( x, y, z \in \text{Ob}(C) \), morphisms

\[
[y, z] \otimes_{\omega} [x, y] \xrightarrow{\mu_{x,y,z}} [x, z]
\]

such that:
ASSOCIATIVITY  

The associativity diagram in $\mathcal{V}$

\[
\begin{array}{ccc}
[y, z] \otimes [x, y] \otimes [w, x] & \xrightarrow{\mu_{x,y,z}} & [x, z] \otimes [w, x] \\
\downarrow & & \downarrow \\
[y, z] \otimes [x, y] & \xrightarrow{\mu_{w,x,y}} & [y, z] \otimes [w, y] \\
\end{array}
\]

commutes for every $w, x, y, z \in \text{Ob}(\mathcal{C})$;

UNITALITY  

The two unitality diagrams in $\mathcal{V}$

\[
\begin{array}{ccc}
1_\omega \otimes [x, y] & \xleftarrow{\eta_{x,y} \otimes \omega} & [y, y] \otimes [x, y] \\
\downarrow & & \downarrow \\
[y, y] \otimes [x, y] & \xrightarrow{\mu_{x,y,y}} & [x, y] \\
\end{array} \quad \text{and} \quad \begin{array}{ccc}
[x, y] \otimes 1_\omega & \xrightarrow{\omega \mu_{x,x,y}} & [x, x] \otimes [x, y] \\
\downarrow & & \downarrow \\
[x, x] \otimes [x, y] & \xleftarrow{\mu_{w,y,z}} & [x, y] \\
\end{array}
\]

commute for every $x, y \in \text{Ob}(\mathcal{C})$.

In other words, the previous definition of a $\mathcal{V}$-category $\mathcal{C}$ is almost identical to that of a category enriched over $\mathcal{V}$ given previously [3.13] except that we do not require $[\cdot, \cdot]$ to be a bifunctor but simply a map between pairs of objects in $\mathcal{C}$ to an object in $\mathcal{V}$ —the hom-objects—nor do we require the composition and unit maps $\mu$ and $\eta$ to be extra/natural transformations.

**Remark 3.30.** Note that one could adapt the extended definition of a category enriched over $\mathcal{V}$ given above [3.12] so as to obtain a $\mathcal{V}$-category with composition maps $m^n$ for all $n \geq -1$ which are not required to be extra/natural transformations. As in the case of categories enriched over $\mathcal{V}$, these two possible definitions (abridged vs. extended) turn out to be equivalent (due to the same mechanism). This has been previously remarked by Leinster [3, 2.2].

To each $\mathcal{V}$-category $\mathcal{C}$ one can associate the corresponding underlying category $\mathcal{C}'$ endowed with the same objects $x, y, \ldots$ and maps of the form $f: 1_\omega \to [x, y]$ as morphisms. Note that the composition of these underlying morphisms—and more importantly its associativity—is due to the comonoid structure of the oplax unit $1_\omega$ [1.20]. Similarly, one can define $\mathcal{V}$-functors as follows.

**Definition 3.31 (\(\mathcal{V}\)-functor).** A $\mathcal{V}$-functor $F: \mathcal{B} \to \mathcal{C}$ between two $\mathcal{V}$-categories $(\mathcal{B}, [-, -], \eta, \mu)$ and $(\mathcal{C}, (-, -), \theta, \nu)$ consists in the following data:
• a map which assigns to each object $x$ of $B$ an object $F(x)$ of $C$;
• a map which assigns to each pair of objects $x,y$ of $B$ a morphism

$$[x,y] \xrightarrow{F_{x,y}} \langle F(x), F(y) \rangle$$

in $\mathcal{V}$,
such that:

**UNIT** the unit diagram in $\mathcal{V}$

$$
\begin{array}{ccc}
1_{\omega} & \downarrow \theta_{F(x)} & \langle F(x), F(x) \rangle \\
\eta_x & & F_{x,x} \\
[x,x] & \rightarrow & \langle F(x), F(x) \rangle \\
\end{array}
$$

commutes for every $x \in \text{Ob}(B)$;

**COMPOSITION** the composition diagram in $\mathcal{V}$

$$
\begin{array}{ccc}
[y,z] \otimes_\omega [x,y] & \xrightarrow{F_{x,y} \otimes_\omega F_{x,y}} & \langle F(y), F(z) \rangle \otimes_\omega \langle F(x), F(y) \rangle \\
\mu_{x,y,z} & & \gamma_{F(x),F(y),F(z)} \\
[x,z] & \xrightarrow{F_{x,z}} & \langle F(x), F(z) \rangle \\
\end{array}
$$

commutes for every $x,y,z \in \text{Ob}(B)$.

Again, the definition of a $\mathcal{V}$-functor differs from the one of a $\mathcal{V}$-enriched functor by the fact that $F$ is not functorial and the components $F_{x,y}$ are not assumed to be natural.

Any $\mathcal{V}$-functor $F: B \to C$ induces an underlying functor $\bar{F}: \bar{B} \to \bar{C}$ between the corresponding underlying categories. The action of $\bar{F}$ is given by the map $x \mapsto F(x)$ on objects and associates to any underlying morphism $f: 1_{\omega} \to [x,y]$, the composite

$$1_{\omega} \xrightarrow{f} [x,y] \xrightarrow{F_{x,y}} \langle F(x), F(y) \rangle$$

interpreted as a morphism in $\bar{C}$.

In order to define natural $\mathcal{V}$-transformations, we shall focus on the case where the category $\mathcal{V}$ is normal so that the structure map $\omega^0(x) \to x$ is invertible for every object $x \in \text{Ob}(\mathcal{V})$. In the normal case, we recover the standard result that the bracket $[-,-]$ is bifunctorial on the underlying category.

**Proposition 3.32** ($[-,-]$ is a bifunctor). Let $\mathcal{V}$ be a normal oplax monoidal category. Let furthermore $(C, [-,-], \eta, \mu)$ be a $\mathcal{V}$-category and let $\bar{C}$ denote its underlying category. The map

$$[-,-]: C^{\text{op}} \times C \rightarrow \mathcal{V}$$

which assigns

• to any pair of objects $x,y \in \text{Ob}(C)$, the object $[x,y]$ of $\mathcal{V}$;
• to any pair of morphisms \( h_1 : 1_\omega \to [y_1, x_1] \) and \( h_2 : 1_\omega \to [x_2, y_2] \), the morphism

\[
[x_1, x_2] \xrightarrow{[h_1, h_2]} [y_1, y_2]
\]

defined as

\[
\begin{array}{ccc}
[x_1, x_2] & \xrightarrow{1_\omega \otimes_\omega [x_1, x_2] \otimes_\omega 1_\omega} & [x_2, y_2] \\
| & \downarrow{[h_1, h_2]} & | \\
& \xrightarrow{h_2 \otimes_\omega \otimes_\omega h_1} & [y_1, x_1] \\
& \xrightarrow{m^2} & [y_1, y_2]
\end{array}
\]

where the first diagonal arrow is either one of the composite

\[
\begin{array}{ccc}
[x_1, x_2] & \xrightarrow{1_\omega \otimes_\omega [x_1, x_2]} & [x_1, x_2] \\
| & \downarrow{1_\omega \otimes_\omega [x_1, x_2] \otimes_\omega 1_\omega} & | \\
& \xrightarrow{1_\omega \otimes_\omega [x_1, x_2] \otimes_\omega 1_\omega} & 1_\omega \otimes_\omega [x_1, x_2] \\
\end{array}
\]

which are identical due to the sequential decomposition conditions;

is a bifunctor.

**Proof.** Similar to the strong monoidal case [21].

**Remark 3.33.** Note that, while the bracket \([\cdot, \cdot]\) is not generically bifunctorial in the non-normal case, the composite \([\cdot, \cdot]_\omega\) is always a bifunctor. This fact arguably constitutes the most important departure from the standard case (wherein one enriches over a strong monoidal category). Indeed, when enriching over an oplax monoidal category, the assignment of a pair of objects \((x, y)\) of a \(\mathcal{V}\)-category \(\mathcal{C}\) to the corresponding hom-object \([x, y]\) of \(\mathcal{V}\) cannot be upgraded to a bifunctor unless the enrichment is normal.

**Definition 3.34 (Natural \(\mathcal{V}\)-transformations).** Let \(\mathcal{V}^\omega\) be a normal oplax monoidal category. Let furthermore \(F, G : B \to C\) be two \(\mathcal{V}\)-functors between two \(\mathcal{V}\)-categories \((B, [-, -], \eta, \mu)\) and \((C, [-, -], \Theta, \nu)\). A natural \(\mathcal{V}\)-transformation \(\alpha : F \Rightarrow G\) is the data of a map assigning to each object \(x\) of \(B\) a morphism

\[
1_\omega \xrightarrow{\alpha_x} \langle F(x), G(x) \rangle
\]

of \(\mathcal{V}\) such that the diagram

\[
\begin{array}{ccc}
[x, y] & \xrightarrow{F_{x,y}} & \langle F(x), F(y) \rangle \\
\downarrow{G_{x,y}} & & \downarrow{(-, \alpha_y)} \\
\langle G(x), G(y) \rangle & \xrightarrow{(\alpha_x, -)} & \langle F(x), G(y) \rangle
\end{array}
\]

commutes for every pair \(x, y \in \text{Ob}(B)\).
\(\mathcal{V}\)-categories, \(\mathcal{V}\)-functors and natural \(\mathcal{V}\)-transformations can be assembled into a 2-category that we shall denote \(\mathcal{V}\)-cat. The construction associating to each \(\mathcal{V}\)-category \(\mathcal{C}\) its underlying category \(\overline{\mathcal{C}}\) naturally extends into a 2-functor \(\mathcal{V}\)-cat \(\rightarrow\) \text{Cat}. In fact, the underlying category of a \(\mathcal{V}\)-category is also naturally endowed with an enrichment structure over \(\mathcal{V}\) due to the bifunctoriality of \([-,-]\), hence one gets a 2-functor 

\[
\mathcal{V}\text{-cat} \rightarrow \text{Cat}_\mathcal{V}
\]

whose image is made of normal categories.

**Definition 3.35** (Normal category enriched over \(\mathcal{V}\)). We shall say that a category \((\mathcal{C},[-,-])\) enriched over \(\mathcal{V}\) is normal if the canonical map

\[
\mathcal{C}(\xi) \rightarrow \mathcal{V}([-\xi\eta\eta_{\xi\eta},\xi])
\]

achieved by sending each \(f : x \rightarrow y\) to either of the equal composite maps

\[
\eta_x \quad [x,x] \quad [-f] \quad [x,y] \\
\eta_y \quad [y,y] \quad [f,-]
\]

is an isomorphism for every \(x,y \in \mathcal{C}\).

**Proposition 3.36.** Let \(\mathcal{V}\) be a normal oplax monoidal category. The 2-category \(\mathcal{V}\)-cat of \(\mathcal{V}\)-categories is a full reflective 2-subcategory

\[
\mathcal{V}\text{-cat} \rightarrow \text{Cat}_\mathcal{V}
\]

of the 2-category \(\text{Cat}_\mathcal{V}\) of categories enriched over \(\mathcal{V}\).

**Proof.** The reflector is simply given by forgetting the underlying category structure of a category enriched over \(\mathcal{V}\) and only remembering its underlying \(\mathcal{V}\)-categorical structure.

This results in an endofunctor \((\mathcal{C},[-,-]) \mapsto (\overline{\mathcal{C}},[-,-])\) of \(\text{Cat}_\mathcal{V}\). Using the construction described in the definition above, one obtains a unit functor

\[
(\mathcal{C},[-,-]) \xrightarrow{u_{\mathcal{C}}} (\mathcal{C},[-,-])
\]

which easily extends into a 2-natural map \(u\). Finally it is enough to observe that by construction, both \(u_{\mathcal{C}}\) and \(u_{\bar{\mathcal{C}}}\) are identities for every \(\mathcal{C}\). \(\square\)

**Remark 3.37.** When \(\mathcal{V}\) is normal, one has a 2-isomorphism

\[
\mathcal{V}\text{-cat} = \text{normal Cat}_\mathcal{V}
\]

between \(\mathcal{V}\)-categories and normal categories enriched in \(\mathcal{V}\). When \(\mathcal{V}\) is not normal, one can no longer use the 2-category of \(\mathcal{V}\)-categories. In this case, it becomes natural to replace the familiar notion of \(\mathcal{V}\)-category with the notion of normal category enriched over \(\mathcal{V}\).
3.6 An example: the theory of operads

Any category enriched over a monoidal category constitutes an example of an oplax enriched monoidal category. In the present subsection, we shall present a genuine example of enrichment for which the oplax monoidal category is not strong.

Let $\mathcal{V}^\otimes$ be a symmetric monoidal category with countable coproducts. Let $\text{Seq}^*(\mathcal{V})$ denote the category of reduced sequences in $\mathcal{V}$ i.e. the category whose objects are sequences $M(1), M(2), \ldots$ of objects of $\mathcal{V}$ indexed from 1.

The category of sequences $\text{Seq}^*(\mathcal{V})$ admits a (normal) oplax monoidal structure $[8, 2.17]$ with unit $1_\triangle := (1_{\mathcal{V}}, \emptyset, \emptyset, \ldots)$ and whose maps $\triangle^1$ and $\triangle^2$ are given by

\[
(M \triangle N)(n) := \bigsqcup_{p} \bigsqcup_{n_1 + \ldots + n_p = n} M(p) \otimes_{\mathcal{V}} N(n_1) \otimes_{\mathcal{V}} \ldots \otimes_{\mathcal{V}} N(n_p)
\]

\[
(M \triangle N \triangle O)(n) := \bigsqcup_{p,q} \bigsqcup_{p_1 + \ldots + p_q = p} \bigsqcup_{n_1 + \ldots + n_p = n} M(q) \otimes_{\mathcal{V}} N(p_1) \otimes_{\mathcal{V}} \ldots \otimes_{\mathcal{V}} N(p_q) \otimes_{\mathcal{V}} O(n_1) \otimes_{\mathcal{V}} \ldots \otimes_{\mathcal{V}} O(n_p)
\]

where all integers in the formulae are positive. The higher maps $\triangle^n$ can be obtained via the combinatorics of the wreath product of planar trees [22]. The structural decomposition natural transformations are obtained by permuta-

A monoid in $\text{Seq}^*(\mathcal{V})^\triangle$ is a reduced planar monochromatic operad in $\mathcal{V}$. In the case where the tensor structure of $\mathcal{V}$ commutes with countable coproducts, the oplax monoidal structure on reduced sequences becomes strong. It is a fundamental tool in the theory of operads.

Let $\mathcal{C}^\otimes$ be a monoidal $\mathcal{V}$-category with enrichment bifunctor $\langle -, - \rangle$. Then $\mathcal{C}$ is naturally a $\text{Seq}^*(\mathcal{V})$-category with enrichment bifunctor

\[
[x, y](n) := \langle x^\otimes n, y \rangle
\]

with unit maps $1_\triangle \to [x, x]$ given by the unit map of the $\mathcal{V}$-enrichment of $\mathcal{C}$ and whose composition maps

\[
[y, z] \triangle [x, y] \longrightarrow [x, z]
\]

arise simply from the composition maps

\[
\langle y^\otimes p, z \rangle \otimes_{\mathcal{V}} \langle x^\otimes n_1, y \rangle \otimes_{\mathcal{V}} \ldots \otimes_{\mathcal{V}} \langle x^\otimes n_p, y \rangle \longrightarrow \langle x^\otimes n, z \rangle
\]

for the tensor structure of $\mathcal{C}$.

For $P$ a reduced planar monochromatic operad in $\mathcal{V}$ viewed as a monoid in $\text{Seq}^*(\mathcal{V})^\otimes$, a morphism of monoids

\[
P \longrightarrow [\Lambda, \Lambda]
\]

is equivalent to endowing the object $\Lambda \in \text{Ob}(\mathcal{C})$ with a $P$-algebra structure. The theory of oplax enriched categories thus allows one to extend the theorem
algebras over operads can be viewed as representations of monoids’ [23, B.1.2] to the case of operads in symmetric monoidal categories that are not closed.

We have chosen to describe the case of reduced planar monochromatic operads for its simplicity. The same arguments extend directly to the general case of symmetric coloured operads with operations in arity 0 [22], though the formulæ for the definition of $\triangleleft$ become more involved.

4 FROM $\mathcal{V}$ TO $\text{Cat}_\mathcal{V}$

Now that we have defined the notion of enrichment over a given oplax monoidal category $\mathcal{V}$, we shall discuss how the enrichment structure varies with the enrichment base. A standard result for categories enriched over a monoidal category states that a lax monoidal functor $f : \mathcal{U} \to \mathcal{V}$ between two monoidal categories induces a 2-functor $f_* : \mathcal{U}\text{-cat} \to \mathcal{V}\text{-cat}$ between the associated categories of (normal) enriched categories [21, 24]. This result can be enhanced to the case where $\mathcal{U}$ and $\mathcal{V}$ are oplax monoidal (and the enrichments may not be normal). In this section, we shall construct a 2-functor $\text{Cat}_{(-)} : \text{Oplax}_{\text{lax}} \to 2\text{-Cat}$. The action of this 2-functor can be summarised in the following diagram.

After reviewing its definition [§ 4.1], we shall show that it is equipped with a lax monoidal structure, corresponding to the external product of enriched categories (i.e. the operation that, given two categories enriched over two different oplax monoidal categories, constructs a category enriched over the product of the latter) [§ 4.2]. Finally we shall deduce that when $D^\lambda_\omega$ is lax-oplax duoidal, the 2-category of $D^\omega_\omega$-enriched categories naturally inherits a lax monoidal structure [§ 4.3].

4.1 Pushforward

PUSHFORWARD OF LAX FUNCTORS Let $\mathcal{U}^\psi, \mathcal{V}^\omega$ be two oplax monoidal categories and let $f : \mathcal{U} \to \mathcal{V}$ be a functor endowed with a lax structure $(l^n)_{n \geq -1}$. We shall describe a 2-functor

$$f_* : \text{Cat}_\mathcal{U} \to \text{Cat}_\mathcal{V}$$

between the corresponding 2-categories of enriched categories. Let $(C, [-, -], \{m^n\})$ be a category enriched over $\mathcal{U}$. From it, we can define a category $(f_! C, [-, -], \{m^n\})$ enriched over $\mathcal{V}$ as follows:

- its underlying category $f_! C$ is the original category $C$;
- the enrichment bifunctor is given by

$$[-, -]_f := f \circ [-, -]$$

i.e. it is the composition of the enrichment bifunctor of $C$ and the lax functor $f$;
the composition maps $m^n_f$ are defined, using the functor $f$ and its lax monoidal structure, as

$$m^n_f : [-,\ldots,-]_{\omega^n f} \xrightarrow{l^n} [-,\ldots,-]_{f \psi^n} \xrightarrow{f(m^n)} [-,-]_f$$

for any integer $n \geq -1$.

The (extra)naturality conditions of $m^n_f$ follow straightforwardly from the (extra)naturality of $m$ and the naturality of $l^n$. The morphism $m^n_0$ verifies the counitality condition of the structure maps of a category enriched over $\mathcal{V}$, i.e. it identifies with the counit structure map $\omega^0 \rightarrow \text{Id}$, as a direct consequence of the unitality condition of the lax monoidal structure of $f$. Moreover, this family of morphisms also verifies the additivity conditions of the structure maps of a category enriched over $\mathcal{V}$, as a consequence of the additivity conditions of the lax monoidal structure $\{l^n\}_{n\geq 1}$ of $f$.

Next, let us define the action of $f$, on 1-morphisms, i.e. enriched functors. Given a $\mathcal{U}$-enriched functor $F : B \rightarrow C$ between two categories $(B,[-,-],\{m^p\})$ and $(C,\langle-,-\rangle,\{n^p\})$ enriched over $\mathcal{U}$, we define the $\mathcal{V}$-enriched functor $f,F : f,B \rightarrow f,C$ as the same underlying functor $F : B \rightarrow C$, equipped with the natural transformation with components

$$(f,F)_{x,y} := f(F_{x,y}) : [x,y]_F \rightarrow (f,F(x),f,F(y))_f$$

for any pair of objects $x,y$ in $B$. The naturality of the lax monoidal structure $l^n$ on the functor $f$, together with the fact that $F$ is a $\mathcal{U}$-enriched functor ensures that the above does define a $\mathcal{V}$-enriched functor. On top of that, it is clear from the functoriality of $f$ that $f_a$ intertwines the composition of $\mathcal{U}$-enriched and $\mathcal{V}$-enriched functors. Finally, a natural $\mathcal{U}$-enriched transformation $a : F \Rightarrow G$ between two natural $\mathcal{U}$-enriched functors $F,G : B \rightarrow C$ is automatically a natural $\mathcal{V}$-enriched transformation between the $\mathcal{V}$-enriched functors $f,F,f,G : f,B \rightarrow f,C$. Indeed, using the above definition, one finds that the $\mathcal{V}$-naturality condition between these $\mathcal{V}$-enriched functors is nothing but the image of the $\mathcal{U}$-naturality condition (by functoriality of $f$), so that we can simply define $f,\alpha = \alpha$.

It is also clear that this definition ensures that $f_a$ preserves the vertical and horizontal compositions of natural $\mathcal{U}$-enriched and $\mathcal{V}$-enriched transformations. As a consequence, we proved the claim made at the beginning of the present section that any lax monoidal functor $f : \mathcal{U} \rightarrow \mathcal{V}$ between two oplax monoidal categories induces a 2-functor $f_* : \text{Cat}_\mathcal{U} \rightarrow \text{Cat}_\mathcal{V}$, referred to as the pushforward of $f$.

**Pushforward of a Monoidal Transformation** Now let $t : f \Rightarrow g$ be a monoidal natural transformation between two functors $f,g : \mathcal{U}^0 \rightarrow \mathcal{V}^0$ with lax monoidal structure $\{k^n\}_{n\geq 1}$ and $\{l^n\}_{n\geq 1}$ respectively. We can define a 2-natural transformation $t_* : f_* \Rightarrow g_*$ between the previously defined pushforward of the lax monoidal functors $f$ and $g$ as follows: it assigns to any category $(\mathcal{C},[-,-],\{m^p\})$ enriched over $\mathcal{U}$, a $\mathcal{V}$-enriched functor $t_* : f,C \rightarrow g,C$.
whose underlying functor is the identity functor, and which is equipped with natural transformations with components

\[(t_C)_{x,y} := t_{[x,y]}: [x,y]_f \to [x,y]_g\]

for any pairs of objects \(x, y\) of \(C\). Indeed, the above does define a \(\mathcal{V}\)-enriched functor due to the monoidality (and naturality) of \(t\). Moreover, upon using the previous definition, one can check that \(t\) satisfies both the 1- and 2-naturality conditions.

**Proposition 4.1.** The operation which sends any oplax monoidal category \(\mathcal{V}\) to the 2-category \(\text{Cat}_\mathcal{V}\), and any lax monoidal functor and monoidal natural transformation to their pushforwards defines a 2-functor

\[\text{Oplax}_{lax}^{\mathcal{V}} \xrightarrow{\text{push}} \text{2-Cat}\]

from the 2-category of oplax monoidal categories to the one of 2-categories.

**Proof.** As we have previously proven that the pushforward of lax monoidal functors and monoidal natural transformations are respectively 2-functors and 2-natural transformations, what is left to show is that the pushforward preserves the composition of 1-cells, preserves the vertical and horizontal compositions of 2-cells and maps identities to identities. Doing so simply amounts to re-writing the various definitions detailed above and checking that they are compatible with the compositions in \(\text{Oplax}_{lax}\) and in \(\text{2-Cat}\), and does not require the use of any particular property other than the associativity of the composition of morphisms in a category. \(\square\)

### 4.2 External product of enriched categories

The category \(\text{Oplax}_{lax}\) admits products. If \(\mathcal{U}^\psi\) and \(\mathcal{V}^\omega\) are two oplax monoidal categories, their product is given by the category \(\mathcal{U} \times \mathcal{V}\) with structural maps

\[(\mathcal{U} \times \mathcal{V})^{n+1} \xrightarrow{\psi \times \omega} \mathcal{U}^{n+1} \times \mathcal{V}^{n+1} \xrightarrow{\mathcal{U} \times \mathcal{V}} \mathcal{U} \times \mathcal{V}\]

for every \(n \geq -1\) and decomposition natural transformations defined similarly.

**Definition 4.2.** The external product is the 2-functor

\[\text{Cat}_\mathcal{U} \times \text{Cat}_\mathcal{V} \xrightarrow{\otimes} \text{Cat}_{\mathcal{U} \times \mathcal{V}}\]

taking as input a \(\mathcal{U}\)-enriched category \((\mathcal{B}, \{-,-\}, \{l^n\})\) and a \(\mathcal{V}\)-enriched category \((\mathcal{C}, \langle -,- \rangle, \{m^n\})\) and returning the \(\mathcal{U} \times \mathcal{V}\)-enriched category \(\mathcal{B} \otimes \mathcal{C}\) defined as the product \(\mathcal{B} \times \mathcal{C}\) with enrichment bifunctor

\[(\mathcal{B}^{\text{op}} \times \mathcal{C}^{\text{op}}) \times (\mathcal{B} \times \mathcal{C}) \xrightarrow{\otimes} (\mathcal{B}^{\text{op}} \times \mathcal{B}) \times (\mathcal{C}^{\text{op}} \times \mathcal{C}) \xrightarrow{\otimes} \mathcal{B} \times \mathcal{C}\]

and composition maps \([l^n \times m^n]\).

The external product of a \(\mathcal{U}\)-enriched functor (resp. natural transformation) with a \(\mathcal{V}\)-enriched functor (resp. natural transformation) is defined similarly.
Proposition 4.3. The external product of enriched categories endows

\[ \text{Oplax}_\text{lax} \xrightarrow{\mathcal{V} \to \text{Cat}_\mathcal{V}} 2\text{-Cat} \]

with the structure of a lax monoidal 2-functor.

Proof. The above proposition can be seen as a direct generalisation in the oplax setting of the standard result according to which the pushforward 2-functor

\[ \text{Mon}_\text{lax} \xrightarrow{\mathcal{V} \to \text{Cat}_\mathcal{V}} 2\text{-Cat} \]

from the 2-category \( \text{Mon}_\text{lax} \) of monoidal categories and lax monoidal functors to the 2-category \( 2\text{-Cat} \) of 2-categories is lax monoidal. The proof in the oplax case is essentially the same as in the monoidal case for which we refer the interested reader to the detailed proof in Cruttwell’s PhD thesis [25]. □

4.3 Lax monoidal structure on \( \text{Cat}_\mathcal{D} \), when \( \mathcal{D} \) is lax-oplax duoidal

When \( \mathcal{D}^\lambda,\omega \) is a lax-oplax duoidal category, the 2-category of \( \mathcal{D}_\omega \)-enriched categories \( \text{Cat}_{\mathcal{D}} \) can be endowed with a lax monoidal structure whose structural 2-functors

\[ \times^n_{\lambda} : \text{Cat}_{\mathcal{D}} \times \cdots \times \text{Cat}_{\mathcal{D}} \xrightarrow{\mathcal{g}^n} \text{Cat}_{\mathcal{D} \times \cdots \times \mathcal{D}} \xrightarrow{(\lambda^n)} \text{Cat}_{\mathcal{D}} \]

are obtained by using the external tensor product of categories and the pushforward of the lax structure of \( \mathcal{D} \).

Theorem 4.4. Let \( \mathcal{D}^\lambda,\omega \) be a lax-oplax duoidal category. Then the lax structure \( \lambda \) induces a lax monoidal 2-category structure \( \times_{\lambda} \) on the 2-category \( \text{Cat}_\mathcal{D} \) of categories enriched over \( \mathcal{D}^\omega \) (i.e. it becomes a lax monoid in \( 2\text{-Cat} \)).

Proof. Lax-oplax duoidal categories can be identified with the lax monoids of the 2-category \( \text{Oplax}_\text{lax} \), meanwhile lax monoidal categories are the lax monoids of \( 2\text{-Cat} \). In the previous section we explained how the external tensor product endows the functor \( \mathcal{V} \mapsto \text{Cat}_\mathcal{V} \) with a lax structure.

Day and Street have shown that lax monoidal 2-functors map lax monoids to lax monoids [6], so that applying the pushforward 2-functor to \( \mathcal{D} \) endows \( \text{Cat}_\mathcal{D} \) with a structure of lax monoid in \( 2\text{-Cat} \) i.e. \( \text{Cat}_\mathcal{D} \) is a lax monoidal 2-category. □

Remark 4.5. In the case of categories enriched over \( \mathcal{D}^\omega \) with only one object (i.e. monoids in \( \mathcal{D}^\omega \)), one recovers the fact that the category of monoids in \( \mathcal{D}^\omega \) inherits a lax monoidal structure from \( \mathcal{D}^\lambda \) [2,5].

Remark 4.6. Note that the notion of lax-strong duoidal category identifies with the one of lax monoid in \( \text{Mon}_\text{lax} \). Hence, the fact that the 2-category of categories enriched over a lax-strong duoidal category is a lax monoidal 2-category can be readily obtained from the fact that the pushforward 2-functor

\[ \text{Mon}_\text{lax} \xrightarrow{\mathcal{V} \to \text{Cat}_\mathcal{V}} 2\text{-Cat} \]

maps lax monoids to lax monoids. The alternative notion of strong-oplax duoidal categories, i.e. a lax-oplax duoidal category for which the lax monoidal
structure is strong monoidal, can be equivalently characterised as a pseudo-monoid in $Oplax_{lax}$. Hence, from the fact that the pushforward 2-functor

$$Oplax_{lax} \xrightarrow{\psi \circ -\text{Cat}_V} 2\text{-Cat}$$

maps pseudo-monoids to pseudo-monoids follow that the 2-category of categories enriched over a strong-oplax duoidal category is strong monoidal.

5 Motivating Example: $(R^e, R^e)$-Bimodules

Let $R$ be a ring and let $R^e := R \otimes_Z R^{op}$ be its enveloping ring. In this section we shall endow the category of $(R^e, R^e)$-bimodules with a lax-strong duoidal structure whose bimonoids are the bialgebroids.

5.1 Strong monoidal structure on $(R^e, R^e)$-bimodules

The category of bimodules over the enveloping ring $R^e$ can naturally be endowed with a tensor structure $\otimes_{R^e}$ which can be lifted into a strong monoidal structure [§ 6.1].

Let us recall that (given a choice of strong monoidal structure on the category of abelian groups) one can describe the strong monoidal structure on the category of $R^e$-bimodules using integrals [9, 10] [26, IX.6]. One can write

$$A \otimes_{R^e} B \otimes_{R^e} \cdots \otimes_{R^e} Z := \int_{(r, s_1), \ldots, (r, s_n)} A_{(r, s_1)} \otimes_{(r, s_1)} B_{(r, s_2)} \otimes \cdots \otimes_{(r, s_n)} Z$$

where $(A, B, \ldots, Z)$ represents any given sequence of $(R^e, R^e)$-bimodules. Here, the integral is taken over elements $(r, s) \in R^e$ whose position as a subscript of a bimodule indicates whether the left or the right $R^e$-action is involved in the computation of the integral.

Remark 5.1. Given an $R^e$-ring $A$, i.e. a ring $A$ together with a morphism of rings $R^e \rightarrow A$, every $A$-module admits a structure of $(R, R)$-bimodule. Thus, the category of $A$-modules can be naturally enriched over the category of $(R^e, R^e)$-bimodules. For each pair of $A$-modules $(M, N)$, the enrichment bifunctor is given by $\text{Hom}_{Z}(M, N)$, with $(R^e, R^e)$-bimodule structure

$$\left[(r, r') \cdot f \cdot (s, s')\right](m) := r \cdot f \cdot (s \cdot m \cdot s') \cdot r'$$

for every pairs $(r, r'), (s, s') \in R^e$, every $f : M \rightarrow N$ and every element $m \in M$.

Notice that in general this enrichment is not normal: the set of weak morphisms between two $A$-modules $M$ and $N$ is in bijection with the set $\text{Hom}_{R^e}(M, N)$.

5.2 Lax monoidal structure on $(R^e, R^e)$-bimodules

On top of the (strong) monoidal structure $\otimes_{R^e}$, the category of $R^e$-bimodules can also be endowed with a strictly normal lax monoidal structure, the restricted tensor product $\times_{R} \subset \otimes_{R}$ of Sweedler [10] and Takeuchi [9].

The restricted tensor product is given by the formula

$$M \times_{R} N := \int_{s} M_{s} \otimes_{R} N_{s} := \int_{s} \int_{r} M_{r} \otimes_{s} N_{s}$$
where $\tilde{r}, \tilde{s}$ denotes elements of the opposite ring $R^\text{op}$, and $\int^s M_i \otimes_R N_i$ is the cointegral whose elements are the combinations $\sum_i m_i \otimes n_i \in M \otimes_R N$ for which $\sum_i m_i \cdot s \otimes n_i = \sum_i m_i \otimes n_i \cdot s$ for every $s \in R$. We shall extend this bifunctor into a fully-fledged lax monoidal structure.

### n-ary Tensors

Given a number of $(R^e, R^e)$-bimodules $(A, B, \ldots, Z)$, their restricted product shall be defined as

$$A \times_R B \times_R \cdots \times_R Z : = \int_{s_1, \ldots, s_n} \int_{r_1, \ldots, r_n} (r_1 A_{s_1}) \otimes (r_1 r_2 B_{s_1, s_2}) \otimes \cdots \otimes (r_n Z_{s_n})$$

which is a new $(R^e, R^e)$-bimodule, with left and right actions given by

$$(r, s) \cdot (a @ b @ \cdots @ y @ z) \cdot (r', s') = (r \cdot a \cdot r') @ b @ \cdots @ y @ (s \cdot z @ s'),$$

for any $(r, s), (r', s') \in R^e$, $a \in A$, $b \in B$, $y \in Y$ and $z \in Z$. Here as usual, $A @ \cdots @ Z$ denotes a chosen strong monoidal lift of the tensor product of abelian groups;

### Unit

The unit object is:

$$\times_R^1 := \text{End}(R) := \text{Hom}_Z(R, R)$$

wherein the $(R^e, R^e)$-bimodule structure of $\text{End}(R)$ is the one described in the previous subsection [5.1];

### Associators

When $q \neq -1$, the natural transformations $\times_R^p(-, \times_R^q, -) \implies \times_R^{p+q}$ are given by

$$\int_{t_1, \ldots, t_p} \int_{r_1, \ldots, r_q} i_1 A_{t_1} \otimes \cdots \otimes i_{p+1} A_{t_{p+1}} \otimes \cdots \otimes t_p Z_{u_p}$$

where $(A, \ldots, Z)$ is any sequence of $(p + q + 1)$ $(R^e, R^e)$-bimodules and where $i$ denotes the position of the insertion of $\times_R^q$ in $\times_R^p$. The first map amounts to the commutativity of integrals with the tensor product of abelian groups together with the distributivity of cointegrals with the same tensor. Lastly the second arrow is the usual distributivity $\int^s \implies \int^s \int^t$ of cointegrals over integrals;

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If $q = -1$, we define:

$$\text{End}(R) \times_R A \times_R \cdots \times_R Z \longrightarrow A \times_R \cdots \times_R Z$$

where $\sum_i \phi_i @ a_i @ \cdots @ z_i \longmapsto \sum_i \phi_i(1_R) @ a_i @ \cdots @ z_i$
\[
A \times_R \cdots \times_R Z \times_R \operatorname{End}(R) \to A \times_R \cdots \times_R Z
\]
\[
\sum_i a_i \otimes \cdots \otimes z_i \otimes \phi_i \mapsto \sum_i a_i \otimes \cdots \otimes \phi_i(1_R) \cdot z_i
\]
and
\[
A \times_R \cdots \times_R M \times_R \operatorname{End}(R) \times_R N \times_R \cdots \times_R Z \to A \times_R \cdots \times_R M \times_R N \times_R \cdots \times_R Z
\]
by
\[
\sum_i \cdots \otimes m_i \otimes \phi_i \otimes n_i \otimes \cdots \to \sum_i \cdots \otimes m_i \otimes (\phi_i(1_R) \cdot n_i) \otimes \cdots
\]
\[
= \sum_i \cdots \otimes (\phi_i(1_R) \cdot m_i) \otimes n_i \otimes \cdots
\]
for any \((R^e, R^e)\)-bimodules \(A, \ldots, M, N, \ldots, Z\). The proof that these maps are \((R^e, R^e)\)-linear is similar to the low arity case \([9, 2.2]\).

**Proposition 5.2.** The structure described above—which extends the restricted tensor product \(\times_R\)—is a (strictly normal) lax monoidal structure on the category of \((R^e, R^e)\)-bimodules.

**Proof.** Sequential and parallel composition axioms follow straightforwardly from integral calculus.

\[5.3\] Lax-strong duoidal structure on \((R^e, R^e)\)-bimodules

We are now going to describe how the strong monoidal structure \(\otimes_{R^e}\) and the lax monoidal structure \(\times_R\) form a lax-strong duoidal structure on the category of \((R^e, R^e)\)-bimodules. For this, we know that it is enough to describe two sequences of maps \(\{\chi^p\}_{p \geq 0}\) and \(\{w^p\}_{p \geq 0}\) \([2.8]\). The difficulty here is in checking that the maps are well defined, the additivity axioms are then straightforward to check.

\((w^p)\) Since \(R^e\) is generated by \((1, 1)\) as a \((R^e, R^e)\)-bimodule, in order to describe the coproducts \(w^p: R^e \to \times^p_{R^e} R^e\), it is enough to give the image of \((1, 1)\). The only constraint being that the image of \((1, 1)\) must be an element on which the left and the right action of \(R^e\) agree.

The map \(w^{-1}: R^e \to \operatorname{End}(R)\) sends \((1, 1)\) to the identity endomorphism and \(w^p\) sends \((1, 1) \in R^e\) to \((1, 1) \otimes \cdots \otimes (1, 1) \in R^e \times_R \cdots \times_R R^e\) for every \(n \geq 0\).

\((\chi^p)\) We shall follow the example of Takeuchi who described the map \(\chi^1\) \([9, 1.12]\). For the case \(p = -1\), the map
\[
\operatorname{End}(R) \otimes_{R^e} \operatorname{End}(R) \to \operatorname{End}(R)
\]
is given by composition of endomorphisms. The case \(p = 0\) is the identity transformation, so let \(p \geq 1\). Let \(M := (M^{(0)}, \ldots, M^{(p)})\) and
$N := (N^{(0)}, \ldots, N^{(p)})$ be two families of $(R^e, R^e)$-bimodules. Consider the composite $(R^e, R^e)$-bimodule map

$$\chi_p^R(\quad) \otimes_R (\otimes^p N)$$

$$\int_{a,b,x_1, \ldots, x_p} g_1 M_a^{(0)} \otimes x_1 x_2 M_1^{(1)} \otimes \cdots \otimes x_p M_b^{(p)} \otimes d N^{(0)} \otimes N^{(1)} \otimes \cdots \otimes b N^{(p)}$$

$$\chi_p^R(\quad) \otimes (\otimes^p N)$$

$$\int_{x_1, \ldots, x_p} (g_1 M_0^{(0)} \otimes_R N^{(0)}) \otimes (x_1 x_2 M_1^{(1)} \otimes_R N^{(1)}) \otimes \cdots \otimes (x_p M^{(p)} \otimes_R N^{(p)})$$

and let us denote it by $\phi_p$. Note that the middle step of this definition (twist) implicitly uses the fact that the symmetric structure on a monoidal category can be lifted to an action of the symmetric group on a strong monoidal category [27].

**Lemma 5.3.** The map $\phi_p$ induces a map

$$\chi_p^R(\quad) \otimes_R (\otimes^p N) \xrightarrow{\chi_p} \chi_p^R(\quad \otimes_R N)$$

of $(R^e, R^e)$-bimodules.

**Proof.** Since tensor products commute with integrals, one can swap the integral out of the formula for the left hand side of the map $\chi_p$ that we are going to define. Let $\sum_j m_i^{(j)} \otimes \cdots \otimes m_i^{(p)}$ be an element of $\chi_p^R(\quad)$, and let $n^{(0)} \otimes \cdots \otimes n^{(p)} \in \otimes^p N$. Then for every $0 \leq j \leq p - 1$ and every $c \in R$,

$$\cdots \otimes m_i^{(j)} \otimes \bar{c} n^{(j)} \otimes m_i^{(j+1)} \otimes n^{(j+1)} \otimes \cdots = \cdots \otimes m_i^{(j)} \otimes n^{(j)} \otimes m_i^{(j+1)} \otimes n^{(j+1)} \otimes \cdots = \cdots \otimes m_i^{(j)} \otimes n^{(j)} \otimes m_i^{(j+1)} c \otimes n^{(j+1)} \otimes \cdots = \cdots \otimes m_i^{(j)} \otimes n^{(j)} \otimes m_i^{(j+1)} c n^{(j+1)} \otimes \cdots$$

in the target of $\phi_p$, where summation symbols are omitted. This shows that $\phi_p$ factors through the quotient given by integration over $c_1, \ldots, c_p$ and that the map

$$\chi_p^R(\quad) \otimes_R (\otimes^p N) \xrightarrow{\chi_p} \otimes^p_R (\quad \otimes_R N)$$

induced by $\phi_p$ is well defined. Finally, the map $\chi_p$ can be obtained as the composition

$$\chi_p^R(\quad) \otimes_R (\int d \otimes^p N) \xrightarrow{\text{distr.}} \int d \chi_p^R(\quad) \otimes_R (\otimes^p N) \xrightarrow{\text{distr.}} \int d \otimes^p_R (\quad \otimes_R N)$$

involving the canonical distributivity of cointegrals over functors.
**Theorem 5.4.** The category of \((R^e, R^e)\)-bimodules endowed with the normal lax structure \(\times_{R^e}\), the strong structure \(\otimes_{R^e}\) and the maps \(\{w^p\}_{p \geq -1}\) and \(\{\chi^p\}_{p \geq -1}\) is a lax-strong duoidal category.

**Proof.** The fact that \(\{w^p\}_{p \geq -1}\) defines a comonoid structure follows straightforwardly from its definition (as repeated insertion of the unit \((1, 1)\) of \(R^e\)). Similarly, the unitality condition for the pair \((\chi^p, w^p)\) to define a lax monoidal structure on the functor \(\times_{R^e}^p\) with respect to \(\otimes_{R^e}\) follows simply from their definition. The associativity condition follows from the property of distributivity and commutativity of \(\text{co/integrals}\), as well as the fact that \(\text{Ab}^\otimes\) is symmetric monoidal and hence its lift as a strong monoidal category is equipped with an action of the symmetric group \([27]\). Finally, the additivity condition for the collection \(\{\chi^p \tau^{1,p}\}_{p \geq -1}\) to define a lax monoidal structure on the functor \(\times_{R^e}^p\) with respect to \(\{\times_{R^e}^p\}_{n \geq -1}\) is verified for the same reasons, while the unitality condition is trivial, as both \(\times_{R^e}^0\) and \(\chi^0\) are identities.

**Corollary 5.5.** The tensor product \(\otimes_{R^e}\) endows the category of \(\times_{R^e}\)-comonoids with a strong monoidal structure.

**Corollary 5.6.** The restricted tensor product \(\times_R\) endows the category of \(R^e\)-rings (the \(\otimes_{R^e}\)-monoids) with a strictly normal lax monoidal structure.

**Remark 5.7.** This last corollary was already proven by Day and Street \([28, 4.1]\).

### 5.4 Bialgebroids

**Definition 5.8 (Bialgebroid) [9, 13].** A bialgebroid over a ring \(R\) consists in a \((R^e, R^e)\)-bimodule \(A\) endowed with:

- Two morphisms of \((R^e, R^e)\)-bimodules

\[
\mu_A : A \otimes_{R^e} A \rightarrow A \quad \text{and} \quad \eta_A : R^e \rightarrow A
\]

making the following associativity and unitality diagrams commute;

- Two morphisms of \((R^e, R^e)\)-bimodules

\[
\delta_A : A \rightarrow A \times_R A \quad \text{and} \quad \epsilon_A : A \rightarrow \text{End}(R)
\]
making the following coassociativity and counitality diagrams

\[
\begin{array}{cccc}
A \times_R A & \xrightarrow{\delta_A} & A & \xrightarrow{\delta_A} \\
\delta_A \times_R \delta_A & \searrow & \delta_A & \swarrow \\
(A \times_R A) \times_R A & \xrightarrow{-\times_R \delta_A} & A \times_R (A \times_R A) & \xrightarrow{\epsilon_A \times_R -} \\
A \times_R A & \xrightarrow{id_A} & A \times_R \text{End}(R) & \xrightarrow{-\times_R \epsilon_A} \\
\end{array}
\]

commute.

In addition, the following compatibility diagrams

\[
\begin{array}{cccc}
A \otimes_R A & \xrightarrow{\mu_A} & (A \times_R A) \otimes_R (A \times_R A) & \xrightarrow{\eta_A} \\
\mu_A \otimes_R \delta_A & \downarrow & \delta_A \otimes_R \mu_A & \searrow \\
(A \otimes_R A) \times_R (A \otimes_R A) & \xrightarrow{\epsilon_A} & A \times_R \text{End}(R) & \xrightarrow{-\times_R \epsilon_A} \\
A \times_R A & \xrightarrow{\eta_A} & A \times_R \text{End}(R) & \xrightarrow{\epsilon_A} \\
\end{array}
\]

are also required to commute.

**Theorem 5.9.** Bialgebroids are the bimonoids of the lax-strong duoidal category \((R^e, R^e)\text{-bimod}^{\otimes_R, \delta_R, \eta_R}\).

**Proof.** This comes from the fact that both \(\times_R\) and \(\otimes_R\) are strictly normal, one then only needs to apply the reconstruction theorem for bimonoids [2.7], modulo the lift of \(\otimes_R\) from a monoidal structure to a strong one.

**Remark 5.10.** The fact that bialgebroids can be described as bimonoids in a lax-oplax duoidal category was considered in the book of Aguiar and Mahajan [17, 6.45] and its corollary—that they can be described as \(\times_R\)-comonoids in the category of \(R^e\)-rings—was proven by Day and Street [28, 4.3].

### 6 Comparison with Other Enrichment Theories

We conclude by displaying a few comments on the relations between the previously introduced theory of enrichment over an oplax monoidal category and...
the theories of enrichment over strong monoidal categories, multicategories, skew-monoidal categories and lax monoidal categories. The corresponding relations are schematised in the following figure [1].

\[
\begin{array}{cccc}
\text{Mon}_\text{lax} & \text{Strong}_\text{lax} & \text{Oplax}_\text{lax} & \text{Multi} \\
\downarrow & \downarrow & \\
\text{Skew}_\text{lax} & \text{Lax}_\text{lax} & \\
\end{array}
\]

**Figure 1**: Relations between enrichment theories

6.1 Enrichment over monoidal categories

We claimed in the introduction that enriching over an oplax monoidal base extends the notion of enrichment over a monoidal base. This is backed by the fact that the definitions of a \( \mathcal{V} \)-category are similar when \( \mathcal{V} \) is either endowed with a monoidal or an oplax monoidal structure.

However, it is not straightforward to relate the 2-category of monoidal categories and lax functors to the one of oplax monoidal categories and lax functors. Indeed, even though there is an obvious forgetful functor

\[
\text{Strong}_\text{lax} \rightarrow \text{Mon}_\text{lax}
\]

one needs a structural result to show the equivalence.

The equivalence was shown by Leinster [14, 3.2.2,3.2.3,3.2.4] using coherence theorems for generalised monoidal categories. Another way of obtaining this result is to use, on the one hand, the classical coherence theorem for monoidal categories to show that each monoidal category admits an essentially unique strong oplax lift given by any choice of parenthesising and, on the other hand, the reconstruction result for lax functors [1.7] given in the first section. This way, one can promote the equivalence result to a 2-equivalence by adding monoidal transformations as the 2-cells on each side.

This makes the theory of categories enriched over oplax monoidal categories a direct extension of the classical theory of categories enriched over a monoidal category, as claimed.

6.2 Enrichment over multicategories

We discussed how the definition of multicategories can be slightly generalised in order for the embedding of oplax monoidal categories into multicategories to become fully faithful [§ 1.8].

One can check that given a normal oplax monoidal category \( \mathcal{V}^\omega \), with associated multicategory \( \mathcal{M} \), the classical notion of category enriched in \( \mathcal{M} \) [3] coincides with the notion of \( \mathcal{V} \)-category we outlined in the present paper. Note the fact that higher arity composition maps also arise in a first definition of an enrichment over a multicategory. However, this extended definition can also be shown to be equivalent to an abridged one containing only a binary composition map and an identity map [3, 2.2], in complete parallel with what we observed for enrichment over oplax monoidal categories.
6.3 Enrichment over skew monoidal categories

Enriching over skew-monoidal categories is in many respects similar to enriching over oplax monoidal categories. Recall that a skew monoidal category [29] is a category endowed with a kind of monoidal structure in which the associator, left and right unitors may not be invertible. As for oplax monoidal categories, one can consider left normal skew-monoidal categories which are skew-monoidal categories wherein the left unitor is invertible.

Initially, the notion of enrichment over a skew-monoidal category \( V \), called a \( V \)-category, was defined by Street [30] in a way similar to that of enrichment over a strong monoidal category, namely as a collection of objects \( x, y, \ldots \) together with hom-objects \( [x, y] \) which belong to \( V \) and endowed with composition maps \( [y, z] \otimes [x, y] \to [x, z] \) and identity maps \( 1 \otimes [x, x] \) obeying some associativity and unitality conditions. However, one is faced with the difficulty that the enrichment structure \( [\cdot, \cdot] \) defines a bifunctor only when the enrichment base \( V \) is left normal. To bypass this problem, another notion of category enriched over a skew monoidal base, referred to as a skew \( V \)-category, was given by Campbell [4], consisting in a category together with a bifunctor \( [\cdot, \cdot] \) and requiring that the composition and identity maps are extra/natural.

Although the notions of skew monoidal categories and oplax monoidal categories both generalise the notion of monoidal category, they are not comparable in general. However when a skew monoidal category is left normal, we expect that its skew structure can be lifted to a normal oplax structure, as in the monoidal case. Whenever \( V^{\omega} \) is an oplax monoidal category admitting an underlying skew monoidal category \( V^{\otimes} \), the \( V^{\otimes, \omega} \)-categories of Street are isomorphic to the \( V^{\omega} \)-categories and the skew \( V^{\otimes, \omega} \)-categories of Campbell are the categories enriched over \( V^{\omega} \).

6.4 Enrichment over lax monoidal categories

Using chiral definitions to the ones given above, one can describe a theory of categories enriched over lax monoidal categories. This has been used for example by Batanin and Weber [15] in order to study higher operads.

Despite their apparent symmetry, there exist several important differences between the theories of lax and oplax enriched categories:

- lax monoidal categories do not generate multicategories and thus categories enriched over lax monoidal categories are not part of the general scheme of ‘categories enriched over multicategories’;
- the 0-part of the enrichment, that is the maps \( [x, y]_1 \to [x, y] \), actually encodes some information. Indeed, recall that a lax monoidal category possesses a unit natural transformation \( 1 \to \lambda^0 \) which sends any object into its image by the endofunctor \( \lambda^0 \), so that the previous map cannot be identified with the aforementioned unit and hence requiring its existence is non-trivial (unlike when the enrichment base is an oplax monoidal category, in which case \( [x, y]_\omega \to [x, y] \) can, and is, identified with the counit);
it is not possible to reconstruct the composition maps \( \mu^n : [x_0, \ldots, x_n]_\lambda \to [x_0, x_n] \) for \( n \geq 2 \) from the \( n \in \{-1, 0, 1\} \) ones. This stems from the fact that the additivity conditions [3.12] are replaced by

\[
\begin{array}{ccc}
[-, \ldots, -]_{\lambda^p(-, -)} \ar[r]^{\lambda^p(-, \mu^q, -)} \ar[d] & [-, \ldots, -]_{\lambda^p} \ar[d]^{\mu^p} \\
[-, \ldots, -]_{\lambda^p+q} \ar[r]^{\mu^p+q} & [-, -]
\end{array}
\]

which cannot be used as a definition of the higher arity composition maps in terms of the lower arity ones, due to the fact that the associator for lax monoidal categories (appearing as the left vertical arrow in the above diagram) is in the opposite direction compared to the associators of an oplax monoidal category.

Acknowledgements

The authors are grateful to Gabriella Böhm for useful correspondence.

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