Approximation by piecewise-regular maps
Marcin Bilski and Wojciech Kucharz

Abstract
A real algebraic variety $W$ of dimension $m$ is said to be uniformly rational if each of its points has a Zariski open neighborhood which is biregularly isomorphic to a Zariski open subset of $\mathbb{R}^m$. We prove that every continuous map from a compact subset of any real algebraic variety into a uniformly rational real algebraic variety can be approximated by piecewise-regular maps.

Keywords: real algebraic variety, piecewise-regular map, continuous rational map, approximation.
MSC: 14P05, 14P99.

1 Introduction

In this paper by a real algebraic variety we mean a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^m$, for some $m$, endowed with the Zariski topology and the sheaf of real-valued regular functions (cf. [2], [13], [14]). Each real algebraic variety is also equipped with the Euclidean topology induced by the standard metric in $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties refer to the Euclidean topology.

The problem of algebraic approximation of continuous maps between real algebraic varieties has been considered by several mathematicians (see [2], [7] and references therein). It is well known that continuous maps into nonsingular real algebraic varieties can be approximated by semialgebraic maps in the compact-open topology. This fact is in general false if we want to approximate by regular maps even for very simple varieties like spheres or projective spaces (cf. [4], [5], [2], [3]). Therefore various intermediate classes of maps have been investigated for approximation properties.

One of such classes is the class of continuous rational maps (see [12]) which on nonsingular varieties coincides with the class of regulus maps (also known as continuous hereditarily rational maps cf. [8], [11]). These maps have attracted a lot of attention in recent years (see [8], [10], [11], [13], [15], [16], [17] and references therein). It has turned out, for example, that every continuous map between spheres can be approximated by regulus maps (see [13]). However, not every continuous map from an arbitrary compact real algebraic variety into a sphere can be approximated by regulus maps (see also [13]).

The problem of approximation of continuous maps from arbitrary compact varieties into spheres has been recently studied in [1]. Such maps can always be approximated by quasi-regulous maps (of any class $C^k$) which are obtained from regulus ones by changing signs (of the components) on some subsets of their domains (cf. [1]).

In the present paper we approximate maps from arbitrary compact varieties into uniformly rational varieties (for definition see Section 2 below) which con-
stitute a large class of varieties containing spheres, Grassmannians (especially interesting from the point of view of the theory of vector bundles), rational non-singular real algebraic surfaces and many others. Relaxing assumptions on target varieties requires enlarging the class of approximating functions. Namely, we work with piecewise-regular maps introduced in [14] (for definition see Section 2 below) the class of which contains regulous and quasi-regulous maps mentioned above as proper subclasses (cf. [1]); approximating functions obtained in the present paper are neither regulous nor quasi-regulous so we do not generalize here the main results of [13] or [1]. But we do generalize Theorems 1.3, 1.5, 1.6 of [14] and their consequences, but not Theorem 1.4 of [14].

Let $L$ be any subset of $\mathbb{R}^n$ and let $W$ be a real algebraic variety. We say that $f : L \rightarrow W$ is of class $C^k$ if it is the restriction of some map of class $C^k$ into $W$ defined in an open neighborhood of $L$. We say that $f : L \rightarrow W$ is a $C^k$ piecewise-regular map if it is a piecewise-regular map and is of class $C^k$. Our main result is the following

**Theorem 1.1** Let $L$ be a compact subset of $\mathbb{R}^n$ and let $W$ be a uniformly rational real algebraic variety. Let $f : L \rightarrow W$ be a continuous map. Then $f$ can be approximated by $C^k$ piecewise-regular maps from $L$ to $W$, where $k$ is an arbitrary nonnegative integer.

Since every real algebraic variety is, by definition, isomorphic to a real algebraic subset of some $\mathbb{R}^n$, the notion of $C^k$ piecewise-regular map defined above on subsets of $\mathbb{R}^n$ has a natural extension to maps defined on subsets of real algebraic varieties. Thus in Theorem 1.1 the space $\mathbb{R}^n$ can be replaced by an arbitrary real algebraic variety.

Approximation, as in [14], is expressed by means of the compact-open topology of the space $C(L, W)$ of all continuous maps from $L$ to $W$. More precisely, by definition, a map $f \in C(L, W)$ can be approximated by $C^k$ piecewise-regular maps if every neighborhood of $f$ (with respect to the compact-open topology) in $C(L, W)$ contains a $C^k$ piecewise-regular map.

The organization of this paper is as follows. In Section 2 we present preliminary material including properties of piecewise-regular maps. In Section 3 the proof of Theorem 1.1 is given.

## 2 Preliminaries

**Definition.** Let $W$ be a real algebraic variety of dimension $n$. A Zariski open subset $W_0 \subset W$ is said to be special if it is biregularly isomorphic to a Zariski open subset of $\mathbb{R}^n$. The variety $W$ is said to be uniformly rational if each point of it has a special Zariski open neighborhood.

**Remark.** Clearly, any uniformly rational real algebraic variety is nonsingular of pure dimension. The question whether every nonsingular rational variety is uniformly rational remains open, see [9] and [13], p. 885, for the discussion
involving complex algebraic varieties.

Let us recall a generalization of the notion of regular map introduced in [14].

**Definition.** Let $V, W$ be real algebraic varieties, $X \subset V$ some (nonempty) subset, and $Z$ the Zariski closure of $X$ in $V$. A map $f : X \to W$ is said to be regular if there is a Zariski open neighborhood $Z_0 \subseteq Z$ of $X$ and a regular map $\tilde{f} : Z_0 \to W$ such that $\tilde{f}|_X = f$.

A *stratification* of a real algebraic variety $V$ is, by definition, a finite collection of pairwise disjoint Zariski locally closed subvarieties (some possibly empty) whose union equals $V$.

**Definition.** Let $V, W$ be real algebraic varieties, $f : X \to W$ a continuous map defined on some subset $X \subset V$, and $S$ a stratification of $V$. The map $f$ is said to be piecewise-$S$-regular if for every stratum $S \in S$ the restriction of $f$ to each connected component of $X \cap S$ is a regular map (when $X \cap S$ is non-empty). Moreover, $f$ is said to be piecewise-regular if it is piecewise-$T$-regular for some stratification $T$ of $V$.

Let us also recall the notion of nonsingular algebraic arc (cf. [14]). A subset $A$ of a real algebraic variety $V$ is said to be a nonsingular algebraic arc if its Zariski closure $C$ in $V$ is an algebraic curve (that is, $\dim(C) = 1$), $A \subset C \setminus \text{Sing}(C)$, and $A$ is homeomorphic to $\mathbb{R}$.

The following result coming from [14] (Theorem 2.9) will be useful in the sequel.

**Theorem 2.1** Let $V, W$ be real algebraic varieties, $X \subset V$ a semialgebraic subset and $f : X \to W$ a continuous semialgebraic map. Then the following conditions are equivalent:

(a) The map $f$ is piecewise-regular
(b) For every nonsingular algebraic arc $A$ in $V$ with $A \subset X$, there exists a nonempty open subset $A_0 \subset A$ such that $f|_{A_0}$ is a regular map.

**Corollary 2.2** Let $M \subset \mathbb{R}^n$ be any semialgebraic subset and let $f : M \to \mathbb{R}$ be a piecewise-regular function. Let $g : M \to \mathbb{R}$ be a continuous function such that $|f| = |g|$. Then $g$ is a piecewise regular function. In particular, the absolute value of every piecewise-regular function on $M$ is a piecewise-regular function.

**Proof.** Let $A$ be any nonsingular algebraic arc in $\mathbb{R}^n$ with $A \subset M$. In view of Theorem 2.1 it is sufficient to check that there exists a nonempty open subset $A_0 \subset A$ such that $g|_{A_0}$ is a regular map. If $A \subset f^{-1}(0)$, then $g$ is constant so it is regular. If $A$ is not contained in the zero-set of $f$ then there is a non-empty open subset $B$ of $A$ such that $g = f$ on $B$ or $g = -f$ on $B$. Since $f, -f$ are piecewise regular, by Theorem 2.1 there is a nonempty open subset $A_0$ of $B$ such that $g$ is regular on $A_0$. $\blacksquare$
Remark. Let $M \subset \mathbb{R}^n$ be any subset. Then the family of all piecewise-regular functions on $M$ constitutes a ring.

The class of functions defined below will be useful to us.

**Definition.** For any open subset $U$ of $\mathbb{R}^n$, let $C^k(U)$ denote the class of all functions $v : U \to \mathbb{R}$ for which $|v|^+ \in C^k(U)$.

The following fact from [1] (Lemma 2.5) will also be useful to us.

**Lemma 2.3** Let $U$ be an open subset of $\mathbb{R}^n$. Let $f \in C^k(U)$, where $l, k \in \mathbb{N}$ with $k \geq 1$ and $l \geq k + 1$. Then for every continuous function $g : U \to \mathbb{R}$ such that for every $x \in U$ either $g(x) = f(x)$ or $g(x) = -f(x)$, we have $g \in C^k(U)$.

## 3 Proof of Theorem 1.1

The following lemma is our main tool.

**Lemma 3.1** Let $S \subset \mathbb{R}^n$ be some bounded set and $k$ a nonnegative integer. Then for every open neighborhood $U$ of $\overline{S}$ in $\mathbb{R}^n$ there exist open semialgebraic neighborhoods $N_1 \subset \subset N_2 \subset \subset U$ of $\overline{S}$ and a piecewise-regular function $\beta : \mathbb{R}^n \to [0, 1]$ of class $C^k$ with the following properties:

1. $\partial N_2$ and $\partial N_1$ are unions of connected components of nonsingular algebraic subvarieties of $\mathbb{R}^n$ of pure codimension 1,
2. $\beta|_{\mathbb{R}^n \setminus N_2} = 0$ and $\beta|_{N_1} = 1$.

**Proof.** Choose an open neighborhood $U$ of $\overline{S}$ in $\mathbb{R}^n$. Without loss of generality we may assume that $\overline{U}$ is compact and semialgebraic. Approximate the continuous function $\text{dist}(. , S)$ by a nonnegative polynomial $P$ on $\overline{U}$ and pick $\varepsilon_2 > \varepsilon_1 > 0$ so that $P_1 = P - \varepsilon_1, P_2 = P - \varepsilon_2$ have the following properties: $\inf_{\partial U} P_1 > \inf_{\partial U} P_2 > 0$ and 0 is a regular value of $P_1$ and $P_2$, and $P_1|_{\overline{S}} < 0, P_2|_{\overline{S}} < 0$.

Now define $N_2 = \{x \in U : P_2(x) < 0\}, N_1 = \{x \in U : P_1(x) < 0\}$ and observe that, by the previous paragraph, (1) clearly holds true.

Let us construct $\beta$. Take $\delta$ with $\varepsilon_1 < \delta < \varepsilon_2$ such that 0 is a regular value of $P - \delta$. Put $F := (P_1 \cdot P_2)^{2^m}$ and $\alpha := \inf_{x \in (P = \delta) \cap U} F(x)$. Choose $\gamma \in (0, \alpha)$ to be a regular value of $F|_{U}$ and define $T = F^{-1}(\gamma)$. Then for every $x \in \partial N_1$ and $y \in \partial N_2$ we have that $x, y$ are in different connected components of $U \setminus T$. Indeed, suppose there is an arc connecting $x, y$ in $U$. We have $P(x) = \varepsilon_1$ and $P(y) = \varepsilon_2$ so there is a point $c$ in the arc such that $P(c) = \delta$. Hence, $F(c) \geq \alpha \geq \gamma$ and $F(x) = F(y) = 0$ so there is $d$ in the arc with $F(d) = \gamma$. This means that the arc intersects $T$, a contradiction.

Define $G = (F - \gamma)^{2^m}$ on $U$ for some nonnegative integer $m$. Then $G^{-1}(0) = T \cap U$. Now define $G_0$ on $U$ by setting $G_0 = -G$ on every connected component of $U \setminus T$ which has a nonempty intersection with $\partial N_2$ and $G_0 = G$ on the other connected components of $U \setminus T$, and $G_0|_{T \cap U} = 0$. By Lemma 2.3 we may assume that $m$ is so large that $G_0$ is of class $C^k$. Moreover, by Corollary
in view of the fact that \( G \) is piecewise-regular, we conclude that \( G_0 \) is also piecewise-regular.

Next define continuous functions

\[
G_1^+ := -|G_0 - \gamma^{2m}| + \gamma^{2m} \quad \text{and} \quad G_1^- := |G_1^+ + \gamma^{2m}| - \gamma^{2m}.
\]

By Corollary 2.2 these functions are piecewise-regular on \( U \). Finally, for \( i \geq 1 \)
define continuous functions

\[
G_{i+1}^+ := -|G_i^- - \gamma^{2m}| + \gamma^{2m} \quad \text{and} \quad G_{i+1}^- := |G_{i+1}^+ + \gamma^{2m}| - \gamma^{2m}.
\]

By inductive application of Corollary 2.2 these functions are piecewise-regular on \( U \).

By construction, we have \((G_0)^{-1}(\kappa \cdot \gamma^{2m}) \subset (G_1^+)^{-1}(\kappa \cdot \gamma^{2m})\) and

\[(G_i^+)^{-1}(\kappa \cdot \gamma^{2m}) \subset (G_{i+1}^+)^{-1}(\kappa \cdot \gamma^{2m})\]

for \( i \geq 1 \) and \( \kappa \in \{-1, 1\} \). Consequently, for every \( i \), we have \( G_i^-(x) = -\gamma^{2m} \)
for \( x \in \partial N_2 \) and \( G_i^-(x) = \gamma^{2m} \) for \( x \in \partial N_1 \).

From the previous paragraph it also follows that if at some point \( x \in U \), \( G_i^- \)
is not of class \( C_k \), then \(|G_i^-(x)| = \gamma^{2m}| \). Indeed, if \(|G_i^-(x)| = \gamma^{2m}| \), then, by
the previous paragraph, \(|G_j^-(x)| \neq \gamma^{2m} \neq |G_j^+(x)| \) for every \( j \leq i \). But \( G_0 \) is of
class \( C_k \) at \( x \) so \( G_j^- \) is of class \( C_k \) at \( x \) for every \( j \leq i \), by construction.

By the fact that \( \overline{U} \) is compact and again by construction, for \( i \) large enough,
\(|G_i^-(x)| \leq \gamma^{2m} \) for every \( x \in U \). Take such \( i \) and set \( \hat{G} = G_i^- \). From what we
have just proved we know that \( \hat{G} \) is of class \( C_k \) on \( U \) possibly outside \( \hat{G}^{-1}(\gamma^{2m}) \cup \hat{G}^{-1}(-\gamma^{2m}) \). Hence for large odd integer \( l \) and large integer \( r \) the function

\[
H = \frac{1}{(2\gamma^{2m})^r} \cdot ((\hat{G} - \gamma^{2m})^l + (2\gamma^{2m})^l)^r
\]

is of class \( C_k \) on \( U \) and satisfies \( 0 \leq H(x) \leq 1 \) for every \( x \in U \). Moreover,
\( H_{|\partial N_1} = 1 \), \( H_{|\partial N_2} = 0 \) and all partial derivatives of \( H \) up to any prescribed order
vanish at every point of \( \partial N_1 \cup \partial N_2 \). Therefore we can define \( \beta|N_2 \setminus N_1 = H|N_2 \setminus N_1 \)
and \( \beta = 0 \) on \( \mathbb{R}^n \setminus N_2 \), and \( \beta = 1 \) on \( N_1 \) \( \blacksquare \)

**Proof of Theorem 1.1.** Let \( \{E_i\}_{i \in I} \) be a finite family of special subsets of \( W \)
such that \( \bigcup_{i \in I} E_i = W \). Let

\[
c := \min \{\sharp J : J \subset I \land f (L) \subset \bigcup_{j \in J} E_j \}.
\]

The proof is by induction on \( c \).

If \( c = 1 \), then there is a Zariski open subset \( E \) of \( W \) such that \( f (L) \subset E \) and
there is a biregular isomorphism \( \phi : D \to E \), where \( D \) is a Zariski open subset
of some \( \mathbb{R}^m \). Then we have a continuous map \( h : L \to D \) such that \( f = \phi \circ h \).
Now it is sufficient to approximate \( h \) on \( L \), using the Weierstrass theorem, by
a polynomial map $\tilde{h}$ into $\mathbb{R}^m$. The map $\phi \circ \tilde{h}$ is a $C^k$ piecewise-regular map approximating $f$.

Let $c > 1$ and let $\{E_1, \ldots, E_c\}$ be a family of special subsets of $W$ such that $f(L) \subset E_1 \cup \ldots \cup E_c$. Note that $f$ can be extended to a continuous function (also denoted by $f$) from an open neighborhood of $\overline{T}$ to $W$ such that $f(\overline{T}) \subset E_1 \cup \ldots \cup E_c$, where $T$ is an open bounded neighborhood of $L$. Then $S = (f(\overline{T}))^{-1}(W \setminus E_c)$ has an open neighborhood $U$ in $\text{dom}(f)$ such that $f(U) \subset E_1 \cup \ldots \cup E_{c-1}$.

By Lemma 3.1 there are open semialgebraic neighborhoods $N_1 \subset \subset N_2 \subset \subset U$ of $S = \overline{T}$ and a piecewise-regular function $\beta : \mathbb{R}^n \to [0,1]$ of class $C^k$ with the following properties:

1. $\partial N_2$ and $\partial N_1$ are unions of connected components of nonsingular algebraic subvarieties of $\mathbb{R}^n$ of pure codimension 1,
2. $\beta|_{\mathbb{R}^n \setminus N_2} = 0$ and $\beta|_{N_1} = 1$.

Since $f(\overline{T}) \subset E_1 \cup \ldots \cup E_{c-1}$ then, by the induction hypothesis, there is a piecewise-regular map $f_1 : U \to W$ of class $C^k$ approximating $f|_U$ as close as we wish.

Let $A \subset \subset N_1$ be an open neighborhood of $S$. Note that $N_2 \setminus N_1 \subset (U \setminus A) =: B$ so $f(B \setminus \overline{T}) \subset E_c$. Shrinking $B$ if necessary (so that it still contains $N_2 \setminus N_1$) we have $f_1(B \cap T) \subset E_c$ because $f_1$ approximates $f$ on $U$. Recall that $E_c$ is special so there is a biregular map $\phi : D \to E_c$, where $D$ is a Zariski open subset of some $\mathbb{R}^m$. By the inclusion $f_1(B \cap T) \subset E_c$, we have a piecewise-regular map $h_1 : B \cap T \to D$ of class $C^k$ such that $f_1|_{B \cap T} = \phi \circ h_1$.

By the definition of $S$ and the choice of $A$, we have $f(\overline{T} \setminus A) \subset E_c$. Consequently, as above, there is a continuous map $h : \overline{T} \setminus A \to D$ such that $f|_{\overline{T} \setminus A} = \phi \circ h$. Now, by the Weierstrass theorem, $h$ can be approximated on $\overline{T} \setminus A$ by a polynomial map $h_2$ into $D$. Note that $B \cap T \subset \overline{T} \setminus A$ and observe that $h_1$ and $h_2$ are close to each other on $B \cap T$. Therefore $\tilde{h} = \beta \cdot h_1 + (1 - \beta) \cdot h_2$ is a piecewise-regular function of class $C^k$ from $B \cap T$ to $D$.

Finally we define a $C^k$ piecewise-regular approximation $\tilde{f}$ of $f$ on $T$ by the formula: $\tilde{f}|_{T \cap N_2} := \phi \circ h_2$ and $\tilde{f}|_{T \cap (N_2 \setminus N_1)} := \phi \circ \tilde{h}$ and $\tilde{f}|_{T \cap N_1} := f_1$ (clearly $T$ could have been chosen to be semialgebraic).

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M. Bilski: Department of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland.
E-mail: Marcin.Bilski@im.uj.edu.pl

W. Kucharz: Department of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland.
E-mail: Wojciech.Kucharz@im.uj.edu.pl