A New Class of Combinatorial Markets with Covering Constraints: Algorithms and Applications

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Abstract

We introduce a new class of combinatorial markets in which agents have covering constraints over resources required and are interested in delay minimization. Our market model is applicable to several settings including scheduling, cloud computing, and communicating over a network. This model is quite different from the traditional models, to the extent that neither do the classical equilibrium existence results seem to apply to it nor do any of the efficient algorithmic techniques developed to compute equilibria seem to apply directly. We give a proof of existence of equilibrium and a polynomial time algorithm for finding one, drawing heavily on techniques from LP duality and submodular minimization. We observe that in our market model, the set of equilibrium prices could be a connected, non-convex set (see figure below). To the best of our knowledge, this is the first natural example of the phenomenon where the set of solutions could have such complicated structure, yet there is a combinatorial polynomial time algorithm to find one. Finally, we show that our model inherits many of the fairness properties of traditional equilibrium models.

Figure 1: An example of a non-convex set of equilibria. See Table 2 for an explanation.

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1 Introduction

In a free market economy, prices naturally tend to find an “equilibrium” under which there is parity between supply and demand. The power of this pricing mechanism is well explored and understood in economics: it allocates resources efficiently since prices send strong signals about what is wanted and what is not, and it prevents artificial scarcity of goods while at the same time ensuring that goods that are truly scarce are conserved [38]. In addition, it ensures that the allocation of goods is Pareto optimal. Hence it is beneficial to both consumers and producers. Furthermore, equilibrium-based mechanisms have been designed even for certain applications which do not involve any exchange of money but require fairness properties such as envy-freeness and the sharing incentive property; a popular one being CEEI\(^1\) [41]. Today, with the surge of markets on the Internet, in which pricing and allocation are done in a centralized manner via computation, an obvious question arises: can we apply insights gained from traditional markets to these new markets to accrue similar benefits? This (in addition to other motivations) has led to a long line of work in the TCS community on the computation of economic equilibria; see Appendix A for more details.

In this paper, we define a broad class of market models that are appropriate for modeling several new markets, including scheduling, cloud computing, and bandwidth allocation in networks. A common feature of our markets is that these are resource allocation markets in which each agent desires a specific amount of resources to complete a task, i.e., each agent has a covering constraint. If the agent does not get all the resources requested, then she will not be able to complete the task and hence has no value for this partial allocation. With several agents vying for the same set of resources, a new parameter that becomes crucially important is the delay experienced by agents. This naturally leads to a definition of supply and demand, as well as pricing and allocation, based on temporal considerations.

We define an equilibrium-based model for pricing and allocation in these markets. Our model is fundamentally different from traditional market models: Each agent needs only a bounded amount of resources to finish her tasks and has no use for more, and her utility, which corresponds to the delay she experiences, also has a finite maximum value, i.e., her “utility function” satiates. On the other hand, traditional models satisfy non-satiation, i.e., no matter what bundle of goods an agent gets, there is a way of giving her additional goods so her utility strictly increases. Non-satiation turns out to be a key assumption in the Arrow-Debreu Theorem, which established existence of equilibrium in traditional markets. Despite this, we manage to give an existence proof for our model. Additionally, we prove that all the above-stated benefits of equilibria, including the fairness properties of CEEI, continue to hold for our model.

We next address the issue of computing equilibria in our model. Rubinstein [47] recently showed that computing an equilibrium in our general model is PPAD-hard. For this reason, we defined a sub-model for which we seek an efficient algorithm. This sub-model is of interest in applications, including the three mentioned above. However, it turns out that equilibria of this sub-model have a different structure than those of models for which polynomial time algorithms have been designed. For instance, we give examples in which the set of equilibrium prices is non-convex. Hence techniques used for designing polynomial time algorithms for traditional models, such as the primal-dual method and convex programming, are not applicable. Our algorithms are based on new ideas: we make heavy use of LP duality and the way optimal solutions to LPs change with changes in certain parameters. Submodular minimization, combined with binary search, is used as a subroutine in this process.

In summary, for the main applications stated above, our market-based model admits equilibrium prices, which can moreover be computed in polynomial time and we show that our algorithm is incentive compatible. In addition, the equilibrium allocations satisfy a range of fairness properties. Considering the many favorable properties of market-based models and the availability of massive computing power for computing equilibria, we believe they will play an important role in markets on the Internet.

\(^1\)Competitive Equilibrium with Equal Incomes
Organization. We define the market model and state our main results in Section 2. In this section, we define the notions of strong feasibility, under which we establish existence of equilibrium, and extensibility, which gives the sub-model for which we give a polynomial time algorithm. We also discuss properties of fairness and incentive compatibility of our solution. In Section 3 we describe the algorithm for a special case in a scheduling setting, in order to convey the main ideas. The algorithm in its full generality, and an overview of the analysis are in Section 4. Section 5 contains all the different examples referred to, and also a description of a run of the algorithm for some examples. The appendices contain more details on related work (A), special cases of our model (D), existence of equilibria (C), connection of our algorithm to Myerson’s ironing in the scheduling case (B), equilibrium characterization for the general model (E), proofs missing from the main paper (F), and fairness and incentive compatibility properties (G).

2 Model and Main Results

We introduce a combinatorial version of the well studied Fisher market model \[6, 19\]. In market \( M \), let \( A \) be a set of \( n \) agents, indexed by \( i \), and \( G \) be a set of \( m \) divisible goods, indexed by \( j \). We represent an allocation of goods to agents using the variables \( x_{ij} \in \mathbb{R}_+ \) for \( i \in A, j \in G \). Each agent \( i \in A \) wants to procure goods that satisfy a set of covering constraints, \( C \), where \( C \) is a set indexing the constraints (\( C \) is the same for all agents for ease of notation).

\[
\forall k \in C, \sum_{j \in G} a_{ijk} x_{ij} \geq r_{ik}, \quad \text{and} \quad \forall j \in G, x_{ij} \geq 0 \quad (CC(i))
\]

The objective of each agent is to minimize the “delay” she experiences, while meeting these constraints. We refer to the term \( d_{ij} \) as the delay faced by agent \( i \) on using good \( j \), and the terms \( r_{ik} \) as the “requirements”; \( d_{ij} \)s and \( r_{ik} \)s are assumed to be non-negative. Agent \( i \) wants an allocation that optimizes the following LP.

\[
\min \sum_{j \in G} d_{ij} x_{ij} \quad \text{s.t.} \quad \forall k \in C, \sum_{j \in G} a_{ijk} x_{ij} \geq r_{ik}, \quad \forall j \in G, x_{ij} \geq 0. \quad (Delay \ LP(i))
\]

We use the notation \( d_i := (d_{ij})_{j \in G}, r_i := (r_{ik})_{k \in C}, A_i := (a_{ijk})_{j \in G, k \in C}, x_i := (x_{ij})_{j \in G} \), and \( X := (x_i)_{i \in A} \). Although our results hold for any LP, the most interesting cases are when the constraints are covering constraints, i.e., the matrix \( A_i \) has only non-negative entries.

We will use a market mechanism to allocate resources. Let \( p_j \in \mathbb{R}_+ \) denote the price per unit amount of good \( j \), and assume agent \( i \) has a total budget of \( m_i \in \mathbb{R}_+ \). Then, as is standard in Fisher markets, the bundle \( x_i \) that the agent may purchase is restricted by,

\[
\sum_{j \in G} p_j x_{ij} \leq m_i. \quad (Budget \ constraint(i))
\]

Allocation \( x_i \) is an optimal allocation (bundle) of agent \( i \) relative to prices \( p := (p_j)_{j \in G} \), if it optimizes LP (Delay LP(i)) with an additional budget constraint (Budget constraint(i)). Each good has a given supply which, after normalization, may be assumed to be equal to 1. The allocation needs to be supply respecting, that is, it has to satisfy the supply constraints:

\[
\forall j \in G, \sum_{i \in A} x_{ij} \leq 1. \quad (Supply \ constraints)
\]

Finally, a supply respecting allocation \( X \) and prices \( p \) are a market equilibrium of \( M \) iff

1. Each agent gets an optimal allocation relative to prices \( p \).
2. If some good \( j \in G \) is not fully allocated, i.e., \( \sum_{i \in A} x_{ij} < 1 \), then \( p_j = 0 \).

The equilibrium condition requires that each agent does the best for herself, regardless of what the other agents do or even what the supply constraints are. From the perspective of the goods, the aim is market clearing (rather than, say, profit maximization)\(^2\). Some goods may not have sufficient demand and therefore we may not be able to clear them. This is handled by requiring these goods to be priced at zero.

In Theorem 3 we obtain characterization of equilibrium in this general model in terms of solutions of a parameterized linear program that has one parameter per agent.

### 2.1 Existence of equilibria.

We show how the above model is a special case of the classic Arrow-Debreu market model with quasi-concave utility functions in Appendix C.1. Unfortunately, these utility functions do not satisfy the “non-satiation” condition required by the Arrow-Debreu theorem for the existence of an equilibrium: utility does not increase beyond a point even if additional goods are allocated. In fact, equilibrium doesn’t always exist for all covering LPs, as shown via a simple example in Section 5, Figure 3. And therefore next we identify conditions under which it does exist; the example in Figure 3 shows how this condition is necessary.

The equilibrium condition requires at a minimum that there exists a supply respecting allocation that also satisfies \( CC(i) \) of all the agents. In fact, it is easy to see that a somewhat stronger feasibility condition is necessary: suppose that a subset of agents all have high budgets while the remaining agents have budgets that are close to 0. Then at an equilibrium, agents in the former set get their “best” goods, which means that whatever supply remains must be sufficient to allocate a feasible bundle for the remaining agents.

We require a similar condition for all minimally feasible allocations, i.e., an allocation \( x_i \) such that reducing amount of any good would make \( CC(i) \) infeasible. We call this condition strong feasibility.

**Definition 1 (Strong feasibility).** Market \( M \) satisfies strong feasibility if any minimally feasible and supply respecting solution to a subset of agents can be extended to a feasible and supply respecting allocation to the entire set. Formally, \( \forall S \subseteq A, \) and \( \forall (x_i)_{i \in S} \) that are minimally feasible for \( (CC(i))_{i \in S} \) and are supply respecting (with \( x_{ij} = 0 \) \( \forall i \in S^c \)), \( \exists \) solutions \( (x_i)_{i \in S^c} \) that are feasible for \( (CC(i))_{i \in S^c} \) and \( (x_i)_{i \in (S \cup S^c)} \) is supply respecting.

**Theorem 1.** [Strong feasibility implies existence of an equilibrium] If \( (CC(i))_{i \in A} \) of market \( M \) satisfies strong feasibility, then \( \exists \) an allocation \( X \) and prices \( p \) that constitute a market equilibrium of \( M \).

The proof of this theorem is in Appendix C. Strong feasibility is quite general in the following sense: it is satisfied if there is a “default” good that has a large enough capacity and may have a large delay but occurs in every constraint with a positive coefficient. In other words, any agent’s covering constraints may all be met by allocating sufficient quantity of the default good.

### 2.2 Efficient computation.

Ideally we would want to design an efficient algorithm for markets with Strong feasibility condition, however this problem turns out to be PPAD-hard [47]. In order to circumvent this hardness, we define a stronger condition called extensibility, and design a polynomial time algorithm to compute a market equilibrium under it. Extensibility requires that any “optimal allocation” to a subset of agents can be “extended” to an “optimal allocation” for a set that includes one extra agent. Hence this is a matroid like condition. For this we first formally define “optimal allocation” for a subset of agents.

\(^2\)However, our algorithm will find an equilibrium where every agent spends all of her budget, and thereby it maximizes the profit automatically.
Definition 2. For any subset of agents $S \subseteq A$, we say that an allocation $X$ is **jointly optimal for $S$** if (i) it satisfies (CC($i$))$_{i \in S}$, (ii) it is supply respecting, and (iii) it minimizes $\sum_{i \in S} d_i \cdot x_i$. (Observe that $X$ may not be optimal for individual agents in $S$.)

Definition 3 (Extensibility). Market $\mathcal{M}$ satisfies **extensibility** if $\forall S \subset A$, given an allocation $X$ that is jointly optimal for $S$, the following holds: for any $i \in S^c$, $\exists$ an allocation $X'$ that is jointly optimal for $S' = S \cup \{i\}$, while not changing the delay of the agents in $S$, i.e., $d_i \cdot x'_i = d_i \cdot x_i$, $\forall i \in S$. In other words, total delay cost of agents in $S'$ can be minimized without changing the delay cost of agents in $S$.

Extensibility seems somewhat stronger than strong feasibility, but the two conditions are formally incomparable; see example in Section 5, Figure 3. In Section 2.3 we show that extensibility condition captures many interesting problems as special cases. Another mild condition we need is that there is enough demand for goods from each agent, otherwise an agent with very little requirement but huge amount of money may drive everyone else out of the market.

Definition 4 (Sufficient Demand). Market $\mathcal{M}$ satisfies sufficient demand if under zero prices, the optimal bundle of each agent contains some good that is demanded more than its supply, i.e., for an optimal solution $x_i$ to (Delay LP($i$)), there exists a good $j$ such that $x_{ij} > 1$.

Even for very simple markets, e.g., Tables 1 and 2 in Section 5, the set of equilibria may turn out to be highly non-convex. Therefore techniques used to obtain polynomial time algorithms for traditional models are not applicable. In Section 4 we design a polynomial time algorithm by making a heavy use of parameterized LP, duality and submodular minimization, and obtain the following result.

Theorem 2. [Extensibility and sufficient demand implies polynomial time algorithm] There is a polynomial time algorithm that computes a market equilibrium allocation $X$ and prices $p$ for any market $\mathcal{M}$ that satisfies extensibility and sufficient demand.

Since the algorithm is quite involved, we first convey the main ideas through a special case of scheduling in Section 3. We show the run of the main algorithm on an example in Section 5, Figure 5.

2.3 Applications

As a consequence of the above theorem we get polynomial time algorithms for the following special cases. (The proofs that these satisfy extensibility are in Appendix D.)

Scheduling. The agents are jobs that require $d$ different types of machines, and the set of machines of type $k$ is $M_k$; the machines are the goods in the market. Each agent needs $r_{ik} \in \mathbb{R}_+$ units of machines in all of type $k$, which is captured by the covering constraint $\sum_{j \in M_k} x_{ijk} \geq r_{ik}, \forall k \in [d]$. All agents experience the same delay $d_{jk}$ from machine $j$ in type $k$. Assume that the number of machines in each type $k$ is greater than the total requirements of the agents, $\sum_{i \in A} r_{ik}$. In reality different machines in this model may represent actually different machines, or the same machine at different times. The main motivation for this problem is scheduling in the cloud computing context, but it also captures other client-server scenarios such as crowdsourcing. An earlier version of this paper designed an algorithm only for this case [20].

Even for this simple case with only one type, we observe that the set of equilibria may form a connected non-convex set. The non-convexity example shown in Section 5, Tables 1 and 2, are instances of this setting.

Restricted assignment with laminar families – Different arrival times. The above basic scheduling setting can be generalized to the following restricted assignment case, where job $i$ is allowed to be processed only on a subset of all the machines $S_{ik} \subseteq M_k$ for type $k$. We need the $S_{ik}$'s to form a laminar \(^3\) family within

\(^3\)A family of subsets is said to be **laminar** if any two sets $S$ and $T$ in the family are either disjoint, $S \cap T = \emptyset$, or contained in one another, $S \subseteq T$ or $T \subseteq S$. 

4
each type, and in addition, we require that the machines in a larger subset have lower delays. That is, if for some two agents $i, i' \in A$, $S_{i'k} \subset S_{ik}$ then $\max_{j \in S_{ik} \setminus S_{i'k}} d_{jk} \leq \min_{j' \in S_{i'k}} d_{j'k}$ for each type $k$. This helps to model the important condition in the cloud computing context that jobs may arrive at different times.

Network Flows. The goods are edges in a network, where each edge $e$ has a certain (fixed) delay $d_e$. Each agent $i$ wants to send $r_i$ units of flow from a source $s_i$ to a sink $t_i$, and minimize her own delay (which is a min-cost flow problem). We show that if the network is series-parallel and the source-sink pair is common to all agents, then the instance satisfies extensibility. This is similar to the basic scheduling example in that there is a sequence of paths of increasing delay, but the difference here is that we need to price edges and not paths. The difficulty is that paths share edges and hence the edge prices should be co-ordinated in such a way that the path prices are as desired. There are networks that are not series-parallel but still the instance satisfies extensibility. We show one such network (and also how our algorithm runs on it) in Figure 5 on page 16. For a general network, an equilibrium may not exist; we give such an example in Figure 3.

A generalization of all these special cases that still satisfies extensibility is as follows: take any number of independent copies of any of these special cases above. E.g., each agent might want some machines for job processing, as well as send some flows through a network or a set of networks, but have a common budget for both together. Our algorithm works for all such cases.

2.4 Properties of equilibria

Fairness: We first discuss an application of our market model to fair division of goods, where there are no monetary transfers involved. This captures scenarios where the goods to be shared are commonly owned, such as the computing infrastructure of a large company to be shared among its users. A standard fair division mechanism is competitive equilibrium from equal incomes (CEEI) [41]. This mechanism uses an equilibrium allocation corresponding to an instance of the market where all the agents have the same budget. This can be generalized to a weighted version, where different agents are assigned different budgets based on their importance.

The fairness of such an allocation mechanism follows from the following properties of equilibria shown in Appendix G.1 for the general model. 1. The equilibrium allocation is Pareto optimal; this an analog of the first welfare theorem for our model. 2. The allocation is envy-free; since each agent gets the optimal bundle given the prices and the budget, he doesn’t envy the allocation of any other agent. 3. Each agent gets a “fair share”: the equilibrium allocation Pareto-dominates an “equal share” allocation, where each agent gets an equal amount of each resource. This property is also known as sharing incentive in the scheduling literature [29]. 4. Incentive compatibility (IC): the equilibrium allocation is incentive compatible “in the large”, where no single agent is large enough to significantly affect the equilibrium prices. In this case, the agents are essentially price takers, and hence the allocation is IC. We also show a version of IC when the market is not large. We discuss this in more detail below.

Incentive Compatibility: In the quasi-linear utility model, an agent maximizes the valuation of the goods she gets minus the payment. In the presence of budget constraints, Dobzinski et al. [21] show that no anonymous IC mechanism can also be Pareto optimal, even when there are just two different goods. In the context of our model, a quasi-linear utility function specifies an “exchange rate” between delay and payments, and wants to minimize a linear combination of the two. We show in Appendix G.3 that the impossibility extends to our model via an easy reduction to the case of Dobzinski et al. [21].

In the face of this impossibility, we show the following second best guarantee in Appendix G.2. For the scheduling application mentioned above, we show that our algorithm as a market based mechanism is IC.

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4 Anonymity is a very mild restriction, which disallows favoring any agent based on the identity.
in the following sense: non-truthful reporting of $m_i$ and $r_i$s can never result in an allocation with a lower delay. A small modification to the payments, keeping the allocation the same, makes the entire mechanism incentive compatible for the model in which agents want to first minimize their delay and subject to that, minimize their payments.

The first incentive compatibility assumes that utility of the agents is only the delay, and does not depend on the money spent (or saved). Such utility functions have been considered in the context of online advertising [5, 23, 42]. It is a reflection of the fact that companies often have a given budget for procuring compute resources, and the agents acting on their behalf really have no incentive to save any part of this budget. In the fair allocation context (CEEI), this gives a truly IC mechanism, since the $m_i$s are determined exogenously, and hence are not private information.

The second incentive compatibility does take payments into account, but gives a strict preference to delay over payments. Such preferences are also seen in the online advertising world, where advertisers want as many clicks as possible, and only then want to minimize payments. The modifications required for this are minimal, and essentially change the payment from a “first price” to a “second price” wherever required.

3 Scheduling on a single machine

Our algorithm for the general setting is quite involved, therefore we first present it for a very special case in a scheduling setting mentioned in Section 2.3. The basic building blocks and the structure of the algorithm and the analysis are reflected in this case. In Section 5, we describe the run of this algorithm on the example in Table 1. We note that the formal proofs are given only for the general case and not for this section.

Suppose that there is just one machine and a good is this machine at a certain time $t \in \mathbb{Z}^+$, which we refer to as slot $t$. The set of goods is therefore $G = \mathbb{Z}^+$ and we index the goods by $t$ instead of $j$ as before. Further, assume that the delay of slot $t$ is just $t$, i.e., $\forall i \in A, d_{it} = t$. Each agent $i$ requires a certain number of slots to be allocated to her, as captured by the covering constraint $\sum_{t \in \mathbb{Z}^+} x_{it} \geq r_i$, for some $r_i \in \mathbb{Z}^+$. We denote the sum of the requirements over a subset $S \subseteq A$ of agents as $r(S) := \sum_{i \in S} r_i$. Recall that the budget of agent $i$ is $m_i$, and similarly $m(S) := \sum_{i \in S} m_i$. We will show that equilibrium prices are characterized by the following conditions.

1. The prices form a piecewise linear convex decreasing curve. Let the linear pieces (segments) of this curve be numbered $1, 2, \ldots, k, \ldots$, from right to left.

2. There is a partitioning of the agents into sets $S^1, S^2, \ldots, S^k, \ldots$, where the number of slots in $k$th segment is $r(S^k)$. Note that since $r_i$s are integers so are the $r(S^k)$s.

3. The sum of the prices of slots in $k$th segment equals $m(S^k)$.

4. For any $S \subseteq S^k$, the total price of the first $r(S)$ slots of the segment $\geq m(S)$, since otherwise these slots would be over demanded. This is equivalent to saying that the total price of the last $r(S)$ slots in this segment $\leq m(S)$.

The above only characterizes equilibrium prices. We will show that Conditions 3 and 4 imply that there exists an allocation of the slots in segment $k$ to the agents in $S^k$ such that both their requirements and budget constraints are satisfied. Such allocations can then be found by solving the following feasibility LP (1). In this LP, segment $k$ corresponds to the interval $[T^k, T^{k-1}]$.

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5For how this equilibrium characterization leads to an analogy with Myerson’s ironing for a special case of this setting, with $r_i = 1$ for all $i \in A$, see Appendix B.
Algorithm 1 Algorithm to compute market equilibrium for scheduling

1: \textbf{Input:} \(A, (m_i)_{i \in A}, (r_i)_{i \in A}\)
2: Initialize \(A' \leftarrow A, p_{\text{low}} \leftarrow 0, T^0 \leftarrow r(A) + 1, \forall t \geq T^0, p_t \leftarrow 0\) and \(k \leftarrow 1\)
3: \textbf{while} \(A' \neq \emptyset\) \textbf{do}
4: \(S^k \leftarrow \text{NextSeg}(p_{\text{low}}, A', (m_i)_{i \in A'}, (r_i)_{i \in A'})\)
5: \(\lambda_{S^k} \leftarrow \frac{2^{m(S^k) - p_{\text{low}}r(S^k)}}{r(S^k)(r(S^k) + 1)}\)
6: \(T^k \leftarrow T^{k-1} - r(S^k)\)
7: \(\forall t \in [T^k, T^{k-1}], \text{set } p_t \leftarrow p_{\text{low}} + (T^{k-1} - t)\lambda_{S^k}\)
8: Compute allocations \(x_i\) for all \(i \in S^k\) by solving LP (1)
9: Update \(p_{\text{low}} \leftarrow p_{T^k}, A' \leftarrow A' \setminus S^k\), and \(k \leftarrow k + 1\)
10: \textbf{end while}
11: Output allocations \(X\) and prices \(p\).

\[
\begin{align*}
\forall i \in S^k & : \sum_{t \in [T^k, T^{k-1}]} x_{it} \geq r_i \\
\forall i \in S^k & : \sum_{t \in [T^k, T^{k-1}]} p_t x_{it} \leq m_i \\
\forall t \in [T^k, T^{k-1}] & : \sum_{i \in S^k} x_{it} \leq 1 \\
\forall i \in S^k, \forall t \in [T^k, T^{k-1}] & : x_{it} \geq 0
\end{align*}
\]  

(1)

We now describe the algorithm, which is formally defined in Algorithm 1. It iteratively computes \(S^k\), starting from \(k = 1\): the last segment that corresponds to the latest slots is computed first, and then the segment to its left, and so on. Inductively, suppose we have computed segments numbered 1 up to \(k-1\). Let \(p_{\text{low}}\) be the price of the earliest slot in segment \(k-1\), and let \(A' = A \setminus \{S^1 \cup \cdots \cup S^{k-1}\}\). For any \(S \subseteq A'\), consider the sum of the prices of \(r(S)\) consecutive slots to the left of this slot, forming a line segment with slope \(-\lambda\) (see Figure 2); this sum is

\[
p_{\text{low}}r(S) + \lambda r(S)(r(S) + 1).
\]

![Figure 2: Prices on a Segment for set \(S \subseteq A'\)](image)

Then for any \(S\), one can solve for the \(\lambda\) where this would be equal to \(m(S)\); we define this as a function of \(S\).

\[
\lambda_S := \frac{2^{m(S) - p_{\text{low}}r(S)}}{r(S)(r(S) + 1)}.
\]
This implies that it will be more convenient to consider a partition of agents rather than a partition of goods. By abuse of description of an equilibrium, therefore some parts that are immediate in that setting may require a proof of goods she gets. Unlike in Section 3, there is no simple ordering among the goods that enables a geometric description of an equilibrium, therefore some parts that are immediate in that setting may require a proof here. Recall that the first step in Section 3 was to find an equilibrium characterization only in terms of prices. This used the geometry of the instance in order to partition the time slots into segments. More generally, it will be more convenient to consider a partition of agents rather than a partition of goods. By abuse of terminology, in this section, by “segment” we refer to a subset of agents. Each agent \(i \in A\) has a parameter \(\lambda_i\), that previously corresponded to the slope of the segment they were in. Similarly, now too, all agents in a

### Algorithm 2 Subroutine NextSeg\((p_{\text{low}}, A', (m_i)_{i \in A'}, (r_i)_{i \in A'})\)

1: Initialize \(\lambda^0 \leftarrow 0, \lambda^1 \leftarrow \max_{i \in A'} m_i\)
2: Define \(f_{\text{low}}(S) := m(S) - p_{\text{low}} r(S) - \lambda r(S)(r(S) + 1)/2\)
3: Set \(S^0 \in \arg \min_{S \subseteq A', S \neq \emptyset} f_{\text{low}}(S)\) and \(S^1 \in \arg \min_{S \subseteq A', S \neq \emptyset} f_{\text{low}}(S)\)
4: while \(S^0 \neq S^1\) do
   5:   Set \(\lambda^* \leftarrow \frac{\lambda^0 + \lambda^1}{2}\) and \(S^* \in \arg \min_{S \subseteq A', S \neq \emptyset} f_{\text{low}}(S)\)
   6:   if \(f_{\text{low}}(\lambda^*)(S^*) > 0\) then Set \(\lambda^0 \leftarrow \lambda^*\) and \(S^0 \leftarrow S^*\)
   7: else Set \(\lambda^1 \leftarrow \lambda^*\) and \(S^1 \leftarrow S^*\)
8:  end if
9: end while
10: Return \(S^0\)

The next segment is defined to be the one with the smallest slope:

\[
S^k = \text{NextSeg}(p_{\text{low}}, A', (m_i)_{i \in A'}, (r_i)_{i \in A'}) := \arg \min_{S \subseteq A'} \lambda_S.
\]

With this definition of the next segment, and with prices for the corresponding slots set to be linear with slope \(-\lambda_S\), it follows that Conditions 3 and 4 are satisfied.

It is not immediately clear how to minimize \(\lambda_S\); the function need not be submodular, for instance. The main idea here is to do a binary search over \(\lambda\), as defined in Algorithm 2. Consider the function \(f_{\text{low}}\lambda\) as defined in line 2 of this algorithm, and notice that \(f_{\text{low}}\lambda\) is decreasing in \(\lambda\). From the preceding discussion, it follows that the segment \(S^*\) we seek is such that \(f_{\text{low}}\lambda(S^*) = 0\) and \(\forall S \subseteq A, f_{\text{low}}\lambda(S) \geq 0\). This implies that \(S^*\) must minimize \(f_{\text{low}}\lambda(S)\) over all subsets of \(A'\). Thus, given any \(\lambda\) and a minimizer of \(f_{\text{low}}\lambda\), we can tell whether the desired \(\lambda_S\) is above or below this \(\lambda\), and a binary search gives us the desired segment. A minimizer of \(f_{\text{low}}\lambda\) can be found efficiently since this is (as we will show) a submodular function.

In addition to the feasibility of LP (1), the main technical aspect of proving the correctness of the algorithm is to show that each agent gets an optimal allocation. This follows essentially from showing Condition 1, that the prices indeed form a piecewise linear convex curve, or equivalently, that the \(\lambda\)'s form an increasing sequence. It is fairly straightforward to see that the running time of the algorithm is polynomial.

### 4 Algorithm under Extensibility

In this section we present the algorithm that proves Theorem 2; we will parallel the presentation in Section 3. We first present equilibrium characterization for the general model in Theorem 3 (complete proof is in Appendix E), and then describe the key ideas in designing the algorithm (the missing proofs and other details of this part are in Appendix F). We show run of our algorithm on a network example in Section 5, Figure 5.

Recall that in our general model, each agent has a delay function and a set of constraints on the bundle of goods she gets. Unlike in Section 3, there is no simple ordering among the goods that enables a geometric description of an equilibrium, therefore some parts that are immediate in that setting may require a proof here. Recall that the first step in Section 3 was to find an equilibrium characterization only in terms of prices. This used the geometry of the instance in order to partition the time slots into segments. More generally, it will be more convenient to consider a partition of agents rather than a partition of goods. By abuse of terminology, in this section, by “segment” we refer to a subset of agents. Each agent \(i \in A\) has a parameter \(\lambda_i\), that previously corresponded to the slope of the segment they were in. Similarly, now too, all agents in a
segment $S^k$ have the same $\lambda_i$. This will also correspond to the optimal dual variable for Budget constraint(i) in the agent’s optimization problem at the equilibrium.

Given a vector of $\lambda_i$s, denoted by $\lambda \in \mathbb{R}_{+}^{|A|}$, next we define a parameterized linear program and its dual. Intuition for this definition comes from the optimal bundle LP of each agent at given prices. In the following, $LP(\lambda)$ has allocation variables $x_{ij}$, the constraint $CC(i)$ for each agent $i$, and the supply respecting constraint for each good $j$. The corresponding dual variables are respectively $\alpha_{ik}$s and $p_j$s, where $p_j$ can be thought of as the price for good $j$.

$$\begin{align*}
\min & : \sum_{i} \lambda_i \sum_{j} d_{ij}x_{ij} \\
\text{s.t.} & : \sum_{j} a_{ijk}x_{ij} \geq \alpha_{ik}, \forall (i,k) \\
& : x_{ij} \leq 1, \forall j \\
& : x_{ij} \geq 0, \forall (i,j)
\end{align*}$$

$$\begin{align*}
\max & : \sum_{i,k} r_{ik}\alpha_{ik} - \sum_{j} p_j \\
\text{s.t.} & : \lambda_i d_{ij} \geq \sum_{k} a_{ijk}\alpha_{ik} - p_j, \forall (i,j) \\
& : p_j \geq 0, \forall j; \quad \alpha_{ik} \geq 0, \forall (i,k)
\end{align*}$$

Remarkably, the next theorem shows that the problem of computing an equilibrium reduces to solving the above LP and its dual for a right parameter vector $\lambda \in \mathbb{R}_{+}^{|A|}$.

**Theorem 3.** For a given $\lambda > 0$ if an optimal solution $X$ of $LP(\lambda)$ and an optimal solution $(\alpha, p)$ of $DLP(\lambda)$ satisfy Budget constraint(i) for all agents $i \in A$ with equality, then they constitute an equilibrium of market $M$.

We note that the proof of Theorem 3 uses only complementary slackness conditions of optimal bundle LP of each buyer, and therefore the theorem holds for the most general model, i.e., without any of the extensibility, enough demand, or strong feasibility assumptions. Theorem 3 gives us the “geometry” of an equilibrium outcome, and is roughly equivalent to Condition 1 from Section 3. It reduces the problem to one of finding a right parameter vector $\lambda$; however there is still the entire $\mathbb{R}_{+}^{|A|}$ to search from. As was done in Section 3, our main goal is to further reduce this task to a sequence of single parameter searches, each involving submodular minimization and binary search.

Theorem 3 is applicable when all agents spend exactly their money at a primal-dual pair of optimal solutions for a given vector $\lambda$. Now the question is to characterize such parameter vectors $\lambda$. Note that, there really is no equivalent of Condition 2 from Section 3, since some goods may be allocated across agents in different segments. This is the source of many of the difficulties we face. Next, in Lemma 2, we derive an (approximate) equivalent of Conditions 3 and 4 from Section 3. This guarantees the existence of (allocation, prices) that satisfy the budget constraints of agents. One difference here is that this is going to be a global condition that involves the entire vector $\lambda$, rather than a local condition that we could apply to a single segment like in Section 3. For this we need a number of properties of optimal solutions of $LP(\lambda)$ and $DLP(\lambda)$ that we show in Lemma 1 next.

Let us define $\text{delay}_i(X) = \sum_{j} d_{ij}x_{ij}$ and $\text{pay}_i(p, X) = \sum_{j} p_jx_{ij}$. Similarly, for a subset of agents $S \subseteq A$, $\text{delay}_S(X) = \sum_{i \in S} \sum_{j} d_{ij}x_{ij}$ and $\text{pay}_S(p, X) = \sum_{i \in S} \sum_{j} p_jx_{ij}$. Let $[d]$ denote the set $\{1, \ldots, d\}$ of indices. By abuse of notation, let us define

$$\lambda(S) = \begin{cases} \\
\text{the } \lambda \text{ value of agents in } S & \text{if all agents in } S \text{ have the same } \lambda_i. \\
\text{undefined} & \text{otherwise} \\
\end{cases}$$

Using extensibility, in the next lemma we show that optimal solutions of $LP(\lambda)$ and $DLP(\lambda)$ satisfy some invariants regarding delays and payments of agents, e.g., we will show that higher the $\lambda$ the better the delay at primal optimal. For a fixed dual optimal the total payment of a segment remains fixed at all optimal allocations, and as the delay of a subset decreases, its payment increases. Recall Definition 2 for “jointly optimal for”.

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Lemma 1. Given \( \lambda \), partition agents by equality of \( \lambda_i \) into sets \( S_1, \ldots, S_d \) such that \( \lambda(S_1) < \cdots < \lambda(S_d) \).

1. At any optimal solution \( X \) of \( \text{LP}(\lambda) \) delay is minimized first for set \( S_d \), then for \( S_{\left(d-1\right)} \), and so on, finally for \( S_1 \). This is equivalent to \( X \) being jointly optimal for each \( T_g, \forall g \in [d] \) where \( T_g = \cup_{q=g}^d S_q \), and for any other optimal solution \( Y \) we have \( \text{delay}_{S_g}(Y) = \text{delay}_{S_g}(X), \forall g \in [d] \).

2. Given two dual optimal solutions \( (\alpha, p) \) and \( (\alpha', p') \), if the first part of dual objective is same at both for some \( g \in [d] \), i.e., \( \sum_{i \in S_g, k} r_{ik} \alpha_{ik} = \sum_{i \in S_g, k} r_{ik} \alpha_{ik}' \), then for any optimal solution \( X \) of \( \text{LP}(\lambda) \), \( \text{pay}_{S_g}(X, p) = \text{pay}_{S_g}(X, p') \).

3. Given two optimal solutions \( X \) and \( X' \) of \( \text{LP}(\lambda) \), and an optimal solution \( (\alpha, p) \) of \( \text{DLP}(\lambda) \), if for any subset \( S \subseteq S_g \) for \( g \in [d] \), \( \text{delay}_S(X) \leq \text{delay}_S(X) \), then \( \text{pay}_S(X, p) \geq \text{pay}_S(X', p) \). The former is strict if the latter is strict too.

In the above lemma, the first claim follows from extensibility. The second and third claim follow from the first claim together with the fact that any pair of primal and dual optimal satisfies complementary slackness.

Recall Conditions 3 and 4 of Section 3 requiring respectively budget balanceness, and that when a subset of agents in a segment are given the “best” allocation, their total payment should be at least their total budget (or else they will over demand some good). Using the first and last part of Lemma 1 the latter can be roughly translated to saying that when the rest of the agents are given the “worst” allocation, the rest underpay in total. Based on this intuition next we define conditions budget balance (BB) and subset condition (SC) in the following.

Definition 5. Given \( (\lambda, p) \), and a set \( S \subseteq A \), we say that condition

- BB is satisfied: If at any optimal solution \( X \) of \( \text{LP}(\lambda) \), we have \( \text{pay}_S(X, p) = m(S) \).
- SC is satisfied: \( \forall T \subseteq S \) let \( X \) to be an optimal solution of \( \text{LP}(\lambda) \) where \( \text{delay}_T \) is maximized. Then, \( m(T) \geq \text{pay}_T(X, p) \).

We will show that if BB and SC are satisfied for each “segment” at any given \( \lambda > 0 \) then \( \lambda \) is the right parameter vector. We will call such a \( (\lambda, p) \) proper, formally defined next.

Definition 6. We say that pair \( (\lambda, p) \) is proper if \( \exists \alpha \) such that \( (\alpha, p) \) is an optimal solution of \( \text{DLP}(\lambda) \), and pair \( \lambda, p \) satisfies BB and SC for subsets \( S_g, \forall g \leq d \), where \( S_1, \ldots, S_d \) is the partition of \( A \) by equality of \( \lambda_i \).

The next lemma shows that the parameter vector \( \lambda \) corresponding to a proper pair would ensure existence of allocation where each agent spends exactly her budget, and thereby will give an equilibrium using Theorem 3.

Lemma 2. If pair \( (\lambda^*, p^*) \) is proper for \( \lambda^* > 0 \) then there exists an optimal solution \( X^* \) of the primal \( \text{LP}(\lambda^*) \) such that \( \text{pay}_i(X^*, p^*) = m_i, \forall i \in A \).

Given such a \( \lambda^* \) and solution \( (\alpha^*, p^*) \) of \( \text{DLP}(\lambda^*) \) that satisfy conditions of Lemma 2, the above lemma ensures existence of allocation that satisfies Budget constraint(\( i \)), \( \forall i \in A \). We derive following feasibility LP in \( X \) variables to compute such an allocation.

\[
\begin{align*}
\hat{X} \text{ is a solution of } \text{LP}(\lambda^*): \quad & \sum_i \lambda_i^* \sum_j d_{ij} x_{ij} = \sum_i \lambda_i^* \sum_j d_{ij} \hat{x}_{ij} \\
\forall (i, k): \quad & \sum_j a_{ijk} x_{ij} \geq r_{ik} \\
\forall j: \quad & \sum_i x_{ij} \leq 1 \\
\forall i: \quad & \sum_j p^*_j x_{ij} = m_i \\
\forall (i, j): \quad & x_{ij} \geq 0
\end{align*}
\]
Algorithm 3 Algorithm to compute market equilibrium under extensibility

1: Input: \( A, (m_i)_{i \in A}, (\text{Delay LP}(i))_{i \in A} \)
2: Initialize \( A' \leftarrow A, p_{\text{cur}} \leftarrow 0, \lambda_{\text{cur}} \leftarrow 0 \) and \( k \leftarrow 1 \)
3: while \( A' \neq \emptyset \) do
4: \( (S^k, \lambda_{\text{new}}^{\text{new}}, p_{\text{new}}) \leftarrow \text{NextSeg}(\lambda_{\text{cur}}, p_{\text{cur}}, A', (m_i)_{i \in A}, (\text{Delay LP}(i))_{i \in A}) \)
5: \( A' \leftarrow A' \setminus S^k \), and \( k \leftarrow k + 1 \)
6: \( \lambda_{\text{cur}} \leftarrow \lambda_{\text{new}}^{\text{new}}, \) and \( p_{\text{cur}} \leftarrow p_{\text{new}}. \)
7: end while
8: Compute allocations \( \alpha \) satisfying Budget constraint(i) for all \( i \in A \), by solving LP (3) for \( p^* = p_{\text{cur}} \)
   and \( \lambda^* = \lambda_{\text{cur}}. \)
9: Output allocations \( X \) and prices \( p_{\text{cur}}. \)

Now our goal has reduced to finding a proper \((\lambda, p)\) pair. That is, if we think of partition of agents by equality of \( \lambda_i \) as “segments”, then we wish to find a vector \( \lambda \) such that BB and SC are satisfied for each “segment”. Our algorithm, defined in Algorithm 3, tries to fulfill exactly this goal. At a high level, like in Section 3, our algorithm will build the segments bottom up, i.e., “segment”. Our algorithm, defined in Algorithm 3, tries to fulfill exactly this goal. At a high level, like in Section 3, our algorithm will build the segments bottom up, i.e., lowest to highest \( \lambda \) segments. We will start by setting all the \( \lambda_i \) to same value, and find lowest \( \lambda \) value where BB and SC are satisfied for a subset. Once found we freeze this subset as a segment and start increasing \( \lambda \) for the rest to find the next segment, and repeat.

In this process of finding the next segment we need to make sure that BB and SC conditions are maintained for the previous segments. In Section 3 we were able to do this by simply fixing the prices of the goods in earlier segments, because goods were not shared across segments. Here, some of the goods allocated to agents in the earlier segments may also be allocated to agents in the later segments, and additionally these allocations are not fixed and may keep changing during the algorithm. (We fix the allocation only at the end.) Furthermore, the prices are required to be dual optimal w.r.t. the \( \lambda \) vector that we eventually find. On the other hand in order to maintain BB and SC conditions for the previous segments we need to ensure that the total payment of previous segments do not change.

The next lemma shows that this is indeed possible by proving that prices of goods bought by agents from previous segments can be held fixed. In fact, we will be able to fix \( \alpha_{ik} \) as well, for agents in the previous segments. The proof involves an application of Farkas’ lemma, leveraging extensibility. During computation of next segment, we will hold fixed the \( \lambda_i \) of agents in segments found so far, and increase the \( \lambda_i \) of the remaining agents. To facilitate this we define \( 1_S \in \{0,1\}^A \) as the indicator vector of \( S \subseteq A \), i.e., \( 1_S(i) = 1 \) if \( i \in S \), and is 0 otherwise.

Lemma 3. Given a \( \lambda \), partition agents into \( S_1, \ldots, S_d \) by equality of \( \lambda_i \), where \( \lambda(S_1) < \cdots < \lambda(S_d) \). For \( R \subseteq S_d \) consider primal optimal \( X \) that is jointly optimal for \( R \), and let \((\hat{\alpha}, \hat{p})\) be a dual optimal. Consider for some \( a > 0 \), the vector \( \lambda' = \lambda + a 1_R \). Then \( \hat{X} \) is optimal in LP(\( \lambda' \)) and there exists an optimal solution \((\alpha', p')\) of DLP(\( \lambda' \)) such that,

\[
\forall j: \quad \forall i \notin R, \forall k, \quad p'_{ij} \geq \hat{p}_j \quad \text{and} \quad \sum_{i \notin R} \hat{x}_{ij} > 0 \Rightarrow p'_{ij} = \hat{p}_j
\]

As discussed above our algorithm will build segments inductively from the lowest to highest \( \lambda \) value, by increasing \( \lambda \) of only the “remaining” agents. Suppose, we have built segments \( S_1 \) through \( S_{k-1} \), and let \( A' = A \setminus \bigcup_{i=1}^{k-1} S_i \) be the remaining agents. Let \( \lambda_{\text{cur}} \) be the current \( \lambda \) vector where \( \lambda_{\text{cur}}(S_1) < \cdots < \lambda_{\text{cur}}(S_{k-1}) < \lambda_{\text{cur}}(A') \). Let \( p_{\text{cur}} \) be the corresponding dual price vector which is optimal for DLP(\( \lambda_{\text{cur}} \)). For ease of notations we define the following.

For any \( a \geq 0 \), define \( \lambda^a = \lambda_{\text{cur}} + a 1_{A'}. \)
Fix an allocation $X^{\text{cur}}$ of $LP(X^{\text{cur}})$. We call an optimal solution $(\alpha, p)$ of $DLP(X^{\alpha})$ valid if prices are monotone w.r.t. $p^{\text{cur}}$ and $X^{\text{cur}}$, in the sense as guaranteed by Lemma 3 (where prices of goods allocated to previous segments are held fixed and prices of the rest of the goods are not decreased), and $\alpha_i$s are fixed for agents outside $A'$. We will call the corresponding prices valid prices. For simplicity we will assume uniqueness of valid prices.\footnote{This is without loss of generality since perturbing the parameters of the market ensures this. A typical way to simulate perturbation is by lexicographic ordering [50].}

Define $p^{\alpha}$ to be the valid price vector at dual optimal of $DLP(X^{\alpha})$ \hspace{1cm} (5)

Since the correctness is proved by induction, the inductive hypothesis is that w.r.t. $(\lambda^{\text{cur}}, p^{\text{cur}})$, both SC and BB are satisfied for $S_1, \ldots, S_{k-1}$, and SC is satisfied for the remaining agents $A'$. The base case is easy with the $\lambda_i$s all set to 0. Our next goal is to find the next segment $S_k \subseteq A'$, a new vector $\lambda^{\text{new}}$ and a new price vector $p^{\text{new}}$ such that the following properties hold.

1. Parameter vector $\lambda^{\text{new}}$ is obtained from $X^{\text{cur}}$ by fixing $\lambda_i$s of agents outside $A'$, increase $\lambda_i$s of agents in $S_k$ by the same amount, and those of agents $A' \setminus S_k$ by some more. The latter increase is to separate $S_k$ from $A'$. That is for some $a \geq 0$ and $\epsilon > 0$, $\lambda^{\text{new}} = \lambda^{\alpha} + \epsilon 1_{A' \setminus S_k}$.

2. Price vector $p^{\text{new}}$ is valid and optimal for $DLP(\lambda^{\text{new}})$.

3. W.r.t. $(\lambda^{\text{new}}, p^{\text{new}})$, $S_1, \ldots, S_k$ satisfy both BB and SC, and $A' \setminus S_k$ satisfies SC.

The computation of the next segment $S_k$ satisfying the above conditions is done by the subroutine NextSeg, which we describe next, and which is formally defined in Algorithm 4. As in Section 3, the basic idea is to reduce this problem to a single parameter binary search. Since Condition SC is satisfied for the

\begin{algorithm}
\caption{Subroutine NextSeg($\lambda^{\text{cur}}, p^{\text{cur}}, A', (m_i)_{i \in A}, (\text{Delay LP}(i))_{i \in A}$)}
\begin{algorithmic}[1]
\STATE Initialize $a^0 \leftarrow 0$, $a^1 \leftarrow \Delta$, where $\Delta = (\sum_{i,j,k} |a_{ijk}| + \sum_{i,k} |r_{ik}| + \sum_{i,j} |d_{ij}| + \sum_i |m_i|)^{2\min|C|}$.
\STATE Define function $f_a$ as in (7).
\STATE Set $S^0 \in \arg \min_{S \subseteq A', S \neq \emptyset} f_{a^0}(S)$ and $S^1 \in \arg \min_{S \subseteq A', S \neq \emptyset} f_{a^1}(S)$.
\WHILE{$S^0 \neq S^1$}
\STATE Set $a^* \leftarrow \frac{a^0 + a^1}{2}$ and $S^* \in \arg \min_{S \subseteq A', S \neq \emptyset} f_{a^*}(S)$.
\IF{$f_{a^*}(S^*) > 0$} \STATE Set $a^0 \leftarrow a^*$ and $S^0 \leftarrow S^*$.
\ELSE \STATE Set $a^1 \leftarrow a^*$ and $S^1 \leftarrow S^*$.
\ENDIF
\ENDWHILE
\STATE $S^* \leftarrow S^0$. Compute $a^*$ by solving feasibility LP for $S^*$ mentioned in Lemma 6 such that $f_{a^*}(S^*) = 0$.
\STATE $A' \leftarrow A' \setminus S^*$.
\WHILE{$A' \neq \emptyset$}
\STATE $S \leftarrow \arg \min_{S \subseteq A', T \neq \emptyset} f_{a^*}(S \cup S^*)$.
\IF{$f_{a^*}(S \cup S^*) > 0$} \STATE break.
\ELSE \STATE set $S^* \leftarrow S^* \cup S$, $A' \leftarrow A' \setminus S$.
\ENDIF
\ENDWHILE
\STATE Set $\lambda^{\text{new}} \leftarrow \lambda^{a^*}$, $\lambda_i^{\text{new}} \leftarrow \lambda_i^{\text{new}} + \epsilon 1_{A'}$, and $p^{\text{new}} \leftarrow$ valid price at $\lambda^{\text{new}}$, where $\epsilon \leftarrow \frac{1}{\Delta}$.
\STATE Return $(S^*, \lambda^{\text{new}}, p^{\text{new}})$.
\end{algorithmic}
\end{algorithm}
remaining agents $A'$ at $(\lambda^\text{cur}, p^\text{cur})$ while BB is not, total payment of $A'$ is less than their total budget $m(A')$. In order to keep track of this surplus budget, consider the following function on $S \subseteq A'$.

$$f_{\lambda, p}(S) = m(S) - \text{pay}_S(X, p),$$

where $X$ is an optimal solution to $LP(\lambda)$ that maximizes delay$_S$. (6)

We translate Condition 3 above in terms of this function, in the following lemma, which essentially reduces the problem to a single parameter search.

**Lemma 4.** Suppose that for some $a \geq 0$,

$$S_k \in \arg\min_{S \subseteq A', S \neq \emptyset} \{f_{\lambda^a, p^a}(S)\}.$$  

Further, suppose that $f_{\lambda^a, p^a}(S_k) = 0$, and $S_k$ be a maximal such set. Then there exists a rational number $\epsilon > 0$ of polynomial-size such that, w.r.t. $(\lambda^\text{new}, p^\text{new})$ as defined above, $S_1, \ldots, S_k$ satisfy both BB and SC, and $A' \setminus S_k$ satisfy SC.

The above lemma reduces the task of finding next segment to finding an appropriate $a$ such that the minimum value of $f_{\lambda^a, p^a}$ is zero under the valid price $p^a$. This requires two things: first we need to find a minimizer of $f_{\lambda^a, p^a}$ for a given $a > 0$, and second we need to find the right value of $a$. The next lemma shows that the first can be done using an algorithm for submodular minimization, and therefore in a polynomial time [51]. For convenience of notation, we define the following functions.

$$f_a(S) := f_{\lambda^a, p^a}(S) \quad \text{and} \quad g(a) := \min_{S \subseteq A', S \neq \emptyset} f_a(S)$$

(7)

**Lemma 5.** Given $a \geq 0$, function $f_a$ is submodular over set $A'$.

Now the question remains, how does one find an $a$ such that the minimum value is 0, i.e., $g(a) = 0$. We will do binary search for the same. In the next lemma we derive a number of properties of $g$ that facilitates binary search, under sufficient demand assumption (Definition 4), while crucially using Lemmas 1 and 3.

**Lemma 6.** Function $g$ satisfies the following: (i) $g(0) \geq 0$. (ii) $f_a(S)$ is continuous and monotonically decreasing in $a$, $\forall S \subseteq A'$, therefore $g$ is continuous and monotonically decreasing. (iii) $\exists a_h > 0$ a rational number of polynomial-size such that $g(a_h) \leq 0$, and $g$ has a zero of polynomial-size. (iv) Given a set $S \subseteq A'$, if $f_a(S) > 0$ and $f_{a'}(S) < 0$ for $a' > a > 0$, then $\exists a^* \geq 0$ such that $f_{a^*}(S) = 0$ and such an $a^*$ can be computed by solving a feasibility linear program of polynomial-size.

The first part follows essentially from the fact that $A'$ satisfies SC condition w.r.t. $(\lambda^\text{cur}, p^\text{cur})$ equivalently $(\lambda^0, p^0)$. For the second part, we show that for any $S \subseteq A'$, function $f_a(S)$ is monotonically decreasing and continuous in $a$. Since min of continuous and decreasing functions is also continuous and decreasing, we get the same property for $g$. It turns out that whatever be the current value of $g(a)$, there is another $a' > a$ where $g(a')$ is strictly smaller (using sufficient demand assumption). This strict decrease property ensures existence of $a$ where $g(a)$ is zero. Setting $a_h$ to higher than any such $a$ would give $g(a_h) \leq 0$ since $f_a(S)$ for all $S$ are monotonically decreasing in $a$, thereby we get the third part. Finally for the fourth part, existence of $a^*$ follows from monotonicity and continuity of $f_a(S)$ in $a$, and using the fact that complementary slackness ensures optimality we construct a feasibility linear program to compute $a^*$, given $S \subseteq A'$, such that $f_{a^*}(S) = 0$.

We initialize our binary search with a lower pivot $a_0 = 0$ and a higher pivot $a_1$ to a value that is guaranteed to be higher than where some set goes tight (third part of Lemma 6). Finally, since submodular minimization, binary search over a polynomial-sized range, and solving linear program all can be done in polynomial-time, we get our main result, Theorem 2, using Lemmas 2, 4 and 6, and Theorem 3.
5 Examples

In this section we show several interesting examples that illustrate important properties of our market and of equilibria. In addition we demonstrate a run of our algorithm on a routing example.

Non-existence of equilibria. Consider the networks (typically used to show Braess’ paradox) in Figure 3, where the label on each edge specifies its (capacity, delay cost). There are two agents, each with a requirement of 1 from $s$ to $t$. Their $m_i$s are 100 and 1 respectively. The network on the left has enough capacity to route two units of flow, but does not satisfy strong feasibility condition and does not have an equilibrium. This demonstrates importance of strong feasibility condition, without which even a simple market may not have an equilibrium.

The network in the middle does satisfy strong feasibility but not extensibility and has an equilibrium. The network on the right satisfies extensibility but not strong feasibility and has an equilibrium. This demonstrates that conditions of strong feasibility and extensibility are incomparable.

Non-convexity of equilibria. Consider a market in the scheduling setting of Section 3, with 6 agents, each with a requirement of 1. Their $m_i$s are 30, 17, 9, 4, 3, 1. Table 1 depicts some of the equilibrium prices for this instance. A bigger example with 9 agents is in Table 2. These examples show that the equilibrium set is not convex, but forms a connected set. Since these are in high dimension, it is not easy to determine the exact shape of the entire equilibrium set, but one can see that it is quite complicated.

A run of the algorithm. We describe the run of our algorithms on simple examples here. The run of Algorithm 1 on the example in Table 1 is as follows. Each row below depicts one iteration, where we find a new segment. We first give the set of agents in this new segment, then the corresponding $\lambda$, and then the prices of the slots determined in this iteration. The last column shows the sets which give the second and third lowest $\lambda$s in that iteration, and hence were not selected.

| $S^1$ | $\lambda_{S^1}$ | $p_0$ | $\lambda_{\{5,6\}} = 1^{1/3}$, $\lambda_{\{4,5,6\}} = 1^{1/3}$, $\ldots$ |
| $S^2$ | $\lambda_{S^2} = 1^{2/3}$, $p_5 = 2^{2/3}$, $p_4 = 4^{1/3}$ | $\lambda_{\{5\}} = 2$, $\lambda_{\{3,4,5\}} = 2^{1/6}$, $\ldots$ |
| $S^3$ | $\lambda_{S^3} = 4^{2/3}$, $p_4 = 9$ | $\lambda_{\{2,3\}} = 5^{7/9}$, $\lambda_{\{1,2,3\}} = 7^{1/6}$ |
| $S^4$ | $\lambda_{S^4} = 8$, $p_2 = 17$ | $\lambda_{\{1,2\}} = 9^{2/3}$ |
| $S^5$ | $\lambda_{S^5} = 13$, $p_1 = 30$ | |
Table 1: An example in the scheduling setting of Section 3 where the set of equilibrium prices is non-convex. There are 6 agents, each with a requirement of 1. Their $m_i$s are 30, 17, 9, 4, 3 and 1. We depict only a subset of all equilibria here. In particular, we depict 6 equilibrium prices, $p_1, p_2, \ldots, p_6$. All prices either along solid lines connecting any two of these points, or in the shaded region are equilibria. However, if any two of these prices are not connected by a solid line, then none of the points on the line joining them is an equilibrium. For example, none of the prices on the line joining $p_1$ and $p_6$, $p_2$ and $p_5$, or $p_3$ and $p_4$ is an equilibrium. There are more equilibrium points not depicted here. As far as we can tell, the shape of the equilibrium set is something akin to a cup, with empty space inside, but forming a single connected region.

The equilibrium price found in this run is the point $p_2$ in Table 1. This price curve is shown in Figure 4. The allocation obtained by solving the feasibility LP (3) is as follows: $x_{11} = 1, x_{22} = 1, x_{33} = 1, x_{44} = 4/5, x_{45} = 1/5, x_{54} = 1/5, x_{55} = 4/5, x_{66} = 1$.

Figure 4: Piecewise linear convex decreasing curve of equilibrium prices obtained by the algorithm for the example in Table 1.

We next describe the run of the algorithm on a network flow example, described in Figure 5. The figure shows the network structure and the edge labels specify (capacity, delay cost). There are five agents with requirements 10, 11, 12, 13, 14 from $s$ to $t$ respectively. Their $m_i$s are 12, 10, 4, 2, 2. This network is not series-parallel, yet it satisfies the extensibility condition, so our algorithm finds an equilibrium.

The run of Algorithm 3 on this example (in Figure 5) is as follows. Once again, each row below depicts one iteration, where we find a new segment. We first give the set of agents in the new segment, then the corresponding $\lambda$, and then the prices of the edges that are fixed in this iteration. The last column shows the second and third lowest $\lambda$s in that iteration.
**Table 2**: An example in the scheduling setting of Section 3 where the set of equilibrium prices is non-convex. There are 9 agents, each with a requirement of 1. Their \( m_i \)'s are 56, 45, 33, 23, 17, 10, 4, 3 and 1. We depict only a subset of all equilibria here. In particular, we depict 20 equilibrium prices, \( p_1, p_2, \ldots, p_{20} \). All prices either along solid lines connecting any two of these points, or in the shaded region are equilibria. However, if any two of these prices are not connected by a solid line, then none of the points on the line joining them is an equilibrium.

| Price | Description |
|-------|-------------|
| \( p_1 \) | \((57, 44, 33, 24, 16, 10, 5, 2, 1)\) |
| \( p_2 \) | \((57, 44/3, 32/3, 24, 16, 10, 5, 2, 1)\) |
| \( p_3 \) | \((57, 44, 33, 24, 16^2/3, 9^2/3, 5, 2, 1)\) |
| \( p_4 \) | \((57, 44, 33, 24, 16, 10, 5, 2^2/3, 1/3)\) |
| \( p_5 \) | \((57^2/3, 44/3, 32, 24, 16, 10, 5, 2/3, 1/3)\) |
| \( p_6 \) | \((57, 44, 33, 24^2/3, 16^2/3, 9, 5, 2, 1)\) |
| \( p_7 \) | \((57, 44, 33, 24, 16, 10, 5^2/3, 2^2/3, 0)\) |
| \( p_8 \) | \((57^2/3, 44^2/3, 32^2/3, 24, 16, 10, 5, 2^2/3, 1/3)\) |
| \( p_9 \) | \((57^2/3, 44^2/3, 32, 24, 16, 10, 5^2/3, 2^2/3, 0)\) |
| \( p_{10} \) | \((57^2/3, 44^2/3, 32, 24, 16, 10, 5, 2^2/3, 1/3)\) |
| \( p_{11} \) | \((57^2/3, 44^2/3, 32, 24, 16, 10, 5^2/3, 2^2/3, 0)\) |
| \( p_{12} \) | \((57^2/3, 44^2/3, 32, 24, 16, 10, 5^2/3, 2^2/3, 1/3)\) |
| \( p_{13} \) | \((57^2/3, 44^2/3, 32, 24, 16, 10, 5^2/3, 2^2/3, 0)\) |
| \( p_{14} \) | \((57^2/3, 44^2/3, 32, 24, 16, 10, 5^2/3, 2^2/3, 0)\) |
| \( p_{15} \) | \((57^2/3, 44^2/3, 32, 24, 16, 10, 5^2/3, 2^2/3, 0)\) |
| \( p_{16} \) | \((57^2/3, 44^2/3, 32, 24, 16, 10, 5^2/3, 2^2/3, 0)\) |
| \( p_{17} \) | \((57^2/3, 44^2/3, 32, 24, 16, 10, 5^2/3, 2^2/3, 0)\) |
| \( p_{18} \) | \((57^2/3, 44^2/3, 32, 24, 16, 10, 5^2/3, 2^2/3, 0)\) |
| \( p_{19} \) | \((57^2/3, 44^2/3, 32, 24, 16, 10, 5^2/3, 2^2/3, 0)\) |
| \( p_{20} \) | \((57^2/3, 44^2/3, 32, 24, 16, 10, 5^2/3, 2^2/3, 0)\) |

**Figure 5**: A network flow example where there are 5 agents with requirements 10, 11, 12, 13, 14 from \( s \) to \( t \) respectively. Their \( m_i \)'s are 12, 10, 4, 2, 2. The network satisfies the extensibility condition, so our algorithm finds an equilibrium.
\[
S^1 = \{3, 4, 5\}, \quad \lambda_{S^1} = \frac{8}{37}, \quad p_{sx} = p_{xt} = p_{wt} = 0, p_{st} = \frac{8}{37}, p_{sw} = \frac{16}{37} \quad (\lambda_{\{4,5\}} = \frac{4}{13}, \quad \lambda_{\{2,3,4,5\}} = \frac{9}{35}) \\
S^2 = \{1, 2\}, \quad \lambda_{S^2} = \frac{478}{1147}, \quad p_{wu} = \frac{478}{1147}, p_{vt} = \frac{478}{1147}, p_{sv} = p_{uv} = p_{ut} = 0 \quad (\lambda_{\{2\}} = \frac{194}{407})
\]

The allocation from the feasibility LP (3):

- Agent 1 sends \(\frac{2012}{239}\) units of flow on path \(s - w - u - v - t\) and \(\frac{378}{239}\) units of flow on path \(s - w - u - t\).
- Agent 2 sends \(\frac{2251}{239}\) units of flow on path \(s - w - u - t\) and \(\frac{378}{239}\) units of flow on path \(s - w - u - v - t\).
- Agent 3 sends 8 units of flow on path \(s - w - t\), \(\frac{5}{2}\) units of flow on path \(s - t\) and \(\frac{3}{2}\) units of flow on path \(s - x - t\).
- Agent 4 sends 2 units of flow on path \(s - w - t\), \(\frac{21}{4}\) units of flow on path \(s - t\) and \(\frac{23}{4}\) units of flow on path \(s - x - t\).
- Agent 5 sends 2 units of flow on path \(s - w - t\), \(\frac{21}{4}\) units of flow on path \(s - t\) and \(\frac{27}{4}\) units of flow on path \(s - x - t\).

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A Related work on computation and applications of market equilibrium

Computation and Complexity: The computational complexity of market equilibrium has been extensively studied for the tradition models in the past decade and a half. This investigation has involved many algorithmic techniques, such as primal-dual and flow based methods [18, 19, 46, 56, 57], auction algorithms [28], ellipsoid [32] and other convex programming based techniques [13], cell-decomposition [16, 17, 55], distributed price update rules [11, 14, 58], and complementary pivoting algorithms [25, 26], to name some of the most prominent. The algorithms have been complemented by hardness results, either for PPAD [10, 12] or for FIXP [22, 27], pretty much closing the gap between the two. Most of these papers focus on traditional utility functions used in the economics literature. A notable exception that considers combinatorial utility functions is [33], that study a market where agents want to send flow in a network, motivated by rate control algorithms governing the traffic in the Internet.

Beyond being an important component in the complexity theory of total functions [39], the computation of market equilibria has been studied by economists for much longer [6, 49, 54]. The classic case for the use of equilibrium computation is counter-factual evaluation of policy or design changes [52], based on the assumption that markets left to themselves operate at an equilibrium.

Fair allocation: Recently, market equilibrium outcomes have been used for fair allocation. Market equilibrium conditions are often considered inherently fair, therefore equilibrium outcomes have been used to allocate resources by a central planner seeking a fair allocation even when there is no actual market or monetary transfers. E.g., the proportional fair allocation, which is well known to be equivalent to the equilibrium allocation in a Fisher market [35], is widely used in the design of computer networks. Exchange of bandwidth in a bittorrent network is modeled as a process that converges to a market equilibrium by Wu and Zhang [58]. Budish [7] proposes “competitive outcome from equal incomes” (CEEI) as a way to allocate courses to students: the allocation is an equilibrium in a market for courses in which the students participate with equal budgets (with random perturbations to break ties). This scheme has been successfully used at the Wharton business school [8]. Cole et al. [15] show that a suitable modification of the Fisher market equilibrium allocation can be used as a solution to a problem of fair resource allocation, without money. The mechanism is truthful, and satisfies an approximate per-agent welfare guarantee. Truthful mechanisms have also been designed for scheduling, where it is the auctioneer who has jobs to be scheduled and the agents are the one providing the required resources e.g., see [37, 44]. This is in contrast to our setting where the agents have scheduling requirements.

Market based mechanisms: There is also a long history of “market based mechanisms”, where a mechanism (with monetary transfers) implements an equilibrium outcome. The New York Stock Exchange uses such a mechanism to determine the opening prices, and copper and gold prices in London are fixed using a similar procedure [48]. There are different ways to do this: use a sample (either historic or random) or a probabilistic model of the population to compute the equilibrium price, and offer these prices to new agents. This is preferable to asking the bidders to report their preferences, computing the equilibrium on reported preferences and offering the equilibrium prices back. The latter leads to obvious strategic issues; Hurwicz [31] shows that strategic behavior by agents participating in such a mechanism can lead to inefficiencies. Babaioff et al. [1] show price of anarchy bounds on such mechanisms. In any case, such mechanisms are “incentive compatible in the large”, meaning that as the market size grows and each agent becomes insignificant enough to affect prices on his own, his best strategy is to accept the equilibrium outcome. Nonetheless such mechanisms have been proposed and used in practice, e.g., for selling TV ads [45].
Budget constraints: Budget constraints in auctions has gained popularity in the last decade due to ad auctions \[4, 21, 24, 40\], but has been studied for quite some time \[9, 36\]. There has also been a recent line of work considering budget constraints in a procurement setting \[2, 53\].

B Relation to Myerson’s ironing

For a special case of the scheduling setting, when \( r_i = 1, \forall i \in A \), we show that equilibrium conditions are equivalent to a set of conditions that are reminiscent of the ironing procedure used in the characterization of optimal auctions by Myerson \[43\]. It is in fact “one higher derivative” analog of Myerson’s ironing. Let’s first restate Myerson’s ironing procedure for the case of a uniform distribution over a discrete support. Suppose that we plot on the \( x \)-axis the quantiles, in the decreasing order of value, and on the \( y \)-axis the corresponding virtual values. This is possibly a non-monotone function, and Myerson’s ironing asks for an ironed function that is monotone non-increasing, and is such that the area under the curve (starting at 0) of the ironed function is always higher than that for the given function. Further, the ironed function given by this procedure is the minimal among all such functions. This means that wherever the area under the curve differs for the two functions, the ironed function is constant. (See Figure 6 on page 22.)

In the special case of our model with a single good and when requirements are all one, the equilibrium price of the good as a function of time is obtained as an ironed analog of the money function: the function \( i \mapsto m_i \), where we assume the \( m_i \)s are sorted in the decreasing order. This money function is monotone non-increasing by definition but it need not be a convex function. The price as a function of time must be a
Lemma 7. If Lemma 9.

Proof. It is easy to see that \( x_{is} \leq \frac{1}{n+1} \), \( \forall i \), and hence the proof follows.

Next we show that equilibria of \( M \) and \( M' \) are related.

Lemma 8. If \( x_i \in OPT_i(p), \forall i \in A \) at prices \( p \geq 0 \) for \( M' \), then \( \sum_i x_{is} < 1 \). That is \( p_s = 0 \) at equilibrium.

Proof. It is easy to see that \( x_{is} \leq \frac{1}{n+1} \), \( \forall i \), and hence the proof follows.

Next we show that equilibria of \( M \) and \( M' \) are related.

Lemma 9. If \( M \) satisfies strong feasibility, then every equilibrium of \( M' \) gives an equilibrium of \( M \).

Proof. Let \( (X^*, p^*) \) respectively be an equilibrium allocation and prices of \( M' \). From Lemma 8, we know that \( p^*_s = 0 \). It suffices to show that \( x^*_{is} = 0, \forall i \in A \), for the lemma to follow.

To the contrary suppose for some agent \( u \), \( x^*_u > 0 \). We will construct another bundle \( x'_u \) that is affordable to agent \( u \) at prices \( p^* \), satisfies CC(\( u \)), and has a lower delay than \( x^*_u \), contradicting optimal bundle condition at equilibrium.

Due to strong feasibility, after all \( i \neq u \) is given their bundle \( x^*_i \), there will be a bundle \( x'_u \) left for \( u \) to buy among goods in \( G \) such that CC(\( u \)) constraints are satisfied. Clearly, one way to construct such \( x'_u \) is

\[ OPT_i(p) = \arg \min \sum_j d_{ij} x_{ij} + d_s x_{is} \]
\[ \text{s.t. } \sum_j a_{ij} x_{ij} + a_s x_{is} \geq r_{ik}, \forall k \in C \]
\[ \sum_j p_j x_{ij} + p_s x_{is} \leq m_i \]
\[ x_{ij} \geq 0, \forall j \in G' \]

The next lemma follows from the construction of market \( M' \).

Lemma 7. If \( M \) satisfies strong feasibility then so does \( M' \).

Price vector \( p \) is said to be at equilibrium, if when every agent is given its optimal bundle, there is no excess demand of any good, and goods with excess supply have price zero. That is, \( (X, p) \) such that,

\[ \forall i \in A, x_i \in OPT_i(p), \text{ and } \forall j \in G', \sum_{i \in A} x_{ij} \leq 1; \text{ } p_j > 0 \Rightarrow \sum_{i \in A} x_{ij} = 1 \]
that the agent keeps buying all goods $j \neq s$ as in $x^*_s$, and starts decreasing $x^*_{us}$ and increasing allocation for some other available good. Note that all such available goods are under-sold at equilibrium and therefore has zero price in $p^*$. Thus payment for $x'_u$ and $x^*_s$ are the same at prices $p^*$. In other words $x'_u$ is affordable at prices $p^*$.

If good $j$ is increased by $\delta_j$ as we go from $x^*_u$ to $x'_u$, then we claim that $\delta_j \leq m!a_s \left( \frac{a_{\text{max}}}{a_{\text{min}}} \right)^m x^*_{us}$, where $m = |G|$. This is because, in a constraint even if coefficient of variable $x_{uj}$ is minimum possible, and it needs to compensate for increase in other goods due to their negative coefficients, this cascade could at most harm by a factor of $m! \left( \frac{a_{\text{max}}}{a_{\text{min}}} \right)^m$. Difference in delay is

$$\sum_j d_{u_j} (x'_{u_j} - x^*_{u_j}) - d_s x^*_{us} = \sum_j d_{u_j} \delta_j - d_s x^*_{us} \leq md_{\text{max}} (\max_j \delta_j) - d_s x^*_{us} \leq \left( m^{(m+1)} d_{\text{max}} a_s \left( \frac{a_{\text{max}}}{a_{\text{min}}} \right)^m - d_s \right) x^*_{us} < 0$$

Due to Lemma 9, to show existence of equilibrium for market $\mathcal{M}$ it suffices to show one for $\mathcal{M}'$. Next to show existence of equilibrium for market $\mathcal{M}'$ it suffices to consider price vectors where $p_s = 0$ due to Lemma 8, and therefore we consider the following set of possible price vectors.

$$P = \{ p \in \mathbb{R}_+^{(m+1)} | p_s = 0; \sum_j p_j \leq M \} \text{ where } M = \sum_i m_i$$

Let us first handle trivial instances. It is easy to see that the feasible set of $x_i$s in LP (8) at $p = 0$ is a superset of the feasible set at any other prices $p$. Therefore, for agent $i$ if $x_i = 0 \in OPT_i(0)$, then she will not buy anything at any prices. In that case, it is safe to discard her from the market. Further, if there is an allocation $X$ satisfying Supply constraints for market $\mathcal{M}'$ such that $x_i \in OPT_i(0), \forall i \in A$, then we get a trivial equilibrium of $\mathcal{M}'$ where all the prices are set to zero; note that in this case zero prices also constitute an equilibrium of market $\mathcal{M}$ by Lemma 9. To show existence for non-trivial instances, w.l.o.g. now on we assume the following for market $\mathcal{M}'$.

**Weak Sufficient Demand (WSD):** If $X$ is such that $\forall i \in A, x_i \in OPT_i(0), \text{ then } x_i \neq 0, \forall i,$ and there exists a good $j \in G'$ such that $\sum_i x_{ij} > 1$. Clearly, $j \neq s$ due to Lemma 8.

**Lemma 10.** For any $p \in P$, $OPT_i(p)$ is non-empty, and assuming WSD, $0 \notin OPT_i(p), \forall i \in A$.

**Proof.** The first part of the proof is easy to see due to the extra good $s$ whose price is zero in $P$. For the second part, to the contrary suppose for $x_i \in OPT_i(p)$ we have $x_i = 0$. By WSD assumption we know that for any $\lambda x_i^0 \notin OPT_i(0)$, we have $\lambda x_i^0 \neq 0$. Further, feasible set of LP (8) is at prices $p$ is a subset of the feasible set of this LP at zero prices. Hence we know that $\sum_j d_{ij}x_{ij}^0 < \sum_j d_{ij}x_{ij} = 0$.

It must be the case that $x_i^0$ is not affordable at prices $p$, i.e., $m_i < \sum_j x_{ij}^0 p_j$. For $\lambda = m_i / \sum_j x_{ij}^0 p_j$, set $x'_i = \lambda x_i^0 + (1 - \lambda)x_i$. Since $x_i$ and $x_i^0$ both satisfy CC(i), so does $x'_i$. And since $x_i = 0$ bundle $x'_i$ is affordable at prices $p$, thereby $x'_i$ is feasible in $OPT_i(p)$. The delay at $x'_i$ is $\sum_i d_{ij}x'_{ij} = \sum_i d_{ij}(\lambda x_{ij}^{0} + (1 - \lambda)x_{ij}) = \lambda \sum_j d_{ij}x_{ij}^0 < 0 = \sum_j d_{ij}x_{ij}$, a contradiction to $x_i$ being optimal bundle at prices $p$. □

Next we will construct a correspondence whose fixed points are exactly the market equilibria of $\mathcal{M}'$. Let $c_{ij}^\text{max}$ be the maximum possible demand of good $j$; we can compute $c_{ij}^\text{max}$ by maximizing $\sum_i x_{ij}$ over the CC(i) constraints of all agents $i \in A$. Define domain

$$D = \{ (X, p) | p \in P; \ X \geq 0; \ \forall j, \ \sum_i x_{ij} \leq c_{ij}^\text{max} \}$$

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Let $\delta = \min_i m_i$. Define correspondence $F : D \to D$ as follows where for a given $(\tilde{X}, \tilde{p}) \in D$, we have $(X', p') \in F(\tilde{X}, \tilde{p})$.

$$\forall i \in A, \ x'_i \in OPT_i(\tilde{p}), \ \text{and} \ p' \in \underset{p \in F, \delta \leq \sum_j p_j \leq \max\{\delta, \sum_{i,j \in G'} x'_{ij}p_j\}}{\arg \max} \sum_{i,j \in G'} \tilde{x}_{ij}p_j$$ (10)

The correspondence is well defined due to Lemma 10. If $F(\tilde{X}, \tilde{p})$ is a convex set, and graph of $F$ is closed, then Kakutani’s Theorem [34] implies that $F$ has a fixed point, i.e., $(X^*, p^*) \in D$ such that $(X^*, p^*) \in F(X^*, p^*)$. Next we show the same.

**Lemma 11.** Correspondence $F$ has a fixed point.

**Proof.** Clearly, $F(\tilde{X}, \tilde{p})$ is a convex set since it is a cross product of solution sets of LPs. The lemma follows using Kakutani’s fixed-point Theorem [34] if graph of $F$ is closed.

Let $(\tilde{X}^t, \tilde{p}^t)$ for $t = 1, 2, \ldots$ be a sequence of points in $D$, and let $(X^t, p^t)$ be another sequence such that $(X^t, p^t) \in F(\tilde{X}^t, \tilde{p}^t)$. Then essentially, $X^t$ and $p^t$ are solutions of LPs that are continuously changing with $(\tilde{X}, \tilde{p})$. Therefore, if $\lim_{t \to \infty}(\tilde{X}^t, \tilde{p}^t) = (X^*, \tilde{p}^*)$ and $\lim_{t \to \infty}(X^t, p^t) = (X^*, p^*)$, then by continuity of parameterized LP solutions, we get that $(X^*, p^*) \in F(X^*, \tilde{p}^*)$, implying graph of $F$ is closed.

**Lemma 12.** If $(X^*, p^*)$ is a fixed-point of $F$ then $\forall j \in G', \sum_{i \in A} x^*_{ij} \leq 1$.

**Proof.** Since $x^*_i \in OPT_i(p^*)$, $\forall i \in A, \sum_{i \in A} x^*_{i} \leq 1$ follows using Lemma 8. Among the rest of the goods, suppose, for $j' \neq s$ we have $\sum_{i \in A} x^*_{ij'} > 1$. Let $U = \max\{\delta, \sum_{i,j} x^*_{ij}p^*_j\}$. Then the optimization problem of (10) is

$$\underset{p \in F, \delta \leq \sum_j p_j \leq \max\{\delta, \sum_{i,j \in G'} x^*_{ij}p_j\}}{\max} \sum_{i,j} x^*_{ij}p_j$$

The above quantity can be made more than $U$ by setting $p_{j'} = U$, and therefore optimal value is strictly more than $U$. However, due to the fixed-point condition $p^*$ is a solution of the above, which implies $\sum_{i,j} x^*_{ij}p^*_j > U$, a contradiction.

**Lemma 13.** Assuming weak sufficient demand (WSD), if $(X^*, p^*)$ is a fixed-point of $F$ then $p^* \neq 0$ and $\sum_{i,j} x^*_{ij}p^*_j \geq \delta$.

**Proof.** For the first part, to the contrary suppose $p^* = 0$. Then $\forall i \in A, x^*_i \in OPT_i(0)$ since $(X^*, p^*)$, and therefore by WSD condition $\exists j \neq s, \ x^*_j > 1$. However $\sum_{i,j} x^*_{ij}p^*_j = 0$. Therefore,

$$\underset{p \in F, \delta \leq \sum_j p_j \leq \max\{\delta, \sum_{i,j} x^*_{ij}p^*_j\}}{\max} \sum_{i,j} x^*_{ij}p_j = \underset{p \in F, \sum_j p_j = \delta}{\max} \sum_{i,j} x^*_{ij}p_j > 0$$

This contradicts the fact that $p^*$ is a maximizer of the above.

For the second part, to the contrary suppose $\sum_{i,j} x^*_{ij}p^*_j < \delta$, then $\max\{\delta, \sum_{i,j} x^*_{ij}p^*_j\} = \delta$. Further, due to Lemma 10 (together with the WSD assumption) we have $\forall i \in A, x^*_i \neq 0$, and therefore at maximum $\sum_j p_j = \delta$. Since $p^*$ is a maximizer of the above, $\sum_j p_j = \delta$.

However since $\sum_{i,j} x^*_{ij}p^*_j < \delta$, we get $\forall i, \sum_j x^*_{ij}p^*_j < m_i$ where $x^*_i$ is feasible in LP (8) at prices $p^*$. Let $X^0$ be a demand vector when all the prices are zero. Due to Lemma 12 and WSD assumption we get that for some agent $i$, $x^*_i \notin OPT_i(0)$. Let $i'$ be this agent. This implies $\sum_j d_{ij}x^*_{ij}p^*_j < \sum_j d_{ij}x^*_{ij}p^*_j$. Despite lower cost at $x^*_i$, she demands $x^*_{i'}$ at prices $p^*$, hence it should be the case that she can not afford $x^*_{i'}$ at those prices. However, since $i'$ is also not spending all the money at prices $p^*$, there exists some $0 < \tau < 1$ such that she can afford $X^0_{i'} = \tau x^*_{i'} + (1 - \tau)x^0_{i'}$ at $p^*$. Since both $x^0_{i'}$ and $x^*_{i'}$ satisfy the CC$(i')$ constraints of LP (8) for agent $i'$, so does $X^0_{i'}$. Thus, $X^0_{i'}$ is a feasible point in $OPT_{i'}(p^*)$ LP, and $\sum_j d_{ij}x^0_{ij} \sum_j d_{ij}x^*_{ij}$, a contradiction to $x^*_{i'} \in OPT_{i'}(p^*)$. 

□
Next we show the main result using Lemmas 11, 12 and 13.

**Theorem 4.** If $\mathcal{M}$ satisfies strong feasibility then it has an equilibrium.

**Proof.** Due to Lemma 11, we know that there exists a fixed-point of correspondence $F$. Let this be $(X^*, p^*)$. We will show that it is a market equilibrium of $\mathcal{M}$. Clearly, optimal allocation condition is satisfied because $x^*_i \in OPT_i(p^*)$. Market clearing remains to be shown, which requires: (a) $\forall j$, $\sum_i x^*_ij \leq 1$, and (b) $p^*_j > 0 \Rightarrow \sum_i x^*_ij = 1$.

(a) follows from Lemma 12. For (b), let $U = \max\{\delta, \sum_{i,j} x^*_ijp^*_j\}$, then due to Lemma 13, $U = \sum_{i,j} x^*_ijp^*_j$. The optimization problem of (10) is

$$\max_{p \in P, \delta \leq \sum_j p_j \leq U} \sum_{i,j} x^*_ijp_j = \max_{p \in P, \delta \leq \sum_j p_j \leq U} \sum_j p_j \sum_i x^*_ij$$

Clearly, at optimal solution of the above $p_j$ is non-zero only where $\sum_i x^*_ij$ is maximum. Since $p^*$ is a solution, if $\exists j$, $\sum_i x^*_ij = 1$, then (b) follows.

On the other hand suppose for all $j$ we have $\sum_i x^*_ij < 1$, then clearly the optimal value of the above is strictly less than $U$. However since $p^*$ is a maximizer it implies that $\sum_{i,j} x^*_ijp^*_j < U$, a contradiction to Lemma 13. □

The next theorem follows using Lemmas 7, 9, and Theorem 4.

**Theorem 1.** [Strong feasibility implies existence of an equilibrium] If $(CC(i))_{i \in A}$ of market $\mathcal{M}$ satisfies strong feasibility, then $\exists$ an allocation $X$ and prices $p$ that constitute a market equilibrium of $\mathcal{M}$.

**Remark 1.** By similar argument, we can show existence of equilibrium for market instances satisfying only extensibility, and sufficient demand conditions (see Definitions 3 and 4). However, since our algorithm returns an equilibrium of such a market, it already gives a constructive proof of existence.

### C.1 Quasi-concave utility functions

In this section, we show that the preferences of agents in our model can be captured by quasi-concave utility functions. Notation: the symbol $\leq$ when used for vectors represents a co-ordinate wise relation, and $<$ represents that at least one of the inequalities is strict. Define the utility of an agent $i$ for an allocation $x_i$ to be the smallest delay of a feasible allocation dominated by $x_i$, times $-1$:

$$U_i(x_i) = -\min \{d_i \cdot x'_i : x'_i \leq x_i & x'_i \text{ is feasible for Delay LP}(i)\}.$$  

If there is no $x'_i \leq x_i$ that is feasible for Delay LP(i) then the utility is $-\infty$. It is easy to check that this utility function is quasi-concave, and induces the same preferences as in our model.

### D Special Cases

In this section, we show how several natural problems satisfy the extensibility condition (Definition 3) in Section 2.
Scheduling. Recall the scheduling problem from Section 2.3. The agents are jobs that require \( d \) different types of machines, and the set of machines of type \( k \) is \( M_k \); the machines are the goods in the market. Each agent needs \( r_{i k} \in \mathbb{R}_+ \) units of machines in all of type \( k \), which is captured by the covering constraint \( \sum_{j \in M_k} x_{i jk} \geq r_{i k}, \forall k \in [d]. \) All agents experience the same delay \( d_{j k} \) from machine \( j \) in type \( k \).

Lemma 14. Scheduling problem satisfies the extensibility condition.

Proof. Consider an arbitrary set of agents \( S \) and an agent \( \hat{i} \) outside of this set. Let \( (x_i)_{i \in S} \) to be a feasible allocation that minimizes the total delay, i.e., \( \sum_{i \in S, j \in M_k, k \in [d]} d_{j k} x_{i jk} \). Since the delays are the same for each agent the total delay minimizes when the agents in \( S \) get \( \sum_{i \in S} r_{i k} \) units of machines of type \( k \) with the smallest delay. Therefore, if we assign the next \( r_{i k} \) units of machines of type \( k \) with the smallest delay to agent \( \hat{i} \) then \( (x_{i})_{i \in (S \cup \hat{i})} \) would be the feasible allocation that minimizes the total delay. Therefore, this problem satisfies the extensibility condition.

Restricted assignment with laminar families.

Lemma 15. Restricted assignment with laminar families satisfies the extensibility condition if the following assumption holds for the instance

- (Monotonicity) \( \forall i, i' \in A, \) such that \( S_{i'} \subset S_i \) then \( \max_{j \in S_i \setminus S_{i'}} d_{j k} \leq \min_{j' \in S_{i'} \setminus S_i} d_{j' k} \) for each type \( k \in [d] \).

Proof. Since the requirement and variables for set of machines of each type is separate, it is enough to show this for \( k = 1 \). Consider an arbitrary set of agents \( T \) and an agent outside of this set \( \hat{i} \). Let \( T' = T \cup \hat{i} \). Let \( (x_i)_{i \in T} \) to be a feasible allocation that minimizes total delay, i.e., \( \sum_{i \in T} d_{i} x_{i} \). For simplicity let \( S'_{i} = \{1, 2, \ldots, m\} \) such that \( d_1 \leq \cdots \leq d_m \). Consider two agent \( i \) and \( i' \). Note that if \( j \in S_i \) and \( j' \in S_{i'} \) then \( \forall j' \in S_i \) such that \( d_{j'} \leq d_{j} \) we have \( j \in S_{i'} \) and vice versa because \( S_i \)s form a laminar family and the monotonicity condition. Therefore, an optimal allocation allocates only a prefix of machines in \( S_i \). Let’s assign to \( \hat{i} \) the next \( r_{i} \) machines with smallest delay in subset \( S_i \). We claim the new allocation \( (x_i)_{i \in S'} \) is minimizing the total delay, i.e., \( \sum_{i \in S'} d_{i} x_{i} \). Let’s prove the claim by contradiction. Suppose the allocation is not optimal. Therefore there exists an optimal allocation \( (x_{i'})_{i \in S'' \cup \hat{i}} \) with less total delay. Suppose allocation \( X' \) has allocated machines \( 1 \) through \( l' \) in set \( S_i \) and allocation \( X \) has allocated machines \( 1 \) through \( l \) in set \( S_i \). Note that in allocation \( X' \) we can assume agent \( \hat{i} \) is getting the last machines in \( X' \) among the machines that has been allocated in \( S_i \) because if there exists agent \( \hat{i} \) that has allocation on the right side of the first machine that is allocated to \( \hat{i} \) then we can swap the allocations so the total delay wouldn’t change. If \( l = l' \) then the delay of agent \( \hat{i} \) is the same in \( X \) and \( X' \). This is a contradiction because then it would conclude the total delay allocation of \( (x_{i'})_{i \in S} \) is less than total delay of \( (x_i)_{i \in S} \) but we assumed \( X \) is an optimal allocation for \( S \). There are two cases.

Case 1. \( l < l' \). Consider allocation \( X' \) after removing agent \( \hat{i} \). Since \( \hat{i} \) is getting the last machines it is easy to see the remaining allocation is optimal for \( S \). Since \( l < l' \) there are agents that have less allocation in \( S_i \) in \( X \) compare to \( X' \). Because of monotonicity assumption they have been allocated to machines with highest delay instead of available machines in \( S_i \). This is a contradiction with the fact \( X \) is a optimal allocation for \( S \).

Case 2. \( l > l' \). This case is very similar to the last case. With the same argument we can argue that this case has contradiction with the fact \( X' \) is an optimal allocation.

Network Flows. Recall that in this setting agent \( i \) wants to send \( r_i \) units of flow from \( s \) to \( t \) in a directed (graph) network where each edge has a capacity and cost per unit flow specified. Here edges are goods, and the covering constraints of agent \( i \) has variable \( f_{ie} \) for each edge \( e \) representing her flow on edge \( e \). The
constraints impose flow conservation at all nodes except \(s\) and \(t\), and that net outgoing and incoming flow at \(s\) and \(t\) respectively is \(r_i\).

**Lemma 16.** A series-parallel network satisfies the extensibility condition.

*Proof.* Consider an arbitrary set of agents \(S\) and an agent outside of this set \(\hat{i}\). Let \((f_i)_{i \in S}\) to be a feasible min cost flow. Let’s remove the allocated capacities from the graph and allocate min cost flow of size \(r_{\hat{i}}\) to agent \(\hat{i}\) in the remaining graph. It is known that this greedy algorithm gives a min cost flow of size \(\sum_{i \in S \cup \hat{i}} r_i\) \cite{3}.

We can consider independent copies of any of these special cases above. E.g., each agent might want some machines for job processing, as well as send some flows through a network, but have a common budget for both together. Note that it would remain extensible because each copy is independent and extensible itself.

### E Equilibrium Characterization

In this section we characterize equilibria of most general market instances. Recall that allocation \(x_i\) of agent \(i\) has to satisfy its covering constraints \(CC(i)\). Next we derive sufficient conditions for prices \(p\) and allocation \(X\) to be an equilibrium. As we discussed in Section 2, given prices \(p\) the optimal bundle of each agent \(i\) is captured by

\[
\text{min } \sum_{j \in G} d_{ij} x_{ij} \\
\text{s.t. } \sum_{j \in G} a_{ijk} x_{ij} \geq r_{ik} \quad \forall k \in C \\
\sum_{j \in G} p_j x_{ij} \leq m_i \\
x_{ij} \geq 0, \quad \forall j \in G
\]

(\(\text{OB-LP}(i)\))

It is well known that solutions of a linear program are exactly the ones that satisfy its complementary slackness conditions \cite{50}. Let \(\beta_{ik}\) and \(\gamma_i\) be the dual variables of first and second of constraints in \(\text{OB-LP}(i)\). Then, corresponding complementary slackness conditions are,

\[
\begin{align*}
\forall k \in C : & \quad \sum_{j \in G} a_{ijk} x_{ij} \geq r_{ik} \quad \bot \quad \beta_{ik} \geq 0 \\
\forall j \in G : & \quad d_{ij} \geq \sum_{k} a_{ijk} \beta_{ik} - \gamma_i p_j \quad \bot \quad x_{ij} \geq 0 \\
& \quad \sum_{j \in G} p_j x_{ij} \leq m_i \quad \bot \quad \gamma_i \geq 0
\end{align*}
\]

(11)

Here \(\bot\) symbol between two inequalities means that both inequalities should be satisfied, and at least one of them has to hold with equality. Next, under a mild condition of *sufficient demand* (see Definition 4), we show that w.r.t. equilibrium prices \(\gamma_i \geq 0, \quad \forall i \in A\). This would also imply that each agent spends all of its budget at equilibrium (thereby maximizing profit of the seller). We note that the following lemma is not needed for Theorem 3, but will be used to show the reverse result in Lemma 19, namely equilibrium prices and allocation gives solutions of the \(LP(\lambda)\) and \(DLP(\lambda)\) for appropriately chosen parameter vector \(\lambda\).

**Lemma 17.** Let \((X^*, p^*)\) be an equilibrium of market \(M\) satisfying sufficient demand, and for all \(i \in A\) let \((\gamma_i^*, \beta_i^*)\) be the corresponding dual variables of \(\text{OB-LP}(i)\) at \(p^*\). Then \(\gamma_i^* > 0, \forall i \in A\). In other words, every agent spends all her money at any equilibrium.

*Proof.* Since \(X^*\) is an equilibrium allocation, no good is over demanded. Therefore, we know that \(x_{ij}^* \leq 1, \forall i \in A\). Let \(p^0\) be all zero price vector. By sufficient demand condition \(x_{ij}^*\) cannot be an optimal allocation of agent \(i\) at prices \(p^0\). We will derive a contradiction to this if \(\gamma_i^* = 0\) for some agent \(i\).

To the contrary suppose for agent \(i \in A\), we have \(\gamma_i = 0\). Since \(x_{ij}^*, \gamma_i^*\) and \(\beta_i^*\) satisfy complementary slackness conditions (12) of \(\text{OB-LP}(i)\) at prices \(p^*\), it is easy to see that they also satisfy these conditions at prices \(p^0\). This would imply that \(x_{ij}^*\) is an optimal allocation of agent \(i\) at prices \(p^0\), a contradiction. \(\square\)
Proof. It suffices to show that \( \forall i \in A \) and \( \alpha_{ik} = \beta_{ik}/\gamma_i \), and write conditions for all the agents together.

\[
\begin{align*}
\forall i \in A, \forall k \in C : & \quad \sum_{j \in G} a_{ijk} x_{ij} \geq r_{ik} \quad \perp \quad \alpha_{ik} \geq 0 \\
\forall i \in A, \forall j \in G : & \quad \lambda_i d_{ij} \geq \sum_k a_{ijk} \alpha_{ik} - p_j \quad \perp \quad x_{ij} \geq 0 \\
\forall i \in A : & \quad \sum_{j \in G} p_j x_{ij} = m_i \quad \text{and} \quad \lambda_i > 0 
\end{align*}
\] (12)

At market equilibrium prices, every agent should get an optimal bundle, and market should clear, i.e., \textbf{Supply constraints} are satisfied and every good with positive price should be fully sold (see Section 2 for the formal definition of market equilibrium). Since optimal allocations at given prices are solutions of \text{OB-LP}(i) for each \( i \), they must satisfy (12). This follows from the fact that primal-dual feasibility and complementary slackness conditions are necessary and sufficient for solutions of a linear program. We get the following characterization.

\textbf{Lemma 18.} If \( \hat{\lambda}, \hat{X}, \hat{p}, \hat{\alpha} \) satisfies (12), and \( \forall j \in G, \sum_{i \in A} \hat{x}_{ij} \leq 1 \quad \perp \quad \hat{p}_j \geq 0 \), then \( \hat{X}, \hat{p} \) constitutes an equilibrium allocation and prices.

Motivated from Lemma 18 we next define parameterized LP that captures complementary slackness conditions of \text{OB-LP}(i) for all the agents together. Suppose we are given \( \lambda_i \)’s, let us define the following linear program parameterized by the \( \lambda \) vector, that we call \( LP(\lambda) \), and its dual \( DLP(\lambda) \) (same as (2) defined in Section 4):

\[
\begin{align*}
\text{min :} & \quad \sum_{i,j} \lambda_i d_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{j \in G} a_{ijk} x_{ij} \geq r_{ik} \quad \forall i \in A, k \in C \\
& \quad \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in G \\
& \quad x_{ij} \geq 0 \quad \forall i \in A, j \in G
\end{align*}
\]

Using the equilibrium characterization of Lemma 18 together with complementary slackness conditions between constraints of \( LP(\lambda) \) and \( DLP(\lambda) \), next we show that solutions of \( LP(\lambda) \) and \( DLP(\lambda) \) exactly capture the equilibria if given appropriate value of parameter vector \( \lambda \).

\textbf{Theorem 3.} For a given \( \lambda > 0 \) if an optimal solution \( X \) of \( LP(\lambda) \) and an optimal solution \( (\alpha, p) \) of \( DLP(\lambda) \) satisfy \textbf{Budget constraint}(i) for all agents \( i \in A \) with equality, then they constitute an equilibrium of market \( \mathcal{M} \).

\textbf{Proof.} It suffices to show that \( (\lambda, \alpha, \hat{p}, \hat{X}) \) satisfy conditions of Lemma 18. Out of these the last one of (12) is already assumed in the hypothesis. For the remaining conditions, let us write the complementary slackness conditions for \( LP(\lambda) \).

\[
\begin{align*}
& \sum_j a_{ijk} x_{ij} \geq r_{ik} \quad \perp \quad \alpha_{ik} \geq 0, \quad \forall i, k \\
& \lambda_i d_{ij} \geq \sum_k a_{ijk} \alpha_{ik} - p_j \quad \perp \quad x_{ij} \geq 0, \quad \forall i, j \\
& \sum_i x_{ij} \leq 1 \quad \perp \quad p_j \geq 0, \quad \forall j
\end{align*}
\] (13) (14) (15)

Conditions (13) and (14) are exactly the first two conditions of (12), and (15) ensures market clearing. Thus proof follows using Lemma 18.
Next we show converse of the above theorem under sufficient demand condition.

**Lemma 19.** If market \( M \) satisfies sufficient demand and \((\hat{X}, \hat{p})\) is its equilibrium, then \( \hat{X} \) and \((\hat{\alpha}, \hat{p})\) gives solution of \( LP(\hat{\lambda}) \) and \( DLP(\hat{\lambda}) \) respectively for some \( \hat{\lambda} \) and \( \hat{\alpha} \).

**Proof.** Due to optimal bundle condition of equilibrium we know that \( \hat{x}_i \) is an optimal solution of OB-LP\((i)\) at prices \( \hat{p} \). Let \( \hat{\beta}_{ik} \) and \( \hat{\gamma}_i \) be the value of corresponding dual variable in (11). Using Lemma 17 we have \( \hat{\gamma}_i > 0, \forall i \). Then it is easy to see using (12) that for \( \hat{\lambda}_i = 1/\hat{\gamma}_i, \forall i \in A \) and \( \hat{\alpha}_{ik} = \hat{\beta}_{ik}/\hat{\gamma}_i, \forall i \in A, k \in C \), \((\hat{X}, \hat{\alpha}, \hat{p})\) satisfy complementary slackness conditions of \( LP(\hat{\lambda}) \) and \( DLP(\hat{\lambda}) \). Therefore the proof follows.

Using the equilibrium characterization given by Theorem 3 crucially, we design a polynomial-time algorithm to find an equilibrium for markets with extensibility (Definition 3) in Section 4.

### F Missing Proofs and Details of Section 4

The proof of first theorem of Section 4, namely Theorem 3 is in Appendix E where we characterize market equilibria. Next we give proof of Lemma 1. For this we basically use the fact that any pair of primal, dual solutions of a linear program has to satisfy complementary slackness [50]. Recall the \( LP(\lambda) \) and \( DLP(\lambda) \) of (2).

\[
\begin{align*}
LP(\lambda) : \\
\min & : \sum_i \lambda_i \sum_j d_{ij} x_{ij} \\
\text{s.t.} & : \sum_j a_{ijk} x_{ij} \geq r_{ik}, \forall k \\
& \sum_i x_{ij} \leq 1, \forall i \\
& x_{ij} \geq 0, \forall (i, j)
\end{align*}
\]

\[
\begin{align*}
DLP(\lambda) : \\
\max & : \sum_{i,k} r_{ik} \alpha_{ik} - \sum_j p_j \\
\text{s.t.} & : \lambda_i d_{ij} \geq \sum_k a_{ijk} \alpha_{ik} - p_j, \forall (i, j) \\
& p_j \geq 0, \forall j; \quad \alpha_{ik} \geq 0, \forall (i, k)
\end{align*}
\]

For any given \( \lambda > 0 \), the optimal solutions of the \( LP(\lambda) \), namely \( X \), and \( DLP(\lambda) \), namely \( (\alpha, p) \) has to satisfy the following complementary slackness conditions.

\[
\begin{align*}
\forall (i, k) : & \quad \alpha_{ik} \geq 0 \quad \perp \sum_j a_{ijk} x_{ij} \geq r_{ik} \\
\forall j : & \quad p_j \geq 0 \quad \perp \sum_i x_{ij} \leq 1 \\
\forall (i, j) : & \quad x_{ij} \geq 0 \quad \perp \lambda_i d_{ij} \geq \sum_k a_{ijk} \alpha_{ik} - p_j
\end{align*}
\]

Recall that the \( \perp \) symbol between two inequalities means that both inequalities should be satisfied, and at least one of them has to hold with equality. Also recall that for subset \( S \subseteq A \), we defined \( \text{delay}_S(X, p) = \sum_{i \in S,j} d_{ij} x_{ij} \) and \( \text{pay}_S(X, p) = \sum_{i \in S,j} x_{ij} p_j \), and for ease of notation we use \( \text{delay}_i \) when \( S = \{i\} \) and similarly \( \text{pay}_i \). Using this we first show a relation between \( \text{delay}_i \) and pay, next.

**Lemma 20.** For a given \( \lambda > 0 \) if \( \hat{X} \) and \((\hat{\alpha}, \hat{p})\) are optimal solutions of \( LP(\lambda) \) and \( DLP(\lambda) \) respectively, while \( X' \) and \((\alpha', p')\) are feasible in \( LP(\lambda) \) and \( DLP(\lambda) \) then, such that,

\[
\begin{align*}
\forall i : & \quad \lambda_i \text{delay}_i(\hat{X}) = \sum_k r_{ik} \hat{\alpha}_{ik} - \text{pay}_i(\hat{X}, \hat{p}) \\
\forall i : & \quad \lambda_i \text{delay}_i(X') \geq \sum_k r_{ik} \alpha'_{ik} - \text{pay}_i(X', p')
\end{align*}
\]

**Proof.** For feasible solutions we have \( \forall (i, k) \), \( \alpha'_{ik} \geq 0 \) and \( \sum_j a_{ijk} x'_{ij} \geq r_{ik} \). Multiplying the two and summing over \( k \) for each \( i \) gives, \( \sum_k \alpha'_{ik} \sum_j a_{ijk} x'_{ij} \geq \sum_k \alpha_{ik} r_{ik}, \forall i \). Another pair of constraints are
\( \forall(i, j), \ x'_{ij} \geq 0 \) and \( \lambda_i d_{ij} \geq \sum_k a_{ijk} \alpha'_{ik} - p'_j \). Multiplying the two gives,

\[
\forall i : \lambda_i \text{delay}(X') = \lambda_i \sum_j x'_{ij} d_{ij} \\
\geq \sum_j x'_{ij} (\sum_k a_{ijk} \alpha'_{ik}) - \sum_j p'_j x'_{ij} \\
= \sum_k \alpha'_{ik} \sum_j a_{ijk} x'_{ij} - \text{pay}_i(X', p') \\
\geq \sum_k \alpha'_{ik} r_{ik} - \text{pay}_i(X', p')
\]

This gives the second part. Since optimal solutions satisfy complementary slackness, all inequalities satisfy with equality in the above for (\( \hat{X}, \hat{\alpha}, \hat{p} \)) and we get the first part. \( \square \)

Recall notation \([n] = \{1, \ldots, n\}\) for any positive integer \(n\), and Definition 2 of an allocation \(X\) being jointly optimal for a subset of agents \(S\).

**Lemma 1.** Given \( \lambda \), partition agents by equality of \( \lambda_i \) into sets \( S_1, \ldots, S_d \) such that \( \lambda(S_1) < \cdots < \lambda(S_d) \).

1. At any optimal solution \( X \) of LP(\( \lambda \)) delay is minimized first for set \( S_d \), then for \( S_{d-1} \), and so on, finally for \( S_1 \). This is equivalent to \( X \) being jointly optimal for each \( T_g \), \( \forall g \in [d] \) where \( T_g = \cup_{q=g}^{d-1} S_q \), and for any other optimal solution \( Y \) we have delay\( S_g(Y) = \text{delay}_{S_g}(X) \), \( \forall g \in [d] \).

2. Given two dual optimal solutions \((\alpha, p)\) and \((\alpha', p')\), if the first part of dual objective is same at both for some \( g \in [d] \), i.e., \( \sum_{i \in S_g,k} r_{ik} \alpha_{ik} = \sum_{i \in S_g,k} r_{ik} \alpha'_{ik} \), then for any optimal solution \( X \) of LP(\( \lambda \)), pay\( S_g(X, p) = \text{pay}_{S_g}(X, p') \).

3. Given two optimal solutions \( X \) and \( X' \) of LP(\( \lambda \)), and an optimal solution \((\alpha, p)\) of DLP(\( \lambda \)), if for any subset \( S \subseteq S_g \) for \( g \in [d] \), delay\( S(X) \leq \text{delay}_{S_g}(X') \), then pay\( S(X, p) \geq \text{pay}_{S_g}(X', p') \). The former is strict iff the latter is strict too.

**Proof.** Note that, \( T_1 = A \), \( T_d = S_d \), and \( S_g = T_g \setminus T_{g-1}, \forall g \in [d-1] \). For the first part, let us rewrite the objective function of LP(\( \lambda \)) as

\[
\sum_{g=2}^{d} (\lambda(S_g) - \lambda(S_{g-1})) \sum_{i \in T_g,j} d_{ij} x_{ij} + \lambda(S_1) \sum_{i \in T_1,j} d_{ij} x_{ij}
\]

Since \( \mathcal{M} \) satisfies extensibility, we can construct a minimum delay allocation \( X^* \) where \( S_d \) gets the best, then next best to \( S_{d-1} \), and so on to finally \( S_1 \). In other words, \( X^* \) is jointly optimal for \( T_g, \forall g \leq d \). Let \( X' \) be an arbitrary optimal solution of LP(\( \lambda \)), not constructed as \( X^* \). Then,

\[
\forall 1 \leq g \leq d, \sum_{i \in T_g,j} d_{ij} x^*_{ij} \leq \sum_{i \in T_g,j} d_{ij} x'_{ij},
\]

To the contrary suppose at least one is strict inequality. Then, since \( \lambda(S_d) > \lambda(S_{d-1}) > \cdots > \lambda(S_1) > 0 \) we have

\[
\lambda(S_1) \sum_{i \in T_1,j} d_{ij} x^*_{ij} + \sum_{g=2}^{d} (\lambda(S_g) - \lambda(S_{g-1})) \sum_{i \in T_g,j} d_{ij} x^*_{ij} < \lambda(S_1) \sum_{i \in T_1,j} d_{ij} x'_{ij} + \sum_{g=2}^{d} (\lambda(S_g) - \lambda(S_{g-1})) \sum_{i \in T_g,j} d_{ij} x'_{ij}
\]

A contradiction to \( X' \) being optimal solution of LP(\( \lambda \)). Since any minimum delay allocation for a subset gives the same total delay, the first part follows.

The second part essentially follows by applying Lemma 20 twice. For optimal pair \( X \) and \((\hat{\alpha}, \hat{p})\), and pair \( X \) and \((\alpha', p')\) we get, \( \forall g \)

\[
\lambda(S_g)\text{delay}_{S_g}(X) = \sum_{i \in S_g,k} r_{ik} \hat{\alpha}_{ik} - \text{pay}_{S_g}(X, \hat{p}), \quad \text{and} \quad \lambda(S_g)\text{delay}_{S_g}(X) = \sum_{i \in S_g,k} r_{ik} \alpha'_{ik} - \text{pay}_{S_g}(X, p')
\]

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implying, \( \text{pay}_{S_d}(X, \hat{p}) = \text{pay}_{S_d}(X, p') \) if and only if \( \sum_{i \in S_d \setminus k} r_{ik}\alpha_{ik} = \sum_{i \in S_d \setminus k} r_{ik}\alpha'_{ik} \).

For the third part, applying Lemma 20 to each agent \( i \in S \) and then taking the sum gives \( \lambda(S_g)\text{delay}_S(\hat{X}) = \sum_{i \in S} r_{ik}\alpha_{ik} - \text{pay}_S(\hat{X}, \hat{p}) \) and \( \lambda(S_g)\text{delay}_S(X') = \sum_{i \in S} r_{ik}\alpha_{ik} - \text{pay}_S(X', p) \). Combining the two, we get \( \lambda(S_g)(\text{delay}_S(\hat{X}) - \text{delay}_S(X')) = \text{pay}_S(X', p) - \text{pay}_S(\hat{X}, \hat{p}) \), thereby the lemma follows. \( \square \)

Next we show existence of a primal-dual solution where every agent spends exactly her money if \( BB \) (budget balance) and \( SC \) (subset condition) of Definition 5 are satisfied at the current value of \( \lambda \) and \( p \).

Recall the definition of proper pair \((\lambda, p)\) (Definition 6).

**Lemma 2.** If pair \((\lambda^*, p^*)\) is proper for \( \lambda^* > 0 \) then there exists an optimal solution \( X^* \) of the primal LP(\( \lambda^* \)) such that \( \text{pay}_i(X^*, p^*) = m_i, \forall i \in A \).

**Proof.** Let \( \tau_i(X) = m_i - \text{pay}_i(X, p^*) \). Without loss of generality suppose agents are ordered such that \( \tau_1(X) \geq \tau_2(X) \geq \cdots \geq \tau_n(X) \). Define \( T_k(X) = \sum_{1 \leq i \leq k} \tau_i(X) \). Let’s define the following potential function for every allocation \( X \). The potential function is \( f(\hat{X}) = \sum_k T_k(X) \).

Let \( X^* \) be an optimal solution of LP(\( \lambda^* \)) that minimizes \( f \). Order the agents such that \( \tau_1(X^*) \geq \tau_2(X^*) \geq \cdots \geq \tau_n(X^*) \). Note that \( T_n(X^*) = 0 \) (condition \( BB \)). Therefore, \( T_i(X^*) \geq 0 \) \( \forall i \in [n] \) in decreasing order. Therefore, \( f(X^*) \geq 0 \). If \( f(X^*) = 0 \) then it must be the case that \( T_i(X^*) = 0 \), \( \forall i \in [n] \) and \( \tau_i(X^*) = 0 \), \( \forall i \in [n] \). This gives \( m_i - \text{pay}_i(X^*, p^*) = 0 \), \( \forall i \) as desired.

To the contrary suppose \( f(X^*) > 0 \). Let \( \hat{X} \) be an optimal allocation of LP(\( \lambda^* \)) where delay of agent 1 is minimum, then of agent 2, and so on, finally of agent \( n \).

**Claim 5.** \( \sum_{i \leq r} \tau_i(\hat{X}) \leq 0, \forall r \in [n] \).

**Proof.** Fix an \( r \in [n] \) and define \( S = \{1, \ldots, r\} \) and \( \bar{S} = \{r+1, \ldots, n\} \). Since the total delay of all the agents is same at both \( X^* \) and \( \hat{X} \), their total payment is also same (first and third part of Lemma 1). Therefore it suffices to show \( \sum_{i \in S} \tau_i(\hat{X}) \geq 0 \) because \( \sum_{1 \leq i \leq n} \tau_i(\hat{X}) = 0 \). Let’s define \( L_g = S \cap S_g \). Note that \( \hat{X} \) is an optimal allocation in which delay of \( \bar{S} \) is maximized. We will show that in an optimal allocation if delay of \( \bar{S} \) is maximized then delay of \( L_g \) is maximized for all \( g \) and so \( m(L_g) \geq \text{pay}_{L_g}(\hat{X}, p^*) \) (SC condition). Therefore, \( m(S) \geq \text{pay}_S(\hat{X}, p^*) \). That completes the proof.

In the following we show that if the delay of \( S \) is maximized then delay of \( L_g \) maximized for all \( g \) in an optimal allocation. Consider an optimal allocation \( X' \) of LP(\( \lambda^* \)) which is constructed by first optimizing for \( S_d \setminus L_d \), then \( L_d \), then \( S_{d-1} \setminus L_{d-1} \) then \( L_{d-1} \) and so on. This is a valid construction due to the extensibility property. We claim that \( X' \) is an optimal allocation which maximizes delay of \( L_g \), \( \forall g \) individually. Total delay of \( S_g \) \( \forall g \) is the same for all optimal allocations (Lemma 1) and delay of \( (S_d \setminus S_g) \cup (S_g \setminus L_g) \) is minimized in \( X' \). Therefore, delay of \( S_g \setminus L_g \) is minimized in \( X' \) and so delay of \( L_g \) is maximized since sum of delays of \( S_g \setminus L_g \) and \( L_g \) is constant among all optimal allocations. \( \square \)

Using the above claim we get \( \exists \hat{r} \) such that \( \sum_{i \leq \hat{r}} \tau_i(\hat{X}) < 0 \) because otherwise \( \tau_i(\hat{X}) = 0, \forall i \) and \( f(\hat{X}) = 0 \). Therefore,

\begin{equation}
\sum_{r \leq n} \sum_{i \leq \hat{r}} \tau_i(\hat{X}) < 0
\end{equation}

Let’s define \( X(\delta) = (1-\delta)X^* + \delta \hat{X} \). Since optimal solutions of an LP forms a convex set, \( X(\delta) \) is an optimal solution of LP(\( \lambda^* \)) for all \( \delta \in [0, 1] \). For every pair \( i \) and \( j \) such that \( i < j \) and \( \tau_i(X^*) = \tau_j(X^*) \), we assume \( \frac{\partial \tau_i(X(\delta))}{\partial \delta} > \frac{\partial \tau_j(X(\delta))}{\partial \delta} \). Note that the assumption is without loss of generality. Therefore, there exists \( \delta \) small enough such that the order of \( \tau_i ' s \) is the same for \( X(\delta) \) as at \( X^* \). Considering that small \( \delta \), we have the following

\[ f(X(\delta)) = \sum_r \sum_{i \leq \hat{r}} \tau_i(X(\delta)) \]

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\[
\begin{align*}
&= \sum_{r} \sum_{i \leq r} (\tau_i(X^*) - \delta(\tau_i(X^*) - \tau_i(\hat{X}))) \\
&= f(X^*) - \delta(f(X^*) - \sum_{r \leq n} \sum_{i \leq r} \tau_i(\hat{X}))) \\
&= (1 - \delta)f(X^*) + \delta(\sum_{r \leq n} \sum_{i \leq r} \tau_i(\hat{X}))) \\
&< f(X^*) \quad \text{(Using (18))}
\end{align*}
\]
Therefore, we get \( f(X(\delta)) < f(X^*) \) which is a contradiction to \( X^* \) being optimal solution where \( f \) is minimized. 

While searching for the next segment during the algorithm, we would like to fix total payment of the segments that already have been created. Note that unlike scheduling on a single machine (Section 3), in this general setting we are not able to fix the allocation of the agents on the previous segments. In fact the allocation will heavily depend on what segments are created later on. However from Lemma 1 we know that the total delay of previous segments will remain unchanged. If delay is fixed then payments can be controlled using the last part of Lemma 1 if prices of goods they buy also remain unchanged. Therefore the only way to ensure that total payments are fixed seems to be, fixing prices of goods they are buying, and also hold their dual variables \( \alpha_{ik} \)s. Next lemma shows that this is indeed possible. In a number of proofs that follows we will use the following version of Farkas’ lemma.

**Lemma 21** (Farkas’ lemma [51]). Given a matrix \( A \), if the system \( Az = b \) and \( z \geq 0 \) is infeasible then there exists vector \( y \) such that \( y^TA \geq 0 \) and \( y^Tb < 0 \).

Recall notation \( 1_S \in \{0, 1\}^{|A|} \) for a subset \( S \subseteq A \) denoting indicator vector of set \( S \).

**Lemma 3.** Given a \( \lambda \), partition agents into \( S_1, \ldots, S_d \) by equality of \( \lambda_i \), where \( \lambda(S_1) < \cdots < \lambda(S_d) \). For \( R \subseteq S_d \) consider primal optimal \( \hat{X} \) that is jointly optimal for \( R \), and let \((\hat{\alpha}, \hat{p})\) be a dual optimal. Consider for some \( a > 0 \), the vector \( X' = \lambda + a1_R \). Then \( \hat{X} \) is optimal in \( LP(X') \) and there exists an optimal solution \((\alpha', p')\) of \( DLP(X') \) such that,

\[
\forall j: \quad p'_j \geq \hat{p}_j \quad \text{and} \quad \sum_{i \notin R} \hat{x}_{ij} > 0 \Rightarrow p'_j = \hat{p}_j \\
\forall i \notin R, \forall k, \quad \alpha'_{ik} = \hat{\alpha}_{ik}.
\]

**Proof.** Due to Lemma 1 it follows that \( \hat{X} \) is an optimal solution of \( LP(X') \). Suppose \( p' = \hat{p} + \delta \), then it is sufficient to show that the following system is feasible.

\[
\delta \geq 0
\]

\[
\forall j \quad s.t. \quad \sum_{i \notin R} \hat{x}_{ij} > 0 \quad \text{or} \quad \sum_i \hat{x}_{ij} < c_j : \quad \delta_j = 0 \quad \text{(19)}
\]

\[
\forall i \in R, \forall j \quad s.t. \quad \hat{x}_{ij} > 0 : \quad \lambda'_{ij}d_{ij} = \sum_k a_{ijk}\alpha'_{ik} - \hat{p}_j - \delta_j \quad \text{(20)}
\]

\[
\forall i \in R, \forall j \quad s.t. \quad \hat{x}_{ij} = 0 : \quad \lambda'_{ij}d_{ij} \geq \sum_k a_{ijk}\alpha'_{ik} - \hat{p}_j - \delta_j \quad \text{(21)}
\]

The proof is by contradiction. Suppose the system is infeasible. We will show a contradiction using Farkas’ lemma (Lemma 21). To convert (21) to equality we add slack variable \( \gamma_{ij} \). In addition, we remove all \( \delta_j \) that are set to zero in (19) from (20) and (21), and remove (19) itself from the system. Let \( T \) denote
the set of goods \( j \) such that \( \sum_{i \in R} \hat{x}_{ij} > 0 \) or \( \sum_{i} \hat{x}_{ij} < c_j \) and \( \bar{T} \) denote the set of goods not in \( T \). The remaining system can be written as follows in \( Az = b \) form in variables \( \delta_{ij} \) and \( \gamma_{ij} \):

\[
\begin{align*}
\forall i & \in R, \forall j \quad \text{s.t.} \quad \hat{x}_{ij} > 0 \& (j \in \bar{T}) : \quad \lambda_i d_{ij} + \hat{p}_j = \sum_k a_{ijk} \alpha_{ik} - \delta_j \\
\forall i & \in R, \forall j \quad \text{s.t.} \quad \hat{x}_{ij} = 0 \& (j \in T) : \quad \lambda_i d_{ij} + \hat{p}_j = \sum_k a_{ijk} \alpha_{ik} - \delta_j + \gamma_{ij} \\
\forall i & \in R, \forall j \quad \text{s.t.} \quad \hat{x}_{ij} > 0 \& (j \in T) : \quad \lambda_i d_{ij} + \hat{p}_j = \sum_k a_{ijk} \alpha_{ik} + \gamma_{ij}
\end{align*}
\]  

Due to Farkas’ lemma if the above system is infeasible then there exists \( y \) such that \( y^T A = 0 \) and \( y^T b < 0 \). That is for variables \( y_{ij}, \forall i \in R, \forall j \):

\[
y^T A \geq 0 \Rightarrow \begin{cases} \\
\forall i \in R, \forall k \quad : \sum_j a_{ijk} y_{ij} \geq 0 \\
\forall i \in R, \forall j \quad \text{s.t.} \quad x_{ij} = 0 : y_{ij} \geq 0 \\
\forall j \in \bar{T} \quad \text{s.t.} \quad \sum_{i \in R} y_{ij} \leq 0 \\
y^T b < 0 \Rightarrow \sum_{i \in R} \lambda_i d_{ij} y_{ij} + \sum_{i \in R} \hat{p}_j y_{ij} < 0
\end{cases}
\]  

Let’s consider two cases.

Case 1. \( \sum_{i \in R,j} d_{ij} y_{ij} \geq 0 \). Since \( \forall i \in R, \lambda(S_d) = \lambda_i < \lambda_i' = \lambda(S_d) + \tau \) and \( y^T b < 0 \) we get

\[
\lambda(S_d) \sum_{i \in R,j} d_{ij} y_{ij} + \sum_{i \in R,j} \hat{p}_j y_{ij} < 0
\]  

On the other hand, note that

\[
\lambda(S_d) \sum_{i \in R,j} d_{ij} y_{ij} + \sum_{i \in R,j} \hat{p}_j y_{ij} = \sum_{i \in R,j} y_{ij} (\lambda_i d_{ij} + \hat{p}_j) \\
\geq \sum_{i \in R,j} y_{ij} (\sum_k a_{ijk} \alpha_{ik}) \\
= \sum_{i \in R} \sum_k \alpha_{ik} \sum_j y_{ij} a_{ijk} \geq 0 \quad (\because (\hat{p}, \hat{\alpha}) \text{ is a solution for DLP(} \lambda))
\]

That is a contradiction.

Case 2. \( \sum_{i \in R,j} d_{ij} y_{ij} < 0 \). We will show that \( \hat{X} \) does not give min-cost allocation to agents of \( R \). Consider \( x_{ij} = \hat{x}_{ij} + \epsilon y_{ij}, \forall i \in R, \forall j \) for a small amount \( \epsilon > 0 \) and \( x_{ij} = \hat{x}_{ij}, \forall i \notin R, \forall j \). Using (23) it follows that \( X \) is feasible in \( LP(\lambda) \). In addition,

\[
\sum_i \lambda_i \sum_j d_{ij} (x_{ij} - \hat{x}_{ij}) = \lambda(S_g) \sum_{i \in R,j} d_{ij} (\epsilon y_{ij}) < 0
\]

This contradicts \( \hat{X} \) being optimal solution of \( LP(\lambda) \).

While creating next segment we need to maintain \( BB \) and \( SC \) conditions for all the previous segments. Lemma 3 ensures that prices of goods bought by agents in previous segments is fixed. If we also manage to ensure that total delay remains unchanged for previous segments, then we will be able to leverage properties from Lemma 1 to show \( BB \) and \( SC \) do remain satisfied for previous segments. The next lemma establishes exactly this.

**Lemma 22.** Given \( \lambda > 0 \) let the partition of agents by equality of \( \lambda_i \) be \( S_1, \ldots, S_{k-1}, A' \) such that \( \lambda(S_1) < \cdots < \lambda(S_{k-1}) < \lambda(A') \). For an \( S_k \subset A' \), let \( X' \geq \lambda \) be such that the induced partition is \( S_1, \ldots, S_k, A' \setminus S_k \) and \( \lambda(S_1) < \cdots < \lambda(S_k) < \lambda(A' \setminus S_k) \). Then for any group \( g < k \), we have \( \text{delay}_{S_g}(X) = \text{delay}_{S_g}(X') \) where \( X \) and \( X' \) are solutions of \( LP(\lambda) \) and \( LP(X') \) respectively. And for any subset \( T \subset S_g \),

\[
\max_{X \text{ optimal of } LP(\lambda)} \text{delay}_T(X) = \max_{X \text{ optimal of } LP(X')} \text{delay}_T(X)
\]
Proof. An optimal solution of both $LP(\lambda)$ and $LP(\lambda')$ first minimizes total delay of $A'$ then of $S_{k-1}$ and so on finally of $S_1$ (Lemma 1). Within $A'$, $LP(\lambda')$ may first minimize for $A' \setminus S_k$ and then of $S_k$. Thus, the optimal solution set may shrink as we go from $\lambda$ to $\lambda'$. However, due to extensibility, if $X$ and $X'$ are optimal solution of $LP(\lambda)$ and $LP(\lambda')$ then, $\text{delay}_{S_k}(X) = \text{delay}_{S_k}(X')$. Further, by Lemma 1 the optimal solution of both $LP(\lambda)$ and $LP(\lambda')$ where delay is maximized essentially minimizes total delay of $A' \bigcup_{q=g+1}^{k-1}(S_q \cup (S_g \setminus T))$ and then of $T$, then of $\bigcup_{q=g}^{k-1}S_q$. By extensibility condition delay $T$ at any such allocation remains the same.

The next lemma shows that if before we start our search for next segment, already created segments $S_1, \ldots, S_{k-1}$ satisfies $BB$ and $SC$ w.r.t. $(\lambda^{\text{cur}}, p^{\text{cur}})$, and the remaining agents satisfy $SC$, then for the value of $a$ where minimum of $f_a$ is zero, the minimizer gives the next segment without ruining the former. Recall that, for a given $a \geq 0$ and $\epsilon > 0$, $\lambda^{\text{new}} = \lambda^a + A1_{A' \setminus S_k}$, and the prices $p^{\text{new}}$ are valid and optimal for $DLP(\lambda^{\text{new}})$, where $1_{A' \setminus S_k}$ is an indicator vector of set $A' \setminus S_k$, and $\lambda^a$ as defined in (4). We will also use notation $p^a$ to denote prices at the valid optimal solution of $DLP(\lambda^a)$ (See (5)), i.e., in the sense guaranteed by Lemma 3.

Lemma 4. Suppose that for some $a \geq 0$,

$$S_k \in \arg \min_{S \subseteq A', S \neq \emptyset} \{ f_{\lambda^a} p^a (S) \}.$$ 

Further, suppose that $f_{\lambda^a} p^a (S_k) = 0$, and $S_k$ be a maximal such set. Then there exists a rational number $\epsilon > 0$ of polynomial-size such that, w.r.t. $(\lambda^{\text{new}}, p^{\text{new}})$ as defined above, $S_1, \ldots, S_k$ satisfy both $BB$ and $SC$, and $A' \setminus S_k$ satisfy $SC$.

Proof. First part of Lemma 1 implies that every optimal solution of $LP(\lambda^{\text{new}})$ minimizes delay of sets $A' \setminus S_k, S_k, \ldots, S_1$ in that sequence, while optimal of $LP(\lambda^{\text{cur}})$ minimizes delay of sets in sequence $A', S_{k-1}, \ldots, S_1$. Therefore clearly the set of optimal solutions of $LP(\lambda^{\text{new}})$ is a subset of the optimal solutions of $LP(\lambda^{\text{cur}})$. This together with the fact that at $(\lambda^{\text{cur}}, p^{\text{cur}})$ $BB$ and $SC$ are satisfied for each $g \leq k-1, S_g$ for any optimal $X'$ of $LP(\lambda^{\text{new}})$, we have $\text{pay}_{S_g}(X', p^{\text{cur}}) = m(S_g)$. Let $\alpha^{\text{cur}}$ be corresponding dual variable, then Lemma 20 gives $\lambda(S_g) \text{delay}_{S_g}(X') = \sum_{i \in S_g, k} r_{ik} \alpha^{\text{cur}} - \text{pay}_{S_g}(X', p^{\text{cur}})$.

Since $p^{\text{new}}$ is a valid solution of $DLP(\lambda^{\text{new}})$, at corresponding valid $(\alpha^{\text{new}}, p^{\text{new}})$ value of $\alpha^{\text{new}}_i$ for each $i \notin A'$ is same as $\alpha^{\text{cur}}_i$. Using Lemma 20 we get $\lambda(S_g) \text{delay}_{S_g}(X') = \sum_{i \in S_g, k} r_{ik} \alpha^{\text{new}}_i - \text{pay}_{S_g}(X', p^{\text{new}}) = \sum_{i \in S_g, k} r_{ik} \alpha^{\text{cur}}_i - \text{pay}_{S_g}(X', p^{\text{cur}})$. This together with the above equality gives $\text{pay}_{S_g}(X', p^{\text{new}}) = \text{pay}_{S_g}(X', p^{\text{cur}}) = m(S)$. Hence $BB$ is satisfied by $S_1, \ldots, S_{k-1}$ at $(\lambda^{\text{new}}, p^{\text{new}})$. By the same reasoning, and using Lemma 22 and third part of Lemma 1 we get that they also satisfy $SC$.

Note that $\lambda^{\text{new}} = \lambda^a$ with $\epsilon$ added to $\lambda$, of agents in $A' \setminus S_k$. And $p^{\text{new}}$ is a valid optimal of $DLP(\lambda^{\text{new}})$ obtained starting from $p^a$ where $\alpha_{ik}$ of agents not in $A' \setminus S_k$, and prices of goods “bought by them” are held fixed (Lemma 3). Since function $f_{\lambda^a} p^a$ keeps track of surplus budget, and $S_k$ is the minimizer of $f_{\lambda^a} p^a$ where surplus budget of $S_k$ is zero at $p^a$ in addition, it follows that for every subset of $S_k$ the surplus budget is non-negative. Thus we get $SC$ for $S_k$ at $(\lambda^{\text{new}}, p^{\text{new}})$ using Lemma 22. For $BB$ note that we have $\lambda^{\text{new}} (S_k) < \lambda^{\text{new}} (A' \setminus S_k)$. Hence, due to first part of Lemma 1, optimal allocations $X$ of $LP(\lambda^{\text{new}})$ will give first to $A' \setminus S_k$ minimum delay, and then next minimum to $S_k$. This is exactly same as maximizing delay $S_k$ (X) among optimal of $LP(\lambda^a)$ (where $\lambda^a(S^*) = \lambda^a (A' \setminus S^*) > \lambda^a (S_g), \forall g \leq k-1$). Due to the fact that $f_{\lambda^a} p^a (S_k) = 0$, at such an allocation we also have $m(S_k) = \text{pay}_{S_k}(X, p^a)$. This will be same as $m(S_k) = \text{pay}_{S_k}(X, p^{\text{new}})$ due to construction of $p^{\text{new}}$ from $p^a$ in Lemma 3. Since at every such allocation delay $S_k$ (X) remains the same, $\text{pay}_{S_k}(X, p^{\text{new}})$ remains the same (Lemma 22).

For the second part, namely $A' \setminus S_k$ satisfies $SC$ w.r.t. $(\lambda^{\text{new}}, p^{\text{new}})$, it suffices to show existence of $\epsilon > 0$ such that $f_{\lambda^{\text{new}}} p^{\text{new}} (T) \geq 0$, $\forall T \subset A' \setminus S_k$. We will show this in Lemma 26 below.
Above lemma implies that if the minimizer of $f_a$ gives zero value, then it forms the next segment. One crucial task therefore is to find a minimizer of $f_a$ efficiently. Next lemma and (7) show that $f_a$ is a submodular function, implying its minimizer can be found in polynomial time.

**Lemma 5.** Given $a \geq 0$, function $f_a$ is submodular over set $A'$.

**Proof.** For ease of notation let us use $f$ to denote function $f_a$, $\lambda = \lambda^a$, $p = p^a$, and $\alpha$ be the dual vector that forms valid solution of $DLP(\lambda^a)$ together with $p^a$. Recall that agent set $A$ is partitioned by equality of $\lambda_i^a$ into sets $S_1, \ldots, S_{(k-1)}$, $A'$ such that $\lambda^a(S_1) < \cdots < \lambda^a(S_{(k-1)}) < \lambda^a(A')$.

Let $S \subset T \subset A'$ and $a \notin T$. Define $S' = S \cup \{a\}$ and $T' = T \cup \{a\}$. It suffices to show the following

$$f(S') - f(S) \geq f(T') - f(T)$$

Let’s recall the following two complementary slackness conditions.

$$\forall i \in A, \forall k \in C : \sum_{j \in G} a_{ijk} x_{ij} \geq r_{ik} \perp \alpha_{ik} \geq 0$$

$$\forall i \in A, \forall j \in G : \lambda_i d_{ij} \geq \sum_k a_{ijk} x_{ik} - p_j \perp x_{ij} \geq 0$$

Using these it is easy to get the following, where $\lambda^*$ is the $\lambda_i$ of agents in $A'$ which we know is the same.

$$\lambda^* \text{delay}_S(X) = \text{pay}_S(X, p) + \sum_{i \in S, k} \alpha_{ik} r_{ik} \quad (26)$$

For set $S \subseteq A'$ let us denote its complement within $A'$ by $\bar{S} = A' \setminus S$. From the first part of Lemma 1 we know that optimal solution of $LP(\lambda)$ will first minimize delay of set $A'$ since it has highest $\lambda$ value. Note that, $\bar{S} = S' \cup \{a\}$. If $X^S$ is an optimal solution of $LP(\lambda)$ where delay of $S$ is maximized then by extensibility it can constructed as follows: within minimization for set $A'$ first minimizing delay for $\bar{S}'$, then for $a$, and lastly for $S$. Then we have:

$$f(S') - f(S) = m_a - \text{pay}_{S'}(X^S, p) + \text{pay}_S(X^S, p)$$

$$= m_a - \sum_{i \in S', k} \alpha_{ik} r_{ik} + \lambda^* \text{delay}_{S'}(X^S) + \sum_{i \in S, k} \alpha_{ik} r_{ik} - \lambda^* \text{delay}_S(X^S) \quad (\text{Using } (26))$$

$$= m_a - \sum_k \hat{\alpha}_a r_{ak} + \lambda^* (\text{delay}_{S'}(X^S) - \text{delay}_S(X^S))$$

$$= m_a - \sum_k \hat{\alpha}_a r_{ak} + \lambda^* \text{delay}_a(X^S) \quad (X^S \text{ construction})$$

Similarly we get,

$$f(T') - f(T) = m_a - \sum_k \hat{\alpha}_a r_{ak} + \lambda^* \text{delay}_a(X^T)$$

where $X^T$ is an optimal solution where delay of $T$ is maximized, which can be constructed by first minimizing delay for $\bar{T}'$, next for $a$, and last for $T$. Recall that $S \subset T$ and so $\bar{T} \subset \bar{S}$. We constructed $X^S$ and $X^T$ by first optimizing for $\bar{S}'$ and $\bar{T}'$ and then adding $a$. Therefore, $\text{delay}_a(X^S) \leq \text{delay}_a(X^T)$ and so

$$f(S') - f(S) \geq f(T') - f(T)$$

The NextSeg subroutine does binary search on the value of $a$ to find the one where minimizer of $f_a$ gives zero. In next few lemmas we show why binary search is the right tool to find this critical value of $a$. Essentially we show Lemma 6 which has four parts and we will show them in separate lemmas.

**Lemma 23.** If $A'$ satisfies SC condition of Definition 5 at $(\lambda^{\text{cur}}, p^{\text{cur}})$ then $f_{\lambda^{\text{cur}}, p^{\text{cur}}}(T) \geq 0, \forall T \subset A'$.

**In other words** $g(0) \geq 0$. 

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Proof. Since $A'$ satisfies SC w.r.t. $(\lambda^{\text{cur}}, p^{\text{cur}})$, by definition of SC condition (Definition 5), it follows that for any $T \subset A'$, $m(T) - \text{pay}_T(X^T, p^{\text{cur}}) \geq 0$, where $X^T$ is an optimal solution of $LP(\lambda^{\text{cur}})$ where delay$_T(X)$ is maximized. Thus, $f_{\lambda^{\text{cur}}, p^{\text{cur}}}(T) = m(T) - \text{pay}_T(X^T, p^{\text{cur}}) \geq 0$. \hfill \Box

Lemma 24. Assuming sufficient demand, for any $T \subset A'$, value $f_a(T)$ monotonically and continuously decreases with increase in $a$. Further, for any given $a \geq 0$, $\exists a' > a$ such that $f_a(A') < f_a(A')$.

Proof. There is a unique valid price vector $p^a$ and $p^{a'}$ constituting optimal solution of $DLP(\lambda^a)$ and $DLP(\lambda^{a'})$. If we apply Lemma 3 for $\lambda = \lambda^a$ and $\lambda' = \lambda^{a'}$, then we get that $p^a \leq p^{a'}$. Note that the partition of agents by equality of $\lambda_i$ is the same at both $\lambda^a$ and $\lambda^{a'}$ and further their ordering by the value of $\lambda(S)$ is also same. Therefore, due to the first part of Lemma 1 every optimal solutions of $LP(\lambda^a)$ are the same $LP(\lambda^{a'})$. Hence, among them the ones minimizing delay$_T(X)$ for any given $T \subset A'$ are the same, say $X^T$ is one of them. Note that delay$_T$ remains the same at all such allocations.

$$f_a(T) = m(T) - \text{pay}_T(X^T, p^a) = m(T) - \sum_{i \in T,j} x_{ij}^T p_{ij}^a \geq m(T) - \sum_{i \in T,j} x_{ij}^T p_{ij}^{a'} = f_a(T)$$

Since solutions of linear programs change continuously with change in parameters, the first part follows.

For the second part, we know that $\forall a' \geq a$, valid prices $p^{a'} \geq p^a$ from the above. It suffices to show that $\exists a' \geq a$ such that $p_{ij}^{a'} > p_{ij}^a$ for some good $j$ that is bought by agents of $A'$. Suppose not, then $p^{a'} = p^a$, $\forall a' \geq a$ since prices of the rest of the goods are fixed anyway. Let $\hat{p} = p^a$, $\hat{\lambda} = \lambda^a$, and $\hat{X}$ be an optimal solution of $LP(\hat{\lambda})$. Note that $\hat{X}$ is an optimal of $LP(\lambda^{a'})$ as well for any $a' > a$. Then, the following system has a solution for any $a' \geq a$, since $\hat{p}$ together with some $\alpha$ gives a optimal solution of $DLP(\lambda^{a'})$ that satisfies complementarity with $\hat{X}$. Here $\alpha$ are variables, and $\hat{\lambda} = \hat{\lambda}(A')$.

$$\forall i \in A', \forall j: \begin{cases} \hat{x}_{ij} > 0 \Rightarrow (\hat{\lambda} + \delta)d_{ij} = \sum_{k} a_{ijk} \alpha_{ik} - \hat{p}_j \\ \forall i \in A', \forall j: \hat{x}_{ij} = 0 \Rightarrow (\hat{\lambda} + \delta)d_{ij} \geq \sum_{k} a_{ijk} \alpha_{ik} - \hat{p}_j \\ \forall i \in A', \forall k: \sum_{j} a_{ijk} \hat{x}_{ij} > r_{ik} \Rightarrow \alpha_{ik} = 0 \end{cases}$$

$$\alpha \geq 0$$

Removing $\alpha_{ik}$ variables that are set to zero, and then applying Farkas’ lemma (Lemma 21) we get that following system in $x^{a'}_{ij}$ variables is infeasible for any $\delta \geq 0$: 

$$\forall i \in A', \forall k: \begin{cases} \sum_{j} a_{ijk} \hat{x}_{ij} = r_{ik} \Rightarrow \sum_{j} a_{ijk} x^{a'}_{ij} \geq 0 \\ \forall i \in A', \forall k: \hat{x}_{ij} = 0 \Rightarrow x^{a'}_{ij} \geq 0 \end{cases}$$

$$(\hat{\lambda} + \delta) \sum_{i \in A', j} d_{ij} x^{a'}_{ij} + \sum_{i \in A', j} \hat{p}_j x^{a'}_{ij} < 0$$

Let $u \in A'$ be an agent and let $x'_u$ be its optimal allocation at zero prices. Since $\forall j, \hat{x}_{uj} \leq 1$ due to feasibility in $LP(\hat{\lambda})$, it can not be optimal at zero prices due to sufficient demand condition. Hence $\sum_{j} d_{uj} x'_u < \sum_{j} d_{uj} \hat{x}_{uj}$. Set $x^{a'}_{ij} = 0$, $\forall i \neq u$, $\forall j$, and $x^{a'}_{ij} = x'_u - \hat{x}_{uj}$, $\forall j$. Clearly $X^a$ satisfies the first set of conditions above for all $i \in A'$, $i \neq u$. For agent $u$ since $x'_u$ is optimal of $(\text{Delay LP}(i))$, $\forall k$ with $\sum_{j} a_{ijk} \hat{x}_{ij} = r_{ik}$, we have

$$\sum_{j} a_{ijk} x^{a'}_{ij} \geq r_{ik} = \sum_{j} a_{ijk} \hat{x}_{ij} \Rightarrow \sum_{j} a_{ijk} (x^{a'}_{ij} - \hat{x}_{ij}) \geq 0 \Rightarrow \sum_{j} a_{ijk} x^{a'}_{ij} \geq 0.$$

Second set of conditions are satisfied by construction. Hence setting $\delta > \frac{|\sum_{i \in A', j} \hat{p}_j x^{a'}_{ij} + \lambda \sum_{i \in A', j} d_{ij} x^{a'}_{ij}|}{|\sum_{i \in A', j} d_{ij} x^{a'}_{ij}|}$, makes the above feasible. A contradiction.

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Lemma 25. There exists a rational \( a_h \geq 0 \) of polynomial size such that \( g(a_h) \leq 0 \). Further, given existence of \( a > 0 \) for set \( S \subseteq A' \) such that \( f_a(S) = 0 \), such an \( a \) can be found by solving a feasibility LP of polynomial size.

Proof. Since value of \( g(a) \leq f_a(S) \), \( \forall S \subseteq A' \), it suffices to show that there exists some rational \( a_h > 0 \) such that \( f_{a_h}(A') \leq 0 \). By definition of \( A^a \) (4), partitions of agents by equality of \( \lambda_i \) and ordering of these groups by value of it, are the same for all \( a \geq 0 \). Let \( \bar{X} \) be an optimal solution of \( LP(\lambda_i) \). From the proof of Lemma 24 we have that \( p^a \) is monotonically increasing. By continuity of parameterized LP solutions we know that it is increasing continuously. Therefore, it suffices to show that there is some \( a \geq a_0 \) such that \( f_a(A') \leq 0 \) or in other words pay \( A'(\bar{X}, p^a) \geq m(A') \).

To the contrary suppose not. Let \( \hat{\lambda} = \lambda^0 \) and \( \hat{p} = p^0 \). Solve the following linear program where \( a, \delta, \alpha \) are variables, and \( \lambda = \hat{\lambda}(A') \), \( c_j = \sum_{i \in A'} \hat{x}_{ij} \),

\[
\begin{align*}
\text{max} : & \quad \sum_j (\hat{p}_j + \delta_j) c_j \\
\text{s.t.} : & \quad \forall j : \quad \sum_{i \in A'} \hat{x}_{ij} > 0 \text{ or } \sum_i \hat{x}_{ij} < 1 \Rightarrow \delta_j = 0 \\
& \quad \forall i \in A', \forall j : \quad \hat{x}_{ij} > 0 \Rightarrow (\hat{\lambda} + a)d_{ij} = \sum_k a_{ijk} \alpha_{ik} - \hat{p}_j - \delta_j \\
& \quad \forall i \in A', \forall j : \quad \hat{x}_{ij} = 0 \Rightarrow (\hat{\lambda} + a)d_{ij} \geq \sum_k a_{ijk} \alpha_{ik} - \hat{p}_j - \delta_j \\
& \quad \forall i \in A', \forall k : \quad \sum_j a_{ijk} \hat{x}_{ij} > r_{ik} \Rightarrow \alpha_{ik} = 0 \quad a \geq 0, \delta \geq 0, \alpha \geq 0
\end{align*}
\]

Note that at any feasible point of the above, we have \( \sum_j (\hat{p}_j + \delta_j) c_j \geq \text{pay}_{A'}(\bar{X}, p^a) \). It has a finite optimal or else there is a \( a \geq 0 \) where \( \text{pay}_{A'}(\lambda^T, p^a) \geq m(T) \). Let \((a^*, \delta^*, \alpha^*)\) be an optimal solution. By construction of above linear program \( p^{a^*} = \hat{p} + \delta \) is the valid price vector at \( \lambda^{a^*} \). By Lemma 24, we can find an \( a' > a^* \) where \( f_{a'}(A') < f_{a^*}(A) \Rightarrow \text{pay}_{A'}(\bar{X}, p^{a'}) > \text{pay}_{A'}(\bar{X}, p^{a^*}) \Rightarrow \sum_j p^{a'}_j c_j > \sum_j p^{a^*}_j c_j \). If \( \alpha^{a'} \) together with \( p^a \) forms a valid optimal of \( DLP(\lambda^{a'}) \), then for \( \delta' = p^{a'} - \hat{p} \) and \( \alpha' = \alpha^{a'} \), point \((a', \delta', \alpha')\) is a feasible point in the above linear program with better objective value. This contradicts optimality of \((a^*, \delta^*, \alpha^*)\) in it.

From the above argument and by monotonicity of \( f_a(A') \) in \( a \) (Lemma 24) it follows that there exists an \( a^*_h > 0 \) such that \( f_{a^*_h}(A') = 0 \). Again since value of \( g(a^*_h) \leq f_{a^*_h}(S), \forall S \subseteq A' \), existence of such an \( a^*_h \) that is a rational number of polynomial sized implies the first part of the lemma. Next we will show this for any \( S \subseteq A' \) thereby proving the first and second part simultaneously.

For \( S \subseteq A' \), existence of \( a \geq 0 \) with \( f_a(S) = 0 \) implies that valid \((p^a = \hat{p} + \delta, \alpha^a)\) of \( DLP(\lambda^a) \) is a feasible point in the following where \( \hat{X} \) is the optimal solution of \( LP(\lambda^a) \) where delay of \( S \) is maximized:

\[
\begin{align*}
\forall j : & \quad \sum_{i \in A'} \hat{x}_{ij} > 0 \text{ or } \sum_i \hat{x}_{ij} < 1 \Rightarrow \delta_j = 0 \\
\forall i \in A', \forall j : & \quad \hat{x}_{ij} > 0 \Rightarrow (\hat{\lambda} + a)d_{ij} = \sum_k a_{ijk} \alpha_{ik} - \hat{p}_j - \delta_j \\
\forall i \in A', \forall j : & \quad \hat{x}_{ij} = 0 \Rightarrow (\hat{\lambda} + a)d_{ij} \geq \sum_k a_{ijk} \alpha_{ik} - \hat{p}_j - \delta_j \\
\forall i \in A', \forall k : & \quad \sum_j a_{ijk} \hat{x}_{ij} > r_{ik} \Rightarrow \alpha_{ik} = 0 \\
& \quad \sum_{i \in S} (\hat{p}_j + \delta_j)( \sum_{i \in S} \hat{x}_{ij} ) = m(S) \\
& \quad a \geq 0, \delta \geq 0, \alpha \geq 0
\end{align*}
\]

Hence existence of rational such \( a \) of polynomial-size, and that it can be found by solving a feasibility LP follow.

The next lemma follows essentially from Lemmas 23, 24, and 25.

Lemma 6. Function \( g \) satisfies the following: (i) \( g(0) \geq 0 \). (ii) \( f_a(S) \) is continuous and monotonically decreasing in \( a \), \( \forall S \subseteq A' \), therefore \( g \) is continuous and monotonically decreasing. (iii) \( \exists a_h \geq 0 \) a rational
number of polynomial size such that \( g(a_h) \leq 0 \), and \( g \) has a zero of polynomial-size. (iv) Given a set \( S \subseteq A' \), if \( f_a(S) > 0 \) and \( f_a'(S) < 0 \) for \( a' > a > 0 \), then \( 3a^* \geq 0 \) such that \( f_{a^*}(S) = 0 \) and such an \( a^* \) can be computed by solving a feasibility linear program of polynomial-size.

Proof. Part (i) follows from Lemma 23. Part (ii) from Lemma 24 and the fact that \( g(a) \) is minimum of \( f_a(S) \) over all subsets \( S \subseteq A' \). Minimum of continuously decreasing functions is also continuously decreasing. Parts (iii) and (iv) from Lemma 25.

From Lemma 4 we know that at the end of NextSeg subroutine, when we have \( S^* \) a maximal minimizer of \( f_{a^*} \) and \( f_{a^*}(S^*) = 0 \) then \( S^* \) together with previous segments \( S_1, \ldots, S_{k-1} \) will satisfy BB and SC at \( (\lambda^{new}, p^{new}) \). Next we show existence of appropriate \( \epsilon > 0 \) so that SC condition is satisfied for \( A' \setminus S^* \). This will complete the missing part of Lemma 4.

Lemma 26. If \( a^* > 0 \) such that \( g(a^*) = 0 \) and \( S^* \subseteq A' \) be maximal set such that \( f_{a^*}(S^*) = 0 \), then \( \exists \epsilon > 0 \) rational of polynomial size such that SC condition is satisfied for \( A' \setminus S^* \) w.r.t. \( (\lambda^{new}, p^{new}) \) where \( \lambda^{new} = \lambda^{*} + \epsilon 1 \), \( p^{new} \) is a valid optimal of DLP(\( \lambda^{new} \)), and \( 1 \) is indicator vector of set \( A' \setminus S^* \).

Proof. To show SC for \( A' \setminus S^* \) w.r.t. \( (\lambda^{new}, p^{new}) \), it suffices to show the following:

- \( f_{\lambda^{new} \cdot p^{new}}(T) \geq 0, \forall T \subseteq A' \setminus S^* \).

Since \( S^* \) was the maximal set where value of function \( f_{a^*} \) is 0 and minimum value of \( f_{a^*} \) is zero, we have that for any \( S \subseteq A' \setminus S^* \) and \( f_{a^*}(S \cup S^*) > 0 \) if there is an \( a^T > a^* \) such that \( f_{a^T}(T) = 0 \), then by applying Lemma 25 it is a rational of polynomial size and can be computed by solving a linear program, otherwise \( a^T = \infty \). Among all of these pick the least one, let \( a^\min \). It has to be strictly more than \( a^* \).

Let \( A'' = A' \setminus S^* \), and fix set \( S \subseteq A'' \). Let \( \lambda = \lambda^* + \epsilon \alpha \), \( \alpha \) be an optimal allocation of LP(\( \lambda^* \)) where \( A'' \setminus \alpha \) gets the best, then \( S^* \) and then the rest of the segments. Let \( \hat{p} = p^{a^*} \). Similar to Lemma 25 solve the following LP to compute maximum value of \( a^S \) such that \( m(S) \geq \hat{p} \) when \( \lambda \) is of only \( A'' \) is increased. Here \( b, \delta, \alpha \) variables, \( c_j^S = \sum_{i \in S} \hat{x}_{ij}, \forall j, \) and \( \hat{\lambda} = \hat{\lambda}_i, i \in A'' \).

\[
\max : b \quad \text{s.t.} \\
\forall j : \sum_{i \in A''} \hat{x}_{ij} > 0 \text{ or } \sum_i \hat{x}_{ij} < 1 \implies \delta_j = 0 \\
\forall i \in A'' \setminus S, \forall j : \hat{x}_{ij} > 0 \implies (\hat{\lambda} + b) \alpha_{ik} - \hat{p}_j - \delta_j \\
\forall i \in A'' \setminus S, \forall j : \hat{x}_{ij} = 0 \implies (\hat{\lambda} + b) \alpha_{ij} \geq R_{ik} \delta_j \implies \forall(i, j) \\
\forall i \in A'' \setminus S, \forall k : \sum_j \hat{x}_{ijk} > \hat{r}_{ik} \implies \alpha_{ik} = 0 \\
\sum_{i \in S} \hat{p}_j + \delta_j c_j^S \leq m(S) \\
\forall j: b \geq 0, \delta \geq 0, \alpha \geq 0
\]

Clearly, \( b = 0, \delta_j = 0, \forall j, \) and \( \alpha_{ik} = \hat{\alpha}_{ik}, \forall(i, k) \) is feasible where \( (\hat{\alpha}, \hat{p}) \) is the valid optimal solution of LP(\( \lambda^* \)). If there is a finite optimal of the above LP then set \( a^S = b \) otherwise set \( a^S = \infty \). Since \( a^T > a^* \), we have \( a^S > 0 \). Taking \( \epsilon \) to be less than \( a^S \), \( \forall S \subseteq A'' \) will suffice, due to monotonicity of \( f_a \) function (Lemma 24). Since \( a^S \) are polynomial size, there is an \( \epsilon \) of polynomial size.

Putting everything together next we argue that at the end of the algorithm all the created segments satisfy BB and SC w.r.t. \( (\lambda^{cur}, p^{cur}) \). In other words, \( (\lambda^{cur}, p^{cur}) \) are proper.

Lemma 27. The \( \lambda^{cur} \) and \( p^{cur} \) obtained at the end of Algorithm 3 are proper (Definition 5).

Proof. Lemmas 4 and 26 imply that if at the beginning of \( k^{th} \) call to NextSeg, w.r.t. \( (\lambda^{cur}, p^{cur}) \), \( S_1, \ldots, S_{k-1} \) satisfies BB and SC, and \( A' \) satisfies SC, then at the end of it, w.r.t. \( (\lambda^{new}, p^{new}) \), \( S_1, \ldots, S_k \) satisfies BB and SC, and \( A' \setminus S_k \) satisfies SC. Applying this inductively, starting from \( k = 1 \) where \( A' = A \),
and resetting \( A' = A' \setminus S_k \) every time, the theorem follows. This is because if we made \( s \) calls in total to \( \text{NextSeg} \) forming segments \( S_1, \ldots, S_s \), and \( A' = \emptyset \) at the end, then all \( s \) segments satisfy \( BB \) and \( SC \) w.r.t. \((\lambda^{cur}, p^{cur})\) at the end. Thus \((\lambda^{cur}, p^{cur})\) are proper.

Next we show correctness of Algorithm 3 using Lemmas 2, 6 and 27, and Theorem 3, and the next theorem follows.

**Theorem 6.** Given a market \( M \) satisfying extensibility and sufficient demand, Algorithm 3 returns its equilibrium allocation and prices in time polynomial in the size of the bit description of \( M \).

**Proof.** The fact that if Algorithm 3 terminates in polynomial time then it returns equilibrium allocation and prices of market \( M \) follows from Lemmas 27 and 2, and Theorem 3.

The question is why should the algorithm terminate, and that too in polynomial-time. Note that, every call to NextSeg reduces size of active set of agents. Therefore, if the instance has \( n \) agents then algorithm makes at most \( n \) calls to NextSeg. Subroutine NextSeg does binary search for value of \( a \) between 0 and a polynomial-sized rational. In each iteration of binary search it minimizes a submodular function, and in each call to the submodular function we will be solving at most constantly many linear programs of polynomial size. Thus submodular minimization can be done in polynomial time. Due to Lemma 6 and in particular Lemma 25 value of \( a \) where \( g(a) \) becomes zero for some set is rational of polynomial-size. Hence overall the binary search has to terminate in polynomial time. The next loop again takes at most \( O(n) \) iterations, with sub-modular minimization in each. Thus the NextSeg subroutine terminates in polynomial-time. Finding allocation satisfying Budget constraint(i) of all the agents \( i \in A \) at the end of the algorithm, given prices \( p^{cur} \) and \( \lambda \) values \( \lambda^{cur} \) is equivalent to solving the feasibility linear program (3). Thus, overall the algorithm terminates in polynomial time.

We get our main result using Theorem 6.

**Theorem 2.** [Extensibility and sufficient demand implies polynomial time algorithm] There is a polynomial time algorithm that computes a market equilibrium allocation \( X \) and prices \( p \) for any market \( M \) that satisfies extensibility and sufficient demand.

### G Fairness and Incentive Compatibility Properties

In this section we show fairness properties of our general model, and incentive compatibility properties of our algorithm as a mechanism in scheduling application. Yet another utility model is that of quasi-linear utilities, where the agent also specifies an “exchange rate” between delay and payments, and wants to minimize a linear combination of the two. We show in Section G.3 that for such a utility model there is no IC mechanism that is also Pareto optimal and anonymous, even with a single good and two agents.

#### G.1 Fairness Properties

The first welfare theorem for traditional market models says that at equilibrium the utility vector of agents is Pareto-optimal among utility vectors at all possible feasible allocations. We first show similar result for our markets that satisfy mild condition of sufficient demand (see Definition 4). For our model the set of feasible delay cost vectors are

\[
D = \{(\text{delay}_1(X), \ldots, \text{delay}_n(X)) \mid x_i \text{ is feasible in } CC(i) \text{ for each agent } i \in A, \\
\text{and } X \text{ satisfies Supply constraints for each good } j \in G\}
\]

**Theorem 7.** Given market \( M \) satisfying sufficient demand, the delay cost vector at any of its equilibrium is Pareto-optimal in set \( D \).
Proof. Let \((X^*, p^*)\) be an equilibrium. Using Lemma 19 we know that for some vector \(\lambda^* > 0\), \(X^*\) is a solution of \(LP(\lambda^*)\). Let \(d_i^* = \text{delay}_i(X^*)\). If there is \(d \in D\) such that \(d_i \leq d_i^*, \forall i \in A\) with at least one strict inequality, then the allocation corresponding to \(d\) is feasible in \(LP(\lambda^*)\) and would give strictly lower objective value, a contradiction.

The next theorem establishes envy-freeness for the general model and follows directly from the equilibrium condition that every agent demands an optimal bundle at given prices.

**Theorem 8.** Equilibrium allocation of a given market \(M\) is envy-free.

Next we show that at equilibrium each agent gets a “fair share”: the equilibrium allocation Pareto-dominates an “equal share” allocation, where each agent gets an equal amount of each resource. This property is also known as sharing incentive in the scheduling literature [29].

**Theorem 9.** Given a market \(M\), let \(X\) be an allocation where agent \(i\) gets \(\frac{m_i}{\sum_{i \in A} m_i}\) amount of each good, i.e., \(x_{ij} = \frac{m_i}{\sum_{i \in A} m_i}, \forall i \in A, \forall j \in G\). Then at any equilibrium \((X^*, p^*)\) of market \(M\), \(\text{delay}_i(X^*) \leq \text{delay}_i(X), \forall i \in A\).

**Proof.** At equilibrium under-sold goods have price zero, and no agent spends more than its budget. This gives

\[
\sum_{i,j} x_{ij}^* p_j^* \leq \sum_i m_i \Rightarrow \sum_j p_j^* \sum_i x_{ij}^* \leq \sum_i m_i \Rightarrow \sum_j p_j^* \leq \sum_i m_i
\]

Therefore, for agent \(i\), bundle \(x_i\) where amount of every good is exactly \(\frac{m_i}{\sum_{i \in A} m_i}\) costs money \(\sum_j x_{ij}^* p_j^* = \frac{m_i}{\sum_{i \in A} m_i} \sum_j p_j^* \leq m_i\). Thus, it is affordable at prices \(p^*\). However, she preferred \(x_i^*\) instead, which implies either \(x_i'\) is not feasible in \(CC(i)\) in which case \(\text{delay}_i(x_i')\) is infinity, or she prefers \(x_i^*\) to \(x_i'\). In either case we get \(\text{delay}_i(X^*) \leq \text{delay}_i(X)\). 

\(\square\)

**G.2 Scheduling: Algorithm as a Truthful Mechanism**

Market based mechanisms are usually not (dominant strategy) incentive compatible (IC), except in the large market assumption where each individual agent is too small to influence the price, and therefore can be assumed to act as a price taker. Somewhat surprisingly, we can show IC, in a certain sense, of the market based mechanism for the special case of our market that corresponds to the scheduling setting presented in Section 3, and its generalization described in 2.3 with multiple machine types.

We show that our market based mechanism is IC in the following sense: non-truthful reporting of \(m_i\) and \(r_{ij}\)s can never result in an allocation with a lower delay cost. A small modification to the payments, keeping the allocation the same, makes the entire mechanism incentive compatible for the setting in which agents want to first minimize their delay and subject to that, minimize their payments.

The first incentive compatibility assumes that utility of the agents is only the delay, and does not depend on the money spent (or saved). Such utility functions have been considered in the context of online advertising [5, 23, 42]. It is a reflection of the fact that companies often have a given budget for procuring compute resources, and the agents acting on their behalf really have no incentive to save any part of this budget. Our model could also be applied to scenarios with virtual currency in which case the agents truly don’t have any incentive to minimize payments.

The second incentive compatibility does take payments into account, but gives a strict preference to delay over payments. Such preferences are also seen in the online advertising world, where advertisers want as many clicks as possible, and only then want to minimize payments. The modifications required for this are minimal, and essentially change the payment from a “first price” to a “second price” wherever required.
Yet another utility model is that of quasi-linear utilities, where the agent also specifies an “exchange rate” between delay and payments, and wants to minimize a linear combination of the two. We show in Section G.3 that for such a utility model there is no IC mechanism that is also Pareto optimal and anonymous, even with a single good and two agents. Pareto optimality is a benign notion of optimality that has been used as a benchmark for designing combinatorial auctions with budget constraints [21, 24, 30]. Anonymity is also a reasonable restriction, which disallows favoring any agent based on the identity. In the face of this impossibility, our mechanism offers an attractive alternative.

G.2.1 Pure delay minimization

Suppose that there are \( j \) independent copies of the basic scheduling setting in Section 3, with the requirement of agent \( i \) for the \( j^{th} \) copy being \( r_{ik} \). In this section we show that our algorithm is actually IC, i.e., the agents have no incentive to misreport \( m_i \)s or \( r_{ik} \)s, assuming that agents only want to minimize their delay cost and don’t care about their payments as long as they are within the budgets. Note that reporting lower \( m_i \) or higher \( r_{ik} \) are the only possible types of misreport. Fixing preferences of all agents except agent \( i \), consider two runs of the algorithm, one where agent \( i \) is truthful and another where she misreports her preferences. In particular, say agent \( i \) either reports a lower budget \( m_i' \), and/or a higher requirements \( r_{ik}' \) for good \( j \).

Consider the first iteration in which the two runs differ, and let \( (S_1, \lambda_1) \) and \( (S_2, \lambda_2) \) be the segments found respectively in the truthful and non-truthful runs in this iteration. For any \( \lambda \), any \( p \), and any set \( S \) that does not contain \( i \), \( f_{p, \lambda}(S) \) remains the same between the two runs; for any set \( S \) that contains \( i \), \( f_{p, \lambda}(S) \) is strictly smaller in the non-truthful run. Hence, \( i \) does not belong to any of the segments found in earlier iterations, and \( S_2 \) necessarily contains \( i \).\footnote{Consider the possibilities where \( i \notin S_2 \) and note that \( S_2 \) cannot be the minimizer in the non-truthful run given that \( S_1 \) is the minimizer in the truthful run.} Further, \( \lambda_2 < \lambda_1 \).

Let \( A' \) be the set of agents who are not in one of the segments found prior to the current iteration. By definition \( A' \) is the same for both the runs, and includes \( i \), as argued in the previous paragraph. Let \( X^1 \) and \( X^2 \) be respectively the allocations output by the algorithm for the truthful and the non-truthful runs. We will show the existence of a weakly feasible allocation \( X' \) such that (1) For every agent \( i' \in A', i' \neq i \), his delay in \( X' \) is no higher than his delay in \( X^1 \), and (2) For agent \( i \), his allocation in \( X' \) is the same as his allocation in \( X^2 \).

This implies that \( i \) is no better off in the non-truthful run, because of the following reasoning. The total delay of all the agents in \( A' \) is minimized in \( X^1 \), therefore the total delay of all the agents in \( A' \) cannot be lower in allocation \( X' \), even when the delay for agent \( i \) is calculated using only his actual requirements. Since no other agent has a higher delay in \( X' \), it is impossible for \( i \) to get a lower delay.

It remains to show the existence of \( X' \) as claimed. We define \( X' \) differently based on whether the agent is in \( S_2 \) or not.

**Case 1:** \( i' \in S_2 \): In this case, \( x_{i'} = x_i^2 \). This satisfies the second requirement since \( i \in S_2 \). Since \( \lambda_2 < \lambda_1 \), every agent in \( S_2 \) faces a smaller price, for every copy \( j \) and every time slot in which she is allocated. For \( i' \neq i \), given the same budget and the same requirements, this actually implies that her delay in \( X^2 \) is strictly smaller than her delay in \( X^1 \).

**Case 2:** \( i' \notin S_2 \): In this case, we first start with the allocation \( X^1 \), in the slots \([1, r_j(B \setminus S_2)]\) for each copy \( j \). Note that these slots have not been allocated at all in Case 1. Consider the total deficit after this allocation. This must be equal to the total amount of slots in \([1, r_j(B \setminus S_2)]\) that are allocated to agents in \( S_2 \) by \( X^1 \), because of feasibility of \( X^1 \). Now re-allocate these empty slots in \([1, r_j(B \setminus S_2)]\) to make up for the remaining requirement of these agents, and note that this can only lower the delay.
G.2.2 Secondary preference for payments

In this section, we consider the utility model where an agent wants to first minimize her flow-time, and subject to that, wants to further minimize her payments. We keep the same allocation as Algorithm 1, but change the payments of some agents, and show that this is IC.

We first define the set of agents whose payments will be modified. Recall that Algorithm 1 outputs a sequence of segments, where each segment corresponds to a pair \((\lambda, S)\). Call an agent marginal if he gets the latest slots in his segment. This includes agents who are in singleton segments, as well as agents who just happen to get such an allocation even though they are in a segment with other agents. We modify the payments of only the marginal agents; all non-marginal agents pay their budget.

**Lemma 28.** Any non-marginal agent gets a strictly higher delay cost for any misreport of his information.

**Proof.** Consider the proof of incentive compatibility for only delay cost minimization in Section G.2.1, and the notation therein. Note that if \(S_2 \neq \{i'\}\), then the delay cost of \(i'\) strictly increases. Now suppose \(S_2 = \{i'\}\). In the new allocation \(f^2\), agent \(i'\) gets the latest slots among all agents in \(B\). Since \(i'\) is not a marginal agent, he was getting a strictly better allocation in \(f^1\), and the lemma follows.

This shows that the mechanism is IC for non-marginal agents, even with their payments equal to the budgets.

We now define the modification to payments for marginal agents. As in Section G.2.1, misreports can still not get a better delay cost for marginal agents, since the allocation remains the same. The only possibility is that misreporting can decrease payments, while keeping the delay cost the same. Marginal agents can decrease their budgets, still get the same allocation, and pay less in the equilibrium payment. This has a limit; at some lower budget declaration, they get “merged” with a previous segment, and any further lowering of the budget will strictly lower their delay cost. **The payment of a marginal agent is defined to be the infimum of all budget declarations for which the lower segments are unaltered, i.e., the run of the algorithm up to the previous segment remains unchanged.**

We now argue that this mechanism is IC, for marginal agents. We only need to consider misreports that don’t change the allocation, since those that do only give a higher delay cost. Among these, misreporting the budget clearly has no effect on the payment. Finally, we argue that reporting a higher \(r_{ik}\) can only lead to a higher payment. This is because the budget at which the agent merges with the previous segment happens at a higher value, as can be seen from the formula for \(\lambda_S\).

G.3 Quasi Linear Utility Model

In this section we consider a quasi linear utility model for the agents. In this model, agents can choose to tradeoff payment for delay cost, as specified by an “exchange rate”, denoted by \(\eta_i\), for agent \(i\). We consider the design of incentive compatible (IC) auctions, that are also Pareto optimal. In the related literature of IC auctions for combinatorial auctions with budget constraints, this has been adopted as the standard notion of optimality. The usual notion of social welfare is ill fitted for the case of budgets.\(^8\)

As in Section 2 let the allocation of agent \(i\) for good \(j\) denoted by \(x_{ij}\), but now we don’t have prices for the slots. Instead we simply have a payment for each agent, denoted by \(\text{payment}(i)\) for agent \(i\). The allocation and the payments are together called the outcome of the auction. Agent \(i\) now wants to minimize the objective

\[
\sum_j d_{ij}x_{ij} + \eta_i\text{payment}(i).
\]

\(^8\)Of course, the revenue objective is also widely considered, and continues to make sense even in the presence of budgets.
A type of an agent is its budget $m_i$, its covering constraints $CC(i)$, and its $\eta_i$. An auction is (dominant strategy) IC if for any agent, misreporting its type does not lead to an outcome with a lower objective, no matter what the other agents report. An outcome is *Pareto optimal* if for no other outcome,

1. all agents, including the auctioneer, are at least as well off as in the given outcome, and
2. at least one agent is strictly better off.

The auctioneer’s objective is to simply maximize the sum of all the payments.

We also restrict the auction to be *anonymous*, which means that the auction cannot rely on the identity of the agents. Formally, an auction is anonymous if it is invariant under all permutations of agent identities.

The main result of this section is an impossibility.

**Theorem 10.** There is no IC, Pareto optimal, and anonymous auction for our scheduling problem with quasi linear utilities, for the case of a single good and two agents.

Since a single good and two agents is the most basic case, an impossibility follows for all generalizations as well.

The theorem follows from a reduction to a combinatorial auction with additive valuations, and an impossibility result of Dobzinski et al. [21]. Consider an auction for a single divisible item, with budget constraints. Agent $i$ has valuation of $v_i$ per unit quantity of the item, and a budget $m_i$. The outcome of the auction is an allocation $x_i$ and payment $\text{payment}(i)$ for agent $i$, such that $\sum_i x_i \leq 1$ and $x_i \in [0, 1]$. The utility of agent $i$ is $v_i x_i - \text{payment}(i)$, and the budget constraint as before is that $p_i \leq m_i$. IC and Pareto optimality are as before, and we need an additional notion of *individual rationality* (IR): $v_i x_i - \text{payment}(i) \geq 0$. Dobzinski et al. [21] showed the following impossibility.

**Theorem 11 (Dobzinski et al. [21]).** There is no IC, Pareto optimal, IR and anonymous auction for auctioning a single divisible good to 2 agents with budget constraints.

**Proof of Theorem 10.** Consider an instance of the scheduling problem of Section 3 with a single machine and two agents, where each agent requires 1 unit of the good. Pareto optimality implies that goods are not wasted, so the entire first two slots are completely allocated. If agent $i$ gets $x_i$ units of the slot $t = 1$, then his delay cost is $x_i + 2(1 - x_i)$. His objective is then

$$x_i + 2(1 - x_i) + \eta_i \text{payment}(i) = 2 - \eta_i \left( \frac{1}{\eta_i} x_i - \text{payment}(i) \right).$$

Minimizing this objective is equivalent to maximizing $\frac{1}{\eta_i} x_i - \text{payment}(i)$, which is exactly as in the single divisible good auction with $v_i = \frac{1}{\eta_i}$. We also show that the IC constraint for the scheduling problem implies the IR constraint for the divisible good case. If the IR constraint is violated, i.e., $\frac{1}{\eta_i} x_i < \text{payment}(i)$, then the value of the objective of agent $i$ for this outcome is strictly smaller than 2. Then the agent is better off stating a budget of 0. This will force his payment to 0. The worst delay cost he can get is 2, so his total objective value is 2.

Therefore, an IC, Pareto optimal, and anonymous auction for our scheduling problem implies an IC, Pareto optimal, IR, and anonymous auction for the divisible good case, and the theorem follows. □