DYNAMICS OF POSITIVE STEADY-STATE SOLUTIONS OF A NONLOCAL DISPERAL LOGISTIC MODEL WITH NONLOCAL TERMS

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Abstract. In this paper, we investigate a class of nonlocal dispersal logistic equations with nonlocal terms

\[
\begin{aligned}
\frac{u_t}{t} &= D_u + u^q \left( \lambda + a(x) \int _{\Omega} b(x)u^p \right), & \text{in } \Omega \times (0, +\infty), \\
u(x, 0) &= u_0(x) \geq 0, & \text{in } \Omega, \\
u &= 0, & \text{on } \mathbb{R}^N \setminus \Omega \times (0, +\infty),
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 1) \) is a bounded domain, \( \lambda \in \mathbb{R}, \ 0 < q \leq 1, \ p > 0, \ a, b \in C(\Omega), \ b \geq 0, \ b \neq 0 \) and \( a \) verifies either \( a > 0 \) or \( a < 0 \). \( D_u = \int _\Omega J(x - y)u(y, t)dy - u(x, t) \) represents the nonlocal dispersal operators, which is continuous and nonpositive. Under some suitable assumptions we establish the existence, uniqueness or multiplicity and stability of positive stationary solution with nonlocal reaction term by using sub-supersolution methods, Leray-Schauder degree theory and Lyapunov-Schmidt reduction and so on.

1. Introduction. In this paper we consider the nonlocal dispersal Logistic equation with nonlocal terms:

\[
\begin{aligned}
\frac{u_t}{t} &= D_u + u^q \left( \lambda + a(x) \int _{\Omega} b(x)u^p \right), & \text{in } \Omega \times (0, +\infty), \\
u(x, 0) &= u_0(x) \geq 0, & \text{in } \Omega, \\
u &= 0, & \text{on } \mathbb{R}^N \setminus \Omega \times (0, +\infty),
\end{aligned}
\]

where \( \Omega \) is a smooth bounded domain of \( \mathbb{R}^N \), the function \( u(x, t) \) represents the population of some species inhabiting a region \( \Omega \) surrounded by an inhospitable area, since the population is subject to homogeneous Dirichlet boundary conditions. The term

\[
D_u(x, t) = \int _\Omega J(x - y)u(y, t)dy - u(x, t)
\]
represents the nonlocal dispersal operator with continuous and nonnegative dispersal kernel $J$. In this context, $\lambda$ is a parameter which represents the intrinsic growth rate of the species, $a(x)$ describes the limiting effects of crowding when $a < 0$ and the intraspecific cooperation when $a > 0$.

The motivation to study problem (1) comes from the problem to model the behavior of a species inhabiting in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, whose the classical logistic equation with Laplacian diffusion is given by

$$
\begin{cases}
  u_t - \Delta u = u(\lambda - K(x)u^p), & \text{in } \Omega \times (0, +\infty), \\
  u = 0, & \text{on } \partial\Omega \times (0, +\infty),
\end{cases}
$$

where $K$ is a positive function denoting the carrying capacity, that is, $K(x)$ describes the limiting effect of crowding of the population. Problems like (3) appear quite often in some models related with population dynamics, one can refer to [18, 19, 20, 28, 26, 27].

Since (3) is a local problem, the crowding effect of the population $u$ at $x$ only depends on the value of the population in the same point $x$, however, the movement of an individual species is sometimes determined by surrounding conditions around the point where the species stays. It seems more realistic to take account of nonlocal effects in the study of species dynamics, see [13, 1, 2, 9, 12, 11]. Usually, this nonlocal effect depends on the value of the population around $x$, that is, the crowding effect depends on the value of $u$. Concerning these reasons, system (3) is replaced by the following more general diffusive logistic population models with nonlocal effect:

$$
\begin{cases}
  u_t - \Delta u = u^p \left( \lambda - \int_{\Omega} K(x, y)u^p(y)dy \right), & \text{in } \Omega \times (0, +\infty), \\
  u = 0, & \text{on } \partial\Omega \times (0, +\infty),
\end{cases}
$$

where $q = 1$ and $p$ is a positive constant, $K: \Omega \times \Omega \rightarrow \mathbb{R}$ is a nonnegative and nontrivial continuous function. Chen and Shi [8] investigated the dynamical behaviors of system (4) when $p = 1$ and the kernel function $K(x, y)$ is a continuous and nonnegative function on $\Omega \times \Omega$ satisfying $\int_{\Omega} K(x, y)u(y)dy > 0$ for all positive continuous functions $u$ on $\Omega$. By means of the implicit function theorem, Chen and Shi [8] obtained the existence of the positive steady-state solution of system (4) when $0 < \lambda - \lambda_1 \ll 1$, where $\lambda_1$ denotes the first eigenvalue of the minus Laplacian operator under homogeneous Dirichlet boundary conditions. Allegretto and Nistri [1] found that (4) possesses a unique positive solution when $\lambda > \lambda_1$ if $K(x, y) = K_\delta(|x - y|)$ is a mollifier in $\mathbb{R}^N$, i.e., $K_\delta(|x - y|) \in C_0^\infty$, $\int_{\mathbb{R}^N} K_\delta(|x - y|)dy = 1$ for any $x$ with

$$K_\delta(|x - y|) = 0 \text{ when } |x - y| \geq \delta,$$

and $K_\delta(|x - y|)$ is bounded away from zero when $|x - y| < \mu < \delta$. Later, Corrêa, Delgado, Suárez, et al. [11] proved that (4) possesses a unique positive solution if $K(x, y)$ is a separable variable, i.e., $K(x, y) = g(x)h(y)$, where $h \geq 0, h \neq 0$ and $g(x) > 0$ in $\Omega$. Sun et al. [37] investigated the existence of positive solutions of system (4) with $K(x, y) = K_1(|x - y|)$ and $\Omega = (-1, 1)$, where $K_1: [0, 2] \rightarrow (0, \infty)$ is a nondecreasing and piecewise continuous function satisfying $\int_{\Omega} K_1(y)dy > 0$. Recently, Corrêa, Delgado and Suárez [11] also studied (4) when $0 < q \leq 1, p > 0$ and obtained the existence, uniqueness or multiplicity and stability of positive steady states solution. Roughly speaking, there are several difficulties that appear.
when one introduces a nonlocal term in (4), which will also appear in this paper. Let us point some of them:

1: In general, (1) has not a variational structure and so we can not apply the powerful tool of variational methods to attack (1).

2: Secondly, if \( q \neq 1 \), that is to say (1) is the non-homogeneous case and hence system (1) does not satisfy a maximum principle, and as main consequences, we can not apply directly some classical methods as sub-supersolutions, that is to say, we shall make some preparation and then use it.

3: Third, the linearized operator of (1) is an integral-differential operator and it will not be self-adjoint. Along these analysis, it is more difficult for us to study (1).

On the other hand, it is well known that the nonlocal dispersal operator \( D \) and variations of it arise from applications as well as pure mathematics \([7, 14, 16, 24, 34]\). In (1), the dispersal of population is nonlocal since the dispersal effect depends on the values of \( u \) in a neighborhood of \( x \) through the term \( \int_\Omega J(x-y)u(y,t)dy \) that has been recently used to model several physical situations. As stated in \([14, 22]\), if \( u(y,t) \) is deemed to be the density at location \( y \) at time \( t \), and \( J(x-y) \) is deemed to be the probability distribution of jumping from \( y \) to \( x \), then \( \int_\Omega J(x-y)u(y,t)dy \) represents the rate at which individuals are arriving to location \( x \) from all other places. The mechanism of dispersal is a major focus of theoretical interest and has received much attention recently. Most continuous models related to dispersal are based upon reaction-diffusion equations, and hence there is a lot of literature on the research of nonlocal dispersal problems recently(see \([4, 21, 23, 25, 29, 30, 32, 34, 35, 36, 38]\)). For example, Bates and Zhao \([5]\) studied the existence, stability and uniqueness of steady states of the nonlinear evolution equation

\[
\begin{align*}
  u_t &= \int_\Omega J(x-y)u(y,t)dy + b(x)u(x,t) + f(x,u(x,t)), \\
  u(x,0) &= \phi(x),
\end{align*}
\]

where \( \phi \in C(\Omega) \), \( J, b \) and \( f \) are sufficiently smooth functions and \( J \) is positive. Later, Sun, Yang and Li\([35]\) obtained a criterion on the existence, uniqueness and stability of steady-states of (5) when \( f(x,u) = \lambda g(x)f(u) \) by using some auxiliary equations and maximum principle. Recently, Sun, Li and Wang \([35]\) also established some results of the positive steady-states (5) when \( b(x) = 1 \) (respectively, \( \int_\Omega J(x-y)dy < 1 \)) which is called nonlocal Dirichlet problem (respectively, called nonlocal Neumann problem). Later, Sun, Li and Wang \([36]\) investigated similar problem when \( f(x,u) = \lambda m(x)u - c(x)u^p \) by using the similar approach with \([35]\). Due to the differences between the Laplacian operator and the dispersal operator, it is difficult for us to prove the existence of principal eigenvalues for nonlocal problem \( Du = \sigma u \) and hence it will bring some obstacle to investigate the dynamics of (1).

A solution of (1) is called a steady-state if it is time independent. We are interested in the steady-state of (1). So we consider the nonlocal equation with nonlocal terms as follows:

\[
\begin{align*}
  -Du &= u^q \left( \lambda + a(x) \int_\Omega b(x)u^p \right), & \text{in } & \Omega, \\
  u(x,0) &= u_0(x) \geq 0, & \text{in } & \Omega, \\
  u &= 0, & \text{on } & \mathbb{R}^N \setminus \Omega,
\end{align*}
\]
Note that the dispersal takes place only in $\Omega$ in both (1) and (6). There is no flux of individuals across the boundary. For convenience, throughout this paper, we make the following assumptions:

(H1): $J \in C(\mathbb{R}^N)$ verifies $J(0) > 0$, $J \geq 0$ and $\int_{\mathbb{R}^N} J(x) dx = 1$;

(H2): $b \in C(\Omega)$, $b \geq 0$, $b \neq 0$;

(H3): $a$ verifies either $a > 0$ or $a < 0$.

Few people considered the dependence of the existence of the nonhomogeneous steady-state solutions upon the changing of the intrinsic growth rate. As Sibly [33] reported, the determinants of the intrinsic growth rate include temperature, PH value and other environmental stressors, where the influence of the temperature is the main factor. Birch [6] found that the species cannot vary but it could maintain its numbers from generation to generation near some so called the optimum temperature point which made the intrinsic growth rate up to the maximum value, while the intrinsic growth rate can vary at other points. Therefore, it is reasonable to regard the intrinsic growth rate as a bifurcation parameter in order to see how the change of the intrinsic growth rate results in the presence of spatially nonhomogeneous steady-state solutions near the trivial steady-state solution of (1). And hence we will take the intrinsic growth rate $\lambda$ as bifurcation parameter. First of all, we consider the homogeneous case of (1), that is, $q = 1$, $p > 0$ and $a < 0$ (or $a > 0$), we will respectively use the sub-supersolution method and Sweeping principle theorem to obtain the existence and uniqueness of the steady-state solution of (1) as $\lambda$ cause some oscillations, besides we also derive the results of stability by analyzing the characterize equation of (1). It is well known that the stability of the steady-state solution $u^*(x)$ of system (1) means the globally asymptotically stable, i.e., there holds

$$\lim_{t \to +\infty} u(x, t; u_0) = u^*(x), \text{ uniformly in } \Omega.$$  

Especially, in view of Lyapunov-Schmidt reduction method, we can obtain the concrete form of the local positive steady-state solutions in a local range of $\lambda$. On the other hand, we consider the nonhomogeneous case of (1) i.e., $0 < q < 1$, $p > 0$ and $a < 0$ (or $a > 0$), we will use the Lerray-Schauder degree theory, the principal eigenvalue of dispersal operator and maximum principle to obtain the existence, uniqueness of the steady-state solution of (1) as $\lambda$ similarly cause some oscillations and the results of stability by analyzing the characterize equation of (1).

Our main results are stated as follows:

Theorem 1.1. Assume $q = 1$.

(1): Assume that $a < 0$, then, there exists a globally asymptotically stable, continuous and positive steady-state solution of (1) if and only if $\lambda > \lambda_1(\Omega)$, where $0 < \lambda_1(\Omega) < 1$ is the principal eigenfunction of the dispersal operator $D$ in $\Omega$ with Dirichlet boundary condition and verifies $u_\lambda \to 0$ uniformly in $\Omega$ as $\lambda \to \lambda_1(\Omega)$. Moreover, the solution satisfies:

$$\left( \frac{\lambda - \lambda_1(\Omega)}{\sup_{x \in \Omega} (-a(x)) \sup_{x \in \Omega} b(x)} \right)^{\frac{1}{p}} \| u_\lambda \|_{L^p(\Omega)} \leq \left( \frac{\lambda}{\inf_{x \in \Omega} (-a(x)) \inf_{x \in \Omega} b(x)} \right)^{\frac{1}{p}},$$

and

$$\| u_\lambda \|_{\infty} \geq \frac{\lambda - \lambda_1(\Omega)}{\sup_{x \in \Omega} (-a(x)) \| b \|_{L^1(\Omega)}},$$

where $\lambda_1(\Omega)$ is the principal eigenvalue of the dispersal operator $D$ in $\Omega$ with Dirichlet boundary condition and verifies $u_\lambda \to 0$ uniformly in $\Omega$ as $\lambda \to \lambda_1(\Omega)$.
for $\lambda > \lambda_1(\Omega)$. Agreement commonly known as \(\inf_{x \in \Omega}(-a(x))\inf_{x \in \Omega}b(x) = +\infty\)
if \(\inf_{x \in \Omega}(-a(x)) = 0\) or \(\inf_{x \in \Omega}b(x) = 0\).

(2): Assume that \(a > 0\). Then, there exists a unique, unstable and positive steady-state solution of (1) if and only if \(\lambda < \lambda_1(\Omega)\) and verifies \(u_\lambda \to 0\)
uniformly in \(\Omega\) as \(\lambda \to \lambda_1(\Omega)\). Moreover, the solution satisfies:
\[
\left(\frac{-\lambda}{\sup_{x \in \Omega}a(x)\sup_{x \in \Omega}b(x)}\right)^{\frac{1}{p}} \leq \|u_\lambda\|_{L^p(\Omega)} \leq \left(\frac{\lambda_1(\Omega) - \lambda}{\inf_{x \in \Omega}a(x)\inf_{x \in \Omega}b(x)}\right)^{\frac{1}{p}}
\]
for \(\lambda < \lambda_1(\Omega)\). Similar agreement commonly known as \(\frac{\lambda_1(\Omega) - \lambda}{\inf_{x \in \Omega}a(x)\inf_{x \in \Omega}b(x)} = +\infty\)
if \(\inf_{x \in \Omega}a(x) = 0\) or \(\inf_{x \in \Omega}b(x) = 0\).

Especially, we can obtain the local concrete form of the solution \(u_\lambda\) if \(q = 1\) and \(p \geq 1, p \in \mathbb{N}\) as follows:

**Corollary 1.** Assume \(q = 1\) and \(p \geq 1, p \in \mathbb{N}\), then there exist a constant \(\epsilon > 0\)
and a continuously differentiable mapping \(\lambda \to \vartheta_\lambda\) from \((\lambda_1(\Omega) - \epsilon, \lambda_1(\Omega) + \epsilon)\) to \(\mathbb{R}\)
such that Eq. (1) has a nontrivial solution
\[
\vartheta_\lambda \varphi_1(x) + h(\vartheta_\lambda \varphi_1(x), \lambda),
\]
which exists for \(\lambda \in (\lambda_1(\Omega) - \epsilon, \lambda_1(\Omega)) \cup (\lambda_1, \lambda_1(\Omega) + \epsilon)\) and satisfies
\[
\lim_{\lambda \to \lambda_1(\Omega)} u_\lambda = 0,
\]
where \(\varphi_1\) is the principal eigenfunction of the dispersal operator \(-D\) in \(\Omega\) with Dirichlet boundary condition.

**Remark 1.** In view of Corollary 1, we observe that the results obtained in Corollary 1 do not contradict with the fact \(u_\lambda \to 0\) as \(\lambda \to \lambda_1(\Omega)\) in Theorem 1.1.

With respect to the nonhomogeneous dispersal case, \(0 < q < 1\), we get:

**Theorem 1.2.** Assume that \(0 < q < 1\) and \(a < 0\). Then, there exists a positive steady-state solution of (1) if and only if \(\lambda > 0\). Moreover, there exist \(\lambda_*\) and \(\lambda^*\)
satisfying \(0 < \lambda_* < \lambda^* < +\infty\) such that

(1): If \(p + q < 1\) and \(\lambda \geq \lambda^*\), then there exists a unique positive steady-state solution.

(2): If \(p + q = 1\) and \(|a|\) small, then there exists a unique and globally asymptotically stable positive steady-state solution.

(3): If \(p + q > 1\) and \(\lambda \leq \lambda_*\), then there exists a unique and globally asymptotically stable positive steady-state solution.

**Theorem 1.3.** Assume that \(a > 0\) and \(0 < q < 1\).

(a): Assume \(p + q < 1\). Then, there exists \(\lambda < 0\) such that (1) admits a positive steady-state solution for \(\lambda \geq \lambda_\lambda\). Furthermore, the positive steady-state solution is unique, globally asymptotically stable if \(\lambda \geq 0\).

(b): Assume that \(p + q = 1\). Then, there exists a constant \(A := \int_\Omega b(x)\omega_\omega^p(x)dx\), where \(\omega_a\) is the unique positive solution of
\[
\begin{cases}
-Du = a(x)u^q, & \text{in } \Omega, \\
u = 0, & \text{on } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

Then
(1): If $A < 1$, then (1) admits a positive, unique and globally asymptotically stable steady-state solution if and only if $\lambda > 0$.

(2): If $A = 1$, then (1) admits a positive, unique infinite steady-state solution if and only if $\lambda = 0$.

(3): If $A > 1$, then (1) admits a positive unstable steady-state solution if and only if $\lambda < 0$.

(c): Assume $p + q > 1$. Then, there exists $\overline{\lambda} < 0$ such that (1) admits a positive solution for $\lambda \leq \overline{\lambda}$.

Remark 2. Here, we would like to mention that the differences and similarities between those in previous works of Corrèa [11] and our paper are the following.

Differences:

1: First, we emphasize that the investigated models are different: the previous one that only is the reaction term nonlocal, whereas its diffusive terms is Laplacian operator. However, the latter one (our paper) that not only is the dispersal term nonlocal but also the reaction term, which brings much difficulties in this paper. It is noted that the dispersal operator is just needed the condition that variable $u$ is continuous in $\overline{\Omega}$, whereas the Laplacian operator requires variable $u$ with two order derivatives in $\Omega$, that is to say, our model is more widely used than the one in [11].

2: Secondly, we derive the concrete form of steady state solutions of (1) under the local range of $\lambda$ (see Corollary 1) for some suitable conditions, however, it was not considered in [11].

3: Finally, we still investigate the stability of the trivial zero solution (see Corollary 2), which was not investigated too in [11].

Similarity: We obtain the existence, uniqueness and stability of steady state solutions of (1) under the global range of $\lambda$ (see Theorem 1.1-1.3) which are agree with the results in Corrèa [11] Theorem 1.1-1.3.

The remaining parts of the paper are structured in the following way. In section 2, we give some spectral theory and some basic sub-supersolution methods and other auxiliary Lemmas. In section 3, we give some results of stability. we prove Theorem 1.1 and Corollary 1 by applying the sub-supersolution method, sweeping principle and Lyapunov-Schmidt reduction in section 4. In section 5, we use the Leray-Schauder degree theory to show the existence of an unbounded continuum of positive solution of (1) for $q < 1$. Finally, section 6 and section 7 are respectively devoted to prove Theorem 1.2 and 1.3.

2. Nonlocal spectral theory and some Lemmas. In this section, we shall introduce some basic results. To begin with, we consider the following eigenvalue equation

$$\int_{\Omega} J(x - y)u(y)dy + h(x)u(x) = \sigma u(x), \text{ in } \Omega,$$

(9)

where $h \in C(\overline{\Omega})$. Since $J$ is continuous, it is obvious that all the positive eigenfunctions associated with a principal eigenvalue are continuous and every eigenvalue $\sigma$ of (9) satisfies $\sigma \in \mathbb{R}$. It is noted that (9) may not possess principal eigenvalue in generally. Here, we give some useful tools to make some important nonlocal estimates.
Coville [10], Shen and Zhang [39] gave some examples to show that (9) may not admit a principal eigenvalue. At the same time, Coville [10] gave a sufficient condition to ensure the existence of the principal eigenvalue of (9), that is

Lemma 2.1. [10] If \( \frac{1}{M - h(x)} \notin L^1_{\text{loc}}(\Omega) \), then there exists a principal eigenvalue of (10), where \( h = \max_{x \in \Omega} h(x) \).

Lemma 2.2. [17] Assume that \( J \) satisfies (H1) and \( b \in C(\Omega) \). If (9) admits a principal eigenvalue \( \sigma_p \), then \( \sigma_p \) is the only eigenvalue possessing a positive eigenfunction and satisfies

\[
\sigma_p = \sup_{u \in L^2(\Omega), u \neq 0} \frac{\int_{\Omega} \int_{\Omega} J(x - y)u(x)u(y)dydx + \int_{\Omega} h(x)u^2(x)dx}{\int_{\Omega} u^2(x)dx}. \tag{10}
\]

Moreover, the supremum can be attained by a strictly positive eigenfunction and \( \sigma_p \) is simple.

Remark 3. 1.: We know that \( \sigma_p \) defined in (10) is Lipschitz continuous with respect to \( h \) by a similar discussion of [10].

2.: In particular, recall [15], if \( h(x) = -1 \), then system (10) admits an eigenvalue \( \lambda_1(\Omega) \) associated to an eigenfunction \( \varphi_1 \in C(\Omega) \) which is positive in \( \Omega \). Moreover, \( \lambda_1(\Omega) \) is simple, unique and verifies \( 0 < \lambda_1(\Omega) < 1 \). It can be variationally characterized as

\[
\lambda_1(\Omega) = 1 - \sup_{0 \neq u \in L^2(\Omega)} \frac{\int_{\Omega} \int_{\Omega} J(x - y)u(x)u(y)dydx}{\int_{\Omega} u^2(x)dx}.
\]

The following lemma gives the existence of the principal eigenfunction of (10), one can refer to [17, 34].

Lemma 2.3. Assume that \( h(x) \in C(\Omega) \) and let \( \sigma_p \) be given in (10). Then (9) admits a principal eigenfunction if and only if \( \sigma_p > \max_{x \in \Omega} h(x) \).

Next, we will investigate an important equation as follows:

\[
-J \ast \bar{u} + m(x)\bar{u} - c(x) \int_{\Omega} d(x)d\bar{u} = \sigma u(x), \tag{11}
\]

where \( m, c, d \in C(\Omega) \) and \( \sigma \in \mathbb{R} \). Since the principal eigenvalue is an important index and we will study the stability to study the system by the tools of principal.

In view of Lemma 2.1, we know that (11) may not admit a principal eigenvalue. If we admit a principal eigenvalue of (11) denoted by \( \sigma_p := \lambda_1(-J \ast + m; c, d) \), then we will give a criteria to ascertain the sign of \( \lambda_1(-J \ast + m; c, d) > 0 \) as below.

Proposition 1. [11]

(1): Assume that there exists a positive function \( \bar{u} \in C(\Omega) \), such that

\[-J \ast \bar{u} + m(x)\bar{u} - c(x) \int_{\Omega} d(x)d\bar{u} > 0 \text{ in } \Omega.
\]

Then \( \lambda_1(-J \ast + m; c, d) > 0 \).

(2): Assume that there exists a positive function \( \underline{u} \in C(\Omega) \), such that

\[-J \ast \underline{u} + m(x)\underline{u} - c(x) \int_{\Omega} d(x)d\underline{u} < 0 \text{ in } \Omega.
\]

Then \( \lambda_1(-J \ast + m; c, d) < 0 \).
Since the sub-supersolution method has been used in reaction-diffusion equations with nonlocal terms, for example [26] and [27], which will be useful in the subsection. Here we generalize these results of reaction-diffusion equations for reaction-dispersal operators without monotonicity assumption, and hence we consider the following system

\[
\begin{align*}
- \int_{\Omega} J(x - y)u(y)dy &= p(x, u, B(u)), & \text{in } \Omega, \\
0 &= u, & \text{on } \mathbb{R}^N \setminus \Omega,
\end{align*}
\]

where \( p : \Omega \times \mathbb{R} \to \mathbb{R} \) a continuous function and \( B : C(\Omega) \to \mathbb{R} \).

First we give the definition of sub-section solution in a reaction-dispersal system.

**Definition 2.4.** A positive function \( \bar{u} \in C(\Omega) \) is said to be a supersolution of (12), if

\[- \int_{\Omega} J(x - y)\bar{u}(y)dy > p(x, \bar{u}, B(\bar{u})) \geq 0.\]

A subsolution is defined similarly by reversing the inequality. We are assuming throughout that all the subsolutions and supersolutions are bounded.

**Lemma 2.5.** Suppose that (H1) is satisfied and \( p \in C^1(\Omega \times \mathbb{R}^2, \mathbb{R}) \), \( p(x, 0, B(0)) = 0 \). If (12) has a positive supersolution \( u \) and subsolution \( \underline{u} \) defined in \( \Omega \) such that \( \underline{u} \leq \bar{u} \), then (12) has a positive solution \( u \) satisfying \( u \in [\underline{u}, \bar{u}] \).

**Proof.** We defined a set \( \mathcal{U} = \{ u \in L^\infty(\Omega) : \underline{u} \leq u \leq \bar{u} \} \). Denote \( p(x, u, B(u)) \) by \( g(x, u) \) and notice the assumption of \( p \in C^1(\Omega \times \mathbb{R}^2, \mathbb{R}) \), we have \( g \in C^1(\Omega \times \mathbb{R}, \mathbb{R}) \), and

\[g_u(x, u) + \beta \gg 0\] (13)

for \((x, u) \in \Omega \times \mathcal{U}\) provided \( \beta \) is sufficiently large.

For such \( \beta \), we defined a mapping \( T : \mathcal{U} \mapsto C(\Omega) \) by \( v = Tu \) if

\[\int_{\Omega} J(x - y)v(y)dy - \beta v(x) = -[g(x, v) + \beta v(x)],\]

which is well defined since the following linear operator

\[Jv(x) := \int_{\Omega} J(x - y)v(y)dy - \beta v(x)\]

is invertible on \( C(\Omega) \). Next, we shall show that \( T \) is monotone increasing, i.e., \( T\omega_1 \leq T\omega_2 \) provided \( \omega_1 \leq \omega_2 \) for \( \omega_1, \omega_2 \in \mathcal{U} \).

Without loss of generality, we assume \( \omega_1 \leq \omega_2 \) and denote \( L\omega = g(x, \omega) + \beta\omega \).

In view of (13), we have

\[L\omega_1 = g(x, \omega_1) + \beta\omega_1 \leq g(x, \omega_2) + \beta\omega_2 = L\omega_2.\]

Considering \( J(T\omega_1) = -L\omega_1 \) and \( J(T\omega_2 - T\omega_1) \leq 0 \). Note the fact that \(-J\) is a linear operator and \((-J)^{-1}\) is a positive operator if \( \beta \) takes a sufficiently large value. Hence, we obtain

\[T\omega_1 \leq T\omega_2.\]

Along the above analysis, we deduce a sequence defined by

\[u_1 = Tu, \quad u_2 = Tu_1, \quad \ldots, \quad u_n = Tu_{n-1}, \quad \ldots\]

which is a monotone decreasing sequence. Similarly, we also obtain another sequence defined by

\[v_1 = Tv, \quad v_2 = Tv_1, \quad \ldots, \quad v_n = Tv_{n-1}, \quad \ldots\]
which is a monotone increasing sequence. Furthermore, we have
\[ u \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq u_1 \leq v, \]
then there exist constant \( u^* \) and \( v^* \), such that
\[ u^* = \lim_{n \to \infty} u_n(x) \quad \text{and} \quad v^* = \lim_{n \to \infty} v_n(x). \]

It is easy to see \( u \leq v^* \leq u^* \leq u \). By the Lebesgue’s dominated convergence theorem, we have
\[ \lim_{n \to \infty} \int_{\Omega} J(x - y)u_n(y)\,dy = \int_{\Omega} J(x - y)u^*(y)\,dy \]
\[ \lim_{n \to \infty} \int_{\Omega} J(x - y)v_n(y)\,dy = \int_{\Omega} J(x - y)v^*(y)\,dy. \]

Due to the continuity of \( g \),
\[ \lim_{n \to \infty} [g(x, u_n(x)) - \beta u_n(x)] = g(x, u^*) - \beta u^*, \]
\[ \lim_{n \to \infty} [g(x, v_n(x)) - \beta v_n(x)] = g(x, v^*) - \beta v^*. \]

Hence, we have
\[ \int_{\Omega} J(x - y)u^*(y)\,dy + g(x, u^*) = 0, \]
and
\[ \int_{\Omega} J(x - y)v^*(y)\,dy + g(x, v^*) = 0. \]

Which completes the proof. \( \square \)

Like the Laplacian operator, the dispersal operator also have the important Maximum principle.

Lemma 2.6. (Maximum Principle) Let \( u \in C(\overline{\Omega}) \) verify \( L_M u(x) \leq 0 \) in \( \Omega \) with \( u \geq 0 \) in \( \mathbb{R}^N \setminus \Omega \). Then either \( u > 0 \) or \( u \equiv 0 \) in \( \overline{\Omega} \), where
\[ L_M u(x) = (J \ast u)(x) - (1 + M)u(x). \]

Proof. Similar the proof of Theorem 7 in [15], we can obtain the result. \( \square \)

Lemma 2.7. [40] The operator \( J : u \mapsto J \ast u \) is compact on \( X = C_0(\overline{\Omega}) = \{ u \in C(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega \} \).

Remark 4. Due to the compactness of the operator \( J \), we can apply the bifurcation theory to establish the existence of the positive steady-states of (1) in the nonhomogeneous case in section 5.

3. Some results of stability. Before giving the proofs of the main results, we prove the stability results.

Proposition 2. 1): Assume \( a > 0 \).
   (a): If \( q = 1 \) any positive steady-state solution of (1) is unstable for any \( p > 0 \).
   (b): If \( p + q < 1 \) and \( \lambda > 0 \) or \( p + q = 1 \) and \( \lambda > 0 \), then, any positive steady-state solution of (1) is globally asymptotically stable.

2): Assume \( a < 0 \).
   (c): If \( q = 1 \), any positive steady-state solution of (1) is globally asymptotically stable for any \( p > 0 \).
Consider a positive steady-state solution $u^*$ of \( (1) \).

\[
\begin{cases}
\int_\Omega J(x - y)v(y)dy - v(x) + H^n(x)v(x) = \sigma^n v(x), & \text{in } \Omega, \\
v(x) = 0, & \text{on } \partial\Omega,
\end{cases}
\]

where $H(x)$ defined by

\[
H(x)v(x) = q(u^*(x))^{q-1}(\lambda + a(x) \int_\Omega b(x)(u^*(x))^pdx)v(x) \\
+ p(u^*(x))^{q-1}a(x) \int_\Omega b(x)(u^*(x))^p v(x)dx.
\]

We can choose a continuous sequence \( \{H^n\}_{n=1}^\infty \) such that $H^n(x) \geq H(x)$, \( \| H - H^n \|_\infty \to 0 \) as $n \to \infty$. In view of Lemma 2.1, the eigenvalue equation

\[
\int_\Omega J(x - y)v(y)dy - v(x) + H^n(x)v(x) = \sigma^n v(x), \quad x \in \Omega
\]

admits a unique principal eigenvalue $\sigma^n_p$ associated with a positive eigenfunction $\varphi^n$ satisfying $\| \varphi^n \|_{L^2(\Omega)} = 1$ for each $n \geq 1$. We have to calculate the sign of the principal eigenvalue $\sigma^n_p$.

First, we will change \( (14) \) into

\[
\int_\Omega J(x - y)v(y)dy - v(x) + [H(x) + (H^n(x) - H(x))]v(x) = \sigma^n v(x),
\]

Next, we are going to apply Proposition 1 with $\overline{v} = u^*$ (respectively, $\underline{v} = u^*$), then \( (14) \) reduces to the following system

\[
u^* (\lambda(q - 1) + (p + q - 1)a(x) \int_\Omega b(x)(u^*(x))^pdx) \\
+ (H^n(x) - H(x))u^*(x) = \sigma^n u^*(x).
\]

Then, we have to ascertain the sign of $\sigma^n_p$. Recall the fact that $\| H - H^n \|_\infty \to 0$ as $n \to \infty$, and there exists a large integer number $n_1$ such that $\sigma^n_p$ is positive for $q = 1$, negative if $q < 1$, $p + q < 1$ and $\lambda \geq 0$ or $0 < q < 1$, $p + q = 1$ and $\lambda > 0$ when $n > n_1$.

Here, we denote $G(x, u) = u^a (\lambda + a(x) \int_\Omega b(x)u^p dx)$.

(a): Homogeneous case: If $q = 1$.

Then $\sigma^n_p > 0$, so we can choose $\varepsilon_0 > 0$ such that

\[
G(x, u^*(x) + \varepsilon) - G(x, u^*(x)) \geq G'_t(x, u^*(x))\varepsilon - \frac{1}{2} \sigma^n_p \varepsilon_0,
\]

for every $0 \leq \varepsilon \leq \varepsilon_0$ and $n \geq N$ where $N$ sufficiently large. Now, we assume that $u^*(x)$ is stable, we take $u_0(x) \geq u^*(x)(u_0 \neq u^*)$ in $\Omega$ such that

\[
\| u(., t; u_0) - u^* \|_\infty \leq \varepsilon_0.
\]

It is obvious that $u(x, t; u_0) \geq u^*$ by comparison principle and there exists $T > 0$ such that $u(x, t; u_0) > u^*(x)$ for every $x \in \overline{\Omega}$, $t > T$ and we have

\[
u(x, t; u_0) = u_0(x) + \int_0^t \left[ \int_\Omega J(x - y)u(y, s)dy - u(x, s) + G(x, u(x, s; u_0)) \right] ds.
\]
For each \( n \geq N \) and \( t > T \), we have

\[
\langle u(x, t; u_0) - u^*(x), \varphi^n(x) \rangle \\
\geq \int_0^t \left[ \int_{\Omega} J(x-y)(u(y, s) - u^*(x))dy - (u(x, s) - u^*(x)) \right] ds, \varphi^n(x) \\
+ \int_0^t [G(x, u(x, s; u_0)) - G(x, u^*(x))] ds, \varphi^n(x) \\
\geq \int_0^t J(x-y)\varphi^n(y)dy - \varphi^n(x), \int_0^t [u(x, s; u_0) - u^*(x)] ds \\
+ G'(x, u^*(x))\varphi^n(x), \int_0^t [u(x, s; u_0) - u^*(x)] ds \\
- \frac{\sigma_n^p}{2} \langle \varphi^n(x), \int_0^t [u(x, s; u_0) - u^*(x)] ds \rangle \\
\geq \sigma_n^p \langle \varphi^n(x), \int_0^t [u(x, s; u_0) - u^*(x)] ds \rangle \\
- \frac{\sigma_n^p}{2} \langle \varphi^n(x), \int_0^t [u(x, s; u_0) - u^*(x)] ds \rangle \\
+ \langle H(x) - H^n(x), \varphi^n(x), \int_0^t [u(x, s; u_0) - u^*(x)] ds \rangle \\
+ \frac{\sigma_n^p}{2} \langle \varphi^n(x), \int_0^t [u(x, s; u_0) - u^*(x)] ds \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product on \( L^2(\Omega) \).

In view of \( \sigma_n^p > 0 \) and \( \|H - H^n\|_\infty \to 0 \) as \( n \to \infty \), we can select \( n' \geq N \) such that for \( n > n' \), we arrive at

\[
\left\langle \left( \frac{\sigma_n^p}{4} + H(x) - H^n(x) \right) \varphi^n(x), \int_0^t [u(x, s; u_0) - u^*(x)] ds \right\rangle > 0.
\]

Then,

\[
\langle u(x, t; u_0) - u^*(x), \varphi^n(x) \rangle \geq \frac{\sigma_n^p}{4} \left\langle \int_0^t [u(x, s; u_0) - u^*(x)] ds, \varphi^n(x) \right\rangle
\]

for \( t > T \) and \( n > n' \). It follows from the above inequality, we obtain

\[
\langle u(x, t; u_0) - u^*(x), \varphi^n(x) \rangle \geq \frac{\sigma_n^p}{4} \left\langle \int_T^{2T} [u(x, s; u_0) - u^*(x)] ds, \varphi^n(x) \right\rangle
\]

for \( t > 2T \) and

\[
\left\langle \int_{2T}^t u(x, t; u_0) - u^*(x), \varphi^n(x) \right\rangle \geq \frac{\sigma_n^p(t - 2T)}{4} \left\langle \int_T^{2T} [u(x, s; u_0) - u^*(x)] ds, \varphi^n(x) \right\rangle
\]

for \( t > 2T \). Since \( \int_{\Omega} (u(x, t_0; u_0) - u^*(x)) \varphi^n(x) dx > 0 \) for every \( T < t_0 < 2T \).

Along the above analysis, we have

\[
\lim_{t \to \infty} \langle u(x, t; u_0) - u^*(x), \varphi^n(x) \rangle = \infty,
\]

which contradicts with (16). This completes the proof of 1) (a).
(b): Non-homogeneous case: If 0 < q < 1.

(i): Assume p + q < 1 and λ ≥ 0.

Since σⁿ is a principal eigenvalue of (14) and σⁿ < 0 for every n ∈ N. Now, we define

\[ u(x, t) = u^{*}(x) + βφ^{n}(x)e^{-γt}. \]

It is clear to see that

\[ \frac{∂u(x, t)}{∂t} - \int_{Ω} J(x, y)u(y, t)dy + u(x, t) - G(x, u(x, t)) = βφ^{n}(x)e^{-γt}[-γ - σ^n + H^n(x)] + \tilde{G}(x), \]

where

\[ \tilde{G}(x) = G(x, u^*(x)) - G(x, u^*(x) + βφ^n(x)e^{-γt}). \]

In view of (15) and Proposition 1, and hence we can find suitable constants α < 0 and N' such that n > N', we have supₙ∈N σⁿ = α < 0 and

\[ -σ^n + (H^n(x) - H(x)) ≥ -\frac{α}{2}. \]

Besides, we can select a small β and 0 < γ < \(-\frac{α}{2}\), and by a simple computation hence we have

\[ \frac{∂u(x, t)}{∂t} - \int_{Ω} J(x, y)u(y, t)dy + u(x, t) - G(x, u(x, t)) ≥ βφ^{n}(x)e^{-γt}[-γ - σ^n + (H^n(x) - H(x)) - C(βφ^n(x)e^{-γt})], \]

for some C > 0.

In a similar way, we can prove \( v(x, t) = u^*(x) - βφ^n(x)e^{-γt} \) satisfies

\[ \frac{∂v(x, t)}{∂t} - \int_{Ω} J(x, y)v(y, t)dy + v(x, t) - G(x, v(x, t)) \]

for β and γ selected as above. Therefore u(x, t) and v(x, t) are super-subsolutions of (1) and we completes the proof.

(ii): Assume p + q = 1, λ > 0.

By a similar discussion as in the proof of (b)(ii), we can deduce the conclusion.

Similar the proof of 1), we can easily obtain 2) and we omit it here. Hence we complete the proof of Proposition 2. □

Notice the fact that \( V_0 = 0 \) is a trivial solution of (1) and by using the proof of Proposition 2, we can easily obtain the stability of \( V_0 = 0 \) as follows:

Corollary 2. 1): Assume a > 0.

(a): If q = 1, then \( V_0 \) is globally asymptotically stable for \( λ < λ_1(Ω) \) and any p > 0.

(b): If q < 1. If p + q > 1 and λ > 0 or \( p + q = 1 \) and λ > 0 or \( p + q < 1 \), then \( V_0 \) is unstable. If \( p + q > 1 \) and λ ≤ 0 or \( p + q = 1 \) and λ < 0, then \( V_0 \) is globally asymptotically stable.

2): Assume a < 0.

(c): If q = 1, then \( V_0 \) is unstable for \( λ > λ_1(Ω) \) and any p > 0.

(d): If q < 1. If p + q > 1 and λ > 0 or \( p + q = 1 \) and λ > 0, then \( V_0 \) is unstable. If \( p + q > 1 \) and λ ≤ 0 or \( p + q = 1 \) and λ < 0 or \( p + q < 1 \), then \( V_0 \) is globally asymptotically stable.
4. Nonexistence, existence and stability of Proof of Theorem 1.1. In this section, we will give the proof of Theorem 1.1.

**Proof.** We will divide the proof into two cases: one is \( a < 0 \) and the other is \( a > 0 \).

**Case 1:** Assume \( a < 0 \). We first denote \( \lambda_1(\Omega) \) be the principal eigenvalue of the following system:

\[
\begin{cases}
\int_\Omega J(x - y)u(y)dy - u(x) = -\lambda u(x), & \text{in } \Omega, \\
u = 0, & \text{on } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

(17)

and prove that if a nontrivial steady-state solution \( u \) to (1) exits, then \( \lambda > \lambda_1(\Omega) \). To this aim, assume \( u \in L^1(\Omega) \) is a positive solution to (1) and \( \varphi_1 \) be a positive eigenfunction associated to \( \lambda_1(\Omega) \) due to Lemma 2.1 and 2.3. Then, we multiply (1) by \( \varphi \) and integrate in \( \Omega \), we have

\[
\int_\Omega \varphi_1(x) \int_\Omega J(x - y)u(y)dydx - \int_\Omega u(x)\varphi_1(x)dx = -\lambda \int_\Omega u(x)\varphi_1(x)dx - \int_\Omega a(x)u(x)dx \int_\Omega b(x)u^p(x)dx.
\]

(18)

Applying Fubini’s theorem in the left side of (18) and recall the fact that \( \varphi_1 \) is an eigenfunction associated to \( \lambda_1(\Omega) \), then

\[
(\lambda - \lambda_1(\Omega)) \int_\Omega u(y)\varphi_1(y)dy = -\int_\Omega a(x)u(x)dx \int_\Omega b(x)u^p(x)dx.
\]

It follows from this equation that necessarily \( \lambda > \lambda_1(\Omega) \) and it is obvious that \( u_\lambda \rightarrow 0 \) as \( \lambda \rightarrow \lambda_1(\Omega) \).

Next, we shall show that there exists a positive steady-state solution to (1) under the condition of \( \lambda > \lambda_1(\Omega) \) by means of sub-supersolution. We show that \( \overline{u} = \varepsilon \varphi_1 \) (where \( \varepsilon \) is small enough) and \( \underline{u} = M \) (where \( M \) is sufficiently large) are a pair of sub-supersolution of (1).

In fact

\[
J \ast (\varepsilon \varphi_1(x) - \varepsilon \varphi_1(x) + \lambda \varepsilon \varphi_1(x) + \varepsilon^{p+1} \varphi_1 a(x)) \int_\Omega b(x)\varphi_1^p(x)dx
\]

\[
= \varepsilon \left[ J \ast \varphi_1(x) - \varphi_1(x) + \varepsilon^p a(x)\varphi_1(x) \int_\Omega b(x)\varphi_1^p(x)dx \right]
\]

\[
= \varepsilon \left[ (\lambda - \lambda_1(\Omega))\varphi_1(x) + \varepsilon^p a(x)\varphi_1(x) \int_\Omega b(x)\varphi_1^p(x)dx \right]
\]

\geq 0,
\]

where the last inequality is valid by taking \( \varepsilon \) sufficiently small since \( p > 0 \). On the other hand, if \( \underline{u} = M \) is a supersolution of (1), then

\[
J \ast M - d(x)M + \lambda M + a(x)M^{p+1} \int_\Omega b(x)dx < 0,
\]

(19)

this is clearly fulfilled by selecting a sufficiently large constant \( M \), where \( a < 0 \) in \( \Omega \). In view of Lemma 2.5, we obtain that (1) admits a positive steady-state solution.

Next, let’s prove that all the positive steady-state solutions of (1) are continuous in \( \Omega \). First of all, we assume that \( u^* \) is a steady-state solution of (1)
and denote \( h(x, u^*) = \lambda + a(x) \int_{\Omega} b(x)(u^*)^p \, dx \), \( f(x, u^*) = u^* \tilde{f}(x, u^*) \). Due to the fact that
\[
J * u^*(x) + (h(x, u^*) - 1)u^*(x) = 0,
\]
and notice that \( u^* \) is strictly positive and bounded in \( \overline{\Omega} \), we may find a constant \( \delta > 0 \) such that
\[
-1 + h(x, u^*(x)) \leq -\delta, \quad \text{for all } x \in \overline{\Omega}.
\]
For any \( x_1, x_2 \in \overline{\Omega} \), we may find that
\[
J*(u^*(x_1) - u^*(x_2)) + \int_{\Omega} [f(x_1, u^*(x_1)) - f(x_2, u^*(x_2))] \, dx
\]
\[
= -(u^*(x_2) - u^*(x_1)) + f(x_2, u^*(x_2)) - f(x_1, u^*(x_1))
\]
\[
= - \left[ -1 + \frac{\partial}{\partial u} f(x_2, \theta u^*(x_1) + (1 - \theta)u^*(x_2)) \right] (u^*(x_1) - u^*(x_2))
\]
where \( 0 \leq \theta \leq 1 \). Without loss of generality, we may assume \( u^*(x_1) \geq u^*(x_2) \) and notice that
\[
\frac{\partial}{\partial u} f(x_2, \theta u^*(x_1) + (1 - \theta)u^*(x_2)) \leq h(x_2, \theta u^*(x_1) + (1 - \theta)u^*(x_2)) \leq h(x_2, u(x_2)).
\]
Hence, we have
\[
- \left[ -1 + \frac{\partial}{\partial u} f(x_2, \theta u^*(x_1) + (1 - \theta)u^*(x_2)) \right] \geq \delta. \tag{20}
\]
In view of (18) and (20), we conclude that \( u^* \) is continuous.
Now, we shall prove that \( u^* \) is stable. Then, consider the sign of the principal eigenvalue of
\[
\begin{cases}
- J * \xi + \xi - (\lambda + a(x) \int_{\Omega} b(x)(u^*)^p \, dx) \xi \\
- \int_{\Omega} (\lambda + a(x) \int_{\Omega} b(x)(u^*)^p \, dx) \xi - pu^* \int_{\Omega} b(x)(u^*)^{p-1} \xi \, dx = \sigma \xi, \quad x \in \Omega,
\end{cases} \tag{21}
\]
\[
\xi = 0, \quad x \in \partial \Omega.
\]
We are going to apply Proposition 2.1 (a), we can easily obtain the stability conclusion.
Finally, we will prove (7). Assume \( u^* \) is a unique stable positive steady-state solution and combine with \( J * u^* > 0 \) and \( a < 0 \), then we have
\[
- \int_{\Omega} J(x - y)u^*(y) \, dy + u^*(x) = u^*(x) \left( \lambda + a(x) \int_{\Omega} b(x)(u^*)^p \, dx \right).
\]
If we multiply the above equality by \( \varphi_1(x) \) and integrate in \( \Omega \), we arrive at
\[
(\lambda - \lambda_1(\Omega)) \int_{\Omega} u^*(x) \varphi_1(x) \, dx = - \int_{\Omega} a(x)u^*(x) \varphi_1(x) \, dx \int_{\Omega} b(x)(u^*)^p \, dx
\]
\[
= -a(\zeta) \int_{\Omega} u^*(x) \varphi_1(x) \, dx \int_{\Omega} b(x)(u^*)^p \, dx
\]
\[
\leq \sup_{x \in \Omega} (-a(x)) \int_{\Omega} u^*(x) \varphi_1(x) \, dx \sup_{x \in \Omega} b(x) \| u \|_{L^p(\Omega)}^p.
\]
and hence \( \| u^* \|_{L^p(\Omega)} \geq \left( \frac{\lambda - \lambda_1(\Omega)}{\sup_{x \in \Omega}(-a(x)) \max_{x \in \Omega} b(x)} \right)^{\frac{1}{p}} \). From the above inequalities, we can also have

\[
(\lambda - \lambda_1(\Omega)) \int_{\Omega} u^*(x) \varphi_1(x) dx = - \int_{\Omega} a(x)u^*(x) \varphi_1(x) dx \int_{\Omega} b(x)(u^*(x))^p dx \\
\leq -a(\zeta) \int_{\Omega} u^*(x) \varphi_1(x) dx \int_{\Omega} b(x)(u^*(x))^p dx \leq \sup_{x \in \Omega} (a(x)) \int_{\Omega} u^*(x) \varphi_1(x) dx \| b \|_{L^1(\Omega)} \| u^* \|_{L^\infty(\Omega)},
\]

where \( \zeta \in \Omega \) and hence \( \| u^* \|_{L^\infty(\Omega)} \geq \left( \frac{\lambda - \lambda_1(\Omega)}{\sup_{x \in \Omega}(-a(x)) \max_{x \in \Omega} b(x)} \right)^{\frac{1}{p}} \).

On the other hand,

\[
(\lambda - 1)u^*(x) = -J \ast u^*(x) - a(x)u^*(x) \int_{\Omega} b(x)(u^*(x))^p dx \\
\geq -u^* - a(x) \int_{\Omega} b(x)(u^*(x))^p dx,
\]

then we obtain \( \| u^* \|_{L^p(\Omega)} \leq \left( \frac{\lambda}{\min_{x \in \Omega}(-a(x)) \inf_{x \in \Omega} b(x)} \right)^{\frac{1}{p}} \), which completes Theorem 1.1 (1).

**Case 2:** If \( a(x) > 0 \). The proof of case 2 is similar to case 1 and we omit it here. This completes the proof of Theorem 1.1.

Next, we will establish the local existence and uniqueness of steady-state solution of (1) if \( 0 < \lambda - \lambda_1(\Omega) \ll 1 \) and \( q = 1, p \geq 1, p \in \mathbb{N} \).

To find solution of (1), we take \( \lambda \) as a bifurcation parameter and Define \( F : \mathbb{X} \times \mathbb{R} \rightarrow \mathbb{Y} \) by

\[
F(v, \lambda)(x) = J \ast v(x) - v + \lambda v + a(x)v \int_{\Omega} b(x)v^p(x) dx,
\]

for \( v \in \mathbb{X} \), where \( \mathbb{X} = L^1(\Omega) \cap C(\Omega) \) and \( \mathbb{Y} = L^2(\Omega) \cap C(\Omega) \), and we use the standard inner product \( \langle u, v \rangle = \int_{\Omega} \tilde{u}(x)v(x) dx \). We shall try to solve \( F(v, \lambda) = 0 \) for \( v \in \mathbb{X} \) and parameter \( \lambda \in \mathbb{R} \). First of all, it is easy to see that, for every fixed parameter value \( \lambda \in \mathbb{R} \), \( F(v, \lambda) = 0 \) always has a trivial solution \( v = 0 \). Namely, \( F(0, \lambda) = 0 \) for all values of the parameter \( \lambda \). If we want to prove the uniqueness of these solutions by the implicit function theorem, we need to compute the Fréchet derivative of \( F \) with respect to \( v \) evaluated at \((0, \lambda)\) which is given by

\[
\mathbb{L}_\lambda v = J \ast v - v + \lambda v, \quad v \in \mathbb{X}.
\]

It is easy to obtain the following result.

**Lemma 4.1.** The kernel of \( \mathbb{L}_\lambda(\Omega) \) is given by \( \text{span}\{\varphi_1\} \), which is denoted by \( \mathbb{K} \).

Our purpose is to find nontrivial solutions to the nonlinear functional equation \( F(v, \lambda) = 0 \) with \( v \) close to 0, and \( \lambda \) close to \( \lambda_1(\Omega) \) in \( \mathbb{R} \). It is easy to see that \( \mathbb{L}_\lambda \) is a self-adjoint operator, i.e., \( \langle u, \mathbb{L}_\lambda v \rangle = \langle \mathbb{L}_\lambda u, v \rangle \). Thus, we have the following decompositions:

\[
\mathbb{X} = \mathbb{K} \oplus \mathbb{X}_1, \quad \mathbb{Y} = \mathbb{K} \oplus \mathbb{Y}_1,
\]

where

\[
\mathbb{X}_1 = \{ y \in \mathbb{X} | \langle y, y \rangle = 0 \text{ for all } v \in \mathbb{K} \},
\]

\[
\mathbb{Y}_1 = \{ y \in \mathbb{Y} | \langle y, y \rangle = 0 \text{ for all } v \in \mathbb{K} \}.
\]
Obviously, the operator $\mathcal{L}_{\lambda_1}(\Omega) : \mathcal{K} \to \mathcal{Y}$ is Fredholm with index zero. $\mathcal{L}_{\lambda_1}(\Omega)|_{\mathcal{X}_1} : \mathcal{X}_1 \to \mathcal{Y}_1$ is invertible and has a bounded inverse.

Now, we use Lyapunov-Schmidt reduction methods as follows. Let $Q$ and $I - Q$ denote the projection operators from $\mathcal{Y}$ onto $\mathcal{Y}_1$ and $\mathcal{K}$, respectively. Thus, $F(v, \lambda) = 0$ is equivalent to the following system:

$$QF(v_1 + v_2, \lambda) = 0, \quad (I - Q)F(v_1 + v_2, \lambda) = 0,$$

where $v_1 \in \mathcal{K}$ and $v_2 \in \mathcal{X}_1$. Notice that $F(0, \lambda_1) = 0$ and $QF(v_2(0, \lambda_1(\Omega))) = \mathcal{L}_{\lambda_1(\Omega)}$. Applying the implicit function theorem, we obtain a continuously differentiable map $h : \mathcal{U} \to \mathcal{X}_1$ such that

$$h(0, \lambda) = 0 \quad \text{and} \quad QF(v_1 + h(v_1), \lambda), \lambda) \equiv 0,$$

where $\mathcal{U}$ is an open neighborhood of $(0, \lambda_1(\Omega))$ in $\mathcal{K} \times \mathbb{R}$. Substituting $v_2 = h(v_1, \lambda)$ into the second equation of (22) gives

$$\tilde{g}(v_1, \lambda) \equiv (I - Q)F(v_1 + h(v_1), \lambda), \lambda) = 0. \quad (24)$$

Thus, each solution to $\tilde{g}(v_1, \lambda) = 0$ in $\mathcal{U}$ one-to-one corresponds to some solution to $F(v, \lambda) = 0$.

For $v_1 = \varphi_1 \in \mathcal{K}$ with $\varphi \in \mathbb{R}$, substituting this into (24) and then calculating the inner product with $\varphi_1$ on $\Omega$, we have $\tilde{g}(\varphi, \lambda) = 0$, where $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}$ is explicitly given by:

$$\tilde{g}(\varphi, \lambda) = \int_{\Omega} \varphi_1(x)F(\varphi_1(x) + h(\varphi_1(x), \lambda), \lambda)dx. \quad (25)$$

Notice that $\tilde{g}(0, \lambda) = 0$, it follows that $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}$ takes the form of

$$\tilde{g}(\varphi, \lambda) = \varphi(\lambda - \lambda_1(\Omega)) + K\varphi^p + O(\varphi^{p+1}),$$

where $p \geq 1$, $p \in \mathbb{N}$ and $K = \int_{\Omega} a(x)\varphi_1^2(x)dx \int_{\Omega} b(x)\varphi_1^p(x)dx < 0$ (respectively $> 0$) if $a < 0$ (respectively $> 0$).

In what follows, we shall investigate the existence of nontrivial zero of $\tilde{g}(\cdot, \lambda)$. In view of $K \neq 0$, then by using the implicit function theorem we see that there exist a constant $\epsilon > 0$ and a continuously differentiable mapping $\varphi : (\lambda_1(\Omega) - \epsilon, \lambda_1(\Omega) + \epsilon) \to \mathbb{R}$, such that

$$\tilde{g}(\varphi, \lambda) \equiv 0 \quad \text{for} \quad \lambda \in (\lambda_1 - \epsilon, \lambda_1 + \epsilon).$$

In fact, we have

$$\varphi = (\frac{\lambda_1 - \lambda}{K})^{\frac{1}{p}} + o(|\lambda - \lambda_1|). \quad (26)$$

Thus, we have the following result.

**Theorem 4.2.** There exist a constant $\epsilon > 0$ and a continuously differentiable mapping $\lambda \to \varphi_1$ from $(\lambda_1(\Omega) - \epsilon, \lambda_1(\Omega) + \epsilon)$ to $\mathbb{R}$ such that Eq. (1) has a nontrivial solution

$$u_\lambda(x) = \varphi_1(x) + h(\varphi_1(x), \lambda),$$

which exists for $\lambda \in (\lambda_1(\Omega) - \epsilon, \lambda_1(\Omega)) \cup (\lambda_1(\Omega), \lambda_1(\Omega) + \epsilon)$ and satisfies

$$\lim_{\lambda \to \lambda_1(\Omega)} u_\lambda = 0.$$

**Remark 5.** Note that $\lambda_1(\Omega)$ is the principle eigenvalues of the operator $-D$ with the associated eigenfunction $\varphi_1(x) > 0$ on $\Omega$. We see that the spatially nonhomogeneous steady-state solution $u_\lambda$ established by Theorem 4.2 is positive (respectively, negative) if $(\lambda_1(\Omega) - \lambda)a(x) > 0$ (respectively, $(\lambda_1(\Omega) - \lambda)a(x) < 0$). In biology, we are just interested in the positive steady-state solution.
5. Some results of the bifurcation of the nonhomogeneous case.

5.1. The results of the bifurcation for $0 < q < 1$ and $p+q \neq 1$. In this section, we will study the properties of an important compact operator when $0 < q < 1$ independently of the sign of $a$, which will appear in our study. First of all, we denote by

$$X = C_0(\Omega) = \{ u \in C(\Omega) : u = 0 \text{ on } \partial \Omega \}, \quad B_r := \{ u \in X : \| u \| < r \}.$$ 

Define

$$F(\lambda, x, u) := Mu^+ + (u^+)^q \left( \lambda + a(x) \int_{\Omega} b(x)(u^+)^p dx \right),$$

where $u^+ = \max\{0, u\}$, and also define

$$K_\lambda : X \mapsto X; \quad K_\lambda(u) := u - (-J \ast +d(x) + M)^{-1}(F(\lambda, x, u)),$$

for a large positive number $M$ which plays important role in making sure the positive $(-J \ast +d(x) + M)^{-1}$ valid.

Now, it is clear that $u$ is a non-negative steady-state solution of (1) if and only if $u$ is a zero of the map $K_\lambda$. We will use Leray-Schauder degree of $K_\lambda$ on $B_r$ with respect to zero, denoted by $\deg(K_\lambda, B_r)$, and the index of the isolated zero $u_0$ of $K_\lambda$, denoted by $i(K_\lambda, u_0)$.

First, we compute the index of the trivial solution for $\lambda < 0$ and $\lambda > 0$.

**Lemma 5.1.** If $\lambda < 0$, then $i(K_\lambda, 0) = 1$.

**Proof.** Assume that $\lambda < 0$ and define the map

$$\mathcal{H}_1 : [0, 1] \times X \mapsto X; \quad \mathcal{H}_1(t, u) := (-J \ast +1 + M)^{-1}(tF(\lambda, x, u)).$$

Next, we shall prove that there exists $0 < \delta < r$ such that

$$u \neq \mathcal{H}_1(t, u) \quad \text{for } u \in \overline{B_\delta}, \; u \neq 0 \text{ and } t \in [0, 1].$$

Suppose on the contrary that there exist two non-negative sequences $\{u_n\}$ and $\{t_n\}$, where $u_n \in X \setminus \{0\}$ and $t_n \in [0, 1]$ with $\| u_n \| \to 0$ such that

$$u_n = \mathcal{H}_1(t_n, u_n).$$

In view of $\| u_n \| \to 0$ as $n \to \infty$, $\lambda < 0$ and $0 < q < 1$, $p > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$-J \ast u_n + u_n + Mu_n = t_n F(\lambda, x, u_n) \leq 0 \text{ in } \Omega, \; u_n = 0 \text{ on } \partial \Omega,$$

for $n > n_0$. By using Lemma 2.6(maximum principle), we conclude $u_n \leq 0$ which is a contradiction.

By applying the admissible property of the homotopy defined by $\mathcal{H}_1$, we have

$$i(K_\lambda, 0) = \deg(I - \mathcal{H}_1(1,.), B_\varepsilon) = \deg(I - \mathcal{H}_1(0,.), B_\varepsilon) = \deg(I, B_\varepsilon) = 1,$$

where $\varepsilon \in (0, \delta]$. Which completes the proof.

Next, we will consider the index of $i(K_\lambda, 0)$ when $\lambda > 0$.

**Lemma 5.2.** If $\lambda > 0$, then $i(K_\lambda, 0) = 0$. 

\[ \Box \]
Proof. Assume that $\lambda > 0$ and $\psi \in X, \psi > 0$, then we define the map

$$\mathcal{H}_2 : [0,1] \times X \mapsto X; \quad \mathcal{H}_2(t,u) := (-J * +1 + M)^{-1}(t\psi + F(\lambda,x,u)).$$

Next, we shall prove that there exists $0 < \delta < r$ such that

$$u \neq \mathcal{H}_2(t,u) \text{ for } u \in \overline{B}_\delta, u \neq 0 \text{ and } t \in [0,1].$$

Suppose on the contrary that there exist two sequences $\{u_n\}$ and $\{t_n\}$, where $u_n \in X \setminus \{0\}$ and $t_n \in [0,1]$ with $\|u_n\|_\infty \to 0$ as $n \to \infty$ such that

$$u_n = \mathcal{H}_2(t_n,u_n).$$

Since $t_n \psi \geq 0$, $M$ is a large positive number, we obtain $u_n \geq 0$. In view of $\|u_n\|_\infty \to 0$, $\lambda > 0$ and $0 < q < 1, p > 0$, we have

$$-J * u_n + Mu_n = F(\lambda,x,u_n) + t_n \psi \geq 0 \text{ in } \Omega, \quad u_n = 0 \text{ on } \partial \Omega,$$

for every $n \in \mathbb{N}$. By using Lemma 2.6 (maximum principle), we conclude $u_n > 0$. Without loss of generality, we assume $M \geq \lambda_1(\Omega)$. Since $\|u_n\|_\infty \to 0$ and $\lambda > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, we obtain

$$-J * u_n + u_n + Mu_n = Mu_n + u_n^q \left(\lambda + a(x) \int \Omega b(x)u_n^p(x)dx\right) + t_n \psi$$

$$> 2Mu_n + t_n \psi,$$

then, we have

$$-J * u_n + u_n - Mu_n > 0. \quad (28)$$

If we multiply (28) by $\varphi_1(x)$ and integrate in $\Omega$, we arrive at

$$-\int \Omega \varphi_1(x) \int \Omega J(x - y)u_n(y)dydx + \int \Omega u_n(x)\varphi_1(x)dydx - M \int \Omega u_n(x)\varphi_1(x)dx > 0.$$

If we apply Fubini’s theorem in the above inequality and use the fact that $\varphi_1$ is an eigenfunction associated to $\lambda_1(\Omega)$ (the symmetry of $J$ is needed here), then we conclude that $\lambda_1(\Omega) > M$. This is a contradiction.

This proves that the homotopy defined by $\mathcal{H}_2$ is admissible. By applying the admissible property of the homotopy defined by $\mathcal{H}_1$, we have

$$i(K_\lambda,0) = deg(I - \mathcal{H}_2(0,\cdot),B_\varepsilon) = deg(I - \mathcal{H}_2(1,\cdot),B_\varepsilon) = 0,$$

where $\varepsilon \in (0,\delta]$.

Notice that the last equality is true because the problem

$$-J * u + u = u^q \left(\lambda + a(x) \int \Omega b(x)u^p(x)dx\right) + \psi$$

has no solution in $\overline{B}_\varepsilon$. And recall that we have shown that $u \neq \mathcal{H}_2(1,u)$ for all $u \in \overline{B}_\delta, u \neq 0$. Which completes the proof. \qed

In view of Lemma 5.1 and 5.2, we have the following result:

**Theorem 5.3.** Assume $0 < q < 1$ and $a \in C(\Omega)$, then $\lambda = 0$ is the only bifurcation point from the zero solution for (1). Moreover, there exists an bounded continuum (closed and connected set) $C_0$ of non-negative steady-state solutions of (1) unbounded in $\mathbb{R} \times X$ emerging from $(\lambda,u) = (0,0)$. 

Proof. That $\lambda = 0$ is a bifurcation point from the trivial solution for the equation $K_{\lambda}(u) = 0$. It follows from a simple modification (see Theorem 4.4 in [3]) of the global bifurcation of the global bifurcation theorem of Rabinowitz [31] Theorem 1.3: The assumptions about the differentiability at $u = 0$ of the mappings involved and the oddness of the multiplicity of the eigenvalue $\lambda = 0$ of the linearized problem at $u = 0$ are placed here by the assertion of Lemma 5.1 and 5.2, and hence we can conclude the existence of a continuum of steady-state solutions of (1) such that meets $(0,0)$ either infinity or $(\hat{\lambda},0)$ with $\hat{\lambda} \neq 0$. We can discard the latter possibility. In fact, there does not exist other bifurcation points in $(-\infty,0)$ (respectively, $(0, +\infty)$) from Lemma 5.1 (respectively, Lemma 5.2). Hence, the existence of an unbounded in $\mathbb{R} \times X$ continuum of steady-state solutions of (1) follows.

We will give the following bifurcation direction as follows:

**Proposition 3.** a): Assume $a < 0$, then the bifurcation is supercritical.

b): Assume $a > 0$ and $0 < q < 1$.

(i): If $p + q < 1$, then the bifurcation is subcritical.

(ii): If $p + q > 1$, then the bifurcation is supercritical.

Proof. a): Assume that $a < 0$, and $(\lambda, u_\lambda)$ is a positive steady-state solution of (1), then we have
\[- J * u_\lambda(x) + u_\lambda(x) = \lambda u_\lambda^q(x) + a(x)u_\lambda^q(x) \int_\Omega b(x)u_\lambda^p(x)dx. \tag{29}\]

If we multiply (29) by $\varphi_1(x)$, integrate in $\Omega$ and assume $\lambda \leq 0$, we arrive at
\[- \int_\Omega \int_\Omega J(x-y)\varphi_1(y)dyu_\lambda(x)dx + \int_\Omega \varphi_1(x)u_\lambda(x)dx = \lambda_1(\Omega) \int_\Omega u_\lambda(x)\varphi_1(x)dx \]
\[= \lambda \int_\Omega u_\lambda^q(x)\varphi_1(x)dx + a(x)u_\lambda^q(x) \varphi_1(x)dx \int_\Omega b(x)u_\lambda^p(x)dx < 0. \]

Recall that $\lambda_1(\Omega) > 0$, which is a contradiction. This implies $\lambda > 0$, that is, the bifurcation is supercritical.

b): (i) If $p + q < 1$ and assume that there exists a sequence $(\lambda_n, u_n)$ of positive solutions to (1) with $\lambda_n \geq 0$ and $\| u_n \|_\infty \to 0$ as $n \to \infty$. Since $\lambda_n \geq 0$ and hence for every $n \geq 1$, we have
\[- J * u_n + u_n \geq a(x)u_n^q \int_\Omega b(x)u_n^p(x)dx. \tag{30}\]

In view of $J * u_n(x) > 0$, then we have
\[u_n > \int_\Omega J(x-y)u_n(x)dy > a(x)u_n^q \int_\Omega b(x)u_n^p(x)dx. \]

Divide both sides at the same time by $u_n^q$ in the above inequality, integrate in $\Omega$ and by using the second mean value theorem, then there exists $\xi \in \Omega$ such
that $b(\xi) > 0$ and
\[
\int_{\Omega} u_n^{1-q}(x)dx > \int_{\Omega} a(x)dx \int_{\Omega} b(x)u_n^p(x)dx
= b(\xi) \int_{\Omega} a(x)dx \int_{\Omega} u_n^p(x)dx.
\]

However, we can choose a sufficiently large $n_0$ such that the last inequality is impossible for $n > n_0$ in view of the fact that $1 - q > p$ and $\| u_n \|_\infty \to 0$ as $n \to \infty$. Which implies $\lambda < 0$ and hence the bifurcation direction is subcritical.

(ii) If $p + q > 1$ and assume that there exists a sequence $(\lambda_n, u_n)$ of positive steady-state solutions to (1) with $\lambda_n \leq 0$ and $\| u_n \|_\infty \to 0$ as $n \to \infty$. Since $\lambda_n \leq 0$ and hence for every $n \geq 1$, we have
\[
-J * u_n + u_n \leq a(x)u_n q \int_{\Omega} b(x)u_n^p(x)dx.
\]

If we multiply (31) by $\varphi_1(x)$ and integrate in $\Omega$, we arrive at
\[
-\int_{\Omega} \varphi_1(x) \int_{\Omega} J(x - y)u_n(y)dydx + \int_{\Omega} J(x - y)u_n(x)\varphi_1(x)dydx
\leq \int_{\Omega} a(x)u_n q(x)\varphi_1(x)dx \int_{\Omega} b(x)u_n^p(x)dx.
\]

If we apply Fubini’s theorem in the above inequality and use the fact that $\varphi_1$ is an eigenfunction associated to $\lambda_1(\Omega)$ (the symmetry of $J$ is needed here) and the second mean value theorem, then there exist $\xi, \theta, \eta \in \Omega$ such that $\varphi_1(\xi), a(\theta) \varphi_1(\theta), b(\eta) > 0$ and
\[
\lambda_1(\Omega) \int_{\Omega} u_n(x)\varphi_1(x)dx = \lambda_1(\Omega)\varphi_1(\xi) \int_{\Omega} u_n(x)dx
\leq \int_{\Omega} a(x)u_n q(x)\varphi_1(x)dx \int_{\Omega} b(x)u_n^p(x)dx.
\]

for every $n \in \mathbb{N}$, which is an absurd. Since $\| u_n \|_\infty \to 0$ as $n \to \infty$ and $p + q > 1$, then we can select an integer $N$ such that
\[
\varepsilon < u_N < k\varepsilon,
\]
where $k > 1$, $\varepsilon < \left( \frac{C_1}{C_2k^{p+q}||\Omega||} \right)^{\frac{1}{p+q}}$ and $C_1 = \lambda_1(\Omega)\varphi_1(\xi)$, $C_2 = a(\theta) \varphi_1(\theta)b(\eta)$. Thus, we obtain
\[
\lambda_1(\Omega)\varphi_1(\xi) \int_{\Omega} u_N(x)dx = C_1 \int_{\Omega} u_N(x)dx
\]
\[
> C_1 \varepsilon ||\Omega||
\]
\[
> C_2k^{p+q}\varepsilon^{p+q}||\Omega||^2
\]
\[
> C_2 \int_{\Omega} u_N^q(x)dx \int_{\Omega} u_N^p(x)dx
\]
\[
= a(\theta) \varphi_1(\theta)b(\eta) \int_{\Omega} u_N^q(x)dx \int_{\Omega} u_N^p(x)dx
\]
which contradicts with (32).
This implies $\lambda < 0$ and hence the bifurcation direction is supercritical.

\[ \square \]

5.2. The results of the bifurcation for $0 < q < 1$ and $p + q = 1$. Next, we will give an important result.

Lemma 5.4. Consider $a \in C(\overline{\Omega})$, $a > 0$, then the following system

\[
\begin{aligned}
& - J \ast u(x) + u(x) - a(x)u^q(x) = 0, x \in \Omega, \\
& u = 0, x \in \partial \Omega,
\end{aligned}
\]  

(33)

exists a unique positive solution denoted by $\omega_a$.

Proof. First, by applying Theorem 5.3, we know that system (1) exists a unbounded continuum of positive steady-state solutions $C_0 \subset \mathbb{R} \times X$ bifurcating from the trivial solution at $\lambda = 0$. Moreover, in view of Proposition 3 b) (ii), then there exists a positive steady-state solution $\omega_a$ of (1) when $\lambda > 0$, $a > 0$ and $0 < q < 1$, $p + q > 1$. We claim that $\pi = M$, $\omega_a = \delta u_\lambda (\lambda > 0)$ are a pair of order super and subsolutions if $M$ large enough and both of $\delta$, $\lambda$ enough small. Indeed, if $0 < q < 1$, we have

\[-J \ast M + M - a(x)M^q(x) > 0,
\]

which implies $M$ is a supersolution of (33). On the other hand, $g$ is a subsolution if

\[-J \ast (\delta u_\lambda)(x) + \delta u_\lambda(x) - a(x)\delta^q(u_\lambda(x))^q = (\delta u_\lambda(x))^q \left[ \lambda + a(x)(\delta^p \int_{\Omega} b(x)(u_\lambda(x))^p dx - 1 \right] < 0.
\]

and this can also be clearly fulfilled by selecting a small enough $\delta$ and $\lambda$ such that

\[ \lambda + a(x) \left( \delta^p \int_{\Omega} b(x)(u_\lambda(x))^p dx - 1 \right) < 0,
\]

since $a > 0$, $b > 0$, small $\lambda > 0$ and $u_\lambda > 0$. Hence thanks to Lemma 2.5 (called the method of sub and supersolutions), we obtain that (33) admits a positive solution denoted $\omega_a$.

We now prove uniqueness. Let $\phi$ be another nontrivial solutions of (33). Assume that $\phi_a(z) = \omega_a(z)$ for some $z \in \overline{\Omega}$. Denote $\theta_a(x) = \omega_a(x) - \phi_a(x)$, then we have

\[ \int_{\Omega} J(x - y)\theta_a(y)dy - \theta_a(x) + qa(x)(\theta_1(x))^{q-1}\theta_a(x) = 0
\]

for some given function $\theta_1(x) > 0$. Thus we obtain

\[ \int_{\Omega} J(z - y)\theta_a(y)dy = J \ast \theta_a(z) = 0.
\]

Since $\Omega$ is bounded, a simple step analysis concludes that $\phi_a(x) \equiv \omega_a(x)$. Without loss of generality, we assume that $\phi_a < \omega_a$ and investigate the following two nonlocal dispersal equations

\[ \int_{\Omega} J(x - y)\omega_a(y)dy - \omega_a(x) + a(x)(\omega_a(x))^q = 0,
\]

\[ \int_{\Omega} J(x - y)\phi_a(y)dy - \phi_a(x) + a(x)(\phi_a(x))^q = 0.
\]

(34)
Denote \( \alpha_0 = \inf\{\alpha > 1 : \omega(x) < \alpha \phi(x) \text{ in } \Omega\} \), we have \( 1 \leq \alpha_0 < \infty \). We take constant \( M > 0 \) such that \( -a(x)s^q - Ms \) is non-increasing in \( [0, \alpha_0 + 1] \). If \( \alpha_0 > 1 \), we know that \( \omega_a(x) \leq \alpha_0 \phi_a(x) \) in \( \Omega \) and we have
\[
\int_{\Omega} J(x - y) \omega_a(y) dy - \omega_a(x) - M \omega_a(x)
\]
\[
= -a(x)(\omega_a(x))^q - M \omega_a(x)
\]
\[
\geq -a(x)[\alpha_0 \phi_a(x)]^q - M \alpha_0 \phi_a(x)
\]
\[
= \alpha_0 \int_{\Omega} J(x - y) \phi_a(y) dy - \alpha_0 \phi_a(x) - M \alpha_0 \phi_a(x).
\]
It follows from maximum principle that
\[
\omega_a(x) < \alpha_0 \phi_a(x) \quad \text{or} \quad \omega_a(x) = \alpha_0 \phi_a(x).
\]
In the case of \( \omega_a(x) < \alpha_0 \phi_a(x) \), we can find \( \alpha' < \alpha_0 \) such that \( \omega_a(x) < \alpha' \phi_a(x) \), which contradicts with the definition of \( \alpha_0 \). On the other case of \( \omega_a(x) = \alpha_0 \phi_a(x) \), and combine with (34), we have
\[
a(x)\alpha_0 (\alpha_0^q - 1) \phi_a(x) = 0
\]
which implies \( \alpha_0 = 0 \) (discard this case) or \( \alpha_0 = 1 \) which is an contradiction due to \( \phi_a < \omega_a \). This implies \( \phi_a = \omega_a \).

**Remark 6.** Suppose \( a > 0, \mu > 0 \) and \( 0 < q < 1 \), and let \( u \) be a unique positive solution of
\[
\begin{cases}
\int_{\Omega} J(x - y) u(y) dy - u(x) = -\mu a(x) u^q(x), & \text{in } \Omega, \\
u = 0, & \text{on } \mathbb{R}^N \setminus \Omega,
\end{cases}
(35)
\]
then \( u = \mu \frac{1}{1+q} \omega_a \). It is easy to see that, if \( u \) is a positive supersolution (respectively subsolution) of (35), then
\[
u \geq \mu \frac{1}{1+q} \omega_a \text{ (respectively } u \leq \mu \frac{1}{1+q} \omega_a \text{) in } \Omega.
\]

**Corollary 3.** Assume \( a > 0 \) and \( 0 < q < 1 \). If \( p + q = 1 \) and a positive steady-state solution \( (\lambda, u_\lambda) \) of (1). Then, if \( A = 1 \) (respectively \( A < 1 \) (respectively \( A > 1 \)) then \( \lambda = 0 \) (respectively \( \lambda > 0 \) (respectively \( \lambda < 0 \)), where
\[
A = \int_{\Omega} b(x) \omega_a^p(x) dx.
\]

**Proof.** If \( p + q = 1 \) and \( u_\lambda \) is a positive steady-state solution of (1) and \( \lambda \leq 0 \), then we have
\[
-J * u_\lambda(x) + u_\lambda(x) = \lambda u_\lambda^q(x) + a(x) u_\lambda^q(x) \int_{\Omega} b(x) u_\lambda^p(x) dx
\]
\[
\leq a(x) u_\lambda^q(x) \int_{\Omega} b(x) u_\lambda^p(x) dx,
\]
which implies \( u_\lambda \) is a subsolution of (35). In view of Remark 6 which tells the relation between \( u_\lambda \) and \( \omega_a \), then we can conclude
\[
u \leq \left( \int_{\Omega} b(x) u_\lambda^p(x) dx \right)^{\frac{1}{1+q}} \omega_a.
\]
Multiply \( b(x) \) and integrate in \( \Omega \) in the above inequality, and using the fact \( p+q = 1 \), we have
\[
1 = \left( \int_\Omega b(x)u^p_\lambda(x)dx \right)^{\frac{1}{p+q}} \leq \int_\Omega b(x)\omega^p_\lambda(x)dx = A.
\]
Similarly, we can obtain \( A \leq 1 \) when \( \lambda \geq 0 \). Which completes the proof. \( \square \)

6. **Proof of Theorem 1.2.** Assume that \( 0 < q < 1 \) and \( a < 0 \). In view of Theorem 5.3, we can obtain the existence of a unbounded continuum of positive steady-state solution \( C_0 \subset \mathbb{R} \times X \) of (1) bifurcating from the trivial solution at \( \lambda = 0 \). Moreover, there does not possess positive solution when \( \lambda \leq 0 \) and \( a < 0 \). Besides, notice the fact that for any positive steady-state solution \( (\lambda, u_\lambda) \) of (1), we obtain that
\[
- \int_\Omega J(x-y)u_\lambda(y)dy + u_\lambda(x) \leq \lambda u^q_\lambda(x)
\]
for \( a < 0 \) and by applying Remark 6, we know
\[
u_\lambda \leq \lambda^\frac{1}{q} \omega_1,
\]
that is, a priori bound of \( u_\lambda \). Hence, there exists a positive of solution of (1) for \( \lambda > 0 \). Recall (36), we deduce that \( \lim_{\lambda \to 0} \| u_\lambda \|_\infty = 0 \).

On the other hand, let \( u_\lambda \) be a positive steady-state solution of (1). In view of (36), we gave
\[
- \int_\Omega J(x-y)u_\lambda(y)dy + u_\lambda(x) = u^q_\lambda(x) \left( \lambda + a(x) \int_\Omega b(x)u^p_\lambda(x)dx \right) \geq \lambda u^q_\lambda(x) \left( 1 + a(x)\lambda^{\frac{p+q-1}{q}} \int_\Omega b(x)\omega^p_\lambda(x)dx \right).
\]

Then, if \( p + q > 1 \) and \( \lambda \) small, or \( p + q = 1 \) and \( |a_L| \) small, or \( p + q < 1 \) and \( \lambda \) large enough, and hence we can conclude that \( -J * u_\lambda + u > 0 \) and then \( u_\lambda \gg 0 \) by applying the maximum principle.

Finally, we will prove the uniqueness of strictly positive steady-state solution of (1). Assume that there exist two strictly positive steady-state solutions \( u \) and \( v \) of (1) and suppose \( u(z) = v(z) \) for some \( z \in \overline{\Omega} \). Denote \( \kappa(x) = u(x) - v(x) \), then we have
\[
\int_\Omega J(x-y)\kappa(y)dy - \kappa(x) + \left[ q(\kappa_1(x))^{q-1}(\lambda + a(x) \int_\Omega b(x)\kappa^p_1(x)dx) \right. \\
\left. + p\kappa^q_1(x)a(x) \int_\Omega b(x)\kappa^{p-1}_1(x)dx \right] \kappa(x) = 0
\]
for some given function \( \kappa_1(x) > 0 \). Thus we obtain
\[
\int_\Omega J(z-y)\kappa(y)dy = J * \kappa(z) = 0.
\]

Since \( \Omega \) is bounded, a simple step analysis concludes that \( u(x) \equiv v(x) \). Without loss of generality, we assume that \( u(x) > v(x) \), then
\[
\int_\Omega b(x)u^p(x)dx > \int_\Omega b(x)v^p(x)dx.
\]
It is easy to see that $u$ is a strictly subsolution of the equation
\[
\begin{cases}
- J * w + w = w^q \left( \lambda + a(x) \int_{\Omega} b(x)v^p(x)dx \right), & \text{in } \Omega,
\end{cases}
\]
and since $v$ is a solution of the equation, we obtain $u < v$, which is a contradiction. Then it is clear that $u \equiv v$. Which completes the proof.

7. **Proof of Theorem 1.3.** Assume $0 < q < 1$ and $a > 0$. In view of Theorem 5.3, we know that there exists a unbounded continuum of positive steady-state solutions $C_0 \subset \mathbb{R} \times X$ of (1) bifurcating from the trivial solution at $\lambda = 0$. In the subsection, we will divide into three cases to finish the proof of Theorem 1.3.

1) If $p + q < 1$.

Recall Proposition 3 b) (i), we observe that the bifurcation direction is subcritical. First, we prove that there does not exist positive steady-state solution for $\lambda$ negative enough. Indeed, if it is not true, then there exists a sequence of positive steady-state solutions $(\lambda_n, u_n)$ of (1) where $\lambda_n \to -\infty$ as $n \to \infty$, and hence
\[
\int_{\Omega} b(x)u_n^p(x)dx = -\int_{\Omega} J(x)u_n(y)dy + u_n(x) \geq 0,
\]
then
\[
\int_{\Omega} b(x)u_n^p(x)dx \to +\infty \quad \text{as} \quad n \to \infty.
\]

On the other hand, since $\lambda_n \leq 0$, we get that
\[
-\int_{\Omega} J(x)u_n(y)dy + u_n(x) \leq a(x)u_n^q(x) \int_{\Omega} b(x)u_n^p(x)dx,
\]
by using Remark 6, we have
\[
\left( \int_{\Omega} b(x)u_n^p(x)dx \right)^{\frac{1-p-q}{4}} \leq \int_{\Omega} b(x)\omega_n^p(x)dx = A,
\]
which is an absurdum since $\int_{\Omega} b(x)u_n^p(x)dx \to +\infty$ as $n \to \infty$.

Next, we will obtain the fact that there exists a constant $R$ such that $\| u \|_{\infty} \leq R$ for $\lambda \leq \Lambda$, where $\Lambda > 0$. Assume the contrary, there exists a sequence of positive steady-state solutions $(\lambda_n, u_n)$ of (1) such $\lambda_n \to \lambda_0 \leq \Lambda$ and $\| u_n \|_{\infty} \to +\infty$ as $n \to \infty$. Here, we denote $U_n = \frac{u_n}{\| u_n \|_{\infty}}$. Then, it is clear to see that
\[
-\int_{\Omega} J(x)u_n(y)dy + U_n(x)
\]
\[
= U_n^q(x) \left( \frac{\lambda_n}{\| u_n \|_{\infty}} + \frac{a(x)}{\| u_n \|_{\infty}^{1-p-q}} \int_{\Omega} b(x)U_n^p(x)dx \right),
\]
and passing to the limit in the above equality, we have $U_n \to U$ as $n \to \infty$ in $C(\overline{\Omega})$ and $U$ satisfying
\[
\begin{cases}
- J * U + U = 0, & \text{in } \Omega,
U = 0, & \text{on } \partial \Omega,
\end{cases}
\]
by applying the maximum principle, we obtain $U = 0$ which is a contradiction since $\| U \|_{\infty} = 1$.

Along the above analysis, there exists a constant $\lambda \in \mathbb{R}$ such that (1) has a positive steady-state solution for $\lambda > \lambda > -\infty$. Now, we consider $\lambda > \lambda$ and denote $u_\lambda$ be a solution of (1) when $\lambda = \lambda$. In fact, it is clear to see that the pair
\((u, \varpi) = (u_\lambda, K\varphi_1)\) is a pair of sub-supersolution of (1) for large \(K\), where \(\varphi_1\) be the principal eigenfunction of (17) associated with the principal eigenvalue \(\lambda_1(\Omega)\). Hence, we obtain the existence of positive steady-state solution of (1) for \(\lambda > \lambda_0\).

We now show the uniqueness of positive steady-state solution of (1) when \(\lambda \geq 0\). On the contrary, we assume that there exist two positive solutions \(u, v\). Similar the proof of Theorem 1.2, we know that if \(u(z) = v(z)\) for some \(z \in \Omega\), then by a simple step analysis we concludes that \(u(x) = v(x)\). Without loss of generality, we assume that \(u(x) < v(x)\), then there exists \(t_0 > 0\) such that \(t_0v \leq u\). Define

\[
t^* := \sup\{t > 0 : \text{ such that } tv \leq u\}.
\]

It is clear that \(t^* \leq 1\) and consider \(\omega := u - t^*v \geq 0\).

Then, using the fact \(t^* \leq 1\) and \(p + q < 1\), we get

\[
-\int_\Omega J(x - y)\omega(y)dy + \omega(x) = u^q(x)\left(\lambda + a(x)\int_\Omega b(x)u^p(x)dx\right) - t^*v^q(x)\left(\lambda + a(x)\int_\Omega b(x)v^p(x)dx\right) = v^q(x)\left[\lambda ((t^*)^q - t^*) + ((t^*)^{p+q} - t^*) a(x)\int_\Omega b(x)u^p(x)dx\right] > 0.
\]

Then by applying the maximum principle, we get that \(\omega > 0\) or \(\omega = 0\). In the case of \(\omega > 0\) that is \(t^*v < u\), then we can find \(t' > t^*\) such that \(t'v \leq u\), which contradicts with the definition of \(t^*\). In the case of \(\omega = 0\) that is \(t^*v = u\), we have \(t^* = 0\) (discard this case) or \(t^* = 1\) which is an absurdum due to \(u < v\).

The above proof confirms the uniqueness.

Finally, we claim that \(\|u_\lambda\|_\infty \to +\infty\) as \(\lambda \to +\infty\). Since \(\varphi = \varepsilon\varphi_1, \varepsilon > 0\) and \(\varphi_1\) be the positive eigenfunction associated to \(0 < \lambda_1(\Omega) < 1\) such that \(\|\varphi_1\|_\infty = 1\), is subsolution of (1) provided of \(\varepsilon \leq \lambda_1^\frac{1}{\lambda_1}\). Then, we have

\[
\lambda_1^\frac{1}{\lambda_1}\varphi_1 \leq u_\lambda.
\]

By Proposition 2, we can easily obtain the stability results. This completes the proof.

2): If \(p + q = 1\).

If we consider the case of \(A < 1\), then \(\lambda > 0\) by applying Corollary 3. It remains to show that \(u\) has a prior bound. If it is not true, then we assume that there exists a sequence of positive steady-state solutions \((\lambda_n, u_n)\) of (1) such that \(\lambda_n \to \lambda_0 > 0\) and \(\|u\|_\infty \to +\infty\) as \(n \to \infty\). Remark:

\[
U_n := \frac{u_n}{\|u_n\|_\infty}.
\]

Then, it is clear that \(U_n\) satisfies

\[
-\int_\Omega J(x - y)U_n(y)dy + U_n(x) = U_n^q(x)\left(\frac{\lambda_n}{\|u_n\|_\infty} + a(x)\int_\Omega b(x)U_n^p(x)dx\right),
\]

and passing to the limit in the above equality, we can arrive at \(U_n \to U\) as \(n \to \infty\) in \(C(\overline{\Omega})\) such that

\[
\left\{
\begin{array}{ll}
-J * U + U = U^q(x)a(x)\int_\Omega b(x)U^p(x)dx, & \text{in } \Omega, \\
U = 0, & \text{on } \partial \Omega,
\end{array}
\right.
\]
and then $A = 1$, which contradicts with $A < 1$. Besides, we can analyze similarly in other cases.

Now, we prove the uniqueness of positive steady-state solution of (1) for $\lambda > 0$. Assume there exist two positive solutions $u$ and $v$. Without loss of generality, there exists $t_0 > 0$ such that $t_0 u \leq v$. Denoted by

$$t^* := \sup\{ t > 0 : \text{ such that } tu \leq v \}.$$ 

We can show that $t^* \geq 1$, then $u \leq v$ and the result follows. We suppose that $t^* < 1$ and consider $\omega := v - t^* u > 0$.

Then, using the fact that $t^* < 1$ and $p + q = 1$ we obtain

$$- \int_{\Omega} J(x - y) \omega(y) dy + \omega(x) = u^q(x) \left( \lambda + a(x) \int_{\Omega} b(x) u^p(x) dx \right) - t^* v^q(x) \left( \lambda + a(x) \int_{\Omega} b(x) v^p(x) dx \right) = v^q(x) [\lambda (t^*)^q - t^*] > 0,$$

then by maximum principle, we have $\omega > 0$ or $\omega = 0$. In the case of $\omega > 0$ that is $t^* v < u$, then we can find $t' > t^*$ such that $t' v \leq u$, which contradicts the definition of $t^*$. In the case of $\omega = 0$ that is $t^* v = u$, we have $t^* = 0$ (discard this case) or $t^* = 1$ which is a contradiction due to $u < v$.

3): If $p + q > 1$.

In view of Proposition 3 b) (ii), it is clear that the bifurcation direction is supercritical. First, we show that there does not exist positive steady-state solution for enough large $\lambda$. Assume that there exists a sequence $(\lambda_n, u_n)$ of positive steady-state solutions of (1) and $\lambda_n \to +\infty$ as $n \to \infty$. Since $a > 0$, then it is easy to see that

$$u_n \geq \lambda_n^{1/p} \omega_1,$$

and denote $r_n := \int_{\Omega} b(x) u_n^p(x) dx \to +\infty$ as $\lambda_n \to +\infty$.

On the other hand, we know that $u_n$ satisfies

$$- \int_{\Omega} J(x - y) u_n(y) dy + u_n(y) = u_n^q(x) (\lambda_n + a(x)r_n).$$

Notice the fact that $r_n^{1/p} \omega_n$ is a subsolution of the above equation, and hence

$$r_n^{1/p} \omega_n \leq u_n,$$

then $r_n^{1/p} A \leq r_n$, which is an absurdum due to $p + q > 1$ and $r_n \to +\infty$ as $n \to \infty$.

Now, we prove the priori bound of $u$. Indeed, if it is not true, then there exist a sequence of positive steady-state solutions $(\lambda_n, u_n)$ such that $\lambda_n \to \lambda_0 < +\infty$ and $\| u_n \|_{\infty} \to +\infty$ as $n \to +\infty$. Hence it is easy to see

$$r_n \to \int_{\Omega} b(x) u_n^p(x) dx \to +\infty$$

as $n \to +\infty$. A similar argument to the used before, we can obtain an absurdum.

Along the above analysis, there exists a constant $\bar{\lambda} < +\infty$ such that the pair $(\bar{u}, \bar{v}) = (\varepsilon \varphi_1, u_\varphi)$ is a pair of sub-supersolution of (1) for $\varepsilon$ small enough and $\lambda < \bar{\lambda}$, where $u_\varphi$ is a solution of (1) at $\lambda = \bar{\lambda}$. Therefore, there exists a positive solution of (1) when $\lambda \leq \bar{\lambda}$.
Here, we will prove that \( \| u_\lambda \|_\infty \to \infty \) as \( \lambda \to -\infty \). Indeed, we have
\[
\lambda + a(x) \int_{\Omega} b(x) u_\lambda^p(x) dx \geq 0,
\]
and so
\[
\int_{\Omega} b(x) u_\lambda^p(x) dx \to +\infty,
\]
hence we obtain \( \| u_\lambda \|_\infty \to +\infty \) as \( \lambda \to -\infty \).

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