Back Reaction of Gravitational Radiation on the Schwarzschild Black Hole

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ABSTRACT

We address some of the issues that appear in the study of back reaction in Schwarzschild backgrounds. Our main object is the effective energy-momentum tensor (EEMT) of gravitational perturbations. It is commonly held that only asymptotically flat or radiation gauges can be employed for these purposes. We show that the traditional Regge-Wheeler gauge for the perturbations of the Schwarzschild metric can also be used for computing physical quantities both at the horizon and at infinity. In particular, we find that the physically relevant components of the EEMT of gravitational perturbations have the same asymptotic behaviour as the stress-energy tensor of a scalar field in the Schwarzschild background, even though some of the metric components themselves diverge.

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1 Introduction

The study of back reaction of gravitational waves on black hole spacetimes has been hampered by technical and conceptual difficulties, from the lack of closed-form expressions for the perturbations to subtle issues of gauge. The classical problem of the back reaction of a metric perturbation propagating out of (or into) a black hole is very interesting, but already complicated enough, and any progress in the classic realm would facilitate tremendously the analysis of quantum effects.

One of the main difficulties in considering the back-reaction of linearized perturbations is the problem of gauge\[1-3\]. We address this issue by considering a consistent perturbative expansion of the Einstein equations in first- and second-order perturbations of the black hole metric. There are gauge transformations related to the parametrization of the dynamical degrees of freedom in each order in perturbation theory, and we use these symmetries judiciously to cast the problem in as simple a manner as possible. The task of identifying physical quantities in the effective energy-momentum tensor (henceforth EEMT) of gravitational perturbations becomes much simpler that way.

The main point of this paper is the possibility of identifying physical quantities in Regge-Wheeler gauge. This issue has also been raised by Gleiser \[4\] for the case of the $\ell = 2$, even-parity gravitational perturbation. We explore the connections between the flux of energy out of the black hole, contained in the $G^\mu_\nu$ equations, and the inner product necessary both for the proper normalization of the gravitational perturbations and for any attempts at quantizing the gravitational fluctuations.

The outline of the paper is as follow: in Section 2 we review the formalism connected with the perturbative expansion of the Einstein equations to second order and define the EEMT of gravitational perturbations, $T^{GW}_{\mu\nu}$. In Section 3 we write the linearized perturbations and consider a scalar product that can fix their normalizations. In Section 4 we give details about our first order gauge choice: the Regge-Wheeler (RW) one. In Section 5 we show how the flux of gravitational waves computed in the RW gauge satisfies all the physical criteria of an energy flux both at infinity and on the horizon. In Section 6 we present a perturbative expansion of the physical degrees of freedom (the Regge-Wheeler and Zerilli functions) that can be used in the expressions for the energy, then we show that it is possible to normalize these
gravitational waves in the Schwarzschild background. Finally, we compute what that energy is in the case of the odd perturbations. We conclude in Section 7.

2 The General Plan

We study back reaction by expanding the Einstein field equations to second order in the initial values of the metric perturbations. The metric of the perturbed black hole is given by the series

\[ g_{\mu\nu} = g_{\mu\nu}^{(0)} + \epsilon \delta g_{\mu\nu} + \epsilon^2 g_{\mu\nu}^{(2)} + \mathcal{O}(\epsilon^3), \tag{1} \]

where \( \epsilon \) is the perturbative parameter, which can be thought of as the amplitude of the metric perturbation on the initial value surface. It is useful for the moment to regard the first order metric perturbations as gravitational waves propagating in the black hole background, and second order metric perturbations as the response (back reaction) effected by the gravity waves.

Both the gravity waves and their back reaction on the metric field can be parameterized in an infinity of ways, reflecting the symmetry of the exact theory under generic gauge transformations. Consistent with the expansion above, we write gauge transformations at first and second order in the form

\[ \tilde{\delta} g_{\mu\nu} = \delta g_{\mu\nu} - \mathcal{L}_{\xi^{(1)}} g_{\mu\nu}^{(0)}, \tag{2} \]

\[ \tilde{g}_{\mu\nu}^{(2)} = g_{\mu\nu}^{(2)} - \mathcal{L}_{\xi^{(1)}} \delta g_{\mu\nu} + \frac{1}{2} \mathcal{L}_{\xi^{(1)}}^2 g_{\mu\nu}^{(0)} - \frac{1}{2} \mathcal{L}_{\xi^{(2)}} g_{\mu\nu}^{(0)}, \tag{3} \]

where \( \mathcal{L} \) is the Lie derivative and \( \xi^{(1)} \), \( \xi^{(2)} \) are two independent vectors. Their significance becomes clearer if we regard this gauge transformation as being generated by a second order coordinate transformation:

\[ \tilde{x}^\mu = x^\mu + \epsilon \xi^{(1)} \mu + \frac{\epsilon^2}{2} \left( \xi^{(1)} \nu \xi^{(1)} \nu + \xi^{(2)} \mu \right) + \mathcal{O}(\epsilon^3). \tag{4} \]

In practical terms, the statement is that the parametrization of the first-order quantities (the gravitational waves) can be carried independently of
the parametrization of the second-order quantities (the back reaction).

Back reaction is the feedback effect driven by the nonlinearities of the Einstein field equations. At second order in the perturbations, we expand Einstein’s equations as

\[ G^{(1)}_{\mu\nu}[g^{(2)}] + G^{(2)}_{\mu\nu}[\delta g] + \ldots = 0 , \]  

where the gravitational waves \( \delta g \) obey the Einstein field equations in vacuum,

\[ G^{(1)}_{\mu\nu}[\delta g] = 0 . \]

The notation should be clear: \( G^{(1)}_{\mu\nu} \) is a differential operator which is linear in the arguments, and for this reason appears in Eq. (6) acting on the first order metric perturbations, and in Eq. (5) acting on the second order metric perturbations. \( G^{(2)}_{\mu\nu} \) is a differential operator which is quadratic in its arguments.

We will consider Eq. (5) for the spherically symmetric (isotropic) second order perturbations. The Schwarzschild background metric in \( x = (t, r, \theta, \phi) \) coordinates is

\[ g^{(0)}_{\mu\nu} = \text{diag}(-\Gamma, 1, r^2, r^2 \sin^2 \theta) , \]

with \( \Gamma \equiv 1 - R/r \) and \( R = 2GM \). The metric to second order is given by

\[ g_{\mu\nu} = g^{(0)}_{\mu\nu}(r) + \epsilon \delta g_{\mu\nu}(x) + \epsilon^2 g^{(2)}_{\mu\nu}(t, r) + \mathcal{O}(\epsilon^3) . \]

We have omitted all second-order anisotropic terms from this expression because at this order in perturbation theory they decouple from the isotropic terms.

Note that by keeping the spherical symmetry in this problem does not imply the time independence of the second order metric, which includes effects from back-reaction: the time independence of the Schwarzschild metric is the consequence of Birkhoff’s theorem, which assumes no source terms for the Einstein equations. Therefore, the metric of a black hole that radiates can no longer be static.

\[ ^1 \text{Note, however, that the parametrization at the first order does affect the second-order quantities through nonlinear terms - e.g., the middle term on the right hand side of (5).} \]
The back-reaction equation we are interested in is the isotropic projection of Eq. (5), which we recast as:

\[ G^{(1)}_{\mu\nu}[g^{(2)}(t, r)] = -\langle G^{(2)}_{\mu\nu}[\delta g] \rangle_\Omega = 8\pi G T^{GW}_{\mu\nu}, \]  

where the angle average is given by

\[ \langle F \rangle_\Omega = \frac{1}{4\pi} \int d\theta d\phi F. \]  

We call \( T^{GW}_{\mu\nu} \) the effective energy-momentum tensor (EEMT) of gravitational waves in the Schwarzschild background.

The gauge transformation at the second order (3) is rewritten as:

\[ \tilde{g}^{(2)}_{\mu\nu}(t, r) = g^{(2)}_{\mu\nu}(t, r) - \langle L \xi^{(1)} \delta g_{\mu\nu} \rangle_\Omega - \frac{1}{2} \mathcal{L} \tilde{\xi}^{(2)} g^{(0)}_{\mu\nu} + \frac{1}{2} \langle \mathcal{L}^2 \xi^{(1)} g^{(0)}_{\mu\nu} \rangle_\Omega, \]  

where now \( \tilde{\xi}^{(2)}_\mu = [\tilde{\xi}^{(2)}_t(t, r), \tilde{\xi}^{(2)}_r(t, r), 0, 0] \), i.e. only the gauge freedom according to the spherical symmetry remains. It will be clear in the next sections, when we explicitly consider the expansion of the perturbations in spherical harmonics, that Eqs. (9) and (11) follow from the averaging of Eqs. (5) and (3) respectively – since the second order quantities appear linearly in the second order relations, only the monopole part of \( g^{(2)}_{\mu\nu} \) and \( \xi^{(2)}_\mu \) survive to the averaging, and these are precisely \( g^{(2)}_{\mu\nu}(t, r) \) and \( \tilde{\xi}^{(2)}_\mu \). The monopole part of the metric perturbation contains four degrees of freedom, but we can use the two components of \( \tilde{\xi}^{(2)}_\mu \) to cancel two of the degrees of freedom, so we are left with two independent degrees of freedom. This is the right number of functions needed to describe a time-dependent isotropic metric, as for the case of an evaporating black hole [6].

Finally, let us comment on three basic points.

First, it is easy to verify that the EEMT obeys conservation equations which are just the Bianchi identities expanded to second order in perturbation theory. These conservation equations are completely independent of the form of the second-order degrees of freedom, since the part of the Bianchi identities which is linear in \( g^{(2)}_{\mu\nu} \) and its derivatives is zero identically (see also [7]).

Second, we do not address the problem of the gauge dependence (or independence) of the EEMT of gravitational waves. Once one has fixed his gauge choices at the first and second order, one can then try to extract
physical information from the analysis of the second order Einstein equations or from considering gauge-invariant observables.

Third, in what follows we consider the problem of linearized perturbations in vacuum. Classically, we would not have any physical solutions without source terms. Here we could think of the gravitational waves as having been generated by quantum effects close to the horizon (Hawking evaporation [8]).

3 Linearized perturbations in vacuum

Consider the Einstein field equations in vacuum, linearized around a background geometry \( g^{(0)}_{\mu \nu} \):

\[
G^{(1)}_{\mu \nu} = \delta g_{\mu \nu | \alpha} - \delta g_{\mu \alpha | \nu} - \delta g_{\nu \alpha | \mu} + \delta g_{\nu | \mu}^{(0)} + g^{(0)}_{\mu \nu} \left( \delta g_{\alpha \beta | \alpha} - \delta g_{| \alpha} - \delta g_{| \alpha} \right) + \delta g_{\mu \nu} R^{(0)} - g^{(0)}_{\mu \nu} \delta g_{\alpha \beta} R^{(0)} = 0
\]

(12)

where \( \delta g \) is the trace of \( \delta g_{\mu \nu} \) and \( | \) denotes the covariant derivative with respect to the background metric. By subtracting from the above equation its trace and considering Ricci flat background spacetimes (such as Schwarzschild) we have

\[
\delta g_{\mu \nu | \alpha} - \delta g_{\mu \alpha | \nu} - \delta g_{\nu \alpha | \mu} + \delta g_{\nu | \mu} = 0,
\]

(13)

and

\[
\delta g_{\alpha \beta | \alpha} - \delta g_{| \alpha} = 0.
\]

(14)

It is easy to see that the last equation is trivial if one uses the transverse (\( \delta g_{\alpha \beta | \alpha} = 0 \)) and traceless (\( \delta g = 0 \)) gauge. By using the relation

\[
\delta g_{\mu \nu | \alpha} = \delta g_{\mu \alpha | \nu} + R^{\alpha}_{\mu \nu} \beta(0) \delta g_{\alpha \beta} + R^{\alpha}_{\nu} \delta g_{\mu \alpha}
\]

(15)

we can switch the order of derivatives in Eq. (13):

\[
\delta g_{\mu \nu | \alpha} - \delta g_{\mu \alpha | \nu} - \delta g_{\nu \alpha | \mu} + 2 R^{\alpha}_{\mu \nu} \beta(0) \delta g_{\alpha \beta} + \delta g_{\nu | \mu} = 0.
\]

(16)

We can associate a scalar product with these equations of motion [10]:
\[
\langle \psi^{\alpha\beta}, \phi_{\alpha\beta} \rangle = -i \int_{\Sigma} d^3x \sqrt{g_\Sigma} n^\mu \\
\times \left[ \psi^{\alpha\beta} \bar{\phi}_{\alpha\beta|\mu} - \psi^{\alpha\beta} \bar{\psi}_{\alpha\beta|\mu} - 2 \left( \psi_{\mu\nu} \bar{\phi}_{\alpha\nu \mid |\alpha} - \bar{\phi}_{\mu\nu} \bar{\psi}_{\alpha\nu \mid |\alpha} \right) \right],
\]

where \( \psi \) and \( \phi \) are two generic (complex) solutions to the equations of motion \((16)\), \( n^\mu \) is the unit vector normal to the spacelike hypersurface \( \Sigma \), \( g_\Sigma \) is the determinant of the induced metric on \( \Sigma \) and \( \bar{\psi}_{\mu\nu} \) is defined by

\[
\bar{\psi}_{\mu\nu} = \psi_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} \psi_\alpha^\alpha.
\]

The normalization of a gravity wave mode should be given by the inner product \((17)\). However, because metric perturbations diverge at infinity in the RW gauge, it is common practice to neglect the Klein-Gordon inner product \((17)\) and instead to normalize the gravity waves through their mass-energy in an asymptotically flat region of space. Since we would like to compute the mass-energy of the gravitational waves from first principles, we will avoid this procedure. Instead, we show how to obtain asymptotic expressions for the gravitational waves in Regge-Wheeler gauge, and use them in the inner product \((17)\), which then becomes well-defined.

### 4 The Regge-Wheeler gauge

Under a gauge transformation \((4)\), the metric perturbations are transformed according to the general laws \((2)\) and \((3)\). In order to choose a gauge we must specify constraints for the metric components that uniquely fix the eight free functions \( \xi^{(1)\mu} \) and \( \xi^{(2)\mu} \) with respect to an arbitrary gauge. The penalty for not fixing the gauge completely is having to deal with unphysical degrees of freedom (which become ghosts after quantization), corresponding to the free functions that were left unconstrained.

To first order in perturbation theory, one of the (infinitely many) choices of coordinates which fixes the gauge completely in the Schwarzschild background is the Regge-Wheeler gauge \([11, 9, 3]\). This choice of gauge is exceptionally convenient, since the tensor structure of the Einstein field equations decouple from their angular dependence. The two degrees of freedom of gravitons in vacuum correspond to two (orthogonal) sets of perturbations,
odd and even (or electric and magnetic), depending on how they transform under parity ($\vec{x} \rightarrow -\vec{x}$).

In the spherically symmetric background (7), the perturbations are expanded in spherical harmonics $Y^m_\ell(\theta, \phi)$ and therefore the modes carry multipole numbers ($\ell, m$). In Regge-Wheeler gauge we have, for the odd metric perturbations:

$$
\delta g^o_{\mu\nu} = \begin{pmatrix}
0 & 0 & \frac{h^\ell_0}{\sin \theta} \frac{\partial Y^m_\ell}{\partial \phi} & \frac{h^\ell_0}{\sin \theta} \sin \theta \frac{\partial Y^m_\ell}{\partial \phi} \\
0 & 0 & \frac{h^\ell_1}{\sin \theta} \frac{\partial Y^m_\ell}{\partial \phi} & \frac{h^\ell_1}{\sin \theta} \sin \theta \frac{\partial Y^m_\ell}{\partial \phi} \\
\text{sym} & \text{sym} & 0 & 0 \\
\text{sym} & \text{sym} & 0 & 0 \\
\end{pmatrix}
$$

(19)

where $h^\ell_0$ and $h^\ell_1$ are functions of $t$ and $r$.

The even parity perturbations are given by

$$
\delta g^e_{\mu\nu} = \begin{pmatrix}
\Gamma H^\ell_0 Y^m_\ell & L^\ell_0 Y^m_\ell & 0 & 0 \\
\text{sym} & \frac{H^\ell_0}{r} Y^m_\ell & 0 & 0 \\
0 & 0 & r^2 K^\ell_0 Y^m_\ell & 0 \\
0 & 0 & 0 & r^2 K^\ell_0 Y^m_\ell \sin^2 \theta \\
\end{pmatrix}
$$

(20)

where $H^\ell_0$, $L^\ell_0$, $H^\ell_2$ and $K^\ell_0$ are also functions of $t$ and $r$.

With the metric perturbations written in this form, the equations of motion can be separated and cast in the simple form (see for example [11, 9, 3])

$$
\frac{\partial^2 \chi^{o,e}_{\mu\nu}}{\partial t^2} - \frac{\partial^2 \chi^{o,e}_{\mu\nu}}{\partial r^*^2} + V^{o,e}(r) \chi^{o,e} = 0,
$$

(21)

where the superscripts $o, e$ correspond to the odd or even perturbations and $r^*$ is the so-called tortoise coordinate,

$$
r^* = r + R \ln \left( \frac{r}{R} - 1 \right).
$$

(22)

The odd and even potentials $V^{o,e}$ are given respectively by:

$$
V^o(r) = \Gamma \left[ \frac{\ell(\ell+1)}{r^2} - \frac{3R}{r^3} \right]
$$

(23)

and
$$V^e(r) = \frac{\Gamma^2 \lambda^2 (\lambda + 1) r^3 + 3 \lambda^2 R r^2 + \frac{2}{3} \lambda R^2 r + \frac{2}{3} R^3}{r^3 (\lambda r + \frac{3}{2} R)^2}, \quad (24)$$

where $\lambda \equiv \ell(\ell + 1)/2 - 1$.

Both the even and odd potentials have the same asymptotic form in the limit $r \to +\infty$:

$$V^{o,e}(r) \simeq \frac{2 \lambda + 1}{r^2}. \quad (25)$$

Both also go to zero on the horizon, but with a different slope:

$$V^o(r) \simeq \frac{2 \lambda - 1}{R^2}, \quad V^e(r) \simeq \frac{2 \lambda^2 + \lambda + 3/4}{\lambda + 3/2}. \quad (26)$$

The two odd functions $h^o_{0m}$ and $h^o_{1m}$ are easily expressed in terms of the Regge-Wheeler function $\chi^o$:

$$\frac{\partial h^o_{0m}}{\partial t} = \frac{\partial}{\partial r^*} (r \chi^o) \quad \text{and} \quad h^o_{1m} = \frac{r}{\Gamma} \chi^o. \quad (27)$$

The even metric perturbations can be written in terms of the Zerilli function $\chi^e$ as well. Since one of the linear equations of motion states that $H^e_{2m} = H^o_{0m} \equiv H^e_{0m}$, there are only three different nonzero metric components:

$$K_{\ell m} = f_\ell(r) \chi^e + \frac{\partial \chi^e}{\partial r^*}$$
$$L_{\ell m} = \frac{\partial}{\partial t} \left[ g_\ell(r) \chi^e + r \frac{\partial \chi^e}{\partial r^*} \right]$$
$$H_{\ell m} = \frac{\partial}{\partial r} \left[ \Gamma g_\ell(r) \chi^e + r \Gamma \frac{\partial \chi^e}{\partial r} \right] - K, \quad (28)$$

where
\[ f_\ell(r) \equiv \frac{\lambda(\lambda + 1)r^2 + 3/2Rr\lambda + 3/2R^2}{r^2(\lambda r + 3/2R)}, \quad (29) \]
\[ g_\ell(r) \equiv \frac{1}{r\Gamma} \frac{\lambda r^2 - 3/2\lambda Rr - 3/4R^2}{\lambda r + 3/2R}. \quad (30) \]

Some remarks are in order. Note that the potentials \( V^{\alpha,e}(r) \) vanish asymptotically both on the horizon and at infinity \( (r^* \to \mp\infty) \). On these asymptotic regions the solutions to the wave equation (21) can be expanded in terms of plane waves in retarded or advanced time,
\[ \chi^{\alpha,e} \sim e^{-i\omega(t\pm r^*)}, \quad (31) \]
where the plus holds for ingoing (towards the horizon) waves and the minus holds for outgoing waves.

This implies that in Regge-Wheeler gauge the odd and even metric perturbations diverge both on the horizon and at infinity: for example, \( h_1 \simeq r^*\chi^o \) when \( r^* \to \infty \) and \( h_1 \simeq Re^{-r^*/R}\chi^o \) when \( r^* \to -\infty \). Of course, this is a coordinate artifact of the Regge-Wheeler gauge, and physical quantities should remain finite in that (or any other) gauge, as long as we keep away from the singularity at the center of the black hole. There are gauges in which physical quantities are manifestly regular at infinity, such as asymptotically flat \(^3\) or radiation gauge \(^9\). Our purpose is to show (see Section 5) that the Regge-Wheeler gauge can also be used to compute physical quantities, even if the metric components diverge.

We can simplify greatly the ansatz for the metric if we adopt the standard procedure of rotating the \( z \)-axis to put each mode \((\ell, m)\) in the state \((\ell, 0)\). The \( \phi \) dependence thus drops out of the metric perturbations \((19)-(20)\), and all subsequent equations involve only Legendre polynomials \( P_\ell(\cos \theta) \) and their derivatives. Of course this simplification does not change our results for the back reaction on the isotropic mode.

If the metric ansatz \((19)-(20)\) is appropriate to describe the anisotropic metric perturbations around the black hole, what is appropriate to describe the back reaction of these perturbations on the isotropic background? Physics dictates that we should expect some mass to be lost by the black hole. However, the mass appears both in \( g_{00} = -(1 - 2GM/r) \) and in \( g_{11} = \)
(1 − 2GM/r)−1. Generically there should be two degrees of freedom describing the (no longer static) isotropic metric, which we define adopting the following gauge choice at the second order:

\[ g_{00} \to -1 + \frac{2GM}{r} - \epsilon^2 \frac{2G\Delta_0 M(r,t)}{r} + \mathcal{O}(\epsilon^3), \quad (32) \]

\[ g_{11} \to \left[ 1 - \frac{2GM}{r} + \epsilon^2 \frac{2G\Delta_1 M(r,t)}{r} \right]^{-1} + \mathcal{O}(\epsilon^3). \quad (33) \]

The interpretation of \( \Delta_0 M \) is related to the ADM mass at infinity, while \( \Delta_1 M \) is related to the location of the horizon. However, sometimes it is useful to make the following approximation: assume that the black hole emits radiation in packets, such that for a spherical shell of radius \( r = \bar{r} \) from the black hole, any given packet is either completely inside or outside the shell. If the gravity waves packet is inside the shell, the metric describing the isotropic background is just the exterior Schwarzschild metric without corrections. If the packet is outside the shell, the mass of the black hole measured by an observer located at \( r = \bar{r} \) has changed by an amount \( \Delta M \) that is now independent of time and radius. In the limit of this approximation we can ignore subtleties with the dynamics that may distinguish \( \Delta_0 M(r,t) \) from \( \Delta_1 M(r,t) \) in a more complex physical situation.

Let us comment on a peculiarity of the gauge choice at second order. There is no way of fixing completely the gauge for the isotropic perturbation, since the remaining freedom is related to a redefinition of time:

\[ t \to t + \frac{\epsilon^2}{2} f(t) + \mathcal{O}(\epsilon^3) \quad (34) \]

This is also related to the fact that time is defined \textit{ad hoc} for the Schwarzschild solution (which for us is the zero order metric of our perturbative analysis). 

5 The flux of gravitational waves

In this section we will focus on the mixed \( t - r \) component of the EEMT of gravitational waves for two main reasons. The first is simplicity: \( G^t_r = R^t_r \)

\footnote{We thank Roberto Casadio for having pointed out this feature to us.}
to second order has a short expression in terms of the perturbations. The second is that this term represents an energy flux. Consider for example a scalar field in the Schwarzschild background. The mixed $t - r$ component of the scalar’s energy-momentum tensor is given by:

$$T^t_r = \frac{1}{\Gamma^2} \varphi_{,t} \varphi_{,r} \left( 2\xi - 1 \right) + 2\frac{\xi}{\Gamma^2} \varphi \left( \varphi_{,tr} - \frac{R}{2r^2} \varphi_{,t} \right),$$

(35)

where $\xi$ is the coupling of the scalar field to the curvature. By considering the proper normalization $\sim 1/r$ in front of the asymptotic spherical plane waves solution for a scalar field one has an asymptotic behaviour $\mathcal{O}(1/\Gamma^2)$ on the horizon and $\mathcal{O}(1/r^2)$ at infinity. We observe that the behaviour of $T^t_r$ on the horizon is not really pathological, indeed it is regular in a freely falling frame.

We now proceed to calculating the mixed $t - r$ component of the effective energy-momentum tensor (EEMT) of gravitational perturbations around a black hole. Since the even and the odd degrees of freedom are orthogonal to each other at this order in perturbation theory, we can consider their contributions to the EEMT separately. For the choice of second order metric coefficient in Eq. (33) one has:

$$R^t_r [g^{(2)}(t, r)] = -\frac{2G\Delta M}{r^2\Gamma}$$

(36)

The odd contribution to the $t - r$ component in Regge-Wheeler gauge is, after averaging over angles:

$$\langle R^t_r \rangle = \sum_{\ell} \frac{\ell(\ell + 1)}{2\ell + 1} \frac{h_1}{2r^4} \left[ r^2(h'_1 - h''_0) - 2r\dot{h}_1 + 2\dot{h}_0 \right],$$

(37)

where a dot and a prime denote derivatives with respect to $t$ and $r$ respectively.

The first term inside square brackets in (37) appears to go as $r^3$ at $r \to \infty$ (since $h_0, h_1 \propto r$ in this limit). If that was so, the radiated mass $\Delta M$ would be divergent. In reality, that term is at most $\propto r^2$ because the leading term cancels upon use of relations (19) and the equations of motion (21). Nevertheless, the $\mathcal{O}(1/r)$ term in $\langle R^t_r \rangle$ still gives a contribution that could
make $\Delta M$ divergent. We show in the next section that these apparently divergent terms in Regge-Wheeler gauge are similar to the terms in the normalization of the gravity waves that are also apparently divergent. However, in both cases the divergences are revealed to be fictious, and the physical information can be retrieved.

On the horizon the leading term in the right hand side in (37) appears to go as $O(1/\Gamma^3)$. Also in this case by using the equation of motion (21) this term vanishes, and so $T^t_{GWr} \sim O(1/\Gamma^2)$, as in the scalar field case (35).

The even contribution to the $t-r$ mixed component has a similar structure:

$$\langle R^t_{r, even}\rangle = \sum_{\ell} \frac{1}{2\ell + 1} \frac{1}{r^2 \Gamma} \left[ 2r^2 \dot{K}'(K - H) + 2r^2 \Gamma K''L \right] +r^2 K'(\dot{K} + \dot{H}) - r^2 \dot{K}H' + 4r \Gamma K'L$$

Again, from (28) there is an apparent divergence in $T^t_{GWr}$ at infinity which is given by terms $O(r^3)$ inside the square brackets in (38). However, the leading term vanishes by using (21), and the physical information can be obtained among the subleading terms. Also in this case the apparent divergence on the horizon $O(1/\Gamma^3)$ vanishes by using (21).  

6 Asymptotics of the Regge-Wheeler and Zerilli functions

Consider now the normalization of the metric perturbations in Regge-Wheeler gauge. Since perturbations with different parity decouple the odd contribution to the inner product (17) can be written as:

$$\langle \psi^\alpha\beta, \phi_{\alpha\beta}\rangle_{odd} = -i \sum_{\ell} \frac{4\ell(\ell + 1)}{2\ell + 1} \int_{\Sigma} dr^s \Gamma$$

Note that all these cancellations of the most divergent terms by using the equations of motions are also a property of the energy momentum tensor of gravitational waves, even before the average over angles.
\[
\times \left[ h_1 \dot{h}_1^* - h_1^* \dot{h}_1 + \frac{h_0^*}{r}(r h_1' + 2 h_1) - \frac{h_0}{r}(r h_1'' + 2 h_1') \right],
\]

where the angle integrations have already been performed. When \( r \to \infty \) the integrand appears to be \( \propto r^2 \). This structure appears in the even contribution to the inner product as well.

We show next that in fact no divergences survive in the normalization conditions once the perturbative solutions to the wave equation (21) are used in the expression above.

From Eq. (21) for the Zerilli (\( \chi^e \)) and Regge-Wheeler (\( \chi^o \)) functions it is clear that at infinity they assume the form of a plane wave in retarded time \( u = t - r^* \) (or advanced time \( v \), in the case of an incoming mode). This suggest an expansion of \( \chi \) in terms of functions of both retarded time and radius of the form\(^3\)

\[
\chi(r,t) = X_0(u) + r^{-1} X_1(u) + r^{-2} X_2(u) + \mathcal{O}(r^{-3}).
\]

(40)

This expansion should be consistent as long as the gravity wave packet does not probe the region where the potential is large, \( r \approx 3GM \). In other words, the approximation breaks down as soon as the reflection and transmission of the waves become important and the very description of a wave packet in terms of purely ingoing (or outgoing) modes breaks down.

We can substitute the ansatz above into the wave equation (21) and solve the hierarchy of equations that ensue. Since the leading order terms in the potentials \( V^o \) and \( V^e \) are the same to order \( r^{-2} \), the solutions for \( \chi^e \) and \( \chi^o \) are identical to each other at that order. The result is, for an ingoing gravity wave of even or odd parity,

\[
\chi^{o,e}(r,t) = \frac{1}{\lambda + 1} \ddot{X}^{e,o}(u) + r^{-1} \dot{X}^{e,o}(u) + r^{-2} \left[ \frac{\lambda}{2} X^{e,o}(u) - \frac{3GM}{2(\lambda + 1)} \dot{X}^{e,o}(u) \right] + \mathcal{O}(r^{-3}).
\]

(41)

Notice that \( X \) is a completely generic function of retarded time, not necessarily a plane wave, reflecting the fact that we have not yet imposed any boundary conditions on the problem. We also stress that \( X \) carries an index \( \ell \), like the mode of metric perturbations.
From now on we focus on the odd perturbations for the sake of simplicity, since the expressions for the even perturbations are too cumbersome, and the physics and the lessons we draw are precisely the same as in the odd case. With the help of relations (27) we can express the metric perturbations in terms of the odd function $X(u)$:

$$h_0 = -\frac{r}{\lambda + 1} \dot{X} - \frac{\lambda}{\lambda + 1} \ddot{X} + r^{-1} \left(-\frac{GM}{2(\lambda + 1)} \dot{X} - \frac{\lambda}{2} X\right) + \mathcal{O}(r^{-2}), \quad (42)$$

$$h_1 = \frac{r}{\lambda + 1} \dot{X} + \left(\frac{2GM}{\lambda + 1} \ddot{X} + \dot{X}\right) + r^{-1} \left(-\frac{4G^2M^2}{\lambda + 1} \dot{X} + GM \frac{4\lambda + 1}{2(\lambda + 1)} \ddot{X} + \frac{\lambda}{2} X\right) + \mathcal{O}(r^{-2}). \quad (43)$$

It is now a matter of algebra to substitute these expressions into Eqs. (37) and (39). The result for the normalization is

$$\langle \psi^{\alpha\beta}, \phi^{\alpha\beta} \rangle_{\text{odd}} = i \frac{8(\lambda + 1)}{2\ell + 1} \int_{\Sigma} dr^* \frac{\lambda}{(\lambda + 1)^2} \partial_t \left(\dot{X}_\psi \dot{X}_{\psi}^* - \dot{X}_\phi \dot{X}_{\phi}^*\right) + \mathcal{O}(r^0). \quad (44)$$

Therefore, the $\mathcal{O}(r^3)$ divergence in the integral (39) has simply disappeared after we used the asymptotic formula (41).

The first term in the expression above would make the integral diverge as $r^2$. However, that term vanishes because it is the time derivative of an integral over the spacelike hypersurface $\Sigma$ (our assumptions are that the gravitational wave packet is either entirely inside or entirely outside $\Sigma$.) There is a physically intuitive way to argue that this divergent term should vanish. In the WKB approximation the $X_a$’s are plane waves, $X_a \propto \exp(-iw_a u)$. In this framework, the phases in the integrand of (44) would interfere destructively.

In either case, the normalization of the odd metric perturbations is given by the next term in perturbation theory:

$$\langle \psi^{\alpha\beta}, \phi^{\alpha\beta} \rangle_{\text{odd,WKB}} = i \frac{24\lambda^2}{(\lambda + 1)(2\ell + 1)} \int_{\Sigma} dr^* (\dot{X}_\psi \dot{X}_{\psi}^* - \dot{X}_\phi \dot{X}_{\phi}^*). \quad (45)$$

If the mode solutions $X_\psi$ and $X_\phi$ are just WKB modes, the integral above gives the expected normalization [10] with a delta function over the frequencies of the modes, $\delta(w_\psi - w_\phi)$. 

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Therefore, it is possible to normalize gravitational waves in Schwarzschild background using the inner product in Regge-Wheeler gauge. As we show below, a similarly divergent time derivative term appears in the equation for the time variation of the mass of the black hole. Again, the next to leading term is the finite one, in agreement with a finite gravitational radiation flux from an object with finite mass. We note that the term in (44) which is proportional to $r$ at infinity contains also a divergence on the horizon: we have just shown how deal with it. The next term in (45) is regular also on the horizon, since an expansion in powers of $\Gamma$ similar to (40) holds close to the horizon.

Let us consider now the flux of energy given by Eq. (37). Substituting the expressions (42) and (43) we obtain the following (apparently) divergent term:

$$\Delta_1 \dot{M} = \sum_\ell \frac{2}{(\lambda + 1)(2\ell + 1)} r \frac{\partial}{\partial t} \ddot{X} \dddot{X}^* + O(r^0).$$  \hspace{1cm} (46)

Some authors have assumed\[3, 4\] that $X$ obeys some asymptotic conditions such that the $O(r)$ term vanishes. It appears to us that the WKB approximation could already be sufficient to ensure the finiteness of the radiated mass.

The radiated mass from odd-parity gravitational waves comes from the next order term, and using (42)-(43) the final result is

$$\Delta_1 M = \sum_\ell \frac{2(\lambda + 1)^2}{(2\ell + 1)} \int dt (\chi^o)^2 + O(r^{-1}),$$  \hspace{1cm} (47)

which is the expression that has been known for a long time \[12\] and was first found using the Landau-Lifshitz pseudo-tensor in the asymptotically flat gauge. Other authors \[3, 4\] have obtained similar results for the even perturbations (in the $\ell = 2$ multipole case), also using the Regge-Wheeler gauge.

7 Conclusions

We have explored the possibility of performing back-reaction computations for the Schwarzschild black-hole in the Regge-Wheeler gauge. The Regge-Wheeler choice has the advantage of fixing completely the gauge, but it has
the disadvantage of divergencies in the metric coefficients both on the horizon and at spatial infinity.

We have shown that divergencies in the gravitational energy flux at infinity and on the horizon, which could arise from choosing the Regge-Wheeler gauge, are in fact nonexistent. Furthermore, the same types of cancellations arise in the evaluation of the inner product, which is related to the equations of motion of the linear perturbations. This could represent a viable check for the amplitude of gravitational waves, independent from the requirement that the energy flux of gravitational waves must be finite at infinity. The investigation of the effect of the back-reaction of gravitational radiation on the horizon is under current investigation.

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