UNIFORM HYPERBOLICITY IN NONFLAT BILLIARDS

MICKAËL KOURGANOFF*

Institut Fourier, Université Grenoble Alpes
100, rue des mathématiques
38610 Gières, France

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Abstract. Uniform hyperbolicity is a strong chaotic property which holds, in particular, for Sinai billiards. In this paper, we consider the case of a nonflat billiard, that is, a Riemannian surface with boundary. Each trajectory follows the geodesic flow in the interior of the billiard, and bounces when it meets the boundary. We give a sufficient condition for a nonflat billiard to be uniformly hyperbolic. As a particular case, we obtain a new criterion to show that a closed surface has an Anosov geodesic flow.

1. Introduction and notations. In this paper, a smooth billiard is a connected compact subset \( D \) of a Riemannian surface \( M \), such that \( D \) has a smooth boundary while \( M \) has no boundary. By “smooth boundary”, we mean that each component of \( \partial D \) is the image of a smooth embedding \( \Gamma : \mathbb{R}/l\mathbb{Z} \to M \), with unit speed, where \( l \) is the length of the component. Each curve \( \Gamma \) is called a wall of \( D \): it has a unit tangent vector \( T \) and a unit normal vector \( N \) pointing toward \( \text{Int} \ D \). A billiard whose walls have negative curvature is said to be dispersing.

Most of the billiards which appear in the literature are flat, and more precisely, in the ambient surface \( M = \mathbb{R}^2 \) or \( M = \mathbb{T}^2 \). Chaotic billiards in general Riemannian surfaces were studied, for example, in \([22]\), \([16]\), and \([24]\). For billiards in surfaces of constant curvature, see also \([3]\) and \([8]\). In this paper, we focus on uniform hyperbolicity (see Definition 1.1) for billiards in general surfaces.

One defines the phase space \( \Omega = T^1(\text{Int} \ D) \), and the billiard flow \( \phi_t : \Omega \to \Omega \), in the following way:

1. As long as it does not hit a wall, the particle follows a geodesic in \( M \);
2. When it arrives to the boundary of the billiard, the particle bounces, following the billiard reflection law: the angle between the particle’s speed vector and the boundary’s tangent line is preserved (Figure 1).

The flow \( \phi_t \) is not defined at all times:

1. It is not defined at times when the particle is on the boundary of the billiard.
   Of course, one could extend the definition to such \( t \), but the flow obtained in this way would not be continuous\(^1\).

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\(^1\)Many authors change the topology of \( \Omega \) in order to make the flow continuous, but it cannot be made differentiable.
2. When the particle makes a grazing collision with a wall at a time \( t_0 > 0 \), i.e. collides with the boundary with an angle \( \theta = 0 \), the flow stops being defined for all times \( t \geq t_0 \). Although one could extend continuously the definition of the trajectory after such a collision, the differentiability of the flow would be lost.

\[ \phi_t \]

**Figure 1.** The billiard reflection law.

We define \( \hat{\Omega} \) as the set of all \((x, v) \in \Omega\) such that the trajectory starting from \((x, v)\) does not contain any grazing collision, in the past or the future. Notice that \( \hat{\Omega} \) is a residual set of full measure, invariant under the flow \( \phi_t \), and that \( \phi_t \) is \( C^\infty \) on \( \hat{\Omega} \).

In the special case where \( D \) has no boundary, the billiard flow is simply the geodesic flow and \( \hat{\Omega} = \Omega = T^1 D \).

**Uniform hyperbolicity.** We define uniform hyperbolicity in the case of billiards. This definition is given in a more abstract framework in [4], but here we adapt it directly to billiard flows.

**Definition 1.1.** The billiard flow \( \phi_t \) is *uniformly hyperbolic* if at each point \( x \in \hat{\Omega} \), there exists a decomposition of \( T_x \hat{\Omega} \), invariant under the flow,

\[ T_x \hat{\Omega} = E^0_x \oplus E^u_x \oplus E^s_x \]

where \( E^0_x = \mathbb{R} \left. \frac{d}{dt} \right|_{t=0} \phi_t(x) \), such that

\[ \|D\phi_t^0|_{E^0_x}\| \leq a\lambda^t, \quad \|D\phi_t^{-t}|_{E^s_x}\| \leq a\lambda^t \]
(for some $a > 0$ and $\lambda \in (0,1)$, which do not depend on $x$).

**Remark.** If the billiard $D$ has no wall (which means that the billiard flow is a geodesic flow), we may use the word *Anosov* instead of *uniformly hyperbolic*.

2. **Results.** In this paper, we give a sufficient condition for a (possibly nonflat) billiard to be uniformly hyperbolic.

2.1. **The case of geodesic flows.** First, let us consider the case where $D$ has no boundary: the billiard flow is simply the geodesic flow on $D$. All surfaces with negative curvature have an Anosov geodesic flow: according to Arnold and Avez [1], the first proof of this fact goes back to 1898 [9]. Later, it was extended to all manifolds with negative sectional curvature (a modern proof is available in [11]). But the negative curvature assumption is not necessary for a geodesic flow to be Anosov. To prove that a geodesic flow is Anosov, one may examine the solutions of the Riccati equation

$$u'(t) = -K(t) - u^2(t)$$

where $K$ is the Gaussian curvature of the surface, and use the following criterion:

**Theorem 2.1.** Let $M$ be a closed surface. Assume that there exists $t_0 > 0$ such that for any geodesic $\gamma : [0,1] \rightarrow M$, and any solution $u$ of the Riccati equation along this geodesic such that $u(0) = 0$, $u$ is well-defined on $[0,t_0]$ and $u(t_0) > 0$. Then the geodesic flow $\phi_t : T^1M \rightarrow T^1M$ is Anosov.

Theorem 2.1 was mentioned in [6] and [18], without details about the proof. In [14], we apply Theorem 2.1 to give new examples of surfaces whose geodesic flow is Anosov while their curvature is not negative everywhere. The genus of such surfaces is necessarily at least 2 [12].

In fact, it is possible to improve this theorem by considering an increasing sequence of times $(t_k)_{k \in \mathbb{Z}} \in \mathbb{R}^2$:

**Theorem 2.2.** Let $M$ be a closed surface. Assume that there exist $m > 0$ and $C > c > 0$ such that for any geodesic $\gamma : \mathbb{R} \rightarrow M$, there exists an increasing sequence of times $(t_k)_{k \in \mathbb{Z}} \in \mathbb{R}^2$ with $c \leq t_{k+1} - t_k \leq C$, such that the solution $u$ of the Riccati equation with initial condition $u(t_k) = 0$ is defined on the interval $[t_k, t_{k+1}]$, and $u(t_{k+1}) > m$. Then the geodesic flow $\phi_t : T^1M \rightarrow T^1M$ is Anosov.

Notice that Theorem 2.1 is immediately deduced from Theorem 2.2 by choosing a constant step $t_{k+1} - t_k$. Theorem 2.2 is used in [15] to obtain a surface of genus 12 embedded in $S^3$ with Anosov geodesic flow.

2.2. **The case of billiards.** Now we consider the general case, in which $D$ may have a boundary.

For billiards, we consider a generalized version of the Riccati equation. We say that $u$ is a solution of this equation if:

1. in the interval between two collisions, $\dot{u}(t) = -K(t) - u(t)^2$;
2. when the particle bounces against the boundary at a time $t$, $u$ undergoes a discontinuity: we have $u(t^+) = u(t^-) - \frac{2\kappa}{\sin \theta}$, where $\kappa$ is the geodesic curvature of the boundary of $D$, and $\theta$ is the angle of incidence.

We are now ready to state the main result of this paper:

\footnote{The notation $u(t^+)$ stands for $\lim_{h \rightarrow 0^+, h>0} u(t+h)$, and likewise $u(t^-) = \lim_{h \rightarrow 0^-, h<0} u(t+h)$. In particular, if $u$ is continuous at $t$, then $u(t^+) = u(t^-) = u(t)$.}
Figure 3. On the left, a dispersing billiard in $T^2$ with infinite horizon. On the right, a dispersing billiard in $T^2$ with finite horizon.

**Theorem 2.3.** Consider a (not necessarily flat) billiard $D$. Assume that there exist positive constants $A, m, c$ and $C$ such that for any trajectory $\gamma$ with $\gamma(0) \in \tilde{\Omega}$, there exists an increasing sequence of times $(t_k)_{k \in \mathbb{Z}} \in \mathbb{R}^\mathbb{Z}$ satisfying $c \leq t_{k+1} - t_k \leq C$, such that for any $k \in \mathbb{Z}$, the solution $u$ of the Riccati equation with initial condition $u(t_k) = 0$ satisfies $u(t^+) \geq -A$ for all $t \in [t_k, t_{k+1}]$, and $u(t_{k+1}^+) > m$. Also assume that for each $k \in \mathbb{Z}$, there is no collision in the interval $(t_k - c, t_k)$, and at most one collision in the interval $(t_k, t_{k+1})$. Then the billiard flow on $D$ is uniformly hyperbolic.

Notice that in the particular case where $D$ has no boundary, Theorem 2.3 becomes exactly Theorem 2.2. Thus we only need to prove Theorem 2.3, which will be done in Section 5.

2.3. Applications. We will explain how Theorem 2.3 can be applied to obtain immediately two famous results: Theorems 2.4 and 2.5. Theorem 2.3 unifies these two theorems, which are both well-known independently. See [15] for a completely new application of Theorem 2.3.

**Theorem 2.4.** Let $M$ be a closed Riemannian surface with nonpositive curvature. Assume that every geodesic in $M$ contains a point where the curvature is negative. Then, the geodesic flow on $M$ is Anosov.

Theorem 2.4 may also be obtained directly, without using Theorem 2.1, from Proposition 3.10 of [7]. Hunt and MacKay [10] used this result to exhibit the first Anosov physical system.

For billiards, we will prove the following counterpart of Theorem 2.4, which is essentially due to Sinai [21]:

**Theorem 2.5.** If $D$ is a smooth dispersing flat billiard in $T^2$ with finite horizon, then the billiard flow is uniformly hyperbolic in $\tilde{\Omega}$.

We say that a billiard has finite horizon if every trajectory hits the boundary at least once.

2.4. Consequences of uniform hyperbolicity. It is shown in [20] that (smooth) volume-preserving Anosov flows are ergodic: every invariant subset has either zero or full measure. It was shown later (see [5] and [12]) that Anosov geodesic flows on surfaces are even exponentially mixing (and then, in all higher dimensions [17]).

As for billiard flows, in the flat case only, Sina proved ergodicity for smooth dispersing billiards with finite horizon in [21]. It was shown in [2] that such flows are exponentially mixing.
The consequences of uniform hyperbolicity in the nonflat case are still unknown.

2.5. Structure of the paper. In Section 3, we prove a cone criterion, following the ideas of Wojtkowski [23]. In Section 4, we study Jacobi fields in (not necessarily flat) billiards. The tools which are introduced in Sections 3 and 4 are used in Section 5 to prove Theorem 2.3. Finally, the two applications are given in Section 6.

3. The cone criterion.

Definition 3.1. Consider a Euclidean space $E$.
A cone$^3$ in $E$ is a set $C$ such that there exist a decomposition $E = F \oplus G$ and a real number $\alpha \geq 0$ such that

$$C = \{(x, y) \in F \oplus G \mid \|x\| \leq \alpha \|y\|\}.$$

The number $\arctan \alpha$ is called the angle of the cone.
Two cones $C_1, C_2$ are said to be supplementary if they correspond to decompositions $E = F_1 \oplus G_1$ and $E = F_2 \oplus G_2$ such that $F_1 = G_2$ and $F_2 = G_1$.

Proposition 3.2. Consider a sequence of invertible linear mappings $A_k : \mathbb{R}^n \to \mathbb{R}^n$, $k \in \mathbb{Z}$, and a sequence of supplementary cones $C_k$ and $D_k$, corresponding to the decomposition $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$. Assume that there exist $a > 0$, $\lambda > 1$ such that for all $k \in \mathbb{Z}$:

1. $A_k(C_k) \subseteq C_{k+1}$ (invariance in the future),
2. $\|A_{k-1} \circ \ldots \circ A_{k-i}(v)\| \geq a\lambda^i \|v\|$ for all $i \geq 0$ and $v \in C_{k-i}$ (expansion in the future),
3. $A^{-1}_k(D_{k+1}) \subseteq D_k$ (invariance in the past),
4. $\|A_{k-1} \circ \ldots \circ A_{k+i-1}(v)\| \geq a\lambda^i \|v\|$ for all $i \geq 0$ and $v \in D_{k+i}$ (expansion in the past).

Then

$$E^u_k = \bigcap_{i=0}^{+\infty} A_{k-1} \circ \ldots \circ A_{k-i}(C_k)$$

is an $m$-dimensional subspace contained in $C_k$, and

$$E^s_k = \bigcap_{i=0}^{+\infty} A^{-1}_{k-1} \circ \ldots \circ A^{-1}_{k+i-1}(D_k)$$

is an $(n-m)$-dimensional subspace contained in $D_k$.

Proof. For all $i \geq 0$, $A_{k-1} \circ \ldots \circ A_{k-i}(C_{k-i})$ is a cone, which contains a vector space $V_i$ of dimension $m$. Thus, the intersection $E^u_k$ contains a vector space $V$ of dimension $m$ (for example, consider a converging subsequence of orthonormal bases of $V_i$). Assume that there exists $w \in E^u_k \setminus V$. Then there exists $v \in V$ and $t \in \mathbb{R}$ such that $v + tw \in \{0\} \times \mathbb{R}^{n-m}$ (notice also that $tw \in E^u_k$). Since $A^{-1}_{k-1} \circ \ldots \circ A^{-1}_{k-1}(tw)$ and $A^{-1}_{k-i} \circ \ldots \circ A^{-1}_{k+i-1}(v)$ lie in $E^u_{k-i}$, Assumption 2 gives us:

$$\|A^{-1}_{k-1} \circ \ldots \circ A^{-1}_{k-1}(tw)\| \leq \frac{1}{a\lambda^i} \|tw\| \to 0, \quad k \to +\infty,$$

$$\|A^{-1}_{k-i} \circ \ldots \circ A^{-1}_{k+i-1}(v)\| \leq \frac{1}{a\lambda^i} \|v\| \to 0, \quad k \to +\infty.$$

$^3$The word “cone” has several different meanings in mathematics: here we take the same definition as [11].
but at the same time, since \( v + tw \in D_k \), Assumption 4 gives:

\[
\| A_{k-1}^{-1} \circ \ldots \circ A_{k-1}^{-1} (v + tw) \| \geq a \lambda^i \| v + tw \| \xrightarrow{k \to +\infty} +\infty,
\]

which contradicts the triangle inequality.

One obtains the result for \( E^*_k \) in the same way.

\[\text{Theorem 3.3.} \]

Let \( A_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \) (with \( k \in \mathbb{Z} \)) be a sequence of \( 2 \times 2 \) matrices, with determinant \( \pm 1 \). Fix \( \epsilon > 0 \), and consider the cone \( C_\epsilon \) of all vectors \( \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \) such that \( \epsilon y \leq x \leq \frac{1}{\epsilon} y \). Assume that for all \( k \), and all \( v = \begin{pmatrix} x \\ y \end{pmatrix} \) with \( xy > 0 \),

\[ A_k v \in C_\epsilon. \]

Then, there exist \( a > 0 \) and \( \lambda > 1 \) such that for all \( k \in \mathbb{Z} \), for all \( i \geq 0 \) and \( v \in C_\epsilon \),

\[
\| A_{k-1} \circ \ldots \circ A_{k-i} (v) \| \geq a \lambda^i \| v \|.
\]

\[\text{Figure 4.} \]

Each \( A_k \) maps the cone \( xy > 0 \) (in grey) into the smaller cone \( C_\epsilon \) (in dark grey).

\[\text{Proof.} \]

On the basis of Wojtkowski’s idea [23], instead of proving expansion directly for the Euclidean norm, we consider the function

\[ N : C_\epsilon \to \mathbb{R}_{\geq 0} \]

\[ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \sqrt{xy}. \]

Notice that \( N \) is equivalent to the Euclidean norm on \( C_\epsilon \), i.e. there exists \( M > 0 \) such that for all \( v \in C_\epsilon \),

\[
\frac{1}{M} \| v \| \leq N(v) \leq M \| v \|,
\]

because \( \frac{\epsilon}{2} (x^2 + y^2) \leq xy \leq \frac{1}{\epsilon} (x^2 + y^2) \) for all \( \begin{pmatrix} x \\ y \end{pmatrix} \in C_\epsilon \).
We are going to show that for all \( k \in \mathbb{Z} \) and \( v \in C_\epsilon \), 
\[ N(A_kv) \geq \frac{1}{1 - \epsilon^2}N(v). \]
With the equivalence of norms, this will complete the proof.

Let \( k \in \mathbb{Z} \). We may assume that \( \det(A_k) = 1 \), by multiplying \( A_k \) by \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) on the left. Moreover, we may assume that all the coefficients of \( A_k \) are positive, by multiplying \( A_k \) by \(-\text{Id.}\)

Notice that the two vectors 
\[ A_k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_k \\ c_k \end{pmatrix} \]
and 
\[ A_k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b_k \\ d_k \end{pmatrix} \]
are in the cone \( C_\epsilon \), by continuity of \( A_k \).

Then for \( v = \begin{pmatrix} x \\ y \end{pmatrix} \in C_\epsilon \):
\[ N(A_kv) = (a_kx + b_ky)(c_kx + d_ky) \]
\[ \geq (a_kd_k - b_kc_k)xy + 2b_yc_kxy \]
\[ \geq (1 + 2b_yc_k)N(v) \]

But \( a_kd_k - b_kc_k = 1 \) and \( a_k \leq \frac{1}{2}b_k, d_k \leq \frac{1}{2}c_k \), so that \( b_yc_k \geq \frac{1}{1 - \epsilon^2} - 1 \).
Finally, \( N(A_kv) \geq \frac{1}{1 - \epsilon^2}N(v) \).

\[ \square \]

4. Jacobi fields.

4.1. Jacobi fields for geodesic flows. The results in this section are standard and will not be proved: see for example [13] for details.

Consider a smooth Riemannian surface \((M, g)\). To show that a geodesic flow is hyperbolic, one has to study how the geodesics move away from (or closer to) each other. Thus, one considers small variations of a given geodesic.

Definition 4.1. Consider a geodesic \( \gamma : (a, b) \to M \). Consider a geodesic variation of \( \gamma \), i.e. a smooth function
\[ f(t, s) : (a, b) \times (c, d) \to M \]
such that \( f(., 0) \) is the geodesic \( \gamma \), and for all \( s \in (c, d) \), \( f(., s) \) is a geodesic.

The vector field \( Y = \frac{\partial f}{\partial s} \) along the curve \( \gamma(t) \) is called an infinitesimal variation of \( \gamma \).

Proposition 4.2. Any infinitesimal variation of \( \gamma \) is a solution of the Jacobi equation:
\[ \ddot{Y}_\gamma = -R(\dot{\gamma}_\gamma, Y)\dot{\gamma}_\gamma \]
where \( R \) is the Riemann tensor. The solutions of the Jacobi equation are called Jacobi fields.

Proposition 4.3. Every Jacobi field along a geodesic \( \gamma \) is an infinitesimal variation of \( \gamma \).

We will now be interested in orthogonal Jacobi fields:

Lemma 4.4. If \( Y(t) \) and \( \dot{Y}(t) \) are orthogonal to \( \dot{\gamma} \) for some \( t \in \mathbb{R} \), then they remain orthogonal for all \( t \in \mathbb{R} \).

From now on, assume that \( M \) has dimension 2, that \( \gamma \) is a unit speed geodesic, and that \( Y \) is a Jacobi field which is orthogonal to \( \dot{\gamma} \). Choose an orientation of the normal bundle of \( \gamma \) in \( M \) (which has dimension 1), i.e. a vector \( e(t) \in T_{\gamma(t)}^1 M \) orthogonal to \( \dot{\gamma}(t) \), so that \( Y(t) \) is identified by one real coordinate, noted \( y(t) = g(Y(t), e(t)). \)
The quantity $\dot{y}$ satisfies

$$\dot{y} = \frac{\partial f}{\partial t} \cdot g(Y, e) + g(Y, \nabla_{\frac{\partial f}{\partial t}} e) = g(\nabla_{\frac{\partial f}{\partial s}} Y, e).$$

Thus:

$$\dot{y} = g(\nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial s}, e) = g(\nabla_{\frac{\partial f}{\partial s}} \dot{\gamma}, e).$$

In other words, $\dot{y}$ measures the infinitesimal variation of the vector $\dot{\gamma}$ with respect to $s$. Thus, when $y$ and $\dot{y}$ have the same sign, the Jacobi field is diverging: the geodesics go away from each other. When $y$ and $\dot{y}$ have opposite signs, the Jacobi field is converging. We will consider the ratio $u = \frac{\dot{y}}{y}$, when it is well-defined (i.e. $y \neq 0$), to measure the convergence rate.

![Diagram](image)

**Proposition 4.5.** When it is well-defined, $u$ is a solution of the Riccati equation:

$$\dot{u}(t) = -K(\gamma(t)) - u^2(t).$$

where $K$ is the Gaussian curvature.

The solutions of this equation are not always defined for all times: it may happen that $u(t)$ blows up to $-\infty$ in positive time (or to $+\infty$ in negative time). This corresponds to the phenomenon of convergence of the wavefront: up to order 1, all the geodesics of the infinitesimal variation “gather at one point”. In most cases, the Jacobi field becomes divergent just after the convergence point (Figure 5).

![Diagram](image)

**Figure 5.** $u$ is not well-defined at the convergence point.

From the fact that $u$ satisfies a differential equation, one infers the following order preserving property:

**Proposition 4.6.** Consider a geodesic $\gamma$ and two Jacobi fields $Y_1(t)$ and $Y_2(t)$ defined on a time interval $[a, b]$. Assume that $y_1(t)$ and $y_2(t)$ do not vanish in this interval (i.e. $u_1(t)$ and $u_2(t)$ are well-defined for $t \in [a, b]$). Then $u_2(b) - u_1(b)$ has the same sign as $u_2(a) - u_1(a)$. 
4.2. Jacobi fields for billiards. Recall that a smooth billiard $D$ is a compact subset of a Riemannian surface $(M,g)$, such that $D$ has a smooth boundary while $M$ has no boundary. We will write $(\cdot \mid \cdot)$ for $g(\cdot, \cdot)$.

Consider a billiard trajectory $\gamma$ and a unit speed variation of this trajectory $f(t,s) : (a,b) \times (c,d) \to D$ (defined for all times $t \in (a,b)$, except for the collision times) such that $f(.,0)$ is the trajectory $\gamma$ and for each $s \in (c,d)$, $f(.,s)$ is a billiard trajectory.

By analogy with the case of geodesic flows, we shall call “Jacobi field” the vector $Y = \frac{d\Gamma}{ds}$ along the curve $\gamma$. Inside the billiard, $Y$ satisfies the equation $\dot{\gamma}(t) = K(t)Y(t)$, where $K(t)$ is the curvature at the point $\gamma(t)$ (if the billiard is flat, then $\dot{\gamma}(t) = 0$). At a (non-grazing) collision time, with an angle of incidence $\theta \in (0,\pi/2]$, $Y$ undergoes a discontinuity, which we are now going to study.

Consider a smooth map $s \mapsto \tau(s)$ such that $\tau(s)$ is a collision time of $f(.,s)$ for all $s \in (c,d)$ (reducing the interval $(c,d)$ if necessary). The collision occurs on some component $\Gamma$ of the boundary $\partial D$: assume that $r \mapsto \Gamma(r)$ is a parametrization by arc length and define $r(s)$ so that $\Gamma(r(s))$ is the point where the collision occurs for each $s \in (c,d)$. The parametrization of $\Gamma$ is chosen so that $\langle \gamma(\tau(0)^-) \mid \frac{dr}{ds}(r(0)) \rangle \geq 0$. As in Section 4.1, choose a section $t \mapsto e_1(t)$ of the unit normal bundle of the trajectory $t \mapsto \gamma(t)$, such that

$$\left\langle e_1(\tau(0)^-) \mid \frac{d\Gamma}{dr}(r(0)) \right\rangle \leq 0 \quad \text{and} \quad \left\langle e_1(\tau(0)^+) \mid \frac{d\Gamma}{dr}(r(0)) \right\rangle \geq 0.$$

Define

$$y_{\perp}(t) = \langle Y(t) \mid e_1(t) \rangle \quad \text{and} \quad y_{\|}(t) = \langle Y(t) \mid \dot{\gamma}(t) \rangle.$$

**Proposition 4.7.** Writing $y_{\perp}^\pm = y_{\perp}(\tau(0)^\pm)$, and defining in the same way $y_{\|}^\pm$, we have:

$$y_{\perp}^+ = -y_{\perp}^- \quad \text{and} \quad y_{\|}^+ = y_{\|}^-.$$

**Proof.** On the one hand,

$$\frac{d}{ds} \Gamma(r(s)) = \frac{y_{\perp}^-}{\sin \theta} \quad \text{and} \quad \frac{d}{ds} \Gamma(r(s)) = -\frac{y_{\perp}^+}{\sin \theta},$$

so $y_{\perp}^+ = -y_{\perp}^-$. On the other hand,

$$\frac{d}{ds} \tau(s) = \frac{-y_{\perp}^-}{\tan \theta} + y_{\|}^- \quad \text{and} \quad \frac{d}{ds} \tau(s) = \frac{y_{\perp}^+}{\tan \theta} + y_{\|}^+, $$

so $y_{\|}^+ = y_{\|}^-$. $\square$

From now on, we consider a perpendicular Jacobi field, that is, we assume that $y_{\|}^- = 0$. Proposition 4.7 implies that $y_{\|}^+ = 0$: in other words, any perpendicular Jacobi field remains perpendicular after a collision. We will write $y(t) = y_{\perp}(t)$ and define $u(t) = \dot{y}(t)/y(t)$.

**Proposition 4.8.** Assume that the geodesic variation $f$ corresponds to an orthogonal Jacobi field.
At a collision,\[\begin{align*}
y^+ &= -y^- \\
y^+ &= -\dot{y}^- + \frac{2\kappa}{\sin \theta} y^- \\
u^+ &= u^- - \frac{2\kappa}{\sin \theta}
\end{align*}\]
where $\kappa$ is the curvature of the boundary and $\theta$ is the angle of incidence.

**Proof.** The first equality was already proved in Proposition 4.7. To obtain the next equality, consider the billiard reflection law:

\[
\left\langle \frac{\partial f}{\partial t}(\tau(s)^+) - \frac{\partial f}{\partial t}(\tau(s)^-) \mid \frac{\partial \Gamma}{\partial r}(r(s)) \right\rangle = 0.
\]

After differentiation with respect to $t$ we obtain:

\[
\left\langle \dot{Y}^+ - \dot{Y}^- \mid \frac{\partial \Gamma}{\partial r}(r(s)) \right\rangle + \left\langle \frac{\partial f}{\partial t}(\tau(s)^+) - \frac{\partial f}{\partial t}(\tau(s)^-) \mid \nabla \frac{\partial \Gamma}{\partial r}(r(s)) \cdot \frac{\partial r}{\partial s} \right\rangle = 0.
\]

We may now compute:

\[
\left\langle \dot{Y}^+ - \dot{Y}^- \mid \frac{\partial \Gamma}{\partial r}(r(s)) \right\rangle = (\dot{y}^+ + \dot{y}^-) \sin \theta,
\]

\[
\left\langle \frac{\partial f}{\partial t}(\tau(s)^+) - \frac{\partial f}{\partial t}(\tau(s)^-) \mid \nabla \frac{\partial \Gamma}{\partial r}(r(s)) \cdot \frac{\partial r}{\partial s} \right\rangle = 2 \sin \theta \cdot \kappa \cdot \frac{-y^-}{\sin \theta}.
\]

Thus:

\[
y^+ = -\dot{y}^- + \frac{2\kappa}{\sin \theta} y^- \\
u^+ = u^- - \frac{2\kappa}{\sin \theta}.
\]

In particular, positively curved walls decrease the value of $u$ (and tend to make the Jacobi field converge), just as the positive curvature of a Riemannian surface. Likewise, negatively curved walls make the quantity $u$ increase, as the negative curvature of a surface.

5. **Proof of Theorem 2.3.** We fix the constants $A$, $c$, $C$ and $m$ which appear in the statement of the theorem, and assume that $A \geq 2$. In this section, we consider times such as $t_a$, $t_b$ or $t_0$, which must not be confused with the times $t_k$ ($k \in \mathbb{Z}$) which appear in the statement of the theorem.

The readers who are only interested in the proof of Theorem 2.2 may skip Lemmas 5.2, 5.3 and 5.4.

**Lemma 5.1.** Assume that $u$ and $v$ are two solutions of the Riccati equation on an interval $[t_a, t_b]$ with $c/3 \leq t_b - t_a \leq 2C$, such that $0 \leq u(t_a) - v(t_a) \leq \exp(-4AC)$. Assume that $u(t) \geq -A$ for all $t \in [t_a, t_b]$. Then

\[u(t_b) - v(t_b) \leq (u(t_a) - v(t_a)) \exp(2A(t_b - t_a)).\]

**Proof.** Let $t_0 = \min \{ t \in [t_a, t_b] \mid u(t) - v(t) \geq 2 \}$ (with $t_0 = t_b$ if this set is empty).

Then for $t \in [t_a, t_b]$, we have $u(t) - v(t) \geq 0$ (by Proposition 4.6) and

\[\dot{u}(t) - \dot{v}(t) = -(u(t) + v(t))(u(t) - v(t)) \leq 2A(u(t) - v(t)).\]
Thus by Grönwall’s lemma
\[ u(t) - v(t) \leq (u(t_0) - v(t_0)) \exp(2A(t - t_0)) \leq 1, \]
so \( t_0 = t_b \) and the result is proved.

From now on we will assume that \( m \leq \min(\exp(-4AC), 1/4) \) and define
\[ \eta = \frac{m^3}{(2K_{\text{max}} + 2)}, c/3, \]
where \( K_{\text{max}} \) is the maximum absolute value of the curvature on \( D \).

**Lemma 5.2.** Assume that \( u \) is a solution of the Riccati equation on an interval \([t_b, t_b + \eta]\), during which no collision occurs. If \( |u(t_b)| \leq 1/2 \), then
\[ |u(t_0 + \eta) - u(t_b)| \leq m^3. \]

**Proof.** Consider \( t_0 = \min \{ t \in [t_b, t_b + \eta] \mid |u(t) - u(t_b)| \geq m^3 \} \) (or \( t_0 = t_b + \eta \) if this set is empty). Then for all \( t \in (t_b, t_0) \):
\[ |u(t) - u(t_b)| = \left| \int_{t_b}^{t} K(x) - u(x)^2 dx \right| \]
\[ \leq \eta(K_{\text{max}} + \eta) \quad \leq \eta(K_{\text{max}} + 1) \quad \leq m^3/2. \]

This implies that \( t_0 = t_b + \eta \) and thus \( |u(t_b + \eta) - u(t_b)| \leq m^3. \)

**Lemma 5.3.** Assume that \( u \) and \( v \) are two solutions of the Riccati equation on an interval \([t_a, t_b]\) with \( c/3 \leq t_b - t_a \leq 2C \), with \( u(t_a) = 0 \) and \( v(t_a + \eta) = 0 \). Assume that \( u(t) \geq -A \) for all \( t \in [t_a, t_b] \). Then \( v(t) \geq u(t_b) - m^2 \).

**Proof.** If \( v(t_a) \geq u(t_a) \) then \( v(t_b) \geq u(t_b) \) (by Proposition 4.6) and there is nothing to prove. Therefore we assume that \( u(t_a) \geq v(t_a) \). Lemma 5.2 implies that \( |u(t_a + \eta)| \leq m^3 \) and Lemma 5.1 shows that \( u(t_b) - v(t_b) \leq m^3 \exp(4AC) \leq m^2 \).

From now on, consider a geodesic \( \gamma \) and the times \( t_k \) given by the assumptions of Theorem 2.3. For each \( k \in \mathbb{Z} \), define \( \hat{t}_k \) in the following way:

- If there is a collision in the interval \([t_k - c/3, t_k]\), define \( \hat{t}_k = t_k + \eta \).
- If not, let \( \hat{t}_k = t_k \).

In the following, if \( t_k \) is itself a collision time, by \( u(t_k) \) we will mean \( u(t_k^+) \).

**Lemma 5.4.** For all \( k \in \mathbb{Z} \), the solution \( u \) of the Riccati equation with initial condition \( u(\hat{t}_k) = 0 \) satisfies \( u(\hat{t}_{k+1}) \geq m/2 \).

**Proof.** Consider the solution \( v \) of the Riccati equation with initial condition \( v(t_k) = 0 \).

First, we prove that \( u(t_{k+1}) \geq m - m^2 \). If \( \hat{t}_k = t_k \), we have \( u = v \) and by assumption \( v(t_{k+1}) \geq m \), so \( u(t_{k+1}) \geq m \). If \( \hat{t}_k = t_k + \eta \), Lemma 5.3 applied to \( v \) and \( u \) gives us \( u(t_{k+1}) \geq v(t_{k+1}) - m^2 \geq m - m^2 \).

Now, we prove that \( u(t_{k+1}) \geq m/2 \). If \( \hat{t}_{k+1} = t_{k+1} \), then \( u(\hat{t}_{k+1}) = u(t_{k+1}) \geq m - m^2 \geq m/2 \). If \( \hat{t}_{k+1} = t_{k+1} + \eta \), then with Lemma 5.2, \( u(t_{k+1}) = u(t_{k+1} + \eta) \geq u(t_{k+1}) - m^3 \geq m - m^2 - m^3 \geq m/2 \).

**Lemma 5.5.** For all \( k \in \mathbb{Z} \), the solution of the Riccati equation with initial condition \( u(\hat{t}_{k+1}) = 0 \) is well-defined on \([\hat{t}_k, \hat{t}_{k+1}]\) and satisfies \( u(\hat{t}_k) \leq -m^2/2 \).
Proof. Consider the solution $v$ of the Riccati equation with initial condition $v(t_0) = 0$, and the solution $w$ of the Riccati equation with initial condition $w(t_0) = -m^2/2$. By Lemma 5.1, $w(t_{k+1}) \geq v(t_{k+1}) - (m^2/2) \exp(4AC) \geq v(t_{k+1}) - m/2$. By Lemma 5.4, $v(t_{k+1}) \geq m/2$ and thus $w(t_{k+1}) \geq 0$.

Now, by Proposition 4.6 applied to $u$ and $w$ between the times $t_k$ and $t_{k+1}$, the solution of the Riccati equation with initial condition $u(t_{k+1}) = 0$ satisfies $u(t_k) \leq -m^2/2$. The lemma is proved.

**Lemma 5.6.** Consider $t_0 \in \mathbb{R}$ and a solution $u$ of the Riccati equation along a trajectory $\gamma$ defined on the interval $[t_0 - \eta, t_0]$. If $\gamma$ has no collision in the time interval $[t_0 - \eta, t_0]$, then $u(t_0) \leq \alpha$, where
\[
\alpha = \sqrt{\frac{1 + e^{-2\sqrt{K_{max}}\eta}}{1 - e^{-2\sqrt{K_{max}}\eta}}}.
\]

Proof. The Riccati equation gives $\dot{u}(t) \leq K_{max} - u(t)^2$.

Notice that whenever $u(t) > \sqrt{K_{max}}$, we have $\dot{u}(t) < 0$. Therefore, the conclusion of the lemma is true if $u(t_0) \leq \alpha$ for some $t \in [t_0 - \eta, t_0]$.

Now we assume that $u(t) \geq \alpha$ for all $t \in [t_0 - \eta, t_0]$. Thus we may write, for $t \in [t_0 - \eta, t_0]$,
\[
\frac{\dot{u}(t)}{K_{max} - u(t)^2} \geq 1
\]
which implies, after integration between $t_0 - \eta$ and $t_0$:
\[
\frac{u(t_0) - \sqrt{K_{max}}}{u(t_0) + \sqrt{K_{max}}} \leq e^{-2\sqrt{K_{max}}\eta} \frac{u(t_0) - \eta}{u(t_0) - \eta} - \frac{\sqrt{K_{max}}}{u(t_0) + \sqrt{K_{max}}} \leq e^{-2\sqrt{K_{max}}\eta}.
\]

Therefore
\[
u(t_0) - \sqrt{K_{max}} \leq e^{-2\sqrt{K_{max}}\eta}(u(t_0) + \sqrt{K_{max})}
\]
and thus
\[u(t_0) \leq \alpha.\]

Therefore for each $(x, v) \in \Omega$, the tangent plane $T_{(x,v)}\Omega$ is the direct sum of a vertical and a horizontal subspace $H_{(x,v)} \oplus V_{(x,v)}$, given by the metric $g$ on $M$. Each of these two spaces is naturally endowed with a norm, respectively $g_H$ and $g_V$: one equips $\Omega$ with the norm $g_T = g_H + g_V$ (in particular, one decides that $H$ is orthogonal to $V$).

Denote by $W_{(x,v)} \subseteq T_{(x,v)}\Omega$ the plane orthogonal to the direction of the flow $\phi_t$, and let $(w, w') \in W_{(x,v)}$. There exists $Y(t)$ a Jacobi field such that $(Y(0), \dot{Y}(0)) = (w, w')$: then the vectors $\dot{Y}(0)$ and $\dot{\gamma}(0)$ are orthogonal, and $(Y(t), \dot{Y}(t)) = D\phi_t(w, w')$ (see [19] for details). Lemmas 4.4 and 4.7 imply that $Y(t)$ remains orthogonal to $\dot{\gamma}(t)$ for all $t$. In particular, the family of planes $(W_{(x,v)})$ (where $(x, v)$ varies in $\Omega$) is invariant under $D\phi_t$.

Consider an element $(x, v) \in \Omega$, and $\gamma$ the billiard trajectory such that $(\gamma(0), \dot{\gamma}(0)) = (x, v)$. Choose an orientation of $H_{(\gamma(t), \dot{\gamma}(t))} \cap W_{(\gamma(t), \dot{\gamma}(t))}$, i.e. a continuous unit vector $e_1(t)$ in $H_{(\gamma(t), \dot{\gamma}(t))} \cap W_{(\gamma(t), \dot{\gamma}(t))}$. It induces naturally an orientation of $V_{(\gamma(t), \dot{\gamma}(t))}$, given by a continuous unit vector $e_2(t)$ in $V_{(\gamma(t), \dot{\gamma}(t))}$. This orthogonal basis of $W_{(\gamma(t), \dot{\gamma}(t))}$ allows us to identify it to the Euclidean $\mathbb{R}^2$.

For $k \in \mathbb{Z}$, set
\[A_k = D_{(\gamma(t_k), \dot{\gamma}(t_k))}\phi_{t_{k+1}-t_k} : W_{(\gamma(t_k), \dot{\gamma}(t_k))} \to W_{(\gamma(t_{k+1}), \dot{\gamma}(t_{k+1}))}.\]
The $A_k$ are linear mappings with determinant $±1$, because the flow $\phi_t$ preserves the Liouville measure.

**Lemma 5.7.** For each $\epsilon > 0$, consider the cones

$$C^\pm_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid cy \leq \pm x \leq \frac{1}{\epsilon} y\} \quad \text{and} \quad C^\pm_0 = \{(x, y) \in \mathbb{R}^2 \mid \pm xy > 0\}.$$

There exists $\epsilon > 0$ such that for all $k \in \mathbb{Z}$,

$$A_k C^+_0 \subseteq C^+_\epsilon \quad \text{and} \quad A_k^{-1} C^-_0 \subseteq C^-_\epsilon.$$

**Proof.** First, we prove $A_k C^+_0 \subseteq C^+_\epsilon$. Since the difference between two solutions of the Riccati equation does not change sign, we only need to see that:

1. The solution of the Riccati equation along $\gamma$ with initial condition $u(\tilde{t}_k) = 0$ is defined on $[\tilde{t}_k, \tilde{t}_{k+1}]$ and satisfies $u(\tilde{t}_{k+1}) \geq \epsilon$. By Lemma 5.4, it is the case for $\epsilon \leq m/2$.

2. Any solution of the Riccati equation along $\gamma$ with $u(\tilde{t}_k) \geq 0$ is defined on $[\tilde{t}_k, \tilde{t}_{k+1}]$ and satisfies $u(\tilde{t}_{k+1}) \leq 1/\epsilon$. It is the case for $\epsilon \leq 1/\alpha$, where $\alpha$ is defined in Lemma 5.6.

Now, let us prove $A_k^{-1} C^-_0 \subseteq C^-_\epsilon$. We need to see that:

1. The solution of the Riccati equation along $\gamma$ with initial condition $u(\tilde{t}_{k+1}) = 0$ is defined on $[\tilde{t}_k, \tilde{t}_{k+1}]$ and satisfies $u(\tilde{t}_k) \leq -\epsilon$. By Lemma 5.5, it is the case for $\epsilon \leq m^2/2$.

2. Any solution of the Riccati equation along $\gamma$ with $u(\tilde{t}_{k+1}) \leq 0$ is defined on $[\tilde{t}_k, \tilde{t}_{k+1}]$ and satisfies $u(\tilde{t}_k) \leq 1/\epsilon$. It is the case for $\epsilon \leq 1/\alpha$, where $\alpha$ is defined in Lemma 5.6 (recall that there is no collision in the interval $[\tilde{t}_k, \tilde{t}_{k+1}]$), according to the assumptions of Theorem 2.3).

Thus the sequences $(A_k)$ and $(A_k^{-1})$ satisfy the assumptions of Theorem 3.3, which provides us with two families of cones: one of them satisfies invariance and expansion in the future, while the other satisfies invariance and expansion in the past. Proposition 3.2 provides distributions $E^s$ and $E^u$ on $\Omega$ which are invariant under the flow $\phi_t$, and satisfy

$$\forall k \in \mathbb{Z}, \quad \|D_{(x,v)}\phi_{t_k}|E^s\| \leq a\lambda^k \quad \text{and} \quad \|D_{(x,v)}\phi_{t_{-k}}|E^s\| \leq a\lambda^k$$

for some $a > 0$ and $\lambda \in (0, 1)$.

To go from this discrete statement to a continuous statement, notice the following:

**Lemma 5.8.** Consider the set $S$ of all $(t, (x,v)) \in [0, 2C] \times T^1M$ such that the geodesic of length $t$ starting from $(x,v)$ is contained in the billiard $D$.

$$\sup_{(t, (x,v)) \in S} \|D\phi_t(x,v)\| < +\infty.$$

**Proof.** The set $S$ is compact.

Therefore, increasing $a$ and $\lambda$ if necessary, we have:

$$\forall t \in \mathbb{R}, \quad \|D_{(x,v)}\phi_t|E^s\| \leq a\lambda^t \quad \text{and} \quad \|D_{(x,v)}\phi_{-t}|E^s\| \leq a\lambda^t.$$

Hence the billiard flow is uniformly hyperbolic, and Theorem 2.2 is proved.
6. Applications.

6.1. Closed surfaces of negative curvature: Proof of Theorem 2.4. In this proof, we will use the lemma:

**Lemma 6.1.** Under the assumptions of Theorem 2.4, there exist $m > 0$ and $t_0 > 0$ such that every unit speed geodesic $\gamma: [0, t_0] \to M$ satisfies:

$$\int_0^{t_0} K(\gamma(t))dt \leq -m.$$

**Proof.** If the conclusion is false, consider a sequence $(\gamma_n)$ of unit speed geodesics defined on $[-n, n]$, such that for all $n$,

$$\int_{-n}^{n} K(\gamma(t))dt \geq -\frac{1}{n}.$$

By the Arzel-Ascoli theorem and a diagonal argument, one may extract a subsequence of $\gamma_n$ which converges uniformly on each $[-n, n]$ to a geodesic defined on $\mathbb{R}$. By dominated convergence, it satisfies $\int_{\mathbb{R}} K(\gamma(t))dt = 0$, which contradicts the assumption. \qed

Now, consider the values of $m$ and $t_0$ given by lemma 6.1, and choose a geodesic $\gamma$. We may assume that $m < 1$ and, by dividing the metric of $M$ by a constant if necessary, that $t_0 < 1$.

Denote by $u$ the solution of the Riccati equation $u'(t) = -K(t) - u^2(t)$ with $u(0) = 0$. Since $u'(t) \geq -u^2(t)$, we have $u(t) \geq 0$ for $t \geq 0$, by comparison with the solution $v$ of the differential equation $v'(t) = -v^2(t)$ with initial condition $v(0) = 0$ (here, $v$ is the zero function). In particular, the solution $u$ does not blow up to $-\infty$.

Set $t_1 = \sup \{t \in [0, 1] \mid u(t) \geq m\}$ (with $t_1 = 0$ if this set is empty). Thus, for all $t \in [t_1, 1]$,

$$u'(t) = -K(t) - u^2(t) \geq -m^2.$$

If $t_1 = 0$, then using the estimate given by Lemma 6.1,

$$u(1) = u(0) + \int_0^1 u'(x)dx$$

$$= \int_0^1 -K(x) - u^2(x)dx$$

$$= -\int_0^1 K(x)dx - \int_0^1 u^2(x)dx$$

$$\geq m - m^2.$$

If $t_1 \neq 0$, then using the fact that $K(t) \leq 0$,

$$u(1) = u(t_1) + \int_{t_1}^1 u'(x)dx \geq u(t_1) + \int_{t_1}^1 -u^2(x)dx \geq m - m^2.$$

In both cases, one gets $u(1) \geq m - m^2$. One may apply Theorem 2.1: the geodesic flow on $M$ is Anosov and Theorem 2.4 is proved.
6.2. Sinai billiards: Proof of Theorem 2.5.

Lemma 6.2. Let $D$ be a flat billiard in $\mathbb{T}^2$ with finite horizon. Then, there exists $t_0$ such that every billiard trajectory in $\bar{\Omega}$ (with unit speed) experiences at least one collision between $t = 0$ and $t = t_0$.

Proof. Assume that the conclusion is false. Then for all $n > 0$, there exists a billiard trajectory $\gamma_n : \mathbb{R} \to \mathbb{T}^2$, without collision on $[-n, n]$: we will write $(x_n, v_n) = (\gamma_n(0), \gamma_n'(0))$. Up to extraction, we may assume that $(x_n, v_n)$ has a limit $(x, v) \in \Omega$. The geodesic of $\mathbb{T}^2$ starting at $(x, v)$ is contained in $D$, so it is periodic (since it cannot be dense in $\mathbb{T}^2$) with period $T$. If it does not intersect the boundary $\partial D$, then this geodesic is a billiard trajectory without collision, so the billiard does not have finite horizon. Thus, we assume that this geodesic intersects $\partial D$, and since $\partial D$ is smooth, there is an open ball $B$ such that $(x, v) \in B \cap \partial D = \emptyset$. Furthermore, there is another ball $B'$ tangent to the geodesic on the other side, such that $B' \cap D = \emptyset$ (otherwise, there is an $x' \in D$ close to $x$ such that the trajectory starting at $(x', v)$ has no collision). If $v_n = v$ for some $n \geq T$, then the trajectory starting at $(x_n, v_n)$ (which has period $T$) has no collision, which again contradicts the finite horizon assumption: thus $v_n \neq v$ for all $n \geq T$. But since $(x_n, v_n)$ tends to $(x, v)$, this implies that there exists $n \geq 2T$ such that the trajectory starting at $(x_n, v_n)$ intersects $B_1$ or $B_2$ in the time interval $[-2T, 2T]$, which contradicts the assumption.

Lemma 6.3. If $D$ is a flat billiard with finite horizon whose walls have negative curvature, then it satisfies the assumptions of Theorem 2.3, where the times $t_k$ are the times of collisions.

Proof. We consider the solution $u$ of the generalized Riccati equation, such that $u(t_k^+) = 0$. On the interval $[t_k, t_{k+1}]$, $u$ is a solution of the equation $u'(t) = -u^2(t)$, so $u$ is equal to 0. Since the walls have positive curvature, $u(t_{k+1}^-) \geq -2\kappa_{\text{max}} > 0$, where $\kappa_{\text{max}}$ is the maximum curvature of the boundary.

Thus, Theorem 2.3 applies and concludes the proof.

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E-mail address: mickael.kourganoff@univ-grenoble-alpes.fr