The sharp lower bound of the lifespan of solutions to semilinear wave equations with low powers in two space dimensions

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Abstract.
This paper is devoted to a proof of the conjecture in Takamura [16] on the lower bound of the lifespan of solutions to semilinear wave equations in two space dimensions. The result is divided into two cases according to the total integral of the initial speed.

§1. Introduction

We consider the initial value problem,

\[\begin{align*}
    u_{tt} - \Delta u &= |u|^p \\
    u(x,0) &= \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), \quad x \in \mathbb{R}^n,
\end{align*}\]

where \(u = u(x,t)\) is an unknown function, \(f\) and \(g\) are given smooth functions of compact support and \(\varepsilon > 0\) is “small.” Let us define a lifespan \(T(\varepsilon)\) of a solution of (1.1) by

\[T(\varepsilon) := \sup\{t > 0 : \exists \text{ a solution } u \text{ of (1.1) for arbitrarily fixed } (f, g)\},\]

where “solution” means a classical one when \(p \geq 2\). When \(1 < p < 2\), it means a weak one, but sometimes the one given by associated integral equations to (1.1) by standard Strichartz’s estimate. See Sideris [14] for instance.

When \(n = 1\), we have \(T(\varepsilon) < \infty\) for any power \(p > 1\) by Kato [8]. When \(n \geq 2\), we have the following Strauss’ conjecture on (1.1) by

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Strauss [15].

\[ T(\varepsilon) = \infty \quad \text{if } p > p_0(n) \text{ and } \varepsilon \text{ is "small" (global-in-time existence)}, \]
\[ T(\varepsilon) < \infty \quad \text{if } 1 < p \leq p_0(n) \text{ (blow-up in finite time)}, \]

where \( p_0(n) \) is so-called Strauss’ exponent defined by positive root of the quadratic equation, \( \gamma(p, n) = 0 \), where

\[ \gamma(p, n) := 2 + (n + 1)p - (n - 1)p^2. \]

That is,

\[ p_0(n) = \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n - 1)}. \]

We note that \( p_0(n) \) is monotonously decreasing in \( n \). This conjecture had been verified by many authors with partial results. All the references on the final result in each part can be summarized in the following table.

| \( n \) | \( p < p_0(n) \) | \( p = p_0(n) \) | \( p > p_0(n) \) |
|---|---|---|---|
| \( n = 2 \) | Glassey [3] | Schaeffer [13] | Glassey [4] |
| \( n = 3 \) | John [7] | Schaeffer [13] | John [7] |
| \( n \geq 4 \) | Sideris [14] | Yordanov & Zhang [19] | Georgiev & Lindblad & Sogge [2] |

In the blow-up case, i.e. \( 1 < p \leq p_0(n) \), we are interested in the estimate of the lifespan \( T(\varepsilon) \). From now on, \( c \) and \( C \) stand for positive constants but independent of \( \varepsilon \). When \( n = 1 \), we have the following estimate of the lifespan \( T(\varepsilon) \) for any \( p > 1 \).

\[
\begin{align*}
&c\varepsilon^{-(p-1)/2} \leq T(\varepsilon) \leq C\varepsilon^{-(p-1)/2} & \text{if } \int_{\mathbb{R}} g(x)dx \neq 0, \\
&c\varepsilon^{-p(p-1)/(p+1)} \leq T(\varepsilon) \leq C\varepsilon^{-p(p-1)/(p+1)} & \text{if } \int_{\mathbb{R}} g(x)dx = 0.
\end{align*}
\]

This result has been obtained by Zhou [20]. Moreover, Lindblad [11] has obtained more precise result for \( p = 2 \),

\[
\begin{align*}
&\exists \lim_{\varepsilon \to +0} \varepsilon^{1/2}T(\varepsilon) > 0 & \text{if } \int_{\mathbb{R}} g(x)dx \neq 0, \\
&\exists \lim_{\varepsilon \to +0} \varepsilon^{2/3}T(\varepsilon) > 0 & \text{if } \int_{\mathbb{R}} g(x)dx = 0.
\end{align*}
\]
Similarly to this, Lindblad \cite{11} has also obtained the following result for $(n, p) = (2, 2)$.

\begin{equation}
\begin{cases}
\exists \lim_{\varepsilon \to +0} a(\varepsilon)^{-1} T(\varepsilon) > 0 & \text{if } \int_{\mathbb{R}^2} g(x) dx \neq 0 \\
\exists \lim_{\varepsilon \to +0} \varepsilon T(\varepsilon) > 0 & \text{if } \int_{\mathbb{R}^2} g(x) dx = 0,
\end{cases}
\end{equation}

where $a = a(\varepsilon)$ is a number satisfying

\begin{equation}
a^2 \varepsilon^2 \log(1 + a) = 1.
\end{equation}

When $1 < p < p_0(n)$ ($n \geq 3$) or $2 < p < p_0(2)$ ($n = 2$), we have the following conjecture.

\begin{equation}
c \varepsilon^{-2p(p-1)/\gamma(p,n)} \leq T(\varepsilon) \leq C \varepsilon^{-2p(p-1)/\gamma(p,n)},
\end{equation}

where $\gamma(p,n)$ is defined by (1.2). We note that (1.8) coincides with the second line in (1.4) if we define $\gamma(p,n)$ by (1.2) even for $n = 1$. All the results verifying this conjecture are summarized in the following table.

| $n$   | lower bound of $T(\varepsilon)$ | upper bound of $T(\varepsilon)$ |
|-------|---------------------------------|---------------------------------|
| $n = 2$ | Zhou \cite{22}                  | Zhou \cite{22}                  |
| $n = 3$ | Lindblad \cite{11}              | Lindblad \cite{11}              |
| $n \geq 4$ | Lai & Zhou \cite{10}           | Takamura \cite{16}              |

We note that, for $n = 2, 3$,

\[\exists \lim_{\varepsilon \to +0} \varepsilon^{2p(p-1)/\gamma(p,n)} T(\varepsilon) > 0\]

is established in this table. When $p = p_0(n)$, we have the following conjecture.

\begin{equation}
\exp\left(c \varepsilon^{-p(p-1)}\right) \leq T(\varepsilon) \leq \exp\left(C \varepsilon^{-p(p-1)}\right).
\end{equation}

All the results verifying this conjecture are also summarized in the following table.

| $n$   | lower bound of $T(\varepsilon)$ | upper bound of $T(\varepsilon)$ |
|-------|---------------------------------|---------------------------------|
| $n = 2$ | Zhou \cite{22}                  | Zhou \cite{22}                  |
| $n = 3$ | Zhou \cite{21}                  | Zhou \cite{21}                  |
| $n \geq 4$ | Lindblad & Sogge \cite{12}     | Takamura & Wakasa \cite{17}     |

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\text{Semilinear Wave Equations in Two Space Dimensions}
In this paper, we are interested in the case of the open part, \( n = 2 \) and \( 1 < p < 2 \). There is no conjecture before Remark 4.1 in Takamura [16] in which the following is implicitly described.

\[
\begin{equation}
\exists \lim_{\varepsilon \to +0} T(\varepsilon)\varepsilon^{(p-1)/(3-p)} > 0 \quad \text{if} \quad \int_{\mathbb{R}^2} g(x)dx \neq 0, \\
\exists \lim_{\varepsilon \to +0} T(\varepsilon)\varepsilon^{2p(p-1)/\gamma(p,2)} > 0 \quad \text{if} \quad \int_{\mathbb{R}^2} g(x)dx = 0.
\end{equation}
\]

Theorem 3.2 and 4.1 in Takamura [16] are the partial result of (1.10), the upper bounds of \( T(\varepsilon) \). Our result is devoted to the lower bounds as follows.

**Theorem 1.** Let \( n = 2, 1 < p < 2 \) and \((f, g) \in C_0^3(\mathbb{R}^2) \times C_0^2(\mathbb{R}^2)\). Then, there exists a positive constant \( \varepsilon_0 = \varepsilon_0(f, g, p, k) \) such that the lifespan \( T(\varepsilon) \) of solutions of (1.1) satisfies that

\[
\begin{equation}
\begin{cases}
T(\varepsilon) \geq c\varepsilon^{-(p-1)/(3-p)} & \text{if} \quad \int_{\mathbb{R}^2} g(x)dx \neq 0, \\
T(\varepsilon) \geq c\varepsilon^{-2p(p-1)/\gamma(p,2)} & \text{if} \quad \int_{\mathbb{R}^2} g(x)dx = 0
\end{cases}
\end{equation}
\]

for \( 0 < \varepsilon \leq \varepsilon_0 \), where \( c \) is a positive constant independent of \( \varepsilon \).

This paper is organized as follows. In the next section, we employ the linear decay estimate and basic lemmas for a priori estimates. In the third section, we prove a priori estimates. The proof of the Theorem 1 is in the final section.

§2. Preliminaries

Throughout this paper, we may assume that \((f, g) \in C_0^3(\mathbb{R}^2) \times C_0^2(\mathbb{R}^2)\) satisfy

\[
\text{supp} \ (f, g) \subset \{x \in \mathbb{R}^2 : |x| \leq k\}, \ k > 1.
\]

Set

\[
\begin{align*}
\phi_{L}(x, t) := & \frac{1}{2\pi} \int_{|x-y| \leq t} \frac{\phi(y)}{\sqrt{t^2 - |x-y|^2}} dy = \frac{t}{2\pi} \int_{|\xi| \leq 1} \frac{\phi(x + t\xi)}{\sqrt{1 - |\xi|^2}} d\xi.
\end{align*}
\]

Then, we note that \( u_L \) satisfies that

\[
\begin{cases}
(u_L)_{tt} - \Delta u_L = 0 & \text{in} \ \mathbb{R}^2 \times [0, \infty), \\
u_L(x, 0) = f(x), \ (u_L)_t(x, 0) = g(x), & x \in \mathbb{R}^2
\end{cases}
\]
in the classical sense, and also that

$$\text{supp } u_L \subset \{(x, t) \in \mathbb{R}^2 \times [0, \infty) : |x| \leq t + k\}. \tag{2.3}$$

We shall employ the following key lemma.

**Lemma 2.1 (Lindblad [11]).** Let $u_L$ be the one in (2.2). Then, there exist positive constants $C_0 = C_0(\|f\|_{W^{3,1}(\mathbb{R}^2)}, \|g\|_{W^{2,1}(\mathbb{R}^2)}, k)$ and $\widetilde{C}_0 = \widetilde{C}_0(k)$ such that $u_L$ satisfies

$$\sum_{|\alpha| \leq 1} |\nabla_x^\alpha u_L(x, t)| \leq \frac{\widetilde{C}_0}{C_0} \left( \frac{\int_{\mathbb{R}^2} g(x) dx}{(t + |x| + 2k)^{1/2}(t - |x| + 2k)^{1/2}} \right) + \frac{C_0}{(t + |x| + 2k)^{1/2}(t - |x| + 2k)^{3/2}}$$

in $\mathbb{R}^2 \times [0, \infty)$.

**Remark 2.1.** This is not exactly Lemma 7.1 in [11], but is basically its refined version. For the sake of completeness, we prove it.

**Proof of Lemma 2.1.** First we note that it is sufficient to show (2.4) for $\alpha = 0$ because $\nabla_x^\alpha$ passes to the integrand in the representation (2.2). Moreover, due to von Wahl [18], or Klainerman [9] as described in Glassey [4], we have that

$$|u_L(x, t)| \leq \frac{C}{\sqrt{1 + t}} (\|f\|_{W^{3,1}(\mathbb{R}^2)} + \|g\|_{W^{1,1}(\mathbb{R}^2)}) \quad \text{in } \mathbb{R}^2 \times [0, \infty),$$

where $C$ is a positive constant independent of $f$ and $g$. Therefore (2.4) is obtained by (2.3) for $-k \leq t - |x| \leq 2k$, or $t \leq 4k$.

From now on, we are concentrated in the case of $\alpha = 0$, $t - |x| \geq 2k$ and $t \geq 4k$. Set $r := |x|$. First we prove (2.4) in the interior domain,

$$D_{\text{int}} := \{(x, t) \in \mathbb{R}^2 \times [0, \infty) : t \geq 2r, t \geq 4k\}.$$

Since

$$|x - y| \leq r + |y| \leq \frac{t}{2} + k \leq t \quad \text{for } (x, t) \in D_{\text{int}} \text{ and } |y| \leq k,$$

we can rewrite $R(g|x, t)$ in (2.2) as

$$R(g|x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{g(y)}{\sqrt{t^2 - |x - y|^2}} dy \quad \text{in } D_{\text{int}}.$$
We shall consider the following estimate:

\[
(2.5) \quad \left| 2\pi R(g|x, t) - \frac{1}{\sqrt{(t + r)(t - r)}} \int_{\mathbb{R}^2} g(y) dy \right| \\
\leq \frac{1}{\sqrt{(t + r)(t - r)}} \int_{\mathbb{R}^2} \frac{|h(x, y, t)|}{\sqrt{t^2 - |x - y|^2}} |g(y)| dy,
\]

where

\[ h(x, y, t) := \sqrt{(t + r)(t - r)} - \sqrt{t^2 - |x - y|^2}. \]

Making use of Taylor expansion in \( y \) at the origin, we get

\[ h(x, y, t) = -\langle x, y \rangle + \theta |y|^2 \sqrt{t^2 - |x - \theta y|^2} \quad \text{with } 0 < \theta < 1. \]

It follows from

\[ |\langle x, y \rangle + \theta |y|^2| \leq (t/2 + k)k \]

and

\[ t - |x - \theta y| \geq t - (r + |y|) \geq t - r - k \geq t/2 - k \geq t/4, \]
\[ t + |x - \theta y| \geq t, \]

for \((x, t) \in D_{\text{int}} \) and \(|y| \leq k\) that

\[ |h(x, y, t)| \leq (1 + 2k/k)k \leq 2k \quad \text{for } (x, t) \in D_{\text{int}} \text{ and } |y| \leq k. \]

Thus, the right hand-side of \((2.5)\) is dominated by

\[ \frac{2k}{\sqrt{(t + r)(t - r)}} \int_{\mathbb{R}^2} \frac{|g(y)| dy}{\sqrt{t^2 - |x - y|^2}} \]

Similarly to the above, it follows from

\[ t - |x - y| \geq t - r - k \geq t - r/2 - r/2 - t/4 \geq (t - r)/2, \]
\[ t + |x - y| \geq t \geq t/2 + r \geq (t + r)/2, \]

for \((x, t) \in D_{\text{int}} \) and \(|y| \leq k\) that

\[ \left| 2\pi R(g|x, t) - \frac{1}{\sqrt{(t + r)(t - r)}} \int_{\mathbb{R}^2} g(y) dy \right| \leq \frac{4k\|g\|_{L^1(\mathbb{R}^2)}}{(t + r)(t - r)} \]

in \( D_{\text{int}} \). We also obtain that

\[ \left| \frac{\partial}{\partial t} R(f|x, t) \right| \leq \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{t|f(y)|}{(t^2 - |x - y|^2)^{3/2}} dy \leq \frac{4\|f\|_{L^1(\mathbb{R}^2)}}{\pi\sqrt{t + r(t - r)^{3/2}}}. \]
in $D_{\text{int}}$. Since $3(t-r) \geq t+r$ holds in $D_{\text{int}}$, summing up all the estimates, we have the desired estimate

$$|u_L(x, t)| \leq \left| \int_{\mathbb{R}^2} g(y) dy \right| \leq \frac{4\sqrt{3}\|f\|_{L^1(\mathbb{R}^2)} + 2k\|g\|_{L^1(\mathbb{R}^2)}}{\pi(t+r)(t-r)}$$

in $D_{\text{int}}$.

Next we prove (2.4) in the exterior domain,

$$D_{\text{ext}} := \{(x, t) \in \mathbb{R}^2 \times [0, \infty) : r + 2k \leq t \leq 2r\}.$$ 

Here we employ the different representation formula from (2.2),

$$R(g|x, t) = \frac{1}{2\sqrt{2\pi r}} \int_0^{\infty} \int_{s=\omega, y=-|y|^2 z/2}^{s=\omega, y=|y|^2 z/2} g(y) dS_y,$$

where $\omega := x/r \in S^1$, $\rho := r - t$ and $z := 1/r$. This is established by (6.2.4) in Hörmander [5]. Due to $z \leq 1/(2k)$ in $D_{\text{ext}}$ and $|y| \leq k$ by (2.1), we get

$$|s| \leq |y| + \frac{|y|^2 z}{2} \leq k + \frac{k^2}{2} \cdot \frac{5}{4k}.$$ 

Since

$$2\sqrt{2\pi \sqrt{r}} R(g|x, t) = \frac{1}{\sqrt{-\rho + \rho^2 z/2}} \int_{\mathbb{R}^2} g(x) dx$$

$$= \int_{-5k/4}^{5k/4} ds \int_{s=\omega, y=-|y|^2 z/2}^{s=\omega, y=|y|^2 z/2} g(y) dS_y,$$

and

$$\left| \frac{1}{\sqrt{s - \rho + \rho^2 z/2}} - \frac{1}{\sqrt{-\rho + \rho^2 z/2}} \right| \leq \frac{\sqrt{-\rho + \rho^2 z/2} - \sqrt{s - \rho + \rho^2 z/2}}{(s - \rho + \rho^2 z/2)(-\rho + \rho^2 z/2)} \leq \frac{|s|}{\sqrt{(s + |\rho|)|\rho|(|\rho| - 5k/4)}}$$

hold, we have that

$$\left| 2\sqrt{2\pi \sqrt{r}} R(g|x, t) - \frac{1}{\sqrt{-\rho + \rho^2 z/2}} \int_{\mathbb{R}^2} g(x) dx \right| \leq \frac{5k}{4|\rho| \sqrt{(|\rho| - 5k/4)}} \|g\|_{L^1(\mathbb{R}^2)}.$$
We note that $\rho \leq -2k$ in $D_{\text{ext}}$. Moreover, we get

$$\left| \frac{\partial}{\partial \rho} \left( 2 \sqrt{\frac{2}{\pi}} \sqrt{r} R(f|x,t) \right) \right| = \left| \frac{\partial}{\partial \rho} \left( \int_{-5k/4}^{5k/4} ds \int_{s=\langle \omega, y \rangle - |y|^{2z/2}} f(y)dS_y \right) \right|$$

$$= \left| \int_{-5k/4}^{5k/4} \frac{1 - \rho z}{2(s - \rho + |\rho|^{2z/2})^{3/2}} ds \int_{s=\langle \omega, y \rangle - |y|^{2z/2}} f(y)dS_y \right| \leq \frac{1 + |\rho|z}{(|\rho| - 5k/4)^{3/2}} \int_{-5k/4}^{5k/4} ds \int_{s=\langle \omega, y \rangle - |y|^{2z/2}} |f(y)|dS_y.$$ 

Hence it follows from $|\rho|z = (t - r)/r \leq 2$ for $(x, t) \in D_{\text{ext}}$ that

$$\left| \frac{\partial}{\partial \rho} \left( 2 \sqrt{\frac{2}{\pi}} \sqrt{r} R(f|x,t) \right) \right| \leq \frac{3}{(|\rho| - 5k/4)^{3/2}} \|f\|_{L^1(\mathbb{R}^2)}.$$ 

Summing up all the estimates and noticing that $\partial/\partial t = -\partial/\partial \rho$, we obtain

$$\left| \frac{\partial}{\partial \rho} \left( 2 \sqrt{\frac{2}{\pi}} \sqrt{r} R(f|x,t) \right) \right| \leq \frac{5k/4\|g\|_{L^1(\mathbb{R}^2)} + 3\|f\|_{L^1(\mathbb{R}^2)}}{2\sqrt{2\pi}(|\rho| - 5k/4)^{3/2}}$$

in $D_{\text{ext}}$. It is trivial that (2.4) in $D_{\text{ext}}$ follows from this inequality. The proof is now complete. \hfill \Box

In what follows, we consider the following integral equations:

(2.6) \hspace{1cm} u(x, t) = u^0(x, t) + L(F)(x, t) \quad \text{for} \quad (x, t) \in \mathbb{R}^2 \times [0, \infty),

where we set $u^0 := \varepsilon u_L$ and

(2.7) \hspace{1cm} L(F)(x, t) := \frac{1}{2\pi} \int_0^t (t - \tau) \int_{|\xi| \leq 1} \frac{F(x + (t - \tau)\xi, \tau)}{\sqrt{1 - |\xi|^2}} d\xi d\tau

for $F \in C(\mathbb{R}^2 \times [0, \infty))$. We note that $u$ in (2.7) solves

\[ \left\{ \begin{array}{ll}
    u_{tt} - \Delta u = F & \text{in} \ \mathbb{R}^2 \times [0, \infty), \\
    u(x, 0) = \varepsilon f(x), \ u_t(x, 0) = \varepsilon g(x), & x \in \mathbb{R}^2
\end{array} \right. \]

when $F \in C^2(\mathbb{R}^2 \times [0, \infty))$. Next two lemmas are useful to handle the radially symmetric functions.
Lemma 2.2 (John [6]). Let $b \in C([0, \infty))$. Then, the identity

\begin{equation}
\int_{|\omega|=1} b(|x + \rho \omega|)dS_\omega = 4 \int_{|\rho - r|}^{\rho + r} \lambda h(\lambda, \rho, r)b(\lambda)d\lambda
\end{equation}

holds for $x \in \mathbb{R}^2$, $r = |x|$ and $\rho > 0$, where $h$ is defined by

\begin{equation}
h(\lambda, \rho, r) := \{(\lambda + r)^2 - \rho^2\}^{-1/2}\{(\lambda - r)^2 - \rho^2\}^{-1/2}.
\end{equation}

See [6] for the proof of this lemma.

Lemma 2.3 (Agemi and Takamura [1]). Let $L$ be a linear integral operator defined by (2.7) and $\Psi = \Psi(|x|, t) \in C([0, \infty)^2)$, $x \in \mathbb{R}^2$. Then we have that

\begin{equation}
L(\Psi)(x, t) = L_1(\Psi)(r, t) + L_2(\Psi)(r, t), \quad r = |x|, \quad x \in \mathbb{R}^2,
\end{equation}

where $L_i(\Psi)$ $(i = 1, 2)$ are defined by

\begin{equation}
L_1(\Psi)(r, t) := \frac{2}{\pi} \int_0^t d\tau \int_{|t - \tau - r|}^{t + r - \tau} \lambda \Psi(\lambda, \tau)d\lambda \int_{|\lambda - r|}^{t - \tau} \frac{\rho h(\lambda, \rho, r)}{\sqrt{(t - \tau)^2 - \rho^2}}d\rho,
\end{equation}

\begin{equation}
L_2(\Psi)(r, t) := \frac{2}{\pi} \int_{|t - \tau - r|}^{t + r - \tau} \lambda \Psi(\lambda, \tau)d\lambda \int_{|\lambda - r|}^{t - \tau} \frac{\rho h(\lambda, \rho, r)}{\sqrt{(t - \tau)^2 - \rho^2}}d\rho,
\end{equation}

where $a_+ = \max\{a, 0\}$. Moreover, the following estimates hold in $[0, \infty)^2$:

\begin{equation}
|L_1(\Psi)(r, t)| \leq \frac{1}{\sqrt{2r}} \int_0^t d\tau \int_{|t - \tau + r|}^{t - \tau} \lambda |\Psi(\lambda, \tau)|d\lambda \frac{1}{\sqrt{\lambda + \tau + t + r}}.
\end{equation}

\begin{equation}
|L_2(\Psi)(r, t)| \leq \frac{1}{\sqrt{2r}} \int_0^{(t - \tau) + t - \tau} d\tau \int_{|t - \tau - r|}^{t - \tau} \lambda |\Psi(\lambda, \tau)|d\lambda \frac{1}{\sqrt{t - r + \lambda - \tau} \sqrt{t - r - \tau - \lambda}}.
\end{equation}

**Proof.** For the sake of completeness, we prove this lemma. Changing variables by $y - x = (t - \tau)\xi$ in (2.7), we obtain that

\begin{equation}
L(\Psi)(x, t) = \frac{1}{2\pi} \int_0^t d\tau \int_{|y - x| \leq t - \tau} \frac{\Psi(|y|, \tau)}{\sqrt{(t - \tau)^2 - |y - x|^2}}dy.
\end{equation}

Introducing polar coordinates, we have that

\begin{equation}
L(\Psi)(x, t) = \frac{1}{2\pi} \int_0^t d\tau \int_{|\omega| = 1} \frac{\rho d\rho}{\sqrt{(t - \tau)^2 - \rho^2}} \int_{|\omega| = 1} \Psi(|x + \rho \omega|, \tau)dS_\omega.
\end{equation}

Thus Lemma 2.2 yields that
\[
L(\Psi)(x,t) = \frac{2}{\pi} \int_0^t d\tau \int_{0}^{t-\tau} \frac{\rho d\rho}{\sqrt{(t-\tau)^2 - \rho^2}} \times \\
\times \int_{|\rho-r|}^{\rho+r} \lambda \Psi(\lambda, \tau) h(\lambda, \rho, r) d\lambda.
\]
(2.15)
Therefore, (2.10) follows from inverting the order of \((\rho, \lambda)\)-integral in (2.15).

The estimates (2.13) and (2.14) are established in the following way. Note that
\[
\lambda + r - \rho \geq \lambda + r - t + \tau, \quad \lambda + r + \rho \geq \lambda + r + |\lambda - r| \geq 2r
\]
for \(\rho \leq t - \tau\) and \(\rho \geq |\lambda - r|\). It is easy to see that
\[
\int_a^b \frac{\rho d\rho}{\sqrt{\rho^2 - a^2} \sqrt{\rho^2 - b^2}} = \frac{\pi}{2} \quad \text{for} \quad 0 \leq a < b.
\]
(2.17)
Hence (2.13) follows from (2.16) and (2.17) with \(a = |\lambda - r|\) and \(b = t - \tau\). Next let \(t > r\). Since we have that
\[
t - \tau - \rho \geq t - \tau - \lambda - r
\]
for \(\rho \leq \lambda + r\) and that
\[
t - \tau + \rho \geq t - \tau + |\lambda - r| \geq t - r - \tau + \lambda
\]
for \(\rho \geq |\lambda - r|, \lambda \leq t - r - \tau\), we obtain (2.14) by (2.18), (2.19) and (2.17) with \(a = |\lambda - r|\) and \(b = \lambda + r\).
\[
\Box
\]

§3. A priori estimate

In this section, we show a priori estimates which play key roles in the classical iteration method as in John [7]. First of all, we define some weighted \(L^\infty\) norms.

For \(r, t \geq 0\), we define the following weighted functions:
\[
w_1(r, t) := \tau_+(r, t)^{1/2} \tau_-(r, t)^{1/2},
\]
(3.1)
\[
w_2(r, t) := \tau_+(r, t)^{1/2} \tau_-(r, t)^{3/2},
\]
(3.2)
\[
w_3(r, t) := \tau_+(r, t)^{p/2 - 1},
\]
(3.3)
where we set
\[
\tau_+(r, t) := \frac{t + r + 2k}{k}, \quad \tau_-(r, t) := \frac{t - r + 2k}{k}.
\]
For these weighted functions, we denote weighted $L^\infty$ norms of $V$ by

\begin{equation}
\|V\|_i := \sup_{(x,t) \in \mathbb{R}^2 \times [0,T]} \{|w_i(|x|,t)|V(x,t)|\},
\end{equation}

where $i = 1, 2, 3$.

The following lemma is one of the most essential estimates.

**Lemma 3.1.** Let $L$ be the linear integral operator defined by (2.7). Assume that $V \in C(\mathbb{R}^2 \times [0,T])$ with supp $V \subset \{(x,t) \in \mathbb{R}^2 \times [0,T] : |x| \leq t + k\}$ and $\|V\|_i < \infty$ ($i = 1, 3$). Then, there exists a positive constant $C_1$ independent of $k$ and $T$ such that

\begin{equation}
\|L(|V|^p)\|_1 \leq C_1 k^2 \|V\|_1^p D_1(T),
\end{equation}

\begin{equation}
\|L(|V|^p)\|_3 \leq C_1 k^2 \|V\|_3^p D_2(T),
\end{equation}

where $D_i(T)$ ($i = 1, 2$) are defined by

\begin{equation}
D_1(T) := \left( \frac{2T + 3k}{k} \right)^{3-p},
\end{equation}

\begin{equation}
D_2(T) := \left( \frac{2T + 3k}{k} \right)^{(p-2)/2}.
\end{equation}

**Proof of Lemma 3.1.** The proof is divided into two pieces according to (3.5) and (3.6). From now on, a positive constant $C$ independent of $\varepsilon$ and $k$ may change from line to line.

**Estimate in (3.5).** It is clear that (3.5) follows from the basic estimate:

\begin{equation}
L(w_1^{-p}) \leq Ck^2 w_1(r,t)^{-1} D_1(T).
\end{equation}

First we shall show a part of (3.9),

\begin{equation}
L_1(w_1^{-p}) \leq Ck^2 w_1(r,t)^{-1} D_1(T).
\end{equation}

Introducing the characteristic variables

\begin{equation}
\alpha = \tau + \lambda, \quad \beta = \tau - \lambda
\end{equation}

in the integral of (2.13), we get

\begin{equation}
|L_1(\Psi)| \leq \frac{C}{\sqrt{r}} \int_{|t-r|}^{t+r} \int_{-k}^{t-r} \left| \Psi \left( \frac{\alpha - \beta}{2}, \frac{\alpha + \beta}{2} \right) \right| \frac{(\alpha - \beta)}{\sqrt{\alpha - t + r}} d\beta.
\end{equation}
Setting $\Psi(\lambda, \tau) = \{w_1(\lambda, \tau)\}^{-p}$, we have

$$L_1(w_1^{-p}) \leq \frac{Ck}{\sqrt{r}} \int_{t-r}^{t+r} \left( \frac{\alpha + 2k}{k} \right)^{1-p/2} \frac{d\alpha}{\sqrt{\alpha - t + r}} \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{-p/2} d\beta$$

$$\leq Ck^2 \tau_+(r,t)^{1-p/2} \tau_+(r,t)^{1-p/2}.$$ 

Thus, recalling (3.11) and (3.7), we get

$$L_1(w_1^{-p}) \leq Ck^2 w_1(r,t)^{-1} \tau_+(r,t)^{-p} \leq Ck^2 w_1(r,t)^{-1} D_1(T).$$

Next we shall show the remaining part of (3.9),

$$L_2(w_1^{-p}) \leq Ck^2 w_1(r,t)^{-1} D_1(T) \quad \text{for} \quad t - r \geq 0.$$ 

Introducing the characteristic variables (3.10) in the integral of (2.14), we get

$$|L_2(\Psi)| \leq C \int_{0}^{t-r} d\alpha \int_{-k}^{t-r} \left| \Psi \left( \frac{\alpha - \beta}{2}, \frac{\alpha + \beta}{2} \right) \right| \frac{\alpha - \beta}{\sqrt{t - r - \alpha}} \frac{\alpha + \beta}{\sqrt{t - r - \beta}}.$$ 

Setting $\Psi(\lambda, \tau) = \{w_1(\lambda, \tau)\}^{-p}$, we have that

\begin{align*}
L_2(w_1^{-p}) &\leq Ck \int_{0}^{t-r} \left( \frac{\alpha + 2k}{k} \right)^{1-p/2} \frac{d\alpha}{\sqrt{t - r - \alpha}} \times \int_{t-r}^{t-r-k} \left( \frac{\beta + 2k}{k} \right)^{-p/2} \frac{d\beta}{\sqrt{t - r - \beta}}.
\end{align*}

First we consider the case of $t - r \geq k$. Then, we get

$$L_2(w_1^{-p}) \leq Ck \tau_+(r,t)^{-p/2} \{J_1(r,t) + J_2(r,t)\},$$ 

where we set

$$J_1(r,t) := \int_{0}^{t-r} \frac{d\alpha}{\sqrt{t - r - \alpha}} \int_{-k}^{(t-r-k)/2} \left( \frac{\beta + 2k}{k} \right)^{-p/2} \frac{d\beta}{\sqrt{t - r - \beta}},$$

$$J_2(r,t) := \int_{0}^{t-r} \frac{d\alpha}{\sqrt{t - r - \alpha}} \int_{-k}^{(t-r-k)/2} \left( \frac{\beta + 2k}{k} \right)^{-p/2} \frac{d\beta}{\sqrt{t - r - \beta}}.$$ 

It is easy to see that

\begin{align*}
J_1(r,t) &\leq \frac{C}{\sqrt{t - r + k}} \int_{0}^{t-r} \frac{d\alpha}{\sqrt{t - r - \alpha}} \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{-p/2} d\beta \\
&\leq Ck \tau_+(r,t)^{-p/2}.
\end{align*}
On the other hand, it follows from \( 1 - \frac{p}{2} > 0 \) that

\[
J_2(r, t) \leq C \tau_-(r, t)^{-p/2} \int_0^{t-r} \frac{d\alpha}{\sqrt{t - r - \alpha}} \int_{(t-r-k)/2}^{t-r} \frac{d\beta}{\sqrt{t - r - \beta}} 
\]

\leq C k \tau_-(r, t)^{1-p/2} \leq C k \tau_+(r, t)^{1-p/2}.

Hence (3.12), (3.13) and (3.14) yield that

\[
L_2(w_1^{3-p}) \leq C k^2 \tau_+(r, t)^{2-p} \leq C k^2 w_1(r, t)^{-1} D_1(T)
\]

for \( t - r \geq k \). In the case of \( 0 < t - r \leq k \), (3.11) yields that

\[
L_2(w_1^{3-p}) \leq C k \int_0^{t-r} \left( \frac{\alpha + 2k}{k} \right)^{-p/2} \frac{d\alpha}{\sqrt{t - r - \alpha}} \times \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{-p/2} \frac{d\beta}{\sqrt{t - r - \beta}}
\]

\leq C k^2 = C k^2 w_1(r, t)^{-1} w_1(r, t) \leq C k^2 w_1(r, t)^{-1} D_1(T).

Therefore (3.9) is now established.

**Estimate in (3.6).** Similarly to the above, we note that (3.6) follows from the basic estimate:

\[
L_1(w_3^{-p}) \leq C k^2 w_3(r, t)^{-1} D_2(T).
\]

Setting \( \Psi(\lambda, \tau) = \{w_3(\lambda, \tau)\}^{-p} \) in (2.13) and introducing (3.10) in the integral of (2.13), we get

\[
L_1(w_3^{-p}) \leq C k \int_{[t-r]}^{t+r} \left( \frac{\alpha + 2k}{k} \right)^{1+(1-p/2)p} \frac{d\alpha}{\sqrt{\alpha - t + r}} \int_{-k}^{t-r} \frac{d\beta}{\sqrt{t - r - \beta}}
\]

\leq C k^2 \tau_+(r, t)^{(1-p/2)p+1} \tau_-(r, t) \leq C k^2 \tau_+(r, t)^{1-p/2+\gamma(2,p)/2}.

Hence we have a part of (3.15),

\[
L_1(w_3^{-p}) \leq C k^2 w_3(r, t)^{-1} D_2(T).
\]

Now, let \( t - r > 0 \). Setting \( \Psi(\lambda, \tau) = \{w_3(\lambda, \tau)\}^{-p} \) in (2.14) and introducing (3.10) in the integral of (2.14), we get

\[
L_2(w_3^{-p}) \leq C k \int_0^{t-r} \left( \frac{\alpha + 2k}{k} \right)^{1+(1-p/2)p} \frac{d\alpha}{\sqrt{t - r - \alpha}} \int_{-k}^{t-r} \frac{d\beta}{\sqrt{t - r - \beta}}
\]

\leq C k \tau_+(r, t)^{(1-p/2)p+1} \int_0^{t-r} \frac{d\alpha}{\sqrt{t - r - \alpha}} \int_{-k}^{t-r} \frac{d\beta}{\sqrt{t - r - \beta}}
\]

\leq C k^2 \tau_+(r, t)^{(1-p/2)p+2}.
Therefore we have that
\[ L_2(w_3^{-p}) \leq Ck^2w_3(r,t)^{-1}D_2(T) \]
for \( t - r \geq 0 \) which yields (3.15). The proof of Lemma 3.1 is complete. □

In order to construct a solution in our weighted \( L^\infty \) space, the following variant to the a priori estimate is required.

**Lemma 3.2.** Let \( L \) be the linear integral operator defined by (2.7) and \( 0 \leq \nu < p \). Assume that \( V, V_0 \in C(R^2 \times [0,T]) \) with \( \text{supp} \ (V, V_0) \subset \{(x,t) \in R^2 \times [0,T] : |x| \leq t + k\} \), and \( \|V_0\|_2, \|V\|_3 < \infty \). Then, there exists a positive constant \( C_2 \) independent of \( k \) such that
\[
\|L(|V_0|^{p-\nu}|V|)\|_3 \leq C_2k^2\|V_0\|^{p-\nu}_2\|V\|^{\nu}_3D_{3,\nu}(T),
\]
where \( D_{3,\nu}(T) \) are defined by
\[
D_{3,\nu}(T) := \begin{cases} 
\left( \frac{2T + 3k}{k} \right)^{\nu(3-p)/2} & \text{if } p > \nu + \frac{2}{3}, \\
\log \frac{2T + 3k}{k} \left( \frac{2T + 3k}{k} \right)^{(7/3-\nu)p/2} & \text{if } p = \nu + \frac{2}{3}, \\
\left( \frac{2T + 3k}{k} \right)^{1-3p/2+(3-p/2)\nu} & \text{if } p < \frac{2}{3} + \nu.
\end{cases}
\]

**Proof.** Similarly to the proof of Lemma 3.1 we note that (3.16) follows from
\[
L(w_2^{-(p-\nu)}w_3^{-\nu}) \leq Ck^2w_3(r,t)^{-1}D_{3,\nu}(T).
\]
A part of this estimate is
\[
L_1(\tau_+^{-(p-\nu)/2+(1-p/2)\nu}\tau_-^{-3(p-\nu)/2}) \leq Ck^2w_3(r,t)^{-1}D_{3,\nu}(T).
\]
Setting
\[
\Psi(\lambda, \tau) = \tau_+^{-(p-\nu)/2+(1-p/2)\nu}\tau_-^{-3(p-\nu)/2}
\]
in (2.13) and introducing (3.10) in the integral in (2.13), we get
\[
L_1(\tau_+^{-(p-\nu)/2+(1-p/2)\nu}\tau_-^{-3(p-\nu)/2}) \leq \frac{Ck}{\sqrt{t-r}} \int_{|t-r|}^{t+r} \left( \frac{\alpha + 2k}{k} \right)^{1-(p-\nu)/2+(1-p/2)\nu} \frac{d\alpha}{\sqrt{\alpha - t + r}} \times \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{-3(p-\nu)/2} d\beta.
\]
Note that
(3.22)
\[ \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{-3(p-\nu)/2} d\beta \leq \begin{cases} 
Ck & \text{if } p > \nu + 2/3, \\
Ck \log \tau_-(r, t) & \text{if } p = \nu + 2/3, \\
Ck\tau_-(r, t)^{1-3(p-\nu)/2} & \text{if } p < \nu + 2/3 
\end{cases} \]
and that the \( \alpha \)-integral in (3.21) is estimated by
\[ Ck \sqrt{r} \tau_+(r, t)^{1-(p-\nu)/2+(1-p/2)\nu}. \]

Therefore (3.19) is follows from
\[ \tau_+^{1-(p-\nu)/2+(1-p/2)\nu} \leq w_3^{-1} \tau_+^{(3-p)\nu/2}. \]

Next, we shall show the remaining part of (3.18),
(3.23)
\[ L_2(\tau_+^{-(p-\nu)/2+(1-p/2)\nu} \tau_-^{-(p-\nu)/2}) \leq C k^2 w_3(r, t)^{-1} D_{3, \nu}(T). \]

Setting \( \Psi(\lambda, \tau) \) such as (3.20) in (2.14) and introducing (3.10) in the integral of (2.14), we get
(3.24)
\[ L_2(\tau_+^{-(p-\nu)/2+(1-p/2)\nu} \tau_-^{-(p-\nu)/2}) \leq C \int_0^{t-r} \left( \frac{\alpha + 2k}{k} \right)^{1-(p-\nu)/2+(1-p/2)\nu} d\alpha \sqrt{t-r-\alpha} \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{-3(p-\nu)/2} d\beta \sqrt{t-r-\beta}. \]

First we consider the case of \( t - r \geq k \). Then we have
\[ L_2(\tau_+^{-(p-\nu)/2+(1-p/2)\nu} \tau_-^{-(p-\nu)/2}) \leq C k w_3(r, t)^{-1} \tau_+(r, t)^{(3-p)\nu/2} \{ K_1(r, t) + K_2(r, t) \}, \]
where we set
\[ K_1(r, t) := \int_0^{t-r} \frac{d\alpha}{\sqrt{t-r-\alpha}} \int_{-k}^{(t-r-k)/2} \left( \frac{\beta + 2k}{k} \right)^{-3(p-\nu)/2} d\beta \sqrt{t-r-\beta}, \]
\[ K_2(r, t) := \int_0^{t-r} \frac{d\alpha}{\sqrt{t-r-\alpha}} \int_{(t-r-k)/2}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{-3(p-\nu)/2} d\beta \sqrt{t-r-\beta}. \]

It follows from (3.22) that
\[ K_1(r, t) \leq C (t-r+k)^{-1/2} \int_0^{t-r} \frac{d\alpha}{\sqrt{t-r-\alpha}} \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{-3(p-\nu)/2} d\beta. \]
Hence (3.23) for \( t - r \geq k \) is established by (3.22). Moreover it is easy to see that

\[
K_2(r, t) \leq C \tau_-(r, t)^{-3(p-\nu)/2} \int_0^{t-r} \frac{d\alpha}{\sqrt{t-r-\alpha}} \int_{(t-r-k)/2}^{t-r} \frac{d\beta}{\sqrt{t-r-\beta}}
\]

\[
\leq Ck\tau_-(r, t)^{1-3(p-\nu)/2}.
\]

Hence (3.23) for \( t - r \geq k \) is established. In the case of \( 0 \leq t - r \leq k \), (3.24) yields that

\[
L_2(\tau_+(p-\nu)/2+(1-p/2)\nu, \tau_-^{-3(p-\nu)/2})
\]

\[
\leq Ck \int_0^{t-r} \frac{(\alpha + 2k)}{k} \frac{d\alpha}{\sqrt{t-r-\alpha}} \times \int_{-k}^{t-r} \frac{(\beta + 2k)}{k} \frac{d\beta}{\sqrt{t-r-\beta}}
\]

\[
\leq Ck^2 = Ck^2 w_3(r, t)^{-1} w_3(r, t) \leq Ck^2 w_3(r, t)^{-1} D_3,\nu(T).
\]

This gives us (3.23) for \( 0 \leq t - r \leq k \) which leads to (3.18). The proof of Lemma 3.2 is complete.

In order to pick up the sufficient condition to construct a solution in our weighted \( L^\infty \) space, we need the following lemma on comparison among the quantities depending on \( T \).

**Lemma 3.3.** Let \( 1 < p < 2 \) and let \( D_2(T) \) and \( D_{3,\nu}(T) \) are the one in (3.8) and (3.17). Then we have

\[
D_{3,1}(T) \leq D_2(T)^{1/p},
\]

\[
D_{3,p-1}(T) \leq D_2(T)^{(p-1)/(p+1)},
\]

\[
D_{3,0}(T) = 1.
\]

**Proof.** All the estimates follow from direct computations. When \( p > 5/3 \), we have that \((3-p)p/2 < \gamma(p, 2)/2\) which yields (3.25). When \( p = 5/3 \), we have that \((7/3 - \nu)\nu/2 = (3-p)\nu/2\) for \( p = \nu + 2/3 \), so that it is sufficient to show that \( p\{\delta + (3-p)/2\} < \gamma(p, 2)/2\) holds for suitable \( \delta > 0 \) to get (3.25). This can be guaranteed by taking \( 0 < \delta < 1/p \). When \( 1 < p < 5/3 \), we have that \( p(4-2p) < \gamma(p, 2)/2 \) holds. Therefore (3.25) is established in all the cases.

The estimate (3.26) follows from \((p+1)(3-p)/2 < \gamma(p, 2)/2\) which is equivalent to \( p > 1 \). The estimate (3.27) is trivial by definition of \( D_{3,\nu}(T) \). The proof is now complete. \( \Box \)
§4. Lower bound of the lifespan

First, we shall show the estimate for (1.11) in the case of
\[ \int_{\mathbb{R}^2} g(x) dx = 0. \]
We consider the following integral equation:
\[ U = L(|u^0 + U|^p) \quad \text{in} \quad \mathbb{R}^2 \times [0, T] \]  
(4.1)
where \( L \) and \( u^0 \) are defined in (2.6). Suppose we obtain the solution \( U(x, t) \) of (4.1). Then, putting \( u = U + u^0 \), we get the solution of (2.6) and its life span is the same as that of \( U \). Thus we have reduced the problem to the analysis of (4.1). In view of (2.6) and (2.7), we note that \( \partial U / \partial t \) can be expressed in \( \nabla_x U \). Hence we consider spatial derivatives of \( U \) only.

We define \( U_l \) by
\[ U_1 = 0, \quad U_l = L(|u^0 + U_{l-1}|^p) \quad \text{for} \quad l \geq 2. \]  
(4.2)
We take \( \varepsilon \) and \( T \) such that
\[ C\varepsilon^{p(p-1)} D_2(T) \leq 1 \]  
(4.3)
where
\[ C := (2^{2p} p) \frac{p}{p-1} \max \{ C_1 k^2 M_0^{p-1}, (C_2 k^2 C_0^{p-1})^p, (C_2 k^2 M_0^{p-2} C_0)^{p-1} \}, \]  
(4.4)
\[ M_0 := 2^p p C_3 k^2 C_0^p \]  
(4.5)
and \( C_3 := \max \{ C_1, C_2 \} \).

In order to get a \( C^1 \) solution of (4.1), we shall show the convergence of \( \{U_l\}_{l \in \mathbb{N}} \) in a function space \( X \) defined by
\[ X := \{ U(x, t) : \nabla_x^\alpha U \in C(\mathbb{R}^2 \times [0, T]) \text{ for } |\alpha| \leq 1, \|U\|_X < \infty, \text{ supp } U \subset \{ (x, t) : |x| \leq t + k \} \} \]  
which is equipped with the norm
\[ \|U\|_X = \sum_{|\alpha| \leq 1} \|\nabla_x^\alpha U\|_3. \]  
We see that \( X \) is a Banach space for any fixed \( T > 0 \). It follows from the definition of the norm (3.4) that there exists a positive constant \( C_T \) depending on \( T \) such that
\[ \|U\|_3 \geq C_T |U(x, t)|, \quad t \in [0, T]. \]
By induction, we shall obtain

\[ \|U_l\|_3 \leq 2M_0\varepsilon^p. \] (4.6)

For \( l = 1 \), (4.6) holds. Assume that \( \|U_{l-1}\|_3 \leq 2M_0\varepsilon^p \) \((l \geq 2)\). Since

\[ |a + b|^p \leq 2^{p-1}(|a|^p + |b|^p) \quad \text{for } p > 1 \text{ and } a, b \in \mathbb{R}, \]

we get from (4.2) that

\[ \|U_l\|_3 \leq 2^{p-1}\{\|L(|u^0|^p)\|_3 + \|L(|U_{l-1}|)^p\|_3\}. \] (4.7)

Making use of (3.16) with \( \nu = 0 \), (3.27) and (2.4), we have

\[ \|L(|u^0|^p)\|_3 \leq C_2k^2\|u^0\|^p_2 \]
\[ \quad \leq C_2k^2C_0^p\varepsilon^p, \] (4.8)

where we used (2.4) with \( \int_{\mathbb{R}^2} g(x)dx = 0 \), (3.2) and (3.4). We see from (3.6) that

\[ \|L(|U_{l-1}|)^p\|_3 \leq C_1k^2\|U_{l-1}\|^p_3D_2(T) \]
\[ \leq C_1k^2(2M_0\varepsilon^p)^pD_2(T). \] (4.9)

Summarizing (4.7), (4.8), (4.9) and (4.5), we get

\[ \|U_l\|_3 \leq M_0\varepsilon^p + 2^{p-1}C_1k^2(2M_0\varepsilon^p)^pD_2(T). \] (4.10)

This inequality shows (4.6) provided (4.3) and (4.4) hold. We shall estimate the differences of \( \{U_l\}_{l \in \mathbb{N}} \). Since

\[ \|a^p - |b|^p\| \leq p(|a|^{p-1} + |b|^{p-1})|a - b| \quad \text{for } p > 1, \]

we obtain from (4.2) that

\[ \|U_{l+1} - U_l\|_3 \leq 2^{p-1}p\{2\|L(|u^0|^{p-1}|U_l - U_{l-1}|)\|_3 \]
\[ + \|L(|U_l|^{p-1} + |U_{l-1}|^{p-1})|U_l - U_{l-1}|)\|_3\}. \] (4.11)

From (3.16) with \( \nu = 1 \), (3.25) and (2.4), we obtain

\[ \|L(|u^0|^{p-1}|U_l - U_{l-1}|)\|_3 \leq C_2k^2\|u^0\|^{p-1}_2\|U_l - U_{l-1}\|_3D_2(T)^{1/p} \]
\[ \leq C_2k^2(C_0\varepsilon)^{p-1}D_2(T)^{1/p}\|U_l - U_{l-1}\|_3. \] (4.12)
We get from (3.6) and (4.6) that
\[
\|L((|U_i|^{p-1} + |U_{l-1}|^{p-1})(U_i - U_{l-1}))\|_3 \\
\leq C_1 k^2 D_2(T)(\|U_i\|_3^{p-1} + \|U_{l-1}\|_3^{p-1})\|U_i - U_{l-1}\|_3 \\
\leq 2C_1 k^2 D_2(T)(2M_0\varepsilon^p)^{p-1}\|U_i - U_{l-1}\|_3.
\]
(4.13)

Hence, we obtain from (4.11), (4.12) and (4.13) that
\[
\|U_{i+1} - U_i\|_3 \leq \frac{1}{2}\|U_i - U_{l-1}\|_3
\]
provided (4.3) and (4.4) hold. Therefore we have
\[
\|U_{i+1} - U_i\|_3 \leq 2^{-l}C_4 \quad \text{for} \quad l \geq 1.
\]
(4.14)

Next, by induction, we shall show the following boundedness of \(\{\partial_i U_l\}\).
\[
\|\partial_i U_l\|_3 \leq 2M_0\varepsilon^p.
\]
(4.15)

Assume that \(\|\partial_i U_{l-1}\|_3 \leq 2M_0\varepsilon^p \quad (l \geq 2)\). From (4.2), we have
\[
\|\partial_i U_l\|_3 \leq 2^{p-1}p\|L\{(|u^0|^{p-1} + |U_{l-1}|^{p-1})(|\partial_i u^0| + |\partial_i U_{l-1}|)\}\|_3.
\]
(4.16)

By using Lemma 3.2 and Lemma 3.3 we shall show
\[
\|L(|u^0|^{p-1}|\partial_i u^0|)\|_3 \leq C_2 k^2 (C_0\varepsilon)^p, \quad \text{(4.17)}
\]
\[
\|L(|U_{l-1}|^{p-1}|\partial_i u^0|)\|_3 \leq C_2 k^2 (2M_0\varepsilon^p)^{p-1}C_0\varepsilon D_2(T)\frac{\varepsilon}{p+1}, \quad \text{(4.18)}
\]
\[
\|L(|u^0|^{p-1}|\partial_i U_{l-1}|)\|_3 \leq C_2 k^2 (C_0\varepsilon)^{p-1}D_2(T)^{1/p}(2M_0\varepsilon^p). \quad \text{(4.19)}
\]

We shall prove only (4.17), since we can prove (4.18) and (4.19) in a similar way. It follows from (3.6) with \(\nu = 0\), (3.27) and (2.4) that
\[
\|L(|u^0|^{p-1}|\partial_i u^0|)\|_3 \leq C_2 k^2 \|u^0\|_2^{p-1}\|\partial_i u^0\|_2 \\
\leq C_2 k^2 (C_0\varepsilon)^p.
\]
We have from (3.6) and (4.6) that
\[
\|L(|U_{l-1}|^{p-1}|\partial_i U_{l-1}|)\|_3 \leq C_1 k^2 \|U_{l-1}\|_3^{p-1}D_2(T)\|\partial_i U_{l-1}\|_3 \\
\leq C_1 k^2 (2M_0\varepsilon^p)^p D_2(T).
\]
(4.20)
Summarizing (4.16), (4.17), (4.18), (4.19) and (4.20), we get from $D_2(T) \geq 1$ that

$$
\| \partial_t U_l \|_3 \leq M_0 \varepsilon^p + 2^{p-1} p C_3 k^2 \{(2M_0)^{p-1} C_0 \varepsilon^2 - p + 1 D_2(T)^{\frac{p-1}{p}}
+ 2M_0 C_0^{p-1} \varepsilon^{2p-1} D_2(T)^{1/p} + (2M_0)^p \varepsilon^p D_2(T)\}. 
$$

(4.21)

Therefore we obtain (4.15) provided (4.3) and (4.4) hold.

Finally, we shall estimate the difference of $\partial_t U_l$. We obtain from (4.2) that

$$
\| \partial_t (U_{l+1} - U_l) \|_3 \leq p \{2^{p-1} \| L((|u^0|^{p-1} + |U_l|^{p-1}) |\partial_t (U_l - U_{l-1})|)\|_3
+ \| L(|U_l - U_{l-1}|^{p-1} |\partial_t (u_0 + U_{l-1})|)\|_3\}. 
$$

(4.22)

We get from (3.16), (3.16) with $\nu = 1$ and (3.20) that

$$
\| L((|u^0|^{p-1} + |U_l|^{p-1}) |\partial_t (U_l - U_{l-1})|)\|_3
\leq C_3 k^2 (\| u^0 \|_2^{p-1} D_2(T)^{1/p} + \| U_l \|_3^{p-1} D_2(T) ) \| \partial_t (U_l - U_{l-1})\|_3
\leq C_3 k^2 \{(C_0 \varepsilon)^{(p-1)} D_2(T)^{1/p} + (2M_0 \varepsilon^p)^{p-1} D_2(T)\} \| \partial_t (U_l - U_{l-1})\|_3.
$$

(4.23)

It follows from (3.16), (3.16) with $\nu = p - 1$ and (3.20) that

$$
\| L(|U_l - U_{l-1}|^{p-1} |\partial_t (u_0 + U_{l-1})|)\|_3
\leq C_3 k^2 \| U_l - U_{l-1} \|_3^{p-1} \{ \| \partial_t u^0 \|_2 D_2(T)^{\frac{p-1}{p+1}} + \| \partial_t U_{l-1} \|_3 D_2(T) \}
\leq C_3 k^2 \{ C_0 \varepsilon D_2(T)^{\frac{p-1}{p+1}} + 2M_0 \varepsilon^p D_2(T)\} \| U_l - U_{l-1} \|_3^{p-1}
\leq C_5 \| U_l - U_{l-1} \|_3^{p-1}.
$$

(4.24)

Summarizing (4.22), (4.23), (4.24) (4.4) and (4.4), we get

$$
\| \partial_t (U_{l+1} - U_l) \|_3 \leq p C_5 \| U_l - U_{l-1} \|_3^{p-1} + \frac{1}{2} \| \partial_t (U_l - U_{l-1})\|_3
\leq C_6 2^{-l(p-1)} + \frac{1}{2} \| \partial_t (U_l - U_{l-1})\|_3.
$$

Hence, we obtain

$$
\| \partial_t (U_{l+1} - U_l) \|_3 \leq C_7 (l + 1) 2^{-l(p-1)} \quad \text{for} \quad l \geq 1.
$$

(4.25)

The estimates (4.14) and (4.25) imply that the $\nabla_x^\alpha U_l$ for $|\alpha| \leq 1$ converge uniformly for $l \to \infty$ towards functions $\nabla_x^\alpha U$, which are continuous in $x$, $t$, where $U$ is a solution of (4.1).
We put
\[ C\varepsilon_0^{p(p-1)}6^{\gamma(2,p)/2} \leq 1. \]

For \(0 < \varepsilon \leq \varepsilon_0\), if we assume that
\[ C\varepsilon^{p(p-1)} \left( \frac{4T}{k} \right)^{\gamma(2,p)/2} \leq 1, \]
then (4.3) holds. Hence, (1.11) in the case of \(\int_{\mathbb{R}^2} g(x)dx = 0\) is proved for \(0 < \varepsilon \leq \varepsilon_0\).

Next, we shall show the estimate for (1.11) in the case of \(\int_{\mathbb{R}^2} g(x)dx \neq 0\). Let \(Y\) be the linear space defined by
\[ Y = \{ u(x,t) : \nabla_x^\alpha u(x,t) \in C(\mathbb{R}^2 \times [0,T]), \| \nabla_x^\alpha u \|_1 < \infty \text{ for } |\alpha| \leq 1, \supp u \subset \{(x,t) : |x| \leq t + k\} \}. \]

We can verify that \(Y\) is complete with respect to the norm
\[ \| u \|_Y = \sum_{|\alpha| \leq 1} \| \nabla_x^\alpha u \|_1. \]

We define the sequence of functions \(\{u_l\}\) by
\[ u_1 = u^0, \quad u_l = u^0 + L(|u_{l-1}|) \text{ for } l \geq 2. \]

Since Lemma 2.1 yields that \(\| u^0 \|_Y \leq C'_0 \varepsilon\) where
\[ C'_0 := \tilde{C}_0 \left| \int_{\mathbb{R}^2} g(x)dx \right| + k^{-1}C_0, \]
we have that \(u^0 \in Y\). We now assume that \(\varepsilon\) is so small that
\[ 2^p pC_1 k^2 (C'_0 \varepsilon)^{p-1} D_1(T) \leq 1. \]

Therefore, as in [7], we see that if (4.26) holds, then there exists a unique local solution of (2.6). By taking \(\varepsilon_0\) small, the lower bound (1.11) with \(\int_{\mathbb{R}^2} g(x)dx \neq 0\) follows immediately from (4.26).
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