ERROR ESTIMATES FOR APPROXIMATIONS OF NONLINEAR
UNIFORMLY PARABOLIC EQUATIONS

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Abstract. We introduce the notion of δ-viscosity solutions for fully nonlinear uniformly parabolic PDE on bounded domains. We prove that δ-viscosity solutions are uniformly close to the actual viscosity solution, with an explicit error of order δα. As a consequence we obtain an error estimate for implicit monotone finite difference approximations of uniformly parabolic PDE.

1. Introduction

We introduce δ-viscosity solutions for the nonlinear parabolic problem

\begin{equation}
    u_t - F(D^2 u) = 0 \quad \text{in } \Omega \times (0, T).
\end{equation}

We prove an estimate between viscosity solutions and δ-viscosity solutions of (1) under the assumption that the nonlinearity F is uniformly elliptic (see (F1) below). As a consequence, we find a rate of convergence for monotone and consistent implicit finite difference approximations to (1). Both results generalize the work of Caffarelli and Souganidis in [8] and [9], who consider the time-independent case.

The nonlinearity F is a continuous function on $S_{n \times n} \times U$, where $S_{n \times n}$ is the set of $n \times n$ real symmetric matrices endowed with the usual order and norm (for $X \in S_{n \times n}$, $||X|| = \sup_{\|v\|=1} |Xv|$). We make the following assumptions:

(F1) F is uniformly elliptic with constants $0 < \lambda \leq \Lambda$, which means that for any $X \in S_{n \times n}$ and for all $Y \geq 0$,

$$
\lambda ||Y|| \leq F(X + Y) - F(X) \leq \Lambda ||Y||;
$$

and,

(F2) $F(0) = 0$.

We also assume:

(U1) $\Omega$ is a bounded subset of $\mathbb{R}^n$.

The main result is an error estimate between viscosity solutions and δ-viscosity solutions of (1). The definition of δ-viscosity solutions is in Section 2.2. Next we present a statement of our main result that has been simplified for the introduction; the full statement is in Section 6.

**Theorem 1.1.** Assume (F1), (F2) and (U1). Assume $u$ is a viscosity solution of (1) that is Lipschitz continuous in $x$ and Hölder continuous in $t$. Assume $\{u_\delta\}_{\delta > 0}$ is a family of Hölder continuous δ-viscosity solutions of (1) that satisfy

$$
    u_\delta = u \quad \text{on the parabolic boundary of } \Omega \times (0, T)
$$

for all $\delta$. There exist positive constants $C$ and $\alpha$ that do not depend on $\delta$ such that for all $\delta$ small enough,

$$
    \|u - u_\delta\|_{L^\infty(\Omega \times (0, T))} \leq C \delta^\alpha.
$$

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We remark that if \( u \) is the solution of a boundary value problem with sufficiently regular boundary data, then the regularity conditions on \( u \) of Theorem 1.1 are satisfied (see Remark 6.2 for more details). We make this an assumption in order to avoid discussing boundary value problems and to keep the setting as simple as possible.

The notion of \( \delta \)-viscosity solutions was introduced by Caffarelli and Souganidis in \([8]\) as a tool for obtaining uniform estimates for viscosity solutions. In \([8]\), \( \delta \)-viscosity solutions were used to establish an error estimate for finite difference approximations of nonlinear uniformly elliptic equations. An error estimate between viscosity and \( \delta \)-viscosity solutions of uniformly elliptic equations was obtained by Caffarelli and Souganidis in \([9]\). This was an important step in establishing a rate for homogenization in random media \([9]\).

The main challenge in obtaining an error estimate between viscosity and \( \delta \)-viscosity solutions, in both the elliptic and parabolic setting, is overcoming the lack of regularity of the viscosity solution \( u \). Indeed, the proof of the error estimate in \([8]\) is based on a regularity result \([8, \text{Theorem A}]\), which says that outside of sets of small measure, solutions of uniformly elliptic equations have second-order expansions with controlled error. We prove a similar result for solutions of equation \((1)\). This is Theorem 3.2 of this paper, and it is an essential part of our proof of Theorem 1.1.

But even once it is known that the solution \( u \) of \((1)\) has second-order expansions with controlled error, it is necessary to further regularize \( u \) and the \( \delta \)-viscosity solution \( u_\delta \) in order establish an estimate for their difference. For this we use the classical inf- and sup- convolutions, along with another regularization of inf-sup type, which we call \( x \)-sup- and \( x \)-inf- convolutions. Both of these regularizations preserve the notions of viscosity and \( \delta \)-viscosity solution. Moreover, we show that the regularity we establish in Theorem 3.2 for solutions \( u \) of \((1)\) is also enjoyed by the \( x \)-inf and \( x \)-sup convolutions of \( u \) (Proposition 4.3).

In this paper we also study implicit finite difference approximations to \((1)\) and prove an error estimate between the solution \( u \) of \((1)\) and approximate solutions to \((1)\). We have simplified the notation in the introduction in order to state this result here; see Section 7 for all the details about approximation schemes and for the full statement of the error estimate.

We write the finite difference approximations as

\[
S_h[u](x,t) = \delta_+^h u(x,t) - \mathcal{F}_h(\delta^2 u(x,t)) = 0 \text{ in } (\Omega \times (0,T)) \cap E.
\]

Here \( E = h\mathbb{Z}^n \times h^2\mathbb{Z} \) is the mesh of discretization, \( S_h[u] \) is the implicit finite difference operator, and \( \delta_+^h \) and \( \delta^2 u \) are finite difference quotients associated to a function \( u \). We assume:

(S1) if \( u_h^1 \) and \( u_h^2 \) are solutions of \((2)\) with \( u_h^1 \leq u_h^2 \) on the discrete boundary of \( \Omega \times (0,T) \), then \( u_h^1 \leq u_h^2 \) in \( \Omega \times (0,T) \); and

(S2) there exists a positive constant \( K \) such that for all \( \phi \in C^3(\Omega \times (0,T)) \) and all \( h > 0 \),

\[
\sup |\phi_t - F(D^2\phi) - S_h[\phi]| \leq K(h + h||D^3\phi||_{L^\infty(\Omega \times (0,T))} + h^2||\phi_{tt}||_{L^\infty(\Omega \times (0,T))}).
\]

Schemes that satisfy (S1) and (S2) are said to be, respectively, monotone and consistent with an error estimate for \( F \). We prove:

**Theorem 1.2.** Assume \((\mathcal{H}), (\mathcal{H}_\infty)\) and \((\mathcal{U}1)\). Assume that \( S_h \) is a monotone implicit approximation scheme for \((1)\) that is consistent with an error estimate. Assume \( u \) is a viscosity solution of \((1)\) that is Lipschitz continuous in \( x \) and Hölder continuous in \( t \) and that \( u_h \) is a Hölder continuous solution of \((2)\). Assume that for each \( h > 0 \),

\[
u_h = u \text{ on the discrete boundary of } \Omega \times (0,T).
\]

There exist positive constants \( C \) and \( \alpha \) that do not depend on \( h \) such that for every \( h \) small enough,

\[
||u - u_h||_{L^\infty(\Omega \times (0,T))} \leq Ch^\alpha.
\]
The convergence of monotone and consistent approximations of fully nonlinear second order PDE was first established by Barles and Souganidis [5]. Kuo and Trudinger later studied the existence of monotone and consistent approximations for nonlinear elliptic and parabolic equations and the regularity of the approximate solutions $v_h$ (see [21, 22, 23, 24]). They showed, both in the elliptic and in the parabolic cases, that if $F$ is uniformly elliptic, then there exists a monotone finite difference scheme $S_h$ that is consistent with $F$, and that the approximate solutions $u_h$ are uniformly equicontinuous. However, obtaining an error estimate remained an open problem.

The first error estimate for approximation schemes was established by Krylov in [17] and [18] for nonlinearities $F$ that are either convex or concave, but possibly degenerate. Krylov used stochastic control methods that apply in the convex or concave case, but not in the general setting. Barles and Jakobsen in [2] and [3] improved Krylov’s error estimates for convex or concave equations. In [19] Krylov improved the error estimate to be of order $h^{1/2}$, but still in the convex/concave case. In addition, Jakobsen [14, 15] and Bonnans, Maroso, and Zidani [6] established error estimates for special equations or for special dimensions. The first error estimate without a convexity or concavity assumption was obtained Caffarelli and Souganidis [6] established error estimates for special equations or for special dimensions. The first error estimate in the convex/concave case. In addition, Jakobsen [14, 15] and Bonnans, Maroso, and Zidani [6] established error estimates for special equations or for special dimensions. The first error estimate without a convexity or concavity assumption was obtained Caffarelli and Souganidis in [8] for nonlinear elliptic equations. To our knowledge, Theorem 1.2 is the first error estimate for nonlinear parabolic equations that are neither convex nor concave.

Our paper is structured as follows. In Section 2 we establish notation, give the definition of $\delta$-viscosity solutions, and state several known results. In Section 3 we prove Theorem 3.2, the regularity result that is essential for the rest of our arguments. In Section 4 we discuss regularizations of inf-sup type and establish Proposition 4.3, a version of Theorem 3.2 for $x$-sup and $x$-inf convolutions. In Section 5 we prove a key estimate between viscosity solutions of (31) and sufficiently regular $\delta$-solutions of (31). Theorems 1.1 and 1.2 are straightforward consequences of this estimate. In Section 6 we give the precise statement and proof of Theorem 1.1. Section 7 is devoted to introducing the necessary notation and stating known results about approximation schemes. The precise statement and proof of the error estimate for approximation schemes is in Section 8. We also include an appendix with known results about inf- and sup-convolutions.

2. Preliminaries

In this section we establish notation, recall the definition of viscosity solutions for parabolic equations, give the definition of $\delta$-viscosity solutions, and recall some known regularity results for solutions of (1).

2.1. Notation. Points in $\mathbb{R}^{n+1}$ are denoted $(x, t)$ with $x \in \mathbb{R}^n$. The parabolic distance between two points is

$$d((x, t), (y, s)) = (|x - y|^2 + |s - t|)^{1/2}.$$ 

We denote the usual Euclidean distance in $\mathbb{R}^{n+1}$ by $d_e((x, t), (y, s)) = (|x - y|^2 + |s - t|^2)^{1/2}$. Throughout the argument we consider parabolic cubes, denoted by

$$Q_r(x, t) = [x - r, x + r]^n \times (t - r^2, t],$$

and forward and backward cylinders,

$$Y_r(x, t) = B_r(x) \times (t, t + r^2) \text{ and } Y_r^-(x, t) = B_r(x) \times (t - r^2, t],$$

where $B_r(x) = \{ y : |x - y| < r \}$ is the open ball in $\mathbb{R}^n$. We write $B_r, Q_r, Y_r$ and $Y_r^-$ to mean $B_r(0), Q_r(0), Y_r(0)$ and $Y_r^-(0)$, respectively.

For $\Omega \subset \mathbb{R}^n$, the parabolic boundary of $\Omega \times (a, b)$ is defined as

$$\partial_p(\Omega \times (a, b)) = (\Omega \times \{ a \}) \cup (\partial \Omega \times (a, b)).$$
The time derivative of a function $u$ is denoted by $u_t$ or $\partial_t u$. The gradient of $u$ with respect to the space variable $x = (x_1, ..., x_n)$ is denoted by $Du = (u_{x_1}, ..., u_{x_n})$, and the Hessian of $u$ with respect to $x$ is denoted by $D^2 u$.

We say that a constant is universal if it is positive and depends only on $\Lambda$, $\lambda$, and $n$. Unless otherwise indicated, all constants depend on $\Lambda$, $\lambda$, and $n$.

We introduce notation for several families of paraboloids.

**Definition 2.1.** Let $M > 0$. We define:

1. the class of convex paraboloids of opening $M$:
   $$\mathcal{P}_M^+ = \left\{ P(x,t) = c + l \cdot x + mt + \frac{M}{2} |x|^2 \text{ where } l \in \mathbb{R}^n, c, m \in \mathbb{R}, |m| \leq M \right\};$$

2. the class of concave paraboloids of opening $M$:
   $$\mathcal{P}_M^- = \left\{ -P(x,t) \big| P \in \mathcal{P}_M^+ \right\};$$

3. the class of paraboloids of arbitrary opening:
   $$\mathcal{P}_\infty = \left\{ P(x,t) = c + l \cdot x + mt + x \cdot Qx^T \text{ where } l \in \mathbb{R}^n, c, m \in \mathbb{R}, Q \in \mathcal{S}_{n \times n} \right\};$$

4. the set of polynomials that are quadratic in $x$ and linear in $t$ with the only mixed term of the form $a \cdot xt$:
   $$\mathcal{P} = \left\{ P(x,t) = c + l \cdot x + mt + a \cdot xt + x \cdot Qx^T \text{ where } l, a \in \mathbb{R}^n, c, m \in \mathbb{R}, Q \in \mathcal{S}_{n \times n} \right\}.$$

We will make use of the following seminorms, norms, and function spaces:

**Definition 2.2.** The class of continuous functions on $U \subset \mathbb{R}^{n+1}$ is denoted $C(U)$. The class $C^2(U)$ is the set of functions $\phi$ that are differentiable in $t$ and twice differentiable in $x$, with $\phi_t \in C(U)$ and $D^2 \phi \in C(U)$.

**Definition 2.3.** For $\eta \in (0, 1]$ and $u \in C(\Omega \times (a, b))$, we define:

$$[u]_{C^{0,\eta}(\Omega \times (a, b))} = \sup_{(x,t), (y,s) \in \Omega \times (a, b)} \frac{|u(x,t) - u(y,s)|}{d((x,t), (y,s))^\eta},$$

$$[u]_{C^{0,\eta}_x(\Omega \times (a, b))} = \sup_{x,y \in \Omega \times (a, b)} \frac{|u(x,t) - u(y,t)|}{|x - y|^\eta},$$

$$[u]_{C^{0,\eta}_t(\Omega \times (a, b))} = \sup_{t,s \in (a,b): x \in \Omega} \frac{|u(x,t) - u(x,s)|}{|t - s|^\eta},$$

and

$$\|u\|_{C^{0,\eta}(\Omega \times (a, b))} = \|u\|_{L^\infty(\Omega \times (a, b))} + [u]_{C^{0,\eta}(\Omega \times (a, b))}.$$
2.2. Viscosity and $\delta$-viscosity solutions. We recall the definition of viscosity solutions for parabolic equations.

**Definition 2.6.**

1. A function $u \in C(\Omega \times (0, T))$ is a **viscosity supersolution** of (1) if for all $(x, t) \in \Omega \times (0, T)$, any $\phi \in C^2(\Omega \times (0, T))$ with $\phi \leq u$ on $Y^-_\rho(x, t)$ for some $\rho > 0$ and $\phi(x, t) = u(x, t)$ satisfies
   
   $$\phi_t(x, t) - F(D^2\phi(x, t)) \geq 0.$$ 

2. A function $u \in C(\Omega \times (0, T))$ is a **viscosity subsolution** of (1) if for all $(x, t) \in \Omega \times (0, T)$, any $\phi \in C^2(\Omega \times (0, T))$ with $\phi \geq u$ on $Y^-_\rho(x, t)$ for some $\rho > 0$ and $\phi(x, t) = u(x, t)$ satisfies
   
   $$\phi_t(x, t) - F(D^2\phi(x, t)) \leq 0.$$ 

3. We say that $u \in C(\Omega \times (0, T))$ is a **viscosity solution** of (1) if $u$ is both a sub- and a super-solution of (1).

**Remark 2.7.** In the above definitions, we require the test function $\phi$ to stay above (or below) $u$ on a backward cylinder $Y^-_\rho(x, t)$. However, this definition is equivalent to the usual one, which requires the test function to stay above (or below) $u$ on a Euclidean open set around $(x, t)$. This equivalence is proven by Crandall, Kocan, and Święch [10, Lemma 1.4] for $L^p$ viscosity solutions. Their proof carries over into our setting with no modifications.

See Section 8 of Crandall, Ishii, and Lions’ [11] for further discussion of viscosity solutions of parabolic equations.

We now introduce $\delta$-viscosity solutions for (1), following the definition of [8] and [9].

**Definition 2.8.** Fix $\delta > 0$.

1. A function $v \in C(\Omega \times (0, T))$ is a **$\delta$-viscosity supersolution** of (1) if for all $(x, t) \in \Omega \times (0, T)$ such that $Y^-_\delta(x, t) \subset \Omega \times (0, T)$, any $P \in {\mathcal{P}}_{\infty}$ with $P \leq v$ on $Y^-_\delta(x, t)$ and $P(x, t) = v(x, t)$ satisfies
   
   $$P_t - F(D^2P) \geq 0.$$ 

2. A function $v \in C(\Omega \times (0, T))$ is a **$\delta$-viscosity subsolution** of (1) if for all $(x, t) \in \Omega \times (0, T)$ such that $Y^-_\delta(x, t) \subset \Omega \times (0, T)$, any $P \in {\mathcal{P}}_{\infty}$ with $P \geq v$ on $Y^-_\delta(x, t)$ and $P(x, t) = v(x, t)$ satisfies
   
   $$P_t - F(D^2P) \leq 0.$$ 

3. We say $v$ is a **$\delta$-viscosity solution** of (1) if it is both a $\delta$-viscosity sub- and super-solution.

From now on we will say “solution” to mean viscosity solution and “$\delta$-solution” to mean $\delta$-viscosity solution.

From the definitions, it is clear that a viscosity solution of (1) is a $\delta$-solution of (1). The main difference between the definitions of viscosity and $\delta$-viscosity solution is that for $v$ to be a $\delta$-supersolution (resp. subsolution), any test polynomial must stay below (resp. above) $v$ on a set of fixed size.

2.3. Known results. We introduce the Pucci extremal operators, which are defined for constants $0 < \lambda \leq \Lambda$ and $X \in {\mathcal{S}}_{n \times n}$ by

$$M^-_{\lambda, \Lambda}(X) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i,$$

$$M^+_{\lambda, \Lambda}(X) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,$$
where the $e_i$ are the eigenvalues of $X$ (see Caffarelli and Cabre \cite{17} and Wang \cite{25}). We also introduce the so-called upper and lower monotone envelopes of a function, which play the role of the convex and concave envelopes in the regularity theory of elliptic equations. We follow the notes of Imbert and Silvestre in \cite{13} Section 4 for our presentation of Definition 2.9, Lemma 2.10 and Proposition 2.11 below.

**Definition 2.9.** Let $\Omega$ be a convex subset of $\mathbb{R}^n$ and let $u : \Omega \times (a, b) \to \mathbb{R}$ be continuous.

1. The **lower monotone envelope** of $u$ is the largest function $v : \Omega \times (a, b) \to \mathbb{R}$ that lies below $u$ and is non-increasing with respect to $t$ and convex with respect to $x$. It is often denoted by $\underline{\Gamma}(u)$.

2. The **upper monotone envelope** of $u$ is the smallest function $v : \Omega \times (a, b) \to \mathbb{R}$ that lies above $u$ and is non-decreasing with respect to $t$ and concave with respect to $x$. It is often denoted by $\overline{\Gamma}(u)$.

We have the following representation formulas for the upper and lower envelopes of $u$ (\cite{13} Section 4):

**Lemma 2.10.** Assume $u \in C(Y^\rho_-^-)$. Then

$$\underline{\Gamma}(u)(x, t) = \sup \{ \zeta \cdot x + h : \zeta \cdot y + h \leq u(s, y) \text{ for all } s \in (a, t), y \in Y^\rho_-^- \}$$

and

$$\overline{\Gamma}(u)(x, t) = \inf \{ \zeta \cdot x + h : \zeta \cdot y + h \geq u(s, x) \text{ for all } s \in (a, t), y \in Y^\rho_-^- \}.$$

The following fact will play a central role in our arguments. In subsection 4.1.1 of the Appendix, we explain how Proposition 2.11 follows from parts of the proof of the parabolic version of the Alexandroff-Bakelman-Pucci estimate as presented in \cite{13} Section 4.1.2. We refer the reader to \cite{13} Section 4 for a history of the parabolic version of the Alexandroff-Bakelman-Pucci estimate and the relevant references.

**Proposition 2.11.** Assume $u \in C(Y^\rho_-^-)$ is such that $u \geq 0$ on $\partial^\rho Y^\rho_-^-$. Assume that there exists a constant $K$ so that $[u]_{C^{0, \alpha}(\partial^\rho Y^\rho_-^-)} \leq K$ and $D^2 u(x, t) \leq K I$ in the sense of distributions for every $(x, t) \in Y^\rho_-^-$. Let $\Gamma$ be the lower monotone envelope of $-u^-$ in $Y^\rho_-^-$. There exists a universal constant $C$ such that

$$\sup_{Y^\rho_-^-} u^- \leq C \rho^{n+\alpha} \|\{ u = \Gamma \} \|^{\frac{1}{n+\alpha}} K.$$

Next we state several regularity results. We will use the following interior Hölder gradient estimate \cite{20} Theorems 4.8 and 4.9.

**Theorem 2.12.** Assume (F1) and (F2). There exist universal constants $\alpha \in (0, 1)$ and $C$ such that if $u \in C(Q_1)$ is a solution of $u_t - F(D^2 u) = 0$ in $Q_1$, then $u \in C^{0, \alpha}(Q_1)$ and $Du \in C^{0, \alpha}(Q_1)$ with

$$\|Du\|_{C^{0, \alpha}(Q_1/2)} + \|u_t\|_{C^{0, \alpha}(Q_1/2)} \leq C(\|u\|_{L^\infty(Q_1)} + 1).$$

The following proposition follows from Theorem 2.12 by a standard rescaling and covering argument.

**Proposition 2.13.** Assume (F1) and (F2). Assume that $V$ is a subset of $\Omega \times (0, T)$ with $d(V, \partial_\Omega \times (0, T)) > r$. There exist universal constants $\alpha \in (0, 1)$ and $C$ such that if $u \in C(\Omega \times (0, T))$ is a solution of $u_t - F(D^2 u) = 0$ in $\Omega \times (0, T)$, then

$$r^{1+\alpha} \|Du\|_{C^{0, \alpha}(V)} + r \|Du\|_{L^\infty(V)} \leq C(\|u\|_{L^\infty(\Omega \times (0, T))} + 1),$$

$$r^{2+\alpha} \|u_t\|_{C^{0, \alpha}(V)} + r^2 \|u_t\|_{L^\infty(V)} \leq C(\|u\|_{L^\infty(\Omega \times (0, T))} + 1).$$
3. The regularity result

In this section, we establish the parabolic version of the regularity result [8, Theorem A], which says that outside of sets of small measure, solutions of uniformly elliptic equations have second-order expansions with controlled error.

**Definition 3.1.** Given $u \in C(Y_1)$, we define the set $\Psi_M(u, Y_1) \subset Y_1$ by

$$\Psi_M(u, Y_1) = \{(x, t) \in Y_1 : \text{there exists } P \in \mathcal{P} \text{ such that for all } (y, s) \in Y_1 \cap \{s \leq t\},$$

$$|u(y, s) - u(x, t) - P(y, s)| \leq nM(|x - y|^3 + |x - y|^2|t - s| + |t - s|^2).$$

In order to state our result, we introduce another family of subsets of $\mathbb{R}^{n+1}$. We define

$$K_r(x, t) = \left[x - \frac{r}{9\sqrt{n}}, x + \frac{r}{9\sqrt{n}}\right]^n \times (t, t + \frac{r^2}{81n})$$

We denote $K_r(0, 0)$ by $K_r$.

We prove the following growth estimate for $|\Psi_M(u, Y_1)|$:

**Theorem 3.2.** Assume $F$ satisfies (I) and (II). Assume $u$ is a solution of $u_t - F(D^2u) = 0$ in $Y_1$ and $Du$ and $u_t$ exist and are continuous in $Y_1$. There exist universal constants $C, M_0$ and $\sigma$ such that for every $M \geq M_0$,

$$K_r(x, t) \setminus \Psi_M(u, Y_1) \leq CM^\sigma \left(||Du||_{L^\infty(Y_1)} + ||u_t||_{L^\infty(Y_1)}\right).$$

**Remark 3.3.** If $u$ is a solution of $u_t - F(D^2u) = 0$ in $\Omega \times (0, T)$ and $Y_1$ is compactly contained in $\Omega \times (0, T)$, then $u$ satisfies the hypotheses of Theorem 3.2 because, according to Proposition 2.13, $Du$ and $u_t$ exist and are continuous in $Y_1$.

**Remark 3.4.** Suppose $(x, t) \in \Psi_M(u, Y_1)$ and $P$ is the paraboloid given by the definition of $\Psi_M(u, Y_1)$. If $u$ is a viscosity solution of $u_t - F(D^2u) = 0$ in $Y_1$, then of course $P_t - F(D^2P) = 0$. On the other hand, suppose $u$ is only a $\delta$-solution of $u_t - F(D^2u) = 0$ in $Y_1$, and $(x, t) \in \Psi_M(u, Y_1)$ with $Y_\delta(x, t) \subset Y_1$. In this case we cannot conclude that $P_t - F(D^2P) = 0$. This is because in the definition of $\Psi_M(u, Y_1)$ we consider paraboloids in the larger class $\mathcal{P}$, while in the definition of $\delta$-solution we allow only paraboloids in the smaller class $\mathcal{P}_\infty$. In Corollary 3.7 we establish a version of Theorem 3.2 with $P \in \mathcal{P}_\infty$ instead of $P \in \mathcal{P}$.

An essential element of our proof is [25, Theorem 4.11], which says that solutions of uniformly parabolic equations have first order expansions with controlled error on large sets. To state this result, we recall the following definitions:

$$\mathcal{G}^M_M(u, Y_1) = \{(x, t) \in Y_1 : \text{there exists } P \in \mathcal{P}_M^+ \text{ with } P(x, t) = u(x, t) \text{ and } P \geq u \text{ on } Y_1 \cap \{s \leq t\}\},$$

$$\mathcal{G}^M(u, Y_1) = \mathcal{G}^M_M(-u, Y_1), \text{ and}$$

$$G_M(u, Y_1) = \mathcal{G}^M_M(u, Y_1) \cap \mathcal{G}^M_M(u, Y_1).$$

We observe that if $(x, t) \in G_M(u, Y_1)$, then there exists $p \in \mathbb{R}^n$ such that for all $(y, s) \in Y_1 \cap \{s \leq t\},$

$$|u(y, s) - u(x, t) - p \cdot (y - x)| \leq \frac{1}{2}M|x - y|^2 + M|t - s|.$$

We now present [25, Theorem 4.11], with notation adapted for our setting:

**Theorem 3.5.** Assume (I) and (II). Assume that $v$ is a solution of

$$v_t - M^-(D^2v) \geq 0,$$

$$v_t - M^+(D^2v) \leq 0$$

for $|x - y|^3 + |x - y|^2|t - s| + |t - s|^2 \leq \frac{r^2}{9\sqrt{n}}$. Then

$$v(x, t) \geq u \text{ on } Y_1 \cap \{s \leq t\}.$$
in $Y_1$. There exist universal constants $C$, $\sigma$, and $M_0$ such that for any $M \geq M_0$,

$$|K_1 \setminus G_M(v, Y_1)| \leq CM^{-\sigma}||v||_{L^{\infty}(Y_1)}^\sigma.$$  

The main idea of the proof of Theorem 3.2 is to apply the growth estimate of Theorem 3.5 to the derivatives of $u$. Formally, upon differentiating $u_t - F(D^2u) = 0$, we obtain that the derivatives $u_x$ and $u_t$ solve the linear parabolic equation $\partial_t u_x - \text{tr}(DF \cdot D^2u_x) = 0$. Therefore, the estimates of Theorem 3.5 apply to $u_x$ and $u_t$, and from this the estimates on $u$ are deduced. To carry out this plan, we will first use the following proposition [26, Theorem 4.6].

**Proposition 3.6.** Assume that $u$ is a solution of $u_t - F(D^2u) = 0$ in $Y_1$. Then $u_t$ and $u_x$, for $i = 1, \ldots, n$, satisfy

$$\begin{cases}
    v_t - M^+(D^2v) \geq 0 \\
    v_t - M^-(D^2v) \leq 0
\end{cases}$$

in $Y_1$.

Next we proceed with the proof of Theorem 3.2. We remark that a result similar to [8, Theorem A] was established later, via similar methods, by Armstrong, Silvestre and Smart [11, Section 5]. Our proof is based on the arguments in [8, Theorem A] and our presentation follows that of [11, Section 5].

We proceed with:

**Proof of Theorem 3.2.** By Proposition 3.6, the derivatives $u_x$, $i = 1, \ldots, n$ and $u_t$ satisfy (6). Therefore, by Theorem 3.5 for any $M \geq M_0$ the size of the “bad sets” of the $u_x$ and of $u_t$ are controlled: we have

$$|K_1 \setminus G_M(u_x, Y_1)| \leq CM^{-\sigma} ||Dv||^\sigma_{L^{\infty}(Y_1)}, \quad i = 1, \ldots, n,$$

$$|K_1 \setminus G_M(u_t, Y_1)| \leq CM^{-\sigma} ||u_t||^\sigma_{L^{\infty}(Y_1)},$$

where $C$ is a universal constant. We define $G_M$ by

$$G_M = \cap_{i=1}^n G_M(u_x, Y_1) \cap G_M(u_t, Y_1).$$

We will show

$$(G_M \cap K_1) \subset (\Psi_M(u, Y_1) \cap K_1),$$

so that, by (3),

$$|K_1 \setminus \Psi_M(u, Y_1)| \leq |K_1 \setminus G_M| \leq CM^{-\sigma} \left(||Dv||^\sigma_{L^{\infty}(Y_1)} + ||u_t||^\sigma_{L^{\infty}(Y_1)}\right).$$

To this end, fix $(x, t) \in G_M \cap K_1$. The goal is to show that $(x, t) \in \Psi(u, Y_1(x, \bar{t})) \cap K_1(x, \bar{t})$; in other words, to produce a polynomial $P$ that is the second order expansion of $u$ at $(x, t)$. Because $(x, t) \in G_M$, there exist $p^1, \ldots, p^{n+1} \in \mathbb{R}^n$ such that for $i = 1, \ldots, n$ and for all $(y, s) \in Y_1 \cap \{s \leq t\}$,

$$|u_{x_i}(y, s) - u_{x_i}(x, t) - p_i \cdot (y - x)| \leq \frac{1}{2} M|x - y|^2 + M |t - s|,$$

and

$$|u_t(y, s) - u_t(x, t) - p^{n+1} \cdot (y - x)| \leq \frac{1}{2} M|x - y|^2 + M |t - s|.$$  

We point out that $Du(x, t) = (p^1, \ldots, p^n)$ and $u_t(x, t) = p^{n+1}$. We define the $n \times n$ matrix $Q$ by $Q_{ij} = p'_i$ and define the paraboloid $P \in \mathcal{P}$ by

$$P(y, s) = Du(x, t) \cdot (y - x) + \frac{1}{2}(y - x)Q(y - x)^T + (s - t)u_t(x, t) + \frac{1}{2} p^{n+1} \cdot (y - x)(s - t).$$
We will now show that this polynomial $P$ is the second order expansion of $u$ at $(x, t)$. We have

$$u(y, s) - u(x, t) = \int_0^1 (y - x) \cdot Du(x + \tau(y - x), t + \tau(s - t)) d\tau + \int_0^1 (s-t)u_t(x + \tau(y - x), t + \tau(s - t)) d\tau.$$ 

Therefore,

$$u(y, s) - u(x, t) - P(y, s) = (y - x) \cdot \int_0^1 (Du(x + \tau(y - x), t + \tau(s - t)) - Du(x, t) - \tau Q(y - x)^T) d\tau + (s-t) \int_0^1 u_t(x + \tau(y - x), t + \tau(s - t)) - u_t(x, t) - \tau p^{n+1} \cdot (y - x) d\tau.$$ 

We take the absolute value both sides of the above and find

$$|u(y, s) - u(x, t) - P(y, s)| \leq |y - x| \int_0^1 \left( \sum_{i=1}^n |u_{x_i}(x + \tau(y - x), t + \tau(s - t)) - u_{x_i}(x, t) - \tau p^j \cdot (y - x)|^2 d\tau \right)^{1/2} + |s - t| \int_0^1 |u_t(x + \tau(y - x), t + \tau(s - t)) - u_t(x, t) - \tau p^{n+1} \cdot (y - x)| d\tau.$$ 

We use (7) to bound each term on the right-hand side of the above, and obtain

$$|u(y, s) - P(y, s)| \leq n|y - x| \int_0^1 \frac{1}{2} M \tau^2 |x - y|^2 + M \tau |t - s| d\tau + |s - t| \int_0^1 \frac{1}{2} M \tau^2 |x - y|^2 + M \tau |t - s| d\tau.$$ 

Integrating in $\tau$ and combining terms yields

$$|u(y, s) - P(y, s)| \leq nM (|x - y|^3 + |x - y|^2 |t - s| + |t - s|^2).$$

\[ \square \]

**Corollary 3.7.** Assume $F$ satisfies (F1) and (F2). Assume $u$ is a solution of $u_t - F(D^2 u) = 0$ in $Y_r$ and $Du$ and $u_t$ exist and are continuous in $Y_r$. There exists a universal constant $C$ such that for any $(x, t) \in \Psi_M(u, Y_r)$, there exists $Q \in P_\infty$ with

$$Q_t - F(D^2 Q) = 0$$

and such that on $Y_r \cap \{ s \leq t \}$,

$$|u(y, s) - u(x, t) - Q(y, s)| \leq \frac{nM}{r^2} (r|x - y|^3 + |x - y|^2 |t - s| + |t - s|^2) + C \frac{M + ||u_t||_{L^\infty(Y_r)} \cdot |x - y|(t - s)}{r}.$$ 

Moreover,

$$|K_r(\bar{x}, \bar{t}) \setminus \Psi_M(u, Y_r(\bar{x}, \bar{t}))| \leq \frac{C r^{n+2}}{M^\sigma} (r^{-\sigma} ||Du||_{L^\infty(Y_r)} + ||u_t||_{L^\infty(Y_r)}).$$

\[ \square \]
Proof. We prove the statement for \( r = 1 \); the general case follows by rescaling: given \( u \) that solves \( u_t - F(D^2 u) = 0 \) in \( Y_r \), the rescaled function \( \hat{u}(\hat{x}, \hat{t}) = \frac{1}{r} u(rx, rt) \) solves the same equation in \( Y_1 \). We omit the details of the rescaling argument and proceed with the proof in the case \( r = 1 \).

Fix \( (x, t) \in \Psi_M(u, Y_1) \). Let \( Q(y, s) = P(y, s) - \frac{1}{2} p^{n+1} \cdot (y - x)(s - t) \), where \( P \) and \( p^{n+1} \) are as in the proof of Theorem 3.2. It holds that for all \( (y, s) \in Y_1 \cap \{ s \leq t \} \),

\[
|u(y, s) - u(x, t) - Q(y, s)| \leq nM(|x - y|^3 + |x - y|^2|t - s| + |t - s|^2) + \frac{1}{2} |p^{n+1}| |y - x||s - t|.
\]

From (7) we see that for any \( (y, s) \in Y_1 \cap \{ s \leq t \} \),

\[
|p^{n+1} \cdot (y - x)| \leq \frac{1}{2} nM|x - y|^2 + M|t - s| + 2||u_t||_{L^\infty(Y_1)}.
\]

Taking \( y \) such that \( |x - y| = \frac{1}{2} \) and \( s = t \) implies

\[
|p^{n+1}| \leq \frac{1}{4} M + 4||u_t||_{L^\infty(Y_1)}.
\]

The bound (8) follows. \( \square \)

4. Two regularizations of inf- sup- type

4.1. Regularization in both the space and time variables. We recall the definitions of the inf-convolution \( v_{\theta, \theta}^- \) and the sup-convolution \( v_{\theta, \theta}^+ \), for \( v \in C(\Omega \times (0, T)) \):

\[
v_{\theta, \theta}^-(x, t) = \inf_{\Omega \times (0, T)} \left\{ v(y, s) + \frac{|x - y|^2}{2\theta} + \frac{|t - s|^2}{2\theta} \right\}
\]

and

\[
v_{\theta, \theta}^+(x, t) = \sup_{\Omega \times (0, T)} \left\{ v(y, s) - \frac{|x - y|^2}{2\theta} - \frac{|t - s|^2}{2\theta} \right\}.
\]

These definitions are quite similar to those used in the regularity theory of elliptic equations, and to the regularizations of [20] Section 4. We use the notation \( v_{\theta, \theta}^- \), instead of the expected \( v_{\theta}^- \), to distinguish these from the \( x \)-inf- and \( x \)-sup- convolutions that we introduce in the next subsection. We summarize the basic properties of inf- and sup- convolutions in Proposition 9.3 of the appendix.

4.2. Regularization in only the space variable. In this subsection we define a regularization of inf-sup type that we call \( x \)-inf and \( x \)-sup convolutions. In the next subsection, we prove that if \( u \) is a solution of \( u_t - F(D^2 u) = 0 \), then the \( x \)-inf and \( x \)-sup convolutions of \( u \) enjoy regularity properties similar to those established in Theorem 3.2 for \( u \).

Definition 4.1. For \( u \in C(\Omega \times (0, T)) \) and \( \theta > 0 \), we define the \( x \)-sup convolution \( u_{\theta}^+ \) and \( x \)-inf-convolution \( u_{\theta}^- \) of \( u \) by

\[
u_{\theta}^+(x, t) = \sup_{y \in \Omega} \left\{ u(y, t) - \frac{|x - y|^2}{2\theta} \right\} \quad \text{and} \quad u_{\theta}^-(x, t) = \inf_{y \in \Omega} \left\{ u(y, t) + \frac{|x - y|^2}{2\theta} \right\}.
\]

For \( u \in C^{0,1}_x(\Omega \times (0, T)) \), we define the set \( U_{x}^\theta \) by

\[
U_{x}^\theta = \left\{ (x, t) \in \Omega \times (0, T) \mid \inf_{y \in \partial \Omega} |x - y| \geq 2\theta |u|_{C^{0,1}_x(\Omega \times (0, T))} \right\}.
\]

In the following proposition we state the facts about \( x \)-inf- and \( x \)-sup- convolutions that we will use in this paper. Their proofs are very similar to those in the elliptic case and for the standard parabolic inf- and sup- convolution (see [9] Chapter 4 or [7] Proposition 5.3) and we omit them.
Proposition 4.2. Assume $u \in C^{0,1}_\theta(\Omega \times (0, T))$. Then:

(1) If $(x^*, t)$ is any point at which the infimum (resp. supremum) is achieved in the definition of $u_\theta^-(x, t)$ (resp. $u_\theta^+(x, t)$), then

$$|x - x^*| \leq 2\theta||Du||_{L^\infty(\Omega \times (0, T))}.$$  

(2) For $(x, t) \in U_\theta^0$, 

$$u_\theta^+(x, t) \leq u(x, t) + 2\theta||Du||^2_{L^\infty(\Omega \times (0, T))}$$

and

$$u_\theta^-(x, t) \geq u(x, t) - 2\theta||Du||^2_{L^\infty(\Omega \times (0, T))}.$$  

(3) In the sense of distributions, $D^2u_\theta^+(x, t) \geq -\theta^{-1}I$ and $D^2u_\theta^-(x, t) \leq \theta^{-1}I$ for all $(x, t) \in \Omega \times (0, T)$.

(4) If $u$ is a subsolution of $u_t - F(D^2u) = c$ in $\Omega \times (0, T)$, then $u_\theta^+$ is a subsolution of $u_t - F(D^2u) = c$ in $U_\theta^0$. If $u$ is a supersolution of $u_t - F(D^2u) = c$ in $\Omega \times (0, T)$, then $u_\theta^-$ is a supersolution of $u_t - F(D^2u) = c$ in $U_\theta^0$.

4.3. Regularity of $x$-inf and $x$-sup convolutions. Next we prove that the extra regularity established in Theorem 4.2 for a solution $u$ of $u_t - F(D^2u) = 0$ carries over to $u_\theta^+$ and $u_\theta^-$. To state this result we introduce the following families of sets:

$$Y_\theta^{1,0}(\bar{x}, \bar{t}) = B(\bar{x}, r - 2\theta||Du||_{L^\infty(\Omega \times (0, T))}) \times (\bar{t}, \bar{t} + r^2)$$

$$K_\theta^{1,0}(\bar{x}, \bar{t}) = \left[\bar{x} - \left(\frac{r}{9\sqrt{n}} - 2\theta||Du||_{L^\infty(\Omega \times (0, T))}\right), \bar{x} + \left(\frac{r}{9\sqrt{n}} - 2\theta||Du||_{L^\infty(\Omega \times (0, T))}\right)\right] \times (\bar{t}, \bar{t} + r^2).$$

(Note that for these definitions to be meaningful, $r$ cannot be too small compared to $\theta$. Whenever we use this notation we make sure this is not the case.)

Proposition 4.3. Assume (F1) and (F2). Suppose $u \in C(\Omega \times (0, T))$ is a solution of $u_t - F(D^2u) = 0$ in $\Omega \times (0, T)$. Assume $Y_\theta^{1,0}(\bar{x}, \bar{t}) \subset \Omega \times (0, T)$ and

$$d(Y_\theta^{1,0}(\bar{x}, \bar{t}), \partial_p(\Omega \times (0, T))) \geq \theta.$$  

There exist universal constants $M_0, \sigma, \alpha$, and $C$ such that for every $M \geq M_0$, there exists a set $\Psi_\theta^+ \subset K_\theta^{1,0}(\bar{x}, \bar{t})$ (resp. $\Psi_\theta^- \subset K_\theta^{1,0}(\bar{x}, \bar{t})$) such that for any $x, t \in \Psi_\theta^+$ (resp. $x, t \in \Psi_\theta^-$), there exists a polynomial $P \in \mathcal{P}_\infty$ such that $P(x, t) = 0$,

$$P_t - F(D^2P) = 0,$$

and if $Y_\theta^-(x, t) \subset Y_\theta^{1,0}(\bar{x}, \bar{t})$, then for all $(y, s) \in Y_\theta^-(x, t)$, we have

$$u_\theta^+(y, s) - u_\theta^+(x, t) \geq P(y, s) - \frac{c(n)}{r^2}M(\rho|x - y|^2 + \rho^2|t - s|)$$

$$- C\rho \left(\frac{M + \theta^{-1}(1 + ||u||_{L^\infty(\Omega \times (0, T))})}{r} (t - s)\right).$$

(resp. $u_\theta^-(y, s) - u_\theta^-(x, t) \leq P(y, s) + \frac{c(n)}{r^2}M(\rho|x - y|^2 + \rho^2|t - s|)$$

$$+ C\rho \left(\frac{M + \theta^{-1}(1 + ||u||_{L^\infty(\Omega \times (0, T))})}{r} (t - s)\right).$$

Moreover,

$$|K_\theta^{1,0}(\bar{x}, \bar{t}) \setminus \Psi_\theta^+| \leq \frac{C r^{n+2}}{M^\sigma \theta^{1+\sigma}}(r^{-\sigma} + \theta^{-\sigma}) (||u||_{L^\infty(\Omega \times (0, T))} + 1)^{1+\sigma}.$$
Although the proof of Proposition 4.3 is quite involved, it is based on the following simple observation: if the supremum in the definition of \( u^+_\theta(x,t) \) is achieved at \((x^*,t)\) and \(u\) is touched from below at \((x^*,t)\) by some function \(\phi\), then \(u^+_\theta\) is touched by below at \((x,t)\) by a translate of \(\phi\). We demonstrate this below.

Let \((x^*,t)\in\Omega\times(0,T)\) be a point at which the supremum is achieved in the definition of \(u^+_\theta(x,t)\), so that

\[
u^+_\theta(x,t) = u(x^*,t) - \frac{|x-x^*|^2}{2\theta}.
\]

Suppose that \(\phi \in C(\Omega \times (0,T))\) is a function with \(u(x^*,t) = \phi(x^*,t)\) and \(u \geq \phi\) on \(\Omega \times (0,t)\). This implies

\[
\frac{\phi}{\phi - u} = \frac{\phi^2 - u\phi}{\phi^2 - \phi u} = \frac{\phi}{\phi - u}.
\]

Fix \(s \leq t\). If \(y \in \Omega\) is sufficiently far from the boundary of \(\Omega\), then we may use \((y + x - x^*)\) as a “test point” in the definition of \(u^+_\theta(y,s)\) and find

\[
u^+_\theta(y,s) \geq u(y + x^* - x,t) - \frac{|x-x^*|^2}{2\theta}.
\]

But, since \(u \geq \phi\) on \(\Omega \times (0,t)\), we have

\[
u^+_\theta(y,s) \geq \phi(y + x^* - x,s) - \frac{|x-x^*|^2}{2\theta}.
\]

Subtracting (13) from the above implies

\[
u^+_\theta(y,s) - \nu^+_\theta(x,t) \geq \phi(y + x^* - x,s) - \phi(x^*,t),
\]

so \(u_\theta^+(y,s)\) is touched from below at \((x,t)\) by \(\phi(y) = \phi(y + x^* - x,s) - \phi(x^*,t) + \nu^+_\theta(x,t)\).

For the proof of Proposition 4.3 we consider the map that takes \(x^*\) to \(x\) and investigate what happens to “good set” \(\Psi_M(u)\) under this map. To do this, we need the following lemma, which says that \(u\) is twice differentiable in \(x\) at any point \(x^*\) where the supremum or infimum is achieved in the regularizations \(u^+_\theta\) or \(u^-_\theta\), and gives a bound on \(D^2u\) at such points.

**Lemma 4.4.** Under the assumptions of Proposition 4.3 let \((x,t)\in Y^\theta_T(\bar{x},\bar{t})\). If \(x^*\) is any point where the supremum (resp. infimum) is achieved in the definition of \(u^+_\theta(x,t)\) (resp. \(u^-_\theta(x,t)\)), then \(u(y,t)\) is twice differentiable in \(y\) at \(x^*\) with

\[
|D^2u(x^*,t)| \leq C \frac{\theta}{\theta} \left(1 + \frac{||u||_{L^\infty(\Omega \times (0,T))}}{\theta} \right).
\]

We postpone the proof of Lemma 4.4 and proceed with the proof of Proposition 4.3.

**Proof of Proposition 4.3.** We will give the proof for \(u^+_\theta\); the arguments for \(u^-_\theta\) are similar. Fix \(M\) and, to simplify the notation, denote \(\Psi_M(u,Y_\tau(\bar{x},\bar{t}))\) by just \(\Psi_M\). We define the set \(C\) by

\[
C = \{(x^*,t)\mid x^*\text{ is a point at which the supremum in the definition of }u^+_\theta(x,t)\text{ is achieved, for some } (x,t) \in Y^\theta_T(\bar{x},\bar{t})\}.
\]

We remark that \(C \subset Y_\tau(\bar{x},\bar{t})\): if \((x^*,t)\in C\), then there exists a corresponding \((x,t)\in Y^\theta_T(\bar{x},\bar{t})\), and, by Proposition 4.2 we have \(|x-x^*| \leq 2\theta||Du||_{L^\infty(\Omega \times (0,T))}\). The definition of \(Y^\theta_T(\bar{x},\bar{t})\) thus implies that \((x,t) \in Y_\tau(\bar{x},\bar{t})\).

By Lemma 4.4 \(Du\) is Lipschitz on \(C\) with

\[
||D^2u||_{L^\infty(C)} \leq C \frac{\theta}{\theta} \left(1 + \frac{||u||_{L^\infty(\Omega \times (0,T))}}{\theta} \right).
\]
For any \((x^*, t) \in C\), the map
\[
y \mapsto \left( u(y, t) - \frac{|x - y|^2}{2 \theta} \right)
\]
has a local maximum at \(x^*\). Therefore,
\[
Du(x^*, t) + \frac{(x - x^*)}{\theta} = 0,
\]
so \(x = x^* - \theta Du(x^*, t)\). We define the map \(T\) by
\[
T(y, t) = (y - \theta Du(y, t), t).
\]
Since \(Du\) is Lipschitz on \(C\), \(T\) is too, and \((14)\) implies that for all \((y, t) \in C\),
\[
|\det DT(y, t)| \leq C \left( \frac{|u||L_{\infty}(\Omega \times (0, T)) + 1|}{\theta} \right).
\]
We define the set \(A\) by
\[
A = T(C \cap (K_r(x, \bar{t}) \setminus \Psi_M)).
\]
By the Area Formula \((12\text{ Chapter 3})\), we have
\[
|A| \leq \int_{C \cap (K_r(x, \bar{t}) \setminus \Psi_M)} |\det DT(y, t)| \, dy \, dt.
\]
The above bound on \(|\det DT(y, t)|\) and the bound \((19)\) on \(|K_r(x, \bar{t}) \setminus \Psi_M|\) therefore imply
\[
|A| \leq \frac{C_{\nu+2}}{M^2} (r^{-\sigma} ||Du||_{L_{\infty}(Y)} + ||u||_{L_{\infty}(Y)}) \left( \frac{|u||L_{\infty}(\Omega \times (0, T)) + 1|}{\theta} \right).
\]
We define the set \(\Psi_M\) by
\[
\Psi_M := K^\theta_{\bar{t}}(x, \bar{t}) \setminus A.
\]
Fix \((x, t) \in K^\theta_{\bar{t}}(x, \bar{t}) \setminus A\). We will now show that there exists a polynomial \(Q \in \mathcal{P}_\infty\) satisfying \((10)\) and \((11)\). To this end, let \((x^*, t)\) be any point at which the supremum in the definition of \(u^\theta_{\bar{t}}(x, t)\) is achieved, so that \((x, t) = T(x^*, t)\).

First we note that since, by Proposition \((12)\),
\[
|x - x^*| \leq 2\theta ||Du||_{L_{\infty}(\Omega \times (0, T))}
\]
and \((11)\), we have \((x^*, t) \in K^\theta_{\bar{t}}(x, \bar{t})\). Then \((x, t) \in T(C \cap (K_r(x, \bar{t}) \setminus \Psi_M)) = A\). However, this is impossible, as we chose \((x, t) \in \Psi_M = K^\theta_{\bar{t}}(x, \bar{t}) \setminus A\). Therefore, \((x^*, t) \in \Psi_M\).

Since \((x^*, t) \in \Psi_M\), there exists a polynomial \(Q \in \mathcal{P}_\infty\) with \(Q_k - F(D^2Q) = 0\) such that on \(Y_r(x, \bar{t}) \cap \{t \leq t\}\),
\[
|u(y, s) - u(x^*, t) - Q(y, s)| \leq \frac{nM}{r^2} (|x - y|^3 + |x - y|^2 |t - s| + |t - s|^2) + C \frac{M + ||u||_{L_{\infty}(Y)}}{r} |x - y|(t - s).
\]
It is easy to check that since \(x^*\) is a point where the supremum is achieved in the definition of \(u^\theta_{\bar{t}}(x, t)\), we have, for all \((y, s) \in Y^{\theta}_{\bar{t}}(x, \bar{t})\),
\[
u^\theta_{\bar{t}}(y, s) - u^\theta_{\bar{t}}(x, t) \geq Q(y, s) - \frac{nM}{r^2} (|x - y|^3 + |x - y|^2 |t - s| + |t - s|^2) - C \frac{M + ||u||_{L_{\infty}(Y)}}{r} |x - y|(t - s),
\]

where $\tilde{Q} \in \mathcal{P}_\infty$ is a paraboloid of the same opening as $Q$. Finally, if $Y_\rho^-(x,t) \subset Y_\rho^\theta(\bar{x},\bar{t})$, then for any $(y,s) \in Y_\rho^\theta(x,t)$,
\begin{equation}
  u_\theta^-(y,s) - u_\theta^+(x,t) \geq \tilde{Q}(y,s) - \frac{nM}{\rho^2}(\rho|x-y|^2 + \rho^2|t-s|) - C\rho M + \frac{||u_t||_{L^\infty(Y_\rho(x,t))}}{r}(t-s).
\end{equation}

Since $d(Y_\rho(x,t), \partial_\rho \Omega \times (0,T)) \geq \theta$, Proposition 2.12 implies
\[ ||u_t||_{L^\infty(Y_\rho(x,t))} \leq \theta^{-2}(||u||_{L^\infty(\Omega \times (0,T))} + 1) \]
and
\[ ||Du||_{L^\infty(Y_\rho(x,t))} \leq \theta^{-1}(||u||_{L^\infty(\Omega \times (0,T))} + 1). \]
We use these bounds in (15) and (16) to complete the proof. \hfill \Box

For the proof of Lemma 4.4, we first need to establish the following basic fact about how parabolic viscosity solutions relate to viscosity solutions in only $x$:

**Lemma 4.5.** If $u$ is a Lipschitz in $t$ viscosity solution of $u_t - F(D^2u) = 0$ in $Y_\rho$, then for any $t \in (0, r^2)$,
\begin{align*}
  \mathcal{M}^-(D^2u(\cdot,t)) &\leq ||u_t||_{L^\infty(Y_\rho)} \text{ in } B_r, \\
  \mathcal{M}^+(D^2u(\cdot,t)) &\geq -||u_t||_{L^\infty(Y_\rho)} \text{ in } B_r.
\end{align*}

**Proof.** We will check that $u(\cdot,t)$ is a subsolution; checking that $u(\cdot,t)$ is a supersolution is analogous.

Assume that $P(y)$ touches $u(y,t)$ from above at $x \in B_r$. Let $(y,s) \in B_r \times (0,t]$. Since $u$ is Lipschitz in $t$, we have
\[ u(y,s) \leq u(y,t) + ||u_t||_{L^\infty(Y_\rho)}|t-s|. \]

Therefore, $P(y) + ||u_t||_{L^\infty(Y_\rho)}(t-s)$ touches $u$ from above in $B_r \times (0,t]$. Because $u$ is a viscosity solution of $u_t - F(D^2u) = 0$, we obtain
\[ -||u_t||_{L^\infty(Y_\rho)} - F(D^2u) \leq 0. \]

The uniform ellipticity of $F$ implies
\[ 0 \geq -||u_t||_{L^\infty(Y_\rho)} - \mathcal{M}^-(D^2P), \]
as desired. \hfill \Box

We also recall the Harnack inequality for elliptic equations (see [7, Theorem 4.3]).

**Theorem 4.6.** Assume that $u : B_s(0) \subset \mathbb{R}^n \to \mathbb{R}$ satisfies $u \geq 0$ in $B_s(0)$ and, for some positive constant $c$,
\begin{align*}
  \mathcal{M}^-(D^2u) &\leq c \text{ in } B_s(0) \\
  \mathcal{M}^+(D^2u) &\geq -c \text{ in } B_s(0).
\end{align*}

There exists a universal constant $C$ so that
\[ \inf_{B_{s/2}(0)} u \geq C \left( \sup_{B_{s/2}(0)} u - s^2 c \right). \]

We are now ready to proceed with:
Proof of Lemma 4.4. We will give the proof for $u_0^+(x,t)$; the argument for $u_0^-(x,t)$ is very similar.

Let $x^*$ be a point at which the supremum in the definition of $u_0^-(x,t)$ is achieved, so that
\begin{equation}
 u(x^*, t) - \frac{|x-x^*|^2}{2\theta} \geq u(y, t) - \frac{|x-y|^2}{2\theta} \text{ for all } y \in \Omega.
\end{equation}
Therefore, as a function of $y$, $u(y, t)$ is touched from above at $x^*$ by the convex paraboloid $Q(y)$, where
\[ Q(y) = u(x^*, t) - \frac{|x-x^*|^2}{2\theta} + \frac{|x-y|^2}{2\theta}. \]
Thus, we have
\begin{equation}
 D^2u(x^*, t) \leq \theta^{-1}I.
\end{equation}
Next we will show, using the Harnack inequality, that $u(\cdot, t)$ is also touched from below at $x^*$ by a paraboloid with bounded opening, thus obtaining a bound from below on $D^2u(x^*, t)$.

There exists an affine function $l(y)$ with $l(x^*) = u(x^*, t)$ and such that
\[ Q(y) = l(y) + \frac{|y-x^*|^2}{2\theta}. \]
There exists $\rho > 0$ such that $B_\rho(x^*) \subset B_r(\bar{x})$. We fix any $0 < s < \rho$ and define the function $w(y)$ by
\[ w(y) = u(y, t) - l(y) - \frac{s^2}{2\theta}. \]
For $y \in B_s(x^*)$,
\[ w(y) \geq Q(y) - l(y) - \frac{s^2}{2\theta} = \frac{|y-x^*|^2}{2\theta} - \frac{s^2}{2\theta} \geq 0. \]
And, by Lemma 4.5, $w$ is a viscosity solution of
\[ M^{-}(D^2w(\cdot, t)) \leq ||u_t||_{L^\infty(Y_s(\bar{x}, \bar{t}))} \text{ in } B_s(x^*) \]
\[ M^{+}(D^2w(\cdot, t)) \geq -||u_t||_{L^\infty(Y_s(\bar{x}, \bar{t}))} \text{ in } B_s(x^*). \]
Therefore, the Harnack inequality implies that there exists a universal constant $C$ such that
\[ \inf_{B_{s/2}(x^*)} w \geq C \left( \sup_{B_{s/2}(x)} w - s^2||u_t||_{L^\infty(Y_s(\bar{x}, \bar{t}))} \right). \]
Since $\sup_{B_{s/2}(x)} w \geq w(x^*) = u(x^*, t) - l(x^*) - \frac{s^2}{2\theta} = -\frac{s^2}{2\theta}$, we obtain
\[ \inf_{B_{s/2}(x^*)} w \geq -Cs^2(\theta^{-1} + ||u_t||_{L^\infty(Y_s(\bar{x}, \bar{t}))}). \]
Thus, for all $y \in B_{s/2}(x^*)$,
\[ u(y, t) - Q(y) = u(y) - l(y) - \frac{|y-x^*|^2}{2\theta} \geq w(y) \geq -Cs^2(\theta^{-1} + ||u_t||_{L^\infty(Y_s(\bar{x}, \bar{t}))}). \]
Since this holds for all $0 < s < \rho$, we have that $(u(\cdot, t) - Q(\cdot))$ is touched from below at $x^*$ by a paraboloid of opening $C(\theta^{-1} + ||u_t||_{L^\infty(Y_s(\bar{x}, \bar{t}))})$. Therefore,
\[ D^2u(x^*, t) \geq (\theta^{-1} + C(\theta^{-1} + ||u_t||_{L^\infty(Y_s(\bar{x}, \bar{t}))}))I. \]
Together with (18), this implies
\[ |D^2u(x^*, t)| \leq C(\theta^{-1} + ||u_t||_{L^\infty(Y_s(\bar{x}, \bar{t}))}). \]
Finally, we use Proposition 2.3 to bound $||u_t||_{L^\infty(Y_s(\bar{x}, \bar{t}))}$ and obtain the desired bound. $\square$
5. A KEY ESTIMATE BETWEEN SOLUTIONS AND SUFFICIENTLY REGULAR δ-SOLUTIONS

In this section we prove a proposition that is a key part of the proofs of the two main results, Theorems 1.1 and 1.2. Once we have proven Proposition 5.1, we will only need to use the basic properties of inf- and sup-convolutions to verify its hypotheses and establish the two main results.

We define the constant \( \zeta \) by

\[
\zeta = \frac{\sigma}{2(3n + 4 + 2\sigma)},
\]

where \( \sigma \) is the constant from Proposition 4.3. We remark that since \( \sigma \) is universal, so is \( \zeta \).

**Proposition 5.1.** Assume (F1), (F2) and (U1) and that \( u \in C^{0,1}(\Omega \times (0, T)) \) is a solution of

\[
u_t - F(D^2u) = 0
\]

in \( \Omega \times (0, T) \). For \( \delta \in (0, 1) \), define the quantity \( r_\delta \) by

\[
r_\delta = 9\sqrt{n}(1 + 2\|Du\|_{L^\infty(\Omega \times (0, T))})\delta^\zeta
\]

and the set \( \tilde{U} \) by

\[
\tilde{U} = \{(x, t) \in \Omega \times (0, T) : t \leq T - r_\delta^2 \text{ and } d((x, t), \partial_\delta \Omega \times (0, T)) > 2r_\delta\},
\]

(1) Assume \( v \in C^{0,\eta}(\Omega \times (0, T)) \) is a \( \delta \)-supersolution of (20) in \( \Omega \times (0, T) \) that satisfies

(a) \( D^2v \leq \delta^{-\zeta}I \) in the sense of distributions, and

(b) \( [v]_{C^{0,\gamma}(\Omega \times (0, T))} \leq 3T\delta^{-\zeta}. \)

There exists a universal constant \( \alpha \), a positive constant \( \bar{c} \) that depends on \( n, \lambda, \Lambda, \text{diam}\Omega, T, \|u\|_{L^\infty(\Omega \times (0, T))} \) and \( \|Du\|_{L^\infty(\Omega \times (0, T))} \), and a positive constant \( \hat{\delta} \) that depends only on \( n \), such that, for all \( \delta \leq \hat{\delta} \),

\[
\sup_{\tilde{U}}(u - v) \leq \bar{c}\delta^\alpha + \sup_{ \partial_{\delta}\tilde{U} } (u - v).
\]

(2) Assume \( w \in C^{0,\eta}(\Omega \times (0, T)) \) is a \( \delta \)-subsolution of (21) in \( \Omega \times (0, T) \) that is twice differentiable almost everywhere in \( \Omega \times (0, T) \), with \( D^2w \geq -\delta^{-\zeta}I \) in the sense of distributions and \( [w]_{C^{0,\gamma}(\Omega \times (0, T))} \leq 3T\delta^{-\zeta}. \). Then for all \( \delta \leq \hat{\delta} \),

\[
\sup_{\tilde{U}}(w - u) \leq \bar{c}\delta^\alpha + \sup_{ \partial_{\delta}\tilde{U} } (w - u),
\]

where \( \alpha, \bar{c} \) and \( \hat{\delta} \) are the same as in part 1 of this proposition.

**Remark 5.2.** Proposition 5.1 holds in any set \( \Omega' \times (a, b) \), where \( \Omega' \) satisfies (U1). In this case, \( \tilde{U} \) becomes

\[
\tilde{U} = \{x \in \Omega' \times (a, b) : t \leq b - r_\delta \text{ and } d((x, t), \partial_\delta \Omega' \times (a, b)) > 2r_\delta\}.
\]

5.1. **Outline.** We seek to control \( \text{sup}(u - v) \) by the size of its contact set with its upper monotone envelope. We first regularize \( u \) by x-sup convolution and obtain \( u_\theta^+ \). We then perturb \( u_\theta^+ \) to make it a strict sub-solution. Let us denote the resulting function by \( \tilde{u} \). We choose this perturbation and the parameter \( \theta \) of the x-sup convolution so that the difference between \( u \) and \( \tilde{u} \) is less than \( c\delta^\alpha \). Thus it will be enough to bound \( \text{sup}(\tilde{u} - v) \). Since \( \tilde{u} \) and \( v \) are sufficiently regular, we are able to apply Proposition 2.11 and find

\[
\sup_{\tilde{U}}(\tilde{u} - v) \leq C|\{\tilde{u} - v = \bar{\Gamma}\}|^{-1/2} K,
\]

where \( \bar{\Gamma} \) is the upper monotone envelope of \( \tilde{u} - v \) and \( K \) is large. To complete the proof, we will show that the contact set is very small compared to \( K \).
The key part of our argument is Proposition 4.3 which states that there exists a subset $\Psi$ of $\Omega \times (0, T)$ on which $u_+^+$ is very close to being a polynomial. This allows us to obtain an upper bound on the size of $\Psi \cap \{\hat{u} - v = \Gamma\}$. Moreover, Theorem 5.2 says that $|\Psi|$ is large, so the size of the remainder of the contact set, $\{\hat{u} - v = \Gamma(\hat{u} - v)\} \cap \Psi^c$, is small. This completes the outline.

5.2. Proof of Proposition 5.1. For the proof of Proposition 5.1 we need a lemma.

Lemma 5.3. Under the assumptions of Proposition 5.1 assume also that $C \subset \hat{U}$ with

$$|C| \geq \tilde{c} \delta^{\zeta(n+1)}.$$

There exists a positive constant $\delta_1$ that depends only on $n$ and a positive constant $C_1$ that depends on $n$, $\lambda$, $\Lambda$, and $\|Du\|_{L^\infty(\Omega \times (0, T))}$, such that, if $\delta \leq \delta_1$, then there exists $(x_0, t_0) \in C$ and a paraboloid $P \in \mathcal{P}_\infty$ with

1. $Y_0^{-}(x_0, t_0) \subset \Omega \times (0, T)$,
2. $P(x_0, t_0) = 0$,
3. $P_t - F(D^2P) = 0$, and
4. for all $(x, t) \in Y_0^-(x_0, t_0)$,

$$u_+^+ + u_0^+ (x, t) \geq P(x, t) - C_1 \zeta^{-1/\sigma} \delta^{1/2} \psi^{(3n+1+2\sigma)}(\|x - x_0\|^2 + (t_0 - t)).$$

We postpone the proof of Lemma 5.3 and proceed with the proof of Proposition 5.1.

Proof of Proposition 5.1. We will give the proof for part 1 of the proposition; the proof of part 2 is very similar. In addition, we assume, without loss of generality, that

$$\sup_{\partial \hat{U}} (u - v) \leq 0.$$  \hfill (22)

For $\zeta$ as given in (22), we regularize $u$ by taking $x$-sup-convolution and denote $u_+^{+\zeta}$ by $u^\zeta$. We also introduce a perturbation $\hat{u}$ of $u^\zeta$, which we define by

$$\hat{u}(x, t) = u^\zeta(x, t) - \delta^{1/4}.$$  \hfill (21)

By Proposition 4.3, $\hat{u}$ is a subsolution of

$$\hat{u}_t - F(D^2\hat{u}) \leq -\delta^{1/4}$$

in $\hat{U}$. By the definition of $\hat{u}$, Proposition 4.2 and (22) we have

$$\sup_{\partial \hat{U}} (\hat{u} - v) \leq \sup_{\partial \hat{U}} (u^\zeta + v) \leq \sup_{\partial \hat{U}} (u - v) + 2\tilde{c}\|Du\|_{L^\infty(\Omega \times (0, T))} \leq c_1 \delta^\zeta,$$

where $c_1$ depends on $\|Du\|_{L^\infty(\Omega \times (0, T))}$. Define $w = -(\hat{u} - v - c_1 \delta^\zeta)$. There exist $\rho > 0$ and $\bar{x}$ such that $\hat{U} \subset Y_\rho^{-}(\bar{x}, T - \rho \delta)$. Since $w \geq 0$ on $\partial \hat{U}$, we may extend $w^-$ by 0 to $Y_\rho^{-}(\bar{x}, T)$. Let $\Gamma$ be the upper monotone envelope of $-w^-$. We point out that $\Gamma = \Gamma$, where $\Gamma$ is the upper monotone envelope of $(\hat{u} - v - c_1 \delta^\zeta)^+$. Moreover, we have

$$\{\hat{u} - v - c_1 \delta^\zeta = \Gamma\} = \{w = \Gamma\}$$

and

$$\sup_{Y_\rho^{-}(\bar{x}, T - \rho \delta)} (w^-) = \sup_{\hat{u}} (\hat{u} - v - c_1 \delta^\zeta).$$

We seek to apply Proposition 2.11 with $w$ instead of $u$ and $Y_\rho^{-}(\bar{x}, T)$ instead of $Y_\rho^\zeta$. 


Since \( d(\hat{U}, \partial_p\Omega \times (0,T)) = 2r \delta \), Proposition \ref{lem:exist_sol} implies that \( u \) is differentiable in \( t \) in \( \hat{U} \), with

\[
\|u_t\|_{L^\infty(\hat{U})} \leq C \delta^{-2}(\|u\|_{L^\infty(\Omega \times (0,T))} + 1) \leq C \delta^{-2c},
\]

where \( C \) depends on \( n, \lambda, \Lambda \), \( \|u\|_{L^\infty(\Omega \times (0,T))} \) and \( \|Du\|_{L^\infty(\Omega \times (0,T))} \). Therefore, \( \|\bar{u}_t\|_{L^\infty(\hat{U})} \leq C \delta^{-2c} \) as well. Together with assumption \((\ref{ass:existSol})\), this implies \( \|w\|_{C^{1,\alpha}(\Omega \times (0,T))} \leq C_0 \delta^{-2c} \), where \( C_0 \) depends on \( n, \lambda, \Lambda, T, \|u\|_{L^\infty(\Omega \times (0,T))} \) and \( \|Du\|_{L^\infty(\Omega \times (0,T))} \). In addition, assumption \((\ref{ass:existSol})\) on \( v \) and the properties of \( x \)-sup convolutions (see item \((3)\) of Proposition \ref{prop:conv}) imply \( D_2^2 w = D^2 (v - \bar{u}) \leq \delta^{-\varepsilon} I \) in the sense of distributions. Therefore, the hypotheses of Proposition \ref{prop:unique} hold with \( K = C_0 \delta^{-2c} \), so we obtain

\[
\sup_{\bar{U}} (\bar{u} - c_1 \delta^c - v) \leq C \inf \{ \bar{u} - c_1 \delta^c - v = \bar{\Gamma} \} \cap \bar{U} \leq \frac{1}{n+1} \delta^{-2c},
\]

where \( \bar{\Gamma} \) is the upper monotone envelope of \( \bar{u} - c_1 \delta^c - v \) and \( C \) depends on \( n, \lambda, \Lambda, T, \text{diam} \bar{U} \), \( \|u\|_{L^\infty(\Omega \times (0,T))} \) and \( \|Du\|_{L^\infty(\Omega \times (0,T))} \). Therefore,

\[
\|\{ \bar{u} - c_1 \delta^c - v = \bar{\Gamma} \} \cap \bar{U} \| \geq C_1 \delta^{2c(n+1)} \left( \sup_{\bar{U}} (\bar{u} - c_1 \delta^c - v) \right)^{(n+1)},
\]

where \( C_1 \) depends on \( n, \lambda, \Lambda, T, \text{diam} \bar{U}, \|u\|_{L^\infty(\Omega \times (0,T))} \) and \( \|Du\|_{L^\infty(\Omega \times (0,T))} \). We claim

\[
\sup_{\bar{U}} (\bar{u} - c_1 \delta^c - v) \leq \left( \frac{\bar{c}}{C_1} \right)^{\frac{1}{n+1}} \delta^c,
\]

where \( \bar{c} \) is given by \((\ref{def:barC})\). If \((\ref{def:barc})\) holds, then we have

\[
\sup_{\bar{U}} (u - v) \leq \sup_{\bar{U}} (\bar{u} - v) + \delta^{1/4} T \leq c_1 \delta^c + \left( \frac{\bar{c}}{C_1} \right)^{\frac{1}{n+1}} \delta^c + \delta^{1/4} T \leq \bar{c} \delta^\alpha,
\]

where \( \bar{c} = c_1 + T + \left( \frac{\bar{c}}{C_1} \right)^{\frac{1}{n+1}} \) and \( \alpha = \min \{1/4, \zeta \} \). Therefore, to complete the proof of the lemma it will suffice to show that \((\ref{def:barc})\) holds. To this end, we proceed by contradiction and assume that \((\ref{def:barc})\) does not hold, in which case we have a lower bound for the right-hand side of \((\ref{def:ineq})\) and thus obtain

\[
\|\{ \bar{u} - c_1 \delta^c - v = \bar{\Gamma} \} \cap \bar{U} \| \geq \bar{c} \delta^{3c(n+1)}.
\]

We now apply Lemma \ref{lem:peap} with \( C = \{ \bar{u} - c_1 \delta^c - v = \bar{\Gamma} \} \cap \bar{U} \) and obtain that there exists a point \((x_0, t_0) \in \{ \bar{u} - c_1 \delta^c - v = \bar{\Gamma} \} \) with

\[
Y_\delta^-(x_0, t_0) \subset \bar{U},
\]

and a paraboloid \( P \) with \( P(x_0, t_0) = 0 \),

\[
P_t - F(D^2 P) = 0,
\]

and

\[
u^+(x, t) - u^+(x_0, t_0) \geq P(y, s) - E \cdot (|x_0 - x|^2 + (t_0 - t)),
\]

where the error \( E \) is given by

\[
E = C_2 \bar{c}^{-1/\sigma} \delta^{1 - \frac{\bar{c}}{2}(3n+4+2\sigma)},
\]

and \( C_2 \) depends on \( n, \lambda, \Lambda, \) and \( \|Du\|_{L^\infty(\Omega \times (0,T))} \). We recall the representation formula for the upper monotone envelope of a function \((\text{Lemma } \ref{lem:existSol})\):

\[
\bar{\Gamma}(w)(x_0, t_0) = \inf \{ \zeta \cdot x + h : \zeta \cdot x + h \geq w(s, x) \text{ for all } t \in (a, t_0), x \in \Omega \}.
\]
Since \((x_0, t_0)\) is in the contact set of \(\tilde{u} - c_1 \delta^\zeta - v\) with its upper monotone envelope, the representation formula implies that there exist \(\zeta \in \mathbb{R}^n\) and \(h \in \mathbb{R}\) such that
\[
\zeta \cdot x_0 + h = \bar{\Gamma}(w)(x_0, t_0) = \tilde{u}(x_0, t_0) - c_1 \delta^\zeta - v(x_0, t_0)
\]
and
\[
\zeta \cdot x + h \geq \tilde{u}(x, t) - c_1 \delta^\zeta - v(x, t)
\]
for all \(x\) and for all \(t \leq t_0\) (so in particular, for all points in \(Y^{-}_\delta(x_0, t_0)\)). Therefore, for all \((x, t) \in Y^{-}_\delta(x_0, t_0)\), we have
\[
v(x, t) \geq \tilde{u}(x, t) - c_1 \delta^\zeta - \zeta \cdot x - h,
\]
with equality holding at \((x_0, t_0)\). By the definitions of \(\tilde{u}\) and \(u^+\), we find
\[
v(x, t) \geq u^+(x_0, t_0) + P(x, t) - E - \delta^{1/4} t - c_1 \delta^\zeta - \zeta \cdot x - h,
\]
with equality at \((x_0, t_0)\). By \((25)\), \(Y^{-}_\delta(x_0, t_0) \subset \Omega \times (0, T)\). Since \(v\) is a \(\delta\)-supersolution of \((20)\) on \(\Omega \times (0, T)\), we find
\[
P_t + E - \delta^{1/4} - F \left( D^2 P - \frac{E}{2} I \right) \geq 0.
\]
Since the nonlinearity \(F\) is uniformly elliptic, we have
\[
P_t + E - \delta^{1/4} - F(D^2 P) + \lambda \frac{E}{2} \geq 0.
\]
But, according to \((20)\), \(P_t - F(D^2 P) = 0\), so we obtain
\[
E + \lambda \frac{E}{2} \geq \delta^{1/4}.
\]
Using the definition of \(E\), we find
\[
C_3 \bar{c}^{-1/\sigma} \delta^{1/4} \geq \delta^{1/4},
\]
where \(C_3\) is a constant that depends on \(n, \lambda, \Lambda, \) and \( \|Du\|_{L^{\infty}(\Omega \times (0, T))}\). By our choice of \(\zeta\), we obtain
\[
C_3 \bar{c}^{-1/\sigma} \delta^{1/2} \geq \delta^{1/4}.
\]
We choose
\[
(28) \quad \bar{c} = C_3^{\sigma}
\]
and find
\[
\delta^{1/2} \geq \delta^{1/4},
\]
which is impossible, so \((24)\) must hold. \(\square\)

Lemma 5.3 follows from Proposition 4.3 and a covering argument. The covering argument is slightly more involved than in the elliptic case because of the need to avoid times \(t\) that are close to the terminal time \(T\).
Figure 1. The sets involved in the proof of Lemma 5.3

Proof of Lemma 5.3. We will denote the $x$-sup convolution $u_{\delta}^+$ by simply $u^+$. We will need the sets:

$$
\tilde{Y}_r(x, t) := Y_{r_\delta}(x, t) = B(x, r_\delta - \delta^\xi|Du|_{L^\infty(\Omega \times (0, T))}) \times (\bar{t}, \bar{t} + r_\delta^2)
$$

$$
\tilde{K}_{r_\delta}(x, t) := K_{r_\delta}(x, t) = [\bar{x} - \delta^\xi, \bar{x} + \delta^\xi] \times (\bar{t}, \bar{t} + \frac{r_\delta^2}{81n})
$$

$$
\tilde{K}_{r_\delta}^T(x, t) := [\bar{x} - \delta^\xi, \bar{x} + \delta^\xi] \times \left(\bar{t} + \frac{r_\delta^2}{162n}, \bar{t} + \frac{r_\delta^2}{81n}\right),
$$

where $\tilde{K}_{r_\delta}^T(x, t)$ is the top half (in terms of $t$) of $\tilde{K}_{r_\delta}(x, t)$. See Figure 1. We remark that there exists a constant $\delta_1$ that depends only on $n$ such that if $\delta \leq \delta_1$, then

$$
(29) \quad \text{if } (x, t) \in \tilde{U} \text{ and } (x_0, t_0) \in \tilde{K}_{r_\delta}^T(x, t), \text{ then } Y_{\delta}^-(x_0, t_0) \subset \tilde{Y}_r(x, t) \cap \Omega \times (0, T).
$$

(We point out that in order for (29) to hold, it is important that $(x_0, t_0) \in \tilde{K}_{r_\delta}^T(x, t)$, and not only in $\tilde{K}_{r_\delta}(x, t)$.) We take $\delta < \delta_1$. We recall that Proposition 4.3 gives a lower bound on the size of $\Psi_{r_\delta}^+$, the set where $u^+$ has second-order expansions from below. For the rest of the argument we will denote $\Psi_{r_\delta}^+$ simply by $\Psi_\delta^+$. We will show that there exists a point in the intersection of $\Psi_\delta^+$ and $\mathcal{C}$ by using the lower bounds on the size of $\mathcal{C}$ and $\Psi_{r_\delta}^+$ (the latter is proved in Proposition 4.3). Since Proposition 4.3 bounds the size of the complement of $\Psi_\delta^+$ inside of sets of the form $\tilde{K}_{r_\delta}(\bar{x}, \bar{t})$, we first cover $\tilde{U}$ by $\{\tilde{K}_{r_\delta}^T(x, t) : (x, t) \in \tilde{U}\}$. It is clear that the finite collection $\{\tilde{K}_{r_\delta}^T(x_j, t_j) : (x_j, t_j) \in \tilde{U}\}_{j=1}^s$ covers $\tilde{U}$, where $s \leq C_0\delta^{-c(n+2)}$, for $C_0$ a constant that depends on $diam\Omega$, $T$, and $n$. Therefore, there exists an $i$ such that

$$
|\mathcal{C} \cap \tilde{K}_{r_\delta}^T(x_i, t_i)| \geq \frac{|\mathcal{C}|}{s}.
$$

The lower bound on $|\mathcal{C}|$ and the upper bound on $s$ gives

$$
|\mathcal{C} \cap \tilde{K}_{r_\delta}^T(x_i, t_i)| \geq \frac{c_0\delta^{c(n+1)}}{C_0\delta^{-c(n+2)}} = \frac{c_0}{C_0}\delta^{c(4n+5)}.
$$

In view of the definition of $\tilde{U}$, we have $d((x_i, t_i), \partial_p \Omega \times (0, T)) \geq 2r_\delta$ and $t_i \leq T - r_\delta^2$; therefore, $Y_{r_\delta}(x_i, t_i) \subset \Omega \times (0, T)$ and

$$
d(Y_{r_\delta}(x_i, t_i), \partial_p \Omega \times (0, T)) \geq r_\delta > \delta^\xi.
$$
Therefore, \( u \) satisfies the hypotheses of Proposition 4.3 in \( Y_r(x, t) \) with \( \theta = \delta^\lambda \) and \( r = r_\delta \). Thus there exists a set \( \Psi_M \) (the "good set" of \( u^+ \)) with
\[
|\tilde{K}_r(x, t) \setminus \Psi_M^+| \leq \frac{C_1 \delta^{-\frac{n+2}{2}}}{M \delta^{(1+\sigma)}} (r_\delta^{-\sigma} + \delta^{-\xi}) \left( \|u\|_{L^\infty(\Omega \times (0, T))} + 1 \right)^{1+\sigma}
\]
where \( C_1 \) is a universal constant. We take
\[
M = \left( \frac{2C_1C_0 \delta^{-\frac{n+2}{2}}}{\tilde{c} \delta^{(3n+4)}} (r_\delta^{-\sigma} + \delta^{-\xi}) \left( \|u\|_{L^\infty(\Omega \times (0, T))} + 1 \right)^{1+\sigma} \right)^{1/\sigma}
\]
so that
\[
|\tilde{K}_r(x, t) \setminus \Psi_M^+| < |C \cap \tilde{K}_r(x, t)|.
\]
Since \( \tilde{K}_r(x, t) \subset \tilde{K}_r(x, t) \), we have
\[
|\tilde{K}_r(x, t) \setminus \Psi_M^+| < |\tilde{K}_r(x, t) \setminus \Psi_M^+| < |C \cap \tilde{K}_r(x, t)|.
\]
Therefore, there exists a point \((x_0, t_0) \in C \cap \Psi_M^+ \cap \tilde{K}_r(x, t) \). Because of (29), we have \( Y_\delta^-(x_0, t_0) \subset Y_r(x, t) \cap \Omega \times (0, T) \). Thus, by Proposition 4.3 there exists a polynomial \( P \in \mathcal{P}_\infty \) such that
\[
P - F(D^2 P) = 0
\]
and, on \( Y_\delta^-(x_0, t_0) \),
\[
u^+ (x, t) - u^+(x_0, t_0) \geq P(y, s) - \frac{c(n)}{r_\delta^\lambda} M (\delta |x - x_0|^2 + \delta^2 |t - t_0|)
\]
\[
- \frac{C_4 \delta^{-\xi}}{r_\delta^\lambda} M + \delta^{-\xi} (1 + \|u\|_{L^\infty(\Omega \times (0, T))}) \right) (t_0 - t).
\]
(30)
There exists a constant \( C_2 \) that depends on \( n \) and \( \|Du\|_{L^\infty(\Omega \times (0, T))} \) such that
\[
\delta^\lambda \leq r_\delta \leq C_2 \delta^\lambda.
\]
Therefore
\[
M \leq C_3 \delta^{-1/\sigma} \delta^{-\frac{3n+4+2\sigma}{2}},
\]
where \( C_3 \) depends on \( n, \lambda, \Lambda, \) and \( \|Du\|_{L^\infty(\Omega \times (0, T))} \). We use this to bound the coefficients of the error terms in (30) and find
\[
u^+ (x, t) - u^+(x_0, t_0) \geq P(y, s) - C_4 \delta^{-1/\sigma} \delta^{1-2\xi - \frac{3n+4+2\sigma}{2}} \left( |x - x_0|^2 + (t_0 - t) \right).
\]

\[\square\]

6. Error estimate

In this section we give the precise statement and the proof of our main result:

**Theorem 6.1.** Assume \((H1), (H2)\) and \((U1)\). Assume \( u \in C^{0,\eta}(\Omega \times (0, T)) \cap C^{0,1}_x(\Omega \times (0, T)) \) is a solution of
\[
u_t - F(D^2 u) = 0 \text{ in } \Omega \times (0, T)
\]
and assume \( \{v_\delta\}_{\delta > 0} \) is a family of \( \delta \)-supersolutions (resp. \( \delta \)-subsolutions) of (31) such that, for some \( M < \infty \) and for all \( \delta > 0 \),
\[
\|v_\delta\|_{C^{0,\eta}(\Omega \times (0, T))} \leq M
\]
and
\[
u_\delta - u_\delta \leq 0 \text{ (resp. } v_\delta - u \leq 0) \text{ on } \partial_p(\Omega \times (0, T)).
\]
There exist a constant $\tilde{\delta}$ that depends only on $n$, a constant $\tilde{\alpha}$ that depends on $n$, $\lambda$, $\Lambda$, and $\eta$, and a constant $\tilde{c}$ that depends on $n$, $\lambda$, $\Lambda$, $\text{diam}\Omega$, $T$, $M$, $\|u\|_{C^{0,\alpha}(\Omega \times (0,T))}$, and $\|Du\|_{L^\infty(\Omega \times (0,T))}$, such that for all $\delta \leq \tilde{\delta}$,

$$u - v_\delta \leq \tilde{c}\delta^{\tilde{\alpha}} \quad (\text{resp. } v_\delta - u \leq \tilde{c}\delta^{\tilde{\alpha}}).$$

**Remark 6.2.** The assumption $u \in C^{0,\eta}(\Omega \times (0,T)) \cap C^{0,1}_x(\Omega \times (0,T))$ is satisfied in the following situation. Assume (F1), (F2), (U1), and that $\partial\Omega$ is regular. Take $g \in C^{1,\alpha}(\partial_p\Omega \times (0,T))$. Then, according to the interior regularity estimates of Theorem 2.12 and the boundary estimates of [26], Section 2], the solution $u$ of the boundary value problem

$$\begin{align*}
\left\{ \begin{array}{ll}
u_t - F(D^2\nu) = 0 & \text{in } \Omega \times (0,T), \\
u = g & \text{on } \partial_p\Omega,
\end{array} \right.
\end{align*}$$

is indeed Lipschitz continuous in $x$ and Hölder continuous in $t$ on all of $\Omega \times (0,T)$, with

$$\|\nu\|_{C^{0,\alpha}(\Omega \times (0,T))}, \|Du\|_{L^\infty(\Omega \times (0,T))} \leq C,$$

where $C$ depends on $n$, $\lambda$, $\Lambda$, $\|g\|_{C^{1,\alpha}(\partial_p\Omega \times (0,T))}$, $\text{diam}\Omega$ and the regularity of $\partial_p\Omega$.

Theorem 6.1 follows easily from Proposition 5.1.

**Proof of Theorem 6.1.** We give the proof of the bound on $\sup(u - v_\delta)$ in the case that $v_\delta$ is a $\delta$-supersolution; the proof of the other case is very similar.

First we regularize $v_\delta$ by taking inf-convolution: we let

$$v(x,t) = \inf_{(y,s) \in \Omega \times (0,T)} \left\{ v_\delta(y,s) + \frac{|x-y|^2}{2\delta^2} + \frac{|t-s|^2}{2\delta^2} \right\}$$

and define the constant

$$\nu = 2\delta^{\tilde{\alpha}/2} \|v\|^{1/2}_{L^\infty(\Omega \times (0,T))} + \delta,$$

and the set

$$U' = \left\{ (x,t) : d((x,t), \partial_p(\Omega \times (0,T))) \geq 2\delta^{\alpha/2} \|v\|^{1/2}_{L^\infty(\Omega \times (0,T))} + \delta \right\}.$$  

We point out that $U'$ is of the form $\Omega' \times (a,b)$, where $\Omega'$ satisfies (U1). Therefore, Proposition 5.1 holds in $U'$ instead of $\Omega \times (0,T)$ (see Remark 6.2). According to item 3 of Proposition 9.4, $v$ is a $\delta$-supersolution of (20) in $U'$. Moreover, by items 3 and 4 of Proposition 9.4, $v$ satisfies the regularity assumptions of Proposition 5.1 in $U'$. Therefore, according to Proposition 5.1

$$\sup_{\partial_p U} (u - v) \leq \tilde{c}\delta^{\tilde{\alpha}} + \sup_{\partial_p U} (u - v),$$

where

$$\tilde{U} = \left\{ (x,t) : d((x,t), \partial_p U') \geq 2r_\delta, \text{ and } t \leq T - r_\delta^2 \right\}$$

and $r_\delta$ is as given in the statement of Proposition 5.1. Since $u \in C^{0,\eta}(\Omega \times (0,T))$, $v \in C^{0,\eta}(\Omega \times (0,T))$, and $u \leq v$ on $\partial_p\Omega \times (0,T)$, there exists a constant constant $c_1 > 0$ that depends on $n$, $M$, and $\|u\|_{C^{0,\eta}(\Omega \times (0,T))}$ such that

$$\sup_{\partial_p U} (u - v_\delta) \leq c_1 \delta^{\tilde{\alpha}}$$

and

$$\sup_{\Omega \times (0,T)} (u - v_\delta) \leq \sup_{\tilde{U}} (u - v) + c_1 \delta^{\tilde{\alpha}}.$$

By item 2 of Proposition 9.4,

$$\sup_{\partial_p U} (u - v) \leq \sup_{\partial_p U} (u - v_\delta) + \|v_\delta\|_{C^{0,\eta}(\Omega \times (0,T))} \left( 2\delta^{\alpha} \|v_\delta\|_{C^{0,\eta}(\Omega \times (0,T))} \right)^{1/2} \leq \sup_{\partial_p U} (u - v_\delta) + c_2 \delta^{\tilde{\alpha}},$$

and

$$\sup_{\partial_p U} (u - v) \leq \sup_{\partial_p U} (u - v_\delta) + \frac{1}{2} \left( 2\delta^{\alpha} \|v_\delta\|_{C^{0,\eta}(\Omega \times (0,T))} \right)^{1/2} \leq \sup_{\partial_p U} (u - v_\delta) + c_3 \delta^{\tilde{\alpha}},$$

for some constants $c_2, c_3 > 0$.
where \( c_2 \) is a constant that depends on \( n \) and \( M \). By the above line and (33) we have
\[
\sup_{\partial_y \mathcal{U}} (u - v) \leq \sup_{\partial_y \mathcal{U}} (u - v_3) + c_2 \delta \frac{\phi^2}{\mathcal{F}_h} \leq c_1 \delta \frac{\phi^2}{\mathcal{F}_h} + c_2 \delta \frac{\phi^2}{\mathcal{F}_h} \leq c_3 \delta \frac{\phi^2}{\mathcal{F}_h},
\]
where \( c_3 \) depends on \( n, M \) and \( \|u\|_{C^{0,0}(\Omega \times (0,T))} \). So, by (34), (32), and the above line, we have
\[
\sup_{\Omega \times (0,T)} (u - v_3) \leq \bar{c} \delta + c_3 \delta \frac{\phi^2}{\mathcal{F}_h}.
\]
We thus take \( \bar{c} = c_3 + \bar{c} \) and \( \bar{\alpha} = \min\{\bar{\alpha}, \frac{\phi^2}{\mathcal{F}_h}\} \) to complete the proof of the theorem.

7. Discrete approximation schemes

We now present the error estimate for finite difference approximation schemes. First, we introduce the necessary notation and assumptions, closely following [8] and [24]. In the next section we give the full statement of the error estimate and its proof. The space-time mesh is denoted by
\[
E = h\mathbb{Z}^n \times h^2 \mathbb{Z} = \{(x, t) | x = (m_1, ..., m_n)h, t = mh^2, \text{ where } m, m_1, ..., m_n \in \mathbb{Z}^n\}.
\]
We fix some \( N > 1 \) and define the subset \( Y \) of \( E \) by
\[
Y = \{y \in h\mathbb{Z}^n | 0 < |y| < hN\}.
\]
Next we introduce finite difference operators for a function \( u \):
\[
\delta_x^- u(x, t) = \frac{1}{h}(u(x, t) - u(x, t - h)),
\]
\[
\delta_y^2 u(x, t) = \frac{1}{|y|^2} (u(x + y, t) + u(x - y, t) - 2u(x, t)), \quad \text{and}
\]
\[
\delta^2 u(x, t) = \{\delta_y^2 u(x, t) | y \in Y\}.
\]
An implicit finite difference operator is an operator of the form
\[
\mathcal{S}_h[u](x, t) = \delta_x^- u(x, t) - \mathcal{F}_h(\delta^2 u(x, t)),
\]
where \( \mathcal{F}_h : \mathbb{R}^Y \rightarrow \mathbb{R} \) is locally Lipschitz. We denote points in \( \mathbb{R}^Y \) by \( r = (r_1, ..., r_{|Y|}) \). We say an operator is monotone if it satisfies:
(S1) there exists a constant \( \lambda_0 \) and \( \Lambda_0 \) such that for all \( i = 1, ..., |Y| \),
\[
\lambda_0 \leq \frac{\partial \mathcal{F}_h}{\partial r_i} \leq \Lambda_0.
\]
This is equivalent to the definition given in the introduction.

A scheme \( \mathcal{S}_h \) is said to be consistent with \( F \) if for all \( \phi \in C^3(\Omega \times (0,T)) \),
\[
\sup_{\mathcal{U}_h} |\phi_t - F(D^2 \phi) - \mathcal{S}_h[\phi]| \rightarrow 0 \text{ as } h \rightarrow 0.
\]
In [24] Section 4], it was shown that if a nonlinearity \( F \) satisfies (11), then there exists a monotone implicit scheme \( \mathcal{S}_h \) that is consistent with \( u_t - F(D^2 u) = 0 \), and the constants \( \lambda_0 \) and \( \Lambda_0 \) depend only on \( n, \lambda \) and \( \Lambda \). In [22], a monotone and consistent approximation scheme for elliptic equations is explicitly constructed, and the construction in the parabolic case is analogous.

In order to obtain an error estimate, we need to make an assumption that quantifies the above rate of convergence. As in [8], we assume:
(S2) there exists a positive constant \( K \) such that for all \( \phi \in C^3(\Omega \times (0,T)) \),
\[
\sup_{\mathcal{U}_h} |\phi_t - F(D^2 \phi) - \mathcal{S}_h[\phi]| \leq K(h + h||D^3 \phi||_{L^\infty(\Omega \times (0,T))} + h^2||\phi_t||_{L^\infty(\Omega \times (0,T))}).\]
Schemes that satisfy (2) are said to be consistent with an error estimate for $F$ in $\Omega \times (0, T)$ with constant $K$.

Given $\Omega \times (0, T) \subset \mathbb{R}^{n+1}$, we denote $(\Omega \times (0, T)) \cap E$ by $\mathcal{U}_h$. We divide $\mathcal{U}_h$ into interior and boundary points relative to the operator $S_h$. Mainly, we define

$$\mathcal{U}_h^i = \{ p \in \mathcal{U}_h | d(p, \partial_p \Omega \times (0, T)) \geq Nh \}$$

to be the interior points and

$$\mathcal{U}_h^b = \mathcal{U}_h \setminus \mathcal{U}_h^i$$

to be the boundary points. Notice that $S_h[u](x, t)$ depends only on $u(x + y, s)$ for $0 \leq |y| < hN$ and for $t - h^2 \leq s \leq t$.

The discrete Hölder seminorm of a function $u$ on $\mathcal{U}_h$ is defined to be

$$[u]_{C^{0, \alpha}(\mathcal{U}_h)} = \sup_{p, q \in \mathcal{U}_h} \frac{|u(p) - u(q)|}{d(p, q)^\alpha}$$

and the discrete Hölder norm is

$$||u||_{C^{0, \alpha}(\mathcal{U}_h)} = ||u||_{L^\infty(\mathcal{U}_h)} + [u]_{C^{0, \alpha}(\mathcal{U}_h)}.$$

In [24] Section 4 it is shown that solutions of the discrete equation $S_h[v_h] = 0$ are uniformly equicontinuous. We summarize this result:

**Theorem 7.1.** Assume that $S_h$ is an implicit monotone finite difference scheme and that $v_h$ is a solution of $S_h[v_h] = 0$. There exists a constant $C$ that depends only on $n$, $\lambda_0$ and $\Lambda_0$ such that for all $h \in (0, 1)$,

$$||v_h||_{C^{0, \alpha}(\mathcal{U}_h)} \leq C.$$

**7.1. Inf and sup convolutions for approximation schemes.** For a mesh function $v : \mathcal{U}_h \to \mathbb{R}$, we define the inf- and sup- convolutions of $v$ at $(x, t) \in \Omega \times (0, T)$ by

$$v_{\theta, \delta}^-(x, t) = \inf_{(y, s) \in \mathcal{U}_h} \left\{ v(y, s) + \frac{|x - y|^2}{2\theta} + \frac{|t - s|^2}{2\theta} \right\}$$

and

$$v_{\theta, \delta}^+(x, t) = \sup_{(y, s) \in \mathcal{U}_h} \left\{ v(y, s) - \frac{|x - y|^2}{2\theta} - \frac{|t - s|^2}{2\theta} \right\}.$$

For $v \in C^{0, \alpha}(\mathcal{U}_h)$, we introduce the quantity

$$\omega(h, \theta) = nh + 2\theta^{1/2} ||v||_{C^{0, \alpha}(\mathcal{U}_h)} (\text{diam} U)^\alpha$$

and the set

$$\mathcal{U}_\theta^h = \{ p \in \Omega \times (0, T) | d_e(p, \partial_p \Omega \times (0, T)) \geq \omega(h, \theta) + Nh \}.$$

In the appendix we summarize the basic properties of inf- and sup- convolutions of mesh functions (see Proposition 8.5).

It is a classical fact of the theory of viscosity solutions that if $u$ is the viscosity solution of $u_t - F(D^2 u) = 0$ in $\Omega \times (0, T)$, then the sup-convolution of $u$ is a subsolution of the same equation. In the following proposition, we establish a similar relationship between solutions $v_h$ of discrete equations and $\delta$-solutions of $u_t - F(D^2 u) = 0$. This is a key step in establishing an error estimate for discrete approximation schemes.

**Proposition 7.2.** Assume that $S_h$ is a monotone and implicit scheme consistent with an error estimate for $F$ with constant $K$. Assume $v \in C^{0, \alpha}(\mathcal{U}_h)$.

1. If $v$ satisfies $S_h[v] \geq 0$ in $\mathcal{U}_h^i$ then $v_{\theta, \delta}^-$ is a $\delta$- supersolution of $u_t - F(D^2 u) = -Kh$ in $\mathcal{U}_\theta^h$, with $\delta = Nh$. 
(2) If \( v \) satisfies \( S_h[v] \leq 0 \) in \( U_h^i \), then \( v_{\alpha,\theta}^+ \) is \( \delta \)-subsolution of
\[
    u_t - F(D^2u) = Kh
\]
in \( U_h^i \), with \( \delta = Nh \).

**Proof.** We give the proof of item (1); the proof of item (2) is analogous.

Let \((x, t) \in U_h^i\) and \((x^*, t^*)\) be a point at which the infimum is achieved in the definition of \( v_{\alpha,\theta}^- \). Assume that \( P \in P_\infty \) is such that \( P \leq v_{\alpha,\theta}^- \) on \( Y_{Nh}(x, t) \), with equality at \((x, t)\). Because \( P \leq v \) on all of \( Y_{Nh}(x, t) \), for each \( y \) with \( 0 < |y| \leq hN \) we have
\[
    \delta_\theta^2 P(x, t) \leq \delta_\theta^2 v_{\alpha,\theta}^-(x, t) \leq \delta_\theta^2 v(x^*, t^*), \quad \text{and}
\]
\[
    \delta_\theta^- P(x, t) \geq \delta_\theta^- v_{\alpha,\theta}^-(x, t) \geq \delta_\theta^- v(x^*, t^*). \tag{35}
\]

The definition of \( U_h^i \) and item (1) of Proposition 8.2 yield \((x^*, t^*) \in U_h^i\); therefore,
\[
    0 \leq S_h[v](x^*, t^*) = \delta_\theta^- v(x^*, t^*) - F_h(\delta_\theta^- v(x^*, t^*)). \tag{36}
\]

Using (35) and the monotonicity of \( F_h \) to bound the right-hand side of the above yields
\[
    0 \leq \delta_\theta^- P(x, t) - F_h(\delta_\theta^- P(x, t)).
\]

That \( S_h \) is consistent with an error estimate implies that
\[
    |P_t(x, t) - F(D^2P(x, t)) - (\delta_\theta^- P(x, t) - F_h(\delta_\theta^- P(x, t)))| \leq Kh.
\]
Together with (36), this implies
\[
    P_t - F(D^2P) \geq -Kh.
\]

\[ \square \]

8. **Error estimate for approximation schemes**

We prove

**Theorem 8.1.** Assume (F1), (F2) and (U1). Assume \( u \in C^{0,\alpha}(\Omega \times (0, T)) \cap C^{0,1}_x(\Omega \times (0, T)) \) is a solution of
\[
    u_t - F(D^2u) = 0 \tag{37}
\]
in \( \Omega \times (0, T) \). Assume \( S_h \) is an implicit monotone scheme consistent with an error estimate for (37) and that \( v_h \) satisfies \( S_h[v_h] \geq 0 \) (resp. \( S_h[v_h] \leq 0 \)) in \( U_h \). Assume that for some constant \( M < \infty \) and for all \( h > 0 \),
\[
    [v_h]_{C^{0,\alpha}(U_h^i)} \leq M
\]
and
\[
    u - v_h \leq 0 \quad \text{(resp. } u - v_h \geq 0 \text{)} \quad \text{on } U_h^i.
\]

There exist a constant \( \tilde{h} \) that depends only on \( n \), a constant \( \alpha \) that depends on \( n, \lambda, \Lambda \) and \( \eta \), and a constant \( \tilde{c} \) that depends on \( n, N, \lambda, \Lambda, \text{diam} \Omega, M, ||u||_{C^{0,\alpha}(\Omega \times (0, T))}, \text{and } ||Du||_{L^\infty(\Omega \times (0, T))} \), such that for all \( h \leq \tilde{h} \),
\[
    \sup_{\Omega \times (0, T)} u - v_h \leq \tilde{c}h^\alpha \quad \text{(resp. } \sup_{\Omega \times (0, T)} v_h - u \leq \tilde{c}h^\alpha \text{)}.
\]

**Remark 8.2.** Suppose that \( u \) is the solution of the boundary value problem
\[
\begin{align*}
    u_t - F(D^2u) &= 0 & \text{in } \Omega \times (0, T), \\
    u &= g & \text{on } \partial_D \Omega,
\end{align*}
\]
and \( v_h \) satisfies
\[
\begin{align*}
    S_h[v_h] &= 0 & \text{in } U_h^i, \\
    v_h &= g & \text{on } U_h^i,
\end{align*}
\]
where $\partial \Omega$ is sufficiently regular and $g \in C^{1,\alpha}(\Omega \times (0, T))$. Then, as explained in Remark 6.2 $u$ satisfies the hypotheses of Theorem 8.1 and $v_h$ satisfies the hypotheses of Theorem 8.1 because of Theorem 7.1.

Remark 8.3. We do not know whether our result holds for explicit or mixed schemes. Our method cannot be used in those cases, since we are able to verify Proposition 7.2 only for implicit schemes.

Theorem 8.1 follows easily from Proposition 5.1.

Proof of Theorem 8.1. We give the proof of the bound on $\sup(u - v_h)$; the proof of the other case is very similar.

First we regularize $v_h$ by taking inf-convolution. We denote $v^- = v^-_{h<\lambda<}$ and define

$$
\nu = \omega(h, \theta) + Nh,
$$

and

$$
U^\nu = \{(x, t) | d((x, t), \partial_p(\Omega \times (0, T))) \geq \omega(h, \theta) + Nh\}.
$$

Then, according to Proposition 5.2, $v^-$ is a $\delta$-supersolution of

$$
v^- - F(D^2 v^-) \geq -h
$$
in $U^\nu$, with $\delta = Nh$. We point out that $U^\nu$ is of the form $\Omega' \times (a, b)$, where $\Omega'$ satisfies (U1). Therefore, Proposition 5.1 holds in $U^\nu$ instead of $\Omega \times (0, T)$ (see Remark 5.2).

Since we proved Proposition 5.1 for $\delta$-solutions with right-hand side 0, we have to perturb $v^-$. We introduce

$$
v(x, t) = v^-(x, t) + ht,
$$
so that $v$ is a $\delta$-supersolution of (37) in $U^\nu$ with $\delta = Nh$. Moreover, by items (3) and (4) of Proposition 9.5, $v$ satisfies the regularity assumptions of Proposition 5.1. Therefore, according to Proposition 5.1

$$
\sup_{U^\nu}(u - v) \leq c(Nh)^{\tilde{\alpha}} + \sup_{\partial_\nu U}(u - v),
$$

where

$$
\tilde{U} = \{(x, t) | d((x, t), \partial_p(\Omega \times (0, T))) \geq \omega(h, \theta) + Nh + 2r_\delta \text{ and } t \leq T - r_\delta\}
$$

and $r_\delta$ is given in the statement of Proposition 5.1. Since $u \in C^{0,\eta}(\Omega \times (0, T))$, $v \in C^{0,\eta}(\Omega \times (0, T))$, and $u \leq v$ on $\partial_p(\Omega \times (0, T))$, there exists a constant $c_1$ that depends on $n$, $N$, $T$, $\|v_h\|_{C^{0,\eta}(\Omega)}$ and $\|u\|_{C^{0,\eta}(\Omega \times (0, T))}$, and a constant $\gamma > 0$ that depends on $n$, $\lambda$, $\Lambda$ and $\eta$, such that

$$
\sup_{\partial_\nu U}(u - v_h) \leq c_1 h^\gamma
$$

and

$$
\sup_{\partial_\nu U}(u - v_h) \leq \sup_{\tilde{U}}(u - v) + c_1 h^\gamma.
$$

By item (2) of Proposition 9.5

$$
\sup_{\Omega \times (0, T)}(u - v_h) \leq \sup_{\tilde{U}}(u - v) + \omega(h, \theta)
$$

so by the above line and (39) we have

$$
\sup_{\partial_\nu U}(u - v) \leq \sup_{\partial_\nu U}(u - v_h) + c_2 h^\gamma \leq c_1 h^\gamma + c_2 \delta^\gamma \leq c_3 h^\gamma,
$$
Lemma 9.1. Proposition 2.11 follows from the following three lemmas.

Proof of Proposition 2.11. We thus take $\bar{c}$ where $\rho = \bar{c}$ and $\bar{\alpha} = \min\{\bar{\alpha}, \gamma\}$ to complete the proof of the theorem. \[\square\]

9. Appendix

9.1. Proof of Proposition 2.11 We outline the proof of Proposition 2.11 which is a slight modification of part of the proof of the parabolic version of the ABP estimate in [13 Section 4.1.2]. We assume $\rho = 1$; the general case follows by rescaling. As in [13 Section 4.1.2], we define the function $G : Y_1 \to \mathbb{R}^{n+1}$ by

$$G(x, t) = (\Gamma(x, t) - x \cdot D\Gamma(x, t), D\Gamma(x, t)).$$

Proposition 2.11 follows from the following three lemmas.

Lemma 9.1. Suppose $u \in C(Y_1^-)$ and let $M = \sup_{Y_1^-} u^-$. Then

$$\{ (h, \xi) \in \mathbb{R}^{n+1} : |\xi| \leq \frac{M}{2} \leq -h \leq M \} \subset G(Y_1^- \cap \{ u = \Gamma \}).$$

Lemma 9.2. If $\Gamma$ is $C^{1,1}$ with respect to $x$ and Lipschitz with respect to $t$, then $G\Gamma$ is Lipschitz in $(x,t)$ and we have

$$\det D_{x,t} G\Gamma = \partial_t \Gamma \det D^2 \Gamma$$

for almost every $(x,t) \in Y_1^-$.\[\square\]

Lemma 9.3. Suppose $u$ is as in the statement of Proposition 2.11. Then $\Gamma$ is $C^{1,1}$ with respect to $x$ and Lipschitz in $t$.

Lemma 9.1 is very similar to [13 Lemma 4.13] – the latter is stated for $u$ a supersolution of $u_t - \mathcal{M}^{-1}(D^2u) \geq f(x)$. However, it is easy to verify that the proof of [13 Lemma 4.13] holds for general functions $u \in C(Y_1^-)$, not only supersolutions. Lemma 9.2 is exactly [13 Lemma 4.4]. The proof of Lemma 9.3 is very similar to that of Lemma [13 Lemma 4.11], which states that if $u$ is a supersolution of $u_t - \mathcal{M}^{-1}(D^2u) \geq f(x)$, then $\Gamma$ is $C^{1,1}$ in $x$ and Lipschitz in $t$. We explain the modifications needed to obtain Lemma 9.3.

Proof of Lemma 9.3 As in the proof of Lemma [13 Lemma 4.11], it will suffice to show that there exists a constant $K$ such that for all $(x,t) \in Y_1^-$, the following holds:

\[\begin{align*}
:\text{if } & P(y,s) = c + \alpha t + p \cdot x + x \cdot X x^T \text{ satisfies } P(x,t) = \Gamma(x,t) \text{ and} \\
&P(y,s) \leq \Gamma(y,s) \text{ for all } (y,s) \text{ in a neighborhood of } (x,t), \\
&\text{then } \alpha \geq -K \text{ and } X \leq KI. 
\end{align*}\]

In the proof of [13 Lemma 4.11], the assumption that $u$ is a supersolution of $u_t - \mathcal{M}^{-1}(D^2u) \geq f(x)$ is used only to show that \[\text{Lemma 4.11 holds for } (x,t) \in \{ u = \Gamma \}. \] Thus, we need to verify \[\text{Lemma 4.11 for such points. To this end, assume } (x,t) \in \{ u = \Gamma \} \text{ and } P(y,s) = c + \alpha t + p \cdot x + x \cdot Mx^T \text{ satisfies } P(x,t) = \Gamma(x,t) \text{ and } P(y,s) \leq \Gamma(y,s) \text{ for all } (y,s) \text{ in a neighborhood of } (x,t). \] Since $\Gamma$ is the lower monotone envelope of $-u^-$ and $(x,t) \in \{ u = \Gamma \}$, we have $P(x,t) = u(x,t)$ and $P(y,s) \leq u(y,s)$ for all $(y,s)$ in a neighborhood of $(x,t)$. Our assumption that $D^2u \leq KI$ in the sense of distributions means that there exists a paraboloid $Q(x)$ of opening $K$ that touches $u(\cdot,t)$ from above at $x$. Therefore, $Q$ touches $P(\cdot,t)$ from above at $x$. Therefore, $X \leq KI$. That $|u|_{C^{0,\gamma}(Y_r^-)} \leq K$ implies $\alpha \geq -K$. \[\square\]
Proof of Proposition 2.11. Lemma 9.1, Lemma 9.2 and the Area Formula [12, Chapter 3], which applies to $\Gamma$ because of Lemma 9.3 imply

$$C M^{n+1} = |\{ (h, \xi) \in \mathbb{R}^{n+1} : |\xi| \leq \frac{M}{2} \leq -h \leq M \}|$$

$$\leq |\Gamma(Y_1^- \cap \{ u = \Gamma \})|$$

(42)

$$\leq \int_{Y_1^- \cap \{ u = \Gamma \}} |\det D_x \Gamma|$$

$$\leq \int_{Y_1^- \cap \{ u = \Gamma \}} -\partial_t \Gamma \det D^2 \Gamma.$$

Because $\Gamma$ is non-increasing in $t$ and convex in $x$, the Alexandroff theorem for functions depending on $x$ and $t$ (see Krylov [10, Appendix 2]) implies that there exists $A \subset Y_1^-$ with $|Y_1^- \cap A| = 0$ such that for every $(x, t) \in A$,

$$|\Gamma(y, s) - \Gamma(x, t) - D\Gamma(x, t) \cdot (y-x) - (y-x) \cdot D^2 \Gamma(x, t)(y-x)^T - \partial_t \Gamma(x, t)(t-s)| \leq o(|x-y|^2 + |t-s|)$$

(in other words, $\Gamma$ is twice differentiable almost everywhere on $Y_1^-$). We claim

(43)

if $(x, t) \in A \cap \{ u = \Gamma \}$, then $-\partial_t \Gamma(x, t) \det D^2 \Gamma(x, t) \leq K^{n+1}$.

Let us fix $(x, t) \in A \cap \{ u = \Gamma \}$. We have $u(x, t) = \Gamma(x, t)$ and $u(y, s) \geq \Gamma(y, s)$ for all $(y, s) \in Y_1^-$. Our assumption that $D^2 u \leq K I$ in the sense of distributions means that there exists a paraboloid $Q(x)$ of opening $K$ that touches $u(\cdot, t)$ from above at $x$. Therefore, $Q$ touches $\Gamma(\cdot, t)$ from above at $x$. Therefore, $D^2 \Gamma(x, t) \leq K I$ and so

$$\det D^2 \Gamma \leq K^n.$$

That $[u]_{C^{0,\eta}(Y_1^-)} \leq K$ implies $\partial_t \Gamma(x, t) \geq -K$. Thus, (43) holds. The bound (4) follows from (43) and (42).

\[\square\]

9.2. Inf and sup convolutions. In the following proposition we state the facts about inf- and sup-convolutions that are used in this paper. Their proofs are very similar to those in the elliptic case (see [9] Propositions 5.3 and 5.5) and we omit them. Given $v \in C^{0,\eta}(\Omega \times (0, T))$ and $\theta > 0$, we define the set

$$U^{\theta, \delta} = \left\{ (x, t) | d_\epsilon((x, t), \partial_p(\Omega \times (0, T))) \geq 2\theta^{1/2} ||v||_{L^\infty(\Omega \times (0, T))}^{1/2} + \delta \right\}.$$
(6) If $v$ is a $\delta$-subsolution of $v_t - F(D^2v) = 0$ in $\Omega \times (0,T)$, then $v_{\theta,\delta}^+\ v_{\theta,\delta}^+$ is a $\delta$-subsolution of $v_t - F(D^2v) = 0$ in $U^{\theta,\delta}$.

9.3. Inf and sup convolutions of mesh functions. In the following proposition we summarize the facts that we need about inf and sup convolutions of mesh functions. The proof is very similar to the elliptic case (see Proposition 2.3 of [8]) so we omit it.

**Proposition 9.5.** Assume $v \in C^{0,\nu}(\mathcal{U}_h)$. Then:

1. If $(x^*, t^*)$ is a point at which the infimum (resp. supremum) is achieved in the definition of $v_{\theta,\theta}^-(x, t)$ (resp. $v_{\theta,\theta}^+(x, t)$), then
   
   \[ d_e((x^*, t^*), (x, t)) \leq \omega(h, \theta). \]

2. Assume $(x, t) \in Q \times (0, T)$ and $(y, s) \in U_h$ is a closest mesh point to $(x, t)$. Then
   \[ v_{\theta,\theta}^-(x, t) \geq v(y, s) - ||v||_{C^{0,\nu}(U_h)}\omega(h, \theta)^\nu \]
   and
   \[ v_{\theta,\theta}^+(x, t) \leq v(y, s) + ||v||_{C^{0,\nu}(U_h)}\omega(h, \theta)^\nu. \]

3. In the sense of distributions, $D^2v_{\theta,\theta}^-(x, t) \leq \theta^{-1}I$ and $D^2v_{\theta,\theta}^+(x, t) \geq -\theta^{-1}I$ for every $(x, t) \in U_h^\theta$.

4. We have $[v_{\theta,\theta}^\pm]_{C^{0,\nu}(\Omega \times (0, T))} \leq 3T\theta^{-1}$.

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