EXISTENCE, BLOW-UP AND EXPONENTIAL DECAY OF SOLUTIONS FOR A SYSTEM OF NONLINEAR WAVE EQUATIONS WITH DAMPING AND SOURCE TERMS

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Abstract. This paper deals with a system of wave equations in one-dimensional consisting of nonlinear boundary / interior damping and nonlinear boundary / interior sources. In particular, our interest lies in the theoretical understanding of the existence, finite time blow-up of solutions and their exponential decay. First, two local existence theorems of weak solutions are established by applying the Faedo-Galerkin method and standard arguments of density corresponding to the regularity of initial conditions. The uniqueness is also obtained in some specific cases. Second, it is proved that any weak solutions possessing negative initial energy has the potential to blow up in finite time. Finally, exponential decay estimates for the global solution is achieved through the construction of a suitable Lyapunov functional. In order to corroborate our theoretical decay, a numerical example is provided here.

1. Introduction

This paper is concerned with the following well-known polynomially damped system of wave equations

\[
\begin{align*}
\mathbf{u}_{tt} - \mathbf{u}_{xx} + \lambda_1 |\mathbf{u}_t|^{r_1-2} \mathbf{u}_t &= f_1(\mathbf{u}, \mathbf{v}) + F_1(x, t), \\
\mathbf{v}_{tt} - \mathbf{v}_{xx} + \lambda_2 |\mathbf{v}_t|^{r_2-2} \mathbf{v}_t &= f_2(\mathbf{u}, \mathbf{v}) + F_2(x, t),
\end{align*}
\]

where for \( i = 1, 2 \), \( \lambda_i > 0 \) the friction terms and \( r_i \geq 2 \) deciding the order of damped parts, are given constants, and \( f_i \) are given interior sources with \( F_i \) the external functions which will be specified later.

Studying systems of wave equations arises naturally within frameworks of material science and physics. Roughly speaking, the system (1.1) is closely related to the Mindlin-Reissner plate equations (see, e.g., [8]) extending the Kirchhoff-Love plate theory for shear deformations, in which they certainly perform three coupled wave and wave-like equations. In mathematical aspects, the systems of wave equations have been gaining considerable attention a long time ago and have been extensively studying by many authors, see [1, 4, 6, 13, 14] and references therein. Based on those previous studies, many interesting results regarding the existence, regularity and the asymptotic behavior of solutions are obtained.

Since 2012, Guo et al. [6] considered the local and global well-posedness of the more general system

\[
\begin{align*}
\mathbf{u}_{tt} - \mathbf{u}_{xx} + g_1(\mathbf{u}_t) &= f_1(\mathbf{u}, \mathbf{v}), \\
\mathbf{v}_{tt} - \mathbf{v}_{xx} + g_2(\mathbf{v}_t) &= f_2(\mathbf{u}, \mathbf{v}),
\end{align*}
\]

over a connected bounded open domain in \( \mathbb{R}^3 \) with nonlinear Robin boundary conditions on \( \mathbf{u} \) and zero boundary conditions on \( \mathbf{v} \). The damping functions \( g_i \) for \( i = 1, 2 \) are expected to be continuous monotone increasing graphs vanishing at the origin and restrictive in growing up at infinity. The interior sources impose themselves on the Nemytskii operator for the supercritical exponent. Using semi-group equipments, the well-posedness of (1.2) is mostly investigated in the relationship of the source and damping terms to the behavior of solutions. One important result included here and later in [13] is that every weak solution blows up in finite time, provides that the initial energy is negative and the sources are more dominant than the damping in the system (1.2).

In [3, 4], Cavalcanti et al. studied the existence of global solutions and the relationship between the asymptotic behavior of the energy and the degenerate wave system/equation with boundary conditions of memory type. The construction of a suitable Lyapunov functional, proves that the energy decays exponentially. The same method is also used in [14] to study the asymptotic behavior of the solutions to a coupled system having integral convolutions as memory terms. The solution of that system decays uniformly in time with rates depending on the speed of decay of the convolutions kernel.

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In recent years, various types of wave equations with linear or nonlinear damping and sources have successfully solved by using Galerkin approximation (see, for instance, [2][11][13][15]). Based on that method associated to a priori estimates, weak convergence, and compactness techniques, and via the construction of a suitable Lyapunov functional, the existence, regularity, blow-up, and exponential decay estimates of solutions for one of such wave equations is showed in [2][11][15]. Moreover, the finite time blow-up of any weak solutions in which negative initial energy fulfills is obtained in [15].

In light of the aforementioned works, we put ourselves into the study of the existence, blow-up, and exponential decay estimate for the system (1.1).

Let \( \Omega = (0, 1) \) and \( \bar{Q}_T = \Omega \times (0, T) \) for \( T > 0 \), a couple of real unknown functions \((u, v)\) is sought for \((x, t) \in \bar{Q}_T = [0, 1] \times [0, T]\). The problem given by (1.1) along with the nonlinear boundary conditions

\[
(1.3) \quad \{eq:main2\} \quad \begin{cases} 
  u(0, t) = 0, -u_x (1, t) + K_1 |u_1 (1, t)|^{p_{1}-2} u_1 (1, t) = \mu_1 |u_t (1, t)|^{q_{1}-2} u_t (1, t), \\
  v(0, t) + K_2 |v_1 (0, t)|^{p_{2}-2} v_1 (0, t) = \mu_2 |v_t (0, t)|^{q_{2}-2} v_t (0, t), v (1, t) = 0, 
\end{cases}
\]

and the initial conditions

\[
(1.4) \quad \{main3\} \quad \begin{cases} 
  u(x, 0) = \tilde{u}_0 (x), u_t (x, 0) = \tilde{u}_1 (x), \\
  v(x, 0) = \tilde{v}_0 (x), v_t (x, 0) = \tilde{v}_1 (x), 
\end{cases}
\]

where for \( i = 1, 2 \) the constants here \( K_i > 0, \mu_i > 0, p_i \geq 2, q_i \geq 2 \) are given and for \( i = 0, 1 \) the given functions \( \tilde{u}_i, \tilde{v}_i \) satisfy conditions specified later.

2. Preliminaries

2.1. Abstract settings. Let us first denote the usual functional spaces used in this paper by the notations \( C^m(\Omega) \), \( W^{m,p} = W^{m,p}(\Omega) \), \( L^p = L^p(\Omega) \), \( H^m = H^{m,2}(\Omega) \) for \( 1 \leq p \leq \infty \) and \( m \in \mathbb{N} \). Let \((\cdot, \cdot)\) be either the scalar product in \( L^2 \) or the dual pairing of a continuous linear functional and an element of a functional space. The notation \( \|\cdot\| \) stands for the norm in \( L^2 \) and we denote by \( \|\cdot\|_X \) the norm in the Banach space \( X \). We call \( X' \) the dual space of \( X \) and denote by \( L^p (0, T; X) \), \( 1 \leq p \leq \infty \) for the Banach space of the real functions \( u : (0, T) \to X \) measurable, such that

\[
\|u\|_{L^p(0,T;X)} = \left( \int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < \infty, \quad \text{for } 1 \leq p < \infty,
\]

and

\[
\|u\|_{L^\infty(0,T;X)} = \text{ess sup}_{0<t<T} \|u(t)\|_X, \quad \text{for } p = \infty.
\]

Let \( u(t), u'(t) = u_t(t), u''(t) = u_{tt}(t), \tilde{u}(t), \tilde{u}'(t) = \tilde{u}_t(t) \), \( \nabla u(t) = u_x(t), \Delta u(t) = u_{xx}(t) \) denote \( u(x, t), \frac{\partial u}{\partial t}(x, t), \frac{\partial^2 u}{\partial x^2}(x, t), \frac{\partial^2 u}{\partial x^2}(x, t) \) respectively.

On \( H^1 \), we use the following norm:

\[
\|u\|_{H^1} = \left( \|u\|^2 + \|u_x\|^2 \right)^{1/2}.
\]

We define

\[
\mathcal{V}_1 = \{ v \in H^1 : v(0) = 0 \}, \quad \mathcal{V}_2 = \{ v \in H^1 : v(1) = 0 \}
\]

two closed subspaces of \( H^1 \). Moreover, the following standard lemmas read the imbedding \( H^1 \) into \( C^0(\bar{\Omega}) \) and the equivalence between two norms, \( \|v_x\| \) and \( \|v\|_{H^1} \) on both \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \).

**Lemma 1.** The imbedding \( H^1 \hookrightarrow C^0(\bar{\Omega}) \) is compact and the following inequality holds

\[
\|v\|_{C^0(\bar{\Omega})} \leq \sqrt{2} \|v\|_{H^1}, \quad \text{for all } v \in H^1.
\]

**Lemma 2.** On \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \), two norm \( v \mapsto \|v_x\| \), and \( v \mapsto \|v\|_{H^1} \) are equivalent. Furthermore,

\[
\|v\|_{C^0(\bar{\Omega})} \leq \|v_x\|, \quad \text{for all } v \in \mathcal{V}_1 \text{ and } \mathcal{V}_2.
\]
For the sake of simplicity, we call \( P \) the problem that contains \([1,1]\) endowed with the conditions \([13,14]\). In addition, we denote the damping terms and also related functions by \( \Psi_r(z) = |z|^{r-2}z \) where \( r \) is a given constant.

2.2. Weak formulation. The weak formulation of the initial-boundary valued problem \( P \) can be given in the following manner:

Find a pair of real unknown solutions \((u, v)\) belonging to the following functional space

\[
\mathcal{W} = \{(u, v) \in L^\infty(0, T; (\mathcal{V}_1 \cap H^2) \times (\mathcal{V}_2 \times H^2)) : (u_t, v_t) \in L^\infty(0, T; \mathcal{V}_1 \times \mathcal{V}_2), (u_{tt}, v_{tt}) \in L^\infty(0, T; L^2 \times L^2)\},
\]

such that \((u, v)\) satisfies the variational equations

\[
\begin{align*}
&\int \langle u_{tt}(t), \phi \rangle + \langle u_x(t), \phi_x \rangle + \lambda_1 \langle \Psi_{r_1}(u(t)), \phi \rangle + \mu_1 \Psi_{q_1}(u(t)) \phi(1) \\
&\quad + \int \langle v_{tt}(t), \tilde{\phi} \rangle + \langle v_x(t), \tilde{\phi}_x \rangle + \lambda_2 \langle \Psi_{r_2}(v(t)), \tilde{\phi} \rangle + \mu_2 \Psi_{q_2}(v(t)) \tilde{\phi}(0) \\
&\quad = K_1 \Psi_{p_1}(u(1, t)) \phi(1) + \langle f_1(u, v), \phi \rangle + \langle F_1(t), \phi \rangle,
\end{align*}
\]

for all \((\phi, \tilde{\phi}) \in \mathcal{V}_1 \times \mathcal{V}_2\), together with the initial conditions

\[
(2.2) \quad \begin{cases} (u(0), u_t(0)) = (\tilde{u}_0, \tilde{u}_1), \quad (v(0), v_t(0)) = (\tilde{v}_0, \tilde{v}_1). \end{cases}
\]

We now make the following assumptions:

\begin{enumerate}
\item [(A1)] \((\tilde{u}_0, \tilde{u}_1) \in (\mathcal{V}_1 \cap H^2) \times \mathcal{V}_1\) and \((\tilde{v}_0, \tilde{v}_1) \in (\mathcal{V}_2 \cap H^2) \times \mathcal{V}_2\);
\item [(A2)] \(F_1, F_2 \in L^1(0, T; L^2)\) such that \(F_1', F_2' \in L^1(0, T; L^2)\);
\item [(A3)] there exists \(F: \mathbb{R}^2 \to \mathbb{R}\) the C\(^2\)-function such that
\end{enumerate}

\[
(2.3) \quad \frac{\partial F}{\partial u}(u, v) = f_1(u, v), \quad \frac{\partial F}{\partial v}(u, v) = f_2(u, v),
\]

and there also exists the constants \(\alpha, \beta > 2\) and \(C > 0\) such that

\[
(2.4) \quad F(u, v) \leq C \left(1 + |u|^\alpha + |v|^\beta \right), \quad \text{for all } u, v \in \mathbb{R};
\]

\begin{enumerate}
\item [(H1)] \((\tilde{u}_0, \tilde{u}_1) \in \mathcal{V}_1 \times L^2\) and \((\tilde{v}_0, \tilde{v}_1) \in \mathcal{V}_2 \times L^2\);
\item [(H2)] \(F_1, F_2 \in L^2(Q_T)\).
\end{enumerate}

Remark 3. There are several examples in which functions \(f_1\) and \(f_2\) satisfy assumption (H3), see e.g. [1, 13]. In particular, the authors in [1] considered

\[
F(u, v) = \alpha |u|^p + 2|uv|^{p+1} + 2\beta |uv|^\beta,
\]

where \(p \geq 3\), \(\alpha > 1\) and \(\beta > 0\). In [13], the authors exploited another analytic one

\[
(2.5) \quad \begin{cases} F(u, v) = \gamma_1 \left(|u|^\alpha + |v|^\beta \right) + \gamma_2 |u|^\alpha |v|^\beta, \end{cases}
\]

where \(\alpha, \beta, \gamma_1\) and \(\gamma_2\) are positive constants with \(\gamma_2 < 2\gamma_1\).

3. The existence and uniqueness of a weak solution

Here we study the existence results involving the uniqueness, of problem \( P \) which differs from that analyzed by the previous works (as showed in the introduction), which consider simpler boundary conditions. It is due to as far as we know there are a few papers which take into account complicated types of boundary conditions, e.g. [1] [13] the two-point boundary conditions (however, the results for a simple type of nonlinear wave equation are usually carried out). Our results included in this section shall therefore extend those previous results with the same strategy, but coupling with necessary modifications.

Now we claim our existence and uniqueness of a weak solution in the following.
Theorem 4. Suppose that (A1)-(A3) hold and the initial datum obey the compatibility relation
\[ u(0) = \phi_0, \quad v(0) = \psi_0, \]
and the time-dependent coefficient functions \( q_j \) are valid, which are
\[ \langle \phi_0, \psi_0 \rangle = 0, \quad \langle \dot{\phi}_0, \dot{\psi}_0 \rangle = 0. \]

If two cases of \((p_1, p_2, q_1, q_2)\) are valid, which are
\[ p_1, p_2 \geq 2 \quad \text{or} \quad \{p_1, p_2 \} \in \{2 \} \cup [3, \infty), \]
and afterwards integrating with respect to the time variable \( t \), we, after some computations, obtain the following
\[ \mathcal{S}_m(t) = \mathcal{S}_m(0) + K_1 \int_0^t \langle \Psi_{p_1}(u_m(s), \dot{u}_m(s)), v_m(s) \rangle \, ds + K_2 \int_0^t \langle \Psi_{p_2}(v_m(s), \dot{v}_m(s)), v_m(s) \rangle \, ds \]
\[ + 2 \int_0^t \left[ \left\langle \frac{\partial F_1}{\partial u}(u_m(s), v_m(s)), u_m(s) \right\rangle + \left\langle \frac{\partial F_2}{\partial v}(u_m(s), v_m(s)), v_m(s) \right\rangle \right] \, ds \]
\[ + 2 \int_0^t \left[ \langle F_1(s), \dot{u}_m(s) \rangle + \langle F_2(s), \dot{v}_m(s) \rangle \right] \, ds \]

where
\[ \mathcal{S}_m(0) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4, \]
\[ S_m(t) = \|\dot{u}_m(t)\|^2 + \|\dot{v}_m(t)\|^2 + \|\nabla u_m(t)\|^2 + \|\nabla v_m(t)\|^2 + 2\lambda \int_0^t \|\dddot{u}_m(s)\|_{L^2} ds \]

\[(3.5) \quad \text{(eq:Sm)} \]

By (3.5) and the third equation of (3.3), there exists a positive constant, says \( S_0 \), that exactly values \( S_m(0) \) for all \( m \in \mathbb{N} \), i.e.

\[ S_m(0) = \|\dot{u}_1\|^2 + \|\dot{v}_1\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 = S_0. \]

We are therefore going to estimate the integrals \( I_k \) for \( k = 1, 4 \) in the right-hand side of (3.4). To do this, let us consider the following elementary inequalities.

**Remark 5** (Young-type inequality). Let \( \delta > 0 \) and \( a, b \geq 0 \) be arbitrarily real numbers and given \( q, q' > 1 \) real constants which are Hölder conjugates of each other. The following inequality holds

\[(3.6) \quad \text{(eq:young)} \quad ab \leq \frac{1}{q}\delta^q a^q + \frac{1}{q'}\delta^{-q'} b^{q'}. \]

**Remark 6.** Given \( N = \frac{1}{2} \max \{ 2; \alpha; \beta; \frac{n(p_1 - 1)}{q_1 - 1}; \frac{n(p_2 - 1)}{q_2 - 1} \} \), then for all \( s \geq 0 \), the inequality \( s^\gamma \leq 1 + s^N \) holds for all \( \gamma \in (0, N] \).

The computations for these kinds of terms \( I_k \) are straightforward. Indeed, estimating \( I_1 \) and \( I_2 \) is the same by using two inequalities of the above remarks. Then, by choosing \( \delta > 0 \) in such a way that

\[ \delta = \min \left\{ \sqrt{\frac{\mu_1 q_1}{2K_1}}; \sqrt{\frac{\mu_2 q_2}{2K_2}} \right\}, \]

it arises naturally for \( C_T > 0 \) representing a bound only depending on \( T \), that

\[(3.7) \quad \text{(eq:esI1)} \quad I_1 + I_2 \leq \frac{1}{2} S_m(t) + C_T \int_0^t [1 + S_m^N(s)] ds. \]

Additionally, for \( C_0 > 0 \) depending only on the initial data \( \bar{u}_0, \bar{v}_0, \bar{u}_1, \bar{v}_1 \), and \( \alpha, \beta \), one can prove that

\[ \|u_m(t)\|_{L^\alpha}^\alpha + \|v_m\|_{L^\alpha}^\beta \leq C_0 + (\alpha + \beta) \int_0^t [1 + S_m^N(s)] ds, \]

the integral \( I_3 \) is essentially estimated by the fact that

\[ I_3 \leq 2 \sup_{|y|, |z| \leq \sqrt{C_0}} |\mathcal{F}(y, z)| + 2C_1 + 2C_1 \left[ C_0 + (\alpha + \beta) \int_0^t [1 + S_m^N(s)] ds \right] \]

\[(3.8) \quad \text{(eq:esI3)} \]

using also the assumption \((A3).\) The last integral can be found easily by the standard Cauchy-Schwartz inequality:

\[(3.9) \quad \text{(eq:esI4)} \quad I_4 \leq \|F_1\|_{L^2(Q_T)}^2 + \|F_2\|_{L^2(Q_T)}^2 + \int_0^t S_m(s) ds \leq C_T + \int_0^t [1 + S_m^N(s)] ds. \]

We hence obtain from (3.7) - (3.9) that for \( 0 \leq t \leq T_m \)

\[ S_m(t) \leq C_T + C_T \int_0^t [1 + S_m^N(s)] ds. \]

We also remark that on the basis of the methods in \([9]\), there exists a constant \( T_\ast > 0 \) depending on \( T \) (but independent of \( m \)) such that
\[(3.10) \quad \{\text{eq:CT} \} \quad S_m(t) \leq C_T, \quad \forall m \in \mathbb{N}, \forall t \in [0, T_*].\]

This result subsequently allows us to take \(T_m = T_*\) for all \(m\).

**Step 2.2. The second estimate.** We now consider the first equation of (3.3). Letting \(t \to 0^+\), then multiplying the equation by \(\tilde{c}_{mj}(0)\) with summing up to \(m\) with respect to \(j\), and importantly by the first compatibility relation (3.11) vanishing the terms \(\mu_1\Psi_{q_1}(\tilde{u}_m(1, 0))\phi_j(1) = \mu_1\Psi_{q_1}(\tilde{u}_1(1))\) and \(K_1\Psi_{p_1}(u_m(1, 0))\phi_j(1) = K_1\Psi_{p_1}(\tilde{u}_0(1))\) in (3.3), we thus obtain

\[
\|\tilde{u}_m(0)\|^2 - \langle \Delta u_m(0), \tilde{u}_m(0) \rangle + \lambda_1 \left\langle |\tilde{u}_1|^{q_1-2} \tilde{u}_1, \tilde{u}_m(0) \right\rangle = \langle f_1(\tilde{u}_0, \tilde{v}_0), \tilde{u}_m(0) \rangle + \langle F_1(0), \tilde{u}_m(0) \rangle.
\]

Relying on the classical inequalities, we have

\[(3.11) \quad \{\text{eq:u:} \} \quad \|\tilde{u}_m(0)\| \leq \|\Delta \tilde{a}_0\| + \lambda_1 \left\| |\tilde{u}_1|^{q_1-1} \right\| + \|f_1(\tilde{u}_0, \tilde{v}_0)\| + \|F_1(0)\|,
\]

then state that there exists a positive constant, says \(C_1\), that exactly values the right-hand side of (3.11) such that

\[
\|\tilde{u}_m(0)\| \leq C_1 \quad \text{for all} \quad m \in \mathbb{N}.
\]

For the second equation of (3.3), one also proves without difficulty using similar arguments that there exists \(C_2 > 0\) in which it bounds \(\|\tilde{v}_m(0)\|\), i.e. \(\|\tilde{v}_m(0)\| \leq C_2\) for all \(m \in \mathbb{N}.

Next, we differentiate (3.3) with respect to \(t\), the first equation becomes

\[(3.12) \quad \{\text{eq:3.10} \} \quad \frac{\langle \tilde{u}_m(t), \phi_j \rangle}{K_1\Psi_{p_1}(u_m(1, t))} = \tilde{u}_m(1, t) \phi_j(1) + \mu_1\Psi_{q_1}(\tilde{u}_m(1, t)) \tilde{u}_m(1, t) \phi_j(1) + \mu_2\Psi_{q_2}(v_m(0, t)) \tilde{v}_m(0, t) \phi_j(0) + \langle F_1(t), \phi_j \rangle,
\]

and the second one is

\[(3.13) \quad \{\text{eq:3.11} \} \quad \frac{\langle \tilde{v}_m(t), \phi_j \rangle}{K_2\Psi_{p_2}(v_m(0, t))} = \tilde{v}_m(0, t) \phi_j(0) + \langle F_2(t), \phi_j \rangle,
\]

for \(1 \leq j \leq m\). Following similar computations above, we multiply the \(j\)-th equation of (3.12) and (3.13), respectively, by \(\tilde{c}_{mj}(t)\) and \(\tilde{d}_{mj}(t)\), then summing with respect to \(j\) up to \(m\), and in addition integrating with respect to the time variable from 0 to \(t\), we shall obtain after some rearrangements

\[
\mathcal{P}_m(t) = \mathcal{P}_m(0) + 2 \int_0^t \left[ \left\langle \frac{\partial^2 F}{\partial u^2}(u_m, v_m) \tilde{u}_m(s), \tilde{u}_m(s) \right\rangle + \left\langle \frac{\partial^2 F}{\partial v \partial u}(u_m, v_m) \tilde{v}_m(s), \tilde{v}_m(s) \right\rangle \right] ds
\]

\[
+ 2 \int_0^t \left[ \left\langle F_1(s), \tilde{u}_m(s) \right\rangle + \left\langle F_2(s), \tilde{v}_m(s) \right\rangle \right] ds
\]

\[
+ 2 \int_0^t \left[ \left\langle K_1\Psi_{p_1}(u_m(1, s)) \tilde{u}_m(1, s), \tilde{u}_m(1, s) \right\rangle + \left\langle K_2\Psi_{p_2}(v_m(0, s)) \tilde{v}_m(0, s), \tilde{v}_m(0, s) \right\rangle \right] ds
\]

\[(3.14) \quad \{\text{eq:Pm:} \} \quad \mathcal{P}_m(t) = \mathcal{P}_m(0) + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4,
\]

where
\[ P_m (t) = \| \ddot{u}_m (t) \|^2 + \| \ddot{v}_m (t) \|^2 + \| \nabla \ddot{u}_m (t) \|^2 + \| \nabla \ddot{v}_m (t) \|^2 \\
+ \frac{8 \lambda_1 (r_1 - 1)}{r_1^2} \int_0^t \left\| \frac{\partial}{\partial s} \left( \frac{1}{\dot{u}_m (x, s)} \right) \right\|^2 ds \\
+ \frac{8 \lambda_2 (r_2 - 1)}{r_2^2} \int_0^t \left\| \frac{\partial}{\partial s} \left( \frac{1}{\dot{v}_m (x, s)} \right) \right\|^2 ds \\
+ \frac{8 \mu_1 (q_1 - 1)}{q_1^4} \int_0^t \left\| \frac{\partial}{\partial s} \left( \frac{1}{\dot{u}_m (1, s)} \right) \right\|^2 ds \\
+ \frac{8 \mu_2 (q_2 - 1)}{q_2^4} \int_0^t \left\| \frac{\partial}{\partial s} \left( \frac{1}{\dot{v}_m (0, s)} \right) \right\|^2 ds. \tag{3.15} \{eq:P_m\} \\
\]

Combining the arguments from the boundedness of \( \| \dot{u}_m (0) \|, \| \dot{v}_m (0) \| \) and the third equation of (3.3) to (3.15), there exists a positive constant, says \( P_0 \), that bounds \( P_m (0) \) for all \( m \in \mathbb{N} \), i.e.

\[ P_m (0) = \| \ddot{u}_m (0) \|^2 + \| \ddot{v}_m (0) \|^2 + \| \nabla \ddot{u}_1 \|^2 + \| \nabla \ddot{v}_1 \|^2 \leq P_0. \tag{3.16} \]

Here the constant \( P_0 \) is dependent of initial data \( \ddot{u}_0, \ddot{v}_0, \dot{u}_1, \dot{v}_1 \), the interior sources \( f_1, f_2 \), the functions \( F_1, F_2 \) and the given constants \( r_1, r_2, \lambda_1, \lambda_2 \).

The technicalities for estimating \( P_m (t) \) are almost similar to (13), so we claim that, by putting

\[ K (T, \mathcal{F}) = \sup_{|y|, |z| \leq \sqrt{T}, |\alpha| = 2} |D^\alpha \mathcal{F} (y, z)|, \]

there are possibilities to estimate the first three integrals, i.e. one can show that

\[ J_1 + J_2 \leq C_T + \int_0^t P_m (s) \, ds, \tag{3.17} \{eq:J1J2\} \]

and by the standard Cauchy-Schwartz inequality, we obtain

\[ J_3 \leq C_T + \int_0^t (\| F_1' (s) \| + \| F_2' (s) \|) P_m (s) \, ds. \tag{3.18} \{eq:J3\} \]

For the last integral \( J_4 \), we shall go through by the following lemma.

**Lemma 7.** If one of the following cases is valid, which is

\[
\begin{align*}
\begin{cases}
p_1, p_2 \geq 2 \\
2 \leq q_1, q_2 \leq 4,
\end{cases}
\text{ or } \begin{cases}
p_1, p_2 \in \{2\} \cup [3, \infty) \\
q_1, q_2 > 4,
\end{cases}
\end{align*}
\]

the integral \( J_4 \) given by (3.14) can be bounded by

\[ 2 \int_0^t \left[ K_1 \psi_{p_1}^\prime (u_m (1, s)) \dot{u}_m (1, s) \ddot{u}_m (1, s) + K_2 \psi_{p_2}^\prime (v_m (0, s)) \dot{v}_m (0, s) \ddot{v}_m (0, s) \right] ds \leq C_T + \frac{1}{2} P_m (t). \tag{3.19} \{eq:J4\} \]

**Proof of Lemma 7.** Since \( J_4 \) can be divided into two separated integrals for which they can be estimated independently. In lieu of taking all possible cases into account, we put myself into the following cases:

- **Case 1:** \( 2 \leq q_1, q_2 \leq 4, p_1 = p_2 = 2 \) and \( 1 \leq q_1, q_2 \leq 4, p_1, p_2 > 2. \)
- **Case 2:** \( q_1, q_2 > 4, p_1, p_2 \geq 3 \) and \( q_1, q_2 > 4, p_1 = p_2 = 2. \)

Now we address below different highly-sophisticated tools corresponding to several definitely complicated terms in each case. This endeavor is well accomplished but not rigorously sharp since our strategy is merely to gain a general uniform bound (3.19).

**Case 1.1:** \( 2 \leq q_1, q_2 \leq 4, p_1 = p_2 = 2 \)

We have
\[ J_1 = 2 \int_0^t \left[ K_1 \Psi'_2 (u_m (1, s)) \dot{u}_m (1, s) \ddot{u}_m (1, s) + K_2 \Psi'_2 (v_m (0, s)) \dot{v}_m (0, s) \ddot{v}_m (0, s) \right] ds \]

\[ = 2K_1 \int_0^t \left[ |\dot{u}_m (1, s)|^{1 - \frac{4q}{q_2}} |\dot{u}_m (1, s)|^{\frac{4q}{q_2} - 1} \ddot{u}_m (1, s) \right] ds \]
\[ + 2K_2 \int_0^t \left[ |\dot{v}_m (0, s)|^{1 - \frac{4q}{q_2}} |\dot{v}_m (0, s)|^{\frac{4q}{q_2} - 1} \ddot{v}_m (0, s) \right] ds \]
\[ \leq \frac{4K_1}{q_1} \int_0^t \left[ |\dot{u}_m (1, s)|^{2 - \frac{4q}{q_2}} \left| \frac{\partial}{\partial s} \left( |\dot{u}_m (1, s)|^{\frac{4q}{q_2} - 1} \ddot{u}_m (1, s) \right) \right| ds \]
\[ + \frac{4K_2}{q_2} \int_0^t \left[ |\dot{v}_m (0, s)|^{2 - \frac{4q}{q_2}} \left| \frac{\partial}{\partial s} \left( |\dot{v}_m (0, s)|^{\frac{4q}{q_2} - 1} \ddot{v}_m (0, s) \right) \right| ds. \]

Due to (3.5) and (3.15) which read

\[ S_m (t) \geq 2\mu_1 \int_0^t |\dot{u}_m (1, s)|^{q_1} ds + 2\mu_2 \int_0^t |\dot{v}_m (0, s)|^{q_2} ds, \]

\[ P_m (t) \geq \frac{8\mu_1 (q_1 - 1)}{q_1} \int_0^t \left| \frac{\partial}{\partial s} \left( |\dot{u}_m (1, s)|^{\frac{4q}{q_2} - 1} \ddot{u}_m (1, s) \right) \right|^2 ds + \frac{8\mu_2 (q_2 - 1)}{q_2} \int_0^t \left| \frac{\partial}{\partial s} \left( |\dot{v}_m (0, s)|^{\frac{4q}{q_2} - 1} \ddot{v}_m (0, s) \right) \right|^2 ds, \]

together with (3.10), they give us the fact that

\[ J_1 \leq \frac{2K_1}{q_1} \int_0^t \left[ \frac{1}{\delta_1} |\dot{u}_m (1, s)|^{4 - q_1} + \delta_1 \left| \frac{\partial}{\partial s} \left( |\dot{u}_m (1, s)|^{\frac{4q}{q_2} - 1} \ddot{u}_m (1, s) \right) \right|^2 \right] ds \]
\[ + \frac{2K_2}{q_2} \int_0^t \left[ \frac{1}{\delta_2} |\dot{v}_m (0, s)|^{4 - q_2} + \delta_2 \left| \frac{\partial}{\partial s} \left( |\dot{v}_m (0, s)|^{\frac{4q}{q_2} - 1} \ddot{v}_m (0, s) \right) \right|^2 \right] ds \]
\[ \leq \frac{2K_1}{q_1 \delta_1} \int_0^t \left[ 1 + |\dot{u}_m (1, s)|^{q_1} \right] ds + \frac{2K_2}{q_2 \delta_2} \int_0^t \left[ 1 + |\dot{v}_m (0, s)|^{q_2} \right] ds + \left( \frac{\delta_1 K_1 q_1}{4\mu_1 (q_1 - 1)} + \frac{\delta_2 K_2 q_2}{4\mu_2 (q_2 - 1)} \right) P_m (t) \]
\[ \leq \frac{2K_1}{q_1 \delta_1} \left[ T + \frac{1}{2\mu_1} S_m (t) \right] + \frac{2K_2}{q_2 \delta_2} \left[ T + \frac{1}{2\mu_2} S_m (t) \right] + \left( \frac{\delta_1 K_1 q_1}{4\mu_1 (q_1 - 1)} + \frac{\delta_2 K_2 q_2}{4\mu_2 (q_2 - 1)} \right) P_m (t) \]

(3.23) \{eq:3.23\}

Here we additionally use two classical inequalities which are \( a^{4-q} \leq 1 + a^q \) for all \( a \geq 0, 2 \leq q \leq 4 \) and \( 2ab \leq \delta a^2 + \delta^{-1}b^2 \) for all \( a, b \geq 0 \) and \( \delta > 0 \). Hereby, to deduce (3.19) from (3.23) we choose

\[ \delta_1 = \delta_2 \leq \frac{2\mu_1 \mu_2 (q_1 - 1) (q_2 - 1)}{K_1 q_1 (q_2 - 1) + K_2 q_2 (q_1 - 1)}. \]

Case 1.2: \( 2 \leq q_1, q_2 \leq 4, p_1, p_2 > 2 \)

By similar and simple computations, we have
\[ J_4 = 2 \int_0^t \left[ K_1 \Psi_{p_1} (u_m (1, s)) \dot{u}_m (1, t) \ddot{u}_m (1, s) + K_2 \Psi_{p_2} (v_m (0, s)) \dot{v}_m (0, s) \ddot{v}_m (0, s) \right] ds \]

\[ = 2 K_1 (p_1 - 1) \int_0^t |u_m (1, s)|^{p_1 - 2} \left| \dot{u}_m (1, s) \right|^{1 - \frac{2}{p_1}} \ddot{u}_m (1, s) \left| \dddot{u}_m (1, s) \right|^{\frac{2}{p_1} - 1} \dot{u}_m (1, s) ds \]

\[ + 2 K_2 (p_2 - 1) \int_0^t |v_m (0, s)|^{p_2 - 2} \left| \dot{v}_m (0, s) \right|^{1 - \frac{2}{p_2}} \ddot{v}_m (0, s) \left| \dddot{v}_m (0, s) \right|^{\frac{2}{p_2} - 1} \dot{v}_m (0, s) ds \]

\[ = \frac{4}{q_1} K_1 (p_1 - 1) \int_0^t |u_m (1, s)|^{p_1 - 2} \left| \dot{u}_m (1, s) \right|^{1 - \frac{2}{p_1}} \ddot{u}_m (1, s) \frac{\partial}{\partial s} \left| \dddot{u}_m (1, s) \right|^{\frac{2}{p_1} - 1} \dot{u}_m (1, s) ds \]

\[ + \frac{4}{q_2} K_2 (p_2 - 1) \int_0^t |v_m (0, s)|^{p_2 - 2} \left| \dot{v}_m (0, s) \right|^{1 - \frac{2}{p_2}} \ddot{v}_m (0, s) \frac{\partial}{\partial s} \left| \dddot{v}_m (0, s) \right|^{\frac{2}{p_2} - 1} \dot{v}_m (0, s) ds. \]

Observe that \( |u_m (1, s)| + |v_m (0, s)| \leq \sqrt{C_T} \) by Lemma 2, (3.5) and (3.10), one continues by estimating the integral \( J_4 \) that

\[ J_4 \leq \frac{4}{q_1} K_1 (p_1 - 1) C_T^{\frac{1}{p_1} - 1} \int_0^t |u_m (1, s)|^{2 - \frac{2}{p_1}} \left| \dot{u}_m (1, s) \right|^{\frac{2}{p_1} - 1} \frac{\partial}{\partial s} \left| \dddot{u}_m (1, s) \right|^{\frac{2}{p_1} - 1} \dot{u}_m (1, s) ds \]

\[ + \frac{4}{q_2} K_2 (p_2 - 1) C_T^{\frac{1}{p_2} - 1} \int_0^t |v_m (0, s)|^{2 - \frac{2}{p_2}} \left| \dot{v}_m (0, s) \right|^{\frac{2}{p_2} - 1} \frac{\partial}{\partial s} \left| \dddot{v}_m (0, s) \right|^{\frac{2}{p_2} - 1} \dot{v}_m (0, s) ds, \]

which gives back to (3.20) in the previous case. By the previously described strategy, (3.19) thus holds.

**Case 2.1:** \( q_1, q_2 > 4, p_1, p_2 \geq 3 \)

We start by using integration by parts

\[ J_4 = 2 \int_0^t \left[ K_1 \Psi_{p_1} (u_m (1, s)) \dddot{u}_m (1, t) \dddot{u}_m (1, s) + K_2 \Psi_{p_2} (v_m (0, s)) \dddot{v}_m (0, s) \dddot{v}_m (0, s) \right] ds \]

\[ = K_1 (p_1 - 1) |u_m (1, t)|^{p_1 - 2} \dddot{u}_m (1, t) - K_1 (p_1 - 2) \int_0^t |u_m (1, s)|^{p_1 - 4} u_m (1, s) \dddot{u}_m (1, s) ds \]

\[ + K_2 (p_2 - 1) |v_m (0, t)|^{p_2 - 2} \dddot{v}_m (0, t) - K_2 (p_2 - 2) \int_0^t |v_m (0, s)|^{p_2 - 4} v_m (0, s) \dddot{v}_m (0, s) ds. \]

Still based on (3.5) and (3.10), we first estimate \( J_4 \) as follows:

\[ J_4 \leq K_1 (p_1 - 1) S_{m}^{\frac{2}{p_1} - 1} \dddot{u}_m (1, t) + K_1 (p_1 - 1) (p_1 - 2) \int_0^t S_{m}^{\frac{2}{p_1} - 3} (s) \dddot{u}_m (1, s) ds \]

\[ + K_2 (p_2 - 1) S_{m}^{\frac{2}{p_2} - 1} \dddot{v}_m (0, t) + K_2 (p_2 - 1) (p_2 - 2) \int_0^t S_{m}^{\frac{2}{p_2} - 3} (s) \dddot{v}_m (0, s) ds \]

\[ \leq C_T \left( |\dddot{u}_m (1, t)|^2 + |\dddot{v}_m (0, t)|^2 \right) + C_T \int_0^t \left[ |\dddot{u}_m (1, s)|^3 + |\dddot{v}_m (0, s)|^3 \right] ds. \]

Secondly, it is immediate to observe that using the inequality \( a^3 \leq 1 + a^4 \) for all \( a \geq 0 \) and \( q \geq 3 \), together with (3.21) and (3.10), let us the following inequality

\[ \int_0^t \left[ |\dddot{u}_m (1, s)|^3 + |\dddot{v}_m (0, s)|^3 \right] ds \leq T + \frac{1}{2} \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) S_m (t) \leq C_T. \]

One also deduces from
\[ \left| \dot{u}_m (1, t) \right|^{q_1} \leq 2 \left| \ddot{u}_m (1) \right|^{q_1} + 2 \int_0^t \frac{\partial}{\partial s} \left( \left| \dot{u}_m (1, s) \right|^{q_1} \right) ds, \]
\[ \left| \dot{v}_m (0, t) \right|^{q_2} \leq 2 \left| \ddot{v}_m (0) \right|^{q_2} + \frac{q_2^2 T}{4 \mu_1 (q_1 - 1)} \mathcal{P}_m (t), \]

along with the elementary inequality \((a + b)^2 \leq 2 (a^2 + b^2)\) for all \(a, b \geq 0\), with Hölder’s inequality and \([3,19]\) that

\[ \left| \dot{u}_m (1, t) \right|^{q_1} \leq 2 \left| \ddot{u}_m (1) \right|^{q_1} + 2 \int_0^t \frac{\partial}{\partial s} \left( \left| \dot{u}_m (1, s) \right|^{q_1} \right) ds \]

\[ \leq 2 \left| \ddot{u}_m (1) \right|^{q_1} + \frac{q_2^2 T}{4 \mu_1 (q_1 - 1)} \mathcal{P}_m (t), \]

and by the similar way, we get

\[ \left| \dot{v}_m (0, t) \right|^{q_2} \leq 2 \left| \ddot{v}_m (0) \right|^{q_2} + \frac{q_2^2 T}{4 \mu_2 (q_2 - 1)} \mathcal{P}_m (t) . \]

Then, following the natural results of inequalities, which are

\[ (a + b)^2 \leq a^2 + b^2, \quad \forall a, b \geq 0, \forall q \geq 2, \]

\[ ab \leq \left( 1 - \frac{2}{q} \right) \delta \frac{a^{q_2}}{b^{q_2}} + \frac{2}{q} \delta \frac{a^{q_2}}{b^{q_2}}, \quad \forall a, b \geq 0, \forall q > 2, \delta > 0, \]

we therefore obtain

\[ C_T \left( \left| \dot{u}_m (1, t) \right|^2 + \left| \dot{v}_m (0, t) \right|^2 \right) \leq C_T \left( 2 \left| \ddot{u}_m (1) \right|^{q_1} + \frac{q_2^2 T}{4 \mu_1 (q_1 - 1)} \mathcal{P}_m (t) \right)^{\frac{2}{q_1}} + C_T \left( 2 \left| \ddot{v}_m (0) \right|^{q_2} + \frac{q_2^2 T}{4 \mu_2 (q_2 - 1)} \mathcal{P}_m (t) \right)^{\frac{2}{q_2}} \leq C_T \left[ 2^{\frac{2}{q_1}} \left| \ddot{u}_m (1) \right|^{2} + 2^{\frac{2}{q_2}} \left| \ddot{v}_m (0) \right|^{2} \right] + C_T \left[ \left( \frac{q_1^2 T}{4 \mu_1 (q_1 - 1)} \right)^{\frac{2}{q_1}} \mathcal{P}_m^{\frac{2}{q_1}} (t) + \left( \frac{q_2^2 T}{4 \mu_2 (q_2 - 1)} \right)^{\frac{2}{q_2}} \mathcal{P}_m^{\frac{2}{q_2}} (t) \right] \leq C_0 + C_T \left[ \left( 1 - \frac{2}{q_1} \right) \delta_1 \frac{a^{q_2}}{b^{q_2}} + \frac{2}{q_1} \delta_1 \frac{a^{q_2}}{b^{q_2}} \right] + C_T \left[ \left( 1 - \frac{2}{q_2} \right) \delta_2 \frac{a^{q_2}}{b^{q_2}} + \frac{2}{q_2} \delta_2 \frac{a^{q_2}}{b^{q_2}} \right] \leq C_T \left( \delta_1, \delta_2 \right) + 2 \left( \frac{\delta_1}{q_1} + \frac{\delta_2}{q_2} \right) \mathcal{P}_m (t) . \]

Hence, to deduce \([3,19]\), we choose \( \delta = \delta_1 = \delta_2 > 0 \) such that \( q_2 \delta \frac{q_2}{q_1} + q_1 \delta \frac{q_1}{q_2} \leq \frac{q_1 q_2}{q_1 q_2} \).

**Case 2.2:** \( q_1, q_2 > 4, p_1 = p_2 = 2 \)

By the same arguments exploited in the previous case, here we can state that
\[ J_4 = 2 \int_0^t \left[ K_1 \Psi'_2 \left( u_m (1, s) \right) \dot{u}_m (1, s) \dot{u}_m (1, s) + K_2 \Psi'_2 \left( v_m (0, s) \right) \dot{v}_m (0, s) \dot{v}_m (0, s) \right] \, ds \]
\[ = K_1 \int_0^t \frac{d}{ds} \left( |\dot{u}_m (1, s)|^2 \right) \, ds + K_2 \int_0^t \frac{d}{ds} \left( |\dot{v}_m (0, s)|^2 \right) \, ds \]
\[ = K_1 \left( |\dot{u}_m (1, t)|^2 - \dot{u}_m^2 (1, 1) \right) + K_2 \left( |\dot{v}_m (0, t)|^2 - \dot{v}_m^2 (0, 0) \right) \]
\[ \leq K_1 |\dot{u}_m (1, t)|^2 + K_2 |\dot{v}_m (0, t)|^2 , \]
also leads to (3.19).
Hence, we complete the proof of Lemma 7.

Now, combining (3.17), (3.18), and (3.19) we are in a great position to give the fact that
\[ P_m (t) \leq 2 P_0 + 6 C_T + 2 \int_0^t (1 + \| F'_1 (s) \| + \| F'_2 (s) \|) P_m (s) \, ds , \]
Thanks to Gronwall’s inequality, we conclude that
\[ P_m (t) \leq (2 P_0 + 6 C_T) \exp \left[ \int_0^t (1 + \| F'_1 (s) \| + \| F'_2 (s) \|) \, ds \right] \leq C_T , \]
for all \( m \in \mathbb{N} \) and \( t \in [0, T_*] \).

**Step 3. Passing to the limit.** The existence of solution on the interval \([0, T_*]\) is now approaching. To summarize, using the Banach-Alaoglu theorem (see, e.g., [3]), the uniform bounds with respect to \( m \), as stated in the above results (3.25), (3.10), (3.15), and (3.24), imply that one can extract a further subsequence (which we relabel with the index \( m \)) necessary such that
\[ (u_m, v_m) \rightharpoonup (u, v) \text{ weak-* in } L^\infty (0, T_*; \mathcal{V}_1 \times \mathcal{V}_2) , \]
\[ (\dot{u}_m, \dot{v}_m) \rightharpoonup (\dot{u}, \dot{v}) \text{ weakly in } L^{r_1} (Q_{T_*}) \times L^{r_2} (Q_{T_*}) \text{ and weak-* in } L^\infty (0, T_*; \mathcal{V}_1 \times \mathcal{V}_2) , \]
\[ (\ddot{u}_m, \ddot{v}_m) \rightharpoonup (\ddot{u}, \ddot{v}) \text{ weak-* in } L^\infty (0, T_*; L^2 \times L^2) , \]
\[ (u_m (1, \cdot), v_m (0, \cdot)) \rightharpoonup (u (1, \cdot), v (0, \cdot)) \text{ weakly in } W^{1,q_1} (0, T_*) \times W^{1,q_2} (0, T_*) , \]
\[ (\ddot{u}_m (1, \cdot), \ddot{v}_m (0, \cdot)) \rightharpoonup (\ddot{u} (1, \cdot), \ddot{v} (0, \cdot)) \text{ weakly in } L^{q_1} (0, T_* \times L^{q_2} (0, T_*) , \]
\[ \left( |\dddot{u}_m (1, \cdot)|^{\frac{q_1}{q_1 - 1}} \dddot{u}_m (1, \cdot), |\dddot{v}_m (0, \cdot)|^{\frac{q_2}{q_2 - 1}} \dddot{v}_m (0, \cdot) \right) \rightharpoonup (\chi_1, \chi_2) \text{ weakly in } H^1 (0, T_*) \times H^1 (0, T_*) , \]
\[ \left( \frac{\partial}{\partial t} \left( |\dddot{u}_m (1, \cdot)|^{\frac{q_1}{q_1 - 1}} \dddot{u}_m (1, \cdot) \right), \frac{\partial}{\partial t} \left( |\dddot{v}_m (0, \cdot)|^{\frac{q_2}{q_2 - 1}} \dddot{v}_m (0, \cdot) \right) \right) \rightharpoonup (\chi_3, \chi_4) \text{ weakly in } L^2 (Q_{T_*}) \times L^2 (Q_{T_*}) . \]

Furthermore, by Aubin-Lions compactness theorem in combination with the imbeddings \( H^2 (0, T_*) \hookrightarrow C^1 ([0, T_*]) , H^1 (0, T_*) \hookrightarrow C^0 ([0, T_*]), W^{1,q_1} (0, T_* \hookrightarrow C^0 ([0, T_*]), W^{1,q_2} (0, T_* \hookrightarrow C^0 ([0, T_*]), it is straightforward to go on extracting from weak convergence results (3.25) - (3.31) a subsequence \( \{(u_m, v_m)\} \) such that
\[ (u_m, v_m) \rightarrow (u, v) \text{ strongly in } L^2 (Q_{T_*}) \times L^2 (Q_{T_*}) \text{ and almost everywhere in } Q_{T_*} , \]
\[ (\dot{u}_m, \dot{v}_m) \rightarrow (\dot{u}, \dot{v}) \text{ strongly in } L^2 (Q_{T_*}) \times L^2 (Q_{T_*}) \text{ and almost everywhere in } Q_{T_*} , \]
\[ (u_m(1,\cdot), v_m(0,\cdot)) \to (u(1,\cdot), v(0,\cdot)) \text{ strongly in } C^0([0,T_*]) \times C^0([0,T_*]), \]  

\[ (\dot{u}_m(1,\cdot), \dot{v}_m(0,\cdot)) \to (\chi_1, \chi_2) \text{ strongly in } C^0([0,T_*]) \times C^0([0,T_*]). \]  

Now we have to show the convergence of the nonlinear terms including damping and interior sources. In fact, using the continuity argument of \( f_1 \), one deduces that

\[ f_1(u_m, v_m) \to f_1(u, v) \text{ almost everywhere in } Q_T. \]

Besides, observe that \( \| f_1(u_m, v_m) \|_{L^2(Q_T)} \) is bounded by \( \sqrt{T_*} \sup_{|y|, |z| \leq \sqrt{C_T}} |f_1(y, z)| \) which cannot go to infinity, and together with a lemma of Lions \cite{10} Lemma 1.3, one continues to obtain the following

\[ f_1(u_m, v_m) \to f_1(u, v) \text{ weakly in } L^2(Q_T), \]

and so is the weak convergence for \( f_2 \), namely

\[ f_2(u_m, v_m) \to f_2(u, v) \text{ weakly in } L^2(Q_T). \]

The weak convergence of damping terms is on its way. Thanks to the inequality

\[ |\Psi_r(z_1) - \Psi_r(z_2)| \leq (r - 1) C^{r-2} |z_1 - z_2|, \quad \forall z_1, z_2 \in [-C, C], C > 0, r \geq 2 \]

in accordance with \eqref{3.15}, \eqref{3.24} and \eqref{3.33}, one easily obtains

\[ (\Psi_{r_1}(u_m), \Psi_{r_2}(v_m)) \to (\Psi_{r_1}(u), \Psi_{r_2}(v)) \text{ strongly in } L^2(Q_{T_*}) \times L^2(Q_{T_*}). \]

It is then worthwhile to mention that \eqref{4.34} gives

\[ (\Psi_{p_1}(u_m(1,\cdot), v_m(0,\cdot))) \to (\Psi_{p_1}(u(1,\cdot), v(0,\cdot))) \text{ strongly in } C^0([0,T_1]) \times C^0([0,T_*]), \]

by the continuity argument of \( \Psi_{p_i} \) for \( i = 1, 2 \), and \eqref{3.35} lets

\[ (\dot{u}_m(1,\cdot), \dot{v}_m(0,\cdot)) \to \left( |\chi_1|^2 \chi_1, |\chi_2|^2 \chi_2 \right) \text{ strongly in } C^0([0,T_*]) \times C^0([0,T_*]). \]

Thus, we take \eqref{3.29} merging with \eqref{3.40} to have

\[ (\dot{u}_m(1,\cdot), \dot{v}_m(0,\cdot)) \to (\dot{u}(1,\cdot), \dot{v}(0,\cdot)) \]

due to the uniqueness of convergence.

In the same vein, we claim

\[ (\Psi_{q_1}(u_m(1,\cdot), v_m(0,\cdot))) \to (\Psi_{q_1}(u(1,\cdot), v(0,\cdot))) \text{ strongly in } C^0([0,T_*]) \times C^0([0,T_*]) \]

from \eqref{3.40} and \eqref{3.41}.

From here on, combining \eqref{3.26}, \eqref{3.27}, \eqref{3.30}, \eqref{3.30}, \eqref{3.37}, \eqref{3.38}, \eqref{3.39}, \eqref{3.35}, \eqref{3.35} is enough to pass to the limit in \eqref{3.3} to show that \((u, v)\) satisfies the problem \((P)\). In addition, one can use \eqref{3.26}, \eqref{3.27}, \eqref{3.32} and \(\text{(A2)}\) to prove that

\[
\begin{align*}
  u_{xx} &= u_t + \lambda_1 \Psi_{r_1}(u_t) - f_1(u, v) - F_1 \in L^\infty(0, T_*; L^2), \\
  v_{xx} &= v_t + \lambda_2 \Psi_{r_2}(v_t) - f_2(u, v) - F_2 \in L^\infty(0, T_*; L^2)
\end{align*}
\]

which verifies \((u, v) \in L^\infty(0, T; (V_1 \cap H^2) \times (V_2 \cap H^2))\) and simultaneously completes the proof of the existence of a local weak solution.
Step 4. Uniqueness of the solution. The existence of solutions on the interval \([0,T_*]\) is established above; therefore, it now remains to show the last statement of theorem, namely the uniqueness. Suppose \((u_1, v_1)\) and \((u_2, v_2)\) are two solutions to \((P)\) on the interval \([0,T_*]\), which is devoted to the case \(q_1 = q_2 = 2\) and \(p_1, p_2 \geq 2\), and they go along with the same initial datum \((\bar{u}_0, \bar{v}_1)\) and \((\bar{u}_0, \bar{v}_1)\), then they must be equal. We also recall that these solutions must belong to the proved functional space properties in (3.2). Define \((u, v) := (u_1 - u_2, v_1 - v_2)\) and based on (2.1) and (2.2), these quantities satisfy the following system of functional equations:

\[
\begin{align*}
\langle u_{tt}(t), \phi \rangle + \langle u_{t}(t), \phi_x \rangle + \lambda_1 \langle \Psi_{r_1} (u_1(t)) - \Psi_{r_1} (\tilde{u}_2(t)), \phi \rangle + \mu_1 u_t (1, t) \phi (1) \\
= K_1 [\Psi_{p_1} (u_1(1,t)) - \Psi_{p_1} (u_2(1,t))] \phi (1) + \langle f_1 (u_1, v_1) - f_2 (u_2, v_2), \phi \rangle,
\end{align*}
\]

(3.43) \{eq:3.43\}

\[
\begin{align*}
\langle v_{tt}(t), \phi \rangle + \langle v_{t}(t), \phi_x \rangle + \lambda_2 \langle \Psi_{r_2} (v_1(t)) - \Psi_{r_2} (\tilde{v}_2(t)), \phi \rangle + \mu_2 v_t (0, t) \phi (0) \\
= K_2 [\Psi_{p_2} (v_1(0,t)) - \Psi_{p_2} (v_2(0,t))] \phi (0) + \langle f_2 (u_1, v_1) - f_2 (u_2, v_2), \phi \rangle,
\end{align*}
\]

(3.44) \{eq:3.44\}

for all \((\phi, \tilde{\phi}) \in \mathbb{V}_1 \times \mathbb{V}_2\) and having zero initial conditions

\[
u (0) = v (0) = u_t (0) = v_t (0) = 0.
\]

Taking into account \((\phi, \tilde{\phi}) = (u_1, v_1)\) in (3.43) and (3.44), then integrating with respect to \(t\), we obtain the following:

\[
\begin{align*}
\mathcal{W} (t) &= 2 \int_0^t \langle f_1 (u_1, v_1) - f_1 (u_2, v_2), u_t (s) \rangle \, ds + 2 \int_0^t \langle f_2 (u_1, v_1) - f_2 (u_2, v_2), v_t (s) \rangle \, ds \\
&\quad + 2K_1 \int_0^t [\Psi_{p_1} (u_1(1,t)) - \Psi_{p_1} (u_2(1,t))] u_t (1, s) \, ds + 2K_2 \int_0^t [\Psi_{p_2} (v_1(0,t)) - \Psi_{p_2} (v_2(0,t))] v_t (0, s) \, ds.
\end{align*}
\]

(3.45) \{eq:W1\}

where

\[
\mathcal{W} (t) = \| u_t (t) \|^2 + \| v_t (t) \|^2 + \| u_x (t) \|^2 + \| v_x (t) \|^2 \\
+ 2\lambda_1 \int_0^t \langle \Psi_{r_1} (\tilde{u}_1 (s)) - \Psi_{r_1} (\tilde{u}_2 (s)), u_t (s) \rangle \, ds + 2\mu_1 \int_0^t |u_t (1, s)|^2 \, ds \\
+ 2\lambda_2 \int_0^t \langle \Psi_{r_2} (\tilde{v}_1 (s)) - \Psi_{r_2} (\tilde{v}_2 (s)), v_t (s) \rangle \, ds + 2\mu_2 \int_0^t |v_t (0, s)|^2 \, ds.
\]

(3.46) \{eq:W1\}

Our procedure below is essentially similar to the above parts: attempt to estimate \(\mathcal{K}_i\) for \(i = 1, 4\) to derive the uniform boundedness of \(\mathcal{W} (t)\) for which we can use Gronwall’s inequality, then formally make the proof of uniqueness self-contained. To handle this, we first state a useful inequality: for all \(r \geq 2\), there exists \(C_r > 0\) such that

\[
\langle \Psi_r (z_1) - \Psi_r (z_2) \rangle (z_1 - z_2) \geq C_r |z_1 - z_2|^r, \quad \forall z_1, z_2 \in \mathbb{R}.
\]

It therefore leads to the fact that

\[
\mathcal{W} (t) \geq \| u_t (t) \|^2 + \| v_t (t) \|^2 + \| u_x (t) \|^2 + \| v_x (t) \|^2 \\
+ 2C_1 \lambda_1 \int_0^t \| u_t (s) \|_{L^r_{x_1}}^r \, ds + C_2 \lambda_2 \int_0^t \| v_t (s) \|_{L^r_{x_2}}^r \, ds \\
+ 2\mu_1 \int_0^t |u_t (1, s)|^2 \, ds + 2\mu_2 \int_0^t |v_t (0, s)|^2 \, ds.
\]

(3.47) \{eq:W2\}

Secondly, we introduce

\[
M = \max_{i=1,2} \left( \| \nabla u_i \|_{L^\infty (0,T;H^1)} + \| \nabla v_i \|_{L^\infty (0,T;H^1)} \right),
\]
\[
C_i(M) = \sup_{|y|, |z| \leq M} \left( \left| \frac{\partial f_i}{\partial y} (y, z) \right| + \left| \frac{\partial f_i}{\partial z} (y, z) \right| \right), \quad i = 1, 2.
\]

Then, it suffices to estimate \( K_i \) for \( i = 1, 4 \). Indeed, applying the Cauchy-Schwartz inequality is straightforward to have

\[
K_1 + K_2 \leq 2 \int_0^t \left( \|f_1(u_1, v_1) - f_1(u_2, v_2)\| + \|f_2(u_1, v_1) - f_2(u_2, v_2)\| \right) ds \\
\leq 2 \int_0^t \left( \|u_1(s)\| + \|v_1(s)\| \right) \left( C_1(M) \|u_1(s)\| + C_2(M) \|v_1(s)\| \right) ds \\
\leq 2 \int_0^t \left( \|u_2(s)\| + \|v_2(s)\| \right) \left( C_1(M) \|u_2(s)\| + C_2(M) \|v_2(s)\| \right) ds
\]

(3.48) \{eq:K1K2\}

To estimate \( K_3 \) and \( K_4 \), we only need to consider two cases, \( p_1 = p_2 = 2 \) and \( p_1, p_2 > 2 \). First, one may easily show that for \( p_1 = p_2 = 2 \),

\[
K_3 + K_4 = 2K_1 \int_0^t u_1(1, s) u_1(1, s) ds + 2K_2 \int_0^t v(0, s) v(0, s) ds \\
\leq \frac{K_1^2}{\mu_1} \int_0^t u_1^2(1, s) ds + \mu_1 \int_0^t u_1^2(1, s) ds + \frac{K_2^2}{\mu_2} \int_0^t v^2(0, s) ds + \mu_2 \int_0^t v_2^2(0, s) ds
\]

(3.49) \{eq:K3K4\}

where we have followed from (3.47) the intrinsic inequality

\[
W(t) \geq u^2(1, t) + v^2(0, t) + 2 \left( \mu_1 \int_0^t |u_1(1, s)|^2 ds + \mu_2 \int_0^t |v_1(0, s)|^2 ds \right).
\]

For \( p_1, p_2 > 2 \), we have

\[
K_3 + K_4 \leq 2 \left( K_1 (p_1 - 1) C_1^{p_1-2} \int_0^t |u_1(1, s)| u_1(1, s) ds + K_2 (p_2 - 1) C_2^{p_2-2} \int_0^t |v(0, s)| v(0, s) ds \right)
\]

(3.50) \{eq:k3k4\}

where we have recalled the following inequality

\[
(3.51) \{eq:3.51\} \quad |\Psi_r(z_1) - \Psi_r(z_2)| \leq (r - 1) C^{r-2} |z_1 - z_2|, \quad \forall z_1, z_2 \in [-C, C], C > 0, r \geq 2.
\]

Combining (3.49) and (3.50), we claim that there exists \( \eta = \eta(p_1, p_2) > 0 \) depending on \( p_1, p_2 \) such that

\[
(3.52) \{eq:finalk3k4\} \quad K_3 + K_4 \leq \eta(p_1, p_2) \int_0^t W(s) ds + \frac{1}{2} W(t).
\]

Therefore, (3.48) and (3.52) together with (3.45), (3.46) and (3.47) imply that

\[
W(t) \leq 2 (2 (C_1(M) + C_2(M) + \eta(p_1, p_2) \int_0^t W(s) ds.
\]

Thus, thanks to Gronwall’s inequality, we have \( W(t) \equiv 0 \) leading to the uniqueness of solution. Hence, this also completes the proof of Theorem 4.
Remark 8. A further fundamental result stemming from the regularity of weak solutions in the above theorem is related to the existence of a strong solution. In fact, (3.2) allows us to show that there exists a pair of strong solutions \((u, v)\) to the problem \((P)\), which satisfies

\[
\begin{align*}
(u, v) &\in L^\infty(0, T_r; (V_1 \cap H^2) \times (V_2 \cap H^2)) \cap C^0([0, T_r]; V_1 \times V_2) \cap C^1([0, T_r]; L^2 \times L^2), \\
(u_t, v_t) &\in L^\infty(0, T_r; V_1 \times V_2) \cap C^0([0, T_r]; L^2 \times L^2), \\
(\phi, \tilde{\phi}) &\in L^\infty(Q_r) \times L^r(Q_r), \\
\int_0^t \langle \tilde{\phi} \rangle \, ds &\in L^r(Q_r), \\
\int_0^t \langle \tilde{\phi} \rangle \, ds &\in L^r(Q_r), \\
\int_0^t \langle \tilde{\phi} \rangle \, ds &\in L^r(Q_r),
\end{align*}
\]

Furthermore, if one rules out in some levels the regularity of initial datum, it then also admits a theorem regarding the existence and uniqueness of a local weak solution.

Theorem 9. Taking \(q_1 = q_2 = 2\) and \(p_1, p_2 \geq 2\) into account, let us assume that (A3) still holds. If the initial datum which obey the compatibility conditions (3.7) only satisfy (H1) and the functions \(F_1, F_2\) are defined in (H2) instead of (A1) and (A2), respectively, the problem \((P)\) has a unique local solution \((u, v)\) such that

\[
\begin{align*}
(u, v) &\in C^0([0, T_r]; V_1 \times V_2) \cap C^1([0, T_r]; L^2 \times L^2), \\
(u_t, v_t) &\in L^r(Q_r),
\end{align*}
\]

for \(T_r > 0\) small enough.

Proof of Theorem 9. In this proof, let us first recall the weaker assumptions on initial datum and miscellaneous functions, that is, \((\tilde{u}_0, \tilde{v}_1) \in V_1 \times L^2\), \((\tilde{v}_0, \tilde{v}_1) \in V_2 \times L^2\) and \((F_1, F_2) \in L^2(Q_r) \times L^2(Q_r)\). Then, we establish sequences \(\{u_{m, m_1}\} \in C^\infty_0(\Omega) \times C^\infty_0(\Omega), \{v_{m, m_1}\} \in C^\infty_0(\Omega) \times C^\infty_0(\Omega)\), and \(\{F_{m, m_2}\} \in C^\infty_0(\Omega) \times C^\infty_0(\Omega)\), satisfying

\[
\begin{align*}
(u_{m, m_1}) &\to (\tilde{u}_0, \tilde{u}_1) \text{ strongly in } V_1 \times L^2, \\
(v_{m, m_1}) &\to (\tilde{v}_0, \tilde{v}_1) \text{ strongly in } V_2 \times L^2, \\
(F_{m, m_2}) &\to (F_1, F_2) \text{ strongly in } L^2(Q_r) \times L^2(Q_r).
\end{align*}
\]

Note that the sequences \(\{u_{m, m_1}\}\) and \(\{v_{m, m_1}\}\), as a result, satisfy themselves the compatibility relation for all \(m \in \mathbb{N}\). So what we can deduce next is the existence of a pair of unique functions \((u_m, v_m)\) for each \(m\) making the conditions in the aforementioned theorem self-propelling. Thus, one easily verifies that such functions \((u_m, v_m)\) for each \(m\) satisfy the variational problem (2.1)-\(2.2\), namely

\[
\begin{align*}
\langle \tilde{u}_m(t), \phi \rangle + \langle \nabla u_m(t), \nabla \phi \rangle + \lambda_1 \langle \Psi_{r_1}(u_m(t), \phi) \rangle + \mu_1 \tilde{u}_m(1, t) \phi(1) \\
= K_1 \Psi_{p_1}(u_m(1, t), \phi(1)) + (f_1(u_m, v_m), \phi) + (F_1(u), \phi), \\
\langle \tilde{v}_m(t), \tilde{\phi} \rangle + \langle \nabla v_m(t), \nabla \tilde{\phi} \rangle + \lambda_2 \langle \Psi_{r_2}(v_m(t), \tilde{\phi}) \rangle + \mu_2 v_m(0, t) \tilde{\phi}(0) \\
= K_2 \Psi_{p_2}(v_m(0, t), \tilde{\phi}(0)) + (f_2(u_m, v_m), \tilde{\phi}) + (F_2(u), \tilde{\phi}),
\end{align*}
\]

for all \((\phi, \tilde{\phi}) \in V_1 \times V_2\), together with the initial conditions

\[
\begin{align*}
(u_m(0), \dot{u}_m(0)) &= (u_{0,m}, u_{1,m}), \\
v_m(0), \dot{v}_m(0) &= (v_{0,m}, v_{1,m}).
\end{align*}
\]

Moreover, the smoothness of \((u_m, v_m)\) on the interval \([0, T_r]\) is said by (3.53) and we recall below the uniform boundedness (independent of \(m\)) of \(S_m(t)\) on \([0, T_r]\) in (3.5) due to the same arguments derived above.

\[
S_m(t) = \|\tilde{u}_m(t)\|_1^2 + \|\tilde{v}_m(t)\|_1^2 + \|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2 + 2\lambda_1 \int_0^t \|\tilde{u}_m(s)\|_{L^r_1}^r \, ds \\
+ 2\lambda_2 \int_0^t \|\tilde{v}_m(s)\|_{L^r_2}^r \, ds + 2\mu_1 \int_0^t |u_m(1, s)|^2 \, ds + 2\mu_2 \int_0^t |v_m(0, s)|^2 \, ds \leq C_T.
\]
where $C_T$ denotes by a positive constant independent of $m$ and $t$ and $t$ moves along the interval $[0, T_*]$. Define $U_{m,k} = u_m - u_k$ and $V_{m,k} = v_m - v_k$, then these quantities satisfy

\[
\left\{ \begin{array}{l}
\langle \dot{U}_{m,k} (t), \phi \rangle + \langle \nabla U_{m,k} (t), \nabla \phi \rangle + \lambda_1 \langle \Psi_{r_1} \left( u_m (t) \right) - \Psi_{r_1} \left( u_k (t) \right), \phi \rangle + \mu_1 \dot{U}_{m,k} (1, t) \phi (1) \\
= K_1 \left[ \Psi_{p_1} \left( u_m (1, t) \right) - \Psi_{p_1} \left( u_k (1, t) \right) \right] \phi (1) + \langle f_1 (u_m, v_m) - f_1 (u_k, v_k), \phi \rangle \quad \text{for all } \phi \in V_1 \times V_2 \quad \text{and the initial conditions are} \\
\langle V_{m,k} (t), \phi \rangle + \langle \nabla V_{m,k} (t), \nabla \phi \rangle + \lambda_2 \langle \Psi_{r_2} \left( v_m (t) \right) - \Psi_{r_2} \left( v_k (t) \right), \phi \rangle + \mu_2 \dot{V}_{m,k} (0, t) \phi (0) \\
= K_2 \left[ \Psi_{p_2} \left( v_m (0, t) \right) - \Psi_{p_2} \left( v_k (0, t) \right) \right] \phi (0) + \langle f_2 (u_m, v_m) - f_2 (u_k, v_k), \phi \rangle \quad \text{for all } \phi, \tilde{\phi} \in V_1 \times V_2 \quad \text{and the initial conditions are} \\
\end{array} \right.
\]

(3.57) \{eq:varium-1\}

It is obvious to obtain the fact that

\[
\begin{align*}
S_{m,k} (t) &= S_{m,k} (0) + \frac{1}{2} \int_0^t \left( \left\langle f_1 (u_m, v_m) - f_1 (u_k, v_k), \dot{U}_{m,k} (s) \right\rangle + \left\langle f_2 (u_m, v_m) - f_2 (u_k, v_k), \dot{V}_{m,k} (s) \right\rangle \right) \, ds \\
&\quad + 2 \int_0^t \left[ \left\langle F_{1m} (s) - F_{1k} (s), \dot{U}_{m,k} (s) \right\rangle + \left\langle F_{2m} (s) - F_{2k} (s), \dot{V}_{m,k} (s) \right\rangle \right] \, ds \\
&\quad + 2 K_1 \int_0^t \left[ |\Psi_{p_1} (u_m (1, s)) - \Psi_{p_1} (u_k (1, s))| \right] \dot{U}_{m,k} (1, s) \, ds \\
&\quad + 2 K_2 \int_0^t \left[ |\Psi_{p_2} (v_m (0, s)) - \Psi_{p_2} (v_k (0, s))| \right] \dot{V}_{m,k} (0, s) \, ds
\end{align*}
\]

(3.58) \{eq:varium-2\}

where

\[
S_{m,k} (0) = \| u_{m1} - u_{1k} \|^2 + \| v_{m1} - v_{1k} \|^2 + \| u_{0m} - u_{0k} \|^2 + \| v_{0m} - v_{0k} \|^2.
\]

(3.59) \{eq:Smk\}

The calculations above follow by a valid replacement of the test functions $\phi$ and $\tilde{\phi}$ by $\dot{U}_{m,k}$ and $\dot{V}_{m,k}$, respectively, in (3.57) and then integrating with respect to $t$. In the similar strategy with classical inequalities such as the Cauchy-Schwartz inequality in combination with (3.51), (3.60) and (3.59), we estimate the terms on the right-hand side of (3.59) and obtain

\[
S_{m,k} (t) \leq C_{m,k} + 2 \left( 1 + \eta (p_1, p_2) + 8 R_T \right) \int_0^t S_{m,k} (s) \, ds,
\]

(3.62) \{eq:finalSmk\}

where the appropriately defined terms are given by

\[
R_T = \max_{i=1,2} \sup_{|y, z| \leq \sqrt{C_T}} \left( \left| \frac{\partial f_i}{\partial y} (y, z) \right| + \left| \frac{\partial f_i}{\partial z} (y, z) \right| \right),
\]

\[
\eta (p_1, p_2) = \frac{K_1^2}{\mu_1} (p_1 - 1)^2 C_T^{-1} + \frac{K_2^2}{\mu_2} (p_2 - 1)^2 C_T^{-2}.
\]
Here we remark that \( C_{m,k} \) approaches zero as \( m \) and \( k \) tend to infinity, and by the aid of Gronwall’s inequality, it follows from (3.62) that for all \( t \in [0, T_*] \)

\[
C_{m,k} = 2S_{m,k} \left( 0 \right) + 2 \| F_{1m} - F_{1k} \|^2_{L^2(\Omega_T)} + 2 \| F_{2m} - F_{2k} \|^2_{L^2(\Omega_T)}.
\]

(3.63) \{eq:Smklast\} \( S_{m,k}(t) \leq C_{m,k}\exp \left( 2T \left( 1 + \eta \left( p_1, p_2 \right) + 8 \mathcal{R}_T \right) \right) \).

Thus, one can show that the right-hand side of (3.63) shall go to zero as \( m \) and \( k \) tend to infinity by the direct argument concerning convergences of \( \{(u_{0m}, u_{1m})\} \) and \( \{(\tilde{v}_{0m}, \tilde{v}_{1m})\} \). Consequently, it gives us the following

\[
\begin{cases}
(u_m, v_m) \to (u, v) \text{ strongly in } C^0([0, T_*]; \mathcal{V}_1 \times \mathcal{V}_2) \cap C^1([0, T_*]; L^2 \times L^2), \\
(\tilde{u}_m, \tilde{v}_m) \to (\tilde{u}, \tilde{v}) \text{ strongly in } L^{r_1}(Q_T) \times L^{r_2}(Q_T), \\
(u_m(1, \cdot), v_m(0, \cdot)) \to (u(1, \cdot), v(0, \cdot)) \text{ strongly in } H^1(0, T_*) \times H^1(0, T_*).
\end{cases}
\]

(3.64) \{eq:3.64\} \( (u_m, v_m) \to (u, v) \) strongly in \( C^0([0, T_*]; \mathcal{V}_1 \times \mathcal{V}_2) \cap C^1([0, T_*]; L^2 \times L^2), \)

(3.65) \( (\tilde{u}_m, \tilde{v}_m) \to (\tilde{u}, \tilde{v}) \) strongly in \( L^{r_1}(Q_T) \times L^{r_2}(Q_T), \)

(3.66) \{eq:3.66\} \( (u_m(1, \cdot), v_m(0, \cdot)) \to (u(1, \cdot), v(0, \cdot)) \) strongly in \( H^1(0, T_*) \times H^1(0, T_*). \)

One important point should be mentioned here is that by (3.56) we can extract a subsequence of \( \{(u_m, v_m)\} \) (still relabel with the old index \( m \)) which reads

(3.67) \{eq:3.67\} \( (u_m, v_m) \to (u, v) \) weak-* in \( L^\infty(0, T_*; \mathcal{V}_1 \times \mathcal{V}_2), \)

(3.68) \{eq:3.68\} \( (\tilde{u}_m, \tilde{v}_m) \to (\tilde{u}, \tilde{v}) \) weak-* in \( L^\infty(0, T_*; L^2 \times L^2), \)

and inheriting from (3.64)-(3.66), one deduces

(3.69) \{eq:3.69\} \( (f_1(u_m, v_m), f_2(u_m, v_m)) \to (f_1(u, v), f_2(u, v)) \) strongly in \( L^2(Q_T) \times L^2(Q_T), \)

(3.70) \{eq:3.70\} \( (\Psi_{r_1}(u_m), \Psi_{r_2}(v_m)) \to (\Psi_{r_1}(u), \Psi_{r_2}(v)) \) strongly in \( L^2(Q_T) \times L^2(Q_T). \)

It is advantageous to state our limit processing now. In fact, pass to the limit in (3.54) associated with (3.55), evidenced by (3.64)-(3.67), we obtain a couple of functions \( (u, v) \) satisfying the variational problem

\[
\begin{aligned}
\frac{d}{dt} \langle u_t(t), \phi \rangle + \langle u_x(t), \phi_x \rangle + \lambda_1 \langle \Psi_{r_1}(u_t(t), \phi) + \mu_1 \Psi_{q_1}(u_t(1, t)) \phi(1) \\
= K_1 \Psi_{p_1}(u(1, t), \phi) + \langle f_1(u, v), \phi \rangle + \langle F_1(t), \phi \rangle,
\end{aligned}
\]

\[
\begin{aligned}
\frac{d}{dt} \langle v_t(t), \phi \rangle + \langle v_x(t), \phi_x \rangle + \lambda_2 \langle \Psi_{r_2}(v_t(t), \phi), \phi(0) \\
= K_2 \Psi_{p_2}(v(0, t), \phi) + \langle f_2(u, v), \phi \rangle + \langle F_2(t), \phi \rangle,
\end{aligned}
\]

for all \( (\phi, \tilde{\phi}) \in \mathcal{V}_1 \times \mathcal{V}_2, \) endowed with (2.22), of \( (P) \) in the current considered case in this theorem such as \( q_1 = q_2 = 2 \) and \( p_1, p_2 \geq 2 \). This also implies the existence of a local solution claimed by the theorem. The uniqueness of such a weak solution is similarly and directly obtained by using the well-known technicalities of regularization procedure investigated by Lions (see, e.g., [12]). Hence, we end up with the proof.

Remark 10. If one has \( N = \frac{1}{2} \max \left\{ \frac{2}{N}; \alpha; \frac{2q_1(p_1 - 1)}{N}, \frac{2q_2(p_2 - 1)}{N} \right\} \leq 1 \) in Remark 6, and considers the assumptions (H1) and (H2), the integral \( S_m(t) \) which was uniformly locally bounded by a constant \( C_T \) depending only on \( T \) (see in (3.10)) can be estimated globally (and uniformly as well) in time, i.e.

\[
S_m(t) \leq C_T, \quad \forall m \in \mathbb{N}, \forall t \in [0, T], \forall T > 0.
\]

Consequently, all arguments used in the proof of Theorem 9 are directly applicable to prove that there exists a global weak solution \((u, v)\) to the problem \((P)\) and satisfying
if the initial energy is negative, then every weak solution of that either it goes to infinity in some specific senses, or it stops being smooth, and so forth. Our main objective here is further statement for the uniqueness in this case.

In this section, the total energy shall be computed in a common way and then it would be a decreasing function along (3.71)  
\[
\begin{aligned}
(eq:regu)
(u, v) &\in L^\infty (0, T; \mathbb{V}_1 \times \mathbb{V}_2), \\
(u_1, v_1) &\in L^\infty (0, T; L^2 \times L^2), \\
u (1, \cdot), v (0, \cdot) &\in H^1 (0, T).
\end{aligned}
\]

Nevertheless, one point should be noticed is that the case \(N = \frac{1}{2} \max \left\{ 2; \alpha; \beta; \frac{q_1 (p_1 - 1)}{q_1 - 1}; \frac{q_2 (p_2 - 1)}{q_2 - 1} \right\} \leq 1 \) does not imply the weak solution obtained here must belong to \( C^0 (0, T; \mathbb{V}_1 \times \mathbb{V}_2) \cap C^1 (0, T; L^2 \times L^2) \) and one cannot also say any further statement for the uniqueness in this case.

4. Finite time blow up

In principle, blow-up phenomenon of a solution to a time-dependent equation is devoted to the study of maximal time domain for which it is defined by a finite length. At the endpoint of that interval, the solution behaves in such a way that either it goes to infinity in some specific senses, or it stops being smooth, and so forth. Our main objective here is to show that if the initial energy is negative, then every weak solution of \((P)\) blows up in finite time. The result here draws from ideas in the treatment of a single wave equation [15, 11] and also, for example, the recent results for systems in [13, 10], but our proofs have to be radically altered.

Let us first make a brief note concerning the so-called total energy. From mathematical point of view, energy method plays a vital role in the study of partial differential equations. That energy is usually used to derive such things as existence, uniqueness of the solution, and whether it depends continuously on the data. For a very fundamental scalar wave equation which was originally discovered by d’Alember t, one may easily find the energy integral defined by, for example, 

\[
\frac{1}{2} \int_\Omega |u_t|^2 \, dx + \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx,
\]

where we basically multiply the equation by \(u_t\) and integrate over \(\Omega\) as before using the divergence theorem.

Physically, the first term in the above energy equation is typically kinetic energy, and the second potential energy. If one imposes mixed boundary conditions, says \(au + bu_x = 0\) on the boundary, where \(a\) and \(b\) have the same sign, the energy can only decrease. If they have opposite sign, the problem is unstable in the sense that the energy will increase. In this section, the total energy shall be computed in a common way and then it would be a decreasing function along the trajectories, starting from a negative initial value.

From here on, we aim to consider the problem \((P)\) in a specific case: linear damping \(r_i = 2\) with \(F_i = 0\), and \(q_i = 2, p_i > 2, K_i > 0, \lambda_i > 0, \mu_i > 0\) for \(i = 1, 2\). Then, we show that the solution of this problem blows up in finite time if 

\[-H(0) = \frac{1}{2} \left( \|\tilde{u}_1\|^2 + \|\tilde{v}_1\|^2 + \|\nabla \tilde{u}_0\|^2 + \|\nabla \tilde{v}_0\|^2 \right) - \left( \frac{K_1}{p_1} |\tilde{u}_0(1)|^{p_1} + \frac{K_2}{p_2} |\tilde{v}_0(0)|^{p_2} \right) - \int_0^1 \mathcal{F}(\tilde{u}_0(x), \tilde{v}_0(x)) \, dx < 0.\]

We also need additional assumptions on interior sources and introduce the total energy associated to the solution \((u, v)\): 
(A3bis) there exists \(\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}\) the \(C^2\)-function such that 

\[
\frac{\partial \mathcal{F}}{\partial u}(u, v) = f_1(u, v), \quad \frac{\partial \mathcal{F}}{\partial v}(u, v) = f_2(u, v),
\]

and there also exists the constants \(\alpha, \beta > 2; d_1, d_2, \tilde{d}_1, \tilde{d}_2 > 0\) such that 

\[
\begin{aligned}
(eq:4.1) \quad d_1 \mathcal{F}(u, v) &\leq uf_1(u, v) + vf_2(u, v) \leq d_2 \mathcal{F}(u, v), &\text{for all } (u, v) &\in \mathbb{R}^2, \\
(eq:4.2) \quad \tilde{d}_1 \left(|u|^\alpha + |v|^\beta\right) &\leq \mathcal{F}(u, v) \leq \tilde{d}_2 \left(|u|^\alpha + |v|^\beta\right), &\text{for all } (u, v) &\in \mathbb{R}^2.
\end{aligned}
\]

(A4) the total quadratic-type energy \(E(t)\) is defined by 

\[
\begin{aligned}
(eq:energy) \quad E(t) &\equiv \frac{1}{2} \left( |\dot{u}(t)|^2 + |\dot{v}(t)|^2 + \|\nabla u(t)|^2 + \|\nabla v(t)|^2 \right) - \left( \frac{K_1}{p_1} |u(1, t)|^{p_1} + \frac{K_2}{p_2} |v(0, t)|^{p_2} \right) - \int_\Omega \mathcal{F}(u(x, t), v(x, t)) \, dx.
\end{aligned}
\]
Define \( H(t) = -E(t) \) and assume that \( H(0) > 0 \), one can show that the time derivative of such a function is never negative along the trajectories in a local time, namely

\[
H'(t) = \lambda_1 \| \dot{u}(t) \|^2 + \lambda_2 \| \dot{v}(t) \|^2 + \mu_1 \| u(1,t) \|^2 + \mu_2 \| v(0,t) \|^2 \geq 0,
\]

for all \( t \in [0, T_\ast) \) where we have multiplied (1.1) by \((u(x,t), \dot{v}(x,t))\) and integrated over \( \Omega \). Together with the fact that \( H(0) > 0 \), we have

\[
0 < H(0) \leq H(t), \quad \forall t \in [0, T_\ast). \tag{4.4}
\]

Observe the final term in the total energy (4.6), it follows from (4.12) that

\[
\frac{1}{2} \left( \| \nabla u(t) \|^2 + \| \nabla v(t) \|^2 \right) \leq \left( \frac{K_1}{p_1} |u(1,t)|^{p_1} + \frac{K_2}{p_2} |v(0,t)|^{p_2} \right) + \int_\Omega F(u(x,t), v(x,t)) \leq \tilde{d}_2 \left( \| u(t) \|_{L^\alpha}^{2\alpha} + \| v \|_{L^\beta}^{2\beta} \right). \tag{4.6}
\]

Combining (4.5), (4.6) and (4.12) gives

\[
0 < H(t) \leq \tilde{d}_2 \left( |u(1,t)|^{p_1} + |v(0,t)|^{p_2} + \| u(t) \|_{L^\alpha}^{2\alpha} + \| v \|_{L^\beta}^{2\beta} \right), \tag{4.7}
\]

for all \( t \in [0, T_\ast) \) and \( \tilde{d}_2 = \max \left\{ \frac{K_1}{p_1}, \frac{K_2}{p_2}, \tilde{d}_2 \right\} \). It also allows us to derive that for all \( t \in [0, T_\ast) \)

\[
\frac{1}{2} \left( \| \nabla u(t) \|^2 + \| \nabla v(t) \|^2 \right) \leq \left( \frac{K_1}{p_1} |u(1,t)|^{p_1} + \frac{K_2}{p_2} |v(0,t)|^{p_2} \right) + \int_\Omega F(u(x,t), v(x,t)) \leq \tilde{d}_2 \left( |u(1,t)|^{p_1} + |v(0,t)|^{p_2} + \| u(t) \|_{L^\alpha}^{2\alpha} + \| v \|_{L^\beta}^{2\beta} \right). \tag{4.8}
\]

Now we construct the following functional

\[
L(t) = H^{1-\xi}(t) + \varepsilon \psi(t), \tag{4.9}
\]

where we define

\[
\psi(t) := \langle u(t), \dot{u}(t) \rangle + \langle v(t), \dot{v}(t) \rangle + \frac{\lambda_1}{2} \| u(t) \|^2 + \frac{\lambda_2}{2} \| v(t) \|^2 + \frac{\mu_1}{2} u^2(1,t) + \frac{\mu_2}{2} v^2(0,t), \tag{4.10}
\]

for \( \varepsilon > 0 \) small enough and \( \xi \in \left( 0, \min \left\{ \frac{\alpha - 2}{2\alpha}, \frac{\beta - 2}{2\beta} \right\} \right) \subset (0, \frac{1}{2}) \).

To show that one can choose \( \varepsilon > 0 \) small enough such that \( L(t) \) is non-decreasing for all \( t \in [0, T_\ast) \), namely

\[
L(t) \geq L(0) > 0, \quad \forall t \in [0, T_\ast), \tag{4.11}
\]

a very clear way is to consider the derivative of such a function. In fact, let us prove the following lemma.

**Lemma 11.** There exists a positive constant \( \gamma \) such that

\[
\{\text{Lemma 11}\} \tag{4.12}
\]

\[
H(t) + \| \dot{u}(t) \|^2 + \| \dot{v}(t) \|^2 + \| \nabla u(t) \|^2 + \| \nabla v(t) \|^2 + \| u(t) \|_{L^\alpha}^{2\alpha} + \| v \|_{L^\beta}^{2\beta} + |u(1,t)|^{p_1} + |v(0,t)|^{p_2}
\]

**Proof.** First, we multiply (1.1) by \((u(x,t), v(x,t))\) and then integrate over \( \Omega \), the derivative of \( \psi(t) \) can be defined as follows:

\[
\psi'(t) = \| \dot{u}(t) \|^2 + \| \dot{v}(t) \|^2 - \left( \| \nabla u(t) \|^2 + \| \nabla v(t) \|^2 \right) + K_1 |u(1,t)|^{p_1} + K_2 |v(0,t)|^{p_2}
\]

\[
+ \langle f_1(u(t), v(t), u(t)) + f_2(u(t), v(t), v(t)) \rangle.
\]

Using this, and the fact from (4.9) that
\[ L'(t) = (1-\xi) H^{-\xi} (t) H'(t) + \varepsilon \left( \|\dot{u}(t)\|^2 + \|\dot{v}(t)\|^2 \right) - \varepsilon \left( \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \right) + \varepsilon (K_1 |u(1,t)|^{p_1} + K_2 |v(0,t)|^{p_2}) + \varepsilon \left( (f_1(u(t), v(t)), u(t)) + (f_2(u(t), v(t)), v(t)) \right), \]

in combination with (4.14) and the following inequality

\[ \langle f_1(u(t), v(t)), u(t) \rangle + \langle f_2(u(t), v(t)), v(t) \rangle \geq d_1 \int_{\Omega} F(u(x,t), v(x,t)) \, dx \geq d_1 \bar{d}_1 \left( \|u(t)\|_{L^\alpha}^\alpha + \|v\|_{L^\beta}^\beta \right), \]

shows that

\[ L'(t) \geq \varepsilon \left( \|\dot{u}(t)\|^2 + \|\dot{v}(t)\|^2 \right) - 2\varepsilon \bar{D}_2 \left( |u(1,t)|^{p_1} + |v(0,t)|^{p_2} + \|u(t)\|_{L^\alpha}^\alpha + \|v\|_{L^\beta}^\beta \right) \]

\[ \geq \varepsilon \left( \|\dot{u}(t)\|^2 + \|\dot{v}(t)\|^2 \right) + \varepsilon \left( \bar{D}_1 - 2\bar{D}_2 \right) \left( |u(1,t)|^{p_1} + |v(0,t)|^{p_2} + \|u(t)\|_{L^\alpha}^\alpha + \|v\|_{L^\beta}^\beta \right), \]

where we have put \( \bar{D}_1 = \min \{K_1, K_2, d_1 \bar{d}_1\} \). If one assumes that \( 2 \max \left\{ \frac{K_1}{p_1}, \frac{K_2}{p_2}, \bar{d}_2 \right\} < \min \{K_1, K_2, d_1 \bar{d}_1\} \), we deduce

\[ (4.14) \quad \{eq:D3\} 0 < \bar{D}_3 = \bar{D}_1 - 2\bar{D}_2 = \min \{K_1, K_2, d_1 \bar{d}_1\} - 2 \max \left\{ \frac{K_1}{p_1}, \frac{K_2}{p_2}, \bar{d}_2 \right\} < \min \{K_1, K_2, d_1 \bar{d}_1\}. \]

Recall from (4.17) that

\[ \bar{D}_2 \left( |u(1,t)|^{p_1} + |v(0,t)|^{p_2} + \|u(t)\|_{L^\alpha}^\alpha + \|v\|_{L^\beta}^\beta \right) \geq H(t), \]

and also thanks to (4.8) and (4.13) with the assumption to get (4.14), it suffices to show that there exists \( \gamma > 0 \) such that (4.12) holds. Hence, we complete the proof. \( \square \)

From this lemma, as said above we obtain (4.11) and realize that the assumption \( 2 \max \left\{ \frac{K_1}{p_1}, \frac{K_2}{p_2}, \bar{d}_2 \right\} < \min \{K_1, K_2, d_1 \bar{d}_1\} \) is additionally necessary to approach the blow-up result. However, before going to that result, let us consider the following supplementary inequalities.

**Lemma 12.** Let \( \xi > 0 \) such that

\[ \begin{cases} 
2 \leq 2/(1-2\xi) \leq \min \{\alpha, \beta\}, \\
2 \leq 2/(1-\xi) \leq \min \{\alpha, \beta, p_1, p_2\}, 
\end{cases} \]

then the following inequalities hold

\[ (4.15) \quad \{eq:4.14b\} \|u(t)\|_{L^\alpha}^{2/(1-2\xi)} + \|u(t)\|_{L^\alpha}^{2/(1-\xi)} + |u(1,t)|^{2/(1-\xi)} \leq 3 \left( \|\nabla u(t)\|^2 + \|u(t)\|_{L^\alpha}^\alpha + |u(1,t)|^{p_1} \right), \quad \forall u \in V_1, \]

\[ (4.16) \quad \{eq:4.25f\} \|v(t)\|_{L^\beta}^{2/(1-2\xi)} + \|v(t)\|_{L^\beta}^{2/(1-\xi)} + |v(0,t)|^{2/(1-\xi)} \leq 3 \left( \|\nabla v(t)\|^2 + \|v(t)\|_{L^\beta}^\beta + |v(0,t)|^{p_2} \right), \quad \forall v \in V_2, \]

for all \( t \in [0, T_1) \).

**Proof.** We only need to prove the first inequality since (4.15) and (4.16) are obviously the same. Put \( s_1 = 2/(1-2\xi) \) and \( s_2 = 2/(1-\xi) \), our strategy is to independently consider each terms on the left-hand side of (4.15) and furthermore, to estimate those quantities we mainly divide into two cases. For clarity, let us first consider two cases for \( \|u\|_{L^\alpha} \).

- \( \|u\|_{L^\alpha} \leq 1 \): One easily deduces from \( 2 \leq s_1 \leq \alpha \) that
  \[ \|u(t)\|_{L^\alpha}^{p_1} \leq \|u(t)\|_{L^\alpha}^{2} \leq \|\nabla u(t)\|^2 + \|u(t)\|_{L^\alpha}^\alpha + |u(1,t)|^{p_1}. \]
- \( \|u\|_{L^\alpha} > 1 \): Similarly, we get
Next, we consider two cases for $\|u\|:$

- $\|u\| \leq 1$: Since $2 \leq s_2 \leq \alpha$,

$$\|u(t)\|^2 \leq \|\nabla u(t)\|^2 + \|u(t)\|_{L^\alpha}^\alpha + |u(1,t)|^{p_1}.$$ (4.19)

- $\|u\| \geq 1$: In the same manner, we have

$$\|u(t)\|^2 \leq \|\nabla u(t)\|^2 + \|u(t)\|_{L^\alpha}^\alpha + |u(1,t)|^{p_1}.$$ (4.19)

Finally, we go through the last term $|u(1,t)|$:

- $|u(1,t)| \leq 1$:

$$|u(1,t)|^2 \leq \|u(t)\|^2 \leq \|\nabla u(t)\|^2 + \|u(t)\|_{L^\alpha}^\alpha + |u(1,t)|^{p_1}.$$ (4.19)

- $|u(1,t)| \geq 1$: It is natural to say $|u(1,t)|^2 \leq |u(1,t)|^{p_1}$.

Combining all of the above inequalities completes the proof.

\[\square\]

**Theorem 13.** Suppose that (A3bis) holds and $H(0) > 0$. If $2 \max \left\{ \frac{K_1}{p_1}, \frac{K_2}{p_2}, d_1 d_2 \right\} < \min \left\{ K_1, K_2, d_1 \bar{d}_1 \right\}$ in (A3bis), then any weak solution $(u, v)$ of (P) blows up in finite time for any $(\tilde{u}_0, \tilde{u}_1)$ and $(\tilde{v}_0, \tilde{v}_1)$ in (H1).

**Proof.** By using an elementary inequality

$$\left( \sum_{i=1}^{7} z_i \right)^r \leq 7^{r-1} \sum_{i=1}^{7} z_i^r, \quad \forall r > 1, x_i \geq 0, i = 1, 7,$$

and thanks to (4.9), (4.10), one can show that

$$L^{1/(1-\xi)}(t) \leq C \left( H(t) + \left| \langle u(t), \dot{u}(t) \rangle \right|^{1/(1-\xi)} + \left| \langle v(t), \dot{v}(t) \rangle \right|^{1/(1-\xi)} \right. \leq \left. \| u(t) \|_{L^\alpha}^{2/(1-\xi)} + \| v(t) \|_{L^\alpha}^{2/(1-\xi)} + u^{2/(1-\xi)}(1,t) + v^{2/(1-\xi)}(0,t) \right),$$ (4.17) \{eq:LXi\}

where $C = 2^{-1/(1-\xi)} \max \left\{ 2^{1/(1-\xi)} \right\}$, $2^{1/(1-\xi)} \in \{0, 1\}$, $(\alpha_1 \varepsilon)^{1/(1-\xi)}, (\alpha_2 \varepsilon)^{1/(1-\xi)}, (\alpha_1 \varepsilon)^{1/(1-\xi)}, (\alpha_2 \varepsilon)^{1/(1-\xi)}$. Moreover, we find

$$\left| \langle u(t), \dot{u}(t) \rangle \right|^{1/(1-\xi)} \leq \| u(t) \|_{L^\alpha}^{1/(1-\xi)} \| \dot{u}(t) \|_{L^\alpha}^{1/(1-\xi)},$$ (4.18) \{eq:4.17\}

where we have used the Cauchy-Schwartz inequality, Hölder’s inequality.

According to Young’s inequality introduced in (3.6), by choosing $\delta = 1, q = \frac{2(1-\xi)}{2\xi}, q' = 2(1-\xi)$ and letting $a = \| u(t) \|_{L^\alpha}^{1/(1-\xi)}, b = \| \dot{u}(t) \|_{L^\alpha}^{1/(1-\xi)}$ one easily obtains from (4.18) that

$$\left| \langle u(t), \dot{u}(t) \rangle \right|^{1/(1-\xi)} \leq c_1 \left( \| u(t) \|_{L^\alpha}^{2/(1-2\xi)} + \| \dot{u}(t) \|^2 \right),$$ (4.19) \{eq:4.19\}

where $c_1 = \max \left\{ \frac{1-2\varepsilon}{2(1-\xi)}, \frac{1}{2(1-\xi)} \right\} < 1$. Similarly, we have for a constant $c_2 \in (0, 1)$

$$\left| \langle v(t), \dot{v}(t) \rangle \right|^{1/(1-\xi)} \leq c_2 \left( \| v(t) \|_{L^\alpha}^{2/(1-\xi)} + \| \dot{v}(t) \|^2 \right).$$ (4.20) \{eq:4.20\}

Therefore, the observability estimate (4.17) along with (4.19), (4.20), (4.15), and (4.16) implies that there always exists a positive constant (here we reuse the notation C for simplicity) such that

$$L^{1/(1-\xi)}(t) \leq C \left( H(t) + \| u(t) \|^2 + \| \dot{u}(t) \|^2 + \| v(t) \|^2 + \| \nabla u(t) \|^2 + \| \nabla v(t) \|^2 \right) + \| u(t) \|_{L^\alpha}^\alpha + \| v(t) \|_{L^\alpha}^\alpha + |u(1,t)|^{p_1} + |v(0,t)|^{p_2}, \quad \forall t \in [0, T_{\ast}).$$ (4.21) \{eq:LL\}
In addition, the estimate (4.12) together with (4.21) allows us to take a positive constant $\bar{C}$ such that

$$L' (t) \geq \bar{C} L^{1/ (1 - \xi)} (t), \quad \forall t \in [0, T_*). \tag{4.22} \label{eq:4.22}$$

Now, integrating (4.22) over $(0, t)$, one has

$$L^{1/ (1 - \xi)} (t) \geq \frac{1}{L^{1/ (1 - \xi)} (0) - \frac{C\xi}{C\xi} t}, \quad t \in \left[0, \frac{1 - \xi}{C\xi} L^{1/ (1 - \xi)} (0)\right),$$

which yields that $L (t)$ definitely blows up in a finite time given by

$$T_* = \frac{1 - \xi}{C\xi} L^{1/ (1 - \xi)} (0).$$

Hence, we complete the proof of the theorem. \hfill \blacksquare

## 5. Exponential decay

While the previous section employs the total energy to find the finite time which produces the blow-up result, the total energy in this section is considered to be in exponential decay. In particular, we study the global solution $(u, v)$ of (P) satisfying (4.17) and corresponding to $r_1 = r_2 = q_1 = q_2 = 2$ and $p_1, p_2 > 2$, and find an related-initial data in the stable set together with suitable and necessary assumptions provided by, for example, external functions, for which the solution decays exponentially. Like the blow-up phenomenon, this sort of results can be seen as an extension of many previous works from the single wave equation in, for example, [7, 10] to the system of equations.

Our result here relies on the construction of a Lyapunov functional by performing a suitable modification of the energy. To this end, for $\delta > 0$, to be chosen later, we define

$$\mathcal{L} (t) = E (t) + \delta \psi (t), \tag{5.1} \label{eq:Lcal}$$

where we have recalled the energy in (A4) and the function $\psi (t)$ in (4.17).

Under some additional conditions, it suffices to see that $\mathcal{L} (t)$ and $E (t)$ are equivalent in the sense that there exist two positive constants $\beta_1$ and $\beta_2$ depending on $\delta$ such that for $t \geq 0$,

$$\beta_1 E (t) \leq \mathcal{L} (t) \leq \beta_2 E (t). \tag{5.2} \label{eq:equi}$$

Before explicitly providing this equivalence, let us first consider the time derivative of the total energy, which can be defined in a similar way for (4.14) by the following lemma.

**Lemma 14.** The time derivative of the total energy satisfies

$$E' (t) \leq -\lambda_* \left( \| \dot{u} (t) \|^2 + \| \dot{v} (t) \|^2 \right) - \mu_* \left( \| \dot{u} (1, t) \|^2 + \| \dot{v} (0, t) \|^2 \right)$$

$$\tag{5.3} \label{eq:E'1}$$

$$- \frac{1}{2} \left( \| F_1 (t) \| + \| F_2 (t) \| \right) + \frac{1}{2} \left( \| F_1 (t) \| + \| F_2 (t) \| \right) \left( \| \dot{u} (t) \|^2 + \| \dot{v} (t) \|^2 \right),$$

$$E' (t) \leq - \left( \lambda_* - \frac{\xi_1}{2} \right) \left( \| \dot{u} (t) \|^2 + \| \dot{v} (t) \|^2 \right) - \mu_* \left( \| \dot{u} (1, t) \|^2 + \| \dot{v} (0, t) \|^2 \right)$$

$$+ \frac{1}{2 \xi_1} \left( \| F_1 (t) \|^2 + \| F_2 (t) \|^2 \right), \tag{5.4} \label{eq:E'2}$$

for all $\xi_1 > 0$, $\lambda_* = \min \{ \lambda_1, \lambda_2 \} > 0$, and $\mu_* = \min \{ \mu_1, \mu_2 \} > 0$.

**Proof.** Let us remark that multiplying (4.14) by $(\dot{u} (x, t), \dot{v} (x, t))$, integrating over $\Omega$ and using integration by parts, we obtain

$$E' (t) = -\lambda_1 \| \dot{u} (t) \|^2 - \lambda_2 \| \dot{v} (t) \|^2 - \mu_1 \| \dot{u} (1, t) \|^2 - \mu_2 \| \dot{v} (0, t) \|^2 + \langle F_1 (t), \dot{u} (t) \rangle + \langle F_2 (t), \dot{v} (t) \rangle. \tag{5.5} \label{eq:E'}$$

To obtain (5.3), we only need to estimate the last two terms. It is straightforward that by using the standard inequalities which read
\[ \langle F_1(t), \dot{u}(t) \rangle \leq \frac{1}{2} \| F_1(t) \| + \frac{1}{2} \| F_1(t) \| \| \dot{u}(t) \|^2, \]
\[ \langle F_2(t), \dot{v}(t) \rangle \leq \frac{1}{2} \| F_2(t) \| + \frac{1}{2} \| F_2(t) \| \| \dot{v}(t) \|^2, \]

it yields
\[ \langle F_1(t), \dot{u}(t) \rangle + \langle F_2(t), \dot{v}(t) \rangle \leq \frac{1}{2} (\| F_1(t) \| + \| F_2(t) \|) + \frac{1}{2} (\| F_1(t) \| + \| F_2(t) \|) \left( \| \dot{u}(t) \|^2 + \| \dot{v}(t) \|^2 \right). \]

For the second estimate (5.4), we also use the same approach to prove. Indeed, by Young’s inequality one deduces the following
\[ \langle F_1(t), \dot{u}(t) \rangle + \langle F_2(t), \dot{v}(t) \rangle \leq \frac{1}{2} (\| F_1(t) \|^2 + \| F_2(t) \|^2) + \frac{\varepsilon_1}{2} \left( \| \dot{u}(t) \|^2 + \| \dot{v}(t) \|^2 \right). \]

Hence, the lemma is proved completely.

Let us next define the following functions \( I_i(t) = I_i(u(t)) \) for \( i = 1, 2 \) and \( J(t) = J(u(t)) \) by rewriting the expression of the total energy \( E(t) \) in (4.3):

\[ E(t) = \frac{1}{2} \left( \| \dot{u}(t) \|^2 + \| \dot{v}(t) \|^2 \right) + J(t), \]

\[ J(t) = \frac{1}{2} \left( 1 - \frac{1}{p_1} - \frac{1}{p_2} \right) \left( \| \nabla u(t) \|^2 + \| \nabla v(t) \|^2 \right) + \frac{I_1(t)}{p_1} + \frac{I_2(t)}{p_2}, \]

\[ I_1(t) = \frac{1}{2} \left( \| \nabla u(t) \|^2 + \| \nabla v(t) \|^2 \right) - K_1 \| u(1,t) \|^{p_1} - \frac{p_1}{2} \int_{\Omega} \mathcal{F}(u(x,t), v(x,t)) \, dx, \]

\[ I_2(t) = \frac{1}{2} \left( \| \nabla u(t) \|^2 + \| \nabla v(t) \|^2 \right) - K_2 \| v(0,t) \|^{p_2} - \frac{p_2}{2} \int_{\Omega} \mathcal{F}(u(x,t), v(x,t)) \, dx. \]

Furthermore, let us provide a further assumption as follows:

\textbf{(H2bis)} \( F_1, F_2 \in L^\infty(\mathbb{R}_+; L^2) \cap L^1(\mathbb{R}_+; L^2) \).

From now on, our main result in this section is established where the proof of the equivalence between \( E(t) \) and \( E(t) \) is also included. It says that if there is an exponential rate of energy decay for external functions \( F_1 \) and \( F_2 \), and influenced by such functions, the related-initial energy function, says \( E_\ast \), is properly suited in a particular set, then the quadratic-type total energy decays exponentially.

\textbf{Theorem 15.} Suppose that (A1), (A3bis) and (H2bis) hold, together with the \( d_2 < \min \{ p_1, p_2 \} \) in (A3bis). Assume that \( I_1(0), I_2(0) > 0 \), and the energy given by (4.3) such that its initial energy satisfies

\[ \eta_\ast = \frac{p_1 + p_2}{2} \gamma_0 \left[ (p_\ast E_\ast)^{\frac{1}{2} - 1} + (p_\ast E_\ast)^{-\frac{1}{2} - 1} \right] + K_1 (p_\ast E_\ast)^{\frac{p_1}{2} - 1} + K_2 (p_\ast E_\ast)^{\frac{p_2}{2} - 1} < 1, \]

where \( p_\ast = \frac{2p_1p_2}{p_1p_2 - p_1 - p_2}, E_\ast = [E(0) + \rho] \exp(2\rho) \) and \( \rho = \frac{1}{2} \int_0^\infty (\| F_1(s) \| + \| F_2(s) \|) \, ds \). Moreover, let us assume the external functions decays exponentially in the sense that

\[ \| F_1(t) \|^2 + \| F_2(t) \|^2 \leq \eta_1 \exp(-\eta_2 t), \quad \forall t \geq 0, \]

where \( \eta_1 \) and \( \eta_2 \) are two positive constants. Then there exist positive constants \( C \) and \( \gamma \) such that

\[ E(t) \leq C \exp(-\gamma t), \quad \forall t \geq 0. \]
Proof. Firstly, we claim that \( I_i (t) \geq 0 \) for \( i = 1, 2 \) and for all \( t \geq 0 \). In fact, since \( I_i (t) \), \( i = 1, 2 \) is continuous and its initial value is positive, thus there exist two positive constants \( T_1 \) and \( T_2 \) such that

\[
I_1 (t) \geq 0, \quad \forall t \in [0, T_1], \text{ and } I_2 (t) \geq 0, \quad \forall t \in [0, T_2]
\]

which also leads to the fact that

\[
J (t) \geq \frac{1}{2} \left( 1 - \frac{1}{p_1} - \frac{1}{p_2} \right) \left( \| \nabla u (t) \|^2 + \| \nabla v (t) \|^2 \right), \quad \forall t \in [0, T]
\]

where \( T = \min \{ T_1, T_2 \} \) or we can say that

\[
(5.13) \quad \| \nabla u (t) \|^2 + \| \nabla v (t) \|^2 \leq \frac{2p_1p_2}{p_1p_2 - p_1 - p_2} J (t) \leq \frac{2p_1p_2}{p_1p_2 - p_1 - p_2} E (t), \quad \forall t \in [0, T].
\]

By (5.3) and thanks to \( I_i (0) > 0 \) for \( i = 1, 2 \) associated with (5.6), we deduce

\[
E (t) \leq E (0) + \frac{1}{2} \int_0^\infty (\| F_1 (s) \| + \| F_2 (s) \|) \, ds + \int_0^t (\| F_1 (s) \| + \| F_2 (s) \|) E (s) \, ds, \quad \forall t \in [0, T],
\]

then using Gronwall’s inequality leads to the following

\[
(5.14) \quad E (t) \leq [E (0) + \rho] \exp (2 \rho) \cdot \exp \left( \int_0^t (\| F_1 (s) \| + \| F_2 (s) \|) \, ds \right)
\]

where we have defined \( \rho = \frac{1}{2} \int_0^\infty (\| F_1 (s) \| + \| F_2 (s) \|) \, ds \).

Therefore, we combine (5.13) with (5.14), and put \( E_* = [E (0) + \rho] \exp (2 \rho) \cdot \frac{2p_1p_2}{p_1p_2 - p_1 - p_2} \) to obtain

\[
(5.15) \quad \| \nabla u (t) \|^2 + \| \nabla v (t) \|^2 \leq p_* E_*, \quad \forall t \in [0, T].
\]

Observe the assumption \((4.2)\) in (A3bis), and also \((5.13)\), it yields

\[
K_1 \| u (1, t) \|^{p_1} + K_2 \| v (0, t) \|^{p_2} \leq K_1 \| \nabla u (t) \|^2 + K_2 \| \nabla v (t) \|^2
\]

\[
(5.16) \quad \| \nabla u (t) \|^2 + K_2 \| \nabla v (t) \|^2 \leq K_1 \| u (1, t) \|^2 + K_2 \| v (0, t) \|^2
\]

and

\[
(5.17) \quad \| \nabla u (t) \|^2 + \| \nabla v (t) \|^2 \leq (p_1 + p_2) \, \tilde{d}_2 \left( \| u (t) \|_{L^\alpha} \| v (t) \|_{L^\beta} + \| v (t) \|_{L^\beta} \right)
\]

Thus, it follows from \((5.16)\) and \((5.17)\) that

\[
K_1 \| u (1, t) \|^2 + K_2 \| v (0, t) \|^2 + \frac{p_1 + p_2}{2} \int_\Omega F (u (x, t), v (x, t)) \, dx \leq \eta_* \left( \| \nabla u (t) \|^2 + \| \nabla v (t) \|^2 \right)
\]

\[
< \left( \| \nabla u (t) \|^2 + \| \nabla v (t) \|^2 \right), \quad \forall t \in [0, T],
\]

where \( \eta_* \) is given by \((5.10)\).

So we now claim that \( I_1 (t) \) and \( I_2 (t) \) are positive for all \( t \in [0, T] \). If we put

\[
T_* = \sup \{ T > 0 : I_1 (t) \text{ and } I_2 (t) \text{ are positive} \forall t \in [0, T] \},
\]

and if \( T_* < \infty \), then (by the continuity of \( I_1 (t) \) and \( I_2 (t) \)) we have \( I_1 (T_*) \) and \( I_2 (T_*) \) are non-negative. Therefore, by the same arguments, it is possible to show that there exists \( T_\ast \) such that \( I_i (t) > 0 \) for \( i = 1, 2 \) for all \( t \in [0, T_\ast] \).

Hence, we can conclude that \( I_i (t) \geq 0 \) \( (i = 1, 2) \) for all \( t \geq 0 \).

We shall next end up with the equivalence of \( \mathcal{L} (t) \) and \( E (t) \). Let us recall that by \((5.1)\), \((5.6)\), \((5.3)\), \((5.4)\), \((5.7)\) and \((4.10)\) the expression of \( \mathcal{L} (t) \) can be rewritten as follows:
\[
\mathcal{L}(t) = \frac{1}{2} \left( \|\dot{u}(t)\|^2 + \|\dot{v}(t)\|^2 \right) + \frac{1}{2} \left( 1 - \frac{1}{p_1} - \frac{1}{p_2} \right) \left( \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \right) + \frac{I_1(t)}{p_1} + \frac{I_2(t)}{p_2} \\
+ \delta \langle u(t), \dot{u}(t) \rangle + \delta \langle v(t), \dot{v}(t) \rangle + \frac{\delta \lambda_1}{2} \|u(t)\|^2 + \frac{\delta \lambda_2}{2} \|v(t)\|^2 + \frac{\delta \mu_1}{2} u^2(1, t) + \frac{\delta \mu_2}{2} v^2(0, t).
\]

It is straightforward to see that

\[
\mathcal{L}(t) \leq \frac{1}{2} (1 + \delta) \left( \|\dot{u}(t)\|^2 + \|\dot{v}(t)\|^2 \right) + \frac{I_1(t)}{p_1} + \frac{I_2(t)}{p_2}
\]

(5.18) \{eq:upper\}

\[
+ \frac{1}{2} \left( 1 - \frac{1}{p_1} - \frac{1}{p_2} + \delta (\lambda_1 + \lambda_2 + \mu_1 + \mu_2) \right) \left( \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \right).
\]

where we have used an elementary inequality

\[
\langle u(t), \dot{u}(t) \rangle + \langle v(t), \dot{v}(t) \rangle \leq \frac{1}{2} \left( \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \right) + \frac{1}{2} \left( \|\nabla v(t)\|^2 + \|\nabla \dot{v}(t)\|^2 \right).
\]

Therefore, one can choose from (5.18) that

\[
\beta_2 = \max \left\{ 1 + \delta, \frac{1 - \frac{1}{p_1} - \frac{1}{p_2} + \delta (\lambda_1 + \lambda_2 + \mu_1 + \mu_2)}{1 - \frac{1}{p_1} - \frac{1}{p_2}} \right\} = \max \left\{ 1 + \delta, 1 + \frac{\delta (1 + \lambda_1 + \lambda_2 + \mu_1 + \mu_2)}{1 - \frac{1}{p_1} - \frac{1}{p_2}} \right\},
\]

to obtain \( \mathcal{L}(t) \leq \beta_2 E(t) \).

Furthermore, one easily finds that

\[
\mathcal{L}(t) \geq \frac{1}{2} (1 - \delta) \left( \|\dot{u}(t)\|^2 + \|\dot{v}(t)\|^2 \right) + \frac{I_1(t)}{2p_1} + \frac{I_2(t)}{2p_2} + \frac{1}{2} \left( 1 - \frac{1}{p_1} - \frac{1}{p_2} - \delta \right) \left( \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \right).
\]

Thus, choosing \( \delta > 0 \) small enough for which

\[
\beta_1 = \min \left\{ 1 - \delta, 1 - \frac{\delta}{1 - \frac{1}{p_1} - \frac{1}{p_2}} \right\} > 0,
\]

most likely we choose \( \delta \in (0, 1 - \frac{\delta}{1 - \frac{1}{p_1} - \frac{1}{p_2}}) \) to obtain \( \mathcal{L}(t) \leq \beta_2 E(t) \). Hereby, our Lyapunov functional \( \mathcal{L}(t) \) is definitely equivalent to the total energy \( E(t) \) for all \( t \geq 0 \).

It remains to consider the functional \( \psi(t) \) which is defined by (4.10). The time derivative of \( \psi(t) \) can be found by multiplying \( \mathbf{1} \) by \( (u(x, t), v(x, t)) \) and then integrating over \( \Omega \). It therefore has the following expression

\[
\psi'(t) = \|u(t)\|^2 + \|v(t)\|^2 - \|\nabla u(t)\|^2 - \|\nabla v(t)\|^2 + K_1 |u(1, t)|^{p_1} + K_2 |v(0, t)|^{p_2}
\]

(5.19) \{eq:5.17\}

\[
+ \langle f_1(u(t), v(t)), u(t) \rangle + \langle f_2(u(t), v(t)), v(t) \rangle + \langle F_1(t), u(t) \rangle + \langle F_2(t), v(t) \rangle.
\]

On the one hand, we have

\[
\langle f_1(u(t), v(t)), u(t) \rangle + \langle f_2(u(t), v(t)), v(t) \rangle \leq d_2 \int_{\Omega} F(u(x, t), v(x, t)) dx,
\]

and by (5.18) and (5.19), we continue to estimate the above inequality

\[
\langle f_1(u(t), v(t)), u(t) \rangle + \langle f_2(u(t), v(t)), v(t) \rangle \leq d_2 \left[ \frac{1}{2} \left( \frac{1}{p_1} + \frac{1}{p_2} \right) \left( \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \right) - \left( \frac{K_1}{p_1} |u(1, t)|^{p_1} + \frac{K_2}{p_2} |v(0, t)|^{p_2} \right) - \left( \frac{I_1(t)}{p_1} + \frac{I_2(t)}{p_2} \right) \right].
\]

(5.20) \{eq:5.18\}

On the other hand, one easily has

\[
\langle f_2(u(t), v(t)), u(t) \rangle + \langle f_2(u(t), v(t)), v(t) \rangle \leq \frac{\varepsilon_2}{2} \left( \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \right) + \frac{1}{2\varepsilon_2} \left( \|F_1(t)\|^2 + \|F_2(t)\|^2 \right), \quad \forall \varepsilon_2 > 0.
\]

(5.21) \{eq:F1F2\}
Thus, we obtain from (5.19)-(5.21) that

\[
\psi'(t) \leq \|\dot{u}(t)\|^2 + \|\dot{v}(t)\|^2 - \left(1 - \frac{\varepsilon_2}{2} - \frac{d_2}{2} \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\right) \left(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\right) + \left(1 - \frac{d_2}{p_1}\right) K_1 |u(1,t)|^{p_1}
\]

(5.22) \{eq:estpsi\}

\[
\left(1 - \frac{d_2}{p_1}\right) K_2 |v(0,t)|^{p_2} - d_2 \left(\frac{I_1(t)}{p_1} + \frac{I_2(t)}{p_2}\right) + \frac{1}{2\varepsilon_2} \left(\|F_1(t)\|^2 + \|F_2(t)\|^2\right).
\]

Now the complete proof is coming. By (5.3), (5.22), and thanks to (5.10) and (5.11), the time derivative of our Lyapunov functional \(\mathcal{L}(t)\) can be estimated as follows:

\[
\mathcal{L}'(t) \leq - \left(\lambda_* - \frac{\varepsilon_1}{2}\right) \left(\|\dot{u}(t)\|^2 + \|\dot{v}(t)\|^2\right) - \mu_* \left(\|\dot{u}(1,t)\|^2 + \|\dot{v}(0,t)\|^2\right) + \frac{1}{2\varepsilon_1} \left(\|F_1(t)\|^2 + \|F_2(t)\|^2\right)
\]

\[
+ \delta \left(\|\dot{u}(t)\|^2 + \|\dot{v}(t)\|^2\right) - \delta \left(1 - \frac{\varepsilon_2}{2} - \frac{d_2}{2} \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\right) \left(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\right)
\]

\[
+ \delta \left(1 - \frac{d_2}{p_1}\right) K_1 |u(1,t)|^{p_1} + \left(1 - \frac{d_2}{p_2}\right) K_2 |v(0,t)|^{p_2}
\]

\[
- \delta d_2 \left(\frac{I_1(t)}{p_1} + \frac{I_2(t)}{p_2}\right) + \frac{\delta}{2\varepsilon_2} \left(\|F_1(t)\|^2 + \|F_2(t)\|^2\right)
\]

\[
\leq - \left(\lambda_* - \delta - \frac{\varepsilon_1}{2}\right) \left(\|\dot{u}(t)\|^2 + \|\dot{v}(t)\|^2\right) - \delta \left(1 - \frac{\varepsilon_2}{2} - \frac{d_2}{2} \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\right) \left(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\right)
\]

\[
+ \delta \left(1 - \frac{d_2}{p_1}\right) K_1 (p_* E_\omega \frac{p_2}{p_1}) \|\nabla u(t)\|^2 + \left(1 - \frac{d_2}{p_2}\right) K_2 (p_* E_\omega \frac{p_2}{p_1}) \|\nabla v(t)\|^2
\]

\[
- \delta d_2 \left(\frac{I_1(t)}{p_1} + \frac{I_2(t)}{p_2}\right) + \frac{\delta}{2} \left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right) \left(\|F_1(t)\|^2 + \|F_2(t)\|^2\right)
\]

(5.23) \{eq:Lc\}

\[
(1 - \eta_\nu) \left(1 - \frac{d_2}{\max\{p_1, p_2\}}\right) - \frac{\varepsilon_2}{2} \left(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\right)
\]

for all \(\delta, \varepsilon_1, \varepsilon_2 > 0\). Here we imply

\[
d_2 < \min\{p_1, p_2\}, \quad 0 < \varepsilon_2 < 2 (1 - \eta_\nu) \left(1 - \frac{d_2}{\max\{p_1, p_2\}}\right).
\]

Then for \(\delta\) small enough, satisfying \(0 < \delta < \lambda_\nu\), and let \(\varepsilon_1 > 0\) such that \(0 < \varepsilon_1 < 2 (\lambda_* - \delta)\), the equivalence of \(\mathcal{L}(t)\) and \(E(t)\) together with (5.23) and (5.11) says that there exists a constant \(\gamma \in (0, \eta_2)\) such that

\[
\mathcal{L}'(t) \leq -\gamma \mathcal{L}(t) + \bar{\eta}_1 \exp(-\eta_2 t), \quad \forall t \geq 0,
\]

and by Gronwall’s inequality, we have

\[
\mathcal{L}(t) \leq \left(\mathcal{L}(0) + \frac{\bar{\eta}_1}{\eta_2}\right) \exp(-\gamma t), \quad \forall t \geq 0,
\]

which leads to (5.12) by the equivalence of \(\mathcal{L}(t)\) and \(E(t)\). Hence, we complete the proof of the theorem. \(\square\)

**A numerical example.** The one-dimensional linear damped system of nonlinear wave equations has been qualitatively investigated in the sense of exponential decays and therefore, in this subsection the emphasis is put on the illustrative framework. We show below a numerical example where we recall that our considered problem in this section is particularly given by

\[
\begin{align*}
u_{tt} - \nu_{xx} + \lambda_\nu \nu_t &= f_1(u, v) + F_1(x, t), \\
u_{tt} - \nu_{xx} + \lambda_\nu \nu_t &= f_2(u, v) + F_2(x, t),
\end{align*}
\]
along with the boundary conditions \([1.3]\) and initial conditions \([1.4]\). We notice that the interior sources \(f_1\) and \(f_2\) have also been indirectly introduced in Remark 3. In fact, the functional \(F \in C^2 (\mathbb{R}^2; \mathbb{R})\) which is observed in \([2.6]\) as an example, can be established by

\[ f_1 (u, v) = \alpha \left( \gamma_1 |u|^\alpha - 2 + \frac{\gamma_2}{2} |u|^\frac{\alpha}{2} - 2 |v|^\frac{\beta}{2} \right) u, \]

\[ f_2 (u, v) = \beta \left( \gamma_1 |v|^\beta - 2 + \frac{\gamma_2}{2} |u|^\frac{\alpha}{2} - 2 |v|^\frac{\beta}{2} \right) v. \]

By these explicit expressions, \([2.3]\) obviously holds. Moreover, we deduce the facts that for all \((u, v) \in \mathbb{R}^2\)

\[ \left( \gamma_1 - \frac{\gamma_2}{2} \right) \left( |u|^\alpha + |v|^\beta \right) \leq F (u, v) \leq \left( \gamma_1 + \frac{\gamma_2}{2} \right) \left( |u|^\alpha + |v|^\beta \right), \]

\[ \min \{ \alpha, \beta \} F (u, v) \leq uf_1 (u, v) + vf_2 (u, v) \leq \max \{ \alpha, \beta \} F (u, v), \]

which imply \(d_1 = \min \{ \alpha, \beta \}, d_2 = \max \{ \alpha, \beta \}, d_1 = \gamma_1 - \frac{\gamma_2}{2}, \) and \(d_2 = \gamma_1 + \frac{\gamma_2}{2}\) in \([1.1], [1.2]\) and if one can choose an appropriate set of constants \(\alpha, \beta, \gamma_1, \) and \(\gamma_2\) such that \(\gamma_2 < 2 \gamma_1, \) and \(d_2 < \min \{ p_1, p_2 \}, \) (A3bis) is clearly valid.

In our example, we take \(\alpha = \beta = 4, p_1 = p_2 = 6\) and \(K_i = 142\) for \(i = 1, 2\) with \(\gamma_1 = \frac{5}{4}, \gamma_2 = \frac{1}{2}\) (so, \(d_2 = 4\) and \(\tilde{d}_2 = 1\)). Next, the initial conditions are defined by

\[ \tilde{u}_0 (x) = x \left( e^9 + 1 \right)^{\frac{-1}{6}}, \quad \tilde{u}_1 (x) = -xe^9 \left( e^9 + 1 \right)^{\frac{-5}{6}}, \]

\[ \tilde{v}_0 (x) = (1 - x) \left( e^9 + 1 \right)^{\frac{1}{7}}, \quad \tilde{v}_1 (x) = (x - 1) e^9 \left( e^9 + 1 \right)^{\frac{-6}{7}}. \]

Then, our external functions are

\[ F_1 (x, t) = \frac{4x^3 - 2x^2 + x}{(e^9 + 1)^{1/4}} - \frac{5e^{9+4t}x}{(e^9 + 1)^{3/4}}, \quad F_2 (x, t) = \frac{(x - 1) (4x^2 - 6x + 3)}{(e^9 + 1)^{3/4}} - \frac{5e^{9+4t} (1 - x)}{(e^9 + 1)^{9/4}}. \]

Finally, the exact solutions here are given by

\[ (5.24) \quad \text{eq: exact} \]

\[ u_{ex} (x, t) = \frac{x}{\sqrt{e^9 + 1}}, \quad v_{ex} (x, t) = \frac{1 - x}{\sqrt{e^9 + 1}}, \]

but they shall be neglected since we want to consider the illustrative approximation solutions.

In order to fulfill all assumptions of Theorem 15, it remains to check \([5.10]\) and \([5.11]\). Obviously, one has \(p_* = 3\) and \(\rho\) is computed by the following

\[ (5.25) \quad \text{eq:6/25} \]

\[ \rho = \frac{1}{2} \int_0^\infty \left( ||F_1 (s)|| + ||F_2 (s)|| \right) ds \]

\[ \leq \int_0^\infty \left[ \frac{142}{105 (e^9 + 1)^{1/2}} + \frac{25e^{2(9+4s)}}{3 (e^9 + 1)^{3/2}} + \frac{19e^{9+4s}}{3 (e^9 + 1)^{3/2}} \right]^{1/2} ds \]

\[ \leq \int_0^\infty \left( \frac{1682}{105} e^{-\frac{8}{2} - 6s} \right)^{1/2} ds < 1.563 \times 10^{-3}, \]

and the initial energy can be estimated as follows:

\[ (5.26) \quad \text{eq:1/4} \]

\[ E (0) = \frac{1}{2} \left( ||\tilde{u}_1||^2 + ||\tilde{v}_1||^2 + ||\nabla \tilde{u}_0||^2 + ||\nabla \tilde{v}_0||^2 \right) - \left( ||\tilde{u}_0 (1)||^6 + ||\tilde{v}_0 (0)||^6 \right) \]

\[ - \frac{3}{4} \int_0^1 \left( |\tilde{u}_0 (x)|^4 + |\tilde{v}_0 (x)|^4 \right) dx - \frac{1}{2} \int_0^1 |\tilde{u}_0 (x)|^2 |\tilde{v}_0 (x)|^2 dx \]

\[ \frac{1}{18} (e^9 + 1)^{-\frac{5}{6}} - (e^9 + 1)^{-\frac{2}{6}} - (e^9 + 1)^{-1} - \frac{1}{60} (e^9 + 1)^{-1} < 0.015. \]
Combining (5.25) and (5.20), the indirect argument of checking (5.11) is made by (5.25), and we obtain that

$$E_* = [E(0) + p] \exp (2p) < 0.017,$$

which leads to

$$\eta_* = 12p, E_* + 2(p, E_*)^2 < 1.$$ 

Therefore, the assumptions needed to check are all satisfied.

At the discretization level for this problem, a uniform grid of mesh-points \((x_k, t_n)\) is used. Here \(x_k = k \Delta x \) and \(t_n = n \Delta t\) where \(k\) and \(n\) are integers and \(\Delta x = \frac{x}{m}, \Delta t = \frac{t}{n}\) the equivalent mesh-widths in space \(x\) and time \(t\), respectively. We shall first consider the following differential system for the unknowns \((U_k(t), V_k(t)) = (u(x_k,t), v(x_k,t))\) for \(k = 0, 1, 2, \ldots, K,\)

\[
\begin{align*}
\frac{dU_k}{dt} &= K^2 (U_{k-1}(t) - 2U_k(t) + U_{k+1}(t)) - U_k(t) + 3U_k^3(t) + V_k^2(t) U_k(t) + F_1(x_k, t), \\
\frac{dV_k}{dt} &= K (U_k^5(t) - U_k(t)) - K^2 (U_K(t) - U_{K-1}(t)) - U_K(t) + 3U_K^3(t) + F_1(x_K, t), \\
\end{align*}
\]

and

\[
\begin{align*}
\frac{dV_0}{dt} &= K (V_0^5(t) - V_0(t)) + K^2 (V_1(t) - V_0(t)) - V_0(t) + 3V_0^3(t) + F_2(x_0, t), \\
\frac{dV_K}{dt} &= K^2 (V_{K-1}(t) - 2V_k(t) + V_{k+1}(t)) - V_k(t) + 3V_k^3(t) + U_k^2(t) V_k(t) + F_2(x_k, t), \\
\frac{dV_{K-1}}{dt} &= K^2 (V_{K-2}(t) - 2V_{K-1}(t)) - V_{K-1}(t) + 3V_{K-1}^3(t) + V_{K-1}^2(t) U_{K-1}(t) + F_2(x_{K-1}, t),
\end{align*}
\]

where \(\langle U_k(t), V_k(t) \rangle\) identically stands for \(\left( \frac{dU_k}{dt}(t), \frac{dV_k}{dt}(t) \right)\) and the initial conditions are

\[
(U_0(0), V_0(0)) = (\tilde{u}_0(x_k), \tilde{v}_0(x_k)), \quad (U_k(0), V_k(0)) = (\tilde{u}_1(x_k), \tilde{v}_1(x_k)), \quad k = 0, 1, 2, \ldots, K.
\]

Here we also notice that the values of \(U_0(t)\) and \(V_K(t)\) are known, so those cases are not considered.

A basic numerical approach is achieved below by using the linear recursive method where the nonlinear terms are linearized. Subsequently, after some rearrangements, we rewrite the linearized differential system to be a differential equation where the solution includes all discrete solutions of the linearized system. Such a unifying way also guarantees that the numerical solution uniquely exists. So by doing this, at the \(m\)-th iterative stage \((m \geq 1)\) the linearized differential system of (5.27)-(5.33) one by one becomes

\[
\begin{align*}
\frac{dU_1^{(m)}}{dt} &= K^2 (U_2^{(m)}(t) - 2U_1^{(m)}(t)) - U_1^{(m)}(t) + 3(U_1^{(m-1)}(t))^3 + 2(U_1^{(m)}(t))^2 U_1^{(m)}(t) + F_1(x_1, t), \\
\frac{dU_k^{(m)}}{dt} &= K^2 (U_{k-1}^{(m)}(t) - 2U_k^{(m)}(t) + U_{k+1}^{(m)}(t)) - U_k^{(m)}(t) + 3(U_k^{(m-1)}(t))^3 + 2(U_k^{(m)}(t))^2 U_k^{(m)}(t) + F_1(x_k, t), \quad k = 2, 3, \ldots, K-1, \\
\frac{dU_{K-1}^{(m)}}{dt} &= K^2 (U_{K-2}^{(m)}(t) - 2U_{K-1}^{(m)}(t)) - U_{K-1}^{(m)}(t) + 3(U_{K-1}^{(m-1)}(t))^3 + 2(U_{K-1}^{(m)}(t))^2 U_{K-1}^{(m)}(t) + F_1(x_{K-1}, t), \\
\end{align*}
\]

\[
\begin{align*}
\frac{dV_0^{(m)}}{dt} &= K (V_0^5(t) - V_0^{(m)}(t)) + K^2 (V_1^{(m)}(t) - V_0^{(m)}(t)) - V_0^{(m)}(t) + 3V_0^{(m)}(t) + F_2(x_0, t), \\
\frac{dV_k^{(m)}}{dt} &= K^2 (V_{k-1}^{(m)}(t) - 2V_k^{(m)}(t) + V_{k+1}^{(m)}(t)) - V_k^{(m)}(t) + 3V_k^{(m)}(t) + U_k^{(m)}(t) V_k^{(m)}(t) + F_2(x_k, t), \\
\frac{dV_{K-1}^{(m)}}{dt} &= K^2 (V_{K-2}^{(m)}(t) - 2V_{K-1}^{(m)}(t)) - V_{K-1}^{(m)}(t) + 3V_{K-1}^{(m)}(t) + V_{K-1}^{(m)}(t) U_{K-1}^{(m)}(t) + F_2(x_{K-1}, t),
\end{align*}
\]
where the solutions\( U(t) \) and the block matrices\( K, B \in \mathbb{R}^{2K \times 2K} \), and the functions\( F_1, F_2 \in \mathbb{R}^{2K} \) defined by

\[
\begin{align*}
\frac{dU^{(m)}}{dt}(t) &= K \left( U^{(m)}(t) \right)^5 + 3 \left( U^{(m-1)}(t) \right)^3 + F_1(x_k, t), \\
\frac{dV^{(m)}}{dt}(t) &= 3 \left( U^{(m-1)}(t) \right)^3 + F_1(x_k, t), \\
\begin{bmatrix}
U_0(t) \\
U_1(t) \\
\vdots \\
U_{K-1}(t) \\
U_K(t)
\end{bmatrix}
&= 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
3 \left( U^{(m-1)}(t) \right)^3 + F_1(x_k, t)
\end{bmatrix}, \\
\begin{bmatrix}
V_0(t) \\
V_1(t) \\
\vdots \\
V_{K-1}(t) \\
V_K(t)
\end{bmatrix}
&= 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
3 \left( V^{(m-1)}(t) \right)^3 + F_1(x_k, t)
\end{bmatrix},
\end{align*}
\]
for all $n \in \mathbb{N}$, then it is not good for our implementation since the objective is considering the system in a large time. Understanding the basic instability coming from stiff systems, we therefore apply the well-known implicit Euler method which reads

$$
\begin{align*}
A^{(m)} &= \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & 0 \\
0 & \cdots & 0 & 1
\end{bmatrix}, \\
B^{(m)} &= \begin{bmatrix}
-K^2 & K^2 & \cdots & 0 \\
0 & -K & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & 0 \\
0 & \cdots & 0 & 1
\end{bmatrix}, \\
A^{(m)} := a_k^{(m)} (t) &= -2K^2 + \left(V^{(m-1)} (t) \right)^2, \\
b_k^{(m)} := b_k^{(m)} (t) &= -2K^2 + \left(U^{(m-1)} (t) \right)^2, \\
k = 1, K - 1, \\
\end{align*}
$$

associated with the initial datum

$$
\begin{align*}
\mathcal{U}^{(m)} (0) &= \begin{bmatrix}
\tilde{u}_0 (x_1) \\
\tilde{u}_0 (x_2) \\
\vdots \\
\tilde{u}_0 (x_{K-1}) \\
\tilde{u}_0 (x_K) \\
\tilde{u}_1 (x_1) \\
\tilde{u}_1 (x_2) \\
\vdots \\
\tilde{u}_1 (x_{K-1}) \\
\tilde{u}_1 (x_K)
\end{bmatrix}, \\
\mathcal{V}^{(m)} (0) &= \begin{bmatrix}
\tilde{v}_0 (x_0) \\
\tilde{v}_0 (x_1) \\
\vdots \\
\tilde{v}_0 (x_{K-2}) \\
\tilde{v}_0 (x_{K-1}) \\
\tilde{v}_1 (x_0) \\
\tilde{v}_1 (x_1) \\
\vdots \\
\tilde{v}_1 (x_{K-2}) \\
\tilde{v}_1 (x_{K-1})
\end{bmatrix}.
\end{align*}
$$

Here the initial guess for our linearization method is simply initial values, i.e. $(\mathcal{U}^{(0)} (t_n), \mathcal{V}^{(0)} (t_n)) = (\mathcal{U}^{(m)} (0), \mathcal{V}^{(m)} (0))$ for all $n = 0, N$. In general, it is remarkable that the system (5.41)-(5.42) might be stiff, then it is not good for our implementation since the objective is considering the system in a large time. Understanding the basic instability coming from stiff systems, we therefore apply the well-known implicit Euler method which reads

$$
\begin{align*}
\mathcal{U}^{(m)}_{n+1} &= \left( I - \frac{T}{N} A^{(m)} (t_{n+1}) \right)^{-1} \left( \mathcal{U}^{(m)}_n + \frac{T}{N} F^{(m)}_1 (t_{n+1}) \right), \\
\mathcal{V}^{(m)}_{n+1} &= \left( I - \frac{T}{N} B^{(m)} (t_{n+1}) \right)^{-1} \left( \mathcal{V}^{(m)}_n + \frac{T}{N} F^{(m)}_2 (t_{n+1}) \right),
\end{align*}
$$

associated with the initial datum

$$
\begin{align*}
\mathcal{U}^{(m)} (0) &= \begin{bmatrix}
\tilde{u}_0 (x_1) \\
\tilde{u}_0 (x_2) \\
\vdots \\
\tilde{u}_0 (x_{K-1}) \\
\tilde{u}_0 (x_K) \\
\tilde{u}_1 (x_1) \\
\tilde{u}_1 (x_2) \\
\vdots \\
\tilde{u}_1 (x_{K-1}) \\
\tilde{u}_1 (x_K)
\end{bmatrix}, \\
\mathcal{V}^{(m)} (0) &= \begin{bmatrix}
\tilde{v}_0 (x_0) \\
\tilde{v}_0 (x_1) \\
\vdots \\
\tilde{v}_0 (x_{K-2}) \\
\tilde{v}_0 (x_{K-1}) \\
\tilde{v}_1 (x_0) \\
\tilde{v}_1 (x_1) \\
\vdots \\
\tilde{v}_1 (x_{K-2}) \\
\tilde{v}_1 (x_{K-1})
\end{bmatrix}.
\end{align*}
$$
for \( n = 0, N - 1 \), where \( I \) stands for the \( 2K \)-by-\( 2K \) identity matrix, and together with the conditions \((5.42)\).

Choosing \( m = 5, T = 20 \), and \( K = N = 50 \), we plot in Fig. 1 the approximation function \((u(x, t), v(x, t))\) solutions to our problem \((P)\) considered in this subsection. We surely check that such functions not only behave like the exact solutions \((5.24)\) (decay exponentially in time), but also completely have the same shapes corresponding to each exact solution. Indeed, let us observe those which are sketched in Fig. 2.

Furthermore, numerical results of the solutions \((u, v)\) together with the exact solutions \((u_{ex}, v_{ex})\) at nodes \((\frac{4}{5}, t_n)\) for \( n = 10; 20; 30 \), and various values of error in the entry-wise norm

\[
\mathcal{E}_{N,K}(u) = \max_{1 \leq k \leq K} \max_{1 \leq n \leq N} |u_{ex}(x_k, t_n) - u(x_k, t_n)|,
\]

\[
\mathcal{E}_{N,K}(v) = \max_{1 \leq k \leq K} \max_{0 \leq n \leq N-1} |v_{ex}(x_k, t_n) - v(x_k, t_n)|,
\]
| $n$ | $u_{ex} \left( \frac{4}{5}, t_n \right)$ | $u \left( \frac{4}{5}, t_n \right)$ | $|u_{ex} \left( \frac{4}{5}, t_n \right) - u \left( \frac{4}{5}, t_n \right)|$ |
|-----|--------------------------------|---------------------------------|---------------------------------|
| 10  | 1.54436330e-03                | 2.91855517e-03                 | 1.37419186e-03                 |
| 20  | 2.82860006e-03                | 7.20714168e-03                 | 4.37851996e-03                 |
| 30  | 5.18076174e-07                | 15.019043e-07                  | 9.83820633e-07                 |
|     | $v_{ex} \left( \frac{4}{5}, t_n \right)$ | $v \left( \frac{4}{5}, t_n \right)$ | $|v_{ex} \left( \frac{4}{5}, t_n \right) - v \left( \frac{4}{5}, t_n \right)|$ |
| 10  | 3.80908276e-04                | 7.29514168e-04                 | 3.43233404e-04                 |
| 20  | 7.07150017e-06                | 1.80147701e-05                 | 1.09432699e-05                 |
| 30  | 1.29519043e-07                | 6.22799676e-07                 | 4.93280633e-07                 |

Table 1. Numerical results at nodes $\left( \frac{4}{5}, t_n \right)$ for $n = 10; 20; 30$.

| $K$  | $N$  | $\mathcal{E}_{N,K} (u)$ | $\mathcal{E}_{N,K} (v)$ |
|------|------|------------------------|------------------------|
| 50   | 50   | 6.68545424e-03         | 6.68150701e-03         |
| 100  | 100  | 3.59475057e-03         | 3.59201931e-03         |
| 150  | 150  | 2.45841870e-03         | 2.45632948e-03         |
| 200  | 200  | 1.86793338e-03         | 1.86628504e-03         |

Table 2. Numerical results for the $l_{\infty}$ norm error $\mathcal{E}_{N,K}$.

are all given in Table 1-Table 2, respectively. To demonstrate the fact that the $l_{\infty}$ norm error $\mathcal{E}_{N,K}$ decreases (and obviously tends to zero) as $K, N$ increase without any attention to the discretization (like CFL conditions), we show in Table 2 the error values when such constants go from 50 to 200. As expected from analysis, our numerical methods used above are reasonable and efficient.

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