A \( p \)-adic analogue of Chan and Verrill’s formula for \( 1/\pi \)

Ji-Cai Liu

Department of Mathematics, Wenzhou University, Wenzhou 325035, PR China
jcliu2016@gmail.com

Abstract. We prove three supercongruences for sums of Almkvist–Zudilin numbers, which confirm some conjectures of Zudilin and Z.-H. Sun. A typical example is the Ramanujan-type supercongruence:

\[
\sum_{k=0}^{p-1} \frac{4k+1}{81^k} \gamma_k \equiv \left( \frac{-3}{p} \right) p \pmod{p^3},
\]

which is corresponding to Chan and Verrill’s formula for \( 1/\pi \):

\[
\sum_{k=0}^{\infty} \frac{4k+1}{81^k} \gamma_k = \frac{3\sqrt{3}}{2\pi}.
\]

Here \( \gamma_n \) are the Almkvist–Zudilin numbers.

Keywords: Supercongruences; Almkvist–Zudilin numbers; Harmonic numbers

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1 Introduction

For \( n \geq 0 \), the following sequence:

\[
\gamma_n = \sum_{j=0}^{n} (-1)^{n-j} \frac{3^{n-3j}(3j)!}{j!^3} \binom{n}{j} \binom{n+j}{j}
\]

are known as Almkvist–Zudilin numbers (see [1] and A125143 in [20]). This sequence appears to be first recorded by Zagier [29] as integral solutions to Apéry-like recurrence equations.

These numbers also appear as coefficients of modular forms. Let \( q = e^{2\pi i \tau} \) and

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)
\]

be the Dedekind eta function. Chan and Verrill [6] showed that if

\[
t_3(\tau) = \left( \frac{\eta(3\tau)\eta(6\tau)}{\eta(\tau)\eta(2\tau)} \right)^4 \text{ and } F_3(\tau) = \frac{(\eta(\tau)\eta(2\tau))^3}{\eta(3\tau)\eta(6\tau)},
\]

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and $|t_3(\tau)|$ is sufficiently small, then

$$F_3(\tau) = \sum_{n=0}^{\infty} \gamma_n t_3^n(\tau).$$

They also constructed some new series for $1/\pi$ in terms of the numbers $\gamma_n$, one of the typical examples is the following formula [6, Theorem 3.14]:

$$\sum_{k=0}^{\infty} \frac{4k + 1}{81k} \gamma_k = \frac{3\sqrt{3}}{2\pi}. \tag{1.1}$$

The above interesting example motivates us to prove the following supercongruence, which was originally conjectured by Zudilin [30, (33)].

**Theorem 1.1** For any prime $p \geq 5$, we have

$$\sum_{k=0}^{p-1} \frac{4k + 1}{81k} \gamma_k \equiv \left( -\frac{3}{p} \right) p \pmod{p^3}, \tag{1.2}$$

where $\left( \frac{\cdot}{p} \right)$ denotes the Legendre symbol.

The supercongruence (1.2) may be regarded as a $p$-adic analogue of (1.1). In the past two decades, Ramanujan-type series for $1/\pi$ as well as related supercongruences and $q$-supercongruences have attracted many experts’ attention (see, for instance, [3, 5, 6, 8–11, 14–16, 24, 26, 28, 30]).

The second result of this paper consists of the following two related supercongruences involving the numbers $\gamma_n$, which were originally conjectured by Z.-H. Sun [21, Conjecture 6.8].

**Theorem 1.2** For any prime $p \geq 5$, we have

$$\sum_{k=0}^{p-1} (4k + 3) \gamma_k \equiv 3 \left( -\frac{3}{p} \right) p \pmod{p^3}. \tag{1.3}$$

**Theorem 1.3** For any prime $p \geq 5$, we have

$$\sum_{k=0}^{p-1} \frac{2k + 1}{(-9)^k} \gamma_k \equiv \left( -\frac{3}{p} \right) p \pmod{p^3}. \tag{1.4}$$

We remark that congruence properties for the Almkvist–Zudilin numbers have been widely investigated by Amdeberhan and Tauraso [2], Chan, Cooper and Sica [4], and Z.-H. Sun [21, 23].

The rest of the paper is organized as follows. In Section 2, we recall some necessary combinatorial identities involving harmonic numbers and prove a preliminary congruence. The proofs of Theorems 1.1–1.3 are presented in Sections 3–5, respectively.
2 Preliminary results

Let

\[ H_n = \sum_{j=1}^{n} \frac{1}{j} \]

denote the \( n \)th harmonic number. The Fermat quotient of an integer \( a \) with respect to an odd prime \( p \) is given by \( q_p(a) = (a^{p-1} - 1)/p \).

In order to prove Theorems 1.1 and 1.2, we need the following two lemmas.

Lemma 2.1 For any non-negative integer \( n \), we have

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n+i}{i} = (-1)^n, \quad (2.1) \]

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n+i}{i} H_i = 2(-1)^n H_n, \quad (2.2) \]

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n+i}{i} H_{n+i} = 2(-1)^n H_n. \quad (2.3) \]

In fact, such identities can be discovered and proved by the symbolic summation package \textit{Sigma} developed by Schneider [19]. One can also refer to [14] for the same approach to finding and proving identities of this type. For human proofs of (2.1) – (2.3), one refers to [18].

Lemma 2.2 For any prime \( p \geq 5 \), we have

\[ \left( -\frac{3}{p} \right) q_p(3) \quad (\text{mod } p). \quad (2.4) \]

Proof. Note that

\[ \sum_{k=0}^{p-1} \frac{(3k)!}{3^{3k} k!^3} (H_{3k} - H_k) \equiv \left( -\frac{3}{p} \right) q_p(3) \quad (\text{mod } p). \quad (2.5) \]

Recall the following identity due to Tauraso [27, Theorem 1]:

\[ \frac{(1/3)_k (2/3)_k}{(1)_k^2} \sum_{j=0}^{k-1} \left( \frac{1}{1/3 + j} + \frac{1}{2/3 + j} \right) = \sum_{j=0}^{k-1} \frac{(1/3)_j (2/3)_j}{(1)_j^2} \cdot \frac{1}{k-j}. \quad (2.6) \]
Substituting (2.6) into (2.5) and exchanging the summation order gives
\[
\sum_{k=0}^{p-1} \frac{(3k)!}{3^{3k}k!^3} (3H_{3k} - H_k) = \sum_{k=0}^{p-1} \sum_{j=0}^{k-1} \frac{(1/3)_j(2/3)_j}{(1)_j^2} \cdot \frac{1}{k-j}
\]
\[
= \sum_{j=0}^{p-2} \frac{(1/3)_j(2/3)_j}{(1)_j^2} H_{p-1-j}
\]
\[
= \sum_{j=0}^{p-2} \frac{(3j)!}{3^{3j}j!^3} (3H_{3j} - H_j) \quad \text{mod } p, \quad (2.7)
\]
where we have utilized the fact that \( H_{p-1-j} \equiv H_j \pmod{p} \). By (2.7), we obtain
\[
\sum_{k=0}^{p-1} \frac{(3k)!}{3^{3k}k!^3} (H_{3k} - H_k) \equiv \frac{1}{3} \left( \sum_{k=0}^{p-2} \frac{(3k)!}{3^{3k}k!^3} H_k - 2 \sum_{k=0}^{p-1} \frac{(3k)!}{3^{3k}k!^3} H_k \right)
\]
\[
\equiv -\frac{1}{3} \sum_{k=0}^{\lfloor p/3 \rfloor} \frac{(3k)!}{3^{3k}k!^3} H_k \quad \text{mod } p, \quad (2.8)
\]
because \((3k)! \equiv 0 \pmod{p}\) for \(k > \lfloor p/3 \rfloor\).

Let \(m = \lfloor p/3 \rfloor\). From [2, Lemma 2.3], we see that for \(0 \leq k \leq m\),
\[
\frac{(3k)!}{3^{3k}k!^3} \equiv (-1)^k \binom{m}{k} \binom{m+k}{k} \pmod{p}. \quad (2.9)
\]
It follows from (2.2), (2.8) and (2.9) that
\[
\sum_{k=0}^{p-1} \frac{(3k)!}{3^{3k}k!^3} (H_{3k} - H_k) \equiv -\frac{1}{3} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{m+k}{k} H_k \pmod{p}
\]
\[
= -\frac{2(-1)^m}{3} H_m.
\]
Finally, noting
\[
(-1)^{\lfloor p/3 \rfloor} = \left( \frac{-3}{p} \right), \quad (2.10)
\]
and the following congruence [13, page 359]:
\[
H_{\lfloor p/3 \rfloor} \equiv -\frac{3}{2} q_p(3) \pmod{p^2}, \quad (2.11)
\]
we complete the proof of (2.4). \(\square\)
3 Proof of Theorem 1.1

We begin with the transformation formula due to Chan and Zudilin [7, Corollary 4.3]:

\[
\gamma_n = \sum_{i=0}^{n} \binom{2i}{i}^2 \binom{4i}{2i} \binom{n + 3i}{4i} (-3)^{3(n-i)}. \tag{3.1}
\]

Using (3.1) and exchanging the summation order, we obtain

\[
\sum_{k=0}^{p-1} \frac{4k+1}{81^k} \gamma_k = \sum_{k=0}^{p-1} \frac{4k+1}{81^k} \sum_{i=0}^{k} \binom{2i}{i}^2 \binom{4i}{2i} \binom{k+3i}{4i} (-3)^{3(k-i)}
\]

\[
= \sum_{i=0}^{p-1} \frac{1}{(-3)^{3i}} \binom{2i}{i}^2 \binom{4i}{2i} \sum_{k=i}^{p-1} \frac{4k+1}{(-3)^{k}} \binom{k+3i}{4i}. \tag{3.2}
\]

Note that

\[
\sum_{k=i}^{n-1} \frac{4k+1}{(-3)^{k}} \binom{k+3i}{4i} = (n-i) \binom{n+3i}{4i} (-3)^{1-n}, \tag{3.3}
\]

which can be easily proved by induction on \(n\). Combining (3.2) and (3.3) gives

\[
\sum_{k=0}^{p-1} \frac{4k+1}{81^k} \gamma_k = 3^{1-p} \sum_{i=0}^{p-1} \frac{p-i}{(-3)^{3i}} \binom{2i}{i}^2 \binom{4i}{2i} \binom{p+3i}{4i}. \tag{3.4}
\]

Furthermore, we have

\[
(-1)^i (p-i) \binom{2i}{i}^2 \binom{4i}{2i} \binom{p+3i}{4i}
\]

\[
= (p-i) \frac{p(p+3i) \cdots (p+1)(p-1) \cdots (p-i)}{i!^4}
\]

\[
\equiv \frac{p(3i)!}{i!^4} (1 + p(H_{3i} - H_{i})) \pmod{p^3}.
\]

Thus,

\[
\sum_{k=0}^{p-1} \frac{4k+1}{81^k} \gamma_k \equiv 3^{1-p} \sum_{i=0}^{p-1} \frac{(3i)!}{3^{3i} i!^3} (1 + p(H_{3i} - H_{i})) \pmod{p^3}.
\]

Finally, noting (2.4) and Mortenson’s supercongruence [17, (1.2)]:

\[
\sum_{i=0}^{p-1} \frac{(3i)!}{3^{3i} i!^3} \equiv \frac{-3}{p} \pmod{p^2},
\]
we arrive at
\[ \sum_{k=0}^{p-1} \frac{4k + 1}{81^k} \gamma_k \equiv p \left( \frac{-3}{p} \right) (3^{1-p} + 3^{1-p} p q_p (3)) \pmod{p^3} \]
\[ = p \left( \frac{-3}{p} \right), \]
as desired.

4 Proof of Theorem 1.2

Recall the following transformation formula [21, (5.1)]:

\[ \gamma_n = \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{2i}{i}^2 \binom{4i}{2i} \binom{n+i}{4i} (-3)^{n-3i}. \tag{4.1} \]

By (4.1), we have
\[ \sum_{k=0}^{p-1} (4k + 3) \gamma_k = \sum_{k=0}^{p-1} (4k + 3) \sum_{i=0}^{\lfloor k/3 \rfloor} \binom{2i}{i}^2 \binom{4i}{2i} \binom{k+i}{4i} (-3)^{k-3i} \]
\[ = \sum_{i=0}^{p-1} \frac{1}{(-3)^{3i}} \binom{2i}{i}^2 \binom{4i}{2i} \sum_{k=i}^{p-1} (-3)^k (4k + 3) \binom{k+i}{4i}. \tag{4.2} \]

It can be easily proved by induction on \( n \) that
\[ \sum_{k=i}^{n-1} (-3)^k (4k + 3) \binom{k+i}{4i} = 3(n - 3i) \binom{n+i}{4i} (-3)^{n-1}. \tag{4.3} \]

It follows from (4.2) and (4.3) that
\[ \sum_{k=0}^{p-1} (4k + 3) \gamma_k = 3^p \sum_{i=0}^{\lfloor p/3 \rfloor} \frac{p - 3i}{(-3)^{3i}} \binom{2i}{i}^2 \binom{4i}{2i} \binom{p+i}{4i}. \]

Note that
\[ (-1)^i (p - 3i) \binom{2i}{i}^2 \binom{4i}{2i} \binom{p+i}{4i} \]
\[ = \frac{(-1)^i p (p+i) \cdots (p+1) (p-1) \cdots (p-3i)}{i!^4} \]
\[ \equiv \frac{p (3i)!}{i!^3} (1 - p (H_{3i} - H_i)) \pmod{p^3}. \]
Thus,

$$\sum_{k=0}^{p-1} (4k + 3) \gamma_k \equiv 3^p p \sum_{i=0}^{[p/3]} (3i)! 3^{3i} \gamma^{3i} (1 - p (H_{3i} - H_i)) \pmod{p^3}. \tag{4.4}$$

Let $m = \lfloor p/3 \rfloor$. Since

$$\frac{(3i)!}{3^{3i}} = (-1)^i \left( \begin{array}{c} -1/3 \\ i \end{array} \right) \left( \begin{array}{c} -1/3 + i \\ i \end{array} \right) = (-1)^i \left( \begin{array}{c} -2/3 \\ i \end{array} \right) \left( \begin{array}{c} -2/3 + i \\ i \end{array} \right),$$

we have

$$\frac{(3i)!}{3^{3i} i!^2} = (-1)^i \left( \begin{array}{c} m - p/3 \\ i \end{array} \right) \left( \begin{array}{c} m - p/3 + i \\ i \end{array} \right)$$

$$= \frac{(-1)^i (m + i - p/3) \cdots (m - i + 1 - p/3)}{i!^2}$$

$$\equiv (-1)^i \left( \begin{array}{c} m \\ i \end{array} \right) \left( \begin{array}{c} m + i \\ i \end{array} \right) \left( 1 - \frac{p}{3} (H_{m+i} - H_{m-i}) \right) \pmod{p^2}. \tag{4.5}$$

Substituting (4.5) into the right-hand side of (4.4) gives

$$\sum_{k=0}^{p-1} (4k + 3) \gamma_k \equiv 3^p p \sum_{i=0}^{m} (-1)^i \left( \begin{array}{c} m \\ i \end{array} \right) \left( \begin{array}{c} m + i \\ i \end{array} \right)$$

$$\times \left( 1 - \frac{p}{3} (H_{m+i} - H_{m-i} + 3H_{3i} - 3H_i) \right) \pmod{p^2}. \tag{4.6}$$

Furthermore, we have

$$H_{3i} = \frac{1}{3} \left( H_i + \sum_{j=1}^{i} \frac{1}{j - 1/3} + \sum_{j=1}^{i} \frac{1}{j - 2/3} \right)$$

$$\equiv \frac{1}{3} \left( H_i + \sum_{j=1}^{i} \frac{1}{m + j} - \sum_{j=1}^{i} \frac{1}{m + 1 - j} \right) \pmod{p}$$

$$= \frac{1}{3} (H_i + H_{m+i} + H_{m-i} - 2H_m). \tag{4.7}$$
It follows from (2.1)–(2.3), (4.6) and (4.7) that
\[
\sum_{k=0}^{p-1} (4k + 3) \gamma_k 
\equiv 3^p p \sum_{i=0}^{m} (-1)^i \binom{m}{i} \binom{m + i}{i} \left(1 - \frac{2p}{3} (H_{m+i} - H_m - H_i)\right) \pmod{p^3}
\]
\[
= 3^p (-1)^m \left(1 + \frac{2p}{3} H_m\right).
\]

Finally, using (2.10) and (2.11), we obtain
\[
\sum_{k=0}^{p-1} (4k + 3) \gamma_k \equiv 3^p \left(-\frac{3}{p}\right) 3^{p-1} (2 - 3^{p-1})
\]
\[
= 3^p \left(-\frac{3}{p}\right) \left(1 - (3^{p-1} - 1)^2\right)
\]
\[
\equiv 3^p \left(-\frac{3}{p}\right) \pmod{p^3},
\]
where we have used the Fermat’s little theorem in the last step.

5 Proof of Theorem 1.3

Recall the following transformation formula [21, Lemma 4.1]:
\[
\gamma_n = \sum_{i=0}^{n} (-9)^{n-i} \binom{2i}{i} \binom{n+i}{2i} \sum_{j=0}^{i} \binom{i}{j} 2 \binom{2j}{j}.
\]
(5.1)

Let
\[
g_n = \sum_{k=0}^{n} \binom{n}{k} 2 \binom{2k}{k}.
\]

By (5.1), we have
\[
\sum_{k=0}^{p-1} \frac{2k + 1}{(-9)^k} \gamma_k = \sum_{k=0}^{p-1} \frac{2k + 1}{(-9)^k} \sum_{i=0}^{k} (-9)^{k-i} \binom{2i}{i} \binom{k+i}{2i} g_i
\]
\[
= \sum_{i=0}^{p-1} \frac{g_i}{(-9)^i} \sum_{k=i}^{p-1} \frac{2k + 1}{2i} \binom{k+i}{2i} \binom{2i}{i}.
\]
(5.2)
Note that
\[
\sum_{k=i}^{n-1} \frac{(2k+1)(k+i)(2i)}{2i} = \frac{n^2}{i+1} \binom{n-1}{i} \binom{n+i}{i}, \tag{5.3}
\]
which can be proved by induction on \(n\). It follows from (5.2) and (5.3) that
\[
\sum_{k=0}^{p-1} \frac{2k+1}{(-9)^k} \gamma_k = p^2 \sum_{i=0}^{p-1} \frac{g_i}{9^i(i+1)} \binom{p-1}{i} \binom{p+i}{i}.
\]
Since
\[
\binom{p-1}{i} \binom{p+i}{i} \equiv (-1)^i \pmod{p^2},
\]
we have
\[
\sum_{k=0}^{p-1} \frac{2k+1}{(-9)^k} \gamma_k \equiv p^2 \sum_{i=0}^{p-1} \frac{g_i}{9^i(i+1)} \pmod{p^3}. \tag{5.4}
\]
From [12, Lemma 2.7], we see that for \(0 \leq i \leq p-1\),
\[
\frac{g_i}{9^i} \equiv \left(\frac{-3}{p}\right) g_{p-1-i} \pmod{p},
\]
and so
\[
\sum_{i=0}^{p-2} \frac{g_i}{9^i(i+1)} \equiv \left(\frac{-3}{p}\right) \sum_{i=0}^{p-2} \frac{g_{p-1-i}}{i+1} = \left(\frac{-3}{p}\right) \sum_{i=1}^{p-1} \frac{g_i}{p-i} \equiv -\left(\frac{-3}{p}\right) \sum_{i=1}^{p-1} \frac{g_i}{i} \pmod{p}.
\]
Using the congruence [25, (1.8)]:
\[
\sum_{i=1}^{p-1} \frac{g_i}{i} \equiv 0 \pmod{p},
\]
we obtain
\[
\sum_{i=0}^{p-2} \frac{g_i}{9^i(i+1)} \equiv 0 \pmod{p}. \tag{5.5}
\]
Furthermore, combining (5.4) and (5.5) gives
\[
\sum_{k=0}^{p-1} \frac{2k + 1}{(-9)^k} \gamma_k \equiv \frac{pg_{p-1}}{9^{p-1}} \quad (\text{mod } p^3).
\]

By [25, Lemma 3.2], we have
\[
g_{p-1} \equiv \left( \frac{-3}{p} \right) (2 \cdot 3^{p-1} - 1) \quad (\text{mod } p^2),
\]
and so
\[
\sum_{k=0}^{p-1} \frac{2k + 1}{(-9)^k} \gamma_k \equiv p \left( \frac{-3}{p} \right) \left( 1 - \frac{(3^{p-1} - 1)^2}{9^{p-1}} \right)
\equiv p \left( \frac{-3}{p} \right) \quad (\text{mod } p^3),
\]
where we have utilized the Fermat’s little theorem.

**Remark.** Z.-H. Sun [21, Conjecture 6.8] also conjectured a companion supercongruence of (1.4):
\[
\sum_{k=0}^{p-1} \frac{2k + 1}{9^k} \gamma_k \equiv \frac{pg_{p-1}}{9^{p-1}} \quad (\text{mod } p^3).
\]

In a similar way, by using (5.1) and the following identity:
\[
\sum_{k=i}^{n-1} (-1)^k (2k + 1) \binom{k + i}{2i} \binom{2i}{i} = (-1)^{n-1} n \binom{n - 1}{i} \binom{n + i}{i},
\]
we can show that
\[
\sum_{k=0}^{p-1} \frac{2k + 1}{9^k} \gamma_k \equiv p \sum_{i=0}^{p-1} \frac{g_i}{9^i} \quad (\text{mod } p^3).
\]

Thus, the conjectural supercongruence (5.6) is equivalent to
\[
\sum_{i=0}^{p-1} \frac{g_i}{9^i} \equiv \left( \frac{-3}{p} \right) \quad (\text{mod } p^2),
\]
which was originally conjectured by Z.-W. Sun [25, Remark 1.1].

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References

[1] G. Almkvist and W. Zudilin, Differential equations, mirror maps and zeta values, in Mirror
symmetry, Vol. V, AMS/IP Studies in Advanced Mathematics, vol. 38 (American Mathematical Society, Providence, RI, 2006), 481–515.

[2] T. Amdeberhan and R. Tauraso, Supercongruences for the Almkvist–Zudilin numbers, Acta
Arith. 173 (2016), 255–268.

[3] H.H. Chan, S.H. Chan and Z. Liu, Domb’s numbers and Ramanujan-Sato type series for
1/π, Adv. Math. 186 (2004), 396–410.

[4] H.H. Chan, S. Cooper and F. Sica, Congruences satisfied by Apéry-like numbers, Int. J.
Number Theory 6 (2010), 89C-97.

[5] H.H. Chan, J. Wan and W. Zudilin, Legendre polynomials and Ramanujan-type series for
1/π, Israel J. Math. 194 (2013), 183–207.

[6] H.H. Chan and H. Verrill, The Apéry numbers, the Almkvist–Zudilin numbers and new
series for 1/π, Math. Res. Lett. 16 (2009), 405–420.

[7] H.H. Chan and W. Zudilin, New representations for Apéry-like sequences, Mathematika 56
(2010), 107–117.

[8] V.J.W. Guo, Proof of a generalization of the (B.2) supercongruence of Van Hamme through
a q-microscope, Adv. in Appl. Math. 116 (2020), Art. 102016.

[9] V.J.W. Guo and W. Zudilin, A q-microscope for supercongruences, Adv. Math. 346 (2019),
329–358.

[10] V.J.W. Guo and J.-C. Liu, Some congruences related to a congruence of Van Hamme,
Integral Transforms Spec. Funct. 31 (2020), 221–231.

[11] V.J.W. Guo and M.J. Schlosser, Some new q-congruences for truncated basic hypergeo-
metric series: even powers, Results Math. 75 (2020), Art. 1.

[12] F. Jarvis and H. Verrill, Supercongruences for the Catalan–Larcombe–French numbers,
Ramanujan J. 22 (2010), 171–186.

[13] E. Lehmer, On congruences involving Bernoulli numbers and the quotients of Fermat and
Wilson, Ann. Math. 39 (1938), 350–360.

[14] J.-C. Liu, Semi-automated proof of supercongruences on partial sums of hypergeometric
series, J. Symbolic Comput. 93 (2019), 221–229.

[15] J.-C. Liu, Some supercongruences arising from symbolic summation, J. Math. Anal. Appl.
488 (2020), Art. 124062.

[16] J.-C. Liu and F. Petrov, Congruences on sums of q-binomial coefficients, Adv. in Appl.
Math. 116 (2020), Art. 102003.

[17] E. Mortenson, Supercongruences between truncated 2F1 hypergeometric functions and their
Gaussian analogs, Trans. Amer. Math. Soc. 355 (2003), 987–1007.

[18] H. Prodinger, Human proofs of identities by Osburn and Schneider, Integers 8 (2008), A10.

[19] C. Schneider, Symbolic summation assists combinatorics, Sém. Lothar. Combin. 56 (2007),
B56b.

[20] N.J.A. Sloane, The on-line encyclopedia of integer sequences, https://oeis.org.

[21] Z.-H. Sun, Congruences for Domb and Almkvist–Zudilin numbers, Integral Transforms
Spec. Funct. 26 (2015), 642–659.
[22] Z.-H. Sun, Super congruences concerning binomial coefficients and Apéry-like numbers, preprint (2020), arXiv:2002.12072.

[23] Z.-H. Sun, New congruences involving Apéry-like numbers, preprint (2020), arXiv:2004.07172.

[24] Z.-W. Sun, Super congruences and Euler numbers, Sci. China Math. 54 (2011), 2509–2535.

[25] Z.-W. Sun, Congruences involving $g_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k$, Ramanujan J. 40 (2016), 511–533.

[26] Z.-W. Sun, New series for powers of $\pi$ and related congruences, Electron. Res. Arch. 28 (2020), 1273–1342.

[27] R. Tauraso, Supercongruences for a truncated hypergeometric series, Integers 12 (2012), A45.

[28] L. Van Hamme, Some conjectures concerning partial sums of generalized hypergeometric series, $p$-adic functional analysis (Nijmegen, 1996), Lecture Notes in Pure and Appl. Math., vol. 192, Dekker, New York, 1997, 223–236.

[29] D. Zagier, Integral solutions of Apéry-like recurrence equations, in Groups and Symmetries, CRM Proceedings and Lecture Notes, Vol. 47 (American Mathematical Society, Providence, RI, 2009), 349–366.

[30] W. Zudilin, Ramanujan-type supercongruences, J. Number Theory 129 (2009), 1848–1857.