Numerical Solution of Two-Dimensional Time Fractional Mobile/Immobile Equation Using Explicit Group Methods

Fouad Mohammad Salama · Umair Ali · Ajmal Ali

Abstract
In this paper, we shall present the development of two explicit group schemes, namely, fractional explicit group (FEG) and modified fractional explicit group (MFEG) methods for solving the time fractional mobile/immobile equation in two space dimensions. The presented methods are formulated based on two Crank-Nicolson (C-N) finite difference schemes established at two different grid spacings. The stability and convergence of order $O(\tau^{2-\alpha} + h^2)$ are rigorously proven using Fourier analysis. Several numerical experiments are conducted to verify the efficiency of the proposed methods. Meanwhile, numerical results show that the FEG and MFEG algorithms are able to reduce the computational times and iterations effectively while preserving good accuracy in comparison to the C-N finite difference method.

Keywords Fractional mobile/immobile equation · Grouping strategy · Crank-Nicolson finite difference · Stability · Convergence · Numerical experiments

Mathematics Subject Classification 35R11, 65N06, 65N12

Introduction
The roots of fractional calculus are nearly as old as those of classical calculus. However, it was only at the turn of the century that interest in fractional calculus exploded. In accordance with this, the first book along with the first conference on fractional calculus and its applications did not appear until 1974. For some researchers, this year is viewed as the beginning of a new era for fractional calculus. Nowadays, a large number of research articles and books are devoted to this topic, and its applications in various areas such as physics, biology, chemistry, finance and economy are witnessing remarkable progress, see [1–11] and the
references therein. The popularity of fractional calculus in all branches of science stems from its successful application in modelling physical phenomena, specifically in the form of fractional differential equations. For instance, Zhang et al. [12] introduced an image enhancement approach for single-pixel imaging that is based on fractional order operators. The authors argue that the new fractional approach strikes a good balance between edge detection and noise suppression, making it useful in areas such as feature extraction and object recognition, in addition to military and industrial applications. Aman et al. [13] considered some fractional-order PDEs to study the memory properties pertaining to the behavior of nanofluids, which cannot be addressed by integer-order PDEs. The authors mentioned that their work has useful applications in diverse fields such as physics, engineering and many others. Khan et al. [14] extended the classical Maxwell equation to its counterpart fractional-order Maxwell model via the emerging definition of the Caputo-Fabrizio derivative. The resulting fractional model was then applied to account for the heat transfer due to convection occurring in a generalized Maxwell fluid. Almeida et al. [15] scrutinized two mathematical models namely, the population growth model and the gross domestic product model, by using fractional differential equations described in the frame of Caputo-Fabrizio derivative and provided some applications to RC-electrical circuits on the basis of fractional differential equations involving the new developed definition. With the help of a fractional-order model, Ali et al. [17] discussed the flow of Jeffrey nanofluid under the effects of thermal radiation and heat/mass transfer. The governing fractional differential equation was described in the Atangana-Baleanu sense. In [18], two epidemic fractional models for describing infection dynamics together with a macroeconomic fractional model were investigated numerically. Uçar et al. [19] yielded approximate numerical solutions for the giving up smoking model expressed in terms of the Atangana-Baleanu fractional differential equation. In another study, Alrabaiah et al. [20] dealt with the phenomenon of tobacco smoking with sniffing class via a system of Caputo-type fractional differential equations. Ndaïrou et al. [21] reported on the mathematical and numerical analysis of a fractional Caputo differential system that approximates the infectious individuals of COVID-19. Some other valuable work on the application of fractional calculus can be found in [22–25].

This article focuses on developing explicit group schemes for the two-dimensional time fractional mobile/immobile equation of the following form:

\[
\frac{\partial u}{\partial t} + \frac{\mathcal{C}_0}{\Gamma(1-\alpha)} \frac{\partial^\alpha u}{\partial x^\alpha} = \Delta u + f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T),
\]

subject to the initial and boundary conditions

\[
u(x, y, 0) = g(x, y), \quad (x, y) \in \Omega
\]

\[
u(x, y, t) = \phi(x, y, t), \quad (x, y, t) \in \partial \Omega \times (0, T),
\]

where \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the Laplacian operator, \( f(x, y, t) \), \( \phi(x, y, t) \) and \( g(x, y) \) are given smooth functions. \( \Omega = \{(x, y)|0 < x < L, \ 0 < y < L\} \) and \( \partial \Omega \) is its boundary. \( \mathcal{C}_0 \frac{\partial^\alpha u}{\partial x^\alpha} \) represents the Caputo fractional derivative defined by

\[
\mathcal{C}_0 \frac{\partial^\alpha u}{\partial x^\alpha}(x, y, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} \frac{\partial u(x, y, \xi)}{\partial \xi} \, d\xi.
\]

The time fractional mobile/immobile Eq. (1) plays a central role in describing various phenomena including heat diffusion and propagation of ocean sounds, among others. Moreover,
it has been verified that the long term limit of continuous time random walks can be controlled by the fractional mobile/immobile equation [26], which interprets the probabilistic nature of the latter. In most cases, fractional partial differential equations (PDEs) are difficult to deal with analytically, and therefore, numerical or approximate analytical solutions become indispensable. In the following, we present some recent numerical developments in solving fractional mobile/immobile equations. Liu and Li [27] presented a Crank-Nicolson (C-N) difference scheme for the one-dimensional mobile/immobile equation with time derivative of variable order. Pourbashash et al. [28] proposed a compact difference scheme for solving the one-dimensional time fractional mobile/immobile equation. The stability and convergence of the numerical scheme were proved using Fourier method. Qiu et al. [29] developed a time two-grid method based on a finite difference approach for the nonlinear time fractional mobile/immobile equation in two dimensions. Yang et al. [26] developed the C-N orthogonal spline collocation method for the two-dimensional time fractional mobile/immobile equation. Jiang et al. [30] considered an alternating direction implicit (ADI) compact scheme to solve the semilinear time fractional mobile/immobile equation in two dimensions. Yin et al. [31] employed the generalized BDF2-θ technique in time and finite element in space to deal with the two-dimensional mobile/immobile equation with Riemann-Liouville time derivative. In addition, Chai et al. [32] constructed the high order compact difference schemes for the one-dimensional and two-dimensional time fractional mobile/immobile equations.

Generally speaking, numerical methods for fractional PDEs are abundant, whereas efficient numerical methods leading to rapid convergence, particularly for time fractional mobile/immobile equations, are relatively sparse. This motivates us to develop fast numerical schemes to solve them. This is useful for long time simulations, especially when attempting to solve multi-dimensional fractional problems [33–36]. It is well known that explicit group methods can diminish the computational complexity and reduce the computational time of numerical algorithms effectively [37–44]. However, numerical approximations based on explicit group methods for fractional mobile/immobile equations are still at an early stage of development. The main goal of this paper is to construct two fractional explicit group methods based on C-N difference schemes for solving Eq. (1). The stability in $l^2$ norm and the convergence order $O(\tau^{2-\alpha} + h^2)$, where $\tau$ and $h$ are respectively the temporal and spatial step sizes, will be proved. Numerical experiments to show that the proposed methods are more efficient than the C-N difference method in terms of execution time are also provided.

The outline of this paper is as follows. Section 2 devotes to the derivation of the C-N finite difference scheme. In Sect. 3, the fractional explicit group methods are constructed. The stability and convergence are rigorously proved in Sects. 4 and 5, respectively. In Sect. 6, some numerical simulations are carried out by using the C-N scheme and the two fractional explicit group methods, and some comparisons between three methods are demonstrated. Finally, a brief summary is given in Sect. 7.

The C-N Difference Scheme

In order to discretize the solution domain, we define $\Omega_{\Delta x, \Delta y} = \{(x_i, y_j)|x_i = i \Delta x, y_j = j \Delta y, \Delta x = L/M_x, \Delta y = L/M_y, 0 \leq i \leq M_x, 0 \leq j \leq M_y\}$ to be a uniform mesh of domain $\Omega$. Likewise, define $\Omega_{\tau} = \{t_k, t_k = k \tau, \tau = T/N, 0 \leq k \leq N\}$ to be a uniform mesh of interval $[0, T]$. $M_x, M_y$ and $N$ are positive integers. Let $u_{i,j}^k$ be the solution value at the grid point $(x_i, y_j, t_k)$. Then, the C-N approximation to Eq. (1) can be expressed as
follows:
\[
\frac{\partial u}{\partial t}_{i,j}^{k+1/2} + \frac{\partial^\alpha u}{\partial t^\alpha}_{i,j}^{k+1/2} = \frac{\partial^2 u}{\partial x^2}_{i,j}^{k+1/2} + \frac{\partial^2 u}{\partial y^2}_{i,j}^{k+1/2} + f_{i,j}^{k+1/2}.
\] (4)

A discrete approximation to the Caputo time fractional derivative $^C_0D_t^\alpha u(x, y, t)$ is given by the formula [45, 46]
\[
\frac{\partial^\alpha u}{\partial t^\alpha}_{i,j}^{k+1/2} = \left[ W_1 u_{i,j}^k + \sum_{m=1}^{k-1} (W_{k-m+1} - W_{k-m}) u_{i,j}^m - W_k u_{i,j}^0 + \sigma \frac{(u_{i,j}^{k+1} - u_{i,j}^k)}{2^{1-\alpha}} \right] + O(\tau^{2-\alpha}),
\] (5)

where
\[
\sigma = \frac{1}{\Gamma(2-\alpha)\tau^\alpha}, \quad W_n = \sigma \left((n+1/2)^{1-\alpha} - (n-1/2)^{1-\alpha}\right).
\]

In addition, utilizing Taylor expansion, we have
\[
\frac{\partial^2 u}{\partial x^2}_{i,j}^{k+1/2} = \frac{1}{2} \left[ \frac{u_{i+1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i-1,j}^{k+1}}{\Delta x^2} + \frac{u_{i+1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i-1,j}^{k+1}}{\Delta x^2} \right] + O(\tau^2 + (\Delta x)^2 + (\Delta y)^2),
\] (6)
\[
\frac{\partial^2 u}{\partial y^2}_{i,j}^{k+1/2} = \frac{1}{2} \left[ \frac{u_{i,j+1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j-1}^{k+1}}{\Delta x^2} + \frac{u_{i,j+1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j-1}^{k+1}}{\Delta x^2} \right] + O(\tau^2 + (\Delta x)^2 + (\Delta y)^2),
\] (7)

and
\[
\frac{\partial u}{\partial t}_{i,j}^{k+1/2} = \frac{u_{i,j}^{k+1} - u_{i,j}^k}{\tau} + O(\tau^2).
\] (8)

Substituting (5-8) into (4), we obtain the following difference equation
\[
\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\tau} + W_1 u_{i,j}^k + \sum_{m=1}^{k-1} (W_{k-m+1} - W_{k-m}) u_{i,j}^m - W_k u_{i,j}^0 + \sigma \frac{(u_{i,j}^{k+1} - u_{i,j}^k)}{2^{1-\alpha}} = \left[ \frac{u_{i+1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i-1,j}^{k+1}}{h^2} + \frac{u_{i+1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i-1,j}^{k+1}}{h^2} \right]
\] (9)
\[
\left[ \frac{u_{i,j+1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j-1}^{k+1}}{h^2} + \frac{u_{i,j+1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j-1}^{k+1}}{h^2} \right] + f_{i,j}^{k+1/2} + O(\tau^{2-\alpha} + h^2),
\]
in which $h = \Delta x = \Delta y$. By disregarding the higher order term in the above equation and replacing $u_{i,j}^k$ with its numerical approximation $U_{i,j}^k$, the following C-N difference scheme is obtained
\[
(1 + 4H_1 + H_2)U_{i,j}^{k+1} = H_1(U_{i+1,j}^{k+1} + U_{i-1,j}^{k+1} + U_{i+1,j}^{k} + U_{i-1,j}^{k}) + H_1(U_{i,j+1}^{k+1} + U_{i,j-1}^{k+1} + U_{i,j+1}^{k} + U_{i,j-1}^{k}) + (1 - \tau W_1 - 4H_1 + H_2)U_{i,j}^{k}.
\]
Fig. 1 Grouping of mesh points for the FEG method with $M_x = M_y = 10$

\[
+ \tau \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) U_{i,j}^m + \tau W_k U_{i,j}^0 + \tau f_{i,j}^{k+1/2},
\]

where $H_1 = \frac{\tau}{2\eta^2}$ and $H_2 = \frac{\tau \sigma}{2\eta^2}$.

**Formulation of Group Methods**

**The Fractional Explicit Group (FEG) Method**

Consider the C-N difference scheme (10). Let the mesh points be grouped in blocks of four points as shown in Fig. 1. Then, Eq. (10) is applied to any group of four points so that the following $(4 \times 4)$ system of equations is obtained

\[
\begin{pmatrix}
Q & -H_1 & 0 & -H_1 \\
-H_1 & Q & -H_1 & 0 \\
0 & -H_1 & Q & -H_1 \\
-H_1 & 0 & -H_1 & Q \\
\end{pmatrix}
\begin{pmatrix}
U_{i,j}^{k+1} \\
U_{i+1,j}^{k+1} \\
U_{i+1,j+1}^{k+1} \\
U_{i,j+1}^{k+1} \\
\end{pmatrix} =
\begin{pmatrix}
\text{rhs}_{i,j} \\
\text{rhs}_{i+1,j} \\
\text{rhs}_{i+1,j+1} \\
\text{rhs}_{i,j+1} \\
\end{pmatrix},
\]

where $Q = 1 + 4H_1 + H_2$,

\[
\text{rhs}_{i,j} = H_1(U_{i-1,j}^{k+1} + U_{i+1,j}^{k} + U_{i-1,j}^{k}) + H_1(U_{i,j-1}^{k+1} + U_{i,j+1}^{k} + U_{i,j-1}^{k})
\]

\[
+ (1 - \tau W_1 - 4H_1 + H_2) U_{i,j}^{k} + \tau \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) U_{i,j}^m
\]

\[
+ \tau W_k U_{i,j}^0 + \tau f_{i,j}^{k+1/2},
\]
where the coefficients matrix in (11) can be inverted to get the four-point FEG equation

\[ \begin{split}
&\text{rhs}_{i+1,j} = H_1(U_{i+2,j}^{k+1} + U_{i+2,j}^k + U_{i,j}^k) + H_1(U_{i+1,j-1}^{k+1} + U_{i+1,j-1}^k + U_{i+1,j}^k + U_{i+1,j}^{k+1}) \\
&\quad + (1 - \tau W_1 - 4H_1 + H_2)U_{i+1,j}^k + \tau \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) U_{i+1,j}^m \\
&\quad + \tau W_k U_{i+1,j}^0 + \tau f_{i+1,j}^{k+1/2},
\end{split} \]

\[ \begin{split}
&\text{rhs}_{i+1,j+1} = H_1(U_{i+2,j+1}^{k+1} + U_{i+2,j+1}^k + U_{i,j+1}^k + U_{i,j+1}^{k+1}) + H_1(U_{i+1,j+2}^{k+1} + U_{i+1,j+2}^k + U_{i,j}^{k+1} + U_{i,j}^k) \\
&\quad + (1 - \tau W_1 - 4H_1 + H_2)U_{i+1,j+1}^k + \tau \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) U_{i+1,j+1}^m \\
&\quad + \tau W_k U_{i+1,j+1}^0 + \tau f_{i+1,j+1}^{k+1/2},
\end{split} \]

The coefficients matrix in (11) can be inverted to get the four-point FEG equation

\[ \begin{pmatrix}
U_{i,j}^{k+1} \\
U_{i+1,j}^{k+1} \\
U_{i,j+1}^{k+1} \\
U_{i,j+1}^{k}
\end{pmatrix}
= \begin{pmatrix}
A_1 & A_2 & A_3 & A_2 \\
A_2 & A_1 & A_2 & A_3 \\
A_3 & A_2 & A_1 & A_2 \\
A_2 & A_3 & A_2 & A_1
\end{pmatrix}
\begin{pmatrix}
\text{rhs}_{i,j} \\
\text{rhs}_{i+1,j} \\
\text{rhs}_{i+1,j+1} \\
\text{rhs}_{i,j+1}
\end{pmatrix}, \tag{12}
\]

where

\[ \begin{align*}
A_1 &= \frac{14H_1^2 + 8H_1H_2 + H_2^2 + 8H_1 + 8H_2 + 1}{(2H_1 + H_2 + 1)(6H_1 + H_2 + 1)(1 + 4H_1 + H_2)} \\
A_2 &= \frac{H_1}{(2H_1 + H_2 + 1)(6H_1 + H_2 + 1)}, \\
A_3 &= \frac{2H_1^2}{(2H_1 + H_2 + 1)(6H_1 + H_2 + 1)(4H_1 + H_2 + 1)}.
\end{align*} \]

Figure 1 depicts the grouping of mesh points into blocks of four points with mesh size \( M_x = M_y = 10 \). It is obvious that the execution of Eq. (12) is only applicable for the grouped points. Therefore, the implementation of the FEG method is carried out by applying Eq. (12) to each group in Fig. 1. Before moving to the next time level, we calculate the remaining ungrouped points near the boundaries by utilizing Eq. (10). At each time level, the computation process is carried out in an iterative manner until the final time level \( N \) is reached. For convenience, the FEG method is defined in Algorithm 1.
The Modified Fractional Explicit Group (MFEG) Method

Another approximation scheme for Eq. (1) is obtained by Taylor’s expansion and considering points at mesh size of $2h$. The C-N difference scheme with $2h$ spacing is as follows

$$\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\tau} + W_1 u_{i,j}^k + \sum_{m=1}^{k-1} (W_{k-m+1} - W_{k-m}) u_{i,j}^m - W_k u_{i,j}^0 + \sigma \left( \frac{u_{i,j}^{k+1} - u_{i,j}^k}{2^{1-\alpha}} \right)$$

$$= \frac{1}{2} \left[ \frac{u_{i,j+2}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j-2}^{k+1}}{4h^2} + \frac{u_{i,j+2}^k - 2u_{i,j}^k + u_{i,j-2}^k}{4h^2} \right]$$

$$+ \frac{1}{2} \left[ \frac{u_{i,j+2}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j-2}^{k+1}}{4h^2} + \frac{u_{i,j+2}^k - 2u_{i,j}^k + u_{i,j-2}^k}{4h^2} \right]$$

$$+ f_{i,j}^{k+1/2} + O(\tau^{2-\alpha} + h^2),$$

(13)

On simplification with $U_{i,j}^k$ being the numerical approximation of $u_{i,j}^k$, the following equation is obtained

$$(1 + 4G_1 + G_2)U_{i,j}^{k+1} = G_1(U_{i+2,j}^{k+1} + U_{i-2,j}^{k+1} + U_{i+2,j}^{k} + U_{i-2,j}^{k}) + G_1(U_{i,j+2}^{k+1} + U_{i,j-2}^{k+1})$$

$$+ U_{i,j+2}^k + U_{i,j-2}^k + (1 - \tau W_1 - 4G_1 + G_2)U_{i,j}^k$$

$$+ \tau \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) U_{i,j}^m + \tau W_k U_{i,j}^0 + \tau f_{i,j}^{k+1/2},$$

(14)

where $G_1 = \frac{\tau}{8h^2}$ and $G_2 = \frac{\tau}{2^{1-\alpha}}$.

Now applying the C-N difference formula with $2h$ spacing (14) to any group of four points will lead to the following $(4 \times 4)$ system

$$\begin{pmatrix}
S & -G_1 & 0 & -G_1 \\
-G_1 & S & -G_1 & 0 \\
0 & -G_1 & S & -G_1 \\
-G_1 & 0 & -G_1 & S
\end{pmatrix}
\begin{pmatrix}
U_{i,j}^{k+1} \\
U_{i+2,j}^{k+1} \\
U_{i,j+2}^{k+1} \\
U_{i,j+2}^{k+1}
\end{pmatrix}
= \begin{pmatrix}
rhs_{i,j} \\
rhs_{i+2,j} \\
rhs_{i+2,j+2} \\
rhs_{i+2,j+2}
\end{pmatrix},$$

(15)

where $S = 1 + 4G_1 + G_2$,

$$rhs_{i,j} = G_1(U_{i-2,j}^{k+1} + U_{i+2,j}^{k+1} + U_{i-2,j}^{k}) + G_1(U_{i,j-2}^{k+1} + U_{i,j+2}^{k+1})$$

$$+ (1 - \tau W_1 - 4G_1 + G_2)U_{i,j}^k$$

$$+ \tau W_k U_{i,j}^0 + \tau f_{i,j}^{k+1/2},$$

$$rhs_{i+2,j} = G_1(U_{i+4,j}^{k+1} + U_{i+4,j}^{k} + U_{i,j}^{k}) + G_1(U_{i+2,j-2}^{k+1} + U_{i+2,j+2}^{k+1} + U_{i+2,j-2}^{k}).$$
where

\( x \)

\[ \text{The skewed C-N difference formula obtained by rotating the following sequence:} \]

\[ 1 \quad 2 \quad 3 \]

\[ \quad 188 \quad 188 \]

\[ \text{The above system can be rewritten in explicit form which results in the four-point MFEG equation} \]

\[ \begin{pmatrix}
U_{i+2,j}^{k+1} \\
U_{i+2,j}^{k+1} \\
U_{i+2,j+2}^{k+1} \\
U_{i+2,j+2}^{k+1}
\end{pmatrix}
= \begin{pmatrix}
B_1 & B_2 & B_3 & B_2 \\
B_2 & B_1 & B_3 & B_2 \\
B_3 & B_2 & B_1 & B_2 \\
B_2 & B_3 & B_2 & B_1
\end{pmatrix}
\begin{pmatrix}
\text{rhs}_{i,j} \\
\text{rhs}_{i+2,j} \\
\text{rhs}_{i+2,j+2} \\
\text{rhs}_{i,j+2}
\end{pmatrix}, \tag{16}
\]

where

\[ B_1 = \frac{14G_1^2 + 8G_1G_2 + G_2^2 + 8G_1 + 8G_2 + 1}{(2G_1 + G_2 + 1)(6G_1 + G_2 + 1)(1 + 4G_1 + G_2)} \]

\[ B_2 = \frac{G_1}{(2G_1 + G_2 + 1)(6G_1 + G_2 + 1)} \]

\[ B_3 = \frac{2G_1^2}{(2G_1 + G_2 + 1)(6G_1 + G_2 + 1)(4G_1 + G_2 + 1)} \]

In the MFEG method, the mesh points of the discretized solution domain Ω are arranged into several groups. Each group comprise four points of type ♦ (shown in Fig. 2). The MFEG Eq. (16) is applied to generate iterations on each group in Fig. 2. Prior going to the next time level, the remaining points of the mesh (□ and ○) can be obtained directly once utilizing the following sequence:

1. The skewed C-N difference formula obtained by rotating the \( x - y \) axis 45° clockwise is used for □ points. With \( K_1 = \tau/4h^2 \) and \( K_2 = (\tau\sigma)/2^{1-\sigma} \), the skewed C-N difference scheme for Eq. (1) is given by

\[ \begin{align*}
(1 + 4K_1 + K_2)U_{i,j}^{k+1} & = K_1(U_{i+1,j-1}^{k+1} - U_{i-1,j+1}^{k+1} + U_{i-1,j+1}^{k+1} + U_{i-1,j-1}^{k+1} + (1 - \tau W_1 - 4K_1 + K_2)U_{i,j}^{k+1} \\
& + \tau \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1})U_{i,j}^{m} + \tau W_k U_{i,j}^0 + \tau f_{i,j}^{k+1/2}.
\end{align*} \tag{17} \]
2. The C-N difference formula (10) is utilized for ○ points.

The MFEG method is illustrated in Algorithm 2. From the described solution procedure, it may be observed that the MFEG method involves only a quarter of the mesh points in the iterative process at each time level. Due to this reduction of points that take part in the iterative process, the MFEG method is expected to save more CPU time in solving Eq. (1).

Algorithm 2 Solution algorithm using the MFEG iterative method

1. Divide the grid points of the discretized solution domain into three types ♦, ○ and □ as depicted in Fig. 2.
2. Arrange all the ♦ points into group of four points.
3. Use Eq. (16) to iterate the solutions at the grouped points ♦ at time level \( k+1 \).
4. Test the convergence. If the iterative solutions converge, move to step 5. Otherwise, repeat the iteration process at the same time level in step 3.
5. The solutions at the residual grid points □ and ○ are computed directly once as follows:
   (a) Use Eq. (17) to compute the solutions at the points of type □.
   (b) For the residual points of type ○, Eq. (10) is employed.
6. Once the targeted time level is reached, print the numerical results.

Stability Analysis

This section is devoted to the stability of the C-N difference schemes (10) and (14), which are the basis of the FEG and MFEG methods, respectively. The following lemma is introduced for the convenience of our analysis

Lemma 1 The coefficients \( W_n \) in Eq. (14) satisfy

1. \( W_{k-m} > W_{k-m+1}, \quad m = 0, 1, 2, \ldots, k - 1. \)
2. \( \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) = W_1 - W_k. \)
Stability of the $h$-Spaced C-N Numerical Scheme

Let $\hat{U}_{i,j}^k$ be the approximate solution of (10), and define

$$\vartheta_{i,j}^k = U_{i,j}^k - \hat{U}_{i,j}^k,$$  \hspace{1cm} (18)

Then by setting (18) into (10), we get

$$(1 + 4H_1 + H_2)\vartheta_{i,j}^{k+1} - H_1(\vartheta_{i+1,j}^{k+1} + \vartheta_{i-1,j}^{k+1}) - H_1(\vartheta_{i,j+1}^{k+1} + \vartheta_{i,j-1}^{k+1})$$

$$= (1 - \tau W_1 - 4H_1 + H_2)\vartheta_{i,j}^{k} + H_1(\vartheta_{i+1,j}^{k} + \vartheta_{i-1,j}^{k}) + H_1(\vartheta_{i,j+1}^{k} + \vartheta_{i,j-1}^{k})$$

$$+ \tau \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) \vartheta_{m,j}^{k} + \tau W_k \vartheta_{i,j}^0.$$  \hspace{1cm} (19)

The Fourier series for $\vartheta^k(x, y)$ is

$$\vartheta^k(x, y) = \sum_{Z_1=-\infty}^{\infty} \sum_{Z_2=-\infty}^{\infty} \lambda^k(Z_1, Z_2)e^{2\pi i (Z_1x + Z_2y/L)},$$

where $I = \sqrt{-1}$ and the Fourier coefficients $\lambda^k(Z_1, Z_2)$ is given by

$$\lambda^k(Z_1, Z_2) = \frac{1}{L^2} \int_0^L \int_0^L \vartheta^k(x, y)e^{-2\pi i (Z_1x + Z_2y/L)} dx dy.$$  \hspace{1cm} (20)

Introducing the following norm

$$\|\vartheta^k\|_2 = \left( \sum_{Z_1=-\infty}^{\infty} \sum_{Z_2=-\infty}^{\infty} |\lambda^k(Z_1, Z_2)|^2 \right)^{1/2} = \left( \int_0^L \int_0^L |\vartheta_{i,j}^k|^2 dx dy \right)^{1/2}. $$

Applying the Parseval’s equality

$$\int_0^L \int_0^L |\vartheta_{i,j}^k|^2 dx dy = \sum_{Z_2=-\infty}^{\infty} \sum_{Z_1=-\infty}^{\infty} |\lambda^k(Z_1, Z_2)|^2,$$

we get

$$\|\vartheta^k\|_2 = \left( \sum_{Z_2=-\infty}^{\infty} \sum_{Z_1=-\infty}^{\infty} |\lambda^k(Z_1, Z_2)|^2 \right)^{1/2}. $$  \hspace{1cm} (21)

With $\gamma_1 = 2\pi Z_1/L$ and $\gamma_2 = 2\pi Z_2/L$, and based on the above analysis, we assume that the solutions of Eq. (19) is of the following form

$$\vartheta_{i,j}^k = \lambda^k e^{i(\gamma_1 \Delta x + \gamma_2 \Delta y)}.$$  \hspace{1cm} (22)

**Lemma 2** Suppose $\lambda^k$ is defined by (20). If $2 + 2H_2 - 2\tau W_1 \geq 0$, then we have

$$|\lambda^{k+1}| \leq |\lambda^0|, \hspace{0.5cm} k = 0, 1, 2, \ldots, N - 1.$$  \hspace{1cm}

**Proof** Setting $\vartheta_{i,j}^k = \lambda^k e^{i(\gamma_1 \Delta x + \gamma_2 \Delta y)}$ into Eq. (19), we obtain

$$(1 + 4H_1 + H_2)\lambda^{k+1} e^{i(\gamma_1 \Delta x + \gamma_2 \Delta y)} - H_1 \lambda^{k+1} (e^{i(\gamma_1 (i+1) \Delta x + \gamma_2 \Delta y)} - e^{i(\gamma_1 (i-1) \Delta x + \gamma_2 \Delta y)})$$

$$+ H_1 \lambda^{k+1} (e^{i(\gamma_1 i \Delta x + \gamma_2 (j+1) \Delta y)} - e^{i(\gamma_1 i \Delta x + \gamma_2 (j-1) \Delta y)}).$$
Utilizing Eq. (27), we get
\[ (1 - \tau W_1 - 4H_1 + H_2)\lambda^k e^{(\gamma_1 i \Delta x + \gamma_2 j \Delta y)} + H_1 \lambda^k (e^{(\gamma_1 (i+1) \Delta x + \gamma_2 j \Delta y)} + e^{(\gamma_1 i \Delta x + \gamma_2 j \Delta y)}) + \tau \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) \lambda^m e^{(\gamma_1 i \Delta x + \gamma_2 j \Delta y)} + \tau W_k \lambda_0 e^{(\gamma_1 i \Delta x + \gamma_2 j \Delta y)}. \] (23)

Upon simplification, we can get
\[ \lambda^{k+1} = \frac{1 - \rho_1 - \rho_2 + H_2}{1 + \rho_1 + \rho_2 + H_2} \lambda^k + \frac{\tau}{1 + \rho_1 + \rho_2 + H_2} \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) \lambda^m - W_1 \lambda^k - W_k \lambda_0 \] (24)
where
\[ \rho_1 = 4H_1 \sin^2 \left( \frac{\gamma_1 \Delta x}{2} \right), \rho_2 = 4H_1 \sin^2 \left( \frac{\gamma_2 \Delta y}{2} \right). \]

We start with \( k = 0 \) in Eq. (27). Since \( \rho_1, \rho_2 \geq 0 \),
\[ |\lambda^1| = \left| \frac{1 - \rho_1 - \rho_2 + H_2}{1 + \rho_1 + \rho_2 + H_2} |\lambda^0| \right| \leq |\lambda^0| \]
Now, assume that \( |\lambda^{s+1}| \leq |\lambda^s|, s = 0, 1, \ldots, k - 1 \). We need to prove this for \( m = k \).
Utilizing Eq. (27), we get
\[ |\lambda^{k+1}| \leq \frac{|1 - \rho_1 - \rho_2 + H_2 - \tau W_1|}{1 + \rho_1 + \rho_2 + H_2} |\lambda^k| + \frac{\tau W_1}{1 + \rho_1 + \rho_2 + H_2} |\lambda^0|. \]
Using induction hypothesis and lemma 1, we obtain
\[ |\lambda^{k+1}| \leq \frac{|1 - \rho_1 - \rho_2 + H_2 - \tau W_1|}{1 + \rho_1 + \rho_2 + H_2} |\lambda^0| + \frac{\tau W_1}{1 + \rho_1 + \rho_2 + H_2} |\lambda^0|. \]
Consider the following two cases:
Case 1. If \( 1 - \rho_1 - \rho_2 + H_2 - \tau W_1 > 0 \), then we have
\[ |\lambda^{k+1}| \leq \frac{1 - \rho_1 - \rho_2 + H_2}{1 + \rho_1 + \rho_2 + H_2} |\lambda^0| \leq |\lambda^0|. \]
Case 2. \( 1 - \rho_1 - \rho_2 + H_2 - \tau W_1 < 0 \), then we have
\[ |\lambda^{k+1}| \leq \frac{-1 + \rho_1 + \rho_2 - H_2 + 2\tau W_1}{1 + \rho_1 + \rho_2 + H_2} |\lambda^0|. \]
Here,
\[ |\lambda^{k+1}| \leq |\lambda^0| \]
\[ \Leftrightarrow -1 + \rho_1 + \rho_2 - H_2 + 2\tau W_1 \leq 1 + \rho_1 + \rho_2 + H_2 \]
\[ \Leftrightarrow 2 + 2H_2 - 2\tau W_1 \geq 0. \]
The proof is completed by mathematical induction. \( \Box \)

**Theorem 1** The C-N difference scheme (10) is stable if \( 2 + 2H_2 - 2\tau W_1 \geq 0 \).
Proof. Utilizing lemma 2 and Parseval’s equality, we obtain

$$\| \vartheta^k \|_2 = \sum_{j=1}^{M_x-1} \sum_{i=1}^{M_y-1} \Delta y \Delta x |\vartheta^k_{i,j}|^2 = \Delta y \Delta x \sum_{j=1}^{M_x-1} \sum_{i=1}^{M_y-1} |\lambda^k e^{I(\gamma_1 \Delta x + \gamma_2 \Delta y)}|^2$$

$$= \Delta y \Delta x \sum_{j=1}^{M_x-1} \sum_{i=1}^{M_y-1} |\lambda^k|^2 \Delta y \Delta x \sum_{j=1}^{M_x-1} \sum_{i=1}^{M_y-1} |\lambda^0|^2$$

$$= \Delta y \Delta x \sum_{j=1}^{M_x-1} \sum_{i=1}^{M_y-1} |\lambda^0 e^{I(\gamma_1 \Delta x + \gamma_2 \Delta y)}|^2 = \| \vartheta^0 \|_2.$$ 

Thus, the difference scheme (10) is stable. \(\square\)

Stability of the 2h-Spaced C-N Numerical Scheme

Let \(\bar{U}_{i,j}^k\) be the approximate solution of (14), and define

$$\zeta_{i,j}^k = U_{i,j}^k - \bar{U}_{i,j}^k.$$ (25)

Substituting (25) into (14), we obtain

$$(1 + 4G_1 + G_2)\zeta_{i,j}^{k+1} - G_1(\zeta_{i,j+1}^{k+1} + \zeta_{i,j-1}^{k+1}) - G_1(\zeta_{i,j+2}^{k+1} + \zeta_{i,j-2}^{k+1})$$

$$= (1 - \tau W_1 - 4G_1 + G_2)\zeta_{i,j}^{k} + G_1(\zeta_{i,j+2}^{k} + \zeta_{i,j-2}^{k}) + G_1(\zeta_{i,j+1}^{k} + \zeta_{i,j-1}^{k})$$

$$+ \tau \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) \zeta_{i,j}^{m} + \tau W_{k} \zeta_{i,j}^{0}.$$ (26)

The Fourier series for \(\zeta^k(x, y)\) is

$$\zeta^k(x, y) = \sum_{Z_1=-\infty}^{\infty} \sum_{Z_2=-\infty}^{\infty} \psi^k(Z_1, Z_2) e^{2\pi I(Z_1x/L + Z_2y/L)},$$

where \(I = \sqrt{-1}\) and the Fourier coefficients \(\psi^k(Z_1, Z_2)\) are given by

$$\psi^k(Z_1, Z_2) = \frac{1}{L^2} \int_0^L \int_0^L \zeta^k(x, y) e^{-2\pi I(Z_1x/L + Z_2y/L)} dx dy.$$ (27)

Introducing the following norm

$$\| \zeta^k \|_2 = \left( \sum_{j=1}^{M_y-1} \sum_{i=1}^{M_x-1} \Delta y \Delta x |\zeta_{i,j}^k|^2 \right)^{1/2} = \left( \int_0^L \int_0^L |\zeta_{i,j}^k|^2 dx dy \right)^{1/2}.$$ 

Applying the Parseval’s equality

$$\int_0^L \int_0^L |\zeta_{i,j}^k|^2 dx dy = \sum_{Z_2=-\infty}^{\infty} \sum_{Z_1=-\infty}^{\infty} |\psi^k(Z_1, Z_2)|^2,$$

we get

$$\| \zeta^k \|_2 = \left( \sum_{Z_2=-\infty}^{\infty} \sum_{Z_1=-\infty}^{\infty} |\psi^k(Z_1, Z_2)|^2 \right)^{1/2}.$$ (28)
Based on the above analysis, we assume that the solution of Eq. (26) is of the following form
\[ \zeta_{i,j}^k = \psi^k e^{i(\gamma_1 \Delta x + \gamma_2 j \Delta y)}. \] (29)

**Lemma 3** Suppose \( \psi^k \) is defined by (27). If \( 2 + 2G_2 - 2\tau W_1 \geq 0 \), then we have
\[ |\psi^{k+1}| \leq |\psi^0|, \quad k = 0, 1, 2, \ldots, N - 1. \]

**Proof** Substituting \( \zeta_{i,j}^k = \psi^k e^{i(\gamma_1 \Delta x + \gamma_2 j \Delta y)} \) into Eq. (26), we get
\[
(1 + 4G_1 + G_2)\psi^{k+1}e^{i(\gamma_1 \Delta x + \gamma_2 j \Delta y)} = G_1\psi^{k+1}(e^{i(\gamma_1 (i+2) \Delta x + \gamma_2 j \Delta y)}
+ e^{i(\gamma_1 (i-2) \Delta x + \gamma_2 j \Delta y)}) - G_1\psi^{k+1}(e^{i(\gamma_1 (i+2) \Delta x + \gamma_2 (j+2) \Delta y)}
+ e^{i(\gamma_1 (i-2) \Delta x + \gamma_2 (j+2) \Delta y)})
+ G_1\psi^k(e^{i(\gamma_1 \Delta x + \gamma_2 j \Delta y)} + e^{i(\gamma_1 \Delta x + \gamma_2 (j+2) \Delta y)})
+ \tau \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) \psi^m - W_1 \psi^k - W_k \psi^0.
\] (30)

After simplification, we can get
\[
\psi^{k+1} = \frac{1 - \kappa_1 - \kappa_2 + G_2}{1 + \kappa_1 + \kappa_2 + G_2} \psi^k
+ \frac{\tau}{1 + \kappa_1 + \kappa_2 + G_2} \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) \psi^m - W_1 \psi^k - W_k \psi^0.
\] (31)

where
\[ \kappa_1 = 4G_1 \sin^2(\gamma_1 \Delta x), \quad \kappa_2 = 4G_1 \sin^2(\gamma_2 \Delta y). \]

Letting \( k = 0 \) in Eq. (31) and using that \( \kappa_1, \kappa_2 \geq 0 \), we obtain
\[ |\psi^1| = \left| \frac{1 - \kappa_1 - \kappa_2 + G_2}{1 + \kappa_1 + \kappa_2 + G_2} \right| |\psi^0| \leq |\psi^0| \]

Now, suppose that \( |\psi^{s+1}| \leq |\psi^0|, \quad s = 0, 1, \ldots, k - 1 \). We must prove it holds for \( m = k \).
Using Eq. (31), we get
\[
|\psi^{k+1}| \leq \left| \frac{1 - \kappa_1 - \kappa_2 + G_2 - \tau W_1}{1 + \kappa_1 + \kappa_2 + G_2} \right| |\psi^k|
+ \left| \frac{\tau}{1 + \kappa_1 + \kappa_2 + G_2} \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) |\psi^k| + |W_k| |\psi^0| \right|.
\]

By the induction hypothesis and lemma 1, we get
\[ |\psi^{k+1}| \leq \frac{|1 - \kappa_1 - \kappa_2 + G_2 - \tau W_1| + \tau W_1}{1 + \kappa_1 + \kappa_2 + G_2} |\psi^0|. \]

Consider the following two cases:

Case 1. If \( 1 - \kappa_1 - \kappa_2 + G_2 - \tau W_1 > 0 \), then we have
\[ |\psi^{k+1}| \leq \frac{1 - \kappa_1 - \kappa_2 + G_2}{1 + \kappa_1 + \kappa_2 + G_2} |\psi^0| \leq |\psi^0|. \]
Case 2. 1 − \kappa_1 − \kappa_2 + G_2 − \tau W_1 < 0, then we have

\[ |\psi^{k+1}| \leq \frac{-1 + \kappa_1 + \kappa_2 - G_2 + 2\tau W_1}{1 + \kappa_1 + \kappa_2 + G_2} |\psi^0|. \]

Here,

\[ |\psi^{k+1}| \leq |\psi^0| \quad \Leftrightarrow -1 + \kappa_1 + \kappa_2 - G_2 + 2\tau W_1 \leq 1 + \kappa_1 + \kappa_2 + G_2 \]

\[ \Leftrightarrow 2 + 2G_2 - 2\tau W_1 \geq 0. \]

This completes the proof. \(\square\)

**Theorem 2** The C-N difference scheme (14) is stable if \(2 + 2G_2 - 2\tau W_1 \geq 0.\)

**Proof** Utilizing lemma 3 and Parseval’s equality, we obtain

\[
\|\xi^k\|_2 = \sum_{j=1}^{M_y-1} \sum_{i=1}^{M_x-1} \Delta y \Delta x |\xi_{i,j}^k|^2 = \Delta y \Delta x \sum_{j=1}^{M_y-1} \sum_{i=1}^{M_x-1} |\psi^k e^{I(\gamma_1 \Delta x + \gamma_2 \Delta y)}|^2
\]

\[
= \Delta y \Delta x \sum_{j=1}^{M_y-1} \sum_{i=1}^{M_x-1} |\psi|^2 \leq \Delta y \Delta x \sum_{j=1}^{M_y-1} \sum_{i=1}^{M_x-1} |\psi^0|^2
\]

\[
= \Delta y \Delta x \sum_{j=1}^{M_y-1} \sum_{i=1}^{M_x-1} |\psi^0 e^{I(\gamma_1 \Delta x + \gamma_2 \Delta y)}|^2 = \|\xi^0\|_2,
\]

which means that the difference scheme (14) is stable. \(\square\)

**Convergence Analysis**

In this section, the convergence of the proposed methods is investigated. By subtracting Eq. (10) from Eq. (9) and defining \(\varepsilon_{i,j}^k = u(x_i, y_j, t_k) - U_{i,j}^k\), the following error equation is obtained

\[
(1 + 4H_1 + H_2)\varepsilon_{i,j}^{k+1} - H_1(\varepsilon_{i+1,j}^{k+1} + \varepsilon_{i-1,j}^{k+1}) - H_1(\varepsilon_{i+1,j+1}^{k+1} + \varepsilon_{i,j-1}^{k+1})
\]

\[
= (1 - \tau W_1 - 4H_1 + H_2)\varepsilon_{i,j}^k + H_1(\varepsilon_{i+1,j}^k + \varepsilon_{i,j+1}^k) + H_1(\varepsilon_{i,j+1}^k + \varepsilon_{i-1,j}^k)
\]

\[+ \tau \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) \varepsilon_{i,j}^m + \tau W_k \varepsilon_{i,j}^0 + \tau R_{i,j}^{k+1/2}.\] (32)

From Eq. (9), there is a positive constant \(C_1\) such that

\[
|R_{i,j}^{k+1/2}| \leq C_1(\tau^{2-\alpha} + h^2). \quad (33)
\]

Similar to the stability analysis, \(\varepsilon^k(x, y)\) and \(R^{k+1/2}(x, y)\) can be expanded in Fourier series as

\[
\varepsilon^k(x, y) = \sum_{Z_2=-\infty}^{\infty} \sum_{Z_1=-\infty}^{\infty} \eta^k(Z_1, Z_2) e^{2\pi i (Z_1 x/L + Z_2 y/L)},
\]
R^{k+1/2}(x, y) = \sum_{Z_2=-\infty}^{\infty} \sum_{Z_1=-\infty}^{\infty} \varphi^k(Z_1, Z_2)e^{2\pi i (Z_1 x/L + Z_2 y/L)},

where the Fourier coefficients \( \eta^k \) and \( \varphi^k \) are

\begin{align*}
\eta^k(Z_1, Z_2) &= \frac{1}{L^2} \int_0^L \int_0^L \varepsilon^k(x, y)e^{-2\pi i (Z_1 x/L + Z_2 y/L)}dxdy, \\
\varphi^k(Z_1, Z_2) &= \frac{1}{L^2} \int_0^L \int_0^L R^{k+1/2}(x, y)e^{-2\pi i (Z_1 x/L + Z_2 y/L)}dxdy.
\end{align*}

By making use of the Parseval’s equality and \( L^2 \) norm, we obtain

\begin{align*}
\|\varepsilon^k\|_2 &= \left(\sum_{j=1}^{M_1-1} \sum_{i=1}^{M_2-1} \Delta y \Delta x |\varepsilon_{i,j}^k|^2 \right)^{1/2} \\
&= \left(\sum_{Z_1=-\infty}^{\infty} \sum_{Z_2=-\infty}^{\infty} |\eta^k(Z_1, Z_2)|^2 \right)^{1/2}, \\
\|R^{k+1/2}\|_2 &= \left(\sum_{j=1}^{M_1-1} \sum_{i=1}^{M_2-1} \Delta y \Delta x |R_{i,j}^{k+1/2}|^2 \right)^{1/2} \\
&= \left(\sum_{Z_1=-\infty}^{\infty} \sum_{Z_2=-\infty}^{\infty} |\varphi^k(Z_1, Z_2)|^2 \right)^{1/2}.
\end{align*}

Next, we suppose the solutions of Eq. (32) are as follows

\( \varepsilon_{i,j}^k = \eta^k e^{i(\gamma_1 \Delta x + \gamma_2 \Delta y)}, \quad R_{i,j}^{k+1/2} = \varphi^k e^{i(\gamma_1 \Delta x + \gamma_2 \Delta y)}. \) (36)

Using the assumptions in (36) and considering (32), we obtain

\begin{align*}
(1 + 4H_1 + H_2)\eta^{k+1} e^{i(\gamma_1 \Delta x + \gamma_2 \Delta y)} - H_1 \eta^{k+1} e^{i(\gamma_1 (i+1) \Delta x + \gamma_2 \Delta y)} \\
+ e^{i(\gamma_1 (i-1) \Delta x + \gamma_2 \Delta y)} - H_1 \eta^{k+1} e^{i(\gamma_1 (i+1) \Delta x + \gamma_2 \Delta y)} + e^{i(\gamma_1 i \Delta x + \gamma_2 \Delta y)} \\
= (1 - \tau W_1 - 4H_1 + H_2)\eta^k e^{i(\gamma_1 \Delta x + \gamma_2 \Delta y)} + H_1 \eta^k e^{i(\gamma_1 (i+1) \Delta x + \gamma_2 \Delta y)} + e^{i(\gamma_1 \Delta x + \gamma_2 \Delta y)} \\
+ \tau \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) \eta^m e^{i(\gamma_1 \Delta x + \gamma_2 \Delta y)} + \tau W_k \eta^0 e^{i(\gamma_1 \Delta x + \gamma_2 \Delta y)}
\end{align*}

After simplification, we have

\begin{align*}
\eta^{k+1} &= \frac{1 - \rho_1 - \rho_2 + H_2}{1 + \rho_1 + \rho_2 + H_2} \eta^k \\
+ \frac{\tau}{1 + \rho_1 + \rho_2 + H_2} \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) \eta^m - W_1 \eta^k + \varphi^{k+1/2},
\end{align*}

where \( \rho_1 \) and \( \rho_2 \) are as defined in the previous section.
Due to the convergence of the series in the right hand side of Eq. (35), there is a positive constant $C_2$ such that
\[ |\varphi^{k+1/2}| = |\varphi^{k+1/2}(Z_1, Z_2)| \leq C_2|\varphi^{1/2}(Z_1, Z_2)| = C_2|\varphi^{1/2}|. \tag{39} \]

**Lemma 4** Suppose $\eta^{k+1}$ and $\varphi^{k+1/2}$ satisfy Eq. (38). If $2 + 2H_2 - \tau(2W_1 - W_k) \geq 0$, then we have
\[ |\eta^{k+1}| \leq C_2(k + 1)\tau|\varphi^{1/2}|, \quad k = 0, 1, \ldots, N - 1. \]

**Proof** Noticing that $\eta^0 = \eta^0(Z_1, Z_2) = 0$. Then, for $k = 0$, we have
\[ |\eta^1| = \left| \frac{1}{1 + \rho_1 + \rho_2 + H_2} \right| |\tau\varphi^{1/2}| \leq \tau|\varphi^{1/2}| \leq C_2 \tau|\varphi^{1/2}|. \]
Now, assume that $|\eta^{s+1}| \leq C_2(s + 1)\tau|\varphi^{1/2}|$, $s = 0, 1, \ldots, k - 1$. We need to prove this for $s = k$. From Eq. (38), we have
\[ |\eta^{k+1}| \leq \left| \frac{1 - \rho_1 - \rho_2 + H_2 - \tau W_1}{1 + \rho_1 + \rho_2 + H_2} \right| |\eta^k| + \left| \frac{\tau}{1 + \rho_1 + \rho_2 + H_2} \right| \left| \sum_{m=1}^{k-1} (W_{k-m} - W_{k-m+1}) |\eta^k| + |\varphi^{k+1/2}| \right|. \]
Utilizing induction hypothesis, lemma 1 and Eq. (39), we get
\[ |\eta^{k+1}| \leq \left[ \frac{1 - \rho_1 - \rho_2 + H_2 - \tau W_1}{1 + \rho_1 + \rho_2 + H_2} \right] |\eta^k| + \frac{1}{1 + \rho_1 + \rho_2 + H_2} C_2 \tau|\varphi^{1/2}|. \]
If $1 - \rho_1 - \rho_2 + H_2 - \tau W_1 > 0$, then we have
\[ |\eta^{k+1}| \leq \left[ \frac{1 - \rho_1 - \rho_2 + H_2 - \tau W_k}{1 + \rho_1 + \rho_2 + H_2} \right] |\eta^k| + \frac{1}{1 + \rho_1 + \rho_2 + H_2} C_2 \tau|\varphi^{1/2}| \leq C_2(k + 1)\tau|\varphi^{1/2}|. \]
If $1 - \rho_1 - \rho_2 + H_2 - \tau W_1 < 0$, then we have
\[ |\eta^{k+1}| \leq \left[ \frac{-1 + \rho_1 + \rho_2 - H_2 + \tau(2W_1 - W_k)}{1 + \rho_1 + \rho_2 + H_2} \right] |\eta^k| + \frac{1}{1 + \rho_1 + \rho_2 + H_2} C_2 \tau|\varphi^{1/2}|. \]
Here,
\[ |\eta^{k+1}| \leq C_2(k + 1)\tau|\varphi^{1/2}| \Leftrightarrow -1 + \rho_1 + \rho_2 - H_2 + \tau(2W_1 - W_k) \leq 1 + \rho_1 + \rho_2 + H_2 \Leftrightarrow 2 + 2H_2 - \tau(2W_1 - W_k) \geq 0. \]
This completes the proof. \qed

**Theorem 3** The C-N difference scheme (10) is convergent and the order of convergence is $O(\tau^{2-\alpha} + h^2)$. 
Proof Using lemma 4 and Parseval’s equality, we obtain

\[ \|\varepsilon^k\|_2^2 = \sum_{j=1}^{M_y-1} \sum_{i=1}^{M_x-1} |\Delta y \Delta x |\varepsilon_{i,j}^k |^2 = \Delta y \Delta x \sum_{j=1}^{M_y-1} \sum_{i=1}^{M_x-1} \left| \eta_j^k e^{i(\gamma_1 \Delta x + \gamma_2 j \Delta y)} \right|^2 \]

\[ = \Delta y \Delta x \sum_{j=1}^{M_y-1} \sum_{i=1}^{M_x-1} |\eta_j^k|^2 \leq C_2^2 (k+1)^2 \tau^2 \Delta y \Delta x \sum_{j=1}^{M_y-1} \sum_{i=1}^{M_x-1} |\varphi^{1/2}|^2 \]

\[ = C_2^2 (k+1)^2 \tau^2 \Delta y \Delta x \sum_{j=1}^{M_y-1} \sum_{i=1}^{M_x-1} \left| \varphi^{1/2} e^{i(\gamma_1 \Delta x + \gamma_2 j \Delta y)} \right|^2 = C_2^2 (k+1)^2 \tau^2 \|R^{1/2}\|_2^2. \]

Noticing that \((k+1)\tau \leq T\) and letting \(C = C_1 C_2 T\). From Eq. (33), we have

\[ \|\varepsilon^k\|_2 \leq C (\tau^{2-\alpha} + h^2), \]

which completes the proof. \(\Box\)

**Theorem 4** The C-N difference scheme (14) is convergent and the order of convergence is \(O(\tau^{2-\alpha} + h^2)\).

**Proof** The proof is similar to Theorem 3. \(\Box\)

**Numerical Experiments**

In this part, three test problems with known exact solutions are simulated on Windows 10 (64 bit) Intel (R) Core (TM) i7-8550U CPU 2.00 GHz, 8GB of RAM utilizing MATLAB software. The numerical results are obtained at three different final times, namely, \(T = 4, T = 8\) and \(T = 12\) with the spatial domain being restricted to \(\Omega = (0,1)^2\). The proposed methods are combined with the Gauss-Seidel iterative scheme with error tolerance of \(10^{-5}\) to solve the test problems of the form in Eq. (1). Throughout the discussion, we let \(u, U_{C-N}, U_{FEG}\) and \(U_{MFEG}\) denote the exact, C-N, FEG and MFEG solutions, respectively. In addition, the computational orders of the presented methods are calculated using the formula [47]

\[ \text{C-Order} = \frac{\log(E_{\infty}(\tau, h_1)/E_{\infty}(\tau, h_2))}{\log(h_1/h_2)} \]

**Test Problem 6.1** In the first problem, consider the source term as follows

\[ f(x, y, t) = \left(2t + \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + 2t^2 \pi^2 \right) \cos(\pi x) \cos(\pi y), \]

where the exact solution is given by

\[ u(x, y, t) = t^2 \cos(\pi x) \cos(\pi y). \]

Tables 1, 2 and 3 show the numerical results of the elapsed time (in seconds), number of iterations (Ite) and maximum absolute error (\(E_{\infty}(\tau, h)\)) for the C-N, FEG and MFEG schemes with \(T = 4, N = 100, \alpha = 0.1, 0.3, 0.7\) for different sizes of space steps, respectively. Figure 3 presents a comparison of the exact solution and other numerical solutions obtained at \(\alpha = 0.3, N = 100, h^{-1} = 42\) and \(y = 0.119\). In addition, the graphical error representation of the C-N scheme for \(T = 4, \alpha = 0.3, N = 100\) and \(h^{-1} = 42\) is depicted in Fig. 4 while
Table 1  Comparison between C-N, FEG and MFEG methods for Test Problem 6.1 with $\alpha = 0.1$, $T = 4$ and $N = 100$

| $h^{-1}$ | C-N | FEG | MFEG |
|---------|-----|-----|------|
|         | Time | Ite | $E_{\infty}(\tau, h)$ | Time | Ite | $E_{\infty}(\tau, h)$ | Time | Ite | $E_{\infty}(\tau, h)$ |
| 10      | 0.4175 | 23  | 2.2757E-02  | 0.3356 | 15  | 2.2754E-02  | 0.0945 | 6   | 8.6466E-02  |
| 18      | 2.3110 | 52  | 6.9678E-03  | 1.0515 | 31  | 6.9591E-03  | 0.2451 | 12  | 2.7489E-02  |
| 26      | 5.8888 | 92  | 3.3755E-03  | 4.1425 | 53  | 3.3677E-03  | 0.4943 | 19  | 1.3548E-02  |
| 34      | 18.2440 | 140 | 1.9430E-03  | 10.1393 | 80  | 1.9427E-03  | 1.1403 | 28  | 7.9289E-03  |
| 42      | 39.2082 | 196 | 1.2113E-03  | 20.9560 | 111 | 1.2394E-03  | 2.6987 | 37  | 5.1846E-03  |

Table 2  Comparison between C-N, FEG and MFEG methods for Test Problem 6.1 with $\alpha = 0.3$, $T = 4$ and $N = 100$

| $h^{-1}$ | C-N | FEG | MFEG |
|---------|-----|-----|------|
|         | Time | Ite | $E_{\infty}(\tau, h)$ | Time | Ite | $E_{\infty}(\tau, h)$ | Time | Ite | $E_{\infty}(\tau, h)$ |
| 10      | 0.3476 | 22  | 2.2785E-02  | 0.2483 | 15  | 2.2778E-02  | 0.0920 | 6   | 8.6576E-02  |
| 18      | 1.6289 | 51  | 6.9747E-03  | 1.0755 | 31  | 6.9638E-03  | 0.2450 | 12  | 2.7522E-02  |
| 26      | 6.5401 | 90  | 3.3776E-03  | 3.8171 | 52  | 3.3693E-03  | 0.6294 | 19  | 1.3561E-02  |
| 34      | 18.1564 | 138 | 1.9419E-03  | 9.9377 | 79  | 1.9423E-03  | 1.2881 | 27  | 7.9373E-03  |
| 42      | 39.9924 | 194 | 1.2115E-03  | 21.2502 | 110 | 1.2385E-03  | 2.1786 | 37  | 5.1877E-03  |

Table 3  Comparison between C-N, FEG and MFEG methods for Test Problem 6.1 with $\alpha = 0.7$, $T = 4$ and $N = 100$

| $h^{-1}$ | C-N | FEG | MFEG |
|---------|-----|-----|------|
|         | Time | Ite | $E_{\infty}(\tau, h)$ | Time | Ite | $E_{\infty}(\tau, h)$ | Time | Ite | $E_{\infty}(\tau, h)$ |
| 10      | 0.4374 | 20  | 2.2831E-02  | 0.3357 | 18  | 2.2823E-02  | 0.1252 | 6   | 8.6801E-02  |
| 18      | 1.5666 | 47  | 6.9786E-03  | 0.924  | 28  | 6.9697E-03  | 0.2853 | 11  | 2.7583E-02  |
| 26      | 4.4705 | 83  | 3.3703E-03  | 2.9016 | 48  | 3.3642E-03  | 0.5462 | 17  | 1.3582E-02  |
| 34      | 11.6037 | 126 | 1.9280E-03  | 6.6387 | 72  | 1.9324E-03  | 1.1091 | 25  | 7.9436E-03  |
| 42      | 28.1122 | 177 | 1.1913E-03  | 13.7709 | 101 | 1.2262E-03  | 1.7506 | 34  | 5.1873E-03  |

its second order convergence can be observed from the results in Table 4. From the data in these tables and figures, the convergence of the presented methods is confirmed. Comparing the resolution of the tested methods, it may be observed that the C-N and FEG schemes have almost the same degree of accuracy while the magnitude of absolute errors for the MFEG method is slightly larger than the said schemes. This is because the iterative process for the C-N and FEG schemes is implemented on a mesh size of $h$ while the iterative process of the MFEG method is carried out on a mesh size of $2h$; hence, increasing the error term by 4. Figure 5 compares the computing times and iteration numbers of the tested methods and shows that the FEG and MFEG methods are more efficient than the C-N difference scheme in terms of execution time and number of iterations.
Fig. 3 Comparison of exact and numerical solutions for Test Problem 6.1 with $\alpha = 0.3$, $T = 4$, $h^{-1} = 42$, $N = 100$ and $y = 0.119$

Fig. 4 Surface error plot of the C-N method for Test Problem 6.1 with $\alpha = 0.3$, $T = 4$, $h^{-1} = 42$ and $N = 100$
Table 4 Maximum errors and computational orders of the C-N method for Test problem 6.1 with $T=1$ and $\tau=0.001$

| $\alpha$ | $h^{-1}$ | $E_\infty(\tau, h)$ | C-Order | $h^{-1}$ | $E_\infty(\tau, h)$ | C-Order | $h^{-1}$ | $E_\infty(\tau, h)$ | C-Order |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 6 | 3.6027E-03 | $-$ | 6 | 3.5845E-03 | $-$ | 6 | 3.5656E-03 | $-$ |
| | 10 | 1.3960E-03 | 1.8560 | 10 | 1.3888E-03 | 1.8562 | 10 | 1.3818E-03 | 1.8557 |
| | 14 | 7.1168E-04 | 2.0024 | 14 | 7.0804E-04 | 2.0022 | 14 | 7.0537E-04 | 1.9984 |
| | 18 | 4.2523E-04 | 2.0492 | 18 | 4.2318E-04 | 2.0480 | 18 | 4.1537E-04 | 2.1071 |

![Fig. 5](image)

(a) Elapsed time  (b) Number of iterations

Fig. 5 The graphs of (a) elapsed time and (b) number of iterations of the proposed methods for Test Problem 6.1 with $\alpha=0.3$

Test Problem 6.2 In the second problem, we consider the source term given by

$$f(x, y, t) = ((1 + \alpha)t^\alpha + \Gamma(2 + \alpha)t - 4t^{1+\alpha}(x-0.5)^2 - 4t^{1+\alpha}(y-0.5)^2 + 4t^{1+\alpha})e^{-(x-0.5)^2-(y-0.5)^2},$$

and the corresponding exact solution

$$u(x, y, t) = e^{-(x-0.5)^2-(y-0.5)^2}t^{1+\alpha}.$$

The initial and boundary conditions are extracted from the exact solution. The exact, C-N, FEG and MFEG solutions at $\alpha = 0.5$, $N = 100$, $h^{-1} = 42$ and $y = 0.5$ are portrayed in Fig. 6, while Fig. 7 demonstrates the surface of absolute errors of the FEG method for $T = 8$ and $\alpha = 0.5$, $N = 100$ and $h^{-1} = 42$. From the figures, it can be observed that the numerical solutions match well with the exact solution. Tables 5, 6 and 7 display the numerical results of the proposed methods with $T = 8$, $N = 100$ and $\alpha = 0.3$, 0.5, 0.7 for various space steps, respectively. Furthermore, Fig. 8 illustrates the computational times and number of iterations pertained to the three tested methods. The elapsed time and iterations number in Tables 5, 6 and 7 and Fig. 8 indicate that the computational cost of the fractional group methods (FEG and MFEG) is lower than the C-N scheme, which show the effectiveness of the former. Table 8 show the numerical errors and computational orders computed using the FEG method. Just as we expect, the second order convergence can be seen from the data recorded in this table.
Fig. 6 Comparison of exact and numerical solutions for Test Problem 6.2 with $\alpha = 0.5$, $T = 8$, $h^{-1} = 42$, $N = 100$ and $y = 0.5$.

Fig. 7 Surface error plot of the FEG method for Test Problem 6.2 with $\alpha = 0.5$, $T = 8$, $h^{-1} = 42$ and $N = 100$. 
Table 5  Comparison between C-N, FEG and MFEG methods for Test Problem 6.2 with $\alpha = 0.3$, $T = 8$ and $N = 100$

| $h^{-1}$ | C-N | FEG | MFEG |
|----------|-----|-----|------|
|          | Time | Itte $E_{\infty}(\tau, h)$ | Time | Itte $E_{\infty}(\tau, h)$ | Time | Itte $E_{\infty}(\tau, h)$ |
| 10       | 0.8609 | 40 | 1.6822E-02 | 0.3782 | 24 | 1.6841E-02 | 0.7086 | 9 | 6.4272E-02 |
| 18       | 3.7968 | 107 | 5.0513E-03 | 2.6769 | 60 | 5.1250E-03 | 0.3748 | 19 | 2.0512E-02 |
| 26       | 14.3701 | 199 | 8.1594E-03 | 8.5991 | 111 | 2.3224E-03 | 1.1356 | 35 | 9.8791E-03 |
| 34       | 42.6815 | 312 | 8.8053E-04 | 22.9559 | 175 | 1.1698E-03 | 2.0979 | 54 | 5.7423E-03 |
| 42       | 57.6235 | 444 | 1.2502E-04 | 43.8745 | 249 | 5.0909E-04 | 4.3400 | 77 | 3.7024E-03 |

Table 6  Comparison between C-N, FEG and MFEG methods for Test Problem 6.2 with $\alpha = 0.5$, $T = 8$ and $N = 100$

| $h^{-1}$ | C-N | FEG | MFEG |
|----------|-----|-----|------|
|          | Time | Itte $E_{\infty}(\tau, h)$ | Time | Itte $E_{\infty}(\tau, h)$ | Time | Itte $E_{\infty}(\tau, h)$ |
| 10       | 0.6385 | 40 | 2.5633E-02 | 0.3369 | 24 | 2.5655E-02 | 0.1324 | 9 | 9.8056E-02 |
| 18       | 3.9289 | 109 | 7.7312E-03 | 2.6144 | 61 | 7.8090E-03 | 0.3197 | 20 | 3.1264E-02 |
| 26       | 15.1236 | 205 | 3.4143E-03 | 8.4977 | 114 | 3.5836E-03 | 1.1475 | 35 | 1.5038E-02 |
| 34       | 43.1109 | 324 | 1.5899E-03 | 22.6807 | 180 | 1.8797E-03 | 2.6300 | 55 | 8.7478E-03 |
| 42       | 57.5005 | 463 | 5.2261E-04 | 45.1226 | 258 | 9.5837E-04 | 4.0916 | 79 | 5.6592E-03 |

Table 7  Comparison between C-N, FEG and MFEG methods for Test Problem 6.2 with $\alpha = 0.7$, $T = 8$ and $N = 100$

| $h^{-1}$ | C-N | FEG | MFEG |
|----------|-----|-----|------|
|          | Time | Itte $E_{\infty}(\tau, h)$ | Time | Itte $E_{\infty}(\tau, h)$ | Time | Itte $E_{\infty}(\tau, h)$ |
| 10       | 0.7414 | 39 | 3.8935E-02 | 0.4136 | 24 | 3.8961E-02 | 0.1449 | 9 | 1.4920E-01 |
| 18       | 2.8969 | 107 | 1.1739E-02 | 1.9721 | 60 | 1.1820E-02 | 0.3785 | 19 | 4.7492E-02 |
| 26       | 10.8329 | 202 | 5.2526E-03 | 6.222 | 112 | 5.4286E-03 | 0.8366 | 35 | 2.2808E-02 |
| 34       | 35.868 | 322 | 2.6066E-03 | 16.4327 | 177 | 2.8908E-03 | 1.6588 | 54 | 1.3244E-02 |
| 42       | 69.2403 | 462 | 1.1268E-03 | 39.3033 | 255 | 1.5682E-03 | 3.3176 | 77 | 8.5538E-03 |

Fig. 8  The graphs of (a) elapsed time and (b) number of iterations of the proposed methods for Test Problem 6.2 with $\alpha = 0.5$
Table 8  Maximum errors and computational orders of the FEG method for Test problem 6.2 with $T=1$ and $\tau=0.001$.

| $\alpha$ | $h^{-1}$ | $E_{\infty}(\tau, h)$ | C-Order | $h^{-1}$ | $E_{\infty}(\tau, h)$ | C-Order | $h^{-1}$ | $E_{\infty}(\tau, h)$ | C-Order |
|----------|----------|------------------------|---------|----------|------------------------|---------|----------|------------------------|---------|
| 0.1      | 6        | 2.9156E-03             | −       | 6        | 2.8109E-03             | −       | 6        | 2.6749E-03             | −       |
|          | 10       | 1.0516E-03             | 1.9963  | 10       | 1.0135E-03             | 1.997   | 10       | 9.6529E-04             | 1.9953  |
|          | 14       | 5.3744E-04             | 1.995   | 14       | 5.1799E-04             | 1.9948  | 14       | 4.9175E-04             | 2.0045  |
|          | 18       | 3.1750E-04             | 2.0943  | 18       | 3.0390E-04             | 2.1219  | 18       | 2.9426E-04             | 2.0433  |

Fig. 9  Comparison of exact and numerical solutions for Test Problem 6.3 with $\alpha=0.7$, $T=12$, $h^{-1}=42$, $N=100$ and $\gamma=0.881$.

Test Problem 6.3  In this problem, the following source term is considered

$$f(x, y, t) = ((1 + \alpha)t^\alpha + \Gamma(2 + \alpha)t - 2t^{1+\alpha})e^{x+y},$$

along with the exact solution

$$u(x, y, t) = t^{1+\alpha}e^{x+y}.$$ 

Again, the initial and boundary conditions can be drawn from the exact solution. In Fig. 9, we sketch the exact solution together with the numerical solutions obtained at $\alpha=0.7$, $N=100$, $h^{-1}=42$ and $\gamma=0.881$, while in Fig. 10 we draw the surface figure of maximum absolute errors of the MFEG method for $T=12$, $\alpha=0.7$, $N=100$ and $h^{-1}=42$. The figures illustrate the accuracy of the proposed methods. In Tables 9, 10 and 11, experimental results of the C-N, FEG and MFEG methods with $T=12$, $N=100$ and $\alpha=0.3, 0.5, 0.7$ for various mesh sizes are recorded, respectively. Both of the elapsed time and the number of iterations versus different mesh sizes are depicted in Fig. 11. Similar to the previous tests, it
Fig. 10 Surface error plot of the FEG method for Test Problem 6.3 with $\alpha = 0.7$, $T = 12$, $h^{-1} = 42$ and $N = 100$

Table 9 Comparison between C-N, FEG and MFEG methods for Test Problem 6.3 with $\alpha = 0.3$, $T = 12$ and $N = 100$

| $h^{-1}$ | C-N | FEG | MFEG |
|---------|-----|-----|------|
|        | Time | Ite | $E_\infty(\tau, h)$ | Time | Ite | $E_\infty(\tau, h)$ | Time | Ite | $E_\infty(\tau, h)$ |
| 10      | 0.9338 | 56  | 8.6771E-03 | 0.4457 | 33  | 8.6925E-03 | 0.1402 | 11  | 3.3764E-02 |
| 18      | 3.9919 | 155 | 2.5894E-03 | 2.5803 | 86  | 2.6599E-03 | 0.3867 | 26  | 1.0715E-02 |
| 26      | 16.5821 | 295 | 1.0450E-03 | 9.1018 | 162 | 1.1852E-03 | 1.1168 | 49  | 5.1784E-03 |
| 34      | 32.0802 | 471 | 3.6668E-04 | 25.2729 | 258 | 5.5033E-04 | 2.4359 | 77  | 3.0052E-03 |
| 42      | 64.4505 | 678 | 3.8970E-04 | 55.5592 | 373 | 2.1681E-04 | 4.7049 | 111 | 1.9285E-03 |

Table 10 Comparison between C-N, FEG and MFEG methods for Test Problem 6.3 with $\alpha = 0.5$, $T = 12$ and $N = 100$

| $h^{-1}$ | C-N | FEG | MFEG |
|---------|-----|-----|------|
|        | Time | Ite | $E_\infty(\tau, h)$ | Time | Ite | $E_\infty(\tau, h)$ | Time | Ite | $E_\infty(\tau, h)$ |
| 10      | 0.7929 | 57  | 1.4381E-02 | 0.4910 | 33  | 1.4393E-02 | 0.1278 | 11  | 5.5926E-02 |
| 18      | 4.2983 | 159 | 4.3691E-03 | 2.5652 | 88  | 4.4432E-03 | 0.3977 | 27  | 1.7753E-02 |
| 26      | 16.7838 | 304 | 1.8992E-03 | 9.7074 | 166 | 2.0478E-03 | 1.1253 | 49  | 8.5798E-03 |
| 34      | 34.1840 | 487 | 8.2457E-04 | 25.7286 | 265 | 1.0484E-03 | 2.4011 | 78  | 4.9996E-03 |
| 42      | 78.4912 | 704 | 2.7888E-04 | 56.3769 | 384 | 5.0444E-04 | 4.4506 | 113 | 3.2386E-03 |
Table 11 Comparison between C-N, FEG and MFEG methods for Test Problem 6.3 with $\alpha = 0.7$, $T = 12$ and $N = 100$

| $h^{-1}$ | C-N | FEG | MFEG |
|----------|-----|-----|------|
|          | Time | It | $E_\infty(\tau, h)$ | Time | It | $E_\infty(\tau, h)$ | Time | It | $E_\infty(\tau, h)$ |
| 10       | 0.7322 | 56 | 2.3677E-02 | 0.4408 | 33 | 2.3693E-02 | 0.1277 | 11 | 9.2293E-02 |
| 18       | 4.1288 | 157 | 7.2080E-03 | 2.5156 | 87 | 7.2878E-03 | 0.3377 | 26 | 2.9239E-02 |
| 26       | 16.0169 | 302 | 3.2214E-03 | 8.8714 | 164 | 3.3752E-03 | 1.1323 | 49 | 1.4102E-02 |
| 34       | 35.1462 | 487 | 1.5531E-03 | 25.0797 | 264 | 1.7958E-03 | 2.3041 | 77 | 8.2001E-03 |
| 42       | 77.8330 | 707 | 6.4924E-04 | 55.6289 | 383 | 9.5072E-04 | 4.4194 | 112 | 5.3181E-03 |

Figure 11 The graphs of (a) elapsed time and (b) number of iterations of the proposed methods for Test Problem 6.3 with $\alpha = 0.7$

Table 12 Maximum errors and computational orders of the MFEG method for Test problem 6.3 with $T = 1$ and $\tau = 0.001$

| $\alpha = 0.1$ | $\alpha = 0.5$ | $\alpha = 0.9$ |
|----------------|----------------|----------------|
| $h^{-1}$       | $E_\infty(\tau, h)$ | C-Order | $E_\infty(\tau, h)$ | C-Order | $E_\infty(\tau, h)$ | C-Order |
| 6              | 3.1728E-03       | –       | 3.1728E-03       | –       | 3.1728E-03       | –       |
| 10             | 1.2360E-03       | 1.8455  | 1.2360E-03       | 1.8455  | 1.2360E-03       | 1.8455  |
| 14             | 6.4392E-04       | 1.9379  | 6.4392E-04       | 1.9379  | 6.4392E-04       | 1.9379  |
| 18             | 3.9126E-04       | 1.9824  | 3.9126E-04       | 1.9824  | 3.9126E-04       | 1.9824  |

can be seen that the FEG and MFEG methods result in simulations with rapid convergence and fewer iterations in comparison to the C-N difference scheme. Among the presented schemes, the MFEG method is clearly the fastest one since it uses the least amount of time consumption for simulating the test problems. The numerical errors and convergence orders of the MFEG method are reported in Table 12. The results are in good agreement with the theoretical analysis.
Conclusion

In the current work, we presented the FEG method and the MFEG method, which are derived based on the combination of two C-N difference schemes with grouping strategies on the standard grid to solve the two-dimensional time fractional mobile/immobile equation. The Fourier analysis method is utilized to prove the stability and convergence in $l^2$ norm. A comparison of the FEG and MFEG methods with the C-N difference scheme is given in terms of computational time, iterations’ number and maximum error. The computational orders of the proposed methods are calculated and found to be in good agreement with the theoretical analysis. Numerical experiments showed that the exact solutions are well matched using all the tested methods. Furthermore, experimental results verified that the FEG and MFEG methods outperform the C-N difference scheme in terms of iterations’ number and execution time, showing their efficiency. Overall, the proposed methods have the advantages of being accurate, computationally efficient and applicable to other types of linear and nonlinear multi-dimensional fractional problems. As an outlook for future, the parallel implementation technique is fairly meaningful to enhance the computational efficiency of the developed methods.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Code availability All numerical simulations have been conducted using Matlab R2018b.

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