Composition operators on Orlicz-Sobolev spaces

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Abstract

The kernel of composition operator $C_T$ on Orlicz-Sobolev space is obtained. Using the kernel, a necessary and a sufficient condition for injectivity of composition operator $C_T$ has been established. Composition operators on Orlicz-Sobolev space with finite ascent as well as infinite ascent have been characterized.

Keywords: Orlicz function, Orlicz space, Sobolev space, Orlicz-Sobolev space, Radon-Nikodym derivative, Composition operators, Ascent.

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1. Introduction and Preliminaries

Let $\Omega$ be an open subset of the Euclidean space $\mathbb{R}^n$ and $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite complete measure space, where $\Sigma$ is a $\sigma$-algebra of subsets of $\Omega$ and $\mu$ is a non-negative measure on $\Sigma$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an N-function [8, 9], i.e., an even, convex and continuous function satisfying $\varphi(x) = 0$ if and only if $x = 0$ with $\lim_{x \to 0} \frac{\varphi(x)}{x} = 0$ and $\lim_{x \to \infty} \frac{\varphi(x)}{x} = \infty$. Such a function $\varphi$ is known as Orlicz function. Let $L^0(\Omega)$ denote the linear space of all equivalence classes of $\Sigma$-measurable functions on $\Omega$, where we identify any two functions are equal if they agree $\mu$-almost everywhere on $\Omega$. The Orlicz space $L^\varphi(\Omega)$ is defined

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as the set of all functions \( f \in L^0(\Omega) \) such that \( \int_\Omega \varphi(\alpha |f|) d\mu < \infty \) for some \( \alpha > 0 \). The space \( L^\varphi(\Omega) \) is a Banach space with respect to the Luxemburg norm defined by
\[
\|f\|_\varphi = \inf \left\{ k > 0 : \int_\Omega \varphi \left( \frac{|f|}{k} \right) \leq 1 \right\}.
\]
If \( \varphi(x) = x^p \), \( 1 \leq p < \infty \), then \( L^\varphi(\Omega) = L^p(\Omega) \), the well known Banach space of \( p \)-integrable function on \( \Omega \) with \( \|f\|_\varphi = \left( \frac{1}{p} \right)^{\frac{1}{p}} \|f\|_p \). An Orlicz function \( \varphi \) is said to satisfy the \( \Delta_2 \)-condition, if there exists constants \( k > 0 \), \( u_0 \geq 0 \) such that \( \varphi(2u) \leq k\varphi(u) \) for all \( u \geq u_0 \). If \( \tilde{L}^\varphi(\Omega) \) denotes the set of all function \( f \in L^0(\Omega) \) such that \( \int_\Omega \varphi(|f|) d\mu < \infty \), then one has \( L^\varphi(\Omega) = \tilde{L}^\varphi(\Omega) \), when the Orlicz function \( \varphi \) satisfies the \( \Delta_2 \) condition. Here, the set \( \tilde{L}^\varphi(\Omega) \) is known as Orlicz class. We define the closure of all bounded measurable functions in \( L^\varphi(\Omega) \) by \( E^\varphi(\Omega) \). Then \( E^\varphi(\Omega) \subset L^\varphi(\Omega) \) and \( E^\varphi(\Omega) = L^\varphi(\Omega) \) if and only if \( \varphi \) satisfies \( \Delta_2 \) condition. For further literature concerning Orlicz spaces, we refer to Kufener, John and Fucik [3], Krasnoselskii & Rutickii [8] and Rao [9].

The Orlicz-Sobolev space \( W^{1,\varphi}(\Omega) \) is defined as the set of all functions \( f \) in Orlicz space \( L^\varphi(\Omega) \) whose weak partial derivative \( \frac{\partial f}{\partial x_i} \) (in the distribution sense) belong to \( L^\varphi(\Omega) \), for all \( i = 1, 2, \cdots, n \). It is a Banach space with respect to the norm:
\[
\|f\|_{1,\varphi} = \|f\|_\varphi + \sum_{i=1}^n \|\frac{\partial f}{\partial x_i}\|_\varphi
\]
If we take an Orlicz function \( \varphi(x) = |x|^p \), \( 1 \leq p < \infty \), then \( W^{1,\varphi}(\Omega) = W^{1,p}(\Omega) \) with \( \|f\|_{1,\varphi} = \left( \frac{1}{p} \right)^{\frac{1}{p}} \|f\|_{1,p} \), i.e., the corresponding Orlicz-Sobolev space \( W^{1,\varphi}(\Omega) \) becomes classical Sobolev space of order one. In fact, these spaces are more general than the usual Lebesgue or Sobolev spaces. For more details on Sobolev and Orlicz-Sobolev spaces, we refer to Adam [11], Rao [9] and Arora, Datt and Verma [13].

Let \( T : \Omega \rightarrow \Omega \) be a measurable transformation, that is, \( T^{-1}(A) \in \Sigma \) for any \( A \in \Sigma \). If \( \mu(T^{-1}(A)) = 0 \) for any \( A \in \Sigma \) with \( \mu(A) = 0 \), then \( T \) is called as nonsingular measurable transformation. This condition implies that the measure \( \mu \circ T^{-1} \), defined by \( \mu \circ T^{-1}(A) = \mu(T^{-1}(A)) \) for \( A \in \Sigma \), is absolutely continuous with respect to \( \mu \) i.e., \( \mu \circ T^{-1} \ll \mu \). Then the Radon-Nikodým theorem implies that there exist a non-negative locally integrable function \( f_T \) on \( \Omega \) such that
\[
\mu \circ T^{-1}(A) = \int_A f_T(x) d\mu(x) \quad \text{for} \ A \in \Sigma.
\]
It is known that any nonsingular measurable transformation \( T \) induces a linear operator (Composition operator) \( C_T \) from \( L^0(\Omega) \) into itself which is defined as
\[
C_T f(x) = f(T(x)), \quad x \in \Omega, \quad f \in L^0(\Omega).
\]
Here, the non-singularity of $T$ guarantees that the operator $C_T$ is well defined. Now, if the linear operator $C_T$ maps from Orlicz-Sobolev space $W^{1,\varphi}(\Omega)$ into itself and is bounded, then we call $C_T$ is a composition operator in $W^{1,\varphi}(\Omega)$ induced by $T$. A major application of Orlicz-Sobolev spaces can be found in PDEs \cite{1}.

The composition operators received considerable attention over the past several decades especially on some measurable function spaces such as $L^p$-spaces, Bergman spaces and Orlicz spaces, such that these operators played an important role in the study of operators on Hilbert spaces. The basic properties of composition operators on measurable function spaces have been studied by many mathematicians. For a flavor of composition operators on different spaces we refer to \cite{5}, \cite{12}, \cite{15} and \cite{16} and the references therein. The boundedness and compactness of the composition operator $C_T$ on Orlicz-Sobolev space $W^{1,\varphi}(\Omega)$ has been characterized in the paper due to Arora, Datt and Verma \cite{13}. Regarding the boundedness of the composition operator $C_T$ on Orlicz-Sobolev space $W^{1,\varphi}(\Omega)$ into itself, we have the following two important results \cite{13}.

**Lemma 1.1.** Let $f_T, \frac{\partial T_k}{\partial x_i} \in L^\infty(\Omega)$ with $\|\frac{\partial T_k}{\partial x_i}\|_\infty \leq M$, for some $M > 0$ and for all $i, k = 1, 2, \cdots, n$, where $T = (T_1, T_1, \cdots, T_n)$ and $\frac{\partial T_k}{\partial x_i}$ denotes the partial derivative (in the classical sense). Then for each $f$ in $W^{1,\varphi}(\Omega)$, we have $C_T(f) \in W^{1,\varphi}(\Omega)$ and if the Orlicz function $\varphi$ satisfies $\Delta_2$ condition, then the first order distributional derivatives of $f \circ T$, given by

$$
\frac{\partial}{\partial x_i}(f \circ T) = \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i},
$$

for $1, 2, \cdots, n$, are in $L^\varphi(\Omega)$.

**Theorem 1.1.** Suppose that conditions are hold as given in the previous lemma. Then the composition operator $C_T$ on Orlicz-sobolev space $W^{1,\varphi}(\Omega)$ is bounded and the norm of $C_T(f)$ satisfies the following inequality:

$$
||C_T(f)||_{1,\varphi} \leq ||f_T||_\infty(1 + nM)||f||_{1,\varphi}.
$$

In the present paper, we are going to present some classical properties of Composition operators on Orlicz-Sobolev space $W^{1,\varphi}(\Omega)$, which have not been proved earlier. We divide this paper into two sections - section 2, 3. In the section 2, we establish a necessary and sufficient condition for the injectivity of the composition operator on Orlicz-Sobolev space. And in the last section, composition operators on Orlicz-Sobolev space $W^{1,\varphi}(\Omega)$ with finite ascent as well as infinite ascent are studied.
2. Composition operator on Orlicz-Sobolev spaces

We begin by the defining Orlicz-Sobolev space. For a given Orlicz function $\phi$, the corresponding Orlicz-Sobolev space is given by

$$W^{1,\phi}(\Omega) = \left\{ f \in L^\phi(\Omega) : \frac{\partial f}{\partial x_i} \in L^\phi(\Omega) \text{ for } i = 1, 2, \ldots, n \right\}.$$ 

Assume that the conditions as given in the lemma 1.1. holds. Let $C_T : W^{1,\phi}(\Omega) \to W^{1,\phi}(\Omega)$ be a nonzero composition operator and $\Omega_o = \{ x \in \Omega : f_T(x) = \frac{d\mu T^{-1}}{d\mu}(x) = 0 \}$. Now consider the subset

$$W^{1,\phi}(\Omega_o) = \left\{ f \in W^{1,\phi}(\Omega) : f(x) = 0 \text{ for } \Omega \setminus \Omega_o \right\}.$$ 

Then the set $\Omega_o$ is obviously measurable. Note that if $\mu(\Omega \setminus \Omega_o) = 0$, then $\frac{d\mu T^{-1}}{d\mu}(x) = 0$ almost everywhere and $\|\frac{d\mu T^{-1}}{d\mu}\|_\infty = 0$. Thus in this case, the corresponding composition operator $C_T$ will be the zero operator (by the theorem 1.1.1). Hence for a nonzero composition operator $C_T$, we have $\mu(\Omega \setminus \Omega_o) > 0$. We start with the following result.

**Lemma 2.1.** If $f \in W^{1,\phi}(\Omega_o)$ then the weak derivative of $f$, $\frac{\partial f}{\partial x_i}(x) = 0$ on $\Omega \setminus \Omega_o$, for every $i = 1, 2, \ldots, n$.

**Proof.** Let $f \in W^{1,\phi}(\Omega_o)$. Then $f \in L^\phi(\Omega)$ and $\frac{\partial f}{\partial x_i} \in L^\phi(\Omega)$ for $1 = 1, 2, \ldots, n$. Since weak derivative of $f$ exists on $\Omega$, hence $f$ has weak derivative on $\Omega \setminus \Omega_o \subset \Omega$. Now $f(x) = 0$ on $\Omega \setminus \Omega_o$ implies that $\int_{\Omega \setminus \Omega_o} f(x) \frac{\partial \phi(x)}{\partial x_i} d\mu = 0$ for all test function $\phi(x) \in C_0^\infty(\Omega)$ and $i = 1, 2, \ldots, n$. We have,

$$\int_{\Omega \setminus \Omega_o} f(x) \frac{\partial \phi(x)}{\partial x_i} d\mu = - \int_{\Omega \setminus \Omega_o} \frac{\partial f(x)}{\partial x_i} \phi(x) d\mu , \forall \phi \in C_0^\infty(\Omega \setminus \Omega_o).$$

Since $C_0^\infty(\Omega \setminus \Omega_o) \subset C_0^\infty(\Omega)$ hence

$$\int_{\Omega \setminus \Omega_o} \frac{\partial f(x)}{\partial x_i} \phi(x) d\mu = - \int_{\Omega \setminus \Omega_o} f(x) \frac{\partial \phi(x)}{\partial x_i} d\mu$$

for all $\phi \in C_0^\infty(\Omega \setminus \Omega_o)$. Now as $\mu(\Omega \setminus \Omega_o) > 0$, it follows that weak derivative of $f$ is zero, i.e., $\frac{\partial f}{\partial x_i}(x) = 0$ on $\Omega \setminus \Omega_o$. 

**Theorem 2.1.** Let $C_T$ be a composition operator on Orlicz-Sobolev space $W^{1,\phi}(\Omega)$. Then $\ker C_T = W^{1,\phi}(\Omega_o)$. 

Proof. Let \( f \in \ker C_T \). Then \( f \circ T = 0 \) in \( W^{1,\varphi}(\Omega) \). This implies that \( \|f \circ T\|_{1,\varphi} = 0 \). But \( \|f \circ T\|_{1,\varphi} = \|f \circ T\|_{\varphi} + \sum_{i=1}^{n} \| \frac{\partial}{\partial x_i} (f \circ T) \|_{\varphi} \). Therefore, we have \( \|f \circ T\|_{\varphi} = 0 \) and \( \| \frac{\partial}{\partial x_i} (f \circ T) \|_{\varphi} = 0 \) for \( i = 1, 2, \ldots, n \). This shows that \( f \circ T = 0 \) in \( L^\varphi \). Hence, there exists \( \alpha > 0 \) such that

\[
0 = \int_\Omega \varphi(\alpha |f \circ T|) d\mu = \int_\Omega \varphi(\alpha |f|) \frac{d\mu \circ T^{-1}}{d\mu} d\mu \tag{2.1}
\]

Suppose \( S_f = \{ x \in \Omega : f(x) \neq 0 \} \). Then, from above it follows that \( \frac{d\mu \circ T^{-1}}{d\mu}|_{S_f} = 0 \).

Since

\[
W^{1,\varphi}(\Omega_0) = \{ f \in W^{1,\varphi}(\Omega) : f(x) = 0 \text{ for } \Omega \setminus \Omega_0 \} = \{ f \in W^{1,\varphi}(\Omega) : S_f \subset \Omega_0 \} = \{ f \in W^{1,\varphi}(\Omega) : \left. \frac{d\mu \circ T^{-1}}{d\mu} \right|_{S_f} = 0 \},
\]

hence, \( f \in W^{1,\varphi}(\Omega_0) \). Therefore, \( \ker C_T \subseteq W^{1,\varphi}(\Omega_0) \).

Conversely, suppose that \( f \in W^{1,\varphi}(\Omega_0) \). Then \( f \in L^\varphi(\Omega) \). Hence there exists \( \alpha > 0 \) such that \( \int_\Omega \varphi(\alpha |f|) d\mu < \infty \), for some \( \alpha > 0 \). Now we have,

\[
\int_\Omega \varphi(\alpha |f \circ T|) d\mu = \int_\Omega \varphi(\alpha |f|) \frac{d\mu \circ T^{-1}}{d\mu} d\mu = \int_{\Omega \setminus \Omega_0} \varphi(\alpha |f|) \frac{d\mu \circ T^{-1}}{d\mu} d\mu + \int_{\Omega_0} \varphi(\alpha |f|) \frac{d\mu \circ T^{-1}}{d\mu} d\mu = 0 \quad [\because f(x) = 0 \text{ on } \Omega \setminus \Omega_0 \text{ and } \frac{d\mu \circ T^{-1}}{d\mu}(x) = 0 \text{ on } \Omega_0] \]

This implies that \( f \circ T = 0 \) in \( L^\varphi(\Omega) \) and hence \( \|f \circ T\|_{\varphi} = 0 \). Now by the lemma 1.1, we have,

\[
\frac{\partial}{\partial x_i} (f \circ T) = \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i}.
\]

Since, \( f \in W^{1,\varphi}(\Omega_0) \) and \( C_T \) is a composition operator on \( W^{1,\varphi}(\Omega) \), hence weak derivative of \( f \circ T \), \( \frac{\partial}{\partial x_i} (f \circ T) \in L^\varphi(\Omega) \), for every \( i = 1, 2, \ldots, n \). Now for some \( \beta > 0 \), we have

\[
\int_\Omega \varphi \left( \beta \left| \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \circ T \right) \frac{\partial T_k}{\partial x_i} \right| \right) d\mu = \int_\Omega \varphi \left( \beta \left| \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(x) \frac{\partial T_k}{\partial x_i}(T^{-1}(x)) \right| \right) \frac{d\mu \circ T^{-1}}{d\mu} d\mu
\]

\[
= \int_{\Omega \setminus \Omega_0} + \int_{\Omega_0}
\]

\[
= 0
\]
as \( \frac{\partial f}{\partial x_k}(x) = 0 \) on \( \Omega \setminus \Omega_0 \) for \( k = 1, 2, \ldots, n \) and \( \frac{d\mu T^{-1}}{d\mu}(x) = 0 \) on \( \Omega_0 \). This shows that \( \frac{\partial}{\partial x_i} (f \circ T) = 0 \) in \( L^\varphi(\Omega) \) and hence \( \| \frac{\partial}{\partial x_i} (f \circ T) \|_\varphi = 0 \) for \( i = 1, 2, \ldots, n \). Thus we have,

\[
\| f \circ T \|_{1, \varphi} = \| f \circ T \|_\varphi + \sum_{i=1}^{n} \left\| \frac{\partial}{\partial x_i} (f \circ T) \right\|_\varphi = 0
\]

Therefore, \( f \circ T = 0 \) in \( W^{1, \varphi}(\Omega) \) and \( W^{1, \varphi}(\Omega_0) \subseteq \ker C_T \). Hence the result follows. \( \square \)

The following theorem gives a necessary and sufficient condition for injectivity of composition operator \( C_T \) on Orlicz-Sobolev space \( W^{1, \varphi}(\Omega) \). We say that \( T : \Omega \to \Omega \) is essentially surjective if \( \mu(\Omega \setminus T(\Omega)) = 0 \).

**Theorem 2.2.** The composition operator \( C_T \) induced by \( T \) on Orlicz-Sobolev space \( W^{1, \varphi}(\Omega) \) is injective if and only if \( T \) is essentially surjective.

**Proof.** Suppose that \( C_T \) is injective. Then \( \ker C_T = \{0\} \). But \( \ker C_T = W^{1, \varphi}(\Omega_0) = \left\{ f \in W^{1, \varphi}(\Omega) : \frac{d\mu T^{-1}}{d\mu}|_{S_f} = 0 \right\} \). Therefore, \( \ker C_T = \{0\} \) implies that \( W^{1, \varphi}(\Omega_0) = \{0\} \). This shows that \( f = 0 \) a.e. if \( \frac{d\mu T^{-1}}{d\mu}|_{S_f} = 0 \). Hence it follows that \( \frac{d\mu T^{-1}}{d\mu} \neq 0 \) a.e. Thus \( \mu(\Omega_0) = 0 \). To complete the proof, it suffices to show that \( \Omega \setminus \Omega_0 = T(\Omega) \). Note that \( \Omega \setminus \Omega_0 = S_{d\mu T^{-1}} \).

Let \( E \subset \Omega \setminus T(\Omega) \). Then \( T^{-1}(E) = \emptyset \) and hence, \( 0 = \mu(T^{-1}(E)) = \int_E \frac{d\mu \circ T^{-1}}{d\mu} d\mu \) implies that \( \frac{d\mu T^{-1}}{d\mu}|_E = 0 \). This shows that \( E \subset \Omega_0 \) and hence \( \Omega \setminus T(\Omega) \subseteq \Omega_0 \). Thus, we have

\[
\mu(\Omega_0) = 0 \Rightarrow \mu(\Omega \setminus T(\Omega)) = 0.
\]

Conversely, assume that \( T \) is essentially surjective so that \( \Omega = T(\Omega) \cup A \), where \( \mu(A) = 0 \). Then, clearly, we have

\[
\ker C_T = \left\{ f \in W^{1, \varphi}(\Omega) : C_T f = 0 \right\} = \left\{ f \in W^{1, \varphi}(\Omega) : f|_{T(\Omega)} = 0 \right\} = \{0\} \quad [\mu(A) = 0]
\]

Therefore, \( C_T \) is injective. \( \square \)

### 3. Ascent of the Composition Operator

F. Riesz in [6] introduced the concept of ascent and descent for a linear operator in a connection with his investigation of compact linear operators. The study of ascent and
descent has been done as a part of spectral properties of an operator ([1], [7]). Before
going to start, let us recall the notion of ascent of an operator on an arbitrary vector
space $X$.

If $H : X \to X$ is an operator on $X$, then the null space of $H^k$ is a $H$-invariant
subspace of $X$, that is, $H(\ker (H^k)) \subseteq \ker (H^k)$ for every positive integer $k$. Indeed,
if $x \in \ker (H^k)$ then $H^k(x) = 0$ and therefore, $H^k(H(x)) = H(H^k(x)) = 0$, i.e., $H(x) \in \ker (H^k)$. Thus we have the following subspace inclusions:

$$\ker (H) \subseteq \ker (H^2) \subseteq \ker (H^3) \subseteq \cdots$$

Following definitions and well known results are relevant to our context ([2], [10], [14]);

**Theorem 3.1.** For an operator $H : X \to X$ on a vector space, if $\ker (H^k) = \ker (H^{k+1})$ for some $k$, then $\ker (H^n) = \ker (H^k)$ for all $n \geq k$.

We now introduce ascent of an operator. The ascent of $H$ is the smallest nat-
ural number $k$ such that $\ker (H^k) = \ker (H^{k+1})$. If there is no $k \in \mathbb{N}$ such that $\ker (H^k) = \ker (H^{k+1})$, then we say that ascent of $H$ is infinite.

Now we are ready to study ascent of the composition operator $C_T$ on Orlicz-Sobolev
space $W^{1,\varphi}(\Omega)$. Observe that if $T$ is a non singular measurable transformation on $\Omega$, then $T^k$ is also non singular measurable transformation for every $k \geq 2$ with respect
to the measure $\mu$. Hence $T^k$ also induces a composition operator $C_{T^k}$. Note that for
every measurable function $f$, $C_{T^k}^k(f) = f \circ T^k = C_{T^k}(f)$. Also we have

$$\cdots \ll \mu \circ T^{-(k+1)} \ll \mu \circ T^{-k} \ll \cdots \ll \mu \circ T^{-1} \ll \mu.$$  

Take $\mu \circ T^{-k} = \mu_k$. Then by Radon-Nikodym theorem, there exists a non-negative locally integrable function $f_{T^k}$ on $\Omega$ so that the measure $\mu_k$ can be represented as

$$\mu_k(A) = \int_A f_{T^k}(x)d\mu(x), \text{ for all } A \in \Sigma$$

where the function $f_{T^k}$ is the Radon-Nikodym derivative of the measure $\mu_k$ with respect
to the measure $\mu$. The following theorem characterizes the composition operators $C_T$
with ascent $k$ on Orlicz-Sobolev spaces $W^{1,\varphi}(\Omega)$.

**Theorem 3.2.** The composition operator on Orlicz space $W^{1,\varphi}(\Omega)$ has ascent $k \geq 1$
if and only if $k$ is the first positive integer such that the measures $\mu_k$ and $\mu_{k+1}$ are equivalent.
Proof. Suppose that \( \mu_k \) and \( \mu_{k+1} \) are equivalent. Then \( \mu_{k+1} \ll \mu_k \ll \mu_{k+1} \). Since \( \mu_k \ll \mu_{k+1} \ll \mu \), hence the chain rule of Radon-Nikodym derivative implies that

\[
\frac{d\mu_k}{d\mu}(x) = \frac{d\mu_k}{d\mu_{k+1}}(x) \cdot \frac{d\mu_{k+1}}{d\mu}(x)
\]

(3.1)

\[
\Rightarrow f_{T^k}(x) = \frac{d\mu_k}{d\mu_{k+1}}(x) \cdot f_{T^{k+1}}(x)
\]

(3.2)

Similarly, \( \mu_{k+1} \ll \mu_k \ll \mu \) implies that

\[
f_{T^{k+1}}(x) = \frac{d\mu_{k+1}}{d\mu_k}(x) \cdot f_{T^k}(x)
\]

(3.3)

Now, the kernel of \( C_T^k \) given by \( ker(C_T^k) = ker(C_T) = W^{1,\varphi}(\Omega_k) \), where \( \Omega_k = \{ x \in \Omega : f_{T^k}(x) = 0 \} \). Similarly, \( ker(C_T^{k+1}) = W^{1,\varphi}(\Omega_{k+1}) \), where \( \Omega_{k+1} = \{ x \in \Omega : f_{T^{k+1}}(x) = 0 \} \). From (3.2) and (3.3) it follows that \( \Omega_k = \Omega_{k+1} \). Therefore we have,

\[
ker(C_T^k) = W^{1,\varphi}(\Omega_k) = W^{1,\varphi}(\Omega_{k+1}) = ker(C_T^{k+1}).
\]

Since \( k \) is the least hence, the ascent of \( C_T \) is \( k \).

Conversely, suppose that ascent of \( C_T \) is \( k \). Now this implies that if \( ker(C_T^k) = W^{1,\varphi}(\Omega_k) \) and \( ker(C_T^{k+1}) = W^{1,\varphi}(\Omega_{k+1}) \), then \( W^{1,\varphi}(\Omega_k) = W^{1,\varphi}(\Omega_{k+1}) \). Hence \( \Omega_k = \Omega_{k+1} \) almost everywhere with respect to the measure \( \mu \). So \( \Omega_k = \{ x \in \Omega : f_{T^k}(x) = 0 \} = \{ x \in \Omega : f_{T^{k+1}}(x) = 0 \} \). It is known that \( \mu_{k+1} \ll \mu_k \). Thus only need to show \( \mu_k \ll \mu_{k+1} \). For this let \( E \in \Sigma \) such that \( \mu_{k+1}(E) = 0 \). Now we have the following cases:

Case-1: When \( E \cap \Omega_k = \emptyset \).

Then \( 0 = \mu_{k+1}(E) = \int_E f_{T^{k+1}}(x)d\mu(x) \) implies that \( \mu(E) = 0 \) as on \( E, f_{T^{k+1}}(x) > 0 \).

As \( \mu_k(E) = \int_E f_{T^k}(x)d\mu(x) \) and \( \mu(E) = 0 \), hence \( \mu_k(E) = 0 \).

Case-2: when \( E \cap \Omega_k \neq \emptyset \).

Then we have,

\[
0 = \mu_{k+1}(E) = \int_E f_{T^{k+1}}(x)d\mu(x)
\]

\[
= \int_{E \setminus (E \cap \Omega_k)} f_{T^{k+1}}(x)d\mu(x) + \int_{E \cap \Omega_k} f_{T^{k+1}}(x)d\mu(x)
\]

\[
= \int_{E \setminus (E \cap \Omega_k)} f_{T^{k+1}}(x)d\mu(x)
\]

Now this implies that \( \mu(E \setminus (E \cap \Omega_k)) = 0 \). Therefore, in either cases \( \mu_{k+1}(E) = 0 \) implies that \( \mu_k(E) = 0 \). Thus \( \mu_{k+1} \ll \mu_k \ll \mu_{k+1} \).

\[
\square
\]

Corollary 3.1. Ascent of the composition operator \( C_T \) on Orlicz-Sobolev spaces is infinite if and only if there does not exist any positive integer \( k \) such that the measures \( \mu_k \) and \( \mu_{k+1} \) are equivalent.
We say that a measurable transformation $T$ is measure preserving if $\mu(T^{-1}(E)) = \mu(E)$ for all $E \in \Sigma$. We also have the following results:

**Corollary 3.2.** 1. If the measure $\mu$ is measure preserving then the ascent of the composition operator $C_T$ on Orlicz-Sobolev space $W^{1,\varphi}(\Omega)$ is 1.

2. If $T$ is a nonsingular surjective measurable transformation such that $\mu(\tau^{-1}(E)) \geq \mu(E)$ for all $E \in \Sigma$, then also the ascent of the composition operator induced by $T$ on Orlicz-Sobolev space is 1.

3. If $T$ is essentially surjective, then also ascent of $C_T$ is equal to 1.

**Conclusions:**

We have proposed and proved a necessary and sufficient condition for the injectivity of composition operator $C_T$. We have also characterized the operator $C_T$ defined on Orlicz-Sobolev space with finite and infinite ascent. Our future plan of work will be to apply these results to a class of non-linear PDEs.

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