Wave scattering by small particles in a medium.

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Abstract

Wave scattering is considered in a medium in which many small particles are embedded. Equations for the effective field in the medium are derived when the number of particles tends to infinity.

1 Introduction

Assume that waves in the medium are described by the equation

\[ Lu := [(a_{ip}(x)u_p)_i + k^2 n(x)]u = 0 \quad \text{in} \quad \mathbb{R}^3, \]  

where over the repeated indices summation is understood, \( u_p := \frac{\partial u}{\partial x_p} \), and the Green function, satisfying the radiation condition, solves the equation:

\[ LG = -\delta(x - y) \quad \text{in} \quad \mathbb{R}^3. \]

If there are \( M \) particles \( D_m \), placed in the medium, situated in a bounded domain \( D \), outside of which

\[ a_{ip}(x) = \delta_{ip}, \quad n(x) = 1, \quad x \in D':= \mathbb{R}^3 \setminus D, \]

where \( a_{ip}(x), n(x) \), are \( C^2 \)-smooth functions, \( \delta_{ip} \) is the Kronecker symbol, and the ellipticity condition holds:

\[ c_1 \sum_{p=1}^{3} |t_p|^2 \leq \sum_{c,p=1}^{3} a_{ip} t_p t_i \leq c_2 \sum_{p=1}^{3} |t_p|^2, \quad c_1 > 0, \]

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where \( t \in \mathbb{C}^3 \) is an arbitrary vector, then the scattering problem consists of solving the equation

\[
LU = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} D_m,
\]

(4)

\[
U|_S = 0 \quad \text{on} \quad S_m, \quad 1 \leq m \leq M,
\]

(5)

\[
U = U_0 + \sum_{m=1}^{M} \int_{S_m} G(x, s) \sigma_m(s) \, ds.
\]

(6)

Here \( LU_0 = 0, u_0 \) is the scattering solution in the absence of particles, i.e., if \( M = 0 \).

By Ramm’s lemma [3], p. 257, one can define the scattering solution \( U_0 \) in the absence of small particles by the relation:

\[
G(x, y) = g(y)U_0(x, \beta)[1 + o(1)], \quad |y| \to \infty, \quad \frac{y}{|y|} = -\beta,
\]

(7)

where \( g(y) := \frac{e^{ik|y|}}{4\pi|y|} \). We assume that \( ka << 1 \), where \( a = \frac{1}{2} \max_{1 \leq m \leq M} \text{diam} D_m \).

The aim of this paper is to develop a general approach to wave scattering in a medium in which many small particles are embedded. Smallness of the particles is understood in the sense \( ka << 1 \). The functions \( n(x) \) and \( a_{ip}(x) \) are assumed practically constant on the scale of the wavelength

\[
k(|\nabla n| + |\nabla a_{ip}|) << 1.
\]

We generalize the approach developed in [4], [5], [6], [7]. Earlier works are [1], [10], [11], to mention a few.

Our basic result is a formula for the wave field in the medium in which small particles are embedded. This field solves equation (4)–(6) and satisfies the radiation condition at infinity:

\[
\frac{\partial(U - U_0)}{\partial r} - ik(U - U_0) = o\left(\frac{1}{r}\right), \quad r := |x| \to \infty.
\]

(8)

We assume that

\[
d \gg a, \quad ka \ll 1,
\]

(9)

where \( d = \min_{m \neq j} \text{dist}(D_m, D_j) \), and the \( \text{dist} \) denotes the distance between two sets. Near any point \( x \in D \), such that \( \min_{1 \leq m \leq M} \text{dist}(x, D_m) >> 1 \), one calculates the wavefield \( U \) by the formula

\[
U(x) = U_0(x) + \sum_{m=1}^{M} G(x, x_m) \int_{S_m} [1 + ik\nu \cdot (s - x_m)] \sigma_m(s) \, ds,
\]

(10)
where $\frac{\nu}{|\nu|} \approx \frac{x-x_m}{|x-x_m|}$, and $\nu$ depends on $x_m, x$, and on the functions $n(x), a_{ip}(x)$. The term $ik\nu \cdot (s-x_m)$ comes from the formulas

$$\int_{S_m} G(x,s)\sigma_m(s)ds = G(x,x_m) \left\{ \int_{S_m} \sigma_m(s)ds + \int_{S_m} \frac{[G(x,s) - G(x,x_m)]}{G(x,x_m)} ds \right\},$$

$$\frac{G(x,s) - G(x,x_m)}{G(x,x_m)} = \int_0^1 \nabla_y G(x,x_m + \tau(s-x_m)) \cdot (s-x_m) d\tau \frac{G(x,x_m)}{G(x,x_m)} : = ik\nu \cdot (s-x_m),$$

where $\nu = \nu(x,x_m), \frac{\nu}{|\nu|} = \frac{x-x_m}{|x-x_m|}$. One may consider $\nu$ as a known vector because $G(x,y)$ is known.

In a generic case, when $Q_m := \int_{S_m} \sigma_m(s)ds \neq 0$, the assumption $ka << 1$ allows one to neglect the term $ik\nu \cdot s = O(ka)$ in (10) and to write (10) as

$$U(x) = U_0(x) + \sum_{m=1}^{M} G(x,x_m)Q_m, \quad Q_m = \int_{S_m} \sigma_m ds, \quad (11)$$

If $Q_m = 0$, then the term $ik\nu \cdot s$ cannot be neglected. We discuss this case in Section 3.

A physical example of such a case is the scattering by acoustically hard particles when the boundary condition is the Neumann one: $\frac{\partial U}{\partial N} \big|_{S_m} = 0, 1 \leq m \leq M$.

## 2 General methodology

Let us first assume that (11) is applicable and calculate $Q_m$. In a neighborhood of $S_j$ one has the exact boundary condition (5), which can be written as:

$$\int_{S_j} G(s,t)\sigma_j(t)dt = - \left( U_0(s) + \sum_{m \neq j} G(s,x_m)Q_m \right) : = u_e(s), \quad (12)$$

where $s \in S_j$ and $u_e$ is the effective field acting on $D_j$. The basic assumption is:

We assume that $u_e(s)$ is practically constant on the distances of order $a$.

As $|s-t| \to 0$, one has

$$G(s,t) = \frac{1}{4\pi |s-t|} [1 + O(ka)], \quad |s-t| \to 0,$$

where $s, t \in S_j$ and we have assumed for simplicity that $a_{ij} = \delta_{ij}$. In the general case one replaces the function $g_0(s,t) := \frac{1}{4\pi |s-t|}$ on the surface $S_j$ by the fundamental solution of the operator $\sum_{i,j=1}^{3} a_{ip}(x_j) \frac{\partial^2}{\partial x_i \partial x_p}$, which can be written explicitly and analytically:

$$G(x,y) = \frac{1}{4\pi \sqrt{\det(a_{ip})}} \frac{1}{|a_{ip}^{-1}(x_i-y_i)(x_p-y_p)|^{1/2}}.$$
where the matrix $a_{ip}^{(-1)}$ is inverse of the matrix $a_{ip}(x)$, $x_i$ is the $i$-th Cartesian component of the vector $x$ (not to be confused with the vector $x_j = x$).

If $a_{ip} = \delta_{ip}$, then $G$ solves the equation

$$G(x, y) = g_0(x, y) + k^2 \int_D g(x, t)n(t)G(t, y)dt, \quad g_0(x, y) = \frac{1}{4\pi|x - y|}. \quad (14)$$

The integral in (14) is $O(1)$ as $|x - y| \to 0$, so (13) follows from (14). We have:

$$G(x, y) = g_0(x, y)[1 + O(|x - y|)], \quad |x - y| \to 0. \quad (15)$$

In this paper $k > 0$ is assumed fixed. The error term in (13) is $O(ka)$, and the small parameter $ka$ is dimensionless. Replacing $G(s, t)$ in (12) by its expression (13) and neglecting the small term $O(ka)$, one gets the following integral equation for $\sigma_j$:

$$\int_{S_j} g_0(s, t)\sigma_j(t)dt = -u_e(x_j), \quad (16)$$

where we may replace $u_e(s)$, $s \in S_j$, by the quantity $u_e(x_j)$ because $|x_j - s| \leq a$, and $u_e$ was assumed practically constant on the distance of order $O(a)$. Equation (16) is an equation for the electrostatic charge density $\sigma_j(t)$ on the surface of a perfect conductor $D_j$, charged to a constant potential $-u_e(x_j)$. The total charge on the surface of this conductor is

$$Q_j = \int_{S_j} \sigma_j dt.$$

One knows from electrostatics that the total charge on the surface of a perfect conductor $D_j$ equals to the product of the electrical capacitance $C_j$ of this conductor and the potential $-u_e(x_j)$, to which this conductor is charged:

$$Q_j = -C_j u_e(x_j). \quad (17)$$

In [4, p.26, formula 5.12], analytical formulas are derived for calculating the electrical capacitances of conductors of arbitrary shapes with any desired accuracy.

Thus, one may consider the capacitance $C_j$ to be known. Equation (11) can be written as

$$U(x) = U_0(x) - \sum_{m=1}^M G(x, x_m)C_mU(x_m), \quad \text{dist} (x, D_m) \gg a. \quad (18)$$

Equation (17) can be considered as a linear algebraic system for finding $Q_m$. Namely, (12) and (17) imply:

$$Q_j = -C_j \left( U_0(x_j) + \sum_{m \neq j} G(x_j, x_m)Q_m \right). \quad (19)$$
Equation (19) is a linear algebraic system for the $M$ unknowns $Q_j$. The matrix elements of this system are known because the Green’s function $G(x, y)$ is known. If the condition
\[ C_j \sum_{m \neq j} |G(x_j, x_m)| < 1, \] (20)
holds, then system (19) can be solved by iterations and the iterative process
\[ Q_j^{(n+1)} = -C_j U_0(x_j) - \sum_{m \neq j} C_j G(x_j, x_m) Q_m^{(n)}, \quad Q_0^{(0)} := -C_j U_0(x_j), \] (21)
converges to $Q_j$ at the rate of a geometric series as $n \to \infty$. If $M \to \infty$ and the limit
\[ \int_{\tilde{D}} C(y) dy = \lim_{M \to \infty} \sum_{D_j \subset \tilde{D}} C_j \] (22)
exists for any subdomain $\tilde{D} \subset D$, where the function $C(y)$ is integrable, then this function is the limiting density of the electrical capacitance of the small particles. If the limit (22) exists, then equation (18) in this limit takes the form:
\[ U(x) = U_0(x) - \int_D G(x, y) C(y) U(y) dy. \] (23)
Here $U(x)$ is the limit of $u_e(x)$ as $M \to \infty$ and it is assumed that (22) holds.

Applying the operator $\nabla + k^2 n(x)$ to (23) and using equation (2) yields the following equation:
\[ [\nabla^2 + k^2 n(x)] U = C(x) U(x). \] (24)
This is a Schrödinger-type equation for the effective (self-consistent) wave field $U$:
\[ [\nabla^2 + k^2 - q(x)] U = 0, \] (25)
where
\[ q(x) := C(x) + k^2 [1 - n(x)]. \] (26)
Thus, in the limit $M \to \infty$ under the assumption (22), we have derived a linear Schrödinger-type equation (25) for the effective wave field $U$. If the small particles are identical, the electrical capacitance of a single conductor with the shape of a particle is $C_0$, and the density of the number of small particles is $N(x)$, then $C(x) = N(x) C_0$, so $N(x)$ can be calculated if $C(x)$ and $C_0$ are known. By the density $N(x)$ of the number of small particles we mean the function $N(x)$, defined by the formula, similar to (22):
\[ \int_D N(x) dx = \lim_{M \to \infty} \sum_{D_j \subset \tilde{D}} 1. \]
Let us summarize our method for calculating the effective field in the medium in which \( M \) small particles are embedded and the Dirichlet condition (5) holds on their surfaces.

This field is calculated by formula (11) at any point \( x \) such that \( \min + m \ \text{dist} \ (x, D_m) \gg a \), and the unknown numbers \( Q_m \) are calculated by solving linear algebraic system (19). If condition (20) holds, then system (19) is uniquely solvable by iterations. Condition (20) always holds if \( M \) is fixed and \( a \) is sufficiently small, because \( C_m = O(a) \).

If \( M \to \infty \) and condition (22) holds, then the field \( U \) can be found from the integral equation (23). Solving this equation is equivalent to solving equation (25) where \( U - U_0 \) satisfies (8). The function \( n(x) \) is known, therefore \( G(x, y) \) is also known.

3 The case when \( Q_m = 0 \).

Consider the same problem as in Section 2 but with the Neumann boundary condition

\[
\frac{\partial u}{\partial N} \bigg|_{S_m} = 0, \quad 1 \leq m \leq M, \tag{27}
\]

in place of the Dirichlet one. Here \( N \) is the unit exterior (i.e., pointing outside of \( D_m \)) normal to \( S_m \), \( 1 \leq m \leq M \). We look for the solution of the same form (6), and the methodology is the same, but now \( Q_m = 0, \ 1 \leq m \leq M \), as we show below. Thus, the term \( ikv \cdot s \) in (10) becomes important. Arguing as in Section 2 we obtain in place of (12) the following equation

\[
\sigma_j(s) = A_j \sigma_j + 2 \frac{\partial u_e(s)}{\partial N}, \quad s \in S_j, \tag{28}
\]

where

\[
A_j \sigma_j := 2 \int_{S_j} \frac{\partial G(s, t)}{\partial N_s} \sigma_j(t) \, dt, \tag{29}
\]

and the known formula from the potential theory:

\[
\frac{\partial}{\partial N_s} \int_{S_j} G(x, t) \sigma_j(t) \, dt = \frac{A_j \sigma_j - \sigma_j}{2}, \tag{30}
\]

was used. Here \( \frac{\partial}{\partial N_s} \) is the normal derivative on \( S_j \) from the outside of \( D_j \).

If \( |s - t| \to 0 \), then

\[
G(s, t) \approx g_0(s, t) = \frac{1}{4\pi |s - t|},
\]

where we again assume for simplicity that \( a_{ip} = \delta_{ip} \). We have

\[
G(x_j, s) = G(x_j, x_m) \left[ 1 + \frac{G(x_j, s) - G(x_j, x_m)}{G(x_j, x_m)} \right], \quad s \in S_m. \tag{31}
\]
Since \(|s - x_m| \leq a\) and \(|x_m - x_j| \gg a\), we have

\[
\frac{G(x_j, s) - G(x_j, x_m)}{G(x_j, x_m)} = \int_0^1 \frac{dG(x_j, x_m + \tau(s - x_m))}{G(x_j, x_m)} d\tau = \int_0^1 d\tau \nabla_y G(x_j, y) \bigg|_{y=x_m+\tau(s-x_m)} \cdot (s - x_m).
\]

Thus,

\[
\frac{G(x_j, s) - G(x_j, x_m)}{G(x_j, x_m)} \approx ik \nu \cdot (s - x_m), \tag{32}
\]

where, by the mean value theorem, one has

\[
iki \nu = \frac{\nabla_y G(x_j, x_m + \bar{\tau}(s - x_m))}{G(x_j, x_m)}, \quad 0 < \bar{\tau} < 1. \tag{33}
\]

If \(|x_j - x_m| >> a\), then

\[
\frac{\nu}{|\nu|} \approx \frac{x_m - x_j}{|x_m - x_j|}, \tag{34}
\]

and \(|\nu|\) depends on the functions \(n\) and \(a_{ip}\). Since \(n(x)\) is known, then \(G(x, y)\) is known, so \(\nu(x_m)\) can be considered as known.

If \(a_{ip} = \delta_{ip}\), then the integral equation for \(G\) can be written as

\[
G(x, y) = g(x, y) - \int_D g(x, \xi) q(\xi) G(\xi, y) d\xi := g - TG, \tag{35}
\]

where \(T\) is the integral operator, defined in (35),

\[
g(x, y) := e^{ik|x-y|} \frac{1}{4\pi|x-y|}, \quad q(y) := k^2[1 - n(y)], \quad q = 0 \quad \text{in} \quad D' := \mathbb{R}^3 \setminus D.
\]

The operator \(T\) is compact in \(L^2(D)\).

The function

\[
\frac{\nabla_y g}{g} \approx ik \frac{y - x}{|y - x|} \quad \text{if} \quad k|y - x| \gg 1.
\]

More precisely, we have used the exact formula:

\[
\nabla_y g = g(x, y) \left(ik - \frac{1}{|y - x|} \right) \frac{y - x}{|y - x|},
\]

and have neglected the term \(\frac{1}{|x-y|}\) compared with \(|ik| = k\).

This is justified if \(k|x - y| \gg 1\).

Equation (35) is uniquely solvable in \(L^2(D)\) because it is of Fredholm type and its homogeneous version has only the trivial solution. Indeed, if \(h\) solves the homogeneous equation (35), then

\[
(\nabla^2 + k^2 - q(x))h = 0 \quad \text{in} \quad \mathbb{R}^3,
\]
where the potential $q$ is compactly supported, $k^2 > 0$, and $h$ satisfies the radiation condition. It is known that this implies $h = 0$. In Kato’s paper [2] a similar but much stronger result is obtained: it is not assumed that $q$ is compactly supported, the potential may satisfy the assumption $|q(x)| = o(|x|^{-1})$ as $|x| \to \infty$.

The unique solution to (35) has the form
\[ G = (I + T_1)g, \]
where $I$ is the identity operator and $T_1 := (I + T)^{-1} - I$ is a linear compact operator in $L^2(D)$.

Therefore
\[ \frac{\nabla_y G}{G} = \frac{(I + T_1) \nabla_y g}{G(x, y)}. \]

If $|x - y| \to \infty$ and $k > 0$ is fixed, then
\[ \frac{G(x, y)}{g(x, y)} = 1 + O \left( \frac{1}{|x - y|} \right) \quad \text{dist} (x, D) \gg 1, \quad \text{dist} (y, D) \gg 1. \tag{36} \]

This follows from equation (35) if one takes into account the following estimates:
\[ |g(x, \xi)| = O \left( \frac{1}{|x - \xi|} \right) \quad \text{as} \quad |x - \xi| \to \infty, \]
\[ |G(\xi, y)| = O \left( \frac{1}{|\xi - y|} \right) \quad \text{as} \quad |\xi - y| \to \infty, \]
and
\[ J := \int_D \frac{dz}{|x - z||z - y|} = O \left( \frac{1}{|x - y|} \right), \quad |x - y| \to \infty. \tag{37} \]

The last estimate holds if $D \subset \mathbb{R}^3$ is a bounded domain. To prove estimate (37), take the origin at the point $y$, note that (37) holds if dist $(y, D) \leq \text{diam} D$, and $J$ decays when $\text{dist} (y, D)$ grows remaining less than $\frac{|x - y|}{2}$, if the gravity center of $D$ moves along the line joining $x$ and $y$.

If $x$ or $y$ remain in $D$ and $|x - y| \to \infty$, then the relation $\frac{G(x, y)}{g(x, y)} \approx 1$ does not hold, in general.

Since $G(x, y)$ is known one may assume that $|\nu| = |\nu(x)| = |\nu(x, y)|$ is known.

Because of (13), the operator $A_j$ in equation (28) can be approximated by the operator
\[ A \sigma = \int_{S_j} \frac{\partial g_0(s, t)}{\partial N_s} \sigma(t) dt. \]

It is known (see [4, p.96, formula (7.21)]) that
\[ \int_{S_j} A \sigma ds = - \int_{S_j} \sigma ds. \tag{38} \]
Therefore equation (28) implies
\[
\int_{S_j} \sigma_j ds = \int_{S_j} \frac{\partial u_e}{\partial N} ds = \int_{D_j} \Delta u_e dx \approx V_j \Delta u_e(x_j), \quad V_j := |D_j|,
\]
where $|D_j|$ is the volume of $D_j$. We had assumed that $u_e(x)$ is $C^2$-smooth. The term $V_j = O(a^3)$, and $\Delta u_e = O(k^2)$, because $\Delta U_0 = -k^2 n(x) U_0 = O(k^2)$. Thus,
\[
V_j \Delta u_e = O(k^2 a^3).
\]
Let us show that the second term in (10), namely, $ik \int_{S_j} \nu \cdot (s - x_j) \sigma ds$, is of the same order of magnitude $O(k^2 a^3)$.

We have assumed that $\nu$ is practically constant on the scale of order $O(a)$. Thus,
\[
V_j \Delta u_e = \nu \sum_{q=1}^{3} \nu_q \int_{S_j} (s - x_j)_q \sigma_j(s) ds = -ik \sum_{q=1}^{3} \nu_q V_j \beta^{(j)}_{pq} \frac{\partial u_e}{\partial x_p}.
\]
Here we have used the following result (see [11, p.98]):

If
\[
h_p = A h_p - 2 N_p,
\]
then
\[
\int_{S_j} (s - x_j)_q h_p ds = V_j \beta^{(j)}_{pq},
\]
where $(s - x_j)_q$ is the $q$-th Cartesian component of the vector $s - x_j$, $V_j$ is the volume of the domain $D_j$, and $\beta^{(j)}_{pq}$ is the magnetic polarizability tensor, defined in [11, p.62, formula (5.62)]. In the cited formula one takes the magnetic constant $\mu_0 = 1$, and the origin is at the point $x_j$, the gravity center of $D_j$. The right-hand side of (10) is of order $O(a^3) = O(k^2 a^3)$ because $k > 0$ is fixed. Thus, if the Neumann boundary condition holds on $S_m$, $1 \leq m \leq M$, then the terms $\int_{S_m} \sigma_m ds$ and $ik \int_{S_m} \nu \cdot (s - x_m) ds$ in (10) are of the same order of smallness as $ka \to 0$, and $k > 0$ is assumed fixed.

Let us compare this conclusion with the case when the Dirichlet boundary condition holds on $S_m$, $1 \leq m \leq M$. In this case, as follows from (17), $Q_m = O(a)$, because $C_m = O(a)$. If $a \to 0$, then $O(a) \gg O(a^3)$. This is the reason for the different physical conclusions in two cases. In the case of the Dirichlet boundary condition the scattering amplitude is of order $O(a)$ and the scattering by a single small particle is isotropic, while in the case of the Neumann boundary condition the scattering amplitude is of order $O(k^2 a^3)$ and the scattering by a single small particle is anisotropic.

Let $M \to \infty$ in the case of the Neumann boundary condition, and assume $a_{ip} = \delta_{ip}$. Equations (10), (39) and (10) yield
\[
\mathcal{U}(x) = \mathcal{U}_0(x) + \int_D G(x, y) \left[ \Delta \mathcal{U}(y) - ik \sum_{p,q=1}^{3} \nu_q(x, y) \frac{\partial \mathcal{U}}{\partial y_p} \beta_{pq}(y) \right] v(y) dy.
\]

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Here we have assumed that for any subdomain $\tilde{D} \subset D$ one has:

$$\int_{\tilde{D}} v(y) dy = \lim_{M \to \infty} \sum_{D_j \subset \tilde{D}} V_j,$$

and then we have passed to the limit $M \to \infty$ in equation (10).

In [4, p.55, formula (5.15)], analytical formulas are given for calculating tensor $\beta_{pq}^{(j)}$ with any desired accuracy for a body $D_j$ of arbitrary shape.

4 Conclusion.

The methodology we have developed for solving many-body wave scattering problem for small particles, embedded in a known medium, has the following new features:

1) Scattering by small particles is considered in a medium.

2) If the number $M$ of the particles is not very large, the scattering problem is reduced to solving linear algebraic systems with matrices, whose entries have physical meaning. Analytical formulas for calculating these entries are obtained in [4]. The reduction to linear algebraic systems bypasses any usage of integral equations, which are usually serving as a basic tool in the scattering theory.

3) If $M \to \infty$, then some integral equations (equations (23) and (41)) are derived for the effective field in the medium in which small particles are embedded.

4) In the case of the Dirichlet boundary condition the relative volume of the embedded particles tends to zero as $a \to 0$, $M \to \infty$.

Indeed, the number of small particles per unit volume of the medium is $O(\frac{1}{d})$, where $d$ is defined in (9), the volume of a single particle is $O(a^3)$, so the relative volume of the small particles is $O(\frac{a^3}{a^3}) \to 0$ as $M \to \infty$, since $a \to 0$ if $M \to \infty$.

In the case of the Neumann boundary condition the relative volume of the small particles tends to a finite non-zero limit $v(y)$ (cf. (42)).

5) The methodology, developed in this paper, can be used in the problems of electromagnetic wave scattering by small particles embedded in a known medium. It can also be used in some nanotechnological problems, consisting of creating “smart” materials with the desired properties, for example, wave-focusing properties (see [8], [9]).

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