Non-Fermi Behavior of the Strongly Correlated Electron Systems

Catalin Pascu Moca

Adviser: Professor M.Crisan

Cluj - 1999
Abstract

(1) The temperature dependence of the specific heat for a marginal Fermi liquid has been calculated. We showed that the expected $T \ln T$ correction is characteristic for the low temperature domain. The high temperature domain has a supplementary correction. The results are in agreement with the non-Fermi behavior of some metallic systems in the low temperature domain.

(2) We calculated the self-energy at $T = 0$ for a two dimensional fermionic system with hyperbolic dispersion. The existence of the saddle points in the energy gives rise to a marginal behavior, a result which has been obtained by numerical calculations.

(3) We present a simple demonstration that the two-dimensional fermionic system with the energy $\varepsilon_k = k_x k_y$ has a non-Fermi behavior. The calculation of the wave function renormalization constant $Z$ were performed using the "poor man’s renormalization" method and we have showed that $Z \to 0$ with the infrared cut-off $\Lambda$ as $Z(\Lambda) = \Lambda^\zeta$ where $\zeta$ is a constant.

(4) The electronic self-energy due to the electron-spin interaction is calculated using the one-loop approximation for the two dimensional system and quasi-two dimensional (anisotropic) model. We analyzed the relevance of the diffusive modes and the temperature dependence of the magnetic correlation length for a possible temperature dependence of the pseudogap.

(5) We study the influence of the amplitude fluctuations of a non-Fermi superconductor on the energy spectrum of the two-dimensional Anderson non-Fermi system. The classical fluctuations give a temperature dependence in the pseudogap induced in the fermionic excitations.

(6) Using the field-theoretical methods we studied the evolution from BCS description of a non-Fermi superconductor to that of Bose-Einstein condensation (BEC) in one loop approximation. We showed that the repulsive interaction between composite bosons is determined by the exponent $\alpha$ of the Anderson propagator in a two dimensional model. For $\alpha \neq 0$ the crossover is also continuous and for $\alpha = 0$ we obtain the case of the Fermi liquid.

(7) Using the renormalization group approach proposed by Millis for the itinerant electron systems we calculated the specific heat coefficient $\gamma(T)$ for the magnetic fluctuations with susceptibility $\chi^{-1} \sim \delta^\alpha + |\omega|^\alpha + f(q)$ near a Lifshitz point. The constant value for $\alpha = 4/5$ and
the logarithmic temperature dependence, specific heat for the non-Fermi behavior, have been obtained in agreement with the experimental data.

This dissertation is based on the following papers:

1. Specific Heat of a Marginal Fermi Liquid *M.Crisan, C.P. Moca Journal of Superconductivity* 9 49 (1996)

2. Marginal Behavior of a Two-Dimensional Fermionic System with Saddle Points in the Energy *C.P. Moca, M. Crisan Journal of superconductivity* 10 3 (1997)

3. Non-Fermi Liquid Behavior of a System with Saddle Points in the Energy *M. Crisan, C. P. Moca Modern Physics Letters B* 9 1753 (1995)

4. An Analytic Approach for the Pseudogap in the Spin Fluctuations Model *C.P. Moca, I.Tifrea, M. Crisan (accepted for publication)*

5. Electron-Fluctuation Interaction in a non-Fermi Superconductor *M. Crisan, C.P. Moca, I. Tifrea Phys.Rev. B* 59 14680 (1999)

6. Field-theoretical Description of the Crossover between BCS and BEC in an non-Fermi Superconductor *C.P. Moca, I. Tifrea M. Crisan (accepted for publication)*

7. Renormalization Group Approach of the Itinerant Electron System Near a Lifshitz Point *C.P. Moca, I. Tifrea, M. Crisan (accepted for publication)*
Acknowledgments

Research presented in this thesis has been deeply influenced by my adviser at "Babes-Bolyai" University, Prof. Mircea Crisan.

I owe a lot to my parents for their support and encouragement. Aside from this dissertation there is a number of people whom I wish to thank. To my colleagues Ionel Tifrea and Ioan Grosu for their support.
## Contents

1 Non-Fermi Behavior and the Fermi Surface 2

1.1 Thermodynamic Properties of the MFL 3

1.1.1 Thermodynamic Potential and Entropy 3

1.1.2 Specific Heat 5

1.2 Marginal Behavior of a System with Saddle Points in the Energy 7

1.2.1 Self-Energy of the 2D Fermi Systems 8

1.2.2 Imaginary Part of the Self-Energy 9

1.2.3 Real Part of the Self-Energy 11

1.3 "Poor Man’s Renormalization" for a System with Saddle Points in the Energy 13

1.3.1 Model 13

1.3.2 Self-Energy and the Renormalization Constant $Z$ 14

1.4 Summary of the Results 17

2 Anderson non-Fermi Model 21

2.1 The Model 21

2.2 A Novel Feature of the non-Fermi Behavior. The Occurrence of the Pseudogap 22

2.2.1 Dynamic Susceptibility 23

2.2.2 Self-Energy of the Electronic Excitations 25

2.2.3 Pseudogap in the Electronic Energy Spectrum 27

2.2.4 Effect of Anisotropy 29

2.3 Electron-Fluctuation Interaction in a non-Fermi Liquid Superconductor 32

2.3.1 Mode Coupling Approximation 33

2.3.2 Electronic Self-Energy 36
Chapter 1

Non-Fermi Behavior and the Fermi Surface

One of the most important problems which was generated by the discovery of cuprates superconductors is the nature of the normal phase. In fact the main question is if a conventional Landau theory of the Fermi liquid (FL) can describe this phase, or as was suggested by the experimental data, this phase is in fact a special metal described by a non-Fermi liquid (NFL).

The first model of a NFL was presented by Varma et.al.[1] and is well known as the marginal-Fermi liquid (MFL) and developed by Kotliar et.al.[2] and Littlewood et.al.[3]. This model is phenomenological but due to its importance for the explanation of the experimental data much effort has been done to give a microscopic foundation for the occurrence of the MFL behavior. The main idea of the MFL model is the occurrence of a linear energy dependence in the imaginary part of the self-energy due to the anomalous frequency and temperature dependence of the density-density (spin-spin) response function. This particular form of the self-energy gives rise to the pair-breaking effects, studied by Horbach et.al.[4] and Bendle et.al.[5].

This particular behavior of the quasiparticle spectrum is essential for the thermodynamics of the system, and the specific heat is one of the most important physical quantities that has to be affected as mentioned by Reizer[6] for the electrons interacting with a transverse electromagnetic field.

For a three dimensional (3D) Fermi liquid, the imaginary part of the self-energy has the form

\[ \text{Im} \Sigma (\omega) \sim \omega^2 \]

and gives a logarithmic correction to the specific heat

\[ C_V = \gamma T + \gamma_{3D} T^3 \ln T. \]

This behavior has been obtained in the framework of the Landau theory, but this correction...
cannot describe the low-temperature behavior of the $^3$He which is considered as the standard Fermi liquid.

Anderson [7] showed that the low temperature dependence of the specific heat for $^3$He can be fitted by a $T \ln T$ dependence and suggested that this dependence is given by coupling of the fermionic quasiparticles to the collectives modes. Balian and Fredkin[8] and Berk and Schriffer[9] showed that the coupling between the fermionic quasiparticles and the triplet paramagnon gives a $T^3 \ln T$ correction in the specific heat.

The correction to the linear dependence of the specific heat for a Fermi liquid has been calculated by Amit et.al.[10] in terms of Landau parameters.

Carneiro and Petrick[11] performed a microscopic calculation of the thermodynamic potential in terms of fully renormalized single-particle propagator considering the effect of the quasiparticle lifetime. This method can be successfully applied for the calculation of the specific heat of the MFL [12]

1.1 Thermodynamic Properties of the MFL

1.1.1 Thermodynamic Potential and Entropy

In this section we calculate the specific heat of a MFL using a similar method with that of Carneiro and Petrick [11]. The thermodynamic potential $\Omega$ is given by the fully renormalized single-particle propagator $G(p, \omega)$ as:

$$\Omega = T \sum_n e^{i \omega_n \eta} \left( - \ln \left[ -G^{-1}(p, \omega_n) \right] - \Sigma(p, \omega_n) G(p, \omega_n) \right)$$  \hspace{1cm} (1.1)

where the propagator $G(p, \omega_n)$ is given by Dyson equation:

$$G^{-1}(p, \omega_n) = G_0^{-1}(p, \omega_n) - \Sigma(p, \omega_n)$$  \hspace{1cm} (1.2)

$\Sigma(p, \omega_n)$ being the self-energy.

The thermodynamic potential $\Omega$ has the property that it is stationary under variation of $G$ at fixed $G_0$:

$$\left( \frac{\delta \Omega}{\delta G} \right)_{G_0} = 0$$  \hspace{1cm} (1.3)

and shows that we can neglect the temperature dependence of the spectral density $A(p, \omega)$. 
\[ A(p, \omega) = -2\text{Im} G(p, \omega) \] (1.4)

in the expression for the entropy \( S(T) \) given by:

\[ S(T) = -\left( \frac{\delta \Omega}{\delta T} \right) |_{\mu} \] (1.5)

Using:

\[ G(p, \omega) = \text{Re} G(p, \omega) - \frac{i}{2} A(p, \omega) \] (1.6)

and:

\[ \Sigma(p, \omega) = \text{Re} \Sigma(p, \omega) - \frac{i}{2} \Gamma(p, \omega) \] (1.7)

equation (1.5) becomes:

\[
S(T) = \sum_p \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\partial n(\omega)}{\partial T} 2\text{Im} \ln \left[ -G^{-1}(p, \omega + i0^+) \right] \\
- \sum_p \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\partial n(\omega)}{\partial T} \text{Re} \Sigma(p, \omega) A(p, \omega) \\
- \sum_p \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\partial n(\omega)}{\partial T} \text{Re} G(p, \omega) \Gamma(p, \omega)
\] (1.8)

where \( n(\omega) \) is the Fermi-Dirac distribution. This equation has been written as:

\[ S(T) = S_{DQ}(T) + S'(T) \] (1.9)

where \( S_{DQ}(T) \) is the dynamical contribution expressed by:

\[
S(T) = \sum_p \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\partial n(\omega)}{\partial T} \left( 2\text{Im} \ln \left[ -G^{-1}(p, \omega + i0^+) \right] - \text{Re} G(p, \omega) \Gamma(p, \omega_n) \right)
\] (1.10)

The second contribution has been identified as the contribution of terms that has vanishing energy denominators. Equation (1.8) can be written as:

\[ S(T) = S_0(T) + S_1(T) + S_2(T) \] (1.11)

where:
\[ S_0(T) = - \sum_p \left[ n(\varepsilon_p^0) \ln(n(\varepsilon_p^0)) + (1 - n(\varepsilon_p^0)) \ln(1 - n(\varepsilon_p^0)) \right] \] (1.12)

and:

\[ S_1(T) = - \sum_p \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\partial n(\omega)}{\partial T} \text{Re} \Sigma(p, \omega) A(p, \omega) \] (1.13)

For \( S_2(T) \) we get:

\[ S_2(T) = - \sum_p \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\partial n(\omega)}{\partial T} \left[ \arctan B(p, \omega) + \frac{B(p, \omega)}{1 + B^2(p, \omega)} \right] \] (1.14)

where:

\[ B(p, \omega) = \frac{\Gamma(p, \omega)}{2\text{Re}G^{-1}(p, \omega)} \] (1.15)

and \( \varepsilon_p^0 \) is the solution of the equation:

\[ \text{Re}G^{-1}(p, \varepsilon_p) = 0 \] (1.16)

If the damping of the quasiparticles is small, \( S_0(T) \) is the entropy of the Fermi liquid and gives a linear contribution in the specific heat of the form:

\[ C_0^0(T) = \gamma T \] (1.17)

In fact such a result will be obtained if we consider that in eq.(1.10) \( G(p, \omega) = G_0(p, \omega) \) where \( G_0(p, \omega) \) is the propagator for a FL, and in this approximation \( S_1(T) \) and \( S_2(T) \) appears as corrections to the linear dependence. These results will be used for the calculation of the specific heat in the following section.

### 1.1.2 Specific Heat

A marginal Fermi liquid can be characterized by the functional form of the frequency and temperature dependence of the single particle self-energy [1]:

\[ \sum(\omega) = \begin{cases} \lambda \omega \ln \frac{T}{\omega_c} - i\frac{\pi}{T} \lambda T & \text{if } \omega < T \\ \lambda \omega \ln \frac{\omega}{\omega_c} - i\frac{\pi}{T} \lambda \omega & \text{if } T < \omega < \omega_c \end{cases} \] (1.18)
where $\lambda$ is a coupling constant and $\omega_c$ is an ultraviolet cut-off. According to Varma and co-
workers [1-3] the normal phase of high-$T_c$ oxides is an MFL that can be understood on the basis of an MFL hypothesis i.e. the total electronic polarizability has a contribution $\mathcal{P}$ representing an electronic excitation with the property:

$$\text{Im} \mathcal{P}(p, \omega) = \begin{cases} -N(0) \frac{\omega}{T} & |\omega| < T \\ -N(0) \text{sgn}(\omega) & |\omega| > T \end{cases}$$

(1.19)

for the leading frequency contribution, where $N(0)$ is the bare electronic density of states. All the universal anomalies as well as the appearance of a 'Fermi surface' within the bound states can be understood from this single hypothesis.

From eq. (1.18) a contribution to the resistivity proportional with $T$ is obtained [1]. The nuclear relaxation time $T_1^{-1}(T)$, tunneling conductance $g(V)$, optical conductivity $\sigma(\omega)$, fit well the experimental data.

The electronic contribution to $C_V$, in the normal state cannot be extracted from the data with high accuracy because the contribution of phonons is very large above the high $T_C$ of these materials. $\kappa(T)$ in the normal state is observed to be nearly temperature independent[13]. If the self-energy from the eq.(1.18) is used to calculate the entropy, logarithmic correction to the linear-dependence are found. The observed $\kappa/T \sigma$ ratio allowed a weak correction.

The spectral density $A(p, \omega)$ can be calculated from the eq.(1.4) and (1.18) as:

$$A(p, \omega) = \frac{\pi \lambda \omega}{[\omega - \varepsilon_p - Re \Sigma(\omega)]^2 + \frac{1}{4} \pi^2 \lambda^2 \omega^2}$$

(1.20)

The contribution given by $S_1(T)$ has the form:

$$S_1(T) = \begin{cases} 2N(0)T \left( F \left( \frac{1}{2} \right) - F \left( 0 \right) \right) \ln \frac{\omega}{T} & \omega < T \\ 2N(0)T \left( F \left( \frac{\omega_c}{T} \right) - F \left( \frac{1}{2} \right) \right) \ln \frac{\omega_c}{T} & \omega < T < \omega_c \end{cases}$$

(1.21)

where $N(0)$ is the density of states and $F(x)$ is defined as:

$$F(x) = 4x - x^2 + x^2 \tanh x + 2 \arctan \left( e^x \right) + 2 \text{Re} L_2(e^x)$$

(1.22)

and:

$$L_2(x) = \int_0^x \frac{dt}{t} \ln |1-t|$$

(1.23)

The contribution given by $S_2(T)$ was calculated using eqs. (1.14), (1.18) as:
\[ S_2(T) = \begin{cases} 
2N(0)T \left( F \left( \frac{1}{2} \right) - F(0) \right) \left( 1 + \lambda \ln \frac{\omega_c}{T} \right) & \omega < T \\
2N(0)T \left( F \left( \frac{\omega_c}{2T} \right) - F \left( \frac{1}{2} \right) \right) \left( 1 + \lambda \ln \frac{\omega_c}{T} \right) & \omega < T < \omega_c 
\end{cases} \]  

(1.24)

Using the results for the entropy we can calculate the specific heat as:

\[ C_V(T) = T \frac{\partial S(T)}{\partial T} \]  

(1.25)

In the following we will consider only the contribution given by the MFL, \( S_0(T) \) giving a linear contribution specific to the FL.

From eqs. (1.21) and (1.24) we get for \( 0 < \omega < T \):

\[ C_V = 2N(0)T \left( F \left( \frac{1}{2} \right) - F(0) \right) \left( 1 + 2\lambda \left( \ln \frac{\omega_c}{T} - 1 \right) \right) \]  

(1.26)

For the limit \( T < \omega < \omega_c \) we get:

\[ C_V = 2N(0)T \left( F \left( \frac{\omega_c}{2T} \right) - F \left( \frac{1}{2} \right) \right) \left( 1 + 2\lambda \left( \ln \frac{\omega_c}{T} - 1 \right) \right) - \\
2N(0)T \left( F \left( \frac{\omega}{2T} \right) - F \left( \frac{\omega_c}{2T} \right) \right) \left( F \left( \frac{\omega_c}{2T} \right) - F \left( \frac{1}{2} \right) \right) \left( 1 + 2\lambda \left( \ln \frac{\omega_c}{T} - 1 \right) \right) \]  

(1.27)

where \( F'(x) = x^2 / \cosh^2(x) \).

From eq.(1.26) we can see that for the MFL the temperature dependence has a correction of the form \( T \ln T \) but in the dominant region \( T < \omega < \omega_c \) a more complicated temperature dependence is added to this correction.

### 1.2 Marginal Behavior of a System with Saddle Points in the Energy

The non-Fermi behavior of electrons with hyperbolic dispersion \( \varepsilon_k = k_x k_y \) has been predicted by News et al. [14] in connection with an electronic mechanism of superconductivity in the high critical temperature superconductors. For such a two-dimensional (2D) system the density of states shows a logarithmic energy dependence, and for a Fermi energy \( E_F \) near the saddle point \( (E_c) \) the critical temperature for the superconducting state has been calculated, and is very sensitive to the new energy scale \( E^* \) given by the density-density excitation. The results from [14] are not realistic because only the electron-hole channel has been considered [15],
but the results remain important for the explanation of the anomalous properties of cuprate superconductors in the normal state. In order to explain the physical properties of the normal state of these superconductors we have to mention that the difficulties are due to the study of a 2D interacting Fermi system and one of the simplest model is to treat it as a Luttinger liquid, which exist for 1D Fermi liquid. However for the (quasi)2D systems which are more realistic for cuprates superconductors, there is no general demonstration for a MFL behavior. As it will be showed the form of the Fermi surface is essential for the marginal behavior. A more realistic model seems to be the model in which the Landau concept for the quasiparticles is still valid and the deviations are due to the anomalous scattering of electrons on the overdamped low energy excitations. Millis, Monien and Pines [15] developed such a phenomenological form of the dynamic susceptibility $\chi(q, \omega)$ which describe the normal and the superconducting phase, the occurrence of the magnetic pseudogap and the influence of the impurities. The pairing has also considered in this model, but the Migdal Theorem discussed by Grosu and Crisan [16] is not valid in the ”hot spots” of the Fermi surface how was recently pointed by Amin and Stamp [17]. Another interesting model has been proposed by Bernard et.al. [18] and is based on the interaction between the electrons with a critical bosonic mode. This interaction lead to a non-Fermi behavior (even to a Varma et.al. [1] marginal behavior) and Crisan and Tataru [19] showed that this is possible even in 3D case. The importance of the van-Hove singularities in the density of states in the 2D Fermi systems for the cuprate superconductors has been pointed by Friedel [20], Labbe and Bok [21] and in a systematic study by the IBM group [22]. In one of their papers [23] it was suggested that the saddle points singularities (called ”hot spots”) of the Fermi surface are responsible for the deviations from the usual metallic state. In the following we calculate the self energy of 2D model with dispersion $\varepsilon_k = k_x k_y / 2m$ at $T = 0$. Such a calculation was also performed in [24] but the integrals were numerically evaluated. We will give an analytic calculation which show the ”marginal behavior” of 2D systems.

### 1.2.1 Self-Energy of the 2D Fermi Systems

We consider the self-energy of a 2D electronic system interacting with the electron-hole excitations which has the polarizability given in [23-24]:

$$Im\chi(q, z) = \frac{2}{\pi\varepsilon_k} \left[ |z + \varepsilon_k| - |z - \varepsilon_k| \right]$$  (1.28)
\[ Re\chi(q, z) = \frac{1}{2\pi} \left[ \ln \left| \frac{4E_c\varepsilon_k}{z^2 - \varepsilon_k^2} \right| - \frac{z}{\varepsilon_k} \ln \left| \frac{z + \varepsilon_k}{z - \varepsilon_k} \right| + 2 \right] \] (1.29)

where \(\varepsilon_k = k_xk_y/2m\). The self-energy at \(T = 0\) is given by the general equations:

\[ Im\Sigma(p, \omega) = \frac{2u^2}{\pi} \int d^2k (2\pi)^2 \int_0^\infty dz Im\chi(p - k, z) ImG(k, z + \omega) \] (1.30)

\[ Re\Sigma(p, \omega) = -\frac{2u^2}{\pi} \int d^2k (2\pi)^2 \int_0^\infty dz Re\chi(p - k, z) ImG(k, z + \omega) \] (1.31)

where \(u\) is the electron-electron interaction which generates the bosonic excitation. We mention that the dispersion

\[ \varepsilon_k = \frac{1}{2m} \left( k_x^2 - k_y^2 \right) \] (1.32)

called the hyperbolic dispersion in [24] is equivalent to the simple case \(\varepsilon_k = k_xk_y/2m\). In order to perform the calculations using eqs.(1.28) and (1.29) we will perform the transformation:

\[ k_x = \frac{k}{2} \left( \lambda + \frac{1}{\lambda} \right) \] (1.33)

\[ k_y = \frac{k}{2} \left( \lambda - \frac{1}{\lambda} \right) \] (1.34)

and the dispersion has the form

\[ \varepsilon_k = \frac{k_xk_y}{2m} \] (1.35)

1.2.2 Imaginary Part of the Self-Energy

Equation (1.30) can be transformed using:

\[ ImG(k, \omega + z) = -\pi\delta(z + \omega - \varepsilon_k) \] (1.36)

as:

\[ Im\Sigma(p, \omega) = \frac{u^2}{\pi^3} \int_0^\infty dk \int_{-\infty}^\infty d\lambda \frac{1}{\lambda} \varepsilon_k \varepsilon_{p-k} \left[ |\varepsilon_k - \omega + \varepsilon_p - k| - |\varepsilon_k - \omega - \varepsilon_p - k| \right] \] (1.37)
Let us calculate this integral first in the limit \( \omega \ll \varepsilon_k \). In this case eq.(1.37) becomes:

\[
Im \Sigma(p, \omega) = \frac{u^2}{\pi^3} \int_0^\infty dk \left[ \varepsilon_k - \omega \right] \int_{-\infty}^\infty \frac{d\lambda}{\lambda \left( p_x^2 + k_x^2 - 2p_xk_x \right) - (p_y^2 + k_y^2 - 2p_yk_y)} \frac{2m}{2m}
\]

and if we take the \( p = (p_0, 0) \) this equation becomes:

\[
Im \Sigma(p, \omega) = \frac{2mu^2}{\pi^3 p} \int_0^\infty dk \left[ \varepsilon_k - \omega \right] \int_{-\infty}^\infty \frac{d\lambda}{\lambda^2 - (p^2 - k^2)/(pk) + 1}
\]

and performing the integration over \( \lambda \) we get:

\[
Im \Sigma(p, \omega) = \frac{2mu^2}{\pi^3 p} \int_0^\infty dk \left[ k^2 - \omega \right] \frac{k^2}{2m} - \omega \left[ \frac{2pk}{p^2 - k^2} \text{arctanh} \left( \frac{2pk - p^2 - k^2}{p^2 - k^2} \right) + \frac{2pk}{p^2 - k^2} \text{arctanh} \left( \frac{p^2 + k^2}{p^2 - k^2} \right) \right]
\]

In the approximation \( \text{arctanh}(x) \approx x \) we calculate the first contribution in eq.(1.40) as:

\[
Im \Sigma_1(p, \omega) = -\frac{2mu^2}{\pi^3 p} \int_0^\infty dk \left[ k^2 - \omega \right] \frac{2pk}{(p+k)^2}
\]

and for \( k \ll p \) eq.(1.41) will be written as:

\[
Im \Sigma_1(p, \omega) = -\frac{2mu^2}{\pi^3 p^2} \int_0^\infty dk \left[ k^2 - \omega \right]
\]

where we neglected the contribution given by \( k^2 + 2pk \ll p^2 \) in the denominator of eq.(1.41). From eq.(1.42) we obtain:

\[
Im \Sigma_1(p, \omega) = \frac{mu^2}{\pi^3} \left[ \frac{p^2}{2m} - \omega \right]
\]

In the same approximation we calculate the contribution from the second term of eq. (1.40) and:

\[
Im \Sigma_2(p, \omega) = \frac{2mu^2}{\pi^3} \left[ \frac{p^2}{2m} - \omega \right]
\]

From these equations the imaginary part of the self-energy becomes:

\[
Im \Sigma(p, \omega) = -\frac{3mu^2}{\pi^3} \left[ \omega - \frac{p^2}{2m} \right]
\]
and for $p = p_F$ we get:

$$Im\Sigma(p, \omega) = -3N(0) \left(\frac{U}{\pi}\right)^2 (\omega - E_F) \quad (1.46)$$

where $N(0) = m/\pi$ and $E_F = p_F^2/2m$

### 1.2.3 Real Part of the Self-Energy

The general expression for $Re\Sigma(p, \omega)$ in the second order of perturbation theory has the general form:

$$Re\Sigma(p, \omega) = -\frac{u^2}{\pi} \int \frac{d^2k}{(2\pi)^2} \int_0^\infty dz Re\chi(p - k, z) ImG(k, z + \omega) \quad (1.47)$$

where $ImG$ will be considered as $\delta$-function and $Re\chi(k, z)$ is given by eq.(1.29). If we perform the integral over $z$ eq.(1.47) becomes:

$$Re\Sigma(p, \omega) = -\frac{u^2}{4\pi} \int_0^\infty dk \int \frac{d\lambda}{\lambda} \left[ \ln \frac{4E_c \varepsilon_{p-k}}{(\varepsilon_k - \omega)^2 - \varepsilon_{p-k}^2} \right. \left. - \frac{\varepsilon_k - \omega}{\varepsilon_{p-k}} \ln \frac{\varepsilon_k - \omega + \varepsilon_{p-k}}{\varepsilon_k - \omega - \varepsilon_{p-k}} + 2 \right] \theta(\varpi - \varepsilon_k + \omega) \quad (1.48)$$

where $\varpi$ is a frequency cut-off. The integral containing the first two terms will be transformed as:

$$Re\Sigma(p, \omega) = -\frac{u^2}{4\pi} \int_0^\infty dk \int \frac{d\lambda}{\lambda} \left[ 1 - \frac{\varepsilon_k - \omega}{\varepsilon_{p-k}} \ln \left| 1 + \frac{\varepsilon_k - \omega}{\varepsilon_{p-k}} \right| - \ln \frac{\varepsilon_{p-k}}{4E_c} \right] \quad (1.49)$$

Using the expansion:

$$\ln(1 + x) = x - \frac{x^2}{2} \quad (1.50)$$

we get:

$$Re\Sigma_1(p, \omega) = -\frac{u^2}{4\pi^4} \int_0^\infty dk \int \frac{d\lambda}{\lambda} \left[ \frac{\varepsilon_k - \omega}{\varepsilon_{p-k}} + \ln \frac{4E_c}{\varepsilon_{p-k}} \right] \quad (1.51)$$

In order to perform the integration over $\lambda$ we calculate
\[ \varepsilon_{p-k} = p^2 + k^2 - 2pk \]  

(1.52)

\[ pk = \frac{pk}{2} \left( \lambda + \frac{1}{\lambda} \right) \]  

(1.53)

where we take \((2m = 1)\) and (1.51) becomes:

\[ \text{Re} \Sigma_1(p, \omega) = -\frac{u^2}{8\pi^3} \int_0^\infty dk \int_0^\infty \frac{\varepsilon_k - \omega}{|p - k(p + k)|} \theta(\omega + \varepsilon_k - \omega_k) \]

\[ = -\frac{u^2}{8\pi^3} \int_0^{\infty} \frac{d\omega}{k^2} \left[ \left( 1 + \frac{p^2 - \omega}{k^2 - p^2} \right) + \frac{4}{k^2 - p^2} \right] \]  

(1.54)

which gives:

\[ \text{Re} \Sigma_1(p, \omega) = \frac{u^2}{8\pi^3} \left[ (\omega + \omega) + (\varepsilon_p - \omega) \ln \left| \frac{\omega + \omega - 1}{\varepsilon_p} \right| - \frac{u^2}{2\pi^3} \ln \left| \frac{\omega + \omega}{\varepsilon_p} - 1 \right| \right] \]  

(1.55)

The last term of eq. (1.49) is:

\[ \text{Re} \Sigma_2(p, \omega) = -\frac{u^2}{2\pi^2} \int_0^\infty dk \int_0^\infty \frac{d\lambda}{\lambda} \theta(\omega - \varepsilon_k + \omega) \]  

(1.56)

we have to consider \(k^2 < \omega + \omega\) and using:

\[ \lambda = \frac{2(k_x + k_y)}{k} \approx \frac{2(k_x + k_y)}{\sqrt{\omega + \omega}} \]  

(1.57)

we obtain that \(\text{Re} \Sigma_2(p, \omega)\) vanished for \(p - p_F \to 0\). The final form of the real part of the self-energy is:

\[ \text{Re} \Sigma(p, \omega) = \frac{u^2}{8\pi^3} \left[ (\omega + \omega) + (E_F - \omega) \ln \left| \frac{\omega}{E_F} - 1 \right| - \frac{u^2}{2\pi^3} \ln \left| \frac{\omega}{E_F} - 1 \right| \right] \]  

(1.58)

We can see that the divergence in the real part has the form:

\[ \text{Re} \Sigma(p, \omega) = (\omega - E_F) \ln \left| \frac{\omega}{E_F} - 1 \right| \]  

(1.59)

These results have been obtained in the one loop approximation. In the following we will show that this model present a non-Fermi behavior using the renormalization group method.
1.3 “Poor Man’s Renormalization” for a System with Saddle Points in the Energy

1.3.1 Model

Using the ”poor man’s renormalization ” we will present a direct demonstration that the saddle points can give the non-Fermi liquid behavior.

We consider, following Newns et.al. [14] a 2D fermionic system with the energy:

$$\varepsilon_k = k_x k_y$$  \hspace{1cm} (1.60)

where \(|k_x| < k_c, |k_y| < k_c\), with \(k_c\) being the cut-off. If the Fermi level is closed to the van-Hove singularity the nearly 2D fermionic system present two important differences to the normal metal. The first one is that the phase space for scattering is much less restricted that in the conventional metal and the lifetime of the quasiparticles becomes of the order of their energy. Such a fermionic system becomes a MFL. The second point is that the presence of the van-Hove singularity close to the Fermi energy is destroying the nesting of the Fermi surface which gives magnetic instabilities which are competing the Cooper instability. The effective interaction between the quasiparticles has been calculated in ref.[14] as:

$$V_{eff} = \frac{1}{\Pi(q,\omega)}$$  \hspace{1cm} (1.61)

where the polarizability \(\Pi(q,\omega)\) is given by:

$$\Pi(q,z) = \frac{f}{2D} \left[ P(z, \frac{q_x}{q_c}) + P(z, \frac{q_y}{q_c}) \right]$$  \hspace{1cm} (1.62)

where \(f\) is a fraction of the area of the Brillouin zone and \(D = k_c^2\). The function \(P(z, x)\) is defined as:

$$P(z, x) = g(1) - g(-1) - g\left(\frac{1}{x}\right) + g\left(-\frac{1}{x}\right) - g\left(1 - \frac{1}{x}\right) + g\left(-1 + \frac{1}{x}\right)$$  \hspace{1cm} (1.63)

where

$$g(u) = (z + u) \ln |z + u|$$  \hspace{1cm} (1.64)
\[ z = \frac{|\nu|}{\varepsilon_q} \]  
(1.65)

Using in eq. (1.63) the expansion:

\[ \ln x \approx 2x - 1 + x + 1 \]  
(1.66)

we can approximate (1.62) by the expression:

\[ P(z, x) = -\frac{4z}{x^2} + 4 \frac{1}{x} - 2 \frac{x}{x^2} \]  
(1.67)

the effective potential has the form

\[
V_{\text{eff}}(q, \nu) = \begin{cases} 
\frac{Z}{2} \left[ V_0 + A |\nu| \sqrt{D} \right] & \nu < \sqrt{E^* \varepsilon_q} \\
V_c & \nu > \sqrt{E^* \varepsilon_q} 
\end{cases}
\]  
(1.68)

where \( V_0 \) is given as \( V_0 = \frac{4D}{Z\varepsilon}, Z_c \) and \( A \) are constants, and \( V_c \) is the saturation value of \( V_{\text{eff}}(q, \nu) \) in the high frequency region and \( E^* = [fV_c - Z\varepsilon V_0]^2 / D \).

### 1.3.2 Self-Energy and the Renormalization Constant \( Z \)

We start with the general expression for the self-energy:

\[
\Sigma(k, \omega) = i \int \frac{d^2q}{(2\pi)^2} \int \frac{d\nu}{2\pi} V_{\text{eff}}(q, \nu) G_0(k - q, \omega - \nu)
\]  
(1.69)

where \( G_0 \) is the Green function for the free fermionic particles and \( V_{\text{eff}} \) is given by eq.(1.61).

The integral over \( \nu \) will be written as:

\[
I(k - q, \omega) = i \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{V_c}{\omega - \nu - \varepsilon_{k-q} + i\delta}
\]

\[
+ i \int_{-\nu_0}^{\nu_0} \frac{d\nu}{2\pi} \frac{V_{1-V_c}}{\omega - \nu - \varepsilon_{k-q} + i\delta}
\]

\[
+ i \int_{-\nu_0}^{\nu_0} \frac{d\nu}{2\pi} \frac{1}{\sqrt{\varepsilon_q}} \frac{1}{\omega - \nu - \varepsilon_{k-q} + i\delta}
\]

where \( \nu_0 = \sqrt{E^* \varepsilon_q} \) and \( E^* \) is the matching energy. In order to calculate \( Z \) the renormalization will be performed only on the real part of the self-energy \( \Sigma^R(k, \omega) = \text{Re} \Sigma(k, \omega) \) and we will be interested only in the contribution:
\[ \text{Re} \, I(k - q, \omega) = -V_c \theta (\nu_0 - |\omega - \varepsilon_{k-q}|) \]

\[ + (V_1 - V_c) \theta (\nu_0 - |\omega - \varepsilon_{k-q}|) \]

\[ + A \frac{|\omega - \varepsilon_{k-q}|}{\sqrt{\varepsilon_q}} \theta (\nu_0 - |\omega - \varepsilon_{k-q}|) \]  

where \( \theta(x) \) is the step function.

Using this result the real part of the self-energy (1.69) is:

\[ \Sigma^R(k, \omega) = \Sigma^R_1(k, \omega) + A \int \frac{d^2q}{(2\pi)^2} \frac{|\omega - \varepsilon_{k-q}|}{\sqrt{\varepsilon_q}} \theta (\nu_0 - |\omega - \varepsilon_{k-q}|) \]  

(1.72)

where \( \Sigma^R_1(k, \omega) \) gives no contribution to the renormalized part. In order to calculate the renormalization constant \( Z \) we will use the "poor man’s renormalization" taking \( \varepsilon_q \) as shown in Fig.1.1.

If we introduce near \( k_c \) the cut-off \( \Lambda \) from eq. (1.72) we calculate \( d\Sigma^R \) as:

\[ d\Sigma^R(\Lambda, k, \omega) = A \frac{d\Lambda}{\Lambda} - \omega + A v_F k \frac{d\Lambda}{\Lambda} \]  

(1.74)

where:

\[ d\Sigma^R(\Lambda, k, \omega) = \Sigma^R(\Lambda, k, \omega) - \Sigma^R(\Lambda - d\Lambda, k, \omega) \]  

(1.75)

In order to obtain the general equation of the renormalized quantities we write the propagator \( G(\Lambda, k, \omega) \) as:

\[ G(\Lambda, k, \omega) = \frac{Z(\Lambda)}{\omega - v_F(\Lambda)k + i\gamma} \]  

(1.76)

and:

\[ 15 \]
Figure 1.1: The domain of integration for the calculation of self-energy

\[ G(\Lambda, k, \omega) = \frac{Z(\Lambda)}{\omega - v_F(\Lambda) k + \Sigma^R(\Lambda, k, \omega) + i \Sigma^I(\Lambda, k, \omega)} \]  \hspace{1cm} (1.77)

where we approximated \( \varepsilon(k) = v_F k \), \( k \) being the momentum measured from the Fermi surface momentum \( k_F \). From these two equations we get:

\[ dZ(\Lambda) (\omega - v_F(\Lambda) k) - dZ(\Lambda) \Sigma^R(\Lambda, k, \omega) - Z(\Lambda) \delta \Sigma^R(\Lambda, k, \omega) = -dv_F(\Lambda) k \]  \hspace{1cm} (1.78)

and in this equation the second term can be neglected since it given no contribution to the renormalized quantities and it is in fact a constant which can be absorbed in \( \omega \). Therefore eq.(1.78) becomes:

\[ \frac{dZ(\Lambda)}{Z(\Lambda)} (\omega - v_F k) - d\Sigma^R(\Lambda) = -dv_F(\Lambda) k \]  \hspace{1cm} (1.79)

on the other hand if \( d\Sigma^R(\Lambda) \) has the general form:
\[- d\Sigma^R(\Lambda) = C_1 \frac{d\Lambda}{\Lambda} \omega + C_2 v_F k \frac{d\Lambda}{\Lambda} \quad (1.80)\]

we get the general equations:

\[\frac{1}{Z(\Lambda)} \frac{dZ(\Lambda)}{d\Lambda} = \frac{C_1}{\Lambda} \quad (1.81)\]

and

\[\frac{1}{v_F(\Lambda)} \frac{dv_F(\Lambda)}{d\Lambda} = \frac{C_1 - C_2}{\Lambda} \quad (1.82)\]

From the eqs.(1.81)-(1.82) we get:

\[v_F(\Lambda) = 0 \quad (1.83)\]

and

\[Z(\Lambda) = \Lambda^\zeta \quad (1.84)\]

Equation (1.83) shows that the Fermi velocity is constant and Eq.(1.84) shows the important result:

\[\lim_{\Lambda \to 0} Z(\Lambda) = 0 \quad (1.85)\]

with an exponent

\[\zeta = \frac{Z_e A \sqrt{D}}{f k_c} \quad (1.86)\]

and using the relation \(D = k_c^2\) we obtain the exponent

\[\zeta = \frac{Z_e \Lambda}{f} \quad (1.87)\]

which depend only on the constants contained in the effective potential.

### 1.4 Summary of the Results

In the first part of this chapter we performed an analytic calculation of the electronic specific heat of MFL and we showed that the \(T \ln \frac{\omega}{T}\) correction predicted in [3] appears only in the
domain \(0 < \omega < T\). For energies in the domain \(T < \omega < \omega_c\) a supplementary correction depending also on \(\ln \frac{\omega}{T}\) appears.

Following the same method in the nonmarginal case the correction in \(C_V\) is \(T^3 \ln T\) if the electron-electron or electron-phonon interactions are considered for 3D FL model [26]. This behavior makes very difficult the separation of MFL contribution as mentioned by Varma et.al.[1-2]. Recently Coffey and Bedell [27] showed that a 2D Fermi liquid has analytic correction in \(T\) of the form \(\delta C_{V}^{2D} = \gamma_{2D} T^2\) and there is no breakdown of the Fermi liquid in 2D. On the other hand we mention that for the uranium compound \(U_{0.2}Y_{0.8}Pd_3\) [28] was found a \(T \ln T\) correction in the specific heat, a result that shows the existence of the non-Fermi behavior in the real systems.

- In the second part of this chapter we showed analytically that the 2D electronic system with hyperbolic dispersion exhibits the marginal behavior \(\text{Im} \sum(p_F, \omega) \sim \omega\) obtained numerically in [24]. The marginal behavior is given by the saddle points of the Fermi surface as was predicted in [23]. We have to mention that the marginal phenomenological model is not restricted to the 2D Fermi systems but the cuprates superconductors are (quasi)2D systems.

- Using the "poor man’s renormalization" method we showed in the last part of this chapter the occurrence of the MFL behavior for 2D fermionic systems with saddle points in the energy. Finally we mention that the NFL behavior is typically for the 1D fermionic systems (Luttinger liquid). The 2D fermionic systems with saddle points in the energy is very realistic for the cuprates superconductors, but cannot be treated analytically without renormalization group methods. More than that even the NFL models based on the coupling to a gauge field [29] or on the superlong range interaction [30] has been treated by this method.
Bibliography

[1] C.M.Varma, P.B. Littlewood, S. Schmitt-Rink, E. Abrahams, A.E. Ruckenstein
*Phys.Rev.Lett.* **63** 1996 (1989)

[2] G. Kotliar, E. Abrahams, A.E. Ruckenstein , C.M. Varma, P.B. Littlewood, S. Schmitt-Rink
*Europhys.Lett.* **15** 655 (1991)

[3] P.B. Littlewood, J. Zaaneu, G. Apple, H. Monien *Phys.Rev.* **B48** 487 (1993)

[4] M.L. Horbach, F.L.J. Vos and W.van Sarlos *Phys.Rev.* **B48** 4061 (1993)

[5] C. Bendle, P.Hertel, J. Apple *Phys.Rev.* **B45** 8062 (1992)

[6] M.Y. Reizer *Phys.Rev.* **B40** 11571 (1984)

[7] P.W. Anderson *Physica* **2** 1 (1965)

[8] R. Balian, D.R. Fredkin *Phys.Rev.Lett.* **15** 480 (1965)

[9] N.F. Berk, J.R. Schriffer *Phys.Rev.Lett.* **17** 433 (1966)

[10] D.J. Amit, J.W. Kane, H. Wegner *Phys.Rev.* **175** 313 (1968)

[11] G.M. Carneiro, C.P. Petrick *Phys.Rev.* **B37** 442 (1988)

[12] R. Chandhury *Can.J.Phys.* **73** 497 (1995)

[13] F. Steglich in Material and Mechanisms of Superconductivity (*North Holland Amsterdam*)(1998)

[14] D.M.Newns, H.R.Krishamurthy, P.C. Patnaik, C.C. Tsuei, C.L.Kane *Phys.Rev.Lett.* **69** 1264 (1992)
[15] A. Millis, H. Monien, D. Pines Phys.Rev. B42 167 (1990) see also B.P. Stjikovic, D. Pines Phys.Rev.Lett. 76 811 (1996)

[16] I. Grosu, M. Crisan Phys.Rev. B49 1296 (1994)

[17] H.H. Amin, P.C.E. Stamp Phys.Rev.Lett. 77 3017 (1996)

[18] P. Bernard, L. Chen, A-M.S. Tramblay Phys.Rev. B47 15127 (1989)

[19] M. Crisan, L. Tataru Phys.Rev. B54 1 (1996)

[20] J. Friedel J. Phys. (Paris) 48 1787 (1987)

[21] J. Labbe, J. Bok, EuroPhys.Lett. 2 1225 (1987)

[22] D.M.Newns, P.C. Pattnaik, C.C. Tsuei Phys.Rev. B43 3075 (1991)

[23] P.C. Pattnaik, C.L. Kane, D.M. Newns, C.C. Tsuei Phys.Rev. B45 5714 (1992)

[24] S. Gopalan, O. Gunnarson, O.K. Andersen Phys.Rev. B46 11798 (1992)

[25] J. Gonzales, F. Guinea EuroPhys.Lett. 34 711 (1996)

[26] D.Coffey, C.P. Petrick Phys.Rev. B37 442 (1988)

[27] D. Coffey, K.S. Bedell preprint 590-1993 Los Alamos Centre for Materials Science

[28] B. Andraka, A.M. Tsvelik Phys.Rev.Lett. 67 2886 (1991)

[29] S.Chakravarty, R.E. Norton, O.F. Syljnasen Phys.Rev.Lett. 74 1423 (1995)

[30] P.M. Bares, X.G. Wen Phys.Rev. B48 8636 (1993)
Chapter 2

Anderson non-Fermi Model

The microscopic description of the superconducting state in cuprates superconductors is a very difficult problem because at the present time it is generally accepted that in the normal state the elementary excitations are not describe by the Fermi liquid theory. However using the BCS-like pairing model the Gorgov equations have been applied to describe the superconducting state in the hypothesis that the normal state is a non-Fermi liquid described by the Anderson model [1]. The superconducting state properties have been discussed by different authors [2-8] and even if these descriptions are phenomenological, it can be a valid starting point for a microscopic model. Recent experimental data (ARPES) showed that these materials presents even more remarkable deviations from the Fermi-liquid behavior due to the occurrence of the pseudogap at the Fermi-surface.

The occurrence of the pseudogap has been explained using different concepts such as the spin fluctuation [9], preformed pairs [10], SO(5) symmetry [11], spin-charge separation [12], and the fluctuation of the order parameter induced pseudogap [13].

2.1 The Model

The non-Fermi liquid behavior of the normal state for the cuprates superconductors proposed by Anderson [1] was developed by different authors in order to describe the superconducting state in the framework of the BCS theory. In the normal state the electrons are describe by the Green function:
\[ G(k, i\omega_n) = \frac{g(\alpha)e^{i\phi}}{\omega_c^\alpha (i\omega_n - \varepsilon_k)^{1-\alpha}} \]  

(2.1)

where 0 < \alpha < 1 and \( g(\alpha) = \pi \alpha / 2 \sin \left( \frac{\pi \alpha}{2} \right) \) and \( \phi = -\pi \alpha / 2 \). The Green function given by eq.(2.1) was given by Abrahams [14] and used by Yin and Chakravarty [15] to study the non-Fermi liquid superconductors. However, the time reverse symmetry is violated unless \( \phi = -\pi \alpha / 2 \). This is because of the invariance with respect to \( \omega \to -\omega \) and \( \varepsilon_k \to -\varepsilon_k \). When \( \varepsilon_k = 0 \), the spectral function is similar to that of Luttinger liquid without spin-charge separation in \( d = 1 \) [16] but is very different when \( \varepsilon_k \neq 0 \). To preserve the equal-time anticommutation relation of the fermions, the spectral function must satisfy the sum rule \( \int_{-\infty}^{\infty} d\omega A(k, \omega) = 1 \), from which relation it can be found the value of \( g(\alpha) \) given above. With the above value of \( g(\alpha) \) the anticommutation relations will be satisfied if \( |\varepsilon_k| < \omega_c \), which is precisely the regime in which a scaling theory is appropriate. The density of states can be calculated from the spectral function [15]. In contrast to the Luttinger liquid the density of states does not vanished as \( \omega \to 0 \). In fact it is unchanged from the Fermi liquid value as \( \alpha \to 0 \). It would agree exactly with the Fermi liquid value, if \( \omega_c \) is chosen to be \( W \) (the band width).

The free action for the model is:

\[ S = \int d\omega d^2 p G^{-1}(p, \omega) \Psi_{p,\sigma}^+(\omega) \Psi_{p,\sigma}(\omega) \]  

(2.2)

where the Green function \( G \) is given by eq.(2.1). If we follow Shankar [17] we see that the four fermions interaction is irrelevant. The spectral anomaly is more stable than the Fermi liquid. In fact in the weak coupling regime it does not even allow a superconducting instability [15]. The coupling has to reach a threshold before superconducting instability occurs.

### 2.2 A Novel Feature of the non-Fermi Behavior. The Occurrence of the Pseudogap.

The recent angle-resolved photoemission spectroscopy (ARPES) experiments confirmed the occurrence of pseudogap in the density of states of the electronic excitations from the normal and superconducting phase of the cuprate superconductors.

Many theoretical approaches and Monte-Carlo simulations [18-19] of strongly correlated electron systems has been performed in order to explain the experimental data. Some common fea-
ture are observed in many of these models, the most important being the fact that the origin of the pseudogap is the interaction between the electronic excitations and fluctuations. This idea leads to the hypothesis that the pseudogap appears in the proximity of a quantum phase transition (QPT) [20-21].

A consistent picture is emerging from the study of the Hubbard model in 2D and 3D using Monte-Carlo calculations and the analytically non-perturbative many-body approach performed by Tremblay group. The main results obtain by authors in Ref.[22-25] is the occurrence of a temperature independent pseudogap for a 2D Hubbard model taking the interaction between the electrons and the renormalized classical fluctuations. In this regime the coherence length is increasing with decreasing of the temperature and a good approach for the model is a QPT.

On the other hand the spin fluctuations are present in both normal and superconducting phase of the cuprates. The Fermi surface evolution and the temperature dependence of the pseudogap have been studied by Chubukov [26-27] using the phenomenological model proposed by Millis Monien and Pines [28] (MMP).

In this section we will perform an analytic calculation of the pseudogap given by the interaction between electrons and magnetic fluctuations. Starting such a problem we expect that a tractable approach will be only one-loop diagram for the self-energy. This approximation does not take the vertex corrections which are important because in such a system probably the Migdal theorem is not valid. However, even such a simple approximation can gives us an idea if the interaction between electrons and spin fluctuations are good candidate to explain pseudogap.

The self-energy is essential determined in the model by the dynamic susceptibility $\chi(k,\omega)$ which will be discussed below. Such a discussion is also usefully because at the present time there are models with $\chi(k,\omega)$ pure diffusive [29] or $\chi(k,\omega)$ describing overdamped fluctuations. Recently it was presented a model with both contributions in $\chi(k,\omega)$ [30].

### 2.2.1 Dynamic Susceptibility

We will consider that the electronic system interacts with magnetic excitations near a quantum critical point (QCP) determined by the condition $J(Q) = -\chi_0^{-1}(Q)$, where $J(Q)$ is the magnetic interaction and $\chi_0^{-1}(Q)$ is the bare susceptibility in $Q = (\pm\pi, \pm\pi)$. The general form for $\chi(q,\omega)$ describing the overdamped antiferromagnetic fluctuations is:
\[ \chi(q, \omega) = \frac{\chi_0(Q)}{\delta + a^2q^2 - i\gamma\omega} \]  

(2.3)

where \(a\) is of the order of the lattice spacing, \(\delta\) is a parameter characterizing the distance from the QCP and \(\gamma = \frac{1}{\omega_{SF}}\), \(\omega_{SF}\) being the characteristic energy of the spin fluctuations. The renormalization group method (RNG) has been applied to the analysis of the phase transition of the itinerant electronic system by Millis [31] taking into consideration the dynamic critical exponent \(z\). The correlation time of the order parameter \(\tau\) and the correlation length \(\xi\) are connected by:

\[ \tau = \xi^z \]  

(2.4)

which shows that \(\tau\) diverges more than \(\xi\). From eqs.(2.3) and (2.4) we get:

\[ \xi^2 = \frac{a^2}{\delta} \]  

(2.5)

\[ \tau = \frac{1}{\gamma\delta} = \frac{\omega_{SF}}{\delta} \]  

(2.6)

so the \(z = 2\), one of the most important predictions obtained in the weak coupling approach. At \(T = 0\) (i.e. for the quantum phase transition) the static and dynamics are mixed in contrast with the case \(T > 0\) where the dynamic behavior affects the static behavior.

The presence of the particle-hole continuum leads to an overdamping of the modes associated with the order parameter. For the superconductors the critical behavior is in the \(z = 2\) universality class. However Pines [32] showed that the phase diagram of the cuprates superconductors is more complicated. The normal phase was divided in two regions: one with \(z = 2\) \((\omega_{SF} \sim \xi^{-2}\) which has no pseudogap and the region with \(z = 1\) which present a weak pseudogap regime \((\omega_{SF} \sim \xi^{-1}, \xi^{-1} = a + bT)\) and another presenting a strong pseudogap regime \((\omega_{SF} \sim \xi^{-1}, \xi^{-1} = \text{const})\) separated by a line \(T^*(x)\) where \(x\) is the doping. The crossover between \(z = 1\) and \(z = 2\) regime was studied by Sachdev et.al. [33] using the scaling analysis in the framework of \(\sigma\) model. The main result is that fermions do not necessary overdamped and destroy the spin waves. At low damping the crossover is the same as in pure \(\sigma\) mode. The difference is for a finite damping of fermions in the quantum disorder regime where the energy scale can drive a crossover from high temperature quantum critical regime \(z = 1\) to low temperature quantum critical regime \(z = 2\).
In this section we will consider for $\chi(q, \omega)$ the expression:

$$
\chi(q, \omega) = \frac{\chi_0(Q)}{\delta + a^2 q^2 - i \gamma \omega - \omega^2 \Delta^2}
$$

where $\Delta$ is the gap in the spin wave spectrum.

### 2.2.2 Self-Energy of the Electronic Excitations

In the one loop approximation the self-energy of the 2D electronic excitations interacting with a bosonic mode has the form:

$$
\Sigma(p, \omega) = g^2 \int \frac{d^2 q}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \text{Im}\chi(q, \omega') \frac{\coth \frac{\omega'}{2T} - \tanh \frac{\epsilon_p + \omega + Q}{2T}}{\omega + \omega' - \epsilon_p + \omega + Q + i\delta}
$$

where $g$ is the coupling constant, $\chi(q, \omega')$ the dynamic susceptibility, $\epsilon_p = \frac{p^2}{2m} - \mu$ and $\mu$ the chemical potential.

There are two limiting cases for the calculation of the self-energy. The Fermi-liquid regime appears for $\omega_{SF} \gg T$ and the non-Fermi liquid regime in the opposite (renormalized classical) regime $\omega_{SF} \ll T$ [25]. Perhaps the best known characteristic of a Fermi liquid is that $\text{Im}\Sigma(k_F, \omega, T = 0) \sim \omega^2$ and $\text{Im}\Sigma(k_F, \omega = 0, T) \sim T^2$.

The key to the understanding the Fermi liquid versus non-Fermi liquid regime is in the relative width in frequency of $\chi''(q, \omega')/\omega'$ versus the width of the combined Bose and Fermi functions. In general the function $n(\omega') + f(\omega + \omega')$ ($n(\omega)$ the Bose-Einstein function, $f(\omega)$ the Fermi-Dirac function) depends on $\omega'$ on a scale $\text{Max}(\omega, T)$ while far from the phase transition the explicit dependence of $\chi''(q, \omega')/\omega'$ is on the scale $\omega_{SF} \sim E_F \gg T$. Hence in this case we can assume that $\chi''(q, \omega')/\omega'$ is constant in the frequency range over which $n(\omega') + f(\omega + \omega')$ differs from zero. Hence we can approximate our expression for the imaginary part of the self-energy with:

$$
\text{Im}\Sigma(p_F, \omega) \cong -g^2 A(p_F) \int \frac{d\omega'}{\pi} \left[ \coth \frac{\omega'}{2T} + \tanh \frac{\omega + \omega'}{2T} \right] \omega'
$$

$$
= -g^2 A(p_F) \left[ \omega^2 + (\pi T)^2 \right]
$$

where:

$$
A(p_F) = \int \frac{d^2 q}{2\pi} \frac{\chi''(q, \omega')}{\omega'}
$$
As we can see in this regime $Im \Sigma(\omega) \sim \omega^2$, so the system behaves as a Fermi liquid. Near an antiferromagnetic phase transition, the spin fluctuation energy is much smaller than the temperature. This is the renormalized classical regime. The condition $\omega_{SF} \ll T$ means that $\chi''(q, \omega')$ is peaked over a frequency interval $\omega' \ll T$ much narrow than the interval $\omega' \sim T$ over which $n(\omega') + f(\omega + \omega')$ changes. This situation is the opposite of that encountered in the Fermi liquid regime. To evaluate $Im \Sigma$ the Fermi factor can now be neglected compared with classical limit of the Bose factor $\frac{T}{\omega}$. Evaluating the integral we obtain the exact value obtained using classical fluctuations [25].

Neglecting the fermionic contribution and using the expansion:

$$\epsilon_{p+q+Q} \cong \epsilon_{p+Q} + v q \cos \theta$$  \hspace{1cm} (2.11)

and Schwinger representation for the propagator:

$$\frac{1}{i} \int_0^\infty dt e^{i(\omega - \epsilon + i\alpha)t} = \frac{1}{\omega - \epsilon + i\alpha}$$  \hspace{1cm} (2.12)

we can write the general form for the self-energy. Let us introduce the structure factor:

$$S(q, T) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \text{Im} \chi(q, \omega') \coth \frac{\omega'}{2T}$$  \hspace{1cm} (2.13)

which describe the behavior of the bosonic fluctuations on different energy scales. Using (2.12) and (2.13) the real and imaginary part $\Sigma' = Re \Sigma$ respectively $\Sigma'' = Im \Sigma$ can be calculated from eq. (2.8) as:

$$\Sigma'(p, \omega) = \frac{g^2}{2\pi} \int_0^\infty dt \sin[(\omega - \epsilon_{p+Q})t] \int_0^\infty dq dqq S(q, T) \int_0^{2\pi} d\theta \frac{d\theta}{2\pi} e^{-ivt\cos \theta}$$  \hspace{1cm} (2.14)

$$\Sigma''(p, \omega) = -\frac{g^2}{2\pi} \int_0^\infty dt \cos[(\omega - \epsilon_{p+Q})t] \int_0^\infty dq dqq S(q, T) \int_0^{2\pi} d\theta \frac{d\theta}{2\pi} e^{-ivt\cos \theta}$$  \hspace{1cm} (2.15)

where $v = \frac{p}{m}$. The integral over $\theta$ can be expressed by the Bessel function:

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-ix\cos \theta}$$  \hspace{1cm} (2.16)

and eqs.(2.14) and (2.15) becomes:

$$\Sigma'(p, \omega) = \frac{g^2}{2\pi} \int_0^\infty dt \sin[(\omega - \epsilon_{p+Q})t] \int_0^\infty dq dqq S(q, T) J_0(qvt)$$  \hspace{1cm} (2.17)
\[ \Sigma''(p, \omega) = -\frac{g^2}{2\pi} \int_0^\infty dt \cos[(\omega - \varepsilon_{p+Q})t] \int_0^\infty dq q S(q, T) J_0(qvt) \] (2.18)

These general expressions will be used to calculate the pseudogap in the electronic energy.

### 2.2.3 Pseudogap in the Electronic Energy Spectrum

In this section we calculate the correction to the self-energy due to the interaction of the electrons with the spin fluctuations for \( \omega \ll T \) using eq.(2.7) for dynamic susceptibility.

From this equation we get:

\[ \text{Im}\chi(q, \omega) = \frac{\chi_0(Q)\gamma\omega}{(\delta + a^2q^2 - \frac{\omega^2}{\Delta^2})^2 + \gamma^2} \] (2.19)

and the structure factor will be approximated as:

\[ S(q, T) = \frac{T\gamma\chi_0(Q)}{\pi} \int_{-\infty}^\infty \frac{d\omega}{(A^2 - \frac{\omega^2}{\Delta^2})^2 + \gamma^2\omega^2} \] (2.20)

where \( A = \delta + a^2q^2 \) as we approximated \( \coth \frac{\omega}{2T} \simeq \frac{2T}{\omega} \). The integral in eq. (2.20) can be performed analytically in the approximation:

\[ \frac{\Delta}{\omega_{SF}} > 2\left(\delta + a^2q^2\right) \] (2.21)

as:

\[ S(q, T) = \frac{T\chi_0(Q)}{\delta + a^2q^2} \] (2.22)

From equations (2.17) and (2.18) using the result:

\[ \int_0^\infty \frac{J_0(qvt)qdq}{\delta + a^2q^2} = \frac{1}{a^2}K_0(tv/\xi) \] (2.23)

and:

\[ \int_0^\infty dx K_0(\beta x) \sin(\alpha x) = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \ln \left| \frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2} + 1} \right| \] (2.24)

\[ \int_0^\infty dx K_0(\beta x) \cos(\alpha x) = \frac{\pi}{\sqrt{\alpha^2 + \beta^2}} \] (2.25)
we obtain the real and imaginary part of the self-energy as:

$$\Sigma'(p, \omega) = \frac{g^2 T \chi_0(Q)}{2 \pi a} \frac{1}{\sqrt{(\omega - \varepsilon_{p+Q})^2 + (\frac{\chi}{T})^2}} \ln \frac{\omega - \varepsilon_{p+Q} + \sqrt{(\omega - \varepsilon_{p+Q})^2 + (\frac{\chi}{T})^2}}{\omega - \varepsilon_{p+Q} - \sqrt{(\omega - \varepsilon_{p+Q})^2 + (\frac{\chi}{T})^2}}$$ (2.26)

$$\Sigma''(p, \omega) = -\frac{g^2 T \chi_0(Q)}{4 \pi a^2} \frac{1}{\sqrt{(\omega - \varepsilon_{p+Q})^2 + (\frac{\chi}{T})^2}}$$ (2.27)

These results are identical with the expressions given by Vilk and Tremblay [25] using for the dynamic susceptibility a simple form $$\chi(q, 0) = (\xi^{-2} + q^2)^{-1}$$. The identity of the results is given by the fact that the details of dynamics are unimportant as long as it does not change the wave vector dependence.

For the case $$|\omega - \varepsilon_{p+Q}| > \frac{\gamma}{\xi}$$ the eqs.(2.26) and (2.27) can be approximated as:

$$\Sigma'(p, \omega) = \frac{g^2 T \xi}{\delta} \frac{\ln(|\omega - \varepsilon_{p+Q}| \xi/v)}{\omega - \varepsilon_{p+Q}}$$ (2.28)

$$\Sigma''(p, \omega) = -\frac{g^2 T \xi}{\omega - \varepsilon_{p+Q}}$$ (2.29)

At this point we mention that the Green function $$G^{-1}(p, \omega) = G_0^{-1}(p, \omega) - \Sigma(p, \omega)$$ will present a correct behavior ( $$G(\omega) \sim 1/\omega$$ ) if $$\Sigma(\omega) \sim 1/\omega$$ if we have:

$$T \ln \frac{\omega_x \xi}{v} = c$$ (2.30)

where $$c$$ is a constant and $$\omega_x$$ is the characteristic energy which is in agreement with the approximations satisfy $$\omega_x \ll T$$. From eq.(2.30) we get:

$$\xi(T) = \frac{v}{\omega_x} \exp \left( \frac{c}{T} \right)$$ (2.31)

and $$\omega_x = \delta U \ll T$$, define the proximity to the phase transition. This simple procedure can be regarded as phenomenology of the Tremblay renormalized classical regime of the fluctuations obtained in the self-consistent way, and gives as a main result a temperature independent for the pseudogap.

However, if we consider the phenomenology proposed by Millis, Monien and Pines and take $$z = 1$$ the pseudogap obtained from eq. (2.28) as:

$$\Delta_{pg}^2 = T \ln \xi$$ (2.32)
will present a temperature dependence for the pseudogap.

### 2.2.4 Effect of Anisotropy

The experimental data showed a highly anisotropic gap. The effect of the interplane anisotropy has been considered by Preosti et al. [34]. Following the method developed in previous section we will show that there is a temperature dependence of the pseudogap if the anisotropy is considered. In this case the structure factor will be taken as:

\[
S_a(q, q_z T) = \frac{T \chi_0(Q)}{\xi^2} \frac{1}{\xi^{-2} + q^2 + \gamma^2 q_z^2}
\]

where \(q^2 = q_x^2 + q_y^2\), \(\gamma = \xi_z / \xi_{xy}\) and \(\xi_{xy} = \xi\). From eq. (2.17) we obtain:

\[
\Sigma'(p, \omega) = \frac{g^2}{2\pi} \int_0^\infty dt \sin[(\omega - \varepsilon_{p+Q})t] \int_0^{q_c} dq q \int_0^{q_c} dq z \frac{S_a(q, q_z, T) J_0(q vt)}{q_{BZ}}
\]

In this expression we perform first the integral over variable \(t\), which is different from zero only if:

\[
|\omega - \varepsilon_{p+Q}| > vq
\]

and we get the result:

\[
\Sigma'(p, \omega) = \frac{g^2}{2\pi} \int_0^{q_c} dq q \int_0^{q_c} dq z \frac{S_a(q, q_z, T)}{q_{BZ} \sqrt{(\omega - \varepsilon_{p+Q})^2 - v^2 q^2}}
\]

which will be written using approximation (2.35) as:

\[
\Sigma'(p, \omega) = \frac{\Delta_{pg}^2(T, \gamma)}{(\omega - \varepsilon_{p+Q})}
\]

where:

\[
\frac{\Delta_{pg}^2(T, \gamma)}{\Delta_{pg}^2(0)} \frac{T_0}{T} = \frac{q_c}{q_{BZ}} A(\xi, \gamma)
\]

\[
T_0 = \frac{2\pi \xi^2}{\chi_0(Q)} \Delta_{pg}(0)
\]
Figure 2.1: Temperature dependence of the pseudogap $\Delta(T)$ for $\gamma = 0.9$ and $\xi^{-1}(T) = \exp[100T_0/T]$

$$A(\xi, \gamma) = \sqrt{1 + (\xi q_c)^2} \arctan \frac{\gamma}{\sqrt{1 + (\xi q_c)^2}} - \frac{\arctan \gamma \xi q_c}{\gamma q_c} + \frac{1}{2} \ln \frac{1 + (\gamma \xi q_c)^2}{1 - (\gamma \xi q_c)^2}$$

(2.40)

We can take a soft cutoff taking $\chi_0(Q) \rightarrow \chi_0(Q)[1 - q_c^{-2}(q^2 + \gamma q_z^2)]$ and the equation for the pseudogap has the form:

$$\frac{\Delta_{pg}^2(T, \gamma)}{\Delta_{pg}^2(0)} \frac{T_0}{T} = \frac{q_c}{q_{BZ}} \left[ 1 + (\xi q_c)^2 \right] A(\xi, \gamma) - \frac{1}{2}$$

(2.41)

If we define the temperature $T^*(\gamma)$ by:

$$\xi(T^*(\gamma)) \rightarrow \infty$$

(2.42)

we can show that:
Figure 2.2: Temperature dependence of the pseudogap $\Delta(T)$ for $\xi^{-1}(T) = 10^{-3} + 10^{-2}T/T_0, \gamma = 0.9$ respectively $\gamma = 1$.

\[
\lim_{\gamma \to 0} T^*(\gamma) = 0 \quad (2.43)
\]

We have to mention that there is an important difference between this result and the result from Ref.[34]. In this model $\xi$ is the correlation length of the magnetic fluctuations, differently from Ref.[34] where $\xi$ is the superconducting correlation length. The condition (2.42) is equivalent to the proximity of the magnetic quantum phase transition. Even if this model is realistic taking into consideration the magnetic fluctuations the matching between the pseudogap and superconducting gap was not solved. An important point of the model is the temperature dependence of the magnetic correlation length. In Fig.2.1 we present the temperature dependence of the pseudogap obtained from eq. (27) and $\xi^{-1}(T) = 10^{-3} + 10^{-2}T$. Using $\gamma = 0.9$ and $\xi^{-1} = 10^{-2} + 10^{-3}T/T_0$ we obtained a dependence of the pseudogap given by eq. (2.38) in Fig.

31
2.3 Electron-Fluctuation Interaction in a non-Fermi Liquid Superconductor

In this section we consider the model introduced in [1] and describe in the first section of this chapter. We consider that the superconducting state appears due to an attractive interaction and is described by the BCS-like order parameter $\Delta_k$ which can be calculated from the Gorkov equations. The fluctuations of this parameter can interact with the electrons and the fermionic spectrum of the elementary excitations changes. Such an effect has been studied in the BCS superconductors by different authors [25], [35] and it was showed that this interaction gives a contribution to the density of states for $T > T_c$ which explained the behavior of the tunnelling measurements.

For a superconductor described by the Gorkov-like equations with the normal state described by eq. (2.1) the propagator of the fluctuations has the expression:

$$D(q, \omega_n) = \frac{1}{V^{-1} + \Pi(q, \omega_n)} \quad (2.44)$$

where $V$ is the attractive interaction between the electrons and $\Pi(q, \omega_n)$ is the polarization operator defined as:

$$\Pi(q, \omega_n) = T \sum_{\omega_l} \int \frac{d^2p}{(2\pi)^2} G(p, \omega_l) G(q-p, \omega_n - \omega_l) \quad (2.45)$$

where $G(p, \omega_l)$ is the Green function related to electrons, which in terms of Dyson equation has the following for:

$$G^{-1}(p, \omega_l) = G_0^{-1}(p, \omega_l) - \Sigma(p, \omega_l) \quad (2.46)$$

where the self-energy is given by:

$$\Sigma(p, \omega_l) = -T \sum_{\omega_l} \int \frac{d^2q}{(2\pi)^2} D(q, \omega_n) G(q-p, \omega_n - \omega_l) \quad (2.47)$$

Equations (2.44)-(2.47) has to be solved self-consistently, but this cannot be done analytically. However, in the mode coupling approximation it can be done and we can calculate the new
energy of the electronic excitations. We mention that in eq. (2.47) the vertex corrections have been neglected in order to developed a simple analytical model.

2.3.1 Mode Coupling Approximation

In this approximation we consider first that $G(p, i\omega_n) \approx G_0(p, i\omega_n)$ and from eq. (2.45) we define the polarization

$$\Pi_0(q, i\omega_m) = \int \frac{d^2k}{(2\pi)^2} S(k, q, i\omega_m)$$

(2.48)

where:

$$S(k, q, i\omega_m) = (-1)^{1-\alpha} T \sum_{\omega_n} \frac{g^2(\alpha)\omega_n^{-2\alpha}}{(i\omega_n - \varepsilon_k)^{1-\alpha}(i\omega_n - i\omega_m - \varepsilon_{q-k})^{1-\alpha}}$$

(2.49)

We performed the analytical calculation of $\Pi_0(q, i\omega_m)$ given by eq. (2.48) and from the eq. (2.44) the propagator for the order parameter fluctuations $h$ as been obtained as:

$$D_0^{-1}(q, i\omega_n) = N(0) A(\alpha) \left\{ C(\alpha) \left[ \left( \frac{T}{\omega_c} \right)^{2\alpha} - \left( \frac{T_c}{\omega_c} \right)^{2\alpha} \right] 
+ \frac{i\omega_n(1-\alpha)}{T} M(\alpha, T, \omega_c, \omega_D) \right. 
+ \frac{v_F g}{2T} \left( 1 - \alpha \right)^2 N(\alpha, T, \omega_c, \omega_D) \left\} \right.$$

(2.50)

where the critical temperature has been obtained as [7]:

$$T_c^{2\alpha} = \frac{1}{C(\alpha)} \left[ D(\alpha) \omega_D^{2\alpha} - \frac{\omega_c^{2\alpha}}{A(\alpha) N(0) V} \right]$$

(2.51)

and the constants from eqs.(2.50) and (2.51) are:

$$A(\alpha) = g^2(\alpha) \frac{\omega_c^{2\alpha}}{\pi} \sin \pi(1-\alpha)$$

(2.52)

$$C(\alpha) = \Gamma^2(\alpha) [1 - 2^{1-2\alpha}] \zeta(2\alpha)$$

(2.53)

$$D(\alpha) = \frac{\Gamma(1-2\alpha) \Gamma(\alpha)}{2\alpha \Gamma(1-\alpha)}$$

(2.54)
\[ M \left( \alpha, \frac{T \omega_D}{\omega_c}, \frac{\omega_D}{\omega_c} \right) = \frac{\Gamma(\alpha - 1) \Gamma(\alpha - 1/2)}{2\sqrt{\pi}} \left[ 1 - 2^{2-2\alpha} \right] \zeta(2\alpha - 1) \left( \frac{T}{\omega_c} \right)^{2\alpha} \]

\[ - \frac{B(2 - 2\alpha, \alpha - 1)}{2(2\alpha - 1)} \left( \frac{\omega_D}{\omega_c} \right)^{2\alpha-1} \left( \frac{T}{\omega_c} \right) \]  

(2.55)

and:

\[ N \left( \alpha, \frac{T \omega_D}{\omega_c}, \frac{\omega_D}{\omega_c} \right) = \left[ \frac{2\Gamma(\alpha - 2) \Gamma(\alpha - 1/2)}{\sqrt{\pi}} + \Gamma^2(\alpha - 1) \right] \]

\[ \times \frac{1 - 2^{3-2\alpha}}{4} \zeta(2 - 2\alpha) \left( \frac{T}{\omega_c} \right)^{2\alpha} \]

\[ - \frac{B(3 - 2\alpha, \alpha - 2)}{4(2\alpha - 2)} \left( \frac{\omega_D}{\omega_c} \right)^{2\alpha-2} \left( \frac{T}{\omega_c} \right)^{2} \]  

(2.56)

\[ B(x, y) = \Gamma(x) \Gamma(y)/\Gamma(x+y) \text{ and } \Gamma(x) \text{ is the Euler function and } \zeta(x) \text{ is the Riemann function.} \]

Using a similar form with one introduced by Schmidt the fluctuation propagator will be written as:

\[ D^{-1}_0 (q, i\omega_n) = N(0) \left[ b(\alpha) \tau(\alpha) + ia(\alpha) \omega_n + \xi^2(\alpha, T) q^2 \right] \]  

(2.57)

where:

\[ \tau(\alpha) = \left( \frac{T}{T_c} \right)^{2\alpha} - 1 \]  

(2.58)

\[ a(\alpha) = M \left( \alpha, \frac{T \omega_D}{\omega_c}, \frac{\omega_D}{\omega_c} \right) \frac{T}{(1 - \alpha)A(\alpha)} \]  

(2.59)

\[ b(\alpha) = A(\alpha) C(\alpha) \left( \frac{T}{\omega_c} \right)^{2\alpha} \]  

(2.60)

and:

\[ \xi(\alpha) = \frac{v_T^2(1 - \alpha)^2}{4T^2} N \left( \alpha, \frac{T \omega_D}{\omega_c}, \frac{\omega_D}{\omega_c} \right) A(\alpha) \]  

(2.61)

In the approximation \( \Sigma \ll \pi T \) the Green function given by eq. (2.46) will be approximated as \( G = G_0 + G_0 \Sigma G_0 \) and \( \Pi \) will be modified by \( \delta \Pi \) also linear in \( \Sigma \). Following Ref.[11] we calculated \( \delta \Pi \) in the "box approximation" as:

34
\[ \delta \Pi = 2T^2 \sum_{\omega_n} \int \frac{d^2p}{(2\pi)^2} G_0^2(p, i\omega_n) G^2(-p, -i\omega_n) \int \frac{d^2q}{(2\pi)^2} D(q, \omega_n = 0) \quad (2.62) \]

where:

\[ D_0^{-1}(q, i\omega_n) = V^{-1} + \Pi(q, i\omega_n) + \delta \Pi(q, i\omega_n) \quad (2.63) \]

The box vertex describes the interaction between fluctuations and becomes important only in the critical regime in the standard superconductors. In order to calculate \( \delta \Pi(q, i\omega_n) \) we introduce:

\[ B_0 = \frac{1}{N(0)} T \sum_{\omega_n} \int \frac{d^2p}{(2\pi)^2} G_0^2(p, i\omega_n) G_0^2(-p, -i\omega_n) \quad (2.64) \]

where \( N(0) = \frac{m}{2\pi} \) is the density of states. If we used for the electronic Green function eq.(2.1) we obtain:

\[ B_0(T) = \frac{B(1/2, 3/2 - 2\alpha)}{\pi} \frac{[2^{3-4\alpha} - 1]}{2^{3-4\alpha}} \frac{\zeta(3 - 4\alpha)}{(\pi T)^{2-4\alpha}} \frac{\omega_c^{-1\alpha}}{g^4(\alpha)} \quad (2.65) \]

If we introduce \( \bar{\tau}(\alpha) = \tau(\alpha) + \delta \Pi/N(0) \) the fluctuation propagator given by eq. (2.57) will be:

\[ D^{-1}(q, i\omega_n) = b(\alpha)\bar{\tau}(\alpha) + ia(\alpha)\omega_n + \xi^2(\alpha, T) q^2 \quad (2.66) \]

where:

\[ \bar{\tau}(\alpha) - \tau(\alpha) = \frac{2B_0(T)}{N(0)} T \int \frac{d^2q}{(2\pi)^2} \frac{1}{\bar{\tau}(\alpha) + \xi^2(\alpha, T) q^2} \quad (2.67) \]

If we perform this integral taking the upper limit \( q_M = 1/\xi(\alpha, T) \) from eq.(2.64) we get:

\[ \bar{\tau}(\alpha) - \tau(\alpha) = \frac{B_0(T)T}{2\pi N(0)\xi^2(\alpha, T)} \ln \frac{1 + \bar{\tau}(\alpha)}{\xi^2(\alpha, T)} \quad (2.68) \]

For realistic parameters ( \( T_c = 100K, \omega_c = 200K \) ) the deference \( \bar{\tau}(\alpha) - \tau(\alpha) \) becomes important only near a critical value of \( \alpha \) defined by \( \xi(\alpha_c) = 0 \). In the BCS limit ( \( \alpha = 0 \) ) this parameter is small and this behavior can be associated with the occurrence of the preformed pairs in the domain \( T_c < T < T^* \), controlled by \( \alpha \). This behavior is in fact due to the occurrence of a pseudogap in the electronic excitations.

35
2.3.2 Electronic Self-Energy

The self energy due to the interaction between electrons and fluctuations is given by eq.(2.47) where \( D(q, i\omega_n) \) is given by eq.(2.66). First we calculate the summation over Matsubara frequencies \( \omega_n \):

\[
S = T \sum_{\omega_l} D(q, i\omega_n) G(q - p, i\omega_n - i\omega_l)
\]

(2.69)

where \( D(q, i\omega_n) \) is given by eq.(2.66). First we calculate the summation over Matsubara frequencies \( \omega_n \):

\[
S = T \sum_{\omega_n} \frac{(-1)^\alpha g(\alpha)\omega_c^{-\alpha}e^{i\pi\alpha/2}}{N(0) (b\tau + i\omega_n + \xi^2q^2) (i\omega_l - i\omega_n + \varepsilon_k)^{1-\alpha}}
\]

(2.70)

transforming this sum in a contour integral which has a pole at \( \Omega(q) = -(b\tau + \xi^2q^2)/a \) and a cut line from \( i\omega_l + \varepsilon_k \) to \( \infty \) in the upper semiplane. From eq. (2.59) we can see that \( a(\alpha) = -|a(\alpha)| \) and \( \Omega(q) = -(b\tau + \xi^2q^2)/|a| \). Performing this integral we obtain:

\[
S = \frac{\omega_c^{-\alpha}}{N(0)} \frac{n(\Omega(q))g(\alpha)e^{i\pi\alpha/2}}{-i\omega_l - \varepsilon_k - \Omega(q)}^{1-\alpha} - \frac{\omega_c^{-\alpha}}{N(0)} \frac{\sin[\pi(1 - \alpha)]}{\pi}
\]

(2.71)

\[
\times \int_{\varepsilon_k}^{\infty} dt \frac{f(t)g(\alpha)e^{i\pi\alpha/2}}{[b\tau - |a| (t + i\omega_l)] e^{\xi^2q^2}(t - \varepsilon_k)^{1-\alpha}}
\]

(2.72)

where \( n(x) \) is the Bose-Einstein function and \( f(x) \) is the Fermi-Dirac function and \( \varepsilon_k = k^2/2m - E_F \). The integral from the second contribution in eq.(2.72) will be performed using the expansion:

\[
f(t) = \sum_{m=0}^{\infty} (-1)^m \exp(-\beta(m+1)t)
\]

(2.73)

and the last term become:

\[
I_1 = \sum_{m=0}^{\infty} \frac{(-1)^m}{|a|} \frac{[\varepsilon_k + \Omega(q)]^{\alpha/2+1}}{[\beta(m+1)]^{\alpha/2}} \exp\left(\frac{\beta(m+1)[\Omega(q) - \varepsilon_k]}{2}\right)
\]

\[
\times \Gamma(\alpha)W_{-\alpha/2,\alpha/2-1/2} \{\beta(m+1)[\varepsilon_k + \Omega(q)]\}
\]

(2.74)

where the Whittaker function \( W_{\lambda,\mu}(z) \) will be approximated as:

\[
W_{\lambda,\mu}(z) \approx e^{-z/2}z^\lambda
\]

(2.75)

These results gives for eq.(2.72) the expression:

\[
S = \frac{\omega_c^{-\alpha}}{N(0)} \frac{n(\Omega(q))g(\alpha)e^{i\pi\alpha/2}}{-i\omega_l - \varepsilon_k - \Omega(q)}^{1-\alpha} + g(\alpha)e^{i\pi\alpha/2} \frac{\omega_c^{-\alpha}}{N(0)}
\]
\[ \times \frac{\sin[\pi(1-\alpha)]}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m \varepsilon_k + i\omega_l + \Omega(q)}{|a| \beta(m+1)^\alpha} \]

\[ \times \Gamma(\alpha) \exp[\beta(m+1)\varepsilon_k] \] (2.76)

In the limit \( k \equiv k_F \) the second term denoted as \( S_2 \) becomes:

\[ S_2 = \frac{\omega_c^{-\alpha} \sin[\pi(1-\alpha)]}{N(0)\pi} \frac{i\omega_l |a| + b\tau + \xi^2 q^2}{|a|^2} \]

\[ \times \Gamma(\alpha)(1-2^{1-\alpha})\zeta(\alpha)g(\alpha)e^{i\pi\alpha/2} \] (2.77)

and if \( T \rightarrow T_c, \omega_l \rightarrow 0, q \rightarrow 0 \) this term can be neglected. This approximation is in fact equivalent with the physical picture proposed by Vilk and Tremblay [25] in which the occurrence of the pseudogap is given by the interaction between the electrons and the classical fluctuations. Indeed, in this regime the first term of eq.(2.76) can be written as:

\[ S \equiv \frac{1}{N(0)} n[\Omega(q)] G(k, -i\omega_l + \Omega(q)) \] (2.78)

and the electronic self-energy becomes:

\[ \Sigma(p, \omega + i0) \equiv -\Delta_{pg}^2 G(k, -i\omega_l) \] (2.79)

where we considered \( \varepsilon_k \gg \Omega(q) \) and:

\[ \Delta_{pg}^2 = \frac{1}{N(0)|a|} \int \frac{d^2q}{(2\pi)^2} n[\Omega(q)] \] (2.80)

will be approximated as:

\[ \Delta_{pg}^2 = \frac{T}{2\pi N(0)|a|} \int_0^{q_M} \frac{q dq}{(b\tau + \xi^2 q^2)/|a|} \] (2.81)

where \( q_M \) is the wave number cutoff. From eq. (2.81) we calculate the temperature dependence of \( \Delta_{pg}^2(T) \) as:

\[ \Delta_{pg}^2 = \frac{T}{4\pi N(0)|a|} \ln \left( 1 + \frac{\xi^2}{b\tau q_M^2} \right) \] (2.82)
The temperature dependence of $\Delta_{pg}$ given by eq.(2.81) using the approximations $\tilde{\tau}(\alpha) \approx \tau(\alpha)$ and $\ln(1 + \frac{\xi^2}{4k_Fq_M}) \approx \ln(\frac{\xi^2}{4k_Fq_M})$ is shown in Fig.2.3 for $\xi q_M \sim k_F \xi \sim 10, T_c = 89K$. 

2.4 Field-Theoretical Description of Crossover Between BCS and BEC in a non-Fermi Superconductor

The problem of the crossover from BCS superconducting state to a Bose-Einstein condensate (BEC) of local pairs [45-46] becomes very important in the context of high temperature superconductors (HTSC). While at the present time there is no quantitative microscopic theory for the occurrence of the superconducting state in the doped antiferromagnetic materials, it is
generally accepted that the superconducting state can be described in terms of a pairing picture. The short coherence length ($\xi \sim 10^{-20}$ A) increased the interest for the problem [48-58] because it showed that the BCS equations of highly overlapping pairs, or the description in terms of composite bosons cannot describe the whole regime between weak and strong coupling. The mean field method developed by different authors [47,48,50,58] and solved analytically in two and three dimension, and the Ginzburg-Landau description [51,52,57] showed that the evolution between the two limits is continues, no singularities during this evolution appearing. The zero temperature coherence length in the framework of field-theoretical method has been studied in Ref. [56]. The problem of the BCS-BEC crossover in arbitrary dimension $d$ using the field-theoretical method has been extensively discussed in Refs. [58-61], where the chemical potential, the number of condensed pairs and the repulsive interaction between pairs have been calculating using the analogy with the field-theoretical description of superfluidity. In this section we apply this method to study the crossover problem for a non-Fermi superconductor described by the Anderson model [1].

### 2.4.1 Weak Coupling Limit

The BCS-like model for the non-Fermi system is described by the Lagrangian

$$L = \psi_\uparrow^+ G^{-1} \psi_\uparrow + \psi_\downarrow^+ (G^{-1})^* \psi_\downarrow - \lambda_0 \psi_\uparrow^+ \psi_\downarrow^+ \psi_\downarrow \psi_\uparrow$$

(2.83)

where the normal state is describe by the Green function (2.1)

If we introduce the two-component fermionic field

$$\Psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}$$

$$\Psi^+ = \begin{pmatrix} \psi_\uparrow^+ \\ \psi_\downarrow^+ \end{pmatrix}$$

(2.84)

the non-interacting part of the Lagrangian (1) is:

$$L_0 = \Psi^+ \begin{pmatrix} G^{-1} & 0 \\ 0 & G^{-1} \end{pmatrix} \Psi$$

(2.85)

In order to calculate the partition function:

$$Z = \int D\Psi^+ D\Psi \exp \left( i \int_x L \right)$$

(2.86)
we will transform the interaction contribution from the Lagrangian (2.83) as:

$$\exp\left( -i\lambda_0 \int_x \psi^+_\uparrow \psi^+_\downarrow \psi_\downarrow \psi_\uparrow \right) = \int \mathcal{D}\Delta^+ \Delta \exp \left( -i \int_x \left( \Delta^+ \psi_\downarrow \psi_\uparrow + \psi^+_\uparrow \psi^+_\downarrow \Delta - \frac{1}{\lambda_0} \Delta^+ \Delta \right) \right)$$

(2.87)

where $\int_x = \int dt \int d^d x$ as in Ref.[58-61] and $\Delta = \lambda_0 \psi_\downarrow \psi_\uparrow$ is a bosonic field. The partition function defined by eq. (2.86) will be expressed using eq. (2.87) in a bilinear form as

$$Z = \int \mathcal{D}\Psi^+ \mathcal{D}\Psi \int \mathcal{D}\Delta^+ \Delta \exp \left( i S_{\text{eff}}[\Delta^+, \Delta] + \frac{1}{\lambda_0} \Delta^+ \Delta \right)$$

(2.89)

where $S_{\text{eff}}[\Delta^+, \Delta]$ is the one loop effective action, which can be written as:

$$S_{\text{eff}}[\Delta^+, \Delta] = -i \text{Tr} \ln \begin{pmatrix} f(\alpha)(p_0 - \varepsilon_p)^{1-\alpha} & -\Delta \\ -\Delta^+ & f(\alpha)(p_0 + \varepsilon_p)^{1-\alpha} \end{pmatrix}$$

(2.90)

and the system as space-time independent the partition function can be written as:

$$Z = Z_0 \exp \left( i \frac{\Delta^+ \Delta}{\lambda_0} \right)$$

(2.92)
$Z_0$ containing the non-interacting contribution, and we get for the renormalization coupling constant $\lambda$ the expression:

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} + \frac{i f^2(\alpha)}{(2\pi)^2} \int \frac{d^2 p}{2\pi} \frac{1}{\left(p_0^2 - |\epsilon|^2\right)^{1-\alpha}}$$

(2.93)

Using the integral:

$$\int_{k_0} \frac{1}{(k_0^2 - E^2 + i\eta)} = i(-1)^l \sqrt{\pi} \frac{\Gamma(l - 1/2)}{\Gamma(l)} \frac{1}{E^{2l-1}}$$

(2.94)

we calculate $\lambda$ as

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} + \frac{1}{\lambda_1}$$

(2.95)

where

$$\lambda_1 = -\frac{4\pi \alpha g^{-2}(\alpha)}{\cos(\pi(\alpha - 1))} \frac{1}{B(1/2, 1/2 - \alpha)} \left(\frac{\omega_c}{\omega_D}\right)^{2\alpha}$$

(2.96)

The expression given by eq. (2.96) is positive for $\alpha < 0.5$. The new coupling constant $\lambda$ has to be also negative in order to have superconductivity ($\lambda < 0$) and this condition is satisfied if $|\lambda_0| < \lambda_1$. If we consider also the condition $|\lambda_0| > \lambda_c$ we get the general condition for the coupling constant $\lambda_0$, $\lambda_c < |\lambda_0| < \lambda_1$, which is satisfied for $0 < \alpha < 0.5$.

We mention that for the weak coupling limit $\lambda_0 \to 0^-$ the BCS limit studied in Ref. [15] is reobtained, but we also showed that the critical constant calculated from the critical temperature is $\lambda_1(\alpha \to 0) = 0$.

In the limit $\lambda_0 \to -\infty$ called the strong coupling limit, we expect an important effect of the non-Fermi character of the electrons in the coupling constant.

### 2.4.2 Strong Coupling Limit

In this limit we consider $\Delta(x) = \overline{\Delta} + \tilde{\Delta}(x)$ and consider the action $S_{eff}[\overline{\Delta}^+, \tilde{\Delta}]$ obtained from eq. (2.90) as:

$$S_{eff} \left( \overline{\Delta}^+, \tilde{\Delta} \right) = -i Tr \left( 1 + \hat{\Delta} \hat{\Delta} \right)$$

(2.97)

where:
\[
\hat{G}^{-1} = \begin{pmatrix}
  f(\alpha)(p_0 - \epsilon_p)^{1-\alpha} & -\Delta \\
  -\Delta^+ & f(\alpha)(p_0 + \epsilon_p)^{1-\alpha}
\end{pmatrix}
\] (2.98)

\[
\hat{\Delta} = \begin{pmatrix}
  0 & \Delta \\
  \Delta^+ & 0
\end{pmatrix}
\] (2.99)

which can be written as:

\[
S_{eff}(\Delta^+, \Delta) = -i \text{Tr} \sum_{l=1}^{\infty} \frac{1}{l} \left[ \hat{G}\hat{\Delta} \right]^l
\] (2.100)

with:

\[
\hat{G}(p_0, p) = \frac{1}{f^2(\alpha) \left( p_0^2 - \epsilon_p^2 \right)^{1-\alpha} - |\Delta|^2} \begin{pmatrix}
  0 & -\Delta \\
  -\Delta^+ & 0
\end{pmatrix}
\] (2.101)

We are interested in quadratic terms in \( \Delta \) and we will take the approximation

\[
S_{eff}(\Delta^+, \Delta) = S_{eff}^{(2)}(0) + S_{eff}^{(2)}(\mathbf{q})
\] (2.102)

which contains the quadratic contributions. The first term in eq. (2.102) has the form

\[
S_{eff}^{(2)}(0) = \frac{1}{2} i \text{Tr} \frac{1}{f^2(\alpha) \left( p_0^2 - \epsilon_p^2 \right)^{1-\alpha} - |\Delta|^2} \left( \Delta^2 \Delta^+ \Delta^+ + \Delta^+ \Delta \Delta + 2 |\Delta|^2 |\Delta|^2 \right)
\]

\[
+ \frac{1}{2} i \text{Tr} \frac{1}{f^2(\alpha) \left( p_0^2 - \epsilon_p^2 \right)^{1-\alpha} - |\Delta|^2} 2 |\Delta|^2
\] (2.103)

which will be approximated, taking in the dominator \( \Delta \approx 0 \) as:

\[
S_{eff}^{(2)}(0) \approx \frac{1}{2} i \text{Tr} \frac{1}{f^2(\alpha) \left( p_0^2 - \epsilon_p^2 \right)^{1-\alpha}} \left( \Delta^2 \Delta^+ \Delta^+ + \Delta^+ \Delta \Delta + 2 |\Delta|^2 |\Delta|^2 \right)
\] (2.104)

the last term giving no contribution to the renormalized coupling constant. Following the same approximation we calculated \( S_{eff}^{(2)}(\mathbf{q}) \) as:

\[
S_{eff}^{(2)}(\mathbf{q}) = \frac{1}{2} i \text{Tr} \frac{1}{f^2(\alpha)(p_0 - \epsilon_p)^{1-\alpha}(p_0 + q_0 - \epsilon_{p+q})^{1-\alpha}} \Delta \Delta^+
\]

\[
+ \frac{1}{2} i \text{Tr} \frac{1}{f^2(\alpha)(p_0 - \epsilon_p)^{1-\alpha}(p_0 + q_0 - \epsilon_{p+q})^{1-\alpha}} \Delta^+ \Delta
\] (2.105)
From eq. (2.104) and (2.105) we have

\[ L(2)(0) = -B(1/2, 3/2 - 2\alpha) \frac{2^{1-4\alpha}}{4\pi f^2(\alpha)} (2m)^{3-4\alpha} \times \]

\[
\int \frac{d^2p}{(2\pi)^2 (p^2 + m\varepsilon_a)^{3-4\alpha}} \left( \Delta^2 \Delta^+ \Delta^+ + \Delta^{+2} \Delta \Delta + 2|\Delta|^2 |\Delta|^2 \right)
\]

and

\[
L(2)(q) = -\frac{\sin(\pi(1 - \alpha))B(\alpha, \alpha)}{4\pi f^2(\alpha)} m^{1 - 2\alpha} \int \frac{d^2p}{(2\pi)^2 (p^2 + m\varepsilon_a + q_0m + q^2/4)^{2 - 2\alpha}} \]

\[
- \sin(\pi(1 - \alpha))B(\alpha, \alpha) \frac{m^{1 - 2\alpha}}{4\pi f^2(\alpha)} m^{1 - 2}\int \frac{d^2p}{(2\pi)^2 (p^2 + m\varepsilon_a - q_0m + q^2/4)^{2 - 2\alpha}}
\]

The integrals from eq. (2.106) and (2.107) can be performed using the formula

\[
\int \frac{1}{(p^2 + A^2)^N} = \frac{\Gamma(N - d/2)}{(4\pi)^{d/2} \Gamma(N) (A^2)^{N-d/2}}
\]

and we obtain

\[
L(2) = -\frac{m}{16\pi^2 f^2(\alpha)} \frac{2^{2-4\alpha}}{1 - 2\alpha} \frac{B(1/2, 3/2 - 2\alpha)}{\varepsilon_a^{2-4\alpha}} \left( \Delta^2 \Delta^+ \Delta^+ + \Delta^{+2} \Delta \Delta + 2|\Delta|^2 |\Delta|^2 \right)
\]

\[ + \frac{m}{16\pi^2 f^2(\alpha)} \frac{\sin(\pi(\alpha - 1))}{2\alpha} \frac{B(\alpha, \alpha)}{(\varepsilon_a + q_0 + q^2/4m)^{2\alpha}} \Delta \Delta^+ \]

\[ + \frac{m}{16\pi^2 f^2(\alpha)} \frac{\sin(\pi(\alpha - 1))}{2\alpha} \frac{B(\alpha, \alpha)}{(\varepsilon_a - q_0 + q^2/4m)^{2\alpha}} \Delta^+ \Delta \]

Using the approximation:

\[
(\varepsilon_a - q_0 \pm q^2/4m)^{2\alpha} = \varepsilon_a^{2\alpha} + 2\alpha \varepsilon_a^{2\alpha - 1} \left( \pm q_0 + \frac{q^2}{4m} \right)
\]

and using the notation

\[
\tilde{\Psi} = \begin{pmatrix} \Delta \\ \Delta^+ \end{pmatrix}
\]

we obtain from eq. (2.109)

\[
L^{(2)} = -\frac{m}{16\pi^2 f^2(\alpha)} \frac{\sin(\pi(1 - \alpha))B(\alpha, \alpha)}{\varepsilon_a^{2\alpha - 1}} \frac{1}{2} \tilde{\Psi} + M \tilde{\Psi}
\]
where

\[
M = \begin{pmatrix}
    q_0 - \frac{q^2}{2m_B} - \mu_0 & -\mu_0 \\
    -\mu_0 & -q_0 - \frac{q^2}{2m_B} - \mu_0
\end{pmatrix} \quad (2.113)
\]

\(m_b = 2m\) being the boson mass and \(\mu_0\) the chemical potential

\[
\mu_0 = \frac{1}{f^2(\alpha)} \frac{2^{2-4\alpha} B(1/2, 3/2 - 2\alpha) |\Delta|^2}{(1 - 2\alpha) \sin(\pi (1 - \alpha)) B(\alpha, \alpha) \varepsilon_\alpha^{2\alpha-1}} \quad (2.114)
\]

the velocity \(c_0\) of the sound mode is:

\[
c_0 = \frac{\mu_0}{m_b} = \frac{1}{f^2(\alpha)} \frac{2^{2-4\alpha} B(1/2, 3/2 - 2\alpha) |\Delta|^2}{(1 - 2\alpha) \sin(\pi (1 - \alpha)) B(\alpha, \alpha) m \varepsilon_\alpha^{2\alpha-1}} \quad (2.115)
\]

and the repulsive interaction \(\lambda_{0b}\) between pairs is:

\[
\lambda_{0b} = \frac{\pi^2}{m} \frac{2^{1-4\alpha} B(1/2, 3/2 - 2\alpha)}{(1 - 2\alpha) [\sin(\pi (1 - \alpha)) B(\alpha, \alpha)]^2} \quad (2.116)
\]

We mention that \(\lim_{\alpha \to 0} \lambda_{0b}(\alpha) = 2\pi/m\) a result identical to the result obtained in Ref. [60] for the 2D case.

### 2.5 Summary of the Results

- In the first part of this chapter we performed a simple calculation of the pseudogap due to the interaction between electrons and magnetic fluctuations. We showed that for the 2D systems, in one loop approximation the self-energy of the electronic excitation can be expressed analytically by the structure factor \(S(q, T)\). The expression for the real and imaginary part of the self-energy given by eqs. (2.17) and (2.18) are very usefully because their behavior as function of \(\omega\) and \(q\) can be easily controlled and predicted if the form of \(S(q, T)\) is known. This calculation make transparent the analytical approach first proposed in Ref.[22-25] using the self consistent treatment.

  We considered in the susceptibility \(\chi(q, \omega)\) the diffusive contribution of the spin waves and we showed that if the condition expressed in eq. (2.21) is satisfied the spin-wave does not change the occurrence of the pseudogap predicted by the coupling with low energy spin fluctuations. The energy scale has been obtained from the condition \(G(\omega) \sim 1/\omega\) (\(\Sigma(\omega) \sim 1/\omega\)) and in fact this condition gives for the coherence length of the magnetic fluctuations an expression
of the form $\xi(T) \sim \exp[C/T]$ used in [25] in the renormalized classical regime, and which is appropriate also for a 2D magnetic system [36]. Recently the RNG equations have been applied for the proximity of the Lifshitz point [37].

- We also tried to consider the effect of the anisotropy in the magnetic fluctuations and following a calculation similar to that of Ref. [34] we calculated a pseudogap which is temperature dependent. This dependence is even more complicated because in different regimes the coherence length $\xi(T)$ for the magnetic fluctuations is different. Using different $T$ dependence for $\xi(T)$ we showed that even for $\xi(T) \sim \exp[C/T]$ there is a temperature dependence in the pseudogap which is not certain from experimental point of view [38].

We mention that the key problem of such model remain the matching between the pseudogap, which in this case has a magnetic origin, and the superconducting gap which should have a d-wave pairing origin. In order to have a picture of this point we propose that the following qualitative behavior of the phase diagram. The total gap $\Delta(T)$ is defined as:

$$
\Delta(T) = \begin{cases} 
\Delta_s(T) + \Delta_{pg}(T), & T < T_c \\
\Delta_{pg}(T), & T_c < T < T^* 
\end{cases}
$$

(2.117)

where $T_c$ is the superconducting critical temperature and $T^*$ is the pseudogap appearing temperature. According to this picture the measured gap below the superconducting critical temperature has two contributions, one from the superconducting gap and the other one from the fluctuation gap.

- In the second part of this chapter we showed that a temperature dependent pseudogap appears in a non-Fermi liquid superconductor due to the interaction between electrons and the fluctuations of the order parameter amplitude. The mode-mode coupling, valid in the weak coupling approximation can give relevant results, even for the intermediate coupling studied by Levin group [39]. The model recently applied by Norman et.al. [40] can be applied for the spin fluctuation model proposed by Chubukov [9] in order to study the temperature dependence of the pseudogap. In Ref. [40] the filling in of the pseudogap due to the increment of the temperature is given by the broadening in the self-energy and is proportional to $T - T_c$. A similar broadening effect proportional to $\tilde{\tau}(\alpha)$ was obtained in our model and this can be seen very easily from eq. (2.77) if in the electronic Green function we take the limit $q = 0$.

Recently such a model for the Fermi liquid superconductor has been studied by Kristoffel and Ord [41] and their temperature dependence is different than our result. However, we mention
that according to their model these authors have to obtain a result similar to the results given in Ref.[25]. The difference is given by the method of performing the integral over $q$ which is not correct in Ref. [41].

Recently Preosti et.al. [34] generalized the method given in Ref. [25] taking into consideration the anisotropy in the dynamic susceptibility due to the interplane pairing. From the temperature dependence of the pseudogap shown in Fig. 2.3 one can see that there is a narrow domain of temperature where this dependence is in fact in agreement with the recent data from Ref. [41]. Increasing the temperature will lead the system to the pure classical regime, where the pseudogap is constant, result in agreement with Ref. [25].

The model analyzed in Ref. [25] and [41] use a constant coupling between electrons and overdamped fluctuations describe by the t-matrix or a $\chi(q,\omega)$ containing an imaginary part. Recently Tchernyshyov [42] showed that using a better approximation for the t-matrix the decay of the Cooper pairs is negligible and a bosonic propagator without damping of excitations gives a pseudogap in the normal state. This idea is also interesting for our model. This simple model neglected the vertex corrections in the electronic self-energy, which for the non-Fermi liquid superconductor are very singular and has to be considered in the transport theory.

At the present time it is generally accepted that the pseudogap appears only in the underdoped cuprates. This fact is reflected in our scenario by the dependence of $\Delta_{pg}$ of $\alpha$ which appears as a parameter of the model. However it was showed [44] that the correlation length depends on $\alpha$ and we can attend the overdamped regime by variation of $\alpha$. This demonstrate that such a model is appropriate for the description of the pseudogap in a non-Fermi-liquid model.

- In the last part of this chapter using the field-theoretical methods we studied the crossover between BCS and BEC in a non-Fermi liquid. The weak coupling case lead to the same results as in the mean field like models. In the strong coupling limit we showed that the pairs form a Bose gas with a repulsive coupling constant which is controlled by $\alpha$. 

46
Bibliography

[1] P.W. Anderson Science 235 1196 (1987) ; Phys.Rev.Lett. 64 1839 (1990) ; 65 2306 (1990)

[2] S. Chakravarty, P.W. Anderson Phys.Rev.Lett. 72 3859 (1994)

[3] X.G. Wen Phys.Rev. B42 6623 (1990)

[4] A.V. Balatsky Philos.Mag.Lett. 68 251 (1993)

[5] A. Sudbo Phys.Rev.Lett. 74 2575 (1995)

[6] A. Sudbo, J.M. Wheatley Phys.Rev. B52 6200 (1995)

[7] V.N. Muthukumar, D. Sa, M. Sardar Phys.Rev. B52 9647 (1995)

[8] I. Grosu, I. Tifrea, M. Crisan, S. Yoksan Phys.Rev. B56 8298 (1997)

[9] A.V. Chubukov, J. Schmalian cond-mat/9711041

[10] V.G. Geshkenbein, L.B. Ioffe, A.I. Larkin Phys.Rev. B55 3173 (1997)

[11] S.C. Zang Science 275 1089 (1997)

[12] P.A. Lee, X.G. Wen Phys.Rev.Lett. 78 4111 (1997)

[13] A.M. Dare, L. Chen, A-M. Tremblay Phys.Rev. B49 4106 (1994)

[14] E. Abrahams (private communications)

[15] L. Yin, S. Chakravarty Int.J.Mod.Phys. B10 805 (1996)

[16] J. Voit Phys.Rev. B47 6740 (1993)

[17] R. Shankar Rev.Mod.Phys. 66 129 (1994)
[18] S.R. White *Phys. Rev.* **B44** 4670 (1991)

[19] N. Bulut, D.J. Scalapino, S.R. White *Phys.Rev.Lett.* **79** 3752 (1997)

[20] R. Ramashvilli, P. Coleman *Phys.Rev.Lett.* **79** 3752 (1997)

[21] A. Rosh, A. Schroder, O. Stockert, H.v. Lohneysen *Phys.Rev.Lett.* **79** 159 (1997)

[22] Y.M. Vilk, L. Chen, A-M.S. Tremblay *Phys.Rev.* **B49** 13267 (1994)

[23] Y.M. Vilk, A-M.S. Tremblay *EuroPhys.Lett.* **33** 159 (1996)

[24] Y.M. Vilk *Phys.Rev.* **B55** 3870 (1997)

[25] Y.M. Vilk, A-M.S. Tremblay *J.Phys. I (Paris)* **7** 1309 (1997)

[26] A.V. Chubukov *Phys.Rev.* **B52** 3840 (1995)

[27] A.V. Chubukov, J. Schmalian *Phys.Rev.* **B57** 11089 (1998)

[28] A.J. Millis, H. Monien, D. Pines *Phys.Rev.* **B42** 176 (1990)

[29] V. Barzykin, D. Pines *Phys.Rev.* **B52** 13585 (1995)

[30] J. Schmalian cond-mat/9810041

[31] A.J. Millis *Phys.Rev.* **B48** 7183 (1993)

[32] D. Pines *Z.Phys.* **B103** 129 (1997)

[33] S. Sachdev, A.V. Chubukov, A. Sokol *Phys.Rev.* **B51** 14874 (1995)

[34] G. Preosti, Y.M. Vilk, M.R. Norman cond-mat/9808298

[35] S. Marcelja *Phys.Rev.* **B1** 2351 (1970)

[36] P. Hazenfrantz, F.Niedermager *Phys.Lett.* **B268** 239 (1991)

[37] R. Ramazashvilli cond-mat/9901191

[38] A. Mourachkine cond-mat/9810161

[39] B. Janko, J. Maly, K. Levin *Phys.Rev.* **B56** 11407 (1997); Q. Chen, I. Kosztin, B.Janko, K. Levin cond-mat/980714
[40] M.R. Norman, M. Randeria, H. Ding, J.C. Campuzano *Phys.Rev.* **B57** 11093 (1998)

[41] N. Kristoffel, T. Ord *Physica C* **298** 37 (1998)

[42] O. Tchernyshyov *Phys.Rev.* **B56** 3372 (1997)

[43] I. Tifrea, I. Grosu, M. Crisan *Physica C* **298** 51 (1998)

[44] D.M. Eagles *Phys.Rev.* **186** 456 (1969)

[45] A.J. Legget *J.Phys. (Paris)* **C7** 19 (1980)

[46] P. Nozieres, S.Schimtt-Rink *J.Low Temp.Phys.* **59** 195 (1985)

[47] R. Fridberg, T.D. Lee *Phys.Rev.* **B40** 6745 (1989)

[48] S. Schimtt-Rink, C.M. Varma, A.E. Ruckenstein *Phys.Rev.Lett.* **63** 445 (1989)

[49] M. Randeria, J.M. Duan, L.Y. Shien *Phys.Rev.* **B41** 327 (1990)

[50] M. Crechster, W. Zwerger *Ann.Phys. (Germany)* **1** 15 (1992)

[51] C.A. R. sa de Melo, M. Randeria, J.R. Engelbrecht *Phys.Rev.Lett.* **71** 3203 (1993)

[52] R. Cote, A. Griffin *Phys.Rev.* **B48** 10404 (1993)

[53] L. Belkhir, M. Randeria *Phys.Rev.* **B49** 6829 (1994)

[54] G. Ropke *Ann.Phys. (Germany)* **3** 134 (1994)

[55] F. Pistolesi, G.C. Strinati *Phys.Rev.* **B53** 15168 (1996)

[56] S. Stintzing, W. Zwerger *Phys.Rev.* **B56** 9004 (1997)

[57] M. Marina, F. Pistolesi, G.C. Strinati *Euro.Phys.J.* **B1** 151 (1998)

[58] A.M.J. Schakel *cond-mat/9811393*

[59] A.M.J. Schakel "*Boulevard of Broken Symmetries*" *cond-mat/9805152*

[60] A.M.J. Schakel "*Time dependent Ginsburg-Landau theory of duality*" *(to be published )*

[61] A.M.J. Schakel *Ph.D.Thesis Univ.of Amsterdam* (1989)

[62] I. Tifrea, M. Crisan *Euro.Phys.J.* **B4** 175 (1998)
Chapter 3

Renormalization Group Approach of Itinerant Electron System near the Lifshitz Point

3.1 Introduction

The occurrence of the non-Fermi behavior in the systems of fermions coupled to a critical fluctuations mode has been suggested in connection with the neutron experiments [1-2] and studied in the framework of many body theory [3-4] in the case of two-dimensional (2D) and three dimensional (3D) models. Recent experiments on the heavy fermions systems showed also a non-Fermi behavior of these materials at low temperatures and it was associated with the proximity of quantum critical point (QCP). The most studied example [5] is CeCu$_{6-x}$Au$_x$ where at the QCP $x = 0.1$ the resistivity increases linearly with temperature $T$ over a wide range of $T$ and the specific heat $C(T)$ is proportional to $T \ln(T_0/T)$. This behavior has been explained [6] by the coupling of 3D fermionic excitations to the 2D critical ferromagnetic fluctuations near the QCP. The inelastic neutron scattering measurements performed on this materials [7-8] showed the following new points in the behavior of this material

- The inelastic neutron scattering data can be fitted with a susceptibility of the form: $\chi^{-1} = C^{-1}[f(q) + (aT - i\omega)^\alpha]$ where $\alpha = 4/5$ and not 1 as is predicted by the mean field approximation
- The quadratic stiffness vanishes, fact which shows that we are dealing with a quantum Lifshitz point (QLP)
The peaks for $x = 0.2$ and $x = 0.3$ can be considered as 2D precursor of 3D order.

The scaling analysis showed [7] that $\gamma(T) = C(T)/T$ has the form

$$
\gamma(T) \sim T^{(D-1/2)a/2-1}
$$

which for $D = 3$ and $\alpha = 4/5$ gives a temperature independent value. This analysis has been performed taking as the most important contribution to $\chi$ the form containing $\omega^\alpha$ and the $q$-dependence of the form $f(q) = Dq_||^2 + Cq_\perp^4$ where $\alpha = 2/z$, $z$ being the critical exponent from the dynamical critical phenomena [9]. In this chapter we will show that using the Hertz [10] renormalization group method (RNG) extended for $T \neq 0$ by Millis [11] we can obtain the $\ln(T_0/T)$ term as a quantum correction to the classical results expressed by eq. (3.1).

In the next section the renormalization group studies of the Gaussian fixed point of magnetic transitions in the metallic phase will be use. This treatment of the Gaussian fixed point essentially lead to equivalent results to the self-consistent renormalization approximation [12]. For the case of ferromagnetic system with isotropic Fermi surface we have for the free susceptibility the following expansion:

$$
\chi(k, i\omega_n) = N(0) \left[ 1 - \frac{1}{3} \left( \frac{k}{2k_F} \right)^2 - \frac{\pi |\omega_n|}{2kv_F} + ... \right] \quad (3.2)
$$

It should be noted that this expansion is possible only for $d > 3$. In $d = 1$ and $d = 2$ the susceptibility is singular at $|k| = 2k_F$, even for an isotropic Fermi surface. Recently the nonanalyticity of $\chi(k, i\omega_n)$ in $k$ for $d < 3$ in the leading order was considered more seriously [14] which led to the conclusion that for the ferromagnetic case, the mean field description is incorrect at $d = 2$ and $d = 3$ in contrast with the result from eq. (3.1).

In the antiferromagnetic case, $\chi(Q + k, i\omega_n)$ is similarly expended for small $k$ and $\omega_n$ if the nesting condition is not satisfied and the spatial dimension satisfies $d \geq 3$. When the nesting condition is satisfied, as in the case of Mott insulator state, this type of simple expansion is not possible.

### 3.2 The Scaling Equations

The main idea of our model is contained in a modified form of the susceptibility given in Ref. [7] as:
\[ \chi^{-1}(q, i\omega_n) = C (f(q) + \delta^{12} + (aT)^{12} + |\omega|^{12}) \] (3.3)

where \( f(q) \) is a smooth function of the wave vector, \( \delta \) is the control parameter (pressure, doping) measuring the distance from QCP and \( \alpha \) is the phenomenological exponent. The real part of this expression satisfies for \( \omega = 0 \) the relation:

\[ \chi'^{-1}(q, T) - \chi'^{-1}(0, T) = C(aT)^{-\alpha} \] (3.4)

which is identical to the relation satisfied by the form proposed in Ref. [7]. The imaginary part of the susceptibility expressed by eq. (3.3) has the form

\[ \chi''(\omega, T) = T^{-\alpha} g \left( \frac{E}{T} \cdot \frac{\delta}{T} \right) \] (3.5)

where:

\[ g(y, x) = C \sin \left[ \arctan \left( \frac{y^{12} \sin(\pi\alpha/2)}{1 + x^{12} + y^{12} \cos(\pi\alpha/2)} \right) \right] \left[ \frac{1}{\left( 1 + x^{12} + y^{12} \cos(\pi\alpha/2) \right)^{1/2} + y^{22} \sin(\pi\alpha/2)} \right]^{1/2} \] (3.6)

and \( f(q) = 0 \). We mention that for \( \delta = 0 \) and \( \alpha = 1 \) we reobtain the mean field results and it can be showed numerically that the scaling function \( g(y, x) \) has the same form with the scaling function \( g(y) \) obtained in Ref. [7].

Using these considerations we consider that in the low temperature approximation the interacting Fermi system can be describe by the effective action:

\[ S_{eff}(\Phi) = S_{eff}^{(2)}(\Phi) + S_{eff}^{(4)}(\Phi) \] (3.7)

where:

\[ S_{eff}^{(2)}(\Phi) = VT \sum_n \int \frac{d^3q}{(2\pi)^3} \left[ \delta^{12} + |\omega_n|^{12} + q_0^{12} + Dq_0^{12} + q_0^{12} \right] |\Phi(q, \omega_n)|^2 \] (3.8)

\[ S_{eff}^{(4)}(\Phi) = uVT^3 \sum_{n_i} \prod_{i=1}^4 \frac{d^3q_i}{(2\pi)^3} \Phi(q_i, \omega_{n_i}) \delta \left( \sum_{i=1}^4 q_i \right) \delta \left( \sum_{i=1}^4 \omega_{n_i} \right) \] (3.9)

with \( \omega_n \) a bosonic frequency, \( u > 0 \) is the coupling constant and \( D \) is the stiffness of the fluctuations.

In order to calculate specific heat we will use the Gaussian form of the free energy obtained from eq. (3.8) as:
\[
F = \int_0^1 \frac{dz}{2\pi} \int_0^1 \frac{d^2q_\parallel}{(2\pi)^2} \int_0^1 \frac{dq_\perp}{2\pi} \coth \frac{z}{2T} \arctan \frac{A\sin \theta}{A\cos \theta + q_\parallel^2 + Dq_\perp^2 + q_\perp^4}
\]

(3.10)

where \( \theta = \arctan(z/\delta^\alpha) \) and \( A^{-\alpha} = (\delta^{2\alpha} + z^2)^{1/2} \). Using the transformations

\[
\begin{align*}
\omega' &= b^{2/\alpha} \omega \\
qu_\parallel' &= bq_\parallel \\
qu_\perp' &= b^{1/2} q_\perp \\
T' &= b^{2/\alpha} T \\
\delta' &= b^{2/\alpha} \delta \\
D' &= b D
\end{align*}
\]

(3.11)

and following the same procedure as in Ref. [11] we obtain the scaling equations:

\[
\frac{dT(b)}{d\ln b} = \frac{2}{\alpha} T(b)
\]

(3.12)

\[
\frac{du(b)}{d\ln b} = \left( \frac{3}{2} - \frac{2}{\alpha} \right) u(b) - u^2(n + 8)f_2
\]

(3.13)

\[
\frac{d\delta^\alpha(b)}{d\ln b} = 2\delta^\alpha(b) + 2u(b)(n + 2)f_1
\]

(3.14)

\[
\frac{dD(b)}{d\ln b} = D(b)
\]

(3.15)

\[
\frac{dF(b)}{d\ln b} = \left( \frac{5}{2} + \frac{2}{\alpha} \right) F(b) + f_3
\]

(3.16)

where \( n \) is the number of the field component, \( f_1 = f_1[T(b), \delta^\alpha(b), D(b)], \ f_2 = f_2[T(b), \delta^\alpha(b), D(b)] \) and \( f_3 = f_3[T(b)] \) are complicated functions but can be approximated in the limit of weak dependence of \( \delta^\alpha(b) \) and \( D(b) \) for \( \delta^\alpha(b), D(b) \ll 1 \). If we are near a QCP which is usually at very low temperature the scaling equations will be linearized, keeping only the linear term in the coupling constant. The renormalization procedure is stopped at

\[
\delta^\alpha(b) = 1
\]

(3.17)

From the linearized eqs. (3.12)-(3.14) we get:

\[
T(b) = T b^{2/\alpha}
\]

(3.18)

\[
u(b) = u b^{3/2 - 2/\alpha}
\]

(3.19)
\[ \delta^\alpha(b) = e^{2\ln b} \left[ \delta^\alpha + 2u(n + 2) \int_0^{\ln b} dx e^{-x(1/2 + 2/\alpha)} f_1(T e^{2x/\alpha}) \right] \]  

(3.20)

These equations will be analyzed in two cases which are in fact the low temperature and high temperature regimes.

In low temperature limit the integral from eq. (3.20) can be approximated as:

\[ I = \int_0^{\ln b} dx e^{-x(1/2 + 2/\alpha)} f_1(T e^{2x/\alpha}) \approx \frac{f_1(0)}{1/2 + 2/\alpha} \]  

(3.21)

and eq. (3.20) becomes

\[ \delta^\alpha(b) = b^2 \left[ \delta^\alpha + \frac{2u(n + 2)f_1(0)}{1/2 + 2/\alpha} + 2Bu(n + 2)T^{1+\alpha/4} \right] \]  

(3.22)

If we introduce the parameter

\[ r_\alpha = \delta^\alpha + \frac{2u(n + 2)f_1(0)}{1/2 + 2/\alpha} \]  

(3.23)

the low temperature regime is defined by

\[ T(b) \ll 1 \]  

(3.24)

and the high temperature regime is defined by

\[ T(b) \gg 1 \]  

(3.25)

From eq. (3.22) using the condition \( \delta^\alpha(\tilde{b}) = 1 \) we get the condition for the low temperature regime as

\[ \frac{T}{r_\alpha^{1/\alpha}} \ll 1 \]  

(3.26)

In the high temperature regime defined now by the inequation (3.26) reversed, we approximate the function \( f_1(T) \) as \( f_1(T) \approx CT \) and introduce the new variable \( v = uT \). The linearized scaling equations are:

\[ \frac{dv(b)}{d\ln b} = \frac{3}{2} v(b) \]  

(3.27)

\[ \frac{d\delta^\alpha(b)}{d\ln b} \approx 2\delta^\alpha(b) + 2Cv(b)(n + 2) \]  

(3.28)
and the gaussian behavior appears if \( \delta^\alpha(b) = 1 \) and \( v(b) \ll 1 \). The general solution of eqs. (3.27) and (3.28) have the form

\[
v(b) = \overline{v} e^{3\ln b/2}
\]

(3.29)

\[
\delta^\alpha(b) = e^{2\ln b \left[ \overline{\delta}^\alpha + 2\overline{C}\overline{v} \right] - 4\overline{C}\overline{v} e^{3\ln b/2}}
\]

(3.30)

with \( \overline{C} = 2(n + 2)C \).

The initial conditions have been determined following the procedure from Ref. [11]

\[
\overline{\delta}^\alpha = T^{-\alpha} \left[ r_\alpha + 2Bu(n + 2)T^{1+\alpha/4} \right]
\]

(3.31)

where \( B \) is a constant defined in Ref.[11]. The condition \( v(b) \ll 1 \) is defined if:

\[
R = \frac{uT}{\left[ r_\alpha + 2(B + C)u(n + 2)T^{1+\alpha/4} \right]^{3/4}} \ll 1
\]

(3.32)

which is in fact the condition for the validity of the Gaussian model (Ginsburg criterion). The critical temperature is well approximated by:

\[
T_c = \left[ \frac{r_\alpha}{2(B + C)u(n + 2)} \right]^{4/(4+\alpha)}
\]

(3.33)

This result for \( T_c(\delta) \) can be obtained from eq. (3.33) as:

\[
T_c \sim \delta^{4\alpha/(4+\alpha)}
\]

(3.34)

and for \( \alpha = 4/5, T_c \sim \delta^{0.67} \).

### 3.3 The Specific Heat

In order to calculate the specific heat we will use eq. (3.10) for the free energy \( F \). Neglecting in the lower approximation the second term we obtain

\[
F(T) = F(b)b^{-2/\alpha-5/2}
\]

(3.35)

The exact solution of eq. (3.16) has the form
\[ F(b) = b^{2/\alpha + 5/2} \int_{0}^{\ln b} dx e^{-(2/\alpha + 5/2)x} f_{3}(Te^{2x/\alpha}) \]  

(3.36)

In order to calculate the temperature dependence of the free energy expressed by eq. (3.36) we consider the variable \( x \) in the domains

\[ 0 < x < \frac{\alpha}{2} \ln \frac{1}{T} \]  

(3.37)

\[ \frac{\alpha}{2} \ln \frac{1}{T} < x < \ln b^* \]  

(3.38)

where \( b^* \) is defined by \( \delta^\alpha(b^*) = 1 \) and from eq. (3.17) was calculated as \( b^* = T^{-\alpha/2} \). Following Ref.[11] in the first domain \( f_{3}(T) \cong C_{3}T \) and in the second domain \( f_{3}(T) \cong DT \). Using these approximations we obtain from eq. (3.36)

\[ b^{-2/\alpha - 5/2}F(b) = \frac{\alpha}{2}T^{1+5\alpha/4} \left[ C_{3} \int_{T}^{1} dT_{1} T_{1}^{-5\alpha/4} + D \int_{1}^{Tb^{2/\alpha}} dT_{1} T_{1}^{-5\alpha/4} \right] \]  

(3.39)

where \( T_{1} = T \exp[2x/\alpha] \). If we take \( \alpha = 4/5 \) from eq. (3.39) we calculate

\[ F(T) = \frac{2}{5}C_{3}T^{2} \ln \frac{1}{T} + \frac{2}{5}DT^{2} - \frac{2}{5}DTb^{*-5/2} \]  

(3.40)

and \( \gamma(T) = C_{v}/T \) as:

\[ \gamma(T) = \gamma_{c0} + \gamma \ln \frac{1}{T} + O \left( \frac{1}{T^{2}} \right) \]  

(3.41)

a result which shows that using RNG for the phenomenological model with \( \alpha = 4/5 \) we obtain the \( \ln(1/T) \) term in \( \gamma(T) \) which is in fact done by the non-Fermi behavior of the model.

Recently Ramazashvilli [13] used the same method studying QLP for such a model with \( \alpha = 1 \). Our results are consistent with the results from Ref. [13], excepting the specific heat coefficient which has \( T^{1/4} \) dependence. The phase diagram can be calculated from the free energy

\[ F \sim (\delta^\alpha + Dq^2 + q^4) \Phi^2 \]  

(3.42)

and for \( D > 0, \delta^\alpha \) we get an ordered phase at \( q = 0 \). A second ordered phase can be obtained for \( D < 0, q = \pm(D/2)^2 \) separates the two ordered phases from disordered phase. The importance
of the $q^2$ and $q^4$ terms in the susceptibility can be also discussed as in Ref. [13] and we get the same result, the only difference being that $\delta(b)$ has to be replace with $\delta^\alpha(b)$.

### 3.4 Summary of the Results

- We calculated the specific heat dependence of a 3D itinerant electron system near a Lifshitz point using the phenomenological model considering the susceptibility of the form $\chi^{-1} \sim f(q) + \delta^\alpha + |\omega|^\alpha + T^\alpha$. This form describes a similar behavior with the phenomenological model proposed in Ref. [7,8] and using the $T \neq 0$ RNG proposed by Millis we showed that the specific heat presents a $\ln T$ contribution for $\alpha = 4/5$ which is typical behavior for a non-Fermi system.

  Our results can be considered as generalization of the recent results obtained in Ref. [13] where the exponent $\alpha$ has the value $\alpha = 1$. The first calculation [11] of the specific heat using RNG methods near a QCP with zero critical temperature used for the dynamic exponent $z = 2$, $z = 3$ or $z = 4$ but no $\ln T$ dependence has been obtained.
Bibliography

[1] P. Benard et.al. *Phys.Rev.* **B47** 15217 (1993)

[2] L. Chen et.al. *Phys.Rev.* **B52** 1152 (1995)

[3] M. Crisan, L. Tataru *J.Supercond.* **8** 341 (1995)

[4] Y.M. Vilk, A-M.S. Tremblay *Phys.Rev.* **B49** 13267 (1994)

[5] H. van Loheysen *J.Phys.Cond.Matt.* **8** 9689 (1996)

[6] A. Rosch et.al. *Phys.Rev.Lett.* **79** 159 (1997)

[7] A. Schroder et.al. *Phys.Rev.Lett.* **80** 5623 (1998)

[8] O. Stockert et.al. *Phys.Rev.Lett.* **80** 5627 (1998)

[9] P. Coleman *cond-mat/9809436*

[10] J.A. Hertz *Phys.Rev.* **B14** 1165 (1976)

[11] A.J. Millis *Phys.Rev.* **B48** 7183 (1993)

[12] T. Moriya *Spin Fluctuation in Itinerant Electron Magnetism* (Springer Heidelberg, 1985)

[13] R. Ramazashvilli *cond-mat/9901191*

[14] D. Belitz, T.R. Kirkpatrik *Phys.Rev.* **B56** 6513 (1997)
Chapter 4

Conclusion

We conclude this thesis presenting the original results obtained and mentioning the unsolved problems in this field.

Chapter 1:

(1) The temperature dependence of a marginal Fermi liquid has been calculated. We showed that the expected $T \ln T$ correction is characteristic for the low temperature domain. The high temperature domain has a supplementary correction. The results are in agreement with the non-Fermi behavior of some metallic systems in the low temperature domain. (results contained in: M. Crisan, C.P. Moca Journal of Superconductivity 9 49 (1996))

(2) We calculated the self-energy at $T = 0$ for a two dimensional fermionic system with hyperbolic dispersion. The existence of the saddle points in the energy gives rise to a marginal behavior, a result which has been obtained by numerical calculations. We present the many-body calculation of the self-energy. We showed that even in a (RPA) approximation this model present a marginal Fermi liquid behavior. (results contained in: C.P. Moca, M. Crisan Journal of superconductivity 10 3 (1997))

(3) In order to give a stronger support to this model we adopted the Renormalization Group Method for this model. The calculation of the wave function renormalization constant $Z$ will be performed using the "poor man’s renormalization" method and we will show that $Z \to 0$ with the infrared cut-off $\Lambda$ as $Z(\Lambda) = \Lambda^\zeta$ where $\zeta$ is a constant. (results contained in: M. Crisan, C. P. Moca Modern Physics Letters B9 1753 (1995))

The last experimental results and the theoretical investigation showed that the non-Fermi behavior is associated with the two novel features of these systems, namely the occurrence of
the pseudogap and the proximity of a Quantum Phase Transition. This problems have been studied by different authors for a normal Fermi-liquid. However we adopted the Anderson non-Fermi model because it can be reduced to the normal model taking the parameter $\alpha$ equal zero.

Chapter 2

(4) The electronic self-energy due to the electron-spin interaction is calculated using the one-loop approximation for the two dimensional system and quasi-two dimensional (anisotropic) model. We analyzed the relevance of the diffusive modes and the temperature dependence of the magnetic correlation length for a possible temperature dependence of the pseudogap. The results contained in this section are in the spirit of the method presented by the Tremblay group for the normal Fermi systems. (results contained in: C.P. Moca, I.Tifrea, M. Crisan (accepted for publication in Journal of Superconductivity)

(5) We study the influence of the amplitude fluctuations of a non-Fermi superconductor on the energy spectrum of the two-dimensional Anderson non-Fermi system. In order to perform such a calculation we had to elaborate Schmid self-consistent model of fluctuations in BCS superconductor for the fluctuations in an Anderson non-Fermi superconductors. The new propagator of fluctuations have been calculated and in the limit of $\alpha = 0$ it gives the results from BCS. These fluctuations are also classical and give a temperature dependence in the pseudogap induced in the fermionic excitations of the Anderson model. (results contained in: M. Crisan, C.P. Moca, I. Tifrea Phys Rev B59 14680 (1999))

In order to study the possibility of the superconducting state in the Anderson model we also studied the problem formulated by the Nozieres and Schmitt-Rink who considered the crossover between weak (BCS) and strong coupling (BEC) using a Fermi liquid model.

(6) Using the field-theoretical methods we studied the evolution from BCS description of a non-Fermi superconductor to that of Bose-Einstein condensation (BEC) in one loop approximation. We showed that the repulsive interaction between composite bosons is determined by the exponent $\alpha$ of the Anderson propagator in a two dimensional model. For $\alpha \neq 0$ the crossover is also continuous and for $\alpha = 0$ we obtain the case of the Fermi liquid.(results contained in:I. Tifrea, C.P. Moca, M. Crisan (presented at the 6th International Conference, Materials and Mechanisms of Superconductivity and High Temperature Superconductors, February 20-25, 2000, Houston, Texas, USA , to be published in Physica C)

The proximity of a Quantum Phase Transition, recently verified by many experimental re-
results, has been studied using the RNG method for the phenomenological model proposed by the group of Hilbert von Lohneysen.

Chapter 3

(7) Using the renormalization group approach proposed by Millis for the itinerant electron systems, in the case of \( d = z = 2.5 \), we calculated the specific heat coefficient \( \gamma(T) \) for the magnetic fluctuations with susceptibility \( \chi^{-1} \approx \delta^\alpha + |\omega|^\alpha + f(q) \) near a Lifshitz point. The constant value for \( \alpha = 4/5 \) and the logarithmic temperature dependence, specific heat for the non-Fermi behavior, have been obtained in agreement with the experimental data. (results contained in: C.P. Moca, I. Tifrea, M. Crisan (accepted for publication in Phys. Rev. B))

Open problems:

In spite of fact that the non-Fermi model proposed by Varma et.al. and Anderson may explain many experimental results, the new very accurate measurements showed that there are still open problems which have to be explained even in a semi-phenomenological model. In this respect a non-Fermi liquid model have to explain quantitatively the

- peak-dip-hump structure of the ARPES lineshape and tunnel spectroscopy.
- the pseudogap spectrum in vortex core of underdoped cuprates superconductors
- the existence of resonances in neutron scattering
- marginal or non-Fermi liquid like relaxation rates in optical conductivity
- nature of the magnetic fluctuations and the magnetic properties of the non-Fermi liquid as well as the transport in (pseudo) two-dimensional non-Fermi systems. In this problem we got some new results studying the electron-hole channel for the non-Fermi Anderson Model (results contained in: M. Crisan, C.P. Moca, I. Tifrea (to be published))