Statistical entropy of a class of regular black holes by brick Wall Model

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Abstract. In the paper we study the entropy of the scalar field in the background of regular black holes (Heyward and nonsingular) by using the brick-wall method suggested by the ’t Hooft. We show the leading term of entropy follows the Bekenstein-Hawking area law and the brickwall method predicts the corrections to the entropy of the regular black hole. The general structure of the coefficient of logarithmic sub-leading corrections for both black holes are same. The coefficients of the logarithmic corrections are $-1/180$ for both regular black holes which differ from the Schwarzschild black hole.

1. Introduction

The study the entropy of black hole is a laboratory to understand the models of quantum gravity. The original proposal of entropy of black hole is given by Bekenstein [1] and proportionality of the black hole is fixed by Hawking [2]

$$S_{BH} = \frac{k_B A}{4 l_P^2}$$

(1)

where $l_P = \frac{\hbar G}{c^3}$ is the Planck length, $k_B$ is a Boltman constant, $G$ is gravitational constant and $c$ is speed of light. This area-entropy law is valid for all black holes, but in the case of regular black hole it violates [3, 4, 5, 6]. Here regular means no black hole singularity. This problem arises due to the inconsistency between the area law and first law of thermodynamics. The first regular black hole solution was found by Bardeen [7] and later by Ayon Betto et al using the nonlinear electromagnetic source [8]. Similarly many black holes have been constructed by introducing nonlinear electromagnetic source [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

Ma et al [20] gave the rectified model of first law for regular black holes. The general structure of the energy-momentum tensor (EMT) of matter fields is the cause of this deviation. On inclusion of black hole mass $M$ in EMT the traditional form of the first law has been reformed with an extra factor.

$$C(M, r_+)dM = T_+ dS,$$

(2)

where $T_+$ is the Hawking temperature and $C(M, r_+)$ is

$$C(M, r_+) = 1 + 4\pi \int_{r_+}^\infty r^2 \frac{\partial T_0}{\partial M} dr$$

(3)

For Bardeen black hole
\begin{align*}
C(M, r_+) &= \frac{r_+^3}{(r_+^2 + g^2)^{3/2}} \\
T_+ &= \frac{r_+^2 - 2g^2}{4\pi r_+(r_+^2 + g^2)} \\
M_+ &= \frac{(r_+^2 + g^2)^{3/2}}{2r_+^2}
\end{align*}

Substituting these values in Eq. 2 of $C(M, r_+)$, we calculate the area law which is shown to fulfill the area law.

In the present work we study the statistical entropy of Bardeen black hole using the brick wall model developed by 't Hooft [21]. Divergence occurs in this model because the number of modes are very close to the event horizon. It has been found that the divergences in the entropy can be controlled by suggesting a cutoff parameter $h_c$ [22].

The paper is classified as follows: we study the general structure of statistical entropy of the black hole by using the brick wall model is presented in Sec. II and entropy of regular black holes with leading and sub-leading corrections in Sec. III. Concluding remarks are included in Sec. IV.

2. Entropy of black holes

We consider the static spherical symmetric line element for the black hole solutions which is given by the following relation

\[ ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2d\Omega^2 \] (7)

Following the ansatz for Maxwell equation $F_{\mu\nu} = 2\delta_\mu^\theta \delta_\nu^\phi Z(r, \theta)$ [8, 3] which leads only non-zero components $F_{\theta\phi} = -F_{\phi\theta} = g \sin \theta$. The equation of the motion of the scalar field in the black hole space-time background (7) is [23, 24, 25, 26, 27, 28]

\[ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}g^{\mu\nu} \partial_\nu \Phi) - m^2 \Phi = 0 \] (8)

Assuming $\Phi = e^{-i\omega t} e^{i\theta \phi} \phi(r)$, then the equation for $f(r)$ reads [29, 30, 31, 32, 33, 34, 35]

\[ \frac{1}{f(r)} E^2 R(r) + \frac{1}{r^2} \partial_r (r^2 f(r) \partial_r R(r)) - \frac{(l+1)}{r^2} R(r) - m^2 R(r) = 0 \] (9)

Now we use the WKB approximation to describe the $r$-dependent radial wave number $k_{EL}(r)$ which is by the relation [21, 29]

\[ f_{EL} \sim e^{\pm \int_0^r k_{EL}(r) dr} \] (10)

where $k_{EL}(r)$ is the radial number and it is given as
The number of radial waves $n_{El}$ satisfies the semi-classical quantization condition is used to find the density of states. The total number of radial modes $n_{El}$ in the brick wall model will be determined by using this relation [21, 29]

$$\pi n_{El} = \int_{r_0+\epsilon}^{L} k_{E\ell}(r) dr$$

(12)

and the total number of the states and the free energy of the system is given by the relation [21, 29]

$$\pi N_E = \sum_{\text{modes}} n_{E\ell} \approx \int^{r_{\text{max}}} (2\ell + 1) d\ell \int_{r_0+\epsilon}^{L} k_{E\ell}(r) dr$$

(13)

$$e^{-\beta F} = \sum_{\text{modes}} e^{-\beta F} = \prod_{n_{El}, l, m} \frac{1}{e^{\beta E} - 1}$$

(14)

and now the free energy of the system is calculated by using the Eq. (14), which is calculated as

$$F = \frac{1}{\beta} \sum_{n_{El}, l, m} \log(1 - e^{-\beta E})$$

(15)

$$= \frac{1}{\beta} \int d\ell(2\ell + 1) \int dE \log(1 - e^{-\beta E})$$

(16)

$$= -\frac{1}{\beta} \int d\ell(2\ell + 1) \int d(\beta E) \frac{n_{El}}{e^{\beta E} - 1}$$

(17)

$$= -\frac{1}{\pi} \int d\ell(2\ell + 1) \int dE \frac{1}{e^{\beta E} - 1} \int_{r_0+\epsilon}^{L} dr k_{E\ell}(r)$$

(18)

The integral can be done explicitly, therefore one gets

$$F = -\frac{2}{3\pi} \int_{0}^{\infty} \frac{1}{e^{\beta E} - 1} dE \int_{r_0+\epsilon}^{L} k_{E\ell}(r) dr$$

$$= -\frac{2}{3\pi} \int_{0}^{\infty} \frac{dE}{e^{\beta E} - 1} \int_{r_0+\epsilon}^{L} \frac{r^2}{f(r)} [E^2 - \mu^2 f(r)] dr$$

We may expand $f(r)$ close to event horizon for non-extremal black hole using the Taylor series

$$f(r) = f(r_+) + f'(r_+) (r - r_+) + f''(r_+) (r - r_+)^2 + ...$$

(19)

where + sign indicates a quantity evaluated at the horizon. The first term is zero as $g''(r_+) = f(r_+) = 0$ and the second term $f'(r)$ can not be zero for the black hole is non-extremal. So that the contribution of the integral near the horizon is
Therefore, the entropy of the black hole with ultraviolet cutoff is given as

\[ S = \frac{\beta^2}{\beta} \frac{\partial F}{\partial \beta} \bigg|_{\beta = \beta^+} \]

\[ = \frac{2}{3\pi f'(r_+)^2} \left\{ \frac{4\pi^4}{15\beta^4} \left[ \frac{1}{\epsilon} - \frac{2}{r_+ - f'(r_+)} \right] \log\left(\frac{L}{\epsilon}\right) + \frac{3\pi^2 m^2}{4\beta^4} f'(r_+) \log\left(\frac{L}{\epsilon}\right) \right\} + O(\epsilon^2) \]

\[ = \frac{2}{3\pi f'(r_+)^2} \left\{ \frac{4\pi^4}{15\beta^4} \left[ \frac{1}{\epsilon} - \frac{2}{r_+ - f'(r_+)} \right] \log\left(\frac{L}{\epsilon}\right) + \frac{3\pi^2 m^2}{4\beta^4} f'(r_+) \log\left(\frac{L}{\epsilon}\right) \right\} + O(\epsilon^2) \]

Then the contribution of the free energy near the horizon is

The entropy can be derived from the free energy by using the following relations

\[ S = \beta^2 \frac{\partial F}{\partial \beta} \bigg|_{\beta = \beta^+} \]

\[ = \frac{2}{3\pi f'(r_+)^2} \left\{ \frac{4\pi^4}{15\beta^4} \left[ \frac{1}{\epsilon} - \frac{2}{r_+ - f'(r_+)} \right] \log\left(\frac{L}{\epsilon}\right) + \frac{3\pi^2 m^2}{4\beta^4} f'(r_+) \log\left(\frac{L}{\epsilon}\right) \right\} + O(\epsilon^2) \]

\[ = \frac{2}{3\pi f'(r_+)^2} \left\{ \frac{4\pi^4}{15\beta^4} \left[ \frac{1}{\epsilon} - \frac{2}{r_+ - f'(r_+)} \right] \log\left(\frac{L}{\epsilon}\right) + \frac{3\pi^2 m^2}{4\beta^4} f'(r_+) \log\left(\frac{L}{\epsilon}\right) \right\} + O(\epsilon^2) \]

Therefore, the entropy of the black hole with \( \beta^+ = 2\pi f'(r_+) \) near the horizon is calculated to be the proper distance of the brick wall from the horizon under near horizon approximation is related to the ultraviolet cutoff as given as

\[ h_\epsilon = \int_{r_+}^{r_+ + \epsilon} f^{-1/2}(r) \, dr \approx \frac{1}{\sqrt{f'(r_+)} \int_{r_+}^{r_+ + \epsilon} (r - r_+)^{-1/2} \, dr - \int_{r_+}^{r_+ + \epsilon} \frac{f''(r_+)}{f^{3/2}(r_+)} (r - r_+)^{1/2} \, dr} \]

\[ = 2 \sqrt{\frac{\epsilon}{f'(r_+)}} + O(\epsilon^{3/2}) \]

So in terms of the proper distance the entropy of the black hole reads
\[ S_{BW} \approx \frac{r_+^2}{90h_c^2} - \left[ \frac{r_+^2}{180} \left( \frac{2f'(r_+)}{r_+} - f''(r_+) \right) - \frac{r_+^2 m^2}{6} \right] \log\left( \frac{r_+}{h_c} \right) \]  

(23)

3. Regular Black hole

3.1. Heyward black hole

The first example is Heyward black hole [36, 19], the metric function of the Heyward black hole is given by

\[ f(r) = 1 - \frac{2Mr^2}{(r^2 + 2gm^2)^{3/2}} \]  

(24)

If the infrared cutoff approaches the event horizon, i.e., if \( L \rightarrow r_+ \), then the entropy is

\[ S_{BW} \approx \frac{r_+^2}{90h_c^2} - \left[ \frac{r_+^2}{180} \left( \frac{18r_+^4}{(r_+^2 + 2gm^2)^2} + \frac{2}{r_+^2} - \frac{16r_+}{r_+^2 + 2gm^2} - \frac{4g^3}{r_+^2 (r_+^2 + 2gm^2)} \right) - \frac{r_+^2 m^2}{6} \right] \log\left( \frac{r_+}{h_c} \right) \]  

(25)

Substituting the value of \( f(r) \) from Eq. (29) into Eq. (25), we get the entropy of the Heyward black hole with logarithmic corrections

\[ S_{BW} \approx \frac{r_+^2}{90h_c^2} - \left[ \frac{r_+^2}{180} \left( \frac{18r_+^4}{(r_+^2 + 2gm^2)^2} + \frac{2}{r_+^2} - \frac{16r_+}{r_+^2 + 2gm^2} - \frac{4g^3}{r_+^2 (r_+^2 + 2gm^2)} \right) - \frac{r_+^2 m^2}{6} \right] \log\left( \frac{r_+}{h_c} \right) \]  

(26)

The leading order term in Eq. (26) is the Bekensten-Hawking area law and the subleading corrections is logarithmic corrections.

3.2. Nonsingular Black hole

The metric function of the nonsingular black hole [37, 38, 39, 40] is given by

\[ f(r) = 1 - \frac{2Me^{-kr}}{r} \]  

(27)

where \( M \) is the mass of the black hole and \( g \) is the magnetic monopole charge, substituting the value of \( f(r) \) from Eq. (26) into Eq. (25), we get!

\[ S_{BW} \approx \frac{r_+^2}{90h_c^2} - \left[ \frac{r_+^2}{180} \left( \frac{4k^2}{r_+^2} - \frac{6k^2}{r_+^2} + \frac{k^2}{r_+^2} \right) - \frac{r_+^2 m^2}{6} \right] \log\left( \frac{r_+}{h_c} \right) \]  

(28)

This is the entropy of the nonsingular black hole. The leading term follow the standard area law. The logarithmic sub-leading corrections are present due to the quantum fluctuations near the horizon. The pre-factor of the both regular black holes are same \(-1/180\) which is different from the singular black holes.

4. Conclusion

In the paper, we have studied the entropy for the scalar field in the presence of the class of regular black holes by using the brick wall model which was proposed by 't Hooft. We show that the brick-wall method predicts the logarithmic sub-leading corrections to the Bekenstein-Hawking entropy. The coefficient of logarithmic sub-leading corrections for regular black hole are same but different for the
singular black hole. The divergence term of the entropy could be properly renormalized by the standard re-normalization of the coupling constant as suggested by Susskind and Uglum [22]

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