Wave-kinetic approach to the Schrödinger–Newton equation

J T Mendonça
IPFN, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisboa, Portugal
E-mail: titomend@tecnico.ulisboa.pt

Abstract
We discuss the Schrödinger–Newton (SN) equation in the context of cold atom physics. For that purpose, we establish the wave-kinetic equation equivalent to the SN equation, which stays valid in different physical scenarios relevant to cold atoms. They include: (1) the usual scenario of matter confined in a self-gravitating field, (2) atomic molasses, confined and cooled by laser beams in a magneto-optical trap (MOT), (3) Bose–Einstein condensates, with or without long range dipolar interactions, and (4) electron states in a quantum plasma. We show that these different systems can be described by a formally identical equation, and they also manifest similar elementary excitations. The wave-kinetic equation is obtained by following the well-known Wigner–Moyal procedure, allowing the representation of quantum states in a classical phase-space. It is particularly well suited to discuss kinetic properties associated with Landau damping, as shown. We also consider generalisation of the SN equation onto the relativistic domain, and its impact on the proposed wave-kinetic description.

1. Introduction

The quantum nature of gravity is one of the major questions in theoretical physics. Is gravity essentially classical, or can the gravitational field be quantised, as all the other physical fields are? No definite answer is known at the moment [1, 2].

The Schrödinger–Newton (SN) equation, sometimes also called the Schrödinger–Poisson equation, has been considered in this context in recent years, namely by Diósi [3], Penrose [4] and others [5, 6], and provides a simple semi-classical model of gravity. This equation can be derived from two different perspectives, one where both matter and gravity are quantised, and the other where only matter is quantised and gravity is assumed as intrinsically classical, the so-called semi-classical approach [7].

The interest of the SN equation is that it suggests the possible use of laboratory tests of quantum gravity, which include search for the existence of equilibrium quantum states and study of the dispersion properties of the elementary excitations. A bridge with present day experiments using cold atoms [8], Bose–Einstein condensates (BECs) [9] and superfluidity of light [10–12] can eventually be explored. Simulation of gravitational phenomena and SN effects using nonlinear optics have in fact been recently discussed [13, 14].

Here we propose a new approach to the SN equation, based on the wave-kinetic theory. We follow the well-known Wigner–Moyal procedure, aimed to provide the representation of quantum states in a classical phase-space [15, 16]. We show that, starting from the SN equation, a generic wave-kinetic equation can be derived, which stays valid in many different scenarios relevant to experimental research in atomic and plasma physics. They include: (1) the usual scenario of matter confined in a self-gravitating field, (2) atomic molasses, confined and cooled by laser beams in a magneto-optical trap (MOT), (3) BECs, with or without long-range dipolar interactions, and (4) electron states in the electrostatic mean-field of a quantum plasma.

We show that, not only these different systems can be described by a formally identical equation, but they also manifest similar elementary excitations. We discuss, as particular cases, the Jeans instability of self-gravitating matter, the hybrid-phonon modes in ultra-cold matter, the Bogoliubov excitations in dipolar condensates and the electron plasma waves in quantum plasmas. Kinetic effects associated with these various
types of elementary excitations, the possible occurrence of Landau damping, kinetic and quantum Jeans instability corrections are also discussed. Finally, we extend our discussion onto the relativistic domain, and comment on the consequences of relativistic effects for the wave-kinetic equation.

2. Basic equations

The SN equation results from the coupling between the usual Schrödinger equation for a quantum particle in the non-relativistic regime, and the Poisson’s equation describing the self-gravitating potential $V_G$ associated with Newton’s law. Our starting equations can therefore be written as

$$i\hbar \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_0 + mV_G \right) \psi,$$

(1)

where $m$ is the mass of the particle, $V_0$ an arbitrary external potential. The Newtonian potential $V_G$ is determined by

$$\nabla^2 V_G = 4\pi mG |\psi|^2,$$

(2)

where $G$ is the gravitational constant. This leads to a single integro-differential equation of the form

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_0 - m^2G \int \frac{|\psi(r', t)|^2}{|r - r'|} dr' \right] \psi,$$

(3)

This equation displays a strong similarity with those pertaining to other systems where long range interactions can occur, such as cold atoms in a MOT, BECs with dipolar interactions, and plasmas. For that reason, we propose to study this equation under a more generic form, namely

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_0 + g \int U(r - r')|\psi(r', t)|^2 dr' \right] \psi,$$

(4)

which reduces to the previous SN equation of a quantum particle in a self-gravitating potential, if we take

$$g = -m^2G, \quad U(r - r') = \frac{1}{|r - r'|},$$

(5)

and the external potential $V_0$ can be assumed as arbitrary. The same equation stands for the electron states in the electrostatic mean-field of a non-relativistic quantum plasma, where we should use [17]

$$g = \frac{e^2}{4\pi\varepsilon_0}, \quad U(r - r') = \frac{1}{|r - r'|}, \quad V_0 = -\frac{e^2}{\varepsilon_0} n_i,$$

(6)

where $e$ is the electron charge and $n_i$ is the ion density (with the ions assumed at rest). On the other hand, we can use equation (4) to describe the mean-field behaviour of ultra-cold atoms in a MOT, using [18]

$$g = \frac{Q}{4\pi}, \quad U(r - r') = \frac{1}{|r - r'|}, \quad V_0 = -Qn_{eq},$$

(7)

where $n_{eq}$ is the equilibrium density imposed by some unspecified confinement force, and $Q = (\sigma_R - \sigma_j) n_i I_a / c$ is the effective atomic charge, resulting from laser cooling beams with intensity $I_a$ [19, 18]. Here, $\sigma_R$ and $\sigma_j$ are the radiation and laser absorption cross-sections, as usually defined in atomic physics. It should be noticed that, in this case, the analogy with a SN system is not perfect, because the confinement force usually depends on the atom velocity and cannot be simply derived from a scalar potential. But the dependence on the atom velocity is only relevant to the atomic cooling process, and is not directly responsible for the collective atom interactions. We can therefore ignore it in our present model.

Finally, the SN equation (4) can be used to describe a BEC with long range dipolar interactions, if we define [9, 20]

$$g = 4\pi\hbar^2am, \quad U(r - r') = \delta(r - r') + \frac{C_{dd}}{8\pi g} \frac{1}{|r - r'|^3}(3\cos^2\varphi - 1),$$

(8)

where $a$ is the scattering length. Here, $C_{dd}$ is the dipolar (electric or magnetic) interaction strength, $\theta$ the angle between the vector $(r - r')$ and the direction of the external polarisation field, and $\varphi$ the orientation angle. The Dirac delta term in the interaction potential describes the usual contact interaction of the condensate, associated with the short range and nearly zero energy atomic collisions characterised by the strength parameter $g$.

This short discussion shows the interest of the proposed new form of the SN equation (4), and the variety of model analogies that can be explored between a purely gravitational problem, associated with the standard SN model, and the various laboratory systems that can be found in cold atoms, condensates and quantum plasmas.
3. Wave-kinetic equation

We now define the Wigner function \( W \equiv W(r, q, t) \) associated with the wavefunction \( \psi(r, t) \), as the Fourier transform of its autocorrelation function, or
\[
W(r, q, t) = \int \psi^*(r - s/2, t) \psi(r + s/2, t) \exp(iq \cdot s) \, ds.
\]
(9)

Applying the well-known Wigner–Moyal procedure [15, 16], and starting from the SN equation (4), we can derive an evolution equation for \( W \), of the form
\[
i\hbar \left( \frac{\partial}{\partial t} + v_q \cdot \nabla \right) W = g \int U(\kappa) n(\kappa, t) \Delta W e^{i\kappa \cdot r} \frac{d\kappa}{(2\pi)^3},
\]
(10)

where \( v_q = h q / m \) is the particle velocity, and \( h q \) is the particle momentum. We also have defined \( \Delta W = [W^- - W^+] \), with \( W^\pm = W(r, q \pm \kappa/2, t) \). In equation (10) we have also used the spectral density components
\[
n(\kappa, t) = \int n(r, t) \exp(-i\kappa \cdot r) \, dr,
\]
(11)

where the density (or probability density) is defined as
\[
n(r, t) = |\psi(r, t)|^2 = \int W(r, q, t) \frac{dq}{(2\pi)^3}.
\]
(12)

The Fourier transform of the long-range interaction potentials, \( U(\kappa) \), can easily be obtained from equations (5) – (8). For the first three cases, we simply have
\[
U(\kappa) = \frac{4\pi}{\kappa^2},
\]
(13)

whereas for the remaining case of a dipolar BEC, we obtain
\[
U(\kappa) = 1 + \frac{C_6}{g} (\cos^2 \theta_e - 1/3),
\]
(14)

where \( \theta_e \) is the angle between the wavevector \( \kappa \) and the polarisation field (we have assumed \( \varphi = 0 \) for simplicity). The first term in this expression results from the short-range atomic collisions at zero energy, which occur inside the BEC. Although, at first sight, the wave-kinetic equation (10) seems counter-intuitive and somewhat hermetic, its use can be found simple and meaningful in many relevant physical situations. This is illustrated next, for the case of elementary excitations in the medium. As an additional remark to equation (10) it is interesting to note that, although it describes quantum effects, it has the mathematical structure of a classical master equation (see, for instance, [21, 22]).

4. Elementary excitations

Let us consider density perturbations which can be excited locally in the medium, on a scale much shorter than the size of the system. In this case we can ignore boundary conditions, as well as the external potential. The perturbations can be described by using \( \tilde{W} = W_0 + \tilde{W} \), where \( W_0 \) represents some given equilibrium state and \( \tilde{W} \) the perturbed quantity, which is assumed to evolve in space and time as \( \exp(i\mathbf{k} \cdot \mathbf{r} - \omega t) \). Linearising the wave-kinetic equation (10) with respect to the perturbed quantities, we can easily get
\[
\tilde{W} = g U(k) \frac{\Delta W_0}{\hbar (\omega - k \cdot v_q)} \tilde{n}(k),
\]
(15)

For mathematical details of the linearisation method, see [20]. Integrating over the \( q \) spectrum, or equivalently, over the particle velocity space, and using equation (12), we can then derive the dispersion relation for the elementary excitations in the medium, as
\[
1 - g U(k) \int \frac{\Delta W_0}{\hbar (\omega - k \cdot v_q)} \frac{dq}{(2\pi)^3} = 0.
\]
(16)

This can also be written in an alternative form, as
\[
1 - g \frac{U(k)}{\hbar} \int W_0(q) \left[ \frac{1}{(\omega_+ - k \cdot v_q)} - \frac{1}{(\omega_- - k \cdot v_q)} \right] \frac{dq}{(2\pi)^3} = 0,
\]
(17)

where we have defined the auxiliary frequencies \( \omega_\pm = \omega \pm \hbar k^2/2m \). It is useful to consider these results in the zero-temperature limit, as determined by the following equilibrium distribution:
\[
W_0(q) = (2\pi)^3 n_0 \delta(q - q_0).
\]

Replacing this in the dispersion relation (17), we can easily get
indeed take place in a quasi 1D con-...\text{plasmas}...\text{processes}. Quantum effects on Jeans instability have previously been considered for self-gravitating dusty gravitational collapse. But it should be compared with those associated with thermal effects and other kinetic...Alternatively, we can say that, for a spherical cloud of matter with uniform density...Here, we recognise the Jeans frequency...therefore useful to consider the particular forms of dispersion in different media. A medium at rest, given the generality of our approach, this dispersion relation applies to a variety of physical situations. It is...quantum corrections, which tend to stabilise the self-gravitating instability for large values of \( k \), or small wavelengths. This stabilising effect is similar to that occurring in attractive BECs [9]. The quantum regime of the Jeans instability will therefore take place for matter density above a certain value, such that...Given the generality of our approach, this dispersion relation applies to a variety of physical situations. It is...\text{instability will only take place for large radii, such that} \[ a^2 \geq \hbar \left( \frac{\pi^2}{n_0 \rho_0 m^3 G} \right)^{1/2}. \] (22) This stabilising effect of the quantum corrections can be seen as a possible signature of quantum effects in a gravitational collapse. But it should be compared with those associated with thermal effects and other kinetic processes. Quantum effects on Jeans instability have previously been considered for self-gravitating dusty plasmas [23]. In the astrophysical context they are usually small, as compared with thermal effects. However, in the context of atomic physics, they could become relevant in the low temperature limit, as discussed below. Considering now the case of electron oscillations in a quantum plasma, we can use equations (6) and (13), to obtain the well-known result for electron oscillations...The quantum effects introduce a dispersion term proportional to \( k^4 \), which in general remains small with respect to the thermal dispersion effects to be discussed below. This is formally identical to the dispersion equation valid for sound waves in ultra-cold matter in a MOT, the so-called hybrid phonons [18], except that the electron plasma frequency is replaced by an effective plasma frequency, valid for the neutral gas, which can be defined by \( \omega_p = Q n_0 / m \). This is, of course, valid for \( Q > 0 \), or \( \sigma_L < \sigma_R \), which is usually the case in typical MOT experiments. When \( \sigma_L > \sigma_R \), the quantity \( Q \) would be negative, and the system would behave like a self-gravitating medium, or an attractive BEC. This is unlikely to occur in the usual three-dimensional atom traps, but can indeed take place in a quasi 1D configuration, as considered by [24, 25]. We can therefore consider the case of a laser-cooled atomic gas in a MOT as an intermediate case, between the quantum plasma and the self-gravitating neutral gas. According to the sign of the quantity \( Q \), it could mimic the dispersion properties of both media. Finally, the case of dipolar BECs can be derived from equation (19) using the appropriate parameters, as defined by equation (8). We get...where the angle-dependent velocity \( c(\theta_k) \) is defined by...Here, we have introduced the parameter \( \eta = C_{dd}/g \), characterising the strength of long-range dipolar interactions, and defined the usual Bogoliubov sound speed as \( c_s = \sqrt{gn_0 / m} \). Quantum fluctuation corrections, and other effects such as those associated with finite-energy collisions could also eventually be included [20]. The different dispersion curves are illustrated in figure 1, showing the similarities as well as the qualitative differences.
between the four considered media. A table with the characteristic parameters of these four media is shown in figure 2.

5. Kinetic effects

Kinetic effects include finite temperature corrections, as well as Landau damping of elementary excitations. Here we mainly focus on the case of a self-gravitating gas, as described by the original SN equation, because the other cases have already been considered in the literature. For that purpose, we go back to the dispersion relation (16) and rewrite it in terms of the parallel velocity distribution

\[ G(q) = \int W_0(q_\perp, q_\parallel) \frac{dq_\perp}{(2\pi)^2}, \]

where we have used the parallel and perpendicular components of the particle velocity and momentum, according to

\[ v_\parallel = \frac{k}{k} + v_\perp, \quad q = \frac{k}{k} + q_\perp. \]
We then get
\[ 1 - \frac{g}{\hbar} \mathcal{U}(k) \int \frac{\Delta G_0}{(\omega - ku) \, 2\pi} \, dq = 0. \] (28)

Normalising the parallel distribution in such a way that \( \int G_0(u) \, du = 1 \), this can also be written in the form
\[ 1 - \frac{g n_0 k^2}{m \omega^2} \mathcal{U}(k) \int \frac{G_0(u) \, du}{(1 - ku / \omega)^2 - \hbar^2 k^4 / 4m^2 \omega^4} = 0. \] (29)

We now consider the expansion of the integrand for \( u \ll \omega / k \), which is valid for excitations with large phase velocities. Assuming even distributions, which are characteristic of nearly equilibrium conditions, and defining the mean square velocity \( \langle u^2 \rangle \) such that
\[ \int u G_0(u) \, du = 0, \quad \int u^2 G_0(u) \, du = \langle u^2 \rangle, \] (30)

this leads to
\[ \omega^2 = \Omega^2 \left( 1 + \frac{3 k^2}{\omega^2} \langle u^2 \rangle \right) + \frac{\hbar^2 k^4}{4m^2 \omega^4}. \] (31)

Here, the characteristic frequency \( \Omega \) is defined by
\[ \Omega^2 = \frac{g n_0 k^2}{m} \mathcal{U}(k). \] (32)

This is the general form of the dispersion relation for elementary excitation at finite temperature. For the self-gravitating medium, we introduce the Jeans frequency \( \omega_J \) such that \( \Omega^2 = -\omega_J^2 \), and we get
\[ \omega^2 \approx -\omega_J^2 + 3 k^2 \langle u^2 \rangle + \frac{\hbar^2 k^4}{4m^2 \omega^4}. \] (33)

which generalises equation (20). This result stays valid as long as the thermal and quantum corrections remain small, and \( |\omega^2| \ll \omega_J^2 \). It clearly shows that, apart from quantum effects, the Jeans instability can also be saturated by thermal effects associated with a finite value of the mean square velocity of the self-gravitating matter, \( \langle u^2 \rangle \approx 0 \). If, instead of a self-gravitating neutral gas we were considering a quantum plasma, the Jeans frequency would be replaced in this dispersion equation (apart from a difference in sign on the first term) by the plasma frequency, with \( \Omega^2 = \omega_p^2 [17] \).

Let us now consider Landau damping. We have seen that the Jeans instability can be saturated by thermal and quantum effects. An additional source of instability saturation is related with a resonant kinetic effect, called Landau damping to be discussed next. But, a more interesting physical situation occurs when Landau damping becomes negative and is reverted into Landau growth. In this case we can use the standard formulation of Landau damping and assume \( \omega = \omega_r + i \Gamma \), where the real part of the frequency is positive, \( \omega_r > 0 \), and the imaginary part is small, \( \omega_r \gg |\Gamma| \). The case of \( |\omega_r| \approx |\Gamma| \) is more complicated and will be ignored. For the purpose of our present analysis, it is useful to rewrite the dispersion relation (28) as \( 1 - \chi(\omega, k) = 0 \), where \( \chi \equiv \chi(\omega, k) \) is the susceptibility of the medium. We can further use \( \chi = \chi_r + i \chi_i \), where the real part is determined by the principal part of the integral in (28) and is determined by
\[ \chi_r(\omega, k) \approx \frac{\Omega^2}{\omega^2} \left( 1 + \frac{3 k^2}{\omega^2} \langle u^2 \rangle \right) - \frac{\hbar^2 k^4}{4m^2 \omega^4}. \] (34)

as seen before, whereas for the imaginary part we have to retain the pole contributions to the integral, and use
\[ \chi_i(\omega, k) = \frac{\Omega^2}{k^2} \int \frac{G_0 - G_0^\dagger}{(\omega - ku) \omega} \, dq. \] (35)

Using \( \omega = \omega_r + i \Gamma \), with \( \omega_r \gg |\Gamma| \), we get
\[ \left( \frac{\partial \chi_r}{\partial \omega} \right) \quad \Gamma = -\frac{\chi_i(\omega_r, k)}{(\partial \chi_r / \partial \omega)_{\omega_r}} \] (36)

This then leads to
\[ \Gamma \approx -\frac{\omega_r}{2} \left[ G_0(u_+ - G_0(u_-) \right]. \] (37)

For an inversion of population, such that \( G_0(u_+) > G_0(u_-) \), the kinetic regime of Jeans instability can eventually occur, for wavenumbers \( k \) where no hydrodynamic instability is possible.
6. Relativistic regime

For relativistic particles, and assuming that spin effects are negligible, the Schrödinger equation can be replaced by a Klein–Gordon equation, and the generic SN equation (4) is transformed into

$$\left[ i\hbar \frac{\partial}{\partial t} - V_0 - g \int \mathcal{L}(r - r')\psi(r', t) \right] \psi = (m^2 c^4 - \hbar^2 c^2 \nabla^2) \psi. $$

(38)

It is well known that, if we assume $$\psi = \psi' \exp(-i\omega_0 t),$$ with $$\omega_0 = mc^2 / \hbar,$$ and the new amplitude $$\psi'$$ slowly evolves in time, such that $$|\partial \psi' / \partial t| \ll |\omega_0 \psi'|,$$ we can easily show that $$\psi'$$ satisfies a Schrödinger equation. Not all the cases associated with the generic SN equation (4) are relevant in the relativistic domain. But, at least two cases can be retained, the self-gravitating matter and the relativistic plasma. The case of a relativistic BEC could also be envisaged. However, the ultra-cold gas in a MOT can never be considered in the relativistic domain and should be disregarded. The interesting thing about the SN equation in the relativistic regime is that we can apply the Wigner–Moyal procedure to equation (38) and obtain a new wave-kinetic equation which strongly resembles the non-relativistic one, and takes the form [26]

$$i\hbar \left( \frac{\partial}{\partial \tau} + u \cdot \nabla \right) W = g \int \mathcal{L}(k) n(k, t) \Delta W e^{i k \tau} \frac{dk}{(2\pi)^3}, $$

(39)

where $$\tau = t / \gamma$$ is the proper time, $$u = v / \gamma$$ is the covariant velocity, and $$\gamma = \sqrt{1 + u^2 / c^2}$$ is the relativistic factor. This formal analogy with the previous wave-kinetic equation (10), is a major advantage of the wave-kinetic approach, because the above discussion of the elementary excitations in a non-relativistic medium can easily be transposed onto the relativistic domain. However, a more detailed analysis of the relativistic case is outside the scope of the present work.

7. Conclusions

In this work, we have considered the wave-kinetic description of the Schrödinger–Newton equation. We have assumed a generic form of the SN equation, which can be applied to many different physical systems, such as matter confined in a self-gravitating field, a laser-cooled atomic gas, BECs with long range dipolar interactions, and quantum plasmas. Following the Wigner–Moyal procedure, we were able to derive an equivalent wave-kinetic equation, which can be used to describe the behaviour of these four different media. In particular, elementary excitations in the media were considered, and a generic dispersion relation was derived. This equation could then be used to describe a variety of different wave phenomena, which include the Jeans instability in a self-gravitating gas, hybrid phonons in a MOT, electron oscillations in a plasma and Bogoliubov oscillations in a dipolar condensate. They all display very similar dispersion properties and quantum corrections, but different stability regimes.

The Jeans instability in the quantum regime was considered, and it was shown that quantum effects tend to stabilise the gravitational instability. Thermal and kinetic effects were also discussed, which include finite temperature dispersion effects and Landau damping. A kinetic regime for the Jeans instability was also shown to exist. Finally, generalisation of the SN equation onto the relativistic domain was briefly discussed. It was shown that, when spin effects are ignored, the relativistic wave-kinetic equation is formally very similar to that derived from the usual SN equation.

Our discussion could be useful to identify some possible laboratory analogues of quantum gravity, using ultra-cold atoms, BECs and quantum plasmas. This could eventually open the way to new experimental approaches to quantum simulations of gravitational interactions and other astrophysical phenomena.

In order to simplify the description of the elementary excitations we have assumed that the medium was homogeneous. This is only valid for local modes, when the scale of the inhomogeneity in the medium is much larger than the mode wavelength. But our approach can also be applied to global modes, where the inhomogeneity and the finite size of the medium can be taken into account. A distinction between local and global modes, in the context of wave-kinetics can be found in [27].

One last remark concerns the search for static equilibria inside a given confining potential, $$V_0.$$ For that purpose, it would be more convenient to use the fluid equations that can be derived from both the SN equation (4) and its wave-kinetic equivalent, equation (9). Neglecting small kinetic effects, and assuming harmonic confinement, this would lead to the well-known Thomas–Fermi equilibrium, in the case of a BEC. Similarly, using a polytropic equation of state, different equilibrium profiles could be established for a laser-cooled gas [28]. The validity of this approach was confirmed by recent detailed experiments in a large MOT [29]. It should be noticed, in this context, that the equilibrium profiles of the laser-cooled gas are described by a generalised Lane–Emden equation [28], very similar to the equation describing matter equilibrium in a gravitational field [30, 31]. Application of this approach to BEC equilibrium and its implications to dark matter
research has also been considered [32, 33]. Possible quantum gravitational effects are usually more pronounced near the edge of the different density profiles, a result well-known in the case of BECs. This leads to the conclusion that the most promising approaches to study quantum processes described by the SN equation would be, either the quantum dispersion effects in elementary excitations as considered in the present work, or the edge effects in equilibrium density profiles as described by a generalised Lane–Emden equation.

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