Generalized Albanese morphisms

Georg Hein

September 27, 2004

Abstract

We define generalizations of the Albanese variety for a projective variety $X$. The generalized Albanese morphisms $\text{alb}_r : X \to \text{Alb}_r(X)$ contract those curves $C$ in $X$ for which the induced morphism $\text{Hom}(\pi_1(X), U(r)) \to \text{Hom}(\pi_1(C), U(r))$ has a finite image up to conjugation. Thus, they may be interpreted as a $U(r)$-version of the Shafarevich morphism. It is shown that for an algebraic surface $X$, we have a regular morphism $\text{alb}_r : X \to \text{Alb}_r(X)$ with the above property.

1 Introduction

The Shafarevich conjecture states that the universal cover $\tilde{X}$ of a projective variety $X$ is holomorphically convex. This means that there exists a proper surjection $\tilde{X} \to Y$ to a Stein space. This implies (see \cite{10}) the existence of a morphism $\text{sh}_X : X \to \text{Sh}(X)$ with connected fibers with the property that a curve $C$ in $X$ is contracted by $\text{sh}_X$, iff the image of $\pi_1(C)$ in $\pi_1(X)$ is finite. The Shafarevich variety $\text{Sh}(X)$ is the quotient of the Stein space $Y$ by the fundamental group $\pi_1(X)$ of $X$.

The Shafarevich conjecture is very hard to tackle in terms of algebraic geometry because the holomorphic map $\tilde{X} \to X$ is not a regular morphism of algebraic varieties unless the fundamental group of $X$ is finite. It is, however, possible to deal with the problem of finding the morphism $\text{sh}_X$. A first version of such a morphism was given by Kollár in \cite{10}. For Kähler manifolds, Campana constructs \cite{3} a generic version of the Shafarevich morphism. Katzarkov (see \cite{8} and \cite{9}) describes a representation theoretical description of $\text{sh}_X$, for one representation $\pi_1(X) \to \text{GL}(r)$. We offer a different approach to such a description for all representations by considering generalized Albanese varieties.

Two constructions for the Albanese variety exist. One construction \cite{21} uses global holomorphic one forms, while another \cite{24} defines the Albanese variety as the dual of the Picard torus. The intermediate Jacobians are a generalization of the first construction. Here we try to show how to generalize the second construction. In the sense of Kollár \cite{10}, where the Albanese variety can be see as the abelian or $U(1)$-version of the Shafarevich morphism, we intend to give an $U(r)$-version of the Shafarevich morphism that is a map $\text{alb}_r(X) : X \to \text{Alb}_r(X)$ having connected fibers, such that an irreducible curve $C$ in $X$ is contracted under $\text{alb}_r(X)$, iff the image of the composition $\text{Hom}(\pi_1(X), U(r)) \to \text{Hom}(\pi_1(C), U(r))$ has a finite image up to conjugation. If the image of $\pi_1(C)$ in $\pi_1(X)$ is finite, then $C$ is contracted by the generalized Albanese morphisms. Therefore, the generalized Albanese morphisms would factor through the Shafarevich morphism.

In section \cite{2} we review the constructions and some properties of the Albanese variety.
The main result of section 3 is the construction of the nef line bundle $L_r$ on $X$ whose properties are investigated in section 4. The main property of $L_r$ is that it has degree zero exactly on those curves $C$ in $X$ for which the morphism $\text{Hom}(\pi_1(X), U(r)) \to \text{Hom}(\pi_1(C), U(r))$ has a finite image up to conjugation (see theorem [48]). Thus, $L_r$ behaves like the pull back of some ample line bundle from $\text{Alb}_r(X)$ to $X$.

In the fifth section, we use the nef reduction to obtain a rational map $\text{alb}_r : X \to \text{Alb}_r(X)$ with the above property for the general fiber. In the sixth section, we go over the construction for the case of a projective algebraic surface $X$. Here we are able to deduce the existence of regular morphisms $\text{alb}_r : X \to \text{Alb}_r(X)$ for all integers $r \geq 1$ (see [62]). In the last section, we give an example of a surface with a classical Albanese variety of dimension one and generalized Albanese varieties of dimension two. Thus, this example shows that the generalized Albanese morphism may “reveal” more than the classical Albanese morphism does.

**Notations.** We work with schemes over the complex numbers. Since we need the restriction of (semi)stable vector bundles to curves, we are required to use the concept of Mumford–Takemoto or slope stability for vector bundles.

## 2 Two constructions for the Albanese variety

### 2.1 The classical construction for the Albanese variety.

Here we assume that $X$ is a connected Kähler manifold. We define the Albanese variety $\text{Alb}(X)$ to be the quotient

$$\text{Alb}(X) := H^0(X, \Omega^1_X)^\vee / H_1(X, \mathbb{Z}) .$$

If we choose a point $x_0 \in X(\mathbb{C})$, then we can define the Albanese morphism $\text{alb}_X : X \to \text{Alb}(X)$ by $x \mapsto \int_{\gamma_x} \gamma_x$ where $\gamma_x$ is a path connecting $x_0$ with $x$.

### 2.2 The $\text{Pic}^0(\text{Pic}^0)$-description of $\text{Alb}(X)$.

Let $X$ be a smooth variety over an algebraically closed field $k$. We consider the Picard torus $\text{Pic}^0(X)$, i.e., the component of $\text{Pic}(X)$ containing $\mathcal{O}_X$. Furthermore, we consider a Poincaré bundle $L$ on $X \times \text{Pic}^0(X)$. This bundle is not unique. To normalize it we choose a point $x_0 \in X(k)$. If we require that $L|_{x_0} \times \text{Pic}^0(X) \cong \mathcal{O}_{\text{Pic}^0(X)}$, then the Poincaré bundle $L$ is uniquely determined. If we consider $L$ as a family of line bundles on $\text{Pic}^0(X)$ parametrized by $X$, then we obtain a morphism from $X$ to the Picard torus of $\text{Pic}^0(X)$:

$$\text{alb}_X : X \to \text{Pic}^0(\text{Pic}^0(X)) =: \text{Alb}(X) .$$

### 2.3 Both constructions coincide for smooth varieties over $\text{Spec}(\mathbb{C})$.

The use of the same notations in the above constructions is justified, because both coincide for a smooth projective variety over $\text{Spec}(\mathbb{C})$. This follows from the universal property of the Albanese variety and the duality between the Albanese variety and the Picard torus (cf. [5]). The following proposition describes the fibers of the Albanese morphism. Since it uses both descriptions (211 and 212), we have to assume that $X/\text{Spec}(\mathbb{C})$ is a smooth projective variety.

### 2.4 Proposition. (Description of the fibers of the Albanese morphisms $\text{alb}_X : X \to \text{Alb}(X)$, for a projective complex manifold $X$ (cf. II.6 in [6]).)

If $i : Z \to X$ is a connected cycle in $X$, then the following conditions are equivalent:

(i) $Z$ is contained in a fiber of the morphism $\text{alb}_X$;
(ii) The image of \(i_* : H_1(Z, \mathbb{Z}) \to H_1(X, \mathbb{Z})\) is finite; 

(iii) The pull back morphism \(\iota^* : H^0(X, \Omega_X^1) \to H^0(Z, \Omega_Z^1)\) is trivial; 

(iv) Let \(\rho : \pi_1(X) \to U(1)\) be a representation of the fundamental group. Then the restriction \(\rho|_{\pi(X)}\) has a finite image; 

(v) If \(L\) is a line bundle on \(X \times S\), then the pull back \(\iota^* L\) on \(Z \times S\) is of the form \(L_1 \boxtimes L_2\), for any Noetherian scheme \(S\).

3 The line bundle \(\mathcal{L}_r\)

3.1 The setup. We fix a smooth projective variety \(X\) of dimension \(n\) with a very ample line bundle \(\mathcal{O}_X(H)\) and a positive integer \(r\). Furthermore, we choose a geometric point \(x_0 \in X\). Let \(M_r = M_X(r, 0, 0, \ldots, 0)\) be the moduli space of \(S\)-equivalence classes of slope semistable rank \(r\) bundles \(E\) with trivial Chern classes in \(H^*(X, \mathbb{Z})\). If \(E\) is a vector bundle parametrized by \(M_r\), then we write \([E]\) for the corresponding point in \(M_r(\mathbb{C})\). By the theorem of Uhlenbeck and Yau (see [16]), \(M_r\) parameterizes flat vector bundles on \(X\) or representations of \(\pi_1(X)\) in \(U(r)\) modulo conjugation. This implies that for \([E] \in M_r\), the restriction \(E|_C\) of \(E\) to any curve \(C \subset X\) is semistable. Moreover, \(M_r\) is a projective scheme provided that we pass to \(S\)-equivalence classes of semistable bundles. This means we identify any vector bundle \(E\) in a short exact sequence \(0 \to E' \to E \to E'' \to 0\) of slope zero bundles with the direct sum \(E' \oplus E''\). Thereafter, we will use the symbol \(M_r\) (or \(M_r(X)\)) for the projective moduli space of \(S\)-equivalence classes of slope semistable bundles on \(X\).

3.2 The line bundle \(\mathcal{O}_M(D_H)\). Using the polarization \(H\) on \(X\), we can define a polarization \(D_H\) on \(M\). We choose a faithfully flat morphism \(\psi : \tilde{M} \to M\), such that we have a universal sheaf \(\tilde{E}\) on \(\tilde{M} \times X\). That means, for any point \(\tilde{m} \in \tilde{M}\), the sheaf \(\tilde{E}_{\tilde{m}} := \tilde{E}|_{\{\tilde{m}\} \times X}\) is a sheaf which belongs to the \(S\)-equivalence class given by \(\psi(\tilde{m})\). The theory of Quot schemes gives the existence of such morphisms. Let \(C = H_1 \cap H_2 \cap \ldots \cap H_{n-1}\) be a complete intersection of \(n - 1\) divisors \(H_i \in |H|\). Furthermore, we take a rank two vector bundle \(F\) on \(C\) with \(\det(F) \cong \omega_C\). We consider the following morphisms:

\[
\begin{array}{ccc}
\tilde{M} \times X & \xrightarrow{\psi} & M \times X \\
\downarrow{\tilde{\pi}} & & \downarrow{\pi} \\
\tilde{M} & \xrightarrow{\psi} & M \\
\end{array}
\]

On \(\tilde{M}\), we define the line bundle \(\mathcal{O}(\tilde{D}_H)\) to be the determinant of cohomology

\[
\mathcal{O}_{\tilde{M}}(\tilde{D}_H) := \det(\tilde{q}_!(\tilde{E} \otimes \tilde{p}^* F))^{-1}.
\]

This line bundle descends to \(M\), i.e. there exists a line bundle \(\mathcal{O}_M(D_H)\) and an isomorphism \(\mathcal{O}_{\tilde{M}}(\tilde{D}_H) \cong \psi^* \mathcal{O}_M(D_H)\). Furthermore, the line bundle \(\mathcal{O}_M(D_H)\) does not depend on the choice of the morphism \(\psi : \tilde{M} \to M\) (see [4]).

3.3 Lemma. The first Chern class \(\tilde{D}_H := c_1(\mathcal{O}_{\tilde{M}}(\tilde{D}_H))\) is given by

\[
\tilde{D}_H = \tilde{q}_*((2c_2(\tilde{E}) - c_1^2(\tilde{E})), \tilde{p}^* H^{n-1}).
\]
In particular, we have $D_{aH} = a^{n-1}D_H$.

**Proof:** This is straightforward computation using the Grothendieck–Riemann–Roch formula

$$
\tilde{D}_H = -\left[ \tilde{q}_*(\text{ch}(\tilde{E}) \cdot \text{Td}(\tilde{q}) \cdot \tilde{p}^*\text{ch}(F)) \right]_1,
$$

and the equalities

$$
\begin{align*}
\text{ch}(\tilde{E}) &= r + c_1(\tilde{E}) + \frac{c_2(\tilde{E}) - 2c_2(\tilde{E})}{2} + \ldots \\
\text{Td}(\tilde{q}) &= \tilde{p}^*\text{Td}(X) \\
\text{ch}(F) &= \text{ch}(\mathcal{O}_X \oplus \mathcal{O}_X(K_X + (n-1)H)) \cdot \text{ch}(\mathcal{O}_H)^{n-1} = 2H^{n-2} + K_X.H^{n-1}.
\end{align*}
$$

Whereas the first two equalities are standard, the last equality follows from the adjunction formula and the fact that a vector bundle $F$ on a curve $C$ is determined in the Grothendieck group $K(C)$ by its rank and determinant. \hfill \square

**3.4 Remark.** The line bundle $\mathcal{O}_{M_r}(D_H)$ is an example of a generalized Theta divisor (see [4] and [7]). This means that if for a given vector bundle $F$ on $C$ we have a dense open subset $U \subset M_r$ such that $[E] \in U$ implies $H^1(E|_C \otimes F) = 0$, then there exists a section $s_F \in \mathcal{O}_{M_r}(D_H)$ which vanishes exactly at the points $\{[E] \in M_r \mid h^1(E|_C \otimes F) > 0\}$. In this case, the divisor $\mathcal{O}_{M_r}(D_H)$ is effective which justifies this notation. It follows that $F$ itself is a semistable bundle. However, the existence of such a bundle $F$ is not clear for $r > 2$.

**3.5 Lemma.** The line bundle $\mathcal{O}_{M_r}(a \cdot D_H)$ is base point free for $a \geq r^2$.

**Proof:** Let $[E] \in M_r$ be a geometric point. Then $E|_C$ is a semistable vector bundle on the curve $C$. Popa shows in [13] that if $a \geq r^2$, then there exists a rank 2$a$ vector bundle $G$ on $C$ with $\det(G) \cong \omega_C^{\otimes a}$ with the property $H^1(C, G \otimes E|_C) = 0$. Thus, the generalized Theta divisor $\theta_G \in H^0(M_r, \det(\tilde{q}(\tilde{E} \otimes \tilde{p}^*G))^{-1})$ associated to $G$ does not pass through the points in $\psi^{-1}[E]$. Analogously to the computations in the proof of lemma 3.3, we see that the line bundle $\det(\tilde{q}(\tilde{E} \otimes \tilde{p}^*G))^{-1}$ is isomorphic to $\mathcal{O}_{M_r}(a \cdot D_H)$.

\hfill \square

**3.6 Lemma.** The line bundle $\mathcal{O}_{M_r}(D_H)$ is ample.

**Proof:** The proof uses the fact that on a projective variety of dimension at least two, the vector bundles $E$ and $E'$ with the same Hilbert polynomial are isomorphic, if and only if their restrictions to a sufficiently big ample divisor $H$ are isomorphic. This follows from the long exact sequence

$$
\begin{array}{cccccc}
\text{Hom}(E, E'(-H)) & \longrightarrow & \text{Hom}(E, E') & \longrightarrow & \text{Hom}(E, E'|_H) & \longrightarrow & \text{Ext}^1(E, E'(-H)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(E' \otimes E'(-H)) & \longrightarrow & \text{Hom}(E|_H, E'|_H) & \longrightarrow & H^1(E' \otimes E'(-H)).
\end{array}
$$

If we have a bounded family of vector bundles as in the case of those parametrized by $M_r$, then we can choose a divisor $H$ such that for any two bundles in this family, the cohomology groups on the left and right hand vanish. The restriction theorem of Mehta and Ramanathan (see [12]) tells us that, for a semistable vector bundle $E$ and $H$ big enough, the formation of graded objects commutes with restriction to $H$. Thus, we obtain an embedding $M_r = M_r(X) \longrightarrow M_r(H)$ . Repeating the argument, we end up with an embedding $M_r \longrightarrow M_r(C)$ for a complete intersection curve $C$. By lemma 3.3 we may assume that this curve $C$ is the curve we considered in the construction of $\mathcal{O}(D_H)$.
By construction, $O(D_H)$ is the pull back of the generalized Theta line bundle on $M_r(C)$. This line bundle is known to be ample by the work of Drezet and Narasimhan (see [11]). Thus, the lemma holds.

3.7 The families $E_{r,i}$. Let $M_{r}^{\text{red}} = \bigcup_{i=1}^{l} M_{r,i}$ be the decomposition of the reduced scheme underlying $M_r$ into its irreducible components, and let $\bar{M}_{r,i}$ be the normalization of the component $M_{r,i}$. We have a morphism $\alpha_i : \bar{M}_{r,i} \to M_r$ and consider the globally generated line bundle (by lemma 3.5) $N_{r,i} := \alpha_i^* O(r^2, D_H)$. Let $C_{r,i}$ be the intersection of $\dim(\bar{M}_{r,i}) - 1$ general global sections of $N_{r,i}$. By Bertini’s theorem, $C_{r,i}$ can be assumed to be a smooth irreducible curve. Thus, by Langton’s theorem (see [11]), we have a universal vector bundle $E_{r,i}$ on $C_{r,i} \times X$. If the universal vector bundle $E_M$ on $M_r \times X$ existed, then $E_{r,i}$ would be the pull back of this bundle to $C_{r,i} \times X$.

3.8 The line bundle $L_{r,i}$. We consider the vector bundle $E_{r,i}$ on $C_{r,i} \times X$ and the morphisms

$$C_{r,i} \leftarrow q C_{r,i} \times X \xrightarrow{p} X.$$ 

We define the line bundle $N_{r,i}$ on the curve $C_{r,i}$ by $N_{r,i} := \det(E_{r,i}|_{C_{r,i} \times \{x_0\}})$. Let $G_{r,i}$ be a vector bundle on $C_{r,i}$ with $\text{rk}(G_{r,i}) = 2r$, and $\det(G_{r,i}) \cong \omega_{C_{r,i}}^* \otimes N_{r,i}^{-2}$. Similar to the definition of $O_{M_r}(D_H)$ in 3.2 we define the line bundle by

$$L_{r,i} := \det(p_!(E_{r,i} \otimes q^* G_{r,i}))^{-1}.$$ 

3.9 Remark. Unfortunately, in contrast to $O_{M_r}(D_H)$, the line bundle $L_{r,i}$ is not independent of the choices. We next give an example for this dependence on the choice of the family $E_{r,i}$.

Let $X$ be a curve, and $E_{r,i}$ be a family of degree zero vector bundles on $X$ parametrized by $C_{r,i}$. For a point $c \in C_{r,i}$, we consider the vector bundle $E_c := E_{r,i}|_{\{c\} \times X}$. Furthermore, we assume that $E_c$ is not stable. Thus, we have a short exact sequence $0 \to E_c' \to E_c \to E_c'' \to 0$ of degree zero vector bundles on $X$. Denote by $E_{r,i}'$ the kernel of the natural surjection $E_{r,i} \to E_c''$. The families $E_{r,i}'$ and $E_{r,i}$ parameterize the same $S$-equivalence classes of vector bundles on $X$. A straightforward computation shows that the resulting line bundles $L_{r,i}$ and $L_{r,i}'$ fulfill

$$L_{r,i}' = \det(E_c'^{2 \text{rk}(E_c''^{\text{op}})} \otimes \det(E_c'')^{-\text{rk}(E_c^\text{op})} \otimes L_{r,i}).$$

Thus, we can only hope that the numerical type of $L_{r,i}$ is well defined. This is the case as we will see in the next section (see Corollary 4.4).

3.10 We end this section by defining the line bundle $L_r$ on $X$ by

$$L_r := \bigotimes_{i=1}^{l} L_{r,i}.$$ 

4 Properties of the line bundles $L_r$

4.1 Relations defined by nef line bundles. Let $L$ be a nef line bundle on a proper variety $X$. This line bundle defines an equivalence relation $\sim_L$ on the geometric points of $X$ as follows:

$$x \sim_L x' \iff \begin{cases} \text{There exists a closed curve } C \subset X \\ \text{with } x \in C, x' \in C, \text{ and } L.C = 0. \end{cases}$$
We define the relation $\preceq$ on nef line bundles by: The condition $L_1 \preceq L_2$ holds if for any curve $C \subset X$ the inequality $L_1.C > 0$ implies $L_2.C > 0$.

We write $L_1 \prec L_2$ if $L_1 \preceq L_2$ holds, and there exists a curve $C \subset X$ with $L_1.C = 0$ and $L_2.C > 0$. If $L_1 \preceq L_2$ holds, then the relation $\sim_{L_2}$ is contained in $\sim_{L_1}$; i.e. if $x \sim_{L_2} x'$, then we have $x \sim_{L_1} x'$. Whenever both relations $L_1 \preceq L_2$ and $L_2 \preceq L_1$ hold, we write $L_1 \sim L_2$. This means that the relations $\sim_{L_1}$ and $\sim_{L_2}$ coincide.

If we have a chain $L_0 \prec L_1 \prec \ldots \prec L_k$ of nef line bundles, then the we have strict inclusions $(\text{NE}(X) \cap L_1^+) \subset (\text{NE}(X) \cap L_2^+) \subset \ldots \subset (\text{NE}(X) \cap L_k^+)$ in the cone of curves. Since the orthogonal complements $L_i^+$ are linear subspaces of $H^2(X, \mathbb{R})$, we deduce that $k \leq \rho(X) = \dim(\text{NE}(X))$.

### 4.2 Theorem

The line bundle $\mathcal{L}_{r,i}$ is nef, i.e. for any morphism $\imath : Y \to X$ of a smooth curve $Y$ to $X$, the degree of the line bundle $\imath^*\mathcal{L}_{r,i}$ is non negative. Furthermore, if the degree of $\imath^*\mathcal{L}_{r,i}$ equals zero, then for all geometric points $y_1$ and $y_2$ of $Y$ there exists an isomorphism $\mathcal{E}_{r,i \times \{y_1\}} \cong \mathcal{E}_{r,i \times \{y_2\}}$, and all the vector bundles on $Y$ parametrized by $C_{r,i}$ are $S$–equivalent.

**Proof:** We divide the proof into steps.

**Step 1:** Reduction to the case where $X$ is a smooth projective curve.

Since all the vector bundles parametrized by $M_r$ restrict to semistable bundles on every closed subscheme of $X$, the pull back $(\text{id} \times \imath)^*\mathcal{E}_{r,i}$ is a family of semistable rank $r$ vector bundles on $Y$ parametrized by $C_{r,i}$. Since the determinant of cohomology commutes with base change, we may assume $X = Y$. Thus, we consider the vector bundle $\mathcal{E}_{r,i}$ on the surface $C_{r,i} \times X$ and the following morphisms to smooth curves

$$C_{r,i} \xrightarrow{q} C_{r,i} \times X \xrightarrow{p} X.$$ 

**Step 2:** The line bundles $L_1$ and $L_2 := \mathcal{L}_{r,i}$.

Let $F$ be vector bundle on $X$ with $\text{rk}(F) = 2r$ and $\det(F) = \omega_X^r$. For a point $x_0 \in X$, we set $N_{r,i} := \det(\mathcal{E}_{r,i}|_{C_{r,i} \times \{x_0\}})$. Let $G$ be a vector bundle on $C_{r,i}$ of rank $2r$ with $\det(G) = \omega_{C_{r,i}}^r \otimes N_{r,i}^{-2}$. We set $L_1 := \det(q^*(\mathcal{E}_{r,i} \otimes p^*F))^{-1}$, and $L_2 := \det(p_!(\mathcal{E}_{r,i} \otimes q^*G))^{-1}$. The nefness property of $\mathcal{L}_{r,i}$ is equivalent to $\deg(L_2) \geq 0$.

**Step 3:** $\deg(L_1) = \deg(L_2)$.

We use the Grothendieck–Hirzebruch–Riemann–Roch theorem to compute the degrees of the line bundles $L_1$ and $L_2$. Let us fix the notations before doing so: By $c_0$ and $x_0$, we denote two geometric points of $C_{r,i}$ and $X$. We use $F_p$ and $F_q$ to name the fibers $p^{-1}(x_0)$ and $q^{-1}(c_0)$. The genera of $C_{r,i}$ and $X$ we denote by $g_C$ and $g_X$. Since we are only interested in the degrees, we may assume $\omega_X = \mathcal{O}_X((2g_X - 2)x_0)$ and $\omega_{C_{r,i}} = \mathcal{O}_X((2g_C - 2)c_0)$. For the same reason, we have $\text{ch}(F) = 2r + 2r(g_X - 1)x_0$ and $\text{ch}(G) = 2r + (2r(g_C - 1) - 2(\int_{C_{r,i} \times X}(F_p.c_1(\mathcal{E}_{r,i}))))c_0$. Furthermore, let $\text{Td}(C_{r,i}) = 1 - (g_C - 1)c_0$ and $\text{Td}(X) = 1 - (g_X - 1)x_0$ be the (numerical) Todd classes.

\[
\deg(L_1) = -\int_{C_{r,i}} \text{ch}(q^*(\mathcal{E}_{r,i} \otimes p^*F))
= -\int_{C_{r,i} \times X} \text{ch}(\mathcal{E}_{r,i} \otimes p^*F)p^*\text{Td}(X)
= -\int_{C_{r,i} \times X} \text{ch}(\mathcal{E}_{r,i})p^*\text{ch}(F)p^*\text{Td}(X)
= -\int_{C_{r,i} \times X} \text{ch}(\mathcal{E}_{r,i})^r\text{ch}(F)\text{Td}(X)
= -\int_{C_{r,i} \times X} \left(r + c_1(\mathcal{E}_{r,i}) + \frac{c_1^2(\mathcal{E}_{r,i}) - 2c_2(\mathcal{E}_{r,i})}{2}\right)p^*(2r)
= r \cdot \int_{C_{r,i} \times X} (2c_2(\mathcal{E}_{r,i}) - c_1^2(\mathcal{E}_{r,i}))
\]
Analogously, we obtain for the degree of $L_2$ that:

$$\deg(L_2) = -\int_{C_{r,i} \times X} \left( r + c_1(\mathcal{E}_{r,i}) + \frac{c_1(\mathcal{E}_{r,i}) - 2c_2(\mathcal{E}_{r,i})}{2} \right) q^*(2r - (2\int_{C_{r,i} \times X} (F_p.c_1(\mathcal{E}_{r,i})c_0)))$$

$$= r \cdot \int_{C_{r,i} \times X} (2c_2(\mathcal{E}_{r,i}) - c_1(\mathcal{E}_{r,i})) + 2 \cdot \left( \int_{C_{r,i} \times X} F_p.c_1(\mathcal{E}_{r,i}) \right) \cdot \left( \int_{C_{r,i} \times X} F_q.c_1(\mathcal{E}_{r,i}) \right)$$

Since $C_{r,i}$ parameterizes a family of $X$ vector bundles of degree zero, the intersection number $\int_{C_{r,i} \times X} F_p.c_1(\mathcal{E}_{r,i})$ equals zero. Thus, we end up with the claimed equality.

**Step 4:** $L_1^\otimes r$ is globally generated. Thus, $\deg(L_2) = \deg(L_1) \geq 0$.

This is a direct consequence of Popa’s (see [13]) result about the base point freeness of certain powers of the generalized Theta line bundles. Here the semistability of all vector bundles parametrized by $C_{r,i}$ is needed. Popa’s result uses Le Potier’s method (cf. [14]) of constructing global sections of the generalized Theta bundle.

To prove $\deg(L_1) \geq 0$, we need fewer premises. Indeed, if at least one point of $C_{r,i}$ parameterizes a semistable vector bundle on $X$, then a power of $L_1$ has a nontrivial section $s$ which is what we need. This argument is the same as in the proof of Lemma 3.5.

If the degree of $L_1$ is positive, then we consider two points $c_1, c_2$ of $C_{r,i}$ with the property that the section $s$ vanishes at $s_1$ but not at $s_2$. It follows, that the $X$ vector bundles parametrized by $c_1$ and $c_2$ are not $S$-equivalent.

From now on, we assume that the degrees of $L_1$ and $L_2$ are zero. For the following steps see also the proof of Theorem I.4 in Faltings article [2] or for the simpler rank two case see Theorem 3.4 in [7].

**Step 5:** For any two geometric points $P$ and $Q$ of $X$, the vector bundles $\mathcal{E}_{r,i}|_{C_{r,i} \times \{P\}}$ and $\mathcal{E}_{r,i}|_{C_{r,i} \times \{Q\}}$ are isomorphic.

We show that, given a point $P \in X$, for almost all points $Q \in X$, we have an isomorphism between $\mathcal{E}_{r,i}|_{C_{r,i} \times \{P\}}$ and $\mathcal{E}_{r,i}|_{C_{r,i} \times \{Q\}}$. From this statement, the assertion of step 5 follows immediately. Let $\mathcal{E}_{c_0} := \mathcal{E}_{r,i}|_{\{c_0\} \times X}$ be the semistable vector bundle on $X$ parametrized by $c_0$. Take a vector bundle $F_P$ on $X$ such that $H^*(X, F_P \otimes \mathcal{E}_{c_0}) = 0$. Such a bundle exists because by Le Potier’s result in [14]. Moreover, it defines a global section in a power of $L_1$ which does not vanish at $c_0$. Since $\deg(L_1) = 0$, this section has an empty vanishing divisor. This implies that $H^*(X, \mathcal{E}_{r,i}|_{\{c\} \times X}) = 0$ for all points $c \in C_{r,i}$. This implies $R^*q_*(\mathcal{E}_{r,i} \otimes p^*F_P)$ is zero. Now we consider a nontrivial extension $F$ in $\text{Ext}^1(k(P), F_P)$

$$(S_P) \quad 0 \to F_P \to F \xrightarrow{\pi} k(P) \to 0.$$ 

The scheme $\mathbb{P}(F)$ parameterizes surjections $\pi : F \to k(Q)$ from $F$ to torsion sheaves of length one. The subset of $\mathbb{P}(F)$ where $H^*(X, \ker(\pi) \otimes \mathcal{E}_{c_0}) = 0$ is open and not empty because it contains $\pi_P$. Thus, for a general point $Q$ of $X$, there exists a short exact sequence

$$(S_Q) \quad 0 \to F_Q \to F \to k(Q) \to 0$$

with $H^*(X, F_Q \otimes \mathcal{E}_{c_0}) = 0$. Applying the functor $R^*q_*(\mathcal{E}_{r,i} \otimes p^*(-))$ to the short exact sequences $(S_P)$ and $(S_Q)$, we obtain that $q_*(p^*F \otimes \mathcal{E})$ is isomorphic to $\mathcal{E}_{r,i}|_{C_{r,i} \times \{P\}}$ and to $\mathcal{E}_{r,i}|_{C_{r,i} \times \{Q\}}$ as well.

**Step 6:** The filtration $F^*(\mathcal{E}_{r,i})$ on the vector bundle $\mathcal{E}_{r,i}$.

If we consider the Harder–Narasimhan filtration on $G := q_*(p^*F \otimes \mathcal{E})$, then the graded summands need not be to be simple bundles. We consider a slight generalization by taking $G_1$ to be a subsheaf of $G$ which is stable of maximal possible slope. Defining $F^1(\mathcal{E}_{r,i}) :=$
G_{1} \otimes q_{*} \text{Hom}(p^{*}G_{1}, \mathcal{E}_{r,i}), \text{ and } F^{i}(\mathcal{E}_{r,i}) := \pi_{1}^{-1}(F^{i-1}(\mathcal{E}_{r,i}/F^{1}(\mathcal{E}_{r,i}))), \text{ where } \pi_{1} \text{ is the surjection from } \mathcal{E}_{r,i} \to \mathcal{E}_{r,i}/F^{1}(\mathcal{E}_{r,i}), \text{ we obtain a filtration } 0 = F^{0}(\mathcal{E}_{r,i}) \subset F^{1}(\mathcal{E}_{r,i}) \subset \ldots \subset F^{k}(\mathcal{E}_{r,i}) = \mathcal{E}_{r,i} \text{ on } \mathcal{E}_{r,i} \text{ with the property that the } j \text{th graded object } \text{gr}^{j}(\mathcal{E}_{r,i}) := F^{j}(\mathcal{E}_{r,i})/F^{j-1}(\mathcal{E}_{r,i}) \text{ is of the form } G_{j} \otimes F_{j}. \text{ By definition the slopes } \mu_{j} := \mu(G_{j}) = \frac{\deg(G_{j})}{\text{rk}(G_{j})} \text{ form a decreasing sequence } \mu_{1} \geq \mu_{2} \geq \ldots \geq \mu_{k}.

Restricting the filtration } F^{*}(\mathcal{E}_{r,i}) \text{ to a fiber } p^{-1}(x) \text{ of } p, \text{ we obtain a filtration of } G \text{ which does not depend on the choice of } x \in X. \text{ The restricted vector bundle } F^{j}(\mathcal{E}_{r,i})|_{p^{-1}(x)} \text{ appears in the Harder–Narasimhan filtration of } G, \text{ if and only if } \mu_{j} > \mu_{j+1}. \text{ Therefore, we use } \text{HNF}^{*}(\mathcal{E}_{r,i}) \text{ to name the subfiltration of the filtration } F^{*}(\mathcal{E}_{r,i}) \text{ consisting of those } F^{j}(\mathcal{E}_{r,i}) \text{ with } \mu_{j} > \mu_{j+1}.

**Step 7: Numerical invariants of the filtration } F^{*}(\mathcal{E}_{r,i}).**

In the Grothendieck group } K(C_{r,i} \times X), \text{ we can identify } \mathcal{E}_{r,i} \text{ with the direct sum of the graded objects } \text{gr}^{j}(\mathcal{E}_{r,i}):

\[ [\mathcal{E}_{r,i}] = \sum_{j=1}^{k} \text{gr}^{j}(\mathcal{E}_{r,i}) = \sum_{j=1}^{k} [G_{j} \otimes F_{j}]. \]

Since the Chern character of the product } G_{j} \otimes F_{j} \text{ is given by

\[ \text{ch}(G_{j} \otimes F_{j}) = q^{*}\text{ch}(G_{j}) \cdot p^{*}\text{ch}(F_{j}) = \text{rk}(G_{j}) \cdot \text{rk}(F_{j}) + [\text{rk}(G_{j})p^{*}c_{1}(F_{j}) + \text{rk}(F_{j})q^{*}c_{1}(G_{j})] + p^{*}c_{1}(F_{j}).q^{*}c_{1}(G_{j}), \]

we deduce the equality \( \int_{C_{r,i} \times X} \text{ch}(\mathcal{E}_{r,i}) = \sum_{j=1}^{k} \deg(G_{j}) \cdot \deg(F_{j}). \) In step 3, we identified the left hand side with \( \frac{1}{2r} \deg(L_{1}). \) Thus, we have

\[ \sum_{j=1}^{k} \deg(G_{j}) \cdot \deg(F_{j}) = 0 \quad (1) \]

The degree \( \deg_{X}(F^{j}(\mathcal{E}_{r,i})|_{p^{-1}(c)}) \text{ of } F^{j}(\mathcal{E}_{r,i}), \text{ restricted to a fiber of } q, \text{ is given by } \deg_{X}(F^{j}(\mathcal{E}_{r,i})|_{p^{-1}(c)}) = \sum_{m=1}^{j} \text{rk}(G_{m}) \cdot \deg(F_{m}). \) Since the restriction of } \mathcal{E}_{r,i} \text{ to a fiber of } q \text{ is semistable of degree zero, we deduce that

\[ A_{j} := \sum_{m=1}^{j} \text{rk}(G_{m}) \cdot \deg(F_{m}) \leq 0, \quad (2) \]

and } \( A_{k} = 0. \) Having in mind that } \( \mu_{j} := \frac{\deg(G_{j})}{\text{rk}(G_{j})}, \) we rewrite equation (1)

\[ 0 = \sum_{j=1}^{k} \mu_{j} \cdot \text{rk}(G_{j}) \cdot \deg(F_{j}) = \sum_{j=1}^{k} A_{j} \cdot (\mu_{j} - \mu_{j+1}), \]

where we set } \( \mu_{k+1} = 0. \) The inequalities (2) and } \( \mu_{j} \geq \mu_{j+1} \text{ imply, therefore, that } A_{j} = 0 \text{ whenever } \mu_{j} > \mu_{j+1}. \)

**Step 8: S–equivalence of all bundles parametrized by } C_{r,i}.**

The conclusion of the preceding steps is, that each quotient } \( \text{gr}^{j}_{\text{HNF}} := \text{HNF}^{j}(\mathcal{E}_{r,i})/\text{HNF}^{j-1}(\mathcal{E}_{r,i}) \) \text{ is on the one hand semistable of degree zero when restricted to the fibers of } q. \text{ On the other hand, the semistable vector bundle on } C_{r,i} \text{ which we obtain by restricting } \text{gr}^{j}_{\text{HNF}} \text{ to a fiber } p^{-1}(x) \text{ of } p \text{ does not depend on the chosen
x ∈ X. Now the situation is symmetric in the sense that $\text{gr}_{HNF}^j$ is semistable on all fibers of $p$ and $q$. Thus, analogously to step 5, we deduce that the restrictions $\text{gr}_{HNF}^j|_{\{c_1\} \times X}$ and $\text{gr}_{HNF}^j|_{\{c_2\} \times X}$ are isomorphic for all geometric points $c_1, c_2 \in X$. In other terms, the direct sums $\oplus_{j=1}^r \text{gr}_{HNF}^j$ of the graded objects gives the same direct sum of semistable vector bundles of degree zero on each fiber of $q$. In short: All vector bundles parametrized by $C_{r,i}$ are $S$–equivalent.

4.3 Theorem. Let $\iota : Y \to X$ be a morphism of a smooth curve $Y$ to $X$. We obtain a morphism $\iota_{M_{r,i}} : \tilde{M}_{r,i}(X) \to M_r(Y)$ by the pull back of vector bundles. The following two conditions are equivalent:

1. The degree of $\iota^*L_{r,i}$ is zero;
2. The morphism $\iota_{M_{r,i}}$ maps $\tilde{M}_{r,i}(X)$ to a point.

Proof: We consider the morphisms $C_{r,i} \xrightarrow{\alpha} \tilde{M}_{r,i}(X) \xrightarrow{\iota_{M_{r,i}}} M_r(Y)$. Theorem 4.2 implies that the degree of $\iota^*L_{r,i}$ is zero, if and only if $\iota_{M_{r,i}}(C_{r,i})$ is a point. This implies the theorem because $\tilde{M}_{r,i}(X)$ is irreducible, and $C_{r,i}$ is the intersection of ample divisors.

4.4 Corollary. The equivalence classes of the nef line bundles $L_{r,i}$ and $L_r$ with respect to $\sim$ (see 4.1) neither depend on the choice of $C_{r,i} \subset \tilde{M}_{r,i}$ nor on the choice of the vector bundle $E_{r,i}$ on $C_{r,i} \times X$. Furthermore, these equivalence classes are independent of the chosen polarization $H$ on $X$.

4.5 Corollary. The line bundle $L_r$ on $X$ is nef. For a morphism $\iota : Y \to X$ of a smooth curve $Y$ to $X$, we have $\deg(\iota^*L_r) = 0$, if and only if the morphism $M_r(X) \to M_r(Y)$ is locally constant.

4.6 Proposition. The line bundles $L_r$ satisfy the inequality $L_r \preceq L_r$, for $r_1 \leq r_2$.

There exists a number $R \in \mathbb{N}$ such that $L_r \preceq L_R$ for all $r$.

Proof: If $r_1 < r_2$, then we have an embedding of moduli spaces $M_{r_1} \to M_{r_2}$ given by $[E] \mapsto [E \oplus C_X^{(r_2-r_1)}]$. Thus, we deduce from theorem 4.4 that the inequality $L_{r_1} \preceq L_{r_2}$ holds. We have seen in 4.4 that in the chain $L_1 \preceq L_2 \preceq \ldots \preceq L_r$ there are at most $\rho(X)$ strict inclusions. This proves the second assertion of the proposition.

4.7 The line bundle $L_\infty$. We use the name $L_\infty$ for the line bundle $L_R$ of the above proposition. When referring to this line bundle, we should be aware that $L_\infty$ is only a class in $\{\text{nef line bundles}\} / \sim$. Considering these equivalence classes (with the discrete topology), we have $\lim_{r \to \infty} L_r \sim L_\infty$.

In the following theorem, we have summarized the results of this section.

4.8 Theorem (Properties of the line bundles $L_r$) Let $X$ be a projective variety. We have an infinite sequence of nef line bundles $L_1 \preceq L_2 \preceq \ldots \preceq L_r \preceq \ldots$ and a nef limit line bundle $L_\infty$ with $\lim_{r \to \infty} L_r \sim L_\infty$ on $X$ such that for any morphism $\iota : Y \to X$ of a smooth curve $Y$ to $X$ the following conditions are equivalent:

1. $\deg(\iota^*L_r) = 0$;
2. The restriction morphism $M_r(X) \to M_r(Y)$ is locally constant;
3. For any connected scheme $Z$ and every vector bundle $E$ on $Z \times X$ parameterizing semistable rank $r$ vector bundles on $X$ with trivial Chern classes, the pull back $(id_Z \times \iota)^*E$ parameterizes only one $S$–equivalence class on $Y$;
4. Modulo conjugation only finitely many $U(r)$ representations of $\pi_1(Y)$ are induced by those of $\pi_1(X)$.
5 The generalized Albanese morphisms

5.1 The construction of the generalized Albanese morphism. If $L_r$ or some power of it were base point free, then it would define a morphism $\psi : X \to \mathbb{P}^m$. Let $\varphi : X \to \text{Alb}_r$ be the Stein factorization of $\psi$, i.e., $\varphi$ is surjective with connected fibers. Two geometric points $x$ and $x'$ of $X$ have the same image under $\varphi$, if and only if $x \sim_{L_r} x'$. By Theorem 1.8, the map $\varphi$ would meet the requirements of a generalized Albanese variety. Indeed, a curve $Y \to X$ would be contracted by $\varphi$, if $\deg_Y (\varphi^* L_r) = 0$. This means (by theorem 1.8) that all families of semistable rank $r$ vector bundles on $X$ with trivial Chern classes become constant when restricted to $Y$, or only finitely many representation classes modulo conjugation of $\pi_1(Y)$ in $U(r)$ are induced by representations of $\pi_1(X)$.

If no line bundle $L_r$ with $L_r \sim L_r$ is base point free (Note, that $L_r \otimes k \sim L_r$ for all $k > 0$), then Tsuji’s nef reduction theorem provides us with a rational version of the generalized Albanese variety up to birational equivalence. In this case we obtain only a birational model of the Albanese morphism and variety.

5.2 Theorem. (see Theorem 2.1 in [2], see also [15]) There exists a dominant rational map $X \to \text{alb}_r(X)$ with connected fibers such that:

1. The line bundle $L_r$ is numerically trivial on all compact fibers $F$ of $\text{alb}_r$ of dimension $\dim(X) - \dim(\text{alb}_r(X))$;
2. For every general point $x \in X$ and every irreducible curve $C$ passing through $x$ with $\dim(\text{alb}_r(C)) > 0$, we have $C.L_r > 0$;
3. There exist compact fibers of $\text{alb}_r$.

Furthermore, the pair $(\text{alb}_r, \text{alb}_r(X))$ is uniquely determined up to birational equivalence.

5.3 The chain of generalized Albanese morphisms. Even though we end up with an infinite sequence of rational morphisms $X \to \text{alb}_r(X)$, for each $r \in \mathbb{N}$, there are at most $\rho(X)$ different generalized Albanese morphisms, since for almost all $r \in \mathbb{N}$, we have $L_r \sim L_{r+1}$. Since $L_r \preceq L_{r+1}$, we get a rational morphism $\text{Alb}_{r+1}(X) \to \text{Alb}_r(X)$. So, we end up with the following commutative diagram:

\[
\begin{array}{c}
\text{Alb}_\infty(X) \leftarrow \text{Alb}_\infty(X) \leftarrow \cdots \\
\downarrow \text{alb}_\infty \quad \downarrow \text{alb}_r \\
\cdots \leftarrow \text{Alb}_{r+1}(X) \leftarrow \cdots \leftarrow \text{Alb}_r(X) \\
\downarrow \cdots \\
\text{alb}_1 \\
\end{array}
\]

5.4 Proposition. (Functoriality) If $\psi : X \to X'$ is a morphism of projective varieties, then we have $\psi^* L'_r \preceq L_r$ for all $r \in \mathbb{N} \cup \{\infty\}$. Therefore, we have commutative diagrams

\[
\begin{array}{c}
X \xrightarrow{\psi} Y \\
\downarrow \text{alb}_r \quad \downarrow \text{alb}_r \\
Y \\
\text{Alb}_r(X) \leftarrow \text{alb}_r(Y).
\end{array}
\]

Proof: Suppose that $\iota : Y \to X$ is a morphism from a smooth curve to $X$ with $\deg(\iota^* \psi^* L'_r) > 0$. This implies by theorem 1.8 that there exists a family $E'$ of rank $r$ vector bundles with trivial Chern classes on $X'$ parametrized by a connected scheme $Z$ such that the pull back $(\text{id}_Z \times (\psi \circ \iota))^* E'$ of $E'$ to $Z \times Y$ parameterizes different $S$-equivalence classes on $Y$. However, then the family $E = (\text{id}_Z \times \psi)^* E'$ also parameterizes
rank $r$ vector bundles on $X$ which pull back to non $S$–equivalent classes. Consequently, again by theorem \ref{thm:non_equiv}, \( \deg(i^*L_r) > 0 \).

5.5 Proposition. If $\psi : X \to X'$ is an étale morphism of projective varieties, then we have $L_r \preceq \psi^*L'_{r-deg(\psi)}$ for all $r \in \mathbb{N}$. Furthermore, $L_\infty \sim \psi^*L'_\infty$.

Proof: The proof is analogous to the preceding one. We simply have to consider the push forward of a family of rank $r$ vector bundles on $X$ to $X'$. The statement about $L_\infty$ is easily obtained from $L_r \preceq \psi^*L'_{r-deg(\psi)} \preceq L_{r-deg(\psi)}$ and the fact that $L_r \sim L_\infty$ for $r \gg 0$.

5.6 Remark. Since the first Albanese variety and morphism exist, it would be enough to have the base point freeness of some power of $L_r$ on every fiber of the Albanese variety to obtain a regular morphism $X_{alb_r} \to \text{Alb}_r(X)$.

This is used to study the generalized Albanese morphisms in the case of algebraic surfaces in the next section.

6 The case of algebraic surfaces

6.1 We consider here the case of a polarized projective algebraic surface $(X, H)$. In this case, we can make a much stronger statement than theorem \ref{thm:albanese}. The generalized Albanese morphism is well understood if $X$ is not of general type (see \ref{thm:general_type}). The rest of this section is devoted to the proof of the

6.2 Theorem. Let $(X, H)$ be a polarized projective surface. Then there exists a surjective morphism $\text{alb}_r : X \to \text{Alb}_r(X)$ with connected fibers, and an effective divisor $D$ on $X$, such that for all morphisms $i : C \to X$ of irreducible curves with $i(C) \not\subset D$ the following conditions are equivalent.

1. $\text{alb}_r(i(C))$ is a point;
2. The associated morphism $\text{Hom}(\pi_1(X), \mathbb{U}(r)) \xrightarrow{i^*} \text{Hom}(\pi_1(C), \mathbb{U}(r))$ modulo conjugation has a finite image;
3. For any base scheme $S$ and any rank $r$ vector bundle $E$ on $X \times S$ such that for each $s E_s$ is semistable with numerically trivial Chern classes, the pull back of $E$ to $C \times S$ is a family of $S$–equivalent vector bundles.

The divisor $D$ of exceptions can be written in the form $D = C_1 + C_2 + \ldots + C_l$, where the $C_i$ are irreducible and form a basis of a proper subspace of the rational Néron–Severi vector space $\text{NS}(X) \otimes \mathbb{Q}$. In particular, we have $l < \dim_{\mathbb{Q}}(\text{NS}(X) \otimes \mathbb{Q})$.

6.3 Preparations for the proof. We consider the nef line bundle $L_r$ on $X$ satisfying the equivalence of theorem \ref{thm:non_equiv}. There are two extreme cases where the proof is a simple remark: When $C.L_r > 0$, for all curves $C$, then we set $\text{alb}_r$ to be the identity morphism of $X$. If $C.L_r = 0$ for all curves, then we set $\text{Alb}_r(X) = \text{Spec}(\mathbb{C})$, and we are finished.

Thus, we assume from now on that $L_r$ is a numerically nontrivial nef line bundle vanishing on a nonempty set $\{C_i\}_{i \in I}$ of irreducible curves. It follows from the construction that all the curves $C_i$ in this family are contracted to points by the classical Albanese morphism.
6.4 Lemma. If \( C \subset X \) is an effective divisor with \( C^2 > 0 \), then \( C.\mathcal{L}_r > 0 \).

Proof: We assume the contrary. Since \( \mathcal{L}_r \) is nef this means that \( C.\mathcal{L}_r = 0 \). The Hodge index theorem (see IV.2.15 in [1]) implies that \( \mathcal{L}_r^2 \leq 0 \) with equality only when \( \mathcal{L}_r \) is torsion. From \( \mathcal{L}_r^2 \geq 0 \) and the assumption that \( \mathcal{L}_r \) is numerically nontrivial we derive a contradiction.

The proof of theorem 6.2 is subdivided into three cases depending on the dimension of \( X \) in the Albanese variety. This is by definition the dimension of \( \text{Alb}_1(X) \).

6.5 The nef reduction for \( \text{Alb}_1(X) = \text{Spec}(\mathbb{C}) \). We consider the curves \( \{C_i\}_{i \in I} \) as vectors in the rational Néron–Severi space \( \text{NS}_\mathbb{Q}(X) \). The dimension of this vector space is the Picard number \( \rho(X) \) of \( X \). The points \( \{C_i\}_{i \in I} \) lie on the hyperplane \( \{C \in \text{NS}_\mathbb{Q}(X) \mid C.\mathcal{L}_r = 0\} \). Suppose there would be a nontrivial linear relation among the \( C_i \). If the number of these curves exceeds \( \rho(X) - 1 \), then we have at least one such relation. We write the linear relation \( a_1C_1 + \ldots + a_mC_m = a_{m+1}C_{m+1} + \ldots + a_MC_M \) with positive rational \( a_i \) and \( C_i \) different from \( C_j \) whenever \( i \neq j \).

After multiplication with a positive integer, we may assume the \( a_i \) to be integers. We set \( D_1 := a_1C_1 + \ldots + a_mC_m \) and \( D_2 := a_{m+1}C_{m+1} + \ldots + a_MC_M \). Since \( D_1 \) and \( D_2 \) coincide in \( \text{NS}_\mathbb{Q}(X) \), their difference is torsion in the Néron–Severi group. So again, after multiplying with an integer, we may assume that the effective Cartier divisor classes \( D_1 \) and \( D_2 \) coincide. (Here we use the fact that the Picard torus, the dual of the Albanese torus, is trivial.)

\[ D_1^2 = D_1.D_2 \geq 0, \quad \text{because} \quad D_1 \quad \text{and} \quad D_2 \quad \text{have no common components}. \]

In view of lemma 6.4 and \( D_1.\mathcal{L}_r = 0 \), we conclude that \( D_1^2 = 0 \). This implies that \( D_1 \) and \( D_2 \) are disjoint. Consequently, the line bundle \( L := \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2) \) has two linearly independent sections which do not intersect. Thus, \( L \) is base point free and defines a morphism whose Stein factorization we denote by \( \text{alb}_r : X \to \text{Alb}_r(X) \).

On the one hand, we have that \( F.\mathcal{L}_r = 0 \) for all fibers of \( \text{alb}_r \). On the other hand, suppose \( C.\mathcal{L}_r = 0 \) for a curve \( C \subset X \). Let \( F \) be an irreducible fiber of \( \text{alb}_r(X) \). If \( C \) were not contained in a fiber, then we would have \( (C + mF)^2 > 0 \) for \( m \gg 0 \). However, we have \( (C + mF).\mathcal{L}_r = 0 \) which contradicts lemma 6.4. Thus, each curve \( C \) with \( C.\mathcal{L}_r = 0 \) is contained in a fiber.

This means that the effective divisor \( D \) of 6.2 can be taken to be the empty set once we have a linear relation between the \( \{C_i\}_{i \in I} \) in \( \text{NS}_\mathbb{Q}(X) \). Since the resulting morphism \( \text{alb}_r \) contracts all these curves, we conclude that \( \text{alb}_r \) does not depend on the chosen linear relation.

6.6 The nef reduction when \( \text{Alb}_1(X) \) is a curve. We consider the morphism \( \text{alb}_1 : X \to \text{Alb}_1(X) \). This morphism is the Stein factorization of the classical Albanese morphism. It follows from \( \mathcal{L}_1 \preceq \mathcal{L}_r \), that each curve \( C \) with \( C.\mathcal{L}_r = 0 \) is contained in a fiber of this morphism. Let \( F \) be the generic fiber of \( \text{alb}_1 \).

If \( F.\mathcal{L}_r = 0 \), then all curves \( C \) with \( C.\mathcal{L}_r = 0 \) are contracted by \( \text{alb}_1 \). Consequently we set \( \text{alb}_r = \text{alb}_1 \) and theorem 6.2 is proven.

We suppose now that \( F.\mathcal{L}_r \) is positive. The set of curves \( \{C_i\}_{i \in I} \) consists of components of reducible fibers of the morphism \( \text{alb}_1 \). We will show that this set is not only finite but linearly independent in \( \text{NS}_\mathbb{Q}(X) \). Indeed, if there were a linear relation, then we would obtain (see 6.3) an effective divisor \( D_1 = a_1C_1 + \ldots + a_mC_m \). This divisor satisfies
$D_1^2 = 0$, $D_1L_r = 0$, and consists of fiber components. This contradicts Zariski’s lemma (see Lemma III.8.2 in [I]), because $F.L_r > 0$ for all fibers $F$. This shows that theorem 6.2 holds when setting $alb_r = id_X$.

6.7 Remark. The finite collection of curves $\{C_i\}_{i \in I}$ must have a negative definite intersection matrix, because of Zariski’s lemma. Thus, by Grauert’s criterion (Theorem III.2.1 in [I]), there exists a contraction of these curves. However, this contraction is not necessarily a projective morphism. If it were, we could take this contraction to be our generalized Albanese morphism $alb_r$.

6.8 The nef reduction for $\dim(Alb_1(X)) = 2$. In this case, the generalized Albanese morphism $alb_1 : X \to Alb_1(X)$ contracts finitely many curves. Among those curves are the $\{C_i\}_{i \in I}$ which are numerically trivial with respect to $L_r$. The intersection matrix of these $\{C_i\}_{i \in I}$ is negative definite. This yields that these curves form a basis of a proper subspace of $\text{NS}_\mathbb{Q}(X)$. Again, just setting $alb_r = id_X$, the assertions of 6.2 are fulfilled. As before remark 6.7 applies.

6.9 Surfaces of Kodaira dimension less than 2. Let $X$ be a projective algebraic surface of Kodaira dimension $\kappa(X) \leq 1$. We assume that $X$ is minimal. This is not a restriction, because the fundamental group of rational curves is zero. The next table gives the generalized Albanese morphism for these surfaces following the Enriques-Kodaira classification (see VI. in [I]). The last two rows are perhaps the most interesting ones. They show that the generalized Albanese morphisms may reveal more of the surface than the classical one. We assume in these two rows that the surface $X$ is not a product of two curves.

| $\kappa(X)$ | class of $X$ | the generalized Albanese morphism |
|-------------|--------------|----------------------------------|
| $-\infty$   | rational surfaces | $X \to \text{Spec}(\mathbb{C})$ |
| $-\infty$   | ruled surfaces  | $X \to \text{Spec}(\mathbb{C})$ |
|             | $X \to B$ with $g(B) \geq 1$ | $X \to B$ |
| 0           | Enriques surfaces | $X \to \text{Spec}(\mathbb{C})$ |
| 0           | K3 surfaces      | $X \to \text{Spec}(\mathbb{C})$ |
| 0           | tori            | $X \to X$ |
| 0           | hyperelliptic surfaces | $\text{Alb}_1(X)$ is an elliptic curve, whereas $\text{Alb}_r(X) \cong X$, for $r > 1$, see also §7 |
| 1           | properly elliptic surfaces | $\text{Alb}_1(X)$ is an algebraic curve, and $\text{Alb}_r(X) \cong X$, for $r > 1$. |

7. An example

7.1 The group $G$. We consider the group $G$ which as a set is $\mathbb{Z}^4$ with group structure given by $(a, b, c, d)(a', b', c', d') = (a + a', b + b', c + (-1)a'c', d + (-1)c'd')$. The group $G$ has the four generators $g_0 = (1, 0, 0, 0)$, $g_1 = (0, 1, 0, 0)$, $g_2 = (0, 0, 1, 0)$, and $g_3 = (0, 0, 0, 1)$ satisfying the six relations:

$$g_0g_1 = g_1g_0, \quad g_0g_2 = g_2^{-1}g_0, \quad g_0g_3 = g_3^{-1}g_0, \quad g_1g_2 = g_2g_1, \quad g_1g_3 = g_3g_1, \quad g_2g_3 = g_3g_2.$$  

$G$ may be described as the semidirect product $(\mathbb{Z}^2) \ltimes (\mathbb{Z}^2)$ where the homomorphism $(\mathbb{Z}^2) \to \text{Aut}(\mathbb{Z}^2)$ is given by $(a, b) \mapsto (-1)^a \text{id}_{\mathbb{Z}^2}$.

7.2 Characters of $G$. It is easy to check that the commutator subgroup $[G, G]$ is generated by $g_2^2$ and $g_3^2$. Thus, we obtain $G/[G, G] \cong \mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})^2$. Consequently, each
group homomorphism \( \tau : G \to \text{U}(1) \) must send \( g_2 \) and \( g_3 \) to \( \pm 1 \). Thus, in a continuous family \( \tau_t \) of such morphisms the images of \( g_2 \) and \( g_3 \) are locally constant. We conclude that there are four families of \( \text{U}(1) \)-representations of \( G \). One is given by

\[
\tau_{t_1,t_2}(g_0) = \exp(t_1 \cdot i) \quad \tau_{t_1,t_2}(g_1) = \exp(t_2 \cdot i) \quad \tau_{t_1,t_2}(g_2) = 1 \equiv \tau_{t_1,t_2}(g_3)
\]

with \((t_1,t_2) \in (\mathbb{R}/2\pi\mathbb{Z})^2\). The other three representations of \( G \) in \( \text{U}(1) \) are obtained by changing the image of \( g_2 \) or \( g_3 \) to \(-1\).

7.3 \( G \) is the fundamental group of a hyperelliptic surface. We consider two elliptic curves \((E_1,0)\) and \((E_2,0)\) with a nonzero two-torsion point \( e_1 \in E_1 \). The fix point free action of \( \mathbb{Z}/2\mathbb{Z} \) on \( E_1 \times E_2 \) generated by \((z_1,z_2) \mapsto (z_1 + e_1,-z_2)\) induces a smooth projective quotient \( X \), and a \( 2:1 \) covering \( p : E_1 \times E_2 \to X \). This \( X \) is a hyperelliptic surface (see [1] V.5). The fundamental group \( \pi_1(X) \) is isomorphic to \( G \) and the commutative subgroup \( 2\mathbb{Z} \times \mathbb{Z}^3 \) of index two corresponds to the cover \( p \). Denoting the \( \mathbb{Z}/2\mathbb{Z} \)-quotient of \( E_1 \) by \( e_1 \) by \( \tilde{E}_1 \), we obtain a morphism \( \alpha : X \to \tilde{E}_1 \). The induced map on the first homology groups

\[
\alpha_* : [H_1(X,\mathbb{Z})] \cong \mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})^2 \longrightarrow [H_1(\tilde{E}_1,\mathbb{Z})] \cong \mathbb{Z}^2
\]

is an isomorphism between the free parts whereas the torsion part of \( H_1(X,\mathbb{Z}) \) is mapped to zero. Thus, the morphism \( \alpha \) is the Albanese morphism for the surface \( X \).

7.4 The generalized Albanese morphism \( \text{alb}_r(X) \) differs from the ordinary Albanese morphism. Let \( \mathcal{L}_2 \) be the line bundle defining the second generalized Albanese morphism \( \text{alb}_2(X) \) on \( X \). And \( \tilde{\mathcal{L}}_1 \) be the corresponding line bundle on \( E_1 \times E_2 \). Since \( \text{Alb}(E_1 \times E_2) = E_1 \times E_2 \), the line bundle \( \tilde{\mathcal{L}}_1 \) is ample. From proposition 5.5 we deduce that \( \tilde{\mathcal{L}}_1 \cong p^* \mathcal{L}_2 \). Thus \( \mathcal{L}_2 \cdot C > 0 \) for all curves \( C \) in \( X \).

Since \( \mathcal{L}_2 \leq \mathcal{L}_r \) for all \( r \geq 2 \), the equivalence relation \( \sim_{\mathcal{L}_r} \) defined by the line bundles \( \mathcal{L}_r \) is the identity relation. Thus, \( \text{alb}_r = \text{id}_X : X \to X \) for all \( r \geq 2 \). On the other hand, the classical Albanese variety has only dimension one.

7.5 A class of examples. The above example is a special case of the following class of examples. Let \( G \) be a finite group with \( |G| \) elements and \( C_1, C_2 \) be two smooth projective curves with a \( G \) action such that

1. The genera \( g_{C_1} \) and \( g_{C_2} \) are positive;
2. \( G \) acts free on \( C_1 \), i.e the quotient map \( C_1 \to C_1/G \) is étale;
3. There are no \( G \)-invariant global sections in \( H^0(C_2,\omega_{C_2}) \). This is equivalent to \( C_2/G \cong \mathbb{P}^1 \).

We obtain a free \( G \)-action on \( C_1 \times C_2 \). Let \( X := (C_1 \times C_2)/G \) be the quotient of this action and \( p : C_1 \times C_2 \to X \) the projection. Since \( C_1 \times C_2 \) is embedded into its Albanese variety, we deduce from proposition 5.5 that \( \mathcal{L}_{|G|}\cdot C > 0 \) for all curves \( C \subset X \). Thus, \( \text{alb}_r = \text{id}_X : X \to X \) for all \( r \geq |G| \), whereas the classical Albanese morphism is just the map to the curve \( C_1/G \).

References

[1] W. Barth; C. Peters; A. Van de Ven, Compact Complex Surfaces, Springer Verlag, Berlin, 1984.
REFERENCES

[2] Th. Bauer; F. Campana; Th. Eckl; S. Kebekus; Th. Peternell; S. Rams; T. Szemberg; L. Wotzlaw, *A reduction map for nef line bundles*, in I. Bauer; F. Catanese; Y. Kawamata; T. Peternell; Y.-T. Siu (editors), *Complex Geometry, Collection of Papers dedicated to Hans Grauert*, p. 27-36, Springer Verlag, 2002.

[3] F. Campana, *Remarques sur le revêtement universel des variétés Kähleriennes compacts*, Bull. Soc. Math. France 122 (1994), p. 255-284.

[4] J.-M. Drezet; M. S. Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques*, Invent. Math. 97 (1989), p. 53-94.

[5] G. Faltings, *Stable G-bundles and projective connections*, J. Alg. Geom. 2 (1993), p. 507-568.

[6] P. Griffiths; J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, John Wiley & Sons, New York, 1978.

[7] G. Hein, *Duality construction of moduli spaces*, Geometria Dedicata 75 (1999), p. 101-113.

[8] L. Katzarkov, *Factorization theorems for the representations of the fundamental groups of quasiprojective varieties and some applications*, preorder 1994.

[9] L. Katzarkov, *On the Shafarevich maps*, Proc. of Symp. in Pure Math. 62.2 (1997) p.173-216.

[10] J. Kollár, *Shafarevich Maps and Automorphic Forms*, Princeton University Press, Princeton NJ, 1995.

[11] S. Langton, *Valuative criteria for families of vector bundles on algebraic varieties*, Ann. Math. 101 (1975), p. 88-110.

[12] V. B. Mehta; A. Ramanathan, *Restriction of stable sheaves and representations of the fundamental group*, Invent. Math. 77 (1984), 163-172.

[13] M. Popa, *Dimension estimates for Hilbert schemes and effective base point freeness on moduli spaces of vector bundles*, Duke Math. Journal 107 (2001), p. 469-495.

[14] J. Le Potier, *Module des fibrés semistables et fonctions thêta*, in M. Maruyama (ed.), *Proc. Symp. Taniguchi Kyoto 1994: Moduli of vector bundles*, Lect. Notes in Pure and Appl. Math. 179 (1996), p. 83-101.

[15] H. Tsuji, *Numerically trivial fibrations*, preorder 2000.

[16] K. Uhlenbeck; S.-T. Yau, *On the existence of of Hermitian Yang-Mills connections on stable vector bundles I & II*, Comm. Pure Appl. Math. 39 (1986) p. 257-293, and 42 (1989) p. 703-707.

Georg Hein
Freie Universität Berlin, Institut für Mathematik II, Arnimallee 3, D-14195 Berlin, Germany
ghein@math.fu-berlin.de