UNOBBSTRUCTEDNESS OF CALABI-YAU ORBIKLEINFOLDS

Z. Ran
Department of Mathematics
University of California
Riverside, CA 92521, USA
ziv@ucrmath.ucr.edu

Abstract. We show that Calabi-Yau spaces with certain types of hypersurface-quotient singularities have unobstructed deformations. This applies in particular to all Calabi-Yau orbifolds nonsingular in codimension 2.

In recent years Calabi-Yau manifolds have attracted a great deal of attention, motivated both by their role in Classification Theory and by their connections with Physics, in particular the phenomenon of Mirror Symmetry. Both the Classification Theory and Physics viewpoints suggest looking more generally at a suitable class of singular Calabi-Yau spaces, e.g. most known Mirror Symmetry constructions involve forming a quotient under a finite group action, thus leading naturally to Calabi-Yau orbifolds and families of such.

Now a basic result about Calabi-Yau manifolds $X$ is the theorem of Bogomolov-Tian-Todorov asserting that $X$ has unobstructed deformations. Some extensions of this theorem to the case of singular $X$ have been considered in [K], [R1], [T], [N]. However, these extensions do not cover, e.g. the case of orbifolds. Thus one is naturally led to ask, as did D. Morrison, whether unobstructedness holds for Calabi-Yau orbifolds, say nonsingular in codimension 2. Our purpose here is to answer this question affirmatively and, in fact, to prove a rather more general statement which at the same time generalizes the result of [R1], allowing (local) quotients of Kleinfolds, whence the term ‘orbiKleinfolds’.

The proof is particularly easy in the orbifold case, requiring little more than a simple combination of (either form of) the ‘dual unobstructedness criterion’ of [R2].

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with the ideas surrounding Schlessinger’s rigidity theorem [S]. The more general orbiKleinfeld case requires more work, which we feel is justified by the interest of the result.

We begin by recalling that a Kleimian singularity is an isolated simple hypersurface singularity. These are classified by Arnold into types A-D-E (cf. [1], p. 132). We generalize this notion as follows.

**Definition 1.** An isolated hypersurface singularity \((X, p)\) is said to be weakly Kleinian provided

(i) \(X\) is cDV (i.e., a general surface section through \(p\) is a DuVal or Kleinian singularity);
(ii) \(X\) admits a \(\mathbb{C}^*\) action with positive weights;
(iii) for a resolution \(\tilde{X} \to X\), all components of the exceptional locus are smooth divisors \(E\) with \(H^{p,q}(E) = 0\) for \(p \neq q\).

**Remark.** Note that (i) implies that \(X\) is a rational singularity, hence \(H^{p,0}(E) = 0, \ p > 0\). In particular, in dimension 3, (i) implies (iii).

Next recall from [R1] that an isolated singularity \((X, p)\) is said to be good if for a resolution \(\hat{X} \to X\) and \(\hat{X}\) the formal neighborhood of the exceptional locus, the map induced by exterior derivative

\[ d : H^i(\Omega^j_{\hat{X}}) \to H^i(\Omega^{j+1}_{\hat{X}}) \]

is injective in the range \(i, j > 0, \ i \neq j, i + j < \dim X\). All 3-fold rational singularities, and \(A_1\) singularities in all dimensions are good. Presumably all Kleinian singularities are good, but this is unproven. The unobstructedness result of [R1] as stated allowed good Kleinian singularities, it was remarked by Namikawa in [N] that the proof covers the good weakly Kleinian case as well.

**Definition 2.** An analytic variety \(X\) is said to be a (weak) OrbiKleinfeld if \(X\) is locally of the form \(V/G\), where \(V\) has (weak) Kleinian singularities and \(G\) is a finite group acting on \(V\) which is ‘small’ in that for all \(g \in G, \ p \in V^g\), the induced action of \(g\) on the Zariski tangent \(T_pV\) has no eigenspace of codimension exactly 1.

**Remarks.**

1. All orbifolds are orbiKleinfelds (cf. [St]).
In dimension 3, all terminal singularities are, by Mori’s classification (cf. [Rd]) (cyclic) quotients of isolated cDV points, hence ‘almost’ weakly orbiKleinian; thus the class of weak orbiKleinfelds is rather more general than that of orbifolds.

It seems reasonable that weak orbiKleinfelds are always Cohen-Macaulay, but I cannot prove this. In any case, the CM property certainly holds in all examples of interest, e.g. orbifolds and, more generally, germs of the form $V(f)/G$ where $f$ is $G$-invariant: indeed $V(f)/G$ is then a Cartier divisor on an orbifold, hence CM.

If $X = V/G$ is a local weak orbiKleinfeld, then $V$ admits a $G$-equivariant retention $\tilde{V}$, yielding an orbifold $X_{orb} = V/G$ with a birational morphism to $X$. This construction clearly globalizes, yielding an ‘orbifold resolution’ $X_{orb} \to X$ for any weak orbiKleinfeld, which is an isomorphism off the (discrete) non-orbifold locus of $X$.

**Definition 3.** A Calabi-Yau (weak) orbiKleinfeld is a compact irreducible (weak) orbiKleinfeld $X$ such that

(i) $X$ admits a resolution of singularities by a Kähler manifold;

(ii) $X$ is Cohen-Macauley;

(iii) the dualizing sheaf $\omega_X \cong \mathcal{O}_X$.

Our main result is the following

**Theorem 4.** Any Calabi-Yau weak orbiKleinfeld nonsingular in codimension 2 has unobstructed deformations.

**Remark.** This generalizes Theorem 1 of [R1]. In dimension 3, unobstructedness of Calabi-Yau spaces with terminal singularities has been proven by Namikawa [N], but not all weak orbiKleinfelds are terminal, nor conversely.

**Proof of theorem.** Let $X$ be a Calabi-Yau weak orbiKleinfeld of dimension $n$, $\Theta = \mathcal{H}om(\Omega^1_X, \mathcal{O}_X)$ its tangent sheaf, and $j : X_{reg} \to X$ the inclusion of the open subset of regular points. In view of our depth hypothesis, results of Schlessinger [S], as exposed in ([A], I.9-10), show that we have an isomorphism of deformation functors
In particular the first-order deformation group

\[ T^1(X) = \text{Ext}^1(\Omega^1_X, \mathcal{O}_X) \cong H^1(X_{\text{reg}}, \Theta); \]

and, more importantly for us obstructions to deforming \( X \) may be taken in the group \( H^2(X_{\text{reg}}, \Theta) \). The following is in essence due to Schlessinger (and is valid more generally for quotients, nonsingular in codimension 2, of isolated hypersurface singularities).

**Lemma 5.** (i) \( R^1 j_* \Theta_{X_{\text{reg}}} \) is supported in the nonorbifold locus of \( X \);

(ii) \( R^2 j_* \Theta_{X_{\text{reg}}} = 0 \).

**Proof.** We work locally on a small open set \( U = V/G \subset X \), where \( V \) is a local hypersurface with one (possibly) singular point \( p \). Put

\[ V_0 = V\{p\} \]

\[ V_{00} = \text{subset of } V_0 \text{ where } G \text{ acts freely}. \]

Note that \( V/V_{00} \) has codimension \( \geq 3 \) in \( X \) and \( U_{\text{reg}} = V_{00}/G \). As in \([A]\), we have

\[ H^i(V_{\text{reg}}, \Theta) = H^i(V_{00}, \Theta_{V_{00}})^G = H^i(V_0, \Theta_{V_0})^G, \quad i = 1, 2, \]

and \( H^i(V_0, \Theta_{V_0}) \) coincides with \( T^i_{V,p} \) and in particular vanishes if \( p \) is regular or \( i = 2 \). \( \square \)

Now put

\[ \tilde{\Omega}_X = j_* \Omega_{X_{\text{reg}}}. \]

This is a complex of reflexive sheaves on \( X \) and by Lemma 2 we have

\[ H^2(X_{\text{reg}}, \Theta_{X_{\text{reg}}}) = H^2(X, j_* \Theta_{X_{\text{reg}}}) \hookrightarrow \text{Ext}^2(\tilde{\Omega}^1_X, \mathcal{O}_X), \]

hence the latter group is an obstruction group for deformations of \( X \). By Grothendieck duality, \( \text{Ext}^2(\tilde{\Omega}^1_X, \mathcal{O}_X) = \text{Ext}^2(\tilde{\Omega}^1_X, \omega_X) \) is dual to \( H^{n-2}(\tilde{\Omega}^1_X) \). Now the following result generalizes simultaneously results of Steenbrink \([St]\) and \(([R1], \text{Proposition } 4)\).
Proposition 6. Let $X$ be a compact weak orbiKleinfold of dimension $n \geq 2$, $\pi_0 : X_{orb} \to X$ an orbifold resolution $\pi_1 : \tilde{X} \to X_{orb}$ a resolution of singularities with $\tilde{X}$ Kähler. Then

(i) $\bar{\Omega}_X = \pi_0^* \bar{\Omega}_{X_{orb}} = (\pi_0 \circ \pi_1)_* \Omega_{\tilde{X}}$;

(ii) $\bar{\Omega}_X$ is a resolution of the constant sheaf $\mathbb{C}_X$;

(iii) the Hodge-De Rham spectral sequence

$$E_1^{p,q} = H^q(X, \bar{\Omega}_X^p) \Rightarrow H^{p,q}(X, \mathbb{C})$$

degenerates at $E_1$ in degrees $\leq n - 1$.

The argument that Proposition 6 implies Theorem 4, based on the ‘$T^2$-injecting criterion’ (Theorem 1.1, (ii) of [R2]) is identical to the corresponding argument in [R1]. In the case of orbifolds, Proposition 6 is due to Steenbrink [St] (and consequently the reader only interested in orbifolds may stop reading here). Our proof of the Proposition combines ideas from [St] and [R1].

Proof of Proposition 6. (i) The assertion is local, and at orbifold points has been proven by Steenbrink ([St], Lemma 1.11). For the nonorbifold points, consider

$$U = V/G \subset X,$$

$$\omega \in \Gamma(U, \hat{\Omega}_X^i) = \Gamma(U_{reg}, \Omega_{reg}^i).$$

By ([St], Lemma 1.8), $\omega$ lifts to $\tilde{\omega} \in \Gamma(V_{reg}, \Omega_{V_{reg}}^i)$. We may assume our desingularization of $X$ is locally over $U$ of the form $\tilde{V}/G$ where $\tilde{V} \to V$ is some desingularization (which will also blow up the nonfree locus of $G$, in addition to $\text{sing} \tilde{V}$).

By [R1], Proposition 4, (i), $\tilde{\omega}$ extends holomorphically to $\tilde{V}$ and being $G$-invariant descends to $\tilde{V}/G$, as required.

(ii) This is a consequence of ([St], Lemma 1.8), ([R1], Proposition 4), and exactness of the functor of $G$-invariants.

(iii) We prove the vanishing of the differentials $d_i^{i+j}$, $i + j < n$, the case of $d_r^{i+j}$, $r \geq 2$, being similar. Denote by $\hat{\bar{\Omega}}_X^i \subset \bar{\Omega}_X^i$ the subsheaf of closed forms and similarly for $\hat{\bar{\Omega}}_{X_{orb}}^i, \bar{\Omega}_{\tilde{X}}^i$. Consider the Leray spectral sequence

$$E_1^{p,q} = H^p(X, R^q \pi_{orb}^* \hat{\bar{\Omega}}_{X_{orb}}^i) \Rightarrow H^{p+q}(X, \bar{\Omega}_{\tilde{X}}^i).$$
and the anlogous one for $\tilde{\Omega}^j_{X_{\text{orb}}}$. Using that $\pi_0$ is an isomorphism off the finite nonorbifold locus of $X$, we get a diagram

\[
\begin{array}{ccc}
H^0(R^{i-1}\pi_0^*\tilde{\Omega}^j_{X_{\text{orb}}}) & \to & H^0(R^{i-1}\pi_0^*\tilde{\Omega}^{j+1}_{X_{\text{orb}}}) \\
\downarrow & & \downarrow \\
\tag{*}
H^i(\tilde{\Omega}^i_X) & \to & H^i(\tilde{\Omega}^{i+1}_X) \\
\downarrow & & \downarrow \\
H^i(\tilde{\Omega}^i_{X_{\text{orb}}}) & \xrightarrow{\alpha_{ij}} & H^i(\tilde{\Omega}^{i+1}_{X_{\text{orb}}})
\end{array}
\]

in which the composite of the middle-row arrow coincides with $d_{i1}^i$. by Steenbrink [St], we have

\[
H^i(\tilde{\Omega}^{j+1}_{X_{\text{orb}}}) = \mathbb{H}^i(F^{j+1}\tilde{\Omega}^j_{X_{\text{orb}}}) = F^{j+1}\mathbb{H}^{i+j+1}(\tilde{\Omega}^j_{X_{\text{orb}}}),
\]

where $F^\cdot$ denotes the Hodge (or stupid) filtration, and it follows easily that $\alpha_{ij}$ vanishes. To prove $d_{i1}^i = 0$ for $i + j < n$, $j \neq i - 2$, it will suffice, by (*), to prove the vanishing of $\beta_{ij}$ in this range. For this we use our goodness hypothesis. Let $\hat{X}_{\text{orb}}$ be the formal neighborhood of the exceptional locus of $\pi_0$ and similarly for $\hat{\tilde{X}}$. Then we have a diagram

\[
\begin{array}{ccc}
H^{i-1}(\hat{X}_{\text{orb}}, \tilde{\Omega}^{j+1}_{X_{\text{orb}}}) & \xrightarrow{\hat{\beta}_{ij}} & H^{i-1}(\hat{X}_{\text{orb}}, \hat{\Omega}^{j+1}_{X_{\text{orb}}}) \\
\pi^* \downarrow & & \pi^* \downarrow \\
H^{i-1}(\hat{\tilde{X}}, \hat{\Omega}^{j+1}_X) & \xrightarrow{\hat{\beta}_{ij}} & H^{i-1}(\hat{\tilde{X}}, \hat{\Omega}^{j+1}_X)
\end{array}
\]

where $\hat{\beta}_{ij}, \hat{\beta}_{ij}$ are the obvious maps. By goodness, $\hat{\beta}_{ij}$ vanishes whenever $i - 1 \neq j+1$, because the supposedly injective exterior-derivative map vanishes on its image. If we can prove the vertical maps $\pi^*$ in (**) are injective, it then follows that $\hat{\beta}_{ij}$, hence $\beta_{ij}$, vanishes. But the injectivity of $\pi^*$ follows from Steenbrink’s duality argument ([St], Proof of Thm. 1.12), applied on the formal scheme $\hat{X}_{\text{orb}}$.

Finally, it remains to prove the vanishing of $d_{11}^{i,i-2}, 2 \leq i \leq \frac{n+1}{2}$. This is done just as in [R1], comparing $H^i(X, \mathbb{C})$ with $H^i(X_{\text{orb}}, \mathbb{C})$ and using the fact that the component of the exceptional locus of $\pi_0$ are quotients of manifolds with only $(p,p)$ cohomology, hence by Steenbrink’s theory are themselves orbifolds with all their cohomology of type $(p,p)$, hence contribute via Gysin only $(p,p)$ classes to $H^i(X_{\text{orb}})$. □

Remark. After this was written, the author became aware of Fujiki’s paper [F], which contains numerous results on symplectic orbifolds and their deformations.
but not including unobstructedness, which Fujiki includes as an hypothesis in some statements. In particular, in his Theorem 4.8, p. 116, the hypothesis ‘$S$ is smooth’ holds automatically.

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