Asymptotics of the probability distributions of the first hitting times of Bessel processes

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Abstract. The asymptotic behavior of the tail probabilities for the first hitting times of the Bessel process with arbitrary index is shown without using the explicit expressions for the distribution function obtained in the authors' previous works.

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1. Introduction and main results

Let $\mathbb{P}_a^{(\nu)}$ be the probability law on the path space $W = C([0, \infty); \mathbb{R})$ of a Bessel process with index $\nu \in \mathbb{R}$ or dimension $\delta = 2(\nu + 1)$ starting from $a > 0$. For $w \in W$ we denote the first hitting time to $b > 0$ by $\tau_b = \tau_b(w)$:

$$\tau_b = \inf\{t > 0; w(t) = b\}.$$

In recent works [4, 5] the authors have shown explicit forms of the distribution function and the density of the distribution of $\tau_b$ under $\mathbb{P}_a^{(\nu)}$ in the case where $0 < b < a$. The other case, which is easier since we do not need to consider a natural boundary, has been known by Kent [6]. See also Gettor-Sharpe [2].

When $\nu > 0$ and $b < a$, it is shown in [3] that there exists a positive constant $C(\nu)$ such that

$$\mathbb{P}_a^{(\nu)}(\tau_b > t) = 1 - \left(\frac{b}{a}\right)^{2\nu} + C(\nu)t^{-\nu} + o(t^{-\nu}).$$

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The constant $C(\nu)$ may be expressed explicitly and we could treat all the cases. However, we need to consider separately the case where $\delta$ is an odd integer and the expression for $C(\nu)$ is different from the other cases. This is because the expression for the distribution function itself is different.

The aim of this note is to show that the constant $C(\nu)$ has the same simple expression also when $\delta$ is an odd integer by considering the asymptotics without using the explicit expressions for the distribution functions obtained in [5].

When $a < b$ and $\nu > 0$, the explicit expressions for $\mathbb{P}_a^{(\nu)}(\tau_b > t)$ has been shown in [6] and, from his result, it is easily shown that the tail probability decays exponentially. Hence we concentrate on the case of $b < a$.

**Theorem 1.** Let $\nu > 0$ and $0 < b < a$. Then, as $t \to \infty$, it holds that

$$
\mathbb{P}_a^{(\nu)}(t < \tau_b < \infty) = b^{2\nu} \left\{ 1 - \left( \frac{b}{a} \right)^{2\nu} \right\} \frac{1}{\Gamma(1 + \nu)(2t)^\nu} + O(t^{-\nu-\varepsilon})
$$

for any $\varepsilon \in (0, \frac{\nu}{1+\nu})$, where $\Gamma$ denotes the usual Gamma function.

**Theorem 2.** Let $\nu < 0$ and $0 < b < a$. Then, as $t \to \infty$, it holds that

$$
\mathbb{P}_a^{(\nu)}(\tau_b > t) = a^{2|\nu|}\left\{ 1 - \left( \frac{a}{b} \right)^{2|\nu|} \right\} \frac{1}{\Gamma(1 + |\nu|)(2t)^{|\nu|}} + O(t^{-|\nu|-\varepsilon})
$$

for any $\varepsilon \in (0, \frac{\nu}{1+\nu})$.

When $\nu = 0$, it is known that

$$
\mathbb{P}_a^{(0)}(\tau_b > t) = \frac{2 \log(a/b)}{\log t} + o((\log t)^{-1}).
$$

This identity has been discussed in [5] and we omit the details.

## 2. Proof of Theorem 1

We assume $\nu > 0$ in this section. At first we give some lemmas. The first one is shown by Byczkowski and Ryznar [1].

**Lemma 3.** There exists a constant $C$ such that

$$
\mathbb{P}_a^{(\nu)}(t < \tau_b < \infty) \leq C t^{-\nu}.
$$

**Lemma 4.** If $0 < b < a$, one has

$$
\mathbb{P}_a^{(\nu)}(\tau_b > t) = 1 - \left( \frac{b}{a} \right)^{2\nu} + \mathbb{E}_a^{(\nu)} \left[ \left( \frac{b}{R_t} \right)^{2\nu} 1_{\{ \inf_{0 \leq s \leq t} R_s > b \}} \right]
$$

for any $t > 0$, where $\mathbb{E}_a^{(\nu)}$ is the expectation with respect to $\mathbb{P}_a^{(\nu)}$ and $\{R_s\}_{s \geq 0}$ denotes the coordinate process.

**Proof.** It is well known that

$$
\mathbb{P}_a^{(\nu)}(\tau_b = \infty) = \mathbb{P}_a^{(\nu)}(\inf_{s \geq 0} R_s > b) = 1 - \left( \frac{b}{a} \right)^{2\nu}.
$$
By the Markov property of Bessel processes, we have for \( t > 0 \)
\[
\mathbb{P}_a^{(\nu)}(\tau_b = \infty) = \mathbb{P}_a^{(\nu)}(\inf_{0 \leq s \leq t} R_s > b \text{ and } \inf_{s \geq t} R_s > b) 
\]
\[
= \mathbb{E}_a^{(\nu)}[\mathbb{P}_R^{(\nu)}(\tau_b = \infty)\mathbf{1}_{\{\inf_{0 \leq s \leq t} R_s > b\}}] 
\]
\[
= \mathbb{E}_a^{(\nu)}[\left\{1 - \left(\frac{b}{R_t}\right)^{2\nu}\right\}\mathbf{1}_{\{\inf_{0 \leq s \leq t} R_s > b\}}], 
\]
which implies \((1)\).

**Lemma 5.** For any \( a > 0 \) and \( p \) with \( 0 < p < 1 + \nu \), it holds that
\[
\frac{\Gamma(1 + \nu - p)}{\Gamma(1 + \nu)} \frac{1}{(2t)^p} e^{-\frac{a^2}{2t}} \leq \mathbb{E}_a^{(\nu)}[\langle R_t \rangle^{-2p}] \leq \frac{\Gamma(1 + \nu - p)}{\Gamma(1 + \nu)} \frac{1}{(2t)^p} + Ct^{-1-p} \tag{2}
\]
for \( t \geq 1 \), where \( C \) is a positive constant independent of \( t \).

**Proof.** By the explicit expression for the transition density of the Bessel process, we have
\[
\mathbb{E}_a^{(\nu)}[\langle R_t \rangle^{-2p}] = \int_0^\infty y^{-2p} \frac{1}{t} \left(\frac{y}{a}\right) \nu ye^{-\frac{a^2}{2t}} I_\nu\left(\frac{ay}{t}\right) dy, 
\]
where \( I_\nu \) is the modified Bessel function of the first kind with index \( \nu \) (cf. \([8]\)) given by
\[
I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^\infty \frac{(z/2)^{2n}}{\Gamma(n+1)\Gamma(1+\nu+n)} \quad (z \in \mathbb{R} \setminus (-\infty, 0)). 
\]
Hence, it is easy to get
\[
\mathbb{E}_a^{(\nu)}[\langle R_t \rangle^{-2p}] = \frac{1}{(2t)^p} e^{-a^2/2t} \sum_{n=0}^\infty \frac{a^{2n} \Gamma(n + \nu + 1 - p)}{\Gamma(n+1)\Gamma(1+n+\nu)(2t)^n} 
\]
and the assertion of the lemma.

**Remark 6.** The moments of \( R_t \) for fixed \( t \) have explicit expressions by means of the Whittaker functions (cf. \([3]\), p.709), but it does not seem useful.

We are now in a position to give a complete proof of Theorem \((1)\).

**Proof of Theorem \((1)\)** By Lemma \((4)\) we have
\[
\mathbb{P}_a^{(\nu)}(t < \tau_b < \infty) = b^{2\nu} \mathbb{E}_a^{(\nu)}[\langle R_t \rangle^{-2\nu}] - b^{2\nu} \mathbb{E}_a^{(\nu)}[\langle R_t \rangle^{-2\nu}\mathbf{1}_{\{\tau_b \leq t\}}]. 
\]
For the first term we have shown in Lemma \((5)\)
\[
\mathbb{E}_a^{(\nu)}[\langle R_t \rangle^{-2\nu}] = \frac{1}{\Gamma(1+\nu)(2t)^\nu} (1 + O(t^{-1})). 
\]
Hence, if we could show
\[
\mathbb{E}_a^{(\nu)}[\langle R_t \rangle^{-2\nu}\mathbf{1}_{\{\tau_b \leq t\}}] = \frac{1}{\Gamma(\nu+1)(2t)^\nu} \left(\frac{b}{a}\right)^{2\nu} + O\left(\frac{1}{t^{\nu+\varepsilon}}\right) \tag{3}
\]
for
for any \( \varepsilon \in (0, \frac{\nu}{\nu+1}) \), we obtain the assertion of the theorem.

For this purpose, we let \( \alpha \in (0, \frac{1}{\nu+1}) \), choose \( p \) satisfying
\[
\frac{1}{1 - \alpha} < p < \frac{1 + \nu}{\nu}
\]
and let \( q \) be such that \( p^{-1} + q^{-1} = 1 \). We divide the expectation on the right hand side of \((3)\) into the sum of
\[
I_1 = \mathbb{E}_{\alpha}^{(\nu)}[(R_t)^{-2\nu}1_{\{\tau_{\alpha} \leq t\}}] \quad \text{and} \quad I_2 = \mathbb{E}_{\alpha}^{(\nu)}[(R_t)^{-2\nu}1_{\{t^{\alpha} < \tau_{\beta} \leq t\}}]
\]
We simply apply the H"older inequality to \( I_2 \). Then we get
\[
I_2 \leq \mathbb{E}_{\alpha}^{(\nu)}[(R_t)^{-2\nu}1_{\{t^{\alpha} < \tau_{\beta} < \infty\}}]^{1/p} \left\{ \mathbb{P}(\nu)(t^{\alpha} < \tau_{\beta} < \infty) \right\}^{1/q}.
\]
and, by Lemmas 3 and 5, we see that there exists a constant \( C_1 \) such that
\[
I_2 \leq C_1 t^{-\nu - \alpha\nu}.
\]
In the following we denote by \( C_i \)'s the constants independent of \( t \).

For \( I_1 \), the strong Markov property of Bessel processes implies
\[
I_1 = \int_{0}^{t^{\alpha}} \mathbb{E}_{\beta}^{(\nu)}[(R_{t-s})^{-2\nu}1_{\{\tau_{\beta} \in ds\}}] = I_{11} + I_{12},
\]
where
\[
I_{11} = \int_{0}^{t^{\alpha}} \frac{1}{2^\nu \Gamma(\nu+1)} \frac{1}{(t-s)^{\nu}} \mathbb{P}(\nu)(\tau_{\beta} \in ds).
\]
Since \( \alpha q < 1 \), Lemma 5 implies
\[
|I_{12}| = |I_1 - I_{11}| \leq \int_{0}^{t^{\alpha}} \frac{C_2}{(t-s)^{\nu+1}} \mathbb{P}(\nu)(\tau_{\beta} \in ds) \leq \frac{C_3}{\nu+1}.
\]
We divide \( I_{11} \) into the sum of
\[
J_1 = \int_{t^{\alpha}}^{t} \frac{1}{2^\nu \Gamma(\nu+1)} \frac{1}{(t-s)^{\nu}} \mathbb{P}(\nu)(\tau_{\beta} \in ds)
\]
and
\[
J_2 = \int_{0}^{t^{\alpha}} \frac{1}{2^\nu \Gamma(\nu+1)} \frac{1}{(t-s)^{\nu}} \mathbb{P}(\nu)(\tau_{\beta} \in ds).
\]
For \( J_1 \) we have by Lemma 5
\[
0 \leq J_1 \leq \frac{C_4}{(t-t^{\alpha})^{\nu}} \mathbb{P}(\nu)(t^{\alpha} < \tau_{\beta} < \infty) \leq \frac{C_5}{t^{\nu+\alpha\nu}}.
\]
For $J_2$ we have
\[
J_2 \leq \frac{1}{2^\nu \Gamma(\nu + 1)(t-t^a)^\nu} \mathbb{P}_a^{(\nu)}(\tau_b \leq t^a)
\]
\[
\leq \frac{1}{\Gamma(\nu + 1)(2t)^\nu} \mathbb{P}_a^{(\nu)}(\tau_b < \infty) \frac{1}{(1-t^{-\nu})^\nu}
\]
\[
\leq \frac{1}{\Gamma(\nu + 1)(2t)^\nu} \left( \frac{b}{a} \right)^{2\nu} \left( 1 + \frac{C_6}{t^{1-\nu}} \right).
\]
On the other hand we have by Lemma 3
\[
J_2 \geq \frac{1}{\Gamma(\nu + 1)(2t)^\nu} \mathbb{P}_a^{(\nu)}(\tau_b \leq t^a)
\]
\[
= \frac{1}{\Gamma(\nu + 1)(2t)^\nu} \left\{ \mathbb{P}_a^{(\nu)}(\tau_b < \infty) - \mathbb{P}_a^{(\nu)}(t^a \leq \tau_b < \infty) \right\}
\]
\[
\geq \frac{1}{\Gamma(\nu + 1)(2t)^\nu} \left( \frac{b}{a} \right)^{2\nu} - \frac{C_7}{t^{\nu+\nu}}.
\]
Combining the above estimates, we obtain
\[
\mathbb{E}_a^{(\nu)}[(R_t)^{-2\nu}1_{\{\tau_b \leq t\}}] = \frac{1}{\Gamma(\nu + 1)(2t)^\nu} \left( \frac{b}{a} \right)^{2\nu} + \frac{1}{t^{\nu+\nu}} O \left( \frac{1}{t^{\nu+\nu}} \right) + O \left( \frac{1}{t^{1-\nu}} \right).
\]
Since
\[0 < \alpha \nu < \frac{\nu}{\nu + 1} < 1 - \alpha < 1\]
and we can choose arbitrary $\alpha$ satisfying this condition,
\[
\mathbb{E}_a^{(\nu)}[(R_t)^{-2\nu}1_{\{\tau_b \leq t\}}] = \frac{1}{\Gamma(\nu + 1)(2t)^\nu} \left( \frac{b}{a} \right)^{2\nu} + O \left( \frac{1}{t^{\nu+\epsilon}} \right)
\]
holds for any $\epsilon \in (0, \frac{\nu}{\nu + 1})$.
Now we have shown (3) and the assertion of Theorem 1.

### 3. Proof of Theorem 2

Theorem 2 is easily obtained from Theorem 1.

We recall explicit expressions for the Laplace transforms of the distributions of $\tau_b$: for $\nu \in \mathbb{R}$, it is known ([2, 5]) that
\[
\mathbb{E}_a^{(\nu)}[e^{-\lambda \tau_b}] = \left( \frac{b}{a} \right)^{\nu} K_\nu(a \sqrt{2 \lambda}) K_\nu(b \sqrt{2 \lambda}), \quad \lambda > 0,
\]
where $K_\nu$ is the modified Bessel function of the second kind. From this identity we easily obtain for $\nu > 0$
\[
\mathbb{P}_a^{(-\nu)}(\tau_b \in dt) = \left( \frac{a}{b} \right)^{2\nu} \mathbb{E}_a^{(\nu)}(\tau_b \in dt).
\]
Hence we get from Theorem 1

\[ \mathbb{P}_{a}^{(-\nu)}(\tau_b > t) = \left( \frac{a}{b} \right)^{2\nu} \mathbb{P}_{a}^{(\nu)}(t < \tau_0 < \infty) \]

\[ = a^{2\nu} \left\{ 1 - \left( \frac{b}{a} \right)^{2\nu} \right\} \frac{1}{\Gamma(1 + \nu)(2t)^\nu} (1 + o(1)). \]

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