Isospectral Flow and Liouville-Arnold Integration in Loop Algebras†

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Abstract. Some standard examples of Hamiltonian systems that are integrable by classical means are cast within the framework of isospectral flows in loop algebras. These include: the Neumann oscillator, the cubically nonlinear Schrödinger systems and the sine-Gordon equation. Each system has an associated invariant spectral curve and may be integrated via the Liouville-Arnold technique. The linearizing map is the Abel map to the associated Jacobi variety, which is deduced through separation of variables in hyperellipsoidal coordinates. More generally, a family of moment maps is derived, embedding certain finite dimensional symplectic manifolds, which arise through Hamiltonian reduction of symplectic vector spaces, into rational coadjoint orbits of loop algebras \( \tilde{g}^+ \subset \tilde{gl}(r)^+ \). Integrable Hamiltonians are obtained by restriction of elements of the ring of spectral invariants to the image of these moment maps; the isospectral property follows from the Adler-Kostant-Symes theorem. The structure of the generic spectral curves arising through the moment map construction is examined. Spectral Darboux coordinates are introduced on rational coadjoint orbits in \( \tilde{gl}(r)^{++} \), and these are shown to generalize the hyperellipsoidal coordinates encountered in the previous examples. Their relation to the usual algebro-geometric data, consisting of linear flows of line bundles over the spectral curves, is given. Applying the Liouville-Arnold integration technique, the Liouville generating function is expressed in completely separated form as an abelian integral, implying the Abel map linearization in the general case.

Keywords. Integrable systems, Liouville-Arnold integration, loop algebras, isospectral flow, spectral Darboux coordinates, Abel map linearization.

†Lectures presented at the VIIIth Scheveningen Conference: Algebraic Geometric Methods in Mathematical Physics, held at Wassenaar, the Netherlands, Aug. 16-21, 1992. To appear in: Springer Lecture Notes in Physics (1993). Research supported in part by the Natural Sciences and Engineering Research Council of Canada and the Fonds FCAR du Québec.

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1 Background Material and Examples

In this first section, we shall examine several examples of integrable Hamiltonian systems that may be represented by isospectral flows on coadjoint orbits of loop algebras. In each case, the flow may be linearized through the classical Liouville-Arnold integration technique. An explicit linearization of the flows can be made in terms of abelian integrals associated to an invariant spectral curve associated to the system. A key element in the integration is the fact that a complete separation of variables occurs within a suitably defined coordinate system - essentially, hyperellipsoidal coordinates, or some generalization thereof. A general theory will then be developed in subsequent sections, based essentially on moment map embeddings of finite dimensional symplectic vector spaces, or Hamiltonian quotients thereof, into the dual space of certain loop algebras, the image consisting of orbits whose elements are rational in the loop parameter.

The origins of this approach may be found in the works of Moser [Mo] on integrable systems on quadrics, Adler and van Moerbeke [AvM] on linearization of isospectral flows in loop algebras and the general algebro-geometric integration techniques of Dubrovin, Krichever and Novikov [KN], [Du]. The theory of moment map embeddings in loop algebras is developed in [AHP], [AHH1]. Its relation to algebro-geometric integration techniques is described in [AHH2], and the use of “spectral Darboux coordinates” in the general Liouville-Arnold integration method in loop algebras is developed in [AHH3]. Some detailed examples and earlier overviews of this approach may be found in [AHH4], [AHH5]. The proofs of the theorems cited here may be found in [AHP], [AHH1-AHH3].

1.1. The Neumann Oscillator

We begin with the Neumann oscillator system ([N], [Mo]), which consists of a point particle confined to a sphere in $\mathbb{R}^n$, subject to harmonic oscillator forces. The phase space is identifiable either with the cotangent bundle or the tangent bundle (the equivalence being via the metric):

$$M = T^*S^{n-1} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x^T x = 1, \ y^T x = 0\} \subset \mathbb{R}^n \times \mathbb{R}^n,$$

(1.1)

where $x$ represents position and $y$ momentum. The Hamiltonian is

$$H(x, y) = \frac{1}{2}[y^T y + x^T A x],$$

(1.2)

where $A$ is the diagonal $n \times n$ matrix

$$A = \text{diag}(\alpha_1, \ldots, \alpha_n) \in \mathbb{M}^{n \times n},$$

(1.3)
with distinct eigenvalues \( \{\alpha_i\}_{i=1,\ldots,n} \) determining the oscillator constants.

Equivalently, we may choose the Hamiltonian as:

\[
\phi(x, y) = \frac{1}{2}[(x^T x)(y^T y) + x^T Ax - (x^T y)^2],
\]

in view of the constraints

\[
x^T x = 1, \quad y^T x = 0.
\]

The unconstrained equations of motion for the Hamiltonian \( \phi \) are:

\[
\frac{dx}{dt} = (x^T x)y - (x^T y)x
\]

and

\[
\frac{dy}{dt} = -(y^T y)x - Ax + (x^T y)y.
\]

Since

\[
\{\phi, x^T x\} = 0,
\]

it is convenient to interpret the relation

\[
n(x) := x^T x = 1,
\]

alone as a first class constraint. The function \( n(x) \) generates the flow

\[
(x, y) \mapsto (x, y + tx),
\]

and is invariant under the \( \phi \)-flow. We may then apply Marsden-Weinstein reduction, quotienting by the flow (1.9). The reduced manifold is identified with a section of the orbits under (1.9) defined by the other constraint:

\[
y^T x = 0.
\]

The integral curves for the constrained system are determined from those for the unconstrained system by orthogonal projection:

\[
(x(t), y(t))_{\text{free}} \mapsto (\hat{x}(t), \hat{y}(t))_{\text{constr.}}
\]

\[
:= (x(t), y(t) - \left( \frac{x^T y(t)}{x^T x(t)} \right) x(t)).
\]

This lifts the projected flow on the Marsden-Weinstein reduced space \( n^{-1}(1)/\mathbb{R} \) to one that is tangential to the section defined by eq. (1.10).
Assuming the constants $\alpha_i$ determining the oscillator strengths are distinct, the integration proceeds (cf. [Mo]) by introducing the Devaney-Uhlenbeck commuting integrals

$$I_i := \sum_{j=1, j \neq i}^{n} \frac{(x_i y_j - y_i x_j)^2}{\alpha_i - \alpha_j} + x_i^2,$$

which satisfy

$$\sum_{i=1}^{n} I_i = x^T x \quad (1.13a)$$

$$\sum_{i=1}^{n} \alpha_i I_i = 2\phi. \quad (1.13b)$$

Define the degree $n - 1$ polynomial $P(\lambda)$ by

$$\frac{P(\lambda)}{a(\lambda)} := \frac{-1}{4} \sum_{i=1}^{n} \frac{I_i}{\lambda - \alpha_i}, \quad (1.14)$$

where

$$a(\lambda) := \prod_{i=1}^{n} (\lambda - \alpha_i),$$

$$P(\lambda) = P_{n-1}\lambda^{n-1} + P_{n-2}\lambda^{n-2} + \cdots + P_0. \quad (1.15)$$

Then on $T^* S^{n-1},$

$$P_{n-1} = -\frac{1}{4} x^T x = -\frac{1}{4},$$

$$P_{n-2} = \frac{1}{2}\phi. \quad (1.16)$$

An equivalent set of commuting integrals consists of the coefficients of the polynomial $\{P_0, \ldots, P_{n-2}\}$. The Liouville-Arnold tori $\mathbf{T}$ are the leaves of the Lagrangian foliation defined by the level sets:

$$P_i = C_i. \quad (1.17)$$

We now proceed to the linearization of the flows through the Liouville-Arnold method. First, introduce hyperellipsoidal coordinates $\{\lambda_\mu\}_{\mu=1, \ldots, n-1}$ and their conjugate momenta, $\{\zeta_\mu\}_{\mu=1, \ldots, n-1}$, which are defined by:

$$\sum_{i=1}^{n} \frac{x_i^2}{\lambda - \alpha_i} = \frac{\prod_{\mu=1}^{n-1} (\lambda - \lambda_\mu)}{a(\lambda)} \quad (1.18a)$$

$$\zeta_\mu = \frac{1}{2} \sum_{i=1}^{n} \frac{x_i y_i}{\lambda_\mu - \alpha_i} = \sqrt{\frac{P(\lambda_\mu)}{a(\lambda_\mu)}}. \quad (1.18b)$$
In terms of these, the canonical 1–form is:

$$\theta = \sum_{i=1}^{n} y_i dx_i \mid_{T \ast S^{n-1}} = \sum_{\mu=1}^{n-1} \zeta_{\mu} d\lambda_{\mu}. \quad (1.19)$$

Restricting this to $T$ determines the differential of the Liouville generating function $S$:

$$\sum_{\mu=1}^{n-1} \zeta_{\mu} d\lambda_{\mu} \mid_{P_i = \text{cst.}} = dS = \sum_{\mu=1}^{n-1} \sqrt{\frac{P(\lambda_{\mu})}{a(\lambda_{\mu})}} d\lambda_{\mu}, \quad (1.20)$$

which, upon integration, gives

$$S = \sum_{\mu=1}^{n-1} \int_{0}^{\lambda_{\mu}} \sqrt{\frac{P(\lambda)}{a(\lambda)}} d\lambda. \quad (1.21)$$

This is seen to be an abelian integral on the (generically) genus $g = n - 1$ hyperelliptic curve $C$ defined by:

$$z^2 + a(\lambda)P(\lambda) = 0. \quad (1.22)$$

The linearizing coordinates conjugate to the invariants $P_j$ are then:

$$Q_j := \frac{\partial S}{\partial P_j} = \frac{1}{2} \sum_{\mu=1}^{n-1} \int_{0}^{\lambda_{\mu}} \frac{\lambda^j d\lambda}{\sqrt{a(\lambda)P(\lambda)}} = b_j t, \quad (1.23)$$

where, for $\phi = 2P_{n-2}$,

$$b_{n-2} = 2, \quad b_j = 0, \quad j < n - 2. \quad (1.24)$$

The map:

$$(\lambda_1, \ldots \lambda_{n-1}) \longrightarrow (Q_1, \ldots Q_{n-1}) \quad (1.25)$$

defined by eq. (1.23) is, up to normalization, the Abel map from the symmetric product $S^{n-1}C$ to the Jacobi variety $\mathcal{J}(C)$ of $C$:

$$A : S^{n-1}C \longrightarrow \mathcal{J}(C) \sim \mathbb{C}^{n-1}/\Gamma, \quad (1.26)$$

where $\Gamma = \text{ is the period lattice.}$

We now turn to the interpretation of such systems as isospectral flows in a loop algebra (cf. [AHP], [AHH3], [AHH4]). Let

$$\mathcal{N}(\lambda) = \lambda Y + \mathcal{N}_0(\lambda), \quad (1.27)$$
where

\[
\mathcal{N}_0(\lambda) := \frac{\lambda}{2} \left( -\sum_{i=1}^n \frac{x_i y_i}{\lambda - \alpha_i} - \sum_{i=1}^n \frac{y_i^2}{\lambda - \alpha_i} \right) \in \tilde{\mathfrak{sl}}(2)^{++}
\]

(1.28a)

\[
Y := \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}.
\]

(1.28b)

We define a map \( \tilde{J}_A : \mathbb{R}^n \times \mathbb{R}^n \to \tilde{\mathfrak{sl}}(2)^{++} \) to the dual space of the positive half of the loop algebra \( \tilde{\mathfrak{sl}}(2) \), relative to the standard splitting \( \tilde{\mathfrak{sl}}(2) = \tilde{\mathfrak{sl}}(2)^{++} + \tilde{\mathfrak{sl}}(2)^{-} \) into holomorphic parts inside and outside a circle \( S^1 \) in the complex \( \lambda \)-plane, containing the \( \alpha_i \)'s in its interior region:

\[
\tilde{J}_A : (x, y) \mapsto \mathcal{N}_0(\lambda) \in \tilde{\mathfrak{sl}}(2)^{++}.
\]

(1.29)

This is a Poisson map with respect to the Lie poisson structure on \( \tilde{\mathfrak{sl}}(2)^{++} \). The Hamiltonian \( \phi \) is then given by restriction of an elementary spectral invariant:

\[
\phi(x, y) = -\text{tr}(\mathcal{N}(\lambda)^2)_0 := -\frac{1}{2\pi i} \oint_{S^1} \text{tr}(\mathcal{N}(\lambda)^2) \frac{d\lambda}{\lambda},
\]

(1.30)

and all the other invariants \( P_j \) may be similarly represented. The equations of motion are seen to be equivalent to the Lax equation:

\[
\frac{d\mathcal{N}}{dt} = [\mathcal{B}, \mathcal{N}],
\]

(1.31)

where

\[
\mathcal{B} := d\phi(\mathcal{N})_+ = \begin{pmatrix} x^T y & \lambda + y^T y \\ -x^T x & -x^T y \end{pmatrix},
\]

(1.32)

and \((d\phi(\mathcal{N})_+\) signifies projection of the element \(d\phi(\mathcal{N}) \in \tilde{\mathfrak{sl}}(2) \) to \( \tilde{\mathfrak{sl}}(2)^{+} \). This is an example of the Adler-Kostant-Symes (AKS) theorem (to be explained more fully in Section 2). The spectral invariants (elements of the AKS ring) are generated by the residues of the rational function

\[
\det \left( \frac{\mathcal{N}(\lambda)}{\lambda} \right) = \frac{P(\lambda)}{a(\lambda)} = -\frac{1}{4} \sum_{i=1}^n \frac{I_i}{\lambda - \alpha_i}.
\]

(1.33)

To see the relation with the standard algebro-geometric linearization methods ([Du], [KN], [AHH2-AHH3]), we begin with the invariant spectral curve:

\[
\det(\mathcal{L}(\lambda) - z I_2) = z^2 + a(\lambda) P(\lambda) = 0,
\]

(1.34)

where

\[
\mathcal{L}(\lambda) := \frac{a(\lambda)}{\lambda} \mathcal{N}(\lambda), \quad z := a(\lambda) \frac{\zeta}{\lambda},
\]

(1.35)
and let
\[ M(\lambda, \zeta) := \left( \frac{N(\lambda)}{\lambda} - \zeta I_2 \right), \tag{1.36} \]
where \( I_2 \) is the \( 2 \times 2 \) unit matrix. Then the columns of the matrix \( \tilde{M}(\lambda, \zeta) \) of cofactors of \( M(\lambda, \zeta) \) are the eigenvectors of \( N(\lambda) \) (\( \zeta \lambda = \text{eigenvalue} \) on \( \mathcal{C} \):
\[ \tilde{M}(\lambda, \zeta) = \left( \begin{array}{c}
\frac{1}{2} \sum_{i=1}^{n} \frac{x_i y_i}{\lambda - \alpha_i} - \zeta \\
- \frac{1}{2} \sum_{i=1}^{n} \frac{x_i^2}{\lambda - \alpha_i} - \frac{1}{2} \sum_{i=1}^{n} \frac{y_i^2}{\lambda - \alpha_i} + \zeta \end{array} \right). \tag{1.37} \]
The hyperellipsoidal coordinates \( \{ \lambda_\mu, \zeta_\mu \}_{\mu=1,...,n-1} \) define the finite part of the zero-divisor:
\[ D = \sum_{\mu=1}^{n-1} p(\lambda_\mu, \zeta_\mu) + p(\infty_1), \tag{1.38} \]
i.e., the zeros of a section of the bundle \( E \to \mathcal{C} \) dual to eigenvector line bundle. This bundle can be shown generically to have degree \( n \), and thus to be an element of the Picard variety \( E \in \text{Pic}^n \). The Abel map then identifies the symmetric product \( S^{n-1}\mathcal{C} \) with the Jacobi variety \( J(\mathcal{C}) \sim \text{Pic}^0 \). The linearity of the flow in \( \text{Pic}^n \) follows from noting that the Lax equation
\[ \frac{dN}{dt} = [d\phi \left[N \right] + N], \tag{1.39} \]
implies a linear exponential form for the transition function
\[ \tau(\lambda, z, t) = \exp(\phi_z(\lambda, z)t). \tag{1.40} \]

1.2 Nonlinear Schrödinger (NLS) Equation

We now apply a similar analysis to the quasi-periodic solutions of the cubically nonlinear Schrödinger equation (cf. [P1], [AHP], [AHH4])
\[ u_{xx} + \sqrt{-1} u_t = 2|u|^2 u. \tag{1.41} \]
Let
\[ N(\lambda) := \frac{\lambda}{2} \left( \begin{array}{cc}
\sum_{j=1}^{n} \frac{|z_j|^2}{\lambda - \alpha_j} & - \sum_{j=1}^{n} \frac{z_j^2}{\lambda - \alpha_j} \\
- \sum_{j=1}^{n} \frac{\bar{z}_j^2}{\lambda - \alpha_j} & - \sum_{j=1}^{n} \frac{|z_j|^2}{\lambda - \alpha_j} \end{array} \right), \tag{1.42} \]
and let \( \omega \) denote the standard symplectic form on \( \mathbb{C}^n \):
\[ \omega = i \; d\mathbf{z}^T \wedge d\mathbf{z} = i \sum_{j=1}^{n} d\bar{z}_j \wedge dz_j \tag{1.43} \]
We define the Poisson map:

\[ \tilde{J} : \mathbb{C}^n \rightarrow \tilde{\mathfrak{su}}(1,1)^{**} \]

and let

\[ \tilde{J} : z \mapsto \mathcal{N}(\lambda), \tag{1.44} \]

and let

\[ \mathcal{L}(\lambda) = \frac{a(\lambda)}{\lambda} \mathcal{N}(\lambda) = L_0 \lambda^{n-1} + L_1 \lambda^{n-2} + \cdots + L_{n-1}. \tag{1.45} \]

The spectral curve is defined by the characteristic equation

\[ \det(\mathcal{L}(\lambda) - zI) = z^2 + a(\lambda)\mathcal{P}(\lambda) = 0, \]

\[ \mathcal{P}(\lambda) := P_0 + P_1 \lambda + \cdots + P_{n-2} \lambda^{n-2}, \tag{1.46} \]

and has genus \( g = n - 2 \). Choosing the AKS Hamiltonians:

\[ H_x = \frac{1}{2} \left[ \frac{a(\lambda)}{\lambda^n} \lambda \text{tr}(\mathcal{N}(\lambda)^2) \right]_0 = -P_{2,n-3} \tag{1.47a} \]

\[ H_t = \frac{1}{2} \left[ \frac{a(\lambda)}{\lambda^n} \lambda^2 \text{tr}(\mathcal{N}(\lambda)^2) \right]_0 = -P_{2,n-4} \tag{1.47b} \]

gives the Lax equations

\[ \frac{d}{dx} \mathcal{L}(\lambda) = [(dH_x)_+, \mathcal{L}(\lambda)] \tag{1.48b} \]

\[ \frac{d}{dt} \mathcal{L}(\lambda) = [(dH_t)_+, \mathcal{L}(\lambda)], \tag{1.48b} \]

where

\[ (dH_x)_+ = \lambda L_0 + L_1 \tag{1.49a} \]

\[ (dH_t)_+ = \lambda^2 L_0 + \lambda L_1 + L_2. \tag{1.49b} \]

Choosing invariant constraints so that:

\[ L_0 = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & \overline{w} \\ w & 0 \end{pmatrix}, \]

\[ L_2 = i \begin{pmatrix} |u|^2 & -\overline{u_x} \\ u_x & -|u|^2 \end{pmatrix}, \tag{1.50} \]

the compatibility conditions:

\[ \frac{\partial (dH_x)_+}{\partial t} - \frac{\partial (dH_t)_+}{\partial x} + [(dH_x)_+, (dH_t)_+] = 0 \tag{1.51} \]
reduce to the NLS equation (1.41). To obtain the quasi-periodic solutions, we introduce the spectral Darboux Coordinates \( \{ q, P, \lambda_\mu, \zeta_\mu \}_{\mu=1,...,n-2} \), analogous to the hyperellipsoidal coordinates above:

\[
\sum_{i=1}^{n} \frac{z_i^2}{\lambda - \alpha_i} = -\frac{2u \prod_{\mu=1}^{n-2}(\lambda - \lambda_\mu)}{a(\lambda)} \tag{1.52a}
\]

\[
\zeta_\mu = -\frac{i}{2} \sum_{i=1}^{n} \frac{|z_i|^2}{\lambda_\mu - \alpha_i} = \sqrt{-\frac{P(\lambda_\mu)}{a(\lambda_\mu)}} \tag{1.52b}
\]

\[
q := \ln(u), \quad P := (L_0)_{22}, \tag{1.52c}
\]

Then the symplectic form may be expressed as

\[
\omega = \sum_{\mu=1}^{n-2} d\lambda_\mu \wedge d\zeta_\mu + dq \wedge dP. \tag{1.53}
\]

As above, the spectral curve is invariant, and the coefficients of the characteristic polynomial generate a complete set of commuting integrals, so we may apply the Liouville-Arnold integration method. The coordinates \( (\lambda_\mu, \zeta_\mu) \) defined by (1.52a,b) again give the finite part of the divisor of zeros of the eigenvectors of \( \mathcal{N}(\lambda) \), while the remaining pair \( (q, P) \) are determined by the additional spectral data at \( \lambda = \infty \). The Liouville generating function in this case becomes

\[
S = \sum_{\mu=1}^{n-1} \zeta_\mu d\lambda_\mu |_{P_i = \text{cst.}} + qP = \sum_{\mu=1}^{n-1} \int_{0}^{\lambda_\mu} \sqrt{-\frac{P(\lambda)}{a(\lambda)}} d\lambda + P \ln u \tag{1.54}
\]

and the linearizing coordinates conjugate to the \( P_i \)'s are

\[
Q_i = \frac{\partial S}{\partial P_i} = \frac{1}{2} \sum_{\mu=1}^{n-2} \int_{0}^{\lambda_\mu} \frac{\lambda' d\lambda}{\sqrt{-a(\lambda)P_2(\lambda)}} = b_i x + c_i t, \quad i = 0, \ldots, n-3, \tag{1.55}
\]

which are abelian integrals of the first kind, and

\[
Q_{2,n-2} = \frac{\partial S}{\partial P_i} = \frac{1}{2} \sum_{\mu=1}^{n-2} \int_{0}^{\lambda_\mu} \frac{\lambda^{n-2} d\lambda}{\sqrt{-a(\lambda)P_2(\lambda)}} - \frac{\ln u}{2P_2} = b_{n-2} x + c_{n-2} t, \tag{1.56}
\]

which is an abelian integral of the third kind, the integrand having simple poles at the two points \( (\infty_1, \infty_2) \) over \( \lambda = \infty \). For the Hamiltonian \( H_x = -P_{n-3} \), we have \( b_i = -\delta_{i,n-3}, c_i = 0 \), while for \( H_t = -P_{n-4} \), \( b_i = 0, c_i = -\delta_{i,n-3} \). An explicit formula for the function \( u(x, t) \) may be obtained in terms of the Riemann
theta function $\theta$ associated to the spectral curve by applying the reciprocity theorem relating the two kinds of abelian integrals (cf. [AHH4]).

$$u(x,t) = \exp(q) = \tilde{K}\exp(bx + ct)\frac{\theta(\mathbf{A}(\infty_2, p) + t\mathbf{U} + x\mathbf{V} - \mathbf{K})}{\theta(\mathbf{A}(\infty_1, p) + t\mathbf{U} + x\mathbf{V} - \mathbf{K})}, \quad (1.57)$$

where $\mathbf{A} : S^{n-2}C \mapsto \mathcal{J}(C) \sim C^{n-2}/\Gamma$ is the Abel map, $\mathbf{U}, \mathbf{V} \in C^{n-2}$, $b, c \in C$ are obtained from the vectors with components $(b_i, c_i)$ on the RHS of eq. (1.55), (1.56) by applying the linear transformation that normalizes the abelian differentials in (1.55), and $\mathbf{K}$ is the Riemann constant.

### 1.3 Sine-Gordon Equation

As a last example, consider the sine-Gordon equation (cf. [HW], [P2], [AA])

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \sin(u). \quad (1.58)$$

Let

$$\mathcal{N}(\lambda) := \lambda Y + \mathcal{N}_0(\lambda), \quad (1.59)$$

where

$$Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{N}_0(\lambda) := 2\lambda \begin{pmatrix} b(\lambda) & c(\lambda) \\ -\bar{c}(\lambda) & -b(\lambda) \end{pmatrix}, \quad (1.60)$$

with $b(\lambda), c(\lambda)$ given by

$$b(\lambda) = \lambda \sum_{i=1}^{p} \left( \frac{-\varphi_i \bar{\gamma}_i}{\alpha_i^2 - \lambda^2} + \frac{\bar{\varphi}_i \gamma_i}{\bar{\alpha}_i^2 - \lambda^2} \right), \quad (1.61a)$$

$$c(\lambda) = \sum_{i=1}^{p} \left( \frac{\alpha_i \bar{\varphi}_i^2}{\alpha_i^2 - \lambda^2} + \frac{\bar{\alpha}_i \varphi_i^2}{\bar{\alpha}_i^2 - \lambda^2} \right), \quad (1.61b)$$

and $\varphi, \gamma \in \mathbb{C}^p$ complex vectors with components $\{\varphi_i, \gamma_i\}_{i=1,\ldots,p}$. (Here $\alpha_i, \bar{\alpha}_i, -\alpha_i, -\bar{\alpha}_i$ are assumed to be distinct.)

Define

$$a(\lambda) := \prod_{i=1}^{p} [(\lambda^2 - \alpha_i^2)(\lambda^2 - \bar{\alpha}_i^2)]. \quad (1.62)$$

The symplectic form on $\mathbb{C}^p \times \mathbb{C}^p$ is given by:

$$\omega = 4 \sum_{i=1}^{p} (d\gamma_i \wedge d\bar{\varphi}_i + d\bar{\gamma}_i \wedge d\varphi_i). \quad (1.63)$$
Again, define a Poisson map:

\[ \tilde{J} : \mathbb{C}^p \times \mathbb{C}^p \to \hat{su}(2)^{++} \]

\[ \tilde{J} : (\varphi, \gamma) \mapsto N_0(\lambda), \quad (1.64) \]

where the twisted loop algebra:

\[ \hat{su}(2)^+ \subset \tilde{su}(2)^+ \subset \tilde{u}(2)^+ \subset \tilde{gl}(2, \mathbb{C})^+ \quad (1.65) \]

is defined as the fixed point set in \( \tilde{sl}(2, \mathbb{C})^+ \) under the involutions:

\[ \sigma_1 : X(\lambda) \mapsto X^\dagger(\bar{\lambda}) \]

\[ \sigma_2 : X(\lambda) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X(-\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.66) \]

Let \( n = 2p \), and

\[ L(\lambda) := \frac{a(\lambda)}{\lambda} N(\lambda) \]

\[ = a(\lambda)Y + L_0 \lambda^{2n-1} + L_1 \lambda^{2n-2} + \cdots + L_{2n-1}. \quad (1.67a) \]

\[ P(\lambda) = P_0 + \lambda^2 P_1 + \cdots + \lambda^{2n-2} P_{n-1} + \lambda^{2n}. \quad (1.67b) \]

The spectral curve is:

\[ \det(L(\lambda) - zI) = z^2 + a(\lambda)P(\lambda) = 0 \]

\[ P(\lambda) = P_0 + \lambda^2 P_1 + \cdots + \lambda^{2n-2} P_{n-1} + \lambda^{2n}. \quad (1.68a) \]

Choosing the AKS Hamiltonians:

\[ H_\xi(X) := \frac{1}{2} \text{tr} \left( \frac{a(\lambda)}{\lambda^2} (X(\lambda) + \lambda Y)^2 \right)_0 = -P_0 \]

\[ H_\eta(X) := -\frac{1}{2} \text{tr} \left( \frac{a(\lambda)}{\lambda^{2N}} (X(\lambda) + \lambda Y)^2 \right)_0 = P_{n-1} \quad (1.69) \]

gives the Lax equations

\[ \frac{d}{d\xi} L(\lambda) = [A, L(\lambda)] \quad (1.70a) \]

\[ \frac{d}{d\eta} L(\lambda) = [B, L(\lambda)], \quad (1.70b) \]

where

\[ -A = dH_\xi(N)_- = \frac{1}{\lambda} (L_{2n-1} + a(0)Y) \]

\[ B = dH_\eta(N)_+ = L_0 + \lambda Y. \quad (1.71) \]
Choosing the level set:

\[ P_0 = \frac{1}{16} \]  

(1.72)

gives

\[ L_{2n-1} + a(0)Y = \frac{1}{4} \begin{pmatrix} 0 & e^{iu} \\ -e^{-iu} & 0 \end{pmatrix}, \]  

(1.73)

where

\[ e^{iu} = a(0)(c(0) - 1), \]  

(1.74)

with \( u \) real. Then the compatibility conditions:

\[ \frac{\partial A}{\partial \eta} - \frac{\partial B}{\partial \xi} + [A, B] = 0 \]  

(1.75)

reduce to the Sine-Gordon equation

\[ u_{xx} - u_{tt} = \sin u, \]  

(1.76)

where

\[ \xi = x + t, \quad \eta = x - t. \]  

(1.77)

The quasi-periodic solutions are obtained as in the previous example. The unreduced spectral curve is defined by

\[ z^2 + \tilde{a}(\lambda) \tilde{P}(\lambda) = 0, \]  

(1.78)

and is again hyperelliptic, with genus \( g = 2n - 1 \). Quotienting by the involution

\[ (z, \lambda) \mapsto (z, -\lambda) \]  

(1.79)

gives a reduced curve with genus \( g = n - 1 \) defined by

\[ z^2 + \tilde{a}(E) \tilde{P}(E) = 0, \]  

(1.80)

where

\[ \lambda^2 := E, \quad \tilde{a}(E); = a(\lambda), \quad \tilde{P}(E) := P(\lambda). \]  

(1.81)

We also introduce the augmented curve, defined by

\[ \tilde{z}^2 + E\tilde{a}(E) \tilde{P}(E) = 0, \]  

(1.82)

of genus \( g = n \), where

\[ \tilde{z} := z\lambda. \]  

(1.83)
The spectral Darboux coordinates are defined by

\[ \tilde{c}(E_\mu) - 1 = 0 \quad (1.84a) \]
\[ \zeta_\mu \sqrt{E_\mu} = 2\tilde{b}(E_\mu), \quad \mu = 1, \ldots, n, \quad (1.84b) \]

where

\[ \tilde{b}(E) := b(\lambda), \quad \tilde{c}(E) := c(\lambda). \quad (1.85) \]

These again are interpreted as zeros of the sections of the dual to the eigenvector line bundle associated to \( N(\lambda) \). The symplectic form is then

\[ \omega = \sum_{\mu=1}^{n} dE_\mu \wedge d\zeta_\mu = -d\theta. \quad (1.86) \]

The Liouville generating function is

\[ S(P_0, \ldots, P_{n-1}, E_1, \ldots, E_n) = \sum_{\mu=1}^{n} \int_{E_0}^{E_\mu} \sqrt{-\tilde{P}(E)/E\tilde{a}(E)} \, dE, \quad (1.87) \]

giving rise to the Abel map linearization:

\[ Q_i = \frac{\partial S}{\partial P_i} = -\frac{1}{2} \sum_{\mu=1}^{N} \int_{E_0}^{E_\mu} \frac{E_i}{\sqrt{-E\tilde{a}(E)} \tilde{P}(E)} \, dE \quad (1.88a) \]
\[ = C_i + 2\delta_{i,0}\xi - 2\delta_{i,n-1}\eta, \quad (1.88b) \]

which only involves abelian integrals of the first kind on the augmented curve. In terms of theta functions, the solution may be expressed as

\[ u = -i \left( \sum_{\mu=1}^{n} \ln(-E_\mu) - \pi \right) \quad (1.89a) \]
\[ = -2i \ln \frac{\Theta(A(p_0, 0) - U\eta - V\xi - K)}{\Theta(A(p_0, \infty) - U\eta - V\xi - K)} + C, \quad (1.89b) \]

where \( U, V \) are again obtained from the coefficients on the RHS of (1.88) by applying the normalizing linear transformation to the abelian differentials appearing in (1.88). Full details for this case may be found in [HW].

In the following two sections, a general approach to integrable systems is developed, yielding all the above results as particular cases, but allowing generalizations to more complex systems of higher rank.
2 Moment Map Embeddings in Loop Algebras

2.1 Phase Space and Loop Group Action

We begin by defining the generalized Moser space (cf. [AHP]) to be the symplectic vector space consisting of pairs \((F, G)\) of rectangular \(N \times r\) matrices:

\[
M = \{(F, G) \in M_{N \times r} \times M_{N \times r}\}
\]  
(2.1)

with symplectic form:

\[
\omega = \text{tr} \, dF^T \wedge dG.
\]  
(2.2)

The loop algebra, denoted \(\tilde{\mathfrak{g}}\), consists of smooth maps from a circle \(S^1\), centred at the origin of the complex \(\lambda\)-plane, into \(\mathfrak{gl}(r), \mathfrak{sl}(r)\), or some subalgebra thereof.

\[
\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(r) \quad \text{(or } \tilde{\mathfrak{sl}}(r)\text{)}
\]

\[
= \{X(\lambda) \in \mathfrak{gl}(r), \lambda \in S^1 \subset \mathbb{C} \cup \infty\}.
\]  
(2.3)

There is a natural splitting of \(\tilde{\mathfrak{g}}\)

\[
\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^+ + \tilde{\mathfrak{g}}^-,
\]  
(2.4)

as a vector space direct sum of the subalgebra \(\tilde{\mathfrak{g}}^+\), consisting of elements \(X(\lambda)\) admitting a holomorphic extension to the interior of \(S^1\), and \(\tilde{\mathfrak{g}}^-\), consisting of elements admitting a holomorphic extension to the exterior, with normalization \(X(\infty) = 0\). We identify \(\tilde{\mathfrak{g}}\) as a dense subspace of its dual space \(\tilde{\mathfrak{g}}^*\) through the pairing

\[
< \mu, X > := \frac{1}{2\pi i} \oint_{S^1} \text{tr} (\mu(\lambda)X(\lambda)) \frac{d\lambda}{\lambda},
\]  
(2.5)

\[
\mu \in \tilde{\mathfrak{g}}^-, \quad X \in \tilde{\mathfrak{g}}^+.
\]

Under this pairing, we have the identification

\[
(\tilde{\mathfrak{g}}^+)^* \sim \tilde{\mathfrak{g}}^-, \quad (\tilde{\mathfrak{g}}^-)^* \sim \tilde{\mathfrak{g}}^+,
\]  
(2.6)

where

\[
\tilde{\mathfrak{g}}^* = \tilde{\mathfrak{g}}^* + \tilde{\mathfrak{g}}^-
\]  
(2.7)

similarly represents a decomposition of \(\tilde{\mathfrak{g}}^*\) into subspaces consisting of elements holomorphic inside and outside \(S^1\), but with the normalization such that elements \(\mu \in \tilde{\mathfrak{g}}^+\) satisfy \(\mu(0) = 0\) (and hence the constant loops are included on \(\tilde{\mathfrak{g}}^-\)).
The loop group $\widetilde{\mathfrak{g}}^+$ is similarly defined to consist of smooth maps $g : S^1 \to Gl(r)$ which admit holomorphic extensions to the interior of $S^1$: We define a Hamiltonian action:

$$\widetilde{\mathfrak{g}}^+ : M \longrightarrow M$$
$$g(\lambda) : (F, G) \longrightarrow (F_g, G_g),$$

where $(F_g, G_g)$ are determined by the decomposition

$$(A - \lambda I)^{-1} F g^{-1}(\lambda) = (A - \lambda I)^{-1} F_g + F_{\text{hol}}$$
$$(2.9a)$$
$$g(\lambda) G^T (A - \lambda I)^{-1} = G^T_g (A - \lambda I)^{-1} + G_{\text{hol}}. $$

(2.9b)

Here $A \in M^{N,N}$ is some fixed $N \times N$ matrix, with eigenvalues in the interior of $S^1$, and $(F_{\text{hol}}, G_{\text{hol}})$ denote the parts of the expressions on the left that are holomorphic in the interior of $S^1$. This Hamiltonian action is generated by the equivariant moment map:

$$\tilde{J}^A : M \longrightarrow \widetilde{\mathfrak{g}}^+ \sim \widetilde{\mathfrak{g}}^-$$
$$\tilde{J}^A(F, G) = \lambda G^T (A - \lambda I)^{-1} F,$$

(2.10)

which is thus a Poisson map with respect to the Lie Poisson structure on $\widetilde{\mathfrak{g}}^{+\ast}$. This map is not injective, its fibres being (generically) the orbits of the subgroup $G_A := \text{Stab}(A) \subset Gl(N)$ acting by conjugation on $A$, and by the natural symplectic action on $M$.

$$g : (F, G) \longrightarrow (gF, (g^T)^{-1}), \quad g \in G_A \subset Gl(N)$$

(2.11)

The relevant phase space is therefore the quotient

$$M/G_A \sim \widetilde{\mathfrak{g}}^*_A,$$

(2.12)

which is identified with a finite dimensional Poisson subspace $\widetilde{\mathfrak{g}}^*_A \subset \widetilde{\mathfrak{g}}^{+\ast}$ consisting of elements that are rational in the loop parameter $\lambda$, with poles at the eigenvalues of $A$.

### 2.2 Simplest Case

We now consider the simplest case, where $A$ is a diagonal matrix:

$$A = \text{diag}(\alpha_1, \ldots, \alpha_1, \ldots, \alpha_k, \ldots, \alpha_k, \ldots, \alpha_n, \ldots, \alpha_n),$$

(2.13)
possibly with multiple eigenvalues \(\{\alpha_i\}_{i=1,\ldots,n}\) of multiplicity \(k_i \leq r\), all in the interior of \(S^1\). The matrices \((F,G)\) are decomposed accordingly as

\[
F = \begin{pmatrix} F_1 & \cdots & \cdots & F_n \end{pmatrix}, \quad G = \begin{pmatrix} G_1 & \cdots & \cdots & G_n \end{pmatrix},
\]

where \((F_i, G_i)\) are the \(k_i \times r\) dimensional blocks corresponding to the eigenvalues \(\alpha_i\). For this case, \(N_0(\lambda)\) has only simple poles:

\[
N_0(\lambda) = \tilde{J}^A(F,G) = -\lambda \sum_{i=1}^{n} \frac{G_i^T F_i}{\lambda - \alpha_i} := \lambda \sum_{i=1}^{n} \frac{N_i}{\lambda - \alpha_i},
\]

with residue matrices \(N_i\) generically of rank:

\[
rk(F_i) = rk(G_i) = k_i.
\]

The \(Ad^*\tilde{G}^+\)–action for this case becomes:

\[
g(\lambda) : N_0(\lambda) \mapsto \lambda \sum_{i=1}^{n} \frac{g(\alpha_i)N_i g^{-1}(\alpha_i)}{\lambda - \alpha_i},
\]

which can be identified with the \(Ad^*\)–action of the direct product group \(Gl(r) \times \cdots \times Gl(r)\) (n times) on \([gl(r)^*]^n\).

The fibres of the map \(\tilde{J}^A\) coincide with the orbits of the block diagonal subgroup:

\[
G_A = \text{Stab}(A) = Gl(k_1) \times Gl(k_2) \times \cdots \times Gl(k_n) \subset Gl(N),
\]

under the action:

\[
(h_1, \ldots, h_i, \ldots, h_n) : (F_i, G_i) \mapsto (h_i F_i, (h_i^T)^{-1} G_i)
\]

This is also a Hamiltonian action, generated by the “dual” moment map:

\[
J_H(F,G) := (F_1 G_1^T, \ldots, F_n G_n^T) \in (gl(k_1) \times \cdots \times gl(k_n))^*.
\]

The \(Ad^*\tilde{G}^+\) orbits are then the level sets of the Casimir invariants:

\[
\text{tr}(F_i G_i^T), \quad k = 1, \ldots, n \quad l = 1, \ldots, k_i.
\]
More generally, the image of the moment map $\tilde{J}^A$ is a Poisson submanifold of $\tilde{g}_A \subset \tilde{g}^+ \subset \tilde{g}$ consisting of elements of the form

$$N_0(\lambda) = \lambda \sum_{i=1}^{n} \sum_{l_i=1}^{p_i} \frac{N_i l_i}{(\lambda - \alpha_i)^{l_i}}, \quad (2.22)$$

where $p_i$ is the dimension of the largest Jordan block of $A$ corresponding to eigenvalue $\alpha_i$.

### 2.3 Dynamics: Isopectral AKS Flows

The Hamiltonian flows to be considered are those generated by elements of the ring of $\text{Ad}^*$-invariant functions $\mathcal{I}(\tilde{g}^*)$, restricted to the translate $\lambda Y + \tilde{g}_A$ of the subspace $\tilde{g}_A$ by a fixed element $\lambda Y \in \tilde{g}^-$, where $Y \in \text{gl}(r)$. (The latter is an infinitesimal character for $\tilde{g}^-$, since it annihilates the commutator of any pair of elements.) We denote the ring of elements so obtained by

$$\mathcal{I}^Y_{\text{AKS}} := \mathcal{I}(\tilde{g}^*)|_{\lambda Y + \tilde{g}_A}, \quad (2.23)$$

and refer to it as the AKS (Adler-Kostant-Symes) ring.

Let

$$N(\lambda) = \lambda Y + N_0(\lambda) \in \lambda Y + \tilde{g}_A. \quad (2.24)$$

We then have the fundamental theorem that underlies the integrability of the resulting Hamiltonian systems, the Adler-Kostant-Symes theorem:

**Theorem 2.1 (AKS).**

1. If $H \in \mathcal{I}^Y_{\text{AKS}}$, Hamilton’s equations are:

$$X_H(N) = \frac{dN}{dt} = [(dH)_+, N] = -[(dH)_-, N] \quad (2.25a)$$

2. If $H_1, H_2 \in \mathcal{I}^Y_{\text{AKS}}$,

$$\{H_1, H_2\} = 0. \quad (2.25b)$$

Thus, all the AKS flows commute, and are generated by isospectral deformations determined by Lax equations of the form (2.25a). In fact, it may be shown ([RS], [AHP], [AHH2]) that on generic coadjoint orbits of the form (2.22), these systems are completely integrable; i.e., the elements of the Poisson commutative ring $\mathcal{I}^Y_{\text{AKS}}$ generate a Lagrangian foliation. Since the map (2.10) with image consisting of elements of the form (2.22) is a Poisson map, and passes to the quotient Poisson space $M/G_A$ to define an injective Poisson map, the same results may be applied to the pullback $\tilde{J}^A \circ H$ of any Hamiltonian in the AKS ring $\mathcal{I}^Y_{\text{AKS}}.$
Corollary 2.2. The results of Theorem 2.1 remain valid if the Hamiltonians $H_1, H_2$ are replaced by $\tilde{J}^A \circ H_1$, $\tilde{J}^A \circ H_2$ on the space $\lambda Y + \tilde{g}_A$, identified with $M/G_A$ through
\[ N(\lambda) = \lambda Y + N_0(\lambda) = \lambda Y + \tilde{J}_A(F,G). \] (2.26)

2.4 Reductions

To obtain interesting examples, one usually must reduce the generic systems described above in a manner that is consistent with the structure of the dynamical equations. This generally consists of Hamiltonian symmetry reductions involving either continuous or discrete symmetry groups. (It may also involve symplectic, or more generally, Poisson constraints.) We briefly summarize the procedure for both types of symmetry reductions below. The discrete Hamiltonian reduction procedure is described in greater detail in [HHM]; the continuous, Marsden-Weinstein reduction is fairly standard [AM].

2.4.1 Discrete Reduction:

We consider discrete groups generated by elements $\sigma$ either of finite order or generating compact orbits, which act on the space $M$ by symplectic diffeomorphisms, and as automorphisms of the loop algebra $\tilde{g}^+$. Let
\[ \sigma : M \longrightarrow M \] (2.27)
be such a symplectomorphism, and
\[ \sigma_g : \tilde{g}^+ \longrightarrow \tilde{g}^+ \] (2.28a)
the corresponding automorphism of $\tilde{g}^+$, with dual Poisson map
\[ \sigma_g^* : \tilde{g}^{++} \longrightarrow \tilde{g}^{++}. \] (2.28b)
We assume that the moment map $\tilde{J}^A$ intertwines these two actions, so that the following diagram commutes
\[ \begin{array}{ccc}
M & \xrightarrow{\sigma} & M \\
\downarrow \tilde{J}^A & & \downarrow \tilde{J}^A \\
\tilde{g}^{++} & \xrightarrow{\sigma_g^*} & \tilde{g}^{++}
\end{array} \] (2.29)
It follows that $\tilde{J}^A$ may be restricted to the fixed point sets
\[ M_\sigma \subset M, \quad \tilde{g}^+ : = \tilde{g}_\sigma^+ \subset \tilde{g}^+, \] (2.30)
and its restriction defines a moment map from the fixed point set $M_\sigma$ to the dual space

$$\tilde{\mathfrak{t}}^+ := \mathfrak{g}_\sigma^+ \subset \tilde{\mathfrak{g}}^+$$

of the subalgebra

$$\tilde{\mathfrak{t}}^+ := \tilde{\mathfrak{g}}^+ \subset \tilde{\mathfrak{g}}^+$$

of fixed elements under $\sigma_\mathfrak{g}$. The results of Theorem 2.1 and Corollary 2.2 may then be applied on the reduced spaces, provided the Hamiltonians in the ring $\mathcal{I}_{AKS}^Y$ are chosen to be invariant under the symmetry $\sigma^*_\mathfrak{g}$.

### 2.4.2 Continuous Hamiltonian Reduction

All the Hamiltonians in the ring $\mathcal{I}_{AKS}^Y$ are invariant under the Hamiltonian group action given by conjugation of $\mathcal{N}(\lambda)$ by $\lambda$-independent elements in the stability subgroup of $Y$:

$$G_Y := \text{Stab}(Y) \subset GL(r), \quad \mathfrak{g}_Y := \text{stab}(Y) \subset \mathfrak{gl}(r).$$

This action is generated by a moment map $J_Y$, given by the leading term $N_0$ of $\mathcal{N}_0$, restricted to $\mathfrak{g}_Y$

$$J_Y := N_0|_{\mathfrak{g}_Y},$$

where

$$\tilde{J}^A = N_0(\lambda) = N_0 + N_1\lambda^{-1} + \ldots.$$  \hspace{1cm} (2.34)

Since the elements of $\mathcal{I}_{AKS}^Y$ are $G_Y$ invariant, $J_Y$ is conserved under all the AKS flows. Fixing a level set

$$J_Y = \mu_0 \in \mathfrak{g}_Y^*,$$

which we assume to be a regular value of $J_Y$, and restricting to the coadjoint orbit $O_{\mathcal{N}_0(\lambda)} \subset (\tilde{\mathfrak{g}}^+)^*$, the reduced space is

$$O_{\text{red}} := J_Y^{-1}(\mu_0)/G_0,$$

where $G_0 \subset G_Y$ denotes the stability subgroup of $\mu_0$. The reduced Hamiltonians $H_{\text{red}}$ on $O_{\text{red}}$ are then given by

$$H_{\text{red}} \circ \pi = H|_{J_Y^{-1}(\mu_0)}, \quad H \in \mathcal{I}_{AKS}^Y,$$

where

$$\pi : J_Y^{-1}(\mu_0) \rightarrow J_Y^{-1}(\mu_0)/G_0$$

denotes the projection map.
2.5 Examples

We now indicate how the loop algebra formulation of examples like those of Section 1 is obtained from the general scheme described above.

2.5.1 Neumann Oscillator (and similar examples) in $\tilde{\mathfrak{sl}}(2, \mathbb{R})^+$

Consider the case $r = 2, k_i = 1, n = N$. A discrete antilinear involution gives the reality conditions:

$$\alpha_i = \overline{\alpha}_i, \quad F = \overline{F}, \quad G = \overline{G}$$  \hspace{1cm} (2.39)

reducing $\mathfrak{gl}(2, \mathbb{C})^*$ to $\mathfrak{gl}(2, \mathbb{R})^*$. The stabilizer of $A = \text{diag}(\alpha_1, \ldots, \alpha_n)$ consists of the diagonal subgroup $G_A = \{\text{diag}(d_1, \ldots, d_n) \subset \text{Gl}(n)\}$ acting as

$$G_A : M \rightarrow M \quad (d_1, \ldots d_n) : \left( \begin{array}{c} \vdots \\ F_i \\ \vdots \end{array} \right), \left( \begin{array}{c} \vdots \\ G_i \\ \vdots \end{array} \right) \mapsto \left( \begin{array}{c} \vdots \\ d_i F_i \\ \vdots \end{array} \right), \left( \begin{array}{c} \vdots \\ d_i^{-1} G_i \\ \vdots \end{array} \right),$$  \hspace{1cm} (2.40)

where $F_i$ and $G_i$ are just $2$–component row vectors. The moment map generating this action is just

$$J_A(F, G) = (F_1 G_1^T, \ldots, F_n G_n^T) \in \mathbb{R}^n,$$  \hspace{1cm} (2.41)

which coincides with the traces of the $2 \times 2$ residue matrices $N_i$ in (2.15). Choosing the zero level set for these, Marsden-Weinstein reduction is equivalent to the subgroup reduction $\tilde{\mathfrak{gl}}(2)^+ \supset \tilde{\mathfrak{sl}}(2)^+$. Choosing an appropriate symplectic section gives the reduced parametrization:

$$F = \frac{1}{\sqrt{2}}(x, y), \quad G = \frac{1}{\sqrt{2}}(y, -x)$$  \hspace{1cm} (2.42)

$$x, y \in \mathbb{R}^n.$$

The reduced symplectic form becomes

$$\omega = dx^T \wedge dy.$$  \hspace{1cm} (2.43)

The reduced moment map is

$$N_0(\lambda) = \tilde{J}^A(F, G) = \frac{\lambda}{2} \left( -\sum_{i=1}^n \frac{x_i y_i}{\lambda - \alpha_i}, -\sum_{i=1}^n \frac{y_i^2}{\lambda - \alpha_i}, \sum_{i=1}^n \frac{x_i y_i}{\lambda - \alpha_i}, \sum_{i=1}^n \frac{x_i y_i}{\lambda - \alpha_i} \right).$$  \hspace{1cm} (2.44)
Viewing this as defined on the symplectic vector space $\mathbb{R}^n \times \mathbb{R}^n$, there is a residual fibration generated by the finite group $(\mathbb{Z}_2)^n$ of reflections in the coordinate hyperplanes. The $\mathfrak{sl}(2)^-$ character $\lambda Y$ may be expressed as:

$$
\lambda Y = \lambda \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \tilde{\mathfrak{sl}}(2)^{-*},
$$

(2.45)

and the resulting AKS flows involve isospectral deformations of elements of the form:

$$
N(\lambda) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + N_0(\lambda).
$$

(2.46)

The Hamiltonians are chosen, as usual, from the AKS ring $I_{\text{AKS}}(\tilde{\mathfrak{sl}}(2)^*)$. In addition to the symmetry reductions already implemented, it is possible to impose further symplectic constraints of the form

$$
f(x, y) = 0, \quad g(x, y) = 0, \quad \{f, g\} \neq 0,
$$

(2.47)

and apply the standard methods for constrained systems. (Provided one of these functions is in the Poisson commutative ring $I_{\text{AKS}}(\tilde{\mathfrak{sl}}(2)^*)$, the constrained Hamiltonians will still commute.) The particular case of the above with

$$
a = 0, \quad b = -\frac{1}{2}, \quad c = 0,
$$

(2.48a)

$$
f := x^T x - 1 = 0, \quad g := y^T x = 0,
$$

(2.48b)

and Hamiltonian (1.30), gives the Neumann oscillator system. The invariant spectral curve is of the form

$$
\det(\mathcal{L}(\lambda) - z\mathbb{I}) = z^2 + a(\lambda)\mathcal{P}(\lambda) = 0,
$$

(2.49)

where

$$
\mathcal{L}(\lambda) := \frac{a(\lambda)}{\lambda}N(\lambda),
$$

(2.50)

and $\mathcal{P}(\lambda)$ is generally a polynomial of degree $n - 1$ or $n$, depending on whether $a^2 + bc$ vanishes or not.

### 2.5.2 The NLS Equation: Reduction to $\tilde{\mathfrak{su}}(1, 1)^+$

Again, we choose $r = 2$, $k_i = 1$, $n = N$. Similarly to the previous example, the zero moment map reduction under

$$
G_A = Stab(A) = \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times
$$

(2.51)
is equivalent the subgroup reduction \( \widetilde{\mathfrak{g}}(2, \mathbb{C})^+ \supset \widetilde{\mathfrak{s}}(2, \mathbb{C})^+ \). Choosing a suitable symplectic section gives the parametrization

\[
F = \frac{1}{\sqrt{2}}(z, w), \quad G = \frac{1}{\sqrt{2}}(w, -z), \quad z, w \in \mathbb{C}^n.
\]

(2.52)

The further reality conditions

\[
\alpha_i = \overline{\alpha}_i, \quad z = i \overline{w}
\]

(2.53)

give the discrete reduction \( \widetilde{\mathfrak{s}}(2, \mathbb{C})^{++} \supset \widetilde{\mathfrak{su}}(1, 1)^{++} \) as the fixed point set under an antilinear involution. On this real subspace, the symplectic form becomes

\[
\omega = idz^T \wedge dz,
\]

(2.54)

and the reduced moment map has the form.

\[
\mathcal{N}_0(\lambda) = \tilde{J}^A(F, G) = \frac{\lambda}{2} \left( \sum_{j=1}^{n} \frac{|z_j|^2}{\lambda - \alpha_j} - \sum_{j=1}^{n} \frac{\overline{z}_j^2}{\lambda - \overline{\alpha}_j} \right). \tag{2.55}
\]

In this case, we choose the character \( \lambda Y \) to vanish, so \( \mathcal{N}(\lambda) \) coincides with \( \mathcal{N}_0(\lambda) \).

The commuting flows are generated by the pair of commuting Hamiltonians:

\[
H_x = \frac{1}{2} \left[ \frac{a(\lambda)}{\lambda^n} \lambda \operatorname{tr}(\mathcal{N}(\lambda)^2) \right]_0 \in \mathcal{I}_{\text{AKS}}^{0}(\widetilde{\mathfrak{su}}(1, 1)^{++}) \tag{2.56a}
\]

\[
H_t = \frac{1}{2} \left[ \frac{a(\lambda)}{\lambda^n} \lambda^2 \operatorname{tr}(\mathcal{N}(\lambda)^2) \right]_0 \in \mathcal{I}_{\text{AKS}}^{0}(\widetilde{\mathfrak{su}}(1, 1)^{++}) \tag{2.56b}
\]

Further invariant constraints are added to ensure that the leading terms of the polynomial matrix (1.45) have the form given in eq. (1.49). Defining \( \mathcal{L}(\lambda) \) again as in eq. (2.50), the resulting invariant spectral curve again has the form

\[
\det(\mathcal{L}(\lambda) - zI) = z^2 + a(\lambda)\mathcal{P}(\lambda) = 0, \tag{2.57a}
\]

\[
\mathcal{L}(\lambda) := \frac{a(\lambda)}{\lambda} \mathcal{N}(\lambda), \tag{2.57b}
\]

where \( \mathcal{P}(\lambda) \) now is of degree \( n - 2 \). The Lax form (1.48a,b) of Hamilton’s equations then follows from the AKS theorem, and the compatibility condition (1.51) gives the NLS equation (1.41).
2.5.3 Higher Rank Case. Two Component Coupled NLS System:
Reduction to $\tilde{su}(1, 2)^+$

As an illustration of a system described by an algebra of higher rank, we consider the case of the coupled two component cubically nonlinear Schrödinger equation (viz. [AHP], [AHH2], [AHH3]):

\[
\begin{align*}
    i u_t + u_{xx} &= 2u(|u|^2 + |v|^2) \\
    iv_t + v_{xx} &= 2v(|u|^2 + |v|^2).
\end{align*}
\]

(2.58a) (2.58b)

In this case, we take $r = 3$ and $k_i = 1$ for all $i$, so $n = N$. The process of discrete and continuous symmetry reduction is applied analogously to the preceding case, giving the sequence $\tilde{gl}(3, \mathbb{C})^{++} \supset \tilde{sl}(3, \mathbb{C})^{++} \supset \tilde{su}(1, 2)^{++}$. The reduced form of the resulting pair of $n \times 3$ matrices $(F, G)$ is

\[
F = (\rho, \eta, \zeta), \quad G = (\rho, -\eta, -\zeta),
\]

(2.59)

where $\eta, \zeta \in \mathbb{C}^n$ is a pair of complex $n$–component column vectors and $\rho \in \mathbb{R}^n$ is a real column vector with components

\[
\rho_i = \sqrt{|\eta_i|^2 + |\zeta_i|^2}, \quad i = 1, \ldots, n.
\]

(2.60)

The reduced symplectic form is

\[
\omega = i(d\bar{\eta}^T \wedge d\eta + d\bar{\zeta}^T \wedge d\zeta),
\]

(2.61)

so the components $(\eta_i, \zeta_i)_{i=1,\ldots,n}$ and their complex conjugates provide a canonical coordinate system on $O_{\mathcal{N}_0}$. The reduced moment map has the form

\[
\mathcal{N}_0(\lambda) = \tilde{J}^A(F, G)
\]

\[
= -i\lambda \sum_{j=1}^{n} \frac{1}{\lambda - \alpha_j} \begin{pmatrix}
\rho_i^2 & \eta_i \rho_i & \zeta_i \rho_i \\
-\eta_i \rho_i & -|\eta_i|^2 & -\eta_i \zeta_i \\
-\zeta_i \rho_i & -\zeta_i \eta_i & -|\zeta_i|^2
\end{pmatrix},
\]

(2.62)

so the coadjoint orbit may be identified with $\mathbb{C}^n \times \mathbb{C}^n$. Again we choose the character $\lambda Y$ to vanish, so $\mathcal{N}(\lambda)$ coincides with $\mathcal{N}_0(\lambda)$. As for the single component NLS equation, the commuting pair of Hamiltonians for the two component CNLS case is chosen to be

\[
H_x = \frac{1}{2} \left[ \frac{a(\lambda)}{\lambda^n} \lambda \text{ tr}(\mathcal{N}(\lambda)^2) \right]_0 \in \mathcal{I}(\tilde{su}(1, 2)^{++})
\]

(2.63a)

\[
H_t = \frac{1}{2} \left[ \frac{a(\lambda)}{\lambda^n} \lambda^2 \text{ tr}(\mathcal{N}(\lambda)^2) \right]_0 \in \mathcal{I}(\tilde{su}(1, 2)^{++}).
\]

(2.63b)
Defining, as before,
\[
L(\lambda) := \frac{a(\lambda)}{\lambda} N_0(\lambda) = L_0 \lambda^{n-1} + L_1 \lambda^{n-2} + \cdots + L_{n-1},
\] (2.64)

further invariant constraints must also be imposed, implying that the leading terms are of the form (cf. [AHP]):
\[
L_0 = \frac{i}{3} \begin{pmatrix}
2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad L_1 = \begin{pmatrix}
0 & \bar{u} & \bar{v} \\
u & 0 & 0 \\
v & 0 & 0
\end{pmatrix},
\]
\[
L_2 = i \begin{pmatrix}
|u|^2 + |v|^2 & -\bar{u}_x & -\bar{v}_x \\
u_x & -|u|^2 & -\bar{v}u \\
v_x & -\bar{u}v & -|v|^2
\end{pmatrix}.
\] (2.65)

The Lax equations generated by the Hamiltonians (2.63a,b) have the same form as eqs. (1.48a,b), and the compatibility conditions (1.51) are equivalent to the CNLS system (2.58a,b). The invariant spectral curve in this case is a three sheeted branched covering of \( \mathbb{P}^1 \) determined by an equation of the form
\[
det(L(\lambda) - zI) = z^3 + a(\lambda)zP(\lambda) + a(\lambda)^2 Q(\lambda) = 0,
\] (2.66)
where \( P(\lambda) \) and \( Q(\lambda) \) are polynomials of degrees \( n-2 \) and \( n-3 \), respectively.

### 3. Spectral Darboux Coordinates and Liouville-Arnold Integration

In this section, the general method of linearization of AKS flows in rational coadjoint orbits will be explained. For simplicity, the spectral properties of the matrix \( A \) will be chosen as in Section 2.2, but the method is equally valid in the more general case (see [AHH3]). It consists of two steps. First, a suitable generalization of the hyperellipsoidal coordinates encountered in the examples of Section 1 is introduced, the spectral Darboux coordinates (Theorem 3.2) associated to the invariant spectral curve \( C \). These consist of families of canonical coordinates on coadjoint orbits \( O_{N_0} \) of the type discussed in the preceding section, which are naturally associated to the spectral data of the matrix \( N(\lambda) \). The second step consists of using a Liouville generating function \( S \) to compute the canonical transformation to coordinates conjugate to the spectral invariants, in which the flow becomes linear. It turns out that for all Hamiltonians in the AKS ring \( T^Y_{\text{AKS}} \) this generating function, defined with respect to the natural isospectral Lagrangian foliation of \( O_{N_0} \), may be expressed within the spectral
Darboux coordinate system in completely separated form (Theorem 3.3). It follows from the construction that this transformation is given in terms of abelian integrals, showing that, in the general case, the Abel map yields a linearization of the flows on the Jacobi variety $\mathcal{J}(\mathcal{C})$ of the spectral curve. One thus arrives at the algebro-geometric linearization results (viz. [Du], [KN], [AvM]) entirely through classical Hamiltonian methods.

### 3.1 Phase Space and Group Actions

In the following, the phase space will initially be thought of as a coadjoint orbit $\mathcal{O}_{N_0}$ within the image of a moment map of the type introduced in Section 2.

$$\tilde{J}^A : M \rightarrow \tilde{\mathfrak{g}}^+$$  \hspace{1cm} (3.1a)

$$\tilde{J}^A : (F, G) \mapsto \lambda G^T (A - \lambda \mathbb{I}_r)^{-1} F.$$ \hspace{1cm} (3.1b)

The image defines a finite dimensional Poisson submanifold

$$\text{Im}(\tilde{J}^A) := \tilde{\mathfrak{g}}_A \subset \tilde{\mathfrak{g}}^+$$ \hspace{1cm} (3.2)

which, in the simplest case, consists of elements of the form

$$\tilde{\mathfrak{g}}_A = \{N_0(\lambda) = \lambda \sum_{i=1}^n \frac{N_i}{\lambda - \alpha_i}\},$$ \hspace{1cm} (3.3)

where the ranks $\{k_i\}_{i=1,...,n}$ of the residue matrices $N_i$ coincide with the multiplicities of the eigenvalues $\{\alpha_i\}_{i=1,...,n}$ of the diagonal $N \times N$ matrix

$$A = \text{diag}(\alpha_i, \ldots, \alpha_i, \ldots \alpha_n).$$ \hspace{1cm} (3.4)

The coadjoint action of the loop group $\tilde{\mathcal{G}}^+$ on $\tilde{\mathfrak{g}}_A$ in this case becomes equivalent to the coadjoint action of $(\mathfrak{gl}(r))^n$ on $(\mathfrak{gl}(r))^n$, obtained by evaluating the group element $g(\lambda) \in \tilde{\mathcal{G}}^+$ at the poles $\lambda = \alpha_i$:

$$g : \{N_i\} \mapsto \{g(\alpha_i) N_i g(\alpha_i)^{-1}\}.$$ \hspace{1cm} (3.5)

It follows that the $\text{Ad}_{\tilde{\mathcal{G}}^+}$-orbits are determined as simultaneous level sets of the Casimir invariants of the separate residue matrices $N_i$ under this action:

$$\mathcal{O}_{N_0} = \{\lambda \sum_{i=1}^n \frac{N_i}{\lambda - \alpha_i} \mid \text{tr} N_i^l = c_{il}, \ l = 1, \ldots, k_i\}.$$ \hspace{1cm} (3.6)
The equations of motion induced by any element of the AKS ring $\phi \in I_{AKS}$ have the Lax form:
\[
\frac{d\mathcal{N}(\lambda)}{dt} = [d\Phi(\mathcal{N})_+, \mathcal{N}],
\]
where
\[
\phi = \Phi|_{\lambda Y + \tilde{g}_A}
\]
is the restriction of the $Ad^*$--invariant element $\Phi \in \mathcal{I}(\tilde{g}^*)$ to the subspace consisting of elements of the form
\[
\mathcal{N}(\lambda) = \lambda Y + \mathcal{N}_0(\lambda), \quad \mathcal{N}_0 \in \tilde{g}_A.
\]
Define the $\mathfrak{gl}(r)$--valued polynomial
\[
\mathcal{L}(\lambda) := \frac{a(\lambda)}{\lambda} \mathcal{N}(\lambda) = a(\lambda)Y + L_0\lambda^{n-1} + \cdots + L_{n-1},
\]
where
\[
a(\lambda) := \prod_{i=1}^{n} (\lambda - \alpha_i)
\]
is the minimal polynomial of $A$. This satisfies the equivalent Lax equation
\[
\frac{d\mathcal{L}(\lambda)}{dt} = [d\Phi(\mathcal{N})_+, \mathcal{L}].
\]
The invariant spectral curve $\mathcal{C}_0$ is then determined by the characteristic equation
\[
\det (\mathcal{L}(\lambda) - z I_r) = 0
\]
which, after suitable compactification, is viewed as an $r$--fold branched cover of $\mathbb{P}^1$, possibly having singularities over the points $\lambda = \alpha_i$ due to the $r - k_i$ fold multiplicity of zero eigenvalues. Other singularities could, of course, also occur, but for simplicity we again place ourselves in a “generic” situation in order to be able to give the main results in as explicit form as possible, and therefore exclude this possibility. The essential results remain valid without such simplifying assumptions, but explicit formulae for the spectral polynomial, genus, dimensions of orbits, and form of the abelian differentials must be modified accordingly.

We assume henceforth, for simplicity, that the spectral curves $\mathcal{C}_0$ have no singularities other than those that arise over $\{\lambda = \alpha_i\}_{i=1,...,n}$, if $k_i < r - 1$, due to the multiple zero eigenvalues of the residue matrices $N_i$. This implies in particular that the $N_i$’s, while not necessarily diagonalizable, must lie on orbits that have the same dimensions as the diagonalizable orbits whose nonzero
eigenvalues are distinct; namely, \( k_i(2r - k_i - 1) \). We also assume that one of two conditions holds, which excludes further singularities over \( \lambda = \infty \):

Case (a) \( Y = 0 \) and \( L_0 \) lies on a \( Gl(r) \) orbit of the same dimension \((r(r - 1))\) as those with simple spectrum.

Case (b) \( Y \neq 0 \) and lies on a \( Gl(r) \) orbit of the same dimension as those with simple spectrum.

**Remark.** An effect of this assumption is to eliminate from consideration the example 2.5.3, which has singularities over \( \lambda = \infty \). However, this case may also be dealt with (viz. [AHH3]), by imposing a further set of symplectic constraints defining a “generic” deformation class of admissible spectral curves.

In some cases, it is not the orbit \( O_{N_0} \) itself that is the relevant phase space, but its reduction under the Hamiltonian action consisting of conjugation by the stability subgroup \( G_Y = \text{Stab}(Y) \subset Gl(r) \):

\[
g : N_0(\lambda) \mapsto gN_0(\lambda)g^{-1}, \quad g \in G_Y.
\]  

The corresponding moment map is just the leading term in \( N_0(\lambda) \):

\[
J(N_0) := L_0 = \sum_{i=1}^{n} N_i,
\]

restricted to the subalgebra \( g_Y := \text{stab}(Y) \subset gl(r) \). Another case of interest, particularly when \( Y = 0 \), consists of restricting to a symplectic submanifold \( O_{N_0}^S \subset O_{N_0} \) determined by the zero level set of the components of \( L_0 \) within the annihilator of a Cartan subalgebra. (This is symplectic at regular elements \( L_0 \).) For future reference, we list the various subcases of interest.

Case (a) \( Y = 0 \). In this case, all the elements \( \phi \in I|_{l_0} \) in the AKS ring are invariant under the full \( Gl(r) \) action (3.14), and all components of \( L_0 \) are conserved. We may therefore reduce by the full group \( Gl(r) \) or any of its subgroups. The two subcases of greatest interest are:

Case (a.1) **Complete reduction at a regular point** \( L_0 = \mu_0 \in gl(r)^* \). The reduced manifold is then

\[
O_{N_0}^{\text{red}} = J^{-1}(\mu_0)/G_0,
\]

where \( G_0 \subset Gl(r) \) is the stabilizer of \( \mu_0 \). The dimension of the reduced orbit is:

\[
\dim O_{N_0}^{\text{red}} = \dim O_{N_0} - (r - 1)(r + 2).
\]

Case (a.2) **Symplectic invariant manifold.** We take the zero level set of all components of \( L_0 \) in the annihilator of a Cartan subalgebra, (e.g., we choose \( L_0 \) to
be diagonal. Denote this submanifold, which is symplectic at all regular values of \( L_0 \), as:

\[
\mathcal{O}_{\mathcal{N}_0}^S := \{ N_0 \in O_{\mathcal{N}_0} | L_0 \in \mathcal{T} \text{ (Cartan subalgebra)} \}. \tag{3.18}
\]

Its dimension is

\[
\dim \mathcal{O}_{\mathcal{N}_0}^S = \dim O_{\mathcal{N}_0} - r(r - 1) = \dim \mathcal{O}_{\mathcal{N}_0}^\text{red} + 2(r - 1). \tag{3.19}
\]

Case (b) \( Y \neq 0 \). In this case, the elements \( \phi \in \mathcal{T}_{\mathcal{g}^\text{red} + \lambda Y} \) of the AKS ring are only invariant under the action of the stabilizer \( G_Y = \text{Stab}(Y) \subset G\text{l}(r) \). Two cases of special interest arise:

Case (b.1) The full orbit \( \mathcal{O}_{\mathcal{N}_0} \) (i.e., no reduction).

Case (b.2) The reduction of \( \mathcal{O}_{\mathcal{N}_0} \) under the full stabilizer \( G_Y \subset G\text{l}(r) \) of a regular element \( Y \in g\text{l}(r) \), taken at a value \( L_0|_{\mathcal{g}Y} = \mu_0 \in \mathfrak{g}^*_Y \). The group \( G_Y \) is a maximal abelian subgroup with \( r - 1 \) dimensional orbits and \( G_0 = G_Y \). The reduced orbit is denoted

\[
\mathcal{O}_{\mathcal{N}_0}^{Y,\text{red}} = J_{\mathcal{g}Y}^{-1}(\mu_0)/G_Y, \tag{3.20}
\]

and has dimension

\[
\dim \mathcal{O}_{\mathcal{N}_0}^{Y,\text{red}} = \dim \mathcal{O}_{\mathcal{N}_0} - 2(r - 1). \tag{3.21}
\]

### 3.2 Structure of the Spectral Curve

The affine part of the spectral curve \( \mathcal{C}_0 \) is determined by the characteristic equation (3.13). Taking into account the ranks of the residue matrices \( \{ N_i \}_{i=1, \ldots, n} \) in (3.3), we see that the characteristic polynomial has the general form

\[
\mathcal{P}(\lambda, z) = \det (\mathcal{L}(\lambda) - zI_r)
\]

\[
= (-z)^r + z^{r-1}P_1(\lambda) + \sum_{j=2}^{r} A_j(\lambda) P_j(\lambda) z^{r-j}, \tag{3.22}
\]

where

\[
A_j(\lambda) := \prod_{i=1}^{n} (\lambda - \alpha_i)^{\max(0, j-k_i)}, \quad \text{rank } \mathcal{L}(\alpha_i) = k_i. \tag{3.23}
\]

This shows that near \( \lambda \sim \infty \), we have

\[
z \sim \lambda^m, \quad m := \begin{cases} 
n & \text{if } Y = 0 \text{ (case (a))} 
n-1 & \text{if } Y \neq 0 \text{ (case (b))}. \end{cases} \tag{3.24}
\]
This suggests a compactification, not within $\mathbb{P}^2$, but rather in the total space of a line bundle over $\mathbb{P}^1 = U_0 \cup U_\infty$ (where $U_0$, $U_\infty$ denote the open disks obtained by deleting $\lambda = \infty$ and $\lambda = 0$, respectively), with coordinate pairs $(\lambda, z)$ over $U_0$ and $(\tilde{\lambda}, \tilde{z})$ over $U_\infty$ related by

$$(\lambda, z) \mapsto (\tilde{\lambda} = \frac{1}{\lambda}, \tilde{z} = \frac{z}{\lambda^m}) \quad \text{(over } U_0 \cap U_\infty).$$

(3.25)

This is just the total space $T$ of the bundle $\mathcal{O}(m) \to \mathbb{P}^1$ whose sheaf of sections consists of homogeneous functions of degree $m$. The transformation (3.25) extends the affine curve $C_0$ defined by (3.13) over $\lambda = \infty$, defining the compactification:

$$C_0 \hookrightarrow C \hookrightarrow T.$$  

(3.26)

The possible spectral curves so arising are branched $r$–sheeted covers of $\mathbb{P}^1$, which within any given orbit of type (3.3), have $z$–values over each $\lambda = \alpha_i$ that are fixed (being Casimir invariants of the coadjoint action (3.5)). Of these, there are $k_i$ nonsingular points $(\lambda = \alpha_i, z = \zeta_{ia})_{a=1,\ldots,k_i}$ corresponding to the nonzero eigenvalues of $\mathcal{L}(\alpha_i)$, and the point $(\lambda = \alpha_i, z = 0)$, which generically is an $r - k_i$–fold ordinary singular point corresponding to the $r - k_i$–fold zero eigenvalue. Figure 3.1 gives a visualization of the spectral curves $C$, embedded in $T$, as branched coverings of $\mathbb{P}^1$, constrained to pass through these points.

**Figure 3.1**

The detailed structure may be expressed more precisely by writing the form of the characteristic polynomial $P(\lambda, z)$ for any spectral curve $C$ in a neighborhood of a given curve $C_R$ as a perturbation of the characteristic polynomial $P_R(\lambda, z)$ defining $C_R$. 

Proposition 3.1 ([AHH3]). In a neighborhood of the point \( N_R \in \mathcal{O}_{N_0} \) with characteristic polynomial \( P_R(\lambda, z) \), the characteristic polynomial has the form:

\[
P(\lambda, z) \equiv P_R(\lambda, z) + a(\lambda) \sum_{j=2}^{r} a_j(\lambda)p_j(\lambda)z^{r-j},
\]

where

\[
a_j(\lambda) = \prod_{i=1}^{n}(\lambda - \alpha_i)^{\max(0,j-k_i-1)}
\]

\[
p_j(\lambda) = \sum_{a=0}^{\delta_j} P_{ja}\lambda^a,
\]

and \( \{p_j(\lambda)\}_{j=2,\ldots,r} \) are polynomials of degree:

\[
\delta_j \equiv \deg p_j(\lambda) = \begin{cases} 
    d_j - j & \text{if } Y = 0 \\
    d_j & \text{if } Y \neq 0
\end{cases}
\]

\[
d_j \equiv \sum_{i=1}^{n} \min(j-1,k_i).
\]

The number of independent spectral parameters \( \{P_{ja}\} \), \( (a = 0, \ldots \delta_j + n - m - 1, \ j = 2, \ldots r) \) is thus:

\[
d = \tilde{g} + r - 1,
\]

where

\[
\tilde{g} = \frac{1}{2}(r - 1)(mr - 2) - \frac{1}{2} \sum_{i=1}^{n}(r - k_i)(r - k_i - 1)
\]

is the genus of the (partially) desingularized spectral curve \( C \) obtained by separating branches at \( \{\alpha_i, 0\} \). In a neighborhood of any generic point on \( \mathcal{O}_{N_0} \), these spectral invariants are all independent.

The complete integrability of the systems under consideration on the various coadjoint orbits \( \mathcal{O}_{N_0} \), and the reductions \( \mathcal{O}_{N_0}^{\text{red}}, \mathcal{O}_{N_0}^{Y,\text{red}} \) and symplectic submanifolds \( \mathcal{O}_{N_0}^{S} \) thereof, may be seen from the following Table of generic dimensions for the various cases discussed above. (Note that the \( P_{ia} \)'s referred to do not include the leading terms of the polynomials \( p_j(\lambda) \) in eq. (3.29).) Recall that the value of the genus \( \tilde{g} \) of the desingularized curve \( C \) given in Proposition 3.1 depends on the value \( m \), which is different in the cases (a) and (b):

\[
\tilde{g} = \tilde{g}(m), \quad m = \begin{cases} 
    n - 1 & \text{for case (a)} \\
    n & \text{for case (b)}
\end{cases}
\]
### Table of Dimensions

| Case | Dimension | \(#P_{1a}'s\) | \(#P'_i's\) |
|------|-----------|----------------|--------------|
| (a)  | \(\dim \mathcal{O}_{N_0} = 2\tilde{g} + (r - 1)(r + 2)\) | \(\tilde{g}\) | \(r - 1\) |
| (a.1)| \(\dim \mathcal{O}_{N_0}^{\text{red}} = 2\tilde{g}\) | \(\tilde{g}\) | 0 |
| (a.2)| \(\dim \mathcal{O}_{N_0}^S = 2(\tilde{g} + r - 1)\) | \(\tilde{g}\) | \(r - 1\) |
| (b.1)| \(\dim \mathcal{O}_{N_0} = 2(\tilde{g} + r - 1)\) | \(\tilde{g}\) | \(r - 1\) |
| (b.2)| \(\dim \mathcal{O}_{N_0}^{Y,\text{red}} = 2\tilde{g}\) | \(\tilde{g}\) | 0 |

Here, the notation \(\{P_i\}_{i=2,...,r}\) is used to denote the components of \(L_0\) evaluated on a basis of the relevant Cartan subalgebra (not including the trivial central element, which is a Casimir invariant, and hence constant on orbits). These are also elements of the ring \(I^Y_{\text{AKS}}\), corresponding to the leading terms in the polynomials \(p_j(\lambda)\) in eq. (3.29) for case (a), and the next to leading terms for case (b) (the leading terms being constant in the latter case), but they are listed separately since, in cases (a.1) and (b.2), they are fixed through the Marsden-Weinstein reduction procedure, and hence do not contribute to the number of independent invariants on the reduced spaces. Moreover, these elements enter again in Section 3.4 when defining the *spectral Darboux coordinates* for these two cases. It follows from the dimensions in the Table and the independence of the commuting invariants that cases (a.1), (a.2), (b.1) and (b.2) all give completely integrable systems.

### 3.3 Spectral Lagrangian Foliation

The Lagrangian foliation given by fixing simultaneous level sets of the invariants in the ring \(I^Y_{\text{AKS}}\) of spectral invariants (i.e., fixing the spectral curve \(C\)) is depicted below in Figure 3.2 for the various cases discussed above.

| \(\mathcal{O}_{N_0}^S\) or \(\mathcal{O}_{N_0}\) (\(\mathcal{O}_{N_0}^{\text{red}}\) or \(\mathcal{O}_{N_0}^{Y,\text{red}}\)) | Liouville-Arnold Tori \(T\) \(\rightarrow\) (isospectral leaves) \(\dim T = \tilde{g} + r - 1\) (or \(\tilde{g}\)) |
|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|
| Admissable curves \(C \subset T\) \(\dim C = \tilde{g} + r - 1\) (or \(\tilde{g}\)) |

**Figure 3.2**

We may summarize the relevant spectral data associated to each element \(N \in \lambda Y + \tilde{g}_A\) as follows:
• A spectral curve \( \mathcal{C} \) \((r\text{-fold branched cover of } \mathbb{P}^1)\) defined by the characteristic equation:

\[
P(\lambda, z) = \det(\mathcal{L}(\lambda) - zI_r) = 0, \tag{3.34}
\]

(after suitable compactification and desingularization). The \(r-1\) points over \(\lambda = \infty\) are determined by the leading terms of the polynomials \(p_j(\lambda)\) of Proposition 3.1.

• An eigenvector subspace: \([V(\lambda, z)] \subset \mathbb{C}^r\) at each point in \(\mathcal{C}\) which, by our genericity assumptions, is one dimensional.

Together, these determine an eigenvector line bundle \(\tilde{E} \to \mathcal{C}\), and its corresponding dual bundle \(E \to \mathcal{C}\). The latter may be shown (viz. [AHH2], [AHH3]) to be generally of degree

\[
\deg(E) = \tilde{g} + r - 1, \tag{3.35}
\]

and hence, by the Riemann-Roch theorem, to have an \(r\)-dimensional space of sections. Conversely, it turns out that this data is sufficient to reconstruct the matrix \(\mathcal{L}(\lambda)\) (and hence \(\mathcal{N}(\lambda)\)) up to conjugation by an element of \(\text{Gl}(r)\); that is, it is sufficient to determine the projected point in the reduced orbit \(\mathcal{O}^{Y,\text{red}}_{\mathcal{N}_0}\) or \(\mathcal{O}^{\text{red}}_{\mathcal{N}_0}\), but not the element \(\mathcal{N}(\lambda)\) itself.

In order to reconstruct the element \(\mathcal{N}(\lambda)\), it is necessary to add some further spectral data, consisting of a framing at \(\lambda = \infty\); that is, a basis of sections \(\{\sigma_i \in H^0(\mathcal{C}, E)\}_{i=1,\ldots,r}\) of the bundle \(E \to \mathcal{C}\), chosen to vanish, e.g. at all but one of the \(r\) points \(\{\infty_i\}_{i=1,\ldots,r}\) over \(\lambda = \infty\) (for the case where the Cartan subalgebra in question consists of the diagonal matrices).

\[
\sigma_i(\infty_j) = 0, \quad i \neq j. \tag{3.36}
\]

This adds \(r - 1\) dimensions to the fibres, (since framings related by \(\{\bar{\sigma}_i = \kappa \sigma_i\}\) are equivalent). Furthermore, the spectrum over \(\lambda = \infty\) in the class of admissible spectral curves must be left undetermined, adding \(r - 1\) dimensions to the base space in Figure 3.2.

More generally, it is insufficient to just consider line bundles, since this excludes the possibility of degeneracy in the spectrum and further singular points. The appropriate generalization consists of a coherent sheaf defined by the exact sequence

\[
0 \longrightarrow \mathcal{O}(-m)^{\oplus r} \xrightarrow{\mathcal{L}^r(\lambda) - zI} \mathcal{O}^{\oplus r} \longrightarrow E \longrightarrow 0, \tag{3.37}
\]

where \(\mathcal{O}(-m)\) denotes the sheaf obtained by pulling back the corresponding sheaf over \(\mathbb{P}^1\) to \(\mathcal{T}\). In the case of bundles, the exact sequence (3.37) just means
that the dual space to the space of eigenvectors over the spectral curve is given by the cokernel of the linear map defined by $L^T(\lambda) - zI_r$. A more complete discussion of the significance of this construction may be found in [AHH2], [AHH3].

### 3.4 Spectral Darboux Coordinates

In this section we give a method for constructing the appropriate Darboux coordinates naturally associated to the spectral data discussed above, in which the Hamiltonians in the spectral ring $I^Y_{\text{AKS}}$ determine a Liouville generating function in completely separated form. First, we shall give a purely computational description of these coordinates in terms of simultaneous solutions of polynomial equations. The significance of this prescription in terms of the eigenvector line bundles of the preceding section will follow.

Let

$$M(\lambda, \zeta) := \frac{N(\lambda)}{\lambda} - \zeta I_r,$$

and denote by $\tilde{M}(\lambda, \zeta)$ the transpose of the matrix of cofactors. Then, over the spectral curve defined by the characteristic equation (3.34), the columns of $\tilde{M}(\lambda, \zeta)$ are the eigenvectors of $L(\lambda)$ (or $N(\lambda)$), and hence, generically, these are all proportional; i.e. $\tilde{M}(\lambda, \zeta)$ has rank 1. Let $V_0 \in \mathbb{C}^r$ be a fixed vector, and denote the solutions to the system of polynomial equations

$$\tilde{M}(\lambda, \zeta)V_0 = 0, \quad V_0 \in \mathbb{C}^r$$

as $\{\lambda_\mu, \zeta_\mu\}_{\mu=1,...}$. Note that, due to the rank condition, there really are only two independent equations here, the other $r-2$ following as linear consequences.

The significance of these equations in relation to the spectral data is quite simple; they are the conditions that a section of the dual eigenvector line bundle $E \to \mathcal{C}$ should vanish. The solutions give the zeros of the components of the eigenvector determined by the vector $V_0$. As is well known in algebraic geometry, giving the divisor of zeros of any section of a line bundle amounts to giving the linear equivalence class of the bundle itself. It follows, since the bundle $E \to \mathcal{C}$ is of degree $\tilde{g} + r - 1$, that there will in general be $\tilde{g} + r - 1$ zeros. However this is not necessarily the number of solutions to (3.39), since some of the zeros may be over $\lambda = \infty$. In fact, we may distinguish two cases of particular interest as follows. In order to characterize the spectrum over $\lambda = \infty$, define

$$\tilde{L}(\lambda) := L(\lambda)/\lambda^m.$$
Case (i). \(V_0\) is an eigenvector of \(\tilde{L}(\infty)\). In this case, \(r - 1\) of the zeros are over \(\lambda = \infty\) (the only point omitted over \(\infty\) being the one corresponding to the eigenvalue of \(V_0\)). Hence, there are only \(\tilde{g}\) finite solutions pairs \(\{\lambda_\mu, \zeta_\mu\}_{\mu = 1, \ldots, \tilde{g}}\), and these are generically independent, when viewed as functions on the phase space \(\mathcal{O}^S_{N_0}\) (case (a)) or \(\mathcal{O}_{N_0}\) (case (b)). Moreover, they are invariant under the action of the reduction group for both cases, since this leaves the space \([V_0]\) invariant, and hence they project to functions on the reduced space \(\mathcal{O}^{\text{red}}_{N_0}\) (case (a)) or \(\mathcal{O}^{Y, \text{red}}_{N_0}\) (case (b)). In view of the dimensions given in the Table of Dimensions, Section 3.2, the projected functions provide coordinate systems on the reduced spaces. On the prereduced spaces, we must supplement these with a further \(r - 1\) pairs of coordinate functions, which we define as follows. Choose a basis where \(L_0\) (case (a)) or \(Y\) (case (b)) is diagonal, and \(V_0 = (1, 0, \ldots, 0)^T\). Then let

\[
q_i := \begin{cases} 
\ln(L_1)_{ii} + \frac{1}{2} \sum_{j=2, j\neq i}^r \ln(p_i - p_j) & \text{for case (a)} \\
\ln(L_0)_{ii} & \text{for case (b)}
\end{cases}
\]

\[
P_i := (L_0)_{ii}, \quad i = 2, \ldots, r.
\]

The pairs \(\{q_i, P_i\}_{i=2, \ldots, r}\) provide the remaining coordinates required.

Case (ii). \(V_0\) is not an eigenvector of \(\tilde{L}(\infty)\) and, furthermore, \(V_0 \notin \text{Im} (\tilde{L}(\infty) - \tilde{z}_i I)\) for any eigenvalue \(\tilde{z}_i(\infty)\) of \(\tilde{L}(\infty)\). In this case, none of the zeros are over \(\lambda = \infty\), and there are generically \(\tilde{g} + r - 1\) independent solution pairs \(\{\lambda_\mu, \zeta_\mu\}_{\mu = 1, \ldots, \tilde{g} + r - 1}\) of eq. (3.39). These then provide a coordinate system on the prereduced space \(\mathcal{O}^S_{N_0}\) (case (a)) or \(\mathcal{O}_{N_0}\) (case (b)).

We then have the following fundamental result.

**Theorem 3.2.** 1. If \(V_0 \notin \text{Im} (\tilde{L}(\infty) - \tilde{z}_i I)\) for any eigenvalue \(\tilde{z}_i(\infty)\) of \(\tilde{L}(\infty)\), the functions \(\{\lambda_\mu, \zeta_\mu\}_{1, \ldots, \tilde{g} + r - 1}\) define a Darboux coordinate system on \(\mathcal{O}^S_{N_0}\) (case (a)) or \(\mathcal{O}_{N_0}\) (case (b)). The orbital symplectic form is therefore:

\[
\omega_{\text{orb}} = \sum_{\mu = 1}^{\tilde{g} + r - 1} d\lambda_\mu \wedge d\zeta_\mu. \tag{3.42a}
\]

2. If \(V_0\) is an eigenvector of \(L_0\) (case (a)) or \(Y\) (case (b)), the functions \(\{\lambda_\mu, \zeta_\mu\}_{1, \ldots, \tilde{g}}\) project to Darboux coordinates on the reduced spaces \(\mathcal{O}^{\text{red}}_{N_0}\) (case (a)) or \(\mathcal{O}^{Y, \text{red}}_{N_0}\) (case (b)), so the reduced orbital symplectic form is:

\[
\omega_{\text{red}} = \sum_{\mu = 1}^{\tilde{g}} d\lambda_\mu \wedge d\zeta_\mu. \tag{3.42b}
\]
3. If \( V_0 \) is an eigenvector of \( L_0 \) (case (a)) or \( Y \) (case (b)), the functions \( \{ \lambda_\mu, \zeta_\mu, q_i, P_i \}_{\mu=1,...,\tilde{g}; i=2,...,r} \) define a Darboux coordinate system on \( \mathcal{O}_{N_0}^{S} \) (case (a)) or \( \mathcal{O}_{N_0} \) (case (b)), so the orbital symplectic form is:

\[
\omega_{\text{orb}} = \sum_{\mu=1}^{\tilde{g}} d\lambda_\mu \wedge d\zeta_\mu + \sum_{i=2}^{r} dq_i \wedge dP_i. \tag{3.42c}
\]

In the following section, we consider a number of elementary examples of the above theorem. We shall see that the resulting *spectral Darboux coordinates* do, indeed, generalize the hyperellipsoidal coordinates that were encountered in the examples of Section 1.

### 3.5 Examples

We begin by considering the simplest possible case; namely, where \( N_0(\lambda) \) has only one pole, at \( \lambda = \alpha_1 \), and \( r = 2 \) or \( 3 \). This just corresponds to coadjoint orbits of the finite dimensional Lie algebras \( \mathfrak{sl}(2) \) and \( \mathfrak{sl}(3) \). Then we consider the case \( \tilde{\mathfrak{sl}}(2, \mathbb{R})^+ \) for arbitrary \( n \), with \( \text{rank}(N_i) = k_i = 1 \) for all \( i = 1, \ldots, n \). This reproduces the hyperellipsoidal coordinates for the finite dimensional examples of Secs. 1.1 and 2.5.1 (cf. [Mo], such as the Neumann oscillator. Finally, we consider the case \( \tilde{\mathfrak{su}}(1, 1)^+ \), which provides the appropriate complex coordinates for the nonlinear Schrödinger equation, as discussed in Secs. 1.2 and 2.5.2.

**(a)** *Single poles: \( n = 1 \)*  
**(a.1)** Take \( g = \mathfrak{sl}(2, \mathbb{R}) \), and (without loss of generality), \( \alpha_1 = 0 \). Then the dimension of a generic orbit is \( \text{dim} \mathcal{O}_{N_0} = 2 \). We parametrize \( N_0(\lambda) \) as follows:

\[
N_0(\lambda) = \frac{\lambda N_1}{\lambda - \alpha_1} = N_1 := \begin{pmatrix}
-a & r \\
u & a
\end{pmatrix}, \tag{3.43}
\]

and choose

\[
Y := \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \quad V_0 = \begin{pmatrix}
1 \\
0
\end{pmatrix}. \tag{3.44}
\]

The characteristic equation is then

\[
\det (L(\lambda) - z\mathbb{I}_r) = z^2 - \lambda^2 - a^2 - ur = 0. \tag{3.45}
\]

In this case, \( V_0 \) is an eigenvector of \( Y \) and the genus of the spectral curve is \( \tilde{g} = 0 \), so there are no \( \{ \lambda_\mu, \zeta_\mu \} \)'s. The single pair of spectral Darboux coordinates is thus

\[
q_2 = \ln u, \quad P_2 = a. \tag{3.46}
\]
It is easily verified that, relative to the Lie Poisson structure, they satisfy

\( \{q_2, P_2\} = 1. \) \hfill (3.47)

(a.2) We consider the same orbit as in (a.1), but choose

\[ Y := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \] \hfill (3.48)

In this case, \( V_0 \) is not an eigenvector of \( Y \). The genus is still \( \tilde{g} = 0 \), but the equation (3.39) now has a finite solution, giving the Darboux coordinate pair

\[ \lambda_1 = -u, \quad \zeta_1 = -\frac{a}{u}. \] \hfill (3.49)

These are verified to also satisfy

\( \{\lambda_1, \zeta_1\} = 0. \) \hfill (3.50)

(a.3) Take \( g = \mathfrak{sl}(3, \mathbb{R}) \), and again, \( \alpha_1 = 0 \). Then the dimension of a generic orbit with \( n = 1 \) is \( \dim \mathcal{O}_{N_0} = 6 \). We parametrize \( N_0(\lambda) \) as:

\[ N_0(\lambda) = N_1 := \begin{pmatrix} -a-b+r&s \\ u&s \\ v & f \end{pmatrix}, \] \hfill (3.51)

and choose

\[ Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad V_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \] \hfill (3.52)

Again, \( V_0 \) is an eigenvector of \( Y \), but the spectral curve has genus \( \tilde{g} = 1 \), and is realized as a 3-fold branched cover of \( \mathbb{P}^1 \). We therefore find one Darboux coordinate pair \( (\lambda_1, \zeta_1) \), corresponding to a finite zero of the eigenvector components, plus two further pairs, \( (q_2, P_2, q_3, P_3) \), corresponding to zeros over \( \lambda = \infty \):

\[ \lambda_1 = \frac{1}{2} \left( b - a - \frac{ev}{u} + \frac{uf}{v} \right), \quad \zeta_1 = \frac{uva + uvb - ev^2 - fu^2}{-uva + uvb - ev^2 + fu^2}. \]

\[ q_2 = \ln u, \quad q_3 = \ln v, \quad P_2 = a, \quad P_3 = b. \] \hfill (3.53)

Again, it is easily verified directly that these form a Darboux system, with nonvanishing Lie Poisson brackets

\[ \{\lambda_1, \zeta_1\} = 1, \quad \{q_2, P_2\} = 1, \quad \{q_3, P_3\} = 1. \] \hfill (3.54)
(b) Now consider the case $\tilde{g}^+ = \tilde{sl}(2, \mathbb{R})^+$, with arbitrary $n$, but ranks $k_i = 1$ for all $i$, and hence $\det(N_i) = 0$ for all the residue matrices $N_i$. For general $Y$, $\mathcal{N}(\lambda)$ then has the form

$$\mathcal{N}(\lambda) = \lambda \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \frac{\lambda}{2} \begin{pmatrix} \sum_{i=1}^n \frac{x_i y_i}{\lambda - \alpha_i} - \sum_{i=1}^n \frac{y_i}{\lambda - \alpha_i} \\ \sum_{i=1}^n \frac{x_i^2}{\lambda - \alpha_i} & \sum_{i=1}^n \frac{x_i y_i}{\lambda - \alpha_i} \end{pmatrix},$$

(3.55)

where $\{x_i, y_i\}_{i=1,...,n}$ form a Darboux system on the reduced Moser space, which is identified with $\mathbb{R}^{2n}/(\mathbb{Z}_2)^N$. In this case, the characteristic equation defining the invariant spectral curve $C$ is

$$\det(\mathcal{L}(\lambda) - z\mathbb{I}_2) = z^2 + a(\lambda)P(\lambda) = 0,$$

(3.56)

where

$$P(\lambda) = -(a^2 + bc)\lambda^n + P_{n-1}\lambda^{n-1} + \ldots.$$

(3.57)

In particular, this gives eq. (1.34) for the case $a = c = 0$, $b = -\frac{1}{2}$. Thus, $C$ is hyperelliptic, a 2–sheeted branched cover of $\mathbb{P}^1$, with $2n - 1$ or $2n$ finite branch points, depending on whether or not $a^2 + bc$ vanishes. The genus is therefore generically $\tilde{g} = n - 1$. The dimension of the coadjoint orbit is $\dim \mathcal{O}_{N_0} = 2n$.

The matrix $\tilde{M}(\lambda, \zeta)$ is

$$\tilde{M} = \begin{pmatrix} -a + \frac{1}{2} \sum_{i=1}^n \frac{x_i y_i}{\lambda - \alpha_i} - \zeta & -b + \frac{1}{2} \sum_{i=1}^n \frac{y_i^2}{\lambda - \alpha_i} \\ -c - \frac{1}{2} \sum_{i=1}^n \frac{x_i^2}{\lambda - \alpha_i} & a - \frac{1}{2} \sum_{i=1}^n \frac{x_i y_i}{\lambda - \alpha_i} - \zeta \end{pmatrix}.$$

(3.58)

Taking

$$V_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

(3.59)

if $c \neq 0$, $V_0$ is not an eigenvector of $Y$, so the full set of $n$ spectral Darboux coordinate pairs $\{\lambda_\mu, \zeta_\mu\}_{\mu=1,...,n}$ are given by:

$$\sum_{i=1}^n \frac{x_i^2}{\lambda_\mu - \alpha_i} + 2c = 0,$$

(3.60a)

$$\zeta_\mu = -a + \frac{1}{2} \sum_{i=1}^n \frac{x_i y_i}{\lambda_\mu - \alpha_i},$$

(3.60b)

$\mu = 1, \ldots, n$.

These are therefore hyperelliptic coordinates $\{\lambda_\mu\}$ and their conjugate momenta $\{\zeta_\mu\}$. In the case $c = 0$, $V_0$ is an eigenvector of $Y$, and eqs. (3.60a,b) are replaced by

$$\frac{1}{2} \sum_{i=1}^n \frac{x_i^2}{\lambda_\mu - \alpha_i} = 0,$$

(3.61a)

$$\zeta_\mu = -a + \frac{1}{2} \sum_{i=1}^n \frac{x_i y_i}{\lambda_\mu - \alpha_i},$$

(3.61b)

$\mu = 1, \ldots, n - 1,$
yielding only \( n - 1 \) pairs of Darboux coordinates \( \{ \lambda_\mu, \zeta_\mu \}_{\mu=1,...,n-1} \), since one of the zeros of the eigenvector components lies over \( \lambda = \infty \). We must therefore complete the system by defining the additional pair

\[
q := \ln \left( \frac{1}{2} \sum_{i=1}^{n} x_i^2 \right), \quad P := \frac{1}{2} \sum_{i=1}^{n} x_i y_i.
\]

(3.62)

It is easily verified directly that

\[
\omega = -d\theta,
\]

(3.63)

where

\[
\theta := \sum_{i=1}^{n} y_i dx_i = \begin{cases} 
\sum_{\mu=1}^{n} \zeta_\mu d\lambda_\mu & \text{if } c \neq 0 \\
\sum_{\mu=1}^{n-1} \zeta_\mu d\lambda_\mu + Pdq & \text{if } c = 0.
\end{cases}
\]

(3.64)

(c) NLS Equation: \( \tilde{\mathfrak{su}}(1,1)^+ \)

Taking the orbit \( O_{N_0} \) in \( \tilde{\mathfrak{su}}(1,1)^{++} \) as parametrized in eqs. (2.54), (2.55), with \( Y = 0 \), the symplectic submanifold \( O_{N_0}^S \subset O_{N_0} \) is defined by the constraint

\[
\sum_{i=1}^{n} z_i^2 = 0.
\]

(3.65)

The spectral Darboux coordinates \( \{ q, P, \lambda_\mu, \zeta_\mu \}_{\mu=1,...,n-1} \) are then given by eqs. (1.52a-c). It is easily verified in this case that the orbital symplectic form restricted to \( O_{N_0}^S \) is

\[
\omega_{\text{orb}} = -d\theta,
\]

(3.66)

where

\[
\theta|_{O_{N_0}} = -i \sum_{j=1}^{n} z_j dz_j|_{O_{N_0}} = \sum_{\mu=1}^{n-2} \zeta_\mu d\lambda_\mu + Pdq,
\]

(3.67)

so \( \{ q, P, \lambda_\mu, \zeta_\mu \}_{\mu=1,...,n-2} \) do, indeed, define a Darboux coordinate system.

A similar construction holds for the case of the sine-Gordon equation (Section 1.3), where the relevant algebra is the twisted loop algebra \( \tilde{\mathfrak{su}}(2)^+ \), obtained by a suitable combination of discrete and continuous reductions. The orbits are parametrized by eqs. (1.60), (1.61a,b), and the relevant spectral Darboux coordinates determined by eqs. (1.84a,b). Details may be found in [HW].

In the last section we explain how, in the general case, these spectral Darboux coordinates lead directly to a linearization of the AKS flows through the Liouville-Arnold integration procedure. In each case the relevant linearizing map turns out to be the Abel map to the Jacobi variety of the spectral curve.
3.6 Liouville-Arnold Integration

Using the spectral Darboux coordinates, we may define the local equivalent of the “canonical” 1–form

\[ \theta := \sum_{\mu=1}^{\tilde{g}} \zeta_\mu d\lambda_\mu + \sum_{i=2}^{r} P_i dq_i \]  

(3.68a)

\[ = \sum_{\mu=1}^{\tilde{g}} \frac{z_\mu}{a(\lambda_\mu)} d\lambda_\mu + \sum_{i=2}^{r} P_i dq_i, \]  

(3.68b)

where

\[ z_\mu := a(\lambda_\mu) \zeta_\mu. \]  

(3.69)

(Note that for the examples given above, this actually is the canonical 1–form on \( \mathbb{R}^{2n}/(\mathbb{Z}_2)^n \), or \( \mathbb{C}^{2n}/(\mathbb{Z}_2)^n \), viewed as the cotangent bundle of \( \mathbb{R}^n/(\mathbb{Z}_2)^n \) and \( \mathbb{C}^n/(\mathbb{Z}_2)^n \), respectively.) On the Liouville-Arnold torus \( T \), defined by taking the simultaneous level sets

\[ P_{ia} = C_{ia}, \quad P_i = C_i \]  

(3.70)

of the spectral invariants, we have

\[ \theta|_T = dS(\lambda_1, \ldots, \lambda_{\tilde{g}}, q_2, \ldots, q_r, P_{ia}, P_i), \]  

(3.71)

where \( S(\lambda_1, \ldots, \lambda_{\tilde{g}}, q_2, \ldots, q_r, P_{ia}, P_i) \) is the Liouville generating function to the canonical coordinates conjugate to the invariants \( (P_{ia}, P_i) \). Eq. (3.71) can be integrated by viewing \( z = z(\lambda, P_{ia}, P_i) \) as a meromorphic function on the Riemann surface of the spectral curve \( \mathcal{C} \).

\[ S(\lambda_\mu, q_i, P_{ia}, P_i) = \sum_{\mu=1}^{\tilde{g}} \int_{\lambda_\mu}^{\lambda_\mu} z(\lambda, P_{ia}, P_i) d\lambda + \sum_{i=2}^{r} q_i P_i, \]  

(3.72)

where \( z_\mu = z(\lambda_\mu, P_{ia}, P_i) \) is essentially determined implicitly by the spectral equation

\[ \mathcal{P}(\lambda_\mu, z_\mu(\lambda_\mu, P_{ia}, P_i)) = 0. \]  

(3.73)

The linearizing coordinates for AKS flows are then given, as usual, by differentiation of \( S \) with respect to the invariants:

\[ Q_{ia} = \frac{\partial S}{\partial P_{ia}} = \sum_{\mu=1}^{\tilde{g}} \int_{\lambda_\mu}^{\lambda_\mu} \frac{1}{a(\lambda)} \frac{\partial z}{\partial P_{ia}} d\lambda = \frac{\partial h}{\partial P_{ia}} t + Q_{ia,0} \]  

(3.74a)

\[ Q_i = \frac{\partial S}{\partial P_i} = \sum_{\mu=1}^{\tilde{g}} \int_{\lambda_\mu}^{\lambda_\mu} \frac{1}{a(\lambda)} \frac{\partial z}{\partial P_i} d\lambda + q_i = \frac{\partial h}{\partial P_i} t + Q_{i,0}, \]  

(3.74b)
where, from the explicit structure (3.27) of the characteristic polynomial given in Proposition 3.1, we obtain, by implicit differentiation, that the integrands of eqs. (3.74a,b) are of the form

\[
\omega_{ia} := \frac{1}{a(\lambda)} \frac{\partial z}{\partial P_{ia}} d\lambda = \frac{a_i(\lambda) z^{-i} \lambda^a}{P_z(\lambda, z)} d\lambda \quad (3.75a)
\]

\[
\omega_i := \frac{1}{a(\lambda)} \frac{\partial z}{\partial P_i} d\lambda = -\sum_{j=2}^{r} \frac{R_{ij} a_j(\lambda)(-z)^{-j} \lambda^{\delta_j - \epsilon}}{P_z(\lambda, z)} d\lambda, \quad (3.75b)
\]

where

\[
R_{ij} := \begin{cases} 
(P_1 - P_i) \sum_{2 \leq i_1 < i_2 < \cdots < i_{j-2} \neq i} P_{i_1} \cdots P_{i_{j-2}} \\
\text{and} \quad \epsilon = 0 & \text{for case (a)} \\
(Y_1 - Y_i) \sum_{2 \leq i_1 < i_2 < \cdots < i_{j-2} \neq i} Y_{i_1} \cdots Y_{i_{j-2}} \\
\text{and} \quad \epsilon = 1 & \text{for case (b)}
\end{cases} \quad (3.76)
\]

The point to note is that the differentials \(\{\omega_{ia}\}, \{\omega_i\}\) appearing in eqs. (3.75a,b) are, respectively, abelian differentials of the first and third kinds on the Riemann surface defined by \(C\), the latter having their poles at the points \((\infty_1, \ldots, \infty_r)\) over \(\lambda = \infty\).

**Theorem 3.3 [AHH3].** The \(\tilde{g}\) differentials \(\{\omega_{ia}\}_{i=1,\ldots,\tilde{g}}\) in eq. (3.75a) form a basis for the space \(H^0(\tilde{S}, K_{\tilde{S}})\) of abelian differentials of the first kind (where \(K_{\tilde{S}}\) denotes the canonical bundle). The linear flow equation (3.74a) may therefore be expressed as:

\[
A(D) = B + Ut, \quad (3.77)
\]

where \(A : \tilde{S}^\sim C \to \tilde{C}^\sim / \Gamma\) is the Abel map, and \(B, U \in \tilde{C}^\sim\) are obtained by applying the inverse of the \(\tilde{g} \times \tilde{g}\) normalizing matrix \(M\), with elements

\[
M_{\mu,(ia)} := \oint_{\alpha,\mu} \omega_{ia}, \quad (3.78)
\]

to the vectors \(C, H \in \tilde{C}^\sim\) with components \(C_{ia}\) and \(-\frac{\partial h}{\partial P_{ia}}\), respectively (the pair \((ia)\) viewed as a single coordinate label in \(\tilde{C}^\sim\)). The \(r-1\) differentials \(\{\omega_i\}_{i=2,\ldots,r}\) in eq. (3.75b) are abelian differentials of the third kind with simple poles at \(\infty_i\) and \(\infty_1\), and residues \(+1\) and \(-1\), respectively.

Comparing this general formula with the specific cases (1.23), (1.55), (1.56), (1.88a,b), for the examples of Section 1, we see that this provides the generalization that was required, expressing all linearized AKS flows on rational coadjoint orbits of \(\mathfrak{sl}(r)+\) and its reductions through the Abel map.
It is possible, moreover, to invert the map expressing these flows, by expressing any symmetric function of the coordinates \( \{ \lambda_\mu, \zeta_\mu \}_{\mu=1,..,\tilde{g}} \) in terms of the Riemann theta function associated to the curve \( C \). For example, in view of eq. (3.74b), the coordinates \( \{ q_i \}_{i=2,..,r} \) themselves are expressed as such symmetric functions through abelian integrals of the third kind. Applying the reciprocity theorem relating the two kinds of abelian integrals (viz. \([AHH3]\)), we obtain:

**Corollary 3.4 [AHH3].** For a suitable choice of constants \( \{ e_i, f_i \}_{i=2,..,r} \), the coordinate functions \( \{ q_i(t) \} \) satisfying eq.(3.74b) are given by:

\[
q_i(t) = \ln \left[ \frac{\theta(B + tU - A(\infty_i) - K)}{\theta(B + tU - A(\infty_1) - K)} \right] + e_i t + f_i, \tag{3.79}
\]

where \( K \in \mathbb{C}_{\tilde{g}} \) is the Riemann constant.

This generalizes the theta function formula (1.57) giving the solution of the NLS equation. Similar formulae exist e.g., for the sine-Gordon equation \([HW]\) and many other systems that can be cast in terms of commuting AKS flows in rational coadjoint orbits of loop algebras. Aside from technical complications resulting, e.g., from the reduction procedure or the imposition of further symplectic constraints, or from the presence of further singularities in the spectral curve, the procedure is largely algorithmic. It provides a very general setting for the explicit application of the Liouville-Arnold integration procedure to a wide class of known - and yet to be discovered - integrable Hamiltonian systems. Moreover, the moment map embedding method makes it possible to treat both the intrinsically finite dimensional systems, and those systems corresponding to finite dimensional sectors of integrable systems of PDE’s (solitons, finite band solutions, etc.) on exactly the same footing.

**Acknowledgements.** Most of the results described here were obtained in collaboration with a number of colleagues and friends, whose important contributions to this work it is a pleasure to acknowledge. My thanks to M. Adams, J. Hurtubise, E. Previato and M.-A. Wisse for their valuable input and help over an extended period. I would also like to thank G. Helminck and the other organizers of the 8th Scheveningen conference for their very kind hospitality and for the cordial and stimulating environment they helped to create.
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