Description of charged scalar system by means of General Relativity approach

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Abstract

The Kähler metric which has been constructed by present author in [1] is used in this paper to find an exact solution of Einstein equations with energy-momentum tensor of special type. That type of $T_{ij}$ admits in particular to use it as energy-momentum tensor of charged scalar field and in case of such system the field function is determined.

Keywords: Ricci-flat Kähler manifold, HyperKähler metric, Einstein equations, Charged scalar field.

1. Let us consider a model consist of two particles where one of them is the center of the inertial frame. The problem to solve is to determine all restrictions which appear by the gravitational interaction reasoning. These restrictions must be applied to the region of space-time in which the observable particle can move.

To start with we turn to the Kähler Ricci-flat 8-metric

$$g_{\alpha\overline{\beta}} = u^m z^\alpha z^{\overline{\beta}} + u' \delta^\alpha_\beta = \frac{a^m}{r^2} (r^m - a^m)^{1-m} z^\alpha z^{\overline{\beta}} + \frac{(r^m - a^m)^{1/m}}{r} \delta^\alpha_\beta, \quad (1)$$

where

$$u' \equiv \frac{du}{dr} = \frac{(r^m - a^m)^{1/m}}{r}, \quad (2)$$

$$u'' \equiv du'/dr, \quad r \equiv \sum_{\alpha=1}^{m} z^\alpha z^{\overline{\alpha}}, \quad m \equiv 4, \quad 0 < a = \text{const.}$$

The metric (1) is determined in [1] for an arbitrary $m = 2n$. It is quite simply to check that for $m = 4$ the metric is hyperKähler.

The first natural way to construct from some Riemannian 4-metric a Lorentz 4-metric is so called ”turn” transformation: $x^4 \rightarrow ix^4$, where in case of 4-metric of type (1) $x^4 = \text{Im} z^2$. But after such turn the signature of $g$ becomes equal 0 or ±4 only. We cannot find Lorentz metric in such a way. By this reason we start from 8-metric (1) which has notable properties: it has holomorphic isometry group equal to SU(4) and since the metric is Kähler and Ricci-flat its holonomy group is subgroup of SU(4).

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Let us come back to the model. The only possible way for time measuring in the model is to use a frame clock (i.e. clock which has been connected with frame body). Therefore such a model must be describe by means of space-time model which become flat Minkovski space in case of gravity absence. The only invariant in Minkovski space is so called interval
\[ r = c^2 t^2 - x^2 - y^2 - z^2. \]

2. "Turn" the gravity on now. The question is how the geometric values of space-time depend on \( r \)? The interval \( r \) must be non-zero since the particle cannot move with speed of light. Let \( a \geq 0 \) be a real number such that \( |r| > a \). Let us find a pseudoRiemann metric in the connected region \( U \) of space-time where \( |x^4 - x^3 - x^2 - x^1| > a, x^4 = ct, x^1 = x, x^2 = y, x^3 = z \).

Using the same approach that in classical two-body problem we can formulate our problem in terms of central field in space-time. This means that all fields in the model must depend on \( r \) only. Moreover, both matter and gravity fields must obey to the Einstein equations.

Since the rest mass of elementary particle is little one the essential gravity impact may be just in case of ultra-relativistic particle velocity, i.e. \( r \) is close to \( a \). In other words, the metric must become Minkovski one when \( r \to \infty \) (or \( a \to 0 \)).

The isometry group of Minkovski metric is Poincare group. But in space-time with central field the translations are prohibited and isometry group become Lorentz group. If we suppose that region \( U \) is oriented that isometry group is SO(3,1).

For an arbitrary Lorentz 4-manifold the structure group of frame bundle is O(3,1). In case of region \( U \) in this manifold has fixed orientation the structure group on the \( U \) is reduced to SO(3,1). In the same time it is well-known that the holonomy group of simply connected manifold is the subgroup of structure group of frame bundle. That is why the restricted holonomy group of \( U \) is subgroup of SO(3,1).

3. Let us consider a complexification \( U_c \) of \( U \), i.e. the set which has the homeomorphsim to \( U \times C \). The complex coordinates \((z^1, \ldots, z^4, \bar{z}^1, \ldots, \bar{z}^4)\) on \( U_c \) be chosen such that \( z^j = x^j + iy^j, \bar{z}^j = x^j - iy^j, j = 1, \ldots, 4 \), where \( x^j \) and \( y^j \) are the real coordinates in \( U \). Then we can construct an analytical continuation of some SO(3,1)-invariant metric \( g \) on \( U \) to the Hermitian metric \( h \) on \( U_c \). The holomorphic isometries group of \( h \) is SU(3,1) and its restricted holonomy group is subgroup of SU(3,1). We can make some kind of "turn" transformation now. Let us put \( z^4 = -z'^4 \) and \( \bar{z}^4 = z'^4 \). Performing such transformation everywhere we find that the new Hermitian metric \( h' \) has SU(4) as holomorphic isometries group and its restricted holonomy group is subgroup of SU(4). We came to the conclusion that \( h' \) has the same properties as [1]. By the Theorem 1 from [1] the \( h' \) is the same as [1].

Making the inverse transformations we can easily find the metric \( g \) on the considered region \( U \) of space-time. This metric has the following form.
\[ g_{ij} = u^{ik} x^k \eta^i_\ell \eta^j_\ell + u^j \eta^i_\ell, \quad i, j = 1, \ldots, 4, \] (3)
where
\[ \eta^i_\ell = \text{diag}(-1, -1, -1, 1), \]
\[ u' = \frac{(r^4 - a^4)^{1/4}}{r}, \quad u'' = du'/dr, \quad r = -(x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2 \]  

(4)

and have signature \((- - + +)\) in case of \(r > a\) or \((+++)\) in case of \(r < -a\).

The following proposition is one of the result of the paper.

**Theorem 1** Let \(U\) be an oriented simply connected region in space-time and there exist some coordinates \((x^1, x^2, x^3, x^4)\) in \(U\) such that the metric \(g\) on \(U\) in these coordinates becomes Minkowski metric (up to sign of signature) when \(|x^1^2 - x^2^2 - x^3^2 - x^4^2| \to \infty\). Then \(g\) has the \((3)\) form. Inverse, let on some region \(U\) of space-time the metric \((3)\) exists. Then \(U\) is oriented and has a central field.

4. The metric \((3)\) is not Ricci-flat. That is way the right side of Einstein equation is non-zero. One can derive that \(T_{ij}\) must be of the next form

\[ T_{ij} = P(r)g_{ij} + Q(r)\eta_{ij}, \]  

(5)

where \(P(r)\) and \(Q(r)\) are determined real functions on \(r\).

Now let us consider for example proton and neutron as particles in abovementioned model and let their spins have inverse directions. Then total spin of the system is zero and total electric charge of the system is non-zero. Therefore the system must be described by complex scalar field. The energy-momentum tensor of complex scalar field \(\phi\) is

\[ T_{ik} = 2\partial_i\phi\partial_k\phi - g_{ik}m^2\phi\phi \]  

(6)

and can be wrote in the form \((3)\), where \(P(r) = 8\phi'\overline{\phi}' + m^2\phi\phi\), \(Q(r) = -8\phi'\overline{\phi}'u'\).

Substitute \((3)\) and \((6)\) into Einstein equations

\[ G_{ij} \equiv r_{ij} - \frac{1}{2}\lambda g_{ij} = \kappa T_{ij}, \]  

(7)

where \(r_{ij}\) is Ricci tensor, \(\lambda\) is scalar curvature and \(\kappa = \frac{8\pi G}{c^4}\). This yields

\[ 8\phi'\overline{\phi}' + m^2\phi\phi u'' = X(r), \]

\[ \phi\phi u'm^2 = Y(r), \]  

(8)

where \(\phi' = \frac{\partial \phi}{\partial r}\). \(X\) and \(Y\) are the following functions

\[ X = \frac{1}{2} \frac{a^4(10r^8 - 7a^4r^4 + 3a^8)}{(r^4 - a^4)^2r^6\kappa}, \]

\[ Y = -\frac{1}{2} \frac{a^4(r^4 - 3a^4)}{(r^4 - a^4)r^5\kappa}. \]

Let

\[ \text{Re} \phi \equiv \mu, \quad \text{Im} \phi \equiv \nu. \]

One can derive from \((8)\) that \(\mu\) and \(\nu\) obey to the following system of equations

\[ \mu'^2 + \nu'^2 = \frac{1}{8}(X - Yu''/u') \equiv A(r), \]
\[ \mu^2 + \nu^2 = \frac{Y}{m^2 d} \equiv B(r). \quad (9) \]

The system has the next general solution.

\[ \mu = \pm F \cos(H + C), \quad \nu = \pm F \sin(H + C), \]

where

\[ F = \sqrt{B}, \quad H = \pm \int \sqrt{4AB - B'^2} \frac{dr}{4B^2}, \]

A and B are determined by (9), C is an arbitrary constant.

The \( \phi \) has the following form

\[ \phi = F(r)e^{i[H(r)+C]}. \quad (10) \]

The formulae (3) and (10) expresses the exact solution of Einstein equation for two-body charged scalar system like for example neutron-proton or neutron-electron.

The geometrical and topological structure of the region \( U \) of space-time and total space-time with considered system is not fully recognized now. But it is clear that none particle can moves in the region where \( |r| \leq a \).

References

[1] Zuev S.V. A 4n-dimensional Kähler Ricci-flat metric, in: Proc. Int. Conf. ”Geometrization of Physics III”, Kazan, 1998. – P. 191.

[2] Zuev S.V. Charged scalar field in curved space-time./ in Recent Problems in Physics. Ed. A.V. Aminova. – Kazan, 1998.