On Germs of Finite Morphisms of Smooth Surfaces

Vik. S. Kulikov

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Abstract—Questions related to deformations of germs of finite morphisms of smooth surfaces are discussed. Four-sheeted finite cover germs $F: (U, o') \rightarrow (V, o)$, where $(U, o')$ and $(V, o)$ are two germs of smooth complex analytic surfaces, are classified up to smooth deformations. The singularity types of branch curves and the local monodromy groups of these germs are also investigated.

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INTRODUCTION

Let $(V, o) = (\mathbb{B}_\varepsilon, o)$ be a ball in $\mathbb{C}^2$ of small radius $\varepsilon > 0$, $(U, o')$ a connected germ of a smooth complex analytic surface, and $F: (U, o') \rightarrow (V, o)$ a germ of a finite holomorphic mapping (below, a finite cover germ) of local degree $\deg o' F = d$, given in local coordinates $z, w$ in $(U, o')$ and $u, v$ in $(V, o)$ by functions

$$u = f_1(z, w), \quad v = f_2(z, w),$$

where $f_i(z, w) \in \mathbb{C}[[z, w]]$ are convergent power series in $U$. Denote by $R \subset (U, o')$ the ramification divisor of $F$ given by the equation

$$J(F) := \det \begin{pmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial w} \end{pmatrix} = 0$$

and by $B = F(R_{\text{red}}) \subset (V, o)$ the germ of the branch curve of $F$.

The germ $F$ defines a homomorphism $F_*: \pi_1(V \setminus B, p) \rightarrow \mathbb{S}_d$ (the monodromy of the germ $F$), where $\mathbb{S}_d$ is the symmetric group acting on the fiber $F^{-1}(p)$. The group $G_F = \text{im} F_*$ is called the (local) monodromy group of $F$. Note that $G_F$ is a transitive subgroup of $\mathbb{S}_d$.

We say that two germs $F_1: (U, o') \rightarrow (V, o)$ and $F_2: (U, o') \rightarrow (V, o)$ are equivalent if they differ from each other by changes of coordinates in $(U, o')$ and $(V, o)$.

The aim of this paper is to investigate the finite cover germs up to deformation equivalence. In short (see Definition 6 in Subsection 1.5), two finite cover germs $F_1$ and $F_2$ of local degree $d$ are deformation equivalent if they can be included in a smooth family of finite cover germs of local degree $d$ that preserves the singularity type of germs of branch curves (see Definition 1 in Subsection 1.1).

The finite cover germs of local degree $\deg o' F = 3$ were investigated in [13]. In the present paper, we investigate the singularity types of germs of branch curves of finite covers of local degree $\deg o' F = 4$, describe their monodromy groups, and give a complete classification of them up to deformation equivalence.

Note that, a priori, the monodromy group $G_F$ of the four-sheeted germ of a finite cover $F$ is one of the following subgroups of $\mathbb{S}_4$: the cyclic group $\mathbb{Z}_4$ generated by a cycle of length 4, the Klein four group $K\mathbb{L}_4$, the dihedral group $\mathbb{D}_4$, the alternating group $\mathbb{A}_4$, and the whole $\mathbb{S}_4$.

Steklov Mathematical Institute of Russian Academy of Sciences, ul. Gubkina 8, Moscow, 119991 Russia.

E-mail address: kulikov@mi-ras.ru
Let $T[f(u, v) = 0] = T(B)$ be the singularity type (see Definition 2 in Subsection 1.1) of a curve germ $B$ given by the equation $f(u, v) = 0$. Below, we will use the following notation:

- $A_n := T[u^2 - u^{n+1} = 0], \ n \geq 0$;
- $D_n := T[(2u - v)(u^2 - v^{n-2}) = 0], \ n \geq 4$;
- $E_6 := T[u^3 - u^4 = 0]$;
- $E_7 := T[v(u^2 - u^3) = 0]$;
- $E_8 := T[u^3 - u^5 = 0]$;
- $T_3(n, \beta) := T[v^3 - u^{3n+\beta} = 0], \ n \geq 0, \ \beta = 1, 2$;
- $T_{3,2k+n,2k} := T[v((v - u^k)^2 - u^{2k+n+1}) = 0], \ k \geq 1, \ n \geq 0$;
- $T_{3,n-1,n} := T[v(u^2 - u^n) = 0], \ n \geq 2$;
- $T_{4,2n_1,2n_2} := [(v^2 - u^{2n_1+1})(u^2 - v^{2n_2+1}) = 0], \ n_1, n_2 \in \mathbb{N}$.

Note that there are several intersections in this notation; for example, $E_6 = T_3(1, 1), E_7 = T_{3,2,3}$, and $E_8 = T_3(1, 2)$. Note also that if $T(B) = T_{3,m,n}$ is the singularity type of a curve germ $B$, then the multiplicity of the singularity of $B$ at the point $o$ is 3 and $B$ splits into two germs, $B = B_1 \cup B_2$, where $B_1$ is a germ of a nonsingular curve, $T(B_1) = T[v = 0] = A_0$, $T(B_2) = A_m$, and the local intersection number $(B_1, B_2)_o$ of the germs $B_1$ and $B_2$ at $o$ is $n$. Similarly, $T(B) = T_{4,2n_1,2n_2}$ means that the multiplicity of the singularity of $B$ at the point $o$ is 4 and $B$ splits into two germs, $B = B_1 \cup B_2$, with $T(B_1) = A_{2n_1}$, $T(B_2) = A_{2n_2}$, and $(B_1, B_2)_o = 4$.

The main result of the paper is the following theorem in which we give a complete classification of four-sheeted finite cover germs up to deformation equivalence and compute their main invariants: the singularity types of branch curves and their monodromy groups.

**Theorem 1.** A finite cover germ $F : (U,o) \rightarrow (V,o), \ deg_o F = 4$, given by functions $u = f_1(z,w)$ and $v = f_2(z,w)$ is either equivalent to one of the germs $F_i = \{u = f_{i,1}(z,w), \ v = f_{i,2}(z,w)\}$ of finite covers listed in Table 1 or deformation equivalent to one of the finite cover germs listed in Table 2. In case $i = 4_2$, any two finite cover germs that are deformation equivalent to the germ $F_{4_2,0_1}$ are equivalent.

**Corollary 1.** The set of monodromy groups $G_F$ of the four-sheeted finite cover germs is the set $\{Z_4, K_{14}, D_4, S_4\}$. In particular, the alternating group $A_4$ is not the monodromy group of any four-sheeted finite cover germ.

Moreover, in Proposition 12 below we prove that the alternating group $A_4$ is not the monodromy group of any finite cover germ. Nevertheless (see Proposition 13), the groups $A_{2n-1}$ for $n \geq 2$ are the monodromy groups of finite cover germs.

Note that all possible deformation types of four-sheeted finite cover germs described in Theorem 1, except for two germs ($F_{4_2,k,m}$ and $F_{4_0,3k,m}$), have different pairs of main invariants. The monodromy groups of the germs $F_{4_0,3k,m}$ and $F_{4_0,3k,m}$ are the same, and the singularity types of their branch curves are also the same, but the deformation types of these covers differ in the singularity types of their ramification divisors. Therefore, the following problem is interesting.

**Table 1**

| $i$   | $F_i$                                           | $T(B_i)$  | $G_{F_i}$ |
|-------|-------------------------------------------------|-----------|-----------|
| 1     | $F_1, = \{u = z, v = w^4\}$                     | $A_0$     | $Z_4$     |
| 2     | $F_2, = \{u = z^2, v = w^2\}$                   | $A_1$     | $K_{14}$  |
| 3     | $F_{3,n} = \{u = z, v = w^4 - 2zw^2\}$          | $A_{4n-1}, n \geq 1$ | $D_4$     |
| 3     | $F_{3,n} = \{u = z^2, v = w^2 - z^{2n+1}\}$     | $D_{2n+3}, n \geq 1$ | $D_4$     |
| 4     | $F_{4,n} = \{u = z, v = w^4 + w^{3n}\}$         | $A_{8n-1}, n \geq 1$ | $S_4$     |
The vertices B transversally for every B resolves the singular point o Denote by finite cover germs of local degree d of finite cover germs, the singularity types of their branch curves, and the monodromy groups of classes of the images of geometric generators of the local fundamental groups of their branch curves are also the same.

In Section 1, we present the main definitions and prove several auxiliary lemmas. In the same section, we also consider different aspects of the relationship between the deformation equivalence of finite cover germs, the singularity types of their branch curves, and the monodromy groups of these germs. The proof of Theorem 1 is given in Section 2. In Section 3, we give examples of finite groups that are not the monodromy groups of any finite cover germs.

1. DEFINITIONS AND PRELIMINARY FACTS

1.1. Equisingular deformations of curve germs. Let

\[ \mathbb{B}_r = \{(u, v) \in \mathbb{C}^2 : |u|^2 + |v|^2 < r^2\} \]

be a ball of radius r centered at the point o and C \( \subset \mathbb{B}_r \) be a reduced curve given by an equation h(u, v) = 0, where h(u, v) is a power series converging in \( \mathbb{B}_r \) with h(0, 0) = 0. In addition, we will assume (see [14, Corollary 2.9]) that the curve C and the boundary \( \partial \mathbb{B}_{r_1} \) of the ball \( \mathbb{B}_{r_1} \subset \mathbb{B}_r \) meet transversally for every \( r_1 \leq r \), and we will say that B is a curve germ in (V, o) if V = \( \mathbb{B}_{r', \varepsilon} \), \( \varepsilon \ll r \), and B = C \( \cap \) V.

Let B_1, \ldots, B_k be the irreducible components of a curve germ B \( \subset V \) and \( \sigma : V_n \to V \), \( \sigma = \sigma_1 \circ \ldots \circ \sigma_n \), be the minimal sequence of \( \sigma \)-processes \( \sigma_i : V_i \to V_{i-1} \) with centers at points that resolves the singular point o of B and is such that \( \sigma^{-1}(B) \) is a divisor with normal crossings. Denote by \( E_{i+k} \subset V_n \) the proper inverse image of the exceptional curve of \( \sigma_i \) and by \( B'_j \subset V_n \) the proper inverse image of \( B_j \).

By definition, the graph \( \Gamma(B) \) of the curve germ B is a weighted graph with n + m vertices \( v_i \). The vertices \( v_i := b_i, i = 1, \ldots, k \), are in one-to-one correspondence with the curves \( B'_1, \ldots, B'_m \), and...
they have weights \( w_i = 0 \). The vertices \( v_{i+k} := e_{i+m}, i = 1, \ldots, n \), are in one-to-one correspondence with the curves \( E_{1+k}, \ldots, E_{n+k} \), and they have weights \( w_{i+k} = (E^2_{i+k})v_e \). Vertices \( v_i \) and \( v_j \) are connected by an edge of \( \Gamma(B) \) if and only if the corresponding curves have a nonempty intersection.

By definition, a family of curve germs is a triple \((\mathcal{V} = V \times D_\delta, \mathcal{B}, \text{pr}_2)\), where \( V = \mathbb{B}_r \subset \mathbb{C}^2 \) is a ball, \( D_\delta = \{t \in \mathbb{C} : |t| < \delta\} \) is a disk in the complex plane, \( \mathcal{B} \) is an effective reduced divisor in \( \mathcal{V} \) given by an equation \( h_r(u,v) = 0 \) with a power series \( h_r(u,v) \in \mathbb{C}\{[r]\}[[u,v]] \) that converges in \( \mathcal{V} \), and the restriction to \( \mathcal{B} \) of the projection \( \text{pr}_2 : \mathcal{V} \to D_\delta \) is a flat holomorphic mapping.

**Definition 1** [19] (see also [18]). A family \((\mathcal{V}, \mathcal{B}, \text{pr}_2)\) is an equisingular deformation of curve germs if \( \text{Sing} \mathcal{B} = \{0\} \times D_\delta \) and there exists a finite sequence of \( n \) monoidal transformations \((\text{blowups}) \tilde{\sigma}_i : \mathcal{V}_i \to \mathcal{V}_{i-1} \) (where \( \mathcal{V}_0 = \mathcal{V} \)) with centers in smooth curves \( S_i \subset \text{Sing} \mathcal{B}_{i-1} \), where \( \mathcal{B}_0 = \mathcal{B} \) and \( \mathcal{B}_{i+1} = \tilde{\sigma}_i^{-1}(\mathcal{B}_i) \), such that

(i) \( \text{Sing} \mathcal{B}_i \) is a disjoint union of sections of \( \text{pr}_2 \circ \tilde{\sigma}_1 \circ \ldots \circ \tilde{\sigma}_{i-1} \) for each \( i \);

(ii) \( \mathcal{B}_n \) is a divisor with normal crossings in \( \mathcal{V}_n \).

Note that in this case the divisor \( \mathcal{B}_n \) has only simple double points as its singular points.

**Definition 2.** We say that two curve germs are equisingular equivalent if they can be embedded in an equisingular deformation of curve germs as fibers of \( \text{pr}_2 \). Let us extend the equisingular equivalence to an equivalence relation and say that two curve germs have the same singularity type if they are equisingular equivalent.

The following two propositions are well known.

**Proposition 1** [18]. Two curve germs \((B_1, o)\) and \((B_2, o)\) are equisingular equivalent if and only if their graphs \( \Gamma(B_1) \) and \( \Gamma(B_2) \) are isomorphic as weighted graphs.

**Definition 3.** We say that a curve germ \((B, o) \subset (V, o)\) is rigid if for any curve germ \((B_1, o) \subset (V, o)\) that is equisingular equivalent to \((B, o)\), there is a biholomorphic mapping \( G : (V, o) \to (V, o) \) such that \( G(B_1) = B \).

**Proposition 2** [1]. A curve germ \((B, o)\) is rigid if and only if it has one of the following singularity types: \( A_n, D_n, E_6, E_7, \) or \( E_8 \).

The embedding \( \mathbb{B}_{r_2} \subset \mathbb{B}_{r_1} \subset \mathbb{B}_r \) of balls, \( r_2 \leq r_1 \leq r \), induces a homomorphism of the fundamental groups, \( i_* : \pi_1(\mathbb{B}_{r_2} \setminus B) \to \pi_1(\mathbb{B}_{r_1} \setminus B) \). The following theorem is well known (see, for example, [5]).

**Theorem 2.** There exists a radius \( r(B) \) such that for all \( \varepsilon \leq r(B) \) the homomorphism \( i_* : \pi_1(\mathbb{B}_\varepsilon \setminus B) \to \pi_1(\mathbb{B}_{r(B)} \setminus B) \) induced by the embedding of balls is an isomorphism.

The group \( \pi_1^{oc}(B, o) := \pi_1(\mathbb{B}_{r(B)} \setminus B) \) is called the local fundamental group of \( B \).

**Definition 4.** We say that an equisingular deformation \((\mathbb{B}_\varepsilon \times D_\delta, \mathcal{B}, \text{pr}_2)\) is strong if \( \varepsilon < r(B_\tau) \) for all \( \tau \in D_\delta \).

Let \( t : [0, 1] = \{0 \leq t \leq 1\} \to D_\delta \) be a smooth path in \( D_\delta \) and \((\mathbb{B}_\varepsilon \times D_\delta, \mathcal{B}, \text{pr}_2)\) a strong equisingular deformation. In this case it is easy to show that

\[
\text{pr}_2 : (\mathbb{B}_\varepsilon \times D_\delta \setminus \mathcal{B}) \times D_\delta [0, 1] \to [0, 1] \tag{1.1}
\]

is a \( C^{\infty}\)-trivial fibration with fiber \( \mathbb{B}_\varepsilon \setminus B_{1(0)} \); in particular, the corresponding fundamental groups \( \pi_1((\mathbb{B}_\varepsilon \times D_\delta \setminus \mathcal{B}) \times D_\delta [0, 1], (p_t, l(t))) \) and \( \pi_1(\mathbb{B}_\varepsilon \setminus B_{l(t)}, p_t) \) are naturally isomorphic for all \( t \in [0, 1] \) and \( p_t \in \mathbb{B}_\varepsilon \setminus B_{l(t)} \).

Below, for every smooth path \( l \) in \( D_\delta \), we fix one of the \( C^{\infty}\)-trivializations of the fibration (1.1) and call it an equipment of \( l \).
1.2. D-automorphisms. In the infinite-dimensional affine space of power series $\mathbb{C}[[u,v]]$, denote by $\mathfrak{B}_T$ the subset consisting of all power series $h(u,v)$ with a nonzero radius of convergence such that the germs of curves given by the equations $h(u,v) = 0$ in $\mathbb{B}_{r(B_h)}$ have the same type of singularity $T$ (i.e., all these germs are equisingular deformation equivalent). Consider the set

$$\mathfrak{B}_T = \{h = (h, \varepsilon, p) \in \mathfrak{B}_T \times \mathbb{R} \times \mathbb{C}^2 : h \in \mathfrak{B}_T, \varepsilon < r(B_h), p \in \mathbb{B}_\varepsilon \setminus B_h\}$$

and define in this set smooth paths $l : [0,1] = \{0 \leq t \leq 1\} \rightarrow \mathfrak{B}_T$ of three types (below, elementary admissible paths; the smoothness of a path $l = (h_t, \varepsilon_t, p_t)$ means that the coefficients of the series $h_t$ and the points $\varepsilon_t$ and $p_t$ depend smoothly on $t$). The first type consists of the paths $l(t) = (h_t, \varepsilon_t, p_t)$ with $h_t = h_0$ and $p_t = p_0$ for all $t \in [0,1]$. The second type consists of the paths $l(t) = (h_t, \varepsilon_t, p_t)$ with $h_t = h_0$ and $\varepsilon_t = \varepsilon_0$ for all $t \in [0,1]$. We call the paths of the first and second types 0-paths. The third type consists of the paths (we call them d-paths) $l(t) = (h_t, \varepsilon_t, p_t)$ for which

(i) $\varepsilon_t = \varepsilon_0$ for all $t \in [0,1]$;

(ii) there exists a smooth path $\tilde{l} : [0,1] \rightarrow D_\delta$ and a strong equisingular deformation $(\mathbb{B}_\varepsilon \times D_\delta, B, \text{pr}_2)$ of the curve germ $B_{h_0}$ such that $B_{h_t} = B \cap \text{pr}_2^{-1}(\tilde{l}(t))$ for all $t \in [0,1]$ and $\{(p_t, t) : t \in [0,1]\}$ is a constant section of the equipment of $\tilde{l}$, i.e., the points $p_t$ do not depend on $t$.

It is obvious that every elementary admissible path $l$ defines an isomorphism from the group $\pi_1(\mathfrak{B}_{t(0)} \setminus B_{h(0)}, p_{(0)})$ to the group $\pi_1(\mathfrak{B}_{t(1)} \setminus B_{h(1)}, p_{(1)})$.

An admissible path $l$ in $\mathfrak{B}_T$ is a finite sequence of elementary admissible paths $(l_1, \ldots, l_n)$ in which the end of the path $l_i$ coincides with the beginning of the path $l_{i+1}$ for $1 \leq i \leq n - 1$.

We fix a point $h_0 = (h_0(u,v), \varepsilon_0, p_0) \in \mathfrak{B}_T$ and call it (and accordingly the curve germ $(B_{h_0}, o)$ given by $h_0(u,v) = 0$) a base representative of the singularity type $T$. Denote by $\Omega_T(h_0)$ the space of all admissible loops in $\mathfrak{B}_T$ that begin at the point $h_0$. Obviously, $\Omega_T(h_0)$ is a semigroup and a natural homomorphism

$$\text{Def} : \Omega_T(h_0) \rightarrow \text{Aut}(\pi_1(\mathfrak{B}_{\varepsilon_0} \setminus B_{h_0}, p_0)) = \text{Aut}(\pi_1^{\text{loc}}(B_{h_0}, o))$$

is well defined. It is easy to see that the image

$$\mathfrak{D}_T := \text{Def}(\Omega_T(h_0))$$

is a group, which we will call the D-automorphism group. Note that $\mathfrak{D}_T$ contains the group of inner automorphisms of the group $\pi_1^{\text{loc}}(B_{h_0}, o)$.

1.3. On the fundamental groups of the complements of curve germs. The Zariski–van Kampen theorem (see below) provides an approach to finding representations of the local fundamental groups of curve germs. It is based on calculating the braid monodromy of the singularity and consists in the following (for a more detailed description, see, for example, [8]). Let a germ $(B, o) \subset (V, o)$ of a reduced holomorphic curve do not contain the germ $u = 0$ and $m$ be the multiplicity of the singularity of $B$ at the point $o$. Assume also that it is given by an equation

$$v^m + \sum_{i=1}^{m} q_i(u)v^{m-i} = 0,$$

where the coefficients $q_i(u)$ are convergent power series in $V = \mathbb{B}_\varepsilon$, $q_i(0) = 0$, and the polynomial $v^m + \sum q_i(u)v^{m-i} \in \mathbb{C}[[u]][v]$ has no multiple factors. Therefore, one can choose a small bidisk $D = D_1 \times D_2 \subset \mathbb{B}_\varepsilon$,

$$D_1 = D_1(\varepsilon_1) = \{u \in \mathbb{C} : |u| \leq \varepsilon_1\}, \quad D_2 = D_2(\varepsilon_2) = \{v \in \mathbb{C} : |v| \leq \varepsilon_2\},$$

where $\varepsilon_1$ and $\varepsilon_2$ are small enough numbers.
such that

(P1) the projection \( pr = pr_1 : B \cap \bar{D} \to \bar{D} \) onto the \( u \)-factor is a proper finite map of degree \( m \);

(P2) \( |v| < \varepsilon_2 \) for every point \((u,v) \in \text{pr}^{-1}(\bar{D}_1)\), and \( o = (0,0) \) is a unique critical point of \( \text{pr} |_{B \cap \bar{D}} \).

Let us choose a point \((p_1,p_2) \in \partial D_1 \times \partial D_2\), \( p_1 = \varepsilon_1, p_2 = e^{3\pi i/2}\varepsilon_2\), and set

\[
K_B := K_B(\varepsilon_1) = \text{pr}_2(\text{pr}^{-1}(p_1)) = \{v_1, \ldots, v_m\} \subset \bar{D}_2.
\]

The fundamental group \( \pi_1(\bar{D}_2 \setminus K_B, p_2) \simeq \mathbb{F}_m \) is a free group of rank \( m \) and is generated by \( m \) loops \( \gamma_1, \ldots, \gamma_m \) around the points \( v_1, \ldots, v_m \).

**Definition 5.** An ordered set \( \{\gamma_1, \ldots, \gamma_m\} \) is called a *good geometric base* of \( \pi_1(\bar{D}_2 \setminus K_B, p_2) \) if the product \( \gamma_1 \cdot \ldots \cdot \gamma_m \) is equal to the element in \( \pi_1(\bar{D}_2 \setminus K_B, p_2) \) represented by the circuit along the circle \( \partial D_2 \) in the positive direction.

Note that \( \text{pr}^{-1}(\varepsilon_1 t), 0 < t \leq 1, \) is a disjoint union of \( m \) paths. Therefore, hereafter, we can identify the groups \( \pi_1(\bar{D}_2 \setminus K_B(\varepsilon_1 t), p_2) \) for different \( t \), and if we need to decrease \( \varepsilon_2 \), we will identify the groups \( \pi_1(\bar{D}_2 \setminus K_B, p_2) \) using the path \( v(t) = e^{3\pi i/2} \varepsilon_2 t \).

An example of a good geometric base of \( \pi_1(\bar{D}_2 \setminus K_B, p_2) \), where \( B \) is given by the equation \( v((v - ku^k)^2 - u^{2k+n}) = 0 \), is shown in Fig. 1, in which

\[
\begin{align*}
v_1 &= k\varepsilon_1^k + \sqrt{\varepsilon_1^{2k+n}}, & v_2 &= k\varepsilon_1^k - \sqrt{\varepsilon_1^{2k+n}}, & v_3 &= 0 & \text{if } k \geq 1, \\
v_1 &= \sqrt{\varepsilon_1^1}, & v_2 &= 0, & v_3 &= -\sqrt{\varepsilon_1^1} & \text{if } k = 0.
\end{align*}
\]

To recall the definition of the braid monodromy \( b_{(B,o)} \) of \( B \), consider the braid group \( \text{Br}_m \). The group \( \text{Br}_m \) has the following presentation. It is generated by elements \( a_1, \ldots, a_{m-1} \) subject to the relations

\[
a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1} \quad \text{for} \quad 1 \leq i \leq m-1, \quad a_ia_k = a_ka_i \quad \text{for} \quad |i-k| \geq 2 \quad \text{(1.3)}
\]

(such generators of the braid group \( \text{Br}_m \) are called *Artin’s* or *standard* generators). The element \( \Delta_m = (a_1a_2\ldots a_{m-1})^m \) is called the *full twist* and belongs to the center of \( \text{Br}_m \).

The counterclockwise oriented loop \( \partial \bar{D}_1 \) starting at \( p_1 \) lifts, via \( \text{pr}^{-1}(\partial \bar{D}_1) \cap B \), to a motion \( \text{pr}_2(\{v_1(t), \ldots, v_m(t)\}) \) of \( m \) distinct points in \( \bar{D}_2 \) that starts and ends at \( K_B \). This motion defines a braid \( b_{(B,o)} \in \text{Br}_m \), which is called the *braid monodromy* of \( (B,o) \) with respect to \( \text{pr} \).

**Lemma 1** [8, Lemma 4.1]. If the germ \( (B,o) \) is given by the equation \( v^2 - u^{n+1} = 0 \) (the singularity type of \( (B,o) \) is \( A_n \)), then \( b_{(B,o)} = a_{n+1}^1 \).

**Lemma 2.** If the germ \( (B,o) \) is given by the equation \( v((v - u^2)^2 - u^{4k+n+1}) = 0 \) (the singularity type of \( (B,o) \) is \( T_{3,4k+n+4k} \)), then \( b_{(B,o)} = \Delta_3^2 a_{n+1}^1 \).

**Proof.** The braid \( b_{(B,o)} \) consists of the three strands

\[
\begin{align*}
v_1(t) &= \varepsilon_2^k e^{4k\pi t} + \sqrt{\varepsilon_1^{2k+n+1}e^{4k+n+1\pi i}}t, \\
v_2(t) &= \varepsilon_2^k e^{4k\pi i} - \sqrt{\varepsilon_1^{2k+n+1}e^{4k+n+1\pi i}}t, & v_3(t) &= 0
\end{align*}
\]
in the space $\mathcal{D}_2 \times \{0 \leq t \leq 1\}$ that start at the points of $K_B \times \{0\} = \{v_1, v_2, v_3\} \times \{0\}$; the strands $v_1(t)$ and $v_2(t)$ make in parallel $2k$ full twists around the strand $v_3(t)$ and make full twists around each other $2k + (n + 1)/2$ times. It is easy to see that the braid consisting of one parallel turn of $v_1(t)$ and $v_2(t)$ around $v_3(t)$ is equal to $a_2 a_1^2 a_2$, and one half-twist of $v_1(t)$ and $v_2(t)$ is the braid $a_1$. It is easy to check that $a_2 a_1^2 a_2$ and $a_1$ commute in $Br_3$ and $(a_2 a_1^2 a_2) a_1^2 = \Delta_3$. □

**Lemma 3** [8, Lemma 4.1]. If the germ $(B,o)$ is given by the equation $v(v^2 - u^n) = 0$ (the singularity type of $(B,o)$ is $T_{3,n-1,n}$), then $b_{(B,o)} = (a_1 a_2 a_1)^n$.

**Proof.** The braid $b_{(B,o)}$ consists of the three strands

\[ v_1(t) = \sqrt{e^{\pi n t}}, \quad v_2(t) = 0, \quad v_3(t) = -\sqrt{e^{\pi n t}} \]

in the space $\mathcal{D}_2 \times \{0 \leq t \leq 1\}$ that start at the points of $K_B \times \{0\} = \{v_1, v_2, v_3\} \times \{0\}$. The strands $v_1(t)$ and $v_3(t)$ make $n$ half-twists around the strand $v_2(t)$. It is easy to see that the braid consisting of one half-twist of $v_1(t)$ and $v_3(t)$ around $v_2(t)$ is equal to $a_1 a_2 a_1 = a_2 a_1 a_2$. □

**Lemma 4.** If the germ $(B,o)$ is given by the equation $(2u - v)(v^2 - u^n-2) = 0$ (the singularity type of $(B,o)$ is $D_n$, $n \geq 4$), then $b_{(B,o)} = \Delta_3 a_2^{n-4}$.

**Proof.** The braid $b_{(B,o)}$ consists of the three strands

\[ v_1(t) = 2e^{2\pi n t}, \quad v_2(t) = \sqrt{e^{\pi n -2}} e^{\pi n t}, \quad v_3(t) = -\sqrt{e^{\pi n-2}} e^{\pi n t}. \]

The first of them makes a full twist around the other two strands, and the last two strands are twisted together $n/2 - 1$ times. □

**Lemma 5** [8, Lemma 4.1]. If the germ $(B,o)$ is given by the equation $v^3 - u^{3n+\beta} = 0$ (the singularity type of $(B,o)$ is $T_3(n, \beta)$), then $b_{(B,o)} = \Delta_3^3 (a_1 a_2)^3$.

**Proof.** The proof is similar. □

Fix a good geometric base $\gamma_1, \ldots, \gamma_m$ of $\pi_1(\mathcal{D}_2 \setminus K_B, p_2)$. The braid group $Br_m$ acts from the right on the free group $\pi_1(\mathcal{D}_2 \setminus K_B, p_2) \simeq F_m$ as follows:

\[ (\gamma_j) a_i = \gamma_j \quad \text{if} \quad j \neq i, i + 1, \quad (\gamma_i) a_i = \gamma_i \gamma_{i+1}^{-1} \gamma_i^{-1}, \quad (\gamma_{i+1}) a_i = \gamma_i. \]

In particular,

\[ (\gamma_i) \Delta_m = (\gamma_1 \ldots \gamma_m) \gamma_i (\gamma_1 \ldots \gamma_m)^{-1} \quad \text{for} \quad i = 1, \ldots, m. \]

The embedding $i: \{p_1\} \times \mathcal{D}_2 \hookrightarrow \mathcal{D}_1 \times \mathcal{D}_2 \hookrightarrow B_\varepsilon$ defines a homomorphism

\[ i_*: \pi_1(\mathcal{D}_2 \setminus K_B, p_2) \to \pi_1(B_\varepsilon \setminus B, p) \simeq \pi^{\text{loc}}_1(B,o). \]

**Zariski–van Kampen theorem.** The homomorphism $i_*: \pi_1(\mathcal{D}_2 \setminus K_B, p_2) \to \pi^{\text{loc}}_1(B,o)$ is an epimorphism. The group $\pi^{\text{loc}}_1(B,o)$ has the following presentation:

\[ \pi^{\text{loc}}_1(B,o) = \langle \gamma_1, \ldots, \gamma_m \mid \gamma_i = (\gamma_i) b_{(B,o)}, \quad i = 1, \ldots, m \rangle, \quad (1.5) \]

where $\gamma_1, \ldots, \gamma_m$ is a good geometric base of $\pi_1(\mathcal{D}_2 \setminus K_B, p_2)$ and $b_{(B,o)}$ is the braid monodromy of $(B,o)$.

The following two lemmas are well known and are easy consequences of Lemmas 1 and 5.

**Lemma 6.** Let $(B,o)$ have a singularity of type $A_n$, where $n = 2k - \delta$ and $\delta = 0$ or 1. Then

\[ \pi^{\text{loc}}_1(B,o) = \langle \gamma_1, \gamma_2 \mid (\gamma_1 \gamma_2)^k \gamma_1^{1-\delta} = (\gamma_2 \gamma_1)^k \gamma_2^{1-\delta} \rangle. \]

(1.6)
Lemma 7. Let \((B_n, \beta, o)\) have a singularity of type \(T_3(n, \beta)\). If \(\beta = 1\), then
\[
\pi^\text{loc}_1(B_n, 1, o) = \{ \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1 = (\gamma_1 \gamma_2 \gamma_3)^{n+1} \gamma_3 (\gamma_1 \gamma_2 \gamma_3)^{-(n+1)}, \\
\gamma_i = (\gamma_1 \gamma_2 \gamma_3)^n \gamma_{i-1} (\gamma_1 \gamma_2 \gamma_3)^{-n}, \ i = 2, 3 \},
\]
and if \(\beta = 2\), then
\[
\pi^\text{loc}_1(B_n, 2, o) = \{ \gamma_1, \gamma_2, \gamma_3 \mid \gamma_i = (\gamma_1 \gamma_2 \gamma_3)^{n+1} \gamma_{i+1} (\gamma_1 \gamma_2 \gamma_3)^{-(n+1)}, \ i = 1, 2, \\
\gamma_3 = (\gamma_1 \gamma_2 \gamma_3)^n \gamma_1 (\gamma_1 \gamma_2 \gamma_3)^{-n} \}.
\]

1.4. WD-subgroups of D-automorphism groups. In this subsection, we use the notation and agreements of the previous one. Denote by \(\mathcal{W}_T\) a subset of \(\mathfrak{B}_T\) consisting of the power series of the form (1.2) and, accordingly,
\[
\mathcal{W}_T = \{ \bar{h} = (h, \varepsilon, (\varepsilon_1, e^{3\pi i/2} \varepsilon_2)) \in \mathfrak{B}_T : h \in \mathcal{W}_T, \ \varepsilon_1, \varepsilon_2 \in \mathbb{R}_+ \},
\]
where the numbers \(\varepsilon_1\) and \(\varepsilon_2\) are such that \(h(u, v)\) has properties (P1) and (P2) in the bidisk \(\mathcal{D} = \mathcal{D}_1(\varepsilon_1) \times \mathcal{D}_2(\varepsilon_2) \subset \mathbb{B}_e\).

A d-path \(l = (h_t, \varepsilon, (\varepsilon_1, e^{3\pi i/2} \varepsilon_2))\) in \(\mathcal{W}_T\) is called a wd-path if the restriction of the equipment of \(l\) to \(\{\partial_2 \mathcal{D} \times \{(t)\} : t \in [0, 1]\}\) is a trivial fibration \(\text{pr}_2 : \partial_2 \mathcal{D} \times [0, 1] \to [0, 1]\), where \(\partial_2 \mathcal{D} = \mathcal{D}_1(\varepsilon_1) \times \partial \mathcal{D}_2(\varepsilon_2)\).

A loop \(l = (l_1, l_1', \ldots, l_k, l_k')\) in \(\mathcal{W}_T\), where \(l_1', \ldots, l_k'\) are 0-paths and \(l_1, \ldots, l_k\) are wd-paths, is called a w-loop.

Fix a point \(\bar{h}_0 = (h_0(u, v), \varepsilon, (\varepsilon_1, e^{3\pi i/2} \varepsilon_2))\) in \(\mathcal{W}_T \subset \mathfrak{B}_T\) as a base representative of the singularity type \(T\) and denote by \(\Omega_{T, \mathcal{W}}(\bar{h}_0)\) the subsemigroup of \(\Omega_{T}(\bar{h}_0)\) generated by the w-loops in \(\mathfrak{B}_T\) that begin at the point \(\bar{h}_0\).

Fix a good geometric base of \(\pi_1(\mathcal{D}_2(\varepsilon_2) \setminus K_{B_{h_0}}, e^{3\pi i/2} \varepsilon_2)\). Due to the identification of the groups \(\pi_1(\mathcal{D}_2(\lambda_2 \varepsilon_2) \setminus K_{B_{h_0}}(\lambda_1 \varepsilon_1), \lambda_2 e^{3\pi i/2} \varepsilon_2)\) for all \(\lambda_1, \lambda_2 \in (0, 1]\) as defined in Subsection 1.3, the motions along w-loops define a homomorphism
\[
\text{def} : \Omega_{T, \mathcal{W}}(\bar{h}_0) \to \text{Br}_m \subset \text{Aut}(\pi_1(\mathcal{D}_2(\varepsilon_2) \setminus K_{B_{h_0}}, e^{3\pi i/2} \varepsilon_2))
\]
such that
\[
(i_*(\gamma))\text{Def}(l) = i_*(\gamma)\text{def}(l)
\]
for all \(l \in \Omega_{T, \mathcal{W}}(\bar{h}_0)\) and \(\gamma \in \pi_1(\mathcal{D}_2 \setminus K_{B_{h_0}}, p_2)\), where
\[
i_* : \pi_1(\mathcal{D}_2 \setminus K_{B_{h_0}}, p_2) \to \pi_1(\mathbb{B}_e \setminus B_{h_0}, p)
\]
is a homomorphism induced by the embedding \(i : \mathcal{D}_2 \setminus K_{B_{h_0}} \hookrightarrow \mathbb{B}_e \setminus B_{h_0}\). The images
\[
\mathcal{D}_{T, \mathcal{W}} := \text{def}(\Omega_{T, \mathcal{W}}(\bar{h}_0)) \quad \text{and} \quad \mathcal{D}_{T, \mathcal{W}} := \text{Def}(\Omega_{T, \mathcal{W}}(\bar{h}_0)) \subset \mathcal{D}_T
\]
are called the WD-automorphism groups (or the groups of Weierstrass D-automorphisms).

Lemma 8. If \(v((v - u^k)^2 - u^{2k+n+1}) = 0\) is the equation of \((B_{h_0}, o)\), \(k \geq 1, n \geq 0\), then the braids \(a_1\) and \(a_2 u_1^2 a_2\) are contained in \(\mathcal{D}_{T_3, 2k+n, 4k, \mathcal{W}} \subset \text{Br}_3\).

Proof. It is easy to see that the loop \(l = \{h_t : 0 \leq t \leq 1\}\) given by
\[
v[(v - u^k)^2 - e^{2\pi i t} u^{2k+n+1}] = 0
\]
in \( \mathfrak{B}_{3,2k+n,2k} \) can be lifted in \( \Omega_{T,3,2k+n,2k} \), and \( \text{def}(\tilde{t}) \) is the standard generator \( a_1 \in Br_3 \). Similarly, the loop \( l = \{t_i : 0 \leq t \leq 1\} \) given by \( v((v - e^{2\pi ti}u^k)^{2} - u^{2k+n+1}) = 0 \) in \( \mathfrak{B}_{3,2k+n,2k} \) gives the element \( a_2a_1^2a_2 \in \mathfrak{B}_{3,2k+n,2k} \).

**Lemma 9.** If \( v(v^2 - u^{2n}) = 0 \) is the equation of \( (B_{h_0}, o) \), then \( \mathfrak{B}_{3,2n-1,2n} = Br_3 \).

**Proof.** It is easy to see that for the lifts \( l_i \) in \( \Omega_{T,3,2n-1,2n} \), their images \( \text{def}(\tilde{t}_i) \) are the standard generators \( a_1 \) and \( a_2 \) of the braid group \( Br_3 \) acting on \( \pi_1(D_2 \setminus K_{B_{h_0}}) \). \( \square \)

### 1.5. Deformation equivalence of germs of covers.

Consider a finite cover germ \( F: (U, o') \rightarrow (V, o) \), \( \deg_{o'} F = d \). Choose local coordinates \( z, w \) in \( (U, o') \) and \( u, v \) in \( (V, o) \). The germ \( F \) is given by two functions

\[
\begin{align*}
u &= f_1(z, w), \\
v &= f_2(z, w),
\end{align*}
\]

where \( f_1(z, w) \in H^0(U, \mathcal{O}_U) \). The ramification divisor \( R \) in \( U \) is defined by the equation

\[
J(F) := \det \left( \begin{array}{ll}
\frac{\partial u}{\partial z} & \frac{\partial u}{\partial w} \\
\frac{\partial v}{\partial z} & \frac{\partial v}{\partial w}
\end{array} \right) = 0,
\]

and let \( B = F(R_{\text{red}}) \subset (V, o) \) be the branch curve of the finite cover germ \( F \).

**Remark 1.** The divisor \( R \subset (U, o') \) and the curve germ \( B \subset (V, o) \) depend only on \( F \) and do not depend on the choice of coordinates in \( (U, o') \) and \( (V, o) \).

**Definition 6.** Let \( \mathcal{U} \) be a complex connected threefold and \( \mathcal{F}: \mathcal{U} \rightarrow (V, o) \times D_\delta \) a finite holomorphic mapping of \( \mathcal{U} \), \( \deg \mathcal{F} = d \), branched along a surface \( B \subset (V, o) \times D_\delta \). Introduce the notation \( U_{\tau_0} = F^{-1}(V \times \{\tau = \tau_0\}) \) and let \( o'_\tau = F^{-1}(o \times \{\tau = \tau_0\}) \) be a single point for every \( \tau_0 \in D_\delta \). The cover \( \mathcal{F} \) is called a **strong deformation** of the finite cover germ \( F_0 = \mathcal{F}_{(U_0, o'_0)}: (U_0, o'_0) \rightarrow (V, o) \times \{0\} \) and the finite cover germs \( F_\tau = \mathcal{F}_{(U_{\tau}, o'_\tau)}: (U_{\tau}, o'_\tau) \rightarrow (V, o) \times \{\tau\} \) are said to be **strong deformation equivalent** to the germ \( F_0 \) if \( ((V, o) \times D_\delta, B, pr_2) \) is a strong equisingular deformation (see Definitions 1 and 4) of the curve germ \( B_0 = B \cap pr_2^{-1}(0) \).

Let us extend the strong deformation equivalence of finite cover germs to an equivalence relation. A finite cover germ \( F: (U, o') \rightarrow (V, o) \) defines a homomorphism

\[
F_* : \pi^1_{1, \text{loc}}(B, o) = \pi_1(V \setminus B, p) \rightarrow S_d,
\]

where \( S_d \) is the symmetric group acting on the fiber \( F^{-1}(p) \). Note that the homomorphism \( F_* \) is defined uniquely only if we fix a numbering of the points of \( F^{-1}(p) \); in the general case it is defined uniquely up to an inner automorphism of \( S_d \). The group \( G_F = \text{im} F_* \) is called the **local monodromy group** of the germ \( F \). The variety \( U \setminus F^{-1}(B) \) is connected, since \( (U, o') \) is an irreducible germ of a smooth surface. Therefore, the group \( G_F \) is a transitive subgroup of \( S_d \). By the Grauert–Remmert–Riemann–Stein theorem (see [4, 17]), the epimorphism \( F_* : \pi_1(V \setminus B) \rightarrow G_F \subset S_d \) uniquely determines the finite cover germ \( F \).

Let \( (V \times D_\delta, B, pr_2) \) be a strong equisingular deformation of a germ \( (B, o) = B \cap pr_2^{-1}(0) \) (see Definition 4). By the Grauert–Remmert–Riemann–Stein theorem, the homomorphism \( F_* : \pi^1_{1, \text{loc}}(B, o) \simeq \pi_1((V \times D_\delta) \setminus B) \rightarrow S_d \) defines a finite \( d \)-sheeted cover \( F: \mathcal{U} \rightarrow V \times D_\delta \), where in the
general case $\mathcal{U}$ is a normal complex analytic variety. However, if $F_*$ is defined by a finite cover germ $F: (U, o') \rightarrow (V, o)$ branched in $(B, o)$, then we have the following theorem.

**Theorem 3.** The cover $F: \mathcal{U} \rightarrow V \times D_\delta$ is a strong deformation of the finite cover germ $F = F|_{F^{-1}((V, o) \times \{0\})}: (U, o') \times \{0\} \rightarrow (V, o) \times \{0\}$.

If $F_1: (U, o') \rightarrow (V, o)$ and $F_2: (U, o') \rightarrow (V, o)$ are strong deformation equivalent germs of finite covers, then their ramification divisors $R_{1, \mathrm{red}}$ and $R_{2, \mathrm{red}}$ are strong equisingular deformation equivalent.

**Proof.** First of all, note that $F^{-1}(V \times \{0\}) = (U, o')$ and $F|_{\mathcal{U}} = F$.

In the notation of Definition 1, the homomorphism $F_*$ defines a finite cover $\tilde{F}: \tilde{\mathcal{U}} \rightarrow \mathcal{V}_n$ branched in $\mathcal{B}_n$. Since $\mathcal{B}_n$ is a divisor with normal crossings and all of its singular points are sections of $\text{pr}_2 \circ \tilde{\sigma}_1 \circ \ldots \circ \tilde{\sigma}_n$, the local fundamental groups of the complement of $\mathcal{B}_n$ at the points of $\mathcal{B}_n$ are abelian and hence, first, $\text{Sing} \tilde{\mathcal{U}}$ is a disjoint union $\bigsqcup \tilde{S}_j$ of sections $\tilde{S}_j$ of $\text{pr}_2 \circ \tilde{\sigma}_1 \circ \ldots \circ \tilde{\sigma}_n \circ \tilde{F}$ lying over $\text{Sing} \mathcal{B}_n$ and, second, at the points of $\tilde{S}_j \subset \text{Sing} \tilde{\mathcal{U}}$ the variety $\tilde{\mathcal{U}}$ is locally biholomorphic to $W_j \times D_\delta$, where $W_j$ is a germ of a two-dimensional cyclic quotient singularity depending on the local monodromy at the points of $\tilde{S}_j \subset \text{Sing} \mathcal{B}_n$ (see [2, Ch. III, Sect. 6] for details). The minimal resolution of singularities $\rho_j: \tilde{W}_j \rightarrow W_j$ defines a resolution of singularities $\rho: \tilde{\mathcal{U}} \rightarrow \tilde{\mathcal{U}}$. Note that $(\tilde{F} \circ \rho)^{-1}(\mathcal{B}_n)$ is a divisor with normal crossings.

By the Stein factorization theorem, there is a holomorphic bimeromorphic mapping $\Sigma: \tilde{\mathcal{U}} \rightarrow \mathcal{U}$ that contracts the divisor $(\tilde{F} \circ \rho)^{-1}(\mathcal{E})$ to the section $F^{-1}(\{o\} \times D_\delta)$ of the projection $\text{pr}_2 \circ F$, where $\mathcal{E}$ is the exceptional divisor of the sequence of monoidal transformations $\tilde{\sigma}_1 \circ \ldots \circ \tilde{\sigma}_n$.

Consider the restriction $\sigma := \Sigma|_{\Sigma^{-1}((U, o'))}: \tilde{\mathcal{U}} = \Sigma^{-1}((U, o')) \rightarrow (U, o')$. The exceptional divisor $E$ of $\sigma$ is $E = \mathcal{E} \cap \tilde{\mathcal{U}}$. By the Zariski theorem, $\sigma$ is a composition of $\sigma$-processes, since $\tilde{\mathcal{U}}$ and $\mathcal{U}$ are nonsingular and $\sigma$ is a bimeromorphic holomorphic mapping. It follows from [6, Ch. 2, Example 6.2.2] and the Nakano contractibility criterion [15] that $\Sigma$ is a composition of monoidal transformations (which are in one-to-one correspondence with the composition of $\sigma$-processes $\sigma$) of smooth threefolds with centers in sections of the projection to $D_\delta$. Therefore, $\tilde{\mathcal{U}}$ is a smooth threefold.

Let $\mathcal{R}$ be the ramification divisor of the cover $F$. Then $R_{0_\mathcal{R}} = \mathcal{R} \cap F^{-1}(\mathcal{B}_n)$ are the ramification divisors of the covers

\[ F_{0_\mathcal{R}} = F|_{F^{-1}(V \times \{\tau = \tau_0\})}: U_{\tau_0} = F^{-1}(V \times \{\tau = \tau_0\}) \rightarrow V \times \{\tau = \tau_0\} \]

for all $\tau_0 \in D_\delta$, where $B_{\mathcal{R}} = \mathcal{B} \cap \text{pr}_2^{-1}(\{\tau = \tau_0\})$. The family $(\mathcal{U}, \mathcal{R}, \text{pr}_2 \circ F)$ is an equisingular deformation of curve germs (see Definition 1), since $\Sigma^{-1}(\mathcal{R}) \subset (\tilde{F} \circ \rho)^{-1}(\mathcal{B}_n)$ is a divisor with normal crossings. \hfill $\square$

Let $\gamma_1, \ldots, \gamma_m$ be a good geometric base of the fundamental group $\pi^\text{loc}_{1}(B, o)$, and let $F_*: \pi^\text{loc}_{1}(B, o) \rightarrow \mathbb{S}_d$ be a homomorphism to the symmetric group $\mathbb{S}_d$ such that $G_F = \text{im} F_*$ is a transitive subgroup of $\mathbb{S}_d$. A collection $\{C_1, \ldots, C_m\}$ of conjugacy classes in $\mathbb{S}_d$ such that $F_*(\gamma_i) \in C_i$ for $i = 1, \ldots, m$ is called a dataset of the homomorphism $F_*$. \vspace{10pt}

**Definition 7.** A homomorphism $F_*$ is said to be sole if it is uniquely defined by its dataset up to inner automorphisms of $\mathbb{S}_d$.

**Definition 8.** We say that two homomorphisms $F_{i*}: \pi^\text{loc}_{1}(B, o) \rightarrow \mathbb{S}_d$, $i = 1, 2$, are equivalent if they differ from each other by an inner automorphism of $\mathbb{S}_d$, and they are deformation equivalent if they differ from each other by a $D$-automorphism in $\mathfrak{D}_T(B)$ and an inner automorphism of $\mathbb{S}_d$. \vspace{10pt}

The following two theorems are simple corollaries of the Grauert–Remmert–Riemann–Stein theorem and Theorem 3.

**Theorem 4.** Two finite cover germs $F_1: (U, o') \rightarrow (V, o)$ and $F_2: (U, o') \rightarrow (V, o)$ of degree $d$, branched along $(B_1, o) \subset (V, o)$ and $(B_2, o) \subset (V, o)$, respectively, are deformation equivalent if and
Consequently, we can assume that the monodromies $F_1: \pi_1 \to S_d$ and $F_2: \pi_1 \to S_d$, where $\pi_1 = \pi_1^{\text{loc}}(B_1, o) = \pi_1^{\text{loc}}(B_2, o)$, are deformation equivalent.

**Theorem 5.** Two deformation equivalent finite cover germs $F_1: (U, o') \to (V, o)$ and $F_2: (U', o') \to (V, o)$ of degree $d$, branched along $(B_1, o) \subset (V, o)$ and $(B_2, o) \subset (V, o)$, respectively, are equivalent if the curve germ $(B_1, o)$ is rigid and the monodromy $F_1: \pi_1^{\text{loc}}(B_1, o) \to S_d$ is sole.

1.6. Examples of homomorphisms to the symmetric group $S_4$. To prove Theorem 1, we will need the following lemmas, the first of which is well-known and will be used in the proofs below without special reference to it.

**Lemma 10.** Let $G$ be a transitive subgroup of the symmetric group $S_d$ generated by transpositions $\tau_1, \ldots, \tau_{d-1}$. Then $G = S_d$ and the product $g = \tau_1 \ldots \tau_{d-1}$ is a cycle of length $d$.

**Lemma 11.** Let $T_{3,8k+2n+4,8k+4}$ be the singularity type of $(B, o)$ and a homomorphism $F_*: \pi_1^{\text{loc}}(B, o) \to S_4$ have the following properties:

(i) the image $F_*(\pi_1^{\text{loc}}(B, o))$ is a transitive subgroup of $S_4$;

(ii) $F_*(\gamma_i) = \tau_i$ are transpositions, where $\gamma_1, \gamma_2, \gamma_3$ is a good geometric base.

Then $F_*$ is sole and $F_*(\pi_1^{\text{loc}}(B_r, o)) = S_4$.

**Proof.** By Lemma 2 and the Zariski–van Kampen theorem, the group $\pi_1^{\text{loc}}(B, o)$ has the following presentation:

$$\pi_1^{\text{loc}}(B, o) = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1 = (\gamma_1 \gamma_2 \gamma_3)^{4k+2} (\gamma_1 \gamma_2)^n \gamma_1 \gamma_2 \gamma_1^{-1} (\gamma_1 \gamma_2)^{-n} (\gamma_1 \gamma_2 \gamma_3)^{(4k+2)},
\gamma_2 = (\gamma_1 \gamma_2 \gamma_3)^{4k+2} (\gamma_1 \gamma_2)^n \gamma_1 \gamma_2 \gamma_1^{-n} (\gamma_1 \gamma_2 \gamma_3)^{-(4k+2)},
\gamma_3 = (\gamma_1 \gamma_2 \gamma_3)^{4k+2} \gamma_3 (\gamma_1 \gamma_2 \gamma_3)^{-(4k+2)} \rangle.$$

(1.9)

Since $F_*(\pi_1^{\text{loc}}(B, o))$ is a transitive subgroup of $S_4$ and $F_*(\gamma_i) = \tau_i$ are transpositions, it follows that $F_*(\pi_1^{\text{loc}}(B, o)) = S_4$ and $\gamma_1 \gamma_2 \gamma_3$ is a cycle of length 4. Therefore, up to conjugation in $S_4$, we can assume that $\gamma_1 \gamma_2 \gamma_3 = (1, 2, 3, 4)$ and $\gamma_3 = (1, 3, 2)$, since $\gamma_3 = (\gamma_1 \gamma_2 \gamma_3)^{4k+2} \gamma_3 (\gamma_1 \gamma_2 \gamma_3)^{-(4k+2)}$. Consequently, $\gamma_1 \gamma_2 = [(1, 2)(3, 4)]$ and we can assume that $\gamma_1 = (1, 2)$ and $\gamma_2 = (3, 4)$ (if $\gamma_1 = (3, 4)$ and $\gamma_2 = (1, 2)$, then we conjugate by $[(1, 3)(2, 4)]$).

**Lemma 12.** Let $(B, o)$ have the singularity type $T_{3,8m+2n,8m}$, and let $F_*: \pi_1^{\text{loc}}(B, o) \to S_4$ be a homomorphism such that

(i) $F_*(\pi_1^{\text{loc}}(B, o))$ is a transitive subgroup of $S_4$;

(ii) $F_*(\gamma_i) = \tau_i$ are transpositions, where $\gamma_1, \gamma_2, \gamma_3$ is a good geometric base.

Then $n = 3k + 1$, $k \in \mathbb{Z}_{\geq 0}$, $F_*(\pi_1^{\text{loc}}(B, o)) = S_4$, and $F_*$ is uniquely defined up to inner automorphisms of $S_4$ and $D$-automorphisms of $\mathcal{D}_{3,8m+6k+2,8m}$.

**Proof.** By Lemma 2 and the Zariski–van Kampen theorem, the group $\pi_1^{\text{loc}}(B, o)$ has the following presentation:

$$\pi_1^{\text{loc}}(B, o) = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1 = (\gamma_1 \gamma_2 \gamma_3)^{4m} (\gamma_1 \gamma_2)^{n} \gamma_1 \gamma_2 \gamma_1^{-1} (\gamma_1 \gamma_2)^{-n} (\gamma_1 \gamma_2 \gamma_3)^{-4m},
\gamma_2 = (\gamma_1 \gamma_2 \gamma_3)^{4m} (\gamma_1 \gamma_2)^{n} \gamma_1 \gamma_2 \gamma_1^{-n} (\gamma_1 \gamma_2 \gamma_3)^{-4m},
\gamma_3 = (\gamma_1 \gamma_2 \gamma_3)^{4m} \gamma_3 (\gamma_1 \gamma_2 \gamma_3)^{-4m} \rangle.$$

(1.10)

Just as in the proof of Lemma 11, since $F_*(\pi_1^{\text{loc}}(B, o))$ is a transitive subgroup of $S_4$ and $F_*(\gamma_i) = \tau_i$ are transpositions, we have $F_*(\pi_1^{\text{loc}}(B, o)) = S_4$, $\gamma_1 \gamma_2 \gamma_3$ is a cycle of length 4, and $\tau_1 \neq \tau_2$. It follows from (1.10) that

$$\tau_1 = (\tau_1 \tau_2)^n \tau_1 \tau_2 \tau_1^{-1} (\tau_1 \tau_2)^{-n}, \quad \tau_2 = (\tau_1 \tau_2)^n \tau_1 (\tau_1 \tau_2)^{-n},$$

(1.11)
and hence \( \tau_1 \) and \( \tau_2 \) do not commute, since otherwise, by (1.11), \( \tau_1 = \tau_2 \). Therefore, up to conjugation in \( S_4 \), we can assume that \( \tau_1 = (1, 2), \tau_2 = (1, 3) \), and \( \tau_1 \tau_2 = (1, 2, 3) \). Now, it is easy to see that these equalities (1.11) hold if and only if \( n = 3k + 1 \).

There are three possibilities for \( \tau_3 \): either \( \tau_3 = (1, 4) \) (denote this homomorphism by \( F_1 \)), or \( \tau_3 = (2, 4) \) (denote this homomorphism by \( F_2 \)), or \( \tau_3 = (3, 4) \) (denote this homomorphism by \( F_3 \)). Note that \( a_1 \in \mathcal{V}_{1,8m+1,8m} \) by Lemma 8. Then the homomorphism \( F_{1*} \) and the homomorphism \( \tilde{F}_{1*} \) sending \( \gamma_1 \) to (1, 3), \( \gamma_2 \) to (2, 3), and \( \gamma_3 \) to (1, 4) differ by the action of the D-automorphism \( i_s(a_1) \). It is easy to check that the homomorphism \( \tilde{F}_{1*} \) coincides with \( F_{2*} \) after the conjugation by (1, 2, 3). Similarly, the homomorphism \( F_{2*} \) and the homomorphism \( \tilde{F}_{2*} \) sending \( \gamma_1 \) to (1, 3), \( \gamma_2 \) to (2, 3), and \( \gamma_3 \) to (2, 4) also differ by the action of the D-automorphism \( i_s(a_1) \). It is easy to check that the homomorphism \( \tilde{F}_{2*} \) coincides with \( F_{3*} \) after the conjugation by (1, 2, 3).

**Lemma 13.** Let \((B, o)\) have the singularity type \( T_{3,8m+2n-1,8m} \), \( m \geq 1, n \geq 1 \), and \( F_*: \pi_1^{\text{loc}}(B, o) \to S_4 \) be a homomorphism such that

(i) \( F_*(\pi_1^{\text{loc}}(B, o)) \) is a transitive subgroup of \( S_4 \);

(ii) \( F_*(\gamma_i) = \tau_i \) are transpositions, where \( \gamma_1, \gamma_2, \gamma_3 \) is a good geometric base.

Then \( F_*(\pi_1^{\text{loc}}(B, o)) = S_4 \) and up to deformation equivalence there exists a unique homomorphism with such properties if \( n = 3k + 1 \) or \( n = 3k + 2, k \in \mathbb{Z}_{\geq 0} \), and there are two homomorphisms satisfying (i) and (ii) if \( n = 3k \).

**Proof.** By Lemma 2 and the Zariski–van Kampen theorem, the group \( \pi_1^{\text{loc}}(B, o) \) has the following presentation:

\[
\pi_1^{\text{loc}}(B, o) = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_i = (\gamma_1 \gamma_2 \gamma_3)^{4m} \gamma_i (\gamma_1 \gamma_2 \gamma_3)^{-n} (\gamma_1 \gamma_2 \gamma_3)^{-4m}, i = 1, 2, \gamma_3 = (\gamma_1 \gamma_2 \gamma_3)^{4m} \gamma_3 (\gamma_1 \gamma_2 \gamma_3)^{-4m} \rangle. \tag{1.12}
\]

Since \( F_*(\pi_1^{\text{loc}}(B, o)) \) is a transitive subgroup of \( S_4 \) and \( F_*(\gamma_i) = \tau_i \) are transpositions, we have \( F_*(\pi_1^{\text{loc}}(B, o)) = S_4 \). Therefore, \( \tau_1 \neq \tau_2 \) and \( \tau_1 \tau_2 \tau_3 \) is a cycle of length 4.

It follows from (1.12) that

\[
\tau_1 = (\tau_1 \tau_2)^n (\tau_1 \tau_2)^{-n}, \quad \tau_2 = (\tau_1 \tau_2)^n (\tau_1 \tau_2)^{-n}. \tag{1.13}
\]

Let \( n = 3k + 1 \) or \( 3k + 2 \). Then it follows from (1.13) that \( \tau_1 \) and \( \tau_2 \) must commute with each other, and hence, up to conjugation, we can assume that \( \tau_1 = (1, 2) \) and \( \tau_2 = (3, 4) \). Then \( \tau_3 \) belongs to the set \( \{(1, 3), (1, 4), (2, 3), (2, 4)\} \). Again, it is easy to see that there is an inner automorphism of \( S_4 \) that leaves \( \tau_1 \) and \( \tau_2 \) fixed and sends \( \tau_3 \) to (1, 3).

Let \( n = 3k \). Up to conjugation in \( S_4 \), we have only two possibilities: either \( \tau_1 = (1, 2) \) and \( \tau_2 = (1, 3) \), or \( \tau_1 = (1, 2) \) and \( \tau_2 = (3, 4) \). The case when \( \tau_1 = (1, 2) \) and \( \tau_2 = (3, 4) \) was considered above, and the case when \( \tau_1 = (1, 2) \) and \( \tau_2 = (1, 3) \) was considered in the proof of Lemma 12.

**Lemma 14.** Let \((B, o)\) have the singularity type \( T_{3,8m-1,8m} \), \( m \geq 1 \), and \( F_*: \pi_1^{\text{loc}}(B, o) \to S_4 \) be a homomorphism such that

(i) \( F_*(\pi_1^{\text{loc}}(B, o)) \) is a transitive subgroup of \( S_4 \);

(ii) \( F_*(\gamma_i) = \tau_i \) are transpositions, where \( \gamma_1, \gamma_2, \gamma_3 \) is a good geometric base.

Then \( F_*(\pi_1^{\text{loc}}(B, o)) = S_4 \) and up to deformation equivalence such a homomorphism \( F_* \) is unique.

**Proof.** By Lemma 3 and the Zariski–van Kampen theorem, the group \( \pi_1^{\text{loc}}(B, o) \) has the following presentation:

\[
\pi_1^{\text{loc}}(B, o) = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_i = (\gamma_1 \gamma_2 \gamma_3)^{4m} \gamma_i (\gamma_1 \gamma_2 \gamma_3)^{-n} (\gamma_1 \gamma_2 \gamma_3)^{-4m}, i = 1, 2, 3 \rangle. \tag{1.14}
\]
Since $F_*(\pi_1^{\text{loc}}(B,o))$ is a transitive subgroup of $S_4$ and $F_*(\gamma_1) = \tau_1$ are transpositions, we have $F_*(\pi_1^{\text{loc}}(B,o)) = S_4$. Therefore, $\tau_1 \tau_2 \tau_3$ is a cycle of length 4 and we can assume that $\tau_1 \tau_2 \tau_3 = (1,2,3,4)$. Since, by definition, the elements of factorization semigroups are finite sequences of group elements considered up to actions on these sequences of braid groups with the number of strands equal to the lengths of the sequences (see, for example, [8–11]), Lemma 14 follows from [9, Theorem 2.1] and Lemma 9. □

**Lemma 15.** Let $A_{4n-1}$ be the singularity type of a curve germ $(B = B_1 \cup B_2, o)$ and a homomorphism $F_* : \pi_1^{\text{loc}}(B,o) \to S_4$ have the following properties:

(i) $F_*(\pi_1^{\text{loc}}(B,o))$ is a transitive group of $S_4$;

(ii) $F_*(\gamma_1) = \tau_1 \tau_2$ is the product of two commuting transpositions $\tau_1$ and $\tau_2$, $\tau_1 \neq \tau_2$, and $F_*(\gamma_2) = \tau_3$ is a transposition, where $\gamma_1$ is a circuit around $B_1$, $\gamma_2$ is a circuit around $B_2$, and $\gamma_1 \gamma_2$ is a good geometric base of $\pi_1^{\text{loc}}(B,o)$.

Then $F_*(\pi_1^{\text{loc}}(B,o)) = \mathbb{D}_4$ is a dihedral subgroup of $S_4$ and $F_*$ is sole.

**Proof.** Up to conjugation in $S_4$, we can assume that $\tau_1 = (1,2)$ and $\tau_2 = (3,4)$. Note that $\tau_3 \neq \tau_i$ for $i = 1,2$, since $F_*(\pi_1^{\text{loc}}(B,o))$ is a transitive subgroup of $S_4$ and $\gamma_1, \gamma_2$ generate the group $\pi_1^{\text{loc}}(B,o)$. Therefore, $\tau_3 = (i_1, i_2)$, where $i_1 \in \{1,2\}$ and $i_2 \in \{3,4\}$. Again, applying a conjugation in $S_4$, we can assume that $\tau_3 = (1,3)$. □

**Lemma 16.** Let the singularity type of a germ $(B = B_1 \cup B_2, o)$ be $(v - 2u = 0$ is the equation of the germ $B_1$ and $v^2 - u^{2k+1} = 0$ is the equation of the germ $B_2$) be $D_{2k+3}$ and a homomorphism $F_* : \pi_1^{\text{loc}}(B,o) \to S_4$ be such that

(i) $F_*(\pi_1^{\text{loc}}(B,o))$ is a transitive subgroup of $S_4$;

(ii) $F_*(\gamma_1)$ is a product of two different commuting transpositions $F_*(\gamma_1)$ and $F_*(\gamma_2)$ are transpositions, where $\gamma_1, \gamma_2, \gamma_3$ is a good geometric base of $\pi_1^{\text{loc}}(B,o)$, $\gamma_2$ and $\gamma_3$ are circuits around $B_2$, and $\gamma_1$ is a circuit around $B_1$.

Then $F_*(\pi_1^{\text{loc}}(B,o))$ is a dihedral subgroup $\mathbb{D}_4$ of $S_4$ and $F_*$ is sole.

**Proof.** By Lemma 4 and the Zariski–van Kampen theorem, the group $\pi_1^{\text{loc}}(B,o)$ has the following presentation:

$$\pi_1^{\text{loc}}(B,o) = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1 = (\gamma_1 \gamma_2 \gamma_3 \gamma_1 (\gamma_1 \gamma_2 \gamma_3)^{-1},$$

$$\gamma_2 = (\gamma_1 \gamma_2 \gamma_3)^{-1},$$

$$\gamma_3 = (\gamma_1 \gamma_2 \gamma_3)^{-1} \rangle. \quad (1.15)$$

Note that $\nu = F_*(\gamma_1) F_*(\gamma_2) F_*(\gamma_3)$ is an even permutation. Therefore, $\nu$ is either a cycle of length 3 or an element of $K_{14}$. However, it follows from (1.15) that $\nu$ cannot be a cycle of length 3, since it should commute with $F_*(\gamma_1)$, and $\nu \neq 1$, since then $F_*(\gamma_2) F_*(\gamma_3) = F_*(\gamma_1)$ and $F_*(\gamma_1), F_*(\gamma_2)$, and $F_*(\gamma_3)$ cannot generate a transitive subgroup of $S_4$. Note also that $F_*(\gamma_2) \neq F_*(\gamma_3)$, since otherwise it follows from (1.15) that $F_*(\gamma_2) = F_*(\gamma_3)$ commutes with $F_*(\gamma_1)$ and hence, again, $F_*(\gamma_1), F_*(\gamma_2)$, and $F_*(\gamma_3)$ cannot generate a transitive subgroup of $S_4$. Consequently, $F_*(\gamma_2) F_*(\gamma_3) \in K_{14}$ and, up to conjugation in $S_4$, we can assume that $F_*(\gamma_1) = [(1,2)(3,4)], F_*(\gamma_2) = (1,3)$, and $F_*(\gamma_3) = (2,4)$. □

**Lemma 17.** Let the singularity type of a germ $(B,o)$ be $T_4(4n + \beta, \beta)$ and a homomorphism $F_* : \pi_1^{\text{loc}}(B,o) \to S_4$ be such that

(i) $F_*(\pi_1^{\text{loc}}(B,o))$ is a transitive subgroup of $S_4$;

(ii) $F_*(\gamma_i), i = 1,2,3$, are transpositions, where $\gamma_1, \gamma_2, \gamma_3$ is a good geometric base of $\pi_1^{\text{loc}}(B,o)$.

Then $F_*(\pi_1^{\text{loc}}(B,o)) = S_4$ and $F_*$ is sole.
such that at the point \( o \) i.e., we can assume that (since we can conjugate by \( H \) where \( H \in \mathbb{C}^{4} \)). By Lemma 7, we have

\[
\nu \in \mathbb{C}^{4} \quad \text{(I)} \quad \nu \in \mathbb{C}^{4} \quad \text{(II)} \quad \nu \in \mathbb{C}^{4}.
\]

The element \( \nu \) acts by conjugation on the set of transpositions in \( S_4 \). There are two orbits of this action:

\[
(1, 2) \mapsto (1, 4) \mapsto (3, 4) \mapsto (2, 3) \mapsto (1, 2) \quad \text{and} \quad (1, 3) \mapsto (2, 4) \mapsto (1, 3).
\]

Therefore, by (i), since we can conjugate by \( \nu \), it follows from (1.16) that we can set \( F_{s(1)} = (1, 2) \), \( F_{s(2)} = (1, 4) \), and \( F_{s(3)} = (3, 4) \) and then check that \( \nu = [(1, 2)] \cdot [(1, 4)] \cdot [(3, 4)]\). □

2. PROOF OF THEOREM 1

Let \( L_1 = \{u_0 = 0\} \) and \( L_2 = \{v_0 = 0\} \) be the axes of some local complex analytic coordinates \( u_0, v_0 \) in \( (V, o) \). Then the local intersection number of the divisors \( M_1 = F^s(L_1) \) and \( M_2 = F^s(L_2) \) at the point \( o' \) is equal to \( (M_1, M_2)_{o'} = \deg_{o'} F = 4 \). Therefore, we have the following possibilities:

(I) either \( M_1 \) or \( M_2 \) is a germ of a nonsingular curve;

(II) \( M_1 \) and \( M_2 \) have a singularity of multiplicity 2 at the point \( o' \).

Let \( R \subset (U, o') \) be the ramification divisor of the finite cover germ \( F \) and \( B = F(R_{\text{red}}) \subset (V, o) \) be the branch curve.

Denote by \( m \subset \mathbb{C}[[z_0, w_0]] \) the maximal ideal in the ring of power series \( \mathbb{C}[[z_0, w_0]] \).

2.1. Case (I). Let \( M_1 \) be nonsingular. Then we can choose local coordinates \( z_0, w_0 \) in \( (U, o') \) such that \( F^s(u_0) = z_0 \) and \( F^s(v_0) = v_0(z_0, w_0) = \sum_{i=0}^{\infty} a_i(z_0)w_0^i \) with \( a_i(z_0) = \sum_{j=0}^{\infty} a_{ij}z_0^j \in \mathbb{C}[[z_0]] \), i.e., we can assume that \( M_1 = \{z_0 = 0\} \). Making the coordinates change \( v_0 \leftrightarrow v_0 - a_0(w_0) \), we can assume that \( a_0(z_0) \equiv 0 \). In addition, we have \( a_{1, 0} = a_{2, 0} = a_{3, 0} = 0 \) and can assume that \( a_{4, 0} = 1 \), since \( (M_1, M_2)_{o'} = 4 \).

The divisor \( R \) is given by the equation

\[
J(F) := \det \begin{pmatrix} 1 & 0 \\ \frac{\partial a_1}{\partial z_0} & \frac{\partial a_0}{\partial z_0} \end{pmatrix} = 0;
\]

i.e., \( R \) is given by the equation

\[
\sum_{i=1}^{\infty} ia_i(z_0)w_0^{i-1} = 0. \tag{2.1}
\]

Let us write equation (2.1) in the form

\[
z_0H_1(z_0, w_0) + 4w_0^3 + H_2(z_0, w_0) = 0, \tag{2.2}
\]

where \( H_1(z_0, w_0) \) is a polynomial of degree 2 and \( H_2(z_0, w_0) \in m^4 \). It follows from (2.2) that \((M_1, R)_{o'} = 3\). Therefore, there are seven possibilities:

(I) \( R = 3R_1 \), where \( R_1 \) is a germ of a smooth curve and \((M_1, R_1)_{o'} = 1\);

(II) \( R = 2R_1 + R_2 \), where \( R_1 \) and \( R_2 \) are germs of smooth curves and \((M_1, R_1, R_2)_{o'} = (M_1, R_2)_{o'} = 1\);

(III) \( R = R_1 \) is reduced and irreducible and \((M_1, R_1)_{o'} = 3\);

(IV) \( R = R_1 + R_2 \), where the germ \( R_1 \) is irreducible, with \((M_1, R_1)_{o'} = 2\) and \( \deg F_{|R_1} = 1 \), and \( R_2 \) is a germ of a smooth curve with \((M_1, R_2)_{o'} = 1\);
(I_{4.2}) \( R = R_1 + R_2 \), where the germ \( R_1 \) is irreducible, with \((M_1, R_1)_{o'} = 2\) and \( \deg F_{|R_1} = 2 \), and \( R_2 \) is a germ of a smooth curve with \((M_1, R_2)_{o'} = 1\);

(I_{5.1}) \( R = R_1 + R_2 + R_3 \), where \( R_1, R_2, \) and \( R_3 \) are germs of smooth curves, \((M_1, R_1)_{o'} = (M_1, R_2)_{o'} = (M_1, R_3)_{o'} = 1\), and the branch curve \( B = F(R) \) consists of three irreducible germs;

(I_{5.2}) \( R = R_1 + R_2 + R_3 \), where \( R_1, R_2, \) and \( R_3 \) are germs of smooth curves, \((M_1, R_1)_{o'} = (M_1, R_2)_{o'} = (M_1, R_3)_{o'} = 1\), and the branch curve \( B = F(R) \) consists of two irreducible germs.

Note that the case when \( R = R_1 + R_2 + R_3 \), where \( R_1, R_2, \) and \( R_3 \) are germs of smooth curves, \((M_1, R_1)_{o'} = (M_1, R_2)_{o'} = (M_1, R_3)_{o'} = 1\), and the germ of the branch curve \( B = F(R) \) is irreducible is impossible, since otherwise \( \deg_{o'} F \geq 6 \), which contradicts the condition \( \deg_{o'} F = 4 \).

**Proposition 3.** In case (I_1), the finite cover germ \( F \) is equivalent to the germ \( F_{11} \).

**Proof.** The germ \( F \) is ramified along \( R_1 \) with multiplicity 4, and the function \( u = z \) is a local parameter on \( R_1 \) at the point \( o' \). Therefore, \( \deg F_{|R_1} = 1 \) and \( B = F(R_1) \) is a smooth curve germ at the point \( o \). Hence, \( \pi^1_{loc}(B, o) \simeq \mathbb{Z} \). Since the germ \( F \) is ramified along \( R_1 \) with multiplicity 4, the monodromy group \( G_F \) of \( F \) is a cyclic group of the fourth order and \( G_F \) is generated in \( S_4 \) by a cycle of length 4. All such subgroups are conjugate in \( S_4 \). Therefore, by Theorem 5, the germ \( F \) is equivalent to the finite cover germ given by the functions \( u = z, v = w^4 \). \( \square \)

**Proposition 4.** In case (I_2), the finite cover germ \( F \) is equivalent to the germ \( F_{4, n} \) for some \( n \in \mathbb{N} \). The singularity type of the germ \( (B, o) \) of the branch curve of \( F_{4, n} \) is \( A_{8n-1} \), and the monodromy group \( G_{F_{4, n}} \) is \( S_4 \).

**Proof.** The finite cover germ \( F \) is ramified along \( R_1 \) with multiplicity 3 and along \( R_2 \) with multiplicity 2. The function \( u = z \) is a local parameter on both \( R_1 \) and \( R_2 \) at the point \( o' \). Therefore, \( B_1 = F(R_1) \) and \( B_2 = F(R_2) \) are smooth curve germs at the point \( o \) and \( \deg F_{|R_1} = \deg F_{|R_2} = 1 \). Note that \( B_1 \neq B_2 \), since \( \deg_{o'} F = 4 \). Let \( (B_1, B_2)_{o} = k \). Therefore, the point \( o \) is a singular point of the branch curve \( B = B_1 \cup B_2 \) of type \( A_{2k-1} \).

By Lemma 6, \( \pi^1_{loc}(B, o) = \langle \gamma_1, \gamma_2 \mid (\gamma_1 \gamma_2)^k = (\gamma_2 \gamma_1)^k \rangle \), where \( \gamma_i \) are circuits around \( B_i, \) \( i = 1, 2 \), and the monodromy group \( G_F \) is generated by a cycle \( \tau_1 = F_4(\gamma_1) \) of length 3 and a transposition \( \tau_2 = F_4(\gamma_2) \). Since \( G_F \) is a transitive subgroup of \( S_4 \), we have \( G_F = S_4 \) and, up to renumbering, can assume that \( \tau_1 = (1, 2, 3) \) and \( \tau_2 = (3, 4) \), i.e., \( F_i \) is sole. We have \( \tau_1 \tau_2 = (1, 2, 4, 3) \) and \( \tau_2 \tau_1 = (1, 2, 3, 4) \). Therefore, \( (\tau_1 \tau_2)^k = (\tau_2 \tau_1)^k \) if and only if \( k = 4n \), and, by Theorem 5, the finite cover germ \( F \) coincides, up to changes of coordinates, with the finite cover germ \( F_{4, n} \) given by

\[
\begin{align*}
    u &= z, \\
    v &= w^4 - z^w w^3.
\end{align*}
\]

Indeed, the ramification divisor \( R \) of \( F_{4, n} \) is given by the equation \( 4w^3 - 3z^w w^2 = 0 \), and hence \( R_1 \) is given by the equation \( w = 0 \) and \( R_2 \) is given by the equation \( 4w = 3z^w \). Then it is easy to see that \( B_1 \) is given by the equation \( v = 0 \) and \( B_2 \) is given by the equation \( 4v^4 + 3w^{4n} = 0 \), i.e., the point \( o \) is a singular point of the branch curve \( B \) of type \( A_{8n-1} \). To complete the proof, note that, by Proposition 2, the singularities of type \( A_m \) are rigid and apply Theorem 5. \( \square \)

**Proposition 5.** In case (I_3), the germ \( F \) is deformation equivalent to the germ \( F_{4, 2, n, \beta} \) for some \( n \in \mathbb{Z}_{\geq 0} \) and \( \beta \neq 0 \).

The germs \( F \) deformation equivalent to the germ \( F_{4, 2, 0, 1} \) are equivalent.

**Proof.** In case (I_3), the germ \( R_1 \subset (U, o') \) is given by an equation of the form

\[
\begin{align*}
    w_0^2 (1 + h_3(z_0)) + \sum_{i=0}^{2} w_0^{i-k_i} (b_i + \bar{h}_i(z_0)) + w_0^3 h_4(z_0, w_0) &= 0,
\end{align*}
\]

\begin{equation}
(2.3)
\end{equation}
where \( \tilde{h}_4(z_0, w_0) \in \mathbb{C}[[z_0, w_0]], \tilde{h}_i(z_0) \in \mathfrak{m} \cap \mathbb{C}[[z_0]] \) for \( i = 0, 1, 2, 3 \), and \( k_i \in \mathbb{N}, b_i \in \mathbb{C}^* \) for \( i = 0, 1, 2 \).

**Lemma 18.** Let an irreducible curve germ \( R_1 \subset (U, o') \) be given by equation (2.3). Then there is a local change of coordinates in \((U, o')\) of the form

\[
z = z_0, \quad w = w_0 + g(z_0),
\]

where \( g(z_0) \in \mathbb{C}[[z_0]] \), such that the equation of \( R_1 \) has the form

\[
w^3(1 + h_3(z)) + w^2z^n(a_2 + h_2(z)) + wz^{2n_1 + \beta_1}(a_1 + h_1(z)) + z^{3m_0 + \beta_0}(a_0 + h_0(z)) + w^4h_4(z, w) = 0, \tag{2.4}
\]

where \( h_4(z, w) \in \mathbb{C}[[z, w]], h_i(z) \in \mathfrak{m} \cap \mathbb{C}[[z]] \) for \( i = 0, 1, 2, 3 \), \( n_i \in \mathbb{Z}_{\geq 0} \) and \( a_i \in \mathbb{C}^* \) for \( i = 0, 1, 2 \), and

\[
\beta_0 = 1 \text{ or } 2, \quad \beta_1 = 0 \text{ or } 1, \quad n_1 \geq n_0,
\]

and if \( n_1 = n_0 \), then \( n_2 \geq n_0 + 1 \) and \( \beta_1 = 1 \). \hfill (2.5)

**Remark 2.** Note that if we change the coordinate \( w_0 \) to \( w_1 = w_0 + g(z_0) \), then, to calculate the Jacobian \( J(F) \) of the finite cover germ \( F \) in the coordinates \( z_0, w_1 \), it suffices to substitute \( w_1 - g(z_0) \) for \( w_0 \) in equation (2.3).

**Proof of Lemma 18.** Let, in equation (2.3), \( k_0 = 3m_0 + \beta_0 \) where \( \beta_0 \) is the remainder of \( k_0 \) divided by 3, \( k_1 = 2m_1 + \beta_1 \) where \( \beta_1 \) is the remainder of \( k_1 \) divided by 2, and \( k_2 := m_2 \).

Let us show that \( m_0 \leq \min\{m_1, m_2\} \), since otherwise the curve germ \( R_1 \) is not irreducible. Indeed, let \( m_1 < m_0 \) and \( m_1 \leq m_2 \) (the case when \( m_2 < m_0 \) and \( m_2 \leq m_1 \) is similar, and therefore we omit its consideration). Consider a sequence of \( \sigma \)-processes \( \sigma: U_{m_0} \to U \) given by the functions \( z_0 = z_{m_1} \) and \( w_0 = w_{m_1}z_{m_1}^{m_1} \). Then the exceptional curve \( E_{m_1} \) of the last \( \sigma \)-process is given by the equation \( z_{m_1} = 0 \), and the proper inverse image \( \sigma^{-1}(R_1) \) of \( R_1 \) is given by the equation

\[
w_{m_1}^3(1 + \tilde{h}_3(z_{m_1})) + \sum_{i=0}^{2} w_{m_1}^i z_{m_1}^{k_i - (3-i)m_1} (b_i + \tilde{h}_i(z_{m_1})) + w_{m_1}^4 z_{m_1}^{m_1} \tilde{h}_4(z_m, w_{m_1}z_{m_1}^{m_1}) = 0.
\]

We rewrite this equation in the form

\[
w_{m_1}^3 + b_0 z_{m_1}^{3(m_0 - m_1) + \beta_0} + b_1 w_{m_1} z_{m_1}^{\beta_1} + b_2 w_{m_1}^2 z_{m_1}^{m_2 - m_1} + (\text{monomials of degree } \geq 4) = 0. \tag{2.6}
\]

It follows from (2.6) that if \( \beta_1 = m_2 - m_1 = 0 \), then \( \sigma^{-1}(R_1) \) intersects \( E_1 \) at the three points \( (z_{m_1}, w_{m_1}) = (0, 0) \) and \( (z_{m_1}, w_{m_1}) = (0, -b_1 \pm \sqrt{b_1^2 - 4b_2}/2) \); if \( \beta_1 = 1 \) and \( m_2 - m_1 = 0 \), then \( \sigma^{-1}(R_1) \) intersects \( E_{m_1} \) at the two points \( (z_{m_1}, w_{m_1}) = (0, 0) \) and \( (z_{m_1}, w_{m_1}) = (0, -b_2) \); if \( \beta_1 = 0 \) and \( m_2 - m_1 > 0 \), then \( \sigma^{-1}(R_1) \) intersects \( E_{m_1} \) at the three points \( (z_{m_1}, w_{m_1}) = (0, 0) \) and \( (z_{m_1}, w_{m_1}) = (0, \pm \sqrt{-b_1}) \), which contradicts the irreducibility of the curve germ \( R_1 \). Finally, if \( \beta_1 = 1 \) and \( m_2 - m_1 > 0 \), then the quadratic part of the left-hand side of equation (2.6) is \( b_1 z_{m_1} w_{m_1} \), which also contradicts the irreducibility of the curve germ \( R_1 \).

Now, let us show that we can assume \( \beta_0 \neq 0 \). Indeed, suppose that \( \beta_0 = 0 \). As above, consider a sequence of \( \sigma \)-process \( \sigma: U_{m_0} \to U \) given by the functions \( z_0 = z_{m_0} \) and \( w_0 = w_{m_0}z_{m_0}^{m_0} \). Then the exceptional curve \( E_{m_0} \) of the last \( \sigma \)-process is given by the equation \( z_{m_0} = 0 \), and the proper inverse image \( \sigma^{-1}(R_1) \) of \( R_1 \) is given by the equation

\[
w_{m_0}^3 + b_0 + b_1 w_{m_0} z_{m_0}^{2(m_1 - m_0) + \beta_1} + b_2 w_{m_0}^2 z_{m_0}^{m_2 - m_0} + (\text{monomials of degree } \geq 4) = 0.
\]
It follows from the irreducibility of the curve germ \( \sigma^{-1}(R_1) \) that
\[
w^3_{m_0} + b_0 + b_1 w_{m_0} z^{2(m_1-m_0)+\beta_1} + b_2 w_{m_0}^2 z^{m_2-m_0} = (w_{m_0} + \sqrt[3]{b_0})^3.
\]
Therefore, if we change the coordinate \( w_0 \) to \( \overline{w}_0 = w_0 + \sqrt[3]{b_0} z^{m_0} \), then we find that \( R_1 \) is given by an equation of the same type as equation (2.3):
\[
\overline{w}_0^3 (1 + \overline{h}_3(z_0)) + \sum_{i=0}^2 \overline{w}_0^i \overline{r}_i (\overline{b}_i + \overline{h}_i(z_0)) + \overline{w}_0^4 \overline{r}_4(z_0, \overline{w}_0) = 0,
\]
in which \( \overline{r}_0 > k_0 \). After a finite number (since the singular point \( o' \) of \( R_1 \) can be resolved by a finite number of \( \sigma \)-processes) of such coordinate changes, we find that the germ \( R_1 \) is given by an equation of the same type as equation (2.4) in which \( \beta_0 = 1 \) or 2 and \( n_0 \leq \min\{n_1, n_2\} \).

Note that the same arguments as above (irreducibility of \( R_1 \) and possibility to carry out \( n_1 \) or \( n_2 \) \( \sigma \)-processes if necessary) show that \( n_2 \geq n_0 + 1 \) and \( \beta_1 > 0 \) if \( n_1 = n_0 \). \( \square \)

In case (I3), by Lemma 18, we can assume that the ramification divisor \( R = R_1 \) of the finite cover germ \( F \) is given by equation (2.4) satisfying conditions (2.5). Then, by Remark 2, the germ \( F \) is given by the functions
\[
u = u^4 (1 + h_3(z)) + 4 w z^{3n_0 + \beta_0} (a_0 + h_0(z)) + \frac{4}{3} w^3 z^{n_2} (a_2 + h_2(z))
\]
\[+ 2 w^2 z^{2n_1 + \beta_1} (a_1 + h_1(z)) + w^5 g(z, w),
\]
where \( w^5 g(z, w) = \int w^4 h_4(z, w) \, dw. \)

To find the singularity type of the branch locus of \( F \), let us show that \( R_1 \) can be given by the parametrization
\[z = \tau^3, \quad w = \tau^{3n_0 + \beta_0} (\sqrt[3]{a_0} + \tau g_1(\tau)), \quad (2.8)
\]
where \( g_1(\tau) \in \mathbb{C}[[\tau]]. \) Indeed, denote by \( \nu: \overline{R}_1 \to R_1 \) the resolution of singularity of the germ \( R_1 \). It follows from equation (2.4) that \( (M_1, R_1)_{o'} = 3 \) and \( (M_2, R_1)_{o'} = 3n_0 + \beta_0 \). Therefore, there exists a local parameter \( \tau \) at the point \( \nu^{-1}(o') \) in \( \overline{R}_1 \) such that \( \nu^{-1}(z) = \tau^3 \) and \( \nu^*(w) = \tau^{3n_0 + \beta_0} \sum_{i=0}^{\infty} c_i \tau^i \).

If we substitute \( \tau^3 \) for \( z \) and \( \tau^{3n_0 + \beta_0} \sum_{i=0}^{\infty} c_i \tau^i + \beta_0 \) for \( w \) in (2.4), then, as a result, we must obtain a power series identically equal to zero. In particular, we find that \( c_0 = \sqrt[3]{a_0} \).

By virtue of (2.5), we have \( n_2 - n_0 > 0 \) and \( n_1 - n_0 + \beta_1 > 0 \). Therefore, if we substitute \( \tau^{3n_0 + \beta_0} (\sqrt[3]{a_0} + \tau g_1(\tau)) \) for \( w \) and \( \tau^3 \) for \( z \) in (2.7), then we find that \( B = F(R_1) \) is given parametrically by equations of the form
\[u = \tau^3, \quad v = \tau^{12n_0 + 4\beta_0} (3a_0 \sqrt[3]{a_0} + \tau g_2(\tau)), \quad (2.9)
\]
where \( g_2(\tau) \in \mathbb{C}[[\tau]]. \) Therefore, the branch curve \( B \) of \( F \) has a singularity of type \( T_3(4n_0 + \beta_0, \beta_0) \) at \( o \).

It is easy to see that \( \deg F|_{R_1} = 1 \) and \( F \) is ramified along \( R_1 \) with multiplicity 2; therefore, the monodromy group \( G_F \) of the germ \( F \) is generated by transpositions, and since \( G_F \) is a transitive subgroup of \( S_4 \), we have \( G_F = S_4 \).

To complete the proof of Proposition 5, it suffices to consider the germs \( F_{4z, z_0, \beta_0} \) given by the functions \( u = z, \ v = w^4 + 4w z^{3n_0 + \beta_0} \) and apply Lemma 17 and Theorems 4 and 5. \( \square \)

The following two propositions will be proved simultaneously.
Proposition 6. In case (I_{5,1}), the germ $F$ is deformation equivalent either to one of the germs $F_{4\cdot k,m}$, $m,k \geq 1$ (the singularity type of $B_{F_{4\cdot k,m}}$ is $T_{3,8m+6k-1,8m}$), or to one of the germs $F_{4\cdot k,m}$, $m,k \geq 1$ (the singularity type of $B_{F_{4\cdot k,m}}$ is $T_{3,8m+2k-1,8m}$), or to one of the germs $F_{4\cdot m}$, $m \geq 1$ (the singularity type of $B_{F_{4\cdot m}}$ is $T_{3,8m-1,8m}$). In all cases, the monodromy groups of the finite cover germs are $S_4$.

Proposition 7. In case (I_{5,2}), the germ $F$ is equivalent to the germ $F_{3,2k}$ for some $k \in \mathbb{N}$. The singularity type of the germ of the branch curve $(B_{F_{3,2k}}, \alpha)$ is $A_{8k-1}$, and the monodromy group $G_{F_{3,2k}}$ is the dihedral group $D_4 \subset S_4$.

Proof. In cases (I_{5,1}) and (I_{5,2}) the ramification divisor $R$ consists of three irreducible germs, $R = R_1 \cup R_2 \cup R_3$. Let us renumber them in such a way that

$$(R_1,R_3)_{\sigma} = (R_2,R_3)_{\sigma} = m \leq (R_1,R_2)_{\sigma} = m + n$$

and choose coordinates $z,w$ such that $w = 0$ is an equation of $R_3$.

Since $(M_1,R_1)_{\sigma} = 1$, $z$ is a local parameter on each germ $R_i$ and we can choose equations of $R_i$ in the following form: $w = f_i(z)$, $i = 1,2$, with

$$f_1(z) = z^n p(z) + z^{m+n} h_1(z), \quad f_2(z) = z^n p(z) + z^{m+n+n_2} h_2(z), \quad n_2 \geq 0, \quad (2.10)$$

if $n \geq 1$, and with

$$f_i(z) = z^m h_i(z) \quad (2.11)$$

if $n = 0$; here $p(z) \in \mathbb{C}[z]$, $\deg p(z) \leq n - 1$, $p(0) \neq 0$, $h_i(z) \in \mathbb{C}[[z]]$, $h_1(0) \neq 0$, $h_2(0) \neq 0$, and $h_1(0) \neq h_2(0)$ if $n_2 = 0$. If $n \geq 1$, then

$$\frac{\partial v}{\partial w} = 4w (w - z^n p(z) - z^{m+n} h_1(z)) (w - z^n p(z) - z^{m+n+n_2} h_2(z)) (1 + g_1),$$

where $g_1 \in \mathfrak{m}$. Consequently, $F$ is given by the functions

$$u = z,$$

$$v = w^4 (1 + g_4) - \frac{4}{3} [2z^n p(z) + z^{m+n} h_1(z) + z^{m+n+n_2} h_2(z)] (1 + g_3) w^3$$

$$+ 2 [p(z)^2 z^{2m} + p(z) (h_1(z) z^{2m+n} + h_2(z) z^{2m+n+n_2}) + h_1(z) h_2(z) z^{2(m+n)}] (1 + g_2) w^2,$$

where $g_i \in \mathfrak{m}$. Similarly, if $n = 0$, then $F$ is given by the functions

$$u = z,$$

$$v = w^4 (1 + g_4) - \frac{4}{3} z^m [h_1(z) + h_2(z)] (1 + g_3) w^3 + 2 z^{2m} h_1(z) h_2(z) (1 + g_2) w^2.$$

Thus, $v = 0$ is an equation of $B_3 = F(R_3)$. To obtain equations of $B_i = F(R_i)$, $i = 1,2$, we substitute $f_i(u)$ from (2.10) or (2.11) for $w$ in (2.12) or (2.13), respectively. After the substitutions, we obtain the equation of $B_i$, $i = 1,2$, in the form

$$v = \frac{p(u)^4}{3} u^{10} (1 + uh_{i,3}(u)) \quad (2.14)$$

if $n > 0$ and

$$v = \frac{h_i(u)^3}{3} (2h_j(u) - h_i(u)) u^{10} (1 + uh_4(u)) \quad (2.15)$$

if $n = 0$, where $\{ j \} = \{ 1,2 \} \setminus \{ i \}$ and $h_{i,3}(u), h_4(u) \in \mathbb{C}[[u]]$. 

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Consider the case when \( B = B_1 \cup B_2 \cup B_3 \) consists of three different germs. Then, in the case \( n > 0 \), it follows from (2.14) that \((B_1, B_3)_o = (B_2, B_3)_o = 4m\) and \((B_1, B_2)_o = 4m + k > 4m\) for some \( k \), since \( p(0) \neq 0 \). Therefore, the singularity type of \((B, o)\) is \( T_{3,8m+2k-1,8m} \).

In particular, it is easy to check that the ramification locus of the germ \( F_{4_3,k,m} \) defined by the functions

\[
u = w = w^4 - \frac{8}{3}z^m(1 + 2z^k)w^3 + 2z^{2m}(1 + 4z^k + 3z^{2k})w^2, \quad m, k \geq 1,
\]

is given by the equation \( w(w - z^m - z^{m+k})(w - z^m - 3z^{m+k}) = 0 \) (i.e., \( R \) has a singularity of type \( T_{3,2m+2k-1,2m} \)) and the singularity type of the branch curve \( B \) is \( T_{3,8m+6k-1,8m} \).

Consider the case when \( n = 0 \). Note that \( 2h_2(0) - h_1(0) \neq 2h_1(0) - h_2(0) \); in particular, the equalities \( 2h_2(0) - h_1(0) = 0 \) and \( 2h_1(0) - h_2(0) = 0 \) cannot hold simultaneously. Therefore, it follows from (2.15) that if \( 2h_1(0) - h_2(0) \neq 0 \) and \( 2h_2(0) - h_1(0) \neq 0 \), then \( B \) has a singularity of type \( T_{3,8m-1,8m} \), and if \( 2h_1(0) - h_2(0) = 0 \) (respectively, if \( 2h_2(0) - h_1(0) = 0 \)), then \( B \) has a singularity of type \( T_{3,8m+2k-1,8m} \) for some \( k \geq 1 \).

In particular, it is easy to check that the ramification locus of the germ \( F_{4_4,k,m} \) defined by the functions

\[
u = w = w^4 - \frac{4}{3}z^m(3 + z^k)w^3 + 4z^{2m}(1 + z^k)w^2, \quad m, k \geq 1,
\]

is given by the equation \( w(w - z^m - z^{m+k})(w - 2z^m) = 0 \) (i.e., \( R \) has a singularity of type \( T_{3,2m-1,2m} \)) and the singular type of the branch curve \( B \) is \( T_{3,8m+2k-1,8m} \).

Again, it is easy to check that the ramification locus of the germ \( F_{4_7,m} \) defined by the functions

\[
u = w = w^4 - \frac{16}{3}z^m w^3 + 6z^{2m}w^2, \quad m \geq 1,
\]

is given by the equation \( w(w - z^m)(w - 3z^m) = 0 \) (i.e., \( R \) has a singularity of type \( T_{3,2m-1,2m} \)) and the singularity type of the branch curve \( B \) is \( T_{3,8m-1,8m} \).

So, in all the cases considered above, the branch curves \( B \) of \( F \) have the singularity types \( T_{3,8m+2k-1,8m} \) for some \( k \in \mathbb{Z}_{\geq 0} \). The germs \( F_{4_5,k,m} \) and \( F_{4_6,3k,m} \) have branch curves of the same singularity types. However, by Theorem 3, the covers \( F_{4_5,k,m} \) and \( F_{4_6,3k,m} \) are not deformation equivalent, since their ramification divisors have different singularity types. Therefore, to complete the proof of Proposition 6, it suffices to apply Lemma 13 and Corollary 4.

If \( F(R_1) = F(R_2) \) in the case when \( n > 0 \) or if \( F(R_3) \) coincides with either \( F(R_1) \) or \( F(R_2) \) in the case when \( n = 0 \), then it follows from (2.14) and (2.15) that \( B \) consists of two irreducible germs and has the singularity type \( A_{8m-1} \), since \( F(R_i) \) are germs of smooth curves. Denote the irreducible components of \( B \) by \( B_1 \) and \( B_2 \), and let \( F^{-1}(B_2) \) be the union of two components of the divisor \( R \). Then \( G_F \) is generated by a transposition \( \tau_1 = F_*(\gamma_1) \), where \( \gamma_1 \) is a circuit around \( B_1 \), and by a product \( P = [\tau_2 \tau_3] = F_*(\gamma_2) \) of two commuting transpositions \( \tau_2 \) and \( \gamma_3 \), where \( \gamma_2 \) is a circuit around \( B_2 \). Therefore, by Lemma 15, \( G_F \) is a dihedral subgroup \( D_3 \subset S_4 \).

Let us show that in this case \( F \) is equivalent to the cover \( F_{3,1,2m} \) given by the functions

\[
u = z, \quad w = w^4 - 2z^{2m}w^2.
\]

Indeed, it is not difficult to see that the ramification locus \( R \) of \( F_{3,1,2m} \) is given by the equation \( w(w - z^m)(w + z^m) \), the germ \( B_1 \) is given by the equation \( v = 0 \), and \( B_2 \) is given by the equation \( v + u^{2m} = 0 \). Therefore, by Lemma 15 and Corollary 3, the cover germ \( F \) is equivalent to \( F_{3,1,2m} \). This completes the proof of Proposition 7. \( \square \)

**Remark 3.** It follows from the proof of Proposition 7 as well as from Lemma 15 and Theorem 3 that if the branch curve \( B \) of the finite cover germ \( F \), \( \deg_F F = 4 \), has a singularity of type \( A_{8m-1} \)}
and the monodromy $F_*$ satisfies the conditions presented in Lemma 15, then the ramification locus $R$ of $F$ consists of three irreducible components.

The following two propositions will also be proved simultaneously.

**Proposition 8.** In case (I_{4,1}), the finite cover germ $F$ is deformation equivalent either to the germ $F_{4,k,m}$ or to the germ $F_{4,1,k,m}$ for some $m \in \mathbb{N}$ and $k \in \mathbb{Z}_{\geq 0}$.

The singularity type of $(B_{4,k,m},o)$ is $T_{3,8m+6k+2,8m}$, and the singularity type of $(B_{4,k,m},o)$ is $T_{3,8m+2k+4,8m+4}$; in both cases $G_F = S_4$.

**Proposition 9.** In case (I_{4,2}), the germ $F$ is equivalent to the germ $F_{3,2k+1}$ for some $k \in \mathbb{Z}_{\geq 0}$.

The singularity type of the germ $(B_{3,2k+1},o)$ is $A_{8k+3}$, and $G_{F_{3,2k+1}}$ is the dihedral group $D_4 \subset S_4$.

**Proof.** According to Remark 1, we can choose coordinates $z, w$ in $(U,o')$ such that $w = 0$ is an equation of $R_2$ and $u = z$, where $u, v$ are coordinates in $(V,o)$. Since the germ $R_1$ is irreducible and $(M_1,R_1)_{o'} = 2$, arguments similar to those used in the proof of Lemma 18 imply that an equation of $R_1$ has the form

$$(w - a_0 z^m (1 + z h_0))^2 - z^{2(m+n)+1} (1 + z h_1) + w z^{m+n+1} h_2 + w^2 z h_3 + \sum_{i=3}^{\infty} w^i h_{i+1} = 0, \quad (2.16)$$

where $h_i \in \mathbb{C}[z]$ for $i \in \mathbb{N}$, $h_0 \in \mathbb{C}[z]$, $\deg h_0 \leq n - 1$, $a_0 \in \mathbb{C}$, $m \in \mathbb{N}$, and $n \in \mathbb{Z}_{\geq 0}$ (here $2m = (R_1,R_2)_{o'}$ and $A_{m+n}$ is the singularity type of the germ $R_1$). Therefore,

$$\frac{\partial v}{\partial w} = 4 \left[ w^3 - 2 a_0 z^m (1 + z h_0) w^2 + a_0^2 z^{2m} (1 + z h_0)^2 w - z^{2(m+n)+1} w \right. \quad (2.16)$$

and hence

$$v = w^4 - \frac{8 a_0}{3} z^m (1 + z h_0) w^3 + 2 a_0^2 (1 + z h_0)^2 w^2 - 2 z^{2(m+n)+2} h_1 w^2 + \frac{4}{3} z^{m+n+1} h_2 w^3 + z h_3 w^4 + \sum_{i=3}^{\infty} \frac{4}{i+2} h_{i+1} w^{i+2}. \quad (2.17)$$

The germ of the ramification curve $R = R_1 \cup R_2$ has the singularity type $T_{3,2m+2n,2m}$ if $a_0 \neq 0$ and $T_{3,2m+2n,2m+2n+1}$ if $a_0 = 0$.

It follows from (2.17) that $B_2 = F(R_2)$ is given by the equation $v = 0$.

Consider the case when $a_0 \neq 0$. As in the proof of Proposition 5, it is easy to show that equation (2.16) implies that $R_1$ can be given parametrically by functions of the following form:

$$z = \tau^2, \quad w = a_0 \tau^{2m} (1 + \tau^2 h_0 (\tau^2)) + \tau^{2m+2n+1} + \sum_{i=2(m+n+1)}^{\infty} c_i \tau^i. \quad (2.18)$$

To obtain a parametrization of $B_1$, we substitute $z(\tau)$ and $w(\tau)$ from (2.18) for $z$ and $w$ in (2.17) and find that $B_1$ has a parametrization of the form

$$u = \tau^2, \quad v = \frac{a_4}{3} \tau^{8m} + \tau^{8m} \sum_{i=1}^{\infty} a_i \tau^i. \quad (2.19)$$

If $a_i = 0$ for all odd $i$, then $\deg F_{R_1} = 2$, $F(R_1)$ is a smooth germ, and $B$ has the singularity type $A_{8m-1}$. Therefore, $F_*(\gamma_1) \subset S_4$ is a product of two commuting transpositions and $F_*(\gamma_2)$ is a
transposition, where $\gamma_1, \gamma_2$ is a good geometric base in which $\gamma_1$ is a circuit around $B_1$ and $\gamma_2$ is a circuit around $B_2$. Then, by Lemma 15, the monodromy group $G_F$ is a dihedral group $\mathbb{D}_4 \subset \mathbb{S}_4$. However, Remark 3 says that this is impossible, since the germ $R$ consists of two irreducible germs.

Let $i_0$ be the smallest number for which $a_{i_0} \neq 0$. Then $\deg F|_{R_1} = 1$ and $B$ has a singularity of type $T_{3;8m+i_0-1,8m}$. It follows from Lemma 12 that $i_0 = 6k + 3$ for some $k \in \mathbb{Z}_{\geq 0}$.

Let us show that in this case the germ $F$ is deformation equivalent to the germ $F_{4k,m}$ given by the functions

$$u = z, \quad v = w^4 - \frac{8}{3}z^m w^3 + 2\alpha_0^2 (z^{2m} - z^{2(m+k)+1}) w^2. \quad (2.20)$$

Indeed, we have

$$\frac{\partial v}{\partial w} = 4 \left[ w^3 - 2z^m w + z^{2m} w^2 - z^{2(m+k)+1} \right],$$

and hence $R_{m,k} = R_1 \cup R_2$, where an equation of $R_2$ is $w = 0$ and an equation of $R_1$ is

$$(w - z^m)^2 - z^{2(m+k)+1} = 0.$$

The germ $R_1$ has the following parametrization:

$$z = r^2, \quad w = r^{2m}(1 - r^{2k+1}), \quad (2.21)$$

and so the curve $R_{m,k}$ has a singularity of type $T_{3;2m+2k,2m}$. The branch curve $B_{m,k}$ has the form $B_1 \cup B_2$, where $B_2$ is given by the equation $v = 0$. In order to obtain a parametrization of $B_1$, we substitute $z(\tau)$ and $w(\tau)$ from (2.21) for $z$ and $w$ in (2.20). As a result, we find that $B_1$ is given parametrically by the functions

$$u = \tau^2,$$

$$v = \tau^{2m} \left( 1 + \tau^{2k+1} \right)^2 \left[ (1 + \tau^{2k+1})^2 - \frac{8}{3} \left( 1 + \tau^{2k+1} \right) + 2 - 2\tau^{4k+2} \right]$$

$$= \frac{1}{3} \tau^{8m} \left( 1 - 2\tau^{4k+2} - 8\tau^{6k+3} - 6\tau^{8k+4} \right). \quad (2.22)$$

Consequently, the curve germ $B_{m,k}$ has a singularity of type $T_{3;8m+6k+2,8m}$. It follows from Lemma 12 and Corollary 4 that $F$ is deformation equivalent to $F_{4k,m}$.

Now, consider the case $a_0 = 0$ (in this case we put $k := m + n$). Then $R = R_1 \cup R_2$ has a singularity of type $T_{3;2k,2k+1}$. Let us write equation (2.16) of $R_1$ in the following form:

$$w^2 - z^{2k+1} + \sum_{m=4k+3}^{\infty} \sum_{2i+(2k+1)j=m} a_{i,j} z^i w^j = 0. \quad (2.23)$$

Then it is easy to see that a parametrization of $R_1$ has the form

$$z = r^2, \quad w = r^{2k+1} + \sum_{n=2k+2}^{\infty} r^n r^n, \quad (2.24)$$

$$\frac{\partial v}{\partial w} = 4 \left[ w^3 - z^{2k+1} w + \sum_{m=4k+3}^{\infty} \sum_{2i+(2k+1)j=m} a_{i,j} z^i w^{j+1} \right].$$
and consequently \( F \) is given by the functions 

\[
  u = z, \quad v = w^4 - 2z^{2k+1}w^2 + w^2 \sum_{m=4k+3}^{\infty} \sum_{j=m}^{\infty} \frac{4}{j+2} \alpha_{i,j} z^i w^j.
\]  \tag{2.25}

To obtain a parametrization of \( B_1 \), we substitute \( z(\tau) \) and \( w(\tau) \) from (2.24) for \( z \) and \( w \) in (2.25) and find that \( B_1 \) has a parametrization of the form 

\[
  u = \tau^2, \quad v = -\tau^{8k+4} + \sum_{m=8k+5}^{\infty} b_m \tau^m.
\]  \tag{2.26}

There are two possibilities: either there is \( m_0 = 8k + 2(n + 2) + 1 \) which is the smallest odd number such that \( b_{m_0} \neq 0 \), or \( b_m = 0 \) for all odd \( m \geq 8k + 5 \).

Let \( b_{8k+2(n+2)+1} \neq 0 \). Then \( B = B_1 \cup B_2 \) has a singularity of type \( T_{3,8k+2n+4,8k+4} \), \( k, n \in \mathbb{Z} \geq 0 \). Let us show that in this case the germ \( F \) is deformation equivalent to the germ \( F_{4,k,n} \) given by the functions 

\[
  u = z, \quad v = w^4 - 2z^{2k+1}w^2 + 12z^{k+n+1}w^3.
\]  \tag{2.27}

The ramification divisor of \( F_{4,k,n} \) is \( R = R_1 \cup R_2 \), where \( R_2 \) is given by the equation \( w = 0 \) and \( R_1 \) is given by the equation 

\[
  w^2 - z^{2k+1} + 3z^{k+n+1}w = 0.
\]  \tag{2.28}

It is easy to check that \( R_1 \) is given parametrically by the functions 

\[
  z = \tau^2, \quad w = \tau^{2k+1} - \frac{3}{2}\tau^{2(k+n+1)} + \tau^{2(k+n)+3} h(\tau),
\]  \tag{2.29}

where \( h(\tau) \in \mathbb{C}[[\tau]] \). To obtain a parametrization of \( B_1 = F_{4,k,n}(R_1) \), let us substitute \( z(\tau) \) and \( w(\tau) \) from (2.29) into (2.27). As a result, we obtain 

\[
  u = \tau^2, \quad v = -\tau^{8k+4} + 6\tau^{8k+2n+5} + \tau^{8k+2n+6} h(\tau),
\]  \tag{2.30}

where \( h_1(\tau) \in \mathbb{C}[[\tau]] \). Consequently, \( B \) has the singularity type \( T_{3,8k+2n+4,8k+4} \) and Proposition 8 follows from Lemma 11 and Corollary 4.

Consider the case when \( b_m = 0 \) for all odd \( m \geq 8k + 5 \). It follows from (2.26) that \( \deg F_{1,R} = 2 \), \( B_1 = F(R_1) \) is a smooth germ, and the singularity type of the germ \( (B,o) \) is \( A_{8k+3} \). Therefore, \( F_* : \pi_{1}^{\loc}(B,o) \rightarrow S_4 \) has the following properties:

\((*1)\) \( G_F = \text{im} F_* \) is a transitive subgroup of \( S_4 \);

\((*2)\) \( F_*(\gamma_1) \) is the product of two commuting transpositions and \( F_*(\gamma_2) \) is a transposition, where \( \gamma_1 \) is a circuit around \( B_1 \) and \( \gamma_2 \) is a circuit around \( B_2 \).

By Lemma 15, \( G_F = D_4 \) is a dihedral subgroup of \( S_4 \). Therefore, by Corollary 4, to complete the proof of Proposition 9, it suffices to show that the branch curve \( B \) of the finite cover germ \( F_{3,2k+1} \) given by the functions \( u = z \) and \( v = w^4 - 2z^{2k+1}w^2 \) has a singularity of type \( A_{8k+3} \) and \( F_{3,2k+1} \) has properties \((*1)\) and \((*2)\).

The ramification divisor \( R = R_1 \cup R_2 \) of \( F_{3,2k+1} \) is given by \( w(w^2 - z^{2k+1}) = 0 \), where \( R_1 \) is given by \( w^2 - z^{2k+1} = 0 \). Therefore, \( R_1 \) has the following parametrization: \( u = \tau^2, v = \tau^{2k+1} \). Consequently, \( F_{3,2k+1,R_1} : R_1 \rightarrow (V,o) \) is given by the functions 

\[
  u = \tau^2, \quad v = -\tau^{8k+4} = (\tau^{2k+1})^4 - 2(\tau^2)^{2k+1}(\tau^{2k+1})^2,
\]  

and hence \( \deg F_{3,2k+1,R_1} = 2 \) and \( B_1 \) is a smooth germ touching the germ \( B_2 \) given by \( v = 0 \) with multiplicity \( 4k + 2 \). Therefore, the branch curve \( B \) of the germ \( F_{3,2k+1} \) has a singularity of type \( A_{8k+3} \) and \( F_{3,2k+1} \) has properties \((*1)\) and \((*2)\). \( \square \)
2.2. Case (II). To complete the proof of Theorem 1, it suffices to prove 

**Proposition 10.** In case (II), the ramification divisor $R$ of the finite cover germ $F$ is the union of two smooth curves, $R = R_1 \cup R_2$, meeting transversally at the point $o'$, and $F$ is ramified along $R$ with multiplicity 2.

There are three possibilities:

(II$_1$) $\deg f_{R_1} = \deg f_{R_2} = 1$, the point $o \in V$ is a singular point of the curves $B_1 = F(R_1)$ and $B_2 = F(R_2)$ of types $A_{n_1}$ and $A_{n_2}$ for some $n_1, n_2 \in \mathbb{N}$, with $(B_1, B_2)_o = 4$, the germ $F$ is deformation equivalent to the germ $F_{4s, n_1, n_2}$, the branch curve $B_{F_{4s, n_1, n_2}}$ has a singularity of type $T_{4,2n_1,2n_2}$, and $G_{F_{4s, n_1, n_2}}$ is $\mathbb{S}_4$;

(II$_2$) up to the numbering, $\deg f_{R_1} = 1$, $\deg f_{R_2} = 2$, the point $o \in V$ is a singular point of the curve germ $B_1 = F(R_1)$ of type $A_{2n}$ for some $n \geq 1$, the curve germ $B_2 = F(R_2)$ is smooth, with $(B_1, B_2)_o = 2$, the germ $F$ is equivalent to the germ $F_{3,2,n}$, the branch curve $B_{F_{3,2,n}}$ has a singularity of type $D_{2n+3}$, and the monodromy group $G_{F_{3,2,n}}$ is a dihedral group $\mathbb{D}_4 \subset \mathbb{S}_4$;

(II$_3$) $\deg f_{R_1} = \deg f_{R_2} = 2$, the curve germs $B_1 = F(R_1)$ and $B_2 = F(R_2)$ are smooth, with $(B_1, B_2)_o = 1$, the germ $F$ is equivalent to the germ $F_{2,1}$, the branch curve $B_{F_{2,1}}$ has a singularity of type $A_1$, and the monodromy group $G_{F_{2,1}}$ is the Klein four group $\mathbb{K}_4 \subset \mathbb{S}_4$.

**Proof.** In case (II), the cover germ $F$ is given by functions of the form

$$u_0 = f_1(z_0, w_0) f_2(z_0, w_0) + h_{\geq 3}(z_0, w_0), \quad v_0 = h_1(z_0, w_0) h_2(z_0, w_0) + h_{\geq 3}(z_0, w_0),$$

where $f_i(z_0, w_0)$ and $h_i(z_0, w_0)$, $i = 1, 2$, are linear forms and $f_{\geq 3}(z_0, w_0), h_{\geq 3}(z_0, w_0) \in \mathbb{m}^3$. Note that the forms $f_i(z_0, w_0)$ and $h_i(z_0, w_0)$ are linearly independent for each pair $(i, j)$, $i, j = 1, 2$, since $(M_1, M_2)_o = 4$.

First, let us show that there are coordinates $z_1, w_1$ in $U$ and $u_1, v_1$ in $V$ such that $F$ is given by the functions

$$u_1 = z_1^2 + g_1(z_1, w_1), \quad v_1 = w_1^2 + g_2(z_1, w_1), \quad (2.31)$$

where $g_1(z_1, w_1), g_2(z_1, w_1) \in \mathbb{m}^3$. Up to the coordinate change $u_0 \leftrightarrow v_0$, we have two possibilities:

1. $f_1(z_0, w_0)$ and $f_2(z_0, w_0)$ are proportional, and
2. $f_1(z_0, w_0)$ and $f_2(z_0, w_0)$ are linearly independent.

In case (1) we can assume (after coordinate changes in $U$ and $V$) that

$$f_1(z_0, w_0) f_2(z_0, w_0) = z_0^2 \quad \text{and} \quad h_1(z_0, w_0) h_2(z_0, w_0) = w_0^2 + a z_0 w_0$$

with some $a \in \mathbb{C}$. After the coordinate changes $z_0 = z_1$, $w_0 = (w_1 - a z_1)/2$ and $u_0 = u_1$, $v_0 = 4 w_1$, we obtain functions of the form (2.31).

In case (2) we can assume (after changes of coordinates in $U$ and $V$) that

$$f_1(z_0, w_0) = z_0, \quad f_2(z_0, w_0) = w_0 \quad \text{and} \quad h_1(z_0, w_0) h_2(z_0, w_0) = a z_0^2 + c z_0 w_0 + b w_0^2,$$

where $c = 0$ and $a \neq 0, b \neq 0$ (if $c \neq 0$ then the change of coordinates $u_0 = u_0, v_0 = v_0 - c u_0$ “kills” the coefficient of $z_0 w_0$). After the changes of coordinates

$$z_1 = \frac{\sqrt{a} z_0 + \sqrt{b} w_0}{2}, \quad w_1 = \frac{\sqrt{b} w_0 - \sqrt{a} z_0}{2} \quad \text{and} \quad u = \frac{u_0 + \sqrt{ab} v_0}{2 \sqrt{ab}}, \quad v = \frac{\sqrt{ab} v_0 - u_0}{2 \sqrt{ab}}$$

we obtain functions of the form (2.31).
We have

\[ J(F) := \det \left( \frac{\partial}{\partial z_1}, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_1} \right) = 4z_1w_1 + S(z_1, w_1), \]

where \( S(z_1, w_1) \in \mathfrak{m}^3 \). Therefore,

\[ J(F) = 4(z_1 + S_1(z_1, w_1))(w_1 + S_2(z_1, w_1)) \]

in the ring \( \mathbb{C}[[z_1, w_1]] \), where \( S_1(z_1, w_1) \in \mathfrak{m}^2 \), and hence \( F \) is ramified along \( R \) with multiplicity 2.

Let us make the change of coordinates \( z_2 = z_1 + S_1(z_1, w_1), \ w_2 = w_1 + S_2(z_1, w_1) \) and write the functions \( u \) and \( v \) in the form

\[ u = z_2^2(1 + z_2\tilde{f}_1(z_2)) + w_2\tilde{g}_1(z_2, w_2), \quad v = w_2^2(1 + w_2\tilde{h}_1(w_2)) + z_2\tilde{g}_2(z_2, w_2), \]

where \( \tilde{f}_1(z_2) \in \mathbb{C}[[z_2]] \) and \( \tilde{h}_1(w_2) \in \mathbb{C}[[w_2]] \). Note that the ramification divisor \( R \) is given by the equation \( z_2w_2 = 0 \). Therefore, \( R = R_1 + R_2 \), where \( R_1 = \{ z_2 = 0 \} \) and \( R_2 = \{ w_2 = 0 \} \).

Finally, if we put

\[ z = z_2\sqrt{1 + z_2\tilde{f}_1(z_2)} \quad \text{and} \quad w = w_2\sqrt{1 + w_2\tilde{h}_1(w_2)}, \]

we find that \( F \) is given by functions of the form

\[ u = z^2 + \sum_{i=3}^{\infty} \alpha_i w^i + zw\tilde{g}_3(z, w), \quad v = w^2 + \sum_{i=3}^{\infty} \beta_i z^i + zw\tilde{g}_4(z, w), \quad (2.32) \]

where \( \tilde{g}_3(z, w), \tilde{g}_4(z, w) \in \mathfrak{m} \). Note that \( z = 0 \) is an equation of \( R_1 \) and \( w = 0 \) is an equation of \( R_2 \). Note also that \( w \) is a local parameter in \( R_1 \) at the point \( o' \) and \( z \) is a local parameter in \( R_2 \).

The restrictions \( F_{|R_1} \) to \( R_1 \) and \( F_{|R_2} \) to \( R_2 \) of \( F \) are specified as follows:

\[ F_{|R_1} = \left\{ u = \sum_{i=3}^{\infty} \alpha_i w^i, \ v = w^2 \right\} \quad \text{and} \quad F_{|R_2} = \left\{ u = z^2, \ v = \sum_{i=3}^{\infty} \beta_i z^i \right\}. \quad (2.33) \]

If \( \alpha_i = 0 \) for all odd \( i \), then it is easy to see that \( \deg F_{|R_1} = 2 \) and \( B_1 = F(R_1) \) is given by the equation

\[ u = \sum_{j=2}^{\infty} \alpha_{2j} w^j. \quad (2.34) \]

Similarly, if \( \beta_i = 0 \) for all odd \( i \), then \( \deg F_{|R_2} = 2 \) and \( B_2 = F(R_2) \) is given by the equation

\[ v = \sum_{j=2}^{\infty} \beta_{2j} w^j. \quad (2.35) \]

If \( i = 2n_1 + 1 \) is the smallest odd index for which \( \alpha_i \neq 0 \), then it follows from (2.33) that \( \deg F_{|R_1} = 1 \), the germ \( B_1 = F(R_1) \) has a singularity of type \( A_{2n_1} \), and the coordinate axis \{ \( u = 0 \) \} is a tangent line of \( B_1 \) at the point \( o \).

Similarly, if \( i = 2n_2 + 1 \) is the smallest odd index for which \( \beta_i \neq 0 \), then \( \deg F_{|R_2} = 1 \), the germ \( B_2 = F(R_2) \) has a singularity of type \( A_{2n_2} \), and the coordinate axis \{ \( v = 0 \) \} is a tangent line of \( B_2 \) at the point \( o \).

If \( i = 2n_1 + 1 \) and \( j = 2n_2 + 1 \) are the smallest odd indices for which \( \alpha_i \neq 0 \) and \( \beta_j \neq 0 \), then the branch curve \( B \) has a singularity of type \( T_{4,2n_1,2n_2} \), the monodromy group \( G_F \) is \( S_4 \), since \( G_F \)
is a transitive subgroup of $S_4$ generated by transpositions, and it is easy to see that the family of finite cover germs given by the functions

$$u = z^2 + \alpha_{2n+1} w^{2n_1+1} + t \left( \sum_{i=1}^{\infty} \alpha_{2i} w^{2i} + \sum_{i=n_1+1}^{\infty} \alpha_{2i+1} w^{2i+1} + z w \tilde{g}_3(z, w) \right),$$

$$v = w^2 + \beta_{2n_2+1} z^{2n_2+1} + t \left( \sum_{j=1}^{\infty} \beta_{2j} z^{2j} + \sum_{j=n_2+1}^{\infty} \beta_{2j+1} z^{2j+1} + z w \tilde{g}_4(z, w) \right),$$

defines a deformation equivalence between $F$ and the germ $F_{4, n_1, n_2}$ given by

$$u = z^2 + \alpha_{2n_1+1} w^{2n_1+1}, \quad v = w^2 + \beta_{2n_2+1} z^{2n_2+1}. $$

If $i = 2n + 1$ is the smallest odd index for which $\beta_i \neq 0$ and if $\alpha_i = 0$ for all odd $i$, then the branch curve $B = B_1 \cup B_2$ has a singularity of type $D_{2n+3}$ and it is easy to see that the family of finite cover germs given by the functions

$$u = z^2 + t \left( \sum_{i=3}^{\infty} \alpha_i w^i + z w \tilde{g}_3(z, w) \right),$$

$$v = w^2 + \beta_{2n+1} z^{2n+1} + t \left( \sum_{i=1}^{\infty} \beta_{2i} z^{2i} + \sum_{i=n+1}^{\infty} \beta_{2i+1} z^{2i+1} + z w \tilde{g}_4(z, w) \right),$$

defines a deformation equivalence between $F$ and the germ $F_{3, n}$ given by

$$u = z^2, \quad v = w^2 + \beta_{2n+1} z^{2n+1}. $$

The group $\pi^1(B, o)$ is generated by circuits $\gamma_1$ around $B_1$ and $\gamma_2, \gamma_3$ around $B_2$. The permutation $F_*(\gamma_1)$ is a product of two commuting transpositions, since $\deg F_{R_1} = 2$, and $F_*(\gamma_i)$, $i = 2, 3$, are transpositions, since $\deg F_{R_2} = 1$. These permutations generate a transitive subgroup in $S_4$. It follows from Lemma 16 that $G_{3, n} \simeq D_4$. Recall also that the germ of the singularity $D_n$ is rigid. Therefore, by Lemma 16 and Corollary 3, the germ $F$ is equivalent to $F_{3, n}$.

If $\alpha_i = 0$ and $\beta_i = 0$ for all odd $i$, then it follows from (2.34) and (2.35) that the singular point of $B = B_1 \cup B_2$ is of type $A_1$. Consequently, $\pi^1(B, o) = \mathbb{Z} \times \mathbb{Z}$ is generated by circuits $\gamma_1$ and $\gamma_2$ around $B_1$ and $B_2$. The permutations $F_*(\gamma_1)$ and $F_*(\gamma_2)$ are products of two commuting transpositions, since $\deg F_{R_1} = \deg F_{R_2} = 2$. These permutations generate a transitive subgroup in $S_4$. Therefore, $G_F$ is the Klein four group $K_4 \subset S_4$. Up to a change of coordinates, the singularity of type $A_1$ is given by the equation $uv = 0$, and for the singularity $(B, o)$ of type $A_1$ there is a unique epimorphism from $\pi^1(B, o)$ to $K_4$. Therefore, the germ $F$ is equivalent to the germ $F_{2, 1}$ given by the functions $u = z^2$ and $v = w^2$. \qed

3. ON THE MONODROMY GROUPS OF FINITE COVER GERMS

3.1. On the transitive subgroups of the symmetric groups. Let $G$ be a finite group. We say that two subgroups $H_1$ and $H_2$ of $G$ are equivalent if there is an inner automorphism $g \in \text{Aut}(G)$ such that $g(H_1) = H_2$. A subgroup $H$ of $G$ is said to be relatively simple if there is no proper nontrivial normal subgroup of $G$ contained in $H$. Denote by $A_G$ a set of representatives of the equivalence classes of relatively simple subgroups of $G$, and let $I_G \subset N$ be the set of indices $i_H = (G : H)$ of the subgroups $H \in A_G$.

We say that an embedding $\varphi: G \to S_d$ is transitive if $\varphi(G)$ is a transitive subgroup of the symmetric group $S_d$.}

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Proposition 11. The set of transitive embeddings of $G$ considered up to conjugations in symmetric groups is in one-to-one correspondence with the set $A_G$. For any $d \in I_G$ there is a transitive embedding $\varphi: G \to S_d$.

Proof. Consider the symmetric group $S_d$ as a group acting on the interval of positive integers $\mathbb{N}_d = \{1, \ldots, d\}$ and denote by $S_{d-1}$ the subgroup of $S_d$ consisting of the permutations $\tau \in S_d$ that leave 1 fixed. Let $G$ be a transitive subgroup of $S_d$. Then $H = G \cap S_{d-1}$ is a subgroup of $G$ of index $(G : H) = d$, since $G$ is a transitive subgroup of $S_d$. Let us show that $H$ is a relatively simple subgroup of $G$. Indeed, assume that a normal subgroup $N$ of $G$ is contained in $H$ and $h \in N$ is a nontrivial element. In this case, for any $i \in \mathbb{N}_d$, there is an element $g_i \in G$ such that $g_i(1) = i$, and therefore $h(i) = i$ for each $i \in \mathbb{N}_d$, since $g_i^{-1} h g_i \in N \subset H$. As a result, we get a contradiction with the assumption that $G$ acts effectively on $\mathbb{N}_d$.

Conversely, let $H$ be a subgroup of $G$. Then $G$ acts on the set of left cosets of $H$ in $G$. This action defines a homomorphism $\varphi: G \to S_d$, where $d = (G : H)$. It is easy to check that $\varphi$ is an embedding if and only if $H$ is a relatively simple subgroup of $G$. \hfill \square

Note that for every finite group $G$ there is at least one transitive embedding, namely, Cayley’s embedding $c: G \hookrightarrow S_{|G|}$ corresponding to the trivial subgroup of $G$.

3.2. Germs of Galois smooth covers. Consider a finite cover germ $F: (U, o') \to (V, o)$, deg$_{o'} F = d$, and its monodromy $F_*: \pi^\text{loc}(B, o) \to G_F \subset S_d$. Let $c: G = G_F \hookrightarrow S_{|G|}$ be Cayley’s embedding. By the Grauert–Remmert–Riemann–Stein theorem, the homomorphism $c \circ F_*: \pi^\text{loc}(B, o) \to S_{|G|}$ defines a germ of a Galois cover $\tilde{F}: (\tilde{U}, \tilde{o}) \to (V, o)$ of degree $\deg_{\tilde{o}} \tilde{F} = \deg_G F = |G|$, where $\tilde{F}$ is a holomorphic finite map and $(\tilde{U}, \tilde{o})$ in the general case is an irreducible germ of a normal complex analytic variety. The group $G$ acts on $(\tilde{U}, \tilde{o})$ in such a way that the quotient variety $(\tilde{U}, \tilde{o})/G$ is $(V, o)$ and $\tilde{F}$ is the quotient map. By Proposition 11, the embedding $G = G_F \hookrightarrow S_{d}$ corresponds to a relatively simple subgroup $H$ of $G$, $(G : H) = d$, and it is well known that we can choose a subgroup $H_F$ of $G$ equivalent to $H$ such that the quotient variety $(\tilde{U}, \tilde{o})/H_F$ is biholomorphic to $(U, o')$ and the cover $\tilde{F}$ is the composition of two covers, $\tilde{F} = F \circ F_{H_F}$, where $F_{H_F}: (\tilde{U}, \tilde{o}) \to (U, o')$ is the quotient map defined by the action of $H_F$ on $(\tilde{U}, \tilde{o})$.

Prior to formulating the statement describing the germs of the Galois smooth covers, let us recall the invariants of the binary linear groups given by Klein [7]. For each of the three groups (tetrahedral, octahedral, and icosahedral ones) there are three invariant forms, which we denote by $\varphi(z, w)$, $\psi(z, w)$, and $\theta(z, w)$, where $\psi(z, w)$ is the Hessian of $\varphi(z, w)$, and $\theta(z, w)$ is the Jacobian of $\varphi(z, w)$ and $\psi(z, w)$. We have

$$
\varphi_4(z, w) = z^4 + 2\sqrt{-3} z^2 w^2 + w^4, \quad \psi_4(z, w) = z^4 - 2\sqrt{-3} z^2 w^2 + w^4,
$$

$$
\theta_6(z, w) = zw(z^4 - w^4)
$$

in the case of the tetrahedral group,

$$
\varphi_6(z, w) = zw(z^4 - w^4), \quad \psi_8(z, w) = z^8 + 14z^4w^4 + w^8,
$$

$$
\theta_{12}(z, w) = z^{12} - 33(z^8w^4 + z^4w^8) + w^{12}
$$

in the case of the octahedral group, and

$$
\varphi_{12}(z, w) = zw(z^{10} + 11z^5w^5 - w^{10}),
$$

$$
\psi_{20}(z, w) = -(z^{20} + w^{20}) + 288(z^{15}w^5 + z^5w^{15}) - 494z^{10}w^{10},
$$

$$
\theta_{30}(z, w) = (z^{30} + w^{30}) + 522(z^{25}w^5 - z^5w^{25}) - 1005(z^{20}w^{10} + z^{10}w^{20})
$$

in the case of the icosahedral group.
Theorem 6 [16]. A germ of a Galois smooth cover $\tilde{F}: (\tilde{U}, \tilde{\sigma}) \to (V, \sigma)$ is equivalent to one of the following rigid germs:

1. $F_1 = \{u = z^2 + zw + w^2, \quad v = z^2w + zw^2\}, \quad T(B_1) = A_2$;
2. $F_2 = \{u = z^{pq} + w^{pq}, \quad v = z^{q}w^{q}\}, \quad T(B_{2,q}) = D_{pq+2}, \quad p > 1, \quad q > 1$;
3. $F_3, n_1, n_2 = \{u = z^{n_1}, \quad v = w^{n_2}\}, \quad T(B_{3,n_1,n_2}) = A_1, \quad n_1, n_2 \in \mathbb{N}$;
4. $F_4 = \{u = \psi_4(z, w), \quad v = \theta_6(z, w)\}, \quad T(B_4) = A_2$;
5. $F_5 = \{u = \varphi^3_4(z, w), \quad v = \theta_6(z, w)\}, \quad T(B_5) = A_3$;
6. $F_6 = \{u = \varphi^3_4(z, w), \quad v = \psi^3_4(z, w)\}, \quad T(B_6) = A_5$;
7. $F_7 = \{u = \varphi^3_4(z, w), \quad v = \psi^3_4(z, w)\}, \quad T(B_7) = D_4$;
8. $F_8 = \{u = \psi_8(z, w), \quad v = \theta_{12}(z, w)\}, \quad T(B_8) = A_2$;
9. $F_9 = \{u = \psi_8(z, w), \quad v = \theta^2_2(z, w)\}, \quad T(B_9) = A_2$;
10. $F_{10} = \{u = \psi_8(z, w), \quad v = \theta_{12}(z, w)\}, \quad T(B_{10}) = A_3$;
11. $F_{11} = \{u = \varphi^3_6(z, w), \quad v = \psi^3_8(z, w)\}, \quad T(B_{11}) = D_4$;
12. $F_{12} = \{u = \varphi_6(z, w), \quad v = \psi_8(z, w)\}, \quad T(B_{12}) = E_7$;
13. $F_{13} = \{u = \varphi^2_5(z, w), \quad v = \theta_{12}(z, w)\}, \quad T(B_{13}) = E_6$;
14. $F_{14} = \{u = \psi_6(z, w), \quad v = \theta^{12}_2(z, w)\}, \quad T(B_{14}) = A_7$;
15. $F_{15} = \{u = \varphi^3_6(z, w), \quad v = \theta_{12}(z, w)\}, \quad T(B_{15}) = D_6$;
16. $F_{16} = \{u = \psi_2(z, w), \quad v = \theta_{30}(z, w)\}, \quad T(B_{16}) = A_2$;
17. $F_{17} = \{u = \psi_2(z, w), \quad v = \theta_{30}(z, w)\}, \quad T(B_{17}) = A_5$;
18. $F_{18} = \{u = \psi^3_2(z, w), \quad v = \theta_{30}(z, w)\}, \quad T(B_{18}) = A_3$;
19. $F_{19} = \{u = \varphi^3_2(z, w), \quad v = \theta_{20}(z, w)\}, \quad T(B_{19}) = D_4$;
20. $F_{20} = \{u = \varphi_2(z, w), \quad v = \theta_{30}(z, w)\}, \quad T(B_{20}) = A_4$;
21. $F_{21} = \{u = \varphi_2(z, w), \quad v = \theta_{30}(z, w)\}, \quad T(B_{21}) = A_3$;
22. $F_{22} = \{u = \varphi_2(z, w), \quad v = \psi_{20}(z, w)\}, \quad T(B_{22}) = E_8$.

In case (1) the monodromy group of $F_1$ is $G_{F_1} = S_3$; in cases (2), $i = 1, 2, 3, G_{F_{i,pq,q}} = G(pq, q, 2)$ (in the notation used in [16]); in cases (j), $3 \leq j \leq 22$, the group $G_{F_j}$ is the group in [16] with number $j$.

Proof. By Cartan’s lemma [3], the action of $G_{\tilde{F}}$ on $\tilde{U}$ can be linearized. Therefore, $G_{\tilde{F}}$ is a subgroup of $GL(2, \mathbb{C})$ generated by reflections. The list of germs of smooth Galois covers in Theorem 6 is in one-to-one correspondence with the list in [16] of finite subgroups of $GL(2, \mathbb{C})$ generated by reflections. The singularity types of the germs of the branch curves $B_i$ of $F_i$, $i = 1, \ldots, 22$, are calculated in [12]. The rigidity of the covers $F_i$ follows easily from Cartan’s lemma.

3.3. Examples of finite groups which can (cannot) be realized as the monodromy groups of finite cover germs. The following two statements are a direct consequence of Theorem 6 and Proposition 11.

Corollary 2. An abelian group $G$ can be realized as the monodromy group of a finite cover germ if and only if for every prime $p$ the number of cyclic $p$-primary factors appearing in the presentation of $G$ as a direct product of cyclic groups is less than three.

The number of inequivalent finite cover germs $F: (U, \sigma') \to (V, \sigma)$ whose monodromy group $G_F$ is an abelian group $G$ is equal to the number of nonisomorphic decompositions $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ (plus one if $G$ is a cyclic group).
Proof. For every abelian group $G$, there exists a unique relatively prime subgroup of $G$, namely, the trivial subgroup. □

Corollary 3. The groups $G = Q_8^k \times \text{Ab}$, $k \in \mathbb{N}$, where

$$Q_8 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = [g_1, g_2], \ g_1^4 = g_2^4 = 1 \rangle$$

is the quaternion group and $\text{Ab}$ is any finite abelian group, cannot be realized as the monodromy groups of finite cover germs.

Proof. For every group $G = Q_8^k \times \text{Ab}$, there exists a unique relatively prime subgroup of $G$, namely, the trivial subgroup, and the groups $G$ are not contained in the list of groups generated by reflections. □

Proposition 12. The alternating group $A_4$ cannot be realized as the monodromy group of a finite cover germ.

Proof. By Proposition 11, if $A_4$ is the monodromy group of the germ $F: (U, o') \to (V, o)$ of some finite cover, $\deg_o' F = d$, then $d$ is either 4, or 6, or 12. By Corollary 1 we have $d \neq 4$, and by Theorem 6 we have $d \neq 12$, since the group $A_4$ is not a group generated by reflections.

Assume that there exists a germ $F$ of a finite cover with $\deg_o' F = 6$. Then the embedding $G_F = A_4 \to S_6$ corresponds to the action of $A_4$ on the left cosets of a relatively simple subgroup $H_F \subset A_4$ of order 2. Consider the Galoisification $F: (\tilde{U}, \tilde{o}) \to (V, o)$ of the cover $F$. The group $A_4$ acts on $(\tilde{U}, \tilde{o})$ in such a way that $(\tilde{U}, \tilde{o})/A_4 = (V, o)$ and $(\tilde{U}, \tilde{o})/H_1 = (U, o')$, where $H_1 = H_F$. The quotient cover $F_{H_F}: (\tilde{U}, \tilde{o}) \to (U, o')$ is a two-sheeted germ of a Galois cover. Therefore, it is branched along a germ of a singular curve $B_{H_F} \subset U$, since $(\tilde{U}, \tilde{o})$ is a germ of a singular (normal) surface by Theorem 6. Hence, the ramification curve $R_{H_F} \subset (\tilde{U}, \tilde{o})$ of $F$ is also a germ of a singular curve. Note also that the curve germ $R_{H_F}$ can be defined as

$$R_{H_F} = \{ p \in \tilde{U} : g(p) = p \text{ for } g \in H_F \}.$$
Obviously, $\deg F_{m,n,k} = m + n + 1$ and $F_{m,n,k}$ is ramified with multiplicity $m + 1$ along the curve germ $R_1$ given by $w = 0$ and with multiplicity $n + 1$ along the curve germ $R_2$ given by $w - z^k = 0$. Therefore, the branch locus of $F_{m,n,k}$ is $B = B_1 \cup B_2$, where $B_1 = F_{m,n,k}(R_1)$ and $B_2 = F_{m,n,k}(R_2)$.

We have

$$v = \sum_{i=0}^{n} \frac{(-1)^{n-i}}{m+i+1} z^{k(n-i)} w^{m+i+1}.$$  

It follows from (3.1) that the restriction $F_{m,n,k}|_{R_i} : R_i \to B_i$ of $F_{m,n,k}$ to $R_i$, $i = 1, 2$, is a biholomorphic mapping, $B_1$ is given by the equation $v = 0$, and $B_2$ is given by the equation

$$v = \sum_{i=0}^{n} \frac{(-1)^{n-i}}{m+i+1} u^{k(n+m+1)}.$$  

Therefore, the singularity type of the curve germ $B$ is $T(B) = A_{2k(m+n+1)}$.

By Lemma 6, the group $\pi_1^{loc}(B, o)$ is generated by two circuits $\gamma_1$ around $B_1$ and $\gamma_2$ around $B_2$, and it follows from the above discussion that $F_*(\gamma_1)$ is a cycle of length $m + 1$ and $F_*(\gamma_2)$ is a cycle of length $n + 1$ in the symmetric group $S_{m+n+1}$. Therefore, up to conjugation in $S_{m+n+1}$, we can assume that $F_*(\gamma_1) = (1, 2, \ldots, m, m + n + 1)$ and $F_*(\gamma_2) = (m + 1, m + 2, \ldots, m + n + 1)$, since $F_*(\gamma_1)$ and $F_*(\gamma_2)$ generate a transitive subgroup $G_{F_{m,n,k}}$ of $S_{m+n+1}$.

**Lemma 19.** Let $G = \langle \tau, \sigma \rangle \subset S_{m+n+1}$ be a subgroup generated by two cyclic permutations

$$\tau = (1, 2, \ldots, m, m + n + 1) \quad \text{and} \quad \sigma = (m + 1, m + 2, \ldots, m + n + 1), \quad m, n \in \mathbb{N}.$$  

Then $A_{m+n+1} \subset G$.

**Proof.** Without loss of generality, we can assume that $m \geq n$. Let us show that $G$ contains all cycles of length 3.

We have $\varrho = \tau \sigma = (1, 2, \ldots, m + n + 1) \in G$,

$$\eta_{m+1-i} = \tau^i \sigma \tau^{-i} = (m + 1 - i, m + 1, \ldots, m + n) \in G \quad \text{for} \quad i = 1, \ldots, m. \quad (3.2)$$  

If $n = 1$, then it follows from (3.2) that $G$ contains a transitive subgroup generated by the transpositions $(j, m + 1)$, $j = 1, \ldots, m$, and $\sigma$. Therefore, $G = S_{m+n+1}$. Hence, we can assume that $2 \leq n \leq m$. If $m = 2$, then a direct verification (which we left to the reader) shows that $\tau = (1, 2, 5)$ and $\sigma = (3, 4, 5)$ generate the group $A_5$. Therefore, we can assume that $m \geq 3$ and $m + n + 1 \geq 5$.

It is easy to check that

$$\eta_{j_1} \eta_{j_2}^{-1} = (j_1, j_2, m + n) \in G \quad \text{for} \quad 1 \leq j_1 < j_2 \leq m,$$

$$(\eta_{j_1} \eta_{j_2}^{-1})(\eta_{j_1} \eta_{j_3}^{-1})^{-1} = (j_1, j_2, j_3) \in G \quad \text{for} \quad 1 \leq j_1 < j_2 < j_3 \leq m.$$  

Therefore,

$$\varrho^{-k} [(\eta_{j_1} \eta_{j_2}^{-1})(\eta_{j_1} \eta_{j_3}^{-1})^{-1}] \varrho^k = (j_1 + k, j_2 + k, j_3 + k) \in G$$  

for $1 \leq j_1 < j_2 < j_3 \leq m$ and $0 \leq k \leq n + 1$, and hence it suffices to prove that if the alternating groups $A_k$ acting on the set $\{1, 2, \ldots, k\}$ and $A_3$ acting on the set $\{k - 1, k, k + 1\}$ are subgroups of $G \subset S_{m+n+1}$, then the alternating group $A_{k+1}$ acting on the set $\{1, 2, \ldots, k, k + 1\}$ is also a subgroup of $G$. However, this is obvious, since, first, the cycles $(j_1, j_2, j_3)$ belong to $G$ for $\{j_1, j_2, j_3\} \subset \{1, 2, \ldots, k\}$; second, the cycles $(j, k - 1, k + 1)$ and $(j, k, k + 1)$ belong to $G$ for $1 \leq j \leq k - 2$, since the cycles $(j, k - 1, k)$ and $(j, k - 1, k + 1)$ belong to $G$; finally, the cycles $(j_1, j_2, k + 1)$ belong to $G$ for $1 \leq j_1 < j_2 \leq k - 2$, since the cycles $(j_1, k, k + 1)$ and $(j_2, k, k + 1)$
belong to $G$. Therefore, to complete the proof of Lemma 19, it suffices to notice that the subgroup $H_{k+1} \subset G$ generated by all cycles $(j_1, j_2, j_3)$ of length 3, $1 \leq j_1 < j_2 < j_3 \leq k + 1$, is a normal subgroup of the group $A_{k+1}$ if $k \geq 4$, and hence $H_{k+1} = A_{k+1}$.

If one of the cycles $(1, 2, \ldots, m, m + n + 1)$ and $(m + 1, m + 2, \ldots, m + n + 1)$ is an odd permutation, then it follows from Lemma 19 that these cycles generate the symmetric group $S_{m+n+1}$, and if both these cycles are even permutations, then by the same lemma they generate the group $A_{m+n+1}$. Therefore, to complete the proof of Lemma 19, it suffices to notice that the subgroup $F_{m,n,k}(m \text{ or } n \text{ is an odd number})$ of finite covers with the monodromy group $G_{F_{m,n,k}} = S_N$. In the second case, if we put $2N = m + n + 2$, then we find that there are at least $(N - 1)/2$ infinite series $(k \in \mathbb{N})$ of the germs $F_{m,n,k}$ (m and n are even numbers) of finite covers with the monodromy group $G_{F_{m,n,k}} = A_{2N-1}$.

The rigidity of the germs $F_{m,n,k}$ follows from the rigidity of the curve germs of singularity type $A_{2k(n+m+1)-1}$ and from Theorem 5, since the monodromy $F_{m,n,k}$ is sole, i.e., it is uniquely (up to conjugation in $S_{m+n+1}$) determined by the triple $(m, n, k)$.

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