Contact interaction between a layered foundation and a system of annular punches with complex base shapes

K E Kazakov\(^1,2\) and S P Kurdina\(^2\)

\(^1\)Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences, Moscow, Russia
\(^2\)Bauman Moscow State Technical University, Moscow, Russia

E-mail: *kazakov-ke@yandex.ru*

Abstract. We study the contact interaction between a system of rigid annular punches and a viscoelastic two-layer foundation. The upper layer is thin compared with the punch width. We study the case where the punch shapes are described by a rapidly varying functions. We use special methods for constructing the solutions, because the standard methods are inefficient.

Introduction

Contact mechanics is a very important part of mechanics of solids, and it is of great practical interest. It can be used to solve many problems, for example, wheel-rail contact, in the calculation of couplings, brakes, tires, sliding and rolling bearings, internal combustion engines, hinges, seals; at stamping, metal working, ultrasonic welding, electric contacts, etc. The contact problems for bodies with coatings were first investigated in the second half of the twentieth century. Several contact problems related to the current paper were considered in [1–6]. These were plane and axisymmetric problems for conformal contact between layered foundations and one or several punches, contact problems for nonuniform foundations, rough punches, and bodies with complex rheology. Complex properties of coatings appears due to the peculiarity of their manufacture (see, for example, [7, 8]). In this paper, we for the first time solve the contact problem for a foundation with a homogeneous coating and a system of annular punches with complex base shapes.

1. Statement of the problem

A viscoelastic basement lies on a rigid basis. This basement consist of two layers: the upper layer made of viscoelastic material at time \(\tau_2\) and the lower layer made of a different viscoelastic material at time \(\tau_1\). Both layers are homogeneous, the height of the upper layer is \(h\) and the height of the lower layer is \(H\). We consider the case of smooth layer-layer and layer-basis contacts. We assume that the upper layer rigidity is less than the rigidity of the lower layer or they are of the same order of magnitude, i.e., Young’s modulus \(E_1(t)\) of the upper layer is less than Young’s modulus \(E_2(t)\) of the upper layer.

At time \(\tau_0 \geq \max\{\tau_1, \tau_2\}\), the forces \(P_i(t)\) start to indent smooth rigid annular punches (figure 1) into the surface of such a foundation. The coating is assumed to be thin compared
with the length of the contact area, i.e., its thickness satisfies the condition \( h \ll (b_i - a_i) \) for all \( i = 1, 2, \ldots, n \), where \( a_i \) and \( b_i \) are the interior and external radii of the \( i \)th punch and \( n \) is the quantity of punches. The base shape of the \( i \)th punch is described by a function \( g_i(r) \) \( (g_i(r) \geq 0, \exists r_{0i} \in [a_i, b_i]: g_i(r_{0i}) = 0) \). We consider the axisymmetric problem.

As a result of this interaction, the punches are immersed in the base at the depths \( \delta_i \) (\( \delta_i \) is the settlement of the \( i \)th punch).

It is easy to show that the system of integral equations and additional conditions for the described problem has the form \((i = 1, 2, \ldots, n)\)

\[
(1 - \nu_1^2)h \left[ \frac{q_i(r, t)}{E_1(t - \tau_1)} - \int_{\tau_0}^{t} \frac{q_i(r, \tau)}{E_1(\tau - \tau_1)} K_1(t - \tau_1, \tau - \tau_1) d\tau \right] \\
+ \frac{2(1 - \nu_2^2)}{H} \sum_{j=1}^{n} \left[ \int_{a_j}^{b_j} k_{ax} \left( \frac{r}{H}, \frac{\rho}{H} \right) \frac{q_j(\rho, t)}{E_2(t - \tau_2)} \rho d\rho \\
- \int_{\tau_0}^{t} K_2(t - \tau_2, \tau - \tau_2) \int_{a_j}^{b_j} k_{ax} \left( \frac{r}{H}, \frac{\rho}{H} \right) \frac{q_j(\rho, \tau)}{E_2(\tau - \tau_2)} \rho d\rho d\tau \right] = \delta_i(t) - g_i(r), \\
2\pi \int_{a_i}^{b_i} q_i(\rho, t) \rho d\rho = P_i(t), \quad K_k(t, \tau) = E_k(\tau) \frac{\partial}{\partial \tau} \left[ \frac{1}{E_k(\tau)} + C_k(t, \tau) \right],
\]

where \( q_i(r, t) \) are contact pressures under the punches, \( \nu_k, K_k(t, \tau), C_k(t, \tau) \) are the Poisson ratios, tensile creep kernels, and tensile creep functions of the upper \((k = 1)\) and lower \((k = 2)\) layers; \( k_{ax}(r/H, \rho/H) \) is the known kernel of the axisymmetric contact problem (see, e.g., [9]):

\[
k_{ax}(r, \rho) = \int_{0}^{\infty} L(u) J_0(ru) J_0(\rho u) \, du,
\]

where

\[
L(u) = \frac{\cosh 2u - 1}{\sinh 2u + 2u}
\]

and \( J_0(u) \) is the Bessel function of the first kind of order zero.
Let us change the variables in (1) and (2) by the formulas

\[(r^*)^2 = \frac{r^2 - a_1^2}{b_1^2 - a_1^2}, \quad (\rho^*)^2 = \frac{\rho^2 - a_2^2}{b_2^2 - a_2^2}, \quad t^* = \frac{t}{\tau_0}, \quad \tau_k^* = \frac{\tau_k}{\tau_0}, \quad \lambda = \frac{H}{b_1 - a_1}, \quad \eta_k = \frac{a_i}{b_1 - a_1}, \quad \zeta_i^2 = \frac{b_i^2 - a_i^2}{(b_1 - a_1)^2}, \quad \delta^* (t^*) = \frac{\delta_i(t) \zeta_i}{b_1 - a_1}, \quad g^i (r^*) = \frac{g_i(r) \zeta_i}{b_1 - a_1}, \quad c^i (t^*) = \frac{E_2(t - \tau_2) 1 - \nu_2^2}{E_1(t - \tau_1) 1 - \nu_2^2} \frac{h}{2(b_1 - a_1)}, \quad q^i(r^*, t^*) = \frac{2q_i(r, t)(1 - \nu_2^2) \zeta_i}{E_2(t - \tau_2)}, \quad P^i(t^*) = \frac{P_i(t)(1 - \nu_2^2) \zeta_i}{\pi E_2(t - \tau_2)(b_i^2 - a_i^2)}, \]

\[F^{ij} f(r^*) = \int_0^1 k^{ij}(r^*, \rho^*) f(\rho^*) \rho^* d\rho^*, \quad V_k^i f(t^*) = \int_1^{t^*} K_k(t^*, \tau^*) f(\tau^*) d\tau^*, \quad k^{ij}(r^*, \rho^*) = \frac{\zeta_i \zeta_j}{\lambda} \text{max} \left( \frac{r}{H}, \frac{\rho}{H} \right), \quad K^{1*}(t^*, \tau^*) = \frac{E_1(t - \tau_1) E_2(\tau - \tau_2)}{E_1(\tau - \tau_1) E_2(t - \tau_2)} K_1(t - \tau_1, \tau - \tau_1) \tau_0, \quad K^{2*}(t^*, \tau^*) = K_2(t - \tau_2, \tau - \tau_2) \tau_0, \quad i, j = 1, 2, \ldots, n. \]

Then, omitting the asterisks, we obtain the system of mixed integral equation and additional conditions in dimensionless form

\[c(t)(I - V_1)q(t, r) + (I - V_2) \sum_{j=1}^n F^{ij} q^j(t, r) = \delta^i(t) - g^i(r), \]

\[\int_0^1 q^i(\rho, t) \rho^* d\rho^* = P^i(t), \quad r \in [0, 1], \quad t \geq 1, \quad i = 1, 2, \ldots, n. \]

It is easy to show that there exist three versions of mathematical statements for the contact problem with a system of punches in the axisymmetric case. We construct the solution in the case with known forces \(P^i(t)\), unknown settlements \(\delta^i(t)\), and contact pressures \(q^i(r, t)\).

2. Operator representation

Assuming

\[q(r, t) = q^i(r, t) \mathbf{i}^i, \quad \delta(t) = \delta^i(t) \mathbf{i}^i, \quad P(t) = P^i(t) \mathbf{i}^i, \quad k(r, \rho) = k^{ij}(r, \rho) \mathbf{i}^i \mathbf{i}^j, \quad Ff(r) = \int_0^1 k(r, \rho) \cdot f(\rho) \rho d\rho. \]

we can represent system with additional conditions (4) as

\[c(t)(I - V_1)q(t, r) + (I - V_2)Fq(t, r) = \delta(t) - g(r), \]

\[\int_0^1 q(\rho, t) \rho d\rho = P(t), \quad r \in [0, 1], \quad t \geq 1. \]

Hereinafter, the summation will be over the repeated upper indices \(i\) and \(j\) from 1 to \(n\) if the left-hand side of the formula is independent of the index.

Thus, we study operator equation (5) with additional condition (6), where \(q(r, t)\) is a vector \(L_2((0, 1), V)\)-function continuous in time \(t\), \(P(t)\) and \(\delta(t)\) are vector functions continuous in time \(t\), and \(g(r)\) is a vector function in \(L_2((0, 1), V)\) (it can be a rapidly varying function). The compact operator \(F\) is a self-adjoint and positive definite operator from \(L_2((0, 1), V)\) to \(L_2((0, 1), V)\).
3. Solution of the problem

We seek \( q(r,t) \) in the form

\[
q(r,t) = Q(r,t) - g(r)(I - V_1)^{-1} \frac{1}{c(t)},
\]

where \( Q(r,t) \) is a new unknown vector function. Then the operator equation (5) and additional condition (6) become

\[
c(t)(I - V_1)Q(r,t) + (I - V_2)FQ(r,t) = \delta(t) + \tilde{c}(t)\bar{g}(r),
\]

\[
\int_0^1 Q(\rho,t)\rho \, d\rho = \tilde{P}(t), \quad r \in [0,1], \quad t \geq 1,
\]

\[
\tilde{c}(t) = (I - V_2)(I - V_1)^{-1} \frac{1}{c(t)} \bar{g}(r) = \int_0^1 k^{ij}(r,\rho)g^j(\rho)\rho \, d\rho,
\]

\[
\tilde{P}(t) = P(t) + (I - V_1)^{-1} \int_0^1 \frac{g(\rho)\rho \, d\rho}{c(t)}.
\]

We seek a solution of Eq. (7) under condition (8) in the class of vector functions continuous in time \( t \) in the Hilbert space \( L_2((0,1), V) \). We use the basis \( \{p_k(r)\}_{k=0,1,2,...} = \{p_k(r)i\}_{k=0,1,2,...} \), where \( \{p_k(r)\}_{k=0,1,2,...} \) is a system of orthogonal polynomials with respect to the \( L_2 \)-norm on the interval \([0,1]\) with weight \( r \), i.e., \( \int_0^1 p_k(r)p_{k'}(r)\rho \, dr = \delta_{k'k} \) (\( \delta_{k'k} \) is the Dirac delta function).

So the system \( \{p_k(r)\}_{k=0,1,2,...} \) satisfies the condition \( \int_0^1 p_k(r) \cdot \tilde{P}^{ij}_k(r)\rho \, dr = \delta_{k'k} \delta_{i'j'} \).

The Hilbert space \( L_2((0,1), V) \) can be represented as a direct sum of orthogonal subspaces \( L_2^{(0)}((0,1), V) \) and \( L_2^{(1)}((0,1), V) \), where \( L_2^{(0)}((0,1), V) \) is the Euclidean space with basis \( \{p_0(r)\}_{i=1,2,...,n} \) and \( L_2^{(1)}((0,1), V) \) is a Hilbert space with basis \( \{p_1(r), p_2(r), ...\}_{i=1,2,...,n} \). The integrand and the right-hand side of (7) can also be represented as an algebraic sum of vector functions continuous in time \( t \) and ranging in \( L_2^{(0)}((0,1), V) \) and \( L_2^{(1)}((0,1), V) \), respectively, i.e.,

\[
Q(r,t) = Q_0(r,t) + Q_1(r,t),
\]

\[
\delta(t) + \tilde{c}(t)\bar{g}(r) = \Delta_0(r,t) + \Delta_1(r,t),
\]

where \( Q_0(r,t), \Delta_0(r,t) \in L_2^{(0)}((0,1), V) \), \( Q_1(r,t), \Delta_1(r,t) \in L_2^{(1)}((0,1), V) \) and

\[
Q_0(r,t) = z_0^i(t)p_0^i(r), \quad \Delta_0(r,t) = \sqrt{2}g^i(t) + \frac{1}{c(t)}g^i(t),
\]

\[
\bar{g}_0 = \sum_{l=0}^{\infty} K_{0l}^i \int_0^1 p_l(\rho)g^l(\rho)\rho \, d\rho, \quad \bar{g}_1 = \bar{g}(r) - \bar{g}_0p_0^i(r),
\]

\[
\tilde{P}_k = \sum_{i=1}^{\infty} K_{kl}^i \int_0^1 k^{ij}(r,\rho)p_m(\rho)\rho \, d\rho,
\]

Note that the representation for \( Q(r,t) \) contains the known term \( Q_0(r,t) \), which is determined by the additional conditions (8)

\[
Q_0(r,t) = z_0^i(t)p_0^i(r), \quad z_0^i(t) = \sqrt{2}P^i(t),
\]

and the term \( Q_1(r,t) \) to be found. Conversely, for the right-hand side, one should find \( \Delta_0(r,t) \), while \( \Delta_1(r,t) \) is known. These specific features permit classifying the obtained problem as a specific case of the generalized projection problem stated and solved in [10].
We can introduce the orthogonal projection operators, mapping the space $L_2((0,1),V)$ onto $L_2^0((0,1),V)$ and $L_2^0((0,1),V)$:

$$P_0 f(r) = \int_0^1 f(\rho) \cdot p_0^l(\rho) \rho \, d\rho, \quad P_1 = I - P_0.$$  

In addition, the following relations hold: $P_1 Q_l(r,t) = Q_l(r,t)$, $P_l [\delta(t) + c(t)\tilde{g}(r)] = \Delta_l(t,r)$ ($l = 0, 1$).

We apply the orthogonal projection operator $P_1$ to Eq. (7). As a result, we obtain the equation for determining $Q_1(r,t)$ with a known right-hand side

$$c(t)(I - V_1)Q_1(r,t) + (I - V_2)P_1 FQ_1(r,t) = -(I - V_2)P_1 FQ_0(r,t) + c(t)\tilde{g}_1(r). \quad (11)$$

It is necessary to construct its solution in the form of a series in the eigenfunctions of the operator $P_1 F$, which is a compact strongly positive self-adjoint operator from $L_2(0,1)$ to $L_2^0((0,1),V)$. The system of eigenfunctions of such an operator is a basis in the space $L_2^1((0,1),V)$, see [11]. The spectral problem for the operator $P_1 F$ can be written as

$$P_1 F \varphi_k(r) = \gamma_k \varphi_k(r), \quad \varphi_k(r) = \sum_{m=1}^\infty \psi_{km}^l P_m(r), \quad k = 1, 2, 3, \ldots$$

Now it is necessary to solve the system for the coefficients $\psi_{km}^l$ and $\gamma_k$

$$\sum_{l=1}^\infty K_{ml}^{ij} \psi_{kl}^j = \gamma_k \psi_{km}^i, \quad k, m = 1, 2, 3, \ldots, \quad i = 1, 2, \ldots, n,$$

where $K_{ml}^{ij}$ are determined by formula (10). We expand the functions $Q_1(r,t)$ and $\tilde{g}_1(r)$ with respect to the basis $\{\varphi_k(r)\}_{k=1,2,3,\ldots}$ in $L_2^1((0,1),V)$, i.e.,

$$Q_1(r,t) = \sum_{k=1}^\infty z_k(t) \varphi_k(r), \quad \tilde{g}_1(r) = \sum_{k=1}^\infty g_k \varphi_k(r),$$

substitute these representations, (9) and (10) into (11), and see that the unknown expansion functions $z_k(t)$ ($k = 1, 2, 3, \ldots$) can be determined by the formula

$$z_k(t) = (I + W_k) \frac{(I - V_2)[g_k(I - V_1)^{-1}c^{-1}(t) - K_{0k}^l z_0^l(t)]}{c(t) + \gamma_k},$$

$$K_{0k}^l = \sum_{m=1}^\infty K_{0m}^{ij} \psi_{km}^j, \quad g_k = \sum_{m=1}^\infty \psi_{km}^i \sum_{l=0}^\infty K_{ml}^{ij} \int_0^1 p_l(\rho) g^j(\rho) \rho \, d\rho, \quad W_k f(t) = \int_1^t R_k^l(t,\tau)f(\tau) \, d\tau,$$

where $R_k^l(t,\tau)$ ($k = 1, 2, 3, \ldots$) are resolvents of the kernel $K_k^l(t,\tau) = [c(t)K^l(t,\tau)]/[c(t) + \gamma_k]$. Note that the obtained solution has the following structure

$$q(r,t) = z_0^l(t)p_0^l(r) + \sum_{m=1}^\infty \left[ \sum_{k=1}^\infty \psi_{km}^l z_k(t) \right] p_m^l(r) - g(r)(I - V_1)^{-1} \frac{1}{c(t)},$$

i.e., one can explicitly distinguish the function $g(r)$, and hence the punch base shape functions $g_i(r)$ related to it by the change of variables (3). The formulas thus obtained permit
deriving efficient analytic solutions for contact problems for layers with thin coatings and a system of punches whose base shapes are described by rapidly varying functions. This can hardly be done by other well-known methods.

Determining the contact pressures under the punches, we can find the unknown settlements of punches. To this end, we must apply the operator $P_0$ to Eq. (7):

$$c(t)(I - V_1)Q_0(r,t) + (I - V_2)P_0[F(Q_0(r,t) + Q_1(r,t))] = \Delta_0(r,t).$$

The obtained relation readily implies the formulas for the settlements $\delta^i(t)$

$$\delta(t) = \sqrt{2} \left\{ c(t)(I - V_1)z_0(t) + (I - V_2) \left[ K_{00}^{ij}z_0^j(t) + \sum_{k=1}^{\infty} K_{0k}^{ij}z_k(t) - g_0^i(I - V_1)^{-1} \frac{1}{c(t)} \right] \right\}.$$

Thus, we have constructed an analytic solution of the axisymmetric contact problem for a viscoelastic layer with a thin coating, which permits efficiently solving problems with complicated functions describing the shapes of the punch bases.

**Conclusions**

We pose and solve axisymmetric contact problems for viscoelastic aging basements with coatings and a system of rigid punches with complicated shapes of their bases. The solution of the problem is obtained analytically, and, in the expressions for the contact stresses, the punch base shape functions are distinguished explicitly, which allows one to perform computations for the real profiles of punch bases, which are described by rapidly varying functions. The explicit formulas were obtained for the settlements of the punches.

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