BOUNDEDNESS OF SPECTRAL MULTIPLIERS OF GENERALISED LAPLACIANS ON COMPACT MANIFOLDS WITH BOUNDARY

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Abstract. Consider a second order, strongly elliptic differential operator \( L \) (maybe a system) on a compact Riemannian manifold \( M \) with smooth boundary, where \( L \) is defined by coercive boundary conditions. We establish the \( L^\infty - \text{BMO} \) and \( L^p \) continuity of \( \psi(\sqrt{-L}) \) for a class of functions \( \psi \) and use it to establish the \( L^p \)-continuity of \( \psi(\sqrt{-L}) \) for a range of exponents \( p \). Finite propagation speed of \( \text{cost}\sqrt{-L} \) is fundamental to our calculations, after the Cheeger-Gromov-Taylor style of analysis (\cite{CGT}). Accordingly we try to derive sufficiency conditions on \( L \) to ensure the finite propagation speed of \( \text{cost}\sqrt{-L} \).

1. Introduction

Consider a compact manifold with boundary \( \overline{M} \) and a second order, negative semidefinite strongly elliptic differential operator \( L \) acting on \( L^2(M) \). Let the domain of \( L \) be \( \mathcal{D}(L) \subset H^2(M) \). Also, assume regular elliptic boundary conditions on \( L \) which additionally make \( L \) into a self-adjoint operator. We want to explicitly point out that often times, our analysis will work when \( L \) is a system. Given a bounded continuous function \( \psi : \mathbb{R} \rightarrow \mathbb{R} \), the spectral theorem defines

\[
\psi(\sqrt{-L}) : L^2(M) \rightarrow L^2(M)
\]

as a bounded operator. Here we consider functions \( \psi \) in the pseudodifferential function class \( S^0_1(\mathbb{R}) \). To recall what this means,

\[
\varphi \in S^0_1(\mathbb{R}) \implies |\varphi^{(k)}(t)| \lesssim (1 + |t|)^{-k}, k = 0, 1, 2, ...
\]

Here our main aim is to establish results of the form

\[
\varphi(\sqrt{-L}) : L^\infty \rightarrow \text{BMO}_L
\]

(see Section 4) and then use interpolation with (1.1) to prove that

\[
\varphi(\sqrt{-L}) : L^p \rightarrow L^p, \quad \forall p \in (1, \infty)
\]

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Here our main tool will be the following functional calculus, as used in \[CGT\], namely

\begin{equation}
\phi(\sqrt{-L}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\phi}(t)e^{it\sqrt{-L}} dt
\end{equation}

Also, since the spectrum of \(\sqrt{-L}\) is discrete and lands inside \([0, \infty)\), it is no loss of generality to assume that \(\phi(\lambda)\) is an even function of \(\lambda\). This reduces (1.5) to

\begin{equation}
\phi(\sqrt{-L}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\phi}(t)\cos t \sqrt{-L} dt
\end{equation}

where \(\cos t \sqrt{-L}\) is the solution operator to the “wave equation” with zero initial velocity, i.e.,

\begin{equation}
\phi(\lambda) = \int_{-\infty}^{\infty} \cos t \sqrt{-L} dt
\end{equation}

(1.7)

\[u(t,x) = \cos t \sqrt{-L} f(x)\]

where

\begin{equation}
\partial_t^2 u - Lu = 0, \quad u(0,x) = f(x), \quad \partial_t u(0,x) = 0
\end{equation}

(1.8)

together with the coercive boundary conditions mentioned above.

Let us also explain the reason why the finite propagation speed of \(\cos t \sqrt{-L}\) is so important. One of the main positive points of using the functional calculus (1.6) is that we can split it into two parts in the following way:

\begin{equation}
\phi(\lambda) = \int_{-\infty}^{\infty} \hat{\phi}(t)\cos t \sqrt{-L} dt
\end{equation}

(1.9)

Then, we can write

\begin{equation}
\phi(\lambda) = \phi^#(\lambda) + \phi^b(\lambda)
\end{equation}

(1.10)

That is

\begin{equation}
\phi(\lambda) = \phi^#(\lambda) + \phi^b(\lambda)
\end{equation}

(1.11)

\[\phi^b(\lambda) \leq (1 + |\lambda|^{-m}), \text{ for some } m > \frac{n}{2}\]

Then, the ellipticity of \(L\) implies

\begin{equation}
\phi^b(\sqrt{-L}) : L^2(M) \rightarrow H^m(M) \subset C(M)
\end{equation}

(1.12)

Now, dual of of \(C(M)\) is all finite Borel measures, and \(L^1\) is a subset of it. In fact, \(L^1\) is the closure of \(L^2\) in it. This gives

\begin{equation}
\phi^b(\sqrt{-L}) : L^1(M) \rightarrow L^2(M)
\end{equation}

(1.13)

This interpolates with (1.1) to give that

\begin{equation}
\phi^b(\sqrt{-L}) : L^p(M) \rightarrow L^p(M) \forall p \in (0, \infty)
\end{equation}

(1.14)
So, for all continuity related aspects, we can be concerned with just $\hat{\varphi}(\sqrt{-L})$ provided we have the finite propagation speed of $\cos t\sqrt{-L}$.

1.1. Main tools. Let us denote by $p(t, x, y)$ the integral kernel of $e^{tL}$, so that
\[ e^{tL} f(x) = \int_M p(t, x, y) f(y) dy. \]
Henceforth, by straightforward analogy, we will refer to $p(t, x, y)$ as the “heat kernel”. Amongst our most important tools will be the following estimates in [G]:

**Proposition 1.1. (Greiner) For some $\kappa \in (0, \infty)$**

\[
|p(t, x, y)| \lesssim t^{-n/2} e^{-\kappa d(x,y)^2/t}, t \in (0, 1], x \in M
\]
and
\[
|\nabla_x p(t, x, y)| \lesssim t^{-n/2-1/2} e^{-\kappa d(x,y)^2/t}, t \in (0, 1], x \in M
\]

We note here that [G] has to use a theory of parabolic layer potentials because of the existence of a non-trivial boundary.

Let us review some literature here. Arguably the most well-studied case has been for $M$ complete (stronger results for compact) without boundary and $L = \Delta$, the Laplace-Beltrami operator. For more details, refer to [CGT], [MMV], [T4], [T5], [T2]. In the papers which deal with a non-compact setting, an additional difficulty is in analysing $\varphi_b(\sqrt{-\Delta})$, because of the failure of the compact Sobolev embedding. Particularly for spaces which do not have bounded geometry, like the hyperbolic space, where the volume of a radius $r$-ball grows exponentially with $r$, one requires more stringent restrictions on $\varphi$, namely, $\varphi$ being holomorphic on a strip around the $x$-axis, satisfying bounds of the form (1.2). This condition was first introduced by the paper [CS]. Also, in [CGT], $\varphi_b(\sqrt{-\Delta})$ was analysed as a pseudodifferential operator, something we cannot do in our present setting, as even the square root of the Dirichlet Laplacian does not admit interpretation as a pseudodifferential operator. Our analysis will be a hybrid of the styles of [DM] which uses mainly probabilistic arguments at a semigroup level on one hand, and the approach of [T5], which follows and refines the results in [MMV] (and also [T2], whose approach is closely similar to [T3]) on the other.

1.2. Outline of the paper. Here, we take the space to describe the overall outline of the paper. In Section 2, we establish some useful information about the “heat kernel” and “Poisson kernel”, which are the integral kernels of the operators $e^{tL}$ and $e^{-t\sqrt{-L}}$ respectively. These are quite parallel to their usual Laplacian counterparts. In Section 3, we investigate some sufficient criteria for $\cos t\sqrt{-L}$ to have finite propagation speed, as that is central to all our calculations. In Section 4, we prove the $L^\infty - \text{BMO}_L$ continuity of $\varphi(\sqrt{-L})$ and use these estimates to prove $L^p$-continuity by interpolation.
2. SOME LEMMAS: GETTING FAMILIAR WITH THE "HEAT" AND "POISSON" KERNELS

In this section, we quickly establish a few properties of the "heat semi-group" $e^{tL}$ and the "Poisson semigroup" $e^{-t\sqrt{-L}}$ in analogy with the corresponding semigroups of the Laplacian.

Clearly, $e^{tL}$ gives a contraction semigroup on $L^2$. For later use, we also establish the following

**Lemma 2.1.**

(2.1) $||e^{tL}||_{L(L^2,L^p)} \lesssim (1 + t^{-n/4-1/2})$, $t > 0$

**Proof.** We will first use the gradient estimate on the "heat"-kernel, namely (1.16) to prove that

(2.2) $\int_M |\nabla_x p(t, x, y)| dy \lesssim t^{-1/2}, t \in (0, 1]$

Now using (1.16), we see that

(2.3) $\int_M |\nabla_x p(t, x, y)| dy \lesssim t^{-n/2-1/2} \int_M e^{-\kappa d(x,y)^2}/t dy$

If we can prove $\int_M e^{-\kappa d(x,y)^2}/t dy \lesssim t^{n/2}$, we are done by the compactness of $M$.

Calling

(2.4) $f(x, t) = \int_M e^{-\kappa d(x,y)^2}/t dy$

we consider the identity mapping $i : (M, g) \rightarrow (M, \sqrt{\kappa}g)$. That gives,

$$
\int_M e^{-\kappa d(x,y)^2}/t dy = \int_M e^{-d(x,z)^2}|J_i| dz \\
\approx t^{n/2} \int_M e^{-d(x,z)^2} dz \\
\approx t^{n/2}
$$

where $J_i$ denotes the Jacobian of the map $i$ and we also use the compactness of $M$ again.

By an absolutely analogous method, we can derive the estimate

(2.5) $\int_M |\nabla_x p(t, x, y)|^2 dy \lesssim t^{-n/2-1}, t \in (0, 1]$

We have, as usual,

$$
e^{tL}f(x) = \int_M p(t, x, y)f(y) dy$$
implying (by differentiating under the integration sign)
\[
\nabla_x e^{tL} f(x) = \int_M \nabla_x p(t, x, y) f(y) dy
\]
\[
\leq ||\nabla_x p||_{L^2} ||f||_{L^2}
\]
which gives, by (2.5),
\[
(2.6) \quad ||e^{tL}||_{L^2(L^2,\text{Lip})} \leq t^{-n/4-1/2}, \quad t \in (0, 1]
\]
Now, if Spec\((-L) \subset [\rho, \infty)\), then for \(t > 0\) we have
\[
(2.7) \quad |\nabla e^{tL} f(x)| \lesssim e^{-t\rho} ||f||_{L^2}
\]
So, putting (2.6) and (2.7) together, we have
\[
||e^{tL}||_{L^2(L^2,\text{Lip})} \lesssim (1 + t^{-n/4-1/2}), \quad t > 0
\]
□

Another proof of the above fact is available in Section 4 of [T 2].
Lastly, for illustrative purposes, we use the above estimates to derive the following conclusions about the Poisson semigroups (also compare [T2], Section 5).

Lemma 2.2.

\[
(2.8) \quad ||e^{-t\sqrt{-L}}||_{\mathcal{L}(L^2,L^\infty)} \lesssim (1 + t^{-n/2})
\]
and
\[
(2.9) \quad ||e^{-t\sqrt{-L}}||_{\mathcal{L}(L^2,\text{Lip})} \lesssim (1 + t^{-n/2-1}),
\]

Proof. We already know that
\[
(2.10) \quad ||e^{tL}||_{\mathcal{L}(L^2,L^\infty)} \lesssim (t^{-n/4} + 1)
\]
Firstly, the estimate on \(e^{-t\sqrt{-L}}\) can be obtained from the Subordination identity
\[
(2.11) \quad e^{-t\sqrt{-L}} \approx \int_0^\infty t e^{-t^2/4s} s^{-3/2} e^s L ds
\]
So,
\[
(2.12) \quad ||e^{-t\sqrt{-L}}||_{\mathcal{L}(L^2,L^\infty)} \lesssim \int_0^\infty t e^{-t^2/4s} s^{-3/2} ||e^s L||_{\mathcal{L}(L^2,L^\infty)} ds
\]
\[
\lesssim \int_0^\infty t e^{-t^2/4s} s^{-3/2} (s^{-n/4} + 1) ds
\]
\[
\lesssim t \int_0^\infty e^{-t^2/4s} s^{-n+6/4} ds + t \int_0^\infty e^{-t^2/4s} s^{-3/2} ds
\]
Calling the first integral above \(I_1\) and the second one \(I_2\), we get
\[
(2.15) \quad I_2 \leq c_0
\]
where $c_n$ is a multiple of $\Gamma\left(\frac{n+1}{2}\right)$ (see \[T6\] for details, particularly, pp 247-248). Similarly,

$$I_1 = t \int_0^\infty e^{-t^2/4s} s^{-\frac{n+2}{2}} \, ds$$

$$\leq c_n/2 \frac{t}{(t^2)^{n/4}}$$

$$\leq c_n/2 \frac{t^{-n/2}}{2}$$

from (5.24) of \[T6\], pp 247. This gives us an estimate on \(\|e^{tL}\|_{L(L^2,L^\infty)}\).

Now we write, when \(t \in (0,1]\),

\[
\|\nabla e^{-t\sqrt{-L}}f\|_{L^\infty} = \|\nabla e^{tL} e^{-t\sqrt{-L}} e^{-tL} f\|_{L^\infty}
\]

\[
\lesssim (t^{-n/4-1/2}) \|e^{-t\sqrt{-L}} e^{-tL} f\|_{L^2}
\]

\[
\lesssim (t^{-n/4-1/2}) \|e^{-t\sqrt{-L}} f\|_{L^2}
\]

the last step coming from elliptic regularity. And when \(t \in [1,\infty)\)

\[
\|\nabla e^{-t\sqrt{-L}}f\|_{L^\infty} = \|\nabla e^{tL} e^{-t\sqrt{-L}} e^{-tL} f\|_{L^\infty}
\]

\[
\lesssim \|e^{-t\sqrt{-L}} e^{-tL} f\|_{L^2}
\]

\[
\lesssim \|e^{-t\sqrt{-L}} f\|_{L^2}
\]

\[
\lesssim \|f\|_{L^2}
\]

the last step coming from elliptic regularity. This proves the lemma. \(\square\)

3. Finite propagation speed of \(\cos t\sqrt{-L}\).

In this section we investigate some sufficient criteria for \(\cos t\sqrt{-L}\) to have finite propagation speed.

One class of operators for which one can readily establish finite propagation speed is those \(L\) which can be written as

\[
-L = \sum_{i=1}^k D_i^* D_i + H
\]

where \(D_i\) are first order elliptic differential operators with either the Dirichlet or Neumann boundary condition (see \[(3.3)\] and \[(3.4)\] below), and \(H\) is a continuous potential for now. For simplicity, we make use of the so-called Davies-Gaffney estimates on the operator \(\cos t\sqrt{-L}\) for \(-L = D^* D + H\), where \(D\) is a first order elliptic operator with either the Dirichlet or Neumann boundary condition. To recall,

**Definition 3.1.** An operator \(L\) satisfies the Davies-Gaffney estimates on a manifold \(M\) if

\[
(c^{tL} u, v) \leq e^{-\frac{t^2}{2}} \|u\|_{L^2} \|v\|_{L^2}
\]
for all $t > 0$, for all pairs of open subsets $U, V$ of $M$, supp $u \subset U$, supp $v \subset V$, sections $u \in L^2(U), v \in L^2(V)$ and $r = d(U, V)$, the metric distance between $U$ and $V$.

We also recall the following

**Lemma 3.2.** For a self-adjoint negative-definite $L$ on $L^2$, satisfaction of the Davies-Gaffney estimates is equivalent to finite propagation speed property.

It might be pointed out that this method is eminently suited to establishing finite propagation speed type results particularly when the manifold has boundary or is less “nice” in some other way, as the theorem above holds in the great generality of metric measure spaces $(X, d, \mu)$, where $\mu$ is a Borel measure with respect to the topology defined by $d$.

So, to establish the finite propagation speed of $\cos t\sqrt{-L}$ in our current setting, we attempt to establish the Davies-Gaffney estimates. Let $-L = D^*D + H$, where $D : H^{s+1}(M, E) \to H^2(M, F)$ be a first-order differential operator\[^1\] between sections of vector bundles. Assume that the symbol $\sigma_D(x, \xi) : E_x \to F_x$ is injective for $x \in M, \xi \in T^*_x M \setminus 0$. Following [12], consider the following generalisation of the Dirichlet condition on $D(D)$:

\begin{equation}
(3.3) \quad u \in D(D) \Rightarrow \beta(x)u(x) = 0, \quad \forall x \in \partial M
\end{equation}

where $\beta(x)$ is an orthogonal projection on $E_x$ for all $x \in \partial M$.

We also consider the following generalisation of the Neumann boundary condition:

\begin{equation}
(3.4) \quad u \in D(D) \Rightarrow \gamma(x)\sigma_D(x, \nu)u(x) = 0, \quad \forall x \in \partial M
\end{equation}

where $\nu(x)$ is the outward unit normal to $\partial M$ and $\gamma(x)$ is an orthogonal projection on $E_x$ for all $x \in \partial M$.

We note that both these boundary conditions have the consequence that

\begin{equation}
(3.5) \quad \langle \sigma_D(x, \nu)v, w \rangle = 0, \quad \forall x \in \partial M
\end{equation}

when $v \in D(D), w \in D^*(D)$.

We have,

**Proposition 3.3.** Let $-L = D^*D + H$ with the generalised Dirichlet or Neumann boundary conditions on $D$, as defined above. Also, assume that \[^3\] below holds. Then $\cos t\sqrt{-L}$ has finite speed of propagation.

**Proof.** Let us start by assuming that $H \geq 0$. We first observe that by the Cauchy-Schwarz inequality

\begin{equation}
(3.6) \quad (e^{tL}u, v) \leq ||\chi Ve^{tL}u||_{L^2}||v||_{L^2}
\end{equation}

\[^1\]Interpreted as operators factoring through first order jet bundles. Note that by the Peetre theorem, differential operators are also exactly the linear operators which decrease support, i.e., have the local property.
where $\chi_V$ represents the characteristic function of $V$.
So, to get to what we want, we are looking to establish a relation like

$$
||\chi_V e^{tL} u||_{L^2} \leq e^{-\frac{r^2}{4t}} ||u||_{L^2}
$$

when $\text{supp} u \subset U$. Also, $\text{supp} v \subset V$, $\text{dist}(U, V) = r$, which is fixed. Now, let $w = \chi_V e^{tL} u$ and $\varphi(x) = \rho \text{dist}(x, U)$, $\rho$ being a constant equalling $\frac{r}{4}$. Then we have

$$
\int_V |w|^2 dx \leq e^{-\rho r} \int_V \langle w, w \rangle e^{\rho \varphi(x)} dx
$$

Let us define

$$
E(t) = \int_M \langle e^{tL} u, e^{tL} u \rangle e^{\rho \varphi(x)} dx
$$

Differentiating, we get

$$
\frac{1}{2} E'(t) = Re \int_M \langle \partial_t e^{tL} u, e^{tL} u \rangle e^{\rho \varphi(x)} dx
$$

$$
= Re \int_M \langle Le^{tL} u, e^{tL} u \rangle e^{\rho \varphi(x)} dx
$$

$$
= -Re \int_M \langle (D^* D + H) e^{tL} u, e^{tL} u \rangle e^{\rho \varphi(x)} dx
$$

$$
= -Re \int_M \langle De^{tL} u, D(e^{tL} u e^{\rho \varphi(x)}) \rangle dx + Re \int_M H \langle e^{tL} u, e^{tL} u \rangle e^{\rho \varphi(x)} dx
$$

$$
+ Re \frac{1}{i} \int_{\partial M} \langle \sigma(x, \nu) e^{tL} u e^{\rho \varphi(x)}, De^{tL} u \rangle dS
$$

$$
= -Re \int_M \langle De^{tL} u, De^{tL} u \rangle e^{\varphi(x)} dx + Re \int_M \langle De^{tL} u, [D, e^{\rho \varphi(x)}] e^{tL} u \rangle dx
$$

$$
+ Re \int_M H \langle e^{tL} u, e^{tL} u \rangle e^{\varphi(x)} dx + Re \frac{1}{i} \int_{\partial M} \langle \sigma(x, \nu) e^{tL} u e^{\rho \varphi(x)}, De^{tL} u \rangle dS
$$

$$
\leq -Re \int_M \langle De^{tL} u, De^{tL} u \rangle e^{\varphi(x)} dx + Re \int_M \langle De^{tL} u, [D, e^{\rho \varphi(x)}] e^{tL} u \rangle dx
$$

$$
\leq \frac{\rho^2}{4} \int_M \langle e^{tL} u, e^{tL} u \rangle e^{\rho \varphi(x)} dx
$$

using the facts that $H \geq 0$ and that under the Dirichlet or the Neumann boundary condition, the last term $\int_{\partial M} \langle \sigma_D(x, \nu) e^{tL} u e^{\rho \varphi(x)}, Du \rangle dS$ disappears.

We now want to say that the above quantity is

$$
\leq \frac{\rho^2}{4} \int_M \langle e^{tL} u, e^{tL} u \rangle e^{\varphi(x)} dx
$$
Now what is the condition that allows this? Let us define \( P = [D, e^{\varphi(x)}] \).
Now we have
\[
4(De^{tL}u, Pe^{tL}u) = 4(e^{\varphi(x)/2}De^{tL}u, e^{-\varphi(x)/2}Pe^{tL}u) \\
\leq 4||e^{\varphi(x)/2}De^{tL}u||^2_L + ||e^{-\varphi(x)/2}Pe^{tL}u||^2_L
\]
So the correct condition seems to demand that
\[
||e^{-\varphi(x)/2}Pe^{tL}u||^2_L \leq \rho^2||e^{\varphi(x)/2}e^{tL}u||^2_L
\]
or,
\[
(3.11) \quad ||e^{-\varphi/2}Pv||^2_{L^2} \leq \rho||e^{\varphi/2}v||^2_{L^2}
\]
So, now we can say
\[
(3.12) \quad E'(t) \leq \rho^2/2E(t)
\]
This gives, by Gronwall’s inequality, \( E(t) \leq e^{\rho^2t/2}E(0) \).
Plugging everything back, we have
\[
\int_V |w|^2 dx \leq e^{\rho^2t^2/2-\rho r}||u||^2_{L^2}
\]
Using \( \rho = r/t \), we have
\[
(3.13) \quad \int_V |w|^2 dx \leq e^{-r^2/4t}||u||^2_{L^2}
\]
This proves what we want, modulo the fact that we need to extend the result to include all functions \( H \) such that \( -L - H \) is positive definite. For this, see below. □

**Remark 3.4.** If \( L \) is the Laplace Beltrami operator with Dirichlet or Neumann boundary condition, then (3.11) holds trivially, which gives us back the special case of finite propagation speed of \( \cos tx \sqrt{-\Delta} \).

Now, we extend the range of \( H \) to \( H \in L^2(\mathcal{M}) \). Towards that end, pick \( H_n \) continuous such that \( H_n \rightarrow H \) and consider \( \mathcal{L}_n \) given by \( -\mathcal{L}_n = D^*D + H_n \).
Let \( -\mathcal{L} = D^*D + H \).
We can see that \( \mathcal{L}_n(u) \rightarrow \mathcal{L}(u) \) for \( u \in \mathcal{D}(L) \) as \( n \rightarrow \infty \). That means, \( \mathcal{L}_n \rightarrow \mathcal{L} \) in the strong resolvent sense as \( n \rightarrow \infty \) (see [RS]). Then, \( \cos tx \) and \( e^{-tx} \) being bounded continuous functions on \( \mathbb{R} \) for all \( t > 0 \), we have
\[
\forall u \in \mathcal{D}(D) \quad \cos t\sqrt{-\mathcal{L}_n}u \rightarrow \cos t\sqrt{-\mathcal{L}}u \\
e^{t\mathcal{L}_n}u \rightarrow e^{t\mathcal{L}}u
\]
The finite propagation speed of \( \cos t\sqrt{-\mathcal{L}} \) now follows by the Davies-Gaffney estimates: if for a fixed pair \( U, V \subset \mathcal{M} \) of open sets \( u, v \) such that \( \text{supp } u \subset U, \text{supp } v \subset V \), \( r = \text{dist } (U, V) \), we have
\[
(3.14) \quad (e^{t\mathcal{L}_n}u, v) \leq e^{-r^2/4t}||u||^2_{L^2}||v||^2_{L^2}
\]
then in the limit, we must have
\[
(e^{tL}u, v) \leq e^{-\frac{t^2}{4}} \|u\|_{L^2} \|v\|_{L^2}
\]
In fact, that we can use finite propagation speed of a family of operators to conclude about the strong limit of the family is one of the nicest properties of the Davies-Gaffney estimates. It seems difficult to find an analogous straightforward argument through the strong convergence of \(\cos t\sqrt{-L_y}\). Indeed, each of the \(\cos t\sqrt{-L_y}\) might have finite propagation speed \(K_y\), such that \(K_y\) increase without bound.

3.1. Sufficiency conditions. We now want to talk about sufficiency conditions ensuring \(\cos t\sqrt{-L}\) with coercive boundary condition has finite speed of propagation. We want to argue that this will happen always, provided \(\cos t\sqrt{-L}\) without any boundary conditions (or with the Dirichlet/Neumann boundary condition) has finite speed of propagation Heuristically, here is how the argument proceeds. Consider initial data supported away from the boundary and look at the solution after a small time \(t\). The function which solves \(Lu = \partial_t^2 u\) on \(M\) subject to \(Bu|_{\partial M} = 0\) will also solve \(Lu = \partial_t^2 u\) when the boundary is not present. Since we have finite propagation speed for \(\cos t\sqrt{-L}\) in the interior, the solution for small time cannot spread far away from the initial support \(K\). Also, it cannot be non-zero on the boundary, as that would violate the fact that \(u \in D(L) \subset H^2(M)\). Similarly, if the initial data is supported on \(K'\) which intersects \(\partial M\), any solution of \(Lu = \partial_t^2 u\) with this initial support and satisfying the boundary condition \(Bu|_{\partial M} = 0\) solves \(Lu = \partial_t^2 u\) if the boundary were not present. So the solution cannot spread far beyond the initial support in the interior, and the spread on the boundary cannot be faster than in the interior, otherwise it would violate that \(u \in D(L) \subset H^2(M)\).

4. \(L^\infty - \text{BMO}_L\) continuity and \(L^p\)-continuity

In this section we investigate the continuity property of \(\varphi(\sqrt{-L})\) on BMO-spaces. It is known that the \(L^p\) continuity results for even \(\varphi(\sqrt{-\Delta})\) cannot be extended to the endpoint cases \(L^1\) and \(L^\infty\). So if we want to have an extension at all, we must use Hardy and BMO spaces respectively as their replacements.

We use the John-Nirenberg bounded mean oscillation spaces, as introduced in [JN], but we interpret them to suit the operator \(-L\) instead of the Laplace-Beltrami operator, following [DY]. We give the following

**Definition 4.1.** \(f \in L^1_{\text{loc}}(M)\) is in \(\text{BMO}_L(M)\) if
\[
\int_B |f(x) - e^{tL}f(x)|\,dx \leq C
\]
where \(\sqrt{t}\) is the radius of the ball \(B\), and \(B\) ranges over all balls in the manifold of radius \(\leq \epsilon\) where \(\epsilon\) satisfies \(\frac{1}{2\sqrt{\epsilon}} \geq \text{diam} M\).
Proof. We have
\[ \frac{1}{B_{2r}(y)} \int_{B_{2r}(y)} |f(x) - e^{2rL}f(x)|dx - \frac{1}{B_r(y)} \int_{B_r(y)} |f(x) - e^{rL}f(x)|dx \]
\[ \leq \frac{1}{B_r(y)} \int_{B_{2r}(y)} |f(x) - e^{2rL}f(x) - \chi_{B_{2r}(y)}f(x) + \chi_{B_r(y)}e^{rL}f(x)|dx \]
\[ \leq \frac{1}{B_r(y)} \int_{B_{2r}(y)} \left| \chi_Af(x) - e^{2rL}f(x) + \chi_{B_r(y)}e^{rL}f(x) \right|dx \]
where \( A \) denoted the “annulus” \( B_{2r}(y) \setminus B_r(y) \), which can be covered by balls of radius \( r \). The last quantity is
\[ \leq \frac{1}{B_r(y)} \int_{B_{2r}(y)} |\chi_Af(x) - \chi_{B_{2r}(y)}e^{2rL}f(x)|dx + \frac{1}{B_r(y)} \int_{B_{2r}(y)} |\chi_{B_r(y)}e^{2rL}f(x) - \chi_{B_r(y)}e^{rL}f(x)|dx \]
\[ \leq \frac{1}{B_r(y)} \int_{B_{2r}(y)} |\chi_{B_r(y)}e^{2rL}f(x) - \chi_{B_r(y)}e^{rL}f(x)|dx \]
\[ \leq \frac{1}{B_r(y)} \int_{B_{2r}(y)} |\chi_{B_r(y)}e^{rL}f(x)|dx + \frac{1}{B_r(y)} \int_{B_r(y)} |e^{2rL}f(x) - e^{rL}f(x)|dx \]
\[ \leq \frac{1}{B_r(y)} \int_{B_r(y)} |e^{2rL}f(x) - e^{rL}f(x)|dx \]
Now once we let \( e^{rL}f(x) = g(x) \), and observe that
\[ \|g\|_{BMO^r} \lesssim \|f\|_{BMO^r} \]
we have that each of the three terms in the last expression can be controlled in terms of \( \|f\|_{BMO^r} \), giving us our result. \( \square \)

Remark 4.3. A doubling property is the main driving force behind the proof of the above lemma. However, since we are working in a compact setting, we have not entered into the intricacies of a proof that would work in more general settings.

At this point, let us recall the main approach of [T5] - [T2] and [MMV]. The analyses in these papers avoided producing a parametrix for (1.6). Instead, they replaced (1.6) by the following
\[ \varphi(\sqrt{-L}) = \frac{1}{2} \int_{-\infty}^{\infty} \varphi_k(t)\mathcal{J}_{k-1/2}(t\sqrt{-L}) \, dt \]
where
\[ J_\nu(\lambda) = \lambda^{-\nu} J_\nu(\lambda) \]
\( J_\nu(\lambda) \) denoting the standard Bessel function, and
\[ \varphi_k(t) = \prod_{j=1}^{k} (-t \frac{d}{dt} + 2j - 2) \hat{\varphi}(t) \]
\( \Phi_k(t) = \frac{1}{2} \int_{-\infty}^{\infty} \psi_k(t) J_{k-1/2}(t \sqrt{-L}) \, dt \)

\[ \text{T5} \] derives (4.3) from (1.6) by an integration by parts argument (see (3.7) - (3.9) of \[ \text{T5} \]). Similarly, we have
\[ \varphi^\#(\sqrt{-L}) = \frac{1}{2} \int_{-\infty}^{\infty} \psi_k(t) J_{k-1/2}(t \sqrt{-L}) \, dt \]
with
\[ \psi_k(t) = \prod_{j=1}^{k} (-t \frac{d}{dt} + 2j - 2) \hat{\varphi}(t) \]
Also, we have
\[ \text{supp } \psi_k \subset [-a, a] \]
Also, (1.2) implies
\[ |(t \partial_t)^j \hat{\varphi}(t)| \lesssim |t|^{-1}, \forall j \in \{0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor + 2 \} \]
which in turn implies
\[ |\psi_k(t)| \lesssim |t|^{-1}, \ 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor + 2 \]
Without any loss of generality, we can scale \( L \) so that the speed of propagation of \( \cos t \sqrt{-L} \) is \( \leq 1 \). Also, let us select \( a = 1 \) in both (4.6) and (1.9).

Using (4.4), we can write
\[ k^\#(x, z) = \int_{0}^{1} \psi_k(t) B_k(s, x, z) \, ds \]
We now derive a technical estimate on \( ||B_k(s, x, \cdot)||_{L^2(B_1(x) \setminus B_{1/2}(x))} \) which we will find essential in the sequel. These estimates are essentially variants of (2.11) in \[ \text{T2} \].

**Lemma 4.4.**
\[ ||B_k(s, x, \cdot)||_{L^2(B_1(x) \setminus B_{1/2}(x))} \lesssim s^{-n/2}, \ s \in (0, 1] \]

**Proof.** We will prove more generally that if \( G : \mathbb{R} \rightarrow \mathbb{R} \) satisfies
\[ |G(\lambda)| \lesssim (1 + |\lambda|)^{-\gamma-1}, \gamma > n/2 \]
then
\[ ||G(s \sqrt{-L})||_{L^2(L^\infty)} \lesssim s^{-n/2}, \ s \in (0, 1] \]
Why is this more general? Because from (4.4) and (4.7), we have

\[ J_{k-1/2}(s\sqrt{-L})f = \int_M B_k(s, x, z)f(z)dz \]
with

\[ |J_{k-1/2}(\lambda)| \lesssim (1 + |\lambda|)^{-k}, k > 0 \]

So, the question reduces to establishing (4.10) and then estimating \( ||g(s, x, .)||_{L^2(B_t(x)\setminus B_{t/2}(x))} \) from (4.10) where \( g \) is the integral kernel of \( G \). Let us first establish (4.10).

Following [T5], we see that we can write

(4.11) \[ G(s\sqrt{-L}) = (1 - s^2L)^{-\sigma}G(s\sqrt{-L})(1 - s^2L)^{\sigma} \]

Now, let \( F(\lambda) = G(\lambda)(1 - \lambda^2)^{\sigma} \). Then, using \( \gamma > n/2 \) and \( 2\sigma = \gamma + 1 \), we see

\[ |F(\lambda)| = |G(\lambda)(1 - \lambda^2)^{\sigma}| \lesssim (1 + |\lambda|)^{-\gamma-1}(1 - \lambda^2)^{\sigma} \]
\[ \lesssim (1 + |\lambda|)^{2\sigma-\gamma-1} \leq C \]

Since \( F \) is bounded, by the spectral theorem, \( F(s\sqrt{-L}) : L^2 \to L^2 \) is continuous. So by virtue of (4.11), our task is reduced to proving that

(4.12) \[ ||(1 - s^2L)^{-\sigma}||_{L^2,L^\infty} \lesssim s^{-n/2}, \sigma > n/4 \]

Now,

\[ (1 - s^2L)^{-\sigma} \simeq \int_0^\infty e^{-r}s^2Lr^{\sigma-1}dr \]

which gives

\[ ||(1 - s^2L)^{-\sigma}||_{L^2,L^\infty} \lesssim \int_0^{s^{-2}} e^{-r}(rs^2)^{-n/4}r^{\sigma-1}dr + \int_{s^{-2}}^{\infty} e^{-r}r^{\sigma-1}dr \]
\[ \lesssim (s^{-n/2} + 1) \lesssim s^{-n/2} \]

This establishes (4.10). That is,

\[ |G(s\sqrt{-L})f(x)| \lesssim s^{-n/2}||f||_{L^2} \]

In particular, with \( f = \delta_x \) and using the compactness of \( M \), we have

\[ ||g(s, x, .)||_{L^2} \lesssim s^{-n/2} \]

With that in place, now, let \( k^\#(x, y) \) be the integral kernel for \( \varphi^\#(\sqrt{-L}) \). Let us define

(4.13) \[ p(t, z, y) = \chi_{B_{2t}(z)}(y)h(t, z, y) \]

where we now use the notation \( h(t, x, y) \) for the heat kernel of \( e^{tL} \), \( \chi \) represents, as usual, the characteristic function of a set, and \( B_r(x) \) denotes the ball of radius \( r \) around the point \( x \). Let \( A_t \) denote the operator generated by \( p(t, x, y) \) and \( k^\#_t(x, y) \) denote the integral kernel for the composite operator \( \varphi^\#(\sqrt{-L})A_t \), the subscript \( t \) denoting the time dependence of the operator.
ϕ#(\sqrt{-L})A_t.

We are in a position to state the main technical lemma of this paper:

**Lemma 4.5. (Key Lemma)** We have

\begin{equation}
\mathop{\sup}_{t>0} \mathop{\sup}_{y \in M} \int_{d(x,y) \geq \sqrt{t}} |k^#(x,y) - k_1(x,y)| \, dx < \infty
\end{equation}

*Proof.* Let \( T_1 \) be an operator with integral kernel \( k_1(x,y) \). Also, let \( T_2 \) be another (time-dependent) operator with integral kernel \( p(t,x,y) \). We are trying to calculate the integral kernel \( k(x,y) \) of the composite operator \( T_1 T_2 \).

Indeed,

\begin{equation}
T_1 T_2 f(x) = \int_M k_1(x,y) T_2 f(y) \, dy
\end{equation}

\begin{equation}
= \int_M k_1(x,y) \int_M p(t,y,z) f(z) \, dz \, dy
\end{equation}

\begin{equation}
= \int_M \int_M k_1(x,y) p(t,y,z) f(z) \, dz \, dy
\end{equation}

So, \( k(x,z) = \int_M k_1(x,y) p(t,y,z) \, dy \), and replacing variables,

\[ k(x,y) = \int_M k_1(x,z) p(t,z,y) \, dz \]

Now we define \( p(t,z,y) \), which will in turn define \( T_2 = A_t \). \( p(t,z,y) \) is given by

\begin{equation}
p(t,z,y) = \chi_{B(y,\sqrt{t})}(z) h(t,z,y)
\end{equation}

Now we have fixed \( t > 0, y \in M \) and we are trying to estimate

\begin{equation}
\int_{M \setminus B(y,\sqrt{t})} |k^#(x,y) - \int_M k^#(x,z) p(t,z,y) \, dz| \, dx
\end{equation}

Call the above quantity \( Q \). Now clearly,

\begin{equation}
Q \leq \int_{M \setminus B(y,\sqrt{t})} |k^#(x,y)| \, dx + \int_{x \in M \setminus B(y,\sqrt{t})} \int_{z \in M} |k^#(x,z) p(t,z,y)| \, dz \, dx
\end{equation}

Call the above quantity \( S \).

For the case when \( t \geq 1 \), we have \( d(x,y) \geq \sqrt{t} \geq 1 \). So, since \( k^#(x,y) \) is supported near the diagonal, we have \( k^#(x,y) = 0 \).\(^2\) Now, when trying to estimate \( B = \int_{x \in M \setminus B(y,\sqrt{t})} \int_{z \in M} |k^#(x,z) p(t,z,y)| \, dz \, dx \), the \( k^#(x,z) \) part causes problems when \( x \) and \( z \) are close together. However, since

\begin{equation}
d(y,z) \leq \frac{1}{2\sqrt{t}} \implies d(x,y) \geq \sqrt{t}
\end{equation}

\(^2\)Note that this tacitly assumes that the propagation speed of \( \cos t \sqrt{-L} \) is 1. There is no loss of generality here, as \( L \) can easily be scaled to achieve this.
we have,

\[(4.22) \quad d(x, z) \geq d(x, y) - d(y, z) \geq \sqrt{t} - \frac{1}{2\sqrt{t}} \geq \frac{1}{2}\]

which now makes the kernel manageable.

\[(4.23) \quad P := \hat{z} \in M \kappa^\#(x, z)p(t, z, y)dz \]

\[(4.24) \quad = \hat{z} \in B_1(x) \setminus B_{1/2}(x) \kappa^\#(x, z)p(t, z, y)dz \]

\[(4.25) \quad \leq \|\kappa^\#(x, \cdot)\|_{L^2(B_1(x) \setminus B_{1/2}(x))}\|p(t, \cdot, y)\|_{L^2} \]

Now, we have,

\[(4.26) \quad \|\kappa^\#(x, \cdot)\|_{L^2(B_1(x) \setminus B_{1/2}(x))} \lesssim \int_1^{1/2} \psi_k(s)\|B_k(s, x, \cdot)\|_{L^2(B_1(x) \setminus B_{1/2}(x))}ds \]

\[(4.27) \quad \lesssim \int_1^{1/2} \frac{1}{s}ds \lesssim [s^{-n/2}]^{1/2} \]

\[(4.28) \quad = 2^{n/2} - 1 \]

By using the semigroup property of \(h(t, z, y)\) (we have to be careful here as \(p(t, x, y)\) itself might not generate a semigroup) we have

\[(4.29) \quad h(t, z, y) = \int_M h(t/2, w, y)h(t/2, z, w)dw \]

\[(4.30) \quad \chi_{B_{1 \sqrt{t}}(y)}(z)h(t, z, y) = \int_M h(t/2, w, y)p(t/2, z, w)dw \]

\[(4.31) \quad p(t, z, y) \leq ||h(t/2, \cdot, y)||_{L^2}||p(t/2, z, \cdot)\|_{L^2} \quad \lesssim (t/2)^{-n} \]

That means,

\[(4.32) \quad ||p(t, \cdot, y)||_{L^2(B_{1 \sqrt{t}}(y))} \lesssim 2^{n/2} \]

See that \((4.29)\) and the three equations following that only work if \(t \in [1, 2]\). However, it can be checked that we can repeat the process and go up in intervals [2, 4], [4, 8], [8, 16], .... by induction using \((4.29)\). On calculation, it is seen that \(||p(t, \cdot, y)||_{L^2(B_{1 \sqrt{t}}(y))}\) remains uniformly bounded; in fact, it decreases sharply as we go up by induction. The reason that this is quite expected is that as \(t\) increases, the ball \(B_{1 \sqrt{t}}(y)\) decreases very fast in size.

Plugging everything back together and using the compactness of \(M\), we see that \(S\) is bounded above.

Now, to bound \(Q\) (which is a function of \(t\) and \(y\)), it suffices to only look at \(t \in (0, \epsilon]\) for some small \(\epsilon\). Why is that? We start by arguing that \(\sup_{y \in M} Q\)
is uniformly bounded in the compact neighbourhood \( t \in [\varepsilon, 1] \), will give us our result. Here is a sketch of how this works. If \( \sup_{t \in [\varepsilon, 1]} \sup_{y \in M} Q \) is unbounded, we select a sequence \((t_n, y_n)\) with the property that \( Q(t_n, y_n) \to \infty \). We get a subsequence, still called \((t_n, y_n)\) such that \( t_n \to t^*, y_n \to y^* \). The finiteness of \( Q(t^*, y^*) \) together with a limiting argument using Lemma 2.3 of [T2] gives us a contradiction.

We choose \( \varepsilon \) small enough such that \( t < \varepsilon \) will mean \( B_{1/\sqrt{t}}(y) = M \). That will imply then, \( p(t, z, y) \) is exactly the heat kernel. Now, we have

\[
\int_M k^\#(x, z)p(t, z, y)dz = \int_M k^\#(z, x)^* p(t, y, z)dz = e^{tL}k^\#(y, x)^* = e^{tL}k^\#(x, y)
\]

where \( L \) is being thought of as \( L_y \), an operator in the \( y \)-variable.

Now all we want is some kind of uniform bound on

\[
|e^{tL}k^\#(., y) - k^\#(., y)|_{L^1(B_1(y) \setminus B_{\sqrt{t}}(y))}
\]

The following is a broad sketch as to how this is done. In spirit this is similar to Lemma (4.4) and the first part of the ongoing proof, so we will not enter into the intricate details. (4.34) is obtained from a uniform bound on

\[
|e^{tL}Lk^\#(s, ., y)|_{L^1(B_1(y) \setminus B_{\sqrt{t}}(y))}
\]

where \( t' \in (0, t] \). This latter quantity, by previous discussion, is

\[
\lesssim t \int_0^{1/t'} \frac{1}{s} \|e^{tL}LB_k(s, ., y)\|_{L^1} ds
\]

which we can prove to be uniformly bounded. To prove this, we use

\[
\|LB_k(s, ., y)\|_{L^2} \lesssim s^{-n/2-2}
\]

and follow the exact same line of reasoning by which [T2] derives uniform bounds on \( \|\nabla y k^\#(., y)\|_{L^1(B_1(y) \setminus B_{\sqrt{t}}(y))} \) from \( \|\nabla y B_k(s, ., y)\|_{L^2(M)} \lesssim s^{-n/2-1} \). For more details, see Lemma 2.2 and Section 5 of [T2].

\[\Box\]

4.1. \( L^\infty - BMO_L \) continuity. To establish \( BMO_L - BMO_L \) or \( L^\infty - BMO_L \) continuity, first we argue that if

\[
\varphi(\sqrt{-L}) = \varphi^b(\sqrt{-L}) + \varphi^\#(\sqrt{-L})
\]

then it is enough to prove the result for \( \varphi^\#(\sqrt{-L}) \). This is because, we have

\[
|\varphi^b(\lambda)| \lesssim (1 + |\lambda|)^{-m} \text{ for some } m > \frac{n}{2}
\]

where \( \dim M = n \). So, as discussed before,

\[
\varphi^b(\sqrt{-L}) : L^1(M) \to L^2(M)
\]
This gives, by duality,

\[(4.38)\quad \varphi^b(\sqrt{-L}) : L^2 \to L^\infty\]

Also, from (1.12) we have

\[(4.39)\quad \varphi^b(\sqrt{-L}) : L^2 \to H^m(M) \subset L^1(M)\]

which gives

\[(4.40)\quad \varphi^b(\sqrt{-L}) : L^2 \to L^1\]

and by duality

\[(4.41)\quad \varphi^b(\sqrt{-L}) : L^\infty \to L^2\]

Now, we have

\[(4.42)\quad ||.||_{L^1} \lesssim ||.||_{L^2} \lesssim ||.||_{BMO_L} \lesssim ||.||_{L^\infty}\]

(4.38) and (4.42) give

\[(4.43)\quad ||\varphi^b(\sqrt{-L})f||_{BMO_L} \leq ||\varphi^b(\sqrt{-L})f||_{L^\infty} \leq ||f||_{L^2}\]

which means

\[(4.44)\quad \varphi^b(\sqrt{-L}) : L^2 \to BMO_L\]

(4.42) and (4.44) give

\[(4.45)\quad \varphi^b(\sqrt{-L}) : L^\infty \to BMO_L\]

So now, we pick \(f \in L^\infty\) and try to show \(\varphi^#(\sqrt{-L})f \in BMO_L\). We have

**Proposition 4.6.**

\(\varphi^#(\sqrt{-L}) : L^\infty \to BMO_L\)

**Proof.**

\[
\int_{B_{r}(\tau(y))} |\varphi^#(\sqrt{-L})f(x) - e^{tL}\varphi^#(\sqrt{-L})f(x)| dx = \int_{B_{r}(\tau(y))} |\varphi^#(\sqrt{-L})\psi(x)f(x) - e^{tL}\varphi^#(\sqrt{-L})\psi(x)f(x)| dx + \int_{B_{r}(\tau(y))} |\varphi^#(\sqrt{-L})(1 - \psi(x))f(x) - e^{tL}\varphi^#(\sqrt{-L})(1 - \psi(x))f(x)| dx = A + B
\]

where \(\psi\) is supported in \(B_{r+\delta}(y)\), and \(\psi(x) = 1\) on \(B_{r}(\tau(y))\). Clearly, by Hölder’s inequality,

\[
A \lesssim ||\varphi^#(\sqrt{-L})\psi f - e^{tL}\varphi^#(\sqrt{-L})\psi f||_{L^2} = ||(I - e^{tL})\varphi^#(\sqrt{-L})\psi f||_{L^2} \lesssim ||\varphi^#(\sqrt{-L})\psi f||_{L^2} \lesssim ||\psi f||_{L^2} \lesssim ||f||_{L^2} \lesssim ||f||_{BMO_L} \lesssim ||f||_{L^\infty}
\]
Also
\[(4.46)\]
\[
|\varphi^#(\sqrt{-L})(I - e^{tL})(1 - \psi(x))f(x)| \leq \int_{M \setminus B_{\sqrt{t}}(x)} |(k^#(x, z) - k_t(x, z))(1 - \psi(z))f(z)|dz
\]
\[
(4.47) \leq ||f||_{L^\infty} \int_{M \setminus B_{\sqrt{t}}(x)} |k^#(x, z) - k_t(x, z)|dz
\]

where \(k_t\) is the integral kernel of \(\varphi^#(\sqrt{-L})e^{tL}\). Now, as \(t\) is very small\(^3\) by the Key Lemma (4.5),
\[
(4.48) \int_{M \setminus B_{\sqrt{t}}(x)} |k^#(x, z) - k_t(x, z)|dz = \int_{M \setminus B_{\sqrt{t}}(x)} |k^#(z, x) - \int_M k^#(z, w)h(t, w, x)dw|dz
\]
\[
(4.49) = \int_{M \setminus B_{\sqrt{t}}(x)} |k^#(z, x) - \int_M k^#(z, w)p(t, w, x)dw|dz
\]

where \(h\) is the heat-kernel. We have already proved that the last quantity is bounded above. That proves
\[
(4.50) \varphi^#(\sqrt{-L}) : L^\infty \to \text{BMO}_L
\]

So, let us see the immediate consequence of Proposition (4.6). By virtue of this, we immediately have \(L^p\)-continuity of \(\varphi(\sqrt{-L})\) upon application of Theorem 5.6 of [DY1]. This result is not new of course. It is established in a more general context in [DOS]. [T2] has a different proof of this same result. However, we believe that this proof is rather different from both of these. Also, it must be remarked that Key Lemma (4.5), in conjunction with Theorem 1 of [DM] also implies directly that \(\varphi(\sqrt{-L})\) is of weak type \((1,1)\), which by application of the Marcinkiewicz interpolation techniques, will give another proof of the \(L^p\)-continuity (also refer to Lemma B.1 of [T2] for a discussion of Theorem 1 of [DM]).

Also note the \(L^\infty - \text{BMO}_L\) result in Theorem 6.2 of [DY1] (actually we became aware of this paper after finishing this work). In any case, note that the conditions used to prove Theorem 6.2 in [DY1] are somewhat stronger than (1.15). So, in comparison, it can be said that we prove similar results is a much more restricted setting; however, here the semigroup \(e^{tL}\) satisfies less stringent conditions.

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\(^3\)This is the main reason for introducing \(\text{BMO}_L^\varepsilon\) instead of just the usual \(\text{BMO}_L\).
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