Bottom-up rewriting for words and terms

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1.1-RECALLS-SEMI-Thue SYSTEMS

A rewrite rule over the alphabet $A$ is a pair

$$ \ell \rightarrow r $$

of words in $A^\ast$.

A semi-Thue system is a pair $(S, A)$ where $S$ a set of rewrite rules built upon the alphabet $A$.

For every $f, g \in A^\ast$,

$$ f \rightarrow_S g $$

iff there exists $\ell \rightarrow r \in S$ and $\alpha, \beta \in A^\ast$ such that

$$ f = \alpha\ell\beta \land g = \alpha r\beta. $$

We call $\rightarrow_S^\ast$ the derivation generated by $S$. 
A rewrite rule is a pair

\[ \ell \rightarrow r \]

of terms in \( \mathcal{T}(\mathcal{F}, \mathcal{V}) \) which satisfy \( \text{Var}(r) \subseteq \text{Var}(\ell) \).

A term rewriting system is a pair \((\mathcal{R}, \mathcal{F})\) where \( \mathcal{F} \) is a signature and \( \mathcal{R} \) a set of rewrite rules over the signature \( \mathcal{F} \). For every \( s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \),

\[ s \rightarrow^* \mathcal{R} t \]

iff there exists \( \ell \rightarrow r \in \mathcal{R} \) a context \( C[\cdot] \) and a substitution \( \sigma \) such that

\[ s = C[\ell\sigma] \land t = C[r\sigma]. \]

We call \( \rightarrow^*_\mathcal{R} \) the derivation generated by \( \mathcal{R} \).
2.1-INTRODUCTION- PROBLEMS

Given a system \( R \) and a set of terms \( T \), we define

\[
\{ s \in T(\mathcal{F}) \mid s \rightarrow^*_R t \text{ for some } t \in T \}
\]

Problem: under which hypothesis over \( R \) does it hold that, for every recognizable set \( T \), \((\rightarrow^*_R)[T]\) is recognizable too.
2.2-INTRODUCTION - MOTIVATIONS

- decidability of the **accessibility** problem for $\rightarrow^R$

- decidability of the **termination** problem

- rational subsets of a monoid $M = A^* / \leftrightarrow^*_S$
  - application to resolution of equations with **rational** constraints in $M$.

- decidability of the **common-ancestor** problem for $\rightarrow^R$

- **sequentiality** problems:
  - computation of a **needed redex** in a term $t$ w.r.t. $R$
  - decidability of the **sequentiality** property for $R$
2.3-INTRODUCTION- KNOWN RESULTS

Two kinds of results.
First kind: *syntactical* condition over $\mathcal{R}$
Generic theorem:
if $\mathcal{R}$ has property $P$, then, for every recognizable set $T$,
$(\rightarrow_\star)^{\mathcal{R}}[T]$ is *recognizable* too.
2.3-INTRODUCTION - KNOWN RESULTS

First kind references (syntax):
- cancellation rules Benois-Sakarovitch, IPL 86
- basic semi-Thue systems Benois, RTA 87
- left-basic semi-Thue systems Sakarovitch, PHD, 79
- ground term-rewriting systems Dauchet-Heuillard-Lescanne-Tison, Inf. and Comput. 87
- linear shallow term-rewriting systems Comon, LICS 95
- linear growing term-rewriting systems Jacquemard, RTA 96
- left-linear growing term-rewriting systems Nagaya-Toyama, Inf. and Comput. 02
- left-linear inverse finite-path overlapping TRS Takai-Kaji-Seki, Sci. Math. Jap., 2006
2.3-INTRODUCTION- KNOWN RESULTS

Second kind: use a special strategy in derivations

Generic theorem (for the strategy $S$):
For every TRS $\mathcal{R}$ and every recognizable set $T$, $(s \rightarrow^{*}_\mathcal{R})[T]$ is recognizable too.
2.3-INTRODUCTION- KNOWN RESULTS

Second kind references (strategy):
- one-pass term rewriting
  Fulop-Jurvanen-Steinby-Vagvolgyi, MFCS 98
- concurrent term rewriting
  Seynhaeve-Tison-Tommasi, FCT 99
- “bottom-up derivations”
  Rety-Vuotto, JSC 05.
We define a new notion of $k$-bottom-up derivation, denoted by $\krightarrow^*_R$.

**Theorem 1** Durand-Sénizergues, RTA’07 Let $\mathcal{R}$ be some linear rewriting system over the signature $\mathcal{F}$, let $\mathcal{T}$ be some recognizable subset of $\mathcal{T}(\mathcal{F})$ and let $k \geq 0$. Then, the set $(\krightarrow^*_R)[\mathcal{T}]$ is recognizable too.

We then introduce the class of Bottom-Up systems as the class of all systems for which the above strategy is complete.

**Theorem 2** Sylvestre, M.th., 2008 For every integer $k \geq 0$, the termination property is decidable for Term Rewriting Systems in SBU($k$).
2.5-INTRODUCTION- METHODS

1- Define a marking operation: the marks are integers that measure the amount of top-down (i.e. “forbidden”) sequence of rules. Bottom-up derivation are those derivations with small marks.

2- Reduce the preservation property to the same property for ground systems: a bottom-up derivation can be simulated by a ground derivation.
3.1-**Marked DERIVATIONS- Unary terms**

\[ M := \max \text{ of the marks} \]

\[ \max(k, M + 1) \]
3.1-Marked DERIVATIONS- Unary terms

A derivation graph.
3.1-Marked DERIVATIONS- Unary terms

A derivation.
3.2-**Marked DERIVATIONS**- General terms

![Diagram of marked derivations]
3.2—Marked DERIVATIONS—General terms
3.2-MARKED DERIVATIONS - GENERAL TERMS

\[ m \]

\[ k \]

\[ \max(k, M + 1) \]
4.1 -BOTTOM-UP DERIVATIONS- BOTTOM-UP DERIVATIONS

**Definition 3** A derivation \( s \rightarrow^*_R t \) is **weakly bottom-up** iff, in the corresponding marked derivation, for every application of rule, the minimum mark of the lhs is 0.

Let \( k \geq 0 \).

**Definition 4** A derivation \( s \rightarrow^*_R t \) is **k bottom-up** iff, in the corresponding marked derivation, for every application of rule, the minimum mark of the lhs is 0 and the maximum mark of the term is \( \leq k \).

Notation:

\[ s \overset{k}{\rightarrow}^*_R t \]

means that there exists a **k bottom-up** derivation from \( s \) to \( t \).
4.1 -BOTTOM-UP DERIVATIONS- BOTTOM-UP DERIVATIONS

A generic non weakly bottom-up derivation.
4.1 -BOTTOM-UP DERIVATIONS- BOTTOM-UP DERIVATIONS

A concrete non weakly bottom-up derivation.
4.1 -BOTTOM-UP DERIVATIONS- BOTTOM-UP DERIVATIONS

A concrete derivation:

- it is weakly bottom-up,
- it is not BU(1),
- it is BU(2),
4.2 -BOTTOM-UP DERIVATIONS- BOTTOM-UP SYSTEMS

**Definition 5** Let $k \geq 0$. A system $(\mathcal{R}, \mathcal{F})$ is called $k$-Bottom-Up iff, it is linear and for every $s, t \in \mathcal{T}(\mathcal{F})$

$$s \rightarrow^{*}_{\mathcal{R}} t \Rightarrow s \overset{k}{\rightarrow}_{\mathcal{R}}^{*} t$$

i.e. if $\mathcal{R}$ is linear and the $k$-BU strategy is complete for $\mathcal{R}$. 
4.3 -BOTTOM-UP DERIVATIONS- KNOWN SUBCLASSES

The following classes of systems are bottom-up:
- every left-basic semi-Thue system is BU(1)
- every linear growing Term Rewriting system is BU(1)
- every linear FPO\(^{-1}\) Term Rewriting system is in \(\bigcup_{k \geq 0} BU(k)\).
5.1-PRESERVATION OF RATIONALITY-**THE RESULT**

**Theorem 1** DS-RTA’07
Let \( \mathcal{R} \) be some linear rewriting system over the signature \( \mathcal{F} \), let \( T \) be some recognizable subset of \( \mathcal{T}(\mathcal{F}) \) and let \( k \geq 0 \). Then, the set \((k \rightarrow^*_{\mathcal{R}})[T]\) is recognizable too.

**Theorem 6** Let \( k \geq 0 \), let \( \mathcal{R} \) be some BU\((k)\) rewriting system over the signature \( \mathcal{F} \) and let \( T \) be some recognizable subset of \( \mathcal{T}(\mathcal{F}) \). Then, the set \(( \rightarrow^*_{\mathcal{R}})[T]\) is recognizable too.
5.2.2-PRESERVATION OF RATIONALITY-Construction of \( S \)

General idea:
- Simulate a bottom-up derivation, by a derivation where the substitutions used have a **bounded depth**
- The deeper part of the substitution is replaced by a **state** of the finite automaton recognizing \( T \).
5.2.2-PRESERVATION OF RATIONALITY-Construction of $S$

Let $d := \max\{dpt(\ell) \mid \ell \to r \in \mathcal{R}\}$.

The system $S$: consists of all the rules

$$\overline{\ell\tau} \to r\overline{\tau}$$

where $\ell \to r$ is a rule of $\mathcal{R}$, $\overline{\tau}, \overline{\tau}$ are marked substitutions, with marks $\leq k$, $dpt(\overline{\tau}) \leq k \cdot d$ and $\overline{\ell\tau} \to r\overline{\tau}$ is a one-step, $k$-bu, marked derivation.
5.2.3-PRESERVATION OF RATIONALITY- SIMULATION LEMMAS

Lifting $S \cup A$

Projecting $R$

\[ \begin{array}{ccc}
\overline{s}' & \stackrel{*}{\longrightarrow} & \overline{t}' \\
A & \downarrow \kappa & A \\
\overline{s} & \stackrel{*}{\longrightarrow} & \overline{t} \\
S \cup A & \end{array} \]

\[ \begin{array}{ccc}
\overline{s} & \stackrel{*}{\longrightarrow} & \overline{t} \\
A & \downarrow \kappa & A \\
\overline{s}' & \ldots \longrightarrow & \overline{t}' \\
S \cup A & \end{array} \]
5.2.4-PRESERVATION OF RATIONALITY- CONCLUSION

Since $A$ and $S$ are *ground* rewriting-systems, it is known that $\rightarrow_{S \cup A}^*$ inverse-preserves recognizability. By the simulation lemmas:

$$\left( k \rightarrow_{R}^* \right)[T] = \left( \rightarrow_{S \cup A}^* \right)[Q_{f}^{\leq k}] \cap T(\mathcal{F})$$

Hence $\left( k \rightarrow_{R}^* \right)[T]$ is recognizable.
6.1-COMPLEXITY/DECIDABILITY- BOTTOM-UP

Theorem 7  The $BU(1)$ property is undecidable for semi-Thue systems.
6.2-COMPLEXITY/DECIDABILITY- STRONG BOTTOM-UP

**Definition 8** Let $k \geq 0$. A system $(\mathcal{R}, \mathcal{F})$ is called **strongly $k$-Bottom-Up** iff, it is linear and for every weakly bottom-up marked derivation

$s = s_0 \rightarrow_{\mathcal{R}} s_1 \rightarrow_{\mathcal{R}} \ldots \rightarrow_{\mathcal{R}} s_i \rightarrow_{\mathcal{R}} \ldots \rightarrow_{\mathcal{R}} s_n = t$

if $s_0$ has only null marks, then all the $s_i$ have all their marks $\leq k$.

**Theorem 9** The SBU($k$) property is **decidable** for Term Rewriting systems.

Follows easily from theorem 1.
6.3 -COMPLEXITY/DECIDABILITY- STRONG BOTTOM-UP

- every left-basic semi-Thue system is \( SBU(1) \)
- every linear growing Term Rewriting system is \( SBU(1) \)
- every linear \( FPO^{-1} \) Term Rewriting system is in \( \bigcup_{k \geq 0} SBU(k) \).
7- TERMINATION

**Theorem 2**  Sylvestre, M.Th. 2008
For every integer $k \geq 0$, the termination property is **decidable** for Term Rewriting Systems in SBU($k$).

**Lemma 10**  Let $(\mathcal{R}, \mathcal{F})$ be **strongly** $k$-Bottom-Up. There exists an infinite derivation for $\rightarrow_{\mathcal{R}}$ iff there exists an infinite derivation for $k^\circ \rightarrow_{\mathcal{R}}$. 
7- TERMINATION

Strengthened lifting and projecting lemma.

Lifting $S$

Projecting $\mathcal{R}$
8- PERSPECTIVES

- Extend the notion and the results to left-linear, non right-linear systems (Current work of Marc Sylvestre in his PHD).
- Study the direct images of rational sets (or context-free sets) by bottom-up rewriting.
- Study a dual notion of top-down derivations