ON THE OSELEDETS-SPLITTING FOR INFINITE-DIMENSIONAL RANDOM DYNAMICAL SYSTEMS

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Dedicated to our friend and colleague Prof. Dr. Igor Dmitrievich Chueshov

Abstract. We investigate the Oseledets splitting for Banach space-valued random dynamical systems based on the theory of center manifolds. This technique gives us random one-dimensional invariant spaces which turn out to be the Oseledets subspaces under suitable assumptions. We apply these results to a stochastic parabolic evolution equation driven by a fractional Brownian motion.

1. Introduction. The aim of this paper is to analyze the long-time behavior of Banach space-valued random dynamical systems by means of the Oseledets multiplicative ergodic theorem (MET). This fundamental result represents a crucial tool for the stability of random systems. In the deterministic case, the stability of autonomous systems strictly depends on the eigenvalues of the linear part. However, for nonautonomous systems, even in the deterministic case, this is a challenging question, see for instance C. Chicone and Y. Latushkin [9], Section 3.

In this paper we are concerned with random dynamical systems arising from random and stochastic partial differential equations on Banach spaces. Since the phase space is not a separable Hilbert space, numerous difficulties encounter.

First of all, for a better comprehension, we recall the main statements of the Oseledets theorem which goes back to V. Oseledets [28]. The precise formulation together with a detailed analysis and examples can be looked up in Section 3.4 in L. Arnold [4]. This setting covers linear random dynamical systems on \( \mathbb{R}^d \) for \( d \geq 1 \). Here, the MET ensures the existence of an invariant splitting (Oseledets subspaces) of \( \mathbb{R}^d \) together with appropriate growth rates (Lyapunov exponents) on each of these subspaces. These assertions can be naturally extended to random dynamical systems on Hilbert spaces, as concluded by D. Ruelle [32] (discrete time) and S. Mohammed, T. Zhang and H. Zhao [26] (continuous time). We emphasize that the construction of the Oseledets subspaces together with the corresponding

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Lyapunov exponents is based on spectral theory for self-adjoint operators, which obviously cannot be transferred to the Banach space-valued case. More precisely, for Hilbert space-valued compact random operators $\Phi : \mathbb{R}^+ \times \Omega \to \mathcal{L}(H)$ satisfying an appropriate integrability condition (see [32], [26], [27]), the limit
\[
\Lambda(\omega) := \lim_{t \to \infty} \left( \Phi^\ast(t, \omega) \circ \Phi(t, \omega) \right)^{1/2} t
\]
exists in the uniform operator topology. Moreover, $\Lambda(\omega)$ is compact, non-negative, self-adjoint and has a discrete spectrum which is constituted by \(\exp(\lambda_1) > \exp(\lambda_2) > \ldots\).

The distinct (nonrandom) numbers $\lambda_1, \lambda_2, \ldots$ are called Lyapunov exponents and build the Lyapunov spectrum. Analogously to the finite-dimensional case, one has an invariant splitting of the Hilbert-space into random subspaces which can be computed from the eigenspaces corresponding to the eigenvalues of $\Lambda(\omega)$. The different growth rates on these subspaces are given by the Lyapunov exponents. Furthermore, if one can group the Lyapunov exponents based on their signs, one can derive the existence of a random stable/unstable subspace of the phase space. Such an invariant splitting is the starting point in the study of random stable and unstable manifolds for the corresponding dynamical system, consult T. Caraballo et al. [6].

When leaving the Hilbert space-valued setting numerous difficulties occur. Obviously, (1) is not even defined in a Banach space. The breakthrough to extend Oseledets theorem to separable Banach spaces was realized in the work of Z. Lian and K. Lu (2010) [22]. Ever since then, numerous authors deal with this issue, see for instance J. Mierczyński and W. Shen [25], T. S. Doan and S. Siegmund [13] and the references specified therein. Moreover, C. González-Tokman and A. Quas [18] gave an alternative proof of the statement of Z. Lian and K. Lu and constructed the Oseledets subspaces for random dynamical systems satisfying a quasi-compactness assumption. Their approach uses duality and it is applicable to reflexive Banach spaces.

In this paper we investigate random dynamical systems which satisfy the assumptions of the MET proved in [22]. After collecting basic results concerning random dynamical systems, measurability of random operators and invariant splittings, Section 3 is entirely devoted to this topic. Here we deal with random parabolic evolution equations, show that their solution operators generate random dynamical systems and verify the integrability condition of the MET. This analysis requires properties of parabolic evolution operators, which are thoroughly discussed in the deterministic case by P. Acquistapace [2], P. Acquistapace and B. Terreni [1] and H. Amann [3]. In Section 4, we provide for a certain type of systems a constructive method of the Oseledets splitting based on the theory of center manifolds. This relies on the technique presented in the work of S.-N. Chow, K. Lu and J. Mallet-Paret [11], [12], which develops a Floquet theory for parabolic differential equations and establishes the existence of periodic unstable and center manifolds. Based on these results a Hartman-Grobman theorem for SPDEs driven by multiplicative Brownian noise is obtained in [21]. Regarding this, we consider a random evolution equation for which we construct random linear subspaces for which the solution is defined for all time points and satisfies suitable growth conditions in Theorem 24. These subspaces are obtained implicitly by constructing invariant manifolds and turn out to be the right setting for the Oseledets theory under certain assumptions, see Theorem 30.
Finally, in Section 5, we show how to apply our results to a linear stochastic evolution equation driven by a fractional Brownian motion. The connection between such equations and the MET is also new. The works [6] and [21] treat the case of a Brownian motion and rely on the transformation of the Stratonovich SPDE by means of the Ornstein-Uhlenbeck process. Here we use the approach in [17] and apply a change of variable formula for Stratonovich integrals with respect to the fractional Brownian motion, see [39]. This immediately gives us the weak (mild) solution for the linear SPDE from which we derive a random dynamical system which fits into the framework of the MET.

2. Preliminaries. In this section we provide basic definitions and results regarding random dynamical systems and the measurability of random operators and of linear random subspaces. We proceed with the definition of random dynamical system. If not further stated $X$ stands for a separable Banach space and $(\Omega, \mathcal{F}, P)$ denotes a probability space.

First of all, in order to quantify uncertainty we describe an appropriate model of the noise.

**Definition 1.** Let $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$ be a family of $P$-preserving transformations having following properties:

(i) the mapping $(t, \omega) \mapsto \theta_t \omega$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F})$-measurable;
(ii) $\theta_0 = \text{Id}_\Omega$;
(iii) $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$.

Then the quadruple $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ is called a metric dynamical system.

**Remark 2.**
(a) For notational simplicity we write $\theta_t \omega$ for $\theta(t, \omega)$.
(b) We always assume that $P$ is ergodic with respect to $(\theta_t)_{t \in \mathbb{R}}$.

The most well-known example of a metric dynamical system is constituted by the Banach space-valued Brownian motion. To this aim let $C_0(\mathbb{R}; X)$ denote the set of continuous $X$-valued functions which are zero at zero equipped with the compact open topology. Furthermore, by taking $P$ as the Wiener measure on $\mathcal{B}(C_0(\mathbb{R}; X))$ and applying Kolmogorov’s theorem about the existence of a continuous version (see for instance Theorem 6.9, p. 76 in J. van Neerven [37]) yields the canonical probability space $(C_0(\mathbb{R}; X), \mathcal{B}(C_0(\mathbb{R}; X)), P)$. To obtain an ergodic metric dynamical system we introduce the Wiener shift, defined by

$$\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t), \text{ for } \omega \in C_0(\mathbb{R}; X).$$

**Definition 3.** A continuous random dynamical system on $X$ over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ is a mapping

$$\varphi : \mathbb{R}^+ \times \Omega \times X \rightarrow X, (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable and satisfies:

(i) $\varphi(0, \omega, \cdot) = \text{Id}_X$ for all $\omega \in \Omega$;
(ii) $\varphi(t + \tau, \omega, x) = \varphi(t, \theta_\tau \omega, \varphi(\tau, \omega, x))$, for all $x \in X$, $t, \tau \in \mathbb{R}^+$ and all $\omega \in \Omega$;
(iii) $\varphi(t, \omega, \cdot) : X \rightarrow X$ is continuous for all $t \in \mathbb{R}^+$.

The second property is referred to as the **cocycle property** and it represents a generalization of the semigroup property. Notice that if the $\omega$-dependence is omitted, (ii) reduces exactly to the semigroup property. Here, the evolution of the noise $(\theta_t \omega)$ has to be additionally regarded.
We now state some basic results concerning the measurability of random operators and their resolvents which are required later on.

**Definition 4.** Let $X$ and $Y$ be two separable Banach spaces. A random operator $T : \Omega \times X \to Y$ is called strongly measurable if $\omega \to T(\omega)x$ is a random variable on $Y$ for every $x \in X$.

The next statement ensures the measurability of the resolvent of a random operator. It can be looked up in A. Bharucha-Reid [5] p. 82.

**Lemma 5.** Let $T : \Omega \times X \to Y$ be a random linear operator and $\lambda > 0$. Then the resolvent operator $R(\lambda; T(\omega)) = (\lambda \text{Id} - T(\omega))^{-1}$ exists for all $\omega \in \{|\lambda| > \|T(\omega)\|\}$ and is strongly measurable. Moreover, $R(\lambda; T(\omega))$ can be expressed as a Neumann power series
\[
R(\lambda; T(\omega)) = (\lambda \text{Id} - T(\omega))^{-1} = \frac{1}{\lambda}(\text{Id} - T(\omega)\lambda^{-1})^{-1} = \frac{1}{\lambda} \sum_{n=0}^{\infty} (\lambda^{-1}T(\omega))^n.
\]

**Remark 6.** Recall the Neumann series representation for a random operator $(\text{Id} - T(\omega))^{-1} = \sum_{n=0}^{\infty} T^n(\omega)$, which converges if $\|T(\omega)\|_{\mathcal{L}(X)} < 1$.

For more details regarding this topic consult A. Skorochod [35] and A. Bharucha-Reid [5].

For the sake of completeness we provide the definition of measurable subspace and invariant splitting. According to Corollary 7.3, p. 59 in [22] we have

**Definition 7.** A family $\{E(\omega)\}_{\omega \in \Omega'}$ of $m$-dimensional vector subspaces of $X$ is called measurable if there exist $(\mathcal{F}, \mathcal{B}(X))$-measurable functions $(v_1, \ldots, v_m) : \Omega' \to X_m$ such that $(v_1(\omega), \ldots, v_m(\omega))$ forms a basis of $E(\omega)$ for each $\omega \in \Omega'$. Here $X_m$ denotes the set of $m$-dimensional subspaces of $X$.

**Remark 8.** Note that for a random set $\{M(\omega)\}_{\omega \in \Omega'}$ we verify that
\[
\omega \mapsto \inf_{y \in M(\omega)} \text{dist}(x, y)
\]
is a random variable on $X$ for every $x \in X$.

Furthermore we note

**Definition 9.** Let $\varphi : \mathbb{R}^+ \times \Omega \times X \to X$ be a random dynamical system. A decomposition $E(\omega) \oplus F(\omega) \subset X$, with $\{E(\omega)\}_{\omega \in \Omega'}$ finite dimensional is said to be invariant if $\varphi(t, \omega)E(\omega) = E(\theta_t \omega)$ and $\varphi(t, \omega)F(\omega) \subset F(\theta_t \omega)$, for each $t \in \mathbb{R}^+$.

We conclude this section with the version of the multiplicative ergodic theorem for compact linear random dynamical systems. This is extracted from Z. Lian and K. Lu [22] (Theorem 3.2, p. 10) and will be required in the next sections.

**Theorem 10.** (Multiplicative Ergodic Theorem for Continuous Time Compact Random Dynamical Systems) Let $\Phi : \mathbb{R}^+ \times \Omega \to \mathcal{L}(X)$ be a continuous time random dynamical system. Assume that $\Phi(1, \cdot) : \Omega \to \mathcal{L}(X)$ is strongly measurable, $\Phi(1, \cdot)$ is injective and compact and the following integrability condition is satisfied
Furthermore let $\rho \in \mathbb{R}^+$ be an invariant subset $\Omega' \subset \Omega$ of full measure such that for each $\omega \in \Omega'$ the one of the following (mutually exclusive) statements holds true:

I) there exist $k$ Lyapunov exponents $\lambda_1 > \ldots > \lambda_j > -\infty$, $j$ finite dimensional invariant finite dimensional subspaces $\{E_i\}_{i=1}^j$ and an infinite dimensional invariant subspace $F(\omega)$, such that

$$X = E_1(\omega) \oplus \ldots \oplus E_j(\omega) \oplus F(\omega);$$

II) there exist infinitely many invariant finite dimensional subspaces $E_j(\omega)$, infinitely many invariant infinite dimensional subspaces $F_j(\omega)$ and infinitely many numbers (Lyapunov exponents)

$$\lambda_1 > \lambda_2 \ldots > \lambda_n \quad \text{with} \quad \lim_{j \to \infty} \lambda_j = -\infty.$$

These subspaces provide an invariant splitting of the phase space $X$, namely for each $j$ following decompositions are valid

$$E_1(\omega) \oplus \ldots E_j(\omega) \oplus F_j(\omega) = X \quad \text{and} \quad F_j(\omega) = E_{j+1}(\omega) \oplus F_{j+1}(\omega).$$

3. Random evolution equations. In this framework let $X_0$ and $X_1$ be two separable Banach spaces such that the embedding $X_1 \hookrightarrow X_0$ is dense. The pair $(X_0, X_1)$ will be called a densely injected Banach couple. We consider the linear nonautonomous evolution equation

$$\begin{cases}
u' = A(\theta_t \omega)u(t), \quad t \geq 0 \\
u(0) = x \in X_0,
\end{cases} \quad (3)$$

where the random linear operators $A(\omega)$, having $X_1 := D(A(\omega))$ are strongly measurable. We are interested in general assumptions under which the solution operator of (3) generates a random dynamical system. Therefore, we suppose that for each $\omega \in \Omega$, $-A(\omega)$ generates an analytic $C_0$-semigroup on $X_0$. For the resolvent $R(\lambda; A(\theta_t \omega))$ we have for all $t \geq 0$ and $\omega \in \Omega$ that

$$||R(\lambda; A(\theta_t \omega))||_{L(X_0)} \leq \frac{M}{|\lambda| + 1}, \quad \text{for all} \ \lambda \in \rho(A(\theta_t \omega)). \quad (4)$$

Furthermore let $\rho \in (0, 1] \nu$ such that $\rho + \nu \geq 1$. In addition assume that for every $\omega$ there exists a positive random variable $C(\omega)$ satisfying for all $t, s \geq 0$ and all $\lambda \in R(\lambda; A(\theta_t \omega))$

$$||\lambda^{\nu} ||A(\theta_t \omega) R(\lambda; A(\theta_t \omega))(A^{-1}(\theta_t \omega) - A^{-1}(\theta_s \omega))||_{L(X_0)} \leq C(\omega)|t - s|^\rho. \quad (5)$$

Remark 11.  

- The analyticity of the $C_0$-semigroup generated by $-A(\omega)$ together with conditions of type (5) are called in the literature the Acquistapace-Terreni assumptions, consult [2]. These allow us to treat the case in which $D(A(\theta_t \omega))$ is not constant.

- For time-independent domains, more precisely if $D(A(\theta_t \omega)) = X_1$, the Kato-Tanabe assumptions (see H. Amann [3], p. 55 and A. Pazy [29], p. 150) are sufficient. In this case, instead of (5) one only assumes the Hölder-continuity of the mapping $t \mapsto A(\theta_t \omega) \in L(X_1, X_0)$

$$||A(\theta_t \omega) - A(\theta_s \omega)||_{L(X_1, X_0)} \leq L(\omega)|t - s|^\rho, \quad \text{for all} \ s, t \geq 0, \quad (6)$$
where \( L(\omega) \) is a positive random variable. This implies
\[
||A(\theta_s \omega)(A(\theta_t \omega)A^{-1}(\theta_s \omega) - \text{Id})||_{\mathcal{L}(X_{1},X_{0})} \leq L(\omega)|t - s|^\rho,
\]
for all \( s, t \geq 0 \) and all \( \omega \in \Omega \).

**Remark 12.** We shortly indicate how the Acquistapace-Terreni conditions can be recovered from the Kato-Tanabe.

**Proof.** Setting \( \nu := 1 \), regarding (4) and (6) we obtain
\[
\begin{align*}
|\lambda||A(\theta_t \omega)R(\lambda; A(\theta_t \omega))(A^{-1}(\theta_t \omega) - A^{-1}(\theta_s \omega))||_{\mathcal{L}(X_{0})} & \\
& \leq |\lambda||R(\lambda; A(\theta_t \omega))||_{\mathcal{L}(X_{0})}||A(\theta_t \omega)(A^{-1}(\theta_t \omega) - A^{-1}(\theta_s \omega))||_{\mathcal{L}(X_{1},X_{0})} \\
& \leq \frac{M|\lambda|}{|\lambda| + 1}||A(\theta_s \omega)A^{-1}(\theta_s \omega) - A(\theta_t \omega)A^{-1}(\theta_s \omega)||_{\mathcal{L}(X_{1},X_{0})} \\
& \leq \frac{M|\lambda|}{|\lambda| + 1}||A(\theta_s \omega) - A(\theta_t \omega)||_{\mathcal{L}(X_{1},X_{0})}||A^{-1}(\theta_s \omega)|| \\
& \leq \bar{C} \frac{M|\lambda|}{|\lambda| + 1}L(\omega)|t - s|^\rho,
\end{align*}
\]
which means that (5) holds true for \( \nu = 1 \) and \( C(\omega) = M L(\omega) \). Here \( \bar{C} \) is a universal constant, which varies from line to line. Note that we assume w.l.o.g that \( 0 \in \rho(A(\theta_t \omega)) \) for all \( t \geq 0 \), which justifies the boundedness of \( ||A^{-1}(\theta_s \omega)|| \) in the previous computation.

In the following, we frequently use the notation \( J_\Delta := \{(s, t) \in [0, \infty) \times [0, \infty) : s \leq t\} \).

**Theorem 13.** Let \((X_0, X_1)\) be a densely injected Banach couple and let \( A \) satisfy (5). Then the nonautonomous evolution equation (3) generates a random dynamical system.

**Proof.** From the above deliberations, applying pathwise Theorem 2.3 in P. Acquistapace [1] we know that \( A(\omega) \) generates a unique parabolic evolution operator. More exactly, there exists a unique map \( U : J_\Delta \times \Omega \rightarrow \mathcal{L}(X_0) \) having following properties:

1. for all \( t \geq 0 \), \( U(t, t, \omega) = \text{Id} \);
2. for all \( 0 \leq r \leq s \leq t \) we have
   \[
   U(t, s, \omega)U(s, r, \omega) = U(t, r, \omega);
   \]
3. for every \( \omega \in \Omega \) and \((t, s) \in J_\Delta\), the map \( U(\cdot, \cdot, \omega) \) is strongly continuous, more precisely \( U(\cdot, \cdot, \omega) \in C(J_\Delta, L_s(X_0)) \);
4. for every \( s < t \), \( U(\cdot, s, \omega) \in C^1([0, \infty) \cap (s, \infty); \mathcal{L}(X_0)) \) and \( \frac{d}{dt} U(t, s, \omega) = A(\theta_t \omega)U(t, s, \omega); \)
5. there exists a positive random variable \( K : \Omega \rightarrow \mathbb{R}^+ \) such that
   \[
   ||A(\theta_t \omega)U(t, s, \omega)||_{\mathcal{L}(X_0)} \leq K(\omega)(t - s)^{-1};
   \]
6. there exists a positive random variable \( \mu : \Omega \rightarrow \mathbb{R}^+ \) and a constant \( K_1 > 0 \) such that
   \[
   ||U(t, s, \omega)|| \leq K_1 e^{\mu(\omega)(t - s)} \quad \text{for all } (t, s) \in J_\Delta.
   \]
Our aim is to derive a random dynamical system. Straightforward, (7) implies that
\[ U(t + \tau, 0, \omega) = U(t + \tau, \omega)U(\tau, 0, \omega). \]
Moreover \( U(t + \tau, \tau, \omega) = U(t, 0, \theta_\tau \omega) \) since \( A(\theta_\tau \omega) = A(\theta_{t+\tau} \omega) \). For \( t \in [0, T] \)
we denote by \( U(t, \omega) := U(t, 0, \omega) \) and show that this gives us a random dynamical system. Indeed, the cocycle property
\[ U(t + \tau, \omega) = U(t, \theta_\tau \omega)U(\tau, \omega) \]
obviously follows from the deliberations above.

For the sake of completeness, the measurability is contained in the next lemma.

**Lemma 14.** The mapping \( U : \mathbb{R}^+ \times \Omega \to \mathcal{L}(X_0) \) is strongly measurable.

**Proof.** We assume without loss of generality that \([0, \infty) \subset \rho(-A(\omega))\).

For \( \varepsilon > 0 \) we consider the Yosida approximations of \( A(\omega) \)
\[ A_\varepsilon(\omega) := A(\omega)(I + \varepsilon A(\omega))^{-1}, \]
and show that they are strongly measurable. Due to the fact that \((I + \varepsilon A(\omega))^{-1}\)
belongs to \( \mathcal{L}(X_0, X_1) \) we conclude by Lemma 5 that the random variable \((I + \varepsilon A(\omega))^{-1}y\), with \( y \in X_0 \)
is measurable with respect to \( \mathcal{B}(X_0) \), recall Lemma 5. This combined with the strong measurability of the mapping \( \omega \to A(\omega) \)
proves the claim. We know that \( u_\varepsilon \) solves the equation
\[ u'_\varepsilon = A_\varepsilon(\theta_\tau \omega)u_\varepsilon, \ u_\varepsilon(0) = x \in X_0 \quad (9) \]
and that \( U_\varepsilon(t, \omega)x = u_\varepsilon(t) \) is the corresponding evolution operator. Since
\[ u_\varepsilon(t) = x + \int_0^t A_\varepsilon(\theta_\tau \omega)u_\varepsilon(\tau)d\tau \]
we can show that \( \omega \to U_\varepsilon(t, \omega)x \) is measurable given that the mapping \( \tau \to ||A_\varepsilon(\theta_\tau \omega)|| \)
is Hölder continuous, see H. Amann [3] (Lemma 6.1.1, p. 75-76) and that \( \omega \to A_\varepsilon(\omega) \) is strongly measurable. This can be achieved by Picard’s iteration method starting with \( u_0^\varepsilon(t) := x \) which is obviously measurable. For more details consult A. Pazy [29] Theorem 5.1, p. 127. Since \( X_0 \) is separable, the pointwise convergence of the Yosida approximations \( U_\varepsilon(t, \omega)y \to U(t, \omega)y \) as \( \varepsilon \to 0 \), see Lemma 6.1.1, p. 75 in [3], yields that the mapping \( \omega \to U(t, \omega)y \) with \( y \in X_0 \) is measurable. From the properties of the evolution operators we know that \( t \to U(t, \omega)y \) is continuous. Due to the Lemma of Castaing and Valadier [7] we have that \( (t, \omega) \to U(t, \omega)y \)
is jointly measurable.

This statement combined with the fact that \( y \to U(t, \omega)y \) is continuous implies that the mapping \( (t, \omega, y) \to U(t, \omega)y \)
is measurable.

**Remark 15.** If additionally for all \( \omega \) the operators \(-A(\omega)\) generate compact analytic semigroups, then the corresponding evolution operators \( U(t, s, \omega) \) are compact for \( t - s > 0 \). We recall that in this case, the resolvent \((\lambda + A(\omega))^{-1}\) is a compact operator for some \( \lambda \in \rho(-A(\omega)) \), which is equivalent to the compactness of the embedding \( D(A(\omega)) = X_1 \hookrightarrow X_0 \).

The estimate (8) in Theorem 13 leads us to the following result.
Lemma 16. The integrability condition of Theorem 10 reduces to
\[ \mathbb{E} \mu < \infty. \]

Proof. Using Theorem 4.4.1, p. 63 in [3] we have that
\[ ||U(t, \omega)|| \leq K_1 e^{\mu(t)}, \quad t \geq 0, \]
where \( \mu(\omega) \) depends on constant \( C(\omega) \) and on the exponent \( \rho \) introduced in (5), more precisely \( \mu(\omega) = K_2(\rho)C(\omega)^{1/\rho} \). Furthermore,
\[ ||U(1, t, \omega)|| = ||U(1 - t, 0, \theta t \omega)|| \leq K_1 e^{\mu(1-t)}, \]
which proves the statement. \( \Box \)

Remark 17. The estimate for the parabolic evolution operator made in (8) relies on a singular version of the Gronwall inequality, consult D. Henry [20], Lemma 7.1.1, p. 188 and H. Amann [3], Theorem 3.3.1, p. 52.

We conclude this section with two examples of random time-dependent differential operators which satisfy the assumptions made above. As already mentioned, emphasize is laid whether the domain \( D(\theta_t \omega) \) changes in time. For the beginning we consider following random partial differential operators in \( L^p(G) \) for \( 1 < p < \infty \), where \( G \subset \mathbb{R}^n \) is a bounded open domain with \( C^\infty \)-boundary \( \partial G \). We set \( X_0 := L^p(G) \).

Example 18. We define the random partial differential operator \( \omega \)-wise
\[ A(\omega, x, D) := \sum_{|\alpha| \leq 2m} a_\alpha(\omega, x)D^\alpha \]
with Dirichlet boundary conditions \( D^\alpha u = 0 \), for \( |\alpha| < m \) on \( \partial G \) and make certain standard assumptions on the coefficients.

1. The operator \( A \) is supposed to be uniformly strongly elliptic in \( G \), which means that there exists \( c(\omega) > 0 \) such that the mapping \( t \mapsto c(\theta_t \omega) \) is Hölder continuous with exponent \( \rho \in (0, 1) \) satisfying
\[ (-1)^m \sum_{|\alpha| = 2m} a_\alpha(\omega, x)\xi_\alpha \geq c(\omega)|\xi|^{2m}, \quad \text{for all } x \in \bar{G} \text{ and } \xi \in \mathbb{R}^n. \]

2. The coefficients build a stochastic process \( (t, \omega) \mapsto a_\alpha(\theta_t \omega, \cdot) \in C^{2m}(\bar{G}) \) and
\[ |a_\alpha(\theta_t \omega, x) - a_\alpha(\theta_s \omega, x)| \leq c_1(\omega)|t - s|^{\rho}, \quad \text{for all } x \in \bar{G} \text{ and } |\alpha| \leq 2m, \]
with \( c_1(\omega) > 0 \).

We define the \( L^p \)-realization of \( A(\cdot, \cdot, D) \) as:
\[ X_1 := D(A_p(\omega)) = W^{2m,p}(G) \cap W_0^{m,p}(G), \quad \text{where } A_p(\omega)u := A(\omega, x, D)u, \quad (12) \]
for \( u \in D(A_p(\omega)) \).

We know from Theorem 3.5, p. 214 in [29] that \( -A_p(\omega) \) generates an analytic \( C_0 \) semigroup in \( X_0 \) for every \( \omega \in \Omega \). We need to show the Hölder continuity of the mapping \( t \mapsto A_p(\theta_t \omega) \) as described in (6). Therefore let \( w \in X_1 \). Then we have
\[ ||A_p(\theta_t \omega) - A_p(\theta_s \omega)||^p = \sup_{w \in X_1, ||w|| = 1} ||(A_p(\theta_t \omega) - A_p(\theta_s \omega))w||_p^p \]
\[ = \sup_{w \in X_1, ||w|| = 1} \left\| \sum_{|\alpha| \leq 2m} (a_\alpha(\theta_t \omega, x) - a_\alpha(\theta_s \omega, x))D^\alpha w \right\|_p^p. \]
Furthermore, since
\[ \left\| \sum_{|\alpha| \leq 2m} (a_\alpha(\theta_t \omega, x) - a_\alpha(\theta_s \omega, x)) D^\alpha w \right\|_p^p = \int_G \left\| \sum_{|\alpha| \leq 2m} (a_\alpha(\theta_t \omega, x) - a_\alpha(\theta_s \omega, x)) D^\alpha w(x) \right\|_p^p dx \]
\[ \leq \int_G C_p \sum_{|\alpha| \leq 2m} \left| (a_\alpha(\theta_t \omega, x) - a_\alpha(\theta_s \omega, x)) D^\alpha w(x) \right|^p dx \]
\[ \leq C_p \sum_{|\alpha| \leq 2m} \sup_{x \in G} \left| a_\alpha(\theta_t \omega, x) - a_\alpha(\theta_s \omega, x) \right|^p \int_G \left| D^\alpha w(x) \right|^p dx \]
\[ \leq C_p \sum_{|\alpha| \leq 2m} \sup_{x \in G} \left| a_\alpha(\theta_t \omega, x) - a_\alpha(\theta_s \omega, x) \right|^p |w|_{L^p}^{2m, p} \]
\[ \leq C_p |w|_{L^p}^{2m, p} \sum_{|\alpha| \leq 2m} \left| a_\alpha(\theta_t \omega) - a_\alpha(\theta_s \omega) \right|_{C^{2m}(\partial G)} \]
together with the imposed Hölder continuity of \((t, \omega) \mapsto a_\alpha(\theta_t \omega, \cdot)\) justifies (6).

Example 19. Consider the second order differential operator
\[ A(\omega, x, D) := \sum_{|\alpha|, |\beta| \leq m} D^\alpha (a_{\alpha, \beta}(\omega, x) D^\beta), \]
with the boundary operator
\[ C(\omega, x, D) := \sum_{|\alpha|, |\beta| \leq m} a_{\alpha, \beta}(\omega, x) \nu_\alpha(x) D^\beta. \]
The coefficients \(a_{\alpha, \beta}\) satisfy the assumptions in the previous example. In this case, taking \(A_p(\omega)\) the \(L^p\)-realization of \(A\), we see that \(D(A_p(\theta_t \omega))\) is obviously no longer constant, but
\[ D(A_p(\theta_t \omega)) = \{ u \in W^{2m, p}(G) : C(\theta_t \omega, x, D)u = 0, \ x \in \partial G \}. \]
In this case, one can show that \(-A_p(\omega)\) generates an analytic \(C_0\)-semigroup on \(X_0\) and that (5) holds true. For more information concerning differential operators with time-dependent domains consult Example 6.2, p. 35 in [31] and the references specified therein. Consequently, we obtain in this case a parabolic evolution operator and by Theorem 13 a random dynamical system. We emphasize that the deliberations made in Lemma 14 remain valid.

4. Oseledets splitting. The aim of this subsection is to provide a constructive method of the Oseledets splitting for parabolic SPDEs their solution operators generate random dynamical systems which are supposed to be hyperbolic, more precisely they have no zero Lyapunov exponents. We present special cases discussed in Section 3 and illustrate following situations in which the linear part of the evolution equation is constituted by the generator of an analytic \(C_0\)-semigroup together with a random nonautonomous bounded perturbation \(B(\theta_t \omega)\) which arises as a multiplication operator. First of all, we investigate the existence of random center manifolds, which turn out to be one-dimensional linear invariant subspaces
of $X$. Thereafter, based on these subspaces, we construct the Oseledets splitting in special cases, see [10], [11], [21].

Motivated from the perturbation theory of analytic $C_0$-semigroups, as described in A. Pazy [29], Section 3.1 and 3.2, p. 76 (in the autonomous case) and D. Henry [20] Chapter 7, p. 188 (in the nonautonomous case), we consider the linear evolution equation for $t \in \mathbb{R}^+$

$$
\frac{du}{dt} = (A + B(\theta_\omega))u(t),
$$

(13)

for which we investigate the existence of random invariant manifolds. Here $A$ is the generator of an analytic $C_0$-semigroup and $B : \Omega \to \mathcal{L}(X)$ is a strongly measurable operator, meaning that $\omega \mapsto B(\omega)x$ is a random variable in $X$ for every $x \in X$. In this case we know from [7] that $\omega \mapsto ||B(\omega)||_{\mathcal{L}(X)}$ is measurable since the unit ball in $X$ contains a countable dense subset. Furthermore, as argued in the previous section, we impose that $t \mapsto B(\theta_\omega)$ is H"older continuous and

$$
||B|| := \sup_{\omega \in \Omega} ||B(\omega)||_{\mathcal{L}(X)} < \infty.
$$

Such operators $B : \Omega \to \mathcal{L}(X)$ can be viewed as random multiplication operators, see also [21].

In order to investigate the Oseledets splitting for (13), we recall the following concept of a basis in Banach spaces, which was introduced by J. Schauder (1927).

**Definition 20.** Let $X$ be a separable Banach space. A Schauder basis is a sequence $(x_n)_{n \geq 1}$, such that every element $x \in X$ admits a unique representation

$$
x = \sum_{n=1}^{\infty} \xi_n x_n, \text{ with } \xi_n \in \mathbb{R}.
$$

Moreover,

$$
X = \text{span}\{x_n : n \geq 1\} = \text{span}\{x_1\} \oplus \text{span}\{x_n : n \geq 2\} = \bigoplus_{n=1}^{\infty} \text{span}\{x_n\}.
$$

Denoting with $P_n$ the canonical projections on the subspace spanned by the first $n$ eigenvectors, more precisely,

$$
P_n x = \sum_{i=1}^{n} \xi_i x_i,
$$

we know according to Theorem 4.13, p. 136 in [19] that these operators are uniformly bounded, namely

$$
M := \sup_{n \in \mathbb{N}} ||P_n||_{\mathcal{L}(X)} < \infty.
$$

For notational simplicity we use the same symbol $M$ to estimate

$$
\sup_{n \in \mathbb{N}} ||\text{Id} - P_n||_{\mathcal{L}(X)} \leq M \text{ as well as } \sup_{n \in \mathbb{N}} ||P_n - P_{n-1}||_{\mathcal{L}(X)} \leq M.
$$

Unlike an ONB in Hilbert spaces, the convergence of the series in the above definition may depend on the order of summation. If this series converges unconditionally, then $(x_n)_{n \geq 1}$ is called an unconditional Schauder basis. More information about Schauder bases or infinite direct sum of Banach spaces (in the sense of Schauder decompositions), see for instance Section 5.6, p. 258 in A. Pietsch [30], C. Heil [19], Chapter 4, p. 125 and the references specified therein.
Keeping this in mind, we let $X := L^p(0,1)$ for $1 < p < \infty$ and consider
\[
\begin{cases}
\frac{du}{dt} = -Au(t) + B(\theta_t \omega)u(t), & t > 0 \\
u(0) = u_0 \in X.
\end{cases}
\tag{14}
\]

**Remark 21.** We call a mild solution of (14) a function which satisfies
\[
u(t) = T(t)u_0 + \int_0^t T(t-s)B(\theta_s \omega)u(s)ds,
\]
for $t > 0$.

More precisely we let $A := -\Delta$ with $D(A) = W^{2,p}(0,1) \cap W^{1,p}_0(0,1)$. Its spectrum contains only isolated points, namely $\sigma_p(A) = \sigma_p(A) = \{-n^2\pi^2 : n \geq 1\}$ and the eigenfunctions corresponding to these eigenvalues $\{e_n := \sin(n\pi x) : n \geq 1\}$ build a Schauder basis in $L^p(0,1)$ which is an ONB in $L^2(0,1)$. Furthermore, we denote by $(T(t))_{t \geq 0}$ the compact analytic semigroup generated by $A$ on $X$. According to Paragraph 2.8, p. 182 in [15], the spectral mapping theorem holds true, namely we have that
\[
\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}, \quad t \geq 0.
\]

Keeping this in mind, we investigate the existence of one-dimensional spaces which consist of all initial conditions for which the solution exists for all $t \in \mathbb{R}$ and is bounded from above by $\gamma_n := (n - 1/2)\pi^2$ and for $n \geq 1$. A similar technique can be found in the deterministic case and in Hilbert spaces in the work of S.-N. Chow, K. Lu and J. Mallet-Paret [11], [12] and the references specified therein.

To this aim we define the Banach space
\[
BC_{\gamma_n} := \{u : \mathbb{R} \to X : \sup_{t \in \mathbb{R}} e^{\pi^2(n^2t - \gamma_n|t|)}\|u(t)\| < \infty\}.
\]

**Definition 22.** The set
\[
M_n(\omega) = \{u_0 : u(t,\omega,u_0) \text{ exists for } t \in \mathbb{R} \text{ and } u(\cdot,\omega,u_0) \in BC_{\gamma_n}\}
\]
is called random pseudo center manifold for (14).

**Remark 23.**
1) The concept manifold will be justified by the following deliberations.
2) We will actually show that $M_n(\omega)$ are linear subspaces, i.e. they are given by graphs of linear mappings. This fact is crucial because we aim to construct for special cases the Oseledets splitting of $X$ based on $M_n(\omega)$ for $n \geq 1$. Therefore, we emphasize once more, that $B$ is assumed to be a linear operator.

The previous deliberations have been conducted in order to establish the next fundamental result.

**Theorem 24.** There exists infinitely many one-dimensional pseudo random center manifolds for the random dynamical system generated by (14) if
\[
||B|| < \frac{\pi^2}{6M}.
\tag{15}
\]

**Proof.** As in the classical theory of center manifolds, we have to analyze the behavior of the solution on the center, stable and unstable subspace.

Particularly, here we fix $n \geq 2$ and take
\[
X^c := \text{span}\{e_n\}
\]
Then we have \( X = X^c \oplus X^u \oplus X^s \) together with the associated projections \( \Pi^c + \Pi^u + \Pi^s = \text{Id} \), which commute with \( (T(t))_{t \geq 0} \).

We first claim that \( u_0 \in M_n(\omega) \) if and only if there exists a function \( u(\cdot, \omega, u_0) \in BC_{\gamma_n} \) which satisfies the integral equation

\[
\begin{align*}
u(t, \omega, u_0) &= T(t)\Pi^u u_0 + \int_0^t T(t-s)\Pi^c B(\theta_s \omega) u(s) \, ds \\
&\quad - \int_t^\infty T(t-s)\Pi^u B(\theta_s \omega) u(s) \, ds + \int_{-\infty}^t T(t-s)\Pi^s B(\theta_s \omega) u(s) \, ds.
\end{align*}
\]

First of all, for \( u_0 \in M_n(\omega) \) we obtain by the variation of constants formula on each subspace:

\[
\begin{align*}
u^c(t) &= T(t)\Pi^u u_0 + \int_0^t T(t-s)\Pi^c B(\theta_s \omega) u(s) \, ds \\
u^u(t) &= T(t-s')\Pi^u u(s', \omega, u_0) - \int_t^{s'} T(t-s)\Pi^u B(\theta_s \omega) u(s) \, ds \\
u^s(t) &= T(t-s')\Pi^s u(s', \omega, u_0) + \int_{s'}^t T(t-s)\Pi^s B(\theta_s \omega) u(s) \, ds.
\end{align*}
\]

For \( t < s' \) and \( s' > 0 \) we have (regarding the definition of \( \Pi^u \) and \( \Pi^s \), the uniformly boundedness of these projections together with the spectral mapping theorem)

\[
||T(t-s')\Pi^u u(s')|| \leq Me^{-\pi^2(n+1)^2(s'-s)}e^{\pi^2(n^2s'+\gamma_n s')}e^{\pi^2(n^2s'+\gamma_n s')}||u(s')||
\]

\[
\leq Me^{-\pi^2(n+1)^2 t}e^{-s'\pi^2(3n-\frac{1}{2})}||u||_{BC_{\gamma_n}},
\]

which tends to 0 for \( s' \to \infty \).

For \( t > s' \) and \( s' < 0 \) a similar estimate yields

\[
||T(t-s')\Pi^u u(s')|| \leq Me^{-\pi^2(n+1)^2(t-s')}e^{\pi^2(n^2s'+\gamma_n s')}e^{-\pi^2(n^2s'+\gamma_n s')}||u(s')||
\]

\[
\leq Me^{-\pi^2(n+1)^2 t}e^{s'\pi^2(n+\frac{1}{2})}||u||_{BC_{\gamma_n}},
\]

which tends to 0 for \( s' \to -\infty \). The above estimates combined with (17), (18) and (19) prove the assertion.

The converse is straightforward. Letting \( u \) be a solution of (16) and knowing that \( \Pi^c + \Pi^u + \Pi^s = \text{Id} \) one immediately concludes that \( u(\cdot, \cdot, u_0) \) with \( u_0 \in M_n(\omega) \) solves (14).

Furthermore, we define for \( \eta \in X^c \) and \( u \in BC_{\gamma_n} \) the mapping \( J(\cdot, \eta, \omega) : BC_{\gamma_n} \to BC_{\gamma_n} \) as the right hand side of the above integral equation, namely:

\[
J(u, \eta, \omega)[t] = T(t)\eta + \int_0^t T(t-s)\Pi^c B(\theta_s \omega) u(s) \, ds
\]
and show that is a uniform contraction in \( u \) with respect to \( \eta \) if the gap condition (15) is satisfied.

Therefore, let \( u, \overline{u} \in BC_{\gamma_n} \) and compute

\[
J(u, \eta, \omega)[t] - J(\overline{u}, \eta, \omega)[t] = \int_0^t T(t - s) \Pi^n B(\theta_s \omega)(u(s) - \overline{u}(s))ds
\]

\[
- \int_t^\infty T(t - s) \Pi^n B(\theta_s \omega)(u(s) - \overline{u}(s))ds + \int_{-\infty}^t T(t - s) \Pi^n B(\theta_s \omega)(u(s) - \overline{u}(s))ds,
\]

consequently

\[
\|J(u, \eta, \omega)[t] - J(\overline{u}, \eta, \omega)[t]\| \leq M\|B\| \int_0^t e^{-n^2 \pi^2 (t-s)} \|u(s) - \overline{u}(s)\|ds
\]

\[
+ M\|B\| \int_t^\infty e^{-(n-1)^2 \pi^2 (t-s)} \|u(s) - \overline{u}(s)\|ds
\]

\[
+ M\|B\| \int_{-\infty}^t e^{-(n+1)^2 \pi^2 (t-s)} \|u(s) - \overline{u}(s)\|ds.
\]

Now,

\[
\|J(u, \eta, \omega) - J(\overline{u}, \eta, \omega)\|_{BC_{\gamma_n}} \leq M\|B\|\|u - \overline{u}\|_{BC_{\gamma_n}} \sup_{t \in \mathbb{R}} \left( \int_0^t e^{-n^2 \pi^2 (t-s)} e^{\pi^2 (-n^2 s + \gamma_n s)} ds + \int_t^\infty e^{-(n-1)^2 \pi^2 (t-s)} e^{\pi^2 (n^2 - \gamma_n)} ds + \int_{-\infty}^t e^{-(n+1)^2 \pi^2 (t-s)} e^{\pi^2 (-n^2 - \gamma_n)} ds\right).
\]

All in all this results in

\[
\|J(u, \eta, \omega) - J(\overline{u}, \eta, \omega)\|_{BC_{\gamma_n}} \leq \frac{3M\|B\|}{\gamma_n^2} \|u - \overline{u}\|_{BC_{\gamma_n}} = \frac{6M\|B\|}{(2n - 1)\pi^2} \|u - \overline{u}\|_{BC_{\gamma_n}}.
\]

Consequently, \( J \) is a uniform contraction with respect to \( \eta \) if (15) holds, which particularly implies that

\[
g := \frac{3M\|B\|}{\gamma_n} < 1, \text{ for all } n \geq 1.
\]

In this case, Banach’s contraction principle yields that for each \( \eta \in X^c \), \( J(\cdot, \eta, \omega) \) has a unique fixed point \( u^*(\cdot, \eta, \omega) \in BC_{\gamma_n} \).

For \( \eta, \overline{\eta} \in X^c \) we have that

\[
\|u^*(\cdot, \eta, \omega) - u^*(\cdot, \overline{\eta}, \omega)\|_{BC_{\gamma_n}} \leq \frac{1}{1 - g} \|\eta - \overline{\eta}\|.
\]
The joint measurability of \((t, \eta, \omega) \mapsto u^*(t, \omega, \eta)\) follows from Lemma III.14 in C. Castaing and M. Valadier [7], since \(u^*(\cdot, \omega, \eta)\) is measurable and \(u^*(t, \omega, \cdot)\) is Lipschitz continuous.

We can show that each \(M_n(\omega), n \geq 1\) can be represented by the graph of a Lipschitz function.

**Lemma 25.** Choosing

\[ ||B|| < \frac{\pi^2}{6M}, \]

we obtain for all \(n \geq 1\) one-dimensional random invariant linear subspaces for (14) given by

\[ M_n(\omega) = \{ \eta + h_n^c(\omega, \eta) : \eta \in X^c \}, \]

where \(h_n^c(\omega, \cdot) : X^c \to X^c \oplus X^u\) is a linear operator defined as

\[ h_n^c(\omega, \eta) := \int_{-T}^{0} T(-s)\Pi^cB(\theta_s\omega)u^*(s, \omega, \eta)ds - \int_{0}^{T} T(-s)\Pi^uB(\theta_s\omega)u^*(s, \omega, \eta)ds. \]

(21)

**Proof.** In order to show the graph structure, recall that \(u^*_0 \in M_n(\omega)\) if and only if there exists \(u^* \in BC_{\gamma_n}\) with \(u^*(0) = u_0\), such that

\[ u^*(t) = T(t)\Pi^c u_0 + \int_{0}^{t} T(t-s)\Pi^cB(\theta_s\omega)u^*(s)ds - \int_{0}^{\infty} T(t-s)\Pi^uB(\theta_s\omega)u^*(s)ds \]

\[ + \int_{-\infty}^{0} T(t-s)\Pi^uB(\theta_s\omega)u^*(s)ds. \]

This leads to

\[ u^*(t) = T(t)\Pi^c u_0 + \int_{0}^{t} T(t-s)\Pi^cB(\theta_s\omega)u^*(s)ds - \int_{0}^{\infty} T(t-s)\Pi^uB(\theta_s\omega)u^*(s)ds \]

\[ + \int_{0}^{t} T(t-s)\Pi^uB(\theta_s\omega)u^*(s)ds + \int_{-\infty}^{0} T(t-s)\Pi^uB(\theta_s\omega)u^*(s)ds \]

\[ + \int_{0}^{t} T(t-s)\Pi^uB(\theta_s\omega)u^*(s)ds, \]

which further entails

\[ u^*(t) = T(t) \left( \Pi^c u_0 - \int_{0}^{\infty} T(-s)\Pi^uB(\theta_s\omega)u^*(s)ds + \int_{-\infty}^{0} T(-s)\Pi^uB(\theta_s\omega)u^*(s)ds \right) \]

\[ + \int_{0}^{t} T(t-s)B(\theta_s\omega)u^*(s)ds, \]

since \(\Pi^c + \Pi^u + \Pi^s = \text{Id.}\)
This means that \( u^* \) solves (14) with the initial condition
\[
\Pi^c u_0 - \int_0^\infty T(-s)\Pi^c B(\theta_s \omega) u^*(s) \, ds + \int_{-\infty}^0 T(-s)\Pi^c B(\theta_s \omega) u^*(s) \, ds \\
= \Pi^c u_0 + h^c_n(\omega, \Pi^c u_0).
\]
In view of Theorem 24 we obtain that \( u_0 \in M_n(\omega) \) if and only if there exists \( u^* \in BC_{\gamma_n} \) such that
\[
u^*(0) = u_0 = \eta + h^c_n(\omega, \eta),
\]
for some \( \eta \in X^c, \) which yields the graph structure of \( M_n(\omega). \)

Moreover, notice that \( h_n(\omega, 0) = 0 \) and \( h_n(\omega, \cdot) \) is Lipschitz continuous in \( \eta. \) The computation made in Theorem 24 in order to establish the gap condition together with (21), yield
\[
||h^c_n(\omega, \cdot)|| \leq \frac{M||B||}{n^2 \pi^2 + \gamma_n - (n-1)^2 \pi^2} + \frac{M||B||}{(n+1)^2 \pi^2 - n^2 \pi^2 - \gamma_n} \frac{1}{1-g},
\]
amely
\[
||h^c_n(\omega, \eta) - h^c_n(\omega, \eta')|| \leq \frac{4M||B||}{(n-1/2)^2 \pi^2 - 3M||B||} ||\eta - \eta'||,
\]
which shows the Lipschitz continuity of \( h^c_n(\omega, \cdot). \)

Remark 26. To point out the linear graph structure of \( M_n(\omega) \) (for \( n \geq 1 \)) we emphasize that \( h^c_n(\omega, \cdot) \) is linear.

This is again due to the fact that \( B(\cdot) \) is a linear operator, consequently \( A + B(\cdot) \) generates a parabolic evolution operator as concluded in the previous section, which implies that the solutions are linear with respect to the initial data.

Lemma 27. The linear subspaces \( M_n(\omega) \) are measurable.

Proof. The statement follows using similar arguments as in the proof of Theorem 3.1 in [14]. Namely, according to Theorem III.9 in [7] we have to prove that for any \( x \in X \) the expression
\[
\omega \mapsto \inf_{y \in X} ||x - (\Pi^c y + h^c_n(\omega, \Pi^c y))||
\]
is measurable. Letting \( X' \) be a dense countable subset of the Banach space \( X \) we infer using the continuity of \( h^c_n(\omega, \cdot) \) that
\[
\inf_{y \in X} ||x - (\Pi^c y + h^c_n(\omega, \Pi^c y))|| = \inf_{y \in X'} ||x - \Pi^c y - h^c_n(\omega, \Pi^c y)||,
\]
which is measurable due to the fact that \( \omega \mapsto h^c_n(\omega, \Pi^c y) \) is measurable for every \( y \in X. \)

We investigate under which assumptions the subspaces arising in this manner can constitute the Oseledets splitting for (14). Note that by construction all the necessary properties are satisfied, particularly the invariance.

To this aim, consider (14) in the space \( X := l^p \) of \( p \)-summable sequences for \( 1 < p < \infty. \) More precisely, we have
\[
\frac{du(t)}{dt} = -Au(t) + B(\theta_t \omega) u(t),
\]
where \( A \) is an infinite diagonal matrix, similar to [21] (Proposition 2.4). The entries are represented by the eigenvalues of a uniformly elliptic differential operator in \( L^p(0, 1), \) as above. For the sake of brevity we write \( u(\cdot) = (u_n(\cdot))_{n \geq 1}. \)
As in (14), $B$ is a bounded multiplication operator. Since we are working with sequences $u = (u_n)_{n \geq 1}$, we mention that

$$B(\omega, u) = (b_n(\omega)u_n)_{n \geq 1},$$

which is well defined on $l^p$ if and only if the sequence $b(\cdot) := (b_n(\cdot))_{n \geq 1}$ belongs to $l_\infty$ and then

$$||B|| := \sup_{\omega \in \Omega} ||b(\omega)||_{l_\infty}.$$

Furthermore, we observe

**Remark 28.** The unit vectors $e_n := (0, \ldots, 1, \ldots)$ for $n \geq 1$ build an unconditional Schauder basis in $l^p$ for $1 < p < \infty$, which is an ONB in $l^2$. Therefore

$$u = \sum_{n=1}^\infty u_n e_n = \sum_{n=1}^\infty <u, e^*_n > e_n,$$

with coefficient functionals $e^*_n$ belonging to $l^q$.

Moreover, regard the following obvious result, which can be looked up for instance in [16], p. 180.

**Lemma 29.** Let $X, Y$ be two Banach spaces and $S$ an isomorphism from $X$ onto $Y$. If $X_1$ is a complemented subspace of $X$ with topological complement $X_2$, then $S(X_1)$ is a complemented subspace of $Y$ with topological complement $S(X_2)$, i.e. if

$$X = X_1 \oplus X_2 \text{ then } Y = S(X_1) \oplus S(X_2).$$

Under the assumptions of Theorem 24 and keeping Lemma 29 in mind we obtain.

**Theorem 30.** We have the splitting

$$X = \bigoplus_{n=1}^\infty M_n(\omega)$$

if

$$\sum_{n=1}^\infty \left( \frac{4M||B||}{(n-1/2)\pi^2 - 3M||B||} \right)^q < 1,$$

where $1/p + 1/q = 1$.

**Proof.** Starting with the invariant, measurable, one-dimensional linear subspace $M_1(\omega)$, we can find according to [22] (p. 67 Step 3) a measurable complementary subspace $F_1(\omega)$, meaning that $X = M_1(\omega) \oplus F_1(\omega)$. Furthermore, we can split $F_1(\omega)$ as $F_1(\omega) = M_2(\omega) \oplus F_2(\omega)$ and so on. Keeping Lemma 25 in mind (which establishes the structure of each $M_n(\omega)$ for $n \geq 1$), we introduce for a sequence $u(\cdot) = (u_n(\cdot))_{n \geq 1} \in l^p$ with $u = \sum_{n=1}^\infty u_n e_n$ (using the same notations as in Remark 28) the linear operator $R : \Omega \times X \to X$

$$R(\omega, u) := \sum_{n=1}^\infty u_n h^c_n(\omega, e_n).$$

Furthermore, we observe that

$$(\text{Id} + R)(\omega, u) = \sum_{n=1}^\infty u_n e_n + h^c_n(\omega, e_n),$$
also to prove the statement, it is necessary to show that \((\text{Id} + R)(\omega, \cdot)\) is an isomorphism on \(X\). Consequently, in order to ensure the invertibility of \((\text{Id} + R)(\omega, \cdot)\) we have to choose \(B\) such that

\[
\left\| R(\omega, \cdot) \right\|_{L(X)} < 1.
\]

Using Hölder’s inequality, we infer that

\[
\left\| R(\omega, u) \right\| \leq \sum_{n=1}^{\infty} |u_n| \left\| h_n^c(\omega, e_n) \right\| \left( \sum_{n=1}^{\infty} |u_n|^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} \left\| h_n^c \right\|_L^{q} \right)^{\frac{1}{q}},
\]

consequently

\[
\left\| R(\omega, u) \right\|_{l^p} \leq \left\| u \right\|_{l^p} \left( \sum_{n=1}^{\infty} \left\| h_n^c \right\|_L^{q} \right)^{\frac{1}{q}},
\]

which means that

\[
\left\| R(\omega, \cdot) \right\|_{l^p} \leq \left( \sum_{n=1}^{\infty} \left\| h_n^c \right\|_L^{q} \right)^{\frac{1}{q}}.
\]

Therefore, the assumption (23) ensures that \(\text{Id} + R(\omega, \cdot)\) is an isomorphism on \(l^p\), which yields the direct sum decomposition (22).

**Remark 31.**  

a) If one considers the evolution equation (14) on \(X = L^p(0, 1)\) for \(1 < p < \infty\) with \(p \neq 2\), one obtains that the trigonometric system builds a Schauder basis, which is not unconditional. Consequently, the computation made in (24) breaks down unless \(p = 2\). Here, one can investigate invariant manifolds as in Theorem 24, but the technique presented above to diagonalize the equation and represent the Oseledets subspaces as one-dimensional center manifolds goes through only in \(L^2\) or in the space of \(p\)-summable sequences \(l^p\) for \(1 < p < \infty\).

b) In \(L^2(0, 1)\) condition (23) rewrites as

\[
\sum_{n=1}^{\infty} \left( \frac{4|B|}{(n - 1/2)\pi^2 - 3|B|} \right)^2 < 1.
\]

This immediately follows using the Cauchy-Schwarz inequality together with the fact that the corresponding projections are orthonormal.

c) Note that one can investigate center manifolds under a classical spectral decomposition assumption of the linear part, see [9] or [8]. The technique of constructing one-dimensional random pseudo center manifolds as in Theorem 24 is required in order to provide a constructive method of the Oseledets splitting.

5. **Applications.** In this section, we deal with a linear stochastic evolution equation driven by a fractional Brownian motion for which we construct the Oseledets subspaces, using the results obtained in Section 4. Unlike in [21] and [6], where SPDEs driven by a multiplicative Brownian noise are treated and the transformation by means of the Ornstein-Uhlenbeck process is required, here we prove a variantion of constants formula for the mild solution, from which we can directly apply the results in Section 4. This relies on the techniques introduced by M. J. Garrido-Atienza, B. Maslowski, J. Šnupárová in [17] who deal with autonomous evolution equations with bilinear fractional noise in Hilbert spaces. The main idea
is to use a change of variable formula for the stochastic integral in the Stratonovich case, which can be looked up in M. Zähle [39] (Theorem 3.1) and S. Samko [33]. More precisely (compare [21]), here we consider
\begin{equation}
\begin{aligned}
du(t) &= (A + B(\theta_\omega))u(t) + Cu(t) \circ dB^H(t) \\
u(0) &= u_0 \in X,
\end{aligned}
\end{equation}
where \(A, B\) and \(X\) are as specified in Example 14 in Section 4. Additionally to the previous section, we assume that \(X\) is a reflexive Banach space. Furthermore, \(C : X \to X\) is a linear operator, which generates a \(C_0\)-group \((T(t))_{t \in \mathbb{R}}\) which commutes with \((T(t))_{t \geq 0}\). In addition \(B^H\) is a one-dimensional fractional Brownian motion with Hurst parameter \(H \in (1/2, 1)\) and the symbol \(\circ\) stands as usually for the Stratonovich differential.

First of all, we shortly recall some foundations on stochastic calculus for the fractional Brownian motion, as introduced by M. Zähle. Namely, the integration in the Stratonovich sense is a special case of the fractional one, more precisely we have that
\begin{equation}
\int_a^b f(x) \circ dg(x) = (-1)^\alpha \int_a^b D^\alpha_{a+} f(x) D^{1-\alpha}_{b-} g_b(x) dx,
\end{equation}
where \(g_b(x) := \mathbf{1}_{(a,b)}(x) \cdot (g(x) - g(b^-))\) and the Weyl-fractional derivatives are defined as
\begin{equation}
D^\alpha_{a+} f(x) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(y) - f(x)}{(x-y)^\alpha + 1} dy \right) \mathbf{1}_{(a,b)}(x),
\end{equation}
respectively
\begin{equation}
D^{1-\alpha}_{b-} g_b(x) = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left( \frac{g(x) - g(b^-)}{(b-x)^{1-\alpha}} + (1 - \alpha) \int_x^b \frac{g(y) - g(x)}{(y-x)^2 - \alpha} dy \right) \mathbf{1}_{(a,b)}(x).
\end{equation}
For more details on pathwise calculus consult [38], [39].

To fix the notations, we recall that \(C^\beta([a, b]; \mathbb{R})\) stands for the Banach space of the real-valued \(\beta\)-Hölder continuous functions. The norm on this space is constituted by
\begin{equation}
\|u\|_{\beta, a, b} = \sup_{t \in [a, b]} |u(t)| + \sup_{a \leq s < t \leq b} \frac{|u(t) - u(s)|}{|t - s|^\beta}.
\end{equation}
In order to investigate (25) we need two results, which again can be looked up in Zähle. The first one gives us an estimate of the stochastic integral.

**Lemma 32.** Let \(f \in C^\beta([a, b]; \mathbb{R})\) and \(g \in C^{\beta'}([a, b]; \mathbb{R})\) such that \(\beta + \beta' > 1\). Then
\begin{equation}
\int_a^b f \circ dg
\end{equation}
is well-defined in the sense of (26). In addition, for any \(a \leq s \leq t \leq b\) there exists a constant \(C = C(b - a, \beta, \beta')\) such that
\begin{equation}
\left| \int_s^t f \circ dg \right| \leq C \|f\|_{\beta, a, b} \|g\|_{\beta', a, b} (t - s)^{\beta'}.
\end{equation}
For our aims, the following change of variable formula is crucial, see Theorem 3.1 in M. Zähle [39]. Here \( I^a_{\alpha+}(L^2([a, b])) \), respectively \( I^{1-\alpha}_b(L^2([a, b])) \) stand for the image of \( L^2[a, b] \) under the right- and left-Riemann Liouville fractional integral, consult [38], [33] and the references specified therein.

**Lemma 33.** Let \( 0 < \alpha < 1/2, f \in I^a_{\alpha+}(L^2([a, b])) \) be bounded and \( g_{b-} \in I^{1-\alpha}_b(L^2([a, b])) \) and

\[
h(t) = h(s) + \int_s^t f(r) \circ dg(r), \quad s < t \in [a, b].
\]

Then for any \( G \in C^1([a, b] \times \mathbb{R}; \mathbb{R}) \) such that \( \frac{\partial G}{\partial x}(t, \cdot) \in C^1([a, b] \times \mathbb{R}; \mathbb{R}) \) and for any \( t_0 \leq t \in [a, b] \), the following change of variable formula is valid

\[
G(t, h(t)) = G(t_0, h(t_0)) + \int_{t_0}^t \frac{\partial G}{\partial x}(s, h(s)) f(s) \circ dg(s) + \int_{t_0}^t \frac{\partial G}{\partial t}(s, h(s)) ds.
\]

Keeping this in mind, we investigate the well-posedness of (25). We write \( A(t, \omega) := A + B(\theta_t \omega) \) and assume for simplicity that \( D(A) \) is not time-dependent. Let \( U_A(\cdot, \cdot, \omega) \) denote the evolution operator generated by \( A \). We introduce the following concept of solution for the nonautonomous equation (25).

**Definition 34.** Given \( T > 0 \), a stochastic process \( u \) is called weak solution of (25) if for any \( x^* \in D(A^*) \) we have

\[
\langle u(t), x^* \rangle = \langle u_0, x^* \rangle + \int_0^t \langle u(r), A^*(\theta_r \omega)x^* \rangle dr + \int_0^t \langle u(r), C^*x^* \rangle \circ d\omega(r),
\]

for all \( t \in [0, T] \), provided that the above integrals are well-defined.

Here we wrote \( \omega \) for \( B^H \) and \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{X, X^*} \) denotes the duality between \( X \) and \( X^* \).

We now state the main result of this section which will later on allow us to apply the deliberations made in Section 4. For notational simplicity, if there is no confusion we drop the \( \omega \) and time dependence and write \( U_{A}(\cdot, \cdot) \) instead of \( U_{A}(\cdot, \cdot, \omega) \), respectively \( A \) for \( A(\cdot, \cdot) \).

**Theorem 35.** There exists a unique weak solution of (25) which is given by

\[
U(t, s, \omega) u_0 = T_C(\omega(t) - \omega(s)) U_A(t, s) u_0, \quad \text{for all } (t, s) \in J_\Delta.
\]

*Proof.* Following the steps in the proof of Theorem 2.2 in [17], we first prove the statement for \( u_0 \in D(A) \) and present the complete proof for \( u_0 \in X \) in Appendix A. To this aim, we take an arbitrary \( \xi \in D(A^*) \) and define \( G : [s, T] \times \mathbb{R} \to \mathbb{R} \) as

\[
G(t, x) := \langle U_A(t, s) u_0, T_C(x) \rangle \xi
\]

and

\[
h(t) = \omega(t) - \omega(s) = \int_s^t \mathbb{I}_{(s, T)}(r) \circ d\omega(r),
\]

which is well-defined by (26), since \( \omega_{T-} \in I^{1-\alpha}_b(L^2[s, T]) \) and the indicator function \( \mathbb{I}_{(s, T)} \in I^a_{\alpha+}(L^2[s, T]) \) for \( \alpha \in (1 - H, 1/2) \). Furthermore we have that \( G \in
$C^1([s, T] \times \mathbb{R}) \to \mathbb{R}$ with $\frac{\partial G}{\partial x}(t, \cdot) \in C^1(\mathbb{R}; \mathbb{R})$. This entails that the change of variable formula (28) is applicable and leads us to

$$G(t, \omega(t) - \omega(s)) = \langle u_0, \xi \rangle + \int_s^t \langle U_A(r, s), T_C(\omega(r) - \omega(s)) C^* \xi \rangle \circ d\omega(r)$$

$$+ \int_s^t \langle A(\theta_r \omega) U_A(r, s) u_0, T_C^*(\omega(r) - \omega(s)) \xi \rangle \circ d\omega(r).$$

This further rewrites as

$$\langle U(t, s, \omega) u_0, \xi \rangle = \langle u_0, \xi \rangle + \int_s^t \langle U(r, s, \omega), C^* \xi \rangle \circ d\omega(r)$$

$$+ \int_s^t \langle U(r, s, \omega) u_0, A^*(\theta_r \omega) \xi \rangle \circ d\omega(r),$$

for all $t, s \in J_A$. To complete the proof, we have to deal with the general case in which $u_0 \in X$. This is crucial because our aim is to obtain a random dynamical system on $X$. The complete proof is presented in Appendix A.

**Theorem 36.** The solution operator of (25) $U : \mathbb{R}^+ \times \Omega \times X \to X$ generates a random dynamical system.

**Proof.** This immediately follows regarding (30) and the proof of Theorem 13. For the sake of completeness, we show the cocycle property

$$U(t + s, 0, \omega) = T_C(\omega(t + s)) U_A(t + s, 0, \omega)$$

$$= T_C(\theta_s \omega(t) - \theta_s \omega(0) + \omega(s)) U_A(t + s, 0, \omega)$$

$$= T_C(\theta_s \omega(t)) T_C(\omega(s)) U_A(t + s, s, \omega) U_A(s, 0, \omega)$$

$$= T_C(\theta_s \omega(t)) U_A(t, 0, \theta_s \omega) T_C(\omega(s)) U_A(s, 0, \omega)$$

$$= U(t, 0, \theta_s \omega) U(s, 0, \omega).$$

Here we use that $U_A$ is the parabolic evolution operator generated by $A + B(\cdot)$ and the $C_0$-semigroup generated by $A$ commutes with the group generated by $C$.

**Remark 37.** Using Theorem 35 together with (30) immediately entails that in order to investigate Lyapunov exponents and Oseledets subspaces for (25) is enough to consider only the evolution operator $U_A$.

**Theorem 38.** The long-time behavior of the random dynamical system generated by the evolution operator of (25) reduces to the long-time behavior of the random dynamical system generated by the solution operator of (14).

**Proof.** Using (30) we can immediately verify the integrability condition of the MET. We have that

$$U(t, 0, \omega) = T_C(\omega(t)) U_A(t, 0, \omega),$$

therefore

$$\log^+ ||U(t, 0, \omega)|| \leq \log^+ ||T_C(\omega(t))|| + \log^+ ||U_A(t, 0, \omega)||$$
From Lemma 16 we know that the integrability condition of Theorem 10 holds true for $U_\mathcal{A}$. Since $||T_C(\omega(t))|| \leq e^{||C|||\omega(t)||}$,
\[ \mathbb{E} \sup_{t \in [0,1]} \log^+ ||T_C(\omega(t))|| < \infty \]
\[ \mathbb{E} \sup_{t \in [0,1]} \log^+ ||T_C(\theta_t \omega(1-t))|| < \infty. \]
Moreover
\[ ||U(t,0,\omega)|| \leq e^{||C|||\omega(t)||} ||U_\mathcal{A}(t,0,\omega)||. \]
Due to the law of iterated logarithm we know that $t \mapsto ||\omega(t)||$ (consult Theorem 2 in M. Marcus [24]) has a subexponential growth as $t \to \pm \infty$, which implies that the long-time behavior of (25) is described by $U_\mathcal{A}$.

Remark 39. Keeping this mind we can construct based on Theorem 24 one-dimensional random invariant subspaces for equation (25) which constitute the Oseledets splitting in $L^2$ respectively $l^p$ for $1 < p < \infty$.

Remark 40. 1) The results are similar to the ones obtained for the Brownian motion in [6] and [21], which is natural since we work with a Stratonovich stochastic differential equation.

2) The results remain valid if one considers in (25) finitely many linear bounded operators $\{C_1, \ldots, C_n\}$ which mutually commute with $A$. This particularly implies that the corresponding groups $\{T_{C_1}, \ldots, T_{C_n}\}$ generated by $C_1, \ldots, C_n$ and the $C_0$-semigroup generated by $A$ also mutually commute.

Appendix A. Proof of Theorem 35. For the sake of completeness, we present the proof of Theorem 35 in full generality. For analytic $C_0$-semigroups this can be found in [17] (proof of Theorem 2.2). Let $u_0 \in X$. Take a sequence $(x_n)_{n \in \mathbb{N}} \in D(A)$ such that
\[ ||x_n - u_0||_X \to 0 \quad \text{as} \quad n \to \infty. \]
As shown
\[ \langle U(t, s)x_n, \xi \rangle = \langle x_n, \xi \rangle + \int_s^t \langle U(r, s), C^* \xi \rangle \circ d\omega(r) \]
\[ + \int_s^t \langle U(r, s)x_n, A^*(\theta_r \omega)\xi \rangle dr, \]
holds true for all $n \in \mathbb{N}$ and $\xi \in D(A^*)$. Here we dropped for notational simplicity the $\omega$-dependence of $U$.

For the deterministic terms appearing above, one can obviously pass to the limit as $n \to \infty$, since
\[ ||\langle x_n, \xi \rangle - \langle u_0, \xi \rangle|| \to 0 \quad \text{as} \quad n \to \infty \]
\[ ||\langle U(t, s)x_n, \xi \rangle - \langle U(t, s)u_0, \xi \rangle|| \to 0 \quad \text{as} \quad n \to \infty \]
\[ \left| \int_s^t \langle U(r, s)x_n, A^*(\theta_r \omega)\xi \rangle dr - \int_s^t \langle U(r, s)u_0, A^*(\theta_r \omega)\xi \rangle dr \right| \to 0 \quad \text{as} \quad t \to \infty. \]
It only remains to estimate the stochastic term. Using (32) we infer that
\[
\left| \int_s^t \langle U(r, s)(x_n - u_0), C^* \xi \rangle \circ d\omega(r) \right| \\
\leq c|\omega||\beta',s,T \| U(\cdot, s)(x_n - u_0), C^* \xi \| \beta',s,T (T - s)^{\beta'}.
\]

Consequently, we have to estimate the \( \beta \)-Hölder norm of the integrand. The first part of the \( \beta \)-Hölder norm easily results in
\[
sup_{\tau \in [s, T]} |\langle U(\tau, s)(x_n - u_0), C^* \xi \rangle| \leq ||U(\cdot, s)|||C^*\xi|\|x_n - u_0|\rightarrow 0 \text{ as } n \rightarrow \infty.
\]

We proceed with the computation for the second part of the \( \beta \)-Hölder norm. For \( s \leq \tau_1 < \tau_2 \leq T \) one has
\[
||\langle U(\tau_2, s) - U(\tau_1, s)(x_n - u_0), C^* \xi, \rangle || = ||(U(\tau_2, s) - U(\tau_1, s))U(\cdot, s)(x_n - u_0), C^* \xi, ||
\]
\[
+ ||T_{\xi}(\omega(\tau_1) - \omega(s))U(\tau_2, s) - U(\tau_1, s)(x_n - u_0), C^* \xi, || = I_1 + I_2.
\]

In order to estimate \( I_1 \), we recall that
\[
T_{\xi}(t)x - T_{\xi}(s)x = \int_s^t T(\tau)Cx d\tau = \int_s^t CT_{\xi}(x) d\tau.
\]
Keeping this in mind for \( T_{\xi} \) and using properties of the duality mapping, further results in
\[
I_1 \leq ||U(\tau_2, s)(x_n - u_0), (T_{\xi}(\omega(\tau_2) - \omega(s)) - T_{\xi}(\omega(\tau_1) - \omega(s)))U(\cdot, s)(x_n - u_0), C^* \xi, ||
\]
\[
\leq ||U(\tau_2, s)||_{\mathcal{L}(X)}||x_n - u_0||_X \int_{\omega(\tau_1) - \omega(s)} |\int_{\omega(\tau_2) - \omega(s)} ||(C^* \xi)|^2 d\tau|
\]
\[
\leq ||U(\tau_2, s)||_{\mathcal{L}(X)}||x_n - u_0||_X ||(C^* \xi)|^2 X \cdot ||\omega||_{\beta',(\tau_1, \tau_2)}(\tau_2 - \tau_1)^{\beta'},
\]
consequently
\[
\sup_{s \leq \tau_1 < \tau_2 \leq T} \frac{I_1}{(\tau_2 - \tau_1)^{\beta'}} \leq ||U(\tau_2, s)|| ||x_n - u_0||_X ||(C^* \xi)|^2 X \cdot ||\omega||_{\beta',(\tau_1, \tau_2)}(T - s)^{\beta' - \beta}.
\]
This converges to 0 as \( n \rightarrow \infty \). Recall that \( \omega \) is \( \beta' \)-Hölder-continuous with \( \beta' > \beta \).

Finally, the estimate of \( I_2 \) relies on estimates for parabolic evolution operators, which are similar to the ones for analytic \( C_0 \)-semigroups, which can be looked up in P. Acquistapace [2] and R. Schnaubelt [34], consult also Lemma 5.1.3 and Section 5.3 in H. Amann [3]. To our aim we use that
\[
||U(\tau_2, s) - U(\tau_1, s)||_{\mathcal{L}(X)}
\]
\[
= ||U(\tau_2, \tau_1) - U(\tau_1, \tau_1)||_{\mathcal{L}(D((-A)^{\beta}, X))} ||U(\tau_1, s)||_{\mathcal{L}(X,D((-A)^{\beta}))}
\]
\[
\leq \widetilde{C} \left( \frac{\tau_2 - \tau_1}{\tau_1 - s} \right)^{\beta},
\]
where \( \widetilde{C} = C(\beta) \).
All in all,
\[ I_2 \leq \frac{(U_A(\tau_2, s) - U_A(\tau_1, s))((x_n - u_0), T_0^s(\omega(\tau_1) - \omega(s))C^\beta_1)}{\leq C \left( \frac{\tau_2 - \tau_1}{\tau_1 - s} \right)^\beta \|x_n - u_0\| \|C^\beta_1\| \|x\| \|\omega\|_{\beta^{'}, \tau_1, \tau_2}(T - s)^{\beta^{'}}}. \]
from which we again obtain that
\[ \left( \frac{\tau_2 - \tau_1}{\tau_1 - s} \right)^\beta \|x_n - u_0\| \|C^\beta_1\| \|x\| \|\omega\|_{\beta^{'}, \tau_1, \tau_2}(T - s)^{\beta^{'}} \leq \frac{I_2}{s \leq \tau_1 < \tau_2 \leq T (\tau_2 - \tau_1)^\beta \leq C \|x_n - u_0\| \|C^\beta_1\| \|x\| \|\omega\|_{\beta^{'}, \tau_1, \tau_2}(T - s)^{\beta^{'}}}. \]
This also tends to 0 as \( n \to \infty \), which concludes the proof for \( u_0 \in X \).

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