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WEAK ERROR IN NEGATIVE SOBOLEV SPACES FOR THE
STOCHASTIC HEAT EQUATION

OMAR ABOURA

Abstract. In this paper, we make another step in the study of weak error of the stochastic heat equation by considering norms as functional.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ a probability space and $T > 0$ a fixed time. $(W(t))_{t \geq 0}$ will be a cylindrical Brownian motion on $L^2(0, 1)$. We consider the stochastic heat equation, written in abstract form in $L^2(0, 1)$:

$$
X(0) = 0, \quad \text{for all } t \in [0, T] \quad X(t, 0) = X(t, 1) = 0 \quad \text{and}
$$

$$
dX(t) = \frac{1}{2} \frac{d^2}{dx^2} X(t) dt + dW(t). \quad (1.1)
$$

It is well known that this equation admits a unique weak solution (from the analytical point of view).

Let $N \in \mathbb{N}^*$ and $h := T/N$. Consider $(t_k)_{0 \leq k \leq N}$ the uniform subdivision of $[0, T]$ defined by $t_k := kh$. We consider the implicit Euler scheme defined as follow:

$$
X^N(t_{k+1}) = X^N(t_k) + \frac{1}{2} \frac{d^2}{dx^2} X^N(t_{k+1}) + \Delta W(k+1), \quad (1.2)
$$

where $\Delta W(k+1) = W(t_{k+1}) - W(t_k)$.

Let $f : L^2(0, 1) \to \mathbb{R}$ be a functional. The strong error is the study of $E \left| X^N(T) - X(T) \right|_{L^2(0, 1)}^2$.

The weak error is the study of $\left| Ef \left( X^N(T) \right) - Ef \left( X(T) \right) \right|$ with respect to the time mesh $h$.

In [6], A. Debussche considers a more general stochastic equation and a more general functional than the one considered here. He obtains a weak error of order $1/2$, which is the double of that proved by [15] for the strong speed of convergence. The novelty of this paper his to prove that for the square of the norm the weak error his better than $1/2$ in negative Sobolev spaces.

2. Preliminaries and main result

Notations. We collect here some of the notations used through the paper. $< . , . >_{L^2(0,1)}$ is the inner product in $L^2(0, 1)$, $H^1_0(0, 1)$ is the Sobolev space of functions $f$ in $L^2(0, 1)$ vanishing in $0$ and $1$ with first derivatives in $L^2(0, 1)$, $H^2(0, 1)$ is the Sobolev space of functions $f$ in $L^2(0, 1)$ with first and second derivatives in $L^2(0, 1)$. Finally, for $m = 1, 2, \ldots$, let $(\epsilon_m(x) = \sqrt{2} \sin(m \pi x)$ and $\lambda_m = \frac{1}{2} \pi^2 m^2$ denote the eigenfunction and eigenvalues of $-\Delta$ with Dirichlet boundary conditions on $(0, 1)$.

An $L^2(0,1)$-valued stochastic process $(X(t))_{t \in [0, T]}$ is said to be a solution of (1.1) if:

$$
X(0) = 0 \quad \text{and for all } g \in H^1_0(0, 1) \cap H^2(0, 1) \text{ we have}
$$

$$
< X(t), g >_{L^2(0,1)} = \int_0^t < X(s), \frac{1}{2} \frac{d^2}{dx^2} g >_{L^2(0,1)} ds + < W(t), g >_{L^2(0,1)}. \quad (1.3)
$$
It is well know that (1.1) admits a unique solution: see [4]. Then \((e_m)_{m \geq 1}\) is a complete orthonormal basis of \(L^2(0, 1)\). If we denote by \(\lambda_m := \frac{1}{2}(\pi m)^2\), \(W_{\lambda_m}(t) := \langle W(t), e_m \rangle_H\) and \(X_{\lambda_m}(t)\) denote the solution of the evolution equation: \(X_{\lambda_m}(0) = 0\) and for \(t > 0:\)

\[dX_{\lambda_m}(t) = -\lambda_m X_{\lambda_m}(t) dt + dW_{\lambda_m}(t).\]

Then the processes \((X_{\lambda_m}(.))_{m \geq 1}\) are independent and \(X(t) = \sum_{m \geq 1} X_{\lambda_m}(t) e_m\) for all \(t \geq 0\).

A sequence of \(L^2(0, 1)\)-valued \((X^N(t_k))_{k=0, \ldots, N}\) is said to be a solution of (1.2) if: \(X^N(t_0) = 0\) and for all \(k = 0, \ldots, N-1\) and for all \(g \in H^1_0(0, 1) \cap H^2(0, 1)\) we have

\[< X^N(t_{k+1}), g >_{L^2(0,1)} = < X^N(t_k), g >_{L^2(0,1)} + h < X^N(t_{k+1}), \frac{1}{2} \frac{d^2}{dx^2} g >_{L^2(0,1)} + \Delta W(k+1), g >_{L^2(0,1)}\]

It is well know that (1.2) has a unique solution and there exists a constant \(C > 0\), independent of \(N\), such that \(E |X^N(T) - X(T)|^2_{L^2(0,1)} \leq C h^\frac{1}{2}\) where \(h = T/N\). Now if we denote by \((X^N_{\lambda_m}(t_k))_{k=0, \ldots, N}\) the solution of: \(X^N_{\lambda_m}(t_0) = 0\) and for \(k = 0, \ldots, N-1\)

\[X^N_{\lambda_m}(t_{k+1}) = X^N_{\lambda_m}(t_k) - \lambda_m h X^N_{\lambda_m}(t_{k+1}) + W_{\lambda_m}(k+1).\]

The random vectors \((X^N_{\lambda_m}(t_k), k = 0, \ldots, N)_{m=1,2,\ldots}\) are independent and \(X^N(t_k) = \sum_{m \geq 1} X^N_{\lambda_m}(t_k) e_m\).

Let \(p \geq 0\); we define the spaces \(H^{-p}\) as the completion of \(L^2(0, 1)\) for the topology induced by the norm \(|u|_{H^{-p}} := \sum_{m \geq 1} \lambda_m^p < u, e_m >^2_H\). The following theorem improves the speed of convergence of \(X^N\) to \(X\) for negative Sobolev spaces.

**Theorem 2.1.** Suppose that \(h < 1\) and let \(p \in \left[0, \frac{1}{2}\right)\). There exists a constant \(C > 0\), independent of \(N\), such that

\[E \left| X^N(T) \right|^2_{H^{-p}} - E \left| X(T) \right|^2_{H^{-p}} \leq C h^{p + \frac{1}{2}}.\]

3. **Proof of the theorem 2.1**

The proof of the theorem will be done in several steps. First we recall the weak error of the Ornstein-Uhlenbeck process. Secondly we prove some technical lemmas. Then we decompose the weak error and analyse each term of these decompositions.

3.1. **Weak error of the Ornstein-Uhlenbeck process.** Let \(\lambda > 0\), \((W_{\lambda}(t))_{t \geq 0}\) be a one dimensional Brownian motion and \((X_{\lambda}(t))_{t \geq 0}\) be the Ornstein Uhlenbeck process solution of the following stochastic differential equation: \(X_{\lambda}(0) = x \in \mathbb{R}\) and

\[dX_{\lambda}(t) = -\lambda X_{\lambda}(t) dt + dW_{\lambda}(t).\]  

In this step, we study two properties associated with this process: the Kolmogorov equation and the implicit Euler scheme.

Let \(\left(X^t_{\lambda}x(s)\right)_{t \leq s \leq T}\) be the solution of (3.1) starting from \(x\) at time \(t\). It is well know that \(X^t_{\lambda}x(T)\) is a normal random variable:

\[X^t_{\lambda}x(T) \sim \mathcal{N} \left( e^{-\lambda(T-t)}x, \frac{1 - e^{-2\lambda(T-t)}}{2\lambda} \right).\]
For \( t \in [0, T] \) and \( x \in \mathbb{R} \) set \( u_\lambda(t, x) := E \left| X_\lambda^{t,x}(T) \right|^2 \). Then \( u_\lambda \) is the solution of the following partial differential equation, called Kolmogorov equation: for all \( x \in \mathbb{R} \), \( u_\lambda(T, x) = |x|^2 \) and for all \((t, x) \in [0, T] \times \mathbb{R}\)
\[
-\frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t, x) - \lambda x \frac{\partial}{\partial x} u(t, x). \tag{3.2}
\]

Since \( X_\lambda^{t,x}(T) \) has a normal law, we can write \( u_\lambda \) explicitly:
\[
u_\lambda(t, x) = \frac{1 - e^{-2\lambda(T-t)}}{2\lambda} + e^{-2\lambda(T-t)}x^2. \tag{3.3}
\]

With this expression we see that \( u_\lambda \in C^{1,2}([0, T] \times \mathbb{R}) \) and we have the following derivatives:
\[
\begin{align*}
\frac{\partial}{\partial x} u_\lambda(t, x) &= 2e^{-2\lambda(T-t)}x, \tag{3.4} \\
\frac{\partial^2}{\partial x^2} u_\lambda(t, x) &= 2e^{-2\lambda(T-t)}, \tag{3.5} \\
\frac{\partial}{\partial t} u_\lambda(t, x) &= -e^{-2\lambda(T-t)} + 2\lambda e^{-2\lambda(T-t)}x^2, \tag{3.6} \\
\frac{\partial}{\partial t \partial x} u_\lambda(t, x) &= 4\lambda e^{-2\lambda(T-t)}x. \tag{3.7}
\end{align*}
\]

The implicit Euler scheme for the Ornstein-Uhlenbeck equation (3.1) starting from 0 at time \( t_0 \), is defined as follow: \( X_\lambda^{N}(t_0) = 0 \) and for \( k = 0, \ldots, N-1 \)
\[
X_\lambda^{N}(t_{k+1}) = X_\lambda^{N}(t_k) - \lambda h X_\lambda^{N}(t_{k+1}) + \Delta W_\lambda(k+1), \tag{3.8}
\]

where \( \Delta W_\lambda(k+1) = W_\lambda(t_{k+1}) - W_\lambda(t_k) \). Since we have the following equation
\[
X_\lambda^{N}(t_{k+1}) = \frac{1}{1+\lambda h} X_\lambda^{N}(t_k) + \frac{1}{1+\lambda h} \Delta W_\lambda(k+1), \tag{3.9}
\]
we see that the scheme is well defined.

**Lemma 3.1.** For \( k = 1, \ldots, N \) we have \( X_\lambda^{N}(t_k) = \sum_{j=0}^{k-1} \frac{\Delta W_\lambda(k-j)}{(1+\lambda h)^{j+1}} \).

**Proof.** We proceed by induction. If \( k = 1 \), we have \( X_\lambda^{N}(t_1) = \frac{1}{1+\lambda h} \Delta W_\lambda(1) \). Suppose the result true until \( k \). Using (3.9), we have
\[
X_\lambda^{N}(t_{k+1}) = \sum_{j=0}^{k-1} \frac{\Delta W_\lambda(k-j)}{(1+\lambda h)^{j+2}} + \frac{1}{1+\lambda h} \Delta W_\lambda(k+1)
= \sum_{l=1}^{k} \frac{\Delta W_\lambda(k-1-l)}{(1+\lambda h)^{l+1}} + \frac{1}{(1+\lambda h)^{k+1}} \Delta W_\lambda(k+1-0),
\]
which concludes the proof. \( \square \)

**Lemma 3.2.** For all \( k = 0, \ldots, N \), we have the following bound \( E \left| X_\lambda^{N}(t_k) \right|^2 \leq \frac{1}{2\lambda} \).

**Proof.** Using the independence of the increments of the Brownian motion and Lemma 3.1, we have
\[
E \left| X_\lambda^{N}(t_k) \right|^2 = \sum_{j=0}^{k-1} \frac{1}{(1+\lambda h)^{2(j+1)}} E \left| \Delta W_\lambda(k-j) \right|^2 = \frac{1}{2h} \sum_{j=0}^{k-1} \frac{1}{(1+\lambda h)^{2(j+1)}}.
\]
Lemma 3.4. The next lemma relates the above processes. 

Let \( a := 1/(1 + \lambda h)^2 \); we deduce that \( E \left[ X^N_\lambda(t_k) \right]^2 = ha \frac{1 - e^k}{1 - a} \). Simple computations yield \( ha/(1 - a) = 1/(2\lambda + \lambda^2 h) \), which implies

\[
E \left[ X^N_\lambda(t_k) \right]^2 = \frac{1}{2\lambda + \lambda^2 h} \left( 1 - \frac{1}{(1 + \lambda h)^{2k}} \right).
\]

This concludes the proof. \( \square \)

For \( t \geq 0 \), we denote \( \mathcal{F}^\lambda_t := \sigma(W_\lambda(s), s \leq t) \) and \( D^{1,2}_\lambda \) the Malliavin Sobolev space with respect to \( W_\lambda \).

**Lemma 3.3.** For all \( k = 1, \ldots, N \), we have \( X^N_\lambda(t_k) \in D^{1,2}_\lambda \cap L^2(\mathcal{F}^\lambda_{t_k}) \).

**Proof.** This is a consequence of Lemma 3.1, the fact that \( L^2(\mathcal{F}^\lambda_{t_k}) \) and \( D^{1,2}_\lambda \) are linear space and for all \( j = 0, \ldots, k - 1 \), \( \Delta W_\lambda(k - j) \in D^{1,2}_\lambda \cap L^2(\mathcal{F}^\lambda_{t_k}) \). \( \square \)

As usual in the study of weak error, we need to use a continuous process that interpolates the Euler scheme. The interpolation process that we use was introduced in [1]. We recall its construction and prove some of its properties.

Let \( k \in \{0, \ldots, N - 1\} \) be fixed. In order to interpolate the scheme between the points \((t_k, X^N_\lambda(t_k))\) and \((t_{k+1}, X^N_\lambda(t_{k+1}))\), we define the process as follows: for \( t \in [t_k, t_{k+1}] \), set

\[
X^N_\lambda(t) := X^N_\lambda(t_k) - \lambda E \left[ X^N_\lambda(t_{k+1}) \middle| \mathcal{F}_t \right] (t - t_k) + W_\lambda(t) - W_\lambda(t_k). \tag{3.10}
\]

In the sequel, we will use the following processes: for \( t \in [t_k, t_{k+1}] \)

\[
\beta^{k,N}_\lambda(t) := -\lambda E \left[ X^N_\lambda(t_{k+1}) \middle| \mathcal{F}_t \right], \tag{3.11}
\]

\[
z^{k,N}_\lambda(t) := -\lambda E \left[ \Delta X_\lambda^N(t_{k+1}) \middle| \mathcal{F}_t \right], \tag{3.12}
\]

\[
\gamma^{k,N}_\lambda(t) := 1 + (t - t_k)z^{k,N}_\lambda(t). \tag{3.13}
\]

The next lemma relates the above processes.

**Lemma 3.4.** Let \( k = 0, \ldots, N - 1 \). For \( t \in [0, T] \), we have

\[
d\beta^{k,N}_\lambda(t) = z^{k,N}_\lambda(t) dW_\lambda(t), \quad z^{k,N}_\lambda(t) = -\frac{\lambda}{1 + \lambda h}, \quad \gamma^{k,N}_\lambda(t) = 1 - (t - t_k)\frac{\lambda}{1 + \lambda h}, \quad dX^N_\lambda(t) = \beta^{k,N}_\lambda(t) dt + \gamma^{k,N}_\lambda(t) dW_\lambda(t).
\]

**Proof.** Using the Clark-Ocone formula and Lemma 3.3, we have

\[
X^N_\lambda(t_{k+1}) = E \left[ X^N_\lambda(t_{k+1}) \middle| \mathcal{F}_t \right] + \int_t^{t_{k+1}} E \left[ \Delta X^N_\lambda(t_{k+1}) \middle| \mathcal{F}_s \right] dW_\lambda(s).
\]

Multiplying by \(-\lambda\), we deduce

\[
-\lambda X^N_\lambda(t_{k+1}) = \beta^{k,N}_\lambda(t) + \int_t^{t_{k+1}} z^{k,N}_\lambda(s) dW_\lambda(s),
\]

which gives the first identity. Applying the Malliavin derivative to (3.9), we have for \( s \in [t_k, t_{k+1}] \)

\[
D_s X^N_\lambda(t_{k+1}) = \frac{1}{T - X(t_{k+1})}.
\]

Multiplying by \(-\lambda\), we deduce the second and third equalities.

Finally, Itô’s formula gives us

\[
d \left( (t - t_k)\beta^{k,N}_\lambda(t) \right) = (t - t_k)z^{k,N}_\lambda(t) dW_\lambda(t) + \beta^{k,N}_\lambda(t) dt,
\]

which concludes the proof. \( \square \)
Lemma 3.5. Let $k \in \{0, \ldots, N - 1\}$. For any $s \in [t_k, t_{k+1}]$, we have

$$E \left| \beta^{k,N}_\lambda(s) \right|^2 \leq 2\lambda, \quad E \left| X^N_\lambda(s) \right|^2 \leq \frac{1}{2\lambda} + h, \quad E\beta^{k,N}_\lambda(s)X^N_\lambda(s) \leq 1.$$  

Proof. Applying the conditional expectation with respect to $\mathcal{F}_s$ on both sides of (3.9) for $s \in [t_k, t_{k+1}]$ we have

$$E \left( X^N_\lambda(t_{k+1}) | \mathcal{F}_s \right) = \frac{1}{1 + \lambda h} \left[ X^N_\lambda(t_k) + (W_\lambda(s) - W_\lambda(t_k)) \right].$$  

Multiplying by $(-\lambda)$ and using (3.11), we obtain

$$\beta^{k,N}_\lambda(s) = -\frac{\lambda}{1 + \lambda h} X^N_\lambda(t_k) - \frac{\lambda}{1 + \lambda h} (W_\lambda(s) - W_\lambda(t_k)). \quad (3.14)$$  

The independence of $\mathcal{F}_{t_k}$ and $W_\lambda(s) - W_\lambda(t_k)$ yields

$$E \left| \beta^{k,N}_\lambda(s) \right|^2 = \frac{\lambda^2}{(1 + \lambda h)^2} E \left| X^N_\lambda(t_k) \right|^2 + \frac{\lambda^2}{(1 + \lambda h)^2} (s - t_k).$$

Using Lemma 3.2, we deduce

$$E \left| \beta^{k,N}_\lambda(s) \right|^2 \leq \frac{\lambda}{2(1 + \lambda h)^2} + \frac{\lambda^2 h}{(1 + \lambda h)^2},$$

which proves the first upper estimate.

Using (3.10) and (3.14), we have for $s \in [t_k, t_{k+1}]$

$$X^N_\lambda(s) = \left( 1 - \frac{\lambda(s - t_k)}{1 + \lambda h} \right) \left[ X^N_\lambda(t_k) + (W_\lambda(s) - W_\lambda(t_k)) \right]. \quad (3.15)$$

Taking the expectation of the square and using the independence of $\mathcal{F}_{t_k}$ and $W_\lambda(s) - W_\lambda(t_k)$, we have

$$E \left| X^N_\lambda(s) \right|^2 = \left( 1 - \frac{\lambda(s - t_k)}{1 + \lambda h} \right)^2 \left[ E \left| X^N_\lambda(t_k) \right|^2 + (s - t_k) \right] \leq E \left| X^N_\lambda(t_k) \right|^2 + h \leq \frac{1}{2\lambda} + h,$$

where the last upper estimates follows from Lemma 3.2.

Multiplying (3.14) and (3.15), taking expectation we obtain

$$E \left( X^N_\lambda(s) \beta^{k,N}_\lambda(s) \right) = -\frac{\lambda}{1 + \lambda h} \left( 1 - \frac{\lambda(s - t_k)}{1 + \lambda h} \right) \left[ E \left| X^N_\lambda(t_k) \right|^2 + (s - t_k) \right].$$

Using Lemma 3.2, we deduce

$$\left| E \left( X^N_\lambda(s) \beta^{k,N}_\lambda(s) \right) \right| \leq \frac{\lambda}{1 + \lambda h} \frac{1}{2\lambda} + \frac{\lambda h}{1 + \lambda h}.$$

This concludes the proof. □

3.2. Some useful analytical lemmas. We at first give a precise upper bound of a series defined in terms of the eigenvalues of the Laplace operator with Dirichlet boundary conditions.

Lemma 3.6. Let $p \in \left[ 0, \frac{1}{2} \right)$. There exists a constant $C > 0$, such that for all $\alpha > 0$, we have

$$\sum_{m \geq 1} \lambda^{-p} e^{-2\lambda m \alpha} \leq C \alpha^{p - \frac{1}{2}}.$$
Proof. The function \((x \in \mathbb{R}_+ \mapsto x^{-2p}e^{-2x^2\alpha})\) is decreasing. So by comparaison, we obtain
\[
\sum_{m \geq 1} m^{-2p}e^{-2m^2\alpha} \leq \int_0^\infty x^{-2p}e^{-2x^2\alpha} \, dx \leq C \alpha^{-\frac{1}{2}} \int_0^\infty y^{-2p}e^{-2y^2} \, dy = C \alpha^{p-\frac{1}{2}}.
\]
Since \(\lambda_m = \frac{1}{2}(\pi m)^2\), we deduce the desired upper estimate. \(\Box\)

Lemma 3.7. Let \(q > 0\). There exists a constant \(C > 0\), such that for all \(\alpha > 0\)
\[
\sum_{m \geq 1} \lambda_m e^{-\lambda_m \alpha} \leq C \left(1 + \frac{1}{\alpha^{q+\frac{1}{2}}} \right).
\]

Proof. Let \(f(x) = x^{2q}e^{-x^2\alpha}\). His derivatives is given by \(f'(x) = 2x^{2q-1}e^{-x^2\alpha}(q - \alpha x^2)\).

Case 1: \(\alpha > q/4\). Then \(f\) is decreasing on \([2, \infty)\) and a standard comparaison argument yields
\[
\sum_{m \geq 1} m^{2q}e^{-m^2\alpha} \leq e^{-\alpha} + 4q e^{-4\alpha} + \sum_{m \geq 3} \int_{m-1}^m x^{2q}e^{-x^2\alpha} \, dx \leq C + \int_0^\infty x^{2q}e^{-x^2\alpha} \, dx \leq C + \alpha^{-q-\frac{1}{2}} \int_0^\infty y^{2q}e^{-y^2} \, dy \leq C(1 + \alpha^{-q-\frac{1}{2}}).
\]

Case 2: \(\alpha \leq q/4\). The function \(f\) is increasing on \([0, \sqrt{q/\alpha})\). So for each \(m = 1, \ldots, \lfloor \sqrt{q/\alpha} \rfloor - 1\), we have
\[
m^{2q}e^{-m^2\alpha} \leq \int_m^{m+1} x^{2q}e^{-x^2\alpha} \, dx.
\]
On the interval \([\sqrt{q/\alpha}, \infty)\), \(f\) is decreasing. So for each integer \(m \geq \lfloor \sqrt{q/\alpha} \rfloor + 2\), we have
\[
m^{2q}e^{-m^2\alpha} \leq \int_{m-1}^m x^{2q}e^{-x^2\alpha} \, dx.
\]
The above upper estimates yield
\[
\sum_{m \geq 1} m^{2q}e^{-m^2\alpha} \leq \sum_{m \leq \lfloor \sqrt{q/\alpha} \rfloor - 1} \int_m^{m+1} x^{2q}e^{-x^2\alpha} \, dx + \sum_{m \geq \lfloor \sqrt{q/\alpha} \rfloor + 2} \int_{m-1}^m x^{2q}e^{-x^2\alpha} \, dx + \sum_{m \in (\lfloor \sqrt{q/\alpha} \rfloor, \lfloor \sqrt{q/\alpha} \rfloor + 1)} m^{2q}e^{-m^2\alpha} \leq C \alpha^{-q-\frac{1}{2}} + \sum_{m \in (\lfloor \sqrt{q/\alpha} \rfloor, \lfloor \sqrt{q/\alpha} \rfloor + 1)} m^{2q}e^{-m^2\alpha} \leq C \alpha^{-q-\frac{1}{2}}.
\]
Now we study each term of the sum in the right hand side. Since \(q \geq \alpha\), we have
\[
\left(\sqrt{\frac{q}{\alpha}}\right)^{2q}e^{-\left(\sqrt{\frac{q}{\alpha}}\right)^2} \leq \left(\frac{q}{\alpha}\right)^q \leq \left(\frac{q}{\alpha}\right)^{q+\frac{1}{2}} \leq C \alpha^{-q-\frac{1}{2}}.
\]
Let 

then there exists a constant 

Therefore, there exists a constant 

as above such that for all 

\[ 0 \leq v(k, m) \leq \lambda_m^{n-p} h^{n+1} e^{-2\lambda_m (T-t_{k+1})}. \]

Then, there exists a constant \( C > 0 \), independent of \( N \), such that 

\[ \sum_{m \geq 1} \sum_{k=0}^{N-2} v(k, m) \leq C h^{p+\frac{2}{q'}}. \]

**Proof.** First we remark that \( T-t_{k+1} = h(N-k-1) \). Using Lemma 3.7, we deduce the existence of \( C \) depending on \( n \) and \( p \), but independent of \( N \), such that for \( k = 0, \ldots, N-2 \) :

\[ \sum_{m \geq 1} v(k, m) \leq C h^{n+1} \left( 1 + \frac{1}{h^{n-p+\frac{2}{q'}} (N-k-1)^{n-p+\frac{2}{q'}}} \right) \]

\[ \leq C \left( h^{n+1} + \frac{h^{p+\frac{2}{q'}}}{(N-k-1)^{n-p+\frac{2}{q'}}} \right). \]

Therefore, there exists a constant \( C \) as above such that

\[ \sum_{m \geq 1} \sum_{k=0}^{N-2} v(k, m) \leq C \left( h^{n+1} + \frac{1}{h^{n-p+\frac{2}{q'}} (N-k-1)^{n-p+\frac{2}{q'}}} \right) \]

\[ \leq C \left( h^{n+1} + \frac{1}{h^{n-p+\frac{2}{q'}} (N-k-1)^{n-p+\frac{2}{q'}}} \right) \leq C h^{p+\frac{2}{q'}}, \]

which concludes the proof. \( \square \)

**3.3. Decomposition of the weak error.** We follow the classical decomposition introduced in [16]. The definition of \( u_\lambda(t, x) \) in section 3.1 yields 

\[ E \left| X^N(T) \right|_{H^{-p}}^2 - E \left| X(T) \right|_{H^{-p}}^2 = \sum_{m \geq 1} \lambda_m^{-p} \left( E \left| X^N_{\lambda_m}(T) \right|_{H^{-p}}^2 - E \left| X_{\lambda_m}(T) \right|_{H^{-p}}^2 \right) \]

\[ = \sum_{m \geq 1} \lambda_m^{-p} \left( Eu_{\lambda_m} \left( T, X^N_{\lambda_m}(T) \right) - u_{\lambda_m} \left( 0, X^N_{\lambda_m}(0) \right) \right). \]

Let \( \delta^N(k, m) := \lambda_m^{-p} \left( Eu_{\lambda_m} \left( t_{k+1}, X^N_{\lambda_m}(t_{k+1}) \right) - Eu_{\lambda_m} \left( t_k, X^N_{\lambda_m}(t_k) \right) \right) \); then

\[ E \left| X^N(T) \right|_{H^{-p}}^2 - E \left| X(T) \right|_{H^{-p}}^2 = \sum_{m \geq 1} \sum_{k=0}^{N-1} \delta^N(k, m). \]
Note that using Lemmas 3.3, 3.4 and (3.4) we deduce that for any \( k = 0, \ldots, N - 1 \)
\[
E \int_{t_k}^{t_{k+1}} \left| \gamma_{\lambda}^{k,N}(t) \frac{\partial u}{\partial x}(t, X_{\lambda}^N(t)) \right|^2 dt < \infty.
\]
From now, we do not justify that the stochastic integral are centered. Itô’s formula and Lemma 3.4, we imply that for \( k = 0, \ldots, N - 1 \)
\[
\delta^N(k, m) = \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} \left\{ \frac{\partial}{\partial t} u_{\lambda m} + \beta_{\lambda m}^{k,N}(t) \frac{\partial}{\partial x} u_{\lambda m} + \frac{1}{2} \left| \gamma_{\lambda m}^{k,N}(t) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda m} \right\} (t, X_{\lambda m}^N(t)) dt
\]
\[
= \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} \left\{ I_{\lambda m}^{k,N}(t) + \frac{1}{2} J_{\lambda m}^{k,N}(t) \right\} dt,
\]
where
\[
I_{\lambda m}^{k,N}(t) := \left( \beta_{\lambda m}^{k,N}(t) + \lambda_m X_{\lambda m}^N(t) \right) \frac{\partial}{\partial x} u_{\lambda m}(t, X_{\lambda m}^N(t)), \quad \text{(3.16)}
\]
\[
J_{\lambda m}^{k,N}(t) := \left( \left| \gamma_{\lambda m}^{k,N}(t) \right|^2 - 1 \right) \frac{\partial^2}{\partial x^2} u_{\lambda m}(t, X_{\lambda m}^N(t)). \quad \text{(3.17)}
\]
This yields the following decomposition:
\[
E \left| X^N(T) \right|^2_{H^{-p}} - E \left| X(T) \right|^2_{H^{-p}} = \sum_{m \geq 1} \delta^N(N - 1, m) + \sum_{m \geq 1} \sum_{k = 0}^{N - 2} \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} I_{\lambda m}^{k,N}(t) dt
\]
\[
+ \frac{1}{2} \sum_{m \geq 1} \sum_{k = 0}^{N - 2} \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} J_{\lambda m}^{k,N}(t) dt. \quad \text{(3.18)}
\]
Now we study each term of this decomposition.

**Lemma 3.9.** There exists a constant \( C \), independant of \( N \), such that
\[
\sum_{m \geq 1} \left| \delta^N(N - 1, m) \right| \leq C h^{p + \frac{1}{2}}.
\]

This study is similar to the third step of [6], page 97.

**Proof.** Using the definition of \( u_{\lambda m}(t, x) \) (3.3) and (3.9), we have
\[
u_{\lambda m}(t_N, X_{\lambda m}^N(t_N)) = \left| X_{\lambda m}^N(t_N) \right|^2 = \frac{1}{(1 + \lambda_m h)^2} \left| X_{\lambda m}^N(t_{N-1}) + \Delta W_m(N) \right|^2,
\]
\[
u_{\lambda m}(t_{N-1}, X_{\lambda m}^N(t_{N-1})) = \frac{1 - e^{-2\lambda_m h}}{2\lambda_m} + e^{-2\lambda_m h} \left| X_{\lambda m}^N(t_{N-1}) \right|^2.
\]
By independence between \( \Delta W_m(N) \) and \( X_{\lambda m}^N(t_{N-1}) \), we have
\[
\delta^N(N - 1, m) = \lambda_m^{-p} \left\{ \frac{1}{(1 + \lambda_m h)^2} - e^{-2\lambda_m h} \right\} E \left| X_{\lambda m}^N(t_{N-1}) \right|^2
\]
\[
+ \frac{h}{\lambda_m (1 + \lambda_m h)^2} \left( 1 - e^{-2\lambda_m h} \right).
\]
Let \( \delta_1(\lambda_m) := \frac{1 - e^{-2\lambda_m h}}{2\lambda_m^{p + h}} \), \( \delta_2(\lambda_m) := \frac{h}{\lambda_m^{p(1 + \lambda_m h)}} \), and
\[
\delta_3(\lambda_m) := \lambda_m^{-p} \left\{ \frac{1}{(1 + \lambda_m h)^2} - e^{-2\lambda_m h} \right\} E \left| X_{\lambda m}^N(t_{N-1}) \right|^2.
\]
With these notations we have
\[ \delta^N(N - 1, m) \leq \delta_1(\lambda_m) + \delta_2(\lambda_m) + \delta_3(\lambda_m). \]

First, we study \( \delta_1(\lambda_m) \). Since \( \frac{1 - e^{-2\lambda h}}{2\lambda} = \int_0^h e^{-2\lambda x} dx \), using Lemma 3.6, we obtain
\[
\sum_{m \geq 1} \delta_1(\lambda_m) = \int_0^h \sum_{m \geq 1} \lambda_m^p e^{-2\lambda m x} dx \leq C \int_0^h x^{p-\frac{1}{2}} dx = Ch^{p+\frac{1}{2}}. \tag{3.19}
\]

Now we study \( \delta_2(\lambda_m) \). Since \((x \in [0, \infty) \mapsto x^{-2p}(1 + x^2h)^2)\) is decreasing, we have for \( p \in [0, \frac{1}{2}) \)
\[
\sum_{m \geq 1} \delta_2(\lambda_m) \leq Ch \int_0^{\infty} \frac{1}{x^{2p}(1 + x^2h)^2} dx \leq Ch^{p+\frac{1}{2}} \int_0^{\infty} \frac{y^{-2p}}{(1 + y^2)^2} dy \leq Ch^{p+\frac{1}{2}}. \tag{3.20}
\]

Finally, we study \( \delta_3(\lambda_m) \). Using Lemma 3.2, we have
\[
\delta_3(\lambda_m) \leq \lambda_m^{-p} \left( \frac{1}{(1 + \lambda_m h)^2} - e^{-2\lambda_m h} \right) \frac{1}{2\lambda_m}. \]

Since \( \frac{1}{(1 + \lambda h)^2} - e^{-2\lambda h} = 2\lambda \int_0^h \left( e^{-2\lambda x} - \frac{1}{(1 + \lambda x)^2} \right) dx \), we have
\[
\delta_3(\lambda_m) \leq \lambda_m^{-p} \int_0^h \left( e^{-2\lambda m x} + \frac{1}{(1 + \lambda m x)^3} \right) dx. \]

Using Lemma 3.6, we have for \( p \in [0, \frac{1}{2}) \)
\[
\sum_{m \geq 1} \lambda_m^{-p} \int_0^h e^{-2\lambda m x} dx \leq C \int_0^h x^{p-\frac{1}{2}} dx \leq Ch^{p+\frac{1}{2}}. \]

Now since \( x \geq 0 \) the map \((y \in \mathbb{R}_{+} \mapsto y^{-2p}(1 + y^2x)^{-3})\) is decreasing, we have for \( p \in [0, \frac{1}{2}) \)
\[
\sum_{m \geq 1} \frac{\lambda_m^{-p}}{(1 + \lambda_m x)^3} \leq C \int_0^{\infty} \frac{1}{y^{2p}(1 + y^2x)^3} dy \leq C_x x^{p-\frac{1}{2}} \int_0^{\infty} \frac{1}{z^{2p}(1 + z^2)^3} dz \leq C x^{p-\frac{1}{2}},
\]
and hence Fubini’s theorem yields
\[
\sum_{m \geq 1} \int_0^h \frac{\lambda_m^{-p}}{(1 + \lambda_m x)^3} dx \leq C \int_0^h x^{p-\frac{1}{2}} dx \leq Ch^{p+\frac{1}{2}}.
\]

The above inequalities imply \( \sum_{m \geq 1} \delta_3(\lambda_m) \leq Ch^{p+\frac{1}{2}} \). This inequality, (3.19) and (3.20) give the stated upper estimate. \( \square \)

**Lemma 3.10.** There exists a constant \( C > 0 \), independent of \( N \), such that
\[
\sum_{m \geq 1} \sum_{k=0}^{N-1} \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} |f^{k,N}_{\lambda_m}(t)| dt \leq Ch^{p+\frac{1}{2}}.
\]

**Proof.** Using Lemma 3.4, we have
\[
|f^{k,N}_{\lambda_m}(t)|^2 - 1 = -\frac{2(t - t_k) \lambda_m}{1 + \lambda_m h} + \frac{|t - t_k|^2 \lambda_m^2}{(1 + \lambda_m h)^2}.
\]

Using (3.5) and (3.17), we have
\[
\lambda_m^{-p} E \int_{t_k}^{t_{k+1}} |f^{k,N}_{\lambda_m}(t)| dt \leq C \left( \lambda_m^{-p} h^2 + \lambda_m^{2-p} h^3 \right) e^{-2\lambda_m (T-t_{k+1})}.
\]

Lemma 3.8 concludes the proof. \( \square \)
Lemma 3.11. There exists a constant $C > 0$, independant of $N$, such that
\[
\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} \left| I_{\lambda_m}^k(t) \right| \, dt \leq C h^{p+\frac{1}{2}}.
\]

Proof. Let $I_{1,\lambda_m}^k(t) := E\beta_{\lambda_m}^{k,N}(t) \frac{\partial}{\partial x} u_{\lambda_m}(t, X_{\lambda_m}^N(t)) + E\lambda_m X_{\lambda_m}^N(t_{k+1}) \frac{\partial}{\partial x} u_{\lambda_m}(0, X_{\lambda_m}^N(t_{k+1}))$ and $I_{2,\lambda_m}^k(t) := -\lambda_m EX_{\lambda_m}^{N,N}(t_{k+1}) \frac{\partial}{\partial x} u_{\lambda_m}(0, X_{\lambda_m}^N(t_{k+1})) + \lambda_m EX_{\lambda_m}^{N,N}(t) \frac{\partial}{\partial x} u_{\lambda_m}(0, X_{\lambda_m}^N(t))$. Using (3.16), we have
\[
EI_{\lambda_m}^k(t) = I_{1,\lambda_m}^k(t) + I_{2,\lambda_m}^k(t).
\]  (3.21)

First we study $I_{1,\lambda_m}^k(t)$. Using (3.4), we know that $\frac{\partial}{\partial x} u_{\lambda_m} \in C^{1,2}$. So using Itô’s formula and Lemma 3.4, we have
\[
d\left[ \beta_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \right] = \left\{ \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} + \left| \beta_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} 
+ z_{\lambda_m}^{k,N}(s) \gamma_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m} \right\} (s, X_{\lambda_m}^N(s)) \, ds

+ \left\{ \beta_{\lambda_m}^{k,N}(s) \gamma_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m} + z_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m} \right\} (s, X_{\lambda_m}^N(s)) \, dW_{\lambda_m}(s).
\]  (3.22)

Using this equation, Lemma 3.4 and the Itô formula we deduce
\[
d \left[ \beta_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \right] = \left\{ \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} + \left| \beta_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} 
+ z_{\lambda_m}^{k,N}(s) \gamma_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m} \right\} (s, X_{\lambda_m}^N(s)) \, ds

+ \left\{ \beta_{\lambda_m}^{k,N}(s) \gamma_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m} + z_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m} \right\} (s, X_{\lambda_m}^N(s)) \, dW_{\lambda_m}(s).
\]  (3.23)

Integrating between $t$ and $t_{k+1}$, taking expectation, and using the fact that $\beta_{\lambda_m}^{k,N}(t_{k+1}) = -\lambda_m X_{\lambda_m}^N(t_{k+1})$, so that $I_{1,\lambda_m}^k(t_{k+1}) = 0$, we obtain
\[
I_{1,\lambda_m}^k(t) = -E \int_{t}^{t_{k+1}} \left\{ \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} + \left| \beta_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} 
+ z_{\lambda_m}^{k,N}(s) \gamma_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m} \right\} (s, X_{\lambda_m}^N(s)) \, ds.
\]  (3.24)

Using (3.7) and Lemma 3.5, we have for $s \in [t, t_{k+1}]
\[
E \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) = 4 \lambda_m e^{-2 \lambda_m (T-s)} E \beta_{\lambda_m}^{k,N}(s) X_{\lambda_m}^N(s) \leq C \lambda_m e^{-2 \lambda_m (T-t_{k+1})},
\]
and hence
\[
\lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_{t}^{t_{k+1}} ds E \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C \lambda_m^{1-p} h^2 e^{-\lambda_m (T-t_{k+1})}.
\]

Using Lemma 3.8, and the above inequality, we deduce
\[
\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_{t}^{t_{k+1}} ds E \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C h^{p+\frac{1}{2}}.
\]  (3.24)

Using (3.5) and Lemma 3.5, we have for $s \in [t_k, t_{k+1}]
\[
E \left| \beta_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) = 4 \lambda_m e^{-2 \lambda_m (T-s)} \leq 4 \lambda_m e^{-2 \lambda_m (T-t_{k+1})},
\]
Therefore, Lemma 3.8 yields
\[
\sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_{t}^{t_{k+1}} ds E \left| \beta_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} (s, X_{\lambda_m}^{N}(s)) \leq C \lambda_m^{1-p} h^2 e^{-2\lambda_m (T-t_{k+1})}.
\]
Thus, integrating between \( s \) and \( t \), we have
\[
\left| \gamma_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} (s, X_{\lambda_m}^{N}(s)) \leq C h^{p+\frac{1}{2}}.
\]
Using equations (3.5) and Lemma 3.4 we have for all \( s \in [t, t_{k+1}] \)
\[
E \left| \gamma_{\lambda_m}^{k,N}(s) \partial_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} (s, X_{\lambda_m}^{N}(s)) = \frac{2\lambda_m}{1 + \lambda_m h} \left( 1 - \frac{(s-t_k)\lambda_m}{1 + \lambda_m h} \right) e^{-2\lambda_m (T-s)} \leq C \lambda_m e^{-2\lambda_m (T-t_{k+1})}.
\]
Using Lemma 3.8, we deduce
\[
\sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_{t}^{t_{k+1}} ds E \left| \gamma_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} (s, X_{\lambda_m}^{N}(s)) \leq C h^{p+\frac{1}{2}}.
\]
Plugging this inequality together with (3.24) and (3.25) into (3.23) gives us
\[
\sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \left| I_{t_k,\lambda_m}^{k,N}(t) \right| \leq C h^{p+\frac{1}{2}}.
\]
Now we study \( I_{t_k,\lambda_m}^{k,N}(t) \). Using Lemma 3.4, equation (3.22) and the Itô formula we have
\[
\frac{dX_{\lambda_m}^{N}(s)}{dx} = \left\{ X_{\lambda_m}^{N}(s) \frac{\partial}{\partial t} u_{\lambda_m} + X_{\lambda_m}^{N}(s) \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m} + \beta_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m} + \left| \gamma_{\lambda_m}^{k,N}(s) \partial_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} (s, X_{\lambda_m}^{N}(s)) ds \]
\[
+ \left\{ \gamma_{\lambda_m}^{k,N}(s) \partial_{\lambda_m}^{k,N}(s) \right\} \frac{\partial^2}{\partial x^2} u_{\lambda_m} (s, X_{\lambda_m}^{N}(s)) dW_{\lambda_m}(s)
\]
So integrating between \( t \) and \( t_{k+1} \) and taking expectation, we obtain
\[
I_{t_k,\lambda_m}^{k,N}(t) = -\lambda_m E \int_{t}^{t+k+1} \left\{ X_{\lambda_m}^{N}(s) \frac{\partial}{\partial t} u_{\lambda_m} + \beta_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m} + X_{\lambda_m}^{N}(s) \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m} + \left| \gamma_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} (s, X_{\lambda_m}^{N}(s)) ds.
\]
Using equation (3.7) and Lemma 3.5, we have for all \( s \in [t, t_{k+1}] \)
\[
\lambda_m E \left| X_{\lambda_m}^{N}(s) \frac{\partial}{\partial t} u_{\lambda_m} (s, X_{\lambda_m}^{N}(s)) \right|^2 \leq \lambda_m^2 (\frac{1}{\lambda_m} + h) e^{-2\lambda_m (T-t_{k+1})}.
\]
Therefore,
\[
\lambda_m^{-p} \int_{t_k}^{t_{k+1}} \lambda_m E \left| X_{\lambda_m}^{N}(s) \frac{\partial}{\partial t} u_{\lambda_m} (s, X_{\lambda_m}^{N}(s)) \right|^2 \leq C \lambda_m^2 (\frac{1}{\lambda_m} + h) e^{-2\lambda_m (T-t_{k+1})}.
\]
and using Lemma 3.8, we deduce
\[
\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_{m}^{-p} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} ds \lambda_{m} E X_{\lambda_{m}}^{N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_{m}}(s, X_{\lambda_{m}}^{N}(s) ) \leq C h^{p+\frac{1}{2}}.
\] (3.28)

The equation (3.4) and Lemma 3.5 yield for all \( s \in [t, t_{k+1}] \)
\[
\lambda_{m} E \beta_{\lambda_{m}}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_{m}}(s, X_{\lambda_{m}}^{N}(s) ) = 2 \lambda_{m} e^{-2\lambda_{m}(T-s)} E \beta_{\lambda_{m}}^{k,N}(s) X_{\lambda_{m}}^{N}(s) \leq C \lambda_{m} e^{-2\lambda_{m}(T-t_{k+1})},
\]
This upper estimate implies
\[
\lambda_{m}^{-p} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} ds \lambda_{m} E \beta_{\lambda_{m}}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_{m}}(s, X_{\lambda_{m}}^{N}(s) ) \leq C \lambda_{m}^{-p} h^{2} e^{-2\lambda_{m}(T-t_{k+1})},
\]
and Lemma 3.8 yields
\[
\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_{m}^{-p} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} ds \lambda_{m} E \beta_{\lambda_{m}}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_{m}}(s, X_{\lambda_{m}}^{N}(s) ) \leq C h^{p+\frac{1}{2}}.
\] (3.29)

Using equation (3.5) and Lemma 3.5, we have for all \( s \in [t, t_{k+1}] \)
\[
\lambda_{m} E X_{\lambda_{m}}^{N}(s) \beta_{\lambda_{m}}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_{m}}(s, X_{\lambda_{m}}^{N}(s) ) \leq C \lambda_{m} e^{-2\lambda_{m}(T-t_{k+1})},
\]
This implies
\[
\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_{m}^{-p} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} ds \lambda_{m} E X_{\lambda_{m}}^{N}(s) \beta_{\lambda_{m}}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_{m}}(s, X_{\lambda_{m}}^{N}(s) ) \leq C h^{p+\frac{1}{2}}.
\] (3.30)

Finally, (3.5) and Lemma 3.4 imply that for all \( s \in [t, t_{k+1}] \)
\[
\lambda_{m} E \left| \gamma_{\lambda_{m}}^{k,N}(s) \right| \frac{2}{2} \frac{\partial^2}{\partial x^2} u_{\lambda_{m}}(s, X_{\lambda_{m}}^{N}(s) ) \leq C \lambda_{m} e^{-2\lambda_{m}(T-t_{k+1})}.
\]
This yields
\[
\lambda_{m}^{-p} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} ds \lambda_{m} E \left| \gamma_{\lambda_{m}}^{k,N}(s) \right| \frac{2}{2} \frac{\partial^2}{\partial x^2} u_{\lambda_{m}}(s, X_{\lambda_{m}}^{N}(s) ) \leq C \lambda_{m}^{-p} h^{2} e^{-2\lambda_{m}(T-t_{k+1})},
\]
and Lemma 3.8 implies
\[
\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_{m}^{-p} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} ds \lambda_{m} E \left| \gamma_{\lambda_{m}}^{k,N}(s) \right| \frac{2}{2} \frac{\partial^2}{\partial x^2} u_{\lambda_{m}}(s, X_{\lambda_{m}}^{N}(s) ) \leq C h^{p+\frac{1}{2}}.
\]

Plugging this inequality together with (3.28) - (3.30) into (3.27), we deduce
\[
\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_{m}^{-p} E \int_{t_k}^{t_{k+1}} | t_{2,\lambda_{m}}^{k,N}(t) | dt \leq C h^{p+\frac{1}{2}}.
\]

This equation together with (3.21) and (3.26) conclude the proof. \( \square \)

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