Refined Solutions of Time Inhomogeneous Optimal Stopping Games via Dirichlet Form

Yipeng Yang∗
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Abstract

The properties of value functions of time inhomogeneous optimal stopping problem and zero-sum game (Dynkin game) are studied through time dependent Dirichlet form. Under the absolute continuity condition on the transition function of the underlying diffusion process and some other assumptions, the refined solutions without exceptional starting points are proved to exist, and the value functions of the optimal stopping and zero-sum game, which are finely and cofinely continuous, are characterized as the solutions of some variational inequalities, respectively.

Key words: Time inhomogeneous Dirichlet form, Optimal stopping, Dynkin game, Variational inequality

AMS subject classifications: 31C25, 49J40, 60G40, 60J60

1 Introduction

Let \( M = (X_t, P_{(s,x)}) \) be a diffusion process on a locally compact separable metric space \( X \). For two finely continuous functions \( g, h \) on \([0, \infty) \times X\) and a constant \( \alpha > 0 \), define the following return functions of optimal stopping games:

\[
J_{(s,x)}(\sigma) = E_{(s,x)}(e^{-\alpha \sigma} g(s+\sigma, X_{s+\sigma}))
\]

(1)

\[
J_{(s,x)}(\tau, \sigma) = E_{(s,x)} \left[ e^{-\alpha (\tau \wedge \sigma)} (g(s+\sigma, X_{s+\sigma})I_{\tau > \sigma} + h(s+\tau, X_{s+\tau})I_{\tau \leq \sigma}) \right].
\]

(2)

The values of the stopping games are defined as \( \tilde{c}_g = \sup_{\sigma} J_{(s,x)}(\sigma) \) and \( \tilde{w} = \sup_{\sigma} \inf_{\tau} J_{(s,x)}(\tau, \sigma) \), respectively. This kind of optimal stopping problems have been continually developed due to its broad application in finance, resource control or production management.

In the time homogeneous case, where \( M, g, h \) are all time homogenous, it is well known that \( \tilde{c}_g \) is a quasi continuous version of the solution of a variational inequality problem

∗Department of Mathematics, University of Missouri-Columbia, Columbia, Missouri, 65211 (yangyip@missouri.edu)
formulated in terms of the Dirichlet form, see Nagai [6]. This result was successfully extended by Zabczyk [12] to Dynkin game (zero sum game) where $\tilde{w}$ was shown to be the quasi continuous version of the solution of a certain variational inequality problem. In their work, there always exist an exceptional set $N$ of starting points of $M$. In 2006, Fukushima and Menda [3] showed that if the transition function of $M$ is absolutely continuous with respect to the underlying measure $m$, then there does not exist the exceptional set $N$, and $\tilde{e}_g$ and $\tilde{w}$ are finely continuous with any starting point of $M$.

However, more work is needed to extend these results to the time inhomogeneous case, especially the characteristics of the value function. Using the time dependent Dirichlet form (generalized Dirichlet form), Oshima [8] showed that under some conditions, $\tilde{e}_g$ (also $\tilde{w}$) is still finely and cofinely continuous with quasi every starting point of $M$, and except on an exceptional set $N$, $\tilde{e}_g$ (also $\tilde{w}$) is characterized as a version of the solution of a variational inequality problem.

Recently, Palczewski and Stettner [9][10] used the penalty method to characterize the continuity of the value function of a time inhomogeneous optimal stopping problem. In their work, the underlying process $M$ is assumed to satisfy the Feller continuity property. Lamberton [4] derived the continuity property of the value function of a one-dimensional optimal stopping problem, and the value function was characterized as the unique solution of a variational inequality in the sense of distributions. However, that result was difficulty to be extended to multi-dimensional diffusions.

In all the afore mentioned work, the property of the time inhomogeneous value functions along the dimension $t$ (time) were not further studied. In this paper, through the time dependent Dirichlet form, it is showed that under the absolute continuity condition on the transition probability function $p_t$ and some other assumptions, the value functions do belong to the functional space $W$, see (3). Further it is showed that Oshima’s [8] results still hold and there does not exist the exceptional set for the starting points of $M$. This result is then applied in Section 4 to the time inhomogeneous optimal stopping games where the underlying process is a multi-dimensional time inhomogeneous Ito diffusion.

2 Time Dependent Dirichlet Form

In this section we define the settings for the time dependent Dirichlet form that are similar to those in [8], although some results from [11], whose notions are different, will be used later. Let $X$ be a locally compact separable metric space and $m$ be a positive Radon measure on $X$ with full support. For each $t \geq 0$, define $(E^{(t)}, F)$ as an $m$-symmetric Dirichlet form on $H = L^2(X; m)$ and for any $u \in F$, we assume that $E^{(t)}(u, u)$ is a measurable function of $t$ and satisfies

$$\lambda^{-1}\|u\|^2_F \leq E^{(t)}_1(u, u) \leq \lambda\|u\|^2_F$$

for some constant $\lambda > 0$, where $E^{(t)}_0(u, v) = E^{(t)}(u, v) + \alpha(u, v)_m$. $\alpha > 0$, and the $F$ norm is defined to be $\|u\|^2_F = E^{(0)}_1(u, u)$. We also assume that $F$ is regular and local in the usual sense [11].
Define $F'$ as the dual space of $F$, then it can be seen that $F \subset H = H' \subset F'$. For each $t$, there exists an operator $L^{(t)}$ from $F$ to $F'$ such that

$$-(L^{(t)}u, v) = E^{(t)}(u, v), \quad u, v \in F.$$ 

Further the $F'$ norm is defined as

$$\|v\|_{F'} = \sup_{\|u\|_{F} = 1} \{(v, u)\},$$

where $(v, u)$ denotes the canonical coupling between $v \in F'$ and $u \in F$.

Define the spaces

$$\mathcal{H} = \{ \phi(t, \cdot) \in H : \|\phi\|_{\mathcal{H}} < \infty \},$$

where

$$\|\phi\|^2_{\mathcal{H}} = \int_{\mathbb{R}} \|\phi(t, \cdot)\|^2_{H} dt,$$

and

$$\mathcal{F} = \{ \phi(t, x) \in F : \|\phi\|_{\mathcal{F}} < \infty \},$$

where

$$\|\phi\|^2_{\mathcal{F}} = \int_{\mathbb{R}} \|\phi(t, \cdot)\|^2_{F} dt.$$

Clearly $\mathcal{F} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{F}'$ densely and continuously, where $\mathcal{H}'$, $\mathcal{F}'$ are the dual spaces of $\mathcal{H}$, $\mathcal{F}$ respectively.

For any $\phi \in \mathcal{F}$, considering $\phi$ as a function of $t \in \mathbb{R}$ with values in $F$, the distribution derivative $\partial \phi / \partial t$ is considered as a function of $t \in \mathbb{R}$ with values in $F'$ such that

$$\int_{\mathbb{R}} \partial \phi / \partial t(t, \cdot) \xi(t) dt = - \int_{\mathbb{R}} \phi(t, \cdot) \xi(t) dt,$$

for any $\xi \in C_0^\infty(\mathbb{R})$. Then we can define the space $\mathcal{W}$ as

$$\mathcal{W} = \{ \phi(t, x) \in \mathcal{F} : \partial \phi / \partial t \in \mathcal{F}', \|\phi\|_{\mathcal{W}} < \infty \},$$

where

$$\|\phi\|^2_{\mathcal{W}} = \left\| \frac{\partial \phi}{\partial t} \right\|^2_{\mathcal{F}'} + \|\phi\|^2_{\mathcal{F}}.$$

Since $\mathcal{F}$ and $\mathcal{F}'$ are Banach spaces, it is easy to see that $\mathcal{W}$ is also a Banach space. Further, $\mathcal{W}$ is dense in $\mathcal{F}$.

We further define the bilinear form $\mathcal{E}$ by

$$\mathcal{E}(\phi, \psi) = \begin{cases} 
-\frac{\partial \phi}{\partial t} + \int_{\mathbb{R}} E^{(t)}(\phi(t, \cdot), \psi(t, \cdot)) dt, & \phi \in \mathcal{W}, \psi \in \mathcal{F}, \\
\langle \frac{\partial \psi}{\partial t}, \phi \rangle + \int_{\mathbb{R}} E^{(t)}(\phi(t, \cdot), \psi(t, \cdot)) dt, & \phi \in \mathcal{F}, \psi \in \mathcal{W},
\end{cases} \quad (4)$$
where \( \langle \frac{\partial \varphi}{\partial t}, \psi \rangle = \int_{\mathbb{R}} \langle \frac{\partial \varphi}{\partial t}, \psi \rangle \, dt \). We call \((\mathcal{E}, \mathcal{F})\) a time dependent Dirichlet form on \(\mathcal{H}\).  

As in \([8]\) we may introduce the time space process \(Z_t = (\tau(t), X_t)\) on the domain \(\mathbb{Z} = \mathbb{R} \times \mathbb{X}\) with uniform motion \(\tau(t)\), then the resolvent \(\mathcal{R}_\alpha f\) of \(Z_t\) defined by

\[
\mathcal{R}_\alpha f(s, x) = \mathcal{E}(s, x) \left( \int_0^\infty e^{-\alpha t} f(s + t, X_{s+t}) dt \right), \quad (s, x) = z, \ f \in \mathcal{H},
\]

satisfies

\[
\left( \alpha - \frac{\partial}{\partial t} - L^{(t)} \right) \mathcal{R}_\alpha f(t, x) = f(t, x), \ \forall t \geq 0.
\]  

Furthermore, \(\mathcal{R}_\alpha f\) is considered as a version of \(G_\alpha f \in \mathcal{W}\), where \(G_\alpha\) is the resolvent associated with the form \(\mathcal{E}_\alpha(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot) + \alpha(\cdot, \cdot)\nu\) and it satisfies

\[
\mathcal{E}_\alpha(G_\alpha f, \varphi) = (f, \varphi)_\nu, \quad \forall \varphi \in \mathcal{F},
\]

where \(d\nu(t, x) = dt dm(x)\). We may write \((\cdot, \cdot)_\nu\) as \((\cdot, \cdot)_{\mathcal{H}}\) to indicate it as the inner product in \(\mathcal{H}\).

We now define \(\mathcal{A}\) as a bilinear form on \(\mathcal{F} \times \mathcal{F}\) by

\[
\mathcal{A}(\varphi, \psi) = \int_{\mathbb{R}} \mathcal{E}^{(t)}(\varphi(t, \cdot), \psi(t, \cdot)) \, dt,
\]

and set \(\mathcal{A}_\alpha(\varphi, \psi) = \mathcal{A}(\varphi, \psi) + \alpha(\varphi, \psi)_{\mathcal{H}}\), then it can be seen that \(\mathcal{E}_\alpha(\varphi, \psi) = -\langle \frac{\partial \varphi}{\partial t}, \psi \rangle + \mathcal{A}_\alpha(\varphi, \psi)\) if \(\varphi \in \mathcal{W}, \psi \in \mathcal{F}\), and \(\mathcal{E}_\alpha(\varphi, \psi) = \langle \frac{\partial \varphi}{\partial t}, \varphi \rangle + \mathcal{A}_\alpha(\varphi, \psi)\) if \(\varphi \in \mathcal{F}, \psi \in \mathcal{W}\). Also notice that if \(\varphi \in \mathcal{W}, \langle \frac{\partial \varphi}{\partial t}, \varphi \rangle = 0\), hence \(\mathcal{E}_\alpha(\varphi, \varphi) = \mathcal{A}_\alpha(\varphi, \varphi)\) in this case, see Corollary 1.1 in \([7]\).

A function \(\varphi \in \mathcal{F}\) is called \(\alpha\)-potential if \(\mathcal{E}_\alpha(\varphi, \psi) \geq 0\) for any nonnegative function \(\psi \in \mathcal{W}\). Denote by \(\mathcal{P}_\alpha\) the family of all \(\alpha\)-potential functions. A function \(\varphi \in \mathcal{F}\) is called \(\alpha\)-excessive if and only if \(\varphi \geq 0\) and \(nG_{n+\alpha}\varphi \leq \varphi\) a.e. for any \(n \geq 0\). For any \(\alpha\)-potential \(\varphi \in \mathcal{F}\), define its \(\alpha\)-excessive modification as

\[
\tilde{\varphi} = \lim_{n \to \infty} nR_{n+\alpha}\varphi.
\]

For any function \(g \in \mathcal{H}\), let

\[
\mathcal{L}_g = \{ \varphi \in \mathcal{F} : \varphi \geq g \ \nu \text{ a.e.} \},
\]

then the following result holds (see Lemma 1.1 in \([8]\)):

**Lemma 2.1.** For any \(\epsilon > 0\) and \(\alpha > 0\), there exists a unique function \(g^\alpha_\epsilon \in \mathcal{W}\) such that

\[
-\left( \frac{\partial g^\alpha_\epsilon}{\partial t}, u \right) + E^{(t)}_\alpha(g^\alpha_\epsilon(t, \cdot), u) = \frac{1}{\epsilon} \left( (g^\alpha_\epsilon(t, \cdot) - g(t, \cdot))^-, u \right)
\]

for any \(u \in \mathcal{F}\).
As a consequence, \( g_\epsilon^\alpha \) solves
\[
E_\alpha(g_\epsilon^\alpha, \psi) = \frac{1}{\epsilon}((g_\epsilon^\alpha - g)^-, \psi), \quad \forall \psi \in \mathcal{F},
\] (9)
see Proposition 1.6 in \([11]\).

By Theorem 1.2 in \([8]\), \( e_g = \lim_{\epsilon \to 0} g_\epsilon^\alpha \) converges increasingly, strongly in \( \mathcal{H} \) and weakly in \( \mathcal{F} \), and furthermore, \( e_g \) is the minimal function of \( \mathcal{P}_\alpha \cap \mathcal{L}_g \) satisfying
\[
\mathcal{A}_\alpha(e_g, e_g) \leq E_\alpha(e_g, \psi), \quad \forall \psi \in \mathcal{L}_g \cap \mathcal{W}.
\] (10)

Given any open set \( A \in \mathbb{Z} \), the capacity of \( A \) is defined by
\[
Cap(A) = E_\alpha(e_{I_A}, \psi), \quad \psi \in \mathcal{W}, \psi = 1 \text{ a.e. on } A.
\]

If \( \varphi \in \mathcal{F} \) is an \( \alpha \)-potential, then there exists a positive Radon measure \( \mu_\alpha^\varphi \) on \( \mathbb{Z} \) such that
\[
E_\alpha(\varphi, \psi) = \int_{\mathbb{Z}} \psi(z) d\mu_\alpha^\varphi(z) \quad \text{for any } \psi \in C_0(\mathbb{Z}) \cap \mathcal{W}.
\]
By Lemma 1.4 in \([8]\), \( \mu_\alpha^\varphi \) does not charge any Borel set of zero capacity. Put \( e_A = e_{I_A} \) and \( \mu_A^\alpha = \mu_{e_A}^\alpha \), then the capacity of the set \( A \) can also be defined by
\[
Cap(A) = \mu_A^\alpha(\bar{A}).
\]

The notion of the capacity is extended to any Borel set by the usual manner. A set is called exceptional if it is of zero capacity. If a statement holds everywhere except on an exceptional set, we say the statement holds quasi-everywhere (q.e.).

3 The Time Inhomogeneous Stopping Games

In this section we will characterize the properties of the value functions \( \tilde{\varepsilon}_g = \sup_{\sigma} J_{s,x}(\sigma) \) and \( \tilde{w} = \sup_{\sigma} \inf_{\tau} J_{s,x}(\tau, \sigma) \) of the time inhomogeneous stopping games. We first assume that the transition probability function \( p_t \) of the process \( X_t \) satisfies the absolute continuity condition:
\[
p_t(x, \cdot) \ll m(\cdot), \quad \forall t.
\] (11)
In fact, the Feller property in \([9]\) implies the absolute continuity condition on \( p_t \), see, e.g., page 165 of \([1]\).

3.1 The Time Inhomogeneous Optimal Stopping Problem

Consider \( \tilde{\varepsilon}_g(z) = \sup_{\sigma} J_{s,x}(\sigma) \) where
\[
J_{s,x}(\sigma) = J_{(s,x)}(\sigma) = E_{(s,x)}(e^{-\alpha \sigma} g(s + \sigma, X_{s+\sigma})).
\] (12)
Oshima showed that (see Theorem 3.1 in [8]) if \( g \in \mathcal{F} \) is quasi continuous and \( \mathcal{L}_g \cap W \neq \emptyset \), then \( \tilde{e}_g(z) \in \mathcal{F} \) is finely and cofinely continuous q.e., and \( e_g \) solves the variational inequality (10). In what follows we give conditions under which \( \tilde{e}_g \in W \) and Oshima’s result holds without the exceptional set.

It is assumed that \( g \in W \) is a finely continuous function on \( Z \) such that

\[
|g(t, x)| \leq \varphi(t, x), \tag{13}
\]

for some finite \( \alpha \)-excessive function \( \varphi \in W \) on \( Z \). We also assume that there exists a constant \( K \) such that

\[
\sup_{\epsilon > 0} \frac{1}{\epsilon} \| (g^\alpha_\epsilon - g)^- \|_{\mathcal{H}} \leq K \| g \|_{\mathcal{H}}, \tag{14}
\]

where \( g^\alpha_\epsilon \) solves (8). In the rest of this section, the notion \( K_i \) for some index \( i \) denotes a constant.

**Lemma 3.1.** Under the assumptions (13) and (14), \( e_g \in W \).

**Proof.** It has been proved that \( e_g \in \mathcal{L}_g \cap P_\alpha \), and \( e_g \in \mathcal{F} \), see Theorem 1.2 of [8] or Proposition 1.7 of [11]. Furthermore,

\[
\sup_{\epsilon} \| g^\alpha_\epsilon - \varphi \|_{\mathcal{F}} \leq K_1 \| \varphi \|_{\mathcal{F}},
\]

and

\[
\sup_{\epsilon} \| g^\alpha_\epsilon \|_{\mathcal{F}} \leq \sup_{\epsilon} \| g^\alpha_\epsilon - \varphi \|_{\mathcal{F}} + \| \varphi \|_{\mathcal{F}} \leq K_1 \| \varphi \|_{\mathcal{F}} + \| \varphi \|_{\mathcal{F}}.
\]

Now that \( g^\alpha_\epsilon \) satisfies

\[
\langle - \frac{\partial g^\alpha_\epsilon}{\partial t}, \psi \rangle + A_\alpha(g^\alpha_\epsilon, \psi) = \frac{1}{\epsilon} \left( (g^\alpha_\epsilon - g)^-, \psi \right)_{\mathcal{H}} , \quad \forall \psi \in \mathcal{F},
\]

we have

\[
\left\| \frac{\partial g^\alpha_\epsilon}{\partial t} \right\|_{\mathcal{F}} = \sup_{\| \psi \|_{\mathcal{F}} = 1} \left\langle \frac{\partial g^\alpha_\epsilon}{\partial t}, \psi \right\rangle
\]

\[
= \sup_{\| \psi \|_{\mathcal{F}} = 1} \left( A_\alpha(g^\alpha_\epsilon, \psi) - \frac{1}{\epsilon} (g^\alpha_\epsilon - g)^-, \psi \right)_{\mathcal{H}}
\]

\[
\leq \sup_{\| \psi \|_{\mathcal{F}} = 1} A_\alpha(g^\alpha_\epsilon, \psi) + \sup_{\| \psi \|_{\mathcal{F}} = 1} \frac{1}{\epsilon} (g^\alpha_\epsilon - g)^-, \psi \right)_{\mathcal{H}} . \tag{15}
\]

By the sector condition,

\[
A_\alpha(g^\alpha_\epsilon, \psi) \leq K_2 \| g^\alpha_\epsilon \|_{\mathcal{F}} \| \psi \|_{\mathcal{F}},
\]

hence

\[
\sup_{\| \psi \|_{\mathcal{F}} = 1} A_\alpha(g^\alpha_\epsilon, \psi) \leq K_2 \| g^\alpha_\epsilon \|_{\mathcal{F}} .
\]
On the other hand, by Cauchy-Schwarz inequality, the following holds:
\[
\frac{1}{\epsilon} ((g_\epsilon^\alpha - g)^-, \psi)_{\mathcal{H}} \leq \frac{1}{\epsilon} \|g_\epsilon^\alpha - g\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}},
\]
and
\[
\|\psi\|_{\mathcal{H}} \leq K_3 \|\psi\|_{\mathcal{F}},
\]
hence
\[
\sup_{\|\psi\|_{\mathcal{F}}=1} \frac{1}{\epsilon} ((g_\epsilon^\alpha - g)^-, \psi)_{\mathcal{H}} \leq \frac{1}{\epsilon} K_3 \|g_\epsilon^\alpha - g\|_{\mathcal{H}}.
\]
Now by taking sup\(_\epsilon\) of (15) and by (14) we get
\[
\sup_{\epsilon} \|\partial g_\epsilon^\alpha / \partial t\|_{\mathcal{F}} \leq K_2 K_1 \|\varphi\|_{\mathcal{W}} + K_2 \|\varphi\|_{\mathcal{F}} + K K_3 \|g\|_{\mathcal{H}} < \infty.
\]
Therefore
\[
\sup_{\epsilon} \|g_\epsilon^\alpha\|_{\mathcal{W}} = \sup_{\epsilon} \left( \|\partial g_\epsilon^\alpha / \partial t\|_{\mathcal{F}} + \|g_\epsilon^\alpha\|_{\mathcal{F}} \right) 
\leq K_2 K_1 \|\varphi\|_{\mathcal{W}} + K_2 \|\varphi\|_{\mathcal{F}} + K K_3 \|g\|_{\mathcal{H}} + K_1 \|\varphi\|_{\mathcal{W}} + \|\varphi\|_{\mathcal{F}},
\]
and as a consequence, \( e_\alpha \in \mathcal{W} \) by Lemma 1.2.12 in [5].

**Corollary 3.1.** There exist constants \( K_4, K_5 \) such that \( \|e_\alpha\|_{\mathcal{W}} \leq K_4 \|\varphi\|_{\mathcal{W}} + K_5 \|g\|_{\mathcal{H}}. \)

**Proof.** This can be seen by Eq. (16) in the proof of Lemma 3.1 and the fact that \( \|\varphi\|_{\mathcal{F}} \leq \|\varphi\|_{\mathcal{W}}. \)

Now we can revise Theorem 1.2 of [8] and get the following result.

**Corollary 3.2.** Under the assumptions (13) and (14), \( e_\alpha = \lim_{\epsilon \to 0} g_\epsilon^\alpha \) converges increasingly, strongly in \( \mathcal{H} \), and weakly in both \( \mathcal{F} \) and \( \mathcal{W} \). Furthermore, \( e_\alpha \) is the minimal function of \( P_\alpha \cap L_\alpha \cap \mathcal{W} \) satisfying
\[
\mathcal{E}_\alpha(e_\alpha, e_\alpha) \leq \mathcal{E}_\alpha(e_\alpha, \psi), \quad \forall \psi \in L_\alpha \cap \mathcal{W}.
\]

**Proof.** Now that \( e_\alpha \in \mathcal{W} \), so \( \langle \partial g_\alpha / \partial t, e_\alpha \rangle = 0 \) (see Lemma 1.1 of [7]), hence \( A_\alpha(e_\alpha, e_\alpha) = \mathcal{E}_\alpha(e_\alpha, e_\alpha) \). The rest of the proof is the same as in [8].

**Theorem 3.1.** Let \( g(z) = g(t, x) \) be a finely continuous function satisfying (13). Assume (14) and the absolute continuity condition (11). Let \( e_\alpha \) be the solution of (17), and \( e_\alpha \) is its \( \alpha \)-excessive regularization. Then
\[
\bar{e}_\alpha(z) = \sup_{\sigma} J_z(\sigma), \quad \forall z = (s, x) \in \mathbb{Z},
\]
where \( J_z(\sigma) = J_{(s,x)}(\sigma) = E_{(s,x)}(e^{-\alpha \sigma} g(s + \sigma, X_{s+\sigma})) \). Furthermore, let the set \( B = \{ z \in \mathbb{Z} : \bar{e}_\alpha(z) = g(z) \} \) and let \( \sigma_B \) be the first hitting time of \( B \) defined by \( \sigma_B = \inf\{ t > 0 : \bar{e}_\alpha(Z_{s+t}) = g(Z_{s+t}) \} \), then
\[
\bar{e}_\alpha(z) = E_{z}[e^{-\alpha \sigma_B} g(Z_{s+\sigma_B})].
\]
Proof. Notice that $\varphi \wedge \tilde{e}_g$ is an $\alpha$-potential dominating $g$, and $\tilde{e}_g$ is the smallest $\alpha$-potential dominating $g$, we get $\tilde{e}_g \leq \varphi \wedge \tilde{e}_g \leq \varphi \nu$-a.e., which implies the finiteness of $\tilde{e}_g$.

Now because $e_g \geq g \nu$ a.e., we have $nR_{n+\alpha}e_g(z) \geq nR_{n+\alpha}g(z)$, $\forall z \in \mathbb{Z}$, $n > 0$, and this implies
\[
\tilde{e}_g(z) \geq \lim_{n \to \infty} nR_{n+\alpha}g(z), \quad \forall z \in \mathbb{Z}.
\]
By the absolute continuity condition and applying the dominated convergence theorem, the following holds,
\[
\lim_{n \to \infty} nR_{n+\alpha}g(z) = g(z), \quad \forall z \in \mathbb{Z},
\]
therefore $\tilde{e}_g(z) \geq g(z)$, $\forall z \in \mathbb{Z}$. Then we have
\[
\tilde{e}_g(z) \geq E_z \left( e^{-\alpha \sigma} \tilde{e}_g(Z_{s+\sigma}) \right) \geq E_z \left( e^{-\alpha \sigma} g(Z_{s+\sigma}) \right),
\]
for any stopping time $\sigma$, which implies $\tilde{e}_g(z) \geq J_z(\sigma)$, $\forall z \in \mathbb{Z}$, hence $\tilde{e}_g(z) \geq \sup_{\sigma} J_z(\sigma)$, $\forall z \in \mathbb{Z}$.

Since $e_g$ is a bounded $\alpha$-potential, there exists a positive Radon measure $\mu^\alpha$ of finite energy such that
\[
\mathcal{E}_\alpha(e_g, w) = \int_{\mathbb{Z}} w(z) \mu^\alpha(dz), \quad \forall w \in C_0(\mathbb{Z}) \cap \mathcal{W},
\]
and $\tilde{e}_g(z) = R_\alpha \mu^\alpha(z)$.

Under the absolute continuity condition (11) of the transition function, there exists a positive continuous additive functional $A_t$ in the strict sense (see Theorem 5.1.6 in [1]) such that
\[
\tilde{e}_g(z) = E_z \left( \int_0^\infty e^{-at} dA_t \right), \quad \forall z \in \mathbb{Z}.
\]
Set $B = \{ z \in \mathbb{Z} : \tilde{e}_g(z) = g(z) \}$, then
\[
\int_{B^c} (\tilde{e}_g(z) - g(z)) \mu^\alpha(dz) = \int_{\mathbb{Z}} (\tilde{e}_g(z) - g(z)) \mu^\alpha(dz) = \mathcal{E}_\alpha(e_g, e_g - g).
\]
Since $e_g$ is an $\alpha$-potential, and $e_g - g$ is nonnegative, $\mathcal{E}_\alpha(e_g, e_g - g) \geq 0$, which implies $\mathcal{E}_\alpha(e_g, e_g) - \mathcal{E}_\alpha(e_g, g) \geq 0$. On the other hand, $e_g$ satisfies (17), which implies $\mathcal{E}_\alpha(e_g, e_g) - \mathcal{E}_\alpha(e_g, g) \leq 0$. Now it can be concluded that $\mathcal{E}_\alpha(e_g, e_g) - \mathcal{E}_\alpha(e_g, g) = 0$, hence $\mu^\alpha(B^c) = 0$. Further we get
\[
E_z \left( \int_0^\infty e^{-at} I_{B^c}(Z_{s+t}) dA_t \right) = R_\alpha(I_{B^c}\mu)(z) = 0, \quad \forall z \in \mathbb{Z}.
\]
By the strong Markov property, we have for any stopping time $\sigma \leq \sigma_B$
\[
\tilde{e}_g(z) = E_z \left[ \int_0^\sigma e^{-at} dA_t \right] + E_z \left[ e^{-\alpha \sigma} \tilde{e}_g(Z_{s+\sigma}) \right], \quad (22)
\]
and because
\[
0 \leq E_z \left[ \int_0^\sigma e^{-at} dA_t \right] \leq E_z \left( \int_0^\infty e^{-at} I_{B^c}(Z_{s+t}) dA_t \right) = 0,
\]
we have \( \tilde{e}_g(z) = E_z[e^{-\alpha\sigma} \tilde{e}_g(Z_{s+\sigma})], \sigma \leq \sigma_B. \) By replacing \( \sigma \) by \( \sigma_B \) and replacing \( \tilde{e}_g(Z_{s+\sigma}) \) by \( g(Z_{s+\sigma_B}) \), we get \( \tilde{e}_g(z) = E_z[e^{-\alpha\sigma} g(Z_{s+\sigma_B})], \) and this together with (20) completes the proof. \( \square \)

**Corollary 3.3.** Under the conditions in Theorem 3.1, \( \tilde{e}_g(z) \) is finely and cofinely continuous for all \( z \in \mathbb{Z} \).

*Proof. Oshima [8] has showed that \( \tilde{e}_g(z) \) is finely and cofinely continuous for q.e. \( z \), and under the conditions in Theorem 3.1 we showed that there does not exist the exceptional set, so \( \tilde{e}_g(z) \) is finely and cofinitely continuous for all \( z \in \mathbb{Z} \). \( \square \)

**Remark 3.1.** Since \( X_t \) is a diffusion process, we can see that \( \tilde{e}_g(z) \) is continuous along the sample paths, and if \( X_t \) is a non-degenerate Ito diffusion, \( \tilde{e}_g(z) \) is continuous. This gives an alternate proof of the continuity of the value function, while Palczewski and Stettner [9] used a penalty method to prove it.

In Palczewski and Stettner’s work [9][10], the optimal policy is to stop the game at the stopping time \( \tilde{\sigma}_B = \inf\{t \geq 0 : \tilde{e}_g(Z_{s+t}) \leq g(Z_{s+t})\} \) or equivalently \( \tilde{\sigma}_B = \inf\{t \geq 0 : \tilde{e}_g(Z_{s+t}) = g(Z_{s+t})\} \). Notice that \( \tilde{\sigma}_B \leq \sigma_B \), by Theorem 3.1 we can see that

\[
\tilde{e}_g(z) \geq E_z[e^{-\alpha\tilde{\sigma}_B} \tilde{e}_g(Z_{s+\tilde{\sigma}_B})] \geq E_z[e^{-\alpha\sigma_B} \tilde{e}_g(Z_{s+\sigma_B})] \geq E_z[e^{-\alpha\sigma_B} g(Z_{s+\sigma_B})] = \tilde{e}_g(z), \forall z,
\]

hence

\[
E_z[e^{-\alpha\tilde{\sigma}_B} \tilde{e}_g(Z_{s+\tilde{\sigma}_B})] = E_z[e^{-\alpha\sigma_B} \tilde{e}_g(Z_{s+\sigma_B})], \forall z,
\]

and as a byproduct, we get the following result:

**Corollary 3.4.** Under the conditions in Theorem 3.1, there does not exist the exceptional set of irregular boundary points of \( B \).

Therefore it is feasible to replace \( \sigma_B \) by \( \tilde{\sigma}_B \) in the results of the rest of this paper.

**Remark 3.2.** In condition (14) which is used to characterize the properties of the value function of optimal stopping, \( g^\alpha \) solves a PDE which involves the generator of the stochastic process, and the part \( (g^\alpha - g)^- \) involves the function \( g \). Since the optimal stopping problem relies on an underlying process \( M \) and the reward function \( g \), condition (14) makes much sense.

### 3.2 The Time Inhomogeneous Zero-sum Game

In this section we will refine the solution of the two-obstacle problem (zero-sum game) in [8].

Let \( g(t, x), h(t, x) \in \mathcal{W} \) be finely continuous functions satisfying

\[
g(t, x) \leq h(t, x), \quad |g(t, x)| \leq \varphi(t, x), \quad |h(t, x)| \leq \psi(t, x), \forall (t, x) \in \mathbb{Z},
\]

(23)

where \( \varphi, \psi \in \mathcal{W} \) are two bounded \( \alpha \)-excessive functions. We also assume that \( g, h \) satisfy the condition (14). Suppose there exist bounded \( \alpha \)-excessive functions \( v_1(t, x), v_2(t, x) \in \mathcal{W} \) such that

\[
g(t, x) \leq v_1(t, x) - v_2(t, x) \leq h(t, x), \forall (t, x) \in \mathbb{Z},
\]

(24)
in which case we say $g$ and $h$ satisfy the **separability condition** \[3\].

Define the sequences of $\alpha$-excessive functions $\{\varphi_n\}$ and $\{\psi_n\}$ inductively by

$$
\varphi_0 = \psi_0 = 0, \psi_n = e_{\varphi_{n-1} - h}, \varphi_n = e_{\psi_{n} + g}, \quad n \geq 1,
$$

then the following holds:

**Lemma 3.2.** Assuming \[24\], then $\varphi_n, \psi_n$ are well defined and $\lim_{n \to \infty} \varphi_n = \bar{\varphi}$, $\lim_{n \to \infty} \psi_n = \bar{\psi}$ converge increasingly, strongly in $\mathcal{H}$ and weakly in both $\mathcal{F}$ and $\mathcal{W}$.

**Proof.** We only need to show the convergence in $\mathcal{W}$ and the rest of this lemma is just Lemma 2.1 in [8]. Firstly $\varphi_0 = 0 \leq v_1$ and $\varphi_0 \in \mathcal{W}$. Suppose $\varphi_{n-1} \in \mathcal{W}$ is well defined and satisfies $\varphi_{n-1} \leq v_1$, then $\varphi_{n-1} - h \leq v_1 - h \leq v_2$. Hence $\psi_n = e_{\varphi_{n-1} - h} \in \mathcal{W}$ is well defined by Lemma 3.1 and we also have $\psi_n \leq v_2$ since $e_{\varphi_{n-1} - h}$ is the smallest $\alpha$-potential dominating $\varphi_{n-1} - h$. Now that $\psi_n + g \leq v_2 + g \leq v_1$, hence $\varphi_n = e_{\psi_{n} + g} \in \mathcal{W}$ is well defined and is dominated by $v_1$.

Notice that $\varphi_0 \leq \varphi_1$. Suppose $\varphi_{n-1} \leq \varphi_n$, then $\psi_n = e_{\varphi_{n-1} - h} \leq e_{\varphi_n - h} = \psi_{n+1}$, hence $\varphi_n = e_{\psi_n + g} \leq e_{\psi_{n+1} + g} = \varphi_{n+1}$. Also by Lemma 3.1 we get

$$
\|\varphi_n\|_\mathcal{W} = \|e_{\psi_n + g}\|_\mathcal{W} \leq K_4 \|v_1\|_\mathcal{W} + K_5 \|\psi_n + g\|_\mathcal{H}.
$$

Notice that $g \leq \psi_n + g \leq v_1$, hence $\|\psi_n + g\|_\mathcal{H}$ is uniformly bounded in $n$, and as a consequence, $\|\varphi_n\|_\mathcal{W}$ is uniformly bounded in $n$. In a similar manner we can show that $\|\psi_n\|_\mathcal{W}$ is uniformly bounded. The convergence of $\varphi_n, \psi_n$ in $\mathcal{W}$ follows by Lemma I.2.12 in [5].

**Corollary 3.5.** Under the separability condition, $\bar{\varphi} = e_{\bar{\psi} + g}$, $\bar{\psi} = e_{\bar{\varphi} - h}$, and they satisfy

$$
E_\alpha(\bar{\varphi}, \bar{\varphi}) \leq E_\alpha(\varphi, w), \quad \forall w \in \mathcal{L}_{\bar{\psi} + g} \cap \mathcal{W},
$$

$$
E_\alpha(\bar{\psi}, \bar{\psi}) \leq E_\alpha(\psi, w), \quad \forall w \in \mathcal{L}_{\bar{\varphi} - h} \cap \mathcal{W}.
$$

**Proof.** Since $\bar{\varphi}$ is an $\alpha$-potential dominating $\bar{\psi} + g$, we get $e_{\bar{\psi} + g} \leq \bar{\varphi}$. On the other hand, $\bar{\varphi} = \lim_{n \to \infty} \varphi_n = \lim_{n \to \infty} e_{\psi_{n} + g} \leq e_{\bar{\psi} + g}$, hence $\bar{\varphi} = e_{\bar{\psi} + g}$. Similarly $\bar{\psi} = e_{\bar{\varphi} - h}$. The proof of \[25\] is immediate by Corollary 3.2.

**Corollary 3.6.** If a pair of $\alpha$-excessive functions $(V_1, V_2)$ satisfy $g \leq V_1 - V_2 \leq h$, then $\bar{\varphi} \leq V_1$, $\bar{\psi} \leq V_2$, and $\bar{w} := \bar{\varphi} - \bar{\psi}$ is the unique function in $\mathcal{J}$ satisfying

$$
E_\alpha(\bar{w}, \bar{w}) \leq E_\alpha(w, w), \quad \forall w \in \mathcal{J}, \quad g \leq w \leq h,
$$

where $\mathcal{J} = \{w = \varphi_1 - \varphi_2 + v : \varphi_1, \varphi_2 \in \mathcal{W} \text{ are } \alpha \text{-potentials}, v \in \mathcal{W}\}$.

**Proof.** Clearly $\varphi_{n-1} - h \leq \psi_n$ and $\psi_n + g \leq \bar{\varphi}_n$, hence $g \leq \bar{\varphi} - \bar{\psi} \leq h$. If $g, h$ satisfy the separability condition with respect to $V_1, V_2$, then we would have $\varphi_n \leq V_1$ and $\psi_n \leq V_2$, and as a consequence $\bar{\varphi} \leq V_1$, $\bar{\psi} \leq V_2$. 

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Now (26) is equivalent to
\[ E_\alpha(\bar{\varphi}, \tilde{\psi}) + E_\alpha(\tilde{\psi}, \bar{\psi}) \leq E_\alpha(\bar{\varphi}, w + \tilde{\psi}) + E_\alpha(\tilde{\psi}, \bar{\varphi} - w), \quad g \leq w \leq h, \]
which holds by (25). Suppose there are two solutions \( \bar{w}_1, \bar{w}_2 \in \mathcal{J} \) satisfying (26). Notice that \( \bar{w}_1 - \bar{w}_2 \in \mathcal{W} \) and \( \langle \partial (\bar{w}_1 - \bar{w}_2), \bar{w}_1 - \bar{w}_2 \rangle = 0 \), so
\[ \langle \partial \bar{w}_1, \bar{w}_2 \rangle + \langle \partial \bar{w}_2, \bar{w}_1 \rangle = 0, \]
and consequently
\[ A_\alpha(\bar{w}_1, \bar{w}_2) + A_\alpha(\bar{w}_2, \bar{w}_1) = E_\alpha(\bar{w}_1, \bar{w}_2) + E_\alpha(\bar{w}_2, \bar{w}_1). \]
Therefore,
\[ A_\alpha(\bar{w}_1 - \bar{w}_2, \bar{w}_1 - \bar{w}_2) = A_\alpha(\bar{w}_1, \bar{w}_1) + A_\alpha(\bar{w}_2, \bar{w}_2) - A_\alpha(\bar{w}_1, \bar{w}_2) - A_\alpha(\bar{w}_2, \bar{w}_1) \]
\[ = A_\alpha(\bar{w}_1, \bar{w}_1) + A_\alpha(\bar{w}_2, \bar{w}_2) - E_\alpha(\bar{w}_1, \bar{w}_2) - E_\alpha(\bar{w}_2, \bar{w}_1) \leq 0, \]
which implies that \( \bar{w}_1 = \bar{w}_2 \) a.e. \( \square \)

Let \( \tilde{\varphi}, \tilde{\psi}, \tilde{w} \) be the \( \alpha \)-excessive modifications of \( \bar{\varphi}, \tilde{\psi}, \bar{w} \), respectively. We further define for arbitrary pair of stopping times \( \tau, \sigma \) the payoff function \( J_z(\tau, \sigma) \) as
\[ J_z(\tau, \sigma) = E_z \left[ e^{-\alpha (\tau \wedge \sigma)} (g(Z_{s+\sigma})I_{\tau > \sigma} + h(Z_{s+\tau})I_{\tau \leq \sigma}) \right], \quad z \in \mathbb{Z}. \quad (27) \]
Then we have the following result:

**Theorem 3.2.** Assuming the separability condition on \( g, h \) and conditions (11) (14). There exists a finite finely and cofinely continuous function \( \tilde{w}(z) \in \mathcal{J} \) satisfying (26) and the identity
\[ \tilde{w}(z) = \sup \inf_{\sigma} J_z(\tau, \sigma) = \inf \sup_{\tau} J_z(\tau, \sigma), \quad \forall z = (s, x) \in \mathbb{Z}, \quad (28) \]
where \( \sigma, \tau \) range over all stopping times. Moreover, the pair \( \hat{\tau}, \hat{\sigma} \) defined by
\[ \hat{\tau} = \inf \{ t > 0 : \bar{w}(Z_{s+t}) = h(Z_{s+t}) \}, \quad \hat{\sigma} = \inf \{ t > 0 : \bar{w}(Z_{s+t}) = g(Z_{s+t}) \} \]
is the saddle point of the game in the sense that
\[ J_z(\hat{\tau}, \hat{\sigma}) \leq J_z(\hat{\tau}, \sigma) \leq J_z(\tau, \sigma) \leq J_z(\hat{\tau}, \sigma), \quad z \in \mathbb{Z}, \]
for all stopping times \( \tau, \sigma \).

**Proof.** We only need to prove (28). By Theorem 3.1 for any \( z \in \mathbb{Z} \) we have
\[ \bar{\varphi}(z) = \sup_{\sigma} E_z[e^{-\alpha \sigma}(\tilde{\psi} + g)(Z_{s+\sigma})] = E_z[e^{-\alpha \sigma}(\tilde{\psi} + g)(Z_{s+\sigma})], \]
\[ \tilde{\psi}(z) = \sup_{\sigma} E_z[e^{-\alpha \tau}(\tilde{\phi} - H)(Z_{s+\tau})] = E_z[e^{-\alpha \tau}(\tilde{\phi} - h)(Z_{s+\tau})], \quad (29) \]
and for any stopping times $\sigma \leq \hat{\sigma}, \tau \leq \hat{\sigma}$,

$$\tilde{\varphi}(z) = E_z[e^{-\alpha \sigma} \tilde{\varphi}(Z_{s+\sigma})], \quad \tilde{\psi}(z) = E_z[e^{-\alpha \tau} \tilde{\psi}(Z_{s+\tau})], \quad \forall z = (s, x) \in \mathbb{Z}. $$

From (22), we could take $\{e^{-\alpha \tau} \tilde{\varphi}(Z_{s+\tau})\}$ and $\{e^{-\alpha \tau} \tilde{\psi}(Z_{s+\tau})\}$ as non-negative $P_z$-supermartingales, therefore, for any $z \in \mathbb{Z}$ and any stopping times $\tau, \sigma$, we have

$$\tilde{\varphi}(z) \geq E_z[e^{-\alpha \sigma} \tilde{\varphi}(Z_{s+\sigma})], \quad \tilde{\psi}(z) \geq E_z[e^{-\alpha \tau} \tilde{\psi}(Z_{s+\tau})].$$

Consequently, for any $z \in \mathbb{Z},$

$$\tilde{w}(z) = \tilde{\varphi}(z) - \tilde{\psi}(z) \leq E_z[e^{-\alpha(\hat{\sigma} \wedge \tau)} \tilde{\varphi}(Z_{s+\hat{\sigma} \wedge \tau})] - E_z[e^{-\alpha(\hat{\sigma} \wedge \tau)} \tilde{\psi}(Z_{s+\hat{\sigma} \wedge \tau})]$$

$= E_z[e^{-\alpha(\hat{\sigma} \wedge \tau)} \tilde{w}(Z_{s+\hat{\sigma} \wedge \tau})] \leq E_z[e^{-\alpha(\tau \wedge \hat{\sigma})}(g(Z_{s+\sigma})I_{\tau>\hat{\sigma}} + h(Z_{s+\tau})I_{\tau<\hat{\sigma}})] = J_z(\hat{\tau}, \hat{\sigma}),$

where the last inequality is due to the fact that $g(z) \leq \tilde{w}(z) \leq h(z), \forall z \in \mathbb{Z}$ and (20). In a similar manner, we can prove that $\tilde{w} \geq J_z(\hat{\tau}, \hat{\sigma})$, and this completes the proof.

### 3.3 Time Inhomogeneous Stopping Game with Holding Cost

Usually the optimal stopping games involve a holding cost function $f \in \mathcal{K}$, see, e.g., [9], and the return functions become

$$J^f_{(s,x)}(\sigma) = E_{(s,x)} \left( \int_0^\sigma e^{-\alpha t} f(s + t, X_{s+t}) dt + e^{-\alpha \sigma} g(s + \sigma, X_{s+\sigma}) \right), \quad (30)$$

and

$$J^f_{(s,x)}(\sigma, \tau) = E_{(s,x)} \left( \int_0^{\sigma \wedge \tau} e^{-\alpha t} f(s + t, X_{s+t}) dt \right) + E_{(s,x)} \left( e^{-\alpha(s \wedge \tau)} (g(s + \sigma, X_{s+\sigma})I_{\sigma<\tau} + h(s + \tau, X_{s+\tau})I_{\tau \leq \sigma}) \right), \quad (31)$$

but this model can be essentially reduced to the classical stopping problem by taking $\hat{g} = g - R_\alpha f$ and $\hat{h} = h - R_\alpha f$ instead of $g$ and $h$ respectively, where $R_\alpha$ is the resolvent and $R_\alpha f$ is considered as a version of $G_\alpha f \in \mathcal{K}$. We assume that conditions (13) (14) also apply to $\hat{g}$ for the optimal stopping game (and similarly conditions (23) (24) apply to $\hat{g}, \hat{h}$ for the zero-sum game).

**Theorem 3.3.** Let $g$ be a finely continuous function satisfying (13). Assume (14) on $g$ and the absolute continuity condition (11) on $p_t$. Let $e_g^f$ be the solution of

$$\mathcal{E}_\alpha(e_g^f, \psi - e_g^f) \geq (f, \psi - e_g^f)\mathcal{K}, \quad \forall \psi \in \mathcal{L}_\alpha \cap \mathcal{K}, \quad (32)$$

and let $\tilde{e}_g^f$ be its $\alpha$-excessive regularization. Then

$$\tilde{e}_g^f(z) = \sup_{\sigma} J^f_{\hat{z}}(\sigma), \quad \forall z = (s, x) \in \mathbb{Z},$$

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where $J^f_z(\sigma)$ is defined as in (30), and $\tilde{e}^f_g(z)$ is finely and cofinely continuous. Furthermore, let the set $B = \{z \in \mathbb{Z} : \tilde{e}^f_g(z) = g(z)\}$ and let $\sigma_B$ be the first hitting time of $B$ defined by

$$\sigma_B = \inf \{t > 0 : \tilde{e}^f_g(Z_{s+t}) = g(Z_{s+t})\},$$

then

$$\tilde{e}^f_g(z) = E_z[e^{-\alpha \sigma_B}g(Z_{s+\sigma_B})].$$

(34)

Proof. Define the function

$$J^f_z(\sigma) = E_z(e^{-\alpha \sigma}g(s + \sigma, X_{s+\sigma})),
$$

where $\hat{g} = g - R_{\alpha}f$, and let $\tilde{e}^f_g = \sup_{\sigma} J^f_z(\sigma)$, then by Theorem 3.1 $\tilde{e}^f_g$ solves

$$E_0(\tilde{e}^f_g, \hat{\psi} - \tilde{e}^f_g) \geq 0, \quad \forall \hat{\psi} \in \mathcal{L}_g \cap \mathcal{W},$$

(35)

and the optimal stopping time is defined by $\sigma_B = \inf \{t > 0 : \tilde{e}^f_g(Z_{s+t}) = \hat{g}(Z_{s+t})\}$. By Dynkin’s formula,

$$E_{s,x}(\int_0^\sigma e^{-at}f(s + t, X_{s+t})dt) = R_{\alpha}f(s, x) - E_{s,x}(e^{-\alpha \sigma}R_{\alpha}f(s + \sigma, X_{s+\sigma})),
$$

which leads to

$$J^f_z(\sigma) = J^f_z(\sigma) + R_{\alpha}f(z),
$$

hence $e^f_g(z) = e^f_g(z) + R_{\alpha}f(z)$.

Now let $e^f_g(z) = e^f_g(z) - R_{\alpha}f(z), \hat{\psi} = \psi - R_{\alpha}f$ in (35) we get

$$E_0(e^f_g - Gf, \psi - e^f_g) \geq 0.
$$

(36)

Since $E_0(Gf, \psi - e^f_g) = (f, \psi - e^f_g)_{\mathcal{X}^\alpha}$, this proves (32). Also notice that the optimal stopping time can be written as $\sigma_B = \inf \{t > 0 : \tilde{e}^f_g(Z_{s+t}) = \hat{g}(Z_{s+t})\}$, and this completes the proof.

Similarly we can modify Theorem 3.2 and get the following result:

**Theorem 3.4.** Let $g, h$ be finely continuous functions satisfying (23) and (24). Assume (14) on $g, h$ and the absolute continuity condition (11) on $\mathcal{F}$. Then there exists a finite finely and cofinely continuous function $\tilde{w}^f \in \mathcal{F}$, $g(z) \leq \tilde{w}^f(z) \leq h(z)$, such that

$$E_\alpha(\tilde{w}^f, w - \tilde{w}^f) \geq (f, w - \tilde{w}^f)_{\mathcal{X}^\alpha}, \quad \forall w \in \mathcal{F}, g \leq w \leq h,
$$

(37)

and

$$\tilde{w}^f(z) = \sup_{\sigma} \inf_{\tau} J_z^f(\tau, \sigma) = \inf_{\sigma} \sup_{\tau} J_z^f(\tau, \sigma), \quad \forall z = (s, x) \in \mathbb{Z},
$$

(38)

where $J_z^f(\tau, \sigma)$ was given by (31) and $\sigma, \tau$ range over all stopping times. Moreover, the pair $\hat{\tau}, \hat{\sigma}$ defined by

$$\hat{\tau} = \inf \{t > 0 : \tilde{w}^f(Z_{s+t}) = h(Z_{s+t})\}, \quad \hat{\sigma} = \inf \{t > 0 : \tilde{w}^f(Z_{s+t}) = g(Z_{s+t})\}
$$

is the saddle point of the game in the sense that

$$J_z^f(\hat{\tau}, \hat{\sigma}) \leq J_z^f(\hat{\tau}, \sigma) \leq J_z^f(\tau, \hat{\sigma}), \quad z \in \mathbb{Z},
$$

for all stopping times $\tau, \sigma$. 

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As an extension of Corollary 3.6, we have the following:

**Corollary 3.7.** The variational inequality (37) has a unique solution.

**Proof.** The case where \( f = 0 \) was proved in Corollary 3.6. For a general \( f \in \mathcal{H} \), notice again that \( (f,w - \bar{w}^f)_{\mathcal{H}} = \mathcal{E}_\alpha(G_\alpha f, w - \bar{w}^f) \), we get

\[
\mathcal{E}_\alpha(\bar{w}^f - G_\alpha f, (w - G_\alpha f) - (\bar{w}^f - G_\alpha f)) \geq 0, \quad \forall w \in \mathcal{J}, \; g \leq w \leq h.
\]

Let \( \hat{\bar{w}}^f = \bar{w}^f - G_\alpha f \), \( \hat{w} = w - G_\alpha f \), \( \hat{g} = g - G_\alpha f \), \( \hat{h} = h - G_\alpha f \), we get

\[
\mathcal{E}_\alpha(\hat{\bar{w}}^f, \hat{w} - \hat{\bar{w}}^f) \geq 0, \quad \forall \hat{w} \in \mathcal{J}, \; \hat{g} \leq \hat{w} \leq \hat{h},
\]

which has a unique solution in view of Corollary 3.6.

\[ \square \]

### 4 Time Inhomogeneous Stopping Games of Ito Diffusion

In this section we are concerned with a multi-dimensional time inhomogeneous Ito diffusion:

\[
dX_t = b(t, X_t)dt + a(t, X_t)dB_t, \; X_s = x, \tag{39}
\]

where

\[
X_t = \begin{pmatrix} X_{1t} \\ \vdots \\ X_{nt} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad a = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}, \quad B_t = \begin{pmatrix} B_{1t} \\ \vdots \\ B_{mt} \end{pmatrix}, \quad m \geq n,
\]

and \( a_{ij}, b_i, i = 1, 2, ..., n, j = 1, 2, ..., m \), are continuous functions of \( t \) and \( X_t \). Define the square matrix \( [A_{ij}] = A = \frac{1}{2}aa^T \). We assume \( A \) is uniformly non-degenerate, and \( a, b \) satisfy the usual Lipschitz conditions so that (39) has a unique strong solution. \( B_t \) in (39) is assumed to be the standard multi-dimensional Brownian motion. Thus we are given a system \( (\Omega, \mathcal{F}, \mathcal{F}_t, X, \theta_t, P_x) \), where \( (\Omega, \mathcal{F}) \) is a measurable space, \( X = X(\omega) \) is a mapping of \( \Omega \) into \( C(\mathbb{R}^n) \), \( \mathcal{F}_t \) is the sigma algebra generated by \( X_s(s \leq t) \), and \( \theta_t \) is a shift operator in \( \Omega \) such that \( X_s(\theta_t \omega) = X_{s+t}(\omega) \). Here \( P_x(x \in \mathbb{R}) \) is a family of measures under which \( \{X_t, t \geq 0\} \) is a diffusion with initial state \( x \).

At each time \( t \), define the infinitesimal generator \( L^{(t)} \) as

\[
L^{(t)} u(x) = \sum_{i=1}^n b_i(t, x) \frac{\partial u}{\partial x_i} + \sum_{i,j} A_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j}. \tag{40}
\]

Let the positive Radon measure \( m(dx) = \rho^{(t)}(x)dx \), where \( \rho^{(t)} \) satisfies

\[
A \nabla \rho^{(t)} = \rho^{(t)} \mu, \quad \forall t, \tag{41}
\]

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and \( \mu_i = b_i - \sum_{j=1}^{n} \frac{\partial A_{i,j}}{\partial x_j}, i = 1, 2, ..., n. \) Notice that when \( a \) and \( b \) in (39) are constants, \( \rho^{(t)} \) reduces to
\[
\rho^{(t)}(x) = e^{(A^{-1}b) \cdot x}.
\]
Thus the associated Dirichlet form \((E^{(t)}, F)\) densely embedded in \( H = L^2(\mathbb{R}^n; m) \) is then given by
\[
E^{(t)}(u, v) = \int_{\mathbb{R}^n} \nabla u(x) \cdot A \nabla v(x) m(dx), \quad u, v \in F,
\]
where
\[
F = \{ u \in H : \ u \ is \ continuous, \ ||u||_F^2 = E^{(0)}_1(u, u) < \infty \}.
\]
Now we can define the sets \( \mathcal{F}, \mathcal{H}, \mathcal{W} \) in the same way as in Section 2 and define the time inhomogeneous Dirichlet form \( \mathcal{E}_\alpha \) as well.

Since \( X_t \) is a non-degenerate Ito diffusion, the absolute continuity condition on its transition function automatically holds, and for the same reason, the fine and cofine continuity notion can be changed to the usual continuity.

Let \( f \in \mathcal{H}, g \in \mathcal{W} \) be continuous functions satisfying the conditions as in Section 3.3 and define the return function \( J^f_z(\sigma) \) as in (30), then we have the following result:

**Theorem 4.1.** Assume (13) (14) on \( g \) and the absolute continuity condition (11) on \( p_t \). Let \( e^f_g \) be the solution of
\[
\mathcal{E}_\alpha(e^f_g, \psi - e^f_g) \geq (f, \psi - e^f_g)_{\mathcal{W}}, \quad \forall \psi \in L_g \cap \mathcal{W},
\]
and let \( \tilde{e}^f_g \) be its \( \alpha \)-excessive regularization. Then
\[
\tilde{e}^f_g(z) = \sup_{\sigma} J^f_z(\sigma), \quad \forall z = (s, x) \in \mathbb{Z},
\]
where \( J^f_z(\sigma) \) is defined as (30), and \( \tilde{e}^f_g(z) \) is continuous. Furthermore, let the set \( B = \{ z \in \mathbb{Z} : \tilde{e}^f_g(z) = g(z) \} \) and let \( \sigma_B \) be the first hitting time of \( B \) defined by \( \sigma_B = \inf\{ t > 0 : \tilde{e}^f_g(Z_{s+t}) = g(Z_{s+t}) \} \), then
\[
\tilde{e}^f_g(z) = E_z[e^{-\alpha \sigma_B} g(Z_{s+B})].
\]

For the zero-sum game of Ito diffusion with the return function \( J^f_z(\sigma, \tau) \) as defined in (31), we have the following result:

**Theorem 4.2.** Let \( g, h \) be continuous functions satisfying (23) and (24). Assume (14) on \( g, h \) and the absolute continuity condition (17) on \( p_t \). Then there exists a finite and continuous function \( \tilde{w}^f \in \mathcal{J}, g(z) \leq \tilde{w}^f(z) \leq h(z) \), such that
\[
\mathcal{E}_\alpha(\tilde{w}^f, w - \tilde{w}^f) \geq (f, w - \tilde{w}^f)_{\mathcal{W}}, \quad \forall w \in \mathcal{J}, \ g \leq w \leq h,
\]
and
\[
\tilde{w}^f(z) = \sup_{\sigma} \inf_{\tau} J^f_z(\tau, \sigma) = \inf_{\tau} \sup_{\sigma} J^f_z(\tau, \sigma), \quad \forall z = (s, x) \in \mathbb{Z},
\]
where $\sigma, \tau$ range over all stopping times and $J^f_z(\sigma, \tau)$ is defined in (31). Moreover, the pair $\hat{\tau}, \hat{\sigma}$ defined by

$$
\hat{\tau} = \inf\{t > 0 : \bar{w}^f(Z_{s+t}) = h(Z_{s+t})\}, \quad \hat{\sigma} = \inf\{t > 0 : \bar{w}^f(Z_{s+t}) = g(Z_{s+t})\}
$$

is the saddle point of the game in the sense that

$$
J^f_z(\hat{\tau}, \sigma) \leq J^f_z(\hat{\tau}, \hat{\sigma}) \leq J^f_z(\tau, \hat{\sigma}), \quad z \in \mathbb{Z},
$$

for all stopping times $\tau, \sigma$.

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