MULTIPlicITIES UNDER BASEchange: FINITE FIELD CASE

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1. INTRODUCTION

Let $G$ and $H$ be two connected algebraic groups over a finite field $\mathbb{F}$ of order $q$ with $H \subset G$. Let $\pi_1$ be an irreducible representation of $G(\mathbb{F})$ and $\pi_2$ of $H(\mathbb{F})$. Assume that both the representations $\pi_1, \pi_2$ have a basechange, denoted as $\pi_E^1, \pi_E^2$, to $E$, where all through the paper, $E$ is the unique quadratic extension of $\mathbb{F}$, with $\pi_E^1$ an irreducible representation of $G(E)$ which is invariant under $\langle \sigma \rangle = \text{Gal}(E/\mathbb{F})$, and $\pi_E^2$ an irreducible representation of $H(E)$ which is invariant under $\langle \sigma \rangle = \text{Gal}(E/\mathbb{F})$. In our paper, we will tacitly assume that $G, H$ are reductive algebraic groups over $\mathbb{F}$, and that $\pi_1, \pi_2$ are uniform representations, i.e., a virtual sum of Deligne-Lusztig representations $R(T, \theta)$ (for varying maximal tori $T \subset G$ and characters $\theta : T(\mathbb{F}) \to \mathbb{C}^\times$). As an example, note that all irreducible representations of $\text{GL}_n(\mathbb{F})$ and $\text{U}_n(\mathbb{F})$ are uniform representations. For such uniform representations, existence of basechange is a well-known theorem due to Digne-Michel [DM].

Our usage of the basechange depends mostly with the Shintani character identity relating twisted character of $\pi_E$ at $g \in G(E)$ with the ordinary character of $\pi$ at the norm of the element $g$ which is a well-defined conjugacy class in $G(\mathbb{F})$, denoted $\text{Nm}(g)$.

The aim of this paper is to relate the multiplicity,

$$m(\pi_1, \pi_2) = \dim \text{Hom}_{H(\mathbb{F})}(\pi_1, \pi_2),$$

with the multiplicity of the basechanged representations:

$$m(\pi_E^1, \pi_E^2) = \dim \text{Hom}_{H(E)}(\pi_E^1, \pi_E^2).$$

It is well-known that basechange allows one to simplify representations, and therefore allows one, in some cases, to calculate $m(\pi_1, \pi_2)$ from the simpler information $m(\pi_E^1, \pi_E^2)$.

This paper itself is inspired by one such application where $G = \text{GL}_{2n}(\mathbb{F})$, and $H = \text{GL}_n(E)$ sitting naturally inside $G$, the representation $\pi_1$ being a cuspidal representation.

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of $G = \text{GL}_{2n}(\mathbb{F})$, and $\pi_2 = \chi$, a one-dimensional representation of $H = \text{GL}_n(\mathbb{E})$ arising from a character $\chi : \mathbb{E}^\times \to \mathbb{C}^\times$ through the determinant map $\det : \text{GL}_n(\mathbb{E}) \to \mathbb{E}^\times$. (We use the same notation $\chi$ to denote a character of $\mathbb{E}^\times$ as well as the associated character of $\text{GL}_n(\mathbb{E})$.) The question that we wish to understand is the multiplicity $m(\pi_1, \chi)$.

Observe that over $\mathbb{E}$, $H(\mathbb{E}) \subset G(\mathbb{E})$ is $\text{GL}_n(\mathbb{E}) \times \text{GL}_n(\mathbb{E}) \subset \text{GL}_{2n}(\mathbb{E})$, and the representation $\pi_1^E$ of $\text{GL}_{2n}(\mathbb{E})$ becomes a principal series representation $\pi \times \pi'$ induced from the $(n, n)$ parabolic of $\text{GL}_{2n}(\mathbb{E})$ with Levi subgroup $\text{GL}_n(\mathbb{E}) \times \text{GL}_n(\mathbb{E})$, where $\pi$ is the cuspidal representation of $\text{GL}_n(\mathbb{E})$ associated to the same character $\theta : \mathbb{F}^\times_{q^{2n}} \to \mathbb{C}^\times$ which is used to define the cuspidal representation $\pi_1$ of $G = \text{GL}_{2n}(\mathbb{F})$.

Basechange thus allows one to reduce a question on cuspidal representations to one on principal series representations which can be treated by ‘geometric’ methods, as we show in this paper in one illustrative case.

### 2. Multiplicity under basechange

We keep the notation introduced in the introduction, thus $H \subset G$ are connected algebraic groups over a finite field $\mathbb{F}$, $\pi_1$ an irreducible representation of $G(\mathbb{F})$ and $\pi_2$ of $H(\mathbb{F})$. We assume that both the representations $\pi_1, \pi_2$ have a basechange, denoted as $\tilde{\pi}_1^E, \tilde{\pi}_2^E$, to $\mathbb{E}$, with $\tilde{\pi}_1^E$ an irreducible representation of $G(\mathbb{E})$ which is invariant under $\langle \sigma \rangle = \text{Gal}(\mathbb{E}/\mathbb{F})$, and $\tilde{\pi}_2^E$ an irreducible representation of $H(\mathbb{E})$ which is invariant under $\langle \sigma \rangle = \text{Gal}(\mathbb{E}/\mathbb{F})$. We will fix an extension $\tilde{\pi}_1^E$ of the irreducible representation $\pi_1^E$ of $G(\mathbb{E})$ to $G(\mathbb{E}) \rtimes \langle \sigma \rangle$; similarly, fix an extension $\tilde{\pi}_2^E$ of the irreducible representation $\pi_2^E$ of $H(\mathbb{E})$ to $H(\mathbb{E}) \rtimes \langle \sigma \rangle$. (In fact, the Shintani character identity fixes a unique extension of $\pi_1^E$ and $\pi_2^E$ to the representation $\tilde{\pi}_1^E$ of $G(\mathbb{E}) \rtimes \langle \sigma \rangle$ and to the representation $\tilde{\pi}_2^E$ of $H(\mathbb{E}) \rtimes \langle \sigma \rangle$.)

The following proposition, much more general than Theorem 1 in [Pr], has the same proof as there.

**Proposition 2.1.** With the notation as above, we have:

$$2m(\tilde{\pi}_1^E, \tilde{\pi}_2^E) = m(\pi_1^E, \pi_2^E) + m(\pi_1, \pi_2).$$

**Proof.** Recall that by the Schur orthogonality theorem, if $V$ is a representation of a finite group $\mathcal{H}$ with character $\Theta$, then

$$\dim V^\mathcal{H} = \frac{1}{|\mathcal{H}|} \sum_{h \in \mathcal{H}} \Theta(h).$$

We will apply the Schur orthogonality theorem to the representation $V$ of $\mathcal{H} = H(\mathbb{E}) \rtimes \langle \sigma \rangle$ which is the restriction of the representation $(\tilde{\pi}_1^E)^\vee \otimes \tilde{\pi}_2^E$ of the group $[G(\mathbb{E}) \times H(\mathbb{E})] \rtimes \langle \sigma \rangle$ to the diagonally embedded subgroup $\mathcal{H} = H(\mathbb{E}) \rtimes \langle \sigma \rangle$.

Note that $\mathcal{H} = H(\mathbb{E}) \rtimes \langle \sigma \rangle = H(\mathbb{E}) \cup H(\mathbb{E}) \cdot \sigma$, hence the sum of characters on $\mathcal{H}$ decomposes as a sum over $H(\mathbb{E})$ and another sum over $H(\mathbb{E}) \cdot \sigma$. By the Shintani character identity which involves the norm mapping

$$\text{Nm} : G(\mathbb{E}) \times H(\mathbb{E}) \to G(\mathbb{F}) \times H(\mathbb{F}),$$

the character of $(\tilde{\pi}_1^E)^\vee \otimes \tilde{\pi}_2^E$ at $(g, h) \cdot \sigma \in [G(\mathbb{E}) \times H(\mathbb{E})] \cdot \sigma$ is the character of the representation $(\pi_1)^\vee \otimes \pi_2$ at the element $(\text{Nm}g, \text{Nm}h) \in G(\mathbb{F}) \times H(\mathbb{F})$. The mapping $\text{Nm} : G(\mathbb{E}) \times H(\mathbb{E}) \to G(\mathbb{F}) \times H(\mathbb{F})$ has the well-known property that the cardinality
of $\sigma$-centralizer of an element $(g, h) \in G(E) \times H(E)$ is the same as the cardinality of the centralizer of $(Nm g, Nm h)$ in $G(F) \times H(F)$ (cf. Lemma 2 in [Pr]), allowing one to conclude the proposition.

**Remark 2.2.** Observe that since $m(\pi_1^E, \pi_2^E)$ is the dimension of the space of $H = H(E) \rtimes \langle \sigma \rangle$ invariant linear vectors in the space $\text{Hom}_C(\pi_1^E, \pi_2^E)$, and $m(\pi_1^E, \pi_2^E)$ is the dimension of the space of $H(E)$ invariant linear vectors in the same space,

$$0 \leq m(\pi_1^E, \pi_2^E) \leq m(\pi_1^E, \pi_2^E).$$

**Corollary 2.3.** With the notation as above,

$$m(\pi_1, \pi_2) \leq m(\pi_1^E, \pi_2^E),$$

and

$$m(\pi_1, \pi_2) \equiv m(\pi_1^E, \pi_2^E) \mod 2.$$  

In particular, if $m(\pi_1^E, \pi_2^E) \leq 1$, then $m(\pi_1, \pi_2) \leq 1$, and

$$m(\pi_1^E, \pi_2^E) = m(\pi_1, \pi_2).$$

The following corollary of Proposition 2.1 is Theorem 1 in [Pr].

**Corollary 2.4.** Let $G$ be a connected reductive group over a finite field $F$, $E/F$ a quadratic extension, and $\pi$ an irreducible uniform representation of $G(E)$, i.e., one which can be expressed as a sum of Deligne-Lusztig representations of $G(E)$ induced from tori. Then the representation $\pi$ has a $G(F)$ fixed vector if and only if $\pi^\sigma \cong \pi^\vee$. If $\pi^\sigma \cong \pi^\vee$, then $\pi$ has a one dimensional space of fixed vectors under $G(F)$, and the representation $\pi \otimes \pi^\sigma$ which is canonically a representation of $[G(E) \times G(E)] \rtimes \mathbb{Z}/2$ has a $G(E) \rtimes \mathbb{Z}/2$ fixed vector.

**Example 2.5.** Here is a simple example of the way multiplicities vary under base change as dictated by Proposition 2.1. If $P(\chi) = P(\chi^{-1})$ denotes the principal series representation of $\text{PGL}_2(F)$ associated to a character $\chi : F^\times \to \mathbb{C}^\times$, $\chi \neq 1$, then it is easy to see that for any 3 non-trivial characters $\chi_1, \chi_2, \chi_3 : F^\times \to \mathbb{C}^\times$, $m(P(\chi_1) \otimes P(\chi_2), P(\chi_3)) = 1$ except when $\chi_3 = \chi_1 \chi_2^\pm 1$ (assuming that $\chi_1 \chi_2^\pm 1 \neq 1$), and that in these exceptional cases, $m(P(\chi_1) \otimes P(\chi_2), P(\chi_3)) = 2$. On the other hand, if $D(\chi) = D(\chi^{-1})$ denotes the cuspidal representation of $\text{PGL}_2(F)$ associated to a character $\chi : E^\times/F^\times \to \mathbb{C}^\times$, $\chi \neq 1$, then it is easy to see that for any 3 non-trivial characters $\chi_1, \chi_2, \chi_3 : E^\times/F^\times \to \mathbb{C}^\times$, $m(D(\chi_1) \otimes D(\chi_2), D(\chi_3)) = 1$ except when $\chi_3 = \chi_1 \chi_2^\pm 1$ (assuming that $\chi_1 \chi_2^\pm 1 \neq 1$), and that in these exceptional cases, $m(D(\chi_1) \otimes D(\chi_2), D(\chi_3)) = 0$.

3. **Linear periods for Principal series**

Following is the main proposition of the paper proved by geometric means.

**Proposition 3.1.** Let $\pi_1, \pi_2$ be two irreducible cuspidal representations of $\text{GL}_n(F)$, $n > 1$ and $\chi_1, \chi_2$ be two characters of $E^\times$. Then the representation $\pi_1 \times \pi_2$ of $\text{GL}_2n(F)$ (parabolically induced from the $(n, n)$-parabolic) has a nonzero vector on which $\text{GL}_n(F) \times \text{GL}_n(F)$ operates by the character $\chi_1 \times \chi_2$ if and only if one of the two conditions hold:

1. $(\pi_1 \otimes \chi_1^{-1})^\vee \cong \pi_2 \otimes \chi_2^{-1}$. 
(2) $n = 2m$ is even, $\pi_1$ contains a vector on which $GL_m(\mathbb{F}) \times GL_m(\mathbb{F})$ operates via $\chi_1 \times \chi_2$ and $\pi_2$ contains a vector on which $GL_m(\mathbb{F}) \times GL_m(\mathbb{F})$ operates via $\chi_1 \times \chi_2$.

The dimension of the space of vector in $\pi_1 \times \pi_2$ on which $GL_n(\mathbb{F}) \times GL_n(\mathbb{F})$ operates by the character $\chi_1 \times \chi_2$ is the sum of dimensions arising from these two options; the first option clearly gives $\dim \leq 1$, and as we will see in Lemma 4.1, the second option also gives $\dim \leq 1$, therefore the dimension of the space of vectors in $\pi_1 \times \pi_2$ on which $GL_n(\mathbb{F}) \times GL_n(\mathbb{F})$ operates by the character $\chi_1 \times \chi_2$ is $\leq 2$. (It may be observed that both the conditions in (1) and (2) are invariant under independent exchange of $\chi_1$ with $\chi_2$, and of $\pi_1$ with $\pi_2$, which is certainly a necessary condition!)

**Proof.** We will calculate the dimension of the space of vector in $\pi_1 \times \pi_2$ on which $H = GL_n(\mathbb{F}) \times GL_n(\mathbb{F})$ operates by the character $\chi_1 \times \chi_2$ by a direct application of the Mackey theory. Recall that Mackey theory gives an answer using the double coset decomposition

$$H \backslash GL_{2n}(\mathbb{F}) / P$$

where $P$ is the $(n, n)$ parabolic in $GL_{2n}(\mathbb{F})$.

Suppose $V = V_1 \oplus V_2$ is the decomposition of a $2n$-dimensional vector space over $\mathbb{F}$ as a direct sum of two $n$-dimensional subspaces, which realizes $H = GL_n(\mathbb{F}) \times GL_n(\mathbb{F})$ as $H = GL(V_1) \times GL(V_2)$. Let $W$ be an $n$-dimensional subspace of $V$ whose stabilizer $P = P(W)$ defines a parabolic subgroup of $GL(V)$.

Clearly, the double cosets in $H \backslash GL(V) / P(W)$ are in bijective correspondence with $GL(V)$-conjugacy classes of triples $(V_1, V_2, W)$ of subspaces of dimension $n$ in $V$ with $V_1 + V_2 = V$. From this it is easy to see that the double cosets in $H \backslash GL(V) / P$ are parametrized by pairs of integers $(r, s)$ with $0 \leq r, s \leq n$ with the only constraint that $r + s \leq n$. The pair $(r, s)$ corresponds to the pair $(\dim(W \cap V_1), \dim(W \cap V_2))$.

To make a detailed calculation, let $V_1, V_2, W, W'$ be subspaces, each of dimension $n$, of a vector space $V$ of dimension $2n$, with the following basis vectors for the subspaces $V_1, V_2, W, W'$ for integers $r \geq 0, s \geq 0, t \geq 0$ with $r + s + t = n$, and $W \oplus W' = V$:

$$V_1 = \{e_1, e_2, \ldots, e_r; g_1, \ldots, g_t; v_1, \ldots, v_s\},$$
$$V_2 = \{f_1, f_2, \ldots, f_s; h_1, \ldots, h_t; w_1, \ldots, w_r\},$$
$$W = \{e_1, \ldots, e_r; f_1, \ldots, f_s; g_1 + h_1, \ldots, g_t + h_t\},$$
$$W' = \{g_1, \ldots, g_t; v_1, \ldots, v_s; w_1, \ldots, w_r\}.$$

We have,

$$\dim(V_1 \cap W) = r,$$
$$\dim(V_2 \cap W) = s,$$

which, since $V_1 + V_2 = V$, implies that,

$$\dim(V_1 \cap [V_2 + W]) = n - s = r + t,$$
$$\dim(V_2 \cap [V_1 + W]) = n - r = s + t.$$

To apply the Mackey theory, we need to calculate $A = [GL(V_1) \times GL(V_2)] \cap P(W)$, and its projection $B$ to $P(W)/U(W)$ where $U(W)$ is the unipotent radical of $P(W)$, so that this double coset gives the representation,
\[ \text{Ind}_{A}^{(\text{GL}(V_1) \times \text{GL}(V_2))} \rho, \]

where \( \rho \) is the representation of the subgroup \( A \) of \( \text{GL}(V_1) \times \text{GL}(V_2) \) operating through the restriction to \( B \) of the representation \( \pi_1 \times \pi_2 \) of \( P(W)/U(W) = \text{GL}(W) \times \text{GL}(W') \).

We now calculate \( A, B \) with

\[ A = [\text{GL}(V_1) \times \text{GL}(V_2)] \cap P(W) = \{ g \in \text{GL}(V)|g(V_1) = V_1, g(V_2) = V_2, g(W) = W \}. \]

Note that an element \( g \in A \) leaves \( (V_1 \cap W), (V_2 \cap W), (V_1 \cap [V_2+W]), (V_2 \cap [V_1+W]) \) invariant.

To understand the subgroup \( B \), we will write an element \( g \in A \subset \text{GL}(V) \) in the basis of \( V = W \oplus W' \) afforded by concatenation of the basis for \( W, W' \) which we recall has the following basis,

\[
\begin{align*}
W &= \{e_1, \cdots, e_r; f_1, \cdots, f_s; g_1 + h_1, \cdots, g_t + h_t\}, \\
W' &= \{g_1, \cdots, g_t; v_1, \cdots, v_s; w_1, \cdots, w_r\},
\end{align*}
\]

in the form:

\[
g = \begin{pmatrix}
* & 0 & B & B & * & 0 \\
0 & * & * & 0 & 0 & * \\
0 & 0 & A & 0 & 0 & -C \\
0 & 0 & 0 & A & * & C \\
0 & 0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & 0 & 0 & * 
\end{pmatrix} \in \text{GL}(V),
\]

where each entry corresponds to a block matrix, for example, the entry at place \((1,1)\) corresponds to the endomorphism of the subspace \( V_1 \cap W = \{e_1, \cdots, e_r\} \); all the entries denoted by \( A, B, C, * \) are arbitrary matrices of appropriate sizes.

It follows that in \( P(W)/U(W) = \text{GL}(W) \times \text{GL}(W') \), \( g \) looks like,

\[
g = \begin{pmatrix}
* & 0 & B \\
0 & * & * \\
0 & 0 & A \\
A & * & C \\
0 & * & 0 \\
0 & 0 & * 
\end{pmatrix} \in \text{GL}(W) \times \text{GL}(W'),
\]

and once again, all the nonzero entries in the matrix are arbitrary (and invertible if necessary).

If \( 0 < r + s < n \), the above subgroup of matrices contains the unipotent subgroup of a nontrivial parabolic in \( \text{GL}(W) \times \text{GL}(W') \). Therefore since we are dealing with parabolic induction arising from a cuspidal data, the double cosets represented by \((r, s)\) with \( 0 < r + s < n \), do not carry any vector left invariant (up to a character) by the subgroup \( H = \text{GL}_n(F) \times \text{GL}_n(F) \).

If \( r = 0, s = 0 \), then the matrix \( g \) above simplifies to,

\[
g = \begin{pmatrix}
\alpha & 0 \\
0 & \alpha 
\end{pmatrix} \in \text{GL}(W) \times \text{GL}(W'),
\]
where $\alpha$ represent an arbitrary matrix in $\text{GL}_n(F)$. In this case, $\text{Ind}_A^{\text{GL}(V_1) \times \text{GL}(V_2)}(\pi_1 \otimes \pi_2)|_A$, has a nonzero vector on which $\text{GL}(V_1) \times \text{GL}(V_2)$ acts by the character $\chi_1 \times \chi_2$ if and only if

$$(\pi_1 \otimes \chi_1^{-1})^\vee \cong \pi_2 \otimes \chi_2^{-1}.$$ 

If $r + s = n$, the matrix $g$ above simplifies to,

$$g = \begin{pmatrix}
*_{r} & 0 \\
0 & *_{s} \\
*s & 0 \\
0 & *_{r}
\end{pmatrix} \in \text{GL}(W) \times \text{GL}(W'),$$

where $*_{r}$, respectively $*_{s}$, represent an arbitrary matrix in $\text{GL}_r(F)$, respectively in $\text{GL}_s(F)$.

Appealing now to the well-known result that a cuspidal representation $\pi$ of $\text{GL}_n(F)$ has a nonzero vector on which the Levi subgroup $\text{GL}_r(F) \times \text{GL}_s(F)$ for $r + s = n$ acts by a character only if $r = s$, in particular $n$ must be even, see for example Proposition 6.10 of [Se], the proof of the proposition is completed after having observed that a block diagonal matrix

$$g = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \in H = \text{GL}_n(F) \times \text{GL}_n(F),$$

(where each of the matrices $A, B, C, D$ are of size $n/2$), when considered as an element of $\text{GL}(W) \times \text{GL}(W')$, looks like:

$$g = \begin{pmatrix}
A & C \\
B & D
\end{pmatrix}.$$

\[\square\]

4. APPLICATION OF BASECHANGE TO LINEAR PERIODS

In this section we use the method of basechange developed in section 2 to study multiplicity questions in conjunction with the result obtained for principal series in the last section, to derive such a result for cuspidal representations.

Lemma 4.1. Let $\pi$ be an irreducible cuspidal representation of $\text{GL}_{2n}(F), n > 1$ and $\chi_1, \chi_2$ be two characters of $F^\times$. Then the space of vectors in $\pi$ on which $\text{GL}_n(F) \times \text{GL}_n(F)$ operates by the character $\chi_1 \times \chi_2$ is of dimension $\leq 1$.

Proof. We will prove the lemma separately for $n$ odd and $n$ even.

Case 1: $n$ odd. By Proposition 3.1 applied to the basechanged representation $\pi^E$, since case (2) in the assertion of the Proposition 3.1 does not arise, we find that

$$m(\pi^E, \chi_1^E \times \chi_2^E) \leq 1,$$
and therefore by Corollary 2.3,
\[ m(\pi, \chi_1 \times \chi_2) \leq 1. \]

**Case 2:** \( n \) even. In this case, as we will notice in Proposition 4.3 applied to \( \pi^E \), case (1) in the assertion of the Proposition 3.1 does not arise. Hence \( m(\pi^E, \chi_1^E \times \chi_2^E) \) arises only through a contribution coming from case (2) in the assertion of the Proposition 3.1, which we can assume is \( \leq 1 \), by an inductive argument on \( n \), therefore
\[ m(\pi^E, \chi_1^E \times \chi_2^E) \leq 1, \]
and therefore by Corollary 2.3,
\[ m(\pi, \chi_1 \times \chi_2) \leq 1. \]

The proof of the lemma is therefore completed. \( \square \)

Exactly the same proof gives a proof of the following Lemma.

**Lemma 4.2.** Let \( \pi \) be an irreducible cuspidal representation of \( \text{GL}_{2n}(\mathbb{F}) \), \( n > 1 \) and \( \chi \) a character of \( \mathbb{E}^\times \). Then the space of vectors in \( \pi \) on which \( \text{GL}_n(\mathbb{E}) \) operates by the character \( \chi \) is of dimension \( \leq 1 \).

The following proposition can be considered as a contribution towards depth-zero case of Conjecture 1 in [PT].

**Proposition 4.3.** Let \( \pi = \pi(\theta) \) be an irreducible cuspidal representation of \( G = \text{GL}_{2n}(\mathbb{F}) \), \( n > 1 \). Assume that \( \pi \) arises from a character \( \theta : \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times \). Let \( \chi : \mathbb{E}^\times \to \mathbb{C}^\times \), a character, thought of as a character of \( H = \text{GL}_n(\mathbb{E}) \) through the determinant map \( \text{det} : \text{GL}_n(\mathbb{E}) \to \mathbb{C}^\times \). Then the representation \( \pi \) of \( \text{GL}_{2n}(\mathbb{F}) \) has a nonzero vector on which \( \text{GL}_n(\mathbb{E}) \) operates by \( \chi \) if and only if \( \theta \) restricted to \( \mathbb{E}^\times \) arises from \( \chi \) restricted to \( \mathbb{F}^\times \) through the norm mapping \( \mathbb{F}_{q^n}^\times \to \mathbb{F}^\times \). In particular the condition that the character \( \chi \circ \text{det} \) of \( \text{GL}_n(\mathbb{E}) \), for \( \chi : \mathbb{E}^\times \to \mathbb{C}^\times \), appears in \( \pi \) depends only on \( \chi \) restricted to \( \mathbb{F}^\times \).

The dimension of the space of linear forms when nonzero is 1.

**Proof:** The proof of the proposition will be based on Proposition 2.1 relating multiplicities under basechange, and the calculation done in Proposition 3.1 regarding the multiplicity for a principal series representation. By Proposition 2.1,
\[
2m(\bar{\pi}^E, \bar{\chi}^E) = m(\pi^E, \chi^E) + m(\pi, \chi)
\]
where \( \chi : \mathbb{E}^\times \to \mathbb{C}^\times \), and \( \chi^E = \chi \times \chi^\sigma : \mathbb{E}^\times \times \mathbb{E}^\times \to \mathbb{C}^\times \), and we know from Lemma 4.1 that \( m(\pi^E, \chi^E) \leq 1 \), thus the only option we have at our disposal is that \( m(\pi^E, \chi^E) = m(\pi, \chi) \). It suffices to calculate when \( m(\pi^E, \chi^E) = 1 \), which is what we do in the rest of the proof.

Suppose that the basechange \( \pi^E \) of \( \pi \) to \( \mathbb{E} \) is \( \pi^E = \pi_1 \times \pi_1^\sigma \) for \( \pi_1 \) a cuspidal representation of \( \text{GL}_n(\mathbb{E}) \), where \( \sigma \) is the Galois action on \( \text{GL}_n(\mathbb{E}) \). If \( \pi \) arises from a character \( \theta : \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times \), then considering the field extension \( \mathbb{F}_{q^n}/\mathbb{E} \), the character \( \theta : \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times \) also gives rise to a cuspidal representation \( \pi_1 = \pi_1(\theta) \) of \( \text{GL}_n(\mathbb{E}) \), defining the representation \( \pi_1 \).

We analyze the condition appearing in Proposition 3.1:
\[
(\pi_1 \otimes \chi^{-1})^\sigma \cong (\pi_1 \otimes \chi^{-1})^\sigma,
\]

(1)
using the fact that cuspidal representations \( \pi_1(\theta_1) \) and \( \pi_2(\theta_2) \) of \( \text{GL}_n(E) \) arising out of characters \( \theta_1, \theta_2 : \mathbb{F}_{q^{2n}}^\times \to \mathbb{C}^\times \) are isomorphic if and only if for some \( \tau \in \text{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{F}) \),
\( \tau(\theta_1) = \theta_2 \). We have been using \( \sigma \) for the nontrivial element of \( \text{Gal}(E/\mathbb{F}) \), but now we will also use \( \sigma \) to denote any element of \( \text{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{F}) \) which projects to this nontrivial element of \( \text{Gal}(E/\mathbb{F}) \). The isomorphism in (1) implies that for some \( \tau \in \text{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{E}) \):

\[
(\theta \cdot \theta^\tau) = (\chi \cdot \chi^\sigma)_{\mathbb{F}_{q^{2n}}},
\]

where \( \chi_{\mathbb{F}_{q^{2n}}} \) denotes the character of \( \mathbb{F}_{q^{2n}}^\times \) arising from the character \( \chi : \mathbb{E}^\times \to \mathbb{C}^\times \) via the norm mapping \( \mathbb{F}_{q^{2n}}^\times \to \mathbb{E}^\times \).

Applying \( \tau \sigma \) to this equality, we have:

\[
(\theta^\tau \cdot \theta^{\tau^2\sigma^2}) = (\chi \cdot \chi^\sigma)_{\mathbb{F}_{q^{2n}}},
\]

therefore these two equations whose right hand sides are the same, give:

\[
\theta = \theta^{\tau^2\sigma^2}.
\]

Since \( \theta \) gives rise to a cuspidal representation of \( \text{GL}_n(E) \), all its Galois conjugate under \( \text{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{E}) \), are distinct, and therefore equation (4) implies that,

\[
(\tau \sigma)^2 = 1,
\]

and therefore, \( n \) must be an odd integer.

Conversely, if \( n \) is odd, we can then take \( \sigma \) to be the unique element of \( \text{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{F}) \) of order 2 (which automatically projects to the nontrivial element of \( \text{Gal}(\mathbb{E}/\mathbb{F}) \)) whose fixed field defines \( \mathbb{F}_{q^n} \). Since \( n \) is odd, \( \tau \) has odd order, and therefore by equation (4), \( \tau = 1 \). In this case, putting \( \tau = 1 \) in the equality in equation (2), we find \( (\theta \cdot \theta^\sigma) = (\chi \cdot \chi^\sigma)_{\mathbb{F}_{q^{2n}}} \), which means that \( \theta \) restricted to \( \mathbb{F}_{q^n}^\times \), arises from \( \chi \) restricted to \( \mathbb{F}^\times \) through the norm mapping \( \mathbb{F}^\times_{q^n} \to \mathbb{F}^\times \), allowing us to complete the proof of the proposition for \( n \) odd.

If \( n \) is even, we have just proved that we cannot have an isomorphism:

\[
(\pi_1 \otimes \chi^{-1})^\vee \not\cong (\pi_1 \otimes \chi^{-1})^\sigma,
\]

and therefore by Proposition 3.1, for the representation \( \pi^E = \pi_1 \times \pi_1^\sigma \) to contain a nonzero vector on which \( \text{GL}_n(E) \times \text{GL}_n(E) \) operates by \( \chi \times \chi^\sigma \), the representation \( \pi_1 \) of \( \text{GL}_n(E) \) must contain a nonzero vector on which \( \text{GL}_{n/2}(E) \times \text{GL}_{n/2}(E) \) operates by \( \chi \times \chi^\sigma \).

Let \( E_2 \) be the quadratic extension of \( E \), and \( \pi_1^{E_2} \) the basechange of the cuspidal representation \( \pi_1 \) of \( \text{GL}_n(E) \) to \( \text{GL}_n(E_2) \). By the multiplicity one result in Lemma 4.1, the representation \( \pi_1 \) of \( \text{GL}_n(E) \) has a nonzero vector on which \( \text{GL}_{n/2}(E) \times \text{GL}_{n/2}(E) \) operates by \( \chi \times \chi^\sigma \) if and only if \( \pi_1^{E_2} \), a representation of \( \text{GL}_n(E_2) \) has a nonzero vector on which \( \text{GL}_{n/2}(E_2) \times \text{GL}_{n/2}(E_2) \) operates by \( (\chi \times \chi^\sigma)^{E_2} \). We can apply Proposition 3.1 to analyze this. If \( n/2 \) is odd, the proof of the proposition is completed, else we continue, and are done by descending induction.

\textbf{Corollary 4.4.} An irreducible cuspidal representation \( \pi = \pi(\theta) \) of \( \text{GL}_{2n}(\mathbb{F}) \) arising from a character \( \theta : \mathbb{F}_{q^{2n}}^\times \to \mathbb{C}^\times \) is distinguished by \( \text{GL}_n(E) \) (i.e., has a vector fixed by \( \text{GL}_n(E) \)) if and only if one of the equivalent conditions hold:

1. \( \theta \) restricted to \( \mathbb{F}^\times_{q^n} \) is trivial,
2. the representation \( \pi \) of \( G = \text{GL}_{2n}(\mathbb{F}) \) is self-dual.
Further, for a character \( \chi : E^\times \to \mathbb{C}^\times \), an irreducible cuspidal representation \( \pi \) of \( \text{GL}_{2n}(F) \) contains the character \( \chi \circ \det : \text{GL}_n(E) \to \mathbb{C}^\times \) if and only if,

\[
\pi \cong \pi^\vee \otimes \chi|_{E^\times}.
\]

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