A Spin Gauge Formulation of Gravity and a New View of Gravity-Matter Interactions

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A first-order formulation of gravity is developed in which the fundamental fields consist of an $SL(2, \mathbb{C})$ connection and two spinor-valued 1-forms. It is shown that the first term of an expansion of the Einstein-Hilbert action leads to an action for these fields which consists of dynamic $L^2$ inner products of their covariant derivatives, resembling reasonable generalisations of the terms found in the actions of typical gauge theories on Minkowski spacetime. If additional terms corresponding to other forces and matter, formulated in the same manner, are then included, this approach may shed new light on interactions of gravity with matter and other force carriers.

INTRODUCTION

Almost all actions of today’s standard models, namely those of Yang-Mills, Dirac, and Klein-Gordon form, can be written in terms of $L^2$ inner products of fields that are considered fundamental and their covariant derivatives with respect to some gauge group. Furthermore, nearly all continuous symmetries demonstrated by physical laws have been built into these actions as local gauge symmetries. Notable exceptions of the above trends, which this letter aims to address, are the standard actions for general relativity, the Einstein-Hilbert action and its variants, and Lorentz symmetry. One of the first steps towards a unified theory of everything may be to come up with an action for the fundamental constituents of forces and matter in which all of the terms have the same form; that of an inner product which is gauge invariant with respect to all fundamental continuous symmetries.

Attempts to formulate gravity as a typical gauge theory, using, say, a Yang-Mills kinetic term for a ‘gravitational connection’, have failed because gravity seems to be universal to a greater extent than the other forces of nature. For instance, even light, which is typically modeled using an electromagnetic potential with zero gravitational charge (mass), seems to be deflected in the presence of a strong gravitational field. It is perhaps of note, however, that recently an action for general relativity which resembles the square root of a Yang-Mills term has been developed [1, 2]. In contrast to that approach, in this letter it is illustrated how gravity may be formulated using a traditional Yang-Mills term together with an additional kinetic-type term that is intimately tied to the geometry of spacetime. This additional term is what distinguishes the theory from gauge theories which may be modeled on a static spacetime, like those for the other forces.

The aim of this letter is to address the above concerns by demonstrating a gauge theoretic action which portrays a dynamical theory for fields which may be used to describe gravity. These fields consist of a connection for the spin group $SL(2, \mathbb{C})$ and two spinor-valued fields which, together, represent an orthonormal spacetime frame. The universality of gravity is explained by the property that the main effect of gravity on matter and force carriers is due to universal interactions with these spinor-valued fields, which are acted upon, in turn, by the connection. It is due to this that the $SL(2, \mathbb{C})$ connection, from which the curvature of spacetime may be determined under certain conditions, may be interpreted as a gravitational potential which causes deformations of the orthonormal frame fields mentioned above. Another useful property of the theory is the way in which metrics with degeneracies, which have been of great interest in recent work on quantum gravity, may arise naturally from the relevant field equations.

THE DYNAMICAL FIELDS

Let $X$ be a smooth spacetime 4-manifold. Weyl spinor fields over $X$ may be described by rules which assign to each event $x \in X$ and spin-frame [3] over $x$ a 2-dimensional complex vector. These spinor fields are called left- (undotted) or right-handed (dotted), if they transform under the standard or conjugate representation of $SL(2, \mathbb{C})$, respectively. Components of left-handed spinor fields with respect to the standard basis $\{e_A\}_{A=1,2}$ of $\mathbb{C}^2$ will be denoted using uppercase indices from the Latin alphabet, like $\alpha^A$, while those of right-handed fields will, in addition, contain an apostrophe, like $\beta^A$. Note that, in this construction, $X$ is not required to admit a spin structure [3] a priori, or a Lorentzian metric for that matter, and so it is not a postulate of the theory that the relevant topological obstructions vanish. Solutions to the field equations for the dynamical variables may, however, imply configurations of spacetime for which all these conditions are met if they result in nondegenerate tetrad fields, which can then be used to define an appropriate soldering form [4].

In the remainder of this letter, only local aspects will be discussed and so references to the theory of bundles
and spin structures necessary for a thorough global analysis will be omitted. Furthermore, only the simpler local expressions of mathematical constructions such as connections on these bundles will be taken into account and all fields on X will be assumed to have compact support.

The dynamical variables of the formulation of gravity to be described here are an SL(2, C) connection, Γ, and two left-handed spinor-valued 1-forms, φ and χ. Given local coordinates ϕ on X, these spinor valued 1-forms take the form ϕμ eA ⊗ dϕμ and χμ eA ⊗ dϕμ.

The connection Γ induces covariant exterior derivatives dΓ which take spinor valued p-forms to spinor valued (p + 1)-forms. The covariant exterior derivative of a left-handed spinor valued p-form α may be written as

\[ d_Γ α = dα - ik ω ∧_p α, \]

where k is the coupling constant and ω = ω^Γ TΓ, for some 1-forms ω^Γ and generators TΓ of SL(2, C) which form a basis for the Lorentz algebra \( \mathfrak{s}(2, C) \cong \mathfrak{o}(1, 3) \), and ∧_p indicates wedge multiplication and the action of \( \mathfrak{s}(2, C) \) on \( C^2 \) via the standard representation ρ. In components, we have

\[ (d_Γ α)^A_{\mu\nu\ldots} = \partial_\mu (α^A_{\nu\ldots}) - ik (ω^A_B)_{[μ} α^B_{ν\ldots]}, \]

where \( (ω^A_B)_{\nu} = \omega^A_B TΓ \) is given by the standard representation ρ. The curvature 2-form Ω of Γ is given by

\[ -\frac{1}{ik} Ω = dω - ik ω ∧_Adω = dω + \frac{k}{2} ω^A ∧ ω^B C^Γ_{AB} TΓ, \]

where ∧_Ad denotes a combination of wedge product and action via the adjoint representation Ad and \( C^Γ_{AB} \) denote the structure constants for the Lorentz algebra \( \mathfrak{s}(2, C) \cong \mathfrak{o}(1, 3) \).

\[ [T_A, T_B] = i C^Γ_{AB} TΓ. \]

In components, we have

\[ \Omega = \frac{1}{2} Ω^Γ_{μν} TΓ dϕ^μ ∧ dϕ^ν = (Ω^A_B)_{μν} dϕ^μ ∧ dϕ^ν, \]

where \( (Ω^A_B)_{μν} = \frac{1}{2} Ω^Γ_{μν} TΓ \) is, again, given by the standard representation ρ (the factor of \( \frac{1}{2} \) is included to match standard gauge theory and relativistic conventions).

\[ -\frac{1}{ik} (Ω^A_B)_{μν} = \partial_μ (ω^A_B)_{ν} - ik (ω^A_C)_{[μ} (ω^C_B)_{ν]}, \]

or

\[ -\frac{1}{ik} Ω^Γ_{μν} = \partial_μ ω^Γ_ν - \partial_ν ω^Γ_μ + k C^Γ_{AB} ω^A_μ ω^B_ν. \]

**INDUCED FIELDS**

Conjugation, denoted by an overbar, on the space of Weyl spinors takes left-handed fields to right-handed ones and vice versa. It additionally induces a connection \( \overline{∇} \) and corresponding covariant exterior derivative which acts on right-handed spinor-valued forms. Conjugation also acts on the space V of tensor products of left- and right-handed spinors and defines an invariant subspace V_R of elements which are called real. Given a spin-frame \( \{ l, o \} \) with \( ε_{AB}^l A^B \), where \( ε \) is the antisymmetric bilinear form on \( C^2 \), it is possible to define a basis for V, given by

\[ \begin{align*}
 v_0^{AA'} &= \frac{1}{√2} (o^A σ^{AA'} + l^A T^{AA'}), \\
v_1^{AA'} &= \frac{1}{√2} (o^A σ^{AA'} + l^A T^{AA'}), \\
v_2^{AA'} &= \frac{1}{√2} (o^A T^{AA'} - l^A ε^{AA'}), \\
v_3^{AA'} &= \frac{1}{√2} (o^A T^{AA'} - l^A T^{AA'}),
\end{align*} \]

which allows one to switch between pairs of spinor indices AA' and Minkowski indices a (denoted by lowercase letters from the Latin alphabet) and which is orthonormal with respect to the Minkowski inner product

\[ η_{ab} = ε_{AB} ε_{A'B'} ε^{AA'} ε^{AB}. \]

The proper orthochronous Lorentz group \( SO(1, 3) \) (of which \( SL(2, C) \) is a double cover), or rather its complexification, is represented on V by the tensor product of the standard and conjugate representations of SL(2, C). Hence the connection ∇ and its conjugate induce a connection which acts on fields with values in V, usually referred to as the spin connection, with covariant exterior derivative \( D = d + A \) which acts on V valued forms, where \( A \) can be thought of as a 1-form with values in \( \mathfrak{o}(1, 3) \). The curvature 2-form of this connection is given, in terms of the connection \( (A^a)_μ \), by

\[ (F^a_μ)_{νρ} = \partial_μ (A^a_ν)_{ρ} + (A^a_ρ)_{[μ} (A^a_ν)]_{ρ}. \]

One such V valued form is the tetrad 1-form θ defined in terms of \( ϕ \) and \( χ \), in a similar fashion to that in [5, 6], by

\[ θ = ϕ^A e_A ⊗ e_V ⊗ dϕ^μ + χ^A e_A ⊗ e_V ⊗ dϕ^μ, \]

i.e., \( θ^AA' = ϕ^A \) and \( θ^AA' = χ^A \). The covariant exterior derivative of θ,

\[ Θ^a_μ : = (Dθ)^a_μ = \partial_μ (θ^a_ν) + (A^a_ρ)_{[μ} (θ^a_ρ)]_{ν}, \]

is a 2-form with values in V called the torsion 2-form. It is said that θ is nondegenerate at an event \( x \in X \) if

\[ \theta \frac{∂}{∂ϕ^μ} \theta \frac{∂}{∂ϕ^ρ} \theta \frac{∂}{∂ϕ^σ} \theta \frac{∂}{∂ϕ^σ} \]

are linearly independent in V. When θ is also real, i.e., \( \overline{θ} = θ \), it provides a linear isomorphism between V and the tangent space at \( x \), and becomes what is known as a soldering form.
The complex metric $g$ on $X$ is then defined, via the tetrad 1-form, by

$$g_{\mu\nu} = \eta_{ab}\theta^a_{\mu}\theta^b_{\nu} = \varepsilon_{AB}(\varphi^A_{\mu}\lambda^B_{\nu} - \chi^A_{\mu}\varphi^B_{\nu}).$$

When $\varphi$ and $\chi$ are such that $\theta$ is real and nondegenerate, then $g$ is also real and nondegenerate and has Lorentz signature $(+---)$. The determinant of $g$ is a perfect square due to its definition by the tetrad fields $\theta$ which are linear in $\varphi$ and $\chi$,

$$\det g = \left(\frac{1}{i}\varepsilon^\mu{}_{\nu\alpha\beta} \varepsilon_{ABCD} \varphi^A_{\mu}\varphi^B_{\nu} \lambda^C_{\alpha}\lambda^D_{\beta}\right)^2$$

and hence

$$\sqrt{-\det g} = \frac{i}{4}\varepsilon^\mu{}_{\nu\alpha\beta} \varepsilon_{ABCD} \varphi^A_{\mu}\varphi^B_{\nu} \lambda^C_{\alpha}\lambda^D_{\beta},$$

where $\epsilon$ is the totally antisymmetric Levi-Civita symbol. If a rank two contravariant tensor $f$ is defined by contracting the product of three $\varphi$'s and three $\chi$'s with antisymmetric symbols like

$$f^{\mu\nu} = \varepsilon^{\mu\nu\beta\gamma} \varepsilon_{ABCD} \varphi^A_{\mu}\varphi^B_{\nu} \lambda^C_{\beta}\lambda^D_{\gamma},$$

then the adjugate of the metric is given by its symmetrization and the inverse metric takes the form

$$g^\mu\nu = \frac{1}{\det g} f^{(\mu\nu)} = \frac{1}{2\det g} (f^{\mu\nu} + f^{\nu\mu}).$$

The inverse tetrads are then given by

$$\theta^a_{\mu} = \theta^a_{\nu} \eta_{ab} g^{\mu\nu},$$

and together with the tetrads can be used to mix Minkowski and spacetime indices.

**FURTHER CONSTRUCTIONS**

The metric $g$ can now be used to define a dynamic Hodge star operator $\star$ which takes $p$-forms to $(4-p)$-forms by contraction with the Levi-Civita tensor $\sqrt{-\det g} \epsilon_{\alpha\beta\gamma\delta}$ using the inverse metric. For instance, the Hodge dual of a scalar valued $p$-form $\alpha$ is given by

$$(\star \alpha)^A_{\nu_1...\nu_p} = \sqrt{-\det g} \epsilon^{\nu_1...\nu_p\mu_1...\mu_q} g^{i_1j_1}...g^{i_qj_q} \alpha^A_{j_1...j_q}.$$

This Hodge star then allows one to define a (possibly degenerate) volume form

$$\star(1) = \sqrt{-\det g} \, d\phi^0 \wedge d\phi^1 \wedge d\phi^2 \wedge d\phi^3,$$

and pointwise inner products $\langle \cdot, \cdot \rangle_{\varphi\chi}$ on the spaces of $p$-forms over $X$ defined by

$$\langle \alpha, \beta \rangle_{\varphi\chi} = (-1)^{p(n-1)+1} \star \left( \alpha \wedge \star \beta \right),$$

which may easily be generalised to pointwise inner products of $p$-forms that are, say, spinor or $V$ valued, by contracting any additional indices with the appropriate bilinear form, such as $\varepsilon, \eta$, or a Killing form. (The subscript $\varphi\chi$ is used to emphasise dependence on $\varphi$ and $\chi$.) When $\varphi$ and $\chi$ are such that $g$ is real and nondegenerate, then $\star \star = (-1)^{p(n-1)} + 1$ and so

$$\star \langle \alpha, \beta \rangle_{\varphi\chi} = \langle \alpha, \beta \rangle_{\varphi\chi} \star (1) = \alpha \wedge \star \beta,$$

or in components,

$$\langle \alpha, \beta \rangle_{\varphi\chi} = \int_X \langle \alpha, \beta \rangle_{\varphi\chi} \star (1).$$

As the above constructions are made up solely of objects such as $g$ and its inverse, which are by definition invariant under local Lorentz transformations, if the forms $\alpha$ and $\beta$ are $SL(2, \mathbb{C})$ gauge covariant, then $\langle \alpha, \beta \rangle_{\varphi\chi}$ is Lorentz invariant.

**THE ACTION**

Now an $SL(2, \mathbb{C})$ gauge invariant action for the fields $\varphi, \chi$ and $\omega$ will be extracted from the Einstein-Hilbert action for the metric $g = g(\varphi, \chi)$ by rewriting it in terms of a different quantity, which is also determined by $\varphi$ and $\chi$; expanding the result as a sum; taking only the simplest term which exhibits local degrees of freedom; and finally restoring Lorentz covariance.

Suppose that $g = g(\varphi, \chi)$ is real and nondegenerate. Firstly, in terms of $A = A(\omega)$, the curvature scalar $R = R(\omega)$ is given by

$$R = \eta^{ab} F^a_{bd} F^{bd}.$$

where the curvature of $A$ with Minkowski indices is given by

$$F^a_{bd} = \partial_d^a \theta^b_d (F^a)_{\mu\nu}.$$

Now, if $\tilde{D} = d + \tilde{\Gamma}$ is covariant exterior derivative corresponding to the Levi-Civita connection defined by $g = g(\varphi, \chi)$, given by

$$\tilde{\Gamma}^p_{\sigma\mu} = \frac{1}{2} \varepsilon^{\rho\Lambda} (\partial_\rho g_{\Lambda\sigma} + \partial_\sigma g_{\Lambda\mu} - \partial_\Lambda g_{\sigma\mu}),$$

then the spin connection form $\tilde{A} = \tilde{A}(\varphi, \chi)$ corresponding to $\tilde{\Gamma} = \tilde{\Gamma}(\varphi, \chi)$ is given by [7]

$$\left( \tilde{A}^p_{\mu} \right)_\rho = \theta^p_{\rho} \tilde{D}_\mu \theta^\rho_a + \theta^a_{\rho} \tilde{\Gamma}^p_{\sigma\mu} \theta^\sigma_b.$$


Next, the torsion scalar \( T = T(\omega, \varphi, \chi) \) corresponding to \( A = A(\omega) \) and \( \theta = \theta(\varphi, \chi) \) is given by

\[
T = (\Sigma, \Theta)_{\varphi \chi} = \frac{1}{2} \eta_{\alpha \beta} g^{\mu \rho} g^{\nu \sigma} \Sigma^\alpha_{\mu \nu} \Theta^\beta_{\rho \sigma},
\]

where

\[
\Sigma^\alpha_{\mu \nu} = \frac{1}{2} \Theta^a_{\mu \nu} - \frac{1}{2} \eta^{ac} \eta_{bd} \left( \Theta^b_{\mu \rho} \Theta^d_{\nu \sigma} - \Theta^b_{\nu \rho} \Theta^d_{\mu \sigma} \right) + \theta^b_{\nu} \delta^a_{\mu} \Theta^b_{\rho \lambda} - \theta^b_{\nu} \Theta^a_{\mu} \delta^b_{\rho \lambda}.
\]

The relationship between all of these objects was shown [7] to be

\[
R = \tilde{R} + 2 \tilde{D}_\mu \left( g^{\mu \rho} \theta^a_{\nu} \Theta^b_{\rho \sigma} \right),
\]

where \( \tilde{R} = \tilde{R}(\varphi, \chi) \) is the curvature scalar corresponding to \( \tilde{A} = \tilde{A}(\varphi, \chi) \).

In teleparallel formulations of gravity, it is required [7] there exist a class of frames where \( A = 0 \) (i.e., \( D = d \)), so that the curvature \( F \) and hence the curvature scalar \( R \) corresponding to \( A \) vanish. The standard action for teleparallel gravity is then (in natural units with \( c = 1 \))

\[
\frac{1}{16\pi G} \int_X T \, (1),
\]

which by (1) is equal to the Einstein-Hilbert action modulo a boundary term, which does not affect the equations of motion. This demonstrates local equivalence between general relativity and teleparallel gravity. Now, given that

\[
T = \frac{1}{4} (\Theta, \Theta)_{\varphi \chi} + \left( \text{more complicated terms involving inverse tetrads} \right),
\]

retaining only the first, simplest, term and switching back to spinor indices then results in the expression

\[
\frac{1}{64\pi G} \int_X \varepsilon_{AB} \varepsilon_{A'B'} \left( D\theta \right)^{AA'} \wedge (D\theta)^{BB'}.
\]

Finally, if it is taken into account that, in the teleparallel case, \( D = d \), one obtains

\[
-\frac{1}{32\pi G} (d\Omega, d\varphi)_{\varphi \chi}.
\]

This expression is not Lorentz invariant. However, replacing the exterior derivatives with their covariant counterparts, which is equivalent to lifting the teleparallel condition; freeing \( \varphi \) and \( \chi \) from the condition that \( g = g(\varphi, \chi) \) is real and nondegenerate; and including a kinetic term for \( \omega \) results in an \( SL(2, \mathbb{C}) \) gauge invariant action,

\[
-\frac{1}{32\pi G} (d\varphi, d\varphi)_{\varphi \chi} - \frac{1}{32\pi G k^2} (\Omega, \Omega)_{\varphi \chi}, \quad (3)
\]

where \( k \) is the coupling constant of \( \nabla \). This action is also naturally diffeomorphism invariant, being an integral of a 4-form over \( X \), and behaves nicely even when the metric defined by \( \varphi \) and \( \chi \) is not everywhere nondegenerate.

Also note that, unlike giving up the teleparallel condition in \( f(T) \) theories, which was shown to result in inconsistencies upon the addition of matter [7], the action principle (3) describes a theory with physically sensible local degrees of freedom. This is because the equations of motion must involve derivatives of the fields \( \varphi \) and \( \chi \), whereas the reasoning in [7] depended on the fact that the relevant equations could be written in a way that did not involve derivatives of the fundamental fields.

### The IMPLIED INTERACTIONS

One possible advantage of using \( L^2 \) inner products as in (3) is that interactions with the frame field defined by \( \varphi \) and \( \chi \) are encapsulated in a way that exhibits the way that spacetime interacts with fields on it. For instance, given an additional gauge potential \( B_\mu \), such as those which reside in electroweak theory or QCD, with coupling constant \( b \) and corresponding curvature (field strength) 2-form \( G_{\mu \nu} = G^T_{\mu \nu} t^r \), where \( t^r \) are generators for the corresponding gauge group; a Yang-Mills kinetic term for \( B \) may be written as

\[
\frac{1}{g^2} \left( G, G \right)_{\varphi \chi} = \frac{1}{b^2} \int_X \text{tr} (G_{\mu \rho} G_{\sigma \nu}) g^{\mu \rho} g^{\sigma \nu} \sqrt{-\det g} d^4x,
\]

where \( \text{tr} \) denotes the Killing form on the Lie algebra of the gauge group (note that when \( g = g(\varphi, \chi) \) is degenerate then \( \det g = 0 \) so the integrand vanishes and the inverse metric is unnecessary, making the above expression well-defined). It may be possible to find a recipe for extracting the interaction vertices that involve deformations of spacetime frames, typically attributed to gravity, directly from such inner products, which may be a step towards a theory which unifies small and large scale phenomena.

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