Scaling behavior of transient dynamics of vortex-like states in self-propelled particles

Pei-fang Wu‡, Wei-chen Guo‡, Bao-quan Ai†, and Liang He§
Guangdong Provincial Key Laboratory of Quantum Engineering and Quantum Materials,
School of Physics and Telecommunication Engineering,
South China Normal University, Guangzhou 510006, China.

Abstract: Nonequilibrium many-body transient dynamics play an important role in the adaptation of active matter systems to environmental changes. However, the generic universal behavior of such dynamics is usually elusive and left as open questions. Here, we investigate the transient dynamics of vortex-like states in a two-dimensional active matter system that consists of self-propelled particles with alignment interactions subjected to extrinsic environmental noise. We identify a universal power-law scaling for the average lifetime of vortex-like states with respect to the speed of the self-propelled particles. This universal scaling behavior manifests strong robustness against the noise, up to the level where environmental fluctuations are large enough to directly randomize the moving directions of particles. Direct experimental observations can be readily performed by related experimental setups operated at a decently low noise level.

Keywords: scaling behavior, vortex, self-propelled particles

I. INTRODUCTION

Active matter composed of many self-propelled agents is a class of intrinsic nonequilibrium (NEQ) systems whose length scale spans from oceanic to subcellular, while the agents can be generally modeled as self-propelled particles [1, 2]. In these NEQ systems, various collective behavior has been identified [3–19], and the transient dynamics of changing from one collective behavior to another, either spontaneously or in reaction to external perturbations, are also observed frequently [20, 22]. Such transient dynamics play an important role in the adaptation of active matter systems to environmental changes. But in most cases, the universal scaling behavior of these transient dynamics is rather elusive and still left as open questions.

A case in point is associated with the vortices or vortex-like states, which frequently appear and undergo transient dynamics with a finite lifetime, resulting in rich dynamical behavior not only in classical [23] and quantum fluids [24], but also in active matter systems [4–28], ranging from animal flocks, over active colloids [29], to bacterial colonies [30–35] and collectively moving microtubules [36–38], etc. In addition, the vortices or vortex-like states have also been observed in active matter systems without alignment interactions, such as active particles in phase-separated configurations, homogeneous dense systems, narrow circular crowns [38–40], etc. Despite generic universal behavior in these many-body transient dynamics being generally quite elusive due to their intrinsic NEQ characteristic, some can indeed be identified, for instance, the famous $-5/3$ power-law scaling for the energy spectrum of turbulence identified by Kolmogorov [41, 42]. Noticing also that certain universal dynamical scaling behavior could emerge in NEQ many-body systems approaching their equilibrium counterparts [13, 14], this thus raises the intriguing question of whether certain universal scaling behavior exists in the transient dynamics of vortex-like states in active matter systems close to equilibrium.

In this work, we address this question for two-dimensional (2D) active matter systems that consist of self-propelled particles with alignment interactions subjected to extrinsic environmental noise [cf. Eq. (2)]. In particular, we focus on the near-equilibrium regime of the system where the speed $v$ of the self-propelled particles is low [45]. By systematically investigating the transient dynamics of vortex-like states in this regime, we identify a universal power-law scaling of the average lifetime $t^*$ of vortex-like states with respect to the speed $v$ of the self-propelled particles, i.e., $t^* \propto v^{-\alpha}$ with $\alpha \simeq 1$ (cf. Fig. 1). This scaling behavior originates from the occurrence of “separated particles” [cf. Fig. 1(c)] and manifests strong robustness against the environmental noise up to the level where environmental fluctuations are large enough to directly randomize the moving directions of particles (cf. Fig. 2). The upper bound of this noise level is further estimated by calculating the survival probability of the vortex-states in the $v \to 0$ limit (cf. Fig. 3).

In regard to the robustness of the power-law scaling against environmental fluctuations, we expect it can be readily observed in current experimental setups such as in animal groups in quasi-2D space, e.g., fishes in a shallow tank, by employing video tracking techniques, or in 2D synthetic active matter systems consisting of self-propelled colloidal particles with alignment interactions [46].

The rest of the paper is organized as follows. In Sec. II we specify the system and model under study. In Sec. III we discuss the power-law scaling behavior of the transient dynamics at low noise levels and the typical behavior of the transient dynamics at high noise levels. In Sec. IV, we discuss possible experimental observations of the power-

* These two authors contributed equally to this work.
† albq@scnu.edu.cn
‡ liang.he@scnu.edu.cn
§ liang.he@scnu.edu.cn

\[ t^* \propto v^{-\alpha} \]
law scaling behavior identified. Finally, we conclude and give an outlook in Sec. IV.

II. SYSTEM AND MODEL

The system under study consists of $N$ self-propelled particles moving in 2D with a constant speed $v$, whose dynamics is modeled by the Vicsek model with extrinsic noise $\{\tilde{\xi}, \tilde{\eta}\}$,

$$x_j(t + \Delta t) = x_j(t) + v_j(t)\Delta t, \quad (1)$$

$$\theta_j(t + \Delta t) = \text{arg}[\sum_{k \in U_j^*} \{e^{i\theta_j(t)} + \eta e^{i\tilde{\xi}_j(t)}\}] \quad (2)$$

Here, $\Delta t$ is a discrete time step, $x_j(t)$, $v_j(t)$ and $\theta_j(t)$ are the position, the velocity and the direction of motion of the $j$th particle at time $t$, respectively. $\xi_j(t) \in [-\pi, \pi]$ is a uniformly distributed random noise with $\eta \in [0, 1]$ being the extrinsic noise level. These particles interact with each other via the alignment interaction, i.e., for instance, the $j$th particle tends to move along the average direction of its neighbors $U_j$ within a circular region of radius $r$. Due to the competition between the alignment interaction and the environmental fluctuations, the steady state of this system assumes an ordered flocking phase at low noise levels and a disordered phase at high noise levels $\{47, 48\}$.

In fact, the above system also assumes a close relationship to the 2D XY model, with the velocity of the particle playing the role of the local spin of the 2D XY model $\{45, 51-53\}$. More specifically, in the low speed limit $v \to 0$, the dynamics of this NEQ system reduce precisely to Monte Carlo dynamics of the 2D XY model. Noticing that at low temperatures (or equivalently, low noise levels), the vortex-type configurations in the 2D XY model are stable due to the topological protection $\{54\}$, one would expect that in the low speed limit, vortex-like states of the self-propelled particles in the 2D space could assume long-diverging lifetime in their transient dynamics. This thus suggests the intriguing possibility that certain universal scaling behavior could exist in the transient dynamics of vortex-like states in the active matter system under study. Indeed, as we shall see in the following, the average lifetime $t^*$ of vortex-like states in this system manifests a universal power-law scaling with respect to the speed $v$ at relatively low noise levels.

III. RESULTS

In the following, we focus on how the average lifetime of the vortex-like state changes with respect to the speed of the self-propelled particles at different noise levels. To this end, we numerically simulate the dynamics of the system with its initial states prepared in the vortex-like states, and monitor their dynamics at different self-propelling speeds $v$ and noise level $\eta$. More specifically, here we follow a simple protocol where the system is first initialized with the type of vortex-like states which can be relatively easily prepared in various experimental setups $\{10\}$, where each of the $N$ self-propelled particles is randomly located on a circle with a radius $R$ with the direction of its velocity aligned with the local tangent direction of the circle [cf. the first plot from the left of Fig. 1(a)]. For each stochastic trajectory of the transient dynamics of the vortex-like state, the lifetime $t_n^*$ for the vortex-like state, with $n$ being the trajectory index, is extracted by monitoring the instant winding number $w(t) = \sum_{j=0}^{N} (\theta_{j+1}(t) - \theta_j(t)) / 2\pi$ of the system, which equals to one for the initial state, i.e., $w(t = 0) = 1$, and jumps to zero at the time point when the vortex-like state disappear, i.e., $w(t = t_n^*) = 0$ [cf. for instance the fourth plot from the left of Fig. 1(a)]. In particular, since the self-propelled particles are randomly located on a circle, the $(N + 1)$th particle is equivalent to the 0th particle. The average lifetime $t^*$ of the vortex-like states is calculated by $t^* = \sum_{n=1}^{N_{\text{traj}}} t_n^* / N_{\text{traj}}$. We remark that at the low noise level below the flocking transition point, the final steady state of the system is a flocking state, i.e., all particles moving in the same direction as shown in the rightmost snapshot in Fig. 1(a). Moreover, Fig. 1(a) only shows the time-evolution that corresponds to one of the stochastic trajectories. For different stochastic trajectories, the directions for the final collective motion are different. If not specified in text, we use $N_{\text{traj}} = 1.5 \times 10^2$ stochastic trajectories to perform ensemble averages and set $r = 1, R = 5, \Delta t = 1$.

A. Power-law scaling behavior in the transient dynamics at low noise levels

Fig. 1(b) shows the speed $v$ dependence of the average lifetime $t^*$ at different system parameters in the low noise level regime. Here, the average lifetime $t^*$ of vortex-like states manifests a power-law dependence of $v$, i.e., $t \propto v^{-\alpha}$, in the low speed limit in double logarithmic coordinates. We further extracted the scaling exponent $\alpha$ from the power-law fitting of the data and find their value are around $\alpha = 1 \{\text{for } \beta = 0, \eta = 0.3\}$, $\alpha = 1.09 \{\text{for } \beta = 0, \eta = 0.4\}$, $\alpha = 0.99 \{\text{for } \beta = 20, \eta = 0.3\}$, $\alpha = 1.09 \{\text{for } \beta = 20, \eta = 0.4\}$, $\alpha = 0.99 \{\text{for } \beta = 30, \eta = 0.3\}$, $\alpha = 1.05 \{\text{for } \beta = 30, \eta = 0.4\}$, and $\rho$ is the initial particle number density on arc, i.e., $\rho = N/(2\pi R)$. This suggests the existence of a universal power scaling $t^* \propto v^{-1}$. 
Indeed, it is natural to expect that the “separated” particles are more inclined to be influenced by environmental fluctuations since the alignment interaction among them is very weak due to the lack of neighbors, hence they are expected to play a key role in causing the disappearance of vortex-like states. Since a portion of self-propelled particles need to travel a certain distance $l$ to separate from each other, this separation process is naturally expected to take a period of $l/\nu$, after which the vortex-like states disappear. This thus suggests that the average lifetime $t^*$ of the vortex-like states should be proportional to $v^{-1}$, which indeed matches the scaling observed in numerical simulations.

Moreover, we also notice that this scaling is robust (the scaling exponent $\alpha$ deviates from 1 by no more than 5%) at even moderate noise levels ($1/2 < \eta/\eta_c < 1$ with $\eta_c \sim 0.6$ being the typical critical noise level of the flocking transition) [cf. the red marks in Fig. 1(b)]. This is consistent with the expectation from the close relationship between the system and the 2D XY model in the low speed limit ($v/(r/\Delta t) \ll 1$), where vortex-type configurations in the 2D XY model are robust against thermal fluctuations due to the topological protection [54]. Noticing that at large enough noise level or high enough temperature ($T/T_c > 1$ with $T_c$ being the critical temperature of the 2D XY model) the 2D XY model assumes a disordered phase [52, 53] with free vortices continuously generated and annihilated in its Monte Carlo dynamics, and from the close relationship between the system and the 2D XY model in the low speed limit [13, 51], we therefore expect that the power-law scaling of the system’s transient dynamics at moderate noise levels should be absent at high enough noise levels ($\eta/\eta_c > 1$), as we shall now discuss in the following.

### B. Transient dynamics at high noise levels

Fig. 2(a) shows the speed $v$ dependence of the average lifetime $t^*$ in the high noise level regime with $\eta = 0.7$. Here, one can notice that the vortex-like states quickly disappear and the average lifetime $t^*$ essentially does not depend on the speed $v$ of self-propelled particles, indicating that the corresponding power scaling behavior is absent in the presence of strong environmental fluctuations. To further investigate the dynamical behavior of the vortex-like states in the high noise level regime, we monitor the evolution of the system configurations. As we can see from Fig. 2(b), which shows a typical configuration of the system at the moment right before the disappearance of the vortex-like state, the moving directions of most self-propelled particles are strongly influenced by the strong environmental fluctuations and become disordered in a short time, resulting in the disappearance of the vortex-like state. This is in sharp contrast to the typical dynamical behavior in the low noise regime where the disappearance of the vortex is accompanied by spatial deformations [cf. Fig. 1(c)].
So far we have seen the transient dynamics of vortex-states show distinct behavior at high and low noise regimes. To further quantitatively distinguish the high noise level regime from the low noise level one, we estimate the upper bound of the noise level above which environmental fluctuations can easily make moving directions of most self-propelled particles disordered, hence resulting in the absence of the scaling observed in the low noise regime. We estimate this upper bound for each set of fixed system parameters by investigating the survival probability \( P \equiv N_w/N_{\text{traj}} \) of the vortex-states with zero speed, i.e., \( v = 0 \), in the presence of environmental noise. Here, \( N_{\text{traj}} \) is the total number of trajectories [usually \( \sim O(10^3) \)] that are initialized with different vortex-like states, and evolve for a fixed long period of time \( T \) [usually \( \sim O(10^3) \)]. \( N_w \) is the number of trajectories with \( w(T) = 1 \), i.e. still remaining in the vortex-like states at \( t = T \). As one can see from Fig. 3(a) that shows the dependence of \( P \) on the noise level \( \eta \) at three sets of system parameters, the survival probability \( P \) undergoes a fast decay once the noise level \( \eta \) exceeds a critical value \( \eta_{\text{UB}} \), indicating that for \( \eta > \eta_{\text{UB}} \) the influences of environmental fluctuations are so large that the power scaling behavior \( t^* \propto v^{-1} \) does not exist anymore. Indeed, as one can see from Fig. 3(b) which shows the extracted exponents from the power law fit the \( v \) dependence of \( t^* \) at different noise levels \( \eta \). For \( \eta > \eta_{\text{UB}} \), \( \alpha \) manifests large deviation from the value \( \alpha = 1 \). See text for more details.

### IV. EXPERIMENTAL OBSERVABILITY

We expect the predicted power-law scaling for the average lifetime of the vortex-like states with respect to the speed \( v \) of self-propelled particles can be observed in current experimental setups. For instance, one can employ the experimental setup presented in Ref. [46], where polymethyl methacrylate spheres are dispersed in hexadecane solution. In the case where the colloid packing fraction is much smaller than the area fraction, the physics of the system is captured by the dynamical model investigated in this work. In the restricted space, the self-propelled colloidal particles can form stable vortex-like state configurations at the boundary [46], and their transient dynamics can thus be investigated after removing the boundary (actually, removing the boundary of the whole system may not be easy to achieve directly in the experimental setup presented in Ref. [46], however, one could further

![Figure 2](image-url)  
**Figure 2.** (a) Absence of the power-law scaling in transient dynamics at a high noise level with \( \eta = 0.7 \). In the presence of strong environmental fluctuations, vortex-like states quickly disappear and the average lifetime \( t^* \) essentially does not depend on the speed \( v \) of self-propelled particles. (b) Typical configuration right before the vortex-like state disappears at a high noise level with \( \eta = 0.7 \). The first plot shows a typical configuration and other plots correspond to the zoom-in of the indexed rectangle area, respectively. In this case, environmental fluctuations are large enough to directly randomize the moving direction of each particle in a short time. See text for more details.

![Figure 3](image-url)  
**Figure 3.** (a) Survival probability \( P \) of the vortex-states as a function of the noise level at zero speed \( v = 0 \). For each set of system parameters, \( N_{\text{traj}} = 10^3 \) different initial configurations of vortex-like states are evolved for a fixed period of time \( T = 10^4 \), respectively. The survival probability \( P \) undergoes a fast decay from 1 to 0 once the noise level \( \eta \) exceeds a critical value \( \eta_{\text{UB}} \) (vertical black lines), indicating that for \( \eta > \eta_{\text{UB}} \) the influences of environmental fluctuations are so large that the power scaling behavior \( t^* \propto v^{-1} \) does not exist anymore. (b) Extracted values for the exponent \( \alpha \) from the power law fit the \( v \) dependence of \( t^* \) at different noise levels \( \eta \). For \( \eta > \eta_{\text{UB}} \), \( \alpha \) manifests large deviation from the value \( \alpha = 1 \). See text for more details.
engineer an additional removable boundary, for instance, a removable ring-shaped barrier in the central region of the experimental setup presented in Ref. \[46\]). Here, the speed of self-propelled particles \(\nu \propto \sqrt{E^2/E_c^2 - 1}\) in experiments \[16, 55\], with \(E_c\) being a critical electrical field and \(E\) being electric field generated by the longitudinal voltage. Therefore, for experiments operated in the low temperature regime (corresponding to low noise levels), we expect that the power-law scaling can be observed experimentally by monitoring the transient dynamics at different speeds of the self-propelled particles tuned by the longitudinal voltage. Moreover, one can also employ systems of bacteria. For instance, in the experiments presented in Ref. \[34\], the vortex-like states can form in the region surrounded by the neighboring pillars. Therefore, if the pillars are further engineered to be removable in this experimental setup, transient dynamics of the vortex-like states can be investigated by removing the pillars.

V. CONCLUSIONS

Despite the generic universal behavior of many-body transient dynamics are generally quite elusive due to their intrinsic NEQ characteristic, some can indeed be identified as the transient dynamics of vortex-like states in self-propelled particles shows: The average lifetime of the vortex-like states in this 2D active matter system manifests a power-law dependence on the speed of the self-propelled particles in a wide system parameter regime. In particular, this scaling behavior is robust against the environmental fluctuations up to the finite noise level where moving directions of self-propelled particles can be directly randomized by the noise, indicating it is promising to be observed directly in related experiments. Moreover, the environmental noise employed in this work is not spatially correlated. In the case where the spatial correlation length of the noise is relatively small compared with the average separation between particles, the dynamical behavior of the system is expected to show no substantial difference from the one reported here. However, since in general, the environmental noise can show some spatial correlations on a length scale larger than the average separation between particles, it is quite intriguing to investigate the transient dynamics of the system in this case. We believe that our work will stimulate further theoretical and experimental efforts in revealing the generic universal behavior of transient dynamics in active matter systems.

ACKNOWLEDGMENTS

This work was supported by NSFC (Grant Nos. 11874017, 12075090, and 12275089), NKRDP (Grant No. 2022YFA1405304), GDSTC (Grant No. 2018A030313853 and No. 2017A030313029), GDUPS (2016), Major Basic Research Project of Guangdong Province (Grant No. 2017KZDXM024), and START grant of South China Normal University.

[1] G. Popkin, Nature 529, 16 (2016).
[2] S. Ramaswamy, Annu. Rev. Condens. Matter Phys. 1, 323 (2010).
[3] M. C. Marchetti, J. F. Joanny, S. Ramaswamy, T. B. Liverpool, J. Prost, M. Rao, and R. A. Simha, Rev. Mod. Phys. 85, 1143 (2013).
[4] C. Bechinger, R. Di Leonardo, H. Löwen, C. Reichhardt, G. Volpe, and G. Volpe, Rev. Mod. Phys. 88, 045006 (2016).
[5] H. Chaté, Annu. Rev. Condens. Matter Phys. 11, 189 (2020).
[6] T. Vicsek and A. Zafeiris, Phys. Rep. 517, 71 (2012).
[7] A. Cavagna and I. Giardina, Annu. Rev. Condens. Matter Phys. 5, 183 (2014).
[8] A. Cavagna, I. Giardina, and T. S. Grigera, Phys. Rep. 728, 1 (2018).
[9] D. J. G. Pearce, A. M. Miller, G. Rowlands, and M. S. Turner, Proc. Natl. Acad. Sci. U.S.A. 111, 10422 (2014).
[10] F. Ginelli, F. Peruani, M.-H. Pilott, H. Chaté, G. Taruault, and R. Bon, Proc. Natl. Acad. Sci. U.S.A. 112, 12729 (2015).
[11] A. Bottinelli, D. T. J. Sumpter, and J. L. Silverberg, Phys. Rev. Lett. 117, 228301 (2016).
[12] N. Bain and D. Bartolo, Science 363, 46 (2019).
[13] V. Schaller, C. Weber, C. Semmrich, E. Frey, and A. R. Bausch, Nature 467, 73 (2010).
[14] C. Dombrowski, L. Cisneros, S. Chatkaew, R. E. Goldstein, and J. O. Kessler, Phys. Rev. Lett. 93, 098103 (2004).
[15] H. P. Zhang, A. Be’er, E.-L. Florin, and H. L. Swinney, Proc. Natl. Acad. Sci. U.S.A. 107, 13626 (2010).
[16] I. Buttinoni, J. Bialké, F. Kümmel, H. Löwen, C. Bechinger, and T. Speck, Phys. Rev. Lett. 110, 238301 (2013).
[17] J. Palacci, S. Sacanna, A. P. Steinberg, D. J. Pine, and P. M. Chaikin, Science 339, 936 (2013).
[18] R. Kürsten, S. Stroteich, M. Z. Hernández, and T. Ihle, Phys. Rev. Lett. 124, 088002 (2020).
[19] R. Kürsten and T. Ihle, Phys. Rev. Lett. 125, 188003 (2020).
[20] J. K. Parrish and L. Edelstein-Keshet, Science 284, 99 (1999).
[21] I. D. Couzin and J. Krause, Adv. Stud. Behav. 32, 1 (2003).
[22] C. Becco, N. Vandewalle, J. Delcourt, and P. Poncin, Phys. A: Stat. Mech. Appl. 367, 487 (2006).
[23] L. D. Landau and E. M. Lifshitz, Fluid Mechanics (Pergamon Press, 1987).
[24] A. Leggett, *Quantum Liquids* (Oxford University Press, 2006).
[25] H. Duan and X. Zhang, Phys. Rev. E 92, 012701 (2015).
[26] H. Chen and Z. Hou, Phys. Rev. E 86, 041122 (2012).
[27] Z. Cheng, Z. Chen, T. Vicsek, D. Chen, and H.-T. Zhang, New J. Phys. 18, 103005 (2016).
[28] A. Costanzo and C. Hemelrijk, J. Phys. D: Appl. Phys. 51, 134004 (2018).
[29] G. Kokot and A. Snezhko, Nat. Commun. 9, 1 (2018).
[30] H. Wioland, F. G. Woodhouse, J. Dunkel, and R. E. Goldstein, Nat. Phys. 12, 341 (2016).
[31] C. Chen, S. Liu, X.-q. Shi, H. Chaté, and Y. Wu, Nature 542, 210 (2017).
[32] J. Dunkel, S. Heidenreich, K. Drescher, H. H. Wensink, M. Bär, and R. E. Goldstein, Phys. Rev. Lett. 110, 28102 (2013).
[33] R. Grosmann, P. Romanczuk, M. Bär, and L. Schimansky-Geier, Phys. Rev. Lett. 113, 258104 (2014).
[34] D. Nishiguchi, I. S. Aranson, A. Snezhko, and A. Sokolov, Nat. Commun. 9, 1 (2018).
[35] S. Liu, S. Shankar, M. C. Marchetti, and Y. Wu, Nature 590, 80 (2021).
[36] T. Vicsek, Nature 483, 411 (2012).
[37] Y. Sumino, K. H. Nagai, Y. Shitaka, D. Tanaka, K. Yoshikawa, H. Chaté, and K. Oiwa, Nature 483, 448 (2012).
[38] L. Caprini, U. Marini Bettolo Marconi, and A. Puglisi, Phys. Rev. Lett. 124, 078001 (2020).
[39] L. Caprini, U. M. B. Marconi, C. Maggi, M. Paoluzzi, and A. Puglisi, Phys. Rev. Res. 2, 023321 (2020).
[40] L. Caprini, C. Maggi, and U. M. B. Marconi, J. Chem. Phys. 154 24, 244901 (2021).
[41] A. N. Kolmogorov, *Dokl. Akad. Nauk SSSR*, 32, 16 (1941), (reprinted in Proc. R. Soc. London A 434, 15 (1991)).
[42] U. Frisch, *Turbulence* (Cambridge University Press, 1995).
[43] L. M. Sieberer, S. D. Huber, E. Altman, and S. Diehl, Phys. Rev. Lett. 110, 195301 (2013).
[44] U. C. Täuber and S. Diehl, Phys. Rev. X 4, 021010 (2014).
[45] J. Toner and Y. Tu, Phys. Rev. Lett. 75, 4326 (1995).
[46] A. Bricard, J.-B. Caussin, D. Das, C. Savoie, V. Chikkadi, K. Shitara, O. Chepizhko, F. Peruani, D. Saintillan, and D. Bartolo, Nat. Commun. 6, 1 (2015).
[47] G. Grégoire and H. Chaté, Phys. Rev. Lett. 92, 025702 (2004).
[48] H. Chaté, F. Ginelli, G. Grégoire, and F. Raynaud, Phys. Rev. E 77, 046113 (2008).
[49] J.-B. Caussin, A. Solon, A. Peshkov, H. Chaté, T. Dauxois, J. Tailleur, V. Vitelli, and D. Bartolo, Phys. Rev. Lett. 112, 148102 (2014).
[50] A. P. Solon, H. Chaté, and J. Tailleur, Phys. Rev. Lett. 114, 068101 (2015).
[51] J. Toner, Y. Tu, and S. Ramaswamy, Ann. Phys. 318, 170 (2005).
[52] J. M. Kosterlitz and D. J. Thouless, J. Phys. C: Solid State Phys. 6, 1181 (1973).
[53] J. M. Kosterlitz, Rep. Prog. Phys. 79, 026001 (2016).
[54] N. D. Mermin, Rev. Mod. Phys. 51, 591 (1979).
[55] S. Q. Lu, B. Y. Zhang, Z. C. Zhang, Y. Shi, and T. H. Zhang, Soft Matter 14, 5092 (2018).