Special metrics and scales in parabolic geometry

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Abstract
Given a parabolic geometry, it is sometimes possible to find special metrics characterised by some invariant conditions. In conformal geometry, for example, one asks for an Einstein metric in the conformal class. Einstein metrics have the special property that their geodesics are distinguished, as unparameterised curves, in the sense of parabolic geometry. This property characterises the Einstein metrics. In this article, we initiate a study of corresponding phenomena for other parabolic geometries, in particular for the hypersurface CR and contact Legendrean cases.

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1 Introduction
Let $M$ be a smooth $n$-manifold. A projective structure on $M$ is an equivalence class of torsion-free connections on the tangent bundle $TM$, where two connections are said to be equivalent if and only if locally they have the same geodesics as unparameterised curves.

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The study of such structures is called projective differential geometry. It is a simple example of a general class of structures called parabolic differential geometries [8]. One can ask whether there is a Riemannian metric on \( M \) such that the geodesics of this metric are the geodesics of the projective structure. It is a classical question known as the metrisability problem for projective structures. It was already known in 1889 that there are obstructions in two dimensions: R. Liouville [28] showed that the problem is governed by an overdetermined linear system of partial differential equations. The two-dimensional case is completely solved in [5]. In all dimensions, the metrisability problem is governed by an overdetermined linear operator [20]. It is a first operator from the projective Bernstein-Gelfand-Gelfand sequence [6, 9], specifically

\[
\sigma_{bc} \mapsto -\text{trace-free part of } \nabla_a \sigma^{bc},
\]

where \( \sigma_{bc} \) is a symmetric tensor of projective weight \(-2\). Metrisability is obstructed in all dimensions [5], and explicit obstructions in three dimensions are given in [16].

A conformal structure on \( M \) is an equivalence class of Riemannian metrics on \( M \), where two such metrics \( g_{ab} \) and \( \widehat{g}_{ab} \) are equivalent if and only if \( \widehat{g}_{ab} = \Omega^2 g_{ab} \) for some positive smooth function \( \Omega \). For \( n \geq 3 \), as we shall henceforth suppose, it is another example of a parabolic differential geometry. Of course, in this case there is no question that there are metrics underlying the structure. But there are special curves in conformal geometry that are invariantly defined by the structure, and which play the rôle of the unparameterised geodesics in projective geometry. These are the conformal circles [1, 31, 33], here to be regarded as unparameterised curves. Unlike the geodesics of a projective structure, they are defined by a higher-order jet at any one point: in a way that will soon be made precise, one needs to know both an initial direction and acceleration to determine a conformal circle. Nevertheless, we can ask the question of whether there is a metric in the conformal class so that all its geodesics are conformal circles. We shall find that this question is equivalent to the classical question of whether the conformal structure admits an Einstein representative. So this is a non-trivial question only for \( n \geq 4 \) with complex projective space \( \mathbb{C}P^2 \), equipped with its Fubini–Study metric, providing a good example where all geodesics are conformal circles. The existence of an Einstein metric in a given conformal class is also governed by a first BGG operator, specifically

\[
\sigma \mapsto \text{the trace-free part of } (\nabla_a \nabla_b \sigma + P_{ab} \sigma),
\]

where \( \sigma \) has conformal weight 1 and \( P_{ab} \) is the Schouten tensor (4). The naïve restriction on a positive smooth function \( \Omega \) such that \( \Omega^2 g_{ab} \) is Einstein is nonlinear and was given by Brinkmann [4, Eq. (2.26)], but LeBrun [25, p. 558] observes that the equation on \( \sigma = \Omega^{-1} \) is linear.

The conformal circles on the round \( n \)-sphere are the genuine round circles for the unit sphere \( S^n \subset \mathbb{R}^{n+1} \) sliced by any plane in \( \mathbb{R}^{n+1} \), not necessarily through the origin. A more congenial viewpoint is to regard the round \( n \)-sphere inside \( \mathbb{R}P^{n+1} \) according to

\[
S^n = \{[x^0, x^1, \ldots, x^n, x^\infty] \text{ s.t. } 2x^0x^\infty = (x^1)^2 + (x^2)^2 + \cdots + (x^n)^2 \}. \tag{1}
\]

Its main feature is that the evident action of \( \text{SO}(n+1, 1) \) is by conformal transformations and that all conformal transformations of \( S^n \) arise in this way (see, for example, [18] for details). In this model, the conformal circles are the intersections of \( S^n \) with projective planes \( \mathbb{R}P_2 \), linearly embedded in \( \mathbb{R}P_{n+1} \).
In particular, it is clear that the great circles play no special rôle and that the space of conformal circles on $S^n$ may be identified as an open subset of $Gr_3(\mathbb{R}^{n+2})$, thus having dimension $3(n-1)$. More specifically, there are three open orbits for the action of $SO(n+1, 1)$ on $Gr_3(\mathbb{R}^{n+2})$ according to whether the corresponding plane in $\mathbb{RP}_{n+1}$ misses, touches, or slices the conformal sphere $S^n \hookrightarrow \mathbb{RP}_{n+1}$. It slices if and only if the quadratic form in (1) restricts to be indefinite on the corresponding 3-plane in $\mathbb{R}^{n+2}$. The moduli space of conformal circles is, therefore, a homogeneous space:

$$SO(n+1, 1)/S(O(n-1) \times O(2, 1)).$$

The corresponding ‘flat model’ for CR geometry arises by considering the action of $SU(n+1, 1)$ on $\mathbb{CP}_{n+1}$. The unique closed orbit is a sphere of real dimension $2n+1$, which we shall realise in the form:

$$S^{2n+1} = \{[z^0, z^1, \ldots, z^n, z^\infty] \text{ s.t. } z^0\bar{z}^\infty + z^\infty\bar{z}^0 = |z^1|^2 + |z^2|^2 + \cdots + |z^n|^2 \}.$$

$SU(n+1, 1)$ acts by CR automorphisms and all such automorphisms arise in this way (see, for example, [19] for details). Now there are two possible geometric constructions of special curves in $S^{2n+1}$. The first is to intersect $S^{2n+1}$ with a linearly embedded $\mathbb{CP}_1 \hookrightarrow \mathbb{CP}_{n+1}$. As verified by Jacobowitz [23], this gives rise to the chains on $S^{2n+1}$, as defined in general CR geometry by Chern and Moser [11]. Thus, in the flat model, the chains may be identified as an open subset of $Gr_2(\mathbb{C}^{n+2})$, therefore having real dimension $4n$ (see also [10]).

A CR manifold $M$ is, in particular, a contact manifold and chains are everywhere transverse to the contact distribution $H \rightarrow M$. There is, however, another class of ‘distinguished curves’ in CR geometry, which are everywhere tangent to $H$. To see them in the flat model, consider the embedding

$$\mathbb{RP}_{n+1} \hookrightarrow \mathbb{CP}_{n+1}$$

induced by the inclusion $\mathbb{R}^{n+2} \subset \mathbb{C}^{n+2}$. It is one of the two orbits for the action of $SL(n+2, \mathbb{R})$ on $\mathbb{CP}_{n+1}$ and its intersection with the CR sphere $S^{2n+1}$ is the conformal sphere $S^n$, as in (1). Thus, we have

$$\mathbb{RP}_{n+1} \subset \mathbb{CP}_{n+1} \cup S^n \subset S^{2n+1}$$

and it is easily verified that $S^n$ is everywhere tangent to the CR contact distribution $H \rightarrow S^{2n+1}$. The conformal circles in $S^n$ give distinguished curves in $S^{2n+1}$, which may be intrinsically defined just in terms of the CR structure on $S^{2n+1}$, the full set of such curves being obtained under the action of $SU(n+1, 1)$. On a general strictly pseudoconvex CR manifold, using the terminology of [8, Remark 5.3.8], these are the ‘distinguished curves of type (c)’. We shall come back to their definition later in this article. Since they will be a main object of study in this article, we shall refer to these curves as ‘contact distinguished’. In
the flat model, as above, they are closely related to conformal circles and, in particular, for
the usual round metric on $S^{2n+1}$, they are circles. In any case, the moduli space of contact
distinguished curves in the flat model is a homogeneous space $SU(n + 1, 1)/S$, where

$$S = \left\{ C = e^{i\theta} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ s.t. } A \in U(n - 1), B \in O(2, 1), \det C = 1 \right\}$$  \hspace{1cm} (2)

It has real dimension

$$(n + 2)^2 - 1 - (1 + (n - 1)^2 + 3 - 1) = 6n - 1.$$  \hspace{1cm} (3)

Contact distinguished curves can be also interesting from the viewpoint of control theory.
Manifolds $M$ underlying a parabolic geometry always carry canonical filtrations (possibly
trivial, as in the conformal case) where $T^{-1}M$ is a bracket-generating distribution in $TM$.
Viewing the distribution $T^{-1}M$ as a control distribution for a suitable system, the bracket-
generating property means that the system is controllable, i.e. for arbitrary point $x$, each
point in a neighbourhood of $x$ can be joined with $x$ by a curve contact to $T^{-1}M$. The
contact distinguished curves are natural candidates for controlling the system (distinguished
curves have the important property that if they are contact at one point then they are contact
everywhere). On the other hand, they are generally determined by higher jets, so there can
be more distinguished curves going in a given direction from a given point $x$. To describe a
suitable family of such curves, it is therefore natural to ask if there is a connection such that
all its contact geodesics are distinguished.

In this article, we set up some general machinery within parabolic geometry and illustrate
our procedures by treating, in detail, conformal geometry and contact Legendrean geometry
(an alternative real form of hypersurface CR geometry). The conformal case is already
understood by other methods (e.g. [1, 31, 33]), which we briefly recall before using symmetry
algebras and the tractor connection to rederive the conformal results in a way that generalises
to all parabolic geometries. The key here is the characterisation of unparameterised distin-
guished curves, due to Doubrov and Žádník [15], in terms of the Cartan connection. We
rephrase their characterisation in terms of tractor connections (and, in an appendix, derive
the equations of Tod [31] for conformal circles by tractor methods). An important point about
such methods is that they are manifestly invariant. Otherwise, one has to check the invariance
of the resulting equations by tedious and unilluminating calculation. In fact, we do not need
the equations themselves in order to understand when (contact) geodesics for a Weyl con-
nection within a parabolic class are distinguished. Much of this article is devoted to carrying
out this procedure in the context of contact Legendrean geometry, culminating in Theorem 3
and its corollary. There are corresponding results in the CR setting, namely Theorem 4 and
its corollary. In both cases, it is interesting to note that there are non-trivial examples, which
may be gleaned from articles of Loboda [29, 30] and, more recently, Kruglikov [24] and
Doubrov-Medvedev-The [13, 14]. In the CR setting, the geometry we identify was already
considered by Leitner [27] who explains how to derive them from Kähler–Einstein metrics.

## 2 Conformal geometry

Fix a Riemannian conformal structure on $M$. We need to define conformal circles on $M$ as
unparameterised curves. Since it is our aim to compare these curves with metric geodesics, it
is convenient to follow Tod [31] in selecting a background metric $g_{ab}$ from the conformal class
and write all differential equations with respect to this metric and its Levi–Civita connection
$\nabla_a$, whilst checking that these equations do not depend on our choice of $g_{ab}$.  

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Let $\gamma \hookrightarrow M$ be a smooth oriented, embedded curve. In the presence of $g_{ab}$, we shall write $U^a$ for the unique unit length vector field defined along and tangent to $\gamma$ in the direction of its orientation. Evidently, if $g_{ab}$ is replaced by $\hat{g}_{ab} = \Omega^2 g_{ab}$, then $U^a$ is replaced by $\hat{U}^a = \Omega^{-1} U^a$. We say that $U^a$ has conformal weight $-1$, equivalently that $\hat{U}^a \equiv g_{ab} U^b$ has conformal weight 1. Again in the presence of a metric $g_{ab}$, let $\partial \equiv U^a \nabla_a$ denote the directional derivative along $\gamma$, where $\nabla_a$ is the Levi–Civita connection of $g_{ab}$. The directional derivative may be applied to any tensor defined along $\gamma$ and, in particular, to the vector field $U^a$. For a given metric $g_{ab}$, we shall refer to $U^a$ as the velocity along $\gamma$ and $\partial U^a$ as the acceleration along $\gamma$.

Then, $\gamma$ is a geodesic for $g_{ab}$ if and only if $\partial U^a \equiv 0$. Notice that, along any curve, since $0 = \partial(U^a U_a) = 2 U^a C_a$, the acceleration is orthogonal to the velocity.

**Lemma 1** If $\omega$ is a smooth 1-form on $M$, then locally it is always possible to find a smooth function $f$ on $M$ such that $\omega = df$ along $\gamma$.

**Proof** In 2-dimensions, we may choose local coordinates $(x, y)$ so that $\gamma$ is the $x$-axis. Then, $\omega|_{\gamma} = \alpha(x) dx + \beta(x) dy$ and we may take

$$f(x, y) = \int_{\gamma} \alpha(x) \, dx + \beta(x) \, y.$$ 

Away from a tubular neighbourhood of $\gamma$, we may extend $f$ arbitrarily by a partition of unity. The $n$-dimensional case is similar. 

**Lemma 2** Locally, it is always possible to find a metric in the given conformal class for which $\gamma$ is a geodesic.

**Proof** If we replace $g_{ab}$ by $\hat{g}_{ab} = \Omega^2 g_{ab}$, then the Levi–Civita connection $\nabla_a$ acting on an arbitrary 1-form $\phi_b$, is replaced by $\hat{\nabla}_a$ acting on $\phi_b$ according to the formula:

$$\hat{\nabla}_a \phi_b = \nabla_a \phi_b - \nabla_b \phi_a \gamma_c \phi_c g_{ab},$$

where $\gamma_a = \nabla_a \log \Omega$. Therefore,

$$\hat{C}_b = \hat{U}^a \hat{\nabla}_a \hat{U}_b = \Omega^{-1} U^a \hat{\nabla}_a (\Omega U_b) = C_b - \gamma_b + U^c \gamma_c U_b,$$

and we aim to enforce $\hat{C}_b = 0$ along $\gamma$ by a suitable choice of $\Omega$. By Lemma 1 we may choose $\Omega$ positive so that $\nabla_a \log \Omega = C_a$ along $\gamma$. Then, $\hat{C}_b = U^c \nabla_c U_b$ along $\gamma$. But $\hat{C}_b$ is also orthogonal to $\gamma$ and hence must vanish.

We shall define the Riemann curvature tensor $R_{abcd}$ by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) X^c = R_{abcd} X^d,$$

using the metric to lower the index $c$ in the usual way. The Schouten tensor is the symmetric tensor $P_{ab}$ such that

$$R_{abcd} = W_{abcd} + P_{ac} g_{bd} - P_{bc} g_{ad} - P_{ad} g_{bc} + P_{bd} g_{ac}$$

where $W_{abcd} = 0$. Following Tod [31], we say that an unparameterised curve $\gamma \hookrightarrow M$ is a conformal circle if and only if

$$\partial C_a = P_{ab} U^b - (C^b C_b + P_{bc} U^b U^c) U_a$$

$\square$
along \( \gamma \). Defined this way, it requires a computation to check conformal invariance: use, for example, the formulae in [17] and that

\[
\hat{P}_{ab} = P_{ab} - \nabla_a \gamma_b + \gamma_a \gamma_b - \frac{1}{2} \gamma^c \gamma_c g_{ab}.
\]

In an appendix, however, we shall derive (5) by tractor calculus, thereby ensuring its conformal invariance. Later in this section, we shall use tractor calculus to avoid the general form of (5), deriving this equation only when it so happens that \( \gamma \) is a geodesic. This special case is all we need in order to prove the following.

**Theorem 1** A curve \( \gamma \hookrightarrow M \) is a conformal circle if and only if there is a metric \( g_{ab} \) in the conformal class for which \( \gamma \) is a geodesic and so that \( P_{ab} U^b = P_{bc} U^b U^c U_a \) along \( \gamma \).

**Proof** If \( \gamma \) is a geodesic and \( P_{ab} U^b = P_{bc} U^b U^c U_a \) along \( \gamma \), then (5) is satisfied and \( \gamma \) is a conformal circle. Conversely, if we choose by Lemma 2 a metric in the conformal class for which \( \gamma \) is a geodesic, then (5) implies that \( P_{ab} U^b = P_{bc} U^b U^c U_a \), as required. \( \square \)

**Corollary 1** The geodesics of a Riemannian metric are all conformal circles if and only if the metric is Einstein.

**Proof** To say that \( P_{ab} U^b = P_{bc} U^b U^c U_a \) is exactly to say that \( U_b \) is an eigenvector of the endomorphism \( P_a^b \) (since if \( P_a^b U_b = \lambda U_a \), then it must be that \( \lambda = U^a P_a^b U_b = P_{bc} U^b U^c U_a \)). All vectors are eigenvectors if and only if \( P_a^b = \lambda \delta_a^b \), where \( \delta_a^b \) is the identity matrix. \( \square \)

**Remark** Smooth curves in a conformal manifold come equipped with a class of preferred parameterisations, well defined up to a projective freedom. If we ask, not only that a geodesic \( \gamma \) of a Riemannian metric be a conformal circle, but also that its arc-length parameterisation be preferred, then we require, in addition, that

\[
U^a \partial_a C_a = 3(U^a C_a)^2 - \frac{3}{2} C^a C_a - P_{ab} U^a U^b
\]

and it follows that \( P_{ab} U^b = 0 \) along \( \gamma \). This is what is proved in [1]. We obtain a different corollary, namely that for all arc-length parameterised geodesics of a metric to be projectively parameterised conformal circles, it is necessary and sufficient that the metric be Ricci-flat. A good example in four dimensions is a Calabi–Yau metric on a K3 surface.

**Remark** This characterisation of parameterised distinguished curves holds true in any parabolic geometry [34, Proposition 3.7].

Following Doubrov and Žádník [15], we may prove Theorem 1 without firstly having to derive the differential equation (5). The main advantage of this alternative proof is that it applies to all parabolic geometries. To implement it, all one needs is the symmetry algebra of an unparameterised curve \( \gamma \) in the flat model \( G/P \) and the Cartan connection, regarded as a \( g \)-valued 1-form \( \omega \) on the total space of the Cartan bundle \( G \to M \). The symmetry algebra \( \text{Sym} L \) of \( L \) is defined to be a certain subalgebra of \( g \) realised as vector fields on \( G/P \) induced by the action of \( G \) on \( G/P \). Specifically,

\[
\text{Sym} L \equiv \{ X \in g \mid X \text{ is tangent to } L \text{ at } p \text{ for all } p \in L \}.
\]

It is evident that \( \text{Sym} L \) is non-trivial if and only if \( L \) is homogeneous. The notion of conformal circles generalises to any parabolic geometry. They are called distinguished curves and [15, Proposition 1] shows that an unparameterised curve \( \gamma \hookrightarrow M \) is distinguished if and only if there is a smooth section \( \gamma \to G \) of the Cartan bundle \( G \to M \) so that \( \gamma^* \omega \) takes values in
Sym$L \subseteq g$ (this is for distinguished curves modelled on $L \hookrightarrow G/P$). If preferred, one could start with this as the definition of distinguished curves in a Cartan geometry.

In what follows, we shall unpack this characterisation of distinguished curves in conformal geometry, supposing, for simplicity, that $\gamma \hookrightarrow M$ is already a geodesic for a chosen metric in the conformal class. By Lemma 2, this is no restriction on $\gamma$ and, in any case, will evidently be sufficient in providing a route to Theorem 1. The object of this exercise is to avoid (5) since this is more difficult to derive. For contact distinguished curves in CR geometry, for example, we do not know, nor need to know, the equation corresponding to (5) in order to establish, in Sect. 5, the analogue of Theorem 1. More precisely, our upcoming Theorem 2 will soon translate the Doubrov–Žádník characterisation [15] of unparameterised distinguished curves for a general parabolic geometry into the language of tractors. But, to apply Theorem 2, we shall need to know the symmetry algebra (6) of a model curve in $G/P$. In the conformal case it is easiest to suppose the initial acceleration vanishes. For completeness, in an appendix we shall use Theorem 2 to derive Eq. (5) for conformal circles in general.

When $\gamma \hookrightarrow M$ is already a geodesic, the model curve in $\mathbb{R}^n$ will be a straight line

$$\mathbb{R} \ni t \mapsto 2t U^b$$

for $U^b$ of unit length (the seemingly spurious factor of 2 being included here so that this straight line can be seen as the limiting case of a circle (32) when, in Appendix, we include acceleration). Hence, the first order of business is to compute the conformal symmetry algebra of (7). The Lie algebra of conformal Killing fields on $\mathbb{R}^n$ is

$$(X^b - F^b_{\ c}x^c + \lambda x^b - Y_c x^c x^b + \frac{1}{2} x_c x^c Y^b) \frac{\partial}{\partial x^b}$$

for constant tensors $X^b, F^b_{\ c}, \lambda, Y_b$ with $F_{bc}$ skew. We see if such a field is everywhere tangent to (7), arriving at the following conclusion.

**Lemma 3** The symmetry algebra of the line (7) consists of those fields (8) satisfying the following linear constraints:

$$X^b = f U^b \quad F^b_{\ c} U^c = 0 \quad Y^b = h U^b$$

where $f$ and $h$ are arbitrary.

Note that this symmetry algebra has dimension $(n-1)(n-2)/2 + 3$, as will remain true when we include acceleration (in Appendix).

To unpack the Doubrov–Žádník characterisation, we may use tractors instead of the Cartan connection. Recall that, in the presence of a metric $g_{ab}$, the standard tractor connection [2] is given by

$$\nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ab} \rho + P_{ab} \sigma \\ \nabla_a \rho - P_a^b \mu_b \end{bmatrix}$$

on the bundle $\mathcal{T} = \bigwedge^0[1] + \bigwedge^1[1] + \bigwedge^0[-1]$ and the invariant inner product is

$$\begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix}, \begin{bmatrix} \tilde{\sigma} \\ \tilde{\mu}_b \\ \tilde{\rho} \end{bmatrix} = \sigma \tilde{\rho} + \mu^b \tilde{\mu}_b + \rho \tilde{\sigma}.$$
Lemma 4  The general ⟨ , ⟩-preserving endomorphism of T has the form:

\[
\Phi \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} X_b \mu_b - \lambda \sigma \\ Y_b \sigma + F_b^c \mu_c - X_b \rho \\ \lambda \rho - Y_b \mu_b \end{bmatrix}
\]

(11)

for unweighted tensors \(X^b, F^b_c, \lambda, Y_b\) with \(F^b_c\) skew.

Proof  An elementary verification. \(\square\)

In fact, the ⟨ , ⟩-preserving endomorphisms of T are, by definition, sections of the adjoint tractor bundle \(\mathcal{A}\), a notion that makes sense in any parabolic geometry: it is the bundle induced from the Cartan bundle \(G \to M\) by the Adjoint representation of \(G\) on \(\mathfrak{g}\). In particular, the symmetry algebra \(\text{Sym}L \subset \mathfrak{g}\) of a homogeneous curve \(L \subset G / P\) induces a collection of preferred subspaces of \(\mathcal{A}\), the conjugates of \(\text{Sym}L\) (under the Adjoint action of \(G\) on \(\mathcal{A}\)). Translating the Doubrov–Žádník result [15] into the language of tractors gives the following.

Theorem 2  In order that \(\gamma \hookrightarrow M\) be an unparameterised distinguished curve modelled on \(L \hookrightarrow G / P\), it is necessary and sufficient that along \(\gamma\) there be a subbundle \(S \subset \mathcal{A}|_\gamma\) whose fibres are everywhere conjugate to \(\text{Sym}L \subset \mathfrak{g}\) and such that \(S\) is preserved by the tractor connection along \(\gamma\).

Remark  Firstly, note that this is a manifestly invariant formulation. Secondly, there is a canonical projection \(\mathcal{A} \to TM\) and if \(\gamma \hookrightarrow M\) is to be a distinguished curve in accordance with Theorem 2, then the image of \(S \subset \mathcal{A}|_\gamma\) in \(TM|_\gamma\) is forced to be the tangent bundle of \(\gamma\).

In the particular case of a Riemannian manifold \(M\) with metric \(g^a_b\), we conclude that a geodesic \(\gamma \hookrightarrow M\) is a conformal circle if and only if we can find a subbundle \(S\) of the endomorphisms of \(T\) along \(\gamma\), having the form (11) and such that

- \(X^a\) is tangent to \(\gamma\),
- fibrewise, the subbundle \(S\) has the form (9),
- \(S\) is preserved by the tractor connection \(\partial = U^a \nabla_a\) along \(\gamma\).

To proceed, it is useful to reformulate the conditions (9) as follows.

Lemma 5  In order that an endomorphism (11) satisfy (9), for some unit vector field \(U^a\), it is necessary and sufficient that

\[
\Phi \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -f U^b \\ \lambda \end{bmatrix}, \quad \Phi \begin{bmatrix} 0 \\ U^b \\ 0 \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ -h \end{bmatrix}, \quad \Phi \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\lambda \\ h U^b \\ 0 \end{bmatrix}.
\]

(12)

Proof  If (9) are satisfied, then all equations of (12) hold with \(X^b = f U^b\) and \(Y^b = h U^b\). Conversely, if (12) hold, then the first two equations determine \(f, U^a, \lambda,\) and \(h\). From (11), the first two constraints from (9) are manifest and we also discover that \(Y^b = h U^b\), which is the final constraint from (9). \(\square\)

Proof of Theorem 1  According to Theorem 2, it suffices to show that the conditions (12) on a tractor endomorphism \(\Phi\) are preserved by the tractor connection along a geodesic \(\gamma\) of the metric \(g^a_b\) if and only if

\[
P^a_b U^b = P^b_c U^b U^c U_a \quad \text{along } \gamma.
\]

(13)
This is a straightforward verification as follows. From (10), the tractor connection \( \partial = U^a \nabla_a \) along \( \gamma \) is given by
\[
\partial \begin{bmatrix}
\sigma \\
\mu_b \\
\rho 
\end{bmatrix} = \begin{bmatrix}
\partial \sigma - U^a \mu_a \\
\partial \mu_b + U_b \rho + U^a P_{ab} \sigma \\
\partial \rho - U^a P_{a b} \mu_b
\end{bmatrix}.
\] (14)

Bearing in mind that \( \partial U^b = 0 \) (since \( \gamma \) is a geodesic) the Leibniz rule now calculates the effect of \( \partial / \Phi_1 \) and, firstly, we find from (12) that
\[
(\partial / \Phi_1) \begin{bmatrix}
0 \\
- f U_b \\
1
\end{bmatrix} = \partial \begin{bmatrix}
f \\
0 \\
- h
\end{bmatrix} = \begin{bmatrix}
\partial f \\
-f U_b + f U^a \sigma \\
-\partial h
\end{bmatrix},
\]
which has the same form with \( f \) and \( \lambda \) replaced by
\[
\tilde{f} = \partial f - \lambda \quad \text{and} \quad \tilde{\lambda} = \partial \lambda + h + P_{ab} U^a U^b,
\]
respectively. Next, we should compute
\[
(\partial / \Phi_1) \begin{bmatrix}
0 \\
U_b \\
0
\end{bmatrix} = \partial \begin{bmatrix}
\Phi \\
0 \\
0
\end{bmatrix} - \Phi \partial \begin{bmatrix}
0 \\
U_b \\
0
\end{bmatrix},
\]
and, from (12) and (14), we find
\[
\partial \begin{bmatrix}
\Phi \\
0 \\
0
\end{bmatrix} = \partial \begin{bmatrix}
f \\
0 \\
- h
\end{bmatrix} = \begin{bmatrix}
\partial f \\
-f U_b + f U^a \sigma \\
-\partial h
\end{bmatrix}
\]
whilst
\[
\Phi \partial \begin{bmatrix}
0 \\
U_b \\
0
\end{bmatrix} = \Phi \begin{bmatrix}
-1 \\
0 \\
-P_{ab} U^a U^b
\end{bmatrix} = \begin{bmatrix}
\lambda \\
(f P_{ac} U^a U^c - h) U_b \\
-\lambda P_{bc} U^b U^c
\end{bmatrix}.
\]
Therefore,
\[
(\partial / \Phi_1) \begin{bmatrix}
0 \\
U_b \\
0
\end{bmatrix} = \begin{bmatrix}
\partial f - \lambda \\
f (P_{bc} U^c - P_{ac} U^a U^c U_b) \\
-\partial h + \lambda P_{bc} U^b U^c
\end{bmatrix},
\]
which has the form required by (12), with \( f \) and \( h \) replaced by
\[
\tilde{f} = \partial f - \lambda \quad \text{and} \quad \tilde{h} = \partial h - \lambda P_{bc} U^b U^c,
\]
respectively, if and only if (13) holds. Finally, we need to compute
\[
(\partial / \Phi_1) \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} = \partial \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} - \Phi \partial \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}.
\]
Well, from (12), we find
\[
\partial \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} = \partial \begin{bmatrix}
-\lambda \\
- h U_b \\
0
\end{bmatrix} = \begin{bmatrix}
-\partial \lambda - h \\
(\partial h) U_b - \lambda P_{ab} U^a \\
-h P_{ac} U^a U^c
\end{bmatrix}.
\]
and
\[
\Phi \partial \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \Phi \begin{bmatrix} 0 & U^a P_{ab} \\ U^a P_{ab} & 0 \end{bmatrix} = \Phi \begin{bmatrix} 0 & P_{ac} U^a U^c U_b \\ P_{ac} U^a U^c U_b & 0 \end{bmatrix},
\]
where we have used (13) to substitute \( U^a P_{ab} = P_{ac} U^a U^c U_b \). From here,
\[
\Phi \partial \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} f P_{ac} U^a U^c & 0 \\ -h P_{ac} U^a U^c & 0 \end{bmatrix}
\]
and so, finally, again using (13) to substitute \( P_{ac} U^a U^c U_b \) for \( P_{ab} U^a \),
\[
(\partial \Phi) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\partial \lambda - h - f P_{ac} U^a U^c & 0 \\ (\partial h) U_b - \lambda P_{ab} U^a & 0 \end{bmatrix} = \begin{bmatrix} \tilde{\lambda} \\ \tilde{h} U_b \end{bmatrix},
\]
as required. \( \square \)

3 The symmetry algebra

In Sect. 2 we defined, by means of (6), the symmetry algebra \( \text{Sym} L \) for any curve \( L \hookrightarrow G/P \). Starting with any \( V \in \mathfrak{g} \), we may consider the curve \( L_V \) in \( G/P \) obtained as the image of \( t \mapsto \exp(t V) \in G \). If \( V \notin \mathfrak{p} \), this curve is non-trivial and homogeneous. By construction, we have \( V \in \text{Sym} L_V \) but the full symmetry algebra \( \text{Sym} L_V \) is generally bigger than \( \langle V \rangle \), the span of \( V \). We expect a completely algebraic procedure to obtain \( \text{Sym} L_V \) from \( V \in \mathfrak{g} \setminus \mathfrak{p} \). It may be given as follows.

**Lemma 6** (from [12]) Consider the series of subalgebras
\[
\mathfrak{p} = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots
\]
defined inductively by
\[
\mathfrak{a}_{\ell+1} = \{ X \in \mathfrak{a}_\ell \mid [X, V] \in \mathfrak{a}_\ell + \langle V \rangle \}
\]
and let \( \mathfrak{a}_\infty = \bigcap_{k=0}^{\infty} \mathfrak{a}_k \). Then, \( \text{Sym} L_V = \mathfrak{a}_\infty + \langle V \rangle \).

In Lemma 3, we computed the symmetry algebra of a straight line in \( \mathbb{R}^n \) directly from the definition (6), and in Lemma 7, we similarly and directly compute the symmetry algebra of a circle in \( \mathbb{R}^n \). These computations may be circumvented by Lemma 6 provided we have a convenient realisation of the Lie algebras \( \mathfrak{g} \supseteq \mathfrak{p} \). In the conformal case, for example, we may identify the conformal Killing fields (8) with matrices
\[
\begin{bmatrix}
\lambda & -Y_c \\
-X^b & F^b c & Y^b \\
0 & X_c & -\lambda
\end{bmatrix} \in \mathfrak{so}(n+1,1)
\]
and the parabolic subalgebra \( \mathfrak{p} \) as matrices of the form:
\[
\begin{bmatrix}
\lambda & -Y_c & 0 \\
0 & F^b c & Y^b \\
0 & 0 & -\lambda
\end{bmatrix}.
\]

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For our element $V \in g \setminus p$, let us take

$$
\begin{bmatrix}
0 & 0 & 0 \\
-U^b & 0 & 0 \\
0 & U_c & 0
\end{bmatrix},
$$
such that $U^b U_b = 1$.

Then, one readily verifies that

$$
a_1 = \left\{ \begin{bmatrix}
\lambda & -Y_c & 0 \\
0 & F^b_c Y^b & 0 \\
0 & 0 & -\lambda
\end{bmatrix} \text{ s.t. } F^b_c U^c = 0 \right\},
$$

and hence that

$$
a_\infty = a_2 = \left\{ \begin{bmatrix}
\lambda & -h U_c & 0 \\
0 & F^b_c h U^b & 0 \\
0 & 0 & -\lambda
\end{bmatrix} \text{ s.t. } F^b_c U^c = 0 \right\},
$$

whence, by Lemma 6, the full symmetry algebra is

$$
s = \left\{ \begin{bmatrix}
\lambda & -h U_c & 0 \\
-\dot{f} U^b & F^b_c h U^b & 0 \\
0 & f U_c & -\lambda
\end{bmatrix} \text{ s.t. } F^b_c U^c = 0 \right\}
$$
in agreement with (9).

In Sect. 4, we shall use Lemma 6 to compute a different symmetry algebra, closely related to the contact distinguished curves in CR geometry, as described in Sect. 1.

### 3.1 The symmetry algebra in parabolic geometry

The purpose of this subsection is to specialise the result of Lemma 6 to the case that the homogeneous space $M = G/P$ is a flat parabolic geometry and the curve $L \hookrightarrow G/P$ starts off, using the notation from [8], in a direction from $T^{-1}M \subseteq TM$. All the examples in this article are modelled in this way, and the general algorithm given in Lemma 6 is especially congenial in this case. To this end, suppose $g$ is semisimple, $p \subset g$ is parabolic, and $V \in g \setminus p$ is nicely positioned with respect to a parabolic grading. Specifically, let us suppose that $g$ is $|k|$-graded:

$$
g = g_{-k} \oplus \cdots \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus \cdots \oplus g_k
$$
as in [8] and that we start with $V \in g_{-1}$. Recall that Lemma 6 defines

$$
a_1 \equiv \{ X \in p \mid [X, V] \in p + \langle V \rangle \}.
$$

Evidently, this is only a constraint on the $g_0$-component of $X$. Then, by induction, the subalgebra $a_{\ell+1}$ constrains only the $g_\ell$-component. In particular, we can see when matters stabilise; specifically $a_\infty = a_{k+1}$.
In the conformal case for example, the Lie algebra $\mathfrak{g} = \mathfrak{so}(n + 1, 1)$ is $|1|$-graded and, employing Dynkin diagram notation from [3],

\[
\begin{align*}
\mathfrak{so}(n + 1, 1) &= 0 1 0 0 \ldots \\
\mathfrak{g} &= 0 1 0 0 \ldots \oplus 0 0 1 0 \ldots \oplus 0 0 0 0 \ldots \\
\mathfrak{g}^{-1} &= \mathfrak{g}_0 \oplus \mathfrak{g}_1
\end{align*}
\]

so $a_\infty = a_2$ is confirmed and we can locate the different pieces of the symmetry algebra (15), namely

\[
\mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1
\]

\[
f U^b \begin{pmatrix} F_{bc} \\ \lambda \end{pmatrix} \quad h U_c.
\]

### 4 Contact Legendrean geometry

This geometry is an alternative real form of strictly-pseudoconvex CR geometry. The flat model is a homogeneous space for $\text{SL}(n+2, \mathbb{R})$ instead of the sphere $S^{2n+1}$ as a homogeneous space for $\text{SU}(n+1, 1)$. Specifically, it is the flag manifold

\[
\mathbb{F}_{1,n+1}(\mathbb{R}^{n+2}) = \{ \mathcal{L} \subset \mathcal{H} \subset \mathbb{R}^{n+2} \mid \dim \mathcal{L} = 1, \dim \mathcal{H} = n + 1 \}
\]

and it is convenient to view elements of $\mathfrak{sl}(n + 2, \mathbb{R})$ in blocks:

\[
\begin{bmatrix}
X^\alpha & C_{\alpha \beta}^\alpha & W^\alpha \\
Y_\beta & e
\end{bmatrix}, \quad \text{where } a + C_{\alpha \beta}^\alpha + e = 0.
\]

Let us take $p$ to be the block upper triangular matrices, more precisely the subalgebra comprising elements of the form:

\[
\begin{bmatrix}
a & Z_\beta & b \\
0 & C_{\alpha \beta}^\alpha & W^\alpha \\
0 & 0 & e
\end{bmatrix}.
\]

For our element in $\mathfrak{g} \setminus p$ let us take

\[
\begin{bmatrix}
0 & 0 & 0 \\
U^\alpha & 0 & 0 \\
0 & V_\alpha & 0
\end{bmatrix}, \quad \text{such that } U^\alpha V_\alpha = 1.
\]

Then, one readily verifies that

\[
a_1 = \left\{ \begin{bmatrix}
a & Z_\alpha & b \\
0 & C_{\alpha \beta}^\alpha & W^\alpha \\
0 & 0 & e
\end{bmatrix} \right\} \quad \text{s.t. } C_{\alpha \beta}^\alpha U^\beta = \frac{a + e}{2} U^\alpha, \quad V_\alpha C_{\alpha \beta}^\alpha = \frac{a + e}{2} V_\beta.
\]
that
\[
\alpha_2 = \begin{bmatrix} a h V_\beta & b & 0 \\ C_\alpha^\beta h U^\alpha & 0 & e \end{bmatrix}
\text{ s.t. } C_\alpha^\beta U_\beta = \frac{a + e}{2} U^\alpha,
\]
\[
V_\alpha C_\alpha^\beta = \frac{a + e}{2} V_\beta,
\]
and finally that
\[
\alpha_\infty = \alpha_3 = \begin{bmatrix} a h V_\beta & 0 & 0 \\ C_\alpha^\beta h U^\alpha & 0 & f U_\alpha \\ 0 & f V_\beta & e \end{bmatrix}
\text{ s.t. } C_\alpha^\beta U_\beta = \frac{a + e}{2} U^\alpha,
\]
\[
V_\alpha C_\alpha^\beta = \frac{a + e}{2} V_\beta,
\]
whence the full symmetry algebra is
\[
s = \begin{bmatrix} a h V_\beta & 0 & 0 \\ f U^\alpha C_\alpha^\beta h U^\alpha & 0 & f V_\beta \\ 0 & f V_\beta & e \end{bmatrix}
\text{ s.t. } C_\alpha^\beta U_\beta = \frac{a + e}{2} U^\alpha,
\]
\[
V_\alpha C_\alpha^\beta = \frac{a + e}{2} V_\beta,
\]
As a consistency check, notice that
\[
\dim s = \begin{cases} 4 & \text{for } \{h, a, e, f\} \\ +(n-1)^2 & \text{for } C_\alpha^\beta \\ -1 & \text{for the whole matrix being trace-free} \end{cases}
\]
\[
n^2 - 2n + 4
\]
and so the moduli space SL(n + 2, R)/S of distinguished curves of type (c) in the flat model has dimension
\[
((n + 2)^2 - 1) - (n^2 - 2n + 4) = 6n - 1,
\]
in agreement with (3).

In this case, the Lie algebra \( g = sl(n + 2) \) is \([2]\)-graded. The case \( n = 3 \) is sufficiently general to see what is happening to \( g = \)
\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
1 & 0 & 1 & -1 \\
1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
\end{array}
\]
This confirms immediately that \( \alpha_\infty = \alpha_3 \) and a step-by-step calculation locates different pieces of the full symmetry algebra, starting with
\[
\left(\begin{array}{c}
U_\alpha \\
V_\beta
\end{array}\right) \in g_{-1} = \begin{array}{cc}
\begin{array}{cccc}
1 & 0 & 1 & -1 \\
1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
\end{array}
\end{array}
\begin{array}{cccc}
\oplus & \oplus & \oplus & \oplus \\
\end{array}
such that \( U_\alpha V_\alpha \neq 0 \).
The symmetry algebra $s$ comprises

$$
\mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2
$$

(18)

where

$$a + C^\alpha \alpha + e = 0, \quad C^\alpha \beta U^\beta = \frac{a+e}{2} U^\alpha, \quad V^\alpha C^\alpha \beta = \frac{a+e}{2} V^\beta.\n$$

### 4.1 Contact Legendrean tractors

To proceed, we need formulæ for contact Legendrean tractors, the standard tractor bundle being modelled on the standard representation of $\text{SL}(n+2, \mathbb{R})$, namely

$$
\begin{align*}
0 & \otimes 0 \otimes 1 \otimes 1 \\
\wedge^0(0, 1) & + \quad E(-1, 0) + \quad \wedge^0(-1, 0)
\end{align*}
$$

where

$$H = E \oplus F = 1_1 \oplus 1_1 \oplus 1_2
$$

whose sections and tractor connection, in directions from $H$, we may write, in a chosen exact scale following the conventions of [8, §5.2.15], as

$$
\nabla_\alpha \left[ \begin{array}{c}
\sigma \\
\mu^\beta \\
\rho
\end{array} \right] = \left[ \begin{array}{c}
\nabla_\alpha \sigma \\
\nabla_\alpha \mu^\beta + \delta^\alpha_\beta \rho + P^\beta_\alpha \sigma \\
\nabla_\alpha \rho + A^\alpha_\beta \mu^\beta + T^\alpha \sigma
\end{array} \right]
$$

(19)

and

$$
\nabla^\alpha \left[ \begin{array}{c}
\sigma \\
\mu^\beta \\
\rho
\end{array} \right] = \left[ \begin{array}{c}
\nabla^\alpha \sigma + \mu^\alpha \\
\nabla^\alpha \mu^\beta + A^\alpha_\beta \sigma \\
\nabla^\alpha \rho + P^\beta_\alpha \mu^\beta + T^\alpha \sigma
\end{array} \right],
$$

(20)

where

$$
\begin{align*}
P^\alpha_\beta & \in \Gamma(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & -2 \\
\end{array}) \\
A^\alpha_\beta & \in \Gamma(\begin{array}{cccc}
-2 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
\end{array}) = \Gamma(\begin{array}{cccc}
\otimes & -2 & 0 & 0 \\
\otimes & 2 & 0 & 0 \\
\otimes & 0 & -1 & 0 \\
\end{array}) \\
A^{\alpha\beta} & \in \Gamma(\begin{array}{cccc}
0 & 0 & 2 & -4 \\
0 & 0 & -4 & -2 \\
1 & -1 & 0 & -1 \\
\end{array}) = \Gamma(\begin{array}{cccc}
\otimes & 0 & 2 & -4 \\
\otimes & 0 & -4 & -2 \\
\otimes & 1 & -1 & 0 \\
\otimes & -1 & 0 & -1 \\
\end{array})
\end{align*}
$$

(21)

are particular parts of the curvature whilst

$$
T_\alpha \in \Gamma(\begin{array}{cccc}
-3 & 1 & 0 & -1 \\
\otimes & 1 & 0 & -1 \\
\otimes & 0 & 1 & -1 \\
\end{array}) \quad \text{and} \quad T^\alpha \in \Gamma(\begin{array}{cccc}
1 & 0 & 1 & -3 \\
\otimes & 0 & 1 & -3 \\
\otimes & 0 & 1 & -3 \\
\end{array})
$$

are given by

$$
T_\alpha = \frac{1}{n+2}(\nabla^\beta A^\alpha_\beta - \nabla_\alpha P^\beta_\beta) \quad \text{and} \quad T^\alpha = -\frac{1}{n+2}(\nabla_\beta A^{\alpha\beta} - \nabla^\alpha P^\beta_\beta).
$$

(22)
We shall also need the dual connection on cotractors:
\[
\nabla_\alpha \begin{bmatrix} \tau \\ \psi_\beta \\ \omega \end{bmatrix} = \begin{bmatrix} \nabla_\alpha \tau - \nu_\alpha \\ \nabla_\alpha \psi_\beta - A_{\alpha \beta} \tau \\ \nabla_\alpha \omega - P_\alpha^\beta \psi_\beta - T_\alpha \tau \end{bmatrix} \quad (23)
\]
and
\[
\nabla^\alpha \begin{bmatrix} \tau \\ \psi_\beta \\ \omega \end{bmatrix} = \begin{bmatrix} \nabla^\alpha \tau \\ \nabla^\alpha \psi_\beta - \delta_\beta^\alpha \omega - P_\alpha^\beta \psi_\beta \\ \nabla_\alpha \omega - A^{\alpha \beta} \psi_\beta - T^\alpha \tau \end{bmatrix}. \quad (24)
\]

### 4.2 Adjoint tractors as endomorphisms

Following our procedure in the conformal setting, we should view the adjoint representation of \( SL(n + 2, \mathbb{R}) \), here written in case \( n = 3 \), as the trace-free endomorphisms of the standard representation. Then, an adjoint tractor of the form (16) acts by
\[
\begin{bmatrix} \sigma \\ \mu^\beta \\ \rho \end{bmatrix} \mapsto \begin{bmatrix} e Y_\beta \\ W^\alpha C^\alpha_\beta X^\alpha \\ b Z_\beta a \end{bmatrix} \begin{bmatrix} \sigma \\ \mu^\beta \\ \rho \end{bmatrix},
\]
which, since
\[
\begin{bmatrix} e Y_\beta \\ W^\alpha C^\alpha_\beta X^\alpha \\ b Z_\beta a \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a Z_\beta b \\ X^\alpha C^\alpha_\beta W^\alpha \\ d Y_\beta e \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},
\]
is consistent with Lie bracket being realised as matrix commutator. From this viewpoint, the symmetry algebra \( s \), as in (17), translates into a preferred class of endomorphisms \( \Phi \) of the standard tractor bundle. We see, for example, that
\[
\Phi \begin{bmatrix} 0 \\ f U^\alpha \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ -a/e U^\alpha \\ 0 \end{bmatrix} \quad \Phi \begin{bmatrix} 0 \\ U^\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} f \\ \frac{a + e}{h} U^\alpha \end{bmatrix} \quad \Phi \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e \\ h U^\alpha \end{bmatrix}. \quad (25)
\]

To characterise \( s \), we also need to encode that \( V_\alpha C^\alpha_\beta = a + e V_\beta \) and for this it is convenient to use cotractors
\[
1 \quad 0 \quad 0 \quad 1 = 1 \quad 1 \quad 0 \quad 0 + 1 \quad 0 \quad 0 \quad 0 + 1 \quad 0 \quad 0 \quad 0, \]
the remaining condition required to characterise \( \Phi \) as an endomorphism of cotractors being that
\[
\Phi \begin{bmatrix} 0 \\ V_\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{a}{h} V_\alpha \\ 0 \end{bmatrix}, \quad (26)
\]
(in addition to
\[
\Phi \begin{bmatrix} 0 \\ f V_\alpha \\ e \end{bmatrix} \quad \text{and} \quad \Phi \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ h V_\alpha \\ 0 \end{bmatrix},
\]
which can act as a check on consistency).
4.3 The distinguished curves

We are in a position to determine, in the contact Legendrean setting, whether a geodesic tangent to the contact distribution and of ‘type (c)’ for an exact Weyl connection is distinguished as an unparameterised curve. To be of ‘type (c)’ is precisely that its tangent vector be of the form \((U^\alpha, V_\beta)\) with \(U^\alpha V_\alpha \neq 0\) and we may suppose without loss of generality that \(U^\alpha V_\alpha \equiv 1\) along \(\gamma\). To invoke Theorem 2, we need a formula for the tractor connection \(\partial = U^\alpha \nabla_\alpha + V_\alpha \nabla^\alpha\) along \(\gamma\). According to (19) and (20), it is

\[
\partial \begin{bmatrix}
\sigma \\
\mu^\beta \\
\rho
\end{bmatrix} = \begin{bmatrix}
\partial \sigma + V_\alpha \mu^\alpha \\
\partial \mu^\beta + U^\beta \rho + U^\alpha P_\alpha^\beta \sigma + V_\alpha A^{\alpha\beta} \sigma \\
\partial \rho + U^\alpha A_\alpha \mu^\beta + V_\alpha P_\beta^\alpha \mu^\beta + U^\alpha T_\alpha \sigma + V_\alpha T^\alpha \sigma
\end{bmatrix}.
\]

As in the conformal case, we employ the Leibniz rule to compute \(\partial/\Phi\). Firstly, from (25) we find that

\[
(\partial/\Phi) \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\rho
\end{bmatrix} - \Phi \begin{bmatrix}
0 \\
U^\beta \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\rho + e/\Lambda - h
\end{bmatrix},
\]

where \(\Lambda \equiv A_\alpha U^\alpha U^\beta + P_\alpha^\beta U^\alpha V_\beta\). This is the same as the first condition from (25) save for the replacements

\[
f \mapsto \tilde{f} \equiv \partial f + \frac{a - e}{2} \quad \text{and} \quad a \mapsto \tilde{a} \equiv \partial a - h + f \Lambda.
\]

Next, we compute

\[
(\partial/\Phi) \begin{bmatrix}
0 \\
U^\beta \\
0
\end{bmatrix} = \partial \begin{bmatrix}
\frac{f}{h} U^\beta \\
\frac{a + e}{2} U^\beta \\
\rho
\end{bmatrix} - \Phi \begin{bmatrix}
0 \\
1 \\
\rho
\end{bmatrix} = \begin{bmatrix}
\partial f + \frac{a + e}{2} U^\beta \\
\partial f + \frac{a + e}{2} U^\beta + f U^\alpha P_\alpha^\beta \\
\partial f + \frac{a + e}{2} U^\beta + f U^\alpha A^{\alpha\beta}
\end{bmatrix} - \begin{bmatrix}
(h + f \Lambda) U^\beta \\
(h + f \Lambda) U^\beta \\
(h + f \Lambda) U^\beta
\end{bmatrix},
\]

which has the form

\[
\begin{bmatrix}
\tilde{f} \\
\frac{a + e}{2} U^\beta \\
\tilde{h}
\end{bmatrix}
\]

if and only if, in addition to \(\tilde{f} = \partial f + \frac{a - e}{2}\) as we already know,

\[
U^\alpha P_\alpha^\beta + V_\alpha A^{\alpha\beta} = \Xi U^\beta
\]

\[
\tilde{h} = \partial h + \frac{a - e}{2} \Lambda + f K, \quad \text{where} \quad K \equiv U^\alpha T_\alpha + V_\alpha T^\alpha
\]

\[
\tilde{e} = \partial e + h + f (2 \Xi - 3 \Lambda)
\]
for some smooth function $\Xi$ (and $\Lambda$ as above). Now, we compute

$$(\partial \Phi) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \Phi \begin{bmatrix} e \\ \epsilon hU^\beta \\ 0 \end{bmatrix} - \Phi \begin{bmatrix} 0 \\ \Xi U^\beta \\ K \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial e + h}{(\partial h)U^\beta + \epsilon \Xi U^\beta} - \frac{a + \epsilon \Xi U^\beta + fKU^\beta}{h\Xi + aK} \\ f\Xi \\ \frac{\partial e + h - f\Xi}{(\partial h + \frac{\epsilon - a}{2} \Xi - fK)U^\beta} \end{bmatrix},$$

which has the form

$$\begin{bmatrix} \tilde{e} \\ \tilde{h}U^\beta \\ 0 \end{bmatrix}$$

if and only if

$$\tilde{e} = \partial e + h - f\Xi \quad \text{so} \quad \Xi = \Lambda$$

and now

$$K = 0$$

and $\tilde{h} = \partial h + \frac{\epsilon - a}{2} \Xi = \partial h + \frac{\epsilon - a}{2} \Lambda$ is confirmed. In summary, we have

$$\tilde{f} = \partial f + \frac{\epsilon - a}{2} \Lambda \quad \tilde{h} = \partial h + \frac{\epsilon - a}{2} \Lambda$$

$$\tilde{a} = \partial a - h + f\Lambda \quad \tilde{e} = \partial e + h - f\Lambda$$

for some smooth function $\Lambda$ and the following non-trivial conditions relating $(U^\alpha, V^\beta)$ and the curvature of our chosen scale:

$$K \equiv U^\alpha T_\alpha + V_\alpha T^\alpha = 0$$

and

$$U^\alpha P_\alpha^\beta + V_\alpha A^{\alpha\beta} = \Lambda U^\beta, \quad \text{where} \quad \Lambda \equiv A_\alpha^\beta U^\alpha U^\beta + P_\alpha^\beta U^\alpha V_\beta.$$
Theorem 3 Suppose that $M$, with contact distribution $H = E \oplus F$, is a contact Legendrean manifold and that $(\nabla_\alpha, \nabla^\alpha)$ is an exact Weyl connection in the contact directions. Suppose $\gamma \hookrightarrow M$ is a geodesic for this connection, everywhere tangent to $H$ with tangent vector $(U^\alpha, V^\alpha)$ such that $U^\alpha V^\alpha = 1$. Then, $\gamma$ is distinguished as an unparameterised curve if and only if the following constraints on curvature

\begin{equation}
U^\alpha T_\alpha + V^\alpha T^\alpha = 0 \quad \text{and} \quad \begin{bmatrix}
P^\beta_{\alpha} & A^{\alpha\beta} \\
A_{\alpha\beta} & P_{\beta\alpha}
\end{bmatrix}
\begin{bmatrix}
U^\alpha \\
V^\alpha
\end{bmatrix} = \Lambda
\begin{bmatrix}
U^\beta \\
V^\beta
\end{bmatrix}
\end{equation}

(27)

are satisfied along $\gamma$ for some scalar function $\Lambda$.  

Corollary 2 All of the unparameterised ‘type (c)’ contact geodesics of an exact Weyl connection on a contact Legendrean manifold are distinguished if and only if

\begin{align}
T_\alpha &= 0, \quad T^\alpha = 0, \quad A_{\alpha\beta} = 0, \quad A^{\alpha\beta} = 0, \quad P^\alpha_{\beta} = \lambda \delta^\alpha_{\beta},
\end{align}

(28)

for some smooth function $\lambda$.

**Proof** It is elementary algebra to verify that the constraints (27) are satisfied for all $(U^\alpha, V^\alpha)$ such that $U^\alpha V^\alpha = 1$ if and only if the equations (28) hold. \qed

Recall that the geometry that we are dealing with here is contact Legendrean equipped with an exact Weyl connection, equivalently a nowhere vanishing scale $\sigma \in \Gamma(\times \cdots \times \mathbb{R}^1)$, as in [8, §5.2.14] where $\theta = 1/\sigma \in \Gamma(\times \cdots \times \mathbb{R}^1) \hookrightarrow \Gamma(\Lambda^1)$ is seen as a choice of contact form. The equations (28) should be regarded as the analogue of the Einstein equations in Riemannian geometry. The following proposition supports this analogy.

**Proposition 1** If (28) hold, then $\lambda$ is constant.

**Proof** Notice that, although $\lambda = \frac{1}{n} P^\alpha_{\alpha}$ is, in the first instance, a section of the line bundle $\times \cdots \times \mathbb{R}^1$, we are working in the presence of a scale $\sigma \in \Gamma(\times \cdots \times \mathbb{R}^1)$, which trivialises this bundle. The formulæ (22) for $T_\alpha$ and $T^\alpha$ now show that $\lambda$ is constant. \qed

**4.4 Examples**

Given the strength of the constraints (28), one might be concerned that they are only satisfied for the flat model. After all, the Einstein equations in three dimensions imply constant curvature and one needs to look in four dimensions to find non-trivial solutions. Indeed, looking ahead to the corresponding and more familiar equations (30) in CR geometry, it turns out that there are no non-trivial solutions to (28) in three dimensions. Hence, we should look in five dimensions and good places to look are homogeneous structures. Fortunately, the homogeneous contact Legendrean structures (with isotropy) have been recently classified by Doubrov, Medvedev, and The [13]. Not only that, but their examples come equipped with a natural choice of contact form. We follow their notation in asserting that the following models (some of which depend on one or two parameters)

N.8, N.7-1, N.7-2, N.6-1, N.6-2

provide non-trivial solutions of (28) with $\lambda = 0$. More interesting is a particular model of the form D.7. Specifically, we may take the contact form $\sigma = du - p \, dx - q \, dy$ with

\begin{equation}
E = \{\sigma, dx, dy\}^\perp \quad \text{and} \quad F = \{\sigma, dp - p^2 \, dx, dq - q^2 \, dy\}^\perp
\end{equation}

(29)
in local coordinates \((x, y, u, p, q)\). This produces a non-trivial solution to (28) with \(\lambda = -1/3\). We remark that the computations here are carried out using MAPLE, which is able to deal with all models from [13] save for D.6-3. Even choosing an explicit parameter, this model remains intractable (with our current program/implementation). The computations themselves are unilluminating and hence omitted.

### 4.5 A first BGG operator

By analogy with the conformal case, one might expect that the existence of a scale for which the equations (28) hold, is governed by a first BGG operator. Acting on scales, there is just one such operator:

\[
\begin{align*}
\sigma &\mapsto \begin{bmatrix}
\nabla_\alpha \nabla_\beta \sigma - A_{\alpha\beta} \sigma \\
\nabla^\alpha \nabla^\beta \sigma - A^{\alpha\beta} \sigma
\end{bmatrix}
\end{align*}
\]

where \(\nabla_\alpha\) and \(\nabla^\alpha\) are parts of the Weyl connection for that scale and \(A_{\alpha\beta}\) and \(A^{\alpha\beta}\) are parts of the corresponding curvature (21). Evidently, if \(\sigma \neq 0\) is in the kernel of this operator and we choose to view it in the scale defined by \(\sigma\), then it becomes

\[
\begin{align*}
1 &\mapsto \begin{bmatrix}
\nabla_\alpha \nabla_\beta 1 - A_{\alpha\beta} 1 \\
\nabla^\alpha \nabla^\beta 1 - A^{\alpha\beta} 1
\end{bmatrix} = \begin{bmatrix}
-A_{\alpha\beta} \\
-A^{\alpha\beta}
\end{bmatrix}
\end{align*}
\]

and we conclude that \(A_{\alpha\beta} = 0\) and \(A^{\alpha\beta} = 0\) in the scale \(\sigma\). These are part of the equations (28). It is unclear whether the remaining equations from (28) are captured by any BGG operator, although the equation \(P_{\alpha\beta} = \lambda \delta_{\alpha\beta}\) seems to be lurking in the formulæ (19) and (20) for standard tractors.

### 5 CR geometry

Computing the symmetry algebra in the CR setting according to Lemma 6 entails exactly the same arithmetic. Specifically, if we take

\[
su(n + 1, 1) = \left\{ \begin{bmatrix}
\lambda & -\bar{r}^t & iq \\
s & C & r \\
i p & -\bar{s}^t & -\bar{\lambda}
\end{bmatrix} \middle| C \text{ is skew Hermitian} \\
\text{trace}(C) + \lambda - \bar{\lambda} = 0 \right\}
\]

then we find that

\[
s = \begin{bmatrix}
x + i\theta & -h\bar{U}^t & 0 \\
fU & M + i\theta \text{Id} & hU \\
0 & -f\bar{U}^t & -x + i\theta
\end{bmatrix} \begin{bmatrix}
M \text{ is skew Hermitian} \\
MU = 0 \\
\text{trace}(M) + (n + 2)i\theta = 0
\end{bmatrix},
\]

which one recognises as the Lie algebra of (2) when \(U\) is a standard basis vector.
Similarly, the investigation of distinguished curves in CR geometry follows exactly the contact Legendrean case. Indeed, one can pursue contact Legendrean geometry in the holomorphic setting and then CR geometry is simply an alternative real form. Thus, one arrives at the following result (with conventions from [8, 22]).

**Theorem 4** Suppose \((M, H, J)\) is a CR manifold of hypersurface type and that \(\theta\) is a choice of contact form. Let \((\nabla_\alpha, \nabla_\bar{\alpha})\) denote the exact Weyl connection in the contact directions that is defined by \(\theta\). Suppose \(\gamma \hookrightarrow M\) is a geodesic for this connection, everywhere tangent to \(H\) with tangent vector \((U_\alpha, V_\bar{\alpha})\) such that \(h_\alpha \bar{\beta} U_\alpha V_\bar{\beta} = 1\), where \(h_\alpha \bar{\beta}\) is the Levi form, viewed in the scale defined by \(\theta\). Then, \(\gamma\) is distinguished as an unparameterised curve if and only if the following constraints on curvature

\[
U_\alpha T_\alpha + V_\bar{\alpha} T_\bar{\alpha} = 0 \quad \text{and} \quad \begin{bmatrix} P_\alpha \beta & A_\alpha \beta \\ A_\bar{\alpha} \bar{\beta} & P_\bar{\alpha} \bar{\beta} \end{bmatrix} \begin{bmatrix} U_\alpha \\ V_\bar{\alpha} \end{bmatrix} = \Lambda \begin{bmatrix} U_\beta \\ V_\bar{\beta} \end{bmatrix}
\]

are satisfied along \(\gamma\) for some scalar function \(\Lambda\) (where indices are raised and lowered using the Levi form \(h_\alpha \bar{\beta}\)).

**Corollary 3** All of the unparameterised type (c) contact geodesics of an exact Weyl connection on a CR manifold of hypersurface type are distinguished if and only if

\[
T_\alpha = 0, \quad T_\bar{\alpha} = 0, \quad A_{\alpha \beta} = 0, \quad A_{\bar{\alpha} \bar{\beta}} = 0, \quad P_\alpha \bar{\beta} = \lambda h_{\alpha \bar{\beta}},
\]

for some smooth function \(\lambda\), which is then necessarily constant.

**Remark** Of course, in the CR setting these curvature are ‘real’ in the sense that

\[
T_\bar{\alpha} = T_\alpha, \quad A_{\bar{\alpha} \bar{\beta}} = A_{\alpha \beta}, \quad P_\alpha \bar{\beta} = P_\bar{\alpha} \alpha,
\]

so the constraints in this corollary amount to

\[
T_\alpha = 0, \quad A_{\alpha \beta} = 0, \quad P_\alpha \bar{\beta} = \lambda h_{\alpha \bar{\beta}}, \quad \lambda \in \mathbb{R}
\]

where \(\lambda \in \mathbb{R}\). The last of these equations was already investigated by Lee [26], who adopts the terminology ‘pseudo-Einstein’ for its solutions in line with the ‘pseudo-Hermitian’ terminology of Webster [32] for a CR structure equipped with a choice of contact form. Lee [26] already remarks in his introduction that this condition is ‘very different from its analogue in Riemannian geometry… such a structure is less rigid than an Einstein structure on a Riemannian manifold.’ The additional equations in (30) seem to restore the analogy with Einstein metrics.

We remark that the constraint \(A_{\alpha \beta} = 0\) has also been considered in the literature. The tensor \(A_{\alpha \beta}\) is called the Webster torsion and its vanishing picks out the *transversally symmetric* pseudo-Hermitian geometries [26], i.e. those for which the associated Reeb field is an infinitesimal CR symmetry. The pseudo-Hermitian geometries enjoying (30) are dubbed ‘TS-pseudo-Einstein’ by Leitner [27] who constructs examples thereof on \(S^1\) bundles over Kähler–Einstein manifolds. In [7], Čap and Gover give a tractor formulation of the TS-pseudo-Einstein structures, which further strengthens the analogy with Einstein metrics in conformal geometry.

We should explain what happens when \(n = 1\), i.e. when \(M\) is three-dimensional. The equation \(P_\alpha \bar{\beta} = \lambda g_{\alpha \bar{\beta}}\) is always satisfied and \(\lambda\) is the Webster scalar curvature. The tensor \(A_{\alpha \beta}\) is equivalent to a complex-valued function (see, for example, [21] for an exposition in three dimensions). These two invariants ‘solve the equivalence problem’ in this context and, in particular, if \(\lambda = 1\) and \(A_{\alpha \bar{\beta}} = 0\), then \(M\) is the sphere \(S^3 \subset \mathbb{C}^2\) equipped with its standard CR structure and contact form.
**Remark** Regarding examples, we may consider the five-dimensional CR manifolds with many symmetries. Following [24], example N.7-2 from Sect. 4.4 has a Levi-definite real form

\[ \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid \text{Im}(z_3) = \log(1 + |z_1|^2 + |z_2|^2)\} \]

satisfying (30) with \( \lambda = 0 \). The more interesting example (29), with \( \lambda = -1/3 \), has a Levi-indefinite version, which appears in [29, 30] as:

\[ \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid \text{Im}(z_3) = \log(1 + z_1 \bar{z}_2 + \log(1 + \bar{z}_1 z_2))\} \]

and in [14] as the tube in \( \mathbb{C}^3 \) over the affine homogeneous surface

\[ \{(x, y, u) \in \mathbb{R}^3 \mid u = \log(xy)\}. \]

As observed in [14], and pointed out to us by Dennis The, this example also has a Levi-definite incarnation, namely as the tube over the affine homogeneous surface

\[ \{(x, y, u) \in \mathbb{R}^3 \mid u = \log(x^2 + y^2)\}. \]

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**Appendix: on the equation for conformal circles**

Of course, as we saw in the flat model, the reason that distinguished curves in conformal geometry are called *conformal circles* is that they are modelled on circles in \( \mathbb{R}^n \). As mentioned in (8), the Lie algebra \( \mathfrak{g} \), to be used in the Doubrov–Žádník [15] characterisation of conformal circles, is the algebra of conformal Killing fields

\[
(X^b - F^b_{\ c} \delta^c + \lambda x^b - Y_c x^c x^b + \frac{1}{2} x_c x^c Y^b) \partial/\partial x^b
\]

for constant tensors \( X^b, F^b_{\ c}, \lambda, Y_b \) with \( F_{bc} \) skew. The circles through 0 \( \in \mathbb{R}^n \) may be parameterised by a pair of vectors

\[
U^a \text{ of unit length, } \quad C^a \text{ such that } U^a C_a = 0.
\]  

(31)

Indeed, it is readily verified that

\[
\mathbb{R} \ni t \mapsto \frac{2}{1 + t^2 C^a C_a} (t U^b + t^2 C^b)
\]

is the circle with velocity \( U^a \) and acceleration \( C^a \) at the origin, where we are allowing straight lines as ‘circles’ with \( C^a = 0 \).

**Lemma 7** The symmetry algebra of the circle (32) is specified by the linear constraints

\[
X^b = f U^b \quad F^b_{\ c} U^c = -f C^b \quad Y^b = h U^b + \lambda C^b + F^b_{\ c} C^c
\]

(33)

where \( f, \lambda, \) and \( h \) are arbitrary.

**Proof** An elementary verification. \( \square \)
Note, in particular, that since the symmetry algebra of a circle is non-trivial (of dimension \((n - 1)(n - 2)/2 + 3\)), circles are homogeneous. It is well known that circles are preserved by conformal transformations. This is why they are suitable model curves in conformal geometry. We also take the opportunity to note that the moduli space of conformal circles in \(S^n\) has dimension given by

\[
\dim \SO(n + 1, 1) - \dim \Sym \text{circle} = \frac{(n + 2)(n + 1)/2 - (n - 1)(n - 2)/2 + 3}{2} = 3(n - 1),
\]
as observed geometrically in Introduction.

To unpack the Doubrov–Žádník characterisation, we shall use the conformal tractor bundle and its connection given, in the presence of a metric, by (10). We shall also need the general form of a tractor endomorphism as in (11) and the tractor directional derivative along \(\gamma\) as in (14). We conclude that \(\gamma \hookrightarrow M\) is a conformal circle if and only if we can find a subbundle \(S\) of the endomorphisms of \(T\) along \(\gamma\), having the form (11) and such that

- \(X^a\) is tangent to \(\gamma\),
- fibrewise, the subbundle \(S\) has the form (33),
- \(S\) is preserved by the tractor connection \(\partial \equiv U^a \nabla_a\) along \(\gamma\).

To proceed, it is useful to reformulate the conditions (33) as follows.

**Lemma 8** In order that an endomorphism (11) satisfy (33), for some fields \(U^a\) and \(C^a\) satisfying (31), it is necessary and sufficient that

\[
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} = \begin{bmatrix}
0 \\
-fU_b \\
\lambda
\end{bmatrix}, \\
\begin{bmatrix}
0 \\
U_b \\
0
\end{bmatrix} = \begin{bmatrix}
f \\
-fC^b_b \\
-h - fC^b_b
\end{bmatrix},
\]

\[
\begin{bmatrix}
1 \\
-C^b_b \\
0
\end{bmatrix} = \begin{bmatrix}
-\lambda \\
hU_b + \lambda C^b_b \\
\lambda C^b_b C^b_b
\end{bmatrix}.
\]

**Proof** If (33) are satisfied, then it is straightforward to verify that all equations of (34) hold. Conversely, notice that the first two equations from (34) determine \(f, U^a, \lambda, C^a\), and \(h\). The first two constraints from (33) are manifest. If we now substitute in (11), then we find that

\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
-\lambda \\
Y^b_b \\
0
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
F^b_c C^c_c \\
-\lambda Y^b_b C^b_b
\end{bmatrix}.
\]

From the second component of the last equation of (34) we conclude that \(Y^b_b = hU^b_b + \lambda C^b_b + F^b_c C^c_c\), which is the final constraint from (33).

In fact, three of the conditions in (34) are automatic as follows.

**Lemma 9** In order that an endomorphism (11) satisfy (33), for some fields \(U^a\) and \(C^a\) satisfying (31), it is necessary and sufficient that

\[
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} = \begin{bmatrix}
0 \\
-fU_b \\
\lambda
\end{bmatrix}, \\
\begin{bmatrix}
0 \\
U_b \\
0
\end{bmatrix} = \begin{bmatrix}
f \\
-fC^b_b \\
* \lambda C^b_b
\end{bmatrix}, \\
\begin{bmatrix}
1 \\
-C^b_b \\
0
\end{bmatrix} = \begin{bmatrix}
* hU^b_b + \lambda C^b_b \\
* 
\end{bmatrix}.
\]

**Proof** In proving Lemma 8, we never used the starred components.

\(\square\)
We aim to interpret Theorem 2 as a system of ordinary differential equations on the velocity $U^a$ and acceleration $C^a$ of $\gamma$. Since $X^a$ should be tangent to $\gamma$ we may write $X^b = fU^b$. This is the first constraint from (33) and, of course, our conventions have been chosen for this to be the case. Our next observation justifies $C^a$ as our choice of notation for the acceleration of both $\gamma$ and the corresponding model circle in $\mathbb{R}^n$.

**Lemma 10** In order to have a subbundle $S$ of $A|_{\gamma}$, constrained by (33) and being preserved by the tractor directional derivative $\partial$ along $\gamma$, it is necessary that the field $C_b$ in (33) be the acceleration $\partial U^b$ of $\gamma$ (defined with respect to our choice of metric $g_{ab}$).

**Proof** According to Lemma 8, the equations (33) are equivalent to (34) and, from (10), the tractor connection $\partial = U^a \nabla_a$ along $\gamma$ is given by

$$\partial \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \partial \sigma - U^a \mu_a \\ \partial \mu_b + U_b \rho + U^a P_{ab} \sigma \\ \partial \rho - U^a P_{ab} \mu_b \end{bmatrix}.$$  

(36)

The Leibniz rule now calculates the effect of $\partial \Phi$, and, in particular, we find from (34) that

$$(\partial \Phi) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \partial \begin{bmatrix} 0 \\ -f U_b \lambda \\ 0 \end{bmatrix} - \Phi \partial \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -f (\partial f - \lambda) U_b + f (\partial U_b - C_b) \\ \partial \lambda + h + f (C^a C_a + P_{ab} U^a U^b) \end{bmatrix}.$$  

Comparison with (34) shows that $C_b = \partial U_b$, as required. \qed

For a full interpretation of Theorem 2 in conformal geometry, we are obliged to investigate what it means for the tractor directional derivative $\partial$ to preserve all the equations from (34). Whilst certainly possible, some equations are automatic and, according to Lemma 9, it suffices to start with an endomorphism $\Phi$ satisfying (34) and investigate what it means for $\partial \Phi$ to satisfy (35). This investigation was begun in the proof of Lemma 10 above and next we should consider

$$(\partial \Phi) \begin{bmatrix} 0 \\ U_b \\ 0 \end{bmatrix} = \partial \left( \Phi \begin{bmatrix} 0 \\ U_b \\ 0 \end{bmatrix} \right) - \Phi \partial \begin{bmatrix} 0 \\ U_b \\ 0 \end{bmatrix},$$  

the right-hand side of which may be computed from (34) and (36). Indeed, we find

$$\partial \left( \Phi \begin{bmatrix} 0 \\ U_b \\ 0 \end{bmatrix} \right) = \partial \begin{bmatrix} f \\ -f C_b \\ -h - f C^b C_b \end{bmatrix} = \begin{bmatrix} \partial f \\ -\partial (f C_b) - h U_b - f C^a C_a U_b + f U^a P_{ab} \\ -\partial (h + f C^a C_a) + f P_{ab} U^a C^b \end{bmatrix}$$

and

$$\Phi \partial \begin{bmatrix} 0 \\ U_b \\ 0 \end{bmatrix} = \Phi \begin{bmatrix} -1 \\ C_b \\ -P_{ab} U^a U^b \end{bmatrix} = \begin{bmatrix} \lambda \\ (f P_{ac} U^a U^c - h) U_b - \lambda C_b \\ -\lambda (C^b C_b + P_{bc} U^b U^c) \end{bmatrix}$$

so

$$(\partial \Phi) \begin{bmatrix} 0 \\ U_b \\ 0 \end{bmatrix} = \begin{bmatrix} \partial f - \lambda \\ -f \lambda C_b - f E_b \\ \ast \end{bmatrix},$$

where $E_b = -f P_{bc} U^c U^b$. \hfill \(\circ\) Springer
where $E_b \equiv \partial C_b - P_{ab} U^a + (C^a C_a + P_{ac} U^a U^c) U_b$ and, for our present purposes, it does not matter what is the last entry. We conclude that $\mathcal{S}$ is preserved along $\gamma$ in accordance with Theorem 2, if and only if $E_b$ is identically zero, which is the conformal circles Eq. (5).

Finally, to complete our investigation of (35), we should compute

\[
(\partial \Phi) \begin{bmatrix} 1 \\ -C_b \\ 0 \end{bmatrix} = \partial \begin{bmatrix} 1 \\ -C_b \\ 0 \end{bmatrix} - \Phi \partial \begin{bmatrix} 1 \\ -C_b \\ 0 \end{bmatrix}.
\]

Well, from (34), we find

\[
\partial \begin{bmatrix} 1 \\ -C_b \\ 0 \end{bmatrix} = \begin{bmatrix} -\lambda h + \lambda C_b \\ \lambda C^a C_a \\ \partial \left( h U_b + \lambda C_b \right) + \lambda C^a C_a U_b - \lambda P_{ab} U^a \\
0 \\ U^a P^b a C_b \\ \partial \left( h U_b + \lambda C_b \right) + \lambda C^a C_a U_b - \lambda P_{ab} U^a \end{bmatrix}.
\]

and

\[
\Phi \partial \begin{bmatrix} 1 \\ -C_b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ U^a P^b a C_b \end{bmatrix} = \Phi \begin{bmatrix} 0 \\ 0 \\ P_{ab} U^a C^b \end{bmatrix}.
\]

where we have just used (5) to rewrite $\partial C_b - P_{ab} U^a$. From here,

\[
\Phi \partial \begin{bmatrix} 1 \\ -C_b \\ 0 \end{bmatrix} = \begin{bmatrix} f(C^a C_a + P_{ac} U^a U^c) \\ -f P_{ac} U^a C^c U_b - f(C^a C_a + P_{ac} U^a U^c) C_b \end{bmatrix}.
\]

Also note that (5) allows us to write

\[
\partial(h U_b + \lambda C_b) + \lambda C^a C_a U_b - \lambda P_{ab} U^a = (\partial h - \lambda P_{ac} U^a U^c) U_b + (\partial \lambda + h) C_b
\]

and so, finally,

\[
(\partial \Phi) \begin{bmatrix} 1 \\ -C_b \\ 0 \end{bmatrix} = \begin{bmatrix} -\partial \lambda - h + f(C^a C_a + P_{ac} U^a U^c) \\ \partial h - \lambda P_{ac} U^a U^c + f P_{ac} U^a C^c U_b \\ (\partial \lambda + h + f(C^a C_a + P_{ac} U^a U^c) C_b \end{bmatrix}.
\]

in accordance with (35). We have proved the following.

**Theorem 5** Let $M$ be a Riemannian manifold. An unparameterised curve $\gamma \hookrightarrow M$ is distinguished in the sense of Theorem 2, modelled on the circles in $\mathbb{R}^n$, if and only if (5) holds along $\gamma$.

**References**

1. Bailey, T.N., Eastwood, M.G.: Conformal circles and parametrizations of curves in conformal manifolds. Proc. Amer. Math. Soc. 108, 215–221 (1990)
2. Bailey, T.N., Eastwood, M.G., Gover, A.R.: Thomas’s structure bundle for conformal, projective and related structures. Rocky Mountain J. Math. 24, 1191–1217 (1994)

\[\text{Springer}\]
3. Baston, R.J., Eastwood, M.G.: The Penrose Transform: its Interaction with Representation Theory. Oxford University Press 1989, and reprinted by Dover Publications (2016)
4. Brinkmann, H.W.: Riemann space conformal to Einstein spaces. Math. Ann. 91, 269–278 (1924)
5. Bryant, R.L., Dunajski, M., Eastwood, M.G.: Metrisability of two-dimensional projective structures. J. Differential Geom. 83, 465–499 (2009)
6. Calderbank, D.M.J., Diemer, T.: Differential invariants and curved Bernstein-Gelfand-Gelfand sequences. J. Reine Angew. Math. 537, 67–103 (2001)
7. Čap, A., Gover, A.R.: CR-tractors and the Fefferman space. Indiana Univ. Math. J. 57, 2519–2570 (2008)
8. Čap, A., Slovák, J.: Parabolic Geometries I: Background and General Theory. American Mathematical Society (2009)
9. Čap, A., Slovák, J., Souček, V.: Bernstein-Gelfand-Gelfand sequences. Ann. Math. 154, 97–113 (2001)
10. Čap, A., Žádník, V.: On the geometry of chains. J. Differential Geom. 82, 1–33 (2009)
11. Chern, S.-S., Moser, J.K.: Real hypersurfaces in complex manifolds. Acta Math. 133, 219–271 (1974)
12. Doubrov, B.M., Komrakov, B.P., Rabinovich, M.M.: Homogeneous surfaces in the three-dimensional affine geometry. Geometry and Topology of Submanifolds VIII, pp. 168–178. World Scientific, Singapore (1996)
13. Doubrov, B.M., Medvedev, A.V., The, D.: Homogeneous integrable Legendrian contact structures in dimension five. J. Geom. Anal. 30, 3806–3858 (2020)
14. Doubrov, B.M., Medvedev, A.V., The, D.: Homogeneous Levi non-degenerate hypersurfaces in \( \mathbb{C}^3 \). Math. Z. 297, 669–709 (2021)
15. Doubrov, B.M., Žádník, V.: Equations and symmetries of generalized geodesics, Differential Geometry and its Applications. Matfyz Press, Prague, pp. 203–216 (2005)
16. Dunajski, M., Eastwood, M.G.: Metrisability of three-dimensional path geometries. Eur. J. Math. 2, 809–834 (2016)
17. Eastwood, M.G.: Uniqueness of the stereographic embedding. Arch. Math. 50, 265–271 (2014)
18. Eastwood, M.G., Graham, C.R.: Invariants of conformal densities. Duke Math. J. 63, 633–671 (1991)
19. Eastwood, M.G., Graham, C.R.: Invariants of CR densities. Proc. Sympos. Pure Math. vol. 52, Part 2, Amer. Math. Soc. 1991, pp. 117–133
20. Eastwood, M.G., Matveev, V.S.: Metric connections in projective differential geometry. In: ‘Symmetries and Overdetermined Systems of Partial Differential Equations’. IMA Volumes vol. 144. Springer, pp. 339–350 (2007)
21. Eastwood, M.G., Neusser, K.: A canonical connection on sub-Riemannian contact manifolds. Arch. Math. (Brno) 52, 277–289 (2016)
22. Gover, A.R., Graham, C.R.: CR invariant powers of the sub-Laplacian. J. Reine Angew. Math. 583, 1–27 (2005)
23. Jacobowitz, H.: Chains in CR geometry. J. Differential Geom. 21, 163–194 (1985)
24. Kruglikov, B.: Submaximally symmetric CR-structures. J. Geom. Anal. 26, 3090–3097 (2016)
25. LeBrun, C.R.: Ambi-twistors and Einstein’s equations. Classical Quantum Gravity 2, 555–563 (1985)
26. Lee, J.M.: Pseudo-Einstein structures on CR manifolds. Amer. J. Math. 110, 157–178 (1988)
27. Leitner, F.: On transversally symmetric pseudo-Einstein and Fefferman-Einstein spaces. Math. Z. 256, 443–459 (2007)
28. Liouville, R.: Sur les invariants de certaines équations différentielle et sur leurs applications. J. l’École Polytechnique 59, 7–76 (1889)
29. Loboda, A.V.: Homogeneous real hypersurfaces in \( \mathbb{C}^3 \) with two-dimensional isotropy groups, (Russian) Tr. Mat. Inst. Steklova 235, 114–142 (2001)
30. Loboda, A.V.: Homogeneous real hypersurfaces in \( \mathbb{C}^3 \) with two-dimensional isotropy groups. Proc. Steklov Inst. Math. 235, 107–135 (2001)
31. Tod, K.P.: Some examples of the behaviour of conformal geodesics. Jour. Geom. Phys. 62, 1778–1792 (2012)
32. Webster, S.M.: Pseudohermitian structures on a real hypersurface. J. Differential Geom. 13, 25–41 (1978)
33. Yano, K.: The Theory of Lie Derivatives and its Applications. North-Holland, Amsterdam (1957)
34. Žádník, V.: Generalized Geodesics. PhD thesis. Masaryk University, Brno (2003)