On the In-Out-Proper Orientations of Graphs

Ali Dehghan

1Systems and Computer Engineering Department, Carleton University, Ottawa, Canada

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Abstract

An orientation of a graph $G$ is in-out-proper if any two adjacent vertices have different in-out-degrees, where the in-out-degree of each vertex is equal to the in-degree minus the out-degree of that vertex. The in-out-proper orientation number of a graph $G$, denoted by $\overrightarrow{\chi}(G)$, is $\min_{D \in \Gamma} \max_{v \in V(G)} |d^+_D(v) - d^-_D(v)|$, where $\Gamma$ is the set of in-out-proper orientations of $G$ and $d^+_D(v)$ is the in-out-degree of the vertex $v$ in the orientation $D$. Borowiecki et al. proved that the in-out-proper orientation number is well-defined for any graph $G$ [Inform. Process. Lett., 112(1-2):1–4, 2012]. So we have $\overrightarrow{\chi}(G) \leq \Delta(G)$, where $\Delta(G)$ is the maximum degree of vertices in $G$. We conjecture that there exists a constant number $c$ such that for every planar graph $G$, we have $\overrightarrow{\chi}(G) \leq c$. Towards this speculation, we show that for every tree $T$ we have $\overrightarrow{\chi}(T) \leq 3$ and this bound is sharp. Next, we study the in-out-proper orientation number of subcubic graphs. By using the properties of totally unimodular matrices we show that there is a polynomial time algorithm to determine whether $\overrightarrow{\chi}(G) \leq 2$, for a given graph $G$ with maximum degree three. On the other hand, we show that it is NP-complete to decide whether $\overrightarrow{\chi}(G) \leq 1$ for a given bipartite graph $G$ with maximum degree three. Finally, we study the in-out-proper orientation number of regular graphs.

Key words: Proper orientation; In-out-proper orientation; In-out-proper orientation number; In-out-degree; Subcubic graphs.

1 Introduction

Let $G$ be a graph and $D$ be an orientation of it. For every vertex $v$ of $G$, we denote the in-degree (out-degree) of $v$ in the orientation $D$ by $d^+_D(v)$ ($d^-_D(v)$, respectively). An orientation of a graph $G$ is called proper if any two adjacent vertices have different in-degrees [1]. The proper orientation number of a graph $G$, denoted by $\chi(G)$, is the minimum of the maximum in-degree taken over all proper orientations of the graph $G$. A proper orientation $D$ of $G$ can be used to form a proper vertex coloring of $G$ by assigning every vertex $v$ of $G$ the color $d^+_D(v)$ [1]. So, we have

$$\chi(G) - 1 \leq \overrightarrow{\chi}(G) \leq \Delta(G).$$  (1)

The proper orientation number of graphs has been studied by several authors, for instance see [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12]. In [4], Araujo et al. asked whether the proper orientation number of a planar graph is bounded. Toward this question, it was shown that if $T$ is a tree, then $\overrightarrow{\chi}(T) \leq 4$ [4]. Also, it was shown that every cactus admits a proper orientation with maximum in-degree at most 7 [5]. Furthermore, it was proved that every bipartite planar graph with minimum degree at least 3 has proper orientation number at most 3 [13].
Let $D$ be an orientation for a given graph $G$. The in-out-degree of the vertex $v$ is defined as $d^+_D(v) = d_D^+(v) - d_D^-(v)$. Note that for a given graph $G$ and orientation $D$, for each vertex $v$ we have

$$-\Delta(G) \leq d^+_D(v) \leq \Delta(G).$$

(2)

Motivated by the proper orientations of graphs we investigate the in-out-proper orientations. An orientation of a graph $G$ is in-out-proper if any two adjacent vertices have different in-out-degrees. The in-out-proper orientation number of a graph $G$, denoted by $\leftarrow\rightarrow\chi(G)$, is $\min_{D \in \Gamma} \max_{v \in V(G)} |d^+_D(v)|$, where $\Gamma$ is the set of in-out-proper orientations of $G$ and $d^+_D(v)$ is the in-out-degree of the vertex $v$ in the orientation $D$. For a given graph $G$, we say that an in-out-proper orientation $D$ is optimal if the maximum of the absolute values of their in-out-degrees is equal to $\leftarrow\rightarrow\chi(G)$.

It is interesting to mention that in-out-proper orientation relates to the flow. In more details, an in-out-proper orientation of a graph $G$ can be thought as a ‘flow’ of $G$ that does not satisfy Kirchhoff’s Current Law. Borowiecki et al. proved that in-out-proper orientation number is well-defined for any graph $G$.\[7]\]

**Theorem 1.** The in-out-proper orientation number is well-defined for any graph $G$.

By Theorem 1 and noting that for a given graph $G$ every in-out-proper orientation defines a proper vertex coloring for $G$, we have

$$\left\lceil \frac{\chi(G) - 1}{2} \right\rceil \leq \leftarrow\rightarrow\chi(G) \leq \Delta(G).$$

(3)

**Example 1.** Let $G$ be a cycle. The degree of each vertex is two, so in each in-out-proper orientation of $G$, the in-out-degree of each vertex is $-2$, $+2$, or $0$. The graph $G$ has at least two adjacent vertices, so $\leftarrow\rightarrow\chi(G) \geq 2$. On the other hand, by Theorem 1, $\leftarrow\rightarrow\chi(G) \leq 2$. Consequently, for every cycle $C_n$ we have $\leftarrow\rightarrow\chi(G) = 2$.

Araujo et al. asked whether the proper orientation number of a planar graph is bounded. We pose the following conjecture for the in-out-proper orientation number of planar graphs.

**Conjecture 1.** There is a constant number $c$ such that for every planar graph $G$, we have $\leftarrow\rightarrow\chi(G) \leq c$.

Towards Conjecture 1, we study the in-out-proper orientation number of trees and show that for every tree $T$ we have $\leftarrow\rightarrow\chi(T) \leq 3$.

**Theorem 2.** For every tree $T$ we have $\leftarrow\rightarrow\chi(T) \leq 3$ and this bound is sharp.

A graph is called subcubic if it has maximum degree at most three. Let $G$ be a subcubic graph. By Theorem 1, we have $\leftarrow\rightarrow\chi(G) \leq 3$. By using the properties of totally unimodular matrices we show that there is a polynomial time algorithm to determine whether $\leftarrow\rightarrow\chi(G) \leq 2$.

**Theorem 3.** There is a polynomial time algorithm to determine whether $\leftarrow\rightarrow\chi(G) \leq 2$, for a given graph $G$ with maximum degree three.

On the other hand, we show that it is NP-complete to decide whether $\leftarrow\rightarrow\chi(G) \leq 1$ for a given bipartite graph $G$ with maximum degree three.

**Theorem 4.** It is NP-complete to decide whether $\leftarrow\rightarrow\chi(G) \leq 1$ for a given bipartite graph $G$ with maximum degree three.

Next, we study the computational complexity of determining the the in-out-proper orientation number of 4-regular graphs. Note that for any 4-regular graph $G$ we have $2 \leq \leftarrow\rightarrow\chi(G) \leq 4$.\[2\]
Theorem 5. It is NP-complete to decide whether $\chi'(G) \leq 2$ for a given 4-regular graph $G$.

Let $G$ be a 4-regular graph with $\chi'(G) \leq 3$ and suppose that $D$ is an optimal in-out-proper orientation. In $G$ the degree of each vertex is four, so the in-out-degree of each vertex is in $\{0, \pm 2\}$. Thus, we have $\chi'(G) \leq 3$ if and only if $\chi'(G) \leq 2$. Thus, by Theorem 5, we have the following corollary.

Corollary 1. It is NP-complete to decide whether $\chi'(G) \leq 3$ for a given 4-regular graph $G$.

The organization of the rest of the paper is as follows: In Section 2, we present some definitions and notations. This is followed in Section 3 by some bounds for the in-out-proper orientation number of subcubic graphs. Next, in Section 4, we prove that the in-out-proper orientation number of each tree is at most three. In Section 5, we focus on the in-out-proper orientation number of subcubic graphs. Section 6 is devoted to the computational complexity of regular graphs. The paper is concluded with some remarks in Section 7.

2 Definitions

In this work, all graphs are finite and simple (i.e. without loops and multiple edges). We follow [14] for terminology and notation where they are not defined here. If $G$ is a graph, then $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. For every $v \in V(G)$, $d_2(v)$ denotes the degree of $v$ in the graph $G$. Also, $\Delta(G)$ denotes the maximum degree of $G$. The distance between two vertices $v$ and $u$, denoted by $\text{distance}(v, u)$, is the length of a shortest path between them.

An orientation $D$ of a graph $G$ is a digraph obtained from the graph $G$ by replacing each edge by just one of the two possible arcs with the same endvertices. For every vertex $v$, the in-degree (the out-degree) of $v$ in the orientation $D$, denoted by $d_D^-(v)$ ($d_D^+(v)$), is the number of arcs with head (tail) $v$ in $D$. Also, the in-out-degree of $v$, denoted by $d_D^{\pm}(v)$, is defined as $d_D^+(v) - d_D^-(v)$. An orientation of a graph $G$ is in-out-proper if any two adjacent vertices have different in-out-degrees. The in-out-proper orientation number of a graph $G$, denoted by $\chi'(G)$, is $\min_{D \in \Gamma} \max_{v \in V(G)} |d_D^+(v)|$, where $\Gamma$ is the set of in-out-proper orientations of $G$.

Let $G$ be a graph. A proper vertex $t$-coloring of $G$ is a function $f: V(G) \to \{1, \ldots, t\}$ such that if $u, v \in V(G)$ are adjacent, then $f(u)$ and $f(v)$ are different. The smallest integer $t$ such that $G$ has a proper vertex $t$-coloring is called the chromatic number of $G$ and denoted by $\chi(G)$. Also, a proper edge $t$-coloring of $G$ is a function $f: E(G) \to \{1, \ldots, t\}$ such that if $e, e' \in E(G)$ have a same endvertex, then $f(e)$ and $f(e')$ are different. The smallest integer $t$ such that $G$ has a proper edge $t$-coloring is called the edge chromatic number (or chromatic index) of $G$ and denoted by $\chi'(G)$.

For a graph $G = (V, E)$, the line graph of $G$ is a graph with the set of vertices $E(G)$ and two vertices are adjacent if and only if their corresponding edges share a common endpoint in $G$.

A matrix $A$ is totally unimodular if every square submatrix of $A$ has determinant 1, 0 or $-1$. The importance of totally unimodular matrices stems from the fact that when an integer linear program has all-integer coefficients and the matrix of coefficients is totally unimodular, then the optimal solution of its relaxation is integral. Therefore, it can be obtained in polynomial time [12].

3 General bounds

For every graph $G$ we have $\chi'(G) \leq \Delta(G)$. It is good to mention that the inequality is tight for any complete graph. For any $n$, each vertex of $K_n$ can have only in-out-degree $n-1$, $n-3$, $\ldots$, $-(n-3)$, $(n-1)$. The number of these values is exactly $n$. Thus, the in-out-proper orientation number of $K_n$ is at least $n-1 = \Delta(K_n)$. 

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Next, we present some observation for the in-out-proper orientation number of graphs.

**Lemma 1.** Let $G$ be a graph with at least one edge and assume that $D$ is an in-out-proper orientation of $G$. Then in the orientation $D$ there is at least one vertex with positive in-out-degree and at least one vertex with negative in-out-degree.

**Proof.** Let $G$ be a graph with at least one edge and assume that $D$ is an in-out-proper orientation of $G$. First, we show that in $D$ there is a vertex with positive in-out-degree. To the contrary assume that the in-out-degree of each vertex is negative or zero. So, we have

$$\sum_{v \in V(G)} d^+_D(v) \leq 0. \quad (4)$$

On the other hand, we have

$$\sum_{v \in V(G)} d^-_D(v) = \sum_{v \in V(G)} d^+_D(v). \quad (5)$$

Thus, by (4) and (5), we conclude that for every vertex $v$ we have $d^+_D(v) = 0$. The graph $G$ has at least one edge, but in $D$ the in-out-degrees of all vertices are zero (so, it is not a proper vertex coloring). Thus $D$ is not an in-out-proper orientation for $G$. This is a contradiction. So, there is a vertex with positive in-out-degree. Similarly, we can show that there is a vertex with negative in-out-degree. $\square$

4 Trees

Next, we study the in-out-proper orientation number of trees and show that for every tree $T$ we have $\chi^+(T) \leq 3$. Also, we show that this bound is sharp.

**Proof of Theorem 4.** First we show that for each tree $T$ we have $\chi^+(T) \leq 3$. Let $T$ be a tree with $n$ vertices and $v$ be a vertex of $T$. Sort the vertices of $T$ according to their distance from $v$ and let $v = v_1, v_2, \ldots, v_n$ be that sorted set. For each vertex $u$, the father of $u$, denoted by $f(u)$, is the unique vertex that is adjacent and closer to the root $v$. Perform the Algorithm 1 and call the resultant orientation $D$.

We have the following properties for the orientation $D$ that we obtained from Algorithm 1.

**Proposition 1.** Let $u$ be a vertex with $d(u) \geq 3$. If $u$ has an even distance from the root $v_1$, then $d^+_D(u) \in \{1, 2, 3\}$. Also, if $u$ has an odd distance from the root $v_1$, then $d^+_D(u) \in \{-1, -2, -3\}$.

**Proof.** Let $u$ be a vertex with $d(u) \geq 3$. If $u$ has an even distance from the root $v_1$, then at Lines 2-3, 5-6, and 8-9, we orient the edges incident with $u$ such that the in-out-degree of $u$ is in $\{1, 2, 3\}$. There is only one other part of the algorithm that we may change the in-out-degree of $u$. That part is Lines 35-36. In that case the in-out-degree of $u$ is one and by reorienting one of the edges that is incident with $u$ we increase the in-out-degree of $u$ by two. Similarly, if $u$ has an odd distance from the root $v$, then at Lines 23-24, 26-27, and 17-18, we orient the edges incident with $u$ such that the in-out-degree of $u$ is in $\{-1, -2, -3\}$. $\square$

**Proposition 2.** Let $u$ be a vertex with $d(u) = 2$. If $u$ has an even distance from the root $v_1$, then $d^+_D(u) \in \{0, 1, 2\}$. Also, if $u$ has an odd distance from the root $v_1$, then $d^+_D(u) \in \{0, -1, -2\}$.

**Proof.** Let $u$ be a vertex with $d(u) = 2$. If $u$ has an even distance from the root $v_1$, then at Lines 2-3, 5-6, and 10-14, we orient the edges incident with $u$ such that the in-out-degree of $u$ is in $\{0, 1, 2\}$. There is only one other part of the algorithm that we may change the in-out-degree of $u$. That part is Lines 31-32. In that case the in-out-degree of $u$ is zero and by reorienting one of the edges that is incident with $u$ we increase the
Algorithm 1

1: for $i = 1$ to $n$ do
2: \hspace{1em} if $i = 1$ then
3: \hspace{2em} Orient the edges incident with $v_1$ such that if $d(v_1)$ is an even number then $d_D^+(v_1) = 2$, and if $d(v_1)$ is an odd number then $d_D^+(v_1) = 1$.
4: \hspace{2em} else if $\text{distance}(v_i, v_1)$ is an even number then
5: \hspace{3em} if the edge $v_if(v_i)$ was oriented from $f(v_i)$ to $v_i$ then
6: \hspace{4em} Orient the set of edges $\{v_iv_j|j > i\}$ such that if $d(v_i)$ is an odd number then $d_D^+(v_i) = 1$, and if $d(v_i)$ is an even number then $d_D^+(v_i) = 2$.
7: \hspace{2em} else if the edge $v_if(v_i)$ was oriented from $v_i$ to $f(v_i)$ then
8: \hspace{3em} if $d(v_i) \geq 3$ then
9: \hspace{4em} Orient the set of edges $\{v_iv_j|j > i\}$ such that $d_D^+(v_i) \in \{1, 2\}$
10: \hspace{3em} else if $d(v_i) = 2$ then
11: \hspace{4em} if $d_D^+(f(v_i)) \neq 0$ then
12: \hspace{5em} Orient the edge $\{v_iv_j|j > i\}$ such that $d_D^+(v_i) = 0$
13: \hspace{4em} else if $d_D^+(f(v_i)) = 0$ then
14: \hspace{5em} Orient the set of edges incident with $v_i$ such that $d_D^+(v_i) = 2$ (note that we reorient the edge $v_if(v_i)$).
15: \hspace{3em} end if
16: \hspace{3em} else if $d(v_i) = 1$ then
17: \hspace{4em} if $d_D^+(f(v_i)) = -1$ then
18: \hspace{5em} Reorient the edge $v_if(v_i)$ from $f(v_i)$ to $v_i$
19: \hspace{3em} end if
20: \hspace{3em} end if
21: \hspace{2em} end if
22: \hspace{2em} else if $\text{distance}(v_i, v_1)$ is an odd number then
23: \hspace{3em} if the edge $v_if(v_i)$ was oriented from $v_i$ to $f(v_i)$ then
24: \hspace{4em} Orient the set of edges $\{v_iv_j|j > i\}$ such that if $d(v_i)$ is an odd number then $d_D^+(v_i) = -1$, and if $d(v_i)$ is an even number then $d_D^+(v_i) = -2$.
25: \hspace{3em} else if the edge $v_if(v_i)$ was oriented from $f(v_i)$ to $v_i$ then
26: \hspace{4em} if $d(v_i) \geq 3$ then
27: \hspace{5em} Orient the set of edges $\{v_iv_j|j > i\}$ such that $d_D^+(v_i) \in \{-1, -2\}$
28: \hspace{4em} else if $d(v_i) = 2$ then
29: \hspace{5em} if $d_D^+(f(v_i)) \neq 0$ then
30: \hspace{6em} Orient the edge $\{v_iv_j|j > i\}$ such that $d_D^+(v_i) = 0$
31: \hspace{5em} else if $d_D^+(f(v_i)) = 0$ then
32: \hspace{6em} Orient the set of edges incident with $v_i$ such that $d_D^+(v_i) = -2$ (note that we reorient the edge $v_if(v_i)$).
33: \hspace{5em} end if
34: \hspace{5em} else if $d(v_i) = 1$ then
35: \hspace{6em} if $d_D^+(f(v_i)) = 1$ then
36: \hspace{7em} Reorient the edge $v_if(v_i)$ from $v_i$ to $f(v_i)$
37: \hspace{6em} end if
38: \hspace{5em} end if
39: \hspace{5em} end if
40: \hspace{4em} end if
41: \hspace{2em} end if
in-out-degree of $u$ by two. So, the final in-out-degree of $u$ is in $\{0, 1, 2\}$. Similarly, if $u$ has an odd distance from the root $v_1$, then at Lines 23-24, 28-32, and 13-14, we orient the edges incident with $u$ such that the in-out-degree of $u$ is in $\{0, -1, -2\}$. 

**Proposition 3.** Let $u$ and $u'$ be two adjacent vertices such that $d(u) = d(u') = 2$. Then $d_D^+(u) \neq d_D^+(u')$.

**Proof.** By Lines 10-14 and Lines 28-32, the algorithm does not produce any two adjacent vertices $u, u'$ such that $d(u) = d(u') = 2$ and $d_D^+(u) = d_D^+(u') = 0$. Thus, by Proposition 2 for any two adjacent vertices $u, u'$ with $d(u) = d(u') = 2$ we have $d_D^+(u) \neq d_D^+(u')$. 

**Proposition 4.** Let $u$ be a vertex with $d(u) = 1$. Then $d_D^+(u) \in \{-1, +1\}$ and $d_D^+(u) \neq d_D^+(f(u))$.

**Proof.** Let $u$ be a vertex with $d(u) = 1$. If $u$ has an even distance from the root $v_1$, then at Lines 2-3, 5-6, and 16-19, we orient the edge incident with $u$ such that the in-out-degree of $u$ is in $\{-1, +1\}$ and also if it is $-1$ then $d_D^+(u) \neq d_D^+(f(u))$. By Propositions 1, 2 we also conclude that if the in-out-degree of $u$ is $1$, then $d_D^+(u) \neq d_D^+(f(u))$. Similarly, if $u$ has an odd distance from the root $v_1$, then at Lines 23-24, and 34-37, we orient the edge incident with $u$ such that the in-out-degree of $u$ is in $\{-1, +1\}$ and if it is $-1$, then $d_D^+(u) \neq d_D^+(f(u))$. By Propositions 1, 2 we conclude that if the in-out-degree of $u$ is $-1$, then $d_D^+(u) \neq d_D^+(f(u))$. This completes the proof.

By Propositions 1, 2, 3, and 4 for every vertex $u$, we have $d_D^+(u) \in \{\pm 3, \pm 2, \pm 1, 0\}$ and for every two adjacent vertices $u, u'$, we have $d_D^+(u) \neq d_D^+(u')$. Thus $D$ is an in-out-proper orientation such that the maximum of absolute values of their in-out-degrees is at most three.

Finally, we show that there is a tree $T$ such that $\chi'(T) = 3$. Consider the tree $T$ with the set of vertices $v_1, v_2, v_3, v_4$ and set of edges $v_1v_2, v_1v_3, v_1v_4$. We have $d(v_2) = d(v_3) = d(v_4) = 1$, so in any orientation of $T$, their in-out-degrees are in $\{\pm 1\}$. On the other hand, the degree of $v_1$ is three, so its in-out-degree is in $\{\pm 1, \pm 3\}$. To the contrary assume that $\chi'(T) < 3$, and let $D$ be an in-out-proper orientation of $T$ such that $d_D^+(v_1) \in \{\pm 1\}$. The in-out-degree of at least one of the vertices $v_2, v_3, v_4$ is $1$ (otherwise $d_D^+(v_1) \notin \{\pm 1\}$) and also the in-out-degree of at least one of the vertices $v_2, v_3, v_4$ is $-1$. So, $D$ has two adjacent vertices with the same in-out-degree. Thus, it is not an in-out-proper orientation. This is a contradiction. So, we conclude that $\chi'(T) = 3$.

**5 Subcubic graphs**

In this section we focus on subcubic graphs. Let $G$ be a subcubic graph. By Theorem 1 we have $\chi'(G) \leq 3$. Next, we show that there is a polynomial time algorithm to determine whether $\chi'(G) \leq 2$. On the other hand, it is NP-complete to decide whether $\chi'(G) \leq 1$ for a given bipartite graph $G$ with maximum degree three.

**Proof of Theorem 5.** Let $G$ be a cubic graph. If $D$ is an optimal in-out-proper orientation, then for any vertex $v$ of degree two we have $d_D^+(v) \in \{0, \pm 2\}$ and also for any vertex $v$ of degree one or three we have $d_D^+(v) \in \{\pm 1\}$. First, we investigate the subcubic graphs without degree two vertices. Then, we present a polynomial time algorithm for subcubic graphs. Let $G$ be a graph such that the degree of each vertex is one or three and without loss of generality assume that $G$ is connected. Also, suppose that $\chi'(G) \leq 2$ and $D$ is an optimal in-out-proper orientation of $G$. Since $d_D^+(v) \in \{\pm 1\}$ for each vertex $v$ and the in-out-degrees form a proper vertex coloring of $G$, $G$ should be bipartite.

**Proposition 5.** Let $G$ be a graph such that the degree of each vertex is one or three. If $\chi'(G) \leq 2$, then $G$ is bipartite.
Consequently, at step one we should check that whether $G$ is bipartite. Next, at step two we want to determine whether it is possible to orient the edges of $G$ such that in-out-degrees of vertices of one partite set of $G$ are 1 and in-out-degrees of vertices of the other partite set of $G$ are $-1$.

**Proposition 6.** Let $G = (X \cup Y, E)$ be a bipartite graph such that the degree of each vertex is one or three. If $\overrightarrow{\chi}(G) \leq 2$, then there is an orientation for the edges of $G$ such that the in-out-degree of each vertex is 1 or $-1$, and the in-out-degrees of all vertices in $X$ are the same, and so are those in $Y$.

It is well-known that there is a polynomial time algorithm to decide whether a given graph is bipartite [14]. Next, we present a polynomial time algorithm for step two. Let $G = (X \cup Y, E(G))$ be a bipartite graph such that the degree of each vertex is one or three. Without loss of generality assume that $X = x_1, x_2, \ldots, x_n$ and $Y = y_1, y_2, \ldots, y_m$. From the graph $G$ we construct a bipartite graph $H$ with vertex set $V(H) = (U_X \cup U_Y) \cup U_0$ and edge set $E(H) = W$. Put $U_X = X$, and $U_Y = Y$. Also, for every edge $x_iy_j \in E(G)$, put $x_{i,j}$ in $U_0$ and the edges $w_{i,j} = x_{i}x_{i,j}, w'_{i,j} = x_{i,j}y_j$ in $W$. See Fig. 1.

Consider the following integer linear program for the graph $H$.

Maximize 1

subject to

$\sum_{x_iy_j \in E(G)} w_{i,j} = 1 \quad \forall x_i \in U_X \text{ s.t. } d_G(x_i) = 1$ 

$\sum_{x_iy_j \in E(G)} w_{i,j} = 2 \quad \forall x_i \in U_X \text{ s.t. } d_G(x_i) = 3$ 

$\sum_{x_iy_j \in E(G)} w'_{i,j} = 0 \quad \forall y_j \in U_Y \text{ s.t. } d_G(y_j) = 1$ 

$\sum_{x_iy_j \in E(G)} w'_{i,j} = 1 \quad \forall y_j \in U_Y \text{ s.t. } d_G(y_j) = 3$ 

$w_{i,j} + w'_{i,j} = 1 \quad \forall x_{i,j} \in U_0$ 

$w_{i,j}, w'_{i,j} \in \{0, 1\} \quad \forall x_iy_j \in E(G)$

Note that we can write the above integer linear program in the following canonical form:

Maximize 1

subject to $Ax = b$ 

$x \in \{0, 1\}^{|E(H)|}$,
where \( A \) is the incidence matrix of \( H \), \( x^T = (w_{1,1}, \ldots, w_{3,3}) \), and \( b \in \{0, 1, 2\}^{|V(H)|} \). For instance, for the graph \( H \) that was shown in Fig. [1], we have \( x^T = (w_{1,1}, w'_{1,1}, w_{1,2}, w'_{1,2}, w_{1,3}, w'_{1,3}) \), \( b = (2, 0, 0, 1, 1, 1) \) and \( A \) is

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\]

For each edge \( x_iy_j \in E(G) \) in [11], we consider two variables \( w_{i,j}, w'_{i,j} \) such that \( w_{i,j}, w'_{i,j} \in \{0, 1\} \). On the other hand, in [10], we have \( w_{i,j} + w'_{i,j} = 1 \), so the value of exactly one of these variables is one and the value of the other variable is zero. We consider the values of \( w_{i,j}, w'_{i,j} \) as an orientation for the edge \( x_iy_j \) in \( G \) such that it is oriented from \( y_j \) to \( x_i \) if and only if \( w_{i,j} = 1 \). So, the values of the variables correspond to an orientation for the graph \( G \). Call that orientation \( D \). By (6) and (7), we ensure that in \( D \) the in-out-degree of each vertex in \( X \) is 1. Also, by (5) and (9), the in-out-degree of each vertex in \( Y \) is –1. Consequently, the above integer linear program is feasible if and only if the graph \( G \) has an in-out-proper orientation such that the in-out-degree of each vertex in \( X \) is 1 and the in-out-degree of each vertex in \( Y \) is –1.

When an integer linear program has all-integer coefficients and the matrix of coefficients is totally unimodular, then the optimal solution of its relaxation is integral. Therefore, it can be obtained in polynomial time [12]. On the other hand, it is shown in [12], Corollary 2.9 in Page 544] that every incidence matrix of a bipartite graph is totally unimodular. So, in our integer linear program the matrix of coefficients is totally unimodular. Consequently, there is a polynomial time algorithm to determine whether the above-mentioned integer linear program is feasible.

Note that there is an orientation of the edges of \( G \) such that the in-out-degree of each vertex in \( X \) is 1 and the in-out-degree of each vertex in \( Y \) is 1 if and only if there is an orientation of the edges of \( G \) such that the in-out-degree of each vertex in \( X \) is 1 and the in-out-degree of each vertex in \( Y \) is –1 (by considering the reverse of the given orientation). Consequently, there is a polynomial time algorithm to determine whether the in-out-proper orientation number of given graph \( G \) with degree set \( \{1, 3\} \) is at most two.

Next, we consider the set of subcubic graphs. Let \( G \) be a subcubic graph. If \( D \) is an optimal in-out-proper orientation, then for any vertex \( v \) of degree two we have \( d^+_D(v) \in \{0, \pm 2\} \) and also for any vertex \( v \) of degree one or three we have \( d^+_D(v) \in \{\pm 1\} \).

Remove all vertices of degree two from the graph \( G \) and call the resultant graph \( G' \). For each vertex \( v \) in \( G' \) if \( d_G(v) \neq d_{G'}(v) \), then put \( d_G(v) - d_{G'}(v) \) isolated vertices and join them to \( v \) (we call these new vertices dummy vertices). Call the resultant graph \( G'' \). Note that in \( G'' \) the degree of each vertex is one or three. See Fig. [2]

Assume that \( \chi'(G) \leq 2 \). Next, we present some necessary conditions for \( G'' \).

**Proposition 7.** Let \( C_1, C_2, \ldots, C_k \) be all the connected components of \( G'' \). For any \( i \in \{1, 2, \ldots, k\} \), \( C_i \) is bipartite and there exists an orientation \( D_i \) of \( C_i \) satisfying

(a) every vertex of \( C_i \) has the in-out-degree 1 or –1, and

(b) for any \( uv \in E(G) \cap E(G'') \), \( d^+_{D_i}(u) \neq d^+_{D_i}(v) \).

**Proof.** By Proposition [5] and Proposition [6] the proof is clear.

In order to complete the proof we do the following steps:
Step 1. Proving that the condition in Proposition 7 is a necessary and sufficient one for $\chi(G) \leq 2$.

Step 2. Showing that the condition in Proposition 7 can be checked in polynomial time.

Step 3. Concluding that Theorem 3 is true by Step 1 and Step 2.

(Proof of Step 1:) We show that the necessary conditions that are presented in Proposition 7 are also sufficient. In other words, we prove that if each connected component of $G''$ is bipartite and also if the graph $G''$ has an orientation such that in each connected component, the in-out-degrees of vertices in different parts (of that bipartite component), except dummy vertices, are different, then we can extend that partial orientation to an in-out-proper orientation of $G$ such that the maximum of absolute values of their in-out-degrees is at most two. To prove that it is enough, we show the following proposition.

**Proposition 8.** Each path $P_n = v_1, v_2, \ldots, v_n$ of length at least two (i.e. $n \geq 3$) has the following four kinds of in-out-proper orientations:

1. The in-out-proper orientation $D_1$ such that $d^+_D(v_1) = d^-_D(v_n) = 1$.
2. The in-out-proper orientation $D_2$ such that $d^+_D(v_1) = d^-_D(v_n) = -1$.
3. The in-out-proper orientation $D_3$ such that $d^+_D(v_1) = 1$ and $d^-_D(v_n) = -1$.
4. The in-out-proper orientation $D_4$ such that $d^+_D(v_1) = -1$ and $d^-_D(v_n) = 1$.

*Proof.* Let $P_n = v_1, v_2, \ldots, v_n$ be a path of length at least two and $D \in \{D_1, D_2, D_3, D_4\}$. Orient the edges $v_1v_2$ and $v_{n-1}v_n$ such that the in-out-degree of $v_1$ is $d^+_D(v_1)$ and the in-out-degree of $v_n$ is $d^-_D(v_n)$. Do Algorithm 2 to orient the remaining edges.

*Algorithm 2*

1: for $i = 2$ to $n-2$ do
2: if $d^+_D(v_{i-1}) = 0$ and $v_{i-1}v_i$ was oriented from $v_{i-1}$ to $v_i$ then
3: Orient $v_{i-1}v_i$ from $v_{i+1}$ to $v_i$
4: else
5: Orient $v_{i-1}v_i$ from $v_i$ to $v_{i+1}$
6: end if
7: end for

By Algorithm 2 there is no two consecutive vertices with the in-out-degree 0. On the other hand, it not possible to have two consecutive vertices with the in-out-degree 2 or $-2$. Moreover, the in-out-degree of $v_n$ is in $\{\pm 1\}$ and the in-out-degree of $v_{n-1}$ is in $\{0, \pm 2\}$. So $D$ is an in-out-proper orientation.

(Proof of Step 2:) To check whether $G''$ has such an orientation we can use the previous mentioned integer linear program with some modifications. In fact, for each dummy vertex $v$ we just remove the corresponding condition in (6) or (8), and then we solve the integer linear program.

(Proof of Step 3:) Having Propositions 6 and 8 and noting that there is a polynomial time algorithm to check Proposition 8 we conclude that there is a polynomial time algorithm to decide whether in-out-proper orientation number of a given subcubic graph is at most two. This completes the proof.

---

Figure 2: The graph $G$ and its corresponding graph $G''$. In the graph $G''$ the degree of each vertex is 1 or 3 and the set of blue vertices are dummy vertices.
Next, we prove that it is NP-complete to decide whether $\chi^-(G) \leq 1$ for a given bipartite graph $G$ with maximum degree three.

**Proof of Theorem** It was shown in \[9\] that the following variant of Not-All-Equal satisfying assignment problem is NP-complete.

**Problem:** Cubic Monotone Not-All-Equal (2,3)-Sat.

**Input:** Set $X$ of variables, collection $C$ of clauses over $X$ such that every clause $c \in C$ has $|c| \in \{2, 3\}$, each variable appears in exactly three clauses and there is no negation in the formula.

**Question:** Is there a truth assignment for $X$ such that every clause in $C$ has at least one true literal and at least one false literal?

Our proof is a polynomial time reduction from Cubic Monotone Not-All-Equal (2,3)-Sat. Let $\Phi$ be an instance with the set of variables $X$ and the set of clauses $C$. We transform it to a bipartite graph $G_\Phi$ with maximum degree three in polynomial time such that $\chi^-(G_\Phi) \leq 1$ if and only if $\Phi$ has a Not-All-Equal truth assignment. We use the auxiliary gadget $I_x$ which is shown in Fig. \[8\] Our construction consists of three steps.

**Step 1.** For each variable $x \in X$ put a copy of the gadget $I_x$ which is shown in Fig. \[8\]

**Step 2.** For each clause $c \in C$ put a vertex $c$ and then for each variable $x$ that appears in the clause $c$ join the vertex $c$ to one of the vertices $x_1, x_2, x_3$ of $I_x$ such that in the resultant graph for each variable $x \in X$ in the gadget $I_x$ the degrees of the variables $x_1, x_2, x_3$ are two. Call the resultant graph $H_\Phi$.

**Step 3.** For each clause $c = (x \lor x') \in C$, without loss of generality assume that $cx_1, cx'_1 \in E(H_\Phi)$. Merge the three vertices $c, x_1, x'_1$ into a new vertex $c'$.

Call the resultant graph $G_\Phi$. The degree of every vertex in the graph $G_\Phi$ is 1, 2 or 3 and the resultant graph is bipartite. Let us now prove that $\chi^-(G_\Phi) \leq 1$ if and only if $\Phi$ has a Not-All-Equal truth assignment.

First, assume that $\chi^-(G_\Phi) \leq 1$. We have the following properties.

**Proposition 9.** Consider the gadget $I_x$ which is shown in Fig. \[8\] Let $D$ be an orientation of $I_x$ such that the in-out-degree of each vertex is in $\{0, \pm 1\}$ and the endvertices of any edge in $I_x$, except three edges incident with the vertices $x_1, x_2, x_3$, have different in-out-degrees, then $d^+_D(x_1) = d^-_D(x_2) = d^-_D(x_3) = 1$ or $d^+_D(x_1) = d^+_D(x_2) = d^-_D(x_3) = -1$.

**Proof.** Note that in the proof of this proposition the notation and colors that we refer are depicted in Fig. \[8\] In the orientation $D$ the in-out-degree of each vertex is in $\{0, \pm 1\}$. On the other hand, the degree of each vertex is one or three, so the in-out-degree of each vertex is in $\{\pm 1\}$. In orientation $D$ the endvertices of any edge, except three edges incident with the vertices $x_1, x_2, x_3$, have different in-out-degrees. Thus, the red vertices have the same in-out-degree and also the blue vertices have the same in-out-degree. Now, two cases can be considered.

**Case 1.** The blue vertices have the in-out-degree 1. Then the red vertices have the in-out-degree $-1$. We
Case 1. We can show that Case 2. The blue vertices have the in-out-degree 1. Let $\Gamma : X \to \{\text{true}, \text{false}\}$ have $d^+_{D}(v_1) = d^+_{D}(v_2) = 1$, so the edges $v_1u_1, v_2u_1$ were oriented form $u_1$ to $v_1$ and $v_2$, respectively. The in-out-degree of $u_1$ is $-1$. Thus, the edge $z_1u_1$ was oriented from $z_1$ to $u_1$. The in-out-degree of $z_1$ is 1 and thus the edge $wz_1$ was oriented from $w$ to $z_1$. We have the same situation for $z_2$. Its in-out-degree is 1 and the edge $wz_2$ was oriented from $w$ to $z_2$. The vertex $w$ is a red vertex and its in-out-degree is $-1$. On the other hand, the edges $wz_1, wz_2$ were oriented from $w$ to $z_1$ and $z_2$. Thus, the edge $wx_2$ was oriented from $w$ to $x_2$ and consequently $d^+_{D}(x_2) = 1$. We have the same conclusion for $x_1$ and $x_3$. Thus, $d^+_{D}(x_1) = d^+_{D}(x_2) = d^+_{D}(x_3) = 1$. 

Case 2. The blue vertices have the in-out-degree $-1$. Then the red vertices have the in-out-degree 1. Similar to Case 1, we can show that $d^+_{D}(x_1) = d^+_{D}(x_2) = d^+_{D}(x_3) = -1$. This completes the proof.

Let $D$ be an optimal in-out-proper orientation of $G_\Phi$. Now, we present a Not-All-Equal truth assignment for the formula $\Phi$. Let $\Gamma : X \to \{\text{true}, \text{false}\}$ be the assignment defined by $\Gamma(x_i) = \text{true}$ if the blue vertices in $I_x$ have the in-out-degree 1, and $\Gamma(x_i) = \text{false}$ if the blue vertices in $I_x$ have the in-out-degree $-1$. 

Next, we prove that $\Gamma$ is a Not-All-Equal truth assignment for $\Phi$. Let $c = (x \lor y \lor r)$ and without loss of generality assume that $cx_1, cy_1, cr_1 \in E(G_\Phi)$. The degree of the vertex $c$ is three, so $d^+_D(c) \in \{\pm 1\}$. Thus, at least one of the edges incident with $c$ was oriented from $c$ to the other endpoint. Note that the other endpoint is one of the vertices $x_1, y_1, r_1$. Also, at least one of the edges incident with $c$ was oriented toward $c$. On the other hand, the degree of vertices $x_1, y_1, r_1$ are two, so $d^+_D(x_1) = d^+_D(y_1) = d^+_D(r_1) = 0$. Thus, $\text{true, false} \in \{\Gamma(x), \Gamma(y), \Gamma(r)\}$. Next, assume that $c = (x \lor y)$. The degree of the vertex $c'$ (that corresponds to the clause $c$ in $C$) is two. So, $d^+_D(c') = 0$. Thus, $\text{true, false} \in \{\Gamma(x), \Gamma(y)\}$.

Now, assume that there is a Not-All-Equal assignment $\Gamma : X \to \{\text{true}, \text{false}\}$ for $\Phi$. For each variable $x \in X$ if $\Gamma(x) = \text{true}$ then orient $I_x$ like Type 2 in Fig. 4 and if $\Gamma(x) = \text{false}$ then orient $I_x$ like Type 1 in Fig. 4 Also, for each clause $c = (x \lor y \lor r)$ orient the edges incident with $c$ such that the in-out-degree of each neighbor of $c$ is 0. Call the resultant orientation $D$. The function $\Gamma$ is a Not-All-Equal assignment, so $D$ is an in-out-proper orientation such that the maximum of absolute values of their in-out-degrees is one.

Figure 4: The two possible orientations of $I_x$. 

![Type 1 and Type 2 orientations](image-url)
This completes the proof. 

\[ \square \]

6 Regular graphs

Next, we study the computational complexity of determining the in-out-proper orientation number of 4-regular graphs.

Proof of Theorem 6. It was shown that it is NP-complete to determine whether the edge chromatic number of a given 3-regular graph is three (see [14]). We reduce this problem to our problem in polynomial time. For a given 3-regular graph \( G \) we construct a 4-regular graph \( H \) such that the edge chromatic number of \( G \) is three if and only if \( \chi_e(H) \leq 2 \).

For a given graph \( G \) with the set of edges \( e_1, e_2, \ldots, e_n \), let \( H \) be the line graph of \( G \) with the set of vertices \( v_1, v_2, \ldots, v_n \), such that \( v_e \) \( v_j \) \( E(H) \) if and only if \( e_i \) and \( e_j \) have a common endvertex. First, assume that the in-out-proper orientation number of \( H \) is two and let \( D \) be an optimal in-out-proper orientation. The orientation \( D \) defines a proper vertex 3-coloring for the vertices of \( H \) using three colors 0, ±2. Thus, \( G \) has a proper edge 3-coloring.

Next, assume that the edge chromatic number of \( G \) is three and let \( f : E(G) \to \{1, 2, 3\} \) be a proper edge 3-coloring of \( G \). Define the function \( h : V(H) \to \{1, 2, 3\} \) such that \( h(e_i) = k \) if and only if \( f(e_i) = k \), for each \( k = 1, 2, 3 \). Let \( K \) be the subset of edges of \( H \) such that for each edge \( v_e, v_j \) \( K \) we have \( \{h(v_e), h(v_j)\} = \{1, 3\} \). In the subgraph \( H \backslash K \) the degree of each vertex is even. In fact the degree of each vertex \( v_e \) with \( h(v_e) = 2 \) is four and also the degree of each vertex \( v_e \) with \( h(v_e) \in \{1, 3\} \) is two. So we can orient the edges in \( H \backslash K \) such that the in-degree of each vertex is equal to its out-degree. Next, for each edge \( v_e, v_j \) \( K \) orient it from \( v_e \) to \( v_j \) if \( h(v_e) = 1 \) and \( h(v_j) = 3 \), otherwise orient it from \( v_j \) to \( v_e \). Consider the union of orientations for \( H \backslash K \) and \( K \) and call the resultant orientation \( D \). In \( D \) the in-out-degree of each vertex \( v_e \) with \( h(v_e) = 1 \) \( h(v_e) = 2 \), \( h(v_e) = 3 \), respectively) is \(-2 \) \((0, 2, \text{respectively}) \). Thus, \( D \) is an in-out-proper orientation such that the maximum of absolute values of their in-out-degree is two. This completes the proof. 

\[ \square \]

7 Conclusions and future research

In this work we studied the in-out-proper orientation number of graphs. We proved that for any graph \( G \), \( \chi_i(G) \leq \Delta(G) \). We conjectured that there exists a constant number \( c \) such that for every planar graph \( G \), we have \( \chi_i(G) \leq c \). Regarding this conjecture, we showed that for every tree \( T \) we have \( \chi_i(T) \leq 3 \) and this bound is sharp. It is interesting to prove constant bounds for other families of planar graphs.

We also studied the in-out-proper orientation number of subcubic graphs. By using the properties of totally unimodular matrices we proved that there is a polynomial time algorithm to determine whether \( \chi_i(G) \leq 2 \), for a given graph \( G \) with maximum degree three. It is interesting to present a polynomial time algorithm for other families of graphs.

It is also interesting to characterize all graphs \( G \) which satisfy \( \chi_i(G) = \chi_i(G) \). It would be interesting to attack this problem for the family of regular graphs.
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