Hyperbolic groups, 4-manifolds and Quantum Gravity

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Abstract. 4-manifolds have special topological properties which can be used to get a different view on quantum mechanics. One important property (connected with exotic smoothness) is the natural appearance of 3-manifold wild embeddings (Alexanders horned sphere) which can be interpreted as quantum states. This relation can be confirmed by using the Turaev-Drinfeld quantization procedure. Every part of the wild embedding admits a hyperbolic geometry uncovering a deep connection between quantum mechanics and hyperbolic geometry. Then the corresponding symmetry is used to get a dimensional reduction from 4 to 2 for infinite curvatures. Physical consequences will be discussed. At the end we will obtain a spacetime representation of a quantum state of geometry by a non-singular fractal space (wild embedding) which is stable in the limit of infinite curvatures.

Keywords: quantum geometry, wild embeddings, large curvature limit, dimensional reduction

1. Introduction
The construction of quantum theories from classical theories, known as quantization, has a long and difficult history. It starts with the discovery of quantum mechanics in 1925 and the formalization of the quantization procedure by Dirac and von Neumann. The construction of a quantum theory from a given classical one is highly non-trivial and non-unique. But except for few examples, it is the only way which will be gone today. From a physical point of view, the world surround us is the result of an underlying quantum theory of its constituent parts. So, one would expect that we must understand the transition from the quantum to the classical world. But we had developed and tested successfully the classical theories like mechanics or electrodynamics. Therefore one tried to construct the quantum versions out of classical theories. In this paper we will go the other way to obtain a quantum field theory by geometrical methods and to show its equivalence to a quantization of a classical Poisson algebra.

The main technical tool will be the noncommutative geometry developed by Connes [1]. Then intractable space like the leaf space of a foliation can be described
by noncommutative algebras. From the physical point of view, we have now an interpretation of noncommutative algebras (used in quantum theory) in a geometrical context. Here we will use this view to discuss a realization of quantum geometry. Main idea is the usage of a wild embedding (as induced by an exotic $\mathbb{R}^3$), another expression for a fractal space. Then we will discuss the natural appearance of a von Neumann algebra (used as observable algebra in quantum mechanics). A similar relation was found by Etesi [2] based on his previous work [3]. Furthermore we will answer the question whether this algebra is a deformation quantization of a classical (Poisson) algebra in a positive manner. As a direct consequence of this correspondence, we will discuss the large (better infinite) curvature limit of the classical space. This limit agrees with the corresponding limit for the quantum space. In particular, we will obtain a dimensional reduction from 4 to 2 for large curvature. A black hole in this theory admits a non-singular solution with constant curvature.

2. From wild embeddings to fractal spaces
In this section we define wild and tame embeddings and construct a $C^*$-algebra associated to a wild embedding. The example of Aleksandrov’s horned ball is discussed.

2.1. Wild and tame embeddings
We call a map $f : N \to M$ between two topological manifolds an embedding if $N$ and $f(N) \subset M$ are homeomorphic to each other. From the differential-topological point of view, an embedding is a map $f : N \to M$ with injective differential on each point (an immersion) and $N$ is diffeomorphic to $f(N) \subset M$. An embedding $i : N \hookrightarrow M$ is tame if $i(N)$ is represented by a finite polyhedron homeomorphic to $N$. Otherwise we call the embedding wild. There are famous wild embeddings like Aleksandrov’s horned sphere or Antoine’s necklace. In physics one uses mostly tame embeddings but as Cannon mentioned in his overview [4], one needs wild embeddings to understand the tame one. As shown by us [5], wild embeddings are needed to understand exotic smoothness.

2.2. $C^*$-algebras associated to wild embeddings
Let $I : K^n \to \mathbb{R}^{n+k}$ be a wild embedding of codimension $k$ with $k = 0, 1, 2$. In the following we assume that the complement $\mathbb{R}^{n+k} \setminus I(K^n)$ is non-trivial, i.e. $\pi_1(\mathbb{R}^{n+k} \setminus I(K^n)) = \pi \neq 1$. Now we define the $C^*$-algebra $C^*(G, \pi)$ associated to the complement $G = \mathbb{R}^{n+k} \setminus I(K^n)$ with group $\pi = \pi_1(G)$. If $\pi$ is non-trivial then this group is not finitely generated. The construction of wild embeddings is usually given by an infinite construction\footnote{This infinite construction is necessary to obtain an infinite polyhedron, the defining property of a wild embedding.} (see Antoine’s necklace or Aleksandrov’s horned sphere). From an abstract point of view, we have a decomposition of $G$ by an infinite union

$$G = \bigcup_{i=0}^{\infty} C_i$$

of level sets $C_i$. Then every element $\gamma \in \pi$ lies (up to homotopy) in a finite union of levels.
The basic elements of the $C^*$-algebra $C^*(G, \pi)$ are smooth half-densities with compact supports on $G$, $f \in C^\infty_c(G, \Omega_{\gamma^1/2})$, where $\Omega_{\gamma^1/2}$ for $\gamma \in \pi$ is the one-dimensional complex vector space of maps from the exterior power $\Lambda^kL$ (dim $L = k$), of the union of levels $L$ representing $\gamma$, to $\mathbb{C}$ such that
\[
\rho(\lambda \nu) = |\lambda|^{1/2} \rho(\nu) \quad \forall \nu \in \Lambda^2L, \lambda \in \mathbb{R}.
\]
For $f, g \in C^\infty_c(G, \Omega_{\gamma^1/2})$, the convolution product $f \ast g$ is given by the equality
\[
(f \ast g)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1)g(\gamma_2)
\]
with the group operation $\gamma_1 \circ \gamma_2$ in $\pi$. Then we define via $f^* (\gamma) = \hat{f}(\gamma^{-1})$ a $*$-operation making $C^\infty_c(G, \Omega_{\gamma^1/2})$ into a $*$-algebra. Each level set $C_i$ consists of simple pieces (in case of Alexanders horned sphere, we will explain it below) denoted by $T$. For these pieces, one has a natural representation of $C^\infty_c(G, \Omega_{\gamma^1/2})$ on the $L^2$ space over $T$. Then one defines the representation
\[
(\pi_x(f) \xi)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1) \xi(\gamma_2) \quad \forall \xi \in L^2(T), \forall x \in \gamma.
\]

The completion of $C^\infty_c(G, \Omega_{\gamma^1/2})$ with respect to the norm
\[
||f|| = \sup \{||\pi_x(f)|| \}
\]
makes it into a $C^*$-algebra $C^\infty_c(G, \pi)$. Finally we are able to define the $C^*$-algebra associated to the wild embedding:

Let $j : K \to S^n$ be a wild embedding with $\pi = \pi_1(S^n \setminus j(K))$ as fundamental group of the complement $M(K, j) = S^n \setminus j(K)$. The $C^*$-algebra $C^\infty_c(K, j)$ associated to the wild embedding is defined to be $C^\infty_c(K, j) = C^\infty_c(G, \pi)$ the $C^*$-algebra of the complement $G = S^n \setminus j(K)$ with group $\pi$.

In [6] we considered the example of Alexanders horned ball $A$ as fractal space [7]. For this example, the group $\pi_1(S^3 \setminus A)$ is a locally free group of infinite rank (and perfect). But the last property implies that this group has the infinite conjugacy class property (icc), i.e. only the identity element has a finite conjugacy class. This property has a tremendous impact on the $C^*$-algebra [8] and its enveloping von Neumann algebra: The enveloping von Neumann algebra $W(C, \pi_1(S^3 \setminus A))$ of the $C^*$-algebra
\[
C^\infty_c(C, \pi_1(S^3 \setminus A))
\]
for the wild embedding $A$ is the hyperfinite factor $II_1$ algebra.
3. Small exotic $\mathbb{R}^4$

The distinguished feature of differential topology of manifolds in dimension 4 is the existence of open 4-manifolds carrying a plenty of non-diffeomorphic smooth structures. In the paper, the special role is played by the topologically simplest 4-manifold, i.e. $\mathbb{R}^4$, which carries a continuum of infinitely many different smoothness structures. Each of them except one, the standard $\mathbb{R}^4$, is called exotic $\mathbb{R}^4$. All exotic $\mathbb{R}^4$ are Riemannian smooth open 4-manifolds homeomorphic to $\mathbb{R}^4$ but non-diffeomorphic to the standard smooth $\mathbb{R}^4$. The standard smoothness is distinguished by the requirement that the topological product $\mathbb{R} \times \mathbb{R}^3$ is a smooth product. There exists only one (up to diffeomorphisms) smoothing, the standard $\mathbb{R}^4$, where the product above is smooth. In the following, an exotic $\mathbb{R}^4$, presumably small if not stated differently, will be denoted as $R^4$.

There are canonical 4-manifolds into which some exotic $R^4$ are embeddable. Here we will use the defining property of small exotic $R^4$: every small exotic $R^4$ is embeddable in the standard $\mathbb{R}^4$ (or in $S^4$). One of the characterizing properties of an exotic $R^4$, which is present in all known examples, is the existence of a compact subset $K \subset R^4$ which cannot be surrounded by any smoothly embedded 3-sphere (and homology 3-sphere bounding a contractible, smooth 4-manifold), see sec. 9.4 in [9] or [10]. The topology of this subset $K$ depends strongly on the $R^4$. Let $R^4$ be the standard $\mathbb{R}^4$ (i.e. $\mathbb{R}^3 \times \mathbb{R}$ smoothly) and let $R^4$ be a small exotic $R^4$ with compact subset $K \subset R^4$ which cannot be surrounded by a smoothly embedded 3-sphere. So, we have the strange situation that an open subset of the standard $\mathbb{R}^4$ represents a small exotic $R^4$.

Now we will describe the construction of this exotic $R^4$. Historically it emerged as a counterexample of the smooth h-cobordism theorem [11, 12]. The compact subset $K$ as above is given by a non-canceling 1-/2-handle pair. Then, the attachment of a Casson handle $CH$ cancels this pair only topologically. A Casson handle is a 4-dimensional topological 2-handle constructed by an infinite procedure. In this process one uses disks with self-intersections (so-called kinky handles) and arrange them along a tree $T_{CH}$: every vertex of the tree is the kinky handle and the number of branches in the tree are the number of self-intersections. Freedman [13] was able to show that every Casson handle is topologically the standard open 2-handle $D^2 \times \mathbb{R}^2$. As the result to attach the Casson handle $CH$ to the subset $K$, one obtains the topological 4-disk $D^4$ with interior $R^4$ over the 1-/2-handle pair was canceled topologically. The 1/2-handle pair cannot cancel smoothly and a small exotic $R^4$ must emerge after gluing the $CH$. It is represented schematically as $R^4 = K \cup CH$. Recall that $R^4$ is a small exotic $R^4$, i.e. $R^4$ is embedded into the standard $R^4$, and the completion $\overline{R^4}$ of $R^4 \subset R^4$ has a boundary given by certain 3-manifold $Y_r$. One can construct $Y_r$ directly as the limit $n \to \infty$ of the sequence $\{Y_n\}$ of some 3-manifolds $Y_n$, $n = 1, 2, \ldots$. Then the entire sequence of 3-manifolds

$$Y_1 \to Y_2 \to \cdots \to Y_\infty = Y_r$$

characterizes the exotic smoothness structure of $R^4$. Every $Y_n$ is embedded in $R^4$ and into $\mathbb{R}^4$. An embedding is a map $i : Y_n \hookrightarrow \mathbb{R}^4$ so that $i(Y_n)$ is diffeomorphic to $Y_n$. Usually, the image $i(Y_n)$ represents a manifold which is given by a finite number of polyhedra (seen as triangulation of $Y_n$). Such an embedding is tame. In contrast, the limit of this sequence $n \to \infty$ gives an embedded 3-manifold $Y_r$ which must be covered by an infinite
number of polyhedra. Then, $Y_r$ is called a wild embedded 3-manifold (see above). By the work of Freedman [13], every Casson handle is topologically $D^2 \times \mathbb{R}^2$ (relative to the attaching region) and therefore $Y_r$ must be the boundary of $D^4$ (the Casson handle trivializes $K$ to be $D^4$), i.e. $Y_r$ is a wild embedded 3-sphere $S^3$. $Y_1$ was described as the boundary of the compact subset $K$ whereas $Y_n$ is given by $0$–framed surgeries along $n$th untwisted Whitehead double of the pretzel knot $9_{46}$. Thus we have a sequence of inclusions

$$\cdots \subset Y_{n-1} \subset Y_n \subset Y_{n+1} \subset \cdots \subset Y_{\infty}$$

with the 3-manifold $Y_{\infty}$ as limit. Let $K_+$ be the corresponding (wild) knot, i.e. the $\infty$th untwisted Whitehead double of the pretzel knot $(-3,3,-3)$ ($9_{46}$ knot in Rolfsen notation). The surgery description of $Y_{\infty}$ induces the decomposition

$$Y_{\infty} = C(K_+) \cup (D^2 \times S^1) \quad C(K_+) = S^3 \setminus (K_+ \times D^2) \quad (1)$$

where $C(K_+)$ is the knot complement of $K_+$. In [14], the splitting of the knot complement was described. Let $K_{9_{46}}$ be the pretzel knot $(-3,3,-3)$ and let $L_{Wh}$ be the Whitehead link (with two components). Then the complement $C(K_{9_{46}})$ has one torus boundary whereas the complement $C(L_{Wh})$ has two torus boundaries. Now according to [14], one obtains the splitting

$$C(K_+) = C(L_{Wh}) \cup T^2 \cdots \cup T^2 \quad C(L_{Wh}) \cup T^2 \quad C(K_{9_{46}}).$$

By general arguments (see [15, 16]) the complement $C(K_+)$ admits a hyperbolic structure, i.e. it is a homogenous space of constant negative curvature. Therefore we obtained the first condition: the sequence of 3-manifolds $Y_1 \to \cdots \to Y_r$ is geometrically a sequence of hyperbolic 3-manifolds! By the same argument, we can also state: The enveloping von Neumann algebra $W(Y_r, \pi_1(C(K_+)))$ of the $C^*$-algebra

$$C^\infty_c(Y_r, \pi_1(C(K_+)))$$

for the wild embedding 3-sphere is the hyperfinite factor $II_1$ algebra.

4. Quantum states from wild embeddings

In this section we will describe a way from a (classical) Poisson algebra to a quantum algebra by using deformation quantization. Therefore we will obtain a positive answer to the question: Does the $C^*$–algebra of a wild (specific) embedding comes from a (deformation) quantization? Of course, this question cannot be answered in most generality, i.e. we use the decomposition of the small exotic $R^4$ into the sequence $Y_1 \to \cdots \to Y_r$. But for this example we will show that the enveloping von Neumann algebra of this wild embedding (wild 3-sphere $Y_r$) is the result of a deformation quantization using the classical Poisson algebra (of closed curves) of the tame embedding. This result shows two things: the wild embedding can be seen as a quantum state and the classical state is a tame embedding. This result was confirmed for another case in [6] so that we will briefly list the relevant results (Turaev-Drinfeld quantization):

- The sequence $Y_1 \to \cdots \to Y_r$ is a sequence of hyperbolic 3-manifolds.
The hyperbolic structure is defined by a homomorphism $\pi_1(Y_i) \to SL(2, \mathbb{C})$ \((\in Hom(\pi_1(Y_i), SL(2, \mathbb{C}))$$) up to conjugation.

Inside of very $Y_i$, there is a special surface $S$ (incompressible surface) inducing a representation $\pi_1(S) \to SL(2, \mathbb{C})$.

The space of all representations $X(S, SL(2, \mathbb{C})) = \text{Hom}(\pi_1(S), SL(2, \mathbb{C})) / SL(2, \mathbb{C})$ has a natural Poisson structure (induced by the bilinear on the group) and the Poisson algebra $(X(S, SL(2, \mathbb{C})), \{, \})$ of complex functions over them is the algebra of observables.

The skein module $K_{-1}(S \times [0, 1])$ (i.e. $t = -1$) has the structure of an algebra isomorphic to the Poisson algebra $(X(S, SL(2, \mathbb{C})), \{, \})$. (see also [17, 18]).

The skein algebra $K_t(S \times [0, 1])$ is the quantization of the Poisson algebra $(X(S, SL(2, \mathbb{C})), \{, \})$ with the deformation parameter $t = \exp(h/4)$. (see also [17]).

To understand these statements we have to introduce the skein module $K_t(M)$ of a 3-manifold $M$ (see [19]). For that purpose we consider the set of links $\mathcal{L}(M)$ in $M$ up to isotopy and construct the vector space $\mathbb{C}\mathcal{L}(M)$ with basis $\mathcal{L}(M)$. Then one can define $\mathbb{C}\mathcal{L}[[t]]$ as ring of formal polynomials having coefficients in $\mathbb{C}\mathcal{L}(M)$. Now we consider the link diagram of a link, i.e. the projection of the link to the $\mathbb{R}^2$ having the crossings in mind. Choosing a disk in $\mathbb{R}^2$ so that one crossing is inside this disk. If the three links differ by the three crossings $L_{oo}, L_o, L_{oo}$ (see figure 1) inside of the disk then these links are skein related. Then in $\mathbb{C}\mathcal{L}[[t]]$ one writes the skein relation $2L_{oo} - tL_o - t^{-1}L_{oo}$. Furthermore let $L \cup O$ be the disjoint union of the link with a circle then one writes the framing relation $L \cup O + (t^2 + t^{-2})L$. Let $S(M)$ be the smallest submodule of $\mathbb{C}\mathcal{L}[[t]]$ containing both relations, then we define the Kauffman bracket skein module by $K_t(M) = \mathbb{C}\mathcal{L}[[t]] / S(M)$.

We list the following general results about this module:

- The module $K_{-1}(M)$ for $t = -1$ is a commutative algebra.
- Let $S$ be a surface then $K_t(S \times [0, 1])$ carries the structure of an algebra.

The algebra structure of $K_t(S \times [0, 1])$ can be simple seen by using the diffeomorphism between the sum $S \times [0, 1] \cup S \times [0, 1]$ along $S$ and $S \times [0, 1]$. Then the product $ab$

\[2 \text{ The relation depends on the group } SL(2, \mathbb{C}).]
of two elements $a, b \in K_i(S \times [0, 1])$ is a link in $S \times [0, 1] \cup_{S} S \times [0, 1]$ corresponding to a link in $S \times [0, 1]$ via the diffeomorphism. The algebra $K_i(S \times [0, 1])$ is in general non-commutative for $t \neq -1$. For the following we will omit the interval $[0, 1]$ and denote the skein algebra by $K_i(S)$.

Now we will present the relation between skein spaces and wild embeddings (in particular to its $C^*$-algebra). For that purpose we will concentrate on the wild embedding $i : S^3 \to R^4$ of $Y_r$, the wild 3-sphere. We will explain now, that the complement $S^3 \setminus i(D^2 \times [0, 1])$ and its fundamental group $\pi_1(S^3 \setminus i(D^2 \times [0, 1]))$ can be described by closed curves around tubes (or annulus) $S^3 \times [0, 1]$.

Let $C$ be the image $C = i(D^2 \times [0, 1])$ decomposed into components $C_i$ so that $C = \cup_i C_i$. Furthermore, let $C_i$ be the decomposition of $i(D^2 \times [0, 1])$ at $i$th level (i.e. a union of $D^2 \times [0, 1]$). The complement $S^3 \setminus C_i$ of $C_i$ with $n_i$ components (i.e. $C_i = \cup_{i}^{n_i} (D^2 \times [0, 1])$) has the same (isomorphic) fundamental group like $\pi_1(\cup_{i}^{n_i} (S^3 \times [0, 1]))$ of $n_i$ components of $S^3 \times [0, 1]$. Therefore, instead of studying the complement we can directly consider the annulus $S^3 \times [0, 1]$ replacing every $D^2 \times [0, 1]$ component.

Let $C'$ be the boundary of $C$, i.e. in every component we have to replace every $D^2 \times [0, 1]$ by $S^1 \times [0, 1]$. The skein space $K_i(S^1 \times [0, 1])$ is a polynomial algebra (see the previous subsection) $\mathbb{C}[[\alpha]]$ in one generator $\alpha$ (a closed curve around the annulus). Let $TL_n$ be the Temperley-Lieb algebra, i.e. a complex $*$-algebra generated by $\{e_1, \ldots, e_n\}$ with the relations

$$e_i^2 = \tau e_i, \quad e_ie_j = e_je_i : |i - j| > 1,$$

and the real number $\tau$. If $\tau$ is the number $\tau = a_0^2 + a_0^{-2}$ with $a_0$ a $4n$th root of unity ($a_0^{4k} \neq 1$ for $k = 1, \ldots, n - 1$) then there is an element $f^{(n)}$ with

$$f^{(n)} A_n = A_n f^{(n)} = 0$$

$$1_n - f^{(n)} \in A_n$$

$$f^{(n)} f^{(n)} = f^{(n)}$$

in $A_n \subset TL_n$ (a subalgebra generated of $\{e_1, \ldots, e_n\}$ missing the identity $1_n$), called the Jones-Wenzl idempotent. The closure of the element $f^{(n+1)}$ in $TL_{n+1} + K_i(S^1 \times [0, 1])$ is given by the image of the map $TL_{n+1} \to K_i(S^1 \times [0, 1])$ which maps $f^{(n+1)}$ to some polynomial $S_{n+1}(\alpha)$ in the generator $\alpha$ of $K_i(S^1 \times [0, 1])$. Therefore we obtain a relation between the generator $\alpha$ and the element $f^{(n)}$ for some $n$.

The wilderness of $Y_r$ is given by a decomposition of $D^2 \times [0, 1]$ into an infinite union of $(D^2 \times [0, 1])$—components $C_i$ (in the notation above). But then we have an infinite fundamental group where every generator is represented by a curve around one $(D^2 \times [0, 1])$—components $C_i$. This decomposition can be represented by a decomposition of a square (as substitute for $D^2$) into (countable) infinite rectangles. Every closed curve surrounding $C_i$ is a pair of opposite points at the boundary, the starting point of the curve and one passing point (to identify the component). Every $C_i$ gives one pair of points. Motivated by the discussion above, we consider the skein algebra $K_i(D^2, 2n)$ with $2n$ marked points (representing $n$ components). This algebra is isomorphic (see [19]) to...
the Temperley-Lieb algebra $TL_n$. As Jones [20] showed: the limit case $\lim_{n \to \infty} TL_n$ (considered as direct limit) is the factor $II_1$. Thus we have constructed the factor $II_1$ algebra as skein algebra.

Therefore we have shown that the enveloping von Neumann algebra

$$W(C, \pi_1(C(K_+)))$$

(=the hyperfinite factor $II_1$ algebra) is obtained by deformation quantization of a classical Poisson algebra (the tame embedding). But then, a wild embedding can be seen as a quantum state.

5. Morgan-Shalen compactification and 2D Einstein-Hilbert action

Above we considered the space $X(Y_i, SL(2, \mathbb{C})$ of hyperbolic structures on the 3-manifold $Y_i$ now denoted by $\mathcal{M}$ depending on $\pi_1 = \pi_1(Y_i)$. Let $\rho : \pi_1 \to SL(2, \mathbb{C})$ be one representation. The character is defined by $\chi_\rho(\gamma) = Tr(\rho(\gamma))$ for a $\gamma \in \pi_1$. The set of all characters forms an algebraic variety which is equivalent to $\mathcal{M}$. By the Ambrose-Singer theorem, the characters (or holonomies in the 3-manifold) are an expression of the curvature of the 3-manifold. Now we will discuss what happens for large curvatures or we will discuss the compactification of the space $\mathcal{M}$. Morgan and Shalen [21] studied a compactification of this space or better they determined the structure of the divergent signals. The compactification $\overline{\mathcal{M}}$ is defined as follows: let $C$ be the set of conjugacy classes of $\Gamma = \pi_1(\mathcal{N})$, and let $\mathbb{P}(C) = \mathbb{P}(\mathbb{R}^C)$ be the (real) projective space of non-zero, positive functions on $C$. Define the map $\vartheta : \mathcal{M} \to \mathbb{P}(C)$ by

$$\vartheta(\rho) = \{ \log(|\chi_\rho(\gamma)| + 2) \mid \gamma \in C \}$$

and let $\mathcal{M}^+$ denote the one point compactification of $\mathcal{M}$ with the inclusion map $\iota : \mathcal{M} \to \mathcal{M}^+$. Finally, $\overline{\mathcal{M}}$ is defined to be the closure of the embedded image of $\mathcal{M}$ in $\mathcal{M} \times \mathbb{P}(C)$ by the map $\iota \times \vartheta$. It is proved in [21] that $\mathcal{M}$ is compact and that the boundary points consist of projective length functions on $\Gamma$ (see below for the definition). Note that in its definition, $\vartheta(\rho)$ could be replaced by the function $\{ \ell_\rho(\gamma) \}_{\gamma \in C}$, where $\ell_\rho$ denotes the translation length for the action of $\rho(\gamma)$ on $\mathbb{H}^3$ (3D hyperbolic space)

$$\ell_\rho(\gamma) = \inf \left\{ dist_{\mathbb{H}^3}(x, \rho(\gamma)x) \mid x \in \mathbb{H}^3 \right\}$$

where $dist_{\mathbb{H}^3}$ denotes the (standard) distance in the 3D hyperbolic space $\mathbb{H}^3$.

Recall that an $\mathbb{R}$-tree is a metric space $(T, d_T)$ such that any two points $x, y \in T$ are connected by a segment $[x, y]$, i.e. a rectifiable arc isometric to a compact (possibly degenerate) interval in $\mathbb{R}$ whose length realizes $d_T(x, y)$, and that $[x, y]$ is the unique embedded path from $x$ to $y$. We say that $x \in T$ is an edge point (resp. vertex) if $T \setminus \{x\}$ has two (resp. more than two) components. A $\Gamma$-tree is an $\mathbb{R}$-tree with an action of $\Gamma$ by isometries, and it is called minimal if there is no proper $\Gamma$-invariant subtree. We say that $\Gamma$ fixes an end of $T$ (or more simply, that $T$ has a fixed end) if there is a ray $R \subset T$ such that for every $\gamma \in \Gamma$, $\gamma(R) \cap R$ is a subray. Given an $\mathbb{R}$-tree $(T, d_T)$, the associated length function $\ell_T : \Gamma \to \mathbb{R}^+$ is defined by

$$\ell_T(\gamma) = \inf_{x \in T} d_T(x, \gamma x)$$
If $\ell_T \neq 0$, which is equivalent to $\Gamma$ having no fixed point in $T$ (cf. [21, 22], Prop. II.2.15), then the class of $\ell_T$ in $P(C)$ is called a projective length function.

Now we are able to formulate the main result:

If $\rho_k \in \mathcal{M}$ is an unbounded sequence, then there exist constants $\lambda_k \to \infty$ (renormalization of the sequence) so that the rescaled length

$$\frac{1}{\lambda_k} \ell_{\rho_k}$$

converge to $\ell_{\rho_\infty} : \Gamma \to Isom(T)$ a representation of $\Gamma$ in the isometry group of the $\mathbb{R}$–tree $T$, i.e. we have the convergence

$$\frac{1}{\lambda_k} \ell_{\rho_k} \Rightarrow \ell_T$$

But what is the meaning of this result? For infinite curvatures, the underlying 3-manifolds degenerate into a tree which agrees with the tree of the wild 3-sphere $Y_r$. To express it differently, trees can be seen as hyperbolic spaces of infinite curvature. This result remained true if we consider the spacetime $Y_r \times [0, 1]$ with a Lorentz structure given by a homomorphism $\pi_1(Y_r \times [0, 1]) \to SL(2, \mathbb{C})$. But then we will obtain a dimensional reduction from $3 + 1$ to $1 + 1$ with the corresponding reduction of the Einstein-Hilbert action of the spacetime $S$

$$\int_S d^2x \sqrt{-g} e^{-2\phi} \left(R + 2(\partial\phi)^2 + 4\lambda^2\right)$$

admitting black hole solutions with no singularity [23]. But how is it related to the quantization via the skein algebra? There is a relation between the Kauffman bracket skein algebra and lattice gauge theory [24]. Again the curvature is related to the holonomies in the lattice and the hyperbolic geometry (as defined by $SL(2, \mathbb{C})$) deformed the usual (Euclidean lattice) to a hyperbolic space. In the limit of infinite curvature we will obtain a tree again, so meeting our result by using the Morgan-Shalen compactification. Expressed differently, the large curvature limit agrees with the quantum description.

6. Conclusion
In this paper we discuss a spacetime representation of a quantum geometry by a fractal space as given by a wild embedding $Y_r$. This wild embedding can be understand as a deformation quantization of a classical state (Poisson algebra). The state will be formed by equivalence classes (skein algebra) of knots (as a basis). The whole construction is consistent with the large curvature limit where the curvature goes to infinity. For this case the underlying space degenerates to a tree (or the spacetime is $1 + 1$ dimensional). By general arguments, this limits agrees with the corresponding limit for the quantum regime. A black hole in this theory has no singularity but constant curvature. The details of the construction will be discussed in forthcoming work.
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